From Chiral quark dynamics with Polyakov loop to the hadron resonance gas model

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Abstract. Chiral quark models with Polyakov loop at finite temperature have been often used to describe the phase transition. We show how the transition to a hadron resonance gas is realized based on the quantum and local nature of the Polyakov loop.

Keywords: finite temperature; heavy quarks; chiral quark models; Polyakov Loop

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INTRODUCTION

The cross-over between the chiral symmetry restoration and deconfinement at a common critical temperature $T_c \sim 200\text{MeV}$ was a first [1, 2, 3] and by now firmly established prediction of lattice QCD [4, 5]. The transition point is characterized by two order parameters. On the one hand the quark condensate $\langle \bar{q}q \rangle$ vanishes smoothly. On the other hand the Polyakov $L_T = \langle \text{tr}_c e^{iA_0/T} \rangle / N_c$ loop, where $A_0$ is a gluon field in the adjoint representation of the SU($N_c$) gauge group, jumps smoothly from zero to one signaling the breaking of the center symmetry $\mathbb{Z}(N_c)$ [6]. Of course, $\langle \bar{q}q \rangle$ and $L_T$ are true order parameters in the opposite limits of vanishing quark masses (chiral limit) and for infinitely quark masses (gluodynamics) respectively, while the true cross-over is defined by inflexion points of both $\langle \bar{q}q \rangle$ and $L_T$. The expectation that a phase transition between a hadronic phase to the quark-gluon plasma phase could be observed on the laboratory has inspired a wealth of work in recent years [7].

There have been many attempts to model the chiral-deconfinement cross-over, the main difficulty lies in properly combining the relevant degrees of freedom for the corresponding order parameter; while well below $T_c$ hadrons provide a complete basis of states, much above $T_c$ just quarks and gluons seem the adequate basis. The cross-over region seems difficult as it marks the coexistence of both degrees of freedom. In this contribution we will focus on the low temperature region of chiral quark models where the crossover is known to occur at higher temperature.

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THE HADRON RESONANCE GAS MODEL

On general grounds, and because quarks and gluons are confined, one expects that at sufficiently low temperatures any observable in QCD should be represented by the relevant hadronic colour neutral states.

For the vacuum energy at finite temperature, and more specifically the trace of the energy momentum tensor, \( \Delta(T) = (\varepsilon - 3p)/T^4 \), the results found on the lattice are expected to be represented by an interacting gas of low-lying stable hadrons (for light \( u \) and \( d \) quarks it would just be a gas of \( \pi, N \) and \( \bar{N} \) states). Most of the interactions in the scattering region generate resonances (such as \( \rho, \omega, \Delta, \text{etc} \) ) which could be represented as narrow states provided the ratio \( \Gamma/M \) is small. This is consistent with the large \( N_c \) limit expectation that \( \Gamma/M = \mathcal{O}(N_c^{-1}) \) while experimentally \( \Gamma/M = 0.12(8) \) \cite{8, 9} for both mesons and baryons listed in the Particle Data Group (PDG) booklet \cite{10}. Thus, in the Hadron Resonance Gas (HRG) model the interactions are represented by a bunch of narrow-looking resonances whose partition function is given by \cite{11, 12, 13, 14, 5}:

\[
\log Z = - \int \frac{d^3xd^3p}{(2\pi)^3} \sum_\alpha \zeta_\alpha z_\alpha g_\alpha \log \left(1 - \zeta_\alpha e^{-\sqrt{p^2 + M_\alpha^2}/T}\right),
\]

(1)

with \( g_\alpha = (2J_\alpha + 1)(2T_\alpha + 1) \) the degeneracy factor, \( \zeta_\alpha = \pm 1 \) for bosons and fermions respectively, \( M_\alpha \) the hadron mass and \( z_\alpha \) the fugacity. The PDG states \cite{10} saturate lattice calculations \cite{15, 16} as also found within a strong coupling expansion for heavy quarks in \cite{17}.

This HRG representation is less obvious for QCD operators involving just gluon fields. However, we have recently shown \cite{18} that a hadronic representation of the Polyakov loop is given by

\[
L_T = \frac{1}{N_c} \langle \text{tr} e^{iA_0}/T \rangle \approx \frac{1}{2N_c} \sum_\alpha g_\alpha e^{-\Delta_\alpha/T},
\]

(2)

where \( g_\alpha \) are the degeneracies and \( \Delta_\alpha \) are the masses of hadrons with exactly one heavy quark (the mass of the heavy quark itself being subtracted). The comparison with the spectrum with \( u, d, s \) light quarks and one extra heavy quark turns out to be rather satisfactory and fairly independent on taking charm, bottom or truly infinite heavy quarks. It is also intriguing since these calculations might provide a handle on deciding the existence of exotic multiquark states \cite{18}.

These HRG approximations are expected to hold at sufficiently low temperatures and agreement with lattice data is observed within the finite lattice uncertainties. In any case, it turns out that many states are needed to saturate both the trace anomaly and the Polyakov loop at temperatures below \( T_c \) as there are no significant gaps in the spectrum.

\[ ^3 \text{A temperature shift of about } T_0 = 10 - 20 \text{MeV is required; } \Delta_{HRG}(T - T_0) = \Delta_{QCD}(T) \text{ in Ref. } [15]. \]
POLYAKOV-CHIRAL-QUARK MODELS AND HEAVY QUARKS

An effective and phenomenologically successful approach to the physics of the phase transition is provided by chiral quark models coupled to gluon fields in the form of a Polyakov loop [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32]. Because most often the venerable Nambu–Jona-Lasinio model has been used, these models are referred to as PNJL models. In this contribution we discuss how PNJL models may be represented as a HRG at low T. Many works remain within a mean field approximation and assume a global Polyakov loop. As we have repeatedly criticized in our previous works [21, 22, 33, 34] this raises the theoretical problem of the undesirable ambiguity of group coordinates on the one hand, but also the practical problem that the adjoint-representation provides a non-vanishing value for the Polyakov loop, contradicting lattice simulations. These difficulties may be overcome [21, 22] by recognizing the local and quantum nature of the Polyakov loop.

We start out from the partition function motivated in [21, 22, 33, 34]

\[ Z = \int D\Omega e^{-S(T, \Omega)}, \]  

(3)

where \( \Omega = e^{iA_0}/T \) and \( D\Omega \) is the invariant SU\( (N_c) \) Haar group integration measure, for each SU\( (N_c) \) variable \( \Omega(x) \) at each point \( x \). Here the action is

\[ S(T, \Omega) = S_q(T, \Omega) + S_G(T, \Omega). \]  

(4)

The fermionic contribution depends on the quarks (and anti-quarks) is obtained from the corresponding fermion determinant. Assuming mass-degenerated quarks for simplicity reads

\[ S_q(T, \Omega) = -2N_f \int \frac{d^3xd^3p}{(2\pi)^3} \left( \text{tr}_c \log \left[ 1 + \Omega(x) e^{-E_p/T} \right] + \text{tr}_c \log \left[ 1 + \Omega^\dagger(x) e^{-E_p/T} \right] \right). \]  

(5)

Here \( E_p = \sqrt{\mathbf{p}^2 + M^2} \) is the energy and \( M \) the constituent quark mass. As one can see the diagonal part of the Polyakov loop corresponds to consider chemical potentials for different color species. Large color gauge invariance is implemented by just averaging over group elements.

The partition function character of the Polyakov loop can be appreciated considering a system with \( N_f \) dynamical quarks and an extra putative heavy quark (not antiquark) of mass \( m_H \) at rest located at a fixed point and with fixed spin and colour \( a = 1, \ldots N_c \). From Eq. (5) the change in the effective action is

\[ S_q(N_f + 1) - S_q(N_f) = -2 \log (1 + \Omega_{aa} e^{-E_b/T}) \approx -2 e^{-m_H/T} \Omega_{aa} \]  

(6)

yielding the partition function

\[ \frac{Z(N_f + 1)}{Z(N_f)} = 1 + (\Omega_{aa})2e^{-m_H/T} = 1 + \frac{1}{N_c} \langle \text{tr}_c \Omega \rangle 2e^{-m_H/T} \]  

(7)
after averaging over color degrees of freedom implied by $D\Omega$. Thus we get

$$
\frac{1}{N_c} \langle \text{tr}_c \Omega \rangle = \lim_{m_H \to \infty} \frac{1}{2} \left[ \frac{Z(N_f + 1)}{Z(N_f)} - 1 \right] e^{m_H/T}
$$

(8)

If we use the HRG, Eq. (1), to evaluate the r.h.s. we reproduce the HRG result, Eq. (2) for the Polyakov loop, providing confidence on the assumed coupling to quarks.

The action $S_G(\Omega)$ would follow from gluodynamics but it is exponentially suppressed at low temperatures and the distribution of $\Omega$ locally coincides with the Haar measure. A convenient model to account for Polyakov loop correlations at different points and compatible with group integration at equal points [35] is [6]

$$
\langle \text{tr}_c \Omega(x) \text{tr}_c \Omega^{-1}(y) \rangle_{S_G} = e^{-\sigma|x-y|/T},
$$

(9)

with $\sigma$ the string tension, including the correct screening of the color charge at large distances. This defines a correlation length and independent confinement domains with volume $V \equiv \frac{8\pi T^3}{\sigma^3}$ which describe the cross-over between deconfinement and chiral symmetry restoration [21, 22, 34].

Using these assumptions let us now consider the calculation of the partition function at low temperatures where to lowest order one gets contributions of just $q\bar{q}$ states

$$
\log Z = (2N_f)^2 \int \frac{d^3x_1 d^3p_1}{(2\pi)^3} \int \frac{d^3x_2 d^3p_2}{(2\pi)^3} e^{-E_1/T} e^{-E_2/T} \langle \text{tr}_c \Omega(x_1) \text{tr}_c \Omega^+(x_2) \rangle_{S_G} + \cdots
$$

(10)

which using Eq. (9) can be rewritten as

$$
\log Z = (2N_f)^2 \int \frac{d^3x_1 d^3p_1}{(2\pi)^3} \int \frac{d^3x_2 d^3p_2}{(2\pi)^3} e^{-H(x_1, p_1; x_2, p_2)/T} + \cdots.
$$

(11)

We recognize the classical partition function of a $q\bar{q}$ system with a Hamiltonian

$$
H(x_1, p_1; x_2, p_2) = \sqrt{p_1^2 + M^2} + \sqrt{p_2^2 + M^2} + \sigma r_{12}.
$$

(12)

Separating CM and relative motion, direct integration gives at low $T$ values

$$
\log Z \approx (2N_f)^2 V \int d^3x e^{-\sigma r/T} \left[ \int \frac{d^3p}{(2\pi)^3} e^{-E_p/T} \right]^2 \approx (2N_f)^2 V V_T \left( \frac{MT}{2\pi} \right)^3 e^{-2M/T}.\quad(13)
$$

Similarly, for the Polyakov loop (heavy quark located at $x_0$) we get

$$
L_T = 2N_f \int \frac{d^3x d^3p}{(2\pi)^3} e^{-H(x,p)/T} \frac{1}{N_c} \langle \text{tr}_c \Omega(x) \text{tr}_c \Omega^{-1}(x) \rangle_{S_G} + \cdots
$$

(14)

whence one can also rewrite the expression as

$$
L(T) = \frac{2N_f}{N_c} \int \frac{d^3x d^3p}{(2\pi)^3} e^{-H(x,p)/T} + \cdots.
$$

(15)
This corresponds to the classical partition function of the one-quark Hamiltonian

\[ H(x, p) = \sqrt{p^2 + M^2 + \sigma r}. \tag{16} \]

where after integration

\[ L(T) = \frac{N_f}{N_c} V_{\sigma} \frac{M^2 T}{\pi^2} K_2 \left( \frac{M}{T} \right) + \cdots \approx \frac{2N_f}{N_c} V_{\sigma} \left( \frac{MT}{2\pi} \right)^{3/2} e^{-M/T}. \tag{17} \]

As it was shown in our previous works these rules provide a satisfactory phenomenological description of the chiral-deconfinement cross-over observed in lattice calculations. The previous lowest order approximations do indeed reproduce the more sophisticated results up to \( T \sim 0.75 T_c \) \[33\].

**QUARK-HADRON DUALITY AT FINITE TEMPERATURE**

While the previous formulation reproduces lattice results in a satisfactory manner at \( T_c \approx 250 \text{MeV} \), within a hadronic phase it should be possible to express *all observables* in terms of purely hadronic properties, a feature absent in the model. To improve on this from Eqs. (11) and (12) we apply standard quantization rules to the relativistic two-body quantum-mechanical problem, which in the CM system \((p_1 + p_2 = 0)\) corresponds to solve a Salpeter equation for scalar particles

\[ \left( 2 \sqrt{p^2 + M^2 + \sigma r} \right) \psi_n = M_n \psi_n. \tag{18} \]

In a more elaborated treatment a two-body Dirac equation should be obtained. Note that at any stage our approach is Pauli-principle preserving at the quark level. Eq. (12), or its Dirac version, describes the interaction of \( q\bar{q} \) pairs to form mesons, each meson with \((2N_f)^2\)-fold degeneracy. The crucial point here is to keep track of the number and labeling of states contributing to the sum in Eq. (11). Thus after quantization and implementation of relativistic invariance, we get

\[ \log Z = \int \frac{d^3x d^3p}{(2\pi)^3} \sum_{\alpha} g_{\alpha} e^{-\sqrt{p^2 + M_\alpha^2}/T} + \cdots \tag{19} \]

Here \( x, p \) represent the former CM coordinates of the \( q\bar{q} \) pair. Note that the result holds even if quark masses are not degenerated.

The expression obtained nicely reproduces the first bosonic term in the expansion of the RGM in Eq. (1). We are assuming that the pion is also contained in the sum although the dynamics leading to such a state is not literally determined by the specific Hamiltonian in Eq. (12). The extension of the previous result to multiquark mesonic and baryonic states will be presented elsewhere \[36\].

The argument can be extended to the Polyakov loop by re-quantization of the heavy-light Hamiltonian Eq. (16) or its Dirac counterpart yielding the eigenvalues \( \Delta_\alpha \) and reproducing indeed the HRG form, Eq. (2), discussed in Ref. [18] as required by quark-hadron duality. This constitutes our main insight which suggests re-analyzing these models after incorporating the connection to the HRG \[36\].
The original Polyakov loop models suggested that $LT$ and $Z$ (and hence $\langle \bar{q}q \rangle_T$) are closely intertwined through their exponential suppression at low temperatures, controlled by the constituent quark mass (Eqs. 15 and 17). However, in the hadronized version, the suppression depends on two not directly related hadron masses, namely, the pion for $Z$ and the lightest meson with a heavy quark for the Polyakov loop. It is also noteworthy that the prefactors, namely, the powers of $T$ present in the original model for both $Z$ and $L(T)$, are removed by the quantization. The origin of these prefactors was the relative motion in the $\bar{q}q$ pair, which produces a continuous spectrum in the classical case, but yields a discrete spectrum after quantization, typical of quantum bound states. Finally, the independence between deconfinement and chiral symmetry restoration corresponding to crossed correlator $\langle \bar{q}q \text{tr} e^{\alpha A/T} \rangle_T \sim \partial L / \partial m_q \approx 0$ becomes evident at low $T$ due to the weak dependence of the Polyakov loop and thus of the static spectrum $\Delta_\alpha$ on the current quark mass. These features are also seen in lattice calculations [5].

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