Separability criteria via sets of mutually unbiased measurements

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Mutually unbiased measurements (MUMs) are generalized from the concept of mutually unbiased bases (MUBs) and include the complete set of MUBs as a special case, but they are superior to MUBs as they do not need to be rank one projectors. We investigate entanglement detection using sets of MUMs and derived separability criteria for d-dimensional multipartite systems, and arbitrary high-dimensional bipartite and multipartite systems. These criteria provide experimental implementation in detecting entanglement of unknown quantum states.

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I. INTRODUCTION

Quantum entanglement as a new physical resource has drawn a lot of attention in the field of quantum information in the past decade [1–10]. It plays a significant role in quantum information processing and has wide applications such as quantum cryptography [2, 11–12], quantum teleportation [1, 4, 13–16], and dense coding [17]. A main task of the theory of quantum entanglement is to distinguish between entangled states and separable states. For bipartite systems, various separability criteria have been proposed such as positive partial transposition criterion [18], computable cross norm or realignment criterion [19], reduction criterion [20], and covariance matrix criterion [21]. For multipartite and high dimensional systems, this problem is more complicated. There are various kinds of classification for multipartite entanglement. For instance, one can discuss it with the notions of k-partite entanglement or k-nonseparability for given partition and unfixed partition, respectively. In [22], Gao et al obtained separability criteria which can detect genuinely entangled and nonseparability n-partite mixed quantum states in arbitrary dimensional systems, and further developed k-separability criteria for mixed multipartite quantum states [23]. In [24], the authors defined k-ME concurrence in terms of all possible k partitions, which is a quantitative entanglement measure that has some important properties. One of the most important property is that $C_{k-ME}$ is zero if and only if the state is k separable. Combining k-ME concurrence with permutation invariance, a lower bound was given on entanglement for the permutation-invariance part of a state that apply to arbitrary multipartite state [25]. At the same time, the concept of “the permutationally invariant (PI) part of a density matrix” is proven to be more powerful because of its basis-dependent property.

Although there have been numerous mathematical tools for detecting entanglement of a given known quantum state, fewer results were obtained of the experimental implementation of entanglement detection for unknown quantum states. In 1960, Schwinger introduced the notion of mutually unbiased bases (MUBs) under a different name [26]. He noted that mutually unbiased bases represent maximally non-commutative measurements, which means the state of a system described in one mutually unbiased base provided no information about the state in another.

Later the term of mutually unbiased bases were introduced in [27], as they are intimately related to the nature of quantum information [28–30]. In [31], the authors availed of mutually unbiased bases and obtained separability criteria in arbitrarily high-dimensional, multipartite and continuous-variable quantum systems. The maximum number $N(d)$ of mutually unbiased bases has been shown to be $d + 1$ when $d$ is a prime power, but the maximal number of MUBs remains open for all other dimensions [32], which limits the applications of mutually unbiased bases. The concept of mutually unbiased bases were generalized to mutually unbiased measurements (MUMs) in [32]. A complete set of $d + 1$ mutually unbiased measurements were constructed [32] in a finite, $d$-dimensional Hilbert space, no matter whether $d$ is a prime power. Recently, Chen, Ma and Fei connected the separability criteria to mutually unbiased measurements [33] for arbitrary d-dimensional bipartite systems. Another method of entanglement detection in bipartite finite dimensional systems were realized using incomplete sets of mutually unbiased measurements [34]. In [34], the author derived entropic uncertainty relations and realized a method of entanglement detection in bipartite finite-dimensional systems using two sets of incomplete mutually unbiased measurements.

In this paper, we study the separability problem via sets of mutually unbiased measurements and propose separability criteria for the separability of d-dimensional
multipartite systems, and arbitrary high dimensional bipartite and m-partite systems.

II. PRELIMINARIES

Two orthonormal bases $B_1 = \{|b_i\rangle\}_{i=1}^d$ and $B_2 = \{|b_{ij}\rangle\}_{j=1}^d$ in Hilbert space $\mathbb{C}^d$ are called mutually unbiased if and only if

$$|\langle b_i | b_{j} \rangle| = \frac{1}{\sqrt{d}}, \quad \forall i, j = 1, 2, \ldots, d.$$ 

A set of orthonormal bases $\{B_1, B_2, \ldots, B_m\}$ of Hilbert space $\mathbb{C}^d$ is called a set of mutually unbiased bases (MUBs) if and only if every pair of bases in the set is mutually unbiased. If two bases are mutually unbiased, they are maximally non-commutative, which means a measurement over one such basis leaves one completely uncertain as to the outcome of a measurement over another one, in the other words, given any eigenstate of one, the eigenvalue resulting from a measurement of the other is completely undetermined. If $d$ is a prime power, then there exist $d + 1$ MUBs, which is a complete set of MUBs, but the maximal number of MUBs is unknown for other dimensions. Even for the smallest non-prime-power dimension $d = 6$, it is unknown whether there exists a complete set of MUBs [27]. For a two qubit separable state $\rho$ and any set of $m$ mutually unbiased bases $B_i = \{|b_{ij}\rangle\}_{j=1}^d, i = 1, 2, \ldots, m$, the following inequality holds [31]. Particularly, for a complete set of MUBs, the inequation above can be simplified as $I_{d+1} \leq 2$.

To conquer the shortcoming that we don’t know whether there exist a complete set of MUBs for all dimensions, Kalev and Gour generalized the concept of MUBs to mutually unbiased measurements (MUMs) [32]. Two measurements on a $d$-dimensional Hilbert space, $\mathcal{P}^{(b)} = \{F_n^{(b)} | F_n^{(b)} \geq 0, \sum_{n=1}^d F_n^{(b)} = 1\}$, $b = 1, 2$, with $d$ elements each, are said to be mutually unbiased measurements (MUMs) [32] if and only if,

$$\text{Tr}(F_n^{(b)}) = 1,$$

$$\text{Tr}(F_n^{(b)} F_{n'}^{(b')}) = \delta_{n,n'} \delta_{b,b'} \kappa + (1 - \delta_{n,n'}) \delta_{b,b'} \frac{1 - \kappa}{d - 1} + (1 - \delta_{b,b'}) \frac{1}{d},$$

Here $\kappa$ is efficiency parameter, and $\frac{1}{d} < \kappa \leq 1$.

A complete set of $d + 1$ MUMs in $d$ dimensional Hilbert space were constructed in [32]. Consider $d^2 - 1$ Hermitian, traceless operators acting on $\mathbb{C}^d$ satisfying

$$\text{Tr}(F_{n,b} F_{n',b'}) = \delta_{n,n'} \delta_{b,b'}.$$ Here, we use the generators of $SU(d)$ were used [32]

$$F_{n,b} = \begin{cases} \frac{1}{\sqrt{d}} \langle |n\rangle |b\rangle \langle b| |n\rangle, & \text{for } n < b, \\ \frac{1}{\sqrt{d}} \langle |n\rangle |b\rangle - \langle b| |n\rangle, & \text{for } b < n, \\ -n|n+1\rangle \langle n+1|, & \text{for } n = b, \end{cases} \quad \text{with } n = 1, 2, \ldots, d - 1. \tag{2}$$

Using such operators, a set of traceless, Hermitian operators $F_n^{(b)}$, $b = 1, 2, \ldots, d + 1$, $n = 1, 2, \ldots, d$, were built as follows [32],

$$F_n^{(b)} = \begin{cases} F_n(b) - (d + \sqrt{d})F_n,b, & n = 1, 2, \ldots, d - 1; \\ (1 + \sqrt{d})F_n(d), & n = d, \end{cases} \tag{3}$$

where $F_n(b) = \sum_{n=1}^{d-1} F_{n,b}$, $b = 1, 2, \ldots, d + 1$. Then one can construct $d + 1$ MUMs explicitly [32]:

$$P_n^{(b)} = \frac{1}{d} I + t F_n^{(b)}, \tag{4}$$

where $t$ is chosen such that $P_n^{(b)} \geq 0$.

These operators $\{F_n^{(b)}\}$ satisfies the conditions [32]:

$$\text{Tr}(F_n^{(b)} F_{n'}^{(b')}) = (1 + \sqrt{d})^2 [\delta_{n,n'}(d - 1) - (1 - \delta_{n,n'})],$$

$$\sum_{n=1}^{d} F_n^{(b)} = 0,$$

$$\text{Tr}(F_n^{(b)} F_{n'}^{(b')}) = 0, \quad \forall b \neq b', \quad \forall n, n' = 1, 2, \ldots, d. \tag{5}$$

Given a set of $M$ MUMs $\mathbb{P} = \{\mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(M)}\}$ of the efficiency $\kappa$ in $d$ dimensions, consider the sum of the corresponding indices of coincidence for the measurements, there is the following bound [33]:

$$\sum_{\mathcal{P} \in \mathbb{P}} C(\mathcal{P} | \rho) \leq \frac{M - 1}{d} + 1 - \kappa + (kd - 1) \text{Tr}(\rho^2), \tag{6}$$

where $C(\mathcal{P}^{(i)} | \rho) = \sum_{n=1}^{d} [\text{Tr}(\mathcal{P}^{(i)} | \rho)]^2$, $\mathcal{P}^{(i)} = \{P_n^{(i)}\}_{n=1}^{d}$, $i = 1, 2, \ldots, M$. For the complete set of $d + 1$ MUMs, we actually have an exact result instead of the inequality [35]:

$$\sum_{b=1}^{d+1} C(\mathcal{P}^{(b)} | \rho) = 1 + \frac{1 - \kappa + (kd - 1) \text{Tr}(\rho^2)}{d - 1}. \tag{7}$$

For pure state the equation can be more simplified as

$$\sum_{b=1}^{d+1} C(\mathcal{P}^{(b)} | \rho) = 1 + \kappa. \tag{8}$$

Corresponding to the construction of MUMs, the parameter $\kappa$ is given by

$$\kappa = \frac{1}{d} + t^2 (1 + \sqrt{d})^2 (d - 1). \tag{9}$$
III. DETECTION OF MULTIPARTITE ENTANGLEMENT

For multipartite systems, the definition of separability is not unique. So we introduce the notion of $k$-separable first. A pure state $|\varphi\rangle\langle\varphi|$ of an $N$-partite is $k$-separable if the $N$ parties can be partitioned into $K$ groups $A_1, A_2, \ldots, A_K$ such that the state can be written as a tensor product $|\varphi\rangle\langle\varphi| = \rho_{A_1} \otimes \rho_{A_2} \otimes \cdots \otimes \rho_{A_K}$. A general mixed state $\rho$ is $k$-separable if it can be written as a mixture of $k$-separable states $\rho = \sum_i \rho_i \rho_i$, where $\rho_i$ is $k$-separable pure states. States that are $N$-separable don’t contain any entanglement and are called fully separable. If a state $\rho$ is not fully separable, then we call it entangled. A state is called $k$-nonseparable if it is not $k$-separable, and a state is 2-nonseparable if and only if it is genuine $N$-partite entangled. Note that the definitions above for $k$-separable mixed states doesn’t require that each $\rho_i$ is $k$-separable under a fixed partition. But in this paper, we consider $k$-separable mixed states as a convex combination of $N$-partite pure states, each of which is $k$-separable with respect to a fixed partition. The notion of fully separable are same in both statements. In the following theorems, we give the necessary conditions of fully separable states. For $k$-separable state for given partition we will discuss it after the theorems.

Firstly, we will give a lemma that is generalized from the AM-GM inequality [36].

**Lemma 1.** For any list of $n$ nonnegative real numbers $x_1, x_2, \ldots, x_n$, we have the following inequality:

$$x_1 x_2 \cdots x_n \leq \left(\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}\right)^{\frac{1}{2}}.$$  \hspace{1cm} (10)

**Proof.** Since the AM-GM inequality [36]

$$\sqrt[n]{a_1 a_2 \cdots a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n},$$

where $a_1, a_2, \ldots, a_n$ are any list of $n$ nonnegative real numbers, and the equality holds if and only if $a_1 = a_2 = \cdots = a_n$. For $x_1, x_2, \ldots, x_n$, we have

$$\sqrt[n]{x_1^2 x_2^2 \cdots x_n^2} \leq \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n},$$

\text{i.e.} \hspace{1cm} \left(\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}\right)^{\frac{1}{2}} \leq \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n},

The function $f(x) = x^a$ is an increasing function when $a \geq 0$ and $x \geq 0$, so for nonnegative real numbers $x_i$, $i = 1, 2, \cdots, n$, we have

$$x_1 x_2 \cdots x_n \leq \left(\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}\right)^{\frac{1}{2}},$$

which completes the proof.

A. Detecting Entanglement for $m$-Qudit Systems

**Theorem 1.** Let $\rho$ be a density matrix in $(\mathbb{C}^d)^\otimes m$ and $\{P_i^{(b)}\}$ be any $m$ sets of $M$ MUMs on $\mathbb{C}^d$ with efficiency $\kappa_i$, where $P_i^{(b)} = \{P_{i,n}^{(b)}\}_{n=1}^{d}$. Define $J(\rho) = \sum_{b=1}^{M} \sum_{n=1}^{d} \text{Tr}(\rho P_{i,n}^{(b)})$. If $\rho$ is fully separable, then

$$J(\rho) \leq \frac{M - 1}{d} + \frac{1}{m} \sum_{i=1}^{d} \kappa_i.$$  \hspace{1cm} (11)

**Proof.** To prove that the inequality is satisfied for all fully separable states, let us verify that it holds for any fully separable pure state $\rho = \otimes_{i=1}^{n} |\psi_i\rangle\langle\psi_i|$ first. Note that

$$J(\rho) = \sum_{b=1}^{M} \sum_{n=1}^{d} \text{Tr}(\rho P_{i,n}^{(b)})$$

and $0 \leq \text{Tr}(P_{i,n}^{(b)}|\psi_i\rangle\langle\psi_i|) \leq 1$, by using Lemma 1, we have

$$J(\rho) \leq \sum_{b=1}^{M} \sum_{n=1}^{d} \left\{ \frac{1}{m} \sum_{i=1}^{d} \left[ \text{Tr}(P_{i,n}^{(b)}|\psi_i\rangle\langle\psi_i|) \right]^2 \right\}$$

$$\leq \sum_{b=1}^{M} \sum_{n=1}^{d} \frac{1}{m} \sum_{i=1}^{d} \left[ \text{Tr}(P_{i,n}^{(b)}|\psi_i\rangle\langle\psi_i|) \right]^2$$

$$= \frac{1}{m} \sum_{b=1}^{M} \sum_{n=1}^{d} \sum_{i=1}^{d} \left[ \text{Tr}(P_{i,n}^{(b)}|\psi_i\rangle\langle\psi_i|) \right]^2$$

By using the relation \(\Box\) for pure state $\rho$, we obtain

$$J(\rho) \leq \frac{M - 1}{d} + \frac{1}{m} \sum_{i=1}^{d} \kappa_i.$$  

The inequality holds for mixed states since $J(\rho)$ is a linear function. This completes the proof.

Especially, when we use the complete sets of MUMs, that is, $M = d + 1$, the inequality becomes

$$J(\rho) \leq 1 + \frac{1}{m} \sum_{i=1}^{d} \kappa_i.$$  \hspace{1cm} (12)

What’s more, when the efficiencies of each set of MUMs are same, the right-hand side of the inequality becomes $1 + \kappa$, and the criterion in Ref. [33] is the special case of our criterion when $m = 2$.

B. Detecting Entanglement for Bipartite Systems and $m$-Partite Systems

For the multipartite system with different dimensions, we have no idea how to detect the separability of states.
using complete sets of MUMs, but with incomplete sets of MUMs, we have the following conclusions.

**Theorem 2.** Let \( \rho \) be a density matrix in \( \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \), and \( \{ P^{(b)}_{i,n} \}_{i=1}^M \) and \( \{ Q^{(b)}_{i,n} \}_{i=1}^M \) be any two sets of M MUMs on \( \mathbb{C}^{d_1} \) and \( \mathbb{C}^{d_2} \) with efficiency \( \kappa_1, \kappa_2 \), respectively, where \( P^{(b)} = \{ P^{(b)}_{i,n} \}_{i=1}^M \) and \( Q^{(b)} = \{ Q^{(b)}_{i,n} \}_{i=1}^M \), \( b = 1, 2, \ldots, M \). Define

\[
J(\rho) = \max_{\{ P^{(b)}_{i,n} \}_{i=1}^M \subseteq \mathbb{C}^{d_1} \otimes \{ Q^{(b)}_{i,n} \}_{i=1}^M \subseteq \mathbb{C}^{d_2}} \sum_{b=1}^M \sum_{n=1}^d \text{Tr}(P^{(b)}_{i,n} \otimes Q^{(b)}_{i,n} \rho).
\]

Here \( d = \min\{d_1, d_2\} \). If \( \rho \) is separable, then

\[
J(\rho) \leq \frac{1}{2} \left( (M - 1) \left( \frac{1}{d_1} + \frac{1}{d_2} \right) + \kappa_1 + \kappa_2 \right) .
\]

**Proof.** We need only consider a pure separable state \( \rho = |\phi\rangle \langle \phi| \otimes |\psi\rangle \langle \psi| \), since \( \sum_{b=1}^M \sum_{n=1}^d \text{Tr}(P^{(b)}_{i,n} \otimes Q^{(b)}_{i,n} \rho) \) is a linear function of \( \rho \). We have

\[
\sum_{b=1}^M \sum_{n=1}^d \text{Tr}(P^{(b)}_{i,n} \otimes Q^{(b)}_{i,n} \rho) = \sum_{b=1}^M \sum_{n=1}^d \text{Tr}(P^{(b)}_{i,n} |\phi\rangle \langle \phi|) \text{Tr}(Q^{(b)}_{i,n} |\psi\rangle \langle \psi|) \]

\[
\leq \sum_{b=1}^M \sum_{n=1}^d \frac{1}{2} \left[ \text{Tr}(P^{(b)}_{i,n} |\phi\rangle \langle \phi|)^2 + \text{Tr}(Q^{(b)}_{i,n} |\psi\rangle \langle \psi|)^2 \right] \]

\[
= \frac{1}{2} \sum_{b=1}^M \sum_{n=1}^d \text{Tr}(P^{(b)}_{i,n} |\phi\rangle \langle \phi|)^2 + \frac{1}{2} \sum_{b=1}^M \sum_{n=1}^d \text{Tr}(Q^{(b)}_{i,n} |\psi\rangle \langle \psi|)^2 \]

\[
\leq \frac{1}{2} \frac{M - 1}{d_1} + \frac{1}{2} \left( \frac{1}{d_1} - 1 \right) \text{Tr}(\rho |\phi\rangle \langle \phi|) \]

\[
= \frac{1}{2} \left( M - 1 \right) \left( \frac{1}{d_1} + \frac{1}{d_2} \right) + \kappa_1 + \kappa_2, \]

where the inequality \( \Box \) is used. This completes the proof.

**Theorem 3.** Suppose that \( \rho \) is a density matrix in \( \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_m} \) and \( P^{(b)} \) are any sets of \( M \) MUMs on \( \mathbb{C}^{d_i} \) with the efficiencies \( \kappa_i \), where \( b = 1, 2, \ldots, M \), \( i = 1, 2, \ldots, m \). Let \( d = \min\{d_1, d_2, \ldots, d_m\} \), and define

\[
J(\rho) = \max_{\{ P^{(b)}_{i,n} \}_{i=1}^M \subseteq \mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_m} \subseteq \mathbb{C}^{d}} \sum_{b=1}^M \sum_{n=1}^d \text{Tr}(\otimes_{i=1}^m P^{(b)}_{i,n} \rho).
\]

If \( \rho \) is fully separable, then

\[
J(\rho) \leq \frac{1}{m} \sum_{i=1}^m \left( \frac{M - 1}{d_i} + \kappa_i \right). \tag{14}
\]

**Proof.** Let \( \rho = \sum_{j} \rho_{ij} \) with \( \sum_{j} \rho_{ij} = 1 \), be a fully separable density matrix, where \( \rho_{ij} = \otimes_{i=1}^m \rho_{ij} \). Since

\[
\sum_{b=1}^M \sum_{n=1}^d \text{Tr}(\otimes_{i=1}^m P^{(b)}_{i,n} \rho_{ij}) \]

\[
= \sum_{b=1}^M \sum_{n=1}^d \text{Tr}(P^{(b)}_{i,n} \rho_{ij}) \]

\[
= \sum_{b=1}^M \sum_{n=1}^d \prod_{i=1}^m \text{Tr}(P^{(b)}_{i,n} \rho_{ij}) \]

\[
\leq \sum_{b=1}^M \sum_{n=1}^d \left( \frac{1}{m} \sum_{i=1}^m \text{Tr}(P^{(b)}_{i,n} \rho_{ij}) \right)^2 \]

\[
\leq \sum_{b=1}^M \sum_{n=1}^d \left( \frac{1}{m} \sum_{i=1}^m \text{Tr}(P^{(b)}_{i,n} \rho_{ij}) \right)^2 \]

\[
= \frac{1}{m} \sum_{i=1}^m \sum_{n=1}^{d_i} \left( \text{Tr}(P^{(b)}_{i,n} \rho_{ij}) \right)^2 \]

\[
\leq \frac{1}{m} \sum_{i=1}^m \left( \frac{M - 1}{d_i} + \kappa_i \right),
\]

which implies that inequality \( \Box \) holds. It is complete.

For Theorem 3, we don’t require the subsystems with the same dimension, so we can use it straightforward to detect \( k \)-nonseparable states with respect to a fixed partition.

**IV. CONCLUSION AND DISCUSSIONS**

In summary we have investigated the entanglement detection using mutually unbiased measurements and presented separability criteria for \( d \) dimensional multipartite systems, and arbitrary high dimensional bipartite and multipartite systems via mutually unbiased measurements. These criteria provide experimental implementation in detecting entanglement of unknown quantum states, and are beneficial for experiments since they require only a few local measurements. One can flexibly use them in practice. For multipartite systems, the definition of separability is not unique. We can detect the \( k \)-nonseparability of \( N \)-partite and high dimensional systems. It would be interesting to study the separability criterion of multipartite systems with different dimensions via complete set of MUMs.
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[1] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. 70, 1895 (1993).
[2] A. Ekert, Phys. Rev. Lett. 67, 661 (1991).
[3] C. H. Bennett, G. Brassard, and N. D. Mermin, Phys. Rev. Lett. 68, 557 (1992).
[4] X. B. Wang, T. Hiroshima, A. Tomita, et al, Phys. Rep. 448, 1 (2007).
[5] H. K. Lo and H. F. Chau, Science 283, 2050 (1999).
[6] F. L. Yan and X. Q. Zhang, Eur. Phys. J. B 41, 75 (2004).
[7] T. Gao, F. L. Yan, and Z. X. Wang, J. Phys. A 38, 5761 (2005).
[8] F. L. Yan, T. Gao, and E. Chitambar, Phys. Rev. A 83, 022319 (2011).
[9] T. Gao, F. L. Yan and Y. C. Li, Europhys. Lett. 84, 50001 (2008).
[10] C. H. Bennett and D. P. DiVincenzo, Nature 404, 247 (2000).
[11] D. Deutsch, A. Ekert, R. Jozzas, C. Macchiavello, S. Popescu, and A. Sanpera, ibid. 77, 2818 (1996).
[12] C. A. Fuchs, N. Gisin, R. B. Griffiths, C. S. Niu, and A. Peres, Phys. Rev. A 56, 1163 (1997).
[13] S. Albeberio and S. M. Fei, Phys. Lett. A 276, 8 (2000).
[14] G. M. D’Ariano, P. Lo Presti, and M. F. Sacchi, ibid. 272, 32 (2000).
[15] S. Albeverio, S.-M. Fei, and W.-L. Yang, Phys. Rev. A 66, 012301 (2002).
[16] T. Gao, Commun. Theor. Phys. 42, 223 (2004).
[17] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. 69, 2881 (1992).
[18] A. Peres, Phys. Rev. Lett. 77, 1413 (1996).
[19] K. Chen, L.-A. WU, Quantum Inf. Comput. 3, 193 (2003).
[20] M. Horodecki, P. Horodecki, Phys. Rev. A. 59, 4206 (1999).
[21] O. Ghne, P. Hyllus, O. Gittsovich, and J. Eisert, Phys. Rev. Lett. 99, 130504 (2007).
[22] T. Gao, Y. Hong, Phys. Rev. A 82, 062113 (2010).
[23] T. Gao, Y. Hong, Y. Lu, and F. L. Yan, Europhys. Lett. 104, 20007 (2013).
[24] Y. Hong, T. Gao, and F. L. Yan, Phys. Rev. A 86, 062323 (2012).
[25] T. Gao, F. L. Yan and S. J. van Enk, Phys. Rev. Lett. 112, 180501 (2014).
[26] J. Schwinger, Pro. Nat. Acad. Sci. U. S. A. 46, 560 (1960).
[27] W. K. Wootters and B. D. Fields, Ann. Phys. (N. Y.) 191, 363 (1989).
[28] S. Wehner and A. Winter, New J. Phys. 12, 025009 (2010).
[29] S. M. Barnett, Quantum Information (Oxford University Press, Oxford, England, 2009).
[30] T. Durt, B.-G. Englert, I. Bengtsson, and K. Życzkowski, Int. J. Quantum. Inform. 08, 535 (2010).
[31] C. Spengler, M. Huber, S. Brierley, T. Adaktylos, and B. C. Hiesmayr, Phys. Rev. A 86, 022311 (2012).
[32] A. Kalev and G. Gour, New J. Phys. 16, 053038 (2014).
[33] B. Chen, T. Ma and S.-M. Fei, Phys. Rev. A 89, 064302 (2014).
[34] A. E. Rastegin, arXiv: 1407.7333v1 [quant-ph] (2014).
[35] B. Chen and S.-M. Fei, arXiv: 1407.6816 [quant-ph] (2014).
[36] http://en.wikipedia.org/wiki/Inequality_of_arithmetic_and_geometric_means