Thermodynamics of Black Holes in Rastall Gravity

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Abstract. A promising theory in modifying general relativity by violating the ordinary energy-momentum conservation law in curved spacetime is the Rastall theory of gravity. In this theory, geometry and matter fields are coupled to each other in a non-minimal way. Here, we study thermodynamic properties of some black hole solutions in this framework, and compare our results with those of general relativity. We demonstrate how the presence of these matter sources amplifies effects caused by the Rastall parameter in thermodynamic quantities. Our investigation also shows that black holes with radius smaller than a certain amount (\( \equiv r_0 \)) have negative heat capacity in the Rastall framework. In fact, it is a lower bound for the possible values of horizon radius satisfied by stable black holes.
1. Introduction

One of the big puzzles in science is the fact that our universe is going through a phase of accelerated expansion. A possible explanation for this is the presence of a dark energy field, one with constant positive energy density and negative pressure, that can provide a kind of matter that allows such acceleration. Despite the fact that the nature of the dark sectors of the cosmos is currently unknown, standard cosmology is very successful in describing the cosmos history. One of these models, based on the presence of a cosmological constant term in Einstein’s gravity, is called $\Lambda CDM$, or $\Lambda$ Cold Dark Matter, where $\Lambda$ is the cosmological constant and plays the role of dark energy. The most promising explanation for the existence of a cosmological constant is the vacuum energy of elementary particles, but calculations tell us that such vacuum energy is more than one hundred orders of magnitude higher than the measured value for $\Lambda$.

Another possibility is that the negative pressure is generated by some peculiar kind of perfect fluid, where the proportion between the pressure and energy density is between $-1$ and $-1/3$. If this perfect fluid is generated by a scalar field, it is generally called quintessence, and it is considered as a hypothetical form of dark energy. Therefore, if this is the case, we must consider that our universe is pervaded by such fluid, and we must study strong gravity objects such as black holes in contact with them. This is the idea presented by Kiselev in [1], and has been generalized to the Rastall model of gravity [2] in Ref. [3].

The idea behind Rastall model is that our laws of conservation, such as conservation of mass/energy, has been probed only in the flat or weak-field arena of spacetime [2]. A new generalization of this theory has recently been proposed, introducing the coupling between matter and gravitational fields in a non-minimal way as an origin for the accelerating phase of the universe [4]. Based on Rastall’s argument, the necessity that the covariant derivative of the energy-momentum tensor to be zero can be relaxed, allowing one to add new terms to the Einstein’s equation. In fact, it has recently been shown that the divergence of the energy-momentum tensor can be non-zero in a curved spacetime [5]. For this theory, several exact solutions has been obtained, both for astrophysical [3,6,13] and cosmological scenarios [14,20].

Comparing thermodynamic quantities and properties of black holes in Rastall gravity with their counterparts in general relativity helps us to be more familiar with the nature of a non-minimal coupling between geometry and matter fields introduced in the Rastall hypothesis. Besides, thermodynamics properties of Kiselev solutions in the general relativity framework have been studied by some authors [21,22]. In fact, thermodynamic properties of diverse black holes have extensively been studied in various theories of gravity [27,51]. Hence, our aim in this paper is to study the thermodynamic quantities and properties of black holes surrounded by a prefect fluid in the Rastall framework. An appealing property that we found concerns the coupling of the Rastall parameter with the energy density of the surrounding fields: they couple in such a way that the fields densities work as an amplifier to Rastall-like deformations, which could
help us to experimentally distinguish between this formalism and general relativity.

This paper is organized as follows. In the next section, we address general remarks on black hole solutions surrounded by a perfect fluid in general relativity and the Rastall theory as well as the thermodynamic quantities of black holes in the Rastall framework. In sections 3 and 4, energy and pressure of solutions surrounded by a quintessence field and cosmological constant, as promising approaches to describe the current accelerating universe, respectively, are studied in details in the Einstein and Rastall frameworks. The case of phantom field is also investigated in section 5. In section 6, we will study the possibility of the occurrence of phase transitions in Rastall black holes. The last section is devoted to a summary and concluding remarks.

2. A black hole surrounded by a perfect fluid in Rastall gravity; general remarks

In this paper we will work with a Schwarzschild-like metric, given by

\[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\Omega^2, \]  

with \( d\Omega^2 = d\theta^2 + \sin^2(\theta)d\phi^2 \), along with a general spherically symmetric energy-momentum tensor. The general expression for time and spatial components of such tensor is given by

\[ T^t_t = A(r), \quad T^t_i = 0, \quad \text{and} \quad T^i_j = C(r)r_jr^i + B(r)\delta^i_j. \] 

For general relativity, Kiselev has shown [1] that if the background is filled by a source with pressure \( p(r) \) and energy density \( \rho(r) \), related to each other by a state parameter \( \omega_q \equiv \frac{p(r)}{\rho(r)} \), then

\[ f(r) = 1 - 2\frac{M_K}{r} - N_K r^{\eta_K} \]  

in which \( M_K \) and \( N_K \) are constants of integration and

\[ \eta_K = -1 - 3\omega_q. \] 

2.1. Black holes surrounded by a perfect fluid in Rastall gravity

Rastall gravity is a theory where the total energy-momentum tensor is not conserved, but its covariant derivative is proportional to the derivative of the Ricci scalar [2]. This means that, in a flat spacetime or as a first approximation of a weak gravitational field, all the known laws of conservation are valid. We should stress that such laws has been tested only in the weak regime of gravity, and it is known that gravity can produce particles via quantum effects, thus breaking some of these laws [2].

Rastall hypothesis can be written as

\[ \nabla_{\mu}T^{\mu\nu} = \lambda \nabla^{\nu}R, \]
where $\lambda$ is the Rastall parameter, and general relativity is recovered in the limit $\lambda \to 0$. From (5), we can write the modified equations of gravity as

$$H^\mu_\nu \equiv G^\mu_\nu + \lambda \kappa \delta^\mu_\nu R = \kappa T^\mu_\nu,$$

where $\kappa$ is Rastall’s gravitational constant. To find solutions to these field equations, one should solve the set of equations (6) for some energy-momentum tensor.

Taking the trace of equation (6), we have

$$R(4\lambda\kappa - 1) = \kappa T,$$

which means that in vacuum one must have $R = 0$ or $\kappa\lambda = 1/4$. As the latter option is not allowed (see [2,52]), the former should be the case, and we must have that all vacuum solutions in GR are also solutions for Rastall gravity.

Applying Kiselev’s approach [1] to the field equations, and considering $\rho(r) = A r^\beta$, where both $A$ and $\beta$ are constants, Heydarzade and Darabi [3] found out

$$\beta = -\frac{3(1 + \omega_q) - 12\kappa\lambda(1 + \omega_q)}{1 - 3\kappa\lambda(1 + \omega_q)},$$

and

$$A = \frac{3N(1 - 4\kappa\lambda)(\kappa\lambda(1 + \omega_q) - \omega_q)}{\kappa(1 - 3\kappa\lambda(1 + \omega_q))^2},$$

where $N$ is an integration constant, related with the surrounding field. Hence, the metric function is given by

$$f(r) = 1 - \frac{2M}{r} - Nr^\eta,$$

with

$$\eta = -\frac{1 + 3\omega_q - 6\kappa\lambda(1 + \omega_q)}{1 - 3\kappa\lambda(1 + \omega_q)},$$

where $M$ is another integration constant, representing the black hole mass. For $\lambda \to 0$ we recover the solution found by Kiselev [1] in the framework of general relativity. It is worthwhile mentioning that similar solutions can also be obtained in Rastall framework for other situations [10,11].

For each choice of equation of state, we can find the metric, as given by [10], along with the constant $A$, given by [3]. To preserve the weak energy condition, i.e., $\rho > 0$, we must have $A > 0$, and this will define the sign of the parameter $N$ as dependent on the parameter $\kappa\lambda$. For a full discussion on each case, see [3].
2.2. Thermodynamic quantities of black holes in Rastall Gravity

To work on the thermodynamic aspects of the Rastall model of gravity, we must define classical thermodynamic quantities, such as energy and entropy. These are local quantities, but for general relativity there is no straightforward way (it is even senseless) to define the local energy of a gravitational field configuration. For the total energy, the most accepted ones are the ADM energy at spatial infinity [53], and the Bondi-Sachs [54, 55] energy at null infinity, both describing an isolated system in an asymptotically flat spacetime. But its local counterpart should be chosen as a useful quantity, defined for the interior of some well-defined boundary, that goes to one of the well-accepted values of energy as the boundary goes to infinity.

Some useful definitions for this quasi-local notion of energy exist in literature [66], and in this paper we will use the results obtained in [52], that uses a generalized Misner-Sharp definition of energy [56], as a suitable definition for energy [67–79] to find the corresponding entropy related with the geometry of the spacetime in Rastall gravity. The unified first law of thermodynamics (UFL) is defined as [80]

\[ dE \equiv A\Psi_a dx^a + W dV, \]

in which, \( \Psi_a = T^b_a \partial_b r + W \partial_a r \) is the energy supply vector, and \( W = -\frac{h_{ab} T_{ab}}{2} \) denotes the work density. Moreover, \( h_{ab} = \text{diag}(-f(r), \frac{1}{f(r)}) \) for metric (11), and in fact, it is the metric on two dimensional hypersurface \((t, r)\). UFL is compatible with the generalization of the Misner-Sharp mass [80], and therefore, one can use the above equation in order to find the generalized Misner-Sharp mass in the gravitational theory under investigation [52, 80]. Defining \( \gamma \equiv \lambda \kappa \), the Newtonian limit leads to [52]

\[ \kappa = \frac{4\gamma - 1}{6\gamma - 1} - \frac{8\pi}{8\pi}, \]

and applying the Unified first law of thermodynamics to the horizon of metric (11), one can get the Misner-Sharp mass content of black holes in Rastall gravity as [1]

\[ E = \frac{6\gamma - 1}{2(4\gamma - 1)} [(1 - 2\gamma)r_H + \gamma r_H^2 f'(r_H)], \]

where \( ' \) means derivative with respect to the coordinate \( r \), and \( r_H \) denotes the horizon radius. In order to obtain the system pressure, one can use the \( r - r \) component of the Rastall field equations (6) to obtain [52]

\[ P(r_H) = \frac{6\gamma - 1}{(4\gamma - 1)8\pi} \left( \frac{1}{r_H} [r_H f'(r_H) - 1] - \frac{\gamma}{r_H^2} [r_H^2 f''(r_H) + 4r_H f'(r_H) - 2] \right). \]

In addition, bearing the first law of thermodynamics \( (dE = TdS - PdV) \) in mind, and using Eqs. (14) and (15), it has been shown that the entropy of black hole is [52]

\[ S = \left( 1 + \frac{2\gamma}{4\gamma - 1} \right) S_o. \]

+ For a general derivation of thermodynamic quantities in Rastall gravity, see [52].
Here, $S_o = A/4$ is the well-known Bekenstein entropy. We can note that, as $\gamma \to 0$, one recovers the formulas valid in general relativity \[52\]. It is useful to note here that since the $\gamma = \frac{1}{4}$ case is not allowed in this theory \[2, 52\], the singularity of the above relations at this value of $\gamma$ is not worrying.

3. Thermodynamics of a black hole surrounded by a quintessence field in Rastall gravity

A quintessence field, with state parameter $\omega_q$ ranging between $-1 < \omega_q < -1/3$, may be responsible for the observed accelerated expansion of the universe \[1, 57\] (for a review, see \[58\]). Here, we will consider the case of a black hole surrounded by the quintessence field with $\omega_q = -2/3$, such that

$$f(r) = 1 - \frac{2M}{r} - N_q r^{\frac{1+2\gamma}{1-\gamma}},$$

which reproduces the solution of Ref. \[1\] for $\gamma = 0$. Applying this state parameter to Eq. \[14\], the Misner-Sharp mass content confined in the horizon is

$$E_q = \frac{(1 - 6\gamma)}{8} \left[ \frac{r_H(1 - \gamma)^2 - N_q \gamma(2 + \gamma) r_H^{\frac{2+3\gamma}{1-\gamma}}}{(\gamma - 1)(\gamma - \frac{1}{4})} \right],$$

which is equal to the Schwarzschild case $E = r_H/2$, for $\gamma = N_q = 0$.

We plotted $E_q$ as a function of the horizon radius ($r_H$) for different values of the Rastall parameter $\gamma$ in Fig. (1). One can see that for small positive values of $\gamma$, the energy grows until a maximum value is reached, then diminishes indefinitely and eventually becomes negative from a finite value of $r_H$. Such behavior is absent in general relativity, where a linear energy growth is observed. The coupling between the Rastall parameter $\gamma$ and the quintessence energy density parameter $N_q$ is essential for this kind of behavior.

Instead, for negative small values of $\gamma$, Rastall gravity furnishes an ever positive contribution to the Misner-Sharp mass. Considering deviations of GR like $\gamma = \pm 0.003$, $\gamma = \pm 0.005$ and $N_q = 0.01$, we plotted it just up to $r_H \sim 10$ km, because this is the order of magnitude of the detected BHs by LIGO \[59, 62\]. In these cases, the energy content within the horizon is increased in comparison to GR for a fixed value of the horizon. The energy density of the quintessence fluid is proportional to $N_q$ (as can be seen in \[3\]) and for GR the formula for the Misner-Sharp mass does not depends on $N_q$, therefore the higher the quintessence energy density, the higher is the difference between GR and Rastall gravity.
Figure 1. Misner-Sharp mass for quintessence field $E_q$ as a function of the horizon $r_H$, in solar mass units. We considered the cases of $\gamma = -0.005$ (blue, dashed line), $\gamma = -0.003$ (blue, dash-dotted line), GR (black, solid line), $\gamma = 0.003$ (red, dash-dotted, thick line) and $\gamma = 0.005$ (red, dashed, thick line). We used $N_q = 0.01$ and $M_\odot G/c^2 \approx 1.48$ km is the solar mass in length units.

The pressure at the horizon is found from (15) as

$$P_q = -\frac{N_q (2 + \gamma)(1 - 6\gamma)}{8\pi (1 - \gamma)^2} r_H^{\frac{2\gamma - 1}{1 - \gamma}}. \tag{19}$$

The quintessence fluid is characterized by presenting a negative pressure (for a positive energy density). This way, it is expected that its presence might be described by the negativity of the system’s pressure. This is a feature presented both in GR and Rastall gravity, however its dependence with the horizon $r_H$ is affected by the Rastall parameter. For the cases that we are considering, its qualitative behavior is preserved (as can be seen in Fig. 2), i.e., the pressure gets suppressed by the BH horizon’s size.

It should be noted that if the Rastall parameter is such that $(2 + \gamma)(1 - 6\gamma) < 0$, its coupling with $N_q$ inverts the sign of the pressure to be positive. The pressure is negative for $-2 < \gamma < 1/6$ and is positive for $\gamma < -2 \cup \gamma > 1/6$. And also the modulus of the pressure grows with the horizon for $1/4 < \gamma < 1$ and decreases for $\gamma < 1/4 \cup \gamma > 1$ (which is the present case).
4. Thermodynamics of a black hole surrounded by a cosmological constant in Rastall gravity

Consider a cosmological constant surrounding the BH, i.e., $\omega_c = -1$, that may also presumably drive the accelerated expansion of the universe [63]. The metric function is the same for GR and Rastall gravity (it does not depend on $\gamma$), and is given by

$$f(r) = 1 - \frac{2M}{r} - N_c r^2.$$ \hspace{1cm} (20)

In fact, it is a solution obtained from other considerations [10][11]. The Misner-Sharp mass confined in the horizon is

$$E_c = \frac{1 - 6\gamma}{2 - 8\gamma} \left(1 - 3\gamma N_c r_H^2\right) r_H.$$ \hspace{1cm} (21)

As can be seen from Eq. (21), even though the metric does not depend on $\gamma$, due to the modified field equations of this theory, the Misner-Sharp mass is deformed. The preservation of the metric function is responsible for avoiding a possible $\gamma$-dependence in the power of $r_H$. However, as in the previous case, an important extra contribution arises from the coupling between the energy density $N_c$ and $\gamma$. For the same reasons of the previous section, and for the same set of parameters, we depict $E_c$ as a function of $r_H$ in Fig. (3). As can be seen, a similar qualitative behavior is found independently on the fluid under consideration. From Eq. (21), for $N_c > 0$, the Misner-Sharp mass is a concave function of $r_H$ for $0 < \gamma < 1/6 \cup \gamma > 1/4$, and is a convex one for $\gamma < 0 \cup 1/6 < \gamma < 1/4$. 

**Figure 2.** The pressure for quintessence field $P_q$ as a function of the horizon $r_H$. We considered the cases of $\gamma = -0.005$ (blue, dashed line), $\gamma = -0.003$ (blue, dash-dotted line), GR (black, solid line), $\gamma = 0.003$ (red, dash-dotted, thick line) and $\gamma = 0.005$ (red, dashed, thick line). We used $N_q = 0.01$. 

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**Analytic details:**

- **Equation 20:** $f(r) = 1 - \frac{2M}{r} - N_c r^2$
- **Equation 21:** $E_c = \frac{1 - 6\gamma}{2 - 8\gamma} \left(1 - 3\gamma N_c r_H^2\right) r_H$
Figure 3. Misner-Sharp mass for quintessence field $E_q$ as a function of the horizon $r_H$, in solar mass units. We considered the cases of $\gamma = -0.005$ (blue, dashed line), $\gamma = -0.003$ (blue, dash-dotted line), GR (black, solid line), $\gamma = 0.003$ (red, dash-dotted, thick line) and $\gamma = 0.005$ (red, dashed, thick line). We used $N_q = 10^{-6}$ and $M_\odot G/c^2 \approx 1.48$ km is the solar mass in length units.

The pressure at the horizon is a constant

$$P_c = -\frac{3}{8\pi} (1 - 6\gamma) N_c.$$

(22)

It becomes positive for $\gamma > 1/6$, whenever $N_C > 0$. It is also apparent that a negative pressure is obtainable for $\gamma > 1/6$ if $N_C < 0$.

5. Thermodynamics of a black hole surrounded by a phantom field in
Rastall gravity

Another interesting fluid that we analyze consists in the so called phantom field with a super-negative equation of state $\omega_p < -1$. For our purposes, we consider $\omega_p = -4/3$. Thus the metric function reads

$$f(r) = 1 - \frac{2M}{r} - N_pr^{\frac{3-2\gamma}{1+\gamma}}.$$

(23)

The Misner-Sharp mass confined in the horizon is

$$E_p = \left(\frac{6\gamma - 1}{8}\right) \left[ r_H(1 - \gamma^2) + N_p\gamma(\gamma - 4)r_H^{\frac{4-2\gamma}{1+\gamma}} \right].$$

(24)

Qualitatively, it behaves similarly to the quintessence case, however with a mass growth governed by an approximately fourth power law, instead of the approximately squared one of the quintessence field, as is depicted in Fig. (4).
The pressure at the horizon is

\[ P_p = \frac{-N_p (4 - \gamma)(1 - 6\gamma)}{8\pi (1 + \gamma)^2} r_H^{\gamma/4}. \]  

(25)

It becomes more negative with the growth of \( r_H \). Also, the distinction between the various values of the Rastall parameter \( \gamma \) becomes more explicit with the increasing of the horizon, as can be seen in Fig. (5).

The pressure is positive for \( 1/6 < \gamma < 4 \) and is negative for \( \gamma < 1/6 \cup \gamma > 4 \). Also, the modulus of the pressure decreases with the horizon for \( \gamma < -1 \cup \gamma > 1/4 \) and increases for \( -1 < \gamma < 1/4 \) (which is the present case).
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Figure 5. The pressure for phantom field $P_q$ as a function of the horizon $r_H$. We considered the cases of $\gamma = -0.005$ (blue, dashed line), $\gamma = -0.003$ (blue, dash-dotted line), GR (black, solid line), $\gamma = 0.003$ (red, dash-dotted, thick line) and $\gamma = 0.005$ (red, dashed, thick line). We used $N_p = 10^{-10}$.

6. Phase transition in black holes

The possibility of occurrence of phase transition [65] for black holes has been studied in various theories of gravity to get more information on the thermodynamic features of black holes [21–51]. Here, we are going to study the phase transitions for black holes in Rastall gravity by focusing on Kiselev counterpart solutions [1] in Rastall gravity [3]. Indeed, these solutions are generally more than a generalization of the Kiselev solutions to the Rastall framework, and can also be valid in some other situations [10, 11].

For the spherically symmetric static metric (1), the radius of event horizon ($r_H$) can be found by solving the $f(r_H) = 0$ equation. Bearing Eq. (10) in mind, one can easily see that for $N > 0$ and $N < 0$, the $\eta = 2$ case recovers the de-Sitter (dS) and anti de-Sitter (AdS) universes, respectively. Moreover, the Reissner-Nordström (RN) universe can be obtained by taking into account the $\eta = -2$ case [10, 11]. In fact, if a black hole is surrounded by a radiation source ($\omega_q = \frac{1}{3}$), then independently of the value of the parameter $\gamma$, we have $\eta = -2$ [10, 11].

Now, since $S = \left(1 + \frac{2\gamma}{4\eta-1}\right)\pi r_H^2$, we have $r_H = \alpha \sqrt{S}$, where $\alpha = \sqrt{\frac{4\gamma-1}{\pi(4\gamma-1)}}$, combined with Eq. (10) to get

$$f'(r_H) \rightarrow f'(S) = \frac{M'}{S} - N'S^{(\eta-1)/2}. \quad (26)$$

in which $N' \equiv \eta\alpha^{\eta-1}N$ and $M' \equiv \frac{2M}{\alpha^2}$. Since the Misner-Sharp definition of energy is fully compatible with the unified first law of thermodynamics [52], we take it as the total thermodynamic energy of system. Therefore, using Eq. (11) and defining $\alpha' \equiv \frac{(1-2\gamma)}{2\pi\alpha} = (1 - 2\gamma)\sqrt{\frac{6\gamma-1}{4\pi(4\gamma-1)^2}}$, we easily reach at

$$E(S) = \alpha' \sqrt{S} + \frac{\gamma}{2\pi}(M' - N'S^{\frac{\eta+1}{\eta}}). \quad (27)$$
The Hawking temperature and the heat capacity can be found as
\[ T(S) = \frac{dE}{dS} = \frac{\tilde{\alpha} - \tilde{N}S^{\frac{\eta}{2}}}{\sqrt{S}}, \quad (28) \]
and
\[ C = \frac{T}{dS/dT} = \frac{2S(\tilde{\alpha} - \tilde{N}S^{\frac{\eta}{2}})}{(\tilde{\alpha} - \tilde{N}S^{\frac{\eta}{2}})(\eta - 1) - \eta \tilde{\alpha}}, \quad (29) \]
where \( \tilde{\alpha} = \frac{\alpha'}{2} \) and \( \tilde{N} = \frac{2N'(\eta+1)}{4\pi} \), respectively. Hence, heat capacity diverges at \( S_0 = \frac{S_m}{(1-\eta)^\frac{\eta}{2}} \), where \( S_m \equiv \left(\frac{\alpha}{\gamma}\right)^\frac{\eta}{2} \), meaning that there can be a second order phase transition at this point \([65]\). In fact, by bearing Eq. (11) in mind, one can see that the value of \( \eta \), and therefore, the possibility of occurrence of a phase transition depends on the values of \( \gamma \) and \( \omega_q \). As a check, we can easily see that the results of considering the Schwarzschild metric can be obtained by applying both the \( \gamma \rightarrow 0 \) and \( \eta \rightarrow 0 \) limits to the above results.

It is interesting to note here that if \( \tilde{\alpha} = 0 \) and \( \eta > \frac{1}{2} \), then \( T \rightarrow 0 \) for \( S \rightarrow 0 \). This means that the second law of thermodynamics is satisfied by this case. Moreover, \( \tilde{N} \) should be negative to meet the \( S > 0 \) condition, meaning that this case requires \( \tilde{N} \leq 0 \). Besides, heat capacity is positive only if \( \eta > 1 \). Additionally, since \( S_0 = 0 \), there is only one phase with \( E = \frac{2\pi}{\gamma} M' - \frac{2N'}{\eta+1} S^{\frac{\eta+1}{2}} \). This way, if \( \eta = 2 \) and \( \gamma = \frac{1}{2} \), we have \( \alpha' = \tilde{\alpha} = 0 \) and \( \omega_q = -1 \). Moreover, Eq. (16) implies \( S = 2S_0 \) in this situation. To clarify the behavior of this case, energy, temperature and heat capacity have been plotted in Fig. (6).

![Figure 6](image_url)

**Figure 6.** Temperature and heat capacity for \( \eta = 2, \tilde{N} = -5 \) and \( \tilde{\alpha} = 0 \). For the energy curve, \( M' = 4\pi \) and \( \gamma = \frac{1}{2} \) compatible with \( \tilde{\alpha} = 0 \) leading to \( \omega_q = -1 \).

Now, for the \( \frac{\tilde{\alpha}}{\tilde{N}} < 0 \) case parallel to \( \tilde{\alpha} > 0 \), since \( \tilde{N} \) is negative, the temperature is positive everywhere, and \( T(S \rightarrow 0) \rightarrow \infty \), meaning that the second law of
thermodynamics is not satisfied. While for $\eta < 1$, the temperature drops to zero for $S \gg 1$. If we have $\eta > 1$, then the temperature have a minimum located at $S = S_0$ for that $T(S_0) = \frac{\eta \tilde{\alpha}}{(\eta - 1) \sqrt{S_0}}$, and it increases as a function of $S$ for $S > S_0$. It is worthwhile mentioning that, for $\eta = 1$, there is no singularity in the behavior of heat capacity, and $T(S \gg 1) \approx -\tilde{N}$. Temperature, energy and heat capacity have been plotted in Figs. (7) and (8), respectively. Here, we only focus on the $M' = \frac{2\pi}{\gamma}$ case combined with the definitions of $M'$, $\alpha$, $\gamma$ and Eq. (13) to reach at $M = \frac{\lambda}{8\pi G}$.

**Figure 7.** Temperature and Energy for $M' = \frac{2\pi}{\gamma}$, $\tilde{N} = -5$ and $\tilde{\alpha} = 1$ while $\eta = 3$.

**Figure 8.** Heat capacity for $\tilde{N} = -5$, $\tilde{\alpha} = 1$ and $\eta = 3$. There is a second order phase transition located at $S_0 = (\frac{1}{100})^{\frac{1}{3}}$.

In Fig. (7), we show that while the temperature is positive for $S < S_0$, its changes are very expressive. Besides, as it is apparent from Fig. (8), the heat capacity is negative
for $S < S_0$, meaning that it is an unstable phase \[51\]. Therefore, black holes of radius $r_H < r_0 \equiv \alpha \sqrt{S_0}$ are unstable and, in fact, $r_0$ is a lower bound for the radius of a stable black holes in this approach.

Based on Eq. (28), if $\tilde{\alpha} > 0$ (or equally $\tilde{\alpha} < 0$), then $T = 0$ for $S = S_m \equiv (\tilde{\alpha} \tilde{\eta})^2$, and thus the system can obtain negative temperatures \[65\]. In fact, this situation is very similar to a system in which magnetic dipoles are located in the direction of the external magnetic field $B$ \[65\]. In Figs. (9) and (10), temperature, heat capacity and energy have been plotted for $\eta = \frac{1}{2}$, leading to $S_0 = 16 S_m$. In this manner, both the heat capacity and temperature are negative while $0 < S < S_m$. They will simultaneously obtain their positive values for $S_m < S < S_0$. For $S > S_0$, although temperature is positive, heat capacity is again negative, thus signalling an unstable state \[51, 65\].

**Figure 9.** Temperature and heat capacity for $\tilde{N} = -5$ and $\tilde{\alpha} = -1$ while $\eta = \frac{1}{2}$ and thus $S_m = \left(\frac{1}{4}\right)^4$.

**Figure 10.** Energy for $M' = \frac{2\pi}{\gamma}$, $\tilde{N} = -5$ and $\tilde{\alpha} = -1$ while $\eta = \frac{1}{2}$. 
Finally, it should again be noted that, unlike Refs. [27–51], we used the Misner-Sharp energy, in full agreement with the unified first law of thermodynamics [52], as the thermodynamic potential in our calculations.

7. Considerations

We studied some thermodynamic properties of black holes in Rastall gravity. The Misner-Sharp mass of Rastall black holes has been used in our approach so to be compatible with the unified first law of thermodynamics. Our investigation shows that the difference between the Misner-Sharp mass of Rastall black holes and their counterparts in Einstein’s gravity will be decreased by reducing the size of black hole. The behavior of the thermodynamic pressure of black holes has also been studied showing that a non-minimal coupling between geometry and matter fields in the Rastall way can lead to notable effects on the pressure of the system. We finally investigated the possibility of occurrence of phase transitions for Rastall black holes. Like general relativity [51], a lower bound for the horizon radius \( r_0 \) was obtained, indicating that the heat capacity of black holes with radius smaller than \( r_0 \) is negative. This means that such black holes are unstable.

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References

[1] V. V. Kiselev, Class. Quant. Grav. 20, 1187 (2003).
[2] P. Rastall, Phys. Rev. D 6, 3357 (1972).
[3] Y. Heydarzade, F. Darabi, Phys. Lett. B 771, 365 (2017).
[4] H. Moradpour, Y. Heydarzade, F. Darabi, I. G. Salako, Eur. Phys. J. C 77, 259 (2017).
[5] T. Josset, A. Perez, Phys. Rev. Lett. 118, 021102 (2017).
[6] M. S. Ma, R. Zhao, Eur. Phys. J. C 77, 629 (2017).
[7] E. R. Bezerra de Mello, J. C. Fabris, B. Hartmann, Class. Quant. Grav. 32, no. 8, 085009 (2015).
[8] A. M. Oliveira, H. E. S. Velten, J. C. Fabris and L. Casarini, Phys. Rev. D 92, no. 4, 044020 (2015).
[9] K. A. Bronnikov, J. C. Fabris, O. F. Piattella and E. C. Santos, Gen. Rel. Grav. 48, no. 12, 162 (2016).
[10] A. M. Oliveira, H. E. S. Velten, J. C. Fabris, L. Casarini, Phys. Rev. D 93, 124020 (2016).
[11] Y. Heydarzade, H. Moradpour and F. Darabi, Can. Jour. Phys.
[12] I. Licata, H. Moradpour, C. Corda, Int. Jour. Geo. Meth. Mod. Phys. Vol. 14, 1730003 (2017).
[13] E. Spallucci, A. Smailagic, arXiv:1709.05795
[14] M. Capone, V. F. Cardone and M. L. Ruggiero, Nuovo Cim. B 125, 1133 (2011).
[15] C. E. M. Batista, M. H. Daouda, J. C. Fabris, O. F. Piattella and D. C. Rodrigues, Phys. Rev. D 85, 084008 (2012).
[16] G. F. Silva, O. F. Piattella, J. C. Fabris, L. Casarini and T. O. Barbosa, Grav. Cosmol. 19, 156 (2013).
[17] A. F. Santos and S. C. Ulhoa, Mod. Phys. Lett. A 30, no. 09, 1550039 (2015).
[18] H. Moradpour, Phys. Rev. Lett. B 757, 187 (2016).
[19] F. F. Yuan and P. Huang, Class. Quant. Grav. 34, no. 7, 077001 (2017).
[20] Z. Haghani, T. Harko, S. Shahidi, arXiv:1707.00939.
[21] G. Q. Li, Phys. Lett. B 735, 256 (2014).
[22] B. Majeed, M. Jamil, P. Pradhan, AHEP, 2015, 124910 (2015).
[23] K. Ghaderi, B. Malakolkalami, Nuc. Phys. B 903, 10 (2016).
[24] K. Ghaderi, B. Malakolkalami, Astrophys. Space Sci. 361, 161 (2016).
[25] K. Ghaderi, B. Malakolkalami, Astrophys. Space Sci. 362, 163 (2017).
[26] Z. Xu, X. Hou, J. Wang, arXiv:1610.05454.
[27] S. W. Hawking, D. N. Page, Commun. Math. Phys. 87, 577 (1983).
[28] J. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998).
[29] E. Witten, Adv. Theor. Math. Phys. 2, 505 (1998).
[30] A. Sahay, T. Sarkar, G. Sengupta, JHEP, 2010, 125 (2010).
[31] R. Banerjee, S. K. Modak, S. Samanta, Eur. Phys. J. C, 70, 317 (2010).
[32] R. Banerjee, S. K. Modak, S. Samanta, Phys. Rev. D 84, 064024 (2011).
[33] Q. J. Cao, Y. X. Chen, K. N. Shao, Phys. Rev. D 83, 064015 (2011).
[34] R. Banerjee, D. Roychowdhury, Phys. Rev. D 85, 044040 (2012).
[35] R. Banerjee, S. Ghosh, D. Roychowdhury, Phys. Lett. B 696, 156 (2011).
[36] R. Banerjee, D. Roychowdhury, JHEP, 2011, 4 (2011).
[37] R. Banerjee, S. K. Modak, D. Roychowdhury, JHEP, 2012, 125 (2012).
[38] S. W. Wei, Y. X. Liu, Eur. Phys. Lett, 99, 20004 (2012).
[39] B. R. Majhi, D. Roychowdhury, Class. Quantum. Grav. 29, 245012 (2012).
[40] W. Kim, Y. Kim, Phys. Lett. B 718, 687 (2012).
[41] Y. D. Tsai, X. N. Wu, Y. Yang, Phys. Rev. D 85, 044005 (2012).
[42] F. Capela, G. Nardini, Phys. Rev. D 86, 024030, (2012).
[43] D. Kubiznak, R. B. Mann, JEHP, 2012, 33 (2012).
[44] C. Niu, Y. Tian, X.-N. Wu, Phys. Rev. D 85, 024017 (2012).
[45] A. Lala, D. Roychowdhury, Phys. Rev. D 86, 084027 (2012).
[46] A. Lala, AHEP, 2013, 918490 (2013).
[47] S. W. Wei, Y. X. Liu, Phys. Rev. D 87, 044014 (2013).
[48] M. Eune, W. Kim, S. H. Yi, JHEP, 2013, 20 (2013).
[49] M. B. J. Poshteh, B. Mirza, Z. Sherkatghanad, Phys. Rev. D 88, 024005 (2013).
[50] J. X. Mo, W. B. Liu, Phys. Lett. B 727, 3361 (2013).
[51] J. X. Mo, W. B. Liu, AHEP, 2014, 739454 (2014).
[52] H. Moradpour, I. G. Salako, Adv. High Energy Phys. 2016, 3492796 (2016).
[53] R. L. Arnowitt, S. Deser, C. W. Misner, Phys. Rev. 116, 1322 (1959).
[54] H. Bondi, M. G. J. van der Burg, A. W. K. Metzner, Proc. Roy. Soc. Lond. A 269, 21 (1962).
[55] R. K. Sachs, Proc. Roy. Soc. Lond. A 270, 103 (1962).
[56] C. W. Misner, D. H. Sharp, Phys. Rev. 136, B571 (1964).
[57] A. Vikman, Phys. Rev. D 71, 023515 (2005).
[58] V. Sahni, Lect. Notes Phys. 653, 141 (2004).
[59] B. P. Abbott et al. [LIGO Scientific and Virgo Collaborations], Phys. Rev. Lett. 116, no. 6, 061102 (2016).
[60] B. P. Abbott et al. [LIGO Scientific and Virgo Collaborations], Phys. Rev. Lett. 116, no. 24, 241103 (2016).
[61] B. P. Abbott et al. [LIGO Scientific and VIRGO Collaborations], Phys. Rev. Lett. 118, no. 22,
[62] B. P. Abbott et al. [LIGO Scientific and Virgo Collaborations], Phys. Rev. Lett. 119, no. 14, 141101 (2017).
[63] S. Perlmutter et al. [Supernova Cosmology Project Collaboration], Astrophys. J. 517, 565 (1999).
[64] R. R. Caldwell, Phys. Lett. B 545, 23 (2002).
[65] R. K. Pathria, P. D. Beale, Statistical Mechanics (Third Edition) (30 Corporate Drive, Suite 400, Burlington, MA 01803, USA 2011).
[66] L. B. Szabados, Living Rev. Rel. 7, 4 (2004).
[67] M. Akbar, R. G. Cai, Phys. Lett. B 648, 243 (2007).
[68] M. Akbar, R. G. Cai, Phys. Lett. B 635, 7 (2006).
[69] M. Akbar, R. G. Cai, Phys. Rev. D 75, 084003 (2007).
[70] R. G. Cai, L. M. Cao, Phys. Rev. D 75, 064008 (2007).
[71] R. G. Cai, L. M. Cao, Nucl. Phys. B 785, 135 (2007).
[72] A. Sheykhi, B. Wang, R. G. Cai, Nucl. Phys. B 779, 1 (2007).
[73] A. Sheykhi, B. Wang, R. G. Cai, Phys. Rev. D 76, 023515 (2007).
[74] A. Sheykhi, J. Cosmol. Astropart. Phys. 05, 019 (2009).
[75] A. Sheykhi, Eur. Phys. J. C 69, 265 (2010).
[76] A. Sheykhi, Class. Quantum. Gravit. 27, 025007 (2010).
[77] R. G. Cai, N. Ohta, Phys. Rev. D 81, 084061 (2010).
[78] A. Sheykhi, Phys. Rev. D 87, 024022 (2013).
[79] A. Sheykhi, M. H. Dehghani, R. Dehghani, Gen. Relativ. Gravit. 46, 1679 (2014).
[80] R. G. Cai, L. M. Cao, Y. P. Hu, N. Ohta, Phys. Rev. D 80, 104016 (2009).