EVALUATION OF THE EINSTEIN’S STRENGTH OF DIFFERENCE SCHEMES FOR SOME CHEMICAL REACTION-DIFFUSION EQUATIONS

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Abstract. In this paper we present a difference algebraic technique for the evaluation of the Einstein’s strength of a system of partial difference equations and apply this technique to the comparative analysis of difference schemes for chemical reaction-diffusion equations. In particular, we analyze finite-difference schemes for the Murray, Fisher, Burgers and some other reaction-diffusion equations, as well as mass balance PDEs of chromatography from the point of view of their strength.

1. Introduction

The concept of strength of a system of partial differential equations (PDEs) was introduced by A. Einstein as a characteristic that provides a measure for the size of the solution space of such a system. In [2] A. Einstein defined the strength of a system of partial differential equations governing a physical field as follows: “... the system of equations is to be chosen so that the field quantities are determined as strongly as possible. In order to apply this principle, we propose a method which gives a measure of strength of an equation system. We expand the field variables, in the neighborhood of a point \( P \), into a Taylor series (which presupposes the analytic character of the field); the coefficients of these series, which are the derivatives of the field variables at \( P \), fall into sets according to the degree of differentiation. In every such degree there appear, for the first time, a set of coefficients which would be free for arbitrary choice if it were not that the field must satisfy a system of differential equations. Through this system of differential equations (and its derivatives with respect to the coordinates) the number of coefficients is restricted, so that in each degree a smaller number of coefficients is left free for arbitrary choice. The set of numbers of 'free' coefficients for all degrees of differentiation is then a measure of the 'weakness' of the system of equations, and through this, also of its 'strength'."

The comparison analysis of the strength of systems of PDEs describing different mathematical models of the same process allows one to obtain an optimal model, that is, to minimize the “arbitrariness” of the corresponding solutions. As an application of this approach, A. Einstein determined that the potential and field formulations of Maxwell equations have different strengths for the dimension four. However, he did not obtain the exact expression of the above-mentioned number of free coefficients as a function of the degree of differentiation. The reason was that

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there were no methods for the strength computation, so Einstein had to evaluate this characteristic "by hand".

Even though there are a number of works on the strength of a system of partial differential equations (in particular, on its relation to Cartan characters), see, for example, [13], [16], [17], [22], [23], [24], [25] and [27], there was no method of its evaluation until 1980 when A. Mikhalev and E. Pankratev [19] showed that the strength of a system of algebraic PDEs (that is, a system of the form $f_i = 0$, $i \in I$, where $f_i$ are multivariate polynomials in unknown functions and their partial derivatives) is expressed by the associated differential dimension polynomial introduced by E. Kolchin [8] (see also [9, Chapter II, Theorem 6]). This observation has led to algorithmic algebraic methods of computation of the strength of a system of algebraic PDEs via computing the differential dimension polynomial of the corresponding differential ideal in the algebra of differential polynomials. The theoretical base for these methods and their detailed description can be found in [10].

Note that A. Einstein [2], K. Mariwalla [13], M. Sue [27] and some other authors who investigated the concept of strength characterized the strength of a system by the "coefficient of freedom", an integer, that is fully determined by the leading coefficient of the differential dimension polynomial. The fact that such a polynomial provides a far more precise description of the strength than its leading term was justified by the result of W. Sit [26] who proved that the set of differential dimension polynomials is well-ordered with respect to the natural order ($f(t) < g(t)$ if and only if $f(r) < g(r)$ for all sufficiently large integers $r$); this result allows one to distinguish two systems of PDEs with the same "coefficient of freedom" by their strength.

Since 1980s the technique of dimension polynomials has been extended to the analysis of systems of algebraic difference and difference-differential equations. In a series of works whose results are summarized in [12] the second author proved the existence and developed some methods of computation of dimension polynomials of difference field extensions and systems of algebraic difference equations. These polynomials determine A. Einstein’s strength of a system of algebraic partial difference equations (we give the details in Section 3 of this work) and, in particular, allow one to evaluate the quality of difference schemes for PDEs from the point of view of their strength.

In this paper we present a method of characteristic sets for inversive difference polynomials and consider applications of this technique to the analysis of difference schemes for reaction-diffusion PDEs. We determine the strengths of systems of partial difference equations that arise from such schemes and perform their comparative analysis. We also perform a similar investigation of difference schemes for mass balance PDEs of chromatography that is one of the main methods for accurate and rapid determination of biologically active organic carboxylic acids in objects such as infusion solutions and blood preservatives, see [3], [4] and [5].

2. Preliminaries

Throughout the paper, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the sets of all non-negative integers, integers, rational numbers, and real numbers, respectively. The number of elements of a set $A$ is denoted by $\text{Card} A$. As usual, $\mathbb{Q}[t]$ denotes the ring of polynomials in one variable $t$ with rational coefficients. By a ring we always mean an associative
ring with unity. All fields considered in the paper are supposed to be of zero characteristic. Every ring homomorphism is unitary (maps unity onto unity), every subring of a ring contains the unity of the ring.

If \( B = \prod_{i=1}^{k} A_i \) is a Cartesian product of \( k \) ordered sets with orders \( \leq_{i}, \cdots \leq_{k} \), respectively \((k \in \mathbb{N}, k \geq 1)\), then by the product order on \( B \) we mean a partial order \( \leq_{P} \) such that \((a_1, \ldots, a_k) \leq_{P} (a'_1, \ldots, a'_k)\) if and only if \( a_i \leq_{i} a'_i \) for \( i = 1, \ldots, k \). In particular, if \( a = (a_1, \ldots, a_k) \), \( a' = (a'_1, \ldots, a'_k) \in \mathbb{N}^k \), then \( a \leq_{P} a' \) if and only if \( a_i \leq_{i} a'_i \) for \( i = 1, \ldots, k \). We write \( a <_{P} a' \) if \( a \leq_{P} a' \) and \( a \neq a' \).

In this section we present some background material needed for the rest of the paper. In particular, we discuss basic concepts and results of difference algebra and properties of dimension polynomials associated with subsets of \( \mathbb{N}^m \) and \( \mathbb{Z}^m \).

2.1. Basic notions of difference algebra. A difference ring is a commutative ring \( R \) together with a finite set \( \sigma = \{\alpha_1, \ldots, \alpha_m\} \) of mutually commuting injective endomorphisms of \( R \) into itself. The set \( \sigma \) is called the basic set of the difference ring \( R \), and the endomorphisms \( \alpha_1, \ldots, \alpha_m \) are called translations. A difference ring with a basic set \( \sigma \) is also called a \( \sigma \)-ring. If \( \alpha_1, \ldots, \alpha_m \) are automorphisms of \( R \), we say that \( R \) is an inversive difference ring with the basic set \( \sigma \). In this case we denote the set \( \{\alpha_1, \ldots, \alpha_m, \alpha_1^{-1}, \ldots, \alpha_m^{-1}\} \) by \( \sigma^* \) and call \( R \) a \( \sigma^* \)-ring. If a difference (\( \sigma \)-) ring \( R \) is a field, it is called a difference (or \( \sigma \)-) field. If \( R \) is inversive, it is called an inversive difference field or a \( \sigma^* \)-field.

Let \( R \) be a difference (inversive difference) ring with a basic set \( \sigma \) and \( R_0 \) a subring of \( R \) such that \( \sigma(R_0) \subseteq R_0 \) for any \( \sigma \in \sigma \) (respectively, for any \( \alpha \in \sigma^* \)). Then \( R_0 \) is called a difference or \( \sigma \)- (respectively, inversive difference or \( \sigma^* \)-) subring of \( R \), while the ring \( R \) is said to be a difference or \( \sigma \)- (respectively, inversive difference or \( \sigma^* \)-) overring of \( R_0 \). In this case the restriction of an endomorphism \( \alpha_i \) on \( R_0 \) is denoted by the same symbol \( \alpha_i \). If \( R \) is a difference (\( \sigma \)-) or an inversive difference (\( \sigma^* \)-) field and \( R_0 \) a subfield of \( R \), which is also a \( \sigma \)- (respectively, \( \sigma^* \)-) subring of \( R \), then \( R_0 \) is said to be a \( \sigma \)- (respectively, \( \sigma^* \)-) subfield of \( R \); \( R \), in turn, is called a difference or \( \sigma \)- (respectively, inversive difference or \( \sigma^* \)-) field extension or a \( \sigma \)- (respectively, \( \sigma^* \)-) overfield of \( R_0 \). In this case we also say that we have a \( \sigma \)- (or \( \sigma^* \)-) field extension \( R/R_0 \).

If \( R \) is a difference ring with a basic set \( \sigma \) and \( J \) is an ideal of the ring \( R \) such that \( \sigma(J) \subseteq J \) for any \( \sigma \in \sigma \), then \( J \) is called a difference (or \( \sigma \)-) ideal of \( R \). If a prime (maximal) ideal \( P \) of \( R \) is closed with respect to \( \sigma \) (that is, \( \sigma(P) \subseteq P \) for any \( \sigma \in \sigma \)), it is called a prime (respectively, maximal) difference (or \( \sigma \)-) ideal of the \( \sigma \)-ring \( R \).

A \( \sigma \)-ideal \( J \) of a \( \sigma \)-ring \( R \) is called reflexive (or a \( \sigma^* \)-ideal) if for any translation \( \alpha \), the inclusion \( \alpha(a) \in J \) \((a \in R)\) implies \( a \in J \). If \( R \) is inversive, then a reflexive \( \sigma \)-ideal (that is, an ideal \( J \) such that \( \alpha(J) = J \) for any \( \alpha \in \sigma^* \)) is also called a \( \sigma^* \)-ideal.

If \( R \) is a difference ring with a basic set \( \sigma = \{\alpha_1, \ldots, \alpha_n\} \), then \( T_{\sigma} \) (or \( T \) if the set \( \sigma \) is fixed) will denote the free commutative semigroup generated by \( \alpha_1, \ldots, \alpha_n \). Elements of \( T_{\sigma} \) will be written in the multiplicative form \( \alpha_{k_1}^{k_1} \cdots \alpha_{k_m}^{k_m} \) \((k_1, \ldots, k_m \in \mathbb{N})\) and considered as injective endomorphisms of \( R \) (which are the corresponding compositions of the endomorphisms of \( \sigma \)). If the \( \sigma \)-ring \( R \) is inversive, then \( \Gamma_{\sigma} \) (or \( \Gamma \) if the set \( \sigma \) is fixed) will denote the free commutative group generated by the set \( \sigma \). It is clear that elements of the group \( \Gamma_{\sigma} \) (written in the multiplicative form}
\[ \alpha_1^{i_1} \cdots \alpha_m^{i_m} \text{ with } i_1, \ldots, i_n \in \mathbb{Z} \] act on \( R \) as automorphisms and \( T_\sigma \) is a subsemigroup of \( \Gamma_\sigma \).

For any \( a \in R \) and for any \( \tau \in T_\sigma \), the element \( \tau(a) \) is called a transform of \( a \).

If the \( \sigma \)-ring \( R \) is inverse, then an element \( \gamma(a) \) \( (a \in R, \gamma \in \Gamma_\sigma) \) is also called a transform of \( a \). An element \( a \in R \) is said to be a constant if \( \alpha(a) = a \) for every \( \alpha \in \sigma \).

If \( J \) is a \( \sigma \)-ideal of a \( \sigma \)-ring \( R \), then \( J^* = \{ a \in R \mid \tau(a) \in J \text{ for some } \tau \in T_\sigma \} \) is a reflexive \( \sigma \)-ideal of \( R \) contained in any reflexive \( \sigma \)-ideal of \( R \) containing \( J \). The ideal \( J^* \) is called the reflexive closure of the \( \sigma \)-ideal \( J \).

Let \( R \) be a difference ring with a basic set \( \sigma \) and \( S \subseteq R \). Then the intersection of all \( \sigma \)-ideals of \( R \) containing \( S \) is denoted by \([S]\). Clearly, \([S]\) is the smallest \( \sigma \)-ideal of \( R \) containing \( S \); as an ideal, it is generated by the set \( T_\sigma S = \{ \tau(a)|a \in S \} \).

If \( J = [S] \), we say that the \( \sigma \)-ideal \( J \) is generated by the set \( S \) called a set of \( \sigma \)-generators of \( J \). If \( S \) is finite, \( S = \{a_1, \ldots, a_k\} \), we write \( J = [a_1, \ldots, a_k] \) and say that \( J \) is a finitely generated \( \sigma \)-ideal of the \( \sigma \)-ring \( R \). (In this case elements \( a_1, \ldots, a_k \) are said to be \( \sigma \)-generators of \( J \)).

If \( R \) is an inversive difference (\( \sigma \)-) ring and \( S \subseteq R \), then the inverse closure of the \( \sigma \)-ideal \( [S] \) is denoted by \([S]^*\). It is easy to see that \([S]^*\) is the smallest \( \sigma^* \)-ideal of \( R \) containing \( S \); as an ideal, it is generated by the set \( \Gamma_\sigma S = \{ \gamma(a)|a \in S \} \).

If \( S \) is finite, \( S = \{a_1, \ldots, a_k\} \), we write \( [a_1, \ldots, a_k]^* \) for \( I = [S]^* \) and say that \( I \) is a finitely generated \( \sigma^* \)-ideal of \( R \). (In this case, elements \( a_1, \ldots, a_k \) are said to be \( \sigma^* \)-generators of \( I \)).

Let \( R \) be a difference ring with a basic set \( \sigma \), \( R_0 \) a \( \sigma \)-subring of \( R \) and \( B \subseteq R \). The intersection of all \( \sigma \)-subrings of \( R \) containing \( R_0 \) and \( B \) is called the \( \sigma \)-subring of \( R \) generated by \( B \) over \( R_0 \), it is denoted by \( R_0\{B\} \). (As a ring, \( R_0\{B\} \) coincides with the ring \( R_0\{\tau(b)|b \in B, \tau \in T_\sigma \} \) obtained by adjoining the set \( \{ \tau(b)|b \in B, \tau \in T_\sigma \} \) to the ring \( R_0 \)). The set \( B \) is said to be the set of \( \sigma \)-generators of the \( \sigma \)-ring \( R_0\{B\} \) over \( R_0 \). If this set is finite, \( B = \{b_1, \ldots, b_k\} \), we say that \( R' = R_0\{B\} \) is a finitely generated difference (or \( \sigma \)-) ring extension (or overring) of \( R_0 \) and write \( R' = R_0\{b_1, \ldots, b_k\} \). If \( R \) is a \( \sigma \)-field, \( R_0 \) a \( \sigma \)-subfield of \( R \) and \( B \subseteq R \), then the intersection of all \( \sigma \)-subfields of \( R \) containing \( R_0 \) and \( B \) is denoted by \( R_0\{B\} \) (or \( R_0\{b_1, \ldots, b_k\} \) if \( B = \{b_1, \ldots, b_k\} \) is a finite set). This is the smallest \( \sigma \)-subfield of \( R \) containing \( R_0 \) and \( B \); it coincides with the field \( R_0\{\tau(b)|b \in B, \tau \in T_\sigma \} \).

The set \( B \) is called a set of \( \sigma \)-generators of the \( \sigma \)-field \( R_0\{B\} \) over \( R_0 \).

Let \( R \) be an inversive difference ring with a basic set \( \sigma \), \( R_0 \) a \( \sigma^* \)-subring of \( R \) and \( B \subseteq R \). Then the intersection of all \( \sigma^* \)-subrings of \( R \) containing \( R_0 \) and \( B \) is the smallest \( \sigma^* \)-subring of \( R \) containing \( R_0 \) and \( B \). This ring coincides with the ring \( R_0\{\gamma(b)|b \in B, \gamma \in \Gamma_\sigma \} \); it is denoted by \( R_0\{B\}^* \). The set \( B \) is said to be a set of \( \sigma^* \)-generators of \( R_0\{B\}^* \) over \( R_0 \). If \( B = \{b_1, \ldots, b_k\} \) is a finite set, we say that \( S = R_0\{B\}^* \) is a finitely generated inversive difference (or \( \sigma^* \)-) ring extension (or overring) of \( R_0 \) and write \( S = R_0\{b_1, \ldots, b_k\}^* \).

If \( R \) is a \( \sigma^* \)-field, \( R_0 \) a \( \sigma^* \)-subfield of \( R \) and \( B \subseteq R \), then the intersection of all \( \sigma^* \)-subfields of \( R \) containing \( R_0 \) and \( B \) is denoted by \( R_0\{B\}^* \). This is the smallest \( \sigma^* \)-subfield of \( R \) containing \( R_0 \) and \( B \); it coincides with the field \( R_0\{\gamma(b)|b \in B, \gamma \in \Gamma_\sigma \} \). The set \( B \) is called a set of \( \sigma^* \)-generators of the \( \sigma^* \)-field extension \( R_0\{B\}^* \) of \( R_0 \). If \( B \) is finite, \( B = \{b_1, \ldots, b_k\} \), we write \( R_0\{b_1, \ldots, b_k\}^* \) for \( R_0\{B\}^* \).

In what follows we often consider two or more difference rings \( R_1, \ldots, R_p \) with the same basic set \( \sigma = \{\alpha_1, \ldots, \alpha_m\} \). Formally speaking, it means that for every
Let $R_1$ and $R_2$ be difference rings with the same basic set $\sigma = \{\alpha_1, \ldots, \alpha_m\}$. A ring homomorphism $\phi : R_1 \to R_2$ is called a difference (or $\sigma$-) homomorphism if $\phi(\alpha(\alpha)) = \alpha(\phi(\alpha))$ for any $\alpha \in \sigma, a \in R_1$. Clearly, if $\phi : R_1 \to R_2$ is a $\sigma$-homomorphism of inverse difference rings, then $\phi(\alpha^{-1}(a)) = \alpha^{-1}(\phi(\alpha))$ for any $\alpha \in \sigma, a \in R_1$. If a $\sigma$-homomorphism is an isomorphism (endomorphism, automorphism, etc.), it is called a difference (or $\sigma$-) isomorphism (respectively, difference (or $\sigma$-) endomorphism, difference (or $\sigma$-) automorphism, etc.). If $R_1$ and $R_2$ are two $\sigma$-overrings of the same $\sigma$-ring $R_0$ and $\phi : R_1 \to R_2$ is a $\sigma$-homomorphism such that $\phi(a) = a$ for any $a \in R_0$, we say that $\phi$ is a difference (or $\sigma$-) homomorphism over $R_0$ or that $\phi$ leaves the ring $R_0$ fixed.

It is easy to see that the kernel of any $\sigma$-homomorphism of $\sigma$-rings $\phi : R \to R'$ is an inverse $\sigma$-ideal of $R$. Conversely, let $g$ be a surjective homomorphism of a $\sigma$-ring $R$ onto a ring $S$ such that $\ker g$ is a $\sigma^*$-ideal of $R$. Then there is a unique structure of a $\sigma$-ring on $S$ such that $g$ is a $\sigma$-homomorphism. In particular, if $I$ is a $\sigma^*$-ideal of a $\sigma$-ring $R$, then the factor ring $R/I$ has a unique structure of a $\sigma$-ring such that the canonical surjection $R \to R/I$ is a $\sigma$-homomorphism. In this case $R/I$ is said to be the difference (or $\sigma$-) factor ring of $R$ by the $\sigma^*$-ideal $I$.

If a difference (inverse difference) ring $R$ with a basic set $\sigma$ is an integral domain, then its quotient field $Q(R)$ can be naturally considered as a $\sigma$-(respectively, $\sigma^*$-) overring of $R$. (We identify an element $a \in R$ with its canonical image $\frac{a}{1}$ in $Q(R)$.) In this case $Q(R)$ is said to be the quotient difference (or quotient $\sigma$-) field of $R$. Clearly, if the $\sigma$-ring $R$ is inverse, then its quotient $\sigma$-field $Q(R)$ is also inverse. Furthermore, if a $\sigma$-field $K$ contains an integral domain $R$ as a $\sigma$-subring, then $K$ contains the quotient $\sigma$-field $Q(R)$.

Let $R$ be a difference ring with a basic set $\sigma = \{\alpha_1, \ldots, \alpha_m\}$, $T_\sigma$ the free commutative semigroup generated by $\sigma$, and $U = \{u_\lambda | \lambda \in \Lambda\}$ a family of elements from some $\sigma$-overring of $R$. We say that the family $U$ is transformally (or $\sigma$-algebraically) dependent over $R$, if the family $T_\sigma(U) = \{\tau(u_\lambda) | \tau \in T_\sigma, \lambda \in \Lambda\}$ is algebraically dependent over $R$ (that is, there exist elements $v_1, \ldots, v_k \in T_\sigma(U)$ and a non-zero polynomial $f(X_1, \ldots, X_k)$ with coefficients in $R$ such that $f(v_1, \ldots, v_k) = 0$). Otherwise, the family $U$ is said to be transformally (or $\sigma$-algebraically) independent over $R$ or a family of difference (or $\sigma$-) indeterminates over $R$. In the last case, the $\sigma$-ring $R\{u_\lambda | \lambda \in \Lambda\}$ is called the algebra of difference (or $\sigma$-) polynomials in the difference (or $\sigma$-) indeterminates $\{u_\lambda | \lambda \in \Lambda\}$ over $R$. If a family consisting of one element $u$ is $\sigma$-algebraically dependent over $R$, the element $u$ is said to be transformally algebraic (or $\sigma$-algebraic) over the $\sigma$-ring $R$. If the set $\{\tau(u) | \tau \in T_\sigma\}$ is algebraically independent over $R$, we say that $u$ is transformally (or $\sigma$-) transcendental over the $\sigma$-ring $R$.

Let $R$ be a $\sigma$-field, $L$ a $\sigma$-overfield of $R$, and $S \subseteq L$. We say that the set $S$ is $\sigma$-algebraic over $R$ if every element $a \in S$ is $\sigma$-algebraic over $R$. If every element of the field $L$ is $\sigma$-algebraic over $R$, we say that $L$ is a $\sigma$-algebraic field extension of the $\sigma$-field $R$. 

$i = 1, \ldots, p$, there is some fixed mapping $\nu_i$ from the set $\sigma$ into the set of all injective endomorphisms of the ring $R_i$ such that any two endomorphisms $\nu_i(\alpha_j)$ and $\nu_i(\alpha_k)$ of $R_i$ commute ($1 \leq j, k \leq n$). We shall identify elements $\alpha_j$ with their images $\nu_i(\alpha_j)$ and say that elements of the set $\sigma$ act as mutually commuting injective endomorphisms of the ring $R_i$ ($i = 1, \ldots, p$).
Proposition 2.1. ([10] Proposition 3.3.7). Let $R$ be a difference ring with a basic set $\sigma$ and $I$ an arbitrary set. Then there exists an algebra of $\sigma$-polynomials over $R$ in a family of $\sigma$-indeterminates with indices from the set $I$. If $S$ and $S'$ are two such algebras, then there exists a $\sigma$-isomorphism $S \rightarrow S'$ that leaves the ring $R$ fixed. If $R$ is an integral domain, then any algebra of $\sigma$-polynomials over $R$ is an integral domain.

The algebra of $\sigma$-polynomials over the $\sigma$-ring $R$ can be constructed as follows. Let $T = T_\sigma$ and let $S$ be the polynomial $R$-algebra in the set of indeterminates $\{y_{i,\tau}\}_{i \in I, \tau \in T}$ with indices from the set $I \times T$. For any $f \in S$ and $\alpha \in \sigma$, let $\alpha(f)$ denote the polynomial from $S$ obtained by replacing every indeterminate $y_{i,\tau}$ that appears in $f$ by $y_{i,\alpha \tau}$ and every coefficient $a \in R$ by $\alpha(a)$. We obtain an injective endomorphism $S \rightarrow S$ that extends the original endomorphism $\alpha$ of $R$ to the ring $S$ (this extension is denoted by the same letter $\alpha$). Setting $y_{i,1} = y_i$ (where 1 denotes the identity of the semigroup $T$) we obtain a set $\{y_i|i \in I\}$ whose elements are $\sigma$-algebraically independent over $R$ and generate $S$ as a $\sigma$-ring extension of $R$. Thus, $S = R\{(y_i)i \in I\}$ is an algebra of $\sigma$-polynomials over $R$ in a family of $\sigma$-indeterminates $\{y_i|i \in I\}$.

Let $R$ be an inversive difference ring with a basic set $\sigma$, $\Gamma = \Gamma_\sigma$, $I$ a set, and $S^*$ a polynomial ring in the set of indeterminates $\{y_{i,\gamma}\}_{i \in I, \gamma \in \Gamma}$ with indices from the set $I \times \Gamma$. If we extend the automorphisms $\beta \in \sigma^*$ to $S^*$ setting $\beta(y_{i,\gamma}) = y_{i,\beta \gamma}$ for any $y_{i,\gamma}$, and denote $y_{i,1}$ by $y_i$, then $S^*$ becomes an inversive difference $\sigma$-ring over $R$ generated (as a $\sigma^*$-overring) by the family $\{(y_i)i \in I\}$. Obviously, this family is $\sigma^*\text{-algebraically independent}$ over $R$, that is, the set $\{\gamma(y_i)|i \in I, \gamma \in \Gamma\}$ is algebraically independent over $R$. (Note that a set is $\sigma^*$-algebraically independent (independent) over an inversive $\sigma$-ring if and only if this set is $\sigma$-algebraically independent (respectively, independent) over this ring.) The ring $S^* = R\{(y_i)i \in I\}$ is called the algebra of inversive difference (or $\sigma^*$) polynomials over $R$ in the set of $\sigma^*$-indeterminates $\{(y_i)i \in I\}$. It is easy to see that $S^*$ is the inversive closure of the ring of $\sigma$-polynomials $S = R\{(y_i)i \in I\}$ over $R$ in the sense that $S^*$ is the smallest inversive $\sigma$-overring of $S$ with the following property: for every $a \in S^*$, there exists $\tau \in T_\sigma$ such that $\tau(a) \in S$. Furthermore, if a family $\{(u_i)i \in I\}$ from some $\sigma^*$-overring of $R$ is $\sigma^*$-algebraically independent over $R$, then the inversive difference ring $R\{(u_i)i \in I\}$ is naturally $\sigma$-isomorphic to $S^*$. Any such overring $R\{(u_i)i \in I\}$ is said to be an algebra of inversive difference (or $\sigma^*$) polynomials over $R$ in the set of $\sigma^*$-indeterminates $\{(u_i)i \in I\}$. We obtain the following analog of Proposition 2.1.

Proposition 2.2. ([10] Proposition 3.4.4). Let $R$ be an inversive difference ring with a basic set $\sigma$ and $I$ an arbitrary set. Then there exists an algebra of $\sigma$-polynomials over $R$ in a family of $\sigma^*$-indeterminates with indices from the set $I$. If $S$ and $S'$ are two such algebras, then there exists a $\sigma^*$-isomorphism $S \rightarrow S'$ that leaves the ring $R$ fixed. If $R$ is an integral domain, then any algebra of $\sigma^*$-polynomials over $R$ is an integral domain.

Let $R$ be a $\sigma$-ring, $R\{(y_i)i \in I\}$ an algebra of difference polynomials in a family of $\sigma$-indeterminates $\{(y_i)i \in I\}$, and $\{(\eta_i)i \in I\}$ a set of elements from some $\sigma$-overring of $R$. Since the set $\{\tau(y_i)|i \in I, \tau \in T_\sigma\}$ is algebraically independent over $R$, there exists a unique ring homomorphism $\phi_\eta : R[\tau(y_i)i \in I, \tau \in T_\sigma] \rightarrow R[\tau(\eta_i)i \in I, \tau \in T_\sigma]$ that maps every $\tau(y_i)$ onto $\tau(\eta_i)$ and leaves $R$ fixed. Clearly, $\phi_\eta$ is a surjective $\sigma$-homomorphism of $R\{(y_i)i \in I\}$ onto $R\{(\eta_i)i \in I\}$; it is called the substitution of
(ηi)i∈I for (yi)i∈I. Similarly, if R is an inversive σ-ring, R{(yi)i∈I}* an algebra of σ*-polynomials over R and (ηi)i∈I a family of elements from a σ*-overring of R, one can define a surjective σ-homomorphism R{(yi)i∈I}* → R{(ηi)i∈I}* that maps every yi onto ηi and leaves the ring R fixed. This homomorphism is also called the substitution of (ηi)i∈I for (yi)i∈I. (It will be always clear whether we talk about substitutions for difference or inversive difference polynomials.) If g is a σ- or σ*- polynomial, then its image under a substitution of (ηi)i∈I for (yi)i∈I is denoted by g((ηi)i∈I). The kernel of a substitution φη is an inversive difference ideal of the σ-ring R{(yi)i∈I} (or the σ*-ring R{(ηi)i∈I})*; it is called the defining difference (or σ-) ideal of the family (ηi)i∈I over R. If R is a σ- (or σ*- ) field and (ηi)i∈I is a family of elements from some σ- (respectively, σ*- ) overfield S, then R{(ηi)i∈I} (respectively, R{(ηi)i∈I}*) is an integral domain (it is contained in the field S). It follows that the defining σ-ideal P of the family (ηi)i∈I over R is a reflexive prime difference ideal of the ring R{(yi)i∈I} (respectively, of the ring of σ*-polynomials R{(yi)i∈I})*. Therefore, the difference field R{(ηi)i∈I} can be treated as the quotient σ-field of the σ-ring R{(yi)i∈I}/P. (In the case of inversive difference rings, the σ*-field R{(ηi)i∈I}* can be considered as a quotient σ-field of the σ*-ring R{(yi)i∈I}*).

Let K be a difference field with a basic set σ and n a positive integer. By an n-tuple over K we mean an n-dimensional vector a = (a1, ..., an) whose coordinates belong to some σ-overfield of K. If the σ-field K is inversive, the coordinates of an n-tuple over K are supposed to lie in some σ*-overfield of K. If each ai (1 ≤ i ≤ n) is σ-algebraic over the σ-field K, we say that the n-tuple a is σ-algebraic over K.

**Definition 2.3.** Let K be a difference (inversive difference) field with a basic set σ and let R be the algebra of σ- (respectively, σ*- ) polynomials in finitely many σ- (respectively, σ*- ) indeterminates y1, ..., yn over K. Furthermore, let Φ = {fj | j ∈ J} be a set of σ- (respectively, σ*- ) polynomials from R. An n-tuple η = (η1, ..., ηn) over K is said to be a solution of the set Φ or a solution of the system of difference algebraic equations fj(y1, ..., yn) = 0 (j ∈ J) if Φ is contained in the kernel of the substitution of (η1, ..., ηn) for (y1, ..., yn). In this case we also say that η annihilates Φ. (If Φ is a subset of a ring of inversive difference polynomials, the system is said to be a system of algebraic σ*-equations.) A system of algebraic difference equations Φ is called prime if the reflexive difference ideal generated by Φ in the ring of σ (or σ*- ) polynomials is prime.

As we have seen, if one fixes an n-tuple η = (η1, ..., ηn) over a σ-field F, then all σ-polynomials of the ring K{y1, ..., yn}, for which η is a solution, form a reflexive prime difference ideal, the defining σ-ideal of η. If η is an n-tuple over a σ*-field K, then all σ*-polynomials g of the ring K{y1, ..., yn}* such that g(η1, ..., ηn) = 0 form a prime σ*-ideal of K{y1, ..., yn}* called the defining σ*-ideal of η over K.

Let Φ be a subset of the algebra of σ-polynomials K{y1, ..., yn} over a σ-field K. An n-tuple η = (η1, ..., ηn) over K is called a generic zero of Φ if for any σ-polynomial f ∈ K{y1, ..., yn}, the inclusion f ∈ Φ holds if and only if f(η1, ..., ηn) = 0. If the σ-field K is inversive, then the notion of a generic zero of a subset of K{y1, ..., yn}* is defined similarly.

Two n-tuples η = (η1, ..., ηn) and ζ = (ζ1, ..., ζn) over a σ- (or σ*- ) field K are called equivalent over K if there is a σ-homomorphism K{η1, ..., ηn} →
A polynomial \( K(\zeta_1, \ldots, \zeta_n) \) (respectively, \( K(\eta_1, \ldots, \eta_n)^* \to K(\zeta_1, \ldots, \zeta_n)^* \)) that maps each \( \eta_i \) onto \( \zeta_i \) and leaves the field \( K \) fixed.

**Proposition 2.4.** ([10] Proposition 3.3.7). Let \( R \) denote the algebra of \( \sigma \)-polynomials \( K\{y_1, \ldots, y_k\} \) over a difference field \( K \) with a basic set \( \sigma \).

(i) A set \( \Phi \subseteq R \) has a generic zero if and only if \( \Phi \) is a prime reflexive \( \sigma \)-ideal of \( R \). If \( (\eta_1, \ldots, \eta_n) \) is a generic zero of \( \Phi \), then \( K(\eta_1, \ldots, \eta_n) \) is \( \sigma \)-isomorphic to the quotient \( \sigma \)-field of \( R/\Phi \).

(ii) Any \( n \)-tuple over \( K \) is a generic zero of some prime reflexive \( \sigma \)-ideal of \( R \).

(iii) If two \( n \)-tuples over \( K \) are generic zeros of the same prime reflexive \( \sigma \)-ideal of \( R \), then these \( n \)-tuples are equivalent.

### 2.2. Numerical polynomials of subsets of \( \mathbb{N}^m \) and \( \mathbb{Z}^m \).

**Definition 2.5.** A polynomial \( f(t) \) in one variable \( t \) with rational coefficients is called numerical if \( f(r) \in \mathbb{Z} \) for all sufficiently large \( r \in \mathbb{Z} \).

Of course, every polynomial with integer coefficients is numerical. As an example of a numerical polynomial with non-integer coefficients one can consider a polynomial \( \left( \frac{t}{k} \right) \) where \( k \in \mathbb{N} \). (As usual, \( \left( \frac{t}{k} \right) \) \((k \geq 1)\) denotes the polynomial \( \frac{t(t-1) \ldots (t-k+1)}{k!} \), \( \left( \frac{t}{0} \right) = 1 \), and \( \left( \frac{t}{k} \right) = 0 \) if \( k < 0 \).)

The following theorem proved in [9] Chapter 0, section 17 gives the “canonical” representation of a numerical polynomial.

**Theorem 2.6.** Let \( f(t) \) be a numerical polynomial of degree \( d \). Then \( f(t) \) can be represented in the form

\[
(2.1) \quad f(t) = \sum_{i=0}^{d} a_i \left( \frac{t + i}{i} \right)
\]

with uniquely defined integer coefficients \( a_i \).

In what follows (until the end of the section), we deal with subsets of \( \mathbb{Z}^m \) \((m \) is a positive integer). If \( a = (a_1, \ldots, a_m) \in \mathbb{Z}^m \), then the number \( \sum_{i=1}^{m} a_i \) will be called the order of the \( m \)-tuple \( a \); it is denoted by \( \text{ord} a \). Furthermore, the set \( \mathbb{Z}^m \) will be considered as the union

\[
(2.2) \quad \mathbb{Z}^m = \bigcup_{1 \leq j \leq 2^m} \mathbb{Z}_j^{(m)}
\]

where \( \mathbb{Z}_1^{(m)}, \ldots, \mathbb{Z}_{2^m}^{(m)} \) are all distinct Cartesian products of \( m \) sets each of which is either \( \mathbb{N} \) or \( \mathbb{Z}_- = \{a \in \mathbb{Z} | a \leq 0\} \). We assume that \( \mathbb{Z}_1^{(m)} = \mathbb{N}^m \) and call \( \mathbb{Z}_j^{(m)} \) the \( j \)th orthant of the set \( \mathbb{Z}^m \) \((1 \leq j \leq 2^m)\).

The set \( \mathbb{Z}^m \) will be considered as a partially ordered set with the order \( \leq \) such that \( (e_1, \ldots, e_m) \leq (e'_1, \ldots, e'_m) \) if and only if \( (e_1, \ldots, e_m) \) and \( (e'_1, \ldots, e'_m) \) belong to the same orthant \( \mathbb{Z}_k^{(m)} \) and the \( m \)-tuple \((|e_1|, \ldots, |e_m|)\) is less than \((|e'_1|, \ldots, |e'_m|)\) with respect to the product order on \( \mathbb{N}^m \).

In what follows, for any set \( A \subseteq \mathbb{Z}^m \), \( W_A \) will denote the set of all elements of \( \mathbb{Z}^m \) that do not exceed any element of \( A \) with respect to the order \( \leq \). (Thus, \( w \in W_A \) if and only if there is no \( a \in A \) such that \( a \leq w \).) Furthermore, for any \( r \in \mathbb{N} \), \( A(r) \) will denote the set of all elements \( x = (x_1, \ldots, x_m) \in A \) such that \( \text{ord} x \leq r \).
The above notation can be naturally applied to subsets of \( \mathbb{N}^m \) (treated as subsets of \( \mathbb{Z}^m \)). If \( E \subseteq \mathbb{N}^m \) and \( s \in \mathbb{N} \), then \( E(s) \) will denote the set of all \( m \)-tuples \( e = (e_1, \ldots, e_m) \in E \) such that \( \text{ord } e \leq s \). Furthermore, we shall associate with a set \( E \subseteq \mathbb{N}^m \) the set \( V_E \subseteq \mathbb{N}^m \) that consists of all \( m \)-tuples \( v = (v_1, \ldots, v_m) \in \mathbb{N}^m \) that are not greater than or equal to any \( m \)-tuple from \( E \) with respect to the product order on \( \mathbb{N}^m \). (Clearly, an element \( v = (v_1, \ldots, v_m) \in \mathbb{N}^m \) belongs to \( V_E \) if and only if for any element \( (e_1, \ldots, e_m) \in E \), there exists \( i \in \mathbb{N}, 1 \leq i \leq m \), such that \( e_i > v_i \).

The following two theorems proved, respectively, in \([9, Chapter 2]\) and \([10, Chapter 2]\) introduce certain numerical polynomials associated with subsets of \( \mathbb{N}^m \) and give explicit formulas for the computation of these polynomials.

**Theorem 2.7.** Let \( E \) be a subset of \( \mathbb{N}^m \). Then there exists a numerical polynomial \( \omega_E(t) \) with the following properties:

(i) \( \omega_E(r) = \text{Card } V_E(r) \) for all sufficiently large \( r \in \mathbb{N} \).

(ii) \( \omega_E \) does not exceed \( m \) and \( \text{deg } \omega_E = m \) if and only if \( E = \emptyset \). In the last case, \( \omega_E(t) = \left( \frac{t + m}{m} \right) \).

The polynomial \( \omega_E(t) \) is called the dimension polynomial of the set \( E \subseteq \mathbb{N}^m \).

**Theorem 2.8.** Let \( E = \{e_1, \ldots, e_q\} \ (q \geq 1) \) be a finite subset of \( \mathbb{N}^m \). Let \( e_i = (e_{i1}, \ldots, e_{im}) \quad (1 \leq i \leq q) \) and for any \( l \in \mathbb{N}, 0 \leq l \leq q \), let \( \Theta(l, q) \) denote the set of all \( l \)-element subsets of the set \( \mathbb{N}_q = \{1, \ldots, q\} \). Furthermore, let \( e_i^j = 0 \) and for any \( \theta \in \Theta(l, q), \theta \neq \emptyset \), let \( \bar{e}_{\theta j} = \max\{e_{ij} \mid i \in \theta\}, 1 \leq j \leq m \). (In other words, if \( \theta = \{i_1, \ldots, i_l\}, \) then \( \bar{e}_{\theta j} \) denotes the greatest \( j \)-th coordinate of the elements \( e_{i_1}, \ldots, e_{i_l} \)). Furthermore, let \( b_\theta = \sum_{j=1}^{m} \bar{e}_{\theta j} \). Then

\[
\omega_E(t) = \sum_{l=0}^{q} (-1)^l \sum_{\theta \in \Theta(l, q)} \left( \frac{t + m - b_\theta}{m} \right)
\]

**Remark.** It is clear that if \( E \subseteq \mathbb{N}^m \) and \( E^* \) is the set of all minimal elements of the set \( E \) with respect to the product order on \( \mathbb{N}^m \), then the set \( E^* \) is finite and \( \omega_E(t) = \omega_{E^*}(t) \). Thus, the last theorem gives an algorithm that allows one to find a numerical polynomial associated with any subset of \( \mathbb{N}^m \); one should first find the set of all minimal points of the subset and then apply Theorem 2.8.

The following two results proved in \([10, Section 2.5]\) describe dimension polynomials associated with subsets of \( \mathbb{Z}^m \).

**Theorem 2.9.** Let \( A \) be a subset of \( \mathbb{Z}^m \). Then there exists a numerical polynomial \( \phi_A(t) \) such that

(i) \( \phi_A(r) = \text{Card } W_A(r) \) for all sufficiently large \( r \in \mathbb{N} \).

(ii) \( \text{deg } \phi_A \leq m \) and the polynomial \( \phi_A(t) \) can be written in the form \( \phi_A(t) = \sum_{i=0}^{m} a_i \left( \frac{t+1}{i} \right) \) where \( a_i \in \mathbb{Z} \) and \( 2^m | a_m \).

(iii) \( \phi_A(t) = 0 \) if and only if \((0, \ldots, 0) \in A. \)

(iv) If \( A = \emptyset \), then \( \phi_A(t) = \sum_{i=0}^{m} (-1)^{m-i}\left( \frac{m}{i} \right) \left( \frac{t+1}{i} \right) \).
Theorem 2.10. With the notation of Theorem 2.9, let us consider a mapping \( \rho : \mathbb{Z}^m \rightarrow \mathbb{N}^{2m} \) such that
\[
\rho((e_1, \ldots, e_m)) = (\max\{e_1, 0\}, \ldots, \max\{e_m, 0\}, \max\{-e_1, 0\}, \ldots, \max\{-e_m, 0\}).
\]
Let \( B = \rho(A) \cup \{\bar{e}_1, \ldots, \bar{e}_m\} \) where \( \bar{e}_i \) (1 \( \leq \) i \( \leq \) m) is a 2n-tuple in \( \mathbb{N}^{2m} \) whose ith and \((m+i)\)th coordinates are equal to 1 and all other coordinates are equal to 0. Then \( \phi_A(t) = \omega_B(t) \) where \( \omega_B(t) \) is the dimension polynomial of the set \( B \subseteq \mathbb{N}^{2m} \) (i. e., the dimension polynomial introduced in Theorem 2.7).

The polynomial \( \phi_A(t) \) is called the dimension polynomial of the set \( A \subseteq \mathbb{Z}^m \). It is easy to see that Theorems 10 and 8 provide an algorithm for computing such a polynomial.

3. Difference Dimension Polynomials

Let \( K \) be a difference field with a basic set \( \sigma = \{\alpha_1, \ldots, \alpha_m\} \). As before, \( T \) will denote the free commutative semigroup of all elements of the form \( \tau = \alpha_1^{k_1} \cdots \alpha_m^{k_m} \) (\( k_i \in \mathbb{N} \) for \( i = 1, \ldots, m \)). We define the order of such an element as \( \text{ord} \tau = \sum_{i=1}^{m} k_i \) and set \( T(r) = \{ \tau \in T \mid \text{ord} \tau \leq r \} \) for every \( r \in \mathbb{N} \). If the difference (\( \sigma \)-)field is inversive, that is, all mappings \( \alpha_i \) are automorphisms of \( K \), then \( \Gamma \) still denotes the free commutative group generated by the set \( \sigma \). If \( \gamma = \alpha_1^{k_1} \cdots \alpha_m^{k_m} \in \Gamma \) \((k_1, \ldots, k_m \in \mathbb{N})\), the order of \( \gamma \) is defined as \( \text{ord} \gamma = \sum_{i=1}^{m} |k_i| \). Furthermore, for any \( r \in \mathbb{N} \), we set \( \Gamma(r) = \{ \gamma \in \Gamma \mid \text{ord} \gamma \leq r \} \).

The following two theorems proved, respectively, in [11] (see also [12], Chapter 4) and [10], Theorem 6.4.8, introduce the concepts and provide some description of dimension polynomials associated with finitely generated difference and inversive difference field extensions.

Theorem 3.1. Let \( K \) be a difference field with a basic set \( \sigma = \{\alpha_1, \ldots, \alpha_m\} \) and let \( L = K(\eta_1, \ldots, \eta_n) \) be a difference field extension of \( K \) generated by a finite set \( \eta = \{\eta_1, \ldots, \eta_n\} \). Then there exists a numerical polynomial \( \phi_\eta(t) \) such that
(i) \( \phi_\eta(r) = \text{tr.deg}_K \{\tau \eta_j \mid \tau \in T(r), 1 \leq j \leq n\} \) for all sufficiently large \( r \in \mathbb{Z} \).
(ii) \( \deg \phi_\eta(t) \leq m \) and \( \phi_\eta(t) \) can be written as
\[
\phi_\eta(t) = \sum_{i=0}^{m} a_i \binom{t+i}{i}
\]
where \( a_0, \ldots, a_m \in \mathbb{Z} \);
(iii) The degree \( d \) of the polynomial \( \phi_\eta(t) \) and the coefficients \( a_m \) and \( a_d \) do not depend on the choice of the system of generators \( \eta \) (clearly, \( a_d \neq a_m \) if and only if \( d < m \), that is, \( a_m = 0 \)). Moreover, \( a_m \) is equal to the difference transcendence degree of \( L \) over \( K \), i. e., to the maximal number of elements \( \xi_1, \ldots, \xi_k \in L \) such that the set \( \{\tau(\xi_i) \mid \tau \in T, 1 \leq i \leq k\} \) is algebraically independent over \( K \) (this characteristic of the \( \sigma \)-field extension is denoted by \( \sigma \)-\text{tr.deg}_K \( L \)).

The numerical polynomial \( \phi_\eta(t) \) is called a difference (or \( \sigma \)-) dimension polynomial of the difference (\( \sigma \)-) field extension \( L/K \).
Theorem 3.2. Let $K$ be an inversive difference field with a basic set $\sigma = \{\alpha_1, \ldots, \alpha_m\}$ and $L$ an inversive difference field extension of $K$ generated by a finite set $\eta = \{\eta_1, \ldots, \eta_n\}$. Then there exists a polynomial $\psi_{\eta|K}(t)$ in one variable $t$ with rational coefficients such that

(i) $\psi_{\eta|K}(r) = \text{tr.deg}_K K(\{\gamma(\eta_j)|\gamma \in \Gamma(r), 1 \leq j \leq p\})$ for all sufficiently large integers $r$.

(ii) $\deg \psi_{\eta|K} \leq n$ and the polynomial $\psi_{\eta|K}(t)$ can be written as

$$\psi_{\eta|K}(t) = \sum_{i=0}^{m} a_i 2^i \binom{t+i}{i}$$

where $a_0, \ldots, a_m \in \mathbb{Z}$.

(iii) The degree $d$ of the polynomial $\psi_{\eta|K}$ and the coefficients $a_m$ and $a_d$ do not depend on the choice of the system of $\sigma^*$-generators $\eta$ of the extension $L/K$. Furthermore, $a_m$ is equal to the difference transcendence degree of $L$ over $K$.

The polynomial $\psi_{\eta|K}(t)$ is called the **inversive difference (or $\sigma^*$-) dimension polynomial** of the $\sigma^*$-field extension $L/K$ associated with the set of $\sigma^*$-generators $\eta = \{\eta_1, \ldots, \eta_n\}$.

Let $K$ be a difference (respectively, inversive difference) field with a basic set $\sigma$ and $R$ an algebra of $\sigma$- (respectively, $\sigma^*$-) polynomials in difference indeterminates $y_1, \ldots, y_n$ over $K$. Let $P$ be a prime inversive difference ideal of $R$ and $\eta = (\eta_1, \ldots, \eta_n)$ a generic zero of $P$. Then the dimension polynomial $\phi_{\eta|K}(t)$ (respectively, $\psi_{\eta|K}(t)$) associated with the $\sigma$- (respectively, $\sigma^*$-) field extension $K(\eta_1, \ldots, \eta_n)/K$ (respectively, $K(\eta_1, \ldots, \eta_n)^*/K$) is called the $\sigma$- (respectively, $\sigma^*$-) **dimension polynomial** of the ideal $P$. It is denoted by $\phi_P(t)$ (respectively, $\psi_P(t)$).

With the above notation, the inversive difference (or $\sigma^*$-) dimension polynomial of a prime system of algebraic difference equations $g_i(y_1, \ldots, y_n) = 0$ ($i \in I$) is defined as the $\sigma^*$-dimension polynomial of the prime $\sigma^*$-ideal $P$ generated by the $\sigma^*$-polynomials $g_i$ in $K(\eta_1, \ldots, \eta_n)^*$. This dimension polynomial has an interesting interpretation as a measure of strength in the sense of A. Einstein. Considering a system of equations in finite differences over a field of functions in several real variables, one can use A. Einstein’s approach to define the concept of strength of such a system as follows. Let

$$A_i(f_1, \ldots, f_n) = 0$$

be a system of equations in finite differences with respect to $n$ unknown grid functions $f_1, \ldots, f_n$ in $m$ real variables $x_1, \ldots, x_m$ with coefficients in some functional field $K$ over $\mathbb{R}$. We also assume that the difference grid, whose nodes form the domain of considered functions, has equal cells of dimension $h_1 \times \cdots \times h_m$ ($h_1, \ldots, h_m \in \mathbb{R}$) and fills the whole space $\mathbb{R}^m$. As an example, one can consider a field $K$ consisting of the zero function and fractions of the form $u/v$ where $u$ and $v$ are grid functions defined almost everywhere and vanishing at a finite number of nodes. (As usual, we say that a grid function is defined almost everywhere if there are only finitely many nodes where it is not defined.)

Let us fix some node $P$ and say that a node $Q$ has order $i$ (with respect to $P$) if the shortest path from $P$ to $Q$ along the edges of the grid consists of $i$ steps (by a step we mean a path from a node of the grid to a neighbor node along the
edge between these two nodes). For example, the orders of the nodes in the two-dimensional case are as follows (a number near a node shows the order of this node).

| 5 | 4 | 3 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 4 | 3 | 2 | 1 | 2 | 3 | 4 |
| 3 | 2 | 1 | 3 | 1 | 2 | 3 |
| 4 | 3 | 2 | 1 | 2 | 3 | 4 |
| 5 | 4 | 3 | 2 | 3 | 4 | 5 |

Let us consider the values of the unknown grid functions $f_1, \ldots, f_n$ at the nodes whose order does not exceed $r$ ($r \in \mathbb{N}$). If $f_1, \ldots, f_n$ should not satisfy any system of equations (or any other condition), their values at nodes of any order can be chosen arbitrarily. Because of the system in finite differences (and equations obtained from the equations of the system by transformations of the form $f_j(x_1, \ldots, x_m) \rightarrow f_j(x_{1} + k_1 h_1, \ldots, x_{m} + k_m h_m)$ with $k_1, \ldots, k_m \in \mathbb{Z}$, $1 \leq j \leq n$), the number of independent values of the functions $f_1, \ldots, f_n$ at the nodes of order $\leq r$ decreases. This number, which is a function of $r$, is considered as the "measure of strength" of the system in finite differences (in the sense of A. Einstein). We denote it by $S_r$.

With the above conventions, suppose that the transformations $\alpha_j$ of the field of coefficients $K$ defined by

$$\alpha_j f(x_1, \ldots, x_m) = f(x_1, \ldots, x_{j-1}, x_j + h_j, \ldots, x_m) \quad (1 \leq j \leq m)$$

are automorphisms of this field. Then $K$ can be considered as an inverse difference field with the basic set $\sigma = \{\alpha_1, \ldots, \alpha_m\}$. Furthermore, assume that the replacement of the unknown functions $f_i$ by $\sigma^*$-indeterminates $y_i$ ($i = 1, \ldots, n$) in the ring $K\{y_1, \ldots, y_n\}^*$ leads to a prime system of algebraic $\sigma^*$-equations (then the original system of equations in finite differences is also called prime). The $\sigma^*$-dimension polynomial $\psi(t)$ of the latter system is said to be the $\sigma^*$-dimension polynomial of the given system in finite differences.

Clearly, $\psi(r) = S_r$ for any $r \in \mathbb{N}$, so the $\sigma^*$-dimension polynomial of a prime system of equations in finite differences is the measure of strength of such a system in the sense of A. Einstein.

The main method of computation of difference (respectively, inverse difference) dimension polynomials is based on the construction of characteristic sets in the algebra difference (respectively, inverse difference) polynomials. In what follows we will consider the case of inverse difference ($\sigma^*$-) polynomials, they reflect finite-difference approximations of PDEs where shifts of arguments can be of any sign.

Let $K$ be an inverse difference field with a basic set of automorphisms (translations) $\sigma = \{\alpha_1, \ldots, \alpha_m\}$ and $\Gamma = \Gamma_\sigma$ the free commutative group generated by $\sigma$. Let $\mathbb{Z}_-$ denote the set of all non-positive integers and let $\mathbb{Z}_1^{(m)}, \mathbb{Z}_2^{(m)}, \ldots, \mathbb{Z}_m^{(m)}$ be all orthants of $\mathbb{Z}^m$. Recall (see section 2) that they are distinct Cartesian products of $m$ factors each of which is either $\mathbb{N}$ or $\mathbb{Z}_-$ (we assume that $\mathbb{Z}_1^{(m)} = \mathbb{N}$). For any $j = 1, \ldots, 2^m$, we set $\Gamma_j = \{\gamma = \alpha_1^{k_1} \ldots \alpha_m^{k_m} \in \Gamma \mid (k_1, \ldots, k_m) \in \mathbb{Z}_j^{(m)}\}$. As before,
for any element $\gamma = \alpha_1^{k_1} \ldots \alpha_m^{k_m} \in \Gamma$, the number $\text{ord} \gamma = \sum_{i=1}^m |k_i|$ will be called the order of $\gamma$.

Let $K\{y_1, \ldots, y_n\}^*$ be the ring of $\sigma^*$-polynomials in $\sigma^*$-indeterminates $y_1, \ldots, y_n$ over $K$ and let $\Gamma Y$ denote the set $\{\gamma y_i | \gamma \in \Gamma, 1 \leq i \leq m\}$ whose elements are called terms (here and below we often write $\gamma y_i$ for $\gamma(y_i)$). By the order of a term $u = \gamma y_j$ we mean the order of the element $\gamma \in \Gamma$. Setting $(\Gamma Y)_j = \{\gamma y_i | \gamma \in \Gamma, 1 \leq i \leq n\}$ $(j = 1, \ldots, 2^n)$ we obtain a representation of the set of terms as a union

$$\Gamma Y = \bigcup_{j=1}^{2^n} (\Gamma Y)_j.$$  

Definition 3.3. A term $v \in \Gamma Y$ is called a transform of a term $u \in \Gamma Y$ if and only if $u$ and $v$ belong to the same set $(\Gamma Y)_j$ $(1 \leq j \leq 2^n)$ and $v = \gamma u$ for some $\gamma \in \Gamma_j$. If $\gamma \neq 1$, $v$ is said to be a proper transform of $u$.

Definition 3.4. A well-ordering of the set of terms $\Gamma Y$ is called a ranking of the family of $\sigma^*$-indeterminates $y_1, \ldots, y_n$ (or a ranking of the set $\Gamma Y$) if it satisfies the following conditions. (We use the standard symbol $\leq$ for the ranking; it will be always clear what order is denoted by this symbol.)

(i) If $u \in (\Gamma Y)_j$ and $\gamma \in \Gamma_j$ $(1 \leq j \leq 2^n)$, then $u \leq \gamma u$.

(ii) If $u, v \in (\Gamma Y)_j$ $(1 \leq j \leq 2^n)$, $u \leq v$ and $\gamma \in \Gamma_j$, then $\gamma u \leq \gamma v$.

A ranking of the $\sigma^*$-indeterminates $y_1, \ldots, y_n$ is called orderly if for any $j = 1, \ldots, 2^n$ and for any two terms $u, v \in (\Gamma Y)_j$, the inequality $\text{ord} u < \text{ord} v$ implies that $u < v$ (as usual, $v < w$ means $v \leq w$ and $v \neq w$).

As an example of an orderly ranking of the $\sigma^*$-indeterminates $y_1, \ldots, y_n$ one can consider the standard ranking defined as follows: $u = \alpha_1^{k_1} \ldots \alpha_m^{k_m} y_i \leq v = \alpha_1^{l_1} \ldots \alpha_m^{l_m} y_j$ if and only if the $(2m + 2)$-tuple $\left(\sum_{\nu=1}^m |k_\nu|, |k_1|, \ldots, |k_m|, k_1, \ldots, k_m, i\right)$ is less than or equal to the $(2m + 2)$-tuple $\left(\sum_{\nu=1}^m |l_\nu|, |l_1|, \ldots, |l_m|, l_1, \ldots, l_m, j\right)$ with respect to the lexicographic order on $\mathbb{Z}^{2m+2}$.

In what follows, we assume that an orderly ranking $\leq$ of the set of $\sigma^*$-indeterminates $y_1, \ldots, y_n$ is fixed. If $A \in K\{y_1, \ldots, y_n\}^*$, then the greatest (with respect to the ranking $\leq$) term that appears in $A$ is called the leader of $A$; it is denoted by $u_A$. If $u = u_A$ and $d = \text{deg}_u A$, then the $\sigma^*$-polynomial $A$ can be written as $A = I_d u^d + I_{d-1} u^{d-1} + \cdots + I_0$ where $I_k (0 \leq k \leq d)$ do not contain $u$. The $\sigma^*$-polynomial $I_d$ is called the initial of $A$; it is denoted by $I_A$.

Definition 3.5. Let $A, B \in K\{y_1, \ldots, y_n\}$. We say that $A$ has higher rank than $B$ and write $\text{rk} A > \text{rk} B$ if either $A \notin K$, $B \in K$, or $u_A$ has higher rank than $u_B$, or $u_A = u_B$ and $\text{deg}_{u_A} A > \text{deg}_{u_B} B$. If $u_A = u_B$ and $\text{deg}_{u_A} A = \text{deg}_{u_B} B$, we say that $A$ and $B$ have the same rank and write $\text{rk} A = \text{rk} B$.

Note that distinct $\sigma^*$-polynomials can have the same rank and if $A \notin K$, then $I_A$ has lower rank than $A$.

Definition 3.6. Let $A, B \in K\{y_1, \ldots, y_n\}$. The $\sigma^*$-polynomial $A$ is said to be reduced with respect to $B$ if $A$ does not contain any power of a transform $\gamma u_B$ ($\gamma \in$...
Definition 3.9. Following partial order on the set of all autoreduced sets. With this assumption we introduce the sets of differential polynomials, see [9, Chapter 2].

Proposition 3.7. Every autoreduced set is finite and distinct elements of an autoreduced set have distinct leaders.

Theorem 3.8. ([12, Theorem 2.4.7]) Let $A = \{A_1, \ldots, A_p\}$ be an autoreduced subset in the ring of $\sigma^*$-polynomials $K\{y_1, \ldots, y_n\}^*$ and let $D \in K\{y_1, \ldots, y_n\}^*$. Furthermore, let $I(A)$ denote the set of all $\sigma^*$-polynomials $B \in K\{y_1, \ldots, y_n\}$ such that either $B = 1$ or $B$ is a product of finitely many polynomials of the form $\gamma(I_{A_i})$ where $\gamma \in \Gamma, i = 1, \ldots, p$. Then there exist $\sigma^*$-polynomials $J \in I(A)$ and $D_0 \in K\{y_1, \ldots, y_n\}$ such that $D_0$ is reduced with respect to $A$ and $JD \equiv D_0(mod \{A\})$.

Note that, with the notation of the last theorem, the process of reduction that leads to the $\sigma^*$-polynomials $J \in I(A)$ and $D_0$ is algorithmic; the steps of the corresponding algorithm can be obtained by mimicking the steps in the proof of Theorem 2.4.1 of [12]. The $\sigma^*$-polynomial $D_0$ is called the remainder of $D$ with respect to $A$. We also say that $D$ reduces to $D_0$ modulo $A$.

In what follows elements of an autoreduced set in $K\{y_1, \ldots, y_n\}^*$ will be always written in the order of increasing rank. With this assumption we introduce the following partial order on the set of all autoreduced sets.

Definition 3.9. Let $A = \{A_1, \ldots, A_p\}$ and $B = \{B_1, \ldots, B_q\}$ be two autoreduced sets of $\sigma^*$-polynomials in $K\{y_1, \ldots, y_n\}^*$. We say that $A$ has lower rank than $B$ and write $rk A < rk B$ if either there exists $k \in \mathbb{N}, 1 \leq k \leq \min\{p, q\}$, such that $rk A_k = rk B_i$ for $i = 1, \ldots, k - 1$ and $rk A_k < rk B_k$, or $p > q$ and $rk A_i = rk B_i$ for $i = 1, \ldots, q$.

Mimicking the arguments of [9, Chapter 1, Section 9], one obtains that every nonempty family of autoreduced subsets of $K\{y_1, \ldots, y_n\}^*$ contains an autoreduced set of lowest rank. In particular, if $\emptyset \neq J \subseteq F\{y_1, \ldots, y_n\}^*$, then the set $J$ contains an autoreduced set of lowest rank called a characteristic set of $J$.

Proposition 3.10. ([12, Proposition 2.4.8]) Let $K$ be an inversive difference field with a basic set $\sigma$, $J$ a $\sigma^*$-ideal of the algebra of $\sigma$-polynomials $K\{y_1, \ldots, y_n\}^*$, and $A$ a characteristic set of $J$. Then

(i) The ideal $J$ does not contain nonzero $\sigma^*$-polynomials reduced with respect to $A$. In particular, if $A \in A$, then $I_A \notin J$.

(ii) If $J$ is a prime $\sigma^*$-ideal, then $J = [A]^* : \Upsilon(A)$ where $\Upsilon(A)$ denotes the set of all finite products of elements of the form $\gamma(I_{A_i}) (\gamma \in \Gamma, A \in A)$.

If $K$ is an inversive difference field, then a $\sigma^*$-ideal of the ring of $\sigma^*$-polynomials $K\{y_1, \ldots, y_n\}^*$ is called linear if it is generated (as a $\sigma^*$-ideal) by
homogeneous linear $\sigma^*$-polynomials, i. e., $\sigma^*$-polynomials of the form $\sum_{i=1}^{p} a_i \gamma_i y_i$, $(a_i \in K, \gamma_i \in \Gamma, 1 \leq k_i \leq n$ for $i = 1, \ldots, p)$. As it is shown in [12 Proposition 2.4.9], every linear $\sigma^*$-ideal in $K\{y_1, \ldots, y_n\}^*$ is prime.

**Definition 3.11.** Let $K$ be an inversive difference $(\sigma^*)$ field and $\mathcal{A}$ an autoreduced set in $K\{y_1, \ldots, y_n\}^*$ that consists of linear $\sigma^*$-polynomials. The set $\mathcal{A}$ is called **coherent** if the following two conditions hold.

(i) If $A \in \mathcal{A}$ and $\gamma \in \Gamma$, then $\gamma A$ reduces to zero modulo $\mathcal{A}$.

(ii) If $A, B \in \mathcal{A}$ and $v = \gamma_1 u_A = \gamma_2 u_B$ is a common transform of the leaders $u_A$ and $u_B (\gamma_1, \gamma_2 \in \Gamma)$, then the $\sigma^*$-polynomial $(\gamma_2 I_B)(\gamma_1 A) - (\gamma_1 I_A)(\gamma_2 B)$ reduces to zero modulo $\mathcal{A}$.

**Theorem 3.12.** ([10 Theorem 6.5.3]) Let $K$ be an inversive difference $(\sigma^*)$ field and $J$ a linear $\sigma^*$-ideal of $K\{y_1, \ldots, y_n\}^*$. Then any characteristic set of $J$ is a coherent autoreduced set of linear $\sigma^*$-polynomials. Conversely, if $\mathcal{A} \subseteq K\{y_1, \ldots, y_n\}^*$ is any coherent autoreduced set consisting of linear $\sigma^*$-polynomials, then $\mathcal{A}$ is a characteristic set of the linear $\sigma^*$-ideal $[\mathcal{A}]^*$.

**Remark 3.13.** Analyzing the proof of the last theorem given in [10], one can see that this proof also works for the case when all elements of a coherent autoreduced set are quasi-linear $\sigma^*$-polynomials, i. e., $\sigma^*$-polynomials that are linear with respect to their leaders. (Of course, in this case the $\sigma^*$-ideal $[\mathcal{A}]^*$ is not linear; moreover, it is not necessarily prime.)

**Proposition 3.14.** With the above notation, let $A$ be a quasi-linear $\sigma$-polynomial in the ring of difference $(\sigma)$-polynomials $K\{y_1, \ldots, y_n\}$. Then

(i) If

$$M = \sum_{i=1}^{p} C_i \tau_i A$$

where $\tau_i \in T$ and $C_i \in K\{y_1, \ldots, y_n\}$ $(1 \leq i \leq p)$, then $M$ contains the leader of some $\tau_i A$.

(ii) The $\sigma$-ideal $[\mathcal{A}]$ of the ring $K\{y_1, \ldots, y_n\}$ is prime.

Furthermore, similar statements hold when the $\sigma$-field $K$ is inversive and $\mathcal{A}$ is a quasi-linear $\sigma^*$-polynomial in $K\{y_1, \ldots, y_n\}^*$. (In this case, $\tau_i \in \Gamma$ and $\tau_i u_A$ are transforms of the leader of $\mathcal{A}$ in the sense of Definition 3.3.)

**Proof.** We will prove our statement in the case when $\mathcal{A}$ is a quasi-linear $\sigma$-polynomial in $K\{y_1, \ldots, y_n\}$ (the proof for a $\sigma^*$-polynomial in $K\{y_1, \ldots, y_n\}^*$ is similar).

Suppose that statement (i) is not true. Let $p$ be the smallest positive integer such that one has equality (3.2) where $M$ does not contain any $u_{\tau_i A} = \tau_i u_A$. Without loss of generality we can assume that $\tau_1 < \cdots < \tau_p$ (so that $u_{\tau_1 A} < \cdots < u_{\tau_p A}$) and $\tau_p A$ does not divide any $C_i$ for $i = 1, \ldots, p - 1$ (otherwise, representation (3.2) is not minimal in the above sense). If one writes $C_i$ $(1 \leq i \leq p - 1)$ as a polynomial of $u_{\tau_p A}$, $C_i = \sum_{i=0}^{d_i} I_i u_{\tau_p A}^i$, where the $\sigma$-polynomials $I_i$ do not contain $u_{\tau_p A}$, then $\deg (C_i - I_i u_{\tau_p A}) d_i > d_i$, so we can obtain a $\sigma^*$-polynomial $D_i$ such that $D_i$ does not contain $u_{\tau_p A}$ and $D_i \equiv C_i \pmod{(\tau_p A)}$. Then $\tau_p A$ divides the $\sigma$-polynomial $M' = M - \sum_{i=1}^{p-1} D_i (\tau_i A)$. However, $M'$ does not contain $u_{\tau_p A}$.
and therefore cannot be divisible by \( \tau_p A \). It follows that \( M' = 0 \), hence 
\[
M = \sum_{i=1}^{p^{-1}} D_i(\tau A),
\]
contrary to the assumption about the minimality of \( p \) in (3.2).

In order to show that the \( \sigma \)-ideal \([A]\) is prime, consider two \( \sigma \)-polynomials \( B, C \in K\{y_1, \ldots, y_n\} \setminus [A] \). Let \( \tau_1 u_A, \ldots, \tau_q u_A \) be all transforms of \( u_A \) that appear in \( B \) such that \( \tau_1 < \cdots < \tau_q \). As above, one can subtract from \( B \) an appropriate linear combination of \( \tau_i u_A \) with coefficients in \( K\{y_1, \ldots, y_n\} \) in order to eliminate \( \tau_i u_A \) in \( B \), so that the difference will contain only transforms \( \tau_i u_A \) with \( i < q \). Repeating this process we arrive at a \( \sigma \)-polynomial \( B' \) such that \( B - B' \in [A] \) and \( B' \) does not contain any \( \tau_i u_A \) (for \( \tau \in T \)). Similarly, we can find \( C' \in K\{y_1, \ldots, y_n\} \) such that \( C' \) does not contain any \( \tau_i u_A \) (for \( \tau \in T \)) and \( C' \in [A] \). Then \( B' C' \in [A] \) and \( B'C' \) contains no term of the form \( \tau u_A \) (for \( \tau \in T \)). This contradiction with the first part of the proposition shows that \( B C \notin [A] \), so the \( \sigma \)-ideal \([A]\) is prime. \( \square \)

The following result is a direct consequence of Theorem 3.12 (taking into account remark 3.13)

**Proposition 3.15.** Let \( K \) be an inversive difference field with a basic set \( \sigma \) and let \( \preceq \) be a preorder on \( K\{y_1, \ldots, y_n\}^* \) such that \( A_1 \preceq A_2 \) if and only if \( u_{A_2} \) is a transform of \( u_A \). Furthermore, let \( A \) be a quasi-linear \( \sigma^* \)-polynomial in \( K\{y_1, \ldots, y_n\}^* \setminus K \) and \( A \Gamma = \{\gamma A \mid \gamma \in \Gamma\} \). Then the set of all minimal (with respect to \( \preceq \)) elements of \( A \Gamma \) is a characteristic set of the \( \sigma^* \)-ideal \([A]^*\).

Theorem 3.12 and Proposition 3.15 imply the following method of constructing a characteristic set of a proper linear \( \sigma^* \)-ideal \( I \) in the ring of \( \sigma^* \)-polynomials \( K\{y_1, \ldots, y_n\}^* \) (a similar method can be used for building a characteristic set of a linear \( \sigma \)-ideal in the ring of difference polynomials \( K\{y_1, \ldots, y_n\} \)). Suppose that 
\[
I = [B_1, \ldots, B_p]^* \quad \text{where} \quad B_1, \ldots, B_p \text{ are linear } \sigma^* \text{-polynomials and } B_1 < \cdots < B_p.
\]

It follows from Theorem 2.4.11 that one should find a coherent autoreduced set \( \Phi \subseteq K\{y_1, \ldots, y_n\}^* \) such that \( \Phi^* = I \). Such a set can be obtained from the set \( B = \{B_1, \ldots, B_p\} \) via the following two-step procedure.

**Step 1.** Constructing an autoreduced set \( A \subseteq I \) such that \([A]^* = I\).

If \( B \) is autoreduced, set \( A = B \). If \( B \) is not autoreduced, choose the smallest \( i \) (\( 1 \leq i \leq p \)) such that some \( \sigma^* \)-polynomial \( B_j \), \( 1 \leq j \leq p \), is not reduced with respect to \( B_i \). Replace \( B_j \) by its remainder with respect to \( B_i \) and arrange the \( \sigma^* \)-polynomials of the new set \( B_1 \) in ascending order. Then apply the same procedure to the set \( B_1 \) and so on. After each iteration the number of \( \sigma^* \)-polynomials in the set does not increase, one of them is replaced by a \( \sigma^* \)-polynomial of lower or equal rank, and the others do not change. Therefore, the process terminates after a finite number of steps when we obtain a desired autoreduced set \( A \).

**Step 2.** Constructing a coherent autoreduced set \( \Phi \subseteq I \).

Let \( A_0 = A \) be an autoreduced subset of \( I \) such that \([A]^* = I\). If \( A \) is not coherent, we build a new autoreduced set \( A_1 \subseteq I \) by adding to \( A_0 \) new \( \sigma^* \)-polynomials of the following types.

(a) \( \sigma^* \)-polynomials \( (\gamma_1 I_{A_1}) \gamma_2 A_2 - (\gamma_2 I_{A_2}) \gamma_1 A_1 \) constructed for every pair \( A_1, A_2 \in A_0 \) such that the leaders \( u_{A_1} \) and \( u_{A_2} \) have a common transform \( \tau = \gamma_1 u_{A_1} = \gamma_2 u_{A_2} \) and \( (\gamma_1 I_{A_1}) \gamma_2 A_2 - (\gamma_2 I_{A_2}) \gamma_1 A_1 \) is not reducible to zero modulo \( A_0 \).

(b) \( \sigma^* \)-polynomials of the form \( \gamma A \ (\gamma \in \Gamma, A \in A_0) \) that are not reducible to zero modulo \( A_0 \).

It is clear that \( \text{rk } A_1 < \text{rk } A_0 \). Applying the same procedure to \( A_1 \) and continuing in the same way, we obtain autoreduced subsets \( A_0, A_1, \ldots \) of \( I \) such that \( \text{rk } A_{i+1} < \text{rk } A_i \).
The following result, whose proof can be extracted from the proof of Theorem 4.2.5 of [12], gives a method of computation of the $\sigma^*$-dimension polynomial associated with a reflexive prime difference ideal of the ring of $\sigma^*$-polynomials $K\{y_1,\ldots,y_n\}^*$. Therefore, it provides a method of computation of the Einstein's strength of a prime system of algebraic partial difference equations. In the next part of the paper this result will be used for the evaluation of the strength of systems of difference equations that represent finite-difference schemes for PDEs describing certain chemical processes.

**Theorem 3.16.** Let $K$ be an inversive difference field with a basic set of automorphisms $\sigma = \{\alpha_1,\ldots,\alpha_m\}$, $R = K\{y_1,\ldots,y_n\}^*$ the ring of $\sigma^*$-polynomials over $K$, and $P$ a reflexive prime difference ideal of $R$. Let $L$ denote the quotient field of $R/P$ treated as the $\sigma^*$-field extension $K\langle \eta_1,\ldots,\eta_n \rangle^*$ of $K$ where $\eta_i$ ($1 \leq i \leq n$) is the canonical image of $y_i$ in $R/P$. Then the $\sigma^*$-dimension polynomial $\psi_{\eta|K}(t)$ of $L/K$ associated with the set of $\sigma^*$-generators $\eta = \{\eta_1,\ldots,\eta_n\}$ (see Theorem 3.2) can be found as follows.

Let $A = \{A_1,\ldots,A_p\}$ be a characteristic set of the $\sigma^*$-ideal $P$ and for every $i = 1,\ldots,n$, let

$$E_i = \{(e_{i1},\ldots,e_{im}) \in \mathbb{Z}^m \mid \alpha_1^{e_{i1}}\ldots\alpha_m^{e_{im}}y_i \text{ is the leader of some element of } A\}$$

(of course, some sets $E_i$ might be empty). Then

$$\psi_{\eta|K}(t) = \sum_{i=1}^{n} \phi_{E_i}(t)$$

where $\phi_{E_i}(t)$ is the dimension polynomial of the set $E_i \subseteq \mathbb{Z}^m$ whose existence is established by Theorem 2.9.

4. **Evaluation of the Einstein’s strength of difference schemes for some reaction-diffusion equations**

**1. The diffusion equation in one spatial dimension** for a constant collective diffusion coefficient $a$ and unknown function $u(x,t)$ describing the density of the diffusing material at given position $x$ and time $t$ is as follows:

$$\frac{\partial u(x,t)}{\partial t} = a \frac{\partial^2 u(x,t)}{\partial x^2}.$$  

($a$ is a constant). Let us compute the strength of difference equations that arise from three most common difference schemes for equation (4.1).

Strength of the forward difference scheme

In order to obtain the forward difference scheme for the diffusion equation (4.1), the occurrences of $\frac{\partial u(x,t)}{\partial x}$ and $\frac{\partial u(x,t)}{\partial t}$ are replaced by $u(x+1,t) - u(x,t)$ and $u(x,t+1) - u(x,t)$, respectively (after the appropriate rescaling, one can use these expressions instead of the standard approximations $\frac{u(x+h,t) - u(x,t)}{h}$ and $\frac{u(x,t) - u(x,t+h)}{h}$ with a small step $h$).
We obtain the equation in finite differences

\[(4.2) \quad u(x, t + 1) - u(x, t) = a(u(x + 2, t) - 2u(x + 1, t) + u(x, t)).\]

Let \( K \) be an inversive difference functional field with basic set \( \sigma = \{\alpha_1 : f(x, t) \mapsto f(x + 1, t), \alpha_2 : f(x, t) \mapsto f(x + 1, t + 1)\} \) \((f(x, t) \in K)\) containing \( a \) and let \( K[y]^* \) be the ring of \( \sigma^* \)-polynomials in one \( \sigma^* \)-indeterminate \( y \) over \( K \). Treating \( y \) as the unknown function \( u(x, t) \) in the equation \((4.2)\), we can write this equation as

\[(4.3) \quad a\alpha_1^2 y - 2a\alpha_1 y - \alpha_2 y + (a + 1)y = 0.\]

Since the left-hand side of the last equation is a linear \( \sigma^* \)-polynomial, it generates a linear (and therefore a prime) \( \sigma^* \)-ideal \( P = [A]^* \) in \( K[y]^* \).

Applying Proposition 3.15, we obtain a characteristic set \( \mathcal{A} = \{A_1, A_2, A_3, A_4\} \) of the ideal \( P \) where

\[
A_1 = A = a\alpha_1^2 y - 2a\alpha_1 y - \alpha_2 y + (a + 1)y,
\]

\[
A_2 = \alpha_1^{-1} A = -\alpha_1^{-1} \alpha_2 y + a\alpha_1 y + (a + 1)\alpha_1^{-1} y + 2ay,
\]

\[
A_3 = \alpha_1^{-1} \alpha_2^{-1} A = a\alpha_1 \alpha_2^{-1} y + (a + 1)\alpha_1^{-1} \alpha_2^{-1} y - \alpha_1^{-1} y - 2a\alpha_2^{-1} y,
\]

\[
A_4 = \alpha_1^{-2} \alpha_2^{-1} A = (a + 1)\alpha_1^{-2} \alpha_2^{-1} y + \alpha_1^{-2} y + 2a\alpha_1 \alpha_2^{-1} y + a\alpha_2^{-1} y.
\]

The leaders of these \( \sigma^* \)-polynomials are \( \alpha_1^2 y, \alpha_1^{-1} \alpha_2 y, \alpha_1 \alpha_2^{-1} y, \) and \( \alpha_1^{-2} \alpha_2^{-1} y, \) respectively (they are written first in the \( \sigma^* \)-polynomials \( A_i \) above). Therefore, the \( \sigma^* \)-dimension polynomial of equation \((4.3)\) is equal to the dimension polynomial of the subset \( E = \{(2, 0), (-1, 1), (1, -1), (-2, -1)\} \) of \( \mathbb{Z}^2 \). Applying the results of theorems 2.10 and 2.8 we obtain that the \( \sigma^* \)-dimension polynomial of equation \((4.3)\) that expresses the Einstein’s strength of the forward difference scheme for \((4.1)\) is

\[
\psi_{Forw}(t) = 5t.
\]

**Strength of the symmetric difference scheme**

The symmetric difference scheme for the diffusion equation \((4.1)\) is obtained by replacing the partial derivatives \( \frac{\partial^2 u(x, t)}{\partial x^2} \) and \( \frac{\partial u(x, t)}{\partial t} \) with \( u(x + 1, t) - 2u(x, t) \) and \( u(x, t + 1) - u(x, t - 1) \), respectively. It leads to the equation in finite differences

\[(4.4) \quad u(x, t + 1) - u(x, t - 1) = a(u(x + 1, t) - 2u(x, t) + u(x - 1, t)).\]

where \( a \) is a constant. As in the case of the forward difference scheme, let \( K \) be an inversive difference functional field with basic set \( \sigma = \{\alpha_1 : f(x, t) \mapsto f(x + 1, t), \alpha_2 : f(x, t) \mapsto f(x, t + 1)\} \) \((f(x, t) \in K)\) and let \( K[y]^* \) be the ring of \( \sigma^* \)-polynomials in one \( \sigma^* \)-indeterminate \( y \) over \( K \) \((y \) is treated as the unknown function \( u(x, t)\); we also assume that \( a \in K \). Then the equation \((4.4)\) can be written as

\[(4.5) \quad a\alpha_1 y + a\alpha_1^{-1} y - \alpha_2 y - \alpha_2^{-1} y - 2ay = 0.\]

By Proposition 3.15, the characteristic set of the \( \sigma^* \)-ideal generated by the \( \sigma^* \)-polynomial \( B = a\alpha_1 y + a\alpha_1^{-1} y - \alpha_2 y - \alpha_2^{-1} y - 2ay \) is \( \{B, \alpha_1^{-1} B\} \). The leaders of \( B \) and \( \alpha_1^{-1} B \) are \( \alpha_1 y \) and \( \alpha_1^{-2} y \), respectively. Now Theorem 2.10 shows that
the strength of the equation (4.5) is expressed by the dimension polynomial \( \omega_E(t) \) where

\[
E = \{(1, 0, 0, 0), (0, 0, 2, 0), (1, 0, 1, 0), (0, 1, 0, 1)\} \subseteq \mathbb{N}^4.
\]

Applying formula (2.3) we obtain that the strength of the equation (4.5), which expresses the symmetric difference scheme for (4.1), is represented by the \( \sigma^* \)-dimension polynomial

\[
\psi_{Symm}(t) = 4t.
\]

Strength of the Crank-Nicholson scheme

The Crank-Nicholson scheme for the diffusion equation with the above interpretation of the shifts of arguments as two automorphisms \( \alpha_1 \) and \( \alpha_2 \) gives the algebraic difference equation of the form

\[
\alpha_1 \alpha_2 y + a_1 \alpha_1^{-1} \alpha_2 y + a_2 \alpha_1 y + a_3 \alpha_2 y + a_4 \alpha_1^{-1} y + a_5 = 0
\]

where \( a_i \) (1 \( \leq i \leq 5 \)) are constants. Applying Proposition 3.15, we obtain that the \( \sigma^* \)-polynomial \( C = \alpha_1 \alpha_2 y + a_1 \alpha_1^{-1} \alpha_2 y + a_2 \alpha_1 y + a_3 \alpha_2 y + a_4 \alpha_1^{-1} y + a_5 \) in the left-hand side of the last equation generates a \( \sigma^* \)-ideal of \( K\{y\}^* \) whose characteristic set consists of the \( \sigma^* \)-polynomials \( C, \alpha_1^{-1} C, \alpha_2^{-1} C, \) and \( \alpha_1^{-1} \alpha_2^{-1} C \). Their leaders are \( \alpha_1 \alpha_2 y, \alpha_1^{-2} \alpha_2 y, \alpha_1 \alpha_2^{-1} y, \) and \( \alpha_1^{-2} \alpha_2^{-1} y \), respectively.

Applying theorem 2.10 and 2.8 to the set \( \{(1, 1), (-2, 1), (1, -1), (-2, -1)\} \subseteq \mathbb{Z}^2 \) we obtain that the strength of the equation (4.6) is expressed by the dimension polynomial

\[
\psi_{\text{Crank-Nicholson}}(t) = 6t - 1.
\]

Thus, the symmetric difference scheme for the diffusion equation has higher strength (that is, smaller dimension polynomial) than the forward difference scheme and the Crank-Nicholson scheme, so the symmetric scheme is the best among these three schemes from the point of view of the Einstein’s strength.

2. Murray, Fisher, Burger and some other quasi-linear reaction-diffusion equations.

Proposition 3.14 allows us to compute the strength of reaction-diffusion equations of the form

\[
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = P \left( u, \frac{\partial u}{\partial x} \right)
\]

where \( u = u(x, t) \) is a function of space and time variables \( x \) and \( t \), respectively, and \( P \left( u, \frac{\partial u}{\partial x} \right) \) is a nonlinear function of \( u \) and \( \frac{\partial u}{\partial x} \). Such equations have recently attracted a lot of attention in the context of chemical kinetics, mathematical biology and turbulence. The following PDEs, that are particular cases of equation (4.7), are in the core of the corresponding mathematical models.

Murray equation [20]:

\[
\frac{\partial^2 u}{\partial x^2} + \mu_1 u \frac{\partial u}{\partial t} + \mu_2 u - \mu_3 u^2, \quad (\mu_1, \mu_2, \mu_3 \text{ are constants}).
\]

Burgers equation [1]:

\[
\frac{\partial^2 u}{\partial x^2} - u \frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 0.
\]
Fisher equation [6]:
\[ \frac{\partial^2 u}{\partial x^2} - u \frac{\partial u}{\partial t} + u(1 - u) = 0. \]

Huxley equation [29]:
\[ \frac{\partial^2 u}{\partial x^2} - u \frac{\partial u}{\partial t} - u(k - u)(u - 1) = 0, \quad k \neq 0. \]

Burgers-Fisher equation [28]:
\[ \frac{\partial^2 u}{\partial x^2} + u \frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} + u(1 - u) = 0. \]

Burgers-Huxley equation [18]:
\[ \frac{\partial^2 u}{\partial x^2} + u \frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} + u(k - u)(u - 1) = 0, \quad k \neq 0. \]

FitzHugh-Nagumo equation [29]:
\[ \frac{\partial^2 u}{\partial x^2} + u \frac{\partial u}{\partial x} + u(1 - u)(a - u) = 0, \quad a \neq 0. \]

The last seven equations are of the form
\[ \frac{\partial^2 u}{\partial x^2} + (au) \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + F(u) = 0 \]
where \( a, b, c \) are constants (\( c \neq 0, ab \neq 0 \)) and \( F(u) \) is a polynomial in one variable \( u \) with coefficients in the ground functional field \( K \). Therefore, the forward difference scheme for equations (4.8) - (4.14) leads to algebraic difference equations of the form
\[ \alpha_1^2 y + (ay + b - 2)\alpha_1 y + c\alpha_2 y + G(y) = 0. \]
(As before, we set \( y = u \), denote the automorphisms of the ground field \( f(x, t) \mapsto f(x + 1, t) \) and \( f(x, t) \mapsto f(x, t + 1) \) by \( \alpha_1 \) and \( \alpha_2 \), respectively, and write the monomials in the left-hand side of the equation in the decreasing order of their highest terms. We also set \( G(y) = F(y) - ay^2 - (b + c - 1)y. \))

Applying Propositions 3.14 and 3.15, we obtain that the \( \sigma^* \)-polynomial \( A = \alpha_1^2 y + (ay + b - 2)\alpha_1 y + c\alpha_2 y + G(y) \) generates a prime \( \sigma^* \)-ideal of \( K\{y\}^* \) (\( \sigma = \{\alpha_1, \alpha_2\} \)). As in the case of equation (4.3), we obtain that the characteristic set of the ideal \( [A]^* \) consists of the \( \sigma^* \)-polynomials \( A, \alpha_1^{-1} A, \alpha_1^{-1}\alpha_2^{-1} A \) and \( \alpha_1^{-2}\alpha_2^{-1} A \) with leaders \( \alpha_1^2 y, \alpha_1^{-1}\alpha_2 y, \alpha_1\alpha_2^{-1} y \) and \( \alpha_1^{-2}\alpha_2^{-1} y \), respectively. Therefore (as in the case of equation (4.3)) the \( \sigma^* \)-dimension polynomial that expresses the Einstein’s strength of the forward difference scheme for each of the equations (4.8) - (4.14) is equal to the dimension polynomial of the set \( \{(2, 0), (-1, 1), (1, -1), (-2, -1)\} \subseteq \mathbb{Z}^2 \), that is,
\[ \psi_{\text{Forw}}(t) = 5t. \]

The symmetric difference scheme for equation (4.15) (and therefore for each of the equations (4.8) - (4.14)) gives an algebraic difference equation of the form
(4.17) \[(ay + b + 1)\alpha_1 y + (1 - ay - b)\alpha_1^{-1} y + c\alpha_2 y - c\alpha_2^{-1} y + F(y) = 0.\]

(Recall that we replace \(\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\) and \(\frac{\partial u}{\partial t}\) with \((\alpha_1 + \alpha^{-1} - 2)u\), \((\alpha_1 - \alpha^{-1})u\) and \((\alpha_2 - \alpha^{-1})u\), respectively.)

Equation (4.17) is not quasi-linear with respect to the standard ranking described after Definition 3.4. However, if one considers a similar ranking with \(\alpha_2 > \alpha_1\), then the \(\sigma^*\)-polynomial \(B = (ay + b + 1)\alpha_1 y + (1 - ay - b)\alpha_1^{-1} y + c\alpha_2 y - c\alpha_2^{-1} y + F(y)\) in the left-hand side of (4.17) is a quasi-linear one with the leader \(\alpha_2 y\). By Propositions 3.14 and 3.15, the \(\sigma^*\)-polynomials \(B\) and \(\alpha_2^{-1}B\) form a characteristic set of the prime \(\sigma^*\)-ideal \([B]^*\) of \(K\{y\}^*\). Since their leaders are, respectively, \(\alpha_2 y\) and \(\alpha_2^{-2} y\), the Einstein’s strength of the symmetric difference scheme for each of the equations (4.8) - (4.14) is expressed by the dimension polynomial \(\psi_{Symm}(t)\) of the set \(\{(1, 0), (0, 2)\} \subseteq \mathbb{Z}^2\). As in the case of equation (4.5), we obtain that

\[\psi_{Symm}(t) = 4t.\]

Thus, one should prefer the symmetric scheme to the forward one while considering the Einstein’s strength of these schemes for PDEs (4.8) - (4.14).

3. **The mathematical model of chemical reaction kinetics with the diffusion phenomena** is described by a system of partial differential equations of the form

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= \frac{\partial^2 u_1}{\partial x^2} - k_1 u_1 u_2 + k_2 u_3, \\
\frac{\partial u_2}{\partial t} &= \frac{\partial^2 u_2}{\partial x^2} - k_1 u_1 u_2 + k_2 u_3, \\
\frac{\partial u_3}{\partial t} &= \frac{\partial^2 u_3}{\partial x^2} + k_1 u_1 u_2 - k_2 u_3.
\end{align*}
\]

(see [14]). Setting \(v_1 = u_1 - u_2\) and \(v_2 = u_1 + u_3\), one can rewrite the system in the form

\[
\begin{align*}
\frac{\partial v_1}{\partial t} &= \frac{\partial^2 v_1}{\partial x^2}, \\
\frac{\partial v_2}{\partial t} &= \frac{\partial^2 v_2}{\partial x^2}, \\
\frac{\partial u_1}{\partial t} &= \frac{\partial^2 u_1}{\partial x^2} - k_1 u_1^2 + k_1 u_1 v_1 + k_2 v_2 - k_2 u_1.
\end{align*}
\]

The forward difference scheme leads to the following system of algebraic difference equations with three \(\sigma^*\)-indeterminates \(y_1\), \(y_2\) and \(y_3\) (they stand for \(v_1\), \(v_2\) and \(u_1\), respectively), where \(\sigma = \{\alpha_1 : f(x,t) \rightarrow f(x+1,t), \alpha_2 : f(x,t) \rightarrow f(x,t+1)\}\) \((f(x,t)\) is an element of an inversive ground functional field \(K\)).
where \( k_1, k_2, k_3 \) are constants in \( K \).

Let \( A, B, \text{ and } C \) be the \( \sigma^* \)-polynomials in the left-hand sides of the first, second and third equations of the last system, respectively. Combining Proposition 2.4.9 of [12] (that states that every linear \( \sigma^* \)-ideal in a ring of \( \sigma^* \)-polynomials is prime) and our Proposition 3.14 we obtain that the \( \sigma^* \)-ideal \( P = [A, B, C]^* \) of the ring \( K\{y_1, y_2, y_3\}^* \) is prime.

Let \( \eta_i \) denote the canonical image of the \( \sigma^* \)-indeterminate \( y_i \) in the \( \sigma^* \)-quotient field \( L \) of \( K\{y_1, y_2, y_3\}^*/P \) \((i = 1, 2, 3)\) and let \( L_1 = K\langle \eta_1 \rangle^* \) and \( L_2 = L_1\langle \eta_2 \rangle^* \) (so that \( L = L_2\langle \eta_3 \rangle^* \)).

For any \( r \in \mathbb{N} \) and \( \zeta \in L \), let \( \Gamma(r)\zeta = \{ \gamma(\zeta) | \gamma \in \Gamma(r) \} \), \( L_{1r} = K(\Gamma(r)\eta_1), \)
\( L_{2r} = K(\Gamma(r)\eta_1 \cup \Gamma(r)\eta_2), \) and \( L_r = K(\Gamma(r)\eta_1 \cup \Gamma(r)\eta_2 \cup \Gamma(r)\eta_3) \). Then Theorem 3.2 and the obvious fact that the fields \( K\langle \eta_1 \rangle^*, K\langle \eta_2 \rangle^* \) and \( K\langle \eta_3 \rangle^* \) are pairwise algebraically disjoint over \( K \) imply that there are numerical polynomials \( \psi_1(t), \psi_2(t) \) and \( \psi_3(t) \) such that \( \psi_1(r) = \text{tr. deg}_K L_{1r}, \psi_2(r) = \text{tr. deg}_K L_{2r} \) and \( \psi_3(r) = \text{tr. deg}_K L_r \) for all sufficiently large \( r \in \mathbb{N} \). Each of the polynomials \( \psi_i(t) \) represents a \( \sigma^* \)-dimension polynomial of the equation of the form (4.3). Indeed, the first two equation of system (4.20) are similar to equation (4.3) and the third equation of the system is a quasi-linear \( \sigma^* \)-equation whose dimension polynomial, according to Proposition 3.14, is equal to the dimension polynomial of (4.3). Since \( \text{tr. deg}_K L_r = \text{tr. deg}_K L_{1r} + \text{tr. deg}_K L_{2r} + \text{tr. deg}_K L_r \), the \( \sigma^* \)-dimension polynomial of the extension \( L/K \) is equal to \( \psi_1(t) + \psi_2(t) + \psi_3(t) \). Now the formula for the strength of the forward difference scheme for the diffusion equation (4.1) shows that the strength of the forward difference scheme for system (4.19) is expressed with the polynomial
\[ \psi_{\text{Forward}}(t) = 15t. \]

Using the above arguments and the results for difference schemes for equation (4.1), we obtain that the strengths of the symmetric and Crank-Nicholson schemes for (4.19) are expressed with the polynomials
\[ \psi_{\text{Symm}}(t) = 12t \quad \text{and} \quad \psi_{\text{Crank–Nicholson}}(t) = 18t - 3, \]
respectively. Therefore, in our case, as in the case of equation (4.1), the symmetric scheme for system (4.19) is the best among these three schemes.

4. The mass balance equations of chromatography for the case of \( N \) components in a column slice (see, [24] p. 24) form the following system of PDEs:
\[
\begin{align*}
\frac{\partial C_1}{\partial t} + F \frac{\partial C_{s1}}{\partial t} + u \frac{\partial C_1}{\partial z} &= D_{L,1} \frac{\partial^2 C_1}{\partial z^2}, \\
\vdots \\
\frac{\partial C_N}{\partial t} + F \frac{\partial C_{sN}}{\partial t} + u \frac{\partial C_N}{\partial z} &= D_{L,N} \frac{\partial^2 C_N}{\partial z^2}.
\end{align*}
\]
where $C_i$ and $C_{s,i}$ ($1 \leq i \leq N$) are the concentrations of the individual components in the mobile phase and in the stationary phase, respectively ($u$ and $D_{L,i}$, $1 \leq i \leq N$, are constants along the column).

Let $K$ be the inversive difference ($\sigma^*$-) functional field over which we consider the system ("ground field"). As before, $\sigma = \{\alpha_1, \alpha_2\}$ where for any function $f(z) \in K$, $\alpha_1 : f(z, t) \mapsto f(z + 1, t)$ and $\alpha_2 : f(z, t) \mapsto f(z, t + 1)$.

The forward difference scheme leads to a system of algebraic difference equations

$$
\begin{aligned}
&\left\{
\begin{array}{l}
D_{L,1} \alpha_2^2 y_1 + a_1 \alpha_1 y_1 - F \alpha_2 y_2 - \alpha_2 y_1 + F y_2 + b_1 y_1 = 0, \\
D_{L,N+1} \alpha_2^2 y_{2N-1} + a_N \alpha_1 y_{2N-1} - F \alpha_2 y_{2N} - \alpha_2 y_{2N-1} + F y_{2N} + b_N y_{2N-1} = 0
\end{array}
\right.
\end{aligned}
$$

where $y_{2i-1}(i = 1, \ldots, N)$ stand for $C_i$ and $C_{s,i}$, respectively, $a_i = -u - 2D_{L,i}$, and $b_i = -D_{L,i} - u - 1$. (The terms in the $\sigma^*$-polynomials in the left-hand sides of the last system are arranged in the decreasing order with respect to the standard ranking.)

Let

$$
A_i = D_{L,i} \alpha_2^2 y_{2i-1} + a_i \alpha_1 y_{2i-1} - F \alpha_2 y_{2i} - \alpha_2 y_{2i-1} + F y_{2i} + b_i y_{2i-1}
$$

($1 \leq i \leq N$), let $P$ denote the linear (and therefore prime) $\sigma^*$-ideal $[A_1, \ldots, A_N]^*$ of the ring $R = K \{y_1, \ldots, y_{2N}\}^*$ and let $\eta_j$ denote the canonical image of the $\sigma^*$-indeterminate $y_j$ in the factor ring $R/P$, so that the quotient field $L$ of $R/P$ can be represented as $L = K \langle \eta_1, \ldots, \eta_{2N} \rangle^*$. Then one can repeat arguments of the analysis of system (4.20) and obtain that the $\sigma^*$-dimension polynomial of system (4.22) is the sum of $N$ equal $\sigma^*$-dimension polynomials associated with the $\sigma^*$-field extensions of $K$ defined by equations $A_i = 0$ ($i = 1, \ldots, N$).

In order to find the $\sigma^*$-dimension polynomial associated with the equation $A_1 = 0$ (and therefore with any equation $A_i = 0$, $1 \leq i \leq N$), we can apply Proposition 3.15 and obtain that $\sigma^*$-polynomials $A_1$, $\alpha_2^{-1} A_1$, $\alpha_2^{-1} \alpha_2^{-1} A_1$ and $\alpha_2^{-2} \alpha_2^{-1} A_1$ with leaders $\alpha_2^2 y_1$, $\alpha_2^{-1} \alpha_2 y_2$, $\alpha_1 \alpha_2 y_2$, respectively, form a characteristic set of the prime $\sigma^*$-ideal $[A_1]^*$ of the ring $K \{y_1, y_2\}^*$. It follows that the desired $\sigma^*$-dimension polynomial is the sum of the dimension polynomials of subsets $\{(2, 0), (1, -1)\}$ and $\{(-1, 1), (-2, -1)\}$ of $\mathbb{Z}^2$. By Theorem 2.10, these polynomials are equal, respectively, to the dimension polynomials $\omega_{E_1}(t)$ and $\omega_{E_2}(t)$ of subsets

- $E_1 = \{(2, 0, 0, 0), (1, 0, 0, 1), (1, 0, 1, 0), (0, 1, 0, 1)\}$
- $E_2 = \{(0, 1, 1, 0), (0, 0, 2, 1), (1, 0, 1, 0), (0, 1, 0, 1)\}$

of $\mathbb{N}^2$. Applying formula (2.3) in Theorem 2.8 we get $\omega_{E_1}(t) = t^2 + 3t + 1$ and $\omega_{E_2}(t) = t^2 + 4t$. It follows that the $\sigma^*$-dimension polynomial of the equation $A_1 = 0$ (as well as of every equation $A_i = 0$, $1 \leq i \leq N$) is $\omega_{E_1}(t) + \omega_{E_2}(t) = 2t^2 + 7t + 1$. Thus, the $\sigma^*$-dimension polynomial that expresses the strength of the forward difference scheme for system (4.21) is as follows:

$$
\psi_{Forw}(t) = 2t^2 + 7t + 1.
$$
Applying the symmetric difference scheme to system (4.21) we obtain \(N\) equations of the form
\[
D_{L,i}(\alpha_{2i-1} - 2 + \alpha^{-1})y_{2i-1} - (\alpha_{2} - \alpha_{2}^{-1})y_{2i-1} - F(\alpha_{2} - \alpha_{2}^{-1})y_{2i} - u(\alpha_{1} - \alpha_{1}^{-1})y_{2i-1} = 0
\]
or
\[
(\alpha_{2} - \alpha_{2}^{-1})y_{2i-1} + F\alpha_{2}y_{2i} - F\alpha_{2}^{-1}y_{2i} + 2D_{L,i}y_{2i-1} = 0.
\]

with \(\sigma^{*}\)-indeterminates in the ring \(K\{y_{1}, \ldots, y_{2N}\}^{*}\).

Denoting the \(\sigma^{*}\)-polynomial in the left-hand side of the last equation by \(B\), one can easily see that \(\{B, \alpha_{1}^{-1}B\}\) is a characteristic set of the linear \(\sigma^{*}\)-ideal \([B]^{*}\) in \(K\{y_{2i-1}, y_{2i}\}^{*} (1 \leq i \leq N)\). The corresponding leaders are \(\alpha_{1}y_{2i-1}\) and \(\alpha_{1}^{-1}y_{2i-1}\). It follows that the strength of the system of difference equations obtained from the symmetric difference scheme for system (4.21) is expressed with the sum of the dimension polynomial of the set \(\{(1, 0), (2, 0)\}\) \(\subseteq \mathbb{Z}^{2}\) (this set corresponds to the leaders containing \(y_{2i-1}\)) and the dimension polynomial of the empty set (that corresponds to the leaders containing \(y_{2i}\)). Using the last part of Theorem 2.9 and the result of the computation of the \(\sigma^{*}\)-dimension polynomial of equation (4.5) we obtain that the strength of the symmetric difference scheme for system (4.21) is represented by the polynomial
\[
\psi_{Symm}(t) = \sum_{i=0}^{2} (-1)^{2-i}2^{i} \binom{2}{i} (t + i) + 4t = 2t^{2} + 6t + 1.
\]

As we see, in this case one should also prefer the symmetric scheme to the forward one; however, there is quite a small difference between these two schemes.

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