A Note on the Extension of the Polar Decomposition for the Multidimensional Burgers Equation

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It is shown that the generalizations to more than one space dimension of the pole decomposition for the Burgers equation with finite viscosity $\nu$ and no force are of the form $u = -2\nu \nabla \log \rho$, where the $\rho$’s are explicitly known algebraic (or trigonometric) polynomials in the space variables with polynomial (or exponential) dependence on time. Such solutions have polar singularities on complex algebraic varieties.

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\textbf{I. INTRODUCTION}

We are interested in the Burgers equation in $\mathbb{R}^n$

$$u_t + u \cdot \nabla u = \nu \nabla^2 u, \quad u = -\nabla \phi,$$

where $u_t$ is the partial derivative of $u$ with respect to time $t$, and $\nu$ is viscosity ($\nu > 0$). This equation is the simplest evolutionary dissipative equation, which is minimally (quadratically) nonlinear and enjoys translational and Galilean invariance. This simplicity and generality of the equation (1) explains its applicability for seemingly different processes occurring in a wide range of physical phenomena. Although originally this equation appeared as a model for Navier–Stokes turbulence \cite{1}, it is mostly used today in cosmology \cite{2}, polymer physics \cite{3}, and nonlinear acoustics \cite{4}. Also this equation is very useful as a testing ground for numerical schemes in hydrodynamics \cite{5}. These features make the Burgers equation important and attractive for physicists.

From the mathematical point of view the Burgers equation is also remarkable, for it is completely integrable \cite{5}, i.e. reducible to a linear problem after equivalent transformation. This follows directly from the Cole-Hopf transformation \cite{6}:

$$\phi = -2\nu \log \theta,$$

which maps (1) into the linear heat equation for the scalar field $\theta$, namely

$$\theta_t = \nu \nabla^2 \theta,$$

from which the solution to the Burgers equation (1) can be obtained explicitly by quadrature.

With a few exceptions, completely integrable PDEs are two dimensional (one dimension for time and one dimension for space) \cite{5}. There is a probable underlying mathematical reason, which hinders integrability in more than one spatial dimension: loosely speaking, it is related to the fact that polynomials with respect to more than one variable generally are not factorizable into non-trivial factors, while in the case of one variable they always are, by virtue of the main theorem of algebra. However, the Burgers equation \cite{4} is integrable in arbitrary number of dimensions. This is obvious, because the Cole-Hopf transformation \cite{6}, which maps (1) to (2), is valid in $\mathbb{R}^n$ for an arbitrary natural $n$. Our goal here is to describe two classes of finite-dimensional exact solutions of the multidimensional Burgers equation (1), which are extensions of the “pole decomposition” of the (1+1)-dimensional Burgers equation (1).

The pole decomposition is a property of PDEs (or integro-PDEs) to have finite dimensional solutions whose degrees of freedom are movable singularities (poles) in the complex plane. Most, if not all, completely integrable models enjoy this property \cite{5}. The most notable examples of pole decomposition in integrable systems can be found in Refs. \cite{8}, \cite{10}–\cite{12}. As said above, the Burgers equation (1) in (1+1)-dimension also enjoys the pole decomposition \cite{5}. (See also \cite{13}, \cite{14} for recent results in this direction.) Namely, it admits “polar” solutions in the form

$$u(t, x) = -2\nu \sum_{k=1}^N \frac{1}{x - z_k(t)},$$

where the poles constitute an $N$-dimensional dynamical system:

$$\frac{dz_l}{dt} = -2\nu \sum_{k \neq l}^N \frac{1}{z_l - z_k}.$$
It is easily checked by substitution into the heat equation that the time dependence of the \( b_K \)'s is

\[
b_K(t) = \sum_{p=0}^{M'} b_{K+2p} (\nu t)^{p_1+p_2+\ldots+p_n},
\]

where \( M' = (m'_1, m'_2, \ldots, m'_n) \), and \( m'_i \) is either \( m_i \) or \( m_i - 1 \) depending on whether \( m_i - k_i \) in (8) is even or odd.

Using (6) we find that our polynomial solutions generate the following rational solutions to the Burgers equation

\[
u_p(t, x) = -2\nu \frac{\sum_{K=0}^{M} \sigma(K) k_p \prod_{l=1}^{n} x_k^l}{x_p \sum_{K=0}^{M} \sigma(K) \prod_{l=1}^{n} x^l},
\]

where \( u_p \) is the \( p \)'th component of the vector field \( u \). Another closely related class of solutions involves trigonometric polynomials

\[
\begin{equation}
P(t, x) = Re \sum_{K=0}^{M} c_K(t) \prod_{l=1}^{n} e^{ik_l x_l}
\end{equation}
\]

with the same \( K \) and \( M \) as in (8). The time-dependence of \( c_K \) is now given by

\[
c_K(t) = c_{K0} \exp \left( -\nu \sum_{l=1}^{n} k_l^2 t \right).
\]

By (6) this generates the following solutions to the Burgers equation

\[
u_p(t, x) = 2\nu \frac{Im \sum_{K=0}^{M} c_K(t) k_p \prod_{l=1}^{n} e^{ik_l x_l}}{Re \sum_{K=0}^{M} c_K(t) \prod_{l=1}^{n} e^{ik_l x_l}}.
\]

Thus we have shown that the Burgers equation (1) in \( \mathbb{R}^n \) possesses exact solutions with a finite number of time dependent parameters generated by the algebraic and trigonometric polynomial solutions of the heat equation in \( \mathbb{R}^n \).

We observe that such solutions, contrary to the one-dimensional case, cannot in general be decomposed into a sum of separate simpler solutions. Indeed, this would correspond to having at all time polynomial solutions of the heat equation which are factorized. Even if the initial polynomial is factorized, the time evolution will in general destroy the factorization. Recently, special solutions possessing the all-time factorization property were found by D. Leshchiner and one of the authors (M. M-W.). We do not yet know how broad is the class of such solutions.

A final remark concerns integrability and explicit characterization of singularities. Knowing explicitly the coefficients of the polynomial solution of the heat equation does not imply that we can explicitly describe the algebraic variety on which the polynomial vanishes. Even in one dimension, if we have a pole decomposition with
more than four poles, we conjecture that Galois theory implies the following: given the initial position, in general it is not possible to find the positions for all times by radicals.

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