Mirror Symmetry, Borcherd-Harvey-Moore
Products and
Determinants of the Calabi-Yau Metrics on K3 Surfaces

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Abstract

In the study of moduli of elliptic curves the Dedekind eta function
\[ \eta = q^{1/24} \prod_{n=1}^{\infty} (1-q^n), \]
where \( q = e^{2\pi i \tau} \) plays a very important role. We will point out the three main properties of \( \eta \).

1. It is well known fact that \( \eta^{24} \) is an automorphic form which vanishes at the cusp. In fact \( \eta^{24} \) is the discriminant of the elliptic curve. 2. The Kronecker limit formula gives the explicit relations between the regularized determinant of the flat metric on the elliptic and \( \eta \). 3. The Fourier expansion of \( \frac{d}{dt} \log \eta(it) \) are positive integers which give the number of elliptic curve that that are covering of the elliptic curve \( E_{\tau} \) of degree \( n \).

Based on the work of Borcherds we construct on the moduli space of K3 surfaces with B-field an automorphic form \( \exp \Phi_{4,20} \) which vanishes on the totally geodesic subspaces orthogonal to \( -2 \) vectors of \( \mathbb{U}^4 \oplus (\mathbb{E}_8(-1))^2 \). We give an explicit formula of the regularized determinants of the Laplacians of Calabi Yau metrics on K3 Surfaces, following suggestions by R. Borcherds. The holomorphic part of the regularized determinants will be the higher dimensional analogue of Dedekind Eta function.

We give explicit formulas for the number of non singular rational curves with a fixed volume with respect to a Hodge metric in the case of K3 surfaces with Picard group unimodular even lattice. The counting of rational curves on special K3 surfaces using the regularized determinants of the Laplacian of CY metrics is related to some results of Bershadsky, Cecotti, Ouguri and Vafa. See \cite{7}.
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1 Introduction

1.1 General Remarks

In the study of moduli of elliptic curves the Dedekind eta function

\[ \eta = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \]

where \( q = e^{2\pi i \tau} \) plays a very important role. We will point out the three main properties of \( \eta \).

1. It is well known fact that \( \eta^{24} \) is a automorphic form which vanishes at the cusp. In fact \( \eta^{24} \) is the discriminant of the elliptic curve.

2. The Kronecker limit formula gives the explicit relations between the regularized determinant of the flat metric on the elliptic and \( \eta \).

3. The Fourier expansion of \( \frac{d}{dt} \log \eta(it) \) are positive integers which give the number of elliptic curve that are covering of the elliptic curve \( E_\tau \) of degree \( n \).

In this paper we will give the analogue of the Dedekind eta function for K3 surfaces. We will show that the main properties stated above of \( \eta \) are satisfied in the case of K3 surfaces.

The study of the moduli space of K3 surfaces recently attracted the attention of string theorists. It is interesting that studies in optics by Fresnel and Hamilton was the first reason to study K3 surfaces. It was A. Weil who outline the main problems in the study of the moduli of K3 surfaces. See [29]. The first main result in the study of moduli of K3 surfaces is due to Shafarevich and Piatetski-Shapiro. See [25]. They proved the global Torelli Theorem for polarized algebraic K3 surfaces. Combining the Theorem of Shafarevich and Piatetski Shapiro with the description of the mapping class group of K3 surface one obtain that the moduli space \( \mathcal{M}_{K3,n} \) of polarized algebraic K3 surfaced with a polarization class \( e \) such that \( \langle e, e \rangle = 2n > 0 \) is a Zariski open set in

\[ \Gamma_{K3,n}^+ \backslash \mathfrak{SO}(2,19)/\mathfrak{SO}(2) \times \mathfrak{SO}(19), \]

where \( \Gamma_{K3,n}^+ \) is an index two subgroup in \( \mathcal{O}_{\Lambda_{K3,n}}(\mathbb{Z}) \) and \( \Lambda_{K3,n} \) is the lattice isomorphic to \(-2n\mathbb{Z} \oplus U^3 \oplus E_8(-1) \oplus E_8(-1)\). In [27] it was proved that every point of \( \mathcal{SO}(3,19)/\mathcal{SO}(2) \times \mathcal{SO}(1,19) \) corresponds to a marked K3 surface. Based on this result in [22] it was proved that the moduli space of Ricci flat metrics on K3 surfaces with a fixed volume is isomorphic to

\[ \mathcal{M}_{KE} := \Gamma^+ \backslash (\mathcal{SO}_0(3,19)/\mathcal{SO}(3) \times \mathcal{SO}(19) - \mathcal{D}_{KE}), \]

where \( \Gamma^+ \) is a subgroup of index 2 in the group of automorphisms of the group of the automorphisms of the Euclidean lattice \( \Lambda_{K3} = \mathbb{U}^3 \oplus E_8(-1) \oplus E_8(-1) \), where

\[ \mathbb{U} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

is the hyperbolic lattice and \( E_8(-1) \) is the standard lattice and \( \mathcal{D}_{KE} \) is the subspace whose points correspond to Ricci flat metrics on orbifolds. Donaldson
proven in [12] that the mapping class group $\Gamma$ of a K3 surface is a subgroup of index 2 in the group of the automorphisms of the Euclidean lattice $\Lambda_{K3}$.

Mirror Symmetry is based on the observation that there are two different models A and B in string theory which define one and the same partition function. The A model is related to the deformation of the Kähler-Einstein metrics. The B-model is related to the deformations of complex structures. To studied mirror symmetry on K3 surfaces we need to define a B-field on a K3 surfaces. It is a class of cohomology $\omega_X (1,1) \in H^{1,1}(X, \mathbb{C})$ of type (1,1) on a K3 surface $X$ such that

$$\int_X \text{Im} \omega \wedge \text{Im} \omega > 0.$$ 

The moduli space of marked K3 surfaces with a B-field is isomorphic to $h_{4,20} := SO_0(4,20)/SO(4) \times SO(20)$. Aspinwall and Morrison proved that the moduli space of Super Conformal Field Theories with supersymmetry (4,4) is described by $\Gamma_B^+ \backslash h_{4,20}$, where $\Gamma_B^+$ is a subgroup of index two in $\mathcal{O}(\Lambda_{K3})$. It is well known that $h_{4,20}$ parametrizes the four-dimensional oriented subspaces in $\mathbb{R}^{4,20}$ on which the bilinear form is strictly positive. See [1]. To a pair $(X, \omega_X (1,1))$ of a marked K3 surface with a B-field $\omega_X (1,1)$ we assign an oriented four dimensional subspace $E_{X, \omega_X (1,1)}$ in

$$H^*(X, \mathbb{Z}) \otimes \mathbb{R} = (H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})) \otimes \mathbb{R}$$ 

on which the bilinear form defined by the cup product is positive. We will assume that $(H^0(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})) = U_0$ and the B-field $\omega_X (1,1)$ we will be identified with

$$(1, -\frac{1}{2} (\omega_X (1,1) \wedge \omega_X (1,1))) \in (H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})). \quad (1)$$

From now on we will consider the B-field $\omega_X (1,1)$ as defined by [1]. The four dimensional subspace $E_{X, \omega_X (1,1)}$ contains the two dimensional subspace $E_{\omega_X}$ spanned by $\text{Re} \omega_X$ and $\text{Im} \omega_X$, where $\omega_X$ is the holomorphic two form on $X$ defined up to a constant and the two dimensional subspace $E_{\omega_X (1,1)}$ spanned by $\text{Re} \omega_X (1,1)$ and $\text{Im} \omega_X (1,1)$, where $\omega_X (1,1)$ is defined by [1]. $E_{\omega_X}$ will the orthogonal to $E_{\omega_X (1,1)}$ in $E_{X, \omega_X (1,1)}$.

Mirror Symmetry is pretty well understood in the case of K3 surfaces. See [1], [13] and [28]. The mirror symmetry is exchanging $E_{\omega_X}$ with $E_{\omega_X (1,1)}$. Special case of mirror symmetry of algebraic K3 surfaces was studied in details in [13].

In this paper we will consider the moduli space of K3 surfaces with B-fields and an automorphic form $\exp (\Phi_{4,20})$ which vanishes on the totally geodesic subspaces that are orthogonal to $-2$ vectors. Such automorphic form exists according to [1]. The restriction of $\exp (\Phi_{4,20})$ on the moduli space $\mathcal{M}_{ell} := \Gamma_{ell} \backslash h_{2,10}$ of elliptic K3 surfaces with a section vanishes on the discriminant locus $\mathcal{D}_{ell} \subset \mathcal{M}_{ell} \subset \Gamma_B^+ \backslash h_{4,20}$ which is defined by the points orthogonal to $-2$ vectors. The mirror $Y$ of the elliptic K3 $X$ with the section has Picard group $\text{Pic}(Y) = U \oplus \mathcal{E}(-1) \oplus \mathcal{E}(-1)$. $\exp (\Phi_{4,20})$ restricted on a line $tL$ in the
Kähler cone $K(Y)$ spanned by the imaginary part $L$ of a Hodge metric, has a Fourier expansion. The Fourier coefficients of $\frac{d}{dt} \log \Phi(it)$ are positive integers and they count the number of rational curves of fixed volume. Thus in the A model of K3 surfaces with Picard group unimodular lattice, the restriction of $\exp (\Phi_{4,20})$ on the Kähler cone counts rational curves. In the B-model the restriction of $\exp (\Phi_{4,20})$ on the moduli space counts vanishing invariant cycles.

The regularized determinants of the Laplacian of Ricci flat metrics $\det (\Delta_{KE})$ acting on $(0,1)$ forms will be a function on the moduli space of Einstein metric $\mathcal{M}_{KE}$. R. Borcherds suggested that one can compute the determinants of the Laplacians of Ricci flat metrics explicitly by using the method of the theta lifts. See [11]. In this paper we will give an explicit expression of the regularized determinants of the Laplacians of CY metrics $\det$ as a function on the moduli space of Einstein metrics $\mathcal{M}_{KE}$.

There are some relations of this paper with the papers [9] and [10].

1.2 Organization of the Paper
In Section 2 we describe some basic property of the symmetric space

$$\mathfrak{h}_{p,q} := \mathbb{SO}_0(p,q)/\mathbb{SO}(p) \times \mathbb{SO}(q).$$

In Section 3 we study the unimodular even indefinite lattices $\Lambda_{p,q}$. We define the discriminant locus $\mathfrak{D}_{p,q}$ in the locally symmetric space $\mathcal{O}^*_+ (\Lambda_{p,q}) \backslash \mathfrak{h}_{p,q}$. We prove in this Section that $\mathfrak{D}_{p,q}$ is irreducible.

In Section 4 we describe the main results about moduli of K3 surfaces.

In Section 5 we study automorphic forms on $\mathcal{O}^*_+ (\Lambda_{p,q}) \backslash \mathfrak{h}_{p,q}$.

In Section 6 we prove the analogue of the Kronecker limit formula, i.e. we gave the explicit formula for the determinant of the Laplacian of Calabi-Yau metrics (Kähler-Einstein metrics) on the moduli space of Kähler-Einstein metrics $\mathcal{O}^*_+ (\Lambda_{3,19}) \backslash \mathfrak{h}_{3,19}$.

In Section 7 we study mirror symmetry of K3 surfaces.

In Section 8 we constructed the analogue of the Dedekind eta function for K3 surfaces and proved its main properties.

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2 Symmetric Space $h_{p,q} := \text{SO}_0(p,q)/\text{SO}(p) \times \text{SO}(q)$

2.1 Global Flat Coordinates on the Symmetric Space $h_{p,q}$

We will need some basic facts about the symmetric space $h_{p,q} := \text{SO}_0(p,q)/\text{SO}(p) \times \text{SO}(q)$.

The following Theorem is standard.

**Theorem 1** There is a one to one correspondence between points $\tau$ in $h_{3,19}$ and all oriented three dimensional $E_\tau$ subspaces in $\Lambda_{K3} \otimes \mathbb{R}$ on which the intersection form on $\Lambda_{K3}$ is strictly positive.

**Theorem 2** Let $\mathbb{R}^{p,q}$ be a $p+q$ dimensional real vector space with a metric with signature $(p,q)$. Let $E_{\tau_0}$ be a $p$–dimensional subspace in $\mathbb{R}^{p,q}$ such that the restriction of the quadratic form on $E_{\tau_0}$ is strictly positive. Let $e_1, \ldots, e_p$ be an orthonormal basis of $E_{\tau_0}$. Let $e_{p+1}, \ldots, e_{p+q}$ be orthogonal vectors to $E_{\tau_0}$ such that $\langle e_i, e_j \rangle = -\delta_{ij}$ for $1 \leq i, j \leq p+q$. Let $E_\tau$ be any $p$–dimensional subspace in $\mathbb{R}^{p,q}$ such that the restriction of the quadratic form in $E_\tau$ is strictly positive. Then there exists a basis $\{g_1(\tau), \ldots, g_p(\tau)\}$ in $E_\tau$ such that

$$g_j(\tau) = e_j + \tau^i_{j} e_i.$$  \hspace{1cm} (2)

**Proof:** Let $f_1(\tau), \ldots, f_p(\tau)$ be an orthonormal basis of $E_\tau$. Let

$$f_i = \sum_{j=p+1}^{p+q} \tau^j_i e_j$$  \hspace{1cm} (3)

for $1 \leq i \leq p$ and $1 \leq j \leq p+q$. Let $(A_{ij}(\tau))$ be the $p \times p$ matrix $\left(\tau^j_i\right)$ for $1 \leq i \leq p$ and $1 \leq j \leq q$ where $\tau^j_i$ are the elements in the expression (3).

**Lemma 3** $\det(A_{ij}(\tau)) \neq 0$.

**Proof:** Suppose that $\det(A_{ij}(\tau)) = 0$. This implies that $rk(A_{ij}(\tau)) < p$. So we can find constants $a_i$ for $i = 1, \ldots, p$ such that at least one of them is non zero and

$$\sum_{i=1}^{p} a_i \left(\sum_{j=1}^{p} \tau^j_i\right) = 0.$$  \hspace{1cm} (4)

Let us consider the vector

$$g(\tau) = \sum_{j=1}^{p} a_j g_j.$$  \hspace{1cm} (5)
Combining (4) and (5) we obtain that
\[ g(\tau) = \sum_{i=p+1}^{p+q} \lambda_i e_i. \]  

(6)

The definition of the vectors \( g_i(\tau) \) and (5) imply that
\[ \langle g(\tau), g(\tau) \rangle = -2 \sum_{i=p+1}^{p+q} |\lambda_i|^2 < 0. \]  

(7)

Clearly \( g(\tau) \) is a non zero vector in \( E_\tau \). So \( \langle g(\tau), g(\tau) \rangle > 0 \). Thus we get a contradiction with (7). Lemma 3 is proved.

Theorem 2 follows directly from Lemma 3.

Lemma 4

There is one to one correspondence between the set of all \( p \times q \) matrices \( (\tau_{ji}) \) for \( 1 \leq i \leq p \) and \( p+1 \leq j \leq p+q \) such that the vectors \( g_i(\tau) \) for \( i = 1, \ldots, p \) defined by (2) spanned a \( p \)-dimensional subspace \( E_\tau \) in \( \mathbb{R}^{p,q} \) on which the restriction of the quadratic form \( \langle u, v \rangle \) is strictly positive and the set of points in \( h_{3,19} \). Thus \( (\tau_{ji}) \) define global coordinates on \( h_{3,19} \).

2.2 Decomposition of \( h_{p,q} \)

Theorem 5

We have the following decomposition of
\[ h_{2,p} = \mathbb{R}^{1,p-1} h_{1,p-1} + \sqrt{-1} h_{1,p-1}. \]  

(8)

Proof: It is a well known fact that \( h_{1,p-1} \) is one of the component \( V^+ \) of the cone \( V := \{ v \in \mathbb{R}^{1,p-1} | \langle v, v \rangle > 0 \} \). Let us consider \( \mathbb{R}^{2,p} = \mathbb{R}^{1,p-1} \oplus \mathbb{R}^{1,1} \). Let us consider the map:
\[ w \in \mathbb{R}^{1,p-1} h_{1,p-1} + \sqrt{-1} h_{1,p-1} \rightarrow \mathbb{P} \left( \left( \mathbb{R}^{1,p-1} \oplus \mathbb{R}^{1,1} \right) \otimes \mathbb{C} \right) \]

defined as follows
\[ \Psi : w = (w_1, \ldots, w_p) \rightarrow \left( w_1, \ldots, w_p, -\frac{\langle w, w \rangle}{2}, 1 \right). \]

It is easy to check that in \( \mathbb{P} \left( \mathbb{R}^{2,p} \otimes \mathbb{C} \right) \) we have \( \langle \Psi(w), \Psi(w) \rangle = 0 \) and \( \langle \Psi(w), \Psi(w) \rangle > 0 \). Thus the image of \( \mathbb{R}^{1,p-1} h_{1,p-1} + \sqrt{-1} h_{1,p-1} \) under the map \( \Psi \) will be \( h_{2,p} \), since \( h_{2,p} \) in \( \mathbb{P} \left( \mathbb{R}^{2,p} \otimes \mathbb{C} \right) \) is given by one of the components of the open set in the quadratic \( \langle w, w \rangle = 0 \) defined by \( \langle w, w \rangle > 0 \). It is very easy to prove that \( \Psi \) is one to one map.

Theorem 6

Suppose that \( p \geq 3 \), and \( q \geq 2 \). Then we have the following decomposition
\[ h_{p,q} = h_{p-1,q-1} \times \mathbb{R}^{p-1,q-1} \times \mathbb{R}_+, \]  

(9)

where \( \mathbb{R}_+ \) is the set of real positive numbers.
Proof: Let us consider in the space $\mathbb{R}^{p,q}$ two vectors $e_{p+q-1}$ and $e_{p+q}$ such that

$$\langle e_{p+q}, e_{p+q} \rangle = \langle e_{p+q-1}, e_{p+q-1} \rangle = 0 \text{ and } \langle e_{p+q-1}, e_{p+q} \rangle = 1.$$  

Clearly the orthogonal complement to the subspace $\{e_{p+q}, e_{p+q}\}$ will be isometric to $\mathbb{R}^{p-1,q-1}$. Let us consider a basis $\{e_1, \ldots, e_{p+q-2}\}$ of $\mathbb{R}^{p,q}$, where $e_1, \ldots, e_{p+q-2}$ is a basis of $\mathbb{R}^{p-1,q-1}$.

There is one to one correspondence between the points $\tau \in h_{p,q}$ and the oriented $p-$dimensional subspaces $E_\tau$ in $\mathbb{R}^{p,q}$ on which the restriction of the bilinear form is strictly positive. The intersection $E_\tau \cap \mathbb{R}^{p-1,q-1}$ will be $(p-1)-$dimensional subspace in $\mathbb{R}^{p-1,q-1}$ on which the bilinear form is strictly positive. Let $f_1$ be a vector in $\mathbb{R}^{p,q}$ orthogonal to $\mathbb{R}^{p-1,q-1} \cap E_\tau$. It is easy to see that the coordinates of $f_1$ can be normalized in such a way that its coordinates in $\mathbb{R}^{p,q}$ are $f_1 = (\mu_1, ..., \mu_{20}, 1, \lambda)$, where $\mu = (\mu_1, ..., \mu_{p+q-2})$ is any vector in $\mathbb{R}^{p-1,q-1}$ and $\lambda > 0$ and $\lambda > \langle \mu, \mu \rangle$. Thus the correspondence $E_\tau \rightarrow (f_1, E_\tau \cap \mathbb{R}^{p-1,q-1})$ establishes the decomposition $\mathbb{R}^{p,q}$. ■

2.3 Definition of the Standard Metric on $h_{p,q}$

Since $h_{p,q} \subset \text{Grass}(p,p+q)$ then the tangent space $T_{\tau_0,h_{p,q}}$ at a point $\tau_0 \in h_{p,q}$ can be identified with $\text{Hom}(E_{\tau_0}, E_{\tau_0}^\perp)$. Thus any tangent vector $A \in T_{\tau_0,h_{p,q}}$ can be written in the form

$$A = \sum_{i=1}^{p} \sum_{j=p+1}^{p+q} \tau_i^j (e_i^* \otimes e_j), \quad (10)$$

where $e_i$ for $i = 1, ..., q$ is an orthonormal basis of $E_{\tau_0}$ and $e_j$ for $j = p+1, ..., p+q$ is an orthonormal basis of $E_{\tau_0}^\perp$. Then we define the metric on $T_{\tau_0,h_{p,q}} = \text{Hom}(E_{\tau_0}, E_{\tau_0}^\perp)$ for $A \in T_{\tau_0,h_{p,q}} = \text{Hom}(E_{\tau_0}, E_{\tau_0}^\perp)$ given by $[10]$ by

$$\|A^2\| = \sum_{i,j} |\tau_i^j|^2. \quad (11)$$

We will call this metric the Bergman metric on $h_{3,19}$.

Lemma 7 The Bergman metric $ds_B^2$ is invariant metric on $h_{p,q}$. It is given in the flat coordinate system by

$$ds_B^2 = \sum_{1 \leq j \leq 3, 1 \leq i \leq 19} (d\tau_i^j)^2 + O(2). \quad (12)$$

Proof: The proof of Lemma 7 follows directly from the definition of the Bergman metric. ■
3 Discriminants in $\mathfrak{H}_{p,q}$

3.1 Definition and Basic Properties of the Discriminant

From now on we will consider the symmetric spaces $\mathfrak{H}_{p,q}$ for which $p - q \equiv 0 \mod 8$. From now on this paper $\Lambda_{p,q}$ will be unimodular even lattice of signature $(q + 8k, q)$. We have the following description all $\Lambda_{p,q}$:

**Theorem 8** Suppose that $\Lambda_{p,q}$ the unimodular even lattice of signature $(p, q)$ for $p - q \equiv 0 \mod 8$. Then

$$\Lambda_{p,q} \cong U \oplus \cdots \oplus U \oplus E_8(-1) \oplus \cdots \oplus E_8(-1).$$

**Definition 9** Define the set $\Delta_{p,q}(e) := \{ \delta \in \Lambda_{p,q} | \langle \delta, \delta \rangle = -2 \}$. Let us define by $O_{p,q}$ the group of the automorphisms of the lattice $\Lambda_{p,q}$. Let $O_{p,q}^+$ be the subgroup of $O_{p,q}$ which preserve the orientation of the positive subspaces of dimension $p$ in $\Lambda_{p,q} \otimes \mathbb{R}$. Then $O_{p,q}^+$ has index two in $O_{p,q}$.

**Definition 10** We know that $\mathfrak{H}_{p,q}$ can be realized as an open set in the Grassmannian $\text{Grass}(p, p + q)$. Let us denote by $\mathfrak{H}_{p,q-1}(\delta)$ the set of all $p$-dimensional subspaces in the orthogonal complement of the vector $\delta$ in $\Lambda_{p,q} \otimes \mathbb{R}$. We will define the discriminant locus $D_{p,q}$ in $O_{p,q}^+ \setminus \mathfrak{H}_{p,q}$ as follows:

$$D_{p,q} := O_{p,q}^+ \setminus \bigcup_{\delta \in \Delta_{p,q}(e)} (\mathfrak{H}_{p,q-1}(\delta)).$$

This definition is motivated by the definition of the discriminant locus in the moduli of algebraic K3 surfaces.

3.2 The Irreducibility of the Discriminant

**Theorem 11** The discriminant locus $D_{p,q}$ in $O_{p,q}^+ \setminus \mathfrak{H}_{p,q}$ is an irreducible divisor, where $\Lambda_{p,q}$ is an even unimodular lattice.

**Proof:** The proof of Theorem 11 will follow if we prove that on the set of vectors $\Delta_{\Lambda_{p,q}}$ the group $O_{\Lambda_{p,q}}^+$ acts transitively. Thus they form one orbit and therefore the discriminant locus $D_{p,q}$ in $O_{p,q}^+ \setminus \mathfrak{H}_{p,q}$ is an irreducible divisor.

The proof that on the set of vectors $\Delta_{\Lambda_{p,q}}$ the group $O_{\Lambda_{p,q}}^+$ acts transitively will use similar ideas as the proof of the irreducibility of the discriminant locus in the moduli space of Enriques surfaces given by Borcherds in [8].

We will proceed by induction on $p$ to prove that the action of $O_{\Lambda_{p,q}}^+$ on the set $\Delta_{\Lambda_{p,q}}$ is transitive. For $p = 0$ the Theorem 11 is obvious. Suppose that Theorem 11 is true for $p$. We will denote by $L$ the lattice

$$U \oplus \cdots \oplus U \oplus E_8(-1) \oplus \cdots \oplus E_8(-1).$$
and by $M$ the lattice $L \oplus U$.

The plan of the proof is the following. We will denote by $R_0$ and $R_1$ the set of norm $-2$ vectors of $M$ which have inner product respectively 0 or 1 with the vector $e = (0, 0, 1) \in L \oplus U = M$. Let $\Gamma_1$ be the group generated by reflections of elements of the set $R_1$ and $\Gamma_2$ be the group generated by reflections of elements of $R_0 \cup R_1$ and $-id$. We will show first that any $-2$ vector of $M$ is conjugate to an element of the set $R_0 \cup R_1$. Then we will show that the group $O_M^+(\mathbb{Z})$ interchange the sets $R_0$ and $R_1$.

**Lemma 12** Any norm $-2$ vector $\delta$ of $M$ is conjugate to an element of $R_0 \cup R_1$ under the group $\Gamma_1$.

**Proof:** The proof of Lemma 12 is based on the following Propositions 13 and 14.

**Proposition 13** Suppose that $v \notin L \subset L \otimes \mathbb{Q}$. Suppose that $x$ is some real number. Then there exists a vector $\vec{\mu} \in L$ such that $|\langle \vec{v}, \vec{\mu} \rangle - x| < 1$.

**Proof:** The proof of Proposition 13 follows the proof of Lemma 2.1 in \cite{8}. Since $\vec{v} \notin L \subset L \otimes \mathbb{Q}$ we can find a primitive isotropic vector $\vec{\rho}$ such that $\langle \vec{\rho}, \vec{v} \rangle$ is not an integer. This is because primitive isotropic vectors span $L$. As the group $O_L(\mathbb{Z})$ acts transitively on norm 0 vectors we can assume that

$$\rho = (0, 0, 1) \in \bigoplus_{p-1} U \oplus \bigoplus_{q} E_{q}(-1) \oplus E_{q}(-1) \oplus U$$

Then $\vec{v} = (\vec{\lambda}, a, b)$ with $a$ not an integer. We will find some $\vec{\mu}$ of the form $\vec{\mu} = (0, m, n)$ with integers $m$ and $n$ such that

$$|\langle \vec{\mu} - \vec{v}, \vec{\mu} - \vec{v} \rangle - x| = |\langle \vec{v}, \vec{v} \rangle - 2(a - m)(b - n) - x| < 1.$$ 

Since $a$ is not an integer we can find some integer $m$ such that $|a - m| < 1$. Whenever we add 1 to $n$, the expression $2(a - m)(b - n)$ is changed by a non zero number less than 2, so we can choose some integer $n$ such that $2(a - m)(b - n)$ is at a distance of less then 1 from any given number $x - \lambda^2$. This proves Proposition 13. \hfill \blacksquare

**Proposition 14** Suppose that $R_1$ is the set of norm $-2$ vectors of $M$ having inner product 1 with $e = (0, 0, 1) \in L \oplus U$. Suppose that $\Gamma_1$ is the subgroup of $O_M(\mathbb{Z})$ generated by reflections of vectors of $R_1$ and the automorphism $-id$. Then any vector $\vec{r} \in M$ is conjugate under $\Gamma_1$ to a vector of the form $(\vec{v}, m, n) \in M$ such that either $m = 0$ or $\frac{\vec{r}}{m} \in L$ and $m > 0$.

**Proof:** We can assume that $\vec{\rho} = (\vec{v}, m, n)$ has the property that $|\langle \vec{\rho}, \vec{\rho} \rangle| = |m|$ is minimal among all conjugates of $\vec{r}$ under $\Gamma_1$, where $\vec{\rho} = (0, 0, 1)$. If $m = 0$ then we are done. So we can assume that $m \neq 0$, and wish to prove that
\( \frac{v}{m} \in L \) and \( m > 0 \). Suppose that \( \frac{v}{m} \notin L \). By Proposition 13 we can find a vector \( \frac{m}{\mu} \in L \) satisfying
\[
\left| \left( \frac{\mu}{m} - \frac{v}{m}, \frac{\mu}{m} - \frac{v}{m} \right) + \left( -\frac{2n}{m} - \frac{\langle v, v \rangle}{m^2} \right) \right| < 1. \tag{13}
\]
Let \( \delta = \left( \frac{\mu}{m}, 1, -\frac{\langle \mu, v \rangle - 2}{m} \right) \). It is easy to see that \( \langle \delta, \delta \rangle = -2 \). Let \( T_\delta(r) = r' = r + \langle r, \delta \rangle \delta \), i.e. \( r' \) is the reflection of \( r \) with respect to the hyperplane of \( \delta \in R_1 \).

Direct computations show that
\[
|\langle r', e \rangle| = |\langle T_\delta(r), e \rangle| = |\langle r, T_\delta(e) \rangle| = |\langle r, e + \delta \rangle| =
\]
\[
\left| m \left( \frac{\mu}{m} - \frac{v}{m}, \frac{\mu}{m} - \frac{v}{m} \right) + \left( -\frac{2n}{m} - \frac{\langle v, v \rangle}{m^2} \right) \right|. \tag{14}
\]
Combining (13) and (14) we deduce that
\[
|\langle r', e \rangle| = \left| m \left( \frac{\mu}{m} - \frac{v}{m}, \frac{\mu}{m} - \frac{v}{m} \right) + \left( -\frac{2n}{m} - \frac{\langle v, v \rangle}{m^2} \right) \right| < m. \tag{15}
\]
We have chosen
\[
|\langle r, \delta \rangle| = m \tag{16a}
\]
and conclude that Proposition 14 is proved.

Proof of Lemma 12: Let \( \delta = (v, m, n) \). By Proposition 14 we can assume that either \( m = 0 \) or \( \frac{v}{m} \in L \). If \( m = 0 \) then Lemma 15 is proved. Suppose that \( \frac{v}{m} \in L \) holds. Then \( \left( \frac{v}{m}, \frac{v}{m} \right) \in \mathbb{Z} \). Thus
\[
\langle \delta, \delta \rangle = -2 = m^2 \left( \frac{\langle v, v \rangle}{m}, \frac{\langle v, v \rangle}{m} \right) + 2mn. \tag{17}
\]
So (17) implies that \(-2\) is divisible by \((m^2, 2m)\). From here we conclude that \( m = 1 \). Lemma 15 is proved.

Let us define the group \( \Gamma_3 \) as the group generated by the automorphisms \( O_L(\mathbb{Z})^+ \) extended to automorphisms of \( M \) by letting them act trivially on \( U \), the group of automorphisms taking \( (v, m, n) \) to \( (v + 2m\lambda, m, n - \langle v, \lambda \rangle - m \langle \lambda, \lambda \rangle) \) for \( \lambda \in L \), and the group of automorphisms given by reflections of norm \(-2\) vectors in \( R_1 \).

Lemma 15 The group \( \Gamma_3 \) acts transitively on the set of vectors of norm \(-2\) in \( M \).

Proof: The proof of Lemma 15 is based on the following Propositions:

Proposition 16 The group \( O_L(\mathbb{Z})^+ \) acts transitively on the set of vectors of norm \(-2\) in \( L \).
Proof: Since by definition \( L = \bigoplus_{p-1} U \oplus E_8(-1) \oplus E_8(-1) \oplus U \) then

Proposition 16 follows from the induction hypothesis. ■

Proposition 17 There exists an element \( \sigma \in \Gamma_3 \) such that if \( \delta = (v, 0, k) \) and \( \delta^2 = -2 \), then \( \sigma(\delta) = (\mu, 0, 0) \).

Proof: The condition \( \langle \delta, \delta \rangle = -2 \) implies that \( \langle v, v \rangle = -2 \). Thus \( v \) is a primitive element in \( L \). Proposition 16 implies that there exists an element \( \sigma \in \mathcal{O}_L(\mathbb{Z})^+ \) such that \( \sigma(\delta) = (e_1 - e_2, 1, -1) \in U \subset L \). Thus easy that \( \lambda = -ke_2 \in L \) such that \( \langle \lambda, v \rangle = -k \). Then from the definition of the group \( \Gamma_3 \) we know that the map

\[
\delta = (v, 0, k) \rightarrow (v, 0, k + \langle \lambda, v \rangle) = (v, 0, 0)
\]

is an automorphism. Proposition 17 is proved. ■

Proposition 18 Suppose that \( \delta \in R_0 \). Then there exists an element \( \sigma \in \Gamma_3 \) such that \( \sigma(\delta) \in R_1 \).

Proof: Proposition 17 implies that without loss of generality we may assume that \( \delta \in L \). Let \( \lambda \in L \), \( \langle \delta, \lambda \rangle \neq 0 \) and \( \langle \lambda, \lambda \rangle \neq 0 \). Let us consider

\[
r = \left( \lambda, 1, -\frac{\langle \lambda, \lambda \rangle + 2}{2} \right) \in L \oplus U = M.
\]

Clearly \( \langle r, r \rangle = -2 \). Then the map \( T_r(\delta) = \delta + \langle r, \delta \rangle r \) is an element of \( \Gamma_3 \) and clearly \( T_r(\delta) \in R_1 \). Proposition 18 is proved. ■

Combining Lemma 12 with Propositions 16, 17 and 18 we derive Lemma 15.

Lemma 15 implies directly Theorem 11. ■

4 Moduli of K3 Surfaces

4.1 Definition of a K3 Surface

A K3 surface is a compact, complex two dimensional manifold with the following properties: i. There exists a non-zero holomorphic two form \( \omega \) on \( X \) without zeroes. ii. \( H^1(X, \mathcal{O}_X) = 0 \).

In [2] and [5], the following topological properties are proved. The surface \( X \) is simply connected, and the homology group \( H_2(X, \mathbb{Z}) \) is a torsion free abelian group of rank 22. The intersection form \( \langle u, v \rangle \) on \( H_2(X, \mathbb{Z}) \) has the properties: 1. \( \langle u, u \rangle = 0 \mod(2) \). 2. \( \det(\langle e_i, e_j \rangle) = -1 \). 3. The symmetric form \( <, > \) has a signature \((3, 19)\).

Theorem 5 on page 54 of [26] implies that as an Euclidean lattice \( H_2(X, \mathbb{Z}) \) is isomorphic to the K3 lattice \( \Lambda_{K3} \), where \( \Lambda_{K3} := \mathbb{U}^3 \oplus (-E_8)^2 \). Every K3 surface is also simply connected.
4.2 Moduli of Marked, Algebraic and Polarized K3 surfaces

**Definition 19** Let $\alpha = \{\alpha_i\}$ be a basis of $H_2(X,\mathbb{Z})$ with intersection matrix $\Lambda_{K3}$. The pair $(X, \alpha)$ is called a marked K3 surface. Let $l \in H^{1,1}(X,\mathbb{R}) \cap H^2(X,\mathbb{Z})$ be the Poincare dual class of a hyperplane section, i.e., an ample divisor. The triple $(X, \alpha, l)$ is called a marked, polarized K3 surface. The degree of the polarization is an integer $2d$ such that $\langle l, l \rangle > 0$.

**Definition 20** The period map $\pi$ for marked K3 surfaces $(X, \alpha)$ is defined by integrating the holomorphic two form $\omega$ along the basis $\alpha$ of $H_2(X,\mathbb{Z})$, meaning

$$\pi(X, \alpha) := (\ldots, \int_{\alpha_i} \omega, \ldots) \in \mathbb{P}^{21}.$$  

The Riemann bilinear relations hold for $\pi(X, \alpha)$, meaning

$$\langle \pi(X, \alpha), \pi(X, \alpha) \rangle = 0 \text{ and } \langle \pi(X, \alpha), \overline{\pi(X, \alpha)} \rangle > 0. \quad (18)$$

Choose a primitive vector $l \in \Lambda_{K3}$ such that $\langle l, l \rangle = 2d > 0$. Let us denote $\Lambda_{K3,l} := \{v \in \Lambda_{K3} | \langle l, v \rangle = 0\}$. Then $\pi(X, \alpha, l) \in \mathbb{P}(\Lambda_{K3,l} \otimes \mathbb{C})$ and it satisfies (18) consists of two components isomorphic to the symmetric space $\mathfrak{h}_{2,19}$. In [25] the following Theorem was proved:

**Theorem 21** The moduli space $\mathcal{M}^{2d}_{K3,mpa}$ of marked, polarized, algebraic K3 surfaces of a fixed degree $2d$ exists and it is embedded by the period map into $\mathfrak{h}_{2,19}$ is an open everywhere dense subset. Let

$$\Gamma_{K3,2d} = \{ \phi \in \text{Aut}^+(\Lambda_{K3}) | \langle \phi(u), \phi(u) \rangle = \langle u, u \rangle \text{ and } \phi(l) = l \},$$

where $l$ is a primitive vector such that $\langle l, l \rangle = 2d > 0$. Then the moduli space $\mathcal{M}^{2d}_{K3,pa}$ of polarized, algebraic K3 surfaces of a fixed degree $2d$ is isomorphic to a Zariski open set in the quasi-projective variety $\Gamma_{K3,2d} \setminus \mathfrak{h}_{2,19}$.

By pseudo-polarized algebraic K3 surface we understand a pair $(X, l)$ where $l$ corresponds to either ample divisor or pseudo ample divisor, which means that for any effective divisor $D$ in $X$, we have $\langle D, l \rangle \geq 0$. Mayer proved the linear system $|3l|$ defines a map: $\phi_{|3l|} : X \to X_1 \subset \mathbb{P}^m$ such that: i. $X_1$ has singularities only double rational points. ii. $\phi_{|3l|}$ is a holomorphic birational map. Let us denote by $\mathcal{M}^{2d}_{K3,ppa}$ the moduli space of pseudo-polarized algebraic K3 surfaces of degree $2d$. From the results proved in [12], [23], [?] and [25] the following Theorem follows:

**Theorem 22** The moduli space of $\mathcal{M}^{2d}_{K3,ppa}$ is isomorphic to the locally symmetric space $\Gamma_{K3,2d} \setminus \mathfrak{h}_{2,19}$. 

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4.3 Discriminant of Pseudo-Polarized K3 Surfaces

The complement of \( \mathcal{M}_{K3,pa}^{2d} \) in \( h_{2,19} \) can be described as follow. Given a polarization class \( e \in \Lambda_{K3} \), set \( T_e \) to be the orthogonal complement to \( e \) in \( \Lambda_{K3} \), i.e. \( T_e \) is the transcendental lattice. Then we have the realization of \( h_{2,19} \) as one of the components of \( h_{2,19} \cong \{ u \in \mathbb{P}(T_e \otimes \mathbb{C}) \mid \langle u, u \rangle = 0 \text{ and } \langle u, u \rangle > 0 \} \). For each \( \delta \in \Delta(e) \), define the hyperplane \( H(\delta) = \{ u \in \mathbb{P}(T_e \otimes \mathbb{C}) \mid \langle u, \delta \rangle = 0 \} \).

Let \( \mathcal{H}_{K3,2d} = \bigcup_{\delta \in \Delta(e)} (H(\delta) \cap h_{2,19}) \).

Let us define the discriminant \( \mathcal{D}_{K3}^{2d} := \Gamma_{K3,2d} \backslash \mathcal{H}_{K3,2d} \). Results from [24], [25], [27] and [23] imply that \( \mathcal{D}_{K3}^{2d} \) is the complement of the moduli space of algebraic polarized K3 surfaces \( \mathcal{M}_{K3,pa}^{2d} \) in the locally symmetric space \( \Gamma_{K3,2d} \backslash h_{K3,2d} \), i.e. \( \mathcal{D}_{K3}^{2d} = (\Gamma_{K3,2d} \backslash h_{K3,2d}) - \mathcal{M}_{K3,pa}^{2d} \).

4.4 Moduli of Elliptic K3 Surfaces with a Section

**Definition 23** We will define an elliptic K3 surface \( X \) as a K3 surface such that there exists a holomorphic map \( \pi : X \to \mathbb{CP}^1 \) such that it has a section \( s \).

The following Theorem is a well known fact:

**Theorem 24** The moduli space of elliptic pseudo polarized K3 surfaces with a polarization class \( e \) is isomorphic to \( \Gamma_{\text{ell},e} \backslash h_{2,18} \), where \( \Gamma_{\text{ell},e} \) is defined as follows.

**Proof:** Theorem 24 follows from Theorem 3.1 proved in [13]. □

4.5 Moduli of Einstein Metrics on K3 Surfaces

Let \( X \) be a K3 surface with a fixed \( C^\infty \) structure. Let us consider the set \( \mathcal{M}_E \) of all metrics \( g \) on \( X \) for which \( \text{Ricci} g = 0 \) with a volume one. We will define the moduli space \( \mathcal{M}_E \) of Einstein metrics as follows: \( \mathcal{M}_E := \mathcal{M}_E / Diff^+(X) \) where \( Diff^+(X) \) is the group of diffeomorphisms of \( X \) preserving the orientation. In [22] the following Theorem was proved:

**Theorem 25** We have the following isomorphism

\[ \mathcal{M}_E \cong O^+(\Lambda_{K3})/SO(3,19)/SO(2) \times SO(19), \]

where \( O^+(\Lambda_{K3}) \) is the group of isomorphisms of the K3 lattice \( \Lambda_{K3} \) which preserve the spinor norm.

**Lemma 26** The moduli space of all marked elliptic K3 surfaces is an everywhere dense subset in \( h_{3,19} \).

**Proof:** We know that the set of all three dimensional vector subspaces \( E \) in \( \Lambda_{K3} \otimes \mathbb{R} \) such that \( E \) contains a non zero vector in \( \Lambda_{K3} \) form an everywhere dense subset in \( h_{3,19} \). Let us denote for any fixed \( v \in \Lambda_{K3} \) such that \( h_{2,19}(v) := \{ E, \text{all oriented three dim } E \subset \Lambda_{K3} \otimes \mathbb{R}, \ v \in E, \text{ and } \langle , \rangle |_{E} > 0 \} \).
The definition of $h_{2,19}(v)$ implies that it is an open set in the Grass$(2,21)$ and thus $h_{2,19}(v) = \text{SO}_0(2,19)/\text{SO}(2) \times \text{SO}(19)$. It is easy to see that we can identify $h_{2,19}(v)$ with the moduli space of marked polarized K3 surfaces with a polarization class $v$. It is easy to see that all $h_{2,19}(v)$ for all primitive $v \in \Lambda_{K3}$ such that $\langle v, v \rangle > 0$ form an everywhere dense subset in $h_{2,19}$. On the other hand it is an easy exercise to see that all elliptic K3 surfaces with polarization vector $v$ and which have a section form an everywhere dense subset in $h_{2,19}(v)$. For the proof of this fact see [21]. From here lemma 26 follows directly.

### 4.6 Moduli of K3 Surfaces with B Fields

**Definition 27** Let $X$ be a K3 surface. A complex closed form $\omega_X(1,1)$ of type $(1,1)$ such that

$$\int_X \text{Im} \omega(1,1) \land \text{Im} \omega(1,1) > 0$$

will be called a B field on $X$. The triple $(X, \alpha, \omega_X(1,1))$, where $\alpha$ is a marking of the K3 surface and $\omega_X(1,1)$ is a B-field will be called a marked K3 surface with a B-field.

The moduli space of marked K3 surfaces with a B field are described by the following Theorem:

**Theorem 28** The moduli space of marked K3 surfaces with B field is isomorphic to $h_{4,20} := \text{SO}_0(4,20)/\text{SO}(4) \times \text{SO}(20)$. See [1].

**Proof:** The proof is based on assigning to each marked K3 surfaces $X$ with a fixed B field a four dimensional oriented subspace in $\Lambda_{\text{ext},K3} \otimes \mathbb{R}$, where

$$\Lambda_{\text{ext},K3} := \Lambda_{K3} \oplus \mathbb{U} = \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8(-1) \oplus \mathbb{E}_8(-1) \cong H^*(M,\mathbb{Z}).$$

Let $\omega_X$ be the holomorphic to form on $X$. Then to the marked K3 surface $(X; \gamma_1, ..., \gamma_22)$ we assign the two dimensional oriented subspace in $\Lambda_{K3} \otimes \mathbb{R} \subset \Lambda_{\text{ext},K3} \otimes \mathbb{R}$ spanned by $\text{Re} \omega_X$ and $\text{Im} \omega_X$. Let us assign to the B field $\omega_X(1,1)$ the vector

$$V_{X,\omega_X(1,1)} := \left( \omega_X(1,1), 1, -\frac{\langle \omega_X(1,1), \omega_X(1,1) \rangle}{2} \right) \in (\Lambda_{K3} \oplus \mathbb{U}) \otimes \mathbb{R}.$$ 

It is easy to check that

$$\langle V_{X,\omega_X(1,1)}, V_{X,\omega_X(1,1)} \rangle = 0 \text{ and } \langle V_{X,\omega_X(1,1)}, \overline{V_{X,\omega_X(1,1)}} \rangle > 0. \quad (19)$$

From (19) we derive that

$$\langle \text{Re} V_{X,\omega_X(1,1)}, \text{Re} V_{X,\omega_X(1,1)} \rangle = \langle \text{Im} V_{X,\omega_X(1,1)}, \text{Im} V_{X,\omega_X(1,1)} \rangle > 0$$

and

$$\langle \text{Re} V_{X,\omega_X(1,1)}, \text{Im} V_{X,\omega_X(1,1)} \rangle = 0.$$
Thus the restriction of the bilinear form of \( \Lambda_{ext,K3} \) to the four dimensional subspace \( E_{X,\omega_X(1,1)} \) spanned by \( \text{Re} \omega_X, \text{Im} \omega_X, \text{Re} V_{X,\omega_X(1,1)}, \text{Im} V_{X,\omega_X(1,1)} \) is strictly positive. So we constructed a map from any marked K3 surface with a B field to \( h_{1,20} := SO_0(4,20)/SO(4) \times SO(20) \). Torelli Theorem for marked K3 surfaces implies that this map is injective.

Let \( E_r \) be a four dimensional positive oriented four dimensional plane in \( (\Lambda_{K3} \oplus U) \otimes \mathbb{R} \). Let us consider the intersection \( E_r \cap (\Lambda_{K3} \otimes \mathbb{R}) \). The orientation of \( E_r \) induces an orientation on the two dimensional positive plane \( E_r \cap (\Lambda_{K3} \otimes \mathbb{R}) \) in \( \Lambda_{K3} \otimes \mathbb{R} \). From the epimorphism of the period map we can conclude that there exists a marked K3 surface \( (X, \alpha) \) such that the two dimensional space \( E_r \cap (\Lambda_{K3} \otimes \mathbb{R}) \) is generated by \( \text{Re} \omega_X \) and \( \text{Im} \omega_X \). Let us denote by \( E_X \) the two dimensional oriented subspace \( E_r \cap (\Lambda_{K3} \otimes \mathbb{R}) \). Let \( E_X^\perp \) be the orthogonal oriented complement to \( E_X \) in \( E_r \). Let \( \omega_1 \) and \( \omega_2 \) be two orthonormal vectors in \( E_X^\perp \). Let us form the class of cohomology \( \Omega = \omega_1 + \sqrt{-1} \omega_2 \in \Lambda_{K3} \oplus U \).

It is easy to check that

\[
\langle \Omega, \Omega \rangle = 0 \text{ and } \langle \Omega, \overline{\Omega} \rangle > 0. \tag{20}
\]

We need the following obvious Lemma:

**Lemma 29** Let \( e_i \) be the standard basis of the hyperbolic lattice \( U \), i.e. \( \langle e_i, e_i \rangle = 0 \) and \( \langle e_1, e_2 \rangle = 1 \). Let us consider \( \Lambda_{K3} \oplus U \). Then we have \( \langle \Omega, e_i \rangle \neq 0 \) for \( i = 1, 2 \).

**Proof:** Suppose that \( \langle \Omega, e_1 \rangle = 0 \). Let \( \Omega_1 := \Omega \cap (\Lambda_{K3} \otimes \mathbb{R}) \). Then combining the assumption \( \langle \Omega, e_1 \rangle = 0 \) with (20) we get that

\[
\langle \Omega_1, \Omega_1 \rangle = 0 \text{ and } \langle \Omega_1, \overline{\Omega_1} \rangle > 0. \tag{21}
\]

Thus (21) implies that \( \text{Re} \Omega_1 \) and \( \text{Re} \omega_1 \) span a two dimensional positive subspace in \( \Lambda_{K3} \otimes \mathbb{R} \) which is orthogonal to the two dimensional positive subspace \( E_X \) in \( \Lambda_{K3} \otimes \mathbb{R} \). So their direct sum will give a four dimensional positive subspace in \( \Lambda_{K3} \otimes \mathbb{R} \). This is impossible since the signature of the quadratic form on \( \Lambda_{K3} \otimes \mathbb{R} \) is \((3, 19)\). Lemma 29 is proved. \( \blacksquare \)

Let us normalize \( \Omega \) such that \( \langle \Omega, e_2 \rangle = 1 \). Then \( \Omega_1 := \Omega \cap (\Lambda_{K3} \otimes \mathbb{R}) \) will satisfy the conditions (21) which imply that \( \langle \text{Im} \Omega_1, \text{Im} \Omega_1 \rangle > 0 \). The definition of \( \Omega \) implies that \( \langle \Omega_1, \text{Re} \Omega_X \rangle = \langle \Omega_1, \text{Im} \Omega_X \rangle = 0 \). Thus \( \Omega_1 \) will be a form of type \((1, 1)\) on \( X \). Thus \( \langle \text{Im} \Omega_1, \text{Im} \Omega_1 \rangle > 0 \) implies that \( \Omega_1 \) is a \( B \)-field on \( X \). Theorem 28 is proved. \( \blacksquare \)

### 5 Automorphic Forms on \( \Gamma \backslash h_{p,q} \) and Theta Lifts

#### 5.1 Automorphic Forms of Weight -2 on \( \Gamma \backslash h_{p,q} \)

In this paper the group \( \Gamma \) will be the group of automorphisms of \( \Lambda_{K3} \) which preserve the spinor norm, i.e. \( \Gamma = O^+_1 (\Lambda_{K3}) \) is a subgroup of index 2 in the group of automorphism \( O_{\Lambda_{K3}} (\mathbb{Z}) \) of the lattice \( \Lambda_{K3} \). It was Donaldson who proved in
that the mapping class group of a K3 surface is isomorphic to $\Gamma$. We will define the one cocycle $\mu(\gamma, \tau)$ of the group $O_{\Lambda_{K3}}(\mathbb{Z})$ with coefficients the non singular $3 \times 3$ matrices with coefficients functions on $\mathfrak{h}_{3,19}$.

Let an element $\gamma \in \Gamma$ be represented by a matrix $(\gamma_{k,l})$ of size $(22 \times 22)$. Let $\tau \in \mathfrak{h}_{3,19}$ then the point $\tau$ is represented by the vectors $g_1(\tau), g_2(\tau)$ and $g_3(\tau)$ in the fix basis $e_1, \ldots, e_{22}$. They span a three dimensional oriented subspace in $\mathbb{R}^{3,19} = \Lambda_{K3} \otimes \mathbb{R}$ on which the intersection form is strictly positive. We know that the point $\tau$ can be represented by the $3 \times 22$ matrix $(E_3, \tau_{ij})$, where $E_3$ is the identity $3 \times 3$ matrix. The action of $\Gamma$ on $\mathfrak{h}_{3,19}$ is described as follow: Take the product of the matrices $(E_3, \tau_{ij}) \times (\gamma_{k,l})$. It can be represented as follows:

$$(E_3, \tau_{ij}) \times (\gamma_{k,l}) = (\mu(\gamma, \tau), \sigma_{\gamma,ij}(\tau)).$$

(22)

where $\mu(\gamma, \tau)$ is $3 \times 3$ matrix $A_{\tau,0}$ defined by (22) and $\sigma_{\gamma,ij}(\tau)$ is some $3 \times 19$ matrix. Theorem [2] implies that the $3 \times 3$ matrix $\mu(\gamma, \tau)$ has rank 3, i.e. $\det(\mu(\gamma, \tau)) \neq 0$.

**Definition 30** Let $\Phi(\tau)$ be a function on $\mathfrak{h}_{3,19}$ such that it satisfies the following functional equation:

$$\Phi(\tau \gamma) = (\det \mu(\gamma, \tau))^k \Phi(\tau).$$

Then we will call $\Phi(\tau)$ an automorphic form of weight $k$.

**Definition 31** Let us recall that according to Theorem [4] to each point $\tau = (\tau^i_j) \in \mathfrak{h}_{3,19}$, $1 \leq j \leq 3$ and $1 \leq i \leq 19$ we assigned the three rows vector $g_i$ of the matrix $(\tau^i_j)$. We will define the function $g(\tau)$ on $\mathfrak{h}_{3,19}$ as follows

$$g(\tau) := \det((g_i, g_j)).$$

(23)

**Theorem 32** The function $g(\tau)$ defined in (23) is an automorphic form of weight $-2$.

**Proof:** We need to compute

$$g(\gamma(\tau)) = \det((\mu(\gamma, \tau) \times g_i(\gamma(\tau)), \mu(\gamma, \tau) \times g_j(\gamma(\tau)))) = ?$$

Theorem [2] and the expression of the matrix $\mu(\gamma, \tau)$ given by (22) imply

$$g(\gamma(\tau)) = \det(\mu(\gamma, \tau) \times (g_i(\gamma(\tau)), g_j(\gamma(\tau))) \times (\mu(\gamma, \tau))^t) = \det(\mu(\gamma, \tau))^2 \det((g_i(\tau), g_j(\tau))) = (\det(\mu(\gamma, \tau))^2) \times g(\tau).$$

Thus we $g(\gamma(\tau)) = \det(\mu(\gamma, \tau))^2 g(\tau)$. So Theorem 32 is proved. ■

6 Regularized Determinants

6.1 Construction of an Automorphic Form with a Zero Set Supported by the Discriminant Locus on $\Gamma\setminus\mathfrak{h}_{3,19}$

The following result follows directly from the results proved in [11].
Let $\Lambda_{p,q}$ be an even unimodular lattice of signature $(p,q)$. Then there exists a non zero automorphic form $\exp(\Phi_{\Lambda_{p,q}}(\tau))$ such that the zero set of $\exp(\Phi_{\Lambda_{p,q}}(\tau))$ coincide with the discriminant $D_{\Lambda_{p,q}} \subset \mathcal{O}^+_{\Lambda_{p,q}}(\mathbb{Z}) \setminus \mathfrak{h}_{p,q}$. Moreover let $\Lambda_{p_1,q_1}$ be an even unimodular sublattice in $\Lambda_{p,q}$. Then

$$
\exp(\Phi_{\Lambda_{p,q}}(\tau)) \big|_{\mathcal{O}^+_{\Lambda_{p_1,q_1}}(\mathbb{Z}) \setminus \mathfrak{h}_{p_1,q_1}} = \exp(\Phi_{\Lambda_{p_1,q_1}}(\tau)).
$$

(24)

We will consider the case of K3 surfaces. We know that $\Lambda_{K3} = \Lambda_{3,19}$. We will study the relations between the non zero automorphic form $\exp(\Phi_{\Lambda_{K3}}(\tau))$ and the regularized determinants.

**Theorem 34** $\Delta_B \Phi_{\Lambda_{K3}}(\tau, \sigma) = 0$.

**Proof:** Any choice of an embedding of the hyperbolic lattice $\mathbb{U} \subset \Lambda_{K3}$ defines a totally geodesic subspace $\mathfrak{h}_{2,18}$ into $\mathfrak{h}_{3,19}$. This follows from Theorem 33. According to the construction of the automorphic form $\exp(\Phi_{\Lambda_{2,18}}(\tau))$ given in [11] it follows that $\Phi_{\Lambda_{2,18}}$ is a holomorphic function on $\mathfrak{h}_{2,18}$. Thus we have $\Delta_B \Phi_{\Lambda_{2,18}} = 0$. All the embeddings $\mathfrak{h}_{2,18} \subset \mathfrak{h}_{3,19}$ corresponding to primitive embeddings $\mathbb{U} \subset \Lambda_{K3}$ form an everywhere dense subset in $\mathfrak{h}_{3,19}$. Since $\mathfrak{h}_{2,18}$ is a totally geodesic subspace in $\mathfrak{h}_{3,19}$ we get that $\Delta_B \Phi_{\Lambda_{K3}}|_{\mathfrak{h}_{2,18}} = \Delta_B \Phi_{\Lambda_{2,18}}$. Thus the restriction of the Bergman Laplacian applied to on $\Phi_{\Lambda_{2,18}}$ is zero on an everywhere dense subset in $\mathfrak{h}_{3,19}$. Thus the continuous function $\Delta_B \Phi_{\Lambda_{K3}}$ is zero on everywhere dense subset in $\mathfrak{h}_{3,19}$. From here we deduce that $\Delta_B \Phi_{\Lambda_{K3}} = 0$. Theorem 34 is proved. ■

### 6.2 Variational Formula

**Theorem 35** The function $\log \det \Delta_{KE} - \log \det (\langle g_i(\tau), g_j(\tau) \rangle)$ is a harmonic function on the moduli space $\mathcal{M}_E$ of Einstein metrics of the K3 surface with respect to the Laplacian corresponding to the Bergman metric.

**Proof:** The proof of Theorem 35 is based on the following Lemmas:

**Lemma 36** Let $\tau_0 \in \mathfrak{h}_{3,19}$. Then there exists a totally geodesic subspace $\mathfrak{h}_{2,19}$ passing through $\tau_0 \in \mathfrak{h}_{3,19}$ and its points correspond to polarized marked K3 surfaces.

**Proof:** We know that each point $\tau \in \mathfrak{h}_{3,19}$ corresponds to a three dimensional subspace $E_\tau \subset H^2(X, \mathbb{R})$ on which the cup product is strictly positive. Let $L \in H^2(X, \mathbb{R})$ be fixed and $\langle L, L \rangle > 0$. Let us consider the following set:

$$
\mathfrak{h}_L := \{\text{three dimensional oriented positive subspaces in } H^2(X, \mathbb{R}) \text{ containing } L\}.
$$

It is easy to see that there is one to one correspondence between the two dimensional oriented positive subspaces in the orthogonal complement $L^\perp = \mathbb{R}^{2,19}$ and $\mathfrak{h}_L$. Thus we get that $\mathfrak{h}_L = \mathfrak{h}_{2,19} = \mathcal{S}O(2,19)/\mathcal{S}O(2) \times \mathcal{S}O(19)$. Lemma 36 is proved. ■
Let us choose an orthonormal basis $e_1, e_2$ and $e_3 = L$ of the three dimensional subspace $E_{\tau_0} \in \mathfrak{h}_L$. Lemma 36 and Corollary 4 imply that the three dimensional subspaces $E_{\tau}$ that correspond to $\tau \in \mathfrak{h}_{2,19} \subset \mathfrak{h}_{3,19}$ are spanned by vectors:

$$f_1 = e_1 + \sum_{i=1}^{19} \tau_i^1 e_i, \quad f_2 = e_2 + \sum_{i=1}^{19} \tau_i^2 e_i \text{ and } f_3 = L = e_3. \tag{25}$$

**Lemma 37** In the coordinate system defined by Corollary 5 and by $(25)$ the totally geodesic subspace is given by the equations $\tau_i^3 = 0$ for $i = 1, \ldots, 19$.

**Proof:** The proof follows directly from $(25)$. ■

We know that $\mathfrak{h}_{2,19}$ is a complex manifold. The complex coordinates on $\mathfrak{h}_{2,19}$ are defined as follows:

$$\rho^i = \tau_i^1 + \sqrt{-1} \tau_i^2. \tag{26}$$

From the epimorphism of the period map we know that $\tau_0$ corresponds to a K3 surface $X_{\tau_0}$ and the class of cohomology of the complex two form $e_1 + \sqrt{-1} e_2$ can be identified with the class of cohomology of the holomorphic two form $\omega_{\tau_0}(2,0)$ on $X_{\tau_0}$. The vector $e_3 = L$ can be identified with the class of cohomology of the imaginary part of a Kähler metric on $X_{\tau_0}$.

We will define the Weil-Petersson metric on $\mathfrak{h}_{2,19}$ as the restriction of the metric on $\mathfrak{h}_{3,19}$ defined by (11).

**Lemma 38** The Weil-Petersson Metric on $\mathfrak{h}_{2,19}$ is a Hermitian metric.

**Proof:** From the expression of the Bergman metric in the coordinates $(\tau_i^j)$ given by (12) and Lemmas 37 it follows that its restriction on $\mathfrak{h}_{2,19}$ is given by

$$ds_B^2|_{\mathfrak{h}_{2,19}} = \sum_{i=1}^{19} \left( (d\tau_i^1)^2 + (d\tau_i^2)^2 \right) + O(2). \tag{27}$$

Combining (26) with (27) we get that

$$ds_B^2|_{\mathfrak{h}_{2,19}} = \sum_{i=1}^{19} (d\rho^i) \otimes (d\rho^i) + O(2). \tag{28}$$

Lemma 38 is proved. ■

**Lemma 39** Let $\tau_0 \in \mathfrak{h}_{3,19}$. Let $\mathfrak{h}_{2,19}$ be the totally geodesic subspace passing through $\tau_0 \in \mathfrak{h}_{3,19}$ and defined by the $L \in E_{\tau}$ as in Lemma 36. Then log det $\langle (g_i(\tau), g_j(\tau)) \rangle|_{\mathfrak{h}_{2,19}}$ is a potential of the Weil-Petersson metric on $\mathfrak{h}_{2,19}$.

**Proof:** Since the matrix $\langle (g_i(\rho), g_j(\rho)) \rangle|_{\mathfrak{h}_{2,19}}$ is symmetric and

$$\langle (g_i(\rho), g_j(\rho)) \rangle|_{\mathfrak{h}_{2,19}} = I_2 + (h_{ij}(\rho))$$
then the following formula is true:

$$\log \det ((g_i(\rho), g_j(\rho))) |_{h_{2,19}} = \sum_{i=1}^{2} \log(1 + \lambda_i),$$

where $\lambda_i$ are the eigen values of the matrix $(h_{ij}(\tau))$. Thus we get

$$\sum_{i=1}^{2} \lambda_i = h_{11}(\tau) + h_{22}(\tau). \quad (29)$$

From the definition of the matrix $((g_i(\tau), g_j(\tau))) |_{h_{2,19}}$ we get that

$$h_{11} = \sum_{i=4}^{22} (\tau_i^1)^2 \text{ and } h_{22} = \sum_{i=4}^{22} (\tau_i^2)^2. \quad (30)$$

Combining (26), (29) and (30) we get that

$$\log \det ((g_i(\rho), g_j(\rho))) |_{h_{2,19}} = \sum_{i=4}^{22} |\rho_i|^2 + O(3). \quad (31)$$

Thus we get from (31) that

$$dd^c (\log \det ((g_i(\rho), g_j(\rho))) |_{h_{2,19}}) = \frac{\sqrt{-1}}{2} \sum_{i=4}^{22} \partial \rho \wedge \overline{\partial \rho} + O(2). \quad (32)$$

From (32) and (28) we conclude the proof of Lemma 39. \[\square\]

**Lemma 40** Let $\Delta_B$ be the Laplacian of the Bergman metric on $h_{3,19}$. Then the restriction of the function $\Delta_B (\log \det \Delta_{KE} - \log \det ((g_i(\tau), g_j(\tau))))$ on each totally geodesic subspace $h_{2,19} \subset h_{3,19}$ is zero.

**Proof:** In [6] the following Theorem was proved:

**Theorem 41** Let $M$ be a CY manifold with a polarization class $L \in H^2(M, \mathbb{Z}) \cap H^{1,1}(M, \mathbb{R})$. Let $\det \Delta_{(0,1)}$ be the regularized determinant of the Laplacian corresponding to the Calabi Yau metric corresponding to the polarization class $L$ and acting on the space of $(0, 1)$ forms. Then $dd^c \log \det \Delta_{(0,1)} = -\text{Im} W.P.$.

Combining Theorem 41 with Lemma 39 we deduce Lemma 40. \[\square\]

It is an obvious fact that the set of three dimensional positive subspaces in $\Lambda_{K3} \otimes \mathbb{R}$ which contain a vector in $\Lambda_{K3} \otimes \mathbb{Q}$ form an everywhere dense subset in $h_{3,19}$. From here it follows that we can find an everywhere dense subset of totally geodesic subsets $h_{2,19}$ in $h_{3,19}$ on which the continuous function

$$\Delta_B (\log \det \Delta_{KE} - \log \det ((g_i(\tau), g_j(\tau))))$$

is zero. Therefore it is zero on $h_{3,19}$. Theorem 35 is proved. \[\square\]
6.3 Relation of Regularized Determinants with Automorphic Forms

**Theorem 42** The following formula holds for the regularized determinant of the Laplacian of the Einstein metrics $\det(\Delta_{KE}(\tau)) = \det(\langle g_i(\tau), g_j(\tau) \rangle) \times |\exp(\Phi_{\Lambda_{K3}}(\tau))|^2$.

**Proof:** According to Theorem 35 the function $\log \det \Delta_{KE} - \log \det(\langle g_i(\tau), g_j(\tau) \rangle)$ is a harmonic function with respect to the Laplacian of the Bergman metric on $h_{3,19}$.

Let us consider the function:

$$\frac{\det \Delta_{KE}}{\det(\langle g_i(\tau), g_j(\tau) \rangle)} = \phi$$

on $h_{3,19}$. According to Theorem 32 the function $\det(\langle g_i(\tau), g_j(\tau) \rangle)$ is an automorphic form of weight $-2$. Therefore the function $\phi$ is an automorphic function of weight 2.

In [17] we proved that $\det \Delta_{KE}$ is a bounded non negative function. Therefore the only zeroes of $\det \Delta_{KE}$ can be located on the discriminant locus $D_{KE}$. We know that $|\exp(\Phi_{\Lambda_{K3}}(\tau))|$ is an automorphic function with a zero set on the discriminant locus $D_{KE}$. Since $D_{KE}$ is an irreducible divisor in $M_{KE}$, by taking suitable power of $\phi$ and $|\exp(\Phi_{\Lambda_{K3}}(\tau))|$, we may assume that the function

$$|\exp(\Phi_{\Lambda_{K3}}(\tau))| = \psi$$

is a non zero function such $\Delta_{B} \log \psi = 0$. Thus we get a harmonic non zero function on $M_{KE}$.

**Lemma 43** $\psi|_{M_{ell}} = \text{const.}$.

**Proof:** Since $\ddc\left( \log \frac{\det(\Delta_{KE}(\tau))}{\det(\langle g_i(\tau), g_j(\tau) \rangle)} |_{M_{ell}} \right) = 0$ we can conclude that

$$\frac{\det(\Delta_{KE}(\tau))}{\det(\langle g_i(\tau), g_j(\tau) \rangle)} |_{M_{ell}} = |\eta|,$$

where $\eta$ is a holomorphic automorphic form defined up to a character $\chi \in \Gamma_{ell}/[\Gamma_{ell}, \Gamma_{ell}]$ and with a zero set $D_{el}$. Since $D_{el}$ is an irreducible divisor, we can conclude that $\eta = \exp(\Phi_{\Lambda_{K3}}(\tau))$. Thus since $\exp(\Phi_{\Lambda_{K3}}(\tau))|_{M_{ell}} = \exp(\Phi_{\Lambda_{ell}}(\tau))$, we get that $\psi|_{M_{ell}} = \text{const.}$ Since any two $M_{ell,1}$ and $M_{ell,2}$ intersect. So the continuous function $\psi$ is a constant on an everywhere dense subset in $M_{KE}$. Thus $\psi$ is a constant. Lemma 43 is proved. ■

Lemma 43 imply Theorem 42. ■
7 Mirror Symmetry, Harvey-Moore-Borcherds Products and Counting Problems

7.1 Mirror Symmetry for K3 Surfaces

Let \((X, \alpha, \omega_X(1,1))\) be a marked K3 surface with a B-field. To define the mirror of \((X, \alpha, \omega_X(1,1))\) we need to fix an unimodular hyperbolic lattice \(U\) in \(H_2(X, \mathbb{Z})\) with generators \(\{\gamma_0, \gamma_1\}\) such that

\[\int_{\gamma_0} \omega_X \neq 0 \text{ and } \int_{\gamma_1} \omega_X \neq 0.\]

Thus we can normalize \(\omega_X\) such that

\[\int_{\gamma_0} \omega_X = 1 \text{ and } \int_{\gamma_1} \omega_X \neq 0.\] (33)

From now on we will consider the set \((X, \alpha, \omega_X(1,1), U)\), where \(U\) is a fixed sublattice in \(H_2(X, \mathbb{Z})\) such that the holomorphic two form satisfies (33). Let \(U^\perp\) be the orthogonal complement of \(U\) in \(H^2(X, \mathbb{Z})\). Let us denote by \(U_0\) the unimodular hyperbolic sublattice \(H_0(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})\) in the cohomology ring \(H^*(X, \mathbb{Z})\). We will assign to the B-field \(\omega_X(1,1)\) the vector

\[\hat{\omega}_X := \left(\omega_X(1,1), 1, -\frac{\omega_X(1,1) \wedge \omega_X(1,1)}{2}\right)\]

in \(H^2(X, \mathbb{Z}) \oplus U_0\).

We will need the following Theorem:

**Theorem 44** There exists a marked K3 surface \(Y\) with a B-field \((Y, \alpha, \omega_Y(1,1))\) such that

i. If we identify \(H^2(Y, \mathbb{Z})\) with \(U^\perp \oplus U_0\), then \([\omega_Y] = [\omega_X(1,1)]\) in \((U^\perp \oplus U_0) \otimes \mathbb{C}\).

ii. If we identify \(H^*(Y, \mathbb{Z})\) with \(H^2(Y, \mathbb{Z}) \oplus U\) then \(\omega_Y(1,1) = [\omega_X]\) in \((H^2(Y, \mathbb{Z}) \oplus U) \otimes \mathbb{C}\), where \(\omega_X\) is normalized as (33).

**Proof:** Let us consider in \((U^\perp \oplus U_0) \otimes \mathbb{C} = \Lambda_{K3} \otimes \mathbb{C}\) the vector \(\omega_X(1,1)\). Then direct computations show that we have \(\langle \hat{\omega}_X, \hat{\omega}_X \rangle = 0\) and \(\langle \hat{\omega}_X, \overline{\hat{\omega}_X} \rangle > 0\). From the epimorphism of the period map for K3 surfaces proved in [27] it follows that there exists a marked K3 surface \((Y, \alpha)\) with a holomorphic two form \(\omega_Y\) such that the class of cohomology \([\omega_Y]\) is the same as the class of cohomology of \(\omega_X(1,1)\). Next we will prove that the class of cohomology \(\omega_X \in H^{1,1}(Y, \mathbb{C})\) satisfies

\[\int_Y \text{Im} \omega_X \wedge \text{Im} \omega_X = \langle \text{Im} \omega_X, \text{Im} \omega_X \rangle > 0.\]

Indeed on \(X\) we have

\[\langle \omega_X, \omega_X(1,1) \rangle = \langle \omega_X, \overline{\omega_X(1,1)} \rangle = 0\] (34)
since \( \omega_X(1,1) \) is a form of type \((1,1)\) and \( \omega_X \) is a form of type \((2,0)\). On the other hand the form \( \omega_X(1,1) \) with respect to the new complex structure \( Y \) on \( X \) it is a form of type \((2,0)\). So (34) means that on \( Y \) \( \omega_X \) is a form of type \((1,1)\).

On the other hand we have

\[
\int_X \omega_X \wedge \overline{\omega_X} = 2 \int_X \Im \omega_X \wedge \Im \omega_X = 2 (\Im \omega_X, \Im \omega_X) > 0. \tag{35}
\]

Thus (35) proves that \( \omega_X \) is a B-field on \( Y \). Theorem 44 is proved.

Now we are ready to define the mirror symmetry:

**Definition 45** We will define the marked surface \((Y, \alpha, \omega_Y(1,1), U)\) constructed in Theorem 44 the mirror of \((X, \alpha, \omega_X(1,1), U_0)\).

### 7.2 Mirror Symmetry and Algebraic K3 Surfaces

Let us consider the Neron-Severi group

\[ M = \text{Pic}(X) := H^2(X, \mathbb{Z}) \cap H^{1,1}(X, \mathbb{R}) \]

We can characterize in another way \( \text{NS}(X) \). It is dual group in \( H^2(X, \mathbb{Z}) \) of the kernel of the functional:

\[ (\omega_X) : H^2(X, \mathbb{Z}) \to \mathbb{C} \]

defined by \( \gamma \to \int \gamma \omega_X \). We define the transcendental classes of homologies \( T(X) \subset H^2(X, \mathbb{Z}) \) on \( X \) as follows: \( T(X) := \ker (\omega_X) \perp \).

We will need the following definition:

**Definition 46** We will say that pairs \((X, M)\) \( M \)-marked K3 surface if \( M \) is the Picard lattice of some algebraic K3 surface together with a primitive imbedding of \( M \) into \( H^2(X, \mathbb{Z}) \).

The following Theorem was proved in [28] or [13]

**Theorem 47** The moduli space \( \mathfrak{M}_M \) of marked pairs \((X, M)\) exists and \( \mathfrak{M}_M \cong \Gamma_M \backslash \mathfrak{H}_{2,20-\rho} \), where \( \rho = \text{rk} M \) and \( \Gamma_M = \{ \phi \in \text{Aut} \Lambda_{K3} | \phi|_M = \text{id} \} \).

Suppose that we consider \( M \) such that \( U \) can be embedded into \( M \perp \). According to a Theorem of Nikulin this is always possible if \( \text{rk} M = \rho \geq 9 \). The construction of mirror symmetry for \( M \) marked K3 surfaces \((X, \alpha, M, U)\), \( \text{where } U \subset M \perp \) was described in [28] and [13] as follows: Let \((X, \alpha, M, U)\) be an algebraic polarized K3 surface. Then Theorem 44 implies the following Corollary:

**Corollary 48** Let \((X, M, U, \omega_X(1,1))\) be \( M \)-marked K3 surface such that \( U \subset T_X \) and the B-field \( \omega_X(1,1) \) satisfies \( \omega_X(1,1)|_{U \perp \subset T_X} = 0 \). Then the mirror \((Y, M_1, U, \omega_Y(1,1))\) satisfies the following conditions: i. \( \text{Pic}(Y) = M_1 = U \perp \subset T_X \). ii. \( T_Y = M \oplus U \cong \text{Pic}(X) \oplus U \).

**Proof:** Corollary 48 follows directly from Theorem 44.

**Remark 49** Some interesting examples and applications of Corollary 48 were discussed in [15].
7.3 The Mirror Map for Marked M-K3

Part of the mirror conjecture states that the

Definition 50 Let $X$ be a K3 surface. We will define the Kähler cone of $K(X)$ of $X$ as follows:

$$K(X) := \{ \omega \in H^{1,1}(X, \mathbb{R}) | \omega = \text{Im} \ g, \text{ and } g \text{ is a Kähler metric on } X \}.$$ 

We will need the characterization of the Kähler cone that is given below. Denote by $\Delta(X) := \{ \delta \in NS(X) | \langle \delta, \delta \rangle = -2 \}$. We will need the following Lemma from [25]:

Lemma 51 Let $\delta \in \Delta(X)$. Then $\delta$ or $-\delta$ can be realized as an effective curve on $X$.

We will denote by $\Delta^+(X) := \{ \delta \in \Delta(X) | \delta \text{ can be realized as an effective curve} \}$. Let us denote by $V := \{ v \in H^{1,1}(X, \mathbb{R}) | \langle v, v \rangle > 0 \}$. Since the restriction of the bilinear form on $H^{1,1}(X, \mathbb{R})$ has a signature $(1,19)$, then $V$ will consist of two components. Let us denote by $V^+$ the component of $V$ which contains a Kähler class.

Each $\delta \in \Delta^+(\Delta)$ generates a reflection $s_\delta$ of $V^+$, where $s_\delta(v) = v + \langle v, \delta \rangle \delta$. Let us denote by $\Gamma(\Delta)$ the subgroup of $O_{\Lambda_{K3}}^+$ generated by $s_\delta$. In [27] the following Theorem was proved:

Theorem 52 The Kähler cone $K(X)$ coincides with the fundamental domain of the group $\Gamma(\Delta)$ in $V^+$ which contains a Kähler class.

Proof: See [27].

Remark 53 According to Theorem [27], $\mathcal{M}_{K3,3} = \Gamma_M \backslash \mathfrak{h}_{220} - \rho$ is the moduli space of M-marked K3 surfaces. Suppose that $U \subset T_X$ is fixed and $M_1 \subset T_X$ is the orthogonal complement of $U$ in $M$. Let $(Y, M_1)$ be some $M_1$ marked K3 surface defined by the primitive embedding $M_1 \subset T_X \subset \Lambda_{K3}$. Let $\mathfrak{h}_{M_1} = M_1 \otimes \mathbb{R} + iK(Y)$, where $K(Y)$ is the Kähler cone of $Y$. Then according to Theorem [27], $\Gamma_M \backslash \mathfrak{h}_{M_1} \cong \mathcal{M}_{K3,3}$. Thus we have a complex analytic covering map:

$$\psi_M : \mathfrak{h}_{M_1} \rightarrow \Gamma_M \backslash \mathfrak{h}_{M_1} = \mathcal{M}_{K3,3}.$$ 

The map $\psi_M^{-1}$ which is multivalued is called the mirror map. It identifies in the case described in this Remark the moduli space of M-marked K3 surfaces with the complexified Kähler cone of the its mirror.

8 Applications of Mirror Symmetry

8.1 Counting Problems on K3

Theorem 54 Let $X$ be an algebraic K3 surface such that $Pic(X)$ is an unimodular lattice. Then we have either $NS(X) = U \oplus E_8(-1)$ or $NS(X) = \ldots$
\( U \oplus E_8(-1) \oplus E_8(-1) \). Let \( l \in NS(X) \) be the polarization class. Let us consider the components \( V^{\perp}_{\text{Enr}} \) and \( V^{\perp}_{\text{ell}} \) of the positive cones in \((U \oplus E_8(-1)) \otimes \mathbb{R}\) and in 
\((U \oplus E_8(-1)) \oplus E_8(-1) \otimes \mathbb{R}\) which contain the polarization vector \( l \). Let us consider the discriminant automorphic forms \( \exp(\Phi_{\text{Enr}}(\tau)) \) and \( \exp(\Phi_{\text{ell}}(\tau)) \) on 
\((U \oplus E_8(-1)) \otimes \mathbb{R} \oplus \sqrt{-1}V^{\perp}_{\text{Enr}}\) and on \(((U \oplus E_8(-1)) \otimes \mathbb{R}) \oplus \sqrt{-1}V^{\perp}_{\text{ell}}\). Then the restriction of the functions \( \exp(\Phi_{\text{Enr}}(\tau)) \) and \( \exp(\Phi_{\text{ell}}(\tau)) \) on the lines \( \sqrt{-1}lt \) are periodic. The Fourier expansions

\[
\frac{d}{dt} (\Phi_{\text{Enr}} (\sqrt{-1}lt)) = -\sum_n a_n \frac{e^{-nt}}{1 - e^{-nt}}
\]

and

\[
\frac{d}{dt} (\Phi_{\text{ell}} (\sqrt{-1}lt)) = -\sum_n b_n \frac{e^{-nt}}{1 - e^{-nt}}
\]

have integer coefficients \( a_n \) and \( b_n \). \( a_n \) and \( b_n \) are equal to the number of non singular rational curves of degree \( n \) on a K3 surface \( X \) with \( NS(X) = U \oplus E_8(-1) \) or \( NS(X) = U \oplus E_8(-1) \oplus E_8(-1) \).

**Proof:** Let us fix a bases \( \{ \gamma_i \} \) and \( \{ \varepsilon_j \} \) of \( U \oplus E_8(-1) \) and \( U \oplus E_8(-1) \oplus E_8(-1) \) respectively. Then we fix the flat coordinates \( \{ \tau^1, ..., \tau^{10} \} \) and \( \{ \tau^1, ..., \tau^{18} \} \) in the symmetric spaces \( h_{2,10} \) and \( h_{2,18} \) respectively represented as tube domains. We will denote by \( \langle \delta, \tau \rangle \) the following expressions:

\[
\langle \delta, \tau \rangle = \sum_{i=1}^{10} \langle \delta, \gamma_i \rangle \tau^i \quad \text{and} \quad \langle \delta, \tau \rangle = \sum_{i=1}^{18} \langle \delta, \varepsilon_i \rangle \tau^i.
\]

Then Harvey-Moore-Borcherds product formula states that there exist automorphic forms on \( \Gamma_{2,10}(h_{2,10}) \) or on \( \Gamma_{2,18}(h_{2,18}) \) which can be represented for some large \( \text{Im} \tau^i \) as the following products.

\[
\exp(\Phi_{\text{Enr}} (\tau)) = \exp(2\pi i \langle \tau, w \rangle) \prod_{\delta \in \Delta_{\text{Enr}}} \left( 1 - \exp \left( 2\pi i \sum_{i=1}^{10} \langle \delta, \gamma_i \rangle \tau^i \right) \right)
\]

and

\[
\exp(\Phi_{\text{ell}} (\tau)) = \exp(2\pi i \langle \tau, w \rangle) \prod_{\delta \in \Delta_{\text{Enr}}} \left( 1 - \exp \left( 2\pi i \sum_{i=1}^{18} \langle \delta, \varepsilon_i \rangle \tau^i \right) \right). \quad (37)
\]

It was proved that \( \exp(\Phi_{\text{Enr}} (\tau)) \) and \( \exp(\Phi_{\text{ell}} (\tau)) \) have an analytic continuation in \( h_{2,10} \) and \( h_{2,18} \) and the zeroes remain the same.

Substituting

\[
\sum_{i=1}^{10} \gamma_i \tau^i = ilt \quad \text{and} \quad \sum_{i=1}^{18} \varepsilon_i \tau^i = ilt
\]

25
in (37) we get
\[
\exp (\Phi_{Enr} (\tau)) = \exp(2\pi i \langle \tau, w \rangle) \prod_{\delta \in \Delta_{Enr}^+} (1 - \exp (-2\pi \langle \delta, l \rangle t))
\]
and
\[
\exp (\Phi_{ell} (\tau)) = \exp(2\pi i \langle \tau, w \rangle) \prod_{\delta \in \Delta_{Enr}^+} (1 - \exp (-2\pi \langle \delta, l \rangle t)). \tag{38}
\]
Let us split the irreducible non-singular on disjoint finite sets \(A_n\), where \(A_n = \{ \delta \in \Delta^+ | \langle \delta, l \rangle = n \}\). Suppose that \(\#A_n = a_n\) in the case of \(\Lambda_{Enr}\) and \(\#A_n = b_n\) in the case \(\Lambda_{ell}\). We can rewrite (38) as follows
\[
\exp (\Phi_{Enr} (\tau)) = \exp(2\pi i \langle \tau, w \rangle) \prod_{\delta \in \Delta_{Enr}^+} (1 - \exp (-2\pi \langle \delta, l \rangle t)) = \exp(2\pi i \langle \tau, w \rangle \prod_{n=1}^{\infty} ((1 - \exp (-2\pi nt))^{a_n}). \tag{39}
\]
In the same way we will get that
\[
\exp (\Phi_{ell} (\tau)) = \exp(2\pi i \langle \tau, w \rangle) \prod_{\delta \in \Delta_{Enr}^+} (1 - \exp (-2\pi \langle \delta, l \rangle t)) = \exp(2\pi i \langle \tau, w \rangle \prod_{n=1}^{\infty} ((1 - \exp (-2\pi nt))^{b_n}). \tag{40}
\]
From (39) and (40) we derive (36) and thus Theorem 54. ■

**Remark 55** We see that in the \(A\) model the automorphic function \(\exp (\Phi_{4,20} (\tau))\) restricted on the Kähler cone when \(\text{Pic}(X)\) is a unimodular lattice counts rational curves. Suppose that in the \(B\) model we represent \(\text{Pic}(Y)\) as a tube domain \(\mathbb{R}^k + iV^+\) modulo action of an arithmetic group. Suppose that the \(\text{Im}\omega_Y \in H^2(Y, \mathbb{Z}) \cap H^{1,1}(Y, \mathbb{R})\). Then the restriction of the automorphic function \(\exp (\Phi_{4,20}(\tau))\) on \(\mathbb{M}_{\text{Pic}(Y)}\) counts vanishing invariant cycles \(\gamma\) such that \(\langle \gamma, \text{Im}\omega_Y \rangle = n\).

### 8.2 The Pluricanonical Canonical Class of the Moduli of Polarized Algebraic K3 Surfaces

**Theorem 56** Let \(l \in \Lambda_{K3}\) be a primitive vector such that \(\langle l, l \rangle = 2n > 0\). Let us denote by \((l)^\perp\) be the sublattice in \(\Lambda_{K3}\) orthogonal to \(Zl\). Then we have
\[
(l)^\perp \cong Zl^* \oplus U^2 \oplus (-E_8)^2,
\]
where \(l^*\) is a primitive vector in \(\Lambda_{K3}\) such that \(\langle l^*, l^* \rangle = -2n < 0\).
Proof: According to [25], the subgroup $\mathcal{O}_{\Lambda K_3}^+$ of index two that preserve the spinor norm acts transitively on the primitive vectors with a fixed positive self intersection. Let us fix $U$ in $\Lambda K_3$ with a basis $e_0$ and $e_1$ such that $\langle e_1, e_1 \rangle = 0$ and $\langle e_1, e_2 \rangle = 1$. Then $l = e_1 + ne_2 \in U$ is a primitive vector such that $\langle l, l \rangle = 2n > 0$. Let $l^* = e_1 - ne_2 \in U$. Clearly $l^*$ is a primitive vector such that $\langle l, l^* \rangle = 0$ and $\langle l^*, l^* \rangle = -\langle l, l \rangle = -2n$.

Then we have

$$\langle l, l^* \rangle = 0 \quad \text{and} \quad \langle l^*, l^* \rangle = -\langle l, l \rangle = -2n.$$ 

Then according to [25] and [12], we have $\mathcal{M}_{K_3,n}$ as follows: Let $\lambda \in \Lambda K_3$, then

$$\mathcal{M}_{K_3,n} = \{ u \in \mathcal{P}(\Lambda K_3 \otimes \mathbb{C}) \mid \langle u, \lambda \rangle = 0 \}.$$ 

Let us define $\mathcal{D}_n$ in $\mathcal{M}_{K_3,n}$ as follows: Let $\lambda \in \Lambda K_3$, then

$$\mathcal{D}_n := \{ u \in \mathcal{P}(\Lambda K_3 \otimes \mathbb{C}) \mid \langle u, \lambda \rangle = 0 \}.$$ 

Then $\mathcal{D}_n := \Gamma_n \setminus \mathcal{D}_n$.

Theorem 58 There exists an automorphic form $\Psi_{19,n}$ on $\mathcal{M}_{K_3,n} = \Gamma_n \setminus h_{2,19}$ such that the zero set of $\Psi_{19,n}$ is $\mathcal{D}_n$.

Proof: According to the results of Harvey, Moore and Borcherds on we can find an automorphic form $[\Psi_{\Lambda K_3}]^2$ on the moduli space of Einstein metrics $\mathcal{O}_{\Lambda K_3}^+ \setminus h_{3,19}$ such that its zeros are exactly on the discriminant locus of $\mathcal{O}_{\Lambda K_3}^+ \setminus h_{3,19}$. Recall that the discriminant locus on $\mathcal{O}_{\Lambda K_3}^+ \setminus h_{3,19}$ is defined as the set of three dimensional positive vector subspaces in $\Lambda K_3 \otimes \mathbb{R}$ perpendicular to $\delta$ such that $\langle \delta, \delta \rangle = -2$ modulo the action of the arithmetic group $\mathcal{O}_{\Lambda K_3}^+$. The moduli space $\mathcal{M}_{K_3,n} = \Gamma_n \setminus h_{2,19}$ can be embedded in $\mathcal{O}_{\Lambda K_3}^+ \setminus h_{3,19}$ as the set of all three dimensional oriented subspaces in $\Lambda K_3 \otimes \mathbb{R}$ containing the polarization vector $l$ modulo the action of $\mathcal{O}_{\Lambda K_3}^+$. The restriction of some power of $\Psi_{\Lambda K_3}$ on $\mathcal{M}_{K_3,n}$ will give us an automorphic form $\Psi_{19,n}$ on $\mathcal{M}_{K_3,n}$. Thus we have the following obvious fact:
Remark 59 The zero set of $\Psi_{19,n}$ is the restriction of the zero set of $\Psi_{\Lambda_{K3}} = \exp(\Phi_{\Lambda_{3,19}})$ on $\mathfrak{m}_{K3,n}$.

Thus we need to compute the projection of the zero set of $\exp(\Phi_{\Lambda_{3,19}})$ on $\Gamma^+ \setminus \mathfrak{h}_{3,19}$ to $\mathfrak{m}_{K3,n} = \Gamma_n \setminus \mathfrak{h}_{2,19}$. Theorem 58 will follow from the following Lemma:

Lemma 60 Let $\delta \in \Lambda_{K3}$ be such that $\langle \delta, \delta \rangle = -2$. Suppose that $\delta \notin \Lambda_{K3,n}$. Then there exists an automorphism $\sigma$ of the lattice $\Lambda_{K3,n}$ such that $\sigma(\delta) \in \mathcal{U}$, i.e. $\Pr_U \sigma(\delta) = l^+$.

Proof: The proof of Lemma 60 is based on the following Propositions:

Proposition 61 Suppose that $\delta = me_1 + m_2e_2 + \mu_\delta$, $\langle \delta, \delta \rangle = -2$ and $\delta$ satisfies (42). Then there exists $\sigma \in \mathcal{O}^+(\mathbb{U}^2 \oplus E_8(-1)^2)$ such that

$$\Pr_{\Lambda,n}(\sigma(\delta)) = n_\delta l^* + \mu_{\sigma(\delta)}, \quad \mu_{\sigma(\delta)} = k_{\sigma(\delta)}(f_1 + m_{\sigma(\delta)}f_2),$$

and

$$\|\mu_\delta\|^2 = \|\mu_{\sigma(\delta)}\|^2$$

(43)

where $k_{\sigma(\delta)} \geq 1$ and $m_{\sigma(\delta)} > 0$.

Proof: The condition that $\delta$ satisfies (42) implies that $\langle \mu_\delta, \mu_\delta \rangle > 0$. From the presentation of $\delta = m_1e_1 + m_2e_2 + \mu_\delta$ it follows that have two possibilities for $\mu_\delta$.

1. $\mu_\delta$ is a primitive vector in $\mathcal{L} = \mathbb{U} \oplus \mathbb{U} \oplus E_8(-1)^2 \oplus E_8(-1)$. According to Theorem 59 all primitive vectors with a fixed positive norm form one orbit under the action of the automorphism group. Thus there is an element $\sigma \in \mathcal{O}_L^+$ such that the primitive element $\sigma(\mu_\delta)$ can be presented as follows:

$$\sigma(\mu_\delta) = f_1 + \frac{\|\mu_\delta\|^2}{2} f_2,$$

where $f_1$ form a basis of $\mathbb{U}$ of isotropic vectors such that $\langle f_1, f_2 \rangle = 1$. 2. $\mu_\delta$ is not primitive. Then the same arguments as in the first case imply (43). Proposition 61 is proved.

Proposition 62 Let $\Lambda_{K3} = \mathbb{U} \oplus \mathcal{L}$ and let $e_1$ and $e_2$ be the isotropic generators of $\mathbb{U}$. Let $l = e_1 + ne_2 \in \mathbb{U}$ and $n > 0$. Suppose that $\delta = n_1e_1 + n_2e_2 + \mu_\delta$ and $\langle \delta, \delta \rangle = -2$. Then there exists an element $\sigma \in \Gamma_n$ such that in the representation

$$\sigma(\delta) = n_1e_1 + n_2e_2 + \mu_{\sigma(\delta)}$$

$$\langle \Pr_U(\sigma(\delta)), \Pr_U(\sigma(\delta)) \rangle < 0,$$

(45)

where $\Pr_U(\sigma(\delta)) = n_1e_1 + n_2e_2$. 

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Proposition 61 implies that without loss of generality we may suppose that
Proposition 63.

Clearly we have
\[ np \]
where
\[ U \]
projection
\[ Pr \]
Thus
\[ \delta \]
Suppose that
\[ \delta, \delta \]
positive number then (46) will imply (45).

Proposition 61 implies that we can choose \( \mu_\delta \) and \( \mu_\delta_1 \) such that \( \langle \mu_\delta, \mu_\delta_1 \rangle = 0 \). Thus
\[ Pr_U(\sigma(2n\delta)) = (nm_1 + m_2)l + 2nk_\delta_1 (nm_1 - m_2)l^\ast. \] (46)

Suppose that \( nm_1 - m_2 \neq 0 \). Then if we choose \( \delta_1 \) such that \( k_\delta_1 \) is a big enough positive number then (46) will imply (47).

Suppose that \( nm_1 = m_2 \). Then \( \delta = m_1(e_1 + ne_2) + \mu_\delta \). Thus
\[ Pr_U(\delta) = m_1l. \]
Let us choose \( \delta_1 = k_\delta l^\ast + \mu_\delta_1 \), where \( \langle \delta_1, \delta_1 \rangle = -2 \) and \( \langle \mu_\delta, \mu_\delta_1 \rangle \neq 0 \). Let us compute
\[ r_{\delta_1}(\delta) = \delta + \langle \delta, \delta_1 \rangle \delta_1 = m_1l + \langle \delta, \delta_1 \rangle (k_\delta l^\ast + \mu_\delta_1). \] (47)
Clearly we have \( \langle \delta, \delta_1 \rangle = \langle \mu_\delta, \mu_\delta_1 \rangle \). Thus (47) implies that
\[ r_{\delta_1}(\delta) = \delta_2 = p_1e_1 + p_2e_2 + \langle \mu_\delta, \mu_\delta_1 \rangle \mu_\delta_1, \]
where \( p_1 - p_2 \neq 0 \). The previous arguments imply Proposition 62.

Let \( G_\kappa \) be the subgroup of \( \Gamma_n \) generated by reflections \( r_\kappa(v) = v + \langle v, \kappa \rangle \kappa \) for all \( \kappa \in \Lambda_{K3,n} \) and \( \langle \kappa, \kappa \rangle = -2 \). Let us consider the orbit \( G_\kappa \delta \) of a fixed \( \delta \in \Lambda_{K3} \) such that \( \langle \delta, \delta \rangle = -2 \) and \( \delta \) satisfies (44). Let \( \delta_{\min} \in \{G_\kappa\delta\} \) be such that
\[ \langle \mu_{\delta_{\min}}, \mu_{\delta_{\min}} \rangle = \min_{\delta \in \{G_\kappa\delta\}} \left( \langle \mu_\delta, \mu_\delta \rangle \right) \geq 0. \] (48)

Proposition 61 implies that without loss of generality we may suppose that
\[ \mu_{\delta_{\min}} = f_1 + \frac{\|\mu_{\delta_{\min}}\|^2}{2}f_2 \text{ or } \mu_{\delta_{\min}} = k_{\delta_{\min}} \left( f_1 + \frac{\|\mu_{\delta_{\min}}\|^2}{2}f_2 \right). \] (49)

Proposition 63 Let \( \mu_{\delta_{\min}} \) be defined as (48). Then \( \langle \mu_{\delta_{\min}}, \mu_{\delta_{\min}} \rangle = 0 \).

Proof: Suppose that Proposition 63 is not true. Then \( \langle \mu_{\delta_{\min}}, \mu_{\delta_{\min}} \rangle = \|\mu_{\delta_{\min}}\|^2 > 0 \). We will show that this assumption leads to a contradiction. We can choose \( \kappa \in \Lambda_{K3,n} \) such that \( \langle \kappa, \kappa \rangle = -2 \) and \( \kappa = k_0l^\ast + \mu_\kappa \), where \( k_0 > 1 \).

The relation \( \langle \kappa, \kappa \rangle = -2 \) implies that \( \|\mu_{\kappa}\|^2 = 2nk_0^2 - 2 > 2n \). Suppose that
\(\mu_{\delta_{\min}}\) is a primitive element of \(L = U \oplus U \oplus E_8(-1) \oplus E_8(-1)\). Without loss of generality we can choose

\[
\mu_\kappa = g_1 + \frac{\|\mu_\kappa\|^2}{2}g_2 - f_2. \tag{50}
\]

Direct computations show that \(Pr_{l,n}(r_{\kappa}(\delta)) = k_1l^* + \mu_{r_{\kappa}(\delta)}\), where

\[
\|\mu_{r_{\kappa}(\delta)}\|^2 = \|\mu_{\delta_{\min}}\|^2 + \langle \mu_{\delta_{\min}}, \mu_\kappa \rangle - 2nk_{\delta_{\min}}k_0 \left( \|\mu_\kappa\|^2 + 2 \langle \mu_{\delta_{\min}}, \mu_\kappa \rangle \right) \geq 0.
\]

So (49) and (50) imply

\[
\langle \mu_\kappa, \mu_{\delta_{\min}} \rangle = \left( f_1 + \frac{\|\mu_{\delta_{\min}}\|^2}{2}, g_1 + \frac{\|\mu_\kappa\|^2}{2}g_2 - f_2 \right) = -1. \tag{52}
\]

Then from (51) and (52) we get that \(\|\mu_{\delta_{\min}}\|^2 > \|\mu_{r_{\kappa}(\delta)}\|^2\). Thus we get a contradiction with \(\|\mu_{\delta_{\min}}\|^2 > 0\) being the minimal value. So \(\langle \mu_{\delta_{\min}}, \mu_{\delta_{\min}} \rangle = 0\). Suppose that \(\mu_{\delta_{\min}}\) is not primitive, i.e. then \(\mu_{\delta_{\min}} = k\mu_{\text{prim}, \delta_{\min}}\) and

\[
\mu_{\text{prim}, \delta_{\min}} = f_1 + \frac{\|\mu_{\text{prim}, \delta_{\min}}\|^2}{2}f_2.
\]

Thus we get

\[
\langle \mu_\kappa, \mu_{\delta_{\min}} \rangle = \left( k \left( f_1 + \frac{\|\mu_{\delta_{\min}}\|^2}{2}f_2 \right), g_1 + \frac{\|\mu_\kappa\|^2}{2}g_2 - f_2 \right) = -k < 0. \tag{53}
\]

Combining (51) and (53) we get that \(\|\mu_{\delta_{\min}}\|^2 > \|\mu_{r_{\kappa}(\delta)}\|^2\). Thus we get a contradiction. Proposition 63 is proved.

Proposition 63 implies Lemma 60.

Lemma 64 The zero set of \(\Psi_{19,n}\) on \(M_{K,3,n}\) is \(D_n\).

Proof: Let \(\delta \in \Lambda_{K,3}\) be such that \(\langle \delta, \delta \rangle = -2\). Let \(Pr_{l,n}(\delta) \in \Lambda_{K,3,n}\) be the orthogonal projection of \(\delta\) on \(\Lambda_{K,3,n}\). If \(Pr_{l,n}(\delta) = \delta \iff \langle l, \delta \rangle = 0\), then it implies that the component

\[
\bigcup_{\langle \delta, \delta \rangle = -2 \& \delta \in \Lambda_{K,3,n}} (b_{2,19} \cap H_\delta)
\]

in the expression defines one the components of \(D_n := \Gamma_n \setminus D_n\) corresponding to the vectors with \(-2\) norm in \(\Lambda_{K,3,n}\).

Suppose that \(\delta \in \Lambda_{K,3}, \langle \delta, \delta \rangle = -2\) and \(Pr_{l,n}(\delta) \neq \delta\). Lemma 64 implies that we can find \(\sigma \in \Gamma_n\) such that \(\sigma(\delta) = m_1e_1 + m_2e_2\). Thus \(Pr_{l,n}(\delta) = k\delta l^*\). Then

\[
\pi (H_\delta \cap b_{2,19}) = \pi (H_{l^*} \cap b_{2,19}) \tag{54}
\]

where \(\pi : b_{2,19} \to \Gamma_n \setminus b_{2,19}\). Thus 64 implies Lemma 64. Theorem 65 is proved.
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