BCS-BEC crossover at finite temperature in spin-orbit coupled Fermi gases

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By adopting a T-matrix-based method within the G0G approximation for the pair susceptibility, we study the effects of the pairing fluctuation on the three-dimensional spin-orbit-coupled Fermi gases at finite temperature. The critical temperatures of the superfluid to normal phase transition are determined for three different types of spin-orbit coupling (SOC): (1) the extreme oblate (EO) or Rashba SOC, (2) the extreme prolate or equal Rashba-Dresselhaus SOC, and (3) the spherical (S) SOC. For EO- and S-type SOC, the SOC dependence of the critical temperature signals a crossover from BCS to BEC state; at strong SOC limit, the critical temperature recovers those of ideal BEC of rashbons. The pairing fluctuation induces a pseudogap in the fermionic excitation spectrum in both superfluid and normal phases. We find that, for EO- and S-type SOC, even at weak coupling, sufficiently strong SOC can induce sizable pseudogap. Our research suggests that the spin-orbit-coupled Fermi gases may open new means to the study of the pseudogap formation in fermionic systems.

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I. INTRODUCTION

The experimental realization of ultracold Fermi gases with tunable interatomic interaction has opened new era for the study of some longstanding theoretical proposals in many-fermion systems. One particular example is the smooth crossover from a Bardeen-Cooper-Schrieffer (BCS) superfluid ground state with largely overlapping Cooper pairs to a Bose-Einstein condensate (BEC) of tightly bound bosonic molecules – a phenomenon suggested many years ago [1–4]. For a dilute Fermi gas in three dimensions (3D) with a short-range interatomic interaction where the effective range \( r_0 \) of the interaction is much smaller than the interatomic distance, such a BCS-BEC crossover can be characterized by the dimensionless gas parameter, \( 1/(k_F a) \), where \( k_F \) is the Fermi momentum and \( a \) is the \( s \)-wave scattering length of the short-range interaction. The BCS-BEC crossover occurs when \( 1/(k_F a) \) is tuned from negative to positive values (the turning point is called unitarity).

This BCS-BEC crossover has been successfully demonstrated in ultracold Fermi gases where the \( s \)-wave scattering length is tuned by means of the Feshbach resonance [5–7]. This has been regarded as one of the key successes in the cold-atom researches and has attracted broad interests due to its special properties. For example, near unitarity, the system is a high-\( T_c \) superfluid: the superfluid to normal transition temperature \( T_c \) is much higher than that of an ordinary BCS superfluid. The normal state near unitarity is strongly affected by many-body effects, e.g., the pair fluctuations which we will thoroughly study, leading to deviations from a Fermi liquid behavior and pseudogap opening, as in (underdoped) cuprate superconductors. Curiously, it is always interesting to look for other mechanisms of realizing the BCS-BEC crossover. Recent experimental breakthrough in generating synthetic non-Abelian gauge field in bosonic gas of \(^{87}\)Rb atoms has opened the opportunity to study the spin-orbit-coupling (SOC) effects in cold atomic gases [8]. In this experiment, two counter-propagating Raman laser beams and a transverse Zeeman field are applied to \(^{87}\)Rb atoms, and three hyperfine levels of \(^{87}\)Rb are coupled by the Raman lasers. By tuning the Zeeman field energy and the Raman laser frequency, two of the three hyperfine levels can become degenerate (which can be interpreted as two spins) and the low energy physics can be described by a model Hamiltonian of a spinor Bose gas coupled to an external spin \( SU(2) \) non-Abelian gauge field which, in their special setup, turns out to induce a SOC. For the fermionic case, some of the recent theoretical results suggested that tuning the SOC may provide an alternative way to realize the BCS-BEC crossover [9–18]. The experimental exploration of the spin-orbit coupled Fermi gases has also achieved remarkable progresses and the Raman scheme designed for generating SOC in \(^{87}\)Rb atoms has been successfully applied to Fermi gases: the spin-orbit coupled \(^{40}\)K and \(^{6}\)Li atoms have been realized at Shanxi University [19] and at Massachusetts Institute of Technology (MIT) [20], respectively.

The SOC of fermions can be induced by a synthetic uniform \( SU(2) \) gauge field, \( A_i^\mu = \lambda_i \delta_i^\mu \), where \( \lambda_i \) will play the roles of the SOC strengths. With this gauge field, the single-particle Hamiltonian reads \( \mathcal{H} = \mathbf{k}^2/(2m) - \mathbf{s} \cdot \mathbf{s}(\mathbf{k}) \) where \( \mathbf{s}(\mathbf{k}) = (\lambda_x k_x, \lambda_y k_y, \lambda_z k_z) \). In a very interesting paper [21], Vyasanakere and Shenoy studied the two-body problem of this Hamiltonian. They paid particular attention to three special types of gauge field configurations: (1) \( \lambda_x = \lambda_y = 0 \) and \( \lambda_z = \lambda \) [called extreme prolate (EP)], (2) \( \lambda_x = \lambda_y = \lambda \) and \( \lambda_z = 0 \) [called extreme oblate (EO)], and (3) \( \lambda_x = \lambda_y = \lambda_z \).
The novel bound state that emerged in the two-body problem suggests that the EO or S SOCs may trigger a new type of BCS-BEC crossover in the many-body problem of fermions. In fact, theoretical studies revealed that for EO or S SOC, even at small negative $k_Fa$, a crossover from the BCS superfluid to the BEC of rashbon states can be achieved by tuning the SOC $\lambda$ to large enough value [9–18]. It was shown that for EO or S SOCs the system enters the rashbon BEC regime at $\lambda \sim v_F$ where $v_F \equiv k_F/m$ is the Fermi velocity. Similar conclusions were also found for 2D Fermi gases with EO SOC [16].

So far, most of the theoretical studies of the BCS-BEC crossover in 3D SOC Fermi gases focused on the zero-temperature ground state based on mean-field theory (MFT). Although the MFT captures some qualitative features of the zero-temperature crossover, it loses the effects of the pairing fluctuation affects the thermodynamics. We note that this is the first systematic study of the 3D spin-orbit-coupled Fermi gases at finite temperature. For 2D spin-orbit-coupled Fermi gases, the possible BKT transition at finite temperature was already studied [16].

The article is organized as follows. In Sec. II, we present a detailed theoretical scheme of the $T$-matrix-based formalism at finite temperature. The numerical results are given in Sec. III. We summarize in Sec. IV. Throughout this article, we use natural units $\hbar = k_B = 1$.

II. T-MATRIX-BASED FORMALISM

We consider a homogenous Fermi gas interacting via a short-range attractive interaction in the spin-singlet channel. In the dilute limit $k_FR_0 \ll 1$ and $m\lambda T \ll 1$, where $r_0$ is the effective interaction range, this system can be described by the following Hamiltonian,

$$H = \int d^3r \psi_\uparrow^\dagger(r) (\mathcal{H}_0 + \mathcal{H}_{SO}) \psi_\uparrow(r) + U \int d^3r \psi_\uparrow^\dagger(r) \psi_\downarrow^\dagger(r) \psi_\downarrow(r) \psi_\uparrow(r),$$

where $\mathcal{H}_0 = -\nabla^2/(2m) - \mu$ is the free single-particle Hamiltonian with $\mu$ being the chemical potential, $\mathcal{H}_{SO} = -i \sum_{\sigma} \lambda_{\sigma\tau} \tau_{\sigma\tau} \delta_{\tau \sigma}$ is the SOC term, and $U < 0$ denotes the attractive s-wave interaction.

Introduce the four-dimensional Nambu-Gorkov spinor $\Psi = (\psi_\uparrow, \psi_\downarrow, \psi_\uparrow^\dagger, \psi_\downarrow^\dagger)^T$. The (imaginary-time) Green’s function of the Nambu-Gorkov spinor is given by

$$S(\tau, r) \equiv -\langle T_\tau \Psi^\dagger(\tau, r) \Psi(0, 0) \rangle = \begin{bmatrix} \mathcal{G}(\tau, r) & \mathcal{F}(\tau, r) \\ \mathcal{F}^\dagger(\tau, r) & \mathcal{G}^\dagger(\tau, r) \end{bmatrix},$$

where $T_\tau$ is the (imaginary-) time-ordering operator and $\tau \equiv it$. It is convenient to work in frequency-momentum space,

$$S(K) = \begin{bmatrix} \mathcal{G}(K) & \mathcal{F}(K) \\ \mathcal{F}^\dagger(K) & \mathcal{G}^\dagger(K) \end{bmatrix},$$

where $K = (k_0 \equiv i\omega_n, k)$ with $\omega_n = (2n + 1)\pi T$ ($n$ integer) being the Matsubara frequency for fermion. The Green’s

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1 The interaction range $r_0$ is about 3.2 nm for $^{40}$K [41] and 2.1 nm for $^6$Li [42]. Thus for these atoms when $k_F a \geq 0.1$ nm$^{-1}$ the dilute limit may be violated. In Shaxi University experiment [19], $m\lambda = 0.008$ nm$^{-1}$ and $k_F$ varies from 0.9m to 1.8m; in MIT experiment [20], $m\lambda = 0.003$ nm$^{-1}$ and $k_F$ varies about $m\lambda$. In both experiments, the dilute conditions are well satisfied.

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[ called spherical (S)]. The EO SOC is physically equivalent to the Rashba SOC which has been famous in condensed matter physics. The EP SOC is physically equivalent to an equal mixture of Rashba and Dresselhaus SOCs. The most surprising finding of Vyasankare and Shenoy was that for EO and S SOCs, even for $a < 0$ where the di-fermion bound state cannot form in the absence of SOC, the di-fermion bound state (referred to as rashbon) always exists and its binding energy is generally enhanced with increased SOC. Meanwhile, the bound state also possesses non-trivial effective mass which is generally larger than twice of the fermion mass $m$. For the two dimensional (2D) case, although a bound state exists for arbitrarily small attraction, it was shown in Ref. [16] that the EO or Rashba SOC can generally enhance the binding energy and the effective mass of the bound state.

In this paper, we study the spin-orbit-coupled Fermi gases at finite temperature. To include the pairing-fluctuation effects and investigate the possible pseudogap phenomena, we will adopt a $T$-matrix formalism based on a $GaG$ approximation for the pair susceptibility which was first introduced by the Chicago group [23–28, 34]. This formalism generalizes the early works of Kadanoff and Martin [39] and Patton [40], and can be considered as a natural extension of the BCS theory since they share the same ground state. Moreover, this formalism allows quasianalytic calculations and gives a simple physical interpretation of the pseudogap emergence. It clearly shows that the pseudogap is due to the incoherent pairing fluctuation. Within this formalism, we can also determine the superfluid critical temperature and study how the pairing fluctuation affects the thermodynamics.
functions have the following properties:
\[
\begin{align*}
\tilde{G}(i\omega_n, k) &= -[G(-i\omega_n, -k)]^T, \\
\tilde{F}(i\omega_n, k) &= +[F(-i\omega_n, k)]^T, \\
\tilde{F}(i\omega_n, k) &= -[\tilde{F}(-i\omega_n, -k)]^T, \\
G(i\omega_n, k) &= +[G(-i\omega_n, k)]^T, \\
\tilde{G}(i\omega_n, k) &= +[	ilde{G}(-i\omega_n, k)]^T.
\end{align*}
\] (2.4)

In the rest of this section, we will introduce the basic method of the \(T\)-matrix. Our strategy will closely follow Refs. [23–28, 34, 37]. The \(T\)-matrix we will adopt is defined as an infinite series of ladder-diagrams in the particle-particle channel by constructing the ladder by one free particle propagator and one full particle propagator. The \(T\)-matrix thus enters the particle self-energy in place of the bare interaction vertex. The equation that defines the \(T\)-matrix, the self-energy equation (or gap equation) as well as the number density equation form a closed set of equations, and should be solved consistently. One can view this approach as the simplest generalization of the BCS theory, which can also be cast formally into a \(T\)-matrix formalism. Let us start with the BCS theory.

### A. BCS Theory

The BCS theory is based on the mean-field approximation to the anomalous self-energy. We start with the mean-field inverse fermion propagator,
\[
S_{mf}^{-1}(K) = \left[ G_0^{-1}(K) + i\sigma y \Delta_{mf} - i\sigma y \Delta_{mf}^{-1}(-K) \right]^{-1},
\]
where the anomalous self-energy \(\Delta_{mf}\) is chosen as a constant and can be used as an order parameter for superfluid phase transition. The inverse free fermion propagator reads,
\[
G_0^{-1}(K) = i\omega_n - \xi_k - \xi_0(k),
\]
with \(\xi_k = k^2/(2m) - \mu\) and \(\xi_0(k) = \sum_{i=1}^{3} \lambda_i \sigma_i k_i\) (\(\lambda_i\) is real). By direct doing the matrix inversion, one obtains
\[
S_{mf}(K) = \left[ G_{mf}(K) - i\sigma y \Delta_{mf} \right]^{-1}
\]
where we introduced \(\lambda|k| = \sqrt{\sum_{i=1}^{3} \lambda_i^2 k_i^2}\) and
\[
\begin{align*}
A_{11}(K) &= \frac{1}{2} \left[ \frac{i\omega_n + \xi_k^+}{(i\omega_n)^2 - (E_k^+)^2} + \frac{i\omega_n + \xi_k^-}{(i\omega_n)^2 - (E_k^-)^2} \right], \\
A_{12}(K) &= \frac{1}{2} \left[ \frac{i\omega_n - \xi_k^+}{(i\omega_n)^2 - (E_k^+)^2} + \frac{i\omega_n - \xi_k^-}{(i\omega_n)^2 - (E_k^-)^2} \right], \\
A_{21}(K) &= \frac{1}{2} \left[ \frac{\Delta_{mf}}{(i\omega_n)^2 - (E_k^+)^2} + \frac{\Delta_{mf}}{(i\omega_n)^2 - (E_k^-)^2} \right], \\
B_{11}(K) &= \frac{1}{2} \left[ \frac{i\omega_n + \xi_k^+}{(i\omega_n)^2 - (E_k^+)^2} - \frac{i\omega_n + \xi_k^-}{(i\omega_n)^2 - (E_k^-)^2} \right], \\
B_{12}(K) &= \frac{1}{2} \left[ \frac{i\omega_n - \xi_k^+}{(i\omega_n)^2 - (E_k^+)^2} - \frac{i\omega_n - \xi_k^-}{(i\omega_n)^2 - (E_k^-)^2} \right], \\
B_{21}(K) &= -B_{12}(K).
\end{align*}
\] (2.17)

Here \(E_k^\pm = \sqrt{(\xi_k^\pm)^2 + \Delta_{mf}^2}\) with \(\xi_k^\pm = \xi_k \pm \lambda|k|\) is the fermion dispersion relation. One can verify that Eqs. (2.13)-(2.16) satisfy Eqs. (2.4)-(2.9).

Then from the standard Green’s function method, the coupled gap and density equations are expressed as
\[
\Delta_{mf} = -\frac{U}{2\beta V} \text{Tr} \sum_K i\sigma y F_{mf}(K)
\]
\[
= \frac{U}{2\beta V} \text{Tr} \sum_K \sum_{k=\pm} \left( 1 - 2n_F(E_k^0) \right),
\]
\[
n = \frac{1}{\beta V} \text{Tr} \sum_K \epsilon_{\alpha \omega_n} G_{mf}(K)
\]
\[
= \frac{1}{\beta V} \sum_{\alpha = \pm} \sum_K \left( (u_{\alpha k}^0)^2 n_F(E_k^0) + (v_{\alpha k}^0)^2 n_F(-E_k^0) \right),
\]
where \(n_F(x) = 1/[\exp(\beta x) + 1]\) is the Fermi-Dirac function, \(\epsilon_{\alpha \omega_n} = \eta \epsilon_{\omega_n}\) with \(\eta \rightarrow 0^+\) is a converging factor for the Matsubara summation, and \((u_{\alpha k}^0)^2 = \frac{1}{2} (1 + \xi_k^0/E_k^0)\) and \((v_{\alpha k}^0)^2 = \frac{1}{2} (1 - \xi_k^0/E_k^0)\) are the Bogoliubov coefficients.

The difference between \(G_0^{-1}(K)\) and \(G_{mf}^{-1}(K)\) defines the mean-field self-energy,
\[
\Sigma_{mf}(K) = G_0^{-1}(K) - G_{mf}^{-1}(K) = -\Delta_{mf}^2 i\sigma y \tilde{G}_0(K) i\sigma y.
\] (2.21)

If we define a \(T\) matrix in the following form,
\[
t_{mf}(Q) = -\Delta_{mf}^2 \delta(Q),
\] (2.22)
where \(Q = (q_0 \equiv i\omega_n, \mathbf{q})\) with \(\omega_n = 2\pi n T (\nu\ \text{integer})\) being the boson Matsubara frequency and \(\delta(Q) = \beta \delta_{\nu \theta_0} \delta^{(3)}(Q)\), Eq. (2.21) can be rewritten in a manner of
\[
\Sigma_{mf}(K) = \frac{1}{\beta V} \sum_Q t_{mf}(Q) i\sigma y \tilde{G}_0(K - Q) i\sigma y
\] (2.23)
This shows that in the BCS theory, the fermion-fermion pairs contribute to the fermion self-energy only through their condensate at zero momentum, and these condensed pairs are associated with a $T$-matrix or propagator (2.22).

Furthermore, if we define the mean-field pair susceptibility as

$$\chi_{mf}\left(Q\right) = \frac{1}{2\beta V} \text{Tr} \sum_{K} \left[ G_{mf}(K) i\sigma_{y} \tilde{\alpha}_{0}(K - Q) i\sigma_{y} \right],$$

(2.24)

we can rewrite the gap equation in the superfluid phase as

$$1 + U\chi_{mf}(0) = 0, \quad T \leq T_{c}. \quad (2.25)$$

This suggests that if one considers the uncondensed pair propagator or $T$ matrix to be of the form

$$t_{\text{pair}} = \frac{U}{1 + U\chi_{mf}(Q)}, \quad Q \neq 0, \quad (2.26)$$

then the gap equation is given by $t_{\text{pair}}^{-1}(0) = 0$.

It is well known that the critical temperature $T_{c}$ in the BCS theory is related to the appearance of a singularity in a $T$ matrix in the form of Eq. (2.26) but with $\Delta_{mf} = 0$. This is the so-called Thouless criterion for $T_{c}$ [43]. But the meaning of Eq. (2.25) is more general as stressed by Kadanoff and Martin [39]. It states that under an asymmetric choice of $\chi_{mf}$, the gap equation is equivalent to the requirement that the $T$ matrix associated with the uncondensed pairs remains singular at zero momentum and frequency for all temperatures below $T_{c}$.

Although the construction of the uncondensed pair propagator (2.26) in BCS scheme is quite natural, the uncondensed pair has no feedback to the fermion self-energy (2.23). In the BCS limit (both $\left|U\right|$ and $\lambda$ are small), such a feedback may not be important, but if the system is strongly coupled (for large $\left|U\right|$ and/or for large $\lambda$ for EO or S SOC), this feedback will be essential. The simplest way to include the feedback effects is to replace $t_{mf}$ in Eq. (2.23) by $t_{mf} + t_{\text{pair}}$. But to make such an inclusion self-consistent, $t_{\text{pair}}$ should be somewhat modified which we now discuss.

B. $G_{0}G$ Formalism at $T \leq T_{c}$

The BCS theory involves the contribution to the self-energy from the condensed pairs only, but generally, in superfluid phase, the self-energy consists of two distinctive contributions, one from the superfluid condensate, and the other from thermal or quantum pair fluctuations. Correspondingly, it is natural to decompose the self-energy into two additive terms,

$$\Sigma(K) = \frac{1}{\beta V} \sum_{Q} t(Q) i\sigma_{y} \tilde{\alpha}_{0}(K - Q) i\sigma_{y}$$

$$= \Sigma_{sf}(K) + \Sigma_{pg}(K), \quad (2.27)$$

with the $T$ matrix accordingly given by

$$t(Q) = t_{sf}(Q) + t_{pg}(Q),$$

$$t_{sf}(Q) = -\Delta_{sf}^{2} \delta(Q),$$

$$t_{pg}(Q) = \frac{U}{1 + U\chi(Q)}, \quad Q \neq 0, \quad (2.28)$$

where the subscript sf and pg indicate that these terms are responsible to the superfluid condensate and pseudogap in fermionic dispersion relation. See Fig. 1 for the Feynman diagrams for $t_{pg}(Q)$ and $\Sigma(K)$.

Comparing with the BCS scheme, $t_{mf}(Q)$ in Eq. (2.23) is replaced by $t(Q)$, and $\Sigma(K)$ now contains the feedback of uncondensed pairs. Inspired by Eq. (2.24), we now choose the pair susceptibility $\chi(Q)$ to be the following asymmetric $G_{0}G$ form,

$$\chi(Q) = \frac{1}{2\beta V} \sum_{K} G(K) i\sigma_{y} \tilde{\alpha}_{0}(K - Q) i\sigma_{y}. \quad (2.29)$$

In spirit of Kadanoff and Martin, we now propose the superfluid instability condition or gap equation as [extension of Eq. (2.25)]

$$1 + U\chi(0) = 0, \quad T \leq T_{c}. \quad (2.30)$$

We stress here that this condition has quite clear physical meaning in BEC regime. The dispersion relation of the bound pair is given by $t_{pg}^{-1}(Q) = 0$, hence $t_{pg}^{-1}(0) \propto \mu \mu_{b}$ with $\mu_{b}$ the effective chemical potential of the pairs. Then the BEC condition requires $\mu_{b} = 0$, and thus $t_{pg}^{-1}(Q) = 0$, for all $T \leq T_{c}$.

The gap equation (2.30) tells us that $t_{pg}(Q)$ is highly peaked around $Q = 0$, so we can approximate $\Sigma_{pg}$ as

$$\Sigma_{pg}(K) \approx -\Delta_{pg}^{2} i\sigma_{y} \tilde{\alpha}_{0}(K) i\sigma_{y}, \quad T \leq T_{c}, \quad (2.31)$$

where we have defined the pseudogap parameter $\Delta_{pg}$ via

$$\Delta_{pg}^{2} = -\frac{1}{\beta V} \sum_{Q \neq 0} t_{pg}(Q). \quad (2.32)$$

The total self-energy now is written in a BCS-type form

$$\Sigma(K) = -\Delta_{sf}^{2} i\sigma_{y} \tilde{\alpha}_{0}(K) i\sigma_{y}, \quad (2.33)$$

but with $\Delta^{2} = \Delta_{sf}^{2} + \Delta_{pg}^{2}$. It is clear that $\Delta_{pg}$ also contributes to the energy gap in fermionic excitation. Physically, the pseudogap $\Delta_{pg}$ below $T_{c}$ can be interpreted as extra contribution to the excitation gap of fermion: an additional energy is needed to overcome the residual binding between fermions in a thermal excited pair to produce fermion-like quasi-particles. One should note that $\Delta_{pg}$ is associated with the thermal fluctuation of the pairs $\Delta_{pg}^{2}(T) \sim \langle \Delta^{2}(T) \rangle - \langle \Delta(T) \rangle^{2} \quad [25, 34]$ hence it does not lead to superfluid (symmetry breaking). Besides, at $T = 0$, the $G_{0}G$ formalism recovers the BCS theory, hence the $G_{0}G$ formalism does not involve quantum fluctuation. We note here that at strong interacting regime the quantum fluctuations could have sizable contributions to certain quantities like the excitation gap. For example, at the unitarity and at $T = 0$, the $G_{0}G$ approach gives $\Delta \approx 0.69 \varepsilon_{F}$ (see, for
According to this Taylor expansion, we apply the pole approximation to the pair propagator or $T$ matrix $t_{pg}(Q)$,

\[ t_{pg}(Q) \simeq \frac{Z^{-1}}{q_{0} - \sum_{i=1}^{3} q_{i}^{2} / (2m_{bi})} \]  

(2.42)

with $q_{0} = i\omega_{n}$, and

\[ T_{\text{mf}} = \sum_{i=1}^{3} \frac{\lambda_{i}^{2} k_{i} (q_{i} - k_{i})}{\lambda_{i}^{2} |q - k|}. \]  

(2.37)

Furthermore, the gap equation (2.30) suggests that we can make the following Taylor expansion for $\chi(Q)$,

\[ \chi(Q) = \chi(0) + Z \left( q_{0} - \sum_{i=1}^{3} \frac{1}{2m_{bi} q_{i}^{2}} \right) + \cdots, \]  

(2.38)

where $Z$ is a pair wave-function renormalization factor and $m_{bi}$ is the effective “boson” mass parameter in the $i$ direction. A straightforward calculation leads to

\[ \chi(0) = -\frac{1}{2V} \sum_{\alpha,s=\pm} \sum_{\mathbf{k}} \frac{s}{2E_{\mathbf{k}}} n_{F}(sE_{\mathbf{k}}), \]  

(2.39)

\[ Z = \frac{\partial \chi(Q)}{\partial q_{0}} \bigg|_{Q=0} = \frac{1}{2V} \sum_{\alpha,s=\pm} \sum_{\mathbf{k}} \frac{s}{2E_{\mathbf{k}}} \frac{n_{F}(E_{\mathbf{k}}^{\alpha}) - n_{F}(sE_{\mathbf{k}}^{\alpha})}{E_{\mathbf{k}}^{\alpha} - s\xi_{\mathbf{k}}^{\alpha}}. \]  

(2.40)

We stress here that in general, the small $Q$ expansion of $t_{pg}^{-1}(Q)$ should contain a term $\propto q_{0}^{3}$. Without this term, Eq. (2.42) does not respect the particle-hole symmetry and thus can work, in principle, only when the system becomes bosonic. At the BCS limit, the system possesses a sharp Fermi
surface, the pair propagator should asymptotically recover the particle-hole symmetry, i.e., the \( q_b^2 \) term should be kept. However, since at BCS limit the pseudogap is expected to be very small, applying Eq. (2.42) does not bring much quantitative difference. Therefore, we will apply Eq. (2.42) to the whole crossover region. We also note that the pole approximation to the pair propagator generally strengthens the uncondensed pairing and thus leads to amplification of the pseudogap effects, however, it remains a very good approximation for our qualitative and semi-quantitative analysis.

Substituting the number density equation, the parameter \( Z \) can be expressed as

\[
Z = \frac{1}{\Delta^2} \left[ \frac{n}{2} - \frac{1}{2V} \sum_{\alpha = \pm} \sum_k n_F(\xi_k^\alpha) \right].
\] (2.43)

The expression in the square bracket of the right-hand side is nothing but the density of the pairs \( n_b \), thus we have \( n_b = Z \Delta^2 \).

Substituting Eq. (2.42) into Eq. (2.32) leads to

\[
\Delta_{pg}^2 = \frac{1}{2V} \sum_q n_B \left[ \frac{3}{2} q_i^2 / (2m_b) \right] = \frac{1}{2} \prod_{i=1}^3 \sqrt{\frac{Tm_b}{2\pi}} \zeta \left( \frac{3}{2} \right),
\] (2.44)

where \( n_B(x) = 1/\left[ \exp(\beta x) - 1 \right] \) is the Bose-Einstein function and a vacuum term was regularized out. It should be stressed that at zero temperature \( \Delta_{pg}^2 = 0 \), hence the \( G_0G \) scheme yields the BCS ground state. It is also worth noting that \( \Delta_{st}^2 = n_b^{\text{uncondensed}} / Z \), and hence \( \Delta_{st}^2 = n_b^{\text{condensed}} / Z \).

Now, Eq. (2.34), Eq. (2.35), as well as Eq. (2.44) are coupled to determine the total excitation gap \( \Delta \), the pseudogap \( \Delta_{pg} \) and the chemical potential \( \mu \) at a given temperature below \( T_c \), and \( T_c \) itself is determined by the vanishing of \( \Delta_{st} \).

### C. \( G_0G \) formalism at \( T \geq T_c \)

Above \( T_c \), Eq. (2.30) does not apply, hence Eq. (2.31) no longer holds. To proceed, we extend our more precise \( T \leq T_c \) equations to \( T > T_c \) in a simplest fashion. We will continue to use Eq. (2.33) to parameterize the self-energy but with \( \Delta = \Delta_{pg} \), and ignore the finite lifetime effect associated with normal state pairs. In the absence of the SOC, it was shown that this is a good approximation when temperature is not very much higher than \( T_c \) [24, 26]. The \( T \) matrix \( t_{pg}(Q) \) at small \( Q \) can be approximated now as

\[
t_{pg}(Q) \simeq \frac{Z^{-1}}{q_0 - \Omega_Q},
\] (2.45)

where \( \Omega_Q = \sum_{i=1}^3 q_i^2 / (2m_b) - \mu_b \). Since there is no condensation in normal state, the effective pair chemical potential \( \mu_b \) is no longer zero, instead, it should be calculated from

\[
Z \mu_b = t^{-1}(0) = \frac{1}{U} + \chi(0) = \frac{1}{U} - \frac{1}{2V} \sum_{\alpha, s = \pm} \sum_k \frac{s}{2E_k} n_F(sE_k^\alpha) - \frac{1}{2} \sum_{x, y, z} T_{xy}^{\alpha}(\Omega_Q).
\] (2.46)

This is used as the modified gap equation. Similarly, above \( T_c \) the pseudogap \( \Delta_{pg} \) is determined by

\[
\Delta_{pg}^2 = \frac{1}{2V} \sum_q n_B(\Omega_q) = \frac{1}{Z} \prod_{i=1}^3 \sqrt{\frac{Tm_b}{2\pi}} \zeta \left( \frac{3}{2} \right) e^{\mu_b / T},
\] (2.47)

where \( \zeta(z) \) is the polylogarithm function. Then Eq. (2.46), Eq. (2.47) and the number equation which remains unchanged determine \( \Delta_{pg}, \mu \) and \( \mu_b \) at \( T > T_c \).

At this point, we comment that at \( T > T_c \), the pseudogap may be closely related to the contact intensity \( C \) which is introduced by Tan [45] through the large momentum tail of the distribution functions, \( n_{\bar{z}, \bar{z}}(k) \rightarrow C/k^4, k \rightarrow \infty \), and underlies a variety of universal thermodynamical relations for the Fermi gases. To see this, we recall the relation [46–48] that

\[ C = - (m^2/\beta V) \sum Q \Gamma_{\text{pair}}(Q) \] with \( \Gamma_{\text{pair}} \) being the full propagator of the pairs. At \( T > T_c \), if we approximate \( \Gamma_{\text{pair}} \) by \( t_{pg} \), we can roughly estimate the contact intensity as

\[ C \sim m^2 \Delta_{pg}^2. \]

### III. RESULTS AND DISCUSSIONS

With all the equations settled down, now we present the predictions obtained by solving them numerically. We will focus on three different types of SOC: EP (\( \lambda_x = \lambda_y = 0, \lambda_z = \lambda \)), EO (\( \lambda_x = \lambda_y = \lambda, \lambda_z = 0 \)), and S (\( \lambda_x = \lambda_y = \lambda_z = \lambda \)). In all these cases, we regularize the UV divergence in the gap equations by introducing the \( s \)-wave scattering length \( a \) through

\[
\frac{1}{U} = \frac{m}{4\pi a} - \frac{1}{V} \sum_k \frac{m}{k^2}.
\] (3.1)

#### A. Analytical Results in the Molecular BEC limit

Let us first examine the molecular BEC limit which can be achieved by either tuning \( 1/(\mu a) \rightarrow +\infty \) for fixed SOC or tuning \( \lambda \rightarrow \infty \) for fixed \( 1/(\mu a) \) for EO and S SOCs. The former case is well studied and here we are mainly interested in the latter case. In the molecular BEC limit, we expect \( \mu < 0 \) and \( |\mu| \gg T_c \). For temperature around or below \( T_c \), we can approximate \( \xi_k^\alpha / T, E_k^\alpha / T \approx \infty \), and all the equations become temperature independent. In this limit, the gap equation determines the chemical potential while the number equation determines the gap. By further expanding the equations in powers of \( \Delta / |\mu| \) and keeping several leading terms, we obtain some analytical results for various quantities at \( T \ll T_c \).
for algebra equation \[ \text{potential}, \]

1. (III) EP case. We find that the EP case is trivial. Increasing problem \[ \text{The effective pair mass becomes anisotropic and is given by} \]

\begin{equation}
\frac{2m}{m_b} \approx \frac{7}{3} - \frac{4}{3} \left( \frac{E_B - m \lambda^2}{E_B} \right)^{3/2} - \frac{2m \lambda^2}{E_B}.
\end{equation}

Other quantities such as $\Delta$ and $Z$ can be evaluated as

\begin{equation}
\Delta^2 \approx \frac{32 \varepsilon_F \sqrt{\varepsilon_F}}{3\pi E_B} \left( \frac{E_B - m \lambda^2}{2} \right)^{3/2},
\end{equation}

\begin{equation}
Z \approx \frac{n}{2 \Delta^2} = \frac{E_B}{8\pi} \left( \frac{m}{E_B - m \lambda^2} \right)^{3/2}.
\end{equation}

II) EO case. The chemical potential is also given by

\begin{equation}
\mu \approx -\frac{E_B}{2},
\end{equation}

where two-body binding energy $E_B$ is determined by the algebra equation \[ \text{[9–11, 21].} \]

\begin{equation}
\sqrt{\frac{E_B}{m \lambda^2}} - \frac{1}{2} \ln \frac{\sqrt{E_B} + \sqrt{m \lambda^2}}{\sqrt{E_B} - \sqrt{m \lambda^2}} = \frac{m}{\lambda a}.
\end{equation}

The effective pair mass becomes anisotropic and is given by

\begin{equation}
m_a^\perp \approx 2m \left[ 1 - \frac{m \lambda^2}{2E_B} - \frac{E_B - m \lambda^2}{2E_B} \ln \frac{E_B - m \lambda^2}{E_B} \right]^{-1},
\end{equation}

\begin{equation}
m_a^\parallel \approx 2m.
\end{equation}

Other quantities such as $\Delta$ and $Z$ can be evaluated as

\begin{equation}
\Delta^2 \approx \frac{8\varepsilon_F (E_B - m \lambda^2)}{3\pi} \left( \frac{E_B}{E_B} \right)^{2/3},
\end{equation}

\begin{equation}
Z \approx \frac{n}{2 \Delta^2} = \frac{E_B}{8\pi (E_B - m \lambda^2)}.
\end{equation}

III) EP case. We find that the EP case is trivial. Increasing $\lambda$ can not induce a BCS-BEC crossover. For large and positive \[ 1/(k_F a), \] the EP SOC only induce a shift for the chemical potential,

\begin{equation}
\mu \approx -\frac{1}{2ma^2} - \frac{m \lambda^2}{2}.
\end{equation}

The pair effective mass is almost isotropic and is given by

\begin{equation}
m_b^\perp \approx m_b^\parallel \approx 2m.
\end{equation}

Other quantities such as $\Delta$ and $Z$ just recover the usual results without SOC,

\begin{equation}
\Delta^2 \approx \frac{8\varepsilon_F k_F}{3\pi ma},
\end{equation}

\begin{equation}
Z \approx \frac{m^2 a}{8\pi}.
\end{equation}

The critical temperature in the BEC limit, $T_{\text{BEC}}$, is determined by the number equation

\begin{equation}
n_B = \frac{1}{V} \sum_k \exp \left[ \frac{\varepsilon_B(k)}{T_{\text{BEC}}} \right] - 1,
\end{equation}

where $\varepsilon_B(k) = \sum_{i=1}^{3} k_i^2/(2m_{bi})$. This leads to $T_{\text{BEC}} = 2\pi [n_B/(\sqrt{\Pi m_{bi}^\lambda(3/2)})]^{3/2}$ in three dimensions. Setting $n_B = n/2$, we obtain

\begin{equation}
T_{\text{BEC}} \approx 0.218 \varepsilon_F \prod_{i=1}^{3} \left( \frac{2m_{bi}}{m_{bi}} \right)^{1/3}.
\end{equation}

Therefore, in the molecular BEC limit, $T_{\text{BEC}}$ is only a function of the combined dimensionless parameter $\eta = 1/(m \lambda a)$. For S SOC we have

\begin{equation}
m_b = \begin{cases} 6m, & \eta \to -\infty \\ 2m, & \eta \to +\infty \\ 2.32m, & \eta \to 0 \end{cases}
\end{equation}

and hence

\begin{equation}
T_{\text{BEC}} = \begin{cases} 0.0726 \varepsilon_F, & \eta \to -\infty \\ 0.218 \varepsilon_F, & \eta \to +\infty \\ 0.188 \varepsilon_F, & \eta \to 0. \end{cases}
\end{equation}

For EO SOC, we obtain

\begin{equation}
m_b = \begin{cases} 4m, & \eta \to -\infty \\ 2m, & \eta \to +\infty \\ 2.40m, & \eta \to 0 \end{cases}
\end{equation}

and

\begin{equation}
T_{\text{BEC}} = \begin{cases} 0.137 \varepsilon_F, & \eta \to -\infty \\ 0.218 \varepsilon_F, & \eta \to +\infty \\ 0.193 \varepsilon_F, & \eta \to 0. \end{cases}
\end{equation}

As we will see, the above obtained $T_{\text{BEC}}$ coincide well with our numerical results in Sec. III B.

**B. Superfluid Critical Temperature**

By numerically solving the set of coupled gap, number density, and pseudogap equations, we can obtain the superfluid order parameter $\Delta_{sf}$, the pseudogap $\Delta_{pg}$, and the fermion chemical potential $\mu$. The superfluid critical temperature $T_c$ is determined by the vanishing of the superfluid order parameter $\Delta_{sf}$. The numerical results for $T_c/\varepsilon_F$ as a function of the gas parameter $1/(k_F a)$ is shown in Fig. 2 for $\lambda = 0$ and $\lambda = v_F$. 
local maximum also appears when one uses the Nozieres-Schmitt-Rink approach to determine \( T_c \) in the absence of SOC. It may be understood by noticing that the BEC critical temperature is increased when repulsive interactions between bosons are turned on [49]. We note that for a Rashba spin-orbit coupled Fermi gas, the superfluid transition temperature has been roughly estimated by approximating the system as a non-interacting mixture of fermions and rashbons and has been found to increase monotonously across the BCS-BEC crossover [11].

\[ T_c/\varepsilon_F \text{ as a function of the SOC } \lambda \text{ is shown in Fig. 3 for fixed } 1/(k_Fa) = -2 \text{ and } 1/(k_Fa) = \infty. \text{ Also shown is the critical temperature predicted by the BCS theory, } T^*/\varepsilon_F, \text{ which is determined by the vanishing of } \Delta_{\text{mi}}. \text{ It is monotonously increasing as } 1/(k_Fa) \text{ or } \lambda/\nu_F \text{ increased. The BCS theory loses the pairing-fluctuation effect and does not give reliable critical temperature particularly at large } 1/(k_Fa) \text{ or } \lambda/\nu_F \text{ where } T_c \text{ is mainly determined by the bosonic degrees of freedom.}

For all three types of SOC, we find that \( T_c \) is a smooth function of \( 1/(k_Fa) \) and \( \lambda/\nu_F \). and the superfluid phase transition is always of second order for the whole crossover region (see next subsection). Also, it can be seen that \( T_c \) is not a monotonous function of \( 1/(k_Fa) \) when \( \lambda \) is small. There is a local maximum in \( T_c \) curve around the unitary point. Simi-

\[ \text{FIG. 2: (Color online) The critical temperature } T_c \text{ scaled by the Fermi energy } \varepsilon_F \text{ as a function of the gas parameter } 1/(k_Fa) \text{ for fixed SOC } \lambda = 0 \text{ and } \lambda = \nu_F. \text{ Also shown is the pair dissociation temperature } T^*. \]

\[ \text{FIG. 3: (Color online) The critical temperature } T_c \text{ and the dissociation temperature } T^* \text{ scaled by the Fermi energy } \varepsilon_F \text{ as a function of the SOC } \lambda/\nu_F \text{ for fixed gas parameters } 1/(k_Fa) = -2 \text{ and } 1/(k_Fa) = 0. \]

From the top panels of Fig. 2 and Fig. 3 we see that the EP SOC does not affect \( T_c \) and \( T^* \). This is consistent with the observation that EP SOC solely does not lead to new novel bound state and the fermion excitation gap does not change [21]. This can be understood by noticing that the EP
SOC in Hamiltonian (2.1) can be gauged away by using the gauge transformation $\psi_1 \to e^{-im\Lambda_z} \psi_1$ and $\psi_1 \to e^{im\Lambda_z} \psi_1$, resulting only a constant shift in the chemical potential, $\mu \to \mu + m\Lambda_z^2/2$.

For the EO and S SOCs, we observe from Fig. 2 that, comparing to the case without SOC, the SOC suppresses $T_c$ for $k_F a$ close to unitarity while increases $T_c$ at the BCS regime. To further understand how $T_c$ is influenced by the SOC, we turn to Fig. 3. From Fig. 3 we see that $T_c$ is not sensitive to $\lambda$ for $\lambda \ll v_F$ and $\lambda \gg v_F$, but becomes sensitive to $\lambda$ for $\lambda \sim v_F$: One can identify a BCS-BEC crossover solely induced by $\lambda$ around $\lambda \approx v_F$. This coincides with previous zero-temperature studies [9–18]. Although at large negative $1/(k_F a)$, $T_c$ increases (almost) monotonously as $\lambda$ grows, near the resonance, we find that $T_c$ is a decreasing function of $\lambda$, in contrast Ref. [50] where the authors predicted an increasing $T_c$ along with $\lambda$. We note that for large enough $\lambda$, our result converges correctly to a universal molecular limit $T_{BCS}$ either near the unitarity or at the BCS regime (see Sec. III A).

It should be stressed that $T^*$ sets a lower bound for the pair dissociation temperature above which the pairs essentially dissociate due to thermal excitations. Previous study in the absence of the SOC shows that it is a good approximation to set $T^*$ on the BCS side or near unitarity as the pair dissociation temperature [24, 26]. So we refer to $T^*$ as the pair dissociation temperature in Fig. 2 and Fig. 3. The region between $T_c$ and $T^*$ is a pseudogap dominated window in which a normal state is no longer described by the Landau Fermi liquid theory. Fig. 2 and Fig. 3 show that even for large negative $1/(k_F a)$ the pseudogap dominated region can be sizable once the SOC is large. Thus, the spin-orbit coupled Fermi gas may provide a new platform to study the formation of pseudogap in fermionic systems.

C. Pseudogap

In this subsection, we focus on S and EO SOCs because EP SOC does not bring qualitatively new features to the temperature dependence of the pseudogap than the $\lambda = 0$ case. In Fig. 4–Fig. 7, we plot $\Delta_s$, $\Delta_{sf}$, and $\Delta_{pg}$ as well as $\Delta_{BCS}$ (in units of $E_F$, the same below) as functions of temperature.

The common feature for all these figures is that $\Delta_{sf}$ monotonically decrease to zero at $T_c$. Below $T_c$, $\Delta_{pg}(T)$ is a monotonically increasing function from zero at $T = 0$ where it vanishes according to $\Delta_{pg} \propto T^{3/4}$ (see Eq. (2.44)). Above $T_c$, $\Delta_{pg}(T)$ is a monotonically decreasing function from its maximum value located at $T_c$. This kind of temperature dependence clearly shows that pseudogap is due to the thermally excited pairs: Below $T_c$ when $T$ goes higher more pairs are excited from the condensate and at $T_c$ all condensed pairs are thermally excited; after that the thermal motion of the pair participants begins to dissociate the pairs and hence $\Delta_{pg}$ (more precisely, $Z\Delta_{pg}^2$) begins to decrease. Although the physical pictures are clear, at temperature much higher than $T_c$ our formalism may fail since the finite life-time of the pairs, which is not included in our formalism, may become important.

By comparing Fig. 5–Fig. 6 to Fig. 4 and by comparing two bottom panels of Fig. 7 to the panel on the top, one can see that although the SOC does not modify the general tendency of the temperature dependence of the gaps, large SOC significantly enlarges the pseudogap window in the normal phase. Such pseudogap window may be detected by RF spectroscopy measurements, which we now turn to study.
D. RF Spectroscopy

The radio-frequency (RF) spectroscopy has been proven to be very successful in probing the fermionic pairing, quasiparticle excitation spectrum, and superfluidity. For an atomic Fermi gas with two hyperfine states, $| \uparrow \rangle$ and $| \downarrow \rangle$, the RF laser drives transitions between one of the hyperfine states (i.e., $| \downarrow \rangle$) and an empty hyperfine state (3) which lies above it by an energy $\omega_{l3}$ (which is set to zero because it can be absorbed into the chemical potential) due to the magnetic field splitting in bare atomic hyperfine levels. The Hamiltonian for RF-coupling may be written as,

$$H_{RF} = V_0 \int d^3 r \left[ \psi_{\downarrow}^\dagger(r) \psi_\downarrow(r) + \text{H.C.} \right]$$

(3.22)

where $\psi_{\downarrow}^\dagger(r)$ is the field operator which creates an atom at the position $r$ and $V_0$ is the strength of the RF drive and is related to a Rabi frequency $\omega_R$ by $V_0 = \omega_R/2$.

Let us now assume that there is no interaction between the third state and the spin-up or spin-down states, i.e., there is no final state effect. This approximation sounds valid for $^{40}K$ atoms, where the s-wave scattering length between the spin-down state and the third hyperfine state is small (i.e., $\sim 200$ Bohr radii) [51]. Within this approximation and taking into account that the third state is not occupied initially, the transfer strength (integrated RF spectrum) per spin-down atom can be written by ($V_0 = 1$),

$$\Gamma(\omega) = \frac{1}{V n_\downarrow} \sum_k A_{\downarrow \downarrow}(k, \xi_k - \omega) n_F(\xi_k - \omega),$$

(3.23)

where $n_\downarrow$ is the number density of the spin-down fermion and $A_{\downarrow \downarrow} = -(1/\pi) \text{Im} G_{\downarrow \downarrow}$ is the spectral function of the spin-down state $^2$. Note that 

$$\int_{-\infty}^{\infty} d\omega \Gamma(\omega) = 1,$$

(3.24)

$^2$ Theoretical predictions for RF spectroscopy of single spin-orbit coupled bound fermion pair and of noninteracting spin-orbit coupled Fermi gas have been reported in Ref. [52]. Recently, the RF spectroscopy of equal Rashba and Dresselhaus spin-orbit coupled Fermi gases has been studied experimentally and theoretically in Ref. [53]. The theory part is based on a formalism similar with what we used here.
that the RF spectra consist of two con-
spectra are calculated in an idealized manner (see Appendix
because
dogap $\Delta_{pg}$ reflected in the positions of the negative branch
peaks. With increasing temperature, more quasi-particles are
excited, leading to a much more pronounced response appear-
ing in the negative branches. Such a temperature-sensitive
feature of the RF spectroscopy may provide a useful way to
experimentally measure the critical temperature and to detect
the existence of pseudogap in the normal phase.

IV. SUMMARY

We have theoretically investigated thermal effects on the
BCS-BEC crossover of spin-orbit coupled Fermi gases. For
this purpose, we have employed a $T$-matrix formalism based
on a $G_0G$ approximation for the pair susceptibility, which was
thoroughly used in the previous studies of Fermi gases with-
out SOC. This formalism extends the standard BCS theory by
appropriately decomposing the excitation gap to a condensa-
tion part and a pseudogap part that characterize the pairing
fluctuations.

Comparing to the BCS theory, our $G_0G$ formalism pre-
dicts lower and more reliable critical temperature $T_c$ for the
superfluid-normal phase transition. The results for $T_c$ have
been presented in Fig. 2 to Fig. 3. At various molecular BEC
limits, our predictions correctly recover the BEC temperature
$T_{BEC}$ of free Bose gases. The pseudogap persists not only
in the superfluid phase ($T < T_c$) but also in a window of
the normal phase ($T_c < T < T^*$) where it represents the
existence of non-condensed, preformed pairs which dissoci-
ate above $T^*$. We have studied how the SOC influences
the emergence of the pseudogap, as shown in Fig. 4-Fig. 7. It is
seen that strong S- or EO- type SOC can significantly enlarge
the pseudogap window in the normal phase. Thus, spin-orbit
coupled Fermi gases provide a new platform to study the pseu-
dogap physics. Experimentally, such pseudogap might be re-
vealed by RF spectroscopy measurements. We have presented
our qualitative predictions on the RF spectra in Fig. 8, which
may be easily tested in future experiments.

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Appendix A: Expressions for the RF spectroscopy

In this appendix, we list some expressions for the ideal-
ized RF spectroscopy for Fermi gases with and without SOC.
Some of them are used in Sec. III D. By “idealized”, we mean
that these expressions neglect the effects due to final state in-
teractions and due to the finite lifetime effects of the uncon-
densed pairs.
(I) Without SOC. The spectral function of spin-down fermion is given by

$$A_{\downarrow \downarrow}^0(k, \omega) = u_k^2 \delta(\omega - E_k) + v_k^2 \delta(\omega + E_k).$$  \hspace{1cm} (A1)

The RF spectrum:

$$\Gamma_0(\omega) = \frac{m^{3/2}}{4\pi^2} \frac{\Delta^2}{\omega^2} \frac{\omega^2 - \Delta^2 + 2\mu \omega}{\omega} n_F \left( \frac{\omega^2 + \Delta^2}{2\omega} \right) \Theta \left( \frac{\omega^2 - \Delta^2 + 2\mu \omega}{\omega} \right).$$  \hspace{1cm} (A2)

(II) S SOC. The spectral function spin-down fermion is given by

$$A_{\downarrow \downarrow}^S(k, \omega) = \frac{1}{2} \sum_{\alpha} \left( 1 - \alpha \frac{k_{z\alpha}}{|k|} \right) [(u_k^{\alpha})^2 \delta(\omega - E_k^\alpha) + (v_k^\alpha)^2 \delta(\omega + E_k^\alpha)].$$  \hspace{1cm} (A3)

The integrated RF spectrum:

$$\Gamma_S(\omega) = \frac{1}{4\pi^2 n_\downarrow} \sum_{\alpha} \int_0^\infty d|k| |k|^2 \left[ (u_k^\alpha)^2 \delta(\xi_k - \omega - E_k^\alpha) + (v_k^\alpha)^2 \delta(\xi_k - \omega + E_k^\alpha) \right] n_F(\xi_k - \omega).$$  \hspace{1cm} (A4)

(III) EO SOC. The spectral function of spin-down fermion is given by

$$A_{\downarrow \downarrow}^{EO}(k, \omega) = \frac{1}{2} \sum_{\alpha} [(u_k^\alpha)^2 \delta(\omega - E_k^\alpha) + (v_k^\alpha)^2 \delta(\omega + E_k^\alpha)].$$  \hspace{1cm} (A5)

The integrated RF spectrum:

$$\Gamma_{EO}(\omega) = \sqrt{\frac{2m}{16\pi^2 n_\downarrow}} \sum_{\alpha} \int_0^\infty dk_z k_{z\alpha} \frac{\Delta^2}{(\omega + \alpha \lambda k_{z\alpha})^2} n_F \left[ \frac{\Delta^2 + (\omega + \alpha \lambda k_{z\alpha})^2}{2(\omega + \alpha \lambda k_{z\alpha})} \right] \left[ \frac{\omega^2 - \Delta^2 - \lambda^2 k_{z\alpha}^2}{2(\omega + \alpha \lambda k_{z\alpha})} - \frac{k_{z\alpha}^2}{2m} + \mu \right]^{-1/2} \times \Theta \left[ \frac{\omega^2 - \Delta^2 - \lambda^2 k_{z\alpha}^2}{2(\omega + \alpha \lambda k_{z\alpha})} - \frac{k_{z\alpha}^2}{2m} + \mu \right].$$ \hspace{1cm} (A6)

(IV) EP SOC. The spectral function of spin-down fermion is given by:

$$A_{\downarrow \downarrow}^{EP}(k, \omega) = \frac{1}{2} \sum_{\alpha} \left( 1 - \alpha \frac{k_{z\alpha}}{|k|} \right) [(u_k^\alpha)^2 \delta(\omega - E_k^\alpha) + (v_k^\alpha)^2 \delta(\omega + E_k^\alpha)].$$  \hspace{1cm} (A7)

The integrated RF spectrum:

$$\Gamma_{EP}(\omega) = \frac{m^2}{8\pi^2 n_\downarrow} \sum_{\alpha} \int_0^\infty dk_z \frac{\Delta^2}{(\omega + \alpha \lambda k_z)^2} n_F \left[ \frac{\Delta^2 + (\omega + \alpha \lambda k_z)^2}{2(\omega + \alpha \lambda k_z)} \right] \Theta \left[ \frac{\omega^2 - \Delta^2 - \lambda^2 k_z^2}{2(\omega + \alpha \lambda k_z)} - \frac{k_z^2}{2m} + \mu \right].$$ \hspace{1cm} (A8)

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