BLOWUP SOLUTIONS FOR THE SHADOW LIMIT MODEL OF A SINGULAR GIERER-MEINHARDT SYSTEM WITH CRITICAL PARAMETERS

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June 15, 2021

Abstract. We consider a nonlocal parabolic PDE, which may be regarded as the standard semi-linear heat equation with power nonlinearity, where the nonlinear term is divided by some Sobolev norm of the solution. Unlike the earlier work in [13] where we consider a subcritical regime of parameters, we focus here on the critical regime, which is much more complicated. Our main result concerns the construction of a blow-up solution with the description of its asymptotic behavior. Our method relies on a formal approach, where we find an approximate solution. Then, adopting a rigorous approach, we linearize the equation around that approximate solution, and reduce the question to a finite dimensional problem. Using an argument based on index theory, we solve that finite-dimensional problem, and derive an exact solution to the full problem. We would like to point out that our constructed solution has a new blowup speed with a log correction term, which makes it different from the speed in the subcritical range of parameters and the standard heat equation.

1. Introduction

In the current work, we consider the following nonlocal parabolic equation

\[
\begin{align*}
\partial_t u &= \Delta u - u + \frac{u^p}{\left(\int_\Omega u^r\,dr\right)\gamma} \quad \text{in} \quad \Omega \times (0,T), \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on} \quad \partial \Omega \times (0,T) \\
u(0) &= u_0 \geq 0 \quad \text{in} \quad \Omega,
\end{align*}
\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) with smooth boundary, and the parameters satisfy \(p > 1, \gamma > 0\) and \(r > 0\). The nonlocal problem (1.1) arises from the shadow system of a singular Gierer-Meinhardt system, as a limiting case when inhibitor’s diffusion dominates activator’s one, cf. [40, 41]. Its dynamical behavior was first considered in [40], where in particular global-in-time existence as well as blowup results were derived according to the range of the involved parameters \(p, r\) and \(\gamma\). The case of an isotropically evolving domain was also considered in [36], where an analytical and numerical study was delivered. In the aforementioned works, it was pointed out that for the limiting problem (1.1), diffusion-driven (Turing) instability occurs, an intriguing phenomenon which was introduced in the seminal paper [70], for a specific range of the parameters. In particular, the observed diffusion-driven (Turing) instability for (1.1) occurs in the form of diffusion-driven blowup under the Turing condition

\[
p - r\gamma < 1,
\]

\(2010\) Mathematics Subject Classification. Primary: 35K05, 35B40; Secondary: 35K55, 35K57.

Key words and phrases. Blowup solution, Blowup profile, Stability, Semilinear heat equation, non variational heat equation.

June 15, 2021.
40. However, in those works, only a rough form of the Turing instability (blowup) patterns is presented and only in the case of a sphere; it is based on known results, cf. [58, 32, 34] for the standard heat equation
\[ \partial_t u = \Delta u + u^p, \tag{1.3} \]
which coincides with (1.1) when \( \gamma = 0 \).

Let us mention here some important results involving to finite time blowup solutions to (1.3). First, we mention the papers [1, 57, 56], where the authors constructed blowup solutions obeying the following profile:
\[ \left\| (T - t)^{-\frac{1}{p-1}} u(x, t) - \left( p - 1 + \frac{(p - 1)^2}{4p} \frac{|x|^2}{(T - t)|\ln(T - t)|} \right)^{-\frac{1}{p-1}} \right\|_{L^\infty} \leq \frac{C}{1 + \sqrt{\ln(T - t)}}. \tag{1.4} \]
We also observe that
\[ u(0, t) \sim \kappa(T - t)^{-\frac{1}{p-1}} := \psi(t), \]
with \( \kappa = (p - 1)^{-\frac{1}{p-1}} \) and \( \psi(t) \) exactly solves\[ \partial_t \psi(t) = \psi^p(t), \]
which is the ODE associated to (1.3). In the literature, if a solution of (1.3) blows up at time \( T > 0 \) and satisfies
\[ \forall t \in [0, T), \| u(., t) \|_{L^\infty} \leq C\psi(t), \]
then, it is called a “Type I” blowup solution. If not, then, it is called a “Type II”. Clearly, the solution shown in (1.4) is of Type I. This classification can naturally be extended to the general equation
\[ \partial_t u = \Delta u + F(u(t)). \]

Regarding the construction of Type I blowup solutions to (1.3), the authors in [1] (also in [57]) used an important method consisting in two main steps: first, a reduction to a finite-dimensional problem, then a topological argument based on index theory to solve the finite-dimensional problem. We also mention that the method has been proven robust in a lot of situations such as [56] for the reconnection of vortex; in [71, 16, 61] for perturbed nonlinear source terms; in [14, 15, 59, 63] for blowup solutions to complex Ginzburg-Landau equations; in [60, 23, 22] for complex-valued heat equations which has no variational structure; in parabolic systems as in [27]; and also in [24] for MEMS models.

In addition to that, a huge literature has been devoted in the last 20 years to the construction of solutions of PDEs with prescribed behavior, beyond the case of parabolic equations such as: Type I anisotropic blowup for the heat equation by Merle et al [54]; Type II blowup for the heat equation by del Pino et al [17, 18, 19, 20], Schwery [68], Collot [10], Merle et al [9], Harada [28, 29], Seki [69]; blowup for nonlinear Schrödinger equation by Merle [48], Martel and Merle [49], Merle et al [51, 53, 52], Raphaël and Szeftel [65]; blowup for wave equations by Côte and Zaag [11], Ming et al [55], Collot [5], Hillairet and Raphaël [30], Krieger et al [43, 42], Ghoul et al [25], Raphaël and Rodnianski [64], Donninger and Schörkhuber [21]; blowup for KdV and gKdV by Martel [47] and Côte [3, 4]; Schrödinger maps by Merle et al [50]; heat flow map by Ghoul et al [26], Raphaël and Schwery [67], Dávila et al [12]; the Keller-Segel system by Ghoul et al [8, 7], Schwery and Raphaël [66], Prandtl’s system by Collot et al [6]; Stefan problem by Hadzic and Raphaël [31]; three-dimensional fluids by Merle et al [52].

Now, we come back to the (nonlocal) equation (1.1) when \( \gamma \neq 0 \). Recently, inspired by the Type I construction mentioned above for equation (1.3), the authors of [13] constructed a blowup...
solution to equation (1.1) via a rigorous analysis, giving the exact form of the blowup profile in some subcritical regime of the parameters, namely when

$$\frac{r}{p - 1} < \frac{N}{2} \quad \text{and} \quad \gamma r \neq p - 1,$$

(1.5)

being in agreement with Turing condition (1.2). Specifically, under (1.5), the authors constructed a blowup solution to (1.1) with the blowup profile (pattern) as follows

$$u(x, t) \sim (\theta^*)^{-\frac{1}{p - 1}} (T - t)^{-\frac{1}{p - 1}} \left( p - 1 + \frac{(p - 1)^2}{4p} \frac{|x|^2}{(T - t)|\ln(T - t)|} \right)^{-\frac{1}{p - 1}} \quad \text{as} \quad t \to T,$$

(1.6)

where

$$\theta^* := \lim_{t \to T} \left( \int_{\Omega} u^r \, dr \right)^{-\gamma}.$$

Note that if the constant \( \theta^* \) is ignored, then the solution has the same structure of blowup solutions, as the solution constructed by [1] and [57] for the standard (local) equation (1.3). In fact, the approach in that work imposed a special analysis to control the non-local term and make it converge to a nonzero constant. This way, we reasonably see that in this regime, equation (1.1) will behave like the standard equation (1.3). In particular, up to some natural scaling, both equations show the same profile (see (1.4) and (1.6)). Inspired by the analysis in [13], it naturally arises the need for the consideration of another regime where the non-local term has a different limit, in particular it converges towards “zero”, leading hopefully to a different blow-up speed. The following is our main result:

**Theorem 1.1.** Let \( \Omega \) be a smooth and bounded domain in \( \mathbb{R}^N \) containing the origin and consider equation (1.1) in the following critical regime:

$$\frac{r}{p - 1} = \frac{N}{2} \quad \text{with} \quad p \geq 3.$$  

(1.7)

Then, there exists \( \gamma_0 \) small such that for all \( \gamma \in (0, \gamma_0) \), we can construct initial data \( u_0 \geq 0 \) such that the solution of (1.1) blows up in finite time \( T(u_0) \), only at the origin. Moreover, we have the following blowup asymptotics:

(i) **Behavior of** \( \theta(t) = \left( \int_{\Omega} u^r \, dr \right)^{-\gamma} \). **It holds that**

$$\theta(t) = \theta_\infty |\ln(T - t)|^{-\beta} \left( 1 + O \left( \frac{1}{\sqrt{|\ln(T - t)|}} \right) \right), \quad \text{as} \quad t \to T,$$

(1.8)

**where**

$$\theta_\infty = \left( \frac{|\Omega| (1 + \frac{N}{2})}{1 - \frac{\gamma}{N}} \left( 2b \frac{\gamma}{\gamma + 2} \right)^{\frac{1 - \frac{\gamma}{2}}{\frac{\gamma}{2} + 1}} \right)^{-\frac{1}{p - 1}},$$

$$\beta = \left( \frac{N}{2} + 1 \right) \frac{\gamma}{1 - \frac{\gamma}{2}} - \frac{b}{2}, \quad b = \frac{(p - 1)^2}{4p} (1 + \beta).$$

(ii) **The intermediate blowup profile.** For all \( t \in (0, T) \), we have

$$\left\| (T - t)^{-\frac{1}{p - 1}} |\ln(T - t)|^{-\nu} u(\cdot, t) - (\theta_\infty)^{-\frac{1}{p - 1}} \varphi_0 \left( \frac{|.|}{\sqrt{|T - t| |\ln(T - t)|}} \right) \right\|_{L^\infty(\Omega)} \leq \frac{C}{1 + \sqrt{|\ln(T - t)|}}$$

(1.9)

**where**

$$\varphi_0(z) = (p - 1 + b |z|^2)^{-\frac{1}{p - 1}} \quad \text{and} \quad \nu = \frac{\beta}{p - 1}.$$
(iii) **The final blowup profile.** It holds that \( u(x, t) \to u^*(x) \in C^2(\Omega \setminus \{0\}) \) as \( t \to T \), uniformly on compact sets of \( \Omega \setminus \{0\} \). In particular, we have

\[
u^*(x) \sim (\theta_\infty)^{-\frac{1}{p'-1}} \left[ 2 \ln |x| \right]^{-\frac{1}{p'-1}} \ln |x|^{\frac{1}{p'-1}} \] as \( x \to 0 \). \quad (1.10)

**Remark 1.2** (Stability). Following the interpretation of \( N+1 \) parameters for the blowup time and the blowup point, originally done in [57], and then applied in [13], we can prove that behaviors \((1.9)\) and \((1.10)\) are stable under perturbation of initial data, the readers can find more details in Remark 2.3 of [13].

**Remark 1.3** (On the smallness of \( \gamma \)). We mention that in the proof of the Theorem, we need \( \gamma \) to be very small. This is due to the limitation of our method, which is inspired by the case of the standard equation \((1.3)\) treated in [1] and [57]. However, regardless of the method, we suspect that the described behavior will not occur when \( \gamma \) large, since \( \theta \) may act against blowup in equation \((1.1)\) in that case.

Furthermore, we would like to point out that the condition on the smallness of parameter \( \gamma \) is consistent with analogous conditions on the exponents of non-local terms that guarantee finite-time blowup for parabolic and hyperbolic non-local problems, for more details see [2, 35, 37, 38, 39, 44, 45, 46] and references therein.

**Remark 1.4** (Novelty). The constructed solution in Theorem 1.1 has the following blowup speed

\[ \|u(t)\|_{L_\infty} \sim \kappa(\theta_\infty)^{-\frac{1}{p'-1}} (T-1)^{-\frac{1}{p'-1}} \ln(T-t)^{\kappa'}, \kappa = (p-1)^{-\frac{1}{p'-1}} \]

which is different from the subcritical regime treated in [13], and also from the case of the standard heat equation \((1.3)\), where no \( \ln(T-t) \) correction appears.

In fact, our problem is non-local, as it involves the following term

\[ \theta(t) = \frac{1}{\int_{\Omega} u^*(t)dx} \]

and thus we first observe that once \( u \) blows up, the non-local integral will affect the solution’s asymptotics. Let us now mention the pioneering papers [13] and [24], where the authors construct blowing up and quenching solutions to equations involving non-local terms, and describe their blowing up and quenching profile respectively. However, in the those works, the authors only handled the regime where the non-local term stays away from 0 and infinity, in the sense that

\[ \theta(t) \to \theta^* > 0, \quad \text{as} \; t \to T. \]

Then, thanks to a natural scaling, the non-local term has no big impact on the solution’s blowing up or quenching behavior. In the current paper, we unveil a new phenomenon, where

\[ \theta(t) \to 0 \]

which clearly affects the nonlinear term, in the sense that

\[ \theta(t) u^p \ll u^p. \]

Naturally, the solution’s behavior will be more affected here than in the subcritical case treated in [13]. In the present paper, we mainly rely on the construction method of [56] (also [1] and [57]), however, we need new ideas to carefully control the behavior of \( \theta(t) \), so that it fits the description given in statement \((i)\) of Theorem 1.1.

**Remark 1.5.** Remarkably we should have that

\[ 1 - \frac{\gamma N}{2} > 0 \implies \gamma < \frac{2}{N} = \frac{p-1}{r} \implies p - r \gamma > 1, \]

and thus we sit in the regime where Turing condition \((1.2)\) is not satisfied. Therefore our main result given by Theorem 1.1 describes the occurrence of a reaction driven blowup.
The Organization: This paper is organized as follows:
- In Section 2, we give a formal approach to derive the behavior of $\theta(t)$, together with the blowup profile.
- In Section 3, we rigorously formulate the problem.
- In Section 4, we give the proof of Theorem 1.1, assuming some technical results.
- In Sections 5 and 6, we give the proofs of the technical results used in Section 4.
- In Sections A, B and C, we give some purely technical computations, which are useful for the proof.

2. Formal approach

In this section, we aim at giving a formal approach which explains how the blowup profile in Theorem 1.1 is derived. Firstly, let us denote

$$\theta(t) = \frac{1}{\left(\int_{\Omega} u^r(x,t)dx\right)^{\gamma}}.$$  \hfill (2.1)

Henceforth, we rewrite equation (1.1) by the following

$$\partial_t u = \Delta u - u + \theta(t)u^p.$$  \hfill (2.2)

Through a formal observation, we focus on the following three interesting situations:

$$\theta(t) \to 0 \text{ as } t \to T,$$  \hfill (2.3)

$$\theta(t) \to \theta^* > 0 \text{ as } t \to T,$$  \hfill (2.4)

$$\theta(t) \to +\infty \text{ as } t \to T.$$  \hfill (2.5)

We point out that (2.5) is excluded by Theorem 3.1 and Remark 3.2 given in [40]. Recently, (2.4) was handled in [13] under some sub-critical regimes (see more in (1.5)). Thus in the current work we provide analytically a construction of blowup solution, satisfying (2.3). In the following, we aim to deliver a formal study deriving the proper blowing behavior.

Here we outline our formal approach into two steps:
- Step 1: A prescribed asymptotic of the solution on $\Omega$. More precisely, we are inspired from [13], and [24] (see also in [56]), to control the solution on the three domains under dynamical hypotheses on the non-local term $\theta(t)$, given in (2.19):
  - Blowup region $P_1$ defined by
    $$P_1(t) = \left\{ x \mid |x| \leq K_0\sqrt{(T-t)|\ln(T-t)|} \right\}. \hfill (2.6)$$
  - Intermediate region $P_2(t)$ defined by
    $$P_2(t) = \left\{ \frac{K_0}{4}\sqrt{(T-t)|\ln(T-t)|} \leq |x| \leq \epsilon_0 \right\}, \hfill (2.7)$$
    for some $\epsilon_0$ small enough.
  - Regular domain $P_3$ defined by
    $$P_3 = \{ x \in \Omega \text{ such that } |x| \geq \epsilon_0 \}. \hfill (2.8)$$
- Step 2: We give a formal justification to show derived behaviors of $u$ on each $P_j$, $j = 1, 2, 3$ which is adapted to (2.3).
2.1. Control of the solution’s asymptotics on $\Omega$

As we mentioned above, in this paragraph we will provide explicit behavior of $u$ on regions $P_1, P_2$ and $P_3$. Now let us assume that $u$ is a blowing up solution in finite time $T$ at the origin $0 \in \Omega$, and (2.3) is satisfied.

**Asymptotic of the intermediate profile in region $P_1$**

Let us introduce the following similarity variable:

$$y = \frac{x}{\sqrt{T-t}}, \quad s = -\ln(T-t) \quad \text{and} \quad W(y, s) = (T-t)^{\frac{1}{p-1}} U(x, t),$$

(2.9)

with

$$U(x, t) := \theta(t)^{\frac{1}{p-1}} u(x, t),$$

(2.10)

which by virtue of (2.1) entails

$$\theta(t) = \left( \int_{\Omega} U^r \, dx \right)^{-\frac{1}{1-p}}.$$  

(2.11)

Next using equation (1.1), $U$ reads

$$\partial_t U = \Delta U + U^p + \left( \frac{1}{p-1} \frac{\theta'(t)}{\theta(t)} - 1 \right) U,$$

(2.12)

where $\theta(t)$ is defined as in (2.11). Using (2.9) and (2.12), $W$ solves

$$\partial_s W = \Delta W - \frac{1}{2} y \cdot \nabla W - \frac{W}{p-1} + W^p + \left( \frac{1}{p-1} \frac{\bar{\theta}'(s)}{\bar{\theta}(s)} - e^{-s} \right) W,$$

(2.13)

where

$$\bar{\theta}(s) = \theta(t(s)), \quad s = -\ln(T-t).$$

(2.14)

Regarding $\theta$’s evolution, we make a hypothesis as follows near the blowup point

$$\frac{\theta'(t)}{\theta(t)} \ll U^{p-1},$$

(2.15)

then (2.12) is considered as a small perturbation to the following

$$\partial_t U = \Delta U + U^p,$$

(2.16)

which has been studied thoroughly in [1, 33, 57], and the references therein. The authors in those works constructed a blowup solution to (2.16), satisfying

$$U \sim \kappa (T-t)^{\frac{1}{p-1}}$$

near the blowup region $P_1$,

from which in conjunction with (2.15) we derive

$$\frac{\theta'(t)}{\theta(t)} \ll (T-t)^{-1},$$

(2.17)

and using the fact that $\bar{\theta}(s) = \theta(t)$ then (2.17) yields

$$\frac{\bar{\theta}'(s)}{\bar{\theta}(s)} \to 0, \quad \text{as} \quad s \to +\infty.$$

We remark that the following situation is impossible:

$$\left| \frac{\bar{\theta}'(s)}{\bar{\theta}(s)} \right| \leq \frac{C}{s^{1+\delta}}, \quad \text{for some} \quad \delta > 0,$$
due to the fact that $\bar{\theta} \to 0$ as $s \to +\infty$. However, it is hard to classify the asymptotic behavior of this quantity and so we only provide a special case so that (2.3) is valid. For example, we consider the following situation

$$\frac{\bar{\theta}'(s)}{\bar{\theta}(s)} = -\beta s^{-1} + O(s^{-1-\delta}), \quad (2.18)$$

for some $\beta, \delta > 0$. In particular, (2.18) has a special solution that

$$\bar{\theta}(s) = \theta_\infty s^{-\beta} \left(1 + O\left(\frac{1}{s^{\delta}}\right)\right), \quad \text{as } s \to +\infty,$$

This also implies

$$\theta(t) = \theta_\infty |\ln(T-t)|^{-\beta} \left(1 + O\left(\frac{1}{|\ln(T-t)|^{\delta}}\right)\right), \quad \text{as } t \to T. \quad (2.19)$$

We remark that (2.19) will rigidly be justified in subsection 2.2. Plugging (2.18) into (2.13), we observe that

$$\left(\frac{\theta'}{\theta} - e^{-s}\right) W = -\frac{\beta}{p-1} \frac{W}{s} + \text{“lower order”}.$$

Hence, we interested in considering the following proxy equation

$$\partial_s W = \Delta W - \frac{1}{2} y \cdot \nabla W - \frac{W}{p-1} + W^p - \frac{\beta}{s(p-1)} W, \quad (2.20)$$

for all $(y,s) \in \Omega_s \times [-\ln(T-t), +\infty)$ and $\Omega_s = e^s \Omega$.

Note that there exists a space-independent solution of (2.20) in the following form

$$W = \kappa + O\left(\frac{1}{s}\right),$$

recalling that $\kappa = (p-1)^{-\frac{1}{p-1}}$.

Now, we consider the linearization around the dominated part $\kappa$ by

$$\bar{W} = W - \kappa,$$

which yields

$$\partial_s \bar{W} = \mathcal{L} \bar{W} + \bar{B}(\bar{W}) - \frac{\beta (\kappa + \bar{W})}{(p-1)s}, \quad (2.21)$$

where

$$\mathcal{L} = \Delta - \frac{1}{2} y \cdot \nabla + Id,$$

$$\bar{B}(\bar{W}) = (\bar{W} + \kappa)^p - \kappa^p - p \kappa^{p-1} \bar{W}.$$ 

In addition to that, we have the fact that for all $|\bar{W}| \leq 1$

$$|\bar{B}(\bar{W}) - \frac{p}{2\kappa} \bar{W}^2| \leq C |\bar{W}|^3.$$ 

We now assume that $\bar{W}$ is radial and expressed by

$$\bar{W}(y,s) = \bar{W}_0(s) + \bar{W}_2(s)(|y|^2 - 2N)$$

(the idea behind this argument, i.e. that the solution approaches a radial profile, and $\bar{W}_2(|y|^2 - 2N)$ is dominated part, can be found in [71]). We expect to have $\bar{W}_0, \bar{W}_2 \to 0$, as $s \to +\infty$.
the above expansion on (2.21), we get
\[ \bar{W}_0'(s) = \bar{W}_0 + \frac{p}{2 \kappa} (\bar{W}_0^2 + 8N \bar{W}_2^2) - \frac{\beta}{(p-1)s} (\kappa + \bar{W}_0) + O(|\bar{W}_0|^3 + |\bar{W}_2|^3), \]
\[ \bar{W}_2'(s) = \frac{4p}{\kappa} \bar{W}_2^2 + \frac{p}{\kappa} \bar{W}_0 \bar{W}_2 - \frac{\beta}{(p-1)s} \bar{W}_2 + O(|\bar{W}_0|^3 + |\bar{W}_2|^3). \]
Solving the above ODE system, we obtain
\[ \bar{W}_0(s) = \frac{\beta \kappa}{(p-1)s} + o \left( \frac{1}{s} \right), \]
\[ \bar{W}_2(s) = -\frac{\kappa}{4ps} (1 + \beta) + o \left( \frac{1}{s} \right), \]
and thus we establish the following inner expansion
\[ W(y, s) = \kappa - \frac{k}{4ps} (1 + \beta) (|y|^2 - 2N) + \frac{\beta \kappa}{(p-1)s} + o \left( \frac{1}{s} \right) \]
\[ = \kappa - \frac{k}{4ps} (1 + \beta) |y|^2 + \frac{N \kappa}{2ps} (1 + \beta) + \frac{\beta \kappa}{(p-1)s} + o \left( \frac{1}{s} \right), \tag{2.22} \]
We now study the outer expansion. By the form of the blowup variable
\[ z = \frac{y}{\sqrt{s}}, \]
we are motivated to seek for a blowup profile in \( z \) as follows
\[ W(y, s) = \varphi_0(z) + \text{lower perturbation, as } s \to +\infty. \]
Plugging into (2.20), we derive
\[ \varphi_0(z) = (p - 1 + b|z|^2)^{-\frac{1}{p+1}}, \tag{2.23} \]
where it naturally requires \( b > 0 \), since \( \varphi_0 \) needs to be global. In particular, by matching to inner expansion (2.22), we obtain
\[ b = \frac{(p-1)^2}{4p}(1 + \beta). \tag{2.24} \]
Thus, we deduce the blowup profile
\[ \varphi(y, s) = \left( p - 1 + b \frac{|y|^2}{s} \right)^{-\frac{1}{p+1}} + \frac{N \kappa}{2ps} (1 + \beta) + \frac{\beta \kappa}{(p-1)s}, \tag{2.25} \]
which is close to the solution
\[ W(y, s) \sim \varphi(y, s), \text{ as } s \to +\infty. \]
The formal result is adapted from [24, 57, 56]. In particular, in [71], the authors obtained the \( W^{1, \infty} \) estimate
\[ \| W - \varphi \|_{W^{1, \infty}} \leq \frac{C}{1 + \sqrt{|\ln(T-t)|}}, \tag{2.26} \]
Let us assume that (2.26) hold. Thus, we derive
\[ \left\| (T-t)^{-\frac{1}{p+1}} \theta^{-\frac{1}{p+1}}(t) u(x, t) - \left( p - 1 + b \frac{|p|^2}{\ln(T-t)} \right)^{-\frac{1}{p+1}} \right\| \leq \frac{C}{1 + \sqrt{|\ln(T-t)|}}, \tag{2.27} \]
for all \( x \in P_1(t) \).

Asymptotic of the intermediate profile in region \( P_2 \):
In region $P_2$, we try to control a rescaled version of $u$. Firstly, we define for all $|x| \leq \epsilon_0$ with $\epsilon_0$ small enough, $t(x)$ as the unique solution of the following equation

$$|x| = \frac{K_0}{4} \sqrt{(T - t(x)) \ln(T - t(x))} \quad \text{with } t(x) < T. \quad (2.28)$$

Note that, $t(x)$ is well defined as long as $\epsilon_0$ is small enough and we have the following asymptotic behavior

$$t(x) \to T, \text{ as } x \to 0.$$ 

For convenience, we introduce

$$\varrho(x) = T - t(x), \quad (2.29)$$

so, it follows

$$\varrho(x) \to 0 \text{ as } x \to 0.$$ 

Next, we assume that $u$ well define on $[0,t_1]$ and we introduce then the re-scaled function

$$\mathcal{U}(x, \xi, \tau) = (T - t(x))^{\frac{1}{p+1}} \theta^{-\frac{1}{p-1}}(t(x))u(X, t), \quad (2.30)$$

where

$$X = x + \xi \sqrt{T - t(x)} \text{ and } t = \varrho(x)\tau + t(x).$$

Note that $\tau$ is considered to belong in $[-\frac{t(x)}{\varrho(x)\tau(x)}, \frac{t_1-t(x)}{\varrho(x)}]$, since $t \in [0,t_1]$, then the problem is well defined. By (2.2), $\mathcal{U}$ satisfies

$$\partial_t \mathcal{U} = \Delta_x \mathcal{U} + \tilde{\theta}(\tau)(\theta(t(x)))^{-1} \mathcal{U}^p - \rho(x)\mathcal{U},$$

where

$$\tilde{\theta}(\tau) = \theta(\tau \varrho(x) + t(x)), \text{ and } \rho(x) \text{ defined as in (2.29).}$$

The readers should understand that $\theta(t') = \theta(0)$ if $t' \leq 0$. We now refer to [56], (see also [24], and [13]) in which the authors studied $\mathcal{U}$’s dynamic on a small region of the local space $(\xi, \tau)$ defined by

$$|\xi| \leq \alpha_0 \sqrt{\ln(\varrho(x))} \text{ and } \tau \in \left[-\frac{t(x)}{\varrho(x)}, 1\right].$$

In particular, the key idea is to show that this flatness is preserved for all $\tau \in [0,1]$ (that is for all $t \in [t(x), T]$), in the sense that the solution does not depend substantially on space. Using that argument, we derive that $\mathcal{U}$ is regarded as a perturbation to $\hat{\mathcal{U}}(\tau)$, where $\hat{\mathcal{U}}(\tau)$ solves the problem

$$\begin{cases}
\partial_t \hat{\mathcal{U}}(x, \tau) = \tilde{\theta}(\tau)\theta^{-1}(t(x))\hat{\mathcal{U}}^p(x, \tau), \\
\hat{\mathcal{U}}(0) = \left(p - 1 + \frac{(p-1)^2 K_0^2}{4p} \right)^{-\frac{1}{p-1}},
\end{cases}$$

and is explicitly given by

$$\hat{\mathcal{U}}(x, \tau) = \left(p - 1 \left(1 - \int_0^\tau \tilde{\theta}(\tau')\theta^{-1}(t(x))d\tau'\right) + \frac{(p-1)^2 K_0^2}{4p} \right)^{-\frac{1}{p-1}}. \quad (2.31)$$

From $t(x)$’s monotonicity, we have that

$$t(x) \leq t \quad \text{and so } \quad T - t(x) \geq T - t.$$ 

and the asymptotic behavior (2.19), which implies $\theta$ is decreasing. Henceforth, we obtain

$$\tilde{\theta}(\tau')\theta^{-1}(t(x)) \leq 1, \forall \tau' \in [0, \tau].$$

Then, we derive

$$|\hat{\mathcal{U}}(\tau)| \leq C,$$

which implies

$$|\mathcal{U}(x, 0, \tau)| \leq C.$$
In addition to that, and relying on the flatness of $\hat{u}$, it is reasonable to assume the following estimate on the gradient
\[
|\nabla \hat{u}(x, 0, \xi)| \leq \frac{C}{\sqrt{\ln \rho(x)}}.
\]
Finally, we use Lemma C.1 and $\theta$'s asymptotic assumed at (2.19) to derive the following $u$'s asymptotic on $P_2$:
\[
\begin{cases}
|u(x, t)| \leq C \left[ |x|^2 \right]^{\frac{1}{p-1}} \ln |x| \left| \theta^\frac{1}{p-1} \right|(t), \\
|\nabla u(x, t)| \leq C \left( |x|^2 \right)^{\frac{1}{p-1}} \ln |x| \left| \theta^\frac{1}{p-1} \right|(t).
\end{cases}
\]
(3.22)

Asymptotic profile in the regular region $P_3$:

Using the well-posedness of the Cauchy problem for equation (2.2), we derive the asymptotic profile of the solution $u$ within that region as a perturbation of initial data $u(0)$. In particular, we obtain some estimates as follows
\[
|u(x, t)| + |\nabla u(x, t)| \leq C,
\]
(3.23)
for all $x \in P_3$.

2.2. On dynamical hypothesis of $\theta$

In the current subsection, we aim to give a justification of ansatz (2.18) by using the formal behavior of the solution in regions $P_j$, $j = 1, 2$ and 3, sketched in (2.27), (2.32) and (2.33). To this end we are inspired by [13], in which the authors have proved an explicit asymptotic to $L^k(\Omega)$, and which is here applied to the critical case $\frac{1}{p-1} = \frac{N}{2}$, see more details in Corollary 2.2. The later result also motivates us to derive the new blowup speed in the current paper. However, the approach developed in [13] needs to be developed further so we can derive a more precise behavior on $\theta$ defined by (2.1); that is actually the main strategy applied in the current work.

We realize that once (2.3) occurs, it immediately follows that
\[
\left( \int_\Omega u^r(x, t)dx \right)^\gamma \to +\infty, \text{ as } t \to T,
\]
provided that $\gamma > 0$. Multiplying (2.2) by $ru^{r-1}$ and integrating over $\Omega$ we derive
\[
\partial_t \|u\|_{L^r(\Omega)} = r \int_\Omega \Delta uu^{r-1}dx + r\theta(t) \int_\Omega u^{p-1+r}dx - r \int_\Omega u^r dx = (1-r)r \int_\Omega |\nabla u|^2 u^{r-2} dx + r\theta(t) \int_\Omega u^{p-1+r}dx - r \int_\Omega u^r dx.
\]
(3.24)

Following a similar tedious calculation as in Section 6 in [13] for the case $\frac{r}{p-1} < \frac{N}{2}$, we use (2.27), (3.22) and (3.23) to obtain the following estimates
\[
\left| \int_\Omega |\nabla u|^2 u^{r-2} dx \right| \leq C(T-t)^{-1} \ln(T-t)^{\frac{N}{2}-1} t^{-\frac{N}{2}} \theta^{-\frac{N}{2}}(t),
\]
(3.25)
\[
\left| \int_\Omega u^r dx \right| \leq C \ln(T-t)^{\frac{N}{2}+1} t^{-\frac{N}{2}} \theta^{-\frac{N}{2}}(t).
\]
(3.26)

It then remains to estimate
\[
I(t) = r\theta(t) \int_\Omega u^{p-1+r}dx = \frac{r}{\|u\|_{L^r}^\gamma} \int_\Omega u^{p-1+r}dx
\]
\[
= r \|u\|_{L^r}^{-\gamma} \int_\Omega u^{p-1+r}dx = r\theta(t) \int_\Omega u^{p-1+r}dx.
\]
Let us note that (2.27) still holds within a larger domain, i.e.
\[ |x| \leq K_0 \sqrt{(T-t) \ln(T-t)| \ln(T-t)|^{\frac{1}{p}}} , \]
provided that (2.26) is valid.

Next, we focus on estimating the following integral
\[ \int_{\Omega} u^{p-1+\gamma} \, dx. \]
Indeed we decompose it as
\[
\int_{\Omega} u^{p-1+\gamma} \, dx = \int_{|x| \leq K_0 \sqrt{(T-t) \ln(T-t)| \ln(T-t)|^{\frac{1}{p}}}} u^{p-1+\gamma} \, dx \\
+ \int_{K_0 \sqrt{(T-t) \ln(T-t)| \ln(T-t)|^{\frac{1}{p}}} \leq |x| \leq \epsilon_0} u^{p-1+\gamma} \, dx \\
+ \int_{|x| \geq \epsilon_0, x \in \Omega} u^{p-1+\gamma} \, dx.
\]
From (2.33), we derive
\[ \left| \int_{|x| \geq \epsilon_0, x \in \Omega} u^{p-1+\gamma} \right| \leq C. \]
Besides that, we use (2.32) to deduce
\[
\int_{K_0 \sqrt{(T-t) \ln(T-t)| \ln(T-t)|^{\frac{1}{p}}} \leq |x| \leq \epsilon_0} u^{p-1+\gamma} \, dx \\
\leq C \theta^{-(1+ \frac{N}{p})} (t) \int_{K_0 \sqrt{(T-t) \ln(T-t)| \ln(T-t)|^{\frac{1}{p}}} \leq |x| \leq \epsilon_0} \left( \frac{|x|^2}{|\ln|x||} \right)^{-1 - \frac{N}{p}} \, dx \\
\leq C \theta^{-(1+ \frac{N}{p})} (t) |\ln(T-t)|^{\frac{N}{p} - 1}.
\]
In addition by virtue of (2.32), (2.33) and the fact that \( \frac{p}{p-1} = \frac{N}{2} \) and \( p \geq 3 \), we derive
\[
\int_{\Omega} u^{p-1+\gamma} = \theta^{-\frac{N}{2}} (t) \left( \int_{0}^{K_0 |\ln(T-t)|^{\frac{1}{2}}} \varphi_0^{p-1+\gamma} (\xi) \xi^{N-1} \, d\xi \right) (T-t)^{-1} |\ln(T-t)|^{\frac{N}{p} - \frac{1}{2}} \\
+ O(\theta^{-\frac{N}{2}} (t)(T-t)^{-1} |\ln(T-t)|^{\frac{N}{p} - \frac{1}{2}}).
\]
On the other hand via the asymptotic behavior (2.19), we obtain the following
\[
\partial_t \|u\|_{L^p} = B \|u\|_{L^p}^{-\gamma} (T-t)^{-1} |\ln(T-t)|^{\frac{N}{p} + \beta\left(\frac{p-1+r}{p-1}\right)} \\
+ O((T-t)^{-1} |\ln(T-t)|^{\frac{N}{p} + \beta\left(\frac{p-1+r}{p-1}\right) - \frac{1}{2}}),
\]
(2.37)
where
\[
B = (\theta_{\infty})^{-\left(\frac{N}{2}+1\right)} |\Omega|^{\gamma} \int_{0}^{\infty} \varphi_0^{p-1+\gamma} (\xi) \xi^{N-1} \, d\xi.
\]
(2.38)
This yields
\[
\|u\|_{L^p} = \left( \frac{(1+\gamma)B}{1+\frac{N}{2} + \beta\left(\frac{p-1+r}{p-1}\right)} \right)^{\frac{1}{1+\gamma}} |\ln(T-t)|^{\left(1+\frac{N}{2} + \beta\left(\frac{p-1+r}{p-1}\right)\right) - \frac{1}{2}} \\
+ O(|\ln(T-t)|^{\left(1+\frac{N}{2} + \beta\left(\frac{p-1+r}{p-1}\right)\right) - \frac{1}{2}}).
\]
(2.39)
Next $\theta$’s definition, via (2.1), yields
\[
\theta(t) = |\Omega|^{\gamma}(\|u\|_{L^r}^r)^{-\gamma},
\]
\[
= |\Omega|^{\gamma} \left( \frac{(1 + \gamma)B}{1 + \frac{N}{2} + \beta\left(\frac{p-1+r}{p-1}\right)} \right)^{\frac{1}{1+\gamma}} |\ln(T-t)|^{\left(1 + \frac{N}{2} + \beta\left(\frac{p-1+r}{p-1}\right)\right)^\frac{1}{1+\gamma}}
\]
\[
+ O \left( |\ln(T-t)|^{\left(1 + \frac{N}{2} + \beta\left(\frac{p-1+r}{p-1}\right)\right)^\frac{1}{1+\gamma} + \frac{1}{2}} \right).
\]

Regarding the parameters $\theta_\infty$ and $\beta$ involved into (2.19), we derive the following system
\[
\begin{cases}
\beta = \left(1 + \frac{N}{2} + \beta\left(1 + \frac{N}{2}\right)\right)^\frac{\gamma}{1+\gamma}, \\
\theta_\infty = |\Omega|^{\gamma} \left( \frac{(1 + \gamma)B}{1 + \frac{N}{2} + \beta\left(\frac{p-1+r}{p-1}\right)} \right)^{\frac{1}{1+\gamma}}
\end{cases}
\]
where $B$ defined as in (2.38). We solve this system to derive the formulas of $\beta$ and $\theta_\infty$:
\[
\theta_\infty = \left( \frac{|\Omega|}{\int_0^\infty \varphi_\theta^{p-1+r}(\xi)\xi^{N-1}d\xi} \right)^{\frac{1}{1+\gamma}},
\]
\[
\beta = \left( \frac{N}{2} + 1 \right)^{\frac{\gamma}{1 - \gamma\frac{N}{2}}}
\]
In particular, via Lemma C.2, we can reformulate (2.41) to
\[
\theta_\infty = \left( \frac{|\Omega| (1 + \frac{N}{2})}{1 - \gamma\frac{N}{2}} \right)^{\frac{\gamma}{1 - \gamma\frac{N}{2}}},
\]
which indeed imposes the following condition
\[
1 - \gamma\frac{N}{2} > 0 \implies \gamma < \frac{2}{N}.
\]
However, the above condition will be restricted drastically in the sequel, where we will need to assume that $\gamma \leq \gamma_0$, with $\gamma_0$ small enough.

3. Formulation of the problem

In this section, we formulate the problem treated by Theorem 1.1. We should point out that the rigorous approach differs at some points from the formal one described in the previous section.

3.1. Similarity variable

Let us consider $u$ be a solution to (1.1), and $U$ is defined as follows
\[
U(x, t) := \theta^{\frac{1}{p-1}}(t)\chi_1(x, t)u(x, t),
\]
where $\theta$ introduced as in (2.1) and $\chi_1$ defined by
\[
\chi_1(x, t) = \chi_0 \left( \frac{|x|}{K_0\sqrt{T-t}|\ln(T-t)|} \right),
\]
with $\chi_0 \in C_0^\infty([0, +\infty))$, satisfying
\[
\text{supp}(\chi_0) \subset [0, 2], \quad 0 \leq \chi_0(x) \leq 1, \forall x \text{ and } \chi_0(x) = 1, \forall x \in [0, 1].
\]
Let us point out that operator $L$ and $\phi$ Next we consider the linearization around the profile, $q$ hence, $(3.10)$ is regarded as a small perturbation term. Now, using again $(2.9)$ we deduce that $W$ solves

$$
\partial_t W = \Delta W - \frac{1}{2} y \cdot \nabla W - \frac{W}{p-1} + W^p + \left( \frac{1}{p-1} \frac{\bar{\theta}(s)}{\theta(s)} - e^{-s} \right) W + \tilde{F}(y, s),
$$

where $\tilde{F}(y, s)$ defined by

$$
\tilde{F}(y, s) := e^{-\frac{p}{p-1}s} F(u, U)(y, s).
$$

Next we consider the linearization around the profile, $\varphi$ given by $(2.25)$:

$$
q := W - \varphi,
$$

hence, $q$ solves

$$
\partial_s q = (L + V)q + B(q) + R(y, s) + G(w, W),
$$

where

$$
L = \Delta - \frac{1}{2} y \cdot \nabla + Id,
$$

$$
V(y, s) = p \left( \varphi^{p-1}(y, s) - \frac{1}{p-1} \right),
$$

$$
B(q) = (q + \varphi)^p - \varphi^p - p\varphi^{p-1} q,
$$

$$
R(y, s) = -\partial_s \varphi - \Delta \varphi - \frac{1}{2} y \cdot \nabla \varphi - \frac{\varphi}{p-1} + \varphi^p,
$$

$$
G(\cdot, s) = \left( \frac{1}{p-1} \frac{\bar{\theta}(s)}{\theta(s)} - e^{-s} \right) (q + \varphi) + \tilde{F}.
$$

In the following we recall some properties of the linear operator $L$ and the potential $V$.

**Operator $L$**

Let us point out that operator $L$ is exactly the same as in [13], [24] and the references therein. The interested readers can find more details about $L$ in those works while here we only present its main properties. Indeed, $L$ is self-adjoint in $\mathcal{D}(L) \subset L^2_\rho(\mathbb{R}^N)$, where

$$
L^2_\rho(\mathbb{R}^N) = \left\{ f \in L^2_{loc}(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} |f(y)|^2 \rho(y) dy < +\infty \right\},
$$

and

$$
\rho(y) := e^{-\frac{|y|^2}{4}} \left(\frac{4\pi}{N}\right)^{\frac{N}{2}}.
$$

The spectrum set is explicitly given by

$$
\text{Spec}(L) = \left\{ 1 - \frac{m}{2} \mid m \in \mathbb{N} \right\}.
$$
The eigenspace corresponding to \( \lambda_m = 1 - \frac{m}{2} \) is given by
\[
E_m = \langle h_{m_1}(y_1), h_{m_2}(y_2), \ldots, h_{m_N}(y_N) \rangle \mid m_1 + \ldots + m_N = m \rangle,
\]
where \( h_{m_i} \) is the (rescaled) Hermite polynomial in one dimension.

**Potential \( V \)**

\( V \) satisfies the following

(i) \( V(., s) \to 0 \) in \( L^2(\mathbb{R}^N) \) as \( s \to +\infty \) and it has some perturbations on \( L \)'s effect.

(ii) \( V(y, s) \) is almost a constant outside the blowup region, i.e for \( |y| \geq K_0 \sqrt{s} \). In particular, we have the following estimate
\[
\sup_{s \geq s_0, |y| \geq C_s} \left| V(y, s) - \left( -\frac{p}{p-1} \right) \right| \leq \epsilon,
\]
for some \( \epsilon > 0, C_\epsilon > 0, \) and \( s_\epsilon \). We also remark that \( -\frac{p}{p-1} < -1 \). Therefore, since the largest eigenvalue of \( L \) is equal to 1, then we have that \( L + V \) has a strictly negative spectrum in the region \( \{ |y| \geq K_0 \sqrt{s} \} \). Thus, we can easily control the solution in that region when \( K_0 \) is large enough.

As it is evident form the above, \( L + V \) has not the same behavior inside and outside the singular domain \( \{ |y| \leq K_0 \sqrt{s} \} \). So, we will use a classical decomposition introduced in [1] (see also [56, 57, 24, 13]). In particular for each \( r \in L^\infty(\mathbb{R}^N) \), we write
\[
r(y) = r_b(y) + r_e(y) \equiv \chi(y, s)r(y) + (1 - \chi(y, s))r(y),
\]
where \( \chi(y, s) \) defined by
\[
\chi(y, s) = \chi_0 \left( \frac{|y|}{K_0 \sqrt{s}} \right),
\]
and \( \chi_0 \) is as introduced by (3.3). Let us remark that
\[
\text{Supp} (r_b) \subset \{ |y| \leq 2K_0 \sqrt{s} \}, \quad \text{Supp} (r_e) \subset \{ |y| \geq K_0 \sqrt{s} \}.
\]
In addition, for all \( r_b \in L^2_\beta(\mathbb{R}^N) \) we write
\[
r_b(y) = r_0 + r_1 \cdot y + y^T \cdot r_2 \cdot y - 2 \text{Tr}(r_2) + r_-(y),
\]
or alternatively
\[
r_b(y) = r_0 + r_1 \cdot y + r_-(y),
\]
where
\[
r_i = (P_\beta(r_b))_{\beta \in \mathbb{N}^N, |\beta|=i}, \quad \forall i \geq 0,
\]
with \( P_\beta(r_b) \) being the projection of \( r_b \) on the eigenfunction \( h_\beta \) defined as follows:
\[
P_\beta(r_b) = \int_{\mathbb{R}^N} r_b \frac{h_\beta}{\|h_\beta\|_{L^2_\beta(\mathbb{R}^N)}^2} \rho dy, \forall \beta \in \mathbb{N}^N,
\]
and
\[
r_\perp = P_\perp(r) = \sum_{\beta \in \mathbb{N}^N, |\beta| \geq 2} h_\beta P_\beta(r_b).
\]
Furthermore, \( r_- \) is defined by
\[
r_- = \sum_{\beta \in \mathbb{R}^N, |\beta| \geq 3} h_\beta P_\beta(r_b).
\]
Finally, we will use the two following expansions
\[
    r(y) = r_0 + r_1 \cdot y + y^T \cdot r_2 \cdot y - 2 \, \text{Tr}(r_2) + r_- + r_e(y). \tag{3.22}
\]
or
\[
    r(y) = r_0 + r_1 \cdot y + r_\perp(y) + r_e(y). \tag{3.23}
\]
Our final goal is to construct a global solution \( q \) on \([s_0, +\infty)\) such that
\[
    \|q(s)|_{L^\infty} \to 0 \text{ as } s \to +\infty. \tag{3.24}
\]

### 3.2. Localization variable

In this part, we aim to formulate the equation of the rescaled solution \( \tilde{U} \) corresponding to \( u \) in region \( P_2 \). Generally, the rigorous problem is similar to the one given at page 8. However, we must to explain some details that help the readers to understand our goal. First, let us consider \( u \) which exists on \([0, t_1]\) for some \( t_1 \in (0, T) \). Using equation (1.1) and \( \dot{u}' \)’s definition given in (2.30), we derive
\[
    \partial_t U = \Delta U + \left( \partial(x, t) \right) \left( \partial(t(x)) \right)^{-1} U' - \rho(x) U, \quad \tau \in \left( -\frac{t(x)}{T - t(x)}, \frac{t_1 - t(x)}{T - t(x)} \right), \tag{3.25}
\]
where \( \theta \) defined as in (2.1). In particular, once \( x \in P_2(0) \), we immediately have \( t(x) \leq 0 \), then we should understand
\[
    \theta(t(x)) = \theta(0), \tag{3.26}
\]
and \( \tilde{\theta} \) be assumed similarly by
\[
    \tilde{\theta}(\tau') = \left\{ \begin{array}{ll}
    \theta(\tau' \rho(x) + t(x)) & \text{if } \tau' \rho(x) + t(x) \geq 0, \\
    \theta(0) & \text{if } \tau' \rho(x) + t(x) \leq 0.
    \end{array} \right. \tag{3.27}
\]
The main goal is the following: for all \( t \in [0, t_1] \) and \( x \in P_2(t) \), we have
\[
    \left| U(x, \xi, \tau(x, t)) - \tilde{U}(x, \tau(x, t)) \right| \leq \delta_0,
\]
where \( \delta_0 \) will be small, and \( \tau(x, t) = \frac{t - t(x)}{T - t(x)} = \frac{t_1 - t(x)}{\rho(x)} \) and
\[
    \dot{U}(x, \tau) = \left( (p - 1) \left[ 1 - \int_0^\tau \tilde{\theta}(\tau') (\theta^{-1}(t(x))) d\tau' \right] + b \frac{K_0^2}{16} \right)^{-\frac{1}{p-1}}. \tag{3.28}
\]

Note that (3.28) makes sense in the following region
\[
    x \in \left[ \frac{K_0}{4} \sqrt{(T - t) \ln(T - t)} \right], \tau' \in [0, \tau(x, t)],
\]
since \( \tilde{\theta}(\tau') (\theta(t(x)))^{-1} \leq 1, \forall \tau' \in [0, \tau] \), with \( \tau \in \left[ -\frac{t(x)}{\rho(x)}, \frac{t_1 - t(x)}{\rho(x)} \right] \), and \( \rho \) defined as in (2.29). This yields
\[
    1 - \int_0^\tau \tilde{\theta}(\tau') (\theta(t(x)))^{-1} d\tau' \geq 1 - \tau \geq 0.
\]
Hence, we get the fact that \( \dot{U}(x, \tau) \) is well defined. In particular, such a \( \dot{U}(\tau) \) solves the following system:
\[
    \begin{cases}
        \partial_\tau \dot{U}(\tau) = \dot{\theta}(\tau) (\theta(t(x)))^{-1} \dot{U}'(\tau), \\
        \dot{U}(0) = \left( (p - 1 + \frac{K_0^2}{16}) \right)^{-\frac{1}{p-1}}.
    \end{cases} \tag{3.29}
\]
4. The existence proof without technical details

The main goal of the current section is to construct $q$, the solution to (3.9), satisfying the asymptotic behavior \((3.24)\). For readers’ convenience, we aim to present the proof of Theorem 1.1 at subsection 4.4 as well as other related complementary results in without technical details.

4.1. Shrinking set

In the sequel, we focus on building a special set where the solution’s behavior is controlled properly, leading to the derivation of the asymptotic behavior \((3.24)\). The construction of this set is inspired by [13], whilst some required modifications should be implemented for the underlying critical regime case.

**Definition 4.1** (Shrinking set). Consider $T,K_0,\epsilon_0,\alpha_0, A, \delta_0, C_0, \eta_0$ and take $t \in [0,T)$ for some $T > 0$. We define the following set

$$ S(T,K_0,\epsilon_0,\alpha_0, A, \delta_0, C_0, \eta_0, t) \quad (S(t) \text{ in short}), $$

as a subset of $C^2(\Omega) \cap C(\bar{\Omega})$, containing all functions $u$ satisfying the following conditions:

(i) **Estimates in $P_1(t)$**: In that region the function $q$ introduced in \((3.8)\) belongs to $V_A(s) \subset L^\infty(\mathbb{R}^N)$, $s = -\ln(T-t)$, with each $r \in V_A(s)$ satisfying the following estimates:

$$ |r_0| \leq \frac{A^3}{s^2}, \quad |q_1| \leq \frac{A}{s^2} \quad \text{and} \quad |r_2| \leq \frac{A^4}{s^2}, $$

$$ |r_-(y)| \leq \frac{A^6}{s^2}(1 + |y|^3) \quad \text{and} \quad |(\nabla r)_\perp| \leq \frac{A^6}{s^2}(1 + |y|^3), \forall y \in \mathbb{R}^N, $$

$$ \|r_\epsilon\|_{L^\infty(\mathbb{R}^N)} \leq \frac{A^7}{\sqrt{s}}, $$

where $r_i, r_-, (\nabla r)_\perp$ and $r_\epsilon$ introduced in \((3.18), (3.20), (3.21), \text{and} (3.16)\), respectively.

(ii) **Estimates in $P_2(t)$**: For all $|x| \in \left[ \frac{K_0}{4} \sqrt{(T-t)} \ln(T-t), \epsilon_0 \right], \tau(x,t) = \frac{t-s(t)}{\varrho(x)}$ and $|\xi| \leq \alpha_0 \sqrt{|\ln \varrho(x)|}$, the following hold:

$$ \left| \mathcal{U}(x,\xi,\tau(x,t)) - \hat{\mathcal{U}}(x,\tau(x,t)) \right| \leq \delta_0, $$

$$ \left| \nabla \mathcal{U}(x,\xi,\tau(x,t)) \right| \leq \frac{C_0}{\sqrt{|\ln \varrho(x)|}}, $$

where $\mathcal{U}, \hat{\mathcal{U}}$ and $\varrho(x)$ defined as in \((2.30), (2.31)\) and \((2.29)\), respectively.

(iii) **Estimates in $P_3(t)$**: For all $x \in \left\{ |x| \geq \frac{\varrho_0}{4} \right\} \cap \Omega$, we have

$$ |u(x,t) - u(x,0)| \leq \eta_0, $$

$$ |\nabla u(x,t) - \nabla e^{t\Delta} u(x,0)| \leq \eta_0, $$

where $e^{t\Delta}$ is the semi-group generated by $\Delta$ with Neumann boundary conditions.

By Definition 4.1(i), we can estimate $q$’s size as follows

**Lemma 4.2** (Growth estimates). We consider $K_0 \geq 1$ and $A \geq 1$. Then, there exists $s_1 = s_1(A,K_0)$ such that for all $s \geq s_1$ and $q \in V_A(s)$, the following hold:

$$ |q(y,s)| \leq \frac{C(K_0)A^7}{\sqrt{s}} \quad \text{and} \quad |q(y,s)| \leq \frac{C(K_0)A^7}{s^2}(1 + |y|^3). $$

In particular, we have

$$ \|q\|_{L^\infty(\{|y| \leq K_0\sqrt{s}\})} \leq C(K_0)\frac{A^6}{s^2}(1 + |y|^3). $$
4.2. Constructing appropriate initial data

In this paragraph, we aim to build initial data $u_0 \in S(0)$ for equation (2.12). Let us consider $\chi_0$ and $\chi_1$ defined as in (3.2) and (3.3), respectively. Next, we introduce $H^*$ as a suitable modification of the final asymptotic profile in the intermediate region in Theorem 1.1:

\[
H^*(x) = \begin{cases} 
\frac{\beta}{\ln |x|} \times \theta \frac{1}{\ln |x|} (2 \ln |x|)^{1/2}, & \forall |x| \leq \min \left( \frac{1}{2} d(0, \partial \Omega), \frac{1}{2} \right), \quad x \neq 0, \\
1, & \forall |x| \geq \frac{1}{2} d(0, \partial \Omega),
\end{cases}
\] where $\beta, \theta$ defined as in (2.42), (2.41), and $\tau_0(x) = -\frac{t(x)}{T-t(x)}$.

Taking $(d_0, d_1) \in \mathbb{R}^{1+N}$, we now define initial data as follows:

\[
u_{d_0, d_1}(x, 0) = T^{-\frac{1}{p-1}} \theta^{-\frac{1}{p-1}} \ln T \ln |x| \left[ \varphi \left( \frac{x}{\sqrt{T}}, -\ln s_0 \right) + \left( d_0 A^3 + A d_1 \cdot \frac{x}{\sqrt{T}} \right) \chi_0 \left( \frac{|x|}{s_0} \right) \right] \chi_1(x),
\]

where $z_0 = \frac{x}{\sqrt{T} \ln T}$, $s_0 = -\ln T$; $\varphi, \chi_0, \chi_1$ and $H^*$ are defined as in (2.25), (3.2), (3.3) and (4.1) respectively.

Note that we can also write the initial data using the similarity variable given by (2.9), that is in terms of $y_0 = \frac{x}{\sqrt{T}} \in \Omega_{s_0}$ and $s_0 = -\ln T$.

- For $U_{d_0, d_1}(0)$: We can use (4.2) to express $U_{d_0, d_1}(0)$ in terms of $\theta(0)$ defined as in (2.1). Then, via (2.10) it follows

\[
U_{d_0, d_1}(0) = \theta^{-1}(0) \chi_1(x, 0) u_{d_0, d_1}(0).
\]

- For $W_{d_0, d_1}(y_0, s_0)$: We consider $U_{d_0, d_1}(0)$ as in (4.3) and use (2.9) to derive $W_{d_0, d_1}(y_0, s_0)$

- For $q_{d_0, d_1}(y_0, s_0)$: We rely on the following definition

\[
q_{d_0, d_1}(y_0, s_0) = w(y_0, s_0) - \varphi(y_0, s_0),
\]

where $\varphi$ defined as in (2.25).

In the following, we construct appropriate initial data of the form (4.2), i.e. initial data which belong to the shrinking set $S(0)$.

**Proposition 4.3 (Constructing initial data).** There exists a positive constant $K_2 > 0$ large enough such that for all $K_0 \geq K_2$ and $\delta_2 > 0$, there exist $C_2(K_0) > 0$ and $\alpha_2(K_0, \delta_3) > 0$ such that for all $\alpha_0 \in (0, \alpha_2)$, there exists $\varepsilon_2(K_0, \delta_3) > 0$ such that for all $\varepsilon_0 \in (0, \varepsilon_2]$ and $A \geq 1$, we can find $T_2(K_0, \delta_2, \varepsilon_0, A, C_2) > 0$ small enough, such that for all $T \leq T_2$ and $s_0 = |\ln T|$, and initial data $u_{d_0, d_1}(0)$ as in (4.2), the following properties hold:

1. For all $|d_0|, |d_1| \leq 2$ the initial data $u_{d_0, d_1}(0)$ satisfy the following estimates:

   - Estimates in $P(1)$ : $q_{d_0, d_1}(s_0)$ defined in (4.4), satisfy

\[
|q_0(s_0)| \leq \frac{A^2}{s_0^3}, \quad |q_{1,j}(s_0)| \leq \frac{A}{s_0^2}, \quad |q_{2,i,j}(s_0)| \leq \frac{1}{s_0^2}, \quad \forall i, j \in \{1, ..., N\},
\]

\[
|q_-(y, s_0)| \leq \frac{1}{s_0^2}(|y|^3 + 1) \quad \text{for } \quad \nabla q_-(y, s_0) \leq \frac{1}{s_0^2}(|y|^3 + 1), \quad \forall y \in \mathbb{R}^N,
\]

and

\[
\|q_e\|_{L^\infty} \leq \frac{1}{\sqrt{s_0}}.
\]
• Estimates in $P_2(0)$: For all $|x| \in \left[ \frac{K_0}{4} \sqrt{T|\ln T|}, \varepsilon_0 \right]$, $\tau_0(x) = -\frac{f(x)}{\theta_0(x)}$ and $|\xi| \leq 2a_0 \sqrt{\ln \varrho(x)}$, we have

$$\left| U(x, \xi, \tau_0(x)) - \hat{U}(x, \tau_0(x)) \right| \leq \delta_3 \text{ and } |\nabla_\xi U(x, \xi, \tau_0(x))| \leq \frac{C_3}{\sqrt{\ln \varrho(x)}},$$

where $U, \hat{U},$ and $\varrho(x)$ are defined as in (2.30), (2.31) and (2.29), respectively. (II) There exits $D_A \subset [-2, 2] \times [-2, 2]^N$ such that the following mapping

$$\Gamma : \mathbb{R}^{1+N} \rightarrow \mathbb{R}^{1+N}, \quad (d_0, d_1) \mapsto (q_0, q_1)(s_0),$$

is affine, one to one from $D_A$ to $\hat{V}_A(s_0)$, where $\hat{V}_A(s)$ defined by

$$\hat{V}_A(s) = \left[ -\frac{A^3}{s^2}, \frac{A^3}{s^2} \right] \times \left[ -\frac{A}{s^2}, \frac{A}{s^2} \right]^N.$$

In addition we have $$\Gamma |_{\partial D_A} \subset \partial \hat{V}_A(s_0),$$

and

$$\text{deg} \left( \Gamma |_{\partial D_A} \right) \neq 0,$$

where $q_0, q_1$ considered as $q_{d_0,d_1}(s_0)$’s components and $q_{d_0,d_1}(s_0)$ defined as in (4.4).

**Proof.** Notably, the shrinking set and initial data are the same as in [56, Lemma 2.4], therefore the proof is basically based on that Lemma. However, the situation in the current work is more complicated due to the presence of non-local term $\theta(t)$. For reader’s convenience, we aim to provide a complete proof at Section 6.

### 4.3. Contribution on the non-local term

In this section, we will show the asymptotic behavior of the non-local term $\theta(t)$ defined by (2.1) which is Proposition 4.5 in the below. Below the.

First, we aim to give some estimates on $u$ once it is trapped in shrinking set $S(t)$.

**Lemma 4.4.** Let us consider $u \in S(K_0, \varepsilon_0, \varepsilon_0, A, \delta_0, C_0, \eta_0, t)$ for all $t \in [0, T)$, defined as in Definition 4.1 where $\eta_0 \ll 1$. Then, the following hold:

1. For all $|x| \leq K_0 \sqrt{(T - t) \ln(T - t)}$, we have

$$\left| u(x, t) - \theta^{-\frac{1}{p-1}}(T - t) - \frac{1}{\theta(t)} \varphi_0 \left( \frac{|x|}{\sqrt{(T - t) \ln(T - t)}} \right) \right| \leq \frac{CA^7(T - t)^{-\frac{1}{p-1}}}{1 + \sqrt{\ln(T - t)}},$$

where $\varphi_0$ defined by (2.23), together with the gradient estimate

$$|\nabla_\xi u(x, t)| \leq \frac{C(K_0)A^7(T - t)^{-\frac{1}{p-1}} \theta^{-\frac{1}{p-1}}(t)}{1 + \sqrt{\ln(T - t)}},$$

In particular, if $p \geq 3$ and $\frac{r}{p-1} = \frac{N}{2}$, then we obtain

$$\left| u^{p-1+r}(x, t) - \theta^{p-1+r} \frac{1}{(T - t)^{\frac{1}{p-1}} \theta^{-\frac{1}{p-1}}(t)} \varphi_0^{p-1+r} \left( \frac{|x|}{\sqrt{(T - t) \ln(T - t)}} \right) \right| \leq CA^7(p-1)(1+\frac{N}{2})(T - t)^{-\frac{p-1+r}{p-1}} \theta^{\frac{p-1+r}{p-1}}(t) \ln(T - t)^{-\frac{p-1}{p-1}(1+\frac{N}{2})} + CA^7(T - t)^{-\frac{p-1+r}{p-1}} \theta^{\frac{p-1+r}{p-1}}(t) \ln(T - t)^{-\frac{N}{2} \varphi_0^{p-2+r}} \left( \frac{|x|}{\sqrt{(T - t) \ln(T - t)}} \right).$$
(ii) For all $|x| \in [\frac{K_0}{4} \sqrt{(T-t)|\ln(T-t)|}, \epsilon_0]$, we have
\[
\frac{1}{C} \|x\|^{-\frac{\mu}{p+\mu}} \ln \|x\|^{\frac{\mu}{p+\mu}} \leq |u(x,t)| \leq C \|x\|^{-\frac{\mu}{p+\mu}} \ln \|x\|^{\frac{\mu}{p+\mu}},
\]
and
\[
|\nabla_x u(x,t)| \leq C(C_0)(\|x\|)^{-\frac{1}{p+\mu} - \frac{1}{2}} \ln \|x\|^{\frac{1}{p+\mu} - \frac{1}{2}},
\]
provided that $K_0 \geq K_0$ and $\epsilon_0 \leq \epsilon_0(K_0)$.

(iii) For all $|x| \geq \epsilon_0$, we have
\[
\frac{1}{2} \leq u(x,t) \leq C(\epsilon_0, \eta_0),
\]
and
\[
|\nabla_x u(x,t)| \leq C(\eta_0, \epsilon_0),
\]
provided that $\eta_0 << 1$.

Proof. The proof mainly bases on estimations provided in Definition 4.1 and it is quite the same as [Lemma 6.2, [13]]. Just to point out that in the current situation $K_0 \sqrt{(T-t)|\ln(T-t)|}$ in [Lemma 6.2, [13]] is now replaced by $K_0 \sqrt{(T-t)|\ln(T-t)|}$, however the same technique applies and (4.8) follows (4.7) in implementing the fundamental inequality $|(a + b)\alpha - b^\alpha| \leq C(\alpha)(b\alpha^{\alpha-1} + b^\alpha)$ with $\alpha = p + r$.

Next, we aim to give the rigorous proof to (2.18) assumed at the formal approach part:

Proposition 4.5 (Dynamics of $\theta$). Let us consider (1.7) and $p \geq 3$ with $\Omega$ being a bounded domain with smooth boundary. Then, there exists $K_0 > 0$ such that for all $K_0 \geq K_0, \delta_0 > 0$, there exists $\alpha_3(K_0, \delta_0) > 0$ such that for all $\alpha_0 \leq \alpha_3$ we can find $\epsilon_3(K_0, \delta_0, \alpha_0) > 0$ such that for all $\epsilon_0 \leq \epsilon_3$ and $A \geq 1, C_0 > 0, \eta_0 > 0$, there exists $T_3 > 0$ such that for all $T \leq T_3$ the following holds: Assuming $U$ is a non negative solution of equation (2.12) on $[0, t_1]$, for some $t_1 < T$ and $U \in S(T, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, t) = S(t)$ for all $t \in [0, t_1]$. Then, the following hold
\[
|\theta(t) - \theta_(\ln(T-t))^{-\beta} - \frac{|\theta(t) - \theta_(\ln(T-t))^{-\beta}|}{\ln(T-t)}^{1-\beta} | \leq CA^T|\ln(T-t)|^{-\beta-\frac{1}{2}},
\]
and
\[
|\theta'(t) - \theta_0^{-\beta}(\ln(T-t))^{-\beta-1} | \leq \frac{\gamma C A^T(T-t)^{-1} |\ln(T-t)|^{-\beta-\frac{1}{2}}},
\]
where $\theta_\infty$ and $\beta$ defined as in (4.41) and (4.42) respectively. In particular, if we define $\tilde{\theta}(s)$ as in (2.14) and take $\gamma \leq A^{-4}$, then we obtain the following estimate
\[
\left| \frac{\tilde{\theta'}(s)}{\tilde{\theta}(s)} + \frac{\beta}{s} \right| \leq \frac{CA^3}{s^2}.
\]

Proof. Let us assume that the hypothesis in Lemma 4.4 holds. We point out that (4.9) is similarly processed as in the formal approach at Subsection 2.2 by virtue of 4.4.

- The proof of (4.9): We recall that by (2.1) we have
\[
\theta(t) = |\Omega|^\gamma \left( \|u(t)\|_{L^r(\Omega)} \right)^{-\gamma}.
\]

Next we estimate $\|u\|_{L^r}$. Indeed, we first decompose the related integral as follows
\[
\|u\|_{L^r} = \int_{\|x\| \leq K_0 \sqrt{(T-t)|\ln(T-t)|}} u^r + \int_{K_0 \sqrt{(T-t)|\ln(T-t)|} \leq \|x\| \leq \epsilon_0} u^r + \int_{\|x\| \geq \epsilon_0, x \in \Omega} u^r.
\]

Using Lemma 4.4 and (4.12), we derive that
\[
\theta(t) \left( \theta^{-\frac{N}{2}}|\ln(T-t)|^{\frac{N}{2}} + |\ln(T-t)|^{1+\frac{N}{2}(1+\beta)} \right)^\gamma \leq 1.
\]
This implies that
\[
\theta(t) \leq C|\ln(T-t)|^{-\gamma(1+\frac{N}{2})(1+\beta)} = C|\ln(T-t)|^\beta,
\]  
(4.13)
since \(\beta\) satisfies (2.40).

We also have
\[
\theta'(t) = \partial_t \left(|\Omega|^{-\gamma}(\|u\|_{L^r(\Omega)}^{-\gamma})\right) = |\Omega|^{-\gamma}(-\gamma)(\|u\|_{L^r(\Omega)}^{-\gamma-1})\partial_t(\|u\|_{L^r(\Omega)}),
\]
and using again (2.1) we obtain
\[
\frac{\theta'}{\theta^{1+\frac{N}{2}}} = |\Omega|^{-1}(-\gamma)\partial_t(\|u\|_{L^r(\Omega)}).
\]

For \(\partial_t(\|u\|_{L^r(\Omega)})\)'s asymptotic, we would like to prove the following
\[
\partial_t(\|u\|_{L^r(\Omega)}) = r\theta^{1-\frac{N}{2}}(t) \left( \int_0^\infty \phi_0^{p-1+r}(k)k^{N-1}dk \right) (T-t)^{-1}|\ln(T-t)|^{\frac{N}{2}}
+ O \left( \theta^{1-\frac{N}{2}}(t)A^T(T-t)^{-1}|\ln(T-t)|^{\frac{N}{2} - \frac{1}{2}} \right), \quad \text{as} \ t \to T.
\]  
(4.14)
Since the computation follows the same steps as in Subsection 2.2 by using now Lemma 4.4 and (4.13), the details are omitted. Note that \(\frac{r}{p-1} = \frac{N}{2}\) then (4.14) takes the form
\[
\partial_t(\|u\|_{L^r(\Omega)}) = r\theta^{1-\frac{N}{2}}(t) \left( \int_0^\infty \phi_0^{p-1+r}(k)k^{N-1}dk \right) (T-t)^{-1}|\ln(T-t)|^{\frac{N}{2}}
+ O \left( \theta^{1-\frac{N}{2}}(t)A^T(T-t)^{-1}|\ln(T-t)|^{\frac{N}{2} - \frac{1}{2}} \right), \quad \text{as} \ t \to T.
\]
Hence, we have
\[
\left(\theta^{1-\frac{N}{2}}(t)^{\frac{1}{N}}\right)' = -\left(1 - \frac{N}{2}\right)|\Omega|^{-\gamma}r \int_0^\infty \phi_0^{p-1+r}(k)k^{N-1}dk(T-t)^{-1}|\ln(T-t)|^{\frac{N}{2}}
+ O \left( A^T\gamma(T-t)^{-1}|\ln(T-t)|^{\frac{N}{2} - \frac{1}{2}} \right), \quad \text{as} \ t \to T.
\]  
(4.15)
which yields
\[
\theta(t) = \left( \frac{r(1 - \frac{N}{2})}{|\Omega|\left(N(\frac{N}{2} + 1)\right)} \right)^{-1} \left( \frac{1}{N} \right)^{\frac{N}{2}} |\ln(T-t)|^{-1} + O \left( A^T|\ln(T-t)|^{\frac{N}{2} - \frac{1}{2}} \right), \quad \text{as} \ t \to T.
\]
Thus, we conclude (4.9).
- The proof of (4.10) follows from (4.9) and (4.15). Finally, for proving (4.11) we use \(\tilde{\theta}\)'s definition together with (4.10) and (4.9).

4.4. Proof of Theorem 1.1

In this section, we provide the proof of our main result Theorem 1.1 skipping many technicalities. All the missing technical details will be given in a separate section later on.

Proposition 4.6 (Existence of a solution belonging to \(S(t)\)). Let \(\Omega\) be a smooth and bounded domain in \(\mathbb{R}^N\) and also assume that condition (1.7) is also satisfied. Then, there exist positive parameters \(T, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0\) such that for all \(\gamma \in (0, A^{-5})\) there exists a pair \((d_0, d_1) \in \mathbb{R}^{1+N}\) such that equation (2.12) with initial data \(u_{d_0, d_1}(0)\), given in (4.2), has a unique solution on \([0, T]\) and \(u(t) \in S(t)\), for all \(t \in [0, T]\), recalling that \(S(t)\) is the shrinking set introduced in Definition 4.1.
**Proof.** Notably Theorem 1.1 arises easily by this proposition. Although, the current proposition is relatively similar to the robustness argument, given in [1, 57, 56, 13], some extra difficulties arising in that case due to the new behavior of the non-local term \( \theta \). Hence, this result adds its own value to the technique of construction of blowup solution in general. For reader’s convenience, we are completing complete proof in Section 5.

### Conclusion of Theorem 1.1

Let us fix positive parameters \( T, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0 \) and \( \eta_0 \) such that Propositions 4.5, and 4.6 hold true. Then, we obtain

\[
u(t) \in S(t) \quad \forall t \in (0, T).
\]

Using (4.9) in Proposition 4.5, we derive

\[
\theta(t) = \theta_\infty \frac{\ln(T-t)}{1 - \beta} + O\left(\frac{\ln(T-t)}{1 - \beta - \frac{1}{\delta}}\right), \quad t \to T.
\]

Next using Definition 4.1 (i), we directly deduce Theorem 1.1 (i) and in particular estimate (1.8). The latter estimate infers that \( u \) blows up in finite time \( T \) at the origin and with speed

\[
u(0) = (\theta_\infty)^{-\frac{1}{\beta - 1}}(T-t)^{-\frac{1}{\beta - 1}} |\ln(T-t)|^{\nu} \left(1 + O_{t \to T} \left(\frac{1}{\sqrt{\ln(|T-t|)}}\right)\right),
\]

recalling that \( \nu = \frac{\beta}{\beta - 1} \).

Besides, from Lemma 4.4(ii), we deduce that \( u \) does not blow up at any \( x \in \Omega \setminus \{0\} \), a single-point occurs.

We now proceed with the proof for item (ii): In fact, the existence of the blowup profile \( u^* \) is quite the same as in [24, Proposition 3.5] and thus we only give a prove (1.10). Consider \( U \) defined as in (2.30) and \( x \in (0, \epsilon_0) \) with \( \epsilon_0 \) small enough. We now apply (1.9) with \( t = t(x) \) to obtain

\[
\sup_{|\xi| \leq \ln \rho(x)} \left| U(x, \xi, 0) - \hat{U}(x, 0) \right| \leq \frac{C}{1 + |\ln \rho(x)|^{1/2}},
\]

where \( \hat{U}(x, \tau) \) defined as in (3.28). By using the argument developed in [24, Proposition 3.5, pages 1307-1310], we obtain

\[
\sup_{|\xi| \leq \ln \rho(x) |\tau \in [0,1]} \left| U(x, \xi, \tau) - \hat{U}(x, \tau) \right| \leq \frac{C}{1 + |\ln \rho(x)|^{1/2}},
\]

as well as

\[
\sup_{|\xi| \leq \ln \rho(x) |\tau \in [0,1]} \left| \partial_\tau U(x, \xi, \tau) \right| \leq C(x).
\]

It follows that

\[
u^* = \lim_{\tau \to 1} \theta_\infty^{\frac{1}{\beta - 1}} (T-t(x))^{\frac{1}{\beta - 1}} |\ln(T-t(x))|^{\nu} \hat{U}(x, 0, \tau)
\]

\[
sim \theta_\infty^{\frac{1}{\beta - 1}} (T-t(x))^{\frac{1}{\beta - 1}} |\ln(T-t(x))|^{\nu} \hat{U}(x, 1),
\]

where \( \hat{U}(x, \tau) \) defined as in (6.7). We now aim to prove that

\[
\int_0^1 \tilde{\theta}(\tau') \theta^{-1} (t(x))d\tau' \sim 1 \quad \text{as} \quad x \to 0,
\]

which is equivalent to

\[
\int_0^1 \tilde{\theta}(\tau') \theta^{-1}_{\infty} |\ln(T-t(x))|^{\beta} d\tau' \sim 1 \quad \text{as} \quad x \to 0,
\]
since, as \( x \to 0 \), we have \( t(x) \to T \), and equality (4.9). Now, we focus on the proof of (4.20). Indeed, using \( \hat{\theta}' \)’s definition given in (3.27) and the fact (4.9) again, we then derive
\[
|\theta(\tau - \theta_\infty') \ln[(T - t(x))(1 - \tau)]|^{-\beta} \lesssim |\ln[(T - t(x))(1 - \tau)]|^{-\beta - \frac{1}{2}}, \forall \tau \in [0, 1].
\]
Therefore it yields
\[
\left| \theta(\tau)'_{\infty} |\ln(T - t(x))|^{\beta} \right| - \frac{1}{1 + \frac{\ln(1 - \tau)}{\ln(T - t(x))}} \lesssim \frac{|\ln(T - t(x))|^{\beta}}{|\ln(T - t(x)) + \ln(1 - \tau)|^{\beta + \frac{1}{2}}}
\]
\[
\lesssim \frac{1}{|\ln(T - t(x))|^{\frac{1}{2}} \left| 1 + \frac{\ln(1 - \tau)}{\ln(T - t(x))} \right|^{\beta}},
\]
since
\[
|\ln(T - t(x)) + \ln(1 - \tau)| \geq |\ln(T - t(x))|.
\]
We now decompose the integral as follows
\[
\int_0^1 \hat{\theta}(\tau')\theta^{-1}_\infty |\ln(T - t(x))|^{\beta} \, d\tau' = \int_0^{1 - e^{-\sqrt{|\ln(T - t(x))|}}} \hat{\theta}(\tau')\theta^{-1}_\infty |\ln(T - t(x))|^{\beta} \, d\tau' + \int_{1 - e^{-\sqrt{|\ln(T - t(x))|}}}^1 \hat{\theta}(\tau')\theta^{-1}_\infty |\ln(T - t(x))|^{\beta} \, d\tau'.
\]
As a matter of fact, the second integral can be bounded by
\[
\int_{1 - e^{-\sqrt{|\ln(T - t(x))|}}}^1 \hat{\theta}(\tau')\theta^{-1}_\infty |\ln(T - t(x))|^{\beta} \, d\tau' \leq Ce^{-\sqrt{|\ln(T - t(x))|}},
\]
since
\[
|\hat{\theta}(\tau)\theta^{-1}_\infty |\ln(T - t(x))|^{\beta}| \leq C, \quad \forall \tau \in [0, 1).
\]
On the other hand, if \( \tau \in [0, 1) \) such that
\[
0 \leq \frac{\ln(1 - \tau)}{\ln(T - t(x))} \leq \frac{1}{|\ln(T - t(x))|^{\frac{1}{2}}},
\]
we equivalently have
\[
0 \geq \ln(1 - \tau) \geq \frac{\ln(T - t(x))}{|\ln(T - t(x))|^{\frac{1}{2}}} = -|\ln(T - t(x))|^{\frac{1}{2}},
\]
which is equivalent again
\[
1 \geq 1 - \tau \geq e^{-|\ln(T - t(x))|^{\frac{1}{2}}} \iff \tau \in \left[ 0, 1 - e^{-|\ln(T - t(x))|^{\frac{1}{2}}} \right].
\]
Then in the above region, we can apply a Taylor expansion to derive
\[
\left| \hat{\theta}(\tau)\theta^{-1}_\infty |\ln(T - t(x))|^{\beta} - 1 \right| \lesssim |\ln(1 - \tau)||\ln(T - t(x))|^{-\frac{1}{2}},
\]
which finally concludes (4.20).
Furthermore, using (4.20), we derive
\[
u(x) \sim \theta^{-1}_\infty^{\frac{1}{\nu - 1}} (T - t(x))^{-\frac{1}{\nu - 1}} |\ln(T - t(x))|^{\nu} \left( \frac{bK^2}{16} \right)^{-\frac{1}{\nu - 1}}, \text{ as } x \to 0
\]
and thus by virtue of Lemma C.1 we obtain

\[ u^*(x) \sim \theta_\infty \frac{1}{p-1} \left[ \frac{b}{2} |x|^2 \ln |x| \right]^{-\frac{1}{p-1}} |\ln |x|| ^{\nu} , \text{ as } x \to 0, \]

recalling that \( \nu = \frac{\beta}{p-1} \), which actually proves statement (iii) of Theorem 1.1.

This concludes the proof of Theorem 1.1.

5. A finite dimensional reduction

The current section deals with the reduction of the problem of controlling \( u(t) \in S(t) \) to a finite dimensional one on controlling only the two positive spectrum modes \( q_0 \) and \( q_1 \) in \( \tilde{V}_A(s) \), introduced in (4.5).

**Proposition 5.1** (Reduction to a finite dimensional problem). There exist positive parameters \( T, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0 \) and \( \eta_0 \) such that if we assume that \( (d_0, d_1) \in D_A \), defined as in (4.3); and with initial data \( u_{d_0, d_1} \), constructed as in (4.2), the solution \( u \) of equation (1.1) exists on \( [0, t_1] \), for some \( t_1 < T \). Furthermore, if we assume that \( u \in S(t) \) for all \( \forall t \in [0, t_1] \) and \( u(t_1) \in \partial S(t_1) \) (see \( S(t) \)'s definition in Definition 4.1), then, the following properties hold:

- (i) At \( s_1 = -\ln(T - t_1) \), we have \( (q_0, q_1)(s_1) \in \partial \tilde{V}_A(s_1) \).
- (ii) We can find \( \nu_0 > 0 \) such that

\[ (q_0, q_1)(s_1 + \nu) \notin \tilde{V}_A(s_1 + \nu), \forall \nu \in (0, \nu_0). \]

In particular, we have the fact that there exists \( \nu_1 > 0 \) such that

\[ u \notin S(t_1 + \nu), \forall \nu \in (0, \nu_1). \]

**Proof.** The proof directly follows from a priori estimates in regions \( P_1, P_2 \) and \( P_3 \) of \( S(t) \), introduced in Definition 4.1. Such a priori estimates are derived in Propositions 5.3, 5.5 and 5.6 below. The main reasoning is basically the same as in [13, 24]. However, since the current situation is more difficult, due to the form of perturbation \( \theta \), some significant modification of the existing technique should be performed. \( \square \)

Let us resume to main arguments on deriving a priori estimates in Propositions 5.3, 5.5 and 5.6:

- In Proposition 5.3 we give estimates such that the bounds in Definition 4.1(i) are satisfied. More precisely, we show that except from the bounds on \( q_0 \) and \( q_1 \), the other estimates are satisfied by stricter bounds.

- In Proposition 5.5 we prove that all estimates in Definition 4.1 (ii) are controlled by stricter bounds.

- Finally, in Proposition 5.6 we again obtain that the required bounds in Definition 4.1(iii) are also satisfied by stricter bounds.

5.1. A priori estimates on \( P_1 \)

We first establish the following result:

**Lemma 5.2.** There exist \( K_4, A_4 > 0 \) such that for all \( K_0 \geq K_4, A \geq A_4 \) and \( \gamma \leq A^{-4} \) and \( l^* > 0 \), we can find \( T_4(K_0, A, l^*) > 0 \) small enough such that for all \( \alpha_0, \delta_0, \epsilon_0, \eta_0, C_0 \) and \( T \leq T_4 \), and \( l \in [0, l^*] \) and under the extra assumptions:

- Initial data \( u_{d_0, d_1} \) as introduced in (4.2) with \( (d_0, d_1) \) as defined in Proposition 4.3 are considered;
• \( u(t) \) belongs to \( S(T, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, t) \) for all \( t \in [T - e^{-\sigma}, T - e^{-(\sigma + l)}] \), for some \( \sigma \geq s_0 \), and \( l \in [0, \ell^*] \).

then, the following estimates hold:

(i) For all \( s \in [\sigma, \sigma + l] \), we have

\[
|q_0'(s) - q_0(s)| \leq \frac{CA}{s^2}, \quad (5.1)
\]

\[
|q_1'(s) - \frac{1}{2} q_1(s)| \leq \frac{C}{s^2}, \quad (5.2)
\]

\[
|q_2'(s) + \frac{2 + \beta}{s} q_2(s)| \leq \frac{CA^3}{s^7}. \quad (5.3)
\]

(ii) For all \( s \in [\sigma, \sigma + l] \), we have two cases:
- If \( \sigma > s_0 \), then

\[
\frac{|q_-(y, s)|}{1 + |y|^3} \leq C \left( A^6 e^{-\frac{s-\sigma}{2}} + A^7 e^{-(s-\sigma)^2} + A^4 e^{(s-\sigma)(1 + (s-\sigma)^2) + (s-\sigma)} \right) \frac{1}{s^2}. \quad (5.4)
\]

- If \( \sigma = s_0 \), then

\[
\frac{|q_-(y, s)|}{1 + |y|^3} \leq \frac{C(1 + s - \sigma)}{s^2}. \quad (5.5)
\]

(iii) For all \( s \in [\sigma, \sigma + l] \), we also have:
- If \( \sigma > s_0 \)

\[
\frac{|
abla q_-(y, s)|}{1 + |y|^3} \leq \frac{C(C_0)}{s^2} \left( A^6 e^{-\frac{s-\sigma}{2}} + e^{-(s-\sigma)^2} + s - \sigma + \sqrt{s - \sigma} \right), \quad (5.6)
\]

- If \( s = s_0 \) then

\[
\frac{|
abla q_-(y, s)|}{1 + |y|^3} \leq \frac{C}{s^3} (1 + s - \sigma + \sqrt{s - \sigma}).
\]

(iv) Finally for all \( s \in [\sigma, \sigma + l] \), we also have:
- If \( \sigma > s_0 \) then

\[
|q_\varepsilon| \leq \frac{C}{\sqrt{s}} \left( A^7 e^{-\frac{s-\sigma}{2}} + A^6 e^{s-\sigma} + A^4 e^{s-\sigma}(1 + (s-\sigma)^2) + 1 + (s-\sigma) \right). \quad (5.7)
\]

- If \( \sigma = s_0 \) then

\[
|q_\varepsilon| \leq \frac{C}{\sqrt{s}} (1 + (s - \sigma)). \quad (5.8)
\]

Proof. The proof is similar to [56, Lemma 3.2]. However, we also need to fine-tune some important estimates corresponding to the current situation. For that reason, we will ignore simple estimates and detailed arguments, and instead we will focus on the derivation of important ones.

(i) In that step, we show estimates (5.1) - (5.3): Let us consider \( P_j, j = 1, 2, 3 \), the projection on the eigenspace \( E_j, j = 1, 2, 3 \); for more details regarding the definitions of \( P_j \) and \( E_j \), see (3.19)
and (3.15). Now, we apply \( P_j \) to (3.9) and make use of Lemmas [A.1-A.4] to obtain

\[
q_0' = q_0 + \left( a - \frac{2bN\kappa}{(p-1)^2} - \frac{\kappa\beta}{(p-1)} \right) \frac{1}{s} + \kappa\tilde{\lambda}(s) + O\left( \frac{1}{s^2} \right),
\]

\[
q_1' = q_1 + O\left( \frac{1}{s^2} \right),
\]

\[
q_2' = \frac{q_2}{s} \left( \frac{8bp}{(p-1)^2} + \frac{ap}{\kappa} - \frac{2Nbp}{(p-1)^2} - \frac{\beta}{(p-1)} \right) - \frac{q_0bp}{(p-1)^2s}
+ \frac{1}{s^2} \left( \frac{bp}{(p-1)^2} \left( \frac{2bN\kappa}{(p-1)^2} - a \right) + \frac{kb}{(p-1)^2} \left( \frac{4bp}{(p-1)^2} - 1 + \frac{\beta}{(p-1)} \right) \right)
+ \lambda q_2 - \tilde{\lambda}(s) \frac{kb}{(p-1)^2s} + O\left( \frac{1}{s^3} \right),
\]

where

\[
b = \frac{(p-1)^2}{4p}(1 + \beta),
\]

\[
a = \frac{N\kappa}{2ps}(1 + \beta) + \frac{\beta\kappa}{(p-1)s},
\]

and

\[
\tilde{\lambda}(s) = \frac{1}{p-1} \left( \frac{\theta'(s)}{\theta(s)} + \frac{\beta}{s} \right).
\]

Then, after some cancellations, we obtain

\[
q_0' = q_0 + \kappa\tilde{\lambda}(s) + O\left( \frac{1}{s^2} \right),
\]

\[
q_1' = \frac{1}{2}q_1 + O\left( \frac{1}{s^2} \right),
\]

\[
q_2' = \frac{2 + \beta}{s}q_2 + \tilde{\lambda}(s)q_2 - \tilde{\lambda}(s) \frac{kb}{(p-1)s} + O\left( \frac{1}{s^3} \right).
\]

Also by Proposition 4.5 we get

\[
\lvert \tilde{\lambda}(s) \rvert \leq \frac{CA^3}{s^2},
\]

provided that \( \gamma \leq A^{-4} \), which concludes item (i).

- We now proceed with the proof of items (ii) and (iv). To this end we write (3.9) under integral form

\[
q(s) = \mathcal{K}(s, \sigma)(q(\sigma)) + \int_\sigma^s \mathcal{K}(s, \tau) (B(q) + R + G)(\tau)d\tau,
\]

where \( \mathcal{K}(s, \sigma) \) is the fundamental solution associated to the linear operator \( \mathcal{L} + \mathcal{V} \), see more details in [1]. In particular, we have the following result (see [1] for the 1-dimensional case, and [62] for the multidimensional one): For all \( t^* > 0 \), there exists \( s^* = s^*(t^*) \) such that for all \( \sigma \geq s^* \) and \( \nu \in L^2(\mathbb{R}^N) \), then, for all \( s \in [\sigma, \sigma + t^*] \), the function \( \psi(s) = \mathcal{K}(s, \sigma)v \) satisfies

\[
\left\| \frac{\psi_-(y, s)}{1 + \lvert y \rvert^\alpha} \right\|_{L^\infty} \leq C e^{s^*-\sigma} \left( \frac{(s-\sigma)^2 + 1}{s} \right) (|v_0| + |v_1| + \sqrt{s}|v_2|)
+ C e^{-\frac{s-\sigma}{2}} \left\| \frac{\psi_-(\cdot)}{1 + \lvert y \rvert^\alpha} \right\|_{L^\infty} + C \frac{e^{-(s-\sigma)^2}}{s^{\frac{3}{2}}} \left\| v_e \right\|_{L^\infty},
\]

\[
(5.10)
\]
and
\[
\|\psi(s)\|_{L^\infty} \leq Ce^{s-\sigma} \left( \sum_{i=0}^{2} s^\frac{i}{2} |v_i| + s^{\frac{2}{3}} \left\| \frac{v_+}{1 + |y|^3} \right\|_{L^\infty} + Ce^{-\frac{s-\sigma}{T}} \|v_e\|_{L^\infty} \right).
\] (5.11)

Now, we apply this result to derive
\[
\left| \int_{s}^{s+T} K(s, \tau) (B(q) + R + G)(\tau) d\tau \right| \leq C \frac{(s - \sigma)}{s^2} (1 + |y|^3),
\]
\[
\left\| \int_{s}^{s+T} K(s, \tau) (B(q) + R + G)(\tau) d\tau \right\|_{L^\infty} \leq C \frac{(s - \sigma)}{\sqrt{s}}.
\]

It remains to obtain the result for \( \psi(s) = K(s, \sigma)q(\sigma) \). Indeed, using the fact that \( q(\sigma) \in V_A(\sigma) \), we obtain
\[
\frac{|\psi_+(y, s)|}{1 + |y|^3} \leq C e^{s-\sigma} \frac{(1 + (s - \sigma)^2)}{s} A^4 \frac{\sigma}{\sigma^2} + Ce^{-\frac{s-\sigma}{\sigma}} \frac{A^6}{\sigma^2} + Ce^{-(s-\sigma)^2}
\]
\[
\leq C \left( e^{-\frac{s-\sigma}{\sigma}} A^6 + e^{-(s-\sigma)^2} A^7 + e^{s-\sigma} (1 + (s - \sigma)^2) A^4 \right) \frac{1}{s^2},
\]
provided that \( \frac{1}{\sigma} \leq \frac{2}{5} \). Similarly, we obtain
\[
\|\psi_e\|_{L^\infty} \leq C \left( A^7 e^{-\frac{s-\sigma}{T}} + e^{s-\sigma} A^6 \right).
\]

Therefore we conclude items (ii) and (iv) for the case \( \sigma > s_0 \). Furthermore, using Proposition 4.3, we also conclude the validity of (ii) and (iv) for the case \( \sigma = s_0 \).

- The proof to item (iii): We similarly proceed as in item (ii) by using [56, Lemma B.2 (ii)] and the second estimate in Definition 4.1(ii).

Next we deal with the following result:

**Proposition 5.3** (A priori estimates in \( P_1(t) \)). There exist \( K_5, A_5 \geq 1 \) such that for all \( K_0 \geq K_5, A \geq A_5, \epsilon_0 > 0, \alpha_0 > 0, \delta_0 \leq \frac{1}{10} U(0), C_0 > 0, \eta_0 > 0, \) and \( \gamma \leq A_{-1}, \) there exists \( T_5(K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0) \) such that for all \( T \leq T_5, \) the following holds: If \( U(t) \) is a non-negative solution of equation (2.12) satisfying \( U(t) \in S(T, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, t) \) for all \( t \in [0, T_5] \) for some \( t_5 \in [0, T) \), and with initial data \( U_{d_0, d_1} \) given in (4.2) for some \( d_0, d_1 \in D_A \) (cf. Proposition 4.3), then for all \( s \in [-\ln T, -\ln(T-T_5)] \), we have the following:

\[
\forall i, j \in \{1, \cdots, n\}, \quad |q_{2,i,j}(s)| \leq A^4 \frac{\sigma}{2s^2},
\]
\[
\left\| q_-(s) \right\|_{L^\infty(\mathbb{R}^N)} \leq A^6 \frac{\sigma}{2s^2}, \quad \left\| \nabla q_-(s) \right\|_{L^\infty(\mathbb{R}^N)} \leq A^7 \frac{\sigma}{2s^2},
\]
\[
\left\| q_e(s) \right\|_{L^\infty(\mathbb{R}^N)} \leq A^7 \frac{\sigma}{2s^2} .
\]

**Proof.** The proof is quite similar to [57, Proposition 3.4] and it arises by using Lemma 5.2. \( \square \)

### 5.2. A priori estimates in \( P_2 \)

In this subsection, we show that the solution satisfies the conditions in \( S(t) \), with strict inequalities.

**Lemma 5.4** (A transmission in \( P_2 \)). We can find \( K_6, A_6, \geq 1 \) such that for all \( K_0 \geq K_6, A \geq A_6, \delta_0 > 0, \) there exist \( \alpha_6(K_0, A, \delta_0), C_6(K_0, A, \delta_0) > 0 \) such that for all \( \alpha_0 \leq \alpha_6, C_0 \geq 2C_6, \) there exists \( \epsilon_6(\alpha_0, A, \delta_0, C_0) > 0 \) such that for all \( \epsilon_0 \leq \epsilon_6, \) and \( \delta_0 \leq \frac{1}{3} \left( p - 1 + \frac{(p - 1)^2 K_0^2}{16} \right)^{-\frac{1}{p-1}} \)}
there exist $T_0(\varepsilon_0, A, \delta_0, C_0)$ and $\eta_0(\varepsilon_0, A, \delta_0, \delta_0, C_0)$ such that for all $T \leq T_0$, we have the following property: assume that $u \in S(T, K_0, \varepsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, t)$, $\forall t \in [0, t^*], \forall t^* \in [0, T]$, and initial data $u(0)$ defined as in (4.2) for some $|d_0|, |d_1| \leq 2$, then the following hold for all $|x| \in \left[\frac{K_0}{4} \sqrt{(T - t^*) \ln(T - t^*)}, \varepsilon_0\right]$.

(i) For all $\xi \in \left[\frac{\eta}{4} \sqrt{\ln \rho(x)}\right]$ and $\tau \in \left[\max\left(0, \frac{\tau(x)}{\rho(x)}\right), \frac{\tau^* - \tau(x)}{\rho(x)} \right]$ the following estimates are valid

\[|\nabla_\xi \dot{U}(x, \xi, \tau)| \leq \frac{C(K_0, C_0) A^2}{\sqrt{\ln \rho(x)}},\]

\[\frac{1}{4} \left(p - 1 + \frac{(p - 1)^2 K_0^2}{16}\right) \leq \frac{\dot{U}(x, \xi, \tau)}{\rho(x)} \leq C(K_0).\]

(ii) For all $|\xi| \leq 2\alpha_0 \sqrt{\ln \rho(x)}$ and $\tau_0 = \max\left(0, \frac{\tau(x)}{\rho(x)}\right)$, we have

\[|\dot{U}(x, \xi, \tau) - \dot{U}(x, \tau_0)| \leq \delta_0 \text{ and } |\nabla_\xi \dot{U}(x, \xi, \tau)| \leq \frac{C_0}{2 \sqrt{\ln \rho(x)}}.\]

Proof. The proof can be found in [56, Lemma 2.6].

Now, we use the parabolic regularity to derive the following:

Proposition 5.5 (A priori estimate in $P_2(t)$). There exist $K_7 \geq 1, A_7 \geq 1$ such that for all $K_0 \geq K_7, A \geq A_7$, there exist $\delta_7 \leq \frac{1}{3} \left(p - 1 + \frac{K_0^2}{16}\right)$ and $C_7(K_0, A)$ such that for all $\delta_0 \leq \delta_7, C_0 \geq C_7$ there exists $\varepsilon_7(K_0, \delta_0, C_0), \alpha_7(K_0, A, C_0)$ such that $\varepsilon_0 \leq \varepsilon_7$ there exists $T_7(K_0, A, \delta_0, C_0, \varepsilon_0), \eta_7 > 0$ such that for all $T \leq T_7, \eta_0 \leq \eta_7$, we have the following property: assume that $u \in S(T, K_0, \varepsilon_0, \alpha_0, A, \delta_0, C_0, t)$ for all $t \in [0, t^*]$ for some $t^* \in [0, T]$, and initial data $u(0)$ defined as in (4.2) for some $|d_0|, |d_1| \leq 2, \varepsilon_0$ then for all $|x| \in \left[\frac{K_0}{4} \sqrt{(T - t^*) \ln(T - t^*)}, \varepsilon_0\right], |\xi| \leq \alpha_0 \sqrt{\ln \rho(x)}$ and $\tau^* = \frac{\tau^* - \tau(x)}{\rho(x)}$, we have

\[|\dot{U}(x, \xi, \tau^*) - \dot{U}(x, \tau^*)| \leq \frac{\delta_0}{2} \text{ and } |\nabla_\xi \dot{U}(x, \xi, \tau^*)| \leq \frac{C_0}{2 \sqrt{\ln \rho(x)}}.\]

Proof. Note that $\dot{U}$ satisfies

\[\dot{\partial}_\tau \dot{U} = \Delta \dot{U} + \tilde{\theta}(\tau) \theta^{-1}(t(x)) U^p - \rho(x) U.\]

By $\theta$’s monotonicity, we conclude that for all $\tau' \in [\tau_0(x), \tau^*(t^*, x)]$

\[\left|\tilde{\theta}(\tau') \theta^{-1}(t(x))\right| \leq 1.\]

We also have

\[\dot{\partial}_\tau \dot{U} = \tilde{\theta}(\tau) \theta^{-1}(t(x)) \dot{U}^p,\]

and if we set

\[\tilde{U}(\tau) := U(\tau) - \dot{U}(\tau),\]

then $\tilde{U}$ satisfies

\[\dot{\partial}_\tau \tilde{U} = \Delta \tilde{U} + \tilde{\theta}(\tau) \theta^{-1}(t(x)) \left[U^p - \dot{U}^p\right] - \rho(x) \dot{U}.$

We remark that

\[\left|\tilde{\theta}(\tau) \theta^{-1}(t(x)) \dot{U}^p - \dot{U}^p\right| \leq C \tilde{U}.\]

We now consider $\chi_{\frac{\gamma}{4}}$ which is smooth, satisfying $\chi_{\frac{\gamma}{4}}(r) = 1, \forall |r| \leq [0, 1]$ and $\chi_{\frac{\gamma}{4}}(r) = 0, \forall |r| \geq \frac{\gamma}{4}$. Next we define

\[\tilde{\chi}(\xi) := \chi_{\frac{\gamma}{4}}\left(\frac{|\xi|}{\alpha_0 \sqrt{\ln \rho(x)}}\right),\]
and

\[ \tilde{u} := \tilde{\chi}u. \]

Hence, we derive

\[ \partial_t \tilde{u} = \Delta \tilde{u} + \tilde{\chi}\tilde{\theta}(\tau)\theta^{-1}(t(x))(U^{p} - \tilde{U}^{p}) + G(x, \xi, \tilde{u}, \tau), \]

where

\[ \left| \tilde{\chi}\tilde{\theta}(\tau)\theta^{-1}(t(x))(U^{p} - \tilde{U}^{p}) \right| \leq C\tilde{u}, \]

and

\[ |G(x, \xi, \tilde{u}, \tau)| \leq \frac{C}{\sqrt{\ln(\rho(x))}}. \]

By virtue of (5.12) we obtain, for all \( \tau \in [\tau_{0}, \tau^{*}] \), the following integral equation:

\[ \tilde{u}(\tau) = e^{(\tau - \tau_{0})\Delta} \tilde{u}(\tau_{0}) + \int_{\tau_{0}}^{\tau} e^{(\tau_{0} - \tau')\Delta} (\tilde{\chi}\tilde{\theta}(\tau')\theta^{-1}(t(x))(U^{p} - \tilde{U}^{p})(\tau') + G(x, \xi, \tilde{u}, \tau'))d\tau'. \]

Taking the \( L^{\infty} \) – norm to the above equation in applying Lemma 5.4, we have

\[ \| \tilde{u}(\tau) \|_{L^{\infty}} \leq C(\delta_{6} + \frac{1}{\sqrt{\ln(\rho(x))}}) + C \int_{\tau_{0}}^{\tau} \| \tilde{u}(\tau') \|_{L^{\infty}}, \]

and via Gronwall’s lemma we finally obtain

\[ \| \tilde{u}(\tau) \|_{L^{\infty}} \leq 3C \left( \delta_{6} + \frac{1}{\sqrt{\ln(\rho(x))}} \right) \leq \frac{\delta_{0}}{2}, \]

provided that \( \delta_{6} \leq \delta_{7}(\delta_{0}) \) and \( \epsilon_{0} \leq \epsilon_{7} \). The proof of the estimate for \( \nabla_{x}\tilde{u} \) follows the same technique and so it is omitted. \( \square \)

### 5.3. A priori estimates on \( P_{3} \)

Next we prove that the conditions in Definition 4.1 (iii) are strictly satisfied. Indeed the following holds:

**Proposition 5.6** (A priori estimate in \( P_{3}(t) \)). Let us consider \( K_{0}, \epsilon_{0}, \alpha_{0}, A, C_{0}, \gamma_{0} > 0 \) and \( \delta_{0}, \in \left( 0, \frac{1}{3} \left( p - 1 + \frac{K_{0}^{2}}{2} \right)^{-\frac{1}{p-1}} \right] \). Then, there exists \( T_{k} > 0 \) small enough such that for all \( T \in (0, T_{k}) \), the following property holds: assume that \( u \) be a non negative solution to (1.1) for all \( t \in [0, t^{*}] \) and it is satisfied that \( u(t) \in S(T, K_{0}, \epsilon_{0}, \alpha_{0}, A, \delta_{0}, C_{0}, \gamma_{0}, t) \) for all \( t \in [0, t^{*}] \) corresponding to initial data \( u_{d_{0}, d_{1}} \) for some \( |d_{0}|, |d_{1}| \leq 2 \), then for all \( x \in \Omega \cap \{|x| \geq \frac{M}{4} \} \), we have

\[ |u(x, t^{*}) - u(0)| \leq \frac{\eta_{0}}{2}, \]

and

\[ \left| \nabla u(x, t^{*}) - \nabla e^{t^{*}\Delta} u(0) \right| \leq \frac{\eta_{0}}{2}, \]

where \( e^{t\Delta} \) is the semi-group associated to \( \Delta \) with Neumann boundary conditions.

**Proof.** We write (1.1) as follows:

\[ u(t) = e^{t\Delta} u(0) + \int_{0}^{t} e^{(t-t')\Delta} \left[ \theta(t')u^{p} - u(t') \right] dt', \]

where \( e^{t\Delta} \) is the semi-group associated to Neumann boundary conditions, see more in [72]. Thus, the proof directly comes from parabolic regularity estimates. \( \square \)
6. Computation on initial data

In the current section we provide the complete proof of Proposition 4.3. Indeed, we first prove that such an initial data considered by Proposition 4.3 will ensure the following asymptotic behavior

\[
\theta(0) = \theta_\infty |\ln(T)|^{-\beta} \left( 1 + O \left( \frac{1}{\sqrt{|\ln T|}} \right) \right) \quad \text{as} \quad t \to T, \tag{6.1}
\]

where \( \theta_\infty \) and \( \beta \) defined as in (2.41) and (2.42), respectively.

Converting to \( \bar{\theta}(s_0), s = -\ln(T - t), s_0 = -\ln T \), defined as in (2.14) then equivalently we must ensure that

\[
\bar{\theta}(s_0) = \theta_\infty s_0^{-\beta} \left( 1 + O \left( \frac{1}{\sqrt{|\ln T|}} \right) \right).
\]

In particular, we use the relation in (2.1), we claim that the following derives (6.1)

\[
\|u_{d_0,d_1}\|_r^r = \theta_\infty^\frac{N}{2} \left[ b \right]^{-\frac{N}{2}} \frac{1}{2(1 + \frac{N}{2}(1 + \beta))} (|\ln T|)^{1 + \frac{N}{2}(1 + \beta)} \left( 1 + O \left( \frac{1}{\sqrt{|\ln T|}} \right) \right). \tag{6.2}
\]

Indeed, let us suppose that (6.2) holds. Then, from using \( \theta \)'s definition we also derive

\[
\theta(0) = |\Omega|^\gamma (\|u_{d_0,d_1}\|_{L^r(\Omega)})^{-\gamma} = |\Omega|^\gamma \theta_\infty^\frac{N}{2} \left[ b \right]^{-\frac{N}{2}} 2^\gamma \left( 1 + \frac{N}{2}(1 + \beta) \right) |\ln T|^{-\gamma(1 + \frac{N}{2}(1 + \beta))} \left( 1 + O \left( \frac{1}{\sqrt{|\ln T|}} \right) \right).
\]

In fact

\[-\gamma \left( 1 + \frac{N}{2}(1 + \beta) \right) = -\beta,
\]

whilst thanks to Lemma (C.2) we have

\[
\int_0^\infty \varphi_0^{p-1+r}(\xi)\xi^{N-1}d\xi = \frac{1}{(p-1)N} b^{-\frac{N}{2}},
\]

and so

\[
\theta_\infty = \left( \frac{|\Omega| (1 + \frac{N}{2})}{2b^{-\frac{N}{2}}} \right)^{\frac{\gamma}{\gamma(1 + \frac{N}{2})}}. \tag{6.3}
\]

The latter infers

\[
|\Omega|^\gamma \theta_\infty^\frac{N}{2} \left[ b \right]^{-\frac{N}{2}} 2^\gamma \left( 1 + \frac{N}{2}(1 + \beta) \right) ^\gamma = \theta_\infty,
\]

which finally implies (6.1).

Now, we start the proof of (6.2). Let us consider \((d_0, d_1) \in [-2, 2]^{1+N}, \frac{r}{p-1} = \frac{N}{2} \) and \( p \geq 3 \). We first provide an estimate for \( \|u_{d_0,d_1}\|_{L^r(\Omega)} \). Recalling its definition we get

\[
\|u_{d_0,d_1}\|_{L^r(\Omega)} = \int_{\Omega} u_{d_0,d_1}^r dx = \int_{|x| \leq K_0 \sqrt{|\ln T|} |\ln T|^{\frac{1}{2}}} u_{d_0,d_1}(0)^r dx + \int_{|x| \geq K_0 \sqrt{|\ln T|} |\ln T|^{\frac{1}{2}}} u_{d_0,d_1}(0)^r dx = I_1 + I_2.
\]
Hence, (6.2) follows the following

\[ I_1 \leq C|\ln T|^{\frac{N}{2} + \frac{r\beta}{p-1}}|\ln |\ln T||, \]

\[ I_2 = \theta_\infty \left[ \frac{b}{2} \right] - \frac{N}{2} \frac{1}{2(1 + \theta_\infty (1 + \beta))} (|\ln T|)^{1 + \frac{N}{2}(1 + \beta)} \left( 1 + O \left( \frac{|\ln |\ln T||}{|\ln T|} \right) \right). \]

- Estimation for \( I_1 \): Let us recall \( u_{d_0,d_1} \) defined as in (4.2), then, for all \(|x| \leq K_0 \sqrt{T} |\ln T||\ln T|^\frac{1}{2}\), we have

\[ u_{d_0,d_1} = T^{-\frac{1}{p-1}} (\theta_\infty |\ln T|^{-\beta})^{-\frac{1}{p-1}} \left[ \varphi \left( \frac{x}{\sqrt{T}} \right) - \ln s_0 \right] + (d_0 A^3 s_0^2 + A s_0^2 d_1 \cdot \frac{x}{T^2}) \chi_0 \left( \frac{|x|}{K_0} \right). \]

We now decompose \( I_1 \) as follows

\[ I_1 = \int_{|x|\leq \frac{K_0}{16} \sqrt{T} |\ln T|} u_{d_0,d_1}^r (0) \, dx + \int_{\frac{K_0}{16} \sqrt{T} |\ln T| \leq |x| \leq K_0 \sqrt{T} |\ln T||\ln T|^\frac{1}{2}} u_{d_0,d_1}^r (0) \, dx. \]

We see that for all \(|x| \leq \frac{K_0}{16} \sqrt{T} |\ln T|\),

\[ |u_{d_0,d_1}^r (0)| \leq C(K_0) T^{-\frac{1}{p-1}} |\ln T|^{\frac{r\beta}{p-1}}. \]

Then we obtain

\[ \left| \int_{|x|\leq \frac{K_0}{16} \sqrt{T} |\ln T|} u_{d_0,d_1}^r (0) \, dx \right| \leq C(K_0) |\ln T|^{\frac{N}{2} + \frac{r\beta}{p-1}}. \]

Now we consider \(|x| \in \left[ \frac{K_0}{16} \sqrt{T} |\ln T|, K_0 \sqrt{T} |\ln T||\ln T|^\frac{1}{2} \right] \) and thus

\[ u_{d_0,d_1}^r (0) = T^{-\frac{1}{p-1}} (\theta_\infty)^{-\frac{1}{p-1}} |\ln T|^{\frac{\beta}{p-1}} \left[ \varphi \left( \frac{|x|}{\sqrt{T} |\ln T|} \right) + \frac{a}{|\ln T|^r} \right]. \]

At this point we recall the following inequality

\[ |(a + b)\alpha| \leq C(\alpha) [a^\alpha + b^\alpha], \]

for \( a, b, \alpha > 0 \). So, we have

\[ \left[ \varphi \left( \frac{|x|}{\sqrt{T} |\ln T|} \right) + \frac{a}{|\ln T|^r} \right] \leq C(r) \left[ \varphi \left( \frac{|x|}{\sqrt{T} |\ln T|} \right) + \frac{1}{|\ln T|^r} \right], \]

which yields

\[ \left| \int_{\frac{K_0}{16} \sqrt{T} |\ln T| \leq |x| \leq K_0 \sqrt{T} |\ln T||\ln T|^\frac{1}{2}} u_{d_0,d_1}^r (0) \, dx \right| \]

\[ \leq |\ln T|^{\frac{N}{2} + \frac{r\beta}{p-1}} \left( \int_{\frac{K_0}{16} \sqrt{T} |\ln T|} \left[ \varphi \left( \frac{|x|}{\sqrt{T} |\ln T|} \right) + \frac{1}{|\ln T|^r} \right] d\xi \right) \]

\[ \leq C |\ln T|^{\frac{N}{2} + \frac{r\beta}{p-1}} |\ln |\ln T||. \]

and finally (6.4) arises.

- Behavior of \( I_2 \): For all \(|x| \in \left[ K_0 \sqrt{T} |\ln T| |\ln T|^\frac{1}{2}, 2K_0 \sqrt{T} |\ln T||\ln T|^\frac{1}{2} \right] \), \( u_{d_0,d_1}^r (0) \) has the following form

\[ u_{d_0,d_1}^r (0) = \chi_1 (x,0) \left( T^{-\frac{1}{p-1}} \theta_\infty^{-\frac{1}{p-1}} |\ln T|^{\frac{\beta}{p-1}} \varphi \left( \frac{|x|}{\sqrt{T} |\ln T|} \right) - H^* (x) \right) + H^* (x). \]
It is easy to see that
\[ |T^{\frac{1}{p-1}} \theta^{-\frac{1}{p-1}} \ln T^{\frac{\beta}{p-1}} \varphi_0 \left( \frac{|x|}{\sqrt{T|\ln T|}} \right) | \leq CH^*(x), \]
which yields
\[ |u_{d_0,d_1}| \leq 2CH^*, \]
and thus we obtain
\[ \int_{K_0 \sqrt{T|\ln T|}| \ln T|^{\frac{x}{2}} \leq |x| \leq 2K_0 \sqrt{T|\ln T|}| \ln T|^{\frac{x}{2}} u_{d_0,d_1}^r(0)dx \]
\[ \leq C \int_{K_0 \sqrt{T|\ln T|}| \ln T|^{\frac{x}{2}} \leq |x| \leq 2K_0 \sqrt{T|\ln T|}| \ln T|^{\frac{x}{2}} (H^r)dx \]
\[ \leq C \int_{K_0 \sqrt{T|\ln T|}| \ln T|^{\frac{x}{2}} \leq |x| \leq 2K_0 \sqrt{T|\ln T|}| \ln T|^{\frac{x}{2}} [x^2]^{-\frac{1}{p-1}} |\ln |x||^{\frac{N}{2}(1+\beta)} dx \]
\[ \leq C |\ln |\ln T||^{1+\frac{N}{2}(1+\beta)}. \]
We also have
\[ \int_{|x| \geq \epsilon_0, x \in \Omega} u_{d_0,d_1}^r dx \leq C(\epsilon_0), \]
and thus it remains to estimate the following integral
\[ \int_{2K_0 \sqrt{T|\ln T|}| \ln T|^{\frac{x}{2}} \leq |x| \leq \epsilon_0} u_{d_0,d_1}^r dx = \int_{2K_0 \sqrt{T|\ln T|}| \ln T|^{\frac{x}{2}} \leq |x| \leq \epsilon_0} (H^r)dx. \]
\[ = \int_{2K_0 \sqrt{T|\ln T|}| \ln T|^{\frac{x}{2}} \leq |x| \leq \epsilon_0} \left[ \theta_{\infty}^{-\frac{b}{2}} \right]^{-\frac{N}{2}} |x|^{-\frac{N}{2}} |2 \ln |x||^{\frac{N}{2}(1+\beta)} dx \]
\[ + O(\| \ln T \|^{\frac{N}{2}(1+\beta)} | \ln |\ln T||) \]
\[ = \theta_{\infty}^{-\frac{b}{2}} \left[ \left( |\ln T| \right)^{1+\frac{N}{2}(1+\beta)} \frac{1}{2(1+\frac{N}{2}(1+\beta))} \right] \left( 1 + O \left( \frac{|\ln |\ln T||}{|\ln T|} \right) \right). \]
Consequently we derive
\[ \int_{2K_0 \sqrt{T|\ln T|}| \ln T|^{\frac{x}{2}} \leq |x| \leq \epsilon_0} u_{d_0,d_1}^r dx \]
\[ = \theta_{\infty}^{-\frac{b}{2}} \left[ \left( |\ln T| \right)^{1+\frac{N}{2}(1+\beta)} \frac{1}{2(1+\frac{N}{2}(1+\beta))} \right] \left( 1 + O \left( \frac{|\ln |\ln T||}{|\ln T|} \right) \right). \]
which concludes (6.5).

In particular, we finish the proof of (6.2).

Now, we start the proof of item (I) of Proposition 6:

- Estimates on the problem in similarity variables: In this part, we estimate the solution expressed in the variable \((y,s)\) as in (2.9). Following the chain of definition \(u_{d_0,d_1}(0) \rightarrow \)
\[ U_{d_0,d_1}(0) \rightarrow W_{d_0,d_1} \rightarrow q(s_0) \] as in (4.3), (4.2) and (4.4), we derive the following:
\[
U_{d_0,d_1}(0) = \theta^\frac{1}{p-\epsilon}(0) \chi_1(0) u_{d_0,d_1}(0)
= (\theta_\infty |\ln T|^{-\beta}) \frac{1}{p-1} \left( 1 + O \left( \frac{1}{\sqrt{|\ln T|}} \right) \right) \chi_1(0) u_{d_0,d_1}(0)
= T^{-\frac{1}{p-1}} \left[ \varphi \left( \frac{x}{\sqrt{T}}, -\ln T \right) + \left( d_0 \frac{A^3}{s_0^2} + \frac{A}{s_0^2} d_1 \cdot y_0 \right) \chi_0 \left( \frac{32 s_0}{K_0} \right) \right] \chi_1^2(0)
+ H^*(x)(1 - \chi_1(0))\chi_1(0)\theta^\frac{1}{p-\epsilon}(0)
+ O \left( \frac{T^\frac{1}{p-1}}{\sqrt{|\ln T|}} \right) \left[ \varphi \left( \frac{x}{\sqrt{T}}, -\ln T \right) + \left( d_0 \frac{A^3}{s_0^2} + \frac{A}{s_0^2} d_1 \cdot y_0 \right) \chi_0 \left( \frac{32 s_0}{K_0} \right) \right] \chi_1^2(0),
\]
where \( y_0 = \frac{x}{\sqrt{T}} \) and \( z_0 = \frac{x}{\sqrt{T|\ln T|}} \). Then, we derive initial data \( W(y_0, s_0) \), a function with \( y_0 \) variable
\[
W_{d_0,d_1}(y_0, s_0) = \left( \varphi(y, s_0) + \left( d_0 \frac{A^3}{s_0^2} + \frac{A}{s_0^2} d_1 \cdot y_0 \right) \chi_0 \left( \frac{32 s_0}{K_0} \right) \right) \chi_1^2(y_0, s_0)
+ T^\frac{1}{p-1} H^*(y_0 \sqrt{T})(1 - \chi_1(y_0, s_0))\chi_1(y_0, s_0)\theta^\frac{1}{p-\epsilon}(0)
+ O \left( \frac{1}{\sqrt{|\ln T|}} \right) \left[ \varphi \left( \frac{x}{\sqrt{T}}, -\ln T \right) + \left( d_0 \frac{A^3}{s_0^2} + \frac{A}{s_0^2} d_1 \cdot y_0 \right) \chi_0 \left( \frac{32 s_0}{K_0} \right) \right] \chi_1^2(y_0, s_0)
+ \varphi(y, s_0) + \left( d_0 \frac{A^3}{s_0^2} + \frac{A}{s_0^2} d_1 \cdot y_0 \right) \chi_0 \left( \frac{32 s_0}{K_0} \right) + \tilde{W}(y_0, s_0),
\]
where \( \tilde{W} \) defined by
\[
\tilde{W}(y_0, s_0) = \left( \varphi(y, s_0) + \left( d_0 \frac{A^3}{s_0^2} + \frac{A}{s_0^2} d_1 \cdot y_0 \right) \chi_0 \left( \frac{32 s_0}{K_0} \right) \right) \left( \chi_1^2(y_0, s_0) - 1 \right)
+ T^\frac{1}{p-1} H^*(y_0 \sqrt{T})(1 - \chi_1(y_0, s_0))\chi_1(y_0, s_0)\theta^\frac{1}{p-\epsilon}(0)
+ O \left( \frac{1}{\sqrt{|\ln T|}} \right) \left[ \varphi \left( \frac{x}{\sqrt{T}}, -\ln T \right) + \left( d_0 \frac{A^3}{s_0^2} + \frac{A}{s_0^2} d_1 \cdot y_0 \right) \chi_0 \left( \frac{32 s_0}{K_0} \right) \right] \chi_1^2(y_0, s_0).
\]
It is easy to have
\[
||\tilde{W}||_{W^{1,\infty}} \leq \frac{C}{\sqrt{|\ln T|}}.
\]
As a matter of fact, \( q(y_0, s_0) \) will be following
\[
q(y_0, s_0) = \left( d_0 \frac{A^3}{s_0^2} + \frac{A}{s_0^2} d_1 \cdot y_0 \right) \chi_0 \left( \frac{32 s_0}{K_0} \right) + \tilde{W}(y_0, s_0).
\]
Up to a small perturbation \( \tilde{W}(y_0, s_0), q(y_0, s_0) \) is the same as in [56, Lemma 2.4].
- Estimate in \( P_2(0) \): Let us consider \( |x| \in \left[ \frac{K_0}{4} \sqrt{T|\ln T|}, \epsilon_0 \right] \), and recall the definition of
\[
\mathcal{U}(x, \xi, \tau_0) = (T - t(x)) \theta^\frac{1}{p-1} (t(x)) u(x, 0),
\]
and
\[
\dot{U}(x, \tau_0(x)) = \left( (p - 1)(1 - \int_0^{\tau_0(x)} \theta(\tau')(\theta(t(x)))^{-1}d\tau') \right)^{-\frac{1}{p-1}},
\]
where
\[
\tau_0(x) = -\frac{t(x)}{T - t(x)} \in [0, 1].
\]
The reader should bear in mind that
\[
\theta(t(x)) = \theta(0) \text{ if } t(x) \leq 0,
\]
with \(\theta(0)\) satisfying (6.1). Now, we observe that for all \(\tau' \in [0, \tau_0(x)]\) and \(|x| \in \left[\frac{K_0}{4} \sqrt{T \ln T}, \epsilon_0\right]\)
\[
\tau' \rho + t(x) \leq 0,
\]
which implies
\[
\dot{U}(x, \tau_0(x)) = \left( (p - 1)(1 - \int_0^{\tau_0(x)} \theta(0)(\theta(0))^{-1}d\tau' + b\frac{K_0^2}{16}) \right)^{-\frac{1}{p-1}}
\]
\[
= \left( (p - 1)(1 - \tau_0(x)) + b\frac{K_0^2}{16} \right)^{-\frac{1}{p-1}}. \tag{6.7}
\]

Now, we will write \(U(\tau_0)\) with \(\tau_0(x) = \frac{-t(x)}{T-t(x)}\) as follows
\[
U(x, \xi, \tau_0) = (T - t(x))^{\frac{1}{p-1}}(\theta(t(x)))^{\frac{1}{p-1}}u(x + \xi \sqrt{T - t(x)}, 0)
\]
\[
= \left( \frac{(T - t(x))\theta(t(x))}{T\theta(\infty) \ln T} \right)^{\frac{1}{p-1}} \left( p - 1 + b\frac{|x + \xi \sqrt{T - t(x)}|^2}{T \ln T} \right)^{\frac{1}{p-1}} \chi_1(x + \xi \sqrt{T - t(x)}, 0)
\]
\[
+ (T - t(x))^{\frac{1}{p-1}} \theta^{\frac{1}{p-1}}(t(x))H^*(x + \xi \sqrt{T - t(x)})(1 - \chi_1(x + \xi \sqrt{T - t(x)}, 0))
\]
\[
= I\chi_1(x + \xi \sqrt{T - t(x)}, 0) + II(1 - \chi_1(x + \xi \sqrt{T - t(x)}, 0)).
\]

Let us mention that, our functions \(U(x, \xi, \tau_0)\) and \(\dot{U}(x, \tau_0(x))\) are similar to the ones at [56, page 1531]. So, we can apply the process to prove the following estimates (see more details in that work):
- For all \(|x| \in \left[\frac{K_0}{4} \sqrt{T \ln T}, 2K_0 \sqrt{T \ln T} \ln T \ln T \frac{1}{2}\right]\) and \(|\xi| \leq 2\alpha_0 \sqrt{\ln \rho(x)}\)
\[
\left| I - \hat{U}(x, \tau_0(x)) \right| \leq \frac{\delta_3}{2}. \tag{6.8}
\]
- For all \(|x| \in \left[K_0 \sqrt{T \ln T} \ln T \ln T \frac{1}{2}, \epsilon_0\right]\) and \(|\xi| \leq 2\alpha_0 \sqrt{\ln \rho(x)}\)
\[
\left| II - \hat{U}(x, \tau_0) \right| \leq \frac{\delta_3}{2}. \tag{6.9}
\]
From (6.8) and (6.9), we conclude
\[
\left| U(x, \xi, \tau_0) - \hat{U}(x, \tau_0) \right| \leq \delta_3.
\]
In addition to that, the technique at [56, pages 1533-1535] can be applied to prove the following:
\[
\left| \nabla_\xi U(x, \xi, \tau_0) \right| \leq \frac{C_3}{\sqrt{\ln \rho(x)}}.
\]
Finally, the proof of the required estimates in \(P_2(0)\) immediately follows.
For item (II) obviously arises from item (I), see more details in [71]. We finish the proof of the Proposition.

A. Necessary estimates

In this part, we aim to give more details on some rough calculations presented in the preceding sections, and support the proof of our main result.

Lemma A.1 (Potential term V). Let us consider V defined as in (3.11). Then, we have the following estimates

\[ V(y, s) = -\frac{pb}{(p-1)^2}(|y|^2 - 2N) + \frac{1}{s} \left( \frac{ap}{\kappa} - \frac{2Nb}{(p-1)^2} \right) + O \left( \frac{1 + |y|^4}{s^2} \right), \]

where

\[ \bar{V}(y, s) \leq C(K_0) \frac{(1 + |y|^2)}{s^2}, \forall |y| \leq 2K_0\sqrt{s}, \]

and

\[ a = \frac{2bN\kappa}{(p-1)^2} + \frac{\kappa \beta}{(p-1)}. \]

Additionally we have

\[ |V(y, s)| \leq C, \text{ for all } y \in \mathbb{R}^N. \]

Proof. The proof arises directly from a Taylor expansion and V’s definition.

Next, we give some estimates on B defined by (3.12).

Lemma A.2 (Quadratic term B). Let us consider B(q) defined as in (3.12) and q \in V_A(s) where V_A introduced in Definition 4.1. Then, the following hold

\[ |B(q)| \leq C(K_0)|q|^2, \forall |y| \leq 2K_0\sqrt{s}, \]

\[ |B(q)| \leq C|q|^\bar{p}, \forall y \in \mathbb{R}^N, \text{where } \bar{p} = \min(p, 2). \]

Proof. The proof is pretty the same as the proof of an analogous result in [57].

Now, we will study on the rest term, R

Lemma A.3 (The rest term R). Let us consider R defined as in (3.13). Then, we have the following estimates

\[ \|R(\cdot, s)\|_{L^\infty} \leq \frac{C}{s}, \tag{A.2} \]

and

\[ R(y, s) = \left( a - 2b \frac{N\kappa}{(p-1)^2} \right) \frac{1}{s} + \frac{a_0}{s^2} \]

\[ + \frac{|y|^2}{s^2} \left( \frac{b}{(p-1)^2} \left( \frac{2bN\kappa}{(p-1)^2} - a \right) + \frac{\kappa b}{(p-1)^2} \left( \frac{4pb}{(p-1)^2} - 1 \right) \right) \]

\[ + O \left( \frac{1 + |y|^4}{s^3} \right). \]

Proof. We mention that R is considered as the remainder term, generated by the blowup profile \( \varphi(y, s) = \varphi_0 \left( \frac{|y|}{\sqrt{s}} \right) + \frac{a}{s} \). Hence, it has the same structure to the one in [73]. For that reason, the proof is similar to Lemma B.5 in that work, and it stems from a Taylor expansion.
Next, we provide some estimates related with term $G$.

**Lemma A.4 (Term $G$).** Let us consider $G$ defined as in (3.14). Then, we have the following expansion

$$G(y, s) = \left( \frac{\tilde{\theta}'(s)}{\tilde{\theta}(s)} \right) \left( \kappa - \frac{\kappa b}{(p - 1)^2} \frac{|y|^2}{s} \right) + \tilde{G},$$  \hspace{1cm} (A.3)

where

$$\left| \tilde{G}(y, s) \right| \leq C \left| \frac{\tilde{\theta}'(s)}{\tilde{\theta}(s)} \right| \left( \frac{1 + |y|^2}{s^2} + |q(y, s)| \right), \forall |y| \leq K_0 \sqrt{s},$$

and $\tilde{a}$ defined as in (A.1).

Furthermore, if we assume that $u \in S(K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, t), \text{ with } t = -\ln(T - t) \text{ and (4.11)}$ holds, then we have the following global bound

$$\|G(\cdot, s)\|_{L^\infty} \leq \frac{C}{s},$$  \hspace{1cm} (A.4)

**Proof.** Now we consider $|y| \leq K_0 \sqrt{s}$ then via $G$’s definition we derive

$$G(y, s) = \frac{\tilde{\theta}'(s)}{(p - 1)\tilde{\theta}(s)} (\varphi + q),$$

where $\varphi(y, s) = \varphi_0 \left( \frac{|y|}{\sqrt{s}} \right) + \frac{2}{s}$ and $a$ defined as in (A.1). Then, by a simple Taylor expansion, we immediately derive (A.3).

It remains to prove (A.4). Indeed, by $G$’s definition we have

$$\text{Supp}(G) \subset \{K_0 s \leq |y| \leq 2K_0 s\}.$$  

We observe that, once (4.11) holds, then

$$\left| \frac{\tilde{\theta}'(s)}{\tilde{\theta}(s)} (q + \varphi) \right| \leq \frac{C}{s}.$$  

So, it is sufficient to prove

$$\|\tilde{F}(\cdot, s)\|_{L^\infty} \leq \frac{C}{s}.$$  \hspace{1cm} (A.5)

Let us recall

$$\tilde{F}(y, s) = e^{-\frac{1}{\sqrt{s}}} \left(-u\theta_{\sqrt{s}}^{\frac{1}{p+1}}(t)\Delta \chi_1 - 2\theta_{\sqrt{s}}^{\frac{1}{p+1}} \nabla \chi_1 \cdot \nabla u + (\theta_{\sqrt{s}}^{\frac{1}{p+1}} u)^p (\chi_1 - \chi_1^p)(x, t)\right),$$

where $s = -\ln(T - t)$ and $y = \sqrt{t - T}$. For all $|y| \in [K_0 s, 2K_0 s]$, using Lemma 4.4 and (C.1), we obtain the following estimates:

$$\left| u\theta_{\sqrt{s}}^{\frac{1}{p+1}}(t)\Delta \chi_1 \right| \leq C \frac{(T - t)^{-\frac{1}{p+1}}}{|\ln(t - t)|},$$

$$\left| \theta_{\sqrt{s}}^{\frac{1}{p+1}}(t)\nabla \chi_1 \cdot \nabla u \right| \leq C \frac{(T - t)^{-\frac{1}{p+1}}}{|\ln(t - t)|}.$$  

In addition, the definition of $\chi_1$ gives

$$\chi_1(x, t) = \chi_0 \left( \frac{|y|}{K_0 s} \right),$$

and thus for all $\frac{|y|}{s} \in [K_0, 2K_0]$, it follows that

$$|\chi_1(1 - \chi_1^{-1})| \leq C,$$
and
\[ |\theta_\alpha^{1-p} u| \leq \frac{(T-t)^{-\frac{1}{\alpha}}}{|\ln(T-t)|}. \]
The latter implies
\[ |e^{-p\theta_\alpha^{1-p}}(\theta_\alpha^{1-p} u)|^p \leq \frac{C}{s} \]
and thus (A.5) follows. This completes the proof of the Lemma. \( \square \)

**B. Parabolic estimates**

Let us recall that \( e^{t\Delta} \) is the semi-group generated by \( \Delta \) associated with Neumann boundary conditions in problem (1.1). It is proved in [72, Lemma 3.3] that the associated heat Kernel \( G(x,x',t) \) satisfies the estimates
\[ |\nabla_x G(x,x',t)| \leq C^{-\frac{N+1}{2}} \exp \left(-C(\Omega)|x-x'|^2 \right), \]
and
\[ e^{t\Delta} u_0 = \int_\Omega G(x,x',t)u_0(x')dx'. \]
In particular, we claim to the following

**Lemma B.1** (Parabolic estimates). Let us consider \( T,K_0,\epsilon_0,\alpha_0,A,\delta_0,C_0,\eta_0 \) be positive constants such that
\[ u \in S(T,K_0,\epsilon_0,\alpha_0,A,\delta_0,C_0,\eta_0), \forall t \in [0,t^*). \]
Then, for all \( x \in \Omega \), we can find \( R_x > 0, C = C(A,K_0,\epsilon_0,\alpha_0,\delta_0,C_0,\eta,T) \) such that
\[ \|\partial_t u(x,t)\|_{B(x,R_x)} \leq C. \]

**Proof.** The proof is similar to [24, Lemma F.1]. We kindly refer the readers to that reference for more details. \( \square \)

**C. Some necessary estimates and integrals**

In this part, we aim to give some fundamental estimates on key quantities. We also provide useful formulas of some key integrals arise in the proofs throughout the manuscript.

We first provide some estimates on \( t(x) \):

**Lemma C.1.** Let us consider \( t(x) \), defined as in (2.28) for all \( |x| \leq \epsilon_0 \), and \( \rho(x) = T-t(x) \). Then, we have
\[ \rho(x) = \frac{8}{K_0^2} \frac{|x|^2}{|\ln|x||} \left( 1 + \frac{|\ln|\ln|x||}{|\ln|x||} \right), \]
and
\[ \ln \rho(x) \sim 2 \ln |x| \left( 1 + \frac{|\ln|\ln|x||}{|\ln|x||} \right), \text{ as } x \to 0. \]
In particular, if \( |x| = K_0\sqrt{T-t}|\ln(T-t)| \), then
\[ \rho(x) = 16(T-t)|\ln(T-t)| \left( 1 + O \left( \frac{|\ln|\ln(T-t)||}{|\ln(t-t)||} \right) \right), \text{ as } t \to T. \] (C.1)

**Proof.** The proof directly follows by (2.29). \( \square \)
Lemma C.2 (Bubble integrals). Let us consider \( N > 0, k \in \mathbb{N}^*, p > 1, \) and \( b > 0, \) we now define

\[
I_{b,p,N,k} = \int_0^\infty (p - 1 + b\xi^2)^{-k - \frac{k}{p}} \xi^{N-1} d\xi.
\]

Then, for all \( k \geq 1, \) we have

\[
I_{b,p,N,1} = \frac{1}{(p - 1)N} b^{-\frac{N}{p}}, \tag{C.2}
\]

\[
I_{b,p,N,k+1} = \frac{1}{p - 1} \frac{k}{k + \frac{N}{2}} I_{b,p,N,k}. \tag{C.3}
\]

Proof. The result follows the integration by parts. Indeed, we firstly handle (C.3). We consider \( k \geq 1, \) then, we write

\[
I_{b,p,N,k} = \int_0^\infty (p - 1 + b\xi^2)^{-k - \frac{k}{p}} \xi^{N-1} d\xi.
\]

Using integral by parts with \( u = (p - 1 + b\xi^2)^{-k - \frac{k}{p}}, \) \( dv = \xi^{N-1} d\xi, \) we obtain

\[
I_{b,p,N,k} = \frac{2}{N} \left( k + \frac{N}{2} \right) \int_0^\infty (p - 1 + b\xi^2)^{-k - \frac{k}{p}} b\xi^2 d\xi
\]

\[
= \frac{2}{N} \left( k + \frac{N}{2} \right) \left[ \int_0^\infty (p - 1 + b\xi^2)^{-k - \frac{k}{p}} \xi^{N-1} d\xi - (p - 1) \int_0^\infty (p - 1 + b\xi^2)^{-k - \frac{k}{p}} \xi^{N-1} d\xi \right]
\]

\[
= \frac{2}{N} \left( k + \frac{N}{2} \right) \left( I_{b,p,N,k} - (p - 1)I_{b,p,N,k+1} \right)
\]

which concludes

\[
I_{b,p,N,k+1} = \frac{1}{p - 1} \frac{k}{k + \frac{N}{2}} I_{b,p,N,k}.
\]

Thus, (C.3) follows.

Next, we prove to (C.2). Let \( K_0 \) be an arbitrary positive constant and we consider the following integral

\[
\int_0^{K_0} (p - 1 + b\xi^2)^{-\frac{k}{p}} \xi^{N-1} d\xi.
\]

Using integration by parts, we obtain

\[
\int_0^{K_0} (p - 1 + b\xi^2)^{-\frac{k}{p}} \xi^{N-1} d\xi = \frac{\xi^N}{N} (p - 1 + b\xi^2)^{-\frac{k}{p}} \bigg|_0^{K_0} - (p - 1) \int_0^{K_0} (p - 1 + b\xi^2)^{-\frac{k}{p}} \xi^{N-1} d\xi
\]

\[
+ \int_0^{K_0} (p - 1 + b\xi^2)^{-\frac{k}{p}} \xi^{N-1} d\xi.
\]

Taking \( K_0 \to +\infty, \) we obtain

\[
I_{b,p,N,1} = \frac{1}{p - 1} \frac{1}{N b^{\frac{k}{p}}},
\]

Finally, we conclude (C.2).
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