THE GROUP OF AFFINE TRANSFORMATIONS OF HOMOGENEOUS SPACES WITH DISCRETE ISOTROPY

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Abstract. We present a method to compute the group of affine transformations of a homogeneous \(G\)-space under specific conditions: when the group \(G\) and the homogeneous \(G\)-space are affine, the natural projection is affine, and the isotropy group \(H\) is discrete. Notably, in many cases one will only need a bi-invariant affine structure on the Lie group \(G\), as we provide conditions to construct an invariant affine structure on the homogeneous \(G\)-space so that the projection is affine. As an application, we calculate the group of the affine transformations of orientable flat affine surfaces and 3-dimensional affine tori.

Keywords: Flat affine manifolds, Affine transformations, Homogeneous spaces, Reductive homogeneous spaces, Invariant connections, Flat affine surfaces, 3-dimensional torus.

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1. Preliminaries

This paper deals with affine transformations of connected manifolds endowed with a linear connection. Throughout this work manifolds are assumed to be connected, real, finite dimensional, and without boundary (unless otherwise specified). Given a manifold \(M\) endowed with a linear connection \(\nabla\), we define an affine transformation of \((M, \nabla)\) as a diffeomorphism \(F\) of \(M\) preserving \(\nabla\), that is, verifying \(F_* \nabla_X Y = \nabla_{F_* X} F_* Y\), for all \(X, Y \in \mathfrak{X}(M)\), where \(\mathfrak{X}(M)\) is the space of smooth vector fields on \(M\). The set of diffeomorphisms preserving \(\nabla\) will be denoted by \(\text{Aff}(M, \nabla)\) and it is known that under the open-compact topology and composition \(\text{Aff}(M, \nabla)\) is a Lie group (see [KoNo] page 229) and is called the group of affine transformations of \((M, \nabla)\). An infinitesimal affine transformation of \((M, \nabla)\) is a smooth vector field \(X\) on \(M\) whose local 1-parameter groups \(\phi_t\) are local affine transformations of \((M, \nabla)\). We will denote by \(\mathfrak{a}(M, \nabla)\) the real vector space of infinitesimal affine transformations of \((M, \nabla)\). The vector subspace \(\text{aff}(M, \nabla)\) of \(\mathfrak{a}(M, \nabla)\) whose elements are complete, with the usual bracket of vector fields, is the Lie algebra of the group \(\text{Aff}(M, \nabla)\) (see [KoNo]).

Recall that the torsion and curvature tensors of a connection \(\nabla\) are defined by

\[
T_{\nabla}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]
\]

\[
K_{\nabla}(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}
\]

for any \(X, Y \in \mathfrak{X}(M)\), where \(\mathfrak{X}(M)\) denotes the space of smooth vector fields on \(M\). If the curvature and torsion tensors of \(\nabla\) are both null, the connection is called flat affine and the pair \((M, \nabla)\) is called a flat affine manifold. These type of manifolds are naturally related to
lagrangian foliations of symplectic manifolds (Theorem 7.8 in [Wei], see also [FOOO]). They are also relevant in integrable systems and mirror symmetry when their holonomy reduces to $GL_n(Z)$ (see [KoSa]).

If $M = G$ is a Lie group, a linear connection on $G$ is called left invariant if every left multiplication is an affine transformation of $(G, \nabla)$, and bi-invariant if both left and right multiplications are affine transformations. Having a left invariant linear connection $\nabla^+$ on a Lie group $G$ is equivalent to have a bilinear product on $\mathfrak{g} = \text{Lie}(G)$ given by $X \cdot Y = (\nabla^+_X + Y^+)\epsilon$, where $X^+, Y^+$ are the left invariant vector fields on $G$ determined respectively by $X$ and $Y$. When the bilinear product is given, the connection is defined by $\nabla_{X^+}Y^+ = (X \cdot Y)^+ = X \cdot Y^+ + f \nabla_{X^+}Y^+$ for every $X, Y, Z \in \mathfrak{g}$. Combining these last two equations, we get that $\nabla^+$ is flat affine if and only if the bilinear product is left symmetric, that is,

$$(X \cdot Y) \cdot Z = (Y \cdot X) \cdot Z = X \cdot (Y \cdot Z) - Y \cdot (X \cdot Z).$$

In this case, the pair $(G, \nabla^+)$ is called a flat affine Lie group and the algebra $(\mathfrak{g}, \cdot)$ a left symmetric algebra (see [Vin2], see also [Kos]). If the connection is bi-invariant and flat affine, the algebra $(\mathfrak{g}, \cdot)$ is associative (see [Med]).

If $p : \hat{M} \to M$ is the universal covering map of a real $n$-dimensional flat affine manifold $(M, \nabla)$. The pullback $\hat{\nabla}$ of $\nabla$ by $p$ is a flat affine structure on $\hat{M}$ and $p$ is an affine map. Moreover, the group of deck transformations, which in this case is isomorphic to $\pi_1(M)$, acts on $\hat{M}$ by affine transformations. There exists an affine immersion $D : (\hat{M}, \hat{\nabla}) \to (\mathbb{R}^n, \nabla^0)$ and a group homomorphism $A : \text{Aff}(\hat{M}, \hat{\nabla}) \to \text{Aff}(\mathbb{R}^n, \nabla^0)$ so that the following diagram commutes

$$\begin{array}{ccc}
\hat{M} & \xrightarrow{D} & \mathbb{R}^n \\
F \downarrow & & \downarrow A(F) \\
\hat{M} & \xrightarrow{D} & \mathbb{R}^n.
\end{array}$$

The map $D$ is called a developing map and it was introduced by Ehresmann (see [Ehr]). In particular for every $\gamma \in \pi_1(M)$, we have $D \circ \gamma = h(\gamma) \circ D$ with $h(\gamma) := A(\gamma)$. The map $h$ is also a group homomorphism called the holonomy representation of $(M, \nabla)$.

It is known that $(G, \nabla^+)$ is a flat affine if and only if there exists an affine étale representation $\rho : \hat{G} \to \text{Aff}(\mathbb{R}^n)$ (see [Kos] and [Med]). This means that the respective action of $\hat{G}$ on $\mathbb{R}^n$ leaves an open orbit $\mathcal{O}$ with discrete isotropy. The open orbit turns out to be the image of a developing map $D : \hat{G} \to \mathbb{R}^n$ and $D$ is a covering map of $\mathcal{O}$.

Finally, given a Lie group $G$, a homogeneous $G$-space $M$ is a manifold admitting a transitive action $\tau : G \times M \to M$. The isotropy group $H$ at any point $p \in M$ is a closed subgroup of $G$, hence it is a Lie subgroup and the set of left cosets $G/H$ admits a unique structure of manifold so that the projection $\pi : G \to G/H$ is a smooth map. Moreover, the manifolds $G/H$ and $M$ are diffeomorphic under the identification $gH \leftrightarrow \tau(g, p)$. A linear connection $\nabla$ on $M$ is called invariant if $\tau_g$ is affine relative to $\nabla$, for every $g \in G$, where $\tau_g : M \to M$ is the map defined by $\tau_g(m) = \tau(g, m)$. Invariant linear connections on a homogeneous $G$-space $M$ are in one-to-one
correspondence with linear maps \( \mathcal{L} : \mathfrak{g} \to \text{End}(\mathfrak{g}/\mathfrak{h}) \) satisfying
\[
\mathcal{L}_X(Y + h) = [X, Y] + h \quad \text{for all } X \in \mathfrak{h}, Y \in \mathfrak{g} \tag{1}
\]
\[
\mathcal{L}_{Ad_h X}(Ad_h Y + h) = \overline{Ad}_h (\mathcal{L}_X(Y + h)), \text{ for all } h \in H \text{ and } X, Y \in \mathfrak{g}. \tag{2}
\]
where \( \mathfrak{g} = \text{Lie}(G) \), \( \mathfrak{h} = \text{Lie}(H) \), \( Ad \) denotes the adjoint map and \( \overline{Ad}_h : \mathfrak{g}/\mathfrak{h} \mapsto \mathfrak{g}/\mathfrak{h} \) the map defined by \( \overline{Ad}_h(X + h) = Ad_h(X) + h \) (see [Vin]).

A homogeneous space \( G/H \) is called reductive if there is a linear subspace \( \mathfrak{m} \) of \( \mathfrak{g} \) so that \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \) and \( Ad_h(\mathfrak{m}) \subseteq \mathfrak{m} \), for every \( h \in H \). This implies that \( ad_h(\mathfrak{m}) \subseteq \mathfrak{m} \), for all \( h \in H \). When \( H \) is connected, the converse is also true. In this special case, invariant linear connections on \( G/H \) are in one-to-one correspondence with bilinear maps \( \theta : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m} \) so that \( Ad_h|_\mathfrak{m} \subseteq \text{Aut}(\mathfrak{m}, \theta) \), for any \( h \in H \) (see [Nom]).

This work is organized as follows. In Section 2 we study homogeneous spaces \( G/H \) where \( G \) is endowed with a left invariant linear connection \( \nabla^+ \) and give conditions on \( \mathfrak{h} = \text{Lie}(H) \) so that \( G/H \) admits an invariant connection determined by \( \nabla^+ \). We also give conditions on the connection so that the natural projection is an affine map. The reductive case is treated separately in Section 3. We use Section 4 to study the group of affine transformations of a homogeneous space \( G/H \) endowed with an invariant connection whose natural projection map is an affine map. We prove that if the connection is bi-invariant and \( H \) is discrete, the group of affine transformations of \( G/H \) is locally isomorphic to the group affine transformations of \( G \) commuting with the action of \( H \) on \( G \). We devote Section 5 to calculate the group of affine transformations of the flat affine two dimensional oriented surfaces. Finally, we finish the work with some further results.

2. Invariant connections on homogeneous \( G \)-spaces from left invariant connections on \( G \)

In this section we give necessary conditions for a left invariant connection on a Lie group, to determine an invariant connection on a homogeneous \( G \)-manifold, see Theorem 1. We also give a version of Proposition 3.3 in [Pos] (see also Lemma 1.8.24 in [Wol]) for homogeneous spaces (see Theorem 2).

Let \( G \) be a Lie group endowed with a left invariant connection \( \nabla^+ \), consider a closed Lie subgroup \( H \) of \( G \) and let \( G/H \) be the respective homogeneous manifold. Set \( \mathfrak{g} = \text{Lie}(G) = T_G G \), \( \mathfrak{h} = \text{Lie}(H) = T_H H \) and denote by \( X^+ \) (respectively \( X^- \)) the left (respectively right) invariant vector field on \( G \) determined by \( X \). Now, consider the bilinear product on \( \mathfrak{g} \) given by \( X \ast Y = (\nabla^+_X Y^-)_e \) and the linear map \( \mathcal{L} : \mathfrak{g} \to \text{End}(\mathfrak{g}/\mathfrak{h}) \) assigning to every \( Y \in \mathfrak{g} \) the map \( \mathcal{L}_Y \) defined by
\[
\mathcal{L}_Y(X + h) = X \ast Y + h = (\nabla^+_X Y^-)_e + h. \tag{3}
\]
Notice that for any \( Y \in \mathfrak{g} \), the map \( \mathcal{L}_Y \) is well defined if and only if \( X \ast Y \in \mathfrak{h} \), for all \( X \in \mathfrak{h} \) if and only if \( \mathfrak{h} \) is a right ideal of the algebra \( (\mathfrak{g}, \ast) \). In this case, according to Vinberg (see [Vin]), the map \( \mathcal{L} \) determines an invariant connection on \( G/H \) if and only if it verifies Equations (1) and (2). The first equation is equivalent to have that \( X \ast Y = [X, Y] \in \mathfrak{h} \) for all \( X \in \mathfrak{h} \). If \( R_h \in \text{Aff}(G, \nabla^+) \) for all \( h \in H \), where \( R_h \) denotes right multiplication by \( h \) on \( G \), that is, \( R_h(g) = gh \). One can easily verify that \( Ad_h(Y \ast X) = Ad_h(Y \ast Ad_h X) \), for any \( X, Y \in \mathfrak{g} \) and \( h \in H \). Hence, Equation (2) holds.
Remark 2.1. A vector field $X \in \mathfrak{X}(G)$ whose flow $\phi^X_t$ commutes with right multiplication $R_h$, for all $h \in H$, determines a vector field $\overline{X}$ on $G/H$ defined by

$$\overline{X}_{\sigma H} = \left. \frac{d}{dt} \right|_{t=0} \phi_t^X(\sigma)H \quad \text{for all } \sigma \in G. \quad (4)$$

As the vector field is invariant under $R_h$ if and only if its flows commute with $R_h$, it follows that the vector field $\overline{X}$ is well defined on $G/H$. It is also easy to see that $\overline{X} = \pi_*(X)$, where $\pi: G \to G/H$ is the natural projection. In particular, for $X \in \mathfrak{g}$, the vector field $\pi_*(X^-)$ is well defined in $G/H$ and will be denoted by $X^\ast$.

Recall that a connection $\nabla$ on the homogeneous space $G/H$ is called invariant if the left action, denoted by $\tau$, of $G$ on $G/H$ preserves the connection. That is, $(\tau_g)_* \nabla_X Y = \nabla_{(\tau_g)_* X} (\tau_g)_* Y$ for all $X, Y \in \mathfrak{X}(G/H)$ and all $g \in G$, where $\tau_g$ is given by $\tau_g (g'H) = (gg')H$. In fact, $\tau_g$ is the map so that the following diagram commutes

$$\begin{array}{ccc}
G & \xrightarrow{L_g} & G \\
\pi \downarrow & & \downarrow \pi \\
G/H & \xrightarrow{\tau_g} & G/H,
\end{array}$$

where $L_g$ denotes left multiplication by $g$ on $G$.

If $R_h \in \text{Aff}(G, \nabla^\ast)$ for any $h \in H$, the vector field $\nabla^+_X Y^-$ is $R_h$-invariant. Hence the vector field $\pi_*(\nabla^+_X Y^-)$ is well defined on $G/H$. Now consider the operator $\nabla$ given by

$$\nabla_{X^\ast} Y^\ast := \pi_*(\nabla^+_X Y^-) \quad (5)$$

By noticing that $(L_g)_*(X^-) = (\text{Ad}_g X^-)$ and that $(\text{Ad}_g X^-) = (\tau_g)_*(X^\ast)$ we get that

$$(\tau_g)_* \nabla_{X^\ast} Y^\ast = (\tau_g)_* \circ \pi_*(\nabla^+_X Y^-) = \pi_* \circ (L_g)_*(\nabla^+_X Y^-) = \nabla_{(\text{Ad}_g X^-)^\ast} \pi_*(\text{Ad}_g Y^-) = \nabla_{(\text{Ad}_g X^-) X^-} \pi_* (\text{Ad}_g Y^-) = (\nabla_{(L_g)_* X^-} (\tau_g)_* X^\ast)_* \nabla_{(\tau_g)_* X^\ast} (\tau_g)_* Y^\ast. \quad (6)$$

That is, $\nabla$ is $\tau_g$-invariant. Moreover, as the map $\phi: \mathfrak{g}/\mathfrak{h} \to T_e H(G/H)$ is defined by $\phi(X + \mathfrak{h}) = \pi_{*,*}(X)$, and by noticing that $\pi_{*,*}(\nabla^+_X Y^-)_e = \phi(\mathcal{L}_X (X + \mathfrak{h}))$, we get that $\nabla$ is the connection on $G/H$ determined by the map defined in (3). From all the previous, we get.

**Theorem 1.** Given a Lie group $G$ endowed with a left invariant linear connection $\nabla^\ast$ and $H$ a Lie subgroup of $G$ verifying

a. $\mathfrak{h} = \text{Lie}(H)$ is a right ideal of the algebra $(\mathfrak{g}, \ast)$ where $\mathfrak{g} = \text{Lie}(G)$ and $\ast$ as defined in (3).

b. $Y \ast X - [X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$ and

c. $R_h \in \text{Aff}(G, \nabla^\ast)$, for all $h \in H$,

then the connection $\nabla$ defined by Equation (5) is an invariant connection on $G/H$.

**Example 2.1.** Consider the group of affine motions of the line $G = \text{Aff}(\mathbb{R})$. This group is diffeomorphic to the group $\mathbb{R}^* \times \mathbb{R}$ with the product $(x, y)(x', y') = (xx', xy' + y)$. Its Lie algebra is $\mathfrak{g} = \text{aff}(\mathbb{R}) = \mathbb{R} e_1 \oplus \mathbb{R} e_2$ with bracket $[e_1, e_2] = e_2$. The 1-dimensional linear subspaces of $\mathfrak{g}$ are given by $\mathfrak{h} = \mathbb{R} f$ with $f = e_1 + \beta e_2$, $\beta \in \mathbb{R}$ or $f = e_2$. Next we exhibit all left invariant linear connections $\nabla^\ast$ on $G$ that determine invariant connections $\nabla$ on $G/H$, with $H$ the Lie subgroup of $G$ of Lie algebra $\mathfrak{h}$. 
If \( h = \mathbb{R}(e_1 + \beta e_2) \) with \( \beta \) fixed, then \( H \) is given by \( H = \{(a, \beta(a-1)) \mid a \neq 0\} \). A calculation shows that all left invariant connections on \( G \) so that conditions a., b., and c. are verified, are given by the bilinear product on \( \text{aff}(\mathbb{R}) \) determined by the table on the left

\[
\begin{array}{c|cc}
  f & e_2 & \ast f & e_2 \\
  e_2 & 0 & 0 & e_2 \\
\end{array}
\]

(7)

the table on the right determines the product \( \ast \) and \( f = e_1 + \beta e_2 \). Notice that all these connections are flat affine. One can verify that \( f^\ast = \pi_e(f^-) = u \frac{\partial}{\partial u} \) and \( e_2^\ast = \frac{\partial}{\partial u} \), where \((u)\) is a system of local coordinates on \( G/H \). Hence from the product \( \ast \) above, the connection \( \nabla \) on \( G/H \) is determined by \( \nabla \frac{\partial}{\partial u} = 0 \), that is, \( \nabla \) is the usual connection on \( G/H \).

Notice also that, as \( \nabla \) is determined by the map \( \mathcal{L} \), it must also satisfy that \( \nabla e_2 f^\ast = e_2^\ast \), that is, \( \nabla \frac{\partial}{\partial u} u \frac{\partial}{\partial u} = \frac{\partial}{\partial u} \) and this is immediately verified using the properties of linear connections.

Now, if \( h = \mathbb{R}e_2 \), the group \( H \) is given by \( H = \{(1, a) \mid a \in \mathbb{R}\} \) and all left invariant connections on \( G \) verifying the conditions of Theorem 1 are determined by the bilinear product obtained from the following table displaying the values on the linear basis \((e_1, e_2)\) of \( \mathfrak{g} \)

\[
\begin{array}{c|cc}
  e_1 & e_2 & \ast e_1 & e_2 \\
  e_2 & (\alpha - \gamma)e_2 & 0 & (\alpha - \gamma + 1)e_2 \\
\end{array}
\]

(8)

The table on the right hand side gives the product \( \ast \). A simple calculation shows that \( e_1^\ast = u \frac{\partial}{\partial u} \) and \( e_2^\ast = 0 \), hence the connection \( \nabla^\alpha \) is determined by \( \nabla u \frac{\partial}{\partial u} u \frac{\partial}{\partial u} = \alpha u \frac{\partial}{\partial u} \). The other condition given by the map \( \mathcal{L} \) is \( \nabla e_1 e_2^\ast = (\gamma - 1)e_2^\ast \), i.e., \( \nabla \frac{\partial}{\partial u} 0 = 0 \) which is obviously satisfied.

**Remark 2.2.** Under the conditions of the previous theorem, if \( \nabla^+ \) is torsion free, Conditions a. and b. of the theorem agree. Hence the connection \( \nabla \) exists if conditions a. and c. are satisfied.

Moreover, if \( \nabla^+ \) is bi-invariant, it determines a left invariant connection on \( G^{\text{op}} \), the opposite Lie group relative to \( G \), hence from Equation (5) we get that the natural projection \( \pi : (G^{\text{op}}, \nabla^+) \to (G/H, \nabla) \) is an affine map. So we get the following.

**Theorem 2.** If \( G \) is a Lie group endowed with a bi-invariant linear connection \( \nabla^+ \) and \( H \) is a Lie subgroup so that conditions a. and b. of the previous theorem hold, then the connection \( \nabla \) defined by Equation (5) is an invariant connection on \( G/H \) so that the natural projection \( \pi : (G^{\text{op}}, \nabla^+) \to (G/H, \nabla) \) is an affine map.

**Proof.** As \( \nabla^+ \) is bi-invariant, right multiplications \( R_g \) belong to \( \text{Aff}(G, \nabla^+) \), for every \( g \in G \), Condition c. of Theorem 1 holds. Hence, by the previous theorem, there is an invariant connection \( \nabla \) on \( G/H \) determined by Equation (5). Moreover, since right invariant vector fields on \( G \) are left invariant vector fields on \( G^{\text{op}} \), the formula \( \nabla^+_X Y^- \) determines a left invariant connection on \( G^{\text{op}} \). Hence Equation (5) means that \( \pi : (G^{\text{op}}, \nabla^+) \to (G/H, \nabla) \) is an affine map. \( \square \)

A similar result was proved in [AbHa] in the context of reductive homogeneous spaces, see Proposition 5.7.
Example 2.2. Consider the group $G = \text{Aff}(\mathbb{R})$ endowed with the connection $\nabla^{+,\alpha}$ determined by the bilinear product on the left hand table in [7], for $\alpha$ fixed. One can verify that the connection $\nabla^{+,1}$ is bi-invariant, hence the projection is an affine map. In particular, notice that the connection $\nabla$ verifies all relations of the table on the right hand side table in [7].

On the other hand, a left invariant connection $\nabla^{+,\alpha,\gamma,\lambda}$ determined by a product on the left hand side table in [8] is bi-invariant if and only if $\lambda = 0$. Hence the map $\pi : (G^{\text{op}}, \nabla^{+,\alpha,\gamma,0}) \to (G/H, \nabla^\alpha)$ is an affine map. However, it can be observed that $\pi$ is affine for any $\alpha$, $\gamma$, and $\lambda$.

Corollary 3. Under the conditions of Theorem [AbHa], if $H$ is a normal subgroup, $\nabla$ is left invariant.

Proof. An easy verification shows that the left multiplication map $L_{gH}$ by $gH$ on $G/H$ coincides with the map $\tau_g$, for any $g \in G$.

Example 2.3. The subgroup $H = \{(1,a) \mid a \in \mathbb{R}\}$ is a normal subgroup of $G = \text{Aff}(\mathbb{R})$, hence the connections $\nabla$ on $G/H$ determined by the table on the left hand table in [8] are left invariant.

3. Reductive case

In this section we exhibit a different proof for Proposition 5.7 in [AbHa]. Recall that a homogeneous space $M \cong G/H$ is called reductive if there is a decomposition of $g$ as a direct sum of vector spaces $g = h \oplus m$ so that $Ad_h(m) \subseteq m$ for all $h \in H$. As $\pi_{*,\epsilon}X = 0$ whenever $X \in h$, and $m$ can be identified with $g/h$, we have an isomorphism $\phi : m \to T_{m}(G/H)$ defined by $\phi(X) = \pi_{*,\epsilon}X$. Moreover, for $X = X_m + X_h \in g$, it holds that $\phi^{-1} \circ \pi_{*,\epsilon}X = X_m$. Consider the product on $m$ defined by

$$X \cdot Y = (X \ast Y)_m, \quad (9)$$

where $X \ast Y = (\nabla^{+,\epsilon}X)_m$. Since $X \ast Y = \phi^{-1} \circ \pi_{*,\epsilon}(X \ast Y)$, Equation (9) defines a bilinear product on $m$. Under these terms we have.

Proposition 4. Let $G$ be a Lie group endowed with a left invariant connection $\nabla^{+}$, let $H$ be a closed Lie subgroup of $G$ and $G/H$ the respective homogeneous manifold. If $G/H$ is reductive, $Ad_h(h) \subseteq h$, and $R_h \in \text{Aff}(G, \nabla^{+})$, for every $h \in H$, then the product defined in Equation (5) determines an invariant connection on $G/H$. The connection is given by the formula

$$\nabla_X \ast Y = \pi_{*,\epsilon}(\nabla^{+,\epsilon}X - Y), \quad \text{for all } X, Y \in m.$$

Moreover, if $\nabla^{+}$ is bi-invariant, the natural projection $\pi : (G^{\text{op}}, \nabla^{+}) \to (G/H, \nabla^\alpha)$ is an affine map.

Proof. First notice that

$$Ad_h(X \ast Y) = Ad_h((X \ast Y)_m + (X \ast Y)_h) = \underbrace{Ad_h((X \ast Y)_m)}_{\in m} + \underbrace{Ad_h((X \ast Y)_h)}_{\in h}$$

hence $Ad_h(X \ast Y)_m = Ad_h((X \ast Y)_m)$. Now, as $L_h, R_h \in \text{Aff}(G, \nabla^{+})$, for any $h \in H$, it follows that $Ad_h(X \ast Y) = Ad_h(X) \ast Ad_h(Y)$. Hence

$$Ad_hX \cdot Ad_hY = (Ad_hX \ast Ad_hY)_m = (Ad_h((X \ast Y))_m = Ad_h((X \ast Y)_m) = Ad_h(X \ast Y)$$

therefore $Ad_h|_m \subseteq \text{Aut}(m, \cdot)$. Hence, from Theorem 8.1 in [Nom] we get that the product in Equation (5) determines an invariant connection on $G/H$. From Equation (6) and by noticing that

$$\pi_{*,\epsilon}(\nabla^{+,\epsilon}X - Y)_\epsilon = \pi_{*,\epsilon}(X \ast Y) = \pi_{*,\epsilon}((X \ast Y)_m + (X \ast Y)_h) = \pi_{*,\epsilon}((X \ast Y)_m) = \pi_{*,\epsilon}(X \ast Y),$$

we obtain
we conclude that the connection determined by Equation (9) is $\nabla$. Finally, by mimicking the proof of Theorem 2, we get that if $\nabla^+$ is bi-invariant, $\pi$ is an affine map.

4. Group of affine transformations in homogeneous spaces

In this section we study the relationship between the group of affine transformations of a Lie group endowed with a linear connection $\nabla$ and a homogeneous $G$-space with a connection $\nabla$ so that $\pi$ is an affine map. We prove that when $H$ is discrete and $\nabla^+$ is bi-invariant, every infinitesimal affine transformation of the $G$-space is a projection of an infinitesimal affine transformation of $G$. We start with the following remark which will be useful in the section.

**Remark 4.1.** Given a smooth map $\phi : G \to G$ commuting with $R_h$, for all $h \in H$, the map $\overline{\phi} : G/H \to G/H$ defined by $\overline{\phi}(gH) = \phi(g)H$ is a smooth map. It is easy to check the well definition of $\overline{\phi}$ and that $\overline{\phi}$ is one-to-one (respectively onto) whenever $\phi$ is one-to-one (respectively onto). Notice that $\overline{\phi}$ is the unique map so that the following diagram commutes

$$
\begin{array}{ccc}
G & \xrightarrow{\phi} & G \\
\pi \downarrow & & \downarrow \pi \\
G/H & \xrightarrow{\overline{\phi}} & G/H.
\end{array}
$$

**Proposition 5.** If $G$ is a Lie group, $G/H$ a homogeneous space both endowed with linear connections $\nabla^+$ and $\nabla$, respectively left invariant and invariant, and $\pi : (G, \nabla^+) \to (G/H, \nabla)$ is an affine map, then we have

$$\{ \overline{\phi} \mid \text{there exists } \phi \in \text{Aff}(G, \nabla^+) \text{ so that } \pi \circ \overline{\phi} = \phi \circ \pi \} \subseteq \text{Aff}(G/H, \nabla)$$

**Proof.** First notice that for $X \in \mathfrak{g}$ we have that

$$\overline{\phi}_* \nabla_X^+ Y^* = \overline{\phi}_* \nabla_X^- Y^- = \pi_* \phi_* \nabla_X^+ Y^* = \pi_* \nabla_X^+ \phi_* Y^-
= \nabla_{\pi_* \phi_* X} \pi_* \phi_* Y^- = \nabla_{\overline{\phi}_* \pi_* X} \overline{\phi}_* \pi_* Y^-
= \nabla_{\overline{\phi}_* X} \overline{\phi}_* Y^*. \quad (12)$$

Now, since the vector fields $X^*$ with $X \in \mathfrak{g}$ generate $\mathfrak{X}(G/H)$, it follows that $\overline{\phi}$ is affine. □

The other inclusion does not hold in general. However, if $H$ is discrete, and $\nabla^+$ bilinear, we can prove the equality in the infinitesimal case. The following lemma is key on its proof.

**Lemma 6.** Let $X \in \mathfrak{X}(G)$ be $R_h$-invariant, for every $h \in H$, if $\pi_*(X) \equiv 0$ and $H$ is discrete, then $X \equiv 0$

**Proof.** If $\phi_t$ is the flow of $X$, by Remark 2, there is well defined vector $\pi_*(X)$ in $\mathfrak{X}(G/H)$ with flow given by $\overline{\phi}_t(gH) = \phi_t(g)H$, for all $g \in G$, and since $\pi_*(X) \equiv 0$, we have that $\overline{\phi}_t \equiv \text{Id}_{G/H}$. It follows that

$$\phi_t(g)H = gH$$

Hence, for every $g \in G$, $g^{-1} \phi_t(g) = h_g(t) \in H$. So, we have that $\frac{d}{dt} \bigg|_{t=0} g^{-1} \phi_t(g) \in \mathfrak{h} = \{0\}$. This means that $h_g(t) = h_g$ does not depend on $t$. Thus, for every $g \in G$, we have

$$X_g = \frac{d}{dt} \bigg|_{t=0} \phi_t(g) = \frac{d}{dt} \bigg|_{t=0} gh_g = 0$$

□
If the homogeneous space $G/H$ is reductive with $\mathfrak{g} = \mathfrak{h} @ \mathfrak{m}$, given a vector field $\overline{X} \in \mathfrak{x}(G/H)$, there always exists a natural lift $L(\overline{X}) \in \mathfrak{x}(G)$ so that $\pi_*(L(\overline{X})) = \overline{X}$ (see [AbHa] pg 247). For this we use the horizontal distribution $\mathcal{D}$ given by $D_g = (L_g)^*_\ast \mathfrak{m}$. That this distribution is $R_h$ invariant for every $h \in H$, follows from the fact that $\mathfrak{m}$ is $Ad_h$ invariant. When $H$ is a discrete subgroup of $G$, the homogeneous space $G/H$ is clearly reductive, so we have the following.

**Proposition 7.** If $G$ is a Lie group endowed with a bi-invariant connection $\nabla^+$ and $H$ is a discrete subgroup of $G$, then there exists an invariant connection $\nabla$ on $G/H$ so that $\pi$ is affine and

$$a(G/H, \nabla) = \{ \overline{X} \in \mathfrak{x}(G/H) \mid L(\overline{X}) \in a(G, \nabla^+) \},$$

where $L(\overline{X})$ is the lift of $\overline{X}$.

**Proof.** Since $\mathfrak{h} = \text{Lie}(H)$ is trivial, conditions a. and b. of Theorem [1] hold, hence by Theorem [2] there exists a connection $\overline{\nabla}$ on $G/H$ so that $\pi$ is an affine map.

Now, for the first inclusion, let $\overline{X} \in a(G/H, \overline{\nabla})$ and $X = L(\overline{X})$ its lift with flow $\phi_t$. Hence the flow $\overline{\phi}_t$ of $\overline{X}$ verifies that $\overline{\phi}_t(gH) = \phi_t(g)H$, for all $g \in G$. Moreover, for any pair of right invariant vector fields $Y^-$ and $Z^-$, we have that

$$\pi_*(\phi_t)_* \nabla^+_{Y^-} Z^- = \overline{\phi}_t \nabla^+_{Y^-} Z^- = \overline{\phi}_t \nabla_{\pi_* Y^-} \pi_* Z^- = \overline{\phi}_t \nabla_{\pi_* Y^-} \pi_* Z^- = \nabla_{\pi_*(\phi_t)_* Y^-} Z^- = \pi_*(\nabla_{Y^-} Z^-) = \nabla_{(\phi_t)_* Y^-} Z^-.$$

Hence, from Lemma [6] we get that $(\phi_t)_* \nabla^+_{Y^-} Z^- = \nabla^+_{(\phi_t)_* Y^-} Z^-$, that is $X \in a(G, \nabla^+)$. For the other inclusion, take $\overline{X} \in \mathfrak{x}(G/H)$ so that $X = L(\overline{X}) \in a(G, \nabla^+)$ and let $\overline{\phi}_t$ and $\phi_t$ be their respective flows. Since $\pi \circ \phi = \overline{\phi} \circ \pi$, we have that

$$(\overline{\phi}_t)_* \nabla_{Y^*} Z^* = (\phi_t)_* \nabla_{\pi_* Y^-} \pi_* Z^- = (\phi_t)_* \nabla_{Y^*} (\pi_* \phi_t)_* Z^- = \pi_* (\phi_t)_* \nabla_{Y^-} (\pi_* \phi_t)_* Z^- = \nabla_{(\overline{\phi}_t)_* Y^-} (\phi_t)_* Z^*.$$

Therefore $X \in a(G/H, \nabla)$. \qed

**Remark 4.2.** Notice that when $H$ is a discrete subgroup of $G$, if a vector field $X \in (G)$ is projectable to a vector field $\overline{X} = \pi_*(X)$, from Lemma [6] we have that $X = L(\overline{X})$. Hence $X$ is invariant under $R_h$ for every $h \in H$. Therefore, from Remark [2.1] we get that a vector field is projectable if and only if it is $R_h$-invariant for every $h \in H$.

Recall that a covering $(\overline{M}, \pi, M)$ is regular if automorphisms of the covering act transitively on its fibers. It is known that when $H$ is discrete, as the action of $H$ on $G$ is smooth, the map $\pi : G \rightarrow G/H$ determines a regular covering (see Example 7.1 in [Lim], see also [Pos] page 34). Hence we have the following.

**Theorem 8.** If $G$ is a Lie group endowed with a bi-invariant connection $\nabla^+$ and $H$ is a discrete subgroup of $G$ then there exists an invariant connection $\overline{\nabla}$ on $G/H$ so that $\pi$ is affine and

$$\text{aff}(G/H, \nabla) = \{ \overline{X} \in \mathfrak{x}(G/H) \mid L(\overline{X}) \in \text{aff}(G, \nabla^+) \},$$

where $L(\overline{X})$ is the lift of $\overline{X}$. 


Proof. If $X \in \mathcal{X}(G/H)$ is so that $L(X) \in \text{aff}(G, \nabla^+)$ with corresponding flows $\phi_t$ and $\phi_t$. Hence $\phi_t \in \text{Aff}(G, \nabla^+)$ for every $t$, and by Remark 4.1 and Proposition 5, $\bar{\phi}_t \in \text{Aff}(G/H, \nabla)$ for every $t$. Thus $\bar{X} \in \text{aff}(G/H, \nabla)$. For the other inclusion, let $\bar{X} \in \text{aff}(G/H, \nabla)$ with flow $\bar{\phi}_t$ and let $\phi_t$ be the flow of $L(\bar{X})$. We claim that $\phi_t$ is a diffeomorphism for every $t$. Let $D$ be a fundamental domain for the right action of $H$ over $G$ and let $D' \subset D$ be a subset containing exactly one representative for every class $gH$, with $g \in G$. Notice that $\phi_t(D')$ also contains exactly one representative for every class because if $\phi_t(g_1)H = \phi_t(g_2)H$, with $g_1, g_2 \in D'$, it follows that $\bar{\phi}_t(g_1H) = \bar{\phi}_t(g_2H)$. Hence, as $\bar{\phi}_t$ is injective, we have that $g_1H = g_2H$, therefore $g_1 = g_2$. Thus $\pi_{D'}$ and $\pi_{\phi_t(D')}$ are both bijective, so $\phi_t/D'$ is also a bijection. The fact that the covering is regular implies that $\phi_t$ is bijective and the claim follows. Finally, from Proposition 7 we conclude that $L(\bar{X}) \in \text{aff}(G, \nabla^+)$. 

Under the conditions of the previous theorem, the map $X \rightarrow \pi_+(X)$ gives a natural isomorphism between $\text{aff}(G, \nabla^+)$, the space of projectable complete infinitesimal affine transformations, and $\text{aff}(G/H, \nabla)$ with inverse $\bar{X} \mapsto L(\bar{X})$. Therefore, using Lie’s third theorem, we have the following.

Corollary 9. Under the hypothesis of the previous theorem, the groups $\text{Aff}(G/H, \nabla)$ and $\{T \in \text{Aff}(G, \nabla^+) \mid T \text{ commutes with } R_h, \text{ for all } h \in H\}$ are locally isomorphic.

5. GROUP OF AFFINE TRANSFORMATIONS OF FLAT AFFINE SURFACES

As an application of Theorem 3 in the following examples we calculate the groups of affine transformations of all flat affine orientable surfaces. We need the following remark.

Remark 5.1. From Lemmas 4 and 5 in [SaFl], we get that the groups of classical affine transformations preserving the upper half plane $H_2$, the quadrant $C_2 = \{(x, y) \mid x, y > 0\}$ and the punctured plane are respectively given by $G_1 = \{T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \mid T(x, y) = (ax + by + c, dy), \ d > 0\}$, $G_2 = \{T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \mid T(x, y) = (ax, by) \text{ with } a, b > 0\}$ and $G_3 = GL_2(\mathbb{R})$.

Example 5.1. (Group of affine transformations of the flat affine tori) It is well known that, up to isomorphism, there are six flat affine homogeneous structures on the 2-torus $\mathbb{T}^2$ (see [NaYa], [ArFu] and [Ben]). These are determined respectively by étale affine representations of $\mathbb{R}^2$, $\rho_i : \mathbb{R}^2 \rightarrow \text{Aff}(\mathbb{R}^2)$, $i = 1, \ldots, 6$ defined by

$$
\rho_1(a, b) = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \end{bmatrix}, \quad \rho_2(a, b) = \begin{bmatrix} 1 & b & a + \frac{1}{2}b^2 \\ 0 & 1 & b \end{bmatrix}, \quad \rho_3(a, b) = \begin{bmatrix} 1 & 0 & a \\ 0 & e^b & 0 \end{bmatrix},
$$

$$
\rho_4(a, b) = \begin{bmatrix} e^a & be^a & 0 \\ 0 & e^a & 0 \end{bmatrix}, \quad \rho_5(a, b) = \begin{bmatrix} e^a & 0 & 0 \\ 0 & e^b & 0 \end{bmatrix}, \quad \rho_6(a, b) = e^a \begin{bmatrix} \cos b & \sin b & 0 \\ -\sin b & \cos b & 0 \end{bmatrix},
$$

where each of the above matrices represent the affine transformation of $\mathbb{R}^2$ whose linear part is given by the first two columns and the last column corresponds to the translation part. Each of these representations determine an action of $\mathbb{Z}^2$ on the open orbit whose quotient is a torus. The actions are as follows:

Case 1. $\mathbb{Z}^2$ acting on the plane by $\xi_1((m, n), (x, y)) = (x + m, y + n)$.

Case 2. The quotient obtained by the action $\xi_2((m, n), (x, y)) = (x + ny + m + n^2/2, y + n)$ of $\mathbb{Z}^2$ on the plane. This torus is isomorphic to the Kupfer torus.

Case 3. The action of $\mathbb{Z}^2$ on the upper half plane $H_2$ given by $\xi_3((m, n), (x, y)) = (x + m, e^ny)$. 

Case 4. $\mathbb{Z}^2$ acting on $H_2$ by $\xi_4((m, n), (x, y)) = e^m(x + ny, y)$. 

Case 5. The action of $\mathbb{Z}^2$ on the quadrant $C_2$ given by $\xi_5((m, n), (x, y)) = (e^m x, e^n y)$.

Case 6. The quotient of the punctured plane $M = \mathbb{R}^2 \setminus \{(0, 0)\}$ by the action of $\mathbb{Z}$ given by $\xi_6(m, (x, y)) = e^m (x, y)$. This is known as the Hopf torus.

The following picture shows each of these tori

(A) Usual torus
(Case 1)
(B) Kuiper torus
(Case 2)
(C) Case 3
(D) Case 4
(E) Case 5
(F) Hopf torus
(Case 6)

FIGURE 1. Regular covering of the 2-torus

For $i = 1, \ldots, 6$, a developing map $D_i$ induces a group structure on $D_i(\mathbb{R}^2)$. In particular, for $i = 6$, taking $D_6(x, y) = e^x(\cos(2\pi y), \sin(2\pi y))$, one obtains the product of nonzero complex numbers. For $i \leq 5$ that, $D_i(\mathbb{R}^2)/D_i(\mathbb{Z}^2)$ is a homogeneous space isomorphic to the torus $T_i$. Notice that for $i = 6$, since $D_6(\mathbb{Z}^2) = \{(e^m, 0)\}$, the torus $T_6$ is isomorphic to $\mathbb{C}^* / \mathbb{Z}$ where $\mathbb{C}^*$ is the group of nonzero complex numbers and $\mathbb{Z}$ is identified with the subgroup $\{(e^m, 0) \mid m \in \mathbb{Z}\}$ of $\mathbb{C}^*$. By letting $\nabla^i$ be the left invariant flat affine connection on $\mathbb{R}^2$ determined by $\rho_i$. By Theorem 8, there exists an invariant connection $\nabla^i$ on the torus $T_i$, $i = 1, \ldots, 6$ and, by Corollary 2, the group of affine transformations is locally isomorphic to the group of classical affine transformations preserving the orbit and commuting with the corresponding action. Using the groups of transformations preserving the upper half plane, the quadrant and the punctured plane given in the previous remark, we get

$\text{Aff}(T_1, \nabla^1)$ is isomorphic to the group of translations of $\mathbb{R}^2$.

$\text{Aff}(T_2, \nabla^2) \cong \{ F : \mathbb{R}^2 \to \mathbb{R}^2 \mid F(x, y) = (x + ay + b, y + a) \}$.

$\text{Aff}(T_3, \nabla^3) \cong \{ F : \mathbb{R}^2 \to \mathbb{R}^2 \mid F(x, y) = (x + a, by), \ b > 0 \} \cong (\mathbb{R}^+ \times (\mathbb{R}, +) \times (\mathbb{R}^+, \cdot)$ where $\mathbb{R}^+$ is the set of positive real numbers.

$\text{Aff}(T_4, \nabla^4) \cong \{ F : \mathbb{R}^2 \to \mathbb{R}^2 \mid F(x, y) = (ax + by, ay), \ a > 0 \} \cong (\mathbb{R}^+, \cdot) \times (\mathbb{R}, +)$.
\[ \text{Aff}(\mathbb{T}_5, \nabla^5) \cong \{ F : \mathbb{R}^2 \to \mathbb{R}^2 \mid F(x, y) = (ax, by), \ a, b > 0 \} \cong (\mathbb{R}^{>0}, \cdot) \times (\mathbb{R}^{>0}, \cdot). \]

\[ \text{Aff}(\mathbb{T}_6, \nabla^6) \cong \text{GL}_2(\mathbb{R}). \]

The first five cases are abelian and 2-dimensional and the last one is non-abelian and 4-dimensional, as it is mentioned without a proof in [NaYa].

Also notice that each group \( \text{Aff}(\mathbb{T}_i, \nabla^i), i = 1, \ldots, 5 \) is compact isomorphic to the torus itself.

**Example 5.2. (Groups of affine transformations of flat affine cylinders)** In all the following cases, the quotient of the given space by the action of \( \mathbb{Z} \), determines an affine cylinder

(i) The plane with the action defined by \( \psi_1(m, x, y) = (x, y + m) \).

(ii) The action on the plane defined by \( \psi_2(m, x, y) = (x + my + m^2/2, y + m) \).

(iii) The upper half plane and the action given by \( \psi_3(m, x, y) = (x, y + m) \).

(iv) The action on the upper half plane defined as \( \psi_4(m, x, y) = (x, e^m y) \).

(v) The upper half plane with the action \( \psi_5(m, x, y) = (x + my, y) \).

(vi) The action on the upper half plane given by \( \psi_6(m, x, y) = (e^{mx}, e^m y) \).

(vii) The quadrant with the action \( \psi_7(m, x, y) = (x, e^m y) \).

(viii) The action \( \psi_8(m, x, y) = (x, y + m) \) on the plane with the connection \( \nabla^6 \). This cylinder is affinely isomorphic to the punctured plane.

Cylinders corresponding to the cases (i) through (vii) are isomorphic to the homogeneous space \( D(\mathbb{R}^2)/D(\mathbb{Z}) \) where \( D \) is a developing map and \( \mathbb{Z} \) is identified with the discrete subgroup \( \{(m, 0) \mid m \in \mathbb{Z} \} \) or \( \{(0, m) \mid m \in \mathbb{Z} \} \). The last one is isomorphic to \( \mathbb{R}^2/\mathbb{Z} \). Hence using Remark [B.], Theorem [8], and Corollary [9] we get that the respective groups of affine transformations are given by

\[ \text{Aff}(C_1, \nabla^1) \cong \{ F : \mathbb{R}^2 \to \mathbb{R}^2 \mid F(x, y) = (ax + b, cy + y + d), \ a \neq 0 \}. \]

\[ \text{Aff}(C_2, \nabla^2) \cong \{ F : \mathbb{R}^2 \to \mathbb{R}^2 \mid F(x, y) = (x + ay + b, y + a) \}. \]

\[ \text{Aff}(C_3, \nabla^3) \cong \{ F : \mathbb{R}^2 \to \mathbb{R}^2 \mid F(x, y) = (x + by + c, cy), \ c > 0 \}. \]

\[ \text{Aff}(C_4, \nabla^4) \cong \{ F : \mathbb{R}^2 \to \mathbb{R}^2 \mid F(x, y) = (x + b, cy), \ a \neq 0, \ c > 0 \}. \]

\[ \text{Aff}(C_5, \nabla^5) \cong \{ F : \mathbb{R}^2 \to \mathbb{R}^2 \mid F(x, y) = (ax + by + c, ay), \ a > 0 \}. \]

\[ \text{Aff}(C_6, \nabla^6) \cong \{ F : \mathbb{R}^2 \to \mathbb{R}^2 \mid F(x, y) = (ax + by, cy), \ a \neq 0, \ c > 0 \}. \]

\[ \text{Aff}(C_7, \nabla^7) \cong \{ F : \mathbb{R}^2 \to \mathbb{R}^2 \mid F(x, y) = (ax, by), \ a, b > 0 \}. \]

\[ \text{Aff}(C_8, \nabla^8) \cong \text{GL}_2(\mathbb{R}) \]

where \( \nabla^i \) is the connection on the cylinder determined from the corresponding connection on \( \mathbb{R}^2 \).

5.1 **Three dimensional Tori.** We use the classification of the three dimensional flat affine structures on \( \mathbb{R}^3 \), due Remm-Goze (see [ReGo]), to find 15 non-isomorphic flat affine structures on the three dimensional torus. Then we apply Theorem [8] to find the algebra of affine transformations and Corollary [9] to find a group locally isomorphic to the group of affine transformations. Remm-Goze’s classification yields the following connections on \( \mathbb{R}^3 \) determined by the given developing maps, which we slightly modified to get appropriate orbits of \( \mathbb{R}^3 \) to use Lemma 4. in [SaPa]. We present them in the following table where we exhibit their corresponding developing maps, a basis for \( \text{aff}(\mathbb{R}, \nabla_i) \), the action determining the torus \( \mathbb{T}_i \), a basis for \( \text{aff}_\pi(\mathbb{R}, \nabla_i) \), i.e., vector fields in \( \text{aff}(\mathbb{R}, \nabla_i) \) commuting with the action, and a matrix group.
locally isomorphic to $\text{Aff}(\mathbb{T}_i, \nabla_i)$, for $i = 1, \ldots, 15$ (the last column of the $3 \times 4$ matrices corresponds to the translation part)

$$D_1(a, b, c) = (e^a, e^{a+b}, e^{a+c}), \quad O_1 = (\mathbb{R}^3)^3$$

$$\begin{cases}x \frac{\partial}{\partial x}, y \frac{\partial}{\partial y}, z \frac{\partial}{\partial z} \\
(m, n, p) \cdot (x, y, z) = \left(e^m x, e^{m+n} y, e^{m+p} z\right) \\
\text{Aff}(\mathbb{T}_1, \nabla_1) \approx \left\{ \begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid a, b, c \neq 0 \right\} \right.$$ 

$$D_2(a, b, c) = (e^a \cos b, e^a \sin b, e^a(e^c - \cos b))$$

$$\begin{cases}x \frac{\partial}{\partial x}, y \frac{\partial}{\partial y}, z \frac{\partial}{\partial z} \\
(m, n, p) \cdot (x, y, z) = \left(\frac{e^m x}{x^2 + 1}, e^m y, e^m(e^p - 1)x + e^{m+p} z\right) \\
\text{Aff}(\mathbb{T}_2, \nabla_2) \approx \left\{ \begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid a, b, c \neq 0 \right\} \right.$$ 

$$D_3(a, b, c) = (ae^c, e^{b+c}, e^c)$$

$$\begin{cases}x \frac{\partial}{\partial x}, y \frac{\partial}{\partial y}, z \frac{\partial}{\partial z} \\
(m, n, p) \cdot (x, y, z) = \left(e^m x + me^p y, e^{m+p} y, e^p z\right) \\
\text{Aff}(\mathbb{T}_3, \nabla_3) \approx \left\{ \begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} | a, c \neq 0 \right\} \right.$$ 

$$D_4(a, b, c) = \left(\frac{a + \frac{b}{2}}{2}\right) e^c, be^c, e^c$$

$$\begin{cases}x \frac{\partial}{\partial x}, y \frac{\partial}{\partial y}, z \frac{\partial}{\partial z} \\
(m, n, p) \cdot (x, y, z) = \left(e^m x + ne^p y + (m + \frac{n^2}{2}) e^p z, e^m y + ne^p y, e^p z\right) \\
\text{Aff}(\mathbb{T}_4, \nabla_4) \approx \left\{ \begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} | a \neq 0 \right\} \right.$$ 

$$D_5(a, b, c) = (ac, be^c, e^c)$$

$$\begin{cases}x \frac{\partial}{\partial x}, y \frac{\partial}{\partial y}, z \frac{\partial}{\partial z} \\
(m, n, p) \cdot (x, y, z) = \left(e^m x + me^p y, e^p y + ne^p y, e^p z\right) \\
\text{Aff}(\mathbb{T}_5, \nabla_5) \approx \left\{ \begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid a \neq 0 \right\} \right.$$ 

$$D_6(a, b, c) = \left(\frac{a + \frac{b}{2}}{2}\right) e^c, e^b c, e^c$$

$$\begin{cases}x \frac{\partial}{\partial x}, y \frac{\partial}{\partial y}, z \frac{\partial}{\partial z} \\
(m, n, p) \cdot (x, y, z) = \left(e^m x, e^m y, z + p\right) \\
\text{Aff}(\mathbb{T}_6, \nabla_6) \approx \left\{ \begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid a, b \neq 0 \right\} \right.$$ 

$$D_7(a, b, c) = (e^a \cos b, e^a \sin b, c)$$

$$\begin{cases}x \frac{\partial}{\partial x}, y \frac{\partial}{\partial y}, x \frac{\partial}{\partial y}, y \frac{\partial}{\partial y}, x \frac{\partial}{\partial z}, y \frac{\partial}{\partial y}, z \frac{\partial}{\partial z} \\
(m, n, p) \cdot (x, y, z) = \left(e^m x, e^m y, z + p\right) \\
\text{Aff}(\mathbb{G}, \nabla_7) \approx \left\{ \begin{bmatrix} a & b & 0 \\ c & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid ad - bc \neq 0 \right\} \right.$$ 

$$D_8(a, b, c) = \left(\frac{a + \frac{b}{2}}{2}\right) e^c, e^b c, e^c$$

$$\begin{cases}x \frac{\partial}{\partial x}, y \frac{\partial}{\partial y}, y \frac{\partial}{\partial y}, y \frac{\partial}{\partial y}, x \frac{\partial}{\partial z}, y \frac{\partial}{\partial y}, z \frac{\partial}{\partial z} \\
(m, n, p) \cdot (x, y, z) = \left(e^m x, y + n, z + p\right) \\
\text{Aff}(\mathbb{T}_8, \nabla_8) \approx \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid a \neq 0 \right\} \right.$$ 

$$D_9(a, b, c) = (be^a, e^a, c)$$

$$\begin{cases}x \frac{\partial}{\partial x}, y \frac{\partial}{\partial y}, z \frac{\partial}{\partial y}, x \frac{\partial}{\partial y}, y \frac{\partial}{\partial y}, y \frac{\partial}{\partial y}, z \frac{\partial}{\partial z} \\
(m, n, p) \cdot (x, y, z) = \left(e^m x + ne^m y, e^m y, z + p\right) \\
\text{Aff}(\mathbb{T}_9, \nabla_9) \approx \left\{ \begin{bmatrix} a & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid a \neq 0 \right\} \right.$$ 

$$D_{10}(a, b, c) = \left(\frac{a + \frac{b}{2}}{2}\right) e^c, e^a, e^c$$

$$\begin{cases}x \frac{\partial}{\partial x}, y \frac{\partial}{\partial y}, y \frac{\partial}{\partial y}, x \frac{\partial}{\partial y}, y \frac{\partial}{\partial y}, y \frac{\partial}{\partial y}, z \frac{\partial}{\partial z} \\
(m, n, p) \cdot (x, y, z) = \left(x + ny + m + \frac{1}{2} n^2, y + n, e^p z\right) \\
\text{Aff}(\mathbb{T}_{10}, \nabla_{10}) \approx \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{bmatrix} \mid c \neq 0 \right\} \right.$$
| $D_{11}(a,b,c) = (a + \frac{1}{2}(b^2 + c^2), b, c)$ | $D_{12}(a,b,c) = (a + \frac{1}{2}(b^2 - c^2), b, c)$ |
|---------------------------------|---------------------------------|
| $\text{aff}(\mathbb{R}^3)$      | $\text{aff}(\mathbb{R}^3)$     |
| $(m,n,p) \cdot (x,y,z) =      | $(m,n,p) \cdot (x,y,z) =      |
| (x + ny + pz + m\frac{1}{2}(n^2 + p^2), y + n, z + p) | (x + ny - pz + m\frac{1}{2}(n^2 - p^2), y + n, z + p) |
| $\left\{ y \partial_{\partial x} + \frac{\partial}{\partial y}, z \partial_{\partial x} + \frac{\partial}{\partial z} \right\}$ | $\left\{ y \partial_{\partial x} + \frac{\partial}{\partial y}, z \partial_{\partial x} + \frac{\partial}{\partial z} \right\}$ |
| $\text{Aff}(\mathbb{T}_{11}, \nabla_{11})_0 \cong \left\{ \begin{array}{c} 1 \quad 0 \quad b \\ 0 \quad 1 \quad a \\ 0 \quad 0 \quad 1 \end{array} \right\}$, $a, b, c \in \mathbb{R}$ | $\text{Aff}(\mathbb{T}_{12}, \nabla_{12})_0 \cong \left\{ \begin{array}{c} 1 \quad 0 \quad b \\ 0 \quad 1 \quad -b \\ 0 \quad 0 \quad 1 \end{array} \right\}$, $a, b, c \in \mathbb{R}$ |
| $D_{13}(a,b,c) = (a + \frac{1}{2}b^2, b, c)$ | $D_{14}(a,b,c) = (a + bc + \frac{1}{6}c^3, b + \frac{1}{2}c^2, c)$ |
| $\text{aff}(\mathbb{R}^3)$ | $\text{aff}(\mathbb{R}^3)$ |
| $(m,n,p) \cdot (x,y,z) =      | $(m,n,p) \cdot (x,y,z) =      |
| (x + ny + m + \frac{1}{2}n^2, y + n, z + p) | (x + py + (n + \frac{1}{2}p^2) + m + np + \frac{1}{6}p^3, y + pz + n + \frac{1}{2}p^2, z + p) |
| $\left\{ y \partial_{\partial x} + \frac{\partial}{\partial y}, z \partial_{\partial x} + \frac{\partial}{\partial z} \right\}$ | $\left\{ y \partial_{\partial x} + z \partial_{\partial y}, z \partial_{\partial x} + \frac{\partial}{\partial y} \right\}$ |
| $\text{Aff}(\mathbb{T}_{13}, \nabla_{13})_0 \cong \left\{ \begin{array}{c} 1 \quad a \quad b \\ 0 \quad 1 \quad a \\ 0 \quad 0 \quad 1 \end{array} \right\}$, $a, b, c \in \mathbb{R}$ | $\text{Aff}(\mathbb{T}_{14}, \nabla_{14})_0 \cong \left\{ \begin{array}{c} 1 \quad a \quad b \\ 0 \quad 1 \quad a \\ 0 \quad 0 \quad 1 \end{array} \right\}$, $a, b, c \in \mathbb{R}$ |

Finally, the usual torus determined by the action $(m,n,p) \cdot (x,y,z) = (x + m, y + n, z + p)$ of $\mathbb{Z}^3$ on $\mathbb{R}^3$. An easy calculation shows that the subspace of $\text{aff}(\mathbb{R}^3)$ commuting with this action has linear basis given by $\left\{ \partial_{\partial x}, \partial_{\partial y}, \partial_{\partial z} \right\}$, so the Lie group $\text{Aff}(\mathbb{R}, \nabla_{15})$ is locally isomorphic to the group $\left\{ T : \mathbb{R}^3 \to \mathbb{R}^3 \mid T(x,y,z) = (x + a, y + b, z + c) \right\}$. More generally, the action $(m_1, \ldots, m_n) \cdot (x_1, \ldots, x_n) = (x_1 + m_1, \ldots, x_n + m_n)$ of $\mathbb{Z}^n$ on $\mathbb{R}^n$ determines an $n$-torus with space of infinitesimal affine transformations isomorphic to the Lie algebra with linear basis $\left\{ \partial_{\partial x_1}, \ldots, \partial_{\partial x_n} \right\}$ and its group of affine transformations $\text{Aff}(\mathbb{R}, \nabla)$ locally isomorphic to the group $\left\{ T : \mathbb{R}^n \to \mathbb{R}^n \mid T(x_1, \ldots, x_n) = (x_1 + a_1, \ldots, x_n + a_n), a_1, \ldots, a_n \in \mathbb{R} \right\}$

6. Further Consequences of the Results

We have the following consequences of the results in Section 4

**Corollary 10.** Under the hypothesis of Theorem 8, if $\nabla^+$ is flat affine, there exists a representation $\rho : \text{Aff}(G/H, \nabla) \to \text{Aff}(\mathbb{R}^n, \nabla^0)$ where $\nabla^0$ is the usual connection of $\mathbb{R}^n$ and $\widehat{\text{Aff}}(G/H, \nabla)$ denote the universal cover of $\text{Aff}(G/H, \nabla)$. Moreover, if $V = \text{aff}_x(G, \nabla^+)$ is the space of projectable complete infinitesimal affine transformations of $G$ relative to $\nabla^+$, we have that $\dim V \geq \dim G$ and $\rho$ admits a point of open orbit. Furthermore, if $\dim V = \dim G$ then $\rho$ is étale.

**Proof.** As $\nabla^+$ is flat affine and $\pi$ is affine, we have that $\nabla$ is flat affine. Corollary 2.4 in [MSV] guarantees the existence of $\rho$ and since right invariant vector fields are projectable, we have that $\dim V \geq \dim G$. The rest if the statements follow from the Corollary 9 and Theorem 2.6 in [MSV].

**Corollary 11.** Under the conditions of Theorem 8, if $(G, \nabla^+)$ is a complete flat affine Lie group then

$$\text{aff}(G/H, \nabla) \cong \{ X \in \mathfrak{a}(G, \nabla^+) \mid X \text{ is } R_h \text{ invariant for all } h \in H \}.$$ 

**Corollary 12.** Let $(G, \nabla^+)$ be a flat affine Lie group with $G$ simply connected and developing map $D : G \to \mathcal{O}$, then $\text{Aff}(\mathcal{O}, \nabla^0)$ is locally isomorphic to the group of transformations of
$\text{Aff}(G, \nabla^+) \text{ commuting with the right action of } I_{S_0} \text{ over } G, \text{ where } I_{S_0} \text{ is the isotropy group at } 0.$

Although the results in this work help to find the group of affine transformations of the homogeneous space from the group of affine transformations of the Lie group, this corollary shows that sometimes maybe easier to calculate the group of affine transformations of the homogeneous space as we show in the following example.

**Example 6.1.** Consider $G = \mathbb{R}^2$ with the flat affine left invariant connection $\nabla^+$ determined by affine étale representation $\rho : \mathbb{R}^2 \to \text{Aff}(\mathbb{R}^2)$ with $\rho(x, y) = e^x \begin{bmatrix} \cos(2\pi y) & -\sin(2\pi y) \\ \sin(2\pi y) & \cos(2\pi y) \end{bmatrix}$.

The orbit $\mathbb{R}^2 \setminus \{(0, 0)\}$ is isomorphic to $\mathbb{R}^2/H$ where $H = \{(0, n) \mid n \in \mathbb{Z}\}$. Using Lemma 2 in [SaFi], we get that the group of affine transformations preserving the orbit is $GL_2(\mathbb{R})$, but this only tells us that the subgroup of $\text{Aff}(G, \nabla^+)$ of transformations commuting with the action by $I_{S_0}$ is isomorphic to $GL_2(\mathbb{R})$. However, in this case the whole group $\text{Aff}(G, \nabla^+)$ is also isomorphic to $GL_2(\mathbb{R})$, see [NaYa].

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