A New Dynamic Scheme via Fractional Operators on Time Scale

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The present work investigates the applicability and effectiveness of the generalized Riemann-Liouville fractional integral operator integral method to obtain new Minkowski, Grüss type and several other associated dynamic variants on an arbitrary time scale, which are communicated as a combination of delta and fractional integrals. These inequalities extend some dynamic variants on time scales, and tie together and expand some integral inequalities. The present method is efficient, reliable, and it can be used as an alternative to establishing new solutions for different types of fractional differential equations applied in mathematical physics.

Keywords: Minkowski inequality, gruss inequality, fractional calculus, rimenn-liouville fractional integral operator, generalized riemann-liouville fractional integral operator, time scale, holder inequality

1. INTRODUCTION

Fractional calculus has also been comprehensively utilized in several instances, but the concept has been popularized and implemented in numerous disciplines of science, technology and engineering as a mathematical model (see [1, 2]). Numerous distinguished generalized fractional integral operators consist of the Hadamard operator, Erdélyi-Kober operators, the Saigo operator, the Gaussian hypergeometric operator, the Marichev-Saigo-Maeda fractional integral operator, and so on.; out of the ones, the Riemann-Liouville fractional integral operator has been extensively utilized by researchers in theory as well as applications (see [1, 3–8]).

Stefan Hilger began the theories of time scales in his doctoral dissertation [9] and combined discrete and continuous analysis (see [10, 11]). From this moment, this hypothesis has received a lot of attention. In the book written by Bohner and Peterson [12] on the issues of time scale, a brief summary is given and several time calculations are performed. Over the past decade, many analysts working in specific applications have proved a reasonable number of dynamic inequalities on a time scale (see [13–15]). Several researchers have created various results relating to fractional calculus on time scales to obtain the corresponding dynamic inequalities (see [16–20]).

Recently, the idea of the fractional-order derivative has been expounded by Bastos et al. [16] via Riemann-Liouville fractional operators on scale versions by considering linear dynamic equations. Another approach on time scales shifts to the inverse Laplace transform [18]. Following such innovator work, the investigation of fractional calculus on time scales created in a mainstream look into research studies on time scales (see [18, 21–29] and references therein). Since the publications in 2015, several researchers made significant contributions to the history of time scales. Sun and Hou [30] employed the fractional q-symmetric systems on time scales. Yaslan and Liceli [29] obtained the three-point boundary value problem with delta Riemann-Liouville fractional
derivative on time scales. Yan et al. [31] adopted the Caputo fractional techniques on differential equations on time scales. Zhu and Wu [32] employed Caputo nabla fractional derivatives in order to find the existence of solutions for Cauchy problems. As certifiable utilities, we refer to the study of calcium ion channels that are impeded with an infusion of calcium-chelator ethylene glycol tetraacetic acid [33]. Actually, physical utilization of initial value-problem solutions in diverse time scales proliferates [10, 34, 35]. For instance, the continuous time scale \( T = \mathbb{R} \), the fractional differential equations that oversee the practices of viscoelastic materials with memory and creep tendencies have been investigated in Chidouh et al. [36].

Integral Inequalities are an excellent way to investigate many scientific fields of research, including engineering, flow dynamics, biology, chaos, meteorology, vibration analysis, biochemistry, aerodynamics and many more. Since the productions of the above outcome in 1883, several works have been published in the literature of time calculus, with varied evidence, various speculations and improvements \[\text{see [13–15, 24, 52] and the references therein}].

Here, we broaden accessible outcomes in the literature [53] by presenting increasingly broad ideas of fractional integral inequalities on time scales in the frame of generalized Riemann-Liouville fractional integral. At that point, we study the dynamic variants of corresponding generalized fractional-order on time scales. We obtain the inequalities Grüss, Minkowski and several others using the delta integrals in arbitrary time scales. For \( \delta = 1 \), the integral will become delta integral and for \( \delta = 0 \), it advances toward turning out to be nabla integral. An astounding audit about the time scale calculus can be found in the paper [54]. The proposed dynamical integral method is reliable and effective to obtain new solutions. This method has more advantages: it is direct and concise. Thus, the proposed method can be extended to solve many systems of non-linear fractional partial differential equations in mathematical and physical sciences. Also, the new exact analytical solutions can be obtained for the generalized ordinary differential equations to obtain new theorems related to stability and continuous dependence on parameters for dynamic equations on time scales.

The present work investigates the applicability and effectiveness of the several dynamic variants that are presented, which are based primarily on the generalized Riemann-Liouville fractional integral operators. We will show that the Grüss and Minkowski type, that we participated in are very specific to the current work. From an application point of view, the results ultimately relate to the study of Young’s inequality, arithmetic, and geometry inequality. Our computed outcomes can be very useful as a starting point of comparison when some approximate methods are applied to this nonlinear space-time fractional equation. Furthermore, there are likewise some occurrences that can be derived from our outcomes.

2. PRELIMINARIES

A non-empty closed subsets \( \mathbb{R} \) of \( T \) is known as the time scale. The well-known examples of time scales theory are the set of real numbers \( \mathbb{R} \) and the integers \( \mathbb{Z} \). Throughout the paper, we refer \( \mathbb{S} \) as time scale and a time-scaled interval is \( \mathbb{S} = [v_1, v_2]_{\mathbb{S}} \). We need the concept of jump operators. The forward jump operator is denoted by the symbol \( \diamond \) and the backward jump operator is denoted by \( \bar{\delta} \), are said through the formulas:

\[ \diamond(t) = \inf\{\lambda \in \mathbb{T} : \rho > t \} \in \mathbb{T}, \quad \bar{\delta}(\omega) = \sup\{\rho \in \mathbb{T} : \rho < \omega \} \in \mathbb{T}. \]

We accumulate as:

\[ \inf \theta := \sup \bar{\theta}, \quad \sup \theta := \inf \bar{\theta}. \]

If \( \diamond(t) > t \), then the term \( t \) is called to be right-scattered and \( \omega \) is right-scattered \( \rho(\omega) < \omega \). The elements that are most likely all the while appropriate-scattered and scattered are known as isolated. The term \( t \) is said to be right dense, if \( \diamond(t) = t \), and \( \omega \) is said to be left dense, if \( \rho(\omega) = \omega \). In addition, the focuses \( t, \omega \) are known to be dense if they are most likely right-dense and left-dense.

The mappings \( \mu, \nu : \mathbb{T} \rightarrow [0, +\infty) \) defined by

\[ \mu(t) := \diamond(t) - t, \quad \nu(t) := t - \bar{\delta}(t) \]

are called the forward and backward graininess functions, respectively.

**Definition 2.1.** [12, 55] “Let \( h : \mathbb{T} \rightarrow \mathbb{R} \) be a real-valued function. Then \( h \) is said to be \( RD \)-continuous on \( \mathbb{T} \) if its left limit at any left dense point of \( T \) is finite and it is continuous on every right dense point of \( T \). All \( RD \)-continuous functions are denoted by \( C_{RD} \).”

**Definition 2.2.** “A function \( F : \mathbb{T} \rightarrow \mathbb{R} \) is called a delta antiderivative of \( h : \mathbb{T} \rightarrow \mathbb{R} \) if \( \int_{v_1}^{t} h(s) \Delta s = F(t) - F(v_1) \) for all \( t \in \mathbb{T} \).

**Theorem 2.1.** [55]. If \( h \in C_{RD} \) and \( t \in \mathbb{T} \), then

\[ \int_{t}^{\diamond(t)} h(s) \Delta s = \mu(t) h(t). \]

**Theorem 2.2.** [55]. Let \( v_1, v_2, v_3 \in \mathbb{T}, \beta \in \mathbb{R} \) and \( h, \omega \in C_{RD}, \) then

(i). \[ \int_{v_1}^{v_2} (h_1(\rho) + h_2(\rho)) \Delta \rho = h_1(v_1) \Delta \rho + h_2(v_2) \Delta \rho; \]

(ii). \[ \int_{v_1}^{v_2} \beta h(\rho) \Delta \rho = \beta \int_{v_1}^{v_2} h(\rho) \Delta \rho; \]
Again, we demonstrate the concept of generalized Riemann-Liouville fractional integral operator which is proposed by \[24\].

**Definition 2.4.** \[24\] “For \(0 < \delta < 1\), let \(\mathbb{T}\) is a time scale and \([v_1, v_2]\) is an interval of \(\mathbb{T}\). Suppose \(F\) be an integrable function on \([v_1, v_2]\) and \(\Phi\) is monotone having a delta derivative \(\Phi^\Delta\) with \(\Phi^\Delta \neq 0\) for any \(\eta \in [v_1, v_2]\). Let \(0 < \delta < 1\), then the (left) generalized fractional integral of order \(\delta\) of \(F\) with respect to \(\Phi\) is defined by

\[
\frac{T^{\eta}_{v_1: v_1: \Phi} J^\delta_{\Phi} (\eta)}{v_1} = \frac{1}{\Gamma(\delta)} \int_{v_1}^\eta (\Phi(\eta) - \Phi(\theta))^{\delta - 1} \Phi^\Delta(\theta) F(\theta) \Delta \theta.
\]

**Remark 2.1.** If \(\mathbb{T} = \mathbb{R}\), then Definitions 2.3 and 2.4 reduces to the well-known Riemann-Liouville and generalized Riemann-Liouville fractional integral, respectively (see \[7\]).

### 3. MINKOWSKI TYPE INEQUALITIES FOR GENERALIZED RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL ON TIME SCALE

This section is inaugurated to establishing generalizations of some reverse Minkowski inequality by introducing the generalized Riemann-Liouville fractional integral on time scale.

**Theorem 3.1.** Let \(\delta, \gamma > 1\), and \(\mathbb{T}\) is a time scale. Suppose \(F, G\) be two positive functions on \([0, \infty)\), and \(\Phi\) is monotone, delta differentiable \(\Phi^\Delta\) with \(\Phi^\Delta \neq 0\) such that for all \(\eta > 0\), \(\frac{T^{\eta}_{\Phi(\eta)} J^\delta_{\Phi} G(\eta)}{\Phi(\eta)} < \infty\), \(\frac{T^{\eta}_{\Phi(\eta)} J^\gamma_{\Phi} F(\eta)}{\Phi(\eta)} < \infty\). If \(0 < m \leq \frac{\mathcal{M}(\Phi)}{\Phi(\eta)} \leq \mathcal{M}, \theta \in [0, \eta]\), then

\[
\left[ \frac{T^{\eta}_{\Phi(\eta)} J^\delta_{\Phi} F(\eta)}{\Phi(\eta)} \right]^\frac{1}{\gamma} \left[ \frac{T^{\eta}_{\Phi(\eta)} J^\gamma_{\Phi} G(\eta)}{\Phi(\eta)} \right]^\frac{1}{\delta} \leq \left( \frac{\mathcal{M}}{\gamma} \right)^{\frac{1}{\delta}} \left( \frac{\mathcal{M}}{\gamma} \right)^{\frac{1}{\delta}} \left( \frac{T^{\eta}_{\Phi(\eta)} J^\gamma_{\Phi} F(\eta)}{\Phi(\eta)} \right)^{\frac{1}{\gamma}} \left( \frac{T^{\eta}_{\Phi(\eta)} J^\gamma_{\Phi} G(\eta)}{\Phi(\eta)} \right)^{\frac{1}{\delta}} \left( \frac{T^{\eta}_{\Phi(\eta)} J^\gamma_{\Phi} F(\eta)}{\Phi(\eta)} \right)^{\frac{1}{\gamma}} \left( \frac{T^{\eta}_{\Phi(\eta)} J^\gamma_{\Phi} G(\eta)}{\Phi(\eta)} \right)^{\frac{1}{\delta}}.
\]

**Proof:** Since \(\frac{T^{\eta}_{\Phi(\eta)} J^\delta_{\Phi} F(\eta)}{\Phi(\eta)} \leq \mathcal{M}, \theta \in [0, \eta], \eta > 0\), we find that

\[
\left( \frac{\mathcal{M}}{\gamma} \right)^{\frac{1}{\delta}} \left( \frac{T^{\eta}_{\Phi(\eta)} J^\gamma_{\Phi} F(\eta)}{\Phi(\eta)} \right)^{\frac{1}{\gamma}} \geq \mathcal{M}^{-\frac{1}{\gamma}} \left( \frac{T^{\eta}_{\Phi(\eta)} J^\gamma_{\Phi} F(\eta)}{\Phi(\eta)} \right)^{\frac{1}{\gamma}}
\]

and

\[
\left( \frac{T^{\eta}_{\Phi(\eta)} J^\gamma_{\Phi} F(\eta)}{\Phi(\eta)} \right)^{\frac{1}{\gamma}} \geq \mathcal{M}^{-\frac{1}{\gamma}} \left( \frac{T^{\eta}_{\Phi(\eta)} J^\gamma_{\Phi} F(\eta)}{\Phi(\eta)} \right)^{\frac{1}{\gamma}}.
\]

Taking product on both sides of (5) \(\frac{\Phi(\eta)}{\Phi(\eta)} \frac{T^{\eta}_{\Phi(\eta)} J^\delta_{\Phi} F(\eta)}{\Phi(\eta)} \leq \mathcal{M}^{-\frac{1}{\gamma}} \left( \frac{T^{\eta}_{\Phi(\eta)} J^\gamma_{\Phi} F(\eta)}{\Phi(\eta)} \right)^{\frac{1}{\gamma}} \frac{T^{\eta}_{\Phi(\eta)} J^\gamma_{\Phi} G(\eta)}{\Phi(\eta)} \), which is positive because \(\theta \in (0, \eta), \eta > 0\), we integrate the resulting identity with respect to \(\theta\) from 0 to \(\eta\) we have

\[
\frac{1}{\Gamma(\delta)} \int_{0}^{\eta} (\Phi(\eta) - \Phi(\theta))^{\delta - 1} \Phi^\Delta(\theta) F(\theta) \Delta \theta 
\]

\[
\geq \mathcal{M}^{-\frac{1}{\gamma}} \int_{0}^{\eta} (\Phi(\eta) - \Phi(\theta))^{\delta - 1} \Phi^\Delta(\theta) F(\theta) \Delta \theta.
\]
which implies that

\[
\frac{T}{0^+;\Phi}\mathcal{J}^\alpha_\theta(F(\theta)\mathcal{J}^\beta_\eta(G(\eta)))^{\frac{1}{2}} \geq M^{\frac{1}{2}} \frac{T}{0^+;\Phi}\mathcal{J}^\alpha_\theta F(\eta). \tag{7}
\]

It follows that

\[
\left(\frac{T}{0^+;\Phi}\mathcal{J}^\alpha_\theta(F(\theta)\mathcal{J}^\beta_\eta(G(\eta)))^{\frac{1}{2}} \right)^{\frac{1}{2}} \geq M^{\frac{1}{2}} \frac{T}{0^+;\Phi}\mathcal{J}^\alpha_\theta F(\eta). \tag{8}
\]

Accordingly, \(mG(\theta) \leq F(\theta)\), \(\theta \in (0, \eta), \eta > 0\), therefore we have

\[
\left(\frac{T}{0^+;\Phi}\mathcal{J}^\alpha_\theta(F(\theta))\right)^{\frac{1}{2}} \geq m^{\frac{1}{2}} (G(\theta))^{\frac{1}{2}}. \tag{9}
\]

Taking product (9) by \(G(\theta)^{\frac{1}{p}}\), we arrive at

\[
(G(\theta))^{\frac{1}{p}} (F(\theta))^{\frac{1}{2}} \geq m^{\frac{1}{p}} G(\theta). \tag{10}
\]

Taking product on both sides of (11)

\[
\frac{1}{\Gamma(\delta)} \int_0^\eta (\Phi(\eta) - \Phi(\theta))^{\delta-1} \Phi^\lambda(\theta) \left(\mathcal{J}^\alpha_\theta(G(\eta))\right)^{\frac{1}{2}} (F(\theta))^{\frac{1}{2}} \Delta \theta \geq m^{\frac{1}{p}} \left(\frac{T}{0^+;\Phi}\mathcal{J}^\alpha_\theta G(\eta)\right)^{\frac{1}{2}}. \tag{11}
\]

Hence, we can write

\[
\left(\frac{T}{0^+;\Phi}\mathcal{J}^\alpha_\theta(F(\theta))\right)^{\frac{1}{2}} \geq m^{\frac{1}{p}} \left(\frac{T}{0^+;\Phi}\mathcal{J}^\alpha_\theta G(\eta)\right)^{\frac{1}{2}}. \tag{12}
\]

Conducting product between (8) and (12), we can draw the desired conclusion easily.

Corollary 3.1. Letting \(\mathbb{T} = \mathbb{R}\), then under the assumption of Theorem 3.1, we have the following inequality in generalized Riemann-Liouville fractional integral:

\[
\left[\mathcal{J}^\alpha_\theta F(\theta)\right]^{\frac{1}{2}} \left[\mathcal{J}^\beta_\eta G(\eta)\right]^{\frac{1}{2}} \leq \left(\frac{T}{0^+;\Phi}\mathcal{J}^\alpha_\theta F(\eta)\right)^{\frac{1}{2}} \left(\frac{T}{0^+;\Phi}\mathcal{J}^\beta_\eta G(\eta)\right)^{\frac{1}{2}}. \tag{13}
\]

**Theorem 3.2.** Let \(\delta, \gamma > 1\), and \(\mathbb{T}\) is a time scale. Suppose \(\mathcal{F}, \mathcal{G}\) be two positive functions on \([0, \infty)\), and \(\Phi\) is monotone, delta differentiable \(\Phi^\lambda\) with \(\Phi^\lambda \neq 0\) such that for all \(\eta > 0\), \(\frac{T}{0^+;\Phi}\mathcal{J}^\alpha_\theta \mathcal{F}(\theta)\) \(\mathcal{F}(\eta)\) \(\mathcal{G}(\eta)\) \(\infty\), \(\frac{T}{0^+;\Phi}\mathcal{J}^\beta_\eta \mathcal{G}(\eta)\) \(\infty\). If \(0 < m \leq \frac{T}{0^+;\Phi}\mathcal{J}^\gamma_\theta \mathcal{G}(\eta)\) \(\mathcal{G}(\eta)\) \(\infty\), then

\[
\left[\frac{T}{0^+;\Phi}\mathcal{J}^\gamma_\theta \mathcal{G}(\eta)\right]^{\frac{1}{2}} \left[\frac{T}{0^+;\Phi}\mathcal{J}^\alpha_\theta \mathcal{F}(\theta)\right]^{\frac{1}{2}} \leq \frac{T}{0^+;\Phi}\mathcal{J}^\gamma_\theta \mathcal{G}(\eta)\). \tag{14}
\]

**Proof:** Replacing \(\mathcal{F}(\theta)\) and \(\mathcal{G}(\theta)\) by \(\mathcal{F}^\beta(\theta)\) and \(\mathcal{G}^\beta(\theta)\), \(\theta \in [0, \eta], \eta > 0\) in Theorem 3.1, we acquire the desired result.

\[
\frac{T}{0^+;\Phi}\mathcal{J}^\gamma_\theta \mathcal{G}(\eta) \geq \frac{T}{0^+;\Phi}\mathcal{J}^\gamma_\theta \mathcal{G}(\eta)\). \tag{15}
\]

**4. GRÜSS TYPE INEQUALITIES VIA GENERALIZED RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL ON TIME SCALE**

Our coming result is the generalization of Grüss type inequality via generalized Riemann-Liouville fractional integral operator on time scale.

**Theorem 4.1.** Let \(\delta, \gamma > 1\), and \(\mathbb{T}\) is a time scale. Suppose there is a positive function \(\mathcal{F}\) on \([0, \infty)\), and \(\Phi\) is monotone, delta differentiable \(\Phi^\lambda\) with \(\Phi^\lambda \neq 0\) such that for all \(\eta > 0\). Assume that the subsequent.

(I) There exist two integrable functions \(\varphi_1, \varphi_2\) on \([0, \infty)\) such that

\[
\varphi_1(\eta) \leq \mathcal{F}(\eta) \leq \varphi_2(\eta), \quad \forall \eta \in [0, \infty). \tag{16}
\]

Then, for \(\eta > 0, \delta, \gamma > 1\), one has

\[
\varphi_1(\eta) \leq \mathcal{F}(\eta) \leq \varphi_2(\eta), \tag{17}
\]

Taking product on both sides of (17)

\[
\left(\frac{T}{0^+;\Phi}\mathcal{J}^\gamma_\theta \mathcal{G}(\eta)\right)^{\frac{1}{2}} \left(\frac{T}{0^+;\Phi}\mathcal{J}^\gamma_\theta \mathcal{G}(\eta)\right)^{\frac{1}{2}} \leq \frac{T}{0^+;\Phi}\mathcal{J}^\gamma_\theta \mathcal{G}(\eta)\). \tag{18}
\]

Taking product on both sides of (19)

\[
\left(\frac{T}{0^+;\Phi}\mathcal{J}^\gamma_\theta \mathcal{G}(\eta)\right)^{\frac{1}{2}} \left(\frac{T}{0^+;\Phi}\mathcal{J}^\gamma_\theta \mathcal{G}(\eta)\right)^{\frac{1}{2}} \leq \frac{T}{0^+;\Phi}\mathcal{J}^\gamma_\theta \mathcal{G}(\eta)\). \tag{19}
\]
\[
\begin{align*}
\mathcal{T}_{\gamma}^{\phi} J^\delta_\eta \phi_2(\eta) & \left( \frac{1}{\Gamma(\gamma)} \int_{0}^{\eta} \left( \Phi(\eta) - \Phi(\lambda) \right)^{\gamma - 1} \Phi^{\Delta}(\lambda) \mathcal{F}(\lambda) \Delta \lambda \\ + & \mathcal{T}_{\gamma}^{\phi} J^\delta_\eta \mathcal{G}(\eta) \right) \\
+ & \mathcal{T}_{\gamma}^{\phi} J^\delta_\eta \omega_1(\eta) + \mathcal{T}_{\gamma}^{\phi} J^\delta_\eta \omega_2(\eta) + \mathcal{T}_{\gamma}^{\phi} J^\delta_\eta \mathcal{F}(\eta) \\
= & \mathcal{T}_{\gamma}^{\phi} J^\delta_\eta \phi_2(\eta) + \mathcal{T}_{\gamma}^{\phi} J^\delta_\eta \mathcal{G}(\eta) + \mathcal{T}_{\gamma}^{\phi} J^\delta_\eta \omega_1(\eta) + \mathcal{T}_{\gamma}^{\phi} J^\delta_\eta \omega_2(\eta) + \mathcal{T}_{\gamma}^{\phi} J^\delta_\eta \mathcal{F}(\eta). 
\end{align*}
\]

(22)

Proof: To prove (A1), from (I) and (II), we have for \( x \in [0, \infty) \) that
\[
(\phi_2(\theta) - \mathcal{F}(\theta))(\mathcal{G}(\lambda) - \omega_1(\lambda)) \geq 0.
\]

Therefore,
\[
\varphi_2(\theta)\mathcal{G}(\lambda) + \omega_1(\lambda)\mathcal{F}(\theta) \geq \omega_1(\lambda)\varphi_2(\theta) + \mathcal{G}(\lambda)\mathcal{F}(\theta).
\]

(24)

Taking product on both sides of (24) \( \frac{\varphi(\phi(\phi)) - \phi(\phi)\frac{\phi(\phi)}{\phi(\phi)}}{\phi(\phi)} \), which is positive because \( \theta \in (0, \eta) \), \( \eta > 0 \), we integrate the resulting identity with respect to \( \theta \) from 0 to \( \eta \) we have
\[
\mathcal{G}(\lambda) \left( \frac{1}{\Gamma(\delta)} \int_{0}^{\eta} \left( \Phi(\eta) - \Phi(\theta) \right)^{\delta - 1} \Phi^{\Delta}(\theta) \varphi_2(\theta) \Delta \theta \\
+ & \omega_1(\lambda) \left( \frac{1}{\Gamma(\delta)} \int_{0}^{\eta} \left( \Phi(\eta) - \Phi(\theta) \right)^{\delta - 1} \Phi^{\Delta}(\theta) \mathcal{F}(\theta) \Delta \theta \\
& \geq \omega_1(\lambda) \left( \frac{1}{\Gamma(\delta)} \int_{0}^{\eta} \left( \Phi(\eta) - \Phi(\theta) \right)^{\delta - 1} \Phi^{\Delta}(\theta) \varphi_2(\theta) \Delta \theta \\
& \geq \omega_1(\lambda) \left( \frac{1}{\Gamma(\delta)} \int_{0}^{\eta} \left( \Phi(\eta) - \Phi(\theta) \right)^{\delta - 1} \Phi^{\Delta}(\theta) \mathcal{F}(\theta) \Delta \theta. 
\end{align*}
\]

(25)

Then we have
\[
\mathcal{G}(\lambda) \left( \frac{1}{\Gamma(\delta)} \int_{0}^{\eta} \left( \Phi(\eta) - \Phi(\theta) \right)^{\delta - 1} \Phi^{\Delta}(\theta) \varphi_2(\theta) \Delta \theta \\
+ \omega_1(\lambda) \left( \frac{1}{\Gamma(\delta)} \int_{0}^{\eta} \left( \Phi(\eta) - \Phi(\theta) \right)^{\delta - 1} \Phi^{\Delta}(\theta) \mathcal{F}(\theta) \Delta \theta \\
& \geq \omega_1(\lambda) \left( \frac{1}{\Gamma(\delta)} \int_{0}^{\eta} \left( \Phi(\eta) - \Phi(\theta) \right)^{\delta - 1} \Phi^{\Delta}(\theta) \varphi_2(\theta) \Delta \theta \\
& \geq \omega_1(\lambda) \left( \frac{1}{\Gamma(\delta)} \int_{0}^{\eta} \left( \Phi(\eta) - \Phi(\theta) \right)^{\delta - 1} \Phi^{\Delta}(\theta) \mathcal{F}(\theta) \Delta \theta. 
\end{align*}
\]

(26)

Again, multiplying both sides of (26) by \( \frac{\Phi(\phi(\phi)) - \phi(\phi)\frac{\phi(\phi)}{\phi(\phi)}}{\phi(\phi)} \), which is positive because \( \lambda \in (0, \eta) \), \( \eta > 0 \), we integrate the resulting identity with respect to \( \lambda \) from 0 to \( \eta \) we have
\[
\begin{align*}
& \mathcal{T}_{\gamma}^{\phi} J^\delta_\eta \phi_2(\eta) \left( \frac{1}{\Gamma(\gamma)} \int_{0}^{\eta} \left( \Phi(\eta) - \Phi(\lambda) \right)^{\gamma - 1} \Phi^{\Delta}(\lambda) \mathcal{G}(\lambda) \Delta \lambda \\
& + \mathcal{T}_{\gamma}^{\phi} J^\delta_\eta \omega_1(\eta) + \mathcal{T}_{\gamma}^{\phi} J^\delta_\eta \omega_2(\eta) + \mathcal{T}_{\gamma}^{\phi} J^\delta_\eta \mathcal{F}(\eta) \\
& = \mathcal{T}_{\gamma}^{\phi} J^\delta_\eta \phi_2(\eta) + \mathcal{T}_{\gamma}^{\phi} J^\delta_\eta \mathcal{G}(\eta) + \mathcal{T}_{\gamma}^{\phi} J^\delta_\eta \omega_1(\eta) + \mathcal{T}_{\gamma}^{\phi} J^\delta_\eta \omega_2(\eta) + \mathcal{T}_{\gamma}^{\phi} J^\delta_\eta \mathcal{F}(\eta). 
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Proof: To prove (A1), from (I) and (II), we have for \( x \in [0, \infty) \) that
\[
(\phi_2(\theta) - \mathcal{F}(\theta))(\mathcal{G}(\lambda) - \omega_1(\lambda)) \geq 0.
\]

Therefore,
\[
\varphi_2(\theta)\mathcal{G}(\lambda) + \omega_1(\lambda)\mathcal{F}(\theta) \geq \omega_1(\lambda)\varphi_2(\theta) + \mathcal{G}(\lambda)\mathcal{F}(\theta).
\]

(24)

Taking product on both sides of (24) \( \frac{\varphi(\phi(\phi)) - \phi(\phi)\frac{\phi(\phi)}{\phi(\phi)}}{\phi(\phi)} \), which is positive because \( \theta \in (0, \eta) \), \( \eta > 0 \), we integrate the resulting identity with respect to \( \theta \) from 0 to \( \eta \) we have
\[
\mathcal{G}(\lambda) \left( \frac{1}{\Gamma(\delta)} \int_{0}^{\eta} \left( \Phi(\eta) - \Phi(\theta) \right)^{\delta - 1} \Phi^{\Delta}(\theta) \varphi_2(\theta) \Delta \theta \\
+ \omega_1(\lambda) \left( \frac{1}{\Gamma(\delta)} \int_{0}^{\eta} \left( \Phi(\eta) - \Phi(\theta) \right)^{\delta - 1} \Phi^{\Delta}(\theta) \mathcal{F}(\theta) \Delta \theta \\
& \geq \omega_1(\lambda) \left( \frac{1}{\Gamma(\delta)} \int_{0}^{\eta} \left( \Phi(\eta) - \Phi(\theta) \right)^{\delta - 1} \Phi^{\Delta}(\theta) \varphi_2(\theta) \Delta \theta \\
& \geq \omega_1(\lambda) \left( \frac{1}{\Gamma(\delta)} \int_{0}^{\eta} \left( \Phi(\eta) - \Phi(\theta) \right)^{\delta - 1} \Phi^{\Delta}(\theta) \mathcal{F}(\theta) \Delta \theta. 
\end{align*}
\]

(25)

Then we have
\[
\mathcal{G}(\lambda) \left( \frac{1}{\Gamma(\delta)} \int_{0}^{\eta} \left( \Phi(\eta) - \Phi(\theta) \right)^{\delta - 1} \Phi^{\Delta}(\theta) \varphi_2(\theta) \Delta \theta \\
+ \omega_1(\lambda) \left( \frac{1}{\Gamma(\delta)} \int_{0}^{\eta} \left( \Phi(\eta) - \Phi(\theta) \right)^{\delta - 1} \Phi^{\Delta}(\theta) \mathcal{F}(\theta) \Delta \theta \\
& \geq \omega_1(\lambda) \left( \frac{1}{\Gamma(\delta)} \int_{0}^{\eta} \left( \Phi(\eta) - \Phi(\theta) \right)^{\delta - 1} \Phi^{\Delta}(\theta) \varphi_2(\theta) \Delta \theta \\
& \geq \omega_1(\lambda) \left( \frac{1}{\Gamma(\delta)} \int_{0}^{\eta} \left( \Phi(\eta) - \Phi(\theta) \right)^{\delta - 1} \Phi^{\Delta}(\theta) \mathcal{F}(\theta) \Delta \theta. 
\end{align*}
\]

(26)
This follows that
\[
\frac{\Gamma}{\alpha} \frac{1}{\alpha} + \frac{1}{\beta} = 0, \quad \forall a, b > 0, \quad \alpha, \beta > 0, \quad \frac{1}{\alpha} + \frac{1}{\beta} = 1, \quad (28)
\]
seting a = \(\mathcal{F}(\lambda)\mathcal{G}(\lambda)\) and b = \(\mathcal{F}(\lambda)\mathcal{G}(\lambda)\), \(\theta, \lambda > 0\), we have
\[
\frac{1}{\alpha} (\mathcal{F}(\lambda)\mathcal{G}(\lambda))^\alpha + \frac{1}{\beta} (\mathcal{F}(\lambda)\mathcal{G}(\lambda))^\beta \geq (\mathcal{F}(\lambda)\mathcal{G}(\lambda))(\mathcal{F}(\lambda)\mathcal{G}(\lambda)).
\]
(29)

Taking product on both sides of (29) \(\left(\Phi(\eta)-\Phi(\eta)\right)^{\lambda-1} \Phi(\eta)\Gamma(\delta)\), which is positive because \(\theta \in (0, \eta)\), \(\eta > 0\), we integrate the resulting identity with respect to \(\theta\) from 0 to \(\eta\) we have
\[
\frac{G^{\alpha}(\lambda)}{\alpha \Gamma(\delta)} \int_0^\eta (\Phi(\eta) - \Phi(\theta))^{\lambda-1} \Phi(\theta) \Gamma(\delta) \mathcal{F}(\theta) \mathcal{G}(\theta) \, d\theta
\]
\[
+ \frac{\mathcal{F}(\lambda)}{\beta \Gamma(\delta)} \int_0^\eta (\Phi(\eta) - \Phi(\theta))^{\lambda-1} \Phi(\theta) \Gamma(\delta) \mathcal{G}(\theta) \Delta \theta
\]
\[
\geq \frac{G(\lambda) \mathcal{F}(\lambda)}{\Gamma(\delta)} \int_0^\eta (\Phi(\eta) - \Phi(\theta))^{\lambda-1} \Phi(\theta) \Gamma(\delta) \mathcal{F}(\theta) \mathcal{G}(\theta) \Delta \theta,
\]
(30)
we get
\[
\frac{G^{\alpha}(\lambda)}{\alpha} \frac{\Gamma}{\beta} \mathcal{F}(\eta) + \frac{\mathcal{F}(\lambda)}{\beta} \mathcal{G}(\eta) \geq \mathcal{G}(\lambda) \mathcal{F}(\lambda) \mathcal{G}(\eta).
\]
(31)
Again, multiplying both sides of (31) by \(\left(\Phi(\eta)-\Phi(\eta)\right)^{\lambda-1} \Phi(\lambda)\), which is positive because \(\lambda \in (0, \eta)\), \(\eta > 0\), we integrate the resulting identity with respect to \(\lambda\) from 0 to \(\eta\) we have
\[
\frac{1}{\alpha \Gamma(\delta)} \int_0^\eta (\Phi(\eta) - \Phi(\theta))^{\lambda-1} \Phi(\lambda) \mathcal{G}(\lambda) \Delta \lambda
\]
\[
+ \frac{1}{\beta \Gamma(\delta)} \int_0^\eta (\Phi(\eta) - \Phi(\theta))^{\lambda-1} \Phi(\lambda) \mathcal{G}(\lambda) \Delta \lambda
\]
(32)
\[ \frac{T}{0^+::\Phi^3_1}\mathcal{F}(\eta)\mathcal{G}(\eta) \geq \int_0^\eta \left( \Phi(\eta) - \Phi(\lambda) \right)^{\gamma-1} \Phi^\Delta(\lambda) \mathcal{G}(\lambda) \Delta \lambda. \] (32)

consequently, we get
\[ \begin{align*}
1 & \frac{T}{0^+::\Phi^3_1} \mathcal{F}(\eta)T^\gamma \geq \frac{T}{0^+::\Phi^3_1} \mathcal{F}(\eta) \mathcal{G}(\eta), \\
\geq & \frac{T}{0^+::\Phi^3_1} \mathcal{F}(\eta) \mathcal{G}(\eta),
\end{align*} \]
which implies (A3). The remaining variants can be proved by adopting the same technique as we did in (A3).

\[ \begin{align*}
(B_3) & \quad a = \frac{\mathcal{F}(\theta)}{\mathcal{G}(\lambda)}, & b = \frac{\mathcal{F}(\theta)}{\mathcal{G}(\lambda)}, \quad \mathcal{F}(\lambda), \mathcal{G}(\lambda) \neq 0, \\
(C_3) & \quad a = \mathcal{F}(\theta) \mathcal{G}(\lambda), & b = \mathcal{F}(\theta) \mathcal{G}(\lambda), \\
(D_3) & \quad a = \mathcal{F}(\lambda) \mathcal{G}(\lambda), & b = \mathcal{G}(\theta), \quad \mathcal{F}(\lambda), \mathcal{G}(\lambda) \neq 0.
\end{align*} \]

Repeating the foregoing argument, we obtain (B3) – (D3). \( \square \)

**Theorem 5.2.** Let \( \delta, \gamma > 1 \), and \( \mathbb{T} \) is a time scale. Suppose \( \mathcal{F}, \mathcal{G} \) be two positive functions on \([0, \infty)\), and \( \Phi \) is monotone, delta differentiable \( \Phi^\Delta \) with \( \Phi^\Delta \neq 0 \) such that for all \( \eta > 0 \), and \( \alpha, \beta > 0 \) satisfying \( \alpha + \beta = 1 \). Then, for \( \eta > 0 \), one has

\[ \begin{align*}
(A_4) & \quad p_{0^+::\Phi^3_1} \mathcal{F}(\eta)T^\gamma \geq \frac{\mathcal{F}(\theta)}{\mathcal{G}(\lambda)}, & b = \frac{\mathcal{F}(\theta)}{\mathcal{G}(\lambda)}, \quad \mathcal{F}(\lambda), \mathcal{G}(\lambda) \neq 0, \\
(B_4) & \quad p_{0^+::\Phi^3_1} \mathcal{F}(\eta)T^\gamma \geq \frac{\mathcal{F}(\lambda)(\mathcal{G}(\lambda)T^\gamma)}{\mathcal{G}(\theta)}, \quad b = \frac{\mathcal{F}(\lambda)(\mathcal{G}(\lambda)T^\gamma)}{\mathcal{G}(\theta)}, \quad \mathcal{F}(\lambda), \mathcal{G}(\lambda) \neq 0. \\
(C_4) & \quad p_{0^+::\Phi^3_1} \mathcal{F}(\eta)T^\gamma \geq \frac{\mathcal{F}(\theta)(\mathcal{G}(\theta)T^\gamma)}{\mathcal{G}(\lambda)}, \quad b = \frac{\mathcal{F}(\theta)(\mathcal{G}(\theta)T^\gamma)}{\mathcal{G}(\lambda)}, \quad \mathcal{F}(\lambda), \mathcal{G}(\lambda) \neq 0. \\
(D_4) & \quad p_{0^+::\Phi^3_1} \mathcal{F}(\eta)T^\gamma \geq \frac{\mathcal{F}(\lambda)(\mathcal{G}(\lambda)T^\gamma)}{\mathcal{G}(\theta)}, \quad b = \frac{\mathcal{F}(\lambda)(\mathcal{G}(\lambda)T^\gamma)}{\mathcal{G}(\theta)}, \quad \mathcal{F}(\lambda), \mathcal{G}(\lambda) \neq 0.
\end{align*} \]

Proof: Taking into account the weighted \( AM - GM \) inequality
\[ \alpha a + \beta b \geq \alpha^2 b^\beta, \quad \forall a, b \geq 0, \quad \alpha, \beta > 0, \quad \alpha + \beta = 1, \] (35)
by setting \( a = \mathcal{F}(\theta) \mathcal{G}(\lambda) \) and \( b = \mathcal{F}(\lambda) \mathcal{G}(\theta) \), \( \lambda, \theta > 0 \), we have
\[ \alpha \mathcal{F}(\theta) \mathcal{G}(\lambda) + \beta \mathcal{F}(\lambda) \mathcal{G}(\theta) \geq (\mathcal{F}(\theta) \mathcal{G}(\lambda))^{\alpha} (\mathcal{F}(\lambda) \mathcal{G}(\theta))^{\beta}. \] (36)

Multiplying both sides of (36) by \( \frac{1}{\Pi(\delta)T^\gamma} \) and integrating the resulting identity from 0 to \( \eta \), we have
\[ \frac{\alpha}{\Pi(\delta)T^\gamma} \int_0^\eta (\Phi(\eta) - \Phi(\lambda))^{\gamma-1} \Phi^\Delta(\lambda) \mathcal{G}(\lambda) \Delta \lambda, \] (37)

we conclude that
\[ \begin{align*}
& \frac{\alpha}{\Pi(\delta)T^\gamma} \int_0^\eta (\Phi(\eta) - \Phi(\lambda))^{\gamma-1} \Phi^\Delta(\lambda) \mathcal{G}(\lambda) \Delta \lambda, \\
& \quad \mathcal{F}(\theta) \mathcal{G}(\lambda), \quad \mathcal{F}(\lambda), \mathcal{G}(\lambda) \neq 0.
\end{align*} \]

Example 5.1. Let \( \delta, \gamma > 1 \), and \( \mathbb{T} \) is a time scale. Suppose \( \mathcal{F}, \mathcal{G} \) be two positive functions on \([0, \infty)\), and \( \Phi \) is monotone, delta differentiable \( \Phi^\Delta \) such that for all \( \eta > 0 \), and \( \alpha, \beta > 0 \) satisfying \( \alpha + \beta = 1 \). Let
\[ m = \min_{0 < b \leq \eta} \frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)}, \quad \mathcal{M} = \max_{0 < b \leq \eta} \frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)}, \] (39)

Then, for \( \eta > 0, \delta, \gamma > 1 \), one has the following inequalities:
\[ \begin{align*}
0 & \leq \frac{T}{0^+::\Phi^3_1} \mathcal{F}(\eta)T^\gamma \geq \frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)}, \quad \mathcal{M} = \max_{0 < b \leq \eta} \frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)}, \\
0 & \leq \frac{T}{0^+::\Phi^3_1} \mathcal{F}(\eta)T^\gamma \geq \frac{\mathcal{F}(\lambda)(\mathcal{G}(\lambda)T^\gamma)}{\mathcal{G}(\theta)}, \quad \mathcal{M} = \frac{\mathcal{M}}{4m, M}, \quad \mathcal{M} = \max_{0 < b \leq \eta} \frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)}, \quad \mathcal{M} = \frac{\mathcal{M}}{4m, M}, \quad \mathcal{M} = \max_{0 < b \leq \eta} \frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)}, \quad \mathcal{M} = \frac{\mathcal{M}}{4m, M}.
\end{align*} \]

Proof: From Equation (39) and the inequality
\[ \int_0^\eta \left( \frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)} - m \right) \left( \mathcal{M} - \frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)} \right) \mathcal{G}(\theta) \frac{\mathcal{G}(\theta)}{\mathcal{F}(\theta)}. \] (40)
then we can write as,
\[
    F^2(\theta) + mMG^2(\theta) \leq (m + M)F(\theta)G(\theta).
\] (41)

Multiplying both sides of (41) by \(\frac{1}{\Gamma(\delta)}(\Phi(\eta) - \Phi(\theta))\Phi^\Delta(\theta)\), which is positive because \(\theta \in (0, \eta)\), \(\eta > 0\) and integrating the resulting identity from 0 to \(\eta\), we have

\[
    \frac{1}{\Gamma(\delta)} \int_0^\eta (\Phi(\eta) - \Phi(\theta))\Phi^\Delta(\theta)F^2(\theta)\Delta\theta
    + mM \frac{1}{\Gamma(\delta)} \int_0^\eta (\Phi(\eta) - \Phi(\theta))\Phi^\Delta(\theta)G^2(\theta)\Delta\theta
\]

\[
    \leq (m + M) \frac{1}{\Gamma(\delta)} \int_0^\eta (\Phi(\eta) - \Phi(\theta))\Phi^\Delta(\theta)F(\theta)G(\theta)\Delta\theta,
\] (42)

implies that

\[
    \frac{T}{\theta^{\alpha-\phi}} J_{\alpha-\phi}^{\Delta\theta} f^2(\eta) + mM \frac{T}{\theta^{\alpha-\phi}} J_{\alpha-\phi}^{\Delta\theta} g^2(\eta) \leq (m + M) \frac{T}{\theta^{\alpha-\phi}} J_{\alpha-\phi}^{\Delta\theta} f(\eta)g(\eta),
\] (43)

on the other hand, it follows from \(mM > 0\) and

\[
    \left( \sqrt{\frac{T}{\theta^{\alpha-\phi}} J_{\alpha-\phi}^{\Delta\theta} f^2(\eta)} - \sqrt{mM \frac{T}{\theta^{\alpha-\phi}} J_{\alpha-\phi}^{\Delta\theta} g^2(\eta)} \right)^2 \geq 0,
\] (44)

that

\[
    2\sqrt{\frac{T}{\theta^{\alpha-\phi}} J_{\alpha-\phi}^{\Delta\theta} f^2(\eta)} \sqrt{mM \frac{T}{\theta^{\alpha-\phi}} J_{\alpha-\phi}^{\Delta\theta} g^2(\eta)} \leq \sqrt{\frac{T}{\theta^{\alpha-\phi}} J_{\alpha-\phi}^{\Delta\theta} f^2(\eta)}
\]

\[
    + \sqrt{mM \frac{T}{\theta^{\alpha-\phi}} J_{\alpha-\phi}^{\Delta\theta} g^2(\eta)}.
\] (45)

then from equation (43) and (45), we obtain,

\[
    4mM \frac{T}{\theta^{\alpha-\phi}} J_{\alpha-\phi}^{\Delta\theta} f^2(\eta) \frac{T}{\theta^{\alpha-\phi}} J_{\alpha-\phi}^{\Delta\theta} g^2(\eta) \leq (m + M)^2 \frac{T}{\theta^{\alpha-\phi}} J_{\alpha-\phi}^{\Delta\theta} f(\eta)g(\eta).
\] (46)

Which implies (1). By some transformation of (1), similarly, we obtain (2) and (3).

6. CONCLUSION

The succinct view of this paper to establish numerous inequalities on an arbitrary time scale for generalized Riemann-Liouville fractional integrals. For the suitable selection of \(\Phi\) on time scale, one can discover numerous novel and existing outcomes as specific cases. This shows the idea of generalized Riemann-Liouville fractional integral is wide and unifying one, yet additionally, improve few consequences in the study on the time scale hypothesis. Numerous variants are explored, when \(T = \mathbb{R}\). Finally, we introduced various dynamic variants by employing generalized Riemann-Liouville fractional integral as an example. Our consequences have potential applications in calcium ion channels, fractional calculus of variations on time scales, involving fractional fundamentalism in mechanics and physics, quantization, control theory, and description of conservative, nonconservative, and constrained systems. The performance of the fractional dynamical integral method is reliable and effective to obtain new solutions. This method has more advantages: it is direct and concise. Thus, the proposed method can be extended to solve many systems of nonlinear fractional partial differential equations in mathematical and physical sciences. Also, the new exact analytical solutions, can be obtained for the generalized ordinary differential equations to obtain new theorems related to stability and continuous dependence on parameters for dynamic equations on time scales. Our computed outcomes can be very useful as a starting point of comparison when some approximate methods are applied to this nonlinear space-time fractional equation.

AUTHOR CONTRIBUTIONS

SR and MA: Conceptualization; SR, MA, and KN: Writing original draft preparation; DB and GR: Formal Analysis; SR and DB: Methodology; KN, DB, and GR: Writing review and Editing.

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REFERENCES

1. Almeida R. A Caputo fractional derivative of a function with respect to another function. *Commun Nonlin Sci Numer Simulat.* (2017) 44:60–81. doi: 10.1016/j.cnsns.2016.09.006
2. Khalil R, Horani MA, Yousef A, Sababheh M. A new definition of fractional derivative. *I J Comp Appl Math.* (2014) 264:65–70. doi: 10.1016/j.cam.2014.01.002
3. Atangana A, Gomez-Aguilar JF. Numerical approximation of Riemann-Liouville definition of fractional derivative: from Riemann-Liouville to Atangana-Baleanu. *Numer Methods Partial Differ Equat.* (2018) 34:1502–23. doi: 10.1002/num.22195
4. Cheng JF, Chu YM. Solution to the linear fractional differential equation using Adomian decomposition method. *Math Prob Eng.* (2011) 2011:587068. doi: 10.1155/2011/587068
5. Cheng JF, Chu YM. On the fractional difference equations of order (2,q). *Abst Appl Anal.* (2011) 2011:497259. doi: 10.1155/2011/497259
6. Cheng JF, Chu YM. Fractional difference equations with real variable. *Abst Appl Anal.* (2012) 2012:918529. doi: 10.1155/2012/918529
7. Kilbas AA, Srivastava HM, Trujillo JJ. Theory and applications of fractional differential equations. *North-Holland Math Stud.* (2006) 204:540.
8. Kumar D, Singh J, Baleanu D. On the analysis of vibration equation involving a fractional derivative with Mittag-Leffler law. *Math Methods Appl Sci.* (2019) 43:443–57. doi: 10.1002/mma.5903
9. Hilger S. Ein Makkettenkalkul mit Anwendung auf Zentrumsmanngteileitu. Universitats-Verlag Wurzburg (1988).
10. Hilger S. Analysis on measure chains a unified approach to continuous and discrete calculus. Results Math. (1990) 18:18–56. doi: 10.1007/BF03232153
11. Hilger S. Differential and difference calculus unified. Nonlin Anal. (1997) 30:2683–95.
12. Bohnen M, Peterson A. Dynamic Equations on Time Scales. Boston, MA: Birkhauser Boston, Inc. (2001).
13. Bohnen EA, Bohnen M, Akin F. Pachpatte inequalities on time scales. J Inequal Pure Appl Math. (2005) 6:1–23.
14. Dinu C. Hermite-Hadamard inequality on time scale. J Inequal Appl. (2008) 2008:24. doi: 10.1155/2008/287947
15. Dinu C. Ostrowski type inequalities on time scales. An Unv Craiova Math Comput Sci Ser. (2007) 34:43–56.
16. Bastos NRO. Fractional calculus on time scales (Ph.D. thesis), University of Aveiro (2012).
17. Bastos NRO, Mozyrska D, Torres DFM. Fractional derivatives and integrals on time scales via the inverse generalized Laplace transform. Int J Math Comput. (2011) 11:1–9.
18. Benkhettou N, Hassani S, Torres DFM. A conformable fractional calculus on arbitrary time scales. J King Saud Univ Sci. (2016) 28:93–8. doi: 10.1016/j.ksus.2015.05.003
19. Gomez-Aguilar JF, Yepez-Martinez H, Escobar-Jimenez RE, Olivares-Peregrino VH, Reyes JM, Sosa IO. Series solution for the time-frational coupled mKDv equation using the homotopy analysis method. Math Prob Eng. (2016) 2016:7047126. doi: 10.1155/2016/7 047126
20. Yepez-Martinez H, Gomez-Aguilar JF, Sosa IO, Reyes JM, Torres-Jimez J. The Feng's first integral method applied to the nonlinear mKDv space-time fractional partial differential equation. Rev Mex Fis. (2016) 62:310–6.
21. Gomez-Aguilar JF, Yepez-Martinez H, Torres-Jimez J, Cordova-Fraga T, Escobar-Jimenez RF, Olivares-Peregrino VH. Homotopy perturbation transformation method for nonlinear differential equations involving to fractional operator with exponential kernel. Adv Differ Equat. (2017) 2017:1–18. doi: 10.1186/s13660-017-0412-7
22. Goswami A, Singh J, Kumar D, Sushila. An efficient analytical approach for fractional equal width equations describing hydro-magnetic waves in cold plasma. Phys A Stat Mech Appl. (2019) 524:563–75. doi: 10.1016/j.physa.2019.04.058
23. Kumar D, Singh J, Ali Qurashi M, Baleanu D. A new fractional SIRS-SI malaria disease model with application of vaccines, anti-malarial drugs, and spraying. Adv Differ Equat. (2019) 2019:278. doi: 10.1186/s13662-019-2199-9
24. Mekhlafi K, Torres DFM. Generalized fractional operators on time scales with applications to dynamic equations. Eur Phy J Spec Top. (2019) 226:3489–99. doi: 10.1140/epjst/e2018-00036-0
25. Morales-Delgado VF, Gomez-Aguilar JF, Yepez-Martinez M, Baleanu D, Escobar-Jimenez RF, Olivares-Peregrino VH. Laplace homotopy analysis method for solving linear partial differential equations using a fractional derivative with and without kernel singular. Adv Differ Equat. (2016) 2016:1–16. doi: 10.1186/s13662-016-0891-6
26. Saad KM, Khader MM, Baleanu D. Numerical solutions of the fractional Fisher’s type equations with Atangana-Baleanu fractional derivative by using spectral collocation methods. Chaos Interdiscipl J Nonlin Sci. (2019) 29:1–13. doi: 10.1063/1.5086771
27. Singh J, Kumar D, Baleanu D. New aspects of fractional Biswas-Milovic model with Mittag-Leffler law. Math Model Nat Phenom. (2014) 19:303. doi: 10.1051/mmnp/2018068
28. Singh J, Kumar D, Baleanu D, Rathore S. On the local fractional wave equation in fractal strings. Math Method Appl Sci. (2019) 42:1588–95. doi: 10.1002/mma.5458
29. Yasar I, Liceli O. Three-point boundary value problems with delta Riemann-Liouville fractional derivative on time scales. Fract. Differ. Calc. (2016) 6:1–16. doi: 10.7155/fdc.06.01
30. Sun M, Hou C. Fractional q-symmetric calculus on a time scale. Adv Differ Equat. (2017) 2017:166. doi: 10.1186/s13662-017-1219-x
53. Nwaeze ER, Torres DFM. Chain rules and inequalities for the BHT fractional calculus on arbitrary timescales. *Arab J Math*. (2017) 6:13–20. doi: 10.1007/s40065-016-0160-2

54. Sheng Q, Fadag M, Henderson J, Davis JM. An exploration of combined dynamic derivatives on time scales and their applications. *Nonlin Anal Real World Appl.* (2006) 7:395–413. doi: 10.1016/j.nonrwa.2005.03.008

55. Agarwal R, Bohner M, Regan D, Peterson A. Dynamic equations on time scales: a survey. *J Comput Appl Math.* (2002) 141:1–26. doi: 10.1016/S0377-0427(01)00432-0

56. Ahmadkhanlu A, Jahanshahi M. On the existence and uniqueness of solution of initial value problem for fractional order differential equations on time scales. *Bull Iranian Math Soc.* (2012) 38:241–52.

57. Kacar E, Kacar Z, Yildirim H. Integral inequalities for Riemann-Liouville fractional integrals of a function with respect to another function. *Iranian J Math Sci Inform.* (2018) 13: 1–13. doi: 10.7508/ijmsi.2018.1.001

58. Tariboon J, Ntouyas SK, Sudsutad W. Some new Riemann-Liouville fractional integral inequalities. *Int J Math Sci.* (2014) 2014:869434. doi: 10.1186/s13661-014-0253-9

59. Kreyszig E. *Introductory Functional Analysis with Applications*. New York, NY: Wiley (1989).

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