Invalidity of the Landauer inequality for information erasure in the quantum regime

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A known aspect of the Clausius inequality is that an equilibrium system subjected to a squeezing $dS < 0$ of its entropy must release at least an amount $|\Delta Q| = T|dS|$ of heat. This serves as a basis for the Landauer principle, which puts a lower bound $T \ln 2$ for the heat generated by erasure of one bit of information. Here we show that in the world of quantum entanglement this law is broken, suggesting that quantum carriers of information can be more efficient than assumed so far.

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1. INTRODUCTION

The laws of thermodynamics are at the basis of our understanding of nature, so it is rather natural that they have applications beyond their original scope, e.g. in computing and information processing \cite{1,2,3,4,5,6,7,8}. The first connection between information storage and thermodynamics was made by von Neumann in the 1950's \cite{2}. His speculation that each logical operation costs at least an amount of energy $T \ln 2$ was too pessimistic. Landauer pointed out that reversible “one-to-one” operations can be performed, in principle, without dissipation; only irreversible operations “many-to-one” operations, like erasure, require dissipation of energy, at an amount at least equal to the von Neumann estimate $T \ln 2$ per erased bit \cite{4,5}. This conclusion is a direct consequence \cite{8} of the Clausius inequality, which connects the change of heat with the change of entropy.

The principal importance of erasure among other information-processing operations arises because it is connected with changes of entropy, and thus cannot be realized in a closed system. One needs to couple the information-carrying system with its environment. Therefore the process is accompanied with changes in heat, to be determined by thermodynamics. It was shown rigorously that all computations can be performed using reversible logical operations only \cite{6}.

Here we will consider thermodynamic aspects of erasure at low temperatures, so low that quantum effects start to play an important role. We choose the simplest example: a one-dimensional Brownian particle in contact with a thermal bath at temperature $T$, subject to an external confining potential. The main new aspect arising at low temperatures is an entanglement of the Brownian particle with the bath. Therefore, even when the total system is in a pure state, the subsystem (the Brownian particle) is in a mixed state. Thus its stationary state cannot be given by equilibrium (Gibbsian) quantum thermodynamics. We stress that this situation is not at all exceptional, since it appears even for a small but generic coupling provided that temperature is low enough.

Our main result will show that when entropy of the particle is decreased by external agents, namely when a part of the information carried by it is erased, the particle can absorb heat in clear contrast with the classical intuition. Later we shall apply this result to show that there is not anything similar to Landauer bound at low temperatures. Indeed, we point out that similar violation occurs for a spin $\frac{1}{2}$ particle coupled to a harmonic bath (spin-boson model). Thus in this respect quantum carriers of information can be more efficient than their classical analogs.

2. ERASURE OF INFORMATION AND GIBBSIAN THERMODYNAMICS

Since information is carried by physical systems, messages are coded by their states, namely every state (or possibly group of states) corresponds to a “letter”. The simplest example is a two-state system, which carries on one bit of information. The basic model of source of information in Shannonian, probabilistic information theory \cite{13,14,15} assumes that the carrier of information can be in different states with certain (so called a priori) probabilities. In other words, the messages of this source appear randomly and the measure of their expectation is given by the corresponding probabilities. In the quantum case this situation is described by a density matrix $\rho$,

$$\rho = \sum_n p_n |n\rangle \langle n|,$$

(2.1)

$$\rho$$

(2.2)
which means that the carrier occupies a state $|n\rangle$ with the a priori probability $p_n$. Moreover, different quantum states are exclusive, $\langle n|m\rangle = \delta_{nm}$.

The fundamental theorem by Shannon [13, 14, 15] states that the information carried by an information source is given by its von Neumann entropy,

$$S_{vN}(\rho) = - \sum_n p_n \ln p_n = - \text{tr}(\rho \ln \rho), \quad (2.3)$$

The physical meaning of this result can be understood as follows. A source which has lower entropy occupies fewer states with higher probability. It can be said to be better known, and therefore the appearance of its results will bring less information. In contrast, a source with higher entropy occupies more states with lower probability. Its messages are less expectable, and therefore bring more information. The rigorous realization of this intuitive arguments appeared here on the information theoretical footing and not as purely thermodynamical quantities [15].

Erasure is an operation which is done by an external agent in order to reduce the entropy of the information carrier. This means that in its final state the carrier brings less information, i.e. some amount of it has been erased. In particular, a complete erasure corresponds to the minimization of entropy. Following standard assumptions [16, 17] we will model external operations by a time-dependent Hamiltonian $H(t)$, the carrier, namely some of its parameters will be varied with time according to given trajectories. If the information carrying system is closed, then its dynamics is described by the von Neumann equation

$$\frac{d\rho}{dt} = i\hbar [\rho(t)H(t) - H(t)\rho(t)] \quad (2.4)$$

The entropy remains constant in time. In order to change it, one has to consider an information carrier, which is an open system. In that case a part of its energy will be controlled (i.e. transferred or received) by its environment as heat. Indeed, the average energy of the carrier $U = \text{tr} H(t)\rho(t)$ changes during a time $dt$ as:

$$dU = dQ + dW = \text{tr}[Hd\rho] + \text{tr}[\rho dH] \quad (2.5)$$

The last term represents the averaged mechanical work $dW$ produced by an external agent [13, 17]. The first term in r.h.s. of Eq. (2.5) arises due to the statistical redistribution in phase space. We shall identify it with the change of heat $dQ$ [16, 17], so Eq. (2.5) is just the first law of thermodynamics.

### 3. QUANTUM BROWNIAN PARTICLE IN CONTACT WITH A BATH

As explained, at low temperatures of the bath the Brownian particle is not described by the quantum Gibbs distribution, except for very weak interaction with the bath. Therefore, its state at low temperatures has to be found from first principles, starting from the microscopic description of the bath and the particle. This program was realized in [8, 10, 21, 22]. In particular, in [9, 10] we investigated statistical thermodynamics of the quantum Brownian particle.

Here we consider a simple example, a harmonic oscillator with Hamiltonian

$$H(p, x) = \frac{p^2}{2m} + \frac{ax^2}{2}, \quad (3.1)$$

where $m$ is the mass, and $a$ is the width. The state of this particle can be described through the Wigner function [17]. In quantum theory this object plays nearly the same role as the common distribution of coordinate and momentum in the classical theory. The stationary Wigner function reads [21, 23, 24, 10]:

$$W(p, x) = \frac{1}{2\pi} \frac{a}{m T_p T_x} \exp\left[-\frac{p^2}{2m T_p} - \frac{ax^2}{2 T_x}\right], \quad (3.2)$$

where $T_x = a(x^2)$ and $T_p = \langle p^2 \rangle / m$ are two effective temperatures, to be discussed a bit later. Eq. (3.2) represents the state of the particle, provided that the interaction with the bath was switched on long time before, so that the particle already came to its stationary state. The effective temperatures $T_p$ and $T_x$ depend not only on the system parameters $m, T$, and $a$, but also on the damping constant $\gamma$, which quantifies the interaction with the thermal bath, and on a large parameter $\Gamma$ which is the maximal characteristic frequency of the bath. In particular, the Gibbsian limit corresponds to $\gamma \to 0$. Then the distribution (3.2) tends to the quantum Gibbsian: $T_x = T_p = \frac{\hbar}{2} \omega_0 \coth(\frac{1}{2} \hbar \omega_0)$,
where $\omega_0 = \sqrt{a/m}$. In the classical limit, which is realized for $\hbar \to 0$ or $T \to \infty$, the dependence on $\gamma$ and $\Gamma$ disappears; both $T_p$ and $T_x$ go to $T$. The appearance of the effective temperatures in the quantum regime can be understood as follows. For $T \to 0$ quantum Gibbs distribution predicts the pure vacuum state for the particle. Due to quantum entanglement this cannot be the case for a non-weakly interacting particle, so must $T_p$, $T_x$ depend on $\gamma$, and, being non-trivial, they have to be obtained from first principles, as the state is not Gibbsian. We will be interested by the so-called quasi-Ohmic limit where $\Gamma$ is the largest characteristic frequency of the problem, the most realistic situation for information storing devices. In this limit one approximately has:

$$T_p = \frac{\hbar}{\pi(\omega_1 - \omega_2)} \left[ (\omega_1^2 - \omega_2^2) + \frac{(\beta\hbar\Gamma)}{2\pi} - \omega_1^2 \psi_1 + \omega_2^2 \psi_2 \right] - T \tag{3.3}$$

$$T_x = \frac{\hbar a}{m(\omega_1 - \omega_2)} \left[ \psi_1 - \psi_2 \right] - T, \quad \psi_{1,2} = \psi \left( \frac{\beta \hbar \omega_{1,2}}{2\pi} \right) \tag{3.4}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ and $\omega_{1,2} = (\gamma/2m)(1 \pm \sqrt{1 - 4am/\gamma^2})$.

The average energy of the Brownian particle

$$U = \int dx dp W(p, x) H(p, x) = \frac{T_p}{2} + \frac{T_x}{2} \tag{3.5}$$

depends on $a$ and $m$. We will need the entropies

$$S_p = -\int dp W(p) \ln W(p) = \frac{1}{2} \ln(mT_p), \tag{3.6}$$

$$S_x = -\int dx W(x) \ln W(x) = \frac{1}{2} \ln \frac{T_x}{a}. \tag{3.7}$$

The expressions for heat and work are generalized from Eqs. (2.5) by simply using the Wigner function $W(p, x)$ instead of $\rho$. One can prove by a direct calculation that quantities $T_p$, $T_x$ do deserve their nomenclature, since the classical Clausius equality can be generalized as

$$dU = dQ + dW = T_p dS_p + T_x dS_x + dW, \tag{3.8}$$

for variation of any parameter. We will be especially interested in variation of the mass and the width of the potential. The corresponding changes of heat read

$$dQ = \frac{1}{2} \left( \frac{\partial T_p}{\partial a} + \frac{\partial T_x}{\partial a} - \frac{T_x}{a} \right) da + \frac{1}{2} \left( \frac{\partial T_p}{\partial m} + \frac{\partial T_x}{\partial m} + \frac{T_x}{m} \right) dm. \tag{3.9}$$

One can show that $\partial Q/\partial a \leq 0$, $\partial Q/\partial m \geq 0$, for all values of parameters including, of course, the classical limit. The work done on the system is

$$\frac{\partial W}{\partial a} = \langle \frac{\partial H}{\partial a} \rangle = \frac{1}{2} \langle x^2 \rangle \geq 0, \quad \frac{\partial W}{\partial m} = \langle \frac{\partial H}{\partial m} \rangle = -\frac{1}{2} \langle p^2 \rangle \leq 0. \tag{3.10}$$

The result for the von Neumann entropy (2.3) reads

$$S_{\text{vN}} = (w + \frac{1}{2}) \ln(w + \frac{1}{2}) - (w - \frac{1}{2}) \ln(w - \frac{1}{2}), \quad w = \sqrt{\frac{mT_pT_x}{\hbar^2a}}. \tag{3.11}$$

### 4. Heat Absorption With(out) Entropy Decrease

Now we will show that there are erasure processes, namely processes where $dS_{\text{vN}} \leq 0$, which are accompanied by an absorption of heat. We noticed already that heat is always absorbed, when the mass is increased. There is a mass-increasing process, where $dS_{\text{vN}} \leq 0$, since one has at very low temperatures: $\partial S_{\text{vN}}/\partial m \sim \partial w/\partial m \leq 0$. An analogous argument can be brought about in the weak-coupling case, $\gamma \to 0$, and low-temperature limit:

$$T_p = \frac{\hbar \omega_0}{2} + \frac{\hbar \gamma}{\pi m} \ln(\frac{\Gamma}{\omega_0 \sqrt{e}}), \quad T_x = \frac{\hbar \omega_0}{2} - \frac{\hbar \gamma}{2\pi m}. \tag{4.1}$$

This also implies $\partial w/\partial m < 0$. Recall that the corresponding expression for $\partial Q/\partial m$ was positive. This just means that for the variation of $m$ we have an interesting case where heat is absorbed when entropy is decreasing. This is a counterexample for the general validity of the Landauer principle.

In another approach we considered the spin-boson model [26], where a spin $\frac{1}{2}$ particle is coupled to a bosonic bath. There it is possible to give a pulse, where the spin is rotated very fast, but, since it is only rotated, entropy is conserved. Nevertheless, there are cases where heat can be extracted from the bath.
5. CONCLUSION

The Landauer principle requires dissipation (release) of $T|dS|$ units of energy as a consequence of erasure of $|dS|$ units of information. This was believed to be the only fundamental energy cost of computational processes \[4, 5, 8\]. Though in practice computers dissipate much more energy, the Landauer principle was considered to put a general physical bound to which every computational device interacting with its thermal environment must satisfy. Indeed, in several physical situations the Landauer principle can be proved explicitly \[8\].

The main purpose of the present paper was to provide counterexamples of this principle, and thus to question its universal validity. In the reported case all general requirements on the information carrier and its interaction with the bath are met. The only new point of our approach is that we were interested by sufficiently low temperatures, where quantum effects are relevant. The Landauer principle appeared to be violated by these effects (in particular, by entanglement). The result occurs in the two most standard models, namely the Caldeira-Leggett model (central oscillator coupled to a harmonic bath) and the spin-boson model (two-level system coupled to such a bath), and therefore should be very general.

Recently the Landauer bound attracted serious attention from the field of applied information science \[27\]. There is a belief that it can be approached by further miniaturization of computational devices. It is hoped that the present paper will help to understand limitations of the Landauer principle itself, which may lead to unexpected mechanisms for computing in the quantum regime.

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