Experimental Verifications of the Casimir Attractive Force between Solid Bodies

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Abstract

A brief review of the recent experimental verifications of the Casimir force between extended bodies is presented. With modern techniques, it now appears feasible to test the force law with precision better than 1%; I will address the issues relating to the theoretical interpretation of experiments at this level of accuracy.

Keywords: Casimir Force, Quantum Electrodynamics, Fluctuations

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I. INTRODUCTION

The force between uncharged conducting surfaces, the so-called “Casimir force,” has been described as one of the least intuitive consequences of quantum electrodynamics. For conducting parallel flat plates separated by distance \( r \), this force per unit area \( A \) has the magnitude \( F(d) \):

\[
F(d)/A = \frac{\pi^2 \hbar c}{240 d^4} = 0.013 \frac{1}{d^4} \text{dyn(\mu m)^4/cm}^2.
\]  

(1)

This relationship can be derived by considering the electromagnetic mode structure between the two plates, as compared to free space, and by assigning a zero-point energy of \( \frac{1}{2} \hbar \omega \) to each electromagnetic mode (photon). The change in total energy density between the plates, as compared to free space, as a function of separation \( d \), leads to the force of attraction.

The only fundamental constants that enter Eq. (1) are \( \hbar \) and \( c \); the electron charge \( e \) is absent, implying that the electromagnetic field is not coupling to matter. The role of \( c \) is to
convert the electromagnetic mode wavelength to a frequency, while \( \hbar \) converts the frequency to an energy.

The term “Casimir effect” is applied to a number of long-range interactions, such as those between atoms or molecules (retarded van der Waals interaction), an atom and a material surface (Casimir-Polder interaction), and the attraction between bulk material bodies. The latter effect is generally referred to as “the” Casimir force and depends only on the bulk properties of the bodies under consideration; I will limit my discussion to the latter effect.

For real materials, Eq. (1) must break down when the separation \( d \) is so small that the mode frequencies are higher than the plasma frequency (for a metal) or higher than the absorption resonances (for a dielectric) of the material used to make the plates; for sufficiently small separation, the force of attraction varies as \( 1/d^3 \), as discussed in particular by Lifshitz \[2\]. In analogy with the attractive forces between atoms, the force in this range is sometimes referred to as the London-van der Waals attraction, while the \( 1/d^4 \) range is referred to as the retarded van der Waals (Casimir) interaction. For the Casimir effect, the crossover distance between the regimes is \( d \sim 100 \) nm, much larger than atomic spacings in the materials, so it still makes sense to describe the materials by their bulk properties (index of refraction); the \( 1/d^3 \) vs. \( 1/d^4 \) interaction is in this case due to the truncation of the mode frequencies that are affected by the changing plate separation. Therefore the crossover between the two regimes appears to be of physically different origin than in the case of the attractive forces between isolated atoms.

The Casimir effect and its calculation represent an electromagnetic waveguide problem, where imperfect materials are used in the construction of the waveguide. The zero-point fields are those associated with the waveguide; these modes do not exist in free space, so the idea that the Casimir effect represents a negative energy density compared to free space is incorrect.

Given that the distances where the force of attraction is sufficiently strong to be experimentally detected are \( d \sim 1000 \) nm or less, an accurate theoretical description of an experimental system must take into account the real material properties as will be discussed in Sec. III below.

Finally, it is interesting to note that there have been only a dozen or so published experimental measurements of the Casimir force, to be compared with the hundreds and hundreds of theoretical papers on the subject. Perhaps very few doubt the strict validity of Eq. (1) or its modification for real materials \[2\]. Because of the unavoidable uncertainties in bulk material and surface properties, verification of Eq. (1) as a test of QED will likely always be inferior to measurements of the Lamb shift or \( g - 2 \) of the electron.

II. EXPERIMENTS

A. Overview

The experimental situation as of 1989 has been reviewed by Sparnaay in the volume prepared in the honor of Dr. Casimir’s 80th birthday \[3\]. Since then, two experiments have been performed, both with significantly better accuracy than had been previously obtained. These two experiments were based on a torsion pendulum balance \[4\] and on atomic force microscopy (AFM) \[5\].
The recent experiments employed techniques that were developed in particular by van Blokland and Overbeek [6] in the measurement of the attractive forces between metallic films. Measurements between metallic films pose difficult problems as compared to dielectric films for which optical techniques can be used for alignment and distance measurements. In the case of metallic films, the distance is determined by measurement of the capacitance between the plates. Alignment is simplified by making one plate convex, in which case the geometry is fully determined by the radius of curvature at the point of closest approach, and the distance between the plates at that point. This technique was first put forward by Deryagin [7] and has found broader application as the Proximity Force Theorem [8]. For this geometry, Eq. (1) becomes

\[ F(d) = 2\pi RE(d) = 2\pi \frac{\pi^2 \frac{1}{240} \frac{\hbar c}{3}}{d^3} \] (2)

where \( R \) is the radius of curvature and \( E(d) \) is the energy per unit area that leads to the force in Eq. (1). It should be noted that the plate area does not enter into Eq. (2). The two recent experiments each used one convex and one flat plate.

B. Torsion Pendulum Experiment

Because I was directly involved in this experiment, I will give an anecdotal account of it; specific details, schematics, and data can be found in [4]. This experiment was started in 1994 as an undergraduate project. Dev Sen, a junior at the University of Washington at that time, was working with our atomic physics group; he heard of the Casimir force in his Electromagnetism course and asked if we might be able to measure it. With no knowledge of previous experimental work, with the exception of the Sparnaay paper of 1958 [9] (which is routinely quoted as “the” verification of Eq. (1)), I proposed an electrostatically balanced hanging (Cavendish) torsion pendulum technique; this is different from the horizontal torsion balance used by van Silfhout [10] (which, at the time, we did not know about) particularly because the angular force constant \( \alpha \) is at least two orders of magnitude smaller for the hanging pendulum. For short times (a few seconds of averaging) our principal noise source was thermal fluctuations (e.g., Brownian motion); in this limit, the signal to noise scales as \( 1/\sqrt{\alpha} \). It appears that this noise was a major limitation of our experiment, particularly when measuring the force at large plate separation.

We constructed the apparatus with junk found around the lab, and the total material construction cost of the apparatus was around $300. The torsion pendulum fiber was about 60 cm in length; mounted on one arm of the pendulum was a flat plate coated with a Cu/Au film, while the other arm served as one electrode of a differential capacitor system that was used in a feedback system to keep the torsion pendulum at a fixed angle. The second plate used for the Casimir force measurement was mounted on a precision mechanical mount and could be moved about 10 \( \mu \)m by use of piezoelectric transducers.

In our first attempts to measure the Casimir force, two flat plates were used. We were never able to properly align the plates, and we eventually gave up when Dev graduated. Some months before I moved to Los Alamos, I decided to give the apparatus one last try. I had the idea to replace one flat plate with a convex plate; I also improved the vacuum and electronic circuitry.
In June of 1996 the apparatus became operational; when the two plates were moved into close proximity, the increase in force over the residual electrostatic interaction was easily evident. Unfortunately, the apparatus was not stable enough to make a measurement; about 20 minutes after adjusting the distance between the plates so that the separation at closest approach would be of order 1 µm or less, the apparatus would drift until the closest approach would increase to over 5 µm. I eventually discovered the source of the drift: the concrete floor of the Physics Department basement would distort while I was standing near the apparatus during adjustment, and the floor would relax to its unstressed position over a 20 minute period. This effect was dealt with by standing in a different position, rather far from the apparatus, while making the initial adjustments; the experiment thereafter became physically (in an anatomical sense) painful.

Data was taken over a one month period, after which the apparatus was calibrated. The final measurement was the radius of curvature of the Cu/Au coated convex plate (lens) in which I made an error. To make this measurement, I used the angular deviation of a laser beam reflected from the lens surface as the lens was translated normally to the beam. Measurements near the center yielded $R = 13.5 \pm 0.5$ cm, while translations across the entire lens yielded $R = 11.3 \pm 0.1$ cm; it did not occur to me that the lens might be aspheric (it was a high quality, precision optical lens of unknown origin or application that was coated with a thin layer of Au on top of a relatively thick Cu layer). I assumed the latter number was more accurate, and the data was described by Eq. (2) with no corrections for finite conductivity; I mistakenly convinced myself that the corrections should be less than 5%. At this point, my work was rudely interrupted by my relocating to Los Alamos, and I published my results in Physical Review Letters.

Subsequently, a number of theorists expressed surprise that no correction due to finite conductivity was required [11], and several expressed an interest in doing a proper calculation. None did the calculation, so I figured out how to do it myself [12]; I knew that the plasma model correction to first order was not applicable (see Sec. IIIa); being upwards of a 30% correction, none of the uncertainties in calibration were large enough to encompass it. However, the variation in the measurement of $R$ was in the back of my mind, and in October of 1997 I made a trip to Seattle and retrieved the lens. Using a mechanical gauge, I measured $R = 12.5$ cm in the region where the Casimir force measurement was made. The accurate finite conductivity correction is small enough that its effect is indistinguishable from a calibration error, or a mismeasurement of $R$. In the end, my data did not agree with the assumption that the film was pure Au, but was better described by a pure Cu film [13]. This result is not so surprising considering that the properties of the film depend on preparation technique, purity, and the possibility that the Au diffused into the Cu layer significantly. A more accurate calculation of the finite conductivity effect would require a direct measurement of the complex permittivity of the films.

In hindsight, it is remarkable that the torsion balance experiment, which was intended as a demonstration, worked as well as it did. The improvement over previous measurements is due to a number of factors, including the high sensitivity of the hanging torsion pendulum and its lack of mechanical hysteresis, larger measurement distances so vibration and mechanical instabilities were less important, improved piezoelectric transducers, and automated data collection so that large amounts of data could be analyzed and averaged.

Much improvement over the present accuracy obtained by this technique is unlikely. The
apparatus was rather unwieldy with its enormous vacuum can and its susceptibility to tilt. The length of the torsion fiber might be significantly shortened, reducing both the intrinsic sensitivity (bad) and sensitivity to external perturbations (good); however we must bear in mind that a factor of ten improvement in sensitivity only extends the measurement distance by factor of about two.

C. Atomic Force Microscopy Experiment

In his 1989 review [3], Sparnaay discusses the possibility of using atomic force microscopy (AFM) to measure the Casimir force; AFM had just been invented at that time [14]. It was not until late 1998 that results from an AFM Casimir experiment were reported by Mohideen and Roy [5].

In this experiment, an Au/Pd + Al coated, 0.3 mm polystyrene sphere is attached to an AFM cantilever. A similarly coated optically polished sapphire plate was attached to a piezoelectric transducer and brought near the sphere. The attractive force was determined by reflecting a laser beam from the cantilever tip; the displacement of the laser beam on a pair of photodiodes produced a difference signal proportional to the cantilever bending angle.

The sensitivity of the apparatus was such that the absolute force could be determined with a fractional error of 1% at \(d = 100 \text{ nm}\), and about 100% at 900 nm.

The use of AFM to measure the Casimir force might be a real breakthrough; this is because the AFM technique is very stable and reproducible. Unfortunately, it is limited to measurements short distance where there are significant theoretical uncertainties in the interpretation of the data. For the first time testing Eq. (1) or its modification for real materials to better than 1% accuracy appears possible. As described in [5], it is anticipated that a factor of 1,000 improvement in sensitivity appears possible, which would extend the separation where the Casimir force can be measured to 1% accuracy to about 1 \(\mu\text{m}\). At this distance, the theoretical uncertainties associated with the corrections for real materials, as described in the next Section, become much less important.

III. CORRECTIONS

A. Imperfect Conductivity

Equation (1) must break down when the plate separation is so small that the mode frequencies being affected when \(d\) is varied are above the material resonance or plasma frequencies. In the case of a simple metal, the real part of the dielectric constant can be approximated by

\[
e'(\omega) = 1 - \omega_p^2/\omega^2
\]

where \(\omega_p\) is the the plasma frequency and is proportional to the effective free electron density in the metal. It is convenient to introduce the plasma wavelength, \(\lambda_p = 2\pi c/\omega_p\). Corrections to Eq. (1), expanded in terms of \(\lambda_p/d\), have been calculated to first order by Haergraves
and by Schwinger et al. [16], and to second order by Bezerra et al. [17]. For flat plates, the corrected force can be written in terms of a Eq. (1) with a multiplicative factor,

$$F'(d) = F(d) \left[1 - \frac{8}{3\pi} \frac{\lambda_p}{d} + \frac{120}{4\pi^2} \left(\frac{\lambda_p}{d}\right)^2\right].$$ (4)

This equation is only valid for $\lambda_p/d << 1$; unfortunately, the Casimir force is large enough to be accurately measured experimentally only in the range $\lambda_p/d \approx 1$ or larger. We are also faced with the problem that Eq. (3) is only approximate.

It is, however, possible to very accurately determine the attractive force as a function of plate separation by numerical calculation, provided we know its complex permittivity as a function of frequency:

$$\epsilon(\omega) = \epsilon'(\omega) + i\epsilon''(\omega)$$ (5)

where $\epsilon'$, $\epsilon''$ are real. With this information, the permittivity along the imaginary axis can then be determined by use of the Kramers-Kronig relation,

$$\epsilon(i\xi) = \frac{2}{\pi} \int_0^\infty \frac{xe''(x)}{x^2 + \xi^2} dx + 1.$$ (6)

This can be used in the Lifshitz expression for the attractive force [2],

$$F'(d) = \frac{\hbar}{2\pi^2 c^3} \int_0^\infty \int_1^\infty p^2 \xi^3 \left(\left[\frac{[s+p]^2}{[s-p]^2}e^{2p\xi d/c} - 1\right]^{-1} + \left[\frac{[s+i\xi p]^2}{[s-i\xi p]^2}e^{2i\xi d/c} - 1\right]^{-1}\right) dp d\xi$$ (7)

where $s = \sqrt{\epsilon(ix/p) - 1 + p^2}$. The numerical calculation for the attractive force between Au, Al, and Cu plates has been recently published [12], and significant deviations from Eq. (4) were found. In particular, for Al with $d \approx 100$ nm Eqs. (4) and (7) differ by about 5%; one should note that including the third order correction to Eq. (4) worsens the deviation. However, these calculations should be considered in light of the notorious variation of bulk and surface properties of materials due to preparation technique, purity, etc. [3,18].

B. Surface Roughness

From the earliest experiments, it was realized that surface roughness would lead to an increase of the apparent Casimir force and therefore cause systematic errors in measurements aimed at verifying Eq. (1). Such effects were observed by van Blokland and Overbeek [4]; roughness has been discussed theoretically by van Bree et al. [19] and more recently in [20].

For high-quality optically polished surfaces, the roughness RMS amplitude $A$ is usually of order $A = 30$ nm or less. For a $1/d^4$ attractive force, the correction to Eq. (1) can be written
\[ F'(d) \approx F(d) \left[ 1 + 4 \left( \frac{A}{d} \right)^2 \right]. \]  

(8)

The correction for the recent torsion balance experiment, at the point of closest approach, is about 1%, while for the AFM experiment, it is about 30%.

The roughness correction was derived in the context of a 1/d^4 force law (this can be easily modified for the spherical plate 1/d^3 case). However, the finite conductivity correction, particularly as given by Eq. (4), effectively has terms containing 1/d^5 and 1/d^6. In principle, the roughness correction should be done for each power law separately, or the average force determined from the accurate calculation, Eq. (7). One should also bear in mind that the simple geometrical averaging procedure isn’t exactly correct; a complete treatment would involved solving the appropriate electromagnetic rough boundary problem. However, the geometrical averaging is correct so long as the period of the roughness is larger than the separation between the plates.

C. Effect of Thin Films on the Plate Surfaces

Either intentionally (Au evaporated onto an Al or Cu coated substrate) or accidentally (formation of oxide layers) every Casimir force measurement has made use of mono- or multilayer coated plates. The calculation of the force for a general film configuration has been given by Spruch and Zhou \cite{21}. The problem that I will consider here is very simple so I will outline its solution in some detail.

A simple geometry that illustrates the effect of a thin material film is shown in Fig. 1; one of two identical perfectly conducting flat plates is coated with a thin layer (thickness \(a\)) of a real substance (Au, for example), and the separation between the perfectly conducting surfaces is \(d + a\). This simplified problem will allow us to determine the qualitative effect of a thin film.

Milonni has presented a complete calculation for the case of materials with no absorption (\cite{22}), and this can be easily adapted to the multilayer case (this calculational technique was first described in \cite{23} and elaborated upon in \cite{24}). Following Milonni, we consider the case where the electromagnetic wave propagation vector in the three materials (vacuum, film, perfect conductor)

\[ K_i^2 = k^2 - \varepsilon_i(\omega) \frac{\omega^2}{c^2} \]  

(9)

where \(k\) is a real number, and \(i = 0, 1, 2\) with 0 (\(\varepsilon_0(\omega) = 1\)) representing the space between the plates and 2 (\(\varepsilon_2(\omega) = \infty\)) the perfect conductor, and we require \(Re(K_i) \geq 0\). For the case considered here, we must allow for a complex \(\varepsilon_i\) in which case the \(K_i\)s can be complex. In the Appendix, the use of the techniques described by Milonni for the case of absorption (complex \(\varepsilon\)) is justified.

There are two type of solutions to the wave equation, one with electric vector parallel to the surfaces (with arbitrary orientation which we choose as the \(y\) axis), \(e_y(z)\) and one with electric vector perpendicular to the surfaces (along the \(z\) axis), \(e_z(z)\). The wave equation is

\[ \frac{d}{dz}e_{y,z}(z) - K_i^2 e_{y,z}(z) = 0 \]  

(10)
and the boundary condition for \( e_z \) are (1) \( de_z/dz \) and \( e \) are continuous, while for \( e_y \) they are (2) \( de_y/dz \) and \( e_y \) are continuous (at the conducting surfaces, \( e_y = 0 \) and \( de_z/dz = 0 \)).

Ignoring unphysical exponentially growing solutions, we have

\[
e_y(z) = A(e^{K_0 z} + e^{-K_0 z}) \quad 0 \leq z \leq d - a
\]
\[
e = Be^{K_1 z} + Ce^{-K_1 z} \quad d - a \leq z \leq d
\]
\[
de_z/dz = 0 \quad z = d
\]
\[
e_y = 0 \quad z = d
\]

where the \( \mp \) sets \( e_y = 0 \) or \( de_z/dz = 0 \) at the conducting boundary located at \( z = 0 \). We therefore have two sets of linear equations involving \( A, B, C \) for the two cases. The condition for non-trivial solutions of these equations is that the determinant of the coefficient matrix is zero, yielding the following two expressions:

\[
f_y (\omega, k, d) = 0
\]
\[
= \left[\left(e^{2K_1 a} + 1\right)K_1 + \left(e^{2K_1 a} - 1\right)K_0\right] e^{2K_0 d} - 1
\]

\[
f_z (\omega, k, d) = 0
\]
\[
= \left[\left(e^{2K_1 a} - 1\right)K_1 + \varepsilon_1(\omega)\left(e^{2K_1 a} + 1\right)K_0\right] e^{2K_0 d} - 1.
\]

The zeroes \( \omega_{ny,nz}(k, d) \) of \( f_{y,z} \) determine the allowed mode “eigen” frequencies.

The zero point energy associated with the plates is determined by assigning energy \( \hbar\omega/2 \) to each mode:

\[
E(d) = \sum_{n,k} \left[\hbar\omega_{ny}(k, d)/2 + \hbar\omega_{nz}(k, d)/2\right]
\]

(in general, the eigen-frequencies are complex, but the imaginary parts cancel as discussed in the Appendix). where the sum over \( k \), in the continuum limit, becomes an integral,

\[
\sum_{k} \rightarrow \left(\frac{L}{2\pi}\right)^2 \int 2\pi k dk
\]

where \( L \) is the transverse dimensions of the plate in the \( x, y \) directions. The theory of complex function can be used to evaluate the sum over eigen-frequencies; specifically, according to the argument theorem [25,26],

\[
\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = N - P
\]

where \( C \) is a closed path in the complex plane, \( N \) and \( P \) are the number of zeros and poles within \( C \), respectively, and the path is counterclockwise. The argument theorem can be modified to give the sum of the zeroes and poles:
\[
\frac{1}{2\pi i} \oint_C z \frac{f'(z)}{f(z)} \, dz = \left[ \sum z_i \right]_{f(z_i)=0} - \left[ \sum z_i \right]_{f(z_i)=\infty} .
\] (17)

Furthermore, \( f'(z)/f(z) = d/dz(\log f(z)) \). The eigen-frequencies of physical interest lie in the right half plane; integrating along the imaginary axis from \( \infty \) to \( -\infty \) and closing the path with a semi-circle at infinity around the right half plane (see [22] for details), and integrating by parts gives

\[
E(d) = \frac{\hbar L^2}{8\pi^2} \int_0^\infty k \, dk \int_{-\infty}^\infty d\xi \left[ \log g_y(\xi, k, d) + \log g_z(\xi, k, d) \right]
\] (18)

where we have set \( \omega = i\xi \), with \( \xi \) real, \( g_{y,z}(\xi, k, d) = f_{x,y}(i\xi, k, d) \), and

\[
K_i = k^2 + \epsilon_i(i\xi)^2/c^2 .
\] (19)

Finally, we note that the poles of Eqs. (12) and (13) do not depend on \( d \) because it only enters in the multiplicative exponential; therefore, Eq. (18) gives the zero point energy up to an additive constant, while the force per unit area is given by

\[
-\frac{\partial}{\partial d} E(d) = -\frac{\hbar}{4\pi^2} \int_0^\infty k \, dk \int_{-\infty}^\infty d\xi K_0 \left[ \frac{1}{g_y(\xi, k, d)} + \frac{1}{g_z(\xi, k, d)} \right]
\] (20)

where (possibly) non-physical \( d \)-independent terms are omitted.

\( F(d) \) for a Au film 35 nm thick is shown in Fig. 2; \( \epsilon_1(i\xi) \) was determined from tabulated optical constants as described in [12].

**IV. CONCLUSION**

In the last few years, two new experiments to verify the existence of the Casimir force between solid bodies have been performed and have shown a new level of accuracy; in particular, atomic force microscopy technique offers great promise for testing Eq. (1) and its modifications for the case of real materials to better than 1% precision. In particular, if the sensitivity of the AFM experiment can be increased substantially, the measurement region can be extended to larger separations where the theoretical uncertainties discussed above are substantially reduced. Perhaps even the effect of finite temperature [27] will be measurable in the not-too-distant future.

**V. APPENDIX**

As discussed by Milonni [22], the contour integration technique used to sum the eigen-frequencies is technically correct only if the eigen-frequencies lie on the positive real axis. There has been much commentary on the extension of this technique to absorptive materials (i.e., complex permittivity) in which case the eigen-frequencies are complex. Barash and Giszburg [24] introduce the idea of an auxiliary system to account for the complex permittivity, with the fundamental eigen-frequencies real. However, the contour integration method gives the correct answer quite simply; the mathematics does not know about the
auxiliary system. I would like to offer a non-rigorous explanation for why the technique works in the case of absorption.

First, for a generalized permittivity \( \epsilon(\omega^*) = \epsilon^*(\omega) \). Therefore, the eigen-frequencies for the general boundary problem case occur in pairs, \( \omega = \pm \omega' + i \omega'' \). We further take the case where \( \text{Re}(K_i) \) are either all positive or all negative which follows from continuation; as \( \omega \to \infty \), all the \( K_i \) become equal because \( \epsilon_i(\omega) \to 1 \). In the case that \( \text{Re}(K_i) > 0 \), representing exponentially damped surface wave, the eigen-frequencies lie in the lower half plane; therefore, \( e^{-i\omega t} \) is damped exponentially in time. For \( \text{Re}(K_i) < 0 \), the eigen-frequencies lie in the upper half plane and represent solutions growing exponentially in time and space. Clearly, the contour integration method as described above should not work without justification regarding branch cuts etc.

The way to see around this problem is instead of considering the eigen-frequencies, one can consider the corresponding eigen-wavenumbers \( K_i \), and \( K_0 \) in particular. The eigen-wavenumbers occur in complex conjugate pairs \( K_0 = K'_0 \pm i K''_0 \) (as can be seen from Eq. (9) and the properties of \( \epsilon(\omega) \) in the complex plane) and by definition the \( K_i \) are in the right half plane for the exponentially damped solutions. Furthermore, if we write the determinant function in terms of \( K_0 \), and using the fact that \( K_0 dK_0 = -\omega d\omega/c^2 \), we find

\[
- c^2 \oint_C K_0 \frac{f'(\omega(K_0))}{f(\omega(K_0))} dK_0 = \oint_C \omega(K_0) \frac{f'(\omega(K_0))}{f(\omega(K_0))} dK_0 \quad (21)
\]

and we can replace \( dK_0 \) by \( d\omega \) because the path is arbitrary in the complex plane (note that in the case of no absorption, the eigenvalues for \( K_0 \) lie on the imaginary axis while those for \( \omega \) lie on the real axis, and the contour must be adjusted accordingly). Because the eigenvalues for \( K_0 \) occur in conjugate pairs, the left side of Eq. (21) is real; therefore, the sum in Eq. (14) is real, as can be seen from Eq. (17) when the integral is taken along the imaginary \( \omega \) axis.
REFERENCES

[1] H.B.G. Casimir, Proc. Kon. ned. Akad. Wetenschap. Ser. B 51, 793 (1948).
[2] E.M. Lifshitz, Sov. Phys. JETP 2, 73 (1956).
[3] A. Sarlemijn adn M.J. Sparnaay, Physics in the Making: Essays on Developments in 20th Century Physics in Honour of H.B.G. Casimir on the Occasion of his 80th Birthday (North-Holland, Amsterdam, 1989). pp. 235-246.
[4] S.K. Lamoreaux, Phys. Rev. Lett. 78, 5 (1997).
[5] U. Mohideen and A. Roy, Phys. Rev. Lett. 81, 4549 (1998).
[6] P.H.G. M van Blokland and J.T.G. Overbeek, J. Chem. Soc. Faraday Trans. 74, 2637 (1978).
[7] B.V. Derjaguin and I.I. Abrikosova, Sov. Phys. JETP 3, 819 (1957).
[8] J. Blocki, J. Randrup, W.J. Swiatecki, and C.F. Tsang, Ann. Phys. 105, 427 (1977).
[9] M.J. Sparnaay, Physica 24, 751 (1958).
[10] A. van Silfhout, Proc. Kon. ned. Acad. Wetenschap. Ser. B 69, 501 (1966).
[11] C. Eberlein, Private Communication 1997.
[12] S.K. Lamoreaux, Phys. Rev. A 59, R3149 (1999).
[13] S.K. Lamoreaux, Phys. Rev. Lett. 81, 5475 (1998).
[14] G. Binnig, C.F. Quate, and Ch. Gerber, Phys. Rev. Lett. 56, 930 (1986).
[15] C.M. Hargreaves, Proc. Kon. ned. Akad. Wetenschap. Ser. B 68, 231 (1965).
[16] J. Schwinger, L.L. DeRaad, Jr., and K.A. Milton, Ann. Phys. 115, 1 (1978).
[17] V.B. Bezzera, G.L. Klimchitskaya, and C. Romero, Mod. Phys. Lett. A 12, 2613 (1997).
[18] F.A. Jenkins and H.E. White, Principles of Optics (McGraw-Hill, New York, 1957). p. 523.
[19] J.L.M.J. van Bree, J.A. Poulis, B.J. Verhaar, and K. Schram, Physica 78, 187 (1974).
[20] M. Bordag, G.L. Klimchitskaya, and V.M. Mostepanenko, Phys. Lett. A 200, 95 (1995).
[21] F. Zhou and L. Spruch, Phys. Rev. A 52, 297 (1995).
[22] P.W. Milonni, The Quantum Vacuum (Academic, San Diego, 1994). pp. 221-227.
[23] N.G. Van Kampen, B.R.A. Nijboer, and K. Schram, Phys. Lett. A 26, 307 (1968).
[24] Yu. S. Barash and V.F. Ginzburg, Sov. Phys. Usp. 18, 305 (1975).
[25] Mary L. Boas, Mathematical Methods in the Physical Sciences (Wiley, New York, 1983). p. 609.
[26] R.V. Churchill, J.W. Brown, and R.F. Verhey, Complex Variables and Applications (McGraw-Hill, New York, 1976). pp. 296-301.
[27] L.S. Brown and G.J. Maclay, Phys. Rev. 184, 1272 (1969).
[28] L.D. Landau and E.M. Lifshitz, Electrodynamics of Continuous Media (Pergamon, Oxford, 1960). Chap. 9.
FIGURES

FIG. 1. Simple geometry for determining the effect of a thin film on a perfectly conducting plate

FIG. 2. The numerically calculated effect of a 35 nm thick Au film on a perfectly conducting surface. Lower curve, no coating; middle curve, Au film; upper curve, perfectly conducting film. The Au film effect is of order 50% of the perfectly conducting film effect in the 100–200 nm range.
$\epsilon_1(\omega)$ for Au

$\epsilon_1 = \infty$

$\epsilon_1 = 1$
\[ \varepsilon_1 = \infty \quad \varepsilon_1(\omega) \quad \varepsilon_1 = \infty \]

d

\[ a \]