On a generalization of a Ramanujan conjecture for binomial random variables

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Abstract

For a binomial random variable $\xi$ with parameters $n$ and $b/n$, it is very well known that the median equals $b$, if $b$ is an integer. In 1968, Jogdeo and Samuels studied a behaviour of the fraction of $P(\xi = b)$ which equals to $1/2 - P(\xi < b)$. They proved its monotonicity in $n$ and posed a question about its monotonicity in $b$. This question is motivated by the solved problem proposed by Ramanujan in 1911 on the monotonicity of the same function but for a Poisson random variable with an integer parameter $b$. In the paper, we give an answer on this question.

1 Introduction

Giving a non-negative integer random variable $\xi$, we call the median of $\xi$

$$\mu(\xi) := \min \left\{ m \in \mathbb{Z}_+ : P(\xi \leq m) \geq \frac{1}{2} \right\}.$$  

Consider a Poisson random variable $\eta_b$ with a positive parameter $b$. It is known \textsuperscript{4} that $b - \ln 2 \leq \mu(\eta_b) < b + 1/3$, and the bounds are best possible. So, for an integer $b$, $\mu(\eta_b) = b$. But this was known long before the result of Choi. Indeed, for an integer $b$, S. Ramanujan \textsuperscript{9} conjectured that

$$y_b := \frac{1/2 - P(\eta_b < b)}{P(\eta_b = b)} \in \left(\frac{1}{3}, \frac{1}{2}\right)$$

and $y_b$ decreases. This was proven independently by G. Szegö in 1928 \textsuperscript{10} and G.N. Watson in 1929 \textsuperscript{11}. And it immediately implies that $\mu(\eta_b) = b$.

The behaviour of $y_b$ was widely studied. Below, we give a very brief history of this study.

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In 1913, in his letter to Hardy, Ramanujan posed a more delicate question: he conjectured that
\[ y_b = \frac{1}{3} + \frac{4}{135(b + \alpha_b)}, \]
where \( \frac{8}{45} \geq \alpha_b \geq \frac{2}{21}. \) This conjecture was proved by Flajolet et al. [7] in 1995. In 2003, S. E. Alm [1] proved that \( \alpha_b \) decreases. In 2004, this result was strengthened by H. Alzer [2]:
\[ y_b = \frac{1}{3} + \frac{4}{135b} - \frac{8}{2835(b^2 + \beta_b)}, \]
where \(-\frac{1}{3} < \beta_b \leq -1 + \frac{4}{\sqrt{21(368-135e)}}\), and the bounds are sharp.

Let us now consider a binomial random variable \( \xi_{b,n} \) with parameters \( n \) and \( b/n \), where \( b \leq n \) are positive integers. It is known [5] that \( |\mu(\xi_{b,n}) - b| \leq \ln 2 \), and so, \( \mu(\xi_{b,n}) = b \). In 1968, K. Jogdeo and S. M. Samuels [8] considered a problem similar to the first Ramanujan conjecture but for binomial random variables. They proved that
\[ z_{b,n} := \frac{1}{2} - \frac{P(\xi_{b,n} < b)}{P(\xi_{b,n} = b)} \in \left( \frac{1}{3}, \frac{2}{3} \right). \]
Moreover, the following result holds.

\textbf{Theorem 1 (K. Jogdeo, S. M. Samuels, 1968)} For every \( b \), \( z_{b,n} \) decreases for \( n \geq 2b \) and \( z_{b,n} \to y_n \) as \( n \to \infty \). Moreover, for all \( n > 2b \), \( 1/3 < z_{b,n} < 1/2 \). For all \( b < n < 2b \), \( 1/2 < z_{b,n} < 2/3 \). Finally, \( z_{b,2b} = 1/2 = z_{b,b} \).

In the paper, they tried to generalize the statement that \( y_b \) decreases but they failed. They also mentioned that, obviously, \( z_{b+1,n} < z_{b,n} \) for all large enough \( n \) but they were unable to make this more precise.

In our paper, we solve the problem proposed by Jogdeo and Samuels on a monotonicity of \( z_{b,n} \) in \( b \). Our main result is the following.

\textbf{Theorem 2} Let \( \varepsilon > 0 \). For \( n \) large enough,

1. if \( n - (1 + \varepsilon)\sqrt{\frac{77}{360}n} > b > (1 + \varepsilon)\sqrt{\frac{77}{360}n} \), then \( z_{b+1,n} > z_{b,n} \).

2. if either \( b > n - (1 - \varepsilon)\sqrt{\frac{77}{360}n} \) or \( b < (1 - \varepsilon)\sqrt{\frac{77}{360}n} \), then \( z_{b+1,n} < z_{b,n} \).
In the paper, we are also interested in a monotonicity of $p_{b,n} := P(\xi_{b,n} < b)$ in $b$. This interest, in particular, has the following motivation. From the result of Szeg"{o} and Watson, it immediately follows that $P(\eta_{b} < b)$ increases (or, in other words, the difference between 1/2 and the probability that $\eta_{b}$ is less than the median decreases). So, for $n$ large enough, $p_{b+1,n} > p_{b,n}$ as well. It is easy to see that, for $n = b + 1$, $0 = p_{b+1,n} < p_{b,n}$, in contrast. For small values of $b$, it can be easily verified that the same inequality holds even for $b + 1 \leq n \leq 3b + 1$. In Appendix A, we show in a very naive way that, for $b \leq 5$, if $n \geq 3b + 2$, then, conversely, $p_{b+1,n} > p_{b,n}$. The second our result states that this is true for all possible values of $b, n$.

**Theorem 3** The following properties hold.

- If $n \geq 3b + 2$, then $p_{b+1,n} > p_{b,n}$.
- If $n \leq 3b + 1$, then $p_{b+1,n} < p_{b,n}$.

The rest of the paper is organized in the following way. In Section 2, we describe main tools. In Section 3, we prove Theorem 3. In Section 4, we prove Theorem 2.

## 2 Main tools

The crucial part of our proofs is a behaviour of the function $g(z) = (1 - z)^{b-1}z^{n-b}$ on $\Delta_{b,n} := [1 - (b + 1)/n, 1 - b/n]$. This is helpful since, as we show in Section 2.1, $g_{b,n} := \int_{\Delta_{b,n}} g(z)dz$ gives the major contribution to $p_{b+1,n} - p_{b,n}$ (a similar observation for $z_{b+1,n} - z_{b,n}$ is obtained in Section 2.2). In Section 2.3, we study an asymptotical behaviour of $g$.

This technique works for proving Theorem 2 when $c\sqrt{n} < b < n/2$ for an appropriate choice of $c > 0$. The case $b \geq n/2$ follows immediately from the observation that $z_{b,n} = 1 - z_{n-b,n}$ (see Section 4). For $b \leq c\sqrt{n}$, we exploit an asymptotical expansion of $z_{b,n}$ that is obtained in Section 2.4.

### 2.1 A useful expression for $p_{b,n}$

This section is devoted to the proof of the following result.

**Claim 1**

$$p_{b+1,n} - p_{b,n} = \frac{\left(\frac{b+1}{n}\right)^b (1 - \frac{b+1}{n})^{n-b} - b \int_{1-(b+1)/n}^{1-b/n} (1 - z)^{b-1}z^{n-b}dz}{b \int_0^{1} (1 - z)^{b-1}z^{n-b}dz}.$$
Proof. Let us rewrite the definition of $p_{b,n}$ in the following way:

$$p_{b,n} = \sum_{i=0}^{b-1} \binom{n}{b-1} \binom{b-1}{i} \frac{(n-b+1)!(b-1-i)!}{(n-i)!} \left( \frac{b}{n} \right)^i \left( 1 - \frac{b}{n} \right)^{n-i}.$$  

Since

$$\frac{(n-b)!(b-1-i)!}{(n-i)!} = \Gamma(n-b+1) \Gamma(b-i) \Gamma(n-i+1) = B(n-b+1, b-i) = \int_0^1 x^{n-b} (1-x)^{b-i-1} dx,$$

we get

$$p_{b,n} = n \binom{n-1}{b-1} \times \int_0^1 \left[ \sum_{i=0}^{b-1} \binom{b-1}{i} (1-x)^{b-i-1} \left( \frac{b}{n} \right)^i \left( 1 - \frac{b}{n} \right)^{n-i} \right] x^{n-b} \left( 1 - \frac{b}{n} \right)^{n-b+1} dx =$$

$$\frac{(n-1)!}{(n-b-1)!(b-1)!} \int_0^1 \left[ \frac{b}{n} + (1-x) \left( 1 - \frac{b}{n} \right) \right]^{b-1} x^{n-b} \left( 1 - \frac{b}{n} \right)^{n-b} dx =$$

$$\frac{n!}{(n-b)!(b-1)!} \int_0^1 \left[ 1 - x \left( 1 - \frac{b}{n} \right) \right]^{b-1} \left[ \frac{b}{n} \right]^{n-b} d \left[ x \left( 1 - \frac{b}{n} \right) \right] =$$

$$\frac{\Gamma(n+1)}{\Gamma(n-b+1) \Gamma(b)} \int_0^{1-b/n} (1-z)^{b-1} z^{n-b} dz = \frac{\int_0^{1-b/n} (1-z)^{b-1} z^{n-b} dz}{\int_0^1 (1-z)^{b-1} z^{n-b} dz}. \quad (1)$$

Therefore,

$$p_{b+1,n} - p_{b,n} = \frac{\int_0^{1-(b+1)/n} (1-z)^{b-1} z^{n-b} dz}{\int_0^1 (1-z)^{b-1} z^{n-b} dz} - \frac{\int_0^{1-b/n} (1-z)^{b-1} z^{n-b} dz}{\int_0^1 (1-z)^{b-1} z^{n-b} dz} =$$

$$= \frac{\int_0^{1-(b+1)/n} (1-z)^b d(z^{n-b})}{\int_0^1 (1-z)^b d(z^{n-b})} - \frac{\int_0^{1-b/n} (1-z)^b d(z^{n-b})}{\int_0^1 (1-z)^b d(z^{n-b})} =$$

$$= \frac{(b+1)^b (1 - \frac{b+1}{n})^{n-b} + b \int_0^{1-(b+1)/n} (1-z)^{b-1} z^{n-b} dz}{b \int_0^1 (1-z)^{b-1} z^{n-b} dz} - \frac{\int_0^{1-b/n} (1-z)^{b-1} z^{n-b} dz}{\int_0^1 (1-z)^{b-1} z^{n-b} dz} =$$

$$= \frac{(b+1)^b (1 - \frac{b+1}{n})^{n-b} - b \int_0^{1-(b+1)/n} (1-z)^{b-1} z^{n-b} dz}{b \int_0^1 (1-z)^{b-1} z^{n-b} dz}. \quad \square$$
2.2 A useful expression for $z_{b,n}$

Using (1), we get

$$z_{b,n} = \frac{1}{2} b \left( \int_{1-b/n}^1 - \int_0^{1-b/n} \right) g(z) dz. \quad (2)$$

Then

$$z_{b+1,n} - z_{b,n} = \frac{n^n}{2} \left( \int_{1-(b+1)/n}^1 - \int_0^{1-(b+1)/n} \right) (1-z)^b d(z^{n-b}) -$$

$$\frac{b \left[ \int_{1-b/n}^1 - \int_0^{1-b/n} \right] g(z) dz}{(b+1)^b (n-b-1)^{n-b-1}} = \frac{n^n}{2(n-b)} \times$$

$$\left( -2 \left( \frac{b+1}{n} \right)^b \left( \frac{n-b-1}{n} \right)^{n-b} + b \left[ \int_{1-(b+1)/n}^1 - \int_0^{1-(b+1)/n} \right] g(z) dz \right) -$$

$$b \left[ \int_{1-b/n}^1 - \int_0^{1-b/n} \right] g(z) dz =$$

$$b \left( 1 + \frac{1}{b} \right)^b \left( 1 - \frac{1}{n-b} \right)^{n-b-1} \left[ \int_{1-b/n}^1 - \int_0^{1-b/n} \right] g(z) dz \right). \quad (3)$$

2.3 Studying a behaviour of $g$

An inductive proof of the following observation is straightforward.

Claim 2 Let $\ell \in \{1, \ldots, \min\{b-1, n-b\}\}$. Then

$$\frac{\partial^\ell g}{\partial z^\ell} = (1-z)^{b-1-\ell} z^{n-b-\ell} \sum_{i=0}^\ell \binom{\ell}{i} (-1)^{\ell-i} z^{\ell-i} (n-1-i)! (n-b)! \frac{(n-b)!}{(n-\ell)! (n-b-i)!}.$$
Let, for every $\ell \in \mathbb{Z}_+$,
\[
g_\ell(z) = \frac{1}{\ell!} \left( z - 1 + \frac{b+1}{n} \right)^\ell \left[ \frac{\partial^\ell g}{\partial z^\ell} \left( 1 - \frac{b+1}{n} \right) \right]
\]
be the $\ell$-th term in the Taylor expansion of $g$, and $g_{\leq 3}(z) = \sum_{\ell=0}^{3} g_\ell(z)$. Finally, let
\[
d_4^-(z) = \frac{\partial^4 g}{\partial z^4} \left( 1 - \frac{b+1}{n} \right), \quad d_4^+(z) = \frac{\partial^4 g}{\partial z^4} \left( 1 - \frac{b}{n} \right).
\]
In the proof of Claim 3, we show that, for every $5 \leq b \leq n/2$, the functions $d_4^-$ and $d_4^+$ are the lower and the upper bounds of $\partial^4 g/\partial z^4$ on $\Delta_{b,n}$ respectively, and, thus,

**Claim 3** For $5 \leq b \leq n/2$ and all $z \in \Delta_{b,n}$,
\[
g_{\leq 3}(z) + g_4^+(z) \geq g(z) \geq g_{\leq 3}(z) + g_4^-(z),
\]
where
\[
g_4^-(z) = \frac{1}{24} \left( z - 1 + \frac{b+1}{n} \right)^4 d_4^-(z), \quad g_4^+(z) = \frac{1}{24} \left( z - 1 + \frac{b+1}{n} \right)^4 d_4^+(z).
\]

**Proof.** For every $\ell \in \{1, \ldots, \min\{b-1, n-b\}\}$, denote
\[
f_\ell = \frac{\partial^\ell g / \partial z^\ell}{(1 - z)^{b-1 - \ell} z^{n-b-\ell}}.
\]
It is easy to see that, for every $\ell$, $\partial f_{\ell+1} / \partial z = -(\ell + 1)(n - \ell - 1)f_\ell$, and
\[
f_2(z) = z^2(n-1)(n-2) - 2z(n-2)(n-b) + (n-b)(n-b-1)
\]
is negative on
\[
\Upsilon := \left( \frac{n-b}{n-1} - \frac{1}{n-1} \sqrt{\frac{(n-b)(b-1)}{n-2}}, \frac{n-b}{n-1} + \frac{1}{n-1} \sqrt{\frac{(n-b)(b-1)}{n-2}} \right).
\]
Let us show that $\Delta_{b,n} \subset \Upsilon$. First,
\[
1 - \frac{b+1}{n} > \frac{n-b}{n-1} - \frac{1}{n-1} \sqrt{\frac{(n-b)(b-1)}{n-2}}
\]
since the difference between the left side and right side of this inequality equals \( \sqrt{\frac{(n-b)(b-1)}{n-2}} - \frac{2n-b-1}{n} \), and

\[
n^2(n-b)(b-1) - (2n-b-1)^2(n-2) = n(b-5)(n(b-n) - b) + n(12n - 15b - 9) + 2b^2 + 2 + 4b > 0
\]
because 5 \(\leq\) b \(\leq\) n/2. Second,

\[
1 - \frac{b}{n} < \frac{n-b}{n-1} + \frac{1}{n-1} \sqrt{\frac{(n-b)(b-1)}{n-2}}
\]
since, in fact, \(1 - \frac{b}{n} < \frac{n-b}{n-1} \). So, \(f_3(z)\) increases on \(\Delta_{b,n}\).

Now, let us show that

\[
f_3(1 - \frac{b}{n}) = (n-b) \left[ \left(2 - 7\frac{n-b}{n}\right) \left(1 - \frac{n-b}{n}\right) + 6\frac{(n-b)^2}{n^3} \right] < 0.
\]
The derivative of the function \((2 - 7(x/n))(1 - (x/n)) + 6(x^2/n^3)\) with respect to \(x\) is negative when \(x < \frac{9n^2}{14n+12}\) and positive when \(x > \frac{9n^2}{14n+12}\). Therefore, negativeness of this function in \(x = \frac{2}{7}n + 1\) and \(x = n - 2\) implies its negativeness for all \(x \in \left[\frac{2}{7}n + 1, n - 2\right]\).

As \(f_3\) increases on \(\Delta_{b,n}\) and negative in \(1 - b/n\), \(f_3'(z)\) is positive on \(\Delta_b\). Moreover, the derivative of \((1 - z)^{b-5}z^{n-b-4}\) with respect to \(z\) is also positive on \(\Delta_b\). Thus, \(\partial^4 g/\partial z^4\) increases on \(\Delta_b\), and this implies the statement of Claim 3. \(\square\)

### 2.4 Behaviour of \(z_{b,n}\) for \(b = O(\sqrt{n})\)

It is very well known (see, e.g., [3]) that

\[
y_b = \frac{1}{3} + \frac{4}{135b} - \frac{8}{2835b^2} + O\left(\frac{1}{b^3}\right).
\]

From this and Stirling’s approximation

\[
b! = \sqrt{2\pi b}b^b e^{-b} \frac{1}{\sqrt{b}} + O(1/b^3)
\]
(see, e.g., [6]), it follows that

\[
P(\eta_b < b) = \frac{1}{2} - \frac{1}{3\sqrt{2\pi b}} - \frac{1}{540\sqrt{2\pi b}b}\sqrt{b} + O\left(\frac{1}{b^2\sqrt{b}}\right).
\]

For a non-negative integer \(\xi\), we denote \(\xi^{(s)} := \xi(\xi - 1) \ldots (\xi - s + 1)\).
Lemma 1 For every \( s \in \mathbb{N} \),
\[
E \left[ \eta_b^{(s)} I(\eta_b < b) \right] = b^s \left( P(\eta_b < b) - \frac{1}{\sqrt{2\pi b}} \left[ s - \frac{2s^3 - s}{12b} \right] \left[ 1 + O \left( \frac{1}{b^2 \sqrt{b}} \right) \right] \right).
\]

Proof.
\[
E \left[ \eta_b^{(s)} I(\eta_b < b) \right] = \sum_{i=0}^{b-1} \frac{i!}{(i-s)! i!} b^i e^{-b} = b^s \sum_{i=s}^{b-1} \frac{b^{i-s}}{(i-s)! i!} e^{-b} = b^s P(\eta_b < b - s) = b^s \left[ P(\eta_b < b) - P(b - s \leq \eta_b < b) \right].
\]

By the Stirling’s approximation,
\[
P(b - s \leq \eta_b < b) = \sum_{i=1}^{s} \frac{b^{i-s} e^{-b}}{(b-i)!} =
\]
\[
= \frac{1}{\sqrt{2\pi b}} \sum_{i=1}^{s} \left( 1 - \frac{i^2}{2b} - \frac{1}{12(b-i)} \right) \left( 1 + O \left( \frac{1}{b^2} \right) \right) =
\]
\[
= \frac{1}{\sqrt{2\pi b}} \left[ s - \frac{2s^3 - s}{12b} \right] \left( 1 + O \left( \frac{1}{b^2} \right) \right). \quad \square
\]

Since, for \( s, k \in \mathbb{N}, s < k \),
\[
\sum_{i=0}^{k} (-1)^i \binom{k}{i} i^{(s)} = 0,
\]

it is straightforward to see that (4) and Lemma 1 imply the following.

Claim 4 For every \( k \in \mathbb{N} \), set
\[
h_1^k = E \left[ (b - \eta_b)^k I(\eta_b < b) \right], \quad h_2^k = E \left[ \eta_b(b - \eta_b)^k I(\eta_b < b) \right].
\]

Then
\[
h_1^1 = \frac{\sqrt{b}}{\sqrt{2\pi}} - \frac{1}{12\sqrt{2\pi b}} + O(b^{-1}), \quad h_2^1 = b \left( \frac{1}{2} - \frac{1}{3\sqrt{2\pi b}} \right) + O(1), \quad h_3^1 = \frac{2b\sqrt{b}}{\sqrt{2\pi}} + O(b),
\]

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\[ h_k^1 = O(b^{k-2}) \text{ for all } k \geq 4; \]
\[ h_1^2 = \frac{b\sqrt{b}}{\sqrt{2\pi}} + O(b), \]
\[ h_k^2 = O(b^k) \text{ for all } k \geq 2. \]

Using Claim 4 we get the following asymptotical expansion of \( z_{b,n} \) when \( b = O(\sqrt{n}) \).

**Claim 5** Let \( C > 0 \) and \( n > Cb^2 \). Then
\[ z_{b,n} = \frac{1}{3} + \frac{4}{135b} + \frac{b}{3n} + O\left(\frac{1}{b\sqrt{b}}\right). \]

**Proof.** First, let us estimate \( p_b - p_{b,n} \). Let \( n > Cb^2 \). Then
\[ p_{b,n} = \sum_{i=0}^{b-1} \binom{n}{i} \left(\frac{b}{n}\right)^i \left(1 - \frac{b}{n}\right)^{n-i} = \]
\[ (1 + O(b^{-2})) \sum_{i=0}^{b-1} \frac{1}{i!} \frac{1}{(n-i)^{n-i}} \sqrt{\frac{n}{n-i}} b^i (n-b)^{n-i} = \]
\[ (1 + O(b^{-2})) \sum_{i=0}^{b-1} \frac{b^i}{i!} e^b \left(1 + \frac{i}{2n}\right) e^{-\frac{(b-i)^2}{2(n-i)}} \frac{(b-i)^3}{3(n-i)^2} = \]
\[ (1 + O(b^{-2})) \sum_{i=0}^{b-1} \frac{b^i}{i!} e^b \left(1 + \frac{i}{2n}\right) e^{-\frac{(b-i)^2}{2n} - \frac{(b-i)^3}{3n^2} + \frac{i}{2n}} \left(1 - \frac{(b-i)^3}{3n^2}\right). \]

Since, for any positive \( x \) and \( y \),
\[ 1 - x(1+y) < e^{-x(1+y)} < e^{-x} < 1 - x + \frac{x^2}{2}, \]
by Claim 4 we get
\[ p_{b,n} = (1 + O(b^{-2})) \sum_{i=0}^{b-1} \frac{b^i}{i!} e^b \left(1 + \frac{i}{2n} - \frac{(b-i)^2}{2n} - \frac{(b-i)^3}{3n^2}\right) = \]
\[ (1 + O(b^{-2})) \left( P(\eta_b < b) + \frac{1}{2n} \mathbb{E} \eta_b I(\eta_b < b) - \frac{1}{2n} h_2^1 - \frac{1}{3n^2} h_3^1 \right). \]

By (4), Lemma 1 and Claim 4, we get
\[ p_{b,n} = \frac{1}{2} - \frac{1}{3\sqrt{2\pi b}} - \frac{1}{540\sqrt{2\pi b\sqrt{b}}} - \frac{\sqrt{b}}{2n\sqrt{2\pi}} + O(b^{-2}). \]

Therefore,
\[ z_{b,n} = \frac{1}{2} - p_{b,n} \left( \frac{n}{b} \right) \left( 1 - \frac{b}{n} \right) = \sqrt{\frac{n - b}{n}} e^{\frac{1}{12} b} \left( \frac{1}{3} + \frac{1}{540b} + \frac{b}{2n} + O \left( \frac{1}{b\sqrt{b}} \right) \right) = \left( 1 - \frac{b}{2n} \right) \left( 1 + \frac{1}{12} b \right) \left( \frac{1}{3} + \frac{1}{540b} + \frac{b}{2n} \right) + O \left( \frac{1}{b\sqrt{b}} \right) = \frac{1}{3} + \frac{4}{135b} + \frac{b}{3n} + O \left( \frac{1}{b\sqrt{b}} \right). \square \]

3 Proof of Theorem 3

For \( b \leq 5 \), the proof is given in Appendix A. The remaining is divided into 3 parts:
- \( 6 \leq b \leq (n - 2)/3 \) (Section 3.1),
- \( (n - 1)/3 \leq b \leq n/2 \) (Section 3.2),
- \( b > n/2 \) (Section 3.3).

3.1 Small values of \( b \)

In this section, we assume that \( 6 \leq b \leq (n - 2)/3 \).

From Claim 3 we get
\[ g_{b,n} \leq g_{b,n}^+ := \sum_{\ell=0}^3 \frac{1}{(\ell + 1)!n^{\ell+1}} \frac{\partial^\ell g}{\partial z^\ell} \left( 1 - \frac{b + 1}{n} \right) + \frac{1}{120n^5} \frac{\partial^4 g}{\partial z^4} \left( 1 - \frac{b}{n} \right). \]

Let us prove that
\[ \left( \frac{b + 1}{n} \right)^b \left( 1 - \frac{b + 1}{n} \right)^{n-b} - bg_{b,n}^+ > 0, \]
which, due to Claim 1 immediately gives us the desired inequality \( p_{b+1,n} > p_{b,n} \). Since
\[ \left( \frac{b + 1}{n} \right)^b \left( 1 - \frac{b + 1}{n} \right)^{n-b} - bg_{b,n}^+ = \]
\[
\frac{1}{24n^6} \left( \frac{b + 1}{n} \right)^{b-4} \left( 1 - \frac{b + 1}{n} \right)^{n-b-2} P_{b,n} - \frac{b}{120n^5} \frac{\partial^4 g}{\partial z^4} \left( 1 - \frac{b}{n} \right),
\]
where \( P_{b,n} = 12b^5 - 16b^4n + 64b^4 + 4b^3n^2 - 71b^3n + 138b^3 + 16b^2n^2 - 112b^2n + 156b^2 + 12bn^2 - 105bn + 94b + 24n^2 - 48n + 24 \), the desired inequality follows from

\[
P_{b,n} > \frac{bn(\partial^4 g/\partial z^4) \left( 1 - \frac{b}{n} \right)}{5 \left( \frac{b+1}{n} \right)^{b-4} \left( 1 - \frac{b+1}{n} \right)^{n-b-2}} = \frac{bn^4}{5(b+1)(n-b-1)^2} \left( \frac{b}{b+1} \right)^{b-5} \left( \frac{n-b}{n-b-1} \right)^{n-b-4} \times \left[ 3(n-b)^2 \left( \frac{b}{n} \right)^2 - 2(n-b) \left( \frac{b}{n} \right) \left( 23 \left( 1 - \frac{b}{n} \right)^2 - 13 \left( 1 - \frac{b}{n} \right) + 3 \right) + 24 \left( 1 - \frac{b}{n} \right)^4 \right].
\]

As \( \ln(1 + x) \leq x \) for all \( x > -1 \), and \( e^x < 1 + 2x \) for all \( x \in (0, 1) \), the expression to the right is less than

\[
\frac{bn^4}{5(b+1)(n-b-1)^2} \left( 1 + \frac{12}{b+1} \right) \left( 3(n-b)^2 \left( \frac{b}{n} \right)^2 + 24 \left( 1 - \frac{b}{n} \right)^4 \right),
\]
so the inequality follows from

\[
5(b+1)^2(n-b-1)^2 P_{b,n} > 3bn^2(b+13)(b^2 + 8)(b-n)^2.
\]

The latter inequality is straightforward (however, for the sake of completeness, we give its proof in Appendix B).

### 3.2 Large values of \( b \)

In this section, we assume that \( (n-1)/3 \leq b \leq n/2 \).

From Claim 3, we get

\[
g_{b,n} \geq g_{b,n}^- := \sum_{\ell=0}^{3} \frac{1}{(\ell+1)!n^{\ell+1}} \frac{\partial^\ell g}{\partial z^\ell} \left( 1 - \frac{b+1}{n} \right) + \frac{1}{120n^5} \frac{\partial^4 g}{\partial z^4} \left( 1 - \frac{b+1}{n} \right).
\]

Here, our goal is to prove that

\[
\left( \frac{b+1}{n} \right)^b \left( 1 - \frac{b+1}{n} \right)^{n-b} - bg_{b,n}^- =
\]
\[
\frac{1}{24n^6} \left( \frac{b+1}{n} \right)^{b-4} \left( 1 - \frac{b+1}{n} \right)^{n-b-2} P_{b,n} - \frac{b}{120n^5 \partial z^4} \left( 1 - \frac{b+1}{n} \right) \]

is negative, where \( P_{b,n} \) is defined in the previous section. Denote \( Q_{b,n} := \)

\[
\left[ 3(n-b-1)^2 \left( \frac{b+1}{n} \right)^2 - 2(n-b-1) \left( \frac{b+1}{n} \right) \left( 23 \left( 1 - \frac{b+1}{n} \right)^2 + 7 \left( 1 - \frac{b+1}{n} \right) - 1 \right) 
+ 96 \left( 1 - \frac{b+1}{n} \right)^3 + 24 \left( 1 - \frac{b+1}{n} \right)^4 \right].
\]

Since
\[
\frac{\partial^4 g}{\partial z^4} \left( 1 - \frac{b+1}{n} \right) = \left( \frac{b+1}{n} \right)^{b-5} \left( 1 - \frac{b+1}{n} \right)^{n-b-4} \quad \text{\( Q_{b,n} > \)}
\]

\[
\left( \frac{b+1}{n} \right)^{b-5} \left( 1 - \frac{b+1}{n} \right)^{n-b-4} \times 
\left[ 3(n-b-1)^2 \left( \frac{b+1}{n} \right)^2 - 2(n-b-1) \left( \frac{b+1}{n} \right) \left( 23 \left( 1 - \frac{b+1}{n} \right)^2 + 7 \left( 1 - \frac{b+1}{n} \right) \right) \right] = 
\left( \frac{b+1}{n} \right)^{b-4} \left( 1 - \frac{b+1}{n} \right)^{n-b-2} (3bn + 46b - 57n + 46),
\]

using the same simple techniques as in the previous case, we get that the negativeness of (6) follows from

\[
5P_{b,n} < bn(3bn + 46b - 57n).
\]

This inequality is proven in Appendix B.

3.3 Above \( n/2 \)

Finally, let us consider \( b > n/2 \). By the definition,

\[
p_{b,n} = 1 - \binom{n}{b} \left( \frac{b}{n} \right)^b \left( 1 - \frac{b}{n} \right)^{n-b} - p_{n-b,n}.
\]

Thus,

\[
p_{b+1,n} - p_{b,n} = \frac{(n-1)!}{b!(n-b-1)!} \frac{(b+1)^b(n-b-1)^{n-b-1}}{n^{n-1}} B + (p_{n-b,n} - p_{n-b-1,n}).
\]
where

\[
B = \left(1 - \frac{1}{b+1}\right)^b \left(1 + \frac{1}{n-b-1}\right)^{n-b-1} - 1 < e^{-\frac{1}{2(b+1)^2}} - \frac{1}{2(n-b-1)^2} - 1 < 0
\]

for all \(b \geq (n+1)/2\) and \(n \geq 6\). So, for \((n+1)/2 < b < 2n/3\), \(p_{b+1,n} < p_{b,n}\).

Let \(n - 6 \geq b \geq 2n/3\). Set \(\tilde{b} = n - b - 1\). In this case \(5 \leq \tilde{b} < n/3\). It makes it possible to apply, for \(\tilde{b}\), Claim 3 and, therefore, inequality (7). From Claim 1, (7) and (10), we get

\[
p_{b+1,n} - p_{b,n} = \\
\left(\begin{array}{c}
n \\
\tilde{b}
\end{array}\right) \times \left[\left(1 + \frac{n - \tilde{b}}{n - \tilde{b} - 1} - \left(\frac{\tilde{b}}{\tilde{b} + 1}\right)^\tilde{b} \left(\frac{n - \tilde{b}}{n - \tilde{b} - 1}\right)^{n-\tilde{b}}\right) \left(\frac{\tilde{b} + 1}{n}\right)^{\tilde{b}} \left(1 - \frac{\tilde{b} + 1}{n}\right)^{n-\tilde{b}} \right. \\
\left. -\tilde{b} \int_{1-(\tilde{b}+1)/n}^{1-\tilde{b}/n} (1-z)^{\tilde{b}-1} z^{n-\tilde{b}} dz \right]< \\
\left(\frac{\tilde{b} + 1}{n}\right)^{\tilde{b}-4} \left(1 - \frac{\tilde{b} + 1}{n}\right)^{n-\tilde{b}-2} \left(\frac{n}{\tilde{b}}\right) \times \left[\frac{1}{24n^6} P_{\tilde{b},n} - \frac{\tilde{b}}{120n^5} (3\tilde{b}n + 46\tilde{b} - 57n + 46) + \right. \\
\left. \left(\frac{n - \tilde{b}}{n - \tilde{b} - 1} - \left(\frac{\tilde{b}}{\tilde{b} + 1}\right)^\tilde{b} \left(\frac{n - \tilde{b}}{n - \tilde{b} - 1}\right)^{n-\tilde{b}}\right) \left(\frac{\tilde{b} + 1}{n}\right)^{\tilde{b}} \left(1 - \frac{\tilde{b} + 1}{n}\right)^{n-\tilde{b}} \right]\]

Since \(\ln(1-x) > -x - x^2/2 - x^3/2\) for \(x \in (0, 1/2]\), and \(\ln(1+x) > x - x^2/2\) for positive \(x\), we get

\[
\frac{n - \tilde{b}}{n - \tilde{b} - 1} - \left(\frac{\tilde{b}}{\tilde{b} + 1}\right)^\tilde{b} \left(\frac{n - \tilde{b}}{n - \tilde{b} - 1}\right)^{n-\tilde{b}} < -\frac{1}{2(\tilde{b} + 1)} + \frac{n - \tilde{b}}{2(n - \tilde{b} - 1)^2}.
\]

Therefore, negativeness of (10) follows from

\[
5p_{\tilde{b},n} - \tilde{b}n(3\tilde{b}n + 46\tilde{b} - 57n) + 60(\tilde{b} + 1)^3(3n\tilde{b} - 2\tilde{b}^2 + 3n - 3\tilde{b} - n^2 - 1) < 0,
\]

which is verified in Appendix B.

We provide the detailed proof of the case \(b = n - 5\) in Appendix C, and all the other cases \(b \in \{n - 4, n - 3, n - 2, n - 1\}\) can be easily solved in the same way.
4 Proof of Theorem 2

The proof is divided into 4 parts:

• \((n - 1)/2 \geq b > (1 + \varepsilon)\sqrt{\frac{31}{60}n}\) (Section 4.1),

• \((1 + \varepsilon)\sqrt{\frac{31}{60}n} \geq b > (1 + \varepsilon)\sqrt{\frac{77}{360}n}\) (Section 4.2),

• \(b < (1 - \varepsilon)\sqrt{\frac{77}{360}n}\) (Section 4.3), and

• \(b \geq n/2\) (Section 4.4), that easily follows from the first three.

4.1 Large values of \(b\)

In this section, we assume that \((n - 1)/2 \geq b > (1 + \varepsilon)\sqrt{\frac{31}{60}n}\).

If \(b = (n - 1)/2\), then, from Theorem 3, for \(n\) large enough,

\[
\frac{z_{b+1,n}}{z_{b,n}} = \frac{1/2 - p_{b+1,n}}{1/2 - p_{b,n}} \frac{b(b - 1)^{n-b}}{(b + 1)(n - b - 1)^{n-b-1}} = \frac{1/2 - p_{b+1,n}}{1/2 - p_{b,n}} > 1.
\]

Now, let \((n - 1)/2 > b > (1 + \varepsilon)\sqrt{\frac{31}{60}n}\). Then

\[
b \ln \left(1 + \frac{1}{b}\right) + (n - b - 1) \ln \left(1 - \frac{1}{n - b}\right) < \]

\[
b \left(\frac{1}{b} - \frac{1}{2b^2} + \frac{1}{3b^3}\right) + (n - b - 1) \left(\frac{1}{n - b} - \frac{1}{2(n - b)^2}\right) - \frac{n - 2b}{2b(n - b)} + \frac{1}{2(n - b)^2} + \frac{1}{3b^2} < 0. \tag{12}
\]

From Theorem 1 and (2),

\[
\left[\int_{1-b/n}^{1} - \int_{0}^{1-b/n}\right] g(z)dz \geq \frac{2}{3b} (b/n)^b (1 - b/n)^{n-b}.
\]

By (3), for \(n\) large enough, we get

\[
\frac{(n - b)(b + 1)^b(n - b - 1)^{n-b-1}}{n^n} (z_{b+1,n} - z_{b,n}) \geq b g_{b,n} -
\]
\[
\left(\frac{b+1}{n}\right)^b \left(\frac{n-b-1}{n}\right)^{n-b} + \frac{1}{3} \left[1 - \left(1 + \frac{1}{b}\right)^b \left(1 - \frac{1}{n-b}\right)^{n-b-1}\right]\left(\frac{b}{n}\right)^b \left(1 - \frac{b}{n}\right)^{n-b} > b_g_{b,n} - \left(\frac{b+1}{n}\right)^b \left(\frac{n-b-1}{n}\right)^{n-b}
\]
\[
\left(1 - \frac{1}{3} \left(e^{\left(-\frac{1}{2(b+1)}\right)} - \frac{1}{3(b+1)^2} - \frac{1}{3(b+1)^3}\right) \right) + (n-b) \left(\frac{1}{n-b-1} - \frac{1}{2(n-b-1)^2}\right)
\]
\[
\left(\frac{n-b-1}{n}\right)^{n-b} \left(1 - \frac{1}{6(b+1)} - \frac{1}{18(b+1)^2} - \frac{1}{6(n-b-1)} + \frac{1}{6(n-b-1)^2}\right).
\]

Denote
\[
A_{b,n} := (n-b) \left(\frac{b+1}{n}\right)^5 \left(1 - \frac{b+1}{n}\right)^3.
\]

Using Claim 3 and (6), we get
\[
A_{b,n}(z_{b+1,n} - z_{b,n}) > \frac{b}{120n^4} Q_{b,n} - \frac{1}{24n^5} \left(\frac{b+1}{n}\right)^2 P_{b,n} + 
\]
\[
\left(\frac{b+1}{n}\right)^4 \left(\frac{n-b-1}{n}\right)^4 \left(\frac{1}{6} + \frac{1}{18(b+1)}\right) - \left(\frac{b+1}{n}\right)^5 \left(\frac{n-b-1}{n}\right)^3 \left(\frac{1}{6} + \frac{1}{6(n-b-1)}\right) ,
\]

where \(Q_{b,n}\) is defined in Section 3.2. Denote \(x := (b+1)/n\). It is easy to see that, for every \(\beta > 0\), there exists \(\delta > 0\) such that, for all \(n\) large enough and \(b > (1 + \varepsilon)\sqrt{\frac{31}{60}} n\),
\[
A_{b,n}(z_{b+1,n} - z_{b,n}) > (1 - \beta) \frac{1}{6} x^5 (1-x)^3 \geq 0, \quad \text{if } x \geq \delta,
\]
\[
A_{b,n}(z_{b+1,n} - z_{b,n}) > (1 - \beta) \left[\frac{1}{6} x^5 - \frac{31}{360} x^3\right] \geq 0, \quad \text{if } x < \delta.
\]

4.2 Medium values of \(b\)

Let \((1 + \varepsilon)\sqrt{\frac{31}{60}} n \geq b > (1 + \varepsilon)\sqrt{\frac{77}{360}} n\).
In this case, the bounds (12) are also true. By Claim 5, for every \( \gamma > 0 \) and \( n \) large enough,
\[
z_{b,n} \geq \frac{1}{3} + \left[ \frac{4}{135b} + \frac{b}{3n} \right] (1 - \gamma).
\]
Thus, for such \( n \), by (2),
\[
\left[ \int_{1-b/n}^{1} - \int_{0}^{1-b/n} \right] g(z) dz \geq \left( \frac{2}{3b} + \left[ \frac{8}{135b^2} + \frac{2}{3n} \right] (1 - \gamma) \right) (b/n)^b (1 - b/n)^{n-b}.
\]
Below, we use the same notations \( x \) and \( A_{b,n} \) as in the previous case. In the same way as in the previous case, from Claim 3, (3) and (6), we get that, for every \( \tilde{\gamma} > \gamma \) and \( n \) large enough,
\[
A_{b,n}(z_{b+1,n} - z_{b,n}) > \frac{b}{120n^4} Q_{b,n} - \frac{1}{24n^5} \frac{b+1}{n} \left( 1 - \frac{b+1}{n} \right)^2 P_{b,n} + \left( \frac{b+1}{n} \right)^4 \left( \frac{n-b-1}{n} \right)^4 \left( \frac{1}{6} + \left( \frac{1}{18} + \frac{2}{135} (1 - \tilde{\gamma}) \right) \frac{1}{b+1} \right) + \frac{1}{6} \left( \frac{b+1}{n} \right)^5 \left( \frac{n-b-1}{n} \right)^4 (1 - \tilde{\gamma}) - \left( \frac{b+1}{n} \right)^5 \left( \frac{n-b-1}{n} \right)^3 \left( \frac{1}{6} + \frac{1}{6(n-b-1)} \right).
\]
Therefore, for every \( \beta > 0 \), large enough \( n \) and \( (1 + \varepsilon) \sqrt{\frac{31}{60}} n \geq b > (1 + \varepsilon) \sqrt{\frac{77}{360}} n \),
\[
A_{b,n}(z_{b+1,n} - z_{b,n}) > (1 - \beta) \left[ \frac{x^5}{3} - \frac{77x^3}{1080n} \right] > 0.
\]

### 4.3 Small values of \( b \)

Let \( b < (1 - \varepsilon) \sqrt{\frac{77}{360}} n \). For constant \( b \) and large enough \( n \), \( z_{b+1,n} < z_{b,n} \) since \( y_{b+1} < y_b \). Below, we consider \( b \) as large as desired.

In this case, the bounds (12) are also true. By Claim 5 and (3), for every \( \gamma > 0 \) and \( n \) large enough,
\[
\left[ \int_{1-b/n}^{1} - \int_{0}^{1-b/n} \right] g(z) dz \leq \left( \frac{2}{3b} + \left[ \frac{8}{135b^2} + \frac{2}{3n} \right] (1 + \gamma) \right) (b/n)^b (1 - b/n)^{n-b}.
\]

Note that, here, we should use \( g_{b,n}^+ \) instead of \( g_{b,n}^- \), but the contribution of the difference between them in (13) is at most \( O(x^2/n^2) \), and so, for some constant \( c \) and every positive \( \beta \),
\[
A_{b,n}(z_{b+1,n} - z_{b,n}) < (1 + \beta) \left[ \frac{x^5}{3} - \frac{77x^3}{1080n} + \frac{x^2}{n^2} \right] < 0
\]
for $n$ large enough.

4.4 Above $n/2$

Finally, let us consider $b \geq n/2$. From (9), we get

$$z_{b,n} = 1 - \frac{1}{2} - \frac{(\binom{n}{b})(b/n)^b(1-b/n)^{n-b} - p_{n-b,n}}{(\binom{n}{b})(b/n)^b(1-b/n)^{n-b}} = 1 - z_{n-b,n}.$$  

Therefore, $z_{b+1,n} > z_{b,n}$ if and only if $z_{n-b+1,n} < z_{n-b,n}$.

Theorem is proved.

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Appendix A. Proof of Conjecture 3 for $b \leq 5$

Claim 6 Let $b \leq 5$. Then

\[ p_{b,n} < p_{b+1,n} \text{ if } n \geq 3b + 2; \]
\[ p_{b,n} > p_{b+1,n} \text{ if } b + 1 \leq n \leq 3b + 1. \]

Remark. All the computations that are skipped below can be worked out using a symbolic computation system such as Mathematica.

Proof. We will provide a detailed proof of Claim 6 for $b = 5$ as the most laborious case and outline similar steps for $b \in \{3, 4\}$. Specifically, for $b = 5$, we want to prove that, for all $n \geq 17$:

\[
n^np_{5,n} = (n - 5)^n + 5n(n - 5)^{n-1} + 25\frac{n(n - 1)}{2}(n - 5)^{n-2} + 125\frac{n(n - 1)(n - 2)}{6}(n - 5)^{n-3} + 625\frac{n(n - 1)(n - 2)(n - 3)}{24}(n - 5)^{n-4} < \\
(n - 6)^n + 6n(n - 6)^{n-1} + 18n(n - 1)(n - 6)^{n-2} + 36n(n - 1)(n - 2)(n - 6)^{n-3} + 54n(n - 1)(n - 2)(n - 3)(n - 6)^{n-4} + \frac{324}{5}n(n - 1)(n - 2)(n - 3)(n - 4)(n - 6)^{n-5} = p_{6,n}n^n. \]

For $b \in \{3, 4, 5\}$, consider the difference $n^n(p_{b+1,n} - p_{b,n})$ and divide it by $(n-b)^n$. Since

\[
\left(1 - \frac{1}{n-b}\right)^{n-b} > e^{-1}\left(1 - \frac{1}{2(n-b)} - \frac{1}{(n-b)^2}\right),
\]

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\[ p_{6,n} > p_{5,n} \] follows from
\[
\frac{899}{5} - \frac{523e}{8} + \frac{20531 \cdot 6 - 9025 \cdot 5e}{60(n-5)} + \frac{80527 \cdot 4 - 24625 \cdot 5e}{40(n-5)^2} + \frac{11009 \cdot 12 - 63125e}{12(n-5)^3} - \frac{20529 - 3125 \cdot 5e}{5(n-5)^4} - \frac{217291}{10(n-5)^5} - \frac{156627}{10(n-5)^6} - \frac{3125}{(n-5)^7} > 0,
\]

\[ p_{5,n} > p_{4,n} \] similarly follows from
\[
\frac{523}{8} - \frac{71e}{3} + \frac{21979 - 168 \cdot 48e}{48(n-4)} + \frac{21683 - 1120 \cdot 8e}{24(n-4)^2} + \frac{61 - 256 \cdot 48e}{48(n-4)^3} - \frac{18091}{12(n-4)^4} - \frac{14723}{12(n-4)^5} - \frac{256}{(n-4)^6} > 0,
\]

and \[ p_{4,n} > p_{3,n} \] follows from
\[
\frac{142 - 51e}{6} + \frac{511 - 189e}{6(n-3)} - \frac{162 - 217}{6(n-3)^2} - \frac{745}{6(n-3)^3} - \frac{731}{6(n-3)^4} - \frac{27}{(n-3)^5} > 0.
\]

To prove these inequalities we consider separately two parts of the summations: the first part contains the constant term and a fraction of next one, and the second part contains all the rest. In particular, when \( b = 5 \), we prove that, for an appropriate choice of \( C > 0 \), the following inequalities hold true:
\[
\frac{899}{5} - \frac{523e}{8} + \frac{20531 \cdot 6 - 9025 \cdot 5e}{60(n-5)} - C > 0.
\]  
(14)

\[
C + \frac{3(80527 \cdot 4 - 24625 \cdot 5e)}{2(n-5)} + \frac{5(11009 \cdot 12 - 63125e)}{(n-5)^2} - \frac{12(20529 - 3125 \cdot 5e)}{(n-5)^3} - \frac{6 \cdot 217291}{(n-5)^4} - \frac{6 \cdot 156627}{(n-5)^5} - \frac{60 \cdot 3125}{(n-5)^6} > 0,
\]  
(15)

If so, summation of (14) with (15) divided by 60(n-5) gives the former inequality.

It is easy to see that, for \( C := 2300 \), both functions on the left sides of (14), (15) increase in \( n \) and are positive for \( n = 20 \). So, for \( b = 5 \) and \( n \geq 20 \), this finishes the proof. In the remaining three cases, it can be computed that
\[
p_{5,17}^{17} < 3.387 \cdot 10^{20} < 3.389 \cdot 10^{20} < p_{6,17}^{17},
\]
\[
p_{5,18}^{18} < 1.619 \cdot 10^{22} < 1.622 \cdot 10^{22} < p_{6,18}^{18},
\]

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p_{5,19}19^{19} < 8.176 \cdot 10^{23} < 8.199 \cdot 10^{23} < p_{6,19}19^{19}.

It remains to check manually that \( p_{5,n} > p_{6,n} \) for every \( n \leq 16 \) (e.g., \( p_{5,16}16^{16} > 7.505 \cdot 10^{18} > 7.503 \cdot 10^{18} > p_{6,16}16^{16} \)), and we skip these simple computations. For the cases \( b = 3 \) and \( b = 4 \) our method proves that \( p_{b,n} < p_{b+1,n} \) for \( n \geq 13 \) and \( n \geq 17 \) (as opposed to claimed \( n \geq 11 \) and \( n \geq 14 \)) respectively. The proof for the rest values of \( n \) can be obtained by the same numerical approach. For \( b = 1 \), the statement is trivial and, for \( b = 2 \), it also can be easily checked by the reader. □

Appendix B. Proof of inequalities in Section 3

Proof of inequality (5)

In this appendix, we prove inequality (5) for \( 6 \leq b \leq \frac{n-2}{3} \).

Set

\[ Q_{b,n} = 156b^2 + 12bn^2 - 105bn + 94b + 24n^2 - 48n + 24, \]

and so

\[ P_{b,n} = 12b^5 - 16b^4n + 64b^4 + 4b^3n^2 - 71b^3n + 138b^3 + 16b^2n^2 - 112b^2n + Q_{b,n}. \]

Surely, for \( 6 \leq \frac{n-2}{3}, 3n^2 - 48n \) is positive. Moreover, \( 16(3b - n)^2 + 5(b - n)^2 = 149b^2 - 106bn + 21n^2 \). Therefore, \( Q_{b,n} > 0 \).

From this,

\[ P_{b,n} > b^2(12b^3 - 16b^2n + 64b^2 + 4bn^2 - 71bn + 138b + 16n^2 - 112n). \]

(16)

Now, let us assume that \( b \geq 39 \).

Since, for \( 39 \leq b \leq \frac{n-2}{3} \), it is true that \((b + 1)^2(n - b - 1)^2 > (b^2 + 8)(b - n)^2\), it is enough to prove that

\[ 5b(12b^3 - 16b^2n + 64b^2 + 4bn^2 - 71bn + 138b + 16n^2 - 112n) - 3n^2(b + 13) > 0. \]

For this polynomial of \( n \) to be positive for all (real) \( n \geq 3b + 2 \), its roots must be less than \( 3b + 2 \). This leads to the inequality \( \frac{B + \sqrt{B^2 - 4AC}}{2A} < 3b + 2 \), where \( A = 20b^2 + 77b - 39, B = -80b^3 - 355b^2 - 560b, C = 60b^4 + 320b^3 + 690b^2 \). The latter inequality is equivalent to

\[ 4(20b^2 + 77b - 39)(28b^3 - 1047b^2 - 1280b - 156) > 0, \]

which is true for \( b \geq 39 \).
Now, let us prove the inequality (5) for \( 6 \leq b \leq 38, \ n \geq 158 \). Since \( b^2 + 8 < (b + 1)^2 \), it is enough to show that

\[
P_{b,n} > \frac{3}{5} b(b + 13)n^2 \left( 1 + \frac{1}{n - b - 1} \right)^2.
\]  

(17)

Since, in the considered range of the parameters,

\[
\frac{3b + 13}{5b} \left( 1 + \frac{1}{n - b - 1} \right)^2 < 2,
\]

the inequality (17) follows from (16) and

\[
12b^3 - 16b^2n + 64b^2 + 4bn^2 - 71bn + 138b + 16n^2 - 112n > 2n^2
\]

(the latter is straightforward since \( n \geq 4b + 6 \)).

In all the remaining cases (\( 6 \leq b \leq 39 \) and \( 3b + 2 \leq n \leq 157 \)), the inequality (5) can be easily verified by computer.

**Proof of inequality (8)**

Here, \( \frac{n-1}{3} \leq b < \frac{n}{2} \). We want to show that \( 5P_{b,n} - bn(3bn + 46b - 57n) < 0 \), or, as in the proof of inequality (3), that the segment \([2b, 3b + 1]\) lies inside \((n_1, n_2)\), where \( n_1 \) and \( n_2 \) are the smallest and the biggest roots of this polynomial of \( n \) respectively. Since \( n_2 > \frac{-B + \sqrt{B^2 - 4AC}}{2A} \), where \( A = 20b^2 + 77b + 129 \), \( B = -80b^3 - 355b^2 - 606b \), \( C = 60b^4 + 320b^3 + 690b^2 + 830b \), the inequality \( n_2 > 3b + 1 \) follows from

\[
4(20b^2 + 77b + 129)(12b^3 - 160b^2 - 1075b - 129) > 0,
\]

which is true for \( b \geq 19 \).

Similarly it can be proven that \( n_1 < 2b \) for \( b \geq 19 \), and the remaining cases (\( 6 \leq b \leq 19, 2b \leq n \leq 3b + 1 \)) can be verified by computer.

**Proof of inequality (11)**

Here, \( 6 \leq \tilde{b} < \frac{n}{3} \). We should prove that \( An^2 + Bn + C < 0 \) where

\[
A = -40(\tilde{b} + 1)^3 + 27(\tilde{b} + 1)^2 + 3(\tilde{b} + 1) + 70,
\]

\[
B = 100(\tilde{b} + 1)^4 - 35(\tilde{b} + 1)^3 - 21(\tilde{b} + 1)^2 - 78(\tilde{b} + 1) - 26,
\]

\[
C = -60(\tilde{b} + 1)^5 + 80(\tilde{b} + 1)^4 + 10(\tilde{b} + 1)^3 + 30(\tilde{b} + 1)^2.
\]

The inequality is true, if the biggest root of the polynomial is less than \( 3\tilde{b} \), or, equivalently, \( 9Ab^2 + 3bB + C < 0 \) which is clearly true for the considered values of \( \tilde{b} \).
Appendix C. Proof of Theorem 3 for \( b = n - 5 \)

Here we will prove that \( p_{n-4,n} < p_{n-5,n} \) for \( n \geq 11 \) (other cases are considered in Appendix A). From (10), we get

\[
n^n(p_{n-4,n} - p_{n-5,n}) = \frac{n(n-1)(n-2)(n-3)(n-4)}{24} (5^n(n-5)^{n-5} - 4^n(n-4)^{n-5}) + n^n(p_{5,n} - p_{4,n}).
\]

By dividing both parts by \((n-4)^n\), substituting

\[
\sum_{i=0}^{4} \binom{n}{i} (5/n)^i (1-5/n)^{n-i} \quad \text{and} \quad \sum_{i=0}^{3} \binom{n}{i} (4/n)^i (1-4/n)^{n-i}
\]

in \( p_{5,n} \) and \( p_{4,n} \) respectively and applying the inequality

\[
\left(1 - \frac{1}{n-4}\right)^k < e^{-1} \left(1 - \frac{k-n+4}{n-4}\right)
\]

for \( n - 5 \leq k \leq n \), we get

\[
\frac{n^n}{(n-4)^n}(p_{n-4,n} - p_{n-5,n}) < \frac{1097}{12e} - \frac{103}{3} + \frac{18649/(24e) - 824/3}{n-4} + \frac{4705/(2e) - 2240/3}{(n-4)^2} + \frac{832/3 - 5225/(8e)}{(n-4)^3} + \frac{24625/(12e) - 256}{(n-4)^4} + \frac{625/e}{(n-4)^5}.
\]

The bound is negative for \( n = 28 \). Moreover, all the terms except the constant are positive for \( n > 4 \), which implies the negativeness of the bound for all \( n \geq 28 \). Other cases (11 \( \leq n \leq 27 \)) can be easily verified by computer.