Legendre functions of fractional degree: transformations and evaluations

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Abstract

Associated Legendre functions of fractional degree appear in the solution of boundary value problems in wedges or in toroidal geometries, and elsewhere in applied mathematics. In the classical case when the degree is half an odd integer, they can be expressed using complete elliptic integrals. In this study, many transformations are derived, which reduce the case when the degree differs from an integer by one-third, one-fourth or one-sixth to the classical case. These transformations, or identities, facilitate the symbolic manipulation and evaluation of Legendre and Ferrers functions. They generalize both Ramanujan’s transformations of elliptic integrals and Whipple’s formula, which relates Legendre functions of the first and second kinds. The proofs employ algebraic coordinate transformations, specified by algebraic curves.

1 Introduction

In applied mathematics and theoretical physics, Legendre or associated Legendre functions occur widely. Their properties are summarized in many places [1, 2]. The most familiar are \( P_\nu^\mu(z) \), \( Q_\nu^\mu(z) \), which are defined if \( z \in (-1,1) \), or more generally on the complex \( z \)-plane with cuts \((-\infty,-1] \text{ and } [1,\infty)\). Mathematicians call these ‘Ferrers functions’ and reserve the term ‘Legendre functions’ for the typographically distinct \( P_\nu^\mu(z) \), \( Q_\nu^\mu(z) \) that are defined if \( z \in (1,\infty) \), or more generally on the \( z \)-plane with cut \((-\infty,1]\). All four functions, said to be of degree \( \nu \) and order \( \mu \), satisfy the same second-order ordinary differential equation with parameters \( \nu \) and \( \mu \), and in the absence of cuts may be multi-valued.

The so-called Legendre polynomials \( P_n^m(z) \), \( Q_n^m(z) \), which are really Ferrers functions of integral degree \( n \) and integral order \( m \), are the most familiar. For \( P_n^0(z) \) in particular, usually written \( P_n(z) \), no cuts are required for single-valuedness, and in fact \( P_n \) equals \( P_n \). Spherical harmonics \( Y_n^m(\theta,\phi) \propto P_n^m(\cos \theta)e^{im\phi} \) arise naturally when separating variables in the Laplace and Helmholtz equations, and are widely used for expansion purposes. (Of course ‘polynomial’ is a misnomer: if \( m \) is odd, \( P_n^m(z) \) includes a \( \sqrt{1-z^2} \) factor, which appears in \( P_n^m(z) \) as \( \sqrt{z^2-1} \).) The use of full ranges for the coordinates, i.e., \( 0 \leq \theta \leq \pi \) and \( 0 \leq \phi \leq 2\pi \), is what causes the quantization of the degree and order to integers. It should be noted that Ferrers functions of half-odd-integer degree and order find application in quantum mechanics [3]. Like the Legendre polynomials, they are elementary functions of \( z \).

Ferrers or Legendre functions in which the degree \( \nu \) is a half-odd-integer and the order is an integer are not elementary but can be expressed in terms of the complete elliptic integrals \( K = K(m) \), \( E = E(m) \), where \( m \) (often denoted \( k^2 \)) is the elliptic modular parameter. (For instance,
P_{-1/2}(z) equals \((2/\pi) K((1-z)/2)\). Ferrers or Legendre functions of unrestricted degree \(\nu \notin \mathbb{Z}\) appear in many contexts, such as two Fourier expansions in the azimuthal coordinate \(\phi\):

\[
\begin{align*}
[\cos \theta + i \sin \theta \sin \phi]^{\nu} &= \sum_{m=-\infty}^{\infty} \frac{\Gamma(\nu + 1)}{\Gamma(\nu + m + 1)} P_{\nu}^m(\cos \theta) e^{im\phi}, \\
[\cosh \xi + \sinh \xi \cos \phi]^{\nu} &= \sum_{m=-\infty}^{\infty} \frac{\Gamma(\nu + 1)}{\Gamma(\nu + m + 1)} P_{\nu}^m(\cosh \xi) e^{im\phi}.
\end{align*}
\]

In (1a), the left side is a generalization to arbitrary \(\nu\) of a standard generating function for spherical harmonics [4] Chap. VII, § 7.3. Equation (1b) is a ‘generalized Heine identity’ which has recently attracted attention [5]. In the case \(\nu = -1/2\), it leads to an alternative to the usual multipole expansion of the \(1/|x-x'|\) potential [6] [7]. Equation (1b) also appears in celestial mechanics, in the analysis (originally) of planetary perturbations [8]. The coefficient of the mode \(\cos(m\phi)\) in the Fourier development of a ‘disturbing function’ \((1 + \alpha^2 - 2\alpha \cos \phi)^{-s}\), where \(s > 0\) is a half-odd-integer, is denoted \(b_{s}^{(m)}(\alpha)\) and called a Laplace coefficient. By (1b), it can be expressed in terms of the Legendre functions \(P_{\nu}^m\), and thus in terms of complete elliptic integrals.

Legendre (rather than Ferrers) functions with \(\nu\) a half-odd-integer and \(\mu\) an integer are commonly called toroidal or ‘anchor ring’ functions, since harmonics including factors of the form \(P_{-1/2}(\cosh \xi), Q_{-1/2}(\cosh \xi)\) appear when solving boundary value problems in toroidal domains, upon separating variables in toroidal coordinates [9] [10]. The efficient calculation of values of toroidal functions, employing recurrences or other numerical schemes, is well understood [11] [12]. There is also a literature focusing on Laplace coefficients, both classical [13] and recent, which makes contact with hypergeometric expansions.

Overview of results.—It is shown that Legendre and Ferrers functions of any degree \(\nu\) differing from an integer by \(\pm 1/r\), for \(r = 3, 4, 6\), can be expressed in terms of like functions of half-odd-integer degree. (The order here must be an integer.) This statement, which leads to unexpected closed-form expressions in terms of complete elliptic integrals, is one consequence of the main results, the large collection of Legendre identities in § 8 which facilitate the rewriting and evaluation of \(P_{-1/r}^{-\alpha}(\cosh \xi), Q_{-1/r}^{-\alpha}(\cosh \xi)\) and \(P_{-1/r}^{-\alpha}(\cos \theta), Q_{-1/r}^{-\alpha}(\cos \theta)\), with \(\alpha \in \mathbb{C}\) arbitrary. (The case when \(\nu\) differs from \(-1/r\) or \(+1/r\) by a non-zero integer is handled by applying well-known differential recurrences, to shift the degree.) The most striking identities may be

\[
\begin{align*}
P_{-1/4}^{-\alpha}(\cosh \xi) &= 2^\alpha \sqrt{\text{sech}(\xi/2)} P_{-\alpha}^{-1/2}(\text{sech}(\xi/2)), \\
P_{-1/6}^{-\alpha}(\cosh \xi) &= 3^{3\alpha/2} \sqrt{\frac{3 \sinh(\xi/3)}{\sinh \xi}} P_{2\alpha - 1/2}^{-\alpha}(\sqrt{\frac{3 \sinh(\xi/3)}{\sinh \xi}} \cosh(\xi/3)).
\end{align*}
\]

which hold if \(\alpha \in \mathbb{C}\) and \(\xi \in (0, \infty)\). As in all Legendre identities derived below, the function arguments on the left and right sides (here trigonometrically parametrized) are algebraically related. These two are of special importance because (1) can be rewritten as

\[
(1 + \text{cos} \phi)^\nu = (1 - x^2)^{-\nu/2} \sum_{m=-\infty}^{\infty} \frac{\Gamma(\nu + 1)}{\Gamma(\nu + m + 1)} P_{\nu}^m(1/\sqrt{1 - x^2}) e^{im\phi},
\]

with \(x = \tanh \xi\) satisfying \(x \in (0, 1)\). This is the little-known Fourier expansion of \((1 + x \cos \phi)^\nu\) (cf. [5] [14]). It follows that if \(\nu = -1/4\) or \(\nu = -1/6\), or more generally if \(\nu\) differs from an integer by \(\pm 1/r\) with \(r = 3, 4, 6\) (as well as the classical case \(r = 2\)), the Fourier coefficients of \((1 + x \cos \phi)^\nu\) can be expressed in terms of complete elliptic integrals. This too is unexpected.

Identities such as (2a), (2b) and the full collection in [8] are closely related to certain function transformations of Ramanujan. In his famous notebooks [15] Ch. 33], he developed a theory
of elliptic integrals with non-classical ‘signature’ \( r = 3, 4, 6 \), and related them to the classical integrals, which have signature 2. His theory yields formulas for the Ferrers functions \( P_{-1/r} \) in terms of \( P_{-1/2} \), or equivalently the classical complete elliptic integral \( K \), with an algebraically transformed argument. (For a compact list of Ramanujan’s transformation formulas that can be written in this way, see [16, Lemma 2.1].) The identities derived here include several of Ramanujan’s rationally parametrized formulas, but such identities as \([20, 21]\) are more general, in that they are formulas for Ferrers or Legendre functions of arbitrary (i.e., non-zero) order \(-\alpha\). The parametrization by trigonometric functions is another novel feature.

Methods.—The technique used below for deriving Legendre identities was developed by considering Whipple’s well-known \( Q \leftrightarrow P \) transformation formula \([2, 3.3(13)]\),

\[
Q_\nu^\mu \left( \left( p^2 + 1 \right)/\left( p^2 - 1 \right) \right) = e^{\pi \nu \xi} \sqrt{\pi/2} \Gamma(\nu + 1) \sqrt{\left( p^2 - 1 \right)/2p} P_{-\nu}^{-\nu-1/2} \left( \left( p^2 + 1 \right)/2p \right),
\]

which holds if \( p \in (1, \infty) \). Whipple’s proof of \([4]\), given in \([17]\), contains the germ of a broadly applicable method. It relies on the arguments on left and right, \( L = \left( p^2 + 1 \right)/\left( p^2 - 1 \right) \) and \( R = \left( p^2 + 1 \right)/2p \), being algebraically related by \( \left( L^2 - 1 \right)/(R^2 - 1) = 1 \). This relation defines an algebraic curve, which is parametrized by \( p \), though it could also be parametrized as \( L = z \) and \( R = z/\sqrt{2z^2 - 1} \), or trigonometrically as \( L = \coth \xi \) and \( R = \cosh \xi \), as is common in the literature.

The point is that the correspondence \( L \leftrightarrow R \) is an algebraic change of the independent variable, which leaves Legendre’s differential equation invariant. What this means is that if \( \mathcal{E}_L \) denotes the second-order differential equation satisfied by the left side as a function of \( p \), and as well by the left side with \( Q_\nu^\mu \) replaced by \( P_\nu^\mu \), and if \( \mathcal{E}_R \) denotes its counterpart coming from the right side; then, \( \mathcal{E}_L, \mathcal{E}_R \) will be the same. Once one has verified this, to prove \([4]\) one needs only to check that the left and right sides are the same element of the (two-dimensional) solution space of \( \mathcal{E}_L = \mathcal{E}_R \). This can be confirmed by examining their asymptotic behavior near singular points.

The many Legendre identities appearing in \([12]\) relating Legendre and Ferrers functions of degree \( \nu = -1/r \), \( r = 3, 4, 6 \), to those of other degrees, are all derived in a similar way, from algebraic curves.

Applications.—Legendre functions of fractional degree occur in many areas of applied mathematics. One lies in mathematical physics: the representation theory of certain Lie algebras \([18]\). Another is geometric-analytic: the spectral analysis of Laplacian-like operators on spaces of negative curvature, which is of interest because of its connection to quantum chaos \([19]\). If \( \Delta_{LB} \) denotes the Laplace–Beltrami operator on the real hyperbolic space \( \mathbb{H}^n \), the associated Green’s function \((-\Delta_{LB} + \kappa^2)^{-1} (x, x')\) will be proportional to \((\sinh d)^{1-n/2} Q_\nu^{n/2-1} (\cosh d)\), where \( d \) is the hyperbolic distance between \( x, x' \) and the degree \( \nu \) depends on \( \kappa^2 \). If \( \kappa^2 = 0 \), then \( \nu = n/2 - 1 \); and more generally, \( \nu \) equals \((n - 1)^2/4 + \kappa^2 - \frac{1}{4} \). (See \([20, 21]\), and \([22]\) for the \( \kappa^2 = 0 \) case.) It follows that this Green’s function on \( \mathbb{H}^n \) can be written in terms of complete elliptic integrals for an infinite, discrete set of values of the ‘energy’ parameter \(-\kappa^2 < 0 \).

Another notable application area is the Tricomi problem, which occurs in two-dimensional transonic potential flow \([23, 24]\). The Tricomi differential equation on the \( \theta-\eta \) (i.e., hodograph) plane, \( \mathcal{L} u = 0 \) with \( \mathcal{L} = \eta D_\eta^2 + D_\eta, \) has many particular solutions expressible in hypergeometric functions \([25]\). Ch. XII]. It is not widely appreciated that the latter are of the special type expressible in terms of Legendre functions, though it has been observed that many solutions can be obtained from a fundamental solution (Green’s function) of \( \mathcal{L} \) that is based on \( P_{-1/6} \). In fact, the so-called Gellerstedt generalization \( \mathcal{L}_k = \eta D_\eta^{k-1} D_\eta^2 + D_\eta^2 \) has a fundamental solution based on \( P_{-1/r}, Q_{-1/r} \), where \( r = 3, 4, 6 \) for \( k = 4, 2, 1 \). Applying such identities as \([20, 21]\) will express such fundamental solutions in terms of complete elliptic integrals.

An additional application area is classical: the separation of variables in boundary value problems, posed on wedge-shaped domains. Along this line, V. A. Fock \([29]\) used toroidal
coordinates in solving a problem on a wedge of opening angle $3\pi/2$, and was led to the fractional-degree functions $P_{-1/6}, Q_{-1/6}$. More recently, a magnetostatic potential has been expanded near a cubic corner in modes of the form $P_n^{\mu}(\cos \theta)e^{im\phi}$, where the (irrational) degree $\nu$ is corner-specific and known only numerically [30]. If one opening angle of the corner is increased to $\pi$, it becomes a right-angled wedge, and the appropriate $\nu$ becomes fractional. In such situations, closed-form representations like [23, 24] can serve as a check on numerical work.

In fluid problems on wedges, fractional-degree Ferrers functions $P_{-1/r}, Q_{-1/r}$ typically appear in the analysis when the wedge angle equals $(1 - \frac{1}{r})\pi$. This includes problems dealing with viscous film coating [31], solidification [32] and vortex layers [33].

Structure of paper.—In section 2, the needed asymptotic behaviors of the Legendre and Ferrers functions are summarized. Section 3 contains the main results: the just-mentioned employs distinct algebraic curves when $r = 3, 4, 6$. In section 6 how to perform integer shifts of the degree $\nu$ is explained. In section 4 explicit formulas are derived, from one of the identities, for $P_{-1/4}^{-1/4} P_{-1/6}^{-1/6}$, $Q_{-1/4}^{-1/3}$ each of which is an elementary (specifically, algebraic) function of its argument. Finally, section 8 displays a curiosity: an isolated identity relating $P_{-1/4}^{-1} P_{-1/4}^{-1}$

2 Normalizations and asymptotics

The (associated) equation of Legendre is the ordinary differential equation

$$\frac{d}{dz} \left[ (1 - z^2) \frac{du}{dz} \right] + \left[ \nu(\nu + 1) - \frac{\mu^2}{1 - z^2} \right] u = 0$$

(5)

on the Riemann sphere $\mathbb{C} \cup \{\infty\}$, with degree parameter $\nu \in \mathbb{C}$ and order parameter $\mu \in \mathbb{C}$. It is invariant under $z \mapsto -z$ and under $\mu \mapsto -\mu$ and $\nu \mapsto -1 - \nu$. It has regular singular points $z = 1, -1$ and $\infty$, with respective pairs of characteristic exponents $\{\pm \mu/2\}$, $\{\pm \mu/2\}$ and $\{-\nu, \nu + 1\}$. This means that locally, any solution is a combination of $(z - 1)^{\pm \mu/2}$, $(z + 1)^{\pm \mu/2}$ and $(1/z)^{\nu}, (1/z)^{\nu+1}$. Or rather, this is the generic behavior. If the difference between the exponents at any singular point is an integer, the local behavior of the dominant solution, which comes from the smaller exponent, will include a logarithmic factor. This is the source of the familiar logarithmic behavior when $z \to 1$ or $z \to -1$ of the ‘polynomials’ $Q_n^{\nu}(z)$ and $Q_n^{\nu}(z)$ (for integer order $\mu$, written here as $m$).

By convention, the Legendre functions $P_\nu^{\mu}, Q_\nu^{\mu}$ are defined so that when $\text{Re} \mu > 0$ and $\text{Re} \nu > -1/2, P_\nu^{-\mu}$ is recessive at $z = 1$, and $Q_\nu^{\mu}$ is recessive at $z = \infty$. (See [34, 31, 32].) That is, each is given by a Frobenius series coming from the larger exponent. But the question of how best to normalize $P_\nu^{\mu}, Q_\nu^{\mu}$ (especially the latter) is vexed. The standard definition of $Q_\nu^{\mu}$ includes factors $e^{\nu \pi i}, \Gamma(\nu + \mu + 1)$ and $\frac{1}{2}$, none of which should arguably be present. Olver [34] felt it wise to introduce a new ‘second Legendre’ function $Q_\nu^{\mu}$ lacking the first two factors, so that the standard $Q_\nu^{\mu}$ equals $e^{\nu \pi i} \Gamma(\nu + \mu + 1) Q_\nu^{\mu}$. In the present paper, an ad hoc function $\tilde{Q}_\nu^{\mu}$ that is defined to lack only the first factor is employed. That is, the standard $Q_\nu^{\mu}$ equals $e^{\nu \pi i} \tilde{Q}_\nu^{\mu}$. Opinion is nearly unanimous that including the factor $e^{\nu \pi i}$ in the definition of $Q_\nu^{\mu}$ was a mistake, since it may cause $Q_\nu^{\mu}(z)$ to be non-real even when $\nu, \mu$ and its argument $z = x > 1$ are all real. (Compare [1].)

The advantage of Olver’s $Q_\nu^{\mu}$ is that like $P_\nu^{\mu}$ and unlike $Q_\nu^{\mu}$ (and $\tilde{Q}_\nu^{\mu}$), it is defined for all $\nu, \mu \in \mathbb{C}$; and for any $n$ not on the cut $(-\infty, 1]$, $Q_\nu^{\mu}(z)$ is an analytic function of $\nu, \mu$. Also, $Q_\nu^{\mu}$ and $Q_{\nu-1}^{\mu}$ are identical, much as $P_{\nu}^{\mu}$ and $P_{\nu-1}^{\mu-1}$ are identical. Generically, $P_{-1}^{\mu}$ and $Q_{-1}^{\mu}$ are
associated respectively with the $+\mu/2$ exponent at $z = 1$ and the $\nu + 1$ exponent at $z = \infty$. The relevant asymptotics at these ‘defining’ singular points are

$$P_\nu^{-\mu}(z) \sim \frac{1}{2^{\mu/2} \Gamma(\mu + 1)} (z - 1)^{\mu/2}, \quad z \to 1,$$

$$Q_\nu^{\mu}(z) \sim \frac{\sqrt{\pi}}{2^{\nu + 1} \Gamma(\nu + 3/2)} (1/z)^{\nu + 1}, \quad z \to \infty.$$  

(6a)

These statements are valid if (respectively) $\mu \neq -1, -2, -3, \ldots$ and $\nu \neq -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}, \ldots$, so that the gamma functions are finite. If either gamma function is infinite, the corresponding Legendre function, when rigorously defined, turns out to be associated to the other characteristic exponent: respectively, $-\mu/2$ or $-\nu$. The correct asymptotics in these two degenerate cases are given in [11]. It should be noted that there are sub-cases of the degenerate cases in which $P_\nu^{-\mu}, Q_\nu^\mu$ are identically zero. Specifically, if $M$ equals $1, 2, 3, \ldots$ then $P_{-M}^M, \ldots, P_{M-1}^M \equiv 0$; and if $N = 1, 2, 3, \ldots$ then $Q_{-N-1/2}^{\pm 1/2}, \ldots, Q_{-N-1/2}^{\pm (N-1/2)} \equiv 0$. The former fact yields a familiar restriction on the order of spherical harmonics, but the latter (dual) fact is less well known.

Many formulas and identities involving $Q_\nu^\mu$ contain such obtrusive factors as $\Gamma(\nu + \mu + 1)$, and the wish to simplify these formulas partially justifies the introduction of the less familiar function $\hat{Q}_\nu^\mu$, and its definition as $\Gamma(\nu + \mu + 1)Q_\nu^\mu$. For example, the formula [34] § 14.1

$$u_n^m(z) = (z^2 - 1)^{m/2} \frac{d^m}{d z^m} u_n(x)$$

(7)

(where $\nu, \mu$ are written as $n, m$ because they are taken to be non-negative integers) holds if $u_n^m$ equals $P_n^\mu$ or $\hat{Q}_n^\mu$; but it does not hold if $u_n^m$ equals $Q_n^m$.

However, the introduction of the useful function $\hat{Q}_\nu^\mu$ comes at a price. Owing to the $\Gamma(\nu + \mu + 1)$ factor, $\hat{Q}_\nu^\mu$ is undefined if $\nu + \mu$ is a negative integer, except in the abovementioned sub-case: if $N = 1, 2, 3, \ldots$, then $\hat{Q}_{-N-1/2}^{\pm 1/2}(z), \ldots, \hat{Q}_{-N-1/2}^{\pm (N-1/2)}(z)$ are defined for $z \notin (-\infty, 1]$; and as analytic functions of $z$, are not identically zero. Informally, this is because in each of these, the product of $\Gamma(\nu + \mu + 1)$ (infinite) with $Q_\nu^\mu$ (zero) is finite and non-zero. This statement can be made rigorous by a limiting argument. The asymptotic behavior of $\hat{Q}_\nu^\mu$ is

$$\hat{Q}_\nu^\mu(z) \sim \frac{\sqrt{\pi} \Gamma(\nu + \mu + 1)}{2^{\nu + 1} \Gamma(\nu + 3/2)} (1/z)^{\nu + 1}, \quad z \to \infty,$$

(8)

if neither gamma function is infinite. By continuity, this fact will suffice for the derivation of Legendre identities that are valid for all choices of parameter for which $\hat{Q}_\nu^\mu$ is defined.

By convention, the Ferrers functions $P_\nu^\mu, Q_\nu^\mu$ are related to the Legendre functions $P_\nu^{\mu}, \hat{Q}_\nu^{\mu}$ on their common domains $\pm \text{Im} z > 0$, i.e., on the upper and lower half-planes, by

$$P_\nu^\mu = e^{x \nu \pi i/2} P_\nu^{\mu},$$

$$Q_\nu^\mu = e^{x \nu \pi i/2} \hat{Q}_\nu^{\mu} \pm \frac{\pi}{2} e^{x \mu \pi i/2} P_\nu^{\mu}.$$  

(9a)

Thus on $(-1, 1), P_\nu^\mu, Q_\nu^\mu$ are combinations of boundary values of the analytic functions $P_\nu^{\mu}, \hat{Q}_\nu^{\mu}$. Equation [9a] is meaningful for all $\nu, \mu$ for which $\hat{Q}_\nu^{\mu}$ is defined, $Q_\nu^\mu$ being defined under the same conditions. The asymptotic behavior of $P_\nu^{\mu}$ is given (when $\mu \neq -1, -2, -3, \ldots$) by

$$P_\nu^{\mu}(z) \sim \frac{1}{2^{\mu/2} \Gamma(\mu + 1)} (1 - z)^{\mu/2}, \quad z \to 1,$$

(10)

and that of $Q_\nu^\mu(z)$ is discussed in section [4]. It should be noted that restricted to $(-1, 1)$, the functions $P_\nu^{\mu}, 2Q_\nu^\mu$ are Hilbert transforms of each other (via ‘Neumann’s integral’) when $\mu = 0,$
and are related by more complicated integral transforms when \( \mu \neq 0 \). This relationship, which is suggested by (10), is why \( Q_{\nu}^\mu, Q_{\nu}^{\hat{\mu}} \) and \( \tilde{Q}_{\nu}^\mu \) should really be defined to be twice as large. But to maintain compatibility with the past, the factor \( \frac{1}{2} \) implicit in their definitions will be kept.

Besides \( P_{\nu}^\mu, Q_{\nu}^\mu \) and \( P_{\nu}^{\hat{\mu}}, \tilde{Q}_{\nu}^{\mu} \), the identities in the next section will be expressed for clarity with the aid of the auxiliary Ferrers function \( \overline{F}_{\nu}^\mu \) defined by \( \overline{F}_{\nu}^\mu(z) = P_{\nu}^\mu(-z) \), or equivalently [2, 3.4(14)]

\[
\overline{F}_{\nu}^\mu = \cos[(\nu + \mu)\pi] P_{\nu}^\mu - \frac{2}{\pi} \sin[(\nu + \mu)\pi] Q_{\nu}^\mu, \quad (11)
\]

and the auxiliary Legendre function \( \overline{P}_{\nu}^\mu \) defined by

\[
\overline{P}_{\nu}^\mu = \cos[(\nu + \mu)\pi] P_{\nu}^\mu - \frac{2}{\pi} \cos(\mu\pi) \sin[(\nu + \mu)\pi] \tilde{Q}_{\nu}^\mu. \quad (12)
\]

The latter \emph{ad hoc} function is less mysterious than it looks: by [2, 3.3(10)], it satisfies

\[
\overline{P}_{\nu}^\mu(x) = \frac{1}{2} \left[ e^{\mu\pi i} P_{\nu}^\mu(-x + i0) + e^{-\mu\pi i} P_{\nu}^\mu(-x - i0) \right], \quad x > 1. \quad (13)
\]

Both \( \overline{F}_{\nu}^\mu \) and \( \overline{P}_{\nu}^\mu \) are solutions of (14), on the respective (Ferrers or Legendre) domain.

## 3 Main results

Theorems 3.1 and their respective corollaries contain the main results: a collection of two dozen algebraic Legendre identities, or transformation formulas. Each Theorem contains a list of identities in rationally parametrized form, and its corollary gives each identity in a trigonometrically parametrized form, which may be more useful in applications.

The identities come from rational curves \( C_r \) and \( C'_r \), where the ‘signature’ \( r \) equals 3, 4 or 6. The following numbering scheme is used. For each \( r \), the identities come in pairs. The two pairs coming from \( C_r \) are labelled \( I_r(i), I_r(i') \) and \( I_r(ii), I_r(ii') \), and the two pairs coming from \( C'_r \) are labelled \( I'_r(i), I'_r(i') \) and \( I'_r(ii), I'_r(ii') \). In each of these pairs with one exception, the Ferrers functions \( P_{-1/r}^{-\alpha}, \overline{P}_{-1/r}^{-\alpha} \) appear on the left. The initial pair \( I_r(i), I_r(i') \) is the exception: on the left it has instead the Legendre functions \( P_{-1/r}^{-\alpha}, \overline{P}_{-1/r}^{-\alpha} \). The identities (2a), (2b) in the Introduction appear here as \( I_4(i), I_6(i) \).

The theorems are ordered so that the case \( r = 4 \) is covered first, since the curves \( C_4, C'_4 \) and their associated identities are relatively simple; then \( r = 6 \); and finally \( r = 3 \). The \( r = 3 \) case closely resembles the \( r = 6 \) case, but is deficient in that the order \( -\alpha \) of the left-hand function must equal zero. The \( r = 3 \) identities given in Theorems 3.5 and 3.6 and their corollaries cannot readily be generalized to non-zero order, and the same is true of the \( r = 4 \) identities coming from \( C'_4 \).

It should be noted that when the order \( -\alpha \) is an integer \( m \), by applying these identities one can express the Ferrers pair \( P_{-1/r}^{-m} \), \( \overline{P}_{-1/r}^{-m} \) (or the Legendre pair \( P_{-1/r}^{-m} \), \( \overline{P}_{-1/r}^{-m} \) appearing on the left of \( I_r(i), I_r(i') \)) in terms of Legendre or Ferrers functions of half-odd-integer degree and integer order. Hence, one can express \( P_{-1/r}^{-m} \), \( Q_{-1/r}^{-m} \) (or \( P_{-1/r}^{-m}, Q_{-1/r}^{-m} \) in terms of complete elliptic integrals. The extension of this result to the case when the degree is not \(-1/r\), but differs from an integer by \(-1/r\) (or by \(+1/r\)) is explained in section 3.

In each identity, the parameter (whether \( \xi, \theta \) or \( p \)) varies over a specified real interval. In fact each identity extends by analytic continuation to the complex domain, to the largest connected open subset of \( \mathbb{C} \) containing this interval on which both sides are defined. The only obstruction to their being defined is the requirement that neither function argument lie on a cut.
3.1 Signature-4 identities

Definition. The algebraic $L$-$R$ curve $\mathcal{C}_4$ is the curve
\[ LR^2 + (R^2 - 2) = 0, \] (14)
which is rationally parametrized by
\[ L = 1 - 2(1 - p^2) = -1 + 2p^2, \quad R = 1/p, \]
and is invariant under $R \mapsto -R$, which is performed by $p \mapsto -p$. An associated prefactor function $A = A(p)$, with limit unity when $p \to 1$ and $(L, R) \to (1, 1)$, is
\[ A(p) = \left[ 2^4 \frac{(R^2 - 1)^2}{(L^2 - 1)^2} \right]^{1/16} = \sqrt{\frac{1}{p}}. \]

Theorem 3.1. For each pair $u, v$ of Legendre or Ferrers functions listed below, an identity
\[ u_{-1/4}^{-\alpha}(L(p)) = 2^\alpha A(p) v_{-1/2}^{-\alpha}(R(p)) \]
of type $I_4$, coming from the curve $\mathcal{C}_4$, holds for the specified range of values of the parameter $p$.

| Label | $u_{-1/4}^{-\alpha}$ | $v_{-1/2}^{-\alpha}$ | $p$ range | $L$ range | $R$ range |
|-------|----------------------|----------------------|-----------|-----------|-----------|
| (i)   | $P_{-1/4}^{-\alpha}$ | $P_{-1/2}^{-\alpha}(2\pi)Q_{-1/2}^{-\alpha}$ | $(1, \infty)$ | $1 < L < \infty$ | $1 > R > 0$ |
| (ii)  | $\csc(\pi/4)P_{-1/4}^{-\alpha}$ | $P_{-1/2}^{-\alpha}(2\pi)Q_{-1/2}^{-\alpha}$ | $(0, 1)$ | $-1 < L < 1$ | $\infty > R > 1$ |

To construct trigonometric versions of these identities, one substitutes $L = \cosh \xi$ and $L = \cos \theta$ into the relation (14), and solves for $R$ as a function of $\xi$ or $\theta$. This yields the following.

Corollary 3.1. The following identities coming from $\mathcal{C}_4$ hold for $\alpha \in \mathbb{C}$, when $\xi \in (0, \infty)$ and $\theta \in (0, \pi)$.

$I_4(i)$: $P_{-1/4}^{-\alpha}(\cosh \xi) = 2^\alpha \sqrt{\text{sech}(\xi/2)P_{-1/2}^{-\alpha}(\text{sech}(\xi/2))}$;

$I_4(\overline{i})$: The same, with $P_{-1/4}^{-\alpha}, P_{-1/2}^{-\alpha}$ replaced by $\csc(\pi/4)P_{-1/4}^{-\alpha}, (2\pi)Q_{-1/2}^{-\alpha}$;

$I_4(ii)$: $P_{-1/4}^{-\alpha}(\cos \theta) = 2^\alpha \sqrt{\text{sec}(\theta/2)P_{-1/2}^{-\alpha}(\text{sec}(\theta/2))}$;

$I_4(\overline{ii})$: The same, with $P_{-1/4}^{-\alpha}, P_{-1/2}^{-\alpha}$ replaced by $\csc(\pi/4)\overline{P}_{-1/4}^{-\alpha}, (2\pi)\overline{Q}_{-1/2}^{-\alpha}$.

Definition. The algebraic $L$-$R$ curve $\mathcal{C}_4$ is the curve
\[ (L - 1)(R + 3)^2 + 2(R - 1)^2 = 0, \] (15)
which is rationally parametrized by
\[ L = 1 - \frac{2(p - 1)^2}{(p + 1)^2} = -1 + 8 \frac{p}{(p + 1)^2}, \quad R = 1 - 2 \frac{p - 1}{p} = -1 + \frac{2}{p}, \]
and is invariant under $R \mapsto 4/(R + 1) - 1$, which is performed by $p \mapsto 1/p$. An associated prefactor function $A = A(p)$, with limit unity when $p \to 1$ and $(L, R) \to (1, 1)$, is
\[ A(p) = \left[ \frac{1}{2^4} \frac{(R - 1)^4(R + 1)^4}{(L - 1)^2(L + 1)^4} \right]^{1/16} = \sqrt{\frac{1 + p}{2p}}. \]
Theorem 3.2. For each pair \(u, v\) of Legendre or Ferrers functions listed below, an identity

\[
u_{-1/4}(L(p)) = A(p) \nu_{-1/2}(R(p))
\]

of type \(I'_4\), coming from the curve \(C'_4\), holds for the specified range of values of the parameter \(p\).

| Label | \(u_{-1/4}\) | \(v_{-1/2}\) | \(p\) range | \(L\) range | \(R\) range |
|-------|--------------|--------------|--------------|-------------|-------------|
| (i)   | \(P_{-1/4}\) | \(P_{-1/2}\) | \((1, \infty)\) | \(1 > L > -1\) | \(1 > R > -1\) |
| (ii)  | \(\frac{1}{2} \csc(\pi/4)P_{-1/4}\) | \(P_{-1/2}\) | \((0, 1)\) | \(-1 < L < 1\) | \(\infty > R > 1\) |
| (iii) | \(\frac{1}{2} \csc(\pi/4)\pi_{-1/4}\) | \((2/\pi)Q_{-1/2}\) | |

Remark. Identities \(I'_4(i), I'_4(ii)\) were found by Ramanujan; see \(\text{[15]}\) Chap. 33, Theorems 9.1 and 9.2] and \(\text{[16]}\) Lemma 2.1, and compare \(\text{[30]}\) Proposition 5.7(a). He used \((p - 1)/(p + 1) \in (0, 1)\) as parameter.

To construct trigonometric versions of these identities, one substitutes \(L = \cos \theta\) into the relation \(\text{[15]}\), and solves for \(R\) as a function of \(\theta\). This yields the following.

Corollary 3.2. The following identities coming from \(C'_4\) hold when \(\theta \in (0, \pi)\).

\[
I'_4(i) : \quad P_{-1/4}(\cos \theta) = \frac{1}{\sqrt{1 + \sin(\theta/2)}} P_{-1/2}\left(1 - 4 \frac{\sin(\theta/2)}{1 + \sin(\theta/2)}\right);
\]

\[
I'_4(ii) : \quad \text{The same, with } P_{-1/4}, P_{-1/2} \text{ replaced by } \frac{1}{2} \csc(\pi/4)P_{-1/4}, P_{-1/2};
\]

\[
I'_4(iii) : \quad P_{-1/4}(\cos \theta) = \frac{1}{\sqrt{1 - \sin(\theta/2)}} P_{-1/2}\left(1 + 4 \frac{\sin(\theta/2)}{1 - \sin(\theta/2)}\right);
\]

\[
I'_4(iii) : \quad \text{The same, with } P_{-1/4}, P_{-1/2} \text{ replaced by } \frac{1}{2} \csc(\pi/4)P_{-1/4}, (2/\pi)Q_{-1/2}.
\]

Remark. The right-hand arguments equal \(-1 + 2 \tan^2((\pi - \theta)/4)\) and \(-1 + 2 \tan^2((\pi + \theta)/4)\), respectively.

### 3.2 Signature-6 identities

**Definition.** The algebraic \(L-R\) curve \(\mathcal{C}_6\) is the curve

\[
(L^2 - 1)(4R^2 - 3)^3 + 27(R^2 - 1) = 0,
\]

which is rationally parametrized by

\[
L = 1 - 54 \frac{p^2 - 1}{(p^2 - 3)^3} = -1 + 2 \frac{p^4(p^2 - 9)}{(p^2 - 3)^3}, \quad R = \frac{3 + \frac{p^2}{4p}}{4p} = \pm 1 \mp \frac{(1 \mp p)(p \mp 3)}{4p}.
\]

It is invariant under \(L \mapsto -L\) and \(R \mapsto -R\), which are performed by \(p \mapsto 3/p\) and \(p \mapsto -p\). An associated prefactor function \(A = A(p)\), with limit unity when \(p \to 1\) and \((L, R) \to (1, 1)\), is

\[
A(p) = \left[3^6 \frac{(R^2 - 1)^2}{(L^2 - 1)^2}\right]^{1/24} = \sqrt{\frac{3 - p^2}{2p}}.
\]

**Theorem 3.3.** For each pair \(u, v\) of Legendre or Ferrers functions listed below, an identity

\[
u_{-\alpha/6}(L(p)) = 3^{3\alpha/2} A(p) \nu_{-\alpha/2}(R(p))
\]

of type \(I_6\), coming from the curve \(\mathcal{C}_6\), holds for the specified range of values of the parameter \(p\).
It is invariant under $p \mapsto -1/6$ and solves for $R$ as a function of $\xi$ or $\theta$. This yields the following.

**Corollary 3.3.** The following identities coming from $C_6$ hold for $\alpha \in \mathbb{C}$, when $\xi \in (0, \infty)$ and $\theta \in (0, \pi)$.

\[ I_6(i) : \quad P_{-1/6}^{-\alpha}(\csc(\pi/6)) = 3^{3\alpha/2} \sqrt{\frac{3 \sinh(\xi/3)}{\sinh \xi}} P_{2\alpha - 1/2}^{-\alpha} \left( \sqrt{\frac{3 \sinh(\xi/3)}{\sinh \xi}} \cosh(\xi/3) \right); \]

\[ I_6(\overline{i}) : \quad The \, same, \, with \, P_{-1/6}^{-\alpha}, \, P_{2\alpha - 1/2}^{-\alpha} \, replaced \, by \, \csc(\pi/6) P_{-1/6}^{-\alpha}, \, (2/\pi) Q_{2\alpha - 1/2}^{-\alpha}; \]

\[ I_6(ii) : \quad P_{-1/6}^{-\alpha}(\cos(\theta)) = 3^{3\alpha/2} \sqrt{\frac{3 \sin(\theta/3)}{\sin \theta}} P_{2\alpha - 1/2}^{-\alpha} \left( \sqrt{\frac{3 \sin(\theta/3)}{\sin \theta}} \cos(\theta/3) \right); \]

\[ I_6(\overline{ii}) : \quad The \, same, \, with \, P_{-1/6}^{-\alpha}, \, P_{2\alpha - 1/2}^{-\alpha} \, replaced \, by \, \csc(\pi/6) P_{-1/6}^{-\alpha}, \, (2/\pi) Q_{2\alpha - 1/2}^{-\alpha}. \]

**Remark.** The right-hand segments equal $[1 + \frac{1}{4} \tan^2(\xi/3)]^{-1/2}$ and $[1 - \frac{1}{4} \tan^2(\theta/3)]^{-1/2}$, respectively.

**Definition.** The algebraic $L$-$R$ curve $C'_6$ is the curve

\[ (L^2 - 1)(R^2 + 3)^3 + 27(R^2 - 1)^2 = 0, \quad (17) \]

which is rationally parametrized by

\[ L = 1 - \frac{54}{(p^2 - 1)^2} \left( \frac{p^2 - 1}{p^2 + 3} \right)^3, \quad R = \frac{3 - p^2}{2p} = \pm 1 \pm \frac{(p + 1)(3 \mp p)}{2p}. \]

It is invariant under $L \mapsto -L$ and $R \mapsto -R$, which are performed by $p \mapsto -3/p$ and $p \mapsto -p$, and in fact under any of the Möbius transformations of $p$ that permute $p = -3, -1, 0, 1, 3, \infty$, the vertices of a regular hexagon on the $p$-sphere. (Each of these, which form a dihedral group of order 12, induces a Möbius transformation of $L$, either $L \mapsto L$ or $L \mapsto -L$, and one of $R$.) An associated prefactor function $A = A(p)$, with limit unity when $p \to 1$ and $(L, R) \to (1, 1)$, is

\[ A(p) = \left[ \frac{3^6}{212} \left( \frac{R^2 - 1}{L^2 - 1} \right)^{1/24} \right] = \sqrt{\frac{3 + p^2}{4p}}. \]

**Theorem 3.4.** For each pair $u, v$ of Legendre or Ferrers functions listed below, an identity

\[ u_{-1/6}^{-\alpha}(L(p)) = 3^{3\alpha/2} \frac{\Gamma(\alpha + \frac{1}{2})}{\sqrt{\pi}} A(p) v_{-1/2}^{-2\alpha}(R(p)) \]

of type $I_6'$, coming from the curve $C'_6$, holds for the specified range of values of the parameter $p$. 

| Label | $u_{-1/6}^{-\alpha}$ | $v_{-1/2}^{-2\alpha}$ | $p$ range | $L$ range | $R$ range |
|-------|----------------------|------------------------|-----------|-----------|-----------|
| (i)   | $P_{-1/6}^{-\alpha}$ | $P_{-1/6}^{-2\alpha}$ | $(1, 3)$  | $1 > L > -1$ | $1 > R > -1$ |
| (ii)  | $\frac{1}{2} \csc(\pi/6) P_{-1/6}^{-\alpha}$ | $P_{-1/6}^{-2\alpha}$ | $(0, 1)$  | $-1 < L < 1$ | $\infty > R > 1$ |
| (iii) | $\frac{1}{2} \csc(\pi/6) P_{-1/6}^{-\alpha}$ | $(2/\pi) \cos(\alpha \pi) Q_{-2\alpha}^{-1/2}$ | |

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Remark. The factors $\frac{1}{2} \csc(\pi/6)$ equal unity and can be omitted; they are included for consistency with the other theorems in this section. When $\alpha = -\frac{1}{2}, -\frac{3}{2}, \ldots$, the gamma function diverges, but each $v_{-2\alpha}^{\mu-1/2}$ is identically zero and the identities are still valid in a limiting sense.

The $\alpha = 0$ case of identities $I_6(i), I_6(\bar{i})$ was found by Ramanujan; see [15, Chap. 33, Theorem 11.1 and Corollary 11.2] and [16, Lemma 2.1], and compare [36, Proposition 5.8]. He used $(p-1)/2 \in (0, 1)$ as parameter. The generalization to arbitrary $\alpha$ was given in hypergeometric notation by Garvan [37, (2.32)].

To construct trigonometric versions of these identities, one substitutes $L = \cos \theta$ into the relation \((17)\), and solves for $R$ as a function of $\theta$. This yields the following.

**Corollary 3.4.** The following identities coming from $E_6'$ hold for $\alpha \in \mathbb{C}$, when $\theta \in (0, \pi)$.

\[
I_6'(i) : \quad P_{-1/6}^{-\alpha}(\cos \theta) = 3^{3\alpha/2} \frac{\Gamma(\alpha + \frac{1}{2})}{\sqrt{\pi}} \sqrt{\frac{\cos(\pi/6)}{\cos(\pi/6 - \theta/3)}} P_{-1/6}^{-\alpha} \left( 1 - 2 \frac{\sin(\theta/3)}{\cos(\pi/6 - \theta/3)} \right);
\[
I_6'(\bar{i}) : \quad \text{The same, with } P_{-1/6}^{-\alpha}, P_{-1/6}^{-2\alpha} \text{ replaced by } \frac{1}{2} \csc(\pi/6) P_{-1/6}^{-\alpha}, P_{-1/6}^{-2\alpha};
\[
I_6'(ii) : \quad P_{-1/6}^{-\alpha}(\cos \theta) = 3^{3\alpha/2} \frac{\Gamma(\alpha + \frac{1}{2})}{\sqrt{\pi}} \sqrt{\frac{\cos(\pi/6)}{\cos(\pi/6 + \theta/3)}} P_{-1/6}^{-2\alpha} \left( 1 + 2 \frac{\sin(\theta/3)}{\cos(\pi/6 + \theta/3)} \right);
\[
I_6'(\bar{\bar{i}}) : \quad \text{The same, with } P_{-1/6}^{-\alpha}, P_{-1/6}^{-2\alpha} \text{ replaced by } \frac{1}{2} \csc(\pi/6) P_{-1/6}^{-\alpha}, (2/\pi) \cos(\alpha \pi) Q_{-1/6}^{-2\alpha}.
\]

Remark. The right-hand arguments equal $\sqrt{3} \cot((\pi + \theta)/3)$ and $\sqrt{3} \cot((\pi - \theta)/3)$, respectively. The unit factors $\frac{1}{2} \csc(\pi/6)$ are included for consistency with the other corollaries in this section.

### 3.3 Signature-3 identities

**Definition.** The algebraic $L$-$R$ curve $E_3$ is the curve

\[
27(4L - 5)^3(R^2 - 1) - 4(L - 1)(L + 1)^3(4R^2 - 3)^3 = 0,
\]

which is rationally parametrized by

\[
L = 1 - 54 \frac{p^2 - 1}{(p^2 - 3)^3} = -1 + 2 p^4 \frac{(p^2 - 9)}{(p^2 - 3)^3}, \quad R = \pm 1 + \frac{(p + 1)(3 \pm p)^3}{8 p^3},
\]

and is invariant under $R \mapsto -R$, which is performed by $p \mapsto -p$. An associated prefactor function $A = A(p)$, with limit unity when $p \to 1$ and $(L, R) \to (1, 1)$, is

\[
A(p) = \left[ \frac{3^6 (R - 3)^3 (R + 1)^2}{24 (L - 1)^3 (L + 1)^6} \right]^{1/24} = \sqrt{\frac{(3 - p^2)^2}{4 p^3}}.
\]

**Theorem 3.5.** For each pair $u, v$ of Legendre or Ferrers functions listed below, an identity

\[
u_{-1/3}(L(p)) = A(p) v_{-1/2}(R(p))
\]

of type $I_3$, coming from the curve $E_3$, holds for the specified range of values of the parameter $p$.

| Label | $u_{-1/3}$ | $v_{-1/2}$ | $p$ range | $L$ range | $R$ range |
|-------|------------|-------------|------------|-----------|-----------|
| (i)   | $P_{-1/3}$ | $P_{-1/2}$  | $(1, \sqrt{3})$ | $1 < L < \infty$ | $1 > R > -\sqrt{3}/2$ |
| (ii)  | $\csc(\pi/3) P_{-1/3}$ | $(2/\pi) Q_{-1/2}$ | $(0, 1)$ | $-1 < L < 1$ | $\infty > R > 1$ |
| (iii) | $\csc(\pi/3) P_{-1/3}$ | $(2/\pi) Q_{-1/2}$ | $(0, 1)$ | $-1 < L < 1$ | $\infty > R > 1$ |
To construct trigonometric versions of these identities, one substitutes $L = \cosh \xi$ and $L = \cos \theta$ into the relation \(\text{(19)}\), and solves for $R$ as a function of $\xi$ or $\theta$. This yields the following.

**Corollary 3.5.** The following identities coming from $\mathcal{C}_3$ hold when $\xi \in (0, \infty)$ and $\theta \in (0, \pi)$.

\[
I_3(i): \quad P_{-1/3}(\cosh \xi) = \sqrt{\frac{3 \sinh(\xi/3) \cosh^2(\xi/6)}{\sinh \xi \cosh^2(\xi/2)}} \\
\quad \times P_{-1/2} \left( \sqrt{\frac{3 \sinh(\xi/3) \cosh^2(\xi/6)}{\sinh \xi \cosh^2(\xi/2)}} \right) [2 \cosh(\xi/3) - \cosh(2\xi/3)] ;
\]

\[
I_3(\overline{i}): \quad \text{The same, with $P_{-1/3}, P_{-1/2}$ replaced by $\csc(\pi/3)P_{-1/3}, (2/\pi)Q_{-1/2}$};
\]

\[
I_3(ii): \quad P_{-1/3}(\cos \theta) = \sqrt{\frac{3 \sin(\theta/3) \cos^2(\theta/6)}{\sin \theta \cos^2(\theta/2)}} \\
\quad \times P_{-1/2} \left( \sqrt{\frac{3 \sin(\theta/3) \cos^2(\theta/6)}{\sin \theta \cos^2(\theta/2)}} \right) [2 \cos(\theta/3) - \cos(2\theta/3)] ;
\]

\[
I_3(\overline{ii}): \quad \text{The same, with $P_{-1/3}, P_{-1/2}$ replaced by $\csc(\pi/3)\overline{P}_{-1/3}, (2/\pi)\overline{Q}_{-1/2}$}.
\]

**Definition.** The algebraic $L$-$R$ curve $\mathcal{C}_3'$ is the curve

\[
27(4L - 5)^3(R^2 - 1)^2 - 4(L - 1)(L + 1)^3(R^2 + 3)^3 = 0,
\]

which is rationally parametrized by

\[
L = 1 - 54 \frac{(p^2 - 1)^2}{(p^2 + 3)^3} = -1 + 2 \frac{p^2(p^2 - 9)^2}{(p^2 + 3)^3}, \quad R = \pm1 \mp \frac{(p \mp 1)(3 \pm p)^3}{8p^4}.
\]

It is invariant under $L \mapsto -L$ and $R \mapsto -R$, which are performed by $p \mapsto 3/p$ and $p \mapsto -p$. An associated prefactor function $A = A(p)$, with limit unity when $p \to 1$ and $(L, R) \to (1, 1)$, is

\[
A(p) = \left[ \frac{3^6 (R - 1)^4(R + 1)^4}{2^{16} (L - 1)^2(L + 1)^6} \right]^{1/24} = \sqrt{\frac{(3 + p^2)^2}{16p^2}}.
\]

**Theorem 3.6.** For each pair $u, v$ of Legendre or Ferrers functions listed below, an identity

\[
u_{-1/3}(L(p)) = A(p) v_{-1/2}(R(p))
\]

of type $I_3'$, coming from the curve $\mathcal{C}_3'$, holds for the specified range of values of the parameter $p$.

| Label | $u_{-1/3}$ | $v_{-1/2}$ | $p$ range | $L$ range | $R$ range |
|-------|------------|------------|------------|-----------|-----------|
| (i)   | $P_{-1/3}$ | $P_{-1/2}$ | $(1, 3)$   | $1 > L > 1$ | $1 > R > 1$ |
| (ii)  | $P_{-1/3}$ | $\frac{1}{\sqrt{\csc(\pi/3)}}P_{-1/3}$ | $(0, 1)$   | $-1 < L < 1$ | $\infty > R > 1$ |

**Remark.** Identities $I_3(i), I_3(\overline{i})$ were found by Ramanujan; see [15] Chap. 33, Theorem 5.6 and Corollary 5.7, and compare [16] Lemma 2.1, and compare [36] Proposition 5.7(b)]. He used $(p - 1)/2 \in (0, 1)$ as parameter.

To construct trigonometric versions of these identities, one substitutes $L = \cos \theta$ into the relation \(\text{(19)}\), and solves for $R$ as a function of $\theta$. This yields the following.
Corollary 3.6. The following identities coming from $C_3$ hold when $\theta \in (0, \pi)$.

\[
I_3(i): \quad P_{-1/3}(\cos \theta) = \sqrt{\frac{\cos(\pi/6)}{2 \cos(\pi/6 - \theta/3) - \cos(\pi/6 + 2\theta/3)}} \times P_{-1/2} \left(1 - \frac{2\sin(\theta/3) + \sin(2\theta/3)}{2\cos(\pi/6 - \theta/3) - \cos(\pi/6 + 2\theta/3)}\right);
\]

\[
I_3(\overline{i}): \quad \text{The same, with } P_{-1/3}, P_{-1/2} \text{ replaced by } \frac{1}{2} \csc(\pi/3)\overline{P}_{-1/3}, \overline{P}_{-1/2};
\]

\[
I_3(ii): \quad P_{-1/3}(\cos \theta) = \sqrt{\frac{\cos(\pi/6)}{2 \cos(\pi/6 + \theta/3) - \cos(\pi/6 - 2\theta/3)}} \times P_{-1/2} \left(1 + \frac{2\sin(\theta/3) + \sin(2\theta/3)}{2\cos(\pi/6 + \theta/3) - \cos(\pi/6 - 2\theta/3)}\right);
\]

\[
I_3(\overline{i}): \quad \text{The same, with } P_{-1/3}, P_{-1/2} \text{ replaced by } \frac{1}{2} \csc(\pi/3)\overline{P}_{-1/3}, (2/\pi)\overline{Q}_{-1/2}.
\]

4 Additional results

This section presents three additional transformation theorems for Legendre functions, based on rational or algebraic transformations of the independent variable, and introduces the fundamental proof technique. Theorem 4.1 relates the four functions $P_{2\alpha-1/2}, P_{2\alpha-1/2}, P^2_{2\alpha-1/2}, \tilde{\tilde{P}}_{2\alpha}$, and Theorem 4.2 (equivalent to Whipple’s transformation formula) relates $P^2_{-\alpha-1/2}, \tilde{\tilde{P}}_{-\alpha}$. Theorem 4.1, or an equivalent, has appeared in the setting of ‘generalized’ (associated) Legendre functions; compare [38 and [39 §4]. It is not well known. Whipple’s formula is more familiar, in part because it has two free parameters rather than one, but the proof indicated below is new. Theorem 4.3 relates the functions $P^2_{2\alpha-1/2}, \tilde{\tilde{Q}}_0$, and is unexpected.

The calculations in the proofs employ the calculus of Riemann P-symbols, which is classical [40]. For any homogeneous second-order ordinary differential equation $Lu = 0$ on the Riemann sphere $\mathbb{P}^1 := \mathbb{C} \cup \{\infty\}$ which is Fuchsian, i.e., has only regular singular points, the P-symbol tabulates the singular points and the two characteristic exponents associated to each point. For example, Legendre’s equation [53] has P-symbol

\[
\left\{ \begin{array}{ccc}
1 & -1 & \infty \\
-\mu/2 & -\mu/2 & -\nu \\
\mu/2 & \mu/2 & \nu + 1
\end{array} \right\},
\]

in which the order of the points and that of the exponents are not significant. As mentioned, $P_{-\mu}^\mu, P_{-\mu}^{-\mu}, P_{\mu}^\mu$ and $\tilde{\tilde{Q}}_0^\nu$ are Frobenius solutions associated respectively with the exponent $\mu/2$ at $z = 1$, the exponent $\mu/2$ at $z = -1$, the exponent $\mu/2$ at $z = 1$ and the exponent $\nu + 1$ at $z = \infty$.

Changes of variable applied to an equation $Lu = 0$ affect its P-symbol in predictable ways. For instance, if $w(z) = (z-z_0)^\nu u(z)$ is a linear change of the dependent variable, the transformed equation $\tilde{\tilde{L}}w = 0$, i.e., $\tilde{\tilde{L}} [(z-z_0)^{-\nu} w] = 0$, will have its exponents at $z = z_0$ shifted upward by $c$ relative to those of $Lu = 0$, and those at $z = \infty$ similarly shifted downward. In interpreting this statement one should note that any arbitrary, i.e. non-singular point has exponents 0, 1.

Any rational map $f: \mathbb{P}^1 \rightarrow \mathbb{P}^1$, i.e., rational change of the independent variable of the form $z = f(\tilde{z})$, will lift a Fuchsian differential equation $Lu = 0$ on $\mathbb{P}^1$ to a Fuchsian equation $\tilde{\tilde{L}}\tilde{u} = 0$ on $\mathbb{P}^1$, the P-symbol of which can be calculated. The simplest case is when $f$ is a homography (also called a linear fractional or Möbius transformation), so that $z = f(\tilde{z}) = (A\tilde{z} + B)/(C\tilde{z} + D)$ with $AD - BC \neq 0$. In this case $f$ provides a one-to-one correspondence between the points of the $z$-sphere and those of the $\tilde{z}$-sphere, and exponents are unaffected by lifting: $\tilde{z}_0 \in \mathbb{P}^1$ is a
singular point of $\tilde{\mathcal{L}} u = 0$ if and only if $f(\tilde{z}_0) \in \mathbb{P}^1_\mathbb{C}$ is one of $\mathcal{L} u = 0$; and the exponents are the same.

More generally, if $f$ is a rational function with $z - z_0 \sim \text{const} \times (\tilde{z} - \tilde{z}_0)^k$, so that $f^{-1}(\tilde{z}_0)$ equals $\tilde{z}_0$ with multiplicity $k$, the exponents at the lifted point $\tilde{z} = \tilde{z}_0$ will be $k$ times those at $z = z_0$. (This assumes that the points $z_0, \tilde{z}_0$ are finite; if $z_0, \tilde{z}_0$ are $\infty$ then $z - z_0, \tilde{z} - \tilde{z}_0$ must be replaced by $1/z, 1/\tilde{z}$.) Thus by an appropriate lifting, and if necessary an additional shifting of exponents (performed by an appropriate linear change of the dependent variable), it may be possible to convert the exponents of a singular point to 0, 1. That is, under rational lifting a singular point of a differential equation may 'disappear': become an ordinary point, in a neighborhood of which each Frobenius solution is analytic.

The calculus of P-symbols is a powerful tool for exploring the effect of changes of variable on Fuchsian differential equations, but a P-symbol does not, in general, uniquely determine such an equation, or even its solution space. If a second-order equation has $m$ specified singular points on the Riemann sphere ($m \geq 3$), it and its two-dimensional solution space are determined by the $2m$ exponents and by $m - 3$ accessory parameters [40]. To prove equality between two second-order differential equations with more than three singular points, which have the same singular point locations and characteristic exponents but which have been obtained by different liftings, one must work out the lifted equations explicitly, and compare them term-by-term.

### 4.1 Homographic identities

Theorem [41] below is based on an algebraic change of variable, from $L$ to $R$, which is relatively simple: it is a homography of the Riemann sphere. The associated curve is denoted $\mathcal{M}$, after Möbius, and the resulting identities are said to be of type $M$.

**Definition.** The algebraic $L$--$R$ curve $\mathcal{M}$ is the curve

$$(L + 1)(R + 1) - 4 = 0, \quad (21)$$

which is rationally parametrized by

$$L = -1 + 2p, \quad R = -1 + 2/p,$$

and is invariant under $L \leftrightarrow R$, which is performed by $p \leftrightarrow 1/p$. An associated prefactor function $A = A(p)$, equal to unity when $p = 1$ and $(L, R) = (1, 1)$, is

$$A(p) = \left[\frac{2}{L + 1}\right]^{1/2} = \left[\frac{R + 1}{2}\right]^{1/2} = \sqrt{\frac{R}{p}}.$$

**Theorem 4.1.** For each pair $u, v$ of Legendre or Ferrers functions listed below, an identity

$$u_{-\alpha/2}^{-2\alpha}(L(p)) = A(p)v_{-\alpha/2}^{-2\alpha}(R(p))$$

of type $M$, coming from the curve $\mathcal{M}$, holds for the specified range of values of the parameter $p$.

| Label | $u_{-\alpha/2}^{-2\alpha}$ | $v_{-\alpha/2}^{-2\alpha}$ | $p$ range | $L$ range | $R$ range |
|-------|---------------------------|---------------------------|----------|----------|----------|
| (i)   | $P_{-\alpha/2}^{-2\alpha}$ | $P_{-\alpha/2}^{-2\alpha}$ | $(1, \infty)$ | $1 < L < \infty$ | $1 > R > -1$ |
| (ii)  | $\frac{(2/\pi)\cos(\alpha\pi)}{Q_{-\alpha/2}^{-2\alpha}}$ | $P_{-\alpha/2}^{-2\alpha}$ | $(0, 1)$ | $-1 < L < 1$ | $\infty > R > 1$ |

To construct trigonometric versions of these identities, one substitutes $L = \cosh \xi$ and $L = \cos \theta$ into the relation (21), and solves for $R$ as a function of $\xi$ or $\theta$. This yields the following.
Corollary 4.1. The following identities coming from $M$ hold for $\alpha \in \mathbb{C}$, when $\xi \in (0, \infty)$ and $\theta \in (0, \pi)$.

\[
M(i): \quad P_{\alpha-1/2}^{-2\alpha}(\cosh \xi) = \mathrm{sech}(\xi/2) P_{\alpha-1/2}^{-2\alpha}(1 - 2 \tanh^2(\xi/2));
\]

\[
M(\overline{i}): \quad \text{The same, with } P_{\alpha-1/2}^{-2\alpha} \text{ replaced by } (2/\pi) \cos(\alpha \pi) \tilde{Q}_{\alpha^{-1/2}}^{-2\alpha} P_{\alpha-1/2}^{-2\alpha};
\]

\[
M(ii): \quad P_{\alpha-1/2}^{-2\alpha}(\cos \theta) = \sec(\theta/2) P_{\alpha-1/2}^{-2\alpha}(1 + 2 \tan^2(\theta/2));
\]

\[
M(\overline{ii}): \quad \text{The same, with } P_{\alpha-1/2}^{-2\alpha}, P_{\alpha-1/2}^{-2\alpha} \text{ replaced by } \tilde{P}_{\alpha^{-1/2}}, (2/\pi) \cos(\alpha \pi) \tilde{Q}_{\alpha^{-1/2}}^{-2\alpha}.
\]

Remark. Owing to the invariance under $L \leftrightarrow R$, the pairs $M(i), M(\overline{i})$ and $M(ii), M(\overline{ii})$ are equivalent, up to parametrization. If $\alpha$ is a half-odd-integer, these identities degenerate or become singular. If $\alpha = -\frac{1}{2}, \frac{3}{2}, . . .$, both sides of each identity equal zero. If $\alpha = \frac{1}{2}, \frac{3}{2}, . . .$, the factor $\cos(\alpha \pi)$ in $M(i)$ and $M(\overline{i})$, which equals zero, is compensated for by the factor $\tilde{Q}_{\alpha^{-1/2}}^{-2\alpha}$, which diverges. These two identities are still valid in a limiting sense, and the singular behavior can be removed by rewriting them in terms of Olver’s function $Q_{\alpha^{-1/2}}^{-2\alpha} = \tilde{Q}_{\alpha^{-1/2}}^{-2\alpha}/\Gamma(-\alpha + 1/2)$, using the reflection formula $\cos(\alpha \pi) = \pi/\Gamma(\alpha + 1/2)\Gamma(-\alpha + 1/2)$.

Proof of Theorem 4.1. Functions $u, v$ satisfy Legendre’s equation (5), of degree $\nu = \alpha - 1/2$ and order $\mu = -2\alpha$, if and only if $u(L(p))$ and $A(p)v(R(p))$ both satisfy a certain second-order differential equation with independent variable $p$, which is obtained by lifting. This can be checked by a P-symbol calculation, noting that the inverse images of the singular points 1, −1, ∞ are $p = 1, 0, \infty$ under $L = L(p)$, and $p = 1, \infty, 0$ under $R = R(p)$. The left and right P-symbols are

\[
\begin{pmatrix}
1 & -1 & \infty \\
\alpha & \alpha & -\alpha + 1/2 \\
-\alpha & -\alpha & \alpha + 1/2
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & \infty \\
\alpha & \alpha & -\alpha + 1/2 \\
-\alpha & -\alpha & \alpha + 1/2
\end{pmatrix},
\]

\[
\left(\sqrt{\frac{1}{p}}ight)
\begin{pmatrix}
1 & -1 & \infty \\
\alpha & \alpha & -\alpha + 1/2 \\
-\alpha & -\alpha & \alpha + 1/2
\end{pmatrix}
= \left(\sqrt{\frac{1}{p}}ight)
\begin{pmatrix}
1 & \infty & 0 \\
\alpha & \alpha & -\alpha + 1/2 \\
-\alpha & -\alpha & \alpha + 1/2
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & \infty & 0 \\
\alpha & \alpha + 1/2 & -\alpha \\
-\alpha & -\alpha + 1/2 & \alpha
\end{pmatrix},
\]

which are equivalent. Put differently, the $L \mapsto R$ homography, followed by a linear change of the dependent variable coming from the prefactor $A(p) = \sqrt{1/p}$, takes Legendre’s equation to itself.

It remains to show that for each $u, v$ listed in the theorem, the functions $u(L(p))$ and $A(p)v(R(p))$ are the same element of the two-dimensional solution space of the lifted equation. First, notice that this is true up to some constant factor, since each of $u, v$ is a Frobenius solution associated with a singular point and one of its exponents, and the points and exponents correspond. In identity $M(\overline{i})$ for example, $u = u(L) = (2/\pi) \cos(\alpha \pi) \tilde{Q}_{\alpha^{-1/2}}^{-2\alpha}(L)$ is a Frobenius solution at $L = \infty$ with exponent $\alpha + 1/2$, and $v = v(R) = \tilde{P}_{\alpha^{-1/2}}^{-2\alpha}(R)$ is one at $R = -1$ with exponent $\alpha$. Examining the above P-symbols reveals that for $M(\overline{i})$, both $u(L(p))$ and $A(p)v(R(p))$, which are Frobenius solutions of the lifted equation, are associated to its singular point $p = \infty$ and the exponent $\alpha + 1/2$; so they must be constant multiples of each other.

Finally, one must check that for each identity, the constant of proportionality equals unity. This follows by comparing the left and right sides near the common singular point. Identity
\( M(\ell) \) is typical. To compare left and right asymptotics, one uses
\[
\tilde{Q}_{\alpha-1/2}^{-2\alpha}(L) \sim \frac{\sqrt{\pi}}{2^{\alpha+1/2} \Gamma(\alpha+1)} (1/L)^{\alpha+1/2}, \quad L \to \infty, \tag{24a}
\]
\[
\tilde{P}_{\alpha-1/2}^{-2\alpha}(R) \sim \frac{\sqrt{\pi}}{2^{\alpha} \Gamma(2\alpha+1)} (1+R)^{\alpha}, \quad R \to -1, \tag{24b}
\]
which come from (\text{6a}). A bit of calculation, using the duplication formula for the gamma function, shows that the two sides behave identically at the \( p = \infty \) singular point; specifically,\[
u(L(p)), A(p)v(R(p)) \sim \frac{\sqrt{\pi}}{\Gamma(2\alpha+1)} (1/p)^{\alpha+1/2}, \quad p \to \infty. \tag{25}\]
The other three identities are proved similarly. \( \square \)

### 4.2 Whipple’s formula

Theorem 4.2 below is a version of Whipple’s \( Q \leftrightarrow P \) transformation formula, but the proof given below is simpler than the original [17]: no integral representations are used. Since the algebraic change of variable from \( L \) to \( R \) was introduced by Whipple, the underlying curve is denoted \( W_2 \).

**Definition.** The algebraic \( L-R \) curve \( W_2 \) is the curve
\[
(L^2 - 1)(R^2 - 1) - 1 = 0, \tag{26}
\]
which is rationally parametrized by
\[
L = \frac{(p+1)^2 + (p-1)^2}{(p+1)^2 - (p-1)^2} = \frac{p^2 + 1}{2p}, \quad R = \frac{p^2 + 1}{p^2 - 1},
\]
and is invariant under \( L \leftrightarrow R \), \( L \to -L \) and \( R \to -R \), which are performed by \( p \to (p+1)/(p-1) \), \( 1/p \) and \( -p \), and under the group they generate (which can be viewed as a dihedral group of order 8, acting on \( p = -1, 0, 1, \infty \)). An associated prefactor function \( A = A(p) \), equal to unity when \( p = 1 + \sqrt{2} \) and \( L = R \), is
\[
A(p) = (L^2 - 1)^{-1/4} = (R^2 - 1)^{1/4} = \sqrt{\frac{2p}{p^2 - 1}}.
\]

**Theorem 4.2.** For each pair \( u, v \) of Legendre functions listed below, an identity
\[
u_{\alpha-1/2}^{-\beta}(L(p)) = \frac{\sqrt{\pi}}{\Gamma(\beta - \alpha + \frac{1}{2})} A(p)v_{\beta-1/2}^{-\alpha}(R(p))
\]
of type \( W_2 \), coming from the curve \( W_2 \), holds for the specified range of values of the parameter \( p \).

| Label | \( u_{\alpha-1/2}^{-\beta} \) | \( v_{\beta-1/2}^{-\alpha} \) | \( p \) range | \( L \) range | \( R \) range |
|-------|------------------|------------------|---------------|----------------|---------------|
| (i)   | \( \sqrt{2} P_{\alpha-1/2}^{-\beta} \) | \( (2/\pi)Q_{\beta-1/2}^{-\alpha} \) | \( 1, \infty \) | \( 1 < L < \infty \) | \( \infty > R > 1 \) |
| (ii)  | \( (2/\pi) \cos((\beta - \alpha)\pi) \tilde{Q}_{\alpha-1/2}^{-\beta} \) | \( \sqrt{2} P_{\beta-1/2}^{-\alpha} \) | \( \infty > L > 1 \) | \( 1 < R < \infty \) | \( 1 < R < \infty \) |

To construct trigonometric versions of these identities, one substitutes \( L = \cosh \xi \) into the relation (26), and solves for \( R \) as a function of \( \xi \); it equals \( \coth \xi \). This yields the following versions, which for the sake of symmetry are expressed in terms of Olver’s function \( Q \) rather than \( \tilde{Q} \). Owing to the \( L \leftrightarrow R \) invariance, they are equivalent to each other up to parametrization, and are also equivalent to Whipple’s \( Q \leftrightarrow P \) formula, eq. (4).
Corollary 4.2. The following identities coming from $\mathcal{W}_2$ hold for $\alpha, \beta \in \mathbb{C}$, when $\xi \in (0, \infty)$.

\[
W_2(i) : \quad P_{\alpha-1/2}^{\beta}(\cosh \xi) = \sqrt{2/\pi} \sqrt{\cosh \xi} \, Q_{\beta-1/2}^{\alpha}(\coth \xi);
\]

\[
W_2(\tilde{e}) : \quad Q_{\alpha-1/2}^{\beta}(\cosh \xi) = \sqrt{\pi/2} \sqrt{\cosh \xi} \, P_{\beta-1/2}^{\alpha}(\coth \xi).
\]

Proof of Theorem 4.2. Similarly to the proof of Theorem 4.1, this follows from lifting Legendre’s equation (5), now of degree $\nu = \alpha - 1/2$ and order $\mu = -\beta$, to the $p$-sphere, along the covering maps $L = L(p)$ and $R = R(p)$. (The latter lifting is followed by a linear change of the dependent variable coming from the prefactor $A(p) = \sqrt{2p/(p^2 - 1)}$. The inverse image of the set of singular points $\{1, -1, \infty\}$ under either $L$ or $R$ is the subset $\{1, -1, 0, \infty\}$ of the $p$-sphere, which comprises the vertices of a square, and the left and right $P$-symbols both turn out to be

\[
\left\{
\begin{array}{cccc}
1 & -1 & \infty & 0 \\
-\beta & -\beta & -\alpha + 1/2 & -\alpha + 1/2 \\
\beta & \beta & \alpha + 1/2 & \alpha + 1/2 \\
\end{array}
\right\},
\]

(27)

when account is taken of the fact that $L^{-1}(1), L^{-1}(-1), R^{-1}(1), R^{-1}(-1)$, which respectively equal $1, -1, 0, \infty$, have double multiplicity. However, since $\{1, -1, 0, \infty\}$ has cardinality greater than three, equality of the $P$-symbols, though necessary, is not sufficient for the lifted equations $\mathcal{E}_L$ and $\mathcal{E}_R$ to equal each other. They take the identical form $\tilde{\mathcal{E}} \tilde{u} = 0$, in fact they are both

\[
\frac{d^2 \tilde{u}}{dp^2} + \frac{2p \, d\tilde{u}}{p^2 - 1 \, dp} + \left[ \frac{1 - 4\alpha^2}{4p^2} - \frac{4\beta^2}{(p^2 - 1)^2} \right] \tilde{u} = 0,
\]

(28)

but verifying this requires a separate calculation.

The identities of the theorem, each based on a pair $(u, v)$, come from the table

\[
\begin{array}{c|cccc}
p & \infty & -1 & 0 & 1 \\
\hline
L(p) & -\infty & -1^* & -\infty/\infty & 1^* \\
R(p) & 1^* & +\infty/-\infty & -1^* & -\infty/\infty \end{array}
\]

(29)

which shows how each of the four intervals $(-\infty, -1), (-1, 0), (0, 1), (1, +\infty)$, into which the real $p$-line is divided by the singular points, is mapped monotonically onto a real $L$-interval and a real $R$-interval. (An asterisk indicates a change of direction.) For each $p$-interval, the possible $(u, v)$ are determined thus: if each of $u$ and $v$ is to be one of $P, P, \tilde{P}, \tilde{Q}, \bar{Q}$, then the defining singular points of $u(L)$ and $v(R)$ (namely, $1$ for $P$ and $-1$ for $\tilde{P}$ and $\infty$ for $\tilde{Q}$) must correspond, in the sense that both must be at the same end of the interval. If so, $u(L(p))$ and $A(p)v(R(p))$ will be the same Frobenius solution in the (two-dimensional) solution space of $\mathcal{E}_L = \mathcal{E}_R$, up to a constant factor.

It should be noted that on the real axis, $P, \tilde{Q}$ are only defined on $(1, \infty)$, and $P, \tilde{P}$ on $(-1, 1)$. But for each $p$-interval in (29) other than $(1, +\infty)$, either $L$ or $R$ ranges between $-1$ and $-\infty$. So the only interval that will work is $(1, +\infty)$; and for it, there are exactly two possibilities for $(u, v)$, namely $(P, \tilde{Q})$ and $(\bar{Q}, P)$. These yield $W_2(i)$ and $W_2(\tilde{e})$. For each identity, the prefactors in the theorem are calculated by requiring agreement between the leading-order behaviors of the left and right sides at the relevant singular point (i.e., at $p = 1$ for $W_2(i)$ and $p = \infty$ for $W_2(\tilde{e})$).

\[
\Box
\]

4.3 Whipple-like relations

Theorem 4.3 below contains an unexpected pair of Whipple-like identities, which are based on an algebraic curve of higher degree. Since its parameterization resembles that of the Whipple curve $\mathcal{W}_2$, with squares replaced by fourth powers, it is denoted $\mathcal{W}_4$, and the resulting identities are said to be of type $\mathcal{W}_4$. 16
Definition. The algebraic $L$–$R$ curve $W_4$ is the curve
\[ 16(L^2 - 1)(R^2 - 1)(4L^2 + 4R^2 - 5) - 1 = 0, \] (30)
which is rationally parametrized by
\[ L = \frac{(p + 1)^4 + (p - 1)^4}{(p + 1)^4 - (p - 1)^4} = \frac{p^4 + 6p^2 + 1}{4p^2 + 1}, \quad R = \frac{p^4 + 1}{p^4 - 1}, \]
and is invariant under $L \leftrightarrow R$, $L \mapsto -L$ and $R \mapsto -R$, which are performed by $p \mapsto (p + 1)/(p - 1)$, $1/p$ and $-p$, and under the group they generate (the same as for $W_2$). An associated prefactor function $A = A(p)$, equal to unity when $p = 1 + \sqrt{2}$ and $L = R$, is
\[ A(p) = \left[ \frac{R^2 - 1}{L^2 - 1} \right]^{1/12} = \sqrt{\frac{2p}{p^2 - 1}}. \]

Theorem 4.3. For each pair $u, v$ of Legendre functions listed below, an identity
\[ u_{2a-1/2}^{-\alpha}(L(p)) = A(p) v_{2\alpha-1/2}^{-\alpha}(R(p)) \]
of type $W_4$, coming from the curve $W_4$, holds for the specified range of values of the parameter $p$.

| Label | $u_{2a-1/2}^{-\alpha}$ | $v_{2\alpha-1/2}^{-\alpha}$ | $p$ range | $L$ range | $R$ range |
|-------|------------------|------------------|---------|---------|---------|
| (i)   | $2P_{2\alpha-1/2}$ | $(2/\pi)Q_{2\alpha-1/2}$ | $(1, \infty)$ | $1 < L < \infty$ | $\infty > R > 1$ |
| (7)   | $(2/\pi)\tilde{Q}_{2\alpha-1/2}$ | $2P_{2\alpha-1/2}$ | | | |

To construct trigonometric versions of these identities, one substitutes $L = \coth \xi$ into the relation (30), and solves for $R$ as a function of $\xi$. This yields the following, which owing to the $L \leftrightarrow R$ invariance, are equivalent up to parametrization.

Corollary 4.3. The following identities coming from $W_4$ hold for $\alpha \in \mathbb{C}$, when $\xi \in (0, \infty)$.

\[ W_4(i) : \quad 2P_{2\alpha-1/2}(\coth \xi) = \sqrt{\sinh(\xi/2)} \left( \frac{2}{\pi} \right) Q_{2\alpha-1/2}^{-\alpha} \left( \frac{\cosh(\xi/2) + \sech(\xi/2)}{2} \right); \]
\[ W_4(7) : \quad (2/\pi)\tilde{Q}_{2\alpha-1/2}^{-\alpha}(\coth \xi) = \sqrt{\sinh(\xi/2)} \left( 2P_{2\alpha-1/2}^{-\alpha} \left( \frac{\cosh(\xi/2) + \sech(\xi/2)}{2} \right) \right). \]

Proof of Theorem 4.3. This closely resembles the proof of Theorem 4.2. The inverse image of the set of singular points $\{1, -1, \infty\}$ under either $L$ or $R$ is now the subset $\{1, -1, \infty, 0, i, -i\}$ of the $p$-sphere, which comprises the vertices of a regular octahedron, and the left and right $P$-symbols both turn out to be
\[ \left\{ \begin{array}{cccccc} 1 & 1 & \infty & 0 & i & -i \\ -2\alpha & -2\alpha & -2\alpha + 1/2 & -2\alpha + 1/2 & -2\alpha + 1/2 & -2\alpha + 1/2 \\ 2\alpha & 2\alpha & 2\alpha + 1/2 & 2\alpha + 1/2 & 2\alpha + 1/2 & 2\alpha + 1/2 \end{array} \right\}. \] (31)

(It is an easy exercise to verify that if the order $-\alpha$ were replaced by $-\beta$, as in Theorem 4.2, the two $P$-symbols would differ; which is why Theorem 4.3 has only one free parameter, namely $\alpha$.) As before, the lifted equations $\mathcal{E}_L, \mathcal{E}_R$ do not merely have the same $P$-symbol: they are both
\[ \frac{d^2\bar{u}}{dp^2} + 2p \frac{d\bar{u}}{dp} + \left[ \frac{(1 - 16\alpha^2)(p^2 - 1)^2}{4p^2(p^2 + 1)^2} - \frac{16\alpha^2}{(p^2 - 1)^2} \right] \bar{u} = 0, \] (32)
by a separate calculation. The remainder of the proof is similar; in fact the table (29), showing how each of the four $p$-intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, $(1, +\infty)$ is mapped monotonically onto a real $L$-interval and a real $R$-interval, is valid without change. \[ \blacksquare \]
Remark. One may wonder how the curve $W_4$ was found, or equivalently the quartic covering maps $p \mapsto L(p), R(p)$. In fact the identities of type $W_4$ were found first, and the curve was engineered to provide a proof. As the reader can verify, they follow from homographic identities (of type $M$, above) by applying Whipple’s transformation formula to both sides.

Many (associated) Legendre functions of half-odd-integer degree and order, including $P_{-1/2}^-, P_{-3/2}^-, P_{-5/2}^-$, have been tabulated for use in quantum mechanics [3]. Up to phase factors (see §2), these are the same as $P_{1/2}^-, P_{5/2}^-, P_{9/2}^-$.

By applying the identity $W_4(\tilde{r})$, one can easily compute $\tilde{Q}_{1/2}^-, \tilde{Q}_{5/2}^-, \tilde{Q}_{9/2}^-$, which have not previously been tabulated.

5 Derivation of main results

The two dozen identities in Theorems §3.1 §3.6 arising from algebraic curves $E_r, E'_r$ with $r = 3, 4, 6$, are proved by the technique developed in §4. The key fact is that in the four identities of each theorem, which come from a single curve, the left and right functions $u(L(p)), A(p)v(R(p))$ satisfy the same second-order differential equation, as functions of the parametrizing variable $p$.

This equality (i.e., $E_L = E_R$) is consistent with $E_L, E_R$ having the same P-symbol, which can be checked ‘on the back of an envelope’; but due to each of these lifted equations having more than three singular points on the $p$-sphere, for full rigor they must be worked out explicitly, and compared.

Once $E_L = E_R$ has been verified, the covering maps $p \mapsto L(p), R(p)$ determine the associated identities: in particular, which of the Legendre functions $P, Q$ or Ferrers functions $P, \overline{P}$ can appear as $u, v$. The algorithm for determining the possible $u, v$ was illustrated in §4. For each real $p$-interval delimited by real singular points, one checks whether the $L$-range or $R$-range is $(-\infty, -1)$; if so, the $p$-interval is rejected. An $L$-range or $R$-range that is $(-1, 1)$, or a subset of it, corresponds to a Ferrers function, and similarly, $(1, \infty)$ corresponds to a Legendre function. For an identity to exist, the left and right functions must be the same Frobenius solution, which means that their defining singular points ($1$ for $P$ and $-1$ for $\overline{P}$ and $\infty$ for $Q$) must appear at the same end of the $p$-interval. Any constant of proportionality needed between the two sides is calculated by considering their asymptotic behavior at this singular point (see §2).

The preceding algorithm suffices to derive or verify all the identities of §3 except for $I_r(\tilde{r})$, $r = 3, 4, 6$. Anomalously, these relate $\tilde{P}, \tilde{Q}$, and a sketch of how they are derived is deferred. The data below on each curve (the singular points of $E_L = E_R$, the characteristic exponent data, and the table of monotone maps from $p$-intervals to $L, R$-intervals) should suffice for the interested reader to confirm all identities other than these. It is exponent differences that are supplied below, since unlike exponent pairs they are unaffected by the replacement of $v(R(p))$ by $A(p)v(R(p))$.

Signature-4 identities.—On the curve $E_4$ viewed as the $p$-sphere, the equation $E_L = E_R$ has singular points $p = -1, 0, 1$. The respective exponent differences are $\alpha, 2\alpha, \alpha$. It also has a ‘removable’ singular point at $p = \infty$, at which the exponent difference is unity, but no Frobenius solution behaves logarithmically. The equation is

$$\frac{d^2 \tilde{u}}{dp^2} + \left( \frac{1}{p+1} + \frac{1}{p-1} \right) \frac{d \tilde{u}}{dp} + \left[ \frac{3}{4(p^2-1)} - \frac{\alpha^2}{p^2(p^2-1)^2} \right] \tilde{u} = 0,$$

by direct computation. The real $p$-intervals and monotonic $p \mapsto L, R$ maps are tabulated as

$$
\begin{array}{c|cccc}
  p & -\infty & -1 & 0 & 1 & +\infty \\
  L(p) & +\infty & 1 & -1^* & 1 & +\infty^* \\
  R(p) & 0 & -1 & -\infty/+\infty & 1 & 0.
\end{array}
$$
The \( p \)-interval \((1, +\infty)\) yields an identity: because \( 1 < p < +\infty \) corresponds to \( 1 < L < +\infty \) and \( 1 > R > 0 \), its left and right functions are \( P, P \). This is identity \( I_4(i) \) of Theorem 3.1. The defining singular points of \( P, P \) (respectively \( L = 1, R = 1 \)) are at \( p = 1 \), i.e., are at the same end, and the prefactor \( 2^\alpha \) in the theorem comes from requiring the two sides to agree at \( p = 1 \).

In the same way, the \( p \)-interval \((0, 1)\) yields both \( I_2(i) \) and \( I_2(ii) \), which respectively relate \( P, P \) and \( P, \bar{P} \). Their defining points are at \( p = 1 \) and \( p = 0 \). The \( p \)-intervals \((-\infty, -1) \) and \((-1, 0)\) also yield identities, but they are related to the ones just found by \( R \leftrightarrow -R \), which is performed by \( p \mapsto -p \).

On the curve \( C_4 \), the equation \( \mathcal{E}_L = \mathcal{E}_R \) has singular points \( p = 0, 1, \infty \). The respective exponent differences are 0, 0, 0. It also has a removable singular point at \( p = -1 \). The equation is

\[
\frac{d^2 \tilde{u}}{dp^2} + \left( -\frac{1}{p+1} + \frac{1}{p} + \frac{1}{p-1} \right) \frac{d\tilde{u}}{dp} + \frac{3}{4} \left[ \frac{1}{(p+1)^2} + \frac{1}{(p+1)} - \frac{1}{p} \right] \tilde{u} = 0, \tag{35}\]

by direct computation. (It is readily verified that if the left and right order parameters equalled \(-\alpha\), as in the \( I_4 \) identities, then \( \mathcal{E}_L = \mathcal{E}_R \) only if \( \alpha = 0 \); which is why Theorem 3.2 includes no free \( \alpha \) parameter.) The real \( p \)-intervals and monotonic \( p \mapsto L, R \) maps are tabulated as

\[
\begin{array}{c|ccccc}
 p & -\infty & -1 & 0 & 1 & +\infty \\
\hline
 L(p) & -1 & -\infty/3 & -1 & 1^* & -1 \\
 R(p) & -1 & -3 & -\infty/+\infty & 1 & -1 \\
\end{array} \tag{36}
\]

The \( p \)-interval \((1, +\infty)\) yields identities \( I_4(i) \) and \( I_4(ii) \) of Theorem 3.2, which respectively relate \( P, P \) and \( \bar{P}, \bar{P} \), and the \( p \)-interval \((0, 1)\) yields both \( I_2(i) \) and \( I_2(ii) \), relating \( P, P \) and \( \bar{P}, \bar{Q} \).

**Signature-6 identities.**—On the curve \( C_6 \) viewed as the \( p \)-sphere, the equation \( \mathcal{E}_L = \mathcal{E}_R \) has singular points \( p = -3, -1, 0, 1, 3, \infty \), which are the vertices of a regular hexagon. The respective exponent differences are \( \alpha, \alpha, 4\alpha, \alpha, \alpha, 4\alpha \). It also has ‘apparent’ singular points at \( p = \pm \sqrt{3} \), at each of which the exponent difference is a non-zero integer other than unity (namely, 2), but no Frobenius solution behaves logarithmically. The equation is

\[
\frac{d^2 \tilde{u}}{dp^2} + \left( \frac{1}{p+3} + \frac{1}{p+1} + \frac{1}{p} + \frac{1}{p-1} + \frac{1}{p-3} - \frac{4p}{p^2-3} \right) \frac{d\tilde{u}}{dp} - \left[ \frac{60p^2}{(p^2-1)(p^2-9)(p^2-3)^2} + \frac{4\alpha^2(p^2-3)^4}{p^2(p^2-1)^2(p^2-9)^2} \right] \tilde{u} = 0, \tag{37}\]

by direct computation. The real \( p \)-intervals and monotonic \( p \mapsto L, R \) maps are tabulated as

\[
\begin{array}{c|cccccc}
 p & -\infty & -3 & -\sqrt{3} & -1 & 0 & \sqrt{3} & +\infty \\
\hline
 L(p) & 1^* & -1 & -\infty/+\infty & -1 & 1^* & +\infty/-\infty & -1 \\
 R(p) & -\infty & -3 & -\sqrt{3}/2^* & -1 & -\infty/+/+\infty & 1 & \sqrt{3}/2^* & 1 \\
\end{array} \tag{38}
\]

The \( p \)-interval \((1, \sqrt{3})\) yields identity \( I_6(i) \) of Theorem 3.3, relating \( P, P \), and the \( p \)-interval \((0, 1)\) yields both \( I_6(i) \) and \( I_6(ii) \), which respectively relate \( P, P \) and \( \bar{P}, \bar{Q} \). The \( p \)-intervals \((-\sqrt{3}, -1), (3, +\infty)\) also yield identities, but they are related to the ones just found by \( L \mapsto -L \) and \( R \mapsto -R \), which are performed by \( p \mapsto 3/p \) and \( p \mapsto -p \).

On the curve \( C_6 \), the equation \( \mathcal{E}_L = \mathcal{E}_R \) has singular points \( p = -3, -1, 0, 1, 3, \infty \). The respective exponent differences are \( 2\alpha, 2\alpha, 2\alpha, 2\alpha, 2\alpha \). It also has apparent singular points at \( p = \pm \sqrt{3} \), with exponent difference 2. The equation is

\[
\frac{d^2 \tilde{u}}{dp^2} + \left( \frac{1}{p+3} + \frac{1}{p+1} + \frac{1}{p} + \frac{1}{p-1} + \frac{1}{p-3} - \frac{4p}{p^2+3} \right) \frac{d\tilde{u}}{dp} - \left[ \frac{15}{(p^2+3)^2} + \frac{\alpha^2(p^2+3)^4}{p^2(p^2-1)^2(p^2-9)^2} \right] \tilde{u} = 0, \tag{39}\]
by direct computation. The real \(p\)-intervals and monotonic \(p \mapsto L, R\) maps are tabulated as

\[
\begin{array}{c|cccccc}
p & -\infty & -3 & -1 & 0 & 3 & +\infty \\
\hline
L(p) & 1^* & -1 & -1^* & -1 & 1^* & 1^* \\
R(p) & +\infty & 1 & -1 & -\infty/+\infty & 1 & -1 & -\infty.
\end{array}
\]  

The \(p\)-interval \((1,3)\) yields identities \(I'_0(i)\) and \(I'_0(\tilde{i})\) of Theorem 3.4 which respectively relate \(P, P\) and \(\overline{P}, \overline{P}\), and the \(p\)-interval \((0,1)\) yields both \(I'_0(ii)\) and \(I'_0(\tilde{ii})\), relating \(P, P\) and \(\overline{P}, \overline{Q}\). The \(p\)-intervals \((-3, -1)\), \((-1, 0)\) also yield identities, but they are related to the ones just found by \(L \mapsto -L\) and \(R \mapsto -R\), which are performed by \(p \mapsto -3/p\) and \(p \mapsto -p\).

**Signature-3 identities.**—On the curve \(C_3\) viewed as the \(p\)-sphere, the equation \(\mathcal{E}_L = \mathcal{E}_R\) has singular points \(p = -3, -1, 0, 1, 3, \infty\). The respective exponent differences are \(0, 0, 0, 0, 0, 0\). It also has removable singular points at \(p = \pm \sqrt{3}\). The equation is

\[
\frac{d^2\tilde{u}}{dp^2} + \left(\frac{1}{p + 3} + \frac{1}{p + 1} + \frac{1}{p} + \frac{1}{p - 1} + \frac{1}{p - 3} - \frac{4p}{p^2 - 3}\right) \frac{d\tilde{u}}{dp} - \left[\frac{96p^2}{(p^2 - 1)(p^2 - 9)(p^2 - 3)^2}\right] \tilde{u} = 0, \\
\]  

by direct computation. (As with \(C'_4\) above, there is no evident generalization to non-zero order \(-\alpha\).) The real \(p\)-intervals and monotonic \(p \mapsto L, R\) maps are tabulated as

\[
\begin{array}{c|cccccc}
p & -\infty & -3 & -\sqrt{3} & -1 & 0 & \sqrt{3} & 3 & +\infty \\
\hline
L(p) & 1^* & -1 & -\infty/+\infty & 1 & -1^* & 1 & +\infty/-\infty & -1 & 1^* \\
R(p) & +\infty & 1 & -\sqrt{3}/2 & -1 & -\infty/+\infty & 1 & -\sqrt{3}/2 & -1 & -\infty.
\end{array}
\]  

The \(p\)-interval \((1, \sqrt{3})\) yields \(I_3(i)\), relating \(P, P\), and the \(p\)-interval \((0,1)\) yields both \(I_3(ii)\) and \(I_3(\tilde{ii})\), which respectively relate \(P, P\) and \(\overline{P}, \overline{Q}\). The \(p\)-intervals \((-\sqrt{3}, -1)\), \((-1, 0)\) also yield identities, but they are related to the ones just found by \(R \mapsto -R\), which is performed by \(p \mapsto -p\).

On the curve \(C'_3\), the equation \(\mathcal{E}_L = \mathcal{E}_R\) has singular points \(p = -3, -1, 0, 1, 3, \infty\). The respective exponent differences are \(0, 0, 0, 0, 0, 0\). It also has removable singular points at \(p = \pm \sqrt{3}i\). By direct computation, the equation is

\[
\frac{d^2\tilde{u}}{dp^2} + \left(\frac{1}{p + 3} + \frac{1}{p + 1} + \frac{1}{p} + \frac{1}{p - 1} + \frac{1}{p - 3} - \frac{4p}{p^2 + 3}\right) \frac{d\tilde{u}}{dp} - \left[\frac{24}{(p^2 + 3)^2}\right] \tilde{u} = 0.
\]  

(As with \(C_3\), there is no evident generalization to non-zero order \(-\alpha\).) The table of real \(p\)-intervals and monotone \(p \mapsto L, R\) maps is the same as for \(C'_6\), and the derivation of identities is similar.

**Finer asymptotics.**—It has now been explained how each identity in Theorems 3.3--5.6 is derived, except for \(I_r(\tilde{i})\), \(r = 3, 4, 6\). Each of these relates a \(\overline{P}, \overline{Q}\), i.e., relates an ad hoc Legendre function on the left (a linear combination of \(P, Q\)) to a Ferrers function of the second kind, on the right. Any identity \(I_r(\tilde{i})\) is anomalous because, as the tables in Theorems 3.3--5.3 and 5.5 show, its \(R\)-interval, over which the Ferrers argument ranges, does not extend the entire way from \(R = 1\) to \(R = -1\). This is why the above proof technique, applied to this \(R\)-range and the corresponding \(p\)-interval, produced only one identity (i.e., \(I_r(i)\)), which came by requiring identical left and right asymptotics at the \(R = 1\) end: at the singular point \(p = 1\). The local behavior at the other end, which is not a singular point, is not given by any simple formula.

This difficulty can be worked around by focusing on the \(p = 1\) end of the relevant \(p\)-interval (which is \((1, \infty)\), \((1, 3)\), \((1, \sqrt{3})\) for \(r = 4, 6, 3\), but employing finer asymptotic approximations. The leading behaviors of \(P'_r(z)\), \(P''_r(z)\) as \(z \to 1\) are given in 65a, 10. Those of \(Q'_r(z)\), \(Q''_r(z)\)
as $z \to 1$ are more difficult to compute. (The point $z = 1$ is not the defining singular point for $\hat{Q}_\nu^\mu$, and the Ferrers function $Q_\nu^\mu$ is not a Frobenius solution at any singular point.) But one can exploit the representation of $\hat{Q}_\nu^\mu$ as a combination of $P_\nu^\mu, P_{\nu+\mu}^{-\mu}$ [2, 3.3(10)], and that of $Q_\nu^\mu$ as a combination of $P_\nu^\mu, P_{\nu+\mu}^{-\mu}$ [2, 3.4(14)]. One finds that if $\mu$ is not an integer and $\nu \pm \mu$ are not negative integers,

$$
\frac{(2/\pi)\sin(\mu\pi)}{\Gamma(\nu + \mu + 1)} Q_\nu^\mu(z) \sim \frac{[(z - 1)/2]^{\mu/2}}{\Gamma(1 - \mu)\Gamma(\nu + \mu + 1)} - \frac{[(z - 1)/2]^{\mu/2}}{\Gamma(1 + \mu)\Gamma(\nu - \mu + 1)}, \quad z \to 1,
$$

(44)

and a similar statement holds with $\hat{Q}_\nu^\mu$ and $z - 1$ replaced by $Q_\nu^\mu$ and $1 - z$, if the first term on the right is multiplied by $\cos(\mu\pi)$. Such asymptotic statements must be interpreted with care: the two terms are the leading terms of distinct Frobenius series, from exponents $-\mu/2, +\mu/2$.

It is easily checked that if in Theorems 3.1, 3.3 and 3.5 the right function $\hat{Q}$, and the left function $\nu$ equals $2/\pi$ times the specified Ferrers function $Q$, and the left function $\alpha$ equals $\csc(\pi/\tau)$ times the specified function $\theta$, the two sides of the identity $I_{r}(\theta)$ will have the same fine asymptotics at $p = 1$: the coefficients of each of the two Frobenius solutions will be in agreement. In fact, it was to obtain this agreement that the ad hoc Legendre function $\tilde{P}$ was defined in [112] as it was, as a certain combination of $P, \tilde{Q}$.

In deriving identity $I_{3}(\theta)$ of Theorem 3.5, a modified approach is needed. This identity relates $\tilde{P}_{1/3}$ to $Q_{-1/2}$, with both functions of order zero (there is no $\alpha$ parameter). In the asymptotic development of $\tilde{Q}_{1/3}(z), Q_{-1/2}(z)$ as $z \to 1$, the Frobenius solutions $(z - 1)^{-\mu/2}, (z - 1)^{\mu/2}$ of (44) are replaced by $1, \ln(z - 1)$; see [2, §3.9.2]. The modifications are straightforward.

6 Elliptic integral representations

The now-proved identities of section 3 joined with differential recurrences for Legendre and Ferrers functions, lead to useful representations in terms of the first and second complete elliptic integrals, $K = K(m)$ and $E = E(m)$, the argument $m$ denoting the elliptic modular parameter.

**Theorem 6.1.** The Legendre functions $P_\nu^{\mu}(\cosh \xi), Q_\nu^{\mu}(\cosh \xi)$ and Ferrers functions $P_\nu^\mu(\cos \theta), Q_\nu^{\mu}(\cos \theta)$, where the degree $\nu$ differs by $\pm 1/r$ ($r = 2, 3, 4, 6$) from an integer and the order $m$ is an integer, can be expressed in closed form in terms of the complete elliptic integrals $K, E$.

**Proof.** The case $r = 2$ is well-known (the Legendre functions of half-integer degree and integer order are the classical toroidal functions). The fundamental representations are

$$
P_{-1/2}(\cosh \xi) = (2/\pi) \sech(\xi/2) K(\tanh^2(\xi/2)), \quad \tilde{Q}_{-1/2}(\cosh \xi) = 2 e^{-\xi/2} K(e^{-2\xi}),
$$

(45a)

$$
P_{-1/2}(\cos \theta) = (2/\pi) K(\sin^2(\theta/2)), \quad \tilde{Q}_{-1/2}(\cos \theta) = K(\cos^2(\theta/2)),
$$

(45b)

and more general $\nu, m$ are handled by applying standard differential recurrences on the degree and order. Let $F_\nu^{\mu}$ denote $P_\nu^{\mu}(\cosh \xi)$ or $Q_\nu^{\mu}(\cosh \xi)$, and $P_\nu^{\mu}$ denote $P_\nu^{\mu}(\cos \theta)$ or $Q_\nu^{\mu}(\cos \theta)$. The order recurrences are

$$
M^\pm F_\nu^{\mu} = s C^\pm F_\nu^{\mu \pm 1},
$$

(46a)

$$
M^\pm F_\nu^{\mu} = \pm C^\pm F_\nu^{\mu \pm 1},
$$

(46b)

where the Legendre and Ferrers ‘ladder’ operators for the order, $M^\pm$ and $M^\pm$, are defined (with $D_\xi = d/d\xi$ and $D_\theta = d/d\theta$) by

$$
M^\pm = D_\xi \mp \mu \coth \xi,
$$

(47a)

$$
M^\pm = D_\theta \mp \mu \cot \theta.
$$

(47b)
The constant of proportionality \( C^- \) equals \((\nu + \frac{1}{2})^2 - (\mu - \frac{1}{2})^2\), \( C^+ \) equals unity and the sign factor \( s \) has the following meaning: \( s = 1, -1 \) for \( F = P, \tilde{Q} \). The degree recurrences are

\[
M_{\pm}^\mu F_\nu^\mu = \left[ \mp (\nu + \frac{1}{2}) + (\mu - \frac{1}{2}) \right] F_\nu^{\mu \pm 1},
\]

\[
M_{\pm}^\mu Z_\nu^\mu = \left[ \mp (\nu + \frac{1}{2}) + (\mu - \frac{1}{2}) \right] Z_\nu^{\mu \pm 1},
\]

(48a)

where the ladder operators for the degree, \( M_{\pm} \), and \( M_{\pm} \), are given by

\[
M_+ = -(\sinh \xi) D_\xi - \left[ \frac{1}{2} \pm (\nu + \frac{1}{2}) \right] \cosh \xi,
\]

\[
M_- = -(\sin \theta) D_\theta - \left[ \frac{1}{2} \pm (\nu + \frac{1}{2}) \right] \cos \theta.
\]

(49a)

By applying these recurrences to any of \( P^\mu_\nu (\cosh \xi), \tilde{Q}^\mu_\nu (\cosh \xi) \) or \( P^\mu_\nu (\cos \theta), Q^\mu_\nu (\cos \theta) \), where \( \nu \) is a half-odd-integer, one can express it in terms of the corresponding \( P_{-1/2}(\cosh \xi), \tilde{Q}_{-1/2}(\cosh \xi) \) or \( P_{-1/2}(\cos \theta), Q_{-1/2}(\cos \theta) \), and its derivatives. By then exploiting formulas \([15a], [15b]\) and the known differentiation formulas for \( \tilde{K} = K(m) \) and \( E = E(m) \), which are

\[
\frac{d}{dm} K = \frac{E - m K}{2m(1 - m)}, \quad \frac{d}{dm} E = \frac{E - K}{2m},
\]

(50)

the desired representation is produced.

The preceding algorithm is easily extended from \( r = 2 \) (the classical case) to \( r = 3, 4, 6 \). Suppose one were given a Legendre function \( P^m_\nu (\cosh \xi), \tilde{Q}^m_\nu (\cosh \xi) \) or \( P^m_\nu (\cos \theta), Q^m_\nu (\cos \theta) \), where \( \nu \) is a half-odd-integer, one can express it in terms of the corresponding \( P_{-1/2}(\cosh \xi), \tilde{Q}_{-1/2}(\cosh \xi) \) or \( P_{-1/2}(\cos \theta), Q_{-1/2}(\cos \theta) \), and its derivatives. The pair of signature-\( r \) identities \( I_r(i), I_r(\tilde{i}) \), which are found in Corollaries \( 3.1, 3.3, 3.5 \) for \( r = 4, 6, 3 \) respectively, will express these Legendre functions in terms of the Ferrers functions \( P_{-1/2}, Q_{-1/2} \). (Recall that \( \tilde{P} \), which appears on the left of \( I_r(i) \), is a linear combination of \( P, \tilde{Q} \); see [12]). Thus the cases \( r = 3, 4, 6 \) reduce to the classical case.

If one were given a Ferrers function, one of \( P^m_\nu (\cos \theta), Q^m_\nu (\cos \theta) \) with \( \nu \in \mathbb{Z} - 1/r \) and \( m \in \mathbb{Z} \), the reduction would be similar, but one of the other three pairs of signature-\( r \) identities (say, the pair \( I_r(i), I_r(\tilde{i}) \)) would be used. The identities in each of these pairs have \( P, \tilde{P} \) on their left sides; but since \( \tilde{P} \) is a combination of \( P, Q \) (see [11]), this is sufficient for reduction.

The only thing that remains to be explained is how to handle the case when the degree \( \nu \) differs by \(+1/r\) rather than \(-1/r\) from an integer. The additional effort required is minor. The functions \( P^\mu_\nu, P^\mu_{\nu + 1} \) are unaffected by the replacement of \( \nu \) by \( -\nu - 1 \), which interchanges the two cases; and for \( Q^\mu_\nu, Q^\mu_{\nu + 1} \), applying the identities [2] 3.3(9) and 3.4(16])

\[
\tilde{Q}^\mu_{\nu - 1} - Q^\mu_\nu = \cos(\nu \pi) \Gamma(\nu + \mu + 1) \Omega(\mu - \nu) \overline{P}^\mu_{\nu - 1},
\]

\[
\sin [\nu - (\nu + \mu) \pi] Q^\mu_{\nu - 1} - \sin [(\nu + \mu) \pi] Q^\mu_{\nu} = -\pi \cos(\nu \pi) \cos(\mu \pi) P^\mu_{\nu}
\]

reduces either case to the other.

The algorithm in this proof is not optimal when \( r = 4, 6 \). For these two values of \( r \), the Ferrers functions \( P^m_{-1/2, r}, P^m_{1/2, r} \) for any \( m \in \mathbb{Z} \) can be reduced directly to the toroidal functions \( P^m_{-1/2, r}, \tilde{Q}^m_{-1/2, r} \) resp. \( P^m_{-1/2, r}, \tilde{Q}^m_{-2m-1/2, r} \) by identities \( I_r(i), I_r(\tilde{i}) \), though there is no analogous reduction for \( r = 3 \) if \( m \) is non-zero. This enhancement for \( r = 4, 6 \) may be of numerical relevance, since the recurrences for Legendre and Ferrers functions are often numerically unstable, and modern schemes for evaluating toroidal functions do not employ them [12].
7 Algebraic Legendre functions

One of the identities of section 3, the signature-6 identity $I_6^{(i)}$ of Theorem 6.4 and Corollary 3.3 can be employed to generate closed-form expressions for $P_{-1/6}^{-1/4}, P_{-1/6}^{-1/4}$ and $\hat{Q}_{-1/4}^{-1/3}$. These turn out to be elementary (specifically, algebraic) functions of their arguments, so the expressions are conceptually as well as practically simpler than the ones for $P_{\nu}^m, \hat{Q}_{\nu}^m, P_{\nu}^m, \hat{Q}_{\nu}^m$ (with $\nu \in \mathbb{Z} \pm 1/2$) that were covered in the last section. No elliptic integrals are involved.

The key fact is that while identity $I_6^{(i)}$ transforms $\hat{P}_{-1/6}^{-\alpha}$ to $\hat{Q}_{-1/2}^{-2\alpha}$, a closed-form expression for $\hat{Q}_{\nu}^m$ (and also $P_{\nu}^m, P_{\nu}^m$ and $Q_{\nu}^m$) is available whenever the order $\mu$ is a half-odd-integer. This expression involves only elementary functions \cite[\S 14.5(iii)]{11}. That any Legendre or Ferrers function with (i) $\mu \in \mathbb{Z} + 1/2$, or (ii) $\nu \in \mathbb{Z}$, can be represented without quadratures is an important result \cite{11}. In this statement, cases (i) and (ii) are related by Whipple’s transformation.

**Theorem 7.1.** The following formulas hold when $\theta \in (0, \pi)$ and $\xi \in (0, \infty)$.

\[
P_{-1/6}^{-1/4}(\cos \theta) = 3^{3/4} \Gamma(5/4)^{-1} (\sin \theta)^{-1/4} \left[ \cos(\theta/3) - \sqrt{\frac{\sin \theta}{3 \sin(\theta/3)}} \right]^{1/4},
\]

\[
P_{-1/6}^{-1/4}(\cosh \xi) = 3^{1/2} \Gamma(5/4)^{-1} (\sinh \xi)^{-1/4} \left[ -\cosh(\xi/3) + \sqrt{\frac{\sinh \xi}{3 \sinh(\xi/3)}} \right]^{1/4}.
\]

Moreover,

\[
\hat{Q}_{-1/4}^{-1/3}(\coth \xi) = C (\sinh \xi)^{-1/4} \left[ -\cosh(\xi/3) + \sqrt{\frac{\sinh \xi}{3 \sinh(\xi/3)}} \right]^{1/4},
\]

where $C = 3^{3/4} \sqrt{\pi/2} \Gamma(5/12)/\Gamma(5/4) = 2^{1/4} 3^{9/8} \Gamma(2/3) \sqrt{3} - 1$ is the constant prefactor.

**Remark.** To obtain explicit formulas when the degree and order differ by integers from those shown in this theorem, one would apply differential recurrences, as in the last section.

**Proof.** An explicit formula for $\hat{Q}_{-1/4}^{-1/2}(\cosh \xi)$ follows from \cite[eq. 14.5.17]{11}. In algebraic rather than trigonometric form, it is

\[
\hat{Q}_{-1/4}^{-1/2}(z) = i Q_{-1/4}^{-1/2}(z) = 4 \sqrt{\pi/2} \left[ (z^2 - 1)^{-1} (z - \sqrt{z^2 - 1}) \right]^{1/4}, \quad z > 1.
\]

(52)

Substituting this into the right side of the $\alpha = 1/4$ case of $I_6^{(i)}$, and performing some lengthy trigonometric manipulations, yields a formula for $P_{-1/6}^{-1/4}(\cos \theta) = P_{-1/6}^{-1/4}(- \cos \theta)$, which upon $\theta$ being replaced by $\theta + \pi$, becomes the one for $P_{-1/6}^{-1/4}(\cos \theta)$ given in the theorem.

The formula for $P_{-1/6}^{-1/4}(\cosh \xi)$ comes by analytic continuation (informally, by setting $\theta = i \xi$). The one for $\hat{Q}_{-1/4}^{-1/3}(\coth \xi)$, with the first value given for the prefactor $C$, then comes by applying Whipple’s transformation. The equality of the two values given for $C$ comes from a gamma-function identity \cite[p. 270]{12}.

The formulas of Theorem 7.1 can be written in algebraic form, since $\cosh(\xi/3), \sinh(\xi/3)$ are algebraic functions of $\cosh \xi$, etc. The significance of $P_{-1/6}^{-1/4}(z), P_{-1/6}^{-1/4}(z)$ and $\hat{Q}_{-1/4}^{-1/3}(z)$ being elementary functions of $z$, expressible in terms of radicals, is the following. Legendre’s differential equation \cite{13} on the Riemann sphere $\mathbb{P}^1$ is of the ‘hypergeometric’ sort, with only three singular points, $z = \pm 1$ and $z = \infty$; and their respective characteristic exponent differences
are $\mu, \mu, 2\nu+1$. It is a classical result of Schwarz (see [2 § 2.7.2], [40] Chap. VII and [43]) that for a differential equation of the hypergeometric sort to have only algebraic solutions, its unordered triple of exponent differences must be one of 15 types, traditionally numbered I–XV. The case when $\nu, \mu = (-1/4, -1/3)$ and $(\mu, \mu, 2\nu+1) = (-1/3, -1/3, 1/2)$ is of Schwarz’s type II, and the case when $\nu, \mu = (-1/6, -1/4)$ and $(\mu, \mu, 2\nu+1) = (-1/4, -1/4, 2/3)$ is of type V.

For each type in Schwarz’s list, there is a (projective) monodromy group: the group of permutations of the branches of an algebraic solution that is generated by loops around the three singular points. (Strictly speaking, the algebraic function here is not a solution of the equation, but the ratio of any independent pair of solutions; which is the import of the term ‘projective.’) For Schwarz’s types II and V, the respective groups are tetrahedral and octahedral: they are isomorphic to the symmetry groups of the tetrahedron and octahedron, which are of orders 12 and 24. It is no accident that as an algebraic function of $\cos \theta$ or $\cosh \xi$, each right side in Theorem 7.1 has a multiple of 12 branches.

An interesting consequence of the formula for $P_{-1/6}^{-1/4}$ is a formula in terms of radicals for an octahedral case of the Gauss hypergeometric function, ${}_2F_1$. Taking into account the relation

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1}\right)^{\mu/2} \frac{2F_1}{(-\nu, \nu+1; 1-\mu; \frac{1-z}{2})},$$

and using Cardano’s formula to solve for $\cosh(\xi/3), \sinh(\xi/3)$ in terms of $z = \cosh \xi$, one deduces

$${}_2F_1\left(\frac{1}{6}, \frac{5}{6}; \frac{3}{4}, 2x\right) = 3^{3/4}(-2x)^{-1/4} \left[-\frac{A^{1/3} + A^{-1/3}}{2} + \sqrt{\frac{1 + A^{2/3} + A^{-2/3}}{3}}\right]^{1/4},$$

$$A = (\sqrt{-2x} + \sqrt{-2(x-1)})^2 / 2.$$  

This holds when $x < 0$, and in fact on the complex $x$-plane with cut $[1, \infty)$, on which the left side is analytic in $x$; provided, that is, that the branch of each radical is appropriately chosen.

It has long been known how to obtain parametric expressions for algebraic hypergeometric functions [40] Chap. VII, and a parametric formula for ${}_2F_1\left(\frac{1}{6}, \frac{5}{6}; \frac{3}{4}, 2x\right)$ has recently been derived [43] eq. (2.8)]. But the explicit formula (54) may be more useful. It does not appear in the best-known data base of closed-form expressions for hypergeometric functions [45], or in the data base generated by Roach [46], which is currently available at www.planetquantum.com.

8 A curiosity

Until this point, each Legendre transformation formula derived in this paper has been related at least loosely to the function transformations in Ramanujan’s theory of signature-$r$ elliptic integrals. More exotic Legendre transformations exist, as the curious theorem and corollary below reveal. They relate $P_{-1/10}^{-1/4}$ to $P_{-1/5}^{-1/5}$ (or $P_{-1/4}^{-1/5}$), despite neither of these functions being an algebraic function of its argument, expressible in terms of complete elliptic integrals, or indeed (by results of Kimura [43]) expressible at all in terms of elementary functions and their integrals.

**Definition.** The algebraic $L-R$ curve $X$ is defined by the rational parametrization

$$L = 1 - \frac{p^2}{1 + p^2} = 1 - \frac{2p^2}{1 + p^2} = -1 + \frac{2}{1 + p^2},$$

$$R = 1 - \frac{2p(2+p)^5}{(1+p^2)(1+11p-p^2)^2} = -1 + \frac{2(1-2p)^5}{(1+p^2)(1+11p-p^2)^2}. $$

```
and is invariant under \((L, R) \mapsto (-L, -R)\), which is performed by \(p \mapsto -1/p\). An associated prefactor function \(A = A(p)\), equal to unity when \(p = 0\) and \((L, R) = (1, 1)\), is

\[
A(p) = \sqrt{\frac{(2 + p)(1 - 2p)}{2(1 + 11p - p^2)}}.
\]

**Theorem 8.1.** For each pair \(u, v\) of Legendre or Ferrers functions listed below, an identity

\[
u_{-1/4}^{-1/10}(L(p)) = \frac{\Gamma(6/5)}{\sqrt{2} \Gamma(11/10)} A(p) v_{-1/4}^{-1/5}(R(p))
\]

of type \(X\), coming from the curve \(X\), holds for the specified range of values of the parameter \(p\).

| Label | \(u_{-1/4}^{-1/10}\) | \(v_{-1/4}^{-1/5}\) | \(p\) range | \(L\) range | \(R\) range |
|-------|-----------------|-----------------|-------------|-------------|-------------|
| (i)   | \(P_{-1/4}^{-1/10}\) | \(P_{-1/4}^{-1/5}\) | \((0, \frac{1}{5})\) | \(1 > L > \frac{4}{5}\) | \(1 > R > -1\) |
| (ii)  | \(P_{-1/4}^{-1/10}\) | \(P_{-1/4}^{-1/5}\) | \((-\frac{1}{5} (5\sqrt{5} - 11), 0)\) | \(11/(5\sqrt{5}) < L < 1\) | \(\infty > R > 1\) |

To construct trigonometric versions of these identities, one substitutes \(L = \cos \theta\) into the parametrization, obtaining \(p = \pm \tan(\theta/2)\), which can be used for \(X(i)\) and \(X(ii)\) respectively, and then writes \(R\) in terms of \(\theta\). This yields the following.

**Corollary 8.1.** The following identities coming from \(X\) hold when \(\theta \in (0, \tan^{-1}(4/3))\) and \(\theta \in (0, \tan^{-1}(2/11))\), respectively.

\[
\begin{align*}
X(i) : & \quad P_{-1/4}^{-1/10} (\cos \theta) = C \sqrt{\frac{4 \cos \theta - 3 \sin \theta}{2 \cos \theta + 11 \sin \theta}} P_{-1/4}^{-1/5} \left(1 - \frac{8 (2 \cos(\theta/2) + \sin(\theta/2))^5}{(2 \cos \theta + 11 \sin \theta)^2} \sin^2(\theta/2)\right); \\
X(ii) : & \quad P_{-1/4}^{-1/10} (\cos \theta) = C \sqrt{\frac{4 \cos \theta + 3 \sin \theta}{2 \cos \theta - 11 \sin \theta}} P_{-1/4}^{-1/5} \left(1 + \frac{8 (2 \cos(\theta/2) - \sin(\theta/2))^5}{(2 \cos \theta - 11 \sin \theta)^2} - \sin^2(\theta/2)\right).
\end{align*}
\]

In both, the constant prefactor \(C\) equals \(\Gamma(6/5)/2 \Gamma(11/10)\).

**Proof of Theorem 8.1** This resembles the proofs in §5 of the main results, and will only be sketched. On the curve \(X\) viewed as the \(p\)-sphere, the equation \(E_L = E_R\) has singular points at \(p = 0, \infty, i, -i\), which are the vertices of a square. The respective exponent differences are \(\frac{5}{6}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\). The lifted equations \(E_L, E_R\) for \(\tilde{u} = \tilde{u}(p) = u(L(p))\) and \(A(p)v(R(p))\) both take the form

\[
\frac{d^2 \tilde{u}}{dp^2} + \frac{1}{p} \frac{d \tilde{u}}{dp} - \left[\frac{1}{100p^2} + \frac{3}{4(p^2 + 1)^2}\right] \tilde{u} = 0,
\]

by a separate computation. The real \(p\)-intervals and monotonic \(p \mapsto L, R\) maps are tabulated as

\[
\begin{array}{c|cccccccc}
\text{Label} & -\infty & -2 & -\frac{1}{2}(5\sqrt{5} - 11) & 0 & \frac{1}{2} & \frac{1}{2}(5\sqrt{5} + 11) & +\infty \\
\hline
L(p) & -1^* & -\frac{1}{5} & 11/(5\sqrt{5}) & 1^* & \frac{1}{5} & -11/(5\sqrt{5}) & -1^* \\
R(p) & -1 & 1 & +\infty^* & 1 & -1 & -\infty^* & -1.
\end{array}
\]

The \(p\)-interval \((0, \frac{1}{5})\) yields the identity \(X(i)\), relating \(P, P\), and the \(p\)-interval \((-\frac{1}{2}(5\sqrt{5} - 11), 0)\) yields \(X(ii)\), relating \(P, P\). The \(p\)-interval \((-\infty, -2)\) also yields an identity, but this third identity, relating \(\overline{P}, \overline{P}\), is related to \(X(i)\) by \((L, R) \mapsto -(L, R)\), which is performed by \(p \mapsto -1/p\).

From the covering map \(R = R(p)\) of \(\overline{P}\) and the table \((57)\), one would expect that \(p = -2\) (in \(R^{-1}(1)\)), \(p = \frac{1}{2}\) (in \(R^{-1}(-1)\)) and \(p = \mp \frac{1}{2}(5\sqrt{5} + 11)\) (in \(R^{-1}(\infty)\)) would also be singular points of \(E_L = E_R\). But they are ordinary points, i.e. singular points that have ‘disappeared’ upon lifting, as explained in §4 on account of their characteristic exponents being \(0, 1\).

Since the cardinality of the set of singular points \(\{0, \infty, i, -i\}\) is greater than three, the equality of \(E_L, E_R\) cannot be verified by an (easy) comparison of their \(P\)-symbols; that they are equal because both are of the form \(\overline{P}(56)\) must be checked explicitly. \(\square\)
The rather mysterious algebraic curve \( X \) was discovered heuristically; although, as one conjectures from a close examination of the resulting identities \( X(i), X(ii) \), their existence is in some way tied to the equations \( 3^2 + 4^2 = 5^2 \) and \( 2^2 + 11^2 = 5^3 \). The existence of other exotic algebraic curves that lead to Legendre or Ferrers identities will be explored elsewhere.

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