Cournot Games with Uncertainty: Coalitions, Competition, and Efficiency

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We investigate the impact of group formations on the efficiency of Cournot games where producers face uncertainties. In particular, we study a market model where producers must determine their output before an uncertainty production capacity is realized. In contrast to standard Cournot models, we show that the game is not efficient when there are many small producers. Instead, producers tend to act conservatively to hedge against their risks. We show that in the presence of uncertainty, the game becomes efficient when producers are allowed to take advantage of diversity to form groups of certain sizes. We characterize the trade-off between market power and uncertainty reduction as a function of group size. Namely, we show that when there are $N$ producers present, competition between groups of size $O(\sqrt{N})$ results in equilibria that are socially optimal.

General Terms: Cournot Competition, Uncertainty, Efficiency, Markets

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1. INTRODUCTION

Cournot games are among the most extensively studied models for oligopolistic competition among multiple firms. A Cournot oligopoly is a model where participants compete with each other by controlling the amount of a homogeneous good that they produce. A market price is determined as a function of the total output of the firms. The profit of a firm is then the product of the market price and their output quantity, less any costs incurred. The producers are assumed to act strategically and rationally to maximize their individual profits. This model was studied by Cournot in [Cournot 1838] in the context of a spring water duopoly. For surveys of such models, see, e.g. [Shapiro 1989; Friedman 1977; Daughety 1988].

In this paper, we consider Cournot competition among firms who face production uncertainty. In the model we consider, firms first commit to an expected level of output; subsequently, actual production is realized, drawn from a distribution parameterized by the firm’s ex ante chosen level. Any shortfall from the precommitted level incurs a penalty. Such a model captures production decisions by firms in environments...
where commitments must be made before all relevant factors influencing production are known.

Electricity markets serve as one motivating example of such an environment. In electricity markets, producers submit their bids before the targeted time of delivery (e.g., one day ahead). However, renewable resources such as wind and solar have significant uncertainty (even on a day-ahead timescale). As a result, producers face uncertainties about their actual production capacity at the commitment stage.

Our paper focuses on a fundamental tradeoff revealed in such games. On one hand, in the classical Cournot model, efficiency obtains as the number of individual firms approaches infinity, as this weakens each firm’s market power (ability to influence the market price through their production choice). On the other hand, this result does not obtain when production uncertainty is present: firms protect themselves against the risk of being unable to meet the prior commitment by under-producing relative to the efficient level.

In considering how to recover efficient performance, we are naturally led to think of coalitions of firms. Informally, if firms pool together, they can mitigate individual uncertainty any one of them may perceive in future production (a law of large numbers effect). Of course, coalitions are not without their downside: coalitions possess greater market power than individual firms. Indeed, this concern is substantial, as coalitions must be of fairly substantial size to mitigate the adverse effects of production uncertainty. As a result we are led to a fundamental question: how many coalitions should be allowed to form, and of what size, if the regulator is interested in maximizing overall market efficiency?

We characterize this tradeoff by studying the efficiency of Cournot competition when producers are allowed to form coalitions. Our main contributions are as follows. First, in the model described above, we characterize equilibrium among competing coalitions, as well as the socially optimal benchmark. Second, as a measure of efficiency, we compare the production output of the firms under Cournot competition with socially optimal output. We characterize an optimal scaling regime for coalition structure (in the limit of many firms) under which the efficiency loss can be made arbitrarily small. That is, there exist coalitions (partition of producers) that achieve essentially efficient reduction in uncertainty, but have no appreciable market power. We also characterize the rate at which efficiency loss vanishes, and establish these results when firms may have correlated uncertainty.

Efficiency and welfare loss in Cournot games have been studied extensively in various contexts. Early empirical analysis of welfare loss was performed by [Harberger 1954; Bergson 1973]. Analytically, at the limit where many firms compete, many authors showed that a competitive limit exists [Frank Jr. and Quandt 1963; Ruffin 1971; Haurie and Marcotte 1985; Novshek and Sommerschein 1978]. The quantification of such a limit was considered in [Anderson and Renault 2003] where the marginal costs of the firms are assumed to be constant. The work of [Johari and Tsitsiklis 2005] showed that for N producers with the same cost function competing for a resource with a differentiable demand curve, the efficiency loss is no more than 1/(2N + 1) when the producers are strategic and price anticipatory. The paper by [Tsitsiklis and Xu 2013] derived a more general bound for convex demand curves and [Guo and Yang 2005] studies the how the loss can be estimated in practice. The loss under asymmetric firms was studied by [Corchon 2008].

Most of the preceding literature concludes that full efficiency is achieved when a large number of producers are competing against each other. In this paper, we show that this is not the case when production uncertainty plays a role in the firms’ profits. Profit maximization under supply (or demand) uncertainty falls under the well studied newsvendor problem in the operations literature. However, most of previous work on
this area is concerned with a single retailer [Stevenson 2009]. Oligopolistic competition is studied in [Yao et al. 2008] for additive demand, and in [Adida and Ratisoontorn 2011] for multiplicative demand; a related model with revenue sharing between different firms is discussed in [Dana and Spier 2001]. To our knowledge, none of the previous works in this area consider efficient coalition formation. Another related research area is contract designs (see, e.g., [McAfee and McMillan, McAfee and McMillan]), where designers impose penalties to ensure that firms operate as expected. In this paper, the penalty is from uncertainties that are intrinsic to the problem.

One closely related work to our own is [Yoshii 1993], where the author studies the role of intermediaries between diversification and competition in a large economy. Their results are derived under the assumption of a common randomness affecting all consumers, whereas in our work each producer faces its own randomness (possibly correlated with others). This latter effect is what creates efficiency gains by allowing coalitions to form.

The remainder of the paper is organized as follows. Section 2 presents our basic model of Cournot competition with two stages: production commitments are made in the first stage, and actual available production capacity is realized in the second stage. The market price is determined in the first stage based on production commitments. In the second stage, the producer is charged a penalty if capacity is short of the firm’s commitment level. We then study the same model assuming firms act in coalitions. As the focus of the paper is on the competition between groups, we do not study the nature of revenue sharing contracts within a group; see, e.g., [Telser 1994; Shapley 1967].

In Section 3, we begin by studying the case where firms face i.i.d. production uncertainty. We first show that as the number of firms \( N \) grows, the efficiency loss does not vanish (due to the adverse effects of production uncertainty). We also show that the other extreme, a grand coalition of all producers, is inefficient (due to excessive exercise of market power). We then study coalitional competition: in particular, we characterize the optimal group size and the optimal rate at which the efficiency loss approaches zero. By balancing the adverse effects of market power with the benefits of reduction in production uncertainty, we show that a coalition size of \( \sqrt{N} \) producers (so \( \sqrt{N} \) coalitions compete in the market) is optimal, and the efficiency loss is no more than \( O(1/\sqrt{N}) \). We show that same result hold under two models of correlated production uncertainty in Section 4. Section 5 concludes the paper.

2. COURNOT COMPETITION WITH PRODUCTION UNCERTAINTY

In this section, we define a two-stage game where multiple producers compete to satisfy the demand for a single resource. The main difference between the two-stage model and classical Cournot competition is that the bidding occurs in the first stage, but each producer has an uncertain production capacity at the time of delivery (second stage).

Suppose there are \( N \) firms. The market operates in two stages as shown in Fig. 1. At the first stage, firm \( i \) chooses a committed production quantity \( x_i \) as its bid into the market. Let \( p(y) \) denote the market price when \( y \) units of aggregate output is committed. Let \( X_i \) denote the capacity constraint on firm \( i \)'s production. Note \( X_i \) is a random variable at the first stage, and is realized in the second stage. Throughout the paper, we will assume \( X_i \)'s are continuous, i.e., they follow a distribution with a continuous probability density function. To focus on the effect of production uncertainty, we assume that each firm does not have a cost for producing the resource.¹

¹Our results would remain unchanged if each firm faced the same constant marginal production cost.

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Fig. 1. The two-stage market model. Firms bid in the first stage, and this determines the price of the good. Capacities are realized in the second stage, and penalties are assessed if a firm’s bid is less than its realized capacity.

If the promised amount \( (x_i) \) is larger than the capacity \( (X_i) \) firm \( i \) is penalized by a cost \( q \) per unit shortfall. Thus the cost of shortfall to firm \( i \) is \( q(x_i - X_i)^+ \). Without loss of generality, we let \( q = 1 \) for the remainder of the paper.

The assumption of a penalty linear in the shortfall is relatively common in the literature on newsvendor problems. This penalty allows us to capture the risk associated with a shortfall, i.e., promising more than what can actually be delivered; indeed, the penalty serves to make a firm risk averse in its choice of commitment level. Our results in Section 3 continue to hold even for penalties of the form \( E[ f(x_i - X_i)^+] \), for convex, increasing \( f \) that satisfy some additional smoothness constraints.

The structure of the penalty makes two important assumptions: first, that the penalty depends only on a firm’s own shortfall (i.e., no inter-firm externalities); and that any excess production capacity cannot be resold in a secondary market. In practice, these assumptions may be violated. For example, in electricity markets, a real-time market is run to balance the realized supply and demand, and the study of such markets remain an important future direction for us. Nevertheless, we believe our model captures the first order effects of production uncertainty on firm behavior, and on the role coalitions play in achieving efficient outcomes.

We use the notation \( x_{-i} \) to denote the quantities of chosen by all firms except \( i \), that is, \( x_{-i} = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \). The expected profit for firm \( i \) is

\[
\pi_i(x_i, x_{-i}) = p \left( \sum_{l=1}^{N} x_l \right) x_i - E[(x_i - X_i)^+].
\]  

(1)

When each firm is price anticipatory, given \( x_{-i} \), firm \( i \) chooses \( x_i > 0 \) to maximize \( \pi_i \).

A Nash equilibrium of the game defined by \( (\pi_1, \ldots, \pi_N) \) is a vector \( x \geq 0 \) such that for all \( i \):

\[
\pi_i(x_i, x_{-i}) \geq \pi_i(\tilde{x}_i, x_{-i}), \text{ for all } \tilde{x}_i \geq 0.
\]  

(2)

To analyze the Nash equilibrium for this game, we make the following assumptions on the price function \( p \); this assumption remains in force for the entire paper.

**Assumption 1.** We assume that:

1. \( p \) is strictly decreasing and \( p(0) > 0 \);
2. \( p(y) \) is concave and differentiable on \( y \geq 0 \) with \( p'(0^+) > 0 \);
3. \( p(y) \to -\infty \) as \( y \to \infty \).

These assumptions are common in the literature (e.g., see [Johari and Tsitsiklis 2005]). The first assumption states that the price decreases as quantity increases and \( p(0) > 0 \)

\[\text{for a real number } z, \text{ let } z^+ \text{ denote the positive part of } z, \text{ i.e., } z^+ = z \text{ for } z > 0, \text{ and } 0 \text{ otherwise.}\]
avoids trivial solutions. Concavity of the demand function is largely made for analytical convenience; we conjecture that our key results on scaling of optimal coalition size continue to hold even with weaker assumptions on demand, e.g., logconcavity. The last assumption is also made for analytical simplicity. In general, the price is zero for large enough $y$. This is analytically undesirable since $p$ may not be globally concave, so in the third assumption we allow $p$ to be negative. This assumption is essentially without loss of generality since the regime of interest is always restricted to aggregate production where $p$ is nonnegative (see Proposition 2.1).

We also make the following assumption on the random variable $X_i$; this assumption is also in force for the entire paper.

**ASSUMPTION 2.** For all $i$, $X_i$ is a continuous random variable with finite mean.

It is now straightforward to show that a Nash equilibrium exists for the game $(\pi_1, \ldots, \pi_N)$ as given in the following result.

**Proposition 2.1.** Suppose $p$ satisfies Assumption 1. Then there exists a Nash Equilibrium $x$ for the game defined by $(\pi_1, \ldots, \pi_N)$. Furthermore, $\sum x_i \leq y_{\max}$ where $y_{\max}$ is the unique point where $p(y_{\max}) = 0$.

**Proof of Proposition 2.1.** We first observe that the strategy space of each firm can be restricted to a compact set, without loss of generality. Since $p$ is decreasing, $p(0) > 0$, and tends to $-\infty$, there is an unique zero crossing point $y_{\max}$ such that:

$$p(y_{\max}) = 0.$$  

(3)

For any vector $x_{-i}$ chosen by other firms, firm $i$ is always no worse off by choosing $0$ rather than choosing $y_{\max}$. Therefore, we may restrict the strategy space of a firm to $[0, y_{\max}]$.

Since $p$ satisfies Assumption 1, $p(\sum x_i) x_i$ is concave. By Assumption 2, $E[(x_i - X_i)^+]$ exists and $-E[(x_i - X_i)^+]$ is concave in $x_i$. By additivity of concave functions, $\pi_i$ is concave for all $x_i \geq 0$.

The game defined by $(\pi_1, \ldots, \pi_N)$ with strategy spaces $([0, y_{\max}], \ldots, [0, y_{\max}])$ is now a concave game: each payoff $\pi_i$ is continuous in $x$ and concave in $x$ and the strategy space of firm $i$ is a nonempty compact set. By Rosen’s existence theorem (see [Rosen 1965]), there is a Nash equilibrium for this game.  

2.1. Social Welfare

We are interested in the efficiency of the Nash equilibrium of the game $(\pi_1, \ldots, \pi_N)$. In this section, we formally define the efficiency metric we consider, by studying the social welfare maximization problem.

Given the price function $p$, we can define aggregate consumer surplus in the usual way as:

$$U(y) = \int_0^y p(z)dz.$$  

(4)

Thus from a social planner’s point of view, the optimal allocation is characterized by solving the following problem:

$$\maximize U \left( \sum_{i=1}^N x_i \right) - E \left[ \left( \sum_{i=1}^N x_i - \sum_{i=1}^N X_i \right)^+ \right]$$  

(5a)

subject to $x_i \geq 0$, $\forall i$.  

(5b)
In the above formulation, the social planner controls the aggregate output of all the firms, but still faces the aggregate uncertainty in $X_i$'s. Note that we allow the social planner to use the production capacity of one firm to offset the shortfall of another firm.

Note that the objective function in (5) only depends on $\sum_{i=1}^{N} x_i$. With a change of variables, we can rewrite (5) as:

$$\begin{align*}
\text{maximize} & \quad U(y) - E \left[ \left( y - \sum_{i=1}^{N} X_i \right)^+ \right] \\
\text{subject to} & \quad y \geq 0.
\end{align*}$$

By Assumption 1, $U$ is differentiable and concave, so the optimal solution to (6) is the unique positive solution to

$$p(y) - \Pr \left( y \geq \sum_{i=1}^{N} X_i \right) = 0. \tag{7}$$

Uniqueness follows because, under our assumptions, the left hand side is strictly decreasing in $y$.

Thus we have the following lemma.

**Lemma 2.2.** A vector $x$ is efficient (i.e., solves the optimization problem (5)) if and only if:

$$\sum_{i=1}^{N} x_i = y'_\text{max}, \tag{8}$$

where $y'_\text{max}$ is the unique solution to (7). Further, $y'_\text{max} \leq y_{\text{max}}$.

Therefore at equilibrium, if the aggregate production of firms is $y'_\text{max}$, the equilibrium is socially optimal.

### 2.2. Efficiency Ratio

Based on Lemma 2.2, we define the efficiency of an equilibrium based on the gap between aggregate output and $y'_\text{max}$. Formally, we have the following definition.

**Definition 2.3.** Consider the Cournot game $(\pi_1, \ldots, \pi_N)$. Let $X$ be the set of all Nash equilibria of the game. The efficiency ratio $r$ is:

$$r = \inf_{x \in X} \frac{\sum_{i=1}^{N} x_i}{y'_\text{max}}. \tag{9}$$

The notion of efficiency in Definition 2.3 is slightly different from the “price of anarchy”, which considers the ratio between utilities of Nash equilibria and the socially optimal outcome (e.g., [Johari and Tsitsiklis 2005]). We choose the efficiency measure in Definition 2.3 because we focus largely in this paper on conditions under which full efficiency obtains, and the ratio in (9) is more straightforward to analyze. By Lemma 2.2, an allocation is efficient if and only if $r = 1$, and we will study conditions under which $r$ approaches one asymptotically in the limit of many firms.

It is also worth noting that the aggregate output is often the key quantity of interest in applications. For example, in electricity markets, the social planner is typically interested in making sure an efficient amount of renewable power injected into the mar-

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*It is straightforward to show, using the fact that $p$ is decreasing, that in any equilibrium firms produce less than $y'_\text{max}$. We omit this standard argument.*
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Therefore the ratio of the total output under Nash equilibria and the maximum possible output $y'_{\text{max}}$ is a more direct measure of interest than the ratio of utilities. Finally, we note that since the utility function $U$ is continuous and has bounded derivative, the ratio $r$ can be easily converted to a bound on the ratio utilities. In particular, as $r$ approaches 1, the aggregate utility of Nash equilibria approaches the socially optimal utility. For the remainder of the paper, therefore, we focus on $r$ as our measure of efficiency.

We define $r$ to be the worse case efficiency among all Nash equilibria, which is similar in spirit to the price of anarchy. However, for games we study in the following sections, there are often an unique Nash equilibrium or all equilibria have similar behavior. So there is not a dichotomy in performance between the difference Nash equilibria that is often present when studying the price of anarchy.

2.3. Deterministic Cournot Games

Before moving on to the main result in Sections 3 and 4, we consider a deterministic version of the Cournot game, i.e., one without production uncertainty. Understanding of this deterministic game provides context for our results; further, our proofs use the deterministic setting as a building block.

In the deterministic setting, we ignore the second stage of the game. Therefore the payoff for firm $i$ is:

$$\pi_i(x_i, \mathbf{x}_{-i}) = p \left( \sum_{i=1}^{N} x_i \right) x_i. \quad (10)$$

Compared with (1), note that the cost for shortfall is omitted. For the rest of the paper, we use overlined variables to represent quantities in the deterministic game.

Consider the game defined by $(\pi_1, \ldots, \pi_N)$. By the same reasoning as in Proposition 2.1, a Nash equilibrium exists for this game. Let $X$ denote the set of all Nash equilibria. Since there is no uncertainty, we measure efficiency with respect to $y'_{\text{max}}$ and analogously define the efficiency ratio $\tau$ as

$$\tau = \inf_{x \in X} \frac{\sum_{i=1}^{N} x_i}{y'_{\text{max}}}. \quad (11)$$

The behavior of $\tau$ as $N$ increases is well understood. As noted in Proposition 2.4, $\tau$ approaches 1 and the game becomes efficient in the limit of many firms. As we show in Section 3 this is no longer true if production uncertainty is present.

**Proposition 2.4 (Corollary 18 in [Johari and Tsitsiklis 2005]).**

$$\lim_{N \to \infty} \tau \to 1.$$

2.4. Coalitions

In this section, we define Cournot competition among coalitions of firms. Given $N$ firms, let $S_1, \ldots, S_K$ be a partition of $\{1, \ldots, N\}$. Let $(x_1, \ldots, x_N)$ be a vector of production levels for each firm. The aggregate production commitment of group $S_k$ is denoted as $x(S_k)$:

$$x(S_k) = \sum_{i \in S_k} x_i.$$
Similarly let $X(S_k) = \sum_{i \in S_k} X_i$ denote the aggregate (random) realized capacity of the group $S_k$. The payoff of the group $S_k$ is defined as:

$$\pi_k(x(S_k)) = P \left( \sum_k x(S_k) \right) x(S_k) - E[(x(S_k) - X(S_k))^+]$$.  \hspace{1cm} \text{(12)}$$

Note that, as for the social planner, a coalition benefits by being able to use the excess production of one member to offset the shortfall of another. Thus the penalty incurred by the coalition is the shortfall between their aggregate realized capacity and aggregate production commitment. Note also that we do not consider the internal profit sharing contracts of each coalition; instead, we assume coalitions are able to optimally maximize the aggregate profit of their members.

Given $S_1, \ldots, S_k$, we can define a Cournot game among coalitions through the payoff functions $(\pi_1, \ldots, \pi_k)$. In this game the action for group $S_k$ is the aggregate production commitment $x(S_k)$; as with profit, we do not focus on how this commitment is divided among the individual firms. The game played by coalitions is a “scaled” version of the original game played by $N$ individual firms. The key difference is that the penalty is not linear in the firms. By Jensen’s inequality,

$$E[(x(S_k) - X(S_k))^+] \leq \sum_{i \in S_k} E[(x_i - X_i)^+]$$.  \hspace{1cm} \text{(13)}$$

It is this reduction in risk that makes coalitions useful, as we describe in the subsequent sections.

3. INDEPENDENT FIRMS

In this section, we consider the efficiency of the two-stage Cournot game when the production uncertainty is i.i.d. across firms. Since we are interested in the large $N$ regime, we need to specify how the random variables $(X_1, \ldots, X_N)$ scale as $N$ increases. Recall that $X_i$ models the realized capacity of production. Since we hold the price function constant as we increase $N$, we should reasonably expect that each firm will produce an infinitesimal amount in the limit. If we do not adjust the production capacity accordingly, then each firm will effectively face no production uncertainty in the large $N$ limit.

To ensure the problem scales correctly, we should adjust the production capacity of each firm down with $N$. Formally, we adjust the scale of the production capacity of each firm according to the following assumption.

ASSUMPTION 3. Let $X$ be a continuous random variable with $E[|X|^3] < \infty$. Let $E[X] = \mu$ and we assume $\mu > y_{\text{max}}$. Suppose there are $N$ firms. The random variables $X_1, \ldots, X_N$ are drawn i.i.d. according to the distribution of $X/N$.

Under the above assumption, the expected total capacity is fixed at $\mu$ and is divided evenly among the $N$ firms. The technical assumption of a bounded third moment avoids random variables with heavy tails, and is largely made for analytical convenience. The assumption $\mu > y_{\text{max}}$ streamlines the proofs, but is not essential.

In this section we are primarily interested in regime where $N \rightarrow \infty$. By Assumption 3 and the law of large numbers, $\sum_1^N X_i \rightarrow \mu$ almost surely. For large $N$, observe that since $\mu > y_{\text{max}}$, the social planner faces no production uncertainty; and therefore the second term in (7) goes to zero, so that $y'_{\text{max}} \rightarrow y_{\text{max}}$ as $N \rightarrow \infty$. For this reason, for the duration of this section, we replace $y'_{\text{max}}$ with $y_{\text{max}}$ in our analysis of the efficiency ratio $r$.

Under Assumption 3 all firms are \textit{ex ante} identical: they have the same profit and face the same production uncertainty. In this section we consider coalitional competi-
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The firms are divided evenly into $K$ groups; since we are interested in large $N$ scenarios, we assume, without loss of generality, $N$ is a multiple of $K$. The two extreme values of $K$ are $K = 1$ and $K = N$: the former corresponds to a grand coalition, while the latter corresponds to competition among individual firms.

The next theorem is the main result of this section, which relates the efficiency ratio to the group size $K$.

**Theorem 3.1.** Suppose there are $N$ firms and $X_1, \ldots, X_N$ satisfies Assumption 3. Let $(S_1, \ldots, S_K)$ be $K$ groups where each group has $N/K$ firms. Let $(x(S_1), \ldots, x(S_K))$ be the solution to the game $(\pi_1, \ldots, \pi_K)$. Then the efficiency ratio scales as:

$$ r = \frac{\sum_{k=1}^{K} x(S_k)}{y_{\text{max}}} = 1 - O\left(\frac{1}{K}\right) - O\left(\frac{K}{N}\right). \tag{14} $$

Before we prove Theorem 3.1, we briefly discuss the result. The last two terms in (14) can be interpreted as the effects of market power and production uncertainty, respectively. Namely, the inefficiency due to market power scales as $1/K$, and it decreases as $K$ grows. On the other hand, the inefficiency due to production uncertainty scales as $K/N$, which decreases as $N/K$ (the number of members in each coalition) grows. Note from (14) that $r$ will approach 1 as long as $K$ and $N/K$ both grow without bound. The maximum rate at which $r$ approaches 1 is found when these two effects are balanced, or when $1/K = K/N$. Therefore, the “optimal” coalition size for market efficiency is $\sqrt{N}$ groups, each of size $\sqrt{N}$.

**Proof of Theorem 3.1**. The strategy of the proof proceeds in two steps. First we consider a deterministic game with $K$ players, and bound the difference between the Nash equilibrium of the game $(\pi_1, \ldots, \pi_K)$ and the Nash equilibrium of the deterministic game. Then we bound the difference between the latter and $y_{\text{max}}$.

Let $(S_1, \ldots, S_K)$ be an equal sized partition of $(1, \ldots, N)$, with each coalition of size $N/K$. Consider a deterministic game defined by $(\pi_1, \ldots, \pi_K)$ where

$$ \pi_k = p\left(\frac{1}{K}\sum_{m=1}^{K} \pi(S_m)\right) \pi(S_k). \tag{15} $$

Let $(\pi(S_1), \ldots, \pi(S_K))$ be a Nash equilibrium of this game. By symmetry, $\pi(S_k)$ are of the same value. We denote this value by $\pi_K$; it satisfies:

$$ p(K\pi_K) + p'(K\pi_K)\pi_K = 0. \tag{16} $$

Let $x(S_1), \ldots, x(S_K))$ be a Nash equilibrium of the game $(\pi_1, \ldots, \pi_K)$. Again, this equilibrium is symmetric. Denote the common production level of each coalition by $x_K$. Similarly, denote $X(S_k)$ by $X_K$. Since $E[(x_K - X_K)^+]$ is increasing in $x_K$, we have $x_K \leq \pi_K$. Therefore we can rewrite $x_K$ as $\pi_K - \Delta$, for some $\Delta \geq 0$ that solves:

$$ p(K(\pi_K - \Delta)) + p'(K(\pi_K - \Delta))(\pi_K - \Delta) - Pr(X_K \leq \pi_K - \Delta) = 0. \tag{17} $$

Subtracting (16) from (17), we have

$$ [p(K\pi_K - K\Delta) - p(K\pi_K)] + [p'(K\pi_K - K\Delta)(\pi_K - \Delta) - p'(K\pi_K)\pi_K] - Pr(X_K \leq \pi_K - \Delta) = 0. \tag{18} $$

Since $p$ is concave and decreasing, $p'(K\pi_K) < p'(K\pi_K - K\Delta) < 0$. Also since $\pi_K$ is positive, the term in the second set of brackets, $p'(K\pi_K - K\Delta)(\pi_K - \Delta) - p'(K\pi_K)\pi_K$ is greater than or equal to zero. Also since $\Delta$ is positive, $Pr(X_K \leq \pi_K - \Delta) < Pr(X_K \leq \pi_K)$. Therefore:

$$ p(K\pi_K - K\Delta) - p(K\pi_K) - Pr(X_K \leq \pi_K) \leq 0. \tag{19} $$

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Since $p$ is concave and decreasing,

$$p(K\bar{X}_K - K\Delta) - p(K\bar{x}_K) \geq K\Delta(-p'(0)). \tag{20}$$

Combining (19) and (20), we obtain a bound on $K\Delta$, or

$$K\Delta \leq \Pr(X_K \leq \bar{x}_K) \leq \Pr(X_K \leq y_{\max}/K). \tag{21}$$

By assumption, $\mu > y_{\max}$; applying Chebyshev’s inequality gives $Pr(X_K \leq y_{\max}/K) = O(K/N)$. Therefore $K\Delta = O(K/N)$.

Now we bound the gap between $K\bar{X}_K$ and $y_{\max}$. Let $\delta = y_{\max} - K\bar{x}_K$. Substituting into (16) gives:

$$p(y_{\max} - \delta) + p'(y_{\max} - \delta) \left(\frac{y_{\max} - \delta}{K}\right) = 0. \tag{22}$$

Since $p$ is decreasing and $\delta$ is positive,

$$p(y_{\max} - \delta) + p'(y_{\max} - \delta) \left(\frac{y_{\max} - \delta}{K}\right) \leq 0. \tag{23}$$

Rearranging yields

$$p(y_{\max} - \delta) \leq -p'(y_{\max} - \delta) \left(\frac{y_{\max} - \delta}{K}\right) \leq -p'(y_{\max}) \left(\frac{y_{\max}}{K}\right), \tag{24}$$

where $(*)$ follows from the fact $p'$ is negative and decreasing ($p$ is decreasing and concave). Since $p$ is concave and decreasing, and $p(y_{\max}) = 0$,

$$p(y_{\max} - \delta) \geq \delta \frac{p(0)}{y_{\max}}. \tag{25}$$

See Figure 2 for an illustration of (25). Combining (24) and (25) gives

$$\delta \frac{p(0)}{y_{\max}} \leq -p'(y_{\max})\frac{y_{\max}}{K}, \tag{26}$$

or

$$\delta \leq \frac{1}{K} - p'(y_{\max})\frac{y_{\max}^2}{p(0)} \tag{27}.$$

Therefore $\delta = O(1/K)$.

Combining the two parts of the proof,

$$y_{\max} - \sum_k x(S_k) = O\left(\frac{1}{K}\right) + O\left(\frac{K}{N}\right). \tag{28}$$

Dividing both sides by $y_{\max}$ gives the desired result:

$$r = 1 - O\left(\frac{1}{K}\right) - O\left(\frac{K}{N}\right), \tag{29}$$

\[\Box\]

The result in Theorem 3.1 can be extended to a more shortfall penalty, as in the following corollary.
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\[ \pi_S(x_S) = p \left( \sum_{i=1}^{N} x_i \right) \left( \sum_{k \in S} x_k \right) - E[f(x_S - X_S)]. \]  

(30)

Under the same conditions as Theorem 3.1, the efficiency ratio still scales as

\[ r = 1 - O \left( \frac{1}{K} \right) - O \left( \frac{K}{N} \right). \]  

(31)

COROLLARY 3.2. Let \( f \) be a convex increasing function with bounded derivative, satisfying \( f(x) = 0 \) for all \( x \leq 0 \). For a group \( S \), let its total profit be given by:

\[ \pi_S(x_S) = p \left( \sum_{i=1}^{N} x_i \right) \left( \sum_{k \in S} x_k \right) - E[f(x_S - X_S)]. \]  

(30)

PROOF OF COROLLARY 3.2 Let \((S_1, \ldots, S_K)\) be a equal sized partition of \((1, \ldots, N)\) (each of size \(N/K\)). Consider the deterministic game \((\pi_1, \ldots, \pi_K)\) played by the groups. Let \((\pi(S_1), \ldots, \pi(S_K))\) be a Nash equilibrium of this game. Let \((x(S_1), \ldots, x(S_K))\) be a Nash equilibrium of the game \((\pi_1, \ldots, \pi_K)\). Since \(E[f(x_K - X_K)]\) is increasing in \(x(S_k)\), \(x_K \leq \pi_K\), we can rewrite \(x_K\) as \(\pi_K - \Delta\) with \(\Delta \geq 0\). Following the proof of Theorem 3.1 we obtain (in place of (21)):

\[ K\Delta \leq \frac{E[f'(x_K - X_K)]}{-p'(0)} - p(0). \]  

(32)

Since \(f'\) is bounded, \(E[f'(x_K - X_K)] \leq B Pr(x_K - X_K \geq 0)\) for some \(B\). Therefore \(\Delta\) scales as \(O(K/N)\). The rest of the proof proceeds exactly as for Theorem 3.1.

It is worthwhile to note that the scaling rates in Theorem 3.1 and Corollary 3.2 represent the asymptotic behavior of firms. Constants of the terms in the scaling rate are determined by the particular distributions of production uncertainty, and the price function. We illustrate the behavior of the efficiency ratio at finite \(N\) with the following two examples.

Example 3.3. Let \(p(y) = 1 - y\). Let \(X\) be normally distributed as \(N(1,1,1)\). Note that \(y_{\max} = 1 < 1.1\). Let \(X_i\) be drawn i.i.d. according to the distribution of \(X/N\). Figure 2 plots the efficiency ratio for two groups sizes: \(\sqrt{N}\) and \(N^{2/3}\). Theorem 3.1 shows that...
ratio for groups of size $\sqrt{N}$ should increase faster than the ratio for groups of size $N^{2/3}$ at large $N$. This observation is validated by Figure 3 but we also see that there is a switch over point between the three ratios.

![Figure 3](image_url)

**Fig. 3.** Efficiency ratio for groups of size $\sqrt{N}$ and $N^{2/3}$.

Some more information can be gained by plotting Figure 3 on semi-log axes. In Figure 4 the x-axis plots the log of the number of firms. We see that there are two regimes. In the first regime (small $N$), uncertainty averaging dominates the rate. Therefore the efficiency for groups size of $N^{2/3}$ grows faster since each group is larger compared to a group size of $\sqrt{N}$. In the second regime, competition is more important, and the efficiency ratio for groups with size $\sqrt{N}$ grows faster.

![Figure 4](image_url)

**Fig. 4.** Semi-log plot of the efficiency ratio for groups of size $\sqrt{N}$ and $N^{2/3}$. In regime 1, uncertainty averaging dominates, and the efficiency ratio for groups of size $N^{2/3}$ grows faster. In regime 2, competition starts to dominate, and the efficiency ratio for groups of size $\sqrt{N}$ ultimately overtakes the ratio for groups of size $N^{2/3}$.

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Example 3.4. In this example we show that the regime switching phenomenon in Figure 4 may not occur. Here we consider a uniform random variable and the effects of competition and averaging are not as distinct as in Example 1.

Again let \( p(y) = 1 - y \). Let \( X \) be a uniform random variable distributed as \( Unif[0, 2] \), with mean \( \mu = 1.1 \). Let \( X_i \) be drawn i.i.d. according to \( X/N \). Figure 5 plots the efficiency ratio for groups of sizes \( \sqrt{N} \) and \( N^{2/3} \). Unlike in Figure 3, the efficiency ratio is always higher for groups of size \( \sqrt{N} \). Also, a regime change is not present in the semi-log plot in Figure 6.

![Graph of efficiency ratio for groups of size \( \sqrt{N} \) and \( N^{2/3} \).](image1)

**Fig. 5.** Efficiency ratio for groups of size \( \sqrt{N} \) and \( N^{2/3} \).

![Semi-log plot of the efficiency ratio for groups of size \( \sqrt{N} \) and \( N^{2/3} \).](image2)

**Fig. 6.** Semi-log plot of the efficiency ratio for groups of size \( \sqrt{N} \) and \( N^{2/3} \). In contrast to Figure 4 there are not regime changes.
4. CORRELATED FIRMS

In this section, we consider two models where firms have correlated production uncertainty.

4.1. Weakly Correlated Firms

The $O(K/N)$ term in (14) results from the law of large numbers. The following corollary recovers the same result, using a version of the law of large numbers for correlated random variables.

**Corollary 4.1.** Let $X$ be a continuous random variable with $E[|X|^3] < \infty$. Let $E[X] = \mu > y_{\text{max}}$. Suppose there are $N$ firms. Assume the random variables $X_1, \ldots, X_N$ each have marginal distribution that is the same as $X/N$. Let $(S_1, \ldots, S_K)$ be $K$ groups where each group has $N/K$ firms. Let $(x(S_1), \ldots, x(S_K))$ be the solution to the game $(\pi_1, \ldots, \pi_K)$.

If

$$\sum_{j=1}^{N} \left| E\left[ \left( X_i - \frac{\mu}{N} \right) \left( X_j - \frac{\mu}{N} \right) \right] \right| \leq \frac{c}{N} \forall i$$

for some $c$ independent of $N$, then the efficiency ratio scales as in (14).

An example of correlated $X_i$'s satisfying the above condition is where $\text{Cov}(X_i, X_j) \leq A|\rho|^{i-j}$, for some finite $A$ and $\rho < 1$. This type of model captures a Hotelling-like geographic structure, where firms with similar indices are more likely to face the same production constraints. It is particularly relevant in electricity markets, where renewable generators located near each other exhibit this behavior.

**Proof of Corollary 4.1.** Consider a group $S_k$. It suffices to show that $\Pr(\sum_{i \in S_k} X_i \leq y_{\text{max}}/K)$ is $O(K/N)$ since the rest of the proof proceeds exactly as that of Theorem 3.1.

Equation (33) implies that

$$\Pr\left( \sum_{i \in S_k} X_i \leq y_{\text{max}}/K \right) \leq \frac{c}{(\mu - y_{\text{max}})^2} \frac{K}{N} = O\left( \frac{K}{N} \right).$$

(See, e.g., [DeGroot and Schervish 2010], for this standard result.)

4.2. Strongly Correlated Firms

Earlier, we considered firms with weakly correlated realized production capacity, in the sense that a firm only have nonzero correlation with a finite number of other firms as $N$ grows. In this section, we consider the case of strongly correlated realized production capacities, where the correlation between all firms remains positive as $N$ grows.

When the firms have correlated capacities, results similar to Theorem 8.1 are difficult to obtain in general since the limiting distribution of $\sum X_i$ does not necessarily concentrate; any such result will depend on the joint distribution of the $X_i$'s. For this section, we assume that the correlation between random variables arises from an additive model, as described in the following assumption.

**Assumption 4.** Let $X$ be a continuous random variable with $E[|X|^3] < \infty$. The random variables $\hat{X}_1, \ldots, \hat{X}_N$ are drawn i.i.d. according to the same distribution as $X/N$. Let $Z$ be a continuous random variable with zero mean, finite variance, bounded...
density function, and independent of $\hat{X}_1, \ldots, \hat{X}_N$. The random variable $X_i$ is given by:

$$X_i = \hat{X}_i + \frac{Z}{N} \text{ for all } i.$$  \hfill (35)

Before discussing the efficiency of the Cournot game under correlated firms, we need to modify the social planner’s problem. Since $X_i$’s are strongly correlated, $\sum_i X_i$ no longer concentrates around its mean. Therefore, even for a social planner, there is some leftover uncertainty in the system, and the social planner pays a cost for any shortfall from the realized production capacity to the committed production level. In the limit of large $N$, therefore, we can view the social planner’s optimization problem as the following:

$$\max_{y \geq 0} U(y) - \mathbb{E}[(y - Z)^+] - \mu.$$  \hfill (36)

The second term in the objective function is motivated by the fact $\sum_i X_i \to \mu + Z$ as $N \to \infty$. As before, let $y'_{\max}$ be the unique solution to (36). Because of the additional uncertainty, $y'_{\max} \leq y_{\max}$.

Again we assume that $N$ firms are divided evenly into $K$ groups. Let $(x(S_1), \ldots, x(S_K))$ be the solution to the two-stage game $(\pi_1, \ldots, \pi_K)$. Define the efficiency ratio $r$ as $\sum_k x(S_k)/y'_{\max}$. Strictly speaking, the mathematical rigorous limiting process should first calculate the ratio between the Nash equilibria and the social optimal outcome for a fixed $N$, then take $N$ to infinity. However, in our case, both quantities are real, nonzero, and positive numbers. Therefore the limit of the ratio equal to the ratio of the limits and we focus on measuring the Nash equilibria to $y'_{\max}$.

Theorem 4.2 states that $r$ has the same large $N$ asymptotic behavior as in Theorem 3.1.

**Theorem 4.2.** Let $\hat{X}_1, \ldots, \hat{X}_N$ and $Z$ be random variables that satisfy Assumption 4. Let $(S_1, \ldots, S_K)$ be a partition of $(1, \ldots, N)$ with size $N/K$ each. Let $(x(S_1), \ldots, x(S_K))$ be the solution to the two-stage game $(\pi_1, \ldots, \pi_K)$. Suppose $\mu > y'_{\max}$. The efficiency ratio behaves as:

$$r = \frac{\sum_k x(S_k)}{y'_{\max}} = 1 - O\left(\frac{1}{K}\right) - O\left(\frac{K}{N}\right).$$  \hfill (37)

The proof can be found in the Appendix.

4.3. Simulation Results

Here we plot the efficiency ratio for correlated firms and compare it to independent firms. Similar to Example 3.3 let $p(y) = 1 - y$. Let $\hat{X}$ be normally distributed as $\mathcal{N}(1.1, 0.7)$ and let $Z$ be normally distributed as $\mathcal{N}(0, 0.71)$; note that with this definition, the variance of $\hat{X} + Z$ is 1. Let $\hat{X}_i$ be drawn i.i.d. according to the same distribution as $\hat{X}/N$. Figure 7 shows the efficiency ratio for groups of size $\sqrt{N}$ on a semi-log plot. As a baseline, we also plot the efficiency ratio where the random variables are drawn i.i.d. with normal distribution $\mathcal{N}(1.1, 1)$.

From Figure 7 we can see that the efficiency ratio approaches 1 much faster if the firms are correlated. This is not unexpected, since production uncertainty is dominated by the common random variable $Z$, and the individual randomness can be averaged out easier. Another way to interpret this result is that since the production uncertainty also reduces the social planner’s optimal welfare, the efficiency ratio is relatively higher.
5. CONCLUSION

In this paper we investigated strategic behavior of firms and coalitions in a Cournot game with production uncertainty. We study the efficiency of Cournot competition by characterizing a fundamental tradeoff: on one hand, market power increases as coalition size grows; on the other hand, the cost of production uncertainty is mitigated as coalition size grows. We show there is a “sweet spot”, in the sense that there exist groups that are large enough to achieve the uncertainty reduction of the grand coalition, but small enough such that they have no significant market power. Namely, when there are $N$ firms, competition between groups of size $O(\sqrt{N})$ results in equilibria that are socially optimal.

These results have important implications for regulators in industries with production uncertainty, such as electricity markets. In particular, our results suggest that within some limits, coalition formation among, e.g., generators of renewables may actually increase overall welfare. We have validated these results in electricity markets in [Zhang et al. 2015], where we empirically study the welfare benefits of coalitions of (a finite number of) wind power generators.

We conclude by noting two important open directions. First, as previously noted, in many real markets (including electricity markets), the penalty for a production shortfall is not exogenous. Rather, a firm may face a “spot” or secondary market, into which it can sell excess capacity, or from which it must buy additional capacity to cover a shortfall. Modeling this two-stage market game remains an important challenge. Second, all our results are asymptotic, though we do characterize the rate of convergence to efficiency with optimal coalition sizes. With finitely many (potentially heterogeneous) firms, the regulator faces the potentially challenging problem of computing optimal coalitions as a benchmark. Developing approaches for this problem remains an open issue.

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APPENDIX

PROOF OF THEOREM 4.2. This proof follows a similar path to that of Theorem 3.1.
First, we define the following intermediate game. Define the payoff function:

\[
\pi_k = p \left( \sum_k x(S_k) \right) x(S_k) - E[(x(S_k) - \frac{1}{K}(Z - \mu))^+].
\] (38)

Let \((\pi(S_1), \ldots, \pi(S_K))\) be a Nash equilibrium of the game defined by \((\pi_1, \ldots, \pi_K)\) (given in (38)). By symmetry, \(\pi(S_k)\) are the same for all \(k\). We denote this value by \(\pi_K\); it satisfies:

\[
p(K\pi_K) + p'(K\pi_K)\pi_K - Pr(\pi_K \geq \frac{1}{K}(Z + \mu)) = 0.
\] (39)

Let \((x(S_1), \ldots, x(S_K))\) be a Nash equilibrium of the game \((\pi_1, \ldots, \pi_K)\). Again by symmetry, all coalitions choose the same production level. Denote this value by \(x_K\). Similarly, denote \(X(S_k)\) as \(X_K\). Since \(E[(x_K - X_K)^+]\) is increasing in \(x(S_k), x_K \leq \pi_K\). Therefore we can rewrite \(x_K\) as \(x_K - \Delta\), for some \(\Delta \geq 0\) that solves:

\[
p(K(\pi_K - \Delta)) + p'(K(\pi_K - \Delta))(\pi_K - \Delta) - Pr(\pi_K - \Delta \geq X_K) = 0.
\] (40)

By definition of \(X_K\), (40) can be written as:

\[
p(K(\pi_K - \Delta)) + p'(K(\pi_K - \Delta))(\pi_K - \Delta) - Pr(\pi_K - \Delta \geq X_K - \Delta) = 0.
\] (41)

Subtracting (39) from (41) and following the steps of the proof in Theorem 3.1, we have

\[
K\Delta(-p'(0)) \leq Pr \left( \pi_K - \Delta \geq \frac{1}{K}Z + \sum_{i=1}^{N/K} \hat{X}_i \right) - Pr \left( \pi_K \geq \frac{1}{K}(Z + \mu) \right).
\]

\[
= Pr \left( K\pi_K - K\Delta \geq Z + K \sum_{i=1}^{N/K} \hat{X}_i \right) - Pr(K\pi_K \geq Z + \mu).
\]

Since \(\Delta \geq 0\),

\[
K\Delta(-p'(0)) \leq Pr \left( K\pi_K \geq Z + K \sum_{i=1}^{N/K} \hat{X}_i \right) - Pr(K\pi_K \geq Z + \mu).
\] (42)

It is convenient to associate the mean \(\mu\) with \(Z\) rather than \(\hat{X}_i\). Define \(Z' = Z + \mu\) and \(\hat{X'}_K = K \sum_{i=1}^{N/K} \hat{X}_i\). With this change of variables, we need to bound

\[
Pr(K\pi_K \geq Z' + \hat{X'}_K) - Pr(K\pi_K \geq Z').
\]
By conditional probability and the independence of $Z'$ and $\hat{X}_K$,

$$\Pr(K\tau_K \geq Z' + \hat{X}_K) = \Pr(K\tau_K \geq Z') = \Pr(K\tau_K \geq Z')$$

$$= \Pr(K\tau_K \geq Z' + \hat{X}_K | \hat{X}_K \leq 0) \Pr(\hat{X}_K \leq 0) + \Pr(K\tau_K \geq Z' + \hat{X}_K | \hat{X}_K > 0) \Pr(\hat{X}_K > 0)$$

$$= \{\Pr(K\tau_K \geq Z') + \Pr(K\tau_K < Z', K\tau_K \geq Z' + \hat{X}_K | \hat{X}_K \leq 0)\} \Pr(\hat{X}_K \leq 0)$$

$$+ \Pr(K\tau_K \geq Z' + \hat{X}_K | \hat{X}_K > 0) \Pr(\hat{X}_K > 0) - \Pr(K\tau_K \geq Z')$$

$$\leq \{\Pr(K\tau_K \geq Z') + \Pr(K\tau_K < Z', K\tau_K \geq Z' + \hat{X}_K | \hat{X}_K \leq 0)\} \Pr(\hat{X}_K \leq 0)$$

$$+ \Pr(K\tau_K \geq Z' + \hat{X}_K | \hat{X}_K > 0) \Pr(\hat{X}_K > 0) - \Pr(K\tau_K \geq Z')$$

$$= \Pr(K\tau_K < Z', K\tau_K \geq Z' + \hat{X}_K | \hat{X}_K \leq 0) \Pr(\hat{X}_K \leq 0)$$

$$\leq \Pr(K\tau_K < Z', K\tau_K \geq Z' + \hat{X}_K | \hat{X}_K \leq 0) \Pr(\hat{X}_K \leq 0) + \Pr(K\tau_K < Z', K\tau_K \geq Z' + \hat{X}_K | \hat{X}_K \leq 0) \Pr(\hat{X}_K \leq 0)$$

$$= \Pr(K\tau_K < Z', K\tau_K \geq Z' + \hat{X}_K).$$

By assumption $Z'$ is a continuous random variable with a bounded density function, and we denote its density by $f_{Z'}$. Let $f_{Z,\max}$ denote the maximum value of $f_{Z'}$. Also, denote the density of $X_k$ by $f_{X_k}$. Then

$$\Pr(K\tau_K < Z', K\tau_K \geq Z' + \hat{X}_K) = \int_{-\infty}^{0} \int_{K\tau_K}^{\infty} f_{Z'}(z)f_{X_k}(x)dz dx$$

$$\leq f_{Z,\max} \int_{-\infty}^{0} (-x)f_{X_k} dx$$

$$\leq f_{Z,\max} \int_{0}^{\infty} x f_{X_k} dx$$

$$\leq f_{Z,\max} \cdot \text{const} \cdot \text{var}(\hat{X}_k),$$

where (a) follows from boundedness of $f_{Z'}$, (b) follows from the symmetry in $\hat{X}'$ and (c) follows from the fact that $\hat{X}_K$ has bounded variance. Therefore $\sum \tau_K - \sum x_k = O(\frac{1}{N}).$

Now we bound the difference between $\sum \tau_K$ and $y_{\max}'. By definition $y_{\max}'$ solves

$$p(y_{\max}') - \Pr(y_{\max}' - Z - \mu > 0) = 0. \tag{43}$$

Write $K\tau_K$ as $y_{\max}' - \delta$ for some $\delta > 0$, and subtracting (43) from (39) gives

$$\{p(y_{\max}' - \delta) - p(y_{\max}')\} + p'(y_{\max}' - \delta) \frac{y_{\max}' - \delta}{\delta} \{\Pr(y_{\max}' - \delta - Z - \mu > 0) - \Pr(y_{\max}' - Z - \mu > 0)\} = 0. \tag{44}$$

Since $p$ is decreasing and concave, and $\delta > 0$, we have $p'(y_{\max}' - \delta)(\delta) > 0$ and $p'(y_{\max}' - \delta) \leq 0$, and $\{\Pr(y_{\max}' - \delta - Z - \mu > 0) - \Pr(y_{\max}' - Z - \mu > 0)\} < 0$. Therefore

$$p(y_{\max}' - \delta) - p(y_{\max}') \leq -p'(y_{\max}') \frac{y_{\max}'}{K\delta}.$$ 

Since $p$ is concave and decreasing, $p(y_{\max}' - \delta) - p(y_{\max}') \geq \frac{\delta}{y_{\max}'} (p(0) - p(y_{\max}'))$ (see Figure 2). Thus:

$$\delta \leq -p'(y_{\max}') \frac{y_{\max}^2}{K(p(0) - p(y_{\max}')).}$$
Combining with the first half of the proof gives the desired result. □