New Estimations for Sturm-Liouville Problems in Difference Equations

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Abstract

In this paper, Sturm-Liouville problem for difference equations is considered with potential function \( q(n) \). The representations of solutions are obtained by variation of parameters method. These solutions are proved, using summation by parts. Also, estimation of asymptotic expansion of the solutions are established.

AMS: 39A13, 34B18.

Keywords: Sturm-Liouville, difference equation, eigenfunction, asymptotic formula, Casoratian.

1. Introduction

Sturm-Liouville operators have been studied for a number of years. Firstly, Sturm Liouville problem developed in a number of articles published by these authors in 1836 and 1837. It is known that the spectral characteristics are spectra, spectral functions, scattering data, norming constants, etc. The representation of solution of Sturm-Liouville problem and asymptotic formulas for eigenfunctions have been obtained by Levitan and Sargsjan [6]. The following problem

\[
L y(t) = -\frac{d^2 y}{dt^2} + q(t) y = \lambda y,
\]

\[y(0) \cos \alpha + y'(0) \sin \alpha = 0\]

\[y(1) \cos \beta + y'(1) \sin \beta = 0\]
is called Sturm-Liouville problem in differential equation. Differential equations are related to difference equations closely.

In general, it is known that difference equations related to recursive relations for a long time. Its actual development appeared by being able to compared to differential equations. Difference equations have many application areas which are problems of physics, mathematics and engineering, vibrating string, economy, population, actuaria and logistics, etc.

As a natural result of comparing the difference equations to differential equations, the theory of linear difference equations have begun to appear in a similar way to the theory of differential equations. Basic theory of linear difference equations improved by De Moivre, Euler, Lagrange, Laplace et al [13]. Hereafter, Hartman [17], Ahlbrandt-Hooker [16], Peterson-Kelley [1], Agarwal [2], Jirari [5], Bender-Orszag [7], Goldberg [12], Elaydi [15], Lakshmikantham, Trigiante and Peterson [13], Mickens [14] and they have contributed to linear difference equations with publications and books.

Especially, in recent years, Sturm-Liouville difference equation has seen a great interest and a lot of study has published, but there are still a lot of things to develop about this subject. Peterson, Kelley, Agarwal, Bender gave a place this subject in studies, also, Jirari studied peculiarly Sturm-Liouville difference equation in his studies.

Atkinson studied discrete and continuous boundary value problems in his book and also he considered self-adjoint second-order difference equations. Jirari contributed Atkinson’s study by Second Order Sturm-Liouville Difference Equations and Orthogonal Polynomials in his thesis.

Hinton and Lewis [9], Clark [19], Shi and Wu [8], Shi and Sun [10], Shi and Chen [18] and Hilscher [20] investigated spectral analysis of second order difference equations and operators. Wang and Shi [11], Ji and Yang [21], Sun and Shi [22] studied eigenvalues of second order difference equations.

In the most of studies for Sturm-Liouville difference equations, $q(n)$ is considered as a real number, solutions and spectral properties for Sturm-Liouville difference equation are investigated by Hamiltoni systems. The following (1.1)–
problem [1], [5]

\[ \Delta (\Delta p(n-1)x(n-1)) + q(n)x(n) + \lambda r(n)x(n) = 0, \quad n = a, \ldots, b \] (1.1)

\[ x(a-1) + hx(a) = 0, \] (1.2)

\[ x(b+1) + kx(b) = 0. \] (1.3)

is called Sturm-Liouville problem in difference equations.

Our study is organized as follows. Fundamental theorems and definitions in Section 2, representations of solutions with two different initial conditions in Section 3 and asymptotic behavior of eigenfunctions are given in Section 4.

2. Preliminaries

**Definition 2.1.** [1] The matrix of Casoratian is given by

\[
w(n) = \begin{pmatrix}
x_1(n) & x_2(n) & \cdots & x_r(n) \\
x_1(n+1) & x_2(n+1) & \cdots & x_r(n+1) \\
\vdots & \vdots & \ddots & \vdots \\
x_1(n+r-1) & x_2(n+r-1) & \cdots & x_r(n+r-1)
\end{pmatrix}
\]

where \( x_1(n), x_2(n), \ldots, x_r(n) \) are given functions. The determinant

\[ W(n) = \det w(n) \]

is called Casoratian. Note that, it is similar to Wronskian determinant.

**Theorem 2.1.** [5] (Wronskian-Type Identity) Let \( y \) and \( z \) be a solutions of (1.1). Then, for \( a \leq n \leq b \)

\[
W[y, z](n) = p(n-1) [y(n) \Delta z(n-1) - z(n) \Delta y(n-1)] - p(n-1) [y(n) z(n-1) - y(n-1) z(n)] \] (2.1)

is a constant (In particular equal to \( W[y, z](a) \)).

**Definition 2.2.** [5] Let’s express (1.1) Sturm-Liouville equation as follows,

\[ Lx(n) = -\lambda x(n), \quad n \in [a, b], \] (2.2)
with initial conditions
\[
\cos \alpha x (a) - \sin \alpha (p (a) \nabla x (a)) = 0,
\]
(2.3)
where \(0 < \alpha, \beta < \pi\), \(\nabla\) is the backward difference operator, \(\nabla x (n) = x (n) - x (n - 1)\), are equivalent to
\[
x (a - 1) + \left( \frac{\cot \alpha}{p (a)} - 1 \right) x (a) = 0,
\]
(2.4)
in other words,
\[
x (a - 1) + h x (a) = 0,
\]
(2.5)
where \(L\) is self-adjoint Sturm-Liouville operator and \(h\) is real number. (2.2) – (2.5) initial value problem is called Sturm-Liouville problem.

**Theorem 2.2.** [1] (Summation by parts) If \(m < n\), then
\[
\sum_{k=m}^{n-1} x (k) \Delta y (k) = [x (k) y (k)]_m^n - \sum_{k=m}^{n-1} \Delta x (k) y (k + 1).
\]
(2.6)

**Theorem 2.4.** [1] If \(y_n\) is an indefinite sum of \(x_n\), then
\[
\sum_{k=m}^{n-1} y (k) = x (n) - x (m).
\]
(2.7)

**Theorem 2.5.** [1], [3] (Annihilator method) Suppose that \(x (n)\) solves following difference equation, \(E\) is the shift operator, \(Ex (n) = x (n + 1)\),
\[
(E^t + p (t - 1) E^{t-1} + ... + p (0)) x (n) = r (n),
\]

and that \(r (n)\) satisfies
\[
(E^m + q (m - 1) E^{m-1} + ... + q (0)) r (n) = 0.
\]
Then \(x (n)\) satisfies
\[
(E^m + q (m - 1) E^{m-1} + ... + q (0)) (E^t + p (t - 1) E^{t-1} + ... + p (0)) x (n) = 0.
\]

3. Main Results
In this paper, we are interested in the representations of solutions of the Sturm-Liouville problem in difference equations with potential function $q(n)$ as follows,

$$\Delta^2 x(n-1) + q(n)x(n) + \lambda x(n) = 0, \quad n = a, \ldots, b \quad (3.1)$$

with initial conditions,

$$x(a-1) + hx(a) = 0, \quad (3.2)$$

where $a, b$ are finite integers with $a \geq 0$, $a \leq b$, $h$ is a real number, $\Delta$ is the forward difference operator, $\Delta x(n) = x(n+1) - x(n)$, $\lambda$ is the spectral parameter, $q(n)$ is a real valued potential function for $n \in [a, b]$. In the general literature, potential function $q(n)$ is taken as real number but we take it as a variable coefficients in a similar way in Levitan and Sargsjan’s study [6].

Levitan and Sargsjan obtained the representation of solution of Sturm-Liouville problem in differential equations and asymptotic formulas for eigenfunctions. In analogous manner, we tried to obtain the representation of solution of Sturm-Liouville problem in difference equations.

A self-adjoint difference operator corresponds to equation (3.1) noted by,

$$Lx(n) = \Delta^2 x(n-1) + q(n)x(n) = -\lambda x(n).$$

In $\ell^2(a,b)$, the Hilbert space of sequences of complex numbers $x(a), \ldots, x(b)$ with the inner product,

$$<x(n),y(n)> = \sum_{n=a}^{b} x(n)y(n),$$

for every $x \in D_L$, let define as follows

$$D_L = \{x(n) \in \ell^2(a,b) : Lx(n) \in \ell^2(a,b), \quad x(0) = -h, \quad x(1) = 1\}.$$

Hence, equation (3.1) can be written as follows

$$Lx(n) = -\lambda x(n).$$

At this section, we present the representation of solution of (3.1) – (3.2) Sturm-Liouville problem by variation of parameters method.
Theorem 3.1. Let define Sturm-Liouville problem in difference equations as follows;

\[ Lx(n) = -\lambda x(n), \quad (3.3) \]

\[ x(0) = -h, \quad x(1) = 1, \quad (3.4) \]

then (3.3) – (3.4) Sturm-Liouville problem has a unique solution for \( x(n) \) as

\[ x(n, \lambda) = \left( \frac{2 - h \left( 2q(0) - 2 + \lambda + \sqrt{\lambda(\lambda - 4)} \right)}{2 \sqrt{\lambda(\lambda - 4)}} \right) \left( \frac{2 - \lambda + \sqrt{\lambda(\lambda - 4)}}{2} \right)^n \]

\[ + \left( \frac{-2 + h \left( 2q(0) - 2 + \lambda - \sqrt{\lambda(\lambda - 4)} \right)}{2 \sqrt{\lambda(\lambda - 4)}} \right) \left( \frac{2 - \lambda - \sqrt{\lambda(\lambda - 4)}}{2} \right)^n \]

\[ - \sum_{i=0}^{n} q(i) x(i) \left( \frac{2 - \lambda - \sqrt{\lambda(\lambda - 4)}}{2 \sqrt{\lambda(\lambda - 4)}} \right)^i \left( \frac{2 - \lambda + \sqrt{\lambda(\lambda - 4)}}{2} \right)^n \]

\[ + \sum_{i=0}^{n} q(i) x(i) \left( \frac{2 - \lambda + \sqrt{\lambda(\lambda - 4)}}{2 \sqrt{\lambda(\lambda - 4)}} \right)^i \left( \frac{2 - \lambda - \sqrt{\lambda(\lambda - 4)}}{2} \right)^n. \]

Where \( \sum_{i=0}^{n} = 0. \)

Proof. If \( x_1(n) \) and \( x_2(n) \) are linearly independent solution for homogen part of (3.1), then it is easily found by characteristic polynomial [1]

\[ x_h(n) = c_1 x_1(n) + c_2 x_2(n), \quad (3.6) \]

\[ x_h(n) = c_1 \left( \frac{2 - \lambda + \sqrt{\lambda(\lambda - 4)}}{2} \right)^n + c_2 \left( \frac{2 - \lambda - \sqrt{\lambda(\lambda - 4)}}{2} \right)^n \quad (3.7) \]

Assume that \(| \lambda - 2 | < 2 \) for existence of eigenvalues of (3.7). By variation of parameters method [1], [7], we take

\[ x_p(n) = c_1(n) x_1(n) + c_2(n) x_2(n), \quad (3.8) \]

If we perform necessary processes, we find \( c_1 \) and \( c_2 \) as follows

\[ c_1(n) = \sum_{i=0}^{n} \frac{q(i) x(i) x_2(i)}{W(x_1(i), x_2(i))}, \quad (3.9) \]

\[ c_2(n) = -\sum_{i=0}^{n} \frac{q(i) x(i) x_1(i)}{W(x_1(i), x_2(i))}. \]
Finally, we obtain general solution

\[ x(n, \lambda) = c_1 \left( \frac{2 - \lambda + \sqrt{\lambda (\lambda - 4)}}{2} \right)^n + c_2 \left( \frac{2 - \lambda - \sqrt{\lambda (\lambda - 4)}}{2} \right)^n \]

\[ + \sum_{i=0}^{n} \frac{q(i) x(i) \left( \frac{2 - \lambda - \sqrt{\lambda (\lambda - 4)}}{2} \right)^i}{W(x_1(i), x_2(i))} \left( \frac{2 - \lambda + \lambda (\lambda - 4) + \sqrt{\lambda (\lambda - 4)}}{2} \right)^n \]

\[ + \sum_{i=0}^{n} \frac{-q(i) x(i) \left( \frac{2 - \lambda + \sqrt{\lambda (\lambda - 4)}}{2} \right)^i}{W(x_1(i), x_2(i))} \left( \frac{2 - \lambda - \lambda (\lambda - 4) - \sqrt{\lambda (\lambda - 4)}}{2} \right)^n. \]

Where, \( W \) Casoratian determinant is a constant by Theorem 2.1,

\[ W(x_1(i), x_2(i)) = -\sqrt{\lambda (\lambda - 4)}. \]

If we use the initial conditions (3.4), then we obtain the representation of the solution of the Sturm-Liouville problem in difference equations as follows,

\[ x(n, \lambda) = \left( \frac{2 - h \left( 2q(0) - 2 + \lambda + \sqrt{\lambda (\lambda - 4)} \right)}{2\sqrt{\lambda (\lambda - 4)}} \right)^n \left( \frac{2 - \lambda + \lambda (\lambda - 4) + \sqrt{\lambda (\lambda - 4)}}{2} \right)^n \]

\[ + \left( \frac{-2 + h \left( 2q(0) - 2 + \lambda - \sqrt{\lambda (\lambda - 4)} \right)}{2\sqrt{\lambda (\lambda - 4)}} \right)^n \left( \frac{2 - \lambda - \lambda (\lambda - 4) - \sqrt{\lambda (\lambda - 4)}}{2} \right)^n \]

\[ - \sum_{i=0}^{n} q(i) x(i) \left( \frac{2 - \lambda - \sqrt{\lambda (\lambda - 4)}}{2} \right)^i \left( \frac{2 - \lambda + \lambda (\lambda - 4) + \sqrt{\lambda (\lambda - 4)}}{2} \right)^n \]

\[ + \sum_{i=0}^{n} q(i) x(i) \left( \frac{2 - \lambda + \sqrt{\lambda (\lambda - 4)}}{2} \right)^i \left( \frac{2 - \lambda - \lambda (\lambda - 4) - \sqrt{\lambda (\lambda - 4)}}{2} \right)^n. \]

Now, let’s show that (3.15) holds Sturm-Liouville problem (3.3) – (3.4). From (3.3)

\[ q(n) x(n) = -\Delta^2 x(n - 1) - \lambda x(n). \] (3.11)

First, let’s take last two term in (3.10) and write equality (3.11) in place of \( q(i) x(i) \). Hence, we obtain

\[ - \sum_{i=0}^{n} \frac{-\Delta^2 x(i-1) - \lambda x(i)}{\sqrt{\lambda (\lambda - 4)}} x_1(n) = \sum_{i=0}^{n} \frac{\Delta^2 x(i-1) x_2(i)}{\sqrt{\lambda (\lambda - 4)}} x_1(n) + \sum_{i=0}^{n} \frac{\Delta x(i) x_2(i)}{\sqrt{\lambda (\lambda - 4)}} x_1(n), \] (3.12)
Second, let’s apply twice summation by parts method to first term at the right hand side of equation (3.12) and (3.13) by Theorem 2.2, we obtain

\[
\sum_{i=0}^{n} \left[ -\Delta^2 x(i-1) - \lambda x(i) \right] x_2(i) x_1(n) = - \sum_{i=0}^{n} \Delta^2 x(i-1) x_1(i) x_2(n) - \sum_{i=0}^{n} \lambda x(i) x_1(i) x_2(n)
\]

(3.13)

and

\[
\sum_{i=0}^{n} \left[ -\Delta^2 x(i-1) - \lambda x(i) \right] x_1(i) x_2(n) = - \sum_{i=0}^{n} \Delta^2 x(i-1) x_2(i) x_1(n)
\]

Since \(x_1\) and \(x_2\) satisfy the homogen part of (3.3), sum expressions at the right hand side of (3.14) and (3.15) equal to zero and hence, (3.14) and (3.15) as follows respectively,

\[
[x_2(n+1) \Delta x(n) - x_2(0) \Delta x(-1)]
\]

(3.16)

\[
-\Delta x_2(n+1) x(n+1) + \Delta x_2(0) x(0)
\]

\[
-\Delta^2 x_2(-1) x(0) \frac{x_1(n)}{\sqrt{\lambda(\lambda-4)}}
\]

(3.17)

\[
- \sum_{i=0}^{n} \left[ \Delta^2 x_1(i-1) + \lambda x_1(i) \right] x(i) x_2(n) \]

\[
- \sum_{i=0}^{n} \left[ \Delta^2 x_1(i-1) x(i) \right] x_2(n)
\]

\[
- \sum_{i=0}^{n} \left[ \Delta^2 x_1(i-1) x(i) \right] x_1(n)
\]

(3.14)

- \Delta x_2(n+1) x(n+1) + \Delta x_2(0) x(0)

- \Delta^2 x_2(-1) x(0) \frac{x_1(n)}{\sqrt{\lambda(\lambda-4)}}

\[
\sum_{i=0}^{n} \Delta^2 x_2(i-1) x_2(i) x_1(n) - \sum_{i=0}^{n} \Delta^2 x_2(i-1) x_1(i) x_2(n)
\]
Finally, performing necessary operations, then we get
\[ -\sum_{i=0}^{n} \frac{q(i)x(i)x_2(i)}{\sqrt{\lambda(\lambda - 4)}} x_1(n) + \sum_{i=0}^{n} \frac{q(i)x(i)x_1(i)}{\sqrt{\lambda(\lambda - 4)}} x_2(n) = \frac{1}{\sqrt{\lambda(\lambda - 4)}} [x(n) \sqrt{\lambda(\lambda - 4)}] \quad (3.18) \]
\[ + x_1(n)x_2(0)x(-1) - x_2(-1)x(0) \]
\[ + x_2(n)(x_1(-1)x(0) - x_1(0)x(-1)) \]

And then, we have the following equation
\[ x(n) = \left( \frac{2-h(2q(0)-2+\lambda+\sqrt{\lambda(\lambda - 4)})}{2\sqrt{\lambda(\lambda - 4)}} \right) x_1(n) + \left( \frac{-2+h(2q(0)-2+\lambda-\sqrt{\lambda(\lambda - 4)})}{2\sqrt{\lambda(\lambda - 4)}} \right) x_2(n) \]
\[ - \sum_{i=0}^{n} \frac{q(i)x(i)x_2(i)}{\sqrt{\lambda(\lambda - 4)}} x_1(n) + \sum_{i=0}^{n} \frac{q(i)x(i)x_1(i)}{\sqrt{\lambda(\lambda - 4)}} x_2(n). \]

Note that, detail of the proof will be given in published paper.

**Theorem 3.2.** Let define Sturm-Liouville problem in difference equations as follows;
\[ Ly(n) = -\lambda y(n), \quad (3.19) \]
\[ y(0) = 1, \quad y(1) = 0, \quad (3.20) \]
then (3.31) – (3.32) Sturm-Liouville problem has a unique solution for \( y(n) \) as

\[ y(n, \lambda) = \left( \frac{2-\lambda+\sqrt{\lambda(\lambda - 4)}-2q(0)}{2\sqrt{\lambda(\lambda - 4)}} \right) \left( \frac{2-\lambda+\sqrt{\lambda(\lambda - 4)}}{2} \right)^n \]
\[ - \sum_{i=0}^{n} q(i) y(i) \left( \frac{2-\lambda-\sqrt{\lambda(\lambda - 4)}}{2} \right)^i \]
\[ + \sum_{i=0}^{n} q(i) y(i) \left( \frac{2-\lambda+\sqrt{\lambda(\lambda - 4)}}{2} \right)^i \left( \frac{2-\lambda-\sqrt{\lambda(\lambda - 4)}}{2} \right)^n. \]

**Proof.** This is proved similarly to the proof of Theorem 3.1.
4. Asymptotic Formulas for Sturm-Liouville Problem in Difference Equations

At this section, we present the asymptotic formulas for the solution of Sturm-Liouville problem. Let’s take (3.3) – (3.4) Sturm-Liouville problem. Then we can give the following Theorem.

**Theorem 4.1.** (3.3) – (3.4) Sturm-Liouville problem has the estimate

\[ x(n) = O(|h|). \]

**Theorem 4.2.** (3.19) – (3.20) Sturm-Liouville problem has the estimate

\[ y(n) = O(1). \]

The proofs will be given in published version of paper.

**Conclusion**

The article has been extended the scope of the representation of solution and asymptotic formulas for the Sturm-Liouville problem in difference equations.
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