Dynamical symmetries of semi-linear Schrödinger and diffusion equations

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Abstract

Conditional and Lie symmetries of semi-linear 1D Schrödinger and diffusion equations are studied if the mass (or the diffusion constant) is considered as an additional variable. In this way, dynamical symmetries of semi-linear Schrödinger equations become related to the parabolic and almost-parabolic subalgebras of a three-dimensional conformal Lie algebra $(\text{conf}_3)_\mathbb{C}$. We consider non-hermitian representations and also include a dimensionful coupling constant of the non-linearity. The corresponding representations of the parabolic and almost-parabolic subalgebras of $(\text{conf}_3)_\mathbb{C}$ are classified and the complete list of conditionally invariant semi-linear Schrödinger equations is obtained. Possible applications to the dynamical scaling behaviour of phase-ordering kinetics are discussed.

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1 Introduction

Symmetries have since a long time played a major rôle in physics. For example, starting with Lie in 1881, the maximal kinematic invariance group of the free diffusion equation, where $\mathcal{M}$ is a constant

$$2\mathcal{M}\partial_t\Phi - \partial_r^2\Phi = 0$$

has been studied, which is the so-called Schrödinger group $\text{Sch}(d)$ \[60] which also arises as dynamical symmetry of the non-relativistic free particle \[53, 40, 4\]. We shall denote its Lie algebra by $\mathfrak{sch}_d$. For $d = 1$ space dimensions, $\mathfrak{sch}_1$ is spanned by the generators

$$
\begin{align*}
X_{-1} &= -\partial_t, \\
Y_{-\frac{1}{2}} &= -\partial_r \\
X_0 &= -t\partial_t - \frac{1}{2} r\partial_r - \frac{x}{2} \\
X_1 &= -t^2\partial_t - tr\partial_r - \frac{M}{2} r^2 - xt \\
M_0 &= -\mathcal{M}
\end{align*}
$$

which allows to write the non-vanishing commutators compactly as $[X_n, X_{n'}] = (n-n')X_{n+n'}, [X_n, Y_m] = (n/2-m)Y_{n+m}, [Y_{\frac{n}{2}}, Y_{-\frac{m}{2}}] = M_0$ where $n, n' \in \{\pm 1, 0\}$ and $m \in \{\pm \frac{1}{2}\}$ (see \[14\] for generalizations to $d > 1$). The free Schrödinger equation is recovered from the analytical continuation $\mathcal{M} = i\mathcal{M}$. It is a well-known fact that the same group also acts as kinematic invariance group of certain non-linear Schrödinger equations of the form

$$2mi\partial_t\Phi - \partial_r^2\Phi = F(t, r, \Phi, \Phi^*)$$

If the potential is chosen in the form $F = c(\Phi\Phi^*)^{2/d} \Phi$, where $c$ is a constant, then eq. (1.3) is invariant under $\mathfrak{sch}_d$ provided the scaling dimension of $\Phi$ is taken to be $x = d/2$, see e.g. \[10, 33, 66\]. Nonlinear Schrödinger equations arise in many physical applications, for recent reviews see \[3, 69\]. We also mention the recently established Schrödinger-invariance of non-relativistic fluid dynamics \[41, 61\]. Mathematically, there has been a lot of effort to establish the existence of solutions with certain regularity properties, see e.g. \[9\], and on the other hand the group classification of non-linear Schrödinger equations has been intensively studied, see \[10, 66, 33, 34, 39, 15, 16, 65, 17\] and references therein. Spectral properties are studied in \[6\]. A great deal is known about the representations of $\mathfrak{sch}_d$ \[62, 59, 26, 24, 30\] and this can be applied to find symbolic solutions of non-linear Schrödinger equations.$^2$

Very similar equations have been studied in attempts to understand the coarsening process which systems undergo after having been quenched from a disordered initial state to below their critical point, see e.g. \[11, 8, 12, 21, 37, 19, 48, 50\] for reviews. In its most simple setting, one considers a ferromagnetic system (e.g. described by an Ising model) which from some initial high-temperature state is rapidly quenched into its ordered phase below its critical temperature $T_c > 0$ where there are at least two competing equilibrium states. Although the system relaxes locally towards one of the equilibrium states, a global relaxation is not possible and this leads to a very slow evolution of the macroscopic observables. From a microscopic point of view, the system’s evolution is characterized by the formation of correlated domains of time-dependent linear size $L(t)$ and one typically finds a power-law behaviour $L(t) \sim t^{1/z}$ where $z$ is called the dynamical exponent. This in turn signals a dynamical scale-invariance).

$^2$Related questions include the dynamical symmetries in de Sitter space \[23, 62\], $q$-deformations \[31\], Chern-Simons theory \[20, 22\], difference equations \[25, 56\] or even the integrable quantum non-linear Schrödinger equation, see e.g. \[13\].
in the system’s evolution. A coarse-grained description is usually given in terms of the order parameter \( \Phi = \Phi(t, r) \), which in the absence of any macroscopic conservation law is assumed to satisfy \[ \frac{\partial \Phi}{\partial t} = \Gamma \nabla^2 \Phi - \frac{dV(\Phi)}{d\Phi} \] \hspace{1cm} (1.4)

where \( \Gamma \) is a kinetic coefficient and \( V(\Phi) \) is the potential which enters into the Ginzburg-Landau free-energy functional and which is assumed to have a double-well structure, i.e. \( V(\Phi) = (1 - \Phi^2)^2 \). For simplicity, we dropped here the thermal noise which is known to be irrelevant for the long-time behaviour we are interested in. The disordered initial state enters through ‘white-noise’ initial conditions and one looks for the long-time behaviour of the solutions of (1.4).

It has turned out to be convenient to characterize this evolution in terms of the two-time autocorrelation \( C(t, s) \) and the associated two-time autoresponse \( R(t, s) \) defined as

\[
C(t, s) := \langle \Phi(t)\Phi(s) \rangle, \quad R(t, s) := \left. \frac{\delta \langle \Phi(t) \rangle}{\delta h(s)} \right|_{h=0}
\] \hspace{1cm} (1.5)

where \( h(s) \) is the time-dependent magnetic field conjugate to \( \Phi \) and \( \langle \cdot \rangle \) denotes the average over the fluctuations in the initial state (and thermal histories). Here, \( t \) is referred to as observation time and \( s \) as waiting time. By definition, the system undergoes ageing if \( C \) or \( R \) depend on both \( t \) and \( s \) and not merely on the difference \( \tau = t - s \). It is well-accepted (although still unproven) that in the ageing regime, that is for times \( t, s \gg t_{\text{micro}} \) and \( t - s \gg t_{\text{micro}} \), where \( t_{\text{micro}} \) is some microscopic time scale, dynamical scaling holds true such that

\[
C(t, s) = s^{-b} f_C(t/s), \quad R(t, s) = s^{-1-a} f_R(t/s)
\] \hspace{1cm} (1.6)

and one would like to be able to compute the scaling functions \( f_{C,R}(y) \). The free diffusion equation is the simplest example of a system undergoing ageing. Motivated by a formal analogy with the well-known conformal invariance in equilibrium critical phenomena, it has been suggested that the Schrödinger group might play a similar role in certain ageing systems. If this working hypothesis is made, explicit forms for the scaling function \( f_R(y) \) and more recently also for \( f_C(y) \) can be derived. These agree with the exact results in several soluble models, such as the spherical model, the 1D Glauber-Ising model, the critical voter model, the bosonic versions of the contact and the pair-contact processes, and the free random walk. Furthermore, these forms also agree with the results of numerical simulations in the 2D and 3D kinetic Ising and XY models and with recent results in the 2D three-states Potts model.

In order to understand whether these observations may be viewed as manifestations of a dynamical symmetry we remark the following:

1. In ageing phenomena, time-translation invariance is broken. Therefore, one should a priori not expect full Schrödinger-invariance but at best invariance under some sub-algebra of \( \mathfrak{sch}_1 \), which does not contain time-translations. For a classification of the Lie subalgebras of \( \mathfrak{sch}_1 \) see [10].

2. If one considers equations as the classical equations of motion of a field-theory, sufficient conditions for the validity of Schrödinger-invariance can be formulated. If that field-theory is local and taking the analogous case of conformal invariance as a guide, one may show from a

\[ ^3 \text{Here we only consider the phase-ordering kinetics of ferromagnetic systems without any macroscopic conservation laws and quenched to below their critical point. Then } z = 2 \text{ has been derived and a necessary condition for the applicability of Schrödinger-invariance is satisfied.} \]
consideration of the energy-momentum tensor that the special Schrödinger-invariance (generated by $X_1$) follows provided that dynamical scaling and in addition Galilei-invariance are satisfied [15].

3. If the deterministic part of the field-theory is Galilei-invariant, then both $C(t, s)$ and $R(t, s)$ in the presence of noise can be expressed in terms of quantities calculable from the noiseless part [61]. Hence it is enough to study the symmetries of the deterministic part, which reduces the problem to understanding the symmetries, especially the Galilei-invariance, of a non-linear partial differential equation, such as (1.3).

At first sight, the problem of finding the Galilei-invariant semi-linear Schrödinger equations (1.3) seems to have been answered long ago and only allows the specific potential $F$ quoted above [33]. Furthermore, complex-valued solutions $\phi$ of eq. (1.3) are required, which apparently excludes applications to kinetic equations such as (1.4) with real-valued solutions. In this paper, we shall propose a way around these difficulties.

We begin by recalling that the Schrödinger algebra $\mathfrak{sch}_d$ in $d$ spatial dimensions is a subalgebra of the complexified conformal algebra $(\mathfrak{conf}_{d+2})^C$ in $d + 2$ dimensions [13, 45]. A simple way of seeing this is to treat the ‘mass’ $M = i m$ as a further dynamical variable and to introduce a new wave function [38, 45]

$$\Phi = \Phi_m(t, r) = \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} d\zeta e^{-im\zeta} \Psi(\zeta, t, r)$$  \hspace{1cm} (1.7)

For notational simplicity, we restrict from now on to the case $d = 1$. Then the generators (1.2) become

$$X_{-1} = -\partial_t, \quad Y_{-\frac{1}{2}} = -\partial_r,$$

$$Y_{\frac{1}{2}} = -t\partial_r - r\partial_\zeta,$$

$$X_0 = -t\partial_t - \frac{1}{2}r\partial_r - \frac{x}{2},$$

$$X_1 = -t^2\partial_t - tr\partial_r - \frac{1}{2}r^2\partial_\zeta - xt,$$

$$M_0 = -\partial_\zeta$$  \hspace{1cm} (1.8)

The free Schrödinger equation (1.1) then becomes $(2\partial_\zeta\partial_t - \partial_\zeta^2)\Psi = 0$ which through a further change of variables can be brought into a massless Klein-Gordon/Laplace equation in three dimensions which has the simple Lie algebra $(\mathfrak{conf}_3)^C \cong \mathfrak{so}(5, \mathbb{C}) \cong B_2$ as dynamical symmetry. It is useful to illustrate this in terms of a root diagram, see figure 1b, from which the correspondence between the roots and the generators of $\mathfrak{sch}_1$ can be read off. Four additional generators should be added in order to get the full conformal algebra $(\mathfrak{conf}_3)^C$ which we take in the form [45]

$$N = -t\partial_t + \zeta\partial_\zeta,$$

$$V_- = -\zeta\partial_\zeta - r\partial_t,$$

$$W = -\zeta^2\partial_\zeta - \zeta r\partial_r - \frac{1}{2}r^2\partial_t - x\zeta,$$

$$V_+ = -2tr\partial_t - 2\zeta r\partial_\zeta - (r^2 + 2\zeta t)\partial_r - 2xr.$$  \hspace{1cm} (1.9)

In applications to ageing phenomena, it turned out that the parabolic subalgebras of $\mathfrak{conf}_3$ play an important rôle.\footnote{The minimal standard parabolic subalgebra $\mathfrak{s}_0$ of a simple complex Lie algebra $\mathfrak{g}$ is spanned by the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and the set of positive roots. A standard parabolic subalgebra $\mathfrak{s} \subset \mathfrak{g}$ is a subalgebra of $\mathfrak{g}$ which contains $\mathfrak{s}_0$ [54].} The complete list of non-isomorphic parabolic subalgebras of $\mathfrak{conf}_3$ is as follows [45].
Figure 1: (a) Root diagram of the complex Lie algebra $B_2$ and the identification of the generators \([1.8,1.9]\) of the complexified conformal Lie algebra \((\text{conf}_3)_C \supset (\text{sch}_1)_C\). The double circle in the center denotes the Cartan subalgebra. The generators belonging to the three non-isomorphic parabolic subalgebras \([45]\) are indicated by the full points, namely (b) $\tilde{\text{sch}}_1$, (c) $\tilde{\text{age}}_1$ and (d) $\tilde{\text{alt}}_1$.

1. $\tilde{\text{sch}}_1$, spanned by the set \(\{X_{-1,0,1}, Y_{-\frac{1}{2}, \frac{1}{2}}, M_0, N\}\).

2. $\tilde{\text{age}}_1$, spanned by the set \(\{X_{0,1}, Y_{-\frac{1}{2}, \frac{1}{2}}, M_0, N\}\).

3. $\tilde{\text{alt}}_1$, spanned by the set \(\{D, X_1, Y_{-\frac{1}{2}, \frac{1}{2}}, M_0, N, V_+\}\).

Here we used the generator $D$ of the full dilatations

\[
D := 2X_0 - N = -t\partial_t - r\partial_r - \zeta\partial_\zeta - x
\]  

In figure 1bcd, we illustrate the definition of these three parabolic subalgebras through their root diagrams.

One can also consider the corresponding subalgebras without the generator $N$, namely

1. $\text{sch}_1$, spanned by the set \(\{X_{-1,0,1}, Y_{-\frac{1}{2}, \frac{1}{2}}, M_0\}\).

2. $\text{age}_1$, spanned by the set \(\{X_{0,1}, Y_{-\frac{1}{2}, \frac{1}{2}}, M_0\}\).

3. $\text{alt}_1$, spanned by the set \(\{D, X_1, Y_{-\frac{1}{2}, \frac{1}{2}}, M_0, V_+\}\).

We call *almost-parabolic subalgebras* those subalgebras of a parabolic subalgebra $\mathfrak{s}$ which are obtained by leaving out one of the generators of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{s}$. Therefore, $\text{sch}_1$, $\text{age}_1$ and $\text{alt}_1$ are almost parabolic, because we merely have to add the generator $N \in \mathfrak{h}$ in order to make them parabolic subalgebras. In view of possible applications to ageing phenomena, we shall be mainly interested in the algebras$^5$ $\text{age}_1$ and $\text{alt}_1$, as well as $\tilde{\text{age}}_1$ and $\tilde{\text{alt}}_1$ because they do not contain the time-translation generator $X_{-1}$.

After these preparations, we can now formulate the questions studied in this paper. For notational simplicity, we restrict ourselves to $d = 1$ space dimension and look for a semilinear and non-derivative extension of the free “Schrödinger equation” of the form

\[
\hat{S}\Psi := \left(2\partial_\zeta\partial_t - \partial_r^2\right)\Psi = F(\zeta, t, r, \Psi, \Psi^*)
\]

which are invariant under one of the subalgebras of \((\text{conf}_3)_C\) as sketched in figure 1bcd. We shall initially take $\Psi$ to be a complex-valued function as is appropriate for the physical context of eq. (1.3) and shall

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$^5$The name of $\text{age}_1$ comes of course from *ageing*, while $\text{alt}_1$ is inspired by its German equivalent, *altern*.
return later to the question whether non-trivial symmetries involving only real-valued functions $\Psi$ are possible, as would appear more natural for equation (1.4). Our main tool will be the construction of new differential-operator representations of the algebras defined above. In section 2, we shall relax the condition of having skew-hermitian representations of the Schrödinger group and of the other algebras of figure 1. More general representations will be constructed and we shall then find the most general semi-linear equation of the form (1.11) invariant under these. On the other hand, non-linear equations of the forms (1.3, 1.4, 1.11) imply the existence of a coupling constant with the nonlinearities, which in existing studies is tacitly admitted to be dimensionless. This hidden assumption leads to rather restricted form of admissible potentials. Hence for a given form of the potential, Schrödinger-invariance should only hold for one special value of the spatial dimensionality $d$, which is in disagreement with the existing numerical and analytical results on ageing quoted above. We shall inquire into the consequences of treating dimensionful coupling constants, which consequently will transform under scaling and special transformations. In section 3, we construct the representations with a dimensionful coupling constant and in section 4 we find the nonlinear Schrödinger equations invariant under these new representations. Section 5 presents our conclusions.

2 Non skew-hermitian representations

2.1 General remarks

We recall first the basic method used in finding non-linear equations with a given symmetry group, as outlined e.g. in [10]. Consider first the linear equation

$$\hat{S}\Phi(t,r) = 0 \quad (2.1)$$

and let $G$ be an one-dimensional Lie group which acts on the equation (2.1) such that solutions are transformed into solutions. In this paper we are going to consider a projective action of the elements $g \in G$ on the solutions $\Phi(t,r)$ such that

$$[T(g)\Phi](t,r) = \mu_g(t,r)\Phi(t',r'). \quad (2.2)$$

where $(t', r') = g(t, r)$. For $G$ to be symmetry group one requires

$$\hat{S}[T(g)\Phi](r,t) = 0 \quad (2.3)$$

for all $\Phi$ satisfying (2.1). It is convenient to consider instead of $G$ its Lie algebra $\mathfrak{g}$ and to expand

$$T(g) = 1 + \varepsilon X + \ldots$$

$$X\Phi = (a(t, r)\partial_t + b(t, r)\partial_r + c(t, r))\Phi \quad (2.4)$$

Now, equation (2.3) implies the operator equation

$$[\hat{S}, X] = \lambda(t, r)\hat{S} \quad (2.5)$$

In the same way, for the nonlinear equation

$$\hat{S}\Phi(t,r) = F(t,r,\Phi,\Phi^*) \quad (2.6)$$
one arrives at the following conditions
\begin{align}
\hat{S}[T(g)\Phi](t,r) &= F(t,r, [T(g)\Phi](t,r), [T(g)\Phi]^*(t,r)) \\
\hat{S}(X\Phi) &= (X\Phi)\partial_\Phi F + (X\Phi)^*\partial_{\Phi^*} F
\end{align} \tag{2.7}
Combining (2.5,2.7) with the explicit form (2.4) of \(X\) gives \[10\]
\begin{align}
[a(t,r)\partial_t + b(t,r)\partial_r - c(t,r)\Phi\partial_\Phi - c^*(t,r)\Phi^*\partial_{\Phi^*} + (c(t,r) + \lambda(t,r))] F(t,r,\Phi,\Phi^*) &= 0 \tag{2.8}
\end{align}
This linear equation can be solved with standard techniques. For Lie algebras with \(\dim g > 1\), one has one such equation for each independent generator. In the same way, further independent variables can be included straightforwardly. Evidently, the symmetry algebra of the quasi-linear equations considered here must be a subalgebra of the symmetry algebra of the linear part.

Throughout, we shall work with the linear Schrödinger operator
\[
\hat{S} := 2M_0X_{-1} - Y_{-\frac{1}{2}} \tag{2.9}
\]
which satisfies
\begin{align}
[\hat{S},M_0] &= [\hat{S},Y_{-\frac{1}{2}}] = [\hat{S},Y_{\frac{1}{2}}] = [\hat{S},X_{-1}] = [\hat{S},N] = 0 \\
[\hat{S},X_0] &= -\hat{S}, \quad [\hat{S},X_1] = -2t\hat{S} - 2 \left( x - \frac{1}{2} \right) M_0 \\
[\hat{S},V_+] &= -4r\hat{S} - 4 \left( x - \frac{1}{2} \right) Y_{-\frac{1}{2}}
\end{align} \tag{2.10}
from which we can read off the eigenvalues \(\lambda(t,r)\). We also see that invariance under the algebras \(\mathfrak{sch}_1\), \(\mathfrak{agc}_1\) or \(\mathfrak{alt}_1\) (or the parabolic extensions) holds on the space of solutions of \(\hat{S}\Phi = 0\), provided \(x = 1/2\).

### 2.2 Representation with a generalized scaling behaviour

For the free Schrödinger equation \(\hat{S}\Phi = 0\), invariance under \(\mathfrak{sch}_1\) implies that the scaling dimension of the solution \(\Phi\) is \(x = \frac{1}{2}\). In order to relax this constraint while conserving the symmetry, we follow \[64\] and modify the generator \(X_1\) in \[1.8\] by adding a term \(-kt\), where \(k\) is some constant. Then we should further add a further generator \(Z := -kZ_0\), where \(Z_0\) is central. From now on, we shall work with a variable ‘mass’ and go over to the conjugate variable \(\zeta\). We require that the commutators which are not in the centre are unchanged and find that with respect to eqs. \[1.8,1.9\] only the following generators are modified
\begin{align}
X_1 &= -t^2\partial_t - tr\partial_r - \frac{1}{2}r^2\partial_\zeta - (x + k)t \\
V_+ &= -2tr\partial_t - 2\zeta r\partial_\zeta - (r^2 + 2\zeta t)\partial_r - 2(x + k)r \\
Z &= -kZ_0
\end{align} \tag{2.11}
The only commutators which are modified are the central ones
\begin{align}
[X_1,X_{-1}] &= 2X_0 + kZ_0 \\
[V_+,Y_{-\frac{1}{2}}] &= 4X_0 - 2N + 2kZ_0
\end{align} \tag{2.12}
and we see that $Z_0$ can be absorbed into a redefinition of the generators $X_0$ or $D$, as expected on general grounds [52]. However, the presence of the parameter $k$ is important for the determination of invariant non-linear equations, as we shall see now.

Consider the non-linear Schrödinger equation (1.11)

$$ (2\partial_\zeta \partial_t - \partial^2_\zeta)\Psi(\zeta, t, r) = \tilde{F}(\zeta, t, r, \Psi, \Psi^*) $$

(2.13)

Eq. (2.8) then gives the following conditions on $\tilde{F}$ for each of the generators:

\[ Y_{-\frac{1}{2}} : \partial_r \tilde{F} = 0 \] (2.14)
\[ X_{-1} : \partial_t \tilde{F} = 0 \] (2.15)
\[ M_0 : \partial_\zeta \tilde{F} = 0 \] (2.16)
\[ Y_{\frac{1}{2}} : (t\partial_r + r\partial_\zeta) \tilde{F} = 0 \] (2.17)
\[ X_0 : 
\left[ \frac{t}{2} \partial_\zeta + \frac{1}{2} r \partial_r - \frac{x}{2} (\Psi \partial_\Psi + \Psi^* \partial_{\Psi^*}) + \left( \frac{x}{2} + 1 \right) \right] \tilde{F} = 0 \] (2.18)
\[ D : [t\partial_t + r\partial_r + \zeta \partial_\zeta - x(\Psi \partial_\Psi + \Psi^* \partial_{\Psi^*}) + (x + 2)] \tilde{F} = 0 \] (2.19)
\[ N : (t\partial_t - \zeta \partial_\zeta) \tilde{F} = 0 \] (2.20)
\[ X_1 : 
\left[ \frac{1}{2} \partial_\zeta + tr\partial_r + \frac{r^2}{2} \partial_\zeta - (x + k)t(\Psi \partial_\Psi + \Psi^* \partial_{\Psi^*}) + (x + k + 2)t \right] \tilde{F} = 0 \] (2.21)
\[ V_+ : [2tr\partial_t + (2t\zeta + r^2)\partial_r + 2r\zeta \partial_\zeta - 2r(x + k)(\Psi \partial_\Psi + \Psi^* \partial_{\Psi^*}) + 2r(x + k + 2)] \tilde{F} = 0 \] (2.22)

We point out that already the invariance of the linear Schrödinger equation under $X_1$ requires $x + k = \frac{1}{2}$. Consequently, the scaling dimension of $\Psi$ which has to be fixed for an invariance is given by $x + k$ and not by $x$ alone. This is a necessary condition for the invariance of the nonlinear equation.

Our results are as follows.

1. The algebra $\mathfrak{sch}_1$. We have to solve the system eqs. \{2.14, 2.15, 2.16, 2.17, 2.18, 2.21\}. A solution only exists in the well-known case $k = 0$, hence $x = \frac{1}{2}$. This in indeed quite analogous to the equation (1.3) with a fixed mass where the same power-law for the potential was found, see e.g. \{10, 4, 33, 66\}. One has

\[ \tilde{F}_{\mathfrak{sch}_1} = \tilde{F}_{\mathfrak{sch}_1}(\Psi, \Psi^*) = \Psi^5 f \left( \frac{\Psi}{\Psi^*} \right). \] (2.23)

where $f$ denotes an arbitray differentiable function. Furthermore, adding the generator $N$ by taking eq. (2.20) into account does not give anything new. Hence the result for the parabolic subalgebra $\mathfrak{sch}_1$ is the same as before, viz. $\tilde{F}_{\mathfrak{sch}_1} = \tilde{F}_{\mathfrak{sch}_1}$.\footnote{In view of the embedding $\mathfrak{sch}_d \subset (\text{conf}_{d+2})_c$ one might also consider the semi-linear conformally invariant Klein-Gordon equation $\Box \phi = F(\phi) = \phi^{(n+2)/(n-2)}$ in $n = d + 2$ dimensions, e.g. \{33, 68\}, which reproduces the form $F \sim \phi^5$ for the case $d = 1$.}

In consequence, no new form of an invariant nonlinear equation is found.

2. The algebra $\mathfrak{apc}_1$. We now have to solve the systems eqs. \{2.14, 2.16, 2.17, 2.18, 2.21\}. There is no further constraint on $k$ and the solution is, using again $x + k = 1/2$

\[ \tilde{F}_{\mathfrak{apc}_1} = \tilde{F}_{\mathfrak{apc}_1}(\Psi, \Psi^*; t) = t^{-4k/(2k+1)} \Psi^{(2k+5)/(2k+1)} f \left( \frac{\Psi}{\Psi^*} \right). \] (2.24)
and we see that there also appears an explicit dependence on time (only the special case \( k = 0 \) was considered in \([10]\)). On the other hand, if we go to the parabolic subalgebra \( \tilde{\mathfrak{g}}_1 \) by adding the generator \( N \) we must have \( k = 0 \) and we find \( \tilde{F}_{\tilde{\mathfrak{g}}_1} = \tilde{F}_{\text{sch}} \), that is the same result as in the Schrödinger case.

3. The algebra \( \mathfrak{alt}_1 \). Here we must solve the equations (2.14, 2.16, 2.17, 2.19, 2.21, 2.22). There are two distinct solutions. The first one exists for all values of \( k \) and reads

\[
\tilde{F}^{(1)}_{\mathfrak{alt}_1} = \tilde{F}^{(1)}_{\mathfrak{alt}_1} (\Psi, \Psi^*; t) = t^{-2} \Psi f \left( \frac{\Psi}{\Psi^*} \right). \tag{2.25}
\]

However, for \( k = 0 \) there is a second solution

\[
\tilde{F}^{(2)}_{\mathfrak{alt}_1} = \tilde{F}^{(2)}_{\mathfrak{alt}_1} (\Psi, \Psi^*; t) = \tilde{F}_{\text{sch}} (\Psi, \Psi^*) \tag{2.26}
\]

Finally, if we go over to \( \tilde{\mathfrak{alt}}_1 \) by adding the generator \( N \), a solution exists only for \( k = 0 \), hence \( \tilde{F}_{\tilde{\mathfrak{alt}}_1} = \tilde{F}_{\text{sch}} \).

A few observations are in order. First, we see that we can trade in some cases the dependence of the potential \( \tilde{F} \) on \( \Psi \) in favour of an explicit dependence on \( t \) such that the total dimension of \( \tilde{F} \) is unchanged. Second, we find that this freedom does not exist for the parabolic subalgebras but only for the almost-parabolic subalgebras \( \mathfrak{g} \mathfrak{c} \mathfrak{t}_1 \) and \( \mathfrak{alt}_1 \) which do not contain the time-translations. Third, and in contrast to the well-known results where the ‘masses’ \( m \) (or \( \mathcal{M} \)) are kept fixed \([10, 33, 66]\), the dependence on the phase \( \Psi/\Psi^* \) is not determined in these representations. In particular, if we take \( f = \text{cste.} \), we even have the possibility to consider a Schrödinger equation with a real solution \( \Psi \) and a non-trivial symmetry, which is impossible if the masses are kept fixed.

While we are not aware of immediate physical applications of the equations found here, these examples suggest that it may be useful to consider the presence of other dimensionful quantities, such as coupling constants, in the nonlinear equations. We shall take this up in the next section.

3 Representations with an additional dimensionful coupling

In our search for more general quasilinear Schrödinger equations with a non-trivial symmetry we now consider explicitly a dimensionful coupling constant \( g \) in the nonlinear term. In what follows, we do consider that \( g \) also changes under the local scale-transformations. Hence the dilatation generator is taken to be of the form

\[
X_0 = -t \partial_t - \frac{1}{2} r \partial_r - yg \partial_y - \frac{x}{2} \tag{3.1}
\]

where \( y \) is the scaling dimension of the coupling \( g \). In addition, we also assume throughout that the generator of space translations \( Y_{\frac{1}{2}} = -\partial_r \) is unchanged.

In this section we construct the remaining generators. Their explicit form will be used in section 4 to find the invariant semilinear equations.
3.1 Subalgebras $\mathfrak{age}_1$ and $\tilde{\mathfrak{age}}_1$

Starting from eqs. (1.8,1.9), we look for extensions of the usual generators and write

\[
\begin{align*}
M_0 &= -\partial_\zeta - L(t, r, \zeta, g)\partial_g \\
Y_{\frac{1}{2}} &= -t\partial_r - r\partial_\zeta - Q(t, r, \zeta, g)\partial_g \\
X_1 &= -t^2\partial_t - tr\partial_r - \frac{1}{2}r^2\partial_\zeta - P(t, r, \zeta, g)\partial_g - xt \\
N &= -t\partial_t + \zeta\partial_\zeta - K(t, r, \zeta, g)\partial_g
\end{align*}
\] (3.2)

The unknown functions $L, Q, P, K$ are determined from the following two requirements:

1. the commutators of the standard realizations of $\mathfrak{age}_1$ and $\tilde{\mathfrak{age}}_1$ are assumed to remain valid.

2. invariance of the linear Schrödinger equation under the new representations.

First, we consider the almost-parabolic subalgebra $\mathfrak{age}_1$. Imposing the commutator relations, our results are as follows. The commutators

\[
[X_0, X_1] = -X_1 \, , \, [X_0, Y_{\frac{1}{2}}] = -\frac{1}{2}Y_{\frac{1}{2}} \, , \, [X_0, M_0] = 0
\] (3.3)

give

\[
P = p_0(\zeta)t^{y+1}p(u, v) \, , \, Q = q_0(\zeta)t^yq(u, v) \, , \, L = l_0(\zeta)t^{y-1}l(u, v)
\] (3.4)

where $u = r^2/t$ and $v = t^y/g$. Next we use

\[
[X_1, Y_{-\frac{1}{2}}] = Y_{\frac{1}{2}} \, , \, [Y_{\frac{1}{2}}, Y_{-\frac{1}{2}}] = M_0 \, , \, [Y_{-\frac{1}{2}}, M_0] = 0 \, , \, [Y_{\frac{1}{2}}, M_0] = 0
\] (3.5)

and obtain

\[
q(u, v) = 2u^\frac{1}{2}\partial_u p(u, v) \, , \, l(u, v) = 2u^\frac{1}{2}\partial_u q(u, v) \, , \, \partial_u l(u, v) = 0 \, , \, p_0(\zeta) = q_0(\zeta) = l_0(\zeta)
\] (3.6)

Hence $l = l(v)$ and

\[
q(u, v) = u^{1/2}l(v) + n(v) \, , \, p(u, v) = \frac{u}{2}l(v) + u^{1/2}n(v) + m(v)
\] (3.7)

where $m(v), n(v)$ are two functions of $v$ to be determined. The last remaining commutators

\[
[X_1, Y_{\frac{1}{2}}] = 0 \, , \, [X_1, M_0] = 0
\] (3.8)

lead to the following system for the yet unknown functions $l(v), n(v), m(v), p_0(\zeta)$, where the prime denotes the derivative with respect to the argument

\[
p_0(\zeta)g(l(v) + vl'(v)) + p_0^2(\zeta)v^2(l(v)m'(v) - m(v)l'(v)) - p_0'(\zeta)m(v) = 0
\] (3.9)

\[
(2y - 1)n(v) + 2ynl'(v) + 2p_0(\zeta)v^2(n(v)m'(v) - m(v)n'(v)) = 0
\] (3.10)

\[
p_0'(\zeta)n(v) - p_0^2(\zeta)v^2(l(v)m'(v) - n(v)l'(v)) = 0
\] (3.11)

Performing a separation of the variables $v$ and $\zeta$ in eq. (3.11), we see that two cases must be distinguished, depending on whether $p_0'(\zeta)$ vanishes or not.
In this case, one has $p_0(\zeta) = \frac{2y}{l_0}$ where $l_0$ is a constant. We find the solutions $l(v) = l_0 v^{-1}$, $n(v) = n_0 v^{(1-2y)/(2y)}$ and $m(v) = cn(v)$ and set $h_0 := n_0/l_0$. Consequently

$$L = 0, \quad Q = 0, \quad P = p_0 t^y m(v)$$

and the function $m(v)$ remains arbitrary.

**Case b)** The MMG-representation with a modified mass generator $M_0$. In this case, we have $p_0(\zeta) = -\frac{2y}{l_0}$, where $l_0$ is a constant. We find the solutions $l(v) = l_0 v^{-1}$, $n(v) = n_0 v^{(1-2y)/(2y)}$ and $m(v) = cn(v)$ and set $h_0 := n_0/l_0$. Consequently

$$L = -\frac{2y}{\zeta} g, \quad Q = -\frac{2y}{\zeta} (rg + h_0 g^{(2y-1)/(2y)})$$

$$P = -\frac{2y}{\zeta} \left[ \frac{r^2 g}{2} + h_0 trg^{(2y-1)/(2y)} + ch_0 t^{3/2} g^{(2y-1)/(2y)} \right]$$

We have hence established the existence of two distinct representations of the algebra $\mathfrak{age}_1$. Explicitly, we always have $Y_{-1/2} = -\partial_r$ and further in the case a) of the NMG-representation

$$M_0 = -\partial_z$$

$$Y_{\frac{1}{2}} = -t \partial_r - r \partial_z$$

$$X_0 = -t \partial_t - \frac{1}{2} r \partial_r - yg \partial_y - \frac{x}{2}$$

$$X_1 = -t^2 \partial_t - tr \partial_r - \frac{1}{2} r^2 \partial_z - p_0 t^y m(t^y/g) \partial_y - xt$$

The NMG-representation of $\mathfrak{age}_1$ is parametrized by the three constants $x, y, p_0$ and an arbitrary function $m(v)$. In the case b) of the MMG-representation we have

$$M_0 = -\partial_z + \frac{2y}{\zeta} g \partial_y$$

$$Y_{\frac{1}{2}} = -t \partial_r - r \partial_z + \frac{2y}{\zeta} (rg + h_0 g^{(2y-1)/(2y)}) \partial_y$$

$$X_0 = -t \partial_t - \frac{1}{2} r \partial_r - yg \partial_y - \frac{x}{2}$$

$$X_1 = -t^2 \partial_t - tr \partial_r - \frac{1}{2} r^2 \partial_z$$

$$+ \frac{2y}{\zeta} \left( \frac{r^2 g}{2} + h_0 trg^{(2y-1)/(2y)} + ch_0 t^{3/2} g^{(2y-1)/(2y)} \right) \partial_y - xt$$

The MMG-representation of $\mathfrak{age}_1$ is parametrized by the four constants $x, y, h_0, c$.

We now look for the most general representation of the parabolic subalgebra $\mathfrak{agc}_1$. From the commutators

$$[X_0, N] = 0, \quad [Y_{-\frac{1}{2}}, N] = 0$$

we obtain

$$K = k_0(\zeta) t^y k(u, v), \quad \partial_u k(u, v) = 0$$

hence $k(u, v) = k(v)$. Next, from the commutators

$$[X_1, N] = X_1, \quad [Y_{\frac{1}{2}}, N] = Y_{\frac{1}{2}}, \quad [M_0, N] = M_0$$
we obtain that the functions \( l(v), n(v), m(v), k(v), p_0(\zeta) \) and \( k_0(\zeta) \) must satisfy the system:

\[
\begin{align*}
 y k_0(\zeta) (k(v) + v k'(v)) - y p_0(\zeta) (m(v) + v m'(v)) + \zeta p_0(\zeta) m(v) \\
 - k_0(\zeta) p_0(\zeta) v^2 (m(v) k'(v) - m'(v) k(v)) &= 0 \quad (3.19) \\
 2 y p_0(\zeta) (n(v) + v n'(v)) - p_0(\zeta) n(v) - 2 \zeta p_0(\zeta) n(v) \\
 + 2 k_0(\zeta) p_0(\zeta) v^2 (n(v) k'(v) - n'(v) k(v)) &= 0 \quad (3.20) \\
 y p_0(\zeta) \left( l(v) + v l'(v) \right) - (p_0(\zeta) + \zeta p_0(\zeta)) l(v) \\
 + k_0(\zeta) p_0(\zeta) v^2 l(v) (k'(v) - l'(v) k(v)) - k_0'(\zeta) k(v) &= 0 \quad (3.21)
\end{align*}
\]

Our results from above for the functions \( l(v), n(v), m(v) \) found for the algebra \( \text{agc}_1 \) lead to:

**Case a)** NMG-representation.

The nontrivial equations are (3.19) and (3.21). The last leads to \( k_0'(\zeta) k(v) = 0 \) which is satisfied if \( k_0(\zeta) = k_0 = \text{cste}. \) (The other case \( k(v) = 0 \) leads to \( K = 0, m(v) = m_0 v^{-1}, m_0 = \text{cste}. \), which one can equivalently obtain by putting \( k_0 = 0 \) in (3.19).)

We find, besides \( Y_{-1/2} = -\partial_r \) and \( M_0 = -\partial_\zeta \) for the NMG-representation of \( \text{agc}_1 \)

\[
\begin{align*}
 Y_{1/2} &= -t \partial_r - r \partial_\zeta \\
 X_0 &= -t \partial_t - \frac{1}{2} r \partial_r - y g \partial_g - \frac{x}{2} \\
 X_1 &= -t^2 \partial_t - tr \partial_r - \frac{1}{2} r^2 \partial_\zeta - p_0 t^{y+1} m \left( \frac{t^y}{g} \right) \partial_g - xt \\
 N &= -t \partial_t + \zeta \partial_\zeta - k_0 t^y k \left( \frac{t^y}{g} \right) \partial_g
\end{align*}
\]

This representation is characterized by the constants \( x, y, p_0, k_0 \) and the functions \( m(v), k(v) \) which must satisfy the condition

\[
y k_0 \frac{d}{dv}(v k(v)) - y p_0 \frac{d}{dv}(v m(v)) - k_0 p_0 v^2 k(v) \frac{d}{dv} \left( \frac{m(v)}{k(v)} \right) = 0 \quad (3.23)
\]

**Case b)** MMG-representation.

In this case the equations (3.19) and (3.20) lead to

\[
y k_0(\zeta) (k(v) + v k'(v)) + p_0(\zeta) m(v) = 0 \quad (3.24)
\]

Two cases must be considered:

1. \( k_0(\zeta) = k_0 = \text{cste}. \) This leads to \( k(v) = \zeta v^{-1}, n(v) = m(v) = 0 \) and the first MMG-realization of the algebra \( \text{agc}_1 \) is

\[
\begin{align*}
 M_0 &= -\partial_\zeta + \frac{2y}{\zeta} \partial_g \\
 Y_\frac{1}{2} &= -t \partial_r - r \partial_\zeta + \frac{2y}{\zeta} (rg) \partial_g \\
 X_0 &= -t \partial_t - \frac{1}{2} r \partial_r - y g \partial_g - \frac{x}{2} \\
 X_1 &= -t^2 \partial_t - tr \partial_r - \frac{1}{2} r^2 \partial_\zeta - \frac{2y}{\zeta} \left( \frac{r^2 g}{2} \right) \partial_g - xt \\
 N &= -t \partial_t + \zeta \partial_\zeta - k_0 g \partial_g
\end{align*}
\]

(3.25)
2. \( k'_0(\zeta) \neq 0 \). We divide the equation \((3.24)\) by \((k_0(\zeta)p_0(\zeta))\) (since \(k_0(\zeta) \neq 0, p_0(\zeta) \neq 0\)), denote \( \mathcal{K} = 1/k_0(\zeta) \) and take the derivative with respect to \( \zeta \) for obtaining
\[
\mathcal{K}'(\zeta) = l_0 \frac{(k(v) + v k'(v))}{m(v)} = c_0 = \text{cste.} \tag{3.26}
\]
with solutions
\[
k_0(\zeta) = \frac{1}{C_0 \zeta}, \quad k(v) = 2y \frac{C_0}{l_0} m(v) = \kappa v^{(1-2y)/(2y)}, \quad \kappa = 2y C_0 \text{ch}_0 \tag{3.27}
\]
The second MMG-realization of the algebra \( \mathfrak{a} \mathfrak{g} \mathfrak{c}_1 \) is
\[
M_0 = -\partial_\zeta + \frac{2y}{\zeta} g \partial_g \\
Y_{\frac{1}{2}} = -t \partial_r - r \partial_\zeta + \frac{2y}{\zeta} \left( r g + h_0 g^{(2y-1)/(2y)} \right) \partial_g \\
X_0 = -t \partial_\zeta - \frac{1}{2} r \partial_r - y g \partial_g - \frac{x}{2} \\
X_1 = -t^2 \partial_\zeta - tr \partial_r - \frac{1}{2} r^2 \partial_\zeta \\
+ \frac{2y}{\zeta} \left( \frac{r^2 g}{2} + h_0 t r g^{(2y-1)/(2y)} + c h_0 t^{3/2} g^{(2y-1)/(2y)} \right) \partial_g - xt \\
N = -t \partial t + \zeta \partial_\zeta - c h_0 \left( \frac{t^2 g^{(2y-1)/(2y)}}{2} \right) \partial_g \tag{3.28}
\]

Having exhausted the constraints from the commutation relation, we now require that the solution space of the linear Schrödinger equation is invariant under the action of these representations. This means
\[
[\hat{S}, X_i] = \lambda_i \hat{S}, \tag{3.29}
\]
for each of the generators, where \( \hat{S} \) is in the representation-independent form \((2.9)\). We must distinguish the two cases

1. a fixed scaling dimension \( x = 1/2 \) for the wave function \( \Psi \).
2. wave functions with arbitrary scaling dimensions.

First, for the NMG-realization \((3.14)\) of the algebra \( \mathfrak{a} \mathfrak{g} \mathfrak{c}_1 \) we have
\[
[\hat{S}, X_0] = -\hat{S} \\
[\hat{S}, M_0] = [\hat{S}, Y_{\frac{1}{2}}] = [\hat{S}, Y_{\frac{3}{2}}] = 0 \\
[\hat{S}, X_1] = -2t \hat{S} - (1-2x) M_0 + M_0 \hat{Q} \tag{3.30}
\]
where
\[
\hat{Q} := [2 p_{01} t^y ((y+1)m(v) + y v m'(v))] \partial_g. \tag{3.31}
\]
In order to satisfy the condition \((3.29)\) one must have
\[
\left( (1-2x) M_0 - M_0 \hat{Q} \right) \Psi = 0 \tag{3.32}
\]
In general, we want to satisfy the \((3.32)\) on the operator level and shall relax this to a condition on the functions in the representation space only if a non-trivial solution cannot be found otherwise.
We begin with the case \( x = \frac{1}{2} \) where we must have \( \hat{Q} \Psi = (2p_{01}t^y((y + 1)m(v) + yvm'(v)))\partial_y \Psi = 0 \). This leads to
\[
m(v) = m_0v^{-(y+1)/y}
\]
(3.33)
The final NMG-realization of \( \mathfrak{agc}_1 \) is in the case at hand
\[
\begin{align*}
Y_{-\frac{1}{2}} &= -\partial_r \\
M_0 &= -\partial_\zeta \\
Y_{\frac{1}{2}} &= -t\partial_r - r\partial_\zeta \\
X_0 &= -t\partial_t - \frac{1}{2}r\partial_r - yg\partial_\zeta - \frac{1}{4} \\
X_1 &= -t^2\partial_t - tr\partial_r - \frac{1}{2}r^2\partial_\zeta - p_{01}m_0g^{(y+1)/y}\partial_g - \frac{1}{2}
\end{align*}
\]
(3.34)
and the invariant linear Schrödinger equation is simply \( (2\partial_\zeta \partial_t - \partial_r^2) \Psi = 0 \).

In the other case of arbitrary scaling dimension \( x \neq \frac{1}{2} \) we have
\[
\hat{Q} \Psi = (1 - 2x)\Psi 
\]
(3.35)
to be satisfied. Here, because of the arbitrary value of \( m(v) \) the condition (3.35) is written in integral form (using \( \Psi(t, r, \zeta, g) \rightarrow \Psi(t, r, \zeta, v) \) where \( v = t^y/g \)):
\[
\Psi(t, r, \zeta, v) = \Psi_0(t, r, \zeta) \frac{1 - 2x}{2p_{01}} \int \frac{dv}{v^2((y + 1)m(v) + yvm'(v))}
\]
(3.36)
and the NMG-realisation of the algebra \( \mathfrak{agc}_1 \) is given by eq. (3.14).

We remember now, that “almost-parabolic subalgebras” were defined through parabolic ones and we anticipate a result of the parabolic subalgebra \( \tilde{\mathfrak{agc}}_1 \). In the NMG-representation, we shall show below that invariance of the linear Schrödinger equation leads to \( k(v) = v^{-1} \). Hence, because of (3.23) we have \( m(v) = v^{-1} \) and
\[
\hat{Q} = 2p_{01}g\partial_g 
\]
(3.37)
for \( x \neq \frac{1}{2} \) and \( Q = 0 \) (\( p_{01} = 0 \)) in the case of canonical scaling dimension \( x = \frac{1}{2} \).

Invariance of the linear Schrödinger equation implies the following condition on the wave function
\[
\hat{Q}\Psi(t, r, \zeta, g) = 2p_{01}g\partial_g\Psi(t, r, \zeta, g) = (1 - 2x)\Psi(t, r, \zeta, g)
\]
(3.38)
which means \( \Psi(t, r, \zeta, g) = g^{(1-2x)/(2p_{01})}\psi(t, r, \zeta) \) and the dependence of \( \Psi \) on \( g \) is determined.

There is another possibility, namely \( M_0\Psi = 0 \). In this case the wave functions do not depend on \( \zeta \). So we have seen that instead of a simple Lie symmetry, we rather have a so-called conditional symmetry as introduced in [33, 55].

Finally, the NMG-representation of the algebra \( \mathfrak{agc}_1 \) for arbitrary scaling dimensions, subject to the auxiliary condition (3.38) is
\[
\begin{align*}
Y_{-\frac{1}{2}} &= -\partial_r \\
M_0 &= -\partial_\zeta \\
Y_{\frac{1}{2}} &= -t\partial_r - r\partial_\zeta \\
X_0 &= -t\partial_t - \frac{1}{2}r\partial_r - yg\partial_\zeta - \frac{x}{2} \\
X_1 &= -t^2\partial_t - tr\partial_r - \frac{1}{2}r^2\partial_\zeta - p_{01}tg\partial_g - xt
\end{align*}
\]
(3.39)
Second, we consider the MMG-realization of (3.15) of the algebra $\mathfrak{age}_1$. From the conditions (3.29) we have $h_0 = 0$ and hence

\[
\begin{align*}
[\hat{S}, X_0] &= -\hat{S} \\
[\hat{S}, M_0] &= [\hat{S}, Y_{-\frac{1}{2}}] = [\hat{S}, Y_{\frac{1}{2}}] = 0 \\
[\hat{S}, X_1] &= -2t \hat{S} - (1 - 2x)M_0.
\end{align*}
\]

(3.40)

which are satisfied automatically in the case $x = \frac{1}{2}$.

**Remark 1:** The subalgebra of $\mathfrak{age}_1$ obtained when leaving out the generator $X_1$ leaves the linear Schrödinger equation invariant for arbitrary $h_0$.

In the case of an arbitrary scaling dimension we must have

\[
M_0 \Psi(t, r, \zeta, g) = \left( \partial_\zeta - \frac{2y}{\zeta} g \partial_g \right) \Psi(t, r, \zeta, g) = 0
\]

(3.41)

which implies $\Psi(t, r, \zeta, g) = \Psi(t, r, \zeta g^{1/2})$ such that the dependence on $\zeta$ and $g$ merely enters through the scaling variable $u := g^{1/2} \zeta$. Again, we merely find a conditional symmetry.

Our final result for the MMG-representation of $\mathfrak{age}_1$ reads, for any value of $x$

\[
\begin{align*}
Y_{-\frac{1}{2}} &= -\partial_r \\
M_0 &= -\partial_\zeta + \frac{2y}{\zeta} g \partial_g \\
Y_{\frac{1}{2}} &= -t \partial_r - r \partial_\zeta + \frac{2y}{\zeta} r g \partial_g \\
X_0 &= -t \partial_t - \frac{1}{2} r \partial_r - yg \partial_g - \frac{x}{2} \\
X_1 &= -t^2 \partial_t - tr \partial_r - \frac{1}{2} r^2 \partial_\zeta + \frac{y}{\zeta} r^2 g \partial_g - xt
\end{align*}
\]

(3.42)

We finish by extending our results to the parabolic subalgebra $\tilde{\mathfrak{age}}_1$. For the NMG-representation we must also satisfy the condition

\[
[\hat{S}, N] = yt^{y-1}k_0(k(v) + vk'(v)) = 0
\]

(3.43)

which gives $k(v) = v^{-1}$ and hence, from (3.23) we find $m(v) = v^{-1}$ for the general case and $m(v) = 0$ in the special case $x = \frac{1}{2}$. The generators of the NMG-representation of $\tilde{\mathfrak{age}}_1$ read

\[
\begin{align*}
Y_{-\frac{1}{2}} &= -\partial_r \\
M_0 &= -\partial_\zeta \\
Y_{\frac{1}{2}} &= -t \partial_r - r \partial_\zeta \\
X_0 &= -t \partial_t - \frac{1}{2} r \partial_r - yg \partial_g - \frac{x}{2} \\
X_1 &= -t^2 \partial_t - tr \partial_r - \frac{1}{2} r^2 \partial_\zeta - p_{01} t g \partial_g - xt \\
N &= -t \partial_t + \zeta \partial_\zeta - k_0 g \partial_g
\end{align*}
\]

(3.44)

**Remark 2:** The case $x = \frac{1}{2}$ is obtained from the above realization by setting $p_{01} = 0$. 

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Table 1: Representations of the almost-parabolic and parabolic subalgebras \( g \subset (\text{conf}_3)_\mathbb{C} \) and their (conditionally) invariant linear Schrödinger equation \( S\Psi = 0 \). The form of \( \hat{S} \) and the auxiliary conditions only depend on the value of \( x \) but are independent of whether a parabolic or an almost-parabolic subalgebra is considered.

| case | \( g \) | representation | \( x \) | auxiliary conditions | \( S \) |
|------|--------|----------------|-----|-------------------|-------|
| 0    | \( \mathfrak{age}_1 \) | NMG \[L = 0, \, Q = 0, \, P = p_{01}m_{0g}^{(y+1)/y} \] | \( = \frac{1}{2} \) | \( \partial_\zeta \Psi = 0 \) | \( 2\partial_\zeta \partial_t - \partial_r^2 \) |
|      |        | NMG \[L = 0, \, Q = 0, \, P = p_{01}t_{y+1}^y m(t^y/g) \] | \( \neq \frac{1}{2} \) | see eq. (3.36) | \( 2\partial_\zeta \partial_t - \partial_r^2 \) |
| 1    | \( \mathfrak{age}_1 \) | NMG \[L = 0, \, Q = 0, \, P = p_{01}t_{g} \] | \( = \frac{1}{2} \) | \( (2p_{01}g\partial_g + (2x - 1))\Psi = 0 \) | \( 2\partial_\zeta \partial_t - \partial_r^2 \) |
|      | \( \tilde{\mathfrak{age}}_1 \) | \( K = k_{0g} \) | \( \neq \frac{1}{2} \) | \( (2\partial_\zeta \partial_t - \partial_r^2)\Psi = 0 \) | \( 2\partial_\zeta \partial_t - \partial_r^2 \) |
| 3    | \( \mathfrak{age}_1 \) | MMG \[L = -2y g/\zeta, \, Q = -2y g r/\zeta, \, P = -y g r^2/\zeta \] | \( = \frac{1}{2} \) | \( (\partial_\zeta - 2y(g/\zeta)\partial_g)\Psi = 0 \) | \( (2\partial_\zeta \partial_t - \partial_r^2)\Psi = 0 \) |
|      | \( \tilde{\mathfrak{age}}_1 \) | \( K = k_{0g} \) | \( \neq \frac{1}{2} \) | \( (\partial_\zeta + sg\zeta^{-1}\partial_g)\Psi = \partial_t \Psi = 0 \) | \( \partial_\zeta \partial_t \) |
| 5    | \( \mathfrak{alt}_1 \) | \( L = sg/\zeta, \, Q = sg r/\zeta, \, P = s g^2 r^2 / 2\zeta \) | \( = \frac{1}{2} \) | \( (2\partial_\zeta \partial_t + 2sg\zeta^{-1}\partial_g\partial_t - \partial_r^2)\Psi = 0 \) | \( \partial_\zeta \partial_t \) |
|      | \( \tilde{\mathfrak{alt}}_1 \) | \( K = k_{0g} \) | \( \neq \frac{1}{2} \) | \( (\partial_\zeta + sg\zeta^{-1}\partial_g)\Psi = \partial_t \Psi = 0 \) | \( \partial_\zeta \partial_t \) |
| 7    | \( \mathfrak{sch}_1 \) | \( L = Q = P = 0 \) | \( = \frac{1}{2} \) | \( \partial_\zeta \Psi = 0 \) | \( 2\partial_\zeta \partial_t - \partial_r^2 \) |
|      | \( \tilde{\mathfrak{sch}}_1 \) | \( K = k_{0g} \) | \( \neq \frac{1}{2} \) | \( (4yg\partial_g + (2x - 1))\Psi = 0 \) | \( 2\partial_\zeta \partial_t - \partial_r^2 \) |
| 8    | \( \mathfrak{sch}_1 \) | \( L = Q = 0, \, P = 2yt_{g} \) | \( = \frac{1}{2} \) | \( \partial_\zeta \Psi = 0 \) | \( 2\partial_\zeta \partial_t - \partial_r^2 \) |

The MMG-representations of \( \tilde{\mathfrak{age}}_1 \) which leave invariant the linear Schrödinger equation coincide with eq. (3.25). For \( x = 1/2 \) there is no restriction on \( \Psi \) while the condition (3.41) must be satisfied if \( x \neq 1/2 \).

We shall need later that for the NMG-representation the linear Schrödinger equation reads simply

\[
(2\partial_\zeta \partial_t - \partial_r^2) \Psi(t, r, \zeta, g) = 0
\]

(3.45)

while for the MMG-representation there arises an additional term

\[
(2\partial_\zeta \partial_t - \frac{4y}{\zeta} g\partial_g\partial_t - \partial_r^2) \Psi(t, r, \zeta, g) = 0
\]

(3.46)

This follows from the explicit representations and the form (2.9) of the Schrödinger operator \( \hat{S} \).

We summarize the results of the classification of this and the following subsections in table 1. For \( \mathfrak{age}_1 \) and \( \tilde{\mathfrak{age}}_1 \) we distinguish five cases. Case 1 is obtained from case 0 by setting \( m(v) = v^{-1} \) and only then there is an extension from \( \mathfrak{age}_1 \) to \( \tilde{\mathfrak{age}}_1 \) which is case 2. The cases 0, 1 and 3 refer to \( \mathfrak{age}_1 \).
and distinguish between the NMG and MMG representations as characterized by the explicit functions $L, Q, P$. For each of them, we give for both $x = 1/2$ and $x \neq 1/2$ the explicit form of the linear Schrödinger operator $\hat{S}$. For $x \neq 1/2$ there may be one or several auxiliary conditions which have to be met as well and in each case will lead to a modified form of the linear Schrödinger operator $\hat{S}$. In these cases, we have hence not found a Lie symmetry but rather a non-classical one, usually referred to as $Q$-conditional symmetry \cite{23}. Finally, the cases 2 and 4 apply to $\tilde{\alpha} \tilde{\gamma}_1$. Here the only new information is the explicit form of $K$ whereas the form of $\hat{S}$ and the auxiliary conditions remain unchanged.

As an example, consider case 2. It refers to the NMG-representation of $\tilde{\alpha} \tilde{\gamma}_1$ and the four functions $P, Q, L, K$ can be read off. Furthermore, one sees that for $x = 1/2$, there is no further condition on $\Psi$ and the linear Schrödinger operator is $\hat{S} = 2\partial_r \partial_t - \partial_r^2$. On the other hand, for $x \neq 1/2$, there are two distinct auxiliary conditions. For each of them, one reads off the corresponding invariant linear Schrödinger operator $\hat{S}$. The entry for case 1 can be read in the same way but the function $K$ is of course not specified. The other cases are understood similarly.

### 3.2 Subalgebras $\mathfrak{alt}_1$ and $\tilde{\mathfrak{alt}}_1$

For these algebras we fix the generators of dilatations $D$ and space translations $Y_{-1/2}$:

\[
Y_{-1/2} = -\partial_r \\
D = -t\partial_t - r\partial_r - \zeta \partial_\zeta - sg\partial_g - x
\]  

where the exponent $s$ describes the scaling behaviour of the coupling $g$. The remaining generators are taken in the following general form

\[
M_0 = -\partial_\zeta - L(t, r, \zeta, g)\partial_g \\
Y_{1/2} = -t\partial_r - r\partial_\zeta - Q(t, r, \zeta, g)\partial_g \\
X_1 = -t^2\partial_t - tr\partial_r - \frac{1}{2}r^2\partial_\zeta - P(t, r, \zeta, g)\partial_g - xt \\
N = -t\partial_t + \zeta \partial_\zeta - K(t, r, \zeta, g)\partial_g \\
V_+ = -2tr\partial_t - 2\zeta r\partial_\zeta - (r^2 + 2\zeta t)\partial_r - F(t, r, \zeta, g)\partial_g - 2xr
\]  

Since this is a subalgebra of $\mathfrak{conf}_3$, we obtain the following results for $\mathfrak{alt}_1$. The commutators

\[
[D, X_1] = -X_1 , \ [D, Y_{1/2}] = 0 , \ [D, M_0] = M_0 , \ [D, V_+] = -V_+
\]  

give

\[
P = ts^{+1}p(u, v, w) , \ Q = ts^q(u, v, w) , \ L = ts^{+1}l(u, v, w) , \ F = ts^{+1}f(u, v, w)
\]  

where

\[
u = \frac{r}{t} , \ v = t^{-s}g , \ w = \frac{\zeta}{t}
\]  

Next we use

\[
[X_1, Y_{-1/2}] = Y_{1/2} , \ [Y_{1/2}, Y_{-1/2}] = M_0 , \ [Y_{-1/2}, M_0] = 0 , \ [Y_{1/2}, M_0] = 0 , \ [V_+, Y_{-1/2}] = 2D
\]  

and find

\[
qu(v, u, w) = u\partial_u p(u, v, w) , \ l(u, v, w) = u\partial_u q(u, v, w) , \ \partial_u l(u, v, w) = 0 \\
q(u, v, w) = ul(v, w) + n(v, w) , \ p(u, v, w) = \frac{u^2}{2}l(v, w) + un(v, w) + m(v, w) , \ f(u, v, w) = 4su + z(v, w).
\]


we have the set of equations

\[ [X_1, Y_{\frac{1}{2}}] = 0 \quad \text{and} \quad [V_+, M_0] = 2Y_{\frac{1}{2}} \quad \text{and} \quad [V_+, Y_{\frac{1}{2}}] = 2X_1 \quad \text{and} \quad [V_+, X_1] = 0 \]  

(3.54)

give the following system for the unknown functions \( l(v, w) \), \( n(v, w) \), \( m(v, w) \) and \( z(v, w) \)

\[
\begin{align*}
\partial_v n + l \partial_v n - n \partial_v l &= 0 \\
(s - 1)l - sv \partial_v l - w \partial_w l - \partial_w m + m \partial_v l - l \partial_v m &= 0 \\
(s - 1)n - sv \partial_v n - w \partial_w n + m \partial_v n - l \partial_v m &= 0 \\
2n + z \partial_v l - l \partial_v z &= 0 \\
2n + z \partial_v l - l \partial_v z - \partial_w z &= 0 \\
2m + 2wl - 2sv + z \partial_v n - n \partial_v z &= 0 \\
2wn - (s + 1)z + sv \partial_v z + w \partial_w z + z \partial_v n - n \partial_v z &= 0
\end{align*}
\]

(3.55)

The solution is readily found

\[ z = 0 \quad n = 0 \quad l(v, w) = vl_1(w) \quad m(v, w) = sv - wvl_1(w) \]

(3.56)

and we arrive at the most general representation of the algebra \( \widetilde{\mathfrak{al}t}_1 \)

\[
D = -t \partial_t - r \partial_r - \zeta \partial_\zeta - sg \partial_g - x \\
Y_{\frac{1}{2}} = -\partial_r \\
M_0 = -\partial_\zeta - t^{-1}l_1(w)g \partial_g \\
Y_{\frac{3}{2}} = -t \partial_t - r \partial_r - t^{-1}w l_1(w)g \partial_g \\
X_1 = -t^2 \partial_t - tr \partial_r - \frac{1}{2} r^2 \partial_\zeta - \left( \frac{t^{-1} r^2}{2} l_1(w) + t(s - w l_1(w)) \right) g \partial_g - xt \\
V_+ = -2tr \partial_t - 2\zeta r \partial_\zeta - (r^2 + 2\zeta t) \partial_r - 2srg \partial_g - 2xr
\]

(3.57)

which depends on the parameters \( x, s \) and arbitrary function \( l_1(w) \).

For the algebra \( \widetilde{\mathfrak{al}t}_1 \) one must satisfy also the commutators

\[
\begin{align*}
[D, N] &= 0 \quad \text{and} \quad [Y_{\frac{1}{2}}, N] = 0 \quad \text{and} \quad [M_0, N] = M_0 \\
[Y_{\frac{3}{2}}, N] &= Y_{\frac{3}{2}} \quad \text{and} \quad [X_1, N] = X_1 \quad \text{and} \quad [V_+, N] = 0
\end{align*}
\]

(3.58)

which gives \( K = t^s k(u, v, w) \) and \( \partial_a k(u, v, w) = 0 \) which means that \( k(u, v, w) = k(v, w) \) and furthermore we have the set of equations

\[
\begin{align*}
2 - s)l + sv \partial_v l + 2w \partial_w l + \partial_w k + l \partial_v k - k \partial_l l &= 0 \\
(1 - s)n + sv \partial_v n + 2w \partial_w n + n \partial_v k - k \partial_v n &= 0 \\
sk - sv \partial_v k - w \partial_w k - sm + sv \partial_v m + 2w \partial_w m + m \partial_v k - k \partial_v m &= 0 \\
(s + 1)z - sv \partial_v z - 2w \partial_w z + z \partial_v k - k \partial_v z &= 0
\end{align*}
\]

(3.59)

Using eq. (3.56) the system (3.59) reduces to the single equation

\[ 2vl_1 + 2vw \partial_w l + \partial_w k - kl_1 + vl_1 \partial_v k = 0 \]

(3.60)

and if we recall that \( D = 2X_0 - N \) we obtain the final result

\[ k(v, w) = k_0' v \quad k_0' + s = 2y \quad l_1(w) = l_0 w^{-1} \quad K = k_0' g \]

(3.61)
where \( k'_0 \) and \( l_0 \) are constants. We therefore have the following realization of \( \widetilde{\mathfrak{alt}}_1 \)

\[
D = -t \partial_t - r \partial_r - \zeta \partial_\zeta - sg \partial_g - x
\]

\[
Y_{-\frac{1}{2}} = -\partial_r
\]

\[
M_0 = -\partial_\zeta - \frac{l_0}{\zeta} g \partial_g
\]

\[
Y_{\frac{1}{2}} = -t \partial_r - r \partial_\zeta - \frac{l_0}{\zeta} g \partial_g
\]

\[
X_1 = -t^2 \partial_t - tr \partial_r - \frac{1}{2} r^2 \partial_\zeta - \left( \frac{l_0}{2\zeta} r^2 + (s - l_0)t \right) g \partial_g - xt
\]

\[
V_+ = -2tr \partial_t - 2\zeta r \partial_\zeta - (r^2 + 2\zeta t ) \partial_r - 2sr g \partial_g - 2xr
\]

\[
N = -t \partial_t + \zeta \partial_\zeta - k'_0 g \partial_q
\]

Next we impose the invariance of the linear Schrödinger equation. We have

\[
[\hat{S}, X_0] = -\hat{S}
\]

\[
[\hat{S}, M_0] = [\hat{S}, Y_{-\frac{1}{2}}] = [\hat{S}, Y_{\frac{1}{2}}] = 0
\]

\[
[\hat{S}, X_1] = -2t \hat{S} - (1 - 2x) M_0
\]

\[
[\hat{S}, V_+] = -4r \hat{S} + 2(1 - 2x) Y_{-\frac{1}{2}}
\]

The conditions (3.29) are satisfied if \( l_0 = s \).

**Remark 3:** If the generator \( X_1 \) is left out from \( \mathfrak{alt}_1 \) and \( \widetilde{\mathfrak{alt}}_1 \), the equation \( \hat{S} \Psi = 0 \) is invariant even for arbitrary \( l_0 \).

In the case with fixed scaling dimension \( x = \frac{1}{2} \) the following realization of algebra \( \mathfrak{alt}_1 \) is a dynamical symmetry of the linear Schrödinger equation

\[
D = -t \partial_t - r \partial_r - \zeta \partial_\zeta - sg \partial_g - \frac{1}{2}
\]

\[
Y_{-\frac{1}{2}} = -\partial_r
\]

\[
M_0 = -\partial_\zeta - \frac{s}{\zeta} g \partial_g
\]

\[
Y_{\frac{1}{2}} = -t \partial_r - r \partial_\zeta - \frac{s}{\zeta} r g \partial_g
\]

\[
X_1 = -t^2 \partial_t - tr \partial_r - \frac{1}{2} r^2 \partial_\zeta - \frac{s}{2\zeta} r^2 g \partial_g - \frac{1}{2} t
\]

\[
V_+ = -2tr \partial_t - 2\zeta r \partial_\zeta - (r^2 + 2\zeta t ) \partial_r - 2sr g \partial_g - 2xr
\]

(3.64)

without any restrictions on the wave function.

For arbitrary \( x \) the conditions (3.63) lead to very strong restrictions on the wave function

\[
M_0 \Psi(\zeta, t, r, g) = (\partial_\zeta + \frac{s}{\zeta} g \partial_g) \Psi(\zeta, t, r, g) = 0
\]

\[
Y_{-1/2} \Psi(\zeta, t, r, g) = -\partial_r \Psi(\zeta, t, r, g) = 0
\]

(3.65)

which means that \( \Psi(\zeta, t, r, g) = \psi(t, g^{1/s}\zeta) \).
Finally, we add the generator $N$ in order to obtain the representation of $\tilde{\mathfrak{al}}_1$. It is of the form $N = -t\partial_t + \zeta\partial_\zeta - k'_0 g\partial_g$ and the condition $[\hat{S}, N] = 0$ is satisfied automatically. Note that for this algebra the linear Schrödinger equation has the form:

$$
\left(2\partial_\zeta \partial_t + \frac{2s}{\zeta} g\partial_g \partial_t - \partial_r^2\right) \Psi(\zeta, t, r, g) = 0
$$

(3.66)

These results are also included in table 1.

### 3.3 Subalgebras $\mathfrak{sch}_1$ and $\tilde{\mathfrak{sch}}_1$

In these algebras we must add to $\mathfrak{age}_1$ and $\tilde{\mathfrak{age}}_1$ the generator of time translation $X_{-1} = -\partial_t$ and satisfy the commutator $[X_1, X_{-1}] = 2X_0$. Inserting this condition into the generators eq. (3.39) we find $p_{01} = 2\gamma$ and $m(v) = v^{-1} = gt^{-y}$.

The representation of $\mathfrak{sch}_1$ is

$$
X_{-1} = -\partial_t, \quad Y_{-\frac{1}{2}} = -\partial_r
$$

$$
M_0 = -\partial_\zeta
$$

$$
Y_{\frac{1}{2}} = -t\partial_r - r\partial_\zeta
$$

$$
X_0 = -t\partial_t - \frac{1}{2}r\partial_r - yg\partial_g - \frac{x}{2}
$$

$$
X_1 = -t^2\partial_t - tr\partial_r - \frac{1}{2}r^2\partial_\zeta - 2ytg\partial_g - xt
$$

(3.67)

and for $\tilde{\mathfrak{sch}}_1$ we have $K = k_0 g$, hence (3.67) holds true, together with

$$
N = -t\partial_t + \zeta\partial_\zeta - k_0 g\partial_g
$$

(3.68)

For the invariance of the linear Schrödinger equation we merely have to consider a single commutator

$$
[S, X_1] = -2t\hat{S} - (1 - 2x)M_0 + M_0 \hat{Q}_{\mathfrak{sch}}, \quad \hat{Q}_{\mathfrak{sch}} = 4yg\partial_g.
$$

(3.69)

The linear Schrödinger equation is invariant if

$$
M_0 \Psi = 0, \quad \Psi(\zeta, t, r, g) = \psi(t, r, g)
$$

(3.70)

which means that either the wave function does not depend on $\zeta$ or else

$$
\hat{Q}_{\mathfrak{sch}} \Psi = (1 - 2x)\Psi, \quad \Psi(\zeta, t, r, g) = g^{(1 - 2x)/4y}\psi(\zeta, t, r)
$$

(3.71)

Even in the case $x = \frac{1}{2}$ we must impose an auxiliary condition on the wave functions.

Again, we include our results in table 1.

### 4 Invariant nonlinear equations

Having classified in the previous section the representations of the parabolic and almost-parabolic subalgebras of $(\text{conf}_3)_C$ which leave the linear Schrödinger equation $\hat{S} \Psi = 0$ invariant, we now construct systematically all semilinear invariant equations $\hat{S} \Psi = F(\zeta, t, r, g; \Psi, \Psi^*)$, along the lines recalled in section 2.1.
4.1 Subalgebras $\mathfrak{a}g\mathfrak{c}_1$ and $\tilde{\mathfrak{a}}g\mathfrak{c}_1$

First we take the almost-parabolic subalgebra (3.14). The potential $F$ must satisfy the system, see eq.(2.8)

$$\partial_r F = 0 , \quad \partial_t F = 0$$

$$(2t\partial_t + 2yg\partial_g - x(\Psi\partial_\Psi + \Psi^*\partial_{\Psi^*}) + (x + 2))F = 0$$

$$(t\partial_t + p_0 t^y m(v)\partial_g - x(\Psi\partial_\Psi + \Psi^*\partial_{\Psi^*}) + (x + 2))F = 0$$

(4.1)

In the general case the solution is:

$$F^0_{\mathfrak{a}g\mathfrak{c}_1} = \Psi_0^{\frac{\partial \zeta}{\partial x}} f_0 \left( \ln \Psi + \int \frac{dv}{v} p_0 t^y m(v) - 2y - \frac{\Psi}{\Psi^*} \right)$$

(4.2)

In the special case $m(v) = v^{-k}$ this leads to the following form of $f_0$

$$f_0 = f_0 \left( \ln \left[ t^{x/2} g^{-x/(2y)} (p_0 t^{-1}) y^{1-k} \Psi \right] - 2y - \frac{\Psi}{\Psi^*} \right)$$

(4.3)

Note that the case $x = 1/2$ can be obtained by putting $k = (y + 1)/y$.

When the operator $\hat{Q}$ is taken in form (3.37) the solution in the NMG-representation is

$$F^1_{\mathfrak{a}g\mathfrak{c}_1} = (t^{p_0 - 2y} g)^{-\frac{x+2}{2(p_0 - y)}} f^1_{\mathfrak{a}g\mathfrak{c}_1} \left( (t^{p_0 - 2y} g)^{\frac{x}{2(p_0 - y)}} \Psi, \Psi^* \right)$$

(4.4)

So, the nonlinear Schrödinger equation is

$$(2\partial_t \partial_t + \partial_r^2)\Psi(\zeta, t, r, g) = F^1_{\mathfrak{a}g\mathfrak{c}_1}$$

(4.5)

For the extention to the parabolic algebra $\tilde{\mathfrak{a}}g\mathfrak{c}_1$, we must add the equation

$$(t\partial_t + k_0 g\partial_g) F = 0$$

(4.6)

which leads to the following equation for $f^1_{\mathfrak{a}g\mathfrak{c}_1}$

$$\frac{p_0 - 2y + k_0}{2(p_0 - y)} \left( x + 2 - xu \frac{\partial}{\partial u} \right) f^1_{\mathfrak{a}g\mathfrak{c}_1} (u, v) = 0.$$ 

(4.7)

where $u = (t^{p_0 - y} g)^{\frac{x}{2(p_0 - y)}} \Psi$ and $v = \Psi / \Psi^*$.

There are two cases:

1. the generic solution $p_0 \neq 2y - k_0$

$$F^1_{\tilde{\mathfrak{a}}g\mathfrak{c}_1} = \Psi_0^{\frac{\partial \zeta}{\partial x}} f_{\text{sch}_1} \left( \frac{\Psi}{\Psi^*} \right)$$

(4.8)

which is the same as for the nonmodified representation of the Schrödinger algebra.

2. a non-generic solution $p_0 = 2y - k_0$. Here the form of the potential is the same as in (4.4) but one has an additional constraint on the parameters of the algebra.
Note that in the canonical case \( x = \frac{1}{2} \) one must also put \( p_{01} = 0 \) in the above results.

We consider now the MMG-realisation (3.32) of the algebra \( \mathfrak{age}_1 \). The potential is found from

\[
\partial_r F = 0, \quad \left( \partial_\zeta - 2y\frac{\partial}{\partial \zeta} \right) F = 0
\]

\[
(2t \partial_t + 2yg \partial_y - x(\Psi \partial_\Psi + \Psi^* \partial_{\Psi^*}) + (x + 2))F = 0
\]

\[
(t \partial_t - x(\Psi \partial_\Psi + \Psi^* \partial_{\Psi^*}) + (x + 2))F = 0
\]

which has the solution

\[
F_{\mathfrak{age}_1}^2 = b^{x+2} f_{\mathfrak{age}_1}^2 \left( b^{-x} \Psi, \frac{\Psi}{\Psi^*} \right), \quad b = t^{-1} \zeta g^{1/(2y)}
\]

The nonlinear equation invariant under the same algebra is

\[
\left( 2\partial_\zeta \partial_t - 4yg \partial_y - \partial_r^2 \right) \Psi(t, r, g) = F_{\mathfrak{age}_1}^2
\]

When we extend this to the parabolic subalgebra \( \widetilde{\mathfrak{age}_1} \) by including the generator \( N \) (see (3.25)), we have

\[
\left( \frac{k_0}{2y} - 2 \right) \left( x + 2 - xu \frac{\partial}{\partial u} \right) f_{\mathfrak{age}_1}^2(u, v) = 0.
\]

where \( u = b^{x} \Psi \) and \( v = \Psi/\Psi^* \). Our results give

1. the generic solution \( k_0 \neq 4y \). We recover the same result as for nonmodified Schrödinger algebra (4.8).

2. a non-generic solution \( k_0 = 4y \). The result is the same as in (4.10), but the central generator \( N = -t \partial_t + \zeta \partial_\zeta - 4yg \partial_y \) is specified.

These results are valid for arbitrary scaling dimension, but for the case \( x = \frac{1}{2} \) there is no restriction on the wave functions.

The results for the non-linear potential \( F \) are collected in the table 2. We give the generic solutions for the cases as defined in table 1. For each of the cases, we had the linear equation \( \hat{S} \Psi = 0 \) together with eventual auxiliary condition(s) for a given value of \( x \). Now the associated non-linear equation is simply \( \hat{S} \Psi = F \), where \( F \) is read from the table. In table 2 we also list the non-generic solutions which are distinguished by the conditions mentioned and for some of which there are modified scaling variables to be used.

### 4.2 Subalgebras \( \mathfrak{alt}_1 \) and \( \widetilde{\mathfrak{alt}_1} \)

For the realization (3.64) of \( \mathfrak{alt}_1 \) (note that for this realization of the algebra one has \( k_0' + s = 2y \) by construction), the potential is found from

\[
\partial_r F = 0, \quad \left( \zeta \partial_\zeta + sg \partial_y \right) F = 0
\]

\[
(t \partial_t + \zeta \partial_\zeta + sg \partial_y - x(\Psi \partial_\Psi + \Psi^* \partial_{\Psi^*}) + (x + 2))F = 0
\]

\[
(t \partial_t - x(\Psi \partial_\Psi + \Psi^* \partial_{\Psi^*}) + (x + 2))F = 0
\]

(4.13)
to have the form

\[ F_{\text{alt}_1} = t^{-x-2} f_{\text{alt}_1} \left( t^x \Psi, \zeta^{-x} g, \frac{\Psi}{\Psi^*} \right) \]  \hspace{1cm} (4.14)

When we want to have also invariance under \( \tilde{\text{alt}}_1 \) the \( N \) operator gives \((t \partial_t - \zeta \partial_\zeta + k'_0 g \partial_g)F_{\text{alt}_1} = 0\). So in this case we have

\[ F_{\text{alt}_1} = a^{-x-2} f_{\text{alt}_1} \left( a^x \Psi, \frac{\Psi}{\Psi^*} \right), \quad a = t \zeta^{s/(s+k'_0)} g^{-1/(s+k'_0)} \]  \hspace{1cm} (4.15)

The nonlinear equation invariant under the same algebra is

\[ \left( 2\partial_\zeta \partial_t + 4y g \zeta \partial_\partial_t - \partial_r^2 \right) \Psi(\zeta, t, r, g) = F_{\text{alt}_1} (F_{\text{alt}_1}) \]  \hspace{1cm} (4.16)

and the results are again included in table 2. We point out that the form of \( F \) agrees with the special solution obtained before for \( \text{age}_1 \)-MMG representation.

4.3 Subalgebras \( \text{sch}_1 \) and \( \tilde{\text{sch}}_1 \)

Now the system is

\[ \partial_t F = 0 \ , \ \partial_\zeta F = 0 \ , \ \partial_r F = 0 \]

\[ (2yg \partial_g - x(\Psi \partial_\Psi + \Psi^* \partial_\Psi^*) + (x+2)) F = 0 \]  \hspace{1cm} (4.17)

which has the solution

\[ F_{\text{sch}_1} = g^{-(x+2)/(2y)} f_{\text{sch}_1} \left( g^{x/(2y)} \Psi, \frac{\Psi}{\Psi^*} \right) \]  \hspace{1cm} (4.18)

The result for the algebra \( \tilde{\text{sch}}_1 \) we obtain by adding the invariance under generator \( N \), which leads to the following equation

\[ \frac{k_0}{2y} \left( x + 2 - xu \frac{\partial}{\partial u} \right) f_{\text{sch}_1}(u, v) = 0 \ , \ u = g^{x/(2y)} \Psi \ , \ v = \Psi/\Psi^* \]  \hspace{1cm} (4.19)
with solutions:

i) in the case \( k_0 \neq 0 \) like (4.8).

ii) in the case \( k_0 = 0 \) like (4.18), but with a non-modified central generator \( N \).

The nonlinear equation has the generic form

\[
(2\partial_\zeta \partial_t - \partial_r^2)\Psi(t, r, \zeta, g) = F_{\text{sch}}
\]

(4.20)

5 Consequences and conclusion

Motivated by the problem to understand the form of the scaling functions of the two-time observables in phase-ordering kinetics, we have been led to reconsider the question of finding semilinear Schrödinger equations which are invariant under a conveniently chosen representation of the Schrödinger group. The main difficulty is that if one considers the ‘mass’ as a fixed constant, Galilei- together with spatial-translation invariance implies that such an equation should be invariant under simple phase shifts which severely restricts the possible form of a non-linear potential and in general is only possible for complex wave functions \( \Phi \). As we have seen, a possible way out of this difficulty is to consider the ‘mass’ as an additional dynamical variable and then to go over to a ‘dual’ formulation with a wave function \( \Psi = \Psi(\zeta, t, r) \), see eq. (1.7). In this way the Schrödinger algebra \( \text{sch}_d \) is actually embedded into a conformal algebra \( \text{conf}_{d+2} \) which naturally leads to the question of finding all semilinear Schrödinger equations \( \hat{S}\Psi = F(\Psi, \Psi^*) \) conditionally invariant under some parabolic or almost-parabolic subalgebra of \( \text{conf}_{d+2} \), see figure 1bcd.

We then considered two extensions by further giving up some other property which is habitually admitted:

1. non-hermitian representations were constructed in section 2 and the corresponding non-linear equations are found, see eqs. (2.23, 2.24, 2.25, 2.26).

2. a dimensionful coupling \( g \) was explicitly introduced into the potential. The classification of the differential operator representations which also leave the linear equation \( \hat{S}\Psi = 0 \) invariant is given in table 1 and the corresponding semi-linear Schrödinger equations \( \hat{S}\Psi = F \) are listed in table 2.

Besides this classification, we think the most remarkable result is that quite generally, \textit{real-valued} solutions of these invariant equations are obtained.

To illustrate the possible impact on the understanding of phase-ordering kinetics, we reconsider eq. (4.4) with \( p_{01} = 2y \) or else eq. (4.18) together with their auxiliary condition. We write the wave function as \( \Psi(\zeta, t, r, g) = g(1-2x)/(4y)\psi(\zeta, t, r) \) where \( \psi \) is a real-valued function which satisfies the equation

\[
(2\partial_\zeta \partial_t - \partial_r^2)\psi = g^{-5/(4y)} f(g^{1/(4y)}\psi) = \psi^5 \tilde{f}(g\psi^4y)
\]

(5.1)

where \( f \) is an arbitrary function and \( f(x) = x^5 \tilde{f}(x) \). Up to the Fourier/Laplace transform with respect to \( \zeta \), this is of the same form as the coarse-grained kinetic equation (1.4) habitually used to describe coarsening. Since quite different functions \( f \) can be described in terms of the same symmetry, this might provide an explanation for the well-established fact [11] that the long-time behaviour of correlators and response functions in phase-ordering kinetics is quite independent of the precise form of the potential \( V(\Phi) \). We shall elaborate on this elsewhere.
On the other hand, one may use these symmetries to reduce the nonlinear Schrödinger equation to a linear equation and hence obtain new explicit solutions, along the lines of [30]. We hope to return to this elsewhere.

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References

[1] S. Abriet and D. Karevski, Eur. Phys. J. B37, 43 (2004).
[2] S. Abriet and D. Karevski, Eur. Phys. J. B41, 79 (2004).
[3] I.S. Aranson and L. Kramer, Rev. Mod. Phys. 74, 100 (2002).
[4] A.O. Barut and R. Raczk, Theory of group representations and applications, Polish Science Publications (Varsovie 1980).
[5] F. Baumann, M. Henkel, M. Pleimling and J. Richert, submitted to J. Phys. A (2005); cond-mat/0504243.
[6] V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov, Adv. Theor. Math. Phys. 7, 711 (2004).
[7] G.W. Bluman and J.D. Cole, J. Math. Mech. 18, 1025 (1969).
[8] J.P. Bouchaud, in M.E.Cates and M.R.Evans (Eds), Soft and Fragile Matter, IOP Press, Bristol (2000).
[9] J. Bourgain, Global solutions of non-linear Schrödinger equations, Am. Math. Society (1999).
[10] C.D. Boyer, R.T. Sharp and P. Winternitz, J. Math. Phys. 17, 1439 (1976).
[11] A.J. Bray, Adv. Phys. 43, 357 (1994).
[12] A.J. Bray, in M.E.Cates and M.R.Evans (Eds), Soft and Fragile Matter, IOP Press, Bristol (2000).
[13] G. Burdet, M. Perrin and P. Sorba: Comm. Math. Phys. 34, 85 (1973);
G. Burdet, J. Patera, M. Perrin et P. Winternitz, Lie subalgebras and Schrödinger algebra (in French), preprint CRM-689 (fév 1997).
[14] V. Caudrelier, M. Minchev and E. Ragoucy, J. Phys. A37, L367 (2004).
[15] R. Cherniha and J.R. King, J. Phys. A33, 267 and 7839 (2000).
[16] R. Cherniha and J.R. King, J. Phys. A36, 405 (2003).
[17] R. Cherniha and M. Henkel, J. Math. Anal. Appl. 298, 487 (2004).
[18] F. Corberi, E. Lippiello and M. Zannetti, Phys. Rev. E65, 046136 (2002).
[19] A. Crisanti and F. Ritort, J.Phys. A36, R181 (2003).
[20] L.F. Cugliandolo, J. Kurchan and G. Parisi, J. Physique I4, 1641 (1994).
[21] L.F. Cugliandolo, *Dynamics of Glassy Systems*; cond-mat/0210312

[22] M. de Montigny, F.C. Khanna, A.E. Santana, E.S. Santos and J.D.M. Vianna, Ann. of Phys. 277, 144 (1999).

[23] P.A.M. Dirac, J. Math. Phys. 4, 901 (1963).

[24] V.K. Dobrev, H.D. Doebner and C. Mrugalla, Reports Math. Phys. 39, 201 (1997).

[25] V.K. Dobrev, H.D. Doebner and C. Mrugalla, Mod. Phys. Lett. A14, 1113 (1999).

[26] H.-D. Doebner and H.-J. Mann, J. Math. Phys. 36, 3210 (1995).

[27] I. Dornic, H. Chaté, J. Chave and H. Hinrichsen, Phys. Rev. Lett. 87, 045701 (2001).

[28] I. Dornic, *Thèse de doctorat*, (Nice et Saclay 2002).

[29] G.V. Dunne, R. Jackiw and C.A. Trugenberger, Ann. of Phys. 194, 197 (1989).

[30] P. Feinsilver, Y. Kocik and R. Schott, Fortschritte Physik 52, 343 (2004).

[31] R. Floreanini and L. Vinet, Lett. Math. Phys. 32, 37 (1994).

[32] C. Fronsdal, Rev. Mod. Phys. 37, 221 (1965); Phys. Rev. D10, 589 (1974); D12, 3819 (1975); D26, 1988 (1982); M. Flato and C. Fronsdal, Phys. Lett. 97B, 236 (1980).

[33] W.I. Fushchich, W.M. Shtelen and N.I. Serov, *Symmetry analysis and exact solutions of equations of nonlinear mathematical physics*, Kluwer (Dordrecht 1993).

[34] W.I. Fushchich and R.M. Cherniha, J. Phys. A28, 5569 (1995).

[35] C. Godrèche and J.M. Luck, J. Phys. A33, 1151 (2000).

[36] C. Godrèche and J.M. Luck, J. Phys. A33, 9141 (2000).

[37] C. Godrèche and J.M. Luck, J. Phys. Cond. Matt, 14, 1589 (2002).

[38] D. Giulini, Ann. of Phys. 249, 222 (1996).

[39] F. Güngör, J. Phys. A32, 977 (1999).

[40] C.R. Hagen, Phys. Rev. D5, 377 (1972).

[41] M. Hassaïne and P.A. Horváthy, Ann. of Phys. 282, 218 (2000); Phys. Lett. A279, 215 (2001).

[42] M. Henkel, J. Stat. Phys. 75, 1023 (1994).

[43] M. Henkel, M. Pleimling, C. Godrèche and J.-M. Luck, Phys. Rev. Lett. 87, 265701 (2001).

[44] M. Henkel, Nucl. Phys. B641, 405 (2002).

[45] M. Henkel and J. Unterberger, Nucl. Phys. B660, 407 (2003).

[46] M. Henkel and M. Pleimling, Phys. Rev. E68, 065101(R) (2003).

[47] M. Henkel and G.M. Schütz, J. Phys. A37, 591 (2004).

25
[48] M. Henkel, Adv. Solid State Phys. 44, 389 (2004).
[49] M. Henkel, A. Picone and M. Pleimling, Europhys. Lett. 68, 191 (2004).
[50] M. Henkel, cond-mat/0503739.
[51] P.C. Hohenberg and B.I. Halperin, Rev. Mod. Phys. 49, 435 (1977).
[52] V.G. Kac and A.K. Raina, Bombay lectures on heighest-weight representations of infinite-dimensional Lie algebras, World Scientific (Singapour 1987).
[53] H.A. Kastrup, Nucl. Phys. B7, 545 (1968).
[54] A.W. Knapp, Representation theory of semisimple groups: an overview based on examples, Princeton University Press (Princeton 1986).
[55] D. Levi and P. Winternitz, J. Phys. A22, 2915 (1989).
[56] D. Levi and P. Winternitz, nlin.SI/0502004.
[57] E. Lippiello and M. Zannetti, Phys. Rev. E61, 3369 (2000).
[58] E. Lorenz and W. Janke, to be published (2005).
[59] A. Medina et P. Revoy, Annales scient. école normale supérieure, 4e série, 18, 533 (1985).
[60] U. Niederer, Helv. Phys. Acta 45, 802 (1972).
[61] L. O’Raifeartaigh and V.V. Sreedhar, Ann. of Phys. 293, 215 (2001).
[62] M. Perroud, Helv. Phys. Acta 50, 233 (1977).
[63] A. Picone and M. Henkel, J. Phys. A35, 5575 (2002).
[64] A. Picone and M. Henkel, Nucl. Phys. B688, 217 (2004).
[65] R.O. Popovich, N.M. Ivanova and H. Eshraghi, J. Math. Phys. 45, 3049 (2004).
[66] G. Rideau and P. Winternitz, J. Math. Phys. 34, 558 (1993).
[67] A.D. Rutenberg and A.J. Bray, Phys. Rev. E51, 5499 (1995).
[68] J. Shatah and M. Struwe, Geometric wave equations, Courant Lecture Notes in Mathematics vol 2, American Mathematical Society (New York 2000).
[69] C. Sulem and P.-L. Sulem, The non-linear Schrödinger equation, Appl. Math. Sci. 139, Springer (1999).
[70] W. Zippold, R. Kühn and H. Horner, Eur. Phys. J. B13, 531 (2000).