The linear complexity of new binary cyclotomic sequences of period $p^n$

Vladimir Edemski

December 12, 2017

Abstract

In this paper, we determine the linear complexity of a class of new binary cyclotomic sequences of period $p^n$ constructed by Z. Xiao et al. (Des. Codes Cryptogr. DOI 10.1007/s10623-017-0408-7) and prove their conjecture about high linear complexity of these sequences.

Keywords: binary sequences, linear complexity, cyclotomy, generalized cyclotomic sequence

Mathematics Subject Classification (2010): 94A55, 94A60

1 Introduction

The linear complexity is an important characteristic of pseudo-random sequence significant for stream ciphers. It is defined as the length of the shortest linear feedback shift register that can generate the sequence [11]. According to the Berlekamp-Massey (B-M) algorithm, it is reasonable to suggest that the linear complexity of a “good” sequence should be at least a half of the period [11].

A generalized cyclotomy with respect to $N = pq$, where $p, q$ are odd primes, was introduced by Whiteman [12] and was extended with respect to odd integers in [3]. Whiteman generalized cyclotomy is not consistent with classical cyclotomy. New generalized cyclotomy which includes classical cyclotomy as a special case, was introduced by Ding and Helleseth in [4]. The unified approach to determine both of Whiteman and Ding-Helleseth generalized cyclotomy was proposed in [8]. Another approach was presented.
in [16], where the order of the generalized cyclotomic classes depends on the choice of parameters (see Section 2).

Using classical cyclotomic classes and generalized cyclotomic classes to construct sequences is a known method for design sequences with a high linear complexity [2]. There are many papers study the linear complexity of the binary and non-binary generalized cyclotomic sequences of period \( p^n \) [1, 6, 7, 10, 13, 14] (see also references therein).

The linear complexity of new binary sequences of period \( p^2 \) for an odd prime \( p \) constructed based on the generalized cyclotomy introduced in [16] was studied in [15]. In conclusion, authors of this paper made a conjecture about the linear complexity of the considered sequences with period \( p^n \). Here, we will try to prove the conjecture from [15].

The remainder of this paper is organized as follows. In Section 2 we recall a definition of new generalized cyclotomic sequences. The linear complexity of these sequences is studied in Section 3.

## 2 Preliminaries

In this section, we first briefly recall the definition of generalized cyclotomic classes with respect to \( p^j \) for any integer \( j \geq 1 \), and definition of new generalized cyclotomic sequences from [15]. Throughout this paper \( \mathbb{Z}_N \) will denote the ring of integers modulo \( N \), where \( N \) is a positive integer. We will also let \( \mathbb{Z}_N^* \) denote the multiplicative group consisting of all invertible elements in \( \mathbb{Z}_N \).

Let \( p \) be an odd prime and \( p = ef + 1 \), where \( e, f \) are positive integers. Denote \( g \) a primitive root modulo \( p^2 \). Then \( g \) is also a primitive root modulo \( p^j \) for each integer \( j \geq 1 \) [9]. It is well known that an order \( g \) by modulo \( p^j \) is equal to \( \varphi(p^j) \), where \( \varphi(\cdot) \) is the Euler’s totient function. Denote \( d_j = \varphi(p^j)/e, j = 1, 2, \ldots, n \) and define

\[
D_0^{(p^j)} = \{ g^{d_j t} \pmod{p^j} : 0 \leq t < e \}, \quad \text{and} \\
D_i^{(p^j)} = g^iD_0^{(p^j)} = \{ g^ix \pmod{p^j} : x \in D_0^{(p^j)} \}, i = 1, 2, \ldots, d_j - 1.
\]

Here and hereafter, we denote by \( x \pmod{N} \) the least nonnegative integer that is congruent to \( x \) modulo \( N \). Cosets \( D_i^{(p^j)} \) are call generalized cyclotomic classes of order \( d_j \) with respect to \( p^j \). Since \( g^{d_j-1(p-1)} \equiv 1 \pmod{p^j} \), it follows
from the definition $D_i^{(p)}$ that
\[ D_i^{(p)} \pmod{p^l} = D_i^{(p)} \pmod{d_i} \text{ for } l : 0 < l \leq j. \] (2)

By [16] we have that $\{D_0^{(p)}, D_1^{(p)}, \ldots, D_{d_j - 1}^{(p)}\}$ forms a partition of $\mathbb{Z}_{p^j}^*$ and for an integer $m \geq 1$
\[ \mathbb{Z}_{p^m} = \bigcup_{j=1}^{m} \bigcup_{i=0}^{d_j - 1} p^{m-j} D_i^{(p)} \cup \{0\}. \]

As in [15] we take $f = 2^r$ with $r \geq 1$ and let $b$ be an integer with $0 \leq b \leq p^{n-1}f - 1$. We now define two family of sets for $m = 1, 2, \ldots, n$
\[ C_0^{(p^m)} = \bigcup_{j=1}^{m} \bigcup_{i=d_j/2}^{d_j-1} p^{m-j} D_i^{(p)} \pmod{d_j} \text{ and } C_1^{(p^m)} = \bigcup_{j=1}^{m} \bigcup_{i=0}^{d_j/2-1} p^{m-j} D_i^{(p)} \pmod{d_j} \cup \{0\}. \] (3)

These are obvious facts that $\mathbb{Z}_{p^m} = C_0^{(p^m)} \cup C_1^{(p^m)}$ and $|C_1^{(p^m)}| = (p^m + 1)/2$. Then we define the generalized cyclotomic binary sequence $s^\infty = (s_0, s_1, s_2, \ldots)$ with period $p^n$ by
\[ s_i = \begin{cases} 0, & \text{if } i \pmod{p^n} \in C_0^{(p^n)}; \\ 1, & \text{if } i \pmod{p^n} \in C_1^{(p^n)}. \end{cases} \] (4)

Here $C_1^{(p^n)}$ is the characteristic set of the sequence $s^\infty$. The linear complexity and other characteristics of these sequences were discussed in [15] for $n = 2$ and authors made a conjecture about the linear complexity of $s^\infty = (s_0, s_1, s_2, \ldots)$ for any $n$ and $p : 2^{(p-1)} \not\equiv 1 \pmod{p^2}$ in conclusion.

Conjecture ([15]). Let $s^\infty$ be a generalized cyclotomic binary sequence of period $p^n$ defined by (4). If $2^{(p-1)} \not\equiv 1 \pmod{p^2}$, then
\[ L = \begin{cases} p^n - \frac{p-1}{2} - \delta \left( \frac{p^{n+1}}{2} \right), & \text{if } 2 \in D_0^{(p)}; \\ p^n - \delta \left( \frac{p^{n+1}}{2} \right), & \text{if } 2 \not\in D_0^{(p)}, \end{cases} \]

where $\delta(t) = 1$ if $t$ is even and $\delta(t) = 0$ if $t$ is odd.

Our goal is a proof of this conjecture. We need some preliminary notation and results before we begin.
3 Linear complexity

It is well known that if \( s^\infty = (s_0, s_1, s_2, \ldots) \) is a sequence of period \( N \), then the linear complexity of this sequence is defined by

\[
L = N - \deg \gcd(x^N - 1, S(x)),
\]

where \( S(x) = s_0 + s_1x + \ldots + s_{N-1}x^{N-1} \) (see, for example [2]). Sometimes, \( S(x) \) is called the generating polynomial of sequence \( s^\infty \). One way to find the linear complexity of a sequence is to examine roots of \( S(x) \). Let \( \beta \) be a primitive \( N \)-th root of unity in the extension of \( \mathbb{F}_2 \) (the finite field of order two). Then latter formula can be rewritten as

\[
L = N - \left| \{ i | S(\beta^i) = 0, \ i = 0, 1, \ldots, N - 1 \} \right|. \tag{5}
\]

In this section, we shall prove the conjecture about the linear complexity of new generalized cyclotomic binary sequences of period \( p^n \) defined in (4). In the following subsections we divide the proof of the conjecture into several lemmas.

3.1 Subsidiary lemmas

In the following two propositions, we present some properties related to \( 2 \pmod{p^j} \), whose validity are easily verified from the definition and basic number theory.

**Lemma 1.** Let \( 2^{(p-1)} \not\equiv 1 \pmod{p^2} \) and let \( c \) be the least positive exponent such that \( 2^c \equiv 1 \pmod{p} \). Then \( 2^k \equiv 1 \pmod{p^{j+1}} \) for \( j \geq 1 \) if and only if \( k \equiv 0 \pmod{cp^j} \).

*Proof.* It is clear that \( c|k \). Let \( k = cdp^j \), where \( \gcd(p, d) = 1 \) and \( l \geq 0 \). By condition of Lemma 1, there exist \( t : 2^c = 1 + pt, \gcd(p, t) = 1 \). Using properties of binomial coefficients we obtain that \( 2^{cd} \equiv 1 + dtp \pmod{p^2} \) and \( 2^{dp^j} \equiv 1 + dtp^{j+1} \pmod{p^{j+2}} \). Hence, if \( 2^k \equiv 1 \pmod{p^{j+1}} \) then \( l \geq j \). To conclude the proof, it remains to note that \( 2^{cp^j} \equiv 1 + cp^{j+1} \pmod{p^{j+2}} \).

**Lemma 2.** Let \( 2^{(p-1)} \not\equiv 1 \pmod{p^2} \) and \( 2 \equiv g^u \pmod{p^2} \). Then \( u \not\equiv 0 \pmod{p} \).

*Proof.* Suppose \( u \equiv 0 \pmod{p} \); then \( g^{u(p-1)} \equiv 1 \pmod{p^2} \) and \( 2^{(p-1)} \equiv 1 \pmod{p^2} \). This contradicts the condition of Lemma 2.

\[ \square \]
According to Lemma 1 we have that $2^j cp F$ and $an order of any nonzero element $K$

Proof. It is well known that if $j$ for $j$ in Lemma 3.
The dimension of the vector space $F$ of $F$ a simple extension of $F$ obtained by adjoining an algebraic element $α _j$.
The dimension of the vector space $F_2(α_j)$ over $F_2$ is then called the degree of $F_2(α_j)$ over $F_2$, in symbols $[F_2(α_j) : F_2]$.

Lemma 3. Let $2^{(p-1)} \neq 1 \pmod{p^2}$. Then

$$[F_2(α_{j+1}) : F_2(α_j)] = p$$

for $j = 1, 2, \ldots, n - 1$.

Proof. It is well known that if $K$ is a finite extension of $F_2$ then $|K| = 2^{[K:F_2]}$ and an order of any nonzero element $K$ divides $|K| - 1$.

As in Lemma 1, let $c$ be a least positive integer such that $2^c \equiv 1 \pmod{p}$. According to Lemma 1 we have that $2^{cp^j} - 1 \equiv 0 \pmod{p^j}$, hence $α_j \in F_{2^{cp^j}}$ and $[F_2(α_j) : F_2] = cp^j$. With similar arguments we get $[F_2(α_{j+1}) : F_2] = cp^{j+1}$. The proposition of this lemma is now established.

\textbf{3.2 Properties of sequence generating polynomial}

Let us introduce the auxiliary polynomials $E_i^{(p^j)}(x) = \sum_{i \in D_i^{(p^j)}} x^i$, $j = 1, 2, \ldots, n; l = 0, \ldots, d_j - 1$. Denote

$$H_k^{(p^j)}(\mod d_j)(x) = \sum_{i=0}^{d_j/2-1} E_{(i+k)}^{(p^j)}(\mod d_j)(x),$$

and

$$T_k^{(p^j)}(\mod d_j)(x) = H_k^{(p^j)}(\mod d_j)(x)+H_k^{(p^j-1)}(\mod d_{j-1})(x^p)+\cdots+H_k^{(p^j)}(\mod d_1)(x^{p^{d_j-1}}),$$

$k = 0, 1, \ldots, d_j - 1$.

Then $H_k^{(p^j)}(\mod d_j)(x) = \sum_{i \in D_i^{(p^j)}} x^i$, $T_k^{(p^j)}(\mod d_j)(x) = \sum_{i \in C_i^{(p^j)}} x^i$ and $S(x) = T_b^{(p^j)}(x) + 1$ by (3) and (1).

In the rest of this paper, the subscripts $i$ in $D_i^{(p^j)}$ and $H_i^{(p^j)}(x)$, $T_i^{(p^j)}(x)$ will be always assumed to be taken modulo the order $d_j$, and the modulo operation will be omitted for simplicity.
According to (6), in order to find the linear complexity of \( s \), it is sufficient to find the values of \( T_b^{(p^n)}(x) \) in the set \( \{ \alpha_n^i, \ i = 0, 1, \ldots, p^n - 1 \} \).

Now we will study the properties of \( T_k^{(p^r)}(x) \).

**Lemma 4.** With the notation in (1), we have \( aD_i^{(p^r)} = D_{i+k}^{(p^r)} \) for \( a \in D_k^{(p^r)} \).

Lemma 4 may be proved similarly as Lemma 1 from [15].

**Lemma 5.** Let \( a \in D_k^{(p^r)} \). Then:

(i) \( E_i^{(p^r)}(\alpha_j^{p^a}) = E_{i+k}^{(p^r)}(\alpha_j^{p^a}) \) for \( 0 \leq l < j \) and \( E_i^{(p^r)}(\alpha_j^{p^a}) = e \) (mod 2) for \( l \geq j \);

(ii) \( H_i^{(p^r)}(\alpha_j^{p^a}) = H_{i+k}^{(p^r)}(\alpha_j^{p^a}) \) for \( 0 \leq l < j \) and \( H_i^{(p^r)}(\alpha_j^{p^a}) = p^{j-1}(p - 1)/2 \) (mod 2) for \( l \geq j \);

(iii) \( T_i^{(p^r)}(\alpha_j^{p^a}) = H_{i+k}^{(p^r)}(\alpha_j^{p^a}) + H_{i+k}^{(p^r-1)}(\alpha_j^{p^a}) + \cdots + H_{i+k}^{(p)}(\alpha_j^{p^a}) + (p^j - 1)/2 \) (mod 2) for \( l < j \);

(iv) \( T_i^{(p^r)}(\alpha_j^{p^a}) = T_{i+k}^{(p^r-1)}(\alpha_j^{p^a}) + (p^j - 1)/2 \) (mod 2) for \( l < j \).

**Proof.** First of all, we prove (i). The assertion in (i) for \( l \geq j \) is clear. Suppose \( l < j \); then by Lemma 1 and definition of \( E_i^{(p^r)} \) we see that

\[
E_i^{(p^r)}(\alpha_j^{p^a}) = E_{i+k}^{(p^r)}(\alpha_j^{p^a}) = \sum_{t \in D_i^{(p^r)}} \alpha_j^{p^t} = \sum_{t \in D_{i+k}^{(p^r)}} \alpha_j^{p^t}.
\]

For conclusion of proof we note that \( D_i^{(p^r)} \) (mod \( p^j - 1 \)) = \( D_{i+k}^{(p^r)} \) by (2).

(ii) By definition and (i) we have that

\[
H_k^{(p^r)}(\alpha_j^{p^a}) = \sum_{i=0}^{d_j/2-1} E_{i+k}^{(p^r)}(\alpha_j^{p^a}) = \sum_{i=0}^{d_j/2-1} E_{i+k}^{(p^r-1)}(\alpha_j^{p^a}) + p^l \sum_{i=0}^{d_j-1/2-1} E_{i+k}^{(p^r)}(\alpha_j^{p^a})
\]

thus second statement follows from (i).

(iii) Using definition of \( T_k^{(p^r)}(x) \) and (ii), we get

\[
T_k^{(p^r)}(\alpha_j^{p^a}) = H_k^{(p^r)}(\alpha_j^{p^a}) + H_k^{(p^r-1)}(\alpha_j^{p^a}) + \cdots + H_k^{(p+1)}(\alpha_j^{p^a}) + H_k^{(p^a)}(1) + \cdots + H_k^{(p)}(1)
\]
or

\[
T_i^{(p^j)}(\alpha_j^{p^i}) = H_i^{(p^{j-l})}(\alpha_{j-l}) + H_i^{(p^{j-2})}(\alpha_{j-l-2}) + \\
\cdots + H_i^{(p)}(\alpha_1) + p^{l-1}(p-1)/2 + p^{l-2}(p-1)/2 + \cdots + (p-1)/2 \pmod{2}.
\]

Thus, the assertion (iii) of this lemma follows from the latter identity.

(iv) Observe that \(T_i^{(p^{j-l})}(\alpha_{j-l}) = H_i^{(p^{j-l-1})}(\alpha_{j-l-1}) + \cdots + H_i^{(p)}(\alpha_1)\) we get (iv) by (ii) and (iii). \(\square\)

Further, since \(\alpha_j\) is the primitive \(p^j\)th root of unity in an extension field of \(\mathbb{F}_2\), we have

\[
0 = \alpha_j^{p^j} - 1 = (\alpha_j - 1)(\alpha_j^{p^{j-1}} + \alpha_j^{p^{j-2}} + \cdots + \alpha_j + 1).
\]

Hence, by (3) and Lemma 5 we get

\[
T_i^{(p^j)}(\alpha_j^a) + T_i^{(p^j)}(\alpha_j^{p^j}) = 1
\]

for any \(a: \gcd(p, a) = 1, j = 1, 2, \ldots, n\).

**Lemma 6.** Let \(2^{(p-1)} \not\equiv 1 \pmod{p^2}\). Then \(T_i^{(p^m)}(\alpha_n^h) \not\in \{0, 1\}\) for all \(h \in \mathbb{Z}_{p^n} \setminus p^{n-1}\mathbb{Z}_p\) and \(k = 0, 1, \ldots, d_n - 1\).

**Proof.** We show that \(T_k^{(p^m)}(\alpha_n^h) \not\in \{0, 1\}\) by contradiction. Suppose there exists \(h = p^{m-m} a\) for \(m > 1, \gcd(p, a) = 1\) such that \(T_k^{(p^m)}(\alpha_n^h) \in \mathbb{F}_2\). Then by Lemma 5 (iv) we get \(T_k^{(p^m)}(\alpha_n^a) \in \mathbb{F}_2\). Without loss of generality by Lemma 5 we can assume that \(k = 0, a = 1\) and \(T_0^{(p^m)}(\alpha_m) = 0\).

Denote \(u = \text{ind}_2 g\) and put \(v = \gcd(u, d_m)\). Then by Lemma 5 we obtain

\[
T_0^{(p^m)}(\alpha_m) = \left( T_0^{(p^m)}(\alpha_m) \right)^2 = T_i^{(p^m)}(\alpha_m). \quad \text{Since there exist integers } y, z: yu + zd_m = v, \text{ it follows that}
\]

\[
0 = T_0^{(p^m)}(\alpha_m) = T_1^{(p^m)}(\alpha_m) = T_2^{(p^m)}(\alpha_m) = \cdots = T_1^{(p^m)}(\alpha_m). \quad (8)
\]

By (7) \(T_0^{(p^m)}(\alpha_m) = 1\), hence \(v\) does not divide \(d_m/2\). Since \(v = \gcd(d_m, u)\) and \(v \not\equiv 0 \pmod{p}\), it follows that \(v = f = 2^r\). With similar arguments as above we get

\[
1 = T_{d_m/2}^{(p^m)}(\alpha_m) = T_{d_m/2+f}^{(p^m)}(\alpha_m) = T_{d_m/2+2f}^{(p^m)}(\alpha_m) = \cdots = T_{d_m/2+(d_m/v-1)f}^{(p^m)}(\alpha_m).
\]
So, \( T_0^{(p^n)}(\alpha_m) + T_{f/2}^{(p^n)}(\alpha_m) = 1 \) and by Lemma 5 (iii) we have that \( H_0^{(p^n)}(\alpha_m) + H_{f/2}^{(p^n)}(\alpha_m) \in \mathbb{F}_2(\alpha_{m-1}). \) Denote \( \gamma = H_0^{(p^n)}(\alpha_m) + H_{f/2}^{(p^n)}(\alpha_m). \) Then \( \gamma \in \mathbb{F}_2(\alpha_{m-1}) \) and

\[
E_0^{(p^n)}(\alpha_m) + \cdots + E_{f/2-1}^{(p^n)}(\alpha_m) + E_{f/p^{m-1}/2}^{(p^n)}(\alpha_m) + \cdots + E_{f/2+f/p^{m-1}/2-1}^{(p^n)}(\alpha_m) = \gamma
\]

(9)

Put \( C = D_0^{(p^n)} + \cdots + D_{f/2-1}^{(p^n)} + D_{f/p^{m-1}/2}^{(p^n)} + \cdots + D_{f/2+f/p^{m-1}/2-1}^{(p^n)} \). Denote \( c = c \pmod{p} \) and define

\[
f(x) = \sum_{c \in C} x^c \alpha_m^{\frac{c}{p^m}} + \gamma.
\]

By (2) we get

\[
C \pmod{p} = D_0^{(p^n)} \pmod{p} + \cdots + D_{f/2-1}^{(p^n)} \pmod{p} + D_{f/p^{m-1}/2}^{(p^n)} \pmod{p} + \cdots + D_{f/2+f/p^{m-1}/2-1}^{(p^n)} \pmod{p} = D_0^{(p)} + \cdots + D_{f/2-1}^{(p)} + D_{f/2}^{(p)} + \cdots + D_{f-1}^{(p)} = \mathbb{Z}_p^*.
\]

Thus \( f(x) = \sum_{i=1}^{p-1} c_i x^i + \gamma \) and \( c_i \in \mathbb{F}_2(\alpha_{m-1}). \) By (9) we obtain that

\[
f(\alpha_m) = \sum_{c \in C} \alpha_m^{\frac{c}{p^m}} \alpha_m^{\frac{c}{p^{m-1}}} + \gamma = \sum_{c \in C} \alpha_m^{\frac{c}{p^m}} \alpha_m^{\frac{c}{p^{m-1}}} + \gamma = 0.
\]

So, \( \alpha_m \) is a root of polynomial \( f(x) \) with coefficients from \( \mathbb{F}_2(\alpha_{m-1}) \) and \( \deg f(x) = p - 1, \) hence \( [\mathbb{F}_2(\alpha_m) : \mathbb{F}_2(\alpha_m)] < p. \) This contradicts Lemma 3 \( \Box \)

### 3.3 Main result

After the preparations in Sect. 3.2, we can now prove the main result of this paper.

**Theorem 7.** Let \( p \) be an odd prime and let \( e \) be a factor of \( p - 1 \) such that \( f = (p-1)/e \) is of the form \( 2^r \) with \( r \geq 1. \) Let \( s^\infty \) be a generalized cyclotomic binary sequence of period \( p^n \) defined by (4). If \( 2^{(p-1)} \neq 1 \pmod{p^2} \), then

\[
L = \begin{cases} 
   p^n - \frac{p-1}{2} - \delta \left( \frac{p^n+1}{2} \right), & \text{if } 2 \in D_0^{(p)}; \\
   p^n - \delta \left( \frac{p^n+1}{2} \right), & \text{if } 2 \notin D_0^{(p)};
\end{cases}
\]

where \( \delta(t) = 1 \) if \( t \) is even and \( \delta(t) = 0 \) if \( t \) is odd.
Proof. By definition of this sequence and (3) \( S(1) = \frac{p^n + 1}{2} \). By Lemmas 5 and 6 we get that \( T_b^{(p^n)}(\alpha_h^1) \notin \{0, 1\} \) for all \( h \in \mathbb{Z}_{p^n} \setminus \mathbb{Z}_{p^n-1} \).

Further, if \( h \in p^n-1\mathbb{Z}_p^* \) then by Lemma 4 we see that \( T_b^{(p^n)}(\alpha_h^1 \cdot a) \in \{0, 1\} \) for \( a \in \mathbb{Z}_p^* \) if and only if \( T_i^{(p)}(\alpha_i^1) \in \{0, 1\} \). The properties of polynomial \( T_i^{(p)}(x) \) were studied in [15]. By [15], if \( T_b^{(p)}(\alpha_i^1) \in \{0, 1\} \) then \( 2 \in D_0^{(p)} \), \( |\{ a : T_b^{(p)}(\alpha_i^1) = 1, a = 1, 2, \ldots, p-1 \} = (p-1)/2 \) and vise versa.

So,

\[
\left| \left\{ i \middle| T_b^{(p^n)}(\alpha_n^i) = 1, \ i \in \mathbb{Z}_{p^n} \right\} \right| = \begin{cases} \frac{p-1}{2} + \delta \left( \frac{p^n+1}{2} \right), & \text{if } 2 \in D_0^{(p)} \smallskip \\ \delta \left( \frac{p^n+1}{2} \right), & \text{if } 2 \notin D_0^{(p)} \end{cases}
\]

where \( \delta(t) = 1 \) if \( t \) is even and \( \delta(t) = 0 \) if \( t \) is odd. This completes the proof of Theorem 7.  

If prime \( p \) satisfying \( 2^{(p-1)} \equiv 1 \pmod{p^2} \) then \( p \) is called a Wieferich prime. By [5] there are only two Wieferich primes, 1093 and 3511, up to \( 6.7 \times 10^{15} \). Therefore, for all other primes \( p < 6.7 \times 10^{15} \) besides 1093 and 3511 the result in Theorem 7 holds.

**Conclusion**

In this paper, the linear complexity of a class of new binary cyclotomic sequences of period \( p^n \) was studied. These almost balanced sequences are constructed by cyclotomic classes using a method presented by Z. Xiao et al. We proved their conjecture about linear complexity of these sequences and show it be high when \( p : 2^{(p-1)} \not\equiv 1 \pmod{p^2} \). In this case the linear complexity of sequence is very close to the period. It is an interesting problem to study other characteristics of these sequences and also the linear complexity for \( f \neq 2^r \).

**References**

[1] Çesmelioglu, A., Meidl, W.: A general approach to construction and determination of the linear complexity of sequences based on cosets, Sequences and Their Applications 6338, 125–138 (2010)
[2] Cusick T.W., Ding C., Renvall A.: Stream Ciphers and Number Theory. Elsevier Science B.V, Amsterdam (1998).

[3] Ding C., Helleseth T.: Generalized cyclotomy codes of length \( p_1^{m_1} p_2^{m_2} \cdots p_t^{m_t} \). IEEE Trans. Inf. Theory 45(2), 467-474 (1999).

[4] Ding C., Helleseth T.: New generalized cyclotomy and its applications. Finite Fields Appl. 4, 140-166 (1998).

[5] Dorais F.G., Klyve D.: A Wieferich prime search up to \( 6.7 \times 10^{15} \). J. Integer Seq. 14(9), (2011). https://cs.uwaterloo.ca/journals/JIS/VOL14/Klyve/klyve3.html

[6] Du X., Chen Z.: A generalization of the Halls sextic residue sequences. Inf. Sci. 222, 784-794 (2013).

[7] Edemskiy, V.: About computation of the linear complexity of generalized cyclotomic sequences with period \( p^{n+1} \), Des. Codes Cryptogr. 61, 251–260 (2011)

[8] Fan, C., Ge, G.: A unified approach to Whiteman’s and Ding-Helleseth’s generalized cyclotomy over residue class rings, IEEE Trans. Inf. Theory 60, 1326–1336 (2014)

[9] Ireland K., Rosen M.: A Classical Introduction to Modern Number Theory, Springer, Berlin (1982)

[10] Kim, Y. J., Song, H. Y. :Linear complexity of prime n-square sequences, IEEE International Symposium on Information Theory, 2405–2408 (2008)

[11] Lidl R., Niederreiter H.: Finite Fields. Addison-Wesley (1983).

[12] Whiteman A.L.: A family of difference sets. Illionis J. Math. 6(1), 107-121 (1962).

[13] Wu C., Chen Z., Du X.: The linear complexity of \( q \)-ary generalized cyclotomic sequences of period \( p^m \), J. Wuhan Univ. 59(2), 129-136 (2013)

[14] Yan T., Li S., Xiao G.: On the linear complexity of generalized cyclotomic sequences with the period \( p^m \), Appl. Math. Lett. 21, 187–193 (2008)
[15] Xiao Z., Zeng X., Li C.: Helleseth T. New generalized cyclotomic binary sequences of period $p^2$. Des. Codes Cryptogr. DOI 10.1007/s10623-017-0408-7

[16] Zeng X., Cai H., Tang X., Yang Y.: Optimal frequency hopping sequences of odd length. IEEE Trans. Inf. Theory 59(5), 3237–3248 (2013).