Multitime hybrid differential games with multiple integral functional

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Abstract
The purpose of this paper to formulate and to prove theorems about multitime differential games based on a multiple integral functional and an $m$-flow as constraint. The most important idea is to use a generating vector field for value functions. The original results include: fundamental properties of multitime upper and lower values, viscosity solutions of multitime (dHJIU) PDEs, representation formula of viscosity solutions for a multitime (dHJ) PDE, and max–min representations. These problems are totally new in the related literature, excepting our papers announced in arXiv.

Keywords Multitime hybrid differential games · Multiple integral cost · Divergence type PDE · Multitime viscosity solution · Multitime dynamic programming

1 Multitime hybrid differential game with multiple integral functional

The single-time theory of hybrid differential game is presented in several papers [4–6] and the single-time theory of viscosity solutions can be found in [1–3,7,8]. Combining these theories with the multi-temporal theories developed in our papers [9–20], we have discovered the theorems of this paper. The main motivation comes from the multitime hybrid control systems arising in many controlled engineering deformation problems (robotic systems, automated highway systems, flight control systems etc).

Let $\Omega_{0,T} \subset \mathbb{R}^m_+$ be the $m$-dimensional parallelepiped fixed by the diagonal opposite points $0 = (0, \ldots, 0)$ and $T = (T^1, \ldots, T^m)$ which is equivalent to the closed
interval $0 \leq t \leq T$ via the product order. Let $t = (t^a) = (t^1, \ldots, t^m) \in \Omega_{0T} \subset \mathbb{R}^m_+$, $\alpha = 1, m$ be an evolution multi-parameter (multi-time), $dt = dt^1 \wedge \cdots \wedge dt^m$ be the volume element ($m$-form) in $\mathbb{R}^m_+$. We add a $C^2$ state vector $x : \Omega_{0T} \rightarrow \mathbb{R}^n$, $x(t) = (x^i(t)), i = 1, n, n \geq m \in C^1$ control vector $u : \Omega_{0T} \rightarrow U \subset \mathbb{R}^p, u(t) = (u^a(t)), a = 1, p$, for the first team of $p$ players (who wants to maximize), a $C^1$ control vector $v : \Omega_{0T} \rightarrow V \subset \mathbb{R}^q, v(t) = (v^b(t)), b = 1, q$, for the second team of $q$ players (who wants to minimize), with $U, V$ compact sets, $u(\cdot) = \Phi(\cdot, \eta_1(\cdot)), v(\cdot) = \Psi(\cdot, \eta_2(\cdot))$, a running cost $L(t, x(t), u(t), v(t))$ as a nonautonomous continuous Lagrangian, a terminal cost (penalty term) $g(x(T))$ and the $C^1$ vector fields $X_\alpha = (X_\alpha^i)$ satisfying the complete integrability conditions (CIC) $D_\beta X_\alpha = D_\alpha X_\beta$ ($m$-flow type problem), where $D_\alpha$ is the total derivative operator with respect to $t^\alpha$.

In this paper, a multitime hybrid differential game is given by a multitime dynamics (PDE system controlled by two vector controllers) and a target including a multiple integral functional. Our new approach is to define and use the generating vector fields and a target including a multiple integral (volume) and a function of the final event (the terminal cost) and whose evolution PDE is an $m$-flow:

\[
\begin{align*}
\min_{u(\cdot)} & \max_{v(\cdot)} I(u(\cdot), v(\cdot)) = \int_{\Omega_{0T}} L(s, x(s), u(s), v(s))ds + g(x(T)), \\
\text{subject to the Cauchy problem} & \\
\frac{\partial x^i}{\partial s^{\alpha}}(s) &= X_\alpha^i(s, x(s), u(s), v(s)), \quad x(0) = x_0, s \in \Omega_{0T} \subset \mathbb{R}^m_+, x \in \mathbb{R}^n.
\end{align*}
\]

Let $[X_\alpha, X_\beta]$ be the bracket of vector fields. Then the piecewise complete integrability conditions (CIC),

\[
(\frac{\partial X_\alpha}{\partial u^a} \delta^\gamma_{\beta} - \frac{\partial X_\beta}{\partial u^a} \delta^\gamma_{\alpha}) \frac{\partial u^a}{\partial s^{\gamma}} + \left( \frac{\partial X_\alpha}{\partial v^b} \delta^\gamma_{\beta} - \frac{\partial X_\beta}{\partial v^b} \delta^\gamma_{\alpha} \right) \frac{\partial v^b}{\partial s^{\gamma}} = [X_\alpha, X_\beta] + \frac{\partial X_\beta}{\partial s^{\alpha}} - \frac{\partial X_\alpha}{\partial s^{\beta}},
\]

are supposed to be satisfied.

We vary the starting multitime and the initial point. We obtain a larger family of similar multitime problems based on the functional

\[
I_{t,x}(u(\cdot), v(\cdot)) = \int_{\Omega_{tT}} L(s, x(s), u(s), v(s))ds + g(x(T))
\]

and the multitime evolution constraint (Cauchy problem for first order PDEs system)

\[
\frac{\partial x^i}{\partial s^{\alpha}}(s) = X_\alpha^i(s, x(s), u(s), v(s)), \quad x(t) = x, s \in \Omega_{tT} \subset \mathbb{R}^m_+, x \in \mathbb{R}^n.
\]
Let us formulate some hypotheses that assure the existence of the solution of a multitime hybrid differential game.

We fix $A = (A_\alpha)$ as a constant 1-form, and denote $t \in \Omega_{0T}, x, \hat{x} \in \mathbb{R}^n, u \in U, v \in V$. We assume that each vector field $X_\alpha : \Omega_{0T} \times \mathbb{R}^n \times U \times V \rightarrow \mathbb{R}^n$ is uniformly continuous and satisfies the boundedness conditions

$$\|X_\alpha(t, x, u, v)\| \leq A_\alpha, \quad \|X_\alpha(t, x, u, v) - X_\alpha(t, \hat{x}, u, v)\| \leq A_\alpha \|x - \hat{x}\|.$$ 

Suppose the functions $g : \mathbb{R}^n \rightarrow \mathbb{R}$, $L : \Omega_{0T} \times \mathbb{R}^n \times U \times V \rightarrow \mathbb{R}$ are uniformly continuous and satisfy the boundedness conditions

$$|g(x)| \leq B, \quad |g(x) - g(\hat{x})| \leq B \|x - \hat{x}\|, \quad |L(t, x, u, v)| \leq C, \quad |L(t, x, u, v) - L(t, \hat{x}, u, v)| \leq C \|x - \hat{x}\|,$$

for constants $B, C$ and all $t \in \Omega_{0T}, x, \hat{x} \in \mathbb{R}^n, u \in U, v \in V$.

### 2 Control sets and value functions

Here we define: control sets, strategies, value functions, and generating vector fields.

**Definition 2.1** Suppose the pair $(u(\cdot), v(\cdot))$ is composed of measurable functions that satisfy CIC. (i) The set $\mathcal{U} = \{ u : \mathbb{R}^n_+ \rightarrow U \}$ is called the control set for the first team of players. (ii) The set $\mathcal{V} = \{ lbracex : \mathbb{R}^n_+ \rightarrow V \}$ is called the control set for the second team of players.

**Definition 2.2** (i) A map $\Phi : \mathcal{V} \rightarrow \mathcal{U}$ is called a strategy for the first team of players, if the equality $v(\tau) = v^*(\tau), t \leq \tau \leq s \leq T$ implies $\Phi[v](\tau) = \Phi[v^*](\tau)$.

(ii) A map $\Psi : \mathcal{U} \rightarrow \mathcal{V}$ is called a strategy for the second team of players, if the equality $u(\tau) = u^*(\tau), t \leq \tau \leq s \leq T$ implies $\Psi[u](\tau) = \Psi[u^*](\tau)$.

Let $\mathcal{A}$ be the set of strategies for the first team of players and $\mathcal{B}$ be the set of strategies for the second team of players.

**Definition 2.3** (i) The function $m(t, x) = \min_{\Psi \in \mathcal{B}} \max_{u(\cdot) \in \mathcal{U}} I_{t, x}[u(\cdot), \Psi[u](\cdot)]$ is called the multitime lower value function.

(ii) The function $M(t, x) = \max_{\Phi \in \mathcal{A}} \min_{v(\cdot) \in \mathcal{V}} I_{t, x}[\Phi[v](\cdot), v(\cdot)]$ is called the multitime upper value function.

To support the Sect. 5, we introduce the most important ingredient, namely the generating vector field (see [15]).

**Definition 2.4** Let $D_\alpha$ be the total derivative with respect to $t^\alpha$ and $B_{hyp}$ be a hyperbolic constant. A vector field $u = (u^\alpha(t, x))$ is called a generating vector field of the function $u(t, x)$, if

$$u(T, x(T)) = B_{hyp} + u(t, x(t)) + \int_{\Omega_T} D_\alpha u^\alpha(s, x(s)) \, ds.$$
We suppose the lower value function \( m(t, x) \) admits a generating lower vector field \( m(t, x) = (m^\alpha(t, x)) \), i.e.,

\[
m(T, x(T)) = c_{hyp} + m(t, x(t)) + \int_{\Omega_{tT}} D\alpha m^\alpha(s, x(s))ds
\]

and the upper value function \( M(t, x) \) is coming from a generating upper vector field \( M(t, x) = (M^\alpha(t, x)) \), i.e.,

\[
M(T, x(T)) = C_{hyp} + M(t, x(t)) + \int_{\Omega_{tT}} D\alpha M^\alpha(s, x(s))ds.
\]

**Remark 2.1** Two multitime Lagrangians which differs by a total divergence term have the same Euler–Lagrange PDEs.

### 3 Multitime dynamic programming optimality conditions

Let us give explicit formulas for lower and upper value functions which represent in fact multitime dynamic programming optimality conditions.

**Theorem 3.1** For each pair of strategies \((\Phi, \Psi)\), the lower and upper value functions can be written respectively in the form

\[
m(t, x) = \min_{\Psi \in B} \max_{u \in U} \left\{ \int_{\Omega_{tt+h}} L(s, x(s), u(s), \Psi[u](s))ds \right\} + m(t + h, x(t + h))
\]

and

\[
M(t, x) = \max_{\Phi \in A} \min_{v \in V} \left\{ \int_{\Omega_{tt+h}} L(s, x(s), \Phi[v](s), v(s))ds \right\} + M(t + h, x(t + h)),
\]

for all \((t, x) \in \Omega_{tT} \times \mathbb{R}^n\) and all \(h \in \Omega_{0T-t}\).

**Proof** To confirm the previous statement for the lower value function, we introduce a new function

\[
w(t, x) = \min_{\Psi \in B} \max_{u \in U} \left\{ \int_{\Omega_{tt+h}} L(s, x(s), u(s), \Psi[u](s))ds \right\} + m(t + h, x(t + h)).
\]

It will be enough to prove that the lower value function \( m(t, x) \) satisfies two inequalities, \( m(t, x) \leq w(t, x) + 2\varepsilon \) and \( m(t, x) \geq w(t, x) - 3\varepsilon \), \( \forall \varepsilon > 0 \).

Let us prove the first inequality. For \( \varepsilon > 0 \), there exists a strategy \( \Upsilon \in B \) such that

\[
w(t, x) \geq \max_{u \in U} \left\{ \int_{\Omega_{tt+h}} L(s, x(s), u(s), \Upsilon[u](s))ds + m(t + h, x(t + h)) \right\} - \varepsilon.
\]
We shall use the state \( x(\cdot) \) which solves the (PDE), with initial condition \( \bar{x} = x(t + h) \), on \( \Omega_{tT} \setminus \Omega_{tt+h} \), for each \( \bar{x} \in \mathbb{R}^n \). The following equality

\[
m(t + h, \bar{x}) = \min_{\Psi \in \mathcal{B}} \max_{u \in \mathcal{U}} \left\{ \int_{\Omega_{tt+h}} L(s, x(s), u(s), \Psi[u](s))ds + g(x(T)) \right\}
\]

holds. Thus there exists a strategy \( \Upsilon_{\bar{x}} \in \mathcal{B} \) such that

\[
m(t + h, \bar{x}) \geq \max_{u \in \mathcal{U}} \left\{ \int_{\Omega_{tt+h}} L(s, x(s), u(s), \Upsilon_{\bar{x}}[u](s))ds + g(x(T)) \right\} - \varepsilon.
\]

Define a new strategy

\[
\Psi \in \mathcal{B}, \Psi[u](s) \equiv \begin{cases} 
\Upsilon[u](s) & \text{if } s \in \Omega_{tt+h} \\
\Upsilon_{\bar{x}}[u](s) & \text{if } s \in \Omega_{tT} \setminus \Omega_{tt+h},
\end{cases}
\]

for each control \( u \in \mathcal{U} \). Consequently, for any \( u \in \mathcal{U} \), we can write

\[
w(t, x) \geq \int_{\Omega_{tT}} L(s, x(s), u(s), \Psi[u](s))ds + g(x(T)) - 2\varepsilon.
\]

Going from side to side and applying maximum, we obtain

\[
\max_{u \in \mathcal{U}} \left\{ \int_{\Omega_{tT}} L(s, x(s), u(s), \Psi[u](s))ds + g(x(T)) \right\} \leq w(t, x) + 2\varepsilon.
\]

By the definition of the lower value function, we have

\[
m(t, x) \leq w(t, x) + 2\varepsilon.
\]

For the reverse inequality, there exists a strategy \( \Psi \in \mathcal{B} \) that satisfies the inequality

\[
m(t, x) \geq \max_{u \in \mathcal{U}} \left\{ \int_{\Omega_{tT}} L(s, x(s), u(s), \Psi[u](s))ds + g(x(T)) \right\} - \varepsilon.
\]

The definition of \( w(t, x) \) implies

\[
w(t, x) \leq \max_{u \in \mathcal{U}} \left\{ \int_{\Omega_{tt+h}} L(s, x(s), u(s), \Psi[u](s))ds + m(t + h, x(t + h)) \right\}
\]

and consequently there exists a control \( u^1 \in \mathcal{U} \) such that

\[
w(t, x) \leq \int_{\Omega_{tt+h}} L(s, x(s), u^1(s), \Psi[u^1](s))ds + m(t + h, x(t + h)) + \varepsilon.
\]
Define a new control
\[ u^* \in U, u^*(s) \equiv \begin{cases} u^1(s) & \text{if } s \in \Omega_{tt+h} \\ u(s) & \text{if } s \in \Omega_{tT} \setminus \Omega_{tt+h} \end{cases} \]
for a control \( u \in U \) and then define a new strategy \( \Psi^* \in B, \Psi^*[u](s) \equiv \Psi[u^*](s), s \in \Omega_{tT} \setminus \Omega_{tt+h} \). We find the inequality
\[
m(t + h, x(t + h)) \leq \max_{u \in U} \left\{ \int_{\Omega_{tt+h}} L(s, x(s), u(s), \Psi^*[u](s)) ds + g(x(T)) \right\}
\]
and so there exists the control \( u^2 \in U \) for which
\[
m(t + h, x(t + h)) \leq \int_{\Omega_{tT} \setminus \Omega_{tt+h}} L(s, x(s), u^2(s), \Psi^*[u^2](s)) ds + g(x(T)) + \varepsilon.
\]
Define a new control
\[ u \in U, u(s) \equiv \begin{cases} u^1(s) & \text{if } s \in \Omega_{tt+h} \\ u^2(s) & \text{if } s \in \Omega_{tT} \setminus \Omega_{tt+h} \end{cases} \]
Then the previous inequalities yield
\[
w(t, x) \leq \int_{\Omega_{tT}} L(s, x(s), u(s), \Psi[u](s)) ds + g(x(T)) + 2\varepsilon,
\]
and so we have the inequality
\[
w(t, x) \leq m(t, x) + 3\varepsilon.
\]
Since \( \varepsilon > 0 \) is arbitrary, this inequality and \( m(t, x) \leq w(x, t) + 2\varepsilon \) complete the proof. \( \square \)

4 Boundedness and continuity of value functions

Now we add boundedness and continuity properties of lower and upper value functions. A basic idea is to replace a multitemporal Cauchy problem with associated curvilinear integral equation.

**Theorem 4.1** The lower value function \( m(t, x) \) and the upper value function \( M(t, x) \) satisfy the boundedness conditions
\[
|m(t, x)|, |M(t, x)| \leq D
\]
\[
|m(t, x) - m(\hat{t}, \hat{x})|, |M(t, x) - M(\hat{t}, \hat{x})| \leq E \text{ vol}(\Omega_{\hat{t}}) + D \|x - \hat{x}\|,
\]
for some constant \( D, E \) and for all \( t, \hat{t} \in \Omega_{0T}, x, \hat{x} \in \mathbb{R}^n \).
Proof Because the two value functions have analogous definitions, we prove only the statement for upper value function $M(t, x)$. Since $|g(x)| \leq B$, $|L(t, x, u, v)| \leq C$, we find

$$|I_{t,x}(u(\cdot), v(\cdot))| = \left| \int_{\Omega_{1T}} L(s, x(s), u(s), v(s))ds + g(x(T)) \right|$$

$$\leq \int_{\Omega_{1T}} |L(s, x(s), u(s), v(s))| ds + |g(x(T))|$$

$$\leq C \int_{\Omega_{1T}} ds + B \leq C \text{vol}(\Omega_{0T}) + B = D \implies |M(t, x)| \leq D,$$

for all $u(\cdot) \in \mathcal{U}$, $v(\cdot) \in \mathcal{V}$.

Let $x_1, x_2 \in \mathbb{R}^n$, $t_1, t_2 \in \Omega_{0T}$. For $\varepsilon > 0$ and the strategy $\Phi \in \mathcal{A}$, we have

$$M(t_1, x_1) \leq \min_{v \in \mathcal{V}} I(\Phi[v], v) + \varepsilon.$$

Define the control

$$\overline{v} \in \mathcal{V}, \overline{v}(s) \equiv \begin{cases} v^1(s) & \text{if } s \in \Omega_{0T} \backslash \Omega_{01} \\ v(s) & \text{if } s \in \Omega_{0T} \backslash \Omega_{02}, \end{cases}$$

for any $v \in \mathcal{V}$ and some $v^1 \in \mathcal{V}$ and for each $v \in \mathcal{V}$, $\Phi \in \mathcal{A}$ (the restriction of $\Phi$ over $\Omega_{0T} \backslash \Omega_{01}$) by $\Phi[v] = \Phi[\overline{v}], s \in \Omega_{0T} \backslash \Omega_{02}$.

Take the control $v \in \mathcal{V}$ such that $M(t_2, x_2) \geq I(\Phi[v], v) - \varepsilon$. By the previous inequality, we deduce $M(t_1, x_1) \leq I(\Phi[\overline{v}], \overline{v}) + \varepsilon$.

We know that the (unique, Lipschitz) solution $x(\cdot)$ of the Cauchy problem

$$\frac{\partial x^i}{\partial s^\alpha}(s) = X^i_\alpha(s, x(s), u(s), v(s)) \quad x(t) = x, \; s \in \Omega_{1T} \subset \mathbb{R}^m_+, \; x \in \mathbb{R}^n,$$

where $i = 1, n$, $\alpha = 1, m$, is the response to the controls $u(\cdot), v(\cdot)$ for $s \in \Omega_{0T}$.

We choose $x_1(\cdot)$ as solution of the Cauchy problem

$$\frac{\partial x^i_1}{\partial s^\alpha}(s) = X^i_\alpha(s, x_1(s), \Phi[\overline{v}], \overline{v}) \quad x_1(t_1) = x_1, \; s \in \Omega_{0T} \backslash \Omega_{01}$$

and $x_2(\cdot)$ as solution of the Cauchy problem

$$\frac{\partial x^i_2}{\partial s^\alpha}(s) = X^i_\alpha(s, x_2(s), \Phi[v], v(s)) \quad x_2(t_2) = x_2, \; s \in \Omega_{0T} \backslash \Omega_{02}.$$

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Equivalently, $x_1(\cdot)$ is solution of curvilinear integral equation

$$x_1(s) = x_1(t_1) + \int_{\Gamma_{t_1}} X_\alpha(\sigma, x_1(\sigma), \Phi[v](\sigma), \overline{v}(\sigma)) \, d\sigma$$

and $x_2(\cdot)$ is solution of curvilinear integral equation

$$x_2(s) = x_2(t_2) + \int_{\Gamma_{t_2}} X_\alpha(\sigma, x_2(\sigma), \Phi[v](\sigma), \overline{v}(\sigma)) \, d\sigma.$$

It follows that

$$\|x_1(t_2) - x_1\| = \|x_1(t_2) - x_1(t_1)\| \leq \|A\| \ell(\Gamma_{t_1 t_2}).$$

Because $v = \overline{v}$ and $\Phi[v] = \Phi[\overline{v}]$, for $s \in \Omega_{0T} \setminus \Omega_{0t_2}$, we find the estimation

$$\|x_1(s) - x_2(s)\| \leq \|x_1(t_1) - x_2(t_2)\| + \left\| \int_{\Gamma_{t_1 t_2}} \cdots \right\| \leq \|A\| \ell(\Gamma_{t_1 t_2}) + \|x_1 - x_2\|,$$

on $t_2 \leq s \leq T$.

The previous inequalities imply

$$M(t_1, x_1) - M(t_2, x_2) \leq I(\Phi[\overline{v}], \overline{v}) - I(\Phi[v], v) + 2\varepsilon$$

$$\leq \left| \int_{\Omega_{t_1 t_2}} L(s, x_1(s), \Phi[\overline{v}](s), \overline{v}(s)) \, ds \right. + \int_{\Omega_{t_2 T}} \left( (L(s, x_1(s), \Phi[v](s), v(s)) - L(s, x_2(s), \Phi[v](s), v(s))) \, ds \right.$$ \n$$+ g(x_1(T)) - g(x_2(T)) + 2\varepsilon \right|$$

$$\leq \int_{\Omega_{t_1 t_2}} |L(s, x_1(s), \Phi[\overline{v}](s), \overline{v}(s))| \, ds$$ \n$$+ \int_{\Omega_{t_2 T}} |(L(s, x_1(s), \Phi[v](s), v(s)) - L(s, x_2(s), \Phi[v](s), v(s)))| \, ds$$ \n$$+ |g(x_1(T)) - g(x_2(T))| + 2\varepsilon$$

$$\leq C \text{vol}(\Omega_{t_1 t_2}) + (C \text{vol}(\Omega_{0T}) + B) \|x_1 - x_2\| + 2\varepsilon.$$

Since $\varepsilon$ is arbitrary, we obtain the inequality

$$M(t_1, x_1) - M(t_2, x_2) \leq E \text{vol}(\Omega_{t_1 t_2}) + D \|x_1 - x_2\|.$$
Let \( \varepsilon > 0 \) and choose the strategy \( \Phi \in \mathcal{A}(t_2) \) such that

\[
M(t_2, x_2) \leq \min_{v \in \mathcal{V}(t_2)} I(\Phi[v], v) + \varepsilon.
\]

For each control \( v \in \mathcal{V} \) and \( s \in \Omega_{0T} \setminus \Omega_{0r_2} \), define the control \( u \in \mathcal{V}, v(s) = v(s) \). For some \( u^1 \in \mathcal{U} \), we define the strategy \( \Phi \in \mathcal{A} \) (the restriction of \( \Phi \) to \( \Omega_{0T} \setminus \Omega_{0r_2} \)) by

\[
\Phi[v](s) = \begin{cases} 
  u^1(s) & \text{if } s \in \Omega_{0r_2} \setminus \Omega_{0r_1} \\
  \Phi[v](s) & \text{if } s \in \Omega_{0T} \setminus \Omega_{0r_2}.
\end{cases}
\]

Now choose a control \( v \in \mathcal{V} \) so that \( M(t_1, x_1) \geq I(\Phi[v], v) - \varepsilon \). By the previous inequality, we have \( M(t_2, x_2) \leq I(\Phi[v], v) + \varepsilon \).

We consider \( x_1(\cdot) \) as solution of the Cauchy problem

\[
\frac{\partial x_1^i}{\partial s^a}(s) = X_a(s, x_1(s), \Phi[v], v(s)), \quad x_1(t_1) = x_1, \quad s \in \Omega_{0T} \setminus \Omega_{0r_1}
\]

and \( x_2(\cdot) \) as solution of the Cauchy problem

\[
\frac{\partial x_2^i}{\partial s^a}(s) = X_a(s, x_2(s), \Phi[v], v(s)), \quad x_2(t_2) = x_2, \quad s \in \Omega_{0T} \setminus \Omega_{0r_2}.
\]

It follows that

\[
\|x_1(t_2) - x_1\| = \|x_1(t_2) - x_1(t_1)\| \leq \|A\| \ell(\Gamma_{t_1t_2}).
\]

For \( s \in \Omega_{0T} \setminus \Omega_{0r_2} \), \( v = v \) and \( \Phi[v] = \Phi[v] \), we find the estimation

\[
\|x_1(s) - x_2(s)\| \leq \|x_1(t_1) - x_2(t_2)\| + \int_{\Gamma_{t_1t_2}} \ldots \|A\| \ell(\Gamma_{t_1t_2}) + \|x_1 - x_2\|,
\]

on \( t_2 \leq s \leq T \).

Thus, the foregoing relations imply

\[
M(t_2, x_2) - M(t_1, x_1) \leq I(\Phi[v], v) - I(\Phi[v], v) + 2\varepsilon
\]

\[
\leq \left| - \int_{\Omega_{1t_2}} L(s, x_1(s), \Phi[v](s), v(s))ds
\right|
\]

\[
+ \int_{\Omega_{2T}} (L(s, x_2(s), \Phi[v](s), v(s))
\]

\[
- L(s, x_1(s), \Phi[v](s), v(s))ds
\]

\[
+ g(x_2(T)) - g(x_1(T)) + 2\varepsilon
\]

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\[
\begin{align*}
\leq & \int_{\Omega_{t_1 t_2}} |L(s, x_1(s), \Phi[v](s), v(s))| \, ds \\
& + \int_{\Omega_{t_2}^T} |L(s, x_2(s), \Phi[v](s), v(s)) \\
& - L(s, x_1(s), \Phi[v](s), v(s))| \, ds \\
& + |g(x_2(T)) - g(x_1(T))| + 2\varepsilon \\
\leq & C \text{ vol}(\Omega_{t_1 t_2}) + (C \text{ vol}(\Omega_{0 T}) + B) \|x_2 - x_1\| + 2\varepsilon.
\end{align*}
\]

Since \( \varepsilon \) is arbitrary, we obtain

\[ M(t_2, x_2) - M(t_1, x_1) \leq E \text{ vol}(\Omega_{t_1 t_2}) + D \|x_1 - x_2\|. \]

By this inequality we proved the continuity of the lower and upper value functions. \( \Box \)

5 Viscosity solutions of multitime (dHJIU) PDEs

The key original idea is that the generating upper vector field \( M = (M^\alpha) \) or the generating lower vector field \( m = (m^\alpha) \) are solutions of (dHJIU) PDEs.

Remark 5.1 The generating upper vector field was introduced by the relation

\[
M(t + h) = M(t) + C_{hyp} + \int_{\Omega_{t+h}} D_\alpha M^\alpha \, ds \\
\Rightarrow M(t) - M(t + h) = -C_{hyp} - \int_{\Omega_{t+h}} D_\alpha M^\alpha \, ds.
\]

The multitime dynamic programming optimality condition gives

\[
M(t) = \max_{\Phi \in A} \min_{v \in V} \left\{ \int_{\Omega_{t+h}} L(s, x(s), \Phi[v](s), v(s)) \, ds \right\} + M(t + h) \\
\Rightarrow M(t) - M(t + h) = \max_{\Phi \in A} \min_{v \in V} \left\{ \int_{\Omega_{t+h}} L(s, x(s), \Phi[v](s), v(s)) \, ds \right\}.
\]

These two equalities suggest a multitime divergence Hamilton–Jacobi–Isaacs–Udriște (dHJIU) PDE.

5.1 PDEs for generating upper vector field, resp. lower vector field

Theorem 5.1 (i) The generating upper vector field \( M = (M^\alpha(t, x)) \) is the viscosity solution of the multitime divergence type upper Hamilton–Jacobi–Isaacs–Udriște (dHJIU) PDE
\[ \frac{\partial M^\alpha}{\partial t^\alpha}(t, x) + \min_{v \in V} \max_{u \in U} \left\{ \frac{\partial M^\alpha}{\partial x^i}(t, x) X^i_\alpha(t, x, u, v) + L(t, x, u, v) \right\} = 0, \]

which satisfies the terminal condition \( M^\alpha(T, x) = g^\alpha(x) \).

(ii) The generating lower vector field \( m = (m^\alpha(t, x)) \) is the viscosity solution of the multitime divergence type lower Hamilton–Jacobi–Isaacs–Udrişte (dHJIU) PDE

\[ \frac{\partial m^\alpha}{\partial t^\alpha} + \max_{u \in U} \min_{v \in V} \left\{ \frac{\partial m^\alpha}{\partial x^i}(t, x) X^i_\alpha(t, x, u, v) + L(t, x, u, v) \right\} = 0, \]

which satisfies the terminal condition \( m^\alpha(T, x) = g^\alpha(x) \).

**Remark 5.2** If we introduce the so-called upper and lower Hamiltonian defined respectively by

\[ H^+(t, x, p) = \min_{v \in V} \max_{u \in U} \left\{ p^\alpha_i(t) X^i_\alpha(t, x, u, v) + L(t, x, u, v) \right\}, \]
\[ H^-(t, x, p) = \max_{u \in U} \min_{v \in V} \left\{ p^\alpha_i(t) X^i_\alpha(t, x, u, v) + L(t, x, u, v) \right\}, \]

then the multitime (dHJIU) PDEs can be written in the form

\[ \frac{\partial M^\alpha}{\partial t^\alpha}(t, x) + H^+(t, x, \frac{\partial M}{\partial x}(t, x)) = 0 \]

and

\[ \frac{\partial m^\alpha}{\partial t^\alpha}(t, x) + H^-(t, x, \frac{\partial m}{\partial x}(t, x)) = 0. \]

The proof is given in the paper [20].

**6 Representation formula of viscosity solution for multitime (dHJ) PDE**

In this section, we want to obtain a representation formula for the viscosity solution \( M = (M^\alpha(t, x)) \) of the multitime (dHJ) PDE (divergence type)

\[ \frac{\partial M^\alpha}{\partial t^\alpha} + H \left( t, x, \frac{\partial M}{\partial x}(t, x) \right) = 0, \quad (t, x) \in \Omega_0 T \times \mathbb{R}^n, \alpha = 1, \ldots, m, \]
\[ M^\alpha(0, x) = g^\alpha(x), \quad x \in \mathbb{R}^n. \]

On the other hand, the upper value function \( M(t, x) \), generated by \( M = (M^\alpha(t, x)) \), satisfies the inequalities

\[ |M(t, x)| \leq D, \quad |M(t, x) - M(\hat{t}, \hat{x})| \leq E \ vol(\Omega_{\hat{t}}) + D \|x - \hat{x}\|, \]

for some constant \( E, D \).
Also, we assume that $g : \mathbb{R}^n \to \mathbb{R}^n$, $H : \Omega_{0T} \times \mathbb{R}^n \times \mathbb{R}^{mn} \to \mathbb{R}$, satisfy the inequalities

$$\|g(x)\| \leq B, \quad \|g(x) - g(\hat{x})\| \leq B\|x - \hat{x}\|$$

and

$$|H(t, x, 0)| \leq K, \quad |H(t, x, p) - H(\hat{t}, \hat{x}, \hat{p})| \leq K(\text{vol}(\Omega_{ij}) + \|x - \hat{x}\| + \|p - \hat{p}\|).$$

The norm of the matrix $p = (p^a_i)$ is $\|p\| = \sqrt{\delta_{ij} \delta_{\alpha\beta} p^a_i p^a_j}$. Otherwise, in this paper, all norms of indexed variables are norms of vectors associated by re-indexing.

**Lemma 6.1** Let $U = B(0, 1) \subset \mathbb{R}^n$, $V = B(0, P) \subset \mathbb{R}^{mn}$, and

$$X_\alpha(u) = Q_\alpha u, \quad Q = (Q^i_\alpha), \quad \|Q\| = K, \quad L(t, x, u, v) = H(t, x, v) - \langle Qu, v \rangle.$$

Suppose the Hamiltonian $H$ is a Lipschitz function. Then, for some constant radius $P > 0$ and for each $t \in \Omega_{0T}, x \in \mathbb{R}^n$, we have

$$H(t, x, p) = \max_{u \in U} \min_{v \in V} \left\{ p^a_i(t) X^i_\alpha(u) + L(t, x, u, v) \right\},$$

if $\|p\| \leq P$.

**Proof** By the assumption $H(t, x, v) - H(t, x, p) \leq K\|p - v\|$, $K = \|Q\|$, by Cauchy-Schwarz formula and by the condition $\|u\| \leq 1$, we have

$$H(t, x, p) = \max_{v \in V} \{H(t, x, v) - K\|p - v\|\}
= \max_{v \in V} \min_{u \in U} \{H(t, x, v) + \langle Qu, p - v \rangle\},$$

for any $x \in \mathbb{R}^n$. \qed

### 6.1 Max–min representation of a Lipschitz function as positive homogeneous function

We apply the idea in title for Hamiltonian $m$-forms.

**Lemma 6.2** Let $H(t, x, p)$ be a Lipschitz $m$-form which is homogeneous in the matrix $p$, i.e., $H(t, x, \lambda p) = \lambda H(t, x, p)$, $\lambda \geq 0$. Then there exist compact sets $U \subset \mathbb{R}^{2n}$, $V \subset \mathbb{R}^{2mn}$ and vector fields $X_\alpha : \Omega_{0T} \times \mathbb{R}^n \times U \times V \to \mathbb{R}^n$, $\alpha = 1, m$, satisfying $\|X_\alpha(x) - X_\alpha(\hat{x})\| \leq A_\alpha\|x - \hat{x}\|$, for each $\alpha$, and such that $H(t, x, p) = \max_{v \in V} \min_{u \in U} \left\{ p^a_i(t) X^i_\alpha(t, x, u, v) \right\}$, for all $t \in \Omega_{0T}, x \in \mathbb{R}^n, p \in \mathbb{R}^{mn}$.

**Proof** Let $u = (u^1, u^2)$ be a $2n$-dimensional control, $v = (v^1, v^2)$ be a $2mn$-dimensional control. We introduce the notations $U = B(0, 1) \times B(0, 1) \subset \mathbb{R}^{2n},$
Multitime hybrid differential games with multiple integral... 1833

\[ V = \mathcal{B}(0, 1) \times \mathcal{B}(0, 1) \subset \mathbb{R}^{2mn}, L(t, x, u^1, v^1) = H(t, x, v^1) - \langle Qu^1, v^1 \rangle, \]

\[ x_\alpha(t, x, u, v) = Qu^1 + Cv^2 + (L(t, x, u^1, v^1) - C)v^2. \]

According to previous Lemma and the assumptions, if \( \|\eta\| = 1 \), we have

\[ H(t, x, \eta) = \max_{v^1 \in V^1} \min_{u^1 \in U^1} \left\{ \langle Qu^1, \eta > + L(t, x, u^1, v^1) \right\}. \]

for \( U^1 = \mathcal{B}(0, 1) \subset \mathbb{R}^n \), \( V^1 = \mathcal{B}(0, 1) \subset \mathbb{R}^{mn} \). For any non-zero matrix \( p = (p^i_j) \), we can write

\[ H(t, x, p) = \|p\| H \left( t, x, \frac{p}{\|p\|} \right) = \max_{v^1 \in V^1} \min_{u^1 \in U^1} \left\{ \langle Qu^1, p > + L(t, x, u^1, v^1)\|p\| \right\}. \]

Then, if we choose \( C > 0 \) such that \( |L| \leq C \), we find

\[ H(t, x, p) = \max_{v^1 \in V^1} \min_{u^1 \in U^1} \left\{ \langle Qu^1, p > + C\|p\| + (L(t, x, u^1, v^1) - C)\|p\| \right\} \]

\[ = \max_{v^1 \in V^1} \min_{u^1 \in U^1} \left\{ \langle Qu^1, p > + \langle Cv^2, p > \right\} \]

\[ + (L(t, x, u^1, v^1) - C) < v^2, p > \right\} = \max_{v \in V} \min_{u \in U} \left\{ < X(t, x, u, v), p > \right\}. \]

Now, interchanging \( \min_{u^1 \in U^1} \) and \( \max_{v^2 \in V^1} \), the result in Lemma follows. \( \square \)

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