Axiomatic definition of quantity as a basis for teaching metrology in Mechanical Engineering

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Abstract. In spite of satisfactory availability of didactical material, there exist some problems in teaching metrology that are conditioned by rather outdated common view on the role and position of this subject in the overall discipline of Mechanical Engineering, as well as in other engineering branches. In fact, in the last ten years, many changes have occurred in metrology both as a science itself and as a practical activity on measurement uniformity support. Terminology, concerning essential elements of measurement process and measurement situation, has been revised to a large extent. In the paper, certain recommendations are given on a shift of the subject teaching paradigm and its contents.

1. Introduction

Metrology (and allied courses like quality control; standardization; and conformity testing), as an academic subject, holds a high position in any engineering educational program and, particularly, in Mechanical Engineering. Common comprehension of this course role in shaping necessary and sufficient set of competencies to provide successful professional activity of an engineer was not indigenous, though, at present, special importance of this role is undisputed [1–4].

Necessary professional competencies of modern-day mechanical engineers are rather versatile. They include abilities such as: participation in development of technical documentation in accordance with standards and engineering specifications; organization and management of metrological assurance of manufacturing process using generic inspection methods of process equipment and production quality; creating subsystems of quality management for technological processes in production floors; carrying out standardization work and preparatory operations for conformity tests of engineering tools, systems, processes, machinery and materials.

These competencies should rely on knowledge of generic metrological concepts; organizational, scientific and methodological foundations of metrological assurance; legal grounds of measurement uniformity and traceability. The knowledge should also be corroborated by such practical skills as to measure object parameters by means of typical methods and instruments; to estimate measurement uncertainties; to prepare equipment and documentation for certification and so on.

The listed knowledge and skills suppose student's comprehension of nature and very fundamentals of quantity notation and allied concepts. However, the issue has been given short shrift by of-the-shelf metrology textbooks, see, e.g. [3, 4]. They mainly are intended to provide understanding of the basics of mechanical measurements and various shop-floor measurement techniques.

Aim of the paper is to show how to introduce in rather artless way both important and contradictory [5] notation of quantity that can be used in teaching engineering metrology in order to explain to stu-
dents what is difference between additive and non-additive, physical and non-physical quantities, etc. We profit here from representational approach to construction of measurement theory, since now, after many years of intensive interdisciplinary debates concerning its merits and demerits, see e.g. [6–12], our strong opinion is that this framework can be a good basis for justification of concepts related to measurement (surely, under creative way of thinking). Let us demonstrate this below as brief sketch.

2. Axiomatic introduction of quantity notation

For our consideration, we use definition of measurement in a wide sense [6]: measurement is the process of empirical objective associating numbers (or symbols) with properties manifestations of objects in order to describe them. Measurement is a mediator between object and subject (see figure 1), that is, object under measurement is not directly observable by a subject. Object under measurement is a body or system, or thing, or phenomenon, or process, etc. which is characterized by one or many properties to be measured. Property is a qualitative determinacy of the object; a property is measured, not an object.

2.1. Empirical and numerical relational systems

Object property can be manifested with different intensity. Example. Specific specimen of a green apple can be light green, green, dark green and so on; just as well a property of tables, length, for the individual table can be manifested as 1 meter or 2 meters.

One can consider manifestations $a_1, a_2, \ldots$ of some property $A$ as elements of appropriate set $A = \{a_1, a_2, \ldots\}$. Then it can be noticed that measurement involves symbolization of binary relations on the set $A$. Thus, the concept of property is based on empirical relation.

![Figure 1. Measurement as mediator between object and subject.](image)

Empirical relational system (ERS) is a set $A = \{a_1, a_2, \ldots\}$ of object property manifestations that possesses an order relation and provides a possibility of observable comparison operation over property manifestations.

Numerical relational system (NRS) is a set $B = \{b_1, b_2, \ldots\}$ of symbols (in particular case, numbers) that possesses an order relation; relations between the set elements are those between numbers including arithmetic operations.

2.2. Operations over property manifestations

As soon as the quantity concept is primary one it is introducing by axiomatic way. For axiomatic definition of quantity, two kinds of operations over property manifestations are necessary and sufficient:

- comparison operation and
- additive operation.

Mathematical operation (in NRS) must conform to physical operation (in ERS).

Comparison operation reveals relations "$>$", "$<$" and "$\approx$" in ERS; and relations "$>$", "$<$" and "$=$" in NRS. The operation implementation, as a rule, is specific for each of investigated properties. Example. (1) To compare two metallic rods by a length it is necessary: to juxtapose the rods; to locate the rods end-to-
end on the one hand; to estimate location of the rods ends on the opposite hand; to shape appropriate judgment. (2) To compare masses of two items one should put them on the different pans of scales.

Additive operation intended to combine (or concatenate) property manifestations using operation of concatenation "∗" in ERS and addition "+" in NRS. Again, to compare manifestation of each particular property one should operate accounting its specific character. Example. (1) When combining lengths of two metallic rods it is necessary the end of one of the rods to fit tightly to the end of other rod and guarantee their collinearity. (2) To combine masses of two items one should put them on the same pan of scales.

2.3. Definition of quantity
Now we can introduce axioms of quantity. Let some property \( A = \{a_1, a_2, \ldots\} \) of some object has manifestations \( a_1, a_2, \ldots \). Corresponding variable \( A \) is an additive quantity and existence of its numerical representation is feasible iff axioms of order and additivity are hold.

Axioms of order guarantee that an order associated with object by assigning numbers is namely that order revealed due to real observation or measurement:

- **Q1. Transitivity** \( a_i \geq a_j \) and \( a_j \geq a_k \Rightarrow a_i \geq a_k \).
- **Q2. Antisymmetry** \( a_i \geq a_j \) and \( a_j \geq a_i \Rightarrow a_i \sim a_j \).
- **Q3. Linearity** \( a_i \geq a_j \) or \( a_j \geq a_i \).

Axioms of additivity provide rules of combining manifestations of investigated property (axiom Q9 introduces a possibility to express property manifestations in terms of unit of measurement):

- **Q4. Associativity** \((a_i + a_j) + a_k \sim a_i + (a_j + a_k)\).
- **Q5. Commutativity** \( a_i + a_j \sim a_j + a_i \).
- **Q6. Monotonicity** \( a_i \geq a_j \Leftrightarrow a_i + a_k \geq a_j + a_k \).
- **Q7. Solvability** \( a_i > a_j \Rightarrow \) there exists such \( a_k \) that \( a_i \sim a_j + a_k \).
- **Q8. Positivity** \( a_i + a_j \geq a_j \).
- **Q9. Archimedian postulate** there exists such \( b \in \mathbb{N} \) that \( ba_i \geq a_i \), where \( 1a_i \sim a_i \); \((b + 1)a_i \sim ba_i \sim a_i \); \( ba_i \) is \( b \)-fold concatenation of \( a_i \) with itself.

The axioms are adopted from [13] and, actually, are axioms proposed by O.L. Hölder [14, 15].

3. Classes of quantities

3.1. Additive quantities
If some variable satisfies axioms Q1–Q9, it is an additive quantity. Example. Weight is an additive quantity as weights can be ordered by their values and each weight manifestation is combined additively with other weight manifestations.

Additive variables are called also quantitative ones as their values (manifestations) always greater or less of each other in definite number of times. In terms of property, which can be qualitatively revealed and be estimated qualitatively or quantitatively, the quantitatively estimated properties are additive.

Among a variety of additive quantities, physical quantities play considerable part. Large volume of data in scientific papers, monographs, reports, and handbooks is represented by just physical quantities. Namely this class of quantities is a subject of conventional metrology.

International vocabulary of metrology (VIM3, [16]) gives the following definition of (physical or chemical) additive quantity:

**quantity** is property of a phenomenon, body, or substance, where the property has a magnitude that can be expressed as a number and a reference
with a note that a reference can be a measurement unit, a measurement procedure, a reference material, or a combination of such.

Physical quantities are expressed in units. Measurement equation for physical quantities has the view:

\[ a = b \, [a], \tag{1} \]

where \( a \) is a magnitude of the physical quantity, \([a]\) is a unit accepted for the given physical quantity, \( b \) is a numerical value of the physical quantity.

Magnitude is a quantitative determinacy being inherent to individual object. In accordance to VIM3, a unit of measurement is real scalar quantity, defined and adopted by convention, with which any other quantity of the same kind can be compared to express the ratio of the two quantities as a number; quantity value is number and unit (reference) together expressing magnitude of a quantity.

**Non-physical quantities.** There usually supposed that physical quantities characterize objects of material world and are investigated in natural and engineering sciences and non-physical quantities are properties of immaterial objects and are studied in humanitarian sciences. However, what is really important whether axioms Q1–Q9 are valid for a given quantity. If an answer to this question is positive, the quantity can be treated as physical. For example, cost (as many other economical, certainly, 'non-physical' quantities) expressed in monetary units can be considered as a physical quantity because axioms Q1–Q9 are hold.

### 3.2. Non-additive quantities

There are quantities, for which only axioms of order Q1–Q3 are valid and additivity axioms are not satisfied. In fact, they ceased to be quantities, that is, entities having magnitude. Such “quantities” are called non-additive or qualitative. **Example.** (1) Ordinal variables: mineral hardness, earthquake intensity, Beaufort wind force. (2) Nominal variables: color, nationality, gender.

### 3.3. Extensive and intensive quantities

**Extensive quantities** are directly observable and accessible to compare; satisfy the distributive rule: if an object \( O \) characterized by value \( a \) of quantity \( A \) is divided into \( k \) parts \( O_1, O_2, \ldots, O_k \), which are represented by values \( a_1, a_2, \ldots, a_k \) of the same quantity \( A \) then \( a = a_1 + a_2 + \ldots + a_k \). **Examples:** length, mass, time, volume, force and others.

Thus, the property of interest does change when the object is divided or combined. An extensive property is one that is additive for independent, non-interacting sub-objects. The property is proportional to the amount of material in the object.

**Intensive quantities** are those the distributive rule is not applied to and expression for \( a \) is \( a = a_1 = a_2 = \ldots = a_k \). An intensive property is a bulk property, meaning that it is a physical property of an object that does not depend on the object size or its amount of material. **Examples:** temperature, velocity,
refractive index, density, hardness, magnetization and others. All specific quantities, i.e. calculated per unit of mass or volume, are intensive.

For intensive quantities axioms of additivity Q4–Q9 are not valid and they are not additive. However, intensive quantities can be measured indirectly, that is, are calculated using measured values of extensive quantities.

4. Measurement and its uncertainty
Measurement is an objective empirical operation \( f: A \rightarrow B \), which maps the property manifestations onto numbers in such a way that the relations between numbers correspond to the relations between empirical elements (see section 2.1). It means that the mapping \( f \) is homomorphic [6].

Under homomorphism \( f \) the set \( A \) is broken up into non-intersected sets of preimages:

\[
A = \bigcup_{b \in B} f^{-1}(b), \tag{2}
\]

where \( f^{-1}(b) \) consists of all elements of \( A \), having the same image in \( f(A) \). This fact allows to introduce the notion of measurement uncertainty into the representational treatment [17]. Figure 3 illustrates this for a case of length measurement using a ruler, where the empirical elements form the series of values of quantity and the numerical objects are the scale numberings. In figure 3 the function \( f \) is a correspondence of each ruler division to a definite numerical score. One can see that inverse mapping \( a' = f^{-1}(b) \) never coincides with preimage \( a \) since there are no empirical conditions able to guarantee validity of the hypothesis \( a' = a \).

5. Possible metrology course outline
Based on the introduced notations course of metrology can have the outline as follows.

**Initial mathematical concepts:** Sets. Set of all subsets, set partitions and other combinatorial configurations. Cartesian product of sets. Binary relations. Main properties of binary relations. Equivalence relations. Order relations. Correspondences. Mappings. Mapping classes.

**Quantities and scales:** Quantity notation. Empirical and numerical relational systems. Object and its properties. Axiomatic definition of quantity. Classes of quantities. Measurement definition. Measurement problems (representation, uniqueness and adequacy). Scale notation. Main types of scale (absolute, ratio, interval, order and nominal scales). Comparative scale characterization. Invariance of feasible scale transformations.

**Quantity calculus:** Main equation of additive quantity; see equation (1). Number, unit, value, magnitude. Principle of representation of additive quantities. Quantity dimension. Dimension properties. Power form of dimensional formulae. Quantity structure. Dimensional analysis. System of units. International System of Units. Number of base units. Principles of establishing system of units. Measurement uniformity. Standards of units.

**Error and uncertainty:** True value of quantity. Systematic and random effects. Sources of uncertainty. Probabilistic model of uncertainty. Normal distribution. Uniform distribution. Estimation of distribution parameters (expectation and standard deviation). Consistency, unbiasedness and efficiency.
of estimators. Interval estimation. Standard normal distribution and Laplace function. Confidence interval and confidence bound.

**Measuring instrument (MI):** Generalized structure of MI. Classes of MI. Metrological characterization of MI. Instrumental error and its components. Accuracy classes. Calibration. Hierarchy of standards. Traceability. Calibration procedures.

**Uncertainty analysis:** Correction of measurements. Recommendations of ISO Guide to Expression of Uncertainty in Measurement (GUM). Type A uncertainties. Type B uncertainties.

6. **Conclusion**

Axiomatic introduction of quantity concept provides a possibility, in consecutive, consistent and understandable for students manner, to lay down all material of the metrology course. Necessary notation of dimension should be also introduced in axiomatic way allowing to study properties of dimensional and dimensionless (dimension one) quantities. In this case, further switching to topics of quantity calculus and systems of quantities (units) will be well-justified.

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