On the eikonal unitarisation at high energies

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Abstract

We consider approaches to the eikonal-like unitarization of elastic amplitude and its generalizations in theories where cross-sections grow with energy, and we discuss corresponding mechanisms of the multiple exchange standing behind it. In particular, we argue that in such theories the weight of the n-fold exchanges can grow with n much faster than for the simplest Glauber eikonal.

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1. Introduction

The eikonal unitarization is a popular method that allows to obtain the amplitude $F$, which satisfies some minimal $s$-channel unitarity conditions from the “non-unitary” amplitude $A$. In its simplest (Glauber) form, this method consists in the replacing of the initial amplitude $A \rightarrow F = i(1 - \exp (iA)) = -i \sum_n (iA)^n / n!$. This corresponds to a summation of contributions from the multiple exchanges, described by the “primary” amplitude $A$, entering with equal weights.

Sometimes one can associate [1] with this approach a picture of a fast particle moving along an almost straight line in a target media with multiple scatterings, and when the full $S$-matrix is the product of $S$-matrices of the individual scattering. However, at high ultrarelativistic energies, the longitudinal “preparation length” of the state of a fast particle is much larger than the size of the interaction region. So, this simple analogy breaks down, and all such multiple exchanges must occur “at the same time”, and they are needed to take into account various effects, such as the effects of screening and also describe new inelastic processes that are not contained in the initial amplitude.

The mechanism of eikonalization can also be connected [2] with the incrementing of a phase of the wave function of a target particle, while passing through it the frozen field of a fast particle.

In the Regge approach [3], the eikonalization corresponds to the summation of contributions of all multi-reggeon exchanges (the non-enhanced Reggeon diagrams). Here the weights of the $n$-reggeon exchange are given by the values of vertices $N_n$ describing the emission of $n$ reggeons by the external particle. Their values are parameters which are defined outside the Regge approach, and can, in principle, be calculated in the underlying “microscopic” theory. And there is no particular reason to assume that these weights have the simple Glauber form, especially when the coupling is strong and nonperturbative as in QCD.

If the bare amplitude $A(s, t)$ grows with energy and becomes large, then basically the high order terms of eikonal series determine the dynamics of high energy processes and their values should be chosen in agreement with the adequate theory. Usually, however, especially in phenomenological calculations, these eikonal weights are taken in a framework of simplest models, without any specification. Here the most popular form is the Glauber eikonal, or its slightly cured form - the quasi-eikonal [5] expression for the weights of the $n$-fold exchange chosen so as to take into account the contribution from diffraction generation. For the same purpose, the matrix generalization of the eikonal is also often used. But when the pomeron contribution increases with the energy (as in QCD) and the higher order eikonal terms become important such simple corrections may not be sufficient.

In this article we review and discuss various questions related to the eikonalization and, in particular, we argue that one should expect a much more rapid growth of weights of higher order exchanges, up to the limiting behavior, when the eikonal series may even begin formally diverge.

Our consideration is carried out in the Regge approach, and in this case there are not
particularly significant the most of a details of the specific field theory lying behind.

The paper is organized as follows.

In section 2 we recall and briefly overview different approaches to eikonalization and also collect number of general formulas for the multipomeron vertices $N_n$ and for cross-sections in which these $N_n$ enter.

In section 3 we discuss some phenomenological models for a generalized eikonal, that were used to describe the experimental data.

In section 4 we consider the structure of $N_n$ vertices in the parton model, in the QCD and in a number of other approaches.

In section 5 we estimate the behavior of $N_n$ vertices in the $n \gg 1$ limit.

Section 6 contains a brief Conclusion

2. The general relations. Review.

The eikonal unitarization of an elastic amplitude corresponds to the summation of contributions from multiple exchanges, described by some primary amplitude $A(s, t)$. In the Regge approach, this procedure is reduced to summing of contributions of all multireggeon exchanges (that is of all non-enhanced reggeon diagrams). At a sufficiently high energy the amplitude $A(s, t)$ can be represented as a pomeron exchange or as a more complex set of reggeon diagrams (Froissarons in the limit of asymptotic energies in QCD and in similar theories). This leads to the expression for the unitarized (eikonalized) elastic amplitude

\[ F(s, t) = \sum_{n=1}^{\infty} F_n(s, t) , \quad y = \ln s/m^2 , \]

\[ F_n(s, t \simeq -k_{\perp}^2) = \frac{-i}{mn!} \int N_n^2(k_{\perp i}) \prod_{i=1}^{n} \frac{d^2k_{\perp i}}{(2\pi)^2} D(s, k_{\perp i}) \delta^2(k_{\perp} - \sum k_{\perp i}) , \]

where the "primary amplitude" $A = F_1 = G(k_{\perp})D(y, k_{\perp})G(k_{\perp})$ is taken in the factorized form, and where $D(s, k_{\perp})$ is the Green function of pomeron $P$ (or of some more generic object like Froissaron $F$). The vertex function $N_n(k_{\perp i})$, entering Eq. (1) and describing the emission of $n$ pomerons with transverse momenta $k_{\perp i}$ by the external particle, can be expressed via integrals of the product of $G$ vertices. Properties of so defined $N_n$ will be considered in this paper. The multiple exchange contribution of type (1) is always contained in a full amplitude, regardless of the structure of the "irreducible amplitude" $D(s, k_{\perp i})$.

For a sufficiently large $s$ the direct interaction between pomerons can become essential, and then the general contribution to $F$, which takes into account all inter-pomeron interactions, can be represented by diagrams Fig.1b, where the block $\Gamma_{mn}$ corresponds to a general

\footnote{For simplification of formulas we do not write explicitly in $\{I\}$ signature factors and suppose that all nontrivial complexity of amplitude $A$ is included in $D$. We also assume that both colliding particles $1+2 \rightarrow$}
transition amplitude of \( n \) pomeron to \( m \). Note in this regard that the average rapidity values \( \langle y_i \rangle \), on pomeron lines, connecting vertices \( N_n \) and \( \Gamma_{mn} \), are not growing when the full rapidity \( Y = \ln s \rightarrow \infty \). Their average values are determined by the magnitude of vertices of inter-pomeron interactions (in particular by the 3\( P \) vertex \( r_{3P} \),... ), so that \( \langle y_i \rangle \sim g/r_{3P} \sim \) the value of a “free path in y of pomeron” in the multipomeron “medium”.

At asymptotic energies the full amplitude \( F \) can be represented (Fig.1c) by the sum of non-enhanced reggeon diagrams with the Froissaron \( \mathcal{F} \) exchange. Note that such non-enhanced diagrams must be included in the full amplitude \( F \) with large weights, because the elastic and diffraction generation cross-sections (which are big at \( s \rightarrow \infty \); \( \sigma_{\text{elast}} \simeq \sigma_{\text{tot}}/2 \) ) are connected with specific unitarity cuts of the full amplitude.

Under rather general assumptions (in fact the same as for the Eq. (1)) the expressions for vertices \( N_n(k_i) \), entering Eq.(1), can be written [4] as an expansion over on mass shell states of diffractive-like beams

\[
N_n(k_i) = \sum_{\nu_1,\nu_2,\ldots,\nu_n} \int G_{\nu_1}(P_{in},p_i^1) G_{\nu_2}(p_i^1,p_j^2) \cdots \\
\cdots G_{\nu_{n-1}}(p_i^{(n-1)},P_{out}) \prod_{i=1}^{n-1} d\Omega_{\nu_i}(p_i^{(1)}),
\]

where \( G_{\nu_1\nu_2}(p_i^1,p_j^2,k_\perp) \) is the transition amplitude for a beam of \( \nu_1 \) particles with momenta \( p_i^1 \) into a beam of \( \nu_2 \) particles with momenta \( p_j^2 \), and with the emission of a “pomeron” with the transverse momentum \( k_\perp \). In (2) the \( d\Omega_{\nu}(p_i) \) is the element of the \( \nu \)

\( 1 + 2 \) are equal, so in all amplitudes enter factors of the type \( N_n^2 \) and not \( N_n^{(1)}N_n^{(2)} \)

\( 2 \) In this case, it may be useful to redefine the input amplitude from pomerson to some approximate froissarons \( A \rightarrow \simeq \mathcal{F} \)

\( 3 \) In Eq(2) it is also assumed that the pomeron-particles amplitudes \( G_{\nu_1\nu_2} \) entering \( N_n \) duo not contain
particles phase-space volume. The expression (2) contains a summation and an integration over all kinematically allowed physical states of particles in beams including full masses of beams. Note that all vertices \( N_n(k_i) \) are real, while multiparticle amplitudes \( G_{\nu_1\nu_2} \) are complex, and they also contain disconnected contributions.

The Glauber eikonal form corresponds to a minimal contribution to \( N_n(k_i) \) from the single-particle state in beams. In this case, all integrations in (2) disappear and the result is the simple factored expression

\[
N_n(k_i) = \prod_{i=1}^{n} g(k_i) ,
\]

where \( g(k) = G_{11}(p, p+k) \) - is the elastic pomeron vertex. This leads to a great simplification and to the exponential form of the elastic S-matrix in the impact parameter representation

\[
S(y, b) = S[v] = \sum_{n=0}^{\infty} \frac{(i v(y,b))^n}{n!} = e^{iv(y,b)} = 1 + iF(y,b) , \tag{4}
\]

\[
v(y,b) = \int d^2 k_\perp e^{ik_\perp\cdot y} g^2(k_\perp) D(y,k_\perp) , \quad y = \ln s/m^2
\]

For a large \( y \) the value of \( \text{Im} \ v(y,b) \) can become large (such as in the case of the supercritical regge pole \( v \sim g^2\vartheta(\Delta) \exp(\Delta y - b^2/4\alpha' y) \), \( \vartheta \approx i, \ \Delta > 0 \)), and then the expression \( (1) \) corresponds to a picture of colliding black discs with the Froissart asymptotic behavior.

The value of \( |S(y, b)|^2 \) is equal to the probability that both fast particles will pass near one another (at the impact parameter \( b \)) without transition to the another states. Therefore, the Glauber expression for \( |S|^2 = e^{-2\text{Im}(v)} \), which follows from \( (4) \), can be interpreted in a simple way. It is given by the Poisson probability that single particles states pass through each other without interaction, and where the value \( 2\text{Im} \ v \) gives the average number of inelastic interactions in such a process. The Glauber form of \( S[v] \) given by \( (4) \) corresponds to that all these \( 2 \text{Im} \ v(y,b) \) interactions are uncorrelated.
The expression (2) for \( N_n(k_1) \) can be represented in the symbolic operator form
\[
N_n(k_1) = \langle P_{in}|\hat{G}(k_1)\hat{G}(k_2)\cdots\hat{G}(k_n)|P_{out}\rangle =
\sum_{\nu_1,\ldots,\nu_{n-1}} \langle P_{in}|\hat{G}(k_1)|\nu_1\rangle \langle \nu_1|\hat{G}(k_2)|\nu_2\rangle \langle \nu_2|\hat{G}(k_3)|\nu_3\rangle \cdots \langle \nu_{n-2}|\hat{G}(k_{n-1})|\nu_{n-1}\rangle \langle \nu_{n-1}|\hat{G}(k_n)|P_{out}\rangle,
\]
as the average of the product of non-local field operators \( \hat{G}(k) \) describing the pomeron emission vertices \( G_{\nu_1\nu_2}(k) \) between the initial and final state of the external particle, and on the next step (6) as the decomposition of this product over the full system of physical states of beams \( |\nu\rangle \). If it is possible to redefine the bases for the beam states \( |\nu\rangle \) so that \( \hat{G}(k)|\nu\rangle = g_\nu(k)|\nu\rangle \), that is to make all vertex operators \( \hat{G}(k) \) diagonal, then the expression (6) is simplified
\[
N_n(k_1) = \sum_\nu w(\nu) \prod_1^n g_\nu(k_i) , \quad w(\nu) = \langle P_{in}|\nu\rangle \langle \nu|P_{out}\rangle ,
\]
where \( w(\nu) \) is the probability to find the fast hadron in the state \( |\nu\rangle \). After that the elastic \( S(b, y) \) matrix in an impact parameter representation can be represented in such a form
\[
S(b, y) = \sum_{\nu_1\nu_2} w(\nu_1) w(\nu_2) \sum_{n=0}^\infty \frac{(i \nu\nu_2(y, b))^n}{n!} = \sum_{\nu_1\nu_2} w(\nu_1) w(\nu_2) e^{i\nu_1\nu_2(y,b)} ,
\]
\[
v_{\nu_1\nu_2}(y, b) = \int d^2k_\perp e^{ibk_\perp} g_{\nu_1}(k_\perp) D(y, k_\perp) g_{\nu_2}(k_\perp) ,
\]
generalizing the Glauber eikonal expression (4). If we assume that the dominant contribution to such a diagonalized vertices (7) is universal
\[
g_{\nu}(k) = g(k)\lambda(\nu) + \tilde{g}(\nu, k) ,
\]
so that the non-factorisable part \( \tilde{g}(\nu, k) \) is relatively small \footnote{Such a factorization corresponds to a situation when the properties of “emitted pomerons” do not depend on the number of particles in beams and the value of \( \lambda(\nu) \) is in fact the number of “average” particle in the beam \( \nu \). This factorized form of \( g_\nu(k_\perp) \) can be expected only for a small \( k_\perp \), since for large \( k_\perp \) the contribution of states \( |\nu\rangle \) of a small transverse size \( \sim k_\perp^{-1} \) can be most essential. However, for large \( k_\perp \) the contributions of higher exchanges are not so significant, and for small \( k_\perp \) the value of \( \beta_n \) is roughly proportional to the number of particles in beams, because the individual pomerons are emitted mainly by the different particles in beams.}
\[
N_n(k_1) \approx \beta_n \prod_1^n g(k_i) , \quad \beta_n = \sum_\nu w(\nu) (\lambda(\nu))^n .
\]
The form (10) for $N_n$ (which is somewhat simplified in comparison with (8)) is such a generalization of the Glauber eikonal which we will mainly consider below. Turning to the impact parameters as in (4), we obtain the generalized eikonal series

$$S(y, b) = \sum_{n=0}^{\infty} \frac{\beta_n^2}{n!} (i v(y, b))^n,$$

where the real weights $\beta_n \geq 1$ can be almost arbitrary. It is useful to present the series (11) in a compact form, resembling the Glauber exponential. One can do this in a number of ways. Here we use one of them. Substituting in (11) coefficients $\beta_n$ in the form

$$\beta_n = \int_0^\infty d\tau \, \tau^n \, \varphi(\tau),$$

we can write $S[v]$ as a superposition of Glauber like eikonal contributions

$$S(y, b) \equiv S[v] = \int_0^\infty d\tau \rho(\tau) \, e^{i v(y, b)} \, \varphi(\tau), \quad \rho(\tau) = \int_0^\infty \frac{d\tau_1}{\tau_1} \, \varphi(\tau_1) \, \varphi(\tau/\tau_1),$$

entering with weights $\rho(\tau)$, where $\tau g$ act as the effective pomeron vertex for the $|\tau>$ state. The normalization condition for $S[v]$ and $w(\nu)$ leads to conditions $\beta_0 = \beta_1 = 1$, and this implies relations

$$\int_0^\infty d\tau \rho(\tau) = \int_0^\infty d\tau \, \tau \rho(\tau) = 1.$$ (14)

Note also the convexity condition

$$\beta_n \beta_m \leq \beta_{n+m},$$

following from the inequality

$$N_n(k_1, \ldots k_n) \, N_m(q_1, \ldots q_m) \leq N_{n+m}(k_1, \ldots k_n, q_1, \ldots q_m),$$

which leads to the following relation for the $S[v]$-matrix as a function of the amplitude $v$

$$S^2[x] \leq S[2x], \quad \text{for Re } x = 0, \, Im \, x < 0,$$

and where the equality is only for the Glauber eikonal case (11).

The Glauber case corresponds to the simple density

$$\rho(\tau) = \delta(\tau - 1),$$ (15)

and for the matrix model with $L$-states the spectral function is

$$\rho(\tau) = \sum_{n=1}^{L} c_n \, \delta(\tau - \tau_n), \quad \tau_n, \, c_n \geq 0, \quad \sum_{n=1}^{L} c_n = \sum_{n=1}^{L} \tau_n c_n = 1.$$ (16)

5Discussion of a more general case is in Section 2c and in Section 3
In addition we also list a number of general relations for cross-sections at given impact parameter, which are expressed in terms of the function \( S[v(y, b)] \) and which are valid for an arbitrary spectral density \( \rho(\tau) \):

the total cross-section

\[
\sigma_{\text{tot}}(y, b) = 2(1 - \Re S[v]) \quad ,
\]  
(17)

the elastic cross-section

\[
\sigma_{\text{el}}(y, b) = 1 - S[|v|]^2 \quad ,
\]  
(18)

the total inelastic cross-section

\[
\sigma_{\text{in}}(y, b) = \sigma_{\text{tot}} - \sigma_{\text{el}} = 1 - |S[v]|^2 \quad ,
\]  
(19)

the pionization cross-section - corresponding to processes when at least one pomeron is s-cut

\[
\sigma_{\pi}(y, b) = 1 - S[2i\Im v] \quad ,
\]  
(20)

the total cross-section of diffraction generation (single - \( \sigma_{sd} \) and double - \( \sigma_{dd} \))

\[
\sigma_{\text{dif}}(y, b) = \sigma_{\text{in}} - \sigma_{\pi} = 2\sigma_d + \sigma_{dd} = S[2i\Im v] - |S[v]|^2 \quad .
\]  
(21)

The expression

\[
\sigma_n(y, b) = \int_0^{\infty} d\tau \rho(\tau) \left( \frac{2\tau \Im v}{n!} \right)^n e^{-2\tau \Im v} \quad , \quad \sigma_\pi = \sum_{n=1}^{\infty} \sigma_n
\]  
(22)

gives the cross-section for \( n \) soft “multiperipheral” beams (these are the contributions of diagrams with \( n \) cut pomerons and of the arbitrary number of uncut pomeron lines) as a superposition of Poisson distributions.

The quantity

\[
\overline{n}(y, b) = \frac{N_n \sigma_n(y, b)}{\sum_n \sigma_n} = 2\Im v(y, b) \quad ,
\]  
(23)

as it follows from (22) and (14), gives the average multiplicity of cut pomerons as a function of \( (y, b) \). It is interesting that this quantity does not depend on the form of \( \rho(\tau) \), i.e. on values of \( \beta_n \).

\[\text{In the Eq. (17 - 22) all cross-sections are given at certain fixed values of the impact parameter and are dimensionless. The corresponding “usual” quantities are given by } \int d^2b \sigma_{\ldots}(y, b)\]
In the field theory approach the eikonal S-matrix is defined \[ \hat{S} \simeq T \exp \left( i \int L_{\text{int}} \right) \], where the T-ordered integration is taken along straight paths of fast particles. It is convenient to describe this scattering in the laboratory frame of one of the colliding particles. The field of the fast charged particle with energy \( E \) is concentrated in the longitudinally compressed thin \( \sim 1/E \) sheet and it is frozen for the time of collision, and therefore it can actually be regarded as a classical perturbation while this field disc passes through the target. Under these conditions, the elastic S-matrix is given by the expression \( \langle \text{out} | \hat{S} | \text{in} \rangle = e^{i\delta(E, b)} \), where \( \delta(E, b) = A(E, b) \) is a phase gained by the wave function of a “target particle” when the fast disk-particle passes through it. This directly leads to the Glauber expression (4) for \( S[A] \).

However, such a simple expression for the \( S \) does not account the possibility that the incoming particle can fly up to the target being in a different “valence” states, and therefore carry a different “classical” fields. Moreover, there can be significant fluctuations of field values on different energy scales ( which corresponds to the parton cascading ). This last one, in particular, leads to a longitudinal spreading ( up to \( \sim 1/m \) ) of the field incident on the target particle.

Some of these effects can be accounted for by summing over all possible valent states (classical fields) of a fast particle approaching the interaction region:

\[
\langle \text{out} | \hat{S} | \text{in} \rangle = \sum_{\alpha, \beta} \langle \text{out} | U(\infty, t_0) | \alpha \rangle \langle \alpha | U(t_0, -t_0) | \beta \rangle \langle \beta | U(-t_0, -\infty) | \text{in} \rangle ,
\]

(24)

where \( |\alpha\rangle, |\beta\rangle \) are the “parton” states of fast particles, involved in the interaction, and where \( t_0 \gg 1/m \), because the preparation time of such states is usually much larger than the interaction time. Leaving here only diagonal states, which correspond to the approximation that one neglects possible changes of the fast particle field in the interaction time interval \((-t_0 \div t_0)\), we obtain

\[
\langle \text{out} | \hat{S} | \text{in} \rangle = \sum_{\alpha} \langle \text{out} | U(\infty, t_0) | \alpha \rangle \langle \alpha | U(t_0, -t_0) | \alpha \rangle \langle \alpha | U(-t_0, -\infty) | \text{in} \rangle \sim
\]

\[
\approx \sum_{\alpha} |\langle \text{in} | U(-\infty, t_0) | \alpha \rangle|^2 \langle \alpha | U(-t_0, t_0) | \alpha \rangle \approx \sum_{\alpha} w_\alpha e^{i\delta_\alpha} ,
\]

(25)

where quantity

\[
w_\alpha = |\langle \text{in} | U(-\infty, t_0) | \alpha \rangle|^2
\]

gives the probability that the fast incoming particle is in the state \( |\alpha\rangle \) directly before the interaction with a target, and the \( \exp(i\delta_\alpha) = \langle \alpha | U(-t_0, t_0) | \alpha \rangle \) - is a factor acquired by the wave function of the target while passing through the field of the fast particle in the state \( |\alpha\rangle \). The expression (25) for \( S \) has the same structure as the spectral representation (13).
The behavior of $S[v]$ at large $|v|$ corresponds to the high energy limit $y \gg 1$, because usually $v(y, b) \sim f(y, b) \exp(\Delta y)$. As follows from (13), the asymptotic behavior of $S[v \to \infty]$ is determined by the spectrum $\rho(\tau)$ at small values of $\tau$. If $\rho(\tau) = 0$ for $\tau < \tau_0$ then $S[v] \sim \exp(-\tau_0 |v|)$. This occurs, for example, in finite matrix models for vertices $N$. 

The general case, when the spectrum $\rho(\tau)$ extends to $\tau = 0$, is more interesting. If $\phi(\tau = 0) \neq 0$, then, as follows from (12), $\rho(\tau) \sim \ln \frac{1}{\tau}$ for $\tau \sim 0$, and we obtain from (13) the asymptotic expression for $S[v]$ at $|v| \gg 1$:

$$S[v] \simeq c_0 \frac{\ln \hat{v}}{\hat{v}} + \frac{c_1}{\hat{v}} + \frac{c_2}{\hat{v}^2} + \ldots, \quad \hat{v} = -i\theta|v| \simeq I\nu v.$$ \hspace{1cm} (26)

Similarly, if $\varphi(\tau \ll 1) \sim \tau^\lambda$, $\lambda > -1$, we get $S[v] \sim 1/\hat{v}^{1+\lambda}$. It is also natural to suppose that $\rho(\tau)$ has no singular contributions of the type $\delta(\tau)$, because they correspond to such components of Fock state of a fast particle that do not interact with the target and cause the asymptotic $S[v \to \infty] \to \text{const}$.

Note that the expression for the eikonal series (11) can be represented in the form

$$S[v] = -\int_{c-i\infty}^{c+i\infty} \frac{dn}{\sin \pi n} \Gamma(1-n)v^n \beta_n^2,$$ \hspace{1cm} (27)

when using the continuation of $\beta_n^2$ to non-integer $n$. The same analytic continuation of $\beta_n$ defines the behavior of the spectrum

$$\varphi(\tau) = -\int_{c-i\infty}^{c+i\infty} \frac{dn}{2\pi i \tau^{n+1}} \beta_n.$$ \hspace{1cm} (28)

The distribution of singularities of the function $\beta_n$ is connected with the properties of spectrum $\varphi(\tau)$. As follows from (27) the behavior (26) is determined by the rightmost singularity of $\beta_2(n)$ for $n = -1$. The expression (11) describing the Glauber eikonal corresponds to $\beta_n^2$ regular for all finite $\tau$. A more general case, when the spectral density $\varphi(\tau \ll 1) \sim \tau^\lambda$, $\lambda > -1$, corresponds to the rightmost singularly $\beta_n$ at $n = -1 - \lambda$.

Eikonal with many different virtuality scales

As was noted above the factorization of the type (9) for vertices of $G_{\nu\nu}(k_\perp)$ should not be expected in the general case, and the dependence of amplitudes $V_{\nu_1\nu_2}(k_\perp)$ on the pomeron momentum $k_\perp$ may be various for different states $\nu$. This is well illustrated by a simple
example of a model, when $V_{\nu_1 \nu_2}$ is the contribution of a certain Regge pole, and when the dependence of vertices from $r$ is taken (for simplicity) in the exponential form

$$V_{\nu_1 \nu_2} = \frac{g_{\nu_1} g_{\nu_2}}{4\pi R^2_{\nu_1 \nu_2}} \exp \left( \Delta \tilde{y} - b^2 / R^2_{\nu_1 \nu_2} \right),$$

where $R^2_{\nu_1 \nu_2} = R^2_{\nu_1} + R^2_{\nu_2} + 4\tilde{y} \alpha'_{\nu_1 \nu_2}$, $\tilde{y} = y - i\Delta/2$. Then for soft beams with a large mass the value of $R^2_{\nu_1 \nu_2}$ can be large. And for those components of $\nu_i$, to which more hard components of pomeron are attached, the value of $R^2_{\nu_1 \nu_2}$ can be relatively small.

One can expect that in theories (like QCD), where the effective coupling decrease with the growth of virtuality, the spectrum of pomeron states would be discrete. Under these conditions, the generalized eikonal series for the diagonalized amplitude (as in (11)) can be approximately written as a sum of contributions arising at the different scales of virtuality

$$F(y, b) = \sum_{\mu} \sum_{n=1}^{\infty} \beta^2_\mu(n) \left( i v_\mu(y, b) \right)^n / n!, \quad \sum_{\mu} \beta^2_\mu = 1, \quad (29)$$

where $v_\mu(y, b)$ is the primary scattering amplitude, on the $\mu$-th scale of virtuality. It is also assumed that in the expression (29) the virtuality of the beam state does not change at vertices of $\hat{G}(\mu_i)$. This corresponds to a relative independence of scattering processes on different virtuality scales.

An example of these amplitudes can be the simple Regge contributions

$$v_\mu(y, b) \simeq v^{(0)}_\mu \exp \left( \Delta_\mu y - b^2 / 4y r^2_\mu \right), \quad \mu = 1, 2, 3, ... \quad (30)$$

coming from the chain of poles [9] with intercepts $\Delta_\mu \sim 1/\mu$ and with the increasing virtualities, corresponding to the average transverse dimensions $r^2_\mu \sim \exp (-c/\mu)$. The amplitudes (30) can be combined to give the BFKL type Pomeron which the QCD coupling running with scale, and for with it is also possible to include the main non-perturbative effects, adjusting the value of first intercepts $\Delta_\mu$.

A similar to (29) type expression for $F$ can be obtained [7] if we consider the high energy scattering process in 5-dimension and connect the average pomeron virtuality with the radial AdS coordinate.

The total $S(y, b)$ matrix corresponding to (29) can be written as the sum of $S_\mu$-matrices related to different scales of virtuality

$$S(y, b) = \sum_{\mu=1}^{\infty} S_\mu(y, b), \quad (31)$$

$$S_\mu(y, b) = \sum_{n=0}^{\infty} \left( \beta_\mu^{(n)} \right)^2 \left( i v_\mu(y, b) \right)^n / n!, \quad \sum_{\mu} \beta_\mu^{(n)} = 1.$$

\[9\] These virtualities are mainly related to transverse dimensions of colliding states and they do not have enough time to change during the collision.
The same expression for \( S \) in the exponential form, similar to (13), is
\[
S(y, b) = \sum_{\mu} \int_{0}^{\infty} d\tau \rho(\tau, \mu) \exp \left( i\tau v_{\mu}(y, b) \right),
\]
where the spectral density \( \rho(\tau, \mu) \) depends now also on the virtuality \( \mu \).

Expressions (31 - 32) correspond to a simple physical picture that the total amplitude for the particles to pass without interaction at the impact parameter \( b \) is given by the superposition of amplitudes to pass without interaction in different virtual states, because these states at a very high energy are prepared during times which are much longer than the time of the collision.

In theories with the \( \Delta_{\mu} \sim 1/\mu \) pomeron sequence when \( y \) increases the gradual inclusion of higher \( S_k \) matrices with a more and more large virtuality take place. The corresponding picture of the Froissart limit will be such that the parton structure of a fast particle can be represented as a system of nested more and more hard disks expanding with the increasing of energy.

3. Some phenomenological models for the generalized eikonal

In this section we briefly review a number of popular models for vertices \( N_n \equiv g^n \beta_n \), generalizing the simple Glauber eikonal. Such models were used to describe the experimental data on hadron elastic and inelastic processes at high energies, and so they reflect many of properties of “correct” \( N_n \) vertices, which unfortunately can not be calculated using the existing now QCD methods.

1) The simplest generalization of the Glauber eikonal is the quasieikonal model [5]. Here eikonal coefficients are chosen in the form
\[
\beta^2_n = C^{n-1} + \delta_{n0}(1 - C^{-1})
\]
where the parameter \( C \) represents a contribution of the average beam in (10) and it is directly related to the cross-section of diffraction generation \( \sigma_{dif}/\sigma_{el} = C - 1 \). From a comparison with various experimental data at not too high energies it follows that \( C \approx 1.2 \). The spectral density has the form
\[
\rho(\tau) = C^{-1}\delta(\tau - C) + (1 - C^{-1})\delta(\tau).
\]
In this case \( S(b, y) = 1 + C^{-1}\exp \left( iCv(b, y) \right) - 1 \). In fact, this model assumes the existence of two states of a fast hadron - the single-particle component interacting with the vertex \( g \), and of a naked component entering in the spectrum at \( \tau = 0 \), and which at all does not participate in the interaction. This, in particular, leads to a nonanalyticity of \( \beta_n \) at \( n = 0 \).
Probably, such non-interacting states corresponding to \( \delta(\tau) \) type contribution to spectrum, can not occur in reasonable field theories. The simplest corrected expression

\[
\beta_n^2 = \frac{n(2 - C) + C}{1 + n} C^{n-1}
\]

has approximately the same properties as (33) but does not violate relations (14) and at large \( n \) give \( \beta_n^2 \simeq C^{n-1}(2 - C) \).

Quite indicative is the one-parameter model, with extremely rapid growth of eikonal coefficients

\[
\beta_n^2 = \frac{\Gamma(n + a)}{a^n \Gamma(a)},
\]

and where the eikonal series converge rather poorly for amplitudes \( v \) growing with energy. The corresponding spectral density is given by

\[
\rho(\tau) = \frac{a}{\Gamma(a)} \tau^{a-1} e^{-a\tau},
\]

and the \( S(b, y) \)-matrix is

\[
S[v] = \left(1 - iv/a\right)^{-a}.
\]

The representation (35) looks particularly simple in the extreme case when \( a = 1 \). All expressions (17 - 22) for cross sections are also strongly simplified in this case:

\[
\sigma_{tot}(y, b) = \frac{2v_2}{1 + v_2}; \quad \sigma_{in} = \frac{2v_2}{1 + 2v_2}; \quad \sigma_{dif} = \frac{|v|^2}{(1 + 2v_2)(1 + 2v_2 + |v|^2)};
\]

\[
\sigma_n(y, b) = \frac{1}{1 + 2v_2} \left(\frac{2v_2}{1 + 2v_2}\right)^n; \quad v_2 = Imv;
\]

The form of \( S[v] \) similar to (35) was discussed many times in the literature. In the next sections we present some dynamical arguments that the behavior of this type can be naturally realized for theories (like QCD) with a growth with the energy cross-sections.

For the not smooth at \( \tau = 0 \) spectrum \( \rho(\tau) = c_1 \delta(\tau) + \rho_1(\tau) \), where the dependence of \( \beta_n \) on \( n \) is nonanalytic, we come to a non-decreasing with \( |v| \to \infty \) contribution to \( S[v] \). An example of such a behavior, which is close to the (34), is the one-parameter model

\[
\beta_n^2 = c^{n-1} n! + \delta_{n0}(1 - c^{-1}) \quad \rho(\tau) = e^{-2c\tau/c} + \delta(\tau)(1 - c^{-1}),
\]

for which

\[
S[v] = \frac{1 + iv(1 - c)}{1 - ivc}
\]

If \( c = 1 \), then this expression obviously coincides with (35) for \( a = 1 \)\(^{10}\). If \( c = 1/2 \), this expression leads \(^{10}\) to an elegant form \( S = (1 + iv/2)/(1 - iv/2) \). For this case \( |S[v]| \leq 1 \), because \( Im v \geq 0 \). For \( |v| \to \infty \) we have \( S[v] \to -1 \).
2) The poly-eikonal model [8] is defined by relations
\[ \beta_n = \prod_{k=1}^{n-1} \xi(n, k) , \quad \xi(n, k) \simeq (1 + \frac{\lambda}{k+a})^{n-k} . \] (37)

It is a simple two-parameter generalization of the Glauber eikonal, where the average multiplicities of particles in beams grow with \( n \), and where each factor \( \xi(n, k) \) corresponds to a beam with \( \sim n - k \) particles. The expression (37) effectively takes into account the non-planar structure of \( N_n \) and also includes the disconnected contributions to \( G_{\nu_1\nu_2} \) vertices. At large \( n \) from (37) one can obtain that \( \beta_n^2 \sim (n!)^\lambda \). The coefficients \( \beta_n^2 \) can be analytically continued in \( n \) so that the only singularity at finite \( n \) is the pole at \( n = -1 - a \).

3) The matrix models for \( N_n \). Here the spectrum of beams is approximated by the discreet sequence of states \( |\nu\rangle \) in Eq. (5). That is all multiparticle beams are replaced by \( L \) resonances, located on the mass shell. In this case vertices \( G_{\nu_1\nu_2} \) describing the transition between states \( |\nu_i\rangle, \ i = 1, 2, ..., L \) are the real symmetric \( L \otimes L \) matrices. Choosing the value of \( L \) large enough and tuning parameters of these matrices, one can obviously get a description of \( N_n \) vertices with any degree of accuracy. Then the diagonalization can always be done explicitly by the orthogonal transformation of basis \( |\nu_i\rangle \), and we get
\[ S(y, b) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\nu_1\nu_2} w_{\nu_1} w_{\nu_2} \left( -iV_{\nu_1\nu_2} \right)^n = \]
\[ = \sum_{\nu_1, \nu_2} w_{\nu_1} w_{\nu_2} e^{-iV_{\nu_1\nu_2}} , \] (39)

where \( w_{\nu} \) is the probability to find a component \( |\nu\rangle \) in the \( |\nu_i\rangle \) state, and
\[ V_{\nu_1\nu_2} = \int d^2k_\perp e^{ibk_\perp} G_{\nu_1\nu_1}(k_\perp) D(y, k_\perp) G_{\nu_2\nu_2}(k_\perp) \] (40)
are diagonalized amplitudes. It is clear that these expressions also follow directly from the general representation (8) for \( S \) if we assume that the spectrum \( w(\nu) \) is discrete.

If we suppose additionally the factorization for vertices \( G_{\nu\nu}(k) \) of the type (10) then
\[ V_{\nu_1\nu_2}(b, y) \simeq \beta_{\nu_1} \beta_{\nu_2} v(b, y) . \]

In this case
\[ \varphi(\tau) = \sum_{i=1}^L c_i \delta(\tau - \tau_i) ; \quad S(y, b) = \sum_{i,j=1}^L c_i c_j e^{i\tau_i \tau_j} v(b, y) . \]
4. $N_n$ vertices in the parton model and scattering of constituents

4a Constituent scattering

It is essential that only the non-planar Feynman diagrams give the contribution to the multipomeron vertices $N_n$. This is usually interpreted so as if all parton chains (corresponding to pomeron amplitudes $v$) must exist simultaneously in the state of fast particles and these chains are attached to the different “valence” partons, at that colliding particles “split”. In such an approach it is possible not to specify all the properties of these primary partons and take into account only the fact that each external hadron is some superposition $\sum a_m |m\rangle$ of states $|m\rangle$ containing $m$ such partons with probabilities $w_m = |a_m|^2$.

Then in the reggeon diagrams with the multi pomeron exchange one can use some pomeron-like amplitude $\lambda^2 v(y, b)$ describing the elastic scattering of valent parton. Here $\lambda = g_0/g$ is the ratio of the parton $g_0$ to hadron $g$ vertices, describing the pomeron emission. Under these assumptions, the amplitude of elastic hadron scattering can be written as

$$F^{(a,b)}[v(y, b)] = \sum_{m_1, m_2}^{\infty} w^{(a)}_{m_1} w^{(b)}_{m_2} f_{m_1, m_2}[v],$$

where $f_{m_1, m_2}[v]$ is the elastic scattering amplitude of partons in states $|m_1\rangle \otimes |m_2\rangle$. It can be expressed in terms of the $\lambda^2 v$ from a purely combinatorial consideration, as the sum over all possible rescatterings of partons

$$f_{m_1, m_2}[v] = -\sum_{n=1}^{\infty} n! C^m_{m_1} C^m_{m_2} (-\lambda^2 v(y, b))^n,$$

Here each binomial coefficient $C^m_n$ corresponds to a selection of $n$ partons participating at scattering, from the available in states $|m_1\rangle \otimes |m_2\rangle$, and the factor $n!$ responds to different possible versions of interaction of these partons.

Substituting the amplitude (42) to (41) and comparing $F^{(a,b)}$ with the generalized eikonal form (11) we obtain such an expression

$$\beta_n = n! \lambda^n \sum_{m=n}^{\infty} C^m_n w_m,$$

connecting vertices $N_n = g^n \beta_n$ with weights of primary parton states. Inverting these relations we obtain expressions for parton probabilities $w_m$ in terms of eikonal factors $\beta_n$:

$$w_m = \frac{1}{m! \lambda^m} \sum_{n=0}^{\infty} \frac{(-\lambda^{-1})^n}{n!} \beta_{n+m}.$$

---

11The amplitude $\lambda^2 v(y, b)$ entering this expression include integrations over the transverse coordinates of partons, with some vertex factors with respect to the center of parton beams.
Substituting in (44) expressions for \( \beta \) in the spectral form (12) we come to the representation of parton probabilities

\[
w_m = \int_0^\infty d\tau \, P_m(\tau/\lambda) \varphi(\tau), \quad P_m(\tau) = \frac{1}{m!} \tau^m e^{-\tau}
\]

as a superposition of the Poisson distributions with the average \( \langle m \rangle = \tau \), and with weights \( \varphi(\tau) \) defined by the spectrum of diagonal states of diffractive beams. It is evident from (45) that Glauber eikonal corresponds to a simple Poisson distribution of number of valent constituents \( w_m \) in the fast hadron with the average \( \langle m \rangle = \lambda^{-1} \).

As follows from (45) the behavior of \( w_m \) at large \( m \) is related to the properties of spectrum \( \varphi(\tau) \) at \( \tau \gg 1 \). If the spectrum \( \varphi(\tau) \) is bounded from above, so that \( \varphi(\tau) = 0 \) for \( \tau > \tau_{\text{max}} \), as it is in the case of finite matrix models, then the tail of \( w_m \) distribution has again the Poisson form \( w_m \sim \tau_{\text{max}}^m / m! \). For a unlimited from above spectrum \( \varphi(\tau) \) the decrease of \( w_m \) can be slower. So for the Gaussian tail, when \( \ln \varphi(\tau) \sim -\tau^2 \), the asymptotics is flatter \( w_m \sim c^m / \sqrt{m!} \). And for the case \( \ln \varphi(\tau) \sim -\tau \), the tail takes power form \( w_m \sim c^m, \quad c < 1 \).

For a large \( n \) the general approximate relation

\[
\beta_n \simeq n! \, \lambda^n \, w_n
\]

follows from (43). Therefore, for Glauber eikonal and for the finite matrix models when \( w_m \sim c^m / m! \), we have that \( \beta_n \sim \lambda^n \), and the Gaussian spectrum for \( \varphi(\tau) \) leads to \( \beta_n^2 \sim n! \).

### 4b Parton picture in Fock representation and eikonal

The approach to high energy scattering based on the parton picture can give a somewhat different view on the role of the multiple scattering included to the eikonalization, since in this case all restrictions, following from the s-unitary, are more simple and clear.

Consider the collision of a fast particle with high energy \( E = me^Y \) in the laboratory frame of another particle, when the elastic amplitude \( \nu \) is given by a reggeon (Pomeron) ladder with \( \Delta > 0 \). In the corresponding parton description the main components of Fock wave function \( |\Psi(Y)\rangle \) of a fast particle correspond to states, created by the parton cascade, which contains \( \sim |\nu| \sim v_0 \exp \Delta Y \) low-energy partons. During the collision mainly these low-energy partons interact with the target particle. This picture is approximately true for all field theories with \( \Delta > 0 \), including QCD.

If the energy \( E \) is not particularly large, or when \( g^2_\rho \) is small, so that \( |\nu| \ll 1 \), then the multiple rescattering (eikonal correction) are not so important, although the elastic scattering and diffraction generation are contained in the \( N^2_c \) eikonal term.

At \( |\nu| \sim 1 \) the eikonal corrections are of the same order as the first term (single exchange).

Here the main “destination” of rescatterings (corresponding to non-enhanced reggeon diagrams) is firstly to take into account various screening in the interaction of target particle
with parton “medium” incoming with the fast particle. They also describe various new processes arising from multiple interactions and the diffraction.

If the energy is so large that \(|v| \gg 1\), then in the eikonal series many terms \((\sim |v|)\) are essential. But for such a large \(y\) the parton merging becomes significant, and as a result the partons sequential saturation takes place, on scales with the virtuality increasing with \(y\). And later the parton system transits to the Froissart phase when the saturated parton disk expands with \(y\), and gradually becomes more and more black as the mean virtuality of partons grows.

To better understand how the S-unitarity is restricted by the eikonalization it is helpful to consider the special case when the colliding energy is very large, so that \(|v| \gg 1\), but the interaction between pomerons is switched off. This means that one must not include the contributions of enhanced reggeon diagrams \(^{12}\). On the parton language this regime corresponds to a model when the partons in cascades are not glued one with another. The parton disk of a fast particle is on average very tightly packed, and the density of low-energy partons infinitely grows with energy. In this case the probability \(W\) for the target particle to tunnel through such fast dense parton disk without interaction is very small. It is important that if the average parton density \(\sim |v(Y, b)|\) is large, the probability \(W\) is approximately determined by the classical physics, without significant interference effects. The average number of interactions of the target particle with such a disk is proportional to its density \(|v(Y, b)|\). And because these interactions are almost independent (up to \(r_{3P}/g\)), then the distribution in a number of interactions will be close to the Poisson form. This implies that the probability that a fast particle does not interact at all with such a disk is

\[
W(Y, b) = e^{-c|v(Y, b)|} \tag{47}
\]

and because \(W(Y, b) = |S[v(Y, b)]|^2\) this \(S(Y, b)\) directly corresponds to the Glauber eikonal \(^{11}\).

It is interesting to note that the minimal Glauber eikonal answer \(^{11}\) is boost-invariant in the parton interaction picture and this signals that here the t-unitarity is properly taken into account. To see this let us consider this tunnelling process in an arbitrary longitudinal system, when the energies of the colliding particles are \(\sim m e^{y_1}, m e^{y_2}, y_1 + y_2 = Y\). Then the probability of \(W(y_1, y_2, b)\) should depend only on \(y_1 + y_2\). If all individual parton interactions are not mutually correlated, then the probability of the parton disk ”\(y_1\)” to go through an another disk ”\(y_2\)” without interaction on the impact parameter \(b\), as follows from \(^{11}\), will be

\[
W(y_1, y_2, b) \sim \exp \left( -c \int d^2b_1 \ln |S(y_1, \vec{b}_1)| \rho(\vec{b} - \vec{b}_1, y2) \right) \sim \exp \left( -c \int d^2b_1 \ v(y_1, \vec{b}_1) \ v(y_2, \vec{b} - \vec{b}_1) \right) \sim e^{-c[v(y_1 + y_2, b)]} \sim W(y_1 + y_2, b) \tag{48}
\]

Such a non-trivial answer arises only because all quantities entering \(^{12}\) behave in a very special way. Namely: \(\ |S| \sim \exp (-c|v|)\); the parton density in the disk \(\rho(b, y) \sim |v(y, b)|\)

\(^{12}\)This mode is almost realized in QCD in a certain range energy, since the \(3P\) vertex is relatively small - \(r(3P)/g(hhP) \ll 1\)
and the amplitude \( v(y, b) \sim y^{-1} \exp \left( -b^2/4y + \Delta y \right) \) has the diffusion structure. If we choose the weights \( \beta^2_n \) in some non-Glauber form, than the boost invariance of \( W \) can be broken. It can be broken also if we take into account the interaction between soft pomerons, because then the saturated soft Froissart disk will be gray.

But the expressions of the type \([47, 48]\) take place only if we neglect the possibility of large fluctuations in the parton disk and consider the collision of particles only in states with the average value of the parton density in disks given by the amplitude \(|v|\). The same will be true when we take into account only local fluctuations of the parton density around the mean value. However, in the parton cascade very large density fluctuations are indeed possible, even such that occupy all disk at once, and which arise mostly from fluctuations at the first steps of the cascade.

The largest fluctuations of this type are such that there are at all no secondary partons in a fast particle state, so that the state contains only the minimal valence components. Such state is completely transparent even for arbitrary large \( E \). Therefore, here the value of \(|S|^2\) is determined directly by the probability of fluctuations bringing the fast particle to such a transparent state. In theories with growing cross-sections of vector type (like QCD) the probability to be in the state without any secondary soft partons is \( w(y) \sim \exp \left( -c_1y \right) \). This is just the probability that the valence partons do not emit any soft photon (gluon ...) in the whole interval of rapidity \( y \). It is essential that such an estimate of \( w(y) \) is valid for any \( y \), including the case of Froissart limit with the soft saturated parton disk.

It is evident that the tunnelling mechanism associated with such fluctuations is much more efficient than that corresponding to a simplest eikonal, it is compared to a tunnelling in an “average” state.

The state without additional partons arise from specific fluctuations in the initial stage of the cascade. Therefore, one may think that the effects of these fluctuations can be taken into account in the reggeon amplitudes, just properly adjusting the coefficients \( N_n \) in eikonal series to get

\[
|S(y, b = 0)|^2 \sim w(y) \sim \exp \left( -c_1y \right) . \tag{49}
\]

It follows from (34) and (35) that for this it is necessary that \( \beta^2_n \sim n! \). Than we will have \(|S| \sim 1/|v|^a \sim \exp \left( -ya\Delta \right)\). So we come to the expression for \( \beta_n \) of the same limiting type that was discussed above.

Note that for the interaction of a projectile with the composite target consisting of \( l \) particles, the probability for such a projectile to tunnel through a dense disk will be \(|S|^{2l}\), if we consider only a mean disk configurations and the independent tunnelling of all its \( l \) components. Or this probability will contain an extra small factor \( \sim r_0^{2l} \), where \( r_0 \) is the size of the region in which all \( l \) target particles should gather, so that their tunnelling looked like the passage of a single particle. These probabilities are parametrically of the same order. At the same time, in configurations where fast particles do not have a secondary parton (it is no soft disk), the probability to pass without interactions will also be the same \(|S|^2\) and does not depend on \( l \).

If one takes into account also hard components in the Froissart disk (like in QCD at
$E \to \infty$) then the estimate (39) will change. Now it is necessary to find the probability that the valence components (quarks) do not emit secondary gluons-partons with arbitrary energies and transverse momenta. This leads (in the language of the model section 2c) to the behavior

$$w(y) \sim \prod_{i=1}^{\sqrt{y}} |S_i(y, b = 0)|^2 \sim \exp(-cy \ln y), \quad S_i \sim e^{-y \Delta_i}, \quad \Delta_i \simeq c/i.$$  

But this quantity is still much bigger than the corresponding Glauber probability

$$w(y) \sim \exp(-|v|) \sim \exp(-c_2 y^{-1} \exp(\Delta y)).$$

4c Remarks on the diffraction generation processes at very high energies

Here we briefly discuss how processes of diffraction generation are connected with the structure of beams entering the $N_n$ vertices.

At very high energies, when $|v| \gg 1$, in the inelastic vertices $N_n$ the beams with the large average number of particles $\nu \sim n \sim |v| \gg 1$ will be significant. Therefore, it may seem that the average number of particles created in the diffraction generation processes can grow in the same way - as $|v| \sim e^{\Delta y}$ - so as grows the average number of cut lines in the non-planar Feynman diagrams that determine the $N_n$ vertices. However, as follows from the relation (21), the contribution to the cross-section of inelastic diffraction processes comes only from such impact parameters, when $|v(y, b)| \sim 1$ and $S \sim 1$, and not from configurations when $1 - |S| \ll 1$ or $|S| \ll 1$\(^{13}\). Therefore in the corresponding eikonal series only the several number of terms will be non small, and this leads to a small average multiplicity of particles in diffraction beams. Such behavior is natural in the parton picture, because for the impact parameters when $|v(y, b)| \gg 1$, all valent components of the initial state are absorbed equally and therefore give a contribution only to the elastic diffraction component. Thus, the diffraction generation processes come only from the border of a dense parton disk.

The interaction between pomerons $P$ slightly changes this picture, because now there appear other types of diffraction processes (of the $3P\ldots$ type), which give the wide distribution over the beam masses.

In the Froissart limit the elastic amplitude $F(Y, b) \simeq i\tilde{\theta}(r^2_0 Y^2 - b^2)$, where $\tilde{\theta}$ is the theta-function with the edge, smeared on the width $y_0 \approx 1$. In this case the dependence of the cross section from the diffraction beam mass ($\sim e^y$) has the form

$$d\sigma_{dif}(Y, y)/dy \sim Y^{-1} \sqrt{(y + y_0)(Y - y + y_0)}.$$  

\(^{13}\text{If the hard components in } v \text{ are also big, then the situation may change somewhat due to the diffraction generation of the hard jets.}\)
This corresponds to the growth of the average mass of diffraction states as \( \sim s^{1/4} \). The full cross-sections of these processes \( \sim Y^{3/2} \), while the cross-section of the diffractive beams with limited mass is \( \sim Y \).

### 4c Comments on peculiarities of the nonperturbative QCD

We have discussed above the properties of \( N_n \) vertices that are not QCD-specific. In the Regge approach main peculiarities of QCD are manifested only in the specific value of parameters of Regge theory. The perturbative QCD dynamics leads to BFKL-like structure of pomeron and corresponds to the inclusion of the large parton virtualities with the increasing energy.

The main nonperturbative field-theoretic effects are firstly reflected in the significant difference between values of regge vertices \( g_P, N_2 \simeq g_P^2 \) ... describing the emission of pomerons by the external hadrons and of vertices like \( R_{3P} \) ... describing the interaction between pomerons themselves.

It is often assumed that this difference is a consequence of the presence of two different scales in the nonperturbative QCD. The "big" radius \( \rho_1 \sim 5 GeV^{-1} \) is defined by the scale of the quark condensate \( \langle q\bar{q} \rangle \), and the second "relatively small" radius \( \rho_2 \sim (0.5 - 1) GeV^{-1} \) can be associated with the value of the correlation lengths in the gluon vacuum [11]. The first scale determines the size of hadrons and also the size and value of vertices \( g_P, N_2, N_3, ... \). The second scale defines the QCD-boundary between perturbative and non-perturbative regimes as well as the average "size of gluons" and of the corresponding (pomeron \( \bigotimes \) pomeron) interaction vertices \( r_{3P} = r_{P-\rightarrow 2P}, r_{2P-\rightarrow 2P} \) ... [14].

The parameter \( \rho_1/\rho_2 \) defines the relative contribution of the non-enhanced (eikonal-like) and enhanced (with inter-pomeron interaction) reggeon diagrams. And the big value of this parameter \( \rho_1/\rho_2 \gg 1 \) in QCD causes that the contribution of the non-enhanced reggeon diagrams dominates in the hadron cross sections up to very high energies, and the \( 3P \) ... screening is not very important even at LHC-energies. Only for the interaction of heavy nuclei, when the number of the effective pomeron exchanges increases in \( A^{1/3} \) times, the fusion of the "thin" pomeron chains (described by \( r_{3P} \)) becomes also important [15].

In such a picture the pomeron lines, entering in \( N_n \) vertices, are attached to the valent quarks or to additional \( q\bar{q} \) pairs, arising from the exchange of light quarks with the \( \langle q\bar{q} \rangle \) condensate. Such \( q\bar{q} \) pars can be present in the valent part of the Fock wave function of fast hadron with some weights \( w_k \).

One can estimate the connection of \( w_k \) with \( N_n \) identifying these weights \( w_k \) with the probabilities of the valent constituents of Section 4a and directly using relations [13 - 16].

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14 The smallness of \( \alpha'_P \) compared to its natural value \( \sim m_{\pi}^{-2} \), and the smallness of the "experimental" value of the pomeron intercept \( \Delta_P \) compared to the perturbative BFKL-value can also be associated with the presence of the two QCD scales

15 Probably, this also explains why the quark-gluon string model [5] describes quantitatively all major observed effects in the high-energy \( hh \) interactions, and with inclusion of simplest \( 3P \) tree diagram corrections also the main \( A \bigotimes A \) effects.

20
Then we come to the following type association

\[ N_n \sim c^n \quad \Leftrightarrow \quad w_m \sim c^n / m! \]

\[ N_n \sim c^n \sqrt{n!} \quad \Leftrightarrow \quad w_m \sim c^n / \sqrt{m!}. \]

Considering the \( w_k \) as a distribution of the number of \( q\bar{q} \) pairs in the nucleons wave function, one can estimate its behavior in various models, for example in the model of “bag”, whose size is fluctuating due to exchange of quarks with the surrounding \( < q\bar{q} > \) condensate. Probably, this mechanism can be effectively taken into account through the contributions to \( N_n \) from the tree diagrams with \( \pi \) and \( \rho \) reggeons (see Section 5).

4d Comments about the specifics of high energy gravitational interactions

The primary amplitude for a gravitational exchange increases very fast with energy \( v \sim s / m_p^2 \), where \( m_p \) is the Planck mass. But for the “existing” energies such amplitudes are too small \( |v| \ll 1 \). The higher eikonal contributions (multiple gravitational exchanges) can become important only at planckian energies, where \( s > m_p^2 \) or for collisions of an ultra-relativistic particle with the energy \( \varepsilon \) with a heavy object of mass \( M \), when the effective \( s \sim 2\varepsilon M > m_p^2 \).

For transplankian particles when \( \varepsilon \gg m_p \), one can expect that all partons in the Fock wave function will have limited energies \( \omega_i \lesssim m_p \). Because of this, in the beam decomposition (2), (5) of the vertices \( N_n \) the many-particle beams with the average number of particles \( n \sim \varepsilon / m_p \) could dominate. For this reason, one can expect that the value of \( N_n \) will increase with the \( n \gg 1 \) much faster than for the ordinary eikonal.

But this does not take place due to the compensation between the contribution to \( N_n \) from the growth with \( \varepsilon \) of the number of particles in beams and by the reduction of effective interaction constants of the individual particles in the beam.

This can be easily seen by considering the general properties of the beam vertices matrix elements of \( \langle \nu_1 | \hat{G}(k) | \nu_2 \rangle \), which enter in (5). In the gravitational case they are proportional to the matrix elements of \( T_{\mu\nu} \) components of energy-momentum \( T_{\mu\nu} \). And these values are determined only by the energy of states \( |\nu_i\rangle \), equal to the full energy \( \varepsilon \) of a fast particle. The relevant value of the \( \varepsilon_1 \ast \varepsilon_2 \) coming from the product of two connected \( \hat{G} \) vertices are included to the factor \( s / m_p \), attributable to each graviton exchange. Therefore, in the case of gravity, the additional factors in \( N_n \) due to an increase with \( n \) of the average number of particles in beams do not appear, and the unitarized amplitude remains fairly close to the minimal Glauber form.

This could be explained a little bit differently. Because of the universality of gravitational interaction, it does not change the relative weights of various components of the Fock wave function at the zero momentum transfer. Therefore, at \( k_{\perp i} = 0 \) all the off-diagonal contributions to vertices \( \langle \nu_i | \hat{G}(k) | \nu_{i+1} \rangle \) disappear and

\[ N_n(k_{\perp i} \sim 0) \simeq m_p^n \prod_{i=1}^{n} (1 + (\rho_n k_{\perp i})^2 + \ldots), \quad (50) \]
where the quantities $\rho_n \sim \rho_0 = \text{the average transverse size of a fast particle } \rho_0 \sim 1/m$.

It is possible that a certain growth of $N_n$ in (50) can take place when $k_{\perp i} \neq 0$, due to a statistical multiplication of beams, similar to the hadron case, so that $\rho_n \sim \rho_0 \sqrt{n}$, and it may again result in a growth of $N_n (k_{\perp i} \sim m) \sim \sqrt{n!}$.

It is interesting to note that the structure of eikonal vertices for gravitons, given by (50), resemble the case of a weak coupling [13] in reggeon theory.

5. Behavior of $N_n$ vertices for large $n$

If the "primary" amplitude $v(y, b)$ increases with energy, then the higher order terms in the eikonal series (11) become more and more significant. For the simplest Glauber eikonal the mean $\langle n \rangle \sim |v|$. A more accurate characteristics of the essential $\langle n \rangle$ is given by the average number of soft jets (cut pomerons) in the full inelastic cross-section (23) as a function of full rapidity $y$ and the impact parameter $b$ given by $\langle \nu(y, b) \rangle = 2 \text{Im } v(y, b)$. The value of this $\langle \nu \rangle$ does not depend on the value of parameters $\beta_n$. At the same time, the structure of distribution of $\nu$ (with the average $\langle \nu(y, b) \rangle$) depends on the behavior of quantities $\beta_n$, and the tail of the distribution over $\nu$ is directly related to the behavior of $\beta_n$ for $n \gg \langle \nu \rangle$. This, in turn, determines the behavior of the distribution of multiplicity of hadrons at $n_h \gg \bar{n}_h$.

As follows from (12) the behavior of $\beta_n$ for $n \gg 1$ depends on properties of the spectrum $\varphi(\tau)$ at large $\tau$. Hence, for example, for the power asymptotic of spectrum of type $\ln \varphi(\tau_0) \sim -\lambda \tau^a$, one can estimate the asymptotic behavior of the eikonal coefficients as $\beta_n \sim (n!)^{1/a}$. If the beam spectrum is cut off at large $\tau$ so that $\varphi(\tau) = 0$ for $\tau > \tau_1 > 1$, we obtain the asymptotic behavior of the quasi-eikonal type $\beta_n \sim \tau_{\text{max}}^{n+1}/n$. For the finite matrix models, we also have eikonal-like asymptotic $\beta_n \sim \tau_{\text{max}}^n$, where $\tau_{\text{max}}$ is the upper value of the spectrum.

To estimate the asymptotic of $N_n$ at large $n$ it is convenient to write expressions for $N_n$ in the symbolic form, as in (3), where the averaging corresponds to the summation over states of particles in beams. Next, for a qualitative estimate one can accept that the main contribution to the matrix elements of vertices $\hat{G}_i$ comes from almost diagonal transitions, and also perform averaging separately for each vertex $\hat{G}$. Then

$$N_n = \langle P_{\text{in}} | \hat{G}(k_1)\hat{G}(k_2)\ldots\hat{G}(k_n) | P_{\text{out}} \rangle \approx \prod_{i=1}^{n} \langle \hat{G}_i \rangle \approx \prod_{i=1}^{n/2} (\langle \hat{G}(\nu_i) \rangle)^2 ,$$

where $\nu_i$ is the average number of particles in i-beam. If we also take that

$$\langle \hat{G}(\nu) \rangle \sim \nu \cdot g ,$$

that is we assume that each pomerion interacts independently with all $\nu_i$ particles in beam,
then we get

\[ N_n \equiv g^n \beta_n \sim g^n \prod_{i=1}^{n/2} \nu_i^2 \sim g^n \exp \left( 2 \int_0^{n/2} d i \ln \nu_i \right). \]  

(51)

It can be expected that for large \( n \) the growth of average numbers of particles in the sequence of these beams is approximately statistical. So, when we move in \( N_n \) along the sequence of such beams - from edge (incoming particles) to the “middle” beam, the values of \( \nu_i \) vary on average as \( \nu_i \sim i^a, \quad a \approx 1/2 \). Then we get from (51), that the vertices \( N_n \) depend on \( n \) as

\[ \beta_n \sim e^{a n (\ln n - 1)} \sim (n!)^a. \]  

(52)

Such an increase for large \( n \) is much faster then the Glauber case \( \beta_n \sim c^n \), and for the simple diffusion case when \( a = 1/2 \), gives the same answer \( \beta_n^2 \sim n! \) as (54).

It is also possible to estimate the behavior of \( N_n \) for large \( n \) in another way - considering the t-channel contributions to \( N_n \), coming from various reggeon diagrams with non-vacuum reggeons. Such a description can be regarded as a t-dual representation with respect to the s-channel beam pictures.\(^{16}\) For simplicity let us consider only the tree diagrams with reggeons \( R \neq \mathcal{P} \). The contribution to \( N_n \) from such diagrams can be obtained from the recursion relation

\[ N_{n+1} \simeq r n N_n, \quad N_1 = g, \]  

(53)

where \( r \) is the value of 3R vertex, and all reggeon lines on the diagrams are taken at a certain average value of the transverse momentum. The solution of (53) has the form

\[ N_n \sim gr^{n-2} n! \]

However, such a rapid growth of \( N_n \) is probably not possible for very large \( n \), since in the corresponding sequence of s-channel beams, when reaching a certain critical particle density,

\(^{16}\)Probably the behavior of \( N_n \) vertices of such a type can be expected in string and dual models
the saturation occurs. This, being applied to the recursion relations \((53)\), may be manifested in a condition that after some \(n > n_{\text{crit}}\) one can attach additional reggeons only to the border (in the transverse plane) of the beam-disk, where the density is not yet saturated, and that contains only \(\sim \sqrt{n}\) partons \(^{17}\). Therefore, the relation \((53)\) is effectively replaced by

\[ N_{n+1} \sim r \sqrt{n} N_n \]

and now it has a solution

\[ N_n \sim \sqrt{n)! \] \hspace{1cm} (54)

So we come again to the fast asymptotic behavior of vertices \(N_n\) of the same type as \((52, 34)\).

Some information about the behavior of vertices \(N_n\) at large \(n\) can be obtained considering

the energy behavior of the probability that a particle with the high energy \(\sim \exp(y)\) be in a Fock state with the minimal possible number of partons. This amplitude defines the behavior of the cross-section for a hard elastic scattering of hadrons \(^{12}\), and it is expressed by the elastic \(S(y, b = 0)\) matrix at the zero impact parameter. It can be seen (Section 4b) from

the comparison between the regge and the parton pictures (this is in fact a constraint from the s-unitarity) that we should have \(S(y, b = 0) \sim \exp(-cy)\), and this corresponds to the growth of \(N_n^2 \sim n!\).

6. Conclusion

One of the goals of this work is to draw attention to the fact that the most simple and very popular Glauber eikonal unitarization can incorrectly adjust the amplitude of process especially in the case when the primary cross-section increases with energy. Although this minimal method leads to amplitudes that already satisfy some conditions following from s-unitarity, it is probably not consistent with other conditions and with t-unitarity.

It can be expected that the weights of multiple exchanges grow significantly faster than in the case of the Glauber eikonal, and it is very likely that they grow extremely fast (\(\sim n!\))

In these conditions the unitarized S-matrix can be asymptotically closer to the form \(S \sim i/v\) , and not to \(S \sim \exp(iv)\), as in the case of Glauber eikonal.

The adequate choice of the s-channel unitarization of elastic amplitudes is also important in a “practical sense” - in connection with various phenomenological models and generators. The weights of the multiple exchanges enter almost all hadron amplitudes at high energies, and so they affect directly the behavior of cross-sections of hadron processes. Examples

\(^{17}\)The same phenomenon can take place also if in the sum of t-channel diagrams for \(N_n\) we include as well all diagrams with loops, that effectively take into account the R-reggeon gluing, leading to the R-saturation.
include the pattern of growth with energy of inclusive cross sections and the shape of the multiplicity distribution of produced particles.

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