On the fate of Lorentz symmetry in relative-locality momentum spaces

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The most studied doubly-special-relativity scenarios, theories with both the speed-of-light scale and a length/inverse-momentum scale as non-trivial relativistic invariants, have concerned the possibility of enforcing relativistically some nonlinear laws on momentum space. For the “relative-locality framework” recently proposed in [arXiv:1101.0931] a central role is played by nonlinear laws on momentum space, with the guiding principle that they should provide a characterization of the geometry of momentum space. Building on previous doubly-special-relativity results I here identify a criterion for establishing whether or not a given geometry of the relative-locality momentum space is “DSR compatible”, i.e. compatible with an observer-independent formulation of theories on that momentum space. I find that given some chosen parametrization of momentum-space geometry the criterion takes the form of an elementary algorithm. I show that relative-locality momentum spaces that fail my criterion definitely “break” Lorentz invariance, i.e. theories on such momentum spaces necessarily are observer-dependent “ether” theories. By working out a few examples I provide evidence that when the criterion is instead satisfied one does manage to produce a relativistic formulation. The examples I use to illustrate the applicability of my criterion also have some intrinsic interest, including two particularly noteworthy cases of $\kappa$-Poincaré-inspired momentum spaces.

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References
I. INTRODUCTION

Freidel, Kowalski-Glikman, Smolin and I recently proposed [1, 2] a class of theories which are most primitively formulated on momentum space, and whose main characteristics are codified in the geometry of momentum space. In that framework the momentum-space metric would primarily govern the form of the on-shell ("dispersion") relation, whereas the momentum-space affine connection governs the law of composition of momenta (and ultimately the law of conservation of "total" momentum in processes). The class of theories one might consider from this perspective is evidently very wide, especially since it appears legitimate [1, 2] to consider choices of affine connection on momentum space which are not Levi-Civita connections (allowing for non-metricity and/or torsion on momentum space). Exploring these possibilities is motivated in part by the fact that several arguments in the quantum-gravity literature could be viewed [1, 2] as pointing toward a role for momentum-space geometry. This was already remarkably emphasized in Born’s 1938 proposal [3] of a role for momentum-space curvature in the study of the quantum-gravity problem, and has more recently resurfaced through several independent arguments (see, e.g., Refs. [4–6]). Moreover, even setting aside the possible role in the study of the quantum-gravity problem, it appears that this novel framework deserves some interest because it provides an opportunity for studying systematically, including a description of interactions among particles, the possibility of a “relativity of spacetime locality”, which had been previously confined [7–9] (also see the more recent Refs. [10, 11]) to the narrow scopes of simple theories of free particles.

In the investigation of this “relative-locality framework” of Refs. [1, 2] it appears likely that, both conceptually [1, 2] and from the viewpoint of phenomenology [12, 13], an important role will be played by the understanding of the fate of Lorentz symmetry in theories formulated on the relative-locality momentum spaces. And this is the main focus of the study I am here reporting. I take as starting point the bulk of results on “doubly-special-relativity (DSR) theories” (see, e.g., Refs. [14–20]). These are relativistic theories with both the speed-of-light scale and a length/inverse-momentum scale as non-trivial relativistic invariants, and their most studied formulations [14–20] concern indeed achieving a (deformed-)relativistic formulation of physical laws introducing novel nonlinearities in momentum space. In particular, the most studied such DSR scenarios allow for novel nonlinearities, governed by the additional (length/inverse-momentum) relativistically-invariant scale, in the on-shell relation and in the law of conservation of “total” momentum, so they provide a very close starting point for the investigation of the fate of Lorentz symmetry in relative-locality momentum spaces, where the conjectured non-trivial geometry of momentum space indeed primarily results [1, 2] in modifications of the on-shell relation and of the law of momentum conservation.

The most significant result I am here reporting is contained in Section III. I introduce two requirements that must be satisfied by the metric and the affine connection on momentum space in order for that momentum-space to be possibly “DSR compatible”, i.e. such that the introduction of a characteristic scale of the geometry of momentum space still allows the formulation of observer-independent laws of physics. I observe that for any given parametrization of momentum-space geometry my two requirements can be expressed in terms of a simple algorithm, which in turn proves useful for assessing very easily whether a given choice of momentum-space geometry (a given choice of the parameters) can be DSR-compatible. I show that when the geometry of a relative-locality momentum space does not satisfy my two requirements then necessarily theories formulated on that momentum space will require the introduction of a preferred “ether” frame, a preferred class of inertial observers. I am unable to offer definite assurance that in all cases when instead the requirements are satisfied it will be possible to achieve a DSR-compatible formulation of theories on that momentum space: the requirements are evidently necessary but not so evidently sufficient. Still by working out (in Secs. IV and V) a few specific illustrative examples I provide some evidence that my requirements might also be sufficient: the illustrative examples I consider are such that the requirements are satisfied and for them I do manage to work out explicitly a formulation of relativistic kinematics which does not require a preferred frame.

Before this main part of the manuscript contained in Secs. III, IV and V I offer in Sec. II a brief review of the general structure of the connection that DSR compatibility establishes between the form of the on-shell relation and the form of the law of energy-momentum conservation.

Sec. VI contains a few remarks on how these results might affect the development of the relative-locality framework.

Sec. VII looks back at the criteria for DSR compatibility introduced in Sec. III and frames them within a simple relativistic theorem.

The closing Sec. VIII summarizes the main results here obtained and attempts to identify some priorities for the next steps.
that could be taken in this research area. I denote the momentum-space relative-locality scale with $\ell$ (an inverse-momentum scale) and I work at leading order in $\ell$. I am assuming that $|\ell|^{-1}$ is roughly of the order of the huge Planck scale, so that a leading-order analysis might be all we need for comparison to data we could realistically imagine to gather over the next few decades.

II. ON-SHELLNESS, MOMENTUM CONSERVATION AND DEFORMATIONS OF LORENTZ SYMMETRY

Before proceeding with the main part of the analysis, let me pause briefly, in this section, for summarizing the main points originally made in Ref. \[14\] concerning the consistency requirements that the relativity of inertial frames imposes on the relationship between the form of the on-shell (dispersion) relation and the form of the law of energy-momentum conservation. This is one of the most used DSR (“deformed special”, or “doubly special”, relativity) results, and plays a pivotal role in the analysis I report in the following sections.

I start this section by revisiting the transition from Galilean Relativity to Special Relativity since this can be of guidance for following then the logical structure of the transition from Special Relativity to DSR.

A. Aside on the transition from Galilean Relativity to Special relativity

Galilean Relativity enforced the relativity of rest (and the associated relativity of velocities). This is a notion that can be formulated without using a reference scale, so in turn the transformation laws from a Galilean-inertial observer to another do not involve any such reference scale.

This is manifest in all laws, and in particular in the on-shell relation

$$E = \frac{p^2}{2m} + m,$$

in the law of composition of velocities

$$\mathbf{u} \oplus \mathbf{v} = \mathbf{u} + \mathbf{v},$$

which we use in particular when connecting the description of a velocity $\mathbf{v}$ for a given observer Alice to the one of an observer Bob, when the relative Alice-Bob velocity is $\mathbf{u}$, and in the law of composition of energy-momentum

$$p_\mu \oplus p'_\mu = p_\mu + p'_\mu,$$

which we use in particular to enforce energy-momentum conservation when processes involving momentum exchange occur.

The transition from Galilean Relativity to Special Relativity enforces the relativity of simultaneity and the associated law of absoluteness of the speed of light. This does not challenge in any way the Galilean law of energy-momentum composition, which is indeed maintained $p_\mu \oplus p'_\mu = p_\mu + p'_\mu$. But the role of the speed of light as a relativistic invariant impose a change of on-shell relation

$$E = \sqrt{c^2 p^2 + c^4 m^2},$$

and a change in the law of composition of velocities

$$\mathbf{u} \oplus \mathbf{v} = \frac{1}{1 + \frac{\mathbf{u} \cdot \mathbf{v}}{c^2}} \left( \mathbf{u} + \frac{1}{\gamma_0} \mathbf{v} + \frac{1}{c^2} \frac{\gamma_0}{1 + \gamma_0 (\mathbf{u} \cdot \mathbf{v})} \mathbf{u} \right),$$

(1)
where as usual $\gamma_u \equiv 1/\sqrt{1 - u \cdot u/c^2}$.

Textbooks for undergraduates often choose to spare students the complexity of the composition law (1), limiting the discussion to the special case of (1) which occurs when $u$ and $v$ are collinear:

$$u \oplus v_{collinear} = \frac{u + v}{1 + \frac{u \cdot v}{c^2}}.$$

But the complexity of (1), which is non-commutative and non-associative, is well understood [21–23] as playing a central role in the logical consistency of Special Relativity. The special-relativistic prescription (1) renders the law of composition of velocities compatible with the principle of relativistic invariance of the speed of light $c$. Evidently insisting on the Galilean law $u \oplus v = u + v$ would have been inconsistent with the presence of an invariant velocity scale: boost transformations of velocity such that they saturate at $c$ could not possibly admit $u \oplus v = u + v$ as an observer-independent prescription. And of course the composition law (1) encodes all the richness of special-relativistic boosts, including the Thomas-Wigner rotations (essentially the fact that in the Lorentz algebra the commutator of non-parallel boosts produces a spatial rotation).

B. DSR-compatibility between on-shellness and momentum conservation

The idea of DSR-deformed Lorentz transformations was put forward [14] as a possible description of certain preliminary theory results suggesting that there might be violations of some special-relativistic laws in certain approaches to the quantum-gravity problem, most notably the ones based on spacetime noncommutativity and loop quantum gravity. The part of the quantum-gravity community interested in those results was interpreting them as a manifestation of a full breakdown of Lorentz symmetry, with the emergence of a preferred class of observers (an “ether”). But it was argued in [14] that departures from Special Relativity governed by a high-energy/short-distance scale may well be compatible with the Relativity Principle, the principle of relativity of inertial observers, at the cost of allowing some consistent modifications of the Poincaré transformations, and particularly of the Lorentz-boost transformations.

The main area of investigation of the DSR proposal has been for the last decade the possibility of introducing relativistically some deformed on-shell relations. The DSR proposal was put forward [14] as a conceptual path for pursuing a broader class of scenarios of interest for fundamental physics, and in particular for quantum-gravity research, including the possibility of introducing the second observer-independent scale primitively in spacetime structure or primitively at the level of the (deformed) de Broglie relation between wavelength and momentum. However, the bulk of the preliminary results from quantum-gravity research concern departures from the special-relativistic on-shell relation, and this in turn became the main focus of DSR research.

My objective in this section is to remind my readers about the line of analysis, originally discussed in Ref. [14], which allows us to conclude that if the on-shell relation involves a relativistically-invariant energy scale $M_* \equiv \ell^{-1}$ and if the relativity of inertial frames is to be preserved, then the scale $\ell$ must also intervene to modify the law of composition of momenta. For my purposes here it indeed suffices to work in leading order in $\ell$ (the scale $c$ from now on is set to 1), focusing for simplicity on the case of a 1+1-dimensional momentum space, and considering for definiteness only the illustrative example of a specific $\ell$-deformed on-shell relation. So let me consider the specific example of the on-shell relation $^1$

$$m^2 = p_0^2 - p_1^2 - \ell^2 p_0^2 p_1^2. \quad (2)$$

---

$^1$ If $\ell$ is of the order of the inverse of the Planck scale, the effects of DSR-deformed Lorentz symmetry in cases where the leading order is $\ell^2$ suppressed appear to be too soft to be appreciated experimentally, even setting a time scale of a few decades from now. This is the main reason why in the remainder of this manuscript I focus on cases where the leading order is only suppressed linearly by $\ell$. In this section however I am merely setting the stage for what follows by establishing the logical connection between the on-shell relation and the law of composition of momenta in a relativistic theory, so it does no harm to contemplate a deformation as weak as the one shown in Eq. (2).
Evidently this law is not Lorentz invariant. If we insist on this law and on the validity of classical (undeformed) Lorentz transformations between inertial observers we clearly end up with a preferred-frame picture, and the Principle of Relativity of inertial frames must be abandoned: the scale \( \ell \) cannot be observer independent, and actually the whole form of (2) is subject to vary from one class of inertial observers to another.

The other option \([14]\) in such cases is the DSR option of enforcing the relativistic invariance of (2), preserving the relativity of inertial frames, at the cost of modifying the action of boosts on momenta. Then in such theories both the velocity scale \( c \) (here mute only because of the choice of dimensions) and the inverse-energy scale \( \ell \) play the same role \([14]\) of invariant scales of the relativistic theory which govern the form of boost transformations.

Evidently if the action of boosts on momenta is non-linearly deformed so that (2) is invariant one must then renounce to the linear law of composition of momenta. In order to exhibit some formulas that illustrate this obvious fact, let me introduce the following deformed boost action

\[
[N, p_0] = p_1 + \frac{3}{2} \ell^2 p_0^2 p_1 + \ell^2 p_1^3, \quad [N, p_1] = p_0 + \frac{\ell^2}{2} p_0^3
\]

which evidently is such to leave invariant the deformed on-shell relation (2):

\[
[N, p_0^2 - p_1^2 - \ell^2 p_0^2 p_1^2] = 0.
\]

Equally evident is the fact that these deformed boosts are relativistically incompatible with the standard linear law of composition of momenta. Let us consider for example the case of a process with two incoming and two outgoing particles \( a + b \rightarrow c + d \). For this case one easily finds that

\[
[N, (p^{(a)} + p^{(b)})_\mu - (p^{(c)} + p^{(d)})_\mu] \neq 0,
\]

even when \( (p^{(a)} + p^{(b)})_\mu = (p^{(c)} + p^{(d)})_\mu \) is enforced.

Following the lessons of what turned out to be necessary for the composition of velocities in going from Galilean Relativity to Special Relativity, we can still look for laws of composition of momenta, \( p \oplus p' \), that would be relativistically compatible with the deformed boosts. A particular example (actually a particularly simple example, see later parts of this manuscript for other strategies of construction of the composition law) is the following

\[
(p \oplus p)_0 = p_0 + p'_0 + \ell^2 p_0^2 p'_1 + \ell^2 p_0 p'_1^2 + 2 \ell^2 p_1 p'_1 (p_0 + p'_0)
\]

\[
(p \oplus p')_1 = p_1 + p'_1 + \frac{\ell^2}{2} p_1 p_0^2 + \frac{\ell^2}{2} p'_1 p'_0^2 + \ell^2 p_0 p'_0 (p_1 + p'_1).
\]

With this prescription for the composition law the relativistic invariance is restored; indeed one can easily verify that when \( (p^{(a)} \oplus p^{(b)})_\mu = (p^{(c)} \oplus p^{(d)})_\mu \) one does have that

\[
[N, (p^{(a)} \oplus p^{(b)})_\mu - (p^{(c)} \oplus p^{(d)})_\mu] = 0
\]

(working again in leading order in \( \ell^2 \), consistently with the approximations made above).

This completes my brief summary of the relativistic consistency between modified on-shell relation and modified energy-momentum composition law first observed in Ref. \([14]\).

This brief summary will suffice to prepare the intuition of the reader for the results reported in the following.

I shall not dwell here on other aspects of DSR research, which however I want to briefly bring to the attention of the reader before closing this section.

It was recently realized that in at least some DSR frameworks the counterpart of the acquired absoluteness of the energy scale \( \ell^{-1} \) is an acquired relativity of spacetime locality \([7–11]\). This fits naturally with the observation that in Special Relativity the counterpart of the acquired absoluteness of the velocity scale \( c \) is an acquired relativity of simultaneity (colloquially...
“relative time”). From the DSR perspective the “relative-locality framework” adopted in the next sections is a particularly promising candidate for organizing logically, in terms of the geometry of momentum space, this relativity-locality features preliminarily characterized in Refs. [7–11], but of course the relative-locality framework may be considered even without DSR-compatibility, as a powerful formalism that can be applied also to cases where a preferred frame does arise.

Another framework that has been much considered from the DSR perspective if the one of Hopf-algebra symmetries (see, e.g., Refs. [24–26]) which in particular naturally accommodates nonlinear actions of the type here shown in Eq. (3).

Another area of active research concerns the DSR description of the properties of macroscopic bodies, composites of a large number of microscopic particles, as discussed in particular in Ref. [16] (also see Ref. [27] for the relative-locality-framework perspective).

III. A “RELATIVISTIC GOLDEN RULE” FOR THE GEOMETRY OF MOMENTUM SPACE

In this section I propose two criteria which can be used to establish whether or not a given relative-locality momentum space is “DSR compatible”, i.e. whether or not it is possible to formulate relativistic theories (observer-independent laws) on that momentum space. The two criteria are labeled, for reasons that shall soon be evident, “no-photon-decay-switch-on constraint” and “no-pair-production-switch-off constraint”. It will be clear for the careful reader that these criteria are applicable to any parametrization of the geometry of momentum space, but for definiteness and simplicity my explicit derivations are focused on a parametrization of the geometry of momentum space which is not completely general. For the momentum space metric I assume that it is such that (in the sense of Refs. [1, 2]) the on-shell (“dispersion”) relation takes the form

$$m^2 = p_0^2 - p_j^2 + \alpha_1 p_0 p_j^2 + \alpha_2 p_0^3 .$$

(6)

While for momentum-space affine connection I assume that it is such that (again in the sense of Refs. [1, 2]) the deformed law of composition of momenta takes the form

\[
\begin{align*}
(k \oplus p)_0 &= k_0 + p_0 + \beta_1 k \cdot \vec{p} + \beta_2 k_0 p_0 , \\
(k \oplus p)_j &= k_j + p_j + \gamma_1 k_0 p_j + \gamma_2 k_j.
\end{align*}
\]

(7)

This is a 6-parameter family of (leading-order) momentum-space geometries, with $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ all assumed to be either of order $\ell$ or completely negligible with respect to $\ell$.

The only other (however very weak) assumption needed for the derivation of my “golden rule” is that the relevant momentum-space theories involve a “vertex/interaction term” [1] potentially eligible for photon decay $\gamma \rightarrow e^+ e^-$

$$p_\gamma = p_+ \oplus p_-$$

(9)

and a “vertex term” eligible for electron-positron pair production from photon-photon collisions $\gamma \Gamma \rightarrow e^+ e^-$

$$p_+ \oplus p_\gamma = p_+ \oplus p_-$$

(10)

A. No-photon-decay-switch-on constraint

The first ingredient of my proposed ‘golden rule” is a “no-photon-decay-switch-on constraint”, essentially amounting to the request that massless particles should not decay$^2$. The photon-decay process $\gamma \rightarrow e^+ e^-$ is forbidden when the geometry of

\footnote{While the second half of my criterion, the requirement discussed in the next subsection (II.B) is completely new, the requirement discussed in this subsection II.A, which is the first half of my criterion, has been used in some previous studies of theories with nonlinearities in momentum space, starting}
momen"mum space is Minkowski/flat. Relative-locality momentum spaces must be such that for low-energy particles the geometry of momentum space is indistinguishable from the Minkowski/flat case, so in any relative-locality momentum space low-energy photons will be stable. Since in a relativistic theory a photon which is low-energy for one (local) observer Alice has different higher energy for another relatively-boosted observer Bob (also local to the photon) it must then be the case that photons of any energy are stable if the relative-locality momentum space is “DSR compatible”, since then evidently the laws that establish whether or not a photon can decay must be observer independent.

A key point for my argument is played by the concept of “decay threshold”. If photons were stable at low energies but could decay at higher energies there should be a threshold value of the photon energy above which the decay is allowed. But photons which are below a threshold energy value for one observer will be above that threshold energy value for other boosted observers. A threshold for photon decay could only be the scale \(|\ell|^{-1}\) of the relative-locality framework, which can be an invariant characteristic of the momentum-space geometry. But if one wants a relative-locality momentum space compatible with the implementation of any form of (possibly deformed) Lorentz invariance there cannot be a threshold for photon decay which is lower than \(|\ell|^{-1}\).

Of course, these requirements are perfectly enforced when momentum space is trivially Minkowski/flat (special relativity), and indeed there the process \(\gamma \rightarrow e^+e^-\) is strictly forbidden. One way to see this technically is to derive the formula that links the opening angle \(\theta\) of the (hypothetical) outgoing electron-positron and the energies \(E_+\) and \(E_-\) respectively of the positron and electron. One finds that the process would require \(\cos\theta > 1\) for any \(E_+, E_-\) combination. In particular, this classic special-relativistic result takes the following form (involving also the electron mass \(m\))

\[
\cos\theta \simeq \frac{2E_+E_- + m^2}{2E_+E_- + m^2\left(\frac{E_+}{E_-} + \frac{E_-}{E_+}\right)}
\]

(11)
in the limit of ultrarelativistic outgoing particles.

Our stated purpose for this subsection evidently requires us to reconsider this classic derivation when the momentum-space is characterized by \((6), (7), (8)\), rather than being trivially Minkowski/flat.

So I start from

\[
\begin{align*}
E_\gamma &= E_+ + E_- + \beta_1 \beta_+ \cdot \beta_- + \beta_2 E_+ E_- \\
\beta_\gamma &= \beta_+ + \beta_- + \gamma_1 E_+ \beta_- + \gamma_2 E_- \beta_+ 
\end{align*}
\]

(12)

(13)

From (13) it follows that

\[
p_\gamma^2 = p_+^2 + p_-^2 + 2p_+ p_- \cos\theta + 2\gamma_1 E_+ E_-^2 + 2\gamma_1 E_- E_+^2 + 2\gamma_1 E_+ E_-^2 + 2\gamma_2 E_+ E_-^2 + \cos\theta.
\]

(14)

Here on the right-hand side I already used the restriction to ultrarelativistic outgoing electron and positron (but only in terms with factors of \(\ell\)). It should be clear that this is where one can look for pathologies since the deformation of geometry of momentum space is negligible at small momenta and I am therefore evidently looking for a possible ultrarelativistic threshold for photon decay into an electron-positron pair.

with the analyses reported in Refs. [25, 24]. Those previous studies however only used this criterion for addressing specific issues within a given proposal of nonlinearities in moment space, whereas here I am using this requirement as a way to investigate the fate of Lorentz symmetry on a large class of theories. Another point to be stressed concerns the emphasis on photon decay: focusing on photon decay is sufficient for my purposes, and allows me keeps the discussion very explicit, but everything I observe here for photon decay applies equally well to some other “forbidden decays” (processes forbidden in special relativity, which might become allowed if Lorentz symmetry is broken) of “light”, but not necessarily massless, particles, such as neutrinos. An example of recent interest [20] involving neutrinos, specifically the process \(\nu \rightarrow \nu' e^+ e^-\), was discussed in Ref. [13].
From \(\text{(14)}\), using \(\text{(6)}\), one obtains
\[
E^2_{\gamma} + (\alpha_1 + \alpha_2)E^3_{\gamma} = E^2_+ + E^2_- - 2m^2 + \left[2E_+E_- + m^2 \left(\frac{E_+}{E_-} + \frac{E_-}{E_+}\right)\right] \cos \theta + (\alpha_1 + \alpha_2)(E^3_+ + E^3_-) +
\]
\[+ 2\gamma E_+E^- + 2\gamma E_+E_-^2 + [(\alpha_1 + \alpha_2)(E_+ + E_-) + 2\gamma E_+ + 2\gamma E_-] E_+E_- \cos \theta\]
\[(15)\]

For the left-hand side of this result one can use \(\text{(12)}\), which leads to
\[
E^2_+ + E^2_- + 2E_+E_- - 2(\beta_2 + \beta_1 \cos \theta)(E_+ + E_-)E_+ + (\alpha_1 + \alpha_2)(E_+ + E_-)^3 =
\]
\[= E^2_+ + E^2_- - 2m^2 + \left[2E_+E_- + m^2 \left(\frac{E_+}{E_-} + \frac{E_-}{E_+}\right)\right] \cos \theta + (\alpha_1 + \alpha_2)(E^3_+ + E^3_-) +
\]
\[+ 2\gamma E_+E^- + 2\gamma E_+E_-^2 + [(\alpha_1 + \alpha_2)(E_+ + E_-) + 2\gamma E_+ + 2\gamma E_-] E_+E_- \cos \theta\]
\[(16)\]

Then taking into account that I am working at leading order in \(\ell\), and assuming again that the outgoing particles are ultrarelativistic (so that \(\ell E_+ \neq 0\) and \(\ell E_- \neq 0\) but \(\ell m \approx 0\) and \(\ell^2 E^2_\pm \approx 0 \approx \ell^2 E^2_{\gamma}\)) one gets
\[
cos \theta \simeq \frac{2E_+E_- + m^2 + 2(\alpha_1 + \alpha_2 + \beta_1 + \beta_2 - \gamma_1 - \gamma_2)E_+E_-}{2E_+E_- + m^2 \left(\frac{E_+}{E_-} + \frac{E_-}{E_+}\right)}
\[(17)\]

So I have established that the “no-photon-decay-switch-on constraint” is
\[
\alpha_1 + \alpha_2 + \beta_1 + \beta_2 - \gamma_1 - \gamma_2 \geq 0
\]\[\text{(18)}\]

If instead \(\alpha_1 + \alpha_2 + \beta_1 + \beta_2 - \gamma_1 - \gamma_2 < 0\) then there would be some high values of \(E_+, E_-\) (and correspondingly \(E_\gamma\)) for which a real \(\theta\) could solve \(\text{(17)}\) \(i.e.\) such that \(|\cos \theta| \leq 1\), and this in turn would mean that photon decay is allowed at those high energies.

It must be noticed that the energies needed for this are ultra-high but below-Planckian: assume for example \(\alpha_1 + \alpha_2 + \beta_1 + \beta_2 - \gamma_1 - \gamma_2 = -\ell\) then photon decay starts to be allowed already at scales roughly of order \((m^2 |\ell|^{-1})^{1/3}\) (which indeed, for \(m\) the electron mass and \(|\ell|^{-1}\) roughly of order the Planck scale, is \(\ll |\ell|^{-1}\)).

### B. No-pair-production-switch-off constraint

For the second requirement, the “no-pair-production-switch-off constraint”, the process I use is electron-positron pair production in photon-photon collisions: \(\gamma + \Gamma \rightarrow e^+ e^-\) (I adopt a convention such that the energy \(\epsilon\) of the photon \(\gamma\) is lower than the energy \(E\) of the photon \(\Gamma\)).

This process if evidently allowed in ordinary special relativity, and it has been established to actually occur in countless experiments, of course in a limited range of so-far-experimentally-accessible values of the energies of the two incoming photons. What could “go wrong” with this pair-production process in absence of (not even some deformation of) Lorentz symmetry? To see this let us gradually lower the value of the energy \(\epsilon\) of the “soft” photon, and ask if there are correspondingly high hard-photon energies compatible with pair production. In special relativity of course the answer is always yes: for given low energy \(\epsilon\) of the photon \(\gamma\) one has that pair production is always allowed if the other photon \(\Gamma\) has energy \(E \geq m^2/\epsilon\). The question “can a photon of energy \(\epsilon\) interact with another photon to produce an electron-positron pair?” always has positive answer, for all values of \(\epsilon\). This is also a necessary consequence of the fact that two relatively boosted observers attribute different energy to a given photon: a relativistic description of such a pair of relatively boosted observers evidently excludes the possibility of a threshold for “pair-production switch-off”.

And it is also evident that such a threshold for pair-production switch-off must not be present in any theory providing a
relativistic description of pairs of relatively-boosted observers. So in order for a chosen geometry of momentum space to be DSR-compatible it must always be possible for a photon of any energy $\varepsilon$ to produce electron-positron pairs in interactions with at least some sufficiently high-energy photons. I shall now show that this imposes another nontrivial requirement on our parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$.

The process $\gamma + \Gamma \rightarrow e^+ e^-$ involves one more particle than the photon-decay process considered in the previous subsection, but the analysis is simplified by the fact that the constraint I am looking for can be established focusing on collinear processes. In fact, the energy $E$ of photons eligible to produce pairs with a photon of given energy $\varepsilon$ is inevitably going to be greater than a certain minimum value of $E$, which I shall denote with $E_{\min}$, for which the process is collinear. So I start from

\[
E_{\min} + \varepsilon = E_+ + E_- + \beta_1 p_+ p_- + \beta_2 E_+ E_-
\]

(19)

\[
p_{\min} - \varepsilon = p_+ + p_- + \gamma_1 E_+ p_- + \gamma_2 E_- p_+
\]

(20)

where I chose to focus on cases where $\varepsilon \ll E_{\min}$ so that $\ell \varepsilon \simeq 0$ even though $\ell E_{\min} \neq 0$. Then one can easily combine (20) and (6), also relying on some of the approximations already exploited in the previous subsection, to establish that

\[
E_{\min} + \alpha_1 + \alpha_2 E_{\min}^2 - \varepsilon = E_+ + E_- - \frac{m^2}{2E_+} + \frac{m^2}{2E_-} + \frac{\alpha_1 + \alpha_2}{2}(E_+^2 + E_-^2) + (\gamma_1 + \gamma_2)E_+ E_-. \quad (21)
\]

A further simplification is obtained by noticing that the zero-th order ($\ell \rightarrow 0$) solution is such that $E_+ = E_- = E_{\min}/2$, and that this zero-th order property can be safely assumed, consistently with the approximations I am adopting, to still hold within terms with already small prefactors of $m^2$ ($\ll E_{\min}^2$) or $\ell$ ($\ll 1/E_{\min}$). So from (21) one has

\[
E_{\min} + \frac{\alpha_1 + \alpha_2}{2} E_{\min}^2 - \varepsilon = E_+ + E_- - 2 \frac{m^2}{E_{\min}} + \frac{\alpha_1 + \alpha_2}{4} E_{\min}^2 + (\gamma_1 + \gamma_2) E_{\min}^2 \quad (22)
\]

and from (19) one has

\[
E_{\min} + \varepsilon = E_+ + E_- + (\beta_1 + \beta_2) \frac{E_{\min}^2}{4} \quad (23)
\]

Combining (22) and (23) one finds

\[
2\varepsilon = 2 \frac{m^2}{E_{\min}} + (\alpha_1 + \alpha_2) \frac{E_{\min}^2}{4} + (\beta_1 + \beta_2) \frac{E_{\min}^2}{4} - (\gamma_1 + \gamma_2) \frac{E_{\min}^2}{4} \quad \text{(24)}
\]

So in summary $E_{\min}$ must satisfy the condition

\[
E_{\min} - (\alpha_1 + \alpha_2 + \beta_1 + \beta_2 - \gamma_1 - \gamma_2) \frac{E_{\min}^3}{4\varepsilon} = \frac{m^2}{\varepsilon} \quad \text{(25)}
\]

This allows us to conclude that in order to avoid the “pair-production switch-off” one must enforce

\[
\alpha_1 + \alpha_2 + \beta_1 + \beta_2 - \gamma_1 - \gamma_2 \leq 0 \quad \text{(26)}
\]

If instead $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 - \gamma_1 - \gamma_2 > 0$ then one could find values of $\varepsilon$ small enough (values of $m^2/\varepsilon$ large enough) that (25) would admit no solution, so that indeed pair-production would be switched off.

And once again it turns out that the issue is not confined to the “Planckian regime”: for example if $\varepsilon$ is $\sim 10^{-5}$ eV in standard special relativity (flat/Minkowski momentum-space geometry) pair production can occur whenever the other photon has energy $\geq 3 \cdot 10^{17}$ eV, whereas with the deformation scheme I am considering, if for example $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 - \gamma_1 - \gamma_2 \simeq \ell$, for $\varepsilon \sim 10^{-5}$ eV the pair-production process is already switched off: if $\varepsilon \sim 10^{-5}$ eV then according to (25) pair production cannot occur for any value of $E$, if $m$ is the electron mass, $|\ell|^{-1}$ is roughly of order the Planck scale, and $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 - \gamma_1 - \gamma_2 \simeq \ell$. 
C. A golden rule

Combining the results derived in the previous two subsections, summarized in Eq. (18) and Eq. (26), I obtain the following constraint on the deformation parameters:

\[ \alpha_1 + \alpha_2 + \beta_1 + \beta_2 - \gamma_1 - \gamma_2 = 0 \]  \hspace{1cm} (27)

Following the logical line of the observations I reported in this section one concludes that for such geometries of relative-locality momentum spaces

IF the geometry of the momentum space violates this constraint (i.e. \( \alpha_1 + \alpha_2 + \beta_1 + \beta_2 - \gamma_1 - \gamma_2 \neq 0 \))

THEN theories on such relative-locality momentum spaces require the introduction of a preferred “ether” frame (the laws of physics on such relative-locality momentum spaces are observer-dependent).

The condition (27) is necessary for a relative-locality momentum space to be DSR-compatible. But is (27) also a sufficient condition for a relative-locality momentum space to be DSR-compatible?

That is, rewriting more explicitly, are we assured that on relative-locality momentum spaces that satisfy (27) it will be possible to introduce laws that involve the scale \( \ell \) characteristic of the momentum-space geometry and yet are observer-independent laws?

The derivations I have offered so far only establish (27) as a necessary condition for DSR-compatibility, but in the following sections, by investigating a few examples of relative-locality momentum spaces, I provide evidence that when (27) is satisfied one does manage to base on such relative-locality momentum spaces laws of physics which are observer independent.

So I conjecture that (27) is also sufficient for DSR-compatibility.

If there are any counter-examples to my conjecture of sufficiency of (27) for DSR compatibility I expect them to be somewhat “pathological” in one way or another: \textit{a posteriori} I can recognize in Eq. (27) a compact summary, a sort of “golden rule”, reflecting a significant part of my experience working for more than a decade with scenarios of Lorentz-symmetry deformation. And even just the established necessity for DSR compatibility of (27) qualifies it as a “golden rule” for the phenomenology of deformations of Lorentz symmetry centered on momentum space: (27) allows one to quickly conclude that a certain geometry of momentum space does not admit observer-independent laws of physics, whereas reaching the same conclusion using more formal tools of investigation can be an endless task.

For the illustrative example of parametrization of momentum space on which I focused my “criteria” (necessary conditions) for DSR compatibility took the shape of the simple algorithmic requirement (27). I expect that for most other parametrizations of the momentum-space geometry my criteria will again produce simple algorithmic recipes. An exception to this expectation of simple algorithmic implementation of the criteria may perhaps be found in scenarios such as the one in Ref. [32], where the deformation is not structured simply as a power series in the components of momentum (in particular Ref. [32] tentatively contemplates deformations involving powers of components of momentum divided by powers of the mass of the particle): in such cases the region of energy scales of interest in physics applications can be outside the region of applicability of the leading-order approximation and, while my criteria would still apply, the algorithmic formulation of my criteria might be unavailable.

IV. SOME EXAMPLES WITH TORSION-FREE MOMENTUM SPACE

The remainder of this manuscript has two goals:
(i) test the reliability of the “golden rule” derived in the previous section, adopted tentatively as a (not only necessary but also) sufficient condition for DSR-compatibility, by looking at momentum spaces where the golden rule is satisfied and checking explicitly that some formulation of deformed Lorentz symmetry is available;
(ii) illustrate some of the peculiarities that can characterize deformed Lorentz symmetry on a relative-locality momentum space.

The examples of relative-locality momentum spaces I consider in this section and the next section are characterized by
being setups which were already of interest during the earliest stages of DSR research. Specifically in this section I start with two examples of torsionless momentum spaces, with in particular one example matching exactly the form of nonlinear laws in momentum space that were considered already in Ref. [14], as an attempt to illustrate the idea that one might try to have a relativistic theory with an observer-independent length/inverse-momentum scale, while preserving the observer-independence of the laws of physics and the overall relativistic nature of the theory. Since relative-locality momentum spaces are always flat in leading order [1, 2] (the lowest-order contribution to curvature of the metric on a relative-locality momentum space is of order $\ell^2$), and these are also torsionless cases, these are evidently not the most interesting scenarios from the novel relative-locality perspective, but I find nonetheless somewhat reassuring that indeed the golden rule does perform well in my two examples of this sort.

Then in the next section I consider two other examples of a traditional type in the DSR literature, examples largely inspired by properties of the $\kappa$-Poincaré Hopf algebra [24–26]. The $\kappa$-Poincaré-inspired nonlinearities for momentum-space laws appeared immediately [14] as a natural candidate for obtaining a DSR-compatible framework. But this $\kappa$-Poincaré opportunity for DSR research has remained in a “sub judice status” mainly as a result that, as already noticed in Ref. [14], some of the attempts to build quantum field theories with $\kappa$-Poincaré structures appeared not to enforce the observer independence of the laws of physics.

The observations I here report in the next section on the $\kappa$-Poincaré-inspired relative-locality momentum space (actually two versions of it) go some way in the direction of establishing more firmly the case for DSR compatibility, and do so while providing further evidence of robustness of the use of my “golden rule” as a sufficient condition for DSR-compatibility. Within the relative-locality framework these $\kappa$-Poincaré-inspired relative-locality momentum spaces are representatives of the case of torsionful momentum spaces, and are therefore rather nontrivial.

As announced, here and in the next section I work again exclusively in leading order in $\ell$. And I work in 1+1 momentum-space dimensions, so that the formulas take their simplest possible form and it will be easier to highlight the conceptual steps.

A. Brief summary of previous results on the “DSR1”

As announced, the first definite example I consider of relative-locality momentum space is actually a setup that was already considered in the DSR literature, and there known as “DSR1”, often used (already in Ref. [14]) as a way to illustrate some of the ingredients that could be used when attempting to produce a DSR framework, with special-relativistic laws deformed by a length/inverse-momentum scale which are however relativistic/observer-independent laws. A key issue of interest in the DSR literature is indeed the compatibility of a given choice of on-shell relation and a given choice of momentum-conservation laws with (possibly deformed) Lorentz invariance.

In this DSR1 setup the on-shell relation takes the form [14]

$$m^2 = p_0^2 - p_1^2 + \ell p_0 p_1,$$  \hspace{1cm} (28)

which is invariant under the DSR1-deformed boost action [14]

$$[N, p_0] = p_1, \quad [N, p_1] = p_0 + \ell p_0^2 + \frac{\ell}{2} p_1^2,$$  \hspace{1cm} (29)

3 The DSR proposal was not exclusively intended [14] for allowing an observer-independence/relativistic description of deformed laws on momentum space. Examples of other options that were and are considered include introducing the second observer-independent scale primitive in spacetime structure or primitively at the level of the (deformed) de Broglie relation between wavelength and momentum [14, 20]. However, since I am here focusing on the relative-locality framework, with its assumption that momentum space is primitive [1, 5], the DSR-research tools I shall need all come from previous attempts to construct a DSR framework centered on nonlinear laws on momentum space.
And an example of DSR1-deformed law of conservation of momentum that would be compatible with the boost action (29) is (14):

\[
p_0^{(a)} + p_0^{(b)} + \ell p_1^{(a)} p_1^{(b)} = p_0^{(c)} + p_0^{(d)} + \ell p_1^{(c)} p_1^{(d)} \tag{30}
\]

\[
p_1^{(a)} + p_1^{(b)} + \ell p_0^{(a)} p_1^{(b)} + \ell p_1^{(a)} p_0^{(b)} = p_1^{(c)} + p_1^{(d)} + \ell p_0^{(c)} p_1^{(d)} + \ell p_1^{(c)} p_1^{(d)} \tag{31}
\]

for an event \(a + b \rightarrow c + d\).

The on-shell relation of this DSR1 model, (28), is evidently of a type that can emerge from a choice of metric on momentum space. But I must still comment on the law of composition of momenta, which according to Refs. [1,2] codifies the affine connection on momentum space. It is nearly self-evident that the conservation laws (30)-(31) implicitly use the composition law

\[
(k \oplus p)_0 = k_0 + p_0 + \ell k_1 p_1 , \quad (k \oplus p)_1 = k_1 + p_1 + \ell k_0 p_1 + \ell p_0 k_1 , \tag{32}
\]

which is commutative and therefore, according to the criteria introduced in Refs. [1,2], corresponds to a torsionless connection 4.

In order to verify explicitly that the composition law (32) corresponds to the conservation law (30)-(31) I must check that (30)-(31) is equivalent to

\[
((p^{(a)} \oplus p^{(b)} + p^{(c)} \oplus p^{(d)}))_\mu = 0 ,
\]

which in light of (32) can be rewritten as

\[
p_0^{(a)} + p_0^{(b)} + p_0^{(c)} + p_0^{(d)} + \ell p_1^{(a)} p_1^{(b)} + \ell p_1^{(a)} p_1^{(c)} + \ell p_1^{(a)} p_1^{(d)} + \ell p_1^{(b)} p_1^{(c)} + \ell p_1^{(b)} p_1^{(d)} + \ell p_1^{(c)} p_1^{(d)} = 0 ,
\]

\[
p_1^{(a)} + p_1^{(b)} + p_1^{(c)} + p_1^{(d)} + \ell p_0^{(a)} p_1^{(b)} + \ell p_0^{(a)} p_1^{(c)} + \ell p_0^{(a)} p_1^{(d)} + \ell p_0^{(b)} p_1^{(c)} + \ell p_0^{(b)} p_1^{(d)} + \ell p_0^{(c)} p_1^{(d)} = 0 ,
\]

which may be viewed as the case of two incoming and two outgoing momenta [1], where \(\ominus\) is the antipode of the \(\oplus\) in (32)

\[
(\ominus p)_0 = -p_0 + \ell p_1 p_1 , \quad (\ominus p)_1 = -p_1 + 2\ell p_0 p_1 . \tag{33}
\]

Indeed one easily checks that \(p \oplus (\ominus p) = 0\), as required for the antipode.

This observation finds confirmation in the fact that

\[
((p^{(a)} \oplus p^{(b)} \oplus [\ominus p^{(c)}]) \oplus [\ominus p^{(d)}])_0 = 0 \iff p_0^{(a)} + p_0^{(b)} + \ell p_1^{(a)} p_1^{(b)} = p_0^{(c)} + p_0^{(d)} + \ell p_1^{(c)} p_1^{(d)}
\]

\[
((p^{(a)} \oplus p^{(b)} \oplus [\ominus p^{(c)}]) \oplus [\ominus p^{(d)}])_1 = 0 \iff p_1^{(a)} + p_1^{(b)} + \ell p_0^{(a)} p_1^{(b)} + \ell p_1^{(a)} p_0^{(b)} = p_1^{(c)} + p_1^{(d)} + \ell p_0^{(c)} p_1^{(d)} + \ell p_1^{(c)} p_1^{(d)}
\]

Moreover, this DSR1 setup, besides being torsionless, can also be mapped by a diffeomorphism onto the flat momentum space of special relativity. Some authors (see, e.g., Refs. [33,34]) have stressed that if the overall theoretical framework introduced on momentum space is diffeomorphism invariant then of course the DSR1 kinematics in such instances would be just reproducing the physical predictions of ordinary special relativity. Not denying the potential appeal of insisting on such a "momentum-space general covariance", I remain interested also in the possibility that the laws of physics on momentum space may not be diffeomorphism invariant, in which case setups such as this DSR1 kinematics and the other setup discussed in the next subsection could be highly nontrivial. Readers who are only interested in setting up theories whose overall structure ensures momentum-space diffeomorphism invariance may well skip this section and proceed to the next Section [V].
where the equivalences are established of course dropping terms which vanish when the conservation laws are enforced (and taking into account that I am working in leading order in $\ell$).

So it is at this point fully established that the geometry of momentum space that matches the prescriptions of the DSR1 model is characterized in terms of the on-shell relation (28) and the composition law (32).

With this observation we are ready to check that the DSR1 setup is consistent with the “golden rule”. For that purpose let me observe that, using the parametrization of the previous section, the DSR1 on-shell relation and composition law correspond to the choice of parameters $\alpha_1 = \ell$, $\alpha_2 = 0$, $\beta_1 = \ell$, $\beta_2 = 0$, $\gamma_1 = \gamma_2 = \ell$, so that the golden rule is satisfied: $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 - \gamma_1 - \gamma_2 = 0$. And indeed it is easy to verify [12] that when the conservation laws (30)–(31) hold for one observer, say Alice, they also hold for any observer boosted according to Eq. (29) with respect to Alice.

We can therefore recognize in the DSR1 model a first success for the possibility of viewing the “golden rule” as a sufficient condition for DSR-compatibility.

### B. Another torsionless example, with abelian energy composition

Let me now consider a slightly different setup, which was not previously discussed in the literature: a setup similar to the one of the previous subsection (in particular with torsionless momentum space) and with the added “simplicity bonus” of undeformed additivity\(^5\) of energy (but not of spatial momenta).

For this second example of torsionless momentum space I take a metric such that the one-shell condition is

$$m^2 = p_0^2 - p_1^2 + 2\ell p_0 p_1^2$$

and for the affine connection I take one such that

$$(k \oplus p)_1 = k_1 + p_1 + \ell k_0 p_1 + \ell p_0 k_1, \quad (k \oplus p)_0 = k_0 + p_0,$$

which is indeed commutative, as required for a torsionless momentum space in the sense of Refs. [1, 2].

I note down the antipode that follows from (35)

\[ (\oplus p)_1 = -p_1 + 2\ell p_0 p_1, \quad (\oplus p)_0 = -p_0 \]

And I also observe that from (35) it follows that

\[ [(k \oplus p) \oplus q)_1 = k_1 + p_1 + q_1 + \ell k_0 (p_1 + q_1) + \ell p_0 (k_1 + q_1) + \ell q_0 (k_1 + p_1), \quad [(k \oplus p) \oplus q)_0 = k_0 + p_0 + q_0 \]

For this other torsionless momentum space that I want to consider I have already laid out the ingredients needed for applying my “golden rule”. From the on-shell relation (34) and the composition law (35) we see that, using the parametrization of the previous section, this is a case with $\alpha_1 = 2\ell$, $\alpha_2 = 0$, $\beta_1 = 0$, $\beta_2 = 0$, $\gamma_1 = \gamma_2 = \ell$, and therefore we have here another case in which the golden rule is satisfied: $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 - \gamma_1 - \gamma_2 = 0$.

So there is no a priori reason to expect that such a setup would require a “preferred-frame formulation” (since the necessary condition for DSR-compatibility is satisfied), and if one assumes that the “golden rule” is also a sufficient condition for DSR-compatibility one must expect that it will be possible to introduce in this setup a satisfactory notion of deformed Lorentz symmetry. Indeed this is the case, as I shall now easily show.

\(^5\) Since my focus is on relativistic properties, postponing to future studies a more detailed discussion of physics applications, I shall not here dwell on the advantages of having undeformed additivity of energy. This is however an issue that was discussed in some detail already in the doubly-special-relativity literature, as seen for example in Ref. [35] and references therein.
Specifically I shall verify that a satisfactory description of deformed Lorentz symmetry of this torsionless momentum space is obtained in terms of the following boost generator

\[ [N, p_0] = p_1 - \ell p_0 p_1, \quad [N, p_1] = p_0 + \ell p_0^2 + \ell p_1^2. \]  \hspace{1cm} (38)

I start by observing that the on-shell relation is invariant:

\[ [N, p_0^2 - p_1^2 + 2\ell p_0 p_1^2] = 0. \]  \hspace{1cm} (39)

And it is only slightly more tedious to check that the boost (38) ensures the covariance of the laws of conservation of momentum based on the composition law (35). I start checking this covariance for the conservation law

\[ (k \oplus p) = 0 \]  \hspace{1cm} (40)

Acting with the boost \( N \) one finds that

\[ [N, (k \oplus p)_0] = [N, (k_0 + p_0)] = k_1 - \ell k_0 k_1 + p_1 - \ell p_0 p_1 = k_1 + p_1 + \ell p_0 k_1 + \ell k_0 p_1 - \ell (p_0 + k_0)(p_1 + k_1) = 0 \]  \hspace{1cm} (41)

where on the right-hand side I of course used the conservation law itself (and took into account that I am working in leading order in \( \ell \)). And similarly one finds that

\[ [N, (k \oplus p)_1] = [N, (k_1 + p_1 + \ell k_0 p_1 + \ell p_0 k_1)] = k_0 + p_0 + \ell p_0^2 + \ell k_0^2 + \ell p_1^2 + 2\ell k_1 p_1 + 2\ell k_0 p_0 = k_0 + p_0 + \ell (p_0 + k_0)^2 + \ell (p_1 + k_1)^2 = 0 \]  \hspace{1cm} (42)

where again on the right-hand side I used the conservation law itself.

The covariance of the conservation law \( (k \oplus p) = 0 \) under the boost (38) is evidently confirmed by (41) and (42).

Let me also check explicitly the covariance of the conservation law

\[ (k \oplus p) \oplus q = 0 \]  \hspace{1cm} (43)

[The covariance of the 4-particle conservation, \( [(k \oplus p) \oplus q] \oplus r = 0 \), and of the general \( N \)-particle conservation is easily checked analogously.]

Acting with the boost \( N \) on the 0-component of (43) one finds that

\[ [N, ((k \oplus p) \oplus q)_0] = [N, (k_0 + p_0 + q_0)] = k_1 - \ell k_0 k_1 + p_1 - \ell p_0 p_1 + q_1 - \ell q_0 q_1 = k_1 + p_1 + q_1 + \ell p_0 k_1 + \ell k_0 p_1 + \ell q_0 k_1 + \ell k_0 q_1 + \ell p_0 q_1 + \ell q_0 p_1 - \ell (p_0 + k_0 + q_0)(p_1 + k_1 + q_1) = 0 \]  \hspace{1cm} (44)

And similarly for the 1-component one finds that

\[ [N, ((k \oplus p) \oplus q)_1] = [N, (k_1 + p_1 + q_1 + \ell k_0(p_1 + q_1) + \ell p_0(k_1 + q_1) + \ell q_0(k_1 + p_1))] = k_0 + p_0 + q_0 + \ell p_0^2 + \ell k_0^2 + \ell q_0^2 + \ell p_1^2 + \ell k_1^2 + 2\ell k_1 p_1 + 2\ell k_0 p_0 + 2\ell k_0 q_0 + 2\ell k_1 q_1 + 2\ell k_0 q_0 + 2\ell p_0 q_0 + 2\ell p_0 q_0 = k_0 + p_0 + q_0 + \ell (p_0 + k_0 + q_0)^2 + \ell (p_1 + k_1 + q_1)^2 = 0 \]  \hspace{1cm} (45)

The covariance of the conservation law \( (k \oplus p) \oplus q = 0 \) under the boost (38) is evidently confirmed by (44) and (45).

So the assumption that golden rule should be a sufficient condition for DSR-compatibility was again successful: also for the momentum space considered in this subsection the golden rule is satisfied and a satisfactory description of deformed Lorentz symmetry was found.
V. DEFORMED LORENTZ SYMMETRY ON MOMENTUM SPACES WITH TORSION

As announced, my next task is to verify the efficacy of the golden rule (and give explicit examples of boost transformations that implement deformed Lorentz symmetry) in cases in which the momentum space has torsion, and specifically for some cases of $\kappa$-Poincaré-inspired momentum spaces. The associated nonlinear laws on momentum space are cases that were immediately perceived \cite{14,15} as promising candidates for the construction of a DSR-compatible framework, even though, as mentioned, they have remained to some extent “sub judice”. Some of the observations I here report, besides providing additional tests of the robustness of my “golden rule”, could be relevant for building a more robust case for considering $\kappa$-Poincaré-inspired nonlinearities on momentum space as a viable candidate for the construction of a DSR framework.

A. Deformed Lorentz symmetry on a momentum space with torsion and modified dispersion

My first example of momentum space with torsion is the “$\kappa$-momentum space”, which, as stressed in Ref. \cite{13}, may deserve special interest from the relative-locality perspective of Refs. \cite{1,2} since it is inspired by some properties of the much-studied $\kappa$-Poincaré Hopf algebra \cite{24–26}.

From the viewpoint of contributing to the development of the relative-locality framework of Refs. \cite{1,2}, the study of deformed boost transformations on this momentum space is particularly significant since this is the only relative-locality momentum space for which a solid relativistic description of distant observers at rest was explicitly obtained \cite{13}.

Following Ref. \cite{13} I characterize $\kappa$-momentum space through a momentum-space metric such that the on-shell (“dispersion”) relation is

$$m^2 = p_0^2 - p_j^2 + \ell p_0 p_j$$

and the torsionful momentum-space affine connection is such that

$$(k \oplus p)_j = k_j + p_j + \ell k_0 p_j, \quad (k \oplus p)_0 = k_0 + p_0.$$  

(47)

This composition law can be qualified as “Majid-Ruegg composition law” \cite{13} (with the associated “Majid-Ruegg connection” on momentum space) since it is primarily inspired by an approach to the description of the $\kappa$-Poincaré Hopf algebra first introduced by Majid and Ruegg in Ref. \cite{25}. I shall compactly refer to it as the “MR composition law”.

I note down the antipode for the MR composition law:

$$\ominus p_j = -p_j + \ell p_0 p_j, \quad \ominus p_0 = -p_0,$$

(48)

which indeed, as verified by direct application of (47), is such that $p \oplus (\ominus p) = 0$.

And I also observe that from (47) it follows that

$$[(k \oplus p) \oplus q]_1 = k_1 + p_1 + q_1 + \ell k_0 p_1 + \ell k_0 q_1 + \ell p_0 q_1, \quad [(k \oplus p) \oplus q]_0 = k_0 + p_0 + q_0$$

(49)

It is easy to verify that also this $\kappa$-momentum space fits the demands of the “golden rule”. In fact, from the on-shell relation (46) and the composition law (47) we see that, using the parametrization of the Sec. III this is a case with $\alpha_1 = \ell$, $\alpha_2 = 0$, $\beta_1 = 0$, $\beta_2 = 0$, $\gamma_1 = \ell$, $\gamma_2 = 0$, and therefore indeed the golden rule is satisfied: $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 - \gamma_1 - \gamma_2 = 0$.

One must therefore expect that also on the $\kappa$-momentum space a satisfactory notion of deformed Lorentz symmetry is available, which indeed is what I shall now describe.

With respect to the torsionless cases considered in the previous section, our torsionful $\kappa$-momentum space is going to require a somewhat more sophisticated type of description of the action of boosts. Let me start by introducing the action of a boost on the momentum of a particle:

$$[N, p_0] = p_1, \quad [N, p_1] = p_0 + \ell p_0^2 + \ell p_1^2$$

(50)
This prescription indeed ensures the invariance of the on-shell relation on the $\kappa$-momentum space:

$$[N, p_0^2 - p_1^2 + \ell p_0 p_1] = 2p_0p_1 - 2p_1(p_0 + \ell p_0^2 + \frac{\ell}{2} p_1^2) + \ell p_1^3 + 2\ell p_0^2 p_1 = 0$$  \hspace{1cm} (51)$$

Evidently the torsion of the $\kappa$-momentum space poses no particular challenge for the description of boosts on the momentum of a single particle: the torsion characterizes composition of momenta and is therefore not felt when boosts act on the momentum of a single particle. The simplicity of the check of invariance of the on-shell relation, (51), confirms this point. However, torsion inevitably affects how boosts act on momenta obtained composing two or more single-particle momenta. Previously, when I examined in Sec. [IV] some torsionless cases, I found that it was possible to simply impose that the boost of a two-particle event $e_{k\oplus p}$ would be governed by

$$[N_{\text{torsionless}}, p \oplus k] = [N^{(p)}_{\text{torsionless}} + N^{(k)}_{\text{torsionless}}, p \oplus k]$$

where on the left-hand side I kept the notation of a generic boost action, while on the right-hand side I decomposed the boost into two pieces, each given in terms of a boost acting exclusively on a certain momentum in the event. Essentially this means that for the torsionless cases that I considered in the previous section I found that one could keep the standard concept of a “total boost” generator obtained by combining trivially the boost generators acting on each individual particle. One can easily retrace the availability of this option to the fact that in the cases I considered in the previous section the law of composition of momenta was symmetric under exchange of the particles.

With torsion in momentum space this simplicity is lost: the lack of symmetry under exchange of particles of the composition law (47) precludes, as one can easily verify, the possibility of adopting a “total boost generator” given by a trivial sum of single-particle boost generators. There is no choice of $N^{(p)}$ capable of ensuring that $[N^{(p)} + N^{(k)} + N^{(q)}, (k \oplus p) \oplus q]_\mu$ vanishes whenever $(k \oplus p) \oplus q)_\mu = 0$.

What does work on the torsionful $\kappa$-momentum space, as I shall show, is adopting

$$N^{(k \oplus p)} = N^{(k)} + N^{(p)} + \ell k_0 N^{(p)}$$

and accordingly

$$N^{((k \oplus p) \oplus q)} = N^{(k)} + N^{(p)} + N^{(q)} + \ell k_0 N^{(p)} + \ell k_0 N^{(q)} + \ell p_0 N^{(q)}$$

Let us verify that indeed these boost actions ensure compatibility with the conservation laws obtained from the MR composition law. For the 0-th component of $k \oplus p = 0$ one finds

$$[N^{(k)} + N^{(p)} + \ell k_0 N^{(p)}, (k \oplus p)_0] = [N^{(k)} + N^{(p)} + \ell k_0 N^{(p)}, k_0 + p_0] = k_1 + p_1 + \ell k_0 p_1 = 0$$  \hspace{1cm} (52)$$

where on the right-hand side I of course used the conservation law itself. Similarly for the 1-component of $k \oplus p = 0$ it turns out that

$$[N^{(k)} + N^{(p)} + \ell k_0 N^{(p)}, (k \oplus p)_1] = [N^{(k)} + N^{(p)} + \ell k_0 N^{(p)}, k_1 + p_1 + \ell k_0 p_1] =$$

$$= k_0 + \ell k_0^3 + \frac{\ell}{2} k_1^2 + p_0 + \ell p_0^2 + \frac{\ell}{2} p_1^2 + \ell k_1 p_1 + 2 \ell k_0 p_0$$

$$= k_0 + p_0 + \ell (p_0 + k_0)^2 + \frac{\ell}{2} (p_1 + k_1)^2 = 0$$  \hspace{1cm} (53)$$

where again on the right-hand side I used the conservation law $(k \oplus p)_\mu = 0$ itself (and took again into account that I am working at leading order in $\ell$).

So the description of boosts given by $N^{(k)} + N^{(p)} + \ell k_0 N^{(p)}$ does ensure the relativistic covariance of the conservation law $k \oplus p = 0$. 




Let me also check that the corresponding description of boosts on momenta composed of 3 single-particle momenta, \( N^{(k)} + N^{(p)} + N^{(q)} + \ell k_0 N^{(p)} + \ell k_0 N^{(q)} + \ell p_0 N^{(q)} \), ensures the relativistic covariance of the torsionful conservation law \((k \oplus p) \oplus q = 0\).

For what concerns the 0-th component one easily finds that

\[
\begin{align*}
[X^{(k)} + N^{(p)} + N^{(q)} + \ell k_0 N^{(p)} + \ell k_0 N^{(q)} + \ell p_0 N^{(q)}, \,(k \oplus p) \oplus q)_0] &= [N^{(k)} + N^{(p)} + N^{(q)} + \ell k_0 N^{(p)} + \ell k_0 N^{(q)} + \ell p_0 N^{(q)}, \,(k \oplus p) \oplus q)]_0 = \\
&= k_1 + p_1 + q_1 + \ell k_0 p_1 + \ell k_0 q_1 + \ell p_0 q_1 = 0
\end{align*}
\]

(54)

And similarly for the 1-component one finds that

\[
\begin{align*}
[X^{(k)} + N^{(p)} + N^{(q)} + \ell k_0 N^{(p)} + \ell k_0 N^{(q)} + \ell p_0 N^{(q)}, \,(k \oplus p) \oplus q)]_1 &= [N^{(k)} + N^{(p)} + N^{(q)} + \ell k_0 N^{(p)} + \ell k_0 N^{(q)} + \ell p_0 N^{(q)}, \,(k \oplus p) \oplus q)_1 = \\
&= k_0 + \ell k_0^2 + p_0 + \ell p_0^2 + \ell k_0 p_1 + p_0 + \ell q_2 + \ell k_0 q_1 + 2\ell k_0 q_0 + 2\ell p_0 q_0 + \ell k_1 p_1 + \ell k_1 q_1 + \ell p_1 q_1 = \\
&= k_0 + p_0 + q_0 + \ell (p_0 + k_0 + q_0)^2 + \ell (p_1 + k_1 + q_1)^2 = 0
\end{align*}
\]

(55)

So I have exposed a description of boosts that provides a satisfactory notion of deformed Lorentz symmetry on the \(k\)-momentum space. And this again should be listed among the successes of the “golden rule”, since, as shown above, the \(k\)-momentum space does fit the demands of the golden rule.

\[\text{B. Deformed Lorentz symmetry on a momentum space with torsion and unmodified dispersion}\]

As a second torsionful example of doubly-special-relativity type deformation of Lorentz symmetry on momentum space, suitable also for testing the “golden rule”, I take another variant of the \(k\)-momentum space. As mentioned the \(k\)-momentum space of the previous subsection is inspired by studies of the \(\kappa\)-Poincaré Hopf algebra, and specifically the work of Majid and Ruegg on the so-called “Majid-Ruegg basis of \(\kappa\)-Poincaré”, and indeed we ended up with adopting in the previous subsection the MR connection/composition law. The second version of \(k\)-momentum space which I now intend to consider is inspired by a different choice of generators for the \(\kappa\)-Poincaré Hopf algebra, a choice of generators proposed in Ref. [36] which is obtained from the Majid-Ruegg generators by acting with a “change of basis” [26, 36].

The Majid-Ruegg basis has been most frequently adopted in the \(\kappa\)-Poincaré literature, mostly because some of the Hopf-algebraic manipulations one is interested in doing turn out to be particularly simple when using the Majid-Ruegg basis. All this however is of little interest for my purposes here, and instead I shall notice that the basis proposed by Agostini, D’Andrea and myself in Ref. [36] provides inspiration for a description of the geometry of momentum space with several intriguing features. Reasoning just as done in Ref. [13] for deriving \(k\)-momentum space from the Majid-Ruegg basis one can easily obtain from the basis proposed in Ref. [36] a formulation of \(k\)-momentum space which (in leading order) is ultimately characterized by the undeformed on-shell relation

\[
m^2 = p_0^2 - p_1^2
\]

(56)

and a torsionful momentum-space affine connection such that

\[
(k \oplus p)_1 = k_1 + p_1 + \ell k_0 p_1 - \ell p_0 k_1, \quad (k \oplus p)_0 = k_0 + p_0,
\]

(57)

which in the following I shall label as the “AAD composition law” as a quick pointer to Ref. [36].
It is interesting to note that the antipode that follows from this AAD composition law is trivial

\[(\ominus p)_{\mu} = -p_{\mu},\]  

which, especially when combined with the fact that the on-shell relation is undeformed, renders this formalization of a k-momentum space particularly simple to handle, while preserving all the elements of complexity of a torsionful momentum space. The simple structure of the AAD composition law is also manifest to some extent when several different momenta are combined, as in the case of the composition of 3 momenta, which is given by

\[[(k \oplus p) \oplus q]_1 = k_1 + p_1 + q_1 + \frac{\ell}{2} k_0 p_1 - \frac{\ell}{2} p_0 k_1 + \frac{\ell}{2} k_0 q_1 - \frac{\ell}{2} q_0 k_1 + \frac{\ell}{2} p_0 q_1 - \frac{\ell}{2} q_0 p_1,\]

\[[(k \oplus p) \oplus q]_0 = k_0 + p_0 + q_0\]  

Before discussing the deformed-Lorentz-symmetry relativistic issues, let me again pause for looking at this scenario with undeformed on-shell relation and AAD composition law fits the demands of the “golden rule”. In fact, using again the parametrization introduced in Sec. III, one sees that undeformed on-shell relation and AAD composition law give \(\alpha_1 = 0, \alpha_2 = 0, \beta_1 = 0, \beta_2 = 0, \gamma_1 = \ell/2, \gamma_2 = -\ell/2\), and therefore indeed the golden rule is satisfied: \(\alpha_1 + \alpha_2 + \beta_1 + \beta_2 - \gamma_1 - \gamma_2 = 0\). Once again, if we trust the “golden rule” as (not only a necessary but also) a sufficient condition for DSR-compatibility we must then expect that this scenario with undeformed on-shell relation and AAD composition law should admit a satisfactory notion of deformed Lorentz symmetry. And indeed this is what I shall now show.

As explained in the previous subsection, even for such torsionful momentum spaces the description of doubly-special-relativity-type deformed boosts on momenta of a single particle is not challenging. For this case with undeformed on-shell relation and AAD composition law I propose

\[ [N, p_0] = p_1 + \frac{\ell}{2} p_0 p_1, \quad [N, p_1] = p_0 + \frac{\ell}{2} p_0^2, \]

which evidently ensures the invariance of the on-shell relation:

\[ [N, p_0^2 - p_1^2] = 2p_0(p_1 + \frac{\ell}{2} p_0 p_1) - 2p_1(p_0 + \frac{\ell}{2} p_0^2) = 0 \]  

Next I must deal again with the fact that torsion inevitably affects how boosts act on momenta obtained composing two or more single-particle momenta.

It is noteworthy that I find that the same prescription used for the other \(\kappa\)-Poincaré-inspired torsionful case considered in the previous subsection,

\[N^{(k \oplus p)} = N^{(k)} + N^{(p)} + \ell k_0 N^{(p)}\]

also works for the the case I am considering in this subsection.

Let me start verifying this on \(k \oplus p = 0\). For what concerns the 0-th component one easily finds

\[ [N^{(k)}, N^{(p)} + \ell k_0 N^{(p)}, (k \oplus p)_0] = [N^{(k)}, N^{(p)} + \ell k_0 N^{(p)}, k_0 + p_0] = k_1 + \frac{\ell}{2} k_0 k_1 + p_1 + \frac{\ell}{2} p_0 p_1 + \ell k_0 p_1 \]

\[= k_1 + p_1 + \frac{\ell}{2} k_0 p_1 - \frac{\ell}{2} p_0 k_1 + \frac{\ell}{2} (k_0 + p_0)(k_1 + p_1) = 0 ,\]

\[6\] Another noteworthy property of \(57\) is \((p \oplus p)_\mu = 2p_\mu\) (trivial composition of “parallel momenta”).
where on the right-hand side I of course used the conservation law itself. Similarly for the 1-component of \( k \oplus p = 0 \) it turns out that

\[
\begin{align*}
[N^{(k)}_{1} + N^{(p)}_{1} + \ell k_0 N^{(p)}_{1}, (k \oplus p)_1] &= [N^{(k)}_{1} + N^{(p)}_{1} + \ell k_0 N^{(p)}_{1}, k_1 + p_1 + \frac{\ell}{2} k_0 p_1 - \frac{\ell}{2} p_0 k_1] = \\
&= k_0 + \frac{\ell}{2} k_0^2 + p_0 + \frac{\ell}{2} p_0^2 + \ell k_0 p_0 \\
&= k_0 + p_0 + \frac{\ell}{2} (k_0 + p_0)^2 = 0
\end{align*}
\]

where again on the right-hand side I used the conservation law \( (k \oplus p)_\mu = 0 \) itself (and took again into account that I am working at leading order in \( \ell \)).

Consistently with the style of analysis I adopted throughout this manuscript let me double-check this consistency between my proposal for boosts and the choice of undeformed on-shell relation and AAD composition law by also considering the case of a 3-particle conservation law, of the form \( (k \oplus p) \oplus q = 0 \). For what concerns the 0-th component one easily finds that

\[
\begin{align*}
[N^{(k)} + N^{(p)} + N^{(q)} + \ell k_0 N^{(p)} + \ell k_0 N^{(q)} + \ell p_0 N^{(q)}, (k \oplus p) \oplus q)_0] &= [N^{(k)} + N^{(p)} + N^{(q)} + \ell k_0 N^{(p)} + \ell k_0 N^{(q)} + \ell p_0 N^{(q)}, k_0 + p_0 + q_0] = \\
&= k_1 + \frac{\ell}{2} k_0 k_1 + p_1 + \frac{\ell}{2} p_0 p_1 + q_1 + \frac{\ell}{2} q_0 q_1 + \ell k_0 p_1 + \ell k_0 q_1 + \ell p_0 q_1 \\
&= k_1 + p_1 + q_1 + \frac{\ell}{2} k_0 p_1 - \frac{\ell}{2} p_0 k_1 + \frac{\ell}{2} k_0 q_1 + \frac{\ell}{2} q_0 k_1 + \frac{\ell}{2} p_0 q_1 - \frac{\ell}{2} q_0 p_1 + \frac{\ell}{2} (k_0 + p_0 + q_0)(k_1 + p_1 + q_1) = 0 .
\end{align*}
\]

And similarly for the 1-component one finds that

\[
\begin{align*}
[N^{(k)} + N^{(p)} + N^{(q)} + \ell k_0 N^{(p)} + \ell k_0 N^{(q)} + \ell p_0 N^{(q)}, (k \oplus p) \oplus q)_1] &= [N^{(k)} + N^{(p)} + N^{(q)} + \ell k_0 N^{(p)} + \ell k_0 N^{(q)} + \ell p_0 N^{(q)}, k_1 + p_1 + q_1 + \frac{\ell}{2} k_0 p_1 - \frac{\ell}{2} p_0 k_1 + \frac{\ell}{2} k_0 q_1 - \frac{\ell}{2} q_0 k_1 + \frac{\ell}{2} p_0 q_1 - \frac{\ell}{2} q_0 p_1] = \\
&= k_0 + \frac{\ell}{2} k_0^2 + p_0 + \frac{\ell}{2} p_0^2 + q_0 + \frac{\ell}{2} q_0^2 + \ell k_0 p_0 + \ell k_0 q_0 + \ell p_0 q_0 \\
&= k_0 + p_0 + q_0 + \frac{\ell}{2} (p_0 + k_0 + q_0)^2 = 0
\end{align*}
\]

So once again the golden rule was successfully tested: for the choice of undeformed on-shell relation and AAD composition law the golden rule is satisfied and indeed I managed to expose a description of doubly-special-relativity-type deformed boosts that ensures the invariance of the undeformed on-shell relation and the covariance of the conservation laws obtained from the AAD composition law.

VI. ON THE MOMENTUM-SPACE METRIC

I have here set the stage for analyzing whether a given geometry of momentum space, as codified in an on-shell (dispersion) relation and in a law of conservation of momenta at interactions, may or may not be DSR-compatible, i.e. compatible with a relativistic formulation of theories on that momentum space. I found that the availability of a relativistic picture requires enforcing the “golden rule”, i.e. a certain consistency between the on-shell relation and the conservation laws (e.g. between the parameters \( \alpha_1, \alpha_2 \) and the parameters \( \beta_1, \beta_2, \gamma_1, \gamma_2 \)). An attractive result of such an analysis would have been to find that this relativistic consistency of combined choices of on-shell relation and conservation laws could have a simple geometric meaning: for example, a case such that given a certain
metric on momentum space (used to codify the on-shell relation) the type of affine connections on momentum space (used to codify the conservation laws) that would allow a relativistic picture would have been, say, metric connections or perhaps non-metric connections of a certain specific type. However, I have not managed to find a way to use the notion of metric and affine connection on momentum space introduced in Refs. [1, 2] for providing such a simple characterization.

There may well be such a characterization which I still did not notice, and in any case the availability of such a characterization can at best be viewed as desirable, but not necessarily available. It is interesting however to speculate about possible alternative geometric pictures, other ways to associate a metric to the on-shell relation and/or to associate an affine connection to the conservation laws. A suitable alternative geometric picture might even lead to a very intuitive geometric characterization of what it takes for a momentum space to be DSR-compatible.

I do not have any particularly compelling proposal at present, but let me illustrate the type of alternative “geometric picture of momentum space” that I have in mind by motivating a possibility in which the affine connection is still associated with the conservation laws exactly as prescribed in Refs. [1, 2], but one would link the metric to the on-shell relation in a way that differs from the one used in Refs. [1, 2]. This alternative proposal finds part of its inspiration in a field-theory argument which I am now going to propose.

Let me start by reminding the reader about the known fact that classical field theories in Minkowski spacetime can be easily formulated as classical field theories on a momentum space with Minkowski metric. For example, a free massless scalar field is described by the momentum-space action

\[-\int d^4k \int d^4p \, \bar{\phi}(k) \eta^{\mu\nu} k_\mu p_\nu \phi(p) \delta(k + p)\]

where \(\eta^{\mu\nu}\) is to be viewed here as a Minkowski metric on momentum space, and from the viewpoint of Refs. [1, 2] the ordinary linear sum of momenta, \(k + p\), codifies the trivial Levi-Civita connection of the Minkowski metric.

Taking as “spacetime field” the \(\phi(x)\) obtained from the momentum-space field \(\bar{\phi}(p)\) by anti-Fourier transform, \(\bar{\phi}(p) = \int d^4x \phi(x) e^{-ikx}\), one indeed obtains with few steps of simple derivation the standard spacetime description of a free massless scalar field:

\[\int d^4x \int d^4y \, \eta^{\mu\nu} (\partial_\mu \phi(x)) (\partial_\nu \phi(y)) \delta^4(x - y)\,.

In light of this it appears natural to assume that on a momentum space which does not have Minkowski geometry a free massless scalar field should be described in terms of the Lagrangian density

\[-\bar{\phi}(k) \, g^{\mu\nu}(p) \, k_\mu p_\nu \, \phi(p) \delta(k \oplus p)\]  

(66)

where \(g^{\mu\nu}\) is the momentum-space metric and \(\oplus\) is the composition law obtained from the momentum-space affine connection following the corresponding prescription of Refs. [1, 2].

For my purposes it is valuable to notice that in particular this should allow one to integrate over \(k\) finding

\[-\bar{\phi}(\ominus p) \, g^{\mu\nu}(p) \, (\ominus p)_\mu p_\nu \, \phi(p)\,.

where \(\ominus\) denotes again the antipode of the composition law. And evidently this is suggesting that the on-shell relation (for massless particles) could be viewed as involving the metric on momentum space together with a momentum and its antipode:

\[0 = -g^{\mu\nu}(p) \, (\ominus p)_\mu p_\nu\]  

(67)

So following this particular line of analysis (but several other possibilities should perhaps be considered) I was led to contemplating a role for the momentum-space metric in the form of the on-shell relation which is alternative to the one of Ref. [1].

At present I do not see any particularly compelling argument to prefer one or another way to link momentum-space metric and on-shell relation. One little observation which could be viewed as favoring the alternative (67) can be based on the two
examples of “κ-Poincaré-inspired momentum spaces” which I considered in Subsections V A and V B. To see this let us start with the “MR κ-momentum space” of Subsection V A: there the on-shell relation is

\[ m^2 = p_0^2 - p_j^2 + \ell p_0 p_j^2 \]  

(68)

and the antipode is given by

\[ (\ominus p)_j = -p_j + \ell p_0 p_j, \quad (\ominus p)_0 = -p_0, \]  

(69)

and intriguingly one can obtain the on-shell relation from the antipode by using the Minkowski metric:

\[ 0 = -\eta^{\mu\nu}(\ominus p)_\mu p_\nu = -(-p_0)p_0 + (-p_j + \ell p_0 p_j)p_j = p_0^2 - p_j^2 + \ell p_0 p_j^2 \]

And the same observation also applies to the “AAD κ-momentum space” of Subsection V B since there the (leading-order) on-shell relation is undeformed

\[ m^2 = p_0^2 - p_j^2 \]

and also the antipode is trivial (in spite of the nontriviality of the AAD composition law (57)),

\[ (\ominus p)_\mu = -p_\mu, \]

so that evidently one obtains the undeformed on-shell relation from a trivial antipode using the prescription (67) with a Minkowski metric on momentum space:

\[ -\eta^{\mu\nu}(\ominus p)_\mu p_\nu = -(-p_0)p_0 + (-p_j)p_j = p_0^2 - p_j^2. \]

It is still not clear to me whether this observation should tell us something about the proper notion of how to connect the on-shell relation with momentum-space metric and/or tell us something that specifically holds for the κ-Poincaré framework. It may even turn out to be ultimately meaningless, but it appeared to be sufficiently intriguing to be worth bringing it to the attention of readers of this manuscript.

VII. A THEOREM WITHOUT FORMULAS

As I am getting close to conclude this study, let me pause for a brief aside, just to show that the criterion I introduced in Sec. II, and the associated “golden rule”, must be understood from a broader perspective as a particular application, of significant practical value, of a simple relativistic theorem which I shall here state and provide the elementary (no formulas) proof for.

The compact (uncareful) version of the theorem states that in a relativistic theory one cannot have one-particle energy thresholds characterizing whether or not a given process is kinematically allowed.

The hypotheses of the theorem are:

- The theory is relativistic, in the sense that there is no preferred-frame description, and reproduces special relativity in the low-energy limit (limit in which all particles have small energies).
- The threshold energy of interest concerns a single specific particle taking part in a process (say, the muon in the out state of \( \pi^+ \to \mu^+ \nu_\mu \)) and marks the separation between a kinematical regime where the process is allowed and a kinematical regime where the process is not allowed (say, when the muon has \( E_\mu > E_{\text{threshold}} \) the process \( \pi^+ \to \mu^+ \nu_\mu \) is not allowed, whereas for \( E_\mu < E_{\text{threshold}} \) the process \( \pi^+ \to \mu^+ \nu_\mu \) is allowed).
- The theory may or may not have an observer-independent energy scale, but if it does the theorem anyway focuses on considering cases where the value of the threshold energy of interest is not an invariant under the boost transformations of the relativistic theory.
• Particle reactions are objective physical processes.

A slightly more careful statement of the theorem is:

⋄ It is not possible to have threshold-energy laws of the type described in these hypotheses, in a relativistic theory fulfilling these hypotheses.

**PROOF:** I can just show that such a threshold-energy law \( E_X > E_{\text{threshold}} \) for a particle of type \( X \), involved in a process of type \( X + A_1 + \cdots + A_n \rightarrow B_1 + \cdots + B_m \) or of type \( A_1 + \cdots + A_n \rightarrow X + B_1 + \cdots + B_m \), is not a relativistic law, within the hypotheses of the theorem. Consider then observer Alice according to which \( E_X \) is just above threshold, \( E_X^{(Alice)} = E_{\text{threshold}} + \varepsilon \), so that according to Alice the process is kinematically allowed (respectively kinematically not allowed). Since the hypotheses ensure that \( E_X \) is not invariant under boosts (even in cases where there is some energy scale which is invariant under boosts) we can safely assume that under the action of an appropriate boost transformation we can reach from Alice an observer Bob for whom the energy of that same particle of type \( X \) is below threshold: \( E_X^{(Bob)} = E_{\text{threshold}} - \varepsilon \), and for whom then, since the hypotheses insist on the case of a relativistic theory, that same process is instead kinematically not allowed (respectively kinematically allowed). All this evidently comes into contradiction with the objectivity of particle reactions. **End of proof.**

The careful reader can easily recognize how this applies in particular to the case of a threshold law for the energy of the photon in the process \( \gamma \rightarrow e^+e^- \), and the associated requirement that one should enforce a “no-photon-decay-switch-on constraint”.

And it is only slightly less immediately obvious that also my “no-pair-production-switch-off constraint” for \( \gamma + \Gamma \rightarrow e^+e^- \) is a particular case of this theorem: if there was a lower threshold for the energy of the soft photon \( \gamma \) below which the process \( \gamma + \Gamma \rightarrow e^+e^- \) was no longer allowed kinematically (because hard photons \( \Gamma \) of no matter how high energy could still not produce electron-positron pairs in interactions with such soft photons \( \gamma \) we would indeed be coming in contradiction with the theorem.

Stating this simple, nearly self-evident, theorem may appear to be a bit of a redundant overkill in light of the discussion here reported, but it was evidently not in the minds of the authors (a few) who have sought “anomalous particle-decay thresholds” in DSR-compatible pictures, since in all such contexts the discussion I gave here of the “no-photon-decay-switch-on constraint” for \( \gamma \rightarrow e^+e^- \) would evidently also apply up to trivial generalization. And it was evidently also not in the minds of the (several) authors who have sought a DSR-compatible photopion-production-switch-off threshold as a way to “explain” the once presumed cosmic-ray GZK-threshold anomaly (for which now there is no longer any evidence [37]): such a photopion-production-switch-off threshold is necessarily obstructed in DSR-compatible frameworks for the same reasons which here led me to exclude a pair-production-switch-off threshold.

I myself had put forward the half of the theorem which is essentially codified by the “no-photon-decay-switch-on constraint” already in Ref. [28] (also see the later study in Ref. [29], and the very recent related study in Ref. [31], essentially generalizing from photons to neutrinos the no-decay-switch-on constraint), but I put in focus only rather recently the other half of the theorem, the one which is essentially contained in my new “no-pair-production-switch-off constraint”.

While the theorem renders obvious the fact that satisfying these two constraints is a necessary condition for DSR-compatibility, it remains of course uncertain whether satisfying these two constraints (enforcing the “golden rule”, within a chosen parametrization) can really provide a sufficient condition for DSR-compatibility, as the few tests I here reported appear to suggest.

**VIII. CLOSING REMARKS**

The recently-proposed “relative-locality framework” has the potential of producing several new physical pictures and of accommodating, reformulating them accordingly, some previously existing proposals. I have here show that, as for other areas of quantum-gravity-inspired research, the DSR concept and the techniques of analysis that have been developed in order to explore it, prove valuable also for the analysis of this powerful framework.
I have here established that some relative-locality momentum spaces necessarily require a formulation of theories with a preferred frame, and I provided a robust and technically simple strategy for identifying such momentum-space geometries. The fact that, within a chosen parametrization of momentum space geometry, my criteria produce a simple algorithmic classification of momentum-space geometries that require a preferred-frame description is particularly convenient for future applications of the results here reported.

The conjecture that my criterion (and the associated “golden rule”) might be not only necessary but also sufficient for DSR-compatibility found some support in the results here reported in Sections IV and V, but evidently needs to be investigated in greater generality.

For what concerns specific models, especially the results I here obtained on the DSR-compatibility of “$\kappa$-momentum spaces” may be of particular interest, since these are evidently among the most natural “laboratories” for a first phase of exploration of the implications of the relative-locality framework.

I have here studied DSR-compatibility of theories on a relative-locality momentum space, but focusing exclusively on momentum-space properties. This is the most fundamental level at which to investigate the availability of a deformed-Lorentz-symmetry picture, since in the “relative-locality framework” momentum space is primitive and spacetime is only a derived entity [1,2]. But of course conceptually the most intriguing questions are to be expected on the side of the actual relativity of spacetime locality, and it will be extremely interesting to investigate the interplay between (deformed-)boost invariance and the relativity of spacetime locality. I have here postponed this task, also assuming that my effort of rendering more robust our understanding of boost transformations on a momentum space with nontrivial geometry would provide solid ground for those further studies.

As I was in the final stages of preparation of this manuscript I became aware of the papers in Refs. [38,39], parts of which were devoted to issues pertaining (deformed) Lorentz transformations with relative locality. None of the points I here made about establishing the general criteria for consistency of the momentum-space geometry with (possibly deformed) Lorentz invariance is found in Refs. [38,39]. Instead Refs. [38,39] each take a specific scenario for momentum space and a specific scenario for boost transformations, attempting to also establish a few starting points for the analysis of the implications for relative spacetime locality. Perhaps the aspect of these two manuscripts which could be most valuable for the future development of the results I here reported is the observation in Ref. [38] suggesting that for a proper description of finite (deformed-)boost transformations in a relative-locality framework it might be necessary to introduce a dependence of the rapidity parameter on the particles momenta. I have here confined my analysis at the complementary level of symmetry generators, setting the stage for describing the case of infinitesimal boost transformations. At least in the $\kappa$-momentum space I here considered in Subsection V A, which was the momentum space considered in Ref. [38], it might be fruitful to contrast and perhaps to complement my observations on symmetry generators and the observations reported in Ref. [38] for the rapidity parameters.

Another issue for future studies, perhaps the most significant one, concerns the interplay between deformed Lorentz symmetry and the implications of relative spacetime locality for cases with several causally-connected events. One should achieve a consistent relativistic description applicable to the case of pairs of distant and relatively boosted observers. As shown in Ref. [13] even just studying distant observers in relative rest is already rather challenging when relative spacetime locality is taken into account for causally-connected events, so the generalization to distant and relatively boosted observers may prove very challenging.

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