Quantum measurements and finite geometry

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Abstract

A complete set of mutually unbiased bases for a Hilbert space of dimension $N$ is analogous in some respects to a certain finite geometric structure, namely, an affine plane. Another kind of quantum measurement, known as a symmetric informationally complete positive-operator-valued measure, is, remarkably, also analogous to an affine plane, but with the roles of points and lines interchanged. In this paper I present these analogies and ask whether they shed any light on the existence or non-existence of such symmetric quantum measurements for a general quantum system with a finite-dimensional state space.

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1 Introduction: Mutually unbiased measurements

I have known Asher Peres since 1979, when he was visiting John Wheeler at the University of Texas at Austin as I was finishing my graduate studies there. In the years since then our collaborations on various problems in quantum mechanics have been among the most enjoyable episodes of my career. One such collaboration took place in 1989 at the Santa Fe Institute. Asher raised the interesting question whether a joint measurement on a composite system could ever discriminate among product states better than a series of separate measurements. Our efforts towards answering this question were fueled in part, and were made much more interesting, by the interaction between our diametrically opposite intuitions on the matter. In those days I did not fully appreciate the depth of Asher’s physical intuition and was confident that I could prove him wrong. It was fun to try, and I learned a lot by trying, and I am grateful for the education. For this and many other reasons, it is a pleasure to dedicate this paper to Asher on the occasion of his seventieth birthday.

In Austin in 1979 Asher and I did not collaborate on any paper, but we did discuss a few physics problems. One of these was the problem of mutually unbiased measurements which is the starting point for the present article.

For a system with an $N$-dimensional state space, a general mixed state is specified by $N^2 - 1$ real parameters. Suppose we are given many copies of such a system and are trying to learn the values of these parameters. If we perform a fixed non-degenerate orthogonal measurement on each of many copies, we will eventually obtain $N - 1$ independent real parameters, namely, the probabilities of $N - 1$ of the outcomes of our measurement. (The last probability is not independent since the probabilities must sum to unity.) By making a different orthogonal measurement on a different set of copies of the system, we can hope to gain another $N - 1$ real parameters, independent of the first set. Thus if we want to obtain all the parameters that define the quantum state, and if we restrict ourselves to orthogonal measurements, the minimum number of distinct measurements we will need is $(N^2 - 1)/(N - 1) = N + 1$. For example, if the system in question is the spin of a spin-1/2 particle, with $N = 2$, we need at least three orthogonal measurements in order to supply enough data to reconstruct the density matrix. It is natural to choose measurements corresponding to three perpendicular spatial axes:
up vs down, right vs left, and in vs out. From the perspective of minimizing the effects of statistical fluctuations, these three are ideal in that they are as different from each other as possible; each one provides information that is maximally independent of the information provided by the others.

The relevant relationship that these three measurements share is “mutual unbiasedness”: each eigenstate of any one of them is an equal-magnitude superposition of the eigenstates of any of the others. In $N$ dimensions, we say that two orthonormal bases $\{|v_1\rangle, \ldots, |v_N\rangle\}$ and $\{|w_1\rangle, \ldots, |w_N\rangle\}$ are mutually unbiased if $|\langle v_i | w_j \rangle|^2 = 1/N$ for each $i$ and $j$. Because of the state-determination problem described above, it is of some interest to find a set of $N + 1$ mutually unbiased bases for an $N$-dimensional state space—we will refer to such a set as a complete set of mutually unbiased bases. Indeed, the problem is more interesting now than it was in the days when Asher and I were discussing it in Austin. For example, mutually unbiased bases are relevant nowadays for quantum cryptography. The original quantum cryptographic schemes used just two mutually unbiased bases in two dimensions [1, 2], but a few years ago Asher, working with Helle Bechmann-Pasquinucci, proposed more general schemes based on multiple unbiased bases in higher dimensions [3].

Is it possible to find $N + 1$ mutually unbiased bases in $N$ dimensions? The answer is yes if $N$ is prime—Ivanovic [4] constructed such bases in 1981—and the answer is again yes if $N$ is any power of a prime [5, 6, 7, 8, 9, 10, 11, 12]. Remarkably, though, the answer is not known for any other values of $N$, not even for $N = 6$. We do know, however, that for any value of $N$ the number of mutually unbiased bases cannot exceed $N + 1$ [4, 13].

In this paper I would like to explore an analogy, noted a few years ago by Zauner [11] and more recently by Klappenecker and Rötteler [12], between the problem of finding mutually unbiased bases and an intriguingly similar problem in combinatorics. It is the well-known problem of finding what are called mutually orthogonal Latin squares. The special case of a complete set of $N + 1$ mutually unbiased bases turns out to be analogous to a finite geometric structure known as an affine plane. This special case is the subject of a recent conjecture of Saniga et al. [14] that we will also consider in this paper.

Another special kind of quantum measurement that has attracted attention lately is what is known as a “symmetric informationally complete positive-operator-valued measure” (SIC POVM) [15]. The second part of this paper discusses this sort of measurement and shows how it too is anal-
ogous to a finite geometric structure. Remarkably, the analogous geometric structure is *again* an affine plane, but with the roles of points and lines interchanged. After presenting these two analogies, we will consider a number of open questions and briefly discuss a connection with the foundations of quantum mechanics.

Both mutually unbiased bases and SIC POVM’s are special cases of “quantum designs”, which have been investigated in the dissertation of Zauner [11]. Indeed, Zauner has already noted not only the two analogies on which I focus in this paper but others as well. What is probably most novel about the work that I present here is a connection—to be specified shortly—between the affine-plane analogies and the discrete phase space of Ref. [22]. As we will see, this connection suggests quantum analogues of the *points* of the finite geometries.

2 Mutually orthogonal Latin squares

The problem of finding mutually unbiased measurements is similar in spirit to the following mathematical problem. For any integer $N \geq 2$, consider a collection of $N^2$ points, with no structure except that the points are distinguishable from each other. Let a “striation” of this set be defined as any partitioning of the $N^2$ points into $N$ disjoint subsets, called “lines”, such that each line consists of $N$ points. Thus a striation defines a set of $N$ lines that are parallel in the sense that no two of them have any points in common. Finally, let us call two striations “mutually unbiased” if each line in either striation has exactly one point in common with each line in the other. A set of four mutually unbiased striations for $N = 3$ is shown in Fig. 1.

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\caption{Four mutually unbiased striations of nine points.}
\end{figure}

I should point out that the above terminology is not standard; I am using it only to make the analogy clear. In the usual formulation, the $N^2$ points are imagined to be arranged in a square (as in Fig. 1), and the vertical and horizontal striations are assumed from the outset. A “Latin square” is
then obtained by specifying a third striation in which each line intersects each row and each column in exactly one point. Two Latin squares are called orthogonal if their respective third striations are mutually unbiased in the above sense. Thus, for example, finding a pair of orthogonal Latin squares amounts to finding four mutually unbiased striations according to the terminology I am using here, because I am including the vertical and horizontal striations in the count.

For a given value of $N$, it is natural to ask how many mutually unbiased striations one can find. Let $M(N)$ be the maximum possible number of such striations. This is a well-studied function, but many questions about it remain unanswered. The following are some of the facts that are known [17].

1. For any $N$, $M(N) \leq N + 1$. (The usual statement is that the maximum number of mutually orthogonal Latin squares is no greater than $N - 1$.)

2. If $N$ is a power of a prime, this upper bound can be achieved; that is, $M(N)$ is exactly equal to $N + 1$. Fig. 1 shows how the bound is achieved for $N = 3$.

3. $M(6) = 3$. [18]

4. If $N - 1$ or $N - 2$ is divisible by four, and if $N$ is not the sum of the squares of two integers, then $M(N)$ is strictly less than $N + 1$. [19]

5. $M(10)$ is strictly less than 11. [20]

It is hard not to notice the parallels with the problem of mutually unbiased bases. The first two items on the above list apply just as well to the maximum number of mutually unbiased bases. Regarding the third item, although we do not know how many mutually unbiased bases one can find for $N = 6$, we can construct three of them, and there is some evidence that no more than three can be found [11, 12, 21]. Regarding the last two items, we simply do not have enough evidence yet to know whether the analogous statements can be made about mutually unbiased bases, but it is not inconceivable that they can.

One wonders, then, whether the two problems are in fact equivalent. On the surface this may seem unlikely because the problems seem to involve different constraints, but the question is worth exploring. If the problems are equivalent, we should be able to find mathematical entities in the unbiased
basis problem that correspond to the lines and points of the unbiased striation problem. Clearly we want each striation of the $N^2$ points to correspond to a basis for the complex vector space, and we want each line in a striation to correspond to an element of the corresponding basis. But what should a point correspond to?

For the special case where $N$ is a power of a prime, there is perhaps a natural interpretation of the points, based on the discrete phase space of Ref. [22]. This phase space is a two-dimensional vector space over the finite field with $N$ elements, so it can be visualized as a square array of $N^2$ points. (There exists an $N$-element field only if $N$ is a power of a prime.) The arithmetic of the field is sufficient to define the concepts “line” and “parallel lines”: a line is the set of points satisfying a linear equation, and two lines are parallel if they have the same slope but different intercepts. One finds that the phase space can be partitioned into $N$ parallel lines in exactly $N+1$ ways. Indeed, this construction is the basis of the standard proof that there exists a complete set of mutually orthogonal Latin squares when $N$ is a power of a prime.

The discrete phase space by itself does not yet provide a quantum mechanical analogue of a “point” in the Latin square problem. This analogue is supplied by the additional structure developed in Ref. [22] for the purpose of representing quantum states as functions on the discrete phase space. This additional structure, called a “quantum net”, assigns to each line $\lambda$ of phase space a pure quantum state, represented by a one-dimensional projection operator $P_\lambda$ in a space of $N$ dimensions. And it assigns to each point $\alpha$ of phase space a Hermitian operator $A_\alpha$ on the same space such that the following properties are satisfied:

1. $\text{Tr}(A_\alpha/N) = 1/N$
2. $\text{Tr}(A_\alpha/N)(A_\beta/N) = (1/N)\delta_{\alpha\beta}$
3. $\sum_{\alpha\in\lambda}(A_\alpha/N) = P_\lambda$

It follows from these properties and from the geometry of the phase space that $\text{Tr}P_\lambda P_\nu = 0$ if $\lambda$ and $\nu$ are parallel lines, and that $\text{Tr}P_\lambda P_\nu = 1/N$ if $\lambda$ and $\nu$ are not parallel. Since there are $N+1$ sets of $N$ parallel lines, the projection operators $P_\lambda$ thus define a complete set of $N+1$ mutually unbiased bases for the state space.
We see then that we can make the following correspondence between the Latin square problem and the unbiased basis problem, when $N$ is a power of a prime.

\[
\begin{align*}
\text{point } \alpha & \leftrightarrow \text{ operator } A_{\alpha}/N \\
\text{line } \lambda & \leftrightarrow \text{ one-dimensional projection } P_{\lambda} \\
\text{striation} & \leftrightarrow \text{ orthonormal basis } \{P_{\lambda_1}, \ldots , P_{\lambda_N}\}
\end{align*}
\]

The operation of composing $N$ points to make a line corresponds, in the quantum setting, to taking the sum of the operators $A_{\alpha}/N$ to obtain a one-dimensional projection operator $P_{\lambda}$.

Does this correspondence imply that when $N$ is a power of a prime, the existence of $N+1$ mutually unbiased bases follows immediately from the existence of $N+1$ mutually unbiased striations in the Latin square problem? No, because it is by no means obvious how to construct the operators $A_{\alpha}$. In Ref. [22] their construction depends on already having in hand a complete set of mutually unbiased bases, which are obtained in a different way. (The method used there to generate these bases is essentially the same as the one discovered independently by Pittinger and Rubin [10].) Perhaps there is an alternative method of constructing these operators such that the two problems can be seen as equivalent. At present I know of no such alternative construction.

It is worth thinking further about the correspondence between geometric objects in the Latin square problem and operators in the quantum problem. In particular, one can make a connection between \textit{cardinalities} of sets of points and \textit{traces} of operators (see also Ref. [11], p. 23). Let $M$ be an operator such as $A_{\alpha}/N$ or $P_{\lambda}$, and let $S_M$ be the set of points that corresponds to that operator, if such a set exists. Let us look for relations of the form

\[
|S_M| = k \text{ Tr } M \quad \text{ and } \quad |S_{M_1} \cap S_{M_2}| = k \text{ Tr } (M_1 M_2),
\]

where $| \cdots |$ indicates the size of the set, “$\cap$” indicates the intersection of the two sets, and $k$ is a constant. Following the correspondence given above, we associate with the operator $A_{\alpha}/N$ the set containing the single point $\alpha$, and with the operator $P_{\lambda}$ the set containing the $N$ points of the line $\lambda$. Then the properties of the $A_{\alpha}$’s listed above lead to the following equations:
1. \( k/N = k \text{Tr}(A_\alpha/N) = |\{\alpha\}| = 1 \)

2. \( (k/N)\delta_{\alpha\beta} = k \text{Tr}(A_\alpha/N)(A_\beta/N) = |\{\alpha\} \cap \{\beta\}| = \delta_{\alpha\beta} \)

3. \( k = k \text{Tr} P_\lambda = |\{\text{points on the line } \lambda\}| = N \)

These conditions are indeed satisfied as long as we choose \( k = N \).

According to these rules, any operator that can be written as the sum of some of the operators \( A_\alpha/N \) corresponds to a set of points. Other operators have no analogues as sets. The identity, being the sum of all \( N^2 \) of the operators \( A_\alpha/N \), corresponds to the complete set of \( N^2 \) points in the Latin square problem. Its trace is \( N \), corresponding to the fact that the cardinality of the set of all the points in the Latin square is \( N^2 \). We will later consider a similar correspondence between traces and cardinalities in connection with our other quantum measurement problem.

Before moving on to that problem, it is interesting to say a little more about the geometry of orthogonal Latin squares. When \( N \) is such that one can find \( N + 1 \) mutually unbiased striations of \( N^2 \) points, one can show that the resulting geometric structure, which includes a total of \( N(N + 1) \) lines, satisfies the following simple rules (for \( N \geq 2 \)):

1. Given any pair of points, there is exactly one line containing both points.

2. Given any line \( \lambda \) and any point \( \alpha \) not lying on \( \lambda \), there is exactly one line through \( \alpha \) that is parallel to \( \lambda \).

3. There exist three noncollinear points.

Any set of points and lines satisfying these rules is called an affine plane. Every affine plane has \( N^2 \) points and \( N(N + 1) \) lines for some value of \( N \), and this value is called the order of the affine plane. The \( N(N + 1) \) lines can be divided into \( N + 1 \) sets of \( N \) parallel lines, and two non-parallel lines always intersect in exactly one point, so that any affine plane defines a complete set of mutually orthogonal Latin squares. Thus the problem of finding a complete set of \( N + 1 \) mutually unbiased bases is analogous to finding an affine plane of order \( N \). At present the only values of \( N \) for which it is known that an affine plane of order \( N \) exists are the powers of primes. And according to the facts we listed earlier, there are some values of \( N \) for which it is known that an affine plane of order \( N \) does not exist, e.g., \( N = 6 \) and \( N = 10 \).
Saniga et al. have recently conjectured that there exists a complete set of mutually unbiased bases in $N$ dimensions if and only if there exists an affine plane of order $N$ \(^1\). As we have discussed, it is not yet clear whether our correspondence between points and operators provides support for this conjecture, but it does provide a direction along which the question might be approached. (See also the discussions of this issue by Zauner \([11]\) and Bengtsson \([23]\).)

### 3 Symmetric informationally complete positive-operator-valued measures

Let us return to the scenario in which we are given many copies of a quantum system and are trying to figure out what quantum state to assign to the system. In the Introduction we restricted our attention to orthogonal measurements, and this restriction led us to the notion of a complete set of mutually unbiased measurements. But other kinds of measurement are certainly possible. For a quantum system with an $N$-dimensional state space, a positive-operator-valued measure (POVM) is a set of positive operators $E_i$ such that $\sum_i E_i = I$, where $I$ is the $N \times N$ identity. A POVM represents a quantum measurement for which the probability of the $i$th outcome is $\text{Tr}(\rho E_i)$, $\rho$ being the system’s density matrix. Note that the operators $E_i$ need not be orthogonal to each other; that is, $\text{Tr}E_iE_j$ need not equal zero when $i \neq j$. Most physicists did not know about POVMs when I met Asher, and indeed his book, *Quantum Mechanics: Concepts and Methods*, is still unusual among quantum mechanics textbooks in explaining or even mentioning POVMs \([24]\).

Whereas the state-reconstruction scheme we discussed earlier required several distinct orthogonal measurements, it is possible to get the same information by means of a single POVM performed on many copies of the system. Since, as before, there are $N^2 - 1$ real parameters to be determined, this single POVM would have to have at least $N^2$ outcomes, thus providing at least $N^2 - 1$ independent probabilities. In the case of a spin-$1/2$ particle, a minimal POVM capable of extracting the necessary parameters would have exactly four outcomes, and the corresponding four probabilities will be max-

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\(^1\)Their paper is couched in terms of projective planes rather than affine planes, but the essential content is the same either way.
imal independence if we make the operators \( E_i \) in some sense as different from each other as possible. The natural choice is to let \( E_i = (1/2)P_i \), where the one-dimensional projectors \( P_i \) correspond to four tetrahedrally related points on the Bloch sphere. The operators \( E_i \) in this case constitute what is called a symmetric informationally complete POVM, or SIC POVM \([15]\).

In \( N \) dimensions, a SIC POVM is a set of \( N^2 \) operators of the form \( E_i = (1/N)P_i \), where the one-dimensional projectors \( P_i \) satisfy the condition

\[
\text{Tr} P_i P_j = \frac{1}{N+1} \quad i \neq j. \tag{2}
\]

One can show that such a set constitutes a POVM and is informationally complete in the sense that the probabilities it provides are sufficient to reconstruct any \( N \times N \) density matrix.

This approach to state-reconstruction leads to another mathematical question about complex vector spaces: for a space of \( N \) dimensions, does there exist a SIC POVM? Like the question about mutually unbiased bases, the answer is known only in certain cases. Here is a summary of our current state of knowledge \([25, 26, 11, 15, 16]\):

1. For \( N = 2, 3, 4, 5, 6, \) and 8, SIC POVM’s exist and explicit expressions for them are known. (In the above discussion we essentially demonstrated the existence of a SIC POVM for \( N = 2 \). The one for \( N = 4 \) \([11, 15]\) is considerably less obvious!)

2. For every \( N \leq 45 \), there is good numerical evidence that a SIC POVM exists \([15]\).

3. There is no known proof that a SIC POVM exists for any value of \( N \) other than those listed in item 1.

Because of the numerical evidence, it is plausible that a SIC POVM exists in every dimension, though if this is indeed the case, it is remarkable that a proof of this existence is so elusive.

4 Affine planes through the looking glass

Let us now try to construct a finite geometric problem analogous to the problem of finding a SIC POVM. As before, we will let lines be the geometric
objects that correspond to one-dimensional projections $P$. But how many points will lie on a line? And how many points will there be altogether? Taking some guidance from our earlier discussion, let us assume relations of the following form between cardinalities of sets of points and traces of operators:

\[ |S_M| = k' \text{Tr} M \quad \text{and} \quad |S_{M_1} \cap S_{M_2}| = k' \text{Tr} (M_1 M_2), \tag{3} \]

where the constant $k'$ is not necessarily equal to the $k$ of Eq. (1).

For each line $\lambda$, the corresponding projection operator $P_\lambda$ has trace equal to 1. Therefore, according to Eq. (3), the number of points on this line should be $k'$. By definition of a SIC POVM, we also have that for two distinct lines $\lambda$ and $\nu$, $\text{Tr} P_\lambda P_\nu = 1/(N + 1)$, from which it follows that $k'/(N + 1)$ is the number of points in the intersection of $\lambda$ and $\nu$. This number must be an integer, so $k'$ must be an integral multiple of $N + 1$. The simplest possibility, then, which also has the pleasing feature that two lines intersect in exactly one point, is to choose $k' = N + 1$. Thus, in our geometry each line will contain $N + 1$ points, and each pair of lines will intersect in one point. How many points should there be altogether? As in the Latin square problem, let us assume that the set of all points corresponds to the identity operator. Then Eq. (3) tells us that the total number of points must be $k' \text{Tr} I = N(N + 1)$. Finally, let us assume for the sake of symmetry that each point lies on the same number of lines as each other point. A simple counting argument then tells us that each point lies on exactly $N$ lines.

To find a geometric model of the SIC POVM problem, then, we are looking for a geometry in which

1. there are exactly $N(N + 1)$ points
2. there are exactly $N^2$ lines
3. each line contains exactly $N + 1$ points
4. each point lies on exactly $N$ lines
5. each pair of lines intersect in exactly one point

It turns out that these conditions precisely describe an affine plane, which we have seen before, except that the roles of points and lines have been reversed. Thus we already know something about the values of $N$ for which we can
find the kind of geometric structure we are looking for: we can find such a structure when $N$ is a power of a prime, and there are other values of $N$ (e.g., $N = 6$ and $N = 10$) for which we know that no such structure exists.

To see how the Latin square problem turns into our current problem when we interchange points and lines, let us consider the case $N = 2$. We start with a square array of four points as shown in Fig. 2(a), where we have also drawn the six lines that define three mutually unbiased striations. Replacing each point with a line and vice versa, while maintaining the coincidence relations between points and lines, we obtain the structure shown in Fig. 2(b). (The circle in the figure counts as one of the lines.) Note that in this structure, the six points can be grouped into three pairs such that the points in each pair are not connected by a line, just as in Fig. 2(a), the six lines can be grouped into three pairs of lines such that the lines in each pair have no point in common.

![Figure 2: (a) The affine plane of order 2. (b) The structure resulting from this affine plane upon interchanging the roles of points and lines.](image)

Note that there is a limitation to the geometric analogy for the SIC POVM problem. For certain small values of $N$ such as 6 and 10, we know that there does not exist an affine plane of order $N$, and yet there does exist a SIC POVM for $N = 6$ [16], and the numerical evidence cited earlier strongly suggests that SIC POVM’s exist for all dimensions up to $N = 45$. Evidently there are ways of constructing SIC POVM’s that have no particular relation to affine planes.

Nevertheless, let us continue with the analogy for those values of $N$ for which affine planes exist, and try to associate a Hermitian operator $B_\alpha$ with each point $\alpha$ of our geometry. (In this discussion it helps to keep Fig. 2(b) in
mind as an example.) As in the Latin square problem, we would like the sum of $B_\alpha$ over each line $\lambda$ to be the projection operator $P_\lambda$ associated with that line. Our rule for relating traces to cardinalities requires that for each point $\alpha$, $\text{Tr} B_\alpha = 1/(N+1)$. What about the traces of products $B_\alpha B_\beta$? We need to choose these so that $\text{Tr}(P_\lambda P_\nu) = 1/(N+1)$ if $\lambda \neq \nu$ and $\text{Tr}(P_\lambda^2) = 1$. One can show that these two relations are guaranteed if we insist on the following trace relations among the $B_\alpha$’s:

1. $\text{Tr}(B_\alpha^2) = \frac{N}{(N+1)^2}$.
2. $\text{Tr}(B_\alpha B_\beta) = \frac{1}{N(N+1)^2}$ if $\alpha \neq \beta$ and $\alpha$ and $\beta$ share a line.
3. $\text{Tr}(B_\alpha B_\beta) = -\frac{1}{(N+1)^2}$ if $\alpha$ and $\beta$ do not share a line.

For the affine plane of order $N$, can we find such a set of operators $B_\alpha$? Here I answer the question in the simplest case $N = 2$, pictured in Fig. 2(b), and leave the question open for other values of $N$. For $N = 2$, we can indeed find such operators. Let each $B_\alpha$ be of the form

$$B_\alpha = \frac{I}{6} \pm \frac{\sigma}{2\sqrt{3}},$$

where $\sigma$ is one of the three Pauli matrices. The six operators $B_\alpha$ are to be assigned to the six points of our geometry in such a way that operators differing only in the sign of the Pauli matrix are assigned to points that do not share a line. One can verify then that the $B_\alpha$’s satisfy the three conditions listed above, as well as the normalization condition $\text{Tr} B_\alpha = 1/3$.

Will these ideas make it easier to find SIC POVM’s, at least for those values of $N$ for which affine planes exist? It is not clear that they will, because there is no obvious method of constructing operators $B_\alpha$ that satisfy the conditions given above. (If one already has a SIC POVM, one can work backward from the projection operators $P_\lambda$ to find such a set of $B_\alpha$’s, but in that case the $B_\alpha$’s must not have been of much help.) But perhaps such a method can be found, at least for certain values of $N$.

5 Discussion

For two problems concerning quantum measurements—one dealing with mutually unbiased bases and the other with SIC POVM’s—we have found analogues in finite geometry. In each case I have suggested (as have other authors) that a line in the geometry is to be associated with a pure state in the
quantum problem. We have found that it also makes sense to associate with each point of the geometry a Hermitian operator, such that the sum of the operators corresponding to all the points on a given line defines the quantum state that is to be associated with that line.

At present there is no evidence that these analogies will help us find either mutually unbiased bases or SIC POVM’s. But it is conceivable that they may end up being a piece of the puzzle. In the SIC POVM problem, for example, if one can find a simple prescription for generating the operator associated with each point, one would be able to generate SIC POVM’s for at least certain values of the dimension of the state space.

These analogies suggest a number of questions. As we have seen, the geometric structure analogous to a complete set of \(N+1\) mutually unbiased bases is identical to the geometric structure analogous to a SIC POVM, except that the roles of points and lines are interchanged. Is there any sense in which the problem of finding \(N+1\) mutually unbiased bases and the problem of finding a SIC POVM are likewise equivalent, at least for those values of \(N\) for which affine planes exist? Suppose, for example, that we are given a complete set of mutually unbiased bases, which is a set of \(N(N+1)\) state vectors (or one-dimensional projection operators). Can we regard these state vectors as the “points” from which we can construct \(N^2\) “lines” that are the elements of a SIC POVM? Or, if we are given a set of \(N^2\) state vectors that define a SIC POVM, can we regard these state vectors as the “points” of a square array, from which we can construct \(N+1\) sets of “parallel lines” that constitute \(N+1\) mutually unbiased bases?

Numerical evidence makes it reasonable to conjecture that SIC POVM’s exist in every dimension, whereas numerical searches have failed to find a complete set of mutually unbiased bases even in six dimensions. Does this mean that the suggestion of equivalence is misleading and that these two problems are in fact entirely unrelated? Does the geometric analogy, which definitely favors certain values of \(N\), have deep significance for the problem of mutually unbiased measurements while being nothing but a red herring for the problem of SIC POVM’s? Or is it a red herring in both cases? Presumably it is only a matter of time, possibly a short time, before we know the answers to these questions.

All of the above questions are mathematical in nature. But it is also interesting to think about these geometric constructions from the perspective of the foundations of quantum mechanics.

Suppose that, for a system with an \(N\)-dimensional state space, the only
pure states available to the system were the $N^2$ states defined by a SIC POVM. Then we could construct a hidden-variable model along the following lines. Let there be exactly $N(N+1)$ hidden “ontic” states available to the system. These are represented by the $N(N+1)$ points of our finite geometry. The “epistemic” states, which we perceive as pure quantum states, are the $N^2$ lines in the geometry, each consisting of $N+1$ ontic states. When we assign one of these epistemic states to the system, we do so (according to this model) because we do not know its ontic state, and the most we can ever know about its ontic state is that it lies on a particular line. (These ideas are very much in the spirit of the toy models of quantum mechanics recently proposed and analyzed by Spekkens [27] and Hardy [28].) We now make a measurement on the state, supposing that the only measurements available to us are yes-no measurements represented by the one-dimensional projection operators that define our SIC POVM. That is, the system is in some state $|\psi_i\rangle$, represented by a line $\lambda_i$ in our geometry, and we are asking whether it will be found to be in some other state $|\psi_j\rangle$, represented by a different line $\lambda_j$. To compute the probability of “yes”, we simply count how many points of $\lambda_i$ are also in $\lambda_j$. The result is this: of the $N+1$ points in $\lambda_i$, exactly one is in $\lambda_j$; so the probability of “yes” is $1/(N+1)$. This is indeed the correct quantum mechanical probability. The agreement is not surprising and is in fact guaranteed by our construction. By making the traces of operators proportional to the cardinalities of the corresponding sets, we ensured that the geometry would produce the correct probabilities.

Can one use a similar construction to generate quantum mechanics itself rather than a limited model of quantum mechanics? By “similar construction” I mean that there is a set of points representing the underlying (but hidden) ontic states, and that what we call pure quantum states are represented by special subsets of these ontic states. Orthogonal states would be represented by disjoint subsets, a complete orthogonal basis would be represented by a partitioning of the whole set, and probabilities would be computed from the sizes of the intersections of subsets. The answer is no; no such model can reproduce all the probabilities given by quantum mechanics. A model of this sort would be a non-contextual hidden variable theory, and such theories are ruled out by the Kochen-Specker theorem [29].² It is true that our symmetric collection of $N^2$ states in $N$ dimensions can be accom-

²Indeed, the kind of model we are considering is a special case and can be ruled out in other ways as well.
modulated within such a model, but in a typical proof of the Kochen-Specker theorem one identifies some other collection of quantum states for which no such model exists. One of the most elegant proofs of the Kochen-Specker theorem, based on a particularly symmetric set of 33 pure states in three dimensions, is due to Asher Peres [30].

It would be interesting to find out how far one can go towards mimicking quantum mechanics with a theory in which pure states are represented as subsets of some larger set of ontic states. In $N$ dimensions, what is the greatest number of vectors one can find such that the squares of their inner products can be obtained from the sizes of the intersections of the corresponding subsets? If $N$ is a power of a prime, we can find at least $N(N + 1)$ such vectors, namely, the elements of all the mutually unbiased bases. Perhaps we can find more. Regardless of the result, however, we cannot go all the way. There are aspects of quantum mechanics that accord with our classical intuition—the relationships among a special collection of states can serve as an example—but the theory as a whole, and the world to which it applies, are profoundly at odds with the framework of classical physics.

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