LEFSCHETZ DISTRIBUTION OF LIE FOLIATIONS

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Abstract. Let \( F \) be a Lie foliation on a closed manifold \( M \) with structural Lie group \( G \). Its transverse Lie structure can be considered as a transverse action \( \Phi \) of \( G \) on \( (M,F) \); i.e., an “action” which is defined up to leafwise homotopies. This \( \Phi \) induces an action \( \Phi^\ast \) of \( G \) on the reduced leafwise cohomology \( \overline{H}(F) \). By using leafwise Hodge theory, the supertrace of \( \Phi^\ast \) can be defined as a distribution \( L_{\text{dis}}(F) \) on \( G \) called the Lefschetz distribution of \( F \). A distributional version of the Gauss-Bonett theorem is proved, which describes \( L_{\text{dis}}(F) \) around the identity element. On any small enough open subset of \( G \), \( L_{\text{dis}}(F) \) is described by a distributional version of the Lefschetz trace formula.

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1. Introduction

Let $\mathcal{F}$ be a $C^\infty$ foliation on a manifold $M$. Let $\text{Diff}(M, \mathcal{F})$ be the group of foliated diffeomorphisms $(M, \mathcal{F}) \to (M, \mathcal{F})$. The elements of $\text{Diff}(M, \mathcal{F})$ that are $C^\infty$ leafwisely homotopic to $\text{id}_M$ form a normal subgroup $\text{Diff}_0(\mathcal{F})$, and let $\overline{\text{Diff}}(M, \mathcal{F})$ denote the corresponding quotient group.

A right transverse action of a group $G$ on $(M, \mathcal{F})$ is an anti-homomorphism $\Phi : G \to \overline{\text{Diff}}(M, \mathcal{F})$. A local representation of $\Phi$ on some open subset $O \subset G$ is a map $\phi : M \times O \to M$ such that $\phi_g = \phi(\cdot, g)$ is a foliated diffeomorphism representing $\Phi_g$ for all $g \in G$. Then $\Phi$ is said to be of class $C^\infty$ if it has a $C^\infty$ local representation on each small enough open subset of $G$.

Recall that the leafwise de Rham complex $(\Omega(\mathcal{F}), d_{\mathcal{F}})$ consists of the differential forms on the leaves which are $C^\infty$ on $M$, endowed with the de Rham derivative of the leaves. Its cohomology $H(\mathcal{F})$ is called the leafwise cohomology. This becomes a topological vector space with the topology induced by the $C^\infty$ topology, and its maximal Hausdorff quotient is the reduced leafwise cohomology $\overline{H}(\mathcal{F})$.

Consider the canonical right action of $\text{Diff}(M, \mathcal{F})$ on $\overline{H}(\mathcal{F})$ defined by pulling-back leafwise differential forms. Since $\text{Diff}_0(\mathcal{F})$ acts trivially, we get a canonical right action of $\overline{\text{Diff}}(M, \mathcal{F})$ on $\overline{H}(\mathcal{F})$. Then any right transverse action $\Phi$ of a group $G$ on $(M, \mathcal{F})$ induces a left action $\Phi^*$ of $G$ on $\overline{H}(\mathcal{F})$.

Suppose from now on that $\mathcal{F}$ is a Lie foliation and the manifold $M$ is closed. It is shown that its transverse Lie structure can be described as a right transverse action $\Phi$ of its structural Lie group $G$ on $(M, \mathcal{F})$. Consider the induced left action $\Phi^*$ of $G$ on $\overline{H}(\mathcal{F})$. For each $g \in G$, we would like to define the supertrace $\text{Tr}^g \Phi_g^*$, which could be called the leafwise Lefschetz number $L(\Phi_g)$ of $\Phi_g$. This can be achieved when $\overline{H}(\mathcal{F})$ is of finite dimension, obtaining a $C^\infty$ function $L(\mathcal{F})$ on $G$ defined by $L(\mathcal{F})(g) = L(\Phi_g)$; the value of $L(\mathcal{F})$ at the identity element $e$ of $G$ is the Euler characteristic $\chi(\mathcal{F})$ of $\overline{H}(\mathcal{F})$, which can be called the leafwise Euler characteristic of $\mathcal{F}$. But $\overline{H}(\mathcal{F})$ may be of infinite dimension, even when the leaves are dense \cite{[1]}, and thus $L(\mathcal{F})$ is not defined in general.

The first goal of this paper is to show that, in general, the role of the function $L(\mathcal{F})$ can be played by a distribution $L_{\text{dis}}(\mathcal{F})$ on $G$, called the Lefschetz distribution of $\mathcal{F}$, whose singularities are motivated by the infinite dimension of $\overline{H}(\mathcal{F})$. 

10.3. Bundles over homogeneous spaces and the Selberg trace formula
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References
The first ingredient to define $L_{\text{dis}}(\mathcal{F})$ is the leafwise Hodge theory studied in [2] for Riemannian foliations; recall that Lie foliations form a specially important class of Riemannian foliations [19]. Fix a bundle-like metric on $M$ whose transverse part is induced by a left invariant Riemannian metric on $G$. For the induced Riemannian structure on the leaves, let $\Delta_F$ be the Laplacian of the leaves operating in $\Omega(\mathcal{F})$. The kernel $\mathcal{H}(\mathcal{F})$ of $\Delta_F$ is the space of harmonic forms on the leaves that are $C^\infty$ on $M$. The metric induces an $L^2$ inner product on $\Omega(\mathcal{F})$, obtaining a Hilbert space $\Omega(\mathcal{F})$. Then $\Delta_F$ is an essentially self-adjoint operator in $\Omega(\mathcal{F})$ whose closure is denoted by $\Delta_F$. The kernel of $\Delta_F$ is denoted by $\mathcal{H}(\mathcal{F})$, and let $\Pi : \Omega(\mathcal{F}) \to \mathcal{H}(\mathcal{F})$ denote the orthogonal projection. In [2], it is proved that $\Pi$ has a restriction $\Pi : \Omega(\mathcal{F}) \to \mathcal{H}(\mathcal{F})$ that induces an isomorphism $\mathcal{H}(\mathcal{F}) \cong \mathcal{H}(\mathcal{F})$, which can be called the leafwise Hodge isomorphism.

Let $\Lambda$ be the volume form of $G$, and let $\phi : M \times O \to M$ be a $C^\infty$ local representation of $\Phi$. For each $f \in C^\infty_c(O)$, consider the operator

$$P_f = \int_G \phi_g^* f(g) \Lambda(g) \circ \Pi$$

in $\Omega(\mathcal{F})$. Our first main result is the following.

**Proposition 1.1.** $P_f$ is of trace class, and the functional $f \mapsto \text{Tr}^s P_f$ defines a distribution on $O$.

It can be easily seen that $\text{Tr}^s P_f$ is independent of the choice of $\phi$, and thus the distributions given by Proposition 1.1 can be combined to define a distribution $L_{\text{dis}}(\mathcal{F})$ on $G$; this is the Lefschetz distribution of $\mathcal{F}$.

Observe that $L_{\text{dis}}(\mathcal{F}) \equiv L(\mathcal{F}) \cdot \Lambda$ when $\mathcal{H}(\mathcal{F})$ is of finite dimension. This justifies the consideration of $L_{\text{dis}}(\mathcal{F})$ as a generalization of $L(\mathcal{F})$; in particular, the germ of $L_{\text{dis}}(\mathcal{F})$ at $e$ generalizes $\chi(\mathcal{F})$.

If the operators $P_f$ are restricted to $\Omega^i(\mathcal{F})$ for each degree $i$, its trace defines a distribution $\text{Tr}^i_{\text{dis}}(\mathcal{F})$, called distributional trace, whose germ at $e$ generalizes the leafwise Betti number $\beta^i(\mathcal{F}) = \dim \mathcal{H}(\mathcal{F})$.

The distributions $L_{\text{dis}}(\mathcal{F})$ and $\text{Tr}^i_{\text{dis}}(\mathcal{F})$ depend on $\Lambda$ and $\mathcal{F}$, endowed with the transverse Lie structure. If the leaves are dense, then the transverse Lie structure is determined by the foliation, and thus these distributions depend only on $\Lambda$ and the foliation. On the other hand, the dependence on $\Lambda$ can be avoided by using top dimensional currents instead of distributions, in the obvious way.

Our second goal is to prove a distributional version of the Gauss-Bonnet theorem, which describes $L_{\text{dis}}(\mathcal{F})$ around $e$. Let $R_\mathcal{F}$ be the curvature of the leafwise metric. Suppose for simplicity that $\mathcal{F}$ is oriented. Then $\text{Pf}(R_\mathcal{F}/2\pi) \in \Omega^p(\mathcal{F})$ ($p = \dim \mathcal{F}$) can be called the leafwise Euler form. This form can be paired with $\Lambda$, considered as a transverse invariant measure, to give a differential form $\omega_\Lambda \wedge \text{Pf}(R_\mathcal{F}/2\pi)$ of top degree on $M$. In particular, if $\dim \mathcal{F} = 2$, then

$$\omega_\Lambda \wedge \text{Pf}(R_\mathcal{F}/2\pi) = \frac{1}{2\pi} K_\mathcal{F} \omega_M,$$
where $K_F$ is the Gauss curvature of the leaves and $\omega_M$ is the volume form of $M$. Let $\delta_e$ denote the Dirac measure at $e$.

**Theorem 1.2** (Distributional Gauss-Bonett theorem). We have

$$L_{\text{dis}}(F) = \int_M \omega_\Lambda \wedge \text{Pf}(R_F/2\pi) \cdot \delta_e$$

on some neighborhood of $e$.

To prove Theorem 1.2, we really prove that

$$L_{\text{dis}}(F) = \chi_\Lambda(F) \cdot \delta_e$$

around $e$, where $\Lambda$ is considered as a transverse invariant measure of $F$, and $\chi_\Lambda(F)$ is the $\Lambda$-Euler characteristic of $F$ introduced by Connes [9]. Then Theorem 1.2 follows from the index theorem of [9].

The third goal is to prove a distributional version of the Lefschetz trace formula, which describes $L_{\text{dis}}(F)$ on any small enough open subset of $G$. For a $C^\infty$ local representation $\phi : M \times O \to M$ of $\Phi$, let $\phi' : M \times O \to M \times O$ be the map defined by $\phi'(x,g) = (\phi_g(x),g)$. The fixed point set of $\phi'$, Fix$(\phi')$, consists of the points $(x,g)$ such that $\phi_g(x) = x$. A point $(x,g) \in \text{Fix}(\phi')$ is said to be leafwise simple when $\phi_g - \text{id} : T_xF \to T_xF$ is an isomorphism; in this case, the sign of the determinant of this isomorphism is denoted by $\epsilon(x,g)$. The set of leafwise simple fixed points of $\phi'$ is denoted by Fix$_0(\phi')$.

Let $\text{pr}_1 : M \times O \to M$ and $\text{pr}_2 : M \times O \to O$ be the factor projections. It is proved that Fix$_0(\phi')$ is a $C^\infty$ manifold of dimension equal to codim $F$. Moreover the restriction $\text{pr}_1 : \text{Fix}_0(\phi') \to M$ is a local embedding transverse to $F$. So $\Lambda$ defines a measure $\Lambda'_\text{Fix}_0(\phi')$ on Fix$_0(\phi')$. Observe that $\text{pr}_2 : \text{Fix}(\phi') \to O$ is a proper map.

**Theorem 1.3** (Distributional Lefschetz trace formula). Suppose that every fixed point of $\phi'$ is leafwise simple. Then

$$L_{\text{dis}}(F) = \text{pr}_{2*}(\epsilon \cdot \Lambda'_\text{Fix}(\phi'))$$

on $O$.

To prove Theorem 1.3, we consider certain submanifold $M'_1 \subset M \times O$ endowed with a foliation $F'_1$, whose leaves are of the form $L \times \{g\}$, where $L$ is a leaf of $F$ and $g \in G$. It is proved that $\text{pr}_2(M'_1)$ is open in some orbit of the adjoint action of $G$ on itself. Let $\text{pr}_1 : M'_1 \to M$ be a local diffeomorphism, and $F'_1 = \text{pr}_1^* F$. So $\Lambda$ lifts to a transverse invariant measure $\Lambda'_1$ of $F'_1$. Moreover the restriction $\phi'_1$ of $\phi'$ to $M'_1$ is defined and maps each leaf of $F'_1$ to itself. For each $f \in C^\infty_c(O)$ supported in an appropriate open subset $O_1 \subset O$, the transverse invariant measure $\Lambda'^{f}_{1} = \text{pr}_{2*}^f f \cdot \Lambda'_1$ is compactly supported. Then the $\Lambda'^{f}_{1}$-Lefschetz number $L_{\Lambda'^{f}_{1}}(\phi'_1)$ is defined according to [14]. Without assuming any condition on the fixed point set, we show that

$$\langle L_{\text{dis}}(F), f \rangle = L_{\Lambda'^{f}_{1}}(\phi'_1).$$
We have that \( \text{Fix}(\phi'_1) \) is a \( C^\infty \) local transversal of \( \mathcal{F}'_1 \). Hence Theorem 1.3 follows from \((2)\) and the foliation Lefschetz theorem of \([14, 24]\).

The numbers \( \chi_\Lambda(\mathcal{F}) \) and \( L_{\Lambda'_1, f}(\phi'_1) \) are defined by using \( L^2 \) differential forms on the leaves, whilst \( L_{\text{dis}}(\mathcal{F}) \) is defined by using leafwise differential forms that are \( C^\infty \) on \( M \). These are sharply different conditions when the leaves are not compact. So \((1)\) and \((2)\) are surprising relations.

By \((2)\), \( L_{\text{dis}}(\mathcal{F}) \) is supported in the union of a discrete set of orbits of the adjoint action. Therefore, when \( \text{codim} \mathcal{F} > 0 \), \( L_{\text{dis}}(\mathcal{F}) \) is \( C^\infty \) just when it is trivial, obtaining the following.

**Corollary 1.4.** If \( \overline{\mathcal{H}}(\mathcal{F}) \) is of finite dimension and \( \text{codim} \mathcal{F} > 0 \), then \( L_{\text{dis}}(\mathcal{F}) \equiv L(\mathcal{F}) = 0 \).

By Corollary 1.4, \( \chi(\mathcal{F}) \) is useless: it vanishes just when it can be defined. Moreover \( \chi_\Lambda(\mathcal{F}) = 0 \) in this case by \((1)\). So, when \( \text{codim} \mathcal{F} > 0 \), the condition \( \chi_\Lambda(\mathcal{F}) \neq 0 \) yields \( \dim \overline{\mathcal{H}}(\mathcal{F}) = \infty \). More precise results of this type would be desirable.

Let \( \dim \mathcal{F} = p \). When the leaves are dense, \( \beta^0(\mathcal{F}) \) and \( \beta^p(\mathcal{F}) \) are finite, and thus \( \text{Tr}^0_{\text{dis}}(\mathcal{F}) \) and \( \text{Tr}^p_{\text{dis}}(\mathcal{F}) \) are \( C^\infty \). On the other hand, when the leaves are not compact, the \( \Lambda \)-Betti numbers of \([9]\) satisfy \( \beta^1_\Lambda(\mathcal{F}) = \beta^p_\Lambda(\mathcal{F}) = 0 \).

Then the following result follows from \((1)\) and Corollary 1.4.

**Corollary 1.5.** If \( \text{codim} \mathcal{F} > 0 \), \( \dim \mathcal{F} = 2 \) and the leaves are dense, then \( \text{Tr}^1_{\text{dis}}(\mathcal{F}) - \beta^1_\Lambda(\mathcal{F}) \cdot \delta_e \) is \( C^\infty \) around \( e \).

In Corollary 1.5, we could say that \( \beta^1_\Lambda(\mathcal{F}) \cdot \delta_e \) is the “singular part” of \( \text{Tr}^1_{\text{dis}}(\mathcal{F}) \) around \( e \).

**Corollary 1.6.** Suppose that \( \text{codim} \mathcal{F} > 0 \) and \( \dim \mathcal{F} = 2 \). If there is a nontrivial harmonic \( L^2 \) differential form of degree one on some leaf, then \( \dim \overline{\mathcal{H}}^1(\mathcal{F}) = \infty \).

It would be nice to generalize Corollary 1.6 for arbitrary dimension. Thus we conjecture the following.

**Conjecture 1.7.** If \( \text{codim} \mathcal{F} > 0 \) and the leaves are dense, then \( \text{Tr}^i_{\text{dis}}(\mathcal{F}) - \beta^i_\Lambda(\mathcal{F}) \cdot \delta_e \) is \( C^\infty \) around \( e \) for each degree \( i \).

The main results were proved in \([3]\) for the case of codimension one. Our results also overlap the corresponding results of \([20]\).

We hope to prove elsewhere another version of Theorem 1.3 with a more general condition on the fixed points, always satisfied by some local representation \( \phi \) of \( \Phi \) defined around any point of \( G \). By \((2)\), what is needed is another version of the Lefschetz theorem of \([14]\), which holds for more general fixed point sets when the transverse measure is \( C^\infty \).

The idea of using such type of trace class operators to define distributional spectral invariants is due to Atiyah and Singer \([5, 30]\). They consider transversally elliptic operators with respect to compact Lie group actions. Further generalizations to foliations and non-compact Lie group actions were
given in [21, 10, 15, 17]. In our case, $\Delta_F$ is not transversally elliptic with respect to any Lie group action or any foliation, but it can be considered as being “transversely elliptic” with respect to the structural transverse action; this simply means that it is elliptic along the leaves of $F$.

2. Transverse actions

Recall that a foliation $F$ on a manifold $M$ can be described by a foliated cocycle, which is a collection $\{U_i, f_i\}$, where $\{U_i\}$ is an open cover of $X$ and each $f_i$ is a topological submersion of $U_i$ onto some manifold $T_i$ whose fibers are connected open subsets of $\mathbb{R}^n$, such that the following compatibility condition is satisfied: for every $x \in U_i \cap U_j$, there is an open neighborhood $U^*_{i,j}$ of $x$ in $U_i \cap U_j$ and a homeomorphism $h^*_{i,j} : f_i(U^*_{i,j}) \to f_j(U^*_{i,j})$ such that $f_j = h^*_{i,j} \circ f_i$ on $U^*_{i,j}$. Two foliated cocycles describe the same foliation $F$ when their union is a foliated cocycle. The leaf topology on $M$ is the topology with a base given by the open sets of the fibers of all the submersions $f_i$. The leaves of $F$ are the connected components of $M$ with the leaf topology. The leaf through each point $x \in M$ is denoted by $L_x$. The pseudogroup on $\bigsqcup T_i$ generated by the maps $h^*_{i,j}$, given by the compatibility condition, is called (a representative of) the holonomy pseudogroup of $F$, and describes the “transverse dynamics” of $F$. Different foliated cocycles of $F$ induce equivalent pseudogroups in the sense of [12, 13].

Another representative of the holonomy pseudogroup is defined on any transversal of $F$ that meets every leaf. It is generated by “sliding” small open subsets (local transversals) along the leaves; its precise definition is given in [12].

When $M$ is a $C^\infty$ manifold, it is said that $F$ is $C^\infty$ if it is described by a foliated cocycle $\{U_i, f_i\}$ which is $C^\infty$ in the sense that each $f_i$ is a $C^\infty$ submersion to some $C^\infty$ manifold.

Let $\Gamma$ be a group of homeomorphisms of a manifold $T$. A foliated cocycle $(U_i, f_i)$ of $F$, with $f_i : U_i \to T_i$, is said to be $(T, \Gamma)$-valued when each $T_i$ is an open subset of $T$, and the maps $h^*_{i,j}$, given by the compatibility condition, are restrictions of maps in $\Gamma$. A transverse $(T, \Gamma)$-structure of $F$ is given by a $(T, \Gamma)$-valued foliated cocycle, and two $(T, \Gamma)$-valued foliated cocycles define the same transverse $(T, \Gamma)$-structure when their union is a $(T, \Gamma)$-valued foliated cocycle. When $F$ is endowed with a transverse $(T, \Gamma)$-structure, it is called a $(T, \Gamma)$-foliation.

Let $F$ and $G$ be foliations on manifolds $M$ and $N$, respectively. Recall the following concepts. A foliated map $f : (M, F) \to (N, G)$ is a map $f : M \to N$ that maps each leaf of $F$ to a leaf of $G$; the simpler notation $f : F \to G$ will be also used. A leafwise homotopy (or integrable homotopy) between two continuous foliated maps $f, f' : (M, F) \to (N, G)$ is a continuous map $H : M \times I \to N$ ($I = [0, 1]$) such that the path $H(x, \cdot) : I \to N$ lies in a leaf of $G$ for each $x \in M$; in this case, it is said that $f$ and $f'$ are leafwisely homotopic (or integrably homotopic).
Suppose from now on that \( F \) and \( G \) are \( C^\infty \). Two \( C^\infty \) foliated maps are said to be \( C^\infty \) leafwisely homotopic when there is a \( C^\infty \) leafwise homotopy between them. As usual, \( T\mathcal{F} \subset TM \) denotes the subbundle of vectors tangent to the leaves of \( \mathcal{F} \), \( \mathfrak{X}(M, \mathcal{F}) \) denotes the Lie algebra of infinitesimal transformations of \((M, \mathcal{F})\), and \( \mathfrak{X}(\mathcal{F}) \subset \mathfrak{X}(M, \mathcal{F}) \) is the normal Lie subalgebra of vector fields tangent to the leaves of \( \mathcal{F} \) (\( C^\infty \) sections of \( T\mathcal{F} \rightarrow M \)). Then we can consider the quotient Lie algebra \( \mathfrak{X}(M, \mathcal{F}) = \mathfrak{X}(M, \mathcal{F})/\mathfrak{X}(\mathcal{F}) \), whose elements are called transverse vector fields. Observe that, for each \( x \in M \), the evaluation map \( ev_x : \mathfrak{X}(M, \mathcal{F}) \rightarrow T_xM \) induces a map \( \overline{ev}_x : \mathfrak{X}(M, \mathcal{F}) \rightarrow T_xM/T_x\mathcal{F} \), which can be also called evaluation map. For any Lie algebra \( g \), a homomorphism \( g \rightarrow \mathfrak{X}(M, \mathcal{F}) \) is called an infinitesimal transverse action of \( g \) on \( (M, \mathcal{F}) \). In particular, we have a canonical infinitesimal transverse action of \( \mathfrak{X}(M, \mathcal{F}) \) on \( (M, \mathcal{F}) \).

Let \( \text{Diff}(M, \mathcal{F}) \) be the group of \( C^\infty \) foliated diffeomorphisms \( (M, \mathcal{F}) \rightarrow (M, \mathcal{F}) \) with the operation of composition, let \( \text{Diff}(\mathcal{F}) \subset \text{Diff}(M, \mathcal{F}) \) be the normal subgroup \( C^\infty \) foliated diffeomorphisms that preserve each leaf of \( \mathcal{F} \), and let \( \text{Diff}_0(\mathcal{F}) \subset \text{Diff}(\mathcal{F}) \) be the normal subgroup of \( C^\infty \) foliated diffeomorphisms that are \( C^\infty \) leafwisely homotopic to the identity map. Then we can consider the quotient group \( \text{Diff}(M, \mathcal{F}) = \text{Diff}(M, \mathcal{F})/\text{Diff}_0(\mathcal{F}) \), whose operation is also denoted by “\( \circ \)”. The elements of \( \text{Diff}(M, \mathcal{F}) \) can be called transverse transformations of \( (M, \mathcal{F}) \). For any group \( G \), an anti-homomorphism \( \Phi : G \rightarrow \text{Diff}(M, \mathcal{F}) \), \( g \mapsto \Phi_g \), is called a right transverse action of \( G \) on \( (M, \mathcal{F}) \). For an open subset \( O \subset G \), a map \( \phi : M \times O \rightarrow M \) is called a local representation of \( \Phi \) on \( O \) if \( \phi_g = \phi(\cdot, g) \in \Phi_g \) for all \( g \in O \). For any leaf \( L \) of \( \mathcal{F} \) and any \( g \in O \), the leaf \( \phi_g(L) \) is independent of the local representative \( \phi \), and thus it will be denoted by \( \Phi_g(L) \). When \( G \) is a Lie group, \( \Phi \) is said to be of class \( C^\infty \) if it has a \( C^\infty \) local representation around each element of \( G \).

Somehow, we can think of \( \text{Diff}(M, \mathcal{F}) \) as a Lie group whose Lie algebra is \( \mathfrak{X}(M, \mathcal{F}) \); indeed, it will be proved elsewhere that, if \( G \) is a simply connected Lie group and \( \mathfrak{g} \) is its Lie algebra of left invariant vector fields, then there is a canonical bijection between infinitesimal transverse actions of \( \mathfrak{g} \) on \( (M, \mathcal{F}) \) and \( C^\infty \) right transverse actions of \( G \) on \( (M, \mathcal{F}) \).

The leafwise de Rham complex \( (\Omega(\mathcal{F}), d_\mathcal{F}) \) is the space of differential forms on the leaves smooth on \( M \) (\( C^\infty \) sections of \( \wedge T\mathcal{F}^* \rightarrow M \)) endowed with the leafwise de Rham differential. It is also a topological vector space with the \( C^\infty \) topology, and \( d_\mathcal{F} \) is continuous. The cohomology \( H(\mathcal{F}) \) of \( (\Omega(\mathcal{F}), d_\mathcal{F}) \) is called the leafwise cohomology of \( \mathcal{F} \), which is a topological vector space with the induced topology. Its maximal Hausdorff quotient \( \overline{\Omega}(\mathcal{F}) = H(\mathcal{F})/\overline{0} \) is called the reduced leafwise cohomology.

By pulling back leafwise differential forms, any \( C^\infty \) foliated map \( f : (M, \mathcal{F}) \rightarrow (N, \mathcal{G}) \) induces a continuous homomorphism of complexes, \( f^* : \Omega(\mathcal{G}) \rightarrow \Omega(\mathcal{F}) \), obtaining a continuous homomorphism \( f^* : \overline{\Omega}(\mathcal{G}) \rightarrow \overline{\Omega}(\mathcal{F}) \). Moreover, if \( f \) is \( C^\infty \) leafwisely homotopic to another \( C^\infty \) foliated map
\[ f' : (M, \mathcal{F}) \to (M, \mathcal{F}) \], then \( f^* = f'^* : \overline{\mathcal{P}}(G) \to \overline{\mathcal{P}}(\mathcal{F}) \) by standard arguments \([7]\). Therefore, for any \( F \in \text{Diff}(M, \mathcal{F}) \) and any \( f \in F \), the endomorphism \( f^* \) of \( \overline{\mathcal{P}}(\mathcal{F}) \) can be denoted by \( F^* \). So any right transverse action \( \Phi \) of a group \( G \) on \( (M, \mathcal{F}) \) induces a left action \( \Phi^* \) of \( G \) on \( \overline{\mathcal{P}}(\mathcal{F}) \) given by \( (g, \xi) \mapsto \Phi_g^* \xi \).

3. Lie foliations

Let \( \mathcal{F} \) be a \( C^\infty \) foliation of codimension \( q \) on a \( C^\infty \) closed manifold \( M \). Let \( G \) be a simply connected Lie group of dimension \( q \), and \( \mathfrak{g} \) its Lie algebra of left invariant vector fields. A transverse Lie structure of \( \mathcal{F} \), with structural Lie group \( G \) and structural Lie algebra \( \mathfrak{g} \), can be described with any of the following objects that determine each other \([11, 19]\):

(L.1) A transverse \((G, G)\)-structure of \( \mathcal{F} \), where \( G \) is identified with the group of its left translations.

(L.2) A \( \mathfrak{g} \)-valued 1-form \( \omega \) on \( M \) such that \( \omega_x : T_x M \to \mathfrak{g} \) is surjective with kernel \( T_x \mathcal{F} \) for every \( x \in M \), and
\[
d\omega + \frac{1}{2} [\omega, \omega] = 0 .
\]

(L.3) A homomorphism \( \theta : \mathfrak{g} \to \overline{\mathfrak{X}}(M, \mathcal{F}) \) such that the composite
\[
\mathfrak{g} \xrightarrow{\theta} \overline{\mathfrak{X}}(M, \mathcal{F}) \xrightarrow{\overline{\pi}_x} T_x M/T_x \mathcal{F}
\]
is an isomorphism for every \( x \in M \).

In (L.1), the elements of \( G \) whose corresponding left translations are involved in the definition of the transverse \((G, G)\)-structure form a subgroup \( \Gamma \), which is called the holonomy group of \( \mathcal{F} \). So the transverse \((G, G)\)-structure is a transverse \((G, \Gamma)\)-structure. In (L.2) and (L.3), \( \omega \) and \( \theta \) can be respectively called the structural form and the structural infinitesimal transverse action.

A \( C^\infty \) foliation endowed with a transverse Lie structure is called a Lie foliation; the terms Lie \( G \)-foliation or Lie \( \mathfrak{g} \)-foliation are used too. If the leaves are dense, then the transverse Lie structure is unique, and thus it is determined by the foliation.

A Lie \( G \)-foliation \( \mathcal{F} \) on a \( C^\infty \) closed manifold \( M \) has the following description due to Fedida \([11, 19]\). There exists a regular covering \( \pi : \tilde{M} \to M \), a fibre bundle \( D : \tilde{M} \to G \) and an injective homomorphism \( h : \text{Aut}(\pi) \to G \) such that the leaves of \( \tilde{\mathcal{F}} = \pi^* \mathcal{F} \) are the fibres of \( D \), and \( D \) is \( h \)-equivariant; i.e.,
\[
D \circ \sigma(\tilde{x}) = h(\sigma) \cdot D(\tilde{x})
\]
for all \( \tilde{x} \in \tilde{M} \) and \( \sigma \in \text{Aut}(\pi) \). This \( h \) is called the holonomy homomorphism. By using the covering space \( \text{ker}(h) \backslash \tilde{M} \) of \( M \) if necessary, we can assume that \( h \) is injective, and thus \( \pi \) restricts to diffeomorphisms of the leaves of \( \tilde{\mathcal{F}} \) to the leaves of \( \mathcal{F} \). The leaf of \( \tilde{\mathcal{F}} \) through each point \( \tilde{x} \in \tilde{M} \) will be denoted by \( \tilde{L}_{\tilde{x}} \).
Given a \((G,G)\)-valued foliated cocycle \(\{U_i,f_i\}\) defining the transverse Lie structure according to (L.1), the \(\mathfrak{g}\)-valued 1-form \(\omega\) of (L.2) and the infinitesimal transverse action \(\theta\) of (L.3) can be defined as follows. For \(x \in U_i\) and \(v \in T_xM\), \(\omega_x(v)\) is the left invariant vector field on \(G\) whose value at \(f_i(x)\) is \(f_i_x(v)\). To define \(\theta\), fix an auxiliary vector subbundle \(\nu \subset TM\) complementary of \(T\mathcal{F}\) (\(TM = \nu \oplus T\mathcal{F}\)). Each \(X \in \mathfrak{g}\) defines a \(C^\infty\) vector field \(X^\nu \in \mathcal{X}(M,\mathcal{F})\) by the conditions \(X^\nu(x) \in \nu_x\) and \(f_i_x(X^\nu(x)) = X(f_i(x))\) if \(x \in U_i\). Then \(\theta(X)\) is the class of \(X^\nu\) in \(\overline{\mathcal{X}}(M,\mathcal{F})\), which is independent of the choice of \(\nu\).

By using Fedida’s geometric description of \(\mathcal{F}\), the definitions of \(\omega\) and \(X^\nu\) can be better understood:

- Let \(\omega_G\) be the canonical \(\mathfrak{g}\)-valued 1-form on \(G\) defined by \(\omega_G(X(g)) = X\) for any \(X \in \mathfrak{g}\) and any \(g \in G\). Then \(\omega\) is determined by the condition \(\pi^*\omega = D^\nu\omega_G\).
- Let \(\bar{\nu} = \pi^{-1}_x(\nu) \subset TM\), which is a vector subbundle complementary of \(T\overline{\mathcal{F}}\). Then, for any \(X \in \mathfrak{g}\), there is a unique \(\overline{X}^\nu \in \mathcal{X}(\overline{M},\overline{\mathcal{F}})\) which is a section of \(\bar{\nu}\) and satisfies \(D_x \circ \overline{X}^\nu = X \circ D\). Since \(D\) is \(h\)-equivariant, \(\overline{X}^\nu\) is \(\text{Aut}(\pi)\)-invariant. Then \(X^\nu\) is the projection of \(\overline{X}^\nu\) to \(M\).

4. STRUCTURAL TRANSVERSE ACTION

Let \(G\) be a simply connected Lie group, and let \(\mathcal{F}\) be a Lie \(G\)-foliation on a closed manifold \(M\). According to Section 2 the structural infinitesimal transverse action corresponds to a unique right transverse action of \(G\) on \((M,\mathcal{F})\), obtaining another description of the transverse Lie structure:

(L.4) A \(C^\infty\) right transverse action \(\Phi\) of \(G\) on \((M,\mathcal{F})\) which has a \(C^\infty\) local representation \(\phi\) around the identity element \(e\) of \(G\) such that the composite

\[
T_xG \xrightarrow{\phi_x^*} T_xM \xrightarrow{D_x^\nu} T_xM/T_x\mathcal{F}
\]

is an isomorphism for all \(x \in M\), where \(\phi_x^* = \phi(x,\cdot)\) and the second map is the canonical projection. This condition is independent of the choice of \(\phi\). This \(\Phi\) is called the structural transverse action.

To describe \(\Phi\), consider Fedida’s geometric description of \(\mathcal{F}\) (Section 3). For any \(g \in G\), take a continuous, piecewise \(C^\infty\) path \(c : I \rightarrow G\) with \(c(0) = e\) and \(c(1) = g\). For any \(\bar{x} \in \overline{M}\), there exists a unique continuous piecewise \(C^\infty\) path \(\overline{c}_x \colon I \rightarrow \overline{M}\) such that

- \(\overline{c}_x^e(0) = \bar{x}\),
- \(\overline{c}_x^e\) is tangent to \(\bar{\nu}\) at every \(t \in I\) where it is \(C^\infty\), and
- \(D \circ \overline{c}_x^e(t) = D(\bar{x}) \cdot c(t)\) for any \(t \in I\).

It is easy to see that such a \(\overline{c}_x^e\) depends smoothly on \(\bar{x}\).

Lemma 4.1. We have \(\sigma \circ \overline{c}_x^e = \overline{c}_{\sigma(\bar{x})}^e\) for \(\bar{x} \in \overline{M}\) and \(\sigma \in \text{Aut}(\pi)\).
Proof. This is a direct consequence of the $h$-equivariance of $D$ and the unicity of the paths $\\sigma_x$. \hfill \square

For each $g \in G$, let $\tilde{\phi}_g : (\tilde{M}, \tilde{\mathcal{F}}) \to (\tilde{M}, \tilde{\mathcal{F}})$ be the $C^\infty$ foliated diffeomorphism given by $\tilde{\phi}_g(\tilde{x}) = \tilde{c}_x^g(1)$. For any $\tilde{x} \in \tilde{M}$ and $\sigma \in \text{Aut}(\pi)$, we have
\[
\sigma \circ \tilde{\phi}_g(\tilde{x}) = \sigma \circ \tilde{c}_x^g(1) = \tilde{c}_x^{\sigma(\tilde{x})}(1) = \tilde{\phi}_g \circ \sigma(\tilde{x})
\]
by Lemma 4.1 yielding $\sigma \circ \tilde{\phi}_g = \tilde{\phi}_g \circ \sigma$. Therefore, there exists a unique $C^\infty$ foliated diffeomorphism $\phi_g : (M, \mathcal{F}) \to (M, \mathcal{F})$ such that $\pi \circ \tilde{\phi}_g = \phi_g \circ \pi$.

Lemma 4.2. The $C^\infty$ leafwise homotopy class of $\phi_g$ is independent of the choice of $c$.

Proof. Let $d : I \to G$ be another continuous and piecewise smooth path with $d(0) = e$ and $d(1) = g$, which defines a $C^\infty$ foliated map $\varphi_g : (M, \mathcal{F}) \to (M, \mathcal{F})$ as above. Since $G$ is simply connected, there exists a family of continuous and piecewise smooth paths $c_s : I \to G$, depending smoothly on $s \in I$, with $c_s(0) = e$, $c_s(1) = g$, $c_0 = e$ and $c_1 = d$. The paths $c_s$ induce a family of $C^\infty$ foliated maps $\phi_{g,s} : (M, \mathcal{F}) \to (M, \mathcal{F})$ as above, defining a $C^\infty$ leafwise homotopy between $\phi_g$ and $\varphi_g$. \hfill \square

Lemma 4.3. The $C^\infty$ leafwise homotopy class of $\phi_g$ is independent of the choice of $\nu$.

Proof. Let $\nu' \subset TM$ be another vector subbundle complementary of $T\mathcal{F}$, which can be used to define a $C^\infty$ foliated map $\phi'_g$ as above. It is easy to find a $C^\infty$ deformation of vector subbundles of $\nu_s \subset TM$ complementary of $T\mathcal{F}$, $s \in I$, with $\nu_0 = \nu$ and $\nu_1 = \nu'$. Then the foliated maps $\phi_{g,s}$, induced by the vector bundles $\nu_s$ as above, define a $C^\infty$ leafwise homotopy between $\phi_g$ and $\phi'_g$. \hfill \square

Therefore, for each $g$, the $C^\infty$ leafwise homotopy class $\Phi_g$ of $\phi_g$ depends only on $g$, $\mathcal{F}$ and its transverse Lie structure. So a map $\Phi : G \to \text{Diff}(M, \mathcal{F})$ is given by $g \mapsto \Phi_g$.

Lemma 4.4. $\Phi$ is a right transverse action of $G$ in $(M, \mathcal{F})$.

Proof. Given $g_1, g_2 \in G$, let $c_1, c_2 : I \to G$ be continuous, piecewise smooth paths such that $c_1(0) = c_2(0) = e$, $c_1(1) = g_1$ and $c_2(1) = g_2$, which are used to define $\phi_{g_1}$ and $\phi_{g_2}$ as above. Let $c : I \to G$ be the path product of $c_1$ and $L_{g_1} \circ c_2$, where $L_{g_1}$ denotes the left translation by $g_1$. We have $c(0) = e$ and $c(1) = g_1g_2$. We can use this $c$ to define $\phi_{g_1g_2}$, obtaining $\phi_{g_1g_2} = \phi_{g_2} \circ \phi_{g_1}$, and thus $\Phi_{g_1g_2} = \Phi_{g_2} \circ \Phi_{g_1}$. \hfill \square

Lemma 4.5. $\Phi$ is $C^\infty$.

Proof. It is easy to prove that each element of $G$ has a neighbourhood $O$ such that there is a $C^\infty$ map $c : I \times O \to G$ so that each $c_g = c(\cdot, g)$ is a path from $e$ to $g$. The corresponding foliated diffeomorphisms $\phi_g$ form a $C^\infty$ representation of $\Phi$ on $O$. \hfill \square
This construction defines the structural transverse action $\Phi$. According to Section 2, $\Phi$ induces a left action $\Phi^*$ of $G$ on $\mathcal{P}(\mathcal{F})$.

**Lemma 4.6.** There is a local representation $\varphi : M \times O \to M$ of $\Phi$ around the identity element $e$ such that $\varphi_e = id_M$.

**Proof.** Construct $\phi$ like in the proof of Lemma 4.5 such that $e \in O$ and $c_e$ is the constant path at $e$. □

Let $\varphi : M \times O \to M$ be a local representation of $\Phi$. A map $\tilde{\varphi} : \tilde{M} \times O \to \tilde{M}$ is called a lift of $\varphi$ if $\pi \circ \tilde{\varphi}_g = \varphi_g \circ \pi$ for all $g \in O$, where $\tilde{\varphi}_g = \tilde{\varphi}(\cdot, g)$. In particular, the above construction of $\phi$ also gives a lift $\tilde{\phi}$. Let $R_g : G \to G$ denote the right translation by any $g \in G$.

**Lemma 4.7.** Any $C^\infty$ lift $\tilde{\varphi} : \tilde{M} \times O \to \tilde{M}$ of each $C^\infty$ local representation $\varphi : M \times O \to M$ of $\Phi$, such that $O$ is connected, satisfies $D \circ \tilde{\varphi}_g = R_g \circ D$ for all $g \in O$.

**Proof.** It is enough to prove the result when $O$ is as small as desired. It is clear that the property of the statement is satisfied by the maps $\phi$ constructed above for connected $O$.

For an arbitrary $\varphi$, if $O$ is small enough and connected, there is some $\tilde{\varphi} : \tilde{M} \times O \to \tilde{M}$ defined by the above construction and some homotopy $H : M \times O \times I \to M$ between $\varphi$ and $\phi$ such that each path $t \mapsto H(x, g, t)$ is contained in a leaf of $\mathcal{F}$. This $H$ lifts to a homotopy $\tilde{H} : \tilde{M} \times O \times I \to \tilde{M}$ between $\tilde{\varphi}$ and $\tilde{\phi}$ so that each path $t \mapsto \tilde{H}(\tilde{x}, g, t)$ is contained in a leaf of $\tilde{\mathcal{F}}$. Then $D \circ \tilde{\varphi} = D \circ \tilde{\phi}$, completing the proof. □

**Corollary 4.8.** $\tilde{\varphi} : \tilde{L} \times O \to \tilde{M}$ is a $C^\infty$ embedding for each leaf $\tilde{L}$ of $\tilde{\mathcal{F}}$.

The transverse Lie structure of $\mathcal{F}$ lifts to a transverse Lie structure of $\tilde{\mathcal{F}}$, whose structural right transverse action is locally represented by the $C^\infty$ lifts of $C^\infty$ local representations of $\Phi$.

5. **The Hodge Isomorphism**

Recall that any Lie foliation is Riemannian [23]. Then fix a bundle-like metric on $M$ [23], and equip the leaves of $\mathcal{F}$ with the induced Riemannian metric. Let $\delta_F$ denote the leafwise coderivative on the leaves operating in $\Omega(\mathcal{F})$, and set $D_F = d_F + \delta_F$. Then $\Delta_F = D^2_F = d_F \circ d_F + d_F \circ \delta_F$ is the leafwise Laplacian operating in $\Omega(\mathcal{F})$. Let $\mathcal{H}(\mathcal{F}) = \ker \Delta_F$ (the space of leafwise harmonic forms which are smooth on $M$). Since the metric is bundle-like, the transverse volume element is holonomy invariant, which implies that $D_F$ and $\Delta_F$ are symmetric, and thus they have the same kernel.

Let $\Omega(\mathcal{F})$ be the Hilbert space of square integrable leafwise differential forms on $M$. The metric of $M$ induces a Hilbert structure in $\Omega(\mathcal{F})$. For any $C^\infty$ foliated map $f : (M, \mathcal{F}) \to (M, \mathcal{F})$, the endomorphism $f^*$ of $\Omega(\mathcal{F})$ is obviously $L^2$-bounded, and thus extends to a bounded operator $f^*$ in $\Omega(\mathcal{F})$. Consider $D_F$ and $\Delta_F$ as unbounded operators in $\Omega(\mathcal{F})$, which are
essentially self-adjoint [8], and whose closures are denoted by $D_F$ and $\Delta_F$ (see e.g. [4, 16]). By [2], $\mathcal{H}(F) = \ker \Delta_F$ is the closure of $\mathcal{H}(F)$ in $\Omega(F)$, and the orthogonal projection $\Pi : \Omega(F) \to \mathcal{H}(F)$ has a restriction $\Pi : \Omega(F) \to H(F)$, which induces a leafwise Hodge isomorphism

$$\Pi(F) \cong H(F).$$

For any $C^\infty$ foliated map $f : (M, F) \to (M, F)$, the homomorphism $f^* : \Pi(F) \to \Pi(F)$ corresponds to the operator $\Pi \circ f^*$ in $\mathcal{H}(F)$ via the Hodge isomorphism. So the left $G$-action on $\mathcal{H}(F)$, defined in Section 4, corresponds to the left $G$-action on $H(F)$ given by $(g, \alpha) \mapsto \Pi \circ \phi^*_g \alpha$ for any $\phi_g \in \Phi_g$.

Since the left action of $G$ on $\mathcal{H}(F)$ is $L^2$-continuous, we get an extended left action of $G$ on $\mathcal{H}(F)$ given by $(g, \alpha) \mapsto \Pi \circ \phi^*_g \alpha$ for any $\phi_g \in \Phi_g$.

These actions on $\mathcal{H}(F)$ and $H(F)$ are continuous on $G$ since $\Phi$ is $C^\infty$.

6. A class of smoothing operators

6.1. Preliminaries on smoothing and trace class operators. Let $\omega_M$ denote the volume forms of $M$. A smoothing operator in $\Omega(F)$ is a linear map $P : \Omega(F) \to \Omega(F)$, continuous with respect to the $C^\infty$ topology, given by

$$(P\alpha)(x) = \int_M k(x, y) \alpha(y) \omega_M(y)$$

for some $C^\infty$ section $k$ of $\bigwedge T_x^*F \otimes \bigwedge T_F$ over $M \times M$; thus

$$k(x, y) \in \bigwedge T_x^*F \otimes \bigwedge T_F \equiv \text{Hom}(\bigwedge T_y^*F, \bigwedge T_x^*F)$$

for any $x, y \in M$. This $k$ is called the smoothing kernel or Schwartz kernel of $P$. Such a $P$ defines a trace class operator in $\mathcal{H}(F)$, and we have

$$\text{Tr} P = \int_M \text{Tr} k(x, x) \omega_M(x).$$

The supertrace formalism will be also used. For any homogeneous operator $T$ in $\mathcal{H}(F)$ or in $\bigwedge T_{x,F}^*$, let $T^\pm$ denote its restriction to the even and odd degree part, and let $T^{(i)}$ denote its restriction to the part of degree $i$. If $T$ is of trace class, then its supertrace is

$$\text{Tr}^s T = \text{Tr} T^+ - \text{Tr} T^- = \sum_i (-1)^i \text{Tr} T^{(i)}. $$

Thus

$$\text{Tr}^s P = \int_M \text{Tr}^s k(x, x) \omega_M(x).$$

Let $W^k\Omega(F)$ denote the Sobolev space of order $k$ of leafwise differential forms on $M$, and let $\| \cdot \|_k$ denote a norm of $W^k\Omega(F)$. A continuous operator $P$ in $\Omega(F)$ is smoothing if and only if $P$ extends to a bounded operator $P : W^k\Omega(F) \to W^l\Omega(F)$ for any $k$ and $l$. 
If an operator $P$ in $\Omega(\mathcal{F})$ has an extension $P : W^k\Omega(\mathcal{F}) \to W^\ell\Omega(\mathcal{F})$, then $\|P\|_{k,\ell}$ denotes the norm of this extension; the notation $\|P\|_k$ is used when $k = \ell$. By the Sobolev embedding theorem, the trace of a smoothing operator $P$ in $\Omega(\mathcal{F})$ can be estimated in the following way: for any $k > \dim M$, there is some $C > 0$ independent of $P$ such that

$$\|\text{Tr } P\| \leq C \|P\|_{0,k}.$$  

6.2. The class $\mathcal{D}$. Let $\mathcal{A}$ be the set of all functions $\psi : \mathbb{R} \to \mathbb{C}$, extending to an entire function $\psi$ on $\mathbb{C}$ such that, for each compact set $K \subset \mathbb{R}$, the set of functions $\{(x \mapsto \psi(x + iy)) \mid y \in K\}$ is bounded in the Schwartz space $S(\mathbb{R})$. This $\mathcal{A}$ has a structure of Frechet algebra, and, in fact, it is a module over $\mathbb{C}[z]$. This algebra contains all functions with compactly supported Fourier transform, and the functions $x \mapsto e^{-tx^2}$ with $t > 0$.

By [25, Proposition 4.1], there exists a “functional calculus map” $\mathcal{A} \to \text{End}(\Omega(\mathcal{F}))$, $\psi \mapsto \psi(D_{\mathcal{F}})$, which is a continuous homomorphism of $\mathbb{C}[z]$-modules and of algebras. Any operator $\psi(D_{\mathcal{F}})$, $\psi \in \mathcal{A}$, extends to a bounded operator in $W^k\Omega(\mathcal{F})$ for any $k$ with the following estimate for its norm: there is some $C > 0$, independent of $\psi$, such that

$$\|\psi(D_{\mathcal{F}})\|_k \leq \int |\hat{\psi}(\xi)| e^{C|\xi|} d\xi,$$

where $\hat{\psi}$ denotes the Fourier transform of $\psi$. Therefore, for any natural $N$, the operator $(\text{id} + \Delta_{\mathcal{F}})^N \psi(D_{\mathcal{F}})$ extends to a bounded operator in $W^k\Omega(\mathcal{F})$ for any $k$ whose norm can be estimated as follows: there is some $C > 0$, independent of $\psi$, such that

$$\|(\text{id} + \Delta_{\mathcal{F}})^N \psi(D_{\mathcal{F}})\|_k \leq \int |(\text{id} - \partial^2_{\xi^2})^N \hat{\psi}(\xi)| e^{C|\xi|} d\xi.$$  

Fix a left-invariant Riemannian metric on $G$, and let $A$ denote its volume form. We can assume that the metrics on $M$ and $G$ agree in the sense that the maps $f_i$ of (L.1) are Riemannian submersions (Section 3). Thus $D : \tilde{M} \to G$ is a Riemannian submersion with respect to the lift of the bundle-like metric to $\tilde{M}$.

A leafwise differential operator in $\Omega(\mathcal{F})$ is a differential operator which involves only leafwise derivatives; for instance, $d_{\mathcal{F}}$, $\delta_{\mathcal{F}}$, $D_{\mathcal{F}}$ and $\Delta_{\mathcal{F}}$ are leafwise differential operators. A family of leafwise differential operators in $\Omega(\mathcal{F})$, $\mathcal{A} = \{A_v \mid v \in V\}$, is said to be smooth when $V$ is a $C^\infty$ manifold and, with respect to $C^\infty$ local coordinates, the local coefficients of each $A_v$ depend smoothly on $v$ in the $C^\infty$-topology. We also say that $\mathcal{A}$ is compactly supported when there is some compact subset $K \subset V$ such that $A_v = 0$ if $v \notin K$. Given another smooth family of leafwise differential operators in $\Omega(\mathcal{F})$ with the same parameter manifold, $B = \{B_v \mid v \in V\}$, the composite $A \circ B$ is the family defined by $(A \circ B)_v = A_v \circ B_v$. Similarly, we can define the sum $A + B$ and the product $\lambda \cdot A$ for some $\lambda \in \mathbb{R}$. 

We introduce the class $\mathcal{D}$ of operators $P : \Omega(\mathcal{F}) \to \Omega(\mathcal{F})$ of the form

$$ P = \int_{O} \phi_{g}^{*} \circ A_{g} \Lambda(g) \circ \psi(D_{\mathcal{F}}), $$

where $O$ is some open subset of $G$, $\phi : M \times O \to M$ is a $C^{\infty}$ local representation of $\Phi$, $A = \{A_{g} \mid g \in O\}$ is a smooth compactly supported family of leafwise differential operators in $\Omega(\mathcal{F})$, and $\psi \in A$.

**Proposition 6.1.** Any operator $P \in \mathcal{D}$ is a smoothing operator in $\Omega(\mathcal{F})$.

**Proof.** Let $P \in \mathcal{D}$ as above. By (5) and since the operator $\phi_{g}^{*}$ preserves any Sobolev space, $P$ defines a bounded operator in $W^{k,\Omega}(\mathcal{F})$ for any $k$.

Let $\varphi : M \times O_{0} \to M$ be a $C^{\infty}$ local representation of $\Phi$ on some open neighborhood $O_{0}$ of the identity element $e$; we can assume that $\varphi_{e} = \text{id}_{M}$ by Corollary 4.8. For any $Y \in \mathfrak{g}$, let $\tilde{Y}$ be the first order differential operator in $\Omega(\mathcal{F})$ defined by

$$ \tilde{Y}u = \frac{d}{dt} \varphi_{\exp tY}^{*}u \bigg|_{t=0}, $$

which makes sense because $\exp tY \in O_{0}$ for any $t > 0$ small enough.

Fix a base $Y_{1},...,Y_{q}$ of $\mathfrak{g}$. Then the second order differential operator $L = -\sum_{j=1}^{q} \tilde{Y}_{j}^{2}$ in $\Omega(\mathcal{F})$ is transversely elliptic. Moreover $\Delta_{\mathcal{F}}$ is leafwise elliptic. By the elliptic regularity theorem, it suffices to prove that $L^{N} \circ P$ and $\Delta_{\mathcal{F}}^{N} \circ P$ belong to $\mathcal{D}$ for any natural $N$. In turn, this follows by showing that $Q \circ P$ and $\tilde{Y} \circ P$ are in $\mathcal{D}$ for any leafwise differential operator $Q$ and any $Y \in \mathfrak{g}$.

We have

$$ Q \circ P = \int_{O} \phi_{g}^{*} \circ B_{g} \Lambda(g) \circ \psi(D_{\mathcal{F}}), $$

where $B_{g} = (\phi_{g}^{*})^{-1} \circ Q \circ \phi_{g}^{*} \circ A_{g}$. Since $\phi_{g}$ is a foliated map, it follows that $\{B_{g} \mid g \in O\}$ is a smooth family of leafwise differential operators, yielding $Q \circ P \in \mathcal{D}$.

For $g \in O$ and $a \in O_{0}$ close enough to $e$, let

$$ F_{a,g} = \phi_{ag} \circ \varphi_{a} \circ \phi_{g}^{-1}. $$

Observe that $F_{e,g} = \text{id}_{M}$ because $\varphi_{e} = \text{id}_{M}$. For each $Y \in \mathfrak{g}$, we get a smooth family $V_{Y} = \{V_{Y,g} \mid g \in O\}$ of first order leafwise differential operators in $\Omega(\mathcal{F})$ given by

$$ V_{Y,g}u = \frac{d}{dt} F_{\exp tY,g}^{*}u \bigg|_{t=0}. $$

Let also $L_{Y}A = \{(L_{Y}A)_{g} \mid g \in O\}$ be the smooth family of leafwise differential operators given by

$$ (L_{Y}A)_{g}u = \frac{d}{dt} A_{\exp(-tY),g}u \bigg|_{t=0}. $$

In particular, if $A_{g}$ is given by multiplication by $f(g)$ for some $f \in C_{c}^{\infty}(G)$, then $(L_{Y}A)_{g}$ is given by multiplication by $(Yf)(g)$.
We proceed as follows:

\[
\int_O \varphi^* \exp t Y \circ \phi^*_g \circ A_g \Lambda(g) = \int_O \phi^*_{\exp t Y \cdot g} F^*_{\exp t Y \cdot g} \circ A_g \Lambda(g) = \int_O \phi^*_g \circ F^*_{\exp t Y \cdot g} \circ A_{\exp t Y \cdot g} \Lambda(g),
\]

yielding

\[
\hat{Y} \circ P = \lim_{t \to 0} \frac{1}{t} \left( \int_O \varphi^* \exp t Y \circ \phi^*_g \circ A_g dg - \int_O \phi^*_g \circ A_g dg \right) \circ \psi(D_F)
\]

\[
= \lim_{t \to 0} \frac{1}{t} \left( \int_O \phi^*_g \circ F^*_{\exp t Y \cdot g} \circ A_{\exp t Y \cdot g} dg - \int O \phi^*_g \circ A_g dg \right) \circ \psi(D_F)
\]

\[
= \int_O \phi^*_g \circ (V_Y \circ A + L_Y A) dg \circ \psi(D_F).
\]

So \( \hat{Y} \circ P \in D \). □

With the above notation, by the proof of Proposition 6.1 and (5), it can be easily seen that, for integers \( k \leq \ell \), there are some \( C, C' > 0 \) and some natural \( N \) such that

\[
\|P\|^{k,\ell} \leq C' \max_{g \in K} \left| (id + \Delta_G)^N \hat{\psi}^i(\xi) \right| e^{C|\xi|} d\xi.
\]

Here, \( C \) depends on \( k \) and \( \ell \), and \( C' \) depends on \( k, \ell \) and \( A \).

6.3. A norm estimate. Let

\[
P = \int O \phi^*_g \cdot f(g) \Lambda(g) \circ \psi(D_F) \in D,
\]

where \( \phi \) and \( \psi \) are like in Section 6.2 and \( f \in C^\infty_c(O) \). In this case, (6) is improved by the following result, where \( \Delta_G \) denotes the Laplacian of \( G \).

**Proposition 6.2.** Let \( K \subset O \) be a compact subset containing \( \text{supp} f \). For naturals \( k \leq \ell \), there are some \( C, C'' > 0 \) and some natural \( N \), depending only on \( K, k \) and \( \ell \), such that

\[
\|P\|^{k,\ell} \leq C'' \max_{g \in K} \left| (id + \Delta_G)^N f(g) \right| \int \left| (id - \partial^2_\xi)^N \hat{\psi}^i(\xi) \right| e^{C|\xi|} d\xi.
\]

**Proof.** Fix an orthonormal frame \( Y_1, \ldots, Y_q \) of \( g \). Consider any multi-index \( J = (j_1, \ldots, j_k) \) with \( j_1, \ldots, j_k \in \{1, \ldots, q\} \). We use the standard notation \( |J| = k \), and, with the notation of the proof of Proposition 6.1, let:

- \( Y_J = Y_{j_1} \circ \cdots \circ Y_{j_k} \) (operating in \( C^\infty(G) \));
- \( \hat{Y}_J = \hat{Y}_{j_1} \circ \cdots \circ \hat{Y}_{j_k} \);
- \( V_J = V_{Y_{j_1}} \circ \cdots \circ V_{Y_{j_k}} \); and
- \( L_J A = L_{Y_{j_1}} \cdots L_{Y_{j_k}} A \) for any smooth family \( A \) of leafwise differential operators in \( \Omega(F) \).

Consider the empty multi-index \( \emptyset \) too, with \( |\emptyset| = 0 \), and define:

- \( Y_{\emptyset} = \text{id}_{C^\infty(G)} \);
\[ \tilde{\Phi}_0 = \text{id}_{\Omega(F)}; \]
\[ V_{0, g} = \text{id}_{\Omega(F)} \] for all \( g \in O \), defining a smooth family \( V_{0, g} \); and
\[ L_0 A = A \] for any smooth family \( A \) of leafwise differential operators in \( \Omega(F) \).

Given any natural \( N \), there is some \( C_1 > 0 \) such that
\[ \| \phi_g^* \|_k \leq C_1, \quad \| (L_J V_{FJ})_g \| \leq C_1, \]
\[ \| (Y_J f)(g) \| \leq C_1 \max_{g \in K} |(\text{id} + \Delta G)^N f(g)|, \]
\[ \| (\text{id} + \phi_g^{*-1} \circ \Delta_F \circ \phi_g^*)_N \circ \psi(\Delta_F) \|_k \leq C_1 \| (\text{id} + \Delta_F)^N \circ \psi(D_F) \|_k \]
for all \( g \in K \) and all multi-indices \( J \) and \( J' \) with \( |J|, |J'| \leq N \).

For any multi-index \( J \), we have
\[ \tilde{\Phi}_J \circ P = \int_O \phi_g^* \circ A_{J, g} \Lambda(g) \circ \psi(D_F), \]
where \( A_J = \{ A_{J, g} \mid g \in G \} \) is the smooth family of leafwise differential operators inductively defined by setting
\[ A_{\emptyset, g} = \text{id}_{\Omega(F)} \cdot f(g), \]
\[ A_{(J, J')} = V_J \circ A_J + L_J A_J. \]

By induction on \( |J| \), we easily get that \( A_J \) is a sum of smooth families of leafwise differential operators of the form
\[ L_{J_1} V_{J_1'} \circ \cdots \circ L_{J_{\ell}} V_{J_{\ell}'} \circ Y_{J''} f, \]
where \( J_1, J_1', \ldots, J_\ell, J'_\ell, J'' \) are possibly empty multi-indices satisfying
\[ |J_1| + |J_1'| + \cdots + |J_\ell| + |J'_\ell| + |J''| = |J|. \]

So there is some \( C_2 > 0 \) such that
\[ \| A_{J, g} \|_k \leq C_2 \max_{g \in K} |(\text{id} + \Delta G)^N f(g)| \]
for all \( g \in K \) and every multi-index \( J \) with \( |J| \leq N \). Hence
\[ \| \tilde{\Phi}_J \circ P \|_k \leq \int_O \| \phi_g^* \|_k \| A_{J, g} \|_k \text{ d}g \| \psi(D_F) \|_k \]
\[ \leq C_1 C_2 \max_{g \in K} |(\text{id} + \Delta G)^N f(g)| \int |\psi(\xi)| e^{C|\xi|} \text{ d}\xi \]
for some \( C > 0 \). On the other hand,
\[ \| (\text{id} + \Delta_F)^N \circ P \|_k \leq \int_O \| (\text{id} + \phi_g^{*-1} \circ \Delta_F \circ \phi_g^*)_N \circ \psi(\Delta_F) \|_k |f(g)| \Lambda(g) \]
\[ \leq C_1 \int \| (\text{id} + \Delta_F)^N \circ \psi(\Delta_F) \|_k |f(g)| \Lambda(g) \]
\[ \leq C_1 \max_{g \in K} |f(g)| \int |(\text{id} - \partial_\xi^2)^N \psi(\xi)| e^{C|\xi|} \text{ d}\xi. \]
for some \( C > 0 \) by (1). Now, the result follows because \( -\sum_{j=1}^{q} \hat{\gamma}_j^2 \) is transversely elliptic, and \( \Delta_F \) is leafwise elliptic. \( \square \)

6.4. **Parameter independence of the supertrace.** Choose an even function in \( A \), which can be written as \( x \mapsto \psi(x^2) \). Take also a \( C^\infty \) local representation \( \phi : M \times O \to M \) of \( \Phi \) and some \( f \in C^\infty_c(O) \). Then consider the one parameter family of operators \( P_t \in D, t > 0 \), defined by

\[
P_t = \int_O \phi^*_g \cdot f(g) \Lambda(g) \circ \psi(t\Delta_F)^2.
\]

**Lemma 6.3.** \( \text{Tr}^* P_t \) is independent of \( t \).

**Proof.** The proof is similar to the proof of the corresponding result in the heat equation proof of the Lefschetz trace formula (see e.g. [28]). We have

\[
\frac{d}{dt} \text{Tr}^* P_t = 2 \text{Tr}^* \int_O \phi^*_g \cdot f(g) \Lambda(g) \circ \Delta_F \circ \psi'(t\Delta_F) \circ \psi(t\Delta_F)
\]

\[
= 2 \text{Tr} \int_O \phi^*_g \cdot f(g) \Lambda(g) \circ d_F \circ \delta_F \circ \psi'(t\Delta_F) \circ \psi(t\Delta_F)
\]

\[
- 2 \text{Tr} \int_O \phi^*_g \cdot f(g) \Lambda(g) \circ d_F \circ \delta_F \circ \psi'(t\Delta_F) \circ \psi(t\Delta_F)
\]

\[
+ 2 \text{Tr} \int_O \phi^*_g \cdot f(g) \Lambda(g) \circ \delta_F \circ d_F \circ \psi'(t\Delta_F) \circ \psi(t\Delta_F)
\]

\[
- 2 \text{Tr} \int_O \phi^*_g \cdot f(g) \Lambda(g) \circ \delta_F \circ d_F \circ \psi'(t\Delta_F) \circ \psi(t\Delta_F).
\]

On the other hand, since the function \( x \mapsto \psi'(x^2) \) is in \( A \), we have

\[
\text{Tr} \int_O \phi^*_g \cdot f(g) \Lambda(g) \circ d_F \circ \delta_F \circ \psi'(t\Delta_F) \circ \psi(t\Delta_F)
\]

\[
= \text{Tr} d_F \circ \int_O \phi^*_g \cdot f(g) \Lambda(g) \circ \psi'(t\Delta_F) \circ \psi(t\Delta_F) \circ \delta_F
\]

\[
= \text{Tr} \psi(t\Delta_F) \circ \delta_F \circ d_F \circ \int_O \phi^*_g \cdot f(g) \Lambda(g) \circ \psi'(t\Delta_F)
\]

\[
= \text{Tr} \int_O \phi^*_g \cdot f(g) \Lambda(g) \circ \psi'(t\Delta_F) \circ \psi(t\Delta_F) \circ \delta_F \circ d_F
\]

\[
= \text{Tr} \int_O \phi^*_g \cdot f(g) \Lambda(g) \circ \delta_F \circ d_F \circ \psi'(t\Delta_F) \circ \psi(t\Delta_F),
\]

where we have used the well known fact that, if \( A \) is a trace class operator and \( B \) is bounded, then \( AB \) and \( BA \) are trace class operators with the same trace. Therefore \( \frac{d}{dt} \text{Tr}^* P_t = 0 \) as desired. \( \square \)

6.5. **The global action on the leafwise complex.** Let \( \mathcal{G} \) be the holonomy groupoid of \( F \). Since the leaves of Lie foliations have trivial holonomy groups, we have

\[
\mathcal{G} \equiv \{(x, y) \in M \times M \mid x \text{ and } y \text{ lie in the same leaf of } F\}.
\]
This is a $C^\infty$ submanifold of $M \times M$ which contains the diagonal $\Delta_M$. Let $d_F$ be the distance function of the leaves of $\mathcal{F}$. For each $r > 0$, the $r$-penumbra of $\Delta_M$ is defined by

$$\text{Pen}_G(\Delta_M, r) = \{ (x, y) \in G \mid d_F(x, y) < r \}.$$ 

Observe that a subset of $G$ has compact closure if and only if it is contained in some penumbra of $\Delta_M$. The product of two elements $(x_1, y_1), (x_2, y_2) \in G$ is defined when $y_1 = x_2$, and it is equal to $(x_1, y_2)$. The space of units of $G$ is $\Delta_M \equiv M$. The source and target projections $s, r : G \to M$ are the restrictions of the first and second factor projections $M \times M \to M$; thus

$$r^{-1}(x) = L_x \times \{ x \}, \qquad s^{-1}(x) = \{ x \} \times L_x$$

for each $x \in M$.

Let $S$ denote the $C^\infty$ vector bundle

$$s^* \bigwedge T\mathcal{F}^* \otimes r^* \bigwedge T\mathcal{F}$$

over $G$; thus

$$S_{(x,y)} \equiv \bigwedge T_x\mathcal{F}^* \otimes \bigwedge T_y\mathcal{F} \equiv \text{Hom} \left( \bigwedge T_y\mathcal{F}^*, \bigwedge T_x\mathcal{F}^* \right)$$

for each $(x, y) \in G$. Let $\omega_F$ be the volume form of the leaves of $\mathcal{F}$ (we assume that $\mathcal{F}$ is oriented). Recall that $C^\infty_c(S)$ is an algebra with the convolution product given by

$$(k_1 \cdot k_2)(x, y) = \int_{L_x} k_1(x, z) \circ k_2(z, y) \omega_F(z)$$

for $k_1, k_2 \in C^\infty_c(S)$ and $(x, y) \in G$. Recall also that the global action of $C^\infty_c(S)$ in $\Omega(\mathcal{F})$ is defined by

$$(k \cdot \alpha)(x) = \int_{L_x} k(x, y) \alpha(y) \omega_F(y)$$

for $k \in C^\infty_c(S), \alpha \in \Omega(\mathcal{F})$ and $x \in M$.

Consider the lift to $\widetilde{M}$ of the bundle-like metric of $M$, and its restriction to the leaves of $\mathcal{F}$. Let $U\Omega(\mathcal{F}) \subset \Omega(\mathcal{F})$ be the subcomplex of differential forms $\alpha$ whose covariant derivatives $\nabla^r \alpha$ of arbitrary order $r$ are uniformly bounded; this is a Frechet space with the metric induced by the seminorms

$$\| \alpha \|_r = \sup \{ \| \nabla^r \alpha(x) \| \mid x \in \widetilde{M} \}.$$ 

Observe that $\pi^*(\Omega(\mathcal{F})) \subset U\Omega(\mathcal{F})$.

The holonomy groupoid $\mathfrak{G}$ of $\mathcal{F}$ satisfies the same properties as $G$, except that, in $\mathfrak{G}$, the penumbras of the diagonal $\Delta_{\mathfrak{M}}$ have compact closure if and only $\mathfrak{M}$ is compact.

The map $\pi \times \pi : \widetilde{M} \times \widetilde{M} \to M \times M$ restricts to a covering map $\widetilde{G} \to G$, whose group of deck transformations is isomorphic to $\text{Aut}(\pi)$: for each $\sigma \in \text{Aut}(\pi)$, the corresponding element in $\text{Aut}(\widetilde{G} \to G)$ is the restriction $\sigma \times \sigma : \widetilde{G} \to G$.
Let \( \tilde{S} \) denote the \( \mathcal{C}^\infty \) vector bundle
\[
\tilde{s}^* \bigwedge T\tilde{F}^* \otimes \tilde{r}^* \bigwedge T\tilde{F}
\]
over \( \tilde{\mathfrak{F}} \), and let \( C^\infty_\Delta(\tilde{S}) \subset C^\infty(\tilde{S}) \) denote the subspace of sections supported in some penumbra of \( \Delta_M \). As above, this set becomes an algebra with the convolution product, and there is a \textit{global action} of \( C^\infty_\Delta(\tilde{S}) \) in \( U\Omega(\tilde{F}) \).

Any \( k \in C^\infty(S) \) lifts via \( \pi \times \pi \) to a section \( \tilde{k} \in C^\infty(\tilde{S}) \). Since \( \pi \) restricts to diffeomorphisms of the leaves of \( \tilde{F} \) to the leaves of \( F \), it follows that \( \tilde{k} \in C^\infty_\Delta(\tilde{S}) \) if \( k \in C^\infty_c(S) \).

Take any \( \psi \in \mathcal{A} \). For each leaf \( L \) of \( F \), denoting by \( \Delta_L \) the Laplacian of \( L \), the spectral theorem defines a smoothing operator \( \psi(\Delta_L) \) in \( \Omega(L) \), and the family
\[
\{ \psi(\Delta_L) \mid \text{L is a leaf of } F \}
\]
is also denoted by \( \psi(\Delta_F) \). By \cite{26} Proposition 2.10, the Schwartz kernels \( k_L \) of the operators \( \psi(\Delta_L) \) can be combined to define a section \( k \in C^\infty(S) \), called the \textit{leafwise smoothing kernel} or \textit{leafwise Schwartz kernel} of \( \psi(\Delta_F) \).

Suppose that the Fourier transform \( \hat{\psi} \) of \( \psi \) is supported in \( [-R, R] \) for some \( R > 0 \). Then, according to the proof of Assertion 1 in \cite{25} page 461, \( k \) is supported in the \( R \)-penumbra of \( \Delta_M \), and thus \( k \in C^\infty_\Delta(S) \). Moreover the operator \( \psi(D_F) \) in \( \Omega(F) \), defined by the spectral theorem, equals the operator given by the global action of \( k \).

Consider also the lift \( \tilde{k} \in C^\infty_\Delta(\tilde{S}) \), whose global action in \( U\Omega(\tilde{F}) \) defines an operator denoted by \( \psi(D_{\tilde{F}}) \). It is clear that the diagram
\[
\begin{array}{ccc}
U\Omega(\tilde{F}) & \xrightarrow{\psi(D_{\tilde{F}})} & U\Omega(\tilde{F}) \\
\pi^* \uparrow & & \uparrow \pi^* \\
\Omega(F) & \xrightarrow{\psi(D_F)} & \Omega(F)
\end{array}
\]
commutes.

Any function \( \psi \in \mathcal{A} \) with compactly supported Fourier transform can be modified as follows to achieve the condition of being supported in \( [-R, R] \). For each \( t > 0 \), let \( \psi_t \in \mathcal{A} \) be the function defined by \( \psi_t(x) = \psi(tx) \).

**Lemma 6.4.** If \( \hat{\psi} \) is compactly supported for some \( \psi \in \mathcal{A} \), then \( \hat{\psi}_t \) is supported in \( [-R, R] \) for \( t \) small enough.

**Proof.** This holds because \( \hat{\psi}_t(\xi) = \frac{1}{t} \hat{\psi}(\frac{\xi}{t}) \).

6.6. **Schwartz kernels.** Let \( \phi, f, \psi \) and \( P \) be like in Section 6.3 such that \( \hat{\psi} \) is compactly supported. Take some \( R > 0 \) so that \( \text{supp} \ \hat{\psi} \subset [-R, R] \). Let \( k \in C^\infty_c(S) \) be the leafwise kernel of \( \psi(D_F) \), and let \( \tilde{k} \in C^\infty_\Delta(\tilde{S}) \) be the lift of \( k \), whose action in \( \Omega(\tilde{F}) \) defines the operator \( \psi(D_{\tilde{F}}) \) (Section 6.5).
Let \( \tilde{\phi} : \tilde{M} \times O \to \tilde{M} \) be a \( C^\infty \) lift of \( \phi \). Define \( \tilde{P} : U\Omega(\tilde{F}) \to U\Omega(\tilde{F}) \) by
\[
\tilde{P} = \int_O \hat{\phi}_g^* : f(g) \Lambda(g) \circ \psi(D\tilde{F}).
\]
The commutativity of the diagram
\[
\begin{array}{cc}
U\Omega(\tilde{F}) & U\Omega(\tilde{F}) \\
\tilde{P} & \tilde{P} \\
\pi^* & \pi^*
\end{array}
\]
follows from the commutativity of (7).

Let \( \omega_{\tilde{F}} \) be the volume form of the leaves of \( \tilde{F} \), which can be also considered as a differential form on \( \tilde{M} \) that vanishes when some vector is orthogonal to the leaves. Thus the volume form of \( \tilde{F} \) is \( \omega_{\tilde{M}} = \Lambda^* \Lambda \wedge \omega_{\tilde{F}} \) with the right choice of orientations. For \( \tilde{x} \in \tilde{M} \) and \( \alpha \in U\Omega(\tilde{F}) \), we have
\[
(\tilde{P}\alpha)(\tilde{x}) = \left( \int_O \tilde{\phi}_g^* \cdot f(g) \Lambda(g) \circ \psi(D\tilde{F})\alpha(\tilde{x}) \right)
\]
\[
= \int_O \tilde{\phi}_g^*((\psi(D\tilde{F})\alpha(\tilde{x})) \cdot f(g) \Lambda(g)
\]
\[
= \int_O \int_{L_{\tilde{x}}} \tilde{\phi}_g^* \circ \tilde{k}(\tilde{\phi}_g(\tilde{x}), \tilde{y})(\alpha(\tilde{y})) \omega_{\tilde{F}}(\tilde{y}) \cdot f(g) \Lambda(g)
\]
\[
= \int_{\phi(L_{\tilde{x}} \times O)} \tilde{\phi}_g^* \circ \tilde{k}(\tilde{\phi}_g(\tilde{x}), \tilde{y})(\alpha(\tilde{y})) \cdot f(g) \omega_{\tilde{F}}(\tilde{y})
\]
by Corollary 4.8, where \( g \in O \) is determined by the condition \( \tilde{y} \in \tilde{\phi}_g(L_{\tilde{x}}) \), which means \( g = D(\tilde{x})^{-1} D(\tilde{y}) \) by Lemma 4.7. So we can say that \( \tilde{P} \) is given by the Schwartz kernel \( \tilde{p} \) defined by
\[
\tilde{p}(\tilde{x}, \tilde{y}) = \begin{cases} 
\tilde{\phi}_g^* \circ \tilde{k}(\tilde{\phi}_g(\tilde{x}), \tilde{y}) \cdot f(g) & \text{if } \tilde{y} \in \tilde{\phi}(L_{\tilde{x}} \times O) \\
0 & \text{otherwise}
\end{cases}
\]
for \( g \in O \) as above. It follows that
\[
p(x, y) = \sum_{\sigma \in \text{Aut}(\pi)} \tilde{p}(\tilde{x}, \sigma(\tilde{y})) ,
\]
where \( \tilde{x} \in \pi^{-1}(x) \), \( \tilde{y} \in \pi^{-1}(y) \), and we use identifications \( T_{\tilde{x}} \tilde{F} \equiv T_x F \) and \( T_{\sigma(\tilde{y})} \tilde{F} \equiv T_y F \) given by \( \pi_* \).

For each \( x \in M \), \( \tilde{x} \in \tilde{M} \) and \( r > 0 \), let \( B_F(x, r) \) and \( B_{\tilde{F}}(\tilde{x}, r) \) be the \( r \)-balls of centers \( x \) and \( \tilde{x} \) in \( L_x \) and \( L_{\tilde{x}} \), respectively. Let \( O_1 \) be an open subset of \( G \) whose closure is compact and contained in \( O \). By the compactness of \( M \times O_1 \), there is some \( R_1 > 0 \) such that
\[
B_F(\phi(x), R) \subset \phi(B_F(x, R_1))
\]
for all $x \in M$ and all $g \in O_1$. So
\begin{equation}
B_{\tilde{\mathcal{F}}}(\tilde{\phi}_g(\tilde{x}), R) \subset \tilde{\phi}_g(B_{\mathcal{F}}(\tilde{x}, R_1))
\end{equation}
for all $\tilde{x} \in \tilde{M}$ and all $g \in O_1$ because $\pi$ restricts to isometries of the leaves of $\mathcal{F}$ to the leaves of $\tilde{\mathcal{F}}$.

**Lemma 6.5.** Each $g \in O$ has a neighborhood $O_1$ as above such that

$$\pi : \tilde{\phi}(B_{\tilde{\mathcal{F}}}(\tilde{x}, R_1) \times O_1) \to M$$

is injective for any $\tilde{x} \in \tilde{M}$.

**Proof.** Since $M$ is compact, there exists a compact subset $K \subset \tilde{M}$ with $\pi(K) = M$. Notice that, if the statement holds for some $\tilde{x} \in \tilde{M}$, then it also holds for all points in the $\text{Aut}(\pi)$-orbit of $\tilde{x}$. So, if the statement fails, there exist sequences $\tilde{x}_i, \tilde{y}_i \in \tilde{M}$ and $\sigma_i \in \text{Aut}(\pi)$ such that $\tilde{x}_i \in K$, $\sigma_i \neq \text{id}_{\tilde{M}}$, and

$$d_{\tilde{M}}(\{\tilde{y}_i, \sigma_i(\tilde{y}_i)\}, \tilde{\phi}_g(B_{\tilde{\mathcal{F}}}(\tilde{x}_i, R_1))) \to 0$$

as $i \to \infty$; observe that $D(\tilde{x}_i)^{-1} D(\tilde{y}_i) \to g$ by Lemma 4.7. Since $K$ is compact, we can assume that there exists $\lim_i \tilde{x}_i = \tilde{x} \in \tilde{M}$, where $d_{\tilde{M}}$ denotes the distance function of $\tilde{M}$. Hence $\tilde{y}_i$ and $\sigma_i(\tilde{y}_i)$ approach $\tilde{\phi}_g(B_{\tilde{\mathcal{F}}}(\tilde{x}, R_1))$. Since $\tilde{\phi}_g(B_{\tilde{\mathcal{F}}}(\tilde{x}, R_1))$ has compact closure, it follows that $\tilde{y}_i$ and $\sigma_i(\tilde{y}_i)$ lie in some compact neighborhood $Q$ of $\tilde{\phi}_g(B_{\tilde{\mathcal{F}}}(\tilde{x}, R_1))$ for infinitely many indices $i$, yielding $\sigma_i(Q) \cap Q \neq \emptyset$. So there is some $\sigma \in \text{Aut}(\pi)$ such that $\sigma_i = \sigma$ for infinitely many indices $i$. In particular, $\sigma \neq \text{id}_{\tilde{M}}$.

On the other hand, since $\tilde{y}_i$ and $\sigma_i(\tilde{y}_i)$ approach $\tilde{\phi}_g(B_{\tilde{\mathcal{F}}}(\tilde{x}, R_1))$, which has compact closure, we can assume that there exist $\lim_i \tilde{y}_i = \tilde{y}$ and $\lim_i \sigma_i(\tilde{y}_i) = \sigma(\tilde{y})$ in $\tilde{\phi}_g(B_{\tilde{\mathcal{F}}}(\tilde{x}, R_1))$, which is contained in the leaf $\tilde{\phi}_g(L_{\tilde{x}})$ (a fiber of $D$). So

$$D(\tilde{y}) = D(\sigma(\tilde{y})) = h(\sigma) \cdot D(\tilde{y}),$$

yielding $h(\sigma) = e$, and thus $\sigma = \text{id}_{\tilde{M}}$ because $h$ is injective. This contradiction concludes the proof. \qed

From now on, assume that $\phi$ satisfies \[10\] and the property of the statement of Lemma 6.5 with some fixed open subset $O_1 \subset O$ which contains the support of $f$.

**Corollary 6.6.** The map $\pi$ is injective on the support of $\tilde{\phi}(\tilde{x}, \cdot)$ for any $\tilde{x} \in \tilde{M}$.

**Proof.** By \[8, 11\] and since $\tilde{k}$ is supported in the $R$-penumbra of $\Delta_{\tilde{M}}$, we get

$$\text{supp}(\tilde{\phi}(\tilde{x}, \cdot)) \subset \tilde{\phi}(B_{\tilde{\mathcal{F}}}(\tilde{x}, R_1) \times O_1)$$

for any $\tilde{x} \in \tilde{M}$, and the result follows from Lemma 6.5. \qed
Corollary 6.7. We have
\[ p(x, y) = \begin{cases} 
\phi_g^* \circ k(\phi_g(x), y) \cdot f(g) & \text{if } y \in \phi_g(B_F(x, R_1)) \\
0 & \text{otherwise},
\end{cases} \]
where \( g \in O_1 \) is determined by the condition \( y \in \phi_g(B_F(x, R_1)) \).

Proof. This is a consequence of (8), (9), Corollary 6.6 and Lemma 6.5. \( \square \)

Corollary 6.8. If \( e \in O_1 \) and \( \phi_e = \text{id}_M \), then
\[ p(x, x) = k(x, x) \cdot f(e). \]

Proof. Since \( \phi_e = \text{id}_M \), the result follows from Corollary 6.7 and the following assertion.

Claim 1. For all \( g \in O_1 \) and \( x \in M \), if \( x \in \phi_g(B_F(x, R_1)) \), then \( g = e \).

By Lemma 6.5, \( \pi : \tilde{\phi}(B_F(\tilde{x}, R_1) \times O_1) \to \phi(B_F(x, R_1) \times O_1) \) is a diffeomorphism. On the other hand, \( \phi : \tilde{L}_x \times O_1 \to \phi(\tilde{L}_x \times O_1) \) is a diffeomorphism as well by Corollary 6.8. It follows that \( \phi : B_F(x, R_1) \times O_1 \to \phi(B_F(x, R_1) \times O_1) \) is also a diffeomorphism, which implies Claim 1 because \( \phi_e(x) = x \). \( \square \)

Lemma 6.9. For \( i \in \{1, 2\} \), suppose that \( x_i \in \phi_{g_i}(B_F(x_i, R_1)) \) for some \( (x_i, g_i) \in M \times O_1 \). If \( x_2 \) is close enough to \( x_1 \), then there is some \( a \in G \) such that \( x_2 \in \Phi_a(L_{x_1}) \) and \( g_2 = a^{-1}g_1a \).

Proof. We have
\[ \Phi_a(L_{x_1}) = \Phi_a \circ \Phi_{g_1}(L_{x_1}) = \Phi_{a^{-1}g_1a} \circ \Phi_a(L_{x_1}) \]
for all \( a \in G \). Therefore, if \( x_2 \) is close enough to \( x_1 \), there is some \( a \in G \) such that \( a^{-1}g_1a \in O_1 \) and
\[ x_2 \in \Phi_a(L_{x_1}) \cap \phi_{a^{-1}g_1a}(B_F(x_2, R_1)). \]
Then the result follows because the condition \( x_2 \in \phi_{g_2}(B_F(x_2, R_1)) \) determines \( g_2 \) in \( O_1 \) by Lemma 6.5. \( \square \)

7. LEFSCHETZ DISTRIBUTION

Let \( \phi : M \times O \to M \) be a \( C^\infty \) local representation of the structural transverse action \( \Phi \) on some open subset \( O \subset G \). For any \( f \in C_c^\infty(O) \) and \( t > 0 \), let \( P_f \) and \( Q_{t,f} \) be the operators in \( \Omega(F) \) defined by
\[ P_f = \int_O \phi_g^* \cdot f(g) \Lambda(g) \circ \Pi, \]
\[ Q_{t,f} = \int_O \phi_g^* \cdot f(g) \Lambda(g) \circ e^{-t\Delta_F}. \]
The operator $Q_{t,f}$ is in the class $D$, and thus it is smoothing by Proposition 6.1.

**Proposition 7.1.** $P_f$ is a smoothing operator.

*Proof.* By [2], $\Pi$ defines a bounded operator in each Sobolev space $W^k\Omega^i(\mathcal{F})$. Hence, $P_f = Q_{t,f} \circ \Pi$ is smoothing because so is $Q_{t,f}$.

By Proposition 7.1, $P_f$ is a trace class operator in the space $\Omega(\mathcal{F})$, and thus so is $P_f^{(i)}$.

**Proposition 7.2.** The functional $f \mapsto \text{Tr} P_f^{(i)}$ is a distribution on $O$.

*Proof.* Since $\Pi$ is a projection in $\Omega(\mathcal{F})$ and $P_f = Q_{t,f} \circ \Pi$, we have
\[
\|P_f^{(i)}\|_{0,k} \leq \|Q_{t,f}^{(i)}\|_{0,k},
\]
and the result follows by (3) and Proposition 6.2.

Proposition 7.1 is given by Propositions 7.1 and 7.2.

Because the endomorphism $\Phi^*_g$ of $\overline{H}(\mathcal{F})$ corresponds to the operator $\Pi \circ \phi^*_g$ in $H(\mathcal{F})$ by the leafwise Hodge isomorphism, the composite $\Pi \circ P_f$ is independent of the choice of $\phi$. Moreover, $\text{Tr} P_f^{(i)} = \text{Tr}(\Pi \circ P_f^{(i)})$. Hence the distributions given by Proposition 7.2 can be combined to form a global distribution $\text{Tr}^{i}_{\text{dis}}(\mathcal{F})$ on $G$; in this notation, $\mathcal{F}$ refers to the foliation endowed with the given transverse Lie structure, which indeed is determined by the foliation when the leaves are dense. Each $\text{Tr}^{i}_{\text{dis}}(\mathcal{F})$ is called a *distributional trace* of $\mathcal{F}$, and define the *Lefschetz distribution* of $\mathcal{F}$ by the formula
\[
L^{\text{dis}}_{\text{dis}}(\mathcal{F}) = \sum_i (-1)^i \text{Tr}^{i}_{\text{dis}}(\mathcal{F}).
\]

**Lemma 7.3.** For any $f \in C^\infty_c(O)$, $\text{Tr} Q_{t,f}^{(i)} \to \text{Tr} P_f^{(i)}$ as $t \to \infty$.

*Proof.* Since $Q_{1,f}$ is smoothing, it defines a bounded operator $W^{-1}\Omega^i(\mathcal{F}) \to W^k\Omega^i(\mathcal{F})$ for any $k$. By [2], $e^{-(t-1)\Delta_F} - \Pi$ is bounded in $W^{-1}\Omega^i(\mathcal{F})$ for $t > 1$ and converges strongly to 0 as $t \to \infty$. From the compactness of the canonical embedding $\Omega^i(\mathcal{F}) \to W^{-1}\Omega^i(\mathcal{F})$, it follows that $e^{-(t-1)\Delta_F} - \Pi$ converges uniformly to 0 as $t \to \infty$ as an operator $\Omega^i(\mathcal{F}) \to W^{-1}\Omega^i(\mathcal{F})$. Therefore $\|Q_{t,f} - P_f\|_{0,k} \to 0$ as $t \to \infty$ for any $k$ because
\[
Q_{t,f} - P_f = Q_{1,f} \circ (e^{-(t-1)\Delta_F} - \Pi).
\]
Then the result follows from (4).

**Corollary 7.4.** $\text{Tr} \circ Q_{t,f} = \text{Tr} \circ P_f$ for all $t$.

*Proof.* This follows from Lemmas 6.3 and 7.3.
8. The distributional Gauss-Bonnet theorem

The holonomy pseudogroup of $\mathcal{F}$ is represented by the pseudogroup on $G$ generated by the left translations given by elements of $\Gamma$. Thus $\Lambda$ can be considered as a holonomy invariant transverse measure of $\mathcal{F}$. To be more precise, take a $(G, \Gamma)$-valued foliated cocycle $\{U_i, f_i\}$ defining the given transverse Lie structure (Section 3). The differential forms $f_i^* \Lambda$ can be combined to get the transverse volume form $\omega_\Lambda$ of $\mathcal{F}$. We can also describe $\omega_\Lambda$ by the condition $D^* \Lambda = \pi^* \omega_\Lambda$. The restriction of $\omega_\Lambda$ to smooth local transversals is the precise interpretation of $\Lambda$ as a holonomy invariant measure on local transversals.

By non-commutative integration theory [9], the holonomy invariant transverse measure $\Lambda$ defines a trace $\text{Tr}_\Lambda$ on the twisted foliation von Neumann algebra $W^*(M, \mathcal{F}, \Lambda T \mathcal{F}^*)$. Consider also the corresponding supertrace $\text{Tr}_\Lambda^\sigma$, equal to $\pm \text{Tr}_\Lambda$, depending on whether the even-odd bigrading is preserved or interchanged.

With the notation of Section 6.5 we have $C^\infty_c(S) \subset W^*(M, \mathcal{F}, \Lambda T \mathcal{F}^*)$; here, each $k \in C^\infty_c(S)$ is identified to the family of operators on the leaves whose Schwartz kernels are the restrictions of $k$, and moreover

$$\text{Tr}_\Lambda(k) = \int_M \text{Tr} k(x, x) \omega_M(x), \quad \text{Tr}_\Lambda^\sigma(k) = \int_M \text{Tr}^\sigma k(x, x) \omega_M(x).$$

For each leaf $L$, let $\Omega(L)$ denote the Hilbert space of $L^2$ differential forms on $L$, let $\mathcal{H}(L) \subset \Omega(L)$ be the subspace of harmonic $L^2$ forms, and let $\Pi_L$ be the orthogonal projection $\Omega(L) \to \mathcal{H}(L)$. The family

$$\Pi_{\mathcal{F}} = \{\Pi_L \mid L \text{ is a leaf of } \mathcal{F}\}$$

defines a projection in $W^*(M, \mathcal{F}, \Lambda T \mathcal{F}^*)$. The notation $\Pi_{\mathcal{F}}^{(i)}$ and $\Pi_{\mathcal{F}}^{(i)}$ is used when we are only considering differential forms of degree $i$. For each leaf $L$, let $S_L = S|_{L \times L}$, and let $k_L, k_L^{(i)} \in C^\infty(S_L)$ denote the Schwartz kernels of $\Pi_L$ and $\Pi_{\mathcal{F}}^{(i)}$. These sections can be combined to define measurable sections $k$ and $k^{(i)}$ of $S$, called the leafwise Schwartz kernels of $\Pi_{\mathcal{F}}$ and $\Pi_{\mathcal{F}}^{(i)}$. Since $k$ and $k^{(i)}$ are $C^\infty$ along the fibers of the source and target projections, their restrictions to the diagonal $\Delta_M$ are measurable, and we have

$$\text{Tr}_\Lambda(\Pi_{\mathcal{F}}^{(i)}) = \int_M \text{Tr} k^{(i)}(x, x) \omega_M(x), \quad \text{Tr}_\Lambda^\sigma(\Pi_{\mathcal{F}}^{(i)}) = \int_M \text{Tr}^\sigma k(x, x) \omega_M(x).$$

According to [9], the $i$th $\Lambda$-Betti number is defined by

$$\beta^i_\Lambda(\mathcal{F}) = \text{Tr}_\Lambda(\Pi_{\mathcal{F}}^{(i)}),$$

and the $\Lambda$-Euler characteristic is given by the formula

$$\chi_\Lambda(\mathcal{F}) = \text{Tr}_\Lambda^\sigma(\Pi_{\mathcal{F}}^{(i)}) = \sum_i (-1)^i \beta^i_\Lambda(\mathcal{F}).$$

**Theorem 8.1.** $L_{\text{dis}}(\mathcal{F}) = \chi_\Lambda(\mathcal{F}) \cdot \delta_e$ in some neighborhood of $e$. 
Like in [25, p. 463], choose a sequence of smooth even functions on \( \mathbb{R} \), written as \( x \mapsto \psi_m(x^2) \) with \( \psi_m(0) = 1 \), whose Fourier transforms are compactly supported and which tend to the function \( x \mapsto e^{-x^2/2} \) in the Schwartz space \( S(\mathbb{R}) \). Let \( k_{m,t} \) be the leafwise Schwartz kernel of \( \psi_m(t\Delta_F)^2 \), which is in \( \mathcal{C}_\infty^\infty(S) \) according to [25]. In [25, p. 463], it is proved that

\[
\text{Tr}_\Lambda^* \psi_m(t\Delta_F)^2 = \chi_{\Lambda}(\mathcal{F}).
\]

Let \( \phi : M \times O \to M \) be any \( \mathcal{C}_\infty^\infty \) local representation of \( \Phi \) on some neighborhood \( O \) of \( e \) such that \( \phi_e = \text{id}_M \), whose existence is given by Lemma 4.6.

For every \( f \in \mathcal{C}_c^\infty(O) \) supported in \( O_1 \), let

\[
Q_{m,t,f} = \int_O \phi^*_g \cdot f(g) \Lambda(g) \circ \psi_m(t\Delta_F)^2 \in \mathcal{D}.
\]

**Lemma 8.2.** \( \text{Tr}_\Lambda^* Q_{m,t,f} = \chi_{\Lambda}(\mathcal{F}) \cdot f(e) \).

**Proof.** By Lemma 6.4 we can apply Corollary 6.8 to \( Q_{m,t,f} \) when \( t \) is small enough, obtaining

\[
\text{Tr}_\Lambda^* Q_{m,t,f} = \int_M \text{Tr}_\Lambda^* k_{m,t}(x,x) \cdot f(e) \omega_M(x) = \text{Tr}_\Lambda^* \psi_m(t\Delta_F)^2 \cdot f(e).
\]

Then the result follows by (12). \( \square \)

Consider the operators \( Q_{t,f} \) and \( P_f \) of Section 7.

**Lemma 8.3.** We have

\[
\lim_{m \to \infty} \text{Tr}_\Lambda^* Q_{m,t,f} = \text{Tr}_\Lambda^* Q_{t,f}
\]

for each \( t \).

**Proof.** Since the function \( x \mapsto \psi_m(tx^2) - e^{-\frac{1}{2}x^2} \) tends to zero in \( \mathcal{A} \) as \( m \to \infty \), we get

\[
\lim_{m \to \infty} \|Q_{m,t,f} - Q_{t,f}\|_{0,k} = 0
\]

for all \( k \) by (3) (or Lemma 6.2), and the result follows from (3). \( \square \)

Theorem 8.1 follows from Lemmas 8.2 and 8.3 and Corollary 7.4.

9. The distributional Lefschetz trace formula

Let \( \mathcal{F}' \) be the foliation of \( M \times G \) whose leaves are the sets \( L \times \{g\} \) for leaves \( L \) of \( \mathcal{F} \) and points \( g \in G \). Lemma 6.9 suggests the following definition: for each \( x \in M \) and \( g \in G \), let

\[
M'_{(x,g)} = \bigcup_{a \in G} (\Phi_a(L_x) \times \{a^{-1}ga\}).
\]

Observe that \( M'_{(x,e)} = M \times \{e\} \). Moreover \( M'_{(x_1,g_1)} = M'_{(x_2,g_2)} \) if and only if \( (x_2,g_2) \in M'_{(x_1,g_1)} \); thus these sets form a partition of \( M \times G \).
Proposition 9.1. The sets $M'_{(x,g)}$ are the leaves of a $C^\infty$ foliation $\mathcal{G}$ on $M \times G$.

Proof. Consider the canonical identity $T_{(x,g)}(M \times G) \equiv T_x M \oplus T_y G$ for each $(x,g) \in M \times G$, and let $\text{Ad} : G \to \text{Aut}(\mathfrak{g})$ denote the adjoint representation of $G$. With the notation of Section 4 consider the $C^\infty$ vector subbundles $\mathcal{V}, \mathcal{W} \subset T(M \times G)$ given by

$$
\mathcal{V}_{(x,g)} = \{(X^\nu(x), (X - \text{Ad}_{g^{-1}}(X))(g)) \mid X \in \mathfrak{g}\},
$$

$$
\mathcal{W}_{(x,g)} = \mathcal{V}_{(x,g)} + T_{(x,g)} \mathcal{F}'.
$$

The distribution defined by $\mathcal{V}$ is not completely integrable. Nevertheless, since $[X^\nu, Y^\nu] - [X, Y]^\nu \in \mathfrak{X}(\mathcal{F}')$ for all $X, Y \in \mathfrak{g}$, it follows that the distribution defined by $\mathcal{W}$ is completely integrable. Thus there is a $C^\infty$ foliation $\mathcal{G}$ on $M \times G$ so that $T\mathcal{G} = \mathcal{W}$. It is easy to check that the leaves of $\mathcal{G}$ are the sets $M'_{(x,g)}$.

Let $\text{pr}_1$ and $\text{pr}_2$ denote the first and second factor projections of $M \times G$ onto $M$ and $G$, respectively.

Proposition 9.2. For each leaf $M'$ of $\mathcal{G}$, we have the following:

(i) the restriction $\text{pr}_1 : M' \to M$ is a covering map; and

(ii) $\text{pr}_2$ restricts to a fiber bundle map of $M'$ to some orbit of the adjoint action of $G$ on itself.

Proof. For any $x \in M$, there is some open neighborhood $P$ of $x$ in $L_x$, and some local representation $\varphi : M \times O \to M$ of $\Phi$ on some open neighborhood $O$ of $e$ such that $\varphi$ restricts to a diffeomorphism of $P \times O$ onto some neighborhood $U$ of $x$. For any $g \in G$ such that $(x,g) \in M'$, the set

$$
\bar{U}_g = \{(\varphi_a(y), a^{-1} ga) \mid y \in P, \ a \in O\}
$$

is an open neighborhood of $(x,g)$ in $M'$, and the restriction $\text{pr}_1 : \bar{U}_g \to U$ is a diffeomorphism. Therefore property (i) follows.

It is clear that $\text{pr}_2(M')$ is an orbit of the adjoint action of $G$ on itself, and that $\text{pr}_2 : M' \to \text{pr}_2(M')$ is a $C^\infty$ submersion; thus its fibers are $C^\infty$ submanifolds. If $(x,g) \in M'$, it can be easily seen that

$$
\text{pr}_2^{-1}(g) \cap M' = \{((\phi_a(y), g) \mid y \in L_x, \ a \in G_g, \ \phi_g \in \Phi_g\},
$$

where $G_g$ is the centralizer of $g$ in $G$. For $\varphi : M \times O \to M$ as above, the set $O' = \{b^{-1}gb \mid b \in O\}$ is an open neighborhood of $g$ in $\text{pr}_2(M')$. Let

$$
F : O' \times (\text{pr}_2^{-1}(g) \cap M') \to \text{pr}_2^{-1}(O') \cap M'
$$

be the map defined by

$$
F(b^{-1}gb; \varphi_a(y), g) = (\varphi_{b^{-1}ab} \circ \varphi_b(y), b^{-1}gb)
$$

for $y \in L_x, \ a \in G_g$ and $b \in O'$. It is easy to see that $F$ is a $C^\infty$ diffeomorphism, which shows property (ii).
Observe that $\mathcal{F}'$ is a subfoliation of $\mathcal{G}$, and, for each leaf $M'$ of $\mathcal{G}$, the restriction $\mathcal{F}'|_{M'}$ is equal to the lift of $\mathcal{F}$ by $pr_1: M' \rightarrow M$.

Let $\phi: M \times O \rightarrow M$ be any $C^\infty$ local representation of $\Phi$. Given $R > 0$, take $R_1 > 0$ and some open subset $O_1$ of $O$ containing $e$ such that (10) and Lemma 6.5 are satisfied. Let

$$S = \{(x, g) \in M \times O_1 \mid x \in \phi_g(B_{\mathcal{F}}(x, R_1))\}.$$  

**Proposition 9.3.** We have:

(i) $S$ is contained in a finite union of leaves of $\mathcal{G}$; and
(ii) the restriction $pr_1: S \rightarrow M$ is injective.

**Proof.** Property (i) is a consequence of Lemma 6.9 and the compactness of $M$. Property (ii) follows from Lemma 6.5. □

Let $\phi': M \times O \rightarrow M \times O$ be the $C^\infty$ diffeomorphism defined by $\phi'(x, g) = (\phi(x, g), g)$. Observe that $\phi'$ is a foliated map $\mathcal{F}'|_{M \times O} \rightarrow \mathcal{F}'|_{M \times O}$.

**Proposition 9.4.** Let $M'$ be a leaf of $\mathcal{G}$. If $\phi'$ preserves some leaf of $\mathcal{F}'|_{M' \cap (M \times O)}$, then it preserves every leaf of $\mathcal{F}'|_{M' \cap (M \times O)}$.

**Proof.** Take some point $(x, g)$ in a leaf $L'$ of $\mathcal{F}'|_{M' \cap (M \times O)}$; thus $L' = L_x \times \{g\}$. Suppose $\phi'(L') \subset L'$, which means $\Phi_g(L_x) = L_x$. Any leaf of $\mathcal{F}'|_{M' \cap (M \times O)}$ is of the form $\Phi_a(L_x) \times \{a^{-1}ga\}$ for some $a \in G$. We have

$$\Phi_{a^{-1}ga} \circ \Phi_a(L_x) = \Phi_{ga}(L_x) = \Phi_a \circ \Phi_g(L_x) = \Phi_a(L_x).$$

So $\phi'$ preserves $\Phi_a(L_x) \times \{a^{-1}ga\}$. □

According to Proposition 9.3, if $O_1$ is small enough, then $S$ is contained in a leaf $M'$ of $\mathcal{G}$; this property is assumed from now on. Let $M'_1 = M' \cap (M \times O_1)$ and $\mathcal{F}'_1 = \mathcal{F}|_{M'_1}$. By Proposition 9.4, $\phi'$ maps each leaf of $M'$ to itself, and thus can be restricted to a map $\phi'_1: M'_1 \rightarrow M'_1$, which is a foliated map $(M'_1, \mathcal{F}'_1) \rightarrow (M'_1, \mathcal{F}'_1)$.

Consider the volume form $\Lambda$ of $G$ as a transverse invariant measure of $\mathcal{F}$. By Proposition 9.2(i), $\Lambda$ lifts to a transverse invariant measure $\Lambda'_1$ of $\mathcal{F}'_1$. Similarly, the Riemannian metric of $M$ lifts to a Riemannian metric of $M'$, which can be restricted to $M'_1$; the volume form of this restriction is denoted by $\omega_{M'_1}$.

Even though the foliated manifolds of [14] are compact, it is clear that its Lefschetz theorem for foliations with transverse invariant measures generalizes to the non-compact case when the transverse invariant measure is compactly supported.

In our case, $M'_1$ may not be compact, but, for every $f \in C_c^\infty(O)$ supported in $O_1$, $\Lambda_{1,f} = pr_{1*}f \cdot \Lambda'_1$ of $\mathcal{F}'_1$ is a compactly supported transverse invariant measure of $\mathcal{F}'_1$. Therefore, according to [14], the $\Lambda_{1,f}$–Lefschetz number $L_{\Lambda_{1,f}}(\phi'_1)$ of $\phi'_1$ can be defined.
Let $\phi$ be the covering map by Proposition 9.2-(i). Like in Section 6.5, there is a global action of $C^\infty(S'_2)$ in $\Omega(F'_1)$. For any $\psi \in \mathcal{A}$ with supp $\psi \subset [-R, R]$, we have defined the leafwise Schwartz kernels $k \in C^\infty(S)$ and $\tilde{k} \in C^\infty(S)$ of $\psi(D_F)$ and $\psi(D_{\tilde{F}})$ in Section 8. Similarly, we can define the leafwise Schwartz kernels $k'_1, k'_2 \in C^\infty(S'_2)$ of $\psi(D_F)$ and $\phi'_1 \circ \Delta_{F'_1}$, respectively. It is easy to see that $k'_1$ can be identified with the lift of $k$ via $pr_1 \times pr_1$. Therefore $k'_1$ is given by

$$\int_{M'_1} \text{Tr}^* k'_1(x, x) \omega_{M'_1}.$$  

For any $\psi \in \mathcal{A}$ with supp $\psi \subset [-R, R]$, we have defined the leafwise Schwartz kernels $k \in C^\infty(S)$ and $\tilde{k} \in C^\infty(S)$ of $\psi(D_F)$ and $\psi(D_{\tilde{F}})$ in Section 8. Similarly, we can define the leafwise Schwartz kernels $k'_1, k'_2 \in C^\infty(S'_2)$ of $\psi(D_F)$ and $\phi'_1 \circ \Delta_{F'_1}$, respectively. It is easy to see that $k'_1$ can be identified with the lift of $k$ via $pr_1 \times pr_1$. Therefore $k'_1$ is given by

$$\int_{M'_1} \text{Tr}^* k'_1(x, x) \omega_{M'_1}.$$  

Choose a sequence of functions $\psi_m$, like in Section 8. Let $k$ and $k_{m,t}$ be the leafwise Schwartz kernels of $\Pi_F$ and $\psi_m(t\Delta_F)^2$, respectively. By [27, Lemma 1.2], $k_{m,t}$ tends to $k$ as $t \to \infty$, and moreover $k_{m,t}$ is uniformly bounded for large $m$ and $t$. Hence, by (14), the leafwise Schwartz kernel $k'_{m,t}$ of $\phi'_1 \circ \psi_m(t\Delta_{F'_1})^2$ tends to $k'_1$ as $t \to \infty$, and $k_{m,t}$ is uniformly bounded for large $m$ and $t$. Therefore

$$\lim_{t \to \infty} \text{Tr}^* \phi'_1 \circ \psi_m(t\Delta_{F'_1})^2 = L_{\Delta_{F'_1}}(\phi'_1)$$

for each $m$ by (13) and the dominated convergence theorem. Furthermore

$$\text{Tr}^* \phi'_1 \circ \psi_m(t\Delta_{F'_1})^2$$

Theorem 9.5. With the above notation and conditions, we have

$$\langle L_{\text{dis}}(F), f \rangle = L_{\mathcal{N}_{t,f}}(\phi'_1)$$

for every $f \in C^\infty_c(O)$ supported in $O_1$.
is independent of $t$ (see [14, Theorem 5.1]). Therefore
\begin{equation}
\text{Tr}^s_{A_{1,f}} (\phi_1^* \circ \psi_m (t \Delta_{\mathcal{F}'_1})) = L_{A_{1,f}}' (\phi_1')
\end{equation}
for all $m$ and $t$.

Let $Q_{m,f}$ be defined like in Section 8.

**Lemma 9.6.** We have
\[
\text{Tr}^s Q_{m,t,f} = L_{A_{1,f}}' (\phi_1').
\]

**Proof.** By Lemma 6.4, the Schwartz kernel $q_{m,t,f}$ of $Q_{m,t,f}$ is given by Corollary 6.7 when $t$ is small enough. So, if $(x, x) \in \text{supp} q_{m,t,f}$ for some $x \in M$, we have
\[
q_{m,t,f}(x, x) = \phi_1^* \circ k_m (\phi_g (x), x) \cdot f(g),
\]
where $g \in O$ is determined by the condition $x \in \phi_g (B_F (x, R_1))$; thus $(x, g) \in \mathcal{S} \subset M_1'$. Therefore, since $\text{pr}_1 : \mathcal{S} \to M$ is injective (Proposition 9.3(ii)),
\[
\text{Tr}^s Q_{m,t,f} = \int_{\mathcal{S}} \text{Tr}^s (\phi_1^* \circ k_m (\phi_g (x), x)) \cdot f(g) \omega_{M_1'}(x, g)
\]
by (15)
\[
= \text{Tr}^s_{A_{1,f}} (\phi_1^* \circ \psi_m (t \Delta_{\mathcal{F}'_1}))
\]
for $t$ small enough. Then the result follows by (15). \square

Theorem 9.3 follows from Lemmas 9.6 and 8.3 and Corollary 7.4.

Now, let us prove Theorem 1.3. Let $\text{Fix}(\phi')$ and $\text{Fix}(\phi_1')$ denote the fixed point sets of $\phi'$ and $\phi_1'$. Observe that $\text{Fix}(\phi') \subset M'$, and thus
\begin{equation}
\text{Fix}(\phi_1') = \text{Fix}(\phi') \cap (M \times O_1).
\end{equation}

It is clear that $\text{pr}_2 : \text{Fix}(\phi') \to O$ is a proper map because $M$ is compact and $\text{Fix}(\phi')$ is closed in $M \times O$. Then $\text{pr}_2 : \text{Fix}(\phi_1') \to O_1$ is proper too by (16).

A fixed point $(x, g)$ of $\phi'$ is said to be leafwise simple if $\phi_g * - \text{id} : T_x \mathcal{F} \to T_x \mathcal{F}$ is an isomorphism. The set of simple fixed points of $\phi'$ is denoted by $\text{Fix}_0(\phi')$. Define $\epsilon : \text{Fix}_0(\phi') \to \{\pm 1\}$ by
\[
\epsilon(x, g) = \text{sign det}(\phi_g * - \text{id} : T_x \mathcal{F} \to T_x \mathcal{F}).
\]

**Lemma 9.7.** $\text{Fix}_0(\phi')$ is a $C^\infty$ regular submanifold of $M'$ whose dimension is equal to $\text{codim} \mathcal{F}$.

**Proof.** Let $\hat{\phi} : M \times O \to M \times M$ be the $C^\infty$ map defined by $\hat{\phi}(x, g) = (x, \phi_g (x))$, and let $\Delta_M$ denote the diagonal in $M \times M$. Then $\text{Fix}(\phi') = \hat{\phi}^{-1}(\Delta_M)$.

There is some open subset $U \subset M \times O$ such that $\text{Fix}_0(\phi') = \text{Fix}(\phi') \cap U$. Then the result follows by showing that the restriction $\hat{\phi} : U \to M \times M$ is transverse to $\Delta_M$. 

Pick any \((x,g) \in \text{Fix}_0(\phi')\). Let \(\Delta_{L_x}\) denote the diagonal in \(L_x \times L_x\).
Consider the canonical identity \(T_{(x,x)}(M \times M) \equiv T_xM \oplus T_xM\). The fact that \(x\) is a simple fixed point of \(\phi_g\) means that
\[
T_xL_x \oplus T_xL_x = \hat{\phi}_s(T_{(x,g)}(L_x \times \{g\})) + T_{(x,x)}\Delta_{L_x}.
\]
Observe that
\[
\mu_x = \phi_s(T_{(x,g)}(\{x\} \times G))
\]
is complementary of \(T_xF\), and
\[
\hat{\phi}_s(T_{(x,g)}(\{x\} \times G)) = 0_x \oplus \mu_x,
\]
where \(0_x\) denotes the zero subspace of \(T_xM\). So
\[
T_xM \oplus T_xM = (T_xL_x \oplus T_xM) + T_{(x,x)}\Delta_M
\]
\[
= (T_xL_x \oplus T_xL_x) + (0_x \oplus \mu_x) + T_{(x,x)}\Delta_M
\]
\[
= \hat{\phi}_s(T_{(x,g)}(M \times G)) + T_{(x,x)}\Delta_M
\]
by (17). \(\square\)

**Proposition 9.8.** Fix \(\phi'\) is a \(C^\infty\) transversal of \(F'|_{M'}\).

**Proof.** By Lemma 9.7 it is enough to prove that \(\text{Fix}_0(\phi')\) is transverse to \(F'|_{M'}\), which follows from the following claim for any point \((x,g) \in \text{Fix}_0(\phi')\).

**Claim 2.** We have
\[
T_{(x,g)}(\text{Fix}_0(\phi')) \cap T_{(x,g)}F' = 0 .
\]
The proof of Claim 2 involves another assertion:

**Claim 3.** We have
\[
T_{(x,g)}(\text{Fix}_0(\phi')) = \ker(\phi_s - \text{pr}_1: T_{(x,g)}M' \to T_xM).
\]
For any \(v \in T_{(x,g)}(\text{Fix}_0(\phi'))\), there is a \(C^\infty\) curve \((x_t, g_t)\) in \(\text{Fix}_0(\phi')\), with \(-\epsilon < t < \epsilon\) for some \(\epsilon > 0\), such that \((x_0, g_0) = (x, g)\) and \(\frac{d}{dt}(x_t, g_t)|_{t=0} = v\). We have \(\phi(x_t, g_t) = x_t = \text{pr}_1(x_t, g_t)\), yielding \(\phi_s(v) = \text{pr}_1(v)\). So
\[
v \in \ker(\phi_s - \text{pr}_1: T_{(x,g)}M' \to T_xM),
\]
obtaining the inclusion “\(\subset\)” of Claim 3.
Since \(\phi_g - \text{id}: T_xF \to T_xF\) is an isomorphism, so is \(\phi_s - \text{pr}_1: T_{(x,g)}F' \to T_xF\). Hence
\[
\ker(\phi_s - \text{pr}_1: T_{(x,g)}M' \to T_xM) \cap T_{(x,g)}F' = 0 ,
\]
yielding Claims 2 and 3 because the inclusion “\(\subset\)” of Claim 3 is already proved. \(\square\)

**Proposition 9.9.** \(\text{pr}_2: \text{Fix}_0(\phi') \to \text{pr}_2(M')\) is a \(C^\infty\) submersion.
Theorem 1.3. The leaves of $\mathcal{F}'$ are contained in the fibers of $pr_2$, the tangent map $pr_2\ast$ induces a homomorphism $pr_2\ast : T(M \times G)/T\mathcal{F}' \rightarrow TG$. Take any $(x,g) \in Fix_0(\phi')$. By Proposition 9.8 and according to the proof of Proposition 9.1, the restrictions of the quotient map $pr_2\ast$ to $pr_2\ast(Fix\mathcal{F}')$ are isomorphisms. Moreover $pr_2\ast$ corresponds to $pr_2\ast$ by these isomorphisms. So

$$pr_2\ast(T_{(x,g)} Fix_0(\phi')) = \{(X - Ad_g^{-1}(X))(g) \mid X \in \mathfrak{g}\} = T_g(pr_2(M'))$$

by the proof of Proposition 9.1.

According to Proposition 9.8, the measure given by $\Lambda'$ on $Fix_0(\phi')$ is denoted by $\Lambda'_{Fix_0(\phi')}$. The direct image $pr_2\ast(\epsilon \cdot \Lambda'_{Fix_0(\phi')})$ is supported in $pr_2(M') \cap O$.

Let $\omega_\Lambda$ be the transverse volume form of $\mathcal{F}$ defined by $\Lambda$. Then the transverse volume form of $\mathcal{F}'|M'$ defined by $\Lambda'$ is $\omega_{\Lambda'} = pr_2\ast \omega_\Lambda$. The restriction of $\omega_{\Lambda'}$ to the $C^\infty$ local transversal $Fix_0(\phi')$ is a volume form, which can be identified to the measure $\Lambda'_{Fix_0(\phi')}$. According to Proposition 9.9, $pr_2\ast(\epsilon \cdot \Lambda'_{Fix_0(\phi')})$ is given by the top degree differential form on $pr_2(M') \cap O$ defined by the integration along the fibers

$$\int_{pr_2} \epsilon \cdot \omega_{\Lambda'|_{Fix_0(\phi')}}$$

By (16), Theorem 9.3 and the Lefschetz theorem of [14], we have

$$\langle L_{dis}(\mathcal{F}), f \rangle = L_{\Lambda'_{\phi'}}(\phi_1') = \int_{Fix(\phi_1')} \epsilon(x,g) f(g) \Lambda'_{Fix(\phi')}\langle x \rangle$$

$$= \int_{O_1 \cap pr_2(M')} f(g) pr_2\ast(\epsilon \cdot \Lambda'_{Fix(\phi')})(g)$$

$$= \langle pr_2\ast(\epsilon \cdot \Lambda'_{Fix(\phi')}), f \rangle,$$

completing the proof of Theorem 1.3.

10. Examples

10.1. Codimension one foliations. Consider the case when $\mathcal{F}$ is a codimension one Lie foliation. So we have $\mathfrak{g} = \mathbb{R}$, $G = \mathbb{R}$, and $\mathcal{F}$ is defined by a closed nonsingular $1$-form $\omega$. The leaves of $\mathcal{G}$ in $M \times \mathbb{R}$ are $M'_s = M \times \{s\}$, $s \in \mathbb{R}$. A global $C^\infty$ representation of $\Phi$ is given by the flow $\phi : M \times \mathbb{R} \rightarrow M$ of an arbitrary vector field $X$ on $M$ such that $\omega(X) = 1$. Then

$$Fix(\phi') = \{(x,s) \in M \times \mathbb{R} \mid \phi_s(x) = x\}.$$
So we have \( \text{Fix}(\phi') \cap M'_s \neq \emptyset \) if and only if either \( s = 0 \) or \( s \) is the period of a closed orbit of the flow \( \phi \). In the latter case, we have

\[
\text{Fix}(\phi') \cap M'_s = \bigcup_c \mathcal{O}_c \times \{s\},
\]

where \( c \) runs over the set of all closed orbits of period \( s \), and \( \mathcal{O}_c \) is the corresponding primitive closed orbit:

\[
\mathcal{O}_c = \{ \phi_t(x) \in M \mid t \in [0, \ell(c)] \}
\]

where \( x \in c \) is an arbitrary point, and \( \ell(c) \) is the length of \( \mathcal{O}_c \). Assume that all closed orbits of \( \phi \) are simple. Then \( \epsilon : \text{Fix}(\phi') \to \{ \pm 1 \} \) is constant on each \( \mathcal{O}_c \times \{s\} \subset \text{Fix}(\phi') \cap M'_s \), and its value on \( \mathcal{O}_c \times \{s\} \) will be denoted by \( \epsilon_s(c) \).

The Lebesgue measure \( \Lambda = dt \) on \( \mathbb{R} \) can be considered as an invariant transverse measure of \( F \). So we have

\[
L_{\text{dis}}(F) = \chi \Lambda(F) \cdot \delta_0
\]

in some neighborhood of 0. The restriction of the transverse volume form \( \omega'_{\Lambda} \) to \( \text{Fix}(\phi') \cap M'_s \) coincides with \( \omega'_{\Lambda} \) on each \( \mathcal{O}_c \). For any component \( \mathcal{O}_c \times \{s\} \subset \text{Fix}(\phi') \cap M'_s \), one can write \( s = k \ell(c) \) for some \( k \neq 0 \), and we see that, on \( \mathbb{R} \setminus \{0\} \),

\[
L_{\text{dis}}(F) = \text{pr}_2^*(\epsilon \cdot \Lambda'|_{\text{Fix}(\phi')}) = \sum_c \ell(c) \sum_{k \neq 0} \epsilon_k \ell(c) \cdot \delta_k \ell(c),
\]

where \( c \) runs over all primitive closed orbits of the flow \( \phi \).

10.2. Suspensions. Let \( X \) be a connected compact manifold, \( \widetilde{X} \) its universal cover, \( G \) a compact Lie group, and \( h : \Gamma = \pi_1(X) \to G \) a homomorphism. Consider the canonical right action of \( \Gamma \) on \( \widetilde{X} \), and the diagonal right action of \( \Gamma \) on \( \widetilde{M} = \widetilde{X} \times G \):

\[
(x, a) \cdot \gamma = (x \cdot \gamma, h(\gamma^{-1}) \cdot a).
\]

Let \( M = \widetilde{M}/\Gamma \) (usually denoted by \( \widetilde{X} \times_\Gamma G \)). The canonical projection \( \pi : \widetilde{M} \to M \) is a covering map. Let \( [x, a] \) be the element of \( M \) represented by each \( (x, a) \in \widetilde{M} \). The foliation \( \mathcal{F} \) on \( \widetilde{M} \) given by the fibers of the second factor projection \( \text{pr}_2 : \widetilde{M} \to G \) gives rise to a foliation \( F \) on \( M \). Let \( \Lambda \) be a left invariant volume form on \( G \), which can be considered as an invariant transverse measure of \( F \) because its holonomy pseudogroup can be represented by the pseudogroup generated by the left translations by elements of \( h(\Gamma) \). The corresponding transverse volume form \( \omega_{\Lambda} \) is defined by the condition \( \pi^* \omega_{\Lambda} = \text{pr}_2^* \Lambda \) of \( \mathcal{F} \), whose restriction to local transversals is another interpretation of \( \Lambda \) as transverse invariant measure of \( F \). It is easy to see that

\[
\chi\Lambda(F) = \text{vol}(G) \cdot \chi\Gamma(\widetilde{X}),
\]
where $\chi_\Gamma(\tilde{X})$ is the $\Gamma$-Euler characteristic of the covering manifold $\tilde{X}$ of $X$ defined by Atiyah [6]. By Atiyah’s $\Gamma$-index theorem [6], we have $\chi_\Gamma(\tilde{X}) = \chi(X)$, where $\chi(X)$ is the Euler characteristic of $X$.

There is a $C^\infty$ global representation $\phi : M \times G \to M$ of the structural transverse action $\Phi$, defined by $\phi([x,a],g) = [x,ag]$.

This $\phi$ is a free action. Therefore

\[ L_{\text{dis}}(\mathcal{F}) = \text{vol}(G) \cdot \chi(X) \cdot \delta_e \]

on the whole of $G$. In particular, if $\chi(X) \neq 0$, then $\dim \overline{H}(\mathcal{F}) = \infty$ for any homomorphism $h : \Gamma \to G$.

We can consider the following concrete example. Let $X$ be a compact oriented surface of genus $g \geq 2$ endowed with a hyperbolic metric. One can show that there exists an injective homomorphism $h : \pi_1(X) \to \text{SO}(3,\mathbb{R})$. One obtains a Lie $\text{SO}(3,\mathbb{R})$-foliation $\mathcal{F}$ whose leaves are dense, simply connected (diffeomorphic to $\mathbb{R}^2$) and isometric to the hyperbolic plane. Assuming that $\text{vol}(G) = 1$, we get

\[ \beta^0_\Lambda(\mathcal{F}) = \beta^2_\Lambda(\mathcal{F}) = 0, \quad \beta^1_\Lambda(\mathcal{F}) = 2g - 2. \]

Since the leaves of $\mathcal{F}$ are dense, we have $\overline{H}^0(\mathbb{R}) \cong \overline{H}^2(\mathbb{R}) \cong \mathbb{R}$, and therefore

\[ \text{Tr}^0_{\text{dis}}(\mathcal{F}) = \text{Tr}^2_{\text{dis}}(\mathcal{F}) = 1. \]

By [10], we get

\[ L_{\text{dis}}(\mathcal{F}) = (2 - 2g) \cdot \delta_e, \]

and

\[ \text{Tr}^1_{\text{dis}}(\mathcal{F}) = (2g - 2) \cdot \delta_e + 2. \]

One can also take any homomorphism of $\Gamma$ to the $n$-torus $\mathbb{R}^n/\mathbb{Z}^n$ to produce a foliation, which has infinite dimensional reduced cohomology of degree one (see [11, Example 2.11]). In this case, we have

\[ \text{Tr}^i_{\text{dis}}(\mathcal{F}) \neq \beta^1_\Lambda(\mathcal{F}) \cdot \delta_e, \]

but $\text{Tr}^i_{\text{dis}}(\mathcal{F}) - \beta^1_\Lambda(\mathcal{F}) \cdot \delta_e$ is $C^\infty$.

10.3. Bundles over homogeneous spaces and the Selberg trace formula. Let $G$ be a simply connected Lie group, $\Gamma$ a discrete cocompact subgroup in $G$, and $\alpha$ an injective homomorphism of $\Gamma$ to the diffeomorphism group $\text{Diff}(X)$ of some compact connected $C^\infty$ manifold $X$. Consider a left action of $\Gamma$ on $\tilde{M} = G \times X$ given by

\[ \gamma \cdot (a,x) = (\gamma a, \alpha(\gamma)(x)). \]

Let $M = \Gamma \backslash (G \times X)$, and let $[a,x]$ be the element of $M$ represented by any $(a,x) \in \tilde{M}$. The canonical projection $\pi : \tilde{M} \to M$ is a covering map. The first factor projection $G \times X \to G$ defines a fiber bundle map $M \to \Gamma \backslash G$, where
whose fibers are the leaves of a foliation $\mathcal{F}$. For each $a \in G$, the leaf of $\mathcal{F}$ through $\Gamma a$ is

$$L_{\Gamma a} = \{ [a, x] \mid x \in X \},$$

which is diffeomorphic to $X$ because $\alpha$ is injective. Consider a left-invariant volume form $\Lambda$ on $G$. It induces a volume form on $\Gamma \backslash G$, denoted by $\Lambda_{\Gamma \backslash G}$, whose pull-back to $M$ via the map $M \to \Gamma \backslash G$ defines a transverse volume form $\omega_{\Lambda}$ of $\mathcal{F}$. Since $M \to \Gamma \backslash G$ is a fiber bundle map with typical fiber $X$, we get

$$\chi_{\Lambda}(\mathcal{F}) = \text{vol}(\Gamma \backslash G) \cdot \chi(X),$$

where $\chi(X)$ is the Euler characteristic of $X$.

The structural transverse action $\Phi_g$ of an element $g \in G$ is given by the leafwise homotopy class of diffeomorphisms $\phi_g : M \to M$ of the form

$$\phi_g([a, x]) = [ag, \beta(x)],$$

where $\beta$ is any diffeomorphism of $X$ homotopic to $\text{id}_X$.

The leaf of the foliation $\mathcal{G}$ through a point $([a, x], b) \in M \times G$ is

$$\{([ag, y], g^{-1}bg) \mid y \in X, g \in G\}.$$

So the leaves of $\mathcal{G}$ are

$$M'_b = \{([g, y], g^{-1}bg) \mid y \in X, g \in G\}, \quad b \in G,$$

with $M'_{b_2} = M'_{b_1}$ when $b_2 \in \text{Ad}(\Gamma)b_1$; thus the leaves of $\mathcal{G}$ are parameterized by the $\Gamma$-conjugacy classes in $G$.

Let $\text{pr}_1$ and $\text{pr}_2$ denote the factor projections of $M \times G$ to $M$ and $G$, respectively. The restriction $\text{pr}_2 : M'_b \to G$ is a bundle map over the orbit

$$\mathcal{O}_b = \{g^{-1}bg \mid g \in G\} \equiv G_b \backslash G$$

of the adjoint representation of $G$ on $G$, where

$$G_b = \{g \in G \mid gb = bg\}$$

is the centralizer of $b$ in $G$.

For each $b \in G$, the restriction $\text{pr}_1 : M'_b \to M$ is a covering map. Indeed, we have $M'_b \equiv \Gamma_b \backslash (G \times X)$, where

$$\Gamma_b = \{ \gamma \in \Gamma \mid \gamma b = b \gamma \} = \Gamma \cap G_b.$$

The leaves of the foliation $\mathcal{F}' = \text{pr}_1^* \mathcal{F}$ on $M'_b$ are described as

$$L_a = \{([a, y], a^{-1}ba) \mid y \in X\}, \quad a \in G,$$

with $L_{a_1} = L_{a_2}$ if and only if $\Gamma_b a_1 = \Gamma_b a_2$. Therefore the leaves of $\mathcal{F}'$ are the fibers of the natural map

$$M'_b \equiv \Gamma_b \backslash (G \times X) \to \Gamma_b \backslash G, \quad ([a, y], a^{-1}ba) \mapsto \Gamma_b a.$$

Take a $C^\infty$ global representation $\phi : M \times G \to M$ of $\Phi$ defined by

$$\phi([a, x], g) = [ag, x].$$

We have

$$\text{Fix}(\phi') = \{([a, x], g) \in M \times G \mid [ag, x] = [a, x]\}.$$
The identity \([ag,x] = [a,x]\) holds if and only if there exists \(\gamma \in \Gamma\) such that \(ag = \gamma a\) and \(\alpha(\gamma)(x) = x\). Hence

\[
\text{Fix}(\phi') = \bigcup_{\gamma \in \Gamma} \{([a,x], a^{-1}\gamma a) \mid x \in X, \alpha(\gamma)x = x, a \in G\}.
\]

We see that if \(\text{Fix}(\phi') \cap M'_b \neq \emptyset\), then one can assume that \(b = \gamma \in \Gamma\) and \(\alpha(\gamma)\) has a fixed point in \(X\). In this case,

\[
\text{Fix}(\phi') \cap M'_\gamma = \bigcup_{k=1}^{d(\gamma)} \{([a,x_k(\gamma)], a^{-1}\gamma a) \mid a \in G\}.
\]

A point \(([a,x], a^{-1}\gamma a) \in \text{Fix}(\phi') \cap M'_\gamma\) is simple if and only if \(x\) is a simple fixed point of \(\alpha(\gamma)\); in this case, we have

\[
\epsilon([a,x], a^{-1}\gamma a) = \text{sign det}(\alpha(\gamma)_* - \text{id} : T_xX \to T_xX),
\]

which is denoted by \(\epsilon_{\alpha(\gamma)}(x)\). Assume that, for any \(\gamma \in \Gamma \setminus \{e\}\), all the fixed points of the diffeomorphism \(\alpha(\gamma)\), denoted by \(x_1(\gamma), x_2(\gamma), \ldots, x_{d(\gamma)}(\gamma)\), are simple. Then

\[
\text{Fix}(\phi') \cap M'_\gamma = \sum_{k=1}^{d(\gamma)} \int_{\Gamma_{\gamma} \setminus G} f(a^{-1}\gamma a) \cdot \epsilon_{\alpha(\gamma)}(x_k(\gamma)) \Lambda_{\Gamma_{\gamma} \setminus G}(\Gamma_{\gamma}a).
\]

By the classical Lefschetz theorem, we have

\[
\sum_{k=1}^{d(\gamma)} \epsilon_{\alpha(\gamma)}(x_k(\gamma)) = L(\alpha(\gamma)),
\]

where

\[
L(\alpha(\gamma)) = \sum_{i=1}^{\dim X} (-1)^i \text{Tr}(\alpha(\gamma)^* : H^i(X) \to H^i(X))
\]

is the Lefschetz number of the diffeomorphism \(\alpha(\gamma)\). It can be easily seen that \(L(\alpha(\gamma))\) depends only on the conjugacy class of \(\gamma\). Take a left invariant Riemannian metric on \(G\) whose volume form is \(\Lambda\). Consider the Riemannian
metric on $G_\gamma \backslash G$ so that the canonical projection $G \to G_\gamma \backslash G$ is a Riemannian submersion, and let $\Lambda_{G_\gamma \backslash G}$ be the corresponding volume form. Then

$$\langle \text{pr}_{2*}(\epsilon \Lambda'|_{\text{Fix}(\phi')}), f \rangle$$

$$= \sum_{\gamma \in \Sigma \setminus \{e\}} L(\alpha(\gamma)) \int_{G_\gamma \backslash G} f(a^{-1} \gamma a) \Lambda_{G_\gamma \backslash G}(G \gamma a)$$

$$= \sum_{\gamma \in \Sigma \setminus \{e\}} L(\alpha(\gamma)) \text{vol}(G_\gamma \backslash G) \int_{G_\gamma \backslash G} f(a^{-1} \gamma a) \Lambda_{G_\gamma \backslash G}(G \gamma a) .$$

Finally, we get the following Selberg type trace formula (cf. [29]):

$$\langle L_{\text{dis}}(F), f \rangle$$

$$= \text{vol}(\Gamma \backslash G) \chi(X) f(e)$$

$$+ \sum_{\gamma \in \Sigma \setminus \{e\}} L(\alpha(\gamma)) \text{vol}(G_\gamma \backslash G) \int_{G_\gamma \backslash G} f(a^{-1} \gamma a) \Lambda_{G_\gamma \backslash G}(G \gamma a) .$$

In the particular case when $G = \mathbb{R}$, $\Gamma = \mathbb{Z}$ and the homomorphism $\alpha$ is given by a diffeomorphism $F$ of a compact manifold $X$, the manifold $M$ is the mapping torus of $F$ and the foliation $F$ is given by the fibers of the natural map $M \to S^1$. Then the formula gives

$$L_{\text{dis}}(F) = \chi(X) \cdot \delta_0 + \sum_{k \in \mathbb{Z} \setminus \{0\}} L(F^k) \cdot \delta_k .$$

10.4. **Homogeneous foliations.** Let $H$ and $G$ be simply connected Lie groups, $\Gamma$ a uniform discrete subgroup in $H$, and $D : H \to G$ a surjective homomorphism so that $\Gamma_1 = D(\Gamma)$ is dense in $G$. Then $M = \Gamma \backslash H$ is a compact manifold, and let $F$ be the foliation on $M$ whose leaves are the projections of the fibers of $D$. If $K = \ker D$, then the leaves of $F$ are the orbits of the right action of $K$ on $M$ induced by the right action on $H$ defined by right translations.

This $F$ is a Lie $G$-foliation whose structural transverse action $\Phi$ is given as follows: for each $g \in G$, $\Phi_g$ is represented by the foliated map $F \to F$ induced by the right multiplication by any element of $D^{-1}(g)$.

The leaf of the foliation $G$ on $M \times G$ through a point $(\Gamma h, a) \in M \times G$ is, by definition,

$$M'_{(\Gamma h, a)} = \{ (\Gamma h_1, g^{-1}a g) \mid g \in G, h_1 \in D^{-1}(D(\Gamma h) g) \} .$$

It is easy to see that there is a bijection between the set of leaves of $G$ and the orbit space $G / \text{Ad}(\Gamma_1)$ of the adjoint action of $\Gamma_1$ on $G$ so that, for $\text{Ad}(\Gamma_1) g_0 \in G / \text{Ad}(\Gamma_1)$, the corresponding leaf is described as

$$M'_{\text{Ad}(\Gamma_1) g_0} = \{ (\Gamma h, g) \in M \times G \mid D(h) g D(h)^{-1} \in \text{Ad}(\Gamma_1) g_0 \} .$$
The first factor projection \( pr_1 : M'_\text{Ad}(\Gamma_1)_{g_0} \rightarrow M \) is a covering map; indeed, \( M'_{g_0} = \Gamma_{g_0} \setminus H \), where \( \Gamma_{g_0} = \Gamma \cap D^{-1}(\Gamma_{g_0}) \), denoting by \( \Gamma_{1,g_0} \) the centralizer of \( g_0 \) in \( \Gamma_1 \).

The leaves of \( F \) can be described as

\[
L_{\Gamma_{1,g_1}} = \{ \Gamma h \in M \mid D(h) \in \Gamma_{1,g_1} \}, \quad g_1 \in \Gamma_1 \setminus G.
\]

By definition, the leaf \( L'_{\Gamma_{1,g_1}} = \text{pr}_1^* (L_{\Gamma_{1,g_1}}) \) of the foliation \( F' = \text{pr}_1^* F \) on \( M'_{g_0} \) consists of all \( (\Gamma h, g) \in M \times G \) such that \( D(h) g D(h)^{-1} \in \text{Ad}(\Gamma_1)_{g_0} \) and \( D(h) \in \Gamma_{1,g_1} \). So it can be parameterized by the elements of \( (\Gamma_{1,g_1}) \times (G/\text{Ad}(\Gamma_1)) \), and it can be described as

\[
L'_{\Gamma_{1,g_1}} = \{ (\Gamma h, g) \in M \times G : D(h) \in \Gamma_{1,g_1}, \ g \in \text{Ad}(g_1) \text{Ad}(\Gamma_1)_{g_0} \}.
\]

We also see that \( \text{pr}_2 (M'_{g_0}) \) is the orbit \( O_{g_0} \) of the adjoint action of \( G \) on \( G \) through \( g_0 \). Moreover, \( \text{pr}_2 : M'_{g_0} \rightarrow O_{g_0} \) is a bundle map, and the fiber of this bundle over \( y \in O_{g_0} \) can be identified with \( \Gamma_{x} \setminus H_x \), where \( x \in H \) is any element such that \( D(x) = y \).

Denote by \( \mathfrak{h} \), \( \mathfrak{g} \) and \( \mathfrak{k} \) the Lie algebras of \( H \), \( G \) and \( K \), respectively. We have a short exact sequence

\[
0 \rightarrow \mathfrak{k} \rightarrow \mathfrak{h} \rightarrow \mathfrak{g} \rightarrow 0.
\]

To construct \( C^\infty \) local representations of \( \Phi \), we choose a splitting of this short exact sequence; that is, a linear map \( s : \mathfrak{g} \rightarrow \mathfrak{h} \) such that \( D_* \circ s = \text{id}_\mathfrak{g} \). So \( s \) is injective and \( s(\mathfrak{g}) \oplus \mathfrak{k} = \mathfrak{h} \). Let \( U \subset \mathfrak{g} \) be an open neighborhood of \( 0 \) in \( \mathfrak{g} \) such that the restriction \( \exp : U \rightarrow \exp(U) \subset G \) of the exponential map to \( U \) is a diffeomorphism. Then, for any \( g \in G \), a \( C^\infty \) local representation \( \phi : M \times O \rightarrow M \) of \( \Phi \) is defined on the open neighborhood \( O = g \exp(U) \) of \( g \) as

\[
\phi(\Gamma h, g \exp Y) = \Gamma hh_1 \exp s(Y), \quad h \in H, \quad Y \in U,
\]

where \( h_1 \in H \) is any element such that \( D(h_1) = g \).

Now fix \( g \in G \) and \( h_1 \in H \) such that \( D(h_1) = g \). By definition,

\[
(\Gamma h, g \exp Y) \in \text{Fix}(\phi') \Leftrightarrow \Gamma hh_1 \exp s(Y) = \Gamma h \Leftrightarrow hh_1 \exp s(Y) h^{-1} \in \Gamma.
\]

We have

\[
D(h) g \exp Y D(h)^{-1} = D(hh_1 \exp s(Y) h^{-1}) \in \Gamma_1,
\]

therefore, we get \( \text{Fix}(\phi') \cap M'_{g_0} \neq \emptyset \) iff \( g_0 \in \Gamma_1 \). In particular, it follows that

\[
\text{pr}_2 (\text{Fix}(\phi')) = \bigcup_{\gamma \in \Sigma} \mathcal{O}_\gamma,
\]

where \( \Sigma \) is a complete set of representatives of the \( \Gamma_1 \)-conjugacy classes in \( \Gamma_1 \). For a fixed class \( \gamma \in \Sigma \), let \( [D^{-1}(\gamma)] \) be the \( \Gamma \)-conjugacy class of the unique element \( \gamma_1 \in \Gamma \) such that \( D(\gamma_1) = \gamma \). Then we have

\[
\text{Fix}(\phi') \cap M'_\gamma = \{ (\Gamma h, g \exp Y) \in (\Gamma \setminus H) \times G \mid hh_1 \exp s(Y) h^{-1} \in [D^{-1}(\gamma)] \}.
\]
For any \((\Gamma h, g \exp Y) \in \text{Fix}(\phi')\), the left translation by \(h\) determines an isomorphism of the tangent space \(T_{\Gamma h} F\) with \(\mathfrak{k}\), and, under this isomorphism, the induced map \((\phi_{h1} \exp s(Y))_* : T_{\Gamma h} F \to T_{\Gamma h} F\) corresponds to the restriction \(\text{Ad}(h_1 \exp s(Y))_* |_{\mathfrak{k}} : \mathfrak{k} \to \mathfrak{k}\) of the differential of the adjoint action of \(g \exp Y \in G\) on \(G\) to \(\mathfrak{k}\). In particular, \((\Gamma h, g \exp Y) \in \text{Fix}(\phi)\) is simple if and only if \(\text{Ad}(h_1 \exp s(Y))_* |_{\mathfrak{k}} : \mathfrak{k} \to \mathfrak{k}\) is an isomorphism. It should be noted that this condition depends only on \(g \exp Y\) and is independent of the choice \(h_1\) and \(s\).

Assume that \(\text{Ad}(h \gamma h^{-1})_* |_{\mathfrak{k}} : \mathfrak{k} \to \mathfrak{k}\) is an isomorphism for any \(\gamma \in \Gamma\) and \(h \in H\). Then the value

\[
\epsilon(\Gamma h, g \exp Y) = \text{sign } \det \left( (\phi_{h1} \exp s(Y))_* - \text{id} : T_{\Gamma h} F \to T_{\Gamma h} F \right)
\]

is the same for any \((\Gamma h, g \exp Y) \in \text{Fix}(\phi') \cap M'_\gamma\), and equals

\[
\epsilon(\gamma) = \text{sign } \det \left( \text{Ad}(\gamma)_* |_{\mathfrak{k}} - \text{id} : \mathfrak{k} \to \mathfrak{k} \right).
\]

Let \(\Lambda\) be a left invariant volume form on \(G\), which can be identified with a transverse volume form of \(F\). Fix \(\gamma \in \Sigma\). Then the transverse volume form \(\Lambda' = \text{pr}_1^* \Lambda\) of \(F\) is given by the lift of \(\Lambda\) to \(M'_\gamma\) by the restriction of the map

\[(\Gamma h, g) \in (\Gamma \backslash H) \times G \mapsto D(h) \in G\]

to

\[M'_\gamma = \{(\Gamma h, g) \in (\Gamma \backslash H) \times G \mid D(h) g D(h)^{-1} \in \text{Ad}(\Gamma_1) \gamma\}\].

As above, take a left invariant Riemannian metric on \(G\) whose volume form is \(\Lambda\). Consider the Riemannian metric on \(G_\gamma \backslash G\) so that the canonical projection \(G \to G_\gamma \backslash G\) is a Riemannian submersion, and let \(\Lambda_{G_\gamma \backslash G}\) be the corresponding volume form. Restricting the form \(\Lambda'\) to \(\text{Fix}(\phi') \cap M'_\gamma\) and integrating it along the fibers of \(\text{pr}_2\), for any \(f \in C^\infty(G)\), we get

\[\langle \chi_{\text{dis}}(F), f \rangle = \langle \text{pr}_2^* (\epsilon \Lambda'), f \rangle = \sum_{\gamma \in \Sigma} \epsilon(\gamma) \text{vol}(\Gamma_\gamma \backslash H_\gamma) \int_{G_\gamma \backslash G} f(g^{-1} \gamma g) \Lambda_{G_\gamma \backslash G}(G_\gamma g)\],

where \(\gamma_0 \in \Gamma\) is the unique element such that \(D(\gamma_0) = \gamma\).

10.5. **Nilpotent homogeneous foliations.** Let \(G\) be a nontrivial simply connected nilpotent Lie group and let \(\Gamma_1 \subset G\) be a finitely generated dense subgroup. By Malcev’s theory \([18]\), there exists a simply connected nilpotent Lie group \(H\), an embedding \(i : \Gamma_1 \to H\) and a surjective homomorphism \(D : H \to G\) such that \(\Gamma = i(\Gamma_1)\) is discrete and uniform in \(H\), and \(D \circ i = \text{id}_{\Gamma_1}\). Consider the corresponding homogeneous foliation on the closed nilmanifold \(M = \Gamma \backslash H\). As above, \(K\) denotes the kernel of \(D\), which is a normal connected Lie subgroup in \(H\), and \(\mathfrak{k}\) denotes the Lie algebra of \(K\). As shown in \([11]\) Theorem 2.10], there is a canonical isomorphism \(\mathcal{F}(F) \cong H(\mathfrak{k})\) (c.f. \([22]\)), and thus \(L_{\text{dis}}(F) = 0\) by Corollary \([14]\). Let us check this triviality.
in another way. It can be easily seen that, under this isomorphism, the action of an element \( g \in G \) on \( \mathcal{P}(\mathcal{F}) \) induced by the structural action \( \Phi \) corresponds to the action \( \text{Ad}_s(h) \) on \( H(\mathfrak{f}) \) induced by the adjoint action of any element \( h \in D^{-1}(g) \). So \( \text{Tr}^\text{dis}_i(\mathcal{F}) \) is a smooth function on \( G \), whose value at \( g \in G \) is the trace of \( \text{Ad}_s(h) \) on \( H^i(\mathfrak{f}) \) with \( h \in D^{-1}(g) \). Since \( H \) is nilpotent, \( \text{Ad}_s(h) \) has a triangular matrix representation whose diagonal entries are equal to 1. So

\[
\text{Tr}^\text{dis}_i(\mathcal{F}) \equiv \dim H^i(\mathfrak{f}) ,
\]

yielding

\[
L^i(\mathcal{F}) \equiv \sum_i (-1)^i \dim H^i(\mathfrak{f}) = \sum_i (-1)^i \dim \bigwedge^i \mathfrak{f} = 0 .
\]

Any local section \( g \mapsto h_g \) of \( D \) on some open subset \( O \subset G \) induces a \( C^\infty \) local representation \( \phi : M \times O \to M \) of the structural action \( \Phi \), where each \( \phi_g \) is induced by the right multiplication by \( h_g \). All the fixed points of \( \phi \) are not simple.

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