Energy-efficient transmission policies for the linear quadratic control of scalar systems

Yifei Sun\(^1\), Samson Lasaulce\(^2\), Michel Kieffer\(^1\), Romain Postoyan\(^2\), Dragan Nešić\(^3\)

\(^1\)Université Paris-Saclay - CNRS - CentraleSupélec - L2S, F-91192 Gif-sur-Yvette, France
\(^2\)CRAN, CNRS-Université de Lorraine, F-54000 Nancy, France
\(^3\)Department of Electrical and Electronic Engineering, The University of Melbourne, Parkville, 3010, Victoria, Australia.

Abstract – This paper considers controlled scalar systems relying on a lossy wireless feedback channel. In contrast with the existing literature, the focus is not on the system controller but on the wireless transmit power controller that is implemented at the system side for reporting the state to the controller. Such a problem may be of interest, e.g., for the remote control of drones, where communication costs may have to be considered. Determining the power control policy that minimizes the combination of the dynamical system cost and the wireless transmission energy is shown to be a non-trivial optimization problem. It turns out that the recursive structure of the problem can be exploited to determine the optimal power control policy. As illustrated in the numerical performance analysis, in the scenario of a dynamics without perturbations, the optimal power control policy consists in decreasing the transmit power at the right pace. This allows a significant performance gain compared to conventional policies such as the full transmit power policy or the open-loop policy.

1 Introduction

The dominant paradigm in system control theory is to assume that information exchanges between the controller(s) and the system(s) to be controlled are perfect. When information exchanges occur over wireless channels, this assumption may be questionable and even not realistic at all. This is one of the reasons why there is an active research area at the interface between control theory and wireless communications. Among representative research works of this approach, we can quote the following papers. The problem of imperfect communication between the various components of a system is addressed in Hespanha et al. (2007). In Delchamps (1990), the problem of imperfect feedback is considered in the case where the noise is caused by the quantization of transmitted data. In Shi et al. (2013), it is shown how a finite communication data rate impacts the controller design. The impact of fast fading wireless channel fluctuations on the control design has been addressed, e.g., in Gatsis et al. (2011), Varma et al. (2020), and Balaghimaloo et al. (2020). The coexistence of several controlled systems sharing the same communication channel prone to interference is considered in Gatsis et al. (2018).

In the present paper, in contrast with the existing literature, the main technical focus is not on the system controller but on the control of the wireless transmit power implemented at the system side for reporting its state to the controller through a wireless feedback channel. This scenario may be of interest, e.g., in the remote control of drones, when the control input is evaluated by a remote controller from measurements of the state of the drone that are transmitted over a wireless channel. The approach proposed has at least three salient features. First, the transmit power is adapted to the wireless feedback channel statistics and the system state. Second, the objective pursued consists of a combination of a system control objective and a communication objective (namely, the wireless transmission energy); managing the wireless transmit power is both relevant in terms of consumed energy and electromagnetic pollution. Third, this adaptation is performed in the presence of an additive perturbation on the (linear) system dynamics and a multiplicative noise for the wireless feedback channel (which corresponds to data packet erasures). This complete framework has not been addressed yet in the literature even in the simple case of scalar linear systems. Good representatives of the closest literature are Willems and Willems (1976); Besson et al. (2000); Primbs and Sung (2009) where the authors also assume a multiplicative noise model for the communication channel but do not focus on the wireless transmit power control problem by both pursuing a system control objective and a wireless transmission energy objective. Rather the cited papers focus on the problem...
of system stability.

The paper is structured as follows. In Section 2 the technical problem to be solved is formulated. Determining the best power control policy is shown to amount to solving a non-trivial multilinear problem. To solve it, we resort to an iterative search technique described in Section 3. Then, in Section 4, we conduct a numerical performance analysis to illustrate the benefits of controlling properly the wireless transmit power. Conclusions and perspectives are provided in Section 5.

2 Assumptions and Problem Formulation

We consider a dynamical system whose state is scalar. The motivation behind this assumption is that the problem appears to be of interest even in this simple case and this makes the introduction of the proposed framework clearer. The system state is denoted by \( x \in \mathbb{R} \) and assumed to evolve according to the discrete-time state equation

\[
x_{t+1} = ax_t + bu_t + d_t,
\]

where \( t \in \{1, \ldots, T\} \), \( T \geq 1 \) being the considered time horizon, \( (a, b) \in \mathbb{R}^2 \), \( u_t \in \mathbb{R} \) is the control input, and \( d_t \sim \mathcal{N}(0, \sigma_d^2) \) is a Gaussian random state perturbation. One assumes that the random variables \( x_t \) and \( d_t \) are independent for all \( t' \geq t \).

![Fig. 1: Communication and control setup](image)

At time (or for data packet) \( t \), the system transmits its state \( x_t \) to a remote controller in charge of computing the control input \( u_t \), see Figure 1. This transmission is performed with power \( p_t \) over a wireless communication channel modeled by a classical baseband additive communication channel of the form \( y_t = h_t x_t + w_t \), where \( h_t \in \mathbb{C} \) represents the coded signal, \( h_t \in \mathbb{C} \) is the channel coefficient, and \( w_t \sim \mathcal{N}(0, \sigma_w^2) \) the i.i.d. Gaussian communication noise. From this communication model, we will only exploit the two following key quantities: \( g_t = |h_t|^2 \) and the communication noise variance \( \sigma_w^2 \). In this paper, no particular assumptions are made on the channel i.i.d. random process \( (g_t)_{t \in \{1, \ldots, T\}} \) except in Prop. 1 and 5 and for simulations, where we assume an exponential p.d.f. with mean \( \overline{g} \) (that is, a classical Rayleigh fading model for \( h_t \)). The message is assumed to be successfully received when the signal-to-noise (SNR) ratio at the receiver is sufficient to allow error-free decoding. This occurs with probability

\[
\pi(p_t) = \Pr \left[ \frac{g_t p_t}{\sigma_w^2} \geq \gamma \right],
\]

where \( g_t \) is the channel gain at time \( t \), \( \sigma^2 \) the variance of the (additive and zero-mean Gaussian) communication noise, and \( \gamma \) is the SNR threshold. Therefore, the controller receives

\[
\tilde{x}_t = x_t z_t
\]

where \( z_t \sim \text{Ber}(\pi_t) \) is a realization of a Bernoulli random variable with parameter

\[
\pi(p_t) = \Pr[z_t = 1].
\]

When the message is too noisy to be decoded successfully, we have that \( z_t = 0 \), which corresponds in practice to a data packet loss. In what follows, the (packet) success probability \( \pi(p_t) \) is denoted by \( \pi_t \) to make the notations simpler.

We assume that \( z_t \) is known at the receiver (this is typical when the communication system uses a cyclic redundancy check to verify the integrity of the received message). When \( z_t = 1 \), a static feedback is evaluated as

\[
u_t = k \tilde{x}_t = k x_t z_t
\]

where \( k \in \mathbb{R} \). When \( z_t = 0 \), the controller is unable to acquire the state \( x_t \) and the chosen control input is \( u_t = 0 \), as proposed, e.g., in Schenato (2009); Quevedo et al. (2014). The control input is then fed back to the system. We assume that the channel from the controller to the system is perfect. This motivation behind this assumption is twofold. It is fully relevant in communications scenarios where there is an asymmetry in terms of resources (e.g., in terms of transmit power, bandwidth, or computation resources). It also makes our analysis more tractable and easier to interpret for a first step into the direction taken in this paper.

We assume that \( x_1 \sim \mathcal{N}(0, \sigma_x^2) \) and consider an optimal controller with finite horizon \( T \). The considered problem is to find a transmission power policy \( p_{1:T} = (p_1, p_2, \ldots, p_T)^T \), to be applied over the control horizon, that minimizes

\[
\overline{J}_{1:T}(p_{1:T}) = \mathbb{E}_{z_{1:T}, d_{1:T}} \left[ \sum_{t=1}^{T} \left( q z_t^2 + r u_t^2 + p_t \right) \right]
\]

with bounded transmission power

\[
0 \leq p_t \leq P_{\text{max}}, t = 1, \ldots, T,
\]

where \( \mathbb{E}_{z_{1:T}, d_{1:T}}[\cdot] \) indicates that the expectation is performed with respect to \( z_1, \ldots, z_T \) and \( d_1, \ldots, d_T \), \( q > 0 \), and \( r > 0 \). The expectation is performed with respect to \( z_{1:T} \), which depends on the transmission powers, and with respect to \( d_{1:T} \). One has \( z_t \) and \( d_j \) independent for all \( i = 1, \ldots, T \) and \( j = 1, \ldots, T \). The presence of the wireless transmission energy term in the above cost allows the transmission to be energy-efficient (for more
details see, e.g., Lasaulce and Tembine (2011)). Technically, the presence of this term makes the problem non-trivial. Indeed, without any energy cost associated with the wireless system, the problem boils down to a classical finite-horizon LQR (linear quadratic regulation) problem and the cost is trivially minimized by transmitting at full power all the time. Because of the presence of the wireless transmission energy cost, a tradeoff needs to be found between the conventional system cost and the communication cost associated with the feedback channel. Technically, the formulated optimization problem turns out to be non-trivial as explained in the next section, which proposes an iterative numerical technique to determine the optimal transmission power control policy.

### 3 Proposed solution

First, let us reformulate the optimization problem associated with (6) and (7). This is the purpose of the following proposition.

**Proposition 1** Assume that the i.i.d. random process \((g_t)_{t \in \{1, \ldots, T\}}\) follows a Rayleigh fading law with mean \(E(g_t) = \overline{g}\). Denote by \(\pi_{1:T} = (\pi_1, \ldots, \pi_T)^T\) the sequence of success probabilities. The problem of minimizing \(\mathcal{J}_{1:T}(\pi_{1:T})\) with respect to \(\pi_{1:T}\) under the constraints (7) can be reformulated as

\[
\begin{align*}
\min_{\pi_{1:T}} & \quad C(\pi_{1:T}) \\
\text{s.t.} & \quad -\frac{\gamma \sigma^2}{\overline{g} \ln \pi_t} - P_{\text{max}} \leq 0, \quad t = 1 \ldots T
\end{align*}
\]

with

\[
C(\pi_{1:T}) = \sigma^2 \sum_{i=2}^{T} (q + r k^2 \pi_i) + \sigma^2 \sum_{i=2}^{T} \prod_{i=1}^{t-1} (a^2 + (b^2 k^2 + 2abk) \pi_i) \\
+ \sigma^2 \sum_{i=2}^{T} \prod_{i=1}^{t-1} (a^2 + (b^2 k^2 + 2abk) \pi_i) \\
- \frac{\gamma \sigma^2}{\overline{g} \ln \pi_t}.
\]

**Proof 1** See Appendix A.1.

From Proposition 1, we see that the cost function \(C(\pi_{1:T})\) is multilinear w.r.t. the success probability vector \(\pi_{1:T}\), which means that, in general, the cost is neither linear, convex, nor quasi-convex. The corresponding problem is therefore non-trivial, see Bao et al. (2013). As will be shown in what follows, the recursive structure of the problem can be exploited to determine the optimal sequence of probabilities of success and therefore the optimal sequence of transmit power levels. For that purpose, one decomposes (6) as

\[
\mathcal{J}_{1:T}(p_{1:T}) = \mathcal{J}_{1:t-1}(p_{1:t-1}) + \mathcal{J}_{t:T}(p_{1:T})
\]

where

\[
\mathcal{J}_{1:t-1}(p_{1:t-1}) = \mathbb{E}_{z_{1:t-1}, d_{1:t-1}} \left( \sum_{k=1}^{t-1} (q x_k^2 + r k^2 x_k^2 z_k + p_k) \right)
\]

and

\[
\mathcal{J}_{t:T}(p_{1:T}) = \mathbb{E}_{z_{1:t}, d_{1:T}} \left( \sum_{k=t}^{T} (q x_k^2 + r k^2 x_k^2 z_k + p_k) \right).
\]

Furthermore, Proposition 2 separates the terms which contains \(p_t\) (or \(\pi_t\)) from \(\mathcal{J}_{t:T}(p_{1:T})\).

**Proposition 2** In (10), \(\mathcal{J}_{t:T}(p_{1:T})\) is expressed as

\[
\mathcal{J}_{t:T}(p_{1:T}) = \mathbb{E}_{z_{1:t-1}, d_{1:t-1}} [x_t^2] \mathcal{F}(p_t) + \sigma^2 \mathcal{T}_s(p_{t+1:T}) + \sum_{t} p_t,
\]

where, for all \(t < T\)

\[
\mathcal{F}(p_t) = (q + r k^2 \pi_t) \\
+ \sum_{\ell=t+1}^{T} (q + r k^2 \pi_\ell) \prod_{i=t}^{\ell-1} (a^2 + (2abk + b^2 k^2) \pi_i)
\]

and for all \(t < T - 1\)

\[
\mathcal{T}_s(p_{t+1:T}) = \sum_{\ell=t+1}^{T} (q + r k^2 \pi_\ell) \\
\times \sum_{i=t}^{\ell-1} \prod_{r=i+1}^{\ell-1} (a^2 + (2abk + b^2 k^2) \pi_r).
\]

**Proof 2** See Appendix A.2.

In (11), \(\mathcal{F}(p_{1:T})\) depends on \(p_t\) (or \(\pi_t\)) while \(\mathcal{T}_s(p_{t+1:T})\) is independent of them. These two terms can be evaluated by Proposition 3.

**Proposition 3** \(\mathcal{F}(p_{1:T})\) and \(\mathcal{T}_s(p_{t+1:T})\) can be evaluated using the following backward recursions

\[
\mathcal{F}(p_t) = (q + r k^2 \pi_t) + (a^2 + \pi_t (2abk + b^2 k^2)) \mathcal{F}(p_{t+1:T})
\]

for all \(t \leq T - 1\) and

\[
\mathcal{T}_s(p_{t:T}) = \mathcal{F}(p_{t:T}) + \mathcal{T}_s(p_{t+1:T})
\]

for all \(t \leq T - 2\).

**Proof 3** See Appendix A.3.

These backward recursions are initialized considering the transmission power \(p_T\) minimizing (6).

**Proposition 4** The transmission power \(p_T\) at time \(T\) minimizing (6) is \(p_T = 0\) and leads to

\[
\mathcal{F}(p_T) = q
\]

and

\[
\mathcal{T}_s(p_T) = q.
\]
Proof 4 See Appendix A.4.

We can then determine $\pi_t$ minimizing (9) when $\pi_{t'}$ is fixed for all $t' = 1, \ldots, T$, $t \neq t'$. This is shown in Proposition 4.

Proposition 5 Assume a Rayleigh fading law with mean $\overline{g}$ for $g_t$. Consider some $t \in \{1, \ldots, T - 1\}$ and assume that $\pi_{t'}$ is fixed for all $t' = 1, \ldots, T, t \neq t'$. The value of $\pi_t$ minimizing (9) with the constraint (13) is either $\pi_t = 0$ or $\pi_t = \min \left( e^{-\frac{\gamma \sigma^2}{\overline{g}}}, \pi^0 \right)$, where $\pi^0$ is such that $e^{-2} < \pi^0$ and

$$E[x_{t-1} a_{t-1} x_t^2] \frac{\partial}{\partial \pi_t} F(p_t ; T) + \frac{\gamma \sigma^2}{\pi^0} - \frac{\gamma \sigma^2}{\overline{g}} = 0.$$  

Proof 5 See Appendix A.5.

Consider a transmission power policy $p_{1:T}$ and its corresponding $x_{1:T}$. From Proposition 3, for all $t = 1, \ldots, T$, one can obtain

$$\pi^*_t = \arg \min_{\pi_t \in \mathcal{I}} C \left( \pi^{(0)}_1, \ldots, \pi^{(0)}_t, \pi_t, \pi^{(0)}_{t+1}, \ldots, \pi^{(0)}_T \right)$$

where $\mathcal{I} = \{0, \min \left( e^{-\frac{\gamma \sigma^2}{\overline{g}}}, \pi^0 \right) \}$.

A set

$$\mathcal{P} = \left\{ \left[ \pi^*_1, \pi^{(0)}_2, \ldots, \pi^{(0)}_T \right], \ldots, \left[ \pi^{(0)}_1, \pi^*_2, \ldots, \pi^{(0)}_T \right] \right\}$$

of associated success vectors is obtained. The vector

$$\pi^{(1)}_{1:T} = \arg \min_{\pi_{1:T} \in \mathcal{P}} C(\pi_{1:T})$$

and the associated transmission power policy $p^{(1)}_{1:T}$ provides a reduced cost. The above process may be repeated as illustrated in Algorithm 1 to obtain an improved transmission power policy.

In the loop of Algorithm 1, each element of the vector $[p^{(k)}_1, \ldots, p^{(k)}_T]$ is replaced with its updated version $[p^{(k+1)}_1, \ldots, p^{(k+1)}_T]$ which induces a smaller cost. The latter property combined with the fact that the cost is bounded guarantees the convergence of the proposed algorithm. The obtained power policy is then obtained by assuming the knowledge of the average power of $x_1$, that is $\mathbb{E}[x_1^2]$, and not the value of $x_1$ itself.

4 Numerical performance analysis

To study the behavior of Algorithm 1 consider a system with $a = 1.1$, $b = -1$, and $k = 1$, as well as a realization $x_1 = 1$. For the communication, $P_{\text{max}} = 3$ and $\gamma \sigma^2 / \overline{g} = 1$. Moreover, $q = 1$, $r = 0.5$, $T = 30$.

Consider a first scenario with $\sigma^2_d = 0$ (perturbation-free case). Figure 2 illustrates $p_t$ for the considered nominal values of the parameters mentioned before and for alternate parameter values where a single change of one component is performed. This illustrates the impact of each parameter on the transmission power policy.
From Figure 2 we observe that for the nominal value of the parameters, transmissions occur at the beginning and stop at \( t = 8 \). Decreasing \( P_{\text{max}} \) leads to transmissions with less power at the beginning to stop at \( t = 15 \). Decreasing \( a \) leads to a more stable open-loop system, requiring less transmissions. Choosing \( k = 1.8 \), which, in close-loop, is less stable, leads to an increase of communications.

If \( q \) increases, the weight of the state in the cost function increases leading to more control effort. On the contrary, a larger value of \( r \) putting more weight on the transmission costs, reduces the number of transmissions.

Let \( a = 1.1, b = -1, k = 1.8, P_{\text{max}} = 3, \gamma \sigma^2 / \gamma = 1 \) and \( \sigma_x^2 = 1 \). Moreover, \( q = 1, r = 0.5 \) and \( T = 30 \). To illustrate the impact of perturbation, Figure 3 shows the power control policy with different values of \( \sigma_d^2 \).

From Figure 3 we observe that, when \( \sigma_d^2 \) increases, there are less time slot where \( p_t \neq 0 \), indicating more communication will occur. This is due to the fact that increasing the perturbation drives the system away from equilibrium and leads an increase need of communications.

Consider \( a = 1.1, b = -1, k = 1.8, P_{\text{max}} = 3, \gamma \sigma^2 / \gamma = 1, q = 1, r = 0.5, \sigma_x^2 = 1 \) and \( \sigma_d^2 = 0.05 \). Different values of the time horizon \( T \) have been considered. The average value over 10000 samples of (6) is compared for three different policies: sending with full transmit power \( P_{\text{max}} = 3 \), open loop policy (sending nothing), sending with power determined by Algorithm 1 see Figure 4.

Figure 4 shows that, when \( T \leq 6 \), the curve of the proposed algorithm overlaps the one of open loop policy. The reason is that the communication between the system and the controller is not worthy, when the control time horizon is not enough large. When \( T > 6 \), the performance of the control policies obtained by the proposed algorithm is the best compared to the algorithm where we always send with full transmit power and the algorithm

**5 Conclusion**

In this paper, we consider a scalar system which control input is evaluated by a remote controller from information sent by the system over a noisy wireless channel. We focus on the optimization of the transmit power implemented at the system side for reporting the state to the controller. We have shown that determining the power control policy that minimizes the combination of the dynamical system cost and the wireless transmission energy is a non-trivial optimization problem. We have proposed an iterative algorithm to evaluate a transmission power policy achieving a trade-off between the system control cost and the energy spent for wireless transmission. In absence of perturbation on the system dynamics, the optimal transmit power is seen to be decreasing with time. The power profile depends on the values for the system and control parameters. The obtained profiles differ significantly from the profiles consisting in transmitting at full power or not transmitting at all (open loop scenario). Significant gains can be observed when comparing the proposed policy to the aforementioned conventional policies. This work will be extended to the vector case and to situations where the wireless resources have to be shared by several system-controller pairs which may generate interference. A significant extension would be to address the challenging case of non-linear systems, which supposes to revisit the proof techniques used in this paper.
Fig. 4: Impact of the choice of the power control policy: For $T = 30$ using the proposed policy allows the combined cost to be divided by 50.

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A Proofs

A.1 Proof of Proposition

For a Rayleigh fading channel model, one has

$$
\pi(p_t) = \int_{0}^{\infty} \frac{1}{g} \exp \left( -\frac{\gamma}{g} \right) dg = \exp \left( -\frac{\gamma \sigma^2}{p_t g} \right). \quad (12)
$$

From (12), one observes that searching for $p_{1:T}$ minimizing $\pi_1$ under the power constraints $\pi_{t}$ is equivalent to searching for $\pi_{1:T}$ minimizing $\pi_1$ under the constraints

$$
-\frac{\gamma \sigma^2}{\ln \pi_{t,g}} \leq p_{\max}, t = 1, \ldots, T. \quad (13)
$$

Moreover, combining (13) and (15), for all $t \geq 1$, one gets

$$
x_{t+1} = (a + bkz_t) x_t + d_t
$$

$$
= \prod_{\ell=1}^{t} (a + bkz_{\ell}) x_1 + \sum_{\ell=1}^{t} \prod_{r=\ell+1}^{t} (a + bkz_r) d_{\ell} \quad (14)
$$
where, by convention, $$\prod_{t=1}^{\ell-1} (a + bk z_r) = 1$$. From (14), one observes that $$x_{t+1}$$ depends on $$x_1$$, on $$z_1, \ldots, z_t$$, and on $$d_1, \ldots, d_t$$. Now, since $$z_t \sim \text{Ber}(\pi_t)$$, one has

$$\mathbb{E}[z_t] = 1 \times \text{Pr}[z_t = 1] + 0 \times \text{Pr}[z_t = 0] = \pi_t$$

and similarly,

$$\mathbb{E}[z_t^2] = \pi_t.$$  \hfill (16)

Moreover, since $$z_t$$ and $$z_{t'}$$ are independent when $$t \neq t'$$,

$$\mathbb{E}[z_t z_{t'}] = \pi_t \pi_{t'}.$$  \hfill (17)

Then, (9) is obtained by introducing (14) in (6), and using (15), (16), and (17) as follows

$$C(\pi_{1:T}) = \mathbb{E}_{z_{1:T} d_{1:T}} \left[ \sum_{t=1}^T \left( q x_t^2 + r k^2 x_t^2 z_t + p_t \right) \right]$$

$$= \mathbb{E}_{z_{1:T} d_{1:T}} \left[ \sum_{t=1}^T \left( q + r k^2 z_t \right) x_t^2 + \sum_{t=1}^T p_t \right]$$

$$= \mathbb{E}_{z_{1:T} d_{1:T}} \left[ \left( q + r k^2 z_1 \right) x_1^2 + \sum_{t=2}^T \left( q + r k^2 z_t \right) \times \left( \prod_{i=1}^{t-1} \left( a + bk z_i \right) x_1 + \sum_{i=1}^{t-1} \left( \prod_{r=i+1}^{t-1} \left( a + bk z_r \right) \right) \right)^2 \right]$$

$$+ \sum_{t=1}^T p_t$$

since $$d_i, d_j$$ are independent if $$i \neq j$$, and $$\mathbb{E}[d_t] = 0$$,

and the last term of the expectation is vanishing,

$$C(\pi_{1:T}) = \mathbb{E}_{z_{1:T} d_{1:T}} \left[ \sum_{t=1}^T \left( q x_t^2 + r k^2 x_t^2 z_t + p_t \right) \right]$$

$$= \mathbb{E}_{z_{1:T} d_{1:T}} \left[ \sum_{t=1}^T \left( q + r k^2 z_t \right) x_t^2 + \sum_{t=1}^T p_t \right]$$

$$= \mathbb{E}_{z_{1:T} d_{1:T}} \left[ \left( q + r k^2 z_1 \right) x_1^2 + \sum_{t=2}^T \left( q + r k^2 z_t \right) \times \left( \prod_{i=1}^{t-1} \left( a + bk z_i \right) x_1 + \sum_{i=1}^{t-1} \left( \prod_{r=i+1}^{t-1} \left( a + bk z_r \right) \right) \right)^2 \right]$$

$$+ \sum_{t=1}^T p_t.$$  \hfill (18)

A.2 Proof of Proposition 2

Developing the cost function, one gets

$$J_{t:T}(p_{1:T}) = \mathbb{E}_{z_{1:T} d_{1:T}} \left[ \sum_{t=1}^T \left( q x_t^2 + r k^2 x_t^2 z_t + p_t \right) \right]$$

$$= \mathbb{E}_{z_{1:T} d_{1:T}} \left[ \left( q + r k^2 z_1 \right) x_1^2 + \sum_{t=1}^T p_t \right]$$

$$= \mathbb{E}_{z_{1:T} d_{1:T}} \left[ \left( q + r k^2 z_1 \right) x_1^2 + \sum_{t=1}^T p_t \right]$$

$$= \mathbb{E}_{z_{1:T} d_{1:T}} \left[ \left( q + r k^2 z_1 \right) x_1^2 + \sum_{t=1}^T p_t \right]$$

$$= \mathbb{E}_{z_{1:T} d_{1:T}} \left[ \left( q + r k^2 z_1 \right) x_1^2 + \sum_{t=1}^T p_t \right]$$

$$+ \sum_{t=1}^T p_t.$$  \hfill (19)
Since the expected value of $x_1^2$ is independent of $z_{t:T}$ and $d_{t:T-1}$, (19) becomes

\[
\mathcal{F}_{1:T}(p_{1:T}) = \mathbb{E}_{z_{t:T}} \left[\left( x_1^2 + \sum_{\ell=t+1}^{T} (q + r k z_{\ell}) + \prod_{i=t}^{\ell-1} (a + b k z_i)^2 + \prod_{i=t}^{\ell-1} d_i^2 \prod_{r=i+1}^{\ell-1} (a + b k z_r)^2 \right) + \sum_{\ell=t}^{T} p_{\ell}\right]
\]

Then, introducing

\[
\mathcal{F}(p_{1:T}) = \mathbb{E}_{z_{1:T}} \left[ \left( q + r k z_1 \right) + \sum_{\ell=t+1}^{T} (q + r k z_{\ell}) + \prod_{i=t}^{\ell-1} (a + b k z_i)^2 + \sum_{i=t}^{\ell-1} d_i^2 \prod_{r=i+1}^{\ell-1} (a + b k z_r)^2 \right] + \sum_{\ell=t}^{T} p_{\ell}.
\]

Using (15) and (16), one obtains:

\[
\mathcal{F}_{1:T}(p_{1:T}) = \mathbb{E}_{z_{1:T}} \left[ \left( x_1^2 + \sum_{\ell=t+1}^{T} (q + r k z_{\ell}) + \prod_{i=t}^{\ell-1} (a + b k z_i)^2 + \prod_{i=t}^{\ell-1} d_i^2 \prod_{r=i+1}^{\ell-1} (a + b k z_r)^2 \right) + \sum_{\ell=t}^{T} p_{\ell}\right]
\]

Then

\[
\mathcal{F}_{1:T}(p_{1:T}) = \mathbb{E}_{z_{1:T}} \left[ \left( q + r k z_1 \right) + \sum_{\ell=t+1}^{T} (q + r k z_{\ell}) + \prod_{i=t}^{\ell-1} (a + b k z_i)^2 + \sum_{i=t}^{\ell-1} d_i^2 \prod_{r=i+1}^{\ell-1} (a + b k z_r)^2 \right] + \sum_{\ell=t}^{T} p_{\ell}.
\]

leading to the backward recursions

\[
\mathcal{F}(p_{t:T}) = \mathbb{E}_{z_{t:T}} \left[ \left( q + r k z_1 \right) + \sum_{\ell=t+1}^{T} (q + r k z_{\ell}) + \prod_{i=t}^{\ell-1} (a + b k z_i)^2 + \sum_{i=t}^{\ell-1} d_i^2 \prod_{r=i+1}^{\ell-1} (a + b k z_r)^2 \right] + \sum_{\ell=t}^{T} p_{\ell}.
\]

A.3 Proof of Proposition 3

From Proposition 2 one has that:

\[
\mathcal{J}_{1:T}(p_{1:T}) = \mathbb{E}_{z_{1:T}} \left[ \left( q + r k z_1 \right) x_1^2 + \sum_{\ell=t+1}^{T} (q + r k z_{\ell}) x_1^2 + \prod_{i=t}^{\ell-1} (a + b k z_i)^2 + \prod_{i=t}^{\ell-1} d_i^2 \prod_{r=i+1}^{\ell-1} (a + b k z_r)^2 \right] + \sum_{\ell=t}^{T} p_{\ell}.
\]

A.4 Proof of Proposition 4

Writing (10) at $t = T - 1$, one gets

\[
\mathcal{J}_{1:T}(p_{1:T}) = \mathcal{J}_{1:T-1}(p_{1:T-1}) + \mathcal{J}_{T:T}(p_{1:T}).
\]

Since $\mathcal{J}_{1:T-1}(p_{1:T-1})$ does not depend on $p_T$, the value of $p_T$ minimizing $\mathcal{J}_{1:T}(p_{1:T})$ has to minimize

\[
\mathcal{J}_{T:T}(p_{1:T}) = \mathbb{E}_{z_{T:T}} \left[ q x_T^2 + r k z_T x_T^2 + p_T \right] = \mathbb{E}_{z_{T:T}} \left[ x_T^2 \right] F(p_T) + p_T,
\]

with $F(p_T) = \mathbb{E}_{z_T} \left[ q + r k^2 z_T \right]$. From the above expressions, one sees that $p_T = 0$ and thus $z_T = 0$ (absence of transmission) minimizes $\mathcal{J}_{T:T}(p_{1:T})$. When $z_T = 0$, one gets $F(0) = q$. 
At time $t = T - 1$
\[ \mathcal{J}_{T-1:T} (p_{1:T}) \]
\[ = \mathbb{E}_{z_{1:T}, d_{1:T-1}} \left[ \frac{1}{2} \left( x_{T-1}^2 + p_T + (q + r k^2 z_{T-1}) x_{T-1}^2 + p_{T-1} \right) \right] \]
\[ = \mathbb{E}_{z_{1:T}, d_{1:T-1}} \left[ \frac{1}{2} \left( (q + b k z_{T-1}) x_{T-1} + d_{T-1} \right)^2 + p_T + (q + r k^2 z_{T-1}) x_{T-1}^2 + p_{T-1} \right] \]
\[ = \mathbb{E}_{z_{1:T}, d_{1:T-1}} \left[ \frac{1}{2} \left( (q + r k^2) x_{T-1} + q d_{T-1} \right)^2 + p_T + (q + r k^2 z_{T-1}) x_{T-1}^2 + p_{T-1} \right] \]
\[ = \mathbb{E}_{z_{1:T}, d_{1:T-1}} \left[ (q + b k z_{T-1}) x_{T-1} + q d_{T-1} \right)^2 + p_T + (q + r k^2 z_{T-1}) x_{T-1}^2 + p_{T-1} \right] \]
\[ = \mathbb{E}_{z_{1:T}, d_{1:T-2}} \left[ x_{T-1}^2 \right] \left( (q + r k^2 \pi_{T-1}) \right) + (a^2 + (b^2 k^2 + 2 b k z_{T-1}) \pi_{T-1}) q + \sigma_\beta^2 q + \sum_{t = T-1} p_t. \]

Using Proposition 3 with $\mathcal{F}(p_T) = q$, one can derive
\[ \mathcal{F}(p_{T-1:T}) = \left( q + r k^2 \pi_{T-1} \right)^2 + \left( a^2 + (2 a b k + b^2 k^2) \pi_{T-1} \right) q \]
and
\[ \mathcal{J}_{T-1:T} (p_{1:T}) = \mathbb{E}_{z_{1:T-2}, d_{1:T-2}} \left[ x_{T-1}^2 \right] \mathcal{F}(p_{T-1:T}) + \sigma_\beta^2 q + \sum_{t = T-1} p_t. \]

From Proposition 2
\[ \mathcal{J}_{T-1:T} (p_{1:T}) = \mathbb{E}_{z_{1:T-2}, d_{1:T-2}} \left[ x_{T-1}^2 \right] \mathcal{F}(p_{T-1:T}) + \sigma_\beta^2 q + \sum_{t = T-1} p_t. \]

A.5 Proof of Proposition 5

To determine the value of $\pi_t$ which minimizes (9), consider (10) and evaluate
\[ \frac{\partial \mathcal{J}_{1:T}}{\partial \pi_t} = \frac{\partial}{\partial \pi_t} \left( \mathcal{J}_{1:t-1} (p_{1:t-1}) + \mathcal{J}_{T:T} (p_{1:T}) \right). \]

Since $\mathcal{J}_{1:t-1} (p_{1:t-1})$ does not depend on $\pi_t$, using Proposition 2 one obtains
\[ \frac{\partial \mathcal{J}_{1:T}}{\partial \pi_t} = \frac{\partial}{\partial \pi_t} \mathcal{J}_{T:T} (p_{1:T}) \]
\[ = \mathbb{E}_{z_{1:t-1}, d_{1:t-1}} \left[ x_{t-1}^2 \right] \frac{\partial}{\partial \pi_t} \mathcal{F}(p_{t-1:T}) \]
\[ + \frac{\partial p_t}{\partial \pi_t} + \sigma_\beta^2 \frac{\partial}{\partial \pi_t} \mathcal{F}(p_{t-1:T}) \]
\[ = \mathbb{E}_{z_{1:t-1}, d_{1:t-1}} \left[ x_{t-1}^2 \right] \left( \frac{\partial}{\partial \pi_t} \mathcal{F}(p_{t-1:T}) \right) \]
\[ + \frac{\gamma \sigma_\beta^2}{\pi_t \ln \pi_t} \]
\[ + \frac{\gamma \sigma_\beta^2}{\pi_t \ln \pi_t}. \] (20)

We can determine $\mathcal{F}(p_{t+1:T})$ using Proposition 3. We also need evaluate $\mathbb{E}_{z_{t-1}, d_{t-1}} \left[ x_{t}^2 \right]$ by forward recursion. From (14), we have
\[ \mathbb{E}_{z_{t-1}, d_{t-1}} \left[ x_{t}^2 \right] = \mathbb{E}_{z_{t-1}, d_{t-1}} \left[ (a + b k z_{t-1}) x_{t}^2 + d_{t}^2 + 2 d_{t} (a + b k z_{t-1}) x_{t} \right] \]
\[ = (a^2 + \pi_t (b^2 k^2 + 2 b k z x_{t-1})) \mathbb{E}_{z_{t-1}, d_{t-1}} \left[ x_{t}^2 \right] + \sigma_\beta^2. \] (21)

The noise does not appear explicitly in (20). But from (21) the additive noise still affect on the derivation $\frac{\partial \mathcal{J}_{1:T}}{\partial \pi_t}$ through $\mathbb{E}_{z_{t-1}} \left[ x_{t}^2 \right]$.

We have a function
\[ \frac{\partial \mathcal{J}_{1:T}}{\partial \pi_t} = \mathbb{E}_{z_{t-1}, d_{t-1}} \left[ x_{t}^2 \right] \times \]
\[ \left( \frac{\partial}{\partial \pi_t} \mathcal{F}(p_{t-1:T}) \right) \]
\[ + \left( \frac{\gamma \sigma_\beta^2}{\pi_t \ln \pi_t} \right). \]

From (13), one has
\[ 0 < \pi_t < e^{-\frac{2 \gamma \sigma_\beta^2}{\pi_t \ln \pi_t}} \leq 1. \]

The first component of $\frac{\partial \mathcal{J}_{1:T}}{\partial \pi_t}$ is independent of $\pi_t$ and the second component $\frac{\gamma \sigma_\beta^2}{\pi_t \ln \pi_t}$ is a function of $\pi_t$. Consequently, $\frac{\partial \mathcal{J}_{1:T}}{\partial \pi_t}$ is minimum when $\frac{\gamma \sigma_\beta^2}{\pi_t \ln \pi_t}$ is minimum. The derivative
\[ \frac{\partial}{\partial \pi_t} \left( \frac{\gamma \sigma_\beta^2}{\pi_t \ln \pi_t} \right) = - \frac{\ln^2 \pi_t - 2 \ln \pi_t}{(\gamma \sigma_\beta^2 \ln \pi_t)^2} \] (22)
vanishes when $\ln \pi_t = 0$ or $\ln \pi_t = -\gamma \sigma_\beta^2$, i.e., when $\pi_t = 1$ or $\pi_t = e^{-2}$. When $\pi_t \in [0, e^{-2}]$, (22) is negative and when $\pi_t \in [e^{-2}, 1]$, (22) is positive. Thus the minimum of $\frac{\partial \mathcal{J}_{1:T}}{\partial \pi_t}$ is obtained for $\pi_t = e^{-2}$ if $e^{-2} < \frac{2 \gamma \sigma_\beta^2}{\pi_t \ln \pi_t}$. Else, the minimum of $\frac{\partial \mathcal{J}_{1:T}}{\partial \pi_t}$ is obtained for $\pi_t = e^{-\frac{2 \gamma \sigma_\beta^2}{\pi_t \ln \pi_t}}$.

The minimum is obtained for $\pi' = e^{-\frac{2 \gamma \sigma_\beta^2}{\pi_t \ln \pi_t}}$.

Assume that the minimum value of the derivative $\frac{\partial \mathcal{J}_{1:T}}{\partial \pi_t}$, is negative, $\exists \delta, \Delta = \| \pi' - \Delta \|$ such that, $\pi' \in \Delta$, $\frac{\partial \mathcal{J}_{1:T}}{\partial \pi_t} < 0$ leading to a decrease of $\mathcal{J}_{1:T}$.

Assume that the minimum value of $\frac{\partial \mathcal{J}_{1:T}}{\partial \pi_t}$ is negative and is obtained when $\pi_t = e^{-\frac{2 \gamma \sigma_\beta^2}{\pi_t \ln \pi_t}}$. The minimum of $\mathcal{J}_{1:T}$ over the interval $[0, e^{-\frac{2 \gamma \sigma_\beta^2}{\pi_t \ln \pi_t}}]$ is then either obtained when $\pi_t = 0$ or when $\pi_t = e^{-\frac{2 \gamma \sigma_\beta^2}{\pi_t \ln \pi_t}}$. Assume now that the minimum value of $\frac{\partial \mathcal{J}_{1:T}}{\partial \pi_t}$ is negative and obtained when $\pi_t = e^{-2} < e^{-\frac{2 \gamma \sigma_\beta^2}{\pi_t \ln \pi_t}}$. The minimum of $\mathcal{J}_{1:T}$ over the interval $[0, e^{-\frac{2 \gamma \sigma_\beta^2}{\pi_t \ln \pi_t}}]$ is then obtained when $\pi_t = 0$, $\pi_t = e^{-\frac{2 \gamma \sigma_\beta^2}{\pi_t \ln \pi_t}}$, or $\pi_t = 0$, where $\pi_0$ is such that $e^{-2} < \pi_0 < e^{-\frac{2 \gamma \sigma_\beta^2}{\pi_t \ln \pi_t}}$ and $\mathbb{E}_{z_{t-1}} \left[ x_{t}^2 \right] \frac{\partial}{\partial \pi_t} \left( \mathcal{F}(p_{t}, \ldots, p_T) \right) + \frac{2 \gamma \sigma_\beta^2}{\pi_t \ln \pi_t} = 0$. 


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