An exact solution for geophysical equatorial edge waves over a sloping beach

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Received 23 May 2012, in final form 6 July 2012
Published 20 August 2012
Online at stacks.iop.org/JPhysA/45/365501

Abstract
We present an explicit solution for the geophysical edge wave problem in the $f$-plane approximation. We also discuss in detail the properties of this solution and compare them with those of the solution found in the absence of Coriolis effects.

PACS numbers: 92.10.ad, 92.10.Ei, 92.10.Hm, 91.50.Cw
Mathematics Subject Classification: 76B15, 74G05, 35B36, 37N10
(Some figures may appear in colour only in the online journal)

1. Introduction

Edge waves are three-dimensional waves of permanent form, which can sometimes be observed close to a sloping beach. These waves travel parallel to the shoreline and they are called trapped waves, since their amplitude is maximal at the shoreline and decays rapidly offshore. Traveling along the beach, they trace nice sinusoidal run-up patterns in the longshore. The edge waves also play an important role in sediment transport in the nearshore.

The edge-wave problem was extensively studied and discussed in recent decades. The first reference to edge waves dates back to the 19th century, when Stokes [39], on the basis of linear water theory, gave a simple solution of a system describing a wave which was bounded in amplitude at the shoreline and decayed in the direction out to the sea. The essential features of this solution were explained by Lamb [28] and later on by Ursell [42], who first gave the complete description of the linear problem. Unaware of Ursell’s work, Roseau [37] developed a systematic approach to construct edge waves based on an integral representation, which converted the problem into a functional equation. Using a similar method, Whitham [43] gave nonlinear corrections to the solution of Ursell and showed how edge waves can be determined systematically. Greenspan [19] found an edge-wave solution in the case of stratified fluid with an exponentially varying density over a sloping beach. Other methods for
the generation of edge wave are given by Evans [16], who constructed edge waves over a sloping beach on which a mixed boundary condition is satisfied.

The investigation of nonlinear edge waves came as a natural extension of the linear theory. The existence of an irrotational weakly nonlinear edge wave propagating parallel to the shore was shown by Whitham [44] by using a formal Fourier series expansion for the full water-wave problem. Yeh [45] studied the properties of nonlinear progressive edge waves, using the fact that the evolution is described by the nonlinear Schrödinger equation.

An interesting result was discovered by Yih [46], and also by Mollo–Christensen [35], which relates the edge-wave problem to Gerstner’s trochoidal wave [1, 5, 18, 20]: making an elementary transformation of coordinates one can demonstrate that Gerstner’s exact solution of the full equations of classical water-wave theory recovers the edge-wave mode obtained by Stokes. An explicit version of this solution was studied by Constantin [2], who showed that this solution inherits all the properties defining the edge waves. Within the context of constant density, it is shown in [40] that the Gerstner solution and the related edge wave propagating along a sloping beach can be adapted to provide explicit free surface flows in incompressible fluids with arbitrary density stratification.

The fact that for Gerstner’s wave fluid particles move on circles is in agreement with the classical description of the particle paths within the framework of linear water wave theory [12, 25, 28, 29, 38]: all water particles trace a circular orbit, the diameter of which decreases with depth. However, it was recently shown within linear theory [11, 33] that for irrotational periodic water waves the particle paths are not closed. Even within the linear water wave theory, the ordinary differential equation system describing the motion of the fluid particles is nevertheless nonlinear and explicit solutions of this system are not available. However, qualitative features of the underlying flow have been obtained in a nonlinear setting in [5, 7, 11, 22, 33, 34, 41]. The particle trajectories and other properties for the flow beneath water waves of finite depth have been established in [3, 8, 10, 15, 23, 24, 32], to mention just some of the contributions.

An one-dimensional model for the edge waves in the presence of strong longshore currents has been analyzed in [21] where the authors find criteria for the existence of edge waves over variable seabed profiles. By means of asymptotic analysis Johnson [26] derived a two-dimensional model equation for the edge-wave problem. This mixed-type elliptic-hyperbolic equation has been studied in [13] in the context of periodic edge waves, and later on in [14] for solitary waves. An overview of the methods and the results that apply to the edge-wave problem can be found in [27].

In the papers mentioned above the Coriolis effects caused by the rotation of the Earth around the polar axis were neglected. Coriolis effects though play an important role when considering the motion of a fluid layer from a reference frame rotating with uniform angular velocity, cf [17, 36]. For waves located close to the equator the Coriolis parameter is constant and the geophysical wave problem is modeled by the so-called $f$-plane approximation. The physical relevance of the $f$-plane approximation for geophysical edge waves near the equator has been recently discussed in [4]. We mention that the $f$-plane approximation for the deep-water wave problem possesses an explicit solution, since Gerstner’s solution [1, 9, 18, 20] has been generalized in [30] to the geophysical context. It is known from [31] that any solution of the $f$-plane approximation which has the property that the pressure is constant along the streamlines has to be a vertical translation of the solution described in [30]. An exact solution for geophysical equatorial water waves in the $\beta$-plane approximation was given in [6]. This solution describes equatorial trapped waves propagating eastward in a stratified inviscid fluid.

In this paper, we present an explicit solution describing geophysical equatorial edge waves propagating over a plane sloping beach with the shoreline parallel to the equator (e.g. the
Bolaangungi beach, at the Molucca Sea in Indonesia). When we neglect Coriolis effects, we re-

discover the edge-wave solution described in [2]. The latter has the property that the wave speed

is independent of the direction of propagation, whereas in the geophysical context solutions

propagating from west to east travel faster then those propagating in the opposite direction.

While the run-up patterns are identical for the geophysical solutions and the solutions studied

in [2], the Coriolis forces influence the asymptotic behavior of the edge waves offshore: geo-

physical solutions traveling from west to east decay faster than the solutions without Coriolis

effects, and, in turn, the latter decay more rapidly than the geophysical edge waves propagating

in the direction east–west. Being three-dimensional solutions of the \(f\)-plane approximation,

our edge waves have a time-dependent vorticity which decays fast away from the shore.

The outline of the paper is as follows: after presenting in section 2 the governing

equations for the geophysical edge-waves problem, we give in section 3, by using a Lagrangian

description, the explicit solution of this system. The properties of the edge-wave solution we

found are discussed and illustrated in section 4.

2. The governing equations for geophysical edge waves

When considering a rotating frame with the origin at a point \(O\) on Earth’s surface, the governing

equation in the \(f\)-plane approximation for a fluid layer localized near the equator is Euler’s

equation

\[
\frac{d\mathbf{u}}{dt} + 2(\mathbf{\Omega} \times \mathbf{u}) = -\nabla P \rho + \mathbf{g}.
\]

Here, \(t\) represents time, \(\mathbf{u} = (u, v, w)\) is the fluid’s velocity and \(\mathbf{\Omega}\) is the rotation vector of

Earth round the polar axis toward east\(^1\). We denoted with \(\rho\) the (constant) density of the water,

\(\mathbf{g}\) is the gravity vector, \(P\) is the pressure and \(d/dt\) is the material time derivative

\[
\frac{dh}{dt} = h_t + h_u u + h_v v + h_w w,
\]

which express the rate of change of the quantity \(h\) associated with the same fluid particle as it

moves about.

In this paper, we consider a sloping beach with an angle \(\alpha\), the \(xy\)-plane is taken to be

parallel to the sloping bed and the \(z\)-axis is normal to it. Moreover, the \(x\)-axis is parallel to the

shoreline and it is tangent to the equator, pointing in the east direction, while the \(y\)-axis and

the rotation vector \(\mathbf{\Omega}\) form an angle equal to \(\alpha\). In the coordinate system \(Oxyz\), the rotation

and the gravity vector have the following representation:

\[
\mathbf{\Omega} = (0, \omega \cos(\alpha), -\omega \sin(\alpha)) \quad \text{and} \quad \mathbf{g} = (0, -g \sin(\alpha), -g \cos(\alpha)),
\]

with \(\omega = 73 \cdot 10^{-6}\text{rad s}^{-1}\) denoting the rotational speed of the Earth and \(g = 9.8\ \text{m s}^{-2}\) is the

gravitational constant.

Therewith, the equation of motion within the fluid layer may be recast in the \(Oxyz\)

coordinates as the following system:

\[
\begin{align*}
    u_t + uu_x + uv_x + uw_x + 2\omega(w \cos(\alpha) + v \sin(\alpha)) &= -P_z/\rho, \\
    v_t + uv_x + vv_x + vw_x - 2\omega u \sin(\alpha) &= -P_z/\rho - g \sin(\alpha), \\
    w_t + uw_x + v w_x + vv_x - 2\omega u \cos(\alpha) &= -P_z/\rho - g \cos(\alpha).
\end{align*}
\]

(2.1)

An additional equation is obtained from the equation of mass conservation and the assumption

of constant density

\[
u_x + v_y + w_z = 0.
\]

(2.2)

Equations (2.1) and (2.2) are the equation of motion within the fluid layer. They are supplemented by suitable boundary conditions, cf [25]:

\(^1\) Taken to be a perfect sphere of radius 6371 km.
(i) at the free surface, which decouples the motion of the water from that of the air, we set $P = P_0$, where $P_0$ is the (constant) atmospheric pressure;

(ii) the kinematic boundary condition expresses the fact that the free surface consists at all times of the same fluid particles;

(iii) the fluid bed is assumed impermeable and therefore the normal component of the velocity vector is required to be zero at the sloping bed.

Summarizing, the description of geophysical edge waves propagating along the sloping beach described above is given by equations (2.1) and (2.2), and the boundary conditions (i)–(iii). We emphasize that in the geophysical context the edge-wave problem depends to a large extent on the orientation of the beach shoreline with respect to the equatorial plane. This aspect is visible for example from the fact that the Coriolis acceleration term $2 \Omega \times \mathbf{u}$ has a different expression when the shoreline is not contained in the equatorial plane:

$$
\begin{align*}
\begin{cases}
  u_t + uu_y + vv_x + uu_x + wu_x + 2\omega(w \cos(\alpha) \cos(\beta) + v \sin(\alpha) \cos(\beta)) = -P/\rho, \\
  v_t + uv_x + vv_y + vv_x - 2\omega(w \sin(\beta) + u \sin(\alpha) \cos(\beta)) = -P/\rho - g \sin(\alpha), \\
  w_t + uw_x + vv_y + wu_y + 2\omega(v \sin(\beta) - u \cos(\alpha) \cos(\beta)) = -P/\rho - g \cos(\alpha),
\end{cases}
\end{align*}
$$

with $\beta$ being the angle between the equator and the beach shoreline and $\alpha$ being the angle between the sloping bed and the still water surface.

3. An exact solution to the geophysical edge-wave problem

In this section, we present an explicit solution of the edge-wave problem (2.1)–(2.2) and (i)–(iii) by generalizing the solution presented in [2] to the geophysical context considered herein. To this end, we adopt a Lagrangian point of view and describe the evolution of each individual fluid particles. First, we introduce the parameter set

$$
\Sigma := \{(a, b, c) : a \in \mathbb{R}, b \leq b_0, 0 \leq c \leq (b_0 - b) \tan(\alpha), \}
$$

where $b_0$ is chosen such that $b_0 \leq 0$, see figure 1. At each moment in time the geophysical edge-wave solution is the diffeomorphic image of $\Sigma$ under the mapping $\Phi(t) := (x(t), y(t), z(t))$, whereby

$$
\begin{align*}
\begin{cases}
  x(t, a, b, c) := a - \frac{1}{k} e^{k(b-c)} \sin(k(a + st)), \\
  y(t, a, b, c) := b - c + \frac{1}{k} e^{k(b-c)} \cos(k(a + st)) + \frac{2\alpha \cot(\alpha)}{g} z(c), \\
  z(c) := \frac{g(1 + \tan(\alpha))}{g + 2\alpha s} - \frac{g \tan(\alpha)}{g + 2\alpha s} e^{k(b_0)} (1 - e^{-2k(1 + \cot(\alpha))}).
\end{cases}
\end{align*}
$$
for \((a, b, c) \in \Sigma\). Hereby, \(k > 0\) is a fixed wavenumber and \(s \in \mathbb{R}\) is the speed at which the wave travels in the direction parallel to the shoreline. As shown in lemma 3.2 below there are two values possible for the wave speed \(s\) of (3.1):

\[
s_{1/2} := \frac{\omega \sin(\alpha) \pm \sqrt{\omega^2 \sin^2(\alpha) + g k \sin(\alpha)}}{k},
\]

If \(s = s_1\), then the wave travels along the equator from west to east, and if \(s = s_2\) it travels from east to west with wave speed \([s_2]\). When we neglect Coriolis effects, that is \(\omega = 0\), and \(s = s_1\) we recover the solution analyzed in [2]. In the geophysical case though the speed of propagation depends on whether the wave travels from east to west or vice versa, whereas, if \(\omega = 0\), the direction of propagation has no influence on the speed of the wave as \(s_1 = -s_2\).

We remark that the \(z\)-coordinate of \(\Phi(t)\) depends only upon \(c\). Each particle within the fluid is uniquely determined by a tuple \((a, b, c)\) and its path is given by \(\Phi(t, a, b, c), t \geq 0\). Our goal is to prove that the motion (3.1) is dynamically possible and that we can associate with it an expression for the pressure \(P\) such that, for \(s = s_1\) or \(s = s_2\), the governing equations and the boundary conditions are satisfied. The free surface of the water will be the image under \(\Phi(t)\) of the upper boundary \(c = (b_0 - b) \tan(\alpha)\) of \(\Sigma\), while the sloping bed of the fluid is the image of the plane \(c = 0\) under \(\Phi(t)\).

In the remainder of this section we prove that indeed equations (3.1) describe a solution of the problem (2.1)–(2.2) and (i)–(iii). For simplicity, we let \(\sigma := k(a + st), s \in \{s_1, s_2\}\). To proceed, we note that the Jacobi matrix of the transformation \(\Phi(t)\) is the given by

\[
\partial \Phi(t) = \begin{pmatrix}
1 - e^{k(b-c)} \cos(\sigma) & -e^{k(b-c)} \sin(\sigma) \\
-e^{k(b-c)} \sin(\sigma) & 1 + e^{k(b-c)} \cos(\sigma) \\
0 & 0
\end{pmatrix} - 1 - e^{k(b-c)} \cos(\sigma) + \frac{2 \omega z'(c)}{g} \tan(\alpha)
\]

whereby

\[
z'(c) = \frac{g(1 + \tan(\alpha))}{g + 2 \omega s} \left(1 - e^{-2k(b_0 - c(1 + \cot(\alpha)))}\right).
\]

Now, we observe that \(s = s_{1/2}\) are the solutions of the quadratic equation

\[
ks^2 - 2 \omega \sin(\alpha) s - g \sin(\alpha) = 0,
\]

meaning that \(g + 2 \omega s > 0\). Particularly, this shows that \(z'(c) > 0\) for \(c \in (0, \infty)\). Since the determinant of the Jacobi matrix \(\det \partial \Phi(t) = (1 - e^{2k(b-c)})z'(c)\) is positive in the interior of \(\Sigma\), we conclude that \(\Phi(t)\) is a local diffeomorphism. Moreover, we have the following lemma.

**Lemma 3.1.** The map \(\Phi(t)\) is a diffeomorphism from \(\Sigma\) to the water region bounded below by the rigid bed \(z = 0\) and above by the free water surface, which is parameterized by

\[
\begin{align*}
x & = a - \frac{1}{k} e^{k(b_0 - b) \tan(\alpha)} \sin(\sigma), \\
y & = b - (b_0 - b) \tan(\alpha) + \frac{1}{k} e^{k(b_0 - b) \tan(\alpha)} \cos(\sigma) + \frac{2 \omega \cot(\alpha)}{g} \z((b_0 - b) \tan(\alpha)), \\
z & = z((b_0 - b) \tan(\alpha)),
\end{align*}
\]

with \(a \in \mathbb{R}, b \leq b_0\) and \(t \geq 0\).
Proof. Since \( z : (0, \infty) \to (0, \infty) \) is a diffeomorphism, it suffices to show that for each fixed \( c \in (0, \infty) \), the mapping \((\tilde{x}(t), \tilde{y}(t))\) with

\[
(\tilde{x}(t, a, b'), \tilde{y}(t, a, b')) := \left( a - \frac{1}{k} e^{ib'} \sin(\sigma), \frac{1}{k} e^{ib'} \cos(\sigma) \right),
\]

with \( a \in \mathbb{R}, \ b' \leq b_0 - \epsilon(1 + \cot(\alpha)) \), is a diffeomorphism. This property though has been established in the context of Gerstner’s two-dimensional solution for the deep-water wave problem [1], and the proof is completed. \( \square \)

Next, we argue that the boundary conditions (i) and (iii) and the conservation of mass (2.2) are fulfilled. To this end, we infer from (3.1) that the fluid particles move on circles located in planes which are parallel to the sloping bed of the ocean. Particularly, since the velocity field is given by \((u, v, w) = (x_t, y_t, z_t)\), the fluid velocity has no normal component to the beach when \( z = 0 \), which proves (iii). Also, the kinematic boundary condition (i) at the wave surface is satisfied by the definition of the wave: the wave surface is the image of the half-plane \( c = (b_0 - b) \tan(\alpha) \) under \( \Phi(t) \) for all \( t \geq 0 \), and consists thus of the same fluid particles. In order to verify the incompressibility condition (2.2), we compute that

\[
\partial \Phi^{-1}(t) = \begin{pmatrix}
\frac{1 + e^{ib(c-c)}}{1 - e^{2ib(c-c)}} & \frac{e^{ib(c-c)}}{1 - e^{2ib(c-c)}} & -\mu e^{ib(c-c)} \\
\frac{1 - e^{ib(c-c)}}{1 - e^{2ib(c-c)}} & \frac{1}{1 - e^{2ib(c-c)}} & \frac{1}{1 - e^{2ib(c-c)}} \\
0 & 0 & \frac{1}{z'(c)}
\end{pmatrix},
\]

where we used the shorthand \( \mu := 2 \cos \cot(\alpha)/g \). By the chain rule (3.3) and (3.7) we then find

\[
u_x + v_y + w_z = x_{xt} + x_{yt} + x_{zt} + y_{xt} + y_{yt} + y_{zt} + z_{xt} + z_{yt} + z_{zt} + z_{zt} = 0
\]

which is the desired relation (2.2).

To finish the proof, we are left to show that there exists a pressure function \( P \) such that the Euler equation (2.1) and the boundary condition (ii) are satisfied.

**Lemma 3.2.** Equations (2.1) and the dynamic boundary condition (ii) are satisfied if the pressure \( P \) is given by the following expression:

\[
P = P_0 + \frac{g \rho \sin(\alpha)}{2k} e^{2ib(c-c)} - g \rho (c \cos(\alpha) + (b - b_0) \sin(\alpha)) = \frac{g \rho \sin(\alpha)}{2k} e^{2ib(c-b_0)\cot(\alpha)}.
\]

(3.8)

**Proof.** Recalling that \( z \) is independent of time, we find that \( w = 0 \) while direct computation shows that

\[
u_x + wu_x + vu_y + wu_z = x_{yt} \quad \text{and} \quad v_x + uw_x + vw_y + uw_z = y_{yt},
\]

see e.g. lemma 3.2 in [30]. Inserting these expressions into the system (2.1), we obtain that the first-order partial derivatives of \( P \) have the following expression:

\[
\begin{align*}
P_x &= -\rho k s^2 e^{ib(c-c)} \sin(\sigma) + 2 \rho \omega s e^{ib(c-c)} \sin(\alpha) \sin(\sigma), \\
P_y &= \rho k s^2 e^{ib(c-c)} \cos(\sigma) - 2 \rho \omega s e^{ib(c-c)} \sin(\alpha) \cos(\sigma) - g \rho \sin(\alpha), \\
P_z &= -2 \rho \omega s e^{ib(c-c)} \cos(\sigma) \cos(\sigma) - g \rho \cos(\alpha).
\end{align*}
\]

(3.9)
Figure 2. The run-up patterns when \( k = 2 \) and (a) \( b_0 = 0 \) (cycloid); (b) \( b_0 < 0 \) (trochoid). These curves separate the emerged part of the sloping beach (located above the curve) from its submerged part.

Back to Lagrangian coordinates, we obtain

\[
\begin{align*}
P_a &= x_a P_x + y_a P_y + z_a P_z = \rho e^{b-kc} \sin(\sigma) (-k^2 + 2\omega \sin(\alpha) + g \sin(\alpha)) = 0, \\
\end{align*}
\]

cf (3.5). Moreover, the same relation (3.5) leads us to

\[
\begin{align*}
P_b &= g \rho \sin(\alpha) e^{2b-kc} - g \rho \sin(\alpha), \\
P_c &= -g \rho \sin(\alpha) e^{2b-kc} - g \rho \cos(\alpha) \sigma'(c) - 2\omega \rho \cos(\alpha) \sigma'(c) + g \rho \sin(\alpha),
\end{align*}
\]

and, taking into account that \( P = P_0 \) at the free surface, we obtain that \( P \) is given by expression (3.8).

4. Properties of the edge-wave solution

In this final section we discuss some of the properties of the edge-wave solution we have presented in the previous section. As we have already seen, due to the rotation of the Earth, the solution of (3.1) which travels from west to east moves at a larger wave speed than the one traveling from east to west along the sloping beach. Despite this property, the alongshore run-up patterns, which separate the submerged part of the sloping beach from the emerged part, are identical in both cases: they are described by a cycloid with upward cusps when \( b_0 = 0 \), and a trochoid when \( b_0 < 0 \), see figure 2. Indeed, setting \( b = b_0 \) in (3.6), we obtain that the run-up pattern at time \( t \geq 0 \) is parametrized by the curve

\[
\mathbb{R} \ni a \mapsto \left( a - \frac{1}{k} \frac{\iota_{kb}}{e^{kb-kc}} \sin(\sigma), b_0 + \frac{1}{k} \frac{\iota_{kb}}{e^{kb-kc}} \cos(\sigma), 0 \right).
\]

We emphasize that the alongshore run-up pattern coincide with those found in [2] in the absence of Coriolis effects.

The edge-wave solution analyzed in [2] has the property that the vorticity vector is independent of time. This property is still true for two-dimensional geophysical waves located near the equator, cf [36, 31]. However, for three-dimensional geophysical waves this feature does not hold in general. The next lemma shows that the flow corresponding to the edge-wave solution (3.1) is rotational and that the vorticity is time dependent and decreases exponentially fast away from the shore.

Lemma 4.1. The vorticity of the water flow (3.1) is given by the following expression:

\[
\text{curl } \mathbf{u} = \left( -\frac{sk \mu e^{k(b-c)} \sin(\sigma)}{1 - e^{2k(b-c)}}, \frac{-sk \mu e^{k(b-c)} + sk \mu e^{k(b-c)} \cos(\sigma)}{1 - e^{2k(b-c)}}, \frac{-2sk \mu e^{k(b-c)}}{1 - e^{2k(b-c)}} \right).
\]

whereby \( \mu = 2\omega \cot(\alpha)/g \) and \( \sigma = k(a + st) \).
Figure 3. The edge-wave solution (3.1) and the corresponding plane toward which it converges away from the shoreline when $s = s_1$, $k = 1$, $b_0 = -0.4$, and $\alpha = \pi/4$:
(a) seen from offshore; (b) seen from the beach. One can see in figure (b) the trochoidal run-up patterns made by the edge wave on the sloping beach.

Proof. Gathering (3.3) and (3.7), it follows by direct computation that
\[
\text{curl } u = \left( \frac{-sk\mu e^{i(b-c)} \sin(\sigma)}{1 - e^{2ikb - c}}, \frac{-sk\mu e^{i(b-c)} \cos(\sigma)}{1 - e^{2ikb - c}}, \frac{-2sk\mu e^{i(b-c)}}{1 - e^{2ikb - c}} \right).
\]
\[\square\]

We are interested now in the amplitude of the edge-wave solution (3.1). Since we deal with waves traveling parallel to the shoreline, it suffices to discuss the problem when setting $t = 0$ (in this case the crest (and trough) lines of the edge waves (3.1) with and without Coriolis effects are in the same planes, cf (4.6)). Letting $c \to \infty$ in (3.1) and using the fact that on the free surface $c = (b_0 - b) \tan(\alpha)$, we conclude that the edge-wave solution (3.1) approaches in the large the plane
\[z = -\frac{\tan(\alpha)}{2k} e^{2kb_0} + (b_0 - y) \tan(\alpha),\] (4.3)
see figures 1 and 3. This property is satisfied also when $\omega = 0$, cf [2], as the angular velocity $\omega$ does not appear in equation (4.3).

In order to compute the elevation of the wave with respect to the plane (4.3) and to compare the result with that for $\omega = 0$, we rotate the coordinate system $Oyz$ counterclockwise with the angle $\alpha$. The new coordinate system $OYZ$ has the property that the $OY$-axis is parallel to the common line of (4.3) and $OYZ$. In these new coordinates, we have $\Phi(t) = (x, Y, Z)$ whereby
\[Y = \left( (b - c) + \frac{1}{k} e^{i(b-c)} \cos(ka) \right) \cos(\alpha) + \frac{2\cos^2(\alpha) - g \sin^2(\alpha)}{g \sin(\alpha)} z(c),\] (4.4)
\[Z = \left( (b - c) + \frac{1}{k} e^{i(b-c)} \cos(ka) \right) \sin(\alpha) + \frac{2 \cos + g}{g} \cos(\alpha) z(c)\] (4.5)
for $(a, b, c) \in \Sigma$, when $t = 0$.

Setting $c = (b_0 - b) \tan(\alpha)$ in (4.5), we obtain that the signed distance of the point $\Phi(0, a, b, c)$ measured from the plane (4.3) is
\[d = \frac{\sin(\alpha)}{2k} \left( e^{2kb_0 \tan(\alpha)} - 2kb_0 \tan(\alpha) + 2 e^{kb_0 \tan(\alpha)} \right) \cos(ka).\] (4.6)
Figure 4. An illustration of the trough and crest lines for geophysical waves and for waves without Coriolis effects. The rigid lines represent crest lines (left) and trough lines (right) of the geophysical edge wave (3.1) when (a) \( s = s_1 \); (b) \( s = s_2 \). The dashed lines represent the corresponding crest (trough) lines of the edge wave (3.1) when Coriolis effects are neglected \( (\omega = 0) \).

We see from this relation that the amplitude of the edge wave decays exponentially as \( b \to -\infty \) (see also figure 3). Moreover, we note from (4.6) that the crest (and trough) lines are orthogonal to the shoreline and parallel to each other, as they correspond to the values \( a = 2m\pi/k \) (and \( a = (2m + 1)\pi/k \), \( m \in \mathbb{Z} \)). Differentiating (4.6) with respect to \( b \), we also see that the crest and trough lines are monotone curves.

Note that we obtain for \( d \) the same relation as in the \( \omega = 0 \) case [2]. Although, it is important to stress that even if the amplitude \( Z \) of the point \( \Phi_0(a, b, c) \) is the same if \( \omega = 0 \) or \( \omega > 0 \), the \( Y \) component of \( \Phi(0, a, b, c) \) is larger in the geophysical context \( (\omega > 0) \) if \( s = s_1 \) and smaller if \( s = s_2 \), cf (4.4). Therefore, the wave (3.1) approaches more rapidly the plane (4.3) if Coriolis effects are taken into account and the wave propagates from west toward east, see figure 4, whereas, if \( s = s_2 \), then, due to the Coriolis effect, the edge wave approaches at a slower rate the plane (4.3) than the corresponding edge-wave solution with \( \omega = 0 \).

Acknowledgments

This research has been supported by the FWF project I544 –N13 ‘Lagrangian kinematics of water waves’ of the Austrian Science Fund.

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