Is a doubly quantized vortex dynamically unstable in uniform superfluids?

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We revisit the fundamental problem of the splitting instability of a doubly quantized vortex in uniform single-component superfluids at zero temperature. We analyze the system-size dependence of the excitation frequency of a doubly quantized vortex through large-scale simulations of the Bogoliubov–de Gennes equation, and find that the system remains dynamically unstable even in an infinite-system-size limit. Perturbation and semi-classical theories reveal that the splitting instability radiates a damped oscillatory phonon as a counterpart of a quasi-normal mode.

Introduction: Vortices appear in many branches of physics. In particular, the structure, stability, and dynamics of vortices in nonlinear fields share common universal features in many physical systems [1]. Quantized vortices are prototypes among those vortices, playing an key role in the fluid dynamics of superfluid helium and Bose–Einstein condensates (BECs) [2–5]. In general, quantized vortices are characterized by the winding number of the phase of the superfluid order parameter around the vortex core. A vortex whose winding number is more than unity is called an $l$-quantized or multiply quantized vortex (MQV). Since the energy of an $l$-quantized vortex is generally larger than the sum of energies of $l$ singly quantized vortices (SQVs), an MQV is energetically unstable to split into SQVs in uniform systems [6]. In fact, MQVs have never been observed in equilibrium. However, this argument does not eliminate the possibility that MQVs survive as a metastable state at very low temperatures when energy dissipation is negligible.

To investigate the splitting instability precisely, we need to analyze the microscopic structure of the vortex core. It is difficult to demonstrate such an analysis in the strongly correlated superfluid $^4$He. Experimentally, there is no established technique to prepare an MQV in helium superfluids as an initial state of the instability problem. The realization of MQV in trapped gases sheds light on this problem, and vortex splitting has been observed [7,8]. The MQV in trapped systems can be dynamically unstable to split into vortices with smaller winding numbers according to the Bogoliubov–de Gennes (BdG) analysis at zero temperature [10,11]. The dynamic instability may occur when the excitation modes have complex frequencies as a result of a coupling or “mixing” between two modes with positive and negative excitation energies. The negative energy mode, called the core mode, is localized at the vortex core, decreasing the angular momentum of the system by $-\hbar l$ in the direction along the core. The positive energy mode is a collective mode of the condensate. The instability depends on the atomic interaction strength in a complicated manner [10–15], obfuscating the underlying physics. Lundh and Nilsen made progress in understanding the splitting instability by employing a perturbation theory; however, no quantitative evaluation was carried out because of the complicated behavior of the imaginary part of excitation frequency (see Fig. 3 in Ref. [15]). Currently, we do not have a definite answer to the question “does the splitting instability occur in bulk superfluids at zero temperature?”. This is partly because long-time numerical simulations with high-spec computers are required for investigating the dynamic stability more precisely. According to the previous studies concerning trapped BECs [10–17], it is not easy to answer this question because the finite-size-effect is essential there. Although Aranson and Steinberg [18] concluded that the lifetime of an MQV may become infinite without a trap in their numerical simulation, its system-size dependence has not been clarified systematically. This problem must be of fundamental importance in quantum fluid dynamics at very low temperatures, and thus, it is essential to understand, e.g., quantum turbulence of helium superfluid [19] and large two-dimensional (2D) BECs [20], where the problems become more complicated if one permits the presence of MQVs.

Here, we consider the most fundamental situation of a doubly quantized vortex (DQV) in a uniform 2D system. We show that a DQV is dynamically unstable in uniform BECs. Our large-scale numerical computation of the BdG equation reveals a nontrivial system-size dependence of the excitation frequency and its asymptotic behavior in the infinite-system-size limit. The nontrivial dependence is well-characterized by the “mixing” between the core mode and phonon with our rescaling perturbation theory. The semi-classical theory, extended to the case of complex eigenvalue, reveals that the instability causes spontaneous radiation and amplification of quasi-normal modes such as damped oscillatory phonons with anomalously long attenuation length. We discuss an analogy between this phenomenon and the rotational superradiance, observed recently as an amplification of
surface water waves by a draining vortex [21].

**Formulation:** We consider BECs in a quasi-2D system at zero temperature when the degrees of freedom along the z-axis are not considered [22]. The condensate is well-described by the order parameter ψ(r, t) that obeys the Gross–Pitaevskii (GP) Lagrangian \( \mathcal{L} = \int d^2x \psi^* \left(i\hbar \partial_t - H - \frac{g}{2} |\psi|^2 \right) \psi \). Here, we used \( H = -\hbar^2 \nabla^2 / (2m_a) - \mu \) with the atomic mass \( m_a \), the chemical potential \( \mu \), and the interaction constant \( g \).

Without loss of generality, a DQV with positive winding number \( l = 2 \) is considered. The stationary state of a DQV is written as \( \psi(r, t) = e^{i\mu t} \psi(r) \). The real amplitude \( f(r) \) obeys the GP equation \( [H_r + 2\hbar^2 / (2m_a r^2) + g f^2] f = 0 \) with \( H_r = -\hbar^2 (\partial_r^2 + r^{-1} \partial_r) / (2m_a) - \mu \).

The dimensionless amplitude \( \tilde{f} \) is characterized by the rescaled length \( \tilde{r} = r / \xi \) with the healing length \( \xi = \hbar / \sqrt{m_a \mu} \); \( \xi \) approaches the asymptotic form \( \tilde{r}^2 \sim 1 - l^2 / (2\tilde{r}^2) \) for \( \tilde{r} \gtrsim 1 \) and \( \tilde{r}^2 \sim \tilde{r}^2l \) for \( \tilde{r} \to 0 \).

To investigate dynamic stability, we introduce a collective fluctuation \( \delta \psi(r, t) = \psi(r, t) - \phi(r) = e^{i\mu t}[\mu(r)e^{i\omega t} - v^*(r)e^{-i\omega t + \omega t}] \). The linearization of the equation of motion of the Lagrangian \( \mathcal{L} \) with respect to the fluctuation leads to the BdG equation for \( \tilde{u} = (u, v)^T \),

\[
h\omega \tilde{u} = \tilde{h} \tilde{u} \equiv \begin{bmatrix} \tilde{h}_+ & -gf^2 \\ gf^2 & \tilde{h}_- \end{bmatrix} \tilde{u} \tag{1}
\]

with \( \tilde{h}_\pm = H_r + \hbar^2 (l^2 + m_a)^2 / 2m_a r^2 + 2g f^2 \). The excitation energy \( \epsilon_\alpha \) for an eigensolution \( (\omega, \tilde{u}) = (\omega_\alpha, \tilde{u}_\alpha) \) is written as \( \epsilon_\alpha = \hbar \omega_\alpha N_{\alpha\alpha} \). Here, excitations are labeled by the integer \( \alpha \) and we define the norm \( N_{\alpha\beta} = 2\pi \int_0^\infty drr^2 \tilde{u}_\alpha r \tilde{u}_\beta = \pm \delta_{\alpha\beta} \) for real eigenvalues with a matrix \( \tilde{\sigma} = \text{diag}(1, -1) \) and the Kronecker’s delta \( \delta_{\alpha\beta} \). From the orthogonality relation \( (\omega_\alpha - \omega_\beta) N_{\alpha\beta} = 0 \) [3], the energy \( \epsilon_\alpha \) becomes zero for a complex eigenvalue, because \( N_{\alpha\alpha} = 0 \) for \( \text{Im}(\omega_\alpha) \neq 0 \). The vortex state is dynamically unstable when there is at least one eigensolution with \( \text{Im}(\omega) > 0 \), because the corresponding excitation is amplified exponentially as \( \propto e^{\text{Im}(\omega)t} \).

**Numerical results:** The eigenvalue problem is numerically analyzed for a cylindrical system of dimensionless radius \( \tilde{R} \) by diagonalizing Eq. (1) with the numerical solution \( f \) of the GP equation with the Neumann boundary condition at \( \tilde{r} = \tilde{R} \). Figure 1 shows the \( \tilde{R}-\)dependence of the dimensionless frequency \( \tilde{\omega} \equiv \hbar \omega / \mu \) of the instability mode with \( m = -1 \) and \( -2 \) when the imaginary part \( \text{Im}(\tilde{\omega}) = 0 \) takes the largest value [22]. The real part is always negative \( \tilde{\omega}_\text{R} \approx \text{Re}(\tilde{\omega}) < 0 \) for \( \text{Im}(\tilde{\omega}) > 0 \).

The eigenvalue is strongly sensitive to \( \tilde{R} \), showing a nearly periodic behavior below \( \tilde{R} \sim 500 \). The similar behavior can be seen in the numerical results for trapped BECs [10–15]. In these studies, the excitation frequency was parameterized by the interaction strength or particle number, which made the problem complicated to analyze.

Figure 1(b) shows that the peak values of \( \tilde{\omega}_1 \) are proportional to \( \tilde{R}^{-1/2} \) for \( \tilde{R} \lesssim 500 \), while \( \tilde{\omega}_1 \) is asymptotic to a finite value for \( \tilde{R} \gtrsim 500 \). This fact concludes that a DQV is dynamically unstable even in the infinite-system-size limit \( \tilde{R} \to \infty \).

**Rescaling perturbation analysis:** To describe our problem more quantitatively, we introduce the perturbation theory for the eigenvalue problem of the BdG equations in a different manner from that in Ref. [15]. We parameterize the chemical potential and the interaction coefficient with a perturbation parameter \( \lambda (\ll 1) \) as \( \mu = \mu_0 (1 + \lambda) \) and \( g \to g_0 (1 + \lambda) \), respectively. Then, the effective system size is represented as \( \tilde{R} = \tilde{R}_0 / \sqrt{1 + \lambda} \) through \( \xi = \xi_0 / \sqrt{1 + \lambda} \). To express explicitly the \( \lambda \)-dependence, we write as \( f \to f_\lambda \) and \( \hbar \to \hbar_\lambda \) in the following.

The perturbation in the BdG theory is represented by
the deviation

$$\delta \hat{h} = \hat{h}_\lambda - \hat{h}_0 = \lambda \left[ -\mu_0 + 2G' + G'' - \mu_0 - 2G' \right]$$  \hspace{1cm} (2)

with $G' = \lim_{\lambda \rightarrow 0} \frac{g_0 f^2 - g_{0 \alpha} a_{\alpha}}{\lambda} \rightarrow \partial_\lambda (g_0 f^2)$. Suppose that the system becomes dynamically unstable when $\lambda$ increases from the unperturbed state ($\lambda = 0$) to a perturbed state ($\lambda \neq 0$). The eigenvector $\vec{u}$ in the perturbed state is described by a linear combination of the eigenvectors $\vec{u}_\alpha$ ($\alpha = 1, 2, \ldots$) in the unperturbed one as $\vec{u} = \sum_\alpha C_\alpha \vec{u}_\alpha$. The coefficient vector $C = [C_1, C_2, \cdots]^T$ obeys the eigenvalue equation $\hbar \mathcal{C} = (\hbar \mathcal{H}_0 + \lambda \mathcal{W}) \mathcal{C}$ with $\mathcal{H}_0 = \text{diag} (\hbar \omega_1, \hbar \omega_2, \ldots)$ and $\mathcal{W}_{\alpha \beta} = \mathcal{W}_{\beta \alpha} \equiv \frac{2 \pi}{\hbar} \int_0^{\infty} dr \vec{u}_\alpha \vec{u}_\beta \delta \hbar \vec{u}_\beta$.

When the frequencies of two modes, namely, $\alpha = 1, 2$, are very close to each other, they may apply a two-mode approximation as in conventional quantum mechanics, i.e., only the contribution from the two modes is considered while neglecting that from other modes [26]. The dynamic instability can occur in case of the opposite norms $N_1 = 11$ [11, 15]. Consider that phonon and the core mode correspond to $\alpha = 1$ and 2, respectively. The core mode, whose angular momentum is $-\hbar$, should have the positive norm $N_2 = 1$ for our case of $l = m = 2$, since the angular momentum carried by the system becomes dynamically unstable when $\omega_1 = 0$ becomes smaller and disappears as $\hat{\omega}_1$ increases from the unperturbed state ($\omega_1 = 0$). The dimensionless energy $\tilde{\varepsilon}_{\text{core}}$ in the unperturbed state of the core mode is given by Eq. (3) with $\tilde{\varepsilon}_{\text{ph}}(k_\text{core}) = -\tilde{\varepsilon}_{\text{core}}$. Therefore, we may write as $\Omega_\text{\infty} \sim \tilde{\varepsilon}_{\text{core}}$ and $1/\tau_\text{\infty} \sim |\tilde{\mathcal{W}}_{\text{mix}}|$.

We can deduce the power-law behavior of $\tilde{\mathcal{W}}_{\text{mix}}$ from the overall profile of $\omega_1$ in Fig. 1(b), $\tilde{\mathcal{W}}_{\text{mix}} \propto R^{1/2}$ for $R \lesssim 500$ (solid line) and $\tilde{\mathcal{W}}_{\text{mix}} \approx 1/\tau_\text{\infty}$ for $R \gtrsim 500$. Such a behavior is anomalous in the sense that the length $R \sim 500 \xi$, around which the power-law behavior changes, is irrelevant to any possible scales in our formalism above [25].

Extended semi-classical analysis: To demonstrate the anomalous behavior beyond the perturbation analysis, we introduce the semi-classical theory for the BdG model, which is available to describe low-energy modes propagating in bulk far from a topological defect or an interface [24, 31]. Here, we extend the theory to our case with complex excitation frequencies. The semi-classical theory starts from the Wentzel–Kramers–Brillouin (WKB) ansatz for the excitation wave function $\vec{u}$ in the first-order approximation, $\vec{u}(r) = e^{iS} \vec{U}$ with $S(r) = S_0(r) + \delta S_1(r)$ and $\vec{U} = (U, V)$. Substituting the ansatz into Eq. (1), we obtain

$$E \vec{U} = \begin{bmatrix} E_+ + \frac{g f^2}{2} & - g f^2 \tau \frac{1}{\hbar} \vec{D} \vec{\sigma} \vec{U} \\ g f^2 & - E_- \end{bmatrix} \vec{U} + \frac{\hbar}{4} D \vec{\sigma} \vec{U},$$  \hspace{1cm} (4)

where we used $E_+ = \frac{\xi^2}{2m_\text{ph}} + (M + L)^2 + 2g f^2 - \mu$, $(E, M, L) = (\hbar \omega, \hbar m, \hbar l)$, $P_\tau = \frac{\xi^2}{2m_\text{ph}}$, and $D(r) = \frac{\xi^2}{2m_\text{ph}} + \frac{1}{2m_\text{ph}} \left( \frac{dP_\tau}{dr} + \frac{d\tau}{dr} \right)$. The zero-th order approximation, called the classical limit, neglects the second term in the right hand side of Eq. (4), yielding $(E - E_+)(E + E_+) + g^2 f^4 = 0$. The first-order correction reduces to the relation $dS_1/dr = -(2\tau)^{-1}$. Considering the bulk region far from the vortex core ($r \gg \xi$) and neglecting terms of $O(\xi^2/r^2)$, we have $E^2 \equiv \frac{\xi^2}{2m_\text{ph}} + 2\mu$ [31]. For our case of a complex eigenvalue $E/\mu = \tilde{\omega}_R + i \tilde{\omega}_I$ with $|\tilde{\omega}_R| \gg \tilde{\omega}_I \geq 0$, the radial momentum $P_\tau$ is written as $P_\tau \xi/\hbar = k + ik$ with $k \ll \tilde{k}$. Then, one obtains

$$\tilde{k} = -|\tilde{\omega}_R| \left( 1 - \frac{\tilde{\omega}_I^2}{8} \right), \quad \tilde{k} = \tilde{\omega}_I \left( 1 - \frac{3\tilde{\omega}_R^2}{8} \right)$$  \hspace{1cm} (5)

![FIG. 2: (Color online) Schematics of a bubble for the splitting instability (a) and the so-called avoided crossing (b).](image_url)
up to the order of \( O(\tilde{\omega}_R^2) \). Here, we choose the sign of \( \tilde{k} \) to be negative since the outgoing phonon has \( \mathcal{N}_{11} < 0 \) in our perturbation analysis. As a result, one obtains \( S(r) = \hbar k \tilde{r} + \hbar \tilde{k} \tilde{r} - \frac{\hbar}{2} \ln \tilde{r} \) + const.

To demonstrate the accuracy of our theory, we describe an observable quantity, i.e., the density fluctuation \( \delta n(r, \theta, t) \equiv |\psi(r, t)|^2 - |\phi(r)|^2 \) induced by the instability mode. The semi-classical solution gives \( \delta n(\rho, \theta) \approx 2 \text{Re}(\rho^* \delta \psi) = 2i/\hbar \left( 1 + \tilde{\kappa} \right) e^{-\tilde{\kappa} \tilde{r} + \omega \tilde{r} + \Theta} \) with a constant \( \Theta \) [see also Fig. 4(a)]. For simplicity, we evaluate the semi-classical result for the cross-section profile

\[
\delta n(r) \equiv \delta n(r, 0, 0) \propto \tilde{r}^{-1/2} e^{-\tilde{k} \tilde{r}} \cos \left[ \tilde{k} \left( \tilde{r} - \tilde{R} \right) \right].
\]  

Here, we took the boundary conditions at \( r = \tilde{R} \) into account. In Fig. 3(b), we compare the radial profile \( \sqrt{\tilde{\kappa}} \delta n(r) \) of the numerical solution with Eq. 6, obtaining a good agreement between them for \( \tilde{r} \gg 1 \).

The semi-classical analysis suggests that the overall \( \tilde{R} \)-dependence of the instability is characterized by the rescaled damping rate in the infinite-system-size limit, \( \tilde{\kappa} \propto \tilde{k} (\tilde{\omega}_R = \tilde{\Omega}_\infty, \tilde{\omega}_l = \tilde{\tau}_\infty^{-1}) \). If the rescaled attenuation length \( \tilde{\kappa}^{-1} \) is much smaller than \( \tilde{R} \), the boundary effect is negligible so that the instability is independent of \( \tilde{R} \). This consideration is helpful to construct an analytic formula for the mixing interaction for describing the overall \( \tilde{R} \)-dependence. We found the simplest interpolating formula between the two limits \( \tilde{\kappa} \rightarrow 0 \) and \( \tilde{\kappa} \rightarrow \infty \),

\[
\tilde{W}_{\text{mix}}^2 = \tilde{\tau}_\infty^{-2} \left[ 1 + 1/(\tilde{\kappa} \tilde{R}) \right]^{1/2}. \tag{7}
\]

The complex frequency of Eq. 3 with Eq. 4 describes the numerical result very well as shown in Fig. 4.

Finally, we make a physical interpretation of such an anomalous damped oscillatory mode by regarding it as a counterpart of a quasi-normal mode, which is typically discussed in the context of gravitational waves from a perturbed black hole (BH) [32]. A perturbed BH evolves towards the unperturbed spherical shape by decreasing its asymmetry exponentially in time; the deviation from a spherical shape is proportional to \( e^{-t/\tau_0} \) with the decay time \( \tau_0 > 0 \). The radiated gravity wave in this decay process is described as a formal solution of linearized differential equations with a complex eigenvalue, namely, the quasi-normal mode. The wave forms a growing oscillation with the growth rate \( \kappa = \left( \epsilon / \mu \right)^{-1} \), where \( \epsilon \) is the speed of light far from the BH. On the contrary, in the case of splitting instability, a DQV is perturbed to split into two SQVs by increasing the asymmetry of the “oscillatory source” exponentially in time; \( d \propto e^{t/\tau_\infty} \). The radiated phonon produces a damped oscillation with the damping rate \( \kappa_\infty \approx (\epsilon c / \tau_\infty)^{-1} \) characterized by the speed \( c_s = \sqrt{\mu/m} \) of phonon, which is consistent with Eq. 4 in the linear dispersion approximation, \( |\tilde{\omega}_R| \approx \tilde{k} \). The relation between dynamic instability and quasi-normal modes has been also discussed in the context of BH physics in Ref. [32].

**Discussion:** The radiation of the quasi-normal mode in the splitting instability produces a double spiral density wave [Fig. 3(a)] in the early stage of the instability development. We have no satisfactory explanation of why \( 1/\tilde{\tau}_\infty \) is so small, although it might be related to the vortex-vortex interaction potential. The quasi-normal mode can be observed experimentally in highly oblate BECs whose size is much larger than the healing length. The instability is induced by an external perturbation, i.e., an external optical potential that violates the rotational symmetry of the initial state with a DQV.

An incident plane wave of phonon, whose energy \( \mu \tilde{\epsilon}_{\text{ph}} \) is close to \( -\mu \tilde{\epsilon}_{\text{core}} \), may trigger the instability. Then, the incident phonon will be amplified due to the exponential growth of the instability mode. This phenomenon is analogous to the rotational superradiance that has been observed recently in classical fluids [21], where incident waves of water surface are amplified by a draining vortex. The experiment in the dissipative classical fluid system did not reveal the mechanism behind the negative energy mode, which should exist as a partner of a positive energy mode to obey the energy conservation law. In our isolated quantum fluid system, the superradiant amplification is caused by a pair nucleation of positive and negative energy modes; the latter is represented by the core mode as a *bound state* whose existence is classically limited inside the so-called “ergo region” \( r < r_e \), and the former can propagate outside. Here, \( r_e / \xi = \sqrt{|\mu|/\tilde{\epsilon}_{\text{ph}}} \) is the effective ergo radius at which the semi-classical energy becomes zero as \( E \approx \mu \tilde{\epsilon}_{\text{ph}} + \hbar m \Omega (r_e) = 0 \) with \( \tilde{\epsilon}_{\text{ph}} = -\tilde{\epsilon}_{\text{core}} \) and the local superfluid velocity \( r \Omega(r) = \hbar / m \tilde{\epsilon}_{\text{ph}} \). Hence, our system could be useful to simulate the BH physics while a similar analogy has been discussed considering a rotating object or a vortex in superfluid systems [34–38].
Acknowledgments

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[24] We estimated the extrapolating values for $\omega_R$ and $\delta_l$ in the limit $\Delta \tau \rightarrow 0$ with the spatial grid size $\Delta \tau$ of the numerical simulation. For details, see Sec. A2 in the Supplemental material.
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Although the power law for $\tilde{R} \lesssim 500$ must reflect the finite-size-effect through the mixing interaction $W_{\text{mix}}$, we have never succeed a quantitative evaluation of $W_{\text{mix}}$ analytically.

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The classical turning point $r = r_0 \sim \xi$ with $P(r = r_0) = 0$ is obtained by taking the higher order term of $O(1/r^2)$. For details of the semi-classical analysis, see Sec. A4 in the Supplemental material.

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Supplemental material

A1. The intervortex potential

Definition of the interaction potential

Since we are interested in the stability of a doubly quantized vortex (DQV), we here discuss the interaction potential between two singly quantized vortices (SQV) in uniform Bose–Einstein condensates (BECs). The results obtained with different methods are summarized in Fig. A1.

First, we define the interaction energy between two vortices. We introduce the energy $E(d)$ of the state $\Psi_d$ with two SQVs as a function of distance $d$ between the two vortices,

$$ E(d) = \int d^2 x \Psi_d^* \left( -\frac{\hbar^2}{2m_\hbar} \nabla^2 + \frac{g}{2} \mid \Psi_d \mid^2 \right) \Psi_d. $$

Measuring the energy from the sum of the energy $E_{\text{bulk}} = \frac{\mu^2}{2g} \pi R^2$ of the bulk state and that $E_2$ of a DQV, the interaction energy $E_{\text{int}}(d)$ between two SQVs may be defined by

$$ E_{\text{int}}(d) \equiv E(d) - E_2 - E_{\text{bulk}}. $$

Estimations in the hydrodynamic and Padé approximations

Conventionally, the dimensionless energy $\tilde{E}_l = E_l g/(\mu^2 \xi^3)$ of a $l$-quantized vortex is estimated as

$$ \tilde{E}_l \simeq n^2 \log \tilde{R}, $$

where we approximate the dimensionless order parameter $\tilde{\psi} = \sqrt{\mu/g} \tilde{\psi}$ for a $l$-quantized vortex as $\tilde{\psi}(0)$ with $\tilde{\psi}(\tilde{x}, \tilde{y}) \simeq e^{i\tilde{\theta}(\tilde{x}, \tilde{y})}$ and $\tilde{\theta}(\tilde{x}, \tilde{y}) \equiv \arctan \left( \tilde{\theta}(\tilde{x}, \tilde{y}) \right)$. For integration, $\tilde{r} = 1$ and $\tilde{\rho} = \tilde{R}$ are chosen as the lower and upper cutoffs, respectively. Equation (A1) shows that the energy $E_l = t^2 E_l$ for $l$-quantized vortex is larger than the energy $lE_1$ for SQVs. In the similar approximation, the interaction energy $\tilde{E}_{\text{int}}(d)$ between two SQVs with the distance $d = \xi d \gg \xi$ can be estimated as

$$ E_{\text{int}}(d) \simeq -2\pi \log(\alpha d), $$

where we approximate the order parameter as a product of two SQV solutions as $\psi_{\text{int}}(\tilde{r}, d) \simeq \psi_1(\tilde{x} - d/2, \tilde{y}) \psi_1(\tilde{x} + d/2, \tilde{y})$. The dimensionless constant $\alpha = O(1)$ in Eq. (A2) depends on the lower cutoff of the integration. The monotonically decreasing structure of $\tilde{E}_{\text{int}}(d)$ in Eq. (A2) supports that a DQV is energetically unfavorable against two SQVs.

We next calculate the interaction energy $\tilde{E}_{\text{int}}(d)$ more precisely at small $d$. A naive estimation of $\tilde{E}_{\text{int}}$ can be done by the product state

$$ \tilde{\psi}_{\text{int}} \simeq \tilde{f}_1(\tilde{x} - d/2, \tilde{y}) \tilde{f}_1(\tilde{x} + d/2, \tilde{y}) e^{i(\tilde{\theta}(\tilde{x} - d/2) + \tilde{\theta}(\tilde{x} + d/2))}, $$

where $\tilde{f}_1(\tilde{x}, \tilde{y})$ is the solution of the amplitude $\tilde{f}(\tilde{r})$ for a SQV at the center $\tilde{x} = \tilde{y} = 0$. Although the product of $\tilde{f}_1$ for two SQVs should be replaced by $\tilde{f}_2$ for a DQV, we do not consider this change within our naive estimation. Within the Padé approximation, $\tilde{f}_1(\tilde{x}, \tilde{y})$ can be obtained as

$$ \tilde{f}_1(\tilde{x}, \tilde{y}) \simeq \sqrt{\frac{a_1 \tilde{r}^2 + a_2 \tilde{r}^4}{1 + b_1 \tilde{r}^2 + a_2 \tilde{r}^4}}, $$

with $\tilde{r}^2 = \tilde{x}^2 + \tilde{y}^2$. Here, $a_1, a_2$, and $b_1$ satisfy $a_1 = (73 + 3\sqrt{201})/176$, $a_2 = (6 + \sqrt{201})/132$, and $b_1 = (21 + \sqrt{201})/48$. For the product state with the amplitude $\tilde{f}_1$ under the Padé approximation, the interaction energy $\tilde{E}_{\text{int}}(d)$ around $d = 0$ becomes

$$ \tilde{E}_{\text{int}}(d) \simeq -0.327 \tilde{d}^2 + O(\tilde{d}^4). $$

The flat structure of $\tilde{E}_{\text{int}}$ at $\tilde{d} = 0$ as $d\tilde{E}_{\text{int}}/dd|_{d=0} = 0$ implies that a DQV is marginally unstable.
Comparison with the numerical results

Figure A1 shows the interaction energy \( \tilde{\xi}_{\text{int}} \) given by the product state with the Padé approximation \( (A3) \) (solid line), that with the numerically obtained amplitude \( f_1 \) (open circles), and Eq. (A2) (dashed line). The most precise result (closed circles) are obtained from the ansatz

\[
\tilde{\psi}_{\text{int}} = \tilde{f}_{\text{int}}(\tilde{r},d)e^{i(\tilde{\xi}d/2)+\theta(\tilde{\xi}d/2)}
\]

by numerically computing \( \tilde{f}_{\text{int}}(\tilde{r},d) \). The exact interaction energy \( \tilde{\xi}_{\text{int}}(d) \) at \( d = 0 \) is flatter than approximated ones with the product states. All data converge to the same behavior except for constants at large \( d \).

FIG. A1: Interaction energy \( \tilde{\xi}_{\text{int}}(d) \) for two SQVs. Closed circles show the numerically obtained values. Open circles and the solid line show values given by the product of two SQV solutions obtained numerically and by the Padé-approximation \( (A3) \) respectively. Dashed line shows values in Eq. (A3), where \( \alpha \approx 0.319 \) is chosen as a fitting parameter.

A2. Technical description on the numerical analysis

Dimensionless equations and the boundary conditions

The stationary solution of a DQV was obtained by employing the method of steepest descent for the Gross–Pitaevskii (GP) equation. By rescaling the order parameter amplitude and the radial coordinate as \( f = \sqrt{\mu/g} \tilde{f} \) and \( r = \xi \tilde{r} \), respectively, the GP equation for the DQV state is reduced to

\[
\left[ -\frac{1}{2} \frac{d^2}{d\tilde{r}^2} - \frac{1}{2r} \frac{d}{d\tilde{r}} - 1 + \frac{2}{\tilde{r}^2} + \tilde{f}^2 \right] \tilde{f} = 0.
\]

This equation was solved numerically under the boundary conditions, \( \tilde{f}(\tilde{r} = 0) = 0 \) and \( \frac{d\tilde{f}}{d\tilde{r}}|_{\tilde{r} = \tilde{R}} = 0 \).

Similarly, the Bogoliubov–de Gennes (BdG) equation \( (1) \) can be reduced to a dimensionless form. The solved equation is

\[
\tilde{\omega} \tilde{u} = \begin{bmatrix} \tilde{h}_+ & -\tilde{f}^2 \\ \tilde{f}^2 & -\tilde{h}_- \end{bmatrix} \tilde{u} \quad (A5)
\]

Spatial profile around the vortex core

Because of the symmetry of Eq. (1), if there is a solution (i) \( (\omega, m, u, v) \) with a complex frequency \( \omega \), we have always other three solutions (ii) \( (-\omega, -m, v, u) \), (iii) \( (\omega, m, u, v) \), and (iv) \( (-\omega, -m, v, u) \). Consider a mode (i) with \( \text{Im}(\omega) > 0 \). Then, the mode (iv) with \( \text{Im}(\omega) > 0 \) is also amplified, while we have \( \text{Im}(\omega) < 0 \) for (ii) and \( \text{Im}(\omega) < 0 \) for (iii). The solution (iv) is physically identical to the partner (i), because the two solutions yield the same fluctuation \( \propto \delta\tilde{\psi} \). For our problem, we may consider only (i) with \( \text{Im}(\omega) > 0 \) by neglecting the solutions (ii-iv).

Figure A2 shows the radial profile of amplitudes \( |u| \) and \( |v| \) for the instability mode \( (m = -2) \) for \( \tilde{R} = 1638.4 \). The dashed curve represents the rescaled profile \( |\tilde{f}(\tilde{r})| \) of the order parameter amplitude.

\[
\tilde{\omega} = \frac{\beta\omega}{\mu} \text{ and } \tilde{h}_\pm = -\frac{1}{2} \frac{d^2}{d\tilde{r}^2} - \frac{1}{2r} \frac{d}{d\tilde{r}} - 1 + \frac{(2\pm m)^2}{4\tilde{r}^2} + 2\tilde{f}^2.
\]

The asymptotic values of \( \tilde{\omega}_R \) and \( \tilde{\omega}_I \)

We have determined the asymptotic values, \( \tilde{\Omega}_\infty \) and \( \tilde{\gamma}_\infty \), by considering the dependence of the values \( \tilde{\omega}_R (< 0) \)
and $\tilde{\omega}_1$ on the numerical grid size $\Delta \tilde{r}$ within the range $1630 \leq \tilde{R} \leq 1638$ (see Fig. [A3]). When $\tilde{R}$ is sufficiently large, $|\tilde{\omega}_R|\tilde{(\tilde{\omega})}$ is simply oscillating like a sinusoidal function of $\tilde{R}$ around its averaged value $\langle |\tilde{\omega}_R|\rangle \langle \tilde{(\tilde{\omega})} \rangle$ with a small amplitude $\delta \tilde{\omega}_{R,1}$. The asymptotic value $\Omega_\infty = -0.438969 \pm 0.000002 (\tilde{\tau}_\infty^{-1} = 0.002429 \pm 0.000002)$ of $\langle \tilde{\omega}_R\rangle$ ($\langle \tilde{(\tilde{\omega})}\rangle$) for the limit $\Delta \tilde{r} \to 0$ is determined by fitting the plot with a quadratic function, $\langle \tilde{\omega}_R\rangle = a_2 \Delta \tilde{r}^2 + \left[\Omega_\infty \right] \langle \tilde{(\tilde{\omega})} \rangle = a_1 \Delta \tilde{r}^2 + \tilde{\tau}_\infty^{-1}$ with the method of least squares: $a_2 = -0.031606 \pm 0.000007$ and $a_1 = -0.000192 \pm 0.000007$. The errors are computed by regarding the small amplitude $\delta \tilde{\omega}_{R,1}$ as the error of the numerical data.

### A3. The two-mode approximation

Let us derive the expression of Eq. (3). The perturbed and unperturbed states obey the BdG equations:

$$
\hbar \omega \tilde{u} = \hbar \lambda \tilde{u} = (\tilde{h}_0 + \delta \tilde{h}) \tilde{u}, \quad (A6)
$$

$$
\hbar \omega_0 \tilde{u}_0 = \hbar \tilde{u}_0, \quad (A7)
$$

respectively. By inserting the expansion $\tilde{u} = \sum_\alpha \tilde{C}_\alpha \tilde{u}_\alpha$ into Eq. (A6), we have

$$
\hbar \omega \sum_\alpha \tilde{C}_\alpha \tilde{u}_\alpha = \sum_\alpha \hbar \omega_\alpha \tilde{C}_\alpha \tilde{u}_\alpha + \sum_\alpha \tilde{C}_\alpha \hbar \delta \tilde{h} \tilde{u}_\alpha. \quad (A8)
$$

By changing the suffix $\alpha$ to $\beta$, multiplying $\tilde{u}_\alpha^1 \tilde{\sigma} \tilde{z}$ from the left side, and integrating by $2 \pi \int_0^\infty r dr \tilde{r}$, we get

$$
\hbar \omega \sum_\beta \tilde{C}_\beta \tilde{N}_{\alpha\beta} = \sum_\beta \hbar \omega_\beta \tilde{C}_\beta \tilde{N}_{\alpha\beta}
$$

$$
+ \sum_\beta \tilde{C}_\beta \int_0^\infty 2 \pi r dr \tilde{u}_0^1 \tilde{\sigma}_z \delta \tilde{h} \tilde{u}_\beta. \quad (A9)
$$

The normalization factor $\tilde{N}_{\alpha\beta}$ is written as $\tilde{N}_{\alpha\alpha} = \pm 1$ and the Kronecker’s delta $\delta_{\alpha\beta}$. Then, dividing Eq. (A9) by $\tilde{N}_{\alpha\alpha}$, we have

$$
\hbar \omega \tilde{C}_\alpha = \hbar \omega_\alpha \tilde{C}_\alpha + \lambda \sum_\beta \tilde{C}_\beta \tilde{W}_{\alpha\beta}, \quad (A10)
$$

where we defined

$$
\tilde{W}_{\alpha\beta} = \frac{2 \pi \tilde{\beta}_{\alpha\alpha}}{\lambda \tilde{N}_{\alpha\alpha}} \int_0^\infty r dr \tilde{u}_0^1 \tilde{\sigma}_z \delta \tilde{h} \tilde{u}_\beta. \quad (A11)
$$

Here, we introduce the two-mode approximation by taking only $\alpha = 1, 2$ to analyze Eq. (A10). The eigenvalue equation is given by

$$
\hbar \omega \left( \begin{array}{cc} C_1 \\ C_2 \end{array} \right) = \left( \begin{array}{cc} \tilde{h}_\omega \tilde{W}_{11} + \lambda \tilde{W}_{12} \\ \lambda \tilde{W}_{21} \end{array} \right) \left( \begin{array}{c} C_1 \\ C_2 \end{array} \right).
$$

By using the notation $\tilde{\varepsilon}_\alpha = (\hbar \omega_\alpha + \lambda \tilde{W}_{\alpha\alpha}) \tilde{N}_{\alpha\alpha}/\mu$ and the relation $\tilde{W}_{21} = \tilde{W}_{12}^*$, the secular equation is written as

$$
(\tilde{N}_{11} \tilde{\varepsilon}_1 - \tilde{\omega}) (\tilde{N}_{22} \tilde{\varepsilon}_2 - \tilde{\omega}) - \lambda^2 \frac{|\tilde{W}_{12}|^2}{\mu} = 0,
$$

whose solution is

$$
\tilde{\omega} = \frac{\tilde{N}_{11} \tilde{\varepsilon}_1 + \tilde{N}_{22} \tilde{\varepsilon}_2}{2} \pm \sqrt{(\tilde{N}_{11} \tilde{\varepsilon}_1 - \tilde{N}_{22} \tilde{\varepsilon}_2)^2 - \frac{\lambda^2 |\tilde{W}_{12}|^2}{\mu}}.
$$

For $\tilde{N}_{11} = -1$ and $\tilde{N}_{22} = 1$, this form is consistent with Eq. (3) by introducing $\tilde{W}_{\text{mix}}^2 = \lambda^2 |\tilde{W}_{12}/\mu|^2$.

### A4. The semi-classical approximation

Starting from the equation $(E - E_+)(E - E_-) + g^2 f^4 = 0$ within the zeroth order approximation, we can calculate the eigenenergy $E$. Here, $E_\pm$ is given by

$$
E_\pm = \frac{P_r^2}{2 m_a} \left( \frac{(M \pm L)^2}{2 m_a r^2} + 2 g f^2 - \mu \right). \quad (A12)
$$

When we consider the bulk region far from the vortex core, the density profile is approximately written as $g f^2 \approx \mu - L^2/(2 m_a r^2)$, so that

$$
E_\pm \approx \frac{1}{2 m_a} \left( \frac{P_r^2}{2 m_a} + \frac{M^2}{r^2} \right) + g f^2 \pm \frac{L M}{m_a r^2}. \quad (A13)
$$

Furthermore, we neglect the higher order term $O \left( r^{-2} \right)$, having then $\mu \approx g f^2$ and $E_\pm = P_r^2/(2 m_a) + \mu$. We eventually get

$$
E^2 = \frac{P_r^2}{2 m_a} \left( \frac{P_r^2}{2 m_a} + 2 \mu \right). \quad (A14)
$$

Let us consider the situation in which $E$ is a complex value. When $E$ is real, the momentum $P_r$ is also a real value, according to Eq. (A11). When $E$ is complex, the momentum $P_r$ should be written as $P_r = \hbar k + i \hbar \kappa$. When $E_R \equiv \text{Re}(E) \gg E_I \equiv \text{Im}(E)$, it is reasonable to assume as $k \gg \kappa$. Substituting $E = E_R + i E_I$ and $P_r = \hbar k + i \hbar \kappa$
into Eq. (A14), and comparing the real and imaginary parts of the both sides of the equation, we have

\[ E_2^R - E_2^I = \frac{\hbar^2 (k^2 - \kappa^2)}{2m_a} \left[ \frac{\hbar^2 (k^2 - \kappa^2)}{2m_a} + 2\mu \right] - \frac{\hbar^4 k^2 \kappa^2}{m_a^2}, \]

\[ 2E_R E_I = \frac{\hbar^2 k \kappa}{m_a} \left[ \frac{\hbar^2 (k^2 - \kappa^2)}{m_a} + 2\mu \right]. \]

By introducing the dimensionless values \( \tilde{\omega}_{R,I} = E_{R,I}/\mu \), \( \tilde{k} = k\xi \), \( \tilde{\kappa} = \kappa \xi \) with \( \xi = \hbar / \sqrt{m_a \mu} \), the above equation can be written as

\[ \tilde{\omega}_R^2 - \tilde{\omega}_I^2 = \frac{\tilde{k}^2 - \tilde{\kappa}^2}{2} \left[ \frac{\tilde{k}^2 - \tilde{\kappa}^2}{2} + 2 \right] - \tilde{k}^2 \tilde{\kappa}^2, \]

\[ 2\tilde{\omega}_R \tilde{\omega}_I = \tilde{k} \tilde{\kappa} (\tilde{k}^2 - \tilde{\kappa}^2 + 2). \]

For \( E_2^R \gg E_2^I \) and \( k^2 \gg \kappa^2 \), the above equations can be further reduced to

\[ \tilde{\omega}_R^2 \approx \frac{\tilde{k}^2}{2} \left( \frac{\tilde{k}^2}{2} + 2 \right), \]

\[ 2\tilde{\omega}_R \tilde{\omega}_I \approx \tilde{k} \tilde{\kappa} (\tilde{k}^2 + 2). \]

By solving these equations, we can get

\[ \tilde{k} \approx \pm |\tilde{\omega}_R| \left( 1 - \frac{\tilde{\omega}_R^2}{8} \right), \]  
(A15)

\[ \tilde{\kappa} \approx \tilde{\omega}_I \left( 1 - \frac{3\tilde{\omega}_R^2}{8} \right). \]  
(A16)