Distinctive Features of Oscillatory Phenomena in Reconstructions of the Topological Structure of Electron Trajectories on Complex Fermi Surfaces

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Abstract—We consider the behavior of classical and quantum oscillations in metals with complex Fermi surfaces near the directions of \(B\) corresponding to changes in the topological structure of the dynamical system describing the semiclassical motion of quasiparticles along the Fermi surface. The transitions through the boundaries of change in this structure are accompanied by sharp changes in the picture of oscillations, the form of which depends in the most essential way on the topological type of the corresponding reconstruction. We list here the main features of such changes for all topological types of elementary reconstructions and discuss the possibilities of experimental identification of such types based on these features.

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1. INTRODUCTION

In this work, we would like to consider features of oscillatory phenomena observed in reconstructions of the topological structure of the system describing the semiclassical motion of electrons on the Fermi surface in the presence of an external magnetic field. As is well known, this system has the form

\[
\mathbf{p} = \frac{\epsilon}{c} (\mathbf{v}_\mathbf{p} \times \mathbf{B}) = \frac{\epsilon}{c} (\nabla \epsilon \mathbf{p} \times \mathbf{B}),
\]

where \(\epsilon(p)\) is the electronic dispersion relation in the crystal for a given conduction band. The relation \(\epsilon(p)\) represents a smooth 3-periodic function in the \(p\)-space with periods equal to the reciprocal lattice vectors. As it is easy to see, the system (1.1) conserves the energy value \(\epsilon(p)\) and the projection of the quasimomentum on the direction of the magnetic field, and, as a consequence, its trajectories are given by the intersections of periodic surfaces \(\epsilon(p) = \text{const}\) by planes, orthogonal to \(\mathbf{B}\) (Fig. 1).

From the physical point of view, points in \(p\)-space, which differ by reciprocal lattice vectors, represent the same physical state, so that the system (1.1) can be considered, in fact, as a system on the three-dimensional torus \(\mathbb{T}^3\) obtained from \(\mathbb{R}^3\) by factorization over reciprocal lattice vectors. The periodic surfaces \(\epsilon(p) = \text{const}\) after such factorization become also compact two-dimensional surfaces embedded in \(\mathbb{T}^3\) (as a rule, in a topologically nontrivial way). As is well known, in the theory of normal metals, among all energy levels, the Fermi energy plays the most important role, and thus the most important is the structure of the trajectories of system (1.1) on the Fermi surface \(\epsilon(p) = \epsilon_F\).

The great importance of the geometry of trajectories of system (1.1) for the theory of galvanomagnetic phenomena in metals was established in the works of the school of I.M. Lifshits in the 1950s (see [1–8]). At this time many important and interesting examples of non-trivial behavior of the trajectories of system (1.1) on complex Fermi surfaces, as well as the corresponding regimes of behavior of magnetic conductivity in strong magnetic fields, were considered. In the general case, the geometry of the trajectories of system (1.1) starts to play a decisive role under the condition \(\omega_B \tau \gg 1\), which also implies sufficient purity of the
the topological structure of system (1.1) can be divided into regions in the space of parameters defining the magnetic field. From the above fact it follows, actually, that usually the space of parameters defining the system (1.1) should be divided into regions in which the topological structure of system (1.1) can be considered unchanged, while at the boundaries of such regions abrupt changes in the structure of (1.1) occur. Besides that, any change in the structure of trajectories of (1.1) on the Fermi surface is always associated with a reconstruction of the structure of closed trajectories on it, which, in fact, also determines the structure of other trajectories.

In this work, we will be primarily interested in the dependency of the topological structure of system (1.1) on the direction of the magnetic field. A typical picture of the boundaries separating different topological structures of (1.1) on the corresponding angular diagram (on the unit sphere $S^2$) was discussed in the most general case in [26], where it was also indicated that the most convenient tool for observing it is the study of oscillatory phenomena (classical or quantum) for different directions of $\mathbf{B}$. The latter circumstance is due to the fact that changes in the topological structure of the system (1.1) always cause the disappearance (and the appearance of new ones) of extreme trajectories, which play a central role in describing oscillatory phenomena in strong magnetic fields (cyclotron resonance, the de Haas–Van Alphen effect, the Shubnikov–de Haas effect, etc.). Thus, the boundaries separating different topological structures of system (1.1) are, in fact, also the boundaries at which abrupt changes in the picture of classical or quantum oscillations occur under small changes of the direction of $\mathbf{B}$.

As was shown in [26], the “net” of boundaries dividing the angular diagram into regions of a fixed topological structure of system (1.1) is generally quite complex and consists of “elementary” segments, each of which corresponds to some “elementary” reconstruction of the structure of system (1.1). The number of “elementary” segments can be generally infinite, in particular, the density of such segments becomes infinite near the directions of $\mathbf{B}$ corresponding to the appearance of open trajectories on the Fermi surface (Fig. 3). The paper [26] also describes all the “elementary” reconstructions of the topological structure of (1.1) on the Fermi surface that arise in the generic situation. Each of these reconstructions corresponds, in particular, to the disappearance and appearance of extreme trajectories of a very special form, determined by its topological type. As we have already said, each of the segments of the reconstructions of the structure of (1.1) (one-dimensional curves at Figs. 2 and 3) corresponds to an elementary reconstruction of a certain topological type, while topological types of reconstructions corresponding to different segments, generally speaking, are different.

The main purpose of this work is to consider the features of observable oscillatory phenomena at the time of a change in the topological structure of the system (1.1) on the Fermi surface. As we will see, each of the “elementary” reconstructions of this structure has its own peculiarities in the behavior of oscillations, which, in particular, can be very useful in the experi-

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**Fig. 2.** Regions on a part of the unit sphere corresponding to different topological structures of system (1.1) on the Fermi surface and the boundaries of reconstructions of this structure separating these regions (schematically).
mental determination of the topological types of such reconstructions.

2. TOPOLOGICAL TYPES OF ELEMENTARY RECONSTRUCTIONS AND THE DISTINCTIVE FEATURES OF THE PICTURE OF OSCILLATORY PHENOMENA FOR RECONSTRUCTIONS OF DIFFERENT TYPES

As we have already said, reconstructions of the topological structure of the system (1.1) on the Fermi surface will mean for us topological reconstructions of the set of closed trajectories on this surface. In fact, as we have already noted above, the knowledge of the set of closed trajectories on the Fermi surface makes it possible to describe also trajectories of other types on it. The set of closed trajectories for generic directions of \( B \) is a finite set of (nonequivalent) cylinders bounded by singular closed trajectories on their bases (Fig. 4). The structure of the set of cylinders of closed trajectories (their position on the Fermi surface and the scheme of their gluing with carriers of other trajectories and with each other) is locally stable for small rotations of \( B \) and can change only for special directions of \( B \) when it becomes a non-generic structure. More precisely, to change the topological structure of (1.1), it is necessary to change the direction of \( B \) so that the height of one (or several) cylinder of closed trajectories becomes zero, i.e. we need the disappearance of a cylinder of closed trajectories, followed by the appearance of a new cylinder of low height (or several cylinders). The sets of directions of \( B \) corresponding to the moment of a reconstruction are one-dimensional curves on the angular diagram (on the unit sphere \( S^2 \)), whose union forms a “net” of directions of \( B \), corresponding to the reconstructions of the structure of (1.1) on the Fermi surface.

As in [26], we will not pay attention here to the disappearance and appearance of “trivial” cylinders of closed trajectories, i.e. cylinders, at least one of the bases of which contracts to a single singular point (Fig. 5), and we will consider only reconstructions of cylinders, both bases of which are “nontrivial” (Fig. 4). In the generic case, we can assume that on each of the bases of such cylinders there is exactly one singular point of the system (1.1), and each of the bases represents one of the figures shown at Fig. 6. At the moment of reconstruction of the system (1.1), a “cylinder of zero height”, containing two singular points of (1.1) connected by singular trajectories, appears on the Fermi surface. For each of the “elementary” reconstructions of the structure of (1.1), the corresponding “zero-height cylinder” represents a flat graph lying in a plane orthogonal to \( B \), and topologically is equivalent to one of the figures shown at Fig. 7. As was shown in [26], to determine the topological type of the “elementary” reconstruction of the system (1.1), it suffices to fix the topological type of the corresponding “cylinder of zero height” and indicate whether the group velocities at two singular points are co-directed, or directed opposite to each other.

The most important circumstance in the situation under consideration is that on each of the cylinders of small height, up to its disappearance, there exist extreme closed trajectories of the system (1.1) (having an extreme circulation period or area in comparison with close trajectories), disappearing together with the corresponding cylinder (Fig. 8). When a new cylinder of closed trajectories appears, new extreme trajectories appear on it, which differ from the disappeared ones in their geometry. As a consequence, each reconstruction of the topological structure of system (1.1) is followed by a sharp change in the picture of oscillatory phenomena in a strong magnetic field, which is a convenient tool for observing the above described “net” of...
directions of $\mathbf{B}$ at the angular diagram. It must be said that extreme closed trajectories arising on cylinders of small height have certain peculiarities in comparison with ordinary extreme trajectories, namely, they contain sections that are very close to singular points of the system (1.1). This circumstance leads, in particular, to an unlimited increase in the circulation period along such trajectories with a decrease in the height of the cylinder, as well as to a number of other features that arise, for example, when observing the phenomenon of cyclotron resonance (see, for example [26]).

In this work, however, we would like to consider in more detail the features of extreme trajectories and the corresponding oscillatory phenomena that arise during each of the elementary reconstructions of the structure of (1.1), which, from our point of view, can be rather useful in the experimental study of a complete picture of reconstructions of the topology of this system on complex Fermi surfaces. As is well known (see, for example, [7, 27, 28]), for description of oscillatory phenomena, in fact, two types of closed extremal trajectories are important, namely, trajectories with an extreme orbital period and trajectories with an extreme area in comparison with trajectories close to them. Trajectories of the first type, as a rule, play a decisive role in the description of classical oscillatory phenomena (classical cyclotron resonance), while trajectories of the second type are important in describing quantum oscillatory phenomena (de Haas–van Alphen effect, Shubnikov–de Haas effect and etc.). Quite often, in reality, the same trajectory can be extreme from both the first and the second point of view, as a rule, this is the case for centrally symmetric sections of the Fermi surface. In most of the situations we consider below, however, this will not be the case, so we need to immediately divide the extreme trajectories into the two indicated types.

As we have already said, we will consider here cylinders of closed trajectories with “nontrivial” bases containing one singular point of the system (1.1). It is easy to see that the circulation period along closed trajectories on each of these cylinders increases infinitely (logarithmically) when approaching each of the bases. As a consequence, each of these cylinders must have at least one extremal trajectory with a minimum orbital period in comparison with trajectories close to it.

As for the area of closed trajectories, it is easy to see that it remains finite at the bases of the cylinders. Its derivative with respect to the distance to the corresponding base, however, goes to infinity (according to the logarithmic law) and can have a positive or negative sign, depending on the geometry of the cylinder. As for trajectories of the first type, this circumstance is also due to the presence of singular points on the bases of the cylinders and is associated with the local geometry of the trajectories near these points. Depending on the signs of the derivative of area with respect to the height on both bases, the cylinder of
closed trajectories may or may not contain extreme trajectories of the second type. Figure 9 shows examples of both cylinders containing extremal trajectories of the second type (a, b) and a cylinder that does not contain such a trajectory (c). It can be noted here that the extreme trajectory at Fig. 9a has a minimal area, while the extreme trajectory at Fig. 9b has the maximum area compared to trajectories close to them.

Thus, it can be seen that any reconstruction of the structure of (1.1) is always accompanied by a sharp change in the picture of oscillations when observing, for example, the classical cyclotron resonance, while the picture of the de Haas–van Alphen or the Shubnikov–de Haas oscillations may contain no abrupt changes (if cylinders of small height on both sides of the reconstruction do not contain trajectories of extreme area). Special note should be made about the reconstructions of (1.1), which have central symmetry. In this case, the central sections of cylinders of small height always represent extreme trajectories of both the first and the second type.

In the most general case, cylinders of small height can contain extreme trajectories of both types, which, however, do not coincide with each other. In this case, although the reconstruction of (1.1) is accompanied by sharp changes in the pictures of oscillations of all types, one can observe a difference in the parameters of the corresponding disappearing or new oscillatory terms. For example, when observing the phenomenon of cyclotron resonance, the orbital period is directly measured along extreme trajectories, which give the main terms in the overall picture of oscillations. At the same time, the orbital period can also be measured for trajectories of extreme area, for example, from the temperature dependence of the corresponding quantum oscillations [7, 29]. It is easy to see that these quantities must coincide in the case when both types of oscillations are generated by the same trajectory and differ if different types of oscillations correspond to different extremal trajectories.

As an example, consider two different reconstructions shown at Figs. 10 and 11. Both reconstructions actually correspond to the same topology of a “cylinder of zero height” (the first of those shown at Fig. 7) and differ only in the directions of group velocities at two saddle singular points of the system (1.1) (oppositely directed and co-directed velocities at singular points).

The reconstruction shown at Fig. 10 may have central symmetry and, thus, it may appear on one part of the Fermi surface (the most common case). Though, the topological structure at Fig. 10 also may not have central symmetry. In this case, it should appear simultaneously on two parts of the Fermi surface, transforming into each other under the central inversion in the $p$-space. Whether or not the structure at Fig. 10 has central symmetry, on the corresponding cylinders of small height, both before and after the reconstruction, trajectories of extreme area appear, and one of them (before the reconstruction) has the minimum area, while the second one (after the reconstruction) has the maximum area in comparison with trajectories close to them. Thus, the reconstruction shown at Fig. 10 should always be accompanied by both a sharp jump in
one of the oscillating terms in classical oscillations (change in the geometry of a trajectory of the extreme period), and a sharp jump in one of the oscillating terms in quantum oscillations (change in the geometry of a trajectory of the extreme area). Both the trajectories of the extreme period and the trajectories of the extreme area have here the shape shown at Fig. 10, while in the case of the presence of central symmetry, they simply coincide. As we said above, in the latter case, the orbital periods measured from the classical oscillations and the temperature dependence of the quantum oscillations of the corresponding oscillatory terms must coincide.

The reconstruction shown at Fig. 11, cannot have central symmetry and its appearance is possible only in pairs, on the sections of the Fermi surface that transform into each other under the central inversion in the $p$-space. The cylinders of low height, both before and after the reconstruction, coincide with the one shown at the last figure of Fig. 9, and do not contain extreme area trajectories. On these cylinders, however, there are always trajectories with an extreme orbital period, the shape of which is shown at Fig. 11. When crossing the boundary of such a reconstruction, therefore, a jump occurs (a sharp change in one of the oscillatory terms) only in the picture of classical oscillations (classical cyclotron resonance, etc.).

Below, Figs. 12–21 present all the remaining topological types of “elementary” reconstructions of system (1.1). In addition to the shape of a reconstruction itself, Figs. 12–21 also show the shape of the lower and upper bases of cylinders of small height before and after the reconstruction ((a, b) and (d, e)), the shape of extreme trajectories on cylinders of small height ((c) and (f)), as well as the structure of the “cylinder of zero height” that appears immediately at the moment of the reconstruction (g). Strictly speaking, the above figures accurately depict the local geometry of the extreme trajectories near the above-mentioned “deceleration sections” on them (colored sections), as well as the topology of their connection by the remaining trajectory sections, but in other details, they can be geometrically more complicated. For the cylinders of small height under consideration, the extremal trajectories of both types are geometrically very close to each other in $p$-space (if both types of trajectories are present on the cylinder), but they can differ significantly from each other in other parameters (for example, the value of the circulation period along the trajectory). As we said above, the main goal of this work is to describe the features of oscillatory (and other) phenomena that make it possible to distinguish between different types of “elementary reconstructions” of the system (1.1) during their experimental observation.

Figures 12–16 represent the reconstructions, during which trajectories of extreme area do not appear and do not disappear on the corresponding cylinders of small height. The extreme trajectories shown at these figures have just the smallest orbital period among all cylinder trajectories. Together with the reconstruction shown at Fig. 11, such reconstructions can be attributed to “reconstructions of the first group.” As we said above, reconstructions of this type are distinguished by the fact that during them we have a sharp change of some of the oscillatory terms only in the picture of classical oscillations.

As it is easy to check, in all the reconstructions shown at Figs. 11–16, group velocities at two saddle singular points are co-directed to each other. Coming back to the description of elementary reconstructions in terms of the topology of “cylinders of zero height” (Fig. 7), it is easy to formulate a simple rule. Namely, for any of the types of “zero-height cylinders” shown at Fig. 7, trajectories of extreme area appear on the corresponding cylinders of small height (both before and after the reconstruction) if the group velocities at its singular points are directed opposite to each other.

The above rule can be easily justified using the well-known formula

$$\frac{\partial S}{\partial p_\gamma} = \frac{eB}{c} \int \psi_\gamma^* \mathbf{v} \cdot dt$$

(where $t$ is the time of motion along the trajectory) for the area of a closed trajectory in $p$-space $S(p)$. Since the singular points near the reconstruction are located at the bases of cylinders of small height, and the time of their passage tends to infinity when approaching the bases of a cylinder, this relation determines the signs of...
\[ \frac{\partial S}{\partial p_x} \text{ near the bases of the cylinders. As is also well known, on trajectories of the extreme area, we have the relation} \]

\[ \oint v_{gr}^z dt = 0. \]

It can be noted that the extreme trajectories shown at Figs. 12, 13, as well as the trajectories shown at Fig. 11, approach the saddle singular points of the system (1.1) three times (twice to one of the singularities and once to the second). Besides that, on different sides of the reconstruction, the multiplicity of the approach of a special trajectory to each of the singular points changes (before the reconstruction, the trajectory approaches twice to one of the singular points, and after the reconstruction, to the other). Each approach to a singular point of (1.1) means the presence of a “deceleration section” on the corresponding part of a trajectory, which gives a finite addition to the trajectory orbital period. The corresponding addition to the orbital period grows logarithmically with decreasing angle \( \alpha \) between the direction of \( B \) and the boundary of a reconstruction and for \( \alpha \ll 1 \) can be written as

\[ \Delta T_i \approx \frac{c}{eBv_{gr}^{\text{asym}}(i)} \frac{1}{\sqrt{G_i}} \ln \frac{1}{\alpha}, \]

where \( v_{gr}^{\text{asym}} \) and \( G_i \) are, respectively, the values of the group velocity and the Gaussian curvature of the Fermi surface at each of the singular points \((i = 1, 2)\).

The total values of \( \Delta T(\alpha) \) can be measured with a sufficiently accurate measurement of the orbital period in classical oscillatory phenomena and a sufficiently close approximation of the direction of \( B \) to the boundary of a reconstruction of the system (1.1). It is easy to see then that when crossing the boundary, the corresponding total addition to the period of circulation along an extreme trajectory changes from the value \( \Delta T(\alpha) = T_1(\alpha) + T_2(\alpha) \) to the value \( T_1(\alpha) + 2T_2(\alpha) \), which distinguishes the reconstructions at Figs. 11–13 among all the reconstructions shown at Figs. 11–16. To distinguish different reconstructions at Figs. 11–13 a test can be used which determines the possibility of deceleration sections falling into the skin layer on each side of the boundary of a reconstruction (see [26]).

For the reconstructions shown at Figs. 14, 15, we have a different situation. Namely, now extreme trajectories have two “deceleration segments” on one side of a reconstruction and four “deceleration segments” on the other. It is easy to see that when crossing the corresponding boundary of the reconstruction the total addition to the orbital period due to the “deceleration segments” changes from the value \( \Delta T(\alpha) = T_1(\alpha) + T_2(\alpha) \) to the value \( 2(T_1(\alpha) + T_2(\alpha)) \), which also makes it possible to distinguish these reconstructions among the six reconstructions of the first group.

To distinguish two reconstructions presented at Figs. 14, 15 we can use, for example, their geometric (and topological) differences in the \( p \)-space, which are also transferred to the coordinate space. For example, for the trajectory shown at Fig. 14c, most of its sections can be located in the skin layer at the sample boundary both before and after the reconstruction. The trajectories of the minimum orbital period on opposite sides of the reconstruction are of the same type (electron or hole).
tions can be established, for example, by the absence or presence of a jump in the direction of $v_{gr}$ in the section falling into the skin layer when crossing the boundary of a reconstruction. We also note here that the measurement of the corresponding direction of $v_{gr}$ is almost always performed in the observation of the classical cyclotron resonance.

The reconstruction shown at Fig. 16, differs from all other reconstructions discussed above. Namely, here each of the extreme trajectories has four “deceleration sections” (on each side of the reconstruction boundary). When crossing the corresponding boundary of the reconstruction, the full addition to the orbital period due to the “deceleration segments” does not change and remains equal to $2(T_1(\alpha) + T_2(\alpha))$. It is easy to see that this property makes it possible to unambiguously identify this reconstruction among all the reconstructions of the first group.

The remaining types of reconstructions of the system (1.1), shown at Figs. 17–21, on the contrary, have the property that they contain trajectories of extreme area on the cylinders of small height, both before and after the reconstruction. Together with the reconstruction shown at Fig. 10, they form the second class of reconstructions, complementing the class of reconstructions shown at Figs. 11–16. All these reconstruc-

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure13}
\caption{Reconstruction of the structure of (1.1) without central symmetry. The trajectories of the extreme area on cylinders of low height are absent both before and after the reconstruction. The trajectories of the minimum orbital period on opposite sides of the reconstruction are of different types (electronic, on the one side of the reconstruction, and hole, on the other).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure14}
\caption{Reconstruction of the structure of (1.1) without central symmetry. The trajectories of the extreme area on cylinders of low height are absent both before and after the reconstruction. The trajectories of the minimum orbital period on opposite sides of the reconstruction are of the same type (electron or hole).}
\end{figure}
tions are experimentally easily distinguishable from those of the first class, since, along with a jump in the picture of classical oscillations, they also have a jump in the picture of quantum oscillations (an abrupt replacement of some oscillation terms by others). For the experimental distinction between the reconstructions of this class, one can also use the above (and also other) features of the oscillatory picture in observing classical oscillations. But, of course, these reconstructions also differ from each other by the features of changes in the picture of quantum oscillations, which we would like to dwell on below.

Note right away that the reconstructions, shown at Figs. 10, 17, 18, differ from the reconstructions shown at Figs. 19–21. Namely, for all reconstructions shown at Figs. 10, 17, 18, the extreme area trajectories have a minimum area on one side of a reconstruction and a maximum area on the other side. In our case, the minimality (maximality) of the trajectory area means that the trajectory area increase (decrease) when approaching the bases of the corresponding cylinder of closed trajectories for a fixed direction of $\mathbf{B}$. This, as we have already said, is due to the presence of singular points at the bases of such cylinders. In fact, for the same reason, the same increase (decrease) in the minimum (maximum) trajectory area (with an infinitely increasing derivative) occurs when the direction of the magnetic field approaches the boundary of a reconstruction of the structure of (1.1) (and a decrease in the height of the corresponding cylinder of closed trajecto-

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**Fig. 15.** Reconstruction of the structure of (1.1) without central symmetry. The trajectories of the extreme area on cylinders of low height are absent both before and after the reconstruction. The trajectories of the minimum orbital period on opposite sides of the reconstruction are of different types (electronic, on the one side of the reconstruction, and hole, on the other).

**Fig. 16.** Reconstruction of the structure of (1.1) without central symmetry. The trajectories of the extreme area on cylinders of low height are absent both before and after the reconstruction. The trajectories of the minimum orbital period on opposite sides of the reconstruction are of the same type (electron or hole).
This circumstance makes it possible to easily identify the trajectories of the minimum and maximum area described by us here, and, in particular, to distinguish experimentally the reconstructions shown at Figs. 10, 17, 18, from the reconstructions shown at Figs. 19–21. As also seen from Figs. 10, 17, 18, in all these cases, the maximum area trajectories have a larger area than the area of the extreme trajectory after the reconstruction. The extreme trajectories on both sides of the reconstruction are of the same (electron or hole) type.

In addition to the above circumstance, it can also be noted that in all the reconstructions shown at Figs. 10, 17, 18, the trajectories of the extreme area are of the same type (electron or hole) before and after the reconstruction. This circumstance can also be easily established experimentally from the behavior of quantum oscillations of transverse (Hall conductivity), which also makes it possible to distinguish these reconstructions from those shown at Figs. 19–21.

As for the distinguishing the reconstructions shown at Figs. 10, 17, 18, from each other, then here, as above, we can immediately note a special feature of the reconstruction at Fig. 17, consisting in the presence of three sections of deceleration on the trajec-
lies of cylinders of low height, both on one side and on the other side of the reconstruction. This circumstance makes it possible to immediately distinguish the reconstruction at Fig. 17 from the other two, for example, by measuring the orbital period along the trajectory when observing classical oscillations or the temperature dependence of quantum oscillations (replacing of $\Delta T(\alpha) = T_1(\alpha) + T_2(\alpha)$ by $T_1(\alpha) + 2T_2(\alpha)$).

But, actually, this fact can be easily established also by simple observation of quantum oscillations from the behavior of the area $S(\alpha)$ of the extreme trajectory near the transition boundary, where its main dependence on $\alpha$ is determined precisely by the approach to singular points of the system (1.1). To distinguish the reconstructions shown at Figs. 10 and 18, it is possible, for example, to investigate the possibility of falling of a deceleration section into the skin layer at the sample boundary [26] in observing the classical cyclotron resonance (it is possible on one side of a reconstruction for the reconstruction at Fig. 10 and it is impossible for the reconstruction at Fig. 18 due to the peculiarities of the geometry of the trajectories).

For the reconstructions shown at Figs. 19, 20, extreme trajectories have a minimum area on both the disappearing and arising cylinders of closed trajectories. In both of these reconstructions, the extreme trajectories have different types (electronic and hole) on opposite sides of a reconstruction. Both these circumstances can be easily established by observing quantum oscillations of various types (the De Haas–Van Alphen effect, the Shubnikov–De Haas effect) and distinguish the reconstructions at Figs. 19, 20 from all other reconstructions. The difference between the reconstructions shown at Figs. 19, and 20, consists, for example, in the number of deceleration sections on the corresponding trajectories before and after the reconstruction. As we have already seen above, this difference can also be easily established by observing both classical and quantum oscillations.

The reconstruction shown at Fig. 21, as it is easy to see, differs in many aspects from all the reconstructions we have considered earlier. In this reconstruction, a pair of extreme trajectories disappear and appear at once. On the one side of the reconstruction, both trajectories are of the same type (electronic or hole), while on the other side, a pair of trajectories of different types appears. This circumstance can be determined, in particular, by observing quantum oscillations of conductivity, which allows one to immediately identify this reconstruction experimentally. We also note here that the reconstruction shown at Fig. 21 may have central symmetry, which, in fact, almost always implies such symmetry in real situations.

All our considerations above was carried out without taking into account the electron spin. It is easy to see that taking into account the spin states leads to the splitting of each of the oscillatory terms into two, in accordance with the direction of the spin along or against the direction of $B$. In addition, in our consideration, we also did not take into account the effect of the Berry phase, which can manifest itself in materials with no inversion center or symmetry with respect to time reversal. The appearance of nonzero Berry curvature in such materials can also lead to a number of interesting effects in the situation under consideration.

In conclusion, we would like to mention here also the phenomenon of magnetic breakdown, which can
be observed in our situation. As is well known, the phenomenon of (intraband) magnetic breakdown is observed in many substances in sufficiently strong magnetic fields and, in particular, can lead to many interesting effects, arising on trajectories of system (1.1) of different geometry (see, for example [7, 30–35]). Thus, it can be expected that the occurrence of a magnetic breakdown on the special extreme trajectories described above should also lead to interesting phenomena, in particular, to significantly affect the quantization of electronic levels for trajectories of the extreme area. It must be said, however, that the magnetic breakdown on the described trajectories can occur only at rather large values of $B$ and only when the direction of the magnetic field approaches the boundary of a reconstruction of the topological structure of system (1.1) very closely (see [26]). As a consequence of this, a clear picture of the corresponding effects should be observed only in very precise experiments, which make it possible to specify the directions of the magnetic field with a very high accuracy (and can be noted only as a certain blurring of the sharp changes in the oscillation picture described above in a very narrow region near the reconstruction boundary.

**Fig. 20.** Reconstruction of the structure of (1.1) without central symmetry. Extreme-area trajectories on low-height cylinders are present both before the reconstruction (the trajectory of the minimum area) and after the reconstruction (the trajectory of the minimum area). The extreme trajectories on opposite sides of the reconstruction are of different types (electronic, on the one side of the reconstruction, and hole-type, on the other).

**Fig. 21.** Reconstruction of the structure of (1.1), which can have central symmetry. Extreme-area trajectories on cylinders of low height are present both before the reconstruction (a pair of trajectories of the minimum area) and after the reconstruction (one trajectory of the minimum area and one trajectory of the maximum area). A pair of extreme trajectories on one side of the reconstruction are of the same type. After the reconstruction, a pair of extreme trajectories of opposite types appear, and the type of the trajectory of a larger area coincides with the type of extreme trajectories before the reconstruction.
at a lower resolution). In this situation, certainly, the dependence of the picture (especially quantum) on oscillations is of great interest. 

3. CONCLUSIONS 

The paper considers the features of oscillatory phenomena in metals near the boundaries of reconstructions of the topological structure of the system describing the adiabatic dynamics of quasiparticles on complex Fermi surfaces. Each elementary reconstruction of such a structure is associated with a change in the picture of closed trajectories on the Fermi surface, which consists in the disappearance of some of the cylinders of closed trajectories and the appearance of new ones. Each of these reconstructions has its own topological structure, and there are a finite number of topological types of such reconstructions. The most important circumstance in each of the reconstructions is the disappearance of a part of closed trajectories with extreme values of the area or the orbital period, which leads to sharp observable changes in the picture of oscillatory phenomena during the reconstruction. The features of such changes are directly related to the geometry of disappearing and arising extreme trajectories, which is determined by the topological type of a reconstruction. The paper presents a detailed comparison of the features of the change in the picture of classical and quantum oscillations at the moment of a reconstruction with its topological type and proposes methods for identifying topological types based on these features. The proposed methods, in our opinion, can be rather useful in studying the geometry of complex Fermi surfaces using classical or quantum oscillations in strong magnetic fields.

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