Tricritical Ising Model near criticality

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\textbf{ABSTRACT:} The most relevant thermal perturbation of the continuous \(d = 2\) minimal conformal theory with \(c = 7/10\) (Tricritical Ising Model) is treated here. This model describes the scaling region of the \(\phi^6\) universality class near the tricritical point. The problematic IR divergences of the naive perturbative expansion around conformal theories are dealt within the OPE approach developed at all orders by the authors. The main result is a description of the short distance behaviour of correlators that is compared with existing long distance expansion (form factors approach) related to the integrability of the model.

SPhT-t96/142
GEF-Th-15
Revised Version 4-1997
PACS:  11.10.Kk,11.25.Hf,11.25.Db,05.70.Jk,64.60.Kw,05.50.+q

Keywords: Tricritical Ising model, critical point, two dimensions, conformal field theories, perturbation theory, IR divergences, Operator Product Expansion.

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1 Introduction

The $d = 2$ minimal conformal quantum field theory whose central charge is equal to $c = 7/10$, known as Tricritical Ising Model (TIM), describes the scaling region near the tricritical point (i.e. the end point of a line where three phases coexist simultaneously) of the universality class corresponding to the Landau-Ginzburg theory with an interaction $\phi^6 [1]$. In particular it can be thought of as a continuum realization of the Ising model with annealed vacancies, see [2], and it describes adsorbed helium on krypton-plated graphite, [3], (for a general reference see [4]).

As is well known in the $d = 2$ minimal conformal theories there exists a finite number of primary operators [5]. In TIM they can be divided in two classes according to their parity with respect to a natural $Z_2$ symmetry: the energy operators are even and the spin operators are odd. At the critical point there is a spontaneous breaking of the $Z_2$ symmetry. TIM displays other symmetries apart from the conformal one: superconformal symmetry [6, 7] and those based on the coset formulation ($su(2)$ and $e_7$ algebras, see [8]).

In this article we study the two point correlation functions of the energy operator and of the spin operator in proximity of the tricritical point. More specifically we consider the model obtained by adding to the conformal action the most relevant energy operator of the critical theory, $\varepsilon$ (of scale dimension $1/5$, see Section 2):

$$A = A_{CFT} + \lambda \int d^2 x \varepsilon(x).$$

We also restrict ourselves to the high temperature phase of the model, that corresponds to positive values of the coupling $\lambda$.

The considered model is integrable and its spectrum and $S$ matrix are known, [11] (see also [14]). In the high temperature phase the massive excitations are given by seven elementary particles. The mass of the fundamental particle is given by:

$$m = C_F \lambda^{5/9},$$

where $C_F$ is [12]:

$$C_F = \left( \frac{2\Gamma(\frac{2}{9})}{\Gamma(\frac{2}{3})\Gamma(\frac{4}{9})} \right) \left( \frac{4\pi^2\Gamma(\frac{2}{9})\Gamma(\frac{4}{9})}{\Gamma(\frac{2}{3})^3\Gamma(\frac{4}{9})} \right)^{5/18} = 3.745372836243954223 \ldots$$

The masses of the other particles can be found in [13] with a more detailed analysis of the model. In the same paper the authors were able to find
the form factors of the model (see \cite{15} for general references and \cite{16} for recent developments) and to calculate the energy-energy correlation function including the excited states given in table 3 of that work. The representation of the correlation function in terms of form factors gives rise to a long distance expansion (the small parameter being $e^{-mr}$) that is well appropriate in the region down to $mr \simeq 1$, but is reasonably less quickly convergent in the $mr << 1$ region.

In this paper we study the energy energy and spin spin correlators by use of a perturbative expansion around the conformal theory: in this case the small parameter is $mr$ and consequently our analysis is complementary to the form factor approach.

The perturbative calculations are performed by using the Operator Product Expansion (OPE) approach in order to overcome the infrared divergences problem typical of the relevant perturbations of massless theories. The idea of this approach came out very early in the general context of quantum field theories \cite{17} (see \cite{18} for similar ideas in QCD), was first introduced in the context of perturbed conformal theories by Al. B. Zamolodchikov in \cite{19} (see also \cite{20} for an analog independent proposal and \cite{21} for an early application of Ref.\cite{19}), and was developed at all orders in a very general context in \cite{22}.

In the OPE approach the terms non analytic in $\lambda$, whose very existence is related to infrared divergences of naive perturbation theory, are segregated in the Vacuum Expectation Values of the operators present in the OPE expansion of correlators

\[
< \Phi_a(r) \Phi_b(0) >_\lambda \sim \sum_c C^{c}_{ab}(r, \lambda) < \Phi_c(0) >_\lambda \tag{1.4}
\]

($\Phi_a$ are deformations of the conformal theory operators, the suffix $\lambda$ refers to the complete theory correlators; the dependence on $\lambda$ of the Wilson coefficients will be omitted in the following). The resulting perturbation theory for the Wilson coefficient $C^c_{ab}(r, \lambda)$ turns out to be IR finite at all orders. The required operators VEV are a non perturbative additional information that should be considered as an external input in this context.

In the specific example that we consider, the mean value of the energy operator $\varepsilon$ will be determined by the Thermodynamic Bethe Ansatz, \cite{12}, and the mean value of the sub-leading energy operator, required for our purposes, will be estimated to a reasonable accuracy by using two exact sum rules that express the scale dimension of the energy and the central charge of the unperturbed conformal theory in terms of moments of the energy energy correlators \cite{9,10} (in the long distance region we use the results of \cite{13} for the correlator).
Our final result is an estimate of the energy and spin spin correlators up to \((mr)^2\). The prediction for the energy energy operator will then be compared with those of [13] in Figure 1.

The plan of the paper is as follows: in Section 2 we collect the required knowledge on the TIM and introduce the first order perturbative formulas for the Wilson coefficients; in Section 3 the final \(O((mr)^2)\) expression for the correlators are reported; in Section 4 the missing VEV of sub-leading energy is estimated by use of exact sum rules and long distance results. Details of calculations are reported in Appendix B while Appendix A reports a very general theorem that turned out to be useful in calculations.

2 Tricritical Ising model

The Tricritical Ising Model (TIM) is the minimal CFT with central charge \(c = \frac{7}{10}\) \((\alpha^2 = \frac{4}{5}\), in notation of [23]). At the critical point we have the following operators:

- the identity \(\Phi_{11} = I, \Delta_I = 0\),
- the magnetization field \(\Phi_{22} = \sigma, \Delta_\sigma = \frac{3}{80}\),
- the sub-leading magnetization operator \(\Phi_{12} = \alpha, \Delta_\alpha = \frac{7}{16}\),
- the energy density \(\Phi_{21} = \varepsilon, \Delta_\varepsilon = \frac{1}{10}\),
- the vacancy operator or sub-leading energy operator \(\Phi_{31} = l, \Delta_l = \frac{3}{5}\),
- the irrelevant field \(\Phi_{41} = \varepsilon'', \Delta_{\varepsilon''} = \frac{3}{2}\).

Using the fusion rules and knowing the Wilson coefficients we can construct the algebra for TIM:

| \(\varepsilon''\) | \(\sigma\) | \(l\) | \(\alpha\) | \(\varepsilon\) |
|---------------|-------------|------|----------|--------------|
| \(\varepsilon\) | \(l\) | \(\alpha + \sigma\) | \(\varepsilon'' + \varepsilon\) | \(\sigma\) | \(I + \varepsilon\) |
| \(\alpha\) | \(\alpha\) | \(\varepsilon + l\) | \(\sigma\) | \(I + \varepsilon''\) | \(\sigma\) |
| \(l\) | \(\varepsilon\) | \(\alpha + \sigma\) | \(I + l\) | \(\sigma\) | \(\varepsilon'' + \varepsilon\) |
| \(\sigma\) | \(\sigma\) | \(\varepsilon + l + \varepsilon''\) | \(\alpha + \sigma\) | \(\varepsilon + l\) | \(\alpha + \sigma\) |
| \(\varepsilon''\) | \(I\) | \(\sigma\) | \(\varepsilon\) | \(\alpha\) | \(l\) |

The corresponding Wilson coefficients can be found in Table 4 of [24] (their normalization: \(\langle \Phi_i \Phi_j \rangle = \delta_{ij}\) will be used everywhere here.)

Let us write here the adimensional Wilson coefficient \(\hat{C}_{\varepsilon \varepsilon}^\alpha (\hat{C} \equiv C|r|^{\dim C})\):

\[
\hat{C}_{\varepsilon \varepsilon}^\alpha = \frac{2}{3} \frac{\Gamma(4/5)\Gamma(2/5)}{\Gamma(1/5)\Gamma(3/5)}
\]
and 

\[ \hat{C}_\epsilon^e = \frac{3}{2} \hat{C}_\epsilon^l \]  

(2.2)

We consider in this paper the following perturbation:

\[ A = A_{CFT} + \lambda \int d^2x \varepsilon(x) \]  

(2.3)

where \( \lambda \sim T - T_c > 0 \) and \( \varepsilon \) is the most relevant energy operator of TIM, with total dimension 1/5.

The idea of the OPE approach (see \[17, 19, 20\]) is to confine contributions non analytic in \( \lambda \) inside the VEV’s and to build up an IR finite perturbation theory (Taylor expansion around \( \lambda = 0 \)) for the Wilson coefficients. Assuming the regularity of the Wilson coefficients in terms of the renormalized coupling \( \lambda \), the asymptotic convergence of OPE and the validity of the Renormalized Action Principle, that in our case looks like

\[- \frac{\partial}{\partial \lambda} \langle [\cdots] \rangle_{\lambda} = \int \langle [: \varepsilon : \cdots] \rangle_{\lambda}, \quad \varepsilon : \equiv \varepsilon - \langle \varepsilon \rangle_{\lambda} \]  

(2.4)

([\cdots] means renormalization, when needed), an IR finite representation for the generic \( n^{th} \) derivative of \( C_{\epsilon \sigma}^e \) with respect to the coupling, evaluated at \( \lambda = 0 \), has been given \[22\], involving integrals of conformal correlators. See \[23\] for an application to the critical Ising model perturbed by a magnetic field.

In our case we will approximate the correlators by use of the following first order expressions (all derivatives below are intended to be evaluated at \( \lambda = 0 \)):

\[ \langle \varepsilon \varepsilon \rangle_{\lambda} = C_{\epsilon \epsilon}^1 + C_{\epsilon \epsilon}^4 < l >_{\lambda} + \lambda \partial_\lambda C_{\epsilon \epsilon}^e < \varepsilon >_{\lambda} + O(\lambda^2) \]  

(2.5)

\[ \langle \sigma \sigma \rangle_{\lambda} = C_{\sigma \sigma}^1 + \lambda \partial_\lambda C_{\sigma \sigma}^1 + C_{\sigma \sigma}^e < \varepsilon >_{\lambda} + \lambda \partial_\lambda C_{\sigma \sigma}^e < \varepsilon >_{\lambda} + C_{\sigma \sigma}^l < l >_{\lambda} + O(\lambda^{5/3}). \]  

(2.6)

At first order we derive the following nontrivial relations, (see also \[19, 20\] for first order formulas):

\[- \partial_\lambda C_{\sigma \sigma}^1 = \int' d^2z < \varepsilon(z) \sigma(r) \sigma(0) - C_{\sigma \sigma}^e \varepsilon(0) > \]  

(2.7)

\[- \partial_\lambda C_{\sigma \sigma}^e < \varepsilon(\infty) \varepsilon(0) >= \int' d^2z < \varepsilon(\infty) \varepsilon(z) \sigma(r) \sigma(0) - C_{\sigma \sigma}^1 - C_{\sigma \sigma}^l (0) > \]  

(2.8)
\[-\partial_{\lambda} C_{\varepsilon\varepsilon}^\varepsilon < \varepsilon(\infty)\varepsilon(0) > =
\]
\[
\int d^2 z < \varepsilon(\infty)\varepsilon(z)(\varepsilon(r)\varepsilon(0) - C_{\varepsilon\varepsilon}^1 - C_{\varepsilon\varepsilon}^l(0)) >
\]
(2.9)

where the prime means any (rotation invariant) IR regularization of the integrals. We observe that the derivative of Wilson coefficients is obtained by adding up a "naive perturbative term" (insertion of the interaction \( f \varepsilon \) in the correlator) and some "IR counterterms" generated by the theory itself in a natural way: both contributions are IR divergent but their sum is finite and gives the wanted result. This structure persists in the general all order formulas. The explicit form of the correlators and the details of the calculation of the IR finite part are reported in Appendix B.

We can parameterize the VEV of \( \varepsilon \) and \( l \) as
\[
< \varepsilon >_\lambda = A_\varepsilon \lambda^{1/9}
\]
(2.10)
\[
< l >_\lambda = A_l \lambda^{2/3}.
\]
(2.11)

Notice that potential adimensional logarithms are absent in the VEV’s because no renormalization effects arises in this case (an operator of dimension \( x \) can mix in this theory only with operators of dimension \( x - \frac{9}{5} k \) for positive \( k \)).

The constant \( A_\varepsilon \) can be fixed by use of results of [12] (Thermodynamic Bethe Ansatz)
\[
A_\varepsilon = \frac{-5}{36} \frac{S(\frac{2}{9})}{S(\frac{4}{9}) S(\frac{2}{9})} \left( \frac{4\pi^2 \Gamma(\frac{2}{9}) \Gamma(\frac{4}{9})^3}{\Gamma(\frac{1}{5})^3 \Gamma(\frac{2}{9})^2} \right) \left( \frac{2\Gamma(\frac{2}{9})}{\Gamma(\frac{4}{9}) \Gamma(\frac{2}{9})} \right)^2
\]
(2.12)
\[
= -1.468395424027621489 \cdots
\]
(2.13)

where \( S(x) = \sin \pi x \) (see e.g. [23] for more details). The quest for an estimate of \( A_l \) is the goal of Section 4.

### 3 Results

We report in this Section the results for the approximate energy energy and spin spin correlators, Eqs.(2.5-2.6). More details on the calculations can be found in Appendix B.

It is convenient to define adimensional quantities \( \hat{C} \equiv C|r|^\text{dim}C \), \( F_{A,B} \equiv < AB >_h |r|^\text{dim}AB \) and introduce the scaling variable \( \tau = mr \), where \( m \) is the fundamental mass of the model, defined in Eq.(1.2).
We have:

\[
F_{\varepsilon\varepsilon} = 1 + A_l \tilde{C}_{\varepsilon\varepsilon}^{l} \left( \frac{\tau}{C_F} \right)^{6/5} + A_\varepsilon \partial_\lambda \tilde{C}_{\varepsilon\varepsilon} \left( \frac{\tau}{C_F} \right)^{2} + O(\tau^{18/5}) \quad (3.1)
\]

\[
F_{\sigma\sigma} = 1 + A_\varepsilon \tilde{C}_{\sigma\sigma}^{e} \left( \frac{\tau}{C_F} \right)^{1/5} + A_l \tilde{C}_{\sigma\sigma}^{l} \left( \frac{\tau}{C_F} \right)^{6/5} + \partial_\lambda \tilde{C}_{\sigma\sigma}^{1} \left( \frac{\tau}{C_F} \right)^{9/5} + A_\varepsilon \partial_\lambda \tilde{C}_{\sigma\sigma} \left( \frac{\tau}{C_F} \right)^{2} + O(\tau^{3}) \quad (3.2)
\]

By use of results in Appendix B we obtain:

\[
- \partial_\lambda \tilde{C}_{\varepsilon\varepsilon}^{e} = n^3 \frac{\Gamma\left(\frac{2}{5}\right)^2 \Gamma\left(\frac{7}{10}\right)^2}{\sqrt{S\left(\frac{3}{5}\right)^2 S\left(\frac{4}{5}\right)^2 S\left(\frac{14}{5}\right)}} \frac{(S\left(\frac{4}{5}\right) - S\left(\frac{8}{5}\right) - S\left(\frac{16}{5}\right))}{2^{6/5} \Gamma\left(\frac{3}{5}\right)^4 \Gamma\left(\frac{3}{5}\right)^4} \quad (3.3)
\]

\[
- \partial_\lambda \tilde{C}_{\sigma\sigma}^{1} = - \frac{S\left(\frac{11}{10}\right)^2 \Gamma\left(\frac{9}{10}\right)^4}{S\left(\frac{6}{5}\right)^2 \Gamma\left(\frac{9}{5}\right)^2} \left( \frac{\Gamma\left(\frac{4}{5}\right) \Gamma\left(\frac{2}{5}\right)^3}{\Gamma\left(\frac{1}{5}\right) \Gamma\left(\frac{3}{5}\right)^3} \right)^{1/2} \quad (3.4)
\]

\[
- \partial_\lambda \tilde{C}_{\sigma\sigma}^{e} = 25 \pi \frac{\Gamma\left(\frac{2}{5}\right) \Gamma\left(\frac{7}{10}\right)^2 S\left(\frac{5}{7}\right)^2 S\left(\frac{7}{5}\right)^2 \left(-S\left(\frac{4}{5}\right) + S\left(\frac{11}{5}\right) + S\left(\frac{12}{5}\right) + S\left(\frac{18}{5}\right) - S\left(\frac{27}{5}\right)\right)}{S\left(\frac{5}{7}\right)^2 S\left(\frac{4}{5}\right)^2 S\left(\frac{8}{5}\right)^2 S\left(\frac{23}{10}\right)^2 S\left(\frac{19}{10}\right)} 2^{12/5} \Gamma\left(\frac{1}{5}\right)^4 \quad (3.5)
\]

All other quantities have been defined previously; for a numerical value of $A_l$ see Eq.(4.11).

4 Sum rules

In this section we will consider two known sum rules satisfied by the exact complete theory.

The sum rule more sensitive to the short distance behavior of the correlators is the one concerning the scale dimension of operators. If we consider, as is the case, a perturbation $\lambda f \varepsilon$ of a conformal field theory (assumed to draw the system towards a massive theory) and an operator $X$ with scale dimension $\Delta_X$, assumed to have a VEV $< X >_\lambda = A_X \lambda^{\Delta_X/(1 - \Delta_\varepsilon)}$, then, by taking the derivative with respect to $\lambda$ of the VEV and by using the Action Principle hypothesis, Eq.(2.4), it is easy to obtain

\[
\frac{\Delta_X}{1 - \Delta_\varepsilon} = -\lambda \int d^2 x < [X(x) \varepsilon(0)] >_{\lambda,\varepsilon} / < X >_{\lambda} \quad (4.1)
\]

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This exact sum rule has been derived first, in a more generalized form, in [10].

By choosing \( X = \varepsilon \) and using the known relation between the trace of energy momentum tensor \( \Theta \) and the perturbing field \( \varepsilon \) (holding at least when \( \Delta_\varepsilon < 1/2 \)),

\[
\Theta = 4\pi \lambda (1 - \Delta_\varepsilon)\varepsilon
\]

we have the equivalent relation:

\[
\int d^2x < \Theta \Theta >_{\lambda c} = -\frac{36\pi^2}{25} \frac{A_\varepsilon}{C_F^2} m^2.
\]

The second sum rule we consider is given by

\[
c \equiv \frac{7}{10} = \frac{3}{4\pi} \int d^2x|x|^2 < \Theta \Theta >_{\lambda c}
\]

and has been obtained in the context of the C-theorem [3].

The short distance behavior of the connected \( \Theta \Theta \) correlator can be expressed in terms of the scaling function \( F_{\varepsilon\varepsilon} \) of Eq.(3.1) as

\[
< \Theta \Theta >_{\lambda c} / m^4 = \left( \frac{18\pi}{5C_F^2} \right)^2 \left( \frac{\tau}{C_F} \right)^{2/5} \left( F_{\varepsilon\varepsilon} - A_\varepsilon^2 \left( \frac{\tau}{C_F} \right)^{2/5} \right)
\]

(Notice that we subtracted \( < \varepsilon >^2 \) to get the connected correlator).

The long distance behavior is given by:

\[
< \Theta \Theta >_{\lambda c} / m^4 \sim \frac{1}{\pi} (f_2^2 K_0(c_2\tau) + f_4^2 K_0(c_4\tau))
\]

where \( c_2 = 2 \cos(5\pi/18), \) \( c_4 = 2 \cos(\pi/18) \) and \( f_2 = 0.9604936853, f_4 = -0.4500141924, [13]. \)

We now proceed in the same way as done in [23], where good results were obtained: we split the integrals in the sum rules at some point \( mr = \Lambda \), evaluating \( mr < \Lambda \) region with the short distance expression Eqs.(3.1),(4.5) and \( mr > \Lambda \) with Eq.(4.6). In particular the integral of the long distance correlator is given by

\[
\int_{|mr|>\Lambda} < \Theta \Theta >_{\lambda c} d^2\tau / m^4 = 2\Lambda(f_2^2/c_2K_1(c_2\Lambda) + f_4^2/c_4K_1(c_4\Lambda))
\]

while for the \( c \) sum rule the integral of the long distance contribution is given by

\[
\int_{|mr|>\Lambda} < \Theta \Theta >_{\lambda c} r^2 d^2x = 2(f_2^2/c_2^2\tilde{K}(c_2\Lambda) + f_4^2/c_4^2\tilde{K}(c_4\Lambda))
\]
where
\[ \tilde{K}(x) = (4x + x^3)K_1(x) + 2x^2K_0(x). \]  

(4.9)

By imposing the sum rule Eq.(4.3) we can recover the unknown constant \( A_l \) as a function of \( \Lambda \). Following \[23\] a good approximation for \( A_l \) can be obtained by looking for a minimal sensitivity point \( \Lambda^* \) of \( A_l \). We obtain in this way
\[ A_l = 3.78 \]  

(4.10)

where an error of 5 – 10\% is estimated by using different variants of the method (imposing validity of Eq.(4.4) or approximate validity of both).

5 Conclusions

By use of results of Sections 3-4, we can give a prediction on the short distance behavior of the energy energy (or \( \Theta \Theta \)) and spin spin correlators of the TIM model near criticality (most relevant energy perturbation). As a by product of our work we get an estimate for the subleading energy VEV, Eqs.(2.11)-(4.10)

The most striking feature of OPE approach used in this paper, is the fact that non analyticities in \( \lambda \), present in the short distance behavior of correlators, are naturally recovered from the VEV and manifest in non negligible fractionary power corrections (in \( \lambda \)) in Eqs.(3.1) - (3.2).

In Figure 1 we compare our predictions for the \( \Theta \Theta \) correlator as a function of \( mr \) with the corresponding long distance prediction of \[13\]. The agreement between the two approaches is good in the intermediate \( 0.5 < mr < 1.5 \) region, where corresponding ”small” parameters are actually of order 1! See \[23, 26, 16\] for similar situations. Moreover the lower \( mr \) region shows a reasonable evidence of convergence of the long distance expansions towards our results. We remark in particular, referring also to an similar comparison made by the authors for the critical Ising Model in magnetic field \[23, \text{fig.2}\], that the weaker the singularity in \( r = 0 \) of a correlator is, the fastest is the convergence of the long distance approximants. We think that, in any case, the two approaches are always complementary.

Finally we want to stress that the OPE approach gives an all order IR finite perturbation theory that is very general (mild and reasonable hypotheses are behind it) and do not uses integrability essentially: the computation of Wilson coefficients can thus reasonably applied to many others physically interesting models in statistical mechanics. The main problem of reaching higher orders is related to the complexity of the required integrals. We think nevertheless that second order expressions when necessary, could be reached,
at least numerically. The ignorance of the expressions of the VEV will result at each finite order in a finite set of universal constant parameterising all the correlators. How to reach a non-perturbative information on those constants, apart from what already known from Thermodynamic Bethe Ansatz, is still an open problem.

Acknowledgements: The authors want to thanks Calin Buia, who participated at the early stages of the work, J.B. Zuber for reading the manuscript, and G. Mussardo and C. Acerbi for giving them a table of numerical data. R.G. thanks H. Navelet, R. Peschanski, M. Bauer for stimulating discussions on the theorem in Appendix A. Part of this work has been done during a visit of the authors to CERN, whose warm hospitality is acknowledged. The work of R.G. is supported by a TMR EC grant, contract No ERB-FMBI-CT-95.0130. R.G. also thanks the INFN group of Genova for the kind hospitality.

Appendix A A useful theorem

We give here a (slight generalisation of) a useful theorem proved in [28], (see also [25, 29] for related work).

Let us consider a double integral on the plane of the form

$$ I = \int d^2 w \sum_{\alpha,\beta=1}^{N} f_\alpha(w)Q_{\alpha\beta}\bar{f}_\beta(w^*) \quad (A.1) $$

where \( \{f_\alpha(w)\}_{\alpha=1}^{N} \) and \( \{\bar{f}_\beta(w^*)\}_{\beta=1}^{N} \) are two sets of linear independent functions, \( Q_{\alpha\beta} \) is a complex, constant, matrix and \(*\) denotes the complex conjugate.

We will assume that \( \bar{f}_\beta(w^*) \) and \( (f_\beta(w))^* \) have the same monodromy structure. In particular we assume firstly that the two sets of functions \( f_\alpha(w) \) and \( g_\beta(w) \equiv (\bar{f}_\beta(w^*))^* \) have the same (algebraic) branch points, \( \{w_k\}_{k=0}^{m+1} \), such

$$ 0 = |w_0| < |w_1| < \cdots < |w_m| < |w_{m+1}| = \infty \quad (A.2) $$

and that are analytic elsewhere. (The relative cuts in the \( w \) plane are taken to stay along the same line \( \gamma \) connecting the \( w_k \)'s and chosen such that \( |w| \) is increasing along it. Cuts in \( w^* \) plane are obtained by \(*\) mapping the cut \( w \) plane.) In the following we will require also that for each choice of \( k = 0, \cdots, m + 1 \), in the limit \( w \to w_k \),

$$ |(w - w_k)f_\alpha(w)| \to 0 \quad (A.3) $$
uniformly on the cut plane. This condition guarantees the convergence of the integral Eq.(A.1) and the validity of some steps of the proof.

Secondly we suppose that $f$ and $g$ have the same related monodromy matrices $M_k$, defined by the requirement

$$f(w_-) = M_k f(w_+).$$  \(\text{(A.4)}\)

$M_k$ above is the same constant matrix for all points $w$ staying in the region of $\gamma$ between $w_{k-1}$ and $w_k$ and $+$ ($-$) means limit from above (below) the cut (similar definition for $g(w)$). In particular $w_{\pm}$ are connected by a path in the cut plane enclosing all $w_j$ with $j < k$.

We assume finally that the matrix $Q$ is invariant by the monodromy group action:

$$\forall k \quad Q = M_k^t Q M_k^*.$$  \(\text{(A.5)}\)

All requirements together enforce the single valuedness of the integrand and the stronger relation

$$f(w'_-) Q \bar{f}(w'_*) = f(w'_+) Q \bar{f}(w'_*)$$  \(\text{(A.6)}\)

for any $|w|, |w'|$ on the region of $\gamma$ between $|w_k|$ and $|w_{k+1}|$ (the same for both).

It follows then, after a simple generalisation of the proof of \cite{28} based on Stokes’ theorem (and using strongly Eq.(A.6)), that we can express $I$ in terms of one dimensional integrals as

$$I = \frac{i}{2} \sum_{k=1}^{m} T^{(k)} \left[ \left( (1 - M_{k+1})^{-1} - (1 - M_k)^{-1} \right)^t Q \right]_{\alpha \beta} \bar{T}^{(k)} \quad \text{(A.7)}$$

where, $^t$ means transposition,

$$T^{(k)} \equiv \int_{C_k} dw f(w)$$  \(\text{(A.8)}\)

$$\bar{T}^{(k)} \equiv \int_{\bar{C}_k} dw^* \bar{f}(w^*)$$  \(\text{(A.9)}\)

and $C_k$ ($\bar{C}_k$) are counter-clockwise (clockwise) circumferences starting at $w_{k+}$ ($w_{k+}^*$) and ending at $w_{k-}$ ($w_{k-}^*$), enclosing all the $w_j$ ($w_j^*$) of lower modulus. We refer to \cite{28} for more details (see also \cite{30} for related integrals).

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Appendix B  More details on computations

The expression for the correlators in Eqs.(2.7-2.9) can be obtained from the results of [25] for the second order correlators of minimal conformal theories. In particular the general structure is

\[
< \phi_{\mu_1}(z_1)\phi_{\mu_2}(z_2)\phi_{\mu_3}(z_3)\phi_{\mu_4}(z_4) > \propto \prod_{i<j} |z_{ij}|^{2\gamma_{ij}} \left( S(a+b+c)S(b)|I_1(a,b,c;\eta)|^2 + S(a)S(c)|I_2(a,b,c;\eta)|^2 \right)
\]

where the ordered multi-index \(\mu = (n, m)\) labels operators and fixes their dimensionality (in our case \(\epsilon\) corresponds to \(\mu = (2, 1)\) while \(\sigma\) to \(\mu = (2, 2)\)), and

\[
S(x) \equiv \sin(\pi x)
\]

\[
\eta \equiv \frac{z_{12}z_{34}}{z_{13}z_{24}}
\]

\[
I_1(a, b, c; \eta) \equiv \frac{\Gamma(-a-b-c-1)\Gamma(b+1)}{\Gamma(-a-c)} {}_2F_1\left(-c, -a-b-c-1; -a-c; \eta\right)
\]

\[
I_2(a, b, c; \eta) \equiv \eta^{1+a+c} \frac{\Gamma(a+1)\Gamma(c+1)}{\Gamma(a+c+2)} {}_2F_1\left(-b, a+1; a+c+2; \eta\right).
\]  \(\text{(B.1)}\)

The relation between the multi-indices \(\mu\) and exponents \(\gamma_{ij}\) and parameters \(a, b, c\) can be found in [25] and will not be reported here. We denote with \(pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; z) \equiv \sum_{k=0}^\infty \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} z^k\) the (generalised) hypergeometric function (where \((a)_k \equiv \Gamma(a+k)/\Gamma(a)\)).

The integrals we are interested in are therefore of the general form:

\[
Z(a, b, c, d, e) \equiv \int d^2\eta d^2z |z|^{2a} |1-z|^{2b} |\eta-z|^{2c} |\eta|^{2d} |1-\eta|^{2e}
\]

\[
= S(a+c)^{-1} \int d^2\eta |\eta|^{2d} |1-\eta|^{2e} \times \left( S(a+b+c)S(b)|I_1(a,b,c,\eta)|^2 + S(a)S(c)|I_2(a,b,c,\eta)|^2 \right).
\]

The second line can be obtained by use of results of [25]. We evaluated the second part of the integral by use of the theorem reported in Appendix A. After the completion of the computation we realized that an equivalent integral could be found in Appendix D of [27] and we checked that the results are the same (modulo some non trivial algebraic relations satisfied by the generalised hypergeometric functions, [32]). We report our expression that, by construction, involves a fewer number of boundary integrals (generalised hypergeometrics) to be computed.
Applying the theorem of Appendix A to our needs, it is easy to realize that in Eq. (A.7) there is only a contribution coming from $I^{(1)}$. If we pose

$$I^{(1)} = 2i(e^{i\pi d} S(d) J_1, e^{i\pi(a+c+d)} S(a+c+d) J_2)$$

and compute the inverse of the monodromy matrix terms, it follows after some algebra that:

$$Z = z_{11} J_1^2 + z_{22} J_2^2 + z_{12} J_1 J_2$$

$$J_1 \equiv \int_0^1 z^d (1-z)^c I_1(a,b,c,z)$$

$$J_2 \equiv \int_0^1 z^d (1-z)^c I_2(a,b,c,z)$$

where $S^{-1}(x) = 1/S(x)$ and

$$J_1 = B(b+1,-a-b-c-1)B(d+1,e+1)F_2(-c,-a-b-c-1,d+1;-a-c,2+d+e;1)$$

$$J_2 = B(a+1,c+1)B(2+a+c+d,e+1)F_2(-b,a+1,2+a+c+d;a+c+2,3+a+c+d+e;1)$$

$$z_{11} = -\frac{1}{4} S(a+c)^{-2} S^{-1}(c+d+e) S^{-1}(a+b+c+d+e) \times$$

$$S(b)S(a+b+c)S(d) \times$$

$$(S(b-c-d) - S(b+c-d) + S(b+c+d) - S(2a+b+c+d)$$

$$- S(b+c+d+2e) + S(2a+b+3c+d+2e))$$

$$z_{22} = \frac{1}{4} S(a+c)^{-2} S^{-1}(c+d+e) S^{-1}(a+b+c+d+e) \times$$

$$S(a)S(c)S(a+c+d) \times$$

$$(S(a+b-d) + S(a-b+d) - S(a+b+d) + S(a+b+2c+d)$$

$$- S(a-b-d-2e) - S(a+b+2c+d+2e))$$

$$z_{12} = 2S(a+c)^{-2} S^{-1}(c+d+e) S^{-1}(a+b+c+d+e) \times$$

$$S(a)S(b)S(c)S(a+b+c)S(d)S(a+c+d)$$

(We assumed the parameters to be real as it is the case of interest).

With the knowledge of $Z(a,b,c,d,e)$ the IR finite part of the regularized "naive" perturbative integrals can be extracted by use of Mellin transform techniques without computing all IR counterterm integrals, as explained in
detail in [23], Section 2-3. Using some known relations of \( _3F_2 \) hypergeometric functions of argument \( z = 1 \), [32], and fixing the normalisation of correlators by imposing clustering properties (see [31]), Eq. (3.5) can be recovered.

**Figure Captions:**

Fig 1: Different estimates of \( < \Theta \Theta >_{\lambda c} \) correlator plotted vs \( mr \): continuous curve is the short distance approach (this paper); dashed curve refers to first two states while dots correspond to the first 11 states in form factor approach, [13].

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