Simultaneous elements of reality for incompatible properties by exploiting locality

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Abstract

We show that the extensions of quantum correlations stemming from a strict interpretation of the criterion of reality of Einstein, Podolsky and Rosen raise the inadequacy of their ideal experiment for the assignment of simultaneous elements of reality to two incompatible properties. Then, we suggest a different physical situation enabling the simultaneous assignment of objective values of two incompatible observables of a spin particle by means of measurements of two compatible properties of a second correlated spin particle.

1 Introduction

In standard quantum theory [1] the condition characterizing the simultaneous measurability of more observables is the commutativity of the corresponding operators; interpretative questions arise when incompatible properties are involved: in general the problem of ascribing simultaneous objective values to non-commuting observables when them both do not undergo an actual measurement has no answer in standard quantum theory.

In the literature other approaches, aiming to extend or complete quantum mechanics, discuss such a question; for instance, in the context of an extended framework of the operational approach [2], unsharp (or fuzzy) observables are introduced, represented as positive operator valued measures; the simultaneous measurability of two unsharp observables is described by the relation - more general than commutativity - of coexistence asking for the existence of a joint unsharp observable whose statistic contains those of the first two.

A different approach is maintained by the hidden variable (HV) theories [3]-[6], assuming that each specimen of the physical system possesses objective values for every

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observable, fixed by certain unknown parameters “which would complete the information carried out by the quantum states” [7], making possible to determine whether two incompatible properties are possessed or not by a specimen of the physical system even when they are not measured; usually, sometimes implicitly, further assumptions other than the HV hypothesis are introduced, whose validity can be questioned [8]-[10].

In this work we are concerned with the question of ascribing the simultaneous values of two non-commuting observables within the framework of Von Neumann approach. In so doing, we take into account those observables having only 1 and −1 as possible outcomes, called two-value observables.

Whenever a functional relation exists between the outcomes of two commuting observables, it can be expressed in terms of empirical quantum implications, defined as follows [11]:

(QI) Quantum Implication. Let A and P be two compatible observables; in the state ψ the correlation A → P holds if and only if in a simultaneous measurement of A and P the occurrence of the outcome 1 for A implies the occurrence of outcome 1 for P.

In a seminal paper of 1935 [12], Einstein, Podolsky and Rosen (EPR) describe a physical situation able to attain simultaneous knowledge about two non-commuting quantities of a system “on the basis of measurements made on another system that had previously interacted with it”; more precisely they infer the simultaneous values of non-commuting quantities by exploiting correlations between commuting ones. It is crucial for their argument the following criterion of reality:

(R) Criterion of Reality. If, without in any way disturbing a system, we can predict with certainty the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.

Since its appearance, two different interpretations of the criterion (R) are specified; EPR interpreted it as follows:

Wide Interpretation. For ascribing reality to P it is sufficient the “possibility” of performing the measurement of A whose outcome would allow for the prediction, with certainty, of the outcome of a measurement of P.

Such a wide interpretation can be replaced by the narrower following one, maintained by Bohr [13]:

Strict Interpretation. To ascribe reality to P the measurement of A, whose outcome would allow for the prediction, must be actually performed.

In the present work we analyze implications of adopting either interpretations of (R) in connection with the question of ascribing simultaneous elements of reality to two incompatible properties; in particular, we prove that the physical situation described by EPR fails in ascribing reality to two non-commuting quantities which are both not measured if we adopt the strict interpretation of (R); then, we suggest a different ideal experiment enabling such an assignment of objective values.

Before describing the logical structure of the work, we point out the crucial role

1They conclude that since quantum mechanics is unable to describe the simultaneous reality of two non-commuting quantities it is not a complete theory.
played by the following principle of locality:

(L) Principle of Locality. Let $R_1$ and $R_2$ be two space-time regions which are separated space-like. The reality in $R_2$ is unaffected by operations performed in $R_1$.

In fact, when considered together, the principle of locality and the criterion of reality entail an extension of the validity of quantum correlations. The analysis in [14] makes evident that when considered with the wide interpretation of (R), (L) implies the following statement:

(EQC) Extension of quantum correlations. Let $A$ and $B$ be two observables whose measurements require operations confined in two space-time regions $R_A$ and $R_B$, respectively, separated space-like from each other. If quantum mechanics predicts the perfect correlation $A \rightarrow B$ and $B \rightarrow A$, in the state $\psi$, between the outcomes of actually performed measurements of $A$ and $B$, then every individual physical system $x$ in the state $\psi$ possesses objective values of $A$ and $B$ which exhibit the same perfect correlation.

A smaller extension, (sEQC), follows by considering (L) together with the strict interpretation of (R):

(sEQC) Strict extension of quantum correlations. Let $A$ and $B$ be two observables whose measurements require operations confined in two space-time regions $R_A$ and $R_B$, respectively, separated space-like from each other. If quantum mechanics predicts the perfect correlation $A \rightarrow B$ and $B \rightarrow A$, in the state $\psi$, between the outcomes of actually performed measurements of $A$ and $B$, then every individual physical system $x$ which actually undergoes a measurement of at least one within $A$ or $B$ possesses objective values of $A$ and $B$ which satisfy the same perfect correlation.

In [14] it has been shown that (EQC) allows for simultaneous knowledge of non-commutative observables but conflicts with locality; on the other hand it cannot be implied by a strict interpretation of the criterion of reality, so that its validity can be questioned. On the contrary, (sEQC) is consistent with locality; does (sEQC) allow for simultaneous knowledge of non-commuting observables? In other words, can the simultaneous knowledge of non-commuting observables be consistent with locality? In the present work we prove that the affirmative answer is valid, by designing an explicit example.

The logical structure of our work is the following: in section 2 we remind the formalism, extensively introduced in [14], in order to express statements (EQC) and (sEQC) in a suitable form. In section 3 we carry out a formal analysis of the physical situation described by EPR [12] showing that by replacing (EQC) by (sEQC) this ideal experiment no longer allows to assign simultaneous element of reality to two incompatible quantities. In section 4, we propose a different ideal experiment enabling to do this by means of a spin measurement of a particle which, in the selected quantum state, turns out to be correlated with a second separated spin particle. In the concluding section 5 we outline some insights of these results with regard to consistent quantum theory ([15]-[19] and reference therein).
2 The formalism

According to standard quantum theory \([\text{I}],\) every two-value observable \(A\) is represented by a self-adjoint operator \(\hat{A}\) of the Hilbert space \(\mathcal{H}\) associated with the physical system, with purely discrete spectrum \(\sigma(\hat{A}) = \{1, -1\}\); every pure state of the system is represented by a state vector \(\psi \in \mathcal{H}\), with \(\|\psi\| = 1\). The probability of obtaining the outcome 1 by measuring \(A\) in the state \(\psi\) is \(p_\psi(A, 1) = \langle \psi | \frac{1}{2} (1 + \hat{A}) \psi \rangle\).

Given a state vector \(\psi\), we define support of \(\psi\) any concrete non empty set \(S(\psi)\) of individual physical systems \((\text{specimens})\) whose quantum state is \(\psi\).

Given a support \(S(\psi)\), in correspondence with a two-value observable \(A\) we introduce the following subsets of \(S(\psi)\):

- by \(A\) we denote the concrete set of specimens of \(S(\psi)\) which actually undergo a measurement of \(A\); by \(A_+\) (resp., \(A_-\)) we denote the set of specimens of \(A\) for which the outcome 1 (resp., -1) of \(A\) has been obtained. Hence, we can assume that \(A_+ \cup A_- = A\) holds;

- by \(\hat{A}\) we denote the set of the specimens in \(S(\psi)\) which objectively possess a value of the observable \(A\), without being measured (for instance as a consequence of (R)); by \(\hat{A}_+\) (resp., \(\hat{A}_-\)) we denote the set of specimens of \(\hat{A}\) which possess the objective value 1 (resp., -1) of \(A\); hence, we can assume that \(\hat{A}_+ \cup \hat{A}_- = \hat{A}\) holds.

We define \(A = \hat{A} \cup A, \hat{A}_+ = \hat{A}_+ \cup A_+, \hat{A}_- = \hat{A}_- \cup A_-\).

Now, we can introduce the following two mappings \(a : A \rightarrow \{1, -1\}\) and \(a : A \rightarrow \{1, -1\}\):

\[
a(x) = \begin{cases} 
1, & \text{if } x \in A_+ \\
-1, & \text{if } x \in A_-
\end{cases}
\]

\[
a(x) = \begin{cases} 
1, & \text{if } x \in A_+ \\
-1, & \text{if } x \in A_-
\end{cases}
\]

The following statements relate the formalism of standard quantum mechanics with the physical concepts so far introduced.

Given a two-value observable \(A\), since for any \(\psi\) we must have \(p_\psi(A, 1) \neq 0\) or \(p_\psi(A, -1) \neq 0\), we derive the following statement.

If \(A\) is a two-value observable then \(\forall \psi, S(\psi)\) exists such that \(A \neq \emptyset\). \((2.0)\)

According to standard quantum theory, the following statements can be assumed to hold for the simultaneous measurability between two observables \(A\) and \(B\):

\[
[\hat{A}, \hat{B}] \neq 0 \quad \text{implies} \quad A \cap B = \emptyset \quad \text{for all} \quad S(\psi). \quad (2.i)
\]

\[
[\hat{A}, \hat{B}] = 0 \quad \text{implies} \quad \forall \psi \ \exists S(\psi) \quad \text{such that} \quad A \cap B \neq \emptyset. \quad (2.ii)
\]

Statement \((2.ii)\) merely asserts that \([\hat{A}, \hat{B}] = 0\) ensures the concrete possibility of performing measurement of \(A\) and \(B\) simultaneously.

The correlation \(A \rightarrow B\) in the quantum state \(\psi\) can be formulated in several equivalent ways:

\(A \rightarrow B\) if and only if \(A_+ \cap B \subseteq B_+\) if and only if \(B_- \cap A \subseteq A_-\), \(\forall S(\psi),\)

if and only if \(a(x) + 1)(b(x) - 1) = 0\) for all \(x \in A \cap B\) whenever \(A \cap B \neq \emptyset\).
2.1 Extensions of quantum correlations

The conditions of locality and reality (R,L) lead to further implications for separated observables. Let $A$ and $B$ be separated two-value observables, written $A \bowtie B$, i.e. observables whose measurements require operations confined in space-like separated regions $R_A$ and $R_B$. As a consequence of the locality condition (L), the following statement holds.

$$A \bowtie B \implies [\hat{A}, \hat{B}] = 0, \text{ hence } S(\psi) \text{ exists such that } A \cap B \neq \emptyset. \quad (3.i)$$

Let us suppose that $A \bowtie B$ holds, and that $A$ is measured on $x \in A$ obtaining $a(x) = 1$, i.e. $x \in A_+$. If the correlation $A \to B$ also holds, then the prediction of the outcome 1 can be considered valid for a measurement of $B$ on the same specimen. Now, by (L) the act of actually performing the measurement of $A$ does not affect the reality in $R_B$; hence the criterion (R) could be applied to conclude that $x \in B$ and $b(x) = 1$:

$$\text{if } A \bowtie B \text{ and } A \to B \text{ then } x \in A_+ \Rightarrow x \in B_+. \quad (3.ii)$$

It is evident that statement (3.ii) simply follows from the strict interpretation of criterion (R). Analogously, if an actual measurement of $B$ yields the outcome $-1$, i.e. if $x \in B_-$, then the strict interpretation of (R) leads us to infer that $x \in A$ and $a(x) = -1$. Therefore it follows that $B_- \subseteq A_- \subseteq A$ and that the correlation $(a(x) = 1) \Rightarrow (b(x) = 1)$ also holds for every $x \in B_-$. Hence, according to the strict interpretation of the criterion (R) the correlation $(a(x) = 1) \Rightarrow (b(x) = 1)$, besides holding for all $x \in A \cap B$, extends to $A_+ \cup B_-$. Thus, from (R,L) and quantum mechanics we infer the following statement.

(sR) Let $A$ and $B$ be space-like separated two-value observables. If $A \to B$ then

$$(a(x) + 1)(b(x) - 1) = 0, \forall x \in (A_+ \cup B_-) \cup (A \cap B). \quad (4.i)$$

The quantum correlation $A \leftrightarrow B$, i.e. $A \to B$ and $B \to A$, in the state $\psi$ means that the correlation $(a(x) = 1) \leftrightarrow (b(x) = 1)$ holds for all $x \in A \cap B$ for all $S(\psi)$. In this case, from (sEQC) we can deduce that $(a(x) = 1) \leftrightarrow (b(x) = 1)$ holds for all $x \in (A_+ \cup B_-) \cup (B_+ \cup A_-) \cup (A \cap B) = A \cup B$ for all $S(\psi)$. Hence, (sR) incorporates the strict extension (sEQC) of quantum correlation $A \leftrightarrow B$ in the state $\psi$:

$$A \bowtie B, A \Leftrightarrow B \text{ imply } A \cup B \subseteq A \cap B \text{ i.e. } a(x) = b(x), \forall x \in A \cup B, \forall S(\psi). \quad (4.ii)$$

The wide interpretation of criterion (R) allows for larger extensions. Indeed it leads us to infer the following statements:

If $A \bowtie B$ and $A \to B$ then $A_+ \subseteq B_-$ and $B_- \subseteq A_-, \forall S(\psi); \quad (5.i)$

If $A \bowtie B$ and $A \leftrightarrow B$ then $A_+ = B_-, B_- = A_-$ and $A = S(\psi), \forall S(\psi). \quad (5.ii)$

The statement (5.ii) is nothing else but (EQC) stated in formal terms. The statement (5.i) says that the correlation “$a(x) = 1$ implies $b(x) = 1$” extends to $A_+ \cup B_-$. 
3 The criterion of reality and the experiment of Einstein, Podolsky and Rosen

In this section we show that while EPR’s argument can be used to infer simultaneous elements of reality for non-commuting properties when (EQC) is adopted, this no longer holds if (sEQC) replaces (EQC). In so doing, we recur to the simplified form of EPR experiment suggested in [20].

The system is made up of a pair of separated non interacting spin-1/2 particles, in the singlet state $\psi$. Let us denote the two spin observables of the first (resp., second) particle along two fixed non parallel directions by $A$ and $B$ (resp., $P$ and $Q$); in $\frac{1}{\sqrt{2}}\hbar$ units, they are two-value observables. According to quantum mechanics, if we actually measure a spin component, $A$ or $B$, of the first particle then the outcome of an actual measurement of the same component for the second particle, $P$ or $Q$ respectively, turns out to be the opposite. Hence, in the state $\psi$, the correlations $A \leftrightarrow -P$ and $B \leftrightarrow -Q$ hold, i.e. for any $S(\psi)$ the following statements hold for concrete outcomes:

\begin{align*}
\text{i)} \quad a(x) &= -p(x) \quad \forall x \in (A \cap P), \\
\text{ii)} \quad b(y) &= -q(y) \quad \forall y \in (B \cap Q). 
\end{align*}

(6.i)

In [12], by means of the criterion of reality, EPR provide the following argument entailing an extension of the validity of such correlations.

EPR’s argument. By measuring either $A$ or $B$ we can predict with certainty, and without in any way disturbing the system, either the value of $P$ or the value of $Q$; so, according to (R), in the first case $P$ is an element of reality, in second one $Q$ is an element of reality, arriving at the conclusion that two physical incompatible observables have simultaneous reality.

Such a statement is noticeably supported by the wide interpretation of (R); in fact, since $A$ and $B$ are non-commuting quantities, they cannot be measured together. As a consequence (EQC) holds, then correlations (6.i) can be extended to the following correlations between objective values.

\begin{align*}
\text{i)} \quad a(x) &= -p(x), \\
\text{ii)} \quad b(x) &= -q(x). 
\end{align*}

$\forall x \in S(\psi)$.

Hence, in spite of the incompatibility between $A$ and $B$ and the consequent impossibility of measuring them together, every specimen $x \in S(\psi)$ possesses values satisfying the correlations predicted by quantum mechanics.

However, the validity of (EQC) is questioned in [14] where it is shown to be responsible for the inconsistency between quantum mechanics and locality claimed by the non-locality theorems of Hardy [21], of Greenberger, Horne, Shimony and Zeilinger [22] and of Bell [3]; moreover, the quoted inconsistency is proved to disappear if (sEQC) replaces (EQC).

For this reason, we investigate the consequences of adopting (sEQC), instead of (EQC), in connection with EPR’s argument. In particular, we prove that the inferred simultaneous reality of both $P$ and $Q$ no longer holds if the strict interpretation of (R) is adopted. In such a case, the extensions of correlations (6.i) are obtained by applying
\( (4.\,i): \)

\begin{align*}
\text{i)} & \quad a(x) = -p(x), \quad \forall x \in A \cup P \equiv X \\
\text{ii)} & \quad b(y) = -q(y), \quad \forall y \in B \cup Q \equiv Y.
\end{align*}

In order to ascribe simultaneous reality to \( P \) and \( Q \), (7.i) and (7.ii) should hold for the same specimen \( x_0 \in X \cap Y \). From (2.i) we derive

\[
X \cap Y = (A \cup P) \cap (B \cup Q) = (A \cap B) \cup (A \cap Q) \cup (P \cap B) \cup (P \cap Q) = (A \cap Q) \cup (P \cap B).
\]

Since \( X \cap Y \neq \emptyset \), by (2.ii) one could think that some specimen \( x_0 \in S(\psi) \) exists, satisfying all the requirements. However, if \( x_0 \in P \cap B \) we are not allowed to ascribe simultaneous reality to \( A \) and \( Q \); indeed, the principle of locality ensures that the reality in \( \mathcal{R}_2 \) is not affected by operations performed in \( \mathcal{R}_1 \) but an actual measurement of \( P \) occurs just in the regions \( \mathcal{R}_2 \); for instance, let us suppose that \( x_0 \in P \cap B \); since \( x_0 \in B \), locality cannot be invoked for deducing that \( a(x_0) = -p(x_0) \) because the measurement of \( B \) could affect the value of \( A \).

We conclude that, by replacing the wide interpretation of the criterion of reality with the strict one, the example of EPR does not allow to ascribe reality to two non-commuting quantities.

### 4 An ideal experiment

In this section we describe an ideal experiment enabling to ascribe simultaneous reality to two incompatible observables when the strict interpretation of the criterion of reality is adopted.

The observables involved are \( 0\text{-}1 \) observables, i.e. having 0 and 1 as possible outcomes, represented in the theory by projection operators. In such a case statements (4.ii) and (5.ii), involved in our argument, while derived for two-value observables, turn out to be valid for 0-1 observables.

As in the previous ideal experiment we exploit two quantum correlations of the type \( A \leftrightarrow P \), for which (sEQC) implies the extension (4.ii); hence, whenever a measurement of \( P \) is actually performed, we can consider “objective” the observable \( A \), i.e. for every specimen of the physical system which undergoes the measurement of \( P \) we are able to infer whether \( A \) is possessed or not possessed by the specimen.

As a consequence, the physical situation described in the rest of this section not only entails the simultaneous reality of two incompatible properties, but also provides their objective values.

The physical system consists of two separated and non-interacting particles, \( I \) and \( II \). Particle \( I \) is a spin-5/2 particle localized in a region \( \mathcal{R}_I \) and described in the Hilbert space \( \mathcal{H}_I \); particle \( II \) is a spin-3/2 particle localized in a region \( \mathcal{R}_{II} \) and described in the Hilbert space \( \mathcal{H}_{II} \); therefore, \( \mathcal{H}_I \otimes \mathcal{H}_{II} \) is the Hilbert space describing the entire system. We adopt the Heisenberg’s picture; in the notation for operators, the suffix \( I \) (resp. \( II \)) denotes an operator of \( \mathcal{H}_I \) (resp., \( \mathcal{H}_{II} \)). By \( A^1_{II} \) we denote the projection operator of \( \mathcal{H}_{II} \) representing the event “the spin component of particle \( II \) in the z-direction is 3/2”; similarly, we define the projections \( A^2_{II}, A^3_{II}, A^4_{II} \) associated to the
values $1/2$, $-1/2$, $-3/2$ of the spin along $z$, respectively. We denote their respective eigenvectors relative to the eigenvalue 1 by $|\frac{5}{2}\rangle_{I}$, $|\frac{1}{2}\rangle_{I}$, $|\frac{3}{2}\rangle_{I}$. By $A_{I}^{1}$ we denote the projection operator of $H_{I}$ representing the event “the spin component in the $z$-direction is $5/2$”; similarly, we define the projections $A_{I}^{2}$, ..., $A_{I}^{6}$ associated to the values $3/2$, $1/2$, $-1/2$, $-3/2$, $-5/2$ of the spin along $z$, respectively. We denote their respective eigenvectors relative to the eigenvalue 1 by $|\frac{5}{2}\rangle_{I}$, $|\frac{3}{2}\rangle_{I}$, $|\frac{1}{2}\rangle_{I}$, $|\frac{3}{2}\rangle_{I}$, $|\frac{1}{2}\rangle_{I}$, $|\frac{3}{2}\rangle_{I}$.

Let us now introduce three projection operators $B_{i}^{j}$, with $i = 1, 2, 3$, where $B_{i}^{j} = |\psi_{i}^{j}\rangle\langle\psi_{i}^{j}|$ and $|\psi_{1}^{j}\rangle = \frac{1}{2}(|\frac{5}{2}\rangle_{I} - |\frac{3}{2}\rangle_{I} + |\frac{1}{2}\rangle_{I} - |\frac{3}{2}\rangle_{I})$, $|\psi_{2}^{j}\rangle = |\frac{5}{2}\rangle_{I}$ and $|\psi_{3}^{j}\rangle = |\frac{3}{2}\rangle_{I}$.

One of two non-commuting observables, $E$ or $G$, can be measured on system $I$, where:

$$
E = E_{I} \otimes 1_{II} = (A_{I}^{1} + A_{I}^{2} + A_{I}^{3}) \otimes 1_{II}
$$

$$
G = G_{I} \otimes 1_{II} = (B_{I}^{1} + B_{I}^{2} + B_{I}^{3}) \otimes 1_{II}
$$

Now we consider the projection operators $T = 1_{I} \otimes (A_{I}^{1} + A_{I}^{2})$ and $Y = 1_{I} \otimes (A_{I}^{2} + A_{I}^{3})$; straightforward calculations show that $T$ and $Y$ represent commuting properties of particle $II$ so that a support $S_{I}(\psi)$ exists such that $T \cap Y \neq \emptyset$; furthermore, their measurements require operations confined in the region $R_{II}$, so that they are separated from, hence commuting (with), both $E$ and $G$.

Let the system be prepared in the entangled state represented by

$$
\psi = \frac{\sqrt{3}}{4}(|\frac{5}{2}\rangle_{II}|\frac{1}{2}\rangle_{II} + |\frac{3}{2}\rangle_{II}|\frac{3}{2}\rangle_{II} + |\frac{1}{2}\rangle_{II}|\frac{3}{2}\rangle_{II} + |\frac{3}{2}\rangle_{II}|\frac{1}{2}\rangle_{II} + |\frac{3}{2}\rangle_{II}|\frac{3}{2}\rangle_{II} + \sqrt{\frac{3}{8}}|\frac{5}{2}\rangle_{II} - |\frac{5}{2}\rangle_{II}|\frac{1}{2}\rangle_{II}).
$$

In the state $\psi$, projection operators $T$ and $E$ turn out to satisfy the condition $E\psi = T\psi$, which is equivalent to the following relation involving conditional probabilities:

$$
p(E|T) = \frac{\langle \psi|ET\psi \rangle}{\langle \psi|T\psi \rangle} = 1 = \frac{\langle \psi|TE\psi \rangle}{\langle \psi|E\psi \rangle} = p(T|E).
$$

According to quantum mechanics this is equivalent to say that in a simultaneous measurement of $T$ and $E$, outcome 1 (resp., 0) for $T$ (resp., $E$) ensures outcome 1 (resp., 0) for $E$ (resp., $T$), i.e. $E \Leftrightarrow T$ and equivalently

$$
e(x) = t(x) \quad \forall x \in E \cap T, \quad \forall S(\psi).
$$

Similarly, equation $G \psi = Y \psi$ holds, entailing $p(G|Y) = p(Y|G) = 1$ for the conditional probabilities, and equivalent to the quantum correlation $G \leftrightarrow Y$, i.e.

$$
g(z) = y(z) \quad \forall z \in G \cap Y, \quad \forall S(\psi).
$$

Now we prove that if we adopt (sEQC), the envisaged physical situation allows to ascribe the simultaneous objective values to two incompatible properties.

Statement (sEQC) incorporates the following extensions of the quantum correlations in the state $\psi$:

$$
i) \quad e(x) = t(x) \quad \forall x \in E \cup T = X, \quad \forall S(\psi);

ii) \quad g(z) = y(z) \quad \forall z \in G \cup Y = Z, \quad \forall S(\psi).
$$

In the state $\psi$, for any $x \in T \cap Y \subseteq X \cap Y$ extensions (10.i) and (10.ii) hold, so that we can conclude that from the outcomes of actual measurements of $T$ and $Y$ we can infer
both the objective values of $E$ and of $G$, in spite of their incompatibility, according to the following table.

|   |   |   |   |
|---|---|---|---|
| T | Y | E | G |
| 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 0 |
| 0 | 1 | 0 | 1 |
| 0 | 0 | 0 | 0 |

Notice that in the present ideal experiment, the act of ascertaining the value of $E$ does not affect the value of the other observable $G$, contrary to what happens in EPR experiment, because the measurement of $T$ and $Y$ are performed in region $\mathcal{R}_{II}$, which is space-like separated from $\mathcal{R}_I$.

5 Simultaneous reality of incompatible properties in the consistent quantum theory

The assignment of reality to the non-commuting properties $E$ and $G$ of section 4 makes them objective, though not measured. Such a result has interesting insights in the context of the consistent quantum theory (CQT) ([15] and references therein). CQT is an extension of standard quantum theory where the basic concept of event (of standard quantum theory) is generalized to that of history, defined as a finite sequence $h = (E_1, E_2, \ldots, E_n)$ of events that the system objectively possesses at respective times $t_1, t_2, \ldots, t_n$. CQT establishes that when a family of histories $\mathcal{C}$ satisfies a criterion of consistency then

(I) the set of all “elementary” histories of $\mathcal{C}$ is a “sample space of mutually exclusive elementary events, one and only one of which occurs” [15].

The occurrence of a history has to be interpreted as follows:

(O) A given history $h = (E_1, E_2, \ldots, E_n)$ occurs if all events $E_1, E_2, \ldots, E_n$ objectively occur at respective times $t_1, t_2, \ldots, t_n$. The occurrence of a history is an objective fact, independent of the performance of a measurement that reveals this occurrence.

The criterion of consistency postulated by CQT is the following:

(C) A family $\mathcal{C}$ is consistent, in the sense of definition above, if and only if it is weakly decohering. i.e. condition $\text{Re}(\text{Tr}(C_{h_1} \rho C_{h_2}^*)) = 0$ holds for all mutually exclusive histories $h_1$ and $h_2$, where $C_{h_i} = E_n \cdot E_{n-1} \cdots E_1$. In this case $p(h) = \frac{1}{N} \text{Tr}(C_{h} \rho C_{h}^*)$ is the probability of occurrence history $h$.

According to CQT, conclusions drawn in two different families, $\mathcal{C}_1$ and $\mathcal{C}_2$, hold together in the case that these families are compatible, i.e. if a third consistent family $\mathcal{C}$ exists such that $\mathcal{C}_1 \cup \mathcal{C}_2 \subseteq \mathcal{C}$.

Moreover, let $h = (E_1, 1)$ a two-time history and let $\mathcal{C}(h)$ be the smallest family containing $h$; the possibility exists of establishing if a system possesses property $E_1$ by means of the measurement of a different observable $E_2$ at time $t_2$. Indeed, history $h_1 = (E_1, E_2)$ is a refinement of $h$; hence, $\mathcal{C}(h_1)$ is a refinement of $\mathcal{C}(h)$, i.e. $\mathcal{C}(h) \subseteq \mathcal{C}(h_1)$.
$C(h_1)$; if standard quantum theory predicts that $p(E_1|E_2) = 1$ then a measurement of $E_2$ with concrete outcome $1$, together with (O), reveals that $h$ occurred.

The analysis of the conceptual problems raised by CQT led some authors to extend the conceptual basis of CQT [18]. Then, for every family $C$ the existence of a support of $C$ is postulated, defined as the concrete set $b(C)$ of all specimens of the physical system such that for each individual $s \in b(C)$ every history of $C$ either occurs or does not occur (briefly, makes sense). Accordingly, a family $C$ is consistent if and only if $b(C) \neq \emptyset$. Given a history $h$ in such a family, by $b_1(h)$ (resp., $b_0(h)$) we denote the subset of those systems for which $h$ occurs (resp., does not occur).

The concept of incompatible families of standard CQT within the formalism of the extended basis becomes:

i) Let $C_1$ and $C_2$ be two incompatible families, then $b(C_1) \cap b(C_2) = \emptyset$.

The possibility claimed in CQT of revealing the occurrence of $E$ by means of the outcome of a measurement of $T$ can be expressed by the following condition of objectification:

ii) for all $s \in b(C(h_E))$, $s \in b_1(h_T)$ implies $s \in b_1(h_E)$.

Moreover, the following statement is assumed to hold in the extended conceptual basis of CQT [18]:

iii) Let $C_1$ and $C_2$ be two families of histories; then $C_1 \subseteq C_2$ implies $b(C_2) \subseteq b(C_1)$.

In the rest of this section we analyze consequences of adopting the extensions of correlations (sEQC) in connection with CQT.

In the previous section we designed a physical situation making objective two incompatible properties, $E$ and $G$, by means of measurements of $(\text{compatible})$ $T$ and $Y$. We can consider the events $E$ and $G$ at a time $t_1$ immediately prior the events $T$ and $Y$ at a time $t_2$; in so doing, two consistent families of histories naturally arise: the families $C(h_E)$ and $C(h_G)$ where $h_E = (E,T)$ and $h_G = (G,Y)$. They are refinements of families $C(h_T)$ and $C(h_Y)$ respectively, generated by histories $h_T = (1,T)$ and $h_Y = (1,Y)$; so that (iii) implies $b(C(h_E)) \subseteq b(C(h_T))$ and $b(C(h_G)) \subseteq b(C(h_Y))$. As a consequence, in general, for a specimen $s \in b_1(h_T)$, condition (ii) does not entail that $s \in b_1(h_E)$, although in the state $\psi$ the correlation $E \leftrightarrow T$ holds, unless $s \in b(C(h_E))$.

Let us suppose that for a specimen $s$ both $T$ and $Y$ occur, i.e. $s \in b_1(h_T) \cap b_1(h_Y)$; since we defined $T = A_1 + A_2$ and $Y = A_1 + A_3$, where $A_i \bot A_j$ for $i \neq j$, $i = 1, 2, 3, 4$, and $\sum_i A_i = 1$, (I) implies that the elementary history $h_{A_1} = (1, A_1)$ occurs for $s$ hence, $s \in b_1(h_{A_1})$.

Standard quantum theory predicts a non-vanishing probability of occurrence for $A_1$, $p(A_1) \neq 0$; as a consequence, $b_1(h_{A_1}) \neq \emptyset$ so that $s_0$ exists such that $s_0 \in b_1(h_{A_1})$; furthermore, (iii) implies $s_0 \in b_1(h_T) \cap b_1(h_Y)$.

In the state $\psi$, the extension of quantum correlations (10) implies that for any $s_0 \in b_1(h_T) \cap b_1(h_Y)$ we can infer both $s_0 \in b_1(h_E)$ and $s_0 \in b_1(h_G)$. Then, we have to conclude that

$$s_0 \in b(C(h_E)) \cap b(C(h_G)) \neq \emptyset. \quad (11)$$

But $[E,G] \neq 0$, hence the families $C(h_E)$ and $C(h_G)$ are incompatible; as a consequence (11) contradicts (i). Thus, the possibility of a double assignment of objective values
two non-commuting properties, provided in the present work, gives rise to interesting interpretative question in connection with CQT, opening the possibility that an individual specimen $s$ of the physical system can simultaneously follow histories $h_E \in C_E$ and $h_G \in C_G$, though no consistent family exists containing both of them.

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