U(1)-GAUGE THEORIES ON G_2-MANIFOLDS

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ABSTRACT. In this paper, we investigate two types of U(1)-gauge field theories on G_2-manifolds. One is the U(1)-Yang-Mills theory which admits the classical instanton solutions, we show that G_2-manifolds emerge from the anti-self-dual U(1) instantons, which is an analogy of Yang’s result for Calabi-Yau manifolds. The other one is the higher-order U(1)-Chern-Simons theory as a generalization of Kähler-Chern-Simons theory. We introduce the notion of higher-order U(1)-instanton, as the vacuum configurations of higher-order U(1)-Chern-Simons theory. By suitable choice of gauge and regularization technique, we calculate the partition function under semiclassical approximation. Finally, to make sure of the invariance at quantum level under the large gauge transformations, we use Deligne-Beilinson cohomology theory to give the higher-order U(1)-Chern-Simons actions (U(1)-BF-type actions) for nontrivial U(1)-principal bundles.

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1. INTRODUCTION

G_2-manifolds appear in the compactification of M-theory or 11-dimensional supergeravity to achieve effective 4-dimensional theory with N = 1 supersymmetry [1, 2]. Mathematically, there are two equivalent approaches to define G_2-manifolds. The first definition treats a G_2 manifold as a 7-dimensional Riemannian manifold with holonomy group as a subgroup of G_2, and the other one defines a G_2-manifold as a 7-dimensional oriented spin manifold with a torsion free G_2-structure which is a special 3-form ϕ (called fundamental 3-form) parallel with respect to the induced Levi-Civita connection. Many examples of 7-dimensional manifold with holonomy group G_2 have also been constructed [3, 4, 5, 6, 7]. To some extent, G_2-manifolds can be viewed as the analog of Calabi-Yau 3-folds:

| Calabi-Yau 3-fold N | G_2-manifold M |
|---------------------|---------------|
| Kähler form ω       | fundamental 3-form ϕ |
| complex structure I  | fundamental 4-form *ϕ |
| I-holomorphic curve in N | associated 3-dimensional submanifold of M |
| special Lagrangian submanifold of N | coassociated 4-dimensional submanifold of M |

It is noteworthy that the metric on a G_2-manifold is totally determined by the fundamental 3-form ϕ via a highly nontrivial manner, hence the fundamental 4-form *ϕ is not independent of the fundamental 3-form ϕ, where *ϕ denote the Hodge star with respect to the induced metric g_ϕ. Conceptually, we should consider which results for Calabi-Yau manifolds can be generalized to G_2-manifolds.

In a series of papers [8, 9, 10, 11, 12, 13, 14, 15, 16, 17] by H.-S. Yang and his collaborators, the authors proposed a kind of emergent gravity, which is achieved by considering the deformation of a symplectic manifold. In this framework, Darboux theorem or the Moser lemma in symplectic geometry is reinterpreted as equivalence principle. From this point of view, a line bundle over a symplectic manifold leads to a dynamical symplectic manifold described by a gauge theory of symplectic gauge fields. Then the quantization of the dynamical symplectic manifold gives rise to a dynamical noncommutative spacetime described by a noncommutative U(1)-gauge theory. A basic idea of Yang’s emergent gravity is to realize the gauge/gravity duality using the Lie algebra homomorphism between noncommutative ***-algebra (gauge theory side) and derivation algebra (gravity side). Then H.-S. Yang showed that the commutative limit of noncommutative anti-self-dual U(1)-instanton equations turns into the equations for spin connections of an emergent Calabi-Yau manifold [18].
In Sec. 2, we will generalize such mechanism of producing Calabi-Yau manifolds to the \( G_2 \)-manifolds. \( G_2 \)-instanton equations were first introduced in [19], which are also divided into self-dual type and anti-self-dual type according to the irreducible representations of \( G_2 \) on 2-forms. In the spirit of taking commutative limit, the anti-self-dual \( U(1) \)-instanton equations give rise to equations satisfied by a collection of local orthogonal frame fields (siebenbein), we will show that these equations force the spin connection to be valued in the Lie algebra \( g_2 \) of \( G_2 \), hence determine a \( G_2 \)-manifold. Namely, we prove the following theorem.

**Theorem 1.1** (\( \Rightarrow \)Theorem 2.2). A 7-dimensional spin Riemannian manifold \( M \) is a \( G_2 \)-manifold if and only if it is an anti-self-dual \( U(1) \)-instanton.

Chern-Simons theory is another important kind of gauge theory [20, 21, 22, 23]. A natural generalization of usual Chern-Simons theory in three dimensions to a \( G_2 \)-manifold \( M \) is to consider the following action which was first introduced in [24]

\[
S_{CS} = \int_M CS_3(A) \wedge \ast \phi, \tag{1.1}
\]

where

\[
CS_3(A) = \text{Tr}(AdA + \frac{2}{3}A^3) \tag{1.2}
\]

is the Chern-Simons 3-form associated to the connection \( A \) on a trivial \( G \)-principal bundle over \( M \) for a compact complex matrix Lie group \( G \). The critical points of the action (1.1) are exactly the anti-self-dual \( G_2 \)-instantons \(^1\), namely the solutions of the equation

\[
F_A \wedge \ast \phi = 0,
\]

where \( F_A = dA + A^2 \) is the curvature 2-form of \( A \). The constructions of anti-self-dual \( G_2 \)-instantons on some special \( G_2 \)-manifolds have been obtained [25, 26, 27, 28, 29]. If one choose a special \( G_2 \)-manifold \( M = CY_3 \times S^1 \), by dimensional reduction, \( S_{CS} \) can be reexpressed as the sum of \( B \)-model 6-brane and \( \bar{B} \)-model 6-brane actions and an extra term related to the stability of the brane [30]. Similar to the work in [31], one may connect the partition function of \( S_{CS} \) with the cohomology of moduli space of Hermitian-Yang-Mills connection on \( CY_3 \) (this work will appear elsewhere).

In the rest part of this paper, we will consider **higher-order Chern-Simons theory** on \( \mathcal{M} = M \times L \) with \( M \) being a \( G_2 \)-manifold and \( L \) standing for \( \mathbb{R} \) or \( S^1 \). Our starting point is the Kähler-Chern-Simons (KCS) theory proposed by Nair and Schiff, so we first briefly introduce this theory. The action is given by

\[
S_{KCS} = \int_{K \times \mathbb{R}} (CS_3(A) \wedge \omega + \text{Tr}(F_A \wedge \phi + F_A \wedge \phi^*)) , \tag{1.3}
\]

where \( K \) is a Kähler surface with the Kähler form \( \omega \), \( A \) is the connection on the trivial \( G \)-principal bundle over \( K \times \mathbb{R} \), \( \phi \) and \( \phi^* \), respectively, are two Lagrange multipliers that are Lie-algebra valued \((2,0)\)-form and \((0,2)\)-form on \( K \) and also 1-forms on \( \mathbb{R} \). Write \( A = A + B \) with \( A \) being a Lie-algebra valued 1-form on \( \mathbb{R} \) and \( B \) being a Lie-algebra valued 1-form on \( K \), then assuming \( K \) closed, the critical configurations of the action (1.3) are solutions of equations

\[
\begin{align*}
\frac{\partial B}{\partial t} &= 0, \\
F_B^{2,0} &= F_B^{0,2} = 0, \\
F_B \wedge \omega &= 0,
\end{align*}
\]

where \( t \) is the coordinate on \( \mathbb{R} \), \( F_B^{2,0} \) and \( F_B^{0,2} \) stand for the \((2,0)\)-part and \((0,2)\)-part of the curvature 2-form \( F_B \) on \( K \) of \( B \), respectively. Note that these equations are equivalent to the Hermitian-Yang-Mills equation on \( K \), in particular, if \( G \) is semisimple, they are also equivalent to the anti-self-dual Yang-Mills instanton equation on \( K \) (see Proposition 1.2.2 in [32]). The quantization moduli space of critical configurations has been investigated in [33, 34, 35].

Now we discuss the \( G_2 \)-analogue of KCS theory. Firstly, the Kähler form \( \omega \) in the action (1.3) is replaced by the fundamental 3-form \( \varphi \), then taking into account the dimensions, \( CS_3(A) \) should be replaced by Chern-Simons 5-form \( CS_5(A) \). More precisely, we consider the so-called higher-order Chern-Simons action,

\[
S_{HCS} = \int_{\mathcal{M}} (\text{Tr}(AdA + \frac{3}{2}A^3dA + \frac{3}{5}A^5)) \wedge \varphi, \tag{1.4}
\]

\(^1\)Some authors call those \( G_2 \)-instantons.
where $A$ is the connection on the trivial $G$-principal bundle over $\mathcal{M}$. Decomposing $A = A + B$ for $A = A_0 dt$, $B = A_\mu dX^\mu$ with local coordinates $t$ on $L$ and $\{X^\mu, i = 1, \ldots, 7\}$ on $M$, the action (1.4) reduces to

$$S_{\text{HCS}} = - \int_L dt \int_M \text{Tr}(B dM B + dM B B + \frac{3}{2} B^3) \wedge \varphi + 3 \int_L dt \int_M \text{Tr}(A_0 F_B^2) \wedge \varphi,$$

(1.5)

where $d_M$ stands for the exterior differential operator on $M$, $\dot{B} = \frac{\partial A_\mu}{\partial t} dX^\mu$, $F_B = dM B + B^2$. By Chern-Weil theory, the integral $\int_M \text{Tr}(F_B^2) \wedge \varphi$ is a topological invariant, hence the action $S_{\text{HCS}}$ has a large gauge symmetry at the quantum level $A_0 \rightarrow A_0 + f(t)$ with $\int_L f(t) dt \in \mathbb{Z}$. Usually, the invariance at quantum level under large gauge transformations requires $\varphi$ should be of integral period, i.e. $\varphi \in H^3(M, \mathbb{Z}) \hookrightarrow H^3(M, \mathbb{Z})$ [36]. In particular, if one picks $G = U(1)$, the action (1.4) is more simple as

$$S_{\text{HCS}} = \int_M A \wedge dA \wedge dA \wedge \varphi,$$

which can be generalized to the higher-order $U(1)$-BF-type action

$$S_{\text{ABC}} = \int_M A \wedge dB \wedge dC \wedge \varphi$$

(1.6)

for one forms $A, B, C$ on $\mathcal{M}$. It would be more appropriate to be called the $U(1)$-ABC action.

In Sec. 3, we introduce the notion of higher-order $G$-instanton. It is described by the equation

$$F_B^2 \wedge \varphi = 0,$$

which is obtained by the variation of the action (1.5) with respect to $A_0$. According to the decomposition of $F_B^2$ into the irreducible representations of $G_2$, we can define self-dual higher-order $G$-instantons and anti-self-dual higher-order $G$-instantons:

$$\begin{cases} F_B^2 = \alpha \wedge \varphi & \text{for some } \alpha \in \Lambda^1(M, \text{Ad}_P), \quad \text{self-dual higher-order } G \text{-instanton;} \\ F_B^2 \wedge \varphi = *_F F_B^2 \wedge \varphi = 0, & \text{anti-self-dual higher-order } G \text{-instanton.} \end{cases}$$

The main results of this section are summarized in the following vanishing-type theorem.

**Theorem 1.2 (Corollary 3.3, Theorem 3.5, Theorem 3.8).** Assume $B$ is a higher-order $U(1)$-instanton, then

1. $B$ is a self-dual instanton $\iff$ $B$ is an anti-self-dual instanton $\iff$ $B$ is a flat connection $\iff$ $F_B \wedge \varphi = 0$;
2. $B$ is a self-dual higher-order $U(1)$-instanton $\iff$ $F_B^2 = 0$;
3. when $B$ is a special higher-order $U(1)$-instanton, $B$ is a flat connection $\iff$ $B \wedge *_F F_B = 0$.

In Sec. 4, we calculate the semiclassical partition function around higher-order $U(1)$-instantons for higher-order $U(1)$-Chern-Simons action over $M \times S^1$. Some crucial ingredients include:

- torus gauge fixing is imposed [37, 38];
- heat kernel regularization technique is used [37, 35];
- Schwarz-Pestun-Witten’s method for calculating partition function of degenerate functional plays an important role [39, 40, 41, 36].

In Sec. 5, we discuss how to write a correct (higher-order) Chern-Simons action which is invariant at quantum level under the large gauge transformations when $G$-principal bundle $P$ is nontrivial. Actually, the story becomes more subtle and complicated. For example, a suitable framework to cope with the nontrivial $U(1)$-principal bundle is the Deligne-Beilinson cohomology theory [42, 43, 44]. We will use the Čech resolution of Deligne complex to give an explicit expression of the higher-order $U(1)$-Chern-Simons action (and more generally, higher-order $U(1)$-BF-type action) for a nontrivial $U(1)$-principal bundle.

2. **$G_2$-manifolds as anti-self-dual $U(1)$-instantons**

There is a standard $G_2$-structure on $\mathbb{R}^7$ consists of a standard Euclidean metric $g_0$ and a 3-form $\varphi_0$ given by

$$\varphi_0 = e^{123} + e^{145} + e^{246} + e^{347} - e^{167} + e^{257} - e^{356},$$

(2.1)

where $\{e_1, \cdots, e_7\}$ is an orthonormal frame that provides a coordinate $\{x^1, \cdots, x^7\}$ on $\mathbb{R}^7$, and $e^{ijk} = e^i \wedge e^j \wedge e^k$ for $e^i$ being the dual basis of $e_i$. The $U(1)$-Yang-Mills functional on $\mathbb{R}^7$ is given by

$$S = \int_{\mathbb{R}^7} d^7 x F_{ij} F_{ij},$$
where \( F_{ij} = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i} \) for the dual vector field \( A = A_i(x)e^i \) on \( \mathbb{R}^7 \). Identifying \( e^i \) with \( dx^i \), the critical point reads

\[
d *_{g_0} F = 0,
\]

where \( F = dA = \frac{1}{2} F_{ij} dx^i \wedge dx^j \), \(*_{g_0}\) denotes the Hodge dual with respect to \( g_0 \). Obviously, if

\[
*_{g_0} F = c F \wedge \varphi_0
\]

for some nonzero constant \( c \), then the above equation holds true automatically. There is a decomposition

\[
F = F_{(1)} + F_{(2)},
\]

where \( F_{(1)}, F_{(2)} \) satisfy

\[
*_{g_0}(F_{(1)} \wedge \varphi_0) = -2F_{(1)},
\]

\[
*_{g_0}(F_{(2)} \wedge \varphi_0) = F_{(2)},
\]

respectively. Therefore, only when \( c = \frac{1}{2} \) or \( c = 1 \) the equation (2.2) admits nontrivial solutions, they are called self-dual \( U(1) \)-instanton or anti-self-dual \( U(1) \)-instanton, respectively, i.e.

\[
\begin{cases}
  *_{g_0} F = -\frac{1}{2} F \wedge \varphi_0, & \text{self-dual } U(1)\text{-instanton}; \\
  *_{g_0} F = F \wedge \varphi_0, & \text{anti-self-dual } U(1)\text{-instanton},
\end{cases}
\]

or more explicitly

\[
\begin{cases}
  F_{ij} = -\frac{1}{2} T_{ij}^{kl} F_{kl}, & \text{self-dual } U(1)\text{-instanton}; \\
  F_{ij} = \frac{1}{2} T_{ij}^{kl} F_{kl}, & \text{anti-self-dual } U(1)\text{-instanton},
\end{cases}
\]

where

\[
T_{ij}^{kl} = \frac{1}{6} \varepsilon_{ijklpqrs} (\varphi_0)^{pqrst}.
\]

We will only focus on anti-self-dual \( U(1) \)-instanton.

Consider a 7-dimensional spin manifold \( M \) equipped with a Riemann metric \( g \). Let \( \{ E_i = E_i^\mu \frac{\partial}{\partial x^\mu}, i = 1, \ldots, 7 \} \) be pointwisely linearly independent local orthogonal frame fields, i.e. so-called "siebenbein", over some neighborhood with local coordinate \( \{ X^\mu, \mu = 1, \ldots, 7 \} \) in \( M \), namely we have

\[
E_i^\mu E_j^\nu \delta^{ij} = g_{\mu \nu},
\]

and let \( \{ E^i = E_i^\mu dx^\mu, i = 1, \ldots, 7 \} \) be the corresponding dual 1-forms which satisfy

\[
E_i^\mu E_j^\nu g^{\mu \nu} = \delta^{ij}.
\]

The spin connection 1-form \( \{ \omega^i_j \} \) on \( M \) is defined by

\[
dE^i + \omega^i_j \wedge E^j = 0,
\]

\[
\omega^i_j + \omega^j_i = 0.
\]

As done in [18], one does the following replacement (so-called commutative limit)

\[
D_i := \frac{\partial}{\partial x_i} + A_i \rightarrow E_i,
\]

then the anti-self-dual \( U(1) \)-instanton equation becomes

\[
[E_i, E_j] + \frac{1}{6} T_{ij}^{kl} [E_k, E_l],
\]

or equivalently,

\[
f_{ij}^k = \frac{1}{6} T_{ij}^{kl} f_{kl}^s,
\]

where \( f_{ij}^k \) is defined by

\[
[E_i, E_j] = f_{ij}^k E_k.
\]

Writing \( \omega^i_j = \omega^i_{kj} E^k \), the equation (2.6) together with (2.10) implies

\[
f_{ij}^k = \omega_{ij}^k - \omega_{ji}^k = : \Omega_{ij}^k,
\]
then the equation (2.9) reduces to

\[ \Omega_{ij}^s = \frac{1}{6} T_{ij}^{kl} \Omega_{kl}^s, \]  
(2.11)

which is explicitly expressed in terms of components as

\[ \Omega_{12}^s = \Omega_{56}^s - \Omega_{47}^s, \]  
(2.12)
\[ \Omega_{13}^s = \Omega_{57}^s + \Omega_{46}^s, \]  
(2.13)
\[ \Omega_{14}^s = -\Omega_{36}^s + \Omega_{27}^s, \]  
(2.14)
\[ \Omega_{15}^s = -\Omega_{37}^s - \Omega_{26}^s, \]  
(2.15)
\[ \Omega_{16}^s = \Omega_{25}^s + \Omega_{34}^s, \]  
(2.16)
\[ \Omega_{17}^s = \Omega_{35}^s - \Omega_{24}^s, \]  
(2.17)
\[ \Omega_{23}^s = \Omega_{67}^s - \Omega_{45}^s. \]  
(2.18)

**Definition 2.1.** Let \( M \) be a 7-dimensional oriented spin manifold equipped with a Riemann metric \( g \). If there exists an open cover \( \{ U_\alpha \} \) of \( M \), and there is a siebenbein \( \{ E_i^{(\alpha)} \}, i = 1, \cdots, 7 \) (compatible with the orientation) over each \( U_\alpha \) such that

\[ [E_i^{(\alpha)}, E_j^{(\alpha)}] = \frac{1}{6} T_{ij}^{kl} [E_k^{(\alpha)}, E_l^{(\alpha)}], \]  
(2.19)

we call \( M \) an anti-self-dual \( U(1) \)-instanton.

The covariant derivative on a spinor \( \eta \) over an open neighborhood \( U \) along the vector field \( E_i \) is defined by

\[ \nabla_{E_i} \eta = E_i(\eta) + \frac{1}{2} \Omega_{jk}^i \Sigma_{jk} \cdot \eta, \]

where \( \cdot \) denotes Clifford multiplication, and \( \{ \Sigma_{ij} = -\Sigma_{ji}, i < j = 1, \cdots, 7 \} \), satisfying the relations

\[ [\Sigma_{ij}, \Sigma_{kl}] = \Sigma_{il} \delta_{jk} + \Sigma_{jk} \delta_{il} - \Sigma_{ik} \delta_{jl} - \Sigma_{jl} \delta_{ik}, \]  
(2.20)

forms a basis of Lie algebra \( \text{spin}(7) \). It follows from the equations (2.12)-(2.18) that

\[ \nabla_{E_i} \eta = E_i(\eta) + \Omega_{56}^i (\Sigma_{56} + \Sigma_{12}) \cdot \eta + \Omega_{47}^i (\Sigma_{47} - \Sigma_{12}) \cdot \eta \]
\[ \quad + \Omega_{57}^i (\Sigma_{57} + \Sigma_{13}) \cdot \eta + \Omega_{46}^i (\Sigma_{46} + \Sigma_{13}) \cdot \eta \]
\[ \quad + \Omega_{36}^i (\Sigma_{36} - \Sigma_{14}) \cdot \eta + \Omega_{27}^i (\Sigma_{27} + \Sigma_{14}) \cdot \eta \]
\[ \quad + \Omega_{37}^i (\Sigma_{37} - \Sigma_{15}) \cdot \eta + \Omega_{26}^i (\Sigma_{26} - \Sigma_{15}) \cdot \eta \]
\[ \quad + \Omega_{25}^i (\Sigma_{25} + \Sigma_{16}) \cdot \eta + \Omega_{34}^i (\Sigma_{34} + \Sigma_{16}) \cdot \eta \]
\[ \quad + \Omega_{35}^i (\Sigma_{35} + \Sigma_{17}) \cdot \eta + \Omega_{24}^i (\Sigma_{24} - \Sigma_{17}) \cdot \eta \]
\[ \quad + \Omega_{67}^i (\Sigma_{67} + \Sigma_{23}) \cdot \eta + \Omega_{45}^i (\Sigma_{45} - \Sigma_{23}) \cdot \eta. \]  
(2.21)

We introduce

\[ V_1 = \Sigma_{56} + \Sigma_{12}, \quad W_1 = \Sigma_{47} - \Sigma_{12}, \]
\[ V_2 = \Sigma_{57} + \Sigma_{13}, \quad W_2 = \Sigma_{46} + \Sigma_{13}, \]
\[ V_3 = \Sigma_{36} - \Sigma_{14}, \quad W_3 = \Sigma_{27} + \Sigma_{14}, \]
\[ V_4 = \Sigma_{37} - \Sigma_{15}, \quad W_4 = \Sigma_{26} - \Sigma_{15}, \]
\[ V_5 = \Sigma_{25} + \Sigma_{16}, \quad W_5 = \Sigma_{34} + \Sigma_{16}, \]
\[ V_6 = \Sigma_{35} + \Sigma_{17}, \quad W_6 = \Sigma_{24} - \Sigma_{17}, \]
\[ V_7 = \Sigma_{67} + \Sigma_{23}, \quad W_7 = \Sigma_{45} - \Sigma_{23}. \]
By the relation (2.20) one easily checks the following nontrivial commutators [45]

\[
\begin{align*}
[V_1, V_2] &= -V_7, & [V_1, V_3] &= V_6 + W_6, \\
[V_1, V_4] &= V_5, & [V_1, V_5] &= -2W_4 \\
[V_1, V_6] &= -V_3 - W_3, & [V_1, V_7] &= V_2, \\
[V_1, W_2] &= W_7, & [V_1, W_3] &= -W_6, \\
[V_1, W_4] &= 2V_5, & [V_1, W_5] &= -W_4, \\
[V_1, W_6] &= W_3, & [V_1, W_7] &= -W_2, \\
[V_2, V_3] &= W_5, & [V_2, V_4] &= 2V_6, \\
[V_2, V_5] &= -V_3 - W_3, & [V_2, V_6] &= -2V_4, \\
[V_2, V_7] &= -V_1, & [V_2, W_1] &= W_7, \\
[V_2, W_3] &= V_5 - W_5, & [V_2, W_4] &= V_6, \\
[V_2, W_5] &= -V_3, & [V_2, W_7] &= -W_1, \\
[V_3, V_4] &= -V_7 - W_7, & [V_3, V_6] &= W_2, \\
[V_3, V_6] &= V_1 + W_1, & [V_3, V_7] &= V_4 - W_4, \\
[V_3, W_1] &= W_6, & [V_3, W_2] &= -2W_5, \\
[V_3, W_4] &= -W_7, & [V_3, W_5] &= 2W_2, \\
[V_3, W_6] &= -W_1, & [V_3, W_7] &= W_4, \\
[V_4, V_5] &= V_1, & [V_4, V_6] &= 2V_2, \\
[V_4, V_7] &= -V_3 - W_3, & [V_4, W_1] &= V_5 - W_5, \\
[V_4, W_2] &= -V_6, & [V_4, W_3] &= -W_7, \\
[V_4, W_5] &= V_1 + W_1, & [V_4, W_6] &= -V_2, \\
[V_4, W_7] &= W_3, & [V_5, V_6] &= -V_7, \\
[V_5, V_7] &= V_6, & [V_5, W_1] &= -W_4, \\
[V_5, W_2] &= V_3, & [V_5, W_3] &= -V_2 + W_2, \\
[V_5, W_4] &= -2V_1, & [V_5, W_6] &= V_7 + W_7, \\
[V_5, W_7] &= -V_6 - W_6, & [V_6, V_7] &= -V_5, \\
[V_6, W_1] &= -W_3, & [V_6, W_2] &= V_4, \\
[V_6, W_3] &= W_1, & [V_6, W_4] &= -V_2, \\
[V_6, W_5] &= V_7 + W_7, & [V_6, W_6] &= V_5 - W_5, \\
[V_7, W_1] &= W_2, & [V_7, W_2] &= -W_1, \\
[V_7, W_3] &= -V_4 + W_4, & [V_7, W_4] &= -V_3 - W_3, \\
[V_7, W_5] &= W_6, & [V_7, W_6] &= -W_5, \\
[W_1, W_2] &= V_7, & [W_1, W_3] &= 2W_6, \\
[W_1, W_4] &= -V_5, & [W_1, W_5] &= -V_4 + W_4, \\
[W_1, W_6] &= -2W_3, & [W_1, W_7] &= V_2, \\
[W_2, W_3] &= -W_5, & [W_2, W_4] &= V_6 + W_6, \\
[W_2, W_5] &= -2V_3, & [W_2, W_6] &= V_4 - W_4, \\
[W_2, W_7] &= V_1, & [W_3, W_4] &= V_7 + W_7, \\
[W_3, W_5] &= -W_2, & [W_3, W_6] &= 2W_1, \\
[W_3, W_7] &= -V_4, & [W_4, W_5] &= V_1, \\
[W_4, W_6] &= -V_2 + W_2, & [W_4, W_7] &= -V_3, \\
[W_5, W_6] &= V_7, & [W_5, W_7] &= V_6 + W_6.
\end{align*}
\]
then we immediately find that \( \{ V_1, \cdots, V_7, W_1, \cdots, W_7 \} \) generate a 14-dimensional Lie subalgebra \( g \) of spin(7).

To show this subalgebra \( g \) is exactly \( g_2 \), we only need recover to \( \varphi_0 \) from the invariant spinor. Dirac gamma matrices satisfying Clifford algebra in 7-dimensional Euclidean space are given by

\[
\begin{align*}
\{ \Gamma_i &= \Gamma^{(6)}_i, \quad i = 1, \cdots, 6; \\
\Gamma_7 &= \sqrt{-1} \Gamma^{(6)}_1 \cdots \Gamma^{(6)}_6, \}
\end{align*}
\]

where \( \Gamma^{(6)}_i, i = 1, \cdots, 6 \), denote the Dirac gamma matrices in six dimensions. Choosing purely imaginary \( 8 \times 8 \) matrices, one explicitly writes Dirac gamma matrices as follows

\[
\begin{align*}
\Gamma_1 &= \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{-1} \\
0 & 0 & \sqrt{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & -\sqrt{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{-1} & 0 \\
0 & 0 & 0 & 0 & 0 & -\sqrt{-1} & 0 & 0 \\
0 & 0 & 0 & -\sqrt{-1} & 0 & 0 & 0 & 0 \\
-\sqrt{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \\
\Gamma_2 &= \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\sqrt{-1} & 0 & 0 & 0 & 0 & \sqrt{-1} \\
\sqrt{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sqrt{-1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{-1} & 0 \\
0 & 0 & 0 & -\sqrt{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\sqrt{-1} & 0 & 0 & 0 \\
0 & -\sqrt{-1} & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \\
\Gamma_3 &= \begin{pmatrix}
0 & \sqrt{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\sqrt{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sqrt{-1} & 0 & 0 \\
0 & 0 & 0 & -\sqrt{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{-1} & 0 & 0 & 0 & \sqrt{-1} & 0 \\
0 & 0 & 0 & 0 & -\sqrt{-1} & 0 & 0 & 0 \\
0 & 0 & -\sqrt{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\sqrt{-1} & 0 & 0 & 0 & 0
\end{pmatrix}, \\
\Gamma_4 &= \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{-1} & 0 \\
0 & 0 & 0 & 0 & -\sqrt{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sqrt{-1} & 0 & 0 \\
0 & 0 & 0 & -\sqrt{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sqrt{-1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\sqrt{-1} & 0 & 0 \\
\sqrt{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{-1} & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\end{align*}
\]
\[
\Gamma_5 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \sqrt{-1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{-1} & 0 \\
0 & 0 & 0 & -\sqrt{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{-1} \\
-\sqrt{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\sqrt{-1} & 0 & 0 & 0
\end{pmatrix},
\]

\[
\Gamma_6 = \begin{pmatrix}
0 & 0 & 0 & 0 & -\sqrt{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\sqrt{-1} & 0 & 0 \\
0 & -\sqrt{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
\sqrt{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{-1} \\
0 & 0 & \sqrt{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\sqrt{-1} & 0 & 0
\end{pmatrix},
\]

\[
\Gamma_7 = \begin{pmatrix}
0 & 0 & 0 & \sqrt{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sqrt{-1} & 0 & 0 \\
-\sqrt{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\sqrt{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\sqrt{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{-1} \\
0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{-1} & 0
\end{pmatrix},
\]

Then \( \Sigma_{ij} \) can be realized as

\[
\Sigma_{ij} = \frac{1}{4} |\Gamma_i, \Gamma_j|,
\]

which provides explicit expressions

\[
V_1 = \frac{1}{2} \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad W_1 = \frac{1}{2} \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
V_2 = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad W_2 = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]
Let $G$ be a connected subgroup of $\text{Spin}(7)$ with Lie algebra $\mathfrak{g}$, then we have the unique $G$-invariant spinor $\eta_0$ up to a constant scalar determined by

$$V_i \cdot \eta_0 = W_i \cdot \eta_0 = 0$$

(2.23)
for any \( i = 1, \ldots, 7 \). Imposing the normalization condition, we write

\[
\eta_0 = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 \\
1 \\
0 \\
0 \\
0 \\
0 \\
-1 \\
\end{pmatrix}.
\]

(2.24)

Define

\[
\psi = \psi_{ijk} e^{ijk} = \sqrt{-\Gamma_{ij}^k} \eta_0 e^{ijk},
\]

(2.25)

where

\[
\Gamma_{ij}^k = \frac{1}{3!} \sum_{\sigma} (-1)^{i|\sigma|} \Gamma_{\sigma(i)} \Gamma_{\sigma(j)} \Gamma_{\sigma(k)}
\]

with \( \sigma \) standing for a permutation. It is clear that \( \psi \) is \( G \)-invariant. A direct calculation shows that the nonzero coefficients \( \psi_{ijk} = \sqrt{-\Gamma_{ij}^k} \Gamma_{ij}^k \) \((i < j < k)\) are given by

\[
\begin{align*}
\psi_{123} &= \psi_{246} = \psi_{167} = \psi_{257} = \psi_{356} = 1, \\
\psi_{134} &= \psi_{145} = -1.
\end{align*}
\]

If one reassigns the frame \( \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\} \) to \( \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\} \), we find that \( \psi \) exactly coincides with \( 6i\varphi_0 \).

If \( M \) is an anti-self-dual \( U(1) \)-instanton, from the above arguments it follows that the condition (2.19) guarantees the spin connection 1-form is valued in \( g_2 \), which implies that the holonomy group of the metric \( g \) lies in \( G_2 \) due to Ambrose-Singer theorem. Conversely, if \( M \) is a \( G_2 \)-manifold, it can be made into an anti-self-dual \( U(1) \)-instanton. As a consequence, we have the following theorem.

**Theorem 2.2.** \( M \) is a \( G_2 \)-manifold if and only if it is an anti-self-dual \( U(1) \)-instanton.

### 3. Higher-order \( U(1) \)-instantons

Varying \( A_0 \) in the action (1.5) leads to the equation

\[
F_B \wedge \varphi = 0.
\]

(3.1)

Here \( F_B \in \Lambda^2(M, \text{Ad}_P) \) for a \( G \)-principal bundle \( P \) (not necessarily trivial) over \( M \) with the adjoint vector bundle \( \text{Ad}_P \). It is known that the spaces \( \Lambda^2(M), \Lambda^3(M) \) and \( \Lambda^4(M) \) of 2-forms, 3-forms and 4-forms on \( M \) have orthogonal decompositions with respect to the metric \( g_\varphi \), respectively,

\[
\begin{align*}
\Lambda^2(M) &= \Lambda^2_{(1)}(M) \oplus \Lambda^2_{(2)}(M), \\
\Lambda^3(M) &= \Lambda^3_{(1)}(M) \oplus \Lambda^3_{(2)}(M) \oplus \Lambda^3_{(3)}(M), \\
\Lambda^4(M) &= \Lambda^4_{(1)}(M) \oplus \Lambda^4_{(2)}(M) \oplus \Lambda^4_{(3)}(M),
\end{align*}
\]

where

\[
\begin{align*}
\Lambda^2_{(1)}(M) &= \{ \beta \in \Lambda^2 : \star_\varphi (\varphi \wedge \beta) = -2\beta \}, \\
\Lambda^2_{(2)}(M) &= \{ \beta \in \Lambda^2 : \star_\varphi (\varphi \wedge \beta) = \beta \}, \\
\Lambda^3_{(1)}(M) &= \{ f \varphi : f \in C^\infty(M) \}, \\
\Lambda^3_{(2)}(M) &= \{ X \star_\varphi \varphi : X \in TM \}, \\
\Lambda^3_{(3)}(M) &= \{ \beta \in \Lambda^3(M) : \varphi \wedge \beta = \star_\varphi \varphi \wedge \beta = 0 \}, \\
\Lambda^4_{(1)}(M) &= \{ f \star_\varphi \varphi : f \in C^\infty(M) \}, \\
\Lambda^4_{(2)}(M) &= \{ \alpha \wedge \varphi : \alpha \in \Lambda^1(M) \}, \\
\Lambda^4_{(3)}(M) &= \{ \eta \in \Lambda^4(M) : \varphi \wedge \beta = \varphi \wedge \varphi = 0 \}.
\end{align*}
\]

Then we introduce the following definition.
Definition 3.1. The connection $B$ on $P$
- is called a higher-order $G$-instanton if $F_B^2 \in \Lambda^2_2(M, \text{Ad}_P) \oplus \Lambda^4_3(M, \text{Ad}_P)$,
- is called a higher-order flat $G$-instanton if $F_B^2 = 0$,
- is called a self-dual higher-order $G$-instanton if $F_B^2 \in \Lambda^1_2(M, \text{Ad}_P)$,
- is called an anti-self-dual higher-order $G$-instanton if $F_B^2 \in \Lambda^4_3(M, \text{Ad}_P)$.

Theorem 3.2. The following two conditions are equivalent
\begin{enumerate}
\item[(1)] $B$ is a higher-order $G$-instanton,
\item[(2)] $2|F_B|_{g_\varphi} = |F_B \wedge \varphi|_{g_\varphi},$
\end{enumerate}
where $|\bullet|_{g_\varphi} = \bullet \wedge \star \bullet$.

Proof. (1) $\Rightarrow$ (2): Let $(F_B)_{(i)}$ be the projection of $F_B$ on $\Lambda^2_2(M, \text{Ad}_P)$. For a higher-order $G$-instanton $B$, writing $F_B = (F_B)_{(1)} + (F_B)_{(2)}$, we obtain
\[
F_B^2 \wedge \varphi = ((F_B)_{(1)} + (F_B)_{(2)}) \wedge ((F_B)_{(1)} + (F_B)_{(2)}) \wedge \varphi = -2(F_B)_{(1)} \wedge \star \varphi (F_B)_{(1)} + (F_B)_{(2)} \wedge \star \varphi (F_B)_{(2)} = 0,
\]
namely,
\[
2|(F_B)_{(1)}|_{g_\varphi} = |(F_B)_{(2)}|_{g_\varphi},
\]
which is equivalent to the identity in (2). Indeed, one has
\[
|F_B \wedge \varphi|_{g_\varphi} = |((F_B)_{(1)} + (F_B)_{(2)}) \wedge \varphi|_{g_\varphi} = | -2(F_B)_{(1)} + (F_B)_{(2)}|_{g_\varphi} = 4|(F_B)_{(1)}|_{g_\varphi} + |(F_B)_{(2)}|_{g_\varphi},
\]
which immediately yields
\[
|F_B \wedge \varphi|_{g_\varphi} = 2(|(F_B)_{(1)}|_{g_\varphi} + |(F_B)_{(2)}|_{g_\varphi}) = 2|F_B|_{g_\varphi}.
\]

(2) $\Rightarrow$ (1): It is known that
\[
(F_B)_{(1)} = \frac{1}{3} F_B - \frac{1}{3} \star \varphi (\varphi \wedge F_B),
\]
\[
(F_B)_{(2)} = \frac{2}{3} F_B + \frac{1}{3} \star \varphi (\varphi \wedge F_B).
\]
By means of (3.2), we have
\[
2|F_B|_{g_\varphi} + 2|F_B \wedge \varphi|_{g_\varphi} - 4F_B^2 \wedge \varphi = 4|F_B|_{g_\varphi} + |F_B \wedge \varphi|_{g_\varphi} + 4F_B^2 \wedge \varphi,
\]
thus
\[
3|F_B \wedge \varphi|_{g_\varphi} - 4F_B^2 \wedge \varphi = 3|F_B \wedge \varphi|_{g_\varphi} + 4F_B^2 \wedge \varphi.
\]
Therefore, $F_B^2 \wedge \varphi = 0$, i.e. $B$ is a higher-order $G$-instanton. \hfill \Box

Corollary 3.3. Assume $G$ is a compact semisimple complex Lie group or $G = U(1)$. If $B$ is a higher-order $G$-instanton, then the following conditions are equivalent
\begin{enumerate}
\item[(1)] $B$ is a self-dual $G$-instanton,
\item[(2)] $B$ is an ant-self-dual $G$-instanton,
\item[(3)] $B$ is a flat connection,
\item[(4)] $F_B \wedge \varphi = 0$.
\end{enumerate}

Consider the formal space $\mathbb{M}_P$ consisting of higher-order $G$-instantons on $P$. If $\alpha \in \Lambda^1(M, \text{Ad}_P)$ satisfies the equation
\[
D_B \alpha \wedge F_B \wedge \varphi + F_B \wedge D_B \alpha \wedge \varphi = 0,
\]
(3.3)
where \( D_B = d_M + [B \wedge \bullet] \), then \( a \) can be treated as a tangent vector at \( B \in \mathbb{M}_P \). One defines a formal 2-form \( \Omega \) on \( \mathbb{M}_P \) by

\[
\Omega|_{B}(a_1, a_2) = \int_M \text{Tr}(F_B \wedge (a_1 \wedge a_2 - a_2 \wedge a_1)) \wedge \varphi
\]

for \( a_1, a_2 \in T_B \mathbb{M}_P \), and defines a formal vector field \( V \) on \( \mathbb{M}_P \) as

\[
V|_{B} = D_B \theta
\]

for a Lie-algebra valued function \( \theta \) on \( M \).

**Proposition 3.4.** Assume \( M \) is closed.

1. \( \Omega \) can be viewed as a formal pre-symplectic form on \( \mathbb{M}_P \).
2. \( V \cdot \Omega = 0 \).

**Proof.** (1) Let \( \mathfrak{d} \) denote the exterior differential operator on \( \mathbb{M}_P \). Then for any \( a_i \in T_B \mathbb{M}_P, i = 1, 2, 3 \), we calculate at \( B \in \mathbb{M}_P \)

\[
(\mathfrak{d} \Omega)|_{B}(a_1, a_2, a_3) = \int_M \text{Tr}(d_M a_1 \wedge a_2 \wedge a_3) \wedge \varphi - \int_M \text{Tr}(d_M a_2 \wedge a_1 \wedge a_3) \wedge \varphi + \int_M \text{Tr}(d_M a_3 \wedge a_1 \wedge a_2) \wedge \varphi
\]

\[
- \int_M \text{Tr}(d_M a_1 \wedge a_3 \wedge a_2) \wedge \varphi + \int_M \text{Tr}(d_M a_2 \wedge a_3 \wedge a_1) \wedge \varphi - \int_M \text{Tr}(d_M a_3 \wedge a_2 \wedge a_1) \wedge \varphi + \int_M \text{Tr}(B \wedge a_1 \wedge a_2 \wedge a_3) \wedge \varphi
\]

\[
- \int_M \text{Tr}(a_1 \wedge B \wedge a_2 \wedge a_3) \wedge \varphi - \int_M \text{Tr}(a_2 \wedge B \wedge a_1 \wedge a_3) \wedge \varphi - \int_M \text{Tr}(a_3 \wedge B \wedge a_2 \wedge a_1) \wedge \varphi
\]

\[
= \int_M \text{Tr}(d_M ((a_1 \wedge a_2 - a_2 \wedge a_1) \wedge a_3)) \wedge \varphi = 0,
\]

which means that \( \mathfrak{d} \Omega = 0 \), i.e. \( \Omega \) is a formal pre-symplectic form on \( \mathbb{M}_P \).

(2) For \( a \in T_B \mathbb{M}_P \), we have

\[
(V \cdot \Omega)|_{B}(a) = \Omega|_{B}(D_B \theta, a)
\]

\[
= \int_M \text{Tr}(F_B \wedge (D_B \theta \wedge a - a \wedge D_B \theta)) \wedge \varphi
\]

\[
= \int_M \text{Tr}(d_M (F_B \wedge (\theta a + a \theta))) \wedge \varphi - \int_M \text{Tr}(F_B \wedge (\theta D_B a + D_B a \theta)) \wedge \varphi = 0,
\]

where the last equality is the result of \( a \) satisfying (3.3). Therefore, \( V \cdot \Omega = 0 \). \( \square \)

From now on, we always assume \( G = U(1) \).

**Theorem 3.5.** If the connection \( B \) is a self-dual higher-order \( U(1) \)-instanton, then \( B \) is a higher-order flat \( U(1) \)-instanton.

**Proof.** We need to show \( F_B^2 = 0 \). Assume \( 0 \neq F_B^2 \in \Lambda^2_\mathbb{C}(M) \), then by definition there exists nonzero \( \alpha \in \Lambda^1(M) \) such that

\[
F_B^2 = \alpha \wedge \varphi.
\]

Pick a point \( p \in M \) such that \( F_B|_p \neq 0 \), and let \( U \) be an open neighbourhood of \( p \) with a local coordinate system such that \( \varphi|_p = \varphi_0 \). Since \( G_2 \) acts transitively on \( S^5 \), by a suitable \( G_2 \)-action, we can put \( \alpha|_p = c e^1 \) for some constant \( c \neq 0 \). Without loss of generality, we assume \( c = 1 \). Then we have

\[
(F_B|_p)^2 = -e^{1246} - e^{1347} + e^{1257} - e^{1356}.
\]

We show that the above equality cannot occur by the following steps.
Step 1: Write \( F_B|_p = e^1 \wedge a + b \) with \( a = \sum_{i \geq 2} a_i e^i \) and \( b = \sum_{i < j, i \geq 2} b_{ij} e^{ij} \), then
\[
e^1_\wedge (F_B|_p)^2 = a \wedge b = -e^{246} - e^{347} + e^{357} - e^{356}.
\] (3.6)

Step 2: We claim all \( a_i \)'s do not vanish. Assume \( a_2 = 0 \), and write \( b = e^2 \wedge d + s \) with \( d = \sum_{i \geq 3} d_i e^i \), \( s = \sum_{i < j, i \geq 3} s_{ij} e^{ij} \), then
\[e^2 \wedge (a \wedge b) = -a \wedge d = -e^{46} + e^{57}.
\]
It follows from that
- \( a_i \neq 0, d_i \neq 0 \) for \( i = 4, 5, 6, 7 \),
- \( a_4 d_5 = a_5 d_4, a_4 d_7 = a_7 d_4, a_5 d_7 - a_7 d_5 = -1 \).

However, one easily checks that the above two conditions are not compatible. This means that \( a_2 \) dose not vanish. Similarly, we can find that \( a_i \neq 0 \) for \( i = 3, \cdots, 7 \).

Step 3: It follows from the equation (3.6) that
\[
\begin{align*}
b_{45} & \quad - b_{35} + b_{34} = 0, \\
b_{46} & \quad - b_{36} + b_{34} = 0, \\
b_{56} & \quad - b_{46} + b_{45} = 0, \\
b_{56} & \quad - b_{36} + b_{35} = -1.
\end{align*}
\]

One finds that the fist three equalities imply \( b_{56} a_6 = b_{56} a_6 + b_{35} a_5 = 0 \), which contradict with the last one.

In conclusion, nonzero \( F_B^2 \) does not lie in \( \Lambda^4_2(M) \). \(\square\)

**Lemma 3.6.** There do not exist a nonzero 1-form \( \alpha \in \Lambda^1(M) \) and a nonzero 2-form \( \beta \in \Lambda^2(M) \) such that \( \alpha \wedge \beta \in \Lambda^1_1(M) \oplus \Lambda^1_2(M) \).

**Proof.** Assume there exist such forms \( \alpha, \beta \). One can always find a point \( p \in M \) with \( \alpha|_p \neq 0, \beta|_p \neq 0 \), then
\[\alpha|_p \wedge \beta|_p = k \varphi|_p + X\wedge(\star \varphi)|_p,\]
where the constant \( k \) and the vector \( X \in TM|_p \) cannot vanish simultaneously. Then we can assume
- Case I: \( \alpha|_p \wedge \beta|_p = e^{123} + e^{145} + e^{246} + e^{347} - e^{167} + e^{257} - e^{356} + e^{256} + e^{346} - e^{247} \),
- Case II: \( \alpha|_p \wedge \beta|_p = e^{123} + e^{145} + e^{246} + e^{347} - e^{167} + e^{257} - e^{356} - e^{256} - e^{346} + e^{247} \),
- Case III: \( \alpha|_p \wedge \beta|_p = e^{123} + e^{145} + e^{246} + e^{347} - e^{167} + e^{257} - e^{356} \),
- Case IV: \( \alpha|_p \wedge \beta|_p = e^{357} + e^{256} + e^{346} - e^{247} \).

We follow the same arguments as in the proof of Theorem 3.5. For the first three cases, write \( \alpha|_p = \sum_{i=1}^7 a_i e^i \) and \( \beta|_p = \sum_{i<j} b_{ij} e^{ij} \).

To show \( a_1 \neq 0 \), we only need to show there are no \( c, d \in \Lambda^1(M)|_p \) such that
\[c \wedge d = e^{23} + e^{45} - e^{67}.
\]

Also write \( c = \sum_{i \geq 2} c_i e^i, d = \sum_{i \geq 2} d_i e^i \) with nonzero \( c_i, d_i \), the above claim follows from that there are no solutions for the following equations \( c_2 d_4 = c_3 d_2, c_3 d_4 = c_4 d_3, c_2 d_3 - c_3 d_2 = 1 \). Similarly, non-vanishing of \( a_2 \) is implied by the fact that there are no \( c, d \in \Lambda^1(M)|_p \) such that
\[c \wedge d = -e^{13} + e^{46} + e^{57} + \delta (e^{56} - e^{47}) \]
for \( \delta = \begin{cases} 1, & \text{Case I}, \\ -1, & \text{Case II}, \\ 0, & \text{Case III}. \end{cases} \)

Therefore, all \( a_i, i = 1, \cdots, 7 \), are nonzero. Then we have
\[
\begin{align*}
b_{23} & \quad - b_{13} + b_{12} a_{12} = 1, \\
b_{24} & \quad - b_{14} + b_{12} a_{12} = 0, \\
b_{34} & \quad - b_{14} + b_{13} a_{13} = 0, \\
b_{34} & \quad - b_{24} + b_{23} a_{23} = 0.
\end{align*}
\]
which admit no solutions. The forth case has been appeared in the proof of Theorem 3.5. Consequently, such 1-forms \( \alpha, \beta \) satisfying our assumption do not exist.

Inspired by Theorem 3.5 and Lemma 3.6, we introduce the following definition.

**Definition 3.7.** A 1-form \( B \in \Lambda^1(M) \) (as a connection on the trivial \( U(1) \)-principal bundle \( P \) over \( M \)) is called a special higher-order \( U(1) \)-instanton if \( B \wedge d_M B \in \Lambda^3_{(3)}(M) \). In particular, \( B \) is called a trivial special higher-order \( U(1) \)-instanton if \( B \wedge d_M B = 0 \).

**Theorem 3.8.** Assume \( B \) is a special higher-order \( U(1) \)-instanton, then

1. \( B \) is an anti-self-dual higher-order \( U(1) \)-instanton,
2. \( B \) is a closed 1-form iff \( B \wedge \ast d_M B = 0 \).

**Proof.** (1) By definition, \( B \) satisfies the conditions

\[
B \wedge d_M B \wedge \varphi = 0, \quad (3.7) \\
\ast \varphi (B \wedge d_M B) \wedge \varphi = 0, \quad (3.8)
\]

The first one immediately implies \( d_M B \wedge d_M B \wedge \varphi = 0 \), and since \( \varphi \) is parallel with respect to the Levi-Civita connection of \( g_\varphi \), the second one leads to \( d_M^\varphi (\ast \varphi (B \wedge d_M B)) \wedge \varphi = \ast \varphi (d_M \wedge d_M B) \wedge \varphi = 0 \), where \( d_M^\varphi \) is the formal adjoint of \( d_M \) with respect to the metric \( g_\varphi \). Therefore, \( B \) is an anti-self-dual higher-order \( U(1) \)-instanton.

(2) Decomposing \( d_M B = C_1 + C_2 \) with \( C_i = (d_M B)_i \), (3.7) is rewritten as

\[
B \wedge d_M B \wedge \varphi = B \wedge (-2 \ast \varphi C_1 + \ast \varphi C_2) \\
= -3B \wedge \ast \varphi C_1 + B \wedge \ast \varphi d_M B \\
= 0,
\]

and (3.8) is rewritten as

\[
B \wedge d_M B \wedge \ast \varphi \varphi = B \wedge C_1 \wedge \ast \varphi \varphi = 0.
\]

On the other hand, since \( C_1 = X \cdot \varphi \) for some vector field \( X \) on \( M \), we have the identities

\[
C_1 \wedge \ast \varphi \varphi = 3 \ast \varphi X^\varphi, \quad \ast \varphi C_1 = \ast \varphi \varphi \wedge X^\varphi,
\]

where \( X^\varphi \) is the dual 1-form of \( X \). Consequently, \( B \) is a special higher-order \( U(1) \)-instanton iff \( B \) satisfies the conditions

\[
3B \wedge \ast \varphi \varphi \wedge X^\varphi = B \wedge \ast \varphi d_M B, \\
B \wedge \ast \varphi X^\varphi = 0,
\]

Thus, if \( B \wedge \ast \varphi d_M B = 0 \), we have

\[
B \wedge \ast \varphi \varphi \wedge X^\varphi = 0, \quad (3.9) \\
B \wedge \ast \varphi X^\varphi = 0, \quad (3.10)
\]

At a point \( p \in M \), we assume \( B|_p = e^1, \varphi|_p = \varphi_0 \), then by (3.9),

\[
X^\varphi|_p \wedge (e^{14567} + e^{12345} - e^{12345}) = 0.
\]

It follows that \( X^\varphi|_p = ce^1 \) for some constant \( c \), however by (3.10), \( c \) must be zero, hence \( C_1 = 0 \). Then Corollary 3.10 implies that \( d_M B = 0 \). \( \square \)

**Example 3.9.** Treat \( \mathbb{R}^7 \) as a \( G_2 \)-manifold with the standard fundamental 3-form \( \varphi_0 \), and consider a 1-form

\[
B = x^2 dx^3 + ax^4 dx^5 + bx^6 dx^7
\]
with constant real numbers $a, b$. Then identifying $e^i \simeq dx^i, i = 1, \cdots, 7$, one easily checks

\[ B \wedge dB = a(x^2e^{345} + x^4e^{235}) + b(x^6e^{237} + x^2e^{367}) + ab(x^4e^{567} + x^6e^{457}), \]
\[ dB \wedge dB = 2ae^{2345} + 2be^{2367} + 2abe^{4567}, \]
\[ dB \wedge \varphi_0 = (1 + a)e^{12345} + (b - 1)e^{12367} + (b - a)e^{14567} \neq 0, \]
\[ dB \wedge *\varphi_0 = (1 + a - b)e^{234567}, \]
\[ B \wedge dB \wedge \varphi_0 = -(b-a)e^{23457} - a(b-1)x^4e^{123567} - b(1+a)x^6e^{123457}, \]
\[ dB \wedge dB \wedge \varphi_0 = (-a + b + ab)e^{123457}, \]
\[ B \wedge dB \wedge *\varphi_0 = dB \wedge dB \wedge *\varphi_0 = 0. \]

Therefore, we find that

- $B$ is an anti-self-dual $U(1)$-instanton iff $b - a = 1$,
- $B$ is an anti-self-dual higher-order $U(1)$-instanton iff $-a + b + ab = 0$,
- $B$ cannot be an anti-self-dual instanton and an anti-self-dual $U(1)$-instanton simultaneously, which can be seen from $dB \neq 0$,
- $B$ is a special anti-self-dual higher-order $U(1)$-instanton iff $a = b = 0$, equivalently, $B$ is a higher-order flat $U(1)$-instanton or $B$ is a trivial special anti-self-dual higher-order $U(1)$-instanton.

**Remark 3.10.** Finding (special) anti-self-dual higher-order $U(1)$-instantons on compact $G_2$-manifolds is an interesting and hard problem.

### 4. Partition Function for Higher-order $U(1)$-Chern-Simons Action

In this section, we only consider the case of $\mathcal{P}$ being a trivial $U(1)$-principal bundle over $\mathcal{M}$, which produces a trivial $U(1)$-principal bundle $\mathcal{P}$ over $\mathcal{M}$. We introduce the ghost fields $\varepsilon, \bar{\varepsilon}$ which lie in $\Lambda^0(\mathcal{M})$ but are fermionic, and $\phi \in \Lambda^0(\mathcal{M})$ which is a Lagrangian multiplier corresponding to $A$, then under the following transformations

\[ \delta A_0 = -\varepsilon := -\frac{\partial \varepsilon}{\partial t}, \]
\[ \delta B = -d_M \varepsilon, \]
\[ \delta \bar{\varepsilon} = \sqrt{-1} \phi, \]
\[ \delta \phi = \delta \varepsilon = 0, \]

the action

\[ S_1 = S_{\text{HCS}} + \int_L dt \int_M d\text{Vol}_{g_\phi}(A_0 \phi - \sqrt{-1} \bar{\varepsilon}), \tag{4.1} \]

where $d\text{Vol}_{g_\phi} = \sqrt{\text{det} g_\phi} dX^1 \wedge \cdots \wedge dX^7$ is the volume form expressed in terms of local coordinate $\{X^i, i = 1, \cdots, 7\}$ compatible with the orientation of $\mathcal{M}$, is invariant up to a total derivative term

\[ -\int_L dt \int_M \frac{\partial}{\partial t} (d_M B \wedge d_M B) \wedge \phi. \]

If one picks $L = \mathbb{R}$, the gauge fixing condition $A_0 = 0$ can be imposed. If $L = S^1$, although the above gauge fixing cannot be reached, one can require [37]

\[ \left\{ \begin{array}{l}
\text{torus gauge fixing for } A_0 : \frac{\partial A_0}{\partial t} = 0, \\
\text{constraint on } \phi : \int_{S^1} d\phi = 0.
\end{array} \right. \]

In the following, we always assume $\mathcal{M}$ is closed and simply-connected, and $L = S^1$, then after gauge fixing we consider the following action

\[ S = -2 \int_{S^1} dt \int_M B \wedge d_M B \wedge \hat{B} \wedge \varphi + 3 \int_{S^1} dt \int_M A_0 d_M B \wedge d_M B \wedge \varphi \]
\[ -\sqrt{-1} \int_{S^1} dt \int_M d\text{Vol}_{g_\phi} \bar{\varepsilon}, \tag{4.2} \]

with the residual gauge symmetries given by

\[ \delta B = d_M \phi, \]
\[ \delta A_0 = c, \]
\[ \delta \varepsilon = \delta \bar{\varepsilon} = 0. \]
for $f \in \Lambda^0(M)$ independent of $t$, $c$ being a constant. The partition function is given by path integral as follows

$$Z_\lambda = \frac{1}{\Vol(G)} \int DA_0 \prod_{n=-\infty}^{\infty} DB_n \prod_{n=-\infty}^{\infty} Dc_n \prod_{n=-\infty}^{\infty} D\bar{c}_n e^{-\lambda^2 S},$$

(4.3)

where $\lambda \in \mathbb{R}$ is the coupling constant, $\Vol(G)$ denotes the formal volume of the group $G$ consisting of the gauge transformation of the action $S$.

By Fourier expansions

$$B = \sum_{n=-\infty}^{\infty} B_n e^{2\sqrt{1-\pi n}t},$$

(4.4)

$$c = \sum_{n=-\infty}^{\infty} c_n e^{2\sqrt{1-\pi n}t}$$

(4.5)

with $\bar{B}_n = B_{-n}$ due to reality of $B$, we have

$$S = 4\pi \sqrt{1-1} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (n + m) \int_M B_n \wedge d_M B_m \wedge B_{-n-m} \wedge \varphi$$

$$+ 3 \sum_{n=-\infty}^{\infty} \int_M A_0 d_M B_n \wedge d_M B_{-n} \wedge \varphi + 2\pi \sum_{n=-\infty}^{\infty} n \int_M d\Vol g \bar{c}_n c_n$$

(4.6)

Choose a background

$$B^b = \sum_{n=-\infty}^{\infty} B^b_n e^{2\sqrt{1-\pi n}t}$$

(4.7)

as the critical point of the action $S$, namely we have

$$d_M B^b \wedge d_M B^b \wedge \varphi = 0,$$

(4.8)

$$d_M B^b \wedge B^b \wedge \varphi = 0,$$

(4.9)

$$d_M A_0 \wedge d_M B^b \wedge \varphi = 0,$$

(4.10)

or equivalently in terms of Fourier modes

$$\sum_{m+n=k} d_M B^b_m \wedge d_M B^b_n \wedge \varphi = 0,$$

(4.11)

$$\sum_{m+n=k} n d_M B^b_m \wedge B^b_n \wedge \varphi = 0,$$

(4.12)

$$d_M A_0 \wedge d_M B^b_m \wedge \varphi = 0,$$

(4.13)

and express $B = B^b + \lambda B$ with

$$B = \sum_{n=-\infty}^{\infty} B_n e^{2\sqrt{1-\pi n}t}$$

(4.14)

then the action $S$ is rewritten as

$$S = 3\lambda^2 [4\pi \sqrt{1-1} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (n + m) \int_M B^b_m \wedge d_M B^b_n \wedge B_{-n-m} \wedge \varphi$$

$$+ \sum_{n=-\infty}^{\infty} \int_M A_0 d_M B^b_n \wedge d_M B^b_{-n} \wedge \varphi]$$

$$+ \lambda^3 [4\pi \sqrt{1-1} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (n + m) \int_M B^b_n \wedge d_M B^b_m \wedge B_{-n-m} \wedge \varphi]$$

$$+ 2\pi \sum_{n=-\infty}^{\infty} n \int_M d\Vol g \bar{c}_n c_n$$

$$=: \lambda^2 S_q + \lambda^3 S_{int} + S_{gh}.$$  

(4.15)
When \( \lambda \to 0 \), \( Z_\lambda \) can be calculated by semiclassical approximation

\[
Z_{sc} = \frac{1}{\text{Vol}(G)} \int DA_0 \prod_{n=-\infty}^{\infty} DB_n \prod_{n=-\infty}^{\infty} D\varepsilon_n \prod_{n=-\infty}^{\infty} Dc_n \prod_{n=-\infty}^{\infty} Dc_n e^{\sqrt{-1}(S_0 + S_0^\lambda)}. \tag{4.16}
\]

Firstly, it is clear that

\[
\int \prod_{n=-\infty}^{\infty} Dc_n \prod_{n=-\infty}^{\infty} D\varepsilon_n e^{\sqrt{-1}S_\lambda} = \text{det}' \left( \frac{\partial}{\partial t} |_{\Lambda^0(M) \otimes \Lambda^0(S^1)} \right),
\]

where the prime above \( \text{det} \) means excluding zero mode. By heat kernel regularization \([37, 35]\), we have

\[
\log \text{det}' \left( \frac{\partial}{\partial t} |_{\Lambda^0(M) \otimes \Lambda^0(S^1)} \right) = \text{Tr}'(e^{-t\Delta^0_M}) \log \frac{\partial}{\partial t} |_{\Lambda^0(M) \otimes \Lambda^0(S^1)}
\]

\[
= \log \text{det}' \left( \frac{\partial}{\partial t} |_{\Lambda^0(S^1)} \right),
\]

where \( \Delta^0_M \) is the Laplacian on \( \Lambda^0(M) \), therefore we get

\[
\int \prod_{n=-\infty}^{\infty} Dc_n \prod_{n=-\infty}^{\infty} D\varepsilon_n e^{\sqrt{-1}S_\lambda} = \text{det}' \left( \frac{\partial}{\partial t} |_{\Lambda^0(S^1)} \right) = \prod_{n>0} (2\pi n)^2 = \left( \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \right)^2 = 1. \tag{4.17}
\]

Next we deal with the path integral

\[
\frac{1}{\text{Vol}(G)} \int DA_0 DB_b D\varepsilon e^{\sqrt{-1}S_\lambda} = \frac{1}{\text{Vol}(G)} \int DA_0 DB_b D\varepsilon e^{\sqrt{-1}f_{\beta,1} dt f_M B \wedge d_M \circ (2B \wedge \varphi \wedge B + d_M A \wedge \varphi \wedge B)}
\]

\[
= \frac{1}{\text{Vol}(G)} \int DA_0 D \prod_{n=-\infty}^{\infty} B_n \prod_{n=-\infty}^{\infty} D\varepsilon_n \prod_{n=-\infty}^{\infty} D\varepsilon_n e^{\sqrt{-1}f_M B_n \wedge d_M \circ (B_n \wedge \varphi \wedge B - m - n) + B_n \wedge (d_M A \wedge \varphi \wedge d_M) B - n \}}. \tag{4.18}
\]

One again applies heat kernel regularization, namely inserts the fact \( e^{\frac{2\pi^2}{\sigma^2}} \) in the operator

\[
D_{A_0, B^b, \varphi} = d_M \circ (2B \wedge \varphi \wedge \frac{\partial}{\partial t} + d_M A \wedge \varphi \wedge)
\]

as

\[
D_{A_0, B^b, \varphi} \to d_M \circ (2e^{\frac{2\pi^2}{\sigma^2}} B \wedge \varphi \wedge e^{\frac{2\pi^2}{\sigma^2}} \frac{\partial}{\partial t} + d_M A \wedge \varphi \wedge e^{\frac{2\pi^2}{\sigma^2}}) \tag{4.19}
\]

then only zero modes of \( B^b \) and \( B \) survive in (4.18). Therefore, we only need to consider

\[
\frac{1}{\text{Vol}(G)} \int DB_b D\varepsilon e^{\sqrt{-1}f_M (B_b \wedge d_M A \wedge d_M (B_b \wedge \varphi))}. \tag{4.20}
\]

We will use Schwarz-Pestun-Witten’s method \([39, 40]\) to formally calculate it.

Introduce the following notations.

- \( \Delta_{\varphi} = d_M^* d_M + d_M d_M^* \), \( \Delta_{\varphi}^\prime = d_M^* d_M \), \( \Delta_{\varphi}'' = d_M d_M^* \),
- \( \Lambda_\bullet(M) \): the space of closed \( \bullet \)-forms,
- \( \Lambda^2_{A_0}(M) = \{ \beta \in \Lambda^2(M) : \beta = \alpha \wedge d_M A_0 \text{ for some } \alpha \in \Lambda^1(M) \} \simeq \Lambda^1(M) \),
- \( \Lambda^2_{A_0}(M) = \{ \beta \in \Lambda^2_{A_0}(M) : \Delta_{\varphi} \beta = \alpha \in \Lambda^2_{A_0}(M) \} \),
- \( E\Lambda^2_{A_0}(M) \): the space of exact 2-forms in \( \Lambda^2_{A_0}(M) \),
- \( \Lambda^1_{A_0}(M) = \{ \alpha \in \Lambda^1(M) : d_M \alpha \in \Lambda^2_{A_0}(M) \} \),
- \( \Lambda^1_{A_0}(M) = \{ \alpha \in \Lambda^1_{A_0}(M) : \Delta_{\varphi} \alpha \in \Lambda^1_{A_0}(M) \} \),
- \( \Lambda^2_{\varphi}(M) = \{ \beta \in \Lambda^2(M) : \beta \wedge \varphi = 0 \} \),
- \( E\Lambda^2_{\varphi}(M) \): the space of exact 2-forms in \( \Lambda^2_{\varphi}(M) \),
- \( \Lambda^1_{\varphi}(M) = \{ \alpha \in \Lambda^1(M) : d_M \alpha \in \Lambda^2_{\varphi}(M) \} \).
Then formally, we have
\[
\frac{1}{\text{Vol}(G)} \int D\mathbb{B}_0 e^{\frac{1}{2} \int_M \mathcal{L}_0(A_0, A_0) + \int_M (\mathcal{L}_0 - \mathcal{L}_1)} = \frac{1}{\text{Vol}(G)} \left[ \text{det}'(9\Delta_\varphi_{\Lambda_{\alpha_0}(M)}) \right]^{-\frac{1}{2}} \left[ \text{det}'(9\Delta_\varphi_{\Lambda_{\beta_0}(M)}) \right]^{\frac{1}{2}} \text{Vol}(E\Lambda^2_{A_0}(M)) \text{Vol}(E\Lambda^2_{\varphi}(M))^{\frac{1}{2}}.
\]
(4.21)

Moreover, we have
\[
\text{Vol}(E\Lambda^2_{A_0}(M)) \text{Vol}(E\Lambda^2_{\varphi}(M)) = \frac{\text{Vol}(\Lambda_{A_0}(M))}{\text{Vol}(\Lambda_{\alpha_0}(M))} \frac{\text{Vol}(H^0(M, \mathbb{R}))}{\text{Vol}(H^1(M, \mathbb{R}))} \left[ \text{det}'(\Delta_\varphi_{\Lambda_{\alpha_0}(M)}) \right]^{\frac{1}{2}} \left[ \text{det}'(\Delta_\varphi_{\Lambda_{\beta_0}(M)}) \right]^{\frac{1}{2}} \text{Vol}(E\Lambda^2_{A_0}(M)) \text{Vol}(E\Lambda^2_{\varphi}(M))^{\frac{1}{2}}.
\]
(4.22)

Let \( G' \subset G \) be the subgroup of \( G \) consisting of gauge transformations on \( \mathbb{B}_0 \) (preserving \( A_0, B_0^0 \)), then \( \text{Vol}(G') \) can be renormalized to be \( \frac{\text{Vol}(\Lambda_{A_0}(M))}{\text{Vol}(\Lambda_{\alpha_0}(M))} \frac{\text{Vol}(H^0(M, \mathbb{R}))}{\text{Vol}(H^1(M, \mathbb{R}))} \). Also, \( \frac{\text{Vol}(H^0(M, \mathbb{R}))}{\text{Vol}(H^1(M, \mathbb{R}))} \) can be renormalized to be 1. Consequently, we arrive at
\[
\frac{1}{\text{Vol}(G)} \int D\mathbb{A}_0 D\mathbb{B}_0^0 D\mathbb{B}_1 e^{\frac{1}{2} \int_M \mathcal{L}_0(A_0, A_0) + \int_M (\mathcal{L}_0 - \mathcal{L}_1)} = \int_{\mathcal{M}^b} D\mathbb{A}_0 D\mathbb{B}_0^0 \frac{[\text{det}'(\Delta_\varphi_{\Lambda_{\alpha_0}(M)})]^{\frac{1}{2}} [\text{det}'(\Delta_\varphi_{\Lambda_{\beta_0}(M)})]^{\frac{1}{2}}}{[\text{det}'(\Delta_\varphi_{\Lambda_{\alpha_0}(M)})]^{\frac{1}{2}} [\text{det}'(9\Delta_\varphi_{\Lambda_{\alpha_0}(M)})]^{\frac{1}{2}} [\text{det}'(9\Delta_\varphi_{\Lambda_{\beta_0}(M)})]^{\frac{1}{2}}},
\]
(4.23)

where \( \mathcal{M}^b \) is the formal the moduli space defined as
\[
\mathcal{M}^b = \{ (B_0^0, A_0) \in \Lambda^1(M) \oplus \Lambda^0(M) : d_M B_0^0 \wedge d_M B_0^0 \wedge \varphi = d_M A_0 \wedge d_M B_0^0 \wedge \varphi = 0 \}.
\]

In addition, we have
\[
\text{det}'(9\Delta_\varphi_{\Lambda^0(M)}) = \text{det}'(9\Delta_\varphi_{\Lambda^1(M)}) = \text{det}'(9\Delta_\varphi_{\Lambda^2(M)}) = \frac{\text{det}'(\Delta_\varphi_{\Lambda^0(M)})}{\text{det}'(9\Delta_\varphi_{\Lambda^0(M)})} = \frac{\text{det}'(\Delta_\varphi_{\Lambda^1(M)})}{\text{det}'(9\Delta_\varphi_{\Lambda^0(M)})} = \frac{\text{det}'(\Delta_\varphi_{\Lambda^2(M)})}{\text{det}'(9\Delta_\varphi_{\Lambda^0(M)})} = \frac{\text{det}'(\Delta_\varphi_{\Lambda^0(M)})}{\text{det}'(9\Delta_\varphi_{\Lambda^0(M)})} = c^{-\dim H^0(M, \mathbb{R})} \text{det}'(\Delta_\varphi_{\Lambda^0(M)})
\]
for some constant \( c \). Finally, we obtain
\[
Z_{\text{nc}} = \frac{1}{9} \int_{\mathcal{M}^b} D\mathbb{A}_0 D\mathbb{B}_0^0 \frac{[\text{det}'(\Delta_\varphi_{\Lambda^0(M)})]^{\frac{1}{2}} [\text{det}'(\Delta_\varphi_{\Lambda^0(M)})]^{\frac{1}{2}}}{[\text{det}'(9\Delta_\varphi_{\Lambda^0(M)})]^{\frac{1}{2}} [\text{det}'(9\Delta_\varphi_{\Lambda^0(M)})]^{\frac{1}{2}}},
\]
(4.24)

5. Higher-order \( U(1) \)-Chern-Simons actions for nontrivial \( U(1) \)-principal bundles

In the final section, we discuss the higher-order \( U(1) \)-Chern-Simons actions for nontrivial \( U(1) \)-principal bundles. As mentioned in Introduction, we need the Deligne-Beilinson cohomology theory. Hence, we first recall some basis materials in this theory. The original Deligne-Beilinson cohomology is defined for algebraic varieties, and the smooth analogy of this theory is also called Cheeger-Simons cohomology [42]. For a compact manifold \( M \), the Deligne complex of sheaves is given by
\[
\text{DC}_{R(\ell)} : 0 \to R(\ell) \to \Lambda^0(X, \ell) \to \Lambda^1(X, \ell) \to \cdots \to \Lambda^\ell(X, \ell),
\]
where \( \Lambda^\bullet(X, \ell) = (2\pi \sqrt{-1})^\ell \Lambda^\bullet(X) \), \( R(\ell) = (2\pi \sqrt{-1})^\ell R \) for a subring \( R \) of \( \mathbb{R} \) and some integer \( \ell \geq 0 \), then the \( q \)-order Deligne-Beilinson cohomology group \( H^q_{\text{DB}}(X, R(\ell)) \) is defined as the \( q \)-th hypercohomology of the Deligne complex \( \text{DC}_{R(\ell)} \), i.e.
$H_{\text{DB}}^q(X, R(\ell)) = \mathbb{H}^q(\mathbf{DC}_{R(\ell)})$. Taking an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of $X$, i.e. $X = \bigcup_{\alpha \in I} U_\alpha$, we consider the Čech resolution of $\mathbf{DC}_{R(\ell)}$:

$$
\begin{align*}
R(\ell) & \xrightarrow{\iota} \Lambda^0(X, \ell) \xrightarrow{d} \Lambda^1(X, \ell) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^\ell(X, \ell) \\
\downarrow & \quad \downarrow \quad \downarrow \quad \downarrow \quad \cdots \quad \downarrow \quad \downarrow \\
C^0(\mathcal{U}, R(\ell)) & \xrightarrow{\iota} C^0(\mathcal{U}, \Lambda^0(X, \ell)) \xrightarrow{d_{\mathcal{U}}} C^0(\mathcal{U}, \Lambda^1(X, \ell)) \xrightarrow{d} \cdots \xrightarrow{d} C^0(\mathcal{U}, \Lambda^\ell(X, \ell)) \\
\downarrow & \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
C^1(\mathcal{U}, R(\ell)) & \xrightarrow{\iota} C^1(\mathcal{U}, \Lambda^0(X, \ell)) \xrightarrow{d_{\mathcal{U}}} C^1(\mathcal{U}, \Lambda^1(X, \ell)) \xrightarrow{d} \cdots \xrightarrow{d} C^1(\mathcal{U}, \Lambda^\ell(X, \ell)) \\
\downarrow & \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
& \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
& \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
& 0 \xrightarrow{\exp_\ell} U^{(1)}_X \xrightarrow{d_{\log}} \Lambda^1(X, 1) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^\ell(M, 1)
\end{align*}
$$

where $\iota$ denotes the natural embedding, $\delta$ denotes the Čech operator, and $C^q(\mathcal{U}, \mathcal{F})$ denotes the space of $q$-dimensional Čech cochains for a sheaf $\mathcal{F}$ over $M$. Therefore

$$
H_{\text{DB}}^q(X, R(\ell)) = \lim_{\mathcal{U}} H^q(\text{Tot}_{\mathcal{U}}(\mathbf{DC}_{R(\ell)})),
$$

where $\text{Tot}_{\mathcal{U}}(\mathbf{DC}_{R(\ell)})$ is the total complex of the Čech resolution of $\mathbf{DC}_{R(\ell)}$ associated to the cover $\mathcal{U}$. In particular, if $\mathcal{U}$ is a simple (good) cover $[44]$, we have $H_{\text{DB}}^q(M, R(\ell)) = H^q(\text{Tot}_{\mathcal{U}}(\mathbf{DC}_{R(\ell)}))$ $[43]$.

For our purpose, we take $R = \mathbb{Z}$. From the following commutative diagram

$$
\begin{align*}
\mathbb{Z}(\ell) & \xrightarrow{\exp_\ell} \Lambda^0(X, \ell) \xrightarrow{d} \Lambda^1(X, \ell) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^\ell(X, \ell) \\
\downarrow & \quad \downarrow \quad \downarrow \quad \downarrow \quad \cdots \quad \downarrow \\
0 & \xrightarrow{\exp_\ell} U^{(1)}_X \xrightarrow{d_{\log}} \Lambda^1(X, 1) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^\ell(M, 1)
\end{align*}
$$

where $U^{(1)}_X$ denotes sheaf of $U(1)$-valued functions over $X$, and $\exp_\ell(f) = e^{(2\pi i f)}$, we find that

$$
H_{\text{DB}}^q(X, \mathbb{Z}(\ell)) \simeq \mathbb{H}^{q-1}(U^{(1)}_X, \ell))
$$

where $U^{(1)}_X(\ell)$ denote the complex $U^{(1)}_X \xrightarrow{d_{\log}} \Lambda^1(X, 1) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^{\ell-1}(X, 1)$. It immediately implies that Deligne-Beilinson cohomology $H_{\text{DB}}^2(X, \mathbb{Z}(1))$ parameterize the isomorphism classes of $U(1)$-principal bundles with connections over $M$ $[43, 44]$. More explicitly, choosing a simple cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of $X$ with index set $I$, an element in $H_{\text{DB}}^2(X, \mathbb{Z}(1))$ is represented by a triple $(\{A_\alpha\}_{\alpha \in I}, \{\Gamma_{\alpha\beta}\}_{\alpha, \beta \in I}, \{\Upsilon_{\alpha\beta\gamma}\}_{\alpha, \beta, \gamma \in I})$ of Čech cochains satisfying

$$
\begin{align*}
\check{A}_\beta - A_\alpha & = d\Gamma_{\alpha\beta} \\
\check{\Gamma}_{\beta\gamma} - \Gamma_{\alpha\gamma} + \Gamma_{\alpha\beta} & = \Upsilon_{\alpha\beta\gamma} \\
\check{\Upsilon}_{\beta\gamma\delta} - \Upsilon_{\alpha\gamma\delta} + \Gamma_{\alpha\beta\delta} & = 0
\end{align*}
$$

where $\{A_\alpha\} \in C^0(\mathcal{U}, \Lambda^1(X, 1)), \{\Gamma_{\alpha\beta}\} \in C^1(\mathcal{U}, \Lambda^0(X, 1)), \{\Upsilon_{\alpha\beta\gamma}\} \in C^2(\mathcal{U}, \mathbb{Z}(1))$. Writing $g_{\alpha\beta} = e^{-\Gamma_{\alpha\beta}}, \Gamma_{\alpha\beta} = -2\pi \sqrt{-1}\Lambda_{\alpha\beta}$, then (5.2) indicates that $\{g_{\alpha\beta}\}$ defines transition functions on a $U(1)$-principal bundle, and (5.1) means that $\{A_\alpha\}$ defines a connection on such $U(1)$-principal bundle.

Now $X = M = M \times S^1$, where $M$ is a closed $G_2$-manifold whose fundamental 3-form $\varphi$ is assumed to be an integral cohomology class. One views $\varphi \in H^3(M, \mathbb{Z})$ as an element of Deligne-Beilinson cohomology $H_{\text{DB}}^3(M, \mathbb{Z}(1))$, hence an element of $H_{\text{DB}}^3(M, \mathbb{Z}(1))$ by natural inclusion. Let $A, B, C \in H_{\text{DB}}^2(M, \mathbb{Z}(1))$ describe isomorphism classes of $U(1)$-principal bundles with connections over $M$. Taking a simple cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of $M$, we represent $A, B, C \in H_{\text{DB}}^2(M, \mathbb{Z}(1))$ as

$$
A = ((\{A_\alpha\}, \{\Gamma_{\alpha\beta}\}, \{\Upsilon_{\alpha\beta\gamma}\}),
$$

$$
B = ((\mathbb{P}_\alpha\ell), \{\Theta_{\alpha\beta}\}, \{\Lambda_{\alpha\beta\gamma}\}),
$$

$$
C = ((\mathbb{C}_\alpha\ell), \{\Psi_{\alpha\beta}\}, \{\Omega_{\alpha\beta\gamma}\}),
$$

and also represent $\varphi \in H_{\text{DB}}^3(M, \mathbb{Z}(1))$ as

$$
\varphi = (\{\chi_{\alpha\beta}\}, \{\tau_{\alpha\beta\gamma}\}, \{2\pi \sqrt{-1}\theta_{\alpha\beta\gamma\delta}\}).
$$
where \( \{2\pi\sqrt{-1}\theta_{\alpha\beta\gamma}\} \in C^3(\mathcal{U}, \mathbb{Z}(1)) \) are determined via the isomorphism

\[
H^3(\mathcal{M}, \mathbb{Z}) \simeq H^3_{\text{Coh}}(\mathcal{M}, \mathbb{Z}),
\]

\[
\varphi \mapsto \{\theta_{\alpha\beta\gamma}\},
\]

and \( \{\chi_{\alpha\beta}\} \in C^1(\mathcal{U}, \Omega^1(\mathcal{M}, 1)) \), \( \{\tau_{\alpha\beta}\} \in C^2(\mathcal{U}, \Omega^0(\mathcal{M}, 1)) \) are determined as follows

\[
\chi_{\alpha\beta} + \sum_{\gamma \in I}(d\tau_{\alpha\beta\gamma})\xi_\gamma = 0,
\]

\[
\tau_{\alpha\beta\gamma} - 2\pi\sqrt{-1}\sum_{\eta \in I}\theta_{\alpha\beta\gamma\eta}\kappa_\eta = 0
\]

for a partition \( \{\xi_\alpha\} \in I \) of unity subordinate to the simple cover \( \mathcal{U} \).

Consider a polyhedral decomposition \( \{(\psi^{(d)}_{\alpha})\}_{\alpha \in I}, \cdots, \{(\psi^{(0)}_{\alpha})\}_{\alpha \in I} \) of \( M \) subordinate to \( \mathcal{U} \), where \( \psi^{(d)}_{\alpha\cdots\alpha_{8-d}} \) is a \( d \)-dimensional submanifold of \( M \) lying in \( \mathcal{U}_{\alpha_0}\cap \cdots \cap \mathcal{U}_{\alpha_{8-d}} \), then we construct the gauge invariant \( U(1) \)-ABC action. We begin with the following action

\[
I_0 = \sum_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} \int_{\psi^{(4)}_{0010234}} \theta_{\alpha_0\alpha_1\alpha_2\alpha_3}\Lambda_{\alpha_0} \wedge d\mathbb{B}_{\alpha_0} \wedge d\mathbb{C}_{\alpha_0}.
\]

Under the local gauge transformation

\[
\Lambda_\alpha \mapsto \Lambda_\alpha + da_\alpha, \quad \mathbb{B}_\alpha \mapsto \mathbb{B}_\alpha + db_\alpha, \quad C_\alpha \mapsto C_\alpha + dc_\alpha,
\]

for \( a_\alpha, b_\alpha, c_\alpha \in C^0(\mathcal{U}, \Omega^1(\mathcal{M}, 1)) \), its variation is given by

\[
\Delta_{bd}I_0 = \sum_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} \int_{\psi^{(4)}_{0010234}} \theta_{\alpha_0\alpha_1\alpha_2\alpha_3}[d\Lambda_{\alpha_0} \wedge d\mathbb{B}_{\alpha_0} \wedge d\mathbb{C}_{\alpha_0}]
\]

\[
= \sum_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} \int_{\psi^{(4)}_{0010234}} [(\theta_{\alpha_0\alpha_1\alpha_2\alpha_3} - \theta_{\alpha_0\alpha_1\alpha_2\alpha_4} + \theta_{\alpha_0\alpha_1\alpha_3\alpha_4} - \theta_{\alpha_0\alpha_2\alpha_3\alpha_4})
\]

\[
\cdot a_{\alpha_0} \wedge d\mathbb{B}_{\alpha_0} \wedge d\mathbb{C}_{\alpha_0} + \theta_{\alpha_1\alpha_2\alpha_3\alpha_4} a_\alpha d\mathbb{B}_{\alpha_1} \wedge d\mathbb{C}_{\alpha_1}]
\]

\[
= \sum_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} \int_{\psi^{(4)}_{0010234}} \theta_{\alpha_1\alpha_2\alpha_3\alpha_4}(b_{\alpha_1} - a_{\alpha_0})dB_{\alpha_0} \wedge dC_{\alpha_0},
\]

which can be eliminated by the variation of the action

\[
I_1 = - \sum_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} \int_{\psi^{(4)}_{0010234}} \theta_{\alpha_1\alpha_2\alpha_3\alpha_4} \Gamma_{\alpha_0\alpha_1 \alpha_2 \alpha_3} \wedge d\mathbb{C}_{\alpha_0}.
\]

under the transformation \( \Gamma_{\alpha\beta} \mapsto \Gamma_{\alpha\beta} + a_\beta - a_\alpha \). However, the action \( I_0 + I_1 \) is not invariant under the large gauge transformation \( \Gamma_{\alpha\beta} \mapsto \Gamma_{\alpha\beta} + z_{\alpha\beta} \) for \( z_{\alpha\beta} \in C^1(\mathcal{U}, \mathbb{Z}(1)) \). Indeed, the variation of \( I_1 \) is given by

\[
\Delta_{lt}I_1 = \Delta_{1ar}I_1 + \Delta_{2ar}I_1
\]

\[
= - \sum_{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4} \int_{\psi^{(3)}_{00102345}} \theta_{\alpha_2\alpha_3\alpha_4\alpha_5}(d\zeta)_{\alpha_0\alpha_1 \alpha_2} d\mathbb{B}_{\alpha_0} \wedge d\mathbb{C}_{\alpha_0}
\]

\[
- \sum_{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4} \int_{\psi^{(3)}_{00102345}} \theta_{\alpha_2\alpha_3\alpha_4\alpha_5} \zeta_{\alpha_1 \alpha_2} (\mathbb{B}_{\alpha_1} d\mathbb{C}_{\alpha_1} - \mathbb{B}_{\alpha_0} d\mathbb{C}_{\alpha_0}).
\]

\( \Delta_{1ar}I_1 \) can be eliminated by variation of the action

\[
I_2 = \sum_{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4} \int_{\psi^{(3)}_{00102345}} \theta_{\alpha_2\alpha_3\alpha_4\alpha_5} \Gamma_{\alpha_0 \alpha_1 \alpha_2} \wedge d\mathbb{C}_{\alpha_0}.
\]
under the transformation $\Upsilon_{\alpha\beta\gamma} \rightarrow \Upsilon_{\alpha\beta\gamma} + (\delta z)_{\alpha\beta\gamma}$. As well as, $\Delta_{\text{inv}}^2 I_1$ can be calculated as

$$
\Delta_{\text{inv}}^2 I_1 = - \sum_{a_0,a_1,a_2,a_3,a_4,a_5} \int_{\Psi_{012345}}^{(3)} \theta_{a_2a_1a_3a_4a_5} \delta z_{a_1a_2} \, d(\Theta_{a_0a_1} dC_{a_0}) = 0
$$

$$
- \sum_{a_0,a_1,a_2,a_3,a_4,a_5} \int_{\Psi_{012345}}^{(2)} \theta_{a_3a_4a_5} (\delta z)_{a_1a_2a_3} \Theta_{a_0a_1} dC_{a_0}
$$

$$
+ \sum_{a_0,a_1,a_2,a_3,a_4,a_5} \int_{\Psi_{012345}}^{(2)} \theta_{a_3a_4a_5} \delta z_{a_1a_2a_3} \Lambda_{a_0a_1a_2} dC_{a_0} = 0
$$

$$
\Upsilon_{\alpha\beta\gamma} \rightarrow \Upsilon_{\alpha\beta\gamma} + (\delta z)_{\alpha\beta\gamma}.
$$

where the notation $\approx$ means that we have omitted the term as the form of $(2\pi \sqrt{-1})^3 \mathbb{Z}$. The three terms on the right hand side of $\approx$ can be eliminated by the variations of the actions

$$
I_3 = - \sum_{a_0,a_1,a_2,a_3,a_4,a_5} \int_{\Psi_{012345}}^{(2)} \theta_{a_3a_4a_5} \Upsilon_{a_1a_2a_3} \Theta_{a_0a_1} dC_{a_0},
$$

$$
I_4 = \sum_{a_0,a_1,a_2,a_3,a_4,a_5} \int_{\Psi_{012345}}^{(1)} \theta_{a_3a_4a_5} \Upsilon_{a_1a_2a_3} \Lambda_{a_0a_1a_2} dC_{a_0},
$$

$$
I_5 = - \sum_{a_0,a_1,a_2,a_3,a_4,a_5} \int_{\Psi_{012345}}^{(0)} \theta_{a_3a_4a_5} \Upsilon_{a_1a_2a_3} \Psi_{a_0a_1} dC_{a_0},
$$

respectively, under the transformation $\Upsilon_{\alpha\beta\gamma} \rightarrow \Upsilon_{\alpha\beta\gamma} + (\delta z)_{\alpha\beta\gamma}$. Similar calculations exhibit $I_2 + I_3$ and $I_4 + I_5$ are invariant under the local gauge transformations. Consequently, the gauge invariant $U(1)$-ABC action reads

$$
S_{\text{ABC}} \approx I_0 + I_1 + I_2 + I_3 + I_4 + I_5
$$

$$
= \sum_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} \int_{\Psi_{\alpha_0123}}^{(5)} \theta_{\alpha_0\alpha_1\alpha_2\alpha_3} a_{\alpha_0} \wedge dB_{\alpha_0} \wedge dC_{\alpha_0} - \sum_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} \int_{\Psi_{\alpha_0123}}^{(4)} \theta_{\alpha_0\alpha_1\alpha_2\alpha_3} \Gamma_{\alpha_0\alpha_1} dE_{\alpha_0} \wedge dC_{\alpha_0} + \sum_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} \int_{\Psi_{\alpha_0123}}^{(3)} \theta_{\alpha_0\alpha_1\alpha_2\alpha_3} \delta z_{\alpha_0\alpha_1\alpha_2} \wedge dC_{\alpha_0}
$$

$$
- \sum_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} \int_{\Psi_{\alpha_0123}}^{(0)} \theta_{\alpha_0\alpha_1\alpha_2\alpha_3} \Upsilon_{\alpha_0\alpha_1\alpha_2} \Theta_{\alpha_0a_1} dC_{\alpha_0} + \sum_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} \int_{\Psi_{\alpha_0123}}^{(1)} \theta_{\alpha_0\alpha_1\alpha_2\alpha_3} \Upsilon_{\alpha_0\alpha_1\alpha_2} \Lambda_{\alpha_0a_1a_2} dC_{\alpha_0} - \sum_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} \int_{\Psi_{\alpha_0123}}^{(0)} \theta_{\alpha_0\alpha_1\alpha_2\alpha_3} \Upsilon_{\alpha_0\alpha_1\alpha_2} \Psi_{\alpha_0a_1}. \tag{5.4}
$$
The Deligne-Beilinson cup product [42]

\[ H^q_{\text{DB}}(X, \mathbb{Z}(\ell)) \times H^t_{\text{DB}}(X, \mathbb{Z}(j)) \to \begin{cases} 
H^{q+t}_{\text{DB}}(X, \mathbb{Z}(\ell + j + 1)), & q = \ell + 1, t \leq j + 1 \text{ or } t = j + 1, q \leq \ell + 1; \\
H^{q+t-1}_{\text{DB}}(X, \mathbb{Z}(\ell + j + 1)), & q \geq \ell + 2, t \leq j + 1 \text{ or } t \geq j + 2, q \leq \ell + 1; \\
H^{q+t}_{\text{DB}}(X, \mathbb{Z}(\ell + j + 1)) \simeq H^{q+t}(X, \mathbb{Z}), & q \geq \ell + 2, t \geq j + 2 \text{ or } t \geq j + 2, q \geq \ell + 2; \\
0, & \text{other cases.}
\end{cases} \]

defines

\[ A \cup B \in H^1_{\text{DB}}(\mathcal{M}, \mathbb{Z}(3)), \quad A \cup B \cup C \in H^5_{\text{DB}}(\mathcal{M}, \mathbb{Z}(5)), \]

\[ \varphi \cup A \cup B \cup C \in H^8_{\text{DB}}(\mathcal{M}, \mathbb{Z}(7)) \simeq \mathbb{R}/\mathbb{Z}. \]

Then the action (5.4) can be briefly written as

\[ S_{ABC} \approx \frac{1}{2\pi \sqrt{-1}} \int_M \varphi \cup A \cup B \cup C - \frac{1}{2\pi \sqrt{-1}} \sum_{\alpha, \alpha_1 \in I} \int_{\mathcal{B}_{\alpha_1}^{(\gamma)}} \chi_{\alpha_0, \alpha_1} dB_{\alpha_0} \wedge dC_{\alpha_0} + \frac{1}{2\pi \sqrt{-1}} \sum_{\alpha, \alpha_1, \alpha_2 \in I} \int_{\mathcal{B}_{\alpha_1}^{(\gamma)}} \tau_{\alpha_0, \alpha_1, \alpha_2} dA_{\alpha_0} \wedge dB_{\alpha_0} \wedge dC_{\alpha_0} \]  

(5.5)

where the extra two terms are obviously gauge invariant.

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