A unified analysis for reaction–diffusion models with application to the spiral waves dynamics of the Barkley model

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Abstract Applying the gradient discretisation method (GDM), the paper develops a comprehensive numerical analysis for nonlinear equations called the reaction–diffusion model. Using only three properties, this analysis provides convergence results for several conforming and non-conforming numerical schemes that align with the GDM. As an application of this analysis, the hybrid mimetic mixed (HMM) method for the reaction–diffusion model is designed, and its convergence is established. Numerical experiments using the HMM method are presented to facilitate the study of the creation of spiral waves in the Barkley model and the ways in which the waves behave when interacting with the boundaries of their generating medium.

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1 Introduction

Some arrhythmias can have fatal consequences if they are permitted to remain in cardiac tissues without undergoing treatment [12, 35]. One method by which it may be possible to develop a better understanding of how arrhythmias develop and the most effective means by which they can be treated is mathematical modelling. A number of scholars have examined the mathematical modelling approaches that are available and, following the application of these methods, they have concluded that cardiac arrhythmias can be traced back to the free rotation of spiral waves. These waves, reactivate an area of tissue at a higher frequency than that associated with the normal sinoatrial node and, thereby, generate a higher than average heartbeat [16, 30, 34, 35, 45]. In some cases, the spiral waves break up into smaller spiral waves that stimulate uncoordinated heart contractions that are referred to as fibrillation. When fibrillation develops in the ventricles of the heart, it causes the heart to tremor and its strength diminishes. Lacking the ability to pump blood at a consistent rate, the heart ultimately suffers from cardiac arrest [26].

It is possible to study the development and behaviour of spiral waves as a form of spatio-temporal solution to reaction–diffusion equations of the general form [10, 36]

$$\begin{align*}
\partial_t \bar{u}(x, t) - \mu \text{div}(\nabla \bar{u}(x, t)) &= f(\bar{u}, \bar{v}), \quad (x, t) \in \Omega \times (0, T), \\
\partial_t \bar{v}(x, t) &= g(\bar{u}, \bar{v}), \quad (x, t) \in \Omega \times (0, T),
\end{align*}$$

for diffusion coefficient $\mu$, with initial conditions and a pure Neumann boundary condition are defined on the domain $\Omega \subset \mathbb{R}^d, d \geq 1$. 

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Although the model has been widely studied theoretically since it entered the general theory of reaction–diffusion equations [43], it is typically very rare for researchers to be able to solve models that pertain to waves of this nature. As such, there is an inherent need for numerical solutions that are accurate and reliable. The existing literature describes the dynamics of spiral waves using the reaction–diffusion models in depth. A variety of mathematical approaches have been employed including finite-difference methods (FDM) [5, 6, 15, 37], pseudo-spectral methods [33, 39, 40], finite-element methods (FEM) [9, 41, 42], and finite-volume methods (FVM) [7, 8, 11, 13, 14, 31, 44]. In this paper, we will focus on the literature related to FVM. In this regard, the 1997 work of Harrild and Henriquez [31] is a particular note. Harrild and Henriquez employed the elements of an FVM-based formulation to examine the phenomenon of conduction in cardiac tissue. Trew et al. [44] described the creation of a novel finite-volume method that could effectively be employed to model bidomain electrical activation in discontinuous cardiac tissue. Coudiré and Pierre [13] examined a 3D FVM that could be effectively used to calculate the electrical activity that could be observed in the myocardium on unstructured meshes and identified stability conditions for two time-stepping methods in distinctive settings. They also generated error estimates by which effective solutions of the monodomain reaction–diffusion systems could be estimated. Bendahmane and Karlsen [8] successfully merged a finite-volume scheme with Dirichlet boundary conditions for the bidomain model, representing a degenerate reaction–diffusion system that models the electrophysiological waves that can be observed in cardiac tissue. Bendahmane et al. [7] employed a comparable FVM with Neumann boundary conditions to demonstrate the existence and uniqueness of the approximate solution for the monodomain and bidomain models for the myocardial tissue electrical activity. Burger et al. [11] presented some fully space-time adaptive multiresolution approaches that were based on a combination of FVM and Barkley’s approach for modelling the complex dynamics of waves in excitable media. More recently in 2017, Coudire and Turpault [14], originated and evaluated a high-order FVM approach in space combined with a high-order strong stability preserving (SSP) Runge–Kutta technique to produce 2D simulations of spiral waves and simple planar waves in cardiac tissue. Their research also yielded error estimates for the given reaction–diffusion systems.

The chief elements of the spiral waves dynamics such as meandering and drift under external perturbations are of great interest. External perturbations can include light-induced drift, dual spiral interaction, and interaction of spirals with a boundary [38]. When we scrutinise spiral-boundary interactions, we can see two varieties of interaction, with the spiral waves being either reflected or annihilated [38]. Examining spiral-boundary interactions and what occurs in order to annihilate the spiral can assist us in creating techniques to prevent arrhythmia generated by the spiral behaviour. For more details on spiral drift due to the boundary effects, see [29, 38, 46] and the references given there.

This paper’s main purpose is to use the gradient discretisation method (GDM) to offer a complete and unified convergence analysis of the reaction–diffusion model. To our knowledge, this analysis provides the first results that are applicable to several conforming and non-conforming methods. The GDM is an abstract setting to study the numerical analysis for linear or nonlinear, steady or time-dependent diffusion partial differential equations (PDEs). Based on a limited number of properties, the GDM can establish convergence of numerical schemes for various models with different boundary conditions. Various studies have established that the GDM covers several families of numerical schemes: conforming, non-conforming and mixed finite elements methods (including the non-conforming, Crouzeix Raviart method and the Raviart Thomas method), the discontinuous Galerkin scheme, the vertex approximate gradient (VAG) scheme, hybrid mimetic mixed methods (which contain hybrid mimetic finite differences, hybrid finite volumes/SUSHI scheme and mixed finite volumes), nodal mimetic finite differences, and finite-volume methods (such as some multi-points flux approximation and discrete duality finite-volume methods). For more details, see [2, 3, 18, 21–25] and the monograph [19] for a complete presentation.

This paper is organised as follows. Section 2 is devoted to the continuous model. Section 3 provides the discrete elements of the GDM and two properties required to analyse the studied model. It also states the discrete problem (the gradient scheme) followed by the main results: the existence of a solution to the scheme and its convergence to a weak solution of the studied model. Section 4 explains the role of the GDM in applying the hybrid mimetic mixed (HMM) methods and establishing its convergence for the considered model. Section 5 is devoted to proving the main novelty of this paper, Theorem 3.6, which is obtained via the compactness technique. In Sect. 6, we study numerically, using the HMM, the propagation of the spiral waves for the Barkley model and their behaviour when they interact with the boundary of the medium where they spread. The paper is completed with a conclusion section.
2 Continuous model

We consider the following nonlinear system of partial differential equations, representing various models such as the Barkley model [6], the FitzHugh–Nagumo model [27], the Aliev and Panfilov model [1], and the Belousov–Zhabotinsky reaction model [47]:

\[
\begin{align*}
\partial_t \tilde{u}(x, t) - \mu \text{div}(\nabla \tilde{u}(x, t)) &= f(\tilde{u}, \tilde{v}), \quad (x, t) \in \Omega \times (0, T), \\
\partial_t \tilde{v}(x, t) &= g(\tilde{u}, \tilde{v}), \quad (x, t) \in \Omega \times (0, T), \\
\nabla \tilde{u}(x, t) \cdot \mathbf{n} &= 0, \quad (x, t) \in \partial \Omega \times (0, T), \\
\tilde{u}(x, 0) &= u_{\text{ini}}(x), \quad x \in \Omega, \tag{2.1} \\
\tilde{v}(x, 0) &= v_{\text{ini}}(x), \quad x \in \Omega, \tag{2.2}
\end{align*}
\]

where \( \mathbf{n} \) is the outer normal to \( \partial \Omega \). In this excitable media system, the unknowns \( \tilde{u}(x, t) \) and \( \tilde{v}(x, t) \) denote the excitation and recovery terms, respectively. A dynamic of a particular case of this model is explained in Sect. 6.

Our analysis focuses on the weak formulation of the above reaction–diffusion model. Several works have studied the existence of solutions of reaction–diffusion models; see [28, 43], for instance. Let us assume the following properties on the data of the model.

**Assumptions 2.1** The assumptions on the data in Problem (2.1)–(2.5) are the following:

1. \( \Omega \) is an open bounded connected subset of \( \mathbb{R}^d \) \( (d \geq 1) \), with a Lipschitz boundary, \( T > 0, \mu \in \mathbb{R} \),
2. \( (u_{\text{ini}}, v_{\text{ini}}) \) are in \( L^\infty(\Omega) \times L^\infty(\Omega) \),
3. the functions \( f, g : \mathbb{R}^2 \to \mathbb{R} \) are Lipschitz continuous with Lipschitz constants \( L_f \) and \( L_g \), respectively.

Under Assumptions 2.1, a pair \( (\tilde{u}, \tilde{v}) \) is said to be a weak solution of Problem (2.1)–(2.5) if the following properties and equalities hold:

\[
\begin{align*}
\tilde{u} &\in L^2(0, T; H^1(\Omega)) \cap C([0, T]; L^2(\Omega)), \quad \tilde{v} \in L^2(0, T; L^2(\Omega)), \\
\forall \tilde{\varphi} &\in L^2(0, T; H^1(\Omega)), \quad \partial_t \tilde{\varphi} \in L^2(0, T; L^2(\Omega)), \quad \tilde{\varphi}(\cdot, T) = 0, \\
\forall \tilde{\psi} &\in L^2(0, T; L^2(\Omega)), \quad \partial_t \tilde{\psi} \in L^2(0, T; L^2(\Omega)), \quad \tilde{\psi}(\cdot, T) = 0,
\end{align*}
\]

\[
\begin{align*}
- \int_0^T \int_\Omega \tilde{u}(x, t) \partial_t \tilde{\varphi}(x, t) \, dx \, dt + \int_0^T \int_\Omega \nabla \tilde{u}(x, t) \cdot \nabla \tilde{\varphi}(x, t) \, dx \, dt, \\
- \int_\Omega u_{\text{ini}}(x) \tilde{\varphi}(x, 0) \, dx = \int_0^T \int_\Omega f(\tilde{u}, \tilde{v}) \tilde{\varphi}(x, t) \, dx \, dt, \tag{2.6a} \\
- \int_0^T \int_\Omega \tilde{v}(x, t) \partial_t \tilde{\psi}(x, t) \, dx \, dt - \int_\Omega v_{\text{ini}}(x) \tilde{\psi}(x, 0) \, dx = \int_0^T \int_\Omega g(\tilde{u}, \tilde{v}) \tilde{\psi}(x, t) \, dx \, dt. \tag{2.6b}
\end{align*}
\]

The existence of a solution to reaction–diffusion models is studied in [43]. Here, the existence of at least one solution, \( (\tilde{u}, \tilde{v}) \), to (2.6), will be a consequence of the convergence analysis of the gradient discretisation method, see Remark 3.7.

3 Discrete problem

As stated in the introduction section, the analysis of numerical schemes for the approximation of solutions to the reaction–diffusion model is performed using the gradient discretisation method. This method first requires reconstructing a set of discrete spaces and operators, called gradient discretisations (GD).
Definition 3.1 (Gradient discretisation for reaction–diffusion model) Let $\Omega$ be an open subset of $\mathbb{R}^d$ (with $d \geq 1$) and $T > 0$. A gradient discretisation for the reaction–diffusion model is $\mathcal{D} = (X_D, Y_D, \Pi_D, \Pi_D', \nabla_D, J_D, J_D', (\Pi_D^t_{Y_D})_{t=0}^{N})$, where

- The two set of discrete unknowns $X_D$ and $Y_D$ are finite dimensional vector spaces on $\mathbb{R}$,
- The function reconstruction $\Pi_D : X_D \rightarrow L^2(\Omega)$ is linear,
- The function reconstruction $\Pi_D' : Y_D \rightarrow L^2(\Omega)$ is linear, and must be defined so that $\|\Pi_D' \|_{L^2(\Omega)}$ is a norm on $Y_D$,
- The gradient reconstruction $\nabla_D : X_D \rightarrow L^2(\Omega)^d$ is linear and must be defined so that
  \[ \|\varphi\|_D = \|\Pi_D \varphi\|_{L^2(\Omega)} + \|\nabla_D \varphi\|_{L^2(\Omega)^d} \]  
  (3.1)

- $J_D : L^\infty(\Omega) \rightarrow X_D$ and $J_D' : L^\infty(\Omega) \rightarrow Y_D$ are linear and continuous interpolation operators for the initial conditions,
- $t^{(0)} = 0 < t^{(1)} < \cdots < t^{(N)} = T$ are discrete times.

Let us introduce some notations to define the space–time reconstructions $\Pi_D \varphi : \Omega \times [0, T] \rightarrow \mathbb{R}$, and $\nabla_D \varphi : \Omega \times [0, T] \rightarrow \mathbb{R}^d$, and the discrete time derivative $\delta_D \varphi : (0, T) \rightarrow L^2(\Omega)$, for $\varphi = (\varphi^{(n)})_{n=0, \ldots, N} \in X_D^N$ and $\psi = (\psi^{(n)})_{n=0, \ldots, N} \in Y_D^N$.

For a.e. $x \in \Omega$, for all $n \in \{0, \ldots, N - 1\}$ and for all $t \in (t^{(n)}, t^{(n+1)})$, let

\[
\begin{align*}
\Pi_D \varphi(x, 0) &= \Pi_D \varphi^{(0)}(x), & \Pi_D \varphi(x, t) &= \Pi_D \varphi^{(n+1)}(x), \\
\nabla_D \varphi(x, t) &= \nabla \varphi^{(n+1)}(x), \\
\Pi_D' \psi(x, 0) &= \Pi_D' \psi^{(0)}(x), & \Pi_D' \psi(x, t) &= \Pi_D' \psi^{(n+1)}(x). 
\end{align*}
\]

Set $\delta t^{(n+\frac{1}{2})} = t^{(n+1)} - t^{(n)}$ and $\delta t_D = \max_{n=0, \ldots, N-1} \delta t^{(n+\frac{1}{2})}$, to define

\[
\begin{align*}
\delta_D \varphi(t) &= \delta_D^{(n+\frac{1}{2})} \varphi := \frac{\Pi_D(\varphi^{(n+1)} - \varphi^{(n)})}{\delta t^{(n+\frac{1}{2})}}, \\
\delta_D' \psi(t) &= \delta_D'^{n+\frac{1}{2}} \psi := \frac{\Pi_D'(\psi^{(n+1)} - \psi^{(n)})}{\delta t^{(n+\frac{1}{2})}}.
\end{align*}
\]

In order to build converging schemes for models including homogeneous Neumann boundary conditions, the elements of a GD must enjoy as much as possible the properties of the continuous space and operators, coercivity, consistency, and limit-conformity. In our model, there is no need for the coercivity property, which gives the discrete Poincaré inequality, since the time derivative in our model plays the same role in establishing the $L^2$ estimate on the discrete solution.

Consistency of the method (refers to an interpolation error), which ensures the accuracy of approximating smooth functions and their gradients by elements is defined from the discrete space, and the convergence of the time steps to zero.

Definition 3.2 (Consistency) If $\mathcal{D}$ is a gradient discretisation in the sense of Definition 3.1, define $S_D : H^1(\Omega) \rightarrow [0, +\infty)$ and $S_D' : L^2(\Omega) \rightarrow [0, +\infty)$ by

\[
\forall \varphi \in H^1(\Omega), \quad S_D(\varphi) = \min_{w \in X_D} \left( \|\Pi_D w - \varphi\|_{L^2(\Omega)} + \|\nabla_D w - \nabla \varphi\|_{L^2(\Omega)^d} \right),  \tag{3.2}
\]

and

\[
\forall \psi \in L^2(\Omega), \quad S_D'(\psi) = \min_{w \in Y_D} \|\Pi_D' w - \psi\|_{L^2(\Omega)}. \tag{3.3}
\]

A sequence $(D_m)_{m \in \mathbb{N}}$ of gradient discretisations is consistent if, as $m \rightarrow \infty$,

- for all $\varphi \in H^1(\Omega)$, $S_{D_m}(\varphi) \rightarrow 0$, and for all $\psi \in L^2(\Omega)$, $S_{D_m}(\psi) \rightarrow 0$,
- for all $w \in L^2(\Omega)$, $D_m J_{D_m} w \rightarrow w$ in $L^2(\Omega)$,
- for all $w \in L^\infty(\Omega)$, $(\Pi_{D_m} J_{D_m} w)_{m \in \mathbb{N}}$ is bounded in $L^\infty(\Omega)$ and converges to $w$ in $L^2(\Omega)$,
• $\delta t_{D_m} \to 0$.

The quantity $W_D$ defined below measures how well the discrete Stokes formula is satisfied. It is only precisely ensured in conforming methods.

**Definition 3.3 (Limit-conformity)** If $D$ is a gradient discretisation in the sense of Definition 3.1, and $H_{\text{div}} = \{ \psi \in L^2(\Omega)^d : \text{div}\psi \in L^2(\Omega), \, \psi \cdot n = 0 \text{ on } \partial \Omega \}$, define $W_D : H_{\text{div}} \to [0, +\infty)$ by

$$\forall \psi \in H_{\text{div}}, \quad W_D(\psi) = \sup_{w \in X_D} \left| \int \langle \nabla_D w \cdot \psi + \Pi_D w \text{div}(\psi) \rangle \, dx \right| / \| w \|_D.$$  

A sequence $(D_m)_{m \in \mathbb{N}}$ of gradient discretisations is limit-conforming if for all $\psi \in H_{\text{div}}$, $W_{D_m}(\psi) \to 0$, as $m \to \infty$.

Finally, dealing with nonlinearity requires the operators $\Pi_D$ and $\nabla_D$ to provide the compactness properties defined below. Note that there is no need for the operator $\Pi_{D'}$ to afford this property.

**Definition 3.4 (Compactness)** A sequence of gradient discretisation $D_m$ in the sense of Definition 3.1 is compact if for any sequence $(\varphi_m)_{m \in \mathbb{N}} \in X_{D_m}$, such that $(\| \varphi_m \|_{D_m})_{m \in \mathbb{N}}$ is bounded, the sequence $(\Pi_{D_m} \varphi_m)_{m \in \mathbb{N}}$ is relatively compact in $L^2(\Omega)$.

Setting the gradient discretisation defined previously in the place of the continuous space and operators in the weak formulation of the model leads to a numerical scheme called a gradient scheme (GS).

**Definition 3.5 (Gradient scheme for (2.6))** Find families $(u^{(n)})_{n=0,\ldots,N} \in X_D^{N+1}$ and $(v^{(n)})_{n=0,\ldots,N} \in Y_D^{N+1}$, such that $u^{(0)} = J_D u_{\text{ini}}$ and $v^{(0)} = J_{D'} v_{\text{ini}}$, and for all $n = 0, \ldots, N - 1$, $u^{(n+1)}$ and $v^{(n+1)}$ satisfy

$$\int_{\Omega} \delta_D^{(n+\frac{1}{2})} u(x) \Pi_D \varphi(x) + \mu \int_{\Omega} \nabla_D u^{(n+1)}(x) \cdot \nabla_D \varphi(x) \, dx$$

$$= \int_{\Omega} f^{(n+1)} \Pi_D \varphi(x) \, dx, \quad \forall \varphi \in X_D,$$

$$\int_{\Omega} \delta_D^{(n+\frac{1}{2})} v(x) \Pi_{D'} \psi(x) = \int_{\Omega} g^{(n+1)} \Pi_{D'} \psi(x) \, dx, \quad \forall \psi \in Y_D,$$

where the discrete reaction terms are defined by

$$f^{(n+1)}_D = \langle \Pi_D u^{(n+1)}(x), \Pi_{D'} v^{(n+1)}(x) \rangle$$

and

$$g^{(n+1)}_{D'} = \langle \Pi_D v^{(n+1)}(x), \Pi_{D'} v^{(n+1)}(x) \rangle.$$

Here, the notations $u^{(n)}$ and $v^{(n)}$ represent an approximation of $\bar{u}$ and $\bar{v}$ at time $t^{(n)}$. The initialisations of $u^{(0)}$ and $v^{(0)}$ are expressed by applying the interpolation operators $J_D$ and $J_{D'}$ to the initial conditions.

The convergence results of this gradient scheme are stated in the following theorem, whose proof is detailed in Sect. 5.

**Theorem 3.6 (Convergence of the GS)** Assume Assumptions (2.1) and let $(D_m)_{m \in \mathbb{N}}$ be a sequence of gradient discretisations in the sense of Definition 3.1, that is consistent, limit-conforming, and compact in the sense of Definitions 3.2, 3.3 and 3.4. For $m \in \mathbb{N}$, let $(u_m, v_m)$ be a solution to the gradient scheme (3.5) with $D = D_m$. Then there exists a weak solution $(\bar{u}, \bar{v})$ of (2.6) and a subsequence of gradient discretisations, still denoted by $(D_m)_{m \in \mathbb{N}}$, such that, as $m \to \infty$,

1. $\Pi_{D_m} u_m$ converges strongly to $\bar{u}$ in $L^\infty(0, T; L^2(\Omega))$,
2. $\Pi_{D_m} v_m$ converges strongly to $\bar{v}$ in $L^\infty(0, T; L^2(\Omega))$,
3. $\nabla_D u_m$ converges strongly to $\nabla \bar{u}$ in $L^2(\Omega \times (0, T))^d$.

**Remark 3.7** There is no requirement to initially assume the existence of a weak solution to the model. As at least one sequence of the gradient discretisation that satisfies the three properties can be constructed, this theorem provides the existence of at least one solution $(\bar{u}, \bar{v})$ to (2.6); the sequence converges to a model solution in the above sense. The limitation of our analysis is linked to the lack of known uniqueness of the continuous solution, which may lead to the approximate solution obtained converging into a non-physical solution. However, physical solutions are those expected to be stable with respect to small perturbations. Since numerical schemes are built on approximations, if a notion of a weak solution can be found that ensures uniqueness, the designed scheme will likely converge to that one.
4 The hybrid mimetic mixed (HMM) methods

We show here that hybrid mimetic mixed (HMM) methods can be expressed as gradient schemes formulation when applied to the reaction–diffusion model. It is shown in [20] that the HMM methods are generic, gathering three different families of methods: the hybrid finite-volume method, the (mixed-hybrid) mimetic finite-difference methods, and the mixed finite-volume methods. In order to construct this mesh-based method, we recall here the notion of polytopal mesh [20].

Definition 4.1 (Polytopal mesh) Let $\Omega$ be a bounded polytopal open subset of $\mathbb{R}^d$ ($d \geq 1$). A polytopal mesh of $\Omega$ is given by $\mathcal{T} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$, where:

1. $\mathcal{M}$ is a finite family of non empty connected polytopal open disjoint subsets of $\Omega$ (the cells) such that $\Omega = \bigcup_{K \in \mathcal{M}} K$. For any $K \in \mathcal{M}$, $|K| > 0$ is the measure of $K$ and $h_K$ denotes the diameter of $K$.
2. $\mathcal{E}$ is a finite family of disjoint subsets of $\partial \Omega$ (the edges of the mesh in 2D, the faces in 3D), such that any $\sigma \in \mathcal{E}$ is a non empty open subset of a hyperplane of $\mathbb{R}^d$ and $\sigma \subset \partial \Omega$. We assume that for all $K \in \mathcal{M}$ there exists a subset $\Delta_K$ of $\partial K$ such that $\partial K = \bigcup_{\sigma \in \mathcal{E}_K} \partial \sigma$. We then set $\mathcal{M}_\sigma = \{ K \in \mathcal{M} : \sigma \in \partial K \}$ and assume that, for all $\sigma \in \mathcal{E}$, $\mathcal{M}_\sigma$ has exactly one element and $\sigma \subset \partial \Omega$, or $\mathcal{M}_\sigma$ has two elements and $\sigma \subset \Omega$. $\mathcal{E}_{\text{int}}$ is the set of all interior faces, i.e. $\sigma \in \mathcal{E}$ such that $\sigma \subset \Omega$, and $\mathcal{E}_{\text{ext}}$ the set of boundary faces, i.e. $\sigma \in \mathcal{E}$ such that $\sigma \subset \partial \Omega$. For $\sigma \in \mathcal{E}$, the $(d-1)$-dimensional measure of $\sigma$ is $|\sigma|$, the centre of mass of $\sigma$ is $\mathbf{x}_\sigma$, and the diameter of $\sigma$ is $h_\sigma$.
3. $\mathcal{P} = (x_K)_{K \in \mathcal{M}}$ is a family of points of $\partial \Omega$ indexed by $\mathcal{M}$ and such that, for all $K \in \mathcal{M}$, $x_K \in K$ ($x_K$ is sometimes called the “centre” of $K$). We then assume that all cells $K \in \mathcal{M}$ are strictly $x_K$-star-shaped, meaning that if $x \in \mathcal{K}$ then the line segment $[x_K, x]$ is included in $K$.

For a given $K \in \mathcal{M}$, let $\mathbf{n}_{K, \sigma} = \mathbf{n}_\sigma$ be the unit vector normal to $\sigma$ outward to $K$ and denote by $d_{K, \sigma}$ the orthogonal distance between $x_K$ and $\sigma \in \partial K$. The size of the discretisation is $h_\mathcal{M} = \sup_{K \in \mathcal{M}} (h_K : K \in \mathcal{M})$.

Let $\mathcal{T}$ be a polytopal mesh. A gradient discretisation $(X_D, Y_D, \Pi_D, \Pi_{D'}, \nabla_D, J_D, J_{D'})$ in the setting of HMM method formats are then constructed by setting

\[
X_D = \{ \varphi = ((\varphi_K)_{K \in \mathcal{M}}, (\varphi_\sigma)_{\sigma \in \mathcal{E}}) : \varphi_K \in \mathbb{R}, \varphi_\sigma \in \mathbb{R} \},
\]
\[
Y_D = \{ \varphi = (\varphi_K)_{K \in \mathcal{M}} : \varphi_K \in \mathbb{R} \},
\]
\[
\forall K \in \mathcal{M} : \Pi_D \varphi = \Pi_{D'} \varphi = \varphi_K \text{ on } K,
\]
\[
\forall w \in L^2(\Omega) : \frac{J_D w}{(w_K)_{K \in \mathcal{M}}, (w_\sigma)_{\sigma \in \mathcal{E}_K}) \in X_D,
\]
\[
\forall w \in L^\infty(\Omega) : \frac{J_{D'} w}{(w_K)_{K \in \mathcal{M}} \in Y_D},
\]
where $w_k = \frac{1}{|K|} \int_K w(x) \, dx$ and $w_\sigma = 0$,
\[
\forall \varphi \in X_D, \forall K \in \mathcal{M}, \forall \mathbf{n}_{K, \sigma} \in \mathcal{E}_K,
\]
\[
\nabla_D \varphi = \nabla_K \varphi + \frac{\sqrt{\alpha}}{d_{K, \sigma}} R_K(\varphi) \mathbf{n}_{K, \sigma} \text{ on } D_{K, \sigma},
\]
where a cell-wise constant gradient $\nabla_K \varphi$ and a stabilisation term $R_K(\varphi)$ are respectively defined by

\[
\nabla_K \varphi = \frac{1}{|K|} \sum_{\sigma \in \partial K} |\sigma| \varphi_\sigma \mathbf{n}_{K, \sigma} \quad \text{and} \quad R_K(\varphi) = (\varphi_\sigma - \varphi_K - \nabla_K \varphi \cdot (\mathbf{x}_\sigma - x_K))_{\sigma \in \mathcal{E}_K}.
\]

Although the gradient $\nabla_K \varphi$ is consistent, it does not satisfy a Poincaré inequality and cannot be directly used in the scheme to approximate the bilinear term $\int_\Omega \nabla u \cdot \nabla \varphi \, dx$. As such, a stabilisation term is required.

The gradient scheme (3.5) coming from such a gradient discretisation can be given by: find $(u^{(n)})_{n=0, \ldots, N} \in X_{D}^{N+1}$ and $(v^{(n)})_{n=0, \ldots, N} \in Y_{D}^{N+1}$, such that $u^{(0)} = J_D u_{\text{ini}}$ and $v^{(0)} = J_{D'} v_{\text{ini}}$, and for all $n = 0, \ldots, N - 1$, $u^{(n+1)}$ and $v^{(n+1)}$ satisfy, for all $\varphi \in X_D$ and for all $\psi \in Y_D$,

\[
\sum_{K \in \mathcal{M}} \frac{|K|}{\delta^{(n+1)}} (u_K^{(n+1)} - u_K^{(n)}) \varphi_K + \mu \sum_{K \in \mathcal{M}} |K| R_K(u^{(n+1)}) \cdot \nabla_K \varphi
\]
\[
+ \mu \sum_{K \in \mathcal{M}} (R_K(\psi))^T \mathbb{W}_K R_K(u^{(n+1)}) = \sum_{K \in \mathcal{M}} \varphi_K \int_K f \left( u_K^{(n+1)} \cdot v_K^{(n+1)} \right) \, dx.
\]
where $\mathbb{B}_K$ is a symmetric positive definite matrix of size $\text{Card}(\mathcal{E}_K)$.

The above HMM scheme can be formulated as a finite-volume scheme (a format used in computational processes). To do so, introduce the linear fluxes $u \mapsto F_{K,\sigma}(u)$ (for $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}_K$) defined in [2]: for all $K \in \mathcal{M}$ and all $u, w \in X_D$, $$\sum_{\sigma \in \mathcal{E}_K} |\sigma| F_{K,\sigma}(u)(w_K - w_{\sigma}) = \mu \int_K \nabla_D u \cdot \nabla_D w \, dx.$$ Then Problem (4.1) can be recast as, for all $n = 0, \ldots, N - 1$, $$\frac{|K|}{\delta t^{(n+\frac{1}{2})}} \left( u^{(n+1)} - u^{(n)} \right) + \mu \sum_{\sigma \in \mathcal{E}_K} |\sigma| F_{K,\sigma}(u^{(n+1)}) = |K| f_{K}^{(n+1)}, \quad \forall K \in \mathcal{M},$$ $$\frac{|K|}{\delta t^{(n+\frac{1}{2})}} \left( v^{(n+1)} - v^{(n)} \right) = |K| g_{K}^{(n+1)}, \quad \forall K \in \mathcal{M},$$ $$F_{K,\sigma}(u^{(n+1)}) + F_{L,\sigma}(u^{(n+1)}) = 0, \quad \forall \sigma \in \mathcal{E}_{\text{int}} \text{ with } \mathcal{M}_{\sigma} = \{K, L\},$$ $$F_{K,\sigma}(u^{(n+1)}) = 0, \quad \forall K \in \mathcal{M}, \forall \sigma \in \mathcal{E}_K \text{ such that } \sigma \subset \partial \Omega,$$

where $f_{K}^{(n+1)} = \frac{1}{|K|} \int_K f(u_{K}^{(n+1)}, v_{K}^{(n+1)}) \, dx$ and $g_{K}^{(n+1)} = \frac{1}{|K|} \int_K g(u_{K}^{(n+1)}, v_{K}^{(n+1)}) \, dx$, for all $K \in \mathcal{M}$.

Now, let us discuss the convergence of the scheme (4.1). Assume the existence of $\theta > 0$ such that, for any $m \in \mathbb{N}$,

$$\theta_{\mathcal{M}} := \max_{K \in \mathcal{M}_m} \left( \max_{\sigma \in \mathcal{E}_K} \frac{h_K}{d_{K,\sigma}} + \text{Card}(\mathcal{E}_K) \right) + \max_{\sigma \in \mathcal{E}_{\text{int}} \mathcal{M}_\sigma = \{K, L\}} \left( \frac{d_{K,\sigma}}{d_{L,\sigma}} + \frac{d_{L,\sigma}}{d_{K,\sigma}} \right) \leq \theta, \quad (4.2)$$

and, for all $K \in \mathcal{M}_m$ and $\gamma \in \mathbb{R}^{\mathcal{E}_K}$,

$$\frac{1}{\theta} \sum_{\sigma \in \mathcal{E}_K} |D_{K,\sigma}| \left| \frac{R_{K,\sigma}(\gamma)}{d_{K,\sigma}} \right|^2 \leq \sum_{\sigma \in \mathcal{E}_K} |D_{K,\sigma}| \left| \frac{(A_{K} R_{K}(\gamma))_{\sigma}}{d_{K,\sigma}} \right|^2 \leq \theta \sum_{\sigma \in \mathcal{E}_K} |D_{K,\sigma}| \left| \frac{R_{K,\sigma}(\gamma)}{d_{K,\sigma}} \right|^2. \quad (4.3)$$

Under the above boundedness assumptions on the mesh regularity parameter $\theta_{\mathcal{M}}$, the limit-conformity and the compactness properties defined in Sect. 3 directly follow from the results in [19,Theorem 13.18]. It remains to prove the consistency. Proving that, for all $\tilde{\varphi} \in H^1(\Omega)$, $\lim_{m \to \infty} S_{D_m}(\tilde{\varphi}) = 0$ is as in the case of the HMM method for PDEs, see [19,Theorem 4.14]. In addition, with the same manner, we can show that, for all $\tilde{\psi} \in L^2(\Omega)$, $S_{D_m}(\tilde{\psi}) \to 0$, as $m \to \infty$. Let $\varphi_m = ((\varphi_K)_{K \in \mathcal{M}}, (\varphi_{\sigma})_{\sigma \in \mathcal{E}}) \in X_{D_m}$ and $\psi_m = (\psi_K)_{K \in \mathcal{M}} \in Y_{D_m}$ be the two interpolants such that $\varphi_m = J_{D_m} u_{\text{ini}}$ and $\psi_m = J_{D_m} v_{\text{ini}}$. Using [19,Estimate (B.11), in Lemma B.6] with $p = 2$, we can shows that $\|u_{\text{ini}} - \Pi_{D_m} J_{D_m} u_{\text{ini}}\|_{L^2(\Omega)} \to 0$ and $\|v_{\text{ini}} - \Pi_{D_m} J_{D_m} v_{\text{ini}}\|_{L^2(\Omega)} \to 0$, as $m \to \infty$, which completes the consistency property. The convergence of the HMM scheme for the reaction–diffusion model is a consequence of Theorem 3.6 if the discrete time steps $\delta t^{(n+\frac{1}{2})}$ tends to 0, as $m \to \infty$.

5 Proof of the convergence results

In order to prove the convergence results, we first establish some preliminary estimates on the solution and its gradient of the scheme.
Lemma 5.1 (Estimates) Under Assumptions 2.1, let $\mathcal{D}$ be a gradient discretisation in the sense of Definition 3.1 and let $(u, v) \in X_{\mathcal{D}} \times Y_{\mathcal{D}}$ be a solution of the gradient scheme (3.5). Then there exists a constant $C_2 \geq 0$ depending only on $L_f, L_g, C_0 \geq \max(f(0), g(0))$, $C_{\text{int}} > \max(\|\Pi_{\mathcal{D}} u(0)\|_{L^2(\Omega)}, \|\Pi_{\mathcal{D}'} v(0)\|_{L^2(\Omega)})$, and $T$, such that

$$\|\Pi_{\mathcal{D}} u\|_{L^\infty(0,T;L^2(\Omega))} + \|\Pi_{\mathcal{D}'} v\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla_{\mathcal{D}} u\|_{L^2(\Omega \times (0,T)^d)} \leq C_2. \quad (5.1)$$

Proof Setting, as test functions in Scheme (3.5), the functions $\varphi := \delta_{t}^{(n+\frac{1}{2})} u^{(n+1)}$ and $\psi := \delta_{t}^{(n+\frac{1}{2})} v^{(n+1)}$ leads to

$$\int_{\Omega} \left( \Pi_{\mathcal{D}} u^{(n+1)}(x) - \Pi_{\mathcal{D}} u^{(n)}(x) \right) \Pi_{\mathcal{D}} u^{(n+1)}(x) \, dx + \mu \int_{t^{(n)}}^{t^{(n+1)}} \int_{\Omega} |\nabla_{\mathcal{D}} u^{(n+1)}(x)|^2 \, dx \, dt,$$

and

$$\int_{\Omega} \left( \Pi_{\mathcal{D}'} v^{(n+1)}(x) - \Pi_{\mathcal{D}'} v^{(n)}(x) \right) \Pi_{\mathcal{D}'} v^{(n+1)}(x) \, dx$$

$$= \int_{t^{(n)}}^{t^{(n+1)}} \int_{\Omega} g_{\mathcal{D}}^{(n+1)} \Pi_{\mathcal{D}'} v^{(n+1)}(x) \, dx \, dt.$$

Apply the inequality $(y - z)^2 \geq \frac{1}{2}(|y|^2 - |z|^2)$ to $y = \Pi_{\mathcal{D}} u^{(n+1)}$ and $z = \Pi_{\mathcal{D}} u^{(n)}$ (resp. $y = \Pi_{\mathcal{D}'} v^{(n+1)}$ and $z = \Pi_{\mathcal{D}'} v^{(n)}$) to obtain

$$\frac{1}{2} \int_{\Omega} \left[ |\Pi_{\mathcal{D}} u^{(n+1)}(x)|^2 - |\Pi_{\mathcal{D}} u^{(n)}(x)|^2 \right] \, dx + \mu \int_{t^{(n)}}^{t^{(n+1)}} \int_{\Omega} |\nabla_{\mathcal{D}} u^{(n+1)}(x)|^2 \, dx \leq \int_{t^{(n)}}^{t^{(n+1)}} \int_{\Omega} \Pi_{\mathcal{D}'} v^{(n+1)}(x) \, dx \, dt,$$

and

$$\frac{1}{2} \int_{\Omega} \left[ |\Pi_{\mathcal{D}'} v^{(n+1)}(x)|^2 - |\Pi_{\mathcal{D}'} v^{(n)}(x)|^2 \right] \, dx \leq \int_{t^{(n)}}^{t^{(n+1)}} \int_{\Omega} \Pi_{\mathcal{D}} u^{(n+1)}(x) \, dx \, dt.$$

Sum on $n = 0,\ldots, m - 1$, for some $m = 0,\ldots, N$:

$$\frac{1}{2} \int_{\Omega} \left[ |\Pi_{\mathcal{D}} u^{(m)}(x)|^2 pm - |\Pi_{\mathcal{D}} u^{(0)}(x)|^2 \right] \, dx + \mu \int_{0}^{t^{(m)}} \int_{\Omega} |\nabla_{\mathcal{D}} u(x)|^2 \, dx \leq \sum_{n=0}^{m-1} \delta_{t}^{(n+\frac{1}{2})} \int_{\Omega} \Pi_{\mathcal{D}} u^{(n+1)}(x) \, dx, \quad (5.2)$$

and

$$\frac{1}{2} \int_{\Omega} \left[ |\Pi_{\mathcal{D}'} v^{(m)}(x)|^2 - |\Pi_{\mathcal{D}'} v^{(0)}(x)|^2 \right] \, dx \leq \sum_{n=0}^{m-1} \delta_{t}^{(n+\frac{1}{2})} \int_{\Omega} g_{\mathcal{D}}^{(n+1)} \Pi_{\mathcal{D}'} v^{(n+1)}(x) \, dx. \quad (5.3)$$

Applying the Cauchy–Schwarz inequality to the right-hand side of both inequalities leads to

$$\frac{1}{2} \int_{\Omega} \left[ |\Pi_{\mathcal{D}} u^{(m)}(x)|^2 - |\Pi_{\mathcal{D}} u^{(0)}(x)|^2 \right] \, dx + \mu \int_{0}^{t^{(m)}} \int_{\Omega} |\nabla_{\mathcal{D}} u(x)|^2 \, dx \, dt.$$
\[
\leq \sum_{n=0}^{m-1} \delta t^{(n+\frac{1}{2})} \| f^{(n+1)} \|_{L^2(\Omega)} \| \Pi Du^{(n+1)} \|_{L^2(\Omega)},
\]

and
\[
\frac{1}{2} \int_{\Omega} \left[ |\Pi D' v^{(m)}(x)|^2 - |\Pi D' v^{(0)}(x)|^2 \right] dx \leq \| g^{(m)} \|_{L^2(\Omega)} \| \Pi D' v^{(m)} \|_{L^2(\Omega)}.
\]

Thanks to the Lipschitz continuous assumptions on \( f \) and \( g \), one writes
\[
\frac{1}{2} \int_{\Omega} \left[ |\Pi D' u^{(m)}(x)|^2 - |\Pi D' u^{(0)}(x)|^2 \right] dx + \mu \int_{0}^{T} \int_{\Omega} |\nabla D'(x, t)|^2 dx dt
\leq \sum_{n=0}^{m-1} \delta t^{(n+\frac{1}{2})} \left[ C_0 \| \Pi D' u^{(n+1)} \|_{L^2(\Omega)} + L_f \| \Pi Du^{(n+1)} \|_{L^2(\Omega)}^2 
+ L_g \| \Pi D' u^{(n+1)} \|_{L^2(\Omega)} \| \Pi D' v^{(n+1)} \|_{L^2(\Omega)} \right],
\]

and
\[
\frac{1}{2} \int_{\Omega} \left[ |\Pi D' v^{(m)}(x)|^2 - |\Pi D' v^{(0)}(x)|^2 \right] dx
\leq \sum_{n=0}^{m-1} \delta t^{(n+\frac{1}{2})} \left[ C_0 \| \Pi D' v^{(n)} \|_{L^2(\Omega \times (0, T))} + L_f \| \Pi Du^{(n+1)} \|_{L^2(\Omega)}^2 
+ L_g \| \Pi D' v^{(n+1)} \|_{L^2(\Omega)} \| \Pi D' u^{(n+1)} \|_{L^2(\Omega)} \right].
\]

Apply the Young’s inequality, with \( \epsilon \) satisfying \( M_1 + \frac{1}{2\epsilon} - \frac{1}{2} > 0 \) and \( M_2 + \frac{1}{2\epsilon} - \frac{1}{2} > 0 \), to the right-hand side of both inequalities to obtain
\[
\frac{1}{2} \int_{\Omega} \left[ |\Pi D' u^{(m)}(x)|^2 - |\Pi D' u^{(0)}(x)|^2 \right] dx + \mu \int_{0}^{T} \int_{\Omega} |\nabla D'(x, t)|^2 dx dt
\leq \sum_{n=0}^{m-1} \delta t^{(n+\frac{1}{2})} \left[ C_0^2 \frac{1}{2\epsilon} + M_1 \| \Pi Du^{(n+1)} \|_{L^2(\Omega)}^2 + \frac{1}{2\epsilon} \| \Pi D' v^{(n+1)} \|_{L^2(\Omega)}^2 \right],
\]

and
\[
\frac{1}{2} \int_{\Omega} \left[ |\Pi D' v^{(m)}(x)|^2 - |\Pi D' v^{(0)}(x)|^2 \right] dx
\leq \sum_{n=0}^{m-1} \delta t^{(n+\frac{1}{2})} \left[ C_0^2 \frac{1}{2\epsilon} + M_2 \| \Pi D' v^{(n+1)} \|_{L^2(\Omega)}^2 + \frac{1}{2\epsilon} \| \Pi D' u^{(n+1)} \|_{L^2(\Omega)}^2 \right],
\]

where \( M_1 := \frac{1}{2} + L_f + \frac{L_2^2}{2} \) and \( M_2 := \frac{1}{2} + L_g + \frac{1}{2\epsilon} \). Now, we can apply the Gronwall inequality [32, Lemma 5.1] to deduce
\[
\frac{1}{2} \int_{\Omega} \left| \Pi D' u^{(m)}(x) \right|^2 dx + \mu \int_{0}^{T} \int_{\Omega} \left| \nabla D'(x, t) \right|^2 dx dt
\leq \left[ \frac{TC^2_0}{2\epsilon} + \frac{1}{2\epsilon} \sum_{n=0}^{m-1} \delta t^{(n+\frac{1}{2})} \| \Pi D' v^{(n+1)} \|_{L^2(\Omega)}^2 + \| \Pi D' u^{(0)}(x) \|_{L^2(\Omega)}^2 \right] \exp \left( \frac{T}{\epsilon} \right),
\]

and
\[
\frac{1}{2} \int_{\Omega} \left| \Pi D' v^{(m)}(x) \right|^2 dx,
\]
Gather the above inequalities together and take the supremum on \( m = 0, \ldots, N \) to establish Estimate (5.1), since the right-hand side can be simplified with terms on the left-hand side in both relations and \( \sup_{m=0,\ldots,N} \int_{\Omega} |\Pi_D U^{(m)}|^2 \, dx = \| \Pi_D U^{(m)} \|^2_{L^2(0,T;L^2(\Omega))} \). (resp. \( \sup_{m=0,\ldots,N} \int_{\Omega} |\Pi_{D'} U^{(m)}|^2 \, dx = \| \Pi_{D'} U^{(m)} \|^2_{L^2(0,T;L^2(\Omega))} \)).

**Corollary 5.2** Assume Assumptions 2.1 and let \( D \) be a gradient discretisation. Then there exists at least one solution \((u, v)\) to the gradient scheme (3.5).

**Proof** At each time step \( n + 1 \), (3.5) provides square nonlinear equations on \( u^{(n+1)} \) and \( v^{(n+1)} \). For a given \( w = (w_1, w_2) \in X_D \times Y_D \), \((u, v) \in X_D \times Y_D \) is the solution to

\[
\begin{align*}
\int_{\Omega} \Pi_D \frac{u^{(n+1)} - u^{(n)}}{\delta t}\varphi(x) + \mu \int_{\Omega} \nabla_D u^{(n+1)}(x) \cdot \nabla_D \varphi(x) \, dx \\
= \int_{\Omega} f(\Pi_D w_1, \Pi_{D'} w_2) \varphi(x) \, dx, \quad \forall \varphi \in X_D, \\
\int_{\Omega} \Pi_{D'} \frac{v^{(n+1)} - v^{(n)}}{\delta t}\psi(x) = \int_{\Omega} g(\Pi_D w_1, \Pi_{D'} w_2) \psi(x) \, dx, \quad \forall \psi \in Y_D.
\end{align*}
\]  

This problem describes a linear square system, whose right-hand side is built from the terms \( \int_{\Omega} f(\Pi_D w_1, \Pi_{D'} w_2) \Pi_D \varphi(x) \, dx, \int_{\Omega} g(\Pi_D w_1, \Pi_{D'} w_2) \Pi_{D'} \psi(x) \, dx, \) and \( \int_{\Omega} \Pi_{D'} v^{(n)} \Pi_{D'} \psi(x) \, dx \).

Using arguments similar to the proof of Lemma 5.1, we can obtain

\[
\begin{align*}
\| \Pi_D u^{(n+1)} \|^2_{L^2(\Omega)} + \| \nabla_D u^{(n+1)} \|^2_{L^2(\Omega)} & \leq C_3 \| f \|^2_{L^2(\Omega)} + \| \Pi_D u^{(n)} \|^2_{L^2(\Omega)}, \\
\| \Pi_{D'} v^{(n+1)} \|^2_{L^2(\Omega)} & \leq C_4 \| g \|^2_{L^2(\Omega)} + \| \Pi_{D'} v^{(n)} \|^2_{L^2(\Omega)},
\end{align*}
\]

where \( C_3 \) and \( C_4 \) are independent of \( u^{(n+1)} \) or \( v^{(n+1)} \). Hence, the kernel of the matrix associated with the above linear square system is reduced to \( \{0\} \), and the matrix is invertible. We then can define the mapping \( T : X_D \times Y_D \to X_D \times Y_D \) by \( T(w) = (u, v) \) with \((u, v) \) is the solution to (5.4). Since \( T \) is continuous, Brouwer’s fixed point establishes the existence of a solution \( u^{(n+1)} \) to the system at time step \( n + 1 \).

We require to establish estimates on the discrete–time derivatives. To do so, we define dual norms on \( \Pi_D(X_D) \subset L^2(\Omega) \) and \( \Pi_{D'}(Y_D) \subset L^2(\Omega) \) as in the following definition.

**Definition 5.3** (Dual norms on \( \Pi_D(X_D) \) and \( \Pi_{D'}(Y_D) \)) Let \( D \) be gradient discretisations. The dual norms \( \| \cdot \|_{*,D} \) on \( \Pi_D(X_D) \) and \( \| \cdot \|_{*,D'} \) on \( \Pi_{D'}(Y_D) \) are given by

\[
\forall w_1 \in \Pi_D(X_D), \\
\| w_1 \|_{*,D} = \sup \left\{ \int_{\Omega} w_1(\varphi) \Pi_D \varphi(x) \, dx : \varphi \in X_D, \| \varphi \|_D = 1 \right\},
\]

and

\[
\forall w_2 \in \Pi_{D'}(Y_D), \\
\| w_2 \|_{*,D'} = \sup \left\{ \int_{\Omega} w_2(\psi) \Pi_{D'} \psi(x) \, dx : \psi \in Y_D, \| \Pi_{D'} \psi \|_{L^2(\Omega)} = 1 \right\},
\]

where the discrete norm \( \| \cdot \|_D \) is defined by (3.1).
Lemma 5.4 (Estimate on the dual norms of $\delta_D u$ and $\delta_D' v$) Assume Assumptions 2.1 holds and let $D$ be gradient discretisations. If $(u, v) \in X_D \times Y_D$ is a solution to the gradient scheme (3.5), then there exists a constant $C_5$ depending only on $C_0 = \max\{f(0), g(0)\}, C_2, L_f, L_g, \Omega$, and $T$, such that
\begin{equation}
\int_0^T \|\delta_D u(t)\|^2_{*,D} \, dt \leq \int_0^T \|\delta_D' v(t)\|^2_{*,D'} \, dt \leq C_5,
\end{equation}
where the dual norms $\| \cdot \|_{*,D}$ and $\| \cdot \|_{*,D'}$ are defined by (5.5) and (5.6).

Proof. Take $\varphi \in X_D$ and $\psi \in Y_D$ as generic test functions in (3.5). Using the Cauchy–Schwarz inequality, we obtain, thanks to assumptions (Assumptions 2.1)
\begin{equation*}
\int_\Omega \delta^{(n+\frac{1}{2})}_D u(x) \Pi_D \varphi(x) \, dx \leq \|\varphi\|_D \left( \mu \|\nabla D u^{(n+1)}\|_{L^2(\Omega \times (0,T))^d} + L_f \|\Pi_D u^{(n+1)}\|_{L^2(\Omega \times (0,T))} \right) + L_f \|\Pi_D' v^{(n+1)}\|_{L^2(\Omega \times (0,T))} + C_0, \right),
\end{equation*}
and
\begin{equation*}
\int_\Omega \delta^{(n+\frac{1}{2})}_D' v(x) \Pi_D' \psi(x) \, dx \leq \|\Pi_D' \psi\|_{L^2(\Omega)} \left( L_g \|\Pi_D u^{(n+1)}\|_{L^2(\Omega \times (0,T))} + L_g \|\Pi_D' v^{(n+1)}\|_{L^2(\Omega \times (0,T))} + C_0 \right).
\end{equation*}
The proof is concluded by taking the supremum over $\varphi \in X_D$ and $\psi \in Y_D$ with $\|\varphi\|_D = \|\Pi_D' \psi\|_{L^2(\Omega)} = 1$, multiplying by $\delta t^{(n+1)}$, summing over $n = 0, \ldots, N - 1$, and using (5.1) to estimate the terms $\|\nabla D u\|_{L^2(\Omega \times (0,T))^d}$, $\|\Pi_D u\|_{L^2(\Omega \times (0,T))}$ and $\|\Pi_D' v\|_{L^2(\Omega \times (0,T))}$. \hfill $\square$

Proof of Theorem 3.6

The proof follows the compactness technique detailed in [17] and it is split into four steps:

Step 1: Compactness results. Due to Estimate (5.1) and to the consistency and the limit-conformity of GD, [19, Lemma 4.8] provides $\bar{u} \in L^2(0, T; H^1(\Omega))$ and $\bar{v} \in L^2(0, T; L^2(\Omega))$, such that, up to a subsequence, $\Pi_D u_m \rightharpoonup \bar{u}$ weakly in $L^2(0, T, L^2(\Omega))$, $\Pi_D' v_m \rightharpoonup \bar{v}$ weakly in $L^2(0, T, L^2(\Omega))$, and $\nabla D u_m \rightharpoonup \nabla \bar{u}$ weakly in $L^2(0, T, L^2(\Omega))$, as $m \to \infty$. Estimate (5.7) with the three properties (consistency, limit-conformity, and compactness) show that the assumptions of [19, Theorem 4.14] are satisfied. This theorem proves that $\Pi_D u_m$ converges strongly to $\bar{u}$ in $L^2(0, T; L^2(\Omega))$, as $m \to \infty$. Estimate (5.1) yields the weak convergence of $\Pi_D' v_m$ to $\bar{v}$ in $L^\infty(0, T; L^2(\Omega))$, as $m \to \infty$.

Let us now prove that $\Pi_D' v_m$ is relatively compact in $L^2(0, T; L^2(\Omega))$. From (3.5b), we note the relation
\begin{equation}
\Pi_D' v_m^{(k+1)} = \Pi_D' J_{D'} v_m^{(k+1)} + \sum_{k=0}^n \delta t^{(k+\frac{1}{2})} g(\Pi_D u^{(k+1)}, \Pi_D' v^{(k+1)}).
\end{equation}
Let both functions $\Pi_D u$ and $\Pi_D' v$ be extended by 0 outside $\Omega$. With this relation, the difference between $\Pi_D' v$ and its space translates can be introduced as, for all $\xi \in \mathbb{R}^d$,
\begin{equation*}
\Pi_D' v^{(n+1)}(x + \xi) - \Pi_D' v^{(n+1)}(x)
\end{equation*}
\begin{align*}
&= \Pi_D' J_{D'} v_m^{(n+1)}(x + \xi) - \Pi_D' J_{D'} v_m^{(n+1)}(x) \\
&+ \sum_{k=0}^n \delta t^{(k+\frac{1}{2})} \left( g(\Pi_D u^{(k+1)}(x + \xi), \Pi_D' v^{(k+1)}(x + \xi)) \\
&- g(\Pi_D u^{(k+1)}(x), \Pi_D' v^{(k+1)}(x)) \right).
\end{align*}
Thanks to the Lipschitz continuity assumption on $g$, this yields, for all $\xi \in \mathbb{R}^d$,
\begin{equation*}
|\Pi_D' v^{(n+1)}(x + \xi) - \Pi_D' v^{(n+1)}(x)|
\end{equation*}
In addition, since \( \Pi_{D_m}^n J_{D_m} v_{\text{ini}}(x + \xi) - \Pi_{D_m}^n J_{D_m}^* v_{\text{ini}}(x) \)

\[ + L \sum_{k=0}^n \delta t^{k+\frac{1}{2}} \| \Pi_{D_m}^n u^{k+1}(x + \xi) - \Pi_{D_m}^n u^{k+1}(x) \| \]

\[ + L \sum_{k=0}^n \delta t^{k+\frac{1}{2}} \| \Pi_{D_m}^n v^{k+1}(x + \xi) - \Pi_{D_m}^n v^{k+1}(x) \|. \quad (5.9) \]

Defining \( T_D : \mathbb{R}^d \to \mathbb{R}^+ \) by

\[ T_D(\xi) = \max_{\psi \in \mathcal{X}_D} \frac{\| \Pi \mathcal{D} \psi(\cdot + \xi) - \Pi \mathcal{D} \psi \|_{L^2(\Omega)}}{\| \nabla \mathcal{D} \psi \|_{L^2(\Omega)^d}}, \quad \text{for all } \xi \in \mathbb{R}^d, \]

it follows that, for all \( k = 0, \ldots, n, \)

\[ \| \Pi_{D_m} u^{k+1}(\cdot + \xi) - \Pi_{D_m} u^{k+1}(\cdot) \|_{L^2(\Omega)} \leq T_D(\xi) \| \Pi_{D_m} u^{k+1}(\cdot + \xi) - \Pi_{D_m} u^{k+1}(\cdot) \|_{L^2(\Omega)^d}. \]

Plug this inequality into (5.9) to attain, for all \( \xi \in \mathbb{R}^d, \)

\[ \| \Pi_{D_m} v^{n+1}(\cdot + \xi) - \Pi_{D_m}^n v^{n+1}(\cdot) \|_{L^2(\Omega)} \leq \| \Pi_{D_m} v^{n+1}(\cdot + \xi) - \Pi_{D_m}^n v^{n+1}(\cdot) \|_{L^2(\Omega)} \]

\[ + L \sum_{k=0}^n \delta t^{k+\frac{1}{2}} T_D(\xi) \| \nabla \mathcal{D} u^{k+1}(\cdot) \|_{L^2(\Omega)^d} \]

\[ + L \sum_{k=0}^n \delta t^{k+\frac{1}{2}} \| \Pi_{D_m}^n v^{k+1}(\cdot + \xi) - \Pi_{D_m} v^{k+1}(\cdot + \xi) \|_{L^2(\Omega)}. \]

Apply the Gronwall inequality [32, Lemma 5.1] to deduce

\[ \| \Pi_{D_m} v^n(\cdot + \xi) - \Pi_{D_m} v^n(\cdot) \|_{L^2(\Omega)} \]

\[ \leq \left( T_D(\xi) \| \nabla \mathcal{D} u(0,T;L^2(\Omega)^d) \| \exp \left( \frac{L_T T_D(\xi)}{1 - T_D(\xi)} \right) \right). \]

Since the sequence \((D_m)_{m \in \mathbb{N}}\) is compact, we have \( T_D(\xi) \to 0, \) as \( |\xi| \to 0, \) thanks to [19, Lemma 2.22]. In addition, since \( \Pi_{D_m}^n J_{D_m} v_{\text{ini}} \to v_{\text{ini}} \) in \( L^2(\Omega), \) as \( m \to \infty, \) then \( \lim_{|\xi| \to 0} \| \Pi_{D_m}^n J_{D_m} v_{\text{ini}}(\cdot + \xi) - \Pi_{D_m}^n J_{D_m} v_{\text{ini}}(\cdot) \|_{L^2(\Omega)} = 0. \) Therefore, the above inequality gives thanks to the estimate (5.1),

\[ \lim_{|\xi| \to 0} \| \Pi_{D_m} v(\cdot + \xi) - \Pi_{D_m} v(\cdot) \|_{L^2(0,T;L^2(\Omega))} = 0. \]

Now, we can use (5.8) and the Lipschitz continuity assumption on \( g \) to get the following bounded estimate, thanks to (5.1), for \( h > 0, \)

\[ \| \Pi_{D_m} v(t + h) - \Pi_{D_m} v(t) \|_{L^2(0,T;L^2(\Omega))} \]

\[ \leq h \left( L_\xi \| \Pi_{D_m} u \|_{L^2(0,T;L^2(\Omega))} + L \| \Pi_{D_m} v \|_{L^2(0,T;L^2(\Omega))} + |g(0,0)| \right), \]

which shows that

\[ \lim_{|h| \to 0} \| \Pi_{D_m} v(\cdot + h) - \Pi_{D_m} v(\cdot) \|_{L^2(0,T;L^2(\Omega))} = 0. \]

By the Kolmogorov theorem, we conclude that the family \((\Pi_{D_m} v_m)_{m \in \mathbb{N}}\) is strongly compact in \( L^2(0,T;L^2(\Omega)) \).
Step 2: \((\bar{u}, \bar{v})\) is a solution to the continuous problem. Take \(\tilde{\varphi} \in L^2(0, T; H^1(\Omega))\) and \(\tilde{\psi} \in L^2(0, T; L^2(\Omega))\) such that \(\partial_t \tilde{\psi}, \partial_t \tilde{\varphi} \in L^2(\Omega \times (0, T))\) and \(\tilde{\psi}(T, \cdot) = \tilde{\varphi}(T, \cdot) = 0\). By [19, Lemma 4.10] (with a slight change), we can find \(\phi_m = (\phi_m^{(n)})_{n=0, \ldots, N_m} \in X^{N_m+1}_m\) and \(w_m = (w_m^{(n)})_{n=0, \ldots, N_m} \in Y^{N_m+1}_m\), such that

\[
\begin{align*}
\Pi_{D_m} \phi_m &\to \tilde{\varphi}, \quad \Pi_{D'_m} w_m \to \tilde{\psi} \text{ strongly in } L^2(0, T; L^2(\Omega)), \\
\nabla_{D_m} \phi_m &\to \nabla \tilde{\varphi} \text{ strongly in } L^2(0, T; L^2(\Omega)^d), \text{ and,} \\
\delta_{D_m} \phi_m &\to \partial_t \tilde{\varphi}, \quad \text{and} \quad \delta_{D'_m} w_m \to \partial_t \tilde{\psi} \text{ strongly in } L^2(\Omega \times (0, T)).
\end{align*}
\]

Set \(\varphi = \delta_{t_m}^{(n+\frac{1}{2})} \phi_m^{(n)}\) and \(\psi = \delta_{t_m}^{(n+\frac{1}{2})} w_m^{(n)}\) as test functions in (3.5) and sum on \(n = 0, \ldots, N_m - 1\) to obtain

\[
\begin{align*}
\sum_{n=0}^{N_m-1} \int_{0}^{T} \left( \Pi_{D_m} u_m^{(n+1)}(x) - \Pi_{D_m} u_m^{(n)}(x) \right) \Pi_{D_m} \phi_m^{(n)}(x) \, dx \\
+ \mu \int_{0}^{T} \int_{\Omega} \nabla_{D_m} u_m(x) \cdot \nabla_{D_m} \phi_m^{(n)}(x) \, dx \, dt \\
= \int_{0}^{T} \int_{\Omega} f_m^{(n+1)} \Pi_{D_m} \phi_m^{(n)}(x) \, dx \, dt, \quad \text{and} \\
\sum_{n=0}^{N_m-1} \int_{0}^{T} \left( \Pi_{D'_m} v_m^{(n+1)}(x) - \Pi_{D'_m} v_m^{(n)}(x) \right) \Pi_{D'_m} w_m^{(n)}(x) \, dx \\
= \int_{0}^{T} \int_{\Omega} g_m^{(n+1)} \Pi_{D'_m} w_m^{(n)}(x) \, dx \, dt.
\end{align*}
\]  

(5.10)  

For two families of real numbers \((a_n)_{n=0, \ldots, N}\) and \((b_n)_{n=0, \ldots, N}\), the form of discrete integration by part [19, Eq. (D.15)] is

\[
\sum_{n=0}^{N-1} (b_{n+1} - b_n)a_n = b_N a_N - b_0 a_0 - \sum_{n=0}^{N-1} b_{n+1}(a_{n+1} - a_n).
\]

Apply this relation to the right-hand sides in (5.10) and (5.11), and use the fact \(\phi^{(N)} = w^{(N)} = 0\) to deduce

\[
\begin{align*}
- \int_{0}^{T} \int_{\Omega} \Pi_{D_m} u_m(x, t) \delta_{D_m} \phi_m(x, t) \, dx \, dt \\
- \int_{0}^{T} \int_{\Omega} \Pi_{D_m} u_m^{(0)}(x) \Pi_{D_m} \phi_m^{(0)}(x) \, dx \\
+ \mu \int_{0}^{T} \int_{\Omega} \nabla_{D_m} u_m(x) \cdot \nabla_{D_m} \phi_m^{(n)}(x) \, dx \, dt \\
= \int_{0}^{T} \int_{\Omega} f_m^{(n+1)} \Pi_{D_m} \phi_m^{(n)}(x) \, dx \, dt, \quad \text{and} \\
- \int_{0}^{T} \int_{\Omega} \Pi_{D'_m} v_m^{(0)}(x) \Pi_{D'_m} w_m^{(0)}(x) \, dx \\
- \int_{0}^{T} \int_{\Omega} g_m^{(n+1)} \Pi_{D'_m} w_m^{(n)}(x) \, dx \, dt \\
= \int_{0}^{T} \int_{\Omega} \Pi_{D'_m} v_m^{(n+1)}(x) \Pi_{D'_m} w_m^{(n)}(x) \, dx \, dt.
\end{align*}
\]

By the space-time consistency, \(\Pi_{D_m} u_m^{(0)} = \Pi_{D_m} J_{D_m} u_{ini} \to u_{ini} \) in \(L^2(\Omega)\) and \(\Pi_{D'_m} v_m^{(0)} = \Pi_{D'_m} J_{D'_m} v_{ini} \to v_{ini} \) in \(L^2(\Omega)\). The strong convergence of \(\Pi_{D_m} u_m\) and \(\Pi_{D'_m} v_m\) and the assumptions on \(f\) and \(g\) lead to \(f(\Pi_{D_m} u_m, \Pi_{D'_m} v_m) \to f(\bar{u}, \bar{v})\) in \(L^2(\Omega \times (0, T))\) and \(g(\Pi_{D_m} u_m, \Pi_{D'_m} v_m) \to g(\bar{u}, \bar{v})\) in \(L^2(\Omega \times (0, T))\). Hence, passing to the limit \(m \to \infty\) in the above both equations implies that the pair \((\bar{u}, \bar{v})\) satisfies the continuous problem (2.6).
Step 3: Strong convergence of $\Pi_{D_m} u_m$ and $\Pi_{D'_m} v_m$ in $L^\infty(0, T; L^2(\Omega))$. Let $s \in [0, T]$ and $(s_m)_{m \in \mathbb{N}}$ be a sequence in $[0, T]$, such that $s_m \to s$, as $m \to \infty$. Let $k(m) \in \{0, \ldots, N_m - 1\}$ such that $s_m \in (t^{(k(m))}, t^{(k(m)+1)})$. It is obvious to have as the discrete estimates (5.2) and (5.3) with $D_m$ and $k_m$,

$$
\frac{1}{2} \int_\Omega (\Pi_{D_m} u(x, s_m))^2 \, dx
\leq \frac{1}{2} \int_\Omega (\Pi_{D_m} J_{D_m} u_{\text{ini}}(x))^2 \, dx - \mu \int_0^{t^{(k(m))}} \int_\Omega (\nabla_{D_m} u(x, t))^2 \, dx \, dt
+ \int_0^{t^{(k(m))}} \int_\Omega f(\Pi_{D_m} u(x, t), \Pi_{D'_m} v(x, t)) \Pi_{D_m} u(x, t) \, dx \, dt,
$$

(5.12)

and

$$
\frac{1}{2} \int_\Omega (\Pi_{D'_m} v(x, s_m))^2 \, dx
\leq \int_0^{t^{(k(m))}} \int_\Omega g(\Pi_{D_m} u(x, t), \Pi_{D'_m} v(x, t)) \Pi_{D'_m} v(x, t) \, dx \, dt
+ \frac{1}{2} \int_\Omega (\Pi_{D'_m} J_{D'_m} v_{\text{ini}}(x))^2 \, dx.
$$

(5.13)

Let $\chi_I$ be the characteristic function of $I$. Thus $\chi_{[0, t^{(k(m))})} \nabla \tilde{u}$ converges strongly to $\chi_{[0, s]} \nabla \tilde{u}$ in $L^2(\Omega \times (0, T))^d$, as $m \to \infty$. With the strong convergences of $\Pi_{D_m} u_m$ and $\Pi_{D'_m} v_m$ to $\tilde{u}$ and $\tilde{v}$, respectively in $L^2(\Omega \times (0, T))$, we have

$$
\mu \int_0^s \int_\Omega (\nabla \tilde{u}(x, t))^2 \, dx \, dt
= \mu \int_0^{t^{(k(m))}} \int_\Omega \chi_{[0, s]}(\nabla \tilde{u}(x, t))^2 \, dx \, dt
= \mu \lim_{m \to \infty} \int_0^T \int_\Omega \chi_{[0, t^{(k(m))})} \nabla \tilde{u}(x, t) \cdot \nabla_{D_m} u_m(x, t) \, dx \, dt
\leq \liminf_{m \to \infty} \left( \|\chi_{[0, t^{(k(m))})} \nabla \tilde{u}\|_{L^2(\Omega \times (0, T))^d} \cdot \mu \|\chi_{[0, t^{(k(m))})} \nabla_{D_m} u_m\|_{L^2(\Omega \times (0, T))^d} \right)
= \|\chi_{[0, s]} \nabla \tilde{u}\|_{L^2(\Omega \times (0, T))^d} \cdot \liminf_{m \to \infty} \mu \|\chi_{[0, t^{(k(m))})} \nabla_{D_m} u_m\|_{L^2(\Omega \times (0, T))^d}.
$$

Divide by $\|\chi_{[0, s]} \nabla \tilde{u}\|_{L^2(\Omega \times (0, T))^d}$ to get

$$
\int_0^s \int_\Omega (\nabla \tilde{u}(x, t))^2 \, dx \, dt \leq \liminf_{m \to \infty} \int_0^{t^{(k(m))}} \int_\Omega (\nabla_{D_m} u_m(x, t))^2 \, dx \, dt.
$$

(5.14)

Pass to limit superior in (5.12) and (5.13) to arrive at, thanks to (5.14):

$$
\limsup_{m \to \infty} \frac{1}{2} \int_\Omega (\Pi_{D_m} u_m(x, s_m))^2 \, dx \leq \frac{1}{2} \int_\Omega u_{\text{ini}}(x)^2 \, dx - \mu \int_0^s \int_\Omega (\nabla \tilde{u}(x, t))^2 \, dx \, dt
+ \int_0^s \int_\Omega f(\tilde{u}(x, t), \tilde{v}(x, t)) \tilde{u}(x, t) \, dx \, dt,
$$

(5.15)

and

$$
\limsup_{m \to \infty} \frac{1}{2} \int_\Omega (\Pi_{D'_m} v_m(x, s_m))^2 \, dx \leq \int_0^s \int_\Omega g(\tilde{u}(x, t), \tilde{v}(x, t)) \tilde{u}(x, t) \, dx \, dt
+ \frac{1}{2} \int_\Omega v_{\text{ini}}(x)^2 \, dx.
$$

(5.16)
Plugging \( \varphi = \bar{u} \chi_{[0,t]}(t) \) and \( \psi = \bar{v} \chi_{[0,s]}(t) \) in Problem (2.6) and integrating by part, one has

\[
\frac{1}{2} \int_\Omega (\bar{u}(x,s))^2 \, dx + \mu \int_0^s \int_\Omega (\nabla \bar{u}(x,t))^2 \, dx \, dt = \frac{1}{2} \int_\Omega u_{\text{ini}}(x)^2 \, dx + \int_0^s \int_\Omega (\bar{f}(x,t) + \nabla \bar{v}(x,t)) \bar{u}(x,t) \, dx \, dt,
\]

(5.17)

and

\[
\frac{1}{2} \int_\Omega (\bar{v}(x,s))^2 \, dx = \frac{1}{2} \int_\Omega v_{\text{ini}}(x)^2 \, dx + \int_0^s \int_\Omega (\bar{g}(\bar{u}(x,t), \bar{v}(x,t)) \bar{v}(x,t) \, dx \, dt.
\]

(5.18)

From (5.15) and (5.17), we have

\[
\limsup_{m \to \infty} \int_\Omega (\Pi_{D_m}u_m(x,s_m))^2 \, dx \leq \int_\Omega \bar{u}(x,s)^2 \, dx.
\]

(5.19)

In addition, (5.16) and (5.18) yield

\[
\limsup_{m \to \infty} \int_\Omega (\Pi_{D_m}v_m(x,s_m))^2 \, dx \leq \int_\Omega \bar{v}(x,s)^2 \, dx.
\]

(5.20)

Estimates (5.1) and (5.7) show that the assumptions of [19, Theorem 4.19] are verified. Then \( (\Pi_{D_m}u_m)_{m \in \mathbb{N}} \) converges to \( \bar{u} \) weakly in \( L^2(\Omega) \) uniformly in \([0, T]\), and \( (\Pi_{D_m}v_m)_{m \in \mathbb{N}} \) converges to \( \bar{v} \) weakly in \( L^2(\Omega) \) uniformly in \([0, T]\). Hence, \( \Pi_{D_m}u_m(\cdot, s_m) \) converges to \( \bar{u}(\cdot, s) \) weakly in \( L^2(\Omega) \), and \( \Pi_{D_m}v_m(\cdot, s_m) \) converges to \( \bar{v}(\cdot, s) \) weakly in \( L^2(\Omega) \), as \( m \to \infty \). Due to Estimates (5.19) and (5.20), these convergence occur in strong sense in \( L^2(\Omega) \). Applying [19, Lemma C.13] gives \( \sup_{t \in [0,T]} \| \Pi_{D_m}u_m(t) - \bar{u}(t) \|_{L^2(\Omega)} \to 0 \) and \( \sup_{t \in [0,T]} \| \Pi_{D_m}v_m(t) - \bar{v}(t) \|_{L^2(\Omega)} \to 0 \), owing to the continuity of \( \bar{u}, \bar{v} : [0, T] \to L^2(\Omega) \).

Step 4: Strong Convergence of \( \nabla \Pi_{D_m}u_m \). Taking \( \bar{\varphi} := u_m \) in (3.5a) and passing to the superior limit gives (choose \( \varphi = \bar{u} \) in (2.6a))

\[
\lim_{m \to \infty} \sup \int_0^T \int_\Omega \nabla \Pi_{D_m}u_m(x,t) \cdot \nabla \Pi_{D_m}u_m(x,t) \, dx \, dt = \int_0^T \int_\Omega f(\bar{u}, \bar{v})\bar{u}(x,t) \, dx \, dt - \int_0^T \int_\Omega \delta_t \bar{u}(x,t)\bar{u}(x,t) \, dx \, dt
\]

\[
= \int_0^T \int_\Omega \nabla \bar{u}(x,t) \cdot \nabla \bar{u}(x,t) \, dx \, dt.
\]

(5.21)

One can write

\[
\int_0^T \int_\Omega \left( \nabla \Pi_{D_m}u_m(x,t) - \nabla \bar{u}(x,t) \right) \cdot \left( \nabla \Pi_{D_m}u_m(x,t) - \nabla \bar{u}(x,t) \right) \, dx \, dt
\]

\[
= \int_0^T \int_\Omega \nabla \Pi_{D_m}u_m(x,t) \cdot \nabla \Pi_{D_m}u_m(x,t) \, dx \, dt - \int_0^T \int_\Omega \nabla \Pi_{D_m}u_m(x,t) \cdot \nabla \bar{u}(x,t) \, dx \, dt
\]

\[
- \int_0^T \int_\Omega \nabla \bar{u}(x,t) \cdot \left( \nabla \Pi_{D_m}u_m(x,t) - \nabla \bar{u}(x,t) \right) \, dx \, dt.
\]

Thanks to the weak convergence of \( \nabla \Pi_{D_m}u_m \) and (5.21), passing to the limit \( m \to \infty \) in each of the terms above completes the proof.
6 Numerical results

We examine here the validity of the HMM method in solving the reaction–diffusion models by exploring the dynamics of the spiral waves of the Barkley model. This model is considered an example of the reaction–diffusion model (2.1)–(2.5) with reaction terms taking the form [6]:

\[
\begin{align*}
    f(\bar{u}, \bar{v}) &= \frac{1}{\rho} \bar{u} (1 - \bar{u}) \left( \bar{u} - \frac{\bar{v} + b}{a} \right), \\
    g(\bar{u}, \bar{v}) &= \bar{u} - \bar{v},
\end{align*}
\]

(6.1)

for \(a, b > 0\). The small parameter \(0 < \rho \ll 1\) represents the time scale separation of fast variable \(\bar{u}\) and slow variable \(\bar{v}\). Before proceeding, we first review the dynamics of the underlying model to understand the mechanism of nucleation and propagation of spiral waves in excitable media.

6.1 Dynamics of the Barkley model

A pair of essential features are shared by all reaction–diffusion models: spatially localised excitation that undergoes diffusion in space and quickly reverts to a recovery state. Precisely, if the stimulus is strong enough, an excitable system will switch from a state of rest to a state of excitement, then quickly revert to a refractory state and back to rest. A new excitement cannot be created until a certain period (refractory time) has gone by. Therefore, an excitable system can support the wave propagation caused by strong disturbances when in a rest state driven by local nonlinearity coupled with diffusion [6, 10]. Figure 1 [4] shows the dynamics of the Barkley model or reaction kinetics when diffusion is not present, with nullclines pictures of \(\bar{u}\) and \(\bar{v}\).

The \(\bar{u}\)-nullclines \((f(\bar{u}, \bar{v}) = 0)\) are represented by three straight lines: \(\bar{u} = 0, \bar{u} = 1, \bar{u} = \frac{\bar{v} + b}{a}\), whereas \(\bar{v}\)-nullclines \((g(\bar{u}, \bar{v}) = 0)\) is the line \(\bar{u} = \bar{v}\), where \(a, b > 0\). Excitable media systems are characterised by dynamic states of excitation and recovery. Thus, by setting a small boundary, say \(\delta\), bordering the line \(\bar{u} = 0\), a given point \((\bar{u}, \bar{v})\) is said to be excited if \(\bar{u} > \delta\) and recovering otherwise.

The detail of local dynamics is given by the physical parameters \(\rho, a, b\). \(\rho\) is selected very small so that the dynamics of the activator \(\bar{u}\) is much faster than the inhibitor \(\bar{v}\) inside the region of excitement. However, \(\bar{u} \approx 0\) within the recovery region and, therefore, the exponential decay of the inhibitor \(\bar{v}\) affects only the local dynamics [6].

Increasing \(a\) would cause a lengthening of the excitation period, and increasing \(\frac{b}{a}\) would raise the excitation threshold [11]. The intersection of all nullclines yields the fixed points \((0, 0)\) and \((1, 1)\). The origin \((0, 0)\) is a stable and excitable fixed point of the model with excitation threshold \(\bar{u}_{th} = \frac{\bar{v} + b}{a}\).

To be precise, the initial conditions found on the left of the threshold \(\bar{u}_{th}\) near the \((0, 0)\) decay straight to the fixed point of origin. Conversely, the initial conditions found on the right of \(\bar{u}_{th}\) undergo a large excursion.
before returning to \((0,0)\); see [6] for more details. When spatial diffusion is added to the reaction kinetics, we gain spiral wave dynamics [11]. This allows us to examine how spiral waves are created, their movement, and how they collapse and disappear. This offers insights into the appearance, modification, and ways of stopping arrhythmias.

6.2 Spiral waves

The wavefronts and backs of spiral waves meet to form a tip that pivots around the core region. If the spiral wave’s core moves, it is considered drifting. In order to examine how spiral waves behave, employing the basis of the HMM method, we consider a pair of examples for the Barkley model at the edge of the simulation domains with no-flux boundary conditions. In both cases, the HMM scheme is tested on a square domain \(\Omega = [-L, L]^2\), divided into a uniform mesh of 3584 conforming triangular elements. Note that the mesh used here satisfies assumptions in Definition 4.1 since it is built on convex cells.

In all test cases, we consider the Barkley model defined in the introduction of this section (6.1). Letting \(x = (x_1, x_2)\), the initial conditions required to initiate the spiral waves are chosen in general form by

\[
\begin{align*}
\tilde{u}_0(x_1, x_2) &= (1 + \exp(4(|x_1| - \alpha_1)))^{-2} - (1 + \exp(4(|x_1| - \alpha_2)))^{-2}, \\
\tilde{v}_0(x_1, x_2) &= \alpha_3, 
\end{align*}
\]

where \(\alpha_1, \alpha_2\) and \(\alpha_3\) are real constants. For the time-discretisation, the forward Euler method is considered with time step \(\delta t > 0\).

**Example 1** For the first example, the parameters of the model are chosen according to [39] as \(\rho = 0.005\), \(a = 0.3\), \(b = 0.01\), and \(L = 30\). The initial conditions are given as

\[
\begin{align*}
\tilde{u}_0(x_1, x_2) &= \begin{cases} 
0, & \text{for } x_1 < 0 \text{ or } x_2 > 5, \\
\bar{u}_0, & \text{otherwise},
\end{cases} \\
\tilde{v}_0(x_1, x_2) &= \begin{cases} 
\bar{v}_0, & \text{for } x_1 < 1 \text{ and } x_2 < 10, \\
0, & \text{otherwise},
\end{cases}
\end{align*}
\]

where \(\bar{u}_0\) and \(\bar{v}_0\) are defined in Eq. (6.2), by setting \(\alpha_1 = 5\), \(\alpha_2 = 1\) and \(\alpha_3 = 0.1\).

Figure 2 shows the way that spiral waves were propagated in this instance, plotted at different times, with the simulation done up to \(t = 100\). It can be seen that the development of the spiral wave involves a core at the medium’s centre, and this develops into a periodically rotating single spiral wave. The trajectory of this spiral wave runs symmetrically from origin to boundary, moving along the length to turn into stationary spirals. The observed behaviour is called the reflection of the spiral at the boundary.

**Example 2** However, in certain instances, the spiral tip may collide with the boundary and then the spiral waves are annihilated instead of being reflected, especially if the initial displacement of the tip is close to the boundary as can be seen in the second example. In this case, we choose the parameters of the model as [39]: \(\rho = 0.0208\), \(a = 0.52\), \(b = 0.05\) and \(\mu = 1\) on the square domain, which is of the length \(L = 7.5\). The initial conditions are given as

\[
\begin{align*}
\tilde{u}_0(x_1, x_2) &= \begin{cases} 
0, & \text{for } x_1 < 0 \text{ or } x_2 > 5, \\
\bar{u}_0, & \text{otherwise},
\end{cases} \\
\tilde{v}_0(x_1, x_2) &= \begin{cases} 
\bar{v}_0, & \text{for } x_1 < -1 \text{ and } x_2 < 3, \\
0, & \text{otherwise},
\end{cases}
\end{align*}
\]

where \(\bar{u}_0\) and \(\bar{v}_0\) are as in Eq. (6.2), by putting \(\alpha_1 = 3\), \(\alpha_2 = 1\) and \(\alpha_3 = 0.25\). The spiral solution for the Barkley model under the considered parameters is shown in Fig. 3 at different time levels, from \(t = 0.2\) to \(t = 3\). In this instance, with a small domain, the spiral commences taking shape at \(t = 1\), although the rotation is not yet complete. Following this, each part of the spiral will drift over to the nearest boundary and be absorbed.
A dormant state will be achieved in both instances as when there are interactions between spiral waves and boundaries within excitable media, the trajectory of the wave is either reflected or destroyed by the interaction with the boundary, according to the parameters that have been chosen. We conclude that our numerical observations for the Barkley model obtained using HMM method are consistent with those observations in the previous studies [9, 39] in terms of spiral waves dynamics.

For the convergence test, we consider Example 2 with the reference solution computed on the fine mesh with $h = 0.02$. To assess the error between the reference solution and an approximate solution at time $t^{(n)}$, we use the following errors on $u$ and $v$

$$E_u := \| \bar{u}(\cdot, t^{(n)}) - \Pi \mathcal{D} u^{(n)} \|_{L^2(\Omega)}$$

and

$$E_v := \| \bar{v}(\cdot, t^{(n)}) - \Pi \mathcal{D} v^{(n)} \|_{L^2(\Omega)}.$$
Table 1 shows the errors on $u$ and $v$ and the corresponding convergence rates with respect to the mesh size. The rates of convergence in $L^2$-norm for the error are 1, which are compatible with the behaviour expectations associated with the lower-order methods, such as the HMM method.

Conclusion

We used the gradient discretisation method to design a complete numerical analysis for a system of reaction–diffusion equations. Based on natural assumptions and a limited number of properties, we established convergence results that apply to different numerical schemes. Our analysis allowed us to design an HMM method for the underlying model and study, through numerical tests, the propagation of the spiral waves for the Barkley model and their behaviour when they interact with the boundary of the medium where they spread.
Table 1 Example 2: errors and convergence rates w.r.t. the mesh size $h$ at time $t = 0.2$

| $h$     | $\|\bar{u}(\cdot, t^{(n)}) - \Pi_{Dh} u^{(n)}\|_{L^2(\Omega)}$ | Rate | $\|\bar{v}(\cdot, t^{(n)}) - \Pi_{Dh} v^{(n)}\|_{L^2(\Omega)}$ | Rate |
|---------|---------------------------------------------------------------|------|---------------------------------------------------------------|------|
| 0.93750 | 0.55004                                                      | -    | 0.05500                                                      | -    |
| 0.46875 | 0.24351                                                      | 1.17550 | 0.03751                                                      | 0.55210 |
| 0.23438 | 0.09090                                                      | 1.42180 | 0.01383                                                      | 1.43920 |

However, establishing the uniqueness of the continuous solution to the reaction–diffusion model considered in this paper and error estimates for numerical approximations of the model are still open questions and would be of interest to consider in our future work.

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