Soliton solutions for higher grading affine Toda from quantum algebras

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Abstract. Based on quantum Lie algebras associated to affine Lie algebras we construct special soliton-like solutions for higher grading affine Toda models.

1. Introduction
The group-theoretical (Lie-algebraic) method [6, 7] of construction and integration of two-dimensional exactly solvable models is one of the most powerful. Using the representation theory for conformal and affine Toda models [9, 7] one is able to reconstruct the general solutions to systems as well as to derive special, in particular, solitonic solutions both in classical and quantum regions. The zero curvature condition applied to elements of connections constructed as sums of products of arbitrary functions with generators of (higher) grading subspaces [5] of a Lie algebra results in systems of equations [3].

Previously, general solution to the higher grading affine Toda equation were obtained in [3]. In [9] group-theoretical specialization for the lower grading subspace were formulated, and classical soliton solutions obtained in frames of algebraic approach to integrable models. In [10] we proposed quantum Lie algebra solutions generating solitons based on further algebraic properties of quantum groups.

In this paper we consider the case of affine Toda models constructed using algebraic elements that belong to grading subspaces up to some number $l$. The group–theoretical approach allows to extract special solitonic solutions from general ones. The lower grading case for the quantized version of the affine Toda models (the sine–Gordon equation) involving the quantum Lie algebra $[1] (sl_2)_q$ was considered in [10]. Here we extended that construction for a higher grading subspace case. Exactly solvable dynamical systems appearing in this the higher grading formulation are generalizations of lower ($l = 1$) cases, and possess more complicated and deep properties describing more complicated physical situations. It is important to take into account next terms in more general equations in order to understand interrelations of ordinary solutions (to equations associated to lower grading algebraic generators) with solutions corresponding to higher grading.

2. Higher grading affine Toda system
In this and the next sections we recall [3] the affine Toda system construction. Consider a two dimensional manifold $\mathcal{M}$ with local coordinates $z_{\pm}$. Up to a gauge transformation, $(1,0)$-component lying in (see subsection 7.1 of Appendix) $\bigoplus_{n=0}^{l} \hat{G}_n$ and $(0,1)$-component in...
\[ \bigoplus_{n=0}^{l} \hat{G}_n \] of a flat connection \( A \) in the trivial holomorphic principal fibre bundle \( \mathcal{M} \times \hat{G} \rightarrow \mathcal{M} \) \((l > 0 \text{ is a fixed integer})\) satisfy the zero curvature condition

\[ \partial_+ A_- - \partial_- A_+ + [A_+, A_-] = 0. \tag{1} \]

The components \( A_{\pm} \) are the following (we keep notations of [3])

\[ A_+ = -B F^+ B^{-1}, \quad A_- = -\partial_+ B B^{-1} + F^- \tag{2} \]

Here \( B \) is a mapping \( \mathcal{M} \rightarrow \hat{G}_0 \) (\( \hat{G}_0 \) is a group with the Lie algebra \( \hat{g}_0 \)) and \( F^\pm (1 \leq m \leq l - 1) \) are mappings to \( \bigoplus_{n=1}^{l} \hat{g}_n \)

\[ F^\pm = E_{\pm l} + \sum_{m=1}^{l-1} F^\pm_m, \tag{3} \]

where \( E_{\pm l} \) are some fixed elements of \( \hat{g}_{\pm l} \) and \( F^\pm_m \in \hat{g}_{\pm m} \), \((1 \leq m \leq l - 1)\). Substituting (2) into (1) one arrives at the equations of motion

\[ \partial_+ (\partial_- B B^{-1}) = [E_{-I}, B E_I B^{-1}] + \sum_{n=1}^{l-1} [F_{n-}, B F_n^+ B^{-1}] \tag{4} \]

\[ \partial_- F^+_m = [E_I, B^{-1} F_{I-m}^- B] + \sum_{n=1}^{l-m-1} [F_{n+m}^+, B^{-1} F_n^- B], \tag{5} \]

\[ \partial_+ F^-_m = -[E_{-I}, B F_{I-m}^+ B^{-1}] - \sum_{n=1}^{l-m-1} [F_{n+m}^-, B F_n^+ B^{-1}]. \tag{6} \]

Since \( Q_s, C \in \hat{g}_0 \) then \( B \) can be parameterized as \( B = b e^{\eta Q_s} e^{\nu C} \) where \( b \) is a mapping to \( G_0 \), the subgroup of \( \hat{G}_0 \) generated by all elements of \( \hat{g}_0 \) other than \( Q_s \) and \( C \). Substituting \( B \) into the equations of motion (4–6) one has

\[ \partial_+ (\partial_- b b^{-1}) + \partial_+ \partial_- \nu C = e^{\eta}[E_{-I}, b E_I b^{-1}] + \sum_{n=1}^{l-1} e^{\nu n} [F_{n-}, b F_n b^{-1}], \tag{7} \]

\[ \partial_- F^+_m = e^{(l-m)\eta} [E_I, b^{-1} F_{I-m}^- B] + \sum_{n=1}^{l-m-1} e^{\nu n} [F_{m+n}^+, b^{-1} F_n^- b], \tag{8} \]

\[ \partial_+ F^-_m = -e^{(l-m)\eta} [E_{-I}, b F_{I-m}^+ b^{-1}] - \sum_{n=1}^{l-m-1} e^{\nu n} [F_{m+n}^-, b F_n^+ b^{-1}], \tag{9} \]

\[ \partial_+ \partial_- \eta Q_s = 0. \tag{10} \]

Now consider the case \( l = 1 \). Let us parameterize the element \( B \) in principal grading of \( \hat{g} \) and take \( b = e^{(\phi(x_+ x_-, \phi) H^0)}, \chi_0(x_\pm) = e^{(\bar{\phi}(x_\pm) H^0)}, \) \((\phi, H^0) = \sum_{i=1}^2 \phi_i H^0_i \). From the equations (7–10) for an infinite dimensional Lie algebra \( \hat{g} \) in the principal grading we obtain the affine Toda field theory systems of equations

\[ \partial_+ \partial_- \phi + \frac{4\mu}{\beta} \sum_{i=1}^r \left( m_i \alpha_i \right) \exp(\beta \alpha_i \cdot \phi) - \frac{\kappa}{\beta} \exp(-\beta \cdot \phi) = 0. \tag{11} \]

The formal general solution to the above equation was introduced in [9]:

\[ e^{-\beta \lambda_i \phi} = e^{-\beta \lambda_i \phi_0} \cdot \frac{\lambda_i (\lambda_i + 1 \mu_{+}(z) \mu_{-}(z) \lambda_i (\lambda_i + 1))}{\lambda_i (\lambda_i + 1 \mu_{+}(z) \mu_{-}(z) \lambda_i (\lambda_i + 1))} = e^{-\beta \lambda_i \phi_0} \cdot \frac{(\lambda_i + 1 \mu_{+}(z) \mu_{-}(z) \lambda_i (\lambda_i + 1))}{(\lambda_i + 1 \mu_{+}(z) \mu_{-}(z) \lambda_i (\lambda_i + 1))} \tag{12} \]
The fact that (12) is indeed a solution to (11) may be checked by using the representation theory of \( \mathcal{G} \). A map \( g : \mathcal{M} \rightarrow G \) appearing in the gradient form of the flat connection \( A_\pm = g^{-1} \partial_\pm g \), may be factorized (according to the Lie algebra decomposition \( \mathcal{G} = \mathcal{G}_- \oplus \mathcal{G}_0 \oplus \mathcal{G}_+ \)) by the modified Gauss decomposition \( g = \mu - \nu + \gamma_0 - \) or \( g = \mu + \nu - \gamma_0 + \) with maps \( \gamma_0 : \mathcal{M} \rightarrow G_0, \mu_\pm, \nu : \mathcal{M} \rightarrow G_\pm \). The grading condition provides the holomorphic property of \( \mu_\pm, \) i.e., they satisfy the initial value problem

\[
\partial_\pm \mu_\pm(z_\pm) = \mu_\pm(z_\pm) \mathcal{E}_\pm(z_\pm), 
\]

with arbitrary functions \( \Phi^\pm_\alpha \) determining the general solution to the system. Note that the summations in (14) are performed over the set of positive roots \( \Delta^+_m \) of \( \mathcal{G} = \sum_{m \in \mathbb{Z}} \mathcal{G}_m \) in the subspace \( \mathcal{G}_m \). Using notations of [3], \( \gamma_0 \equiv \gamma_0 + \gamma_0 \), and \( \gamma_0(x_\pm) : \mathcal{M} \rightarrow G_0 \) are arbitrary mappings. The element \( B \) is parameterized as \( B^{-1} = (\gamma_0(x_+))^{-1} \gamma_0(\gamma_0(x_-)) \).

### 3. Generalization of the affine Toda equations

Consider for example the case of \( sl_2 \) in principal grading (see subsection 7.1 of Appendix). The mapping \( B \) is parameterised as

\[
B = e^{\varphi \hat{H}^0 + \psi \hat{C}} e^{\eta Q} = e^{\varphi \hat{H}^0 + \psi \hat{C}} e^{\eta Q}, 
\]

where \( \hat{H}^0 = H^0 - \frac{1}{2} C \) is Cartan element, and \( \nu = \varphi - \frac{1}{2} \varphi \). The case \( l = 3 \) delivers a generalization of (11). In this case \( E_3 = a_+ E^1_+ + a_- E^2_-, E_{-3} = a_+ E^{-2}_+ + a_- E^{-1}_- \), [3] and therefore \( [E_3, E_{-3}] = 3a_+ a_- C \equiv \beta C \). Introduce the notations

\[
E_1 = a_+ E^0_+ + a_- E^1_-, \quad E_{-1} = a_+ E^{-1}_+ + a_- E^0_-, \quad f_1 = a_+ E^0_+ - a_- E^1_-, \quad f_{-1} = a_+ E^{-1}_+ - a_- E^0_-, \\
f_2 = -\sqrt{a_+ a_-} H^1, \quad f_{-2} = -\sqrt{a_+ a_-} H^{-1}.
\]

The fields of the model are in (15) as well as matter fields \( \psi^{i R/L} \), \( i = 1, 2 \), \( \chi \) defined as

\[
F^+_1 = \psi^2 R f_1 + \chi E_1, \quad F^+_2 = \psi^1 R f_2, \quad F^-_1 = \psi^1 L f_{-1} + \chi E_{-1}, \quad F^-_2 = \psi^2 L f_{-2},
\]

\( \psi^{i R/L}, \ i = 1, 2 \), are components of two the Dirac fields \( \psi^i \). It follows then that

\[
[E_3, [E_{-3}, E_{\pm 1}]] = 0, \quad [E_3, [E_{-3}, \hat{H}^0]] = 4a_+ a_- \hat{H}^0, \quad [E_{3}, [E_{-3}, f_i]] = 4a_+ a_- f_i, \ i = \pm 1, \pm 2.
\]
Using (7–10), the equations of motion are

$$\partial_+ \partial_- \varphi = -a_+ a_- \left((e^{2\varphi} - e^{-2\varphi}) \left(e^{3\eta} - e^{\eta} \left(\psi_L^1 \psi_R^2 - \chi_+ \chi_-\right)\right) - e^{\eta} \left(e^{2\varphi} + e^{-2\varphi}\right) \left(\psi_L^2 \chi_+ - \psi_R^1 \chi_+\right)\right),$$

$$\partial_+ \partial_- \nu = -a_+ a_- \left((e^{2\varphi} + e^{-2\varphi}) \left(\frac{3}{2} e^{3\eta} - \frac{1}{2} e^{\eta} \left(\psi_L^1 \psi_R^2 - \chi_+ \chi_-\right)\right) - \frac{1}{2} e^{\eta} \left(e^{2\varphi} - e^{-2\varphi}\right) \left(\psi_L^1 \chi_- + \psi_R^2 \chi_-\right) + 2e^{2\eta} \psi_L^1 \psi_R^1\right),$$

$$\partial_- \psi_R^1 = \sqrt{a_+ a_-} e^{\eta} \left(\psi_L^1 \left(e^{2\varphi} + e^{-2\varphi}\right) - \chi_\left(e^{2\varphi} - e^{-2\varphi}\right)\right),$$

$$\partial_- \psi_R^2 = \sqrt{a_+ a_-} e^{\eta} \left(-2\psi_R^1 + \frac{1}{2} e^\eta \psi_R^2 \left(\psi_L^1 \left(e^{2\varphi} - e^{-2\varphi}\right) + \chi_\left(e^{2\varphi} + e^{-2\varphi}\right)\right)\right),$$

$$\partial_- \chi_+ = \sqrt{a_+ a_-} e^{\eta} \psi_R^1 \left(\chi_- \left(e^{2\varphi} - e^{-2\varphi}\right) - \psi_L^2 \left(e^{2\varphi} + e^{-2\varphi}\right)\right),$$

$$\partial_- \chi_- = \sqrt{a_+ a_-} e^{\eta} \psi_L^2 \left(\chi_+ \left(e^{2\varphi} - e^{-2\varphi}\right) + \psi_R^2 \left(e^{2\varphi} + e^{-2\varphi}\right)\right),$$

$$\partial_+ \partial_- \eta = 0.$$  

The general solutions to the matter fields $F_{ij}^\pm$ may be written in the following form. For $m = 1$ in (7–10) one has [3]

$$\langle i | F_1^+ | i; i \rangle = \sum_{i=0}^\infty k_i (\phi_i - \phi) e^{\mu_+ \partial_+} \left(\langle i | \mu_+^{-1} \mu_- | i; i \rangle \langle i | \mu_+^{-1} \mu_- \eta | i \rangle\right).$$

Here $| i; i \rangle$ denotes an element of the Verma module which is result of the action of the lowering generator on the highest state vector.

4. Solitonic solutions from general solutions

In [9] it was shown how to extract solitonic solutions from the formal general solutions of the affine Toda field equations. Let's take $\mu_0^\pm = 1$ in (12) to be a constant function. Then the mappings $\mu_\pm$ are $\mu_\pm = \mu_0^\pm e^{\pm \hat{\epsilon}_\pm}$ with $\mu_0^\pm$ being some fixed mappings independent of $\hat{\epsilon}_\pm$. Next take $\hat{\epsilon}_\pm$ in (14) as $\hat{\epsilon}_\pm = E_{\pm 1} + \sum_{N=1}^{l\pm} c_N E_{\pm N}$ where $E_{\pm}$ are elements of a Heisenberg subalgebra of $\widehat{G}$, namely $[E_+, E_-] = \Omega C$. One can consider principal of homogenous Heisenberg subalgebras for that purpose. In this paper we only deal with the principal case while the homogeneous case will be discussed elsewhere. Thus, we arrive at a special solution to (11)

$$e^{-\beta \lambda_i \phi} = e^{-\beta \lambda_i \phi_0 \left(\langle \lambda_i | e^{\beta \lambda_i \phi} \langle \lambda_i | \lambda_i \rangle^{(1)}\right) \left(\lambda_i e^{\beta \lambda_i \phi} \langle \lambda_i | \lambda_i \rangle^{(1)}\right)^{-1}} - \lambda_i e^{\beta \lambda_i \phi} \langle \lambda_i | \lambda_i \rangle^{(1)}.$$ 

In order to compute these solutions explicitly we have to remove $\hat{\epsilon}_\pm$-dependence from (26) moving $\hat{\epsilon}_+$ to the right and $\hat{\epsilon}_-$ to the left. Then we should find such a $\mu_0 = \prod_{i=1}^N e^{\hat{\epsilon}_i}$ so that $\nu_i$ would be eigenvectors with respect to the adjoint action of $\hat{\epsilon}_\pm$, i.e., $[E_\pm, \nu_i] = \omega_i \nu_i$. Then it turns out [9] that resulting expressions provide us with solitonic solutions to the equations under considerations while parameters $\omega_i$ characterize solitons.

In the next sections we will explain the affinization $(sl_2)_q$ of the quantum Lie algebra $(sl_2)_q$ [1] and then apply those formulations to the construction of $q$-deformed solutions to the quantised [8] affine Toda equations in the specific example of the higher grading the sine–Gordon equation.
5. Quantum Lie algebra
5.1. Universal enveloping algebra $U_q(sl_2)$

The complex Lie algebra $sl_2$ is generated by three elements $X_+,$ $X_-$ and $H$ satisfying

$$HX^\pm - X^\pm H = \pm 2X^\pm, \quad X^+X^- - X^-X^+ = H.$$

The Lie bracket is skew-symmetric and satisfies Jacobi identity. The quantized enveloping algebra $U_q(sl_2)$ [2], [4] is an associative algebra generated by $X^+, X^-, h$ with $q$-deformed commutator relations

$$X^+X^- - X^-X^+ = \frac{q^H - q^{-H}}{q - q^{-1}}, \quad HX^\pm - X^\pm H = \pm 2X^\pm,$$

It possesses a Hopf algebra structure with deformed adjoint action

$$(ad_{X^\pm})_q a = X^\pm a q^{H/2} - q^{1/2} a X^\pm, \quad (ad_H)_q a = H a - a H, \quad (27)$$

(for all $a \in U_q(sl_2)$).

5.2. Quantum algebra $(sl_2)_q$

In [1] new generators in the quantized universal enveloping algebra $U_q(\mathfrak{sl}_2)$ were introduced as

$$X^\pm_h = \sqrt{\frac{2}{q + q^{-1}}} q^{H/2} X^\pm, \quad H_h = \frac{2}{q + q^{-1}} \left( qX^+X^- - q^{-1}X^-X^+ \right). \quad (28)$$

It forms a three-dimensional subspace $(sl_2)_q = X^+_h, X^-_h, H_h$ in $U_q(sl_2)$, closed under the quantum Lie bracket

$$[a, b]_h = (ad_a)_q b, \quad (29)$$

$a, b \in (sl_2)_q$. These generators possesses the following commutation relations

$$[H_h, X^\pm_h]_h = \pm 2q^{1/2} X^\pm_h, \quad [X^\pm_h, H_h]_h = \mp 2q^{1/2} X^\pm_h, \quad [X^+_h, X^-_h]_h = H_h,$$

$$[X^-_h, X^+_h]_h = -H_h \quad [H_h, H_h]_h = 2(q - q^{-1}) H_h, \quad [X^\pm_h, X^\pm_h]_h = 0. \quad (30)$$

We should mention that $(sl_2)_q$ is not a Lie algebra in the standard sense. The generators $(28)$ do not satisfy Jacobi identity and $q$-deformed Lie bracket is not skew-symmetric. Although $q$-analog of Jacobi identity for $(sl_2)_q$ is still absent, nevertheless (29) is $q$-skew-symmetric, [1]. Under $q$-skew-symmetry we mean a symmetry under $q$-conjugation, (that we will denote by tilde) an automorphism of $(sl_2)_q$ defined by $q \mapsto \frac{1}{q}$. Then for any element of $(sl_2)_q (a X^+_h + b X^-_h + c H_h) = \tilde{a} X^+_h + \tilde{b} X^-_h + \tilde{c} H_h, \quad (a, b, c \in C)$ and $q$-deformed Lie bracket satisfies

$$(a, b)]_h = [\tilde{b}, \tilde{a}]_h.$$  

5.3. Affinization of the $(sl_2)_q$

In [10] an affinized version, the principal Heisenberg subalgebra, an eigenvalue vertex operator, and representations of the quantum algebra $(sl_2)_q$ were considered. In this section and subsection 7.2 of Appendix we recall these constructions and then apply them in section 6 to the construction of solutions for the higher grading the sine–Gordon equation. Let $\mathcal{G} = (sl_2)_q$ and $\mathcal{L} = C[t, t^{-1}]$ be the algebra of Lourant polynomials in $t$ and let $\mathcal{L}(\mathcal{G}) = \mathcal{L} \otimes_C \mathcal{G}$. Introduce a complex vector space $(sl_2)_0^q : \mathcal{L}(\mathcal{G}) = \mathcal{L}(\mathcal{G}) \oplus C K \oplus C d$. This is a loop algebra $\mathcal{L}(\mathcal{G})$ completed with the derivation $d$ (acting as $t \frac{d}{dt}$ in $\mathcal{L}$ and trivially on $K$) extended by a one-dimensional center $K$ corresponding to $C$-valued $q$-deformed 2-cocycle on $\mathcal{L}(\mathcal{G}) \Psi_q(a, b) = (x|y)_h \Phi(P, Q), \quad \Psi_q \in C^2(\mathcal{G}, K)$,
Then one can calculate the adjoint action
\[
\Phi(P, Q) = \text{Res} t \frac{dP}{dt} Q. \quad \text{Here} \quad (x|y)_h \quad \text{is a quantum bilinear form on} \quad (sl_2)_q \quad \text{[1]} \quad \text{and} \quad P, Q \quad \text{are polynomials in} \quad t.
\]
We define \( q \)-deformed Lie bracket in this algebra as
\[
[t^m \otimes x \otimes \omega K \otimes \nu d, \quad t^n \otimes y \otimes \omega K \otimes \nu_1 d]_h =
(t^{m+n} \otimes [x, y]_h + \nu m^nt \otimes y - \nu_1 m^mt \otimes x) \otimes m\delta_{m+n,0}(x|y)_hK.
\]
(31)
where \( x, y \in G, \nu, \omega, \nu_1, \omega_1 \in C \).

Now we introduce generators that constitute an affinization of quantum algebra \((sl_2)_q\)
\[
H_1 = 1 \otimes H_h, \quad H_0 = 1 \otimes (K - H_h), \quad e_1 = 1 \otimes X^+_h, \quad e_0 = t \otimes X^-_h, \quad f_1 = 1 \otimes X^-_h, \quad f_0 = t^{-1} \otimes X^+_h.
\]
Then one can calculate the adjoint action
\[
[H_0, e_0]_h = 2e_0q^{-1}, [H_0, f_0]_h = -2f_0q^{-1}, [H_0, e_1]_h = -2e_1q, [H_0, f_1]_h = 2f_1q^{-1},
\]
\[
[H_1, e_1]_h = 2e_1q, [H_1, f_1]_h = -2f_1q^{-1}, [H_1, e_0]_h = 2e_0q^{-1}, [H_1, f_0]_h = 2qf_0,
\]
\[
[H_0, H_1]_h = -2(q - q^{-1})H_1, [H_0, H_0]_h = 2(q - q^{-1})H_1, [H_1, H_1]_h = 2(q - q^{-1})H_1,
\]
\[
[e_0, H_0]_h = -2qe_0, [e_0, H_1]_h = 2qe_0, [e_1, H_0]_h = 2e_1q, [e_1, H_1]_h = -2e_1q,
\]
\[
[f_0, H_0]_h = 2q^{-1}f_0, [f_0, H_1]_h = -2qf_0, [f_1, H_1]_h = 2f_1q, [f_1, H_0]_h = -2qf_1.
\]

5.4. Heisenberg subalgebra and eigenvector of the \( q \)-deformed adjoint action
The key point in the construction of solitonic solution to the affine Toda equation is the existence of eigenvectors with respect to elements of the Heisenberg subalgebra of the underlying affine algebra. Let
\[
qE_{+1} = 1 \otimes X^+_h + t \otimes X^-_h, \quad qE_{-1} = 1 \otimes X^-_h + t^{-1} \otimes X^+_h
\]
so that the following series
\[
A_{2m} = (q + q^{-1})^m t^m \otimes (-H_h), \quad A_{2m+1} = (q + q^{-1})^m t^m \otimes q^{-1}X^+_h - t^{-1} \otimes qX^-_h + (X^-_h X^+_h)_h,
\]
\[
F_q = \sum_{k = -\infty}^{+\infty} \zeta^k A_k,
\]
(32)
is an eigenvector of \( qE_{+1} \) with respect to the quantum affine Lie bracket (31). Using \( F_q \) we can find \( q \)-deformation of the solitonic solution (26) corresponding to the affinization of the quantum algebra \((sl_2)_q\).

6. Solution to the affine Toda equations corresponding to \((sl_2)_q^l\)
In this section we will use results of the section 5. We propose \( q \)-deformed solitonic solutions to the affine Toda equations associated with the affine Lie algebra \( \hat{sl}_2 \), i.e., the sine–Gordon equation. Those solutions will be constructed by means of the Heisenberg subalgebra operators and the eigenvector \( F \) involved in the exponent in the mapping \( \mu_0 \). Using properties of the highest weight representation of the affinized quantum algebra \((sl_2)_q\) one can check (similar to [9], [8]) that (12) is a solution to the affine Toda equation (11) when we replace its ingredients by their quantum group counterparts. A consideration in the case of the sine–Gordon equation is given in [9].

First consider the case of \( l = 1 \). Starting from the general solution (12) to the affine Toda equations for \( \hat{sl}_2 \) we take \( \gamma_0^2 = 1 \) and substitute the vectors of the fundamental highest weight representation of \( \hat{sl}_2 \) by the fundamental highest weight representation vectors of \((sl_2)_q^l\)
\[
e^{-\beta \lambda_1 \cdot \phi} = e^{-\beta \lambda_1 \cdot \phi_0} \frac{1}{(\lambda_0|\mu_+^{\lambda_1^1}(z_+)\mu_-(z_)\lambda_0(q_0^l(t))^m/t)} \frac{1}{(\lambda_0|\mu_-^{\lambda_1^1}(z_-)\mu_+(z_)\lambda_0(q_0^l(t))^m/t)}.
\]
(33)
where \( |i\rangle_q^{(1)} = |\lambda_i\rangle_q^{(1)} = \nu_i^{(i)} \) denotes the highest vector in the \( i \)-th fundamental representation of \((sl_2)_q^t\). Taking \( \mathcal{E}_\pm = qE_{\pm 1} \) in the solitonic solution and inserting exponential of the eigenoperator \( F_q \) as a group element, we obtain [10]

\[
e^{-\beta \lambda_i} \phi = e^{-\beta \lambda_i} \phi_0 \frac{\langle j | e^{-\beta \lambda_i} q^\mathcal{E}_\pm e^{\beta \lambda_i} | j \rangle_q^{(1)}}{\langle j | e^{\beta \lambda_i} q^\mathcal{E}_\pm e^{-\beta \lambda_i} | j \rangle_q^{(1)}}.
\]

Then it follows that

\[
e^{-\beta \lambda_i} \phi = e^{-\beta \lambda_1} \phi_0 \frac{\langle j | \exp \left( q e^{-2z+\zeta-2\frac{1}{2}z\zeta^{-1}} \mathcal{F}_q \right) \exp \left( -z_j \right) | j \rangle_q^{(1)}}{\langle j | \exp \left( q e^{-2z+\zeta-2\frac{1}{2}z\zeta^{-1}} \mathcal{F}_q \right) \exp \left( -z_j \right) | 0 \rangle_q^{(1)}}.
\]

So for \( j = 0 \)

\[
e^{-\beta \lambda_1} \phi = e^{-\beta \lambda_1} \phi_0,
\]

and for \( j = 1 \) this gives

\[
e^{-\beta \lambda_1} \phi = e^{-\beta \lambda_1} \phi_0 \exp \left( -q e^{-2z+\zeta-2\frac{1}{2}z\zeta^{-1}} \sum_{m=-\infty}^{+\infty} \zeta^{2m} t^m (q + q^{-1})^m \right).
\]

### 6.1. The case \( l = 3 \)

The affine Toda equation system (11) is a particular example of more general construction described in the first section. Namely equations (11) correspond to \( l = 1 \) grading subspace. The case \( l = 3 \) leads us to a generalization of (11). Using the general form of solution to the affine Toda equation for \( l > 1 \) (12) we can form a class of \( q \)-deformed solutions constructed with by means of the eigenvector \( F_q \) (32). We take in (12)

\[
\mathcal{E}_\pm = qE_{\pm 1} + qE_{\pm 3}.
\]

where \( qE_{+3} = t \otimes X_h^3 + t^2 \otimes X_h^3 \), \( qE_{-3} = t^{-1} \otimes X_h^3 + t^{-2} \otimes X_h^3 \). Then we arrive at a solution for Toda type field \( \phi \)

\[
e^{-\beta \lambda_1} \phi = e^{-\beta \lambda_1} \phi_0 \exp \left( -q e^{-2z+\zeta-2\frac{1}{2}z\zeta^{-1}} \sum_{m=-\infty}^{+\infty} \zeta^{2m} t^m (q + q^{-1})^m \right).
\]

Now we want to find \( q \)-deformed solitonic solutions for matter fields \( F_{\pm n}^+ \) just in the same way as we did it for Toda type fields. We substitute elements (38) to (25) and take the state vectors from (42) from the basis of the fundamental highest weight representation of \((sl_2)_q^t\). Finally the \( q \)-solitonic solutions are

\[
f_1^+ = \exp \left( \sum_{n=0}^{+\infty} k_{1t} \phi_0 \right) e^{\mu_0} \partial_+ \left( q | 1 \rangle \ exp \left( q e^{-2z+\zeta-2\frac{1}{2}z\zeta^{-1}} \sum_{m=-\infty}^{+\infty} \zeta^{2m} t^m (q + q^{-1})^m \right) \right.
\]

\[
\times \left( \sum_{m=-\infty}^{+\infty} \zeta^{2m+1} t^m (q + q^{-1})^m \right) \left( 1 - 2qq^{-\left(\lambda (1) - 2\right)} \right) | 1 ; 1 \rangle_q + \sum_{k=0}^{n-1} \left( 1 - 2qq^{-\left(\lambda (1) - 2\right)} \right) q^{-1} q^{-\left(\lambda (1) - 2\right)}
\]

\[
\times \left( \sum_{m=-\infty}^{+\infty} \zeta^{2m+1} t^m (q + q^{-1})^m \right) q^{-1} q^{-\left(\lambda (1) - 2\right)}
\]

\[
\times \left( \exp (1 - 2qq^{-\left(\lambda (1) - 2\right)}) | 1 ; 1 \rangle_q + \frac{\exp (1 - 2qq^{-\left(\lambda (1) - 2\right)}) - 1}{(1 - 2qq^{-\left(\lambda (1) - 2\right)}) - 1} | 1 \rangle_q \right)
\].
$$f_0^+ = \exp \left( \sum_{l=0}^{1} k_{1l}(\phi^- - \phi) l \right) e^{i\alpha_1} \partial_+ \exp \left( Q e^{-2\alpha_+(1+\gamma)\zeta - 2\gamma\zeta - (1+\frac{1}{2})\zeta^{-1} \sum_{m=-\infty}^{\infty} \zeta^m e^m (q + q^{-1})^m} \right) \times q^{-1} q^{-\lambda(0)} \left( \exp \left( 1 - 2q^{-1} q^{2+\lambda(0)} \right) [0; 0]_q + \exp \left( 1 - 2q^{-1} q^{2+\lambda(0)} \right) Q \right) \times (1 - 2q^{-1} q^{2+\lambda(0)})^{-1} [0]_q \right).$$

Similar relations can be obtained for all higher cases of $l$ elsewhere.

7. Appendix

7.1. Affine Kac–Moody algebras

Here we recall facts about affine Kac–Moody algebras [5, 3]. Consider an untwisted affine Kac–Moody algebra $\mathcal{G}$ endowed with an integral grading $\mathcal{G} = \bigoplus_{n \in \mathbb{Z}} \mathcal{G}_n$, and denote $\mathcal{G}_\pm = \bigoplus_{n>0} \mathcal{G}_\pm$. By an affine Lie algebra we mean a loop algebra corresponding to a finite dimensional simple Lie algebra $\mathcal{G}$ of rank $r$, extended by the center $C$ and the derivation $D$. According to [5], integral gradings of $\mathcal{G}$ are labelled by a set of co-prime integers $s = (s_0, s_1, ..., s_r)$, and the grading operators are given by

$$Q_s \equiv H_s + N_s D - \frac{1}{2N_s} \text{Tr} \left( (H_0)^2 \right).$$

Here $H_s \equiv \sum_{a=1}^{r} s_a \lambda_a \cdot H^0$, $N_s \equiv \sum_{a=1}^{r} s_a \psi \cdot \psi = \sum_{a=1}^{r} m_a \alpha_a$, $m_a = 1$. $H^0$ is an element of Cartan subalgebra of $\mathcal{G}$; $\alpha_a, a = 1, 2, ..., r$, are its simple roots; $\psi$ is its maximal root; $m_a$ the integers in expansion $\psi = \sum_{a=1}^{r} m_a \alpha_a$; and $\lambda_a$ are the fundamental co–weights satisfying the relation $\alpha_a \cdot \lambda_b = \delta_{ab}$.

The principal grading operator $Q_{\text{ppal}}$ is given by (40) where $N_s = h$ is Coxeter number. Therefore $\mathcal{G}_0 = \{ H_0^a, a = 1, 2, ..., r; C; Q_{\text{ppal}} \}$, $\mathcal{G}_m = \{ E^0_{\alpha(m)}, E^1_{\alpha(h-m)} \}$, $\mathcal{G}_{-m} = \{ E^{0}_{-\alpha(m)}, E^{-1}_{\alpha(h-m)} \}$ where $0 < m < h$, and $\alpha^{(m)}$ are positive roots of height $m$. The element $B$ is parameterized as $B = e^{i\alpha_1} H^0 e^{i\psi} C e^{iQ_{\text{ppal}}} = e^{i\alpha_1} H^0 e^{i\psi} C e^{iQ_{\text{ppal}}}$, where $H^0$ is defined in [3] as $H^0 = H_0^a - \frac{1}{N_s} \text{Tr} (H_0^a H_0^a) C = H^0 - \frac{2}{\alpha_2} s_a C$, and $\nu = \psi - \frac{1}{2} \delta \cdot \varphi$, with $\delta = \sum_{a=1}^{r} \lambda_a \alpha_a$, and $\lambda_a$ being the fundamental weights of $\mathcal{G}$. Let us denote by $H^0, E^a_2, D, C$ the Chevalley basis generators of $sl_2$. The commutation relations are

$$[H^m, H^n] = 2m C \delta_{m+n,0}, \quad [H^m, E^n_a] = \pm 2E^{m+n}_a, \quad [E^m_+, E^n_-] = H^{m+n} + m C \delta_{m+n,0}, \quad [D, T^{m}] = m T^{m}, \quad T^{m} = H^{m}, E^{m}_\pm.$$ 

The grading operator for the principal grading ($s = (1, 1)$) is $Q \equiv \frac{1}{2} H^0 + 2D$. Then the eigensubspaces are $\mathcal{G}_0 = \{ H^0, C, Q \}$, $\mathcal{G}_{2n+1} = \{ E^m_+, E^{m-1}_- \}$, $n \in \mathbb{Z}$, $\mathcal{G}_{2n} = \{ H^n \}$, $n \in \{ \mathbb{Z} - 0 \}$.

7.2. Highest weight representations of $(sl_2)_q$

The highest weight vector $v_0$ of the fundamental representation of the quantum Lie algebra $(sl_2)_q$ [10] satisfies the relations $H_h v_0 = v_0$, $X^+_h v_0 = 0$, $X^-_h v_0 = v_1$. Note that because of the definition of generators $(sl_2)_q$ (28) we have

$$[H, X^+_h] = [H, X^-_h] = 2X^+_h, \quad [H, X^-_h] = -2X^-_h, \quad [H, H_h] = [H, H_h] = [H, H_h] = 0.$$ 

The element $H$ lies in Cartan subalgebra of $(sl_2)_q$. Therefore $H_h (H v_0) = (H v_0)$, $H_h v_0 = \lambda v_0$. Using the definitions (28) we find that $H_h = \frac{2}{q + q^{-1}} (q X^+ X^- - q^{-1} X^- X^+) = q^H (X^+_h X^-_h - X^-_h X^+_h)$, and the action of $H_h$ on $v_0$ in such a form gives $q^H (X^+_h X^-_h - X^-_h X^+_h) v_0 = v_0$, and finally $X^+_h v_1 = q^H v_0$, where $\lambda$ can be found from $\frac{2}{q + q^{-1}} (q X^+ X^- - q^{-1} X^- X^+) v_0 = \frac{2}{q + q^{-1}} [H] v_0 = v_0$, where $[H] v_0 = v_0$. 
with \( \frac{2q(q^\lambda q^{-\lambda})}{(q+q^{-1})(q-q^{-1})} = 1 \). This procedure can be recurrently continued for all \( v_n \) in the basis of the representation. In section 7 we introduced a quantum affine algebra \((sl_2)_q^T\) as an affinization of \((sl_2)_q\). The highest weight vector of the \( i \)-th \((i = 1, 2)\) fundamental representation of \((sl_2)_q^T\) possesses the properties similar to the properties of the highest weight vector of fundamental representation of \((sl_2)_q\). The action of \( h_{1,2}, e_{1,2} \) and \( f_{1,2} \) generators on highest weight vectors \( v_0^{(1)} \) and \( v_0^{(0)} \) is given by

\[
\begin{align*}
h_{1}v_0^{(1)} &= v_0^{(1)}, \quad h_{1}v_0^{(0)} = 0, \quad h_{0}v_0^{(1)} = 0, \quad h_{0}v_0^{(0)} = v_0^{(0)}, \quad e_{1}v_0^{(1)} = e_1v_0^{(1)} = e_0v_0^{(0)} = e_0v_0^{(1)} = f_0v_0^{(1)} = f_1v_0^{(0)} = 0, \\
f_0v_0^{(1)} &= f_1v_0^{(0)} = 0, \quad f_0v_0^{(0)} = v_1^{(0)}, \quad f_1v_0^{(0)} = v_1^{(0)},
\end{align*}
\]

where superscripts correspond to representation and subscripts are vector basis numbers. In the same way as for \((sl_2)_q\)

\[
\begin{align*}
e_1v_1^{(1)} &= q^{-H}v_0^{(1)}, \quad e_1v_1^{(1)} = q^{-\lambda(1)}v_0^{(1)}, \quad h_1v_1^{(1)} = \lambda(1)v_0^{(1)}, \quad e_0v_1^{(0)} = 0 \quad h = 1 \otimes H, \\
e_0v_1^{(1)} = q^{-H}v_0^{(1)}, \quad e_0v_1^{(0)} = q^{-\lambda(0)}v_0^{(0)}, \quad h_0v_1^{(0)} = \lambda(0)v_0^{(0)}, \quad e_0v_1^{(1)} = 0, \\
H_1v_1^{(1)} &= (1 - 2qq^{-\lambda(1)} - 2)v_1^{(1)}, \quad H_1v_1^{(0)} = 2q^{-1}q^{2\lambda(0)}v_1^{(0)}.
\end{align*}
\]

8. Conclusions

In this paper we discussed affine Toda models formulated with the inclusion of higher then one grading subspaces for affine Lie algebras. The zero curvature condition applied to elements of connection involving higher gradign Lie algebra generators lead us to more suffisticated and interesting systems of equations describing dynamical systems that are closer (with respect to ordinary affine Toda models) to phenomena in nature. In particular, these systems of equations take into account interactions of original Toda fields with matter fields. The group-theoretical way to treat higher grading affine Toda models allows us to find special solitonic solutions. In the quantum situation, corresponding methods using properties of quantum groups and quantum Lie algebras [1] make possible constructions of special quantum soliton solutions.

9. References

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