Component Edge Connectivity of Hypercube-like Networks *

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Abstract

As a generalization of the traditional connectivity, the $g$-component edge connectivity $c_{λg}(G)$ of a non-complete graph $G$ is the minimum number of edges to be deleted from the graph $G$ such that the resulting graph has at least $g$ components. Hypercube-like networks (HL-networks for short) are obtained by manipulating some pairs of edges in hypercubes, which contain several famous interconnection networks such as twisted cubes, Möbius cubes, crossed cubes, locally twisted cubes. In this paper, we determine the $(g + 1)$-component edge connectivity of the $n$-dimensional HL-networks for $g \leq 2\lceil \frac{n}{2} \rceil$, $n \geq 8$.

Key words: HL-networks; component connectivity; component edge connectivity.

1 Introduction

As we all know, the interconnection network is an important part of the multiprocessor system. For convenience, we usually model the interconnection network by a graph, with vertices representing processors and edges representing links between processors. There are many parameters to evaluate the reliability of a network. The traditional connectivity $κ(G)$ of the graph $G$ is the most crucial one among them. Generally speaking, the larger the $κ(G)$ of the network is, the better its reliability is. However, the traditional connectivity only indicates when the network will break but does not further depict the properties of components, which makes it impossible to accurately evaluate the reliability of the network. In order to further describe the properties of components, a lot of more precise connectivity have been proposed, such as extra connectivity [10], super connectivity [2] and restricted connectivity [8]. In 1984, Chartrand et al. [4] and Sampathkumar [14] respectively introduced $g$-component connectivity $cκg(G)$ and $g$-component edge connectivity $cλg(G)$ of the graph $G$.

A $g$-component (edge) cut of a non-complete $G$ is a vertex (edge) set to be deleted from the graph $G$ such that the resulting graph has at least $g$ components. The $g$-component (edge) connectivity of $G$, written $cκg(G)$ ($cλg(G)$), is the minimum size of the $g$-component (edge) cut of $G$. Obviously, the component (edge) connectivity is a generalization of the traditional connectivity and $κ(G) = cκ2(G)$ ($λ(G) = cλ2(G)$). More importantly, compared with the traditional connectivity, the component (edge) connectivity can be better satisfied in practical applications. Thus, many researches have been focused on the component connectivity of some famous networks in recent years. (see, for example [5, 15, 16, 18]). But there are relatively few papers about component edge connectivity. Zhao et al. [17] determined the $(g + 1)$-component edge connectivity of hypercubes for $g \leq 2\lceil \frac{n}{2} \rceil$, $n \geq 7$. Based on this nice work, we shall be centered on the $(g + 1)$-component edge connectivity of hypercube-like networks for $g \leq 2\lceil \frac{n}{2} \rceil$, $n \geq 8$.

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As a popular topology for the design of the multiprocessor system, hypercubes have many excellent properties including symmetry, relatively diameter, good connectivity and recursive scalability. Improving these properties has led to the evolution of hypercube variants. A lot of hypercube variants, such as twisted cubes [11], crossed cubes [7] and Möbius cubes [6], have been introduced successively. Many of the properties of these networks are identical with that of the hypercubes. In particular, their diameter is shorter than that of the hypercubes. In order to conduct a unified study on these variants, Vaidya et al. proposed hypercube-like networks (in brief, HL-networks), which contain all of the networks mentioned above. Hence, HL-networks attracted considerable attention in the past (see, for example [3, 12]). This class of networks are sometimes called BC-networks [9]. We will use the term HL-networks in this paper.

The recursive definition of the HL-networks is as follows:
\[ HL_1 = \{K_1\} \text{ and } HL_n = \{G_{n-1} \oplus G_{n-1}^* | G_{n-1}, G_{n-1}^* \in HL_{n-1}\}, \]
where the symbol “\( \oplus \)” represents the perfect matching operation that connects \( G_{n-1} \) and \( G_{n-1}^* \) using some disjoint edges. It’s easy to get that \( HL_0 = \{K_1\} \), \( HL_1 = \{K_2\} \), \( HL_2 = \{C_4\} \), and \( HL_3 = \{Q_3, G(8, 4)\} \), where \( C_4 \) is a cycle of length 4, and \( Q_3 \) and \( G(8, 4) \) are shown in Figure 1. The 2-dimensional HL-network \( G_n \) is \( n \) regular, and it has \( 2^n \) vertices and \( 2^{2n-1} \) edges.

Next, we will introduce some notations and definitions which will be used in this paper. Let \( G = (V(G), E(G)) \) be a graph. The size of \( G \) is the number of edges of \( G \). The degree of \( v \), denoted by \( d_G(v) \), is the number of edges incident to \( v \) in \( G \). For a subset \( X \) of \( V(G) \), \( G[X] \) is the subgraph induced by \( X \). If \( D_i \) and \( D_j \) are two disjoint subgraphs of \( G \), then we use \( E(D_i, D_j) \) to denote the set of edges between the subgraphs \( D_i \) and \( D_j \). Similarly, we use \( E(v, D_i) \) to denote the set of edges between \( v \) and \( D_i \), where \( v \in V(G) \) but \( v \notin V(D_i) \). For any \( v \in V(G) \), \( G - v \) denotes a subgraph obtained by deleting \( v \) and edges incident to \( v \). Similarly, \( G - D_i \) denotes a subgraph obtained by removing all vertices in \( V(D_i) \) and all edges incident to vertices of \( V(D_i) \), where \( D_i \) is a subgraph of \( G \).

The rest of this paper is organized as follows. In section 2, we shall determine the maximum size of the subgraph induced by \( g \) vertices in HL-networks. In section 3, we shall determine the \((g + 1)\)-component edge connectivity of HL-networks by applying the results of section 2.

![Figure 1: Two 3-dimensional HL-networks: (a) \( Q_3 \) (b) \( G(8, 4) \)](image)

2 The maximum size of the subgraph induced by \( g \) vertices

It is a classical problem to determine the size (the number of edges) of the subgraph that satisfies some given property in a graph. For instance, Erdős has studied an interesting problem in [1]: What is the maximum size of the subgraph without cycles of length 4 in hypercubes. In this section, we shall determine the maximum size of the subgraph induced by \( g \) vertices in HL-networks. Furthermore, its application in the next section helps us find the \((g + 1)\)-component edge connectivity of HL-networks.

Let \( e_g \) be the maximum size of the subgraph induced by \( g \) vertices in the \( n \)-dimensional HL-network \( G_n \), that is, \( e_g = \max \{|E(G_n[X])| : X \subseteq V(G_n) \text{ and } |X| = g\} \). By the mathematical principle of
For any integer \( g \) can be written as the sum of the exponents of 2, that is, \( g = \sum_{i=0}^{s} 2^{t_i} \), where \( t_0 = \lfloor \log_2 g \rfloor \), \( t_i = \lfloor \log_2 (g - \sum_{j=0}^{i-1} 2^{t_j}) \rfloor \) for \( i \geq 1 \). Li and Yang have determined \( e_g \) of hypercubes in [13]. In order to avoid confusion, we use \( e_g(Q_n) \) to denote \( e_g \) of hypercubes, where \( e_g(Q_n) = \sum_{i=0}^{s} t_i 2^{t_i-1} + \sum_{i=0}^{s} i 2^{t_i} \). Furthermore, the function \( e_g(Q_n) \) has the following property.

**Lemma 2.1** [13][17] If \( g_0 \leq g_1 \), then \( e_{g_0 + g_1}(Q_n) \geq e_{g_0}(Q_n) + e_{g_1}(Q_n) + g_0 \).

Next, we shall give an algorithm to find the subgraph \( H \) induced by \( g \) vertices in the \( n \)-dimensional HL-network \( G_n \) such that \( |E(H)| = \sum_{i=0}^{s} t_i 2^{t_i-1} + \sum_{i=0}^{s} i 2^{t_i} \).

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**Algorithm-MS**

**Input:** An \( n \)-dimensional network \( G_n \), two integer \( g \) (suppose \( g = \sum_{i=0}^{s} 2^{t_i} \)) and \( s \).

**Output:** A vertex set \( V = V(D_0) \cup V(D_1) \cup \ldots \cup V(D_s) \) and a subgraph \( H = G_n[V] \).

**Initialization:** \( i \leftarrow 0 \), \( V \leftarrow \emptyset \), where \( \emptyset \) is an empty set.

**Iteration:** As long as \( i \leq s \), we take a \( t_i + 1 \)-dimensional subcube \( T_i \) from \( D_{i-1}^s \) \((D_{s-1}^i = G_n)\). Then \( T_i \) can be written as \( D_i \oplus D_i^s \), where \( D_i \) and \( D_i^s \) are two \( t_i \)-dimensional subcubes. Add \( V(D_i) \) to \( V \), \( i \leftarrow i + 1 \).

For ease of understanding, we list \( D_i \) as follows (see Figure 2 for \( s = 2 \)):

- \( D_0 \) \((t_0\)-dimensional subcube\);
- \( D_1 \) \((t_1\)-dimensional subcube taken from \( D_0^g \));
- \( D_2 \) \((t_2\)-dimensional subcube taken from \( D_1^g \));
- \ldots
- \( D_s \) \((t_s\)-dimensional subcube taken from \( D_{s-1}^g \)).

We need to verify that the subgraph \( H \) found by the algorithm-MS that satisfies \( |V(H)| = g \) and \( |E(H)| = \sum_{i=0}^{s} t_i 2^{t_i-1} + \sum_{i=0}^{s} i 2^{t_i} \).

Note that \( H = G_n[V(D_0) \cup V(D_1) \cup \ldots \cup V(D_s)] \) and \( |V(D_i)| = 2^{t_i} \). Thus,

\[
V(H) = \sum_{i=0}^{s} |V(D_i)| = \sum_{i=0}^{s} 2^{t_i} = g.
\]

By the choice of \( D_i \), one has \( |E(D_i, D_j)| = |V(D_j)| = 2^{t_j} \) for \( i < j \). Thus,

\[
E(H) = \sum_{i=0}^{s} |E(D_i)| + \sum_{i=0}^{s-1} \sum_{j=i+1}^{s} |E(D_i, D_j)| = \sum_{i=0}^{s} t_i 2^{t_i-1} + \sum_{i=0}^{s} i 2^{t_i}.
\]

**Theorem 2.2** For any \( G_n \in HL_n \), suppose that \( X \subseteq V(G_n) \) with \( |X| = g = \sum_{i=0}^{s} 2^{t_i} \), then \( e_g = \sum_{i=0}^{s} t_i 2^{t_i-1} + \sum_{i=0}^{s} i 2^{t_i} \).
Lemma 2.5
If \( g \leq 2^{n-2} \), then \( (n-2)g - 2e_g \geq 0 \).

Proof: If \( g \leq 2^{n-2} \), then we can take a subgraph \( H \) from an \((n-2)\)-dimensional HL-network \( G_{n-2} \) such that \( |V(H)| = g \) and \( |E(H)| = e_g \). Note that \( G_{n-2} \) is an \((n-2)\)-regular graph. Thus, \( (n-2)g - 2e_g \geq 0 \).

Lemma 2.4 \( e_{i+1} = e_i + s + 1 \), where \( i = 2^{t_0} + 2^{t_1} + \ldots + 2^{t_s} \).

Proof: If \( t_s > 0 \), then \( i + 1 = 2^{t_0} + 2^{t_1} + \ldots + 2^{t_s} + 2^0 \). Note that \( e_i = \sum_{i=0}^{s} t_i 2^{t_i-1} + \sum_{i=0}^{s} i 2^{t_i} \). We have that
\[
e_{i+1} = \sum_{i=0}^{s} t_i 2^{t_i-1} + 0 \cdot 2^{0-1} + \sum_{i=0}^{s} i 2^{t_i} + (s + 1) \cdot 2^0 = e_i + s + 1.
\]
Otherwise, \( t_s = 0 \). Then there is an integer \( r \) \((0 \leq r \leq s)\) such that \( t_j = s - j \) for all \( r \leq j \leq s \). In other words, \( i = 2^{t_0} + \ldots + 2^{t_{r-1}} + 2^{s-r} + 2^{s-r-1} + \ldots + 2^1 + 2^0 \). Then \( i + 1 = 2^{t_0} + \ldots + 2^{t_{r-1}} + 2^{s-r} + 2^{s-r-1} + \ldots + 2^1 + 2^0 + 2^0 = 2^{t_0} + \ldots + 2^{t_{r-1}} + 2^{s-r+1} \).

Using the algorithm-MS, we take two subgraph \( H_1 \) and \( H_2 \) from \( G_n \), where \( H_1 \) and \( H_2 \) are induced by \( i \) and \( i + 1 \) vertices respectively. Clearly, \( |E(H_1)| = e_i \) and \( |E(H_2)| = e_{i+1} \). We can assume that
\[
H_1 = G_n[V(D_0) \cup \ldots \cup V(D_{r-1}) \cup V(D_r) \cup V(D_{r+1}) \cup \ldots \cup V(D_s)];
\]
\[
H_2 = G_n[V(D_0) \cup \ldots \cup V(D_{r-1}) \cup V(D'_r)].
\]
where \( D_i \) is a \( t_i \)-dimensional subcube, \( D'_r \) is an \((s-r+1)\)-dimensional subcube.

In fact, \( |V(H_1)| = i \) and \( |V(H_2)| = i + 1 \). In other words, \( H_1 \) has one less vertex than \( H_2 \). Let the vertex be \( v \). From (1), we can see that the first \( r \) subcubes \( D_i \) in \( H_1 \) and \( H_2 \) are the same. Thus, one has \( v \subseteq V(D'_r) \). Clearly, \( d_{H_2}(v) = s - r + 1 \). In addition, there is only one edge between \( v \) and every \( D_i \) in \( H_2 \) for \( 0 \leq i \leq r - 1 \). Thus, \( d_{H_2}(v) = r + (s-r+1) = s + 1 \). We have that
\[
|E(H_2)| = |E(H_1)| + d_{H_2}(v),
\]
that is, \( e_{i+1} = e_i + s + 1 \).

Lemma 2.5 If \( i \leq j \), then \( e_{i+1} + e_j \leq e_{i+j} \).

Proof: Suppose \( i = 2^{t_0} + \ldots + 2^{t_s} \), then \( e_{i+1} = e_i + s + 1 \) by Lemma 2.4. Note that \( e_g(Q_n) = e_g \). By the Lemma 2.1, \( e_{i+j} \geq e_i + e_j + i = e_{i+1} + e_j + i - (s + 1) \geq e_{i+1} + e_j \).
3 Component edge connectivity of HL-networks

In this section, we shall apply Theorem 2.2 to determine the \((g+1)\)-component edge connectivity
of HL-networks.

For any \(X \subseteq V(G_n)\), we use \(E_X\) to denote a set of edges in which each edge has exactly one
endpoint in \(X\).

**Lemma 3.1** For any \(G_n \in HL_n\), let \(X \subseteq V(G_n)\) with \(|X| = g = \sum_{i=0}^{s} 2^{i}\). Then \(|E_X| \geq ng - 2e_g\).
Moreover, \(ng - e_g\) is strictly increasing (respect to \(g\)) for \(g \leq 2^{\lceil\frac{n}{2}\rceil}\), \(n \geq 2\).

**Proof:** Note that \(G_n\) is an \(n\)-regular graph. We find that \(|E_X| = n|X| - 2|E(G_n[X])| \geq ng - 2e_g\).
According to Lemma 2.4, if \(g \leq 2^{\lceil\frac{n}{2}\rceil}\) and \(n \geq 2\), then \(e_{g+1} - e_g = s + 1 < n\). Hence, we have that
\((n(g + 1) - e_{g+1}) - (ng - e_g) = n - (e_{g+1} - e_g) > 0\), which indicate that \(ng - e_g\) is strictly increasing
for \(g \leq 2^{\lceil\frac{n}{2}\rceil}\), \(n \geq 2\). \(\square\)

**Theorem 3.2** For any \(G_n \in HL_n\), \(c_{\lambda+1}(G_n) = ng - e_g\) for \(g \leq 2^{\lceil\frac{n}{2}\rceil}\), \(n \geq 8\).

**Proof:** We first prove that \(c_{\lambda+1}(G_n) \geq ng - e_g\) by constructing a \((g+1)\)-component edge-cut \(F\)
such that \(|F| = ng - e_g\). Using the algorithm-MS, we can get a subgraph \(H\) such that \(|V(H)| = g\) and
\(|E(H)| = e_g\). Let \(F = E(V(H)) \cup E(H)\), then we have that \(|F| = ng - e_g\). Moreover, \(G_n - F\) has
at least \(g + 1\) components. Thus, \(c_{\lambda+1}(G_n) \geq ng - e_g\).

Next, we shall prove that \(c_{\lambda+1}(G_n) \geq ng - e_g\). Assume that \(F\) is the smallest \((g+1)\)-component
edge-cut and \(G_n - F\) has exactly \(g + 1\) components. Denote the \(g + 1\) components in \(G_n - F\) by \(C_1, C_2, ..., C_{g+1}\). Without loss of generality, we can assume that \(|V(C_1)| \leq |V(C_2)| \leq ... \leq |V(C_{g+1})|\).

**Case 1:** Suppose that \(|V(C_{g+1})| < 2^{n-2}\).

In this case, \(|V(C_i)| < 2^{n-2}\) for all \(i\). Note that \(\sum_{i=1}^{g+1} |V(C_i)| = 2^n\). Then we can find an integer
\(j\) such that \(\sum_{i=1}^{j} |V(C_i)| < 2^{n-2}\) but \(2^{n-2} \leq \sum_{i=1}^{j+1} |V(C_i)| < 2^{n-1}\). Let \(X = \bigcup_{i=1}^{j+1} V(C_i)\) with
\(|X| = m = \sum_{i=0}^{s} 2^{i}\). Clearly, \(2^{n-2} \leq m = \sum_{i=0}^{s} 2^{i} < 2^{n-1}\). It follow that \(t_0 = n - 2\). We suppose that \(m' = m - 2^{t_0}\), then \(m' < 2^{n-2}\). By Lemma 2.3, we have that \((n - 2)m' - 2e_{m'} \geq 0\). Combining with Lemma 3.1, we have that
\(|E_X| \geq nm - 2e_m\)
\(= n^2 t_0 + n(2^{t_0} + ... + 2^{s}) - t_0 2^{t_0} - (\sum_{i=1}^{s} t_i 2^{i} + \sum_{i=1}^{s} 2 \cdot i \cdot 2^{i})\)
\(= (n - t_0) 2^{t_0} + (n\sum_{i=1}^{s} 2^{i} - (\sum_{i=1}^{s} t_i 2^{i} + \sum_{i=1}^{s} 2 \cdot i \cdot 2^{i}))\)
\(= (n - t_0) 2^{t_0} + nm' - 2e_{m'} - 2m'\)
\(\geq (n - t_0) 2^{t_0}\)
\(= (n - t_0) \cdot 2^{(t_0 - \lceil\frac{n}{2}\rceil)} \cdot 2^{\lceil\frac{n}{2}\rceil}\).

Note that \(t_0 = n - 2\). Then \((n - t_0) \cdot 2^{(t_0 - \lceil\frac{n}{2}\rceil)} > \frac{3n}{4} \) for \(n \geq 8\). we have that
\(|F| \geq |E_X| > \frac{3n}{4} \cdot 2^{\lceil\frac{n}{2}\rceil} \geq n \cdot 2^{\lceil\frac{n}{2}\rceil} - e_2 \geq ng - e_g\).

The last inequality holds, because \(ng - e_g\) is strictly increasing (respect to \(g\)) for \(g \leq 2^{\lceil\frac{n}{2}\rceil}\) by Lemma 3.1.

**Case 2:** Suppose that \(|V(C_{g+1})| \geq 2^{n-2}\).

Let \(|V(C_i)| = m_i\) for \(1 \leq i \leq g\) and let \(m = \sum_{i=1}^{g} m_i\). Set \(X = \bigcup_{i=1}^{g} V(C_i)\). If \(m \geq 2^{n-2}\), we can get that \(|F| \geq |E_X| > ng - e_g\) by using the proof similar to case 1. So we assume \(m < 2^{n-2}\). If \(m = g\), then the component \(C_i\) (\(1 \leq i \leq g\)) is an isolated vertex. Thus, \(|F| \geq ng - e_g\). If \(m > g\), then let \(F' = F \cap E(G_n[X])\). We have that \(|F| \geq \sum_{i=1}^{g} |E_{V(C_i)}| = |E_{V(C_1)}| + ... + |E_{V(C_g)}| - |F'|.\) Since \(|E_{V(C_i)}| \geq nm_i - 2|E(C_i)|\) and \(|F'| \leq e_m - \sum_{i=1}^{g} E(C_i)|),\) we have that
Then we find that

\[ |F| \geq \left| \bigcup_{i=1}^{\frac{n}{2}} E(V(C_i)) \right| = \left| E(V(C_1)) \right| + \ldots + \left| E(V(C_{\frac{n}{2}})) \right| - |F'| \]

\[ \geq \sum_{i=1}^{\frac{n}{2}} (nm_i - 2|E(C_i)|) - \epsilon_m - \sum_{i=1}^{\frac{n}{2}} |E(C_i)| \]

\[ = nm - \epsilon_m - \sum_{i=1}^{\frac{n}{2}} |E(C_i)| \]

\[ \geq nm - \epsilon_m - \sum_{i=1}^{\frac{n}{2}} \epsilon_{m_i}. \]

By Lemma 2.5, \( \epsilon_{m_1} + \epsilon_{m_2} + \ldots + \epsilon_{m_g} \leq \epsilon_{m_1 + m_2 - 1} + \epsilon_{m_3} + \ldots + \epsilon_{m_g} \leq \ldots \leq \epsilon_{m - g + 1} \). Thus, we have that \( |F| \geq nm - \epsilon_m - \sum_{i=1}^{\frac{n}{2}} \epsilon_{m_i} \geq nm - \epsilon_m - \epsilon_{m - g + 1} \).

Next, we shall show that \( nm - \epsilon_m - \epsilon_{m - g + 1} > ng - \epsilon_g \).

Note that \( G_n = G_{n-1} \oplus G_{n-1}^* \). We take a subgraph \( H_1 \) of \( m \) vertices from an \((n-2)\)-dimensional subcube in \( G_{n-1} \) by using the algorithm-\( MS \), since \( m < 2^{n-2} \). Clearly, \( |E(H_1)| = \epsilon_m \). We take a subgraph \( H_2 \) of \( m - g + 1 \) vertices from \( H_1 \) by using the algorithm-\( MS \), since \( m - g + 1 \leq m \). So \( |E(H_2)| = \epsilon_{m - g + 1} \). Use \( v_m, v_{m-1}, \ldots, v_{g-1}, v_g \) to denote the vertices of \( H_2 \). By the proof of Lemma 2.4, there exists a vertex in \( H_2 \), say \( v_g \), such that \( d_{H_2}(v_g) = s + 1 \), where \( m - g = 2^{t_0} + 2^{t_1} + \ldots + 2^{t_s} \). Use \( v_1, v_2, \ldots, v_{g-1} \) to denote vertices of \( V(H_1) - V(H_2) \). We assume that (see Figure 3)

\[
\begin{align*}
f_0 &= |E(H_1 - H_2)|; & f_4 &= |E(v_g, G_n - H_1)|; \\
f_1 &= |E(H_1 - H_2, G_n - H_1)|; & f_5 &= |E(G_n - H_1, H_2 - v_g)|; \\
f_2 &= |E(H_1 - H_2, H_2 - v_g)|; & f_6 &= |E(v_g, H_2 - v_g)|; \\
f_3 &= |E(v_g, H_1 - H_2)|.
\end{align*}
\]

Then we find that

\[ f_0 + f_1 + f_2 + f_3 + f_4 + f_5 = (f_1 + f_4 + f_5) + (f_0 + f_2 + f_3) \]

\[ = |E(V(H_1))| + (|E(H_1)| - |E(H_2)|) \]

\[ = nm - 2|E(H_1)| + (|E(H_1)| - |E(H_2)|) \]

\[ = nm - \epsilon_m - \epsilon_{m - g + 1}. \]

Let \( S = \{v_1, v_2, \ldots, v_g\} \). Note that \( |E(G_n[S])| = f_0 + f_3 \leq \epsilon_g \), and \( |E_S| + 2|E(G_n[S])| = (f_1 + f_2 + f_4 + f_6) + 2(f_0 + f_3) = ng \). Thus,

\[ f_0 + f_1 + f_2 + f_3 + f_4 + f_5 = (f_1 + f_2 + f_4 + f_5) + (f_0 + f_2 + f_3) \geq ng - \epsilon_g. \]

Moreover, each vertex of \( H_1 \) has at least two neighbors out of \( H_1 \) since \( H_1 \) be taken from an \((n-2)\)-dimensional subcube. Thus, \( f_5 > |V(H_2 - v_g)| = m - g \). Note that \( f_6 = d_{H_2}(v_g) = s + 1 \) and \( m - g = 2^{t_0} + 2^{t_1} + \ldots + 2^{t_s} \). It is not difficult to get that \( f_5 > m - g \geq s + 1 = f_6 \). Thus, \( nm - \epsilon_m - \epsilon_{m - g + 1} > ng - \epsilon_g \) by comparing (2) with (3).

To sum up, \( c\lambda_{g+1}(G_n) = ng - \epsilon_g \).
Corollary 3.3 Let $F$ be a $(g + 1)$-component edge-cut of the $n$-dimensional HL-network $G_n$ and $|F| = c\lambda_{g+1}(G_n)$, then $G_n - F$ contains $g$ isolated vertices for $g \leq 2\lceil \frac{n}{2} \rceil$, $n \geq 8$.

Proof: By the proof of Theorem 3.2, $|F| = c\lambda_{g+1}(G_n)$ if and only if $m = g$ in case 2, that is, $g$ components are isolated vertices. \qed

4 Conclusion

Component edge connectivity is an generation of the traditional connectivity. In this paper, we studied the $(g + 1)$-component edge connectivity of HL-networks. $c\lambda_{g+1}(G_n) = ng - e_g$ for $g \leq 2\lceil \frac{n}{2} \rceil$, $n \geq 8$. But for $g > 2\lceil \frac{n}{2} \rceil + 1$, The problem has not been solved.

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