Local equivalence of representations of $\text{Diff}^+(S^1)$ corresponding to different highest weights

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Abstract

Let $c, h$ and $\tilde{c}, \tilde{h}$ be two admissible pairs of central charge and highest weight for $\text{Diff}^+(S^1)$. It is shown here that the positive energy irreducible projective unitary representations $U_{c,h}$ and $U_{\tilde{c},\tilde{h}}$ of the group $\text{Diff}^+(S^1)$ are locally equivalent. This means that for any $I \subset S^1$ open proper interval, there exists a unitary operator $W_I$ such that $W_I U_{c,h}(\gamma) W_I^* = U_{\tilde{c},\tilde{h}}(\gamma)$ for all $\gamma \in \text{Diff}^+(S^1)$ which act identically on $I^c \equiv S^1 \setminus I$ (i.e. which can “displace” or “move” points only in $I$). This result extends and completes earlier ones that dealt with only certain regions of the “$c, h$-plane”, and closes the gap in the full classification of superselection sectors of Virasoro nets.

1 Introduction

The highest weight projective unitary representations of the group of orientation preserving diffeomorphisms $\text{Diff}^+(S^1)$ of the unit circle $S^1 = \{ z \in \mathbb{C} | |z| = 1 \}$ play a fundamental role in conformal quantum field theory. We postpone the detailed description of the representation $U_{c,h}$ associated to an admissible pair of the central charge $c > 0$ and highest weight $h \geq 0$ (and how it is obtained from the unitary representation $L_{c,h}$ of the Virasoro algebra through the use of the stress-energy field $T_{c,h}$) to the preliminaries, but note here that they are all irreducible and pairwise inequivalent: that is, if $(c, h)$ and $(\tilde{c}, \tilde{h})$ are both admissible pairs and $W$ is a unitary such that $W U_{c,h}(\gamma) W^* = U_{\tilde{c},\tilde{h}}(\gamma)$ for all $\gamma \in \text{Diff}^+(S^1)$, then $(c, h) = (\tilde{c}, \tilde{h})$ and $W$ is a multiple of the identity. However, some of these representations might be locally equivalent. This means, that even with $(c, h) \neq (\tilde{c}, \tilde{h})$ it can happen that for any open proper interval of the circle $I \subset S^1$, the restrictions to the subgroup formed by the diffeomorphisms localized in $I$ are unitarily equivalent; i.e. that for any $I \subset S^1$ there exists a unitary $W_I$ such that $W_I U_{c,h}(\gamma) W_I^* = U_{\tilde{c},\tilde{h}}(\gamma)$ for all $\gamma \in G_I = \{ \gamma \in \text{Diff}^+(S^1) | \gamma|_{S^1 \setminus I} = \text{id}_{S^1 \setminus I} \}$.

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Local equivalence can be also formulated at the level of self-adjoint generators: as it will be explained in the preliminaries, the unitary $W_I$ establishes a local equivalence (relative to $I \Subset S^1$) between $U_{c,h}$ and $\tilde{U}_{\tilde{c},\tilde{h}}$ if and only if

$$W_I T_{c,h}(f) W_I^* = T_{\tilde{c},\tilde{h}}(f)$$

for all $f \in C^\infty(S^1, \mathbb{R})$ with support in $I$.

The question of local equivalence comes up naturally when studying superselection sectors of conformal field theory in the setting [8] of Haag-Kastler nets. When $(c, 0)$ is admissible, the collection of von Neumann algebras

$$\mathcal{A}_c(I) = \{U_{c,0}(\gamma) | \gamma \in G_I\}'' (I \Subset S^1)$$

(3)

together with the representation $U_{c,0}$ form a conformal chiral net: the Virasoro net $\text{Vir}_c$ at central charge $c$. It is a highly important model since every conformal chiral net of von Neumann algebras contains a Virasoro net as an irreducible subsystem; a fact which for example enabled complete classification [12] of conformal chiral nets with central charge $c < 1$ and a partial one [4, 18] at central charge $c = 1$.

From the point of view of the Virasoro subnet, the full conformal net with central charge $c$ as well as its superselection sectors are all locally normal representations of $\text{Vir}_c$. This is why it is so crucial to understand and classify the superselection sectors (i.e. the locally normal irreducible representations) of $\text{Vir}_c$.

A locally normal irreducible representation $\pi$ of any conformal net $(\mathcal{A}, U)$ — and in particular, of $(\mathcal{A}_c, U_{c,0})$ — is automatically diffeomorphism covariant with positive energy [17]. This means that we have a strongly continuous projective unitary positive energy representation $U^\pi$ of $\text{Diff}^+(S^1)$ such that

$$\pi_I(U(\gamma)) = U^\pi(\gamma)$$

(4)

for every $\gamma \in G_I$. In case we deal with a Virasoro net, then $U^\pi$ must be irreducible and hence — up to unitary equivalence — it must coincide with one of the highest weight representations $U_{\tilde{c},\tilde{h}}$; see e.g. [3] Theorem A.2] and the references there given for the classification of strongly continuous projective unitary positive energy representations of $\text{Diff}^+(S^1)$. As $\pi_I$ is always unitarily implementable — see e.g. [8] Lemma 4.4 — we finally arrive to the conclusion: sectors of $\text{Vir}_c$ are in one-to-one correspondence of the highest weight projective unitary representations of $\text{Diff}^+(S^1)$ that are locally equivalent to $U_{c,0}$ in the sense we introduced it here, c.f. [4, Proposition 2.1].

As is explained in the preliminaries, the central charge $c$ is “locally detectable”. That is, if $U_{c,h}$ and $\tilde{U}_{\tilde{c},\tilde{h}}$ are locally equivalent, then $c = \tilde{c}$. So we can discuss the problem for each possible value of the central charge in a separate manner.

The case $c < 1$ has been “completely cleared”: because of the coset construction of Goddard, Kent and Olive [9], we know that for each value of the highest weight $h$ for which the pair $(c, h)$ is admissible, $U_{c,h}$ indeed defines a superselection sector of $\text{Vir}_c$ (that is, it is locally equivalent to $U_{c,0}$); see the more detailed explanation at [4 Section 2.4].
Actually, for the $c < 1$ case, not only that we have a complete list of sectors, but even their fusion rules and statistical dimensions are well-understood, see more in [12].

Using stress-energy constructions in the vacuum representation space of the U(1) current algebra, Buchholz and Schulz-Mirbach proved [2] local equivalence of $U_{c,h}$ and $U_{c,0}$ for all values of $c \geq 1$ and $h \geq \frac{c-1}{24}$. Some results concerning fusion rules and statistical dimensions of these sectors can be found in [15, 5, 4, 18]; however, our knowledge is not complete.

As noted by Buchholz, using tensorial products, one can show that if the admissible pairs $(c, h)$ and $(\tilde{c}, \tilde{h})$ are “good” in the sense that $U_{c,h}$ is locally equivalent to $U_{c,0}$ and likewise, $U_{\tilde{c},\tilde{h}}$ is locally equivalent to $U_{\tilde{c},0}$, then it follows that also $U_{c+\tilde{c},h+\tilde{h}}$ is “good” in that it is locally equivalent to $U_{c+\tilde{c},0}$; see the details at [4, Section 2.4]. In this way the region where local equivalence of $U_{c,h}$ and $U_{c,0}$ can be shown enlarges; in particular all values of $c \geq 2$ and $h \geq 0$ will fall in. However, for example when $1 < c < \frac{1}{2} + \frac{7}{10}$, this method will surely not give anything since the two smallest possible values of the central charge are $\frac{1}{2}$ and $\frac{7}{10}$.

Thus, after many years the problem was first noted, some regions of the $c, h$-plane could still not be covered till now.

In this paper it is shown that if $c > 1$, then $U_{c,h}$ is locally equivalent to $U_{c,h_c}$ for any $h < h_c = \frac{c-1}{24}$. Together with the listed earlier results this completely settles the question of local equivalence and shows that for any two admissible pairs $(c, h)$ and $(\tilde{c}, \tilde{h})$, the representations $U_{c,h}$ and $U_{\tilde{c},\tilde{h}}$ are locally equivalent if and only if $c = \tilde{c}$. In particular, each highest weight representation $U_{c,h}$ gives a locally normal irreducible representation of Vir$_c$ (and there are no other ones).

The proof relies on two main ingredients. First, just as the method of Buchholz and Schulz-Mirbach, it uses realizations of the Virasoro algebra in the U(1) current (or as it is also called: the Heisenberg) algebra. Second, that under certain conditions, the dependence of expectation values of the form

$$\langle \Psi_{c,h}, e^{iT_{c,h}(f_1)} \ldots e^{iT_{c,h}(f_n)} \Psi_{c,h} \rangle$$

on $h$ (where $f_1, \ldots, f_n \in C^\infty(S^1, \mathbb{R})$ are considered as fixed functions and $\Psi_{c,h}$ is the normalized highest weight vector) can be shown to be complex analytic. It is the method of analytic continuations that will ultimately allow us to access the region not covered by previous arguments.

2 Preliminaries

A unitary “highest weight” representation of Virasoro algebra with central charge $c > 0$ and highest weight $h \geq 0$ consists of a complex scalar product space $V_{c,h}$ and a collection of linear operators $L_n^{c,h}$ ($n \in \mathbb{Z}$) acting on $V_{c,h}$ such that we have the Virasoro algebra commutation relations

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n,-m}I \quad (n, m \in \mathbb{Z})$$

the unitarity condition

$$\langle u, L_n v \rangle = \langle L_{-n}u, v \rangle \quad (u, v \in V_{c,h}, n \in \mathbb{Z}),$$

and the selection rules

$$\langle L_n u, v \rangle = 0 \quad \text{for} \quad n < 0 \quad \text{and} \quad n \geq h + |c|/2.$$
and an up-to-phase unique normalized vector (which we shall refer to as the normalized highest weight vector) \( \Psi_{c,h} \in V_{c,h}, \| \Psi_{c,h} \| = 1 \) such that

\[
L_{0}^{c,h} \Psi_{c,h} = 0 \quad L_{n}^{c,h} \Psi_{c,h} = 0 \quad \text{for all } n > 0
\]

and \( V_{c,h} \) is the smallest subspace containing \( \Psi_{c,h} \) and invariant for all operators \( L_{n}^{c,h} (n \in \mathbb{Z}) \). These representations are completely determined by the listed properties, are irreducible, and for different values of the central charge and highest weight are pairwise inequivalent in the following sense. If both \( L_{n}^{c_{1},h_{1}} \) and \( L_{n}^{c_{2},h_{2}} \) form a unitary highest weight representation of the Virasoro algebra with central charges \( c_{1} \) and \( c_{2} \), highest weights \( h_{1} \) and \( h_{2} \) and normalized highest weight vectors \( \Psi_{c_{1},h_{1}} \) and \( \Psi_{c_{2},h_{2}} \), respectively, then there exists an invertible linear map \( W \) such that \( W L_{n}^{c_{1},h_{1}} W^{-1} = L_{n}^{c_{2},h_{2}} \) for all \( n \in \mathbb{Z} \) if and only if \( c_{1} = c_{2} \) and \( h_{1} = h_{2} \) and in this case \( W \) can be uniquely normalized so that \( W \Psi_{c_{1},h_{1}} = \Psi_{c_{2},h_{2}} \). Moreover, with this normalization \( W \) is actually a scalar product preserving linear isomorphism.

When such a unitary highest weight representation exists, we will say that \( (c, h) \) is an \textit{admissible pair}; this happens if and only if either \( c \geq 1 \) and \( h \geq 0 \) or there exists an \( m \in \mathbb{N}, m \geq 3 \) and a \( p, q \in \{1, \ldots, m+1\}, q < p \) such that

\[
c = 1 - \frac{6}{m(m+1)} \quad \text{and} \quad h = \frac{(m+1)p - mq - 1}{4m(m+1)}.
\]

For more details, references and background on the representation theory of the Virasoro algebra, we refer to the book [11]. Here we are only interested by how such a representation “integrates” into a projective unitary representation of \( \text{Diff}^{+}(S^1) \).

Let \( (c, h) \) be an admissible pair and \( \mathcal{H}_{c,h} = \bigvee_{c,h} \) be the Hilbert space obtained by the completion of \( V_{c,h} \). It can be shown that for every smooth function \( f : S^1 \to \mathbb{R} \) is a smooth function with Fourier components

\[
\hat{f}_n = \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-i n \theta} d\theta / 2\pi, \quad (n \in \mathbb{Z})
\]

the sum

\[
T_{c,h}^{0}(f) = \sum_{n \in \mathbb{Z}} \hat{f}_n L_{n}^{c,h}
\]

is absolute convergent on every vector of the dense subspace \( V_{c,h} \subset \mathcal{H}_{c,h} \). The obtained operator is closable, its closure \( T_{c,h}(f) = \overline{T_{c,h}^{0}(f)} \) is self-adjoint. We shall refer to \( T_{c,h} \) as the stress-energy field. It turns out that \( V_{c,h} \) is included in the domain of every product of the form \( T_{c,h}(f_1) T_{c,h}(f_2) \ldots T_{c,h}(f_n) \) and with

\[
[f, g] := f \partial_{\theta} g - f \partial_{\theta} g \quad \text{and} \quad (f, g) := \int_{-\pi}^{\pi} \left( \frac{d}{d\theta} f(e^{i\theta}) + \left( \frac{d}{d\theta} \right)^3 f(e^{i\theta}) \right) g(e^{i\theta}) e^{i \theta} d\theta / 2\pi
\]
Lemma 2.1. Let \((c, h)\) and \((\tilde{c}, \tilde{h})\) be two admissible pairs, \(I \Subset S^1\) an open proper interval and \(W_I\) a unitary operator. Then the two conditions

1. \(W_I T_{c, h}(f) W_I^* = T_{\tilde{c}, \tilde{h}}(f)\) for all \(f \in C^\infty(S^1, \mathbb{R})\) with support in \(I\),
are equivalent, and any of them implies that $c = \tilde{c}$.

Proof. The first condition clearly implies the second one because exponentials of vector fields with support in $I$ generate a dense subgroup in $G_I$, see [13] Section V.2. Conversely, assume the second property. Then whenever $f \in C^\infty(S^1, \mathbb{R})$ has a support in $I$ and $t$ is a real number, $W_I e^{it\mathcal{C}}(f) W_I^*$ must be a multiple of $e^{it\mathcal{C}}(f)$ and hence $W_I T_{c,\mathcal{C}}(f) W_I^*$ and $T_{c,\mathcal{C}}(f)$ can only differ in an additive constant. Suppose now that $f_1, f_2, g_1, g_2 \in C^\infty(S^1, \mathbb{R})$ have their support in $I$. Then using that additive constants do not matter inside a commutator, we find that

$$W_I ([T_{c,\mathcal{C}}(f_1), T_{c,\mathcal{C}}(g_1)] - [T_{c,\mathcal{C}}(f_2), T_{c,\mathcal{C}}(g_2)]) W_I^*$$

$$= [T_{c,\mathcal{C}}(f_1), T_{c,\mathcal{C}}(g_1)] - [T_{c,\mathcal{C}}(f_2), T_{c,\mathcal{C}}(g_2)].$$

(17)

On the other hand, it is easy to see that one can choose $f_1, f_2, g_1, g_2$ such that $[f_1, g_1] = [f_2, g_2]$ but $f_1, g_1 \neq f_2, g_2$. Then by the relations discussed at eq. (14), the closure of the left hand side is equal to $\frac{n}{12}((f_1, g_1) - (f_2, g_2))I$, whereas that of the right hand side is equal to $\frac{n}{12}((f_1, g_1) - (f_2, g_2))I$, implying that $c = \tilde{c}$. Once we know that $c = \tilde{c}$, we can use similar arguments (relying on commutators) to show that $W_I T_{c,\mathcal{C}}([g_1, g_2]) W_I^*$ is precisely equal (no additive constant) to $T_{\tilde{c},\mathcal{C}}([g_1, g_2])$. However, elementary analysis shows that every $f \in C^\infty(S^1, \mathbb{R})$ with support in $I$ can be written as the sum of at most 2 commutators; i.e. that with suitable choice of the local functions $f_1, f_2, g_1, g_2$ we have $f = [f_1, f_2] + [g_1, g_2]$.

$\Box$

3 Realizations of the Virasoro algebra using currents

A unitary representation of the $U(1)$ current (or as it also called: the Heisenberg algebra) consist of a complex scalar product space $V$ and a collection of linear operators $J_n$ ($n \in \mathbb{Z}$) acting on $V$ and satisfying the commutation relation

$$[J_n, J_m] = n\delta_{n,-m}I \quad (n, m \in \mathbb{Z})$$

(18)

and the unitarity condition

$$\langle u, J_n v \rangle = \langle J_{-n} u, v \rangle \quad (u, v \in V, n \in \mathbb{Z}).$$

(19)

In what follows we shall suppose that we deal with the vacuum representation of $U(1)$ current algebra; that is, we have an (up-to-phase unique) element of unit length $\Omega \in V$ (called the vacuum vector) such that

$$J_n \Omega = 0 \quad \text{for every} \quad n \geq 0$$

(20)
and $V$ is the minimal subspace containing $\Omega$ and invariant to all operators $J_n$ ($n \in \mathbb{Z}$). It then follows that the seemingly infinite sum appearing in

$$L_n = \frac{1}{2} : J^2 :_n \equiv \frac{1}{2} \left( \sum_{m=-\infty}^{-1} J_m J_{n-m} + \sum_{m=0}^{\infty} J_{n-m} J_m \right)$$

(21)

actually results in only finitely many non-zero terms whenever it is applied to a vector of $V$, and defines a unitary representation of the Virasoro algebra with unit central charge. Moreover, the hermitian $L_0$ is diagonalizable with nonnegative integer eigenvalues:

$$V = \bigoplus_{k=0}^{\infty} V_k \quad \text{where} \quad V_k = \text{Ker}(L_0 - kI).$$

(22)

Further, $J$ is covariant with respect to this Virasoro algebra representation:

$$[L_n, J_m] = -m J_{n+m} \quad (n, m \in \mathbb{Z}).$$

(23)

In a similar manner to how it was done in the preliminaries, one can smear $J$ and $T$ with smooth test functions and for an $f \in C^\infty(S^1, \mathbb{R})$ introduce

$$J(f) = \sum_{n \in \mathbb{Z}} \hat{f}_n J_n, \quad T(f) = \sum_{n \in \mathbb{Z}} \hat{f}_n J_n$$

(24)

which will again turn out to be self-adjoint operators. Moreover, one finds that every vector of $V$ is actually analytic for $J(f)$, the Weyl-operator $e^{iJ(g)}$ leaves invariant the dense subspace

$$\cap_{n \in \mathbb{N}} \left( T_0^n \right)$$

(25)

of smooth vectors and thus one can use convergent series to show that with $\partial_\theta g$ defined as the function $z = e^{i\theta} \mapsto \frac{d}{d\theta} g(e^{i\theta})$, we have the transformation rules

$$e^{iJ(g)} J(f) e^{-iJ(g)} = J(f) + \int_{-\pi}^{\pi} (\partial_\theta g)(e^{i\theta}) f(e^{i\theta}) \frac{d\theta}{2\pi} I$$

(26)

and

$$e^{iJ(g)} T(f) e^{-iJ(g)} = T(f) + J((\partial_\theta g)f) + \int_{-\pi}^{\pi} \frac{(\partial_\theta g)(e^{i\theta})^2}{2} f(e^{i\theta}) \frac{d\theta}{2\pi} I,$$

(27)

see the details for example at [3, Section 4.2].

The following construction is well-known and has been used others, see e.g. [11, 6, 4] (though note also that at [11, Section 3.4], the formula is given not on the vacuum space of the U(1) current). However, in part because of differences in conventions and notations, in part because of self-containment here we briefly recall the main idea.

**Proposition 3.1.** For any pair of values $\alpha, \beta \in \mathbb{C}$ the operators $\tilde{L}^{\alpha,\beta}_0 = L_0 + \frac{1}{2}(\alpha^2 + \beta^2)I$ and

$$\tilde{L}^{\alpha,\beta}_n = L_n + \alpha J_n + i\beta n J_n \quad (n \in \mathbb{Z}, n \neq 0)$$

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form a representation of the Virasoro algebra with central charge $c^{\alpha,\beta} = 1 + 12\beta^2$. When $c^{\alpha,\beta} > 1$ and $h^{\alpha,\beta} := \frac{1}{2}(\alpha^2 + \beta^2) > 0$, then one can redefine the scalar product on $V$ such that $\tilde{L}^{\alpha,\beta}_n$ becomes a unitary highest weight representation with lowest energy $h^{\alpha,\beta}$ and normalized highest weight vector $\Omega$. If $\alpha, \beta \in \mathbb{R}$, then the above holds with no need of redefinition of the scalar product.

Proof. Straightforward check shows that $\tilde{L}^{\alpha,\beta}_n$ indeed form a representation of the Virasoro algebra with central charge $c^{\alpha,\beta}$. The unitarity in case of real $\alpha, \beta$ parameters is also clear. Now let $M$ be the smallest invariant subspace for this representation that contain the vector $\Omega$. Since $\tilde{L}^{\alpha,\beta}_0\Omega = h^{\alpha,\beta}\Omega$ and $\tilde{L}^{\alpha,\beta}_n\Omega = 0$ for all $n > 0$, we have that $\Omega$ is a highest weight vector and the restriction of the representation to $M$ must factor through the Verma module corresponding to central charge $c^{\alpha,\beta}$ and lowest energy $h^{\alpha,\beta}$. However, as is known [11, Proposition 8.2], for $c^{\alpha,\beta} > 1$ and $h^{\alpha,\beta} > 0$ the Verma module is irreducible, so actually the restriction of our representation to $M$ is equivalent to the Verma one. Thus the only thing that remains to be shown is that $M$ is the full space $V$.

Since $\tilde{L}^{\alpha,\beta}_0 = L_0 + h^{\alpha,\beta} I$, our subspace must be a direct sum $M = \oplus_{k=0}^{\infty} M_k$ where $M_k \subset V_k$ for each $k$. By what we have established $\dim(M_k)$ must be equal to the dimension of the $k^{th}$ energy level of the Verma modul (corresponding to the value $k + h^{\alpha,\beta}$), which is the number of partitions of $k$. However, this is also the dimension of $V_k$; hence the inclusion $M_k \subset V_k$ is actually an equality and the proof is finished. \qed

By what has been explained in the preliminaries, when $\alpha, \beta \in \mathbb{R}$, the representation $\tilde{L}^{\alpha,\beta}$ gives rise to a stress-energy field $\tilde{T}^{\alpha,\beta}$ for which one has that

$$\tilde{T}^{\alpha,\beta}_\alpha(f)v = T(f)v + \alpha J(f)v + \beta J(f')v + \frac{\alpha^2 + \beta^2}{2} \int_{-\pi}^{\pi} f(v$$

for every smooth vector $v$ and $f \in C^\infty(S^1, \mathbb{R})$.

### 4 Dependence of expectations on lowest weight

In what follows we shall fix a smooth function $g : S^1 \to \mathbb{R}$ with the property that $\partial_\theta g|_I = 1$ for a certain open proper interval $I \subset S^1$. (Note that on the full circle it is not possible to require the derivative to be constant 1.) Then by a straightforward computation using the transformation rules (26) and (27), the formula (28) and the fact that Weyl-operators preserve the set of smooth vectors and our fields in question are all essentially self-adjoint on an even smaller set, we find that for any $\alpha \in \mathbb{R}$ and smooth function $f : S^1 \to \mathbb{R}$ with support in $I$

$$e^{i\alpha J(g)}\tilde{T}^{\alpha,\beta}_\alpha(f)e^{-i\alpha J(g)} = \tilde{T}^{\alpha,\beta}_\alpha(f).$$

(Here we really mean equality with domains, not just an equality on a dense set). In what follows, we shall also fix a collection $f_1, \ldots, f_n$ of smooth, real-valued functions on $S^1$ with
Lemma 4.1. Suppose $c > 1$. Then the function

$$
\left(\frac{c-1}{24}, \infty\right) \ni h \mapsto F(t, c, h)
$$

extends in an analytic manner to the set $\{z \in \mathbb{C} | \text{Re}(z) > \frac{c-1}{24}\}$. 

Proof. With $\beta = \sqrt{\frac{c-1}{12}}$ and $\alpha = \sqrt{2h - \beta^2} = \sqrt{2(h - \frac{c-1}{24})}$ we have that

$$
F(t, c, h) = \langle \Psi_{c,h}, e^{itT_{c,h}(f_1)} \ldots e^{itT_{c,h}(f_n)} \Psi_{c,h} \rangle = \langle \Omega, e^{it\alpha J(g)} \Omega \rangle = \langle e^{it\alpha J(g)} \Omega, e^{it\alpha J(g)} \rangle. \tag{30}
$$

The claim then follows because $\Omega$ is an analytic vector for $J(g)$. \hfill \square

Lemma 4.2. $F(t, c, h) F(t, \tilde{c}, \tilde{h}) = F(t, c + \tilde{c}, h + \tilde{h})$.

Proof. The claim can be justified by considering tensorial products. The restriction of the representation $U_{c,h} \otimes U_{\tilde{c}, \tilde{h}}$ to the minimal invariant subspace containing the vector $\Psi_{c,h} \otimes \Psi_{\tilde{c}, \tilde{h}}$ is unitarily equivalent to $U_{c + \tilde{c}, h + \tilde{h}}$ as $\Psi_{c,h} \otimes \Psi_{\tilde{c}, \tilde{h}}$ is a normalized highest weight vector for $U_{c,h} \otimes U_{\tilde{c}, \tilde{h}}$ with energy $h + \tilde{h}$. So we may write

$$
\langle \Psi_{c+\tilde{c}, h+\tilde{h}}, U_{c+\tilde{c}, h+\tilde{h}}(\gamma) \Psi_{c+\tilde{c}, h+\tilde{h}} \rangle = \langle \Psi_{c,h} \otimes \Psi_{\tilde{c}, \tilde{h}}, \left(U_{c,h}(\gamma) \otimes U_{\tilde{c}, \tilde{h}}(\gamma)\right) \Psi_{c,h} \otimes \Psi_{\tilde{c}, \tilde{h}} \rangle = \langle \Psi_{c,h}, U_{c,h}(\gamma) \Psi_{c,h} \rangle \langle \Psi_{\tilde{c}, \tilde{h}}, U_{\tilde{c}, \tilde{h}}(\gamma) \Psi_{\tilde{c}, \tilde{h}} \rangle \tag{31}
$$

which however has the disadvantage, that — since we deal with a projective, rather than a true representation — the quantities appearing in it are only defined “up-to-phase” (i.e. a unit complex multiple). Nevertheless, using products of exponentials of the form $e^{itT_{c,h}(f)}$ (rather than projective unitary operators of the form $U_{c,h}(\gamma)$) we can obtain similar equalities without the ambiguity of phases; and this is exactly what we wanted to justify. \hfill \square

Corollary 4.3. Let $c$ be greater than 1. Then there exists an $\epsilon > 0$ such that for every $t \in (-\epsilon, \epsilon)$, the function

$$
\mathbb{R}^+ \ni h \mapsto F(t, c, h)
$$

has an analytical extension to the half plane $\{z \in \mathbb{C} | \text{Re}(z) > 0\}$. 

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Proof. Let us choose a \( c_0 > 1 \) and an \( h_0 > \frac{c_0^2-1}{24} \). Since \( t \mapsto F(t, c_0, h_0) \) is continuous and \( F(0, c_0, h_0) = 1 \), there exists an \( \epsilon > 0 \) such that \( F(t, c_0, h_0) \neq 0 \) for any \( t \in (-\epsilon, \epsilon) \). Then, for such \( t \) values, using our previous lemma we have that

\[
F(t, c, h) = \frac{F(t, c + c_0, h + h_0)}{F(t, c_0, h_0)}.
\] (33)

However, by lemma the right hand side — and hence also the left hand side — has an analytical extension to the region \( \text{Re}(h + h_0) > \frac{c_0^2-1}{24} \). In particular, all values of \( h \) for which \( \text{Re}(h) > 0 \) are inside of this region.

Proposition 4.4. Let \( c \) be greater than 1. Then there exists an \( \epsilon > 0 \) such that for every \( t \in (-\epsilon, \epsilon) \) and \( h \in (0, \frac{c_0^2-1}{24}) \), we have

\[
F(t, c, h) = \langle \eta_{s, r}, e^{i t T_{0, \beta}}(f_1) \ldots e^{i t T_{0, \beta}}(f_n) \eta_r \rangle
\]

where \( \eta_r = e^{r J(\lambda)} \Omega \) and \( s = \sqrt{2(\frac{c_0^2-1}{24} - h)} \).

Proof. The right hand side of the claim is analytic in \( s \), and by eq. (33), for every \( s = i \alpha \), \( \alpha \in \mathbb{R} \) it is equal to \( F(t, c, \frac{c_0^2}{24} + \frac{1}{2} \alpha^2) \). Thus the claim follows from the uniqueness of analytical continuations and our previous corollary.

Corollary 4.5. Let \( c > 1 \), \( h \in (0, h_c) \) where \( h_c = \frac{c_0^2}{24} \) and fix an \( I \subseteq S^1 \). Then there exist two vectors \( \zeta_L, \zeta_R \in \mathcal{H}_{c,h_c} = V_{c,h_c} \) such that

\[
\langle \Psi_{c,h}, e^{i T_{c,h}(f_1)} \ldots e^{i T_{c,h}(f_n)} \Psi_{c,h} \rangle = \langle \zeta_L, e^{i T_{c,h_c}(f_1)} \ldots e^{i T_{c,h_c}(f_n)} \zeta_R \rangle
\]

for any collection \( f_1, \ldots, f_n \in C^\infty(S^1, \mathbb{R}_0^+) \cup C^\infty(S^1, \mathbb{R}_0^-) \) of functions with support in \( I \).

Proof. We shall first deal with the case when all functions involved are nonnegative. By the result of Fewster and Hollands and the assumed nonnegativity, the self-adjoint operators \( T_{c,h}(f_j) \) and \( \tilde{T}_{0, \beta}(f_j) \) appearing in the previous proposition are all bounded from below. Thus for both sides of that equation, there exists an extension (for the “\( t \)” variable) which is continuous and analytical in the upper complex half plane. It follows that the equality there deduced for \( t \in (-\epsilon, \epsilon) \) actually holds for all \( t \in \mathbb{R} \). Then the claimed equality of our corollary follows by setting \( t = 1 \) and considering that with \( \beta = \sqrt{\frac{c_0^2}{12}} \), the stress-energy field \( T_{0, \beta} \) given on the Hilbert space of the U(1) current algebra is a unitary equivalent realization of \( T_{c,h_c} \).

Let us now shortly discuss the more general case when some of the functions involved are nonpositive, whereas possibly some others are nonnegative. Say for simplicity that \( n = 2 \), and \( f_1 \geq 0 \) and \( f_2 \leq 0 \). Then by what has been already established, the functions \( G_1 \) and \( G_2 \) defined by the formulas

\[
G_1(t) = \langle \Psi_{c,h}, e^{i T_{c,h}(f_1)} e^{i t T_{c,h}(f_2)} \Psi_{c,h} \rangle, \tag{34}
\]

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Moreover, because of the spectrum condition, both $G_i$ and $G_j$ have continuous extensions that are analytic — this time on the lower complex half plane. As before, it follows that $G_1(1) = G_2(1)$ and hence our statement remains true even when some of the functions involved are nonpositive rather than nonnegative. 

\section{Proof of local equivalence}

When $c, h$ is an admissible pair, we shall set

$$A^0_{c,h}(I) = \text{Alg}^\ast\{e^{T_{c,h}(f)}| f \in C^\infty(S^1, \mathbb{R}_0^+), \text{Supp}(f) \subset I\}$$

for every open proper interval $I \subset S^1$. Here by “$\text{Alg}^\ast$” we mean the “star algebra generated”; i.e. the linear span (without taking closures) of products of finite many of the generating unitaries and their inverses. When in need of \textit{von Neumann} algebras, we shall consider the double commutant

$$A_{c,h}(I) = (A^0_{c,h}(I))''.$$  

Evidently, the above defined algebras satisfy \textit{isotony}, that is, $A_{c,h}(I_1) \subset A_{c,h}(I_2)$ whenever $I_1 \subset I_2$. Slightly less evidently, but we also have the important \textit{covariance} relation $U_{c,h}(\gamma)A_{c,h}(I)U^*_{c,h}(\gamma) = A_{c,h}(\gamma(I))$. This is because of the transformation formula mentioned in the preliminaries: $U_{c,h}(\gamma)T_{c,h}(f)U_{c,h}(\gamma)^* = T_{c,h}((\gamma^\prime f) \circ \gamma^{-1}) + a \text{ constant times} \gamma$ the identity. Since $\gamma^\prime$ is a strictly positive function, $(\gamma^\prime f) \circ \gamma^{-1}$ remains a nonnegative function. Finally, we also have \textit{irreducibility:}

$$\bigcap_{I \in S^1} A^\prime_{c,h}(I) = C I.$$  

This is because of the mentioned simplicity of Diff$^+(S^1)$. Indeed, exponentials of vector fields corresponding to functions in $C^\infty(S^1, \mathbb{R}_0^+) \cup C^\infty(S^1, \mathbb{R}_0^-)$ which are “localized” (i.e. whose support is contained in some open proper interval) evidently form a normal subgroup containing nontrivial elements. Hence this subgroup is actually the full group; thus for any $\gamma \in \text{Diff}^+(S^1)$ there exists an $n \in \mathbb{N}$, some open proper intervals $I_1, \ldots, I_n \subset S^1$ and $f_1, \ldots, f_n \in C^\infty(S^1, \mathbb{R}_0^+) \cup C^\infty(S^1, \mathbb{R}_0^-)$ such that the support of $f_j$ is contained in $I_j$ — implying that $e^{iT_{c,h}(f_j)} \in A^0_{c,h}(I_j)$ — and

$$U_{c,h}(\gamma) = e^{iT_{c,h}(f_1)} \ldots e^{iT_{c,h}(f_n)}.$$  

Thus the explained irreducibility property is a direct consequence of the irreducibility of the representation $U_{c,h}$.

\textbf{Corollary 5.1.} \textit{Let $I \subset S^1$ be an open proper interval. Then the lowest energy vector $\Psi_{c,h}$ is cyclic and separating for $A_{c,h}(I)$ (and hence also for the dense subalgebra $A^0_{c,h}(I)$).}
Proof. This is essentially the Reeh-Schlieder theorem; the proof can be done almost exactly as for example in [3]. We have every ingredient like isotony, covariance and irreducibility. Although in the cited paper the authors seemingly also make use of the invariance of the vacuum vector, a closer inspection reveals that the argument works with any vector as long as it is analytical in some strip along the real line for the self-adjoint generators of certain one-parameter groups \( t \mapsto U_{c,h}(\gamma_t) \). By [1] Theorem 3.3] this condition is always satisfied for any vector which is analytical for \( L_0^{c,h} \); in particular it holds for any eigenvector of \( L_0^{c,h} \).

For \( h = 0 \) the algebras \( A_{c,0}(I) (I \subseteq S^1) \) form a local conformal net on \( S^1 \) and hence they are all type \( \III_1 \) factors, see [3]. Note that one usually introduces local algebras by setting

\[
A_c(I) = \{ U_{c,0}(\gamma) \mid \gamma \in G_I \}
\]

which evidently contains the algebra we use: \( A_c(I) \supset A_{c,0}(I) \). However, our choice also forms a conformal net, so using Haag-duality [8], one has that

\[
A_c(I)' = A_c(S^1 \setminus I) \supset A_{c,0}(S^1 \setminus I) = A_{c,0}(I)'
\]

implying that we have containment also in the other direction and so in return that

\[
A_c(I) = A_{c,0}(I).
\]

Another important thing to note is that the introduced algebras are type \( \III_1 \) factors also in case \( c \geq 1 \) and \( h = h_c = \frac{c-1}{2\pi} \). This is because by [2], in this case we already know to have a unitary operator \( W_I \) such that \( W_I T_{c,0}(f) W_I^* = T_{c,h_c}(f) \) for all \( f \in C^\infty(S^1, \mathbb{R}) \) with support in \( I \), implying that \( W_I A_{c,0}(I) W_I^* = A_{c,h_c}(I) \).

For simplicity, from now — with the exception of the last theorem — we shall fix a single \( c > 1 \) and an \( h \in (0, h_c) \) for once and all, so that we will not need to repeat this act at every single statement. We have the following simple operator algebraic fact.

Lemma 5.2. Since \( A_{c,h_c}(I) \) is a type \( \III \) factor in standard form, any normal state on \( A_{c,h_c}(I) \) can be represented by a vector which is also cyclic for \( A_{c,h_c}(I) \).

Proof. Modular theory — see e.g. [10] Sect. 10] — ensures the existence of some representing vector \( \zeta \). Now let \( P' \) be the ortho-projection onto \( A_{c,h_c}(I)' \). Evidently, we have that \( P' \in A_{c,h_c}(I)' \), which — still by modular theory — is also a type \( \III \) factor. Hence there exists a partial isometry \( V' \in A_{c,h_c}(I)' \) such that \( V''V' = P' \) while \( V'V'' = I \). It is an exercise to check that \( V'\zeta \) is a cyclic vector for \( A_{c,h_c}(I) \) giving the same state as \( \zeta \).

Proposition 5.3. Let \( I \) be an open proper interval of \( S^1 \) and \( \zeta_L, \zeta_R \) the two vectors in \( H_{c,h_c} \) given by corollary 4.3. Then there exists a single vector \( z \) which is cyclic for \( A_{c,h_c}(I) \) and satisfies

\[
\langle \zeta_L, A\zeta_R \rangle = \langle \zeta, A\zeta \rangle
\]

for all \( A \in A_{c,h_c}(I) \).
Proof. By the equation given in the cited corollary where the two vectors $\zeta_L, \zeta_R$ were introduced, $\langle \zeta_L, \cdot \rangle$ is actually a state (i.e. a positive, normalized functional) on the dense subalgebra $\mathcal{A}_{c,h}^0(I)$. Using Kaplansky's density theorem, which ensures that every positive element of $\mathcal{A}_{c,h}^0(I)$ is the strong limit of a sequence of positives in $\mathcal{A}_{c,h}^0(I)$ — see e.g. [10, Theorem 3.10] — we conclude that $\langle \zeta_L, \cdot \rangle$ is also a state on $\mathcal{A}_{c,h}^0(I)$. By the form it is given, it is evidently a normal state. Thus our claim follows directly from the previous lemma.

Collecting what we have established and using usual constructions we arrive to the following conclusion.

Corollary 5.4. The formula
\[ e^{itc,h(f_1)} \ldots e^{itc,h(f_n)} \Psi_{c,h} \mapsto e^{itc,h(f_1)} \ldots e^{itc,h(f_n)} \zeta \]
where $f_1, \ldots, f_n \in C^\infty(S^1, \mathbb{R}_0^+) \cup C^\infty(S^1, \mathbb{R}_0^-)$ have all their supports in a certain $I \Subset S^1$ and $\zeta$ is the vector appearing in corollary 5.3 defines a unitary operator $K_{h,I}$ such that
\[ K_{h,I} e^{itc,h(f)} K_{h,I}^* = e^{itc,h(f)} \]
for all $f \in C^\infty(S^1, \mathbb{R}_0^+) \cup C^\infty(S^1, \mathbb{R}_0^-)$ with support in $I$.

Lemma 5.5. The unitary operator $K_{h,I}$ appearing in the last corollary satisfies the relation
\[ K_{h,I} T_{c,h}(f) K_{h,I}^* = T_{c,h}(f) \]
not only for functions $f \in C^\infty(S^1, \mathbb{R}_0^+) \cup C^\infty(S^1, \mathbb{R}_0^-)$ with support in $I$, but actually for any $f \in C^\infty(S^1, \mathbb{R})$ with support in $I$.

Proof. For any $f \in C^\infty(S^1, \mathbb{R})$ with support in $I$ we can find two nonnegative smooth functions $f_1, f_2 \geq 0$ with support still in $I$ such that $f = f_1 - f_2$. Of course the relations $K_{h,I} T_{c,h}(f_j) K_{h,I}^* = T_{c,h}(f_j)$ ($j = 1, 2$) evidently follow from our last corollary. Then using eq. (14),
\[ K_{h,I} T_{c,h}(f) K_{h,I}^* = K_{h,I} \left( T_{c,h}(f_1) - T_{c,h}(f_2) \right) K_{h,I}^* = \frac{K_{h,I} T_{c,h}(f_1) K_{h,I}^* - K_{h,I} T_{c,h}(f_2) K_{h,I}^*}{T_{c,h}(f_1) - T_{c,h}(f_2)} = T_{c,h}(f) \]
which is what we wanted to prove.

Collecting all we have obtained so far, we can now state the main result of this paper.

Theorem 5.6. Let $c, h$ and $\tilde{c}, \tilde{h}$ be two admissible pairs of central charges and highest weights. Then $U_{c,h}$ is locally equivalent to $U_{\tilde{c},\tilde{h}}$ if and only if $c = \tilde{c}$.
Proof. As was already noted at lemma 2.1, $c = \tilde{c}$ is a necessary condition of local equivalence. Moreover, for each admissible pair of the central charge $c$ and highest weight $h$, the pair $(c,0)$ is also admissible. Clearly then, it is enough to prove the local equivalence between $U_{c,h}$ and $U_{c,0}$; if we can do that for any of the possible $h$-values, than — passing through $U_{c,0}$ — we can also conclude the local equivalence of $U_{c,h}$ and $U_{c,\tilde{h}}$.

As was explained in the introduction, apart from the region $\{1 < c, 0 < h < h_c = \frac{c-1}{24}\}$ we already know that for any $I \subset S^1$ there exists a unitary operator $W_{h,I}$ such that
\[ W_{h,I} T_{c,0}(f) W_{h,I}^* = T_{c,h}(f) \] (44)
for all $f \in C^\infty(S^1, \mathbb{R})$ with support in $I$. On the other hand, even if $c, h$ is in this “bad” region, we can use 1) the unitary $K_{h,I}$ constructed in this section and 2) the fact that $c, h_c$ lies outside of the “bad” region so we already have a unitary $W_{h_c,I}$. Then
\[ W_{h_c,I} T_{c,0}(f) W_{h_c,I}^* = T_{c,h_c}(f) = K_{h,I} T_{c,h}(f) K_{h,I}^* \] (45)
implying that with $W_{h,I} := K_{h,I}^* W_{h_c,I}$ we satisfy eq. (44). Thus, the existence of a “suitable” unitary $W_{h,I}$ is ensured in all cases. 

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