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Particle in the Electromagnetic Wave with Cylindrical Symmetry, an Analog of the Volkov Problem

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Abstract

Cylindrically symmetric analogue of the Volkov problem is examined. In presence of the field of a cylindrical electromagnetic wave, classical motion of a non-relativistic particle on a cylindrical surface is described exactly in terms of elliptic functions.

1. Particle in a plane wave, the Volkov problem

Problems of mathematical physics constitute a basis for simulating processes in modern physics. Expanding the range of such problems is useful in many ways. In particular, in the context of investigating the properties of nanostuctures may be of interest the transport of electrical charges in a very narrow cylindrical conductive layer in presence of an external electromagnetic wave with cylindrical symmetry. As shown in this study, such a system has many common mathematical properties of the well-known Volkov problem [1] on the motion of electric charge in the ordinary plane electromagnetic wave.

Let us start with the potential for a plane wave propagating along positive direction of the $z$-axis

$$A_1(x^3) = A\cos\frac{2\pi}{T}(t - \frac{x^3}{c}) = A\cos(\omega t - kx^3) ,$$

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\[ A_0 = 0, \ A_2 = 0, \ A_3 = 0, \ k = \frac{\omega}{c}; \]

\[ F_{01} = -Ak \sin(\omega t - kx^3), \quad F_{31} = +Ak \sin(\omega t - kx^3). \quad (1.1) \]

Transforming this field to the cylindrical coordinates

\[ x^1 = r \cos \phi, \quad x^2 = r \sin \phi, \quad x^3 = z, \]

\[ dS^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 - dz^2, \quad (1.2a) \]

we get

\[ A_r = \frac{\partial x^1}{\partial r} A_1 = +A \cos \phi \cos(\omega t - kz), \quad A_0 = 0, \]

\[ A_\phi = \frac{\partial x^1}{\partial \phi} A_1 = -A r \sin \phi \cos(\omega t - kz), \quad A_z = 0; \quad (1.2b) \]

\[ F_{0r} = -A k \cos \phi \sin(\omega t - kz), \quad F_{0\phi} = +A k r \cos \phi \sin(\omega t - kz), \quad F_{0z} = 0, \]

\[ F_{\phi z} = kA r \sin \phi \sin(\omega t - kz), \quad F_{zr} = kA \cos \phi \sin(\omega t - kz), \quad F_{r\phi} = 0. \quad (1.2c) \]

Next we use the shortening notation for the phase: \( \omega t - kz = \Omega; \) then a plane wave is described by the relations

\[ A_r = +A \cos \phi \cos \Omega, \quad A_\phi = -A r \sin \phi \cos \Omega, \]

\[ F_{0r} = -A k \cos \phi \sin \Omega, \quad F_{0\phi} = +A k r \cos \phi \sin \Omega, \]

\[ F_{\phi z} = kA r \sin \phi \sin \Omega, \quad F_{zr} = kA \cos \phi \sin \Omega. \quad (1.3) \]

Let us consider the problem of a particle in the plane wave, using Lagrangian formalism (from the very beginning specifying the problem in cylindrical coordinates) [2]

\[ L = \frac{m}{2} \left( -g_{ik} V^i V^k \right) - \frac{e}{c} g_{ik} A^i V^k = \]

\[ = \frac{m}{2} \left( V^r V^r + r^2 V^\phi V^\phi + V^z V^z \right) - \frac{e}{c} A \cos \phi \cos \Omega V^r + \frac{e}{c} A r \sin \phi \cos \Omega V^\phi. \quad (1.4) \]

Euler–Lagrange equations take the form

\[
\frac{d}{dt} \left( V^r - \frac{eA}{mc} \cos \phi \cos \Omega \right) = r V^\phi \ V^\phi + \frac{eA}{mc} \sin \phi \cos \Omega V^\phi,
\]

2
\[
\frac{d}{dt} \left( r^2 V^\phi \right) + \frac{eA}{mc} r \sin \phi \cos \Omega = \frac{eA}{mc} \sin \phi \cos \Omega V^r + \frac{eA}{mc} \cos \phi \cos \Omega V^\phi ,
\]
\[
\frac{d}{dt} V^z = -\frac{eA}{mc} k \cos \phi \sin \Omega V^r + \frac{eA}{mc} k \cos \phi \sin \Omega V^\phi .
\]

The system (1.5) is equivalent to
\[
\frac{dV^r}{dt} - rV^\phi V^\phi = -\frac{eA}{m} k \cos \phi \sin \Omega + \frac{eA}{mc} k \cos \phi \sin \Omega V^z ,
\]
\[
\frac{dV^\phi}{dt} + \frac{2}{r} V^r V^\phi = \frac{eA}{m} k \sin \phi \sin \Omega - \frac{eA}{mc} k \sin \phi \sin \Omega V^z ,
\]
\[
\frac{dV^z}{dt} = -\frac{eA}{mc} k \cos \phi \sin \Omega V^r + \frac{eA}{mc} k \cos \phi \sin \Omega V^\phi .
\]

(1.5)

A simpler system of equations can be obtained by starting with Cartesian coordinates
\[
m \frac{dV^1}{dt} = eE^1 + e(V^2 B^3 - V^3 B^2) ,
\]
\[
m \frac{dV^2}{dt} = eE^2 + e(V^3 B^1 - V^1 B^3) ,
\]
\[
m \frac{dV^3}{dt} = eE^3 + e(V^1 B^2 - V^2 B^1) .
\]

(1.7)

Taking into account
\[
E^1 = F_{01} = -Ak \sin(\omega t - kx^3) ,
\]
\[
eB^2 = -F_{31} = -Ak \sin(\omega t - kx^3) ,
\]
equations (1.7) are written
\[
\frac{dV^1}{dt} = -\frac{eA}{m} k \sin \Omega + \frac{eA}{mc} k V^3 \sin \Omega ,
\]
\[
\frac{dV^2}{dt} = 0 , \quad \frac{dV^3}{dt} = -\frac{eA}{mc} k V^1 \sin \Omega .
\]

(1.8)

Systems (1.8) and (1.6) must be equivalent, which is readily verified by direct recalculation, using the formulas
\[
V^1 = -r \sin \phi V^\phi + \cos \phi V^r , \quad V^2 = r \cos \phi V^\phi + \sin \phi V^r .
\]

(1.9)
Next we will analyze the equations (1.8). According to (1.8), along the $x^2$ the particle moves with constant velocity. In the plane of the 1–3, its motion is described by equations

$$\frac{1}{c} \frac{dV^1}{dt} = - \frac{eA}{mc} k \sin \Omega + \frac{eA}{mc} \frac{V^3}{c} \sin \Omega,$$

$$\frac{1}{c} \frac{dV^3}{dt} = - \frac{eA}{mc} k \sin \Omega \frac{V^1}{c}.$$  \hspace{1cm} (1.10)

Let us introduce the variables

$$q = \frac{eA}{mc} k, \quad [q] = \frac{1}{sec}, \quad v^1 = \frac{V^1}{c}, \quad v^2 = \frac{V^2}{c}$$ \hspace{1cm} (1.11)

then equations (1.10) can be written as

$$\frac{dv^1}{dt} = -q \sin \Omega + q v^3 \sin \Omega, \quad \frac{dv^3}{dt} = -q \sin \Omega v^1.$$ \hspace{1cm} (1.12)

You can find an approximate solution of this equation, considering the motion for sufficiently small time intervals (see [3]):

$$v^3 << 1, \quad kz = \frac{\omega}{c} \int_0^t \frac{dz}{dt} dt = \omega \int \frac{V^3}{c} dt << \omega t$$ \hspace{1cm} (1.13)

that is, by imposing an additional constraint $\Omega = \omega t - kz \approx \omega t$. This system of equations (1.12) takes the form

$$\frac{dv^1}{dt} = -q \sin \omega t, \quad \frac{dv^3}{dt} = -q \sin \omega t v^1.$$ \hspace{1cm} (1.14)

Integrating the first equation, we get

$$v^1 = \frac{q}{\omega} \cos \omega t + (v^1(0) - \frac{q}{\omega}),$$

$$\frac{x}{c} = \frac{q}{\omega^2} \sin \omega t + (v^1(0) - \frac{q}{\omega}) t + \frac{x_0}{c}.$$ \hspace{1cm} (1.15)

After that, it is easy to integrate the second equation

$$\frac{dv^3}{dt} = -q [\frac{q}{\omega} \cos \omega t + (v^1(0) - \frac{q}{\omega})] \sin \omega t,$$

$$v^3 = -\frac{q^2}{2\omega^2} \sin^2 \omega t + \frac{q}{\omega} (v^1(0) - \frac{q}{\omega}) \cos \omega t + \left[ v^3(0) - \frac{q}{\omega} (v^1(0) - \frac{q}{\omega}) \right],$$
\[
\frac{z}{c} = -\frac{q^2}{2\omega^2}\left(\frac{t}{2} - \frac{\sin 2\omega t}{4\omega}\right) + \frac{q}{\omega^2} (v^1_0 - \frac{q}{\omega}) \sin \omega t + [v^3_0 - \frac{q}{\omega} (v^1_0 - \frac{q}{\omega})] t + \frac{z_0}{c} .
\]

(1.16)

Let us turn to the system (1.12) in the general case, and translate it to a new time variable \( \tau \)
\[
\tau = t - \frac{z}{c} , \quad t = \tau + \frac{z}{c} , \quad dt = d\tau + \frac{dz}{c} .
\]

(1.17a)

Generalized velocities transform according to
\[
v^1 = c^{-1} \frac{dx^1}{d\tau} + c^{-1} \frac{dz}{d\tau} = \frac{\dot{v}^1}{1 + \dot{v}^3} , \quad v^2 = c^{-1} \frac{dx^2}{d\tau} + c^{-1} \frac{dz}{d\tau} = \frac{\dot{v}^2}{1 + \dot{v}^3} ,
\]
\[
\dot{v}^1 = c^{-1} \frac{dx^1}{d\tau} , \quad \dot{v}^2 = c^{-1} \frac{dx^2}{d\tau} , \quad \dot{v}^3 = c^{-1} \frac{dx^3}{d\tau} ,
\]
\[
v^3 = \frac{c^{-1} \frac{dz}{d\tau}}{1 + \frac{dz}{d\tau}} = \frac{\dot{v}^3}{1 + \dot{v}^3} , \quad \dot{v}^3 = \frac{v^3}{1 - v^3} ;
\]

(1.17b)

operator of differentiation with respect to time is converted according to
\[
\frac{d}{dt} f = \frac{d}{d\tau} f = \frac{1}{1 + \dot{v}^3} \frac{d}{d\tau} f .
\]

(1.17c)

The system of equations (1.12) takes the form
\[
\frac{1}{1 + \dot{v}^3} \frac{d}{d\tau} \frac{\dot{v}^1}{1 + \dot{v}^3} = -q \sin \omega \tau + q \sin \omega \tau \frac{\dot{v}^3}{1 + \dot{v}^3} ,
\]
\[
\frac{1}{1 + \dot{v}^3} \frac{d}{d\tau} \frac{\dot{v}^3}{1 + \dot{v}^3} = -q \sin \omega \tau \frac{\dot{v}^1}{1 + \dot{v}^3} .
\]

(1.18)

The previous equation can be rewritten as
\[
\frac{1}{q \sin \omega \tau} \frac{1}{1 + \dot{v}^3} \frac{d}{d\tau} \left(\frac{-\dot{v}^1}{1 + \dot{v}^3}\right) = 1 - \frac{\dot{v}^3}{1 + \dot{v}^3} = \frac{1}{1 + \dot{v}^3} ,
\]
\[
\frac{1}{q \sin \omega \tau} \frac{1}{1 + \dot{v}^3} \frac{d}{d\tau} \frac{\dot{v}^3}{1 + \dot{v}^3} = \left(\frac{-\dot{v}^1}{1 + \dot{v}^3}\right) .
\]

(1.20)

Allowing for the second equation, from the first one we get
\[
\left(\frac{1}{q \sin \omega \tau} \frac{1}{1 + \dot{v}^3} \frac{d}{d\tau}\right) \left(\frac{1}{q \sin \omega \tau} \frac{1}{1 + \dot{v}^3} \frac{d}{d\tau}\right) \frac{\dot{v}^3}{1 + \dot{v}^3} = \frac{1}{1 + \dot{v}^3} \quad (1.21a)
\]
or

\[\text{5}\]
\[
\frac{d}{d\tau} \left[ \frac{1}{\sin \omega \tau} \frac{1}{1 + \hat{v}^3} \frac{d}{d\tau} \frac{\hat{v}^3}{1 + \hat{v}^3} \right] = q^2 \sin \omega \tau . \tag{1.21b}
\]

After integrating over \( \tau \) we have
\[
\frac{1}{1 + \hat{v}^3} \frac{d}{d\tau} \frac{\hat{v}^3}{1 + \hat{v}^3} = (-\frac{q^2}{\omega} \cos \omega \tau + \lambda) \sin \omega \tau . \tag{1.21c}
\]

Note that in the limit \( \hat{v}^3 \ll 1 \), \( kz \ll \omega t \), \( \tau \approx t \), \( \hat{v}^3 \approx v^3 \), the previous equation is simplified
\[
\frac{d}{dt} v^3 = \left(-\frac{q^2}{\omega} \cos \omega t + \lambda \right) \sin \omega t , \tag{1.22}
\]

which coincides with the first equation in (1.16).

We return to equation (1.21c):
\[
\frac{1}{1 + \hat{v}^3} \frac{d}{d\tau} \frac{\hat{v}^3}{1 + \hat{v}^3} = (-\frac{q^2}{\omega} \cos \omega \tau + \lambda) \sin \omega \tau . \tag{1.23a}
\]

It is a differential equation of first order with separated variables
\[
\frac{d \hat{v}^3}{(1 + \hat{v}^3)^2} = (-\frac{q^2}{\omega} \cos \omega \tau + \lambda) \sin \omega \tau \frac{d\tau}{\omega} , \tag{1.23b}
\]

which is easily integrated (we introduce a constant integration)
\[
-\frac{1}{2(1 + \hat{v}^3)^2} = -\frac{q^2}{\omega} \frac{\sin^2 \omega \tau}{2\omega} - \frac{\lambda}{\omega} \cos \omega \tau - \frac{\Lambda}{2\omega^2} \tag{1.23c}
\]

and further
\[
\hat{v}^3 = -1 \pm \frac{\omega}{\sqrt{q^2 \sin^2 \omega \tau + 2\lambda \omega \cos \omega \tau + \Lambda}} . \tag{1.23d}
\]

From (1.23d) one can obtain an expression for the particle velocity \( v^3 \) – see (1.17b):
\[
\frac{1}{c} \frac{dz}{dt} = v^3 = \frac{\hat{v}^3}{1 + \hat{v}^3} ,
\]

\[
\frac{dz}{dt} = c \pm \frac{c}{\omega} \sqrt{q^2 \sin^2 \omega(t - z/c) + 2\lambda \omega \cos \omega(t - z/c) + \Lambda} . \tag{1.24a}
\]

If we assume that at the time \( t = 0 \) coordinate \( z = z_0 = 0 \), then from (1.24a) it follows
\[
\dot{z}_0 = c \pm \frac{c}{\omega} \sqrt{2\lambda \omega + \Lambda} . \tag{1.24b}
\]
Variant with the lower sign must be discarded as not physical, because it involves an initial velocity greater than the rate of of light. We return to equation (1.23d)

\[ c^{-1} \frac{dz}{d\tau} = -1 \pm \frac{\omega}{\sqrt{q^2 \sin^2 \omega \tau + 2 \lambda \omega \cos \omega \tau + \Lambda}}, \]

so we get

\[ u = \omega \tau = \omega (t - z/c), \]

\[ \frac{z}{c} = -\tau \pm \int \frac{du}{\sqrt{q^2 + \Lambda - q^2 \cos^2 u + 2 \lambda \omega \cos u}}. \]  \hspace{1cm} (1.25)

This integral reduces to an elliptic one (in more detail we consider the analogous integral in the next section). As a result we get the function of \( z(t) \) in an implicit transcendental form – this is a known property of the classical problem of motion of a charged particle in a plane electromagnetic wave (Volkov [1]).

In the next section we will construct a cylindrically symmetric analog of such a system, treating a particle in the simplest plane electromagnetic wave with cylindrical symmetry.

2. Particle in the presence of a cylindrically symmetric electromagnetic wave

We start from a simple solution of the Maxwell’s equations in cylindrical coordinates \((ct, r, \phi, z)\)

\[ A_0 = 0, \quad A_r = 0, \quad A_\phi = a \cos(\omega t - kz), \quad A_z = 0; \]  \hspace{1cm} (2.1)

it meets the following electromagnetic tensor

\[ F_{0\phi} = -ka \sin(\omega t - kz), \quad F_{z\phi} = +ka \sin(\omega t - kz). \]  \hspace{1cm} (2.2)

It is easy to check that here indeed we have the solution of the Maxwell’s equations

\[ \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \sqrt{-g} F^{\alpha\beta} = 0, \]

which is reduced to one non-trivial equation

\[ \beta = \phi, \quad -c \partial_t \frac{1}{r^2} F_{0\phi} + \partial_z \frac{1}{r^2} F_{z\phi} = 0 \quad \implies \]
\[-c^{-1}\partial_t[-ka \sin(\omega t - kz)] + \partial_z[ka \sin(\omega t - kz)] = 0 ,\]

that is an identity
\[+k^2a \cos(\omega t - kz) - k^2a \cos(\omega t - kz) = 0 .\]

It is useful to convert the electromagnetic 4-potential to Cartesian coordinates
\[A_1 = -\frac{x^2}{(x^1)^2 + (x^2)^2} a \cos(\omega t - kz) ,\]
\[A_2 = +\frac{x^1}{(x^1)^2 + (x^2)^2} a \cos(\omega t - kz) . \tag{2.3}\]

It corresponds to the electromagnetic tensor
\[cB^1 = F_{23} = -\frac{x^1}{(x^1)^2 + (x^2)^2} ka \sin(\omega t - kz) ,\]
\[cB^2 = F_{31} = -\frac{x^2}{(x^1)^2 + (x^2)^2} ka \sin(\omega t - kz) ,\]
\[E^1 = F_{01} = -\frac{x^2}{(x^1)^2 + (x^2)^2} ka \sin(\omega t - kz) ,\]
\[E^2 = F_{02} = -\frac{x^1}{(x^1)^2 + (x^2)^2} ka \sin(\omega t - kz) ,\]
\[cB^3 = F_{12} = 0 , \quad E^3 = F_{03} = 0 . \tag{2.4}\]

This is a wave propagating along the axis \(z\), it is singular on the axis \(z\) (when \(r = 0\)). In the plane of the fixed value of the variable \(z\), at each point the magnetic field is directed toward the center \((0,0)\) along the radius, and the electric field at each point in the plane is directed along the vector \(e_\phi\). Amplitude of the electric and magnetic fields varies according to the same law
\[E \sim \frac{1}{r} ak \sin(\omega t - kz) , \quad cB \sim \frac{1}{r} ak \sin(\omega t - kz) .\]

Consider a particle in that field. The Lagrangian of the system is given by (examining the problem of a non-relativistic particle)
\[L = \frac{m}{2} (... \text{expression omitted due to length}) = \frac{m}{2} [V^\alpha V^\alpha + r^2 V^\phi V^\phi + V^z V^z] - \frac{e}{c} a \cos(\omega t - kz) V^\phi . \tag{2.5}\]
Euler–Lagrange equations
\[
\frac{d}{dt} \frac{\partial L}{\partial V^r} = \frac{\partial L}{\partial r}, \quad \frac{d}{dt} \frac{\partial L}{\partial V^\phi} = \frac{\partial L}{\partial \phi}, \quad \frac{d}{dt} \frac{\partial L}{\partial V^z} = \frac{\partial L}{\partial z},
\]
here take the form
\[
\frac{dV^r}{dt} = r V^\phi V^\phi,
\]
\[
\frac{d}{dt} \left[ r^2 V^\phi - \frac{ea}{mc} \cos(\omega t - kz) \right] = 0 \quad \Rightarrow \quad \frac{dI}{dt} = 0,
\]
\[
\frac{dV^z}{dt} = -\frac{ea}{mc} k \sin(\omega t - kz) V^\phi.
\] (2.6)

From (2.6) we get a much more simpler systems, if consider the motion at sufficiently small time intervals:
\[
kz = \frac{\omega c}{\int^t \frac{dz}{dt} dt} = \omega \int \frac{V^3}{c} dt << \omega t \quad (2.7a)
\]
that is, by imposing an additional constraint
\[
\omega t - kz \approx \omega t. \quad (2.7b)
\]
Then (2.6) gives (we use the notation \(ea/mc = b\))
\[
\frac{dV^r}{dt} = r V^\phi V^\phi, \quad \quad V^\phi = \frac{I + b \cos \omega t}{r^2},
\]
\[
\frac{dV^z}{dt} = -bk \sin \omega t V^\phi. \quad \quad (2.8)
\]
and then we get three equations with separated variables:
\[
\frac{d^2 r}{dt^2} = \frac{(I + b \cos \omega t)^2}{r^3},
\]
\[
\frac{dV^z}{dt} = -bk \frac{\sin \omega t (I + b \cos \omega t)}{r^2(t)},
\]
\[
V^\phi = \frac{I + b \cos \omega t}{r^2(t)}. \quad \quad (2.9)
\]
Unfortunately, such a system hardly can be solved as well. One can try to simplify the original system by imposing additional constraint. The most interesting from a physical point of view, is the condition \(r = r_0 = \text{const.}\). It means that a particle can only move along a given cylindrical surface. In
this case the electric component of the external wave speed varies linearly along the vector $\mathbf{e}_\phi$, and the magnetic field changes the linear velocity along the axis $z$.

With this condition, instead of (2.6) we will have a simpler Lagrangian and two new equations of motion:

$$
L = \frac{m}{2} \left[ r_0^2 V^\phi V^\phi + V^z V^z \right] - \frac{e}{c} a \cos(\omega t - kz) V^\phi ,
$$

$$
\frac{d}{dt} \left[ r_0^2 V^\phi - b \cos(\omega t - kz) \right] = 0 ,
$$

$$
\frac{dV^z}{dt} = -bk \sin(\omega t - kz) V^\phi . \tag{2.10}
$$

Variable $\phi$ varies with time according to

$$
V^\phi = \frac{I + b \cos(\omega t - kz)}{r_0^2} . \tag{2.11a}
$$

The second equation in (2.10) takes the form differential equation for $z(t)$

$$
\frac{dV^z}{dt} = -\frac{bk}{r_0^2} \sin(\omega t - kz) \left[ I + b \cos(\omega t - kz) \right] . \tag{2.11b}
$$

First we will find the solution of equations (2.11) in the approximation of the velocity much smaller than the speed of light, and not too large time intervals

$$
kz = \frac{\omega}{c} \int_0^t \frac{dz}{dt} dt = \omega \int \frac{V^3}{c} dt << \omega t , \quad \omega t - kz \approx \omega t .
$$

In this case (2.11) can be simplified

$$
V^\phi = \frac{I + b \cos \omega t}{r_0^2} , \quad V_0^\phi = \frac{I + b}{r_0^2} ,
$$

$$
\frac{dV^z}{dt} = -\frac{bk}{r_0^2} \sin \omega t \left( I + b \cos \omega t \right) . \tag{2.12}
$$

These equations are easily integrated:

$$
\phi(t) = \phi_0 + \frac{I}{r_0^2} t + \frac{b}{r_0^2 \omega} \sin \omega t , \tag{2.13a}
$$

$$
V^z = \frac{bk}{r_0^2} \left[ \frac{b}{2\omega} \sin^2 \omega t - \frac{I}{\omega} \cos \omega t \right] + (V_0^z - \frac{bkI}{r_0^2 \omega}) ,
$$
\[
z(t) = \int \left\{ -\frac{bk}{r_0^2} \left[ \frac{b}{2\omega} \frac{1 - \cos 2\omega t}{2} - \frac{I}{\omega} \cos \omega t \right] + \left( V_0^z - \frac{bkI}{r_0^2 \omega} \right) \right\} dt = \\
= -\frac{bk}{r_0^2} \left[ \frac{b}{2\omega} \left( \frac{1}{2} t - \frac{\sin 2\omega t}{4\omega} \right) - \frac{I}{\omega^2} \sin \omega t \right] + \left( V_0^z - \frac{bkI}{r_0^2 \omega} \right) t + z_0 .
\]

Character of the motion is as follows: along the \( z \) axis the particle moves with a certain average constant speed, with two superimposed oscillatory movement in time with frequencies \( \omega \) and \( 2\omega \); on variable \( \phi \) is also the movement at a constant (angular) velocity and when imposed oscillating motion.

Let us return to the analysis of equation (2.11b) without approximation of small intervals of time. It is convenient to introduce a new time variable

\[
\tau = t - \frac{z}{c}, \quad t = \tau + \frac{z}{c}, \quad dt = d\tau + \frac{dz}{c} .
\]

Generalized velocity \( V^z \) is transformed according to

\[
V^z = \frac{dz}{d\tau + c^{-1}dz} = \frac{\hat{V}^z}{1 + c^{-1}V^z} , \quad \hat{V}^z = \frac{V^z}{1 - c^{-1}V^z} ;
\]

operator of differentiation with respect to time is converted according to

\[
\frac{d}{dt} f = \frac{d}{d\tau} \frac{1}{1 + c^{-1}V^z} d\tau f = \frac{1}{1 + c^{-1}V^z} \frac{d}{d\tau} f .
\]

Equation (2.11b) takes the form

\[
\frac{1}{1 + c^{-1}\hat{V}^z} \frac{d}{d\tau} \frac{c^{-1}\hat{V}^z}{1 + c^{-1}V^z} = -\frac{bk}{c} \frac{\sin \omega \tau (I + b \cos \omega \tau)}{r_0^2} ,
\]

or (go to the dimensionless velocity \( \hat{v}^z = c^{-1}\hat{V}^z \))

\[
d \left( \frac{1}{(1 + \hat{v}^z)^2} \right) = \frac{2b\omega}{c^2 r_0^2} \sin \omega \tau (I + b \cos \omega \tau) d\tau .
\]

In fact, this equation (up to notation) coincides with (1.23b) – an equation arising in the case of particles in the field of an ordinary plane wave. After integration of (2.15a)

\[
\frac{1}{(1 + \hat{v}^z)^2} = \frac{2b\omega}{c^2 r_0^2} \left[ \frac{b}{2\omega} \sin^2 \omega \tau - \frac{I}{\omega} \cos \omega \tau \right] + C
\]
and further
\[
\frac{1}{1 + \dot{v}^2} = \pm \sqrt{\frac{2b\omega}{c^2 \omega^2} \left[ \frac{b}{2\omega} \sin^2 \omega \tau - \frac{I}{c} \cos \omega \tau \right]} + C = \pm \sqrt{\Gamma(\tau)} . \tag{2.15b}
\]

The equation for the velocity \( \dot{v}^2 \) can be integrated:
\[
c^{-1} \frac{dz}{d\tau} = \dot{v}^2 = 1 \pm \frac{1}{\sqrt{\Gamma(\tau)}} \quad \implies \quad z(\tau) = -c\tau \pm \int \frac{d\tau}{\sqrt{\Gamma(\tau)}} . \tag{2.15c}
\]

There will arise an implicitly defined function \( z(t) \) with use of elliptic functions. Indeed, consider the integral
\[
J = \int \frac{dw}{\sqrt{a \cos^2 w + b \cos w + c}} ; \tag{2.16a}
\]
introducing a variable
\[
-\cos w = u , \quad dw = \frac{du}{\pm \sqrt{1 - u^2}} ,
\]
we get
\[
J = \int \frac{du}{\sqrt{(1 - u^2)(au^2 + bu + c)}} . \tag{2.16b}
\]

It is convenient to introduce yet another variable through a linear fractional transformation
\[
u = \rho x + \sigma , \quad 1 - u^2 = \frac{(1 - \rho^2)x^2 + 2(1 - \rho \sigma)x + (1 - \sigma^2)}{(1 + x)^2} , \tag{2.17a}
\]
\[
au^2 + bu + c = \frac{(a\rho^2 + b\rho + c)x^2 + [2a\rho \sigma + b(\rho + \sigma) + 2c]x + (a\sigma^2 + b\sigma + c)}{(1 + x)^2} .
\]
The required coefficients are determined by conditions
\[
1 - \rho \sigma = 0 , \quad 2a \rho \sigma + b(\rho + \sigma) + 2c = 0 , \quad \rho = -\frac{(a + c) + \sqrt{(a + c)^2 - b^2}}{b} ,
\]
\[12\]
\[ \sigma = -(a + c) - \sqrt{(a + c)^2 - b^2} \div b. \] (2.17b)

As a result, the integral \( J \) is the following:

\[ J = \frac{1}{A} \int \frac{dx}{\sqrt{1 - \rho^2 x^2} \sqrt{1 + m^2 x^2}}, \]

\[ A = \frac{a + \rho b + \rho^2 c}{\rho^2 (\rho^2 - 1)} , \quad m^2 = \frac{\rho^2 (a + \rho b + c)}{a + \rho b + \rho^2 c}. \] (2.18)

Making the change of variables

\[ \rho x = \sqrt{1 - U^2}, \quad k = \frac{m}{\sqrt{\rho^2 + m^2}}, \]

we reduce the integral to the canonical form of the elliptic integral:

\[ J = -\frac{1}{\sqrt{A (\rho^2 + m^2)}} \int \frac{dU}{\sqrt{1 - U^2} \sqrt{1 - k^2 U^2}}. \] (2.19)

Therefore, as a result, we produce solution \( z(t) \) in an implicit form

\[ z(\tau) = -c\tau \pm \frac{1}{\omega} \left[ -\frac{1}{\sqrt{A (\rho^2 + m^2)}} \int \frac{dU}{\sqrt{1 - U^2} \sqrt{1 - k^2 U^2}} \right]. \] (2.20)

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