MAPS BETWEEN LOCAL PICARD GROUPS

JÁNOS KOLLÁR

Let $X$ be a scheme and $x \in X$ a point. The local Picard group of $X$ at $x$, denoted by $\text{Pic}^{\text{loc}}(x, X)$, is the Picard group of the punctured neighborhood $\text{Spec}_X \mathcal{O}_{x, X} \setminus \{x\}$. Our aim is to study the pull-back map on the local Picard group in two situations.

**Question 1** (Normalization). Let $\pi : \bar{X} \to X$ denote the normalization and $\bar{x}_i \in \bar{X}$ the preimages of $x$. What is the kernel of the pull-back map $\pi^* : \text{Pic}^{\text{loc}}(x, X) \to \sum_i \text{Pic}^{\text{loc}}(\bar{x}_i, \bar{X})$?

**Question 2** (Restriction to a divisor). Let $x \in D \subset X$ be an effective Cartier divisor. What is the kernel of the restriction map $r^*_X : \text{Pic}^{\text{loc}}(x, X) \to \text{Pic}^{\text{loc}}(x, D)$?

In both cases we are interested in conditions that guarantee that the kernels of these maps are “small.” Here “small” can mean trivial, or finite or “naturally” a subgroup of a linear algebraic group.

Grothendieck’s local Lefschetz-type theorem [Gro68, XI.3.16] is the first major result on Question 2. He proves that $r^*_D$ is injective if $\text{depth}_x X \geq 4$. Conjecture 1.2 in [Kol13a] asserts that this can be relaxed to $\text{depth}_x X \geq 3$ and $\text{dim} X \geq 4$. This was proved in [Kol13a] when $X$ is log canonical and in [BdJ13] when $X$ is normal.

I was led to Question 1 while investigating stable varieties and their moduli. The method of [Kol13b] studies first the normalization $\bar{X}$ of a stable variety $X$ and then descends the information from $\bar{X}$ to $X$. It is especially important to know when the canonical divisor of $X$ is $\mathbb{Q}$-Cartier; this was settled in [Kol11]. For semi-log canonical varieties Question 1 is answered in [Kol13b, Sec.5.7].

In many applications, the natural assumption is that $X$ satisfies Serre’s condition $S_2$ only. To understand this case, one needs to know that, as in the global situation, the local Picard group is not just a group but a group scheme. The local definition is more involved than the projective one and the natural setting turns out to be the following.

**Condition 3.** $X$ is a scheme that is essentially of finite type over a field $k$, $x \subset X$ is a 0-dimensional closed subscheme, $X$ satisfies Serre’s condition $S_2$ and has pure dimension $\geq 3$ (that is, every associated prime of $\mathcal{O}_X$ has the same dimension $\geq 3$).

[Bou78] defines a local Picard functor $\text{Pic}^{\text{loc}}(x, X)$ and proves that, under the above assumptions, it is represented by the local Picard scheme $\text{Pic}^{\text{loc}}(x, X)$; see Definition 21 and [Bou78, Thm.1.ii] for details. Frequently $\text{Pic}^{\text{loc}}(x, X)$ has infinitely many connected components. The identity component, usually denoted by $\text{Pic}^{\text{loc-\tau}}(x, X)$, is a commutative, connected algebraic group of finite type but it need not be a linear algebraic group. The union of those components of $\text{Pic}^{\text{loc}}(x, X)$ that become torsion modulo $\text{Pic}^{\text{loc-\tau}}(x, X)$ is denoted by $\text{Pic}^{\text{loc-\tau}}(x, X)$. 
The following theorem gives a unified answer to both questions.

**Theorem 4.** Let $X$ and $Y$ be schemes that satisfy Condition 3 and $\pi : Y \to X$ a finite morphism. Let $x \subset X$ be a 0-dimensional closed subscheme, $y \subset Y$ the preimage of $x$ and $r^Y_X : \text{Pic}^{\text{loc}}(x, X) \to \text{Pic}^{\text{loc}}(y, Y)$ the natural pull-back map.

Assume that $\pi(Y)$ contains the support of a complete intersection subscheme $x \subset Z \subset X$ of dimension $\geq 3$. Then

$$(r^Y_X)^{-1} \text{Pic}^{\text{loc}-\tau}(y, Y) = \text{Pic}^{\text{loc}-\tau}(x, X).$$

5 (Local Néron–Severi groups). One can restate the above result in terms of the local Néron–Severi groups

$$\text{NS}^{\text{loc}}(x, X) := \text{Pic}^{\text{loc}}(x, X)/\text{Pic}^{\text{loc}-\tau}(x, X).$$

The local Néron–Severi group is a finitely generated abelian group if $\text{char } k = 0$ but only the prime-to-$p$ part is known to be finitely generated if $\text{char } k = p > 0$; see Definition 22 for details. Theorem 4 is equivalent to saying that the kernel of the natural pull-back map $\text{NS}^{\text{loc}}(x, X) \to \text{NS}^{\text{loc}}(y, Y)$ is torsion.

The global variant of this was proved in [Kle66, p.305].

Theorem 4 implies that $\ker r^Y_X \subset \text{Pic}^{\text{loc}-\tau}(x, X)$ but for both Questions considered above we can say more. Our answers are optimal in characteristic 0 but in characteristic $p > 0$ I have not been able to exclude a possibly infinite, discrete, $p\infty$-torsion quotient. The precise statements are the following.

**Theorem 6.** Assume that $(x, X)$ satisfies Condition 3 in particular $\dim X \geq 3$. Let $\pi : Y \to X$ be a finite surjection and $y \subset Y$ the preimage of $x$. Then the kernel of the pull-back map $\pi^* : \text{Pic}^{\text{loc}}(x, X) \to \text{Pic}^{\text{loc}}(y, Y)$ is

1. a linear algebraic group if $\text{char } k = 0$ and
2. the extension of a discrete $p\infty$-torsion group by a linear algebraic group if $\text{char } k = p > 0$.

I do not know any example where the kernel of $\pi^*$ is not a linear algebraic group.

We have to be careful with the scheme-theoretic formulation of Question 2 since $D \subset X$ need not be $S_2$, thus $\text{Pic}^{\text{loc}}(x, D)$ need not exist. Nonetheless for now I use the suggestive notation $r^X_D : \text{Pic}^{\text{loc}}(x, X) \to \text{Pic}^{\text{loc}}(x, D)$; see Definition 21 for its precise meaning.

**Theorem 7.** Assume that $(x, X)$ satisfies Condition 3 and $\dim X \geq 4$. Let $x \subset D \subset X$ be an effective Cartier divisor. Then the kernel of the restriction map $r^X_D : \text{Pic}^{\text{loc}}(x, X) \to \text{Pic}^{\text{loc}}(x, D)$ is

1. a unipotent algebraic group if $\text{char } k = 0$ or $X$ is normal and
2. an extension of a discrete $p\infty$-torsion group by a unipotent algebraic group if $\text{char } k = p > 0$.

As before, I do not know any example where the kernel of $r^X_D$ is not unipotent.

**Remark 8.** In both cases the dimension restrictions are optimal. For example, for the non-normal surface $S := (xyz = 0) \subset \mathbb{A}^3$ with normalization $\pi : \bar{S} \to S$ we have

$$\ker [\text{Pic}^{\text{loc}}(0, S) \xrightarrow{\pi^*} \text{Pic}^{\text{loc}}(\bar{0}, \bar{S})] \cong \mathbb{Z}^3.$$
For the ordinary 3-fold node $X := (x^2 + y^2 + z^2 + t^2 = 0) \subset \mathbb{A}^4$ and $D := (t = 0)$ we have

$$\ker [\text{Pic}^\text{loc}(x, X) \xrightarrow{r^X_D} \text{Pic}^\text{loc}(x, D)] \cong \mathbb{Z}.$$  

There are also many instances where the kernels in $(6.1)$ and $(7.1)$ are positive dimensional; see Examples $[26]$ $[30]$ $[\text{Kol13a} \text{ Exmp.12}]$ and $[\text{BdJ13 Exms.1.31–35}]$. It is also important to assume in $(7)$ that $D$ is a Cartier divisor. For instance, let $H \subset \mathbb{P}^n$ be a hyperplane and $P \subset \mathbb{P}^n \setminus H$ a finite set of points. Let $(v, X)$ be a cone over the blow-up $B_P \mathbb{P}^n$ with vertex $v$ and $v \in D \subset X$ the cone over $H$. Then

$$\ker [\text{Pic}^\text{loc}(v, X) \xrightarrow{r^X_p} \text{Pic}^\text{loc}(v, D)] \cong \mathbb{Z}^{|P|}.$$  

If $\text{depth}_{x, X} \geq 3$ then $\text{Pic}^\text{loc-}\sigma(x, X)$ is trivial and we get the following results.

**Theorem 9.** Notation and assumptions as in Theorems $[26]$ $[27]$ Assume in addition that $\text{depth}_{x, X} \geq 3$. Then

1. $r^X_D : \text{Pic}^\text{loc}(x, X) \rightarrow \text{Pic}^\text{loc}(x, D)$ is injective,
2. $\ker(\pi^*)$ is finite if $\text{char} \ k = 0$ and
3. the prime-to-$p$ part of $\ker(\pi^*)$ is finite if $\text{char} \ k = p > 0$.

**Theorem 11.** Using the above notation, assume in addition that $D_t := D|_{X_t}$ is Cartier for every $t \in T$. Then $D$ is a Cartier divisor on $X$ iff the self-intersection number $(D_t^3)$ is independent of $t \in T$.

**12 (The complex analytic case).** Even if $k = \mathbb{C}$, the proofs of Theorems $[26]$ $[27]$ proceed by reduction to positive characteristic. There are non-isolated complex analytic singularities that do not lie on any algebraic variety. Our proof does not apply to them, but the conclusions are hopefully valid.

Over $\mathbb{C}$ one can use the first Chern class to realize $\text{NS}^\text{loc}(x, X)$ as a subgroup of $H^2(\text{link}(x, X), \mathbb{Z})$, the second cohomology of the link of $(x, X)$, see $[31]$. Thus it would be natural to try to prove the analytic case by showing that the kernels of the corresponding maps between these cohomology groups are torsion. This is, however, not true.

Example $[32]$ shows a semi-log canonical hypersurface singularity $(x, X)$ of dimension 3 whose normalization $\pi : (\tilde{x}, \tilde{X}) \rightarrow (x, X)$ is also a hypersurface singularity and

$$\ker [\pi^* : H^2(\text{link}(x, X), \mathbb{Z}) \rightarrow H^2(\text{link}(\tilde{x}, \tilde{X}), \mathbb{Z})] \cong \mathbb{Z}^2.$$  

For Theorem $[7]$ similar examples are given in $[\text{Kol13a} \text{ Sec.5}]$. 

10 (Numerical criteria for Cartier divisors). Let $T$ be an irreducible, regular, 1-dimensional scheme and $f : X \rightarrow T$ a flat, projective morphism of relative dimension $n$. Assume for simplicity that $f$ has normal fibers.

Let $D$ be a divisor on $X$ such that $D_t := D|_{X_t}$ is Cartier for every $t \in T$. In general $D$ need not be Cartier. For example, let

$$X = (x^2 - y^2 + z^2 - t^2 = 0) \subset \mathbb{A}^3_{xyz} \times \mathbb{A}^1_t \quad \text{and} \quad D = (x - y = z - t = 0).$$

$D$ is Cartier, except at the origin, where it is not even $\mathbb{Q}$-Cartier. However $D_0$ is a line on a quadric cone, hence $2D_0 = (x - y = 0)$ is Cartier. Thus $2D$ is Cartier on every fiber but it is not Cartier.

We discuss several criteria in Section $[11]$. The following is an easy-to-state special case of Theorem $[39]$. 

**Theorem 11.** Using the above notation, assume in addition that $D_t$ is ample for every $t \in T$. Then $D$ is a Cartier divisor on $X$ iff the self-intersection number $(D_t^3)$ is independent of $t \in T$. 

12 (The complex analytic case). Even if $k = \mathbb{C}$, the proofs of Theorems $[26]$ $[27]$ proceed by reduction to positive characteristic. There are non-isolated complex analytic singularities that do not lie on any algebraic variety. Our proof does not apply to them, but the conclusions are hopefully valid. 

Over $\mathbb{C}$ one can use the first Chern class to realize $\text{NS}^\text{loc}(x, X)$ as a subgroup of $H^2(\text{link}(x, X), \mathbb{Z})$, the second cohomology of the link of $(x, X)$, see $[31]$. Thus it would be natural to try to prove the analytic case by showing that the kernels of the corresponding maps between these cohomology groups are torsion. This is, however, not true. 

Example $[32]$ shows a semi-log canonical hypersurface singularity $(x, X)$ of dimension 3 whose normalization $\pi : (\tilde{x}, \tilde{X}) \rightarrow (x, X)$ is also a hypersurface singularity and

$$\ker [\pi^* : H^2(\text{link}(x, X), \mathbb{Z}) \rightarrow H^2(\text{link}(\tilde{x}, \tilde{X}), \mathbb{Z})] \cong \mathbb{Z}^2.$$  

For Theorem $[7]$ similar examples are given in $[\text{Kol13a} \text{ Sec.5}]$. 

1. Outline of the proofs

The proofs of Theorems 4–9 have 4 major components.

- The proof of Theorem 7 when $X$ is normal and $\text{char } k > 0$. This was done in [BdJ13].
- The proof of Theorem 6 over finite fields. This is done in Section 5. These two together imply Theorem 4 over finite fields.
- A lifting argument that derives the general assertions from the finite field cases. This is done in Sections 6–9.
- Dealing with torsion elements in $\text{ker } r_X$. The nonexistence of prime-to-$p$ torsion follows from [Gro68, XIII.2.1]; see Paragraph 17. The $p^\infty$-torsion is excluded in Section 10 using a global argument.

The key to the results is to find a good answer to the following.

Question 13. Let $(x, X)$ be a local scheme and $L$ a line bundle on $X \setminus \{x\}$. How can one check if $L$ is in $\text{Pic}^\tau_{\text{loc}}(x, X)$?

In the global case, when $Y$ is a proper scheme over a field $k$, there is a simple numerical criterion: a line bundle $L$ is in $\text{Pic}^\tau(Y)$ iff $L$ has degree 0 on every reduced, irreducible curve $C \subset Y$.

For the local Picard group there are no proper curves to work with and I do not know any similar numerical criterion to identify $\text{Pic}^\tau_{\text{loc}}(x, X)$ in general. (If $X$ is normal and it has a resolution of singularities $X' \to X$ then one can work on $X'$ and use the exceptional curves. This was used in [Bou78] to prove that $\text{NS}^\tau_{\text{loc}}(x, X)$ is finitely generated.)

Over a finite field the $\mathbb{F}_q$-points of a finite type group scheme form a finite group, which gives the following.

Claim 14. Let $(x, X)$ be a local scheme over a finite field $\mathbb{F}_q$ that satisfies Condition $\mathfrak{3}$ and $L$ a line bundle on $X \setminus \{x\}$. Then $L \in \text{Pic}^\tau_{\text{loc}}(x, X)$ iff $L$ is torsion.

This leads to a somewhat roundabout way of proving that a line bundle $L$ is in the connected component of a Picard group:

- reduce everything to finite fields,
- check that we get torsion line bundles and
- lift back to the original setting.

The last step is the critical one; let us see it in more detail for local schemes $(x, X)$ of finite type over a field $k$. We may assume that $k$ is finitely generated over its prime field, thus we can view $k$ as the function field of an integral scheme $S$ that is of finite type over $\mathbb{Z}$. By a suitable choice of $S$ we may even assume that we have

1. a scheme flat and of finite type $X_S \to S$ with a section $\sigma : S \to X_S$ and
2. a line bundle $L_S$ on $X_S \setminus \sigma(S)$ such that
3. over the generic fiber we recover $(x, X)$ and $L$.

The key technical result that we need is the following.

Claim 15. $L \in \text{Pic}^\tau_{\text{loc}}(x, X)$ iff the set of closed points

$$\{ s \in S \text{ such that } L_S|_{X_s} \text{ is torsion} \}$$

is Zariski dense in $S$.

This is quite easy to prove if the local Picard groups of the fibers $(x_s, X_s)$ are themselves fibers of a “reasonable” group scheme $\text{Pic}^\tau_S(\sigma, X)$. Even in the proper
case, such relative Picard groups exist only under some restrictions; see [BLR90, Chap.8] for a detailed discussion. In the local case, the existence of such a group scheme $\text{Pic}^\text{loc}(\sigma, X)$ is not known. We prove that, at least after replacing $S$ by a dense open subset of red $S$, there is a good enough approximation of $\text{Pic}^\text{loc-\tau}(\sigma, X)$ to make the rest of the proof work; see Section 8.

Constructing $\text{Pic}^\text{loc-\tau}(\sigma, X)$ is relatively easy if the generic point $s_g \in S$ has characteristic 0 since then $\text{Pic}^\text{loc-\tau}(x_g, X_g)$ is a smooth algebraic group. However, in positive characteristic we need to understand the obstruction theory of the Picard functor. A delicate technical point is that the obstruction theory is governed by $H^2_s(X, \mathcal{O}_X)$ which is usually infinite dimensional. A section of a coherent sheaf over $S$ vanishes at the generic point iff it vanishes at a Zariski dense set of closed points, but this is no longer true for quasi-coherent sheaves; see Example 71. Thus the general theory does not exclude the possibility that the dimension of $\text{Pic}^\text{loc-\tau}(x_s, X_s)$ jumps at every closed point.

Once this issue is settled, we complete the proofs as follows.

Since $S$ is of finite type over $\mathbb{Z}$, the residue fields at closed points are all finite. Over $\mathbb{F}_q$, we prove Theorem 6 for the normalization $\pi : \tilde{X} \to X$ by factoring it as

$$\pi : \tilde{X} \xrightarrow{\pi_3} X^{\text{wn}} \xrightarrow{\pi_2} \text{red } X \xrightarrow{\pi_1} X$$

where $X^{\text{wn}}$ is the weak-normalization of $X$. The nontrivial part is finiteness of the kernel for $\pi_3^*$. This is established in Proposition 10 using the quotient theory of [Kol12] and Seifert $\mathbb{G}_m$-bundles as in [Kol13a, 9.53]. Besides proving Theorem 6 over $\mathbb{F}_q$, this reduces the proof of Theorem 7 in positive characteristic to the case when $X$ is normal. The latter was done in [BdJ13].

For all the theorems we use Claim 15 to pass to arbitrary base fields $k$.

16. Here we show that Theorem 6 is implied by the special case when $Y = \tilde{X}$. To see this, let $\sigma : \tilde{Y} \to Y$ be the normalization of $Y$. It is enough to show that the kernel of the composite

$$\text{Pic}^\text{loc}(x, X) \xrightarrow{\pi_*} \text{Pic}^\text{loc}(y, Y) \xrightarrow{\sigma^*} \text{Pic}^\text{loc}(\tilde{y}, \tilde{Y})$$

is contained in $\text{Pic}^\text{loc-\tau}(x, X)$. The map $\sigma^* \circ \pi^*$ is also the composite of

$$\text{Pic}^\text{loc}(x, X) \xrightarrow{\pi_*} \text{Pic}^\text{loc}(\tilde{x}, \tilde{X}) \xrightarrow{\pi_*} \text{Pic}^\text{loc}(\tilde{y}, \tilde{Y}).$$

We already know that the kernel of $\pi^*$ is contained in $\text{Pic}^\text{loc-\tau}(x, X)$.

If $q : U \to V$ is a finite surjection between irreducible normal schemes, then for any line bundle $L$ on $V$ we have $L^\text{deg}_{U/V} \cong \text{norm}_{U/V} q^* L$. Thus the kernel of $\pi^*$ is torsion and so the kernel of $\sigma^* \circ \pi^* = \pi^* \circ \pi^*$ is also contained in $\text{Pic}^\text{loc-\tau}(x, X)$.

17. Notation and assumptions as in Theorem 7. Here we show that once we know that $\ker(r^X_D) \subset \text{Pic}^\text{loc-\tau}(x, X)$, the remaining claims about unipotence follow.

Let $L \in \text{Pic}^\text{loc}(x, X)$ be a nontrivial line bundle such that $L|_D \cong \mathcal{O}_D$ and $L^m \cong \mathcal{O}_X$ for some $m > 0$ not divisible by $\text{char } k$. Choose $m$ to be the smallest with these properties. Then $L$ determines a degree $m$, irreducible, cyclic cover $\tau : \tilde{X} \to X$ that induces a trivial cover on $D \setminus \{x\}$. Thus $\tilde{D} := \tau^{-1}(D)$ is a Cartier divisor such that $\tilde{D} \setminus \{\tilde{x}\}$ has $m + 1$ connected components. By [Gro68, XIII.2.1] this is impossible.

Thus $\ker(r^X_D)$ is torsion free if char $k = 0$ and contains only $p^\infty$-torsion if char $k = p > 0$. Using Lemma 18 this implies that once $\ker(r^X_D)$ is known to be of finite
type, it is unipotent if char $k = 0$ and unipotent up-to $p^\infty$-torsion if char $k = p > 0$.
(Note that a finite $p^\infty$-torsion group is unipotent if char $k = p > 0$, so ker$(\psi_D)$ is unipotent unless the $p^\infty$-torsion part is not finitely generated.)

**Lemma 18.** Let $G$ be a locally of finite type, commutative algebraic group over a
field $k$. Assume that $G$ is torsion free if char $k = 0$ and contains only $p^\infty$-torsion if char $k = p > 0$.
Then $G^o$ is unipotent and $G/G^o$ is torsion free if char $k = 0$ and contains only $p^\infty$-torsion if char $k = p > 0$.

Proof. A connected commutative algebraic group $H$ has a unique connected
subgroup $0 \subset H_1 \subset H$ such that $H/H_1$ is an Abelian variety and $H_1 \cong H_m + H_u$ is
a linear algebraic group where $H_m$ is multiplicative and $H_u$ is unipotent.

The existence of $H_1$ is called Chevalley’s theorem; the first published proof is in
[Bar55], see [BSU13] for a modern treatment. The decomposition $H_1 \cong H_m + H_u$
is in most books on linear algebraic groups, see for instance [Bar91, Thm.4.7].

Note that both Abelian varieties and multiplicative groups contain many torsion
elements. These lift back to torsion elements in $H$ using the following elementary
observation.

Let $H$ be a group and $K \subset H$ a central subgroup. Assume that $\hat{h} \in H/K$ is
$m$-torsion and $K$ is $m$-divisible. Then $\hat{h}$ lifts to an $m$-torsion element $h \in H$. $\square$

2. Definition of Local Picard Groups

The literature is very inconsistent, there are at least four variants called the local
Picard group by some authors.

**Definition 19** (Local Picard group). Let $X$ be a scheme and $x \in X$ a point.
Assume for simplicity that $X$ is excellent and $\text{depth}_x \mathcal{O}_X \geq 2$.

The local Picard group $\text{Pic}^\text{loc}(x, X)$ is a group whose elements are $S_2$
sheaves $F$ on some neighborhood $x \in U \subset X$ such that $F$ is locally free on $U \setminus \{x\}$. Two such
sheaves give the same element if they are isomorphic over some neighborhood of $x$.
The product is given by the $S_2$-hull of the tensor product.

One can also realize the local Picard group as $\text{Pic}(\text{Spec} \mathcal{O}_{x,X} \setminus \{x\})$ or as the
direct limit of $\text{Pic}(U \setminus \{x\})$ as $U$ runs through all open Zariski neighborhoods of $x$.
Usually it is necessary to take the limit.

If $X$ is normal and $X \setminus \{x\}$ is smooth (or locally factorial) then $\text{Pic}^\text{loc}(x, X)$ is
isomorphic to the divisor class group of $\mathcal{O}_{x,X}$.

In many contexts it is more natural to work with the étale-local Picard group
$\text{Pic}^{\text{et-loc}}(x, X) := \text{Pic}(\text{Spec} \mathcal{O}_{x,X}^h \setminus \{x\})$ where $\mathcal{O}_{x,X}^h$ is the Henselization of the
local ring $\mathcal{O}_{x,X}$. Alternatively, $\text{Pic}^{\text{et-loc}}(x, X)$ is the direct limit of $\text{Pic}^{\text{loc}}(x', X')$
as $(x', X')$ runs through all étale neighborhoods of $(x, X)$. Usually it is necessary to
take the limit; see [27].

Even for isolated singularities over $\mathbb{C}$, it is quite hard to understand the relation-
ship between $\text{Pic}^{\text{loc}}(x, X)$ and $\text{Pic}^{\text{et-loc}}(x, X)$. By [PS94] there are many
singularities such that $\text{Pic}^{\text{loc}}(x, X) = 0$ yet $\text{Pic}^{\text{et-loc}}(x, X)$ is large.

Example [27] shows that, for some rather simple singularities, one always has
$\text{Pic}^{\text{loc}}(x, X) \neq \text{Pic}^{\text{et-loc}}(x, X)$.

**Definition 20** (Picard group of a local ring). Let $(R, m)$ be a semilocal ring.
Assume for simplicity that $R$ is excellent and $\text{depth}_m R \geq 2$. We can define its
local Picard group $\text{Pic}^{\text{loc}}(R, m)$ purely algebraically as follows. Its elements are isomorphism classes of finite $R$-modules $M$ such that $\text{depth}_m M \geq 2$ and $M_r$ is locally free of rank $1$ over $R_r$ for every non–zerodivisor $r \in m$. The product is given by the $S_2$-hull of the tensor product.

If $(R, m)$ is the (semi)local ring of a point $x$ (or of a $0$-dimensional subscheme) on a scheme $X$ then $\text{Pic}^{\text{loc}}(R, m) = \text{Pic}^{\text{loc}}(x, X)$.

In particular, $\text{Pic}^{\text{loc}}(x, X)$ does not depend on our choice of the base scheme.

Assume next that $(x, X)$ is essentially of finite type over $(s, S)$. Then $k(x)$ is a finitely generated field extension of $k(s)$. Pick a transcendence basis $t_1, \ldots, t_m \in k(x)/k(s)$, lift these back to $t_1, \ldots, t_m \in \mathcal{O}_{x,X}$ and localize $\mathcal{O}_{s,S}[t_1, \ldots, t_m]$ at the generic point of its intersection with $m_x$ to get $(m_R, R)$. We can now view $\mathcal{O}_{x,X}$ as an essentially of finite type $R$-algebra. The advantage is that now $\mathcal{O}_{x,X}/m_x$ is a finite extension of $R/m_R$. Thus $\mathcal{O}_{x,X}$ is the localization of a finite type $R$-algebra at a closed point.

That is, in the study of local Picard groups on schemes of finite type, it is sufficient to work with closed points.

**Definition 21** (Local Picard functor and scheme). [Bou78] Let $k$ be a field and $(x, X)$ a local, Noetherian $k$-scheme. Set $U := \text{Spec}_x \mathcal{O}_{x,X} \setminus \{x\}$.

For a local $k$-algebra $A$ consider the pre-sheaf

$$A \mapsto \text{Pic}((U \times_k \text{Spec} A)^b)$$

where the superscript $b$ denotes the Henselization. Sheafifying in the étale topology gives the local Picard functor $\text{Pic}^{\text{loc}}(x, X)$.

Thus the local Picard functor works with objects $(\pi : (X, \tilde{X}) \to (x, X), \tilde{L})$ where $\pi$ is étale and $\tilde{L}$ is a line bundle on $\tilde{X} \setminus \tilde{x}$.

By [Bou78 Thm.1.ii], if $\text{depth}_x X \geq 2$ and $H^1(U, \mathcal{O}_U) \cong H^2_\sigma(X, \mathcal{O}_X)$ is finite dimensional, then the local Picard functor is represented by a $k$-group scheme $\text{Pic}^{\text{loc}}(x, X)$ that is locally of finite type. The tangent space of $\text{Pic}^{\text{loc}}(x, X)$ at the identity is naturally isomorphic to $H^2_\sigma(X, \mathcal{O}_X)$.

If $H^2_\sigma(X, \mathcal{O}_X) = 0$ then $\text{Pic}^{\text{loc}}(x, X)$ is essentially the same as $\text{NS}^{\text{loc}}(x, X)$. The interesting case is when $0 < \dim_k H^2_\sigma(X, \mathcal{O}_X) < \infty$. This holds if

(i) $k(x)$ is finite over $k$,

(ii) $X$ is $S_2$ and

(iii) $X$ is pure of dimension $\geq 3$.

(These conditions are almost necessary, see (76).)

**Algebraic equivalence** is given by

(a) a connected $k$-scheme $T$ with two points $t_1, t_2$,

(b) an étale morphism $\pi : Y \to X \times T$ such that the injection $\{x\} \times T \to X \times T$ lifts to $\sigma : \{x\} \times T \to Y$ and

(c) a line bundle $L_Y$ on $Y \setminus \sigma(\{x\} \times T)$.

Let a subscript $i$ denote restriction to the fiber over $t_i$. The two line bundles

$$\left(\pi_1 : (y_1, Y_1) \to (x, X), L_1\right) \text{ and } \left(\pi_2 : (y_2, Y_2) \to (x, X), L_2\right)$$

are declared *algebraically equivalent*. Note that $L_1, L_2 \in \text{Pic}^{\text{loc}}(x, X)$ are algebraically equivalent if they are algebraically equivalent after some field extension $K \supset k$. All line bundles algebraically equivalent to the trivial bundle $\mathcal{O}_X$ form the identity component of $\text{Pic}^{\text{loc}}(x, X)$, denoted by $\text{Pic}^{\text{loc}-}(x, X)$. As usual,
If $H^2_x(X, \mathcal{O}_X)$ is infinite dimensional then $\text{Pic}^\text{loc}(x, X)$ does not exist, but, as long as $\text{depth}_x X \geq 2$, the unit section of $\text{Pic}^\text{loc}(x, X)$ is represented by a finitely presented closed immersion

$$\text{Pic}^\text{loc}(x, X) \subset \text{Pic}^\text{loc}(x, X) / \text{Pic}^\text{loc-\tau}(x, X).$$

If $H^2_x(X, \mathcal{O}_X)$ is infinite dimensional then $\text{Pic}^\text{loc}(x, X)$ does not exist, but, as long as $\text{depth}_x X \geq 2$, the unit section of $\text{Pic}^\text{loc}(x, X)$ is represented by a finitely presented closed immersion by \[\text{Bou78}\] Thm.1.i.

Assume now that $X$ is $S_2$, $\text{dim} X \geq 4$ and $x \in D \subset X$ is a Cartier divisor. Set $\tilde{D} := \text{Spec}_D j_\ast \mathcal{O}_D/x$ and let $\tilde{x} \subset \tilde{D}$ be the preimage of $x$. Then $\text{depth}_x \tilde{D} \geq 2$ hence there is a subgroup scheme $K(X|D) \subset \text{Pic}^\text{loc}(x, X)$ representing those line bundles that become trivial when restricted to $D \setminus \{x\} \cong \tilde{D} \setminus \{\tilde{x}\}$. This $K(X|D)$ gives the precise definition of $\ker[\gamma_x : \text{Pic}^\text{loc}(x, X) \to \text{Pic}^\text{loc}(x, D)]$ used in Theorem 4.

**Definition 22** (Local Néron-Severi group). Let $k$ be a field and $(x, X)$ a local, Noetherian $k$-scheme such that $\text{Pic}^\text{loc}(x, X)$ exists. The quotient

$$\text{NS}^\text{loc}(x, X) := \text{Pic}^\text{loc}(x, X) / \text{Pic}^\text{loc-\tau}(x, X)$$

is called the local Néron-Severi group. The analytic methods show that if $k$ has characteristic $0$ then $\text{NS}^\text{loc}(x, X)$ is a finitely generated abelian group, see \[\text{Bou78}\]. Combining the method of \[\text{Bou78}\] with \[\text{dJ96}\] implies that, even in characteristic $p > 0$, the local Néron-Severi group is finitely generated if $X$ is normal and finitely generated modulo $p^\infty$-torsion in general. Our computations show that the $p^\infty$-torsion part has bounded exponent, but say nothing about finite generation. I do not know any examples satisfying Condition 3 where $\text{NS}^\text{loc}(x, X)$ is not finitely generated.

The above is probably not a completely agreed-upon definition; another candidate is the usually smaller

$$\text{Pic}^\text{loc}(x, X) / (\text{Pic}^\text{loc}(x, X) \cap \text{Pic}^\text{loc-\tau}(x, X)).$$

23 (Comparing Pic and Pic). Let $(x, X)$ be a local, Noetherian scheme over a field $k$. By definition there is a natural injection

$$\text{Pic}^\text{loc}(x, X) \hookrightarrow \text{Pic}^\text{loc}(x, X)(k)$$

which is usually not a surjection. If $k$ is algebraically closed, then, essentially by definition,

$$\text{Pic}^\text{et-\text{loc}}(x, X) \cong \text{Pic}^\text{loc}(x, X)(k).$$

If, in addition, $X$ is complete or henselian then

$$\text{Pic}^\text{loc}(x, X) \cong \text{Pic}^\text{loc}(x, X)(k).$$

3. Examples of local Picard groups

Here we discuss a series of examples of local Picard groups. They show that the assumptions of Theorems \[\text{Bou78}\] and \[\text{dJ96}\] are essentially optimal.

**Example 24.** Let $(0, X)$ be an isolated singularity with a resolution $\rho : Y \to X$ that is an isomorphism over $X \setminus \{0\}$. Let $\{E_i : i \in I\}$ be the exceptional divisors. There is a natural exact sequence

$$0 \to \sum \mathbb{Z}[E_i] \to \text{Pic}(Y) \to \text{Pic}^\text{loc}(0, X) \to 0.$$  \[\text{24.1}\]

(Left exactness follows from \[\text{KM98}\] 3.39, but there are many other ways to prove it.)
Assume in addition that there is an effective, exceptional divisor $E$ such that $O_Y(-E)$ is $p$-ample. (Any resolution obtained by repeatedly blowing up subschemes whose support lies over $\{0\}$ has this property.) Let $L$ denote the line bundle $O_X(-E)|_E$; it is ample by assumption.

Let $nE$ denote the subscheme of $X$ defined by $O_X(-nE)$. There is an exact sequence

$$0 \to L^n \to O_{(n+1)E}^* \to O_{nE}^* \to 1. \quad (24.2)$$

Since $L$ is ample, $H^1(E, L^n) = H^2(E, L^n)$ for $n \gg 1$. Thus the natural restriction maps $\text{Pic}((n + 1)E) \to \text{Pic}(nE)$ are isomorphisms for $n \gg 1$. If $O_X$ is complete then

$$\text{Pic}(Y) \cong \text{Pic}(nE) \quad \text{for} \quad n \gg 1. \quad (24.3)$$

(This probably needs Grothendieck’s existence theorem [Gro60 III.5.1.4].) Thus

$$\text{Pic}(0, X) = \text{Pic}(nE)/\sum_i \mathbb{Z}[E_i] \quad \text{for} \quad n \gg 1. \quad (24.4)$$

For normal surfaces over a field of characteristic $0$, (24.4) is used in [Mum61] to define the local Picard scheme. Note, however, that the scheme structures of the two sides can differ in positive characteristic, as we see next.

**Example 25 (Cones).** Let $W \subset \mathbb{P}^N_k$ be a smooth projective variety and $(0, X)$ the affine cone over $W$; see [Ko13B 3.8] for our conventions on cones. By blowing up the vertex we get a resolution $p : Y \to X$ where

$$X = \text{Spec}_k \sum_{r \geq 0} H^0(W, O_W(r)) \quad \text{and} \quad Y = \text{Spec}_W \sum_{r \geq 0} O_W(r).$$

Thus (24.3) suggests that

$$\dim T_0 \text{Pic}(Y) \cong H^1(Y, O_Y) = \sum_{r \geq 0} H^1(W, O_W(r)). \quad (24.1)$$

On the other hand, $\dim T_0 \text{Pic}_{\text{loc}}(0, X)$ equals $H^1(U, O_U)$ where $U = X \setminus \{0\}$. Since $U = \text{Spec}_W \sum_{r=0}^\infty O_W(r)$, we see that

$$\dim T_0 \text{Pic}_{\text{loc}}(0, X) = \sum_{r=0}^\infty H^1(W, O_W(r)). \quad (25.1)$$

Compared with (25.1) we see extra summands for $r < 0$. This is not a problem in characteristic $0$ where these vanish by Kodaira’s theorem. However, the two formulas can give different answers in positive characteristic.

Assume next that the characteristic is $0$. Then the above considerations give an exact sequence

$$0 \to \sum_{r \geq 0} H^1(W, O_W(r)) \to \text{Pic}_{\text{loc}}(0, X) \to \text{Pic}(W)/\mathbb{Z}[O_W(1)] \to 0. \quad (25.3)$$

Let $H \subset W$ a smooth hyperplane section and $D \subset X$ the cone over it. A similar computation gives an exact sequence

$$0 \to \sum_{r > 0} H^1(H, O_H(r)) \to \text{Pic}_{\text{loc}}(0, D) \to \text{Pic}(H)/\mathbb{Z}[O_H(1)] \to 0. \quad (25.4)$$

Thus we see that the restriction maps $\text{Pic}_{\text{loc}}(0, X) \to \text{Pic}_{\text{loc}}(0, D)$ has a positive dimensional kernel if the maps $H^1(W, O_W(r)) \to H^1(H, O_H(r))$ are not all injective, cf. [BdJ13 Exmps.1.31–35].
Example 26 (Weighted cones). In (25) let \((t = 0)\) be an equation of \(H \subset W\). We can view \(t\) as a map \(X \to \mathbb{A}^1\). After base-change to \(t = s^m\) we get a new singularity \((0, X_m)\) containing the same \((0, D)\) as a hyperplane section. Thinking of \(X\) as \(\text{Spec} \sum_r t^r H^0(W, \mathcal{O}_W(r))\), we now have
\[
X_m = \text{Spec} \sum_r t^r H^0(W, \mathcal{O}_W([r/m])).
\]
Computing as in (25) we obtain that
\[
H^2(X_m, \mathcal{O}_{X_m}) \cong \sum_{r \geq 0} H^1(W, \mathcal{O}_W([r/m])) \cong \bigoplus_m H^2(X, \mathcal{O}_X).
\]
Thus, if \(\text{char } k = 0\) then \(\dim \text{Pic}^{\text{loc}}(0, X_m) = m \cdot \dim \text{Pic}^{\text{loc}}(0, X)\) and so
\[
\dim \ker [\text{Pic}^{\text{loc}}(0, X_m) \to \text{Pic}^{\text{loc}}(0, D)]
\]
grows with \(m\), save when \(\text{Pic}^{\text{loc}}(0, X)\) is 0-dimensional.

Example 27. Let \(S\) be a normal, projective surface with a single isolated singularity at \((0, S)\). Assume that \(\text{Pic}^{\text{loc}}(0, S)\) has no Abelian subvarieties. We claim that the image \(\text{Pic}(S \setminus 0) \to \text{Pic}^{\text{loc}}(0, S)\) is finitely generated.

To see this let \(\pi : T \to S\) be a resolution of singularities. We can factor \(\text{Pic}(S \setminus 0) \to \text{Pic}^{\text{loc}}(0, S)\) as \(\text{Pic}(S \setminus 0) \to \text{Pic}(T) \to \text{Pic}^{\text{loc}}(0, S)\). Since \(\text{Pic}(T)\) is an Abelian variety, its image in \(\text{Pic}^{\text{loc}}(0, S)\) is also an Abelian variety, thus trivial. Hence \(\text{Pic}(T) \to \text{Pic}^{\text{loc}}(0, S)\) is the constant map and so \(\text{Pic}(T) \to \text{Pic}^{\text{loc}}(0, S)\) factors through \(\text{NS}(T) \to \text{Pic}^{\text{loc}}(0, S)\).

Here are some concrete equations with the above properties.

(24) Cusps, for example \((xyz + x^4 + y^4 + z^4 = 0)\) or \((z^2 = x^2(x^2 + y^2) + y^2)\). For these \(\text{Pic}^{\text{loc}}(0, S) \cong \mathbb{G}_m\).

(27) Let \(C\) be an irreducible curve with a single node whose normalization \(E := \overline{C}\) is a smooth elliptic curve. There is an extension
\[
1 \to \mathbb{G}_m \to \text{Pic}^0(C) \to \text{Pic}^0(E) \cong E \to 0
\]
The extension is non-split, even more, \(\text{Pic}^0(C)\) does not contain any compact curves iff the difference of the 2 preimages of the node is non-torsion on \(\text{Pic}(E)\). For example, the exceptional divisor of the minimal resolution of \((z^2 = x^2(x^4 + cy^4)+y^7)\) is such a curve; the non-torsion condition holds for very general \(c \in \mathbb{C}\).

In the non-normal case, several new phenomena occur.

Example 28 (Non-normal surfaces). We compute the local Picard group for three of the simplest non-normal surfaces. Let \(S := (xy = 0) \subset \mathbb{A}^3\) be the union of two planes in \(\mathbb{A}^3\), \(T := (z^2 = 0) \subset \mathbb{A}^3\) the double plane in \(\mathbb{A}^3\) and \(W := (y^2 - x^3 = 0) \subset \mathbb{A}^3\) the product of a cuspidal cubic with a line. \(S, T\) and \(W\) are all \(S_2\).

Let \(s \in S\) denote the origin; then \(\text{depth}_s \mathcal{O}_S = 2\). \(S\) has two irreducible components \(S_x = (x = 0) \cong \mathbb{A}^2\) and \(S_y = (y = 0) \cong \mathbb{A}^2\). The normalization \(\overline{S}\) is the disjoint union \(S_x \amalg S_y\). Thus \(\text{Pic}^{\text{loc}}(\overline{s}, \overline{S}) \cong 0\). We claim that
\[
\text{Pic}^{\text{loc}}(s, S) \cong \ker [\text{Pic}^{\text{loc}}(s, S) \to \text{Pic}^{\text{loc}}(\overline{s}, \overline{S})] \cong \mathbb{Z}. \quad (28.1)
\]
To see this let \(L\) be an invertible sheaf on \(S \setminus s\). Both \(L|_{S_x}\) and \(L|_{S_y}\) are trivial; let \(\sigma_x, \sigma_y\) be generating sections. Then \(\sigma_x, \sigma_y\) restrict to global sections of \(L\) on the punctured \(z\)-axis. Their quotient is a regular function on the punctured \(z\)-axis; its order of pole or zero at the origin gives the isomorphism \(\text{Pic}^{\text{loc}}(s, S) \cong \mathbb{Z}\).
Next let \( t \in T \) denote the origin; then \( \text{depth}_t \mathcal{O}_T = 2 \). The normalization of \( T \) is \( \overline{T} = (z = 0) \subset \mathbb{A}^3 \). There is an exact sequence
\[
0 \to \mathcal{O}_{\overline{T} \setminus t} \to \mathcal{O}_{T \setminus t} \to \mathcal{O}_{\overline{T} \setminus t}^* \to 1
\]
where \( e(g) = 1 + zg \). Taking cohomology and using that \( \text{Pic}^{\text{loc}}(t, \overline{T}) = 0 \) gives the isomorphism
\[
\text{Pic}^{\text{loc}}(t, T) \cong H^1(\overline{T} \setminus t, \mathcal{O}_{\overline{T} \setminus t}).
\]
The latter can be naturally identified with \( \sum_{m \geq 2} H^1(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}(-m)) \). Thus
\[
\text{Pic}^{\text{loc}}(t, T) \cong \ker\left[ \text{Pic}^{\text{loc}}(t, T) \xrightarrow{\pi^*} \text{Pic}^{\text{loc}}(\overline{t}, \overline{T}) \right] \cong (\text{infinite dimensional vectorspace}).
\]

Let \( w \in W \) denote the origin. We can write \( W = \text{Spec} k[t^2, t^3, z] \) with normalization \( W \cong \text{Spec} k[t, z] \). Let \( C \subset W \) denote the \( z \)-axis; this is the singular locus of \( W \). There is an exact sequence
\[
1 \to \mathcal{O}_W^* \to \mathcal{O}_W^* \xrightarrow{d} \mathcal{O}_C \to 0
\]
where \( d(f_0(z) + f_1(z)t + \cdots) = f_1(z)/f_0(z) \). Then we have the exact sequence
\[
H^0(\overline{W} \setminus \{w\}, \mathcal{O}_W^*) \xrightarrow{d} H^0(\mathbb{P} \setminus \{w\}, \mathcal{O}_C) \to \text{Pic}(W \setminus \{w\}) \to \text{Pic}(\overline{W} \setminus \{w\}) = 1.
\]
Note that \( d \) factors as
\[
H^0(\overline{W} \setminus \{w\}, \mathcal{O}_W^*) = H^0(\overline{W}, \mathcal{O}_W^*) \xrightarrow{d} H^0(\mathbb{P}, \mathcal{O}_C) \to H^0(C, \mathcal{O}_C).
\]
This shows that
\[
\text{Pic}^{\text{loc}}(w \in W) \cong H^1_w(\mathcal{O}_C) \cong k[z, z^{-1}]/k[z] \cong (\text{infinite dimensional vectorspace}).
\]

**Example 29.** As a slight variation of example (28), let \( X \subset \mathbb{A}^{2n+1} \) be the union of the linear spaces \( X_1 = (x_1 = \cdots = x_n = 0) \) and \( X_2 = (x_{n+2} = \cdots = x_{2n+1} = 0) \). Here \( X_1 \cap X_2 \cong \mathbb{A}^1 \) and the normalization is \( \overline{X} = X_1 \amalg X_2 \). Here \( X \) is not \( S_2 \) if \( n \geq 2 \) but depth\( \text{loc} \) \( X \) is \( 2 \) where \( x \in X \) denotes the origin.

As in the previous example, we see that
\[
\text{Pic}^{\text{loc}}(x, X) \cong \ker\left[ \text{Pic}^{\text{loc}}(x, X) \xrightarrow{\pi^*} \text{Pic}^{\text{loc}}(\overline{x}, \overline{X}) \right] \cong \mathbb{Z}.
\]

**Example 30 (Demi-normal varieties).** Fix a ground field \( k \) an let \( X_i \) be cones over the Segre embedding of \( \mathbb{P}^1 \times \mathbb{P}^n \) with vertices \( v_i \). Note that the \( X_i \) have rational singularities at the origin and \( \text{Pic}^{\text{loc}}(v_i, X_i) \cong \mathbb{Z} \).

Pick distinct points \( p_j \in \mathbb{P}_1 \) and let \( D_{ij} \subset X_i \) be the cone over \( \{p_j\} \times \mathbb{P}^n \). Note that \( D_{ij} \cong \mathbb{A}^{n+1} \).

The first example \( Y_2 \) is obtained by gluing \( X_1 \) to \( X_2 \) using the natural identifications \( D_{1j} \cong D_{2j} \) for \( j = 1, 2 \). Then \( Y_2 = X_1 \amalg X_2 \) and so \( \text{Pic}^{\text{loc}}(\overline{v}, \overline{Y}_2) \cong \mathbb{Z}^2 \). We claim that
\[
\text{Pic}^{\text{loc}}(v, Y_2) \cong k^* + \mathbb{Z}^2 \quad \text{and} \quad \ker\left[ \text{Pic}^{\text{loc}}(v, Y_2) \xrightarrow{\pi^*} \text{Pic}^{\text{loc}}(\overline{v}, \overline{Y}_2) \right] \cong k^*.
\]

The extra \( k^* \) is obtained as follows. Set \( D_i = D_{i1} + D_{i2} \subset X_i \). Take the trivial bundles \( \mathcal{O}_{X_i} \) and an isomorphism
\[
\phi^0 : \mathcal{O}_{D_i \setminus v} \cong \mathcal{O}_{D_2 \setminus v}.
\]
to get an element of $\text{Pic}^{\text{loc}}(v, Y_2)$. Two such isomorphisms give the same line bundle if they differ by multiplication by sections of $H^0(\mathcal{X}_1 \setminus v, \mathcal{O}_{\mathcal{X}_1 \setminus v})$. Since $\dim D_{ij} \geq 2$, the isomorphism $\phi^0$ extends to a pair of isomorphisms

$$
\phi_j : \mathcal{O}_{D_{ij}} \cong \mathcal{O}_{D_{ij}} \quad \text{for } j = 1, 2.
$$

These give isomorphisms

$$
\phi_j(v) : k \cong k(v_1) \cong k(v_2) \cong k.
$$

The quotient $\phi_1(v)/\phi_2(v)$ gives a well-defined element of $k^*$.

The second example $Y_3$ is obtained by gluing $X_1$ to $X_2$ using the natural identifications $D_{1j} \cong D_{2j}$ for $j = 1, 2, 3$. We claim that here the kernel is a direct product:

$$
\text{Pic}^{\text{loc}}(v, Y_3) \cong k^{n+1} \times k^* + \mathbb{Z}^2 \quad \text{and}
$$

$$
\ker[\text{Pic}^{\text{loc}}(v, Y_3) \xrightarrow{\sim} \text{Pic}^{\text{loc}}(\bar{v}, \bar{Y}_3)] \cong k^{n+1} \times (k^*)^2.
$$

To see this, set $D_i = D_{11} + D_{12} + D_{13} \subset X_i$. The trivial bundles $\mathcal{O}_{X_i}$ can be glued using a section of $H^0(D_i \setminus v, \mathcal{O}_{D_i \setminus v})$. Two such sections give the same line bundle iff they differ by multiplication by a section of $H^0(X_i \setminus v, \mathcal{O}_{X_i \setminus v})$. The previous considerations explain the $(k^*)^2$ but now we also get new conditions from sections of the form $1 + (\text{linear functions})$. On $D_i$, we have $n+1$ independent linear functions on each $D_{ij}$, giving $3n+3$ independent linear functions all together. On $X_i \subset \mathbb{A}^{2n+2}$ we only have $2n+2$ independent linear functions. This accounts for the $k^{n+1}$ part.

In principle we could get more conditions by considering $1 + (\text{quadratic functions})$, but it is easy to see that this does not happen for 3 or 4 copies of $D_{ij}$. For more copies, we get conditions coming from higher degree polynomials.

**Example 31** (Pinch point). Let $k$ be any field and $S := (x^2 = y^2z) \subset \mathbb{A}^3$ the pinch point. Its normalization is $\pi : \bar{S} := \mathbb{A}_{\mathbb{A}^2}^n \to S$ given by $(u,v) \mapsto (uv,u,v^2)$.

The line $(x = z = 0)$ generates $\text{Pic}^{\text{loc}}(0, S) \cong \mathbb{Z}/2$.

Note that $\pi$ is a homeomorphism if $\text{char } k = 2$. This suggests that in positive characteristic there may not be a perfect analog of the first Chern class as a mapping to topological cohomology.

A more general version is the following.

**Example 32.** Start with $(\bar{X}, \bar{D}) \cong (\mathbb{A}^n, \mathbb{A}^{n-1})$ and let $\bar{x} \in \mathbb{A}^{n-1}$ be the origin. Let $\tau : \mathbb{A}^{n-1} \to \mathbb{A}^{n-1}$ be coordinate-wise multiplication by a primitive $r$th root of unity. Assume that the characteristic does not divide $r$.

Construct $\pi : \bar{X} \to X$ by identifying the points in the $\tau$-orbits with each other. Thus $X$ is an affine variety, even CM. For $r = 2$ it has only double normal crossing singularities along $D := \pi(\bar{D})$. We claim that

$$
\text{Pic}^{\text{loc}}(x, X) = \ker[\text{Pic}^{\text{loc}}(x, X) \xrightarrow{\sim} \text{Pic}^{\text{loc}}(\bar{x}, \bar{X})] \cong \mathbb{Z}/r.
$$

For any $r$th root of unity $\epsilon$, the $\tau$-action on $\bar{D}$ can be lifted to $\mathcal{O}_{\bar{D}}$ as $g(x) \mapsto \epsilon \cdot g(\tau^{-1}(x))$. Taking the quotient we get line bundles $L(\epsilon)$ over $X$ corresponding to the $r$th roots of unity.

Choose coordinates such that $\bar{D} = (z_n = 0)$. A local trivialization of $L(\epsilon)$ would correspond to an invertible function $\phi(z_1, \ldots, z_n)$ such that $\phi(\epsilon z_1, \ldots, z_{n-1}, 0) = \epsilon \phi(z_1, \ldots, z_{n-1}, 0)$. This would imply $\phi(0, \ldots, 0) = \epsilon \phi(0, \ldots, 0)$, thus $\phi(0, \ldots, 0) = 0$ if $\epsilon \neq 1$. Therefore $\phi$ is not invertible near the origin if $n \geq 2$. 

4. Analytic local Picard groups

**Definition 33** (Analytic local Picard groups). Let $X$ be a complex analytic space and $x \in X$ a point. Assume for simplicity that $\text{depth}_x \mathcal{O}_X \geq 2$.

Let $W \subset X$ be the intersection of $X$ with a small (open) ball around $x$. The analytic local Picard group $\text{Pic}^\text{an-loc}(x, X)$ can be defined as in [19] using (analytic) $S_2$ sheaves on $W$. By [Art69], if $X$ is an algebraic variety over $\mathbb{C}$ then there is a natural isomorphism

$$\text{Pic}^\text{et-loc}(x, X) \cong \text{Pic}^\text{an-loc}(x, X^\text{an}).$$

By [Siu69], if $X$ is $S_2$ and has pure dimension $\geq 3$ then

$$\text{Pic}^\text{an}(W \setminus \{x\}) \cong \text{Pic}^\text{an-loc}(x, X).$$

Note that if $\dim X = 2$ then $\text{Pic}^\text{an}(W \setminus \{x\})$ is infinite dimensional but $\text{Pic}^\text{an-loc}(x, W)$ is finite dimensional if $X$ is normal.

**34** (Exponential sequence). Let $U$ be a complex space. Then $\text{Pic}(U) \cong H^1(U, \mathcal{O}_U)$ and the exponential sequence

$$0 \to \mathbb{Z}_U \xrightarrow{2\pi i} \mathcal{O}_U \xrightarrow{\exp} \mathcal{O}_U^* \to 1$$

gives an exact sequence

$$H^1(U, \mathcal{O}_U) \to \text{Pic}(U) \xrightarrow{\xi_1} H^2(U, \mathbb{Z}).$$

Let $X$ be a complex space, $x \in X$ a point and set $U := X \setminus \{x\}$. A piece of the local cohomology exact sequence is

$$H^1(X, \mathcal{O}_X) \to H^1(U, \mathcal{O}_U) \to H^2_x(X, \mathcal{O}_X) \to H^2(X, \mathcal{O}_X).$$

Thus if $X$ is Stein then we have an isomorphism

$$H^1(U, \mathcal{O}_U) \cong H^2_x(X, \mathcal{O}_X).$$

For local questions we should replace $X$ by a contractible open neighborhood $x \in W \subset X$. Then $W \setminus \{x\}$ is homotopy equivalent to $\text{link}(x, X)$, which is the intersection of $X$ with a small sphere centered at $x$. Thus $H^2(W \setminus \{x\}, \mathbb{Z}) = H^2(\text{link}(x, X), \mathbb{Z})$.

Combining with [Siu69] and [Art69] we obtain the following well known result.

**Proposition 35.** Let $(x, X)$ be a $\mathbb{C}$-scheme of finite type that is $S_2$ and has pure dimension $\geq 3$. Then taking the first Chern class gives an exact sequence

$$0 \to \text{Pic}^\text{loc-ø}(x, X) \to \text{Pic}^\text{loc}(x, X) \xrightarrow{\xi_1} H^2(\text{link}(x, X), \mathbb{Z}).$$

This suggests a topological way to proving Theorem 6 in characteristic 0: we should prove that the kernel of the pull-back map

$$H^2(\text{link}(x, X), \mathbb{Z}) \to H^2(\text{link}(\bar{x}, \bar{X}), \mathbb{Z})$$

is torsion. However, Examples 36-37 show that these maps can have non-torsion kernel, even for hypersurface singularities. We start with a projective example and then take a cone over it to get a local example.

**Example 36** (A singular K3 surface). Let $g_4(x_0, x_1, x_2)$ be a general quartic form. Then

$$S := (u^2 = x_0^2 g_4(x_0, x_1, x_2)) \subset \mathbb{P}^3(1, 1, 1, 3)$$
is a K3 surface with a double line \( L = (x_0 = u = 0) \). Its normalization \( \tilde{S} \to S \) is the smooth Del Pezzo surface of degree 2
\[
\tilde{S} := (v^2 = g_4(x_0, x_1, x_2)) \subset \mathbb{P}^3(1, 1, 1, 2).
\]
Thus \( H^2(\tilde{S}, \mathbb{Z}) = \text{Pic}(\tilde{S}) \cong \mathbb{Z}^8 \).

The preimage of the line \( L \) is a smooth elliptic curve \( E = (x_0 = 0) \). We claim that
\[
\ker[H^2(S, \mathbb{Z}) \xrightarrow{\sim} H^2(\tilde{S}, \mathbb{Z})] \cong H^1(E, \mathbb{Z}) \cong \mathbb{Z}^2. \tag{36.1}
\]
To see this we start with the short exact sequence
\[
0 \to \mathbb{Z}_L \to \pi_*\mathbb{Z}_E \to Q \to 0
\]
which shows that
\[
H^1(L, Q) \cong H^1(E, \mathbb{Z}) \quad \text{and} \quad H^2(L, Q) = 0. \tag{36.2}
\]
The sheaf \( Q \) also sits in the short exact sequence
\[
0 \to \mathbb{Z}_S \to \pi_*\mathbb{Z}_S \to Q \to 0
\]
and from this we get an exact sequence
\[
0 \to H^1(L, Q) \to H^2(S, \mathbb{Z}) \to H^2(\tilde{S}, \mathbb{Z}) \to H^2(L, Q). \tag{36.3}
\]
Putting (36.2) and (36.3) together gives (36.1).

**Example 37.** Using the previous notation, set
\[
X := (u^2 = x_0^2g_4(x_0, x_1, x_2)) \subset \mathbb{A}^4.
\]
Its normalization \( \tilde{X} \to X \) is given as
\[
\tilde{X} := (v^2 = g_4(x_0, x_1, x_2)) \subset \mathbb{A}^4.
\]
Set \( U := X \setminus \{0\} \) and \( \tilde{U} := \tilde{X} \setminus \{0\} \). Note that \( U \) (resp. \( \tilde{U} \)) is a Seifert \( \mathbb{C}^* \)-bundle over \( S \) (resp. \( \tilde{S} \)). This implies that
\[
H^2(U, \mathbb{Z}) \cong \mathbb{Z}^7 \quad \text{and} \quad H^2(U, \mathbb{Z}) \cong \mathbb{Z}^9.
\]
(The Seifert \( \mathbb{C}^* \)-bundle structure gives these modulo torsion. Since both \( X, \tilde{X} \) are hypersurface singularities, \( U, \tilde{U} \) are simply connected, thus the above \( H^2 \) are torsion free.) Thus \( \pi^* \) is not injective on \( H^2 \) and
\[
\mathbb{Z}^2 \cong H^1(E, \mathbb{Z}) \cong \ker[H^2(U, \mathbb{Z}) \xrightarrow{\sim} H^2(\tilde{U}, \mathbb{Z})]. \tag{37.1}
\]

By contrast, we claim that \( \text{Pic}^{\text{loc}}(0, X) = 0 \) for very general \( g_4 \). To see this note first that \( \mathbb{Q} \)-Cartier divisors \( D \) on \( S \) correspond to Cartier divisors \( \tilde{D} \) on \( \tilde{S} \) for which \( \tilde{D}|_E \) is invariant under the Galois involution of \( E \to L \).

Since \( \tilde{S} \) is a Del Pezzo surface, the pair \( (E, \tilde{S}) \) is obtained from a pair \( (E, \mathbb{P}^2) \) by blowing up \( 7 \) points \( P_1, \ldots, P_7 \in E \). Thus the image of \( \text{Pic}(\tilde{S}) \to \text{Pic}(E) \) is generated by \( \mathcal{O}_{\mathbb{P}^2}(1)|_E \) and the \( \mathcal{O}_E(P_i) \). For very general choice of the \( P_i \) these are independent in \( \text{Pic}(E) \) and the only Galois invariant divisor classes are given by the multiples of \( \mathcal{O}_{\mathbb{P}^2}(3)|_E \otimes \mathcal{O}_E(-P_1 - \cdots - P_7) \). This is also the pull-back of the hyperplane class under the projection \( S \to \mathbb{P}^2 \). Therefore \( \text{Pic}^{\text{loc}}(0, X) = 0 \).
5. Schemes over finite fields

We start with the following strengthening of Theorem 6 over finite fields.

**Theorem 38.** Let $k$ be a finite field and $X$ an excellent $k$-scheme of pure dimension $\geq 3$ that is $S_2$ (or at least topologically $S_2$, see Definition 45). Let $x \in X$ be a point with finite residue field $k(x)$. Let $\pi : \bar{X} \to X$ denote the normalization and $\bar{x}$ the preimage of $x$. Then the kernel of the pull-back map

$$\pi^* : \text{Pic}^{\text{loc}}(x, X) \to \text{Pic}^{\text{loc}}(\bar{x}, \bar{X})$$

is torsion.

Note that, by our Examples 28–30, the assumptions that $\dim X \geq 3$, $X$ be topologically $S_2$ and $k(x)$ be finite are all necessary.

**Proof.** We factor the normalization $\pi : \bar{X} \to X$ as

$$\pi : \bar{X} \xrightarrow{\pi_3} X^{\text{wn}} \xrightarrow{\pi_2} \text{red } X \xrightarrow{\pi_1} X$$

and show that for each step the kernel of the pull-back map on the local Picard groups is torsion.

For $\pi_1 : \text{red } X \to X$ this is done in Lemma 39 and for $\pi_2 : X^{\text{wn}} \to \text{red } X$ in Corollary 44. Both of these results are well known and hold in much greater generality. The most delicate part is finiteness of the kernel for $\pi_3^* : \text{Pic}^{\text{loc}}(x^{\text{wn}}, X^{\text{wn}}) \to \text{Pic}^{\text{loc}}(\bar{x}, \bar{X})$.

This is established in Proposition 49, using the quotient theory of [Kol12] and Seifert $\mathbb{G}_m$-bundles as in [Kol13b, 9.53]. □

**Lemma 39.** Let $k$ be a field of positive characteristic and $X$ a Noetherian $k$-scheme. Then the kernel of the restriction map $\text{Pic}(X) \to \text{Pic}(\text{red } X)$ is killed by some power of $\text{char } k$.

**Proof.** Let $J \subset \mathcal{O}_X$ be an ideal sheaf such that $J^2 = 0$. Set $X_J := \text{Spec } \mathcal{O}_X/J$. Note that $j \mapsto 1 + j$ identifies $J$ with the kernel of $\mathcal{O}_X^* \to \mathcal{O}_X^*$. This gives an exact sequence

$$H^1(X, J) \to H^1(X, \mathcal{O}_X^*) \to H^1(X_J, \mathcal{O}_X^{*})).$$

Since $H^1(X, J)$ is a $k$-vector space, it is killed by multiplication by $\text{char } k$.

Using this for the powers of the ideal sheaf defining $\text{red } X$ we see that the kernel of the restriction map $\text{Pic}(X) \to \text{Pic}(\text{red } X)$ is killed by multiplication by a power of $\text{char } k$. □

The above proof even shows the following stronger result.

**Corollary 40.** Let $k$ be a field of positive characteristic and $(x, X)$ a local Noetherian $k$-scheme that is $S_2$ and has pure dimension $\geq 3$. Then the kernel of the restriction map $\text{Pic}^{\text{loc}}(x, X) \to \text{Pic}^{\text{loc}}(x, \text{red } X)$ is a connected unipotent group scheme. □

**Definition 41.** Let $X$ be a scheme. A partial weak normalization is a finite morphism $\pi : X' \to X$ such that

1. $X'$ is reduced,
2. $\pi$ is a finite, universal homeomorphism and
3. the induced morphism $X' \to \text{red } X$ is an isomorphism at all generic points.
X is called *weakly normal* if \( X \) is reduced and its sole partial weak normalization is the identity \( X \cong X \).

If \( X \) is excellent, more generally, if the normalization \( \tilde{X} \) is finite over \( X \), then there is a unique maximal partial weak normalization \( X^{wn} \to X \), called the *weak normalization* of \( X \). Note that \( X^{wn} \) is weakly normal.

If \( X \) has residue characteristic 0, then the weak normalization agrees with the seminormalization. See [Kol96, Sec.1.7.2] for details.

A weakly normal scheme is \( S_2 \) iff it is connected in codimension 1 as in [Har62]. Therefore, if \( X \) is \( S_2 \) then so is its weak normalization.

For the proof of the next lemma, see [Kol97, Sec.6] or [Kol12, Prop.35].

**Lemma 42.** Let \( k \) be a field of positive characteristic and \( g : Y \to X \) a finite morphism of Noetherian \( k \)-schemes. The following are equivalent.

1. \( g \) is a universal homeomorphism.
2. \( Y \) is dominated by a Frobenius twist \( F_q : X_q \to Y \to X \) for some power \( q \) of \( \text{char} \ k \).

**Corollary 43.** Let \( k \) be a field of positive characteristic and \( g : Y \to X \) a finite, universal homeomorphism of Noetherian \( k \)-schemes. Then the kernel of the pull-back map \( g^* : \text{Pic}(X) \to \text{Pic}(Y) \) is killed by some power of \( \text{char} \ k \).

**Proof.** By Lemma 42 there is a \( q = (\text{char} \ k)^m \) such that \( g \) factors as \( F_q : X_q \to Y \to X \).

Thus it is enough to prove that the kernel of the pull-back map \( F_q^* : \text{Pic}(X) \to \text{Pic}(X_q) \) is killed by \( q \).

If a line bundle \( L \) is given by an open cover \( \{U_i\} \) and transition functions \( \{\phi_{ij}\} \) then \( F_q^*L \) can be given by the open cover \( \{F_q^{-1}U_i\} \) and transition functions \( \{F_q^*\phi_{ij}\} \).

As an abstract scheme \( X_q \) is naturally isomorphic to \( X \) and under this isomorphism \( F_q^*\phi_{ij} = \phi_{ij}^q \). Therefore the Frobenius pull-back \( F_q^*L \) is isomorphic to \( L^{\otimes q} \) under this isomorphism. Thus the kernel of \( F_q^* : \text{Pic}(X) \to \text{Pic}(X_q) \) is precisely the set of \( q \)-torsion elements.

Since the weak normalization of an excellent scheme is a finite, universal homeomorphism, as a special case we get the following.

**Corollary 44.** Let \( k \) be a field of positive characteristic and \( X \) an excellent \( k \)-scheme with weak-normalization \( \tau : X^{wn} \to X \). Then the kernel of the pull-back map \( \tau^* : \text{Pic}(X) \to \text{Pic}(X^{wn}) \) is killed by some power of \( \text{char} \ k \).

In contrast with (40), the kernel of \( \tau^* \) need not be connected, as shown by Example 31.

The last step of the proof uses the concept of topologically \( S_2 \) schemes and the theory of finite, set theoretic equivalence relations developed in [Kol12].

**Definition 45** (\( S_2 \) and topologically \( S_2 \)). Recall that a scheme \( X \) is \( S_2 \) if a finite morphism \( g : Y \to X \) is an isomorphism provided

1. there is a closed subset \( Z \subset X \) of codimension \( \geq 2 \) such that \( g \) is an isomorphism over \( X \setminus Z \) and
2. \( Y \) has no associated primes supported in \( g^{-1}(Z) \).

Similarly, a scheme \( X \) is *topologically \( S_2 \)* if a finite morphism \( g : Y \to X \) is a finite, universal homeomorphism provided
(1') there is a closed subset $Z \subset X$ of codimension $\geq 2$ such that $g$ is a finite, universal homeomorphism over $X \setminus Z$ and
\( (2') \) $Y$ has no irreducible components supported in $g^{-1}(Z)$.

It is not hard to see that a pure dimensional scheme $X$ is topologically $S_2$ iff the following holds.

(3') Let $\pi : U \to X$ be an étale morphism from a connected scheme $U$ and $Z \subset U$ a closed subscheme of codimension $\geq 2$. Then $U \setminus Z$ is connected.

These imply that an $S_2$ scheme is topologically $S_2$, a weakly normal scheme is $S_2$ iff it is topologically $S_2$ and the weak normalization of an $S_2$ scheme is topologically $S_2$. We also use that by [Gro68, XIII.2.1] a Cartier divisor in a topologically $S_2$ scheme is also topologically $S_2$.

46 (Set theoretic equivalence relations). (For more details, see [Kol12].)

Let $X$ be an excellent scheme, $\pi : \tilde{X} \to X$ its normalization and $R$ the normalization of $\tilde{X} \times_X \tilde{X}$. Together with the coordinate projections $\sigma_1, \sigma_2 : R \xrightarrow{\sim} \tilde{X}$ we have a finite, set theoretic equivalence relation.

Let $Q \subset R$ be a closed subscheme such that $\sigma_1|_Q, \sigma_2|_Q : Q \xrightarrow{\sim} \tilde{X}$ is also a set theoretic equivalence relation. By [Kol12, Lem.1.7], the geometric quotient $\tilde{X}/Q$ exists and the geometric fibers of $\tilde{X} \to X/Q$ are precisely the $Q$-equivalence classes.

In particular, if $Q = R$ then we get that $\tilde{X}/R \to X$ is a finite, universal homeomorphism. If $X$ is weakly normal then $\tilde{X}/R = X$.

Finally we study what happens as we go from a weakly normal scheme to its normalization.

**Proposition 47.** Let $k$ be a field of characteristic $p > 0$ and $X$ an excellent $k$-scheme of pure dimension $\geq 3$ that is weakly normal and $S_2$. Let $\pi : \tilde{X} \to X$ denote the normalization. Let $x \in X$ be a closed point with residue field $k(x)$. Then there is a $k$-torus $T$ and a linear representation
\[
I_x : \ker [\Pic^\loc(x, X) \xrightarrow{\pi^*} \Pic^\loc(\tilde{x}, \tilde{X})] \to T
\]
whose kernel is $p^\infty$-torsion.

Proof. Let $L$ be a line bundle on $X \setminus \{x\}$ such that $\pi^*L$ is trivial on $\tilde{X} \setminus \{\tilde{x}\}$.

We can view $L$ as a $\mathbb{G}_m$-bundle over $X \setminus \{x\}$, thus $\pi^*L$ is a trivial $\mathbb{G}_m$-bundle over $\tilde{X} \setminus \{\tilde{x}\}$. It extends to a trivial $\mathbb{G}_m$-bundle $\tilde{L}$ over $\tilde{X}$. Since $X$ is excellent, $\tilde{X}$ is finite over $X$ hence $\tilde{L}$ is of finite type over $X$.

Let $R$ be the normalization of $\tilde{X} \times_X \tilde{X}$. Then $\sigma_1, \sigma_2 : R \xrightarrow{\sim} \tilde{X}$ is a set-theoretic finite equivalence relation. Since $X$ is weakly normal, the geometric quotient $\tilde{X}/R$ equals $X$; see Paragraph[46]

Let $r \subset R$ denote the union of the reduced preimages of $\tilde{x}$. Since $\pi^*L$ is pulled-back from $X$, we have isomorphisms
\[
\phi_{R \setminus r} : \sigma_1^*(\pi^*L) \cong \sigma_2^*(\pi^*L) \quad \text{over } R \setminus r.
\]

By assumption $\pi^*L$ is trivial, thus $\phi_{R \setminus r}$ is can be viewed as an isomorphism of two trivial $\mathbb{G}_m$-bundles on $R \setminus r$.

Let $R^{(1)} \subset R$ be the union of the irreducible components of dimension $\geq \dim X - 1$ and $R^{(1)} \subset R$ the equivalence relation generated by $R^{(1)}$. We check at the end that $\tilde{X}/R = \tilde{X}/R^{(1)}$. [171]
Since $R^{[1]}$ is normal and every irreducible component has dimension $\geq 2$, $\phi_{R \setminus r}$ extends to an isomorphism

$$\phi_{[1]}^{[1]} : \sigma_1^*(L) \cong \sigma_2^*(L) \quad \text{over } R^{[1]}.$$ (47.2)

The isomorphisms (47.2) define a $G_m$-equivariant extension of the relation $R^{[1]}$ to a finite relation on $L$. Let $\tilde{R}_L$ denote the equivalence relation generated by this extension. In general, such an extension is a pro-finite equivalence relation.

In our case the only non-finiteness can occur over $x \in X$. For $\bar{x}_i \in \bar{x}$ let $G_m(\bar{x}_i)$ denote the multiplicative group scheme of the residue field $k(\bar{x}_i)$. For a point $r_\ell \in R^{[1]}$ lying over $x$ set $\bar{x}_i := \sigma_1(r_\ell)$ and $\bar{x}_j := \sigma_2(r_\ell)$. (We allow $i = j$.) Then $\phi_{[1]}^{[1]}$ gives an isomorphism

$$\phi_{ij\ell} : G_m(\bar{x}_i) \cong G_m(\bar{x}_j)$$ (47.3)

that is defined over $k(r_\ell)$.

It is enough to prove the Theorem after a finite field extension. We can thus replace $k$ by the composite of the above $k(\bar{x}_i)$ and $k(r_\ell)$.

Let $\Gamma_x$ denote the graph whose vertices are the points $\bar{x}_i$ and to each $r_\ell$ we add an edge connecting $\bar{x}_i$ and $\bar{x}_j$. Fixing a base point $\bar{x}_0$, compositions of the above $\phi_{ij\ell}$ define a homomorphism

$$I_x(L) : H_1(\Gamma_x, \mathbb{Z}) \to G_m(k).$$ (47.4)

The construction is compatible with field extensions and Henselization, thus we get a homomorphism

$$I_x : \ker \left[ \text{Pic}^{\text{loc}}(x, X) \xrightarrow{\pi^*} \text{Pic}^{\text{loc}}(\bar{x}, X) \right] \to \text{Hom}(H_1(\Gamma_x, \mathbb{Z}), G_m).$$ (47.5)

Assume next that $L$ is in the kernel of $I_x$. Then $\tilde{R}_L$ is a finite equivalence relation. More precisely, each $\tilde{R}_L$-equivalence class over $x$ contains exactly 1 point of $G_m(\bar{x}_0)$.

Since $L$ is of finite type over $X$, by [Kol12 Thm.6] the geometric quotient $\tilde{L}/\tilde{R}_L$ exists and it is a Seifert $G_m$-bundle over $\bar{X}/R^{[1]}$ by [Kol13b 9.48]. (The statement there assumes that two other conditions (HN) and (HSN) are also satisfied. These are, however, used only to ensure that the geometric quotient $\tilde{L}/\tilde{R}_L$ exists. In our case existence is assured by [Kol12 Thm.6], the rest of the proof then works.) By [Kol13b 9.53], a power $L^{\otimes q}$ of $L$ descends to a $G_m$-bundle over $\bar{X}/R^{[1]}$ for some $q = p^r$.

Finally, let $R^{[2]} \subset R$ denote the union of all irreducible components of dimension $\leq \dim X - 2$. Let $Z \subset X$ be the image of $R^{[2]}$. Then $Z$ has codimension $\geq 2$ in $X$ and $R$ agrees with $R^{[1]}$ over $X \setminus Z$. Thus $\bar{X}/R^{[1]} \to X$ is a finite, universal homeomorphism over $X \setminus Z$. Since $X$ is $S_2$, this implies that $\bar{X}/R^{[1]} \to X$ is a finite, universal homeomorphism over $X$. By construction, it is an isomorphism at all generic points. Since $X$ is weakly normal, this implies that $\bar{X}/R^{[1]} = X$. Thus $L^{\otimes m}$ extends to a $G_m$-bundle over $X$, hence it is trivial in $\text{Pic}^{\text{loc}}(x, X)$.

48. Note that we have proved that $I_x$ is a homomorphism of algebraic groups but the proof did not establish that its image is a closed algebraic subgroup of $T$. (For instance we did not exclude the possibility that $\ker \pi^* \cong \mathbb{Z}$ and $I_x$ is an injection.) However, once we know that $\ker \pi^*$ in Theorem 6 is of finite type, Proposition 47 implies that $\ker \pi^*$ is in fact linear.
Corollary 49. Let \( k \) be a finite field and \( X \) an excellent \( k \)-scheme of pure dimension \( \geq 3 \) that is weakly normal and \( S_2 \). Let \( x \in X \) be a closed point. Then the kernel of the pull-back map

\[
\pi^* : \text{Pic}^\text{loc}(x, X) \to \text{Pic}^\text{loc}(\bar{x}, \bar{X})
\]

is torsion.

In contrast with the previous steps, the order of the torsion kernel need not be a power of \( \text{char} \, k \).

Proof. Let \( L \) be a line bundle on \( X \setminus \{x\} \) such that \( \pi^* L \) is trivial on \( \bar{X} \setminus \{\bar{x}\} \). In \([47]4\) we constructed a homomorphism

\[
I_x(L) : H_1(\Gamma_x, \mathbb{Z}) \to \mathbb{G}_m(k).
\]

If \( k \) is finite, then \( \mathbb{G}_m(k) \) is a torsion group. Thus \( I_x(L^m) \) is the trivial homomorphism for some \( m > 0 \). By \([47]\) this implies that \( L^m \) is \( p^\infty \)-torsion, hence \( L \) is a torsion element of \( \text{Pic}^\text{loc}(x, X) \). \( \square \)

We can now prove the following form of Theorem 4 over finite fields.

Theorem 50. Let \( k \) be a finite field, \( X, Y \) excellent \( k \)-schemes of pure dimension \( \geq 3 \) that are topologically \( S_2 \) and \( \pi : Y \to X \) a finite morphism. Let \( x \subset X \) be a finite subscheme and \( y \subset Y \) the preimage of \( x \).

Assume that \( \pi(Y) \) contains the support of a complete intersection subscheme \( x \subset Z \subset X \) of dimension \( \geq 3 \).

Then the kernel of the natural map \( \text{Pic}^\text{loc}(x, X) \to \text{Pic}^\text{loc}(y, Y) \) consists of torsion elements.

Proof. We use induction on the dimension of \( X \). If \( \dim X \leq 3 \) then in fact \( X = Z \) and so \( \pi \) is a finite surjection. This case is treated in Theorem \([38]\) and Paragraph \([16]\).

Thus assume that \( \dim X \geq 4 \). If we replace \( Y \) by some \( Y' \to Y \), the kernel only increases. Thus we may assume that \( \pi(Y) = \text{Supp} \, Z \), \( Y \) is normal and \( Y \to X \) factors through the normalization \( \bar{X} \to X \). Using Theorem \([38]\) we may thus also assume that \( X \) is normal.

Since \( Z \subset X \) is a complete intersection, it is also a complete intersection inside some Cartier divisor \( D \subset X \). As we noted in Definition \([13]\) the divisor \( D \) is also topologically \( S_2 \) by \([\text{Gro68} \text{ XIII.2.1}]\). Using \([\text{BdJ13}]\) we can thus replace \( X \) by \( D \) and finish by induction. \( \square \)

6. USING THE RELATIVE PICARD GROUP

Theorems \([31]\) assert that certain line bundles are contained in \( \text{Pic}^\text{loc-\tau}(x, X) \). We plan to prove such results by first establishing the claim over finite fields and then going back to arbitrary fields. As we already noted in Question \([13]\) we need a method to decide when a line bundle on \( X \setminus \{x\} \) is contained in \( \text{Pic}^\text{loc-\tau}(x, X) \).

51 (General set-up). Theorems \([31]\) and \([7]\) can be formulated as follows.

Step 0 (Starting point). Let \( p : (y, Y) \to (x, X) \) be a finite morphism of local \( k \)-schemes and \( L \in \text{Pic}^\text{loc}(x, X) \) such that \( \pi^* L \) is in \( \text{Pic}^\text{loc-\tau}(y, Y) \). By passing to a power of \( L \), we may assume that \( \pi^* L \) is in \( \text{Pic}^\text{loc-\tau}(y, Y) \), that is, \( \pi^* L \) is algebraically equivalent to the trivial bundle.

We would like to prove that, under suitable conditions, \( L \in \text{Pic}^\text{loc-\tau}(x, X) \).
Step 1 (Spreading out). There is an integral $\mathbb{Z}$-scheme of finite type $S$ such that we have the following.

(i) There are $S$-schemes of finite type $Y_S \to S$ and $X_S \to S$ with sections $\sigma_Y : S \to Y_S$ and $\sigma_X : S \to X_S$.
(ii) There is a finite morphism $p_S : Y_S \to X_S$ such that $\sigma_X = p_S \circ \sigma_Y$ and $\text{red} p_S^{-1}(\sigma_X(S)) = \sigma_Y(S)$.
(iii) There is a line bundle $L_S$ on $X_S \setminus \sigma_X(S)$ such that, for every $s \in S$ the restriction $p_S^* L_S|_{X_s}$ is algebraically equivalent to the trivial bundle on $Y_s$.
(iv) There is a map to the generic point $\text{Spec} k \to S$ such that
\[(p : Y \to X, L) \cong \text{Spec} k \times_S (p_S : Y_S \to X_S, L_S).\]

There are several known results that say that certain good properties of the generic fiber are inherited by all fibers, at least over a dense open subset; see \cite{Gro60} IV.12 for long lists. For example, if the generic fiber is $S_2$ and has pure dimension $d$ then, possibly after shrinking $S$, we may assume that every fiber is $S_2$ and has pure dimension $d$.

Step 2 (Over finite fields). For every closed point $s \in S$ we have a finite morphism $p_s : (y_s, Y_s) \to (x_s, X_s)$ and $L_s \in \text{Pic}^{\text{loc}}(x_s, X_s)$ such that $p_s^* L_s$ is algebraically equivalent to the trivial bundle.

This is an instance of the original problem over the residue field $k(s)$. Since $S$ is $\mathbb{Z}$-scheme of finite type, these residue fields are all finite. Thus Theorem \[50\] implies that, under our assumptions, $L_s^{m_s}$ is trivial for some $m_s > 0$.

If $m_s = m$ is independent of $s$ then it is reasonable to expect that $L^m \cong O_X$. However, this is usually not the case. If $\text{Pic}^{\text{loc}}(x, X)$ has positive dimension and char $k = 0$ then there are non-torsion line bundles $L \in \text{Pic}^{\text{loc}}(x, X)$. For these, the $m_s$ are not even bounded.

Step 3 (Lifting; easy case). If depth$_x X_s \geq 3$ then \[\text{Gro68} \ \text{XI.3.16}\] implies that $L_s^{m_s} \cong O_X$ and we are done. This happens precisely when $\text{Pic}^{\text{loc}}(x, X) \cong \text{Spec} k$, thus $\text{Pic}^{\text{loc}}(x, X)$ is identified with the torsion subgroup of $\text{NS}^{\text{loc}}(x, X)$.

Step 4 (Lifting; hard case). At this point we can forget about $Y$ and $\pi : Y \to X$ in the original set-up. Thus from now on we have only the following data.

(i) A local $k$-scheme $(x, X)$ and $L \in \text{Pic}^{\text{loc}}(x, X)$.
(ii) A finitely generated $\mathbb{Z}$-subalgebra $A \subset k$ and its spectrum $S$.
(iii) An $S$-scheme of finite type $X_S \to S$ with a section $\sigma_X : S \to X_S$. We set $x_s := \sigma_X(s)$ for $s \in S$.
(iv) A line bundle $L_S$ on $X_S \setminus \sigma_X(S)$ such that $L_s \in \text{Pic}^{\text{loc}}(x_s, X_s)$ is torsion for every closed point $s \in S$.
(v) An isomorphism $(x, X, L) \cong \text{Spec} k \times_S (\sigma_X(S), X_s, L_s)$.

The proofs of Theorems \[4, 6\] and \[7\] will be completed by the next result.

**Theorem 52.** Let $S$ be an integral scheme whose closed points are dense in $S$. Let \( f : X \to S \) be a flat morphism whose fibers are $S_2$ and have pure dimension $\geq 3$. Let $\sigma : S \to X$ be a section and $L$ a line bundle on $X \setminus \sigma(S)$.

Assume that, for a dense set of closed point $s \in S$, there is a natural number $m_s \in \mathbb{N}$ such that $L^{m_s} \cong O_{U_s}$, where $U_s := X_s \setminus \{x_s\}$.

Then $L_{k(S)} \in \text{Pic}^{\text{loc}}(x_{k(S)}, X_{k(S)})$. 
Remark 53. It is possible that the assumption on closed points being dense is not necessary. By Proposition 58 this holds if the universal deformation in Proposition 55 admits an algebraization. I am able to prove only a weaker version of this: an algebraization of $\text{Pic}^{\text{loc}}$ over the generic fiber gives an algebraization of the universal deformation over an open subset of $S$. This is why I need to assume that closed points are dense in $S$; an assumption that always holds in our applications.

7. Formal deformation theory of $\text{Pic}^{\text{loc}}$

54. Let $(s, S)$ be a local scheme, $(y, Y)$ a local Henselian scheme and $g : Y \to S$ a flat, affine morphism.

Fix a line bundle $L$ on $Y \setminus \{y\}$. Let $(A, m_A)$ be a local Artin $\mathcal{O}_S$-algebra with residue field $k = A/m_A$. Set $Y_A := Y \times_S \text{Spec} A$ and $U_A := Y_A \setminus \{y\}$.

Let $\text{Def}_S(L, A)$ denote the set of isomorphism classes of line bundles on $U_A$ whose restriction to $U_k$ is isomorphic to $(\text{the pull-back of}) \ L$.

We check the conditions of [Sch08] for the pro-representability of the functor $A \mapsto \text{Def}_S(L, A)$. Consider an extension

$$0 \to M \to B \to A \to 0$$

such that $M^2 = 0$. Correspondingly there is an exact sequence

$$0 \to M \otimes \mathcal{O}_{U_s} \to \mathcal{O}_{U_B} \to \mathcal{O}_{U_A} \to 1.$$ (54.1)

Pick any $\gamma_A \in H^0(U_A, \mathcal{O}_{U_A}^*)$. If the fibers of $f$ are $S_2$ then $\gamma_A$ extends to a section $\gamma'_A \in H^0(Y_A, \mathcal{O}_{Y_A})$. Since $f$ is affine, we can lift $\gamma'_A$ to $\gamma_B \in H^0(Y_B, \mathcal{O}_{Y_B})$ and then restrict to $\gamma_B \in H^0(U_B, \mathcal{O}_{U_B}^*)$. Thus

$$H^0(U_B, \mathcal{O}_{U_B}^*) \to H^0(U_A, \mathcal{O}_{U_A}^*)$$

is surjective. (54.3)

Thus the following is a piece of the long exact cohomology sequence of (54.2)

$$0 \to M \otimes H^1(U_s, \mathcal{O}_{U_s}) \to H^1(U_B, \mathcal{O}_{U_B}^*) \to H^2(U_s, \mathcal{O}_{U_s})$$

where $\text{obs}$ is called the obstruction map. If (54.1) splits then the liftings form a principal homogeneous space under $M \otimes H^1(U_s, \mathcal{O}_{U_s})$; the latter is independent of $A$. Furthermore, if $\dim X_s \geq 3$ then $H^1(U_s, \mathcal{O}_{U_s})$ is finite dimensional by (76).

It remains to understand liftings to $D := B \times_A C$ where $B \to A$ and $C \to A$ are maps of Artin $\mathcal{O}_S$-algebras. A line bundle $L_D$ on $U_D$ is determined by a line bundle $L_B$ on $U_B$, a line bundle $L_C$ on $U_C$ plus an isomorphism $\phi : L_C|_{Y_A} \cong L_B|_{Y_A}$. Any different isomorphism is given by $\gamma_A \cdot \phi$ where $\gamma_A \in H^0(U_A, \mathcal{O}_{U_A}^*)$. By (54.3) one can lift $\gamma_A$ to $\gamma_B \in H^0(U_B, \mathcal{O}_{U_B}^*)$. Thus instead of changing $\phi$ by $\gamma_A$ we can change $L_B$ by the isomorphism $\gamma_B : L_B \to L_B$ to conclude that $L_D$ does not depend on $\phi$. That is

$$\text{Def}_S(L, B \times_A C) \cong \text{Def}_S(L, B) \times_{\text{Def}_S(L, A)} \text{Def}_S(L, C).$$

We have thus proved the following.

Proposition 55. Let $(s, S)$ be a local scheme, $(y, Y)$ a local Henselian scheme and $g : Y \to S$ a flat morphism with $S_2$ fibers of dimension $\geq 3$. Let $L$ be a line bundle on $Y_s \setminus \{y\}$. Then $A \mapsto \text{Def}_S(L, A)$ is pro-representable by a complete, local, Noetherian $\mathcal{O}_S$-algebra $\text{Def}_S(L)$. □
Definition 56. Let $S$ be a (non-local) scheme and $f : X \to S$ a flat morphism with $S_2$ fibers of pure dimension $\geq 3$. Let $\sigma : S \to X$ be a section. For every $s \in S$ we are especially interested in two deformation functors.

\[ \text{(56.1)} \quad \text{Def}_{(s,S)}(O_{X_s}) \] is obtained by applying Proposition 55 to $(s, S) := \text{the localization of } S \text{ at } s \text{ and } (y, Y) := \text{the Henselization of } X \text{ at } \sigma(s). \] This is an $O_{s,S}$-algebra parametrizing deformations (over Artin algebras over $O_{s,S}$) of the trivial line bundle $O_{X_s}$ (pulled back to $Y_s \setminus \{y\}$).

\[ \text{(56.2)} \quad \text{Def}_{k(s)}(O_{X_s}) \] is obtained by applying Proposition 55 to $S := \{s\}$ and $(y, Y) := \text{the Henselization of the fiber } X_s \text{ at } \sigma(s).$ This is a $k(s)$-algebra parametrizing deformations (over Artin algebras over the residue field $k(s)$) of the trivial line bundle $O_{X_s}$ (pulled back to $Y_s \setminus \{y\}$).

Note that

\[ \text{Def}_{k(s)}(O_{X_s}) \cong \text{Def}_{(s,S)}(O_{X_s}) \otimes_S k(s). \] (56.3)

Definition 57 (Universal families). Let $S$ be a scheme, $f : X \to S$ a flat morphism with $S_2$ fibers and $\sigma : S \to X$ a section. A family in $\text{Pic}_{\text{loc}}$ is given by

1. a morphism $p : P \to S$,
2. an étale morphism $g : Y \to X \times_S P$,
3. a lifting of the closed subscheme $(\sigma, 1_P) : P = S \times_S P \hookrightarrow X \times_S P$ to $\sigma_P : P \to Y$ and
4. a line bundle $L_Y$ on $Y \setminus \sigma_P(P)$.

For a point $z \in P$ the corresponding line bundle on $Y_{p(z)} \setminus \sigma_P(z)$ is denoted by $L_z$.

These data give a finite type family in $\text{Pic}_{\text{loc}}$ if $P$ is of finite type over $S$.

It is probably not necessary but for technical reasons it is easier to assume that

5. $\text{red } p : \text{red } P \to \text{red } S$ is smooth.

This always holds in characteristic 0. In positive characteristic it can be achieved after a purely inseparable, dominant (but not necessarily finite) base change $S' \to S$ and by passing to an open subset $P' \subset P$.

The above family is a deformation of the trivial bundle if, in addition,

6. there is a section $p : S \to P$ such that the restriction of $L_Y$ to $Y_p \setminus \sigma_P(P)$ is isomorphic to the structure sheaf where $Y_p := g^{-1}(\pi_2^{-1}(\rho(S)))$.

For every $s \in S$ we can localize at $s$ and get a deformation of $O_{Y_s}$ as in (56.1).

We say that the above family is universal at a line bundle $L_s$ on $X_s \setminus \sigma(s)$ corresponding to a point $z \in P_s$ if the induced map

\[ \text{Def}_{(s,S)}(L_s) \to \hat{\mathcal{O}}_{z,P} \] is an isomorphism. (57.7)

Proposition 58. Let $(s,S)$ be an integral, local scheme with function field $K$. Let $f : X \to S$ be a flat morphism with $S_2$ fibers, $\sigma : S \to X$ a section and $L$ a line bundle on $X \setminus \sigma(S)$. Assume that

1. there is a finite type family as in (57.1–6) that is universal at $O_{X_s}$, and
2. $L_s \cong O_{X_s}$.

Then $L_K \in \text{Pic}_{\text{loc} - \tau}(x_K, X_K)$.

Proof. For every Artin $O_{X_s}$-algebra $A$, the line bundle $L$ defines a deformation of $O_{X_s}$ over $A$ since $L_s \cong O_{X_s}$. This gives a formal deformation of $L_s \cong O_{X_s}$ over $S$, the completion of $S$ at $s$.

By assumption $\text{Def}_{(s,S)}(O_{X_s}) \cong \hat{\mathcal{O}}_{\rho(s), P}$, thus there is a section $u : \hat{S} \to P$ such that $(u^*L_Y)|_{X_A}$ is isomorphic to $L|_{X_A}$ for every Artin $O_{s,S}$-algebra $A$. Proposition...
then implies that $u^*L_Y \cong L|_S$. Since $S$ is integral, the section $u$ factors through red $P$.

Now we use that red $P \to S$ is smooth [57]. Thus $\hat{O}_{\rho(s),\text{red}}$ is a formal power series ring over $\hat{O}_{s,S}$ and so each formal section of red $P \to S$ can be given by an ideal $(t_1-g_1, \ldots, t_m-g_m)$ where $g_i \in \hat{O}_{s,S}$. Thus any two sections are algebraically equivalent. This implies that $L_K$ and $\mathcal{O}_{X_K}$ are algebraically equivalent over $k(\hat{S})$, hence also over $K$. □

**Proposition 59.** Let $(s, S)$ be a local scheme with maximal ideal $m$. Let $f : X \to S$ be a scheme, flat over $S$ with $S_2$-fibers. Let $X_n := \text{Spec}_X \mathcal{O}_X/m^{n+1}\mathcal{O}_X$ be the $n$th infinitesimal neighborhood of $X_0 := X_s$. Let $Z \subset X$ be a subscheme that is finite over $S$ and $j : X \setminus Z \to X$ and $j_n : X_n \setminus Z_n \to X_n$ the natural injections. Let $L$ be a locally free sheaf on $X \setminus Z$ and $L_n := L|_{X_n \setminus Z_0}$. Assume that one of the following holds.

1. $(j_n)_*(L_n)$ is locally free for every $n \geq 0$.
2. $(j_0)_*(L_0)$ is locally free and $R^1(j_0)_*(L_0) = 0$.

Then $j_*L$ is locally free in a neighborhood of $Z_0$.

Proof. We may assume that $\mathcal{O}_S$ is $m$-adically complete and, possibly after passing to a smaller neighborhood of $Z_0$, we may assume that $f$ is affine and $(j_0)_*(L_0) \cong \mathcal{O}_X$. For every $n$ we have an exact sequence

$$0 \to (m^0_0/m_0^{n+1}) \otimes L_0 \to L_n \to L_{n-1} \to 0.$$ 

Pushing it forward we get an exact sequence

$$0 \to (m^0_0/m_0^{n+1}) \otimes (j_0)_*(L_0) \to (j_n)_*(L_n) \xrightarrow{\rho} (j_{n-1})_*(L_{n-1}) \to$$

$$\to (m^0_0/m_0^{n+1}) \otimes R^1(j_0)_*(L_0).$$

If $(j_n)_*(L_n)$ is locally free then so is its restriction to $X_{n-1}$ and $r_n$ gives a map of locally free sheaves

$$r_n : (j_n)_*(L_n)|_{X_{n-1}} \to (j_{n-1})_*(L_{n-1})$$

that is an isomorphism on $X_{n-1} \setminus Z_{n-1}$. Since depth$_{Z_{n-1}} X_{n-1} \geq 2$, this implies that $\bar{r}_n$ is an isomorphism and so $r_n$ is surjective. The vanishing of $R^1(j_0)_*(L_0)$ also implies that $r_n$ is surjective. Thus each $(j_n)_*(L_n)$ is locally free along $X_n$ and the constant 1 section of $(j_0)_*(L_0) \cong \mathcal{O}_{X_0}$ lifts back to a nowhere zero global section of $\lim (j_n)_*(L_n)$. Hence $\lim (j_n)_*(L_n) \cong \mathcal{O}_X$. 

Furthermore, we have a natural map $j_*L \to \lim (j_n)_*(L_n) \cong \mathcal{O}_X$ that is an isomorphism on $X \setminus Z$. Since depth$_Z j_*L \geq 2$, this implies that $j_*L \cong \mathcal{O}_X$. □

The examples below show that going from formal triviality of deformations to actual triviality is not automatic.

**Example 60.** Let $(e, E) \cong (e, E')$ be an elliptic curve. Set $X := (E \setminus \{e\}) \times E'$ and $p : X \to E'$ the second projection. Let $\Delta \subset X$ be the diagonal and $L = \mathcal{O}_X(\Delta)$.

For $p \in E' \setminus \{e\}$ the line bundle $L|_{X_p}$ is a nontrivial element of

$$\text{Pic}(X_p \setminus \{e\}) \cong \text{Pic}(E \setminus \{e\}) \cong \text{Pic}^0(E).$$

but $L|_{X_e}$ is trivial.
For $m \in \mathbb{N}$ let $X_m \subset X$ denote the $m$th infinitesimal thickening of the fiber $X_1 := X_e$. We have exact sequences

$$H^1(X_1, \mathcal{O}_{X_1}) \to H^1(X_{m+1}, \mathcal{O}_{X_{m+1}}) \to H^1(X_m, \mathcal{O}_{X_m}) \to H^2(X_1, \mathcal{O}_{X_1}).$$

Since $X_1 \cong E \{e\}$ is affine, this shows that

$$\text{Pic}(X_m \{e\}) \cong \text{Pic}(E \{e\}) \cong \text{Pic}^0(E).$$

Thus $L|_{X_m}$ is trivial for every $m$.

**Example 61.** Consider the family of smooth, affine surfaces

$$S := (x^2 + y^2 = 1 + t^2 z^2) \subset A_3 \times A_1.$$

Set $D := (x - 1 = y - tz = 0)$ and $L := \mathcal{O}_S(D)$.

$S_t$ is a hyperboloid for $t \neq 0$, thus $\text{Pic}(S_t) \cong \mathbb{Z}$ is generated by $L_t$. For $t = 0$ we get a cylinder and $\text{Pic}(S_t) \cong \mathbb{Z}/2$ is generated by $L_0$. As in the previous example, we see that $L^2$ is trivial on all infinitesimal neighborhoods of $S_0$.

8. **Existence of universal families**

62. Assume that we have a field $K$ that is finitely generated over its prime field and a local scheme $(x, X)$ of finite type over $K$. Write $P_K := \text{Pic}^\text{loc-}\sigma(x, X)$.

As in (57.1–4) there is a universal family

$$(Y_K \xrightarrow{g_K} X_K \times P_K \to P_K, L_{Y_K}).$$

(62.1)

Everything in (62.1) can be defined over a finitely generated subring of $A \subset K$, thus there is an integral scheme $S$ of finite type over Spec $\mathbb{Z}$ such that (62.1) is the generic fiber of a family

$$(Y_S \xrightarrow{g_S} X_S \times P_S \to P_S, L_{Y_S}).$$

(62.2)

The statement and the proof of the next result closely follow [Art74]. I go through the details for two reasons. I always found [Art74] rather concise and, more importantly, not all the assumptions of [Art74] are satisfied in our case. There are two main differences. The automorphisms groups of our objects are all infinite dimensional, but, as it was already observed in [Bou78], this does not seem to cause any problems. A more difficult point is that the obstruction spaces are also infinite dimensional.

For the knowledgeable reader, Theorem 78 is the only part not contained in [Art74], [Bou78].

**Theorem 63** (Openness of universality). Using the above notation, there is a dense open subset $T \subset S$ such that

$$(Y_T \xrightarrow{g_T} X_T \times P_T \to P_T, L_{Y_T})$$

is everywhere universal (as in Definition 57).

**Remark 64.** One can imagine that

$$P_T \xrightarrow{\sim} \text{Pic}^\text{loc-}\sigma(\sigma_T, X_T),$$

but we do not claim this. The main reason is that there are families of line bundles $L$ over $X_S \setminus \sigma(S)$ such that $L_s \in \text{Pic}^\text{loc-}\sigma(x_s, X_s)$ at the generic point but not at some special points. Consider for example the family

$$X_s := (xy = uv(u + v + s)) \subset \mathbb{C}^4 \quad \text{and} \quad D_s := (x = u = 0) + (x = v = 0).$$
For \( s \neq 0 \) we see that \( D_s \sim (x = 0) \) is trivial in \( \text{Pic}^{\text{loc}}(0, X_s) \). For \( s = 0 \) we can use [Kol91, 2.2.7] to see that \( \text{Pic}^{\text{loc}}(0, X_0) \cong \mathbb{Z}^2 \) with \( (x = u = 0) \) and \( (x = v = 0) \) as generators. Thus \( D_0 \) gives a non-torsion element in \( \text{Pic}^{\text{loc}}(0, X_0) = \text{NS}^{\text{loc}}(0, X_0) \). I do not know whether such points \( s \in S \) can be Zariski dense or not.

Thus \( P_T \) should be viewed as an open neighborhood of the zero section in the (possibly nonexistent) \( \text{Pic}^{\text{loc}}(\sigma_T, X_T) \).

Proof. By generic flatness we may assume that \( P \to S \) is flat.

First we prove that \( \text{Def}_{(s, S)}(\mathcal{O}_{Y_s}) \to \tilde{\mathcal{O}}_{\rho(s), P} \) is an isomorphism if the map between the fibers over the points \( \text{Def}_{(s, S)}(\mathcal{O}_{Y_s}) \to \tilde{\mathcal{O}}_{\rho(s), P} \) is an isomorphism. This is completely general and follows from (67).

Consider the tangent map

\[
t_{P_K/K} : T_{\rho(K), P_K} \to R^1 j_K^* \mathcal{O}_{U_K}
\]

defined in (65). By assumption \( t_{P_K/K} \) is an isomorphism. We prove in (66) that the tangent maps

\[
t_{P_s/s} : T_{\rho(s), P_s} \to R^1 j_s^* \mathcal{O}_{U_s}
\]

are isomorphisms for all \( s \) in a Zariski open subset of \( S \).

If \( P \to S \) is smooth then a simple algebra lemma (68) shows that \( \text{Def}_{(s, S)}(\mathcal{O}_{Y_s}) \to \tilde{\mathcal{O}}_{\rho(s), P} \) is an isomorphism and we are done. If \( \text{Pic}^{\text{loc}}(x, X) \) is a smooth group scheme then, possibly after shrinking \( S \), we may assume that \( P \to S \) is smooth. A group scheme over a field of characteristic 0 is always smooth, thus the proof of Theorem (63) and hence also the proofs of Theorems (64) are complete if char \( k = 0 \).

Otherwise we need a more detailed study of obstruction theory; this is started in (69).

65 (Tangent map). We continue with the notation of (62). Let \( I_{S, P} \subset \mathcal{O}_P \) be the ideal sheaf of \( \rho(S) \subset P \). We identify \( S \) with \( \rho(S) \) and set \( R := \text{Spec}_P \mathcal{O}_P/I_{S, P}^2 \).

The ideal sheaf of \( S \subset R \) is denoted by \( I_S \). These data are encoded in a diagram

\[
\begin{array}{ccc}
Y_S & \hookrightarrow & Y_R \\
\downarrow & & \downarrow g \uparrow \sigma \\
S & \hookrightarrow & R \\
\| & & \downarrow \\
S & = & S
\end{array}
\]

where \( g \) is flat,

\[
\text{where } \sigma \text{ is a section and } I_S^2 = 0
\]

Set \( U_R := Y_R \setminus \sigma(R) \) with natural injection \( j : U_R \to Y_R \) and \( U_S := Y_S \setminus \sigma(S) \). Let \( L_R \) be a line bundle on \( U_R \) and \( L_S := L_R|_{U_S} \). There is an exact sequence

\[
0 \to I_S \otimes_S L_S \to L_R \to L_S \to 0.
\]

Pushing it forward by \( j \) we get

\[
0 \to I_S \otimes_S j_* L_S \to j_* L_R \to j_* L_S \overset{\partial}{\to} I_S \otimes_S R^1 j_* L_S.
\]

Assume now that \( L_S \cong \mathcal{O}_U \) and \( Y_S \to S \) has \( S_2 \) fibers. Then \( j_* L_S \cong \mathcal{O}_{Y_S} \) and the exact sequence becomes

\[
0 \to I_S \otimes_S \mathcal{O}_{Y_S} \to j_* L_R \to \mathcal{O}_{Y_S} \overset{\partial}{\to} I_S \otimes_S R^1 j_* \mathcal{O}_{Y_S}.
\]

Here \( \partial \) factors through \( \mathcal{O}_{Y_S} \to \mathcal{O}_Z \cong \mathcal{O}_S \) thus, if \( I_S \) is locally free over \( \mathcal{O}_S \), \( \partial \) is equivalent to the tangent map

\[
t_{R/S} : \text{Hom}_S(I_S, \mathcal{O}_S) \to R^1 j_* \mathcal{O}_{Y_S}.
\]
(Note that $\mathcal{H}om_S(I_S, \mathcal{O}_S)$ is isomorphic to the relative tangent sheaf of $R/S$ restricted to $S \cong \sigma(S)$.)

If we started with a set-up as in (62) then $R^1j_*\mathcal{O}_{Y_S} \cong R^1j_*\mathcal{O}_{X_S}$ and we get the following.

**Proposition 66.** Let $(Y_S \stackrel{\phi_S}{\rightarrow} X_S \times \mathbb{P}_S \rightarrow \mathbb{P}_S, S \stackrel{\phi}{\rightarrow} \mathbb{P}_S, L_{Y_S})$ be a deformation as in (62). Assume that $I_{S,P}/I_{S,P}^2$ is free over $S$ and $R^1j_*\mathcal{O}_{X_S}$ is free and commutes with base change. Then the tangent map

$$l_{P/S}: \mathcal{H}om_S(I_{S,P}, \mathcal{O}_S) \rightarrow R^1j_*\mathcal{O}_{X_S}$$

has constant rank over a dense open subset of $S$. 

We have used the following commutative algebra lemmas.

**Lemma 67.** Let $(m, S)$ be a local ring and $\phi : (m_1, R_1) \rightarrow (m_2, R_2)$ a map of local $S$-algebras.

1. If $R_1$ is complete and $\bar{\phi} : R_1/mR_1 \rightarrow R_2/mR_2$ is surjective then $\phi$ is surjective.
2. If $\phi$ is surjective, $\bar{\phi}$ is an isomorphism and $R_2$ is flat over $S$ then $\phi$ is an isomorphism.

**Proof.** For every $r \geq 1$ we have a commutative diagram

$$
\begin{array}{ccc}
(m^r/m^{r+1}) \otimes R_1/mR_1 & \rightarrow & (m^r/m^{r+1}) \otimes R_2/mR_2 \\
\downarrow & & \downarrow \\
m^rR_1/m^{r+1}R_1 & \xrightarrow{\alpha_r} & m^rR_2/m^{r+1}R_2.
\end{array}
$$

The vertical arrows are surjective hence so is $\alpha_r$. By induction on $r$ we obtain that $\bar{\alpha}_r : R_1/m^{r+1}R_1 \rightarrow R_2/m^{r+1}R_2$ are surjective.

Since $R_1$ is complete, we can pass to the inverse limit to conclude that $R_1 = \varprojlim R_1/m^{r+1}R_1 \rightarrow \varprojlim R_2/m^{r+1}R_2$ are surjective.

This factors through the injection $R_2 \hookrightarrow \varprojlim R_2/m^{r+1}R_2$ thus $\phi : R_1 \rightarrow R_2$ is surjective.

For part (2), let $J$ be the kernel of $\phi$. We have an exact sequence

$$0 \rightarrow J \rightarrow R_1 \xrightarrow{\delta} R_2 \rightarrow 0.$$ 

Since $R_2$ is flat over $S$, tensoring with $S/mS$ is also exact, thus we get

$$0 \rightarrow J/mS J \rightarrow R_1/mSR_1 \xrightarrow{\delta} R_2/mSR_2 \rightarrow 0.$$ 

This implies that $J/mSJ = 0$ thus $J = 0$ by the Nakayama lemma.

**Lemma 68.** Let $(R, m)$ be a complete local ring and $(S, n)$ a regular, local $(R, m)$-algebra. Then $R = S$ iff the natural maps $R/m \rightarrow S/n$ and $m/m^2 \rightarrow n/n^2$ are isomorphisms.

**Proof.** If $R = S$ then clearly $R/m \rightarrow S/n$ and $m/m^2 \rightarrow n/n^2$ are isomorphisms.

Conversely, assume that $R/m \rightarrow S/n$ and $m/m^2 \rightarrow n/n^2$ are isomorphisms. By induction on $r$ we see that the natural maps $R/m^r \rightarrow S/n^r$ are surjective. Among local rings $(A, m_A)$ with fixed embedding dimension $\dim_A/m_A/m_A^2$, the length of $S/n^r$ is the largest possible since $(S, n)$ is regular. Thus each $R/m^r \rightarrow S/n^r$ is an isomorphism. Since $(R, m)$ is complete this implies that $R = S$. 

□
69 (Obstruction map). Let $S$ be a base scheme, $R \to S$ a flat scheme with a section $\rho : S \to R$ with ideal sheaf $I_S \subset \mathcal{O}_R$. Let $T \subset R$ be a subscheme with ideal sheaf $I_T$. Assume that $\rho(S) \subset T$ and $I_S I_T = 0$.

Let $g : Y_R \to R$ be a flat morphism with a section $\sigma : R \to Y_R$. By restriction we get $Y_S \to S$ and $Y_T \to T$. These data are summarized in the following diagram.

$$
\begin{array}{ccc}
Y_S & \hookrightarrow & Y_T & \hookrightarrow & Y_R \\
\downarrow & & \downarrow \uparrow \sigma & & \\
S & \hookrightarrow & T & \hookrightarrow & R \\
\| & & \downarrow & & \\
S & = & S & = & S
\end{array}
$$

where $g$ is flat and $\sigma$ is a section and (69.1)

Finally write $U_R := Y_R \setminus \sigma(R)$, $U_T := Y_T \setminus \sigma(T)$, $U_S := Y_S \setminus \sigma(S)$ and $L_T$ be a line bundle on $U_T$. We would like to understand when $L_T$ extends to a line bundle $L_R$ on $U_R$.

We have an exact sequence

$$0 \to I_T \otimes_S \mathcal{O}_{U_S} \xrightarrow{m-1+1} \mathcal{O}_{U_R}^* \to \mathcal{O}_{U_T}^* \to 1. \quad (69.2)$$

This gives

$$R^1 j_* \mathcal{O}_{U_R}^* \to R^1 j_* \mathcal{O}_{U_T}^* \xrightarrow{\partial} I_T \otimes_S R^2 j_* \mathcal{O}_{U_S}. \quad (69.3)$$

The line bundle $L_T$ corresponds to a section $\mathcal{O}_S \to R^1 j_* \mathcal{O}_{U_T}^*$; composing with $\partial$ gives

$$[L_T] : \mathcal{O}_S \to I_T \otimes_S R^2 j_* \mathcal{O}_{U_S}. \quad (69.4)$$

Thus $L_T$ extends to a line bundle $L_R$ iff $[L_T] = 0$.

If $I_T/I_S I_T$ is free over $S$ then $[L_T]$ is equivalent to a map, called the obstruction,

$$\text{obs}(L_T, R) : \text{Hom}_S(I_T, \mathcal{O}_S) \to R^2 j_* \mathcal{O}_{U_S}. \quad (69.5)$$

and $L_T$ extends to a line bundle $L_R$ iff $\text{obs}(L_T, R) = 0$.

For us the following two consequences are especially important.

Claim 69.6. Assume that $S$ is integral, $I_T/I_S I_T$ is free over $S$ and $R^2 j_* \mathcal{O}_{X_S}$ is free and commutes with base change over $S$.

i) The obstruction map $\text{obs}(L_T, R)$ has constant rank and commutes with base change over a dense open subset of $S$.

i) If $\text{obs}(L_T, R)$ is injective then $L_T$ can not be extended over any subscheme $T \subset T' \subset R$. \hfill \Box

70 (End of the proof of Theorem 63). We return to the setting of (69.1) and, in addition, we choose a scheme $W_K \supset P_K$ such that $W_K$ is smooth at the identity and has the same tangent space as $P_K$. (Thus $W_K = P_K$ if $P_K$ is smooth.)

As before, everything is defined over a finitely generated subring $A \subset K$, thus there is an integral scheme $S$ of finite type over $\text{Spec} \ Z$ such that (62.1) is the generic fiber of a family

$$\{Y_S \xrightarrow{\rho_S} X_S \times P_S \to P_S, W_S \supset P_S, L_{Y_S}\}. \quad (70.1)$$

In order to apply (69.6), let $\hat{\mathcal{O}}_{S,W}$ denote the completion of the structure sheaf of $W_S$ along the section $\rho(S)$, $\hat{I}_S \subset \hat{\mathcal{O}}_{S,W}$ the ideal sheaf of $\rho(S)$ and $\hat{I}_P \subset \hat{\mathcal{O}}_{S,W}$ the ideal sheaf of $P_S$. Finally set $T := \text{Spec}_S \hat{\mathcal{O}}_{S,W}/\hat{I}_P$ and $R := \text{Spec}_S \hat{\mathcal{O}}_{S,W}/\hat{I}_S \hat{I}_P$.

Shrinking $S$ if necessary, the following conditions can be satisfied:

i) $T, R$ are flat over $S$ and $I_T/I_S I_T$ is free over $S$. 

Thus we see that $g$ has no multiple prime factors. Let $\tilde{\mathcal{V}}$ be any open set. By contrast, $R^1 f_*\mathcal{O}_{X_S}$ is free and commutes with base change. (This follows by applying Theorem 78 to $Y_S \to P_S$ and the sheaf $F = \mathcal{O}_{Y_S}$.)

Thus, by Claim 69.6, the line bundle $L|_T$ does not extended over any subscheme $T \subsetneq T' \subset R$. Therefore $P_S \to S$ is universal along the zero section $\rho : S \to P_S$. Since $P_S \to S$ is a group scheme, this implies that it is everywhere universal. □

9. Quasi-coherent higher direct images

A coherent sheaf over a reduced scheme is free over a dense open set, but there are (even locally free) quasi-coherent sheaves that are not free over any dense open subset. Even worse, a nonzero section may vanish at every closed point.

**Example 71.** Let $M \subset \mathbb{Q}$ be the $\mathbb{Z}$-submodule consisting of all $m/n$ such that $n$ has no multiple prime factors. Let $\tilde{M}$ be the corresponding quasi-coherent sheaf over $\text{Spec}\mathbb{Z}$. Then $\tilde{M}$ is locally free of rank 1, but it is not free. In fact, every global section of it vanishes at all but finitely many points of $\text{Spec}\mathbb{Z}$.

Our aim is to show that such bad behavior does not happen for certain higher direct images.

**72 (Direct image functors).** We study functors $\mathcal{H}$ with the following properties.

Given a morphism $f : X \to S$ and a quasi-coherent sheaf $F$ on $X$, $\mathcal{H}(F)$ is a quasi-coherent sheaf on $S$ and for every $p : T \to S$ there are base-change maps

$$p^*\mathcal{H}(F) \to \mathcal{H}(p^*_X F).$$

This is called an isomorphism for every $p : T \to S$. We say that $\mathcal{H}$ is free if $\mathcal{H}(F)$ is a free quasi-coherent sheaf.

We say that $\mathcal{H}(F)$ is generically free and commutes with base change if there is a nonempty open set $S^0 \subset S$ such that $\mathcal{H}(F^0)$ is free and commutes with base change where $F^0$ denotes the restriction of $F$ to $X^0 := f^{-1}(S^0)$.

The Cohomology and Base change theorem for proper morphisms implies that if $S$ is reduced, $f : X \to S$ is proper and $F$ is a coherent sheaf on $X$ then $R^i f_* F$ is generically free and commutes with base change for every $i$. In our applications we are especially interested in some cases where $\mathcal{H}(F)$ is only quasi-coherent, even though $F$ itself is coherent. Let us start with an example where higher direct image is either not generically free or does not commute with base change.

**Example 73.** Let $E$ be an elliptic curve and $L$ the Poincaré bundle on $E \times E$. Set $X := \text{Spec}_{E \times E} \sum_{i \geq 0} L^i$ and $g : X \to E \times E \to E$ a projection. Note that $g$ is smooth, has fiber dimension 2 but it is neither affine nor proper.

If $e \in E$ is not a torsion point then $H^0(X_e, \mathcal{O}_{X_e}) = H^1(X_e, \mathcal{O}_{X_e}) = 0$. If $e \in E$ is a torsion point of order exactly $m$ then there are natural identifications

$$H^0(X_e, \mathcal{O}_{X_e}) \cong H^1(X_e, \mathcal{O}_{X_e}) \cong \sum_{i \in m\mathbb{N}} k(e).$$

Thus we see that $g_* \mathcal{O}_X = 0$ is free but it does not commute with base change over any open set. By contrast, $R^1 g_* \mathcal{O}_X$ is a sum of skyscraper sheaves of infinite rank supported at the torsion points. Thus it is not generically free but it does commute with base change.
The following are some basic examples that we use.

**Theorem 74** (Generic freeness). Let \( f : X \to S \) be an affine morphism of finite type, \( S \) reduced and \( F \) a coherent sheaf on \( X \). Then \( f_* F \) is generically free and commutes with base change.

Proof. Frequently this is stated as “Generic flatness” under the above assumptions, \( F \) is flat over a dense, open subscheme \( S^0 \subset S \); see, for example [Mum66 Lect.8]. It is stated as “Generic freeness” in [Eis95, Sec.14.2], but without the commutation with base change. However, both proofs show the stronger forms.

**Example 75.** Let \( X = \mathbb{A}^n \cong \text{Spec}_S \mathcal{O}_S[x_1, \ldots, x_n] \), \( Z \subset X \) the zero section and \( j : X \setminus Z \hookrightarrow X \) the natural injection. Then \( f_* \circ R^i j_* \mathcal{O}_X \mid Z \) is free and commutes with base change.

Note that \( f_* \circ j_* \mathcal{O}_X \setminus Z = f_* \mathcal{O}_X \) if \( n \geq 2 \) and, for \( i \geq 1 \), the sheaf \( R^i j_* \mathcal{O}_X \setminus Z \) is supported on \( Z \cong S \), thus \( f_* \) is an isomorphism. Furthermore, the only nonzero case is \( R^{n-1} j_* \mathcal{O}_X \setminus Z \) which can be identified with the quasi-coherent sheaf freely generated by

\[
\frac{1}{x_1^{a_1} \cdots x_n^{a_n}} \quad \text{for all} \quad a_1 \geq 1, \ldots, a_n \geq 1.
\]

(This is equivalent to the computation of the groups \( H^{n-1}(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(m)) \) done in [Har77, III.5.1], see especially the 4th displayed formula on p.226.) For more explicit references see [Eis95 p.685].

More generally, if \( f : X \to S \) is smooth along a section \( Z \subset X \) and \( j : X \setminus Z \hookrightarrow X \) is the natural injection then \( f_* \circ R^i j_* \mathcal{O}_X \setminus Z \) is generically free for \( i \geq 1 \).

We use the following basic coherence theorem.

**Theorem 76.** [Gro68 VIII.2.3] Let \( X \) be an excellent scheme, \( Z \subset X \) a closed subscheme, \( U := X \setminus Z \) and \( j : U \hookrightarrow X \) the open embedding. Assume in addition that \( X \) is locally embeddable into a regular scheme. For a coherent sheaf \( G \) on \( U \) and \( n \in \mathbb{N} \) the following are equivalent.

1. \( R^i j_* G \) is coherent for \( i < n \).
2. \( \text{depth}_u G \geq n \) for every point \( u \in U \) such that \( \text{codim}_u(Z \cap \bar{u}) = 1 \). \( \square \)

We need some elementary properties of direct image functors.

**Lemma 77.** Let \( 0 \to \mathcal{H}_1(F) \to \mathcal{H}_2(F) \to \mathcal{H}_3(F) \to 0 \) be an exact sequence of direct image functors.

1. If \( \mathcal{H}_1(F) \), \( \mathcal{H}_3(F) \) are free then so is \( \mathcal{H}_2(F) \).
2. If \( \mathcal{H}_2(F) \) is free, \( \mathcal{H}_1(F) \) is coherent and \( S \) is reduced then \( \mathcal{H}_3(F) \) is generically free.
3. If \( \mathcal{H}_1(F) \), \( \mathcal{H}_3(F) \) commute with base change then so does \( \mathcal{H}_2(F) \). \( \square \)

The following is the main new result of this section. For our applications we need the case \( F = \mathcal{O}_X \) and \( r = 1, 2 \) but the proof of the general version is similar.

**Theorem 78.** Let \( S \) be a reduced scheme, \( f : X \to S \) an affine morphism of finite type and \( F \) a coherent sheaf on \( X \). Let \( Z \subset X \) be a section, \( U := X \setminus Z \) and \( j : U \hookrightarrow X \) the natural injection. Assume that \( F \mid_U \) is flat over \( S \) and each \( F \mid_U \) is pure dimensional and \( r_S \).

Then \( f_* \circ R^i j_* (F \mid_U) \) is generically free and commutes with base change. \( \square \)

---

1K. Smith pointed out that the assumptions can be considerably weakened; see [Smi14].
Note that if each $F_s$ is $S_{r+1}$ then $f_s \circ R^rj_*(F|_U)$ is coherent by (76) but otherwise $f_s \circ R^rj_*(F|_U)$ almost always has infinite rank.

Proof. By induction on $r$, starting with $r = 0$. Here we do not assume that $F_s$ is pure dimensional. If $\dim F_s \leq 1$ then $\text{Supp} F$ is affine and $f_s \circ j_*(F|_U) \cong (f|_U)_*(F|_U)$ is generically free and commutes with base change by (74).

If every associated prime of $F_s$ has dimension $\geq 2$ then $j_*(F|_U)$ is coherent by (76). We claim that it generically commutes with base change. A sheaf $G$ on $X$ such that $G|_U \cong F$ equals $j_*(F|_U)$ iff it has no associated primes supported on $Z$ and $\text{depth}_X G \geq 2$. Both of these are open conditions in flat families by [Gro60 IV.12.1.6]. Since $\text{Supp} j_*(F|_U)$ is affine over $S$, we are done as above.

Even if $F_s$ is not pure dimensional, we have an exact sequence

$$0 \to F^{\leq 1} \to F|_U \to Q \to 0$$

where $F^{\leq 1}$ is the largest subsheaf with $\leq 1$-dimensional support; thus every associated prime of $Q_s$ has dimension $\geq 2$. By pushing forward, we get

$$0 \to (f|_U)_*(F^{\leq 1}) \to f_s \circ j_*(F|_U) \to f_s \circ j_* Q \to R^1j_*F^{\leq 1} = 0.$$ 

Thus $f_s \circ j_*(F|_U)$ is is generically free and commutes with base change by (76).1

Thus assume that $r \geq 1$. In this case $R^rj_*(F|_U)$ is supported on $Z \cong S$, thus $f_s$ is an isomorphism and we can drop it from the notation.

Let $n$ be the fiber dimension of $\text{Supp} F \to S$. If $n \leq r$ then $R^rj_*(F|_U) = 0$ and we are done. Thus assume from now on that $n \geq r + 1$. We can also replace $X$ by $\text{Supp} F$.

After possibly shrinking $S$, we can use Noether normalization to obtain a finite morphism $g : X \to \mathbb{A}^n_S$ such that $Z$ is mapped isomorphically to the 0-section $Z_0 \subset \mathbb{A}^n_S$. Let $j_0 : U_0 := \mathbb{A}^n_S \setminus Z_0 \to \mathbb{A}^n_S$ be the natural injection.

After possibly further shrinking $S$ we may also assume that $g^{-1}(Z_0)$ is the disjoint union of $Z$ and of another closed subscheme $Z'$. Set $U' := X \setminus Z'$ with natural injection $j' : U' \to X$. Then

$$R^r(j_0)_*(g_*(F|_{U_0})) = (g_*R^rj_*(F|_U)) + (g_*R^rj'_*(F|_{U'})).$$

Thus if $R^r(j_0)_*(g_*(F|_{U_0}))$ is generically free and commutes with base change then the same holds for its direct summands. Since $R^rj_*(F|_U)$ is supported on $Z$ and $g|_Z : Z \to Z_0$ is an isomorphism, $g_*R^rj_*(F|_U)$ is naturally isomorphic to $R^rj_*(F|_U)$.

Thus we have reduce everything to the case when $f : X \cong \mathbb{A}^n_S \to S$ is smooth with integral fibers and $F_s$ is torsion free of rank say $m$ for every $s \in S$.

Over an integral scheme, a torsion free coherent sheaf of rank $m$ is isomorphic to a subsheaf of a free sheaf of rank $m$. Hence there is an exact sequence

$$0 \to F|_U \to O^m_U \to Q \to 0$$

where $\dim Q \leq n - 1$. We also need that $Q$ is $S_{r-1}$, see for example [Kol13b 2.60]. In particular, if $r \geq 2$ then $Q$ has pure dimension $n - 1$ but if $r = 1$ then $Q$ need not be pure dimensional.

Applying $j_*$ we get an exact sequence

$$R^{r-1}j_*O^m_U \to R^{r-1}j_*Q \to R^rj_*(F|_U) \to R^rj_*O^m_U \to R^rj_*Q.$$ 

Depending on the values of $(r, n)$, several of the terms vanish. (78)1 If $r > 1$ and $n \geq r + 2$ then $R^{r-1}j_*O_U = R^rj_*O_U = 0$, thus

$$R^{r-1}j_*Q \cong R^rj_*(F|_U)$$
and we are done by induction.

(78.2) If \( r > 1 \) and \( n = r + 1 \) then \( R^{-1}j_*O_U = 0 \) and \( R^rj_*Q = 0 \), thus we have the exact sequence

\[
0 \rightarrow R^{-1}j_*Q \rightarrow R^rj_*(F|_U) \rightarrow R^rj_*O_U^m \rightarrow 0.
\]

Here \( R^{-1}j_*Q \) and \( R^rj_*O_U^m \) are generically free and commute with base change by induction and (75). Thus \( R^rj_*(F|_U) \) is generically free and commutes with base change by (77.1).

(78.3) If \( r = 1 \) and \( n \geq 3 \) then \( R^1j_*O_U = 0 \) and we have the exact sequence

\[
O_X^m \rightarrow j_*Q \rightarrow R^1j_*(F|_U) \rightarrow 0.
\]

We can use (77.2) to show that \( R^1j_*(F|_U) \) is generically free and commutes with base change.

(78.4) Finally, if \( r = 1 \) and \( n = 2 \) then we can use the exact sequence

\[
0 \rightarrow \text{coker}[O_X^m \rightarrow j_*Q] \rightarrow R^1j_*(F|_U) \rightarrow R^1j_*O_U^m \rightarrow 0.
\]

\( \square \)

10. Restriction of torsion bundles

Here we complete the proof of Theorem 9. We assume that \( \text{depth} \cdot X \geq 3 \), hence \( \text{Pic}^\text{loc}(x, X) \) is 0-dimensional and a 0-dimensional linear algebraic group is finite. This shows (92–3). Thus the only new claim is that \( r^X_D : \text{Pic}^\text{loc}(x, X) \rightarrow \text{Pic}^\text{loc}(x, D) \) is injective (91).

We have already noted in Paragraph 17 that every torsion element in the kernel of \( r^X_D \) has \( p \)-power order where \( p \) is the characteristic. The proof relied on the observation that, for \( p \nmid m \), a torsion element of order \( m \) in \( \text{Pic}^\text{loc}(x, X) \) corresponds to a degree \( m \) étale cover of \( X \setminus \{x\} \) and then used [Gro68, XIII.2.1].

Here we develop another approach to deal with torsion elements in the kernel. The advantage is that for this method the characteristic does not matter. A disadvantage is that the main step works best for proper morphisms, thus it applies only when \( X \) is essentially of finite type.

Theorem 79. Let \( T \) be the spectrum of a DVR with closed point \( 0 \in T \) and generic point \( \eta \in T \). Let \((x, X)\) be a local scheme and \( f : (x, X) \rightarrow (0, T) \) a flat morphism that is essentially of finite type. Let \( x \in Z_0 \subset X_0 \) be a closed subscheme such that \( \text{depth}_{Z_0} X_0 \geq 2 \). Then

\[
\text{ker}[r^X_0 : \text{Pic}(X \setminus Z_0) \rightarrow \text{Pic}(X_0 \setminus Z_0)] \text{ is torsion free.}
\]

Before we start the proof, we recall some results on pull-back and push-forward of sheaves.

80 (Push forward and restriction). Let \( X \) be an excellent scheme, \( Z \subset X \) a closed subscheme and \( U := X \setminus Z \) with natural injection \( j : U \rightarrow X \).

Let \( F \) be a coherent sheaf on \( U \) such that \( \text{codim}_P(\bar{P} \cap Z) \geq 2 \) for every associated prime \( P \in U \) of \( F \). Then \( j_*F \) is coherent by [Gro60, IV.5.11.1]. Moreover, it is the unique coherent sheaf \( G \) on \( X \) such that \( G|_U \cong F \) and \( \text{depth}_X G \geq 2 \).

Let \( f : X \rightarrow S \) be a flat morphism to a regular, 1-dimensional scheme. For \( s \in S \) the restriction of \( j \) is denoted by \( j_s : U_s \rightarrow X_s \). There are natural maps

\[
r_s : (j_*F)|_{X_s} \rightarrow (j_s)_*(F|_{X_s}).
\]

Note that \( j_*F|_{X_s} \) has depth \( \geq 1 \) along \( Z \cap X_s \), in particular, it has no embedded points supported in \( Z \cap X_s \). Thus \( r_s \) is an injection that is an isomorphism over
Let $U_s$. By [Gro60] IV.12.1.6, $j_*F|_{X_s}$ has depth $\geq 2$ along $Z \cap X_s$ for general $s \in S$. Thus $r_s$ is an isomorphism for general $s \in S$.

We are mainly interested in the case when $T = S$ is the spectrum of a DVR with closed point $0 \in T$ and generic point $g \in T$. Let $f : X \to T$ be a flat morphism and $Z_0 \subset X_0$ a closed subscheme such that $\text{depth}_{Z_0} X_0 \geq 2$. This implies that $\text{codim}_{X_0} Z_0 \geq 2$, thus $\dim X_0 \geq 2$ (unless $Z_0$ is empty). Set $U := X \setminus Z_0$ and $U_0 := X_0 \setminus Z_0$ with natural open injections $j : U \to X$ and $j_0 : U_0 \to X_0$.

Let $L_U$ be a line bundle on $U$ and $L_{U_0} := L_U|_{U_0}$ its restriction to $U_0$. We are interested in the sheaves $L$ when it is an isomorphism along $Z$.

Note that $\dim Z$ and ample on $X$ where $L$ along $Z$. As in (80.1) there is a natural map $\chi : j_*L_U|_{X_0} \to (j_0)_*L_{U_0}$.

These sheaves are locally free on $U$ (resp. $U_0$) and have depth $\geq 2$ along $Z_0$. If $L_{U_0} \cong \mathcal{O}_{U_0}$ then $(j_0)_* L_{U_0} \cong \mathcal{O}_{X_0}$; here we use the assumption that depth of $Z_0$, $X_0 \geq 2$.

As in (80.1) there is a natural map

$$r_0 : j_*L_U|_{X_0} \to (j_0)_*L_{U_0}.$$  \hfill (80.2)

Note that $r_0$ an injection that is an isomorphism over $U_0$. We aim to understand when it is an isomorphism along $Z_0$ and then relate this to (79) through a series of local fineness criteria for $j_*L_U$.

**Lemma 81.** Notation and assumptions as in (80). Then $j_*L_U$ is locally free iff

1. $(j_0)_* L_{U_0}$ is locally free and
2. $r_0 : j_*L_U|_{X_0} \to (j_0)_*L_{U_0}$ is an isomorphism.

Proof. If $j_*L_U$ is locally free then $j_*L_U|_{X_0}$ is locally free hence it has depth $\geq 2$ along $Z_0$. Thus $(j_0)_* L_{U_0} = j_*L_U|_{X_0}$ is locally free.

Conversely, if $j_*L_U|_{X_0}$ is locally free then $j_*L_U$ is locally free near $X_0$ by the Nakayama lemma. \hfill $\square$

**82 (Numerical inequalities).** Notation and assumptions as in (80). Assume in addition that $f$ is proper and has relative dimension $n$. By semicontinuity

$$H^0(X_0, (j_0)_* L_{U_0}) \geq H^0(X_0, j_*L_U|_{X_0}) \geq H^0(X_g, L_g)$$ \hfill (82.1)

where $L_g := L_U|_X$. In particular, if $L_g$ is ample on $X_g$ and $(j_0)_* L_{U_0}$ is locally free and ample on $X_0$ then, applying (82.1) to powers of $L_U$ we conclude that

$$(c_1((j_0)_* L_{U_0})^n) \geq (c_1(L_g)^n).$$ \hfill (82.2)

One can be more precise if $\dim Z_0 = 0$. Then $\text{coker} r_0$ is artinian, thus

$$\chi(X_0, (j_0)_* L_{U_0}) = \chi(X_0, j_*L_U|_{X_0}) + \text{length}(\text{coker} r_0).$$ \hfill (82.3)

Since $j_*L_U$ is flat over $T$,

$$\chi(X_0, j_*L_U|_{X_0}) = \chi(X_g, j_*L_U|_{X_0}) = \chi(X_g, L_g).$$

Combining these we see that

$$\chi(X_0, (j_0)_* L_{U_0}) = \chi(X_g, L_g) + \text{length}(\text{coker} r_0).$$ \hfill (82.4)

**Corollary 83.** Notation and assumptions as in (80). Assume in addition that $f$ is proper and $\dim Z_0 = 0$. Then $j_*L_U$ is locally free iff

1. $(j_0)_* L_{U_0}$ is locally free and
2. $\chi(X_0, (j_0)_* L_{U_0}) = \chi(X_g, L_g)$.

Proof. By (82.4) $\chi(X_0, (j_0)_* L_{U_0}) = \chi(X_g, L_g) + \text{length}(\text{coker} r_0)$. Thus $r_0$ is an isomorphism if $\chi(X_0, (j_0)_* L_{U_0}) = \chi(X_g, L_g)$ hence (81) applies. \hfill $\square$

The following is a global version of Theorem 79 and a key step of its proof.
Lemma 84. Notation and assumptions as in (30). Assume in addition that $f$ is proper and $\dim Z_0 = 0$. Then $j_*, L_U$ is locally free iff

1. $(j_0)_* L_{U_0}$ is locally free and
2. $j_*(L_U^r)$ is locally free for some $r > 0$.

Proof. Note that if $(j_0)_* L_{U_0}$ is locally free then so is

$((j_0)_* L_{U_0})^m \cong (j_0)_*(L_{U_0}^m)$ for every $m \in \mathbb{Z}$.

We use (30) for $L_{U_0}^m$ for every $m$. We obtain that the following are equivalent:

3. $j_*, L_U$ is locally free,
4. $j_*(L_U^m)$ is locally free for every $m \in \mathbb{Z}$ and
5. $\chi(X_0, (j_0)_* L_{U_0})^m = \chi(X_g, L_g^m)$ for every $m \in \mathbb{Z}$.

Both Euler characteristics are polynomials in $m$, hence they agree iff they agree for infinitely many values of $m$. Since $j_*(L_U)$ is locally free we know that

$\chi(X_0, (j_0)_* L_{U_0})^m = \chi(X_g, L_g^m)$ for every $m \in \mathbb{Z}$.

Thus $\chi(X_0, (j_0)_* L_{U_0}) = \chi(X_g, L_g)$ and so $j_*, L_U$ is locally free by (30). □

Next we remove the properness assumption from (30).

Lemma 85. Notation and assumptions as in (30). Assume in addition that $f$ is of finite type and $\dim Z_0 = 0$. Then $j_*, L_U$ is locally free iff

1. $(j_0)_* L_{U_0}$ is locally free and
2. $j_*(L_U^r)$ is locally free for some $r > 0$.

Proof. The assertions are local on $X$, hence we may assume that $X$ is affine. Since $f$ is of finite type, there is a compactification $\hat{X} \supset X$ such that $f$ extends to a flat morphism $\hat{f} : \hat{X} \to T$. We intend to use Lemma 84 the only problem is that we do not know how to extend $L_U$ to $\hat{X}$ and what happens along $\hat{X} \setminus X$.

Thus we need to improve the compactification $\hat{X}$. In order to do this, choose an effective Cartier divisor $D_U$ on $U$ such that $L_U \cong O_U(-D_U)$. Let $\hat{D} \subset \hat{X}$ denote its closure and $I_{\hat{D}} := O_{\hat{X}}(-\hat{D})$ its ideal sheaf. Let $\hat{X}$ be the scheme obtained by gluing $\hat{X}_1 := X$ and the blow-up

$\hat{X}_2 := I_{\hat{D}}(\hat{X} \setminus Z_0)$

along $\hat{X}_1 \supset U \cong B_{I_{\hat{D}}} \subset \hat{X}_2$. By construction $f$ extends to a flat morphism $\hat{f} : \hat{X} \to T$. Note that $\hat{f}$ is proper (but not necessarily projective). Also, even if $f$ has $S_2$ fibers, the central fiber of $\hat{f}$ need not be $S_2$. Let $\hat{X}_0$ denote the fiber of $\hat{f}$ over $0 \in T$. Set $\hat{U} := \hat{X} \setminus Z_0$ and $\hat{U}_0 := \hat{X}_0 \setminus Z_0$ with natural injections $\hat{j} : \hat{U} \hookrightarrow \hat{X}$ and $\hat{j}_0 : \hat{U}_0 \hookrightarrow \hat{X}_0$.

Let $I_{\hat{D}} \subset O_{\hat{X}}$ denote the inverse image ideal sheaf of $I_{\hat{D}}$. Then $I_{\hat{D}}$ is locally free on $\hat{U} := \hat{X} \setminus Z_0$; denote its restriction by $L_{\hat{U}}$. By construction $L_{\hat{U}}|_{\hat{U}} \cong L_U$ and hence $j_*, L_{\hat{U}}$ is locally free. Similarly, $L_{\hat{U}_0}$ agrees with $L_{U_0}$ over $U_0 \subset X_0 \subset \hat{X}_0$, thus $(j_0)_*, L_{\hat{U}_0}$ is locally free.

Thus Lemma 84 applies to $\hat{f} : \hat{X} \to T$ and $L_{\hat{U}}$ to conclude that $j_*, L_{\hat{U}}$ is locally free. Therefore $j_*, L_U$ is also locally free. □

86 (Proof of Theorem 84). We use the notation of (30). Let $L_U$ be a line bundle on $U$ such that $L_{U_0} := L_U|_{U_0} \cong O_{U_0}$ and $L_U^r \cong O_U$ for some $r > 0$. 


If \( r_0 : j_*L_U|_{X_0} \to (j_0)_*L_{U_0} \) is an isomorphism then the constant 1 section of \( L_{U_0} \cong \mathcal{O}_{U_0} \) lifts to a nowhere zero section of \( L_U \) and so \( L_U \cong \mathcal{O}_U \). Thus it is enough to prove that \( r_0 : j_*L_U|_{X_0} \to (j_0)_*L_{U_0} \) is an isomorphism. The latter is a local question on \( X \).

By localizing at a generic point of the set where \( r_0 \) is not known to be an isomorphism we are reduced to the case when \( Z_0 \) is a single closed point of \( X_0 \). As in Definition 20, we are further reduced to the case when \( f : X \to T \) is of finite type and the latter was treated in Lemma 55.

\[ \square \]

11. Numerical criteria for relative Cartier divisors

**Definition 87.** Let \( T \) be a regular 1-dimensional scheme and \( f : X \to T \) a flat morphism with \( S_2 \) fibers. A generically flat family of divisors \( D \) on \( X \) is given by

1. an open set \( U \subset X \) such that \( \text{codim}_{X_t}(X_t \setminus U) \geq 2 \) for every \( t \in T \) and
2. a relative Cartier divisor \( D_U \) on \( U \); that is, a Cartier divisor whose restriction to every fiber is a Cartier divisor.

For each \( t \in T \) set \( U_t := X_t \cap U \). Then \( D_U|_{U_t} \) extends uniquely to a divisor on \( X_t \); we denote it by \( D_t \) and call it the restriction of \( D \) to \( X_t \).

If \( |H| \) is a base point free linear system on \( X \) then the restriction of \( D \) to a general \( H \subset |H| \) is again generically flat family of divisors.

We say that \( D \) is a fiber-wise Cartier family of divisors if each \( D_t \) is a Cartier divisor. As we noted in (80),

\[ r_t : \mathcal{O}_X(D)|_{X_t} = (j_*\mathcal{O}_U(D_U))|_{X_t} \to (j_{t*})(\mathcal{O}_{U_t}(D_{U_t})) = \mathcal{O}_{X_t}(D_t) \]

is an isomorphism for general \( t \in T \); thus \( D \) is Cartier except possibly along finitely many closed fibers.

Our main interest is to find conditions that guarantee that a fiber-wise Cartier family of divisors is everywhere Cartier.

We say that \( D \) is a fiber-wise ample family of divisors if each \( D_t \) is an ample Cartier divisor.

Combining Theorems 89 and 79 we have the following.

**Theorem 88.** Let \( T \) be an irreducible, regular, 1-dimensional scheme and \( f : X \to T \) a flat morphism that is essentially of finite type and has \( S_2 \) fibers.

Let \( D \) be a generically flat family of fiber-wise Cartier divisors on \( X \). Assume that there is a closed subscheme \( W \subset X \) such that \( \text{codim}_{X_t}(X_t \cap W) \geq 3 \) for every \( t \in T \) and \( D \) is Cartier on \( U := X \setminus W \).

Then \( D \) is a Cartier divisor on \( X \).

\[ \square \]

It remains to understand what happens when the putative non-Cartier locus has codimension 2. For projective morphisms we have the following numerical criteria.

**Theorem 89.** Let \( T \) be an irreducible, regular, 1-dimensional scheme and \( f : X \to T \) a flat, projective morphism of relative dimension \( n \) with \( S_2 \) fibers.

Let \( D \) be a generically flat family of fiber-wise Cartier divisors on \( X \) and \( H \) an \( f \)-ample Cartier divisor on \( X \). The following are equivalent.

1. \( D \) is a Cartier divisor on \( X \).
2. \( \langle D_t^2, H_t^{n-2} \rangle \) is independent of \( t \in T \).

If \( D \) is fiber-wise ample, then these are further equivalent to

3. \( \langle D_t^n \rangle \) is independent of \( t \in T \).
Proof. If $D$ is Cartier then all the intersection numbers $(D_t^i \cdot H_i^{n-i})$ are independent of $t \in T$. Thus (1) $\Rightarrow$ (2) and (3).

To see the converse we may assume that $T$ is local with closed point $0 \in T$ and generic point $g \in T$. Let $Z_0 \subset X$ be the smallest closed subset such that $D$ is Cartier on $X \setminus Z_0$. Note that $Z_0 \subset X_0$ since $D$ is Cartier on $X_g$. We can choose $U := X \setminus Z_0$ as the open set in (87).

Assume first that $n = 2$, thus (2) and (3) coincide. For each $t \in T$, the Euler characteristic is a quadratic polynomial

$$
\chi(X_t, \mathcal{O}_{X_t}(mD_t)) = a_t m^2 + b_t m + c_t,
$$

and we know from Riemann–Roch that $a_t = \frac{1}{2}(D_t^2)$ and $c_t = \chi(X_t, \mathcal{O}_{X_t})$. Furthermore, (82.4) implies that

$$
a_0 m^2 + b_0 m + c_0 \geq a_g m^2 + b_g m + c_g \quad \text{for every } m \in \mathbb{Z}. \quad \text{(89.4)}
$$

For $m \gg 1$ the quadratic terms dominate, which gives that

$$
(D_0^2) = 2a_0 \geq 2a_g = (D_g^2). \quad \text{(89.5)}
$$

Assume now that $(D_0^2) = (D_g^2)$. Then $a_0 = a_g$ thus (89.5) implies that

$$
b_0 m + c_0 \geq b_g m + c_g \quad \text{for every } m \in \mathbb{Z}. \quad \text{(89.6)}
$$

For $m \gg 1$ this implies that $b_0 \geq b_g$ and for $m \ll -1$ that $-b_0 \geq -b_g$. Thus $b_0 = b_g$ and $c_0 = \chi(X_0, \mathcal{O}_{X_0}) = \chi(X_g, \mathcal{O}_{X_g}) = c_g$ also holds since $f$ is flat. Therefore we have equality in (89.4). Using (89) we see that

$$
r_0 : j_* \mathcal{O}_U(D_U)|_{X_0} \to (j_0)_* \mathcal{O}_{U_0}(D_0)
$$

is an isomorphism and hence $\mathcal{O}_X(D) = j_* \mathcal{O}_U(D_U)$ is locally free.

In order to prove (2) $\Rightarrow$ (1) for $n \geq 3$ we use induction on $n$. A suitable multiple $|mH|$ provides an embedding $X \subset \mathbb{P}^N_U$; let $X' \subset X$ be a general hyperplane section. Then $f' := f|_{X'} : X' \to T$ is a flat, projective morphism of relative dimension $n - 1$ and $Z' := Z \cap X'$ is a closed subscheme such that $\text{codim}_{X'}(X' \cap Z') \geq 2$ for every $t \in T$.

Furthermore, $D' := D|_{X'}$ is a divisor, Cartier over $U' := X' \setminus Z'$ whose support does not contain any irreducible component of a fiber of $f'$ and $D'|_{X_t'}$ is Cartier for every $t$. Finally $(D_t^i \cdot H_t^{n-i}) = m(D_t^i \cdot H_t^{n-2})$ is independent of $t \in T$.

Since $f$ has $S_2$ fibers, a hyperplane section of it usually has only $S_1$ fibers. However, by [Gro60 IV.12.1.6], a general hyperplane section again has $S_2$ fibers. Thus, by induction, $D' = D|_{X'}$ is a Cartier divisor. As we noted in (87), since $X'$ is general, this implies that $D$ is Cartier along $X'$. Hence there is a closed subscheme $W \subset X$ such that $W \cap X_t$ is 0-dimensional for every $t \in T$ and $D$ is Cartier on $X \setminus W$. Thus $D$ is Cartier by Theorem (88).

Assume finally that $D$ is fiber-wise ample. Choose $m > 0$ such that $mD_0 - H_0$ and $mD_t - H_t$ are ample. Thus $B := mD - H$ is also fiber-wise ample and

$$
m^a(D_t^n) = \sum_{i=0}^a (B_t^i \cdot H_t^{n-i}).
$$

We can apply (82.2) to the restriction of $B$ to an intersection of $n - i$ general members of $H$ to obtain that $(B_0^i \cdot H_0^{n-i}) \geq (B_0^i \cdot H_0^{n-i})$ for every $i$. However, $(D_0^n) = (D_g^n)$ by assumption (3), thus $(B_0^i \cdot H_0^{n-i}) = (B_0^i \cdot H_0^{n-i})$ for every $i$. We can use this for $i = 2$ and the already established (2) $\Rightarrow$ (1) to conclude that $D$ is Cartier. Thus $mD = B + H$ is also Cartier. By the same argument $(m + 1)D$ is also Cartier; this implies that $D$ is Cartier.
12. Open problems

Finite generation of $\text{NS}^\text{loc}(x, X)$.

The main unsolved problem is the finite generation of $\text{NS}^\text{loc}(x, X)$. This is known if $X$ is normal or if the characteristic is 0 and $\dim X \geq 3$. In positive characteristic, our methods do not seem to distinguish a unipotent subgroup of $\text{Pic}^\text{loc-œ}(x, X)$ from a discrete $p$-group.

For surfaces over $\mathbb{C}$, even stronger results should hold; see (93).

Local Picard group of excellent schemes.

While our theorems settle only the geometric cases, I see no reason why they should not hold in general.

Conjecture 90. Let $X$ be an excellent scheme that is $S_2$ and has pure dimension $\geq 3$. Let $x \in X$ be a closed point. Then $\ker \left[ \pi^* : \text{Pic}^\text{loc}(x, X) \to \text{Pic}^\text{loc}(x, X) \right]$ is a linear algebraic group.

Conjecture 91. Let $X$ be an excellent scheme that is $S_2$ and has pure dimension $\geq 4$. Let $x \in X$ be a closed point and $x \in D \subset X$ an effective Cartier divisor. Then $\ker \left[ r^*_X : \text{Pic}^\text{loc}(x, X) \to \text{Pic}^\text{loc}(x, D) \right]$ is a unipotent algebraic group.

Note that not even the existence of $\text{Pic}^\text{loc}(x, X)$ is known in the above generality, but algebraic equivalence can be defined as in (21.a–c).

Local Picard group of surfaces.

The construction of [Bou78] does not apply to surfaces; in fact, the functorial approach always gives the “wrong” answer. For normal surface singularities over $\mathbb{C}$, [Mum61] constructed a finite dimensional local Picard group $\text{Pic}^\text{loc}(s, S)$. The examples (28.3–4) show that in many non-normal cases there is no finite dimensional local Picard group.

Conjecture 92. Let $(s, S)$ be an $S_2$ surface. Then there is a finite dimensional local Picard group $\text{Pic}^\text{loc}(s, S)$ iff $S$ is seminormal.

Conjecture 93. Let $(x, X)$ be a local $\mathbb{C}$-scheme of finite type that is $S_2$ and has pure dimension 2. Then taking the first Chern class gives an exact sequence

$$0 \to \text{Pic}^\text{loc-œ}(x, X) \to \text{Pic}^\text{loc}(x, X) \xrightarrow{c_1} H^2(\text{link}(x, X), \mathbb{Z}).$$

The normal case is discussed in [Mum61]. It would be nice to have something similar in positive characteristic, but I do not know what should replace $H^2(\text{link}(x, X), \mathbb{Z})$. See also Example 81.

In connection with (27), one can ask the following.

Question 94. Let $(0, S^\text{an})$ be a normal, analytic surface singularity over $\mathbb{C}$ such that $\text{Pic}^\text{loc-œ}(0, S)$ is proper. Is there an algebraic model $(0, S)$ such that $\text{Pic}^\text{loc}(0, S) = \text{Pic}^\text{loc}(0, S^\text{an})$?

Second cohomology of links.

One can refine the topological approach of Section 4 using the mixed Hodge structures on the cohomology groups of the links $H^2(\text{link}(x, X), \mathbb{C})$ and $H^2(\text{link}(x, X), \mathbb{C})$ [PS08, Sec.6.2]. Since the Chern class of a line bundle has pure Hodge type $(1, 1)$, the following would imply most of Conjecture 90 for analytic spaces.
Conjecture 95. Let $X$ be a complex analytic space that is $S_2$ and has pure dimension $\geq 3$. Let $x \in X$ be a point. Then the pull-back map

$$\pi^*: H^2(\text{link}(x, X), \mathbb{C}) \to H^2(\text{link}(\bar{x}, \bar{X}), \mathbb{C})$$

is injective on the weight 2 graded piece of the mixed Hodge structure.

Obstruction theory for $\text{Pic}^{loc}$.

The usual obstruction space for $\text{Pic}^{loc}(x, X)$ is $H^3_x(X, O_X)$. This is infinite dimensional if $\dim X = 3$ or if $\dim X \geq 4$ and $X \setminus \{x\}$ is not $S_3$.

Already [Art74] observed that an obstruction theory is an external construct imposed on a functor and there could be different “natural” obstruction theories that work for the same functor. Theorem 78 suggests that one should be able to develop a finite dimensional obstruction theory for $\text{Pic}^{loc}$. I do not know how to formulate such a theory.

Acknowledgments. I thank B. Bhatt, J. de Jong and K. Smith for answering my questions and suggesting many improvements. Partial financial support was provided by the NSF under grant number DMS-0968337.

References

[Art69] Michael Artin, Algebraic approximation of structures over complete local rings, Inst. Hautes Études Sci. Publ. Math. (1969), no. 36, 23–58. MR MR0268188 (42 #3087)

[Art74] , Versal deformations and algebraic stacks, Invent. Math. 27 (1974), 165–189. MR MR0399094 (53 #2945)

[Bar55] Iacopo Barsotti, Un teorema di struttura per le varietà gruppali, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 18 (1955), 43–50. MR 0076427 (17,897b)

[BdJ13] B. Bhatt and A. J. de Jong, Lefschetz for local Picard groups, ArXiv e-prints (2013).

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud, Néron models, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 21, Springer-Verlag, Berlin, 1990. MR 1045822 (91i:14034)

[Bor91] Armand Borel, Linear algebraic groups, second ed., Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991. MR 1102012 (92d:20001)

[Bou78] Jean-François Boutot, Schéma de Picard local, Lecture Notes in Mathematics, vol. 632, Springer, Berlin, 1978. MR 492263 (81j:14008)

[BSU13] Michel Brion, Preema Samuel, and V. Uma, Lectures on the structure of algebraic groups and geometric applications, CMS Lecture Series in Mathematics, vol. 1, Hindustan Book Agency, New Delhi; Chennai Mathematical Institute (CMI), Chennai, 2013. MR 3088271

[dJ96] A. Johan de Jong, Smoothness, semi-stability and alterations, Inst. Hautes Études Sci. Publ. Math. (1996), no. 83, 51–93. MR 1423020 (98e:14011)

[Eis95] David Eisenbud, Commutative algebra, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry. MR 1322960 (97a:13001)

[Gro60] Alexander Grothendieck, Éléments de géométrie algébrique. I–IV., Inst. Hautes Études Sci. Publ. Math. (1960), no. 4,8,11,17,20,24,28,32.

[Gro68] , Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2), North-Holland Publishing Co., Amsterdam, 1968, Augmenté d’un exposé par Michèle Raynaud, Séminaire de Géométrie Algébrique du Bois-Marie, 1962, Advanced Studies in Pure Mathematics, Vol. 2. MR 0476737 (57 #16294)

[Har62] Robin Hartshorne, Complete intersections and connectedness, Amer. J. Math. 84 (1962), 497–508. MR 0412547 (26 #116)

[Har77] , Algebraic geometry, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157 (57 #3116)

[Kle66] Steven L. Kleiman, Toward a numerical theory of ampleness, Ann. of Math. (2) 84 (1966), 293–344. MR MR0206099 (34 #5834)
[KM98] János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[Kol91] János Kollár, *Flips, flops, minimal models, etc*, Surveys in differential geometry (Cambridge, MA, 1990), Lehigh Univ., Bethlehem, PA, 1991, pp. 113–199. MR 1144527 (93b:14059)

[Kol96] János Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 32, Springer-Verlag, Berlin, 1996.

[Kol97] János Kollár, *Quotient spaces modulo algebraic groups*, Ann. of Math. (2) **145** (1997), no. 1, 33–79.

[Kol11] János Kollár, *Sources of log canonical centers*, arXiv:1107.2863, 2011.

[Kol12] János Kollár, *Quotients by finite equivalence relations*, Current developments in algebraic geometry, Math. Sci. Res. Inst. Publ., vol. 59, Cambridge Univ. Press, Cambridge, 2012, With an appendix by Claudiu Raicu, pp. 227–256. MR 2931872

[Kol13a] János Kollár, *Grothendieck-Lefschetz type theorems for the local Picard group*, J. Ramanujan Math. Soc. **28A** (2013), 267–285. MR 3115196

[Kol13b] János Kollár, *Singularities of the minimal model program*, Cambridge Tracts in Mathematics, vol. 200, Cambridge University Press, Cambridge, 2013, With the collaboration of Sándor Kovács.

[Mum61] David Mumford, *The topology of normal singularities of an algebraic surface and a criterion for simplicity*, Inst. Hautes Études Sci. Publ. Math. (1961), no. 9, 5–22.

[Mum66] David Mumford, *Lectures on curves on an algebraic surface*, With a section by G. M. Bergman, Annals of Mathematics Studies, No. 59, Princeton University Press, Princeton, N.J., 1966. MR 0209285 (35 #187)

[PS94] A. J. Parameswaran and V. Srinivas, *A variant of the Noether-Lefschetz theorem: some new examples of unique factorisation domains*, J. Algebraic Geom. **3** (1994), no. 1, 81–115. MR 1242007 (95d:13025)

[PS08] Chris A. M. Peters and Joseph H. M. Steenbrink, *Mixed Hodge structures*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 52, Springer-Verlag, Berlin, 2008. MR MR2393625

[Sch68] Michael Schlessinger, *Functors of Artin rings*, Trans. Amer. Math. Soc. **130** (1968), 208–222.

[Siu69] Yum-tong Siu, *Extending coherent analytic sheaves*, Ann. of Math. (2) **90** (1969), 108–143. MR 0245837 (39 #7143)

[Smil4] Karen E. Smith, *(in preparation)*, 2014.

Princeton University, Princeton NJ 08544-1000

collar@math.princeton.edu