ON OPERATOR NORMS FOR HYPERBOLIC GROUPS

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To the memory of Uffe Haagerup

Abstract. We estimate the operator norm of radial non-negative functions on hyperbolic groups. As a consequence, we show that several forms of Haagerup’s inequality are optimal.

1. Introduction

Let $\Gamma$ be a non-elementary hyperbolic group. As with any group, it is a difficult problem to compute, or even estimate the operator norm $\|\lambda(a)\|$, where $\lambda$ is the left-regular representation on $\ell^2\Gamma$ and $a$ is an element in the group algebra $\mathbb{C}\Gamma$. Haagerup’s inequality gives an upper bound for the operator norm, and it is stated and used in either one of the following forms:

(\text{H}_\circ) \quad \|\lambda(a)\| \lesssim (n+1)\|a\|_2$ whenever $a \in \mathbb{C}\Gamma$ is supported on the sphere of radius $n$;

(\text{H}_* \circ) \quad \|\lambda(a)\| \lesssim (n+1)^{3/2}\|a\|_2$ whenever $a \in \mathbb{C}\Gamma$ is supported on the ball of radius $n$;

(\text{H}_s) \quad \|\lambda(a)\| \lesssim \|a\|_{2,s}$ for $s > 3/2$, where $\|a\|_{2,s} = (\sum (|\gamma|+1)^s |a(\gamma)|^2)^{1/2}$ is the $s$-weighted $\ell^2$-norm.

The upper-bounding quantity requires the choice of a word-length $|\cdot|$, given by some finite and symmetric generating set for $\Gamma$. The notation $\lesssim$ means inequality up to positive multiplicative constants that only depend on $\Gamma$ and the choice of generating set, and $\asymp$ is the corresponding equivalence.

Haagerup’s original result [8] concerned free groups, equipped with the standard word-length. The extension to hyperbolic groups is due to Jolissaint [10] and de la Harpe [9]. Note that the spherical inequality (\text{H}_\circ) implies the ball inequality (\text{H}_*) and the weighted inequality (\text{H}_s), by obvious applications of the Cauchy - Schwartz inequality. The weighted inequality (\text{H}_s) is intimately related to certain Sobolev-type phenomena for the reduced group $\mathbb{C}^\ast$-algebra of $\Gamma$. The Haagerup inequality for hyperbolic groups is an instance of what we now call the property of Rapid Decay. See [15] for a very recent overview.

The starting point of this note is the following natural question: are the above Haagerup inequalities essentially sharp? For the spherical inequality (\text{H}_\circ), this is known to be the case for free groups. Cohen [3] has computed the operator norm of the spherical element

$$\sigma_n = \sum_{|\gamma|=n} \gamma \in \mathbb{C}\Gamma$$

when $\Gamma$ is free and endowed with the standard word-length. Cohen’s computation immediately implies that $\|\lambda(\sigma_n)\| \asymp (n+1)\|\sigma_n\|_2$. This suggests that (\text{H}_\circ) ought to be essentially sharp for any hyperbolic $\Gamma$, and it is fairly easy to devise a combinatorial argument - quasifying that of Cohen - showing that this is indeed the case. On the other hand, whether (\text{H}_*) and (\text{H}_s) are essentially sharp is a less obvious matter. Relaxing the support condition on $a$ makes the
operator norm of a harder to estimate. But the fact that \( (H_\bullet) \) and \( (H_s) \) are reasonable by-products of \( (H_\circ) \), which we just claimed to be essentially sharp, gives a hint of what the answer might be.

The main result of this note is an estimate for the operator norm of radial elements in \( \mathcal{C} \Gamma \) with non-negative coefficients.

**Theorem.** Let \( a = \sum a_k \sigma_k \in \mathcal{C} \Gamma \), where \( a_k \geq 0 \). Then \( \| \lambda(a) \| \asymp \sum (k+1) \| a_k \sigma_k \|_2 \).

**Corollary.** The following hold.

i) For the spherical element \( \sigma_n \), we have \( \| \lambda(\sigma_n) \| \asymp (n+1) \| \sigma_n \|_2 \). In particular, \( (H_\circ) \) is essentially sharp.

ii) For the ball element \( \beta_n = \sum_{k \leq n} a_k \sigma_k \), where \( a_k \geq 0 \) is such that \( \| a_k \sigma_k \|_2 = k+1 \), we have \( \| \lambda(a) \| \asymp (n+1) \| \beta_n \|_2 \). In particular, \( (H_s) \) is essentially sharp.

iii) For the radial element \( a = \sum_{k \leq n} a_k \sigma_k \), where \( a_k \geq 0 \) is such that \( \| a_k \sigma_k \|_2 = (k+1)^{-2} \), we have \( \| \lambda(a) \| \asymp \log(n+1) \| a \|_{2,3/2} \). In particular, \( (H_e) \) is essentially sharp.

The above statements directly follow from the main theorem, except for part ii) which needs in addition Lemma 3 below.

Parts iii) and iv) are more than sufficient to settle a problem raised in [14, Section 4]. The degree of rapid decay is \( 3/2 \), not just for free groups as originally asked, but for hyperbolic groups in general.

### 2. Proof of the main theorem

That \( \| \lambda(a) \| \asymp \sum (k+1) \| a_k \sigma_k \|_2 \) follows from the spherical Haagerup inequality \( (H_\circ) \). We need to show that \( \| \lambda(a) \| \asymp \sum (k+1) \| a_k \sigma_k \|_2 \). A combinatorial argument, within \( \Gamma \), might be possible, but it appears to be difficult. We use instead an analytic approach, involving the boundary of \( \Gamma \).

Consider the Cayley graph of \( \Gamma \) with respect to the given generating set, and let \( \partial \Gamma \) denote its boundary. The Gromov product on \( \Gamma \), given by the formula \( (x, y) = \frac{1}{2}(|x| + |y| - |x^{-1}y|) \), is extended in a somewhat ad-hoc way to the boundary by setting

\[
(x, \xi) = \inf_{x_i \to \xi} \lim inf (x, x_i), \quad (\xi, \xi') = \inf_{x_i \to \xi, x_i' \to \xi'} \lim inf (x_i, x_i').
\]

With this convention, the hyperbolic inequality \( (x, y) \geq \min\{(x, z), (y, z)\} - \delta \) extends from \( \Gamma \) to \( \Gamma \cup \partial \Gamma \).

We now recall some facts about the metric - measure structure of \( \partial \Gamma \). For small enough \( \varepsilon > 0 \) there is a visual metric \( d_\varepsilon \) on the boundary \( \partial \Gamma \), satisfying \( d_\varepsilon(\xi, \xi') \asymp \exp(-\varepsilon \langle \xi, \xi' \rangle) \). Different choices of the visual parameter \( \varepsilon \) yield comparable Hausdorff measures. Let \( \mu \) be a probability measure in the comparability class of visual Hausdorff measures. By [4], \( \mu \) is Ahlfors-regular: the measure of any \( r \)-ball with respect to the visual metric \( d_\varepsilon \) satisfies \( \mu(r \text{-ball}) \asymp r^{e(\Gamma)/\varepsilon} \) for \( 0 \leq r \leq \text{diam} \partial \Gamma \). Here, the critical exponent \( e(\Gamma) \) with respect to the chosen word-length is the finite, positive number given by \( e(\Gamma) = \inf\{ s > 0 : \sum \gamma \exp(-s|\gamma|) < \infty \} \). For more details, see [3] as well as the discussions in [12, Sec.15], [8, Sec.5].

The boundary representation of \( \Gamma \) with respect to \( \mu \) is the unitary representation on \( L^2(\partial \Gamma, \mu) \) defined as follows:

\[
\pi(\gamma)f = (d(\gamma, \mu)/d\mu)^{1/2} \gamma.f
\]

**Lemma 1.** The boundary representation \( \pi \) is weakly contained in the regular representation \( \lambda \).

**Proof.** This follows [2,13] from the fact that the action of \( \Gamma \) on \( \partial \Gamma \) is amenable [1, 7, 11].
Consequently,

\[ \|\lambda(a)\| \geq \|\pi(a)\| \geq \langle \pi(a), 1 \rangle. \]

where \(1\) is the constant function equal to \(1\) on \(\partial \Gamma\).

The next step concerns the asymptotics of the spherical function \(\gamma \mapsto \langle \pi(\gamma), 1 \rangle\). In what follows, we put \(q := \exp(e(\Gamma))\).

**Lemma 2.** \(\langle \pi(\gamma), 1 \rangle \asymp (|\gamma| + 1)^{-1/2}\).

**Proof.** By [4], \(\mu\) is quasi-conformal: \(d(\gamma_* \mu)/d\mu (\xi) \asymp \exp(e(\Gamma) (2(|\gamma| - |\xi|))\) for \(\mu\)-almost all \(\xi \in \partial \Gamma\). The ad-hoc definition of the extended Gromov product offers no guarantee that the map \(\xi \mapsto (\gamma, \xi)\) is measurable. But the same procedure which yields the visual metric \(d_\varepsilon\) on \(\partial \Gamma\) also yields a companion metric-like map \(\xi \mapsto (\gamma, \xi)\). Thus, up to replacing it by a comparable continuous map, we may indeed assume that \(\xi \mapsto q(\gamma, \xi)\) is measurable.

Now we have

\[ \langle \pi(\gamma), 1 \rangle = \int_{\partial \Gamma} (d(\gamma_* \mu)/d\mu)^{1/2} q^{-1/2} \int_{\partial \Gamma} q(\gamma, \xi) d\xi. \]

To estimate the last integral, we apply the following general formula: if \((X, \nu)\) is a probability space and \(f\) is a non-negative measurable function, then

\[ \int_X \exp(f) = 1 + \int_0^\infty \nu\{x \in X : f(x) \geq t\} \exp(t) dt. \]

In our case, \(f\) is the map \(\xi \mapsto (\log q)(\gamma, \xi)\). After a change of variable \(t \mapsto (\log q) t\), we get

\[ \int_{\partial \Gamma} q(\gamma, \xi) d\xi = 1 + (\log q) \int_0^\infty \mu(S(t)) q^t dt \]

where \(S(t) = \{\xi \in \partial \Gamma : (\gamma, \xi) \geq t\}\). We claim that the following hold:

- \(S(t)\) is empty for \(t > |\gamma|\);
- \(\mu(S(t)) \ll q^{-t}\) for all \(t \geq 0\);
- there is a constant \(C > 0\), depending only on \(\Gamma\) and the choice of generating set, such that \(\mu(S(t)) \gg q^{-t}\) for all \(0 \leq t \leq |\gamma| - C\).

These facts immediately yield that

\[ \int_{\partial \Gamma} q(\gamma, \xi) d\xi \asymp |\gamma| + 1. \]

The first point is simply a rewriting of the fact that \((\gamma, \xi) \leq |\gamma|\) for all \(\xi \in \partial \Gamma\). For the second point, we note that diam \(S(t) \prec \exp(-t\varepsilon)\), since \((\xi, \xi') \geq \min\{(\gamma, \xi), (\gamma, \xi')\} - \delta \geq t - \delta\) whenever \(\xi, \xi' \in S(t)\). By Ahlfors-regularity, \(\mu(S(t)) \ll \exp(-t\varepsilon) = q^{-t}\). Finally, let us settle the third point. There is a constant \(C' > 0\) such that each element of \(\Gamma\) has distance at most \(C'\) to some geodesic ray based at the identity. Up to additive constants which can be absorbed into \(C'\), the distance from \(\gamma\) to a geodesic ray can be written as \(|\gamma| - (\gamma, \omega)\), where \(\omega \in \partial \Gamma\) is the boundary point corresponding to the ray. Thus \((\gamma, \omega) \geq |\gamma| - C'\). Now let \(t \leq |\gamma| - C' - \delta\). If \((\xi, \omega) \geq t + \delta\) then \((\gamma, \xi) \geq \min\{(\gamma, \omega), (\xi, \omega)\} - \delta \geq \min\{|\gamma| - C', t + \delta\} - \delta = t\). In other words, \(S(t)\) contains a ball of radius \(\varepsilon\) \(\exp(-\varepsilon t)\) around \(\omega\). Ahlfors-regularity, once again, yields \(\mu(S(t)) \gg \exp(-\varepsilon t)\) and

\[ \mu(S(t)) \gg \exp(-\varepsilon t) = q^{-t}. \]

The last ingredient is the asymptotical behaviour of spheres in \(\Gamma\).

**Lemma 3.** \(\#\{\gamma \in \Gamma : |\gamma| = k\} \asymp q^k\).

**Proof.** This is, essentially, a result from [4].
We now complete the proof of the theorem. In light of the last two lemmas, we have:

\[ \langle \pi(\sigma_k)1, 1 \rangle = \sum_{|\gamma|=k} \langle \pi(\gamma)1, 1 \rangle \approx \sum (k + 1)^{-k/2} \]

Therefore

\[ \|\lambda(a)\| \geq \langle \pi(a)1, 1 \rangle = \sum a_k \langle \pi(\sigma_k)1, 1 \rangle \approx \sum (k + 1) a_k \|\sigma_k\|_2 \]

as desired.

**Remark.** Instead of using the boundary amenability, which yields Lemma 1, we can also argue as follows. By Lemma 2 the spherical function \( \gamma \mapsto \langle \pi(\gamma)1, 1 \rangle \) is \( p \)-summable for every \( p > 2 \). Let \( \pi' \) be the restriction of \( \pi \) to the closed subspace spanned by \( \{\pi(\gamma)1 : \gamma \in \Gamma\} \). Now [5, Thm.1] says that \( \pi' \) is weakly contained in \( \lambda \), and the rest of the argument goes through with \( \pi' \) instead of \( \pi \).

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