Ferromagnetic behavior in the strongly interacting two-component Bose gas

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We investigate the low-temperature behavior of the integrable one-dimensional two-component spinor Bose gas using the thermodynamic Bethe ansatz. We find that for strong coupling the characteristics of the thermodynamics at low temperatures are quantitatively affected by the spin ferromagnetic states, which are described by an effective ferromagnetic Heisenberg chain. The free energy, specific heat, susceptibility, and local pair correlation function are calculated for various physical regimes in terms of temperature and interaction strength. These thermodynamic properties reveal spin effects which are significantly different than those of the spinless Bose gas. The zero-field susceptibility for finite strong repulsion exceeds that of a free spin paramagnet. The critical exponents of the specific heat $c_v \sim T^{1/2}$ and the susceptibility $\chi \sim T^{-2}$ are indicative of the ferromagnetic signature of the two-component spinor Bose gas. Our analytic results are consistent with general arguments by Eisenberg and Lieb for polarized spinor bosons.

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I. INTRODUCTION

Experiments with ultracold quantum gases are opening up exciting new possibilities for testing and exploring quantum effects in many-body systems (for recent reviews, see Refs. [1–3]). These include experiments on effectively one-dimensional (1D) quantum Bose gases of $^{87}$Rb atoms in which the interaction strength between atoms is tunable [4–8]. The experiments provide a striking example of realizing an integrable quantum many-body problem. They demonstrate the explicit fermionization of bosons and provide a direct test of theoretical results obtained for the integrable 1D interacting (spinless) Bose gas [9,10]. Another frontier of activity involves spinor Bose gases of alkali atoms in which hyperfine states comprise the pseudospins [11–13]. In these systems quantum collisional effects can produce spatiotemporal spin oscillations (spin waves) [14–17]. The observation of collective dynamics of spin waves and spin-state segregation in trapped spinor Bose gases has stimulated a wide range of interest in studying magnetism, topological spin defects, and novel quantum phase transitions in spinor Bose gases [18,19].

Two-component spinor Bose gases have been experimentally created in a magnetic trap by rotating two hyperfine states so that the two atomic hyperfine states make up a pseudospin doublet [20], e.g., the $|F=2, m_F=-1\rangle$ and $|F=1, m_F=1\rangle$ hyperfine states of $^{87}$Rb. In general, spin-independent $s$-wave scattering dominates interactions in alkali atomic gases. In 1D, the two-component Bose gas with spin-independent $s$-wave scattering can be exactly solved, as in the spinless model, by means of the Bethe ansatz [21,22]. In contrast to Fermi gases, ferromagnetic order emerges in spinor Bose gases as long as the interaction is fully spin independent [15,23]. The low-energy excitations of the model split into collective excitations carrying charge and collective excitations carrying spin. The charge excitations are phonons whereas the spin excitations have quadratic dispersion connected to spin-wave excitations [24,25]. Spin dynamics in the 1D ferromagnetic Bose gas have been studied recently [26]. Girardeau’s Fermi-Bose mapping has been used to study the 1D spinor Bose gases [27]. In general the two-component interacting Bose gas provides a tunable testing ground for observing the phenomenon of spin-charge separation [28].

Quantum gases with multispin states are expected to exhibit even richer quantum effects than their single component counterparts [1–3]. Universal features appearing in the low-temperature behavior of strongly interacting spinor Bose gases should differ significantly from those of spinless Bose gases and the antiferromagnetic behavior of Fermi gases due to their fundamentally different statistical signatures. One way to calculate the thermodynamics of integrable many-body systems is via the thermodynamic Bethe ansatz (TBA) [29–33], introduced by Yang and Yang [34] for the 1D Bose gas. However, it is a challenging problem to derive exact TBA results for the thermodynamics of 1D quantum many-body systems. Our aim here is to obtain universal characteristics of ferromagnetic behavior for the 1D two-component strongly interacting Bose gas of ultracold atoms via the TBA method. We will see that the ferromagnetic phase associated with the spin degrees of freedom may separate from the gas phase in the strongly repulsive regime due to spin-charge separation. The low-temperature behavior is dominated by the spin ferromagnetic states, which are described by an effective ferromagnetic Heisenberg chain. In this way we make contact with the known results for the thermodynamics of the ferromagnetic Heisenberg chain [35,36] to derive analytic expressions for the free energy, specific heat, susceptibility, and local pair correlation function for the strongly interacting two-component Bose gas in terms of temperature and interaction strength. These thermodynamic properties reveal some novel spin effects. Our explicit results are consistent with general arguments by Eisenberg and Lieb [23] for polarized spinor bosons.
This paper is set out as follows. In Sec. II we present the Bethe ansatz solution of the 1D two-component interacting Bose gas. The ground-state properties are also calculated. In Sec. III we introduce the TBA for the spinor Bose gas in order to study the thermodynamics at low temperatures, including the analysis of spin-charge separation. The ferromagnetic ground state is studied by means of the solution of the TBA equations in Sec. IV. We discuss low-temperature ferromagnetic behavior for the 1D strongly interacting Bose gas of atoms in Sec. V. The local pair correlation function is studied at low temperatures in Sec. VI. Section VII is devoted to a brief summary and concluding remarks.

II. THE TWO-COMPONENT SPINOR BOSE GAS

The Hamiltonian describing a δ-function interacting gas of $N$ bosons of mass $m$ constrained by periodic boundary conditions to a line of length $L$ with internal degrees of freedom is

$$
\mathcal{H} = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + g_{\text{1D}} \sum_{1 \leq i < j \leq N} \delta(x_i - x_j). \tag{1}
$$

For the ultracold atomic gases [37], the coupling constant $g_{\text{1D}}$ can be written in terms of the scattering strength $c = 2/a_{\text{1D}}$ as $g_{\text{1D}} = \hbar^2 c/m$. The effective 1D scattering length $a_{\text{1D}}$ can be related to the 3D scattering length for bosons or fermions confined in a 1D geometry. The dimensionless coupling constant $\gamma = c/n = mg_{1D} / (\hbar^2 n)$ is convenient for physical analysis. Here $n = N/L$ is the linear density. We take $2m = \hbar = 1$ for simplicity in the following equations. However, we restate them where appropriate in discussing the thermodynamics of the model. The wave functions of Hamiltonian (1) for the spinor Bose gas are symmetric under exchange of spatial and internal spin coordinates between two particles. We shall see that this statistical signature triggers rather novel ferromagnetic behavior in the degenerate quantum spinor Bose gas. The interaction is attractive for $g_{\text{1D}} < 0$ and repulsive for $g_{\text{1D}} > 0$. However, one should note that there is no thermodynamic limit for the attractive case in Bose gases.

For $M$ spin-down bosons, the Bethe ansatz equations (BAEs) for the two-component Bose gas are of the form [21,22]

$$
\exp(i k_j L) = \prod_{\ell=1}^{N} \frac{k_j - k_\ell + i c}{k_j - k_\ell - i c} \prod_{\alpha=1}^{M} \frac{\lambda_\alpha - k_j - 1/2 i c}{\lambda_\alpha - k_j + 1/2 i c}, \tag{2}
$$

$$j = 1, \ldots, N,$n

$$
\prod_{\ell=1}^{N} \frac{\lambda_\alpha - k_\ell - 1/2 i c}{\lambda_\alpha - k_\ell + 1/2 i c} = - \prod_{\rho=1}^{M} \frac{\lambda_\alpha - \lambda_\rho - 1/2 i c}{\lambda_\alpha - \lambda_\rho + 1/2 i c}, \tag{3}
$$

$$\alpha = 1, \ldots, M$$

in terms of which the energy eigenspectrum is given by $E = \Sigma_{j=1}^{N} k_j^2$. In the thermodynamic limit, i.e., $N, L \to \infty$ with $N/L$ finite, these equations can be written as coupled integral equations in terms of the particle and hole root densities $\rho(k)$ and $\rho^h(k)$ [$\sigma(\lambda)$ and $\sigma^h(\lambda)$] for the charge (spin) degrees of freedom, respectively. These are

$$
\rho(k) + \rho^h(k) = \frac{1}{2\pi} + \frac{1}{2\pi} \int_{-Q}^{Q} \frac{2c \rho(k')}{c^2 + (k-k')^2} dk' + \frac{1}{2\pi} \int_{-B}^{B} \frac{2c \sigma(\lambda)}{c^2 + (k-k')^2} d\lambda, \tag{4}
$$

$$\sigma(\lambda) + \sigma^h(\lambda) = \frac{1}{2\pi} \int_{-Q}^{Q} \frac{c \rho(k)}{c^2 + (\lambda-k)^2} dk + \frac{1}{2\pi} \int_{-B}^{B} \frac{2c \sigma(\lambda)}{c^2 + (\lambda-k)^2} d\lambda'. \tag{5}
$$

The integration limits $Q$ and $B$ are determined by $N/L = \int_{-Q}^{Q} \rho(k) dk$ and $M/L = \int_{-B}^{B} \sigma(\lambda) d\lambda$. At zero temperature, the ground state corresponds to the configuration $\sigma(\lambda) = \rho^h(\lambda) = 0$ leading to a ferromagnetic ground state. For the ground state there are therefore no holes in the charge degrees of freedom and no quasiparticles in the spin degrees of freedom. However, as the temperature increases spin strings become involved in the thermal equilibrium states. We shall investigate this ground-state configuration via analysis of the string solutions to the TBA.

III. THE THERMODYNAMIC BETHE ANSATZ

For finite temperatures each of the $N$ quasimomenta $k_j$ are real due to the repulsive interaction. However, the spin quasimomenta form complex strings of the form $[38] \lambda_\alpha = \Lambda_\alpha + i(n+1-2j)c/2$ for $j = 1, \ldots, N$. Here the number of strings $\alpha = 1, \ldots, N$, $\Lambda_\alpha$ on the real axis denotes the position of the center of a length-$n$ string. The number of $n$-strings $N_n$ satisfies the relation $M = \Sigma_{n=m}^{N_n} n$. It is assumed that the distribution of Bethe roots along the real axis is dense enough to pass to the continuum limit. After performing a standard calculation with the string solutions and introducing the convolution integral $(f \ast g)(\lambda) = \int_{-\infty}^{\infty} f(\lambda - \lambda') g(\lambda') d\lambda'$, the BAE (2) become

$$
\rho(k) + \rho^h(k) = \frac{1}{2\pi} + \frac{1}{2\pi} \int_{-Q}^{Q} \frac{2c \rho(k')}{c^2 + (k-k')^2} dk' - \sum_{n=1}^{\infty} a_n \ast \sigma(\lambda), \tag{6}
$$

$$\sigma(\lambda) + \sigma^h(\lambda) = \frac{1}{2\pi} \int_{-Q}^{Q} \frac{c \rho(k)}{c^2 + (\lambda-k)^2} dk - \sum_{n=1}^{\infty} T_{nm} \ast \sigma_m(\lambda), \tag{7}
$$

where

$$
\sigma_{n}(\lambda) + \sigma_{n}(\lambda) = a_n \ast \rho(\lambda) - \sum_{m=1}^{\infty} T_{nm} \ast \sigma_m(\lambda), \tag{8}
$$

$$
\rho(k) + \rho^h(k) = \frac{1}{2\pi} + \frac{1}{2\pi} \int_{-Q}^{Q} \frac{2c \rho(k')}{c^2 + (k-k')^2} dk' - \sum_{n=1}^{\infty} a_n \ast \sigma(\lambda), \tag{9}
$$

$$\sigma(\lambda) + \sigma^h(\lambda) = \frac{1}{2\pi} \int_{-Q}^{Q} \frac{c \rho(k)}{c^2 + (\lambda-k)^2} dk - \sum_{n=1}^{\infty} T_{nm} \ast \sigma_m(\lambda), \tag{10}
$$

\begin{align*}
\sigma_{n}(\lambda) + \sigma_{n}(\lambda) &= a_n \ast \rho(\lambda) - \sum_{m=1}^{\infty} T_{nm} \ast \sigma_m(\lambda), \tag{11}
\end{align*}

The integration limits $Q$ and $B$ are determined by $N/L = \int_{-Q}^{Q} \rho(k) dk$ and $M/L = \int_{-B}^{B} \sigma(\lambda) d\lambda$. At zero temperature, the ground state corresponds to the configuration $\sigma(\lambda) = \rho^h(\lambda) = 0$ leading to a ferromagnetic ground state. For the ground state there are therefore no holes in the charge degrees of freedom and no quasiparticles in the spin degrees of freedom. However, as the temperature increases spin strings become involved in the thermal equilibrium states. We shall investigate this ground-state configuration via analysis of the string solutions to the TBA.
\[ T_{nm}(\lambda) = \begin{cases} a_{|n-m|}(\lambda) + 2a_{|n-m|/2}(\lambda) + \cdots + 2a_{n+m-2}(\lambda) + a_{n+m}(\lambda), & \text{for } n \neq m, \\ 2a_{2}(\lambda) + 2a_{4}(\lambda) + \cdots + 2a_{2n-2}(\lambda) + a_{2n}(\lambda), & \text{for } n = m, \end{cases} \]  

and \[ a_{\mu}(\lambda) = \frac{1}{2\pi} \frac{nc}{(nc/2)^2 + \lambda^2}. \]  

The equilibrium states at finite temperature \( T \) are described by the equilibrium particle and hole densities \( \rho(k) \) and \( \rho^h(k) \) of the charge degrees of freedom and the equilibrium string densities \( \sigma_{\mu}(\lambda) \) and \( \sigma_{\mu}^h(\lambda) \) of the spin degrees of freedom. Here \( n = 1, 2, \ldots, \infty \). The partition function \( Z = \text{tr}(e^{-H/T}) \) is defined by

\[ Z = \sum_{\rho, \rho^h, \sigma_{\mu}, \sigma_{\mu}^h} W(\rho, \rho^h, \sigma_{\mu}, \sigma_{\mu}^h)e^{-E(\rho, \rho^h, \sigma_{\mu}, \sigma_{\mu}^h)/T}, \]  

where the densities satisfy (4) with \( W(\rho, \rho^h, \sigma_{\mu}, \sigma_{\mu}^h) \) the number of states corresponding to the given densities. Introducing the combinatorial entropy \( S = \ln W(\rho, \rho^h, \sigma_{\mu}, \sigma_{\mu}^h) \), the grand partition function is \( Z = e^{-GT} \), where the Gibbs free energy \( G = E - \mu N - H(\mathcal{N}_1 - \mathcal{N}_i)/2 - TS \). Here \( \mu \) is the chemical potential and \( \mathcal{N}_i (\mathcal{N}_1) \) denotes the number of the particles with up (down) spin. Recall that \( \mathcal{N}_1 = M \). The energy per unit length is defined by

\[ E/L = \int_{-\infty}^{\infty} k^2 \rho(k)dk - m^2 H. \]  

Here \( H \) is the external magnetic field and \( m^2 = (\mathcal{N}_1 - \mathcal{N}_i)/2 \) denotes the atomic magnetic momentum (where the Bohr magneton \( \mu_B \) and the Landé factors are absorbed into the magnetic field \( H \)). The magnetization per unit length in the \( z \) direction is thus given by

\[ m^\uparrow = \frac{1}{2} \int_{-\infty}^{\infty} \rho(k)dk - \sum_n n \int_{-\infty}^{\infty} \sigma_{\mu}(\lambda)d\lambda. \]  

Now the equilibrium states are determined by the minimization condition of the Gibbs free energy \( [30,34] \), i.e., the condition \( \delta(E - \mu n - TS) = 0 \), which gives rise to the coupled nonlinear integral equations (the TBA equations) \( [39] \)

\[ e(k) = k^2 - \mu - \frac{1}{2} H - \frac{T}{2\pi} \int_{-\infty}^{\infty} \frac{2c}{c^2 + (k-k')^2} \ln(1 + e^{-\epsilon(k')/T}) \]
\[ - T \sum_{n=1}^{\infty} a_n(\lambda - \lambda') \ln[1 + \eta_n^{-1}(\lambda')]. \]  

In the above equations, we have only kept terms to order \( 1/c \) for the dressed energy \( e(k) \). We have also made the variable changes \( \lambda \rightarrow c\lambda/2 \) and \( \eta(\lambda) \rightarrow \eta(c\lambda/2) \). Thus the function \( a_n(\lambda) \) defined in (6) becomes \( a_n(\lambda) = \frac{a_n(\lambda)}{n! n^2} \), which is Takahashi’s notation for the ferromagnetic Heisenberg chain [30]. It will be clearly seen from the TBA Eqs. (13) that the low-temperature behavior of the strongly interacting two-component boson model is determined by the hardcore Bose gas state and ferromagnetic spin-wave fluctuations described by the ferromagnetic Heisenberg chain with an effective coupling strength \( J = 2P(T, H)/c > 0 \). We shall see through the TBA Eqs. (13) that the temperature-dependent part of this
We can find a relation between the pressure $P(T,H)$ and chemical potential $\mu$ by iteration. This provides a starting point to calculate the thermodynamics of the strongly interacting two-component Bose gas at low temperatures, including the zero-temperature limit.

IV. THE GROUND STATE AND THE SPINLESS BOSE GAS

Pure dynamical interaction drives the spinless Bose gas into distinct quantum phases of matter: from the quasi-Bose-Einstein condensate to the Tonks-Girardeau phase. As stated in the Introduction, this elegantly simple Bethe ansatz solved 1D quantum many-body system [9,10] is testable in experiments on trapped quantum gases of ultracold atoms [4–7]. It is natural to expect that spin dynamics in the interacting two-component spinor Bose gases would lead to significantly different quantum effects than those of the spinless Bose gas. At $T \to 0$, it is suitable to use the dressed energy formalism in the TBA Eqs. (16), i.e.,

$$
\xi_1(\lambda) = 2\pi f s(\lambda) + s^* \xi_2^{\ast}(\lambda), \quad \xi_n(\lambda) = s^* [\xi_{n-1}(\lambda) + \xi_{n+1}(\lambda)],
$$

where $\lim_{n \to \infty} \xi_n/n = H$. Here the dressed energy is defined as $\xi_n(\lambda) = T \ln \eta_n(\lambda)$, with $\xi_n(\lambda)$ [$\xi_n^{\ast}(\lambda)$] denoting the dressed energy for $\xi_n(\lambda) \geq 0$ [$\xi_n(\lambda) < 0$]. The free energy is now

$$
f_{XXX}(T,H) = \frac{1}{2}H + \sum_{n=1}^{\infty} a_n(\lambda) \xi_n(\lambda) d\lambda = J \ln 2 - \int_{-\infty}^{\infty} d\lambda s(\lambda) \xi_1^{\ast}(\lambda).
$$

For the fermionic case $J > 0$, the solution of the TBA (19) is given by [30]

$$
\xi_n(\lambda) = \xi_n^{\ast}(\lambda) = 2\pi J a_n(\lambda) + Hn,
$$

where $n = 1, 2, \ldots, \infty$. Here we see that for $T = 0$, $f_{XXX} = -H/2$. Therefore it follows that the fully polarized state forms a ferromagnetic ground state. In this way the TBA gives rise to a direct proof of the existence of the ferromagnetic ground state.

In order to understand spin effects in the spinor Bose gas, we first discuss the low-temperature behavior of the spinless Bose gas. It was for this model that Yang and Yang introduced the TBA formalism, with the result

$$
e(k) = e^0(k) - \frac{T}{2\pi} \left[ \int_{-\infty}^{\infty} dk' \frac{2c}{c^2 + (k-k')^2} \ln(1 + e^{-e(k')/T}) \right],
$$

which is the special case of the TBA Eq. (10) for the spinless Bose gas. At zero temperature, one can obtain physical quantities, such as the ground-state energy per unit length $E_0$, chemical potential $\mu$, pressure $P_0$, and the cutoff momentum $Q$. In the strong-coupling limit the results for these quantities are

$$
E_0 \approx \frac{1}{3\pi^2} \left[ 1 - \frac{4}{\gamma} \right], \quad \mu_0 = n^2 \pi^2 \left[ 1 - \frac{16}{3\gamma} \right].
$$
The macroscopic velocity is

\[ v_c = \sqrt{2 \frac{\partial P_0}{\partial n}} = \frac{\hbar \pi n}{m \pi} \left(1 - \frac{4}{\gamma}\right). \]

(24)

The low-temperature thermodynamics can also be calculated directly from the pressure (18). For the spinless case, the function \( A(T, 0) = \mu (1 + 2^{\mu(T, 0)}). \) Substituting \( A(T, 0) \) into Eq. (18) and using the relation \( \partial P(T, 0)/\partial \mu = n \) gives the chemical potential

\[ \mu = \mu_0 \left[ 1 + \frac{\pi^2}{12} \left(1 - \frac{16}{3\gamma}\right) \left(\frac{K_B T}{\mu_0}\right)^2 + \frac{\pi^4}{36} \left(1 - \frac{32}{5\gamma}\right) \left(\frac{K_B T}{\mu_0}\right)^4 \right]. \]

(25)

Here \( \mu_0 = n^2 \pi^2 (1 - \frac{16}{3\gamma}) \) coincides with the result given in Eq. (23). In terms of the degenerate temperature \( \tau = K_B T / T_d \), the free energy per unit length follows from relation (12) as

\[ F(\tau) = E_0 \left[ 1 - \frac{\tau^2}{4 \pi^2} \left(1 + \frac{8}{\gamma}\right) - \frac{\tau^4}{60 \pi^4} \left(1 + \frac{16}{\gamma}\right) \right]. \]

(26)

The ground-state energy \( E_0 \) agrees with the result given in Eq. (23). The expression (18) provides a simple way to derive the thermodynamics. In addition the results (25) and (26) obtained for the spinless Bose gas via the TBA are in good agreement with results derived from generalized exclusion statistics [41]. The results (25) and (26) characterize the low-temperature behavior of the 1D spinless Bose gas induced by the dynamical interaction in the strong-coupling regime. The specific heat \( c_v \) and the entropy \( S \) follow from the free energy (26) as

\[ c_v = -T L \frac{\partial^2 F(T, 0)}{\partial T^2} = \frac{N K_B \tau}{6 \left(1 - \frac{4}{\gamma}\right)} + \frac{N K_B \tau^3}{15 \pi^2 \left(1 - \frac{12}{\gamma}\right)}, \]

(27)

\[ S = -T L \frac{\partial F(T, 0)}{\partial T} = \frac{N K_B \tau}{6 \left(1 - \frac{4}{\gamma}\right)} + \frac{N K_B \tau^3}{45 \pi^2 \left(1 - \frac{12}{\gamma}\right)}, \]

(28)

which coincide with the results given in Refs. [41,44]. Here the degenerate temperature \( \tau = K_B T / T_d \), with \( T_d = \frac{\hbar n^2}{2 m \pi} \).

The total energy per unit length follows from the relation \( E(T, 0) = F(T, 0) + ST \), with the result

\[ E(\tau) = E_0 \left[ 1 + \frac{\tau^2}{4 \pi^2} \left(1 + \frac{8}{\gamma}\right) + \frac{\tau^4}{20 \pi^4} \left(1 + \frac{16}{\gamma}\right) \right]. \]

(29)

The ground-state energy \( E_0 \) is as given in Eq. (23). From Eq. (26) we see that

\[ F(T) = F(0) - \frac{\pi C (K_B T)^2}{6 \hbar v_c} + O(T^2) \]

(30)
as expected from conformal field theory arguments for a critical system, i.e., for a system with massless excitations [42] Here the central charge \( C = 1 \) and \( v_c \) is given by Eq. (24). Similarly the finite-size corrections [43] are given by

\[ E(L, N) - L e_{\infty} = -\frac{\hbar \pi C v_c}{6 L} + O(1/L^2). \]

(31)

Here \( E(L, N) \) is the finite size ground state energy and \( e_{\infty} \) is the energy per unit length in the thermodynamic limit.

Furthermore, at low temperatures the strongly interacting spinless 1D Bose gas can be viewed as a system of ideal particles obeying nonmutual generalized exclusion statistics (GES) with statistics parameter \( \alpha = 1 - 2/\gamma \). These particles obey GES interpolating between bosons and fermions [46–48]. For 1D interacting many-body systems the pairwise dynamical interaction between identical particles is inextricably related to their statistical interaction through scattering. GES is thus the result of collective behavior exhibited in 1D quantum many body systems. From the GES approach, the free energy and the total energy per unit length are given by [41]

\[ F(\tau) = E_0 \left[ 1 - \frac{\tau^2}{4 \pi^2} \left(1 + \frac{2}{\gamma}\right) + \frac{3 \xi(3) \tau^3}{2 \gamma \pi^6} \left(1 + \frac{4}{\gamma}\right) - \frac{\tau^4}{60 \pi^4} \left(1 + \frac{4}{\gamma}\right) \right], \]

(32)

\[ E(\tau) = E_0 \left[ 1 + \frac{\tau^2}{4 \pi^2} \left(1 + \frac{2}{\gamma}\right) - \frac{3 \xi(3) \tau^3}{2 \gamma \pi^6} \left(1 + \frac{4}{\gamma}\right) + \frac{\tau^4}{20 \pi^4} \left(1 + \frac{4}{\gamma}\right) \right]. \]

(33)

Here \( \xi(3) = \Sigma_{n=1}^{\infty} n^{-3} \).

V. LOW-TEMPERATURE FERROMAGNETIC BEHAVIOR

The low-temperature behavior of the spinor Bose gas, triggered by the ferromagnetic spin-spin interaction, is intimately related to the thermodynamic behavior of the ferromagnetic Heisenberg chain, which has been extensively studied via various methods, e.g., numerics [49,50], spin-wave theory [36], and the TBA approach with extrapolation [51]. Although there has been a wide range of interest in the ferromagnetic Heisenberg chain, realization of ferromagnet chains are relatively rare [52]. Most recent interest in the ferromagnetic Heisenberg chain has been from the perspective of string theory [53,54]. Obtaining exact analytical results for the thermodynamics of this model still provides a number of open challenges. For one, it is extremely hard to solve the infinitely many Eqs. (16) involved in the TBA. Nevertheless, Takahashi and his co-workers [36,51] have given some results for the free energy and susceptibility which are generally accepted. Schlottmann [35] has also predicted the leading order of the specific heat and zero-field susceptibility via analysis of the string solutions to the TBA Eqs. (16). With the help of these known results for the ferromagnetic Heisenberg chain, we show here that, in the strong coupling regime, the ferromagnetic state induced by the internal spin-spin interaction significantly affects the low-temperature behavior of the two-component spinor Bose gas.
FIG. 1. (Color online) Magnetization (normalized by the linear density \( n \)) vs magnetic field \( H \) (with Bohr magneton \( \mu_B=1 \)) for different temperatures (in units of \( K_B \)). In the strong coupling limit \( \gamma \to \infty \) two-component spinor bosons exhibit free spin behavior at low temperatures.

**A. Paramagnet: \( \gamma \to \infty \)**

We first consider the extreme case \( \gamma \to \infty \) or, say, \( \gamma \gg 1/K_B T \). In this case the driving term in the TBA Eqs. (16) vanishes as \( \gamma \to \infty \). Thus the string solutions are given by [30]

\[
\eta_n(\lambda) \approx \left[ \frac{\sinh \left( \frac{(n+1)\lambda}{2} \right)}{\sinh \left( \frac{\lambda}{2} \right)} \right]^2 - 1 \tag{34}
\]

which are known as free spin solutions. In this case, the particles with down spins are unable to exchange their positions with the particles with up spins. The spins are thus frozen locally and the spin-spin exchange interaction vanishes.

In this case, the statistical interaction is completely suppressed due to the strong repulsion. In this sense, for \( \gamma \to \infty \) both the spinor Bose gas and the Fermi gas behave similar to a free spin paramagnet. In this case, the spins are very sensitive to external magnetic fields, with a small field able to polarize all atoms. From Eq. (14) we obtain

\[
\epsilon(k) = \frac{\hbar^2 k^2}{2m} - \mu - K_B T \ln \left( 2 \cosh \frac{H}{2K_B T} \right). \tag{35}
\]

Under the condition \( \gamma \gg 1/K_B T \) and \( K_B T < T_d \), the free energy (12) gives

\[
F(\tau, h) = \frac{\hbar^2}{32m} \pi^2 n^3 \left[ 1 - \frac{\tau^2}{4 \pi} - \frac{\tau^4}{60 \pi^2} - \frac{3 \tau^6}{4 \pi^3} \ln \left( 2 \cosh \frac{h}{2 \tau} \right) \right], \tag{36}
\]

where we have set \( h = H/T_d \).

It follows that in the strong-coupling limit \( \gamma \to \infty \) the magnetic properties of the two-component spinor Bose gas are those of su(2) free spins with a divergent susceptibility \( \chi = \frac{n}{4K_B T} [1 - \tanh^2 (H/(2K_B T))] \) at low temperatures. The magnetization is \( m = \frac{\hbar}{2} \tan h[H/(2K_B T)] \). Figure 1 shows the free spin behavior in the magnetization as a function of the magnetic field at different temperatures.

The specific heat and entropy follow from Eq. (36) as

\[
c_v = \frac{1}{6} N K_B \tau + \frac{N K_B \tau^3}{15 \pi^2} + \frac{N K_B h^2}{4 \pi^2} \left[ 1 - \tanh^2 \left( \frac{h}{2 \tau} \right) \right], \tag{37}
\]

\[
S = \frac{1}{6} N K_B \tau + \frac{N K_B \tau^3}{45 \pi^2} + N K_B \ln \left[ 2 \cosh \left( \frac{h}{2 \tau} \right) \right] - \frac{N K_B h}{2 \tau} \tanh \left( \frac{h}{2 \tau} \right). \tag{38}
\]

In the absence of external field (\( H=0 \)) the specific heat behaves similar to that of a free Fermi gas. However, the free spins make a contribution \( N K_B \ln 2 \) to the free Fermi entropy. Figure 2 shows a plot of the specific heat in the presence of magnetic field. We see clearly here in the strong-coupling limit that the specific heat is sensitive to the external field. We also note that our calculations differ from those in Refs [22,39] where the spin degrees of freedom were ignored.

**B. Ferromagnetic: \( 1 \ll \gamma \ll 1/K_B T \)**

For finite temperatures the solutions of the TBA Eqs. (16) vary from the string solutions (21) to the free spin solution (34) [30,35]. The spin-spin exchange interactions are enhanced as the interaction strength \( \gamma \) decreases from the strong coupling limit. In this section we explore the ferromagnetic behavior of the spinor Bose gas with finitely strong interaction, i.e., in the regime \( 1 \ll \gamma \ll 1/K_B T \) or, more precisely speaking, \( \frac{\tau}{2P(t,0)} < 1/K_B T \). In this regime, the known results for the free energy of the ferromagnetic Heisenberg chain at low temperatures obtained by Takahashi and colleagues are applicable to the ferromagnetic state associated with the free energy (15). The result is [36,51]
The degenerate temperature to the free energy of the spinor Bose gas. Schlottmann's method we find from the TBA Eqs. mating the infinite set of TBA Eqs. for the ferromagnetic Heisenberg chain by approximating the energies for the spinless Bose gas. For further comparison the symbols show the energies (32) and (33) derived for the spinless Bose gas from an ideal gas of particles obeying nonmutual GES [41]. In general the curves reveal the universal characteristics of the energies at low temperatures.

\[ f_{XXX}(T,0) = J \left[ -1.042 \left( \frac{K_B T}{J} \right)^{3/2} + \left( \frac{K_B T}{J} \right)^2 - 0.9 \left( \frac{K_B T}{J} \right)^{5/2} \right] \]

with effective coupling strength \( J = 2P(T,0)/c \). At low temperature, the effective coupling approaches the constant value \( J = 4E_F/3\gamma \) as a result of the temperature-dependent part in the coupling only making an \( O[(K_B T)^3] \) contribution to the free energy of the spinor Bose gas. Schlottmann [35] calculated the leading terms of the specific heat and free energy for the ferromagnetic Heisenberg chain by approximating the infinite set of TBA Eqs. (16). Using Schlottmann’s method we find from the TBA Eqs. (16) that the leading order in the free energy is proportional to \( (K_B T)^{3/2} \).

With the help of Takahashi’s result for the ferromagnetic Heisenberg chain (39), we may calculate the thermodynamics of the spinor Bose gas within finitely strong interaction, which we now do.

Substituting \( A(T,0) = \mu + 2P(T,0)/c - f_{XXX}(T,0) \) into Eq. (18) we have

\[ P(T,0) = \frac{1}{\sqrt{\pi n^2 \nu^2}} \left[ \frac{2\hat{\mu}}{3} \left[ 1 + 2\frac{c}{\sqrt{\frac{K_B T}{c}}} + \frac{\hat{\mu}^2}{8} \left( \frac{K_B T}{\hat{\mu}} \right)^2 \right] \right] \]

where \( \hat{\mu} = \mu - f_{XXX}(T,0) \). For the regime \( \frac{c}{2P(T,0)} < 1/K_B T \) calculation of the chemical potential via the relation \( \partial P(T,0)/\partial \mu = n \) gives

\[ \mu = \mu_0 \left\{ 1 + \mu_1 \left( \frac{\gamma K_B T}{\mu_0} \right)^{3/2} + \mu_2 \left( \frac{\gamma K_B T}{\mu_0} \right)^2 + \mu_3 \left( \frac{\gamma K_B T}{\mu_0} \right)^{5/2} + \frac{\pi^2}{12} \left( 1 - \frac{16}{3\gamma} \right) \left( \frac{K_B T}{\mu_0} \right)^2 \right\} \]

where

\[ \mu_1 = \frac{1.042 \times \sqrt{3}}{4\gamma} \left( 1 - \frac{17}{3\gamma} \right) \]

\[ \mu_2 = -\frac{3}{2\gamma} \left( 1 - \frac{7}{3\gamma} \right) \]

\[ \mu_3 = \frac{0.9 \times 21 \sqrt{3}}{16\gamma} \left( 1 - \frac{11}{7\gamma} \right) \]

Some algebra gives the free energy \( F(\tau,0) \) and total energy \( E(\tau,0) \) per unit length

\[ F(\tau,0) = E_0 \left[ 1 - \frac{1.042 \times 3\sqrt{3}}{2\gamma n^2} \left( 1 + \frac{7}{\gamma} \right) (\gamma \tau)^{3/2} \right] \]

\[ + \frac{9}{4\gamma n^2} \left( 1 + \frac{10}{\gamma} \right) (\gamma \tau)^{5/2} - \frac{0.9 \times 9 \sqrt{3}}{8\gamma n^5} \left( 1 + \frac{13}{\gamma} \right) \]

\[ \times (\gamma \tau)^{3/2} - \frac{\tau^2}{4\pi^2} \left( 1 + \frac{8}{\gamma} \right) \]

\[ , \]

\[ \]
We see explicitly that the ferromagnetic behavior of the spin exchange interaction dominates the thermodynamics of the strongly interacting two-component Bose gas.

It is of interest to note that the leading temperature-dependent term in the energy expressions is $O(T^{3/2})$ for the spinor Bose gas compared to $O(T^5)$ for the spinless Bose gas. In the strong interaction regime $1 \ll \gamma < \mu_0/K_BT$ this leads to quantitatively different low-temperature behavior compared to the spinless Bose gas. Figure 3 shows the total energy and the free energy in the strong interaction regime as a function of temperature for the spinor Bose gas, the spinless Bose gas and an ideal gas obeying GES. Significantly different characteristics of low-temperature behavior for the spinor Bose gas and the spinless Bose gas are depicted. For the spinless Bose gas with strong coupling, the low-temperature thermodynamics is known to coincide with that of ideal particles obeying nonmutual GES with statistics parameter $\alpha=1-2/\gamma$. Figure 3 hints that the strongly interacting spinor Bose gas might also be equivalent to a gas of ideal particles obeying nonmutual GES. It is an open question as to what the statistics parameter $\alpha$ for the spinor Bose gas might be.

To conclude this section the specific heat and entropy are given by

\[ E(\gamma,0) = E_0 \left[ 1 + \frac{1.042 \times 3\sqrt{3}}{4\pi^3} \left( 1 + \frac{7}{\gamma} \right) (\gamma\tau)^{3/2} - \frac{9}{4\pi^4} \left( 1 + \frac{10}{\gamma} \right) (\gamma\tau)^2 + \frac{0.9 \times 27\sqrt{3}}{10\gamma^3} \left( 1 + \frac{13}{\gamma} \right) \right] \times \left( \gamma\tau \right)^{5/2} + \frac{\tau^2}{4\pi^4} \left[ 1 + \frac{8}{\gamma} \right]. \]  

\[ S = \frac{1.042 \times 3\sqrt{3}(\gamma\tau)^{1/2}}{4(1-\frac{3}{\gamma})\pi} - \frac{3(\gamma\tau)}{2(1-\frac{3}{\gamma})\pi^2} + \frac{0.9 \times 15\sqrt{3}(\gamma\tau)^{1/2}}{16(1-\frac{3}{\gamma})\pi^3} + \frac{\tau}{6(1-\frac{3}{\gamma})}, \]  

which differ significantly from the corresponding spinless Bose gas results (27) and (28) and also from free spin case (37) and (38). Figure 4 shows the specific heat as a function of the temperature. The specific heat exponent $c_v \approx T^{\alpha}$ indicates that $a=-0.5$ for spinor Bose gas for the regime $1 \ll \gamma < \mu_0/K_BT$ whereas $a=-1$ for the Lieb-Lininger gas. For $\gamma \gg 1/K_BT$, the spinor Bose gas and the Lieb-Lininger gas both have $a=-1$ for the absence of the external field.

C. Susceptibility: $1 \ll \gamma < 1/K_BT$

Now we consider the effect of a small external field $(H \ll T)$ within the regime $1 \ll \gamma < 1/K_BT$. Here we adapt the known free energy result for the ferromagnetic chain in the presence of an external field [36,51], namely,

\[ f_{XXX}(T,H) = f_{XXX}(T,0) - \frac{H^2J}{8(K_BT)^2} \left[ \frac{1}{6} + 0.5826 \left( \frac{K_BT}{J} \right)^{1/2} + 0.678 \left( \frac{K_BT}{J} \right) \right], \]  

with $f_{XXX}(T,0)$ as given in Eq. (39). Repeating the procedure of the previous section with the free energy (46) gives the result.
function of the degenerate temperature $\tau$ for different values of interaction strength $\gamma$. In the strong-coupling limit $\gamma \to \infty$ the two-component spinor bosons behave similar to a paramagnet with susceptibility $\chi \sim T^{-1}$. For finitely strong interaction, the ferromagnetic susceptibility behaves as $\chi \sim T^{-2}$.

$F(\tau, h) = F(\tau, 0) - \frac{E_g h^2}{\tau^2} \left( 1 - \frac{2}{\gamma} \right) n \left( T_{cd} \right) \int \frac{d\gamma}{2\pi i} \log(1 - \gamma e^{-i\rho \gamma}) + 0.5826 \times \frac{\sqrt{3}}{4 \gamma \pi} (1 - \frac{1}{\gamma})(\gamma \tau)^{1/2} + 0.678 \times \frac{3}{8 \gamma \pi^2} \left( 1 - \frac{4}{\gamma}(\gamma \tau) \right)$

with $F(\tau, 0)$ as given in Eq. (42).

For small magnetic field ($h < \tau$) the susceptibility per unit length

$\chi = -\frac{\partial^2 F(T, H)}{\partial H^2} = \frac{\pi^2}{T_{cd}} \left( 1 - \frac{6}{\gamma} \right) + \frac{0.5826 \times \sqrt{3} \pi}{6 \gamma \tau^{3/2}} \times \left( 1 - \frac{3}{\gamma} \right) + \frac{0.678}{4 \tau} \left( \frac{3}{\gamma} \right)^{1/2}$

is indeed greater than that of free spins, for which $\chi = n/(2K_B T)$. This result is also consistent with Eisenberg and Lieb’s general argument for polarized spinor bosons [23]. Figure 5 shows the zero-field susceptibility as a function of temperature for different values of the interaction strength. The susceptibility diverges as $\tau \to 0$. The susceptibility decreases with increasing interaction toward the free spin paramagnetic susceptibility is the lowest curve (solid line). The susceptibility exponent defined by $\chi \sim T^{-b}$ is $b=2$ for the regime $\gamma < \mu_0/K_B T$ with $b=1$ for the paramagnet.

VI. LOCAL PAIR CORRELATION

The local pair correlation function for the 1D Bose gas has been determined experimentally in a gas $^{87}$Rb atoms as a function of the interaction strength by measuring photoassociation rates [55]. In general local two-particle correlations can be used to study phase coherence behavior and classify various finite-temperature regimes in 1D interacting quantum gases [56,57]. In the grand canonical description, two-particle pair correlation are given in terms of the field operator $\psi$ and the free energy $f(\gamma, T)$ by [29,56]

$g^{(2)}(0) = \langle \psi^\dagger \psi \rangle = \frac{2m}{\hbar^2 n} \frac{\partial f(\gamma, T)}{\partial \gamma} |_{n, T}$

(49)

At zero temperature the local pair correlation is $g^{(2)} = 1$ for the weakly interacting Bose gas. On the other hand, $g^{(2)} \to 0$ as $\gamma$ increases into the Tonks-Girardeau regime, indicative of free fermionic behavior. In this regime, the long-range behavior is characterized by the one-body correlation function $g^{(1)}(x) = \langle \psi^\dagger(x) \psi(0) \rangle \sim \gamma^2$, corresponding to the momentum distribution $n(\rho) \approx 1/\sqrt{\gamma}$. Here $p$ is the momentum. More generally, in terms of the Luttinger parameter $K$ [58], $g^{(1)}(x) \approx 1/\sqrt{2K}$ and $n(\rho) \approx 1/p^{(1/2K)}$. In the weak-coupling limit $K = \pi f/\sqrt{\gamma}$, which leads to a power-law decay in the one point correlation.

The pair correlation function for the spinless Bose gas

$g^{(2)}(0) = \frac{4\pi^2}{3 \gamma^2} \left( 1 + \frac{\gamma^2}{4\pi^2} + \frac{\gamma^4}{20\gamma^2} \right)$

(50)

follows from the TBA result (26), which coincides with the result given in Refs. [56,57]. It is evident that the dynamical interaction dramatically reduces pair correlation due to decoherence between individual wave functions of colliding particles. On the other hand, increasing temperature slowly enhances local pair correlation. At temperatures $\tau \ll 1$ ($T \ll T_{cd}$), the local pair correlation approaches free Fermi behavior as $\gamma \to \infty$, as was quantitatively demonstrated in the experimental observations of the pair correlation function for a gas of interacting $^{87}$Rb atoms confined to 1D [55].

For the two-component spinor Bose gas considered here, the ferromagnetic spin-spin exchange interaction results in a different temperature-dependent pair correlation function. For this model, the local pair correlation function

$g^{(2)}(0) \approx \frac{4\pi^2}{3 \gamma^2} \left[ 1 - \frac{1.042 \times 3 \sqrt{3} (\gamma \tau)^{3/2}}{16\pi^3} \left( 1 - \frac{3}{\gamma} \right) + \frac{9(\gamma \tau)^2}{16\pi^3} \right]$

$- \frac{0.9 \times 2(\gamma \tau)^{3/2}}{64\pi^5} 
\left[ 1 + \frac{3}{\gamma} + \frac{\gamma^2}{4\pi^2} \right]$

(51)

follows from Eq. (42) in the regime $\mu_0/K_B T > \gamma > 1$. We see for the spinor Bose gas the local pair correlation again quickly decays with respect to the dynamical interaction $\gamma$. Figure 6 shows the local pair correlation for the spinless and the spinor Bose gases as a function of the interaction strength at different temperatures. In contrast to the spinless Bose gas, where the temperature enhances local pair correlation due to thermal fluctuations, for the spinor Bose gas the local pair correlation decreases with increasing temperature due to the ferromagnetic spin-spin exchange interaction.

VII. CONCLUSION

In this paper we have studied the thermodynamics of the integrable 1D two-component Bose gas via the thermody-
Analytic low-temperature results were obtained for the free energy, total energy, specific heat, entropy, pressure, susceptibility, and pair correlation function in the strongly interacting regimes $\gamma \gg 1/K_BT$ and $1 \ll \gamma \ll \mu_0/K_BT$. Where appropriate, comparison was made with corresponding thermodynamic properties of the integrable 1D spinless Bose gas. Our key finding is that the temperature-induced ferromagnetic spin-spin exchange interaction triggers a number of novel quantum effects in the thermodynamic properties of the spinor Bose gas at low temperatures. In the regime $1 \ll \gamma \ll \mu_0/K_BT$ the specific-heat exponent for the spinor Bose gas following from $c_v \sim T^{1/2}$ is different than that of the spinless Bose gas for which $c_v \sim T$. In this regime, the susceptibility exponent is given by $\chi \sim T^{-2}$ which exceeds that of free spin paramagnet for which $\chi \sim T^{-1}$. In contrast to the spinless Bose gas, where the pair correlation function increases with increasing temperature, the two-component spinor Bose gas pair correlation function decreases with increasing temperature as a result of the temperature-induced spin-spin ferromagnetic exchange interaction.

In general these exact results should be relevant to understanding ferromagnetic behavior and spin effects in two-component spinor Bose gases of cold atoms, for which the interaction strength can in principle be tuned. However, the introduction of precise thermometry into these systems to measure universal temperature-dependent effects provides a number of challenges. However, this is a worthwhile goal.

As we have seen, for strong coupling the characteristics of the thermodynamics of the spinor Bose gas at low temperatures are described by an effective ferromagnetic Heisenberg spin chain. There is a remarkable three-way correspondence [53,54] between the ferromagnetic Heisenberg chain, the limit of weakly coupled planar $\mathcal{N}=4$ supersymmetric Yang-Mills theory and the limit of free strings on $\text{AdS}_5 \times S^5$. This triality between gauge theory, string theory, and the thermodynamics of the ferromagnetic Heisenberg chain has recently been used to calculate the Hagedorn temperature of the string theory in agreement with the Hagedorn deconfinement temperature calculated on the gauge theory side [54]. It now appears that we can add a further connection with the thermodynamics of the strongly interacting two component spinor Bose gas.

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