Weak prime $L$–fuzzy filters of semilattices

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Abstract

The concept of weak prime $L$–fuzzy filter of a semilattice $S$ is introduced and example are given. A characterization of weak prime $L$–fuzzy filters is established and prime filters of $S$ are identified with weak prime $L$–fuzzy filters. Also, minimal weak prime $L$–fuzzy filters are characterized.

Keywords: Bounded semilattice, $L$–fuzzy filter, prime $L$–fuzzy filter, weak prime $L$–fuzzy filter, frame.

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1. Introduction

Zadeh, in his pioneering work [11] introduced the notion of a fuzzy subset $A$ of a non-empty set $X$ as a function from $X$ into $[0, 1]$. Rosenfield [6] applied this notion to develop the theory of groups. Goguen [1] generalized and continued the work of Zadeh and realized that the unit interval $[0, 1]$ is not sufficient to take the truth values of general fuzzy statements. Therewith, several researchers took interest to the fuzzyfication of algebraic structures. In which, Kuroki [2], Liu [3], Malik and Mordersan [4], and Mukherjee and Sen [5] are engaged in fuzzifying various concepts and obtained significant results of algebras.

Further, Swamy and Swamy [10] have introduced the concept of a fuzzy prime ideal of a ring and developed the theory of fuzzy ideals by assuming truth values in a complete lattice $L$ satisfying the infinite meet distributive law, such lattices are called frames. The concept of prime ideal is vital in the study of structure theory of distributive lattices. In [8], the authors have introduced and studied the notion of $L$–fuzzy filters of a semilattice $S$ with truth values in a frame $L$. It is proved that $S$ is distributive iff the lattice $\mathcal{F}(S)$ of all filters of $S$ is distributive iff the lattice $\mathcal{F}_L(\mathcal{F}(S))$ of all $L$–fuzzy filters of $S$ is distributive. In [9], the authors have introduced the concept of prime $L$–fuzzy filters of a bounded semilattice $S$, which are meet-prime elements in the lattice $\mathcal{F}_L(\mathcal{F}(S))$. Further, in [7] the authors have introduced the notion of $L$–fuzzy ideals of a semilattice $S$ and obtained significant results on this.

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The aim of this paper is to study the $L$-fuzzy filters $A$ of a bounded semilattice $S$ for which each $\alpha$-cut $A_\alpha$ i.e., $A_\alpha = \{x \in S : A(x) \geq \alpha\}$ is either a prime filter of $S$ or whole semilattice $S$. This paper consists of four sections. In the second section we recall some definitions and certain results. In third section we introduce the concept of a weak prime $L$-fuzzy filter (WPLF) of a bounded semilattice $S$ and characterize these. Fourth section deals with minimal weak prime $L$-fuzzy filters (Minimal WPLFs).

Throughout this paper, $S$ stands for a bounded semilattice $(S, \wedge, 0, 1)$ unless otherwise stated. And, $L$ stands for a non-trivial frame $(L, \wedge, \vee, 0, 1)$; i.e., a complete lattice satisfying the infinite meet distributive law

\[
\alpha \wedge \left( \bigvee_{\beta \in T} \beta \right) = \bigvee_{\beta \in T} (\alpha \wedge \beta),
\]

for all $\alpha \in L$ and any $T \subseteq L$. Here the operations $\wedge$ and $\vee$ are supremum and infimum in the lattice $L$. An element $1 \neq c \in L$ is said to be meet-prime if, for any $a, b \in L$ and $a \wedge b \leq c$ imply $a \leq c$ or $b \leq c$.

2. Preliminaries

In this section we collect basic definitions and certain results from [8, 9], that we need in sequel.

A semilattice (meet-semilattice) is an algebra $S = (S, \wedge)$ satisfying the axioms

1. $x \wedge x = x$;
2. $x \wedge y = y \wedge x$; and
3. $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, for all $x, y, z \in S$.

If we define $x \leq y$ if $x \wedge y = x$, then $\leq$ is a partial order on $S$ in which $x \wedge y$ is the inf$(x,y)$ in $S$. A non-empty subset $F$ of $S$ is said to be a final segment of $S$ if, for any $x \in F$, $y \in S$ and $x \leq y$ implies $y \in F$. A filter $F$ of a semilattice $S$ is a final segment $F$ of $S$ such that $x \wedge y \in F$ for all $x, y \in F$. The principal filter generated by an element $a$ of $S$, i.e., the set $\{x \in S : x \geq a\}$ will be denoted by $[a]$. A proper filter $P$ of a semilattice $S$ is said to be prime if whenever two filters $G$ and $H$ are such that $\phi \neq G \cap H \subseteq P$ imply either $G \subseteq P$ or $H \subseteq P$ (or equivalently, if, for any $a, b$ are such that $a \not\in P$ and $b \not\in P$ imply the existence of $x \in S$ such that $a \leq x, b \leq x$ and $x \not\in P$).

**Definition 2.1.** Let $X$ be an non-empty set and $L$ a frame. Any function $A : X \to L$ is called an $L$-fuzzy subset of $X$. For any $L$-fuzzy subset $A$ of $X$ and $\alpha \in L$, $A_\alpha$ denotes $\alpha$-cut of $A$, i.e.,

\[
A_\alpha = \{x \in X : \alpha \leq A(x)\}.
\]

**Definition 2.2.** For any $L$-fuzzy subsets $A$ and $B$ of $X$, define

\[
A \leq B \iff A(x) \leq B(x), \quad \text{for all } x \in X.
\]

Then $\leq$ is a partial order on the set of $L$-fuzzy subsets of $X$ and is called the point wise ordering.

**Result 1.** Let $A$ and $B$ be $L$-fuzzy subsets of $X$. Then

\[
A \leq B \iff A_\alpha \subseteq B_\alpha, \quad \text{for all } \alpha \in L.
\]

**Definition 2.3.** A proper $L$-fuzzy subset $A$ of $X$ is a non-constant $L$-fuzzy subset of $X$, i.e., $A(\alpha) \neq 1$ for some $x \in X$.

**Definition 2.4.** An $L$-fuzzy subset $A$ of $S$ is said to be an $L$-fuzzy filter of $S$ if,

\[
A(x_0) = 1, \quad \text{for some } x_0 \in S,
\]

and

\[
A(x \wedge y) = A(x) \wedge A(y), \quad \text{for all } x, y \in S.
\]
Result 2. The following are equivalent to each other, for any \(L\)-fuzzy subset \(A\) of \(S\),

1. \(A\) is an \(L\)-fuzzy filter of \(S\).
2. \(A(x_0) = 1\) for some \(x_0 \in S\), \(A(x \land y) \geq A(x) \land A(y)\) and \(x \leq y \Rightarrow A(y) \geq A(x)\).
3. \(A_\alpha\) is a filter of \(S\), for all \(\alpha \in L\).

Result 3. Let \(A\) be a fuzzy filter of \(S\) and \(X\) a non-empty subset of \(S\), and \(x, y \in S\). We have

1. \(x \in [X] \Rightarrow A(x) \geq \bigwedge_{i=1}^{m} A(a_i)\) for some \(a_1, a_2, \ldots, a_m \in X\), where

\[
[X] = \{a \in S : \bigwedge_{i=1}^{n} x_i \leq a \text{ for some } x_i \in X\}.
\]

2. \(x \in [y] \Rightarrow A(x) \geq A(y)\).
3. If \(S\) is bounded then \(A(0) < 1\) and \(A(1) = 1\).

Result 4. Let \((S, \land)\) be a bounded semilattice and \(F_L(F(S))\) denote the lattice all \(L\)-fuzzy filters of \(S\). Then the following are equivalent to each other:

1. \(F_L(F(S))\) is a distributive.
2. \(F(S)\) is a distributive, where \(F(S)\) denotes the lattice of filters of \(S\).
3. \(S\) is distributive.

Definition 2.5. A proper \(L\)-fuzzy filter \(A\) of a bounded semilattice \(S\) is said to be prime \(L\)-fuzzy filter of \(S\) if, for any \(L\)-fuzzy filters \(B\) and \(C\) of \(S\),

\[B \land C \leq A \Rightarrow B \leq A\] or \(C \leq A\),

where \((B \land C)(x) = B(x) \land C(x)\).

Result 5. Let \(A\) be an \(L\)-fuzzy filter of \(S\). Then \(A\) is prime \(L\)-fuzzy filter of \(S\) if and only if, the following are satisfied.

1. \(|\text{Im}(A)| = 2\), i.e., \(A\) is two-valued.
2. For any \(x \in S\), either \(A(x) = 1\) or \(A(x)\) is meet-prime element in \(L\).
3. \(A_1\) is a prime filter of \(S\).

Result 6. Let \(A\) be an \(L\)-fuzzy filter of \(S\). Then \(A\) is a prime \(L\)-fuzzy filter of \(S\) iff there exists a prime filter \(P\) of \(S\) and a meet-prime element \(\alpha\) in \(L\) such that \(A = A^\alpha_P\), where

\[
A^\alpha_P(x) = \begin{cases} 1 & \text{if } x \in P, \\ \alpha & \text{if } x \notin P. \end{cases}
\]

3. Weak prime \(L\)-Fuzzy filters (WPLF)

Let us recall that an \(L\)-fuzzy subset \(A\) of \(S\) is an \(L\)-fuzzy filter of \(S\) iff \(A_\alpha\) is a filter of \(S\) for each \(\alpha \in L\).

Definition 3.1. A proper \(L\)-fuzzy filter \(A\) of \(S\) is called a weak prime \(L\)-fuzzy filter (WPLF), if for each \(\alpha \in L\), \(A_\alpha\) is a prime filter of \(S\) or \(A_\alpha = S\).
Example 3.2. Consider the semilattice \( S \) whose Hasse-diagram is as depicted in Figure 1 and \( L = [0, 1] \), the closed interval of real numbers which is a frame in which, for any \( x, y \in L \),
\[
    x \lor y = \max\{x, y\}, \quad x \land y = \min\{x, y\}.
\]

Figure 1: Hasse-diagram of Semilattice \( S \).

Clearly \( \{a\}, \{b\} \) and \( \{a, b, c\} \) are all prime filters of \( S \). Now, define \( A : S \to L \) as follows:
\[
    A = ((0, 0), (c, 0.5), (b, 0.5), (a, 1)).
\]

Then \( A \) is a WPLF; since, the \( \alpha \)-cuts of \( A \) are
\[
    A_0 = S,
    A_1 = \{a\},
    A_{0.5} = \{a, b, c\},
    A_{\alpha} = \{a\}, \quad \text{for any } \alpha \in (0.5, 1),
\]
and
\[
    A_{\alpha} = \{a, b, c\}, \quad \text{for any } \alpha \in (0, 0.5).
\]

Theorem 3.3. Let \( A \) be a proper \( L \)-fuzzy filter of \( S \). If \( A \) is a WPLF of \( S \), then \( \text{Im}(A) \) is a chain.

Proof. Let \( a \) and \( b \in S \) and put \( \alpha = A(a) \lor A(b) \). Then,
\[
    x \in [a] \cap [b] \Rightarrow a \leq x \quad \text{and} \quad b \leq x
    \Rightarrow A(a) \leq A(x) \quad \text{and} \quad A(b) \leq A(x)
    \Rightarrow \alpha = A(a) \lor A(b) \leq A(x)
    \Rightarrow x \in A_{\alpha}.
\]
Therefore \( [a] \cap [b] \subseteq A_{\alpha} \). Since \( A_{\alpha} \) is prime, \( [a] \subseteq A_{\alpha} \) or \( [b] \subseteq A_{\alpha} \).
\[
    [a] \subseteq A_{\alpha} \Rightarrow a \in A_{\alpha} \Rightarrow \alpha = A(a) \lor A(b) \leq A(a)
    \Rightarrow A(b) \leq A(a).
\]
Similarly, \( [b] \subseteq A_{\alpha} \Rightarrow A(a) \leq A(b) \). Thus \( \text{Im}(A) \) is a chain in \( L \).

The converse of above theorem is not true. For, consider the following example.

Example 3.4. Consider two lattices \( S \) and \( L \) whose Hasse-diagrams are given in Figure 2 and Figure 3 respectively, where \( S = \{0, c, a, b, 1\} \) and \( L = \{0, s, 1\} \).
Clearly for any L-fuzzy filter $A$ of $S$, $\text{Im}(A)$ is a chain. Define $A : S \to L$ as

$$A = \{(0,0), (c,s), (a,s), (b,s), (1,1)\}.$$  

Then the $\alpha$-cuts of $A$ are $A_0 = S$, $A_s = \{c, a, b, 1\}$ and $A_1 = \{1\}$, which are filters of $S$. Therefore $A$ is an L-fuzzy filter of $S$. However $A$ is not WPLF because $A_1$ is not prime since $[a] \cap [b] = \{1\}$. The following gives a characterization of WPLFs.

**Theorem 3.5.** For any L-fuzzy filter $A$ of $S$, the following are equivalent:

1. $A$ is a WPLF of $S$.
2. For any $a$ and $b \in S$,

$$\bigwedge \{A(x) : x \in [a] \cap [b]\} = A(a) \lor A(b).$$

3. For any $a$ and $b \in S$,

$$\bigwedge \{A(x) : x \in [a] \cap [b]\} = A(a) \lor A(b),$$

and

$$\text{Im}(A)$$

is a chain in $L$.

**Proof.** First note that for any $a$ and $b \in S$,

$$A(a)$$

and $A(b) \leq \bigwedge \{A(x) : x \in [a] \cap [b]\}$.

(1) $\Rightarrow$ (2): Let $a$ and $b \in S$ and put $\alpha = \bigwedge \{A(x) : x \in [a] \cap [b]\}$. Then $\alpha \leq A(x)$ for all $x \in [a] \cap [b]$, so that $[a] \cap [b] \subseteq A_\alpha$. By (1) $A_\alpha$ is a prime filter of $S$ and hence $[a] \subseteq A_\alpha$ or $[b] \subseteq A_\alpha$. So that $a \in A_\alpha$ or $b \in A_\alpha$, i.e., $\alpha \leq A(a)$ or $\alpha \leq A(b)$. This implies $\alpha = A(a)$ or $A(b)$.

(2) $\Rightarrow$ (3): Let $a$ and $b \in S$. Then, by (2),

$$\bigwedge \{A(x) : x \in [a] \cap [b]\} = A(a) \lor A(b),$$

and hence $A(b) \leq A(a)$ or $A(a) \leq A(b)$. Therefore $\text{Im}(A)$ is a chain in $L$. Also, by (2) and since $A(a)$, $A(b)$ are lower bounds of $\{A(x) : x \in [a] \cap [b]\}$, it follows that

$$\bigwedge \{A(x) : x \in [a] \cap [b]\} = \max\{A(a), A(b)\} = A(a) \lor A(b).$$
Let $\alpha \in L$. Such that $A_\alpha \neq S$. Let $G$ and $H$ be two filters of $S$ such that $G \not\subseteq A_\alpha$ and $H \not\subseteq A_\alpha$. Then, $\alpha \not\subseteq A(a)$ and $\alpha \not\subseteq A(b)$ for some $a, b \in S$. By (3), $A(a) \leq A(b)$ or $A(b) \leq A(a)$. Hence,

$$\alpha \not\subseteq \max(A(a), A(b)) = A(a) \lor A(b).$$

Also, by (3),

$$\alpha \not\subseteq \bigwedge \{A(x) : x \in [a] \cap [b]\}.$$ 

Hence $\alpha \not\subseteq A(x)$ for some $x \in [a] \cap [b]$. This implise $G \cap H \not\subseteq A_\alpha$. Hence $A_\alpha$ is a prime filter of $S$. Thus $A$ is a WPLF of $S$. 

Now, we slightly generalize an $\alpha$-level L–fuzzy filter $A_\alpha^F$ corresponding to a filter $F$ (see Result 6).

**Definition 3.6.** For any filter $F$ of $S$ and $\alpha, \beta \in L$, define an L–fuzzy subset $A_{\alpha, \beta}^F$ of $S$ as follows:

$$A_{\alpha, \beta}^F(x) = \begin{cases} 
1 & \text{if } x = 1, \\
\alpha & \text{if } 1 \not\in x \in F, \\
\beta & \text{if } x \not\in F.
\end{cases}$$

Note that $A_{1, \beta}^F = A_\beta^F$ and $A_{1, 0}^F = \chi_r$, the characteristic function corresponding to $F$.

The following is straight forward verification.

**Lemma 3.7.** Let $F$ be a proper filter of $S$ and $\alpha, \beta \in L$. Then

$$A_{\alpha, \beta}^F$$

is an L–fuzzy filter of $S$ iff $\beta \leq \alpha$.

and, in the case, $A_{\alpha, \beta}^F$ is proper iff $\beta < 1$.

**Theorem 3.8.** For any proper filter $P$ of $S$, the following are equivalent:

(1) $P$ is a prime filter of $S$.

(2) $A_{1, \beta}^P$ is a WPLF of $S$ for each $\beta < 1$.

(3) $\chi_r$ is a WPLF of $S$.

**Proof.** (1) $\Rightarrow$ (2): Suppose $P$ is prime and let $\beta < 1$ in $L$. Put $A = A_{1, \beta}^P$. Then,

$$A(x) = \begin{cases} 
1 & \text{if } x \in P, \\
\beta & \text{if } x \not\in P.
\end{cases}$$

Let $a$ and $b \in S$. Then,

$$a \in P \lor b \in P \Rightarrow A(a) = 1 \text{ or } A(b) = 1 \text{ and } [a] \cap [b] \subseteq P$$

$$\Rightarrow A(a) = 1 \text{ or } A(b) = 1 \text{ and } A(x) = 1 \text{ for all } x \in [a] \cap [b]$$

$$\Rightarrow \bigwedge \{A(x) : x \in [a] \cap [b]\} = 1 = A(a) \lor A(b).$$

$$a \not\in P \land b \not\in P \Rightarrow A(a) = \beta = A(b) \text{ and } [a] \cap [b] \not\subseteq P$$

$$\Rightarrow A(a) = \beta = A(b) \text{ and } x \not\in P \text{ for some } x \in [a] \cap [b]$$

$$\Rightarrow \bigwedge \{A(x) : x \in [a] \cap [b]\} = \beta = A(a) = A(b).$$


Therefore
\[ \bigwedge \{ A(x) : x \in [a] \cap [b] \} = A(a) \text{ or } A(b). \]

Thus $A$ is WPLF.

(2) $\Rightarrow$ (3): It is clear by the fact that $\chi_p = A_{1,0}^P$.

(3) $\Rightarrow$ (1): Suppose $\chi_p$ is WPLF. Let $a$ and $b \in S$ such that $a \not\in P$ and $b \not\in P$. Then $\chi_p(a) = 0 = \chi_p(b)$. By supposition and hence by Theorem 3.5,
\[ \bigwedge \{ \chi_p(x) : x \in [a] \cap [b] \} = \chi_p(a) \text{ or } \chi_p(b). \]

So that
\[ \bigwedge \{ \chi_p(x) : x \in [a] \cap [b] \} = 0. \]

Hence $\chi_p(x) = 0$ for some $x \in [a] \cap [b]$. (for, $\chi_p(x) = 1$ for all $x \in [a] \cap [b] \Rightarrow \chi_p(a) = 1$ or $\chi_p(b) = 1$; a contradiction). Therefore $x \not\in P$. So that $[a] \cap [b] \not\subseteq P$. Thus $P$ is prime. \(\square\)

Lemma 3.9. For any bounded semilattice $S$, the following are equivalent:

(1) [1] is a meet-prime element in the lattice $\mathcal{F}(S)$ of all filters of $S$.

(2) For any $1 \neq a$ and $1 \neq b \in S$, there exists $1 \neq c \in S$ such that $c \supseteq a$ and $b$, i.e., $c \in [a] \cap [b]$.

Theorem 3.10. Let $P$ be a proper filter of $S$ and suppose that [1] is a meet-prime element in the lattice $\mathcal{F}(S)$ of filters of $S$. Then $P$ is prime iff $A_{\alpha, \beta}^P$ is WPLF for all $1 \neq \beta \leq \alpha$ in $L$.

Proof. Suppose $P$ is prime and $1 \neq \beta \leq \alpha \in L$. Put $A = A_{\alpha, \beta}^P$. Then $A$ is a proper L–fuzzy filter of $S$ (by Lemma 3.7). Let $a$ and $b \in S$. Then $A(a)$ and $A(b) \leq A(x)$ for all $x \in [a] \cap [b]$. Let $\gamma \in L$ such that $\gamma \leq A(x)$ for all $x \in [a] \cap [b]$. Now,
\[ a = 1 \text{ or } b = 1 \Rightarrow A(a) = 1 \text{ or } A(b) \text{ and hence } \bigwedge \{ A(x) : x \in [a] \cap [b] \} = A(a) \text{ or } A(b) \]
\[ a \not\in P \text{ and } b \not\in P \Rightarrow A(a) = \beta = A(b) \text{ and } [a] \cap [b] \not\subseteq P \]
\[ \Rightarrow A(a) = \beta = A(b) \text{ and } A(x) = \beta \text{ for some } x \in [a] \cap [b] \]
\[ \Rightarrow \gamma \leq A(x) = \beta = A(a) = A(b) \]
\[ \Rightarrow \bigwedge \{ A(x) : x \in [a] \cap [b] \} = A(a) = A(b), \]
and
\[ 1 \neq a \in P, 1 \neq b \in P \Rightarrow A(a) = \alpha = A(b) \text{ and there exists } 1 \neq c \in S \text{ such that } c \in [a] \cap [b] \subseteq P \]
\[ \Rightarrow \gamma \leq A(c) = \alpha = A(a) = A(b) \]
\[ \Rightarrow \bigwedge \{ A(x) : x \in [a] \cap [b] \} = A(a) = A(b). \]

Thus, by Theorem 3.5, $A$ is WPLF. \(\square\)

Finally in this section we discuss an inter-relationship between prime L–fuzzy filters (refer Result 6) and WPLFs.

Theorem 3.11. Every prime L–fuzzy filter of $S$ is WPLF.

Proof. Let $B$ be a Prime L–fuzzy filter of $S$. Then, $B = A_{\alpha}^P$ for some prime filter $P$ of $S$ and a meet-prime element $\alpha$ in $L$. Since $P$ is prime and $\alpha < 1$, we have $A_{\alpha}^P$ is a WPLF of $S$ (by Theorem 3.8). Thus $B$ is WPLF. \(\square\)
The converse of the above theorem is true. For, consider the example given in the following.

**Example 3.12.** Let $S$ be the 5-element lattice $\{0, b, c, a, 1\}$ represented by the Hasse-diagram given below Figure 4 and $L$ be the 3-element chain $\{0, s, 1\}$ with $0 < s < 1$.

Figure 4: Hasse-diagram of 5-element lattice $S$.

Define $A : S \rightarrow L$ by $A = \{(0,0), (b,s), (c,0), (a,s), (1,1)\}$. Then $A$ is a proper $L$–fuzzy filter of $S$. Here the $\alpha$-cuts of $A$ are $A_0 = S$, $A_s = \{b,a,1\}$ and $A_1 = \{1\}$, which are prime filters of $S$. Hence $A$ is WPLF. But $A$ is not Prime $L$–fuzzy filter since $A$ is not two-valued.

### 4. Minimal WPLF

By a minimal prime filter $M$ of $S$, we mean that there is no prime filter $Q$ of $S$ such that $Q \subset M$ and analogously, a minimal WPLF is a minimal element in the set of all WPLFs under the point-wise partial ordering.

**Theorem 4.1.** Let $A$ be a WPLF of $S$. If $A$ is a minimal WPLF of $S$, then $A_1$, i.e., 1-cut of $A$ is a minimal prime filter of $S$.

**Proof.** Suppose that $A$ is a minimal WPLF of $S$. Then $A_1 = \{x \in S : A(x) = 1\}$ is a prime filter of $S$. To prove $A_1$ is minimal, let $Q$ be a prime ideal of $S$ such that $Q \subset A_1$. Then, choose $x \in A_1$ such that $x \notin Q$. Since $Q$ is prime and hence by Theorem 3.8, $\chi_Q$ is a WPLF of $S$ and $\chi_Q(x) < A(x)$. Therefore $\chi_Q \not\leq A$. This shows that $A$ is not minimal; a contradiction. Thus $A_1$ is a minimal prime filter of $S$.

Converse of above theorem is not true. For example, in Example 3.2, $A$ is an WPLF and $A_1 = \{a\}$ which is a minimal prime filter of $S$. But $A$ is not minimal. If we define $B : S \rightarrow L$ by $B = \{(0,0), (c,0.25), (b,0.25), (a,1)\}$, then $B$ is a WPLF of $S$ and $B \not\leq A$.

**Theorem 4.2.** Let $A$ be a WPLF of $S$ and $(1)$ is a meet-prime element in the lattice $\mathcal{F}(S)$ filters of $S$. Then, $A$ is a minimal WPLF of $S$ iff, $A_\alpha$ is a minimal prime filter of $S$, for each $\alpha \in L$.

**Proof.** Assume that $A$ is a minimal WPLF of $S$. If $A_\beta$ is not a minimal prime filter of $S$ for some $0 < \beta < 1$. Then, there exists a prime filter $P$ of $S$ such that $P \subset A_\beta$. Now, define $B : S \rightarrow L$ by

$$
B(x) = \begin{cases} 
1 & \text{if } x = 1, \\
\beta & \text{if } 1 \neq x \in P, \\
0 & \text{if } x \notin P.
\end{cases}
$$

Clearly $B = A_{\beta,0}$. By Theorem 3.5, $B$ is a WPLF of $S$. As $P \subset A_\beta$, choose $y \in A_\beta$ such that $y \notin P$. Then, $\beta \leq A(y)$, and $B(y) < A(y)$.
Also $B(x) \leq A(x)$ for all $x \neq y \in S$. Therefore $B \not\leq A$; a contradiction to our assumption. Thus $A_\alpha$ is a minimal prime filter of $S$ for all $\alpha \in L$.

Conversely, assume that $A_\alpha$ is a minimal prime filter of $S$ for all $\alpha \in L$. If $B$ is a WPLF of $S$ such that $B \leq A$. Then, $B_\alpha \subseteq A_\alpha$ for all $\alpha \in L$. By assumption, $A_\alpha = B_\alpha$ for all $\alpha \in L$. Hence $B = A$. Thus $A$ is a minimal WPLF of $S$. □

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