Abstract

Given $D$ and $H$ two digraphs, $D$ is $H$-coloured iff the arcs of $D$ are coloured with the vertices of $H$. After defining what do we mean by an $H$-walk in the coloured $D$, we characterise those $H$, which we call panchromatic patterns, for which all $D$ and all $H$-colourings of $D$ admit a kernel by $H$-walks. This solves a problem of Arpin and Linek from 2007 [1].

1 Introduction

The notion of kernel, as introduced by Von Neumann in 1944 [7], has been generalised in several directions, see for example [3, 4, 5]. Here, we study yet another generalisation introduced by Sands et al. [6] which arises naturally when the arcs of a digraph are coloured (see also [2]); such a generalisation was deeply studied too by Arpin and Linek [1].

Following Arpin and Linek [1], let $B_3$ be the class of digraphs $H$ with the property that for every digraph $D$ and every $H$-colouring of $D$ (to be defined) there exists an $H$-kernel by walks in $D$. We call those digraphs in $B_3$ panchromatic patterns.

The aim of this paper is to characterise panchromatic patterns. For, we first prove a technical lemma (see Section 1) that allow us to add, under controlled circumstances, an arc to a panchromatic pattern preserving this property. This allow us to settle a question raised by Arpin and Linek [1] which characterises all panchromatic patterns of order 3.

Then, we introduce the notion of a bicomplete digraph and prove (see Lemma 7) the sufficiency for such digraphs to be panchromatic patterns. The rest of the paper, after
some preliminaries in Section 2, is devoted to prove the necessity of such a property to be a panchromatic pattern and settle the desired characterisation.

2 Preliminaries

By a digraph $D = (V, A)$ we mean a finite non-empty set of vertices $V$ and a set of (directed) arcs $A \subseteq V \times V$. Given another digraph $H$, by an $H$ colouring of $D$, we mean a map $\varsigma : A(D) \to V(H)$ from the arcs of $D$ to the vertices of $H$ — we think on the vertices of $H$ as colours assigned to the arcs of $D$, hence the name. Given such a colouring, a walk $W = x_0, x_1, \ldots, x_k$ in $D$ is called an $H$-walk if $\varsigma(W) = \varsigma(x_0, x_1), \varsigma(x_1, x_2), \ldots, \varsigma(x_{k-1}, x_k)$ is a walk in $H$. A subset $K \subset V(D)$ is called an $H$-kernel if it is both, $H$-independent and $H$-absorbent; viz., there are no $H$-walks between any pair of different vertices in it, and given any vertex out of it, there is an $H$-walk into such a subset.

Suppose that $H$ is a looped digraph with the following properties:

1. $V(H) = \bigsqcup_{i=1}^{n} C_i$,
2. $(x, y) \in A(H)$, whenever $x \neq y$ and $x, y \in C_i$ for some $i$, and
3. $C_i \times C_j \subseteq A(H)$ whenever $i \neq j$ and $C_i \times C_j \cap A(H) \neq \emptyset$ or $i = j$ and $(x, x) \in A(H)$ for some $x \in C_i$.

Then, the digraph $H'$ with vertices $V(H') = \{C_1, \ldots, C_n\}$ and arcs $(C_i, C_j) \in A(H')$ if and only if $(C_i \times C_j) \cap A(H) \neq \emptyset$ is called a contraction of $H$ and $H$ is an expansion of $H'$.

A digraph $H$ is called a panchromatic pattern if given any digraph $D$ and any $H$-colouring of $D$, we can find an $H$-kernel of $D$. The aim of this paper is to characterise the class $B_3$ of panchromatic patterns. For, we will use the following results of Arpin and Linek [1], without further reference:

Lemma 1

1. If $H \in B_3$, then every vertex of $H$ has a loop (is looped),
2. If $H \in B_3$, and $H'$ is an induced subdigraph of $H$, then $H' \in B_3$,
3. Let $H'$ be a contraction of $H$. $H \in B_3$ if and only if $H' \in B_3$.

Lemma 2 Let $W = x_0, x_1, \ldots, x_k$ be a walk in $H$ such that

1. for all $x_j \in W$, with $0 \leq j \leq k - 1$, there is a colour $c_j \in V(H)$ such that $(x_j, c_j) \not\in A(H)$,
2. $(x_k, x_0) \not\in A(H)$.
Then, \( H \not\in \mathcal{B}_3 \).

**Lemma 3** The digraphs depicted in Figure 1 are all elements of \( \mathcal{B}_3 \).

![Figure 1: Panchromatic patterns of order 3](image)

**Lemma 4** Non of the digraphs depicted in Figure 2 are elements of \( \mathcal{B}_3 \).

![Figure 2: Some non-panchromatic patterns of order 3](image)

**Lemma 5** If \( H \in \mathcal{B}_3 \), then \( H \) does not contain odd directed cycles in its complement.
3 Main Lemmas

Lemma 6 Let $H$ be a digraph, with all its vertices looped, and $a = (u, v) \notin A(H)$ a pair of vertices at distance 2 from $u$ to $v$. If $H \cup a \notin B_2$, then $H \notin B_3$.

Proof. Let $H$ be a looped digraph and $u, z, v$ a path in $H$ from $u$ to $v$, two non adjacent vertices of $H$. Let $H' = H \cup (u, v)$.

First we will show that for every $D$ and every $H'$-colouring of $D$, there exists a digraph $D'$ and an $H$-colouring of $D'$, such that $D$ admits an $H'$-kernel by walks if (and only if) $D'$ admits an $H$-kernel by walks. For, let $D'$ be constructed from $D$ as follows: for each walk $(x, y, z)$ in $D$, with arcs coloured by $u$ and $v$, in that order, we add a new vertex $\hat{y}$ in $D'$ and the symmetric arrows $(y, \hat{y})$ and $(\hat{y}, y)$, each of them coloured with $z$. We denote by $\hat{Y} = \{\hat{y}\}$ the set of all those new vertices in $D'$ and by $Y$ their corresponding neighbours in $D$ (see Figure 3).

![Figure 3: The construction](image)

Let $K' \subset V(D')$ be an $H$-kernel by walks. We can construct an $H'$-kernel by walks in $D$ with the following set

$$K := K' \cup \{y \in V(D) : \hat{y} \in K'\} \setminus \hat{Y}.$$ 

To see the independence of $K$, consider a couple of vertices $\alpha, \beta \in K$. If there is an $H'$-walk from $\alpha$ to $\beta$ and such a walk never uses a vertex in $Y$, such a walk is an $H$-walk and it would contradict the independence of $K'$ in $D'$. So, let us suppose that such walk passes through a vertex $y \in Y$. Then, we can construct an $H$-walk in $D'$ adding the neighbour $\hat{y}$ of $y$ and the arrows $(y, \hat{y})$ and $(\hat{y}, y)$ in the position of the vertex $y$ (see Figure 4).

We have to be careful and note that, if $\alpha$ or $\beta$ are vertices of $Y$, we need only to add the arcs $(\hat{y}, y)$ or $(y, \hat{y})$, respectively, at the beginning or the end of the $\alpha\beta$ walk to obtain an $H$-walk in $D'$ and contradicting the independence of $K'$.

Analogously, we can show the absorbency of $K$ by omitting the occurrences of vertices in $\hat{Y}$ in the $H$-walks of $D'$ and using the $H$-absorbency of $K'$. 

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Finally, seeking for a contradiction, suppose that $H \in \mathcal{B}_3$. Since $H' \notin \mathcal{B}_3$, then there exists a $D$ and an $H'$-colouring of $D$ which do not admit an $H'$-kernel by walks. Let $D'$ be constructed and coloured as before. Since $H \in \mathcal{B}_3$ then $D'$ admits an $H$-kernel by walks. Therefore, by the previous argument, $D$ would admit an $H'$-kernel by walks contradicting the hypothesis of the theorem, and concluding the proof.

As a consequence of this lemma, we can now decide that the following two digraphs in Figure 5 are not in $\mathcal{B}_3$, which was previously unknown (see again [1]).

Corollary 6.1 The patterns in Figure 5 are not panchromatic.

Proof. We first concentrate on Figure 5a. For, due to Lemma 1.3, it is enough to show that the solid arrows in Figure 6 induce a non-panchromatic digraph. Consider such a digraph and add the doted arc $(x, w)$. Due to lemma 6, it is enough now to show that this new digraph is not a panchromatic pattern; this last follows from the fact that the subdigraph induced by $x, w$ and $z$ is not a panchromatic pattern (see Figure 2d).

Analogously, to show that Figure 5b is not a panchromatic pattern, we extend it with a vertex $w$ (see figure 7), add the dotted arc, and find the subdigraph induced by $x, w$ and $z$, which we know is not in $\mathcal{B}_3$.

We say that a looped digraph $H$ is bicomplete if it has the following three properties:
1. $V(H) = X \sqcup Y$,
2. $H[X]$ and $H[Y]$ are complete digraphs, and
3. $\forall x \in X$ and $y \in Y : (y, x) \in A(H)$.

Note that in a bicomplete digraph some of the $(x, y)$ arcs may belong to $A(H)$.

Lemma 7 If $H$ is bicomplete, then $H \in \mathcal{B}_3$.

Proof. Let $H$ be a bicomplete digraph. On the one hand, let us suppose that there is no arc $(x, y) \in A(H)$, with $x \in X$ and $y \in Y$. Consider the digraph

$$H' := (V(H) \cup \{z\}, A(H) \cup \{(x, z), (z, x), (y, z), (z, y), (z, z) : x \in X, y \in Y\}).$$

It is easy to see that $H'$ can be contracted to the digraph in Figure 1a, which is in $\mathcal{B}_3$, and therefore by the Lemma 1.3, $H'$ and all its induced subdigraphs (Lemma 1.2) belongs to $\mathcal{B}_3$ — in particular, $H \in \mathcal{B}_3$.

On the other hand, consider a new pair $a = (x, y)$. We can add the arc $a$ to $H$ preserving its panchromaticity (i.e., $H \cup a \in \mathcal{B}_3$). For, observe that there is a path of length 2 $(x, z, y)$ in $H'$ and, by the previous Lemma 6 we garantee that $H' \cup (x, y) \in \mathcal{B}_3$. Since $H \cup a$ is an induced subdigraph of $H' \cup a$, by Lemma 1.2 we guarantee its panchromaticity.

Thus, we can recursively add arcs from $X$ to $Y$ and in each step we preserve the panchromaticity of the pattern.

\[ \blacksquare \]
4 Main Theorem

Due to Sands et al. \cite{6} we know that the digraph $2K_1$ consisting in two looped vertices belongs to $B_3$; moreover, due to Lemma \[13\], any expansion of $2K_1$ belongs to $B_3$. Furthermore, due to lemma \[7\] we know that every bicomplete digraph is also in $B_3$. We will now show that these are all digraphs in $B_3$.

**Theorem 1** $H$ is a panchromatic pattern if and only if $H$ is bicomplete or $H$ can be contracted to $2K_1$.

**Proof.** The sufficiency is the content of Lemma \[7\] and Sauer et al. \cite{6}, respectively. For the necessity, let us suppose that $H \in B_3$ is a panchromatic pattern and that it is not an expansion of $2K_1$. Let us denote by $G = H^c$ the complementary digraph of $H$.

**Claim.** If $(u, v) \in A(G)$ is an asymmetric arc, then $d^+_G(v) = 0$.

For, suppose that there exist a $y \neq u$ such that $a = (v, y) \in A(G)$. If $a$ is an asymmetric arc (i.e., $(y, v) \not\in A(G)$), then, depending on the relationship between $u$ and $y$, one of the digraphs in Figure 2.b,e,f,g appears as an induced subdigraph of $H$, contradicting its panchromaticity. If $a$ is symmetric (i.e., $(y, v) \in A(G)$), depending on the relationship between $u$ and $y$, one of the digraphs in Figure 2.a,b,c,d appears as an induced subdigraph of $H$, contradicting again its panchromaticity. In any case, $a$ cannot belong to $G$, concluding the proof of the claim.

As an immediate consequence of this claim we conclude that: *if $H \in B_3$ then every cycle of length at least 4 is symmetric.*

Recall that Arpin and Linek \cite{1} proved that no digraph in $B_3$ has odd directed cycles in its complement.

**Claim.** The underling graph of $G$ has no odd cycles; i.e., $G$ is bipartite.

Searching for a contradiction, let us suppose that the underling graph of $G$ has an odd cycle. Such a cycle cannot be symmetric since it would contain a directed odd cycle, contradicting Lemma \[5\]; therefore it has an asymmetric arc. Due to the previous claim, it is easy to see that Figure 5.a, without its loops, must be part of the supposed cycle. From here we conclude that one of the digraphs in Figure 5.a or Figure 2.a,b,c,h must be induced subdigraphs of $H$ contradicting its panchromaticity.

From here, we have two cases to analyse; namely, if $G$ has directed cycles of length at least 4, or not.

**Case 1.** *If $G$ contains a directed cycle of length at least 4, then $G$ is a bipartite complete digraph, and therefore $H$ is an extension of $2K_1$.*

For, recall that every directed cycle of length at least 4 is symmetric. Therefore every non-trivial strongly connected component of $G$ is symmetric (since in a strongly connected digraph each arc is in a directed cycle).
Claim. If $S$ is a strongly connected component of $G$ with a directed cycle of length at least 4, then $S$ is a bipartite complete digraph (i.e., all arcs are symmetric and it has all arcs between two independent sets of vertices).

By the previous claim, $G$ is bipartite and therefore the induced subgraph of $S$ is bipartite too. Let $V(S) = A \sqcup B$ be the bipartition of $S$. We first show that $S$ has a cycle of length exactly 4. Let $\gamma$ be a directed girth of $S$. Searching for a contradiction, suppose $|\gamma| > 4$; since there are no odd cycles, we have that $|\gamma| \geq 6$. This induces a $P_4 = (x, w, v, u)$ subgraph in $G$. Then, we have the path $(w, u, x, v)$ induced in $H$ and by Lemma 2 we have that $H \not\in B_3$. Thus, let $(u, v, w, x)$ be the cycle of length 4 guaranteed by the previous argument. We further suppose that $u$ and $w$ are in $A$.

Observe that every vertex in $B$ is adjacent to $u$ (by a symmetric arc); for, if there exists a vertex $z$ in $B$ non adjacent to $u$, then the induced subgraph by $u, z$ and $v$ is isomorphic to Figure 5b, contradicting the panchromaticity of $H$. Analogously $w$ is adjacent to each vertex in $B$; moreover, $v$ and $x$ are adjacent to all vertices in $A$. Furthermore, each vertex $y \in A \setminus \{u, w\}$ is adjacent to all vertices in $B$; for, observe that if there is a vertex $z \in B \setminus \{v, x\}$ non adjacent to $y$, then the subdigraph induced by $w, y$ and $z$ is isomorphic to Figure 5b, which contradicts the panchromaticity of $H$, concluding the proof of the claim.

Claim. Every connected component of $G$ is strongly connected.

For, let $S$ be as before with its 4-cycle $(u, v, w, x)$ and bipartition $V(S) = A \sqcup B$, and suppose that the connected component of $S$, in the weak sense, is bigger. Then either an asymmetric arc goes into $S$ from its complement, or there is an asymmetric arc from $S$ to its complement. Due to our first claim, there cannot be an “incoming” arc $(y, s)$, with $s \in S$ and $y \in G \setminus S$, so let us suppose there is an “outgoing” arc $(s, y)$. With out loose of generality, suppose that $s \in B$. Then, the subdigraph induced by $w, s$ and $y$ in $H$, depending in the relationship between $w$ and $y$, is isomorphic to one of the Figures 2b,c,h or Figure 5a, contradicting the panchromaticity of $H$ and concluding the claim.

Claim. The underlying graph of $G$ is connected.

Let $S$ be as before, suppose there is another component of $G$, and let $y$ be a vertex in $G \setminus S$. Then the subdigraph of $H$ induced by $w, x$ and $y$ is isomorphic to Figure 5b, which contradicts the panchromaticity of $H$.

Therefore, we have that $G$ is isomorphic to $S$ which we have shown to be a complete bipartite and $H$ is an extension of $2K_1$ concluding case 1.

Case 2. If $G$ does not have a directed cycle of length at least 4, then $H$ is a bicomplete digraph.

Recall that we had proved that the underlying graph of $G$ is bipartite; we will work with its bipartition $V(G) = A \sqcup B$.

Claim. If $G$ contain cycles (viz., symmetric arcs), then all of them pass through a single vertex $x \in V(G)$.
If $G$ is acyclic, we are done; so, let us suppose that $G$ has a symmetric arc $\{u,v\}$, where $u \in A$ and $v \in B$. We will show that either every cycle of $G$ contain $u$ or all of them contain $v$. First of all, suppose there is a cycle (of length 2 — or a symmetric arc, if you will) that does not contain either $u$, nor $v$; call such an arc $\{z,w\}$ where $z \in A$ and $w \in B$. Then the symmetric arc $\{v,z\}$ must exist since the complement of $v, w$ and $z$ must be in Figure 1. Analogously, the pair $\{u,w\}$ form a symmetric arc; therefore we have the cycle $(u,v,z,w)$ which contradict the fact that every cycle is of length 2.

Thus, we can suppose now that every cycle either contains $u$, or it contains $v$. Suppose there are cycles $\{z,v\}$ and $\{u,w\}$ in $G$. Since we don’t have cycles of length 4, $z$ and $w$ must be independent, therefore we have that the complement of $u, z$ and $w$ is Figure 5.b, contradicting the panchromaticity of $H$ and concluding the proof of the claim.

Claim. There is a partition of $V(G \setminus x) = A \sqcup B$ (possibly degenerated; i.e., $B = \emptyset$), where $x$ is that vertex contained in all symmetric arcs, such that all arcs go from $A$ to $B$.

For, simply observe that $G \setminus x$ is acyclic and that, as proved earlier, the final vertex of every asymmetric arc has out-degree zero; so, let $A$ be the set of initial vertices of all arcs, and $B$ its complement.

We end the proof showing that such a vertex $x \in V(G)$, indeed, does not exist.

Claim. If $H$ is not an extension of $2K_1$, then the digraph $G$ is acyclic.

For, suppose we have such a vertex $x \in V(G)$ and the partition $V(G \setminus x) = A \sqcup B$, where all arcs go from $A$ to $B$. By definition of $x$, there is a symmetric arc $\{x,y\}$ in $G$. If $y$ is a vertex in $A$, then there are symmetric arcs $\{x,a\}$ for all vertices in $A$ since otherwise the complement of $x,y$ and the nonadjacent $a \in A$ would induce a digraph isomorphic to Figure 2.d or one of the two digraphs in Figure 5. Furthermore, $B$ has to be empty since otherwise $x, a$ and $b$ would induce a subgraph isomorphic to Figure 2.b,c,h or Figure 5.a,b. So, $H$ would be an isolated vertex union a complete digraph, which is an extension of $2K_1$. Analogously, there are no symmetric arcs from $x$ to $B$.

Finally, since $G$ is bipartite and acyclic, and all out degrees of final vertices are 0, then either $H$ is bicomplete or an extension of $2K_1$, which completes the proof.

References

[1] P. Arpin and V. Linek; Reachability problems in edge-colored digraphs. Discrete Mathematics 307:2276–2289 (2007).

[2] H. Galeana-Sánchez and R. Strausz; On panchromatic digraphs and the panchromatic number. Graphs and Combinatorics 31:115–125 (2015).

[3] G. Hahna, P. Ille and R.E. Woodrow; Absorbing sets in a arc-coloured tournaments. Discrete Math. 283:93–99 (2004).

[4] V. Linek and B. Sands; A note on paths in edge-coloured tournaments. Ars Combin. 44:225–228 (1996).
[5] K.B. Reid; Monotone reachability in arc-coloured tournaments. *Ars Combin.* **44**:225–228 (1996).

[6] B. Sands, N. Sauer and R. Woodrow; On monochromatic paths in edge-colored digraphs. *Journal of Combinatorial Theory, series B* **33**:271–275 (1982).

[7] J. V. Neumann and O. Morgenstern; Theory of games and economic behavior. *Princeton University Press* (1944).