Formation time distribution of dark matter haloes: theories versus N-body simulations

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ABSTRACT

This paper uses numerical simulations to test the formation time distribution of dark matter haloes predicted by the analytic excursion set approaches. The formation time distribution is closely linked to the conditional mass function and this test is therefore an indirect probe of this distribution. The excursion set models tested are the extended Press-Schechter (EPS) model, the ellipsoidal collapse (EC) model, and the non-spherical collapse boundary (NCB) model. Three sets of simulations (6 realizations) have been used to investigate the halo formation time distribution for halo masses ranging from dwarf-galaxy like haloes ($M = 10^{-3}M_\ast$, where $M_\ast$ is the characteristic non-linear mass scale) to massive haloes of $M = 8.7 M_\ast$. None of the models can match the simulation results at both high and low redshift. In particular, dark matter haloes formed generally earlier in our simulations than predicted by the EPS model. This discrepancy might help explain why semi-analytic models of galaxy formation, based on EPS merger trees, under-predict the number of high redshift galaxies compared with recent observations.

Key words: methods: N-body simulations – cosmology: theory – dark matter – galaxies: haloes – galaxies: formation

1 INTRODUCTION

In the present preferred cosmological model, dark matter haloes form hierarchically through accretion and merging of smaller structures that grow from a Gaussian initial density field. This process is modeled by Press-Schechter theory (Press & Schechter 1974), which simply assumes a region with the mass over-density above a certain threshold will turn around and eventually collapse to form a bound object. Bond et al. (1991) developed an excursion set approach and used it to derive the number density of collapsed dark matter haloes more rigorously. Lacey & Cole (1993) used the excursion set theory to predict the merger rate at which small objects merge with each other to form larger objects, the conditional probability of its progenitor for a parent halo, as well as the survival probability of haloes. The predictions for the halo formation time distribution have been tested against N-body simulations [with 128$^3$ particles] (Lacey & Cole 1994). The conditional probability predicted by the excursion set theory (or EPS theory) has been widely used to plant merger trees (Kauffmann & White 1993)

Lacey & Cole (1994) [Somerville & Kolatt 1999] so as to construct semi-analytical models of galaxy formation, and to study the clustering of dark matter halos (Mo & White 1996; Mo, Jing, & White 1997). However the EPS theory may fail to model the details of halo formation. For instance, the predicted mass function was not found to fit the simulation results very well (e.g. Lee & Shandarin 1998, Sheth & Tormen 1999). Tormen (1998) also reported that the excursion set predictions did not well fit the conditional mass function of sub-clumps in simulations. In addition, since the EPS theory failed to correctly describe the spatial distribution of small haloes in high resolution numerical simulations (Jing 1998, 1999; Lee & Shandarin 1999; Sheth & Tormen 1999), it may also fail to predict the halo formation time distribution. There are already some indications that the distribution of halo formation time in N-body simulations is not consistent with predictions (van den Bosch 2002a).

In fact, it is an approximation that the spherical collapse assumed in the EPS formalism. This process can be better modeled by a triaxial turn-around model (Bond & Myers 1996; Sheth, Mo & Tormen 2001; Sheth & Tormen 2002). One of such models is the so-called Ellipsoidal Collapse model or EC model. By taking the ellipsoidal collapse into account, Sheth, Mo and Tormen (2001)
found that the modified mass function can fit N-body simulations well. Sheth & Tormen (2002) pointed out that the conditional mass function was not universal since it was not consistent with their simulation results at every redshift. Another model is the Non-spherical Collapse Boundary model (or NCB model shortly) proposed by Chiueh & Lee (2001). It relates the halo formation to the collapse of the Zel’dovich pancakes. A recent version of the model was presented by Lin, Chiueh & Lee (2002). Both the EC and the NCB models were calibrated to fit the spatial two-point correlation function of halos (Jing 1998) and the mass function (Sheth & Tormen 1999) over a large range of halo mass.

This paper uses three sets of high resolution N-body simulations [with 256^3 or 512^3 particles] to study the distribution of halo formation redshift and make comparison with theory predictions. It is organized as follows. In section 2 we present the theory predictions by three analytical models. The simulations are described briefly in section 3 together with the method to find the halo formation redshift. There the simulation results are compared to the predictions. Main conclusions and discussion are given in section 4.

2 THEORETICAL PREDICTIONS FOR THE DISTRIBUTION OF HALO FORMATION REDSHIFT

The halo formation redshift is defined as the redshift at which its main progenitor has accumulated half of the halo mass. According to the EPS theory, the probability that a volume of mass \(M_1\), which is within the region of mass \(M_2\) collapsed at redshift \(z_2\), collapsed to form a progenitor at redshift \(z_1\) is given by the conditional mass function, equation (1):

\[
T(S_1, z_1|S_2, z_2) = \sum_{n=0}^{\infty} \frac{\left((S_2 - S_1)^n\right)}{n!} \frac{\partial^n B(S_1, z_1) - B(S_2, z_2)}{\partial S_1^n}
\]

where the moving barrier \(B(S, z) = \sqrt{\alpha S^3(1 + \beta (S/a_0))^\alpha}\) with \(S_c \equiv \delta_c^2(z)\). The parameters are adopted from the best fitting of the mass function with N-body simulations, \(\alpha = 0.707, \beta = 0.615\) [Sheth, Mo & Tormen 2001; Sheth & Tormen 2002].

While for the NCB model (Chiueh & Lee 2001), the conditional probability is a fitting formula [Lin et al. 2002] which reads as:

\[
\frac{d\mu'}{d\mu} = 2A(\kappa)(1 + \frac{1}{\mu'^2})^{1/2}{\exp}\left(-\frac{\mu'^2}{2}\right)d\mu',
\]

where

\[
A = 0.322 + \frac{0.178}{\kappa},
\]

\[
\kappa = x \equiv (\sqrt{\pi/2}/\delta_c(z_1) - 0.25)/\kappa,
\]

and

\[
\frac{d\mu}{dS} = \frac{\sqrt{2\pi S}}{\kappa} \left(1 - \frac{q(\kappa)}{\kappa}\right),
\]

Here \(\kappa\) is the separation of two boundaries, which is defined as \(\delta_c(z_1)/\delta_c(z_2)\). Refer to the appendix of Lin et al. (2002) for further explanation of the fitting procedure for these parameters. Note that we use the uncorrected \(\mu\) [eq.(29) of Lin et al.] rather than the corrected \(\mu'\) [eq.(33) of Lin et al.]. There are two reasons for this choice. First, the halo formation distribution is incorrectly normalized when using the corrected \(\mu'\). Second, the correction for \(\mu'\) could be mass dependent.

Integrating equation (2) over the mass range \(M_2/2 < M_1 < M_2\) gives the probability \(P(< t_1)\) that its formation time is earlier than \(t_1\) or its formation redshift is larger than \(z_1\). For a halo with mass \(M_0\) formed at the present, we set \(M_2 = M_0, t_2 = t_0, z_2 = z_0 = 0\). Therefore the probability can be written as

\[
P(< t_1) \equiv P(> z_1) = \int_{S_0}^{S_h} M(S_1) f(S_1 | S_0) dS_1,
\]

with \(S_h = S(M_0/2)\) [Lacey & Cole 1993]. In this integration, the conditional probability functions (1), (4) and (5) are used for the EPS, EC and NCB models respectively. The accumulative probability \(P(> z_1)\) and thereby its redshift distribution \(\frac{dP}{dz}\), can be calculated numerically.

The predictions of the halo formation redshift distribution for 3 typical masses are shown in Fig. We assume a LCDM cosmogony (with \(\sigma_8 = 0.9\) whose parameters will

\[
\delta_c(z_1)/\delta_c(z_2) \equiv \left(\frac{S_{c,1}}{S_{c,2}}\right)\frac{\sqrt{2\pi S_{c,1}}}{\kappa_{c,1}} \left(1 - \frac{q(\kappa_{c,1})}{\kappa_{c,1}}\right) = \frac{\sqrt{2\pi S_{c,2}}}{\kappa_{c,2}} \left(1 - \frac{q(\kappa_{c,2})}{\kappa_{c,2}}\right).
\]
be given in next section. The left and right panels show respectively the formation redshift distribution and the probability that a halo formed at redshift larger than \( z_f \). The solid, dotted and dashed lines represent the EPS, EC and NCB predictions respectively. As can be seen, for haloes with a small mass of \( 10^{-3}M_* \), the predictions of the three models differ dramatically from each other. At low redshift the NCB prediction is close to the EPS one, however the predictions by the NCB and EC model are coincident at high redshift. For haloes with masses of 0.1M* and 10M*, the EC and NCB results only have a small difference. For the large mass, there is almost no big difference in the prediction among the three models. In general, the \( dP/dz_f \) profiles of the EC and NCB models are broader and have lower peaks than those of the EPS model. Compared to the EPS model, the EC or the NCB model predicts a larger fraction of haloes formed at high redshift and the EC model predicts a smaller fraction at low redshift.

### 3 SIMULATIONS

Three samples of N-body simulations are used to study the formation redshifts of dark haloes with mass ranging from \( 10^{-3} M_* \) to 8.7 \( M_* \). Each simulation has at least 30 outputs, and can be used to trace the formation of a halo accurately. The cosmological model is the currently popular flat low-density model with the density parameter \( \Omega_0 = 0.3 \) and the cosmological constant \( \Lambda_0 = 0.7 \) (LCDM). The shape parameter of the linear density power spectrum is \( \Gamma = \Omega_0 h = 0.2 \). The characteristic mass \( M_* = 9.55 \times 10^{12} h^{-1} M_\odot \) for LCDMa, LCDMc simulations and \( 1.66 \times 10^{13} h^{-1} M_\odot \) for LCDMb simulations. Other parameters of the simulations are listed in Table 1, where \( \sigma_8 \) is the amplitude of the power spectrum, \( N \) is the total number of particles in the realization, \( m_p \) is the mass of a particle and \( z_i \) is the initial redshift of the simulations.

The simulation data were generated on the VPP5000 Fujitsu supercomputer of the national Astronomical Observatory of Japan with a vectorized-parallel P^3M code (Jing & Suto 1998, Jing & Suto 2002).

#### 3.1 The formation of small dark matter haloes

We use the LCDMa simulations to study the formation of small dark haloes. For each simulation, 56 out of the 169 outputs are used. Only those haloes with 100 particles or more at the present are included to assure a reliable identification of haloes (i.e., at the half-mass formation redshift, they have 50 particles at least). The haloes with more than 1200 particles will not be considered, since these haloes are not abundant enough to have a reliable determination of the distribution of the halo formation redshift. Thus the dark haloes we study here have a mass range between \( 7.73 \times 10^9 \) and \( 9.28 \times 10^{10} h^{-1} M_\odot \) (\( \sim 10^{-3} - 10^{-2} M_* \)). These haloes are “small” and represent the typical dark haloes of dwarf galaxies.

Dark matter haloes are identified with the spherical over-density method (Jing & Suto 2002). We adopt the following method to find the formation redshift of a halo. At the beginning we pick up a halo as a parent halo in the final output (\( z = 0 \)), and find out the particles within its virial radius. Then the member particles are traced in the outputs from high redshift to low redshift step by step. We calculate the fraction of the member particles in all progenitors, and select the progenitor with the maximum number of particles as the main progenitor. Generally, the members of the main progenitor will increase with time because of merging and accretion, although in a few circumstances its mass may decrease because of mass loss due to the tidal stripping by nearby haloes and/or unbound particles. When the main progenitor has reached at least half of the parent halo’s mass for the first time, we define the corresponding redshift as the formation redshift (half-mass formation redshift \( z_f \)) of the selected parent. Alternatively, we can go along the merger tree from redshift 0 to high redshift, and we can define the formation redshift as the time when the mass of the main progenitor first drops below half of the parent halo’s mass. We find that these two definitions give almost an identical distribution of the halo formation time distribution, as serious tidal stripping of the main progenitor happens only rarely. In the following, we will adopt the first definition and compare the simulations with the three analytical models.

The results are shown in Fig. 2(a)-(c). The solid points are the result measured from the simulations, and the vertical error bars present the 1σ error of the mean value derived from the scatter between the realizations. The solid, short-dashed, and long-dashed lines are the predictions of the EPS, EC and NCB analytical models. For the model predictions, we plotted two lines for halo at the lower mass end (the

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1. There is only a slight shift of the profile, but the conclusions change little.
alytical models. Different panels show the results for haloes of
simulation results, and the lines are the predictions of the thr ee ana-
dwarf-galaxy like haloes. The points with error bars are the sim-
The distribution of the halo formation redshift for
Figure 2.
Table 1. Model parameters for simulations

| Model  | $\sigma$ | $N_p$  | box-size $h^{-1}$ Mpc | $m_p$  | time-steps | $z_f$ | outputs | realizations |
|--------|---------|--------|----------------------|--------|------------|------|---------|-------------|
| LCDMa  | 0.9     | 256$^3$| 25                   | $7.73 \times 10^7$ | 5000    | 72   | 169     | 2           |
| LCDMb  | 1.0     | 256$^3$| 100                  | $4.95 \times 10^9$ | 600     | 36   | 30      | 3           |
| LCDMc  | 0.9     | 512$^3$| 300                  | $1.67 \times 10^{10}$ | 1200   | 36   | 60      | 1           |

Figure 2. The distribution of the halo formation redshift for
dwarf-galaxy like haloes. The points with error bars are the sim-
ulation results, and the lines are the predictions of the three ana-
lytical models. Different panels show the results for haloes of $N_p$
particles. The total number of simulated haloes (2 realizations)
used for the analysis are 2334/2295, 1747/1786, 505/456 for the
results in panel (a)-(c) respectively.

curve with higher peak) and the higher mass end (the curve
with lower peak) respectively. We plot both of them, be-
cause there may be a bias when using the prediction for the
mean halo mass, as the formation probability is invariant in
terms of $\sigma(M)^2$(rather than $M$). For the narrow ranges of
halo mass chosen in this study, this bias is sufficiently small
as the predictions at the upper and lower mass limits are
very close. To elucidate another possible bias caused by the
limited output resolution, we also add a right-hand “error”
bar for the simulation results. This bias is introduced by the
fact the halo formation time in simulation is measured to be
the redshift at which the fraction of the member particles is
larger than 0.5 for the first time, however the real formation
redshift should fall between this output and the earlier one.
This uncertainty is shown by the right-hand “error” bars.

As the figure shows, at low redshift ($\leq 1$) the EPS and
NCB models predict more haloes than the simulations, while
the EC predictions are consistent with the simulation re-
results. However at high redshift, the EPS model fits the sim-
ulation results better than the EC and NCB models. In every
panel, the formation redshift distribution of the simulations
peaks at a higher redshift than the EPS prediction $^2$ and
has a narrower profile (but with a higher peak) than the EC
and NCB predictions.

3.2 The formation of sub-$M_*$ dark matter haloes

The LCDMb simulations are used to study the formation of
sub-$M_*$ dark matter haloes. Again the number of particles in a
halo spans from 100 to 1200, or the corresponding halo mass
ranges from $4.95 \times 10^9 \, h^{-1} M_\odot$ to $5.94 \times 10^{12} \, h^{-1} M_\odot$ (i.e.
$\sim 0.03 - 0.36 M_\ast$). The definition of haloes is slightly different
from that used in the last subsection. Here distinct haloes
were found using the FOF method with a bonding length
0.2 times of the mean particle separation. We have tested
the results for one realization against the two halo identi-
fication algorithms, and found that the two identifications
give nearly identical results. We calculate the halo formation
redshift as in the last subsection for the 30 outputs, and plot
the results in Fig.3.

The results found for the sub-$M_*$ haloes are quite simi-
lar to those for smaller haloes, but we can see continuous
changes with halo mass of the model predictions relative to
the simulation results. The EPS model predicts too many
haloes of low formation redshifts again, while the EC and
NCB models predict relatively well the fraction of these
haloes. On the other hand, the EC and NCB models pre-
dict too many haloes of high formation redshift. Note that
for the mass range chosen here the EC and NCB predictions
are close to each other (cf. Fig.1).

3.3 The formation of large dark matter haloes

We use a simulation (LCDMc) with 512$^3$ particles (Jing
2002) to study the formation of large dark haloes. The cor-
responding halo mass ranges from $1.67 \times 10^{12} \, h^{-1} M_\odot$ to
$8.35 \times 10^{13} \, h^{-1} M_\odot$ ($\sim 0.17 - 8.74 M_\ast$). We did the same

$^2$ For the theoretical predictions of the halo formation redshift,
a test on the infrared cut-off of wavenumber $k$ due to a finite
simulation box has been done. The cut-off can modify the $M -
\sigma^2(M)$ relation so that it can change the distribution of the halo
formation redshift. However, for the small haloes considered here,
the wavenumber cutoff has negligible effect on their formation
redshift.
The formation of dark matter haloes

3. Resolution tests

We consider halos with at least 100 particles, so the identification of the halos is generally secure. There are still a couple of issues one should consider about the simulation resolutions. If the simulations are not started sufficiently early, some non-linear collapse could be missed or delayed, and the initial distribution of particles might still have an appreciable effect at the high redshift. Another worry is that some low mass haloes may be missed because of the force softening adopted in the simulations. To check these issues, we performed another two simulations that were run until $z \approx 3.18$. These two simulations have the same model parameters and the same initial fluctuations (including the phase) as the first realization of the LCDMb simulations, but one simulation was started at an earlier epoch $z_i = 72$ and the other adopts force softening $\eta = 78$ kpc that is twice large of the value used in the LCDMb simulation (39 kpc). The two simulations have also the data output at $z \approx 4.78$. In Figure 5 we plot the mass functions $\nu f(\nu)$ of haloes at both redshifts for these two simulations, and compared them with that of the corresponding LCDMb realization, where $\nu \equiv [\delta_c(z)/\sigma(m)]^2$. Except for very massive haloes that are rare objects (so their mass functions have large statistical fluctuations), the agreement of the mass functions among the three simulations is nearly perfect, especially for the mass range of 50 to 600 member particles that we are inter-
simulations: the dots for the simulation with $z$ Figure 5.

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mass function and the spatial correlation of small haloes known that the EPS theory fails accurately describing the observations of faint blue galaxies (van den Bosch 2002b). It is not unexpected since the excursion set approaches roughly failure for the analytical models at different halo mass. This discrepancy between our results and the EPS prediction more severely at $z\sim1$. If the merger trees used by the semi-analytical models are replaced by the merger trees from the simulations, one would expect that the dwarf galaxies will become generally redder, which contradicts with the observations more seriously. There are several ways to solve this discrepancy. First, the observed faint blue galaxies may not constitute a representative sample of small haloes. This could happen if many of red dwarf galaxies, because of their low star formation rate, have escaped from being detected. Second, star formation within small haloes is significantly delayed due to heating or re-ionization (see Mo & Mao 2002, and reference therein). Third, faint blue galaxies may experience a recent interaction with nearby galaxies, which will trigger star formation. We have traced the trajectories of small haloes, and found that about 10% ~ 15% of the small haloes once pass the central part (with distance less than half of a virial radius) of a bigger halo which is at least 3 times more massive than the small one. These haloes should have experienced a strong interaction. If gas in the small halo would not be stripped off, these strong interactions could trigger star formation and change the colour of the dwarf galaxies to blue. However these processes are so complex that more efforts are needed to work out the problem of dwarf galaxies.

From the results of the LCDMb simulations, we found an earlier formation time for sub-$M_\odot$ haloes (about $10^{12}$ solar mass) than the EPS prediction. This may indicate that dwarf galaxies are formed slightly earlier in the numerical simulations than in the EPS model. This result can explain a recent finding of Cimatti et al. (2002). Cimatti et al. measured the redshift distribution of galaxies with $K<20$, and compared it with the predictions of semi-analytical models of galaxy formation based on the EPS merger trees. They found that the galaxies in the observation formed earlier than in the galaxy formation models of about 0.3 in redshift. Qualitatively this can be seen in our Figure 3. Of course, the formation times of haloes and galaxies are not the same thing, but they should have a similar trend.

There seems to be discrepancy between our results and those of van den Bosch (2002a) who found that the formation time distribution of the haloes in the GIF simulation deviates from the EPS prediction more severely at a larger mass. In fact, he checked for the haloes in two mass ranges: $5.6 \times 10^{11} \leq M \leq 1.1 \times 10^{12} h^{-1} M_\odot$ and $2.0 \times 10^{12} \leq M \leq 2.0 \times 10^{14} h^{-1} M_\odot$ (his Figure 4), corresponding to the number of particles $40 \leq N_p \leq 80$ and $140 \leq N_p \leq 14000$ respectively. The number of haloes in each mass bin is dominated by the lower mass end, especially in the larger mass bin. For the small haloes of $M \leq 1.1 \times 10^{12} h^{-1} M_\odot$, it is likely that his formation time is underestimated (i.e. formation delayed) because of the lim-

![Figure 5](image_url)
The formation of dark matter haloes

Bardeen et al. (1980). In practice \( \sigma^2(r_0) \) was calculated up to an overall constant which is fixed by the choice of \( \sigma \equiv \sigma(8h^{-1}\text{Mpc}) \).

Collapsed halos are taken to be regions in the linear density field with the density contrast greater than some critical density contrast, \( \delta_c \). In practice we take into account the linear growth by holding the variance \( \sigma \) fixed and increasing the density thresholds at high redshift,

\[
\delta_c(t(z)) = (1 + z) \frac{g(\Omega_m)}{g(\Omega(z))} \delta_c.
\]

The growth rate can be approximated as

\[
g(\Omega(z)) = \frac{5}{2} \Omega \left[ \frac{1}{70} + \frac{209}{140} \Omega - \frac{\Omega^2}{140} + \Omega^{4/7} \right]^{-1}
\]

and

\[
\Omega(z) = \Omega_m \frac{(1 + z)^3}{1 - \Omega_m (1 + z)^{4/3} \Omega_m}
\]

for a flat \( \Lambda \) universe (Carroll, Press & Turner 1992).

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REFERENCES

Bardeen J., Bond J.R., Kaiser N., Szalay A.S., 1986, ApJ, 304, 15
Bond J.R., & Myers S., 1996, ApJS, 103, 1
Bond J.R., Cole S., Efstathiou G., Kaiser N., 1991, ApJ, 379, 440
Carroll S.M., Press W.H., & Turner E.L., 1992, ARA&A, 30, 499
Chiueh T., & Lee J., 2001, ApJ, 555, 83
Cimatti A., et al., 2002, A&A Letter, 391, L1
Jing Y.P., 1999, ApJ, 503, L9
Jing Y.P., 1999, ApJ, 515, L45
Jing, Y.P. 2002, MNRAS, 335, L89
Jing Y.P., & Suto Y., 1998, ApJ, 494, L5
Jing Y.P., & Suto Y., 2002, ApJ, 574, 538
Kauffmann G., & White S.D.M., 1993, MNRAS, 261, 921
Lacey C., & Cole S., 1993, MNRAS, 262, 627
Lacey C., & Cole S., 1994, MNRAS, 271, 676
Lee J., & Shandarin S.F., 1998, ApJ, 500, 14
Lee J., & Shandarin S.F., 1999, ApJ, 517, 5L
Lin L., Chiueh T., & Lee J., 2002, ApJ, 574, 527
Mo H.J., & Mao S., 2002, MNRAS, 333, 768
Mo, H. J., Jing, Y. P., & White, S. D. M. 1996, MNRAS, 282, 347
Mo, H. J., Jing, Y. P., & White, S. D. M. 1997, MNRAS, 284, 189
Press W., & Schechter P., 1974, ApJ, 187, 425
Sheth R.K., Mo H.J., & Tormen G., 2001, MNRAS, 323, 1
Sheth R.K., & Tormen G., 1999, MNRAS, 308, 119
Sheth R.K., & Tormen G., 2002, MNRAS, 329, 64
Somerville R.S., & Kolatt T.S., 1999, MNRAS, 305, 1
Tormen G., 1998, MNRAS, 297, 648
van den Bosch F.C., 2002a, MNRAS, 331, 98
van den Bosch F.C., 2002b, MNRAS, 332, 456