SATELLITES OF INFINITE RANK IN THE SMOOTH
CONCORDANCE GROUP

MATTHEW HEDDEN AND JUANITA PINZÓN-CAICEDO

Abstract. We conjecture that satellite operations are either constant or have
infinite rank in the concordance group. We reduce this to the difficult case of
winding number zero satellites, and use SO(3) gauge theory to provide a gen-
eral criterion sufficient for the image of a satellite operation to generate an in-
finites rank subgroup of the smooth concordance group \( C \). Our criterion applies
widely, notably to scores of unknotted patterns for which the corresponding
operators on the topological concordance group are zero. We raise some ques-
tions and conjectures regarding satellite operators and their interaction with
concordance.

1. Introduction

Oriented knots are said to be concordant if they cobound a properly embedded
cylinder in \([0,1] \times S^3\). One can vary the regularity of the embeddings, and typically
one considers either smooth or locally flat continuous embeddings. Either choice
defines an equivalence relation under which the set of knots becomes an abelian
group using the connected sum operation. These concordance groups of knots are
intensely studied, with strong motivation provided by the profound distinction be-
tween the groups one defines in the topologically locally flat and smooth categories,
respectively. Indeed, many questions pertaining to 4–manifolds with small topol-
yogy (like the 4–sphere) can be recast or addressed in terms of concordance. Despite
the efforts of many mathematicians, the concordance groups are still rather poorly
understood. For instance, in both categories, it is still unknown whether the groups
possess elements of any finite order other than two.

Some of the most powerful tools for analyzing concordance groups come from satel-
lite operations. To define these, consider a knot in a solid torus \( P \subset S^1 \times D^2 \).
Assign to an arbitrary knot \( K \) the image of \( P \) under the canonical identification
of a neighborhood of \( K \) with \( S^1 \times D^2 \). The corresponding satellite knot is denoted
\( P(K) \). Since framings of a properly embedded annulus in \([0,1] \times S^3\) are naturally
identified with framings of either circle on the boundary, it follows that the satellite
operator \( P \) descends to concordance classes. Thus for any knot \( P \subset S^1 \times D^2 \) we
obtain a self-map on the (smooth or topological) concordance group:
\[ P : \mathcal{C} \rightarrow \mathcal{C}. \]

It is important to note that these maps are typically not homomorphisms. Indeed,
the first author has conjectured that they essentially never are:

Conjecture 1. The only homomorphisms on the concordance groups induced by
satellite operators are the zero map, the identity, and the involution coming from
orientation reversal.
Despite their conjectural disregard for addition, satellite operations have nonetheless proved to be extremely useful for studying the structure of these groups and figure prominently in many applications and structural theorems like [CT07, CO93, Liv00, CHL11b, ChH11a, HKL16, HLR12, CHH13, CHP17, Hom15, Lev16, CDR14, Liv01, Liv83, Lit84, DR16], to cite only a few.

Particularly noteworthy is work of Cochran-Harvey-Leidy [CHL11b], which conjectured a fractal nature of the concordance group derived from the abundance of satellite operators. As evidence, they introduced the notion of robust doubling operators and showed that they interact well with the Cochran-Orr-Teichner filtration [COT03]. In particular, a robust doubling operator has infinite rank and is injective on large subsets of concordance. More work in this direction came from [CDR14] which showed that many winding number non-zero operators are injective on homological variants of the concordance group; [CHP17] provided further evidence that concordance is a fractal set by defining metrics using gropes with respect to which winding zero satellites are contractions. Despite these efforts, it remains unknown whether there exists any injective winding zero satellite operator. Interesting questions can be made regarding surjectivity as well. It is easy to show that a winding zero operator is never surjective, as its image consists of knots with bounded genus. By using an additive concordance invariant with values bounded by the genus, like the Ozsváth-Szabó-Stipsicz $\Upsilon$ invariant, one can show that its image cannot even generate concordance [OSS17, Liv17, Wan16]. Much more subtle is the winding one case, addressed by Levine [Lev16], who showed that there are winding one satellite operators whose image does not contain zero. These results motivate us to make the following conjecture:

**Conjecture 2.** The image of every non-constant satellite operator has infinite rank.

Since, in light of Conjecture 1, we have no reason to expect the image to be a subgroup, rank should be interpreted as the rank of the subgroup generated by the image. It is relatively easy to verify the conjecture in the case of non-zero winding number patterns, since the algebraic concordance class is additive in an appropriate sense under satellites. We make this precise in Proposition 8 below. Conjecture 2 then reduces to the winding number zero case, which is significantly harder. The purpose of this article is to provide a general criterion to guarantee that such an operator has infinite rank. To state our result, we recall that the rational linking number between curves $\gamma, \eta$ in a rational homology sphere is defined to be $1/d$ times the algebraic intersection number of $\gamma$ with a $2$-chain whose boundary maps to $d\eta$. Its reduction modulo $\mathbb{Z}$ is the linking form on first homology. Note that the rational linking number assigns a number to a framed curve, defined to be the linking of the curve with a push-off defined by the framing. In these terms, we have our main theorem

**Theorem 3.** Let $P \subset S^1 \times D^2$ be a pattern with winding number zero, and let $J$ denote a framed lift of $\partial D^2$ to the branched double cover $\Sigma(P(U))$. If the rational linking number of $J$ in $\Sigma(P(U))$ is non-zero, then $P : C \to C$ has infinite rank.

It is interesting to compare our result to that of Cochran-Harvey-Leidy [CHL11b]. Robust doubling operators have the property that the Blanchfield self-pairing of a lift of $\partial D^2$ to the infinite cyclic cover of $P(U)$ is non-trivial. If the homology
class \([J] \in H_1(\Sigma(P(U)))\) has non-trivial self-pairing under the \(\mathbb{Q}/\mathbb{Z}\)-valued linking form, it follows easily that its lift to the infinite cyclic cover will have non-trivial Blanchfield self-pairing. We do not, however, require any conditions on isotropic submodules of linking forms as in [CHL11b]. Moreover, our results extend to the case where the branched cover \(\Sigma(P(U))\) is a homology sphere (with trivial linking form). In particular, patterns with \(P(U)\) an unknot are of primary interest. In such cases the image of the satellite operator consists of topologically slice knots, so that \(P\) acts as zero on the topological concordance group. The methods of [CHL11b] do not apply in this setting, being manifestly topological.

Our result is proved in the context of \(SO(3)\) gauge theory, and uses instanton moduli spaces of adapted bundles over 4–manifolds in conjunction with the Chern-Simons invariants of flat connections on the 3–manifolds arising as cross sections of their ends. This technique was pioneered by Furuta [Fur90] and Fintushel-Stern [FS85], and later refined by the first author and Kirk [HK12, HK11] for the purpose of studying such questions. Indeed, our result should be viewed as a vast generalization of the main theorem of [HK12], which showed that the Whitehead doubling operator has infinite rank, and of [PC17], which showed that some generalizations of the Whitehead doubling operator also have infinite rank. The technique is inherently smooth; indeed, as remarked above, any unknotted pattern will be zero on topological concordance. It is also worth emphasizing that it seems very difficult, if not impossible, to prove a result of this form using any other known smooth concordance invariants. For instance, \(\Upsilon\) cannot prove that an infinite set of knots whose genera are bounded (like the image of any of our patterns) are independent in concordance [Liv17, Theorem 9.2]; it seems similarly unlikely that any invariants derived from the stable equivalence class of the knot Floer homology complex can prove such a result [Hom17]. While the correction terms of branched covers [MO07, Jab12] or knot Floer homology of the branch loci therein [GRS08] should contain a great deal of concordance information, they are quite challenging to compute. Similarly, the various concordance invariants coming from Khovanov homology and its generalizations [Ras10, Lob09, LL16] are extremely difficult to compute for families and are not expected to behave predictably under satellites.

We use the instanton cobordism obstruction to show that, given a pattern satisfying the rational linking number hypothesis, an infinite collection of torus knots can be chosen such that their images under \(P\) are \(\mathbb{Z}\)-linearly independent. While we use an independent set of knots (a subset of torus knots) to prove our theorem, we suspect that this was unnecessary. In fact, we make a rather bold strengthening of our conjecture:

**Conjecture 4.** For any non-constant winding number zero operator \(P\), there exists a knot \(K\) for which the set \(\{P(nK)\}_{n \in \mathbb{Z}}\) has infinite rank.

Thus we expect satellite operators to expand the concordance group, in the sense that the image of a finite rank subgroup will often have infinite rank. This again drives home the expectation that they are not homomorphisms. It would be very interesting to verify that Whitehead doubles of the \(n\)-fold connected sums of the trefoil knot are independent in concordance when \(n > 0\), or to find any knot for which the Whitehead double of both \(K\) and \(-K\) are non-zero in concordance.
Finally, we remark that in 3-dimensional topology, JSJ theory applied to knot complements tells us that the decomposition of a knot as an iterated satellite is quite rigid. It is reasonable to ask whether there are 4-dimensional remnants of this rigidity. As a sample, given a winding zero satellite operator $P$, we can define the $P$-filtration of the concordance group to be the descending filtration whose $i$-th term is the subgroup generated by $i$-fold iterated satellites with pattern $P$:

$$
\ldots \subseteq \langle P^2(C) \rangle \subseteq \langle P(C) \rangle \subseteq C.
$$

**Question 5.** If $P$ is a non-constant winding zero operator, does each associated graded group of the $P$-filtration have infinite rank?

Since the genera of knots in the image of $P$ are bounded, the first quotient $C/\langle P(C) \rangle$ is of infinite rank [Wan16]. Again, it would be interesting to understand even the much simpler question of whether iterated Whitehead doubles of the trefoil are independent. Kyungbae Park [Par18] has shown that the first two are. In the case of non-zero winding number, Wenzhao Chen recently gave a criterion for iterates of an operator to be independent and has shown that infinitely many iterated satellites with the Mazur pattern are independent [Che]. As a final question, we have

**Question 6.** If $P$ and $Q$ are winding number zero satellite operators inducing isomorphic filtrations of $C$, are $P$ and $Q$ concordant in the solid torus?

**Outline:** The next section verifies Conjecture 2 for satellites with non-zero winding number, then turns to topological constructions essential for the proof of our main result. Section 3 offers a quick overview of the instanton cobordism obstruction derived from instanton moduli spaces and Chern-Simons invariants. Section 4 uses this obstruction in conjunction with the topological constructions from Section 2 to prove the main result. Finally, Section 5 contains some examples that illustrate the way our result can be applied in practice. In particular, we show how to check whether an operator satisfies the hypothesis of Theorem 3 and how to use this method to easily produce examples of topologically slice operators with infinite rank on smooth concordance.

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2. Topological Preliminaries and Key Constructions

In this section we introduce notation and then verify Conjecture 2 in the case of satellite operators with non-zero winding number. The proof breaks down completely in the case of winding number zero satellites, but it inspires the rough idea that one should look for invariants of winding zero satellite knots that remember the companion and yet are robust enough to distinguish infinitely generated subgroups of the topologically slice subgroup of the smooth concordance group. For
us, these will be gauge theoretic properties and invariants of the 2-fold branched covers which behave well with respect to definite cobordisms.

To begin, let us provide a more precise definition of a satellite

**Definition 7.** Let $P \subset V := S^1 \times D^2$ be an oriented knot in the solid torus. Consider an orientation preserving embedding $h : V \to S^3$ taking $V$ to a tubular neighborhood of a knot, $K$, so that a longitude of $V$ i.e. $S^1 \times \{ \ast \in \partial D^2 \}$ is sent to the canonical longitude of $K$. The knot $h(P)$ is called the satellite knot with pattern $P$ and companion $K$, and is denoted $P(K)$. The winding number of the satellite is defined to be the algebraic intersection number of $P$ with $\ast \times \partial D^2$.

We now turn to verifying that patterns with non-zero winding number induce operators with infinite rank on concordance.

**Proposition 8.** Let $P \subset S^1 \times D^2$ be a pattern for a satellite operator with non-zero winding number, $w$. Then $P : \mathcal{C} \to \mathcal{C}$ has infinite rank.

**Proof.** We appeal to the jump function of the Tristram-Levine signature. Let $\sigma_K(\zeta)$ denote the signature of the Hermitian matrix $(1 - \zeta)V_K + (1 - \zeta)\overline{V_K}$, where $V_K$ is a Seifert matrix for a knot $K$ and $\zeta = e^{2\pi it} \in S^1$. The signature jump function is defined as

$$\delta_K(\zeta) = \frac{1}{2} \left( \lim_{s \to \zeta^+} \sigma_K(s) - \lim_{s \to \zeta^-} \sigma_K(s) \right).$$

For any fixed $\zeta \in S^1$, the jump function provides a homomorphism from the (topological) concordance group to $\mathbb{Z}$.

Litherland proved a formula for the signature function of a satellite knot [Lit79]. Expressed in terms of the jump function, it reads as

$$\delta_P(K)(\zeta) = \delta_P(U)(\zeta) + \delta_K(\zeta^w),$$

where $w$ is the winding number.

Litherland also showed that the jump functions of torus knots are independent in the additive group of functions on the circle [Lit79]. In particular, as we vary through the set of torus knots, there are infinitely many distinct $\zeta \in S^1$ for which $\delta_{T_{p,q}}(\zeta) \neq 0$ for some $(p, q)$. Since the function $\zeta \to \zeta^w$ is finite to one, the same holds for the functions $\delta_{T_{p,q}}(\zeta^w)$ as we vary over $p, q$. It then follows from Equation (1) that the same holds for the jump functions of satellites of torus knots with pattern $P$. This immediately implies the operator defined by $P$ has infinite rank, as the non-zero values of the signature jump function occur only at roots of the Alexander polynomial, and there are only finitely many such roots for any finite set of knots.

The proof of the above proposition clearly breaks down in the winding number zero case, as the signature function of the satellite forgets the companion knot entirely. This is not simply a deficit of the signature function. Indeed, in the case that $P$ is unknotted, or even merely an Alexander polynomial one knot, the topological concordance class of $P(K)$ is trivial for any $K$ by work of Freedman [FQ90]. Thus, far subtler techniques are required. For the remainder of this section, we pave our
way to apply an instanton cobordism obstruction to the problem at hand by constructing some explicit cobordisms.

Recall, then, that closed oriented 3–manifolds $Y_0$ and $Y_1$ are oriented cobordant if there exists a compact, oriented 4–manifold $W$ with oriented boundary $\partial W = -Y_0 \sqcup Y_1$, where we follow the “outward normal first” convention for orientations induced on the boundary. The manifold $W$ is called a cobordism from $Y_0$ to $Y_1$, and $Y_0$ (resp. $Y_1$) will be referred to as the “incoming” (resp. “outgoing”) boundary component. We will construct cobordisms whose incoming boundary component is the branched double cover of a satellite knot, and whose outgoing boundary contains a manifold obtained by Dehn surgery on the companion as a prime summand. These cobordisms allow us to isolate the companion knot $K$ from $(P(K))$, and thereby utilize desirable gauge theoretic properties of surgeries on the former to obstruct sliceness of the latter in the spirit of the signature argument.

The 2–fold cover of $S^3$ branched over a satellite $P(K)$ will be denoted by $\Sigma(P(K))$. One can easily show that $\Sigma(P(K))$ splits into the union of the 2-fold cover of $N(K) \cong h(V) \cong S^1 \times D^2$ branched over $h(P)$ and a 2-fold covering space of the knot complement $S^3 \setminus N(K)$; see, for example [Sei50] or [LM85]. The isomorphism type of this latter covering space depends only on the parity of the winding number. In the case that it is even, the cover is the disjoint union of two copies of $S^3 \setminus N(K)$. In this case, then, if $J_1, J_2$ are the lifts of $* \times \partial D^2$ to $\Sigma(P(U))$, the map $h$ from Definition 7 induces an identification

$$\Sigma(P(K)) \cong \Sigma(P(U)) \setminus N(J_1 \sqcup J_2) \cup \frac{1}{h}(S^3 \setminus N(K)),$$

where the gluing map $\tilde{h}$ is induced by $h$. It identifies each copy of $\mu_K$ with $J_1$ or $J_2$, and each corresponding copy of $\lambda_K$ with the corresponding meridian of $J_1$ or $J_2$. Thus, the 2-fold branched cover of $S^3$ over a satellite knot with even winding number is determined by the 2-fold cover of $S^3$ branched over $P(U)$, the two lifts $J_1$ and $J_2$ of the meridian of $V$, and the knot exterior $S^3 \setminus N(K)$.

Next, we briefly recall the definition of the rational linking number between curves in a rational homology sphere, $\Sigma$ (see Cha and Ko [CK02, Section 3] for a nice reference). Suppose $\gamma, \eta$ are two such curves. As the first $\mathbb{Q}$–homology of $\Sigma$ vanishes, we can find 2-chains $a, b$ with boundary equal to $n\gamma$ and $m\gamma$, respectively, for integers $m, n > 0$ (the order of the homology classes $[\gamma], [\eta]$ would suffice for $m,n$). Now the rational linking number of $\gamma$ and $\eta$ is defined to be:

$$lk_\Sigma(\gamma, \eta) := \frac{1}{n}(a \cdot \eta) = \frac{1}{m}(\gamma \cdot b) \in \mathbb{Q},$$

where $\cdot$ denotes algebraic intersection number. The linking number is independent of the choices of $a, b, m, n$, and can be viewed as a simultaneous generalization of the ordinary linking number between null-homologous curves and the linking pairing on first homology. Indeed, the value of $lk_\Sigma(\gamma, \eta)$ modulo $\mathbb{Z}$ depends only on the homology classes $[\gamma], [\eta]$ and equals the $\mathbb{Q}/\mathbb{Z}$–valued linking pairing on $H_1(\Sigma)$. The linking number itself depends on more than just the homology classes but it is, like ordinary linking numbers, an invariant of the concordance class of the link $\gamma \sqcup \eta$. Finally, we observe that while there is no definition of a self-linking number
of a single curve, the linking number assigns a well-defined rational number to a framed curve, defined to be the linking number between a curve and its framing. Here we identify framings of a curve $\gamma$ with embedded parallel copies of $\gamma$ through the tubular neighborhood theorem.

The following proposition provides the cobordisms requisite for our strategy.

**Proposition 9.** If $P \subset S^3 \times D^2$ is a pattern with winding number zero, then $\Sigma(P(K))$ is cobordant to a 3–manifold containing $S^3_{q/p}(K)$ as one of its prime summands, with $p, q$ arbitrary relatively prime nonzero integers. Moreover, if the rational linking number of (either) framed lift of $\partial D^2$ to $\Sigma(P(U))$ satisfies $lk_{\Sigma}(J_1, J_1) < -p/q$, then the cobordism can be taken to be negative definite.

**Proof.** Referring to our decomposition of $\Sigma(P(K))$ in Equation (2), let $\eta$ be a simple closed curve in $T = \partial N(J_1)$. Form a smooth 4–manifold $W$ by attaching a 2-handle to $[0, 1] \times \Sigma(P(K))$ along $\eta$ with framing given by a pushoff of $\eta$ along the torus $T$. A well-known argument shows that the “outgoing” component of $\partial W$ is diffeomorphic to a connected sum $S^3_{q/p}(K) \# Y$ where $Y$ is the result of a Dehn filling on $\Sigma(P(K)) \setminus (S^3 \setminus N(K))$ (see [Gor83, Lemma 7.2] or [Gor09, Section 2.2]). The intersection form of $W$ depends on the rational linking number of $J_1$, framed by a lift of the canonical framing of the meridian of the solid torus $\ast \times \partial D^2$, viewed as a knot in $S^3$. Indeed, since $W$ is obtained from a $\mathbb{Q}$–homology sphere by attaching exactly one 2-handle, its intersection form can be easily computed using the framing of the 2-handle, which in this case is given by $lk_{\Sigma(P(K))}(\eta, \eta')$, the linking number in $\Sigma(P(K))$ of $\eta$ and its pushoff $\eta'$ along the torus $T$.

To compute $lk_{\Sigma(P(K))}(\eta, \eta')$ we will use a formula of Cha and Ko [CK02, Theorem 3.1] which computes linking numbers in rational homology spheres described as integral surgery on a framed link $L \subset S^3$. To obtain a surgery description for $\Sigma(P(K))$, we first realize the cover $\Sigma(P(U))$ as surgery on a link $L$ in $S^3$ obtained from a Seifert surface for $P(U)$ as described in [AK80]. Now let $\Gamma_1, \Gamma_2$ be two copies of a surgery description for the companion knot $K$. By this, we mean that we realize $S^3 \setminus N(K)$ by surgery on a link in a solid torus, where the meridian of the solid torus is the longitude for $K$, and all of the components of the surgery link have framings in $\{\pm 1\}$ and linking number zero with $K$. Such a description can be obtained from an unknotting sequence for $K$, by realizing the crossing changes by surgery. Denote by $J_1, J_2$ the lifts of $\partial D^2$ and consider the link $L \sqcup J_1 \sqcup J_2$. The surgery description of $\Sigma(P(K))$ is obtained from $L \sqcup J_1 \sqcup J_2$ by replacing the tubular neighborhood $N(J_i)$ with the aforementioned surgery presentation for $S^3 \setminus N(K)$. Here, the meridian of $N(J_i)$ identified with a longitude for $\Gamma_i$ and a longitude of $N(J_i)$ with a meridian of $\Gamma_i$. Figure 3 in the final section illustrates this procedure with an example.

To make the computations easier, endow the torus $T = \partial N(J_1)$ with a basis given by a meridian-longitude pair so that $\eta$ is a $(p, q)$ curve in $T$. Let $B_1, B_2$ be linking matrices for $\Gamma_1, \Gamma_2$ respectively, and let $S$ be a Seifert matrix for the Seifert surface used to get a surgery description for $\Sigma(P(U))$ as described by Akbulut-Kirby in [AK80]. Then the matrix

$$A = B_1 \oplus B_2 \oplus (S + S^T)$$
is the linking matrix for the surgery presentation for $\Sigma(P(K))$. For a link $L \subset S^3$ and a knot $K \subset S^3$, denote by $lk_{S^3}(K, L)$ the row vector whose $i$-th entry is $lk(K, L_i)$ where $L_i$ is the $i$-th component of $L$. [CK02, Theorem 3.1] then gives:

$$lk_{S^3}(\eta, \eta') = lk_{S^3}(\eta, \eta') - lk_{S^3}(\eta, \Gamma_1 \sqcup \Gamma_2 \sqcup L)A^{-1}lk_{S^3}(\eta', \Gamma_1 \sqcup \Gamma_2 \sqcup L)^T.$$ 

Since the curves in the surgery description for $S^3 \setminus N(K)$ have linking number zero with any curve in $T$, it follows that

$$lk_{S^3}(\eta, \Gamma_1 \sqcup \Gamma_2 \sqcup L)A^{-1}lk_{S^3}(\eta', \Gamma_1 \sqcup \Gamma_2 \sqcup L)^T = lk_{S^3}(\eta, L)(S + S^T)^{-1}lk_{S^3}(\eta', L)^T.$$ 

Similarly, since a meridian disk for $J_i$ is disjoint from $L$, we have

$$lk_{S^3}(\eta, L) = qlk_{S^3}(J_1, L) = qlk_{S^3}(\eta', L).$$

Therefore the previous computation reduces to

$$lk_{S^3}(\eta, \eta') = pq - q^2lk_{S^3}(J_1, L)(S + S^T)^{-1}lk_{S^3}(J_1, L)^T = pq + q^2l.$$ 

The last equality is obtained by noticing that

$$l := lk_{S^3}(J_1, J_1) = -lk_{S^3}(J_1, L)(S + S^T)^{-1}lk_{S^3}(J_1, L)^T.$$ 

From this it follows that the intersection form for $W$ has matrix presentation

$$Q_W \cong \langle pq + q^2l \rangle,$$

and so $W$ is negative definite whenever $pq + q^2l < 0$ or equivalently if $l < -\frac{p}{q}$ whenever $q \neq 0$. □

The relevant case of Proposition 9 that will be used to prove Theorem 3 is included as follows in the form of a corollary.

**Corollary 10.** Let $P \subset S^1 \times D^2$ be a pattern with winding number zero, and let $K$ be a knot which can be unknotted by changing only positive crossings. If $lk_{\Sigma}(J_i, J_i) < 0$, then there exists relatively prime odd positive integers $p, q$ and a negative definite cobordism $W$ from $\Sigma(P(K))$ to $S^3_{q/p}(K) \# Y$, where $Y$ is a $\mathbb{Z}/2$-homology sphere which depends only on $P$ and the pair $(p, q)$. 

Knots which can be unknotted by changing positive crossings are abundant e.g. positive knots. For our purposes, the positive torus knots $T_{r,n}$ will be sufficient.

**Proof.** Choose $\eta$ to be the $(p, q)$ curve in $T = \partial N(J_1)$ with $p/q$ lying in the interval $(0, -lk_{\Sigma}(J_1, J_1))$. The cobordism described in the proof of Proposition 9 has intersection form with matrix presentation $pq + q^2lk_{\Sigma}(J_1, J_1)$, which is negative since $0 < p/q < -lk_{\Sigma}(J_1, J_1)$. Now the other summand in the outgoing end of the cobordism is the 3-manifold resulting from Dehn filling the complement in $\Sigma(P(K))$ of one of the lifts of $S^3 \setminus N(K)$. In terms of the decomposition of Equation (2), this is a Dehn filling of the manifold

$$\Sigma(P(U)) \setminus N(J_1 \sqcup J_2) \cup_k \left( S^3 \setminus N(K) \right)$$

(noting we’ve removed one of the copies of the complement of $K$ in our decomposition). Since $K$ can be unknotted by a series of positive-to-negative crossing changes, there is a negative definite cobordism (rel boundary) from $S^3 \setminus N(K)$ to $S^3 \setminus N(U) \cong S^1 \times D^2$. (c.f. [HK12, Lemma 3.5], or [CG88]). Applying this cobordism to the remaining copy of $S^3 \setminus N(K)$ in our decomposition, we obtain a cobordism whose outgoing manifold is the connected sum of $S^3_{q/p}(K)$ with a Dehn
filling of the manifold $\Sigma(P(U)) \setminus N(J_1)$; this latter manifold clearly depends only on $P$ and $(p,q)$. Since we picked $(p,q)$ odd, it will be a $\mathbb{Z}/2$–homology sphere. □

3. Instanton Obstruction to Sliceness

In this section we survey works of Furuta, Fintushel-Stern, and Hedden-Kirk which use moduli spaces of anti-self-dual (ASD) connections on $SO(3)$ bundles over 4–manifolds with cylindrical ends to study the 3–dimensional $\mathbb{Z}/2$–homology cobordism group. These techniques provide an obstruction to the existence of negative definite 4–manifolds whose boundary is a given disjoint union of 3–manifolds. To state it, recall that the relative Chern-Simons invariant $cs(\alpha, \beta)$ between flat $SO(3)$ connections $\alpha, \beta$ on a closed oriented 3–manifold, is defined as the integral

$$cs(\alpha, \beta) := -\frac{1}{8\pi} \int_{[0,1] \times Y} Tr(F(A_t) \wedge F(A_t)) \in \mathbb{R}/\mathbb{Z},$$

where $A_t$ is any path of connections between $\alpha$ and $\beta$. The right hand side is the Chern-Weil integrand for the first Pontryagin class. Integrality of the first Pontryagin number of a bundle on a closed 4–manifold implies that, modulo $\mathbb{Z}$, the integral is independent of the chosen path, and invariant under the gauge group actions on $\alpha$ and $\beta$. The obstruction is phrased in term of the minimal Chern-Simons invariant. For a $\mathbb{Z}/2$–homology 3–sphere $Y$, this is defined as

$$\tau(Y) := \min\{cs(\alpha, \theta) \mid \alpha \text{ flat connection on } Y \in (0,1],$$

where $\theta$ is the trivial connection on the unique (trivial) $SO(3)$ bundle on $Y$, and where we have identified $\mathbb{R}/\mathbb{Z}$ with $[0,1]$ in the obvious way (we could, in fact, lift the relative and minimal Chern-Simons values to $\mathbb{Z}/4\mathbb{Z}$, but will have no need to do so for our purposes. See Section 2.2 of [HK11] for more details.) The following theorem is a restatement of results of Furuta [Fur90], Fintushel-Stern [FS85], and Hedden-Kirk [HK11] that will be used to establish Theorem 3.

**Theorem 11** (Furuta [Fur90], Fintushel-Stern [FS85], Hedden-Kirk [HK11]). Consider a family $\{\Sigma_i\}_{i=1}^N$ of oriented $\mathbb{Z}/2$–homology 3-spheres. Let $(p, q)$ and $(r, s)$ be two pairs of relatively prime and positive integers and suppose $\Sigma_N = S^3_{q/p}(T_{r,s})$. If

$$(3) \quad \frac{q}{rs(prs - q)} < \min\left\{\frac{1}{r}, \frac{1}{s}, \frac{1}{prs-q}, \tau(\pm \Sigma_1), \ldots, \tau(\pm \Sigma_{N-1})\right\},$$

then there does not exist a smooth 4–manifold $X$ with $H^1(X, \mathbb{Z}/2) = 0$ and negative definite intersection form, whose oriented boundary is given by

$$\partial X = \prod_{i=1}^N a_i \Sigma_i, \quad \text{with } a_i \in \mathbb{Z}, a_N > 0.$$

In the above, $a_i \Sigma_i$ means the disjoint union of $a_i$ copies of $\Sigma_i$, endowed with the given orientation if $a_i > 0$ and opposite otherwise. We sketch a proof of the theorem, which is by way of contradiction.

**Sketch of proof:** The manifold $\Sigma_N = S^3_{q/p}(T_{r,s})$ is diffeomorphic to $\Sigma(r,s,prs - q)$, where $\Sigma(r,s,prs - q)$ is the link of the surface singularity

$$z_0^r + z_1^s + z_2^{prs-q} = 0,$$
and the bar means we reverse its orientation. Now $\Sigma(r, s, prs - q)$ is clearly Seifert fibered, and bounds two closely related negative definite smooth 4–manifolds. The first is the resolution $R$ of the singularity giving rise to it. The second is the smooth negative definite 4–manifold $W$ obtained from the mapping cylinder of the Seifert fibration $\Sigma(r, s, prs - q) \to S^2$ by excising neighborhoods of the singularities that arise from the singular fibers (thus $W$ has three additional lens space boundary components). Over $W$ one can construct a non-trivial $SO(3)$ bundle $E$ from which one defines an associated moduli space $M$ of ASD connections. The virtual dimension of this moduli space equals 1 whenever $(p, q)$ and $(r, s)$ are pairs of relatively prime positive integers, as computed by the Neumann-Zagier formula [NZ85]. Moreover, the moduli space is non-empty, possessing a unique singular point that corresponds to an explicit reducible connection. The singular point has a neighborhood diffeomorphic to a half open interval $[0, \epsilon)$.

Assume that a negative definite 4–manifold $X$ as in the theorem exists. One can glue it to the disjoint union of $W$ and $(a_N - 1)$ copies of the resolution $R$ along $a_N \Sigma_N$ to obtain

$$ \hat{X} = X \cup W \cup (a_N - 1)R. $$

The fact that $X$ and $R$ are negative definite allows one to argue that the extension $\hat{E}$ of $E \to W$ by the trivial bundle on $X$ and $R$ also possesses a non-empty moduli space $\hat{M}$ of ASD connections, with the same dimension as $M$. As before, the singular points of $\hat{M}$ correspond precisely to reducible connections on $\hat{E}$, the number of which can be computed in terms of the order of the torsion subgroup of $H_1(X; \mathbb{Z})$ and the number of even factors in it. In particular, the number of singular points is odd. This implies that the moduli space $M$ must be non-compact, as a compact 1–manifold has an even number of boundary points (which we identify with singular points under identifications of their neighborhoods with half-open intervals).

Now failure of compactness in moduli spaces of ASD connections on manifolds with cylindrical ends occurs only through bubbling or by energy escaping down the ends in the form of broken flowlines for the gradient of the Chern-Simons functional. Each of these phenomena require a certain quanta of analytic energy which, for an ASD connection, is given by the integral over $\hat{X}$ of the integrand defining the Chern-Simons invariant; for points in the moduli space $\hat{M}$, the energy is $q/rs(prs - q)$. Bubbling cannot occur since this quantity is less than 4, and the assumptions in Equation (3) guarantee that a sequence cannot diverge to a broken flowline. Indeed, the minimal Chern-Simons invariants of $\pm \Sigma_i$ and the lens space boundary components of $W$ shown there provide a lower bound for the amount of energy carried by a broken flowline. It follows that the moduli space is compact, a contradiction which rules out the existence of $X$. \hfill $\square$

The following corollary explains how negative definite cobordisms can be used to study linear independence in the $\mathbb{Z}/2$–homology cobordism group. Recall that the $\mathbb{Z}/2$–homology cobordism group $\Theta_{\mathbb{Z}/2}$ consists of equivalence classes of oriented $\mathbb{Z}/2$–homology 3–spheres, where two such are equivalent if they cobound a homology cylinder. Addition is given by connected sum.
Corollary 12 (Furuta [Fur90], Fintushel-Stern [FS85], Hedden-Kirk [HK11]). Let \( \{ \Sigma_i \}_{i=1}^N \) be a family of oriented \( \mathbb{Z}/2 \)-homology 3-spheres. Suppose \( \Sigma_N \) is cobordant via a negative definite cobordism with \( H^1(\mathbb{Z}; \mathbb{Z}/2) = 0 \) to \( S^3_{q/p}(T_{r,s}) \# Y \), where \( Y \) is any \( \mathbb{Z}/2 \)-homology 3–sphere and \((p,q)\) and \((r,s)\) are pairs of relatively prime positive integers with \( q \) odd. If

\[
\frac{q}{rs(prs - q)} < \min \left\{ \frac{1}{r}, \frac{1}{s}, \frac{1}{prs - q}, \tau(\pm \Sigma_1), \ldots, \tau(\pm \Sigma_{N-1}), \tau(Y) \right\},
\]

then \( \Sigma_N \) has infinite order in \( \Theta_{\mathbb{Z}/2} \) and is independent from the other manifolds:

\[
\langle \Sigma_N \rangle \cap \langle \Sigma_1, \ldots, \Sigma_{N-1} \rangle = \{0\},
\]

where \( \langle - \rangle \) denotes the subgroup of \( \Theta_{\mathbb{Z}/2} \) generated by \(-\).

Proof. Suppose, to the contrary, that either the intersection of the subgroups in question is non-trivial or \( \Sigma_N \) has finite order in \( \Theta_{\mathbb{Z}/2} \). This implies there exist integers \( c_1, \ldots, c_N \) with \( c_N > 0 \) such that

\[
c_N \Sigma_N \# (c_1 \Sigma_1 \# \cdots \# c_{N-1} \Sigma_{N-1}) = \partial Q,
\]

for \( Q \) a smooth 4–manifold with the same \( \mathbb{Z}/2 \)-homology groups as the 4–ball. Here, we temporarily use the notation \( c_i \Sigma_i \) to denote the connected sum (instead of disjoint union) of \( c_i \) copies of \( \Sigma_i \), with opposite its given orientation if \( c_i < 0 \).

Form the 4–manifold

\[
X_0 = Q \cup_{c_N \Sigma_N} c_N Z,
\]

where \( c_N Z \) is the boundary connected sum of \( c_N \) copies of the negative definite cobordism from \( \Sigma_N \) to \( S^3_{q/p}(T_{r,s}) \# Y \). Attaching 3–handles to \( X_0 \) along all of the connected sum spheres yields a smooth negative definite 4–manifold \( X \) with \( H^1(X; \mathbb{Z}/2) = 0 \), and boundary

\[
\partial X = c_N S^3_{q/p}(T_{r,s}) \prod_{i=1}^{N-1} c_i Y \prod_{i=1}^{N-1} -c_i \Sigma_i, \quad \text{with} \quad c_i \in \mathbb{Z}, c_N > 0.
\]

Relabelling as appropriate, we arrive at a contradiction to Theorem 11. \( \square \)

There is a well-known homomorphism from the concordance group to the \( \mathbb{Z}/2 \)-homology cobordism group,

\[
\Sigma : \mathcal{C} \to \Theta_{\mathbb{Z}/2},
\]

defined by sending the concordance class of a knot \( K \) to the homology cobordism class of its branched 2–fold cover \( \Sigma(K) \). This homomorphism, in conjunction with Theorem 11, provides a tool for showing that the image of a satellite operator has infinite rank. Indeed, to show that an operator \( P \) has infinite rank, we only need to argue that the composition \( \Sigma \circ P \) does. For this, it suffices to find an infinite collection of torus knots \( \{ T_{r_i,s_i} \}_{i=1}^{\infty} \) for which the branched covers of their satellites \( \{ \Sigma(P(T_{r_i,s_i})) \}_{i=1}^{\infty} \) are independent in \( \Theta_{\mathbb{Z}/2} \). We accomplish this in the next section.

4. Proof of the Main Result

We are now in a position to use the instanton obstruction from the previous section in conjunction with the cobordisms constructed in Section 2 to prove our main theorem. For convenience, we restate it here.
Proceeding as before, choose $K_i$, $i = 1, \ldots, K$. For any pair of relatively prime positive integers $p, q$, satisfying $l < -p/q < 0$. In terms of these, we define the function of two variables

$$\rho(r,s) = \frac{q}{rs(prs - q)}.$$

Corollary 10 provides a negative definite cobordism from $\Sigma(P(T_{r,s}))$ to $S_{q/p}(T_{r,s})\#Y$ for any choice of torus knot (or any knot, for that matter). We therefore choose a pair of relatively prime positive integers $r_1, s_1$ so that

$$\rho(r_1, s_1) < \min \left\{ \tau(Y), \frac{1}{r_1}, \frac{1}{s_1}, \frac{1}{prs - q} \right\}.$$

Letting $K_1 = T_{r_1, s_1}$, and $\Sigma_1 = \Sigma(P(K_1))$, the hypothesis for Corollary 12 are met, thereby showing that $\Sigma_1$ has infinite order in $\Theta_{\mathbb{Z}/2}$ i.e. $\langle \Sigma_1 \rangle \cong \mathbb{Z}$.

Now choose a pair of relatively prime positive integers $r_2, s_2$ so that

$$\rho(r_2, s_2) < \min \left\{ \tau(Y), \tau(\pm \Sigma_1), \frac{1}{r_1}, \frac{1}{s_2}, \frac{1}{prs - q} \right\}.$$

Corollary 10 again gives a negative definite cobordism, now from $\Sigma_2 = \Sigma(P(T_{r_2, s_2}))$ to $S_{q/p}(T_{r_2, s_2})\#Y$. Corollary 12, together with our previous choices, gives $\langle \Sigma_2 \rangle \cong \mathbb{Z}$, and $\langle \Sigma_2 \rangle \cap \langle \Sigma_1 \rangle = 0$, so that $\langle \Sigma_2, \Sigma_1 \rangle \cong \mathbb{Z}^2$.

In general, suppose knots $K_1, \ldots, K_{N-1}$ have been chosen so that

$$\langle \Sigma_1, \Sigma_2, \ldots, \Sigma_{N-1} \rangle \cong \mathbb{Z}^{N-1}.$$

Proceeding as before, choose $r_N, s_N$ so that

$$\rho(r_N, s_N) < \min \left\{ \tau(Y), \tau(\pm \Sigma_1), \ldots, \tau(\pm \Sigma_{N-1}), \frac{1}{r_N}, \frac{1}{s_N}, \frac{1}{prs - q} \right\}.$$ 

For $K_N = T_{r_N, s_N}$ and $\Sigma_N = \Sigma(P(K_N))$, Corollary 12 together with Equation (5) shows that $\langle \Sigma_1, \Sigma_2, \ldots, \Sigma_N \rangle \cong \mathbb{Z}^N$.

This recursive procedure defines a family $\{K_i\}_{i=1}^{\infty}$. Any finite subset of $\{P(K_i)\}_{i=1}^{\infty}$ is of full rank in $\mathcal{C}$, since we have shown its image under the homomorphism.
\[ \Sigma : \mathcal{C} \to \Theta_{\mathbb{Z}/2} \] has full rank. Since the finite subset can be chosen arbitrarily, \( \{ P(K_i) \}_{i=1}^\infty \) is an independent family in \( \mathcal{C} \).

Having treated the case when \( lk_{\Sigma}(J,J') \) is negative, we notice that if this quantity is positive for a pattern \( P \) then its mirror \( m(P) \) has rational linking number negative. Thus, if \( \{ m(P)(K_i) \}_{i=1}^\infty \) is an independent family, then \( \{ P(m(K_i)) \}_{i=1}^\infty \) is independent as well.

5. Examples

The hypothesis required of a winding number zero pattern by our theorem is rather mild, and easily verified in terms of linear algebra as we now explain. Given a pattern knot \( P \subset S^1 \times D^2 \) with winding number zero, we can find a Seifert surface \( S \) for \( P \) contained within the solid torus. Considering this surface inside \( S^3 \) via an unknotted embedding of \( S^1 \times D^2 \), we have its Seifert form \( S \), given in terms of a basis \( \{ a_i, b_i \}_{i=1}^g \) for \( H_1(F;\mathbb{Z}) \cong \mathbb{Z}^{2g} \). There is an Alexander dual basis \( a_i^*, b_i^* \) for \( H_1(S^3 \setminus F;\mathbb{Z}) \cong \mathbb{Z}^{2g} \), concretely given as follows: for a curve \( e \subset F \subset S^3 \) in the basis, its dual \( e^* \subset S^3 \setminus F \) is a curve linking \( e \) exactly once, and linking no other basis curve. To check whether the pattern knot satisfies our hypothesis, one expresses \( \partial D^2 \) in terms of the Alexander dual basis, by a vector we denote \( lk(\partial D^2) \).

The quantity
\[
\ell = -lk(\partial D^2)(S + S^T)^{-1}lk(\partial D^2)^T
\]
is identified with the self-linking number of a framed lift of \( \partial D^2 \) to the branched double cover. For a generic winding zero pattern it will be non-zero, and our theorem will show that the corresponding operator on concordance has infinite rank.

This calculation of the linking number also yields a concrete method for producing infinite rank winding zero satellite operators. To do this, consider any embedding of a surface \( \iota : F \hookrightarrow S^3 \) with a single boundary component. The boundary \( \iota(\partial F) \) will be a (possibly trivial) knot, \( P \). Now consider any unknotted curve \( \gamma \subset S^3 \setminus \iota(F) \). Then the complement of \( \gamma \) is homeomorphic to a solid torus, with \( P \) embedded therein with winding number zero (since \( \iota(F) \) is a Seifert surface for \( P \) in the complement). One can express \( \gamma \), as above, as a vector in terms of the Alexander dual basis for \( H_1(S^3 \setminus \iota(F);\mathbb{Z}) \cong \mathbb{Z}^{2g} \), and compute the linking number of its framed lift to the branched double cover of \( P \) by the same formula. This provides a far-reaching method for producing satellite operators with infinite rank. We can also easily specify embeddings of surfaces whose Seifert forms have trivial Alexander polynomial (for instance, by taking higher genus Seifert surfaces for an unknot, as in the case of the Whitehead doubling operator). Finding appropriate \( \gamma \) in these cases will produce satellite operators with infinite rank on smooth concordance, but which represent the zero map on topological concordance.

To illustrate this last point, let \( F \) have genus one and let \( \{ a, b \} \) be the basis for \( H_1(F;\mathbb{Z}) \) consisting of the cores of the 1-handles in a handle decomposition for \( F \). Imposing the condition that the Alexander polynomial of \( \partial F \) be trivial implies the Seifert form for \( F \) has matrix (in terms of the given basis)
\[
S = \begin{bmatrix} n & m \\ m-1 & l \end{bmatrix}, \text{ where } nl = m(m-1).
\]
Then
\[ lk_\Sigma(J, J') = (x, y) \left[ \begin{array}{cc} 2l & 1 - 2m \\ 1 - 2m & 2n \end{array} \right] \left( \begin{array}{c} x \\ y \end{array} \right), \]
where \( x = lk(\gamma, a) \) and \( y = lk(\gamma, b) \). It follows that \( lk_\Sigma(J, J') \neq 0 \) as long as \( (x, y) \) is not an integral multiple of the elements
\[
\begin{cases}
(1, 0) & \text{if } l = 0 \\
(0, 1) & \text{if } n = 0 \\
\left( \frac{m}{\gcd(m, l)}, \frac{l}{\gcd(m, l)} \right) & \text{or } \left( \frac{m-1}{\gcd(m-1, l)}, \frac{l}{\gcd(m-1, l)} \right) \text{ if } n, l \neq 0
\end{cases}
\]

Figure 1 gives a simple example of the previous discussion.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{A genus 1 example acting as zero on topological concordance.}
\end{figure}

5.1. **Twist knots (including Whitehead double).** Let \( P_k \) be the twist knot with \( k \) full twists and consider the curve \( \gamma \) shown in Figure 2. Viewing \( P_k \) as a knot in the solid torus \( S^3 \setminus N(\gamma) \) defines a pattern for a satellite operator. Notice that \( P_0 \cup \gamma \) defines the pattern for the positive untwisted Whitehead double. The Seifert surface \( F_k \) specified by Figure 2 has Seifert matrix
\[
S_k = \left[ \begin{array}{cc} k & 0 \\ -1 & -1 \end{array} \right].
\]
The algorithm described in [AK80] to produce a surgery description for \( \Sigma(P_k(U)) \) yields the framed link \( L \) shown to the right of Figure 2, and it follows from the algorithm itself that the linking vector of \( J_1 \) (a lift of \( \gamma \)) with \( L \) is precisely the linking vector of \( \gamma \) with the cores of the 1-handles of \( F_k \). Thus \( \vec{lk}(J_1, L) = (1, 0) \) and \( l = lk_\Sigma(J_1, J'_1) = -\frac{2}{4k-1} \neq 0 \) so that \( P_k \cup \gamma \) is of infinite rank for all \( k \).

The diagram in the left of Figure 3 gives a description for \( S^3 \setminus N(T_{2,3}) \) as surgery along a curve in an unknotted solid torus. The diagram in the right of Figure 3 is the Kirby diagram for \( \Sigma(P_k(T_{2,3})) \) obtained from a surgery presentation for \( T_{2,3} \) and a surgery presentation for \( \Sigma(P_k(U)) \).

5.2. **Iterated Doubles.** Let \( D^r = D \circ D \circ \ldots \circ D \) be the pattern for the \( r \)-th iterated positive untwisted Whitehead double, and let \( D^r \cup \gamma \) be the 2-component link which defines the relevant embedding of the unknot in the solid torus. The Seifert surface in Figure 4 shows that \( D^r \cup \gamma \) is a boundary link and therefore \( l = 0 \). Thus, our main theorem fails to apply. Results of the second author [PCss], however, show that the operator \( D^r : \mathcal{C} \rightarrow \mathcal{C} \) nonetheless has infinite rank.
A Seifert surface for $P_k$ in $S^3 \setminus N(\gamma)$.
Shown is the case $k = 2$.

Kirby diagram for $\Sigma(P_k)$ obtained from the Seifert surface for $P_k$.

Figure 2. The link $J \sqcup \tau_4$.

Surgery description for the right handed trefoil, and Kirby diagram for $\Sigma(P_k(T_{2,3}))$.

Figure 3. Surgery description for the right handed trefoil, and Kirby diagram for $\Sigma(P_k(T_{2,3}))$.

Figure 4. The link $(D \circ D) \sqcup \gamma$ and a Seifert surface for $D(D(U))$ in $S^3 \setminus N(\gamma)$.

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Department of Mathematics, Michigan State University, East Lansing, MI 48823
E-mail address: mhedden@math.msu.edu

Department of Mathematics, North Carolina State University, Raleigh, NC 20608
E-mail address: jpinzon@ncsu.edu