Quantum state merging for arbitrarily-small-dimensional systems

Hayata Yamasaki and Mio Murao

Abstract—Quantum technology in the near future will facilitate information processing using quantum computers on the small and intermediate scale of up to several dozens of qubits. We investigate entanglement cost required for one-shot quantum state merging, aiming at quantum state transformation on this scale. In contrast to existing coding algorithms achieving nearly optimal approximate quantum state merging on a large scale, we construct algorithms for exact quantum state merging so that the algorithms are applicable to any given state of an arbitrarily-small-dimensional system. In the algorithms, entanglement cost can be reduced depending on a structure of the given state derived from the Koashi-Imoto decomposition. We also provide an improved converse bound for exact quantum state merging achievable for qubits. Our results are applicable to distributed quantum information processing and multipartite entanglement transformation on a small and intermediate scale.

Index Terms—Quantum state merging, multipartite entanglement transformation, small and intermediate scale.

I. INTRODUCTION

The ERA of small- and intermediate-scale quantum computers of up to several dozens of qubits is approaching due to advances in quantum technology. There exists, however, technical difficulty in increasing the number of low-noise qubits built in one quantum device [1]. For further scaling up, distributed quantum information processing using multiple quantum devices connected by a network for quantum communication is considered to be promising [2], [3]. Aimed at efficient quantum information processing, coding algorithms for quantum communication tasks in such a distributed setting should be designed to be suitable for transferring quantum states on this small and intermediate scale.

Quantum state merging [4], [5] is a task playing crucial roles in distributed quantum information processing [6]–[8] and multipartite entanglement transformations [9]–[13]. Originally, state merging, or state redistribution [14], [15] as a generalized task including state merging, was introduced in the context of quantum Shannon theory, and it has applied to the analyses of various tasks in quantum Shannon theory such as derivation of a capacity of noisy quantum channels [16]–[23]. In the task of state merging formulated using the framework of local operations and classical communication (LOCC) in the

original paper [4], two spatially separated parties A and B initially share an entangled resource state $|\psi^{AB}\rangle$ with the Schmidt rank $K$ denoted by the top blue circles to transfer the reduced state of $|\psi^{RAB}\rangle$ on A to B and obtain $|\psi^{RB'B}\rangle$ while $|\Phi^+\rangle$ with the Schmidt rank $L$ denoted by the bottom blue circles is also obtained.

Fig. 1. Exact state merging of a given state $|\psi^{RAB}\rangle$ denoted by the red circles. Parties A and B perform LOCC assisted by a maximally entanglement resource state $|\Phi^+_K\rangle^{AB}$ with the Schmidt rank $K$ denoted by the top blue circles to transfer the reduced state of $|\psi^{RAB}\rangle$ on A to B and obtain $|\psi^{RB'B}\rangle$ while $|\Phi^+\rangle$ with the Schmidt rank $L$ denoted by the bottom blue circles is also obtained.

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by quantum teleportation [25] to transfer the reduced state from \( A \) to \( B \). This trivial algorithm does not require the classical description, and as the result, it requires the same entanglement cost for every given state. Given the classical description, state merging is expected to consume less amount of entanglement than quantum teleportation in general. In state merging, entanglement cost can even be negative when the algorithm provides a net gain of shared entanglement.

While this type of asymptotic scenarios are well-established in quantum Shannon theory, there have also been studied zero-error scenarios [26], which are originally established in a classical setting by Shannon [27] and first introduced into a quantum setting in Ref. [28]. In the zero-error scenarios of classical source coding with decoder side information, optimal zero-error code design is proven to be \( NP \)-hard [29]. However, in classical coding theory, explicit construction of zero-error coding algorithms such as Shannon coding [30] and Huffman coding [31], if not necessarily optimal, establishes a foundation of theoretical analyses as well as practical applications. In this direction, explicit zero-error coding algorithms for classical source coding with decoder side information are given in Refs. [29], [32]–[37].

Aside from this regime where infinitely many copies of \( |\psi\rangle^{RAB} \) are given, another regime is the one-shot regime where only a single copy is given. The scenarios in the one-shot regime can also be classified into two scenarios: one is an exact scenario with zero error, and the other is an approximate scenario in which a nonzero error is tolerated for reducing entanglement cost. Analysis in the one-shot regime clarifies the structure of algorithms achieving the task at a single-copy level and is more relevant to practical situations such as distributed quantum information processing.

However, the existing algorithms for one-shot quantum state merging or redistribution [38]–[41] achieve near optimality only on a large scale relevant to a branch of quantum Shannon theory, one-shot quantum information theory, where the smooth conditional min- and max-entropies [50], [51] are used to evaluate entanglement cost. These algorithms also cause a nonzero approximation error in fidelity since the vital techniques for these algorithms, namely, one-shot decoupling [42] and the convex-split lemma [43], cannot avoid errors. As higher fidelity is pursued in state merging of a fixed single copy of \( |\psi\rangle^{RAB} \), entanglement cost required for the algorithms diverges to infinity. Hence, there always exists a region of error close to zero where the algorithms do not contribute to reducing the entanglement cost. Moreover, in cases where the system size for the reduced state of \( |\psi\rangle^{RAB} \) on \( A \) is as small as up to a few dozens of qubits, the algorithms require more entanglement cost than quantum teleportation even if the error tolerance is reasonably large (see Remark 2 in Sec. III-A for more discussion). In this sense, strategies in state merging to exploit the classical description of \( |\psi\rangle^{RAB} \) for reducing entanglement cost have not yet been established for arbitrarily-small-dimensional systems or arbitrarily high fidelity.

In this paper, we explicitly construct algorithms for one-shot state merging which have the following features:

1) Applicable to any state including small- and intermediate-scale states;
2) Achieving zero error, which satisfies arbitrarily high fidelity requirement;
3) Retaining the essential feature of state merging, that is, exploiting classical description of \( |\psi\rangle^{RAB} \) for reducing entanglement cost.

The tasks of one-shot state merging investigated in this paper are achieved exactly, that is, without approximation, which we call **exact state merging** (Fig. 1). Entanglement cost of our algorithms for exact state merging is not larger than, and can be strictly smaller than, the optimal entanglement cost of its inverse task, exact state splitting, depending on a decompositon of \( |\psi\rangle^{RAB} \) referred to in Ref. [52] as the Koashi-Imoto decomposition [52]–[55]. In the same way as the asymptotic scenarios, the entanglement cost of our algorithm can even be negative. In addition to providing achievability bounds for any state of an arbitrarily-small-dimensional system, we improve the existing converse bound given in terms of the conditional min- and max-entropy [38] and show that our converse bound is achievable when the state to be merged is represented by qubits.

This paper is organized as follows. In Sec. III we introduce definitions of exact state merging and provide a summary of the Koashi-Imoto decomposition. In Sec. III, we present our main results: Theorems 5 and 6 for achievability of exact state merging and Theorem 7 for converse. Implications are discussed in Sec. IV. Our conclusion is given in Sec. V. In Appendix A, exact state splitting is also analyzed, where Theorem 14 yields the optimal entanglement cost of exact state splitting.

II. Preliminaries

In this section, after presenting our notations in Sec. III-A, we define exact state merging in Sec. III-B. Then, we introduce the Koashi-Imoto decomposition in Sec. III-C.

### A. Notations

We represent a system indexed by \( X \) as a Hilbert space denoted by \( \mathcal{H}^X \). The set of density operators on \( \mathcal{H}^X \) is denoted by \( \mathcal{D}(\mathcal{H}^X) \). The set of bounded operators on \( \mathcal{H}^X \) is denoted by \( \mathcal{B}(\mathcal{H}^X) \). Superscripts of an operator or a vector represent the indices of the corresponding Hilbert spaces, e.g., \( \psi^{R_A} \in \mathcal{D}((\mathcal{H}^R \otimes \mathcal{H}^A)) \) for a mixed state and \( |\psi\rangle^{RAB} \in \mathcal{H}^R \otimes \mathcal{H}^A \otimes \mathcal{H}^B \) for a pure state. We may write an operator corresponding to a pure state as \( \psi^{RAB} := |\psi\rangle \langle \psi|^{RAB} \). A reduced state may be represented by superscripts if obvious, such as \( \psi^{R_A} := T^B \psi^{RAB} \). The identity operator and the identity map on \( \mathcal{H}^X \) are denoted by \( I^X \) and id\(^X\), respectively. In particular, to explicitly show the dimension of an identity operator, we use subscripts if necessary, e.g. the identity operator on \( \mathcal{H}_D^R \) of dimension \( D \) may be denoted by \( I^D_D \).

### B. Definition of exact state merging

Exact state merging involves three parties \( A, B, \) and \( R \), where \( R \) is a reference to consider purification. Let \( A \) have \( \mathcal{H}^A \) and \( \mathcal{H}^\overline{A} \), \( B \) have \( \mathcal{H}^B \), \( \mathcal{H}^\overline{B} \), and \( \mathcal{H}^\overline{D} \), and \( R \) have \( \mathcal{H}^R \), where
dim $\mathcal{H}^A = \dim \mathcal{H}^B$. We assume that the parties $A$ and $B$ can freely perform LOCC assisted by a maximally entangled resource state on $\mathcal{H}^A \otimes \mathcal{H}^B$ initially shared between $A$ and $B$. Regarding a formal definition of LOCC, refer to Ref. [56]. Note that $A$ and $B$ cannot perform any operation on $\mathcal{H}^R$.

We define the task of exact state merging as illustrated in Fig. 1. Initially, a possibly mixed state $\psi_{AB}^{RAB}$ is shared between $A$ and $B$, where $\psi_{AB}^{RAB}$ is a task for $A$ and $B$ to exactly transfer the reduced state $\psi_A$ from $A$ to $B$ and obtain $\psi_{R'B'B}^{RAB}$. The given state $\psi_{R'B'B}^{RAB}$ may have entanglement between $A$ and $B$, and hence $A$ and $B$ may also be able to distill this entanglement. Let $K$ denote the Schmidt rank of an initial resource state

$$|\psi_K^{AB}\rangle := \frac{1}{\sqrt{K}} \sum_{l=0}^{K-1} |l\rangle^A \otimes |l\rangle^B$$

with the Schmidt rank $L$ to be used in the future. If $\log_2 K - \log_2 L \geq 0$, $\log_2 K - \log_2 L$ is regarded as the amount of net entanglement consumption in exact state merging, and otherwise $\log_2 L - \log_2 K$ is regarded as the amount of net entanglement gain. In cases where $\log_2 K > 0$ and $\log_2 L > 0$, a part of entanglement in the initial resource state is interpreted to be used catalytically, where an initial resource state with larger $\log_2 K$ may be used to decrease $\log_2 K - \log_2 L$. We call this setting the catalytic setting. On the other hand, simply minimizing the amount of entanglement of the initial resource state may also be useful especially in the one-shot regime. Thus, we also consider another setting of fixing $\log_2 L = 0$ as a variant of exact state merging, where the catalytic use of shared entanglement is forbidden. We call such a task non-catalytic exact state merging.

**Definition 1. Exact state merging.** Exact state merging of a purified given state $|\psi_{RAB}\rangle$ is a task for parties $A$ and $B$ to achieve a transformation

$$id^R \otimes M \left( \psi_{RAB}^{RAB} \otimes \Phi_K^{AB} \right) = \psi_{R'B'B}^{RAB} \otimes \Phi_L^{AB}$$

by an LOCC map $M : \mathcal{B} \left( \mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^A \otimes \mathcal{H}^B \right) \rightarrow \mathcal{B} \left( \mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^A \otimes \mathcal{H}^B \right)$. The definition of non-catalytic exact state merging is also obtained by setting $\log_2 L = 0$ in the above definition.

The entanglement cost of exact state merging in the catalytic setting is defined as $\log_2 K - \log_2 L$, and that of non-catalytic exact state merging is defined as $\log_2 K$. If $\log_2 L \geq \log_2 \dim \mathcal{H}^A$, there exists a trivial algorithm for exact state merging by quantum teleportation to transfer $\psi_A$ from $A$ to $B$. Our results given in Sec. III provide algorithms at less entanglement cost.

**C. Koashi-Imoto decomposition**

We introduce the Koashi-Imoto decomposition. For a pure state $|\psi_{RAB}\rangle$ and a positive semidefinite operator $\Lambda^R \geq 0$ on $\mathcal{H}^R$, let $\psi_A (\Lambda^R) \in D (\mathcal{H}^A)$ denote a state obtained by performing a measurement of $\psi_{RA}^R$ on $\mathcal{H}^R$ and post-selecting an outcome corresponding to $\Lambda^R$, that is,

$$\psi_A (\Lambda^R) := \frac{\text{Tr}_R \left( \Lambda^R \otimes \mathbb{1}^A \right) \psi_{RA}^R}{\text{Tr} \left( \left( \Lambda^R \otimes \mathbb{1}^A \right) \psi_{RA}^R \right)}.$$

In addition, for a completely positive and trace-preserving (CPTP) map $T : B (\mathcal{H}^A) \rightarrow B (\mathcal{H}^A)$, we let $\mathcal{H}^{A'}$ denote an auxiliary system for the Stinespring dilation of $T$, and we write the Stinespring dilation as $T (\rho) = Tr_{A'} U_T \rho U_T^\dagger$, where $U_T$ is an isometry from $\mathcal{H}^A$ to $\mathcal{H}^{A'}$.

The Koashi-Imoto decomposition is first introduced in Ref. [53] to characterize a CPTP map $T$ which does not change any state in a given set $\{ \psi_i^A \in D (\mathcal{H}^A) : i \in I \}$. Note that the index set $I$ can be an infinite set. The Koashi-Imoto decomposition of a set of states is presented in the following lemma, of which an algorithmic proof is given in Ref. [53], and alternative proofs are given in Refs. [54], [55] through an operator-algebraic approach. Note that the Koashi-Imoto decomposition satisfying the second condition in the following lemma is uniquely determined, which is said to be maximal in Ref. [55].

**Lemma 2.** (Theorem 3 in Ref. [53], Theorem 9 in Ref. [54], and Lemma 6 in Ref. [55]) **Koashi-Imoto decomposition of a set of states.** Given any set $\{ \psi_i^A \in D (\mathcal{H}^A) : i \in I \}$, there exists a unique decomposition of $\mathcal{H}^A$

$$\mathcal{H}^A = \bigoplus_{j=0}^{J-1} \mathcal{H}^{A^L}_j \otimes \mathcal{H}^{A^R}_j$$

such that

1) For each $i \in I$, $\rho_A^i$ is decomposed into

$$\rho_A^i = \bigoplus_{j=0}^{J-1} p(j) \omega_{j}^{A^L} \otimes \delta_{i,j},$$

where $p(j)$ is a probability distribution and for each $j \in \{ 0, \ldots, J-1 \}$, $\omega_j^{A^L} \in D (\mathcal{H}^{A^L}_j)$ is independent of $i$, and $\delta_{i,j} \in D (\mathcal{H}^{A^L}_j)$ depends on $i$.

2) For any CPTP map $T : B (\mathcal{H}^A) \rightarrow B (\mathcal{H}^A)$, if $T$ leaves $\psi_i^A$ invariant for each $i \in I$, that is, $T (\psi_i^A) = \psi_i^A$, then the isometry $U_T$ for the Stinespring dilation of $T$ is decomposed into $U_T = \bigoplus_{j=0}^{J-1} U_j^{A^L} \otimes 1^{A^R}_j$, where, for each $j \in \{ 0, \ldots, J-1 \}$, $U_j^{A^L}$ is an isometry from $\mathcal{H}^{A^L}_j$ to $\mathcal{H}^{A^L}_j \otimes \mathcal{H}^{A^R}$ satisfying $Tr_{A'} U_T \omega_j^{A^L} U_T^\dagger = \omega_j^{A^L}$.

Applying the above lemma to the set of states $\{ \psi_A (\Lambda^R) : \Lambda^R \geq 0 \}$, where we regard the operator $\Lambda^R$ as the dual of the set, we obtain the Koashi-Imoto decomposition of a bipartite state $\psi_{RA}$ as presented in the following lemma shown in Ref. [54].
**Lemma 3.** (in Proof of Theorem 6 in Ref. [53]) Koashi-Imoto decomposition of a bipartite state. Given any bipartite state \( \psi^{RA} \in D (H^R \otimes H^A) \), there exists a decomposition of \( H^A \)

\[
H^A = \bigoplus_{j=0}^{J-1} H^{a_L} \otimes H^{a_R}
\]

such that \( \psi^{RA} \) is decomposed into

\[
\psi^{RA} = \sum_{j=0}^{J-1} p(j) |j\rangle \langle j|^{a_0} \otimes \omega_j^{a_L} \otimes \phi_j^{Ra_R},
\]

where \( p(j) \) is a probability distribution.

Introducing an auxiliary system \( H^{a_0} \), we can also write the above decomposition as

\[
(1^R \otimes U^A) \psi^{RA} (1^R \otimes U^A)^\dagger = \sum_{j=0}^{J-1} p(j) |j\rangle \langle j|^{a_0} \otimes \omega_j^{a_L} \otimes \phi_j^{Ra_R},
\]

where \( H^{a_0} \), \( H^{a_L} \), and \( H^{a_R} \) satisfy

\[
\dim H^{a_0} = J,
\]

\[
\dim H^{a_L} = \max_j \left\{ \dim H^{a_L}_j \right\},
\]

\[
\dim H^{a_R} = \max_j \left\{ \dim H^{a_R}_j \right\},
\]

\( U^A \) is an isometry from \( H^A \) to \( H^{a_0} \otimes H^{a_L} \otimes H^{a_R} \), and \( \{|j\rangle : j = 0, \ldots, J-1 \} \) is the computational basis of \( H^{a_0} \).

Considering a purification \( |\psi\rangle^{RAB} \) of the bipartite state \( \psi^{RA} \) in Lemma 3, we obtain the Koashi-Imoto decomposition of the tripartite pure state \( |\psi\rangle^{RAB} \) as presented in the following lemma shown in Ref. [53].

**Lemma 4.** (Lemma 11 in Ref. [52]) Koashi-Imoto decomposition of a tripartite pure state. Given any tripartite pure state \( |\psi\rangle^{RAB} \), there exists a decomposition of \( H^A \) and \( H^B \)

\[
H^A = \bigoplus_{j=0}^{J-1} H^{a_L} \otimes H^{a_R}, \quad H^B = \bigoplus_{j=0}^{J-1} H^{b_L} \otimes H^{b_R},
\]

such that \( |\psi\rangle \) is decomposed into

\[
|\psi\rangle = \sum_{j=0}^{J-1} \sqrt{p(j)} |j\rangle \langle j|^{a_0} \otimes |\omega_j\rangle^{a_L} \otimes |\phi_j\rangle^{Ra_R},
\]

where \( p(j) \) is a probability distribution.

Introducing auxiliary systems \( H^{a_0} \) and \( H^{b_0} \), we can also write the above decomposition as

\[
1^R \otimes U^A \otimes U^B |\psi\rangle^{RAB} = \sum_{j=0}^{J-1} \sqrt{p(j)} |j\rangle^{a_0} \otimes |j\rangle^{b_0} \otimes |\omega_j\rangle^{a_L} \otimes |\phi_j\rangle^{Ra_R} \otimes |\omega_j\rangle^{b_L} \otimes |\phi_j\rangle^{Ra_R},
\]

where \( H^{a_0} \), \( H^{b_0} \), and \( H^{a_R} \) satisfy

\[
\dim H^{a_0} = J,
\]

\[
\dim H^{b_0} = J,
\]

\[
\dim H^{a_R} = \max_j \left\{ \dim H^{a_L}_j \right\},
\]

\[
\dim H^{b_R} = \max_j \left\{ \dim H^{b_L}_j \right\},
\]

\( U^B \) is an isometry from \( H^B \) to \( H^{b_0} \otimes H^{b_L} \otimes H^{b_R} \), \( \{|j\rangle^{b_0} : j = 0, \ldots, J-1 \} \) is the computational basis of \( H^{b_0} \), and the other notations are the same as those in Eq. (1).

Consequently, to obtain the Koashi-Imoto decomposition of a given pure state \( |\psi\rangle^{RAB} \), apply the algorithm presented in Ref. [53] or the operator-algebraic theorems used in Refs. [54], [55] to the set of states \( \{ |\psi^A (\Lambda^R) : \Lambda^R \geq 0 \} \), and then follow the above argument.

## III. MAIN RESULTS

In this section, we first provide an algorithm achieving exact state merging in Sec. III-A. Then, we also analyze the converse bound for exact state merging in Sec. III-B.

### A. Achievability bound for exact state merging applicable to arbitrarily-small-dimensional systems

We provide algorithms for exact state merging applicable to any state of an arbitrarily-small-dimensional system, using the Koashi-Imoto decomposition introduced in Sec. II-C. Given any state \( |\psi\rangle^{RAB} \), Lemma 3 implies that \( H^A \) and \( H^B \) are uniquely decomposed into

\[
H^A = \bigoplus_{j=0}^{J-1} H^{a_L}_j \otimes H^{a_R}_j, \quad H^B = \bigoplus_{j=0}^{J-1} H^{b_L}_j \otimes H^{b_R}_j,
\]

and \( |\psi\rangle^{RAB} \) is uniquely decomposed into

\[
|\psi\rangle^{RAB} = \sum_{j=0}^{J-1} \sqrt{p(j)} |\omega_j\rangle^{a_L \otimes b_L} \otimes |\phi_j\rangle^{Ra_R \otimes b_R},
\]

where \( p(j) \) is a probability distribution. Also, for each \( j \in \{0, \ldots, J-1\} \), we write the reduced state of \( |\omega_j\rangle^{a_L \otimes b_L} \) on \( H^{a_L}_j \) as

\[
\omega_j^{a_L} := \text{Tr}_{b_L} |\omega_j\rangle \langle \omega_j|^{a_L}, \quad \omega_j^{b_L} = \sum_{l=0}^{\text{rank} \omega_j^{a_L} - 1} \lambda_l^{a_L} |l\rangle \langle l|,
\]

where \( l \in \{0, \ldots, \text{rank} \omega_j^{a_L} - 1\} \), the right hand side represents the spectral decomposition, and we let \( \lambda_0^{a_L} \) denote the largest eigenvalue of \( \omega_j^{a_L} \). Using the Koashi-Imoto decomposition, we provide an algorithm for exact state merging, which yields the following theorem.

**Theorem 5.** An achievability bound of entanglement cost of exact state merging applicable to arbitrarily-small-dimensional systems. Given any pure state \( |\psi\rangle^{RAB} \) and any \( \delta > 0 \), there exists an algorithm for exact state merging of \( |\psi\rangle^{RAB} \) achieving

\[
\log_2 K - \log_2 L \leq \max_j \left\{ \log_2 \left( \lambda_0^{a_L} \dim H^{a_R}_j \right) \right\} + \delta,
\]

where \( K \) and \( L \) are the entanglement costs for exact state merging applicable to arbitrarily-small-dimensional systems.
where the notations are the same as those in Eqs. (3), (4), and (5).

As for non-catalytic exact state merging, the entanglement cost \( \log_2 K \) of the initial resource state can be reduced compared to \( \log_2 K \) required for the algorithm in the catalytic setting in Theorem 5. Note that, however, \( \log_2 K \) for non-catalytic exact state merging may be more than the net entanglement cost \( \log_2 K - \log_2 L \) required for the algorithm in the catalytic setting in Theorem 5.

**Theorem 6.** An achievable bound of entanglement cost of non-catalytic exact state merging applicable to arbitrarily-small-dimensional systems. Given any pure state \( |\psi\rangle^{RAB} \), there exists an algorithm for non-catalytic exact state merging of \( |\psi\rangle^{RAB} \) achieving

\[
\log_2 K = \max_j \left\{ \log_2 \left[ \lambda_0^{a_j} \dim \mathcal{H}^{a_j} \right] \right\},
\]

where \([ \cdots ]\) is the ceiling function, and the other notations are the same as those in Theorem 5.

**Proof of Theorem 5** We construct an algorithm for exact state merging of \( |\psi\rangle^{RAB} \) achieving Inequality (6). We define

\[
j_0 := \arg\max_j \left\{ \log_2 \left( \lambda_0^{a_j} \dim \mathcal{H}^{a_j} \right) \right\}, \quad D^{a_j} := \dim \mathcal{H}^{a_j} \quad \text{for each } j \in \{0, \ldots, J-1\}.
\]

Hence, we have

\[
\lambda_0^{a_j} \leq D^{a_j} \lambda_0^{a_j} \lambda_0^{a_j}
\]

and since \( \lambda_0^{a_j} \in \mathbb{Q} \), there exist integers \( K_j \) and \( L_j \) such that the right hand side of the above inequality is written as

\[
\frac{D^{a_j}}{L_j} \lambda_0^{a_j} = K_j
\]

Therefore, we obtain

\[
\frac{\lambda_0^{a_j}}{K_j} \leq \frac{1}{L_j}.
\]

For each \( j \in \{0, \ldots, J-1\} \), the majorization condition for LOCC convertibility between bipartite pure states guarantees that there exists an LOCC map represented by a family of operators \( \{M_{j,m} \otimes U_{j,m}\}_{m_1} \) achieving, for each \( m_1 \),

\[
(M_{j,m} \otimes U_{j,m}) \left( |\omega_j\rangle^{a_j} \otimes |\Phi_{K_j}^{+}\rangle^{\lambda_j} \right) = |\Phi_{L_j}^{+}\rangle^{\lambda_j},
\]

where \( \{M_{j,m} \otimes U_{j,m}\}_{m_1} \) represents \( A \)'s measurement from \( \mathcal{H}^a \otimes \mathcal{H}^A \) to \( \mathcal{H}^A \) with outcome \( m_1 \) satisfying the completeness \( \sum_m M_{j,m} M_{j,m}^\dagger = 1 \), and \( U_{j,m} \) represents \( B \)'s isometry from \( \mathcal{H}^{b_j} \otimes \mathcal{H}^B \) to \( \mathcal{H}^B \) conditioned by \( m_1 \). Regarding an explicit form of \( \{M_{j,m} \otimes U_{j,m}\}_{m_1} \), refer to Refs. 57, 58.

**Subprocess 2: Quantum teleportation to transfer the quantum part.** Sending the full reduced state \( \phi_j^{R}\rangle := Tr_{R}\langle \phi_j^{R} \) is inefficient, and hence, we adopt a compression method instead of sending the full reduced state \( \phi_j^{R} \). Consider \( A \)'s auxiliary system \( \otimes_{j=0}^{J-1} \mathcal{H}^{(a_j)^R} \), where \( \dim \mathcal{H}^{(a_j)^R} = D^{a_j} \). In our algorithm, \( |\phi_j\rangle^{R_{a_j}b_j} \) is compressed into

\[
|\phi_j\rangle^{R_{a_j}b_j} = U_j |\phi_j\rangle^{R_{a_j}b_j},
\]

where \( U_j \) is an isometry from \( \mathcal{H}^{a_j} \) to \( \mathcal{H}^{(a_j)^R} \), and \( |\phi_j\rangle^{R(a_j)^R} \) represents the same state as \( |\phi_j\rangle^{R_{a_j}b_j} \). Quantum teleportation [25] to send states of \( \mathcal{H}^{(a_j)^R} \) consists of \( A \)'s projective measurement in the maximally entangled basis \( \{|\Phi_{j,m_2}\rangle_{m_2}\}_{m_2} \) on \( \mathcal{H}^{(a_j)^R} \otimes \mathcal{H}^A \) with outcome \( m_2 \) and \( B \)'s generalized Pauli correction \( \sigma_j m_2 \) from \( \mathcal{H}^B \) to \( \mathcal{H}^{(b_j)^R} \) conditioned by \( m_2 \), where \( \mathcal{H}^{(b_j)^R} \) is \( B \)'s auxiliary system corresponding to \( \mathcal{H}^{a_j} \). The map for quantum teleportation is represented by \( \{|\Phi_{j,m_2} \otimes \sigma_j m_2\rangle_{m_2}\}_{m_2} \), which traces out the post-measurement state of \( A \) and achieves, for each \( m_2 \),

\[
(|\Phi_{j,m_2} \otimes \sigma_j m_2\rangle_{m_2} \left( |\phi_j\rangle^{R_{a_j}b_j} \otimes |\Phi_{K_j}^{+}\rangle^{\lambda_j} \right) = |\phi_j\rangle^{R(b_j)^R} \right).
\]
**Subprocess 3: Coherently merging the classical part by a measurement.** As for the classical part \( \mathcal{H}^{a_0} \), a measurement should be performed by \( A \) to merge the classical part without breaking coherence between \( B \) and \( R \). This contrasts with the algorithm proposed in Ref. [59] for transferring a state drawn from a given ensemble, in which a projective measurement onto each of the subspaces of the Koashi-Imoto decomposition indexed by \( j \) in Lemma 2 destroys superposition of states among different subspaces. In our algorithm, \( A \)’s measurement on \( \mathcal{H}^{a_0} \) is a projective measurement with outcome \( m_3 \) in the Fourier basis \( \{ | m_3 \rangle \}_{m_3} \) defined in terms of the computational basis \( \{ | j \rangle^{a_0} \}_{j} \), that is, for each \( m_3 \),

\[
| m_3 \rangle^{a_0} := \sum_{j=0}^{J-1} \exp \left( \frac{i\pi jm_3}{J} \right) | j \rangle^{a_0}.
\]

We combine Subprocesses 1–3 using controlled measurements and controlled isometries. Regarding \( A \)’s measurement, the measurements used in Subprocesses 1 and 2 are performed by extending each measurement to a measurement controlled coherently by the computational-basis state \( | j \rangle^{a_0} \). Regarding Subprocess 1 for the redundant part, the controlled version of the measurement is given by \( \sum_{j=0}^{J-1} | j \rangle^{a_0} \otimes M_{j,m_1} \), and regarding Subprocess 2 for the quantum part, given by \( \sum_{j=0}^{J-1} | j \rangle^{a_0} \otimes (\Phi_{j,m_2} | U_j^\prime \rangle \rangle \). The measurement in Subprocess 3 for the classical part is also represented in terms of the computational basis as \( \sum_{j=0}^{J-1} \langle m_3 |^{a_0} \langle j |^{a_0} = \sum_{j=0}^{J-1} \exp \left( \frac{-i\pi jm_3}{J} \right) | j \rangle^{a_0} \). Combining these three together, we obtain \( A \)’s measurement \( \{ M_{m_1,m_2,m_3} \}_{m_1,m_2,m_3} \) given by

\[
M_{m_1,m_2,m_3} = \sum_{j=0}^{J-1} \exp \left( \frac{-i\pi jm_3}{J} \right) | j \rangle^{a_0} \otimes (\Phi_{j,m_2} | U_j^\prime M_{j,m_1} \rangle \rangle .
\]

The completeness of this measurement follows from

\[
\sum_{m_1,m_2,m_3} M_{m_1,m_2,m_3}^\dagger M_{m_1,m_2,m_3} = \sum_{m_1,m_2,m_3} \sum_{j,j'} \exp \left( \frac{i\pi jm_3 (j'-j)}{J} \right) | j' \rangle \langle j | \otimes \left[ M_{j,m_1}^\dagger | U_j^\prime \rangle \langle \Phi_{j,m_2} | U_j^\prime M_{j,m_1} \rangle \right]
\]

\[
= \sum_{j} | j \rangle \langle j | \otimes \left[ \sum_{m_1,m_2} M_{j,m_1}^\dagger | U_j^\prime \rangle \langle \Phi_{j,m_2} | U_j^\prime M_{j,m_1} \rangle \right] = 1,
\]

where \( \mathbb{I} \) is the identity operator on \( \mathcal{H}^{a_0} \otimes \mathcal{H}^{a_L} \otimes \mathcal{H}^{R} \otimes \mathcal{H}^{\mathcal{K}} \).

As for \( B \)’s state, the isometries in Subprocesses 1 and 2 are also controlled coherently by the computational-basis state \( | j \rangle^{b_0} \). Regarding Subprocess 1 for the redundant part, the controlled version of the isometry is given by \( \sum_{j=0}^{J-1} | j \rangle^{b_0} \otimes U_{j,m_1} \), and regarding Subprocess 2 for the quantum part, given by \( \sum_{j=0}^{J-1} | j \rangle^{b_0} \otimes \sigma_{j,m_2} \). The originally given state of \( \mathcal{H}^{a_0} \otimes \mathcal{H}^{a_L} \otimes \mathcal{H}^{b_L} \) can be recovered from \( B \)’s classical part \( \mathcal{H}^{b_0} \) of the post-measurement state of \( A \)’s measurement in Subprocess 3 by \( \sum_{j=0}^{J-1} \exp \left( \frac{i\pi jm_3}{J} \right) | j \rangle^{(b_0)'} \otimes | j \rangle^{(b_0)} \otimes | \omega_j \rangle | b_L \rangle \), where \( \mathcal{H}^{(b_0)'} \otimes \mathcal{H}^{(b_0) L} \) is \( B \)’s auxiliary system corresponding to \( \mathcal{H}^{a_0} \otimes \mathcal{H}^{a_L} \). Combining these three together, we obtain \( B \)’s isometry \( U_{m_1,m_2,m_3} \) given by

\[
U_{m_1,m_2,m_3} = \sum_{j=0}^{J-1} \exp \left( \frac{i\pi jm_3}{J} \right) | j \rangle^{(b_0)'} \otimes | j \rangle^{(b_0)} \otimes | \omega_j \rangle | b_L \rangle \otimes \sigma_{j,m_2} U_{j,m_1} .
\]

Consequently, for any combination \( (m_1, m_2, m_3) \), the LOCC map represented by a family of operators \( \{ M_{m_1,m_2,m_3} \otimes U_{m_1,m_2,m_3} \}_{m_1,m_2,m_3} \) acts as

\[
(M_{m_1,m_2,m_3} \otimes U_{m_1,m_2,m_3}) \left( | j \rangle^{a_0} \otimes | j \rangle^{b_0} \otimes | \omega_j \rangle | a_L b_L \rangle \otimes | \phi_j \rangle \right) R^{a_R} \otimes | \Phi^+_L \rangle \langle \Phi^+_L \rangle_{\mathcal{B}}.
\]

For each \( j \), the entanglement cost is evaluated by

\[
\log_2 D^{a_R} + \log_2 K_j - \log_2 L_j
\]

\[
= \log_2 \left( K_j \right) \binom{D^{a_R}}{L_j}
\]

\[
= \log_2 \left( \lambda_{0_L}^{a_L} D^{a_L_{0_L}} \right),
\]

which is independent of \( j \). Choosing \( K \) as the least common multiple of the integers \( \{ D^{a_R}, K_0, \ldots, D^{a_{R-1}}, K_{j-1} \} \), we obtain an LOCC map represented by

\[
\{ \left[ M_{m_1,m_2,m_3} U^A \right] \otimes \left[ \left( T^{B_R} \right)^{\dagger} \otimes \left( T^{B} \right)^{\dagger} \right] U_{m_1,m_2,m_3} U^{B} \} \}_{m_1,m_2,m_3,}
\]

which achieves, for each \( (m_1, m_2, m_3) \),

\[
[M_{m_1,m_2,m_3} U^A] \otimes \left[ \left( T^{B_R} \right)^{\dagger} \otimes \left( T^{B} \right)^{\dagger} \right] U_{m_1,m_2,m_3} U^{B} \]

\[
\psi^{RAB} \otimes | \Phi^+_K \rangle \langle \Phi^+_K |_{\mathcal{B},}
\]

\[
= | \psi^{R' B'} \rangle \otimes | \Phi^+_L \rangle \langle \Phi^+_L |_{\mathcal{B}},
\]

where \( U^A \) and \( U^B \) are those in Eq. (2), and \( \left( T^{B_R} \right)^{\dagger} \) from \( \mathcal{H}^{(b_0)'} \otimes \mathcal{H}^{(b_0) L} \otimes \mathcal{H}^{(b_0) R} \to \mathcal{H}^{B'} = \oplus_{j=0}^{J-1} \mathcal{H}^{(b_L)_j} \otimes \mathcal{H}^{(b_R)_j} \) acts in the same way as \( (U^A)^{\dagger} \). The entanglement cost of the LOCC map represented by Eq. (8) is given by

\[
\log_2 K - \log_2 L
\]

\[
= \log_2 \left( \lambda_{0_L}^{a_L} D^{a_L_{0_L}} \right)
\]

\[
\leq \log_2 \left( \lambda_{0_L}^{a_L} D^{a_L_{0_L}} \right) + \delta
\]

\[
= \max_{j} \left\{ \log_2 \left( \lambda_{0_L}^{a_L} \dim \mathcal{H}^{b_L}_0 \right) \right\} + \delta,
\]

which yields the conclusion. Q.E.D.
Proof of Theorem 6. We construct an algorithm for non-catalytic exact state merging of $|\psi\rangle^{RAB}$ achieving Eq. (7). We define, for each $j \in \{0, \ldots, J-1\}$,
\[ D^a_j := \dim H^a_j. \]
We omit identity operators, such as $1_R$, in the following for brevity. The core idea of the algorithm is similar to that in Theorem 5 using the Koashi-Imoto decomposition in the form of Eq. (2). The rest of the proof is given in the same way as the proof of Theorem 5 where Subprocess 2 and Subprocess 3 are the same as those in Theorems 5 and Subprocess 1 is modified as follows since we do not use the resource state catalytically in the entanglement distillation from the redundant part in Subprocess 1.

Subprocess 1: For each $j \in \{0, \ldots, J-1\}$, it holds that
\[ \lambda_0^j D^a_j \leq \left| \lambda_0^j D^a_j \right| \leq \max_j \left\{ \left| \lambda_0^j D^a_j \right| \right\}. \]
Then, given the resource state $|\Phi^+_K\rangle$, where
\[ K = \max_j \left\{ \lambda_0^j D^a_j \right\}, \]
we have
\[ \frac{\lambda_0^j}{K} \leq \frac{1}{D^a_j}. \]
For each $j \in \{0, \ldots, J-1\}$, the majorization condition for LOCC convertibility between bipartite pure states [57] guarantees that there exists an LOCC map represented by a family of operators $\{M_{j,m_1} \otimes U_{j,m_1}\}_{m_1}$ achieving, for each $m_1$,
\[ (M_{j,m_1} \otimes U_{j,m_1})(|\omega_j\rangle^{n_k} b^L \otimes |\Phi^+_K\rangle^{x \otimes b}) = \left| \Phi^+_K\right|^{x \otimes b} \]
where $\{M_{j,m_1}\}_{m_1}$ represents $A$’s measurement from $H^A \otimes H^X$ to $H^A$ with outcome $m_1$ satisfying the completeness $\sum_{m_1} M_{j,m_1}^\dagger M_{j,m_1} = 1$, and $U_{j,m_1}$ represents $B$’s isometry from $H^B \otimes H^Y$ to $H^Y$ conditioned by $m_1$.

In the same way as Theorem 5 A’s combined measurement $\{(m_1, m_2, m_3)\}_{m_1,m_2,m_3}$, where the post-measurement state is traced out, is given by
\[ \langle m_1, m_2, m_3 \rangle \]
\[ = \sum_{j=0}^{J-1} \exp \left( -\frac{i\pi j m_3}{J} \right) \langle j|^{a_0} \otimes [\Phi^+_{j,m_2} U_{j,m_1}]. \]
Also, $B$’s combined isometry $U_{m_1,m_2,m_3}$ is given by
\[ U_{m_1,m_2,m_3} \]
\[ = \sum_{j=0}^{J-1} \exp \left( \frac{i\pi j m_3}{J} \right) |j\rangle^{(b_0)} \otimes |j\rangle^{b_0} \otimes |\omega_j\rangle^{(b'_L) k'} \]
\[ \otimes \sigma_{j,m_2 U_{j,m_1}.} \]
Consequently, we obtain an LOCC map represented by
\[ \left\{ \left[ \left( (U_{B'})^\dagger \otimes (U_B)^\dagger \right) U_{m_1,m_2,m_3} U_B \right] \right\}_{m_1,m_2,m_3}, \]
which achieves, for any combination $(m_1, m_2, m_3)$,
\[ \left[ \langle m_1, m_2, m_3 | U^A \rangle \otimes \left[ \left( (U_{B'})^\dagger \otimes (U_B)^\dagger \right) U_{m_1,m_2,m_3} U_B \right] \right] \]
\[ = |\psi\rangle^{RAB} \otimes |\Phi^+_K\rangle^{x \otimes b} \]
where $U_A$, $U_B$, and $U_{B'}$ are the same as those in Eq. (8). The entanglement cost of the LOCC map represented by Eq. (9) is given by
\[ \log_2 K \]
\[ = \max_j \left\{ \log_2 \left[ \lambda_0^j D^a_j \right] \right\} \]
\[ = \log_2 \max_j \left\{ \log_2 \left[ \lambda_0^j \dim H^a_j \right] \right\}, \]
which yields the conclusion.

Q.E.D.

Remark 1. Comparison between exact state merging and splitting. Entanglement cost of exact state merging is not larger than that of its inverse task, that is, exact state splitting analyzed in Appendix A. For any $|\psi\rangle^{RAB}$,
\[ \max_j \left\{ \log_2 \lambda_0^j (\dim H^a_j) \right\} \leq \log_2 \text{rank} \psi^A, \]
\[ \max_j \left\{ \log_2 \left[ \lambda_0^j (\dim H^a_j) \right] \right\} \leq \log_2 \text{rank} \psi^A, \]
where the right hand sides are the optimal entanglement cost of exact state splitting obtained in Theorem 14 in Appendix A-B and the notations are the same as those in Theorems 5 and 6. These inequalities can be derived from dim $H^a_j \leq$ rank $\psi^A$ and $\lambda_0^j \leq 1$, where the former inequality holds by construction of the Koashi-Imoto decomposition. Moreover, as shown in Implication 1 in Sec. IV, entanglement cost of exact state merging can be strictly smaller than that of spitting.

Remark 2. Usefulness of the algorithms for exact state merging on a small and intermediate scale. We discuss the cases where the obtained exact algorithms for one-shot state merging outperforms the existing approximate algorithms [38–49].

For a given approximation error $\epsilon > 0$, the approximate algorithms for one-shot state merging of $|\psi\rangle^{RAB}$ yield a final state $|\psi_{\text{final}}\rangle$ satisfying $F^2(|\psi\rangle \langle \psi|, |\psi_{\text{final}}\rangle : |\psi\rangle_{\text{final}} \rangle \geq 1 - \epsilon^2$, where $F$ represents the fidelity. While some of the existing algorithms are fully quantum algorithms implemented by local operations and quantum communication assisted by shared entanglement, we replace the quantum communication in a fully quantum algorithm with quantum teleportation to obtain an entanglement-assisted LOCC algorithm corresponding to the fully quantum algorithm and compare entanglement cost $E(\psi)$ in the LOCC framework.

Our algorithms for exact state merging of $|\psi\rangle^{RAB}$ require at most as much entanglement cost as quantum teleportation of $\psi^A$, and when the system size for $\psi^A$ is small, our algorithms cost less than the existing approximate algorithms. Regarding the existing algorithms, the achievability bounds of $E(\psi)$ of the corresponding entanglement-assisted LOCC algorithms can be calculated from the analyses in Refs. [38], [40–44].
Given $\epsilon > 0$, these achievability bounds are in the form $E(\psi) = \cdots + O(\log \frac{1}{\epsilon})$ as $\epsilon \to 0$, which diverges to infinity as higher fidelity is pursued. For example, from Theorem 4 in Ref. [44], the achievability bound of $E(\psi)$ of one-shot state merging of $|\psi\rangle^{RAB}$ within an error $\epsilon > 0$ is given by

\[ H^\epsilon_{\max}(A|B)_\psi + 2 \log_2 \frac{1}{\epsilon^4} + 3, \]

where $\epsilon = 8\epsilon_1 + \sqrt{3\epsilon_4}$, and the first term is represented by the smooth conditional max-entropy \[50], \[51]. To achieve $\epsilon = 0.02$, the second and third terms amount to

\[ 2 \log_2 \frac{1}{\epsilon^4} + 3 > 28.7. \]

Note that $\epsilon = 0.02$ guarantees, in the task of state discrimination of $|\psi\rangle$ and $\psi_{\text{final}}$, the optimal success probability $P_{\text{succ}} = \frac{1}{2} + \frac{1}{4} \| \psi - \psi_{\text{final}} \|_1$ ≤ 51%, which is obtained from the Fuchs-van de Graaf inequalities $\frac{1}{2} \| \psi - \psi_{\text{final}} \|_1 \leq \frac{1}{2} \sqrt{I - F^2}$ [60]. Thus, given $|\psi\rangle^{RAB}$ where $\dim \mathcal{H}^A \leq 2^{28}$, even if $H^\epsilon_{\max}(A|B)_\psi = 0$, the approximate algorithm requires more entanglement cost than our algorithms and even than quantum teleportation.

B. Improved converse bound for exact state merging

We provide a converse bound of entanglement cost of exact state merging. This converse bound improves the existing converse bound in terms of conditional max-entropy originally shown in Ref. [38]. In this section, after showing our bound, we compare the bound with the existing bound and then discuss the tightness of the bound.

Given any $|\psi\rangle^{RAB}$, we analyze the converse bound for exact state merging of a maximally entangled state corresponding to $|\psi\rangle^{RAB}$ instead of analyzing $|\psi\rangle^{RAB}$ itself, which is justified as follows. To define the maximally entangled state corresponding to $|\psi\rangle^{RAB}$, let $D$ denote the Schmidt rank of $|\psi\rangle^{RAB}$ with respect to bipartition between $\mathcal{H}^R$ and $\mathcal{H}^A \otimes \mathcal{H}^B$, and the Schmidt decomposition of $|\psi\rangle^{RAB}$ is written as

\[ |\psi\rangle^{RAB} = \sum_{l=0}^{D-1} \sqrt{\lambda_l} |l\rangle^R \otimes |\psi_l\rangle^{AB}, \]

where $\lambda_l > 0$ for each $l \in \{0, \ldots, D - 1\}$. Entanglement cost of exact state merging of $|\psi\rangle^{RAB}$ is the same as that of the corresponding $D$-dimensional maximally entangled state

\[ |\Phi^+_{D}(\psi)\rangle^{RAB} := \sum_{l=0}^{D-1} \frac{1}{\sqrt{D}} |l\rangle^R \otimes |\psi_l\rangle^{AB}, \]

which is shown in Proposition 13 in Appendix B. Thus, in the following analysis of converse bounds for exact state merging, we assume that $\psi^R = \frac{1}{\sqrt{D}}$ holds for a given state $|\psi\rangle^{RAB}$.

We provide our converse bound for exact state merging as follows.

**Theorem 7.** A converse bound of entanglement cost of exact state merging. For any algorithm for exact state merging of $|\psi\rangle^{RAB}$ where $\psi^R = \frac{1}{\sqrt{D}}$,

\[ \log_2 K - \log_2 L \geq \log_2 \left( \lambda_0^B D \right), \]

where $\lambda_0^B$ is the largest eigenvalue of $\psi^R$. For any algorithm for non-catalytic exact state merging of $|\psi\rangle^{RAB}$ where $\psi^R = \frac{1}{\sqrt{D}}$,

\[ \log_2 K \geq \log_2 \left[ \lambda_0^B D \right], \]

where $[\cdots]$ is the ceiling function, and $\lambda_0^B$ is the same as that in Eq. (11).

**Proof of Eq. (11):** Any algorithm for exact state merging transforms $|\psi\rangle^{RAB} \otimes |\Phi^+_{\ell}(K)\rangle^{AB}$ into $|\psi\rangle^{RAB} \otimes |\Phi^+_{\ell}(L)\rangle^{AB}$ by LOCC. Hence, with respect to the bipartition between $\mathcal{H}^R \otimes \mathcal{H}^A \otimes \mathcal{H}^B$, the majorization condition for LOCC convertibility between bipartite pure states \[57] yields

\[ \frac{\lambda_0^B}{K} \leq \frac{1}{D}. \]

Therefore,

\[ \log_2 K - \log_2 L \geq \log_2 \left( \lambda_0^B D \right). \]

Q.E.D.

**Proof of Eq. (12):** From the same argument as the above, we obtain

\[ \frac{\lambda_0^B}{K} \leq \frac{1}{D}. \]

Hence,

\[ K \geq \lambda_0^B D, \]

and since $K$ is an integer, we have

\[ K \geq \left\lceil \lambda_0^B D \right\rceil. \]

Therefore,

\[ \log_2 K \geq \log_2 \left[ \lambda_0^B D \right]. \]

Q.E.D.

A converse bound of entanglement cost of exact state merging in terms of the conditional max-entropy has been derived in Ref. [38] as follows.

**Lemma 8.** (Corollary 4.12. in Ref. [38]) An existing converse bound of entanglement cost of exact state merging. For any algorithm for exact state merging of $|\psi\rangle^{RAB}$,

\[ \log_2 K - \log_2 L \geq H_{\max}(A|B)_\psi, \]

where the right hand side is the conditional max-entropy \[50], \[51].

For states in the form (10), our converse bound in Theorem 7 is at least as tight as the existing bound in Lemma 8 as stated in the following proposition. Moreover, Implication 3 in Sec. IV shows a case where our bound is strictly tighter than the existing bound.

**Proposition 9.** Comparison of converse bounds of entanglement cost of exact state merging. For any pure state $|\psi\rangle^{RAB}$ where $\psi^R = \frac{1}{\sqrt{D}}$,

\[ \log_2 \left( \lambda_0^B D \right) \geq H_{\max}(A|B)_\psi, \]

where the notations are the same as those in Theorem 7 and Lemma 8.
Proposition 10. A necessary and sufficient condition for non-catalytic exact state merging by one-way LOCC. Given any pure state \( |\psi\rangle^{RAB} \) where \( \psi^R = \frac{1}{\sqrt{D}} \), there exists one-way LOCC map \( \mathcal{M}^{A\rightarrow B} \) from \( A \) to \( B \) achieving

\[
\text{id}^R \otimes \mathcal{M}^{A\rightarrow B} \left( |\psi\rangle^{RAB} \otimes \Phi_K^{+\mathcal{H}B} \right) = |\psi\rangle^{RB'B} \]

if and only if there exists a mixed-unitary channel \( \mathcal{U}(\rho) = \sum_m p(m) U_m \rho U^\dagger_m \) [60], where \( p(m) \) is a probability distribution and \( U_m \) for each \( m \) is a unitary, achieving

\[
\text{id}^R \otimes \mathcal{U} \left( |\psi\rangle^{RB} \otimes \frac{1}{\sqrt{D}} \right) = |\psi\rangle^{RB} \otimes \frac{1}{\sqrt{D}} \sum_{l=0}^{D-1} |l\rangle^R \otimes |l\rangle^B. \tag{13}
\]

Proof: If part: Assume that

\[
|\psi\rangle^{RB} \otimes \frac{1}{\sqrt{D}} = \sum_m p(m) \left( |\psi\rangle^{R} \otimes |l\rangle^B_m \right) \Phi_D^{+RB} \left( |\psi\rangle^{R} \otimes |l\rangle^B_m \right)^\dagger.
\]

A purification yields

\[
\left( |\psi\rangle^{R} \otimes |l\rangle^B \right) \left( \Phi_D^{+RB} \otimes \mathcal{H}^\beta \right) = \sum_m \sqrt{p(m)} |m\rangle^A \otimes \mathcal{U} \left( |\psi\rangle^{R} \otimes |l\rangle^B \right) \Phi_D^{+RB},
\]

where \( \mathcal{H}^A \) is \( A \)'s auxiliary system, and \( \mathcal{U} \) is an isometry performed by \( A \). Hence, a one-way LOCC map from \( A \) to \( B \) represented by \( \left\{ \left( |m\rangle^A \otimes |l\rangle^B \right) \left( \Phi_D^{+RB} \otimes \mathcal{H}^\beta \right) \right\} \), where the post-measurement state of \( A \) is traced out, achieves, for each \( m \),

\[
|\psi\rangle^{R} \otimes \left( \left( |m\rangle^A \otimes |l\rangle^B \right)^\dagger \right) \left( \Phi_D^{+RB} \otimes \mathcal{H}^\beta \right) \propto |\psi\rangle^{R} \otimes |l\rangle^B,
\]

and \( |\psi\rangle^{R} \otimes |l\rangle^B \) on the right hand side can be transformed into \( |\psi\rangle^{RB'B} \) by \( B \)'s local isometry.

Only if part: Assume that there exists \( A \)'s positive operator-valued measure (POVM [23]) \( \{ \Lambda_m \} \) on \( \mathcal{H}^A \otimes \mathcal{H}^B \) satisfying for each \( m \)

\[
\text{Tr}_A \left[ \left( |\Lambda_m\rangle \langle \Lambda_m | \right) \left( |\psi\rangle^{RAB} \otimes \Phi_K^{+AB} \right) \right] = p(m) \left( |\psi\rangle^{R} \otimes U_m^B \right) \Phi_D^{+RB} \left( |\psi\rangle^{R} \otimes U_m^B \right)^\dagger,
\]

where \( p(m) \) is a probability distribution, and \( U_m^B \) is \( B \)'s unitary correction conditioned by \( m \). Note that \( \Phi_D^{+RB} \) on the right hand side can be transformed into \( |\psi\rangle^{RB'B} \) by \( B \)'s local isometry. Then, we obtain

\[
|\psi\rangle^{RB} \otimes \frac{1}{\sqrt{D}} \sum_{l=0}^{D-1} |l\rangle^B \otimes |l\rangle^B = \psi^{RB'B} \otimes \Phi^{+RB'}, \tag{Q.E.D.}
\]

Note that it is straightforward to generalize the above proof of Proposition 10 on non-catalytic exact state merging to the catalytic setting, that is,

\[
\text{id}^R \otimes \mathcal{M}^{A\rightarrow B} \left( |\psi\rangle^{RAB} \otimes \Phi_K^{+\mathcal{H}B} \right) = |\psi\rangle^{RB'B} \otimes \Phi^{+RB'},
\]

which is also shown for quantum state redistribution in the approximate scenarios [62].

For qubits, our bound in Theorem 7 is tight enough to provide the optimal entanglement cost as stated in the following. Note that an equivalent condition in terms of Schmidt coefficients of \( |\psi\rangle^{AB} \) in Eq. (10) is also given in Theorem II.1. in Ref. [63].

Theorem 11. Optimal entanglement cost of non-catalytic exact state merging for qubits. Consider any three-qubit pure state \( |\psi\rangle^{RAB} \in \mathbb{C}^2 \otimes \mathbb{C}^2 \) where \( \psi^R = \frac{1}{\sqrt{2}} \), non-catalytic exact state merging of \( |\psi\rangle^{RAB} \) is achievable if and only if

\[
\log_2 K \geq \log_2 \left[ \lambda_0^{B} \right],
\]

where the notations are the same as those in Theorem 7. Equivalently, non-catalytic exact state merging of \( |\psi\rangle^{RAB} \) where \( \psi^R = \frac{1}{\sqrt{2}} \) is achievable at entanglement cost \( \log_2 K = 0 \) if and only if \( \psi^B = \frac{1}{\sqrt{2}} \), and otherwise entanglement cost \( \log_2 K = 1 \) is required.

Proof: If part: We assume that \( \psi^B = \frac{1}{\sqrt{2}} \) and show the existence of an LOCC algorithm for exact state merging of \( |\psi\rangle^{RAB} \) achieving \( \log_2 K = 0 \) since otherwise quantum teleportation of \( \psi^A \) achieves \( \log_2 K = 1 \). To show the
existence of the LOCC algorithm, Proposition 10 implies that it is sufficient to prove the existence of a mixed-unitary channel \( U \) achieving
\[
\text{id}^R \otimes U^B \left( \Phi_2^{RB} \right) = \psi^{RB}.
\]

Note that \( \mathcal{H}^B \) in Eq. (13) in Proposition 10 is simply written as \( \mathcal{H}^B = \mathcal{H}^B \) in this proof.

Given \( \psi^{RB} \) where \( \psi = \frac{1}{\sqrt{2}} \), we can regard \( \psi^{RB} \) as a normalized operator of the Choi operator \([60]\) of a CPTP map \( U^B \). Tracing out \( R \) for \( \psi^{RB} \) yields
\[
U^B \left( \frac{1}{2} I^B \right) = \psi^B = \frac{1}{2} I^B,
\]
that is, \( U^B \) is unital. Since any unital channel on a qubit is a mixed-unitary channel \([60]\), \( U^B \) is a mixed-unitary channel, which yield the conclusion. Q.E.D.

As for qudits of more than two dimension, our bound in Theorem 5 is not necessarily achievable since there exists an example of non-catalytic exact state merging which does not satisfy the equality of \( [12] \). We show a three-qutrit state of which any one-way LOCC algorithm for non-catalytic exact state merging fails to achieve
\[
\log_2 K = \log_2 \left[ \lambda_0^B D \right],
\]
where the notations are the same as those in Theorem 5.

**Proposition 12.** Impossibility of achieving the converse bound of entanglement cost of non-catalytic exact state merging for qudits. There exists a three-qutrit pure state \( |\psi\rangle^{RAB} \in (C^3)^{\otimes 3} \) where \( \psi^R = \frac{1}{\sqrt{D}} \) and \( D = 3 \), such that non-catalytic exact state merging of \( |\psi\rangle^{RAB} \) cannot be achieved by any one-way LOCC algorithm at entanglement cost
\[
\log_2 K = \log_2 \left[ \lambda_0^B D \right],
\]
where the notations are the same as those in Theorem 5.

**Proof:** Consider a CPTP map
\[
\mathcal{N}(\rho) = \frac{1}{2} \left( \text{Tr} \rho I - \frac{1}{2} \rho^T \right),
\]
where \( \rho^T \) is transpose of \( \rho \) with respect to the computational basis. The Choi operator of \( \mathcal{N} \) is written as
\[
J(\mathcal{N}) := \left( \frac{1}{\sqrt{2}} |21\rangle - \frac{1}{\sqrt{2}} |12\rangle \right) \left( \frac{1}{\sqrt{2}} |21\rangle - \frac{1}{\sqrt{2}} |12\rangle \right)^\dagger + \\
\frac{1}{\sqrt{2}} |02\rangle - \frac{1}{\sqrt{2}} |20\rangle \left( \frac{1}{\sqrt{2}} |02\rangle - \frac{1}{\sqrt{2}} |20\rangle \right)^\dagger + \\
\frac{1}{\sqrt{2}} |10\rangle - \frac{1}{\sqrt{2}} |01\rangle \left( \frac{1}{\sqrt{2}} |10\rangle - \frac{1}{\sqrt{2}} |01\rangle \right)^\dagger.
\]
This map \( \mathcal{N} \) is a unital channel but not a mixed-unitary channel \([60], [64]\).

Consider
\[
\psi^{RB} = \frac{J(\mathcal{N})}{3}.
\]
A purification of \( \psi^{RB} \) is
\[
|\psi\rangle^{RAB} = \frac{1}{\sqrt{3}} |0\rangle^A \otimes \left( \frac{1}{\sqrt{2}} |21\rangle - \frac{1}{\sqrt{2}} |12\rangle \right)^{RB} + \\
\frac{1}{\sqrt{3}} |1\rangle^A \otimes \left( \frac{1}{\sqrt{2}} |02\rangle - \frac{1}{\sqrt{2}} |20\rangle \right)^{RB} + \\
\frac{1}{\sqrt{3}} |2\rangle^A \otimes \left( \frac{1}{\sqrt{2}} |10\rangle - \frac{1}{\sqrt{2}} |01\rangle \right)^{RB}.
\]
For this state,
\[
\psi^R = \frac{1}{3} I^R,
\]
\[
\psi^B = \frac{1}{3} I^B.
\]
Hence,
\[
\log_2 \left[ \lambda_0^B D \right] = 0,
\]
where \( D = 3 \).

We assume that there exists a one-way LOCC algorithm for non-catalytic exact state merging of \( |\psi\rangle^{RAB} \) at entanglement cost \( \log_2 K = 0 \) to derive a contradiction. Due to Proposition 10, this assumption is equivalent to the existence of a mixed-unitary channel \( U \) such that
\[
\text{id}^R \otimes U^B \left( \Phi_3^{RB} \right) = \psi^{RB} = \frac{J(\mathcal{N})}{3},
\]
where, in the same way as Eq. (14), \( \mathcal{H}^B \) in Eq. (13) in Proposition 10 is written as \( \mathcal{H}^B \). Therefore, \( \mathcal{N} = U \) is necessary, which contradicts to the fact that \( \mathcal{N} \) is not a mixed-unitary channel, and we obtain the conclusion. Q.E.D.

IV. IMPLICATIONS

We discuss implications of our main results. In the following, we omit \( \otimes \). We define
\[
|+\rangle := \frac{1}{\sqrt{2}} \left( |0\rangle + |1\rangle \right),
\]
\[
|\Psi^\pm\rangle := \frac{1}{\sqrt{2}} \left( |0\rangle |1\rangle \pm |1\rangle |0\rangle \right),
\]
\[
|\Phi^\pm\rangle := \frac{1}{\sqrt{2}} \left( |0\rangle |0\rangle \pm |1\rangle |1\rangle \right).
\]

**Implication 1.** Reduced entanglement cost of exact state merging by performing a measurement on the classical part followed by classical communication. Consider a three-qutrit pure state
\[
|\psi\rangle^{RAB} = \frac{1}{\sqrt{3}} \sum_{l=0}^{2} |l\rangle^R |l\rangle^A |l\rangle^B.
\]
Quantum teleportation of \( \psi^A \) requires \( \log_2 3 \) ebits, that is, \( |\Phi^+_3\rangle \) for an initial resource state. Note that exact state splitting analyzed in Appendix A also requires \( \log_2 3 \) ebits due to Theorem 14 in Appendix A-B. By contrast, the algorithms for exact state merging of \( |\psi\rangle^{RAB} \) in Theorems 5 and 6 achieve
\[
\log_2 K - \log_2 L = 0 < \log_2 3 \text{ and } \log_2 K = 0 < \log_2 3,
\]
respectively.
Implication 2. Negative entanglement cost of exact state merging by entanglement distillation from the redundant part. Consider a pure state
\[
|\psi\rangle_{RAB} = \frac{1}{\sqrt{3}} \left( |0\rangle^R |\Psi^+\rangle^{A_1B_1} |\Phi^+\rangle^{A_2B_2} |\Phi^+\rangle^{A_3B_3} + |1\rangle^R |0\rangle^{A_1} |0\rangle^{B_1} |\Phi^+\rangle^{A_2B_2} |\Phi^+\rangle^{A_3B_3} + |2\rangle^R |2\rangle^{A_1} |2\rangle^{B_1} |0\rangle^{A_2} |0\rangle^{B_2} |\Psi^-\rangle^{A_3B_3} \right),
\]
where each of \(\mathcal{H}^A = \mathcal{H}^{A_1} \otimes \mathcal{H}^{A_2} \otimes \mathcal{H}^{A_3}\) and \(\mathcal{H}^B = \mathcal{H}^{B_1} \otimes \mathcal{H}^{B_2} \otimes \mathcal{H}^{B_3}\) is of \(3 \times 2 \times 2 = 12\) dimension. Quantum teleportation of \(\psi^A\) requires \(\log_2 K - \log_2 L = -1 < 0\) and \(\log_2 K = 0\), respectively. The former negative entanglement cost leads to a net gain of shared entanglement.

Implication 3. Improvement in converse bounds of entanglement cost of exact state merging. Consider a three-qubit pure state
\[
|\psi\rangle_{RAB} = \frac{1}{\sqrt{2}} \left( |0\rangle^R |\Psi^+\rangle^{AB} + |1\rangle^R |0\rangle^A |0\rangle^B \right).
\]
The algorithms for exact state merging of \(|\psi\rangle_{RAB}\) in Theorems 5 and 6 require \(\log_2 K - \log_2 L = 1\) and \(\log_2 K = 1\), respectively. Since \(\psi^B \neq \frac{1}{\sqrt{2}}\), the latter equality for non-catalytic exact state merging is optimal due to Theorem 11. As for the former, this example shows the difference between the converse bounds of entanglement cost of exact state merging in Theorem 7 and Lemma 8. In this case,
\[
\log_2 (\lambda_0 D) = \log_2 \frac{3}{2} > 0.5849, \quad H_{\max}(A|B)_{\psi} < 0.5432,
\]
where the notations are the same as those in Theorem 7 and Lemma 8 and the value of \(H_{\max}(A|B)_{\psi}\) is calculated by a semidefinite programming (SDP) using Split Conic Solver (SCS) [63] and YALMIP [66]. These calculations imply that our converse bound in Theorem 7 can be strictly tighter than the existing converse bound obtained from Lemma 8.

Implication 4. Asymmetry between \(A\) and \(B\) in exact state merging. Consider a three-qubit pure state
\[
|\psi\rangle_{RAB} = \frac{1}{\sqrt{2}} \left( |0\rangle^R |0\rangle^A |0\rangle^B + |1\rangle^R |1\rangle^A |+\rangle^B \right).
\]
The algorithms for exact state merging of \(|\psi\rangle_{RAB}\) in Theorems 5 and 6 require \(\log_2 K - \log_2 L = 1\) and \(\log_2 K = 1\), respectively. Since \(\psi^B \neq \frac{1}{\sqrt{2}}\), the latter equality for non-catalytic exact state merging is optimal due to Theorem 11. In contrast, interchange \(A\) and \(B\) for \(|\psi\rangle_{RAB}\) to consider
\[
|\psi\rangle_{RAB} = \frac{1}{\sqrt{2}} \left( |0\rangle^R |0\rangle^A |0\rangle^B + |1\rangle^R |+\rangle^A |1\rangle^B \right).
\]
In the same way as the above case of \(|\psi\rangle_{RAB}\), the algorithms for exact state merging of \(|\psi\rangle_{RAB}\) in Theorems 5 and 6 require \(\log_2 K - \log_2 L = 1\) and \(\log_2 K = 1\), respectively. However, since \(\psi^B = \frac{1}{\sqrt{2}}\), Theorem 11 implies that there exists an algorithm for non-catalytic exact state merging of \(|\psi\rangle_{RAB}\) achieving \(\log_2 K = 0 < 1\). Indeed, \(|\psi\rangle_{RAB}\) can also be written as
\[
|\psi\rangle_{RAB} = \frac{1}{\sqrt{2}} \left( \left| 1 + \frac{\sqrt{2}}{4} \right| 0 \rangle + |1\rangle \right) \left| \Phi^-\rangle^{RB} + \left| 1 - \frac{\sqrt{2}}{4} \right| 0 \rangle + |1\rangle \right) \left| \Phi^+\rangle^{RB} ,
\]
and hence, \(A\)'s measurement in basis
\[
\{(1 + \frac{\sqrt{2}}{4}) |0\rangle + |1\rangle, (1 - \frac{\sqrt{2}}{4}) |0\rangle + |1\rangle \}
\]
yields a maximally entangled state between \(R\) and \(B\).

These cases imply that the difference in entanglement costs between the optimal algorithm and the algorithms presented in Theorems 5 and 6 may arise depending on whether the quantum part of the Koashi-Imoto decomposition can be merged at less entanglement cost than performing quantum teleportation. Note that the optimal algorithm obtained in Theorem 11 works only for qubits, and Proposition 12 implies that extension to qudits is not straightforward.

V. CONCLUSION

We constructed exact algorithms for one-shot state merging, which work for any state of an arbitrarily-small-dimensional system and satisfy arbitrarily high fidelity requirements. The algorithms retain the essential feature of state merging; that is, entanglement cost can be reduced by exploiting a structure of a given state. This feature arises because the Koashi-Imoto decomposition of the given state shows the classical part, the nonclassical (quantum) part, and the redundant part of the state, and the redundant part can be used for entanglement distillation while the classical part can be merged by a measurement followed by classical communication of the measurement outcome. In addition to achievability bounds for an arbitrarily-small-dimensional system derived from the algorithms, we provided an improved converse bound of entanglement cost of exact state merging, which is proven to be optimal when a purification of the state to be merged is a three-qubit state. Our results yield the first algorithms for one-shot quantum state merging applicable even to small- and intermediate-scale states, and further research will be needed to establish a general strategy for achieving exact state merging at the minimum cost.

Being complementary to existing algorithms achieving nearly optimal one-shot state merging on a large scale, our analysis opens the way to another direction for future research on a small and intermediate scale. As investigated in our accompanying paper [67], the algorithms in this paper serve as essential tools for analyzing exact transformation of a multiparty entangled state shared among spatially separated parties connected by a communication network. For more practical situations, it would also be useful to extend our results to approximate state merging by considering smoothed versions [50] of the results, which we leave for future works.
APPENDIX A

EXACT STATE SPLITTING

In this section, we analyze entanglement cost of exact state splitting, which is an inverse task of exact state merging. After giving the definition in Sec. A-A, we proceed to provide the result in Sec. A-B.

A. Definition of exact state splitting

Exact state splitting is an inverse task of exact state merging involving three parties A, B and R, where R is a reference to consider purification. By convention, for exact state splitting, we assign A as a sender and B as a receiver. Let A have systems $\mathcal{H}^A$, $\mathcal{H}^{A'}$, and $\mathcal{H}^{\overline{A}}$, B have $\mathcal{H}^B$ and $\mathcal{H}^\overline{B}$, and R have $\mathcal{H}^R$, where $\dim \mathcal{H}^{A'} = \dim \mathcal{H}^B$. We assume that A and B can freely perform local operations and classical communication (LOCC) assisted by a maximally entangled resource state initially shared between $\mathcal{H}^{\overline{A}}$ and $\mathcal{H}^\overline{B}$. We write the maximally entangled resource state as

$$\Phi^{\overline{AB}} := \frac{1}{\sqrt{K}} \sum_{l=0}^{K-1} |l\rangle_A \otimes |l\rangle_B,$$

where $K$ denotes the Schmidt rank of the resource state. Note that A and B cannot perform any operation on $\mathcal{H}^R$.

We define the task of exact state splitting as illustrated in Fig. 2. Initially, a possibly mixed state $\psi^{A'A'}$ is given to A, where a purification of $\psi^{A'A'}$ is represented by $|\psi^{RAA'}\rangle$. Exact state splitting of $|\psi^{RAA'}\rangle$ is a task for A and B to transfer the reduced state $\psi^A$ from A to B and obtain $|\psi^{RAB}\rangle$.

Definition 13. Exact state splitting. Exact state splitting of a purified given state $|\psi^{RAA'}\rangle$ is a task for parties A and B to achieve a transformation

$$\text{id}^R \otimes S \left( \psi^{RAA'} \otimes \Phi^{\overline{AB}} \right) = \psi^{RAB}$$

by an LOCC map $S : B \left( \mathcal{H}^A \otimes \mathcal{H}^{A'} \otimes \mathcal{H}^{\overline{A}} \otimes \mathcal{H}^\overline{B} \right) \rightarrow B \left( \mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^{\overline{A}} \otimes \mathcal{H}^\overline{B} \right)$.

The entanglement cost of exact state splitting is defined as $\log_2 K$. If there is a sufficiently large amount of entanglement in the resource state, there exists a trivial algorithm for exact state splitting by quantum teleportation to transfer $\psi^A$ from A to B. In contrast, our algorithm presented in Sec. A-B exploits classical description of $|\psi^{RAA'}\rangle$ to reduce the entanglement cost, and it is shown to be an optimal algorithm achieving the minimal entanglement cost.

B. Optimal algorithm for exact state splitting

We derive a formula for the minimal entanglement cost of an algorithm for exact state splitting. For exact state splitting of $|\psi^{RAA'}\rangle$, the following theorem yields the minimal entanglement cost and an optimal algorithm.

Theorem 14. Optimal entanglement cost of exact state splitting. Given any pure state $|\psi^{RAA'}\rangle$, exact state splitting of $|\psi^{RAA'}\rangle$ is achievable if and only if

$$\log_2 K \geq \log_2 \text{rank} \psi^A.$$

Proof: If part: We construct an LOCC algorithm achieving

$$\log_2 K = \log_2 \text{rank} \psi^A.$$

To achieve Eq. (15), we provide a method for compressing $\psi^A$. Consider the Schmidt decomposition of the given state $|\psi^{RAA'}\rangle$ with respect to the bipartition between $\mathcal{H}^R \otimes \mathcal{H}^A$ and $\mathcal{H}^{A'}$, that is,

$$|\psi^{RAA'}\rangle = \sum_{l \in R_{\psi}} \sqrt{\lambda_l^\psi} |l\rangle^A \otimes |l\rangle^{A'},$$

where $R_{\psi} := \{0, \ldots, \text{rank} \psi^A - 1\}$, each $\sqrt{\lambda_l^\psi} > 0$ is a nonzero Schmidt coefficient, and $\{|l\rangle^A : l \in R_{\psi}\}$ and $\{|l\rangle^{A'} : l \in R_{\psi}\}$ are subsets of the Schmidt bases of $\mathcal{H}^R \otimes \mathcal{H}^A$ and $\mathcal{H}^{A'}$, respectively, corresponding to the nonzero Schmidt coefficients. Let $\mathcal{H}^{A''}$ be A’s auxiliary system satisfying $\dim \mathcal{H}^{A''} = \text{rank} \psi^A$ and $\{|l\rangle^{A''} : l \in R_{\psi}\}$ be the computational basis of $\mathcal{H}^{A''}$. Consider an isometry $U_{\text{split}}$ from $\mathcal{H}^{A'}$ to $\mathcal{H}^{A''}$ satisfying $|l\rangle^{A''} = U_{\text{split}} |l\rangle^{A'}$ for each $l \in R_{\psi}$. By performing $U_{\text{split}}$, $\psi^A$ is compressed into a state on $\mathcal{H}^{A''}$, that is,

$$|\psi^{RAA''}\rangle := \text{id}^R \otimes U_{\text{split}} |\psi^{RAA'}\rangle = \sum_{l \in R_{\psi}} \sqrt{\lambda_l^\psi} |l\rangle^A \otimes |l\rangle^{A''}.$$

By performing $U_{\text{split}}^\dagger$, we can recover the given state $|\psi\rangle$ from the compressed state $|\psi^{RAA''}\rangle$. 

Fig. 2. Exact state splitting of a given state $|\psi^{RAA'}\rangle$ denoted by the red circles. Parties $A$ and $B$ perform LOCC assisted by a maximally entangled resource state $\Phi^{\overline{AB}}$ with the Schmidt rank $K$ denoted by the blue circles to transfer the reduced state $\psi^A$ from A to B and obtain $|\psi^{RAB}\rangle$. 

Proof: If part: We construct an LOCC algorithm achieving

$$\log_2 K = \log_2 \text{rank} \psi^A.$$
The LOCC algorithm achieving Eq. \([\text{15}]\) is as follows. First, \(A\) performs \(U_{\text{split}}\) to transform the given state \(|\psi\rangle^{RAA'}\) into the compressed state \(|\psi\rangle^{RAA''}\). Next, the reduced state \(\rho^{RA''}\) is sent from \(A\) to \(B\) by quantum teleportation using the resource state satisfying Eq. \([\text{15}]\). After performing quantum teleportation, \(B\) performs \(U_{\text{split}}^\dagger\) on the system for the received state to recover \(|\psi\rangle^{RAB}\).

Only if part: We use LOCC monotonicity of the Schmidt rank \([68]\). The Schmidt rank of \(|\psi\rangle^{RAA''} \otimes |\Phi_K^+ \rangle_{\text{AB}}\) between the party \(B\) and the other parties \(R\) and \(A\) is \(K\). After performing an LOCC map \(\text{id}^{R} \otimes S\), the Schmidt rank of \(|\psi\rangle^{RAB}\) between the party \(B\) and the other parties \(R\) and \(A\) is rank \(\rho^{AA'}\). Since the Schmidt rank of pure states is monotonically non-increasing under LOCC, it holds that \(K \geq \text{rank } \rho^{AA'}\). Therefore, we obtain \(\log_2 K \geq \log_2 \text{rank } \rho^{AA'}\).

Q.E.D.

Remark 3. Asymptotic limit of exact state splitting. Given any \(|\psi\rangle^{RAA'}\), from our exact algorithm for one-shot state splitting in Theorem \([\text{14}]\) we can derive the rate of entanglement cost required for asymptotic state splitting of \(|\psi\rangle\) in the LOCC framework as follows. Note that the asymptotic rate derived in the following, that is, \(H(A')_\psi\), is optimal \([\text{14}]\), where \(H\) denotes the quantum entropy \([23]\).

For large \(n\), \(|\psi\rangle^n\) can be approximated by \(|\tilde{\psi}^n\rangle := (\mathbb{1}^{RA} \otimes \Pi_{\delta}^A)^n |\psi\rangle^n \approx |\tilde{\psi}^n\rangle\), where \(\Pi_{\delta}^A\) is the projector onto the \(\delta\)-typical subspace of \((|\psi\rangle^A)^n\) \([23]\), and \((\mathbb{1}^{RA} \otimes \Pi_{\delta}^A)^n\) is the identity operator on \((H^R \otimes H^A)^n\). Then, the entanglement cost \(\log_2 K\) of exact state splitting of \(|\tilde{\psi}^n\rangle\) yields \(H(A')_\psi\) required for the asymptotic state splitting of \(|\psi\rangle\) because

\[
\frac{1}{n} \log_2 K = \frac{1}{n} \log_2 \text{Tr}_{RRA} |\tilde{\psi}^n\rangle \langle \tilde{\psi}^n| \\
\leq \frac{1}{n} \log_2 \text{Tr}_{RRA} |\tilde{\psi}^n\rangle \langle \tilde{\psi}^n| \\
= H(A')_\psi + O\left(\frac{1}{n}\right) \quad \text{as } n \to \infty.
\]

Appendix B

Generality of exact state merging of a maximally entangled state

In this section, we show that, given any \(|\psi\rangle^{RAB}\), entanglement cost of exact state merging of \(|\psi\rangle^{RAB}\) is the same as that of a maximally entangled state corresponding to \(|\psi\rangle^{RAB}\) as stated in the following proposition.

Proposition 15. Generality of exact state merging of a maximally entangled state. Given any pure state

\[
|\psi\rangle^{RAB} = \sum_{l=0}^{D-1} \sqrt{\lambda_l} |l\rangle^R \otimes |\psi_l\rangle^{AB},
\]

where the right hand side is the Schmidt decomposition, \(D\) is the Schmidt rank, and \(\lambda_l > 0\) for each \(l \in \{0, \ldots, D-1\}\), exact state merging of \(|\psi\rangle^{RAB}\) is achievable if and only if exact state merging of the corresponding maximally entangled state of Schmidt rank \(D\)

\[
|\Phi_D^+(\psi)\rangle^{RAB} := \sum_{l=0}^{D-1} \frac{1}{\sqrt{D}} |l\rangle^R \otimes |\psi_l\rangle^{AB}
\]

is achievable. Also, non-catalytic exact state merging of \(|\psi\rangle^{RAB}\) is achievable if and only if non-catalytic exact state merging of \(|\Phi_D^+(\psi)\rangle^{RAB}\) is achievable.

Proof: We prove the statement in the catalytic setting while the statement on non-catalytic exact state merging follows from the same argument setting \(\log_2 L = 0\).

If part: Consider an LOCC map \(\mathcal{M}\) by \(A\) and \(B\) achieving exact state merging of \(|\Phi_D^+(\psi)\rangle^{RAB}\), that is,

\[
\text{id}^R \otimes \mathcal{M} \left( \Phi_D^+(\psi) \otimes \Phi_K^+_{\text{AB}} \right) = \Phi_D^+(\psi) \otimes \Phi_L^+_{\text{L}}.
\]

The left hand side and the right hand side are written as

\[
\text{id}^R \otimes \mathcal{M} \left( \Phi_D^+(\psi) \otimes \Phi_K^+_{\text{AB}} \right) = \sum_{l,l'} \frac{1}{D} \langle l' |^R \otimes \mathcal{M} \left( |\psi_l\rangle \langle \psi_l|^{AB} \otimes \Phi_K^{+\text{AB}} \right),
\]

\[
\Phi_D^+(\psi) \otimes \Phi_L^+_{\text{L}} = \sum_{l,l'} \frac{1}{D} \langle l' |^R \otimes |\psi_l\rangle \langle \psi_l|^{B'\text{B}} \otimes \Phi_L^{+\text{AB}}.
\]

Due to the linear independence, we obtain

\[
\mathcal{M} \left( |\psi_l\rangle \langle \psi_l|^{AB} \otimes \Phi_K^{+\text{AB}} \right) = |\psi_l\rangle \langle \psi_l|^{B'\text{B}} \otimes \Phi_L^{+\text{AB}}
\]

for any \(l\) and \(l'\). Therefore, due to linearity, we obtain

\[
\text{id} \otimes \mathcal{M} \left( |\psi\rangle^{RAB} \otimes \Phi_K^{+\text{L}} \right) = |\psi\rangle^{RAB} \otimes \Phi_L^{+\text{L}}.
\]

Only if part: The converse can be shown by interchanging \(\Phi_D^+(\psi)\) and \(\psi\) in the above argument. Q.E.D.

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