CONSTRUCTION OF $p$-ADIC COVARIANT QUANTUM FIELDS IN THE FRAMEWORK OF WHITE NOISE ANALYSIS

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Abstract. In this article we construct a large class of interacting Euclidean quantum field theories, over a $p$-adic space time, by using white noise calculus. We introduce $p$-adic versions of the Kondratiev and Hida spaces in order to use the Wick calculus on the Kondratiev spaces. The quantum fields introduced here fulfill all the Osterwalder-Schrader axioms, except the reflection positivity.

1. Introduction

In this article, we construct interacting Euclidean quantum field theories, over a $p$-adic spacetime, in arbitrary dimension, which satisfy all the Osterwalder-Schrader axioms [35] except for reflection positivity. More precisely, we present a $p$-adic analogue of the interacting field theories constructed by Grothaus and Streit in [13].

The basic objects of an Euclidean quantum field theory are probability measures on distributions spaces, in the classical case, on the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$. In conventional quantum field theory (QFT) there have been some studies devoted to the optimal choice of the space of test functions. In [17], Jaffe discussed this topic (see also [31] and [44]); his conclusion was that, rather than an optimal choice, there exists a set of conditions that must be satisfied by the candidate space, and any class of test functions with these properties should be considered as valid. The main condition is that the space of test functions must be a nuclear countable Hilbert one. This fact constitutes the main mathematical motivation the study of QFT on general nuclear spaces.

A physical motivation for studying QFT in the $p$-adic setting comes from the conjecture of Volovich stating that spacetime has a non-Archimedean nature at the Planck scale, [50], see also [46]. The existence of the Planck scale implies that below it the very notion of measurement as well as the idea of ‘infinitesimal length’ become meaningless, and this fact translates into the mathematical statement that the Archimedean axiom is no longer valid, which in turn drives to consider models based on $p$-adic numbers. In the $p$-adic framework, the relevance of constructing quantum field theories was stressed in [49] and [47]. In the last 35 years $p$-adic QFT has attracted a lot of attention of physicists and mathematicians, see e.g. [1, 7-10, 13, 18-19, 24-30, 33-36, 41-42, 46-53], and the references therein.

A $p$-adic number is a sequence of the form

$$x = x_{-k}p^{-k} + x_{-k+1}p^{-k+1} + \ldots + x_0 + x_1p + \ldots,$$

with $x_{-k} \neq 0$.

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where \( p \) denotes a fixed prime number, and the \( x_j \)s are \( p \)-adic digits, i.e. numbers in the set \( \{0, 1, \ldots, p - 1\} \). There are natural field operations, sum and multiplication, on series of form (1.1). The set of all possible \( p \)-adic sequences constitutes the field of \( p \)-adic numbers \( \mathbb{Q}_p \). The field \( \mathbb{Q}_p \) cannot be ordered. There is also a natural norm in \( \mathbb{Q}_p \) defined as \( |x|_p = p^k \), for a nonzero \( p \)-adic number \( x \) of the form (1.1). The field of \( p \)-adic numbers with the distance induced by \( |·|_p \) is a complete ultrametric space. The ultrametric property refers to the fact that \( |x - y|_p \leq \max \{ |x - z|_p, |z - y|_p \} \) for any \( x, y, z \) in \( \mathbb{Q}_p \). As a topological space, \( (\mathbb{Q}_p, |·|_p) \) is completely disconnected, i.e. the connected components are points. The field of \( p \)-adic numbers has a fractal structure, see e.g. [2], [49]. All these results can be extended easily to \( \mathbb{Q}_N^p \), see Section 2.

In [52], see also [28, Chapter 11], the second author introduced a class of non-Archimedean massive Euclidean fields, in arbitrary dimension, which are constructed as solutions of certain covariant \( p \)-adic stochastic pseudodifferential equations, by using techniques of white noise calculus. In particular a new non-Archimedean Gel’fand triple was introduced. By using this new triple, here we introduce non-Archimedean versions of the Kondratiev and Hida spaces, see Section 4. The non-Archimedean Kondratiev spaces, denoted as (\( \mathcal{H}_\infty \))\(^1 \), (\( \mathcal{H}_\infty \))\(^{-1} \), play a central role in this article.

Formally an interacting field theory with interaction \( V \) has associated a measure of the form

\[
d\mu_V = \frac{\exp \left( - \int_{\mathbb{Q}_p^N} V(\Phi(x)) \, d^N x \right) \, d\mu}{\int \exp \left( - \int_{\mathbb{Q}_p^N} V(\Phi(x)) \, d^N x \right) \, d\mu},
\]

where \( \mu \) is the Gaussian white noise measure, \( \Phi(x) \) is a random process at the point \( x \in \mathbb{Q}_p^N \). In general \( \Phi(x) \) is not an integrable function rather a distribution, thus a natural problem is how to define \( V(\Phi(x)) \). For a review about the techniques for regularizing \( V(\Phi(x)) \) and the construction of the associated measures, the reader may consult [12], [13], [40], [44] and the references therein.

Following [13], we consider the following generalized white functional:

\[
\Phi_H = \exp^{\diamond} \left( - \int_{\mathbb{Q}_p^N} H^{\diamond}(\Phi(x)) \, d^N x \right),
\]

where \( H \) is analytic function at the origin satisfying \( H(0) = 0 \). The Wick analytic function \( H^{\diamond}(\Phi(x)) \) of process \( \Phi(x) \) coincides with the usual Wick ordered function : \( H(\Phi(x)) \) : when \( H \) is a polynomial function. It turns out that \( H^{\diamond}(\Phi(x)) \) is a distribution from the Kondratiev space (\( \mathcal{H}_\infty \))\(^{-1} \), and consequently, its integral belong to (\( \mathcal{H}_\infty \))\(^{-1} \), if it exists. In general we cannot take the exponential of \( - \int H^{\diamond}(\Phi(x)) \, d^N x \), however, by using the Wick calculus in (\( \mathcal{H}_\infty \))\(^{-1} \), see Section 4.3.6 we can take the Wick exponential \( \exp^{\diamond}(\cdot) \).

In certain cases, for instance when \( H \) is linear or is a polynomial of even degree, see [18], and if we integrate only over a compact subset \( K \) of \( \mathbb{Q}_p^N \) (the space cutoff), the function \( \Phi_H \) is integrable, and we have a direct correspondence between (1.2)
and (1.3), i.e.

$$\Phi_H d\mu = \left\{ \exp \left( -\int_K H^\circ (\Phi (x)) d^N x \right) \right\} d\mu.$$ 

In general the distribution $\Phi_H$ is not necessarily positive, and for a large class of functions $H$, there are no measures representing $\Phi_H$. It turns out that $\Phi_H$ can be represented by a measure if and only if $-H(it) + \frac{1}{2} t^2$, $t \in \mathbb{R}$, is a Lévy characteristic, see Theorem 2. These measures are called generalized white noise measures.

Generalized white measures were considered in [52], in the $p$-adic framework, and in the Archimedean case in [3]-[4]. Euclidean random fields over $\mathbb{Q}_p$ were constructed by convolving generalized white noise with the fundamental solutions of certain $p$-adic pseudodifferential equations. These fundamental solutions are invariant under the action of a $p$-adic version of the Euclidean group, see Section 5.6.

For all convoluted generalized white noise measures such that their Lévy characteristics have an analytic extension at the origin, we can give an explicit formula for the generalized density with respect to the white noise measure, see Theorem 3. In addition, there exists a large class of distributions $\Phi_H$ of type (1.3) that do not have an associated measure, see Remark 12. We also prove that the Schwinger functions corresponding to convoluted generalized functions satisfy Osterwalder-Schrader axioms (axioms OS1, OS2, OS4, OS5 in the notation used in [13]) except for reflect positivity, see Lemma 2. Theorems 3, 4, just like in the Archimedean case presented in [13].

The $p$-adic spacetime $(\mathbb{Q}_p^N, q (\xi))$ is a $\mathbb{Q}_p$-vector space of dimension $N$ with an elliptic quadratic form $q (\xi)$, i.e. $q (\xi) = 0 \Leftrightarrow \xi = 0$. This spacetime differs from the classical spacetime $(\mathbb{R}^N, \xi_1^2 + \cdots + \xi_N^2)$ in several aspects. The $p$-adic spacetime is not an ‘infinitely divisible continuum’, because $\mathbb{Q}_p^N$ is a completely disconnected topological space, the connected components (the points) play the role of ‘spacetime quanta’. Since $\mathbb{Q}_p$ is not an ordered field, the notions of past and future do not exist, then any $p$-adic QFT is an acausal theory. The reader may consult the introduction of [33] for an in-depth discussion of this matter. Consequently, the reflection positivity, if it exists in the $p$-adic framework, requires a particular formulation, that we do not know at the moment. The study of the $p$-adic Wightman functions via the reconstruction theorem is an open problem.

Another important difference between the classical case and the $p$-adic one comes from the fact that in the $p$-adic setting there are no elliptic quadratic forms in dimension $N \geq 5$. We replace $q (\xi)$ by an elliptic polynomial $l (\xi)$, which is a homogeneous polynomial satisfying $l (\xi) = 0 \Rightarrow \xi = 0$. For any dimension $N$ there are elliptic polynomials of degree $d \geq 2$. We use $\| l (\xi) \|_p^2$ as a replacement of $\| q (\xi) \|_p$. This approach is particularly useful to define the $p$-adic Laplace equation that the (free) covariance function $C_p (x - y)$ satisfies, this equation has the following form:

$$(L_\alpha + m^2) C_p (x - y) = \delta (x - y), \quad x, y \in \mathbb{Q}_p^N,$$

where $\alpha > 0$, $m > 0$ and $L_\alpha$, is the pseudodifferential operator

$L_\alpha \varphi (x) = F^{-1}_{\xi \to x} (l (\xi))_p^\alpha F_{x \to \xi \varphi},$

here $F$ denotes the Fourier transform. The QFTs presented here are families depending on several parameters, among them, $p$, $\alpha$, $m$, $l (\xi)$. 

The \( p \)-adic free covariance \( C_p(x - y) \) may have singularities at the origin depending on the parameters \( \alpha, d, N \), and has a ‘polynomial’ decay at infinity, see Section 5.5.3. The \( p \)-adic cluster property holds under the condition \( \alpha d > N \). Under this hypothesis the covariance function does not have singularities at the origin. Since \( \alpha \) is a ‘free’ parameter, this condition can be satisfied in any dimension. We think that the condition \( \alpha d > N \) is completely necessary to have the cluster property due to the fact that our test functions do not decay exponentially at infinity, see Remark 1

2. \( p \)-Adic Analysis: Essential Ideas

In this section we collect some basic results about \( p \)-adic analysis that will be used in the article. For an in-depth review of the \( p \)-adic analysis the reader may consult [2], [15], [19].

2.1. The field of \( p \)-adic numbers. Along this article \( p \) will denote a prime number. The field of \( p \)-adic numbers \( \mathbb{Q}_p \) is defined as the completion of the field of rational numbers \( \mathbb{Q} \) with respect to the \( p \)-adic norm \( | \cdot |_p \), which is defined as

\[
|x|_p = \begin{cases} 
0 & \text{if } x = 0 \\
p^{-\gamma} & \text{if } x = p^\gamma \frac{a}{b},
\end{cases}
\]

where \( a \) and \( b \) are integers coprime with \( p \). The integer \( \gamma := ord(x) \), with \( ord(0) := +\infty \), is called the \( p \)-adic order of \( x \).

Any \( p \)-adic number \( x \neq 0 \) has a unique expansion of the form

\[
x = p^{ord(x)} \sum_{j=0}^{\infty} x_j p^j,
\]

where \( x_j \in \{0, \ldots, p-1\} \) and \( x_0 \neq 0 \). By using this expansion, we define the fractional part of \( x \in \mathbb{Q}_p \), denoted \( \{x\}_p \), as the rational number

\[
\{x\}_p = \begin{cases} 
0 & \text{if } x = 0 \text{ or } ord(x) \geq 0 \\
p^{ord(x)} \sum_{j=0}^{-ord(x)-1} x_j p^j & \text{if } ord(x) < 0.
\end{cases}
\]

In addition, any non-zero \( p \)-adic number can be represented uniquely as \( x = p^{ord(x)} ac(x) \) where \( ac(x) = \sum_{j=0}^{\infty} x_j p^j, x_0 \neq 0 \), is called the angular component of \( x \). Notice that \( |ac(x)|_p = 1 \).

We extend the \( p \)-adic norm to \( \mathbb{Q}_p^N \) by taking

\[
||x||_p := \max_{1 \leq i \leq N} |x_i|_p, \text{ for } x = (x_1, \ldots, x_N) \in \mathbb{Q}_p^N.
\]

We define \( ord(x) = \min_{1 \leq i \leq N} \{ord(x_i)\} \), then \( ||x||_p = p^{-ord(x)} \). The metric space \((\mathbb{Q}_p^N, || \cdot ||_p)\) is a complete ultrametric space. For \( r \in \mathbb{Z} \), denote by \( B_r^N(a) = \{x \in \mathbb{Q}_p^N ; ||x - a||_p \leq p^r \} \) the ball of radius \( p^r \) with center at \( a = (a_1, \ldots, a_N) \in \mathbb{Q}_p^N \), and take \( B_r^N(0) := B_r^N \). Note that \( B_r^N(a) = B_r(a_1) \times \cdots \times B_r(a_N) \), where \( B_r(a_i) := \{x \in \mathbb{Q}_p; ||x - a_i||_p \leq p^r \} \) is the one-dimensional ball of radius \( p^r \) with center at \( a_i \in \mathbb{Q}_p \). The ball \( B_r^N \) equals the product of \( N \) copies of \( B_0 = \mathbb{Z}_p \), the ring of \( p \)-adic integers of \( \mathbb{Q}_p \). We also denote by \( S_r^N(a) = \{x \in \mathbb{Q}_p^N ; ||x - a||_p = p^r \} \) the sphere of radius \( p^r \) with center at \( a = (a_1, \ldots, a_N) \in \mathbb{Q}_p^N \), and take \( S_r^N(0) := S_r^N \).
We notice that $S_1^0 = \mathbb{Z}_p^\times$ (the group of units of $\mathbb{Z}_p$), but $(\mathbb{Z}_p^\times)^N \subseteq S_0^N$. The balls and spheres are both open and closed subsets in $\mathbb{Q}_p^N$. In addition, two balls in $\mathbb{Q}_p^N$ are either disjoint or one is contained in the other.

As a topological space $(\mathbb{Q}_p^N, \| \cdot \|_p)$ is totally disconnected, i.e. the only connected subsets of $\mathbb{Q}_p^N$ are the empty set and the points. A subset of $\mathbb{Q}_p^N$ is compact if and only if it is closed and bounded in $\mathbb{Q}_p^N$, see e.g. [49 Section 1.3], or [2 Section 1.8]. The balls and spheres are compact subsets. Thus $(\mathbb{Q}_p^N, \| \cdot \|_p)$ is a locally compact topological space.

We will use $\Omega(p^{-r}||x - a||_p)$ to denote the characteristic function of the ball $B^N(a)$. We will use the notation $1_A$ for the characteristic function of a set $A$. Along the article $d^Nx$ will denote a Haar measure on $(\mathbb{Q}_p^N, +)$ normalized so that $\int_{\mathbb{Q}_p^N} d^N x = 1$.

### 2.2. Some function spaces.

A complex-valued function $\varphi$ defined on $\mathbb{Q}_p^N$ is called locally constant if for any $x \in \mathbb{Q}_p^N$ there exist an integer $l(x) \in \mathbb{Z}$ such that

$$\varphi(x + x') = \varphi(x) \text{ for } x' \in B^N_l(x).$$

A function $\varphi : \mathbb{Q}_p^N \rightarrow \mathbb{C}$ is called a Bruhat-Schwartz function (or a test function) if it is locally constant with compact support. The $\mathbb{C}$-vector space of Bruhat-Schwartz functions is denoted by $\mathcal{D} := \mathcal{D}(\mathbb{Q}_p^N)$. Let $\mathcal{D}' := \mathcal{D}'(\mathbb{Q}_p^N)$ denote the set of all continuous functional (distributions) on $\mathcal{D}$.

We will denote by $\mathcal{D}_{\mathbb{R}} := \mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^N)$, the $\mathbb{R}$-vector space of test functions, and by $\mathcal{D}'_{\mathbb{R}} := \mathcal{D}'_{\mathbb{R}}(\mathbb{Q}_p^N)$, the $\mathbb{R}$-vector space of distributions.

Given $\rho \in [0, \infty)$, we denote by $L^\rho := L^\rho(\mathbb{Q}_p^N) := L^\rho(\mathbb{Q}_p^N, d^N x)$, the $\mathbb{C}$-vector space of all the complex valued functions $g$ satisfying $\int_{\mathbb{Q}_p^N} |g(x)|^\rho d^N x < \infty$, and $L^\infty := L^\infty(\mathbb{Q}_p^N) = L^\infty(\mathbb{Q}_p^N, d^N x)$ denotes the $\mathbb{C}$-vector space of all the complex valued functions $g$ such that the essential supremum of $|g|$ is bounded. The corresponding $\mathbb{R}$-vector spaces are denoted as $L^\rho_{\mathbb{R}} := L^\rho_{\mathbb{R}}(\mathbb{Q}_p^N) = L^\rho_{\mathbb{R}}(\mathbb{Q}_p^N, d^N x)$, $1 \leq \rho \leq \infty$.

Set

$$\mathcal{C}_0(\mathbb{Q}_p^N, \mathbb{C}) := \left\{ f : \mathbb{Q}_p^N \rightarrow \mathbb{C}; \text{ } f \text{ is continuous and } \lim_{||x||_p \rightarrow \infty} f(x) = 0 \right\},$$

where $\lim_{||x||_p \rightarrow \infty} f(x) = 0$ means that for every $\epsilon > 0$ there exists a compact subset $B(\epsilon)$ such that $|f(x)| < \epsilon$ for $x \in \mathbb{Q}_p^N \setminus B(\epsilon)$. We recall that $(\mathcal{C}_0(\mathbb{Q}_p^N, \mathbb{C}), \| \cdot \|_{L^\infty})$ is a Banach space. The corresponding $\mathbb{R}$-vector space will be denoted as $\mathcal{C}_0(\mathbb{Q}_p^N, \mathbb{R})$.

### 2.3. Fourier transform.

Set $\chi_p(y) := \exp(2\pi i y)_p$ for $y \in \mathbb{Q}_p$. The map $\chi_p(\cdot)$ is an additive character on $\mathbb{Q}_p$, i.e. a continuous map from $(\mathbb{Q}_p, +)$ into $S$ (the unit circle considered as multiplicative group) satisfying $\chi_p(x_0 + x_1) = \chi_p(x_0) \chi_p(x_1)$, $x_0, x_1 \in \mathbb{Q}_p$. The additive characters of $\mathbb{Q}_p$ form an Abelian group which is isomorphic to $(\mathbb{Q}_p, +)$, the isomorphism is given by $\xi \rightarrow \chi_p(\xi x)$, see e.g. [2 Section 2.3].

Given $x = (x_1, \ldots, x_N), \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{Q}_p^N$, we set $x \cdot \xi := \sum_{j=1}^N x_j \xi_j$. If $f \in L^1$ its Fourier transform is defined by

$$(\mathcal{F}f)(\xi) = \int_{\mathbb{Q}_p^N} \chi_p(\xi \cdot x) f(x) d^N x, \text{ for } \xi \in \mathbb{Q}_p^N.$$
We will also use the notation $F_{x-\xi} f$ and $\hat{f}$ for the Fourier transform of $f$. The Fourier transform is a linear isomorphism from $D(\mathbb{Q}_p^N)$ onto itself satisfying
\begin{equation}
(F(F f))(\xi) = f(-\xi),
\end{equation}
for every $f \in D(\mathbb{Q}_p^N)$, see e.g. [2] Section 4.8. If $f \in L^2$, its Fourier transform is defined as
\[
(F f)(\xi) = \lim_{k \to \infty} \int_{\|x\|_p \leq p^k} \chi_p(\xi \cdot x) f(x) d^N x, \quad \text{for } \xi \in \mathbb{Q}_p^N,
\]
where the limit is taken in $L^2$. We recall that the Fourier transform is unitary on $L^2$, i.e. $\|f\|_{L^2} = \|F f\|_{L^2}$ for $f \in L^2$ and that (2.1) is also valid in $L^2$, see e.g. [45 Chapter III, Section 2].

The Fourier transform $F[W]$ of a distribution $W \in D'(\mathbb{Q}_p^N)$ is defined by
\[
(F[W], \varphi) = (W, F[\varphi]) \quad \text{for all } \varphi \in D(\mathbb{Q}_p^N).
\]
The Fourier transform $W \to F[W]$ is a linear isomorphism from $D'(\mathbb{Q}_p^N)$ onto itself. Furthermore, $W = F[F[W]](-\xi)$. We also use the notation $F_{x-\xi}W$ and $\hat{W}$ for the Fourier transform of $W$.

**3. $p$-Adic White Noise**

In this section we review some basic aspects of the white noise calculus in the $p$-adic setting. For an in-depth exposition on the white noise calculus on arbitrary nuclear spaces the reader may consult [5, 11, 15, 10, 37]. We will use white noise calculus on the nuclear spaces $H_{\infty}$ introduced by Zúñiga-Galindo in [52], see also [28, Chapters 10, 11].

**3.1. A class of non-Archimedean nuclear spaces.**

**3.1.1. $H_{\infty}$, a non-Archimedean analog of the Schwartz space.** We denote the set on non-negative integers by $\mathbb{N}$, and set $[\xi]_p := \max(1, \|\xi\|_p)$ for $\xi \in \mathbb{Q}_p^N$. We define for $\varphi, \theta \in D(\mathbb{Q}_p^N)$, and $l \in \mathbb{N}$, the following scalar product:
\[
\langle \varphi, \theta \rangle_l = \int_{\mathbb{Q}_p^N} [\xi]_p^l \overline{\varphi}(\xi) \theta(\xi) d^N \xi,
\]
where the overbar denotes the complex conjugate. We also set $\|\varphi\|_l := \langle \varphi, \varphi \rangle_l$. Notice that $\|\cdot\|_l \leq \|\cdot\|_m$ for $l \leq m$. We denote by $H_l(\mathbb{C}) := H_l(\mathbb{Q}_p^N, \mathbb{C})$ the complex Hilbert space obtained by completing $D(\mathbb{Q}_p^N)$ with respect to $\langle \cdot, \cdot \rangle_l$. Then $H_l(\mathbb{C}) \hookrightarrow H_m(\mathbb{C})$ for $l \leq m$. Now we set
\[
H_{\infty}(\mathbb{C}) := H_{\infty}(\mathbb{Q}_p^N, \mathbb{C}) = \bigcap_{l \in \mathbb{N}} H_l(\mathbb{C}).
\]
Notice that $H_{\infty}(\mathbb{C}) \subset L^2$. With the topology induced by the family of seminorms $\{\|\cdot\|_l\}_{l \in \mathbb{N}}$, $H_{\infty}(\mathbb{C})$ becomes a locally convex space, which is metrizable. Indeed,\[
d(f, g) := \max_{l \in \mathbb{N}} \left\{ 2^{-l} \frac{\|f - g\|_l}{1 + \|f - g\|_l} \right\}, \quad \text{with } f, g \in H_{\infty}(\mathbb{C}),\]
is a metric for the topology of $H_{\infty}(\mathbb{C})$. The projective topology $\tau_p$ of $H_{\infty}(\mathbb{C})$ coincides with the topology induced by the family of seminorms $\{\|\cdot\|_l\}_{l \in \mathbb{N}}$. The space $H_{\infty}(\mathbb{C})$ endowed with the topology $\tau_p$ is a countably Hilbert space in the
sense of Gel'fand-Vilenkin. Furthermore, \((\mathcal{H}_\infty(\mathbb{C}), \tau_p)\) is metrizable and complete and hence a Fréchet space, cf. [28, Lemma 10.3], see also [52].

The space \(\mathcal{H}_\infty(\mathbb{C})\) with respect to \(d\), and since \(\mathcal{D}(\mathbb{Q}_p^N)\) is nuclear, then \(\mathcal{H}_\infty(\mathbb{C})\) is a nuclear space, which is continuously embedded in \(C_0(\mathbb{Q}_p^N, \mathbb{C})\), the space of complex-valued bounded functions vanishing at infinity. In addition, \(\mathcal{H}_\infty(\mathbb{C}) \subset L^1 \cap L^2\), cf. [28, Theorem 10.15].

**Remark 1.** (i) We denote by \(\mathcal{H}_l(\mathbb{R}) := \mathcal{H}_l(\mathbb{Q}_p^N, \mathbb{R})\) the real Hilbert space obtained by completing \(\mathcal{D}(\mathbb{Q}_p^N)\) with respect to \(\langle \cdot, \cdot \rangle_l\). We also set \(\mathcal{H}_\infty(\mathbb{Q}_p^N, \mathbb{R}) := \mathcal{H}_\infty(\mathbb{R}) = \bigcap_{l \in \mathbb{N}} \mathcal{H}_l(\mathbb{R})\). In the case in which the ground field \((\mathbb{R} \text{ or } \mathbb{C})\) is clear, we shall use the simplified notation \(\mathcal{H}_l, \mathcal{H}_\infty\). All the above announced results for the spaces \(\mathcal{H}_l(\mathbb{C}), \mathcal{H}_\infty(\mathbb{C})\) are valid for the spaces \(\mathcal{H}_l(\mathbb{R}), \mathcal{H}_\infty(\mathbb{R})\). In particular, \(\mathcal{H}_\infty(\mathbb{R})\) is a nuclear countably Hilbert space.

(ii) The following characterization of the space \(\mathcal{H}_\infty(\mathbb{C})\) is very useful:

\[
\mathcal{H}_\infty(\mathbb{C}) = \left\{ f \in L^2(\mathbb{Q}_p^N) \; | \; \|f\|_l < \infty \text{ for any } l \in \mathbb{N} \right\} = \left\{ W \in \mathcal{D}'(\mathbb{Q}_p^N) \; | \; \|W\|_l < \infty \text{ for any } l \in \mathbb{N} \right\},
\]

cf. [28, Lemma 10.8]. An analog result is valid for \(\mathcal{H}_\infty(\mathbb{R})\).

(iii) The spaces \(\mathcal{H}_l(\mathbb{R}), \mathcal{H}_l(\mathbb{C})\), for any \(l \in \mathbb{N}\), are nuclear and consequently they are separable, cf. [11, Chapter I, Section 3.4].

The spaces \(\mathcal{H}_\infty(\mathbb{Q}_p^N, \mathbb{C})\) and \(\mathcal{H}_\infty(\mathbb{Q}_p^N, \mathbb{R})\) were introduced in [52], see also [28]. These spaces are invariant under the action of a large class of pseudodifferential operators.

3.1.2. The dual space of \(\mathcal{H}_\infty\). For \(m \in \mathbb{N}\), and \(W \in \mathcal{D}'(\mathbb{Q}_p^N)\) such that \(\hat{W}\) is a measurable function, we set

\[
\|W\|_{-m}^2 := \int_{\mathbb{Q}_p^N} |\xi|^{-m} \left| \hat{W}(\xi) \right|^2 \, d^N\xi.
\]

Then

\[
\mathcal{H}_{-m}(\mathbb{C}) := \mathcal{H}_{-m}(\mathbb{Q}_p^N, \mathbb{C}) = \left\{ W \in \mathcal{D}'(\mathbb{Q}_p^N) \; | \; \|W\|_{-m} < \infty \right\}
\]

is a complex Hilbert space. If \(\mathcal{X}\) is a locally convex, we denote by \(\mathcal{X}^*\) the dual space endowed with the strong dual topology or the topology of the bounded convergence. We denote by \(\mathcal{H}_m^*(\mathbb{C})\) the dual of \(\mathcal{H}_m(\mathbb{C})\) for \(m \in \mathbb{N}\), we identify \(\mathcal{H}_m^*(\mathbb{C})\) with \(\mathcal{H}_{-m}(\mathbb{C})\), by using the bilinear form:

\[
\langle W, g \rangle = \int_{\mathbb{Q}_p^N} \overline{W}(\xi) g(\xi) \, d^N\xi \text{ for } W \in \mathcal{H}_{-m}(\mathbb{C}) \text{ and } g \in \mathcal{H}_m(\mathbb{C}).
\]

Then

\[
\mathcal{H}_\infty^*(\mathbb{Q}_p^N, \mathbb{C}) := \mathcal{H}_\infty^*(\mathbb{C}) = \bigcup_{m \in \mathbb{N}} \mathcal{H}_{-m}(\mathbb{C})
\]

\[
= \left\{ W \in \mathcal{D}'(\mathbb{Q}_p^N) \; | \; \|W\|_{-m} < \infty \text{ for some } m \in \mathbb{N} \right\}.
\]

We consider \(\mathcal{H}_\infty^*(\mathbb{C})\) endowed with the strong topology. We use [82] as pairing between \(\mathcal{H}_\infty^*(\mathbb{C})\) and \(\mathcal{H}_\infty(\mathbb{C})\). By a similar construction one obtains the space \(\mathcal{H}_\infty^*(\mathbb{R}) := \mathcal{H}_\infty^*(\mathbb{Q}_p^N, \mathbb{R})\). The above announced results are also valid for \(\mathcal{H}_\infty^*(\mathbb{R})\). If there is no danger of confusion we use \(\mathcal{H}_\infty^*\) instead of \(\mathcal{H}_\infty^*(\mathbb{C})\) or \(\mathcal{H}_\infty^*(\mathbb{R})\).
Remark 2. (i) For complex and real spaces, $||·||_{±1}$ denotes the norm on $\mathcal{H}_l$ and $\mathcal{H}_{l−1}$. We denote by $\langle ·, · \rangle$ the dual pairings between $\mathcal{H}_{l−1}$ and $\mathcal{H}_l$ and between $\mathcal{H}_∞$ and $\mathcal{H}_∞^*$. We preserve this notation for the norm and pairing on tensor powers of these spaces.

(ii) If $\mathcal{X}_i \in A$ is a family of locally convex spaces, we denote by $\varprojlim_{i \in \mathbb{N}} \mathcal{X}_i$ the projective limit of the family, and by $\varinjlim_{i \in \mathbb{N}} \mathcal{X}_i$ the inductive limit of the family.

(iii) If $\mathcal{N}$ is a nuclear space, which is the projective limit of the Hilbert spaces $\mathcal{H}_i$, $i \in \mathbb{N}$, the $n$-th symmetric tensor product of $\mathcal{N}$ is defined as $\mathcal{N}^{⊗n} = \varprojlim_{i \in \mathbb{N}} \mathcal{H}_i^{⊗n}$. This is a nuclear space. The dual space is $\mathcal{N}^{∗⊗n} = \varprojlim_{i \in \mathbb{N}} \hat{\mathcal{H}}_i^{⊗n}$.

3.2. Non-Archimedean Gaussian measures. The spaces

$$\mathcal{H}_∞(\mathbb{R}) \hookrightarrow L_2(\mathbb{Q}_p^N) \hookrightarrow \mathcal{H}_∞^*(\mathbb{R})$$

form a Gel’fand triple, that is, $\mathcal{H}_∞(\mathbb{R})$ is a nuclear countably Hilbert space which is densely and continuously embedded in $L_2^N$ and $||g||_0^2 = \langle g, g \rangle_0$ for $g \in \mathcal{H}_∞(\mathbb{R})$. This triple was introduced in [22], see also [28] Chapter 10. The inner product and the norm of $(L_2^N(\mathbb{Q}_p^N))^{⊗n} \simeq L_2^N(\mathbb{Q}_p^{Nn})$ are denoted by $\langle ·, · \rangle_0$ and $||·||_0$. From now on, we consider $\mathcal{H}_∞^{⊗n}(\mathbb{R})$ as subspace of $\mathcal{H}_∞^*(\mathbb{R})$, then $\langle ·, · \rangle_{\mathcal{H}_∞^{⊗n}(\mathbb{R})} = n! \langle ·, · \rangle_0$.

We denote by $\mathcal{B} := \mathcal{B}(\mathcal{H}_∞(\mathbb{R}))$ the $\sigma$-algebra generated by the cylinder subsets of $\mathcal{H}_∞^*(\mathbb{R})$. The mapping

$$C : \mathcal{H}_∞(\mathbb{R}) \rightarrow \mathbb{C}
\quad f \mapsto e^{-\frac{i}{2}||f||_0^2}$$

defines a characteristic functional, i.e. $C$ is continuous, positive definite and $C(0) = 1$. By the Bochner-Minlos theorem, see e.g. [4], [10], there exists a probability measure $\mu$, called the canonical Gaussian measure on $(\mathcal{H}_∞^*(\mathbb{R}), \mathcal{B})$, given by its characteristic functional as

$$\int_{\mathcal{H}_∞^*(\mathbb{R})} e^{i \langle W, f \rangle} d\mu(W) = e^{-\frac{i}{2}||f||_0^2}, \quad f \in \mathcal{H}_∞(\mathbb{R}).$$

We set $(L_2^N) := L^2((\mathcal{H}_∞^*(\mathbb{R}), -\mu; \mathbb{C})$ to denote the complex vector space of measurable functions $\Psi : \mathcal{H}_∞^*(\mathbb{R}) \rightarrow \mathbb{C}$ satisfying

$$||\Psi||_2^2 = \int_{\mathcal{H}_∞^*(\mathbb{R})} |\Psi(W)|^2 d\mu(W) < \infty.$$

The space $(L_2^N) := L^2((\mathcal{H}_∞^*(\mathbb{R}), -\mu; \mathbb{R})$ is defined in a similar way. The pairing $\mathcal{H}_∞^*(\mathbb{R}) \times \mathcal{H}_∞^*(\mathbb{R})$ can be extended to $\mathcal{H}_∞^*(\mathbb{R}) \times L^2(\mathbb{Q}_p^N)$ as an $(L_2^N)$-function on $\mathcal{H}_∞^*(\mathbb{R})$, this fact follows from

$$\int_{\mathcal{H}_∞^*(\mathbb{R})} |\langle W, g \rangle|^2 d\mu(W) = ||g||_0^2,$$

see e.g. [27] Lemma 2.1.5. If $g \in L_2^N$, then $W \rightarrow \langle W, g \rangle$ belongs to $(L_2^N)$. Let $f \in \mathcal{H}_∞(\mathbb{R})$ and $W_f(J) := \langle J, f \rangle$, $J \in \mathcal{H}_∞^*(\mathbb{R})$. Then $W_f$ is a Gaussian random variable on $(\mathcal{H}_∞^*(\mathbb{R}), -\mu)$ satisfying

$$\mathbb{E}_\mu(W_f) = 0, \quad \mathbb{E}_\mu(W_f^2) = ||f||_0^2.$$
Then the linear map
\[ \mathcal{H}_\infty(\mathbb{R}) \rightarrow (L^2_\mathbb{R}) \]
\[ f \rightarrow Wf \]
can be extended to a linear isometry from \( L^2(\mathbb{Q}_p^N) \) to \( (L^2_\mathbb{R}) \).

### 3.3. Wick-ordered polynomials

Let \( P_n(\mathbb{R}) \), respectively \( P_n(\mathbb{C}) \), be the vector space of finite linear combinations of functions of the form
\[ W \mapsto \langle W, f \rangle_n = \langle W \otimes^n, f \otimes^n \rangle \]
where \( f \) runs over \( \mathcal{H}_\infty(\mathbb{R}) \), respectively \( \mathcal{H}_\infty(\mathbb{C}) \). Notice that \( P_n(\mathbb{C}) = P_n(\mathbb{R}) + iP_n(\mathbb{R}) \). An element of the direct algebraic sums
\[ P(\mathbb{R}) := \bigoplus_{n=0}^{\infty} P_n(\mathbb{R}), \ P(\mathbb{C}) := \bigoplus_{n=0}^{\infty} P_n(\mathbb{C}) \]
is called a polynomial on the Gaussian space \( \mathcal{H}_\infty^* \). These functions are not very useful because they do not satisfy orthogonality relations. This is the main motivation to introduce and utilize the Wick-ordered polynomials.

For \( W \in \mathcal{H}_\infty^*(\mathbb{R}) \) and \( f \in \mathcal{H}_\infty^*(\mathbb{R}) \), we define the Wick-ordered monomial as
\[
\langle : W \otimes^n : f \otimes^n \rangle = \sum_{k=0}^{[\frac{n}{2}]} \frac{n!}{k!(n-2k)!} \left( \frac{-1}{2} \langle f, f \rangle_0 \right)^k \langle W, f \rangle^{n-2k}_0
\]
where \( H_n \) denotes the \( n \)-th Hermite polynomial. Then:
\[ \Phi(W) = \sum_{n=0}^{\infty} \langle : W \otimes^n : \phi_n \rangle, \]
where \( \phi_n \) belong to the symmetric \( n \)-fold algebraic tensor product \( \mathcal{H}_\infty^*(\mathbb{R}) \) of \( \mathcal{H}_\infty(\mathbb{R}) \), respectively \( \mathcal{H}_\infty(\mathbb{C}) \), and the sum symbol involves only a finite number of non-zero terms. A function of type (3.4) is called a Wick-ordered polynomial. For two polynomials \( \Phi, \Psi \in P(\mathbb{C}) \) given respectively by (3.4) with \( \phi_n \in (\mathcal{H}_\infty(\mathbb{C})) \), and by
\[ \Psi(W) = \sum_{n=0}^{\infty} \langle : W \otimes^n : \psi_n \rangle, \] it holds that
\[ \int_{\mathcal{H}_\infty^*(\mathbb{R})} \Phi(W) \Psi(W) d\mu(W) = \sum_{n=0}^{\infty} n! \langle \phi_n, \psi_n \rangle_0, \]
where \( \langle \cdot, \cdot \rangle_0 \) denotes the scalar product in \( (L^2(\mathbb{Q}_p^N))^\otimes_n \). In particular,
\[ ||\Phi||^2_{L^2} = \sum_{n=0}^{\infty} n! ||\phi_n||^2_0, \]
where \( ||\cdot||_0 \) denotes the norms of \( (L^2(\mathbb{Q}_p^N))^\otimes_n \), see e.g. \[ \text{Proposition 2.2.10].} \)
Consequently, each \( \Psi \in P(\mathbb{C}) \) is uniquely expressed as a Wick-ordered polynomial.
Remark 3. We denote by \( I_n(f_n) \) the linear extension to \( (L^2(Q_p^N))^\otimes_n \) of the map \( f_n \to \langle : W^\otimes : ; f_n \rangle, W \in \mathcal{H}_\infty^* (\mathbb{R}) \), then
\[
I_n(f^\otimes_n) = \|f\|^n_0 H_n(\|f\|^{-1} W_f), \quad f \in L^2,
\]
and
\[
\int_{\mathcal{H}_\infty^* (\mathbb{R})} I_n(f_n)I_m(g_m) d\mu = \delta_{nm} n! \langle f_n, g_m \rangle_0, \quad f_n \in L^2_\otimes, g_m \in L^2_\otimes^{\otimes m}.
\]
We shall also use \( \langle : W^\otimes ; ; f_n \rangle \) to denote \( I_n(f_n) \) formally. In this case the symbol \( \langle , \rangle \) should not be confused with the bilinear form on \( H_\infty^* \times H_\infty^* \).

3.4. Wiener-Itô-Segal isomorphism. Let \( \Gamma (L^2(Q_p^N)) \) be the space of sequences \( f = \{ f_n \}_{n \in \mathbb{N}}, f_n \in (L^2(Q_p^N))^\otimes_n, \) such that
\[
\|f\|^2_{\Gamma (L^2(Q_p^N))} := \sum_{n=0}^\infty n! \|f_n\|^2_0 < \infty.
\]
The Hilbert space \( \Gamma (L^2(Q_p^N)) \) is called the Boson Fock Space on \( L^2(Q_p^N) \). The Wiener-Itô-Segal theorem asserts that for each \( \Phi \in (L^2(Q_p^N)) \) there exists a sequence \( \phi = \{ \phi_n \}_{n \in \mathbb{N}} \) in \( \Gamma (L^2(Q_p^N)) \) such that \( \|\phi\| \leq \|\Phi\| \) holds in the \( (L^2(Q_p^N))^\otimes_n \)-sense, but with \( \phi_n \in (L^2(Q_p^N))^\otimes_n, \) see Remark 3. Conversely, for any \( \phi = \{ \phi_n \}_{n \in \mathbb{N}} \in \Gamma (L^2(Q_p^N)), \) \( \Phi = \{ \phi_n \}_{n \in \mathbb{N}} \) defines a function in \( (L^2(Q_p^N))^\otimes_n \). In this case
\[
\|\Phi\|^2_{(L^2(Q_p^N))} = \sum_{n=0}^\infty n! \|\phi_n\|^2_0 = \|\Phi\|^2_{\Gamma (L^2(Q_p^N))},
\]
see e.g. \cite{37} Theorem 2.3.5, \cite{39}.

4. Non-Archimedean Kondratiev Spaces of Test Functions and Distributions

In this section we introduce non-Archimedean versions of Kondratiev-type spaces of test functions and distributions.

4.1. Kondratiev-type spaces of test functions. We define for \( l, k \in \mathbb{N}, \) and \( \beta \in [0, 1] \) fixed, the following norm on \( (L^2(C)) \):
\[
\|\Phi\|^2_{l,k,\beta} = \sum_{n=0}^\infty (n!)^{1+\beta} 2^{nk} \|\phi_n\|^2_l,
\]
where \( \Phi \) is given in \( \mathcal{3.4} \), and \( \|\cdot\|_l \) denotes the norm on \( \mathcal{H}_l^\otimes_n \).

We now define
\[
\mathcal{H}_{l,k,\beta} = \left\{ \Phi (W) = \sum_{n=0}^\infty \langle : W^\otimes : ; \phi_n \rangle \in \left( L^2(C) \right) ; \|\Phi\|^2_{l,k,\beta} < \infty \right\}.
\]
The space \( \mathcal{H}_{l,k,\beta} \) is a Hilbert space with inner product
\[
\langle \Phi, \Psi \rangle_{l,k,\beta} = \sum_{n=0}^\infty (n!)^{1+\beta} 2^{nk} \langle \phi_n, \psi_n \rangle_l,
\]
where \( \Phi, \Psi \in (L^2(C)) \) are as in \( \mathcal{3.4} \) and \( \langle , \rangle_l \) denotes the inner product on \( \mathcal{H}_l^\otimes_n \).
The Kondratiev space of test functions $(\mathcal{H}_\infty)\beta$ is defined to be the projective limit of the spaces $\mathcal{H}_{l,k,\beta}$:

$$(\mathcal{H}_\infty)\beta = \lim_{l,k \in \mathbb{N}} \mathcal{H}_{l,k,\beta}.$$ 

As a vector space $(\mathcal{H}_\infty)\beta = \cap_{l,k \in \mathbb{N}} \mathcal{H}_{l,k,\beta}$. The space of test functions $(\mathcal{H}_\infty)\beta$ is a nuclear countable Hilbert space, which is continuously and densely embedded in $(L_2^\infty)$. Moreover, $(\mathcal{H}_\infty)\beta$ and its topology do not depend on the family of Hilbertian norms $\{\|\cdot\|_l\}_{l \in \mathbb{N}}$, see e.g. [20, Theorem 1], [16, Chapter IV, Theorem 1.4].

The construction used to obtain the spaces $(\mathcal{H}_\infty)\beta$ can be carried out starting with an arbitrary nuclear space $\mathcal{N}$. For $0 \leq \beta \leq 1$, the spaces $(\mathcal{N})^{\beta}$ were studied by Kondratiev, Leukert and Streit in [23], [21], [20], see also [16, Chapter IV]. In the case $\beta = 0$ and $\mathcal{N} = \mathcal{S}$, the Schwartz space in $\mathbb{R}^n$, the space $(\mathcal{N})^0$ is the Hida space of test functions, see e.g. [15].

4.2. Kondratiev-type spaces of distributions. Let $\mathcal{H}_{-l,-k,-\beta}$ be the dual with respect to $(L^2_\infty)$ of $\mathcal{H}_{l,k,\beta}$ and let $(\mathcal{H}_\infty)^{-\beta}$ be the dual with respect to $(L^2_\infty)$ of $(\mathcal{H}_\infty)^\beta$. We denote by $\langle \cdot, \cdot \rangle$ the corresponding dual pairing which is given by the extension of the scalar product on $(L^2_\infty)$. We define the expectation of a distribution $\Phi \in (\mathcal{H}_\infty)^{-\beta}$ as $E_\mu(\Phi) = \langle (\Phi, 1) \rangle$.

The dual space of $(\mathcal{H}_\infty)^{-\beta}$ is given by

$$(\mathcal{H}_\infty)^{-\beta} = \bigcup_{l,k \in \mathbb{N}} \mathcal{H}_{-l,-k,-\beta},$$

see [16, Chapter IV, Theorem 1.5]. We will consider $(\mathcal{H}_\infty)^{-\beta}$ with the inductive limit topology. In particular, we know that every distribution is of finite order, i.e. for any $\Phi \in (\mathcal{H}_\infty)^{-\beta}$ there exist $l,k \in \mathbb{N}$ such that $\Phi \in \mathcal{H}_{-l,-k,-\beta}$.

The chaos decomposition introduces a natural decomposition of $\Phi \in (\mathcal{H}_\infty)^{-\beta}$ into generalized kernels $\Phi_n \in (\mathcal{H}_\infty^\ast(C))^{\hat{\otimes} n}$. Let $\Phi_n \in (\mathcal{H}_\infty^\ast(C))^{\hat{\otimes} n}$ be given. Then there is a distribution, denoted as $(\Phi_n, \cdot \otimes^n :)$ in $(\mathcal{H}_\infty)^{-\beta}$ acting on $\Psi \in (\mathcal{H}_\infty)^\beta$

$$(\Psi = \sum_{n=0}^\infty \langle (\cdot \otimes^n, \psi_n), \Psi \rangle), \quad \text{with} \quad \psi_n \in (\mathcal{H}_\infty(C))^{\hat{\otimes} n},$$

as

$$\langle \langle (\Phi_n, \cdot \otimes^n :), \Psi \rangle \rangle = n! \langle \Phi_n, \psi_n \rangle.$$ 

Any $\Phi \in (\mathcal{H}_\infty)^{-\beta}$ has a unique decomposition of the form

$$\Phi = \sum_{n=0}^\infty \langle (\Phi_n, \cdot \otimes^n :), \Phi_n \in (\mathcal{H}_\infty^\ast(C))^{\hat{\otimes} n},$$

where the series converges in $(\mathcal{H}_\infty)^{-\beta}$; in addition, we have

$$\langle (\Phi, \Psi) \rangle = \sum_{n=0}^\infty n! \langle \Phi_n, \psi_n \rangle, \quad \Psi \in (\mathcal{H}_\infty)^\beta.$$ 

Now, $\mathcal{H}_{-l,-k,-\beta}$ is a Hilbert space, that can be described as follows:

$$\mathcal{H}_{-l,-k,-\beta} = \left\{ \Phi \in (\mathcal{H}_\infty)^{-\beta}; \|\Phi\|_{-l,-k,-\beta} < \infty \right\},$$

where

$$\|\Phi\|_{-l,-k,-\beta}^2 = \sum_{n=0}^\infty (n!)^{1-\beta} 2^{-nk} \|\Phi_n\|^2_l,$$

(4.1)
see [16] Chapter IV, Theorem 1.5).

Remark 4. Notice that
\[(H_\infty)^1 \subset \cdots \subset (H_\infty)^\beta \subset \cdots \subset (H_\infty)^0 \subset (L_2^\infty) \]
\[\subset (H_\infty)^{-0} \subset \cdots \subset (H_\infty)^{^{-\beta}} \subset \cdots \subset (H_\infty)^{-1} .\]

Following Kondratiev, Leukert and Streit, in this article we work with the Gelfand triple \((H_\infty)^1 \subset (L_2^\infty) \subset (H_\infty)^{-1} .\)

4.3. The S-transform and the characterization of \((H_\infty)^{-1} .\)

4.3.1. The S-transform. We first consider the Wick exponential:
\[
: \exp (W, g) := \exp \left( \langle W, g \rangle - \frac{1}{2} \|g\|^2_0 \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle W \otimes^n, g^\otimes_n \rangle ,
\]
for \(W \in H_\infty^\prime (\mathbb{R}) , g \in H_\infty (\mathbb{C}) .\) Then \(: \exp (W, g) \in (L_2^\infty) \) and its \(l, k, 1\)-norm is given by
\[
\| : \exp (\cdot, g) : \|_{l,k,1}^2 = \sum_{n=0}^{\infty} (n!)^2 2^{n k} \left\| \frac{1}{n!} g^\otimes_n \right\|_l^2 = \sum_{n=0}^{\infty} \left( 2^k \|g\|_l^2 \right)^n .
\]
This norm is finite if and only if \(2^k \|g\|_l^2 < 1 ,\) i.e. \(: \exp (W, g) \in H_{l,k,\beta} \) if and only if \(g\) belongs to the following neighborhood of zero:
\[
U_{l,k} = \left\{ f \in H_\infty (\mathbb{C}) ; \|f\|_l < \frac{1}{2^k} \right\} .
\]

Therefore the Wick exponential does not belong to \((H_\infty)^1 ,\) i.e. it is not a test function, in contrast to usual white noise analysis.

Let \(\Phi \in (H_\infty)^{-1} ,\) then there exist \(l, k\) such that \(\Phi \in H_{-l,-k,-1} .\) For all \(f \in U_{l,k} ,\) we define the (local) S-transform of \(\Phi\) as
\[
S_\Phi (f) = \langle \Phi, : \exp (\cdot, f) : \rangle = \sum_{n=0}^{\infty} \langle \Phi_n, f^\otimes_n \rangle .
\]
Hence, for \(\Phi \in H_{-l,-k,-1} ,\) [12] defines the S-transform for all \(f \in U_{l,k} .\)

4.3.2. Holomorphic functions on \(H_\infty (\mathbb{C}) .\) Let \(V_{l,\epsilon} = \{ f \in H_\infty (\mathbb{C}) ; \|f\|_l < \epsilon \} \) be a neighborhood of zero in \(H_\infty (\mathbb{C}) .\) A map \(F : V_{l,\epsilon} \to C\) is called holomorphic in \(V_{l,\epsilon} ,\) if it satisfies the following two conditions: (i) for each \(g_0 \in V_{l,\epsilon} , g \in H_\infty (\mathbb{C}) \) there exists a neighborhood \(V_{g_0,g} \) in \(\mathbb{C} \) around the origin such that the map \(z \to F (g_0 + zg) \) is holomorphic in \(V_{g_0,g} .\) (ii) For each \(g \in V_{l,\epsilon} \) there exists an open set \(U \subset V_{l,\epsilon} \) containing \(g\) such that \(F (U) \) is bounded.

By identifying two maps \(F_1\) and \(F_2\) coinciding in a neighborhood of zero, we define \(Hol_0(H_\infty (\mathbb{C}))\) as the space of germs of holomorphic maps around the origin.

4.3.3. Characterization of \((H_\infty)^{-1} .\) A key result is the following: the mapping
\[
S : (H_\infty)^{-1} \to Hol_0(H_\infty (\mathbb{C}))
\]
\[
\Phi \mapsto S_\Phi
\]
is a well-defined bijection, see [20] Theorem 3, [16] Chapter IV, Theorem 2.13].
4.3.4. Integration of distributions. Let \((\mathcal{E}, \mathcal{A}, \nu)\) be a measure space, and

\[
\mathcal{E} \rightarrow (\mathcal{H}_\infty)^{-1},
\]

\[
l \rightarrow \Phi_l.
\]

Assume that there exists an open neighborhood \(\mathcal{V} \subset \mathcal{H}_\infty(\mathbb{C})\) of zero such that (i) \(S\Phi_1, l \in \mathcal{E}\), is holomorphic in \(\mathcal{V}\); (ii) the mapping \(l \rightarrow S\Phi_1(g)\) is measurable for every \(g \in \mathcal{V}\); and (iii) there exists a function \(C(l) \in L^1(\mathcal{E}, \mathcal{A}, \nu)\) such that \(|S\Phi_1(g)| \leq C(l)\) for all \(g \in \mathcal{V}\) and for \(\nu\)-almost \(l \in \mathcal{E}\). Then there exist \(l_0, k_0 \in \mathbb{N}\) such that \(\int_{\mathcal{E}} \Phi_1 d\nu(l)\) exists as a Bochner integral in \(\mathcal{H}_{-l_0, -k_0, -1}\), in particular,

\[
(4.3) \quad S \left( \int_{\mathcal{E}} \Phi_1 d\nu(l) \right)(g) = \int_{\mathcal{E}} S\Phi_1(g) d\nu(l), \text{ for any } g \in \mathcal{V},
\]

cf. [20, Theorem 6], [16, Chapter IV, Theorem 2.15].

4.3.5. The Wick product. Given \(\Phi, \Psi \in (\mathcal{H}_\infty)^{-1}\), we define the Wick product of them as

\[
\Phi \circ \Psi = S^{-1}(S\Phi S\Psi).
\]

This product is well-defined because \(Hol_0(\mathcal{H}_\infty(\mathbb{C}))\) is an algebra. The map

\[
(\mathcal{H}_\infty)^{-1} \times (\mathcal{H}_\infty)^{-1} \rightarrow (\mathcal{H}_\infty)^{-1},
\]

\[
(\Phi, \Psi) \rightarrow \Phi \circ \Psi
\]

is well-defined and continuous. Furthermore, if \(\Phi \in \mathcal{H}_{-l_1, -k_1, -1}, \Psi \in \mathcal{H}_{-l_2, -k_2, -1}\), and \(l := \max\{l_1, l_2\}, k := k_1 + k_2 + 1\), then

\[
\|\Phi \circ \Psi\|_{-l, -k, -1} \leq \|\Phi\|_{-l_1, -k_1, -1} \|\Psi\|_{-l_2, -k_2, -1},
\]

cf. [20] Proposition 11. The Wick product leaves \((\mathcal{H}_\infty)^{-1}\) invariant. By induction on \(n\), we can define the Wick powers:

\[
\Phi^{\circ n} = S^{-1}((S\Phi)^n) \in (\mathcal{H}_\infty)^{-1}.
\]

Consequently \(\sum_{n=0}^{m} a_n \Phi^{\circ n} \in (\mathcal{H}_\infty)^{-1}\).

4.3.6. Wick analytic functions in \((\mathcal{H}_\infty)^{-1}\). Assume that \(F\) is an analytic function in a neighborhood of the point \(z_0 = \mathbb{E}_\mu(\Phi)\) in \(\mathbb{C}\), with \(\Phi \in (\mathcal{H}_\infty)^{-1}\). Then \(F^{\circ}(\Phi) = S^{-1}(F(S\Phi))\) exists in \((\mathcal{H}_\infty)^{-1}\), cf. [20, Theorem 12]. In addition, if \(F\) is analytic in \(z_0 = \mathbb{E}_\mu(\Phi)\), with power series \(F(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n\), then the Wick series \(\sum_{n=0}^{\infty} c_n (\Phi - z_0)^{\circ n}\) converges in \((\mathcal{H}_\infty)^{-1}\) and \(F^{\circ}(\Phi) = \sum_{n=0}^{\infty} c_n (\Phi - z_0)^{\circ n}\).

5. Schwinger Functions and Euclidean Quantum Field Theory

5.1. Schwinger functions.

**Definition 1.** Let \(f_1, \ldots, f_n \in \mathcal{H}_\infty(\mathbb{R}), n \in \mathbb{N}\). The \(n\)-th Schwinger function corresponding to \(\Phi \in (\mathcal{H}_\infty)^{-1}\), with \(\mathbb{E}_\mu(\Phi) = 1\), is defined as

\[
S_n^\Phi(f_1 \otimes \cdots \otimes f_n)(W) = \begin{cases} 
1 & \text{if } n = 0 \\
\langle \langle \Phi, (W, f_1) \cdots (W, f_n) \rangle \rangle & \text{if } n \geq 1,
\end{cases}
\]

for \(W \in \mathcal{H}_\infty^*(\mathbb{R})\).
Lemma 3. We first notice that

\[ T \Phi (g) = \exp \left( -\frac{1}{2} \| g \| ^2 _0 \right) S \Phi (ig) \]

for \( \Phi \in (H_\infty )^{-1} \) and \( g \in \mathcal{U} \), where \( \mathcal{U} \) is neighborhood of zero in \( H_\infty (C) \). The Schwinger functions can be computed by using the \( T \)-transform:

**Lemma 1** ([23] Proposition III.3). Let \( f_1, \ldots, f_n \in H_\infty (R) \), \( n \in N \). The \( n \)-th Schwinger function corresponding to \( \Phi \in (H_\infty )^{-1} \) is given by

\[ S_n^\Phi (f_1 \otimes \cdots \otimes f_n) = (i)^{n} \frac{\partial ^n}{\partial t_1 \cdots \partial t_n} T \Phi (t_1 f_1 + \cdots + t_n f_n) |_{t_1=\cdots=t_n=0}. \]

**Lemma 2.** For each distribution \( \Phi \in (H_\infty )^{-1} \), with \( E_\mu (\Phi ) = 1 \), the Schwinger functions \( \{S_n^\Phi \}_{n \in N} \) satisfy the following conditions:

(OS1) the sequence \( \{S_n^\Phi \}_{n \in N} \) with \( S_n^\Phi \in (H_\infty (C))^{\otimes n} \), satisfies

\[ |S_n^\Phi (f_1 \otimes \cdots \otimes f_n)| \leq KC^n n! \prod _{l=1} ^n || f_l || _l, \]

for some \( l, k \in N \), where \( K = \sqrt{\int_0 (2^{-k}) || \Phi || _{l=1-k-1} } \), here \( I_0 \) is the modified Bessel function of order zero, which satisfies \( I_0 (2^{-k}) < 1.3 \), \( C = c2^k \), and for any \( f_1, \cdots, f_n \in H_\infty (R) \);

(OS4) for \( n \geq 2 \) and all \( \sigma \in S_n \), the permutation group of order \( n \), it holds that

\[ S_n^\Phi (f_1 \otimes \cdots \otimes f_n) = S_n^\Phi (f_{\sigma (1)} \otimes \cdots \otimes f_{\sigma (n)}), \]

for any \( f_1, \cdots, f_n \in H_\infty (R) \).

**Proof.** Estimation (OS1) is given in the proof of Theorem 2 in [22]. The Schwinger functions \( \{S_n^\Phi \} \) are symmetric by definition.

5.2. A white-noise process. For \( t \in Q_p \), \( \vec{x} \in Q_p ^{N-1} \), we set \( x = (t, \vec{x}) \). We denote by \( \delta _x := \delta _{(t, \vec{x})} \), the Dirac distribution at \( (t, \vec{x}) \).

**Lemma 3.** \( \delta _{(t, \vec{x})} \in (H_\infty )^{-1} \).

**Proof.** We first notice that

\[ \left| \| \delta _{(t, \vec{x})} \| _{l=1} ^2 \right| ^2 = \int_{Q_p ^l} \frac{d^N \xi}{| \xi | _p ^l} < \infty \text{ for } l > N, \]

which implies that \( \delta _{(t, \vec{x})} \in H_{-l}(C) \) for all \( l > N \), see [31]. Now, we define \( \{ \Phi _n \}_{n \in N} \), with \( \Phi _n \in (H_\infty ^*_C) ^{\otimes n} \), as \( \Phi _n = 0 \) if \( n \neq 1 \) and \( \Phi _1 = \delta _{(t, \vec{x})} \). Then

\[ \sum _n \langle \Phi _n, : W ^{\otimes n} : \rangle = \langle \delta _{(t, \vec{x})}, : W : \rangle \in (H_\infty )^{-1}. \]

In addition, for \( \psi \in H_\infty (C) \), we have

\[ \langle \langle \langle \delta _{(t, \vec{x})}, : W : , \psi \rangle \rangle \rangle = \langle \delta _{(t, \vec{x})}, \psi \rangle = \int_{Q_p ^{l}} \chi _p (-\xi \cdot x) \hat{\psi} (x) d^N \xi \]

\[ = \psi (t, \vec{x}), \]
where we used that $\psi$ is a continuous function in $L^1 \cap L^2$, see Section 3.1.1 and Theorem 10.15.

We now set
\[
\Phi(t, \vec{x}) := \left\langle \delta(t, \vec{x}), : W : \right\rangle \in (\mathcal{H}_\infty)^{-1}.
\]
Then $\Phi(t, \vec{x})$ is a white-noise process with $\mathbb{E}_\mu(\Phi(t, \vec{x})) = 0$.

We now set
\[
\Phi(t, \vec{x}) := \sum_{k=0}^{\infty} \frac{1}{k!} H_k \Phi(t, \vec{x})^\otimes_k \in (\mathcal{H}_\infty)^{-1}.
\]
Assume that $H(z) = \sum_{k=0}^{\infty} \frac{1}{k!} H_k z^k, z \in U \subset \mathbb{C}$, is a holomorphic function in $U$, an open neighborhood of $0 = \mathbb{E}_\mu(\Phi(t, \vec{x}))$. By [20, Theorem 12], see also Section 4.3.6, we can define
\[
SH^\otimes(\Phi(x)) = \sum_{k=0}^{\infty} \frac{1}{k!} H_k \left\langle \delta^\otimes_k, : W^\otimes_k : \right\rangle \in (\mathcal{H}_\infty)^{-1}.
\]

Our next goal is the construction of the potential
\[
\int_{\mathbb{Q}_p^N} H^\otimes(\Phi(x)) d^N x
\]
as a white-noise distribution. This goal is accomplished through the following result:

**Theorem 1.** (i) Let $H$ be a holomorphic function at zero such that $H(0) = 0$. Then (5.3) exists as a Bochner integral in a suitable space of $(\mathcal{H}_\infty)^{-1}$.

(ii) The distribution
\[
\Phi_H := \exp \left( -\int_{\mathbb{Q}_p^N} H^\otimes(\Phi(x)) d^N x \right)
\]
is an element of $(\mathcal{H}_\infty)^{-1}$.

(iii) The $T$-transform of $\Phi_H$ is given by
\[
T\Phi_H(g) = \exp \left( -\int_{\mathbb{Q}_p^N} H(i g(x)) + \frac{1}{2} (g(x))^2 \ d^N x \right)
\]
for all $g$ in a neighborhood $U \subset \mathcal{H}_\infty(\mathbb{C})$ of the zero. In particular, $\mathbb{E}_\mu(\Phi_H) = 1$.

**Proof.** (i) The result follows from the discussion presented in Section 4.3.3, see also [20, Theorem 6], as follows. Let $r > 0$ be the radius of convergence of the Taylor series of $H$ at the origin. We set $C(N) := \sqrt{\int_{\mathbb{Q}_p^N} \frac{d^N x}{|x|^l}},$ for a fixed $l > N$, and
\[
U_0 := \left\{ g \in \mathcal{H}_\infty(\mathbb{C}) : \|g\|_l < \frac{r}{C(N)} \right\}.
\]
Then, for $g \in U_0$ we have
\[
SH^\otimes(\Phi(x))(g) = \sum_{k=1}^{\infty} \frac{1}{k!} H_k \left\langle \delta^\otimes_k, g^\otimes_k \right\rangle = \sum_{k=1}^{\infty} \frac{1}{k!} H_k g(x)^k
\]
\[
= \sum_{k=1}^{\infty} \frac{1}{k!} H_k \left\{ \frac{g(x)}{r} \right\}^k r^k.
\]
By Claim A, \(|g(x)| < 1\), and from (5.4) we obtain that

\[
(5.5) \quad |SH^\diamond (\Phi(x))(g)| \leq |g(x)| \sum_{k=1}^{\infty} \frac{1}{k!} |H_k| r^{k-1} \in L^1(\mathbb{Q}_p^N),
\]

because \(H_\infty(\mathbb{C}) \subset L^1(\mathbb{Q}_p^N)\), cf. [20, Theorem 10.15]. Estimation (5.5) implies the
holomorphy of \(SH^\diamond(\Phi(x))(g)\) for any \(g \in \mathcal{U}_0\). Since \(SH^\diamond(\Phi(x))(g)\) is measurable
by [20, Theorem 6], we conclude that (5.3) is an element of \((\mathcal{H}_\infty)^{-1}\).

**Claim A.** \(\mathcal{U}_0 \subset \mathcal{U} := \{g \in \mathcal{H}_\infty(\mathbb{C}) ; \|g\|_{L^\infty} < r\}\).

The Claim follows from the fact that
\[\|g\|_{L^\infty} \leq C(N) \|g\|_1, \text{ for } g \in \mathcal{H}_\infty(\mathbb{C}).\]

This last fact is verified as follows: by using that \(g \in L^1(\mathbb{Q}_p^N) \cap L^2(\mathbb{Q}_p^N)\), and the Cauchy-Schwarz inequality, we have
\[
|g(x)| = \left| \int_{\mathbb{Q}_p^N} \chi_p(-\xi \cdot x) \hat{g}(\xi) d^N\xi \right| \leq \int_{\mathbb{Q}_p^N} |\hat{g}(\xi)| d^N\xi
= \int_{\mathbb{Q}_p^N} \frac{1}{|\xi|^2} \left\{ |\xi|^2 |\hat{g}(\xi)| \right\} d^N\xi \leq C(N) \|g\|_1.
\]

(ii) Since \(\exp\) is analytic in a neighborhood of \(0 = \mathbb{E}_\mu(\Phi(t, x))\), then
\[
\exp^\diamond \left( - \int_{\mathbb{Q}_p^N} H^\diamond(\Phi(x)) d^N x \right) = S^{-1} \left( \exp \left( S \left( - \int_{\mathbb{Q}_p^N} H^\diamond(\Phi(x)) d^N x \right) \right) \right),
\]
and by (i), \(- \int_{\mathbb{Q}_p^N} H^\diamond(\Phi(x)) d^N x \in (\mathcal{H}_\infty)^{-1}\), and then its \(S\)-transform is analytic
at the origin, and its composition with \(\exp\) gives again an analytic function at the
origin, whose inverse \(S\)-transform gives an element of \((\mathcal{H}_\infty)^{-1}\), cf. [20, Theorem 12].

(iii) The calculation of the \(T\)-transform uses (5.2), \(\exp^\diamond(\cdot) = S^{-1}(\exp(S(\cdot)))\), and (5.4) as follows:
\[
(T\Phi_H)(g) = \exp \left( - \frac{1}{2} \|g\|_0^2 \right) \exp \left( S \left\{- \int_{\mathbb{Q}_p^N} H^\diamond(\Phi(x)) d^N x \right\} (ig) \right)
= \exp \left( - \frac{1}{2} \|g\|_0^2 \right) \exp \left( - \int_{\mathbb{Q}_p^N} \langle H^\diamond(\Phi(x))(\cdot), \exp(\cdot, ig \cdot) \rangle d^N x \right)
= \exp \left( - \frac{1}{2} \|g\|_0^2 \right) \exp \left( - \int_{\mathbb{Q}_p^N} SH^\diamond(\Phi(x))(ig) d^N x \right)
= \exp \left( - \int_{\mathbb{Q}_p^N} H(ig(x)) + \frac{1}{2} g(x)^2 \right) d^N x.
\]

In particular \(\mathbb{E}_\mu(\Phi_H) = T\Phi_H(0) = 1\). \(\square\)

5.3. **Pseudodifferential Operators and Green Functions.** A non-constant homogenous polynomial \(l(\xi) \in \mathbb{Z}_p[\xi_1, \cdots, \xi_N]\) of degree \(d\) is called **elliptic** if it satisfies \(l(\xi) = 0 \Leftrightarrow \xi = 0\). There are infinitely many elliptic polynomials, cf. [53].
Lemma 24. A such polynomial satisfies
\begin{equation}
(5.6) \quad C_0 (\alpha) \| \xi \|_p^{\alpha d} \leq |l (\xi)|_p^\alpha \leq C_1 (\alpha) \| \xi \|_p^{\alpha d},
\end{equation}
for some positive constants \( C_0 (\alpha) \), \( C_1 (\alpha) \), cf. \[33\] Lemma 25. We define an elliptic pseudodifferential operator with symbol \( |l (\xi)|_p^\alpha \), with \( \alpha > 0 \), as
\begin{equation}
(5.7) \quad (L, h) (x) = F_{\xi \to x}^{-1} \left( \frac{1}{|l (\xi)|_p^\alpha + m^2} \right),
\end{equation}
for \( h \in \mathcal{D}(\mathbb{Q}_p^N) \). We define \( G := G (x; m, \alpha) \in \mathcal{D}'(\mathbb{Q}_p^N) \), with \( \alpha > 0 \), \( m > 0 \), to be the solution of
\begin{equation}
(5.8) \quad (L + m^2) G = \delta \in \mathcal{D}'(\mathbb{Q}_p^N).
\end{equation}
We will say that the Green function \( G (x; m, \alpha) \) is a fundamental solution of the equation
\begin{equation}
(5.9) \quad G (x; \alpha, m) = F_{\xi \to x}^{-1} \left( \frac{1}{|l (\xi)|_p^\alpha + m^2} \right).
\end{equation}
Notice that by \( (5.6) \), we have
\[ \frac{1}{|l (\xi)|_p^\alpha + m^2} \in L^1 (\mathbb{Q}_p^N, d^N \xi) \quad \text{for} \quad \alpha d > N, \]
and in this case, \( G (x; \alpha, m) \) is an \( L^\infty \)-function.

There exists a Green function \( G (x; \alpha, m) \) for the operator \( L + m^2 \), which is continuous and non-negative on \( \mathbb{Q}_p^N \setminus \{0\} \), and tends to zero at infinity. The equation
\begin{equation}
(5.10) \quad (L + m^2) u = g,
\end{equation}
with \( g \in \mathcal{H}_\infty (\mathbb{R}) \), has a unique solution \( u(x) = G (x; \alpha, m) * g(x) \in \mathcal{H}_\infty (\mathbb{R}) \), cf. \[28\] Theorem 11.2.

As a consequence one obtains that the mapping
\begin{equation}
(5.11) \quad G_{\alpha, m} : \mathcal{H}_\infty (\mathbb{R}) \rightarrow \mathcal{H}_\infty (\mathbb{R})
\end{equation}
g \( \to \ G (x; \alpha, m) * g(x) \),
is continuous, cf. \[28\] Corollary 11.3.

**Remark 5.** For \( \alpha > 0 \), \( \beta > 0 \), \( m > 0 \), we set
\begin{equation}
(5.12) \quad (L_{\alpha, \beta, m} h) (x) = F_{\xi \to x}^{-1} \left( \left| l (\xi) \right|_p^\alpha + m^2 \right)^{\beta} F_{\xi \to \xi} h,
\end{equation}
for \( h \in \mathcal{D}(\mathbb{Q}_p^N) \). We denote by \( G (x; \alpha, \beta, m) \) the associated Green function. By using the fact that
\[ C_0 (\alpha, \beta, m) \left| \xi \right|_p^{\alpha \beta d} \leq \left| l (\xi) \right|_p^\alpha + m^2 \leq C_1 (\alpha, \beta, m) \left| \xi \right|_p^{\alpha \beta d}, \]
all the results presented in this section for operators \( L + m^2 \) can be extended to operators \( L_{\alpha, \beta, m} \). In particular,
gives rise to a continuous mapping. As operators on $\mathcal{H}_\infty(\mathbb{R})$, we can identify $G_{\alpha,\beta,m}$ with the operator $(L_\alpha + m^2)^{-\beta}$, which is a pseudodifferential operator with symbol $(|\xi|_p^\alpha + m^2)^{-\beta}$.

**Remark 6.** The mapping

$$G_{\alpha,\beta,m}^\otimes \colon \mathcal{H}_\infty^\otimes \to \mathcal{H}_\infty^\otimes$$

is well-defined and continuous. By using (37, Proposition 1.3.6), any element $h$ of $\mathcal{H}_\infty^\otimes$ can be represented as an absolutely convergent series of the form $h = \sum_i f_i \otimes g_i$, consequently, $\sum_i G_{\alpha,\beta,m}(f_i) \otimes G_{\alpha,\beta,m}(g_i)$ is an element of $\mathcal{H}_\infty^\otimes$, which implies that $G_{\alpha,\beta,m}^\otimes$ is a well-defined mapping. On the other hand, the space $\mathcal{H}_\infty^\otimes$ is locally convex, the topology is defined by the seminorms

$$\|h\|_{l,k} = \inf \sum_i \|f_i\|_l \cdot \|g_i\|_k, \quad h \in \mathcal{H}_\infty \otimes_{al} \mathcal{H}_\infty,$$

where the infimum is taken over all the pairs $(f_i, g_i)$ satisfying $h = \sum_j f_j \otimes g_j$. The continuity of $G_{\alpha,\beta,m}^\otimes$ is equivalent to

$$\|(G_{\alpha,\beta,m}^\otimes - 1) h\|_{l,k} \leq C \|h\|_{l',k'},$$

where the indices $l', k'$ depend on $l, k$. This condition can be verified easily using the continuity of $G_{\alpha,\beta,m}$.

**Remark 7.** We denote by $Tr$ (the trace), which is the unique element of $\mathcal{H}_\infty^\otimes$ determined by the formula

$$\langle Tr, f \otimes g \rangle = \langle f, g \rangle_0, \text{ for } f, g \in \mathcal{H}_\infty.$$

We define $(G_{\alpha,\beta,m}^\otimes - 1) Tr \in \mathcal{H}_\infty^\otimes$ as

$$\langle (G_{\alpha,\beta,m}^\otimes - 1) Tr, f \otimes g \rangle = \langle Tr, (G_{\alpha,\beta,m}^\otimes - 1) (f \otimes g) \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the pairing between $\mathcal{H}_\infty^\otimes$ and $\mathcal{H}_\infty^\otimes$. For a general construction of this type of operators the reader may consult (29, Theorem 9.11).

### 5.4. Lévy characteristics.

We recall that an infinitely divisible probability distribution $P$ is a probability distribution having the property that for each $n \in \mathbb{N} \setminus \{0\}$ there exists a probability distribution $P_n$ such that $P = P_n * \cdots * P_n$ ($n$-times). By the Lévy-Khinchine Theorem, see e.g. (32), the characteristic function $C_P$ of $P$ satisfies

$$C_P(t) = \int_{\mathbb{R}} e^{ist} dP(s) = e^{F(t)}, \quad t \in \mathbb{R},$$

where $F : \mathbb{R} \to \mathbb{C}$ is a continuous function, called the Lévy characteristic of $P$, which is uniquely represented as follows:

$$F(t) = iat - \frac{\sigma^2 t^2}{2} + \int_{\mathbb{R} \setminus \{0\}} \left( e^{ist} - 1 - \frac{ist}{1 + s^2} \right) dM(s), \quad t \in \mathbb{R},$$

where $a, \sigma \in \mathbb{R}$, with $\sigma \geq 0$, and the measure $dM(s)$ satisfies

$$\int_{\mathbb{R} \setminus \{0\}} \min(1, s^2) \, dM(s) < \infty.$$
On the other hand, given a triple \((a, \sigma, dM)\) with \(a \in \mathbb{R}\), \(\sigma \geq 0\), and \(dM\) a measure on \(\mathbb{R} \setminus \{0\}\) satisfying (5.14), there exists a unique infinitely divisible probability distribution \(P\) such that its Lévy characteristic is given by (5.13).

Let \(f\) be a Lévy characteristic defined by (5.13). Then there exists a unique probability measure \(P_f\) on \((\mathcal{H}_\infty^\prime (\mathbb{R}), \mathcal{B})\) such that the ‘Fourier transform’ of \(P_f\) satisfies

\[
(5.15) \quad \int_{\mathcal{H}_\infty^\prime (\mathbb{R})} e^{(W,f)} dP_f (W) = \exp \left\{ \int_{Q_1^2} F (f(x)) d^N x \right\}, \quad f \in \mathcal{H}_\infty (\mathbb{R}),
\]

cf. [52, Theorem 5.2], alternatively [28, Theorem 11.6].

We will say that a distribution \(\Theta \in (\mathcal{H}_\infty)^{-1}\) is represented by a probability measure \(P\) on \((\mathcal{H}_\mathbb{R} (\infty), \mathcal{B})\) if

\[
(\Theta, \Psi) = \int_{\mathcal{H}_\infty^\prime (\mathbb{R})} \Psi (W) dP (W) \quad \text{for any } \Psi \in (\mathcal{H}_\infty)^1.
\]

We will denote this fact as \(dP = \Theta d\mu\). In this case \(\Theta\) may be regarded as the generalized Radon-Nikodym derivative \(\frac{d\Theta}{d\mu}\) of \(P\) with respect to \(\mu\).

By using this result, Theorem \(\Pi\)(iii), and assuming that

\[
(5.17) \quad F (t) = -H(it) - \frac{1}{2} t^2, \quad t \in \mathbb{R}
\]
is a Lévy characteristic, there exists a probability measure \(P_H\) on \((\mathcal{H}_\mathbb{R} (\infty), \mathcal{B})\) such that

\[
(5.18) \quad T \Phi_H (f) = \int_{\mathcal{H}_\infty^\prime (\mathbb{R})} \exp (i (W, f)) dP_H (W), \quad f \in \mathcal{H}_\infty (\mathbb{R}).
\]

**Theorem 2.** Assume that \(H\) is a holomorphic function at the origin satisfying \(H(0) = 0\). Then \(dP_H = \Phi_H d\mu\) if and only if \(F (t)\) is a Lévy characteristic.

**Proof.** Assume that \(F (t)\) is a Lévy characteristic. By (5.13), we have

\[
(5.19) \quad T \Phi_H (\lambda f) = \int_{\mathcal{H}_\infty^\prime (\mathbb{R})} \exp (\lambda i (W, f)) dP_H (W) = (\langle \Phi_H, \exp (\lambda i (W, f)) \rangle),
\]

for any \(\lambda \in \mathbb{R}\).

In order to establish (5.16), it is sufficient to show that (5.16) holds for \(\Psi\) in a dense subspace of \((L^2_\mathbb{R})\), we can choose the linear span of the exponential functions of the form \(\exp \alpha (W, f)\) for \(\alpha \in \mathbb{C}, f \in \mathcal{H}_\infty (\mathbb{R})\), cf. [15 Proposition 1.9]. On the other hand, since \(\Phi_H \in \mathcal{H}_{-l,-k-1} (\mathbb{C})\) for some \(l, k \in \mathbb{N}\), and \((L^2_\mathbb{C})\) is dense in \(\mathcal{H}_{-l,-k-1} (\mathbb{C})\), it is sufficient to establish (5.16) when \(\Phi_H \in (L^2_\mathbb{C})\). Now the result follows from (5.19) by using the fact that

\[
\lambda \rightarrow T \Phi_H (\lambda f) = \int_{\mathcal{H}_\infty^\prime (\mathbb{R})} \exp (\lambda i (W, f)) d\mu (W), \quad \lambda \in \mathbb{R},
\]

has an entire analytic extension, cf. [15 Proposition 2.2].

Conversely, assume that \(dP_H = \Phi_H d\mu\), then by Theorem \(\Pi\)(iii), we have

\[
(5.20) \quad \int_{\mathcal{H}_\infty^\prime (\mathbb{R})} e^{i(W,f)} dP_H (W) = \int_{\mathcal{H}_\infty^\prime (\mathbb{R})} e^{i(W,f)} \Phi_H (W) d\mu (W)
\]

\[
= \langle \langle \Phi_H, e^{i (\cdot, f)} \rangle \rangle = T \Phi_H (f) = \exp \left\{ \int_{Q_1^2} F (f(x)) d^N x \right\},
\]
for \( f \in \mathcal{H}_\infty(\mathbb{R}) \). We now take \( f (x) = t 1_{\mathbb{Z}_p^N} (x) \), where \( t \in \mathbb{R} \) and \( 1_{\mathbb{Z}_p^N} \) is the characteristic function of \( \mathbb{Z}_p^N \). By using that \( \tilde{H}(0) = 0 \), we have

\[
(5.21) \quad \exp \left\{ \int_{\mathbb{Q}_p^N} F (f (x)) d^N x \right\} = \exp F (t).
\]

Now, we consider the random variable:

\[
\langle \cdot, 1_{\mathbb{Z}_p^N} \rangle : (\mathcal{H}_\infty^*(\mathbb{R}), B, P_H) \to (\mathbb{R}, B(\mathbb{R}))
\]

\[
W \to \langle W, 1_{\mathbb{Z}_p^N} \rangle,
\]

with probability distribution \( \nu \langle \cdot, 1_{\mathbb{Z}_p^N} \rangle (A) = P_H \{ W \in \mathcal{H}_\infty^*(\mathbb{R}) ; \langle W, 1_{\mathbb{Z}_p^N} \rangle \in A \} \), where \( A \) is a Borel subset of \( \mathbb{R} \). Then, by (5.20) - (5.21),

\[
(5.22) \quad \int_{\mathcal{H}_\infty^*(\mathbb{R})} e^{i t (W, f)} dP_H (W) = \int_{\mathbb{R}} e^{i t z} d\nu \langle \cdot, 1_{\mathbb{Z}_p^N} \rangle (z) = \exp F (t).
\]

We call these measures generalized white noise measures. The moments of the measure \( P_H \) are the Schwinger functions \( \{ S_n^{G_H} \}_{n \in \mathbb{N}} \).

Since \( (G_{\alpha,m}, f) (x) := G (x; \alpha, m) * f (x) \) gives rise to a continuous mapping from \( \mathcal{H}_\mathbb{R}(\infty) \) into itself, then, the conjugate operator \( \tilde{G}_{\alpha,m} : \mathcal{H}_\mathbb{R}^*(\infty) \to \mathcal{H}_\mathbb{R}^*(\infty) \) is a measurable mapping from \( (\mathcal{H}_\mathbb{R}^*(\infty), B) \) into itself. For the sake of simplicity, we use \( \tilde{G} \) instead of \( \tilde{G}_{\alpha,m} \) and \( G \) instead of \( G (x; \alpha, m) \). We set \( P_H^G \) to be the image probability measure of \( P_H \) under \( \tilde{G} \), i.e. \( P_H^G \) is the measure on \( (\mathcal{H}_\mathbb{R}^*(\infty), B) \) defined by

\[
(5.23) \quad P_H^G (A) = P_H \left( \tilde{G}^{-1} (A) \right), \text{ for } A \in B.
\]

The Fourier transform of \( P_H^G \) is given by

\[
(5.24) \quad \int_{\mathcal{H}_\mathbb{R}^*(\mathbb{R})} e^{i t (W, f)} dP_H^G (W) = \exp \left\{ \int_{\mathbb{Q}_p^N} f \left\{ \int_{\mathbb{Q}_p^N} G (x - y; \alpha, m) \right. f (y) d^N y \right\} d^N x \right\},
\]

for \( f \in \mathcal{H}_\mathbb{R}(\infty) \), where \( f \) is given as in (5.17), cf. [52, Proposition 6.2], alternatively [28, Proposition 11.12]. Finally, (5.24) is also valid if we replace \( G = G (x; \alpha, m) \) by \( G (x; \alpha, \beta, m) \).

5.5. The free Euclidean Bose field. An important difference between the real and \( p \)-adic Euclidean quantum field theories comes from the ‘ellipticity’ of the quadratic form \( q_N (\xi) = \xi_1^2 + \cdots + \xi_N^2 \). In the real case \( q_N (\xi) \) is elliptic for any \( N \geq 1 \). In the \( p \)-adic case, \( q_N (\xi) \) is not elliptic for \( N \geq 5 \). In the case \( N = 4 \), there is a unique elliptic quadratic form, up to linear equivalence, which is \( \xi_1^2 - s \xi_2^2 - p \xi_3^2 + s \xi_4^2 \), where \( s \in \mathbb{Z} \setminus \{ 0 \} \) is a quadratic non-residue, i.e. \( \left( \frac{s}{p} \right) = -1 \).
5.5.1. The Archimedean free covariance function. The free covariance function $C(x - y; m) := C(x - y)$ is the solution of the Laplace equation
\[
(-\Delta + m^2) C(x - y) = \delta(x - y),
\]
where $\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$. As a distribution from $S'(\mathbb{R}^N)$, the free covariance is given by
\[
C(x - y) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \exp\left(-ik \cdot (x - y)\right) d^N k,
\]
where $k, x, y \in \mathbb{R}^N$, $d^N k$ is the Lebesgue measure of $\mathbb{R}^N$, $k^2 = k \cdot k$, and $k \cdot x = \sum_{i=1}^{N} k_i x_i$. Notice that the quadratic form used in the definition of the Fourier transform $k^2$ is the same as the one used in the propagator $\frac{1}{k^2 + m^2}$, this situation does not occur in the $p$-adic case. In particular the group of symmetries of $C(x - y)$ is the $SO(N, \mathbb{R})$. The function $C(x - y)$ has the following properties (see [12 Proposition 7.2.1]):

(i) $C(x - y)$ is positive and analytic for $x - y \neq 0$;
(ii) $C(x - y) \leq \exp(-m \|x - y\|)$ as $\|x - y\| \to \infty$;
(iii) for $N \geq 3$ and $m \|x - y\|$ in a neighborhood of zero,
\[
C(x - y) \sim \|x - y\|^{-N + 2},
\]
(iv) for $N = 2$ and $m \|x - y\|$ in a neighborhood of zero,
\[
C(x - y) \sim -\ln(m \|x - y\|).
\]

5.5.2. The Archimedean free Euclidean Bose field. Take $H_m$ to be the Hilbert space defined as the closure of $S(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|_m$ induced by the scalar product
\[
(f, g)_m := \int_{\mathbb{R}^N} f(x) (-\Delta + m^2)^{-1} g(x) d^N x = \left(f, (-\Delta + m^2)^{-1} g\right)_{L^2(\mathbb{R}^N)}.
\]
Then $S(\mathbb{R}^N) \hookrightarrow H_m \hookrightarrow S'(\mathbb{R}^N)$ form a Gel'fand triple. The probability space $(S'(\mathbb{R}^N), \mathcal{B}, \nu)$, where $\nu$ is the centered Gaussian measure on $\mathcal{B}$ (the $\sigma$-algebra of cylinder sets) with covariance
\[
\int_{S'(\mathbb{R}^N)} \langle W, f \rangle \langle W, g \rangle \, d\nu(W) = \left(f, (-\Delta + m^2)^{-1} g\right)_{L^2(\mathbb{R}^N)},
\]
for $f, g \in S(\mathbb{R}^N)$, jointly with the coordinate process $W \rightarrow \langle W, f \rangle$, with $f \in S(\mathbb{R}^N)$, is called the free Euclidean Bose field of mass $m$ in $N$ dimensions.

5.5.3. The non-Archimedean free covariance function. The $p$-adic free covariance $C_p(x - y; m) := C_p(x - y)$ is the solution of the pseudodifferential equation
\[
(L_\alpha + m^2) C(x - y) = \delta(x - y),
\]
where $L_\alpha$ is the pseudodifferential operator defined in (5.7). As a distribution from $\mathcal{D}'(\mathbb{Q}_p^N)$, the free covariance is given by
\[
C_p(x - y) = \int_{\mathbb{Q}_p^N} \frac{\chi_p(-\xi \cdot (x - y))}{|\xi|_p^d + m^2} d^N \xi,
\]
where $k, x, y \in \mathbb{Q}_p^N$, $d^N \xi$ is the Haar measure of $\mathbb{Q}_p^N$, $\chi(k)$ is an elliptic polynomial of degree $d$, and $k \cdot x = \sum_{i=1}^{N} k_i x_i$. In this case $\chi(k) \neq k \cdot k$, and then the symmetries of
$C_p(x - y)$ form a subgroup of the $p$-adic orthogonal group attached to the quadratic form $k \cdot k$. There are other possible propagators, for instance

\[
\frac{1}{\|l(k)\|_p + m^2}, \quad \alpha > 0.
\]

For a discussion on the possible scalar propagators, in the $p$-adic setting, the reader may consult [41].

The function $C_p(x - y)$ satisfies (see [52] Proposition 4.1, or [23] Proposition 11.1):

(i) $C_p(x - y)$ is positive and locally constant for $x - y \neq 0$;
(ii) $C_p(x - y) \leq C \|x - y\|_{p}^{-\alpha d - N}$ as $\|x - y\|_p \to \infty$;
(iii) for $0 < \alpha d < N$ and $\|x - y\|_p \leq 1$,

\[
C_p(x - y) \leq C \|x - y\|_{p}^{\alpha d - N};
\]
(iv) for $N = \alpha d$ and $\|x - y\|_p \leq 1$,

\[
C_p(x - y) \leq C_0 - C_1 \ln \|x - y\|_p.
\]

5.5.4. The non-Archimedean free Euclidean Bose field. Take $H_m$ to be the Hilbert space defined as the closure of $D_B(\mathbb{Q}_p^N)$ with respect to the norm $\| \cdot \|_m$ induced by the scalar product

\[
(f, g)_m := \int_{\mathbb{Q}_p^N} \overline{f}(\xi) \hat{g}(\xi) \frac{d^N \xi}{\|l(\xi)\|_p^\alpha + m^2} = \left( f, (L_\alpha + m^2)^{-1} g \right)_{L_2(\mathbb{Q}_p^N)}.
\]

By using that

\[
C_0 [\xi]^{[d\alpha]}_{p} \leq \|l(\xi)\|_p^\alpha + m^2 \leq C_1 [\xi]^{[d\alpha]}_{p},
\]

where $[t] = \min \{ m \in \mathbb{Z}; m \geq x \}$ and $[t] = \max \{ m \in \mathbb{Z}; m \leq x \}$, we have

\[
\mathcal{H}_{-[d\alpha]}(\mathbb{R}) \subset H_m \subset \mathcal{H}_{-[d\alpha]}(\mathbb{R}).
\]

Then $\mathcal{H}_\infty(\mathbb{R}) \subset H_m \subset \mathcal{H}_\infty(\mathbb{R})$ from a Gel'fand triple. The probability space $(\mathcal{H}_\infty(\mathbb{R}), \mathcal{B}, \nu_{d,\alpha})$, where $\nu_{d,\alpha}$ is the centered Gaussian measure on $\mathcal{B}$ (the $\sigma$-algebra of cylinder sets) with covariance

\[
\int_{\mathcal{H}_\infty(\mathbb{R})} \langle W, f \rangle \langle W, g \rangle dv_{d,\alpha}(W) = \left( f, (L_\alpha + m^2)^{-1} g \right)_{L_2(\mathbb{Q}_p^N)},
\]

for $f, g \in \mathcal{H}_\infty(\mathbb{R})$, jointly with the coordinate process $W \rightarrow (W, f)$, with fixed $f \in \mathcal{H}_\infty(\mathbb{R})$, is called the non-Archimedean free Euclidean Bose field of mass $m$ in $N$ dimensions.

If $N = 4$ and $d = 2$, then there is a unique elliptic quadratic form up to linear equivalence. If $N \geq 5$ and $l(\xi)$ is an elliptic polynomial of degree $d$, then $\|l(\xi)\|_p^{\frac{2}{d}}$ is a homogeneous function of degree 2 that vanishes only at the origin. We can use this function as the symbol for a pseudodifferential operator, such operator is a $p$-adic analogue of $-\Delta$ in dimension $N$.

If we use the propagator $\frac{1}{\|l(k)\|_p + m^2}$ instead of $\frac{1}{\|l(k)\|_p^\alpha + m^2}$, similar results are obtained due to the fact that

\[
C_0 [k]^{[d\alpha]}_{p} \leq \left( \|l(k)\|_p + m^2 \right)^\alpha \leq C_1 [k]^{[d\alpha]}_{p}.
\]
We prefer using propagator \( \frac{1}{|i(k)|_p^2} \) because the corresponding ‘Laplace equation’ has been studied extensively in the literature. On the other hand, \( \frac{\partial u(x,t)}{\partial t} + L_\alpha u(x,t) = 0 \), with \( x \in \mathbb{Q}_p^N \), \( t > 0 \), behaves like a ‘heat equation’, i.e. the semigroup associated to this equation is a Markov semigroup, see [53 Chapter 2], which means that \(-L_\alpha\) can be considered as \( p\)-adic version of the Laplacian.

5.6. Symmetries. Given a polynomial \( a(\xi) \in \mathbb{Q}_p[\xi_1, \cdots, \xi_n] \) and \( \Lambda \in GL_N(\mathbb{Q}_p) \), we say that \( \Lambda \) preserves \( a \) if \( a(\xi) = a(\Lambda \xi) \), for all \( \xi \in \mathbb{Q}_p^N \). By simplicity, we use \( \Lambda x \) to mean \( [\Lambda_{ij}] x^T \), \( x = (x_1, \cdots, x_N) \in \mathbb{Q}_p^N \), where we identify \( \Lambda \) with the matrix \( [\Lambda_{ij}] \).

Let \( q_N(\xi) = \xi_1^2 + \cdots + \xi_n^2 \) be the elliptic quadratic form used in the definition of the Fourier transform, and let \( I(\xi) \) be the elliptic polynomial that appears in the symbol of the operator \( L_\alpha \). We define the homogeneous Euclidean group of \( \mathbb{Q}_p^N \) relative to \( q(\xi) \) and \( I(\xi) \), denoted as \( E_0(\mathbb{Q}_p^N) := E_0(\mathbb{Q}_p^N; q, I) \), as the subgroup of \( GL_N(\mathbb{Q}_p) \) whose elements preserve \( q(\xi) \) and \( I(\xi) \) simultaneously. Notice that if \( \mathbb{O}(q_N) \) is the orthogonal group of \( q_N \), then \( E_0(\mathbb{Q}_p^N) \) is a subgroup of \( \mathbb{O}(q_N) \).

We define the inhomogeneous Euclidean group, denoted as \( E(\mathbb{Q}_p^N) := E(\mathbb{Q}_p^N; q, I) \), to be the group of transformations of the form \((a, \Lambda) x = a + \Lambda x \), for \( a, x \in \mathbb{Q}_p^N \), \( \Lambda \in E_0(\mathbb{Q}_p^N) \).

In the real case \( q_N = I(\xi) \) and thus the homogeneous Euclidean group is \( SO(N, \mathbb{R}) \). In the \( p\)-adic case, \( E_0(\mathbb{Q}_p^N; q, I) \) is a subgroup of \( \mathbb{O}(q_N) \), in addition, it is not a straightforward matter to decide whether or not \( E_0(\mathbb{Q}_p^N; q, I) \) is non trivial. For this reason, we approach the Green kernels in a different way than do in [13], which is based on [43].

Notice that \((a, \Lambda)^{-1} x = \Lambda^{-1}(x-a)\). Let \((a, \Lambda)\) be a transformation in \( E(\mathbb{Q}_p^N) \), the action of \((a, \Lambda)\) on a functional \( f \in \mathcal{H}_\infty \) is defined by

\[
((a, \Lambda) f) (x) = f( (a, \Lambda)^{-1} x ) , \quad x \in \mathbb{Q}_p^N ,
\]

and on a functional \( W \in \mathcal{H}_\infty^* \), by

\[
\langle (a, \Lambda) W, f \rangle := \langle W, (a, \Lambda)^{-1} f \rangle , \quad f \in \mathcal{H}_\infty(\mathbb{R}) .
\]

These definitions can be extended to elements of the spaces \( \mathcal{H}_\infty^* \) and \( \mathcal{H}_\infty^* \), by taking

\[
(a, \Lambda) (f_1 \otimes \cdots \otimes f_n) := (a, \Lambda)^{-1} f_1 \otimes \cdots \otimes (a, \Lambda)^{-1} f_n .
\]

In general, if \( F : \mathcal{H}_\infty^* \rightarrow \mathcal{X} \) is a linear \( \mathcal{X} \)-valued functional, where \( \mathcal{X} \) is a vector space, we define

\[
((a, \Lambda) F) (f_1 \otimes \cdots \otimes f_n) = F((a, \Lambda) (f_1 \otimes \cdots \otimes f_n)) ,
\]

and we say that \( F \) is Euclidean invariant if and only if \((a, \Lambda) F = F \) for any \((a, \Lambda) \in E(\mathbb{Q}_p^N) \).

**Definition 2.** We call a distribution \( \Phi = \sum_{n=0}^\infty \langle \Phi_n, \cdot \rangle \in (\mathcal{H}_\infty)^{-1} \), with \( \Phi_n \in \mathcal{H}_\infty^{\otimes n} \), Euclidean invariant if and only if the functional \( \langle \Phi_n, \cdot \rangle \) is Euclidean invariant for any \( n \in \mathbb{N} \).

It follows from this definition that \( \Phi \in (\mathcal{H}_\infty)^{-1} \) is Euclidean invariant if and only if \( S \Phi \) and \( T \Phi \) are Euclidean invariant.
6. Schwinger functions and convoluted white noise

We set $G := G (x; m, \alpha)$ for the Green function \((5.9)\). For $\Phi \in (\mathcal{H}_\infty)^{-1}$, we define $\Phi^G$ as
\[
\left( T \Phi^G \right) (g) = (T \Phi) (G * g), \quad g \in \mathcal{U},
\]
where $\mathcal{U}$ is an open neighborhood of zero. Since $G : \mathcal{H}_\infty (\mathbb{R}) \to \mathcal{H}_\infty (\mathbb{R})$, see \((5.11)\), is linear and continuous, cf. [28, Corollary 11.3], by the characterization theorem, cf. [20, Theorem 3], or Section 4.3.3, $\Phi^G$ is a well-defined and unique element of $(\mathcal{H}_\infty)^{-1}$.

Remark 8. By using that $\langle \delta_x, G * f \rangle = \langle G * \delta_x, f \rangle$ for any $f \in \mathcal{H}_\infty$, we have that the white-noise process introduced in Section 5.2 satisfies
\[
\langle \langle G * \Phi (x), \Psi \rangle \rangle = \langle \langle \Phi (x), G * \Psi \rangle \rangle,
\]
because $\langle \langle \Phi (x), G * \Psi \rangle \rangle = \langle \delta_x, G * \psi_1 \rangle$, where $\Psi = \sum_{n=0}^{\infty} \langle \cdot \otimes^n ; \psi_n \rangle, \psi_n \in \mathcal{H}_\infty^\otimes$.

We denote by $\{S^{H, G}_n\}_{n \in \mathbb{N}}$ the Schwinger functions attached to $\Phi^G_H$.

Theorem 3. With $H$ and $\Phi_H$ as in Theorem 1, then distribution $\Phi^G_H \in (\mathcal{H}_\infty)^{-1}$ is Euclidean invariant and is given by
\[
\Phi^G_H = \exp \left( \int_{\mathbb{Q}^N_p} H^\otimes (G * \Phi (x)) \, d^N x + \frac{1}{2} \langle (G^\otimes^2 - 1) \, Tr, : \otimes^2 : \rangle \right),
\]
where $Tr \in (\mathcal{H}_\infty^* (\mathbb{Q}^N_p, \mathbb{C}))^\otimes^2$ denotes the trace kernel defined by $\langle Tr, f \otimes g \rangle = \langle f, g \rangle_0, f, g \in \mathcal{H}_\infty^* (\mathbb{Q}^N_p, \mathbb{R})$. The Schwinger functions $\{S^{H, G}_n\}_{n \in \mathbb{N}}$ satisfy the conditions (OS1) and (OS4) given in Lemma 2 and
\[
\text{(OS2) (Euclidean invariance) } S^{H, G}_n ((a, \Lambda) f) = S^{H, G}_n (f), \quad f \in (\mathcal{H}_\infty (\mathbb{C}))^\otimes^n,
\]
for any $(a, \Lambda) \in E (\mathbb{Q}^N_p)$.

Proof. By definition \((6.1)\) and Theorem 1(iii), we have
\[
\left( T \Phi^G_H \right) (g) = \exp \left( \int_{\mathbb{Q}^N_p} H (iG * g (x)) + \frac{1}{2} (G * g (x))^2 \, d^N x \right).
\]
On the other hand, by taking the $T$-transform in \((6.2)\) and using \((5.4)\) and Remarks 6-8, we obtain
\[
\left( T \Phi^G_H \right) (g) = \exp \left( \frac{-1}{2} \|g\|_0^2 \right) \times 
\exp \left\{ -S \left( \int_{\mathbb{Q}^N_p} H^\otimes (G * \Phi (x)) \, d^N x \right) (ig) - \frac{1}{2} S (\langle (G^\otimes^2 - 1) \, Tr, : \otimes^2 : \rangle (ig) \right\}
\]
\[ \exp \left( \frac{-1}{2} ||g||^2 \right) \times \exp \left\{ -\int_{\mathbb{Q}_p^n} H \left( iG \ast g \left( x \right) \right) d^N x \right\} \exp \left\{ \frac{1}{2} S \left( \langle (G \ast \cdot) - 1 \rangle Tr, \cdot \rangle \ast g \rangle \right) (ig) \right\} \]

\[ = \exp \left( \frac{-1}{2} ||g||^2 \right) \exp \left\{ -\int_{\mathbb{Q}_p^n} H \left( iG \ast g \left( x \right) \right) d^N x - \frac{1}{2} \langle (G \ast \cdot) - 1 \rangle Tr, g \ast g \rangle \right\} \]

\[ = \exp \left( \frac{-1}{2} ||g||^2 \right) \exp \left\{ -\int_{\mathbb{Q}_p^n} H \left( iG \ast g \left( x \right) \right) d^N x - \frac{1}{2} \langle Tr, G \ast g \otimes G \ast g - g \otimes g \rangle \right\} \]

(6.4) \[ = \exp \left\{-\int_{\mathbb{Q}_p^n} H \left( iG \ast g \left( x \right) \right) d^N x - \frac{1}{2} \langle G \ast g \otimes G \ast g \rangle_0 \right\} \]

Formula (6.2) follows from (6.3)-(6.4). Since \( \mathbb{E}(6.4) \), we have (6.4). The probability measure on \( \exp \left\{ -\int_{\mathbb{Q}_p^n} H \left( iG \ast g \left( x \right) \right) d^N x - \frac{1}{2} \langle G \ast g \otimes G \ast g \rangle_0 \right\} \)

Remark 9. (i) Set \( G_{1/2} := G_{\frac{1}{2}} \), \( G_{1/2} \) for \( f \in \mathcal{H}_\infty(\mathbb{R}) \). By taking \( H = 0 \), we obtain the free Euclidean field. Indeed, for \( f \rightarrow \exp \left\{-\frac{1}{2} \langle G_{1/2} \ast f, G_{1/2} \ast f \rangle_0 \right\} \) defines a characteristic functional. Let denote by \( \nu_{G_{1/2}} \) the probability measure on \( (\mathcal{H}_\infty(\mathbb{R}), \mathcal{B}) \) provided by the Bochner-Minlos theorem. Then

\[ \left( T_{\Phi_{0}^{G_{1/2}}} \right) (g) = \exp \left\{-\frac{1}{2} \langle G_{1/2} \ast g, G_{1/2} \ast g \rangle_0 \right\} \]

\[ = \left\langle \Phi_{0}^{G_{1/2}}, \exp i \langle \cdot, g \rangle \right\rangle = \int_{\mathcal{H}_\infty(\mathbb{R})} \exp i \langle W, g \rangle d\nu_{G_{1/2}} (W). \]

(ii) Assuming that \( f \) (t), see [5.17], is a Lévy characteristic, Theorem 3 implies that the probability measure \( P_{H}^{G} \), see [5.2A], admits \( \Phi_{H}^{G} \) as a generalized density with respect to to white noise measure \( \mu \), i.e. \( P_{H}^{G} = \Phi_{H}^{G} \mu \). Indeed, by [5.2A] and (6.4), we have

\[ \int_{\mathcal{H}_\infty(\mathbb{R})} e^{i(W,f)} d\mathbb{P}_{H}^{G} (W) = \exp \left\{ \int_{\mathbb{Q}_p^n} F \left( G \left( x; \alpha, m \right) \ast f \left( x \right) \right) yd^N x \right\} \]

\[ = \exp \left\{ -\int_{\mathbb{Q}_p^n} H \left( iG \ast f \left( x \right) \right) d^N x - \frac{1}{2} \langle G \ast f, G \ast f \rangle_0 \right\} = \left( T_{\Phi_{H}^{G}} \right) (f) \]

\[ = \left\langle \Phi_{H}^{G}, \exp i \langle \cdot, f \rangle \right\rangle. \]

6.1. Truncated Schwinger functions and the cluster property. We denote by \( P^{(n)} \) the collection of all partitions \( I \) of \( \{1, \ldots, n\} \) into disjoint subsets.

Definition 3. Let \( \left\{ S_{n}^{H,G} \right\}_{n \in \mathbb{N}} \) be a sequence of Schwinger functions, with \( S_{0}^{H,G} = 1 \), and \( S_{n}^{H,G} \in \mathcal{H}_\infty(\mathbb{Q}_p^n, \mathbb{C}) \) for \( n \geq 1 \). The truncated Schwinger functions
\[
\left\{ S_{n,T}^{H,G} \right\}_{n \in \mathbb{N}} \text{ are defined recursively by the formula}
\]

\[
S_{n}^{H,G} (f_1 \otimes \cdots \otimes f_n) = \sum_{I \in P^{(n)}} \prod_{j_1, \ldots, j_l} S_{i_1}^{H,G} (f_{j_1} \otimes \cdots \otimes f_{j_l}),
\]

for \( n \geq 1 \). Here for \( \{j_1, \ldots, j_l\} \in I \) we assume that \( j_1 < \ldots < j_l \).

**Remark 10.** By the kernel theorem, the sequence \( \left\{ S_{n}^{H,G} \right\}_{n \in \mathbb{N}} \) uniquely determines the sequence \( \left\{ S_{n,T}^{H,G} \right\}_{n \in \mathbb{N}} \) and vice versa. All the \( S_{n}^{H,G} \) are Euclidean (translation) invariant if and only if all the \( S_{n,T}^{H,G} \) are Euclidean (translation) invariant. The same equivalence holds for ‘temperedness’ (i.e., membership to \( \mathcal{H}_{\infty}^{-1} \)).

**Definition 4.** Let \( a \in \mathbb{Q}_p^N, a \neq 0 \), and \( \lambda \in \mathbb{Q}_p \). Let \( T_{a,\lambda} \) denote the representation of the translation by \( a, \lambda \) on \( \mathcal{H}_{\infty}(\mathbb{Q}_p^N, \mathbb{R}) \). Take \( n, m \geq 1, f_1, \ldots, f_n \in \mathcal{H}_{\infty}(\mathbb{Q}_p^N, \mathbb{R}) \).

\((OS5)/(\textbf{Cluster property})\) A sequence of Schwinger functions \( \left\{ S_{n}^{H,G} \right\}_{n \in \mathbb{N}} \) has the cluster property if for all \( n, m \geq 1 \), it verifies that

\[
\lim_{|\lambda|_p \to \infty} \left\{ S_{m+n}^{H,G} (f_1 \otimes \cdots \otimes f_m \otimes T_{a,\lambda} (f_{m+1} \otimes \cdots \otimes f_{m+n})) \right\} = S_{m}^{H,G} (f_1 \otimes \cdots \otimes f_m) S_{n}^{H,G} (f_{m+1} \otimes \cdots \otimes f_{m+n}).
\]

\((\textbf{Cluster property of truncated Schwinger functions})\) A sequence of truncated Schwinger functions \( \left\{ S_{n,T}^{H,G} \right\}_{n \in \mathbb{N}} \) has the cluster property, if for all \( n, m \geq 1 \), it verifies that

\[
\lim_{|\lambda|_p \to \infty} S_{m+n,T}^{H,G} (f_1 \otimes \cdots \otimes f_m \otimes T_{a,\lambda} (f_{m+1} \otimes \cdots \otimes f_{m+n})) = 0.
\]

**Remark 11.** In the Archimedean case, it is possible to replace \( \lim_{|\lambda|_p \to \infty} (\cdot) \) in (6.5) and (6.6) by \( \lim_{|\lambda|_p \to \infty} |\lambda|^m (\cdot) \) for arbitrary \( m \), cf. [3, Remark 4.4]. This is possible because Schwartz functions decay at infinity faster than any polynomial function. This is not possible in the \( p \)-adic case, because the elements of our ‘\( p \)-adic Schwartz space \( \mathcal{H}_{\infty}(\mathbb{Q}_p^N, \mathbb{R}) \)’ only have a polynomial decay at infinity. For instance, consider the one-dimensional \( p \)-adic heat kernel \( Z(x; t) = \mathcal{F}_x^{-1} \left( e^{-|\xi|^p} \right) \), for \( t > 0 \), and \( \alpha > 0 \), which is an element of \( \mathcal{H}_{\infty}(\mathbb{Q}_p, \mathbb{R}) \). The Fourier transform \( e^{-|\xi|^p} \) of \( Z(x; t) \) decays faster that any polynomial function in \( |\xi|^p \). However, \( Z(x; t) \) has only a polynomial decay at infinity, more precisely,

\[
Z(x; t) \leq C \frac{t^\alpha}{(t^p + |x|^p)^{\alpha+1}}, \quad t > 0, \quad x \in \mathbb{Q}_p,
\]

cf. [19] Lemma 4.1).

**Lemma 4.** Let \( H(z) = \sum_{n=0}^{\infty} H_n z^n, z \in U \subset \mathbb{C}, \) and \( G \) as in Theorem [3] and \( f_1, \ldots, f_n \in \mathcal{H}_{\infty}(\mathbb{Q}_p^N, \mathbb{R}) \). Assume that \( I(t) = -H(it) - \frac{1}{t^2}, t \in \mathbb{R} \) is a Lévy characteristic, then the truncated Schwinger functions are given by

\[
S_{n,T}^{H,G} (f_1 \otimes \cdots \otimes f_n) = \begin{cases} 
-H_n \int_{\mathbb{Q}_p^N} \prod_{i=1}^{n} G * f_i (x) \, d^n x & \text{for } n \geq 2 \\
(-H_2 + 1) \int_{\mathbb{Q}_p^N} G * f_1 (x) G * f_2 (x) \, d^n x & \text{for } n = 2.
\end{cases}
\]
Proof. The result follows from the formula for the Schwinger functions given in Theorem 7.7 in [52], and the uniqueness of the truncated Schwinger functions. The coefficients in front of the integrals in (6.7) are the $n$-th derivatives of the Lévy characteristic divided by $i^n$. For the general $H$ as in Theorem 3 these coefficients are the $n$-th derivatives of $-\left(H(it)+\frac{1}{2}t^2\right)$, $z \in U$.

Lemma 5. Assume that $\alpha d > N$. Let $\Phi, H, G$ as in Theorem 3. Then the sequence of truncated Schwinger functions $\left\{S_{n,T}^{H,G}\right\}_{n \in \mathbb{N}}$ has the cluster property.

Proof. Fix $a \in \mathbb{Q}_p^N$ and take $\lambda \in \mathbb{Q}_p$, $m, n \geq 1$, $f_1, \cdots, f_{m+n} \in \mathcal{H}_\infty\left(\mathbb{Q}_p^N, \mathbb{R}\right)$. By Lemma 4, we have

$$\left|S_{n,T}^{H,G} (f_1 \otimes \cdots \otimes f_n) \otimes T_{a\lambda} (f_{m+1} \otimes \cdots \otimes f_{m+n})\right| = \left|-H_{m+n}\right| \int_{\mathbb{Q}_p^N} \prod_{i=1}^{m} (G * f_i) (x) \prod_{i=m+1}^{m+n} T_{a\lambda} (G * f_i) (x) \right|$$

We now use that $G * f_i \in \mathcal{H}_\infty\left(\mathbb{Q}_p^N, \mathbb{R}\right)$ and that $\mathcal{H}_\infty\left(\mathbb{Q}_p^N, \mathbb{R}\right) \subset C_0\left(\mathbb{Q}_p^N, \mathbb{R}\right)$ to get

$$\left|S_{n,T}^{H,G} (f_1 \otimes \cdots \otimes f_n) \otimes T_{a\lambda} (f_{m+1} \otimes \cdots \otimes f_{m+n})\right| \leq \left|H_{m+n}\right| \prod_{i=1}^{m} \|G * f_i\|_{L^\infty} \prod_{i=m+1}^{m+n-1} \|T_{a\lambda} (G * f_i)\|_{L^\infty} \int_{\mathbb{Q}_p^N} \|T_{a\lambda} (G * f_{m+n}) (x)\| d^N x.$$ 

Now, the announced result follows from the following fact:

Claim. If $\alpha d > N$, for any $f \in \mathcal{H}_\infty\left(\mathbb{Q}_p^N, \mathbb{R}\right)$, it verifies that

$$\lim_{|\lambda|_p \to \infty} \int_{\mathbb{Q}_p^N} G (x - \lambda a - y) |f_{m+n} (y)| \ d^N y = 0.$$ 

Since $\alpha d > N$, by the Riemann-Lebesgue theorem, $G \in C_0\left(\mathbb{Q}_p^N, \mathbb{R}\right)$, and consequently $G (x - \lambda a - y) |f_{m+n} (y)| \leq \|G\|_{L^\infty} |f_{m+n} (y)| \in L^1\left(\mathbb{Q}_p^N\right)$. Now the Claim follows by applying the dominated convergence theorem. 

Theorem 4. With $H, G$ and $\Phi^G_H \in (\mathcal{H}_\infty)^{-1}$ as in Theorem 3. If $\alpha d > N$, then the sequence of Schwinger functions $\left\{S_{n,T}^{H,G}\right\}_{n \in \mathbb{N}}$ has the cluster property (OS5).

Proof. In [3, Theorem 4.5] was established that the cluster property and the truncated cluster property are equivalent. By using this result, the announced result follows from Lemma 5.

Remark 12. The class of Schwinger functions $\left\{S_{n,T}^{H,G}\right\}_{n \in \mathbb{N}}$ corresponding to a distribution $\Phi^G_H \in (\mathcal{H}_\infty)^{-1}$ as in Theorem 3 differs of the class of Schwinger functions corresponding to the convoluted generalized white noise introduced in [52]. In order to explain the differences, let us compare the properties of the Levy characteristic used in [52] with the properties of the function $H$ used in this article, where $F(t) = -H(it) - \frac{1}{2}t^2$, $t \in U \subset \mathbb{R}$. We require only that function $H$ be holomorphic at zero and $H(0) = 0$, as in [13]. This only impose a restriction in choosing the coefficients in front of the integrals corresponding to the $n$-th truncated Schwinger function, see (6.7). On the other hand in [52], the author requires the condition that the measure $\mu$ has finite moments of all orders. This implies that $F$ belongs to $C^\infty(\mathbb{R})$, but $F$ does not have to have a holomorphic extension. Furthermore,
since $\exp sF(t)$ is positive definite for any $s > 0$, cf. [52, Proposition 5.5], and by using $F(0) = 0$ and a result due Schoenberg, cf. [6, Theorem 7.8], we have $-F(t) : \mathbb{R} \to \mathbb{C}$ is a negative definite analytic function. Since $| -F(t) | \leq C |t|^2$ for any $|t| \geq 1$, [6, Corollary 7.16], we conclude that $-F(t)$ is a polynomial of the degree at most 2, and then $H_n = 0$ for $n \geq 3$.

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