Moment bounds for autocovariance matrices under dependence

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Abstract

The goal of this paper is to obtain expectation bounds for the deviation of large sample autocovariance matrices from their means under weak data dependence. While the accuracy of covariance matrix estimation corresponding to independent data has been well understood, much less is known in the case of dependent data. We make a step towards filling this gap, and establish deviation bounds that depend only on the parameters controlling the “intrinsic dimension” of the data up to some logarithmic terms. Our results have immediate impacts on high dimensional time series analysis, and we apply them to high dimensional linear VAR(d) model, vector-valued ARCH model, and a model used in Banna et al. (2016).

Keywords: Autocovariance matrix, effective rank, weak dependence, τ-mixing.

Mathematical subject classification (2000): 60E15, 60F10.

1 Introduction

Consider a sequence of $p$-dimensional mean-zero random vectors $\{Y_t\}_{t \in \mathbb{Z}}$ and a size-$n$ fraction $\{Y_i\}_{i=1}^n$ of it. This paper aims to establish moment bounds for the spectral norm deviation of lag-$m$ autocovariances of $\{Y_i\}_{i=1}^n$, $\hat{\Sigma}_m := (n-m)^{-1} \sum_{i=1}^{n-m} Y_{i+m} Y_i^T$, from their mean values.

A first result at the origin of such problems concerns product measures, with $m = 0$ and $\{Y_i\}_{i=1}^n$ independent and identically distributed (i.i.d.). For this, Rudelson (1999) derived a bound on $E\|\hat{\Sigma}_0 - E\hat{\Sigma}_0\|$, where $\|\cdot\|$ represents the spectral norm for matrices. The technique is based on symmetrization and the derived maximal inequality is a consequence of a concentration inequality on a “symmetrized” version of $p \times p$ symmetric and deterministic matrices, $A_1, \ldots, A_n$ (cf. Oliveira (2010)). That is, for any $x \geq 0$,

$$
P\left(\left\| \sum_{i=1}^n \epsilon_i A_i \right\| \geq x\right) \leq 2p \exp\{-x^2/(2\sigma^2)\}, \quad \sigma^2 := \left\| \sum_{i=1}^n A_i^2 \right\|,
$$

where $\{\epsilon_i\}_{i=1}^n$ are independent and taking values $\{-1, 1\}$ with equal probability. The applicability of this technique then hinges on the assumption that the data are i.i.d..

Later, Vershynin (2012), Srivastava and Vershynin (2013), Mendelson and Paouris (2014), Lounici (2014), Bunea and Xiao (2015), Tikhomirov (2017), among many others, derived different
types of deviation bounds for $\hat{\Sigma}_0$ under different distributional assumptions. For example, Lounici (2014) and Bunea and Xiao (2015) showed that, for such $\{Y_i\}_{i=1}^n$ that are subgaussian and i.i.d.,

$$E \|\hat{\Sigma}_0 - \Sigma_0\| \leq C\|\Sigma_0\|\left\{\sqrt{r(\Sigma_0)\log(ep)\frac{n}{n}} + \frac{r(\Sigma_0)\log(ep)}{n}\right\}. \tag{1.2}$$

Here $C > 0$ is a universal constant, $\Sigma_0 := EY_1Y_1^T$, and $r(\Sigma_0) := \text{Tr}(\Sigma_0)/\|\Sigma_0\|$ is termed the “effective rank” (Vershynin, 2012) where $\text{Tr}(X) := \sum_{i=1}^p X_{i,i}$ for any real $p \times p$ matrix $X$.

Statistically speaking, Equation (1.2) is of rich implications. For example, combining (1.2) with Davis-Kahan inequality (Davis and Kahan, 1970) suggests that the principal component analysis (PCA), a core statistical method whose aim is to recover the leading eigenvectors of $\Sigma_0$, could still produce consistent estimators even if the dimension $p$ is much larger than the sample size $n$, as long as the “intrinsic dimension” of the data, quantified by $r(\Sigma_0)$, is small enough. See Section 1 in Han and Liu (2018) for more discussions on the statistical performance of PCA in high dimensions.

The main goal of this paper is to give extensions of the deviation inequality (1.2) to large autocovariance matrices, where the matrices are constructed from a high dimensional structural time series. Examples of such time series include linear vector autoregressive model of lag $d$ (VAR($d$)), vector-valued autoregressive conditionally heteroscedastic (ARCH) model, and a model used in Banna et al. (2016). The main result appears below as Theorem 2.1, and is nonasymptotic in its nature. This result will have important consequences in high dimensional time series analysis. For example, it immediately yields new analysis for estimating large covariance matrix (Chen et al., 2013), a new proof of consistency for Brillinger’s PCA in the frequency domain (cf. Chapter 9 in Brillinger (2001)), and we envision that it could facilitate a new proof of consistency for the PCA procedure proposed in Chang et al. (2018).

The rest of the paper is organized as follows. Section 2 characterizes the settings and gives the main concentration inequality for large autocovariance matrices. In Section 3, we present applications of our results to some specific time series models. Proofs of the main results are given in Section 4, with more relegated to an appendix.

## 2 Main results

We first introduce the notation that will be used in this paper. Without further specification, we use bold, italic lower case alphabets to denote vectors, e.g., $u = (u_1, \cdots, u_p)^T$ as a $p$-dimensional real vector, and $\|u\|_2$ as its vector $L_2$ norm. We use bold, upper case alphabets to denote matrices, e.g., $X = (X_{i,j})$ as a $p \times p$ real matrix, and $I_p$ as the $p \times p$ identity matrix. Throughout the paper, let $c, c', C, C', C''$ be generic universal constants, whose actual values may vary at different locations. For any two sequences of positive numbers $\{a_n\}, \{b_n\}$, we denote $a_n = O(b_n)$ if there exists an universal constant $C$ such that $a_n \leq Cb_n$ for all $n$ large enough. We write $a_n \asymp b_n$ if both $a_n = O(b_n)$ and $b_n = O(a_n)$ hold.

Consider a time series $\{Y_t\}_{t \in \mathbb{Z}}$ of $p$-dimensional real entries $Y_t \in \mathbb{R}^p$ with $\mathbb{R}, \mathbb{Z}$ denoting the sets of real and integer numbers respectively. In the sequel, the considered time series does not need to be stationary nor centered, and we are focused on a size-$n$ fraction of it. Without loss of generality, we denote this fraction to be $\{Y_i\}_{i=1}^n$. 

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As described in the introduction, the case of independent \( \{Y_t\}_{t=1}^n \) has been discussed in depth in recent years. We are interested here in the time series setting, and our main emphasis will be to describe nontrivial but easy to verify cases for which Inequality (1.2) still holds. The following four assumptions are accordingly made, with the notation that

\[
S^p := \{ x \in \mathbb{R}^p : \| x \|_2 = 1 \}, \quad \hat{S}^p := \{ x \in \mathbb{R}^p : | x_1 | = \cdots = | x_p | = 1 \},
\]

and

\[
\| X \|_{L(p)} := (\mathbb{E}|X|^p)^{1/p}, \quad \| X \|_{\psi_2} := \inf \{ k \in (0, \infty) : \mathbb{E}[\exp\{ (|X|/k)^2 \}] - 1 \leq 1 \}
\]

for any random variable \( X \).

(A1) Assume that

\[
\kappa_1 := \sup_{t \in \mathbb{Z}} \sup_{u \in S^{p-1}} \| u^T Y_t \|_{\psi_2} < \infty, \quad \kappa_s := \sup_{t \in \mathbb{Z}} \sup_{u \in \hat{S}^{p-1}} \| u^T Y_t \|_{\psi_2} < \infty.
\]

(A2) Assume, for any integer \( j \), there exists a sequence of random vectors \( \{ \tilde{Y}_t \}_{t > j} \) which is independent of \( \sigma(\{ Y_t \}_{t \leq j}) \), identically distributed as \( \{ Y_t \}_{t > j} \), and for any integer \( k \geq j + 1 \),

\[
\| Y_k - \tilde{Y}_k \|_2 \leq \gamma_1 \kappa_1 \exp\{-\gamma_2 (k - j - 1)\}
\]

for some constants \( \gamma_1, \gamma_2, \epsilon > 0 \).

(A3) Assume, for any integer \( j \), there exists a sequence of random vectors \( \{ \tilde{Y}_t \}_{t > j} \) which is independent of \( \sigma(\{ Y_t \}_{t \leq j}) \), identically distributed as \( \{ Y_t \}_{t > j} \), and for any integer \( k \geq j + 1 \),

\[
\sup_{u \in S^{p-1}} \| (Y_k - \tilde{Y}_k)^T u \|_{L(1+\epsilon)} \leq \gamma_3 \kappa_s \exp\{-\gamma_4 (k - j - 1)\}
\]

for some constants \( \gamma_3, \gamma_4 > 0 \).

(A4) Assume there exists an universal constant \( c > 0 \) such that, for all \( t \in \mathbb{Z} \) and for all \( u \in \mathbb{R}^p \),

\[
\| u^T Y_t \|_{\psi_2}^2 \leq c \mathbb{E}(u^T Y_t)^2.
\]

Two observations are in order. We first reveal the relationship between \( \kappa_1, \kappa_s \) and the effective rank highlighted in (1.2). As a matter of fact, it is easy to see, as \( Y_t \sim N(0, \Sigma_0) \), \( \kappa_1^2 \) and \( \kappa_s^2 \) scale at the same orders of \( \| \Sigma_0 \| \) and \( \text{Tr} (\Sigma_0) \), and the same observation applies to all subgaussian distributions with the additional condition (A4), which is identical to Assumption 1 in Lounici (2014). Hence, \( \kappa_s^2/\kappa_1^2 \) could be pictured as a generalized “effective rank”. In the following, this important ratio will be denoted as \( r^* \).

Secondly, we note that Assumptions (A2) and (A3) are characterizing the intrinsic coupling property of the sequence. In practice, such couples can be constructed from time to time. Consider, for example, the following causal shift model,

\[
Y_t = H_t(\xi_t, \xi_{t-1}, \xi_{t-2}, \ldots),
\]

where \( \{ \xi_t \}_{t \in \mathbb{Z}} \) consists of independent elements with values in a measurable space \( \mathcal{X} \) and \( H_t : \mathcal{X}^{\mathbb{Z}^+} \to \mathbb{R}^p \) is a vector-valued function. Then it is natural to consider

\[
\tilde{Y}_t = H_t(\xi_t, \ldots, \xi_{j+1}, \tilde{\xi}_j, \tilde{\xi}_{j-1}, \ldots)
\]
The following is the main result of this paper.

**Theorem 2.1.** Let \( \{Y_t\}_{t \in \mathbb{Z}} \) be a sequence of random vectors satisfying Assumptions (A1)-(A3) and recall \( r^* = \kappa_2^2/\kappa_1^2 \). Assume \( \gamma_1 = O(\sqrt{r^*}) \) and \( \gamma_3 = O(1) \). Then, for any integer \( n \geq 2 \) and \( 0 \leq m \leq n - 1 \), we have

\[
E \| \hat{\Sigma}_m - \hat{\Sigma}_m \| \leq C \kappa_1^2 \left\{ \sqrt{\frac{r_* \log ep}{n - m}} + \frac{r_* \log ep (\log np)^3}{n - m} \right\}
\]

for some constant \( C \) only depending on \( \epsilon, m, \gamma_2, \gamma_4 \). If in addition, \( \{Y_t\}_{t \in \mathbb{Z}} \) is secondarily stationary of mean-zero random vectors and Assumption (A4) holds, then

\[
E \| \hat{\Sigma}_m - \hat{\Sigma}_m \| \leq C' \| \Sigma_0 \| \left\{ \sqrt{\frac{r(\Sigma_0) \log ep}{n - m}} + \frac{r(\Sigma_0) \log ep (\log np)^3}{n - m} \right\}
\]

for some constant \( C' \) only depending on \( \epsilon, c, m, \gamma_2, \gamma_4 \).

We first comment on the temporal correlatedness conditions, Assumptions (A2) and (A3). We note that they correspond exactly to the \( \delta \)-measure of dependence introduced in Chapter 3 of Dedecker et al. (2007), for the sequence \( \{Y_t\}_{t \in \mathbb{Z}} \) and \( \{u^TY_t\}_{t \in \mathbb{Z}} \) respectively. In addition, as will be seen soon, our measure of dependence is also very related to the \( \tau \)-measure introduced in Dedecker and Prieur (2004). In particular, ours is usually stronger than, but as \( \epsilon \to 0 \), reduces to the \( \tau \)-measure. Lastly, our conditions are also quite connected to the functional dependence measure in Wu (2005), on which many moment inequalities in real space have been established (cf. Liu et al. (2013) and Wu and Wu (2016)). However, it is still unclear if a similar matrix Bernstein inequality could be developed under Weibiao Wu’s functional dependence condition.

Secondly, we note that one is ready to verify that Inequality (2.1) gives the exact control of the deviation from the mean. Actually, Inequality (2.1) is nearly a strict extension of the results in Lounici (Lounici, 2014) and Bunea and Xiao (Bunea and Xiao, 2015) to weak data dependence up to some logarithmic terms.

Admittedly, it is still unclear if Inequality (2.1) could be further improved under the given conditions. Recently, in a remarkable series of papers (Koltchinskii and Lounici, 2017a,b,c), Koltchinskii and Lounici showed that, for subgaussian independent data, the extra multiplicative \( p \) term on the righthand side of Inequality (2.1) could be further removed. The proof rests on Talagrand’s majorizing measures (Talagrand, 2014) and a corresponding maximal inequality due to Mendelson (Mendelson, 2010). In the most general case, to the authors’ knowledge, it is still unknown if Talagrand’s approach could be extent to weakly dependent data, although we conjecture that, under stronger temporal dependence (e.g., geometrically \( \phi \)-mixing) conditions, it is possible to recover Koltchinskii and Lounici’s result without resorting to the matrix Bernstein inequality in the proof of Theorem 2.1.

Nevertheless, we make a first step towards eliminating these logarithmic terms via the following theorem. It shows, when assuming a Gaussian sequence is observed, one could further tighten the upper bound in Inequality (2.1) by removing all logarithm factors. The obtained bound is thus tight in view of Theorem 2 in Lounici (2014) and Theorem 4 in Koltchinskii and Lounici (2017a).
Theorem 2.2. Let \( \{Y_t\}_{t \in \mathbb{Z}} \) be a stationary mean-zero Gaussian sequence that satisfies Assumptions (A2)-(A3) with \( \gamma_1 = O(\sqrt{r(\Sigma_0)}), \gamma_3 = O(1) \), and \( \epsilon > 1 \). Then, for any integer \( n \geq 2 \) and \( 0 \leq m \leq n - 1 \),
\[
\mathbb{E} \| \hat{\Sigma}_m - \Sigma_m \| \leq C \| \Sigma_0 \| \left( \sqrt{\frac{r(\Sigma_0)}{n-m}} + \frac{r(\Sigma_0)}{n-m} \right)
\]
for some constant \( C > 0 \) only depending on \( \epsilon, m, \gamma_2, \gamma_4 \).

In a related track of studies, Bai and Yin (1993), Srivastava and Vershynin (2013), Mendelson and Paouris (2014), and Tikhomirov (2017), among many others, explored the optimal scaling requirement in approximating a large covariance matrix for heavy-tailed data. For instance, for i.i.d. data and as \( \Sigma_0 \) is identity, Bai and Yin (Bai and Yin, 1993) showed that \( \| \hat{\Sigma}_0 - \Sigma_0 \| \) will converge to zero in probability as long as \( p/n \to 0 \) and 4-th moments exist. Some recent developments further strengthen the moment requirement. These results cannot be compared to ours. In particular, our analysis is focused on characterizing the role of “effective rank”, a term of strong meanings in statistical implications and a feature that cannot be captured using these alternative procedures.

3 Applications

In this section, we examine the validity of Assumptions (A1)-(A4) in Section 2 under three models, a stable VAR(d) model, a model proposed by Banna et al. (2016), and an ARCH-type model.

We first consider such \( \{Y_t\}_{t \in \mathbb{Z}} \) that is a random sequence generated from VAR(d) model, i.e.,
\[
Y_t = A_1 Y_{t-1} + \cdots + A_d Y_{t-d} + E_t,
\]
where \( \{E_t\}_{t \in \mathbb{Z}} \) is a sequence of independent vectors such that for all \( t \in \mathbb{Z} \) and \( u \in \mathbb{R}^p \),
\[
\|u^T E_t\|_{\psi_2} \leq c' \|u^T E_t\|_{L(2)}
\]
for some universal constant \( c' > 0 \). In addition, assume \( \sup_{t \in \mathbb{Z}} \sup_{u \in \mathbb{R}^{p-1}} \|u^T E_t\|_{\psi_2} < D_1 \) for some universal positive constant \( D_1 < \infty \), \( \|A_k\| \leq a_k < 1 \) for all \( 1 \leq k \leq d \), and \( \sum_{k=1}^d a_k < 1 \), where \( \{a_k\}_{k=1}^d \), \( \rho_1 \) are some universal constants.

Under these conditions, we have the following theorem.

Theorem 3.1. The above \( \{Y_t\}_{t \in \mathbb{Z}} \) satisfies Assumptions (A1)-(A4) with
\[
\gamma_1 = C(\kappa_s/\kappa_1)(\|A\|/\rho_1)^K, \gamma_2 = \log(\rho_1^{-1}), \gamma_3 = C'd(\|A\|/\rho_1)^K, \gamma_4 = \log(\rho_1^{-1}).
\]
Here we denote
\[
\bar{\Lambda} := \begin{bmatrix} a_1 & a_2 & \cdots & a_{d-1} & a_d \\ 1 & 0 & \cdots & 0 & 0 \\ & & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix},
\]
\( \rho_1 \) is a universal constant such that \( \rho(\bar{\Lambda}) < \rho_1 < 1 \) whose existence is guaranteed by the assumption that \( \sum_{k=1}^d a_k < 1 \) (cf. Lemma 4.10 in Section 4), \( K \) is some constant only depending on \( \rho_1 \), and \( C, C' > 0 \) are some constants only depending on \( \epsilon \).

We secondly consider the following time series generation scheme whose corresponding matrix version has been considered by Banna, Merlevède, and Youssef (Banna et al., 2016). In detail, let
{Y_t}_{t \in \mathbb{Z}} be a random sequence generated by

\[ Y_t = W_t E_t, \]

where \( \{E_t\}_{t \in \mathbb{Z}} \) is a sequence of independent random vectors independent of \( \{W_t\}_{t \in \mathbb{Z}} \) such that for all \( t \in \mathbb{Z} \) and \( u \in \mathbb{R}^p, \|u^T E_t\|_{\psi_2} \leq c' \|u^T E_t\|_{L(2)} \) for some universal constant \( c' > 0 \). In addition, we assume

\[
\sup_{t \in \mathbb{Z}} \sup_{u \in \mathbb{R}^{p-1}} \|u^T E_t\|_{\psi_2} \leq \kappa'_1 \quad \text{and} \quad \sup_{t \in \mathbb{Z}} \sup_{v \in \mathbb{R}^{p-1}} \|v^T E_t\|_{\psi_2} \leq \kappa'_s
\]

for some constants \( 0 < \kappa'_1, \kappa'_s < \infty \), \( \{W_t\}_{t \in \mathbb{Z}} \) is a sequence of uniformly bounded \( \tau \)-mixing random variables such that \( \max_{t \in \mathbb{Z}} |W_t| \leq \kappa_W \), and

\[
\tau(k; \{W_t\}_{t \in \mathbb{Z}}, \cdot) \leq \kappa_W \gamma_5 \exp\{-\gamma_6(k - 1)\}
\]

for some constants \( 0 < \gamma_5, \gamma_6, \kappa_W < \infty \).

**Theorem 3.2.** The above \( \{Y_t\}_{t \in \mathbb{Z}} \) satisfies Assumptions (A1)-(A4) with

\[
\gamma_1 = C \kappa'_s \kappa_W \gamma_5^{\frac{1}{1+\epsilon}} / \kappa_1, \quad \gamma_2 = \gamma_6/(1+\epsilon), \quad \gamma_3 = C' \kappa'_s \kappa_W \gamma_5^{\frac{1}{1+\epsilon}} / \kappa_1, \quad \gamma_4 = \gamma_6/(1+\epsilon)
\]

for some constants \( C, C' > 0 \) only depending on \( \epsilon \).

Lastly, we consider a vector-valued ARCH-model with \( \{Y_t\}_{t \in \mathbb{Z}} \) being a random sequence generated by

\[ Y_t = A Y_{t-1} + H(Y_{t-1}) E_t, \]

where \( H: \mathbb{R}^p \rightarrow \mathbb{R}^{p \times p} \) is a matrix-valued function and \( \{E_t\}_{t \in \mathbb{Z}} \) is a sequence of independent random vectors such that

\[
\sup_{t \in \mathbb{Z}} \sup_{u \in \mathbb{R}^{p-1}} \|u^T E_t\|_{\psi_2} \leq \kappa'_1 \quad \text{and} \quad \sup_{t \in \mathbb{Z}} \sup_{v \in \mathbb{R}^{p-1}} \|v^T E_t\|_{\psi_2} \leq \kappa'_s
\]

for some constants \( 0 < \kappa'_1, \kappa'_s < \infty \). Assume further that \( \|A\| \leq a_1 \) and the function \( H(\cdot) \) satisfies

\[
\sup_{u, v \in \mathbb{R}^p} \|H(u) - H(v)\| \leq \frac{a_2}{\kappa'_s} \|u - v\|_2
\]

for some universal constant \( a_1 < 1, a_2 > 0 \) such that \( a_1 + a_2 < 1 \).

**Theorem 3.3.** If the above \( \{Y_t\}_{t \in \mathbb{Z}} \) satisfies Assumption (A1), it satisfies Assumptions (A2)-(A3) with

\[
\gamma_1 = C \kappa_s / \kappa_1, \quad \gamma_2 = - \log(a_1 + a_2), \quad \gamma_3 = C' \max(\kappa_s \kappa'_s / \kappa'_1, 1), \quad \gamma_4 = \log(a_1 + a_2)^{-1}
\]

for some constants \( C, C' > 0 \) only depending on \( \epsilon \). If we further assume the above \( \{Y_t\}_{t \in \mathbb{Z}} \) to be a stationary sequence and \( \sup_{u \in \mathbb{R}^p} \|H(u)\| < D_2 \) for some universal constant \( D_2 < \infty \), then \( \{Y_t\}_{t \in \mathbb{Z}} \) satisfies Assumption (A1).
4 Proofs

4.1 Proof of Theorem 2.1

Proof of Theorem 2.1. The proof depends mainly on the following tail probability bound of deviation of the sample covariance from its mean.

Proposition 4.1. Let \( \{ Y_t \}_{t \in \mathbb{Z}} \) be a sequence of random vectors satisfying (A1)-(A3). For any integer \( n \geq 2 \), integer \( 0 \leq m \leq n - 2 \) and real number \( 0 < \delta \leq 1 \), define

\[
M_\delta := C \max \left\{ \left( \frac{\kappa_*}{\kappa_1} \right)^2 \log \frac{n - m}{\delta}, \left( \frac{\kappa_*}{\kappa_1} \right)^2, \frac{2\kappa_*\gamma_1}{\kappa_1} \right\}.
\]

Then for any \( x \geq 0 \),

\[
\mathbb{P} \left[ \| \hat{\Sigma}_m - \mathbb{E} \hat{\Sigma}_m \| \geq \kappa_1^2 \{ x + \sqrt{\delta/(n - m)} \} \right] \leq 2p \exp \left\{ - \frac{C'(n - m)^2x^2}{A_1(n - m) + A_2M_\delta^2 + A_3(n - m)M_\delta} \right\} + \delta,
\]

with

\[
A_1 := \frac{\kappa_*\gamma_1/\kappa_1 + (\kappa_*/\kappa_1)^2(\gamma_3 + 2m + 1) + 2m + 1}{1 - \exp\left\{ - \min\left( \frac{\kappa_*}{\kappa_1} \gamma_2, \gamma_4 \right) \right\}}, \quad A_2 := \frac{453^2}{\gamma_2}, \quad A_3 := \frac{2 \log(n - m)}{\log 2} \max \left\{ 1, 8m + \frac{48 \log(n - m)p}{\gamma_2} \right\}
\]

for some constants \( C, C' > 0 \) only depending on \( \epsilon \).

Without loss of generality, let \( m = 0 \). Taking \( x = \sqrt{\frac{r_* \log \epsilon}{n} t} \), \( \delta = x^{-\gamma} \) for some \( \gamma > 1 \), \( \gamma_1 = O(\sqrt{r_*}) \), and \( \gamma_3 = O(1) \) in Proposition 4.1, we obtain

\[
\mathbb{P} \left( \| \hat{\Sigma}_0 - \mathbb{E} \hat{\Sigma}_0 \| \geq C_1 \kappa_1^2 \sqrt{\frac{r_* \log \epsilon}{n} t} \right) \leq 2p \exp \left\{ - \frac{C_2(\log \epsilon) t / \{ \log \left( \sqrt{\frac{r_* \log \epsilon}{n} t} \right) \}^2}{1 + r_* (\log n)^2 + \sqrt{r_* \log \epsilon} t (\log \epsilon n)^3} \right\} + x^{-\gamma}
\]

for some constants \( C_1, C_2 > 0 \) only depending on \( \epsilon, \gamma_2, \gamma_4 \).

If \( 1 + \frac{r_* (\log n)^2}{n} \geq \frac{r_* \log \epsilon (\log \epsilon n)^6}{n^2} \), we have

\[
\mathbb{E} \left\| \hat{\Sigma}_0 - \mathbb{E} \hat{\Sigma}_0 \| \right\|^2 \leq 1 + \frac{r_* (\log n)^2}{n} + \int_{1 + \frac{r_* (\log n)^2}{n}}^\infty 2p \exp \left\{ - \frac{C_2(\log \epsilon) t / \{ \log \left( \sqrt{\frac{r_* \log \epsilon}{n} t} \right) \}^2}{1 + r_* (\log n)^2} \right\} dt
\]

\[
+ \int_{1 + \frac{r_* (\log n)^2}{n}}^\infty 2p \exp \left\{ - \frac{C_2(\log \epsilon) \sqrt{t} / \{ \log \left( \sqrt{\frac{r_* \log \epsilon}{n} t} \right) \}^2}{\sqrt{r_* \log \epsilon} (\log \epsilon n)^6} \right\} dt
\]

\[
\leq C_3 \left( 1 + \frac{r_* (\log n)^2}{n} + \frac{r_* \log \epsilon (\log \epsilon n)^6}{n} \right).
\]

This gives that

\[
\mathbb{E} \| \hat{\Sigma}_0 - \mathbb{E} \hat{\Sigma}_0 \| \leq C_4 \kappa_1^2 \left\{ \frac{r_* \log \epsilon}{n} + \frac{r_*^2 (\log \epsilon)^2 (\log \epsilon n)^6}{n^2} \right\}.
\]
On the other hand, if \( 1 + \frac{r_*(\log n)^2}{n} \leq \frac{r_* \log \exp (\log np)^6}{n} \),
\[
\frac{\mathbb{E} \| \hat{\Sigma}_0 - \mathbb{E} \hat{\Sigma}_0 \|^2}{(C_1 \kappa_1^2 \sqrt{r_* \log \exp (\log np)^6})^2} \leq \frac{r_* \log \exp (\log np)^6}{n} + \int_{\frac{r_* \log \exp (\log np)^6}{n}}^{\infty} 2p \exp \left[- \frac{C_2(\log \exp) \sqrt{t}}{\log (\sqrt{r_* \log \exp (\log np)^6})^2} \right] dt \\
\leq C_5 \frac{r_* \log \exp (\log np)^6}{n}.
\]
This renders
\[
\mathbb{E} \| \hat{\Sigma}_0 - \mathbb{E} \hat{\Sigma}_0 \|^2 \leq C_5 \kappa_1^4 \left\{ \frac{r_* (\log np)^2 (\log np)^6}{n^2} \right\}.
\]
Combining two cases gives us the final result by using the simple fact that \( \mathbb{E} \| \hat{\Sigma}_0 - \mathbb{E} \hat{\Sigma}_0 \| \leq (\mathbb{E} \| \hat{\Sigma}_0 - \mathbb{E} \hat{\Sigma}_0 \|^2)^{\frac{1}{2}} \). This completes the proof of the first part of Theorem 2.1.

Notice that under Assumptions (A1), (A4), zero-mean, and stationarity, we have \( \kappa_1^2 \propto \| \Sigma_0 \| \) and \( \kappa_2^2 \propto \text{Tr}(\Sigma_0) \). Thus plugging in Theorem 2.1 finishes the proof. \( \square \)

Now we prove Proposition 4.1 under Assumptions (A1)-(A3). In the proof, the cases for covariance and autocovariance matrices are treated separately. In addition, the proof depends on a Berstein-type inequality for \( \tau \)-mixing random matrices and some related lemmas, whose proofs are presented later.

Given a sequence of random vectors \( \{Y_t\}_{t \in \mathbb{Z}} \), denote \( X_t := Y_t Y_t^T \) for all \( t \in \mathbb{Z} \). Then for any constant \( M > 0 \), we introduce the following “truncated” version of \( X_t \):
\[
X_t^M := \frac{M \wedge \|X_t\|}{\|X_t\|} X_t,
\]
where \( a \wedge b := \min(a, b) \) for any two real numbers \( a, b \).

For any integer \( m > 0 \), we denote \( Z_t^{(m)} := Y_t Y_{t+m}^T \) for all \( t \in \mathbb{Z} \). For the sake of clarification, the superscript “\( (m) \)” is dropped when no confusion is possible. Then the truncated version is
\[
Z_t^M := \frac{M \wedge \|Z_t\|}{\|Z_t\|} Z_t
\]
for any \( M > 0 \).

We further define the “variances” for \( \{X_t^M\}_{i=1}^n \) and \( \{Z_t^M\}_{i=1}^{n-m} \) as
\[
\nu_X^2 := \sup_{K \subseteq \{1, \ldots, n\}} \frac{1}{\text{card}(K)} \lambda_{\max} \left\{ \mathbb{E} \left( \sum_{i \in K} X_i^M - \mathbb{E} X_i^M \right)^2 \right\},
\]
\[
\nu_Z^2 := \sup_{K \subseteq \{1, \ldots, n-m\}} \frac{1}{\text{card}(K)} \left\| \mathbb{E} \left( \sum_{i \in K} Z_i^M - \mathbb{E} Z_i^M \right)^2 \right\|.
\]
Here \( \lambda_{\max}(X) \) and \( \lambda_{\min}(X) \) denote the largest and smallest eigenvalues of \( X \) respectively.

**Proof of Proposition 4.1.** We first assume \( \kappa_1 = 1 \). We consider two cases.

**Case I:** When \( m = 0 \), \( \{X_t\}_{t \in \mathbb{Z}} \) is a sequence of symmetric random matrices. We have,
\[
P \left\{ \frac{1}{n} \left\| \sum_{i=1}^n (X_i - \mathbb{E} X_i) \right\| \geq x \right\}
\]
Lemma 4.2. \( \|X_i - X_i^M - \mathbb{E}X_i^M + \mathbb{E}X_i^M - \mathbb{E}X_i \| \geq x \) 

\[
= \mathbb{P}\left\{ \frac{1}{n} \left\| \sum_{i=1}^{n} (X_i - X_i^M - \mathbb{E}X_i^M + \mathbb{E}X_i^M - \mathbb{E}X_i) \right\| \geq x \right\}
\]

\[
\leq \mathbb{P}\left\{ \frac{1}{n} \left\| \sum_{i=1}^{n} (X_i^M - \mathbb{E}X_i^M + \mathbb{E}X_i^M - \mathbb{E}X_i) \right\| + \frac{1}{n} \left\| \sum_{i=1}^{n} (X_i - X_i^M) \right\| \geq x \right\}
\]

\[
\leq \mathbb{P}\left\{ \left\| \sum_{i=1}^{n} (X_i^M - \mathbb{E}X_i^M) \right\| \geq nx \right\} + \mathbb{P}\left\{ \left\| \sum_{i=1}^{n} (X_i - X_i^M) \right\| > 0 \right\}
\]

\[
\leq \mathbb{P}\left[ \lambda_{\max} \left\{ \sum_{i=1}^{n} (X_i^M - \mathbb{E}X_i^M) \right\} \geq nx - \sum_{i=1}^{n} \left\| \mathbb{E}X_i^M - \mathbb{E}X_i \right\| \right] + 
\]

\[
\mathbb{P}\left[ \lambda_{\min} \left\{ \sum_{i=1}^{n} (X_i^M - \mathbb{E}X_i^M) \right\} \leq -nx + \sum_{i=1}^{n} \left\| \mathbb{E}X_i^M - \mathbb{E}X_i \right\| + \sum_{i=1}^{n} \mathbb{P}(X_i \neq X_i^M) \right].
\]  \( (4.1) \)

We first show that the difference in expectation between the “truncated” \( X_i^{M_\delta} \) and original one \( X_i \) can be controlled with the chosen truncation level \( M_\delta \). For this, we need the following lemma.

**Lemma 4.2.** Let \( \{Y_i\}_{i \in \mathbb{Z}} \) be a sequence of \( p \)-dimensional random vectors under Assumption \( \text{(A1)} \). Then for all \( t \in \mathbb{Z} \) and for all \( x \geq 0 \),

\[
\mathbb{P}(\|Y_i\|_2^2 \geq 2\kappa_x^2 + 8\kappa_x^2(x + \sqrt{x})) \leq \exp(-Cx)
\]

for some arbitrary constant \( C > 0 \).

By applying Lemma 4.2, we obtain that for all \( i \in \{1, \ldots, n\} \),

\[
\|\mathbb{E}X_i^{M_\delta} - \mathbb{E}X_i\| = \left\| \mathbb{E}(1 - \frac{M_\delta}{\|X_i\|})X_i1_{\{\|X_i\| > M_\delta\}} \right\|
\]

\[
\leq \sup_{u, v \in \mathbb{S}^{p-1}} \mathbb{E}[u^T X_i v | 1_{\{\|X_i\| > M_\delta\}}]
\]

\[
\leq \sup_{u, v \in \mathbb{S}^{p-1}} \{\mathbb{E}(u^T Y_i v^T v)^2\}^{\frac{1}{2}} \{\mathbb{P}(\|X_i\| > M_\delta)\}^{\frac{1}{2}}
\]

\[
\leq \sqrt{\delta/n},
\]

where the last line followed by Assumption \( \text{(A1)} \), Lemma 4.2, and the chosen \( M_\delta \).

The second step heavily depends on a Bernstein-type inequality for \( \tau \)-mixing random matrices. The theorem slightly extends the main theorem of Banna et al. (2016) in which the random matrix sequence is assumed to be \( \beta \)-mixing. Its proof is relegated to the Appendix.

**Theorem 4.3.** Consider a sequence of real, mean-zero, symmetric \( p \times p \) random matrices \( \{X_i\}_{i \in \mathbb{Z}} \) with \( \|X_i\| \leq M \) for some positive constant \( M \). In addition, assume that this sequence is \( \tau \)-mixing (see, Appendix Section A.1 for a detailed introduction to the \( \tau \)-mixing coefficient) with geometric decay, i.e.,

\[
\tau(k; \{X_i\}_{i \in \mathbb{Z}}, \| \cdot \|) \leq M\psi_1 \exp\{-\psi_2(k - 1)\}
\]

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Moreover, for some constant $C > 0$, we have
\[
\mathbb{P}\left\{ \lambda_{\max}\left( \sum_{i=1}^{n} X_i \right) \geq x \right\} \leq p \exp\left\{ -\frac{x^2}{8(15^2n^2 + 60^2M^2/\psi^2)} + 2xM\bar{\psi}(\psi_1, \psi_2, n, p) \right\},
\]
where
\[
\nu^2 := \sup_{K \subseteq \{1, \ldots, n\}} \frac{1}{\text{card}(K)} \lambda_{\max}\left\{ \mathbb{E}\left( \sum_{i \in K} X_i \right)^2 \right\}
\]
and $\bar{\nu}(\psi_1, \psi_2, n, p) := \frac{\log n}{\log 2} \max\left\{ 1, \frac{8\log(\psi_1 n^6 p)}{\psi_2} \right\}$.

In order to apply Theorem 4.3, we need the following two lemmas. Lemma 4.4 is to show that the sequence of “truncated” matrices $\{X_i^M\}$ under Assumptions (A1)-(A2) is a $\tau$-mixing random sequence with geometric decay. Lemma 4.5 calculates the upper bound for $\nu^2$ term in Theorem 4.3 for $\{X_i^M\}_{t \in \mathbb{Z}}$.

**Lemma 4.4.** Let $\{Y_t\}_{t \in \mathbb{Z}}$ be a sequence of random vectors under Assumptions (A1)-(A2). Then $\{X_t^M\}_{t \in \mathbb{Z}}, \{X_t^M - \mathbb{E}X_t^M\}_{t \in \mathbb{Z}}, \{Z_t^M\}_{t \in \mathbb{Z}},$ and $\{Z_t^M - \mathbb{E}Z_t^M\}_{t \in \mathbb{Z}}$ are all $\tau$-mixing random sequences. Moreover,
\[
\tau(k; \{X_t^M\}_{t \in \mathbb{Z}}; \| \cdot \|) \leq C_1\kappa_1\kappa_* \exp\{-\gamma_2(k-1)\},
\]
\[
\tau(k; \{X_t^M - \mathbb{E}X_t^M\}_{t \in \mathbb{Z}}; \| \cdot \|) \leq C_1\kappa_1\kappa_* \exp\{-\gamma_2(k-1)\},
\]
\[
\tau(k; \{Z_t^M\}_{t \in \mathbb{Z}}; \| \cdot \|) \leq C' \exp\{\gamma_2 \min(k, m)\} \max(\gamma_1\kappa_1\kappa_* \gamma_1, \kappa^2) \exp\{-\gamma_2(k-1)\},
\]
\[
\tau(k; \{Z_t^M - \mathbb{E}Z_t^M\}_{t \in \mathbb{Z}}; \| \cdot \|) \leq C' \exp\{\gamma_2 \min(k, m)\} \max(\gamma_1\kappa_1\kappa_* \gamma_1, \kappa^2) \exp\{-\gamma_2(k-1)\}
\]
for $k \geq 1$ and some constants $C, C' > 0$ only depending on $\epsilon$.

**Lemma 4.5.** Let $\{Y_t\}_{t \in \mathbb{Z}}$ be a sequence of random vectors under Assumptions (A1)-(A3). Take $M \geq C_1\kappa_1\kappa_*$ for some constant $C > 0$ only depending on $\epsilon$. Then we obtain
\[
\nu^2 \leq C' \kappa^2 \left\{ \kappa_1^2 + \kappa_1\kappa_* \gamma_1 + \kappa^2 (\gamma_3 + 2) \right\} \frac{1}{1 - \exp\{-\min(\frac{5\kappa^2 + 18\kappa_1}{4\kappa^2}, \gamma_1, \gamma_4)\}},
\]
\[
\nu^2 \leq C'' \kappa^2 \left\{ (2m + 1)\kappa^2 + \kappa_1\kappa_* \gamma_1 + \kappa^2 (\gamma_3 + 2m + 2) \right\} \frac{1}{1 - \exp\{-\min(\frac{5\kappa^2 + 18\kappa_1}{4\kappa^2}, \gamma_1, \gamma_4)\}}
\]
for some constants $C', C'' > 0$ only depending on $\epsilon$.

Therefore, by applying Theorems 4.3, Lemma 4.4, and Lemma 4.5 with the chosen $M_\delta$, we obtain for any $x > 0$,
\[
\mathbb{P}\left\{ \lambda_{\max}\left\{ \frac{1}{n} \sum_{i=1}^{n} (X_i^M - \mathbb{E}X_i^M) \right\} \geq x + \sqrt{\delta/n} \right\} \leq p \exp\left\{ -\frac{n^2x^2}{A_1n + A_2M^2_\delta + A_3nxM_\delta} \right\},
\]
where
\[
A_1 := \frac{C_1\kappa_1\gamma_1 + \kappa^2_2 (\gamma_3 + 2) + 1}{1 - \exp\{-\min(\frac{5\kappa^2 + 18\kappa_1}{4\kappa^2}, \gamma_1, \gamma_4)\}}, \quad A_2 := \frac{453^2}{\gamma_2}, \quad \text{and} \quad A_3 := \frac{2\log n}{\log 2} \max\left\{ 1, \frac{48\log(np)}{\gamma_2} \right\}
\]
for some constant $C > 0$ only depending on $\epsilon$. 

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Similarly, notice that \( \lambda_{\min}(\sum_{j=1}^{n} X_j^{M_k}) = \lambda_{\max}(\sum_{j=1}^{n} X_j^{M_k}) \). Hence the same argument renders the same upper bound

\[
P\left[ \lambda_{\min}\left\{ \frac{1}{n} \sum_{i=1}^{n} (X_i^{M_k} - \mathbb{E}X_i^{M_k}) \right\} \leq -(x + \sqrt{\delta/n}) \right] \leq p \exp \left( -\frac{n^2x^2}{A_1n + A_2M_\delta^2 + A_3nxM_\delta} \right) \tag{4.3}\]

with same constants as above.

For the last term of (4.1), with the choice of \( M_\delta \) and Lemma 4.2, we obtain

\[
\sum_{i=1}^{n} \mathbb{P}(X_i \neq X_i^{M_\delta}) = \sum_{i=1}^{n} \mathbb{P}(\|X_i\| > M_\delta) \leq \delta. \tag{4.4}
\]

Combining (4.2), (4.3), and (4.4), we obtain

\[
\mathbb{P}(\|\Sigma_0 - \mathbb{E}\Sigma_0\| \geq x + \sqrt{\delta/n}) \leq 2p \exp \left( -\frac{n^2x^2}{A_1n + A_2M_\delta^2 + A_3nxM_\delta} \right) + \delta
\]

with the constants \( A_1, A_2, A_3 \) defined above.

**Case I:** Now we consider the case when \( 0 < m \leq n - 2 \). Since \( Z_t := Y_tY_t^T \) is not symmetric for all \( t \in \mathbb{Z} \), by applying matrix dilation (See Tropp (2015), Section 2.1.16 for more details), we define the symmetric version of \( Z_{t}^{M} \) as

\[
\tilde{Z}_{t}^{M} := \begin{bmatrix} 0 & Z_t^{M} \\ (Z_t^{M})^T & 0 \end{bmatrix}.
\]

Observe that \( \lambda_{\max}(\tilde{Z}_{t}^{M}) = \|\tilde{Z}_{t}^{M}\| = \|Z_t^{M}\| \). By Lemma 4.4, \( \{\tilde{Z}_{t}^{M}\}_{t \in \mathbb{Z}} \) and \( \{\tilde{Z}_{t}^{M} - \mathbb{E}\tilde{Z}_{t}^{M}\}_{t \in \mathbb{Z}} \) are also sequences of \( \tau \)-mixing random matrices. Define

\[
\nu_{Z_t}^2 := \sup_{K \subseteq \{1, \ldots, n-m\}} \frac{1}{\text{card}(K)} \lambda_{\max}\left\{ \mathbb{E}\left( \sum_{i \in K} Z_i^{M} - \mathbb{E}Z_i^{M} \right)^2 \right\}.
\]

Notice that \( \nu_{Z_t}^2 \) and \( \nu_{Z_t}^2 \) have the same upper bound since spectral norm of block diagonal matrix is less than or equal to the spectral norm of each block.

Now we apply similar arguments in Case I to \( \{\tilde{Z}_{t}\}_{t \in \mathbb{Z}} \) and \( \{\tilde{Z}_{t}^{M}\}_{t \in \mathbb{Z}} \).

\[
P\left\{ \frac{1}{n-m} \left\| \sum_{i=1}^{n-m} (Z_i - \mathbb{E}Z_i) \right\| \geq x \right\}
\leq P\left[ \lambda_{\max}\left\{ \sum_{i=1}^{n-m} (Z_i^{M} - \mathbb{E}Z_i^{M}) \right\} \geq (n-m)x - \sum_{i=1}^{n-m} \left\| \mathbb{E}Z_i - \mathbb{E}Z_i^{M} \right\| \right] + \sum_{i=1}^{n-m} P(Z_i \neq Z_i^{M}).
\]

The rest is straightforward by using Theorem 4.3, Lemma 4.2, Lemma 4.4, and Lemma 4.5, and we thus finish the rest of the proof.

Lastly, we consider \( \kappa_1 \neq 1 \). Notice that for any sequence \( \{Y_t\}_{t \in \mathbb{Z}} \) satisfying Assumptions (A1)-(A3), the sequence \( \{Y_t/\kappa_1\}_{t \in \mathbb{Z}} \) will satisfy Assumptions (A1) automatically and Assumptions (A2)-(A3) with \( \kappa_1 = 1 \). Hence, applying the above to \( \{Y_t/\kappa_1\}_{t \in \mathbb{Z}} \) renders the results. This completes the proof of Proposition 4.1. 

\[\square\]
4.2 Proof of Theorem 2.2

Proof. The proof of Theorem 2.2 consists of two cases.

Case I. When \( m = 0 \), we first state a more general result of Gaussian process. Proposition 4.6 considers a general Gaussian process without further assumptions on the covariance and autocovariance matrices. The proof modifies that of Theorem 5.1 in van Handel (2017) with dependence between observations taken into account.

Proposition 4.6. Let \( \{Y_t\}_{t \in \mathbb{Z}} \) be a stationary sequence of mean-zero Gaussian random vectors with autocovariance matrices \( \Sigma_m \) for \( 0 \leq m \leq n - 1 \). Then

\[
\mathbb{E} \| \hat{\Sigma}_0 - \Sigma_0 \| \leq \frac{2}{n} \left\{ 2 \left( \| \Sigma_0 \|_* + 2 \sum_{m=1}^{n-1} \| \Sigma_m \|_* \right) + \sqrt{2n \| \Sigma_0 \| \left( \| \Sigma_0 \|_* + 2 \sum_{m=1}^{n-1} \| \Sigma_m \|_* \right)} \right\}
\]

\[
+ \sqrt{2n \left( \| \Sigma_0 \| + 2 \sum_{m=1}^{n-1} \| \Sigma_m \| \right) \text{Tr}(\Sigma_0)} ,
\]

where \( \| \cdot \|_* \) is the matrix nuclear norm.

The rest of the proof is to show the geometric decay of spectral norm and nuclear norm of autocovariance matrices under Assumptions (A2)-(A3) in order to apply Proposition 4.6. It is obvious that \( \kappa^2_1 \asymp \| \Sigma_0 \| \) and \( \kappa^2_* \asymp \text{Tr}(\Sigma_0) \) when the process is a centered stationary Gaussian process. We first prove the geometric decay of spectral norm of autocovariance matrices. For any \( 0 \leq m \leq n - 1 \) and any integer \( j \), by Assumption (A3), there exists \( \tilde{Y}_{1+m} \) that is identically distributed as \( Y_{1+m} \), independent of \( Y_1 \), and

\[
\sup_{u \in \mathbb{S}^{p-1}} \| (Y_{1+m} - \tilde{Y}_{1+m})^T u \|_{L(1+\epsilon)} \leq \gamma_3 \sqrt{\| \Sigma_0 \|} \exp\{-\gamma_4(m - 1)\} .
\]

Therefore,

\[
\| \Sigma_m \| = \| \mathbb{E} Y_1 Y_1^T \|
\]

\[
= \| \mathbb{E} Y_1 (Y_{1+m} - \tilde{Y}_{1+m} + \tilde{Y}_{1+m})^T \|
\]

\[
= \| \mathbb{E} Y_1 (Y_{1+m} - \tilde{Y}_{1+m})^T \|
\]

\[
\leq \sup_{u, v \in \mathbb{S}^{p-1}} | \mathbb{E} u^T Y_1 (Y_{1+m} - \tilde{Y}_{1+m})^T v |
\]

\[
\leq C \| \Sigma_0 \| \exp\{-\gamma_4(m - 1)\} ,
\]

where the last inequality is followed by Assumption (A3) and \( \gamma_3 = O(1) \) for some constant \( C > 0 \) only depending on \( \epsilon, \gamma_3 \).

Similarly, by Assumption (A2), there exists \( \tilde{Y}_{1+m} \) that is identically distributed as \( Y_{1+m} \), independent of \( Y_1 \), and

\[
\| \| Y_{1+m} - \tilde{Y}_{1+m} \|_2 \|_{L(1+\epsilon)} \leq \gamma_1 \sqrt{\| \Sigma_0 \|} \exp\{-\gamma_2(m - 1)\} .
\]

\[
\| \Sigma_m \|_* = \sqrt{\text{Tr}(\Sigma_m^T \Sigma_m)}
\]

\[
= \sqrt{\text{Tr}\left\{ \mathbb{E} (Y_{1+m} - \tilde{Y}_{1+m}) Y_1^T \mathbb{E} Y_1 (Y_{1+m} - \tilde{Y}_{1+m}) \right\} .}
\]
\[ \leq \sqrt{\text{Tr}\{\mathbb{E}(Y_{1+m} - \tilde{Y}_{1+m})Y_1^T(Y_{1+m} - \tilde{Y}_{1+m})^T\}} \]

\[ = \sqrt{\text{Tr}\{\mathbb{E}Y_1^TY_1(Y_{1+m} - \tilde{Y}_{1+m})(Y_{1+m} - \tilde{Y}_{1+m})^T\}} \]

\[ = \sqrt{\mathbb{E}||Y_1||^2||Y_{1+m} - \tilde{Y}_{1+m}||^2} \]

\[ \leq ||Y_1||_2||Y_{1+m} - \tilde{Y}_{1+m}||_2 ||Y_{1+m} - \tilde{Y}_{1+m}||_2 \]

\[ \leq C \text{Tr}(\Sigma_0) \exp\{-\gamma_2(m-1)\}, \]

where the third line is followed by the fact that \( \mathbb{E}(Y_{1+m} - \tilde{Y}_{1+m})Y_1^TY_1(Y_{1+m} - \tilde{Y}_{1+m})^T \leq \mathbb{E}Y_1^TY_1(Y_{1+m} - \tilde{Y}_{1+m})(Y_{1+m} - \tilde{Y}_{1+m})^T \) ("\( \preceq \)" is the Loewner partial order of Hermitian matrices), and both matrices are positive semi-definite, and the last line by Assumption (A2) and \( \gamma_1 = O(\sqrt{r(\Sigma_0)}) \). Indeed, for any \( u \in \mathbb{R}^p, \mathbb{E}\{u^T(Y_{1+m} - \tilde{Y}_{1+m})\}^2(Y_1^TY_1) = \sum_{j=1}^p \mathbb{E}\{u^T(Y_{1+m} - \tilde{Y}_{1+m})\}^2Y_{1j}^2 \) and \( \mathbb{E}\{u^T(Y_{1+m} - \tilde{Y}_{1+m})\}Y_1^TY_1(Y_{1+m} - \tilde{Y}_{1+m})^Tu = \sum_{j=1}^p \mathbb{E}\{u^T(Y_{1+m} - \tilde{Y}_{1+m})Y_{1j}\}^2 \). The result follows.

**Case II.** When \( m > 0 \), we denote \( \overline{Y}_i : (Y_i^T Y_i^T)^T \) for \( 1 \leq i \leq n - m \). It is obvious that \( \{\overline{Y}_i\} \) is a centered stationary Gaussian process satisfying Assumptions (A2)-(A3). Denote \( \overline{\Sigma}_0 := \mathbb{E}\overline{Y}_i\overline{Y}_i^T \) and notice that \( \overline{\Sigma}_m \) is the off-diagonal block submatrix of \( \overline{\Sigma}_0 \). By Case I and the fact that spectral norm of submatrix is bounded above by that of the full matrix, we obtain

\[ \mathbb{E}\|\overline{\Sigma}_m - \overline{\Sigma}_m\| \leq C\|\overline{\Sigma}_0\| = \left(\frac{r(\overline{\Sigma}_0)}{n-m} + \frac{r(\overline{\Sigma}_0)}{n-m}\right). \]

Notice that \( \|\overline{\Sigma}_0\| \leq \|\overline{\Sigma}_0\| \leq \|\overline{\Sigma}_0\| + \|\overline{\Sigma}_m\| \leq 2\|\overline{\Sigma}_0\| \) since \( \overline{\Sigma}_0 - \overline{\Sigma}_m \) is positive semi-definite. This completes the proof. \( \square \)

**Proof of Proposition 4.6.** The proof heavily depends on the following observation. Denote \( \mathbf{Y} := (Y_1 \ldots Y_n) \) and let \( \tilde{\mathbf{Y}} \) be an independent copy of \( \mathbf{Y} \). Then

\[ \mathbb{E}\|\tilde{\Sigma}_0 - \Sigma_0\| \leq \frac{2\mathbb{E}\|\mathbf{YY}^T\|}{n}. \]

This is exactly Lemma 5.2 in van Handel (2017) by noticing that the result holds without independence assumption.

Now we state the following two core lemmas used to complete the proof.

**Lemma 4.7.** We have

\[ \mathbb{E}\|\tilde{\Sigma}_0 - \Sigma_0\| \leq \frac{2\sqrt{2}}{n} \left\{ \mathbb{E}\|\mathbf{Y}\| \cdot \sqrt{\text{Tr} \left( \Sigma_0 + 2 \sum_{d=1}^{n-1} \tilde{\Sigma}_d \right)} + \sqrt{2 \left( \|\Sigma_0\| + 2 \sum_{d=1}^{n-1} \|\Sigma_d\| \right) \cdot n \text{Tr}(\Sigma_0)} \right\} \]

where \( \tilde{\Sigma}_d := (U_d A_d U_d^T + V_d A_d V_d^T)/2 \). Here \( U_d, V_d, A_d \) are left singular vectors, right singular vectors and singular values of \( \Sigma_d \) for all \( 1 \leq d \leq n - 1 \) respectively.

**Lemma 4.8.** We have

\[ \mathbb{E}\|\mathbf{Y}\| \leq \sqrt{2 \text{Tr} \left( \Sigma_0 + 2 \sum_{d=1}^{n-1} \tilde{\Sigma}_d \right)} + \sqrt{2n\|\Sigma_0\|} \]
where $\tilde{\Sigma}_d$ for all $1 \leq d \leq n - 1$ are defined in Lemma 4.7.

The proof of Proposition 4.6 completes by combining Lemma 4.7 and Lemma 4.8.

4.3 Proofs of auxiliary lemmas

Proof of Lemma 4.2. By Lemma A.2 in Bunea and Xiao (2015), we have $\mathbb{E}\|Y_t\|_2^2 \leq (2k)^k\kappa_*^2$ for $t \in \mathbb{Z}$. Hence

$$\|\|Y_t\|_2^2 - \mathbb{E}\|Y_t\|_2^2\|_2 \leq 2\|\|Y_t\|_2^2\|_2 \leq 4\|\|Y_t\|_2^2\|_2 \leq 8\kappa_*^2.$$ 

Thus by property of subexponential random variable and Chernoff inequality, we have for any $x \geq 0$,

$$\mathbb{P}(\|Y_t\|_2^2 - \mathbb{E}\|Y_t\|_2^2 \geq x) \leq \exp\left\{-C \min\left(\frac{x^2}{64\kappa_*^4}, \frac{x}{8\kappa_*^2}\right)\right\},$$

for some arbitrary constant $C > 0$. Obviously, we have for all $x \geq 0$,

$$\mathbb{P}\{\|Y_t\|_2^2 \geq 2\kappa_*^2 + 8\kappa_*^2(x + \sqrt{x})\} \leq \exp(-Cx)$$

for some arbitrary constant $C > 0$. This completes the proof.

Proof of Lemma 4.4. We first show that $\{X_t\}_{t \in \mathbb{Z}}$ is a sequence of $\tau$-mixing random vectors with geometric decay. Under Assumption (A2) (without loss of generality, take $j = 0$), there exists a sequence of random vectors $\{\tilde{Y}_t\}_{t > 0}$ which is independent of $\sigma(\{Y_t\}_{t \leq 0})$, identically distributed as $\{Y_t\}_{t > 0}$, and for any integer $t \geq 1$,

$$\|\|Y_t - \tilde{Y}_t\|_2 \|_{L(1+\epsilon)} \leq \gamma_1\kappa_1\exp\{-\gamma_2(t - 1)\}$$

for some constant $\epsilon > 0$. Then for any $m \geq 0$,

$$\mathbb{E}\|Y_t Y_{t+m} - \tilde{Y}_t \tilde{Y}_{t+m}\| = \mathbb{E}\|Y_t Y_{t+m} - \tilde{Y}_t Y_{t+m} + \tilde{Y}_t Y_{t+m} - \tilde{Y}_t \tilde{Y}_{t+m}\|$$

$$\leq \mathbb{E}\|Y_t (Y_{t+m} - \tilde{Y}_{t+m})\| + \mathbb{E}\|Y_t - \tilde{Y}_t\| \|Y_{t+m}\|_2 \|_{L(1+\epsilon)} + \mathbb{E}\|Y_{t+m}\|_2 \|_{L(1+\epsilon)} \|Y_t - \tilde{Y}_t\|_2 \|_{L(1+\epsilon)}$$

$$\leq C\gamma_1\kappa_1\kappa_* \exp\{-\gamma_2(t - 1)\},$$

where the fourth line is followed by Hölder’s inequality and the fact that

$$\sup_{t \in \mathbb{Z}} \|\|Y_t\|_2 \|_{L(\alpha)} \leq \sup_{t \in \mathbb{Z}} \sup_{u \in \mathbb{S}^{p-1}} \|u^T Y_t\|_2 \leq \sup_{t \in \mathbb{Z}} \sup_{u \in \mathbb{S}^{p-1}} \sqrt{\alpha}\|u^T Y_t\|_2 \leq \sqrt{\alpha}\kappa_*$$

for any $\alpha \geq 1$. Here $C > 0$ is some constant only depending on $\epsilon$.

Now define $\tilde{X}_t := \tilde{Y}_t Y_t^T$ for any integer $t > 0$. It is obvious that $\{\tilde{X}_t\}_{t > 0}$ is independent of $\{X_t\}_{t \leq 0}$ and identically distributed as $\{X_t\}_{t > 0}$. By applying Lemma A.1, for any indices $0 < k \leq t_1 < \cdots < t_\ell$, we obtain

$$\tau(\sigma(\{X_t\}_{t \leq 0}), (X_{t_1}, \ldots, X_{t_\ell}); \|\cdot\|) \leq \sum_{i=1}^{\ell} \mathbb{E}\|X_{t_i} - \tilde{X}_{t_i}\| \leq C\gamma_1\kappa_1\kappa_* \ell \exp\{-\gamma_2(k - 1)\}.$$
By definition of \( \tau \)-mixing coefficient, this yields
\[
\tau(k; \{X_t\}_{t \in \mathbb{Z}}, \| \cdot \|) \leq C \gamma_1 \kappa_1 \kappa_* \exp\{-\gamma_2(k-1)\}
\]
for some constant \( C > 0 \) only depending on \( \epsilon \).

Now we proceed to prove \( \tau \)-mixing properties for the “truncated version”. The following lemma is needed.

**Lemma 4.9.** Let \( u_1, u_2, v_1, v_2 \in \mathbb{R}^p \) for \( p \geq 1 \) with unit length under \( \ell_2 \)-norm and \( \sigma_u \geq 0 \). Then the function
\[
f(\sigma_v) = \|\sigma_v v_1 v_2^\top - \sigma_u u_1 u_2^\top\|
\]
is non-decreasing in the range \( \sigma_v \in [\sigma_u, \infty) \). In particular, for any \( M \geq 0 \) such that \( M \leq \sigma_u \), \( M \leq \sigma_v \), we have
\[
\|M v_1 v_2^\top - M u_1 u_2^\top\| \leq \|\sigma_v v_1 v_2^\top - \sigma_u u_1 u_2^\top\|.
\]

Now consider three cases.

1. When \( \|X_t\| \leq M \) and \( \|\tilde{X}_t\| \leq M \), \( \|X_t^M - \tilde{X}_t^M\| = \|X_t - \tilde{X}_t\| \).

2. When \( \|X_t\| \leq M \) and \( \|\tilde{X}_t\| > M \), we have
   \[
   X_t^M = X_t = \frac{Y_t}{\|Y_t\|_2} \frac{Y_t^\top}{\|Y_t\|_2} \quad \text{and} \quad \tilde{X}_t^M = M \frac{\tilde{Y}_t}{\|\tilde{Y}_t\|_2} \frac{\tilde{Y}_t^\top}{\|\tilde{Y}_t\|_2}.
   \]
   Since \( \frac{Y_t}{\|Y_t\|_2} \) have unit length and \( \|Y_t\|_2^2 \leq M < \|\tilde{Y}_t\|_2^2 \), we have \( \|X_t^M - \tilde{X}_t^M\| \leq \|X_t - \tilde{X}_t\| \) by Lemma 4.9. By symmetry, the same argument also applies to the case where \( \|X_t\| > M \) and \( \|\tilde{X}_t\| \leq M \).

3. When \( \|X_t\| > M \) and \( \|\tilde{X}_t\| > M \), we have \( X_t^M = M \frac{Y_t}{\|Y_t\|_2} \frac{Y_t^\top}{\|Y_t\|_2} \) and \( \tilde{X}_t^M = M \frac{\tilde{Y}_t}{\|\tilde{Y}_t\|_2} \frac{\tilde{Y}_t^\top}{\|\tilde{Y}_t\|_2} \).

Again by Lemma 4.9, we have \( \|X_t^M - \tilde{X}_t^M\| \leq \|X_t - \tilde{X}_t\| \).

By combining three cases, \( \|X_t^M - \tilde{X}_t^M\| \leq \|X_t - \tilde{X}_t\| \) always holds, and hence \( \mathbb{E}\|X_t^M - \tilde{X}_t^M\| \leq \mathbb{E}\|X_t - \tilde{X}_t\| \) for any \( t \geq 1 \). Hence for any indices \( 0 < k < t_1 < \cdots < t_i \) by Lemma A.1, we have
\[
\tau(\sigma(\{X_t^M\}_{t \leq 0}), (X_{t_1}^M, \ldots, X_{t_i}^M); \| \cdot \|) \leq C \gamma_1 \kappa_1 \kappa_* \ell \exp\{-\gamma_2(k-1)\}
\]
for some constant \( C > 0 \) only depending on \( \epsilon \). By definition of \( \tau \)-mixing coefficient, this yields
\[
\tau(k; \{X_t^M\}_{t \in \mathbb{Z}}, \| \cdot \|) \leq C \gamma_1 \kappa_1 \kappa_* \exp\{-\gamma_2(k-1)\}
\]
for some constant \( C > 0 \) only depending on \( \epsilon \). Notice that \( \mathbb{E}\|X_t^M - \mathbb{E}X_t^M - (\tilde{X}_t^M - \mathbb{E}\tilde{X}_t^M)\| = \mathbb{E}\|X_t^M - \tilde{X}_t^M\| \) since \( \mathbb{E}\tilde{X}_t^M = \mathbb{E}X_t^M \) for any \( t \geq 1 \). The \( \tau \)-mixing property stated above applies to \( \{X_t^M - \mathbb{E}X_t^M\}_{t \in \mathbb{Z}} \) directly.

Similar arguments apply to \( \{Z_t^M\}_{t \in \mathbb{Z}} \) and \( \{Z_t^M - \mathbb{E}Z_t^M\}_{t \in \mathbb{Z}} \) so we omit the details. This completes the proof.

**Proof of Lemma 4.5.** The proof consists of two steps.

**Step I.** We first provide an upper bound for \( \nu_k^2 \). Without loss of generality, we only consider \( \|\mathbb{E}(X_0 - \mathbb{E}X_0)(X_k - \mathbb{E}X_k)\| \) for \( k \geq 0 \). Under Assumptions (A2)-(A3), there exists \( \tilde{Y}_k \) where \( \tilde{Y}_k \)
is independent of $\sigma(\{Y_i\}_{i \leq 0})$, identically distributed as $Y_k$, and
\[
\|Y_k - \bar{Y}_k\|_{L(1+\epsilon)} \leq \gamma_1 \kappa_1 \exp\{-\gamma_2 (k-1)\},
\]
\[
\|(Y_k - \bar{Y}_k)^T u\|_{L(1+\epsilon)} \leq \gamma_3 \kappa_1 \exp\{-\gamma_4 (k-1)\}
\]
for constants $\gamma_1, \gamma_2, \gamma_3, \gamma_4 > 0$ in Assumptions (A2)-(A3).

For $k = 0$, we have
\[
\|E X_0 X_0 - E X_0 E X_0\| \leq C(\kappa_1^4 + \kappa_1^2 \kappa_4^2)
\]
by Assumption (A1) for some universal constant $C > 0$. For $k > 0$, we obtain
\[
\|E X_0 X_k - E X_0 E X_k\| = \|E X_0 X_k - E X_0 \bar{X}_k\|
\]
\[
= \|E X_0 (X_k - \bar{X}_k)\|
\]
\[
\leq \sup_{u,v \in \mathcal{S}_n} \|E u^T Y_0^T (Y_k^T - \bar{Y}_k^T) v\|
\]
\[
\leq \sup_{u,v \in \mathcal{S}_n} \|E u^T Y_0^T Y_k^T - \bar{Y}_k^T v + u^T Y_0^T (Y_k^T - \bar{Y}_k^T) v\|
\]
\[
\leq \sup_{u,v \in \mathcal{S}_n} \{E |Y_0^T Y_k^T|^{\frac{3(1+\epsilon)}{2}} \|u^T Y_0\|_{L(\frac{3(1+\epsilon)}{2})} \|Y_k^T - \bar{Y}_k^T\|_{L(1+\epsilon)} + \}
\]
\[
\{E |u^T \bar{Y}_k^T v|^{\frac{3(1+\epsilon)}{2}} \|Y_0\|_{L(\frac{3(1+\epsilon)}{2})} \|Y_k^T - \bar{Y}_k\|_{L(1+\epsilon)}\}
\]
\[
\leq C \kappa_4 \kappa_3 \kappa_4 + \kappa_1 \gamma_1 \exp\{-\min(\gamma_2, \gamma_4)(k-1)\}
\]
where the first line is followed by $E X_k = E \bar{X}_k$, fifth line by Hölder’s inequality, and sixth line by Assumptions (A1)-(A3) for some constant $C > 0$ only depending on $\epsilon$.

Hence for any $K \subseteq \{1, \ldots, n\}$,
\[
\frac{1}{\text{card}(K)} \lambda_{\max} \left\{ E \left( \sum_{i \in K} X_i - E X_i \right)^2 \right\}
\]
\[
\leq \frac{1}{\text{card}(K)} \left\| \sum_{i,j \in K} E (X_i - E X_i)(X_j - E X_j) \right\|
\]
\[
\leq \frac{1}{\text{card}(K)} \sum_{i,j \in K} \|E (X_i - E X_i)(X_j - E X_j)\|
\]
\[
\leq C \left[ \kappa_1^4 + \kappa_1^2 \kappa_4^2 + \frac{\kappa_1^2 \kappa_4 (\kappa_3 + \kappa_1 \gamma_1)}{\text{card}(K)} \sum_{i,j \in K, i \neq j} \exp\{-\min(\gamma_2, \gamma_4)(|i - j| - 1)\} \right]
\]
\[
\leq C \left[ \frac{\kappa_1^4 (\kappa_3 + \kappa_1 \gamma_1) + \kappa_1^2 (\kappa_3 + 1)}{1 - \exp\{-\min(\gamma_2, \gamma_4)\}} \right].
\]

**Step II.** We first bound $\nu_k^2$. By definition, we have
\[
\left\| E \left( \sum_{i \in K} X_i^M - E X_i^M \right)^2 \right\| = \left\| \sum_{i,j \in K} E (X_i^M - E X_i^M)(X_j^M - E X_j^M) \right\| = \left\| \sum_{i,j \in K} (E X_i^M X_j^M - E X_i^M E X_j^M) \right\|
\]
Without loss of generality, we consider $\left\| E X_0^M X_k^M - E X_0^M E X_k^M \right\|$ for $k \geq 0$. Let $X_k^M$ be defined as
in the proof of Lemma 4.4. Then $\widetilde{X}_k^M$ is independent of $\widetilde{X}_0^M$ and distributed as $X_k^M$. Hence
\[
\|EX_0^M X_k^M - EX_0^M \bar{X}_k^M\| = \|EX_0^M X_k^M - EX_0^M \bar{X}_k^M\|
\]
Then we could rewrite
\[
\|EX_0^M X_k^M - EX_0^M \bar{X}_k^M\| = \|EX_0^M \bar{X}_k^M - EX_0^M \bar{X}_k^M\|
\]
where $\zeta_i = \frac{M^\lambda\|X_i\|}{\|X_i\|}$, $\zeta_i = \frac{M^\lambda\|X_i\|}$. Since $\zeta_0$, $\zeta_k$ are bounded by 1, we have
\[
\|EX_0^M \bar{X}_k^M - \bar{X}_k^M \zeta_0\| = \|EX_0 \bar{X}_k^M - \bar{X}_k^M \zeta_0\|
\]
where the last inequality is from result in Step I for some constant $C > 0$ only depending on $\epsilon$.

On the other hand, by applying Hölder’s inequality, we have
\[
\|EX_0 \bar{X}_k^M \zeta_0\| = \sup_{u,v \in \mathbb{S}^{p-1}} \{\|u^T Y_0^\top \bar{Y}_k^T v\| \|u^T \bar{Y}_k\| \|\bar{Y}_k\| \|Y_0\| \|\bar{Y}_k\| \|\bar{Y}_k\| \}
\]
where the first line follows by Hölder’s inequality and the last line by Assumption (A1) for some constant $C > 0$ only depending on $\epsilon$.

Next, we need to bound $\|\zeta_k - \zeta_k\|_{L \frac{5(1+\epsilon)}{5+\epsilon}}$. For the sake of presentation clearness, we denote $a_k := \|X_k\|$ and $\bar{a}_k := \|\bar{X}_k\|$, and rewrite
\[
\|\zeta_k - \zeta_k\|_{L \frac{5(1+\epsilon)}{5+\epsilon}}
\]
where the last inequality follows by the fact that $\| \cdot \|_{L \frac{5(1+\epsilon)}{5+\epsilon}}$ is a norm for $\epsilon > 0$.
For the first term, we have
\[
\left\| M \left(\frac{1}{a_k} - \frac{1}{\bar{a}_k}\right) \mathbf{1}\{a_k > M, \bar{a}_k > M\}\right\|_{L^2(\frac{5(1+\epsilon)}{6+\epsilon})} \leq \frac{1}{M} \left\{ \mathbb{E}\left[ a_k - \frac{M}{a_k} \right] \mathbf{1}\{a_k > M, \bar{a}_k > M\}\right\} \leq \frac{1}{M} \left\{ \mathbb{E}\left[ \|X_k - \bar{X}_k\|^2 \right] \right\}^{\frac{5+\epsilon}{5(1+\epsilon)}} \leq C \gamma_1 \kappa_1 \kappa_0 \exp\{-\gamma_2(k-1)\}/M,
\]
where the last inequality is followed by Lemma 4.4 for some constant \( C > 0 \) only depending on \( \epsilon \).

With the chosen \( M \geq C \gamma_1 \kappa_1 \kappa_0 \), we have
\[
\left\| M \left(\frac{1}{a_k} - \frac{1}{\bar{a}_k}\right) \mathbf{1}\{a_k > M, \bar{a}_k > M\}\right\|_{L^2(\frac{5(1+\epsilon)}{6+\epsilon})} \leq \exp\{-\gamma_2(k-1)\}.
\]

For the second term, taking any \( \epsilon_k > 0 \), we have
\[
\left\| \left(1 - \frac{M}{a_k}\right) \mathbf{1}\{a_k > M, \bar{a}_k \leq M\}\right\|_{L^2(\frac{5(1+\epsilon)}{6+\epsilon})} = \left\| \left(1 - \frac{M}{M + \epsilon_k}\right) \mathbf{1}\{M < a_k \leq M + \epsilon_k, \bar{a}_k \leq M\}\right\|_{L^2(\frac{5(1+\epsilon)}{6+\epsilon})} + \left\| \left(1 - \frac{M}{a_k}\right) \mathbf{1}\{a_k > M + \epsilon_k, \bar{a}_k \leq M\}\right\|_{L^2(\frac{5(1+\epsilon)}{6+\epsilon})} \leq \frac{\epsilon_k}{M} + \left\{ \mathbb{P}(|a_k - \bar{a}_k| > \epsilon_k) \right\}^{\frac{5+\epsilon}{5(1+\epsilon)}}.
\]

By Markov inequality and Lemma 4.4, we have
\[
\mathbb{P}(|a_k - \bar{a}_k| > \epsilon_k) \leq \frac{\mathbb{E}\left[ ||X_k - \bar{X}_k||\right]}{\epsilon_k} \leq \frac{C \gamma_1 \kappa_1 \kappa_0 \exp\{-\gamma_2(k-1)\}}{\epsilon_k}
\]
for some constant \( C > 0 \) only depending on \( \epsilon \). Taking \( \epsilon_k = C \gamma_1 \kappa_1 \kappa_0 \exp\{-\frac{5+\epsilon}{6\epsilon+10}\gamma_2(k-1)\} \), we obtain
\[
\left\| \left(1 - \frac{M}{a_k}\right) \mathbf{1}\{a_k > M, \bar{a}_k \leq M\}\right\|_{L^2(\frac{5(1+\epsilon)}{6+\epsilon})} \leq 2 \exp\left\{ \frac{5+\epsilon}{6\epsilon+10}\gamma_2(k-1) \right\}.
\]

The third term follows by symmetry. Putting together, we have for \( k > 0 \),
\[
\left\| \zeta_k - \tilde{\zeta}_k \right\|_{L^2(\frac{5(1+\epsilon)}{6+\epsilon})} \leq C \exp\left\{ \frac{5+\epsilon}{6\epsilon+10}\gamma_2(k-1) \right\},
\]
\[
\left\| \mathbb{E} X_0 \bar{X}_k \zeta_0 (\zeta_k - \tilde{\zeta}_k) \right\| \leq C \kappa_1^2 \kappa_0^2 \exp\left\{ -\frac{5+\epsilon}{6\epsilon+10}\gamma_2(k-1) \right\},
\]
\[
\left\| \mathbb{E} X_0^M \bar{X}_k^M - \mathbb{E} X_0^M \mathbb{E} \bar{X}_k^M \right\| \leq C \kappa_1^2 \kappa_0 \kappa_1 \kappa_0 \gamma_1 + \kappa_0^2 (\gamma_3 + 1) \exp\left\{ -\min\left( \frac{5+\epsilon}{6\epsilon+10}\gamma_2, \gamma_4 \right)(k-1) \right\}
\]
for some constant \( C > 0 \) only depending on \( \epsilon \). Hence for any \( K \subseteq \{1, \ldots, n\} \),
\[
\frac{1}{\text{card}(K)} \lambda_{\text{max}} \left\{ \mathbb{E} \left( \sum_{i \in K} X_i^M - \mathbb{E} X_i^M \right)^2 \right\}
\]

\[ \leq \frac{1}{\text{card}(K)} \max_{i,j \in K} \frac{\text{E}(X^i - \text{E}X^i)(X^j - \text{E}X^j)}{\text{E}(X^i - \text{E}X^i)(X^j - \text{E}X^j)} \]

\[ \leq \frac{1}{\text{card}(K)} \sum_{i,j \in K} \frac{\max_{i,j \in K} \text{E}(X^i - \text{E}X^i)(X^j - \text{E}X^j)}{\text{E}(X^i - \text{E}X^i)(X^j - \text{E}X^j)} \]

\[ \leq \sum_{i,j \in K} \frac{\max_{i,j \in K} \text{E}(X^i - \text{E}X^i)(X^j - \text{E}X^j)}{\text{E}(X^i - \text{E}X^i)(X^j - \text{E}X^j)} \exp \left\{ -\min \left( \frac{5 + \epsilon}{6\epsilon + 10\gamma_2, \gamma_4} (|i - j| - 1) \right) \right\} \]

for some constant \( C > 0 \) only depending on \( \epsilon \).

Similar arguments apply to \( \nu^2_M \), so we omit the details. This completes the proof. \( \square \)

**Proof of Lemma 4.9.** Fix \( u_1, u_2, v_1, v_2 \in \mathbb{R}^p \) with unit length and \( \sigma_u \geq 0 \). For any \( \sigma_v \geq \sigma_u \), we perform singular value decomposition for matrix \( X(\sigma_v) := \sigma_u u_1 u_2^T - \sigma_v v_1 v_2^T \). According to Equation (8) in Brand (2006), the non-zero singular values of \( X(\sigma_v) \) are identical to those of

\[ S(\sigma_v) = \begin{bmatrix} \sigma_u - \sigma_v u_1^T v_1 u_1^T v_2 & u_2 \\ v_2^T v_1 - u_1 u_1^T v_1 & \sigma_v^2 \| v_1 - u_1 u_1^T v_1 \|_2 \| v_2 - u_2 u_2^T v_2 \|_2 \end{bmatrix}. \]

For simplicity, denote \( w = u_1^T v_1 v_2 u_2, \tilde{v}_1 = v_1 - u_1 u_1^T v_1, \tilde{u}_1 = v_2 - u_2 u_2^T v_2 \). Hence \( S(\sigma_v) \) could be rewritten as

\[ S(\sigma_v) = \begin{bmatrix} \sigma_u - \sigma_v w & -\sigma_v u_1^T v_1 \| \tilde{v}_2 \|_2 \\ \sigma_v u_2^T v_2 \| \tilde{v}_1 \|_2 & \sigma_v^2 \| \tilde{v}_1 \|_2 \| \tilde{v}_2 \|_2 \end{bmatrix}. \]

Using the calculation on Page 86 in Blinn (1996), \( \|S(\sigma_v)\| = Q(\sigma_v) + R(\sigma_v) \), where

\[ Q(\sigma_v) := \sqrt{\left( \sigma_u - \sigma_v w + \sigma_v \| \tilde{v}_1 \|_2 \| \tilde{v}_2 \|_2 \right)^2 + \sigma_v^2 (u_1^T v_1 \| \tilde{v}_2 \|_2 + u_2^T v_2 \| \tilde{v}_1 \|_2)^2}/2, \]

\[ R(\sigma_v) := \sqrt{\left( \sigma_u - \sigma_v w - \sigma_v \| \tilde{v}_1 \|_2 \| \tilde{v}_2 \|_2 \right)^2 + \sigma_v^2 (u_1^T v_1 \| \tilde{v}_2 \|_2 - u_2^T v_2 \| \tilde{v}_1 \|_2)^2}/2. \]

We are left to show that both \( Q \) and \( R \) are non-decreasing function of \( \sigma_v \in [\sigma_u, \infty] \). By differentiating \( Q, R \) with respect to \( \sigma_v \), we obtain

\[ \frac{dQ}{d\sigma_v} = c_Q(\sigma_v) \left\{ \sigma_u \left( \| \tilde{v}_1 \|_2 \| \tilde{v}_2 \|_2 - w \right) + \sigma_v \left( w^2 + \| \tilde{v}_1 \|_2^2 \| \tilde{v}_2 \|_2^2 + \left( u_1^T v_1 \right)^2 \| \tilde{v}_2 \|_2^2 + \left( u_2^T v_2 \right)^2 \| \tilde{v}_1 \|_2^2 \right) \right\}, \]

\[ \frac{dR}{d\sigma_v} = c_R(\sigma_v) \left\{ -\sigma_u \left( \| \tilde{v}_1 \|_2 \| \tilde{v}_2 \|_2 + w \right) + \sigma_v \left( w^2 + \| \tilde{v}_1 \|_2^2 \| \tilde{v}_2 \|_2^2 + \left( u_1^T v_1 \right)^2 \| \tilde{v}_2 \|_2^2 + \left( u_2^T v_2 \right)^2 \| \tilde{v}_1 \|_2^2 \right) \right\} \]

for some nonnegative constants \( c_Q(\sigma_v), c_R(\sigma_v) \).

By simple algebra, we have \( w^2 + \| \tilde{v}_1 \|_2^2 \| \tilde{v}_2 \|_2^2 + \left( u_1^T v_1 \right)^2 \| \tilde{v}_2 \|_2^2 + \left( u_2^T v_2 \right)^2 \| \tilde{v}_1 \|_2^2 = 1 \) so that

\[ \frac{dQ}{d\sigma_v} = c_Q(\sigma_v) \left\{ \sigma_u \left( \| \tilde{v}_1 \|_2 \| \tilde{v}_2 \|_2 - w \right) + \sigma_v \right\}. \]

Moreover, since \( u_1, u_2, v_1, v_2 \in \mathbb{R}^p \) are all length 1, we have \( |w| \leq 1 \) by Cauchy-Schwartz. Hence by the fact that \( \sigma_v \geq \sigma_u \geq 0 \), we have \( \frac{dQ}{d\sigma_v} \geq 0 \). On the other hand, denote \( a := u_1^T v_1 \) and \( b := u_2^T v_2 \) and again by Cauchy-Schwartz we have \( |a| \leq 1, |b| \leq 1 \). In addition, we have

\[ \| \tilde{v}_1 \|_2 = \sqrt{(v_1 - u_1 u_1^T v_1)(v_1 - u_1 u_1^T v_1)} \]
Now we take conditional expectation w.r.t. \( \|\bar{v}_1\|_2 \) since \( \|S(\sigma_v)\| = Q(\sigma_v) + R(\sigma_v) \) is a non-decreasing function with respect to \( \sigma_v \). Applying the monotonicity property proved above, we have \( \|\sigma_v v_1 \| \leq \|\sigma_v u_1 \| \). This completes the proof.

**Proof of Lemma 4.7.** By the observation in the proof of Proposition 4.6, we have

\[
\mathbb{E}\|\tilde{\Sigma}_0 - \Sigma_0\| \leq \frac{2}{n} \mathbb{E}\|YY^T\| = \frac{2}{n} \mathbb{E}\left( \sup_{u, v \in \mathbb{S}^{p-1}} \sum_{k=1}^{n} u^T Y_k \bar{Y}_k^T v \right) = \frac{2}{n} \mathbb{E}\left( \sup_{u, v \in \mathbb{S}^{p-1}} W_{u,v} \right).
\]

Now consider

\[
\begin{align*}
(W_{u,v} - W_{u',v'})^2 &= \left( \sum_{k=1}^{n} u^T Y_k \bar{Y}_k^T v - \sum_{k=1}^{n} u'^T Y_k \bar{Y}_k^T v' \right)^2 \\
&= \left( \sum_{k=1}^{n} u^T Y_k \bar{Y}_k^T v - \sum_{k=1}^{n} u'^T Y_k \bar{Y}_k^T v + \sum_{k=1}^{n} u'^T Y_k \bar{Y}_k^T v' - \sum_{k=1}^{n} u'^T Y_k \bar{Y}_k^T v' \right)^2 \\
&= \left( \sum_{k=1}^{n} (u - u')^T Y_k \bar{Y}_k^T v + \sum_{k=1}^{n} u'^T Y_k \bar{Y}_k^T (v - v') \right)^2 \\
&\leq 2 \left( \sum_{k=1}^{n} (u - u')^T Y_k \bar{Y}_k^T v \right)^2 + 2 \left( \sum_{k=1}^{n} u'^T Y_k \bar{Y}_k^T (v - v') \right)^2 \\
&= 2 \sum_{k=1}^{n-1} \sum_{|j-k|=d} (u - u')^T Y_j \cdot (u - u')^T Y_k \cdot v^T \bar{Y}_j \cdot v^T \bar{Y}_k \\
&\quad + 2 \sum_{d=0}^{n-1} \sum_{|j-k|=d} u'^T Y_j \cdot u'^T Y_k \cdot (v - v')^T \bar{Y}_j \cdot (v - v')^T \bar{Y}_k.
\end{align*}
\]

Now we take conditional expectation \( \mathbb{E}_Y := \mathbb{E}(:\bar{Y}) \).

\[
\mathbb{E}_Y(W_{u,v} - W_{u',v'})^2 \leq 2(u - u')^T \Sigma_0 (u - u') \sum_{j=1}^{n} v^T \bar{Y}_j \bar{Y}_j^T v + 2 \sum_{d=1}^{n-1} (u - u')^T (\Sigma_d + \Sigma_d^T)(u - u') \sum_{(j-k)=d} v^T \bar{Y}_j \cdot v^T \bar{Y}_k.
\]

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Taking expectation with respect to $\tilde{Y}$, where

$$
E(\tilde{Y}) = E(Y) = 0,
$$

and using the fact that $\tilde{Y}$ is an independent copy of $Y$, we obtain,

$$
E \sup_{u,v} W_{u,v} \leq \sqrt{2} E \|Y\| \cdot \sqrt{\text{Tr} \left( \Sigma_0 + 2 \sum_{d=1}^{n-1} \tilde{\Sigma}_d \right) + \sqrt{2} \left( \|\Sigma_0\| + 2 \sum_{d=1}^{n-1} \|\Sigma_d\| \right) \cdot \sqrt{n \text{Tr}(\Sigma_0)}}.
$$

This completes the proof of Lemma 4.7. 

\[\Box\]

**Proof of Lemma 4.8.** Define $W_{u,v} := u^T Y v$. Then

$$
E(W_{u,v} - W_{u',v'})^2 = E(u^T Y v - u'^T Y v')^2
$$
\begin{align*}
&\leq 2\mathbb{E}((u - u')^T Y v)^2 + 2\mathbb{E}(u'^T Y (v - v'))^2 \\
= & 2 \sum_{i,j} (u - u')^T \Sigma_{i-j} (u - u') v_i v_j + 2 \sum_{i,j} u'^T \Sigma_{i-j} u'(v_i - v'_i)(v_j - v'_j).
\end{align*}

In addition, define
\begin{align*}
\Sigma_L := 
\begin{bmatrix}
\Sigma_0 & \Sigma_1 & \cdots & \Sigma_{n-1} \\
\Sigma_1^T & \Sigma_0 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\Sigma_{n-1}^T & \Sigma_{n-2}^T & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix},
\Sigma_{L,u} := 
\begin{bmatrix}
u^T & 0 & \cdots & 0 \\
0 & u^T & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & u^T \\
\end{bmatrix},
\Sigma_u := (u^T \Sigma_0 u) 1_n 1_n^T,
\Sigma^\circ := \|\Sigma_0\| 1_n 1_n^T.
\end{align*}

Since $\Sigma_L$ is a positive semi-definite matrix, we have
$$
\Sigma_{L,u} \preceq \Sigma_u \preceq \Sigma^\circ
$$
for all $u \in \mathbb{S}^{p-1}$ where “$\preceq$” is the Loewner partial order of Hermitian matrices. Hence
$$
\mathbb{E}(W_{u,v} - W_{u',v'})^2 \leq 2\left\| \left( \Sigma_0 + 2 \sum_{d=1}^{n-1} \overline{\Sigma}_d \right)^{\frac{1}{2}} (u - u') \right\|^2 + 2\|\Sigma_0\|\|v - v'\|1_n 1_n^T (v - v').
$$

Then define the following Gaussian process:
$$
Y_{u,v} := \sqrt{2} u^T \left( \Sigma_0 + 2 \sum_{d=1}^{n-1} \overline{\Sigma}_d \right)^{\frac{1}{2}} g + \sqrt{2} \|\Sigma_0\|^{\frac{1}{2}} v^T g',
$$
where $g, g' \in \mathbb{R}^p, g' \in \mathbb{R}^n$ are independent Gaussian random vectors with mean $0$ and covariance matrices $I_p$ and $1_n 1_n^T$ respectively. Thus by previous inequality, we have
$$
\mathbb{E}(W_{u,v} - W_{u',v'})^2 \leq \mathbb{E}(Y_{u,v} - Y_{u',v'})^2.
$$

Hence by Slepian-Fernique inequality, we have
\begin{align*}
\mathbb{E}\sup_{u,v \in \mathbb{S}^{p-1}} W_{u,v} &\leq \mathbb{E}\sup_{u,v \in \mathbb{S}^{p-1}} Y_{u,v} \\
= & \sqrt{2} \mathbb{E}\sup_{u \in \mathbb{S}^{p-1}} u^T \left( \Sigma_0 + 2 \sum_{d=1}^{n-1} \overline{\Sigma}_d \right)^{\frac{1}{2}} g + \sqrt{2} \|\Sigma_0\|^{\frac{1}{2}} \mathbb{E}\sup_{v \in \mathbb{S}^{p-1}} v^T g' \\
\leq & \sqrt{2} \left\| \left( \Sigma_0 + 2 \sum_{d=1}^{n-1} \overline{\Sigma}_d \right)^{\frac{1}{2}} g \right\| + \sqrt{2} \|\Sigma_0\|^{\frac{1}{2}} \mathbb{E}\|g'\| \\
\leq & \sqrt{\frac{2}{n}} \mathbb{E}\left( \left( \Sigma_0 + 2 \sum_{d=1}^{n-1} \overline{\Sigma}_d \right)^{\frac{1}{2}} g \right) + \sqrt{2} \|\Sigma_0\|^{\frac{1}{2}} \cdot \mathbb{E}\|g'\|.
\end{align*}

This completes the proof of Lemma 4.8. \hfill \square

4.4 Proof of results in Section 3

Proof of Theorem 3.1. We first examine Assumptions (A1) and (A4). First of all, we will study VAR(1) model, i.e., $Y_t = A Y_{t-1} + E_t$. Notice that for VAR(1), we could rewrite the original
sequence as a moving-average model, i.e., \( Y_t = \sum_{j=0}^{\infty} A^j E_{t-j} \). For any \( u \in \mathbb{R}^p \), we have

\[
\| u^T Y_t \|_{\psi_2} = \left\| \sum_{j=0}^{\infty} u^T A^j E_{t-j} \right\|_{\psi_2} \\
\leq C \left( \sum_{j=0}^{\infty} \| u^T A^j E_{t-j} \|_{\psi_2}^2 \right)^{\frac{1}{2}} \\
\leq C d^c \left( \sum_{j=0}^{\infty} \| u^T A^j E_{t-j} \|_{L(2)}^2 \right)^{\frac{1}{2}} = C d^c \| u^T Y_t \|_{L(2)}
\]

for some universal constant \( C > 0 \). Here the second line and last equality are followed by the fact that \( \{ E_t \}_{t \in \mathbb{Z}} \) is a sequence of independent random vector, and the third line by the moment assumption on \( \{ E_t \}_{t \in \mathbb{Z}} \). Since \( Y_{t-1} \) is a stable process when \( \| A \| < 1 \), \( \| u^T Y_t \|_{\psi_2} \leq C d^c \| u^T Y_t \|_{L(2)} < \infty \) for all \( u \in \mathbb{R}^p \).

Denote \( \tilde{Y}_t := (Y_t^T \ldots Y_{t-d}^T)^T \) and \( \tilde{E}_t := (E_t^T 0^T \ldots 0^T)^T \). For \( \{ \tilde{Y}_t \}_{t \in \mathbb{Z}} \) generated from a VAR(1) model, \( \{ \tilde{Y}_t \}_{t \in \mathbb{Z}} \) is a VAR(1) process, i.e., \( \tilde{Y}_t = \tilde{A} \cdot \tilde{Y}_{t-1} + \tilde{E}_t \). Thus by previous argument, taking any \( v \in \mathbb{R}^{d+1} \) where only the first \( p \) digits are non-zero and denoting \( v' \in \mathbb{R}^p \) to be first-\( p \) part of \( v \), we have \( \| v' \tilde{Y}_t \|_{\psi_2} = \| v^T \tilde{Y}_t \|_{\psi_2} \leq C \| v^T \tilde{Y}_t \|_{L(2)} = C \| v^T \tilde{Y}_t \|_{L(2)} < \infty \) for some constant \( C > 0 \) only depending on \( c' \) where the last inequality is followed by the fact that \( \{ Y_t \} \) is a stable process (see Lemma 4.10). Assumptions (A1) and (A4) are verified.

Then we examine Assumption (A2). Without loss of generality, take \( j = 0 \) in Assumption (A2). Let \( \{ \tilde{Y}_t \}_{t=1-d}^0 \) be a sequence of random vectors independent of \( \{ Y_t \}_{t \leq 0} \) and identically distributed as \( \{ Y_t \}_{t=1-d}^0 \). Define \( \tilde{Y}_t = A_1 \tilde{Y}_{t-1} + \cdots + A_d \tilde{Y}_{t-d} + E_t \) for every \( t > 0 \). It is obvious that \( \{ \tilde{Y}_t \}_{t \leq 0} \) is independent of \( \{ Y_t \}_{t \leq 0} \) and identically distributed as \( \{ Y_t \}_{t \geq 0} \). Moreover, for any \( t \geq 1 \), we have

\[
\| Y_t - \tilde{Y}_t \|_{L(1+\epsilon)}^2 = \{ \mathbb{E} \| A_1 Y_{t-1} + \cdots + A_d \tilde{Y}_{t-d} + E_t - (A_1 \tilde{Y}_{t-1} + \cdots + A_d \tilde{Y}_{t-d} + E_t) \|_{2+\epsilon}^2 \}^{\frac{1}{1+\epsilon}} \\
\leq \{ \mathbb{E} \| A_1 (Y_{t-1} - \tilde{Y}_{t-1}) + \cdots + A_d (Y_{t-d} - \tilde{Y}_{t-d}) \|_{2+\epsilon}^2 \}^{\frac{1}{1+\epsilon}} \\
\leq \sum_{k=1}^{d} a_k \{ \mathbb{E} \| Y_{t-k} - \tilde{Y}_{t-k} \|_{2+\epsilon}^2 \}^{\frac{1}{1+\epsilon}},
\]

where the third line follows by \( \| \cdot \|_{L(1+\epsilon)} \) is a norm for \( \epsilon > 0 \). Denoting \( \phi_t = \| Y_t - \tilde{Y}_t \|_{L(1+\epsilon)} \), we have \( \phi_t \leq \sum_{k=1}^{d} a_k \phi_{t-k} \). Let \( u \) be the unit vector with 1 at first position and 0 elsewhere. Then by iteration, we have

\[
u^T (\phi_1, \ldots, \phi_{d+1})^T \leq \nu^T \tilde{A} \nu^T (\phi_0, \ldots, \phi_{1-d})^T \leq \| \tilde{A} \| \| (\phi_0, \ldots, \phi_{1-d})^T \|_2.
\]

Note that \( \phi_t = C \kappa_t \) for \( t \leq 0 \) by Assumption (A1) for some constant \( C > 0 \) only depending on \( \epsilon \). By the following lemma, we could choose some arbitrary \( \rho_1 \) such that \( \rho(\tilde{A}) < \rho_1 < 1 \).

**Lemma 4.10.** For \( \tilde{A} \) defined above, \( \rho(\tilde{A}) < 1 \) if and only if \( \sum_{k=1}^{d} a_k < 1 \), where \( \rho(\tilde{A}) \) is the spectral radius of \( \tilde{A} \).

By Gelfand’s formula, there exists a \( K > 0 \), such that for all \( t \geq K \), \( \| \tilde{A}^t \| < \rho_1^t \). For \( t < K \), we
\[ \phi_t \leq 2d\kappa_s \left( \frac{\|\overline{A}\|}{\rho_1} \right)^K \rho_1^t. \]

For \( t \geq K \), we have \( \phi_t \leq Cd\kappa_s\rho_1^t \) for some constant \( C > 0 \) only depending on \( \epsilon \). Taking \( \gamma_1 = Cd(\kappa_s/\kappa_1)(\|\overline{A}\|/\rho_1)^K \) for some constant \( C > 0 \) only depending on \( \epsilon \) and \( \gamma_2 = \log(\rho_1^{-1}) \) verifies Assumption (A2).

Lastly, we verify Assumption (A3). Following the same construction as in verifying Assumption (A2), we have for any \( u \in \mathbb{R}^p \),
\[
\| (Y_t - \tilde{Y}_t)^T u \|_{L(1+\epsilon)}
= (\mathbb{E}(\{ A_1 Y_{t-1} + \cdots + A_d Y_{t-d} + E_t - (A_1 \tilde{Y}_{t-1} + \cdots + A_d \tilde{Y}_{t-d} + E_t) \})^Tu^{1+\epsilon} \right)^{\frac{1}{1+\epsilon}}
\leq (\mathbb{E}(\{ A_1 Y_{t-1} + \cdots + A_d Y_{t-d} - (A_1 \tilde{Y}_{t-1} + \cdots + A_d \tilde{Y}_{t-d}) \})^Tu^{1+\epsilon} \right)^{\frac{1}{1+\epsilon}}
\leq \sum_{k=1}^d a_k (\mathbb{E}(Y_{t-k} - \tilde{Y}_{t-k})^Tu_k^{1+\epsilon} \right)^{\frac{1}{1+\epsilon}},
\]
for \( u_k := A_k u/\| A_k u \|_2 \), \( k \in \{1, \ldots, d\} \). The result follows as we follow the same arguments to verify Assumption (A2). This completes the proof of Theorem 3.1.

Lemma 4.10 provides sufficient and necessary conditions for matrix \( \overline{A} \) to have spectral radius strictly less than 1. The proof is as follows.

**Proof of Lemma 4.10.** The result is well known and here we include a proof merely for completeness. First of all, we prove the sufficient condition. A key observation is that the characteristic equation \( \det(\overline{A} - \lambda I_d) = 0 \) for matrix \( \overline{A} \)
\[ f(\lambda) = \lambda^d - a_1\lambda^{d-1} - \cdots - a_{d-1}\lambda^1 - a_d = 0. \]
Assume \( \sum_{j=1}^d a_j \geq 1 \). We obtain \( f(1) = 1 - \sum_{j=1}^d a_j \leq 0 \) and \( f(\infty) = \infty \). By continuity of \( f(\lambda) \), there exists at least one root whose modulus is greater than or equal to 1. This contradicts with the fact that \( \rho(\overline{A}) \) is strictly less than 1.

Secondly, we prove the necessary condition. Suppose there exists a root \( z \in \mathbb{C} \) (the set of complex numbers) of \( f(\lambda) \) such that \( |z| \geq 1 \). Here \( |z| \) is the modulus of \( z \). Then
\[ |z|^d = |a_1 z^{d-1} + \cdots + a_{d-1} z^1 + a_d| \leq a_1 |z|^{d-1} + \cdots + a_{d-1} |z|^1 + a_d. \]
Since \( |z| \geq 1 \), we have \( |z|^{\overline{\epsilon}} \leq |z|^d \) for \( 0 \leq k \leq d - 1 \). Hence \( |z|^d \leq (a_1 + \cdots + a_d)|z|^d \) implies \( a_1 + \cdots + a_d \geq 1 \). This contradicts the fact that \( \sum_{j=1}^d a_j \) is strictly less than 1. This completes the proof.

**Proof of Theorem 3.2.** First of all, we verify Assumptions (A1) and (A4). It is trivial that Assumptions (A1) and (A4) are satisfied if \( W_t \) is 0 almost surely for all \( t \in \mathbb{Z} \). If \( W_t \neq 0 \) almost surely, then for all \( u \in \mathbb{R}^p, \| u^T Y_t \|_{\psi_2} \leq \| W_t \|_{L(\infty)} \| u^T E_t \|_{\psi_2} \leq c^W \kappa W \| u^T E_t \|_{L(2)} \leq c^W \inf_{t \in \mathbb{Z}} \| W_t \|_{L(2)} \| u^T Y_t \|_{L(2)} < \infty \). This verifies Assumptions (A1) and (A4).

For Assumption (A2), without loss of generality, take \( j = 0 \). Since \( \{ W_t \}_{t \in \mathbb{Z}} \) is a sequence of uniformly bounded \( \tau \)-mixing random variables, we could find \( \{ \tilde{W}_t \}_{t \geq 0} \) which is independent of
for some constant $C > 0$ only depending on $\epsilon$. Taking $\gamma_1 = C\kappa'_s\kappa W\gamma_5^{1/\epsilon}/\kappa_1$ and $\gamma_2 = \frac{1}{1+\epsilon}\gamma_6$ verifies Assumption (A2).

For Assumption (A3), without loss of generality, take $j = 0$. Let $\{\tilde{Y}_t\}_{t > 0}$ be the same construction as above. For any integer $t \geq 1$,

$$
\sup_{u \in \mathbb{S}^{p-1}} \| (Y_t - \tilde{Y}_t)^T u \|_{L(1+\epsilon)} = \sup_{u \in \mathbb{S}^{p-1}} \{ \mathbb{E}[|W_t E_t - \tilde{W}_t E_t|^{1+\epsilon}] \}^{1/(1+\epsilon)}
\leq (\mathbb{E}[W_t - \tilde{W}_t] \cdot |W_t - \tilde{W}_t|^{1+\epsilon})^{1/(1+\epsilon)} (\mathbb{E}[|E_t|^2]^{1+\epsilon})^{1/(1+\epsilon)}
\leq C\kappa'_s\kappa W\gamma_5^{1/\epsilon} \exp \left\{ -\frac{1}{1+\epsilon}\gamma_6(t-1) \right\}
$$

for some constant $C > 0$ only depending on $\epsilon$. Taking $\gamma_3 = C\kappa'_s\kappa W\gamma_5^{1/\epsilon}/\kappa_1$ and $\gamma_4 = \frac{1}{1+\epsilon}\gamma_6$ verifies Assumption (A2). This completes the proof of Theorem 3.2. □

**Proof of Theorem 3.3.** We first verify Assumptions (A2) and (A3). Without loss of generality, take $j = 0$ in Assumption (A2). Let $\tilde{Y}_0$ be a random vector independent of $\{Y_t\}_{t \leq 0}$ and identically distributed as $Y_0$. Define $\tilde{Y}_t = A\tilde{Y}_{t-1} + H(\tilde{Y}_{t-1})E_t$ for every $t \geq 1$. It is obvious that $\{\tilde{Y}_t\}_{t > 0}$ is independent of $\{Y_t\}_{t \leq 0}$ and identically distributed as $\{Y_t\}_{t > 0}$. We obtain for any $t \geq 1$,

$$
\|Y_t - \tilde{Y}_t\|_{L(1+\epsilon)} = \mathbb{E}[|A(Y_{t-1} + H(Y_{t-1})E_t) - (A\tilde{Y}_{t-1} + H(\tilde{Y}_{t-1})E_t)|^{1+\epsilon}] \|Y_{t-1} - \tilde{Y}_{t-1}\|_{L(1+\epsilon)}
\leq \mathbb{E}[|A(Y_{t-1} + H(Y_{t-1})E_t) - (A\tilde{Y}_{t-1} + H(\tilde{Y}_{t-1})E_t)|^{1+\epsilon}] \|Y_{t-1} - \tilde{Y}_{t-1}\|_{L(1+\epsilon)}
\leq (a_1 + a_2)\|Y_{t-1} - \tilde{Y}_{t-1}\|_{L(1+\epsilon)}.
$$

By iteration, we obtain

$$
\|Y_t - \tilde{Y}_t\|_{L(1+\epsilon)} \leq (a_1 + a_2)^t \mathbb{E}[|Y_0 - \tilde{Y}_0|^{1+\epsilon}] \leq C\kappa_s(a_1 + a_2)^t
$$

for some constant $C > 0$ only depending on $\epsilon$. Taking $\gamma_1 = C\kappa_s/\kappa_1$ and $\gamma_2 = -\log(a_1 + a_2)$ verifies Assumption (A2).

For Assumption (A3), following the construction above, we have for any $u \in \mathbb{S}^{p-1}$ and $t \geq 1$,

$$
\| (Y_t - \tilde{Y}_t)^T u \|_{L(1+\epsilon)} = \mathbb{E}[|A(Y_{t-1} + H(Y_{t-1})E_t) - (A\tilde{Y}_{t-1} + H(\tilde{Y}_{t-1})E_t)|^{1+\epsilon}] \|Y_{t-1} - \tilde{Y}_{t-1}\|_{L(1+\epsilon)}
\leq \mathbb{E}[|A(Y_{t-1} + H(Y_{t-1})E_t) - (A\tilde{Y}_{t-1} + H(\tilde{Y}_{t-1})E_t)|^{1+\epsilon}] \|Y_{t-1} - \tilde{Y}_{t-1}\|_{L(1+\epsilon)}
\leq (a_1 + a_2)\|Y_{t-1} - \tilde{Y}_{t-1}\|_{L(1+\epsilon)}.$$

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where \( v := Au/\|Au\|_2 \in \mathbb{S}^{p-1} \). By iteration, we obtain

\[
\|(Y_t - \tilde{Y}_t)^\top u\|_{L(1+\epsilon)} \leq C\{\kappa_1a_1' + 2\kappa_s \kappa_1'/\kappa_s' a_2 \sum_{\ell=0}^{t-1} a_1'(a_1 + a_2)'^{1-\ell}\} \leq C(a_1 + a_2)'\max(\kappa_s, \kappa_1')
\]

for some constant \( C > 0 \) only depending on \( \epsilon \). Taking \( \gamma_3 = C\max(\frac{\kappa_s + \kappa_1'}{\kappa_s \kappa_1'}, 1) \) and \( \gamma_4 = -\log(a_1 + a_2) \) verifies Assumption \textbf{(A3)}.

By further assuming that \{\( Y_t \)\} is a stationary process and \( H(\cdot) \) is uniformly bounded, we have that for all \( t \in \mathbb{Z} \), \( \sup_{u \in \mathbb{S}^{p-1}} \|u^\top Y_t\|_{\psi_2} \leq \|A\| \sup_{u \in \mathbb{S}^{p-1}} \|u^\top Y_{t-1}\|_{\psi_2} + D_2 \sup_{u \in \mathbb{S}^{p-1}} \|u^\top E_t\| \). By stationarity, this renders \( \kappa_1 = \sup_{u \in \mathbb{S}^{p-1}} \|u^\top Y_t\|_{\psi_2} \leq \frac{1}{1-\|A\|} D_2 \kappa'_1 < \infty \). Similar argument applies to \( \kappa_s \). This verifies Assumption \textbf{(A1)} under additional assumptions and completes the proof of Theorem 3.3.

\[ \square \]

Appendix

\section{Proof of Theorem 4.3}

In this appendix we present the proof of Theorem 4.3, which slightly extend the Bernstein-type inequality proven by Banna et al. (2016) in which the random matrix sequence is assumed to be \( \beta \)-mixing. The proof is largely identical to theirs, and we include it here mainly for completeness.

In the following, \( \tau_k \) abbreviates \( \tau(k) \) for \( k \geq 1 \). If a matrix \( X \) is positive semidefinite, denote it as \( X \succeq 0 \). For any \( x > 0 \), we define \( h(x) = x^{-2}(e^x - x - 1) \). Denote the floor, ceiling, and integer parts of a real number \( x \) by \( \lfloor x \rfloor \), \( \lceil x \rceil \), and \( [x] \). For any two real numbers \( a, b \), denote \( a \lor b := \max\{a, b\} \). Denote the exponential of matrix \( X \) as \( \exp(X) = I_p + \sum_{q=1}^{\infty} X^q/q! \). Letting \( \sigma_1 \) and \( \sigma_2 \) be two sigma fields, denote \( \sigma_1 \lor \sigma_2 \) to be the smallest sigma field that contains \( \sigma_1 \) and \( \sigma_2 \) as sub-sigma fields.

A roadmap of this appendix is as follows. Section A.1 formally introduces the concept of \( \tau \)-mixing coefficient. Section A.2 contains the construction of Cantor-like set which is essential for decoupling dependence matrices. Section A.3 develops a major decoupling lemma for \( \tau \)-mixing random matrices and will be used in Section A.5 to prove Lemma A.4. Then Section A.4 finishes the proof of Theorem 4.3.

\subsection{Introduction to \( \tau \)-mixing random sequence}

This sections rigorously introduces the \( \tau \)-mixing coefficient. Consider \((\Omega, \mathcal{F}, \mathbb{P})\) to be a probability space, \( X \) an \( L_1 \)-integrable random variable taking value in a Polish space \((\mathcal{X}, \| \cdot \|_\mathcal{X})\), and \( \mathcal{A} \) a sigma algebra of \( \mathcal{F} \). The \( \tau \)-measure of dependence between \( X \) and \( \mathcal{A} \) is defined to be

\[
\tau(\mathcal{A}, X; \| \cdot \|_\mathcal{X}) = \sup_{g \in \Lambda(\| \cdot \|_\mathcal{X})} \left\{ \int g(x)\mathbb{P}_{X|\mathcal{A}}(dx) - \int g(x)\mathbb{P}_X(dx) \right\} \|_{L(1)},
\]

where \( \mathbb{P}_X \) is the distribution of \( X \), \( \mathbb{P}_{X|\mathcal{A}} \) is the conditional distribution of \( X \) given \( \mathcal{A} \), and \( \Lambda(\| \cdot \|_\mathcal{X}) \) stands for the set of 1-Lipschitz functions from \( \mathcal{X} \) to \( \mathbb{R} \) with respect to the norm \( \| \cdot \|_\mathcal{X} \).
The following two lemmas from Dedecker and Prieur (2004) and Dedecker et al. (2007) characterize the intrinsic “coupling property” of $\tau$-measure of dependence, which will be heavily exploited in the derivation of our results.

**Lemma A.1** (Lemma 3 in Dedecker and Prieur (2004)). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $X$ be an integrable random variable with values in a Banach space $(\mathcal{X}, \| \cdot \|_\mathcal{X})$ and $A$ a sigma algebra of $\mathcal{F}$. If $Y$ is a random variable distributed as $X$ and independent of $A$, then

$$\tau(A, X; \| \cdot \|_\mathcal{X}) \leq \mathbb{E}\|X - Y\|_\mathcal{X}.$$ 

**Lemma A.2** (Lemma 5.3 in Dedecker et al. (2007)). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $A$ be a sigma algebra of $\mathcal{F}$, and $X$ be a random variable with values in a Polish space $(\mathcal{X}, \| \cdot \|_\mathcal{X})$. Assume that $\int \|x - x_0\|_\mathcal{X} \mathbb{P}_X(dx)$ is finite for any $x_0 \in \mathcal{X}$. Assume that there exists a random variable $U$ uniformly distributed over $[0, 1]$, independent of the sigma algebra generated by $X$ and $A$. Then there exists a random variable $\tilde{X}$, measurable with respect to $A \vee \sigma(X) \vee \sigma(U)$, independent of $A$ and distributed as $X$, such that

$$\tau(A, X; \| \cdot \|_\mathcal{X}) = \mathbb{E}\|X - \tilde{X}\|_\mathcal{X}.$$ 

Let $\{X_j\}_{j \in J}$ be a set of $\mathcal{X}$-valued random variables with index set $J$ of finite cardinality. Then define

$$\tau(A, \{X_j \in \mathcal{X} \}_{j \in J}; \| \cdot \|_\mathcal{X}) = \sup_{g \in \Lambda(\| \cdot \|_\mathcal{X}')} \left\{ \int g(x)\mathbb{P}_{\{X_j \}_{j \in J}}(dx) - \int g(x)\mathbb{P}_{\{X_j \}_{j \in J}, A}(dx) \right\},$$

where $\mathbb{P}_{\{X_j \}_{j \in J}}$ is the distribution of $\{X_j \}_{j \in J}$, $\mathbb{P}_{\{X_j \}_{j \in J}, A}$ is the conditional distribution of $\{X_j \}_{j \in J}$ given $A$, and $\Lambda(\| \cdot \|_\mathcal{X}')$ stands for the set of $1$-Lipschitz functions from $\mathcal{X} \times \cdots \times \mathcal{X}$ to $\mathbb{R}$ with respect to the norm $\|x\|_\mathcal{X}' := \sum_{j \in J} \|x_j\|_\mathcal{X}$ for any $x = (x_1, \ldots, x_J) \in \mathcal{X}^{\text{card}(J)}$.

Using these concepts, for a sequence of temporally dependent data $\{X_t\}_{t \in \mathbb{Z}}$, we are ready to define measure of temporal correlation strength as follows,

$$\tau(k; \{X_t \}_{t \in \mathbb{Z}}, \| \cdot \|_\mathcal{X}) := \sup_{i > 0} \max_{1 \leq \ell \leq i} \frac{1}{\ell} \sup_{a + k \leq j_1 < \cdots < j_{\ell}} \tau(\sigma(X^a_{\infty}), \{X_{j_1}, \ldots, X_{j_\ell}\}; \| \cdot \|_\mathcal{X}),$$

where the inner supremum is taken over all $a \in \mathbb{Z}$ and all $\ell$-tuples $(j_1, \ldots, j_{\ell})$. $\{X_t \}_{t \in \mathbb{Z}}$ is said to be $\tau$-mixing if $\tau(k; \{X_t \}_{t \in \mathbb{Z}}, \| \cdot \|_\mathcal{X})$ converges to zero as $k \to \infty$. In Dedecker et al. (2007) the authors gave numerous examples of random sequences that are $\tau$-mixing.

**A.2 Construction of Cantor-like set**

We follow Banna et al. (2016) to construct the Cantor-like set $K_B$ for $\{1, \ldots, B\}$. Let $\delta = \frac{\log 2}{2 \log B}$ and $\ell_B = \sup \{k \in \mathbb{Z}^+ : \frac{B(1-\delta)^{k-1}}{2^k} \geq 2 \}$. We abbreviate $\ell := \ell_B$. Let $n_0 = B$ and for $j \in \{1, \ldots, \ell\}$, $n_j = \left\lfloor \frac{B(1-\delta)^j}{2^j} \right\rfloor$ and $d_{j-1} = n_{j-1} - 2n_j$.

We start from the set $\{1, \ldots, B\}$ and divide the set into three disjoint subsets $I^1_1, I^1_0, I^2_1$ in order with $\text{card}(I^1_1) = \text{card}(I^2_1) = n_1$ and $\text{card}(I^1_0) = d_0$. Specifically, $I^1_1 = \{1, \ldots, n_1\}$, $I^1_0 = \{n_1 + 1, \ldots, n_1 + d_0\}$, $I^2_1 = \{n_1 + d_0 + 1, \ldots, 2n_1 + d_0\}$, where $B = 2n_1 + d_0$. Then we divide $I^1_1, I^2_1$. 

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ties are equal and even. $I_1^d$ is divided into three disjoint subsets $I_1^1, I_1^2, I_2^d$ in the same way as the previous step with \( \text{card}(I_2^d) = \text{card}(I_2^2) = n_2 \) and \( \text{card}(I_1^d) = d_1 \). We obtain $I_1^1 = \{n_2 + 1, \ldots, n_2 + d_1\}$, $I_2^2 = \{n_2 + d_1 + 1, \ldots, 2n_2 + d_1\}$, where $n_1 = 2n_2 + d_1$. Similarly, $I_1^2$ is divided into $I_1^3, J_1^2, J_2^2$ with \( \text{card}(I_1^3) = \text{card}(J_2^2) = n_2 \) and \( \text{card}(J_1^2) = d_1 \). We obtain $I_1^3 = \{2n_2 + d_0 + d_1 + 1, \ldots, 3n_2 + d_0 + d_1\}$, $J_1^2 = \{3n_2 + d_0 + d_1 + 1, \ldots, 3n_2 + d_0 + 2d_1\}$, $I_2^4 = \{3n_2 + d_0 + 2d_1 + 1, \ldots, 4n_2 + d_0 + 2d_1\}$, where $B = 4n_2 + d_0 + 2d_1$. Suppose we iterate this process for $k$ times $(k \in \{1, \ldots, \ell\})$ with intervals $I_{k,i}^1, i \in \{1, \ldots, 2^k\}$. For each $I_{k,i}^1$, we divide it into three disjoint subsets $I_{k+1,i}^2, J_{k+1,i}^2, I_{k+1,i+1}^2$ in order with \( \text{card}(I_{k+1,i}^2) = \text{card}(J_{k+1,i}^2) = n_{k+1} \) and \( \text{card}(I_{k+1,i+1}^2) = d_k \). More specifically, if $I_{k,i}^1 = \{a_{k,i}^1, \ldots, b_{k,i}^1\}$, then $I_{k+1,i}^2 = \{a_{k,i}^1, \ldots, a_{k,i}^1 + n_{k+1} - 1\}$, $I_{k,i}^2 = \{a_{k,i}^1 + n_{k+1}, \ldots, a_{k,i}^1 + n_{k+1} + d_k - 1\}$, $J_{k+1,i}^2 = \{a_{k,i}^1 + n_{k+1} + d_k, \ldots, a_{k,i}^1 + 2n_{k+1} + d_k\}$. After \( \ell \) steps, we obtain $2^{\ell}$ disjoint subsets $I_{\ell,i}^1, i \in \{1, \ldots, 2^{\ell}\}$ with \( \text{card}(I_{\ell,i}^1) = n_{\ell} \). Then the Cantor-like set is defined as

\[
K_B = \bigcup_{i=1}^{2^\ell} I_{\ell,i}^1,
\]

and for each level $k \in \{0, \ldots, \ell\}$ and each $j \in \{1, \ldots, 2^k\}$, define

\[
K_k^j = \bigcup_{i=(j-1)2^{\ell-k}+1}^{j2^{\ell-k}} I_{\ell,i}^1.
\]

Some properties derived from this construction are given by Banna et al. (2016):

1. $\delta \leq \frac{1}{2}$ and $\ell \leq \frac{\log B}{\log 2}$;
2. $d_j \geq \frac{B\delta(1-\delta)^j}{2^{j+1}}$ and $n_{\ell} \leq \frac{B(1-\delta)^\ell}{2^{\ell+1}}$;
3. Each $I_{\ell,i}^1, i \in \{1, \ldots, 2^\ell\}$ contains $n_{\ell}$ consecutive integers, and for any $i \in \{1, \ldots, 2^{\ell-1}\}$, $I_{\ell-1,i}^2$ and $I_{\ell-1,i+1}^2$ are spaced by $d_{\ell-1}$ integers;
4. \( \text{card}(K_B) \geq \frac{B}{2} \);
5. For each $k \in \{0, \ldots, \ell\}$ and each $j \in \{1, \ldots, 2^k\}$, \( \text{card}(K_k^j) = 2^{\ell-k} n_{\ell} \). For each $j \in \{1, \ldots, 2^{\ell-1}\}$, $K_{k-1}^{j-1}$ and $K_{k-1}^j$ are spaced by $d_{\ell-1}$ integers;
6. $K_0^1 = K_B$ and $K_\ell^j = I_{\ell,i}^1$ for $j \in \{1, \ldots, 2^\ell\}$.

### A.3 A decoupling lemma for $\tau$-mixing random matrices

This section introduces the key tool to decouple $\tau$-mixing random matrices using Cantor-like set constructed in Section A.2. Let \( \{X_j\}, j \in \{1, \ldots, n\} \), be a sequence of $p \times p$ symmetric random matrices. In addition, $\mathbb{E}(X_j) = 0$ and $\|X_j\| \leq M$ for some positive constant $M$ and for all $j \geq 1$. For a collection of index sets $H_1^k, k \in \{1, \ldots, d\}$, we assume that their cardinalities are equal and even. Denote $\{X_j\}_{j \in H_1^k}$ to be the set of matrices whose indices are in $H_1^k$. Assume $\{X_j\}_{j \in H_1^1}, \ldots, \{X_j\}_{j \in H_1^d}$ are mutually independent, while within each block $H_1^k$ the matrices are possibly dependent. For each $k$, decompose $H_1^k$ into two disjoint sets $H_2^{2k-1}$ and $H_2^{2k}$.
with equal size, containing the first and second half of $H_4^k$ respectively. In addition, we denote $\tau_0 := \tau \{ \sigma(\{X_j\}_{j \in H_2^{2k-1}}), \{X_j\}_{j \in H_2^{2k}}; \| \cdot \| \} \text{ for some constant } \tau_0 \geq 0 \text{ and for all } k \in \{1, \ldots, d\}. \text{ For a given } \epsilon > 0, \text{ we achieve the following decoupling lemma.}

**Lemma A.3.** We obtain for any $\epsilon > 0$,

$$E \text{ Tr exp} \left( t \sum_{k=1}^{d} \sum_{j \in H_1^k} X_j \right) \leq \sum_{i=0}^{d} \left( \frac{d}{i} \right) (1 + L_1 + L_2)^{d-i}(L_1)^i E \text{ Tr exp} \left\{ (-1)^i t \left( \sum_{k=1}^{2d} \sum_{j \in H_1^k} \tilde{X}_j \right) \right\},$$

$$E \text{ Tr exp} \left( -t \sum_{k=1}^{d} \sum_{j \in H_1^k} X_j \right) \leq \sum_{i=0}^{d} \left( \frac{d}{i} \right) (1 + L_1 + L_2)^{d-i}(L_1)^i E \text{ Tr exp} \left\{ (-1)^{i+1} t \left( \sum_{k=1}^{2d} \sum_{j \in H_2^k} \tilde{X}_j \right) \right\},$$

where

$$L_1 := p t \epsilon \exp(t \epsilon), \quad L_2 := \exp\{\text{card}(H_1^1)tM\} \tau_0 / \epsilon,$$

and $\{\tilde{X}_j\}_{j \in H_1^k}, \ k \in \{1, \ldots, 2d\}$, are mutually independent and have the same distributions as $\{X_j\}_{j \in H_1^k}, \ k \in \{1, \ldots, 2d\}$.

**Proof.** We prove this lemma by induction. For any $k \in \{1, \ldots, d\}$, we have $H_4^k = H_2^{2k-1} \cup H_2^{2k}$ and hence $\sum_{j \in H_1^k} X_j = \sum_{j \in H_2^{2k-1}} X_j + \sum_{j \in H_2^{2k}} X_j$.

By Lemma A.2, for each $k \in \{1, \ldots, d\}$, we could find a sequence of random matrices $\{\tilde{X}_j\}_{j \in H_2^{2k}}$ and an independent uniformly distributed random variable $U_k$ on $[0, 1]$ such that

1. $\{\tilde{X}_j\}_{j \in H_2^{2k}}$ is measurable with respect to the sigma field $\sigma(\{X_j\}_{j \in H_2^{2k-1}}) \vee \sigma(\{X_j\}_{j \in H_2^{2k}}) \vee \sigma(U_k)$;

2. $\{\tilde{X}_j\}_{j \in H_2^{2k}}$ is independent of $\sigma(\{X_j\}_{j \in H_2^{2k-1}})$;

3. $\{\tilde{X}_j\}_{j \in H_2^{2k}}$ has the same distribution as $\{X_j\}_{j \in H_2^{2k}}$;

4. $\mathbb{P}(\|\sum_{j \in H_2^{2k}} X_j - \sum_{j \in H_2^{2k}} \tilde{X}_j\| > \epsilon_k) \leq \mathbb{E}(\|\sum_{j \in H_2^{2k}} X_j - \sum_{j \in H_2^{2k}} \tilde{X}_j\|) / \epsilon_k \leq \tau_0 / \epsilon_k$ by Markov’s inequality and the fact that $\tau_0 = \sum_{j \in H_2^{2k}} \mathbb{E}(\|X_j - \tilde{X}_j\|)$.

To make notation easier to follow, we set equal value to $\epsilon_k$ for $k \in \{1, \ldots, d\}$ and denote it as $\epsilon$. Moreover, we denote the event $\Gamma_k = \{\|\sum_{j \in H_2^{2k}} X_j - \sum_{j \in H_2^{2k}} \tilde{X}_j\| \leq \epsilon\}$ for $k \in \{1, \ldots, d\}$.

For the base case $k = 1$,

$$E \text{ Tr exp} \left( t \sum_{k=1}^{d} \sum_{j \in H_1^k} X_j \right) = E \left\{ 1_{\Gamma_1} \text{ Tr exp} \left( t \sum_{k=1}^{d} \sum_{j \in H_1^k} X_j \right) \right\} + E \left\{ 1_{(\Gamma_1)^c} \text{ Tr exp} \left( t \sum_{k=1}^{d} \sum_{j \in H_1^k} X_j \right) \right\} = I + II.$$

We have

$$I = E \left[ 1_{\Gamma_1} \text{ Tr exp} \left\{ t \left( \sum_{j \in H_2^1} X_j + \sum_{j \in H_2^1} \tilde{X}_j + \sum_{k=2}^{d} \sum_{j \in H_1^k} X_j \right) \right\} \right]$$

$$\leq E \text{ Tr exp} \left\{ t \left( \sum_{j \in H_2^1} X_j + \sum_{j \in H_2^1} \tilde{X}_j + \sum_{k=2}^{d} \sum_{j \in H_1^k} X_j \right) \right\}$$

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By linearity of expectation and the facts that $\text{Tr}(X) \leq p\|X\|$ and $\|\exp(X) - \exp(Y)\| \leq \|X - Y\|$ exp($\|X - Y\|\)) exp(||Y||), we obtain

\[
\mathbb{E}\left(1_{\Gamma_1}\left[\text{Tr}\exp\left\{t\left(\sum_{j \in H^1_2} X_j + \sum_{j \in H^2_2} X_j + \sum_{k=2}^d \sum_{j \in H^k_2} X_j\right)\right\} - \text{Tr}\exp\left\{t\left(\sum_{j \in H^1_2} X_j + \sum_{j \in H^2_2} \tilde{X}_j + \sum_{k=2}^d \sum_{j \in H^k_2} X_j\right)\right\}\right]\right)
\leq \mathbb{E}\left(1_{\Gamma_1} p\left\|\exp\left\{t\left(\sum_{j \in H^1_2} X_j + \sum_{j \in H^2_2} X_j + \sum_{k=2}^d \sum_{j \in H^k_2} X_j\right)\right\} - \exp\left\{t\left(\sum_{j \in H^1_2} X_j + \sum_{j \in H^2_2} \tilde{X}_j + \sum_{k=2}^d \sum_{j \in H^k_2} X_j\right)\right\}\right\|\right)
\leq \mathbb{E}\left(1_{\Gamma_1} p\left\|t \sum_{j \in H^2_2} (X_j - \tilde{X}_j)\right\| \exp\left\{\left\|t \sum_{j \in H^2_2} (X_j - \tilde{X}_j)\right\|\right\} \exp\left\{\left\|t \left(\sum_{j \in H^1_2} X_j + \sum_{j \in H^2_2} \tilde{X}_j + \sum_{k=2}^d \sum_{j \in H^k_2} X_j\right)\right\|\right\}\right).
\]

By spectral mapping theorem, for a symmetric matrix $X$ with $\|X\| \leq M$, we have $\exp(\|X\|) \leq \|\exp(X)\| \lor \|\exp(-X)\| \leq \|\exp(X)\| + \|\exp(-X)\|$. Moreover, since $\exp(X)$ is always positive definite for any matrix $X$ and $\|X\| \leq \text{Tr}(X)$ for any positive definite symmetric matrix $X$, we obtain $\|\exp(X)\| \leq \text{Tr}(\exp(X))$ and $\|\exp(-X)\| \leq \text{Tr}(\exp(-X))$. In addition, since we have $\|\sum_{j \in H^2_2} (X_j - \tilde{X}_j)\| \leq \epsilon$ on $\Gamma_1$, we could further bound the inequality above by

\[
\mathbb{E}\left[1_{\Gamma_1} p t\epsilon \exp(te)\left\|\exp\left\{t\left(\sum_{j \in H^1_2} X_j + \sum_{j \in H^2_2} \tilde{X}_j + \sum_{k=2}^d \sum_{j \in H^k_2} X_j\right)\right\}\right\|\right]
\leq p t\epsilon \exp(te) \mathbb{E} \text{Tr} \exp\left\{t\left(\sum_{j \in H^1_2} X_j + \sum_{j \in H^2_2} \tilde{X}_j + \sum_{k=2}^d \sum_{j \in H^k_2} X_j\right)\right\}
+ \mathbb{E} \text{Tr} \exp\left\{-t\left(\sum_{j \in H^1_2} X_j + \sum_{j \in H^2_2} \tilde{X}_j + \sum_{k=2}^d \sum_{j \in H^k_2} X_j\right)\right\}.
\]

Putting together, we reach

\[
I \leq \{1 + p t\epsilon \exp(te)\}\mathbb{E} \text{Tr} \exp\left\{t\left(\sum_{j \in H^1_2} X_j + \sum_{j \in H^2_2} \tilde{X}_j + \sum_{k=2}^d \sum_{j \in H^k_2} X_j\right)\right\}
+ p t\epsilon \exp(te) \mathbb{E} \text{Tr} \exp\left\{-t\left(\sum_{j \in H^1_2} X_j + \sum_{j \in H^2_2} \tilde{X}_j + \sum_{k=2}^d \sum_{j \in H^k_2} X_j\right)\right\}.
\]

We then aim at $II$. For this, the proof largely follows the same argument as in Banna et al. (2016). Omitting the details, we obtain

\[
II \leq \exp\{\text{card}(H^1_1) M\}(\tau_0/\epsilon)\mathbb{E} \text{Tr} \exp\left\{t\left(\sum_{j \in H^1_2} X_j + \sum_{j \in H^2_2} \tilde{X}_j + \sum_{k=2}^d \sum_{j \in H^k_2} X_j\right)\right\}.
\]

(A.1)

(A.2)
Denote $L_1 := pte \exp(\epsilon t)$ and $L_2 := \exp(\{\text{card}(H_1^k)\}tM)\tau_0/\epsilon$. Combining (A.1) and (A.2) yields
\[
\mathbb{E} \text{Tr} \exp \left( t \sum_{k=1}^{d} \sum_{j \in H_1^k} X_j \right) \\
\leq (1 + L_1 + L_2) \mathbb{E} \text{Tr} \exp \left\{ t \left( \sum_{j \in H_2^1} X_j + \sum_{j \in H_2^2} \tilde{X}_j + \sum_{k=2}^{d} \sum_{j \in H_1^k} X_j \right) \right\} \\
+ L_1 \mathbb{E} \text{Tr} \exp \left\{ -t \left( \sum_{j \in H_2^1} X_j + \sum_{j \in H_2^1} \tilde{X}_j + \sum_{k=2}^{d} \sum_{j \in H_1^k} X_j \right) \right\} \\
= \sum_{i=0}^{1} \binom{1}{i} (1 + L_1 + L_2)^{1-i}(L_1)^i \mathbb{E} \text{Tr} \exp \left\{ (-1)^i t \left( \sum_{j \in H_2^1} X_j + \sum_{j \in H_2^2} \tilde{X}_j + \sum_{k=2}^{d} \sum_{j \in H_1^k} X_j \right) \right\}.
\]
This finishes the base case.

The induction steps are followed similarly and we omit the details. By iterating $d$ times, we arrive at the following inequality:
\[
\mathbb{E} \text{Tr} \exp \left( t \sum_{k=1}^{d} \sum_{j \in H_1^k} X_j \right) \\
\leq \sum_{i=0}^{1} \binom{d}{i} (1 + L_1 + L_2)^{d-i}(L_1)^i \mathbb{E} \text{Tr} \exp \left\{ (-1)^i t \left( \sum_{k=1}^{2d} \sum_{j \in H_2^k} X_j + \sum_{k=1}^{d} \sum_{j \in H_2^k} \tilde{X}_j \right) \right\},
\]
where $\{X_j\}_{j \in H_2^{2k-1}}, k \in \{1, \ldots, d\}$ and $\{\tilde{X}_j\}_{j \in H_2^{2k}}, k \in \{1, \ldots, d\}$ are mutually independent. In addition, they have the same distributions as $\{\tilde{X}_j\}_{j \in H_2^{2k-1}}, k \in \{1, \ldots, d\}$ and $\{X_j\}_{j \in H_2^{2k}}, k \in \{1, \ldots, d\}$, respectively. For the sake of simplicity and clarity, we add an upper tilde to the matrices with indices in $H_2^{2k-1}, k \in \{1, \ldots, d\}$, i.e., $\{\tilde{X}_j\}_{j \in H_2^{2k-1}}$ is identically distributed as $\{X_j\}_{j \in H_2^{2k-1}}$ for $k \in \{1, \ldots, d\}$. Hence (A.3) could be rewritten as
\[
\mathbb{E} \text{Tr} \exp \left( t \sum_{k=1}^{d} \sum_{j \in H_1^k} X_j \right) \leq \sum_{i=0}^{1} \binom{d}{i} (1 + L_1 + L_2)^{d-i}(L_1)^i \mathbb{E} \text{Tr} \exp \left\{ (-1)^i t \left( \sum_{k=2}^{2d} \sum_{j \in H_2^k} \tilde{X}_j \right) \right\},
\]
where $\{\tilde{X}_j\}_{j \in H_2^k}, k \in \{1, \ldots, 2d\}$ are mutually independent and their distributions are the same as $\{X_j\}_{j \in H_2^k}, k \in \{1, \ldots, 2d\}$.

By changing $X$ to $-X$, we immediately get the following bound:
\[
\mathbb{E} \text{Tr} \exp \left( -t \sum_{k=1}^{d} \sum_{j \in H_1^k} X_j \right) \leq \sum_{i=0}^{1} \binom{d}{i} (1 + L_1 + L_2)^{d-i}(L_1)^i \mathbb{E} \text{Tr} \exp \left\{ (-1)^{i+1} t \left( \sum_{k=1}^{2d} \sum_{j \in H_2^k} \tilde{X}_j \right) \right\}.
\]
This completes the proof of Lemma A.3. \hfill \Box

\subsection{Proof of Theorem 4.3}

\textbf{Proof.} Without loss of generality, let $\psi_1 = \tilde{\psi}_1$.

\textbf{Case I.} First of all, we consider $M = 1$.  

\pagebreak
Step I (Summation decomposition). Let $B_0 = n$ and $U_j^{(0)} = X_j$ for $j \in \{1, \ldots, n\}$. Let $K_{B_0}$ be the Cantor-like set from $\{1, \ldots, B_0\}$ by construction of Section A.2, $K_{B_0}^c = \{1, \ldots, B_0\} \setminus K_{B_0}$, and $B_1 = \text{card}(K_{B_0}^c)$. Then define

$$U_j^{(1)} = X_{i_j},$$

where $i_j \in K_{B_0}^c = \{i_1, \ldots, i_{B_1}\}$.

For each $i \geq 1$, let $K_{B_i}$ be constructed from $\{1, \ldots, B_i\}$ by the same Cantor-like set construction. Denote $K_{B_i}^c = \{1, \ldots, B_i\} \setminus K_{B_i}$ and $B_{i+1} = \text{card}(K_{B_i}^c)$. Then

$$U_j^{(i+1)} = U_{k_j}^{(i)},$$

where $k_j \in K_{B_i}^c = \{k_1, \ldots, k_{B_{i+1}}\}$.

We stop the process when there is a smallest $L$ such that $B_L \leq 2$. Then we have for $i \leq L - 1$, $B_i \leq n 2^{-i}$ because each Cantor-like set $K_{B_{i+1}}$ has cardinality greater than $B_{i}/2$. Also notice that $L \leq \lfloor \log n / \log 2 \rfloor.$

For $i \in \{0, \ldots, L - 1\}$, denote

$$S_i = \sum_{j \in K_{B_i}} U_j^{(i)}$$

and $S_L = \sum_{j \in K_{B_{L-1}}^c} U_j^{(L)}$.

Then we observe

$$\sum_{j=1}^{n} X_j = \sum_{i=0}^{L} S_i.$$

Step II (Bounding Laplacian transform). This step hinges on the following lemma, which provides an upper bound for the Laplace transform of sum of a sequence of random matrices which are $\tau$-mixing with geometric decay, i.e., $\tau(k) \leq \psi_1 \exp(-\psi_2(k-1))$ for all $k \geq 1$ for some constants $\psi_1, \psi_2 > 0$.

Lemma A.4. For a sequence of $p \times p$ matrices $\{X_i\}, \ i \in \{1, \ldots, B\}$ satisfying conditions in Theorem 4.3 with $M = 1$ and $\psi_1 \geq p^{-1}$, there exists a subset $K_B \subseteq \{1, \ldots, B\}$ such that for $0 < t \leq \min\{1, \frac{\psi_2}{8\log(\psi_1 B^{p}))}\}$,

$$\log \mathbb{E} \text{Tr} \exp \left( t \sum_{j \in K_B} X_j \right) \leq \log p + 4h(4)Bt^2\psi^2 + 151 \left[ 1 + \exp \left( \frac{1}{\sqrt{p}} \exp \left( -\frac{\psi_2}{64t} \right) \right) \right] t^2 \psi_2 \exp \left( -\frac{\psi_2}{64t} \right).$$

For each $S_i, i \in \{0, \ldots, L - 1\}$, by applying Lemma A.4 with $B = B_i$, we have for any positive $t$ satisfying $0 < t \leq \min\{1, \frac{\psi_2}{8\log(\psi_1 n 2^{-i}) p})\}$,

$$\log \mathbb{E} \text{Tr} \exp(t S_i) \leq \log p + t^2(C_1 2^{-i} n + C_{2,i})$$

where $C_1 := 4h(4)\psi^2, C_{2,i} := 302 \cdot 2^{\psi_2} / \psi_2 n^{\psi_2}$.

Denote

$$\bar{f}(\psi_1, \psi_2, i) := \min\left\{ 1, \frac{\psi_2}{8\log(\psi_1 n 2^{-i}) p})\right\}.$$

For any $0 < t \leq \bar{f}(\psi_1, \psi_2, i)$, we obtain

$$\log \mathbb{E} \text{Tr} \exp(t S_i) \leq \log p + \frac{t^2(C_1 2^{-i} n + C_{2,i})}{1 - t/\bar{f}(\psi_1, \psi_2, i)} \leq \log p + \frac{t^2(C_1^2(2^{-i} n)^2 + C_{2,i}^2)}{1 - t/\bar{f}(\psi_1, \psi_2, i)}.$$
For $S_L$, since $B_L \leq 2$, for $0 < t \leq 1$,
\[ \log \mathbb{E} \text{Tr} \exp(tS_L) \leq \log p + t^2 h(2t)\lambda_{\max}\{\mathbb{E}(S_L^2)\} \leq \log p + \frac{2t^2\nu^2}{1-t}. \]
Denote $\sigma_i := C_1^2(2^{-i}n)^{1/2} + C_2^2$, $\sigma_L := \sqrt{2\nu}$, $\kappa_i := 1/\tilde{f}(\psi_1, \psi_2, i)$, and $\kappa_L := 1$. Summing up, we have
\[ \sum_{i=0}^{L} \sigma_i = \sum_{i=0}^{L-1} \{C_1^2(2^{-i}n)^{1/2} + C_2^2\} + \sqrt{2\nu} \leq 15\sqrt{n}\nu + 60\sqrt{1/\psi_2}, \]
\[ \frac{\sum_{i=0}^{L} \kappa_i}{\log 2} \max \{1, \frac{8\log(\psi_1n^6p)}{\psi_2}\} := \tilde{\psi}(\psi_1, \psi_2, n, p). \]
Hence by Lemma 3 in Merlevède et al. (2009), for $0 < t \leq \{\tilde{\psi}(\psi_1, \psi_2, n, p)\}^{-1}$, we have
\[ \log \mathbb{E} \text{Tr} \exp \left( t \sum_{j=1}^{n} X_j \right) \leq \log p + \frac{t^2 \left( 15\sqrt{n}\nu + 60\sqrt{1/\psi_2} \right)^2}{1 - t\tilde{\psi}(\psi_1, \psi_2, n, p)}. \]

**Step III (Matrix Chernoff bound).** Lastly by matrix Chernoff bound, we obtain
\[ \mathbb{P}\left\{ \lambda_{\max} \left( \sum_{j=1}^{n} X_j \right) \geq x \right\} \leq p \exp \left\{ - \frac{x^2}{8(15^2n\nu^2 + 60^2/\psi_2) + 2x\tilde{\psi}(\psi_1, \psi_2, n, p)} \right\}. \]

**Case II.** We consider general $M > 0$. It is obvious that if $\{X_t\}_{t \in \mathbb{Z}}$ is a sequence of $\tau$-mixing random matrices such that $\tau(k; \{X_t\}_{t \in \mathbb{Z}}, \| \cdot \|) \leq M \psi_1 \exp(-\psi_2(k-1))$, then $\{X_t/M\}_{t \in \mathbb{Z}}$ is also a sequence of $\tau$-mixing random matrices such that $\tau(k; \{X_t/M\}_{t \in \mathbb{Z}}, \| \cdot \|) \leq \psi_1 \exp(-\psi_2(k-1))$ and $\|X_t/M\| \leq 1$. Then applying the result of Case I to $\{X_t/M\}_{t \in \mathbb{Z}}$, we obtain
\[ \mathbb{P}\left\{ \lambda_{\max} \left( \sum_{j=1}^{n} X_j/M \right) \geq x \right\} \leq p \exp \left\{ - \frac{x^2}{8(15^2n\nu^2_M + 60^2/\psi_2) + 2x\tilde{\psi}(\psi_1, \psi_2, n, p)} \right\}, \]
where $\nu^2_M := \sup_{K \subseteq \{1, \ldots, n\}} \frac{1}{\text{card}(K)} \lambda_{\max}\left\{ \mathbb{E}\left( \sum_{i \in K} X_i/M \right)^2 \right\} = \nu^2/M^2$ for $\nu^2$ defined in Theorem 4.3. Thus
\[ \mathbb{P}\left\{ \lambda_{\max} \left( \sum_{j=1}^{n} X_j \right) \geq x \right\} \leq p \exp \left\{ - \frac{x^2}{8(15^2n\nu^2 + 60^2M^2/\psi_2) + 2xM\tilde{\psi}(\psi_1, \psi_2, n, p)} \right\}. \]
This completes the proof of Theorem 4.3. \qed

### A.5 The proof of Lemma A.4

**Proof.** Let $K_B$ be constructed as in Section A.2 for any arbitrary $B \geq 2$ and $M = 1$.

**Case I.** If $0 < t \leq 4/B$, by Lemma 4 in Banna et al. (2016), we have
\[ \mathbb{E} \text{Tr} \left\{ \exp \left( t \sum_{i \in K_B} X_i \right) \right\} \leq \exp \left[ t^2h\left( t\lambda_{\max}\left( \sum_{i \in K_B} X_i \right) \right) \lambda_{\max}\left\{ \mathbb{E}\left( \sum_{i \in K_B} X_i \right)^2 \right\} \right]. \]
By Weyl’s inequality, $\lambda_{\max}(\sum_{i \in K_B} X_i) \leq B$ since $\text{card}(K_B) \leq B$, and by definition of $\nu^2$ in Theorem 4.3, we have $\lambda_{\max}\{\mathbb{E}(\sum_{i \in K_B} X_i)^2\} \leq B\nu^2$. Therefore, we obtain $h(t\lambda_{\max}(\sum_{i \in K_B} X_i)) \leq h(tB) \leq$
Similarly, we obtain

$$h(4) \text{ and } \mathbb{E} \operatorname{Tr} \exp \left( t \sum_{i \in K_B} X_i \right) \leq p \exp \left\{ t^2 h(4) B \nu^2 \right\}. \quad (A.4)$$

**Case II.** Now we consider the case where $4/B < t \leq \min \{ 1, \frac{\psi_2}{8 \log (1/\ell_B \nu)} \}$.

**Step I.** Let $J$ be a chosen integer from $\{0, \ldots, \ell_B \}$ whose actual value will be determined later. We will use the same notation to denote Cantor-like sets as in Section A.2. By Lemma A.3 and similar induction argument as in Banna et al. (2016), we obtain

$$\mathbb{E} \operatorname{Tr} \exp \left( t \sum_{j \in K_0^1} X_j \right) \leq \sum_{i_1=0}^{2^j} \cdots \sum_{i_J=0}^{2^J-1} \left( \prod_{k=1}^{J} A_{k,i_k} \right) \mathbb{E} \operatorname{Tr} \exp \left\{ (-1)^{\sum_{k=1}^{J} i_k} t \left( \sum_{i=1}^{2^J} \sum_{j \in K_j^\ell} \tilde{X}_j \right) \right\} \quad (A.5)$$

where $\{ \tilde{X}_j \}_{j \in K_j^\ell}$ for $i \in \{1, \ldots, 2^J \}$ are mutually independent and have the same distributions as $\{ X_j \}_{j \in K_j^\ell}$ for $i \in \{1, \ldots, 2^J \}$, and

$$A_{k,i_k} := (2^{k-1})_{i_k} \left( 1 + L_{k,1} + L_{k,2} \right)^{2k-1-i_k} (L_{k,1})^{i_k},$$

$$\epsilon_k := (2pt)^{-\frac{1}{2}} \left\{ 2^{\ell-k} n_\ell \exp (t2^{\ell-k+1} n_\ell) \tau_{d_\ell-1+1} \right\}^{\frac{1}{2}},$$

$$L_{k,1} := (pt/2)^{-\frac{1}{2}} \exp (t\epsilon_k) \left\{ 2^{\ell-k} n_\ell \exp (t2^{\ell-k+1} n_\ell) \tau_{d_\ell-1+1} \right\}^{\frac{1}{2}},$$

$$L_{k,2} := (2pt)^{-\frac{1}{2}} \exp (t\epsilon_k) \left\{ 2^{\ell-k} n_\ell \exp (t2^{\ell-k+1} n_\ell) \tau_{d_\ell-1+1} \right\}^{\frac{1}{2}}.$$

**Step II:** Now we choose $J$ as follows:

$$J = \inf \left\{ k \in \{0, \ldots, \ell \} : \frac{B(1-\delta)^k}{2^k} \leq \min \left\{ \frac{\psi_2}{8 \ell^2}, B \right\} \right\}.$$ 

We first bound $\mathbb{E} \operatorname{Tr} \exp \left\{ t \left( \sum_{i=1}^{2^J} \sum_{j \in K_j^\ell} \tilde{X}_j \right) \right\}$ and $\mathbb{E} \operatorname{Tr} \exp \left\{ -t \left( \sum_{i=1}^{2^J} \sum_{j \in K_j^\ell} \tilde{X}_j \right) \right\}$. From (A.5) we obtain $2^J$ sets of $\{ \tilde{X}_j \}$ that are mutually independent. To make notation less confusing, we will remove the upper tilde from $\tilde{X}_j$ for all $j$. Denote the number of matrices in each set $K_j^i$ to be $q := 2^{\ell-j} n_\ell$. For each set $K_j^i$, $i \in \{1, \ldots, 2^J \}$, we divide it into consecutive sets with cardinality $\bar{q}$ and potentially a residual term if $q$ is not divisible by $\bar{q}$. More specifically, we have $2\bar{q} \leq q$ and $m_{q,\bar{q}} := [q/2\bar{q}]$. The value $\bar{q}$ will be determined later.

Then each set $K_j^i$ contains $2m_{q,\bar{q}}$ numbers of sets with cardinality $\bar{q}$ and one set with cardinality less than $2\bar{q}$. For each $K_j^i$, $i \in \{1, \ldots, 2^J \}$, denote these consecutive sets described above by $Q_k^i$, $k \in \{1, \ldots, 2m_{q,\bar{q}} + 1\}$. Given this notation, we could rewrite the bound as the following:

$$\mathbb{E} \operatorname{Tr} \exp \left( t \sum_{i=1}^{2^J} \sum_{j \in K_j^i} X_j \right)$$

$$= \mathbb{E} \operatorname{Tr} \exp \left( \sum_{i=1}^{2^J} \sum_{k=1}^{2m_{q,\bar{q}}+1} \sum_{j \in Q_k^i} X_j \right) = \mathbb{E} \operatorname{Tr} \exp \left( \sum_{i=1}^{2^J} \sum_{j \in Q_{2k-1}^i} X_j + \sum_{k=1}^{2m_{q,\bar{q}}+1} \sum_{j \in Q_{2k-1}^i} X_j \right).$$

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Since $\text{Tr} \exp(\cdot)$ is convex, by Jensen’s inequality, we have

$$
\mathbb{E} \text{Tr} \exp \left( t \sum_{i=1}^{2^j} \sum_{j \in K_j} X_j \right) \leq \frac{1}{2} \mathbb{E} \text{Tr} \exp \left( 2t \sum_{i=1}^{2^j} \sum_{k=1}^{m_q,q} \sum_{j \in Q_{2k}} X_j \right) + \frac{1}{2} \mathbb{E} \text{Tr} \exp \left( 2t \sum_{i=1}^{2^j} \sum_{k=1}^{m_q,q+1} \sum_{j \in Q_{2k-1}} X_j \right).
$$

Since the number of odd index sets is always equal to or one more than that of the even index sets, the upper bound of $\frac{1}{2} \mathbb{E} \text{Tr} \exp \left( 2t \sum_{i=1}^{2^j} \sum_{k=1}^{m_q,q} \sum_{j \in Q_{2k}} X_j \right)$ will always be less than or equal to that of $\frac{1}{2} \mathbb{E} \text{Tr} \exp \left( 2t \sum_{i=1}^{2^j} \sum_{k=1}^{m_q,q+1} \sum_{j \in Q_{2k-1}} X_j \right)$. Hence we only need to provide an upper bound for $\mathbb{E} \text{Tr} \exp \left( 2t \sum_{i=1}^{2^j} \sum_{k=1}^{m_q,q+1} \sum_{j \in Q_{2k-1}} X_j \right)$. Our goal is then to replace all $\{X_j\}_{j \in Q_{2k-1}}$ in the last inequality by mutually independent copies $\{\tilde{X}_j\}_{j \in Q_{2k-1}}$ with same distributions for $k \in \{1, \ldots, 2m_q,q + 1\}$, $i \in \{1, \ldots, 2^j\}$. Again we will proceed by induction. We first show

$$
\mathbb{E} \text{Tr} \exp \left( 2t \sum_{i=1}^{2^j} \sum_{k=1}^{m_q,q+1} \sum_{j \in Q_{2k-1}} X_j \right)
\leq \sum_{i=0}^{1} \tilde{A}_i \mathbb{E} \text{Tr} \exp \left\{ (-1)^i 2t \left( \sum_{k=1}^{m_q,q+1} \sum_{j \in Q_{2k-1}} \tilde{X}_j + \sum_{i=2}^{2^j} \sum_{k=1}^{m_q,q+1} \sum_{j \in Q_{2k-1}} X_j \right) \right\},
$$

where the constants $\tilde{A}_i$, will be specified later. For each $\{X_j\}_{j \in Q_{2k-1}}$, $k \in \{1, \ldots, m_q,q + 1\}$, we could find a sequence of $\{\tilde{X}_j\}_{j \in Q_{2k-1}}$, $k \in \{1, \ldots, m_q,q + 1\}$ that are mutually independent with each other. More specifically, let $\tilde{X}_j_{j \in Q_{2k-1}^1} = \{X_j\}_{j \in Q_{2k-1}^1}$. By applying Lemma A.2 on $\{\tilde{X}_j\}_{j \in Q_{2k-1}^1}$ and $\{X_j\}_{j \in Q_{2k-1}^2}$ with a chosen $\tilde{\epsilon} > 0$, we could find a sequence of random matrices $\{\tilde{X}_j\}_{j \in Q_{2k-1}^2}$ such that for each $j_0 \in Q_{2k-1}^1$, we have

1. $\tilde{X}_{j_0}$ is measurable with respect to $\sigma(\{\tilde{X}_j\}_{j \in Q_{2k-1}^1} ) \lor \sigma(X_{j_0}) \lor \sigma(U_{j_0})$;

2. $\tilde{X}_{j_0}$ is independent of $\sigma(\{X_j\}_{j \in Q_{2k-1}^2})$;

3. $\tilde{X}_{j_0}$ has the same distribution as $X_{j_0}$;

4. $P(\|\tilde{X}_{j_0} - X_{j_0}\| \geq \tilde{\epsilon}) \leq \mathbb{E}(\|\tilde{X}_{j_0} - X_{j_0}\|)/\tilde{\epsilon} \leq \tau_{q+1}/\tilde{\epsilon}$ by Markov’s inequality.

For each $j_0 \in Q_{2k-1}^1$, $U_{j_0}$ is independent with $\{\tilde{X}_j\}_{j \in Q_{2k-1}^1}$ and $X_{j_0}$. In addition, since there are at least $\tilde{q}$ number of matrices between $\tilde{X}_{j_0}$ and $X_{j_0}$ by our construction, we have $\tau(\sigma(\{\tilde{X}_j\}_{j \in Q_{2k-1}^1}, X_{j_0}; \|\cdot\|) \leq \tau_{q+1}$. Note that $\{\tilde{X}_j\}_{j \in Q_{2k-1}^1}$ is independent with $\{\tilde{X}_j\}_{j \in Q_{2k-1}^1}$ but not mutually independent within the set $Q_{2k-1}^1$.

Following the induction steps similar to the previous step and without redundancy, we obtain

$$
\mathbb{E} \text{Tr} \exp \left( 2t \sum_{i=1}^{2^j} \sum_{k=1}^{m_q,q+1} \sum_{j \in Q_{2k-1}} X_j \right)
\leq \sum_{i=0}^{1} \tilde{A}_i \mathbb{E} \text{Tr} \exp \left\{ (-1)^i 2t \left( \sum_{k=1}^{m_q,q+1} \sum_{j \in Q_{2k-1}} \tilde{X}_j + \sum_{i=2}^{2^j} \sum_{k=1}^{m_q,q+1} \sum_{j \in Q_{2k-1}} X_j \right) \right\},
$$

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where

\[
\tilde{\epsilon} := (4pt)^{-\frac{1}{2}} \{\exp(2tq)\tau_{q+1}\}^\frac{1}{2},
\]

\[
\tilde{L}_1 := \frac{1}{2} (4pt)^{\frac{1}{2}} q \exp(2tq\tilde{\epsilon}) \{\exp(2tq)\tau_{q+1}\}^\frac{1}{2},
\]

\[
\tilde{L}_2 := (4pt)^{\frac{1}{2}} q \{\exp(2tq)\tau_{q+1}\}^\frac{1}{2},
\]

\[
\tilde{A}_{i_1} := \left( \frac{1}{i_1} \right) (1 + \tilde{L}_1 + \tilde{L}_2)^{1-i_1}(\tilde{L}_1)^{i_1},
\]

This completes the base case.

Iterating the above calculation, we arrive at the following bound:

\[
\mathbb{E} Tr \exp \left( 2t \sum_{i=1}^{2^j} \sum_{k=1}^{m_{q,\tilde{\epsilon}}+1} \sum_{j \in \mathcal{Q}_{2k-1}} X_j \right)
\]

\[
\leq \sum_{i_1=0}^{1} \cdots \sum_{i_{2j}=0}^{1} \left( \prod_{r=1}^{2j} \tilde{A}_{i_r} \right) \mathbb{E} Tr \exp \left\{ (-1)^{\sum_{r=1}^{2^j} i_r} 2t \sum_{i=1}^{2^j} \sum_{k=1}^{m_{q,\tilde{\epsilon}}+1} \sum_{j \in \mathcal{Q}_{2k-1}} \tilde{X}_j \right\}, \tag{A.6}
\]

where \( \{\tilde{X}_j\}_{j \in \mathcal{Q}_{2k-1}} \) for \((i, k) \in \{1, \ldots, 2^j\} \times \{1, \ldots, m_{q,\tilde{\epsilon}} + 1\}\) are mutually independent and identically distributed as \( \{X_j\}_{j \in \mathcal{Q}_{2k-1}} \) for \((i, k) \in \{1, \ldots, 2^j\} \times \{1, \ldots, m_{q,\tilde{\epsilon}} + 1\}\), and

\[
\tilde{\epsilon} := (4pt)^{-\frac{1}{2}} \{\exp(2tq)\tau_{q+1}\}^\frac{1}{2},
\]

\[
\tilde{L}_1 := \frac{1}{2} (4pt)^{\frac{1}{2}} q \exp(2tq\tilde{\epsilon}) \{\exp(2tq)\tau_{q+1}\}^\frac{1}{2},
\]

\[
\tilde{L}_2 := (4pt)^{\frac{1}{2}} q \{\exp(2tq)\tau_{q+1}\}^\frac{1}{2},
\]

\[
\tilde{A}_{i_r} := \left( \frac{1}{i_r} \right) (1 + \tilde{L}_1 + \tilde{L}_2)^{1-i_r}(\tilde{L}_1)^{i_r}.
\]

Let \( \tilde{q} := [2/t] \wedge [q/2] \). \( \{\tilde{X}_j\}_{j \in \mathcal{Q}_{2k-1}} \) for \((i, k) \in \{1, \ldots, 2^j\} \times \{1, \ldots, m_{q,\tilde{\epsilon}} + 1\}\) are mutually independent with mean 0 and \( 2^j \sum_{k=1}^{m_{q,\tilde{\epsilon}}+1} \text{card}(\mathcal{Q}_{2k-1}) \leq B \). Moreover by Weyl’s inequality, for \((i, k) \in \{1, \ldots, 2^j\} \times \{1, \ldots, m_{q,\tilde{\epsilon}} + 1\}\), we have

\[
2\lambda_{\text{max}} \left( \sum_{j \in \mathcal{Q}_{2k-1}} \tilde{X}_j \right) \leq 2\tilde{q} \leq \frac{4}{i}.
\]

By Lemma 4 in Banna et al. (2016), we obtain

\[
\mathbb{E} Tr \exp \left( 2t \sum_{i=1}^{2^j} \sum_{k=1}^{m_{q,\tilde{\epsilon}}+1} \sum_{j \in \mathcal{Q}_{2k-1}} \tilde{X}_j \right) \leq p \exp\{4h(4)Bt^2\nu^2\}, \tag{A.7}
\]

\[
\mathbb{E} Tr \exp \left( -2t \sum_{i=1}^{2^j} \sum_{k=1}^{m_{q,\tilde{\epsilon}}+1} \sum_{j \in \mathcal{Q}_{2k-1}} \tilde{X}_j \right) \leq p \exp\{4h(4)Bt^2\nu^2\}. \tag{A.8}
\]
Plugging (A.7) and (A.8) into (A.6) and using the fact that \( \sum_{i=0}^{\infty} A_i = 1 + 2L_1 + L_2 \), we obtain

\[
\mathbb{E} \text{Tr} \left( 2t \sum_{i=1}^{2^j} \sum_{k=1}^{m_{q,i+1}} \sum_{j \in Q_{2k-1}} X_j \right) \leq (1 + 2L_1 + L_2) \sum_{j=0}^{2^j-1} p \exp \{ 4h(4)Bt^2 \nu^2 \}. \tag{A.9}
\]

By replacing \( X \) by \( -X \), we obtain

\[
\mathbb{E} \text{Tr} \exp \left( -2t \sum_{i=1}^{2^j} \sum_{k=1}^{m_{q,i+1}} \sum_{j \in Q_{2k-1}} X_j \right) \leq (1 + 2L_1 + L_2) \sum_{j=0}^{2^j-1} p \exp \{ 4h(4)Bt^2 \nu^2 \}. \tag{A.10}
\]

Combining (A.5) with (A.9) and (A.10), we get

\[
\mathbb{E} \text{Tr} \left( t \sum_{j \in K_B} X_j \right) \leq \sum_{j=0}^{2^j-1} \left( \prod_{k=1}^{J} A_{k,i_k} \right) \sum_{j=0}^{2^j-1} \left( 1 + 2L_1 + L_2 \right) \sum_{j=0}^{2^j-1} p \exp \{ 4h(4)Bt^2 \nu^2 \}
\]

\[
= \left\{ \prod_{k=1}^{J} \left( 1 + 2L_1 + L_2 \right) \right\} \sum_{j=0}^{2^j-1} \left( 1 + 2L_1 + L_2 \right) \sum_{j=0}^{2^j-1} p \exp \{ 4h(4)Bt^2 \nu^2 \}, \tag{A.11}
\]

where the last equality follows by \( \sum_{j=1}^{2^j-1} A_{k,i_k} = (1 + 2L_1 + L_2) \sum_{j=0}^{2^j-1} p \exp \{ 4h(4)Bt^2 \nu^2 \} \).

By using \( \log(1 + x) \leq x \) for \( x \geq 0 \), we have

\[
\log \mathbb{E} \text{Tr} \left( t \sum_{j \in K_B} X_j \right) \leq \sum_{j=0}^{2^j-1} \left( 2L_1 + L_2 \right) \sum_{j=0}^{2^j-1} \left( 2L_1 + L_2 \right) + \log \left( \sum_{j=0}^{2^j-1} p \exp \{ 4h(4)Bt^2 \nu^2 \} \right).
\]

(A.12)

For simplicity, we denote \( I = \sum_{k=1}^{J} \sum_{j=1}^{2^j-1} \left( 2L_1 + L_2 \right) \), \( II = 2L_1 + L_2 \) in (A.12).

**Step III:** Following calculations similar to Banna et al. (2016), we obtain

\[
I \leq \frac{32\sqrt{2}}{\log 2} \left[ 1 + \exp \left( \frac{1}{\sqrt{2}p} \exp \left( - \frac{\psi_2}{16t} \right) \right) \right] \frac{t^2}{\psi_2} \exp \left( - \frac{\psi_2}{32t} \right), \tag{A.13}
\]

and

\[
II \leq 128 \left[ 1 + \exp \left( \frac{1}{p} \exp \left( - \frac{\psi_2}{32t} \right) \right) \right] \frac{t^2}{\psi_2} \exp \left( - \frac{\psi_2}{64t} \right). \tag{A.14}
\]

Hence by combining (A.4), (A.12), (A.13) and (A.14), we obtain for \( 0 < t \leq \min \{ 1, \frac{\psi_2}{8 \log(\psi_1 B^p)} \} \),

\[
\log \mathbb{E} \text{Tr} \left( t \sum_{j \in K_B} X_j \right) \leq \log p + 4h(4)Bt^2 \nu^2 + 151 \left[ 1 + \exp \left( \frac{1}{\sqrt{p}} \exp \left( - \frac{\psi_2}{64t} \right) \right) \right] \frac{t^2}{\psi_2} \exp \left( - \frac{\psi_2}{64t} \right).
\]

This completes the proof of Lemma A.4. \( \square \)
References

Bai, Z. and Yin, Y. (1993). Limit of the smallest eigenvalue of a large dimensional sample covariance matrix. *The Annals of Probability*, 21(3):1275–1294.

Banna, M., Merlevède, F., and Youssef, P. (2016). Bernstein-type inequality for a class of dependent random matrices. *Random Matrices: Theory and Applications*, 5(2):1650006.

Blinn, J. (1996). Consider the lowly $2 \times 2$ matrix. *IEEE Computer Graphics and Applications*, 16(2):82–88.

Brand, M. (2006). Fast low-rank modifications of the thin singular value decomposition. *Linear Algebra and its Applications*, 415(1):20–30.

Brillinger, D. R. (2001). *Time Series: Data Analysis and Theory*. Siam.

Bunea, F. and Xiao, L. (2015). On the sample covariance matrix estimator of reduced effective rank population matrices, with applications to fPCA. *Bernoulli*, 21(2):1200–1230.

Chang, J., Guo, B., and Yao, Q. (2018+). Principal component analysis for second-order stationary vector time series. *The Annals of Statistics (in press)*.

Chen, X., Xu, M., and Wu, W. B. (2013). Covariance and precision matrix estimation for high-dimensional time series. *The Annals of Statistics*, 41(6):2994–3021.

Davis, C. and Kahan, W. M. (1970). The rotation of eigenvectors by a perturbation. iii. *SIAM Journal on Numerical Analysis*, 7(1):1–46.

Dedecker, J., Doukhan, P., Lang, G., Leon, J., Louhichi, S., and Prieur, C. (2007). *Weak Dependence: With Examples and Applications*. Springer-Verlag New York.

Dedecker, J. and Prieur, C. (2004). Coupling for $\tau$-dependent sequences and applications. *Journal of Theoretical Probability*, 17(4):861–885.

Han, F. and Liu, H. (2018+). ECA: High-dimensional elliptical component analysis in non-Gaussian distributions. *Journal of the American Statistical Association (in press)*.

Koltchinskii, V. and Lounici, K. (2017a). Concentration inequalities and moment bounds for sample covariance operators. *Bernoulli*, 23(1):110–133.

Koltchinskii, V. and Lounici, K. (2017b). New asymptotic results in principal component analysis. *Sankhya A*, 79(2):254–297.

Koltchinskii, V. and Lounici, K. (2017c). Normal approximation and concentration of spectral projectors of sample covariance. *The Annals of Statistics*, 45(1):121–157.

Liu, W., Xiao, H., and Wu, W. B. (2013). Probability and moment inequalities under dependence. *Statistica Sinica*, 23(3):1257–1272.
Lounici, K. (2014). High-dimensional covariance matrix estimation with missing observations. *Bernoulli*, 20(3):1029–1058.

Mendelson, S. (2010). Empirical processes with a bounded $\psi_1$ diameter. *Geometric and Functional Analysis*, 20(4):988–1027.

Mendelson, S. and Paouris, G. (2014). On the singular values of random matrices. *Journal of the European Mathematical Society*, 16:823–834.

Merlevède, F., Peligrad, M., and Rio, E. (2009). Bernstein inequality and moderate deviations under strong mixing conditions. In *High Dimensional Probability V: the Luminy Volume*, pages 273–292. Institute of Mathematical Statistics.

Oliveira, R. (2010). Sums of random Hermitian matrices and an inequality by Rudelson. *Electronic Communications in Probability*, 15:203–212.

Rudelson, M. (1999). Random vectors in the isotropic position. *Journal of Functional Analysis*, 164(1):60–72.

Srivastava, N. and Vershynin, R. (2013). Covariance estimation for distributions with $2+\epsilon$ moments. *The Annals of Probability*, 41(5):3081–3111.

Talagrand, M. (2014). *Upper and Lower Bounds for Stochastic Processes: Modern Methods and Classical Problems*. Springer.

Tikhomirov, K. (2017). Sample covariance matrices of heavy-tailed distributions. *International Mathematics Research Notices*, page rnx067.

Tropp, J. A. (2015). An introduction to matrix concentration inequalities. *Foundations and Trends in Machine Learning*, 8(1-2):1–230.

van Handel, R. (2017). Structured random matrices. In *Convexity and Concentration*, volume 161, pages 107–156. Springer.

Vershynin, R. (2012). Introduction to the non-asymptotic analysis of random matrices. In *Compressed Sensing*, pages 210–268. Cambridge University Press.

Wu, W. B. (2005). Nonlinear system theory: Another look at dependence. *Proceedings of the National Academy of Sciences of the United States of America*, 102(40):14150–14154.

Wu, W. B. and Wu, Y. N. (2016). Performance bounds for parameter estimates of high-dimensional linear models with correlated errors. *Electronic Journal of Statistics*, 10(1):352–379.