MANIFOLDS ADMITTING A METRIC WITH CO-INDEX OF SYMMETRY 4

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Abstract. By a recent result, it is known that compact homogeneous spaces with co-index of symmetry 4 are quotients of a semisimple Lie group of dimension at most 10. In this paper we determine exactly which ones of these spaces actually admit such a metric. For all the admissible spaces we construct explicit examples of these metrics.

1. Introduction

The problem of classifying the $G$-invariant Riemannian metrics on a given homogeneous manifold $M = G/H$ is a difficult one. Even in the case $M = G/\{e\}$ of a Lie group with a left invariant metric, this problem is far from being solved. What makes more sense is to impose some geometric constrains and restrict ourselves to a more manageable class. For instance, we know exactly which Lie groups admit a bi-invariant metric, and how a bi-invariant metric looks like in such a group. More generally, if we ask for parallel tensor curvature, we end up with Cartan’s classification of the symmetric spaces [Car26].

One possible way to approach this general problem, is by trying to classify homogeneous spaces according to their index of symmetry, first introduced in [ORT14]. This proves to be a fruitful way to address the issue, leading to very interesting examples and strong structure results. Let us say quickly that the index of symmetry is a geometric invariant, which measures how far is a homogeneous Riemannian manifold from being a symmetric space. More precisely, the index of symmetry of a homogeneous Riemannian manifold $M = G/H$ can be defined as the maximum number $i_s(M)$ of linearly independent Killing fields which are parallel at a given point of $M$. Associated to this concept there is a $G$-invariant distribution on $M$ called the distribution of symmetry, whose rank equals $i_s(M)$, which is integrable with totally geodesic leaves. Moreover, the leaves of the distribution of symmetry are isometric to a globally symmetric space, called the leaf of symmetry of $M$. The distribution of symmetry was computed for compact naturally reductive spaces in [ORT14] and for naturally reductive nilpotent Lie groups in [Reg18a]. In [Pod15], Podestá computed the index of symmetry for Kähler metrics on generalized flag manifolds, showing that the leaf of symmetry is a Hermitian symmetric space. There is also a classification of left invariant metrics on 3-dimensional unimodular Lie groups according to their index of symmetry [Reg18b]. Although there is some work in the non compact setting, the most important structure results related to the index of symmetry appear almost exclusively in the compact case (mainly because of the existence of a bi-invariant metric on the full isometry group). In particular, in the work [BOR17] the classification
of compact homogeneous spaces with co-index of symmetry less or equal than 3 is given (the co-index of symmetry of M is \( \dim M - \delta(M) \)). Namely, there are no spaces with co-index 1 (this is also the case for non compact spaces according to [Reg18b]); all spaces with co-index of symmetry 2 are covered by \( SU(2) \) with certain left invariant metrics; and the spaces with co-index 3 arise as certain \( SO(4) \)-invariant metrics on \( SO(4)/SO(2) \) (for the standard inclusion of \( SO(2) \) into \( SO(4) \)). In particular, in these cases the underlying manifold supporting such metrics is the same, up to a cover. These results rely on a more general theorem proved in [BOR17] which gives a bound on the dimension of \( M \) in terms of its co-index of symmetry. More precisely, if \( M \) is compact homogeneous (without symmetric factors) of co-index of symmetry \( k \), then there exists a transitive semisimple Lie group \( G' \) such that

\[
\dim G' \leq \frac{k(k + 1)}{2}.
\]  

(1.1)

This is the reason why in the above cases there is only one possible space admitting such metrics. The next logical step is to study spaces with co-index of symmetry 4. But in this case the situation is more complicated, as there are several possibilities for the group \( G' \). The goal of this paper is to determine which homogeneous spaces \( G'/H' \), with \( G' \) as in (1.1), admit a metric of co-index of symmetry \( k = 4 \). By a simple inspection one can easily derive a list of all the spaces \( G'/H' \) which could admit a metric of co-index 4. Actually the list is somewhat shorter than one expects, as in the extreme case where \( \dim G' = 10 \), the isotropy group must have positive dimension. From this list we can exclude the spaces \( SO(5)/SO(2) \) and \( SO(5)/(SO(3) \times SO(2)) \). In order to do that, we need to study the isotropy representation and the transvection group of the possible leaf of symmetry (which have dimension 5 and 2 respectively). For all the remaining cases we give explicit metrics with co-index 4. Some families of examples are constructed from the classification given in [BOR17] for co-index 3 and the classification of naturally reductive spaces of dimension 6 [AFF15]. Another argument used in the construction of the metrics comes from the so-called double symmetric pairs \( G_1 \supset G_2 \supset G_3 \), where \( G_1/G_2 \) and \( G_2/G_3 \) are symmetric pairs. This trick is used in [ORT14], where perturbing the normal homogeneous metric on \( G_1/G_3 \), one sometimes gets a metric with leaf of symmetry \( G_2/G_3 \). This argument does not always work, as one has to prove every time that the proposed metric is not symmetric. Some examples of this were known, but we can give a new one associated with double symmetric pair \( SO(5) \supset SO(4) \supset SO(2) \times SO(2) \).

Here the leaf of symmetry is a product of spheres.

2. Preliminaries

We use this section to fix some notation and review the structure theory concerning the index of symmetry of a compact homogeneous space. The main references for this section are [ORT14] and [BOR17]. Let \( M = G/H \) be a compact homogeneous space, where \( G = I(M) \) is the full isometry group of \( M \). Let \( \mathfrak{g} \) be the Lie algebra of \( G \), which is naturally identified with the algebra \( \mathfrak{r}(M) \) of Killing vector fields on \( M \). We also denote by \( \mathfrak{h} \) the Lie algebra of the full isotropy group \( H \). Given \( q \in M \), we define the Cartan subspace at \( q \) as

\[
\mathfrak{p}^q = \{ X \in \mathfrak{g} : (\nabla X)_q = 0 \},
\]

where \( \nabla \) is the Levi-Civita connection of \( M \). The elements in \( \mathfrak{p}^q \) are called transvections at \( q \). The symmetric isotropy algebra at \( q \) is defined by

\[
\mathfrak{t}^q = \text{span}_\mathbb{R}\{ [X,Y] : X, Y \in \mathfrak{p}^q \}.
\]
It is easy to see that \( q^q \) is contained in \( h \). Let us define
\[
g^q = \mathfrak{f}^q \oplus \mathfrak{p}^q,
\]
which is an involutive subalgebra of \( g \). We denote by \( G^q \) the connected Lie subgroup of \( G \) with Lie algebra \( g^q \). The distribution of symmetry \( s \) of \( M \) is defined by
\[
q \mapsto s_q = \{ X_q : X \in \mathfrak{p}^q \}
\]
and it is a \( G \)-invariant autoparallel distribution of \( M \) (that is, integrable with totally geodesic leaves). The rank \( i_s(M) \) of the distribution \( s \) is known as the index of symmetry of \( M \), and the co-index of symmetry of \( M \) is defined as \( ci_s(M) = \dim M - i_s(M) \). The integral manifold \( L(q) \) of \( s \) by \( q \) is a totally geodesic submanifold of \( M \), and moreover, it is extrinsically a globally symmetric space. The leaves of the distribution of symmetry form a foliation \( \mathcal{L} \) on \( M \) called the foliation of symmetry of \( M \). Since all the leaves of the foliation of symmetry are isometric, we will refer to \( L(q) \) as the leaf of symmetry of \( M \). Let us denote
\[
g^s = \{ X \in g : X \in s \},
\]
which is an ideal of \( g \) and let \( G^s \) be the corresponding normal subgroup of \( G \).

**Remark 2.1.** The following facts hold (see [BOR17]).

(1) The groups \( G^s \) and \( G^q \) act almost effectively on the leaf of symmetry \( L(q) \).
(2) If \( \tilde{G}^q = \{ g|_{L(q)} : g \in G^q \} \) and \( \tilde{K}^q = \{ h|_{L(q)} : h \in H \} \), then the Lie algebra of \( \tilde{K}^q \) is \( \mathfrak{f}^q \) (restricted to the leaf of symmetry) and \( G^q/K^q \) is a symmetric presentation for \( L(q) \).

The most important general result for compact homogeneous spaces related to these topics is the following theorem.

**Theorem 2.2 ([BOR17]).** Let \( M = G/H \) a compact, simply connected homogeneous Riemannian space with \( G = I(M) \) and co-index of symmetry \( k \). Assume that \( M \) does not split of a symmetric de Rham factor. Then \( k \geq 2 \) and there exists a Lie group \( G' \) with the following properties.

(1) \( G' \) is a semisimple normal subgroup of \( G \).
(2) \( G' \) is transitive on \( M \).
(3) \( g = g' \oplus g' \) (direct sum of ideals), where \( g' \) is the Lie algebra of \( G' \).
(4) \( \dim G' \leq k(k + 1)/2 \).
(5) If \( \dim G' = k(k + 1)/2 \) then the universal cover of \( G' \) is \( \text{Spin}(k + 1) \).
(6) If \( k \geq 3 \) and \( \dim G' = k(k + 1)/2 \), the isotropy group of \( G' \) has positive dimension.

In particular, the item 4 of the above theorem gives us a bound on the dimension of \( M \) in terms of its co-index of symmetry. Finally, recall that the Lie algebra \( \mathfrak{g}^q \) of \( G^q \) (which is isomorphic to \( g^q \)) can be decomposed as a sum of ideals
\[
\mathfrak{g}^q = \mathfrak{g} \oplus \mathfrak{g}^q,
\]
where \( \mathfrak{g}^q \) is the restriction of \( g' \cap g^q \) to \( L(q) \) and \( \mathfrak{g} \) is the restriction of \( g^q \cap g^q \). (Recall that \( g^q \cap g^q \) could contain an ideal which acts trivially on \( L(q) \).)

In the case of co-index 4, Theorem 2.2 says that the underling manifold, up to a cover, is one of the following:
The rest of the article is devoted to decide which ones of the above manifolds does actually admit an invariant metric with co-index of symmetry 4.

3. Inadmissible manifolds

**Theorem 3.1.** There is not any SO(5)-invariant metric on \( M = SO(5)/SO(2) \) with co-index of symmetry equal to 4.

**Proof.** Since \( \dim M = 9 \) and \( ci(M) = 4 \), the leaf of symmetry \( L(q) \) is a symmetric space of dimension 5. Since we are working locally, we can assume that \( L(q) \) is product of a simply connected symmetric space of the compact type and a (possibly trivial) torus. So, the different possibilities for \( L(q) \) are \( S^5, S^4 \times T^1, S^3 \times S^2, S^3 \times T^2, S^2 \times S^2 \times T^1, S^2 \times T^3 \) and \( T^5 \).

Let us first look at the case \( L(q) = S^5 \). Here \( so(6) = \mathfrak{g}^d = \mathfrak{g} \oplus \mathfrak{g}^q \) is a simple Lie algebra, and hence one of these two ideals must be trivial. Since \( so(6) \) is the full isometry Lie algebra of \( L(q) \), and from Theorem 2.2, \( G' \) has isotropy group of positive dimension, we conclude that \( \mathfrak{g} = 0 \). This implies that \( so(6) = \mathfrak{g}^q \), which acts effectively on \( L(q) \), must be contained in \( \mathfrak{g}' = so(5) \). A contradiction. For the case \( L(q) = S^4 \times T^1 \) we argue similarly. Here \( \mathfrak{g}^d = so(5) \oplus \mathbb{R} \), so in the decomposition \( \mathfrak{g}^d = \mathfrak{g} \oplus \mathfrak{g}^q \) we must have \( \mathfrak{g} = \mathbb{R} \) and \( \mathfrak{g}^q = \mathfrak{g}' = so(5) \), and hence \( \mathfrak{g}' \) could not be transitive on \( M \), which is absurd.

Assume now that \( L(q) = S^3 \times S^2 \), and hence \( \mathfrak{g}^q = so(4) \oplus so(3) \) as a direct sum of ideals, where the first summand corresponds to the full isometry Lie algebra of \( S^3 \) and the second one is the isometry algebra of \( S^2 \). Since \( G' \) is transitive on \( M \), \( \mathfrak{g}^q \) splits as the direct sum of two ideals \( \mathfrak{g}^3 \oplus so(3) \), where \( \mathfrak{g}^3 \) is the Lie algebra of a transitive isometry subgroup of \( S^3 \) and the second summand is the Lie algebra of the full isometry group of \( S^2 \). We claim that \( \mathfrak{g} = 0 \) in the decomposition \( \mathfrak{g}^q = \mathfrak{g} \oplus \mathfrak{g}^q \). Otherwise, we must have that \( \mathfrak{g} \simeq so(3) \) and, up to an isometry of \( M \), \( \mathfrak{g}^q \) decomposes in the following manner. If we identify the sphere \( S^3 \) with the unit quaternions, then we can present \( \mathfrak{g}^q = so(3)^f \oplus so(3)^r \) as a direct sum of ideals isomorphic to \( so(3) \), where \( so(3)^f \) and \( so(3)^r \) are the Lie algebras of the left and right multiplications respectively on \( S^3 \). Without lose of generality we can assume that \( \mathfrak{g} = so(3)^f \) and \( \mathfrak{g}^q = so(3)^r \oplus so(3) \). Let us denote by \( SO(4) = SO(3)^f \times SO(3)^r \) (almost direct product) the isometry group of the factor \( S^3 \) of \( L(q) \). Since \( \mathfrak{g} \subset \mathfrak{g}^q \), we have that \( SO(3)^f \) leaves invariant the factor \( S^3 \) of any other leaf of symmetry. This implies that \( SO(3)^r \) does so, and hence \( so(3)^r \) must be contained in \( \mathfrak{g} \), a contradiction from assuming \( \mathfrak{g} \neq \{0\} \). So, \( \mathfrak{g}^q = \mathfrak{g}^q = so(4) \oplus so(3) \) is the direct sum of the Lie algebras of transvections of \( S^3 \) and \( S^2 \). This says that the dimension of the isotropy group of \( G' \) is greater or equal than 2, which is a contradiction. This excludes the case \( L(q) = S^3 \times S^2 \).

The cases \( S^3 \times T^2, S^2 \times S^2 \times T^1, S^2 \times T^3 \) and \( T^5 \) can be disregarded all at once with the following argument. In such cases the leaf of symmetry is a symmetric space of rank at least 3, and \( G' = SO(5) \) must contain a subgroup which is transitive on \( L(q) \), but this is impossible since \( SO(5) \) has rank 2.

□

**Remark 3.2.** Recall that the proof of Theorem 3.1 is independent of the choice of the inclusion \( SO(2) \hookrightarrow SO(5) \), for which there are infinitely many geometric possibilities.
Proposition 3.3. There is not any SO(5)-invariant metric on \( M = SO(5)/(SO(3) \times SO(2)) \) with co-index of symmetry equal 4.

Proof. Since \( \dim M = 6 \), if the metric has co-index 4, then the leaf of symmetry \( L(q) \) must be locally isometric to the sphere \( S^2 \) or the torus \( T^2 \). This implies that \( \dim \mathfrak{g}^q \leq 3 \) and \( \mathfrak{g} = \{0\} \) in the decomposition \( \mathfrak{g}^q = \mathfrak{g} \oplus \mathfrak{g}^q \). On the other hand, we have that the isotropy group \( SO(3) \times SO(2) \) of \( G' = SO(5) \) leaves invariant \( L(q) \) and hence, \( \mathfrak{so}(3) \oplus \mathfrak{so}(2) \subset \mathfrak{g}^q \). This is impossible, since the action of \( G^q \) on \( L(q) \) is almost effective. \( \square \)

Remark 3.4. As a matter of fact, the case of \( M = SO(5)/SO(4) \), which is diffeomorphic to the sphere \( S^4 \), is not even under consideration because co-index 4 means that \( i_s(M) = 0 \), and we are only interested in the cases where the distribution of symmetry is non-trivial. Nevertheless, this situation is also impossible, since is a well-known fact that the only \( SO(5) \)-invariant metric on \( S^5 \) is the round one (up to scaling). This follows, for instance, from the fact that \( SO(5)/SO(4) \) is an isotropy irreducible space (see [Wol68]).

4. Examples of spaces with co-index of symmetry 4

In this section we present an example of a metric with co-index of symmetry 4 for each of the manifolds which were not excluded in Section 3.

4.1. Double symmetric pairs. For the first two examples we use a construction given in [ORT14] using double symmetric pairs. Let us review briefly this argument. Let us as consider a triple \( G \supset G' \supset K' \) where \( G \) is a compact Lie group and \( G', K' \) are compact subgroups of \( G \). Assume that \( G' \) is simple and \( (G', K') \) is a symmetric pair (which cannot be of the group type). Let \( (-,-) \) be an Ad(\( G \))-invariant inner product on the Lie algebra \( \mathfrak{g}' \) of \( G' \). Denote by \( \mathfrak{g}' = \mathfrak{g}' \oplus \mathfrak{p}' \) the Cartan decomposition of \( (G', K') \), where \( \mathfrak{g}' \) is the Lie algebra of \( K' \). Recall that, since \( G' \) is simple, the restriction of \( (-,-) \) to \( \mathfrak{g}' \) is a multiple of the Killing form of \( \mathfrak{g}' \), and so \( \mathfrak{g}' \) is orthogonal to \( \mathfrak{p}' \) with respect to \( (-,-) \). Let \( \mathfrak{m} \) be orthogonal complement of \( \mathfrak{g}' \) with respect to \( (-,-) \). Since \( \mathfrak{p}' \subset \mathfrak{m} \), we have an orthogonal decomposition \( \mathfrak{m} = \mathfrak{m}' \oplus \mathfrak{p}' \), where \( \mathfrak{m}' = (\mathfrak{p}')^\perp \cap \mathfrak{m} \). Now, we define an inner product \( \langle -, - \rangle \) on \( \mathfrak{m} \) by asking:

\[ \langle \mathfrak{m}', \mathfrak{p}' \rangle = 0, \quad \langle -, - \rangle|_{\mathfrak{m}' \times \mathfrak{m}'} = \langle -, - \rangle|_{\mathfrak{m'} \times \mathfrak{m}'}, \quad \langle -, - \rangle|_{\mathfrak{p}' \times \mathfrak{p}'} = 2\langle -, - \rangle|_{\mathfrak{p}' \times \mathfrak{p}'}.
\]

Endow \( M = G/K' \) with the \( G \)-invariant Riemannian metric induced by the inner product \( \langle -, - \rangle \) on \( \mathfrak{m} = T_{eH} M \). We denote such metric with same symbol \( \langle -, - \rangle \). It follows from the results in [ORT14] that the \( G \)-invariant distribution induced by \( \mathfrak{p}' \) is contained in the distribution of symmetry of \( M \). Moreover, if \( G/G' \) is an irreducible symmetric space (with the normal homogeneous metric) and \( G/K' \) is not a locally symmetric space, then the distribution of symmetry of \( M \) is exactly the distribution induced by \( \mathfrak{p}' \) and the leaf of symmetry is isometric to \( G'/K' \).

4.2. The case of \( SO(5)/SO(3) \). This case case was already treated in [ORT14]. Consider the standard inclusions \( SO(5) \supset SO(4) \supset SO(3) \). The construction given in Subsection 4.1 does not apply directly, since \( SO(4) \) is not simple, but this difficulty can be avoided by noticing that the restriction of the Killing form of \( \mathfrak{so}(5) \) to \( \mathfrak{so}(4) \) is a multiple of the Killing form of \( \mathfrak{so}(4) \). The above construction gives an \( SO(5) \)-invariant metric on \( SO(5)/SO(3) \) with leaf of symmetry isometric to the sphere \( S^3 = SO(4)/SO(3) \). (One should check that \( SO(5)/SO(3) \) is not a symmetric space.) Recall that, after a rescaling of the metric, \( SO(5)/SO(3) \) is isometric to the unit tangent bundle of the 4-sphere of curvature 2 (with the Sasaki metric).
4.3. **The case** $SO(5)/\langle SO(2) \times SO(2) \rangle$. Now consider the standard inclusions $SO(5) \supset SO(4) \supset SO(2) \times SO(2)$. We have here the same situation as in the above case where the Killing form of $\mathfrak{so}(4)$ is a scalar multiple of the restriction of the Killing form of $\mathfrak{so}(5)$, so the construction from double symmetric pairs applies. Recall that $SO(4)/\langle SO(2) \times SO(2) \rangle$ is the Grassmannian $G_2^+ (\mathbb{R}^4)$ of oriented 2-planes in $\mathbb{R}^4$, which is isometric to the product of round spheres $S^2 \times S^2$. So, the metric of Subsection 4.1 gives us a $SO(5)$-invariant metric on $SO(5)/\langle SO(2) \times SO(2) \rangle$, with leaf of symmetry $G_2^+ (\mathbb{R}^4)$, provided it is not symmetric.

**Lemma 4.1.** *With the* $SO(5)$-*invariant metric defined in the above paragraph, the space* $M = SO(5)/\langle SO(2) \times SO(2) \rangle$ *is not a locally symmetric space.*

**Proof.** Let us consider the universal covering $\tilde{M} = Spin(5)/\langle Spin(2) \times Spin(2) \rangle$ of $M$, where $Spin(2) \simeq SO(2)$. It is enough to prove that $\tilde{M}$ is not a globally symmetric space. Assume that $\tilde{M}$ is a symmetric space. Recall that, since $\tilde{M}$ is compact and simply connected, it cannot have a flat factor.

Let us prove first that $\tilde{M}$ must be irreducible. In fact, let $\tilde{M} = M_1 \times \cdots \times M_k$ be the de Rham decomposition of $\tilde{M}$, where $M_i$ is a compact, simply connected, irreducible symmetric space. Since, $Spin(5)$ is simple, projecting down the group $Spin(5) \subset I(\tilde{M})$ to $I(M_i)$ we get a transitive subgroup of $I(M_i)$ isomorphic to $Spin(5)$ (since the kernel of this projection is a normal subgroup of $Spin(5)$ and $M_i$ is simply connected). In particular, since $\dim \tilde{M} = 8$, no factor $M_i$ in the decomposition of $\tilde{M}$ can be a symmetric space of the group type. Let us denote by $n_i$ the dimension of $M_i$. Since $10 = \dim Spin(5) \leq \dim I(M_i) \leq n_i(n_i + 1)/2$, we conclude that $k = 2$, and $n_1 = n_2 = 4$. This implies that $\tilde{M} = S^4 \times S^4$ and $I_0(\tilde{M}) = Spin(5) \times Spin(5)$ (almost effective action). This is a contradiction, because no subgroup of $I(\tilde{M})$, isomorphic to $Spin(5)$ can be transitive in $S^4 \times S^4$.

So $\tilde{M}$ is a simply connected, compact irreducible symmetric space which is not of the group type. Thus the only possibilities are $\tilde{M} = G_1^+(\mathbb{H}^3)$ or $\tilde{M} = G_2^+ (\mathbb{R}^6) = SO(6)/\langle SO(2) \times SO(4) \rangle$. Since we note before that $\tilde{M}$ has a totally geodesic submanifold isometric to $G_2^+ (\mathbb{R}^4)$, we can easily exclude the case $\tilde{M} = G_1^+(\mathbb{H}^3)$, which is a rank one symmetric space. The case $\tilde{M} = G_2^+ (\mathbb{R}^6)$ is also impossible, because $Spin(5)$ could not act transitively on $\tilde{M}$.

So, $\tilde{M}$ is not a symmetric space, which concludes the proof of the lemma. $\square$

**Remark 4.2.** We remark the work of Podestá [Pod15] on constructing invariant metrics on generalized flag manifold, which applies to the homogeneous manifold $SO(5)/\langle SO(2) \times SO(2) \rangle$. He deals with Kähler and the leaves of symmetry is always an irreducible Hermitian symmetric space. So, our example is different from the ones given by Podestá, since in our case the leaf of symmetry is $G_2^+ (\mathbb{R}^4) \simeq S^2 \times S^2$. In particular, the metric is not Kähler.

4.4. **The case** of $SO(4)$. Let us work, for simplicity, in the universal covering group of $SO(4)$ presented as $SU(2) \times SU(2)$. We present several families of left invariant metrics on with co-index of symmetry 4. First of all, we recall the classification of homogeneous spaces with co-index of symmetry 2, which are all left invariant metrics on $SU(2)$ (see [BOR17] or [Reg18b]). Let us denote by

$$
X_1 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_2 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad X_3 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}
$$

(4.1)
the standard basis of $\mathfrak{su}(2)$. Any left invariant metric on $SU(2)$, up to isometric automorphism, is represented in the basis (4.1) by the symmetric definite positive matrix

$$M(\lambda, \mu, \nu) = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}, \quad \lambda \geq \mu \geq \nu > 0,$$

(4.2)

being the round metric on $SU(2)$ the one with $\lambda = \mu = \nu$. The left invariant metrics on $SU(2)$ with co-index of symmetry 2 are, up to isometry and scaling, the associated with the matrices $M(\lambda, \lambda - 1, 1)$, with $\lambda > 2$; $M(\lambda, 1, 1)$, with $\lambda > 1$; and $M(1, 1, \nu)$, with $0 < \nu < 1$. The last two families parameterise the so-called Berger spheres. Let us denote by $SU(2)_{\lambda, \mu, \nu}$ the group $SU(2)$ endowed with the left invariant metric represented by $M(\lambda, \mu, \nu)$.

From the previous comments, one can easily construct a large number of examples of left invariant metrics on $SU(2) \times SU(2)$ with co-index of symmetry 4. Namely, denote by $(\lambda, \mu, \nu)$ one of the triples $(\lambda, \lambda - 1, 1)$, $(\lambda, 1, 1)$ or $(1, 1, \nu)$ with the restrictions imposed above, and similarly assume that $(\lambda', \mu', \nu')$ takes the form $(\lambda', \lambda' - 1, 1)$, $(\lambda', 1, 1)$ or $(1, 1, \nu')$. So, one can form six 2-parameter families of spaces $SU(2)^{\lambda, \mu, \nu} \times SU(2)^{\lambda', \mu', \nu'}$ with co-index of symmetry 4. Note that these spaces are Riemannian products, but they do not split of a symmetric de Rham factor and so they satisfies the hypothesis of Theorem 2.2.

We present another family of examples, which appears in the classification of naturally reductive spaces of dimension up to 6 given by Agricola, Ferreira and Friedrich [AFF15], and whose index of symmetry is computed by using the results in [ORT14]. We present $SU(2) \times SU(2)$ as the homogeneous manifold $G/H$ where $G = SU(2) \times SU(2) \times SU(2)$ modulo the diagonal subgroup $H = \{(g, g, g) : g \in SU(2)\}$. Denote by $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and $\mathfrak{h} = \{(X, X, X) : X \in \mathfrak{su}(2)\}$ their respective Lie algebras. Put

$$m_1 = \{\tilde{X} = (X, aX, bX) : X \in \mathfrak{su}(2), a, b, \in \mathbb{R}\},$$

$$m_2 = \{\tilde{Y} = (Y, cY, dY) : Y \in \mathfrak{su}(2), c, d, \in \mathbb{R}\}.$$

If we ask $0 \neq (a - 1)(d - 1) - (b - 1)(c - 1) = \det \begin{pmatrix} 1 & 1 & 1 \\ 1 & a & b \\ 1 & c & d \end{pmatrix}$ then $m = m_1 \oplus m_2$ is a reductive complement of $\mathfrak{h}$ and, for each $\lambda > 0$, the inner product on $m$ defined by

$$\langle (\tilde{X}_1, \tilde{Y}_1), (\tilde{X}_2, \tilde{Y}_2) \rangle = -\frac{1}{2} \left( \text{trace}(X_1X_2) + \frac{1}{\lambda^2} \text{trace}(Y_1Y_2) \right)$$

induces a naturally reductive metric on $G/H$. It is easy to see that the set of fixed vectors of the isotropy representation is a 2-dimensional subspace of $m$. So in the generic case (when the metric is not symmetric), it follows from [ORT14] that the co-index of symmetry is equal to 4.

4.5. The case of $SO(3) \times SO(3) \times SO(3)$. We can form metrics with co-index of symmetry 4 in $SO(3) \times SO(3) \times SO(3)$ by taking the product of the bi-invariant (symmetric) metric on the first factor and one of the metrics presented in the above case on the others two factors. So, we have a rank 5 distribution of symmetry in a 9-dimensional homogeneous space. Recall that this example is not exactly in the hypothesis of Theorem 2.2 it has co-index of symmetry 4 though. Sadly, we have not been able to find an irreducible example yet.
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