Viscous Dark Energy in $f(T)$ Gravity

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Abstract
We study the bulk viscosity taking dust matter in the generalized teleparallel gravity. We consider different dark energy models in this scenario along with a time dependent viscous model to construct the viscous equation of state parameter for these dark energy models. We discuss the graphical representation of this parameter to investigate the viscosity effects on the accelerating expansion of the universe. It is mentioned here that the behavior of the universe depends upon the viscous coefficients showing the transition from decelerating to accelerating phase. It leads to the crossing of phantom divide line and becomes phantom dominated for specific ranges of these coefficients.

Keywords: $f(T)$ gravity; Viscosity; Effective equation of state.
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1 Introduction

Dark energy (DE) seems to play an important role of an agent that drives the present acceleration of the universe with the help of large negative pressure. An effective viscous pressure can also play its role to develop the dynamical history of an expanding universe [1]-[4]. It is found [4] that viscosity effects are viable at low redshifts, which observe negative pressure for the cosmic expansion with suitable viscosity coefficients. In general, the universe inherits dissipative processes [5], but perfect fluid is an ideal fluid with zero viscosity.

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Although, perfect fluid is mostly used to model the idealized distribution of matter in the universe. This fluid in equilibrium generates no entropy and no frictional type heat because its dynamics is reversible and without dissipation. The dissipative processes mostly include bulk and shear viscosities. The bulk viscosity is related with an isotropic universe whereas the shear viscosity works with anisotropy of the universe. The CMBR observations indicate an isotropic universe, leading to bulk viscosity where the shear viscosity is neglected [6]. Long before the direct observational evidence through the SN Ia data, the indication of a viscosity dominated late epoch of accelerating expansion of the universe was already mentioned [7].

The origin of the bulk viscosity in a physical system is due to its deviations from the local thermodynamic equilibrium. Thus the existence of bulk viscosity may arise the concept of accelerating expansion of the universe due to the collection of those states which are not in thermal equilibrium for a small fraction of time [8]. These states are the consequence of fluid expansion (or contraction). The system does not have enough time to restore its equilibrium position, hence an effective pressure takes part in restoring the system to its thermal equilibrium. The measurement of this effective pressure is the bulk viscosity which vanishes when it restores its equilibrium [9]-[12]. So, it is natural to assume the existence of a bulk viscous coefficient in a more realistic description of the accelerated universe today.

Physically, the bulk viscosity is considered as an internal friction due to different cooling rates in an expanding gas. Its dissipation reduces the effective pressure in an expanding fluid by converting kinetic energy of the particles into heat. Thus, it is natural to think of the bulk viscous pressure as one of the possible mechanism that can accelerate the universe today. However, this idea needs a viable mechanism for the origin of the bulk viscosity, although there are many proposed best fit models.

Many models have been suggested to discuss the vague nature of DE. During the last decade, the holographic dark energy (HDE), new agegraphic dark energy (NADE), their entropy corrected versions and correspondence with other DE models have received a lot of attention. The HDE model is based on the holographic principle which states that the number of degrees of freedom in a bounded system should be finite and has a relationship with the area of its boundary [13]. Moreover, in order to reconcile the validity of an effective local quantum field, Cohen et al. [14] provided a relationship between the ultraviolet (UV) and the infrared (IR) cutoffs on the basis of
limit set by the formation of a black hole. This is given by \[ \rho_\Lambda = 3c^2 M_p^2 L^{-2}, \] where constant $3c^2$ is used for convenience, $M_p^2 = (8\pi G)^{-1}$ is the reduced Planck mass and $L$ is the IR cutoff. This model has been tested by using different ways of astronomical observations [17]-[20]. Also, it has been discussed widely in various frameworks such as in the general relativity, modified theories of gravity and extra dimensional theories [21]-[29].

The NADE model was developed in view of the Heisenberg uncertainty principle with general relativity. This model exhibits that DE originates from the spacetime and matter field fluctuations in the universe. In this model, the length measure is taken as the conformal time instead of age of the universe and its energy density is $\rho_\Lambda = \frac{3\eta^2}{\kappa^2 \eta^2}$ where $\eta$ is the conformal time. The causality problem occurs in the usual HDE model, while it is avoided here. Many people have explored the viability of this model through different observations [17]-[20, 30].

Another proposal to discuss the accelerating universe is the modified gravity theories [31]. The $f(T)$ gravity is the generalization of teleparallel gravity by replacing the torsion scalar $T$ with differentiable function $f(T)$, given by

\[ L_T = \frac{e}{2\kappa} T + L_m \Rightarrow L_{f(T)} = \frac{e}{2\kappa} f(T) + L_m, \]

where $\kappa$ is the coupling constant and $e = \sqrt{-g}$. This leads to second order field equations formed by using Weitzenböck connection which has no curvature but only torsion. The equation of state (EoS) parameter, $\omega = p/\rho$, is used to explore the cosmic expansion. Bengochea and Ferraro [32] tested power-law $f(T)$ model for accelerated expansion of the universe. They performed observational viability tests and concluded that this model exhibits radiation, matter and DE dominated phases. Incorporating exponential model along with power-law model, Linder [33] investigated the expansion of the universe in this theory. He observed that power-law model depends upon its parameter while exponential model acts like cosmological model at high redshift.

Bamba et al. [34] discussed the EoS parameter for exponential, logarithmic as well as combination of these $f(T)$ models and they concluded that the crossing of phantom divide line is observed in combined model only. Karami and Abdolmaleki [35] constructed this parameter for HDE, NADE
and their entropy corrected models in the framework of \( f(T) \) gravity. They found that the universe lies in phantom or quintessence phase for the first two models whereas phantom crossing is achieved in entropy corrected models. Sharif and Rani [36] described the graphical representation of k-essence in this modified gravity with the help of EoS parameter. Some other authors [37, 38] explored the expansion of the universe with different techniques in \( f(T) \) gravity. Also, the effects of viscous fluid in modified gravity theories [39–41] are analyzed to display accelerating expansion.

In this paper, we construct the viscous EoS parameter for different viable DE models in the framework of \( f(T) \) gravity with pressureless matter. For this purpose, we consider a time dependent viscous model with its constant viscous reduction to explore the DE era in general fluid. The graphical behavior indicates the acceleration of the universe for suitable viscous coefficients. The scheme of paper is as follows: Section 2 provides basic formalism and discussion about the field equations of \( f(T) \) gravity. In section 3, the viscous EoS parameter is constructed for different DE models. Also, we discuss the graphical behavior of this parameter for these models. The last section summarizes the results.

## 2 The Field Equations

The \( f(T) \) theory of gravity (as the generalization of the teleparallel gravity) is uniquely determined by the tetrad field \( h^{\mu}_{\alpha}(x) \) [42]. It is an orthonormal set of four-vector fields defined on Lorentzian manifold. The metric and tetrad fields can be related as

\[
g_{\mu\nu} = \eta_{ij} h^i_\mu h^j_\nu, \tag{3}\]  

where \( \eta_{ij} = \text{diag}(1, -1, -1, -1) \) is the Minkowski metric for the tangent space. Here we use Greek alphabets \( (\mu, \nu, \rho, ... = 0, 1, 2, 3) \) to denote space-time components while the Latin alphabets \( (i, j, k, ... = 0, 1, 2, 3) \) are used to describe components of tangent space. The non-trivial tetrad field \( h_i \), yielding non-zero torsion, can be written as

\[
h_i = h^i_\mu \partial_\mu, \quad h^j = h^j_\nu dx^\nu, \tag{4}\]  

satisfying the following properties

\[
h^i_\mu h^\mu_j = \delta^i_j, \quad h^i_\mu h^\nu_i = \delta^\nu_\mu. \tag{5}\]
The variation of Eq. (2) with respect to the tetrad field leads to the following field equations [37, 44]

\[ e^{-1} \partial_{\mu}(e S_{i}^{\mu \nu}) + h_{i}^{\lambda} T^{\rho \mu \lambda} S_{\rho}^{\nu \mu} f_{T} + S_{i}^{\mu \nu} \partial_{\mu}(T) f_{TT} + \frac{1}{4} h_{i}^{\nu} f = \frac{1}{2} \kappa^{2} h_{i}^{\nu} T_{\rho}^{\nu}, \]  

(6)

where \( f_{T} = \frac{df}{dT} \), \( f_{TT} = \frac{d^{2}f}{dT^{2}} \).

The torsion scalar is defined as

\[ T = S_{\rho}^{\mu \nu} T_{\rho}^{\mu \nu}, \]  

(7)

where \( S_{\rho}^{\mu \nu} \) and torsion tensor \( T_{\rho}^{\mu \nu} \) are given as follows

\[ S_{\rho}^{\mu \nu} = \frac{1}{2}(K_{\rho}^{\mu \nu} + \delta_{\rho}^{\mu} T^{\theta \nu}_{\theta} - \delta_{\rho}^{\nu} T^{\theta \mu}_{\theta}), \]  

(8)

\[ T_{\rho}^{\lambda} = \Gamma_{\rho}^{\lambda \nu} - \Gamma_{\lambda}^{\rho \nu} = h_{i}^{\lambda}(\partial_{\nu} h_{i}^{\mu} - \partial_{\mu} h_{i}^{\nu}), \]  

(9)

\[ K_{\rho}^{\mu \nu} = -\frac{1}{2}(T_{\mu \rho}^{\nu} - T_{\nu \rho}^{\mu} - T_{\rho}^{\mu \nu}), \]  

(10)

which are antisymmetric. The energy-momentum tensor for perfect fluid is

\[ T_{\rho}^{\nu} = (\rho + p) u_{\nu} u_{\rho} - p \delta_{\nu}^{\rho}, \]  

(11)

where \( u_{\nu} \) is the four-velocity in comoving coordinates, \( \rho \) and \( p \) denote the total energy density and pressure of fluid inside the universe.

The flat homogenous and isotropic FRW universe is described by

\[ ds^{2} = dt^{2} - a^{2}(t)(dx^{2} + dy^{2} + dz^{2}), \]  

(12)

where \( a(t) \) is the scale factor such that \( a(t) = 1/1 + z \) in the form of redshift \( z \). The corresponding tetrad components are [34]-[38]

\[ h_{i}^{\mu} = diag(1, a, a, a), \]  

(13)

which obviously satisfies Eq. (6). Using Eqs. (7) and (12), the torsion scalar turns out in the form of Hubble parameter \( H \) as \( T = -6H^{2}, \quad H = \frac{\dot{a}}{a} \). The corresponding modified Friedmann equations become

\[ -2T f_{T} + f = 2\kappa^{2} \rho, \]  

(14)

\[ -8T \dot{H} f_{TT} + (2T - 4\dot{H}) f_{T} - f = 2\kappa^{2} p. \]  

(15)
For the realistic model, we take viscosity term which introduces the effective pressure in the energy-momentum tensor \([45]\), i.e.,

\[
T_\nu^\rho = (\rho + p_{\text{eff}})u_\nu u_\rho - p_{\text{eff}}\delta_\nu^\rho,
\]
defined by

\[
p_{\text{eff}} = p - 3H\xi(t),
\]
here \(\xi\) is the time dependent bulk viscosity function. To avoid the violation of the second law of thermodynamics, \(\xi(t) > 0\).

The field equations (14) and (15) may be rewritten as

\[
\rho_m + \rho_T = \frac{3H^2}{k^2}, \quad p_T = -\frac{1}{k^2}(2\dot{H} + 3H^2).
\]
We assume here the pressureless (dust) matter, i.e., \(p_m = 0\) and the expressions for torsion contributions \(\rho_T\), \(p_T\) and effective pressure become

\[
\rho_T = \frac{1}{2k^2}(2TfT - f - T), \quad p_T = -\frac{1}{2k^2}(-8T\dot{H}f_{TT} + (2T - 4\dot{H})f_T
- f + 4\dot{H} - T), \quad p_{\text{eff}} = p_T - 3H\xi(t).
\]

It is noted that if we insert \(f(T) = T\) in Eq.(17) with non-viscous case, we arrive at the usual Friedmann equations in general relativity. The corresponding viscous EoS parameter becomes

\[
\omega_{\text{eff}} = -1 + \frac{2\kappa^2(\rho_m + 3H\xi(t))}{2\kappa^2\rho_m + 2Tf_T - f - T} + \frac{4\dot{H}(2Tf_{TT} + f_T - 1)}{2\kappa^2\rho_m + 2Tf_T - f - T}.
\]
The phantom and quintessence regions are mostly described with the help of constant EoS parameter such as, \(-1 < \omega < -1/3\), which corresponds to the quintessence era whereas phantom era is referred to \(\omega < -1\) and the phantom divide line is given by \(\omega = -1\). If we consider a torsion dominated universe, then Eq.(17) reduces to

\[
\frac{3H^2}{k^2} = \rho_T.
\]
Inserting the above value in the energy conservation equation for torsion, it follows that \((\omega_{\text{eff}} \rightarrow \omega_T)\)

\[
\omega_T = -1 + \frac{\kappa^2\xi(t)}{H} - \frac{2\dot{H}}{3H^2}.
\]
The EoS parameter $\omega_T$ describes a vacuum, phantom dominated or quintessence dominated universe for $\dot{H} = \frac{3}{2}H\kappa^2\xi(T)$, $\dot{H} > \frac{3}{2}H\kappa^2\xi(T)$ or $\dot{H} < \frac{3}{2}H\kappa^2\xi(T)$ respectively for viscous case. For the non-viscous case ($\xi(t) = 0$), these conditions reduce to $\dot{H} = 0$, $\dot{H} > 0$ or $\dot{H} < 0$.

3 Viscous Fluid and Dark Energy Models

Viscous models have interesting insights about the evolution of the expanding universe. Here we consider a simple time dependent bulk viscous model as follows \[46, 47\]

$$\xi(t) = \xi_0 + \xi_1 \frac{\dot{a}}{a} + \xi_2 \ddot{a} = \xi_0 + \xi_1 H + \xi_2 (\dot{H} + H^2), \quad (24)$$

where $\xi_0$, $\xi_1$ and $\xi_2$ are positive coefficients. The cosmological evolution can be explored for different values of these coefficients \[47\]-\[49\]. This bulk viscosity model is motivated due to the terms involved, i.e., viscosity is related to the velocity and acceleration which give the phenomenon of scalar expansion in fluid dynamics. The viscous model having constant $\xi_0$ and velocity term $\dot{a}$ are discussed in \[46\], thus a linear combination of these two with acceleration term $\ddot{a}$ may give more physical results.

In general, the existence of viscosity coefficients in a fluid is due to the thermodynamic irreversibility of the motion. If the deviation from reversibility is small, the momentum transfer between various parts of the fluid can be taken to be linearly dependent on the velocity derivatives. This case corresponds to the constant viscous model. When viscosity is proportional to Hubble parameter the momentum transfer involves second order quantities in the deviation from reversibility leading to more physical results. The proper choices of their coefficients may lead to the crossing of phantom divide line.

To determine the evolution of effective EoS parameter incorporating $f(T)$ and viscous models, we assume the Hubble parameter in the form \[50, 51\]

$$H(t) = \frac{h}{(t_s - t)^\gamma}. \quad (25)$$

Here $h$ and $t_s$ are positive constants, the constant $\gamma$ is either positive or negative and $t < t_s$ is guaranteed for the accelerated expansion of the universe due to the violation of strong energy condition ($\rho + 3p \geq 0$). For $\gamma = 1$, it leads to the scale factor $a(t) = a_0(t_s - t)^{-h}$ which ends up the universe with
future finite time Big Rip singularity. Using Eq. (25) with $\gamma = 1$, the torsion scalar becomes

$$T = -\frac{6h^2}{(t_s - t)^2}$$

with $t_s - t = (z + 1)^{1/2}$. Also, taking the value of $H$, the energy conservation equation, $\dot{\rho}_m + 3H\rho_m = 0$ for dust matter yields the solution

$$\rho_m = \rho_{m0}(t_s - t)^{3h} = \rho_{m0}(1 + z)^3,$$  \hspace{1cm} (26)

where $\rho_{m0}$ is an arbitrary constant.

In the following, we discuss three DE models by taking into account of the viscosity.

### 3.1 The First Model

First, we consider the following DE model \[35, 52\]

$$f(T) = \beta \sqrt{T} + (1 - \alpha)T,$$  \hspace{1cm} (27)

where $\alpha$ and $\beta$ are constants. For $\beta = 0$, this model leads to the teleparallel gravity. It is interesting to note that the model (27) is the result of correspondence between energy densities of $f(T)$ and HDE model. In flat FRW universe, the IR cutoff $L$ in Eq. (1) becomes the future event horizon $R_h = a \int_0^\infty \frac{dt}{a}$ resulting the HED energy density. Using Eq. (25) in the correspondence, $\alpha$ takes the form

$$\alpha = c^2 \left(1 + \frac{1}{h}\right)^2,$$  \hspace{1cm} (28)

and $\beta$ is an integration constant. Replacing $f(T)$ and viscous models in Eq. (21), the viscous EoS parameter takes the form

$$\omega_{\text{eff}} = \frac{\kappa^2 \rho_{m0}(1 + z)^{3+\frac{4}{h}} + 3h\kappa^2[\xi_0(1 + z)^{2/3} + \xi_1 h(1 + z)^{1/3} + \xi_2 h(h + 1)]}{(\kappa^2 \rho_{m0}(1 + z)^{3+\frac{4}{h}} + 3\alpha h^2)(1 + z)^{\frac{1}{h}}}$$

$$- \frac{2\alpha h}{\kappa^2 \rho_{m0}(1 + z)^{3+\frac{4}{h}} + 3\alpha h^2} - 1.$$  \hspace{1cm} (29)

The graphical behavior of time dependent viscous EoS parameter with respect to redshift is shown in Figure 1. We draw this parameter by taking arbitrary values of the coefficients ($\xi_0$, $\xi_1$, $\xi_2$) of viscous model, where $\alpha$ depends upon the constant $c$ which is 0.818 for flat model \[35\]. Also, we fix the redshift range from 0 to 5 to discuss the behavior of the universe at low
redshifts. The left graph in Figure 1 shows the evolution of the universe initially from matter dominated era for higher values of $z$ and then converges to quintessence era at $z = 1.6$ for $\xi_0 = 0.005$ and $(\xi_1, \xi_2) = 0.1$. The phantom divide line is being crossed by the $\omega_{eff}$ as $z$ approaches to zero. By decreasing $\xi_1$ and $\xi_2$ from 0.1, the universe remains in phantom dominated era (shown in the right graph).

For the constant viscous case, we take $\xi_1 = 0 = \xi_2$ in Eq.(24), thus the constant viscous EoS parameter becomes

$$\omega_{eff} = \frac{\kappa^2 \rho_m(1 + z)^{3 + \frac{2}{h}} + 3h\kappa^2 \xi_0(1 + z)^{\frac{1}{h}} - 2\alpha h}{\kappa^2 \rho_m(1 + z)^{3 + \frac{2}{h}} + 3\alpha h^2} - 1. \quad (30)$$

Figure 2 represents the same behavior as indicated by time dependent viscous EoS parameter. However, the phantom crossing for the constant viscous coefficient occurs at $\xi_0 = 0.82$, it shows phantom behavior for $\xi_0 < 0.82$ (right graph).

### 3.2 The Second Model

Assuming the exponential $f(T)$ model \[53, 54\]

$$f(T) = Te^{kT}, \quad (31)$$
where \( b \) is an arbitrary constant. Inserting \( f(T) \) and viscous models in Eq.\((21)\), the viscous EoS parameter takes the form

\[
\omega_{eff} = -1 + \left[ 2\kappa^2 \rho_m (1+z)^3 + \frac{6h\kappa^2}{(1+z)^{\frac{3}{2}}} \left( \xi_0 + \frac{\xi_1 h}{(1+z)^{\frac{1}{2}}} + \frac{\xi_2 h (1+h)}{(1+z)^{\frac{3}{2}}} \right) 
+ \frac{4h}{(1+z)^{\frac{3}{2}}} \left( -\frac{6h^2}{(1+z)^{\frac{3}{2}}} \exp\left(-\frac{6h^2 b}{(1+z)^{\frac{3}{2}}} \right) (2b^2 + 5b - \frac{(1+z)^{\frac{3}{2}}}{6h^2}) \right)
- 1 \right] \left[ 2(1+z)^3 + \frac{36h^4}{(1+z)^{\frac{3}{2}}} \exp\left(-\frac{6h^2 b}{(1+z)^{\frac{3}{2}}} \right) \right]^{-1}.
\]

Figure 3 represents the graphical behavior of time dependent viscous \( w_{eff} \) versus \( z \). In the left graph, the plot shows the evolution of the universe from matter to DE phase for higher values of redshift, approximately for \( z > 2.37 \). At \( z = 2.37 \) for particular values \( \xi_0 = 0.005 \) and \( (\xi_1, \xi_2) = 0.2 \), the EoS parameter indicates the quintessence era and approaches to \(-1\) as \( z \to 0 \). As we decrease the values of \( \xi_1, \xi_2 \), the \( \omega_{eff} \) represents the phantom era of the universe.

Now for constant viscous EoS parameter, we take \( (\xi_1, \xi_2) = 0 \) in Eq.\((32)\)
Figure 3: Plot of time dependent viscous $\omega_{eff}$ versus $z$ for exponential model with $b = 0.05$, $h = 2$, $\kappa^2 = 1 = \rho_m0$. In the left graph, we take $\xi_0 = 0.005$, $\xi_1 = 0.2 = \xi_2$ and in the right graph, $\xi_0 = 0.005$, $\xi_1 = 0.02 = \xi_2$.

yields

$$\omega_{eff} = -1 + \left[ 2\kappa^2\rho_m0(1+z)^3 + \frac{6h\kappa^2\xi_0}{(1+z)^{\frac{3}{2}}} + \frac{4h}{(1+z)^{\frac{3}{2}}} \left( -\frac{6h^2}{(1+z)^{\frac{3}{2}}} \right) \right]^{-1} \left[ 2(1+z)^3 \right.$$  

$$\times \exp\left(-\frac{6h^2b}{(1+z)^{\frac{3}{2}}} \left( 2b^2 + 5b - \frac{(1+z)^2}{6h^2} \right) - 1 \right) - 1 \left. \right]^{-1}.$$  

$$+ \frac{36h^4}{(1+z)^{\frac{3}{2}}} \exp\left(-\frac{6h^2b}{(1+z)^{\frac{3}{2}}} \right)^{-1}.$$  

Its plot versus $z$ is in Figure 4, showing same behavior as that of time dependent case. Approximately, the universe meets the quintessence era at $z < 2.1$ and converges towards $\omega_{eff} = -1$ as $z$ approaches to zero (in left graph). In right graph, the evolution of EoS parameter represents the phantom era of the universe for $z \leq 0.5$ by decreasing the value of $\xi_0$, i.e., $\xi_0 < 1.6$.

### 3.3 The Third Model

Finally, we take the model

$$f(T) = \epsilon \sqrt{T} + T + \frac{\gamma}{1+2h} T^{1+h},$$

which includes linear and nonlinear terms of torsion scalar and $\epsilon$, $\gamma$ are constants. Similar to the first model [27], this model comes through the
correspondence of NADE model with \( f(T) \) gravity. The energy density of the NADE model inherits the conformal time \( \eta = \int_{t_s}^{t} \frac{dt}{a} \). Incorporating the correspondence, here \( \epsilon \) is an integration constant and \( \gamma \) is

\[
\gamma = \frac{6n^2 a_0^2 (1 + h)^2}{(-6h^2)^{1+h}},
\]

(35)

where \( n = 2.716 \) for flat universe. Replacing Eq. (34) in (21), the viscous EoS parameter becomes

\[
\omega_{\text{eff}} = \frac{2 \kappa^2 \rho_m(1 + z)^{3 + \frac{3}{2}} + 6h \kappa^2 [\xi_0 (1 + z)^{2/h} + \xi_1 h (1 + z)^{1/h} + \xi_2 h (h + 1)]}{(2 \kappa^2 \rho_m(1 + z)^{5 + \frac{3}{2}} + \gamma (-6h^2)^{1+h}) (1 + z)^{\frac{1-3h}{2h}}} + \frac{4 \gamma h (1 + h) (-6h^2)^h (1 + z)^{2-h}}{2 \kappa^2 \rho_m (1 + z)^{5 + \frac{3}{2}} + \gamma (-6h^2)^{1+h}} - 1.
\]

(36)

The graphical behavior of time dependent viscous \( \omega_{\text{eff}} \) is given in Figure 5. Initially, it shows the deceleration phase \( (\omega_{\text{eff}} > -\frac{1}{3}) \) of the universe for higher values of \( z \). As we decrease the value of redshift up to 0.4, it meets the quintessence region for the particular values \( \xi_0 = 0.05 \) and \( (\xi_1, \xi_2) = 4.2 \), and crossing of the phantom divide line takes place for \( z \) tends to zero. The right graph indicates that the universe remains in this era for \( (\xi_1, \xi_2) < 4.2 \).
Figure 5: Plot of time dependent viscous $\omega_{\text{eff}}$ versus $z$ for third model with $n = 2.716$, $h = 2$, $a_0 = 1$, $\kappa^2 = 1 = \rho_{m0}$. In the left graph, $\xi_0 = 0.05$, $\xi_1 = 4.2 = \xi_2$ and in the right graph $\xi_0 = 0.05$, $\xi_1 = 2.6 = \xi_2$.

The constant viscous model for this case is

$$\omega_{\text{eff}} = \frac{2\kappa^2 \rho_{m0} (1 + z)^{5 + \frac{2}{3} + h} + 6h\kappa^2 \xi_0 (1 + z)^{2 + \frac{1}{3} + h}}{2\kappa^2 \rho_{m0} (1 + z)^{5 + \frac{2}{3} + h} + \gamma(-6h^2)^{1+h}} - 1 + \frac{4\gamma h (1 + h)(-6h^2)^{h}(1 + z)^{2-h}}{2\kappa^2 \rho_{m0} (1 + z)^{5 + \frac{2}{3} + h} + \gamma(-6h^2)^{1+h}}.$$

(37)

Figure 6 shows its plot versus redshift. It provides the crossing of phantom divide line for a high value $\xi_0 = 32$, whereas $\xi_0 \leq 32$ corresponds to the phantom region for decreasing $z$ of the accelerating expansion of the universe.

### 4 Outlook

Viscous models have been discussed in cosmological evolution of the universe as compared to the ideal perfect fluid. The term of shear viscosity vanished when a completely isotropic universe is assumed and only the bulk viscosity contributes for the accelerating universe to get negative pressure. In this paper, we have considered viscosity by taking dust matter in the framework of $f(T)$ gravity. We have taken three different viable DE models and a time dependent viscous model to construct the viscous EoS parameter for these models. The graphical representation is also developed by considering arbitrary values of the coefficients in viscous model for a specific expression.
of Hubble parameter. The results and the comparison with non-viscous case are given as follows.

All the three models in viscous fluid indicates the behavior of the universe from matter dominated phase to quintessence era and then converges to phantom era of the DE dominated phase for decreasing $z$. It shows the phantom universe by taking the particular values of viscous coefficients. The constant viscous cases also exhibit phantom behavior. The non-viscous case $\xi = 0$ shows a universe which always stays in phantom for $h > 0$ or quintessence for $h > 1$ regions [35]. However, the third model has resulted the phantom phase of the universe for the higher values of viscous coefficients as compared to the first and second $f(T)$ models. In each case, the time dependent case shows the phantom crossing by taking small values of viscous coefficients while constant viscous case needs higher values for crossing.

The combination of torsion and viscosity influences the accelerating expansion of the universe in such a way that it strictly depends upon the viscous coefficients of the model. We have to fix the ranges for these coefficients in order to get our desired results. We conclude that the viscosity model leads to different behavior of the accelerating universe in DE era under the effects of viscous fluid. On the other hand, viscosity may result the crossing of the phantom divide line and phantom dominated universe [6, 39, 55] as shown in Figures 1 and 6. In the non-viscous case [35], the universe remains in the phantom and quintessence eras for the relevant scale factors. Beyond the ideal situation, we remark that the DE era of the universe in a real fluid may be observed and hence accelerating expansion of the universe is achieved.
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