Minkowski decomposition and generators of the moving cone for toric varieties

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Abstract. We prove that for smooth projective toric varieties, the Okounkov body of a $T$-invariant pseudo-effective divisor with respect to a $T$-invariant flag decomposes as a finite Minkowski sum of indecomposable polytopes. We prove that these indecomposable polytopes form a finite Minkowski base and that they correspond to the rays in the secondary fan. Moreover, we present an algorithm to find the Minkowski base.

Introduction

The definition of Okounkov bodies originates from papers due to A. Okounkov from the middle of the 1990s (for instance [Ok96]). More recently, Lazarsfeld and Mustaţă [LM09] and independently Kaveh and Khovanskii [KK12] initiated an intensive research on the topic recording strong relations of the construction to properties of linear series that were not observed at first.

The idea of the construction of Okounkov bodies is to associate simple geometric objects to linear series on normal varieties. This idea comes as a generalization of the toric case, where to each divisor $D$ there is associated a polytope $P_D$. As for the toric, also in the general case Okounkov bodies are convex bodies which encode several properties of linear series, as for example, their volume. The idea of the construction is quite natural, even though it is very hard to determine them.

One possibility introduced for the case of surfaces in [L-S12] to simplify the construction is that of finding “minimal elements” generating all the possible bodies. As for the surface case, the philosophy of Zariski decomposition plays an important role in the generalization of the construction we are going to give in this paper. First, since the global sections of a divisor are asymptotically the same as those of its movable part, it is enough to consider the bodies associated to movable divisors. Second, for a Mori Dream Space $X$, movable divisors generate a cone $\text{Mov}(X)$, that can be subdivided into chambers representing the nef cone of different flipped models of $X$ ([HK00]). A possible approach to construct these minimal bodies is to find indecomposable polytopes associated to the extremal rays of the nef cone (since the cone is not round) of each model and ask whether these elements are enough to reconstruct all the possible bodies. This will yield a Minkowski decomposition of the polytope. The set of indecomposable elements is called Minkowski base.

In [L-SS13], the second author and P. Łuszcz-Świdecka prove that for smooth projective surfaces whose pseudo-effective cone is rational polyhedral, the Okounkov

2000 Mathematics Subject Classification: Primary 14C20; Secondary 14M25.
Keywords. Okounkov body, Toric variety, Movable cone, Secondary fan.
The first author was partially supported by WCMCS, Warsaw.
The second author was supported by DFG grant BA 1559/6-1.
body of a big divisor with respect to a general flag decomposes as the Minkowski sum of finitely many simplices and segments arising as Okounkov bodies of nef divisors. In the higher dimensional case, as mentioned above, it is obvious that nef divisors will not be enough to obtain a similar result, since applying a flip for the variety will imply a change of the nef cone but not of the Okounkov body of the divisors. The aim is instead to look at Minkowski indecomposable polytopes arising from divisors in the movable cone, $\text{Mov}(X)$.

Among the prominent objects studied in relation with Okounkov bodies there are toric varieties. For instance in [A13], it is shown that varieties having an ample divisor with a rational polyhedral Okounkov body with respect to some flag, admit a flat degeneration to a toric variety.

As a first step in the generalization for higher dimensional varieties of the result in [L-SS13], in this paper we study the case of Okounkov bodies on toric varieties constructed with respect to torus-invariant flags. One of the main tools that we are going to use is given in [LM09], where the authors identify the polytopes arising as toric polytopes and those coming up as Okounkov bodies. This will be the starting point of our work. The main result we prove is:

**Theorem 3.1.** The set of all $T$-invariant divisors $D$ on a smooth projective toric variety $X$ such that there exists a small modification $f : X \rightarrow X'$ and a divisor $D'$ spanning an extremal ray of $\text{Nef}(X')$ such that $D = f^*(D')$ forms a Minkowski base with respect to $T$-invariant flags.

Moreover, we give an effective algorithm finding the Minkowski base for a projective toric variety of arbitrary dimension, whose elements will correspond to the extremal rays of the chambers decomposing the movable cone $\text{Mov}(X)$. This will also give a complete description of the secondary fan (or GKZ decomposition) for the movable cone of the given toric variety [CLS11 14.4]. Moreover this correspondence implies that the Minkowski base is unique up to numerical equivalence and scaling.

We give a full description, both in words and as a semi-code, of the algorithm and we prove its correctness. We show that the algorithm finds all the indecomposable polytopes arising as Okounkov bodies of movable $T$-invariant divisors with respect to a $T$-invariant flag and every polytope in the output of the algorithm is indecomposable. Then this set of polytopes corresponds to the Minkowski base.

In sections 1 and 2, we recall the main definitions and properties of Okounkov bodies and toric varieties, and give a general definition of a Minkowski base, emphasizing the relations among the different structures.

In section 3, we prove the main theorem and discuss the algorithm.

In section 4, we give examples on how the algorithm works in the case of $\mathbb{P}^2$ blown-up in two points and $\mathbb{P}^3$ blown-up in two intersecting lines. In particular, there will come out the difference between varieties admitting or not admitting flips and how choosing a $T$-invariant cone for the flag that is canceled by a flip can require extra computations.

Finally, in section 3.2 we present a method to find the Minkowski decomposition of the Okounkov body for a given divisor using the output of the algorithm. This will yield the position of the class of its moving part in the movable cone.

**Acknowledgements.** The authors would like to thank Alex K"uonya and Tomasz Szemberg for helpful discussions and suggestions. The authors are grateful to the Pedagogical University of Cracow and IMPAN in Warsaw. We warmly thank these institutions for their hospitality that allowed the beginning of this collaboration.
1. Okounkov bodies and Minkowski decomposition

We start with a short reminder of the construction of Okounkov bodies. Let $X$ be a smooth projective variety over the complex numbers of dimension $n$. Suppose that $D$ is a Cartier divisor. To such a divisor we associate a convex polytope called the Okounkov body. The construction of the polytope requires an admissible flag, i.e. a finite sequence of subvarieties

$$Y_i : X = Y_0 \supseteq Y_1 \supseteq \ldots \supseteq Y_{n-1} \supseteq Y_n = \{ pt \}$$

such that $\text{codim} Y_i = i$ and $Y_n$ is a smooth point for every $Y_i$. In the above setting we construct a valuation-like function

$$\nu = \nu_{Y_i,D} : H^0(X, \mathcal{O}_X(D)) \ni s \mapsto (\nu_1(s), \ldots, \nu_n(s)) \in \mathbb{Z}^n \cup \{ \infty \}$$

in the following way. For $0 \neq s \in H^0(X, \mathcal{O}_X(D))$ let $\nu_1(s) = \text{ord}_{Y_1}(s)$. After choosing a local equation of $Y_1$ in $X$, the section $s$ determines another section $\tilde{s}_1 \in H^0(X, \mathcal{O}_X(D - \nu_1 Y_1))$ that does not vanish identically along $Y_1$. Taking the restriction $s_1 \in H^0(Y_1, \mathcal{O}_{Y_1}(D - \nu_1 Y_1))$ we define $\nu_2(s) = \text{ord}_{Y_2}(s_1)$. Continuing the procedure we obtain the vector $\nu(s) = (\nu_1(s), \ldots, \nu_n(s))$, which will be called the valuation vector of the section $s$.

Define

$$\Gamma(D)_m = \text{Im}\left\{ H^0(X, \mathcal{O}_X(mD)) \ni s \mapsto (\nu_1(s), \ldots, \nu_n(s)) \in \mathbb{Z}^n \right\}.$$ 

Then the Okounkov body for $D$ is defined as

$$\Delta(D) = \text{closed convex hull} \left\{ \frac{1}{m} \Gamma(D)_m \right\}.$$ 

It is worth to point out that the above construction can be mimicked for graded linear series (with minor changing of assumptions), see [KK12]. Okounkov bodies encode several geometric and algebraic information about divisors, for instance their volume.

Determining the Okounkov body of a given divisor with respect to a given flag is in general a quite difficult computational task. There are two cases in which it is possible to bypass the construction described above and to determine Okounkov bodies more directly, the case of surfaces and that of toric varieties [LM09]. In the case of surfaces, Zariski decomposition of big divisors can be used to produce two functions which bound the Okounkov body. In practice it turns out that this approach is still quite involved, since many Zariski decompositions have to be determined. This fact motivated research in the direction of Minkowski decomposition.

The idea, stemming from [L-S12] is to find a (preferably finite) set of divisors such that the knowledge of their Okounkov bodies suffices to determine the Okounkov bodies of arbitrary big divisors. In the case of surfaces this has been accomplished in [L-SS13] by extensive use of the so called Bauer-Küronya-Szemberg decomposition of the big cone [BKS04]. The result in the surface case is the following.

**Theorem 1.1** ([L-SS13]). Let $X$ be a smooth complex projective surface such that $E(X)$ is rational polyhedral. Suppose that we fix a general flag $(x, L)$, where $L$ is a big and nef divisor and $x$ is a general point. Then there exists a finite set of nef
divisors \( \text{MB}(X) = \{P_1, \ldots, P_s\} \) such that for every big and nef \( \mathbb{Q} \)-divisor \( D \) there exist non-negative numbers \( a_i \geq 0 \) such that

\[
D = \sum_{i=1}^{s} a_i P_i \quad \text{and} \quad \Delta(D) = \sum_{i=1}^{s} a_i \Delta(P_i).
\]

This result is constructive, i.e., its proof contains an algorithm, which allows us to construct Okounkov bodies with respect to a general flag as above.

**Example 1.2.** Consider \( X \) the blow up of \( \mathbb{P}^2 \) in two points with exceptional divisors \( E_1, E_2 \). From the construction in the proof of the above result we obtain the set of divisors

\[
\{H, H - E_1, H - E_2, 2H - E_1 - E_2\}
\]

as the set \( \text{MB}(X) \) in this case. Here \( H \) denotes the pullback of the class of a line in \( \mathbb{P}^2 \). Let us consider the big and nef divisor \( D = 7H - 2E_1 - 2E_2 \). We can write \( D = 3H + 2(2H - E_1 - E_2) \) and, considering a flag \( Y_\bullet = \{l, x\} \) given by a general line and a general point, obtain the following decomposition of Okounkov bodies

\[
\Delta(D) = 3\Delta(H) + 2\Delta(2H - E_1 - E_2).
\]

In order to generalize the idea to higher dimensions, we fix the following definition.

**Definition 1.3.** Let \( X \) be a smooth projective variety of dimension \( n \) and \( Y_\bullet : X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_{n-1} \supseteq Y_n = \{\text{pt}\} \) an admissible flag on \( X \). A collection \( \{D_1, \ldots, D_r\} \) of pseudo-effective divisors on \( X \) is called a Minkowski base of \( X \) with respect to \( Y_\bullet \) if

- for any pseudo-effective divisor \( D \) on \( X \) there exist non-negative numbers \( \{a_1, \ldots, a_r\} \) and a translation \( \varphi : \mathbb{R}^n \to \mathbb{R}^n \) such that
  \[
  \varphi(\Delta_{Y_\bullet}(D)) = \sum a_i \Delta_{Y_\bullet}(D_i)
  \]
- the Okounkov bodies \( \Delta_{Y_\bullet}(D_i) \) are indecomposable in the sense of Minkowski decomposition.

Note that in the result on the surface case cited above the set \( \text{MB} \) constitutes a Minkowski base, since the Okounkov body of any pseudo-effective divisor on a surface is just a translate of the Okounkov body of its positive part, which is nef, and thus can be decomposed into bodies coming from the divisors in \( \text{MB} \). Furthermore, it
follows from the construction that these divisors have as Okounkov bodies either lines or triangles, i.e., indecomposable polytopes.

In the present paper we prove that for a smooth projective toric variety $X$ a Minkowski basis with respect to any $T$-invariant flag is given by the set of all movable divisors which span an extremal ray of the nef cone $\text{Nef}(X')$ for some small modification of $X$. In particular, the Minkowski base is independent of the flag.

2. Toric varieties and Okounkov bodies

The main references for this section will be [CLS11] and [LM09].

Consider a normal projective toric variety $X = X_\Sigma$ corresponding to a complete fan $\Sigma$ in $N_\mathbb{R}$ (with no torus factor), with $\dim N_\mathbb{R} = n$. Recall that every $T_N$-invariant Weil divisor is represented as a sum

$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho,$$

where $\rho$ is a one-dimensional subcone (a ray), and $D_\rho$ is the associated $T_N$-invariant prime divisor. $D$ is Cartier if for every maximal dimensional subcone $\sigma \in \Sigma(n)$, $D|_{U_\sigma}$ is locally the divisor of a character $(\text{div}(\chi_{m_\sigma})$ with $m_\sigma \in N^\vee = M)$.

To every divisor we can associate a polyhedron:

$$P_D = \{ m \in M_\mathbb{R} | \langle m, u_\rho \rangle \geq -a_\rho \text{ for every } \rho \in \Sigma(1) \},$$

whose integral points represent the global sections of the divisor. In particular through the toric polyhedron we can detect whether a given divisor is globally generated, hence nef.

**Theorem 2.1** ([CLS11] 6.3.12). For a divisor $D$ on a toric variety $X$ the following properties are equivalent.

1. $D$ is nef.
2. $\mathcal{O}_X(D)$ is generated by global sections.
3. $m_\sigma \in P_D$ for all $\sigma \in \Sigma(n)$.

**Example 2.2.** Let us consider the blow-up of $\mathbb{P}^2$ in one point. The fan $\Sigma$ is given by 4 rays, i.e., $D_1 = H - E_1$, $D_2 = H$, $D_3 = H - E_1$ and $E_1$. Let us consider the $T$-invariant divisor $D = D_1 + D_2 + D_3 + E_1$ and construct the associated polytope $P_D$ in the dual space.
Toric polyhedra are a great tool for detecting properties of $T$-invariant divisors. For example, these are some important results that we are going to use in this work.

**Theorem 2.3 ([CLS11] 6.2.8).** Let $D$ be a basepoint free Cartier divisor on a complete toric variety, and let $X_D$ be the toric variety of the polytope $P_D \subseteq M_\mathbb{R}$. Then the refinement $\Sigma$ of $\Sigma_{P_D}$ induces a proper toric morphism

$$\varphi : X_\Sigma \rightarrow X_{P_D}.$$  

Furthermore, $D$ is linearly equivalent to the pullback via $\varphi$ of the ample divisor on $X_{P_D}$ coming from $P_D$.

In particular we can deduce the following.

**Remark 2.4.** The polytope of the pullback of a $T$-invariant divisor by a toric morphism is isomorphic to the original polytope.

More connected to Minkowski decomposition, but still consequences of [23], we have the following statements.

**Proposition 2.5 ([CLS11] 6.2.13).** Let $P$ and $Q$ be lattice polytopes in $M_\mathbb{R}$. Then:

1. $Q$ is an $\mathbb{N}$-Minkowski summand of $P$ if and only if $\Sigma_P$ refines $\Sigma_Q$.
2. $\Sigma_{P+Q}$ is the coarsest common refinement of $\Sigma_P$ and $\Sigma_Q$.

**Corollary 2.6 ([CLS11] 6.2.15).** Let $P$ be a full dimensional lattice polytope in $M_\mathbb{R}$. Then a polytope $Q \subseteq M_\mathbb{R}$ is an $\mathbb{N}$-Minkowski summand of $P$ if and only if there is a torus invariant basepoint free Cartier divisor $A$ on $X_P$ such that $Q = P_A$.

**Lemma 2.7 ([F93] Section 3.4).** If $D$ and $E$ are nef $T$-invariant divisors, then

$$P_{D+E} = P_E + P_D.$$  

At last, the following statement will be the key ingredient for the construction of our algorithm.

**Proposition 2.8 ([LM09] 6.1).** Let $X$ be a smooth projective toric variety, and let $Y_\ast$ be a flag given as a complete intersection of a set of $T$-invariant divisors generating a maximal cone $\sigma$. Given any big line bundle $\mathcal{O}_X(D)$ on $X$, such that $D|_{d_\sigma} = 0$, then

$$\Delta(\mathcal{O}_X(D)) = \varphi(P_D),$$  

where $\varphi$ is an $\mathbb{R}$-linear map.

In practice, what the proposition tells us to do is to choose a vertex of a toric polyhedron, corresponding to a $T$-invariant cone, and translate the polyhedron so that the vertex is at the origin. After this, mapping the generators of the cone to the coordinate axis gives the Okounkov body with respect to the complete intersection flag constructed with the $T$-invariant divisors corresponding to the generators.

In particular, in the case of toric varieties, studying toric polytopes is equivalent to understanding Okounkov bodies constructed with some special flag.
Example 2.9. Let us keep the same assumptions of Example 2. Following Proposition 2.8 consider the element \( D' = 2D_2 + D_1 \in |D| \). Then we can find the Okounkov body \( \Delta(D) \) with \( T \)-invariant flag \( Y_\bullet = \{ D_3, D_3 \cap E_1 \} \) via the linear function \( \varphi \).

3. Minkowski bases for toric varieties

In this section we will characterize the Minkowski base elements for a smooth projective toric variety of arbitrary dimension and present an algorithm to obtain them, given as data the complete fan defining the toric variety.

3.1. Characterization of Minkowski base elements

Let \( X \) be a smooth projective toric variety of dimension \( n \). First note that a Minkowski base for any given \( T \)-invariant flag will also be a Minkowski base for any other \( T \)-invariant flag. This is due to Theorem 2.8 which says that the (transpose of the) Okounkov body of a big divisor \( D \) on \( X \) with respect to a flag \( Y_\bullet \) is the linear transform of the polytope \( P_D \). Now, Minkowski summation obviously respects linear transformation, so both properties of a Minkowski base can be checked immediately on the level of \( T \)-invariant polytopes, which is what we will always do in the rest of the paper. Our aim is to prove the following.

Theorem 3.1. The set of all \( T \)-invariant divisors \( D \) on a smooth projective toric variety such that there exists a small modification \( f : X \rightarrow X' \) and a divisor \( D' \) spanning an extremal ray of \( \text{Nef}(X') \) such that \( D = f^*(D') \) forms a Minkowski base with respect to \( T \)-invariant flags.

Remark 3.2. The theorem can be rephrased in the following way: finding a Minkowski base for a toric variety is equivalent to finding the rays in the secondary fan contained in the moving cone. (For the definition and an analysis of the secondary fan see for example [CLS11, 14.4]).

For convenience, we split the proof up into two parts.
Proof. Let $D$ be a $T$-invariant pseudo-effective divisor on $X$. There exists a small modification $f : X \to X'$ such that $D$ admits a CKM-Zariski decomposition considered as a divisor on $X'$. We can thus assume that $D$ admits a CKM-Zariski decomposition $D = P + N$ on $X$ (since a small modification does not change the polytope). Now, all sections in $H^0(X, \mathcal{O}_X(D))$ come from $H^0(X, \mathcal{O}_X(P))$ and so the Okounkov body of $D$ is merely a translate of the body of $P$, which is a nef divisor. We conclude the proof of the first part of the theorem by proving that the polytope corresponding to $P$ can be decomposed as a Minkowski sum of divisors spanning extremal rays of $\text{Nef}(X)$. But this immediately follows from the fact that we can write $P$ as a non-negative $\mathbb{Q}$-linear combination of these divisors together with Lemma 2.7 stating the additivity of polytopes for nef divisors.

The proof of the Theorem then follows from Proposition 3.3. □

Let us prove that the set of divisors spanning extremal rays of the nef cone on some small modification also satisfies the second condition for a Minkowski base.

**Proposition 3.3.** For a $T$-invariant movable divisor $D$ the polytope $P_D$ is indecomposable if and only if there exists a small modification $f : X \to X'$ and divisor $D'$ spanning an extremal ray of $\text{Nef}(X')$ with $D = f^*(D')$, or, in other words, if $D$ spans a ray in the secondary fan of $X$.

**Remark 3.4.** This result is slightly stronger than what we need in order to prove the theorem. In fact, it shows that if we require elements of a Minkowski base to be movable, then the Minkowski base is unique and it corresponds exactly to the set of rays in the secondary fan. Therefore, finding all indecomposable polytopes coming from movable divisors recovers this Minkowski base. This is exactly what the algorithm given in the next section does.

**Proof.** First note that we can assume $D$ to be nef on $X$, since being movable, it is the pullback of a nef divisor under a small modification $f : X \to X'$ and the polytopes agree, since $f$ does not alter the rays in the fan defining $X$, but only changes higher dimensional cones.

Let us consider $X_{P_D}$, the variety given by the normal fan of $P_D$. Note that $\Sigma$ is a refinement of $\Sigma_{P_D}$, in particular we obtain a proper toric morphism $\varphi : X \to X_{P_D}$.

Let us now suppose that $D$ spans an extremal ray of $\text{Nef}(X)$ and that $P = Q + N$ is a Minkowski decomposition. Since $P_D$ is a full dimensional polytope with respect to $X_{P_D}$, this means that there exist nontrivial basepoint free $T$-invariant, hence nef, Cartier divisors $A$ and $B$ on $X_{P_D}$ such that $Q = P_A$ and $N = P_B$.

But now we have that $\varphi^*(A), \varphi^*(B)$ are nef divisors on $X$ and since toric polytopes remain invariant via pullback, we have

$$P_D = P_{\varphi^*(A)} + P_{\varphi^*(B)} \quad \text{and} \quad D = \varphi^*(A) + \varphi^*(B)$$

hence we get a contradiction.

For the opposite implication, assume $D$ does not lie in an extremal ray of $\text{Nef}(X)$. We can then find non-trivial nef divisors $A$ and $B$ such that $D = A + B$. Now for globally generated divisors we know that the polytope of a sum coincides with the Minkowski sum of the polytopes, i.e.,

$$P_D = P_A + P_B,$$

showing that $P_D$ is decomposable. □
3.2. How to find a Minkowski base

Let us consider a toric variety $X = X_\Sigma$, $\dim(X) = n$ and let $d = \#(\Sigma(1)) - n$, where as usual $\Sigma(1)$ denotes the set of rays in $\Sigma$. We will first informally describe the idea of the algorithm determining the Minkowski base of $X_\Sigma$ consisting of movable divisors.

A formal description of the algorithm follows afterwards. In the following, $R_m$ will always be a set of rays of the fan $\Sigma$ and $R^*_m$ denotes the set of half-spaces dual to given rays in $R_m$. Furthermore, $P_m$ will be the set of points corresponding to the vertices of $R^*_m$. $\Delta_i = CH(P_m)$ will denote the convex hull generated by the points in $P_m$.

Let $H_t(\rho_i) = \{x \in \mathbb{R} | \langle x, \rho_i \rangle \geq -t \}$ and $\overline{H}_t(\rho_i) = \{x \in \mathbb{R} | \langle x, \rho_i \rangle = -t \}$.

- Fix a cone $\sigma \in \Sigma(n)$ and consider its dual cone $\sigma^\vee$.
  Let $R_1 = \{\rho \in \Sigma(1) \setminus \sigma(1)\}$, $R^*_1 = \{\rho^\vee | \rho \in \sigma(1)\}$.
  Set $d = \#(R_1)$, $R_1 = \{\rho_1\}_i$.

- ($\ast$) For every $\rho_i \in R_1$, we consider the half-spaces $H_1(\rho_i)$.
  Define : $P_2 = \overline{H}_0(\rho_i) \cap \{\rho | \rho \in \sigma(1)\}$; $R_2 = R_1 \setminus \{\rho_i\}$; $m = 2$.

- ($\ast \ast$) For every $\rho_i \in R_m$ do either of the following:
  - Let $t = \max(0, \min\{t' \in \mathbb{R} | H_{t'}(\rho_i) \supseteq \Delta_{m-1}\})$;
    set $R_{m+1} = R_m \setminus \{\rho_i\}$, $R^*_m = R^*_{m-1} \cap H_t(\rho_i)$;
    If $m = d$, check if $\Delta_m$ corresponds to a divisor. If it does, return the divisor (this is a Minkowski base element).
    If $m < d$, set $m = m + 1$ and go to ($\ast \ast$).
  - Set $R_{m+1} = R_m \setminus \{\rho_i\}$, $R^*_m = R^*_{m-1} \cap H_0(\rho_i)$;
    If $m = d$, check if $\Delta_m$ corresponds to a divisor. If it does, return the divisor (this is a Minkowski base element).
    If $m < d$, set $m = m + 1$ and go to ($\ast \ast$).

- Iterate until all possible combinations in ($\ast$) and ($\ast \ast$) are exhausted.

We now give a formal description of the above sketch which we in fact implemented for actual computations.

Algorithm 3.5. Algorithm TMB

Input: fan $\Sigma$ with $n + d$ rays
Output: Minkowski base for $X_\Sigma$

Variables:
- $R$, an array of length $d$; each entry is a set of rays.
- $M$, an array of length $d$; each entry is a set of rays.
- $P$, an array of length $d$; each entry is a set of points.
- $\Delta$, an array of length $d$; each entry is a polyhedron.

Uses:
- CorrespondsToDivisor checks if the given polytope arise as a polytope of a $T$-invariant divisor

for $k$ from 1 to $d$ do
  $R_k \leftarrow \Sigma(1)$;
  $M_k, \Delta_k \leftarrow \emptyset$;
  $P_k \leftarrow \emptyset$;
end do;
for \( l \) from 1 to \( d \) do
    \( R_1 \leftarrow \Sigma(1) \setminus \rho_l \); \\
    \( \Delta_1 \leftarrow H_1(\rho_l) \cap \sigma^\nu ; \)
    if (CorrespondsToDivisor(\( \Delta_1 \))) then \\
        \( TMB \leftarrow TMB \cup \{ \Delta_1 \} \);
    end if;
    \( P_2 \leftarrow \text{Vertices}(\Delta_1) ; \)
    while (\( R_2 \neq \emptyset \)) do
        \( m \leftarrow \max \{ n \mid P_n \neq \emptyset \} ; \)
        if (\( M_m = \emptyset \)) then \\
            Pick \( \rho \in R_m ; \)
            \( R_m \leftarrow R_m \setminus \rho ; \)
            for \( k \) from \( m + 1 \) to \( d \) do \\
                \( R_k \leftarrow R_{m-1} \setminus \rho ; \)
                \( M_k \leftarrow \emptyset ; \)
            end do;
            \( P_m \leftarrow \text{Vertices}(\Delta_{m-1}) ; \)
            \( M_m \leftarrow \rho ; \)
        end if;
        \( t \leftarrow \min_{\rho \in P_m} (\text{Solve}_s(\rho)) ; \)
        if (\( t > 0 \)) then \\
            \( \Delta_m \leftarrow \Delta_{m-1} \cap H_t(\rho) ; \)
            if (CorrespondsToDivisor(\( \Delta_m \))) then \\
                \( TMB \leftarrow TMB \cup \{ \Delta_m \} ; \)
            end if;
            \( P_m \leftarrow \{ 0 \} ; \)
        else \\
            \( \Delta_m \leftarrow \Delta_{m-1} \cap H_0(\rho) ; \)
            if (CorrespondsToDivisor(\( \Delta_m \))) then \\
                \( TMB \leftarrow TMB \cup \{ \Delta_m \} ; \)
            end if;
            \( s \leftarrow \max \{ n \leq m-1 \mid R_n \neq \emptyset \} ; \)
            if (\( m < s \)) then \\
                \( P_m \leftarrow \emptyset ; \)
                \( P_{m+1} \leftarrow \text{Vertices}(\Delta_m) ; \)
                \( M_m \leftarrow \emptyset ; \)
            else \\
                \( M_m \leftarrow \emptyset ; \)
                \( P_m \leftarrow \emptyset ; \)
                if (\( P_s = \emptyset \)) then \\
                    \( P_s \leftarrow \text{Vertices}(\Delta_{s-1}) ; \)
                end if;
            end if;
        end if;
    end do;
end do;
return TMB;
3.3. How the algorithm works

In this subsection we will prove correctness of the algorithm. Note that it clearly terminates.

**Proposition 3.6.** Every polytope $\Delta$ in the output of the algorithm is indecomposable (as a sum of polytopes having the origin as a vertex).

*Proof.* The idea behind the algorithm is quite straightforward: every “slope” appearing in some toric polyhedron has to occur in the polytope of at least one of the Minkowski base elements. Let us recall the steps of the algorithm:

- We fix a $T$-invariant flag, i.e., we choose a set of $d = \dim(X)$ rays that generate one of the cones $\sigma \in \Sigma(d)$. This cone will correspond to a cone $\sigma^\vee$ in the dual space $M_R$. Let us denote by $\Sigma_0$ the fan (not complete) generated by the rays of $\sigma$.

- In the first step we fix a slope, i.e., a ray $\rho_1 \in \Sigma(1) \setminus \sigma(1)$. To this ray corresponds an hyperplane $H_1(\rho_1)$.

- $H_1(\rho_1)$ intersects an edge of $\sigma^\vee$ if and only if the corresponding hyperplanes generate a convex cone in $N_R$. If it intersects all the rays, then we found a simplex, hence a Minkowski base element. Either way, we call the resulting polyhedron $\Delta_1$ and the corresponding fan $\Sigma_1$.

- If $\Delta_1$ is not a (bounded) polytope we need to add an additional ray, say $\rho_2$, into $\Sigma_1$, i.e., intersect our polyhedron with an additional half-space. There are two different ways in which we intersect in the algorithm: either take $\Delta_2$ to be the intersection of $\Delta_1$ with the half-space $H_t(\rho_2)$, where $t$ is the minimal positive number such that all vertices of $\Delta_1$ are contained in $H_t(\rho_2)$ or with the half-space $H_0(\rho_2)$.

- We keep intersecting until all $\rho \in \Sigma_1 \setminus \sigma(1)$ are exhausted. Let us denote by $\Delta_k$ the region bounded by the already existing hyperplanes after $k$ steps.

We first prove inductively that in each step of the algorithm, the fan $\Sigma_k$ remains minimal, in the sense that it is not the refinement of any other fan with the same convex span. In particular this means that as soon $\Delta_k$ is bounded, then it is indecomposable by Proposition 2.5.

$\Sigma_0$ is clearly minimal. Let us suppose that $\Sigma_k$ for $k \geq 0$ is a minimal fan. When adding a face to the dual polyhedron $\Delta_k$ that is tangent to a vertex of $\Delta_k$, this is equivalent to:

- cancel in the fan all the cones corresponding to faces completely contained in the complement of the half-space, so that the resulting fan is minimal again

- cancel in the fan the cone corresponding to the vertex (and all of its faces but the rays)

- add a ray that is not contained in the convex span of the rays we are left with

- construct all the possible cones with the new ray and the rays of the last cone we canceled
• if there is a containment of cones, then the ray of the inner one that is properly contained gets canceled from the new fan (we do not admit star subdivisions)

Is the new fan minimal? Yes, in fact:

• by construction we cannot cancel any ray for the new cones we have, since we already canceled all the star subdivisions

• none of the other rays can get canceled by minimality of the fan at the previous stage.

□

Remark 3.7. Of course, not every indecomposable $T$-invariant polytope is found by the algorithm, but only the convex ones with a vertex on the origin. In case the variety admits a flip then it is possible that not all the indecomposable polytopes corresponding to extremal rays of the movable cone will have the origin as a vertex for a fixed flag. To overcome this issue, it will be enough to go through all the possible flags, given by a different $T$-invariant maximal cone, run again the algorithm and compare the possibly extra elements with the base found in the previous steps (Example 4.2). Note that if we know that one of the cones cannot be flipped, then the associated flag will give the complete set of Minkowski base elements running just once the algorithm (Example 4.3).

Proposition 3.8. Let $X = X_\Sigma$ be a toric variety and let $Y_\bullet$ be a flag given by the complete intersection of the generators of a maximal subcone. Let $D$ be a $T$-invariant divisor such that no element of the chosen flag is contained in its base locus. Then if $P_D$ is indecomposable then $D$ is in the output of the algorithm.

Proof. This is given by the construction of the polytopes in the algorithm.

Denote by $\sigma$ the cone in $M_\mathbb{R}$ corresponding to the flag $Y_\bullet$. Without loss of generality, we can assume $D$ to be a member of its linear series with $D|_\sigma = 0$. This guarantees $P_D$ to be contained in the dual cone $\sigma^\vee$. Write $D = \sum a_i D_i$ where the $D_i$’s are the prime $T$-invariant divisors corresponding to the rays $\rho_1, \ldots, \rho_s$ of $\Sigma_X$. We can assume by reordering that the first $s$ of the $D_i$’s are those which correspond to facets of $P_D$. We describe how the algorithm finds $P_D$.

Let us define $\Delta_1$ to be $\sigma^\vee$. As the first step we intersect $\Delta_1$ with all the half-spaces $H_0(\rho_i)$ where $P_D$ is contained in $\overline{H_0(\rho_i)}$, say for all $s + 1 \leq i \leq s'$ to obtain $\Delta_2$. Note that $P_D$ is an indecomposable full dimensional lattice polytope in the intersection of $N_\mathbb{R}$ with these half-spaces, thus in the dual lattice $M'$ of this intersection its normal fan is not the refinement of any complete fan by Proposition 2.5.

Let now $\Delta_3$ be the intersection of $\Delta_2$ with $H_{a_j}(\rho_j)$ for which $a_j = 0$ and $j \leq s$. This will give the smallest cone containing $P_D$.

Since the fan is convex and complete, we can choose a ray $\rho_1$ that is contained in a cone adjacent to $\Delta_3^\vee$ and such that the induced completion in the span of the fan is contained in $\Sigma_{P_D}$. Up to rescaling we can assume that $a_1 = 1$. This will be the starting point for the iterative part of the algorithm. Let us define $P_1 = \{\text{vertices of } \overline{H_1(D_1)} \cap \Delta_3\}$. Then, choosing the ray $\rho_2$ with the same criterion, if $t = \inf\{s|H_s(\rho_2) \supseteq P_1\}$, then by construction $t = a_2$. Iterating this step with increasing each time the number of intersection points the polytope will be fully reconstructed.
4. Examples

4.1. The blow-up of $\mathbb{P}^2$ in two points

Let us consider the blow-up of $\mathbb{P}^2$ in two points. The fan $\Sigma$ is given by 5 rays, corresponding to divisor classes $D_1 = H - E_1$, $D_2 = H - E_2$, $D_3 = H - E_1 - E_2$, $E_1$ and $E_2$. Let us fix the flag given by $Y_\bullet = \{ D_3, D_3 \cap E_1 \}$.

We can now run the algorithm. In the following table it is recorded what is saved in each step in $R$, $M$, $P$ and $D$.

| $R$ | $M$ | $P$ | $D$ |
|-----|-----|-----|-----|
| $[D_1, D_2, E_2, [D_1, D_2, E_2, [D_1, D_2, E_2]]$ | $[\emptyset, [\emptyset], [\emptyset]]$ | $[\emptyset]$ | $\varnothing$ |
| $[D_2, E_2, [D_2, E_2, [D_2, E_2]]$ | $[D_1, [\emptyset], [\emptyset]]$ | $[\emptyset, [0, 0], (0, 0), (0, 0)]$ | $D_1$ |
| $[D_2, E_2, [E_2, [E_2]]$ | $[D_2, [\emptyset], [\emptyset]]$ | $[\emptyset, [0, 0], [0, 0], (0, 0)]$ | $D_1$ |
| $[D_2, E_2, [E_2, [E_2]]$ | $[D_2, [\emptyset], [\emptyset]]$ | $[\emptyset, [0, 0], [0, 0], (0, 0)]$ | $D_1$ |
| $[D_2, E_2, [E_2, [E_2]]$ | $[D_2, [\emptyset], [\emptyset]]$ | $[\emptyset, [0, 0], [0, 0], (0, 0)]$ | $D_1$ |
| $[D_2, E_2, [E_2, [E_2]]$ | $[D_2, [\emptyset], [\emptyset]]$ | $[\emptyset, [0, 0], [0, 0], (0, 0)]$ | $D_1$ |
| $[D_2, E_2, [D_2, [D_2]]$ | $[D_2, [\emptyset], [\emptyset]]$ | $[\emptyset, [0, 0], [0, 0], (0, 0)]$ | $D_1$ |
| $[D_2, E_2, [E_2, [E_2]]$ | $[D_2, [\emptyset], [\emptyset]]$ | $[\emptyset, [0, 0], [0, 0], (0, 0)]$ | $D_1$ |
| $[D_2, E_2, [D_2, [D_2]]$ | $[D_2, [\emptyset], [\emptyset]]$ | $[\emptyset, [0, 0], [0, 0], (0, 0)]$ | $D_1$ |
| $[D_2, E_2, [E_2, [E_2]]$ | $[D_2, [\emptyset], [\emptyset]]$ | $[\emptyset, [0, 0], [0, 0], (0, 0)]$ | $D_1$ |
| $[D_2, E_2, [D_2, [D_2]]$ | $[D_2, [\emptyset], [\emptyset]]$ | $[\emptyset, [0, 0], [0, 0], (0, 0)]$ | $D_1$ |
| $[D_1, E_2, [D_1, E_2, [D_1, E_2]]$ | $[D_2, [\emptyset], [\emptyset]]$ | $[\emptyset, [0, 0], (1, 1), [\emptyset]]$ | $D_1$ |
| $[D_1, E_2, [E_1, [E_2]]$ | $[D_2, [\emptyset], [\emptyset]]$ | $[\emptyset, [0, 0], [0, 0], (0, 0)]$ | $D_2 + E_2$ |
| $[D_1, E_2, [E_1, [E_2]]$ | $[D_2, [\emptyset], [\emptyset]]$ | $[\emptyset, [0, 0], [0, 0], (0, 0)]$ | $D_2$ |
| $[D_1, E_2, [E_1, [E_2]]$ | $[D_2, [\emptyset], [\emptyset]]$ | $[\emptyset, [0, 0], [0, 0], (0, 0)]$ | $D_2$ |

...
When there is no output for $D$ (in the table: $-$), it means that there is no divisor corresponding to the particular configuration of generating hyperplanes.

To make the table more readable, we made pictures for the steps marked with a number.

In this case $\Delta = P_D$ with $D = D_2 + E_2 \in \text{MBE}$.

**Remark 4.1.** In the 2-dimensional case there is no need of repeating the algorithm since the nef and the movable cone coincide.
4.2. The blow-up of $\mathbb{P}^3$ in two intersecting lines

Let us consider the blow-up of $\mathbb{P}^3$ in two intersecting lines. The fan $\Sigma$ contains 6 rays which are spanned by the points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(-1,-1,-1)$, $(0,-1,-1)$, $(-1,0,-1)$, and correspond to divisor classes $D_1 = H - E_1$, $D_2 = H - E_2$, $D_3 = H$, $D_4 = H - E_1 - E_2$, $E_1$ and $E_2$ respectively. Let us fix the flag given by $Y_\bullet = \{D_1, D_1 \cap D_2, D_1 \cap D_2 \cap E_1\}$.

Since this variety admits flips (reversing the order of the blow-ups) and we have chosen a cone that can be flipped, we expect not to find all the generators of the moving cone at the first stage.
Remark 4.2. Since we have chosen a flag based on a cone that can be flipped, there are two special features appearing:

- There are extremal rays of the movable cone missing. In fact not every movable divisor will have χ^0d as local section, whereas this is true for nef divisors. This will be remedied by looking at another T-invariant flag and completing with the new Minkowski base elements found.

- The divisor \( D_4 + D_3 = 2H - E_1 - E2 \) still corresponds to an irreducible polytope, since it is only decomposable by movable divisors \( 2H - E_1 - E2 = (H - E_1) + (H - E_2) \) and additivity of polytopes is only respected by nef divisors.
4.3. Flip invariant flag

In the setting of the previous example, we see that choosing a flag not affected by any of the possible flips of the variety we obtain all the generators just in one step.

Choosing the flag \( Y_\bullet = \{ D_1, D_1 \cap D_2, D_1 \cap D_2 \cap D_3 \} \) we obtain all base elements.

| \( R \) | \( M \) | \( P \) | \( D \) |
|---|---|---|---|
| \([E_1, E_2], [E_2], [\emptyset]\) | \([D_4], [E_1], [E_2]\) | \([\emptyset], [\emptyset], [D_4]\) | \( D_4 + E_1 + E_2 \) |
| \([E_1, E_2], [E_2], [\emptyset]\) | \([D_4], [E_1], [E_2]\) | \([\emptyset], [\emptyset], [D_4]\) | \( D_4 + E_1 \) |
| \([E_1, E_2], [E_2], [\emptyset]\) | \([D_4], [E_1], [E_2]\) | \([\emptyset], [\emptyset], [D_4]\) | \( D_4 + E_2 \) |
| \([E_1, E_2], [E_2], [\emptyset]\) | \([D_4], [E_1], [E_2]\) | \([\emptyset], [\emptyset], [D_4]\) | \( - \) |
| \([D_4, E_2], [D_4], [\emptyset]\) | \([E_1], [E_2], [D_4]\) | \([\emptyset], [\emptyset], [D_4]\) | \( 2D_4 + E_1 + E_2 \) |
| \( \ldots \) | \( \ldots \) | \( \ldots \) | \( \ldots \) |

5. Finding the decomposition

Let us now turn to the problem of finding the Minkowski decomposition of a given \( T \)-invariant big divisor \( D \) once the Minkowski base for \( X \) is determined by the algorithm.

Two different equivalent approaches are possible: we can either determine the cone of the secondary fan in which the movable part \( M_D \) of \( D \) lies and then write \( M_D \) as a non-negative linear combination of its extremal rays, or we can invest knowledge of the Okounkov body \( \Delta_{Y_\bullet}(D) \) with respect to some \( T \)-invariant flag and go backwards by finding Minkowski summands of this polytope. Note that the first option can be quite challenging since it presumes knowledge of the whole structure of the secondary fan as well as of the decomposition of \( D \) into fixed and movable part. On the other hand the input for the second option comes naturally, we just have to describe how to find the Minkowski summands of a polytope, but this is given by Proposition 2.5 as follows.

Given the divisor \( D \), we consider its polytope \( P_D \) and the corresponding dual fan \( \Sigma_D \). Now, a Minkowski base element \( B \) is a summand in the Minkowski decomposition of \( D \) if and only if its fan \( \Sigma_B \) is refined by \( \Sigma_D \). This is a straightforward check. Note also that the fans \( \Sigma_D \) and \( \Sigma_B \) only dependent on the linear equivalence classes. We thus obtain all classes of Minkowski base elements with representative \( B_i \) having positive coefficient \( a_i \) in the Minkowski decomposition \( M_D = \sum a_i B_i \). Note that from this data we can read off important information that in general can be hard to obtain: in particular for a moving divisor \( D \) the above procedure immediately tells us in which cone of the secondary fan \( D \) lies.
Example 5.1. Let us consider $X$ the blow-up of $\mathbb{P}^2$ in two points. We denote the exceptional divisors with $E_1, E_2$ and the class of the pullback of a general line by $H$. The corresponding fan is spanned by the following rays in $\mathbb{R}^2$.

\[
\begin{array}{c|c}
\Sigma \in N & D_2 \\
E_2 & D_1 \\
D_3 & E_1
\end{array}
\]

It is easy to see that the nef cone $\text{Nef}(X)$ is spanned by the classes $M_1 := H$, $M_2 := H - E_1$ and $M_3 := H - E_2$. By Theorem 3.1 these classes form a Minkowski base with respect to torus-invariant flags. The normal fans of their polytopes are depicted below.

Let us now decompose the $T$-invariant divisor $D := D_1 + D_2 + E_1$. It has the following polytope $P_D$ and corresponding normal fan $\Sigma_{P_D}$.

Now, $\Sigma_{P_D}$ is the refinement of the fans $\Sigma_{M_1}$ and $\Sigma_{M_3}$ but not of $\Sigma_{M_2}$. Thus $D$ decomposes as a positive linear combination of $M_1$ and $M_3$. We easily obtain the decomposition $D \sim M_1 + M_3 = H + (H - E_2)$. 

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