Subsolutions and Hopf’s Theorem of Fractional $p$–Laplacian

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Abstract

In this article, we prove that $(−Δ)^s_p u(x)$ is bounded in the unit ball $B_1 \subset \mathbb{R}^n$, where $u(x) = (1 − |x|^2)^s_+$. And we then introduce a Hopf’s theorem for $u \in C^{1,1}_{loc} \cap \mathcal{L}_{sp}$ by subsolution and comparison principle method.

Keywords: Fractional $p$-Laplacian operator; Subsolution; Hopf’s theorem

1 Introduction and main results

$(−Δ)^s_p u(x)$ is the fractional $p$-Laplacian nonlocal operator, which is of the form

$$
(−Δ)^s_p u(x) = C_{n,s,p} \text{P.V.} \int_{\mathbb{R}^n} \frac{|u(x)−u(y)|^{p−2}[u(x)−u(y)]}{|x−y|^{n+sp}} \, dy
$$

$$
= C_{n,s,p} \lim_{\varepsilon→0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{|u(x)−u(y)|^{p−2}[u(x)−u(y)]}{|x−y|^{n+sp}} \, dy,
$$

where P.V. is the Cauchy principle value.

In order that the integral on the right hand is well defined, we require that

$$
u \in C^{1,1}_{loc} \cap \mathcal{L}_{sp},
$$

where

$$
\mathcal{L}_{sp} = \left\{ u \in L^{p−1}_{loc} \ | \ \int_{\mathbb{R}^n} \frac{1+u(x)|^{p−1}}{1+|x|^{n+sp}} < \infty \right\}.
$$

When $p = 2$, $(−Δ)^s_2$ becomes the fractional Laplacian operator, Caffarelli and Silvestre [3] introduced an extension method to cope with the nonlocality of the operator. Silvestre [13], Ros-oton and Serra [12] discussed about the regularity of equations involving the fractional Laplacian operator. Chen, C.Li and Y.Li [5], Chen, C.Li and Qi [7] introduced direct method of moving planes and moving spheres for the fractional Laplacian operator.

When $p \neq 2$, $(−Δ)^s_p$ is no longer a linear operator. For $p = 2$, we know that $(−Δ)^s u(x) = \text{const}$ for $x \in B_1$, where $u(x) = (1 − |x|^2)^s_+$. This is proved by Getoor [9] Theorem 5.2 with Fourier transforms

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and hypergeometric functions. But for $p > 2$, Fourier transform does not work anymore due to the nonlinearity when $p > 2$, and we could not find out any hypergeometric function to exploit. So people even do not know whether $(-\Delta)^s_p u(x)$ is bounded. In fact, in 1 dimension, we can claim $(-\Delta)^s_p u(x)$ is not constant by numerical computation. In this article, we will prove that $(-\Delta)^s_p u(x)$ is bounded, which plays a pivotal role in the proof of Hopf’s theorem. To our knowledge, this is the first time to prove the boundedness of $(-\Delta)^s_p u(x)$. Our proof is based on rigorous analysis on the singular term of $(-\Delta)^s_p u(x)$, more precisely, we find the exact coefficient of the singular term and we prove the coefficient identically equals to 0. It is worth mentioning that the fact that $(-\Delta)^s_p u(x)$ is bounded itself deserves certain attention. And Hopf’s theorem plays a critical part in classical theories, so we believe our results will definitely promote the study of the Fractional $p$-Laplacian.

Chen and C.Li [4] proved a boundary estimate for $(-\Delta)^s_p$ operator, which is a key part in the moving plane method. And the boundary estimate plays the role of Hopf’s theorem to some degree. Del pezzo and Quaas [8], Chen, C.Li and Qi [6] have proved some Hopf’s theorem for certain Hölder continuity up to the boundary on bounded domain. In this paper, we will prove the coefficient identically equals to 0. It is worth mentioning that the fact that $(-\Delta)^s_p u(x)$ is bounded itself deserves certain attention. And Hopf’s theorem plays a critical part in classical theories, so we believe our results will definitely promote the study of the Fractional $p$-Laplacian.

Assume that $u$ is lower semicontinuous on $\Omega$, where $\Omega$ is any bounded domain with a uniform interior ball condition (e.g. a domain of class $C^{1,1}$) and $c(x) \geq 0$ is bounded. Then there is a constant $C = C(\Omega, u) > 0$, such that

$$\liminf_{x \to \partial \Omega} \frac{u(x)}{[\text{dist}(x, \partial \Omega)]^s} \geq C.$$  \hspace{1cm} (1)

Theorem 1.1 (Subsolutions). Let $s \in (0, 1)$, $p > 2$, $u(x) = (1 - |x|^2)^s_+$, then $(-\Delta)^s_p u(x)$ is bounded for any dimension.

Theorem 1.2 (Hopf’s theorem). For any $u(x) \in C^{1,1}_{loc}(\Omega) \cap L^p(\mathbb{R}^n)$, where $p > 2$, $s \in (0, 1)$. We assume that $u$ is lower semicontinuous on $\overline{\Omega}$ and pointwisely satisfies

$$\begin{cases}
(-\Delta)^s_p u + c(x) u \geq 0, & x \in \Omega, \\
u > 0, & x \in \Omega, \\
u = 0, & x \in \mathbb{R}^n \setminus \Omega,
\end{cases}$$  \hspace{1cm} (2)

where $\Omega$ is any bounded domain with a uniform interior ball condition (e.g. a domain of class $C^{1,1}$) and $c(x) \geq 0$ is bounded. Then there is a constant $C = C(\Omega, u) > 0$, such that

$$\liminf_{x \to \partial \Omega} \frac{u(x)}{[\text{dist}(x, \partial \Omega)]^s} \geq C.$$  \hspace{1cm} (1)

Theorem 1.3. Assume that $\Omega$ is any domain (bounded or not) with a uniform two-sided ball condition (e.g. a domain of class $C^{1,1}$), $s \in (0, 1)$, $p \geq 2$, and $u \in C^{1,1}_{loc}(\Omega) \cap L^p$ is a bounded solution of

$$\begin{cases}
(-\Delta)^s_p u = f(x, u), & x \in \Omega, \\
u = 0, & x \in \mathbb{R}^n \setminus \Omega,
\end{cases}$$  \hspace{1cm} (2)

where $f(x, u)$ is bounded, then there is a $v_0 \in (0, s)$, such that $u \in C^{v_0}(\mathbb{R}^n)$. Moreover,

$$[u]_{C^{v_0}(\mathbb{R}^n)} \leq C(v_0) \left[1 + \|u\|_{L^\infty(\Omega)} + C\|f\|_{L^\infty(\Omega)} \right].$$
This paper is organized as follows. Section 2 is devoted to show \((-\Delta)^s_p u(x)\) is bounded for 1 dimension. Section 3 is intended to show boundedness of \((-\Delta)^s_p u(x)\) operator for higher dimensions. Section 4 is contributed to prove Hopf’s theorem\([1,2]\). The constant \(C\) may vary from line to line or even in the same line. Section 5 is contributed to prove the global Hölder regularity for bounded solutions. The constant \(C\) may vary from line to line.

2 \(n = 1\)

In this section, we will prove that \((-\Delta)^s_p u(x)\) is bounded in the ball \(B_1\), where \(u(x) = (1 - x^2)^s\). Due to the symmetry, we only need to consider \(x \in (0, 1)\). Moreover, we only need to think about the case that \(x\) is close to 1. For some \(\delta > 0\) fixed, and \(x < 1 - \delta\), we will treat this case in Section 3 for all dimensions.

Step 1. Firstly we give a general estimate for \((-\Delta)^s_p u(x)\) and \(x\) is close to 1. For simplicity, we omit the constant \(C_{n,s,p}\).

\[
(-\Delta)^s_p u(x) = \int_{-\infty}^{\infty} \frac{(1 - x^2)^s(y^{(p-1)})}{(x-y)^{1+sp}} dy + \int_{-1}^{1} \frac{[(1 - x^2)^s - (1 - y^2)^s]^{p-1}}{(x-y)^{1+sp}} dy + \int_{1}^{\infty} \frac{(1 - x^2)^s(y^{(p-1)})}{(y-x)^{1+sp}} dy
\]

\[
+ \lim_{\epsilon \to 0} \left\{ \int_{-\infty}^{\infty} \frac{[(1 - x^2)^s - (1 - y^2)^s]^{p-1}}{(x-y)^{1+sp}} dy + \int_{1}^{\infty} \frac{(1 - x^2)^s(y^{(p-1)})}{(y-x)^{1+sp}} dy \right\}
\]

\[
= \int_{1-x}^{\infty} \frac{(1 - x^2)^s(y^{(p-1)})}{z^{1+sp}} dz + \int_{2x}^{1+x} \frac{[(1 - x^2)^s - (1 - (x-z)^2)^s]^{p-1}}{z^{1+sp}} dz + \int_{1-x}^{\infty} \frac{(1 - x^2)^s(y^{(p-1)})}{z^{1+sp}} dz
\]

\[
+ \lim_{\epsilon \to 0} \left\{ \int_{-\infty}^{\epsilon} \frac{[(1 - x^2)^s - (1 - x^2)^s]^{p-1}}{z^{1+sp}} dz + \int_{-\infty}^{\epsilon} \frac{(1 - x^2)^s(y^{(p-1)})}{z^{1+sp}} dy \right\}
\]

\[
= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\]

Where

\[
I_1 + I_6 = (1 - x^2)^s((p-1)) \left[ \int_{1-x}^{\infty} \frac{1}{z^{1+sp}} dz + \int_{1}^{\infty} \frac{1}{z^{1+sp}} dz \right]
\]

\[
= (1 - x^2)^s((p-1)) \left[ \frac{1}{sp(1+x)^sp} + \frac{1}{sp} \right].
\]

\[
|I_2| + |I_5| \leq \int_{2x}^{1+x} \frac{1}{z^{1+sp}} dz + \int_{1}^{2x} \frac{1}{z^{1+sp}} dz = \int_{1}^{1+x} \frac{1}{z^{1+sp}} dz = \frac{1}{sp} \left[ 1 - \frac{1}{(1+x)^sp} \right].
\]
So $I_1, I_2, I_5, I_6$ are uniformly bounded. Then we only need to consider $I_3, I_4$.

$$I_3 = \lim_{e \to 0} \int_{e}^{1-x} \frac{\left(1 - x^2 \right)^s - \left(1 - (x + z)^2 \right)^s}{z^{1+s}} - \left[\left(1 - (x - z)^2 \right)^s - \left(1 - x^2 \right)^s\right]^{p-1} \, dz$$

$$= (1 - x^2)^s (p-1) \lim_{e \to 0} \int_{e}^{1-x} \left[1 - \left(1 - \frac{2xz}{1+x^2} - \frac{x^2}{1+x^2}\right)^s\right]^{p-1} - \left[\left(1 + \frac{2xz}{1+x^2} - \frac{x^2}{1+x^2}\right)^s - 1\right]^{p-1} \, dz$$

$$= \frac{(2x)^sp}{(1 - x^2)^s} \lim_{e \to 0} \frac{\int_{e}^{1-x} \left[1 - \frac{1 - k - \frac{1-x^2}{4x^2} k^2}{s}\right]^{p-1} - \left[\left(1 + k - \frac{1-x^2}{4x^2} k^2\right)^s - 1\right]^{p-1}}{k^{1+sp}} \, dk$$

$$:=(2x)^sp \frac{(1 - x)^s(1 + x)^s I'_3}{s p}$$

where we used the substitution $k = \frac{2x}{1 - x^2}z$.

Similarly, we deal with $I_4$.

$$I_4 = \int_{1-x}^{1} \frac{(1 - x^2)^s (p-1) - \left(1 - (x - z)^2 \right)^s - \left(1 - x^2 \right)^s}{z^{1+s}} \, dz$$

$$= (1 - x^2)^s (p-1) \left[\frac{1}{s p (1 - x)^s} - \frac{1}{s p}\right] - \int_{1-x}^{1} \left[\left(1 - (x - z)^2 \right)^s - \left(1 - x^2 \right)^s\right]^{p-1} \, dz$$

$$= (1 - x^2)^s (p-1) \left[\frac{1}{s p (1 - x)^s} - \frac{1}{s p}\right] - \frac{(2x)^sp}{(1 - x^2)^s} \int_{1-x}^{1} \left[\frac{1 + k - \frac{1-x^2}{4x^2} k^2}{s}\right]^{p-1} \, dk$$

$$= \frac{1}{s p (1 - x)^s} \left[(1 + x)^s (p-1) - \frac{1}{s p (1 - x)^s} \int_{1-x}^{1} \left[\left(1 + k - \frac{1-x^2}{4x^2} k^2\right)^s - 1\right]^{p-1} \, dk\right] - \frac{(1 - x^2)^s (p-1)}{s p}$$

$$:=(1 - x)^s \left[(1 + x)^s (p-1) - \frac{1}{s p (1 - x)^s} I'_4\right] - \frac{(1 - x^2)^s (p-1)}{s p}.$$
This is because

\[
\limsup_{x \to 1} (1 - x)^{-s} \left| \int_{\frac{2x}{1+x}}^{1} \frac{\left[ 1 - \left( 1 - k - \frac{1-x^2}{4x^2} k^2 \right)^s \right]^{p-1} - \left[ 1 - \left( 1 - k \right)^s \right]^{p-1}}{k^{1+sp}} \right| \int_{x=1}^{1} (1 - x)^{-s} \left( 1 - \frac{x}{2} \right)^{sp} \frac{1 - x}{1 + x} = 0.
\]

Next we are going to prove there is a constant \( C > 0 \), such that

\[
-C \leq (1 - x)^{-s} \int_{0}^{1} \frac{\left[ 1 - \left( 1 - k - \frac{1-x^2}{4x^2} k^2 \right)^s \right]^{p-1} - \left[ 1 - \left( 1 - k \right)^s \right]^{p-1}}{k^{1+sp}} \right| \int_{0}^{1} (1 - x)^{-s} \left( 1 - \frac{x}{2} \right)^{sp} \frac{1 - x}{1 + x} \leq C.
\]

First we consider the difference,

\[
(1 - x)^{-s} \int_{0}^{1} \frac{\left[ 1 - \left( 1 - k - \frac{1-x^2}{4x^2} k^2 \right)^s \right]^{p-1} - \left[ 1 - \left( 1 - k \right)^s \right]^{p-1}}{k^{1+sp}} \right| \int_{0}^{1} (1 - x)^{-s} \left( 1 - \frac{x}{2} \right)^{sp} \frac{1 - x}{1 + x} \leq C.
\]

And similarly,

\[
(1 - x)^{-s} \int_{0}^{1} \frac{\left[ 1 - \left( 1 + k - \frac{1-x^2}{4x^2} k^2 \right)^s \right]^{p-1} - \left[ 1 - \left( 1 + k \right)^s \right]^{p-1}}{k^{1+sp}} \right| \int_{0}^{1} (1 - x)^{-s} \left( 1 - \frac{x}{2} \right)^{sp} \frac{1 - x}{1 + x} \leq C.
\]

Secondly, we estimate the term \((1 - x)^{-s} I_4'\),

\[
I_4' = \int_{\frac{2x}{1+x}}^{2x} \left( 1 + k - \frac{1-x^2}{4x^2} k^2 \right)^{-s} - 1 \left| \frac{k^{1+sp}}{1 + x} \right|.
\]
Next we will prove

\[(1 - x)^{-s} \int_1^{1 + x} \left[ (1 + k - \frac{1-x^2}{4x} k^2)^s - 1 \right]^{p-1} \frac{dk}{k^{1+sp}} \leq (1 - x)^{-s} \int_1^{1 + x} \left[ (1 + k - \frac{1-x^2}{4x} k^2)^s - 1 \right]^{p-1} \frac{dk}{k^{1+sp}} + C.\]

This is due to

\[\limsup_{x \to 1} (1 - x)^{-s} \int_1^{1 + x} \left[ (1 + k - \frac{1-x^2}{4x} k^2)^s - 1 \right]^{p-1} \frac{dk}{k^{1+sp}} \leq \limsup_{x \to 1} C (1 - x)^{-s} \frac{1 - x}{1 + x} = 0,\]

and

\[\limsup_{x \to 1} (1 - x)^{-s} \int_1^{1 + x} \left[ (1 + k - \frac{1-x^2}{4x} k^2)^s - 1 \right]^{p-1} \frac{dk}{k^{1+sp}} \leq \limsup_{x \to 1} C (1 - x)^{-s} \int_1^{1 + x} \frac{k^{s(p-1)}}{k^{1+sp}} \, dk\
\leq \limsup_{x \to 1} C (1 - x)^{-s} \frac{1 - x^2}{(2x)^{1+s}} \left( \frac{1}{1 - x} - \frac{2x}{1 - x^2} \right) = 0.\]

Now we claim that there is a positive constant \(C > 0\), such that

\[-C \leq (1 - x)^{-s} \int_1^{1 + x} \left[ (1 + k - \frac{1-x^2}{4x} k^2)^s - 1 \right]^{p-1} \frac{dk}{k^{1+sp}} - (1 - x)^{-s} \int_1^{1 + x} \left[ (1 + k)^s - 1 \right]^{p-1} \frac{dk}{k^{1+sp}} \leq C.\]

Because \(x\) is close to 1, we have

\[(1 - x)^{-s} \int_1^{1 + x} \left[ (1 + k - \frac{1-x^2}{4x} k^2)^s - 1 \right]^{p-1} \frac{dk}{k^{1+sp}} \leq (1 - x)^{-s} \int_1^{1 + x} \frac{Ck^{s(p-2)} \left[ (1 + k)^s - \left(1 + k - \frac{1-x^2}{4x} k^2 \right)^s \right]}{k^{1+sp}} \, dk\
\leq (1 - x)^{-s} \int_1^{1 + x} \frac{Ck^{s(p-2)} (1 + k)^s}{k^{1+sp}} \left[1 - \left(1 - \frac{1-x^2}{4x} k^2 \right)^s\right] \, dk\
\leq (1 - x)^{-s} \int_1^{1 + x} \frac{Ck^{s(p-2)} (1 + k)^s}{k^{1+sp}} \left[1 - \left(1 - \frac{1-x^2}{4x} k^2 \right)^s\right] \, dk\
\leq (1 - x)^{-s} \int_1^{1 + x} \frac{Ck^{s(p-2)} (1 - x)k^{1+s}}{k^{1+sp}} \, dk\
= (1 - x)^{1-s} \int_1^{1 + x} C k^{-s} \, dk \leq C (1 - x)^{1-s} \frac{1}{(1 - x)^{1-s}} = C,
\]

where we used the estimate

\[\frac{1 - x^2}{4x^2} \frac{k^2}{1 + k} \leq \frac{1 + x}{4x^2(2 - x)} \to \frac{1}{2} \quad \text{if} \quad x \to 1.\]
And there is a positive constant $C$, such that

$$-C \leq (1 - x)^{-s} \int_1^{-1} \frac{[(1 + k)^s - 1]^{p-1}}{k^{1+sp}} \, dk - (1 - x)^{-s} \int_1^\infty \frac{[(1 + k)^s - 1]^{p-1}}{k^{1+sp}} \, dk \leq C.$$  

This is because when $x$ is close to 1, we have

$$(1 - x)^{-s} \int_1^\infty \frac{[(1 + k)^s - 1]^{p-1}}{k^{1+sp}} \, dk$$

$$\leq (1 - x)^{-s} \int_1^{\infty} k^{(p-1)} \, dk$$

$$= (1 - x)^{-s} \int_1^{\infty} k^{-s-1} \, dk \leq C.$$

Step 3. We will prove the singular term is bounded uniformly. By the above simplification, the singular term of $(-\Delta)^p u(x)$ is

$$(1 - x)^{-s} \frac{(2x)^p}{(1 + x)^s} \left\{ \int_0^1 \left[ 1 - (1 - k)^s \right]^{p-1} - \left[ (1 + k)^s - 1 \right]^{p-1} \, dk + \frac{(1 + x)^p}{sp(2x)^p} - \int_1^\infty \frac{[(1 + k)^s - 1]^{p-1}}{k^{1+sp}} \, dk \right\}.$$  

Furthermore, there is a constant $C > 0$,

$$-C \leq (1 - x)^{-s} \frac{(1 + x)^p}{sp(2x)^p} - (1 - x)^{-s} \frac{1}{sp} \leq C.$$  

Because

$$(1 - x)^{-s} \frac{1}{sp} \left[ \frac{(1 + x)^p}{(2x)^p} - 1 \right] = (1 - x)^{-s} \frac{1}{sp} \left[ \left( 1 + \frac{1 - x}{2x} \right)^p - 1 \right]$$

$$= (1 - x)^{-s} \frac{1}{sp} \left[ \frac{1 - x}{2x} + o \left( \frac{1 - x}{2x} \right) \right] \leq C.$$  

Hence we only need to prove the following integration identity:

$$\frac{1}{sp} + \int_0^1 \frac{[1 - (1 - k)^s]^{p-1} - [(1 + k)^s - 1]^{p-1}}{k^{1+sp}} \, dk - \int_1^\infty \frac{[(1 + k)^s - 1]^{p-1}}{k^{1+sp}} \, dk = 0. \quad (5)$$  

So we begin to prove the above identity. For any $\varepsilon \in (0, 1)$ fixed, we have

$$\int_0^1 \frac{[1 - (1 - k)^s]^{p-1} - [(1 + k)^s - 1]^{p-1}}{k^{1+sp}} \, dk - \int_1^\infty \frac{[(1 + k)^s - 1]^{p-1}}{k^{1+sp}} \, dk$$

$$= \int_0^\varepsilon \frac{[1 - (1 - k)^s]^{p-1} - [(1 + k)^s - 1]^{p-1}}{k^{1+sp}} \, dk + \int_1^\varepsilon \frac{[1 - (1 - k)^s]^{p-1} - [(1 + k)^s - 1]^{p-1}}{k^{1+sp}} \, dk$$

$$- \int_1^\infty \frac{[(1 + k)^s - 1]^{p-1}}{k^{1+sp}} \, dk$$

$$= \int_0^\varepsilon \frac{[1 - (1 - k)^s]^{p-1} - [(1 + k)^s - 1]^{p-1}}{k^{1+sp}} \, dk + \int_\varepsilon^1 \frac{[1 - (1 - k)^s]^{p-1} - [(1 + k)^s - 1]^{p-1}}{k^{1+sp}} \, dk - \int_1^\infty \frac{[(1 + k)^s - 1]^{p-1}}{k^{1+sp}} \, dk$$

We notice that

$$\int_\varepsilon^1 \frac{[1 - (1 - k)^s]^{p-1}}{k^{1+sp}} \, dk = \int_{1-\varepsilon}^{\infty} \frac{(1 + t)^{s-1} - [(1 + k)^s - 1]^{p-1}}{t^{1+sp}} \, dt,$$
where we used the change of variable \[ t = \frac{k}{1 - k}. \]

Then

\[
\int_0^1 \frac{[1 - (1 - k)^s]}{k^{1 + sp}} - \left[ \frac{1}{k^{1 + sp}} \right]^{1 - k} \, dk - \int_t^\infty \frac{[1 + k]^s}{k^{1 + sp}} \, dk
\]

\[
= \int_0^1 \frac{[1 - (1 - k)^s]}{k^{1 + sp}} - \left[ \frac{1}{k^{1 + sp}} \right]^{1 - k} \, dk - \int_t^\infty \frac{(1 + t)^{s-1}}{t^{1 + sp}} \, dt
\]

\[
= \int_0^e \frac{[1 - (1 - k)^s]}{k^{1 + sp}} - \left[ \frac{1}{k^{1 + sp}} \right]^{1 - k} \, dk - \int_e^{\frac{e}{1 - k}} \frac{(1 + k)^s}{k^{1 + sp}} - \frac{1}{sp} \frac{[(1 + t)^{s-1}]^{1 - k}}{t^{1 + sp}} \, dt
\]

\[
= H_1 + H_2 + H_3.
\]

Now we consider the three terms separately and let \( e \) goes to 0.

\[ H_1 = \int_0^e \frac{[1 - (1 - k)^s]}{k^{1 + sp}} - \left[ \frac{1}{k^{1 + sp}} \right]^{1 - k} \, dk \]

\[
\leq \int_0^e C k^{s(p-2)} \left[ 2 - (1 - k)^s - (1 + k)^s \right] \, dk
\]

\[
\leq \int_0^e C k^{s(p-2)} \left[ 2 - \left( 1 - sk - \frac{s(1-s)}{2} k^2 + o(k^2) \right) - \left( 1 + sk - \frac{s(1-s)}{2} k^2 + o(k^2) \right) \right] \, dk
\]

\[
\leq \int_0^e C k^{s(p-2)} \left[ k^2 + o(k^2) \right] \, dk \to 0 \quad \text{as} \quad e \to 0.
\]

\[ H_2 = \int_0^{\frac{e}{1 - k}} \frac{[(1 + k)^s - 1]}{k^{1 + sp}} \, dk - \int_0^{\frac{e}{1 - k}} \frac{[(1 + sk + o(k)) - 1]}{k^{1 + sp}} \, dk \leq \int_0^{\frac{e}{1 - k}} \frac{C k^{p-1}}{k^{1 + sp}} \, dk
\]

\[
= \left\{ \begin{array}{ll}
\int_0^{\frac{e}{1 - k}} C k^{p-s-2} \, dk & \leq C \left( \frac{e}{1 - k} \right)^{p-s-1} \to 0 \quad \text{as} \quad e \to 0, \quad \text{if} \quad p - sp > 1; \\
\int_0^{\frac{e}{1 - k}} C k^{p-1} \, dk & \leq C \left( \frac{e}{1 - k} \right)^{p-1} \to 0 \quad \text{as} \quad e \to 0, \quad \text{if} \quad p - sp = 1; \\
\int_0^{\frac{e}{1 - k}} C k^{p-1} \, dk & \leq C \left( \frac{e}{1 - k} \right)^{p-1} \to 0 \quad \text{as} \quad e \to 0, \quad \text{if} \quad p - sp < 1.
\end{array} \right.
\]

\[ H_3 = -\frac{1}{sp} \frac{[(1 + t)^{s-1}]^{1 - k}}{t^{1 + sp}} \bigg|_{t = \frac{e}{1 - k}} = -\frac{1}{sp} + \frac{1}{sp} \frac{[1 - (1 - e)^s]^p}{e^{sp}} \to -\frac{1}{sp} \quad \text{as} \quad e \to 0.
\]

Therefore we have completed the proof of the identity. And hence we have proved the boundedness of \((-\Delta)^p u(x)\) for 1 dimension.
3 \ n \geq 2

The purpose of this section is to prove that \((-\Delta)^{s}u(x)\) is bounded in the ball \(B_{1}\) for higher dimensions, where \(u(x) = (1 - |x|^2)^{s}_{+}\).

Step 1. Due to the symmetry, we assume \(x = (x,0,\ldots,0) \in B_{1}\) is close to \((1,0,\ldots,0)\), and \(y = (y_{1},y_{2},\ldots,y_{n}) =: (y,\bar{y})\). We omit the constant \(C_{n,s,p}\) for simplicity.

\[
(-\Delta)^{s}u(x) = \lim_{\epsilon \to 0} \int_{R^{n}\setminus B_{\epsilon}(x)} \frac{|u(x) - u(y)|^{p-2}[u(x) - u(y)]}{|x - y|^{n+s}} \, dy
\]

\[
= \lim_{\epsilon \to 0} \int_{\{y \in R^{n}: |x - y| \geq \epsilon\}} \frac{|(1 - x^2)^{s} - (1 - |y|^{2})_{+}^{s}|^{p-2}[|1 - x^{2}|^{s} - (1 - |y|^{2})_{+}^{s}]}{|x - y|^{n+s}} \, dy
\]

\[
= \lim_{\epsilon \to 0} \int_{\{y \in R^{n}: |x - y| \geq \epsilon\}} \frac{|(1 - x^2)^{s} - (1 - |y|^{2})_{+}^{s}|^{p-2}[|1 - x^{2}|^{s} - (1 - |y|^{2})_{+}^{s}]}{|(x - y)|^{2} + |\bar{y}|^{2}} \, dy.
\]

Set \(z = (z_{1},\bar{z})\), where \(z_{1} = x - y_{1}\), \(\bar{z} = \bar{y}\). Then

\[
(-\Delta)^{s}u(x) = \lim_{\epsilon \to 0} \int_{\{z \in R^{n}: |z| \geq \epsilon\}} \frac{|(1 - x^2)^{s} - (1 - (x - z_{1})^2 - |z|^{2})_{+}^{s}|^{p-2}[|1 - x^{2}|^{s} - (1 - (x - z_{1})^2 - |z|^{2})_{+}^{s}]}{(z_{1}^{2} + |z|^{2})^{n+s}} \, dz
\]

\[
= (1 - x^{2})^{s(p-1)} \lim_{\epsilon \to 0} \int_{\{z \in R^{n}: |z| \geq \epsilon\}} \frac{1 - \left(1 + \frac{2z_{1}}{1 - x^{2}} - \frac{|z|^{2}}{1 - x^{2}}\right)^{s} + |z|^{n+s}}{\left(1 - \left(1 + \frac{2z_{1}}{1 - x^{2}} - \frac{|z|^{2}}{1 - x^{2}}\right)^{s}\right)_{+}^{s}} \, dz.
\]

Let \(w = \frac{2z_{1}}{1 - x^{2}}\), where \(w_{1} = \frac{2z_{1}}{1 - x^{2}}\) and \(\bar{w} = \frac{2z_{1}}{1 - x^{2}}\), then

\[
(-\Delta)^{s}u(x) = \frac{(2x)^{sp}}{(1 - x^{2})^{s}} \lim_{\epsilon \to 0} \int_{\{w \in R^{n}: |w| \geq \frac{2z_{1}}{1 - x^{2}}\}} \frac{1 - \left(1 + w_{1} - \frac{1 - x^{2}}{4z_{1}^{2}}|w|^{2}\right)^{s} + |w|^{n+s}}{\left|1 - \left(1 + w_{1} - \frac{1 - x^{2}}{4z_{1}^{2}}|w|^{2}\right)^{s}\right|_{+}^{s}} \, dw.
\]
Now we start to estimate the coefficient term.

\[
\int_{|w| \geq \frac{2\pi}{1 - \epsilon^2}} \left| \frac{1 - \left(1 + w_1 - \frac{1 - x^2}{4\pi^2} |w|^2 \right)^s}{|w|^{n+sp}} \right|^{p-2} \left[ \frac{1 - \left(1 + w_1 - \frac{1 - x^2}{4\pi^2} |\bar{w}|^2 \right)^s}{|w|^{n+sp}} \right] \, dw
\]

\[
= \int_{-\infty}^{\infty} \int_{|w|^2 \geq \frac{4\pi^2}{1 - \epsilon^2}} \left| \frac{1 - \left(1 + w_1 - \frac{1 - x^2}{4\pi^2} w_1^2 - \frac{1 - x^2}{4\pi^2} |\bar{w}|^2 \right)^s}{(w_1^2 + |\bar{w}|^2)^{n+sp}} \right|^{p-2} \frac{1 - \left(1 + w_1 - \frac{1 - x^2}{4\pi^2} w_1^2 - \frac{1 - x^2}{4\pi^2} |\bar{w}|^2 \right)^s}{(w_1^2 + |\bar{w}|^2)^{n+sp}} \, d\bar{w} \, dw
\]

\[
= C \int_{-\infty}^{\infty} \int_{\rho^2 \geq \frac{4\pi^2}{1 - \epsilon^2}} \left(1 + \rho^2 \right)^{p-2} \rho^{n-2} d\rho
\]

\[
= C \int_{-\infty}^{\infty} \int_{\rho^2 \geq \frac{4\pi^2}{1 - \epsilon^2}} \left(1 + \rho^2 \right)^{p-2} \rho^{n-2} d\rho
\]

\[
=: J_1 + J_2.
\]

Step 2. In this part, we will prove \( \lim_{\epsilon \to 0} J_1 = 0. \) We omit the constant for simplicity.

\[
J_1 = \int_{-\infty}^{\infty} \int_{\rho^2 \geq \frac{4\pi^2}{1 - \epsilon^2}} \left(1 + \rho^2 \right)^{p-2} \rho^{n-2} d\rho
\]

\[
= \int_{-\infty}^{\infty} \int_{\rho^2 \geq \frac{4\pi^2}{1 - \epsilon^2}} \left(1 + \rho^2 \right)^{p-2} \rho^{n-2} d\rho
\]

\[
+ \int_{-\infty}^{\infty} \int_{\rho^2 \geq \frac{4\pi^2}{1 - \epsilon^2}} \left(1 + \rho^2 \right)^{p-2} \rho^{n-2} d\rho
\]

\[
=: J_{11} + J_{12}.
\]
Firstly, we give an estimate on \( J_{12} \),

\[
|J_{12}| \leq \int_{-\frac{2x}{1-x^2}}^{\frac{2x}{1-x^2}} dw_1 \int_{\frac{2\epsilon^2}{1-\epsilon^2}}^{\epsilon^2} \left[ 1 - \left(1 + w_1 - \frac{1-x^2}{4x^2}w_1^2 - \frac{1-x^2}{4x^2}w_1^2 \right)^s \right] \frac{1}{\rho^{n-2}} d\rho
\]

\[
\leq \int_{-\frac{2x}{1-x^2}}^{\frac{2x}{1-x^2}} dw_1 \int_{\frac{2\epsilon^2}{1-\epsilon^2}}^{\epsilon^2} \frac{\rho^{n-2}}{(w_1^2 + \rho^2)^{\frac{n+1}{2}}} d\rho
\]

\[
\leq \int_{-\frac{2x}{1-x^2}}^{\frac{2x}{1-x^2}} dw_1 \int_{\frac{2\epsilon^2}{1-\epsilon^2}}^{\epsilon^2} \frac{1}{\rho^{2+2\epsilon}} d\rho
\]

\[
\leq C(x) \int_{-\frac{2x}{1-x^2}}^{\frac{2x}{1-x^2}} dw_1 \rightarrow 0 \quad \text{as} \quad \epsilon \rightarrow 0.
\]

Secondly, we estimate \( J_{11} \) by taylor expansion,

\[
J_{11} = -\int_{-\frac{2x}{1-x^2}}^{\frac{2x}{1-x^2}} dw_1 \int_{\frac{2\epsilon^2}{1-\epsilon^2}}^{\epsilon^2} \left[ 1 - \left(1 + w_1 - \frac{1-x^2}{4x^2}w_1^2 - \frac{1-x^2}{4x^2}w_1^2 \right)^s \right] \frac{1}{\rho^{n-2}} d\rho
\]

\[
+ \int_{-\frac{2x}{1-x^2}}^{\frac{2x}{1-x^2}} dw_1 \int_{\frac{2\epsilon^2}{1-\epsilon^2}}^{\epsilon^2} \left[ 1 - \left(1 + w_1 - \frac{1-x^2}{4x^2}w_1^2 - \frac{1-x^2}{4x^2}w_1^2 \right)^s \right] \frac{1}{\rho^{n-2}} d\rho
\]

\[
+ \int_{-\frac{2x}{1-x^2}}^{\frac{2x}{1-x^2}} dw_1 \int_{\frac{2\epsilon^2}{1-\epsilon^2}}^{\epsilon^2} \left[ 1 - \left(1 + w_1 - \frac{1-x^2}{4x^2}w_1^2 - \frac{1-x^2}{4x^2}w_1^2 \right)^s \right] \frac{1}{\rho^{n-2}} d\rho
\]

\[
\leq \int_{-\frac{2x}{1-x^2}}^{\frac{2x}{1-x^2}} dw_1 \int_{\frac{2\epsilon^2}{1-\epsilon^2}}^{\epsilon^2} C \left[ 1 - \frac{1-x^2}{4x^2}w_1^2 + O(w_1^2) \right] \frac{1}{\rho^{n-2}} d\rho + 2C \int_{-\frac{2x}{1-x^2}}^{\frac{2x}{1-x^2}} dw_1 \int_{\frac{2\epsilon^2}{1-\epsilon^2}}^{\epsilon^2} \frac{1}{\rho^{n-2}} d\rho.
\]
Moreover, we have

\[
\int_0^{\frac{2\pi}{2}} \int_0^{2\pi} \left( \sqrt{\frac{4\pi^2}{1-x^2}} \right) \frac{\cos^2 \theta}{\sqrt{1-w_1^2}} \right) \frac{1-x^2}{w_1^2} \rho^{p-2} d\rho \leq C \int_0^{2\pi} \left( \sqrt{\frac{4\pi^2}{1-x^2}} \right) \frac{1-x^2}{w_1^2} \rho^{p-2} d\rho
\]

\[
\leq C \int_0^{2\pi} \left( \sqrt{\frac{4\pi^2}{1-x^2}} \right) \frac{1-x^2}{w_1^2} \rho^{p-2} d\rho
\]

\[
= C \int_0^{2\pi} \left( \sqrt{\frac{4\pi^2}{1-x^2}} \right) \frac{1-x^2}{w_1^2} \rho^{p-2} d\rho
\]

And

\[
\int_0^{\frac{2\pi}{2}} \int_0^{2\pi} \left( \sqrt{\frac{4\pi^2}{1-x^2}} \right) \frac{1-x^2}{w_1^2} \rho^{p-2} d\rho \leq C \int_0^{2\pi} \left( \sqrt{\frac{4\pi^2}{1-x^2}} \right) \frac{1-x^2}{w_1^2} \rho^{p-2} d\rho
\]

\[
\leq \int_0^{\frac{2\pi}{2}} \int_0^{2\pi} \left( \sqrt{\frac{4\pi^2}{1-x^2}} \right) \frac{1-x^2}{w_1^2} \rho^{p-2} d\rho
\]

\[
\leq C \int_0^{2\pi} \left( \sqrt{\frac{4\pi^2}{1-x^2}} \right) \frac{1-x^2}{w_1^2} \rho^{p-2} d\rho
\]

\[
\leq C \int_0^{2\pi} \left( \sqrt{\frac{4\pi^2}{1-x^2}} \right) \frac{1-x^2}{w_1^2} \rho^{p-2} d\rho
\]

Hence \( f_1 \to 0 \) as \( \epsilon \to 0 \).
Step 3. We work on $J_2$ partly,

\[
J_2 = \int_{|w_1| \geq \frac{2\kappa}{1 + \kappa^2}} dw_1 \int_0^1 \frac{1 - \left(1 + w_1 - \frac{1 - x^2}{4 \pi^2} w_1^2 - \frac{1 - x^2}{4 \pi^2} \rho^2\right)^s}{(w_1^2 + \rho^2)^{\frac{n+\rho}{2}}} \rho^{n-2} \, d\rho
\]

\[
= \int_{\frac{2\kappa}{1 + \kappa^2}}^\infty dw_1 \int_0^1 \frac{1 - \left(1 + w_1 - \frac{1 - x^2}{4 \pi^2} w_1^2 - \frac{1 - x^2}{4 \pi^2} \rho^2\right)^s}{(w_1^2 + \rho^2)^{\frac{n+\rho}{2}}} \rho^{n-2} \, d\rho
\]

\[\quad + \int_{\frac{2\kappa}{1 + \kappa^2}}^\infty dw_1 \int_0^\infty \frac{1 - \left(1 + w_1 - \frac{1 - x^2}{4 \pi^2} w_1^2 - \frac{1 - x^2}{4 \pi^2} \rho^2\right)^s}{(1 + \rho^2)^{\frac{n+\rho}{2}}} \rho^{n-2} \, d\rho.
\]

Set $y = \frac{\rho}{\sqrt{1 + \rho^2}}$, then

\[
J_2 = \int_{\frac{2\kappa}{1 + \kappa^2}}^\infty \frac{1}{w_1^{1+sp}} \, dw_1 \int_0^1 \frac{1 - \left(1 + w_1 - \frac{1 - x^2}{4 \pi^2} (1 + y^2) w_1^2\right)^s}{(1 + y^2)^{\frac{n+\rho}{2}}} y^{n-2} \, dy
\]

\[\quad + \int_{\frac{2\kappa}{1 + \kappa^2}}^\infty \frac{1}{w_1^{1+sp}} \, dw_1 \int_0^\infty \frac{1 - \left(1 + w_1 - \frac{1 - x^2}{4 \pi^2} (1 + y^2) w_1^2\right)^s}{(1 + y^2)^{\frac{n+\rho}{2}}} y^{n-2} \, dy
\]

\[=: J_{21} + J_{22}.
\]

Where

\[
J_{22} = \int_{\frac{2\kappa}{1 + \kappa^2}}^\infty \frac{1}{w_1^{1+sp}} \, dw_1 \int_0^1 \frac{1 - \left(1 + w_1 - \frac{1 - x^2}{4 \pi^2} (1 + y^2) w_1^2\right)^s}{(1 + y^2)^{\frac{n+\rho}{2}}} y^{n-2} \, dy
\]

\[\quad + \int_{\frac{2\kappa}{1 + \kappa^2}}^\infty \frac{1}{w_1^{1+sp}} \, dw_1 \int_0^\infty \frac{y^{n-2}}{(1 + y^2)^{\frac{n+\rho}{2}}} \, dy + \int_{\frac{2\kappa}{1 + \kappa^2}}^\infty \frac{1}{w_1^{1+sp}} \, dw_1 \int_0^\infty \frac{y^{n-2}}{(1 + y^2)^{\frac{n+\rho}{2}}} \, dy
\]

\[=: \text{①} + \text{②} + \text{③}.
\]
$$J_{21} = \int_0^\infty \frac{1}{\gamma_1^{1-s}} \frac{1}{w_1^{1+s}} dw_1 \int_0^\infty \left| 1 - \left(1 + w_1 - \frac{1-x^2}{4x^2} (1 + y^2) w_1^2 \right)^s \right|^{p-2} \left[ 1 - \left(1 + w_1 - \frac{1-x^2}{4x^2} (1 + y^2) w_1^2 \right)^s \right]^{y''-2} dy$$

$$= - \int_0^{\frac{2\pi}{1-s}} \frac{1}{w_1^{1+s}} \frac{1}{\sqrt{\frac{4\pi^2 \gamma_1}{1-x^2} w_1^2}} \left[ 1 - \left(1 + w_1 - \frac{1-x^2}{4x^2} (1 + y^2) w_1^2 \right)^s \right]^{p-1} \left[ 1 - \left(1 + w_1 - \frac{1-x^2}{4x^2} (1 + y^2) w_1^2 \right)^s \right]^{y''-2} dy dw_1$$

$$+ \int_0^{\frac{2\pi}{1-s}} \frac{1}{w_1^{1+s}} \frac{1}{\sqrt{\frac{4\pi^2 \gamma_1}{1-x^2} w_1^2}} \left[ 1 - \left(1 + w_1 - \frac{1-x^2}{4x^2} (1 + y^2) w_1^2 \right)^s \right]^{p-1} \left[ 1 - \left(1 + w_1 - \frac{1-x^2}{4x^2} (1 + y^2) w_1^2 \right)^s \right]^{y''-2} dy dw_1$$

$$=: 1' + 2' + 3' + 4' + 5' + 6'.$$

Step 4. In this step, we will figure out $1 + 1'$,

$$1 + 1' = \int_0^{\frac{1}{2}} \frac{1}{w_1^{1+s}} \frac{1}{\sqrt{\frac{4\pi^2 \gamma_1}{1-x^2} w_1^2}} \left[ 1 - \left(1 + w_1 - \frac{1-x^2}{4x^2} (1 + y^2) w_1^2 \right)^s \right]^{p-2} \left[ 1 - \left(1 + w_1 - \frac{1-x^2}{4x^2} (1 + y^2) w_1^2 \right)^s \right]^{y''-2} dy dw_1$$

$$+ \int_0^{\frac{1}{2}} \frac{1}{w_1^{1+s}} \frac{1}{\sqrt{\frac{4\pi^2 \gamma_1}{1-x^2} w_1^2}} \left[ 1 - \left(1 + w_1 - \frac{1-x^2}{4x^2} (1 + y^2) w_1^2 \right)^s \right]^{p-1} \left[ 1 - \left(1 + w_1 - \frac{1-x^2}{4x^2} (1 + y^2) w_1^2 \right)^s \right]^{y''-2} dy dw_1$$

$$+ \int_0^{\frac{2\pi}{1-s}} \frac{1}{w_1^{1+s}} \frac{1}{\sqrt{\frac{4\pi^2 \gamma_1}{1-x^2} w_1^2}} \left[ 1 - \left(1 + w_1 - \frac{1-x^2}{4x^2} (1 + y^2) w_1^2 \right)^s \right]^{p-1} \left[ 1 - \left(1 + w_1 - \frac{1-x^2}{4x^2} (1 + y^2) w_1^2 \right)^s \right]^{y''-2} dy dw_1$$

$$=: (I) + (II) + (III) - (IV).$$

Step 5. From now on, we will provide estimates item by item. In this part, we claim that $3', 5', 6'$ are bounded when multiplied by $(1 - x)^{-s}$ for $x$ close to 1, i.e. we can throw away the
three terms.

\[(1 - x)^{-s} (\mathbb{A}' + \mathbb{B}') \leq C(1 - x)^{-s} \int_{\frac{2x}{1+x^2}}^{\frac{1}{1+x^2}} \frac{1}{w_1^{1+s p}} \int_{0}^{\infty} \frac{y^{n-2}}{(1 + y^2)^{\frac{n+s p}{2}}} dy \, dw_1\]

\[\leq C(1 - x)^{-s} \int_{\frac{2x}{1+x^2}}^{\frac{1}{1+x^2}} \frac{1}{w_1^{1+s p}} \, dw_1\]

\[\leq C(1 - x)^{-s} (\frac{1 - x^2}{4x^2})^{sp} \leq C(1 - x)^{s(p-1)}.\]

\[(1 - x)^{-s}(\mathbb{C}') \leq C(1 - x)^{-s} \int_{\frac{2x}{1+x^2}}^{\frac{1}{1+x^2}} \frac{1}{w_1^{1+s p}} \int_{0}^{\infty} \frac{y^{n-2}}{(1 + y^2)^{\frac{n+s p}{2}}} dy \, dw_1\]

\[\leq C(1 - x)^{-s} \int_{\frac{2x}{1+x^2}}^{\frac{1}{1+x^2}} \frac{1}{w_1^{1+s p}} \, dw_1\]

\[\leq C(1 - x)^{-s} (\frac{1 - x^2}{2x})^{sp} \leq C(1 - x)^{s(p-1)}.\]

Step 6. We assert that \(\mathbb{A}\) can be replaced by

\[
\int_{1}^{\infty} \frac{1}{w_1^{1+s p}} \, dw_1 \int_{0}^{\infty} \frac{y^{n-2}}{(1 + y^2)^{\frac{n+s p}{2}}} \, dy = \frac{1}{sp} \int_{0}^{\infty} \frac{y^{n-2}}{(1 + y^2)^{\frac{n+s p}{2}}} \, dy. \tag{6}
\]

This is because

\[\int_{1}^{\infty} \frac{1}{w_1^{1+s p}} \, dw_1 \int_{0}^{\infty} \frac{y^{n-2}}{(1 + y^2)^{\frac{n+s p}{2}}} \, dy \]

\[\leq C(1 - x)^{-s} \int_{\frac{2x}{1+x^2}}^{\frac{1}{1+x^2}} \frac{1}{w_1^{1+s p}} \, dw_1\]

\[= C(1 - x)^{-s} (\frac{1 + x}{2x})^{sp} \left[ 1 - (\frac{1 - x}{1 + x})^{sp} \right] \]

\[\leq C(1 - x)^{-s} (\frac{1 + x}{2x})^{sp} \left[ C \cdot sp \frac{1 - x}{1 + x} \right] \leq C(1 - x)^{1-s}.
\]

Step 7. We will prove that \(\mathbb{E}\) can be replaced by

\[
\int_{\frac{1}{2}}^{1} \frac{1 - (1 - w_1)^{s}}{w_1^{1+s p}} \, dw_1 \int_{0}^{\infty} \frac{y^{n-2}}{(1 + y^2)^{\frac{n+s p}{2}}} \, dy.
\]

Firstly, we will prove that \(\mathbb{E}\) can be substituted by

\[
\int_{\frac{1}{2}}^{\frac{2x}{1+x^2}} \frac{1}{w_1^{1+s p}} \int_{0}^{\frac{2x}{1+x^2} \frac{1}{w_1^{1+s p}}} \frac{1 - (1 - w_1)^{s}}{w_1^{1+s p}} \, y^{n-2} \, dy \, dw_1.
\]

15
Since

\[(1-x)^{-s}\int_{\frac{x}{2}}^{\infty} \frac{1}{w_{1}+sp} \int_{0}^{\infty} \frac{[1-(1-w_{1})^{s}]^{p-1}}{(1+y^{2})^{\frac{n+sp}{2}}} y^{n-2} dy dw_{1}\]

\[\leq C(1-x)^{-s}\int_{\frac{x}{2}}^{\infty} \frac{1}{w_{1}+sp} \int_{0}^{\infty} \frac{[1-(1-w_{1})^{s}]^{p-1}}{(1+y^{2})^{\frac{n+sp}{2}}} y^{n-2} dy dw_{1}\]

\[\leq C(1-x)^{-s}\int_{\frac{x}{2}}^{\infty} \frac{1}{w_{1}+sp} \int_{0}^{\infty} \frac{[1-(1-w_{1})^{s}]^{p-1}}{(1+y^{2})^{\frac{n+sp}{2}}} y^{n-2} dy dw_{1}\]

\[\leq C\int_{\frac{x}{2}}^{\infty} \frac{1}{w_{1}+sp} \int_{0}^{\infty} \frac{[1-(1-w_{1})^{s}]^{p-1}}{(1+y^{2})^{\frac{n+sp}{2}}} y^{n-2} dy dw_{1} \leq C.\]

Moreover, (III) can be replaced by

\[\int_{\frac{3x}{2}}^{\infty} \frac{1}{w_{1}+sp} \int_{0}^{\infty} \frac{[1-(1-w_{1})^{s}]^{p-1}}{(1+y^{2})^{\frac{n+sp}{2}}} y^{n-2} dy dw_{1}.\]

Due to the fact that

\[(1-x)^{-s}\int_{\frac{x}{2}}^{\infty} \frac{1}{w_{1}+sp} \int_{0}^{\infty} \frac{[1-(1-w_{1})^{s}]^{p-1}}{(1+y^{2})^{\frac{n+sp}{2}}} y^{n-2} dy dw_{1}\]

\[\leq C(1-x)^{-s}\int_{\frac{x}{2}}^{\infty} \frac{1}{w_{1}+sp} \int_{0}^{\infty} \frac{[1-(1-w_{1})^{s}]^{p-1}}{(1+y^{2})^{\frac{n+sp}{2}}} y^{n-2} dy dw_{1}\]

\[= C(1-x)^{-s}\int_{\frac{x}{2}}^{\infty} \frac{1}{w_{1}+sp} \int_{0}^{\infty} \frac{[1-(1-w_{1})^{s}]^{p-1}}{(1+y^{2})^{\frac{n+sp}{2}}} y^{n-2} dy dw_{1}\]

\[\leq C(1-x)^{-s}\int_{\frac{x}{2}}^{\infty} \frac{1}{w_{1}+sp} \int_{0}^{\infty} \frac{[1-(1-w_{1})^{s}]^{p-1}}{(1+y^{2})^{\frac{n+sp}{2}}} y^{n-2} dy dw_{1}\]

\[\leq C(1-x)^{-s}\int_{\frac{x}{2}}^{\infty} \frac{1}{w_{1}+sp} \int_{0}^{\infty} \frac{[1-(1-w_{1})^{s}]^{p-1}}{(1+y^{2})^{\frac{n+sp}{2}}} y^{n-2} dy dw_{1}\]

\[\leq C\int_{\frac{x}{2}}^{\infty} \frac{1}{w_{1}+sp} \int_{0}^{\infty} \frac{[1-(1-w_{1})^{s}]^{p-1}}{(1+y^{2})^{\frac{n+sp}{2}}} y^{n-2} dy dw_{1} \leq C.\]

Furthermore, (III) can be replaced by

\[\int_{\frac{1}{2}}^{1} \frac{1}{w_{1}+sp} \int_{0}^{\infty} \frac{[1-(1-w_{1})^{s}]^{p-1}}{(1+y^{2})^{\frac{n+sp}{2}}} y^{n-2} dy dw_{1}.\]
This is due to
\[
(1 - x)^{-s} \int_0^1 \frac{1}{w_1^{1+sp}} \int_0^\infty \frac{[1 - (1 - w_1)^s]^{p-1}}{(1 + y^2)^{\frac{n+sp}{2}}} y^{n-2} dy dw_1
\]
\[
\leq C(1 - x)^{-s} \int_0^1 \frac{1}{w_1^{1+sp}} dw_1
\]
\[
\leq C(1 - x)^{-s} \left( \frac{1 + x}{2x} \right)^{sp} \left[ 1 - \left( 1 - \frac{1 - x}{1 + x} \right)^{sp} \right]
\]
\[
\leq C(1 - x)^{-s} \left( \frac{1 + x}{2x} \right)^{sp} \left[ C \cdot sp \frac{1 - x}{1 + x} \right] \leq C(1 - x)^{1-s}.
\]

Step 8. In this part, we will simplify the form of (I). We are going to prove that (I) can be replaced by
\[
\int_0^1 \frac{1}{w_1^{1+sp}} \left[ \frac{1 - (1 - w_1)^s}{1 + w_1} - \frac{(1 + w_1)^s - 1}{w_1} \right]^{p-1} \left( \frac{y^{n-2}}{(1 + y^2)^{\frac{n+sp}{2}}} \right) dy.
\] (7)

Firstly, we will prove that (I) can be reduced to
\[
\int_0^1 \frac{1}{w_1^{1+sp}} \int_0^{\frac{1}{2} \frac{1}{(1 + y^2)^{\frac{n+sp}{2}}}} \left[ 1 - (1 - w_1)^s - 1 \right]^{p-1} \left( 1 + y^2 \right)^{n-2} dy dw_1.
\]

On the one hand,
\[
(1 - x)^{-s} \int_0^1 \frac{1}{w_1^{1+sp}} \int_0^{\frac{1}{2} \frac{1}{(1 + y^2)^{\frac{n+sp}{2}}}} \left[ 1 - \left( 1 - w_1 - \frac{1 - x}{4x^2} (1 + y^2) w_1^2 \right)^s \right]^{p-1} \left( 1 - (1 - w_1)^s \right)^{-1} \left( 1 + y^2 \right)^{n-2} dy dw_1
\]
\[
= (1 - x)^{-s} \int_0^1 \frac{1}{w_1^{1+sp}} \int_0^{\frac{1}{2} \frac{1}{(1 + y^2)^{\frac{n+sp}{2}}}} \left[ 1 - \left( 1 - w_1 - \frac{1 - x}{4x^2} (1 + y^2) w_1^2 \right)^s \right]^{p-1} \left( 1 - (1 - w_1)^s \right)^{-1} \left( 1 + y^2 \right)^{n-2} dy dw_1
\]
\[
+ (1 - x)^{-s} \int_0^1 \frac{1}{w_1^{1+sp}} \int_0^{\frac{1}{2} \frac{1}{(1 + y^2)^{\frac{n+sp}{2}}}} \left[ 1 - \left( 1 - w_1 - \frac{1 - x}{4x^2} (1 + y^2) w_1^2 \right)^s \right]^{p-1} \left( 1 - (1 - w_1)^s \right)^{-1} \left( 1 + y^2 \right)^{n-2} dy dw_1
\]
\[
\leq C(1 - x)^{-s} \int_0^1 \frac{1}{w_1^{1+sp}} \int_0^{\frac{1}{2} \frac{1}{(1 + y^2)^{\frac{n+sp}{2}}}} \left[ 1 - \left( 1 - w_1 - \frac{1 - x}{4x^2} (1 + y^2) w_1^2 \right)^s \right]^{p-1} \left( 1 + y^2 \right)^{n-2} dy dw_1
\]
\[
+ C(1 - x)^{-s} \int_0^1 \frac{1}{w_1^{1+sp}} \int_0^{\frac{1}{2} \frac{1}{(1 + y^2)^{\frac{n+sp}{2}}}} \left[ 1 - \left( 1 - w_1 - \frac{1 - x}{4x^2} (1 + y^2) w_1^2 \right)^s \right]^{p-1} \left( 1 + y^2 \right)^{n-2} dy dw_1
\]
\[
\leq C(1 - x)^{-s} \int_0^1 \frac{1}{w_1^{1+sp}} \int_0^{\frac{1}{2} \frac{1}{(1 + y^2)^{\frac{n+sp}{2}}}} \left[ 1 - \left( 1 - w_1 - \frac{1 - x}{4x^2} (1 + y^2) w_1^2 \right)^s \right]^{p-1} \left( 1 + y^2 \right)^{n-2} dy dw_1
\]
\[
+ C(1 - x)^{-s} \int_0^1 \frac{1}{w_1^{1+sp}} \int_0^{\frac{1}{2} \frac{1}{(1 + y^2)^{\frac{n+sp}{2}}}} \left[ 1 - \left( 1 - w_1 - \frac{1 - x}{4x^2} (1 + y^2) w_1^2 \right)^s \right]^{p-1} \left( 1 + y^2 \right)^{n-2} dy dw_1
\]
\[
\leq C(1 - x)^{-s} \int_0^1 \frac{1}{w_1^{1+sp}} \int_0^{\frac{1}{2} \frac{1}{(1 + y^2)^{\frac{n+sp}{2}}}} \left[ 1 - \left( 1 - w_1 - \frac{1 - x}{4x^2} (1 + y^2) w_1^2 \right)^s \right]^{p-1} \left( 1 + y^2 \right)^{n-2} dy dw_1.
\]
Further,

$$(1 - x)^{-s} \int_{\frac{1}{2} \omega}^{1} \frac{1}{w_1^{1 + sp}} \int_{0}^{\frac{4x^2}{1 - x^2} \frac{1}{w_1^{1 - 1}}} w_1^{p - 2} (1 - w_1)^{s} \left[ 1 - \left( 1 - \frac{1 - x^2}{4x^2} (1 + y^2) \frac{w_1^{2}}{1 - w_1^{2}} \right)^{s} \right] y^{-2} dy dw_1$$

$$\leq C (1 - x)^{-s} \int_{\frac{1}{2} \omega}^{1} \frac{1}{w_1^{1 + sp}} \int_{0}^{\frac{4x^2}{1 - x^2} \frac{1}{w_1^{1 - 1}}} w_1^{p - 2} (1 - w_1)^{s} \left[ s \frac{1 - x^2}{4x^2} (1 + y^2) \frac{w_1^{2}}{1 - w_1^{2}} \right] y^{-2} dy dw_1$$

$$\leq C (1 - x)^{-s} \int_{\frac{1}{2} \omega}^{1} \frac{1}{w_1^{1 + sp - 1}} \int_{0}^{\frac{4x^2}{1 - x^2} \frac{1}{w_1^{1 - 1}}} \frac{1}{(1 + y^2)^{\frac{n + p}{2}}} dy dw_1$$

$$\leq \begin{cases} C (1 - x)^{-s} \int_{\frac{1}{2} \omega}^{1} \frac{1}{w_1^{1 + sp - 1}} \int_{0}^{\frac{4x^2}{1 - x^2} \frac{1}{w_1^{1 - 1}}} \frac{1}{(1 + y^2)^{\frac{n + p}{2}}} dy dw_1 & \text{if } sp > 2; \\
C (1 - x)^{-s} \int_{\frac{1}{2} \omega}^{1} \frac{1}{w_1^{1 + sp - 1}} \int_{0}^{\frac{4x^2}{1 - x^2} \frac{1}{w_1^{1 - 1}}} \frac{1}{(1 + y^2)^{\frac{n + p}{2}}} dy dw_1 & \text{if } sp \leq 2. 
\end{cases}$$

And,

$$(1 - x)^{-s} \int_{\frac{1}{2} \omega}^{1} \frac{1}{w_1^{1 + sp}} \int_{0}^{\frac{4x^2}{1 - x^2} \frac{1}{w_1^{1 - 1}}} w_1^{p - 1} \left[ 1 + w_1 \right]^{s - 1} \left[ 1 + w_1 - \frac{1 - x^2}{4x^2} (1 + y^2) w_1^{2} \right]^{s - 1} y^{-2} dy dw_1$$

$$\leq C (1 - x)^{-s} \int_{\frac{1}{2} \omega}^{1} \frac{1}{w_1^{1 + sp}} \int_{0}^{\frac{4x^2}{1 - x^2} \frac{1}{w_1^{1 - 1}}} \frac{1}{(1 + y^2)^{\frac{n + p}{2}}} y^{-2} dy dw_1$$

$$\leq C \int_{\frac{1}{2} \omega}^{1} \frac{1}{w_1^{1 + sp}} \int_{0}^{\frac{4x^2}{1 - x^2} \frac{1}{w_1^{1 - 1}}} \frac{1}{(1 + y^2)^{\frac{n + p}{2}}} dy dw_1 \leq C.$$

On the other hand, similarly, we have

$$(1 - x)^{-s} \int_{\frac{1}{2} \omega}^{1} \frac{1}{w_1^{1 + sp}} \int_{0}^{\frac{4x^2}{1 - x^2} \frac{1}{w_1^{1 - 1}}} w_1^{p - 2} \left[ 1 + w_1 \right]^{s - 1} \left[ 1 + w_1 - \frac{1 - x^2}{4x^2} (1 + y^2) w_1^{2} \right]^{s - 1} y^{-2} dy dw_1$$

$$\leq C (1 - x)^{-s} \int_{\frac{1}{2} \omega}^{1} \frac{1}{w_1^{1 + sp}} \int_{0}^{\frac{4x^2}{1 - x^2} \frac{1}{w_1^{1 - 1}}} \frac{1}{(1 + y^2)^{\frac{n + p}{2}}} y^{-2} dy dw_1$$

$$= C (1 - x)^{-s} \int_{\frac{1}{2} \omega}^{1} \frac{1}{w_1^{1 + sp}} \int_{0}^{\frac{4x^2}{1 - x^2} \frac{1}{w_1^{1 - 1}}} \frac{1}{(1 + y^2)^{\frac{n + p}{2}}} y^{-2} dy dw_1$$

$$\leq C (1 - x)^{-s} \int_{\frac{1}{2} \omega}^{1} \frac{1}{w_1^{1 + sp}} \int_{0}^{\frac{4x^2}{1 - x^2} \frac{1}{w_1^{1 - 1}}} \frac{1}{(1 + y^2)^{\frac{n + p}{2}}} y^{-2} dy dw_1$$

$$\leq C (1 - x)^{-s} \int_{\frac{1}{2} \omega}^{1} \frac{1}{w_1^{1 + sp}} \int_{0}^{\frac{4x^2}{1 - x^2} \frac{1}{w_1^{1 - 1}}} \frac{1}{(1 + y^2)^{\frac{n + p}{2}}} dy dw_1 \leq \begin{cases} C (1 - x)^{-s}, & \text{if } sp > 2; \\
C (1 - x)^{-s}, & \text{if } sp \leq 2. 
\end{cases}$$
Moreover, we derive from the above arguments that (I) can be replaced by

\[
\int_0^1 \frac{1}{w_1^{1+sp}} \int_0^\infty \frac{4s^2}{1-y^2} \left[ 1 - (1 - w_1)^s \right]^{p-1} - \left[ (1 + w_1)^s - 1 \right]^{p-1} y^{n-2} dy \, dw_1.
\]

Furthermore, we conclude this step by the following estimate.

\[
(1-x)^{-s} \int_0^{1-x} \frac{1}{w_1^{1+sp}} \int_0^\infty \left[ 1 - (1 - w_1)^s \right]^{p-1} - \left[ (1 + w_1)^s - 1 \right]^{p-1} y^{n-2} dy \, dw_1
\]

\[
= (1-x)^{-s} \int_0^{1-x} \frac{1}{w_1^{1+sp}} \int_0^\infty \frac{1}{(1+y^2)^{n+sp}} y^{n-2} dy \, dw_1
\]

\[
\leq C(1-x)^{-s} \int_0^{1-x} \frac{1}{w_1^{1+sp}} \left[ 1 - (1 - w_1)^s \right]^{p-1} - \left[ (1 + w_1)^s - 1 \right]^{p-1} y^{n-2} dy \, dw_1
\]

\[
\leq C(1-x)^{-s} \int_0^{1-x} \frac{1}{w_1^{1+sp}} \left[ 2 - (1 - w_1)^s - (1 + w_1)^s \right] \frac{1}{(1+y^2)^{n+sp}} y^{n-2} dy \, dw_1
\]

\[
\leq C(1-x)^{1+sp(p-2) \over 2} \int_0^{1-x} \frac{w_1^{p-2} - (1 - w_1)^s - (1 + w_1)^s}{w_1^{1+sp}} \frac{1}{(1+y^2)^{n+sp}} y^{n-2} dy \, dw_1
\]

Step 9. We demonstrate that \( \mathcal{Z} \) can be dropped, i.e. \( (1-x)^{-s} \mathcal{Z} \) is bounded when \( x \) is close to 1.

\[
(1-x)^{-s} \int_0^{1-x} \frac{1}{w_1^{1+sp}} \int_0^\infty \left( \frac{2s^2}{1-y^2} \right) \frac{y^{n-2}}{(1+y^2)^{n+sp}} dy \, dw_1
\]

\[
= (1-x)^{-s} \int_0^{1-x} \frac{1}{w_1^{1+sp}} \int_0^\infty \left( \frac{2s^2}{1-y^2} \right) \frac{y^{n-2}}{(1+y^2)^{n+sp}} dy \, dw_1
\]

\[
= (1-x)^{-s} \int_0^{1-x} \frac{1}{w_1^{1+sp}} \int_0^\infty \left( \frac{2s^2}{1-y^2} \right) \frac{y^{n-2}}{(1+y^2)^{n+sp}} dy \, dw_1
\]

For one thing,

\[
(1-x)^{-s} \int_0^{1-x} \frac{1}{w_1^{1+sp}} \int_0^\infty \left( \frac{2s^2}{1-y^2} \right) \frac{y^{n-2}}{(1+y^2)^{n+sp}} dy \, dw_1
\]

\[
\leq C(1-x)^{-s} \int_0^{1-x} \frac{1}{w_1^{1+sp}} \left[ \left( \frac{2s^2}{1-y^2} \right) \frac{y^{n-2}}{(1+y^2)^{n+sp}} \right] dy \, dw_1
\]

\[
\leq C(1-x)^{1+sp(p-2) \over 2}.
\]
For another thing,

\[(1 - x)^{-s} \int \frac{2^d}{w_1^{1+sp}} \frac{1}{w_1} \int_{1+x}^{\infty} \frac{y^{n-2}}{(1+y^2)^{\frac{1+sp}{2}}} dy dw_1 \leq C \int \frac{2^d}{w_1^{1+sp}} \frac{1}{w_1} \int_{1+x}^{\infty} \frac{y^{n-2}}{(1+y^2)^{\frac{1+sp}{2}}} dy dw_1 \]

\[\leq C(1 - x)^{-s} \int \frac{2^d}{w_1^{1+sp}} \frac{1}{w_1} \int_{1+x}^{\infty} \frac{y^{n-2}}{(1+y^2)^{\frac{1+sp}{2}}} dy dw_1 \]

\[\leq C(1 - x)^{-s} \int \frac{2^d}{w_1^{1+sp}} \frac{1}{w_1} \int_{1+x}^{\infty} \frac{y^{n-2}}{(1+y^2)^{\frac{1+sp}{2}}} dy dw_1 \]

\[\leq C \left( \frac{2^d}{1+x - w_1} \right)^{-s} \leq C.\]

Step 10. We will prove that \((1 - x)^{-s} \Theta'\) is bounded when \(x\) is close to 1.

\[(1 - x)^{-s} \int \frac{2^d}{w_1^{1+sp}} \frac{1}{w_1} \int_{1+x}^{\infty} \frac{y^{n-2}}{(1+y^2)^{\frac{1+sp}{2}}} dy dw_1 \]

\[= (1 - x)^{-s} \int \frac{2^d}{w_1^{1+sp}} \frac{1}{w_1} \int_{1+x}^{\infty} \frac{y^{n-2}}{(1+y^2)^{\frac{1+sp}{2}}} dy dw_1 \]

\[= (1 - x)^{-s} \int \frac{2^d}{w_1^{1+sp}} \frac{1}{w_1} \int_{1+x}^{\infty} \frac{y^{n-2}}{(1+y^2)^{\frac{1+sp}{2}}} dy dw_1 \]

\[+ (1 - x)^{-s} \int \frac{2^d}{w_1^{1+sp}} \frac{1}{w_1} \int_{1+x}^{\infty} \frac{y^{n-2}}{(1+y^2)^{\frac{1+sp}{2}}} dy dw_1.\]

On the one hand,

\[(1 - x)^{-s} \int \frac{2^d}{w_1^{1+sp}} \frac{1}{w_1} \int_{1+x}^{\infty} \frac{y^{n-2}}{(1+y^2)^{\frac{1+sp}{2}}} dy dw_1 \]

\[\leq C(1 - x)^{-s} \int \frac{2^d}{w_1^{1+sp}} \frac{1}{w_1} \int_{1+x}^{\infty} \frac{1}{(w_1 + \frac{2^d}{w_1})^{\frac{1+sp}{2}}} dw_1 \]

\[\leq C(1 - x)^{-s} \int \frac{2^d}{w_1^{1+sp}} \frac{1}{w_1} \int_{1+x}^{\infty} \left( w_1 + \frac{2^d}{1+x} \right)^{-\frac{1+sp}{2}} dw_1 \]

\[\leq \begin{cases} C(1 - x)^{-\frac{1+sp}{2}} & \text{if } sp > 1; \\
C(1 - x)^{-\frac{1+sp}{2}} \log \frac{1}{1-x}, & \text{if } sp = 1; \\
C(1 - x)^{-\frac{1+sp}{2}} \frac{1}{(1-x)^{1-sp}} = C(1 - x)^{s(p-1)} & \text{if } sp < 1. \end{cases} \]
On the other hand,

\[(1 - x)^{-s} \int_{\frac{4x}{1 - x^2}}^{\frac{2x}{1 - x^2}} \frac{1}{w_1^{1+sp}} \int_{\frac{4x}{1 - x^2}}^{\frac{2x}{1 - x^2}} \frac{1}{w_1^{1+sp}} \left( 1 - \frac{y}{(1 + y^2)^{\frac{n+sp}{2}}} \right)^{p-1} \frac{y^{n-2}}{(1 + y^2)^{\frac{n+sp}{2}}} dy \, dw_1 \leq C(1 - x)^{-s} \int_{\frac{2x}{1 - x^2}}^{\frac{4x}{1 - x^2}} \frac{1}{w_1^{1+sp}} \, dw_1 \leq C(1 - x)^{s(p-1)}.
\]

Step 11. This part is intended to prove that \((1 - x)^{-s} (2^p)'\) is bounded when \(x\) is close to 1.

\[
(1 - x)^{-s} \int_{\frac{2x}{1 - x^2}}^{\frac{4x}{1 - x^2}} \frac{1}{w_1^{1+sp}} \int_{\frac{2x}{1 - x^2}}^{\frac{4x}{1 - x^2}} \frac{1}{w_1^{1+sp}} \left[ 1 - \left( 1 + w_1 - \frac{x^2}{4x^2} (1 + y^2) w_1^2 \right)^{s} \right]^{p-1} y^{n-2} \, dy \, dw_1
\]

We will estimate the two terms separately.

\[
(1 - x)^{-s} \int_{\frac{2x}{1 - x^2}}^{\frac{4x}{1 - x^2}} \frac{1}{w_1^{1+sp}} \int_{\frac{2x}{1 - x^2}}^{\frac{4x}{1 - x^2}} \frac{1}{w_1^{1+sp}} \left[ 1 - \left( 1 + w_1 - \frac{x^2}{4x^2} (1 + y^2) w_1^2 \right)^{s} \right]^{p-1} y^{n-2} \, dy \, dw_1
\]

\[
= (1 - x)^{-s} \int_{\frac{2x}{1 - x^2}}^{\frac{4x}{1 - x^2}} \frac{1}{w_1^{1+sp}} \int_{\frac{2x}{1 - x^2}}^{\frac{4x}{1 - x^2}} \frac{1}{w_1^{1+sp}} \left[ 1 - \left( 1 + w_1 - \frac{x^2}{4x^2} (1 + y^2) w_1^2 \right)^{s} \right]^{p-1} y^{n-2} \, dy \, dw_1
\]

\[
+ (1 - x)^{-s} \int_{\frac{4x}{1 - x^2}}^{\frac{2x}{1 - x^2}} \frac{1}{w_1^{1+sp}} \int_{\frac{4x}{1 - x^2}}^{\frac{2x}{1 - x^2}} \frac{1}{w_1^{1+sp}} \left[ 1 - \left( 1 + w_1 - \frac{x^2}{4x^2} (1 + y^2) w_1^2 \right)^{s} \right]^{p-1} y^{n-2} \, dy \, dw_1
\]

\[
\leq C(1 - x)^{-s} \int_{\frac{2x}{1 - x^2}}^{\frac{4x}{1 - x^2}} \frac{1}{w_1^{1+sp}} \int_{\frac{4x}{1 - x^2}}^{\frac{2x}{1 - x^2}} \frac{1}{w_1^{1+sp}} \left[ 1 - \frac{x^2}{4x^2} (1 + y^2) w_1^2 \right]^{p-1} y^{n-2} \, dy \, dw_1 + C(1 - x)^{s(p-1)}.
\]
Moreover,

\[
(1 - x)^{-s} \int_1^{2x} \frac{1}{w_1^{1+sp}} \int \sqrt{\frac{4x^2 - w_1}{1-x^2 - w_1^2}} \frac{1 - \left(1 + w_1 - \frac{1}{4x^2}(1 + y^2)w_1^2\right)^{s}}{(1 + y^2)^{1+sp}} y^{n-2} dy \, dw_1
\]

\[
\leq C(1 - x)^{p-s-1} \int_1^{2x} w_1^{2p-sp-3} \int \sqrt{\frac{4x^2 - w_1}{1-x^2 - w_1^2}} \frac{1 - \left(1 + y^2\right)^{1-(p-1)}}{(1 + y^2)^{1+sp}} \, dw_1
\]

\[
\leq \left\{
\begin{aligned}
& C(1 - x)^{p-s-1} \int_1^{2x} w_1^{2p-sp-3} \frac{1 - \left(1 + y^2\right)^{1-(p-2)}}{(1 + y^2)^{1+sp}} \, dw_1, & \text{if } p - \frac{sp}{2} - 2 \geq 0; \\
& C(1 - x)^{p-s-1} \int_1^{2x} w_1^{2p-sp-3} \frac{1 - \left(1 + y^2\right)^{1-(p-2)}}{(1 + y^2)^{1+sp}} \sqrt{\frac{2}{1-x^2 - w_1^2}} \, dw_1, & \text{if } p - \frac{sp}{2} - 2 < 0.
\end{aligned}
\right.
\]

And

\[
(1 - x)^{-s} \int_1^{2x^2} \frac{1}{w_1^{1+sp}} \frac{1}{\sqrt{\frac{4x^2 - w_1}{1-x^2 - w_1^2}}} \, dw_1
\]

\[
\leq C(1 - x)^{1+s(p-2)} \frac{1}{w_1^{1+sp}} \, dw_1
\]

\[
\leq \left\{
\begin{aligned}
& C(1 - x)^{1+s(p-2)}, & \text{if } sp > 1; \\
& C(1 - x)^{1+s(p-2)} \log \frac{1}{1-x}, & \text{if } sp = 1; \\
& C(1 - x)^{s(p-1)}, & \text{if } sp < 1.
\end{aligned}
\right.
\]

In addition,

\[
(1 - x)^{-s} \int_1^{2x} \frac{1}{w_1^{1+sp}} \int \sqrt{\frac{4x^2 - w_1}{1-x^2 - w_1^2}} \frac{1 - \left(1 + w_1 - \frac{1}{4x^2}(1 + y^2)w_1^2\right)^{s}}{(1 + y^2)^{1+sp}} y^{n-2} dy \, dw_1
\]

\[
= (1 - x)^{-s} \int_1^{2x} \frac{1}{w_1^{1+sp}} \int \sqrt{\frac{4x^2 - w_1}{1-x^2 - w_1^2}} \frac{1 - \left(1 + w_1 - \frac{1}{4x^2}(1 + y^2)w_1^2\right)^{s}}{(1 + y^2)^{1+sp}} y^{n-2} dy \, dw_1
\]

\[
+ (1 - x)^{-s} \int_1^{2x^2} \frac{1}{w_1^{1+sp}} \frac{1}{\sqrt{\frac{4x^2 - w_1}{1-x^2 - w_1^2}}} \frac{1 - \left(1 + w_1 - \frac{1}{4x^2}(1 + y^2)w_1^2\right)^{s}}{(1 + y^2)^{1+sp}} \, dw_1
\]

\[
\leq C(1 - x)^{-s} \int_1^{2x^2} \frac{1}{w_1^{1+sp}} \frac{1}{\sqrt{\frac{4x^2 - w_1}{1-x^2 - w_1^2}}} \frac{1}{(1 + y^2)^{1+sp}} \, dw_1 + C(1 - x)^{s(p-1)}.
\]
Further,

\[(1 - x)^{-s} \int_{\frac{2x^2}{1 - x^2}}^{\frac{2x^2}{1 - x^2}} \frac{1}{w_1^{1+sp}} \frac{1}{\sqrt{\frac{4x^2}{1 - x^2} \frac{1 + w_1}{w_1}}} dw_1\]

\[\leq C(1 - x)^{\frac{1+sp}{2}} \int_{\frac{2x^2}{1 - x^2}}^{\frac{2x^2}{1 - x^2}} \left( \frac{1}{2} + w_1 \right)^{-\frac{1+sp}{2}} dw_1\]

\[\leq \begin{cases} 
C(1 - x)^{\frac{1+sp}{2}} & \text{if } sp > 1; \\
C(1 - x)^{\frac{1+sp}{2}} \log \left( \frac{1}{1 - x} \right) & \text{if } sp = 1; \\
C(1 - x)^{sp-1} & \text{if } sp < 1.
\end{cases}\]

Step 12. We plan to prove that \((1 - x)^{-s} (II)\) is bounded for \(x\) close to 1.

\[(1 - x)^{-s} \int_{\frac{2x^2}{1 - x^2}}^{\frac{2x^2}{1 - x^2}} \frac{1}{w_1^{1+sp}} \left[ 1 - \left( 1 - w_1 - \frac{1 - x^2}{4x^2} (1 + y^2) w_1^2 \right)^{s-1} \right] \frac{y^{-2}}{1 + y^2} dy dw_1\]

\[= (1 - x)^{-s} \int_{\frac{2x^2}{1 - x^2}}^{\frac{2x^2}{1 - x^2}} \frac{1}{w_1^{1+sp}} \left[ 1 - \left( 1 - w_1 - \frac{1 - x^2}{4x^2} (1 + y^2) w_1^2 \right)^{s-1} \right] \frac{y^{-2}}{1 + y^2} dy dw_1\]

\[+ (1 - x)^{-s} \int_{\frac{2x^2}{1 - x^2}}^{\frac{2x^2}{1 - x^2}} \frac{1}{w_1^{1+sp}} \left[ 1 - \left( 1 - w_1 - \frac{1 - x^2}{4x^2} (1 + y^2) w_1^2 \right)^{s-1} \right] \frac{y^{-2}}{1 + y^2} dy dw_1\]

\[= (1 - x)^{-s} \int_{\frac{2x^2}{1 - x^2}}^{\frac{2x^2}{1 - x^2}} \frac{1}{w_1^{1+sp}} \left[ 1 - \left( 1 - w_1 - \frac{1 - x^2}{4x^2} (1 + y^2) w_1^2 \right)^{s-1} \right] \frac{y^{-2}}{1 + y^2} dy dw_1\]

\[+ (1 - x)^{-s} \int_{\frac{2x^2}{1 - x^2}}^{\frac{2x^2}{1 - x^2}} \frac{1}{w_1^{1+sp}} \left[ 1 - \left( 1 - w_1 - \frac{1 - x^2}{4x^2} (1 + y^2) w_1^2 \right)^{s-1} \right] \frac{y^{-2}}{1 + y^2} dy dw_1\]

\[\leq C(1 - x)^{-s} \int_{\frac{2x^2}{1 - x^2}}^{\frac{2x^2}{1 - x^2}} \frac{1}{w_1^{1+sp}} \left[ 1 - \left( 1 - w_1 - \frac{1 - x^2}{4x^2} (1 + y^2) w_1^2 \right)^{s-1} \right] \frac{y^{-2}}{1 + y^2} dy dw_1\]

\[+ C(1 - x)^{-s} \int_{\frac{2x^2}{1 - x^2}}^{\frac{2x^2}{1 - x^2}} \frac{1}{w_1^{1+sp}} \left[ 1 - \left( 1 - w_1 - \frac{1 - x^2}{4x^2} (1 + y^2) w_1^2 \right)^{s-1} \right] \frac{y^{-2}}{1 + y^2} dy dw_1 + C(1 - x)^{-s} \int_{\frac{2x^2}{1 - x^2}}^{\frac{2x^2}{1 - x^2}} \frac{1}{w_1^{1+sp}} \left[ 1 - \left( 1 - w_1 - \frac{1 - x^2}{4x^2} (1 + y^2) w_1^2 \right)^{s-1} \right] \frac{y^{-2}}{1 + y^2} dy dw_1\]

\[\leq C(1 - x)^{p-s-1} \int_{\frac{2x^2}{1 - x^2}}^{\frac{2x^2}{1 - x^2}} w_1^{2p-3} \left[ 1 - \left( 1 - w_1 - \frac{1 - x^2}{4x^2} (1 + y^2) w_1^2 \right)^{s-1} \right] \frac{y^{-2}}{1 + y^2} dy dw_1 + C(1 - x)^{\frac{1+sp}{2}}.\]
Furthermore,

\[(1 - x)^{p-s-1} \int_{\frac{2}{1+x^2}}^{\frac{4s}{1+x^2}} \frac{1}{w_1^{1+sp}} \sqrt{\frac{4s^2 + w_1 - 1}{w_1}} \int_0^1 (1 + y^2)^{p-1} dydw_1 \]

\[\leq \begin{cases} 
C(1 - x)^{p-s-1} \int_{\frac{2}{1+x^2}}^{\frac{4s}{1+x^2}} \frac{1}{w_1^{1+sp}} (4s^2 + \frac{1}{1-x} w_1) \left( 1 - \frac{1}{4s^2} (1 + y^2 w_1^2) \right)^{p-2} \frac{1}{1+x^2} \sqrt{\frac{4s^2 + w_1 - 1}{w_1}} \, dw_1 \leq C(1 - x)^{\frac{1+s(p-2)}{2}}, & \text{if } p - \frac{sp}{x} - 2 \geq 0; \\
C(1 - x)^{p-s-1} \int_{\frac{2}{1+x^2}}^{\frac{4s}{1+x^2}} \frac{1}{w_1^{1+sp}} (4s^2 + \frac{1}{1-x} w_1) \left( 1 - \frac{1}{4s^2} (1 + y^2 w_1^2) \right)^{p-2} \frac{1}{1+x^2} \sqrt{\frac{4s^2 + w_1 - 1}{w_1}} \, dw_1 \leq C(1 - x)^{\frac{1+s(p-2)}{2}}, & \text{if } p - \frac{sp}{x} - 2 < 0.
\end{cases}
\]

Step 13. We finally prove that (IV) can be reduced to

\[\int_\frac{1}{2}^\infty \frac{[(1 + w_1)^s - 1]^{p-1}}{w_1^{1+sp}} \int_0^\infty \frac{y^{n-2}}{(1 + y^2)^{\frac{n+sp}{2}}} dydw_1. \tag{8}\]

Firstly, we prove that (IV) can be replaced by

\[\int_\frac{1}{2}^\infty \frac{1}{w_1^{1+sp}} \int_0^\infty \frac{[1 + w_1]^{s-1} - 1}{(1 + y^2)^{\frac{n+sp}{2}}} y^{n-2} dydw_1.
\]

By considering the difference,

\[(1 - x)^{-s} \int_\frac{1}{2}^\infty \frac{1}{w_1^{1+sp}} \int_0^\infty \frac{[1 + w_1]^{s-1} - 1}{(1 + y^2)^{\frac{n+sp}{2}}} y^{n-2} dydw_1 \]

\[= (1 - x)^{-s} \int_\frac{1}{2}^\infty \frac{1}{w_1^{1+sp}} \int_0^\infty \frac{[1 + w_1]^{s-1} - 1}{(1 + y^2)^{\frac{n+sp}{2}}} y^{n-2} dydw_1 \]

\[+ (1 - x)^{-s} \int_\frac{1}{2}^\infty \frac{1}{w_1^{1+sp}} \int_0^\infty \frac{[1 + w_1]^{s-1} - 1}{(1 + y^2)^{\frac{n+sp}{2}}} y^{n-2} dydw_1 \]

\[\leq C(1 - x)^{-s} \int_\frac{1}{2}^\infty \frac{1}{w_1^{1+sp}} \int_0^\infty \frac{[1 + w_1]^{s-1} - 1}{(1 + y^2)^{\frac{n+sp}{2}}} \left( 1 - \frac{1}{4s^2} (1 + y^2 w_1^2) \right)^{p-1} \, dw_1 \]

\[+ (1 - x)^{-s} \int_\frac{1}{2}^\infty \frac{1}{w_1^{1+sp}} \int_0^\infty \frac{[1 + w_1]^{s-1} - 1}{(1 + y^2)^{\frac{n+sp}{2}}} \left( 1 - \frac{1}{4s^2} (1 + y^2 w_1^2) \right)^{p-1} \, dw_1 \]

\[+ C(1 - x)^{s(p-1)}.
\]
Moreover, we begin to evaluate the first two terms separately,

\[
(1 - x)^{-s} \int_0^\frac{1}{2} \frac{1}{w_1^{1+s+p}} \left( \int_0^{\frac{4s^2}{1-s^2} \frac{1}{w_1}} \frac{(1 + w_1)^{s-2} \left((1 + w_1)(1 + w_1 - \frac{1 - x^2}{4s^2}(1 + y^2)w_1^2)\right)^{s}}{(1 + y^2)^{\frac{p+1}{2}}} \right) y^{n-2} dy \ dw_1
\]

\[
\leq C(1 - x)^{-s} \int_0^\frac{1}{2} \frac{1}{w_1^{1+s+p}} \left( \int_0^{\frac{4s^2}{1-s^2} \frac{1}{w_1}} \frac{(1 + w_1)^{s-2} \left((1 + w_1)(1 + w_1 - \frac{1 - x^2}{4s^2}(1 + y^2)w_1^2)\right)^{s}}{(1 + y^2)^{\frac{p+1}{2}}} \right) y^{n-2} dy \ dw_1
\]

\[
\leq C(1 - x)^{-s} \int_0^\frac{1}{2} \frac{1}{w_1^{1+s+p}} \left( \int_0^{\frac{4s^2}{1-s^2} \frac{1}{w_1}} \frac{(1 + w_1)^{s-2} \left((1 + w_1)(1 + w_1 - \frac{1 - x^2}{4s^2}(1 + y^2)w_1^2)\right)^{s}}{(1 + y^2)^{\frac{p+1}{2}}} \right) y^{n-2} dy \ dw_1
\]

= C(1 - x)^{-s} \int_0^\frac{1}{2} \frac{1}{w_1^{1+s+p}} \left( \int_0^{\frac{4s^2}{1-s^2} \frac{1}{w_1}} \frac{1}{(1 + y^2)^{\frac{p+1}{2}}} \right) \ dw_1 \leq C(1 - x)^{\frac{s(p-2)}{2}}, \text{ if } sp \geq 2;

\[
C(1 - x)^{-s} \int_0^\frac{1}{2} \left( \frac{4s^2}{1-s^2} \frac{1}{w_1}\right)^{1-\frac{4p}{s}} \ dw_1 \leq C(1 - x)^{\frac{s(p-2)}{2}}, \text{ if } sp < 2.
\]

Moreover,

\[
(1 - x)^{-s} \int_0^\frac{1}{4} \frac{1}{w_1^{1+s+p}} \left( \int_0^{\frac{4s^2}{1-s^2} \frac{1}{w_1}} \frac{(1 + w_1)^{s-1} - \left((1 + w_1 - \frac{1 - x^2}{4s^2}(1 + y^2)w_1^2)\right)^{s}}{(1 + y^2)^{\frac{p+1}{2}}} \right) y^{n-2} dy \ dw_1
\]

\[
= (1 - x)^{-s} \int_0^\frac{1}{4} \frac{1}{w_1^{1+s+p}} \left( \int_0^{\frac{4s^2}{1-s^2} \frac{1}{w_1}} \frac{(1 + w_1)^{s-1} - \left((1 + w_1 - \frac{1 - x^2}{4s^2}(1 + y^2)w_1^2)\right)^{s}}{(1 + y^2)^{\frac{p+1}{2}}} \right) y^{n-2} dy \ dw_1
\]

\[
+ (1 - x)^{-s} \int_0^\frac{1}{4} \frac{1}{w_1^{1+s+p}} \left( \int_0^{\frac{4s^2}{1-s^2} \frac{1}{w_1}} \frac{(1 + w_1)^{s-1} - \left((1 + w_1 - \frac{1 - x^2}{4s^2}(1 + y^2)w_1^2)\right)^{s}}{(1 + y^2)^{\frac{p+1}{2}}} \right) y^{n-2} dy \ dw_1
\]

\[
\leq C(1 - x)^{-s} \int_0^\frac{1}{4} \frac{1}{w_1^{1+s+p}} \left( \int_0^{\frac{4s^2}{1-s^2} \frac{1}{w_1}} \frac{1}{(1 + y^2)^{\frac{p+1}{2}}} \right) \ dw_1
\]

\[
+ C(1 - x)^{-s} \int_0^\frac{1}{4} \frac{1}{w_1^{1+s+p}} \left( \frac{2s^2}{1-s^2} \frac{1}{w_1}\right)^{\frac{4p}{s}} \ dw_1
\]

\[
\leq C(1 - x)^{-s} \int_0^\frac{1}{4} \frac{1}{w_1^{1+s+p}} \left( \int_0^{\frac{4s^2}{1-s^2} \frac{1}{w_1}} \frac{(1 + w_1)^{s-2} \left(1 + w_1 - \frac{1 - x^2}{4s^2}(1 + y^2)w_1^2\right)^{s}}{(1 + y^2)^{\frac{p+1}{2}}} \right) y^{n-2} dy \ dw_1 + C(1 - x)^{\frac{s(p-2)}{2}} \int_0^\frac{1}{4} \frac{1}{w_1^{1+s+p}} \ dw_1
\]

\[
\leq C(1 - x)^{-s} \int_0^\frac{1}{4} \frac{w_1^{1-2s}}{(1 + w_1)^{1-s}} \left( \int_0^{\frac{4s^2}{1-s^2} \frac{1}{w_1}} \frac{1}{(1 + y^2)^{\frac{p+1}{2}}} \right) \ dw_1 + C(1 - x)^{\frac{s(p-2)}{2}}.
\]
Besides,

\[(1 - x)^{1-s} \int_{\frac{1}{2}}^{2x^{-p}} \frac{w^{-1-s}}{(1 + w)^{1-s}} \int_{0}^{\sqrt{\frac{2x^{2}}{1 + x^{2}w^{-1}}} - 1} \frac{1}{(1 + y^{2})^{\frac{p}{2}}} dydw_{1}\]

\[\leq C(1 - x)^{1-s} \int_{\frac{1}{2}}^{2x^{-p}} w^{-s} \int_{0}^{\sqrt{\frac{2x^{2}}{1 + x^{2}w^{-1}}} - 1} \frac{1}{(1 + y^{2})^{\frac{p}{2}}} dydw_{1}\]

\[\leq \begin{cases} (1 - x)^{1-s} \int_{\frac{1}{2}}^{2x^{-p}} w^{-s} dw_{1} \leq C, & \text{if } sp \geq 2; \\ C(1 - x)^{1-s} \int_{\frac{1}{2}}^{2x^{-p}} \left( \frac{2x^{2}}{1 + x^{2}w^{-1}} \right)^{1-\frac{sp}{2}} dw_{1} \leq C(1 - x)^{\frac{sp}{2}} \int_{\frac{1}{2}}^{2x^{-p}} w^{\frac{sp}{2}-1} dw_{1} \leq C, & \text{if } sp < 2. \end{cases}\]

Secondly, we claim that (IV) can be substituted by

\[\int_{\frac{1}{2}}^{2x^{-p}} \frac{1}{w^{1+sp}} \int_{0}^{\infty} \left[ \frac{(1 + w)^{s} - 1}{(1 + y^{2})^{\frac{p}{2}}} \right] y^{n-2} dydw_{1}.\]

Since

\[(1 - x)^{-s} \int_{\frac{1}{2}}^{2x^{-p}} \frac{1}{w^{1+sp}} \int_{0}^{\infty} \left[ \frac{(1 + w)^{s} - 1}{(1 + y^{2})^{\frac{p}{2}}} \right] y^{n-2} dydw_{1}\]

\[\leq C(1 - x)^{-s} \int_{\frac{1}{2}}^{2x^{-p}} \left[ \frac{(1 + w)^{s} - 1}{w^{1+sp}} \right] \frac{1}{(4x^{2} \frac{1}{1 - x^{2}w^{-1}})^{\frac{p}{2}}} dydw_{1}\]

\[\leq C(1 - x)^{\frac{sp}{2}} \int_{\frac{1}{2}}^{2x^{-p}} w_{1}^{s(p-1)} dw_{1}\]

\[\leq C(1 - x)^{\frac{sp}{2}} \int_{\frac{1}{2}}^{2x^{-p}} w_{1}^{s(p-1)} dw_{1} \leq C.\]

Thirdly, (IV) can be replaced by

\[\int_{\frac{1}{2}}^{\infty} \frac{1}{w^{1+sp}} \int_{0}^{\infty} \left[ \frac{(1 + w)^{s} - 1}{(1 + y^{2})^{\frac{p}{2}}} \right] y^{n-2} dydw_{1}.\]

Due to the fact that

\[(1 - x)^{-s} \int_{\frac{1}{2}}^{2x^{-p}} \frac{1}{w^{1+sp}} \int_{0}^{\infty} \left[ \frac{(1 + w)^{s} - 1}{w^{1+sp}} \right] y^{n-2} dydw_{1} \leq C(1 - x)^{-s} \int_{\frac{1}{2}}^{2x^{-p}} w_{1}^{s(p-1)} dw_{1} \leq C.\]

Step 14. Overall, our conclusion is that the singular term for \(x\) close to 1 is

\[(1 - x)^{-s} \int_{0}^{\infty} \left( \frac{y^{n-2}}{(1 + y^{2})^{\frac{p}{2}}} \int_{0}^{\frac{1}{2}} \left[ \frac{1}{w^{1+sp}} \right] + \int_{0}^{\infty} \left[ \frac{1}{w^{1+sp}} \right] \right) dw_{1} \leq C.\]
Hence we only need to prove the following identity,

\[ \frac{1}{sp} + \int_0^1 \frac{[(1 - (1 - w_1)^s) - [(1 + w_1)^s - 1]^{p-1}}{w_1^{1+sp}} - \int_1^\infty \frac{[(1 + w_1)^s - 1]^{p-1}}{w_1^{1+sp}} dw_1 = 0. \] (9)

The proof of the identity is located in Step 3 in Section 2.

Step 15. For all \( x \in (0, 1 - \delta) \) with \( \delta > 0 \). By \[ \|\nabla^2 u(x)\| \leq C \int |x-y|^{n+sp} dy + C|\nabla^2 u(x)|^{p-2} \int_{B_r(x)} \frac{|x-y|^{p-2}}{|x-y|^{n+sp}} dy \leq C. \] (10)

Hence we have proved the boundedness of \( (-\Delta)^{\delta} p u(x) \) for higher dimensions.

4 Hopf’s theorem

We first give an example of a function in \( \{ C^{1,1}_{loc} \cap \mathcal{L}sp \} \setminus \mathcal{W}^{s,p} \) for bounded domain \( \Omega \), which means that the function space \( C^{1,1}_{loc} \cap \mathcal{L}sp \) is different from the function space \( \mathcal{W}^{s,p}(\Omega) \), where

\[ \mathcal{W}^{s,p}(\Omega) := \{ u \in L^p_{loc}(\mathbb{R}^n) \mid \exists U \owns \Omega \text{ such that } \int_{U \times U} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \ dx \ dy < +\infty \}. \]

Example 4.1. We set

\[ u(x) = \begin{cases} \frac{1}{|x|^{p}}, & \text{if } x \in B_1^+(0) := \{ x \in B_1(0) \mid x_n > 0 \}; \\ 0, & \text{if } x \in \mathbb{R}^n \setminus B_1^+(0). \end{cases} \] (11)

then \( u \in C^{1,1}_{loc}(B_1^+(0)) \) and \( u \) is lower semicontinuous,

\[ \int_{\mathbb{R}^n} \frac{|1 + u(x)|^{p-1}}{1 + |x|^{n+sp}} \ dx = \int_{B_1^+(0)} \frac{1}{1 + |x|^{n+sp}} \ dx + \int_{\mathbb{R}^n \setminus B_1^+(0)} \frac{1}{1 + |x|^{n+sp}} \ dx \]

\[ \leq C(p) \int_{B_1^+(0)} \frac{1}{1 + |x|^{n+sp}} \ dx + C(p) \int_{\mathbb{R}^n} \frac{1}{1 + |x|^{n+sp}} \ dx \]

\[ \leq C(p) \int_{B_1^+(0)} \frac{1}{|x|^{(p-1)}} \ dx + C(p) \]

\[ \leq C(p, n) \int_0^1 \rho^{n-t(p-1)-1} \ d\rho + C(p, n) \]

\[ < +\infty \text{ if } t < \frac{n}{p-1}. \]

So \( u(x) \in C^{1,1}_{loc}(B_1^+(0)) \cap \mathcal{L}_{sp}(\mathbb{R}^n) \) when \( t < \frac{n}{p-1} \). However, when \( t \geq \frac{n}{p-1} \), \( u(x) \notin L^p(B_1^+(0)) \).

So for \( p \in (\frac{n}{p-1}, \frac{n}{p-1}) \), \( u(x) \in \{ C^{1,1}_{loc} \cap \mathcal{L}_{sp} \} \setminus \mathcal{W}^{s,p} \). Hence \( C^{1,1}_{loc} \cap \mathcal{L}_{sp} \) is different from the space \( \mathcal{W}^{s,p} \).

From the example above, it is meaningful to investigate the Hopf’s theorem for \( u(x) \in C^{1,1}_{loc} \cap \mathcal{L}_{sp} \) in the pointwise sense.
Proof of Hopf’s theorem. Since \( u \) satisfies a uniform interior ball condition, we assume the uniform radius is \( 10\epsilon \). Then for every \( z \in \partial \Omega \), there is a ball \( B_\epsilon(y) \subset \Omega \) centered at \( y \in \Omega \) with \( \{z\} = \partial B_\epsilon(y) \cap \partial \Omega \). And \( \delta(x) := \text{dist}(x, \partial \Omega) = |x-z| \) for all \( x \in [y,z] \). Without the loss of generality, we can relocate the origin to \( y \) so that \( \delta(x) = \epsilon - |x| \).

We set \( \psi(x) = (1 - |x|^2)^s \), \( \psi_\epsilon(x) = \psi(x) / \epsilon^{2s} \), \( x \in B_\epsilon \),

then by theorem \( \ref{thm:hopf} \) there is a constant \( C_{0\epsilon} \), such that

\[
(-\Delta)_p^s \psi \leq C_0, \quad (-\Delta)_p^s \psi_\epsilon(x) = \frac{(-\Delta)_p^s \psi(x)}{\epsilon^{sp}} \leq C_0 / \epsilon^{sp}.
\]

Since \( u > 0 \) in \( \Omega \), we consider a region \( D := \{x \in \Omega : \text{dist}(x, \partial \Omega) \geq 3\epsilon\} \) which has a positive distance with \( B_\epsilon \). We can say \( C_D := \inf_D u(x) > 0 \). Then we set \( u_-(x) = u\chi_D + \epsilon \psi_\epsilon \), where \( \epsilon < C_D \) is to be decided later. Then suppose \( |t|^{p-1} = |t|^{p-2}t \) and for any \( x \in B_\epsilon \),

\[
(-\Delta)_p^s u_-(x) = C_{ns,p} P.V. \int_{\mathbb{R}^n} \frac{[\epsilon \psi_\epsilon(x) - u_-(y)]^{p-1}}{|x-y|^{n+sp}} dy
\]

\[
= C_{ns,p} P.V. \left\{ \int_{B_\epsilon(y)} \frac{[\epsilon \psi_\epsilon(x) - \epsilon \psi_\epsilon(y)]^{p-1}}{|x-y|^{n+sp}} dy + \int_{D} \frac{[\epsilon \psi_\epsilon(x) - u(y)]^{p-1}}{|x-y|^{n+sp}} dy 
\right.
\]

\[
+ \int_{\mathbb{R}^n \setminus (D \cup B_\epsilon(y))} \frac{[\epsilon \psi_\epsilon(x)]^{p-1}}{|x-y|^{n+sp}} dy \right\}
\]

\[
= \epsilon^{p-1} (-\Delta)_p^s \psi_\epsilon(x) + C_{ns,p} P.V. \left\{ \int_{D} \frac{[\epsilon \psi_\epsilon(x) - u(y)]^{p-1}}{|x-y|^{n+sp}} dy - \int_{D} \frac{[\epsilon \psi_\epsilon(x)]^{p-1}}{|x-y|^{n+sp}} dy \right\}
\]

\[
\leq C_{0\epsilon} \epsilon^{p-1} + C_{ns,p} P.V. \left\{ \int_{D} \frac{[\epsilon \psi_\epsilon(x)]^{p-1}}{|x-y|^{n+sp}} dy - \int_{D} \frac{[\epsilon \psi_\epsilon(x)]^{p-1}}{|x-y|^{n+sp}} dy \right\}
\]

\[
\leq C_{0\epsilon} \epsilon^{p-1} + C_{ns,p} 2^{-p} P.V. \left\{ \int_{D} \frac{[-u(y)]^{p-1}}{|x-y|^{n+sp}} dy \right\}
\]

\[
\leq C_{0\epsilon} \epsilon^{p-1} - C_{ns,p} 2^{-p} C_D \right\}
\]

\[
\leq C_{0\epsilon} \epsilon^{p-1} - C_{ns,p} 2^{-p} C_D C(q),
\]

where we used the inequality in lemma \( \ref{lem:inequality} \). \( G(t) - G(s) \leq 2^{2-p} G(t-s) \) if \( t < s \). So we can choose \( \epsilon_0 \) such that \( (-\Delta)_p^s u_-(x) + c(x) u_- \leq 0 \) for any \( x \in B_\epsilon \). Then by comparison principle,

\[
u(x) \geq u_-(x) = \epsilon_0 \psi_\epsilon(x) = \frac{\epsilon_0 (\epsilon^2 - |x|)^s}{\epsilon^{2s}} = \frac{\epsilon_0 (\epsilon + |x|)^s (\epsilon - |x|)^s}{\epsilon^{2s}} = \frac{\epsilon_0 (\epsilon + |x|)^s}{\epsilon^{2s}} \delta(x)^s, \forall x \in [y,z].
\]

Lemma 4.1. Suppose \( G(t) = |t|^{p-2} t \), \( p \geq 2 \), then \( G(t) - G(s) \leq 2^{2-p} G(t-s) \), if \( t < s \).

Proof. We only need to prove \( G(s) - G(t) \geq 2^{2-p}. \)
We assume $s \neq 0$, then
\[
\frac{G(s) - G(t)}{G(s - t)} = \left| \frac{s}{s - t} \right|^{p - 2} \frac{s - t}{|s - t|^{p - 2}} = \frac{1 - |\frac{s}{s - t}|^{p - 2} \frac{1}{s - t}}{|1 - \frac{1}{s - t}|^{p - 2}(1 - \rho)}.
\]

We set
\[
F(\rho) = \frac{1 - |\rho|^{p - 2} \rho}{|1 - \rho|^{p - 2}(1 - \rho)}.
\]

where $\rho \neq 1$, and $\lim_{\rho \to \pm \infty} F(\rho) = 1$. Then
\[
F'(\rho) = \frac{(p - 1)(1 - |\rho|^{p - 2})}{|1 - \rho|^p}.
\]

So for $p \geq 2$, $F(\rho)$ attains its minimum at $\rho = -1$, i.e. $F(\rho) \geq F(-1) = 2^{2-p}$.

\[
\square
\]

5 Global Hölder Regularity of Bounded Solutions

Theorem 5.1. Assume that $\Omega$ is any domain (bounded or not) with a uniform two-sided ball condition (e.g. a domain of class $C^{1,1}$), $s \in (0, 1)$, $p \geq 2$, and $u \in C^{1,1}_{loc}(\Omega) \cap L^s_{\Omega}$ is a bounded solution of

\[
\begin{cases}
(-\Delta)_p^s u = f(x, u), & x \in \Omega, \\
u = 0, & x \in \mathbb{R}^n \setminus \Omega,
\end{cases}
\]

where $f(x, u)$ is bounded, then there is a $\nu_0 \in (0, s)$, such that $u \in C^\nu(\mathbb{R}^n)$. Moreover,

\[
[u]_{C^\nu(\mathbb{R}^n)} \leq C(\nu_0) \left[ 1 + \|u\|_{L^\infty(\Omega)} + C\|f\|_{L^\infty(\Omega)} \right].
\]

Remark: When $u > 0$ in $\Omega$, the boundary condition can be reduced to $u \leq 0$ in $\mathbb{R}^n \setminus \Omega$. See [11] for more information about the ball condition.

Proof. First we repeat the last part of the proof of [11] Theorem 2, there are $\epsilon_0, \nu_0$, such that

\[
|u(x)| \leq c \left[ dist(x, \partial \Omega) \right]^{\nu_0}, \quad \forall x \in V := \{x \in \Omega | dist(x, \partial \Omega) < \epsilon_0\}.
\]

And by the boundeness of $u$, we have

\[
|u(x)| \leq c \left[ dist(x, \partial \Omega) \right]^{\nu_0}, \quad \forall x \in \Omega.
\]

Now we consider $x \in W := \{x \in \Omega \mid dist(x, \partial \Omega) \geq \epsilon_0\}$, then $B_{2\epsilon_0}(x) \subset \subset \Omega, \forall x \in W$.

We can prove that if $u \in C^{1,1}_{loc} \cap L^s_{\Omega}$, then $u \in W_{loc}^{s,p}$. In fact, for any domain $\Omega' \subset \subset \Omega$,

\[
\int_{\Omega' \times \Omega'} \frac{|u(x) - u(y)|^p}{|x - y|^{n + sp}} \, dx \, dy = \int_{\Omega'} \left( \int_{\Omega'} \frac{\nabla u(y) \cdot (x - y) + o(|x - y|)^p}{|x - y|^{n + sp}} \, dx \right) \, dy
\]

\[
\leq C(\Omega') \int_{\Omega'} \left( \int_{\Omega'} \frac{|x - y|^p}{|x - y|^{n + sp}} \, dx \right) \, dy \leq C(\Omega').
\]

29
By [10] Proposition 2.12, Lemma 2.5, \( u \) is a weak solution of \((-\Delta)^{\nu} u = f\) in \( B_{\frac{1}{4^n}}(x) \), \( \forall x \in W \). Then by [2] Theorem 1.4] we have for this \( v_0 \),

\[
[u]_{C_0^0(B_{\frac{1}{4^n}}(x))} \leq \frac{C(v_0)}{\epsilon_0 v_0} \left[ \left| u \right|_{L^\infty(B_{\frac{1}{4^n}}(x))} + \left( \epsilon_0^{sp} \int_{\mathbb{R}^n \setminus B_{\frac{1}{4^n}}(x)} \frac{|u|^{p-1}}{|x-y|^{n+sp}}dy \right)^{\frac{1}{p-1}} + \left( \epsilon_0^{sp} \left\| f \right\|_{L^\infty(B_{\frac{1}{4^n}}(x))} \right)^{\frac{1}{p-1}} \right]
\]

\[
\leq \frac{C(v_0)}{\epsilon_0 v_0} \left[ \left| u \right|_{L^\infty(\Omega)} + \epsilon_0^{sp} \left\| f \right\|_{L^\infty(\Omega)}^{\frac{1}{p-1}} \right] \leq C(v_0, \epsilon_0) \left[ \left| u \right|_{L^\infty(\Omega)} + \left\| f \right\|_{L^\infty(\Omega)}^{\frac{1}{p-1}} \right].
\]

So \( \forall x, y \in W \), we have \( \frac{|u(x) - u(y)|}{|x-y|^{\nu}} \leq C(v_0, \epsilon_0) \left[ \left| u \right|_{L^\infty(\Omega)} + \left\| f \right\|_{L^\infty(\Omega)}^{\frac{1}{p-1}} \right]. \)

The next part is similar to the [10] Section 5.2. Now we consider that \( x, y \in V \). Again by [2] Theorem 1.4], we use \( \delta_y \) to denote \text{dist}(x, \partial \Omega),

\[
[u]_{C_0^0(B_{\delta_y}(x))} \leq \frac{C(v_0)}{\delta_y v_0} \left[ \left| u \right|_{L^\infty(B_{\delta_y}(x))} + \left( \delta_y^{sp} \int_{\mathbb{R}^n \setminus B_{\delta_y}(x)} \frac{|u|^{p-1}}{|x-y|^{n+sp}}dy \right)^{\frac{1}{p-1}} + \left( \delta_y^{sp} \left\| f \right\|_{L^\infty(B_{\delta_y}(x))} \right)^{\frac{1}{p-1}} \right]
\]

\[
\leq \frac{C(v_0)}{\delta_y v_0} \left[ \delta_y^{\nu_0} + \left( \delta_y^{sp} \int_{\Omega \setminus B_{\frac{\delta_y}{4^n}}(x)} \frac{C\delta_y^{\nu_0(p-1)}}{|x-y|^{n+sp}}dy \right)^{\frac{1}{p-1}} + \left( \delta_y^{sp} \left\| f \right\|_{L^\infty(B_{\delta_y}(x))} \right)^{\frac{1}{p-1}} \right]
\]

\[
\leq C(v_0) \left( 1 + \left\| f \right\|_{L^\infty(\Omega)}^{\frac{1}{p-1}} \right) + \frac{C(v_0)}{\delta_y v_0} \left( \delta_y^{sp} \int_{\Omega \setminus B_{\delta_y}(x)} \frac{C(\delta_y + |x-y|^{\nu_0(p-1)})}{|x-y|^{n+sp}}dy \right)^{\frac{1}{p-1}}
\]

\[
\leq C(v_0) \left( 1 + \left\| f \right\|_{L^\infty(\Omega)}^{\frac{1}{p-1}} \right) + \frac{C(v_0)}{\delta_y v_0} \left( \delta_y^{sp} \int_{\Omega \setminus B_{\delta_y}(x)} \frac{C|x-y|^{\nu_0(p-1)}}{|x-y|^{n+sp}}dy \right)^{\frac{1}{p-1}}
\]

\[
\leq C(v_0) \left( 1 + \left\| f \right\|_{L^\infty(\Omega)}^{\frac{1}{p-1}} \right) + \frac{C(v_0)}{\delta_y v_0} \left( \delta_y^{sp} \int_{\Omega \setminus B_{\delta_y}(x)} \frac{C|x-y|^{\nu_0(p-1)}}{|x-y|^{n+sp}}dy \right)^{\frac{1}{p-1}}
\]

\[
\leq C(v_0) \left( 1 + \left\| f \right\|_{L^\infty(\Omega)}^{\frac{1}{p-1}} \right) + \frac{C(v_0)}{\delta_y v_0} \left( \delta_y^{sp} \int_{\Omega \setminus B_{\delta_y}(x)} \frac{C|x-y|^{\nu_0(p-1)}}{|x-y|^{n+sp}}dy \right)^{\frac{1}{p-1}}
\]

\[
\leq C(v_0) \left( 1 + \left\| f \right\|_{L^\infty(\Omega)}^{\frac{1}{p-1}} \right) + \frac{C(v_0)}{\delta_y v_0} \left( \delta_y^{sp} \int_{\Omega \setminus B_{\delta_y}(x)} \frac{C|x-y|^{\nu_0(p-1)}}{|x-y|^{n+sp}}dy \right)^{\frac{1}{p-1}}
\]

\[
\leq C(v_0) \left( 1 + \left\| f \right\|_{L^\infty(\Omega)}^{\frac{1}{p-1}} \right).
\]

Now for \( x, y \in V \) and without the loss of generality, we assume \( \delta_x \geq \delta_y \).

In the case \( |x-y| < \frac{\delta_x}{4^n} \), i.e. \( y \in B_{\frac{\delta_x}{4^n}}(x) \), so \( \frac{|u(x) - u(y)|}{|x-y|^{\nu}} \leq C(v_0) \left( 1 + \left\| f \right\|_{L^\infty(\Omega)}^{\frac{1}{p-1}} \right) \);

In the case \( |x-y| \geq \frac{\delta_x}{4^n} \geq \frac{\delta_y}{4^n} \), then \( |u(x) - u(y)| \leq |u(x)| + |u(y)| \leq C(\delta_x^{\nu_0} + \delta_y^{\nu_0}) \leq C|x-y|^{\nu_0} \).

Hence our conclusion is \( u \in C_0^0(\mathbb{R}^n) \). □
Conflict of interest statement

The authors report no conflicts of interest. The authors alone are responsible for the content and writing of this article.

References

[1] Simona Barb. *Topics in geometric analysis with applications to partial differential equations*. PhD thesis, University of Missouri–Columbia, 2009.

[2] Lorenzo Brasco, Erik Lindgren, and Armin Schikorra. Higher holder regularity for the fractional p-laplacian in the superquadratic case. *Advances in Mathematics*, 338:782–846, 2018.

[3] Luis Caffarelli and Luis Silvestre. An extension problem related to the fractional laplacian. *Communications in partial differential equations*, 32(8):1245–1260, 2007.

[4] Wenxiong Chen and Congming Li. Maximum principles for the fractional p-laplacian and symmetry of solutions. *Advances in Mathematics*, 335:735–758, 09 2018.

[5] Wenxiong Chen, Congming Li, and Yan Li. A direct method of moving planes for the fractional laplacian. *Advances in Mathematics*, 308:404–437, 2017.

[6] Wenxiong Chen, Congming Li, and Shijie Qi. A hopf lemma and regularity for fractional p-laplacians, 05 2018.

[7] Wenxiong Chen, Yan Li, and Ruobing Zhang. A direct method of moving spheres on fractional order equations. *Journal of Functional Analysis*, 272(10):4131–4157, 2017.

[8] Leandro M Del Pezzo and Alexander Quaas. A hopf’s lemma and a strong minimum principle for the fractional p-laplacian. *Journal of Differential Equations*, 263(1):765–778, 2017.

[9] R. K. Getoor. First passage times for symmetric stable processes in space. *Transactions of the American Mathematical Society*, 101(1):75–90, 1961.

[10] Antonio Iannizzotto, Sunra Mosconi, and Marco Squassina. Global holder regularity for the fractional p-laplacian. *arXiv preprint arXiv:1411.2956*, 2014.

[11] Yan Li Lingyu Jin. A hopf’s lemma and the boundary regularity for the fractional p-laplacian, 2019.

[12] Xavier Ros-Oton and Joaquim Serra. The dirichlet problem for the fractional laplacian: regularity up to the boundary. *Journal de Mathématiques Pures et Appliquées*, 101(3):275–302, 2014.

[13] Luis Silvestre. Regularity of the obstacle problem for a fractional power of the laplace operator. *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, 60(1):67–112, 2007.