When Slepian Meets Fiedler: Putting a Focus on the Graph Spectrum

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Abstract—Network models play an important role in studying complex systems in many scientific disciplines. Graph signal processing is receiving growing interest as to design novel tools to combine the analysis of topology and signals. The graph Fourier transform, defined as the eigendecomposition of the graph Laplacian, allows extending conventional signal-processing operations to graphs. One main feature is to let emerge global organization from local interactions; i.e., the Fiedler vector has the smallest non-zero eigenvalue and is key for Laplacian embedding and graph clustering. Here, we introduce the notion of Slepian graph signals, by maximizing energy concentration in a predefined subgraph for a given spectral bandlimit. We also establish a link with classical Laplacian embedding and graph clustering, for which the graph Slepian design can serve as a generalization.

Index Terms—Graph signal processing, Graph Laplacian, Slepian functions

I. INTRODUCTION

GRAPHS are powerful mathematical models to represent complex data structures such as networks [1]. Spectral graph theory supports an important class of methods for studying graph topology as well as analyzing graph signals. Graph partitioning problems [2] can be solved as the convex relaxation of the graph-cut criterion, which reverts to an eigendecomposition of the graph Laplacian. Thresholding the eigenvector with the smallest non-zero eigenvalue—commonly called the Fiedler vector—leads to a graph bipartition [3]. These eigenvectors also solve the embedding problem; i.e., mapping the nodes onto a line (or higher-dimensional space) such that the distance between connected nodes is minimized [3]. In addition, the same eigenvectors are also used to define the graph Fourier transform (GFT), which has attracted a lot of attention from the signal processing community as many Fourier-domain operations have been generalized to graphs using the GFT [5]. One notable example is the spectral graph wavelet transform [6] that defines graph-domain wavelets by using a window function in the spectral domain.

In this Letter, we propose the graph extension of Slepian functions, which were introduced on regular domains by Slepian and colleagues [7], [8] to find a trade-off between temporal and spectral energy concentration. These functions have been extended to several other domains, including spherical ones in the context of geophysics [9]. Balancing the spread of graph signals in the original and the spectral domain relates to extensions of uncertainty principles [10], which have been explored by Tsitsvero et al. [11] based on a Slepian-type of constraints. Here, we propose graph Spleipians as a useful transformation for analyzing graph signals by establishing a further link with Laplacian embedding and graph clustering. In particular, the Slepian design can be formulated w.r.t. energy concentration, but also as a modified version for the embedded distance.

II. DEFINING SLEPIAN FUNCTIONS ON GRAPHS

A. Slepian’s Time-Frequency Concentration Problem

In seminal work by Slepian, Landau, and Pollak [7], [8], the fundamental problem of optimally concentrating a signal jointly in temporal and spectral domains was solved. Here we will briefly review the 1-D discrete-discrete case, which is directly relevant for the generalization that we will introduce in this work. We define the unitary discrete Fourier transform (DFT) for a length-$N$ signal by the $N \times N$ matrix

$$F = \frac{1}{\sqrt{N}} [e^{j\omega l(k-1)}]_{k,l}, \quad k, l = 1, \ldots, N,$$

where $\omega_l = 2\pi(l - [(N - 1)/2])/N$. Then, the signal represented as a vector $f$ can be Fourier transformed as $\hat{f} = F^H f$, where $\cdot^H$ indicates the Hermitian transpose. Due to Parseval, we have that $\|f\|^2 = \|\hat{f}\|^2$, where the $\ell_2$-norm is defined as $\|f\|^2 = \langle f, f \rangle = f^H f$.

The problem at hand is now to optimally concentrate a strictly band-limited signal $g$ in a predefined interval defined by the set of indices $S$ that does not necessarily need to be contiguous. Finding such an optimal signal in terms of energy concentration in $S$ can be formulated as maximizing

$$\mu = \frac{\sum_{k \in S} |g[k]|^2}{\sum_{k=1}^{N} |g[k]|^2}.$$

By definition, the band-limited signal can be represented by a linear combination of a truncated Fourier basis; i.e., we write $g = W F^H \tilde{g}$, where $W$ of size $N \times N_W$ selects the first $N_W$ Fourier vectors with smallest $|\omega_l|$. In addition, we introduce the $N \times N$ diagonal selection matrix $S$ with elements $S_{k,k} = 1/0$ that indicate the presence of index $k$ in $S$; so the number of selected indexes is $\text{trace}(S) = N_S$. This allows us now to rewrite Eq. (1) as the Rayleigh quotient

$$\mu = \frac{\sum_{k=1}^{N} |\tilde{g}[k]|^2}{\sum_{k=1}^{N} |\tilde{g}[k]|^2},$$

where $C = W^T F^H$ is the concentration matrix of size $N_W \times N_W$. Finding the optimal solution translates into an eigendecomposition problem, $C \tilde{s}_k = \mu_k \tilde{s}_k$, where $\mu_k$ represents the energy concentration of the Slepian vector $\tilde{s}_k = W F \tilde{s}_k$ in $S$. 

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We further consider the normalized graph Laplacian $D$ the diagonal degree matrix with elements $D_{i,i} = \sum_{j=1}^{N} A_{i,j}$. The selected interval is chosen centered at $k = 256$ with length $N_2 = 129$. The bandlimited space contains $N_W = 17$ Fourier basis vectors ($|\omega| \leq \pi/32$). In Fig. 1 the first Slepians vector $s_1$ has a Gaussian shape and is well localized in the interval. The second Slepians vector $s_2$ is orthogonal to the first one (both over the original domain and the selected interval as shown above) and still well localized. We then show $s_5$ which has a weaker energy concentration in the interval. By convention, the eigenvalues of the Slepians eigendecomposition are sorted according to decreasing energy concentration $1 > \mu_1 \geq \mu_2 \geq \ldots > 0$ in $S$. The eigenvalue spectrum presents a sudden transition between well-localized and poorly-localized eigenvectors; this phase transition has been well studied and occurs at the time-bandwidth product or Shannon number $K = N_W \cdot N_S / N$. The inset of Fig. 1 illustrates this phenomenon that happens around $K = 4.3$ for this example.

### B. Graph Fourier Spectrum

For an undirected weighted graph with $N$ nodes, the connectivity is characterized by a $N \times N$ symmetric adjacency matrix $A$, where the elements $A_{i,j}$ indicate the edge weights between nodes $i, j = 1, \ldots, N$. Here we assume the graph is connected and the edge weights are positive. Next to the graph and its topology, we can also consider graph signals that define a mapping from the nodes to a length-$N$ vector associating a value with every node; e.g., the graph topology can model the presence of communication links between nodes, while the graph signal can represent measures taken at the nodes, e.g., the graph topology can model the presence of communication links between nodes, while the graph signal can represent measures taken at the nodes.

The graph Laplacian is defined as $L = D - A$, where $D$ is the diagonal degree matrix with elements $D_{i,i} = \sum_{j=1}^{N} A_{i,j}$. We further consider the normalized graph Laplacian $L = D^{-1/2}L'D^{-1/2}$ that factors out differences in degree and thus is only reflecting relative connectivity.

Spectral methods for graphs are based on the insight that the eigendecomposition of the graph Laplacian gives access to a GFT. The eigenvectors $u_k$ of $L$ minimize

$$\lambda = \frac{u^T L u}{u^T u}$$

and can be ordered according to increasing eigenvalues $\lambda_1 = 0 \leq \lambda_2 \leq \ldots \leq \lambda_N$. The eigenvectors play the role of basis vectors of the graph spectrum, and the associated eigenvalues of frequencies $[12]$.

As an illustration, we build the graph of a mesh structure with $4'567$ nodes and $13'650$ edges ($9'078$ faces), see Fig. 2. In Fig. 3 the first $9$ Laplacian eigenvectors with smallest eigenvalues are represented. These graph signals correspond to our intuition about low-frequency Fourier basis vectors that represent slow variations along the principle “axes of variation”.

### C. Slepians on Graphs

To generalize Slepians to graphs, we follow the approach by [11], which is based on introducing selectivity and band-
width. First, selectivity can be easily specified by a subset of nodes \( S \) that contains the \( N_S \) nodes in which we want the energy concentration to be optimal. Similar to the classical 1-D case, we can represent the subgraph by the selection matrix \( S \). Second, for the notion of “bandwidth”, we propose to restrict the spectrum to the eigenvectors with the \( N_W < N \) smallest eigenvalues \( \lambda \). We can then consider the truncated graph spectrum matrix \( UW \) of size \( N \times N_W \); i.e., any bandlimited graph signal can be represented by \( g = UW \hat{g} \).

Similar to the 1-D case outlined before, finding the linear combination of eigenvectors within the bandlimit \( N_W \) and with maximal energy in \( S \) reverts to optimizing the Rayleigh quotient

\[
\mu = \frac{\hat{g}^T C \hat{g}}{\hat{g}^T \hat{g}},
\]

where \( C = W^T U^T S U W \) is the concentration matrix. Since \( L \) is real and symmetric, \( U \) is real as well and we revert to the regular transpose \( ^T \).

We reuse[1] the mesh example to demonstrate the Slepian design in Fig. 4a. We selected a subgraph that corresponds to the head of the animal (i.e., 1’534 nodes), see Fig. 2. The bandwidth ratio \( N_W/N \) is a design choice, which we choose here as \( 9/4’567 = 0.2\% \) to build upon the previously shown part of the Laplacian spectrum. We ordered the Slepian vectors according to increasing eigenvalues. The vector \#9 with highest eigenvalue (0.995) has a strong positive signal in the lower part of the head, in particular, the snout region. The second most-concentrated Slepian \#8 highlights left-versus-right contrast in the head, while the third one has strong signal in the upper part of the head. Then the energy concentration drops sharply and the following Slepian vectors have almost no more signal in the head.

III. LINK WITH LAPLACIAN EMBEDDING AND GRAPH CLUSTERING

The GFT plays a central role in Laplacian embedding and graph clustering. Specifically, in Laplacian embedding, the aim is to find a mapping from the nodes onto a line such that strongly connected nodes stay as close as possible, which can be expressed as[3]:

\[
\arg \min_{\hat{g}} \sum_{i,j=1}^{N} A_{i,j} (g_i - g_j)^2 = \arg \min_{\hat{g}} \hat{g}^T L \hat{g},
\]

where \( \hat{g}^T \hat{g} = 1 \) and \( \hat{g}^T 1 = 0 \). The solution is the eigenvector of the Laplacian with the smallest non-zero eigenvalue, the Fiedler vector[3]. The optimization[5] is also related to the classical graph cut problem that consists of partitioning the graph into clusters of nodes such that the cut size is minimal. In this setting, \( g_i = \pm 1 \) indicate the labels of the nodes, which can be relaxed to take any real values. For instance, in Fig. 3 the Fiedler vector is \#2, which means that 1-D Laplacian embedding projects nodes according to the posterior-anterior axis. An optimal graph cut would separate the front and the back of the animal.

We now use the Slepian design to generalize the criterion[5] as to find a bandlimited solution that is optimized for a selected set of nodes. To reveal this link, we first rewrite the quadratic form[5] as

\[
g^T L g = g^T U A U^T g = \hat{g}^T A \hat{g}.
\]

By observing the identity \( A = \Lambda^{1/2} U^T U \Lambda^{1/2} \), we propose to generalize the criterion by introducing the bandwidth selection in the spectral, and node selection in the graph domain, respectively:

\[
\xi = \hat{g}^T \Lambda^{1/2} W^T U^T S U W \Lambda^{1/2} \hat{g},
\]

which boils down to solving the eigendecomposition of the modified concentration matrix

\[
C_{emb} = \Lambda^{1/2} C \Lambda^{1/2}.
\]

This demonstrates that Laplacian embedding can be generalized as a Slepian problem with the additional weighting of the Laplacian eigenvalues. It is interesting to note that the eigenvalues \( \xi \) of \( C_{emb} \) represent the modified embedded distance, or, equivalently, a “frequency” that is localized in the subgraph \( S \), whereas the eigenvalues \( \mu \) of \( C \) represent the energy concentration in the subgraph.

In Fig. 4b, we show the solution for the modified Slepian design. In addition, we also compute for each vector the Laplacian embedded distance and energy concentration according to Eqs. [3] and [4], respectively. For comparison, we also include this information in Figs. 3 and 4a, where the values that are true eigenvalues are typeset in bold.

The eigenvector \#1 is the trivial (constant) solution, followed by a number of eigenvectors that have very low modified embedded distance since they are not well localized in the subgraph. The eigenvectors \#8 and \#9 have the highest modified embedded distance and highest energy concentration, but they appear in different order as compared to the first design (Fig. 4b), since these criteria drive for a different ranking. It is also interesting to notice that the sharp transition in energy concentration between \#6 and \#7 for the first design, is much less pronounced for the second design, but both vectors have a lower modified embedded distance (which was optimized).

Finally, the Laplacian eigenvector \#9 (Fig. 3) had already a high energy concentration in the subgraph to start with, and it is reused in both Slepian designs as \#8 and \#9, respectively. Finally, we note that for both designs the Slepian vectors are orthonormal over the entire graph as well as orthogonal over the interval \( \mathcal{S} \); i.e., it can be readily shown that \( g_i^T g_l = \delta_{k-l} \) as well as \( g_i^T S g_l = \mu_k \delta_{k-l} \) and \( g_i^T S g_l = \xi_k \delta_{k-l} \) for the first and second design, respectively, where \( \delta \) is the Kronecker delta.

IV. DISCUSSION AND CONCLUSION

We proposed two designs to obtain Slepian graph signals that are parametrized by the subgraph selection and the bandwidth. The latter provides a handle on the amount of “global” information that can be used to optimize energy concentration or embedded distance in the subgraph. Interestingly,

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1It is interesting to note that the classical 1-D case of Subsec. III-A can be implemented using the chain graph where every node is connected to its two neighbors, which corresponds to a circulant adjacency matrix. The GFT then becomes equivalent to the DFT.
when $N_W$ approaches $N$ and without subgraph selection, the concentration matrix of the first design becomes identity, while for the second one it becomes equivalent to the conventional Laplacian.

In practice, for large graphs, we want to apply the design to a limited number of eigenvectors $N_W$ that can be obtained using efficient large-scale solvers [13] implemented in widely available software libraries such as ARPACK. The Slepian optimization itself is a low-dimensional eigendecomposition of a matrix of size $N_W \times N_W$.

Future extensions of the Slepian design can also strike a balance between energy concentration and modified embedded distance, for instance, by considering the blended concentration matrix
\[
C_{\alpha} = (1 - \alpha)C + \alpha C_{emb}, \quad 0 \leq \alpha \leq 1. \tag{9}
\]

Finally, the proposed Slepian designs can serve further methodological developments in this active field, such as being adapted for spectral definitions of directed graphs [14], selective filtering or interpolation of graph signals [5], [15], or to generalize multi-resolution approaches [6], [16], [17]. The flexibility of the method, to easily recompute the Slepian vectors for different choices of the subgraph, can also enable gradual refinements and thus extend existing approaches to discover hierarchical graph structure; e.g., [18], [19].

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