Review Article

Nontrivial Solutions for the $2n^{\text{th}}$ Lidstone Boundary Value Problem

Yaohong Li,1 Jiafa Xu2, and Yongli Zan3

1School of Mathematics and Statistics, Suzhou University, Suzhou 234000, Anhui, China
2School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, China
3School of Mathematics and Statistics, Qilu University of Technology (Shandong Academy of Sciences), Jinan 250353, Shandong, China

Correspondence should be addressed to Jiafa Xu; xujiafa292@sina.com

Received 1 September 2020; Revised 28 September 2020; Accepted 17 October 2020; Published 4 November 2020

1. Introduction

In this paper, we investigate the existence of nontrivial solutions for the following $2n^{\text{th}}$ Lidstone boundary value problem with a sign-changing nonlinearity:

\[
\begin{aligned}
(-1)^n u^{(2n)}(t) &= f(t, u(t), -u''(t), \ldots, (-1)^{n-1} u^{(2n-2)}(t)), \quad 0 < t < 1, \\
u^{(2i)}(0) &= u^{(2i)}(1) = 0, \\
i &= 0, 1, \ldots, n-1,
\end{aligned}
\]

where the nonlinearity $f$ satisfies the following condition:

(i) $f \in C([0, 1] \times \mathbb{R}^n, \mathbb{R})$ and there exist three nonnegative functions $a(t), b(t),$ and $K(x_1, x_2, \ldots, x_n) = 0$ such that

\[f(t, x_1, x_2, \ldots, x_n) \leq a(t) - b(t)K(x_1, x_2, \ldots, x_n),\]

for $t \in [0, 1], x_i \in \mathbb{R}, i = 1, 2, \ldots, n.$

The Lidstone boundary value problem arises in many different areas of applied mathematics and physics. When $n = 2$, problem (1) describes the deformation of an elastic beam in which both ends are simply supported. Recently, this problem has been extensively studied, and the authors refer the reader to [1–11] and references cited therein. For example, in [1], the authors used a cone-theoretic fixed point theorem to study the existence of nontrivial solutions for the nonlinear Lidstone boundary value problem:
\[
y^{(2m)}(t) = \lambda a(t)f\left(y(t), \ldots, y^{(2\beta)}(t), \ldots, y^{(2(m-1))}(t)\right), \quad 0 < t < 1,
y^{(2\alpha)}(0) = y^{(2\alpha)}(1),
\]

where \((-1)^m f > 0\) is continuous and \(a\) is nonnegative. In [2], the authors investigated the existence and uniqueness of positive solutions for the following generalized Lidstone boundary value problem:

\[
\begin{align*}
(-1)^m u^{(2m)} &= f\left(t, u, u''(0), \ldots, (-1)^{\nu_i-1} u^{(2m-2)}(0)\right), \\
\alpha_0 u^{(2\beta)}(0) - \beta_0 u^{(2\beta+1)}(0) &= 0 (i = 0, 1, 2, \ldots, n - 1), \\
\alpha_1 u^{(2\beta)}(1) - \beta_1 u^{(2\beta+1)}(1) &= 0 (i = 0, 1, 2, \ldots, n - 1),
\end{align*}
\]

(4)

where \(\alpha_i \geq 0, \beta_i \geq 0 (j = 0, 1)\) and \(\alpha_0 \alpha_1 + \alpha_1 \beta_1 + \alpha_0 \beta_0 > 0\). In view of symmetry, these results demonstrate that problem (4) is essentially identical with Dirichlet boundary condition (1).

Meanwhile, we also note that there are a large number of papers in the literature devoted to sign-changing nonlinearities, and some results can be found in a series of papers [12–32] and the references cited therein. For example, in [12], the authors studied the following higher-order nonlinear fractional boundary value problem involving Riemann–Liouville fractional derivatives:

\[
\begin{align*}
D_{\alpha}^\nu u(t) &= -f\left(t, u(t), D_{\alpha}^p u(t), D_{\alpha}^{p+1} u(t), \ldots, D_{\alpha}^{p+n} u(t)\right), \quad 0 < t < 1, \\
u(0) &= u'(0) = \cdots = u^{(n-1)}(0) = 0,
\end{align*}
\]

(5)

where \(f\) is a sign-changing nonlinearity. Under some appropriate conditions involving the eigenvalues of the relevant linear operators, they utilized the topological degree to obtain a nontrivial solution for (5). In [13], the authors adopted the similar method in [12] to study the existence of nontrivial solutions for the following system of fractional \(q\)-difference equations with \(q\)-integral boundary conditions:

\[
\begin{align*}
D_{\nu}^\alpha x(t) + f_1(t, y(t)) &= 0, \quad t \in (0, 1), \\
D_{\nu}^\beta y(t) + f_2(t, x(t)) &= 0, \quad t \in (0, 1), \\
x(0) &= 0, D_{\nu} x(0) = 0, D_{\nu}^\alpha x(1) = \int_0^t h(t) D_{\nu}^\alpha x(t) dt, \\
y(0) &= 0, D_{\nu} y(0) = 0, D_{\nu}^\beta y(1) = \int_0^t h(t) D_{\nu}^\beta y(t) dt,
\end{align*}
\]

(6)

where \(\alpha \in (2, 3), \nu \in (1, 2),\) and \(D_{\nu}^\alpha\) is the \(\alpha\)-order Riemann–Liouville’s fractional \(q\)-derivative.

Inspired by the aforementioned works, in this paper, we study the existence of nontrivial solutions for (1) where the nonlinearity \(f\) is sign-changing. Under some conditions involving the eigenvalues of the relevant linear operators, we use the topological degree to obtain our results.

2. Preliminaries

Let \(E = C([0, 1], \mathbb{R}), \|v\| = \max_{t \in [0, 1]} |v(t)|, P = \{v \in E: v(t) \geq 0, \forall t \in [0, 1]\}, B_r = \{v \in E: \|v\| < r\} \) for \(r > 0\). Clearly, \((E, \|\cdot\|)\) is a real Banach space and \(P\) is a solid cone in \(E\). In (1), let \((-1)^\nu u^{(2m-2)}(t) = v(t)\) and from [2, P224], we can obtain that (1) is equivalent to the following integral equation:

\[
v(t) = \int_0^1 G_1(t, s)f\left(s, \int_0^1 G_{n-1}(s, r)v(r) dr, \ldots, \int_0^1 G_1(s, r)v(r) dr, v(s)\right) ds,
\]

(7)

where

\[
\begin{align*}
G_1(t, s) &= \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1, \\
\end{cases} \\
G_i(t, s) &= \int_0^1 G_1(t, r)G_{i-1}(r, s) dr, \quad i = 2, 3, \ldots,
\end{align*}
\]

(8)

Next, we provide a lemma, which expresses some vital properties of the functions \(G_i(i = 1, 2, \ldots)\).

Lemma 1 (i) \(G_i\) are nonnegative continuous functions on \([0, 1]^2\), and \(G_i(t, s) > 0, (t, s) \in (0, 1)^2\)

(ii) \(G_3\) has the inequalities \(t(1-t)G_3(s, t) \leq G_3(t, s) \leq G_3(s, s), \forall t, s \in [0, 1]\)

(iii) \(G_i(t, s) = G_i(s, t), \forall t, s \in [0, 1]\)

(iv) \[\int_0^1 G_i(t, s) \sin \pi t ds = (1/\pi^2)^i \sin \pi t, t \in [0, 1], \quad \text{and} \]

\[\int_0^1 G_i(t, s) \sin \pi nt ds = (1/\pi^2)^i \sin \pi ns, s \in [0, 1]\]

Proof. We only prove (iii) and (iv). For (iii), \(i = 1\) holds obviously. From the definition of \(G_i\), we have
Using the symmetry of $\alpha$ this completes the proof.

For (iv), when $i = 1$, we have
\[
\int_0^1 G_1(t,s)\sin \pi s \, ds = \int_0^1 s(1-t)\sin \pi s \, ds + \int_t^1 (1-s)\sin \pi s \, ds = \frac{1}{\pi} \sin \pi t.
\]
(10)

Noting that $G_1(t,s) = G_1(t,s)$, we have
\[
\int_0^1 G_1(t,s)\sin \pi t \, ds = \int_0^1 G_1(t,s)\sin \pi t \, dt = \frac{1}{\pi} \sin \pi s.
\]
(11)

When $i \geq 1$, we have
\[
\int_0^1 G_i(t,s)\sin \pi ds = \int_0^1 \cdots \int_0^1 G_1(t,\tau_{i-1}) \cdots G_1(\tau_3,\tau_2) G_1(\tau_2,\tau_1) G_1(\tau_1,s) \sin \pi s ds dr_{i-1} \cdots dr_2 dr_1
\]
\[
= \frac{1}{\pi^i} \int_0^1 \cdots \int_0^1 G_1(t,\tau_{i-1}) \cdots G_1(\tau_3,\tau_2) G_1(\tau_2,\tau_1) \sin \pi \tau dr_{i-1} \cdots dr_2 dr_1
\]
(12)

Using the symmetry of $G_i$, we easily have
\[
\int_0^1 G_i(t,s)\sin \pi t dt = \left(\frac{1}{\pi^i}\right)^i \sin \pi s.
\]
(13)

This completes the proof.

Let $\alpha_i \geq 0$ with $\sum_{i=1}^n \alpha_i^2 \neq 0$. Then, we have the following equations:
\[
\int_0^1 G_{n_1,\ldots,n_n}(t,s)\sin \pi s ds = \left[\alpha_1\left(\frac{1}{\pi^i}\right)^n + \cdots + \alpha_n\left(\frac{1}{\pi^i}\right)^1\right] \sin \pi t,
\]
\[t \in [0,1],
\]
(14)
\[
\int_0^1 G_{n_1,\ldots,n_n}(t,s)\sin \pi t dt = \left[\alpha_1\left(\frac{1}{\pi^i}\right)^n + \cdots + \alpha_n\left(\frac{1}{\pi^i}\right)^1\right] \sin \pi s,
\]
\[s \in [0,1],
\]
(15)

where
\[
G_{n_1,\ldots,n_n}(t,s) = \alpha_1 G_n(t,s) + \cdots + \alpha_n G_1(t,s), \text{ for } t,s \in [0,1].
\]
(16)

**Lemma 2.** Let $(Lv)(t) = \int_0^1 G_1(t,s)v(s) ds$, for $t \in [0,1]$. Then, if $v \in P$, we have $Lv \in P_0$, where
\[
P_0 = \{v \in P : \|v(t)\| \geq t(1-t)\|v\|, \forall t \in [0,1]\}.
\]
(17)

This is a direct result from Lemma 1 (ii), so we omit its proof.

**Remark 1.** $\sin \pi t \in P_0$, for $t \in [0,1]$.

**Lemma 3** (see [33], Theorem 1 [3]). Let $\Omega$ be a bounded open set in a Banach space $E$ and $T : \Omega \rightarrow E$ be a continuous compact operator. If there exists $x_0 \in E \setminus \{0\}$ such that
\[ x - Tx \neq \mu x_0, \quad \forall x \in \partial \Omega, \mu \geq 0, \]
then the topological degree \( \text{deg}(I - T, \Omega, 0) = 0. \)

**Lemma 4** (see [33], Lemma 4 [1]). Let \( \Omega \) be a bounded open set in a Banach space \( E \) with \( 0 \in \Omega \) and \( T: \Omega \rightarrow E \) be a continuous compact operator. If

\[ Tx \neq \mu x, \quad \forall x \in \partial \Omega, \mu \geq 1, \]
then the topological degree \( \text{deg}(I - T, \Omega, 0) = 1. \)

### 3. Main Results

Define the operator \( A: E \rightarrow E \) by

\[ (Av)(t) = \int_0^1 G_1(t, s)f(s, \int_0^1 G_{n-1}(s, r)v(r)dr, \ldots, \int_0^1 G_1(s, r)v(r)dr, v(s))ds \]

Moreover, the continuity of \( f \) implies that \( A \) is completely continuous and the existence of solutions for (1) is equivalent to that of fixed points of \( A. \)

Now, we list some assumptions for the functions \( f \) and \( K: \)

(C1) There exist \( \beta_i \geq 0 \) (i = 1, 2, \ldots, n) with \( \sum_{i=1}^{n} \beta_i^2 \neq 0 \) such that

\[ K(x_1, x_2, \ldots, x_n) \rightarrow \infty \quad \beta_i |x_1| + \beta_2 |x_2| + \cdots + \beta_n |x_n| = 0. \]

We denote by

\[ \lambda_{\beta_1, \beta_2, \ldots, \beta_n} = \beta_1 \left( \frac{1}{\pi^2} \right)^n + \cdots + \beta_n \left( \frac{1}{\pi^2} \right)^1, \]

\[ \lambda_{\gamma_1, \gamma_2, \ldots, \gamma_n} = \gamma_1 \left( \frac{1}{\pi^2} \right)^n + \cdots + \gamma_n \left( \frac{1}{\pi^2} \right)^1. \]

**Theorem 1.** Suppose that (C0)–(C3) hold. Then, (1) has at least one nontrivial solution.

**Proof.** We divide the following two steps:

(i) **Step 1.** By (C3), there exist \( \varepsilon_i \in (0, \lambda_{\gamma_1, \gamma_2, \ldots, \gamma_n}) \) and \( r > 0 \) such that

\[ |f(t, x_1, x_2, \ldots, x_n)| \leq \lambda_{\gamma_1, \gamma_2, \ldots, \gamma_n} (x_1) + |y_2| |x_2| + \cdots + |y_n| |x_n| \leq r. \]

Substituting this into (20), we have

\[ \|(Av)(t)\| \leq \int_0^1 G_1(t, s) \left| f(s, \int_0^1 G_{n-1}(s, r)v(r)dr, \ldots, \int_0^1 G_1(s, r)v(r)dr, v(s)) \right| ds \]

\[ \leq \lambda_{\gamma_1, \gamma_2, \ldots, \gamma_n} \int_0^1 G_1(t, s) \left| \int_0^1 G_{n-1}(s, r)v(r)dr \right| + \cdots + \int_0^1 G_1(s, r)v(r)dr + |v_n| v(s) \right| ds \]

\[ \leq (\lambda_{\gamma_1, \gamma_2, \ldots, \gamma_n} - \varepsilon) \int_0^1 G_1(t, s) \left| y_1 \int_0^1 G_{n-1}(s, r)v(r)dr \right| + \cdots + \int_0^1 G_1(s, r)v(r)dr + |v_n| v(s) \right| ds \]

Now for this \( r \), we claim

\[ Av \neq \lambda v, \quad \forall \lambda \in \partial B, \lambda \geq 1. \]

Suppose the contrary. Then, there exist \( v_0 \in \partial B \) and \( \lambda_0 \geq 1 \) such that \( Av_0 = \lambda_0 v_0 \). Therefore, we obtain

\[ |v_0(t)| \leq \lambda_0 |v_0(t)| = |(Av_0)(t)| \leq (\lambda_{\gamma_1, \gamma_2, \ldots, \gamma_n} - \varepsilon) \int_0^1 G_{\gamma_1, \gamma_2, \ldots, \gamma_n}(t, s)v_0(s)ds. \]

Multiply by \( \sin \pi t \) on both sides and integrate over \( [0, 1] \) and use (15) to obtain...
\[ \int_0^1 |v_0(t)| \sin \pi t \, dt \leq \left( \lambda_1 \gamma_1 \gamma_2 - \gamma_n \right) - \varepsilon_1 \int_0^1 \int_0^s G_1(y, s) \, |v_0(s)| \sin \pi t \, ds \]

(28)

This indicates that \( \int_0^1 |v_0(t)| \sin \pi t \, dt = 0 \), and thus, \( |v_0(t)| \equiv 0, t \in [0, 1] \). This contradicts to \( v_0 \in \partial B_r \). Consequently, (26) holds and Lemma 4 yields that

\[ \deg(I - A, B_r, 0) = 1. \]

(29)

**Step 2.** By virtue of (C2), there exist \( \varepsilon_2 > 0 \) and \( X_0 > 0 \) such that

\[ f(t, x_1, x_2, \ldots, x_n) \geq \left( \lambda_{\beta_1 \beta_2 \ldots \beta_n} + \varepsilon_2 \right) \left( |\beta_1 x_1| + |\beta_2 x_2| + \cdots + |\beta_n x_n| \right) - a(t) - b(t)K(x_1, x_2, \ldots, x_n) \]

\[ \geq \left( \lambda_{\beta_1 \beta_2 \ldots \beta_n} + \varepsilon_2 \right) \left( |\beta_1 x_1| + |\beta_2 x_2| + \cdots + |\beta_n x_n| \right) - a(t) - b(t) \left( |\beta_1 x_1| + |\beta_2 x_2| + \cdots + |\beta_n x_n| \right) \]

\[ \geq \left( \lambda_{\beta_1 \beta_2 \ldots \beta_n} + \varepsilon_2 - \|b\| \varepsilon \right) \left( |\beta_1 x_1| + |\beta_2 x_2| + \cdots + |\beta_n x_n| \right) - a(t). \]

(30)

Noting that when \( t \in [0, 1], x_1, x_2, \ldots, x_n \) are bounded, we can let

\[ C_{X_1} = \max_{0 \leq s \leq 1, \beta_1 |x_1| + \beta_2 |x_2| + \cdots + \beta_n |x_n| \leq X_1} \left| f(t, x_1, x_2, \ldots, x_n) \right| \]

\[ + \left( \lambda_{\beta_1 \beta_2 \ldots \beta_n} + \varepsilon_2 - \|b\| \varepsilon \right) X_1, \]

(33)

Then, we can obtain

\[ f(t, x_1, x_2, \ldots, x_n) \geq \left( \lambda_{\beta_1 \beta_2 \ldots \beta_n} + \varepsilon_2 - \|b\| \varepsilon \right) \left( |\beta_1 x_1| + |\beta_2 x_2| + \cdots + |\beta_n x_n| \right) - a(t) - C_{X_1} \]

(34)

\[ S_1 = \frac{\int_0^1 \left( \lambda_{\beta_1 \beta_2 \ldots \beta_n} + \varepsilon_2 - \|b\| \varepsilon \right) (y, y) \left( a(y) + K^* b(y) + C_{X_1} \right) \, dy}{\left( \varepsilon_2 - \|b\| \varepsilon \right) \left( 1 - \|b\| \max_{0 \leq t \leq 1} \int_0^t G_{\beta_1 \ldots \beta_n} (t, r) \, dr \right) - \varepsilon_2 - \|b\| \varepsilon \int_0^1 b(y) \sum_{n=1}^N \int_0^1 G_{\beta_1 \ldots \beta_n} (y, d) \, dy}, \]

(36)

and \( G_{\varphi}(y, t) \equiv 1 \). Now, we claim

\[ v - Av \neq \mu \varphi, \quad \forall v \in \partial B_R, \mu \geq 0, \]

(37)

where \( \varphi(t) = \sin \pi t, t \in [0, 1] \). Suppose that (37) is not satisfied. Then, there exist \( v_1 \in \partial B_R \) and \( \mu_1 > 0 \) such that

\[ v_1 - Av_1 = \mu_1 \varphi. \]

(38)
Let
\[
\bar{v}(t) = \int_0^1 G_1(t,s) \left[ a(s) + b(s)K \left( \int_0^{t} G_{n-1}(s,r)v_1(r)dr, \ldots, \int_0^{t} G_1(s,r)v_1(r)dr, v_1(s) \right) + C_{X_1} \right] ds.
\]
(39)

Then, from Lemma 2, we have \( \bar{v} \in P_0 \). Now, we estimate the norm of \( \bar{v} \). Noting that \( v_1 \in \partial B_R(\|v_1\| = R) \), we obtain
\[
\bar{v}(t) \leq \int_0^1 G_1(t,s) \left[ a(s) + b(s) \left( \lambda_1^{\mu_1} \int_0^{t} G_{n-1}(s,r)v_1(r)dr + \ldots + \lambda_n^{\mu_1} \int_0^{t} G_1(s,r)v_1(r)dr + \beta_n \|v_1(s)\| + K^* \right) + C_{X_1} \right] ds
\]
\[
\leq \int_0^1 G_1(t,s) \left[ a(s) + K^* b(s) + C_{X_1} \right] ds + \|b\| \|v_1\| \max_{t \in [0,1]} \int_0^1 G_{n-1}(t,r)v_1(r)dr
\]
\[
< R.
\]
(40)

We calculate \( v_1 + \bar{v} \). By (38) we have
\[
v_1(t) + \bar{v}(t) = (Av_1)(t) + \mu_1 \varphi(t) + \bar{v}(t)
\]
\[
= \int_0^1 G_1(t,s) \left[ f(s, \int_0^{t} G_{n-1}(s,r)v_1(r)dr, \ldots, \int_0^{t} G_1(s,r)v_1(r)dr, v_1(s) \right]
\]
\[
+ a(s) + b(s)K \left( \int_0^{t} G_{n-1}(s,r)v_1(r)dr, \ldots, \int_0^{t} G_1(s,r)v_1(r)dr, v_1(s) \right) + C_{X_1} \right] ds + \mu_1 \varphi(t).
\]
(41)

Using (C0), Lemma 2 and Remark 1, we find
\[
v_1 + \bar{v} \in P_0,
\]
(42)

Therefore, (34) enables us to calculate
\[
(Av_1)(t) + \bar{v}(t) \geq \left( \lambda_1^{\mu_1} \varphi_1 + \varepsilon_2 - \|b\| \varepsilon_1 \right) \int_0^1 G_1(t,s) \left[ \beta_1^{\mu_1} \int_0^{t} G_{n-1}(s,r)v_1(r)dr + \ldots + \beta_n^{\mu_1} \int_0^{t} G_1(s,r)v_1(r)dr + \beta_n \|v_1(s)\| \right] ds
\]
\[
- \int_0^1 G_1(t,s) \left( a(s) + C_{X_1} \right) ds
\]
\[
+ \int_0^1 G_1(t,s) \left( a(s) + b(s)K \left( \int_0^{t} G_{n-1}(s,r)v_1(r)dr, \ldots, \int_0^{t} G_1(s,r)v_1(r)dr, v_1(s) \right) + C_{X_1} \right) ds
\]
\[
\geq \left( \lambda_1^{\mu_1} \varphi_1 + \varepsilon_2 - \|b\| \varepsilon_1 \right) \int_0^1 G_1(t,s) \left[ \beta_1^{\mu_1} \int_0^{t} G_{n-1}(s,r)v_1(r)dr + \ldots + \beta_n^{\mu_1} \int_0^{t} G_1(s,r)v_1(r)dr + \beta_n \|v_1(s)\| \right] ds
\]
\[
= \left( \lambda_1^{\mu_1} \varphi_1 + \varepsilon_2 - \|b\| \varepsilon_1 \right) \int_0^1 G_{n-1}(t,s)v_1(s) ds
\]
\[
\geq \left( \lambda_1^{\mu_1} \varphi_1 + \varepsilon_2 - \|b\| \varepsilon_1 \right) \int_0^1 G_{n-1}(t,s)v_1(s) ds.
\]
(43)
Consequently, we have
\[(\lambda_{\beta_1, \beta_2, \ldots, \beta_n} + \varepsilon_2 - \|b\|e) \int_0^1 G_{\beta_1, \ldots, \beta_n}(t, s) (v_1(s) + \bar{v}(s)) \, ds\]
\[= (\lambda_{\beta_1, \beta_2, \ldots, \beta_n} + \varepsilon_2 - \|b\|e) \int_0^1 G_{\beta_1, \ldots, \beta_n}(t, s) (v_1(s) + \bar{v}(s)) \, ds\]
\[= \lambda_{\beta_1, \beta_2, \ldots, \beta_n} \int_0^1 G_{\beta_1, \ldots, \beta_n}(t, s) (v_1(s) + \bar{v}(s)) \, ds + (\varepsilon_2 - \|b\|e) \int_0^1 G_{\beta_1, \ldots, \beta_n}(t, s) \bar{v}(s) \, ds\]

(44)

Now, we estimate
\[(\varepsilon_2 - \|b\|e) \int_0^1 G_{\beta_1, \ldots, \beta_n}(t, s) (v_1(s) + \bar{v}(s)) \, ds - (\lambda_{\beta_1, \beta_2, \ldots, \beta_n} + \varepsilon_2 - \|b\|e) \int_0^1 G_{\beta_1, \ldots, \beta_n}(t, s) \bar{v}(s) \, ds\]
\[\geq (\varepsilon_2 - \|b\|e) \int_0^1 G_{\beta_1, \ldots, \beta_n}(t, s) (v_1(s) + \bar{v}(s)) \, ds\]
\[\leq (\lambda_{\beta_1, \beta_2, \ldots, \beta_n} + \varepsilon_2 - \|b\|e) \int_0^1 G_{\beta_1, \ldots, \beta_n}(t, s) \bar{v}(s) \, ds\]
\[= \lambda_{\beta_1, \beta_2, \ldots, \beta_n} \int_0^1 G_{\beta_1, \ldots, \beta_n}(t, s) (v_1(s) + \bar{v}(s)) \, ds + (\varepsilon_2 - \|b\|e) \int_0^1 G_{\beta_1, \ldots, \beta_n}(t, s) \bar{v}(s) \, ds\]

(45)

As a result, by means of (43) and (44), we obtain
\[(Av_1)(t) + \bar{v}(t) \geq \lambda_{\beta_1, \beta_2, \ldots, \beta_n} \int_0^1 G_{\beta_1, \ldots, \beta_n}(t, s) (v_1(s) + \bar{v}(s)) \, ds\]
\[(46)\]

Combining this with (38), we have
\[v_1(t) + \bar{v}(t) = (Av_1)(t) + \mu_1 \varphi(t) + \bar{v}(t)\]
\[\geq \lambda_{\beta_1, \beta_2, \ldots, \beta_n} \int_0^1 G_{\beta_1, \ldots, \beta_n}(t, s) (v_1(s) + \bar{v}(s)) \, ds + \mu_1 \varphi(t)\]
\[\geq \mu_1 \varphi(t)\]

(47)

Define \(\mu^* = \sup S = \sup\{\mu > 0 : v_1 + \bar{v} \geq \mu \varphi\}\). Then, \(S \neq \emptyset\) (\(\mu_1 \in S\)) and \(\mu^* \geq \mu_1\). Hence, from (14), we have
\[(\mu^* - \mu_1) \varphi(t) = \mu_1 \varphi(t)\]
Therefore, the operator $A$ has at least one fixed point in $\overline{B_r} \setminus \overline{B_e}$. Equivalently, (1) has at least one nontrivial solution. This completes the proof.

4. Conclusion

In this paper, we use the topological degree to study the nontrivial solutions for the $2n$th Lidstone boundary value problem (1). To the best of our knowledge, there are few works that deal with the problem where the nonlinear terms may be unbounded and sign-changing. Moreover, it is remarked that the main result is discussed under some conditions concerning the first eigenvalues corresponding to the relevant linear operators. These mean that our main result is an improvement in some related works.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors’ Contributions

The study was carried out in collaboration among all authors. All authors read and approved the final manuscript.

Acknowledgments

This work was supported by University Natural Science Foundation of Anhui Provincial Education Department (Grant nos. KJ2019A0666 and KJ2018A0452), the Foundation of Suzhou University (Grant no. 2016XJGG13), Technology Research Foundation of Chongqing Educational Committee (Grant nos. KJQN201800533 and KJQN202000528), and Qilu University of Technology (Shandong Academy of Sciences) Young Doctor Cooperative Funding (Grant no. 2017BSHZ012).

References

[1] P. Eloe, “Nonlinear eigenvalue problems for higher order lidstone boundary value problems,” Electronic Journal of Qualitative Theory of Differential Equations, vol. 2, no. 2, 8 pages, 2000.

[2] Z. Yang, “Existence and uniqueness of positive solutions for a higher order boundary value problem,” Computers & Mathematics with Applications, vol. 54, no. 2, pp. 220–228, 2007.

[3] R. P. Agarwal and P. J. Y. Wong, “Lidstone polynomials and boundary value problems,” Computers & Mathematics with Applications, vol. 17, no. 10, pp. 1397–1421, 1989.

[4] R. P. Agarwal, D. O’Regan, and S. Staněk, “Singular lidstone boundary value problem with given maximal values for solutions,” Nonlinear Analysis: Theory, Methods & Applications, vol. 55, no. 7-8, pp. 859–881, 2003.

[5] Z. Bai and W. Ge, “Solutions of 2n-th Lidstone boundary value problems and dependence on higher order derivatives,” Journal of Mathematical Analysis and Applications, vol. 279, no. 2, pp. 442–450, 2003.

[6] J. M. Davis, P. W. Eloe, and J. Henderson, “Triple positive solutions and dependence on higher order derivatives,” Journal of Mathematical Analysis and Applications, vol. 237, no. 2, pp. 710–720, 1999.

[7] J. Ehme and J. Henderson, “Existence and local uniqueness for nonlinear lidstone boundary value problems,” Journal of Inequalities in Pure and Applied Mathematics (JIPAM), vol. 1, no. 8, p. 2000.

[8] Y. Ma, “Existence of positive solutions of lidstone boundary value problems,” Journal of Mathematical Analysis and Applications, vol. 314, no. 1, pp. 97–108, 2006.

[9] Y.-M. Wang, “On 2n-th-order lidstone boundary value problems,” Journal of Mathematical Analysis and Applications, vol. 312, no. 2, pp. 383–400, 2005.

[10] Z. Yang, “Existence of positive solutions for a system of generalized lidstone problems,” Computers & Mathematics with Applications, vol. 60, no. 3, pp. 501–510, 2010.

[11] X. Hao, D. O’Regan, and J. Xu, “Positive solutions for a system of 2n-th-order boundary value problems involving semi-positone nonlinearities,” Journal of Inequalities and Applications, vol. 2020, p. 20, 2020.

[12] K. Zhang, D. O’Regan, J. Xu, and Z. Fu, “Nontrivial solutions for a higher order nonlinear fractional boundary value problem involving riemann-liouville fractional derivatives,” Journal of Function Spaces, vol. 2019, Article ID 2381530, 11 pages, 2019.

[13] Y. Li, J. Liu, D. O’Regan, and J. Xu, “Nontrivial solutions for a system of fractional q-difference equations involving q-integral boundary conditions,” Mathematics, vol. 8, no. 5, p. 828, 2020.

[14] Z. Fu, S. Bai, D. O’Regan, and J. Xu, “Nontrivial solutions for an integral boundary value problem involving riemann-liouville fractional derivatives,” Journal of Inequalities and Applications, vol. 2019, p. 104, 2019.

[15] B. Liu, J. Li, and L. Liu, “Nontrivial solutions for a boundary value problem with integral boundary conditions,” Boundary Value Problems, vol. 2014, p. 15, 2014.

[16] K. Zhang, D. O’Regan, and Z. Fu, “Nontrivial solutions for boundary value problems of a fourth order difference equation with sign-changing nonlinearity,” Advances in Difference Equations, vol. 2018, p. 370, 2018.

[17] W. Fan, X. Hao, L. Liu, and Y. Wu, “Nontrivial solutions of singular fourth-order Sturm-Liouville boundary value problems with a sign-changing nonlinear term,” Applied Mathematics and Computation, vol. 217, no. 15, pp. 6700–6708, 2011.

[18] J. Xu, J. Jiang, and D. O’Regan, “Positive solutions for a class of p-laplacian hadamard fractional-order three-point boundary value problems,” Mathematics, vol. 8, no. 3, p. 308, 2020.

[19] H. Zhang, Y. Li, and J. Xu, “Positive solutions for a system of fractional integral boundary value problems involving hadamard-type fractional derivatives,” Complexity, vol. 2019, p. 11, Article ID 2671539, 2019.

[20] B. Liu and Y. Liu, “Positive solutions of a two-point boundary value problem for singular fractional differential equations in Banach space,” Journal of Function Spaces, vol. 2013, p. 9, Article ID 585639, 2013.

[21] Y. Liu and H. Yu, “Bifurcation of positive solutions for a class of boundary value problems of fractional differential inclusions,” Abstract and Applied Analysis, vol. 2013, p. 8, Article ID 942831, 2013.
[22] Y. Liu, "Positive solutions using bifurcation techniques for boundary value problems of fractional differential equations," *Abstract and Applied Analysis*, vol. 2013, p. 7, Article ID 162418, 2013.

[23] T. Qi, Y. Liu, and Y. Cui, "Existence of solutions for a class of coupled fractional differential systems with nonlocal boundary conditions," *Journal of Function Spaces*, vol. 2017, p. 9, Article ID 6703860, 2017.

[24] T. Qi, Y. Liu, and Y. Zou, "Existence result for a class of coupled fractional differential systems with integral boundary value conditions," *The Journal of Nonlinear Sciences and Applications*, vol. 10, no. 07, pp. 4034–4045, 2017.

[25] Y. Wang, Y. Liu, and Y. Cui, "Multiple solutions for a nonlinear fractional boundary value problem via critical point theory," *Journal of Function Spaces*, vol. 2017, Article ID 8548975, 8 pages, 2017.

[26] Y. Wang, Y. Liu, and Y. Cui, "Multiple sign-changing solutions for nonlinear fractional kirchhoff equations," *Boundary Value Problems*, vol. 2018, p. 193, 2018.

[27] Y. Liu and D. O'Regan, "Controllability of impulsive functional differential systems with nonlocal conditions," *Electronic Journal of Differential Equations*, vol. 19410 pages, 2013.

[28] Y. Liu, "Bifurcation techniques for a class of boundary value problems of fractional impulsive differential equations," *Journal of Nonlinear Sciences and Applications*, vol. 09, no. 04, pp. 340–353, 2015.

[29] Y. Wang, Y. Liu, and Y. Cui, "Infinitely many solutions for impulsive fractional boundary value problem with p-laplacian," *Boundary Value Problems*, vol. 2018, p. 94, 2018.

[30] J. Li, Y. Cheng, and Z. Li, "Superconvergence of the composite rectangle rule for computing hypersingular integral on interval," *Numerical Mathematics: Theory, Methods and Applications*, vol. 13, no. 3, pp. 770–787, 2020.

[31] J. Li and Y. Cheng, "Linear barycentric rational collocation method for solving second-order volterra integro-differential equation," *Computational and Applied Mathematics*, vol. 39, no. 2, p. 92, 2020.

[32] J. Li and Y. Cheng, "Linear barycentric rational collocation method for solving heat conduction equation," *Numerical Methods for Partial Differential Equations*, in press, 2020.

[33] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Orlando, FL, USA, 1988.