EXTENSION OF COHOMOLOGY CLASSES AND
HOLOMORPHIC SECTIONS DEFINED ON SUBVARIETIES

XIANGYU ZHOU, LANGFENG ZHU

Abstract. In this paper, we obtain two extension theorems for cohomology classes and holomorphic sections defined on analytic subvarieties, which are defined as the supports of the quotient sheaves of multiplier ideal sheaves of quasi-plurisubharmonic functions with arbitrary singularities. The first result gives a positive answer to a question posed by Cao-Demailly-Matsumura, and unifies a few well-known injectivity theorems. The second result generalizes and optimizes a general $L^2$ extension theorem obtained by Demailly.

1. Introduction and main results

Let $(X, \mathcal{O}_X)$ be a complex manifold, $\mathcal{J} \subset \mathcal{O}_X$ be a coherent ideal sheaf, and $L$ be a holomorphic line bundle on $X$. Let $Y := V(\mathcal{J})$ be the zero variety of $\mathcal{J}$ endowed with the structure sheaf $\mathcal{O}_Y := (\mathcal{O}_X / \mathcal{J})|_Y$. Then $(Y, \mathcal{O}_Y)$ may be non-reduced.

The extension problem for cohomology classes defined on $(Y, \mathcal{O}_Y)$ is to find appropriate conditions on $X$, $\mathcal{J}$ and $L$ such that the natural homomorphism

$$H^q(X, \mathcal{O}_X(K_X \otimes L)) \rightarrow H^q(Y, \mathcal{O}_Y(K_X \otimes L)) = H^q(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{O}_X / \mathcal{J})$$

is surjective, where $q$ is a nonnegative integer. The surjectivity is equivalent to the injectivity of the homomorphism

$$H^{q+1}(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{J}) \rightarrow H^{q+1}(X, \mathcal{O}_X(K_X \otimes L)).$$

This extension problem is related to vanishing theorems and injectivity theorems. If $X$ is Stein, this problem is solved by Cartan’s Theorem B. For other cases, there have been a great number of important works related to this problem, and various advanced techniques have been developed.

In this paper, we present and develop some new idea and technique based on existed advanced techniques to obtain new results on this problem. There are two main points which seem to be different from the previous related works. One is that we introduce an idea to approximate two weight functions simultaneously (see Lemma 3.9, Subsection 4.2 and Subsection 5.2). Another one is that we take an idea such that the limit process for weight functions is done prior to other limit processes after solving $\bar{\partial}$-equations (see Subsection 4.4 and Subsection 5.4).

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The first result in this paper (Theorem 1.1) is an extension theorem for cohomology classes defined on a not necessarily reduced analytic subvariety, which is defined as the support of the quotient sheaf of multiplier ideal sheaves of quasi-plurisubharmonic functions with arbitrary singularities. This result gives a positive answer to a question posed by Cao-Demailly-Matsumura in [6] (see Remark 1.2 below), and also unifies a few well-known injectivity theorems (see Remark 1.3 below).

Let us first recall some definitions (see [39], [8], [10], [51], [11], [42], [12], [6], [13], etc).

Let $X$ be a complex manifold. A function $\varphi : X \rightarrow [-\infty, +\infty]$ on $X$ is said to be quasi-plurisubharmonic (quasi-psh) if $\varphi$ is locally the sum of a plurisubharmonic function and a smooth function.

If $\varphi$ is a quasi-psh function on $X$, the multiplier ideal sheaf $\mathcal{I}(e^{-\varphi})$ is the ideal subsheaf of $\mathcal{O}_X$ defined by

$$\mathcal{I}(e^{-\varphi})_x = \{ f \in \mathcal{O}_{X,x} : \exists U \ni x \text{ such that } \int_U |f|^2 e^{-\varphi} d\lambda < +\infty \},$$

where $U$ is an open coordinate neighborhood of $x$, and $d\lambda$ is the Lebesgue measure with respect to the coordinates on $U$. It is well known that $\mathcal{I}(e^{-\varphi})$ is coherent.

Let $L$ be a holomorphic line bundle over $X$. A singular Hermitian metric $h$ on $L$ is simply a Hermitian metric which can be expressed locally as $e^{-\varphi_U}$ on $U$ such that $\varphi_U$ is quasi-psh, where $U \subset X$ is a local coordinate chart such that $L|_U \simeq U \times \mathbb{C}$. It has a well-defined curvature current $\sqrt{-1} \Theta_{L,h} := \sqrt{-1} \partial \bar{\partial} \varphi_U$ on $X$.

In the definition of a singular Hermitian metric, $\varphi_U$ is required to be quasi-psh here, while $\varphi_U$ is only required to be in $L^1_{\text{loc}}$ in [8] and [11].

Our first result is the following extension theorem, which was announced in [57].

**Theorem 1.1.** Let $X$ be a holomorphically convex complex $n$-dimensional manifold possessing a Kähler metric $\omega$, $\psi$ be an $L^1_{\text{loc}}$ function on $X$ which is locally bounded above, and $(L,h)$ be a holomorphic line bundle over $X$ equipped with a singular Hermitian metric $h$. Assume that $\alpha > 0$ is a positive continuous function on $X$, and that the following two inequalities hold on $X$ in the sense of currents:

(i) $\sqrt{-1} \Theta_{L,h} + \sqrt{-1} \partial \bar{\partial} \psi > 0,$

(ii) $\sqrt{-1} \Theta_{L,h} + (1 + \alpha) \sqrt{-1} \partial \bar{\partial} \psi \geq 0.$

Then the homomorphism induced by the natural inclusion $\mathcal{I}(he^{-\psi}) \hookrightarrow \mathcal{I}(h),$

$$H^q(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(he^{-\psi})) \rightarrow H^q(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h))$$

is injective for every $q \geq 0$. In other words, the homomorphism induced by the natural sheaf surjection $\mathcal{I}(h) \rightarrow \mathcal{I}(he^{-\psi}),$

$$H^q(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)) \rightarrow H^q(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}))$$

is surjective for every $q \geq 0$.

**Remark 1.1.** The quotient sheaf $\mathcal{I}(h)/\mathcal{I}(he^{-\psi})$ is supported on an analytic subvariety $Y \subset X$, which is the zero set of the ideal sheaf

$$\mathcal{J}_Y := \mathcal{I}(he^{-\psi}) : \mathcal{I}(h) = \{ g \in \mathcal{O}_X : g \cdot \mathcal{I}(h) \subset \mathcal{I}(he^{-\psi}) \}.$$
When $h$ is smooth, we have $\mathcal{I}(h) = \mathcal{O}_X$ and $\mathcal{I}(he^{-\psi}) = \mathcal{I}(e^{-\psi}) = \mathcal{J}_Y$. Then $\mathcal{O}_Y = (\mathcal{I}(h)/\mathcal{I}(he^{-\psi}))(\mathcal{J}_Y)$ and Theorem 1.1 implies that
\[ H^0(X, \mathcal{O}_X(K_X \otimes L)) \longrightarrow H^0(Y, \mathcal{O}_Y(K_X \otimes L)) \]
is surjective.

**Remark 1.2.** Theorem 1.1 was proved in [6] in the case when $\psi$ is a quasi-psh function with neat analytic singularities, here, a quasi-psh function $\varphi$ on $X$ is said to have analytic singularities if every point $x \in X$ possesses an open neighborhood $U$ on which $\varphi$ can be written as
\[ \varphi = c \log \sum_{1 \leq j \leq J_0} |g_j|^2 + u, \]
where $c$ is a nonnegative number, $g_j \in \mathcal{O}_X(U)$ and $u$ is a bounded function on $U$. If $u$ is further assumed to be a smooth function on $U$, $\varphi$ is said to have neat analytic singularities (see [12]).

The general case when $\psi$ is a quasi-psh function with arbitrary singularities was posed as a question in Remark 3.10 in [6]. Theorem 1.1 gives an affirmative answer to this question.

**Remark 1.3.** Theorem 1.1 also unifies some injectivity theorems in previous important works (see [54], [33], [16], [53], [41], etc, especially the recent works [9], [18], [19], [36]). In Section 2, we will show how to deduce these injectivity theorems from Theorem 1.1. A key point in deducing them is that $\psi$ is neither required to have analytic singularities nor required to be quasi-psh in Theorem 1.1. We will also discuss the application of Theorem 1.1 to vanishing theorems in Section 2.

Some $L^2$ extension theorems and their important applications have been obtained since the establishment of the celebrated Ohsawa-Takegoshi $L^2$ extension theorem in [44]. In recent years, optimal $L^2$ extension theorems ([3], [21], [22], [23]) have been established since the utilization of the method of the undetermined function with ODE initiated in [27] and [60] (see also [56]). As an application of the optimal $L^2$ extension, inequality part of the Suita conjecture has been solved ([3], [21]). At that time very few other applications and connections of the optimal $L^2$ extension theorem existed, till some unexpected applications (including the proof of the full version of the Suita conjecture and the geometric meaning of the optimal $L^2$ extension) were found in [23]. Actually, at the present time optimal $L^2$ extension theorems have many more interesting applications (see [5], [25], [26], [29], [42], [43], [45], [58], [59], etc).

For the above extension problem, it is desirable to obtain some optimal $L^2$ estimate. The second result in this paper (Theorem 1.2) is an $L^2$ extension theorem with an optimal estimate for holomorphic sections with an estimate, which generalizes and optimizes a general $L^2$ extension theorem in [12] (see Remark 1.5 below). First let’s recall some notions and notations as below.

Let $X$ be a complex $n$-dimensional manifold possessing a smooth Hermitian metric $\omega$, $\psi$ be an $L^1_{\text{loc}}$ function on $X$ which is locally bounded above, and $(L, h)$ be a holomorphic line bundle over $X$ equipped with a singular Hermitian metric $h$. Assume that $\sqrt{-1} \Theta_{L, h} + \sqrt{-1} \partial \bar{\partial} \psi \geq \gamma$ on $X$ in the sense of currents for some continuous real $(1,1)$-form $\gamma$ on $X$. 

In this paper, we don’t assume that $\psi$ has analytic singularities and that $\psi$ is quasi-psh, although $\psi$ was assumed to be a quasi-psh function with analytic singularities in [40], [23] and [12].

Following Definition 2.11 in [12], the restricted multiplier ideal sheaf $I_\psi(h)$ is defined to be the set of germs $f \in I(h)_x \subset O_{X,x}$ such that there exists a coordinate neighborhood $U$ of $x$ satisfying

$$\lim_{t \to -\infty} \int_{\{y \in U: t < \psi(y) < t+1\}} |f|^2 e^{-\psi} d\lambda < +\infty,$$

where $U$ is small enough such that $h$ can be written as $e^{-\psi}$ with respect to a local holomorphic trivialization of $L$ on a neighborhood of $\mathcal{U}$, and $d\lambda$ is the $n$-dimensional Lebesgue measure on $U$. It is obvious that $I_\psi(h) \supset I(h^{-\psi}).$

Denote by $Y$ the zero set of the ideal sheaf $J_Y := I(h^{-\psi}) : I(h)$ (cf. Remark 1.1). Let $f$ be an element in

$$H^0(X, O_X(K_X \otimes L) \otimes I_\psi(h)/I(h^{-\psi})).$$

Then $f$ is actually supported on $Y$. We define a positive measure $|f|^2_{\omega,\cdot} dV_{\omega,\cdot}[\psi]$ (a purely formal notation, cf. [40], [23] and (2.10) in [12]) on $Y$ as the minimum element of the partially ordered set of positive measures $d\mu$ satisfying

$$\int_Y g d\mu \geq \lim_{t \to -\infty} \int_{\{x \in X: t < \psi(x) < t+1\}} g |\hat{f}|^2_{\omega,\cdot} e^{-\psi} dV_{\omega,\cdot}$$

for any nonnegative continuous function $g$ on $X$ with $\text{supp } g \subset X$, where $\hat{f}$ is a smooth extension of $f$ to $X$ such that $\hat{f} - f \in O_X(K_X \otimes L) \otimes O_X I(h^{-\psi}) \otimes O_X C^\infty$ locally for any local holomorphic representation $\hat{f}$ of $f$. It is not hard to check that the upper limit on the right hand side of the above inequality is independent of the choice of $\hat{f}$.

It is useful to consider $L^2$ estimates with variable factors. Let us recall the following definition in [23].

**Definition 1.1** ([23]). Let $\alpha_0 \in (-\infty, +\infty]$ and $\alpha \in (0, +\infty)$. If $\alpha = +\infty$, $\frac{1}{\alpha}$ is defined to be 0. When $\alpha_0 \neq +\infty$, let $R_{\alpha_0, \alpha}$ be the class of functions defined by

$$\left\{ R \in C^\infty(\infty, \alpha_0); R > 0, \ R \text{ is decreasing near } -\infty, \lim_{t \to -\infty} e^t R(t) < +\infty, \ C_R := \int_{-\infty}^{\alpha_0} \frac{1}{R(t)} dt < +\infty \text{ and } \right.$$  

$$\int_t^{\alpha_0} \left( \frac{1}{\alpha R(\alpha_0)} + \int_{t_2}^{\alpha_0} \frac{dt_1}{R(t_1)} \right) dt_2 + \frac{1}{\alpha^2 R(\alpha_0)} < R(t) \left( \frac{1}{\alpha R(\alpha_0)} + \int_t^{\alpha_0} \frac{dt_1}{R(t_1)} \right)^2$$

for all $t \in (\infty, \alpha_0)$.

When $\alpha_0 = +\infty$, we replace $R \in C^\infty(-\infty, \alpha_0]$ with the assumptions

$$R \in C^\infty(-\infty, +\infty), \ R(+\infty) := \lim_{t \to +\infty} R(t) = +\infty \text{ and } \lim_{t \to +\infty} \frac{R(t)}{R(t)} > \frac{1}{\alpha}$$

in the above definition of $R_{\alpha_0, \alpha}$. \qed
Remark 1.4. The number $\alpha_0$, $\alpha$ and the function $R(t)$ are equal to the number $A$, $\delta$ and the function $\frac{1}{c_A(t-D)}$ defined just before the main theorems in [23]. If $\alpha_0 \neq +\infty$ and $R$ is decreasing on $(-\infty, \alpha_0]$, the longest inequality in the definition of $\mathcal{R}_{\alpha_0, \alpha}$ holds for all $t \in (-\infty, \alpha_0]$. \hfill \Box

Theorem 1.2. Let $\alpha_0 \in (-\infty, +\infty)$, $\alpha \in (0, +\infty]$, and $R \in \mathcal{R}_{\alpha_0, \alpha}$. Let $X$ be a weakly pseudoconvex complex $n$-dimensional manifold possessing a Kähler metric $\omega$, $\psi$ be an $L^1_{\text{loc}}$ function on $X$ satisfying $\sup \psi < \alpha_0$ for every relatively compact set $\Omega \subset X$, and $(L, h)$ be a holomorphic line bundle over $X$ equipped with a singular Hermitian metric $h$. Denote by $Y$ the zero set of the ideal sheaf $\mathcal{J}_Y := \mathcal{I}(he^{-\psi})$; $\mathcal{I}(h)$ (cf. Remark 1.1). Assume that the following two inequalities hold on $X$ in the sense of currents:

(i) $\frac{1}{\sqrt{-1}}\sqrt{-1} \partial \bar{\partial} \psi > 0$,

(ii) $\frac{1}{\sqrt{-1}}\sqrt{-1} \partial \bar{\partial} \psi \geq 0$.

Then for every section $f \in H^0(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}_\psi(h)/\mathcal{I}(he^{-\psi}))$ such that

\begin{equation}
\int_Y |f|^2_{\omega, h} dV_{X, \omega}[\psi] < +\infty,
\end{equation}

there exists a section $F \in H^0(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}_\psi(h))$ which maps to $f$ under the morphism $\mathcal{I}_\psi(h) \rightarrow \mathcal{I}_\psi(h)/\mathcal{I}(he^{-\psi})$, such that

\begin{equation}
\int_X |F|^2_{\omega, h} e^\psi R(\psi) dV_{X, \omega} \leq \left( \frac{1}{\alpha R(\alpha_0)} + C_R \right) \int_Y |f|^2_{\omega, h} dV_{X, \omega}[\psi].
\end{equation}

Moreover, the restriction morphism

$$H^0(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}_\psi(h)) \longrightarrow H^0(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}_\psi(h)/\mathcal{I}(he^{-\psi}))$$

is surjective.

Remark 1.5. Theorem 1.2 was proved in [12] for an explicit function $R$ with a non-optimal $L^2$ estimate in the case when $\psi = (m_p - m_{p-1})\varphi$ for a quasi-psh function $\varphi$ on $X$ with neat analytic singularities, where $m_p$ are jumping numbers (see Theorem 2.13 in [12] for details). Theorem 1.2 gives an optimal $L^2$ estimate of Theorem 2.13 in [12]. In fact, the constant $\frac{1}{\alpha R(\alpha_0)} + C_R$ in the $L^2$ estimate (1.2) is optimal since it is reached in some special cases of Theorem 1.2 (see [23]). \hfill \Box

Remark 1.6. By using the methods in [23] and [2], Hosono [31] obtained an optimal $L^2$ estimate of Theorem 2.13 in [12] in the case when $X$ is a bounded pseudoconvex domain in $\mathbb{C}^n$, $Y$ is a closed complex submanifold, $\psi$ is a negative psh Green-type function continuous on $X \setminus Y$ with poles along $Y$, and $(L, h)$ is a trivial bundle with a continuous Hermitian metric $h$. \hfill \Box

2. Applications of Theorem 1.1

In this section, we will show that Theorem 1.1 implies several recent injectivity theorems obtained in [34], [36], [19], [18] and [37]. We will also discuss an application of Theorem 1.1 to vanishing theorems.

By Theorem 1.1, we can obtain the following result, which unifies some well-known injectivity theorems.
Theorem 2.1. Let $X$ be a holomorphically convex Kähler manifold. Let $(F, h_F)$ and $(M, h_M)$ be two holomorphic line bundles over $X$ equipped with singular Hermitian metrics $h_F$ and $h_M$ respectively. Assume that the following two inequalities hold on $X$ in the sense of currents:

(i) $\sqrt{-1}\Theta_{F,h_F} \geq 0$,

(ii) $\sqrt{-1}\Theta_{F,h_F} \geq b\sqrt{-1}\Theta_{M,h_M}$ for some $b \in (0, +\infty)$.

Then, for a non-zero global holomorphic section $s$ of $M$ satisfying $\sup_{\Omega} |s|_{h_M} < +\infty$ for every relatively compact set $\Omega \subset \subset X$, the following map $\beta$ induced by the tensor product with $s$

$$H^q(X, \mathcal{O}_X(K_X \otimes F) \otimes \mathcal{I}(h_F)) \xrightarrow{\beta} H^q(X, \mathcal{O}_X(K_X \otimes F \otimes M) \otimes \mathcal{I}(h_F h_M))$$

is injective for every $q \geq 0$.

Proof. Let $\psi := 2\log |s|_{h_M}$. Then $\psi$ is an $L^1_{\text{loc}}$ function on $X$ which is locally bounded above.

Let $U = \{U_i\}_{i \in I}$ be a Stein covering of $X$.

It is easy to see that the following maps induced by the tensor product with $s$

$$H^0(U_i, \mathcal{O}_X(K_X \otimes F) \otimes \mathcal{I}(h_F)) \longrightarrow H^0(U_i, \mathcal{O}_X(K_X \otimes F \otimes M) \otimes \mathcal{I}(h_F h_M e^{-\psi}))$$

are isomorphisms. Hence they induce isomorphisms between the spaces of Čech cochains $C^p(U, \mathcal{O}_X(K_X \otimes F) \otimes \mathcal{I}(h_F))$ and $C^p(U, \mathcal{O}_X(K_X \otimes F \otimes M) \otimes \mathcal{I}(h_F h_M e^{-\psi}))$.

Since these isomorphisms between the spaces of Čech cochains commute with the Čech coboundary mappings, it follows from Leray’s theorem that the following map $\sigma$ induced by the tensor product with $s$

$$H^q(X, \mathcal{O}_X(K_X \otimes F) \otimes \mathcal{I}(h_F)) \xrightarrow{\sigma} H^q(X, \mathcal{O}_X(K_X \otimes F \otimes M) \otimes \mathcal{I}(h_F h_M e^{-\psi}))$$

is an isomorphism.

Let $\iota$ be the following map induced by the natural inclusion $\mathcal{I}(h_F h_M e^{-\psi}) \longrightarrow \mathcal{I}(h_F h_M)$

$$H^q(X, \mathcal{O}_X(K_X \otimes F \otimes M) \otimes \mathcal{I}(h_F h_M e^{-\psi})) \xrightarrow{\iota} H^q(X, \mathcal{O}_X(K_X \otimes F \otimes M) \otimes \mathcal{I}(h_F h_M)).$$

Then we get the injectivity of $\iota$ by applying Theorem 1.1 to the case when $L := F \otimes M$, $h_L := h_F h_M$ and $\alpha := b$.

Therefore, the map $\beta$ in Theorem 2.1 is injective by the relation $\beta = \iota \circ \sigma$. Hence we get Theorem 2.1. \qed

Remark 2.1. Theorem 2.1 was proved in [34] in the case when $X$ is compact, and the metrics $h_F$, $h_M$ are both smooth on some Zariski open subset of $X$ (see Theorem 1.5 in [34]). \qed

Remark 2.2. Theorem 2.1 unifies the two main results Theorem 1.2 and Theorem 1.3 in [37]. More precisely, Theorem 2.1 was proved in [37] under one of the following two additional assumptions:

(1) $(M, h_M) = (F^m, h_F^m)$ for some nonnegative integer $m$ (see Theorem 1.2 in [37], and see also Theorem 1.3 in [36] for the case when $X$ is compact);

(2) $h_M$ is smooth and $\sqrt{-1}\Theta_{M,h_M} \geq 0$ (see Theorem 1.3 in [37], and see also Theorem A in [18] for the case when $X$ is compact). \qed
Remark 2.3. Theorem 2.1 was also proved in [19] in the case when \( X \) is compact, \( \sqrt{-\Theta}_{M,h_{\mu}} \geq 0 \) and \( h_{F} = h_{L}^{b} = h_{\Delta} \) for some \( b \in (0, +\infty) \) and some effective \( \mathbb{R} \)-divisor \( \Delta \) on \( X \), where \( h_{\Delta} \) is the singular Hermitian metric defined by \( \Delta \) (see Theorem 1.3 in [19]). Thus the dlt extension theorem 1.4 in [19] for compact Kähler manifolds also holds for holomorphic convex Kähler manifolds by using Theorem 2.1 and the same arguments as in [19] (see also [48], [49], [50] and [15] for the background of the dlt extension problem). \( \square \)

Injectivity theorems have been used to obtain vanishing theorems in many previous important works. Using Theorem 2.1, we can get a vanishing theorem (Theorem 2.2). Before stating the result, let’s recall some notions and notations.

For any holomorphic line bundle \((L, h_{L})\) equipped with a singular Hermitian metric \( h_{L} \) over a compact complex manifold \( X \), denote by \( H^{0}_{\text{bdd}, h_{L}}(X, L) \) the space of the holomorphic sections of \( L \) with bounded norms. Namely,

\[
H^{0}_{\text{bdd}, h_{L}}(X, L) := \{ s \in H^{0}(X, L) : \sup_{X} |s|_{h_{L}} < +\infty \}.
\]

Let \( \{h_{k}\}_{k=1}^{+\infty} \) be a sequence of singular Hermitian metrics on \( L \). The generalized Kodaira dimension \( \kappa_{\text{bdd}}(L, \{h_{k}\}_{k=1}^{+\infty}) \) of \( L \) is defined to be \( -\infty \) if \( H^{0}_{\text{bdd}, \{h_{k}\}_{k=1}^{+\infty}}(X, L^{k}) = 0 \) for any positive integer \( k \) which is large enough. Otherwise, \( \kappa_{\text{bdd}}(L, \{h_{k}\}_{k=1}^{+\infty}) \) is defined to be

\[
\sup \left\{ m \in \mathbb{Z} : \lim_{k \to +\infty} \frac{\dim H^{0}_{\text{bdd}, \{h_{k}\}_{k=1}^{+\infty}}(X, L^{k})}{k^{m}} > 0 \right\}.
\]

For more details about the generalized Kodaira dimension in the case when \( h_{k} = h_{L} \) \((\forall k \in \mathbb{Z}^{+})\) for some fixed metric \( h_{L} \) on \( L \), one can see Section 5.2 in [34] and Section 4 in [36].

Theorem 2.2. Let \( X \) be a complex \( n \)-dimensional projective manifold, and \((F, h_{F})\) be a holomorphic line bundle over \( X \) equipped with a singular Hermitian metric \( h_{F} \) satisfying \( \sqrt{-\Theta}_{F, h_{F}} \geq 0 \) on \( X \) in the sense of currents. Let \( Q \) be a holomorphic line bundle over \( X \), and \( \{h_{k}\}_{k=1}^{+\infty} \) be a sequence of singular Hermitian metrics on \( Q \). Assume that the following two inequalities hold on \( X \) in the sense of currents:

\[(i) \quad \sqrt{-\Theta}_{F, h_{F}} \geq \varepsilon_{k} \sqrt{-\Theta}_{Q, h_{k}} \quad (\forall k \in \mathbb{Z}^{+}) \]

\[(ii) \quad \sqrt{-\Theta}_{F, h_{F}} + k \sqrt{-\Theta}_{Q, h_{k}} \geq -C \omega \quad (\forall k \in \mathbb{Z}^{+}) \]

for some sequence of positive numbers \( \{\varepsilon_{k}\}_{k=1}^{+\infty} \) and some positive number \( C \) and some smooth positive \((1, 1)\)-form \( \omega \) on \( X \).

Then

\[H^{q}(X, \mathcal{O}_{X}(K_{X} \otimes F) \otimes \mathcal{I}(h_{F})) = 0 \quad \text{for} \quad q > n - \kappa_{\text{bdd}}(Q, \{h_{k}\}_{k=1}^{+\infty}),\]

where \( \kappa_{\text{bdd}}(Q, \{h_{k}\}_{k=1}^{+\infty}) \) is the generalized Kodaira dimension of \( (Q, \{h_{k}\}_{k=1}^{+\infty}) \).

Proof. The proof is similar to that of Theorem 1.4 (2) in [34].

Theorem 2.2 holds trivially if \( \kappa_{\text{bdd}}(Q, \{h_{k}\}_{k=1}^{+\infty}) \leq 0 \). Hence we assume that \( \kappa_{\text{bdd}}(Q, \{h_{k}\}_{k=1}^{+\infty}) \) is positive.

For a contradiction, we assume that there exists a non-zero cohomology class \( \xi \in H^{q}(X, \mathcal{O}_{X}(K_{X} \otimes F) \otimes \mathcal{I}(h_{F})) \) for some \( q > n - \kappa_{\text{bdd}}(Q, \{h_{k}\}_{k=1}^{+\infty}) \).

Let \( k \) be a positive integer. Then, the following map induced by the tensor product with \( \xi \)

\[H^{0}_{\text{bdd}, \{h_{k}\}_{k=1}^{+\infty}}(X, Q^{k}) \to H^{q}(X, \mathcal{O}_{X}(K_{X} \otimes F \otimes Q^{k}) \otimes \mathcal{I}(h_{F}(h_{k}^{k})))\]
is a linear map.
Since $\sqrt{-1}\Theta_{F,h_F} \geq 0$ and $\sqrt{-1}\Theta_{F,h_F} \geq \varepsilon_k \sqrt{-1}\Theta_{Q,h_k}$ hold on $X$, applying Theorem 2.1 to the above linear map in the case when $(M,h_M) := (Q^k, (h_k)^k)$ and $b := \frac{\varepsilon_k}{b}$, we get that the above linear map is injective by the assumption $\xi \neq 0$.

Hence

$$\dim H^0_{\text{bd}(h_k)}(X, Q^k) \leq \dim H^q(X, \mathcal{O}_X(K_X \otimes F \otimes Q^k) \otimes \mathcal{I}(h_F(q))) .$$

By Nadel’s vanishing theorem (see [39] or Theorem 5.11 in [11]) and the assumption (ii) in Theorem 2.2, it follows from Lemma 3.7 that

$$\dim H^q(X, \mathcal{O}_X(K_X \otimes F \otimes Q^k) \otimes \mathcal{I}(h_F(q))) = O(\kappa^{n-q}) \quad \text{as} \quad k \to +\infty .$$

Hence

$$\dim H^0_{\text{bd}(h_k)}(X, Q^k) = O(\kappa^{n-q}) \quad \text{as} \quad k \to +\infty .$$

By the definition of the generalized Kodaira dimension, the above equality is a contradiction to the inequality $q > n - \kappa_{\text{bd}}(Q, \{h^k\}_{k=1}^\infty)$. Hence we get Theorem 2.2.

\[\square\]

Remark 2.4. Theorem 2.2 contains the following two cases.

The first case is when $h_k = h_Q$ (\forall k \in \mathbb{Z}^+) for some fixed singular metric $h_Q$ on $Q$. Then the curvature assumptions (i) and (ii) are equivalent to the assumption

$$\sqrt{-1}\Theta_{F,h_F} \geq \varepsilon \sqrt{-1}\Theta_{Q,h_Q} \geq 0 \text{ for some positive number } \varepsilon .$$

If $h_Q$ is further a smooth metric with strictly positive curvature, then Theorem 2.2 is just Nadel’s vanishing theorem.

The second case is when $Q$ is numerically effective. Then there exists a sequence of smooth metrics $\{h_k\}_{k=1}^{\infty}$ such that the curvature assumption (ii) holds. In this case, $\kappa_{\text{bd}}(Q, \{h^k\}_{k=1}^\infty)$ is just the usual Kodaira dimension of $Q$.

\[\square\]

Remark 2.5. In the case when $\varepsilon_k = 1$ and $h_k = h_Q$ (\forall k \in \mathbb{Z}^+) for some fixed smooth metric $h_Q$ on $Q$, Theorem 2.2 was proved in [34] under the additional assumption that $h_F$ is smooth on some Zariski open subset of $X$ (see Theorem 1.4 (2) in [34]).

\[\square\]

Remark 2.6. Theorem 2.2 was proved in [36] in the case when $Q = F$ and $h_k = h_F$ (\forall k \in \mathbb{Z}^+) (see Theorem 4.5 in [36], and see also Theorem 1.4 (1) in [34], Theorem 5.2 in [34], Theorem 1.2 in [35]).

\[\square\]

Remark 2.7. By the vanishing theorem obtained in [4] and the strong openness property of multiplier ideal sheaves obtained in [24], one can get

$$H^q(X, \mathcal{O}_X(K_X \otimes F) \otimes \mathcal{I}(h_F)) = 0 \quad \text{for} \quad q > n - \text{nd}(F, h_F).$$

Here $\text{nd}(F, h_F)$ is the numerical dimension defined in [55] and [4], which depends on the pair $(F, h_F)$. Here, $\kappa_{\text{bd}}(Q, \{h^k\}_{k=1}^\infty)$ rather than $\text{nd}(F, h_F)$ is used in Theorem 2.2.

\[\square\]
3. Some results used in the proofs

In this section, we recall and obtain some results which will be used in the proofs of the main results in the present paper.

**Lemma 3.1** (Proposition 3.12 in [12]). Let $X$ be a complete Kähler manifold equipped with a (non necessarily complete) Kähler metric $\omega$, and let $(Q,h)$ be a holomorphic vector bundle over $X$ equipped with a smooth Hermitian metric $h$. Assume that $\tau$ and $A$ are smooth and bounded positive functions on $X$ and let

$$B := [\tau \sqrt{-1} \Omega_{Q,h} - \sqrt{-1} \partial \bar{\partial} \tau - \sqrt{-1} A^{-1} \partial \tau \wedge \bar{\partial} \tau, A].$$

Assume that $\delta \geq 0$ is a number such that $B + \delta I$ is semi-positive definite everywhere on $\wedge^{n,q} T^*_X \otimes Q$ for some $q \geq 1$. Then there exists an approximate solution $u \in L^2(X, \wedge^{n,q} T^*_X \otimes Q)$ such that $\partial u = 0$ and

$$\int_X (B + \delta I)^{-1} g, g_{\omega,h} dV_{X,\omega} < +\infty,$$

there exists an approximate solution $u \in L^2(X, \wedge^{n,q-1} T^*_X \otimes Q)$ and a correcting term $v \in L^2(X, \wedge^{n,q} T^*_X \otimes Q)$ such that $\partial u + \sqrt{\delta} v = g$ and

$$\int_X |u|^2_{\omega,h} dV_{X,\omega} + \int_X |v|^2_{\omega,h} dV_{X,\omega} \leq \int_X (B + \delta I)^{-1} g, g_{\omega,h} dV_{X,\omega}.$$

**Lemma 3.2** (Theorem 4.4.2 in [30]). Let $\Omega$ be a pseudoconvex open set in $\mathbb{C}^n$, and $\varphi$ be a plurisubharmonic function on $\Omega$. For every $w \in L^2_{(p,q+1)}(\Omega, e^{-\varphi})$ with $\partial w = 0$ there is a solution $s \in L^2_{(p,q)}(\Omega, \text{loc})$ of the equation $\partial s = w$ such that

$$\int_{\Omega} \frac{|s|^2}{(1 + |z|^2)^2} e^{-\varphi} d\lambda \leq \int_{\Omega} |w|^2 e^{-\varphi} d\lambda,$$

where $d\lambda$ is the $2n$-dimensional Lebesgue measure on $\mathbb{C}^n$.

**Lemma 3.3** (Theorem 1.5 in [7]). Let $X$ be a Kähler manifold, and $Z$ be an analytic subset of $X$. Assume that $\Omega$ is a relatively compact open subset of $X$ possessing a complete Kähler metric. Then $\Omega \setminus Z$ carries a complete Kähler metric.

**Lemma 3.4** (Lemma 6.9 in [7]). Let $\Omega$ be an open subset of $\mathbb{C}^n$ and $Z$ be a complex analytic subset of $\Omega$. Assume that $u$ is a $(p,q-1)$-form with $L^2_{\text{loc}}$ coefficients and $g$ is a $(p,q)$-form with $L^1_{\text{loc}}$ coefficients such that $\partial u = g$ on $\Omega \setminus Z$ (in the sense of currents). Then $\partial u = g$ on $\Omega$.

**Lemma 3.5** (The open mapping theorem, cf. [47]). Let $T : F_1 \rightarrow F_2$ be a linear map between Fréchet spaces $F_1$ and $F_2$. If $T$ is continuous and surjective, then $T$ is open.

**Lemma 3.6** (Theorem 2 of Section D in Chapter II of [28]). Let $U$ be an open neighborhood of the origin $0$ in $\mathbb{C}^n$, and let $G_1, \ldots, G_k$ be holomorphic functions on $U$. Denote by $\mathcal{O}_K$ the ring of germs of holomorphic functions on the set $K$ for any closed set $K \subset \mathbb{C}^n$, and denote by $\mathcal{G}$ the ideal of $\mathcal{O}_0$ generated by the germs of $G_1, \ldots, G_k$ at $0$. Then there exists an open neighborhood $V \subset U$ of $0$ and a positive number $C$, such that every $F \in \mathcal{O}_V$ whose germ at $0$ belongs to $\mathcal{G}$ can be written in the form

$$F = \sum_{j=1}^k a_j G_j \text{ as germs on } V,$$
where \( a_j \in \mathcal{O}_X \) and \( \sup_{\mathfrak{V}} |a_j| \leq C \sup_{\mathfrak{V}} |F| \).

**Lemma 3.7** (Lemma 4.3 in [34]). Let \( X \) be a complex \( n \)-dimensional projective manifold, and \( Q \) be a holomorphic line bundle on \( X \). Let \( \mathcal{G} \) be a coherent analytic sheaf on \( X \), and \( \{ \mathcal{I}_k \}_{k=1}^{+\infty} \) be ideal sheaves on \( X \) such that there exists a very ample line bundle \( \mathcal{A} \) on \( X \) satisfying

\[
H^q(X, \mathcal{A}^m \otimes \mathcal{G} \otimes Q^k \otimes \mathcal{I}_k) = 0 \text{ for any positive integers } q, m, k.
\]

Then for any \( q \geq 0 \), we have

\[
\dim H^q(X, \mathcal{G} \otimes Q^k \otimes \mathcal{I}_k) = O(k^{n-q}) \text{ as } k \to +\infty.
\]

**Theorem 3.8** (Theorem 6.1 in [9]). Let \( X \) be a complex manifold equipped with a Hermitian metric \( \omega \), and \( \Omega \subset\subset X \) be an open subset. Suppose that the Chern curvature tensor of \( T_X \) satisfies

\[
\left( \frac{\sqrt{-1}}{2\pi} \Theta_{T_X} + \omega \otimes \text{Id}_{T_X} \right) (\kappa_1 \otimes \kappa_2, \kappa_1 \otimes \kappa_2) \geq 0 \quad (\forall \kappa_1, \kappa_2 \in T_X \text{ with } \langle \kappa_1, \kappa_2 \rangle = 0)
\]
on a neighborhood of \( \overline{\Omega} \), for some continuous nonnegative \((1,1)\)-form \( \omega \) on \( X \). Assume that \( \varphi \) is a quasi-psh function on \( X \), and let \( \gamma \) be a continuous real \((1,1)\)-form such that \( -\sqrt{-1} \partial \overline{\partial} \varphi \geq \gamma \) in the sense of currents. Then there is a family of functions \( \varphi_{\varsigma, \rho} \) defined on a neighborhood of \( \overline{\Omega} \) (\( \varsigma \in (0, +\infty) \)) and \( \rho \in (0, \rho_1) \) for some positive number \( \rho_1 \) independent of \( \gamma \), such that

(i) \( \varphi_{\varsigma, \rho} \) is quasi-psh on a neighborhood of \( \overline{\Omega} \), smooth on \( \overline{\Omega} \setminus E_\varsigma(\varphi) \), increasing with respect to \( \varsigma \) and \( \rho \) on \( \overline{\Omega} \), and converges to \( \varphi \) on \( \overline{\Omega} \) as \( \rho \to 0 \),

(ii) \( \sqrt{-1} \partial \overline{\partial} \varphi_{\varsigma, \rho} \geq \gamma - \pi \varsigma \omega - \delta_\rho \omega \) on \( \Omega \),

where \( E_\varsigma(\varphi) := \{ x \in X : \nu(\varphi, x) \geq \varsigma \} \) (\( \varsigma > 0 \)) is the \( \varsigma \)-upperlevel set of Lelong numbers, and \( \{ \delta_\rho \} \) is an increasing family of positive numbers such that \( \lim_{\rho \to 0} \delta_\rho = 0 \).

**Remark 3.1.** Although Theorem 3.8 is stated in [9] in the case \( X \) is compact, almost the same proof as in [9] shows that Theorem 3.8 holds in the noncompact case while uniform estimates are obtained only on the relatively compact subset \( \Omega \). \( \square \)

We need to use the proof of Theorem 3.8 to obtain an approximation lemma below. Let us first review the construction of \( \varphi_{\varsigma, \rho} \) in [9].

Select a smooth cut-off function \( \theta : \mathbb{R} \to \mathbb{R} \) such that

\[
\theta(t) > 0 \text{ for } t < 1, \quad \theta(t) = 0 \text{ for } t \geq 1, \quad \text{and} \quad \int_{v \in \mathbb{C}^n} \theta(|v|^2) d\lambda(v) = 1.
\]

We set

\[
\varphi_\rho(x) = \frac{1}{\rho^{2n}} \int_{\{\zeta \in T_{X,x} : |\zeta| < \rho\}} \varphi(\text{exp}^x_\zeta) \theta\left(\frac{|\zeta|^2}{\rho^2}\right) d\lambda(\zeta), \quad \rho > 0,
\]

where \( \text{exp}^x_\zeta \) is a smooth map modified from the exponential map, and \( d\lambda(\zeta) \) denotes the Lebesgue measure on the Hermitian space \( (T_{X,x}, \omega(x)) \).

Let \( \rho \) be a small positive number. For \( w \in \mathbb{C} \) with \( |w| = \rho \), we have

\[
\varphi_\rho(x) = \Phi(x, w) = \Phi(x, \rho)
\]

with

\[
\Phi(x, w) := \int_{\{\zeta \in T_{X,x} : |\zeta| < 1\}} \varphi(\text{exp}^x_\zeta(\omega \zeta)) \theta(|\zeta|^2) d\lambda(\zeta).
\]

(3.1)
In [9], Demailly proved that there is a positive number $K$ and a positive number $\rho_0$ such that

(a) $\Phi(x, w)$ is smooth over $\Omega \times \{0 < |w| < \rho_0\}$ and $\lim_{\rho \to 0^+} \Phi(x, \rho) = \varphi(x)$ for any $x \in \Omega$,

(b) $\Phi(x, \rho) + K\rho^2$ is convex and increasing in $\log \rho$ when $\rho \in (0, \rho_0)$ and $x \in \Omega$,

(c) $\Phi(x, \rho)$ is quasi-psh on a neighborhood of $\Omega$ for any fixed $\rho \in (0, \rho_0)$.

By following an idea of Kiselman [32], $\varphi_{\varsigma, \rho}$ was defined in [9] to be the Legendre transform

$$\varphi_{\varsigma, \rho}(x) = \inf_{0 < |w| < 1} \left( \Phi(x, \rho |w|) + |\rho| |w| + \frac{\rho}{1 - |w|^2} - \varsigma \log |w| \right).$$

Then it is clear that $\varphi_{\varsigma, \rho}$ is increasing in $\varsigma$ and $\rho$, and that

$$\lim_{\rho \to 0^+} \varphi_{\varsigma, \rho}(x) = \lim_{|w| \to 0} \Phi(x, w) = \varphi(x).$$

By a lot of computations, Demailly obtained estimates for $\sqrt{-1}\partial \bar{\partial} \Phi$ and proved that $\varphi_{\varsigma, \rho}$ satisfies the conclusion of Theorem 3.8.

Now, by using the construction of $\varphi_{\varsigma, \rho}$, we prove the following approximation lemma, which plays an important role in the proofs of Theorem 1.1 and Theorem 1.2.

**Lemma 3.9.** Let $X$ be a complex manifold equipped with a Hermitian metric $\omega$, and $\Omega \subset \subset X$ be an open subset. Suppose that the Chern curvature tensor of $T_X$ satisfies

$$\left( \frac{\sqrt{-1}}{2\pi} \Theta_{T_X} + \varpi \otimes \text{Id}_{T_X} \right)(\kappa_1 \otimes \kappa_2, \kappa_1 \otimes \kappa_2) \geq 0 \quad (\forall \kappa_1, \kappa_2 \in T_X \text{ with } (\kappa_1, \kappa_2) = 0)$$

on a neighborhood of $\overline{\Omega}$, for some continuous nonnegative $(1, 1)$-form $\varpi$ on $X$. Let $\varphi_1$ be a quasi-psh function on $X$, and $\varphi_2$ be an $L^1_{\text{loc}}$ function on $X$ which is bounded above. Assume that $\varphi_1 + \varphi_2$ and $\varphi_1 + (1 + \alpha)\varphi_2$ are quasi-psh on $X$ such that the following two inequalities hold on $X$ in the sense of currents:

$$\sqrt{-1}\partial \bar{\partial} \varphi_1 + \sqrt{-1}\partial \bar{\partial} \varphi_2 \geq \gamma_1,$$

$$\sqrt{-1}\partial \bar{\partial} \varphi_1 + (1 + \alpha)\sqrt{-1}\partial \bar{\partial} \varphi_2 \geq \gamma_2,$$

where $\gamma_1$ and $\gamma_2$ are continuous real $(1, 1)$-forms on $X$, and $\alpha$ is a positive number.

Let

$$\Sigma := \{ \varphi_1 + \varphi_2 = -\infty \} \cup \{ \varphi_1 + (1 + \alpha)\varphi_2 = -\infty \}$$

and

$$\Sigma_{\varsigma} := E_{\varsigma}(\varphi_1 + \varphi_2) \cup E_{\varsigma}(\varphi_1 + (1 + \alpha)\varphi_2),$$

where $E_{\varsigma}(\varphi) := \{ x \in X : \nu(\varphi, x) \geq \varsigma \}$ ($\varsigma > 0$) is the $\varsigma$-upperlevel set of Lelong numbers for a quasi-psh function $\varphi$ on $X$. Then there are two family of upper semicontinuous functions $\{ \varphi_{1, \varsigma, \rho} \}$ and $\{ \varphi_{2, \varsigma, \rho} \}$ defined on a neighborhood of $\overline{\Omega}$ with values in $[-\infty, +\infty]$ ($\varsigma \in (0, +\infty)$ and $\rho \in (0, \rho_1)$ for some positive number $\rho_1$) independent of $\gamma_1$ and $\gamma_2$, such that

(i) $\varphi_{1, \varsigma, \rho} + \varphi_{2, \varsigma, \rho}$ and $\varphi_{1, \varsigma, \rho} + (1 + \alpha)\varphi_{2, \varsigma, \rho}$ are quasi-psh on a neighborhood of $\overline{\Omega}$ up to their values on $\Sigma_{\varsigma}$, smooth on $\overline{\Omega} \setminus \Sigma_{\varsigma}$, increasing with respect to $\varsigma$ on $\overline{\Omega} \setminus \Sigma$, increasing with respect to $\rho$ on $\overline{\Omega} \setminus \Sigma_{\varsigma}$, and converge to $\varphi_1 + \varphi_2$ and $\varphi_1 + (1 + \alpha)\varphi_2$ on $\overline{\Omega} \setminus \Sigma_{\varsigma}$ respectively as $\rho \to 0$ (we say that a function
\( f \) is quasi-psh on an open set \( \Omega_1 \) up to its values on a set \( \Sigma_1 \) if \( f = g \) on \( \Omega_1 \setminus \Sigma_1 \) for some quasi-psh function \( g \) on \( \Omega_1 \).

(i) \( \varphi_{1,\rho} \geq \varphi_1 \) on \( \Omega, \varphi_{2,\rho} \leq \sup_\Omega \varphi_2 \) on \( \Omega \), and \( \partial_\rho \varphi_{k,\rho} \in L^1_{\Omega} \) for \( k = 1, 2 \).

(ii) \( \sqrt{10} \partial_\rho \varphi_{1,\rho} + \sqrt{10} \partial_\rho \varphi_{2,\rho} \geq \gamma_1 - \pi \omega - \delta_\omega \) on \( \Omega \).

(iii) \( \sqrt{10} \partial_\rho \varphi_{1,\rho} + (1 + \alpha) \sqrt{10} \partial_\rho \varphi_{2,\rho} \geq \gamma_2 - \pi \omega - \delta_\omega \) on \( \Omega \).

where \( \{\delta_\rho\} \) is an increasing family of positive numbers such that \( \lim_{\rho \rightarrow 0} \delta_\rho = 0 \).

**Proof.** As in (3.1), we set

\[
\Phi_k(x, w) = \int_{\{\zeta \in T_{X,x} : |\zeta| < 1\}} \varphi_k(\exp \chi(w, \zeta)) \theta(|\zeta|^2) d\lambda(\zeta), \quad k = 1, 2.
\]

Let

\[
\Upsilon_{1,\rho}(x) := \inf_{0 < |w| < 1} \left( \Phi_1(x, \rho|w|) + \Phi_2(x, \rho|w|) + \rho|w| + \frac{\rho}{1 - |w|^2} - \zeta \log |w| \right)
\]

and

\[
\Upsilon_{2,\rho}(x) := \inf_{0 < |w| < 1} \left( \Phi_1(x, \rho|w|) + (1 + \alpha)\Phi_2(x, \rho|w|) + \rho|w| + \frac{\rho}{1 - |w|^2} - \zeta \log |w| \right).
\]

If \( x \notin \Sigma_\zeta \), define

\[
\varphi_{1,\rho}(x) = \frac{1 + \alpha}{\alpha} \Upsilon_{1,\rho}(x) - \frac{1}{\alpha} \Upsilon_{2,\rho}(x)
\]

and

\[
\varphi_{2,\rho}(x) = \frac{1}{\alpha} \Upsilon_{2,\rho}(x) - \frac{1}{\alpha} \Upsilon_{1,\rho}(x).
\]

If \( x \in \Sigma_\zeta \), define

\[
\varphi_{1,\rho}(x) = \lim_{\zeta \to x} \varphi_{1,\rho}(y) \quad \text{and} \quad \varphi_{2,\rho}(x) = \lim_{\zeta \to x} \varphi_{2,\rho}(y).
\]

Hence we have

\[
\varphi_{1,\rho}(x) + \varphi_{2,\rho}(x) = \Upsilon_{1,\rho}(x)
\]

and

\[
\varphi_{1,\rho}(x) + (1 + \alpha)\varphi_{2,\rho}(x) = \Upsilon_{2,\rho}(x)
\]

for \( x \notin \Sigma_\zeta \).

Therefore, (i), (iii) and (iv) holds by Theorem 3.8. It is also easy to see that \( \varphi_{1,\rho} \) and \( \varphi_{2,\rho} \) are upper semicontinuous.

Let \( A : (0, 1) \longrightarrow \mathbb{R} \) and \( B : (0, 1) \longrightarrow \mathbb{R} \) be two functions such that

\[
\inf_{0 < t < 1} A(t) > -\infty.
\]

The simple property

\[
\inf_{0 < t < 1} \left( A(t) + B(t) \right) - \inf_{0 < t < 1} A(t) \geq \inf_{0 < t < 1} B(t)
\]

implies that

\[
\varphi_{1,\rho}(x) \geq \inf_{0 < |w| < 1} \left( \Phi_1(x, \rho|w|) + \rho|w| + \frac{\rho}{1 - |w|^2} - \zeta \log |w| \right)
\]

for any \( x \in \Omega \setminus \Sigma_\zeta \).

Hence \( \varphi_{1,\rho} \geq \varphi_1 \) on \( \Omega \setminus \Sigma_\zeta \) when \( \rho \) is small enough by the conclusion (i) in Theorem 3.8. Then \( \varphi_{1,\rho} \geq \varphi_1 \) on \( \Omega \) by the quasi-plurisubharmonicity of \( \varphi_1 \) and the definition of \( \varphi_{1,\rho} \) on \( \Sigma_\zeta \).
The simple property
\[
\inf_{0 < t < 1} (A(t) + B(t)) - \inf_{0 < t < 1} A(t) \leq \sup_{0 < t < 1} B(t)
\]
implies that
\[
\varphi_{2, \kappa, \rho}(x) \leq \sup_{0 < |w| < 1} \Phi_2(x, \rho|w|)
\]
for any \( x \in \overline{\Omega} \setminus \Sigma_c \).

Since it is easy to see that \( \Phi_2(x, w) \leq \sup_{\mathcal{X}} \varphi_2 \) for any \( x \in \overline{\Omega} \) when \( |w| \) is small enough, we get \( \varphi_{2, \kappa, \rho}(x) \leq \sup_{\mathcal{X}} \varphi_2 \) for any \( x \in \overline{\Omega} \setminus \Sigma_c \) when \( \rho \) is small enough. Hence \( \varphi_{2, \kappa, \rho} \leq \sup \varphi_2 \) on \( \overline{\Omega} \) when \( \rho \) is small enough by the definition of \( \varphi_{2, \kappa, \rho} \) on \( \Sigma_c \).

Since \( \Upsilon_{1, \kappa, \rho} \) and \( \Upsilon_{2, \kappa, \rho} \) are quasi-psh functions on a neighborhood of \( \overline{\Omega} \) by Theorem 3.8, and the first partial derivatives of any quasi-psh function are in \( L^1_{loc} \), we get that \( \partial \varphi_{k, \kappa, \rho} \in L^1_{\Omega} \) on \( \Omega \) for \( k = 1, 2 \).

Therefore, we get (ii). \( \square \)

Let \( X \) be an \( n \)-dimensional complex analytic space, and \( \mathcal{F} \) be a coherent analytic sheaf over \( X \). Then there is a natural topology on the cohomology groups \( H^q(X, \mathcal{F}) \) (\( 0 \leq q \leq n \)).

In fact, let \( \mathcal{U} = \{ U_i \}_{i \in I} \) be a Stein covering of \( X \). Since \( \mathcal{F} \) is coherent, \( \mathcal{F} \) is locally isomorphic to a quotient sheaf of some direct sum \( \mathcal{O}_{X}^{\oplus k} \). Since the space of sections \( \Gamma(U, \mathcal{O}_{X}^{\oplus k}) \) for any Stein open subset \( U \subset X \) can be endowed with the topology of local uniform convergence of holomorphic sections, there is a natural quotient topology on \( \Gamma(V, \mathcal{F}) \) for any Stein open subset \( V \subset X \). Then we consider the product topology on the spaces of \( \check{\text{C}}ech \) cochains \( C^q(\mathcal{U}, \mathcal{F}) = \prod_{V} \Gamma(U_{i_0 \cdots i_k}, \mathcal{F}) \) and the quotient topology on \( \check{\text{H}}^q(\mathcal{U}, \mathcal{F}) \), where \( U_{i_0 \cdots i_k} \) denotes \( U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_k} \). Since Leray’s Theorem shows that the sheaf (or \( \check{\text{C}}ech \) cohomology group \( \check{\text{H}}^q(X, \mathcal{F}) \) is isomorphic to \( \check{\text{H}}^q(\mathcal{U}, \mathcal{F}) \), \( \check{\text{H}}^q(X, \mathcal{F}) \) can be endowed with the resulting topology, which is in fact independent of the choice of the Stein covering \( \mathcal{U} \) (cf. Section 4.C of Chapter IX in [14]).

The following lemma is a topological result for spaces of sections of a coherent analytic sheaf.

**Lemma 3.10** (Lemma 12 of Section A in Chapter VIII of [28]). Let \( X \) be a complex analytic space, and \( \mathcal{F} \) be a coherent analytic sheaf over \( X \). Let \( x \in X \), and \( \mathcal{M} \) be a submodule of \( \mathcal{F}_x \). Then for any open neighborhood \( U \) of \( x \),
\[
M_U := \{ F \in H^0(U, \mathcal{F}) : \text{the germ of } F \text{ at } x \text{ belongs to } \mathcal{M} \}
\]
is a closed subset of \( H^0(U, \mathcal{F}) \).

If \( X \) is a holomorphically convex complex analytic space, there exists a topological isomorphism between cohomology groups as stated in the following lemma.

**Lemma 3.11** (Lemma II.1 in [46]). Let \( X \) and \( S \) be complex analytic spaces, and let \( \pi : X \rightarrow S \) be a proper holomorphic surjection. Let \( \mathcal{F} \) be a coherent analytic sheaf on \( X \) and suppose \( S \) is a Stein space. Then the direct image map
\[
H^q(X, \mathcal{F}) \rightarrow H^0(S, R^q\pi_*\mathcal{F})
\]
is an isomorphism of topological vector spaces for each \( q \geq 0 \). In particular, each \( H^q(X, \mathcal{F}) \) is Hausdorff.
Let $X$ be an $n$-dimensional complex manifold, $\psi$ be an $L^1_{\text{loc}}$ function on $X$ which is locally bounded above, and $(L, h)$ be a holomorphic line bundle over $X$ equipped with a singular Hermitian metric $h$. Assume that $\sqrt{-1}\Theta_{L, h} + \sqrt{-1}\partial\bar{\partial}\psi \geq \gamma$ on $X$ in the sense of currents for some continuous real $(1, 1)$-form $\gamma$ on $X$.

Now we define topologies on the $L^2_{\text{loc}}$ Dolbeault cohomology groups with respect to the singular metric $h$.

Let $L^{n,q}_{(2), h}$ $(0 \leq q \leq n)$ denote the sheaf over $X$ whose stalk over a point $x \in X$ consists of germs of Lebesgue measurable $L$-valued $(n, q)$-forms $u$ such that

$$\int_U |u|^2_h d\lambda + \int_U |\partial u|^2_h d\lambda < +\infty$$

for some open coordinate neighborhood $U$ of $x$, where $d\lambda$ is the Lebesgue measure with respect to the coordinates on $U$. Then

$$0 \to \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h) \xrightarrow{i} L^{n,0}_{(2), h} \xrightarrow{\partial_h} L^{n,1}_{(2), h} \xrightarrow{\partial_h} L^{n,2}_{(2), h} \xrightarrow{\partial_h} \cdots$$

is a fine resolution of $\mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)$ by Lemma 3.2, where $i$ is the inclusion homomorphism.

Let $U \subset X$ be an arbitrary Stein open coordinate subset. We define semi-norms $\| \bullet \|_{h, K}$ on the space of sections $\Gamma(U, L^{n,q}_{(2), h})$ by

$$\|u\|_{h, K} := \left( \int_K |u|^2_h d\lambda + \int_K |\partial u|^2_h d\lambda \right)^{1/2},$$

where $K$ is any compact subset of $U$. Then $\Gamma(U, L^{n,q}_{(2), h})$ together with the family of semi-norms $\| \bullet \|_{h, K}$ becomes a Fréchet space (the Fréchet topology is independent of the choice of the coordinates on $U$). Then the following induced sequence

$$(3.2) \quad 0 \to \Gamma(U, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)) \xrightarrow{i} \Gamma(U, L^{n,0}_{(2), h}) \xrightarrow{\partial_h} \Gamma(U, L^{n,1}_{(2), h}) \xrightarrow{\partial_h} \Gamma(U, L^{n,2}_{(2), h}) \xrightarrow{\partial_h} \cdots$$

is exact by Lemma 3.2. The homomorphism $\partial_h$ is continuous for each $q$ by the definitions of the semi-norms $\| \bullet \|_{h, K}$, and the homomorphism $i$ is continuous by Lemma 3.6, where $\Gamma(U, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h))$ is endowed with the Fréchet topology of locally uniform convergence of holomorphic sections.

The corresponding Dolbeault complex is

$$\Gamma(X, L^{n,0}_{(2), h}) \xrightarrow{\partial} \Gamma(X, L^{n,1}_{(2), h}) \xrightarrow{\partial} \Gamma(X, L^{n,2}_{(2), h}) \xrightarrow{\partial} \cdots,$$

where we also endow $\Gamma(X, L^{n,q}_{(2), h})$ with the Fréchet topology defined similarly as the topology of $\Gamma(U, L^{n,q}_{(2), h})$. Let

$$\partial_{-1} : 0 \to \Gamma(X, L^{n,0}_{(2), h})$$

be the zero map. Then the $L^2_{\text{loc}}$ Dolbeault cohomology group with respect to the singular metric $h$ defined by

$$H^{n,q}_{(2)}(X, L, h) = \frac{\ker \partial_q}{\text{Im} \partial_{q-1}} \quad (0 \leq q \leq n)$$

is endowed with the quotient topology.

For the sheaf (or Čech) cohomology group and the $L^2_{\text{loc}}$ Dolbeault cohomology group, there is a topological isomorphism as stated in the following lemma.
Lemma 3.12. Under the topologies defined above, the group isomorphism
\[ H^q(X, \mathcal{O}_X(KX \otimes L) \otimes \mathcal{I}(h)) \simeq H^{n,q}_{(2)}(X, L, h) \quad (0 \leq q \leq n) \]
is a topological isomorphism.

Proof. The topological isomorphism for \( q = 0 \) follows easily from Lemma 3.6.

Since the homomorphisms in the exact sequence (3.2) are continuous, the topological isomorphism for \( q \geq 1 \) can be proved in the same way as in Proposition 12 in [1], where the topological isomorphism between the Čech cohomology group and the usual \( C^\infty \) Dolbeault cohomology group was proved. □

Similarly, we can define topologies on the \( L^2_{\text{loc}} \) Dolbeault cohomology groups with respect to the quotient sheaf \( \mathcal{I}(h)/\mathcal{I}(he^{-\psi}) \).

In fact,
\[ 0 \to \mathcal{O}_X(KX \otimes L) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}) \xrightarrow{i} \mathcal{L}^{n,0}_{(2),h}/\mathcal{L}^{n,0}_{(2),he^{-\psi}} \xrightarrow{\partial} \mathcal{L}^{n,1}_{(2),h}/\mathcal{L}^{n,1}_{(2),he^{-\psi}} \xrightarrow{\partial} \cdots \]
is a fine resolution of \( \mathcal{O}_X(KX \otimes L) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}) \) by Lemma 3.2, where \( i \) is the inclusion homomorphism.

Let \( U \subset X \) be an arbitrary Stein open coordinate subset. We define semi-norms \( \| \bullet \|_{h,K} \) on the space of sections \( \Gamma(U, \mathcal{L}^{n,q}_{(2),h}/\mathcal{L}^{n,q}_{(2),he^{-\psi}}) \) by the quotient topology induced from the isomorphism
\[ \Gamma(U, \mathcal{L}^{n,q}_{(2),h}/\mathcal{L}^{n,q}_{(2),he^{-\psi}}) \simeq \Gamma(U, \mathcal{L}^{n,q}_{(2),he^{-\psi}})/\Gamma(U, \mathcal{L}^{n,q}_{(2),he^{-\psi}}), \]
where \( K \) is any compact subset of \( U \), i.e.,
\[ \| u' \|_{h,K} := \inf \{ \| u \|_{h,K} : u \in \Gamma(U, \mathcal{L}^{n,q}_{(2),h}) \text{ and } u \text{ is in the equivalent class } u' \}. \]
Then \( \Gamma(U, \mathcal{L}^{n,q}_{(2),h}/\mathcal{L}^{n,q}_{(2),he^{-\psi}}) \) together with the family of semi-norms \( \| \bullet \|_{h,K} \) becomes a Fréchet space. Then the following induced sequence
\[ 0 \to \Gamma(U, \mathcal{O}_X(KX \otimes L) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi})) \xrightarrow{i} \Gamma(U, \mathcal{L}^{n,0}_{(2),h}/\mathcal{L}^{n,0}_{(2),he^{-\psi}}) \xrightarrow{\partial} \Gamma(U, \mathcal{L}^{n,1}_{(2),h}/\mathcal{L}^{n,1}_{(2),he^{-\psi}}) \xrightarrow{\partial} \cdots \]
is exact by Lemma 3.2. The homomorphism \( \partial_1 \) is continuous for each \( q \) by the definitions of the semi-norms \( \| \bullet \|_{h,K} \), and the homomorphism \( i \) is continuous by Lemma 3.6, where
\[ \Gamma(U, \mathcal{O}_X(KX \otimes L) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi})) \simeq \frac{\Gamma(U, \mathcal{O}_X(KX \otimes L) \otimes \mathcal{I}(h))}{\Gamma(U, \mathcal{O}_X(KX \otimes L) \otimes \mathcal{I}(he^{-\psi}))} \]
is endowed with the quotient topology induced by the Fréchet topology of locally uniform convergence of holomorphic sections.

The corresponding Dolbeault complex is
\[ \Gamma(X, \mathcal{L}^{n,0}_{(2),h}/\mathcal{L}^{n,0}_{(2),he^{-\psi}}) \xrightarrow{\partial_1} \Gamma(X, \mathcal{L}^{n,1}_{(2),h}/\mathcal{L}^{n,1}_{(2),he^{-\psi}}) \xrightarrow{\partial_2} \Gamma(X, \mathcal{L}^{n,2}_{(2),h}/\mathcal{L}^{n,2}_{(2),he^{-\psi}}) \xrightarrow{\partial_3} \cdots, \]
where we also endow \( \Gamma(X, \mathcal{L}^{n,q}_{(2),h}/\mathcal{L}^{n,q}_{(2),he^{-\psi}}) \) with the Fréchet topology defined similarly as the topology of \( \Gamma(U, \mathcal{L}^{n,q}_{(2),h}/\mathcal{L}^{n,q}_{(2),he^{-\psi}}) \). Let
\[ \partial_{-1} : 0 \to \Gamma(X, \mathcal{L}^{n,0}_{(2),h}/\mathcal{L}^{n,0}_{(2),he^{-\psi}}) \]
be the zero map. Then the $L^2_{bc}$ Dolbeault cohomology group with respect to the quotient sheaf $\mathcal{I}(h)/\mathcal{I}(he^{-\psi})$ defined by

$$H^{n,q}_{(2)}(X,L,h/he^{-\psi}) = \frac{\text{Ker} \, \bar{\partial}_q}{\text{Im} \, \bar{\partial}_{q-1}} \quad (0 \leq q \leq n)$$

is endowed with the quotient topology.

Similarly, we have

**Lemma 3.13.** Under the topologies defined above, the group isomorphism

$$H^q(X,\mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi})) \simeq H^{n,q}_{(2)}(X,L,h/he^{-\psi}) \quad (0 \leq q \leq n)$$

is a topological isomorphism.

**Remark 3.2.** It is not hard to obtain the following commutative diagram

$$H^q(X,\mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)) \xrightarrow{p_X} H^q(X,\mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}))$$

$$\downarrow i_X \quad \downarrow j_X$$

$$H^{n,q}_{(2)}(X,L,h) \xrightarrow{P_X} H^{n,q}_{(2)}(X,L,h/he^{-\psi}),$$

where $i_X$, $j_X$ are the isomorphisms in Lemma 3.12 and Lemma 3.13 respectively, and $p_X$, $P_X$ are the natural homomorphisms. □

### 4. Proof of Theorem 1.1

In this section, we will denote the sheaf $\mathcal{O}_X(K_X \otimes L)$ simply by $\mathcal{K}$. Then $\mathcal{K} \otimes \mathcal{I}(h)$ and $\mathcal{K} \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi})$ are coherent analytic sheaves.

Since $X$ is a holomorphically convex complex manifold, Remmert’s reduction theorem implies that there exists a proper holomorphic surjection $\pi : X \to S$ such that $S$ is a normal Stein space. Then by Grauert’s direct image theorem, the $q$-th direct image sheaf $R^q\pi_*\mathcal{F}$ over $S$ of any coherent analytic sheaf $\mathcal{F}$ over $X$ is coherent and we have the group isomorphism (cf. [20])

$$H^q(X,\mathcal{F}) \simeq H^q(S, R^q\pi_*\mathcal{F}).$$

Therefore, by the Stein property of $S$ and Cartan’s Theorem B, in order to prove the surjectivity statement of Theorem 1.1, it is enough to prove the surjectivity of the sheaf homomorphism

$$P : R^q\pi_*\mathcal{K} \otimes \mathcal{I}(h) \to R^q\pi_*\mathcal{K} \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}).$$

(4.1)

The proof will be divided into the following five subsections.

Let $y_0 \in S$ be an arbitrary fixed point and take

$$f \in R^q\pi_*\mathcal{K} \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi})|_{y_0}.$$ 

Then there exists a Stein neighborhood $S_0 \subset S$ of $y_0$ such that the germ $f$ has a representation (still denoted by $f$) in $H^q(S_0, R^q\pi_*\mathcal{K} \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}))$, i.e.,

$$f \in H^q(X_0, \mathcal{K} \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi})),$$

where $X_0 := \pi^{-1}(S_0)$. 
4.1. Construction of a smooth representation \( \tilde{f} \) of \( f \) from \( Y \cap X_0 \) to \( X_0 \), where \( Y \) is defined in Remark 1.1. 

Let \( \mathcal{U} = \{ U_i \}_{i \in I} \) be a locally finite Stein covering of \( X_0 \). By Leray’s theorem, \( f \) is represented by a Čech \( q \)-cocycle \( \{ c_{i_0 \cdots i_q} \} \) with 
\[
c_{i_0 \cdots i_q} \in \Gamma \left( U_{i_0 \cdots i_q}, \mathcal{K} \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}) \right),
\]
where \( U_{i_0 \cdots i_q} \) denotes \( U_{i_0} \cap \cdots \cap U_{i_q} \).

Since the natural homomorphism 
\[
\Gamma \left( U_{i_0 \cdots i_q}, \mathcal{K} \otimes \mathcal{I}(h) \right) \to \Gamma \left( U_{i_0 \cdots i_q}, \mathcal{K} \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}) \right)
\]
is surjective by the Stein property of \( U_{i_0 \cdots i_q} \), we can assume that 
\[
c_{i_0 \cdots i_q} \in \Gamma \left( U_{i_0 \cdots i_q}, \mathcal{K} \otimes \mathcal{I}(h) \right).
\]

By the explicit expression of the isomorphism between Čech cohomology groups and \( C^\infty \) Dolbeault cohomology groups (cf. [1], [14] or [6]), \( f \) is represented by an \((n, q)\)-form 
\[
\tilde{f} := \sum_{i_0 \cdots i_q} c_{i_0 \cdots i_q} \wedge \xi_{i_0} \bar{\partial} \xi_{i_0} \wedge \cdots \wedge \bar{\partial} \xi_{i_q-1},
\]
where \( \{ \xi_i \}_{i \in I} \) is a partition of unity subordinate to \( \mathcal{U} \).

Let the symbols \( L_{n,q}^{(2),h} \) and \( \bar{\partial} \) be as in Section 3 (cf. the arguments before Lemma 3.12 and Lemma 3.13). Then \( \tilde{f} \) is smooth on \( X_0 \), \( \tilde{f} \in \Gamma(X_0, L_{n,q}^{(2),h}) \) and \( \bar{\partial} \tilde{f} = 0 \) in \( \Gamma(X_0, L_{n,q+1}^{(2),h}/L_{n,q+1}^{(2),he^{-\psi}}) \), i.e., 
\[
\bar{\partial} \tilde{f} \in \Gamma(X_0, L_{n,q+1}^{(2),h}).
\]

Let \( h_0 \) be any fixed smooth metric of \( L \) on \( X \). Then \( h = h_0 e^{-\phi} \) for some global function \( \phi \) on \( X \), which is quasi-psh by the assumption in the theorem.

Let \( S_1 \subset S_0 \) be any fixed Stein neighborhood of \( y_0 \) and let \( X_1 := \pi^{-1}(S_1) \). Then we have 
\[
\int_{X_1} |\bar{\partial} \tilde{f}|_{\omega, h_0}^2 e^{-\phi - \psi} dV_{X_1, \omega} < +\infty.
\]

4.2. Approximation of singular weights.

By the assumptions in Theorem 1.1, the following two inequalities hold on \( X_0 \):
\[
\sqrt{-1} \bar{\partial} \phi + \sqrt{-1} \bar{\partial} \psi \geq -\sqrt{-1} \Theta_{L,h_0}
\]
and
\[
\sqrt{-1} \bar{\partial} \phi + (1 + \alpha) \sqrt{-1} \bar{\partial} \psi \geq -\sqrt{-1} \Theta_{L,h_0},
\]
where \( \alpha \) can be assumed to be a positive number since \( X_0 \subset X \).

By the two curvature inequalities above, \( \phi + \psi \) and \( \phi + (1 + \alpha) \psi \) are equal to quasi-psh functions on \( X_0 \) almost everywhere. Without loss of generality, we can assume that they are quasi-psh on \( X_0 \).

Since there must exist a continuous nonnegative \((1,1)\)-form \( \varpi \) on \( X \) such that 
\[
\left( \frac{\sqrt{-1}}{2\pi} \Theta_{T_X} + \varpi \otimes \text{Id}_{T_X} \right)(\kappa_1 \otimes \kappa_2, \kappa_1 \otimes \kappa_2) \geq 0 \quad (\forall \kappa_1, \kappa_2 \in T_X)
\]
holds on a neighborhood of \( X_1 \), by applying Lemma 3.9 to the case \( \varphi_1 := \phi \), \( \varphi_2 := \psi \), \( \gamma_1 = \gamma_2 := -\sqrt{-1} \Theta_{L,h_0} \) and \( \Omega := X_1 \), we obtain two family of upper semicontinuous functions \( \{ \phi_{\varsigma, \rho} \} \) and \( \{ \psi_{\varsigma, \rho} \} \) \((\varsigma \in (0, +\infty) \) and \( \rho \in (0, \rho_1) \) for some...
positive number \( \rho_1 \) defined on a neighborhood of \( X_1 \) satisfying the conclusion of Lemma 3.9.

Let \( n_1 \) be a positive integer such that \( n_1 \omega_1 \geq \omega \) on \( X_1 \), and let \( \psi := \frac{\delta_{n_1}}{n_1} \), where \( \delta_n \) is as in the conclusion of Lemma 3.9. Denote \( \phi_{\psi, \rho}, \psi_{\psi, \rho} \) and \( \Sigma_{\psi} \) (cf. Lemma 3.9) simply by \( \phi_{\rho}, \psi_{\rho} \) and \( \Sigma_{\rho} \) respectively. It is obvious that \( \Sigma_{\rho} \subset X \) for any \( \rho \in (0, \rho_1) \) (\( \Sigma \) is defined as in Lemma 3.9). Then we have

(i) \( \phi_{\rho} \) and \( \psi_{\rho} \) are smooth on \( X_1 \setminus \Sigma_{\rho} \), \( \lim_{\rho \to 0} \phi_{\rho} = \phi \) on \( X_1 \setminus \Sigma \), and \( \lim_{\rho \to 0} \psi_{\rho} = \psi \)
onumber

on \( X_1 \setminus \Sigma \),

(ii) \( \phi_{\rho} + \psi_{\rho} \) is quasi-psh on a neighborhood of \( X_1 \) up to its values on \( \Sigma_{\rho} \), increasing with respect to \( \rho \) on \( X_1 \setminus \Sigma \), and converges to \( \phi + \psi \) on \( X_1 \setminus \Sigma \) as \( \rho \to 0 \),

(iii) \( \phi_{\rho} \geq \phi \) on \( X_1 \), \( \psi_{\rho} \leq \sup \psi \) on \( X_1 \), and \( \partial \psi_{\rho} \in L^1 \) on \( X_1 \),

(iv) \( \sqrt{-1} \Theta_{L, h_0} + \sqrt{-1} \partial \overline{\partial} \phi_{\rho} + \sqrt{-1} \partial \overline{\partial} \psi_{\rho} \geq -2 \delta_\rho \omega \) on \( X_1 \),

(v) \( \sqrt{-1} \Theta_{L, h_0} + \sqrt{-1} \partial \overline{\partial} \phi_{\rho} + (1 + \alpha) \sqrt{-1} \partial \overline{\partial} \psi_{\rho} \geq -2 \delta_\rho \omega \) on \( X_1 \),

where \( \{ \delta_\rho \} \) is an increasing family of positive numbers such that \( \lim_{\rho \to 0} \delta_\rho = 0 \).

4.3. Construction of additional weights and twist factors.

For any \( t \in (-\infty, 0) \), let \( \sigma_t \) be the smooth function on \( \mathbb{R} \) defined by

\[
\sigma_t(s) := \log(e^s + e^t).
\]

Without loss of generality, we can assume that \( \sup \psi < 0 \). Then it follows from the property (iii) in Subsection 4.2 that there exists a negative number \( t_0 \) such that \( \sigma_t(\psi_{\rho}) < 0 \) on \( X_1 \) for any \( t \in (-\infty, t_0) \) and for any \( \rho \in (0, \rho_1) \).

Let \( \zeta \) and \( \chi \) be the solution of the following system of ODEs defined on \( (-\infty, 0) \):

\[
\chi(t) \zeta'(t) - \chi'(t) = 1,
\]

(4.2)

\[
\left( \chi(t) + \frac{\chi'(t)}{\chi(t) \zeta''(t) - \chi''(t)} \right) e^{\zeta(t) + t} = \frac{1}{\alpha} + 1,
\]

(4.3)

where we assume that \( \zeta \) and \( \chi \) are all smooth on \( (-\infty, 0) \), and that \( \inf_{t<0} \chi(t) = \frac{1}{\alpha}, \chi' > 0 \) and \( \chi' < 0 \) on \( (-\infty, 0) \). By the similar calculation as in [23] or [58], we can solve the system of ODEs and get the solution

\[
\begin{aligned}
\zeta(t) &= \log \frac{1}{\alpha} + \frac{1}{\alpha} + 1 - e^t, \\
\chi(t) &= \frac{1}{\alpha} e^{-t} - (\frac{1}{\alpha} + 1 + t) + e^t.
\end{aligned}
\]

Let \( h_{t, \rho} \) be the new metric on the line bundle \( L \) over \( X_1 \setminus \Sigma_{\rho} \) defined by

\[
h_{t, \rho} := h_0 e^{-\phi_{\rho} + \psi_{\rho} - \zeta(\sigma_t(\psi_{\rho}))}.
\]

Let \( \tau_{t, \rho} := \chi(\sigma_t(\psi_{\rho})) \) and \( A_{t, \rho} := \frac{(\chi'(\sigma_t(\psi_{\rho})))^2}{\chi(\sigma_t(\psi_{\rho})) \zeta''(\sigma_t(\psi_{\rho})) - \chi''(\sigma_t(\psi_{\rho}))} \). Set \( B_{t, \rho} = [\Theta_{t, \rho}, \Lambda] \) on \( X_1 \setminus \Sigma_{\rho} \), where

\[
\Theta_{t, \rho} := \tau_{t, \rho} \sqrt{-1} \Theta_{L, h_{t, \rho}} - \sqrt{-1} \partial \overline{\partial} \tau_{t, \rho} - \sqrt{-1} \frac{\partial \tau_{t, \rho} \wedge \overline{\partial} \tau_{t, \rho}}{A_{t, \rho}}.
\]
We want to prove

\[
\Theta_{t,\rho}|_{X_1 \setminus \Sigma_{\rho}} \geq \frac{e^{\psi_{t}+t}}{(e^{\psi_{t}} + e^{t})^2} \sqrt{-1} \partial \psi_{\rho} \wedge \bar{\partial} \psi_{\rho} - 2\chi(\sigma_t(\psi_{\rho}))\delta_{\rho,\omega}.
\]

It follows from (4.2) that

\[
\Theta_{t,\rho}|_{X_1 \setminus \Sigma_{\rho}} = \chi(\sigma_t(\psi_{\rho}))(\sqrt{-1}T_{L,h_0} + \sqrt{-1}J_{\partial} \psi_{\rho} + \sqrt{-1}J_{\bar{\partial}} \bar{\psi}_{\rho}) + \left(\chi(\sigma_t(\psi_{\rho}))\right)'(\sigma_t(\psi_{\rho})) \sqrt{-1}J_{\bar{\partial}} \sigma_t(\psi_{\rho})
\]

\[
= \chi(\sigma_t(\psi_{\rho}))(\sqrt{-1}T_{L,h_0} + \sqrt{-1}J_{\partial} \psi_{\rho} + \sqrt{-1}J_{\bar{\partial}} \bar{\psi}_{\rho}) + \sqrt{-1}J_{\bar{\partial}}(\sigma_t(\psi_{\rho}))
\]

\[
= \chi(\sigma_t(\psi_{\rho}))(\sqrt{-1}T_{L,h_0} + \sqrt{-1}J_{\partial} \psi_{\rho} + \sqrt{-1}J_{\bar{\partial}} \bar{\psi}_{\rho}) + \left(\frac{e^{\psi_{t}}}{\alpha(e^{\psi_{t}} + e^{t})}\right) \cdot \alpha \sqrt{-1}J_{\bar{\partial}} \psi_{\rho}
\]

\[
\geq \frac{e^{\psi_{t}}}{\alpha(e^{\psi_{t}} + e^{t})}(\sqrt{-1}T_{L,h_0} + \sqrt{-1}J_{\partial} \psi_{\rho} + \sqrt{-1}J_{\bar{\partial}} \bar{\psi}_{\rho} + 2\delta_{\rho,\omega}) - 2\chi(\sigma_t(\psi_{\rho}))\delta_{\rho,\omega}
\]

\[
\geq -2\chi(\sigma_t(\psi_{\rho}))\delta_{\rho,\omega}
\]

on $X_1 \setminus \Sigma_{\rho}$. Hence we get (4.4) as desired.

We choose an increasing family of positive numbers $\{\rho_t\}_{t \in (-\infty, t_0)}$ such that $\rho_t < \rho_1$ for any $t$, \( \lim_{t \to -\infty} \rho_t = 0 \), and

\[
2\chi(t)\delta_{\rho_t} < \frac{e^{2t}}{q + 1} \text{ for any } t.
\]

Since $\sigma_t(\psi_{\rho}) \geq t$ on $X_1$ and $\chi$ is decreasing, we have $\chi(\sigma_t(\psi_{\rho})) \leq \chi(t)$ on $X_1$. Then it follows from (4.4) and (4.5) that

\[
\Theta_{t,\rho}|_{X_1 \setminus \Sigma_{\rho}} \geq \frac{e^{\psi_{t}+t}}{(e^{\psi_{t}} + e^{t})^2} \sqrt{-1} \partial \psi_{\rho} \wedge \bar{\partial} \psi_{\rho} - \frac{e^{2t}}{q + 1} \omega, \quad \forall \rho \in (0, \rho_1).
\]

Hence

\[
B_{t,\rho} + e^{2t} \geq \left[\frac{e^{\psi_{t}+t}}{(e^{\psi_{t}} + e^{t})^2} \sqrt{-1} \partial \psi_{\rho} \wedge \bar{\partial} \psi_{\rho}, \Lambda \right] = \frac{e^{\psi_{t}+t}}{(e^{\psi_{t}} + e^{t})^2} T_{\partial \psi_{\rho}} T_{\partial \psi_{\rho}}^* \geq 0
\]

holds on $X_1 \setminus \Sigma_{\rho}$ for any $\rho \in (0, \rho_1]$ as an operator on $(n, q + 1)$-forms, where $T_{\partial \psi_{\rho}}$ denotes the operator $\partial \psi_{\rho} \wedge \bullet$ and $T_{\partial \psi_{\rho}}^*$ is its Hilbert adjoint operator.
4.4. Construction of suitably truncated forms and solving $\tilde{\partial}$ globally with $L^2$ estimates.

It is easy to construct a smooth function $\theta : \mathbb{R} \rightarrow [0, 1]$ such that $\theta = 1$ on $(-\infty, 0]$, $\theta = 0$ on $[1, +\infty)$ and $|\theta'| \leq 2$ on $\mathbb{R}$.

Define $g_{t, \rho} = \tilde{\partial}(\theta(\psi - t)\tilde{f})$ on $X_1 \setminus \Sigma_{\rho}$, where $\tilde{f}$ is as in Subsection 4.1. Then $\tilde{\partial}g_{t, \rho} = 0$ on $X_1 \setminus \Sigma_{\rho}$ and

$$g_{t, \rho} = g_{1, t, \rho} + g_{2, t, \rho},$$

where $g_{1, t, \rho} := \theta'(\psi - t)\tilde{\partial}\psi \wedge \tilde{f}$ and $g_{2, t, \rho} := \theta(\psi - t)\tilde{\partial}\tilde{f}$.

The Cauchy-Schwarz inequality and (4.6) imply that, on $X_1 \setminus \Sigma_{\rho}$,

$$(4.7) \quad \int_{X_1 \setminus \Sigma_{\rho}} \langle (B_{t, \rho} + 2e^{2t})^{-1}g_{1, t, \rho}, g_{1, t, \rho} \rangle_{\omega, h_{t, \rho}} dV_{X, \omega} \leq 2\int_{X_1} \frac{2}{e^{2\psi - t}} \int_{X_1} |\tilde{f}|^2 e^{-\phi_{\psi}(t < \psi < t + 1)} dV_{X, \omega} \leq \frac{4(e + 1)^2}{e^t} \int_{X_1} \frac{1}{e^{2t}} \int_{X_1} |\tilde{f}|^2 e^{-\phi_{\psi}(t < \psi < t + 1)} dV_{X, \omega}$$

for any $\rho \in (0, \rho_l]$, where $I_{\psi < \psi < t + 1}$ is the characteristic function associated to the set $\{t < \psi < t + 1\}$.

Since $\theta = 0$ on $[1, +\infty)$, $\zeta \geq 0$ and $\phi_\rho + \psi_\rho \geq \phi + \psi$ on $X_1 \setminus \Sigma$ (cf. the property (iii) in Subsection 4.2), we get

$$\int_{X_1 \setminus \Sigma_{\rho}} (\frac{1}{e^{2t}} g_{2, t, \rho} \cdot g_{2, t, \rho})_{\omega, h_{t, \rho}} dV_{X, \omega} \leq \frac{1}{e^{2t}} \int_{X_1} |\tilde{\partial}f|_{\omega, h_0}^2 e^{-\phi_{\psi}(t < \psi < t + 1)} dV_{X, \omega} \cdot$$

Therefore, it follows from (4.7) that

$$C_{\rho}(t) := \int_{X_1 \setminus \Sigma_{\rho}} \langle (B_{t, \rho} + 2e^{2t})^{-1}g_{1, t, \rho}, g_{1, t, \rho} \rangle_{\omega, h_{t, \rho}} dV_{X, \omega} \leq \frac{8(e + 1)^2}{e^t} \int_{X_1} |\tilde{\partial}f|_{\omega, h_0}^2 e^{-\phi_{\psi}(t < \psi < t + 1)} dV_{X, \omega} + \frac{2}{e^{2t}} \int_{X_1} |\tilde{\partial}f|_{\omega, h_0}^2 e^{-\phi_{\psi}(t < \psi < t + 1)} dV_{X, \omega}$$

for any $\rho \in (0, \rho_l]$.

Since we have obtained

$$\int_{X_1} |\tilde{f}|_{\omega, h_0}^2 e^{-\phi} dV_{X, \omega} < +\infty$$

and

$$\int_{X_1} |\tilde{\partial}f|_{\omega, h_0}^2 e^{-\phi_{\psi}} dV_{X, \omega} < +\infty$$
in Subsection 4.1, it follows from the property \( \lim_{\rho \to 0} \psi_\rho = \psi \) on \( X_1 \setminus \Sigma \) (cf. property (i) in Subsection 4.2) and Fatou’s lemma that

\[
C(t) := \lim_{\rho \to 0} C_\rho(t) \leq \frac{8(e + 1)^2}{e^t} \int_{X_1} |\tilde{f}|^2_{\omega, h_0} e^{-\phi_\rho} (t \leq \psi \leq t + 1) dV_{\omega, \omega} + \frac{2}{e^{2t}} \int_{X_1} |\tilde{\partial} \tilde{f}|^2_{\omega, h_0} e^{-\phi_\rho} (t \leq \psi \leq t + 1) dV_{\omega, \omega}
\]

and

\[
\lim_{t \to -\infty} e^{2t} C(t) = 0.
\]

Since \( X_1 \) is a holomorphically convex Kähler manifold, \( X_1 \) carries a complete Kähler metric. Hence \( X_1 \setminus \Sigma \) carries a complete Kähler metric by Lemma 3.3. Then by Lemma 3.1, there exists an \( L \)-valued \((n, q)\)-form \( u_{t, \rho} \) and an \( L \)-valued \((n, q + 1)\)-form \( v_{t, \rho} \) such that

\[
(4.10) \quad \tilde{\partial} u_{t, \rho} + \sqrt{2} e^t v_{t, \rho} = g_{t, \rho} \quad \text{on} \quad X_1 \setminus \Sigma
\]

and

\[
\int_{X_1 \setminus \Sigma} \left| u_{t, \rho} \right|^2_{\omega, h_0} e^{-\phi_\rho} (t \leq \psi \leq t + 1) dV_{\omega, \omega} + \int_{X_1 \setminus \Sigma} \left| v_{t, \rho} \right|^2_{\omega, h_0} e^{-\phi_\rho} (t \leq \psi \leq t + 1) dV_{\omega, \omega} \leq C_\rho(t)
\]

for any \( \rho \in (0, \rho_1) \).

Since \( \sup \frac{\phi_\rho}{\rho} < +\infty \) (cf. the property (ii) in Subsection 4.2), we get that

\( u_{t, \rho} \in L^2 \) and \( v_{t, \rho} \in L^2 \). Since \( \tilde{\partial} \tilde{\psi}_\rho \in L^1 \) on \( X_1 \) (cf. the property (i) in Subsection 4.2), we have \( g_{t, \rho} = g_{1, t, \rho} + g_{2, t, \rho} \in L^1 \) on \( X_1 \). Then it follows from (4.10) and Lemma 3.4 that

\[
(4.12) \quad \tilde{\partial} u_{t, \rho} + \sqrt{2} e^t v_{t, \rho} = g_{t, \rho} = \tilde{\partial} (\theta(\psi_\rho - t) \tilde{f}) \quad \text{on} \quad X_1.
\]

Since \( \lim_{\rho \to 0} \psi_\rho = \psi \) on \( X_1 \setminus \Sigma \) (cf. the property (i) in Subsection 4.2), \( \theta(\psi_\rho - t) \tilde{f} \)
converges to \( \theta(\psi - t) \tilde{f} \) in \( L^2 \) as \( \rho \to 0 \) by Lebesgue’s dominated convergence theorem. Hence \( \tilde{\partial} (\theta(\psi_\rho - t) \tilde{f}) \) converges to \( \tilde{\partial} (\theta(\psi - t) \tilde{f}) \) in the sense of currents as \( \rho \to 0 \).

Let \( t \) be fixed and let \( \{\rho_j\}_{j=1}^{+\infty} \) be a decreasing sequence of positive numbers such that \( \rho_2 < \rho_1 \) and \( \lim_{j \to +\infty} \rho_j = 0 \).
Since $\phi_{\rho_j} + \psi_{\rho_j}$ decreases to $\phi + \psi$ on $X_1 \setminus \Sigma$ as $j \to +\infty$ (cf. the property (ii) in Subsection 4.2), by extracting weak limits of $\{u_{t,\rho_j}\}_{j=1}^{+\infty}$ as $j \to +\infty$, it follows from (4.11) and the diagonal argument that there exists an $L^2$ $L$-valued $(n,q)$-form $u_t$ such that a subsequence $\{u_{t,\rho_{j_r}}\}_{r=1}^{+\infty}$ of $\{u_{t,\rho_j}\}_{j=1}^{+\infty}$ converges to $u_t$ weakly in $L^2_{(n,q)}(X_1, e^{-\phi_{\rho_j} - \psi_{\rho_j}}dV_{X,\omega})$ for any positive integer $i$ as $r \to +\infty$.

Hence the Banach-Steinhaus Theorem implies that, for any positive integer $i$,

$$\int_{X_1} \frac{\alpha \epsilon |u_t|^2_{\omega, h_0} e^{\phi_{\rho_j} - \psi_{\rho_j}}}{1 + \alpha} dV_{X,\omega} \leq \lim_{r \to +\infty} \int_{X_1} \frac{\alpha \epsilon |u_{t,\rho_{j_r}}|^2_{\omega, h_0} e^{\phi_{\rho_j} - \psi_{\rho_j}}}{1 + \alpha} dV_{X,\omega}$$

$$\leq \lim_{r \to +\infty} \int_{X_1} \frac{\alpha \epsilon |u_{t,\rho_{j_r}}|^2_{\omega, h_0} e^{\phi_{\rho_{j_r}} - \psi_{\rho_{j_r}}}}{1 + \alpha} dV_{X,\omega}.$$ 

Then Fatou’s lemma implies that

$$\int_{X_1} \frac{\alpha \epsilon |u_t|^2_{\omega, h_0} e^{-\phi - \psi}}{1 + \alpha} dV_{X,\omega} \leq \lim_{r \to +\infty} \int_{X_1} \frac{\alpha \epsilon |u_{t,\rho_{j_r}}|^2_{\omega, h_0} e^{-\phi_{\rho_{j_r}} - \psi_{\rho_{j_r}}}}{1 + \alpha} dV_{X,\omega}$$

Then Fatou’s lemma implies that

$$\int_{X_1} \frac{\alpha \epsilon |u_t|^2_{\omega, h_0} e^{-\phi - \psi}}{1 + \alpha} dV_{X,\omega} \leq \lim_{r \to +\infty} \int_{X_1} \frac{\alpha \epsilon |u_{t,\rho_{j_r}}|^2_{\omega, h_0} e^{-\phi_{\rho_{j_r}} - \psi_{\rho_{j_r}}}}{1 + \alpha} dV_{X,\omega}.$$ 

Similar weak limit argument can also be applied to subsequences of $\{v_t, \rho_{j_r}\}_{j=1}^{+\infty}$.

In conclusion, by (4.12), (4.11) and (4.8), there exist $L^2$ forms $u_t$ and $v_t$ such that

$$\partial u_t + \sqrt{2} \epsilon v_t = \partial (\theta(\psi - t)) \text{ on } X_1$$

and

$$\int_{X_1} \frac{\alpha \epsilon |u_t|^2_{\omega, h_0} e^{-\phi - \psi}}{1 + \alpha} dV_{X,\omega} + \int_{X_1} \frac{|v_t|^2_{\omega, h_0} e^{-\phi - \psi}}{1 + \alpha} dV_{X,\omega} \leq C(t).$$

(4.9) and (4.14) imply that

$$\lim_{t \to -\infty} \int_{X_1} \sqrt{2} \epsilon v_t |v_t|^2_{\omega, h_0} e^{-\phi - \psi} dV_{X,\omega} \leq \lim_{t \to -\infty} 2(1 + \alpha) e^{2t} C(t) = 0.$$ 

4.5. Final conclusion.

Some ideas in this subsection come from [6].

By Lemma 3.12, Lemma 3.13 and the commutative diagram in Remark 3.2, we will identify the sheaf (or Čech) cohomology groups with the $L^2_{\text{loc}}$ Dolbeault cohomology groups, i.e., $i_{X_1} = \text{Id}$, $j_{X_1} = \text{Id}$ and $p_{X_1} = P_{X_1}$.

Since $H^{q+1}(X_1, K \otimes \mathcal{I}(h))(0 \leq q \leq n)$ is Hausdorff by Lemma 3.11, Lemma 3.12 implies that the image of the map

$$\partial_q : \Gamma(X_1, \mathcal{L}_{(2), h}^{n,q}) \to \Gamma(X_1, \mathcal{L}_{(2), h}^{n,q+1}) \quad (0 \leq q \leq n)$$

is closed. Hence

$$\Gamma(X_1, \mathcal{L}_{(2), h}^{n,q}) \xrightarrow{\partial_q} \text{Im } \partial_q \quad (0 \leq q \leq n)$$

is a continuous linear surjection between Fréchet spaces. Therefore, the open mapping theorem (Lemma 3.5) implies that this map is open. Hence $\partial_q$ induces a topological isomorphism of Fréchet spaces

$$\frac{\Gamma(X_1, \mathcal{L}_{(2), h}^{n,q})}{\text{Ker } \partial_q} \cong \text{Im } \partial_q \quad (0 \leq q \leq n).$$
Let $w_t := \sqrt{2}e^t v_t$. Then (4.13) and (4.14) implies that

$$w_t = \bar{\partial}(\theta(\psi - t)\tilde{f} - u_t) \in \text{Im} \bar{\partial}_t.$$  

Since (4.15) implies that

$$\lim_{t \to -\infty} \int_{X_1} |w_t|^2_{\omega,h} dV_{X,\omega} = 0,$$

it follows from (4.16) that there exists a sequence of $L$-valued $(n,q)$-forms $s_t \in \Gamma(X_1, L^{n,q}_{(2),h})$ such that $\bar{\partial}s_t = w_t$ on $X_1$ and

$$\lim_{t \to -\infty} \|s_t\|_{h,K} = 0$$

for any compact set $K$ in open coordinate charts of $X_1$, where $\| \cdot \|_{h,K}$ is defined just before Lemma 3.12.

Let $\tilde{f}_t := \theta(\psi - t)\tilde{f} - u_t - s_t$. Then $\bar{\partial}\tilde{f}_t = 0$. Hence

$$\tilde{f}_t \in \text{Ker} \bar{\partial}_t \subset \Gamma(X_1, L^{n,q}_{(2),h}).$$

As explained just before Lemma 3.13, the Fréchet topology on

$$\Gamma(X_1, L^{n,q}_{(2),h}/L^{n,q}_{(2),he^{-\psi}})$$

is defined by the semi-norms $\| \cdot \|_{h,K}$ for any compact set $K$ in open coordinate charts of $X_1$, and the topology of the $q$-th $L^2_{\text{loc}}$ Dolbeault cohomology group

$$H^{n,q}_{(2)}(X_1, L, he^{-\psi})$$

is obtained as the quotient topology induced from the semi-norms $\| \cdot \|_{h,K}$. Let $\| \cdot \|_{h,K}'$ denote the induced semi-norms on $H^{n,q}_{(2)}(X_1, L, he^{-\psi})$. Then $\| \cdot \|_{h,K}'$ is smaller than $\| \cdot \|_{h,K}$ in some sense.

Since (4.14) and the construction in Subsection 4.1 imply that

$$\theta(\psi - t)\tilde{f} - u_t - \tilde{f} \in L^2(X_1, \wedge^{n,q}T_X^* \otimes L, he^{-\psi})$$

and

$$\bar{\partial}(\theta(\psi - t)\tilde{f} - u_t - \tilde{f}) = w_t - \bar{\partial}\tilde{f} \in L^2(X_1, \wedge^{n,q+1}T_X^* \otimes L, he^{-\psi}),$$

we get

$$\theta(\psi - t)\tilde{f} - u_t - \tilde{f} \in \Gamma(X_1, L^{n,q}_{(2),he^{-\psi}}).$$

Then

$$\tilde{f}_t - \tilde{f} = -s_t \mod \Gamma(X_1, L^{n,q}_{(2),he^{-\psi}}).$$

Hence

$$\|\tilde{f}_t - \tilde{f}\|_{h,K}' \leq \|\tilde{f}_t - \tilde{f}\|_{h,K} \leq \|s_t\|_{h,K} \to 0 \text{ as } t \to -\infty,$$

where $K$ is any compact subset in open coordinate charts of $X_1$.

Hence Lemma 3.11 implies that $\tilde{f}$ belongs to the closure of $H^0(S_1, \text{Im} P)$ in $H^0(S_1, R^0\pi_*(K \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}))$, where $P$ is the sheaf homomorphism in (4.1) and $\text{Im} P$ is the image of $P$.

Since

$$H^0(S_1, \text{Im} P) = \bigcap_{y \in S_1} \{ F \in H^0(S_1, R^0\pi_*(K \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi})) :$$

the germ of $F$ at $y$ belongs to $(\text{Im} P)_y \},$$
Lemma 3.10 implies that $H^0(S_1, \text{Im } P)$ is a closed subset of

$$H^0(S_1, R^0\pi_*(K \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}))).$$

Hence $f \in H^0(S_1, \text{Im } P)$. Since $f$ is arbitrary, $P$ is surjective. Thus Theorem 1.1 is proved.

5. Proof of Theorem 1.2

We will only give the proof when $\alpha \in (0, +\infty)$. The proof for the case $\alpha = +\infty$ is almost the same.

The proof will be divided into five subsections. Some arguments in the proof will be similar as those in Section 4.

5.1. Construction of a smooth extension $\tilde{f}$ of $f$ from $Y \cap X$ to $X$.

Since $f \in H^0(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}_\phi(h)/\mathcal{I}(he^{-\psi}))$, there exists a locally finite covering $\mathcal{U} = \{U_i\}_{i \in I}$ of $X$ by coordinate balls, and a family of holomorphic sections

$$f_i \in \Gamma(U_i, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}_\phi(h)) \quad (i \in I)$$

such that $f$ is the images of $\{f_i\}_{i \in I}$ under the natural morphisms

$$\Gamma(U_i, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}_\phi(h)) \rightarrow \Gamma(U_i, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}_\phi(h)/\mathcal{I}(he^{-\psi})), \quad i \in I.$$ 

Hence

$$f_i - f_j \in \Gamma(U_i \cap U_j, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(he^{-\psi})), \quad \forall i, j \in I.$$

Let $\{\xi_i\}_{i \in I}$ is a partition of unity subordinate to $\mathcal{U}$, and let

$$\tilde{f} := \sum_{i \in I} \xi_i f_i.$$

Then $\tilde{f}$ is smooth on $X$, and

$$\partial \tilde{f}|_{U_i} = \partial \tilde{f} - \partial f_j = \partial(\sum_{i \in I} \xi_i f_i) - \partial(\sum_{i \in I} \xi_i f_j) = \sum_{i \in I} \partial \xi_i \wedge (f_i - f_j), \quad \forall j \in I.$$

Since $X$ is weakly pseudoconvex, there exists a smooth psh exhaustion function $\Psi$ on $X$. Let $X_k := \{x \in X : \Psi(x) < k\} \ (k \in \mathbb{Z}^+ \text{ and } k \geq 3)$, we choose $\Psi$ such that $X_3 \neq \emptyset$.

Let $h_0$ be any fixed smooth metric of $L$ on $X$. Then $h = h_0 e^{-\phi}$ for some global function $\phi$ on $X$, which is quasi-psh by the assumption in the theorem. Then we have

\begin{equation}
(5.1) \quad \int_{X_k} |\tilde{f}|^2 e^{-\phi} dV_{X, \omega} < +\infty
\end{equation}

and

\begin{equation}
(5.2) \quad \int_{X_k} |\partial \tilde{f}|^2 e^{-\phi} dV_{X, \omega} < +\infty.
\end{equation}

In the following subsections, $k$ will be fixed until the end of the proof ($k \in \mathbb{Z}^+$ and $k \geq 3$).
5.2. Approximation of singular weights.

This subsection is almost the same as Subsection 4.2. We need only to replace $X_0$ and $X_i$ in Subsection 4.2 by $X_{k+1}$ and $X_k$ respectively, and we obtain two family of upper semicontinuous functions $\{\phi_{\rho}\}_{\rho \in \{0, \rho_1\}}$ and $\{\psi_{\rho}\}_{\rho \in \{0, \rho_1\}}$ ($\rho_1$ is some positive number) defined on a neighborhood of $X_k$ satisfying the same properties $(i) - (v)$ as in Subsection 4.2.

5.3. Construction of additional weights and twist factors.

Let $\varrho : \mathbb{R} \to [0, +\infty)$ be the function defined by

$$\varrho(t) = \begin{cases} \frac{1}{|t|} & \text{if } |t| < 1 \\ 0 & \text{if } |t| \geq 1. \end{cases}$$

Let $\varepsilon \in (0, \frac{1}{2})$ and let $\varrho_\varepsilon : \mathbb{R} \to [0, +\infty)$ be the function defined by $\varrho_\varepsilon(t) = \frac{\varepsilon}{\varepsilon} \varrho(\frac{\varepsilon}{\varepsilon} t)$. Then $\varrho_\varepsilon$ is smooth on $\mathbb{R}$ with support contained in $[-\frac{1}{4}, \frac{1}{4}]$ and $\int_{-\infty}^{\infty} \varrho_\varepsilon(t) dt = 1$.

Let $\{\sigma_{\varepsilon,t}\}_{\varepsilon \in (0, \frac{1}{4})}, t \in (-\infty, -1)$ be the family of functions on $\mathbb{R}$ defined by (see Section 5 in [23])

$$\sigma_{\varepsilon,t}(s) = \int_{-\infty}^{s} \left( \int_{-\infty}^{t_2} \frac{1}{1 - 2\varepsilon} (\varrho_{t+\varepsilon,t+1-\varepsilon}) * \varrho_{\varepsilon}(t_1) dt_1 \right) dt_2 - \int_{-\infty}^{0} \left( \int_{-\infty}^{t_2} \frac{1}{1 - 2\varepsilon} (\varrho_{t+\varepsilon,t+1-\varepsilon}) * \varrho_{\varepsilon}(t_1) dt_1 \right) dt_2,$$

where $\varrho_{t+\varepsilon,t+1-\varepsilon}$ is the characteristic function associated to the interval $(t + \varepsilon, t + 1 - \varepsilon)$, and the notation * denotes the convolution of two functions.

Then we have

$$\sigma'_{\varepsilon,t}(s) = \int_{-\infty}^{s} \frac{1}{1 - 2\varepsilon} (\varrho_{t+\varepsilon,t+1-\varepsilon}) * \varrho_{\varepsilon}(t_1) dt_1$$

and

$$\sigma''_{\varepsilon,t} = \frac{1}{1 - 2\varepsilon} \varrho_{t+\varepsilon,t+1-\varepsilon} * \varrho_{\varepsilon}.$$

Hence for any fixed $\varepsilon \in (0, \frac{1}{4})$ and any fixed $t \in (-\infty, -1)$, $\sigma_{\varepsilon,t}$ is a smooth increasing convex function, $\sigma_{\varepsilon,t}(0) = 0$, $0 \leq \sigma'_{\varepsilon,t} \leq 1$, $0 \leq \sigma''_{\varepsilon,t} \leq \frac{1}{2\varepsilon}$, $\sigma'_{\varepsilon,t} = 0$ on $(-\infty, t + \frac{1}{4\varepsilon}]$, and $\sigma'_{\varepsilon,t} = 1$ on $[t + 1 - \frac{1}{4\varepsilon}, +\infty)$. In particular,

$$\sigma_{\varepsilon,t}(s) = \begin{cases} a_{\varepsilon,t} \geq t & \text{if } s \leq t \\ s & \text{if } s \geq t + 1, \end{cases}$$

where $a_{\varepsilon,t}$ is a constant depending only on $\varepsilon$ and $t$.

Since $\sup_{\Omega} \psi < \alpha_0$ for any $\Omega \subseteq \subseteq X$ by assumption, we have

$$\alpha_k := \sup_{X_{k+1}} \psi < \alpha_0.$$

Then it follows from the property $(iii)$ in Subsection 4.2 that

$$\sup_{X_k} \psi_{\rho} \leq \alpha_k, \quad \forall \rho \in (0, \rho_1).$$

Let $t_0 := \min\{\sup_{\Omega} \psi - 1, -1\}$. Then for any $\varepsilon \in (0, \frac{1}{4})$, any $t \in (-\infty, t_0)$ and any $\rho \in (0, \rho_1)$, we have

$$t \leq \sigma_{\varepsilon,t}(\psi_{\rho}) \leq \sigma_{\varepsilon,t}(\alpha_k) = \alpha_k \quad \text{on } X_k.$$
Let \( \zeta \) and \( \chi \) be the solution to the following system of ODEs defined on \((-\infty, \alpha_0)\):

\[
\begin{align*}
\chi(t)\zeta'(t) - \chi'(t) &= 1, \\
\left(\chi(t) + \frac{(\chi'(t))^2}{\chi(t)\zeta''(t) - \chi''(t)}\right) e^\zeta(t) &= \left(\frac{1}{\alpha R(\alpha_0)} + \frac{\varepsilon}{R(t)}\right),
\end{align*}
\]

where we assume that \( \zeta \) and \( \chi \) are both smooth on \((-\infty, \alpha_0)\), and that \( \inf_{t<\alpha_0} \zeta (t) = 0, \inf_{t<\alpha_0} \chi (t) \geq \frac{1}{\alpha}, \zeta' > 0 \) and \( \chi' < 0 \) on \((-\infty, \alpha_0)\). By the similar calculation as in [23] or [58], we can solve the system of ODEs and get the solution

\[
\begin{align*}
\zeta(t) &= \log \left(\frac{1}{\alpha R(\alpha_0)} + \varepsilon R(t)\right) - \log \left(\frac{1}{\alpha R(\alpha_0)} + \int_t^{\alpha_0} dt \right), \\
\chi(t) &= \frac{\int_t^{\alpha_0} \frac{1}{\alpha R(\alpha_0)} + \int_t^{\alpha_0} \frac{dt}{R(t)}}{\frac{1}{\alpha R(\alpha_0)} + \int_t^{\alpha_0} \frac{dt}{R(t)}}.
\end{align*}
\]

Let \( h_{\varepsilon, t, \rho} \) be the new metric on the line bundle \( L \) over \( X_k \setminus \Sigma_{\rho} \) defined by

\[
h_{\varepsilon, t, \rho} := h e^{-\phi - \psi - \zeta(\sigma_{\varepsilon, t}(\psi))}.
\]

Let \( \tau_{\varepsilon, t, \rho} := \chi(\sigma_{\varepsilon, t}(\psi)) \) and \( A_{\varepsilon, t, \rho} := \frac{\chi'(\sigma_{\varepsilon, t}(\psi))}{\chi(\sigma_{\varepsilon, t}(\psi))} \). Set

\[
B_{\varepsilon, t, \rho} = [\Theta_{\varepsilon, t, \rho}, \Lambda] \text{ on } X_k \setminus \Sigma_{\rho},
\]

where

\[
\Theta_{\varepsilon, t, \rho} := \tau_{\varepsilon, t, \rho} \sqrt{-1} \Theta_L h_{\varepsilon, t, \rho} - \sqrt{-1} \partial \bar{\partial} \tau_{\varepsilon, t, \rho} - \sqrt{-1} \frac{\partial \bar{\partial} \tau_{\varepsilon, t, \rho} \wedge \bar{\partial} \bar{\partial} \tau_{\varepsilon, t, \rho}}{A_{\varepsilon, t, \rho}}.
\]

We want to prove

\[
\Theta_{\varepsilon, t, \rho} = \sigma''_{\varepsilon, t}(\psi) \sqrt{-1} \partial \bar{\partial} \psi - 2\chi(\sigma_{\varepsilon, t}(\psi)) \delta_{\rho, \omega}.
\]

It follows from (5.4) that

\[
\Theta_{\varepsilon, t, \rho} |_{X_k \setminus \Sigma_{\rho}} \geq \sigma''_{\varepsilon, t}(\psi) \sqrt{-1} \partial \bar{\partial} \psi - 2\chi(\sigma_{\varepsilon, t}(\psi)) \delta_{\rho, \omega}.
\]

Since \( 0 \leq \sigma'_{\varepsilon, t} \leq 1 \) on \( R \) and \( \chi \geq \inf_{t<\alpha_0} \chi (t) \geq \frac{1}{\alpha} \) on \((-\infty, \alpha_0)\), it follows from the properties (iv) and (v) in Subsection 4.2 that

\[
\begin{align*}
\chi(\sigma_{\varepsilon, t}(\psi)) &\left(\sqrt{-1} \Theta_{L, h_0} + \sqrt{-1} \partial \bar{\partial} \phi + \sqrt{-1} \partial \bar{\partial} \psi\right) = \chi(\sigma_{\varepsilon, t}(\psi)) \left(\sqrt{-1} \Theta_{L, h_0} + \sqrt{-1} \partial \bar{\partial} \phi + \sqrt{-1} \partial \bar{\partial} \psi + 2\delta_{\rho, \omega} - 2\chi(\sigma_{\varepsilon, t}(\psi)) \delta_{\rho, \omega}\right) \\
&= \chi(\sigma_{\varepsilon, t}(\psi)) \left(\sqrt{-1} \Theta_{L, h_0} + \sqrt{-1} \partial \bar{\partial} \phi + \sqrt{-1} \partial \bar{\partial} \psi + 2\delta_{\rho, \omega}\right) - 2\chi(\sigma_{\varepsilon, t}(\psi)) \delta_{\rho, \omega}
\end{align*}
\]
on $X_k \setminus \Sigma_\rho$. Hence we get (5.6) as desired.

Let $\delta(t)$ be a positive increasing function defined on $(-\infty, t_0)$, such that

$$\lim_{t \to -\infty} \delta(t) = 0.$$  

The explicit expression of $\delta(t)$ will be determined later (cf. (5.11)).

We choose an increasing family of positive numbers $\{ \rho_t \}_{t \in (-\infty, t_0)}$ such that $\rho_t < \rho_1$ for any $t$, $\lim_{t \to -\infty} \rho_t = 0$, and

$$2 \chi(t) \delta_{\rho_t} < \delta(t) \text{ for any } t.$$  

(5.7)

Since $\chi$ is decreasing, we have $\chi(\sigma_{\varepsilon,t}(\psi_{\rho})) \leq \chi(t)$ on $X_k$ by (5.3). Then it follows from (5.6) and (5.7) that

$$\Theta_{\varepsilon,t,\rho} |_{X_k \setminus \Sigma_\rho} \geq \sigma''_{\varepsilon,t}(\psi_{\rho}) \sqrt{-1} \partial \psi_{\rho} \wedge \bar{\partial} \psi_{\rho} - \delta(t) \omega, \quad \forall \rho \in (0, \rho_1].$$

Hence

$$B_{\varepsilon,t,\rho} + \delta(t) I \geq [\sigma''_{\varepsilon,t}(\psi_{\rho}) \sqrt{-1} \partial \psi_{\rho} \wedge \bar{\partial} \psi_{\rho}, \Lambda] = \sigma''_{\varepsilon,t}(\psi_{\rho}) \partial \psi_{\rho} \bar{\partial} \psi_{\rho} \geq 0$$

holds on $X_k \setminus \Sigma_\rho$ for any $\rho \in (0, \rho_1]$ as an operator on $(n, q + 1)$-forms, where $\bar{\partial} \psi_{\rho}$ denotes the operator $\bar{\partial} \psi_{\rho} \wedge \bullet$ and $T_{\partial \psi_{\rho}}$ is its Hilbert adjoint operator.

### 5.4. Construction of suitably truncated forms and solving $\bar{\partial}$ globally with $L^2$ estimates.

Define $g_{\varepsilon,t,\rho} = \bar{\partial} ((1 - \sigma_{\varepsilon,t}(\psi_{\rho})) \hat{f})$ on $X_k \setminus \Sigma_\rho$, where $\hat{f}$ is as in Subsection 5.1. Then $\bar{\partial} g_{\varepsilon,t,\rho} = 0$ on $X_k \setminus \Sigma_\rho$ and

$$g_{\varepsilon,t,\rho} = g_{1,\varepsilon,t,\rho} + g_{2,\varepsilon,t,\rho},$$

where $g_{1,\varepsilon,t,\rho} := -\sigma''_{\varepsilon,t}(\psi_{\rho}) \bar{\partial} \psi_{\rho} \wedge \hat{f}$ and $g_{2,\varepsilon,t,\rho} := (1 - \sigma_{\varepsilon,t}(\psi_{\rho})) \bar{\partial} \hat{f}$.

The Cauchy-Schwarz inequality and (5.8) imply that, on $X_k \setminus \Sigma_\rho$,

$$\langle B_{\varepsilon,t,\rho} + 2 \delta(t) I \rangle^{-1} g_{1,\varepsilon,t,\rho} \omega_{h_{\varepsilon,t,\rho}} \leq (1 + \varepsilon) \langle B_{\varepsilon,t,\rho} + 2 \delta(t) I \rangle^{-1} g_{1,\varepsilon,t,\rho} \omega_{h_{\varepsilon,t,\rho}} + (1 + \frac{1}{\varepsilon}) \langle B_{\varepsilon,t,\rho} + 2 \delta(t) I \rangle^{-1} g_{2,\varepsilon,t,\rho} \omega_{h_{\varepsilon,t,\rho}}$$

for any $\rho \in (0, \rho_t]$.

Since $\zeta \geq 0$ and $\phi_{\rho} \geq \phi$ on $X_k$, we obtain from (5.8) that

$$\int_{X_k \setminus \Sigma_\rho} \langle B_{\varepsilon,t,\rho} + \delta(t) I \rangle^{-1} g_{1,\varepsilon,t,\rho} \omega_{h_{\varepsilon,t,\rho}} dV_{X,\omega} \leq \int_{X_k} \sigma''_{\varepsilon,t}(\psi_{\rho}) |\hat{f}|^2_{L^2} e^{-\phi_{\rho} - \psi_{\rho}} dV_{X,\omega} \leq \frac{1}{1 - 2 \varepsilon} \int_{X_k} |\hat{f}|^2_{L^2} e^{-\phi_{\rho} - \psi_{\rho}} \Pi_{t + \frac{2}{\varepsilon} < \psi_{\rho} < t + 1 - \frac{2}{\varepsilon}} dV_{X,\omega}$$

for any $\rho \in (0, \rho_1]$, where $\Pi_{t + \frac{2}{\varepsilon} < \psi_{\rho} < t + 1 - \frac{2}{\varepsilon}}$ is the characteristic function associated to the set $\{ x \in X_k : t + \frac{2}{\varepsilon} < \psi_{\rho}(x) < t + 1 - \frac{2}{\varepsilon} \}$. 

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Hence for any \( \rho \) and \( n \),
\[
\| \tilde{f} \|_{\omega, h_0}^2 e^{-\phi - \psi} \leq (t + 2^\varepsilon < \psi < t + 1 - 4^\varepsilon) dV_{\omega, \omega}.
\]
Thus, \( \tilde{f} \) is a weakly pseudoconvex Kähler manifold,\( C(t) := \int_{X_k} \langle (B_{\varepsilon, t, \rho} + 2\delta(t)1)^{-1} g_{\varepsilon, t, \rho}, g_{\varepsilon, t, \rho} \rangle_{\omega, t, \rho} dV_{\omega, \omega}
\]
\leq 1 + 2^\varepsilon \int_{X_k} |\tilde{f}|_{\omega, h_0}^2 e^{-\phi - \psi} dV_{\omega, \omega}
+ (1 + 1 - 2^\varepsilon \delta(t) \int_{X_k} |\tilde{f}|_{\omega, h_0}^2 e^{-\phi - \psi} dV_{\omega, \omega}.
\]
Since \( \lim_{\rho \to 0} \psi = \phi \) on \( X_k \setminus \Sigma \), it follows from (5.10), (5.2) and Fatou’s lemma that
\[
\int_{X_k \setminus \Sigma} |\tilde{f}|_{\omega, h_0}^2 e^{-\phi - \psi} dV_{\omega, \omega} \leq 1 + 2^\varepsilon \int_{X_k} |\tilde{f}|_{\omega, h_0}^2 e^{-\phi - \psi} dV_{\omega, \omega}
\]
\leq 1 + 2^\varepsilon \int_{X_k} |\tilde{f}|_{\omega, h_0}^2 e^{-\phi - \psi} dV_{\omega, \omega}.
\]
Hence
\[
\lim_{t \to \infty} C(t) \leq \frac{1 + \varepsilon}{1 - 2^\varepsilon} \int_Y |f|_{\omega, h}^2 dV_{\omega, \omega}[\psi].
\]
Since \( X_k \) is a weakly pseudoconvex Kähler manifold, \( X_k \) carries a complete Kähler metric. Hence \( X_k \setminus \Sigma \) carries a complete Kähler metric by Lemma 3.3. Then by Lemma 3.1, there exists an \( L \)-valued \( (n, 0) \)-form \( u_{k, \varepsilon, t, \rho} \) and an \( L \)-valued \( (n, 1) \)-form \( v_{k, \varepsilon, t, \rho} \) such that
\[
\tilde{u}_{k, \varepsilon, t, \rho} + \sqrt{2\delta(t)} v_{k, \varepsilon, t, \rho} = g_{\varepsilon, t, \rho} \text{ on } X_k \setminus \Sigma,
\]
and
\[
\int_{X_k \setminus \Sigma} |u_{k, \varepsilon, t, \rho}|_{\omega, h_0}^2 e^{-\phi - \psi - \zeta(\sigma_{\varepsilon, t}(\psi))} dV_{\omega, \omega}
\]
\leq C(t) for any \( \rho \in (0, \rho_t] \).
Since (5.5) implies that
$$e^{-\zeta(\sigma_{\varepsilon,t}(\psi))} = \frac{1}{\tau_{\varepsilon,t,\rho} + A_{\varepsilon,t,\rho}} = \frac{1}{\frac{1}{\alpha R(t_0)} + C_R} R(\sigma_{\varepsilon,t}(\psi))$$
and (5.3) implies that
$$e^{-\zeta(\sigma_{\varepsilon,t}(\psi))} \geq e^{-\zeta(\alpha_k)},$$
we get
(5.14)
$$\int_{X_k} \left| \frac{u_{k,\varepsilon,t,\rho}}{\alpha R(t_0) + C_R} \right|^2 e^{-\phi_0 - \psi_0} dV_{\omega,\alpha} + e^{-\zeta(\alpha_k)} \int_{X_k} \left| v_{k,\varepsilon,t,\rho} \right|^2 e^{-\phi_0 - \psi_0} dV_{\omega,\alpha} \leq C_\rho(t)$$
for any $\rho \in (0, \rho_t]$.

Since $R$ is decreasing near $-\infty$, (5.3) implies that $R(\sigma_{\varepsilon,t}(\psi)) \leq R(t)$ for any $\varepsilon \in (0, t_0)$, any $t \in (-\infty, t_0]$ and any $\rho \in (0, \rho_t)$. Then (5.14) and the property $\sup \sup (\phi_0 + \psi_0) < +\infty$ imply that $u_{k,\varepsilon,t,\rho} \in L^2$ and $v_{k,\varepsilon,t,\rho} \in L^2$.

The property $\partial \tilde{\psi}_\rho \in L^1$ on $X_k$ implies that $g_{\varepsilon,t,\rho} = g_{1,\varepsilon,t,\rho} + g_{2,\varepsilon,t,\rho} \in L^1$ on $X_k$.

Then it follows from (5.13) and Lemma 3.4 that
(5.15)
$$\partial u_{k,\varepsilon,t,\rho} + \sqrt{2\delta(t)} v_{k,\varepsilon,t,\rho} = g_{\varepsilon,t,\rho} = \partial(1 - \sigma_{\varepsilon,t}(\psi)) \tilde{f} \quad \text{on} \quad X_k.$$  

Since $\lim_{\rho \to 0} \psi_\rho = \psi$ on $X_k \setminus \Sigma$, $(1 - \sigma'_{\varepsilon,t}(\psi)) \tilde{f}$ converges to $(1 - \sigma'_{\varepsilon,t}(\psi)) \tilde{f}$ in $L^2$ as $\rho \to 0$ by Lebesgue’s dominated convergence theorem. Hence $\partial((1 - \sigma'_{\varepsilon,t}(\psi)) \tilde{f})$ converges to $\partial((1 - \sigma'_{\varepsilon,t}(\psi)) \tilde{f})$ in the sense of currents as $\rho \to 0$.

Let $t$ be fixed and let $\{\rho_j\}_{j=2}^{+\infty}$ be a decreasing sequence of positive numbers such that $\rho_2 < \rho_t$ and $\lim_{j \to +\infty} \rho_j = 0$.

Since $\lim_{j \to +\infty} \psi_{\rho_j} = \psi$ on $X_k \setminus \Sigma$ and $\phi_{\rho_j} + \psi_{\rho_j}$ decreases to $\phi + \psi$ on $X_k \setminus \Sigma$ as $j \to +\infty$,
$$e^{-\phi_{\rho_j} - \psi_{\rho_j}} \sup_{p \geq j} R(\sigma_{\varepsilon,t}(\psi_{\rho_j}))$$
increases to $\frac{e^{-\phi - \psi}}{R(\sigma_{\varepsilon,t}(\psi))}$ on $X_k \setminus \Sigma$ as $j \to +\infty$.

By extracting weak limits of $\{u_{k,\varepsilon,t,\rho_j}\}_{j=2}^{+\infty}$ as $j \to +\infty$, it follows from (5.14) and the diagonal argument that there exists an $L^2$ $L$-valued $(n, 0)$-form $u_{k,\varepsilon,t}$ such that a subsequence $\{u_{k,\varepsilon,t,\rho_j}\}_{j=2}^{+\infty}$ of $\{u_{k,\varepsilon,t,\rho_j}\}_{j=2}^{+\infty}$ converges to $u_{k,\varepsilon,t}$ weakly in
$$L^2((n, 0)) \left( \frac{X_k}{X_k} \right. e^{-\phi_{\rho_r} - \psi_{\rho_r}^t} dV_{\omega,\alpha} \left. \sup_{p \geq i} R(\sigma_{\varepsilon,t}(\psi_{\rho_r})) \right)$$
for any positive integer $i$ as $r \to +\infty$.

Hence the Banach-Steinhaus Theorem implies that, for any positive integer $i$,
$$\int_{X_k} \left| \frac{u_{k,\varepsilon,t,\rho_i}}{\alpha R(t_0) + C_R} \right|^2 e^{-\phi_{\rho_i} - \psi_{\rho_i}^t} dV_{\omega,\alpha} \sup_{p \geq i} R(\sigma_{\varepsilon,t}(\psi_{\rho_i})) \leq \lim_{r \to +\infty} \int_{X_k} \left| \frac{u_{k,\varepsilon,t,\rho_r}}{\alpha R(t_0) + C_R} \right|^2 e^{-\phi_{\rho_r} - \psi_{\rho_r}^t} dV_{\omega,\alpha} \sup_{p \geq i} R(\sigma_{\varepsilon,t}(\psi_{\rho_r})) \leq \lim_{r \to +\infty} \int_{X_k} \left| \frac{u_{k,\varepsilon,t,\rho_r}}{\alpha R(t_0) + C_R} \right|^2 e^{-\phi_{\rho_r} - \psi_{\rho_r}^t} dV_{\omega,\alpha} \sup_{p \geq j} R(\sigma_{\varepsilon,t}(\psi_{\rho_r}))$$
Then Fatou’s lemma implies that
\[
\int_{X_k} \frac{|u_{k,t}|^2_{\omega,h_0} e^{-\phi - \psi} dV_{X,\omega}}{R(\sigma_{\epsilon,t}(\psi))} \leq \lim_{t \to +\infty} \int_{X_k} \frac{|u_{k,t}|^2_{\omega,h_0} e^{-\phi - \psi} dV_{X,\omega}}{\sup_{p \geq t} R(\sigma_{\epsilon,t}(\psi_p))} \\
\leq \lim_{r \to +\infty} \int_{X_k} \frac{|u_{k,t}|^2_{\omega,h_0} e^{-\phi - \psi} dV_{X,\omega}}{\sup_{p \geq jr} R(\sigma_{\epsilon,t}(\psi_p))}.
\]

Similar weak limit argument can also be applied to subsequences of \(\{v_{k,\epsilon,t,\rho_j}\}_{j=2}^{\infty}\).

In conclusion, by (5.15), (5.14) and (5.10), there exist \(L^2\) forms \(u_{k,\epsilon,t}\) and \(v_{k,\epsilon,t}\) such that
\[
\dot{\partial} u_{k,\epsilon,t} + \sqrt{2\delta(t)} v_{k,\epsilon,t} = \dot{\partial} (1 - \sigma'_{\epsilon,t}(\psi)) \bar{f} \quad \text{on} \quad X_k
\]
and
\[
\int_{X_k} \frac{|u_{k,t}|^2_{\omega,h_0} e^{-\phi - \psi} dV_{X,\omega}}{\bar{R}(\sigma_{\epsilon,t}(\psi))} + e^{-\zeta(\alpha_k)} \int_{X_k} |v_{k,t}|^2_{\omega,h_0} e^{-\phi - \psi} dV_{X,\omega} \leq C(t).
\]

(5.12) and (5.17) imply that
\[
\lim_{t \to -\infty} \int_{X_k} \sqrt{2\delta(t)} v_{k,\epsilon,t} \leq \lim_{t \to -\infty} 2e^{\zeta(\alpha_k)} \delta(t) C(t) = 0.
\]

Define \(F_{k,\epsilon,t} = -u_{k,\epsilon,t} + (1 - \sigma'_{\epsilon,t}(\psi)) \bar{f}\) on \(X_k\). Then (5.16) implies that
\[
\bar{\partial} F_{k,\epsilon,t} = \sqrt{2\delta(t)} v_{k,\epsilon,t} \quad \text{on} \quad X_k.
\]

Since \(R\) is decreasing near \(-\infty\), \(R(\sigma_{\epsilon,t}(s)) \leq R(s)\) for all \(s \in (-\infty, \alpha_0)\) when \(t\) is small enough. Then (5.17) implies that
\[
\int_{X_k} \frac{|F_{k,t}|^2_{\omega,h_0} e^{-\phi}}{e^\phi R(\psi)} dV_{X,\omega} \leq (1 + \epsilon) \int_{X_k} \frac{|u_{k,\epsilon,t}|^2_{\omega,h_0} e^{-\phi}}{e^\phi R(\sigma_{\epsilon,t}(\psi))} dV_{X,\omega} + \frac{1 + \epsilon}{\epsilon} \int_{X_k} \frac{|1 - \sigma'_{\epsilon,t}(\psi)) \bar{f}|^2_{\omega,h_0} e^{-\phi}}{e^\phi R(\psi)} dV_{X,\omega} \\
\leq (1 + \epsilon) \left( \frac{1}{\alpha R(\sigma_0)} + CR \right) C(t) + \bar{C}(t)
\]
when \(t\) is small enough, where
\[
\bar{C}(t) := \frac{1 + \epsilon}{\epsilon} \int_{X_k} \frac{|\bar{f}|^2_{\omega,h_0} e^{-\phi} \mathbb{1}_{\{\psi < t + 1\}}}{e^\phi R(\psi)} dV_{X,\omega}.
\]

Since (1.1) implies that
\[
\lim_{t \to -\infty} \int_{X_k} |\bar{f}|^2_{\omega,h_0} e^{-\phi - \psi} \mathbb{1}_{\{\psi < t + 1\}} dV_{X,\omega} < +\infty,
\]
there exists a positive number \(C_1\) such that
\[
\int_{X_k} |\bar{f}|^2_{\omega,h_0} e^{-\phi - \psi} \mathbb{1}_{\{t-j < \psi < t+1-j\}} dV_{X,\omega} \leq C_1
\]
for all nonnegative integer \(j\) when \(t\) is small enough.
Since $R$ is decreasing near $-\infty$, we get
\[
\bar{C}(t) \leq \frac{1 + \varepsilon}{\varepsilon} \sum_{j=0}^{+\infty} \frac{1}{R(t + 1 - j)} \int_{X_k} |\tilde{f}|^2 e^{-\phi} e^{-\psi} \chi_{(t-j \leq \psi < t+1-j)} dV_{X,\omega}
\]
\[
\leq \frac{1 + \varepsilon}{\varepsilon} C_{1} \int_{-\infty}^{t+2} \frac{1}{R(s)} ds
\]
when $t$ is small enough. Hence
\begin{equation}
(5.21) \quad \lim_{t \to -\infty} \bar{C}(t) = 0.
\end{equation}

Since $\lim_{s \to -\infty} \varepsilon R(s) < +\infty$, we obtain from (5.20), (5.12) and (5.21) that
\begin{equation}
(5.22) \quad \int_{X_k} |F_{k,\varepsilon}|^2 e^{-\phi} dV_{X,\omega} \leq C_2
\end{equation}
for some positive number $C_2$ independent of $t$ when $t$ is small enough.

By extracting weak limits of $\{F_{k,\varepsilon,t} \in (-\infty, h_0)\}$ as $t \to -\infty$, it follows from (5.20), (5.22), (5.12) and (5.21) that there exists a sequence of negative numbers $\{t_j\}_{j=1}^{+\infty}$ and an $L$-valued $(n,0)$-form $F_{k,\varepsilon}$ such that $\lim_{j \to +\infty} t_j = -\infty$, $F_{k,\varepsilon,t_j} \to F_{k,\varepsilon}$ weakly in both $L^2_{(n,0)}(X_k, e^{-\phi} dV_{X,\omega})$ and $L^2_{(n,0)}(X_k, dV_{X,\omega})$, and
\begin{equation}
(5.23) \quad \int_{X_k} |F_{k,\varepsilon}|^2 e^{-\phi} dV_{X,\omega} \leq \frac{1 + \varepsilon}{1 - 2\varepsilon} \left( \frac{1}{\alpha R(a_0)} + C_R \right) \int_{V} |f|^2 e^{-\phi} dV_{X,\omega} \psi \].
\end{equation}

It follows from (5.18) and (5.19) that $\bar{\partial}F_{k,\varepsilon} = 0$ on $X_k$. Thus
\[
F_{k,\varepsilon} \in H^0(X_k, O_X(K_X \otimes L)).
\]

In Subsection 5.5, we will prove that $F_{k,\varepsilon} \in H^0(\{X_k, O_X(K_X \otimes L) \otimes T'(h)\})$ and that $F_{k,\varepsilon}$ maps to $f$ under the morphism $T'_0(h) \to T'_0(h)/T'(h-e^{-\psi})$.

5.5. Solving $\bar{\partial}$ locally with $L^2$ estimates and the end of the proof.

Let $\{U_i\}_{i \in I}$ be the covering of $X$ as in Subsection 5.1. Let $i \in I$ be any fixed index. Let $V$ be an arbitrary relatively compact coordinate ball contained in $U_i \cap X_k$ such that $L$ is trivial on $V$. Then $f_i$, $\tilde{f}_i$, $u_{k,\varepsilon,t}$, $v_{k,\varepsilon,t}$, $F_{k,\varepsilon,t}$ and $F_{k,\varepsilon}$ can be regarded as forms with values in $\mathbb{C}$ when they are restricted to $V$.

It follows from (5.17) that
\[
\int_{V} |v_{k,\varepsilon,t}|^2 e^{-\phi} d\lambda_n \leq C_3
\]
for some positive number $C_3$ independent of $t$ when $t$ is small enough, where $d\lambda_n$ is the $n$-dimensional Lebesgue measure on $V$.

Since $\bar{\partial}v_{k,\varepsilon,t} = 0$ on $V$ by (5.16), applying Lemma 3.2 to the $(n,1)$-form
\[
\sqrt{2\delta(t)} v_{k,\varepsilon,t} \in L^2_{(n,1)}(V, e^{-\phi - \psi}),
\]
we get an $(n,0)$-form $w_{k,\varepsilon,t}$ such that $\bar{\partial}w_{k,\varepsilon,t} = \sqrt{2\delta(t)} v_{k,\varepsilon,t}$ on $V$ and
\begin{equation}
(5.24) \quad \int_{V} |w_{k,\varepsilon,t}|^2 e^{-\phi} d\lambda_n \leq C_4 \delta(t)
\end{equation}
for some positive number $C_4$ independent of $t$. 

Hence

\[ (5.25) \quad \int_V |w_{k,\varepsilon,t}|^2 d\lambda_n \leq C_5 \delta(t) \]

for some positive number \( C_5 \) independent of \( t \).

Now define \( G_{k,\varepsilon,t} = -u_{k,\varepsilon,t} - w_{k,\varepsilon,t} + (1 - \sigma'_{\varepsilon,t}(\psi)) \tilde{f} \) on \( V \). Then

\[ G_{k,\varepsilon,t} = F_{k,\varepsilon,t} - w_{k,\varepsilon,t} \quad \text{and} \quad \partial G_{k,\varepsilon,t} = 0. \]

Hence \( G_{k,\varepsilon,t} \) is holomorphic on \( V \). Furthermore, we get from (5.22) and (5.25) that

\[ (5.26) \quad \int_V |G_{k,\varepsilon,t}|^2 d\lambda_n \leq C_6 \]

for some positive number \( C_6 \) independent of \( t \).

Since \( \sigma_{\varepsilon,t} \geq t \) and \( R \) is decreasing near \(-\infty\), we can obtain that \( R(\sigma_{\varepsilon,t}(\psi)) \leq R(t) \) on \( X_k \) when \( t \) is small enough. Then we obtain from (5.17) that

\[ \int_V |u_{k,\varepsilon,t}|^2 e^{-\phi - \psi} d\lambda_n \leq C_7 R(t) \]

for some positive number \( C_7 \) independent of \( t \).

Therefore, we get

\[ \int_V |u_{k,\varepsilon,t} + w_{k,\varepsilon,t}|^2 e^{-\phi - \psi} d\lambda_n \leq 2C_7 R(t) + 2C_4 \delta(t). \]

Since \( G_{k,\varepsilon,t} - f_i = -u_{k,\varepsilon,t} - w_{k,\varepsilon,t} + (1 - \sigma'_{\varepsilon,t}(\psi)) \sum_{j \in I} \xi_j (f_j - f_i) - \sigma'_{\varepsilon,t}(\psi) f_i \)

and \( \sigma'_{\varepsilon,t}(\psi) = 0 \) on \( V \cap \{ \psi < t \} \), we have

\[ (5.27) \quad G_{k,\varepsilon,t} - f_i \in \mathcal{I}(he^{-\psi})_x, \quad \forall x \in V. \]

Since \( w_{k,\varepsilon,t_j} \rightarrow 0 \) in \( L^2 \) by (5.25) and \( F_{k,\varepsilon,t_j} \rightarrow F_{k,\varepsilon} \) weakly in \( L^2 \) as \( j \rightarrow +\infty \), we get \( G_{k,\varepsilon,t_j} \rightarrow F_{k,\varepsilon} \) weakly in \( L^2 \) as \( j \rightarrow +\infty \). Hence it follows from (5.26) and routine arguments with applying Montel’s theorem that a subsequence of \( \{G_{k,\varepsilon,t}\}_{j=1}^{+\infty} \) converges to \( F_{k,\varepsilon} \) uniformly on compact subsets of \( V \). Then it follows from (5.26), (5.27) and Lemma 3.6 that

\[ (5.28) \quad F_{k,\varepsilon} - f_i \in \mathcal{I}(he^{-\psi})_x \]

for any \( x \in V \) and thereby for any \( x \in U_i \cap X_k \).

Since \( \phi \) is locally bounded above and \( \lim_{s \rightarrow -\infty} e^{\phi} R(s) < +\infty \), applying Montel’s theorem and extracting weak limits of \( \{F_{k,\varepsilon}\}_{k \geq 3, k \in \mathbb{Z}, \varepsilon \in (0, \varepsilon_0)} \), first as \( \varepsilon \rightarrow 0 \), then as \( k \rightarrow +\infty \), we obtain from (5.23), (5.28) and Lemma 3.6 a section \( F \in H^0(X, \mathcal{O}_X(K_X \otimes L)) \) such that

\[ \int_X \frac{|F|^2}{{\omega_h}^2} dV_{X,\omega} \leq \left( \frac{1}{\alpha R(\alpha_0)} + C_R \right) \int_Y |f|^2 dV_{\omega,\hat{h}} \]

and

\[ F - f_i \in \mathcal{I}(he^{-\psi})_x, \quad \forall x \in U_i \cap X. \]

Hence \( F \in H^0(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}_\varepsilon(h)) \), and \( F \) maps to \( f \) under the morphism \( \mathcal{I}_\varepsilon(h) \rightarrow \mathcal{I}_\varepsilon(h) / \mathcal{I}(he^{-\psi}) \).

The last surjectivity statement in the conclusion of Theorem 1.2 follows by replacing the metric \( h \) with a new metric \( h_1 := he^{-\phi(\Psi)} \), where \( \Psi \) is the smooth psh
exhaustion function on $X$ (cf. Subsection 5.1), and $\Phi : \mathbb{R} \to [0, +\infty)$ is some smooth increasing convex function. In fact, in order to obtain a global holomorphic extension $F$, the key point in the whole proof above is the existence of a constant $C_0$ (independent of $\rho$, $t$ and $k$) satisfying $\lim_{t \to -\infty} C(t) \leq C_0$ (cf. (5.12)). It is not hard to see that such a constant $C_0$ exists if $\Phi$ increases fast enough.

In conclusion, Theorem 1.2 is proved.

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Xiangyu Zhou: Institute of Mathematics, AMSS, and Hua Loo-Keng Key Laboratory of Mathematics, Chinese Academy of Sciences, Beijing 100190, China; School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China

E-mail address: xyzhou@math.ac.cn

Langfeng Zhu: School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China

E-mail address: zhulangfeng@amss.ac.cn