Energy corrections of the two dimensional Dunkl harmonic oscillator in the 
Non-Commutative phase-space

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Abstract

In this paper, we examine the harmonic oscillator problem in non-commutative phase space (NCPS) by
using the Dunkl derivative instead of the habitual one. After defining the Hamilton operator, we use the
perturbation method to derive the binding energy eigenvalues. We find eigenfunctions that correspond to
these eigenvalues in terms of the Laguerre functions. We observe that the Dunkl-Harmonic Oscillator (DHO)
in the NCPS differs from the ordinary one in the context of providing additional information on the even
and odd parities. Therefore, we conclude that working with the Dunkl operator could be more appropriate
because of its rich content.

Keywords: Non-commutative phase space; Dunkl operator; harmonic oscillator; perturbation method; even
and odd parities.

1 Introduction

In the last three decades, the study of physical problems with the concept of non-commutativity of the coor-
dinates based on various motivations has received a great deal of attention [1-28]. Such as the effect of NC energy
levels of a quantum particle under the action of the Newtonian gravitational potential of the Earth [29], Weyl-
Wigner formalism in the context of a phase space noncommutative extension of quantum mechanics [30], effect
of phase-space noncommutativity on some quantum properties such as quantum beating, quantum information
and decoherence [31], and related applications have been well addressed in [32-37]. In Ref. [3], the authors
presented one of the first formulations of examining the non-commutativity of the coordinate space within the
usual non-relativistic quantum mechanics. Since the formulation of NCPS is complex, most of the quantum
mechanical problems, for example, the central potential problems, can be solved within perturbation methods
[5, 6]. In Refs. [7-9], the authors investigated the Klein-Gordon and Pauli-Dirac oscillator dynamics in NCPS.
Recently in a very interesting paper authors considered a charged harmonic oscillator in NCPS and showed that
if a uniform magnetic field exists, then a minimum length and a minimum momentum uncertainty may arise [10].

Lately, we observe an increasing interest in studies dealing with the Dunkl derivative. This is not surprising, since the Dunkl derivative operator has a rich structure consisting of a combination of differential and discrete parts with the reflection operator term [38, 39]. In fact, the history of this deformation dates back to the middle of the last century. In 1951, Yang employed the reflection operator, which was first defined by Wigner one year ago [40], to solve one-dimensional quantum harmonic oscillator [41]. In 2013, in a series of two papers, Genest et al. considered an isotropic Dunkl-harmonic oscillator in the plane by displacing the Dunkl derivative with the partial derivative and discussed the superintegrability of the model [42, 43]. Then, they revisited the problem in three spatial dimensions [44]. Two years later, they examined the non-relativistic Dunkl-Coulomb problem in the same context [45]. In 2019, Mota et al. used this approach in the relativistic regime and obtained an exact solution of the Dunkl-Dirac oscillator which is under the influence of an external magnetic field [46]. Last year, they solved the Dunkl-Klein-Gordon equation in two dimensions with the Coulomb potential [47]. Very recently, the Dunkl-Klein-Gordon oscillator solutions are found in two and three dimensions, respectively in [48, 49, 50].

We believe that the Dunkl derivative formalism in a fuzzy space could provide very interesting results in theoretical physics. With this motivation, we consider a two-dimensional Dunkl harmonic oscillator in noncommutative space and intend to derive the energy eigenvalues and their corresponding eigenfunctions within perturbation methods. We organize the manuscript as follows: In section 2, we construct the two dimensional Dunkl-Hamiltonian operator of the harmonic oscillator in the NCPS. Then, in section 3, we employ perturbation techniques to obtain eigenenergies and their corresponding eigenstates. In the final section, we conclude the manuscript.

2 Two dimensional non-commutative Dunkl harmonic oscillator

The time-independent perturbation theory is based on the assumption in which the Hamiltonian of the system can be divided into two parts, such that an exact solution of one of them can be obtained and the other part can be approximated with the help of these solutions. Therefore, we have to derive the Hamiltonian of the system we are considering. To this end, at first we need to express the non-commutative Hamiltonian, and then, extend it by replacing the partial derivatives with the Dunkl ones. Accordingly, we start by writing the non-commutative harmonic oscillator Hamiltonian in two dimensions

\[ H_{NC} = \frac{1}{2m} \left[ (p_{xNC})^2 + (p_{yNC})^2 \right] + \frac{1}{2} m \omega^2 \left[ (x_{NC})^2 + (y_{NC})^2 \right], \]  

where the non-commutative momentum and position operators are defined as [10]

\[ p_{xNC} = p_x + \frac{\eta}{2} y, \] \hspace{1cm} (2a)
\[ p_{yNC} = p_y - \frac{\eta}{2} x, \] \hspace{1cm} (2b)
\[ x_{NC} = x - \frac{\theta}{2} p_y, \] \hspace{1cm} (2c)
\[ y_{NC} = y + \frac{\theta}{2} p_x. \] \hspace{1cm} (2d)
Here, $\eta$ and $\theta$ are the NCPS parameters. Then, we substitute these operators in Eq. (1), and express the non-commutative Hamilton operator in the form of

$$H^{NC} = \left[\frac{1 + m^2 \omega^2 x^2}{2m} \right] (p_x^2 + p_y^2) + \left[\frac{x^2 + m^2 \omega^2 y^2}{2m} \right] (x^2 + y^2) + \left[\frac{\eta + m^2 \omega^2 \theta}{2m} \right] (yp_x - xp_y).$$

Next, we change the partial derivatives with the Dunkl ones [42]. It is worth noting that partial derivatives only appear in the momentum operators, $p_x$ and $p_y$. Therefore, we consider

$$p_x \rightarrow p_{x}^{\mu_1} = \frac{\hbar}{i} D_{x}^{\mu_1} = \frac{\hbar}{i} \left[ \frac{\partial}{\partial x} + \frac{\mu_1}{x} (1 - R_1) \right],$$

$$p_y \rightarrow p_{y}^{\mu_2} = \frac{\hbar}{i} D_{y}^{\mu_2} = \frac{\hbar}{i} \left[ \frac{\partial}{\partial y} + \frac{\mu_2}{y} (1 - R_2) \right],$$

where the Wigner parameters, $\mu_1$ and $\mu_2$, are positive two constants [45]. Besides, the reflection operators, $R_1$ and $R_2$, satisfy

$$R_1 f(x, y) = f(-x, y), \quad R_2 f(x, y) = f(x, -y), \quad R_1^2 = 1, \quad R_1 x = -x R_1, \quad R_2 y = -y R_2.$$

By using the Dunkl-momentum operators defined in equations (4) and (5), we construct all necessary operators of the Hamiltonian, such as

$$p_x^2 + p_y^2 \rightarrow -\hbar^2 \left[ \frac{\partial^2}{\partial x^2} + \frac{2 \mu_1}{x} \frac{\partial}{\partial x} - \frac{\mu_1}{x^2} (1 - R_1) + \frac{\partial^2}{\partial y^2} + \frac{2 \mu_2}{y} \frac{\partial}{\partial y} - \frac{\mu_2}{y^2} (1 - R_2) \right],$$

and the $z$-component of the Dunkl-angular momentum operator

$$yp_x - xp_y \rightarrow \frac{\hbar}{i} \left[ \frac{\partial}{\partial x} - \frac{x}{y} \mu_1 (1 - R_1) - \frac{x}{y} \mu_2 (1 - R_2) \right].$$

After performing some simple algebra, we express the non-commutative Dunkl-harmonic oscillator Hamiltonian as

$$H_{Dunkl} = \mathcal{G}_1 H_0(x, y) + \mathcal{G}_2 H_1(x, y)$$

where

$$H_0(x, y) = \frac{\partial^2}{\partial x^2} + \frac{2 \mu_1}{x} \frac{\partial}{\partial x} - \frac{\mu_1}{x^2} (1 - R_1) + \frac{\partial^2}{\partial y^2} + \frac{2 \mu_2}{y} \frac{\partial}{\partial y} - \frac{\mu_2}{y^2} (1 - R_2) + \mathcal{G}_3 (x^2 + y^2) + \mathcal{G}_4 L_z,$$

$$H_1(x, y) = \frac{\hbar}{i} \frac{\mu_1 (1 - R_1) - x}{y} \mu_2 (1 - R_2),$$

$$\mathcal{G}_1 = -\frac{\hbar^2 (1 + m^2 \omega^2 x^2)}{2m}, \quad \mathcal{G}_2 = \frac{\hbar (\eta + m^2 \omega^2 \theta)}{2m}, \quad \mathcal{G}_3 = -\frac{(\omega^2 + m^2 \omega^2 x^2)}{\hbar^2 (1 + m^2 \omega^2 y^2)}, \quad \mathcal{G}_4 = -\frac{\mathcal{G}_2}{\mathcal{G}_1}.$$

Obviously, for $\mu_1 = \mu_2 = \theta = \eta = 0$, we return to the ordinary case.

### 3 First-order perturbation

In this section, we intend to derive the binding energy of the two dimensional Dunkl-harmonic oscillator via the first-order perturbation theory. In order to state the impact of noncommutativity, first we revisit the well-known commutative case, where the Dunkl-Hamiltonian satisfies

$$H_0(x, y) \psi^{R_1, R_2}_{n_x, n_y} (x, y) = E^{(0)}_{n_x, n_y} \psi^{R_1, R_2}_{n_x, n_y} (x, y).$$

$$H_0(x, y) \psi^{R_1, R_2}_{n_x, n_y} (x, y) = E^{(0)}_{n_x, n_y} \psi^{R_1, R_2}_{n_x, n_y} (x, y).$$

(13)
We assume that the wavefunction can be separable to
\[ \psi_{n_x,n_y}^{(R_1,R_2)}(x,y) = \psi_{n_x}^{R_1}(x)\psi_{n_y}^{R_2}(y). \]  
\( \text{(14)} \)

Then, we substitute it into equation (13). Straightforwardly, we obtain the following eigenenergies
\[ E_{n_x,n_y}^{(R_1,R_2)} = \mathcal{G}_1 \mathcal{G}_4 L_z - \mathcal{G}_1 \sqrt{\mathcal{G}_3} \left( 4(n_x + n_y + 1) + \mathcal{K}_1 + \mathcal{K}_2 \right), \]  
\( \text{(15)} \)

and eigenstates
\[ \psi_{n_x}^{R_1}(x) = e^{-\frac{1}{2}N_{z_x} x^2} x^{\frac{1}{2}(-2\mu_1 + \mathcal{K}_1 + 1)} L_{n_x}^{\frac{1}{2}\mathcal{K}_1} \left( \sqrt{\mathcal{G}_3} x^2 \right), \]  
\( \text{(16)} \)

\[ \psi_{n_y}^{R_2}(y) = e^{-\frac{1}{2}N_{z_y} y^2} y^{\frac{1}{2}(-2\mu_2 + \mathcal{K}_2 + 1)} L_{n_y}^{\frac{1}{2}\mathcal{K}_2} \left( \sqrt{\mathcal{G}_3} y^2 \right), \]  
\( \text{(17)} \)

where \( \mathcal{K}_i = \sqrt{4\mu_i^2 - 4\mu_i R_i + 1} \), for \( i = 1, 2 \). Here, \( L_n^\alpha(u) \) denotes the associated Laguerre polynomial. According to the first-order perturbation theory, we evaluate the unperturbed energy with the help of equation (9) as follows:
\[ (R_1, R_2) : \quad E_{n_x,n_y}^{(0)} = \langle n_x,n_y | \mathcal{G}_1 H_0 | n_x,n_y \rangle = \mathcal{G}_1^2 \left[ \mathcal{G}_4 \hbar \mu - \sqrt{\mathcal{G}_3} \left( 4(n_x + n_y + 1) + \mathcal{K}_1 + \mathcal{K}_2 \right) \right], \tag{18} \]

where \( L_z = \hbar \mu \). Due to the presence of the parity operators \( (R_1, R_2) \) in equation (18), it is possible to obtain degenerate energy eigenvalues for different parity modes.

odd – odd : 
\[ E_{0,0}^{(0)} = \mathcal{G}_1^2 \left( \mathcal{G}_4 \hbar \mu - 2\sqrt{\mathcal{G}_3} (\mu_1 + \mu_2 + 3) \right), \tag{19} \]

\[ E_{0,1}^{(0)} = \mathcal{G}_1^2 \left( \mathcal{G}_4 \hbar \mu - 2\sqrt{\mathcal{G}_3} (\mu_1 + \mu_2 + 5) \right) = E_{1,0}^{(0)}, \tag{20} \]

\[ E_{1,1}^{(0)} = \mathcal{G}_1^2 \left( \mathcal{G}_4 \hbar \mu - 2\sqrt{\mathcal{G}_3} (\mu_1 + \mu_2 + 7) \right). \tag{21} \]

even – even : 
\[ E_{0,0}^{(0)} = \mathcal{G}_1^2 \left( \mathcal{G}_4 \hbar \mu - 2\sqrt{\mathcal{G}_3} (\mu_1 + \mu_2 + 1) \right), \tag{22} \]

\[ E_{0,1}^{(0)} = \mathcal{G}_1^2 \left( \mathcal{G}_4 \hbar \mu - 2\sqrt{\mathcal{G}_3} (\mu_1 + \mu_2 + 3) \right) = E_{1,0}^{(0)}, \tag{23} \]

\[ E_{1,1}^{(0)} = \mathcal{G}_1^2 \left( \mathcal{G}_4 \hbar \mu - 2\sqrt{\mathcal{G}_3} (\mu_1 + \mu_2 + 5) \right). \tag{24} \]

even – odd = (odd – even) : 
\[ E_{0,0}^{(0)} = \mathcal{G}_1^2 \left( \mathcal{G}_4 \hbar \mu - 2\sqrt{\mathcal{G}_3} (\mu_1 + \mu_2 + 2) \right), \tag{25} \]

\[ E_{0,1}^{(0)} = \mathcal{G}_1^2 \left( \mathcal{G}_4 \hbar \mu - 2\sqrt{\mathcal{G}_3} (\mu_1 + \mu_2 + 4) \right) = E_{1,0}^{(0)}, \tag{26} \]

\[ E_{1,1}^{(0)} = \mathcal{G}_1^2 \left( \mathcal{G}_4 \hbar \mu - 2\sqrt{\mathcal{G}_3} (\mu_1 + \mu_2 + 6) \right). \tag{27} \]

Essentially, in general we can write
\[ E_{n_x,n_y;even,even}^{(0)} = E_{n_y,n_x;even,even}^{(0)}, \tag{28} \]

\[ E_{n_x,n_y;even,odd}^{(0)} = E_{n_y,n_x;odd,even}^{(0)}, \tag{29} \]

\[ E_{n_x,n_y;odd,even}^{(0)} = E_{n_y,n_x;odd,even}^{(0)}. \tag{30} \]
If we consider weak non commutativity, then we can determine energy eigenvalue corrections to the binding energy. According to the first-order perturbation theory, these corrections have to be calculated with the help of equation (11) via

\[(R_1, R_2) : \Delta E_{n_x, n_y}^{(1)}(1) = G_2(n_x, n_y) \frac{y}{x} \mu_1 (1 - R_1) - \frac{x}{y} \mu_2 (1 - R_2) |n_x, n_y\].

We find

\[(odd - odd) : \Delta E_{0,0}^{(1)} = 2 \alpha_1 \left( \mu_1 - \mu_2 \right).\]  
\[(odd - odd) : \Delta E_{0,1}^{(1)} = \frac{\alpha_1}{2} \left( \mu_1 (4\mu_2 - 1)\mu_2 + 9 - \mu_2 (4\mu_2 + 1) + 5 \right).\]  
\[(odd - odd) : \Delta E_{1,0}^{(1)} = \frac{\alpha_1}{2} \left( \mu_1 (4\mu_1 + 1)\mu_1 + 5 - \mu_2 (4\mu_1 - 1)\mu_1 + 9 \right).\]  
\[(odd - odd) : \Delta E_{1,1}^{(1)} = \frac{\alpha_1}{8} \left( \mu_1 (4\mu_1 (\mu_1 + 1) + 5) (4\mu_2 - 1)\mu_2 + 9 - \mu_2 (4\mu_1 - 1)\mu_1 + 9) (4\mu_2 (\mu_2 + 1) + 5) \right).\]

\[(even - even) : \Delta E_{n_x, n_y}^{(1)} = 0.\]

\[(even - odd) : \Delta E_{0,0}^{(1)} = -2 \alpha_2 \mu_2.\]  
\[(even - odd) : \Delta E_{0,1}^{(1)} = -\frac{\alpha_2 \mu_2}{2} \left( 4\mu_2 + 1)\mu_2 + 5 \right).\]  
\[(even - odd) : \Delta E_{1,0}^{(1)} = -\frac{\alpha_2 \mu_2}{2} \left( 4\mu_1 - 1)\mu_1 + 5 \right).\]  
\[(even - odd) : \Delta E_{1,1}^{(1)} = -\frac{\alpha_2 \mu_2}{8} \left( 4\mu_1 - 1)\mu_1 + 5 \right) (4\mu_2 + 1)\mu_2 + 5 \right).\]

\[(odd - even) : \Delta E_{0,0}^{(1)} = 2 \alpha_2 \mu_1.\]  
\[(odd - even) : \Delta E_{0,1}^{(1)} = \frac{\alpha_2 \mu_1}{2} \left( 4\mu_2 - 1)\mu_2 + 5 \right).\]  
\[(odd - even) : \Delta E_{1,0}^{(1)} = \frac{\alpha_2 \mu_1}{2} \left( 4\mu_1 + 1)\mu_1 + 5 \right).\]  
\[(odd - even) : \Delta E_{1,1}^{(1)} = \frac{\alpha_2 \mu_1}{8} \left( 4\mu_1 + 1)\mu_1 + 5 \right) (4\mu_2 - 1)\mu_2 + 5 \right).\]

Here, \(\alpha_1 = \frac{G_2}{\nu_1}\), and \(\alpha_2 = \frac{G_2}{\nu_2}\). These results hold for the situations where \(\eta < 2m\omega, \theta < \frac{1}{m\omega}, \text{ and } \mu_i > -\frac{1}{2}\).

4 Conclusion

In this manuscript, we examine two dimensional Dunkl-harmonic oscillator problem in a non-commutative phase space. In the context of time-independent perturbation theory, we derive eigenenergies and the corresponding eigenstates in terms of associated Laguerre polynomials. We show that considering Dunkl operator leads to different results than its ordinary state according to the even and odd parities. Our results also points out the degeneracy in energy eigenvalues.

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Data Availability Statements

The authors declare that the data supporting the findings of this study are available within the article.

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