Abstract
In this paper we study maximal chains in certain lattices constructed from powers of chains by iterated lax colimits in the 2-category of posets. Such a study is motivated by the fact that in lower dimensions, we get some familiar combinatorial objects such as Dyck paths and Kreweras walks.

Keywords  Bijection · Catalan number · Chain · Distributive lattice · Dyck path · Fixed point · Hermite history · Hypercube · Involution · Kreweras walk · Lattice · Lax colimit · Maximal chain number · Noetherian form · Partition · Poset · Walk · Word

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Introduction
The well-known combinatorial objects, Dyck paths, can be interpreted as maximal chains in lattices (in the sense of [2]) given by the following Hasse diagrams:

Each of the lattices $D_m$ can be constructed as a ‘lax colimit’, in the 2-category of posets, of the diagram

$C_0 \longrightarrow C_1 \longrightarrow C_2 \longrightarrow C_3 \longrightarrow C_4 \longrightarrow \cdots \longrightarrow C_m$
where \( C_i \) stands for the chain with \( i + 1 \) elements and each homomorphism \( C_i \to C_{i+1} \) is an inclusion of down-closed sub-join-semilattices. A lax colimit ‘stacks’ the chains above each other, turning each assignment \( x \mapsto y \) given by a homomorphism into the relation \( x < y \) in the lax colimit. This process can be visualized in the case of the diagram above with \( n = 3 \) as follows:

One may now consider variations of this construction, where chains are replaced with other lattices. For instance, when we replace each \( C_i \) with its cartesian square \( C_i^2 \), we get the following lattices:

Each of these lattices decomposes as a lax colimit as follows (the example shown is for \( n = 3 \)):

Maximal chains in these lattices are in bijection with Kreweras walks [3, 10] — an observation originally due to Sarah Selkirk (private communication). This gives rise to the following
question: *which combinatorial objects arise as maximal chains in lattices stacked by means of lax colimits?* For instance, noting that the two examples above correspond to the second and the third rows in a (commutative) diagram of powers of chains (see Fig. 1), it becomes interesting to explore other sequences of homomorphisms arising from the same diagram.

Stacking lattices in the first row of the diagram in Fig. 1 simply gives the sequence of chains (the second row) — not so interesting in its own right. However, stacking lattices in the sequence of chains, so lattices in the second row, gives the Dyck situation described above. Maximal chains in these lattices (i.e., the lattices $D_m$) can be counted, as it is well known, by the Catalan numbers:

$$\frac{(2m)!}{m!(m+1)!}$$

The third row gives the Kreweras case. This already is a highly nontrivial combinatorial situation; for example, a bijective proof for the formula

$$\frac{(3m)!4^m}{(m+1)!(2m+1)!}$$

that counts Kreweras walks (i.e., maximal chains in the lattices $K_m$) was found not long ago [1]. This might mean that getting similar numbers for stacking lattices along the subsequent rows can be vastly difficult. In this paper we solve the orthogonal problem: we find the maximal chain numbers for stacking lattices along the columns of the diagram in Fig. 1. The first column is again trivial, as it is identical (or rather, isomorphic) to the first row. The sequence of numbers here is simply the sequence of all positive natural numbers $1, 2, 3, 4, \ldots$. The second column stacks ‘hypercubes’. We show that the corresponding numbers are given by the odd double factorials:

$$\frac{(2n)!}{2^n n!} = (2n - 1)!!$$

We first prove this by making use of the technique of representing weighted Dyck paths as involutions with no fixed points, from the lecture series of X. Viennot [13], which then suggests a direct bijection with maximal chains in stacks of hypercubes. After this, it becomes evident how to deal with the remaining columns: involutions with no fixed points, which

![Homomorphism lattice of powers of chains](image-url)

*Fig. 1* Homomorphism lattice of powers of chains
are the same as 2-partitions, get replaced with $m$-partitions (of an $mn$ element set), giving us the numbers

$$\frac{(mn)!}{(m!)^m n!}$$

for maximal chains in the stacks along the $m$-th column of Fig. 1.

These results are obtained in Section 1 of this paper (see Theorems 1 and 2). In Section 2, we turn our attention to iterating the process of stacking and find the following:

- Maximal chain numbers for $k$-th iteration of stacking lattices in the first column of Fig. 1 are the $k$-dimensional Catalan numbers from [11]:

$$\frac{(k-1)!(kn)!}{n!(n+1)!\ldots(n+k-1)!}$$

- Maximal chains in $k$-th iteration of stacking hypercubes (i.e., lattices in the second column of Fig. 1) are in bijection with the combinatorial objects discussed in [4] in the case of $k = 2$ (see the entry A213275 by Alois P. Heinz in the OEIS for the general $k$): words $w$ of length $(k+1)n$ in an alphabet $\{a_1, \ldots, a_n\}$ with $n$ distinct letters, such that each letter occurs $k+1$ times in the word, and for each prefix $z$ of $w$, either $a_i$ does not occur in $z$ or if it does, then for each $j > i$, it occurs more or the same number of times as $a_j$. We establish this in Theorem 4. No explicit formulas for counting these combinatorial objects seem to be known.

- Maximal chains in iteration of stacking lattices in the third column onwards in Fig. 1 appear to be new combinatorial objects. At least, we could not find the corresponding numbers, generated on a computer, in the OEIS (see Table 1).

All three of these results arise as applications of a ‘representation theorem’ (Theorem 3), which gives an embedding of $k$-iterated stacking of $m$-fold hypercubes into $(k + m)$-fold hypercubes, where by an $r$-fold hypercube we mean a lattice of the form $C_r^n$ (i.e., a lattice in the $r$-th column of Fig. 1). We formulate this theorem in Section 2, but defer its proof to Section 4. The representation theorem also helped with the visualization of the corresponding stacked lattices — see Figs. 3 and 4, where $\Sigma^k_n C_m^n$ denotes the $k$-iterated stacking of $m$-fold hypercubes up to dimension $n$.

In Section 3, we expand on the link with lax colimits in the 2-category of posets, and establish some algebraic properties of this construction to get an insight as to what kinds of lattices may arise when considering stacking of lattices in Fig. 1? In particular, we show that first of all, they are indeed lattices, and secondly, they are in fact distributive lattices. We do not know, however, whether distributivity of these lattices can play a role in the combinatorial investigation of their maximal chains. Section 3 also prepares a way to Section 4, which is devoted to the proof of the representation theorem mentioned earlier. The proof is based on decomposing lax colimits of chains of homomorphisms into another kind of (weighted) colimits in category theory, which we call ‘lax pushouts’ (although they do not form a particular type of lax colimits). This method enables one to almost trivialize the geometric complexity of the stacked lattices of $m$-fold hypercubes. A similar method can be used to prove a representation theorem (Theorem 7) for iterated stacking of lattices along the rows of Fig. 1, which we formulate in the last Section 5. As did the previous representation theorem for stacked lattices along the columns of Fig. 1, this representation theorem allows
Table 1 The maximal chain numbers $\#\Sigma^k C^n_m$ and the corresponding entry code (indicated in brackets) in the On-Line Encyclopedia of Integer Sequences (accessed 28/05/2022). For the program that was used to compute these numbers, see [8]

$n = 0, 1, 2, 3, 4, 5 \ldots$

$m = 0$

\[
\begin{align*}
  k = 0 & : 1, 1, 1, 1, 1, \ldots(A000012) \\
  k = 1 & : 1, 1, 1, 1, 1, \ldots(A000012) \\
  k = 2 & : 1, 1, 2, 5, 14, 42, \ldots(A000108 – \text{Catalan numbers}) \\
  k = 3 & : 1, 1, 5, 42, 6006, \ldots(A005789 – 3\text{-dimensional Catalan numbers}) \\
  k = 4 & : 1, 1, 14, 42, 24024, 1662804, \ldots(A005790 – 4\text{-dimensional Catalan numbers})
\end{align*}
\]

$m = 1$

\[
\begin{align*}
  k = 0 & : 1, 1, 2, 6, 24, 120, \ldots(A000142 – \text{factorial numbers}) \\
  k = 1 & : 1, 1, 3, 15, 105, 945, \ldots(A001147 – \text{double factorial numbers}) \\
  k = 2 & : 1, 1, 7, 106, 2575, 87595, \ldots(A213863 – \text{no explicit formula}) \\
  k = 3 & : 1, 1, 19, 1075, 115955, 19558470, \ldots(A213864 – \text{no explicit formula}) \\
  k = 4 & : 1, 1, 56, 13326, 7364321, 7236515981, \ldots(A213865 – \text{no explicit formula})
\end{align*}
\]

$m = 2$

\[
\begin{align*}
  k = 0 & : 1, 1, 6, 90, 2520, 113400, \ldots(A000680 – \frac{(2n)!}{2^n}) \\
  k = 1 & : 1, 1, 10, 280, 15400, 1401400, \ldots(A025035 – \frac{(3n)!}{(3!)^n n!}) \\
  k = 2 & : 1, 1, 25, 2305, 482825, 183500625, \ldots(\text{no entry}) \\
  k = 3 & : 1, 1, 71, 25911, 25754021, 52213860026, \ldots(\text{no entry}) \\
  k = 4 & : 1, 1, 216, 345651, 1848745731, 23070700145026, \ldots(\text{no entry})
\end{align*}
\]

$m = 3$

\[
\begin{align*}
  k = 0 & : 1, 1, 20, 1680, 369600, 168168000, \ldots(A014606 – \frac{(3n)!}{(3!)^n}) \\
  k = 1 & : 1, 1, 35, 5775, 2627625, 2546168625, \ldots(A025036 – \frac{(4n)!}{(4!)^n n!}) \\
  k = 2 & : 1, 1, 91, 51821, 94597041, 404793761526, \ldots(\text{no entry}) \\
  k = 3 & : 1, 1, 266, 621831, 5616763761, 134269580611026, \ldots(\text{no entry}) \\
  k = 4 & : 1, 1, 827, 8721245, 438307511209, 66953592509190248, \ldots(\text{no entry})
\end{align*}
\]

one to easily generate on a computer the first few terms in the corresponding sequences of maximal chain numbers, and just as before, we quickly encounter new sequences — see Table 2.

This paper is aimed at readers of diverse background. While we bring together combinatorics, lattice theory and category theory, we took particular care in the presentation to make the paper accessible to non-experts of each of these fields (moreover, the paper uses only basic concepts from these fields). We hope that our work will entice further research on combinatorics of maximal chains of stacked lattices (or posets, more generally). Among questions left unresolved in this paper is the question of finding explicit formulas for those integer sequences from Tables 1 and 2 that do not appear in OEIS, or proving that they do not exist. We anticipate these questions to be highly non-trivial.
Table 2 The maximal chain numbers \( \#\Sigma_m^k \) and the corresponding entry code (indicated in brackets) in the On-Line Encyclopedia of Integer Sequences (accessed 28/05/2022). For the program that was used to compute these numbers, see [8]

\[
m = 0, 1, 2, 3, 4, 5, \ldots
\]

\[
\begin{align*}
n = 0 : \\
k = 0 & : 1, 1, 1, 1, 1, \ldots \text{(A000012)} \\
k = 1 & : 1, 1, 1, 1, 1, \ldots \text{(A000012)} \\
k = 2 & : 1, 1, 2, 5, 14, 42, \ldots \text{(A000108 – Catalan numbers)} \\
k = 3 & : 1, 1, 5, 42, 606, \ldots \text{(A005789 – 3-dimensional Catalan numbers)} \\
k = 4 & : 1, 1, 14, 624, 24024, 1662804, \ldots \text{(A005790 – 4-dimensional Catalan numbers)}
\end{align*}
\]

\[
\begin{align*}
n = 1 : \\
k = 0 & : 1, 1, 1, 1, 1, \ldots \text{(A000012)} \\
k = 1 & : 1, 1, 2, 5, 14, 42, \ldots \text{(A000108 – Catalan numbers)} \\
k = 2 & : 1, 1, 5, 42, 606, \ldots \text{(A005789 – 3-dimensional Catalan numbers)} \\
k = 3 & : 1, 1, 14, 624, 24024, 1662804, \ldots \text{(A005790 – 4-dimensional Catalan numbers)} \\
k = 4 & : 1, 1, 42, 606, 1662804, 701149020, \ldots \text{(A005791 – 5-dimensional Catalan numbers)}
\end{align*}
\]

\[
\begin{align*}
n = 2 : \\
k = 0 & : 1, 2, 6, 20, 70, 252, \ldots \text{(A000680 – \( (2m)!/(m!)^2 \))} \\
k = 1 & : 1, 2, 16, 2816, 46592, \ldots \text{(A006335 – Kreweras)} \\
k = 2 & : 1, 2, 46, 2240, 160504, 14594568, \ldots \text{(no entry)} \\
k = 3 & : 1, 2, 140, 30108, 11721144, 6625780016, \ldots \text{(no entry)} \\
k = 4 & : 1, 2, 444, 448272, 1024045836, 3936970992944, \ldots \text{(no entry)}
\end{align*}
\]

\[
\begin{align*}
n = 3 : \\
k = 0 & : 1, 6, 90, 1680, 34650, 756756, \ldots \text{(A006480 – \( (3m)!/(m!)^3 \))} \\
k = 1 & : 1, 6, 288, 24444, 2738592, 361998432, \ldots \text{(A340540)} \\
k = 2 & : 1, 6, 918, 363984, 234506712, 203517798360, \ldots \text{(no entry)} \\
k = 3 & : 1, 6, 2988, 5753484, 22547430432, 137927632096368, \ldots \text{(no entry)} \\
k = 4 & : 1, 6, 9936, 96198840, 2404039625820, 109858268535649608, \ldots \text{(no entry)}
\end{align*}
\]

It is worthy to note that the sequence A340540 from Table 2 has been added to OEIS while the final version of this paper was in preparation. An explicit formula for that sequence is not known either.

1 Stacking Hypercubes

Consider the sequence \( C_1^\infty = (C_1^0, C_1^1, C_1^2, C_1^3, \ldots) \) of hypercubes: each \( C_1^n \) denotes the \( n \)-th cartesian power of the 1-chain \( C_1 \). Thus, each \( C_1^n \) is an \( n \)-dimensional cube. The numbers
\#C_1^n \text{ of maximal chains in the members of the sequence } C_1^\infty \text{ are of course given by the factorials }
\#C_1^n = n! \quad (n \geq 0).

In general, we will write \#P for the number of maximal chains in a (finite) poset \( P \).

We will stack the hypercubes \( C_1^i \), as described in the Introduction, along the obvious embeddings \( C_1^i \rightarrow C_1^{i+1} \), visualized below in the case when \( i = 3 \):

The resulting lattices will be denoted by \( \Sigma_n C_1^n = \Sigma_n C_1^n \). See Section 3 for a formal definition of these lattices (and the stacking process, in general), and see the second row in Fig. 3 for illustrations of \( \Sigma_n C_1^n \) in the case when \( n = 0, 1, 2, 3, 4 \).

**Theorem 1** The number of maximal chains in the lattice \( \Sigma_n C_1^n \) obtained by stacking the hypercubes \( C_1^0 \rightarrow C_1^1 \rightarrow \cdots \rightarrow C_1^n \) is given by odd double factorials (where \((-1)!! = 1\)):
\[
\#\Sigma_n C_1^n = (2n - 1)!!.
\]

**Proof** It is well known (see e.g. [13]) that odd double factorials count the number of involutions on a set with \( 2n \) elements with no fixed points (which are the same as 2-partitions). To see how, notice that such an involution can be represented as a list of distinct elements of \( 2n \) (permutation), where each consecutive pair of odd and neighboring even entry in the list is an involuted pair. There are \( (2n)! \) such lists, but we get them too often: for each pair, we must divide by 2, so altogether by \( 2^n \), and then the order of the pairs does not matter, so we must divide by \( n! \). This gives
\[
\frac{(2n)!}{2^n n!} = (2n - 1)!!.
\]

We will now show that the number of maximal chains in \( \Sigma_n C_1^n \) is the weighted sum of the number of \( 2n \)-step Dyck paths, which is known to count the number of involutions of a \( 2n \) element set having no fixed points (we will recall how, on an example). This will then conclude the proof.

A walk along a maximal chain in \( \Sigma_n C_1^n \) visits each cube \( C_1^i \), makes some steps in the cube, and moves on to the next cube (where in the last cube, the walk reaches the top element of \( C_1^n \)).
Let $i_j$ represent height reached in the cube $C^j_i$ before the exit ($i_0 = 0$). Then $i_j$ is also the height at which the next cube $C^{j+1}_i$ is entered (except when $j = n$, in which case there is no next cube). The maximum height that can be reached in $C^j_i$ is $j + 1$. The number of possible paths that can be taken inside $C^{j+1}_i$ after entering it at height $i_j$ and before existing it at height $i_{j+1}$ is the falling factorial

$$(j + 1 - i_j)(j + 1 - i_j - 1) \cdots (j + 1 - i_j - (i_{j+1} - i_j - 1)) = (j + 1 - i_j)^{i_{j+1} - i_j}.$$ 

Note that in general, the falling factorial is defined by $n^k = \frac{n!}{(n-k)!} = (n-0)(n-1) \cdots (n-(k-1))$. The number of maximal chains in $\Sigma_n C^n_1$ is thus given by

$$\# \Sigma_n C^n_1 = \sum_{0=i_0 \leq i_1 \leq \cdots \leq i_n=n} \prod_{j=0}^{n-1} (j + 1 - i_j)^{i_{j+1} - i_j}.$$ 

The indices in this summation form a Dyck path. Here is an example, with $n = 7$ (ignore the numbers in bold font for now):

This is a weighted Dyck path, with the weights given by those factors of the falling factorials in the corresponding summand of the summation above, where $i_{j+1} \neq i_j$.
(when $i_{j+1} = i_j$, the corresponding falling factorial is $(j + 1 - i_j)^{i_{j+1}-i_j} = (j + 1 - i_j)^0 = 1$). In the case of the example considered, these factors are:

$$1^0 \cdot 2^0 \cdot 3^1 \cdot 4^2 \cdot 2^2 \cdot 1^1 = 3^1 \cdot 4^2 \cdot 2^2 \cdot 1^1 = 3 \cdot (4 \cdot 3 \cdot 2) \cdot (2 \cdot 1) \cdot 1.$$ 

The weights are shown on the diagram in bold font. Each bold-face number represents number of choices when going upward in the given cube. A more traditional way of drawing this is:

![Diagram](image)

Notice that the weights at each level match with the height of the level. In his video book, X. Viennot ($n!$–garden, part (b)) constructs a fixed-point free involution on the set \{1, 2, \ldots, 14\} from such a weighted Dyck path (which he calls ‘Hermite history’). First, he draws the following, which indicates whether at each position the Dyck path makes an ‘up-step’ or a ‘down-step’.

![Diagram](image)

He will pair each up-step-node with a down-step-node, but which one? We have choices! Copy the weights from the Dyck path:

![Diagram](image)

As we can see, one may connect 11 with one of 12, 13, 14. The boldface number 3 now tells us that we have 3 options for such a pairing. Write below the bold-face numbers which options will be selected:

![Diagram](image)

Now we work off the up-step-nodes from right to left, and connect with our choice. For 11 we choose 1, therefore our first choice (from the right), which is node 14:
We then move to the next node, which is labeled 8, and connect it with the 3rd option (from the right), which is node 10:

Now we move to the next node 7 and connect it with the first option (from the right), which is node 13:

And so forth:

The goal is achieved; we constructed an involution.

To see the bijection between 2-partitions of a 2n-element set and maximal chains in the lattice $\Sigma_n C_n^2$ more directly (without passing through weighted Dyck paths, as we did in the proof above), we first embed $\Sigma_n C_n^2$ in $C_n^2$ as follows. Represent elements of $C_2$ as 0, 1 and 2, in the increasing order. Then elements of $C_2^2$ can be represented as strings of 0,1,2 of length $n$. On the other hand, if we write 1 and 2 for the elements of $C_1$, then elements of each $C_1^j$ can be represented as strings of 1,2 of length $j$. Embed $C_1^j$ into $C_2^j$ by adding in front 0's to fill up each string of length $j$ to a string of length $n$. A walk along a maximal chain in $\Sigma_n C_n^2$ now becomes a walk along a maximal chain in $C_n^2$ which passes through only those strings where a zero never follows a nonzero entry. Each time we make a step along such walk, we have two choices: either to increment any of the nonzero entries in
the string, or to increment the right-most 0 entry to 1. This second choice corresponds to moving to the next cube, whereas the first one corresponds to moving up in the same cube. There are altogether $2n$ steps to make. Pair each $j$-th step of the first type to the $j'$-th step of the second type, when at both steps the $i$-th entry was incremented. Note that each step can only be incremented twice and the step of the first type incrementing the $i$-th entry will always succeed the step of the second type incrementing the $i$-th entry (first we would have to move to the cube that has the dimension along which we want to make a step, before such step can be made). Let us illustrate this by displaying the walk corresponding to the 2-partition shown at the end of the proof above (where $n = 14$):

This argument easily generalizes, by replacing $C_1$ with $C_{m-1}$, to get a bijection between maximal chains in $\Sigma_n C_{m-1}$ and $m$-partitions of a set with $mn$ elements. We therefore get the following theorem, which subsumes the previous one:

**Theorem 2** For any natural number $m \geq 1$, there is a bijection between the set of maximal chains in the lattice $\Sigma_n C_{m-1}$ obtained by stacking $C_0 \to C_1 \to \cdots \to C_{m-1}$, and the set of $m$-partitions of a set with $mn$ elements. Therefore,

$$\#\Sigma_n C_{m-1} = \frac{(mn)!}{(m!)^n n!}.$$ 

### 2 Iterated Stacking

The process of stacking lattices in a sequence can be iterated: the stacked lattices produce a sequence of lattices, whose members can be stacked. We write $\Sigma_n^k L_n$ for the result of $k$-th iteration, with $\Sigma_n^0 L_n = \Sigma_n L_n$ and $\Sigma_n^0 L_n = L_n$. Each homomorphism $\Sigma_n^k L_n \to \Sigma_{n+1}^k L_{n+1}$ used for the next iteration is given by the universal property of lax colimit and the homomorphism $\Sigma_{n+1}^{k-1} L_n \to \Sigma_{n+1}^{k-1} L_{n+1}$ from the previous iteration (see Section 3). The sequence $C_1, C_2, \ldots$ of chains, with the inclusions $C_i \to C_{i+1}$ we have been considering, can be obtained by stacking copies of the trivial chain $C_0$ along the identity maps $C_0 \to C_0 \to \cdots \to C_0$. Thus,

$$\Sigma_n C_0 = C_n.$$
We then get
\[
\#\Sigma^2_n C_0 = \#\Sigma_n C_n = \frac{(2n)!}{n!(n+1)!}.
\]

A natural question arises: what happens if we go higher in iteration? Geometric inspection of \(\Sigma^k_n C_0\) shows that it is isomorphic to the portion of \(C^n_k\) consisting of points with non-decreasing coordinates; that is, all points whose coordinates satisfy
\[
x_1 \leq x_2 \leq \cdots \leq x_n.
\]

Hence maximal chains in \(\Sigma^k_n C_0\) are counted by
\[
\#\Sigma^k_n C_0 = \frac{(k-1)!(kn)!}{n!(n+1)!(n+k-1)!},
\]

nothing other than \(k\)-dimensional Catalan numbers [11].

So then, what do we get if we iterate stacking of hypercubes? Adopting ideas from the discussion before Theorem 2, we can represent each \(\Sigma_n^k C^n_m\) as a subposet of \(C^n_{k+m}\), consisting of those elements whose coordinates \((x_1, \ldots, x_n)\) satisfy the following condition:

\[\star\] for each \(i \in \{1, \ldots, n−1\}\), if \(x_i+1 \in \{0, \ldots, k−1\}\) then \(x_i \leq x_i+1\).

This representation turns out to be an embedding of lattices:

**Theorem 3** (representation of \(k\)-iterated stacking of \(m\)-fold hypercubes) The poset \(\Sigma_n^k C^n_m\) is isomorphic to the sublattice of \(C^n_{k+m}\) consisting of those elements that satisfy \((\star)\).

The formal proof of this theorem is given in Section 4.

Note that the Dyck situation and the hypercube one come together with \(\Sigma_n^k C^n_m\): we get the first by letting \(m = 0\) and \(k = 2\), and the second by letting \(k = 1\) and \(m = 1\). The representation from the theorem above has been used to create drawings of stacked lattices in Figs. 2, 3 and 4. In each row of each figure, the hollow vertices represent points mapped from the previous term.

The case \(k = 0\) is trivial. This is when no stacking is taking place. So in this case,
\[
\#\Sigma^0_n C^n_m = \frac{(mn)!}{(m!)^n}
\]

is the well-known number of maximal chains of \(C^n_m\). The proof is a straightforward. Walking up in the lattice \(C^n_m\) requires \(mn\) many steps. Write the steps out in a sequence where the first \(m\) many steps are from a walk in the first dimension, the second \(m\) many steps in the second dimension, etc. In each group, the order of steps is insignificant, so for each group, divide \((mn)!\) by all possible permutations of the group – that is, by \(m!\). There are \(n\) many such divisions that need to take place.

It appears that except those cases discussed in this paper, no explicit formulas are known for \(\Sigma_n^k C^n_m\), and moreover, when \(k > 1\) and \(m > 1\), these combinatorial objects have not been considered in the literature. Table 1 gives some of the numbers \(\#\Sigma_n^k C^n_m\) generated by a computer. As indicated there (as well as in the Introduction), the numbers corresponding to the iterated stacking of hypercubes, i.e., the case when \(m = 1\), do show up on the On-Line Encyclopedia of Integer Sequences with an interpretation given in the following theorem.

In the case when \(k = 2\), these combinatorial objects are in bijection with certain tree-child networks, as explained in [4].

**Theorem 4** There is a bijection between maximal chains in the lattice \(\Sigma_n^k C^n_1\) and words \(w\) of length \((k+1)n\) in an alphabet \([a_1, \ldots, a_n]\) with \(n\) distinct letters, such that each letter
Fig. 2 First, second and third iteration of stacking the trivial lattice — in the second and third iteration, maximal chains in these lattices are given by Catalan numbers and 3-dimensional Catalan numbers, respectively.
Fig. 3 Stacking hypercubes — zeroth, first and second iteration
Fig. 4  Lattices $\Sigma^n_k C_2$ for $k \in \{0, 1, 2\}$ and $n \in \{0, 1, 2, 3, 4\}$
occurs $k + 1$ times in the word, and for each prefix $z$ of $w$, either $a_i$ does not occur in $z$ or if it does, then for each $j > i$, it occurs more or the same number of times as $a_j$.

Proof Let $w$ be such a word. As we read letters encountered in $w$ from left to right, record the letters that remain unused as an $n$-tuple $(x_1, \ldots, x_n)$, where $x_i$ is the number of remaining occurrences of $a_i$ in the rest of the word. The following example (where $n = 2$ and $k = 2$) illustrates this, where letters left of the dot are those that have been read.

\[
\begin{array}{c}
.2a_1a_1a_2a_1a_2 \quad (3, 3) \\
a_2a_1a_1a_2a_1a_2 \quad (3, 2) \\
a_2a_1a_1a_2a_1a_2a_2 \quad (2, 2) \\
a_2a_1a_1a_2a_1a_2a_2a_2 \quad (1, 2) \\
a_2a_1a_1a_2a_1a_2a_2a_2 \quad (1, 1) \\
a_2a_1a_1a_2a_1a_2a_2a_2a_2 \quad (0, 1) \\
a_2a_1a_1a_2a_1a_2a_2a_2a_2 \quad (0, 0)
\end{array}
\]

With every next reading, exactly one of the terms in $(x_1, \ldots, x_n)$ decrements. We get a bijection between all words $w$ of length $(k + 1)n$ in the alphabet $\{a_1, \ldots, a_n\}$ and maximal chains in the lattice $C_{k+1}^n$ — the points of the chain, as we descend down the chain, are given by the sequence of $n$-tuples $(x_1, \ldots, x_n)$. If we could show that the condition $(\ast)$ on $(x_1, \ldots, x_n)$ is equivalent to the requirement on the prefix $z$ read (which resulted in the tuple $(x_1, \ldots, x_n)$) given in the theorem, then we would be done. However, as we will now see, this equivalence actually fails. First, note the following:

- $x_i = k + 1$ if and only if $a_i$ does not occur in $z$.
- $x_i \leq x_j$ if and only if the number of occurrences of $a_i$ in $z$ is more or the same as the number of occurrences of $a_j$.

So the requirement in the theorem on a prefix $z$ translates to the following requirement on the corresponding tuple $(x_1, \ldots, x_n)$.

$(\ast')$ For each $i \in \{1, \ldots, n\}$, either $x_i = k + 1$ or $x_i \leq x_j$ for each $j > i$.

To see that $(\ast)$ is not equivalent to $(\ast')$, consider the tuple $(3, 0)$ (for $n = 2$ and $k = 2$). For this tuple, $(\ast')$ holds, but $(\ast)$ does not. To see what to do next, we look at an illustration of each of the two types of tuples, seen as points in $C_{k+1}^n$, in the case when $n = 2$ and $k = 2$:

![Tuples allowed by $(\ast)$](image1)

![Tuples allowed by $(\ast')$](image2)

To prove the theorem it is sufficient to show what these pictures suggest: that $(\ast)$ implies $(\ast')$ for individual points (tuples), and points satisfying $(\ast')$ but not $(\ast)$, can never be encountered on a maximal chain all of whose points satisfy $(\ast')$.
Suppose \((\ast)\) holds for a point \((x_1, \ldots, x_n)\) and \(x_i \neq k + 1\) for some \(i \in \{1, \ldots, n\}\). Assume the contrary to what \((\ast')\) requires: that \(x_i > x_j\) for some \(j > i\). Since \(x_i \leq k\), we get \(x_j < k\) and iteratively applying \((\ast)\) results in \(x_i \leq x_{i+1} \leq \cdots \leq x_j\). This contradicts the assumption \(x_i > x_j\). So \((\ast')\) follows from \((\ast)\). Conversely, suppose \((\ast')\) holds. Let \(x_{j+1} \in [0, \ldots, k - 1]\) and suppose unlike what \((\ast)\) requires, we have \(x_i > x_{i+1}\). Then \((\ast')\) forces \(x_i = k + 1\). Suppose the point \((x_1, \ldots, x_n)\) is encountered in some maximal chain. Then at some descend along the chain there is a point \((y_1, \ldots, y_n)\) with either \(y_{i+1} = x_{i+1} - 1\) and \(y_i = x_i\), or \(y_{i+1} = x_{i+1}\) and \(y_i = x_i - 1 = k\). The second case would violate \((\ast')\). For the first case, we repeat the same argument. Eventually, we end up with the second case, and so a maximal chain containing a point that satisfies \((\ast')\) but not \((\ast)\), will also contain a point that does not satisfy \((\ast')\). This completes the proof.

\[\square\]

3 Some Conceptual Remarks on Stacking Lattices

As remarked in the Introduction, the process of stacking lattices in a sequence comes from a construction of ‘lax colimit’ in category theory. In this section we elaborate a bit on this remark. We will be concerned with the 2-category of posets, which we denote by \(\text{Pos}\). Note that this is equivalent to a subcategory of the 2-category of categories, consisting of those categories where any two parallel morphisms are equal and any isomorphism is an identity morphism. More explicitly:

- **Objects** in \(\text{Pos}\) are posets — partially ordered sets, i.e., sets equipped with a reflexive, transitive and antisymmetric binary relation.
- **Morphisms** are monotone maps between posets, i.e., maps which preserve the relation.
- **Composition of morphisms** is defined by composition of maps.
- **For any two posets** \(L\) and \(M\), the category structure on the set of morphisms from \(L\) to \(M\) is a poset structure given by setting \(f \leq g\) when \(f(x) \leq g(x)\) for each \(x \in L\).

In other words, a 2-cell between two morphisms \(f\) and \(g\) is a relation \(f \leq g\) (it either does not exist, or is unique, for a given \(f\) and \(g\)).

Given a sequence

\[
M_0 \rightarrow f_0 \rightarrow M_1 \rightarrow f_1 \rightarrow M_2 \rightarrow f_2 \rightarrow M_3 \rightarrow f_3 \rightarrow M_4 \rightarrow f_4 \rightarrow \cdots \rightarrow f_{n-1} \rightarrow M_n
\]

of objects and morphisms in \(\text{Pos}\), we define its *lax sum* as a specialization of the notion of a lax colimit of a diagram in a 2-category. It is given by an object \(L = \Sigma_n M_n\) in \(\text{Pos}\), equipped with morphisms \(\iota_j^n: M_j \rightarrow L\), where \(j \in \{0, \ldots, n\}\), such that the following conditions hold:

- \(\iota_j^n \leq \iota_{j+1}^n \circ f_j\) for each \(j \in \{1, \ldots, n-1\}\).
- For any object \(L'\) and morphisms \(\iota'_j\) (where \(j \in \{0, \ldots, n\}\)) such that \(\iota'_j \leq \iota'_{j+1} \circ f_j\) for each \(j \in \{1, \ldots, n-1\}\), there exists a unique morphism \(u: L \rightarrow L'\) such that \(u \circ \iota_j^n = \iota'_j\) for each \(j\).
This property can be pictured as follows:

In the picture, each occurrence of the symbol \( \leq \) stands for a 2-cell between the composites of the surrounding diagram, indicating that the surrounding diagram lax commutes. In other words, they represent the relations \( \iota^i_j \leq \iota^i_{j+1} \circ f_j \) and \( \iota^{i+1}_j \leq \iota^{i+1}_{j+1} \circ f_j \).

As any object defined by a universal property, lax sum is unique up to an isomorphism: two lax sums of the same diagram will be connected by morphisms in both directions from the above universal property, which will turn out to be inverses of each other.

Concretely, a lax sum of a sequence displayed above can be constructed as follows:

- Start by taking the disjoint union of all the posets \( M_j \). Let \( (x, j) \) denote the representative of \( x \in M_j \) in the disjoint union.
- To turn the disjoint union of posets into a poset, equip it with the relations \( (x, j) \leq (y, j + i) \) for each
  \[
  (f_{j+i-1} \circ \cdots \circ f_j)(x) \leq y.
  \]
  This includes the possibility \( i = 0 \), in which case \( (f_{j+i-1} \circ \cdots \circ f_j)(x) \) is defined as \( x \).
- Each map \( \iota^n_j \) is then defined by \( \iota^n_j(x) = (x, j) \).

It is not difficult to show that this construction has the universal property required from a lax sum. We call this the concrete lax sum. Note that a concrete lax sum always exists and any lax sum is canonically isomorphic to a concrete one.

There is also another (equivalent) conceptual interpretation of a lax sum. Thinking of the given sequence of posets as a functor from the chain \( C_n \) seen as a category, into the category of categories, the lax sum is nothing other than the category (which in this case happens to be a poset) arising from the (dual of) standard Grothendieck construction in the theory of fibrations [7].

A sequence

\[
M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} M_4 \xrightarrow{f_4} \cdots
\]

of posets and monotone maps (can be an infinite sequence or a finite one) gives rise to another such sequence,
\[ \Sigma_0 M_0 \xrightarrow{f'_0} \Sigma_1 M_1 \xrightarrow{f'_1} \Sigma_2 M_2 \xrightarrow{f'_2} \Sigma_3 M_3 \xrightarrow{f'_3} \Sigma_4 M_4 \xrightarrow{f'_4} \cdots \]

where each \( f'_n \) arises from the universal property of lax sum \( \Sigma_n M_n \), as shown in the following picture:

As a morphism between concrete lax sums, the definition of \( f'_n \) is simple:

\[ f'_n(x, n) = (x, n), \]

for all \( x \in M_n \).

Iterating the process above gives the diagram in Fig. 5, where we do not distinguish in notation the \( \iota' \)'s arising at different iterations. The posets in the \( i \)-th iteration are denoted by \( \Sigma_i^j M_j \), where \( \Sigma_i^0 M_j = M_j \).

We may think of \( \Sigma_n M_n \) as the \( n \)-th ‘partial sum’ of the infinite ‘series’ of posets. Note that \( \Sigma_0 M_0 = M_0 \). The process of stacking lattices considered in this paper is given by this notion of ‘partial sum’. Recall that a lattice is a poset having binary joins and meets. The fact that in all situations considered in this paper, stacking lattices results in a lattice, comes from the following result:

**Theorem 5** For any sequence \( f_i : M_i \to M_{i+1} \) of posets, the following hold:

1. Each \( f'_i : \Sigma_i M_i \to \Sigma_{i+1} M_{i+1} \) is order-reflecting (hence injective) and has down-closed image. Moreover, each \( \iota^n_j : M_j \to \Sigma_n M_n \) is order-reflecting (and hence injective).
2. If \( M_n \) has empty meet (top element), then so does \( \Sigma_n M_n \), and \( \iota^n_n : M_n \to \Sigma_n M_n \) preserves empty meet.
3. If each \( M_i \) has empty join (bottom element) and each \( f_i : M_i \to M_{i+1} \) preserves it, then each \( \Sigma_n M_n \) has empty join and \( \iota^n_0 : M_0 \to \Sigma_n M_n \), as well as each \( f'_i : \Sigma_i M_i \to \Sigma_{i+1} M_{i+1} \), preserves empty join.
4. If each \( M_i \) has binary joins and each \( f_i : M_i \to M_{i+1} \) preserves them, then each \( \Sigma_n M_n \) has binary joins, and moreover, each \( f'_i : \Sigma_i M_i \to \Sigma_{i+1} M_{i+1} \), as well as each \( \iota^n_j : M_j \to \Sigma_n M_n \), preserves binary joins.
Fig. 5 Iterated lax sums for a sequence $f_i: M_i \to M_{i+1}$ of poset morphisms

(5) If each $M_i$ has binary meets and each $f_i: M_i \to M_{i+1}$ is order-reflecting with down-closed image, then each $\Sigma_n M_n$ has binary meets and each $\iota^n: M_j \to \Sigma_n M_n$ preserves binary meets (as does each $f_i$ and $f'_i$).

Proof Recall that $\iota^n_j(x) = (x, j)$, and: $(x, j) \leq (y, k)$, when $j \leq k$ and $(f_k \circ \cdots \circ f_j)(x) \leq y$. Furthermore, recall that each $f'_i: \Sigma_i M_i \to \Sigma_{i+1} M_{i+1}$ is defined by

$$f'_i(x, j) = (x, j), \quad j \leq i.$$  

It is then clear that the image of each $f'_i$ is down-closed and order-reflecting, as well as that each $\iota^n_j$ is order-reflecting, so we have (5) — note that an order-reflective monotone map is always injective.

When $M_n$ has top element $t$, the top element in $\Sigma_n M_n$ is given by $(t, n)$. This yields (5).

Suppose $M_0$ has bottom element. If each $f_i$ preserves the bottom element, then $(b, 0)$ is the bottom element of $\Sigma_n M_n$, where $b$ is the bottom element of $M_0$. This yields (5).

Suppose now each $M_i$ has binary joins and each $f_i$ preserves them. Consider two elements $(x, j)$ and $(y, k)$ of $\Sigma_n M_n$, with $j \leq k$. Define

$$(x, j) \lor (y, k) = ((f_{k-1} \circ \cdots \circ f_j)(x) \lor y, k),$$

where the join appearing on the right hand side of the equality is the join in $M_k$. We will now show that $(x, j) \lor (y, k)$ is the join of $(x, j)$ and $(y, k)$ in $\Sigma_n M_n$. It is easy to see that $(x, j) \leq (x, j) \lor (y, k)$ and $(y, k) \leq (x, j) \lor (y, k)$. Suppose $(x, j) \leq (z, l)$ and
(x, k) ≤ (z, l). Then k ≤ l and

\[(f_{l-1} \circ \cdots \circ f_j)(x) \vee (f_{l-1} \circ \cdots \circ f_k)(y) \leq z.\]

On the other hand, since each f preserves joins,

\[(f_{l-1} \circ \cdots \circ f_k)((f_{k-1} \circ \cdots \circ f_j)(x) \vee y) = (f_{l-1} \circ \cdots \circ f_j)(x) \vee (f_{l-1} \circ \cdots \circ f_k)(y).\]

This shows (x, j) ∨ (y, k) ≤ (z, l). We have thus proved that Σ_nM_n has binary joins. The formula for join in Σ_nM_n established above guarantees that each f_j', as well as each ι_j^n, preserve joins. This proves (5).

Suppose each M_i has binary meets, and, each f_i is order-reflecting with down-closed image. Then, each f_i is injective and preserves binary meets. Thanks to (5), the same is true for each f_i'. We now show that each Σ_nM_n has binary meets. Consider two elements (x, j) and (y, k) in Σ_nM_n. Then x ∈ M_j and y ∈ M_k, where j ≤ n and k ≤ n. Without loss of generality, assume j ≤ k. Consider the meet

\[z = (f_{k-1} \circ \cdots \circ f_j)(x) \wedge y\]

in M_k. Let z' be the unique element of M_j such that

\[(f_{k-1} \circ \cdots \circ f_j)(z') = z.\]

We will prove

\[(z', j) = (x, j) \wedge (y, k).\]

Note that (z', j) ≤ (y, k). Moreover, since z ≤ (f_{k-1} \circ \cdots \circ f_i)(x), we get z' ≤ x by the order-reflection property of the composite f_k-1 \circ \cdots \circ f_i. So (z', j) ≤ (x, j). Now suppose (w, l) ≤ (x, j) and (w, l) ≤ (y, k). Then l ≤ j. Moreover,

\[(f_{j-1, \ldots, f_l})(w) ≤ x \text{ and } (f_{k-1, \ldots, f_l})(w) ≤ y.\]

This implies (f_{k-1, \ldots, f_l})(w) ≤ z, which by order-reflection gives

\[(f_{j-1, \ldots, f_l})(w) ≤ z'.\]

Then (w, l) ≤ (z', j), as desired. So (z', j) is indeed the meet of (x, j) and (y, k). In the case when j = k, we get z' = x \wedge y, from which it follows at once that ι_j^n preserves binary meets. This proves (5).

The sequences of lattices that we dealt with in the paper satisfy all assumptions of the theorem above. Let us call such sequence of lattices a lattice series. Thus, a lattice series is an infinite sequence

\[M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} M_4 \xrightarrow{f_4} \cdots\]

of lattices and homomorphisms of join semi-lattices, such that f_i’s are order-reflecting and have down-closed images, and hence also preserve binary meets and are injective. A lattice series is a model of the self-dual axiomatic context for isomorphism theorems described in [5]: it is an example of a ‘noetherian form’ [12]. The following construction, used already in the proof of the theorem above, is related to ‘diagram chasing’ in a noetherian form. We introduce below a short-hand notation for it.

For x ∈ M_j and k ≥ j, define

\[x^k = \begin{cases} x, & k = j, \\ (f_{k-1} \circ \cdots \circ f_j)(x), & k > j. \end{cases}\]
For \( y \in M_k \), if \( y = (f_{k-1} \circ \cdots \circ f_j)(x) \) for \( x \in M_j \) and \( j \leq k \), then this \( x \), necessarily unique, will be denoted by \( x = y^{-j} \). According to Theorem 5 above, the sequence of partial sums of a lattice series is again a lattice series. The proof of Theorem 5 shows that the joins and the meets in the poset \( \Sigma_n M_n \) can be expressed in terms of joins and meets in each \( M_i \) as follows:

- \((x, j) \lor (y, k) = (x^k \lor y, k)\) when \( j \leq k \),
- \((x, j) \land (y, k) = ((x^k \land y)^{-j}, j)\) when \( j \leq k \).

We furthermore have the following rules; the last three of these rules are consequences of the properties of the homomorphisms \( f_i \) in a lattice series:

- \((x^i)^k = x^k\) when \( j \leq k \) and \( x \in M_i \), where \( i \leq j \),
- \((x^{-i})^j = x^{-j}\) when \( i \leq j \leq k \) and \( x \in M_k \) is such that \( x^{-i} \) is defined,
- \((x^k)^{-i} = x^{-i}\) when \( i \leq j \leq k \) and \( x \in M_i \),
- \((x^{-i})^j = x^{-j}\) when \( i \leq j \leq k \) and \( x \in M_j \) with \( x^{-i} \) defined,
- \((x \lor y)^j = x^j \lor y^j\) when \( x, y \in M_i \), where \( i \leq j \),
- \((x \lor y)^j = x^j \lor y^j\) when \( x, y \in M_i \) with \( i \leq j \),
- \((x \lor y)^{-j} = x^{-j} \lor y^{-j}\) when \( x, y \in M_k \) with \( j \leq k \), and \((x \lor y)^{-j}\) is defined.

With these rules we can establish the following:

**Theorem 6** If in a lattice series, each lattice is a distributive lattice, i.e., the identity
\[
x \land (y \lor z) = (x \land y) \lor (x \land z)
\]
holds in it, then each partial sum of the series is also a distributive lattice.

**Proof** A lattice is distributive if and only if the identity
\[
(x \land y) \lor (y \land z) \lor (z \land x) = (x \lor y) \land (y \lor z) \land (z \lor x)
\]
holds in it (see e.g. [2]). Since this identity is symmetric in \( x, y, z \), to establish it in the partial sum of a lattice series it is sufficient to prove it for \((x, j), (y, k), (z, l)\) where \( j \leq k \leq l \). This can be done as follows:

\[
\begin{align*}
((x, j) \lor (y, k)) \land ((y, k) \lor (z, l)) \land ((z, l) \lor (x, j)) \\
= (x^k \lor y, k) \land (y^l \lor z, l) \land (z \lor x^l, l) \\
= (x^k \lor y, k) \land ((y^l \lor z) \lor (z \lor x^l), l) \\
= (((x^k \lor y)^l \land (y^l \lor z) \land (z \lor x^l))^{-k}, k) \\
= (((x^l \lor y^l) \land (y^l \lor z) \land (z \lor x^l))^{-k}, k) \\
= (((x^l \lor y^l) \lor (y^l \lor z) \lor (z \lor x^l))^{-k}, k) \\
= ((x^l \lor y^l)^{-k} \lor (z \lor x^l)^{-k} \lor (y^l \lor z)^{-k}, k) \\
= (((x^l \lor y^l)^{-j} \lor (z \lor x^l)^{-j})^k \lor (y^l \lor z)^{-k}, k) \\
= (((x^k \lor y)^{-j} \lor (z \lor x^l)^{-j})^k \lor (y^l \lor z)^{-k}, k) \\
= ((x^k \lor y)^{-j} \lor (z \lor x^l)^{-j}, j) \lor ((y^l \lor z)^{-k}, k) \\
= ((x^k \lor y)^{-j}, j) \lor ((y^l \lor z)^{-k}, k) \lor ((z \lor x^l)^{-j}, j) \\
= ((x, j) \land (y, k)) \lor ((y, k) \land (z, l)) \lor ((z, l) \land (x, j)).
\end{align*}
\]
All lattices in Fig. 1 are distributive. Moreover, all maps there are join-preserving and order-reflecting, with down-closed images. So Theorems 5 and 6 allow us to conclude that iterated stacking of these lattices along rows or columns (or in fact, along any path of consecutive arrows, for that matter), will always give rise to distributive lattices. Note that the fact that the lattices in Fig. 1 are distributive itself follows from the theorems above: $C_m$ can be obtained by staking $C_0$ along identity morphisms; then, $C_m^n$ is a cartesian power of a distributive lattice, and so is distributive.

**Remark 7** The link with noetherian forms briefly mentioned above suggest to look for lattice series that arise as series of lattices of substructures of group-like structures, given by sequences of group homomorphisms. A combinatorial investigation of such lattice series, similar to what we did in the present paper for the columns of Fig. 1, would be interesting. These lattices would be far from being distributive, unlike the ones considered in the present paper. Lattice series of the present paper all live inside the ‘noetherian form of distributive lattices’, studied in depth in [6]. The series $C_{1}^\infty$ can be realized as a series of substructures in another noetherian form — that of sets and partial bijections: the lattice $C_1^n$ is of course nothing other than the lattice (Boolean algebra) of all subsets of an $n$-element set.

## 4 Proof of Theorem 3

The proof of Theorem 3 that we present below relies on further analysis of the concept of lax sum. We will show first that each of the squares in Fig. 5 is a ‘co-comma’ diagram of posets seen as categories, which we simply call a ‘lax pushout’; note however that it is not an instance of a lax colimit — rather, it is an instance of (another) type of ‘indexed colimit’ (also called a ‘weighted colimit’) in a 2-category [9]. This will provide another way of constructing the posets $\Sigma_n^k M_n$: the morphisms $i_0^k$ are identity morphisms and so we can start by forming the top left square, the one next to it, and so on in each row successively.

A diagram

\[
\begin{array}{c}
K \\ \downarrow g \\
N \\
\downarrow f' \\
L
\end{array}
\xrightarrow{f} \begin{array}{c}
M \\ \downarrow g' \\
L
\end{array}
\]

in Pos is a lax pushout if and only if:

- $f'$ and $g'$ are order-reflecting morphisms of posets (hence injective) whose images are disjoint, with the union of images giving the entire $L$,
- the image of $f'$ is down-closed and for any $a \in N$ and $b \in M$, we have $f'(a) \leq g'(b)$ if and only if there exists $c \in K$ such that $a \leq g(c)$ and $f(c) \leq b$.

It is easy to see that such a diagram has the following universal property (that is a specialization of the construction of a co-comma object in a general 2-category):

- the square lax commutes, i.e., there is a 2-cell from $f' \circ g$ to $g' \circ f$ (thus, $f' \circ g \leq g' \circ f$),

\[ Springer \]
• for any other lax commuting square over \( f, g \),

\[
\begin{array}{c}
K \\
\downarrow g \\
N \\
\downarrow f''
\end{array}
\begin{array}{c}
\rightarrow \\
\downarrow \leq \\
\downarrow \\
\rightarrow L'
\end{array}
\begin{array}{c}
M \\
\downarrow g'' \\
\downarrow \\
\rightarrow L'
\end{array}
\]

there is a unique morphism \( u \) such that \( u \circ f' = f'' \) and \( u \circ g' = g'' \), as shown on the picture:

\[
\begin{array}{c}
K \\
\downarrow g \\
N \\
\downarrow f''
\end{array}
\begin{array}{c}
\rightarrow \\
\downarrow \leq \\
\downarrow \\
\rightarrow L'
\end{array}
\begin{array}{c}
M \\
\downarrow g' \\
\downarrow \leq \\
\downarrow u \\
\downarrow \rightarrow L'
\end{array}
\begin{array}{c}
\rightarrow \\
\downarrow f''
\end{array}
\]

Just as any other (weighted) colimit, a lax pushout is unique up to a canonical isomorphism. It exists for any given \( f \) and \( g \) as we can construct one concretely as follows:

• Let the elements of \( L \) be pairs of the form \((a, 1)\), where \( a \in N \), as well as pairs of the form \((b, 2)\), where \( b \in M \).
• Define \((a, 1) \leq (a', 1)\) when \( a \leq a' \) in \( N \).
• Define \((b, 2) \leq (b', 2)\) when \( b \leq b' \) in \( N \).
• Define \((a, 1) \leq (b, 2)\) when there exists \( c \in K \) such that \( a \leq g(c) \) and \( f(c) \leq b \).
• Define \( f' \) and \( g' \) by the identities \( f'(a) = (a, 1) \) and \( g'(b) = (b, 2) \).

**Lemma 8** Each square in Fig. 5 is a lax pushout.

**Proof** It suffices to prove this for the squares in the first row, since the rest of the rows are obtained by iteration. So we prove that for each \( n \), the following square is a lax pushout:

\[
\begin{array}{c}
\Sigma_n \Sigma_n \rightarrow \\
\downarrow f'_n \\
\Sigma_n \rightarrow \\
\downarrow \\
\Sigma_n M_n \\
\downarrow f_n \\
M_n \\
\downarrow \\
M_{n+1} \\
\downarrow f_n \\
M_{n+1}
\end{array}
\begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow
\end{array}
\begin{array}{c}
\uparrow \\
\uparrow \\
\uparrow \\
\uparrow \\
\uparrow \\
\uparrow \\
\uparrow
\end{array}
\begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow
\end{array}
\begin{array}{c}
\uparrow \\
\uparrow \\
\uparrow \\
\uparrow \\
\uparrow \\
\uparrow \\
\uparrow
\end{array}
\]

\( \Sigma_n \Sigma_n \rightarrow \Sigma_n M_n \rightarrow M_{n+1} \rightarrow M_{n+1} \rightarrow \Sigma_{n+1} M_{n+1} \)
For this, we need to check the following:

(i) $f'_n$ an order-preserving morphism of posets (and hence is injective), having down-closed image.

(ii) $\iota_{n+1}^n$ is an order-reflecting morphism of posets (and hence injective).

(iii) The images of $f'_n$ and $\iota_{n+1}^n$ are disjoint, and their union is the entire $\Sigma_{n+1}M_{n+1}$.

(iv) For any $a \in \Sigma_nM_n$ and $b \in M_{n+1}$, we have $f'_n(a) \leq \iota_{n+1}^n(b)$ if and only if there exists $c \in K$ such that $a \leq \iota_{n}^n(c)$ and $f_n(c) \leq b$.

We know (i) and (ii) already from Theorem 5. Since $f'_n$ is defined by $f'_n(x, j) = (x, j)$, where $j \leq n$ and $x \in M_j$, and $\iota_{n+1}^n(x) = (x, n + 1)$ where $x \in M_{n+1}$, the images of these two functions are clearly disjoint. Their union is the entire $\Sigma_{n+1}M_{n+1}$ since the latter only consists of elements of the form $(x, j)$, where $x \in M_j$ with $j \leq n$. So we have (iii). For any $(x, j) \in \Sigma_nM_n$ and $y \in M_{n+1}$, we have $f'_n(x, j) \leq \iota_{n+1}^n(y)$ if and only if $(x, j) \leq (y, n + 1)$. This is the case if and only if $x^{n+1} \leq y$ (see the previous section for this notation). If $x^{n+1} \leq y$, then define $c = x^n \in M_n$. We get $(x, j) \leq (c, n) = \iota_n^n(c)$ and $f_n(c) = x^{n+1} \leq y$. Conversely, if there exists $c \in M_n$ such that $(x, j) \leq \iota_n^n(c) = (c, n)$ and $f_n(c) \leq y$, then $x^n \leq c$ and so $x^{n+1} = f_n(x^n) \leq f_n(c) \leq y$, which gives $(x, j) \leq (y, n + 1)$. The proof is now complete.

We are now ready to prove Theorem 3.

First, we show the following (which, actually, does not require any of what we have done above):

**Lemma 9** The elements of $C_{k+m}^n$ satisfying $(\ast)$ are closed under meets and joins in $C_{k+m}^n$.

**Proof** Suppose both $(x_1, \ldots, x_n)$ and $(x'_1, \ldots, x'_n)$ satisfy $(\ast)$. Their meet in $C_{k+m}^n$ is given by

$$(x_1, \ldots, x_n) \wedge (x'_1, \ldots, x'_n) = (\min(x_1, x'_1), \ldots, \min(x_n, x'_n)).$$

Suppose $\min(x_{i_1+1}, x'_{i_1+1}) \in \{1, \ldots, k - 1\}$. Without loss of generality, assume $x_{i_1+1} \leq x'_{i_1+1}$. Then $x_i \leq x_{i+1}$ and so

$$\min(x_i, x'_i) \leq x_{i+1} = \min(x_{i+1}, x'_{i+1}).$$

This proves that tuples satisfying $(\ast)$ are closed under meets. The join in $C_{k+m}^n$ is given by

$$(x_1, \ldots, x_n) \vee (x'_1, \ldots, x'_n) = (\max(x_1, x'_1), \ldots, \max(x_n, x'_n)).$$

Suppose $\max(x_{i_1+1}, x'_{i_1+1}) \in \{1, \ldots, k - 1\}$. Then both $x_{i_1+1}$ and $x'_{i+1}$ belong to $\{1, \ldots, k - 1\}$ and the conclusion

$$\max(x_i, x'_i) \leq \max(x_{i+1}, x'_{i+1})$$

is immediate by the fact that both tuples satisfy $(\ast)$.

\[\square\]
To prove Theorem 3, it remains to show that each poset $\Sigma^k_m C^n$ is isomorphic to the sublattice of $C^n$ consisting of $n$-tuples satisfying $(\ast)$. Consider the diagram

$$
\begin{align*}
C^0_{0+m} &\to C^1_{0+m} &\to C^2_{0+m} &\to C^3_{0+m} &\to \cdots \\
&\downarrow &\downarrow &\downarrow & \\
C^0_{1+m} &\to C^1_{1+m} &\to C^2_{1+m} &\to C^3_{1+m} &\to \cdots \\
&\downarrow &\downarrow &\downarrow & \\
C^0_{2+m} &\to C^1_{2+m} &\to C^2_{2+m} &\to C^3_{2+m} &\to \cdots \\
&\downarrow &\downarrow &\downarrow & \\
C^0_{3+m} &\to C^1_{3+m} &\to C^2_{3+m} &\to C^3_{3+m} &\to \cdots \\
&\vdots &\vdots &\vdots & \\
&\vdots &\vdots &\vdots & \\
&\vdots &\vdots &\vdots & \\
\end{align*}
$$

of lattices $C^n$, where each vertical arrow $C^n_{k+m} \to C^n_{k+1+m}$ maps $(x_1, \ldots, x_n)$ to $(x_1 + 1, \ldots, x_n + 1)$ and each horizontal arrow $C^n_{k+m} \to C^n_{k+1+m}$ maps $(x_1, \ldots, x_n)$ to $(0, x_1, \ldots, x_n)$. Each square in this diagram lax commutes: left-bottom composite is below the top-right composite. Indeed:

$$(0, x_1 + 1, \ldots, x_n + 1) \leq (1, x_1 + 1, \ldots, x_n + 1).$$

It is easy to see that both the horizontal and the vertical morphisms preserve the property $(\ast)$. So the diagram above restricts to the following diagram, where $C^n_{k+m}^\ast$ denotes the sublattice of $C^n_{k+m}$ determined by the property $(\ast)$, while the morphisms are defined in the same way as above:

$$
\begin{align*}
C^0_{0+m}^\ast &\to C^1_{0+m}^\ast &\to C^2_{0+m}^\ast &\to C^3_{0+m}^\ast &\to \cdots \\
&\downarrow &\downarrow &\downarrow & \\
C^0_{1+m}^\ast &\to C^1_{1+m}^\ast &\to C^2_{1+m}^\ast &\to C^3_{1+m}^\ast &\to \cdots \\
&\downarrow &\downarrow &\downarrow & \\
C^0_{2+m}^\ast &\to C^1_{2+m}^\ast &\to C^2_{2+m}^\ast &\to C^3_{2+m}^\ast &\to \cdots \\
&\downarrow &\downarrow &\downarrow & \\
C^0_{3+m}^\ast &\to C^1_{3+m}^\ast &\to C^2_{3+m}^\ast &\to C^3_{3+m}^\ast &\to \cdots \\
&\vdots &\vdots &\vdots & \\
&\vdots &\vdots &\vdots & \\
&\vdots &\vdots &\vdots & \\
\end{align*}
$$

Lemma 8 and the fact that lax pushouts are unique up to a canonical isomorphism, reduces the proof of Theorem 3 to showing that each square in the above diagram is a lax pushout.
Indeed, for if it is so, we will be able to recursively create isomorphisms $d_{k,n,m}: \Sigma^k_n C^m \rightarrow C^{n*}_{k+m}$, the diagonal arrows in the following diagram, starting with identity morphisms along the top and the left side of the picture: note that when $n = 0$ or $k = 0$, we have $\Sigma^k_n C^m = C^{n*}_{k+m}$, and so we can set the $d_{0,n,m}$ and $d_{k,0,m}$ diagonal morphisms to be the identity morphisms.

So the final step in the proof is given by the following:

**Lemma 10** For any $k, n, m$, the following square is a lax pushout:

$$(x_1, \ldots, x_n)$$

$$(0, x_1, \ldots, x_n)$$

$$(x_1 + 1, \ldots, x_n + 1)$$

$$(0, x_1 + 1, \ldots, x_n + 1 + 1)$$

$$(y_1, \ldots, y_{n+1})$$

$$(y_1 + 1, \ldots, y_{n+1} + 1)$$

$$(z_1, \ldots, z_n)$$

$$(0, z_1, \ldots, z_n)$$
Proof The bottom map is clearly an order-reflecting morphism of posets, and it is easy to see that its image is down-closed. The right map is also obviously an order-reflecting morphism of posets. The images of these two maps certainly do not intersect. Consider any tuple \((x_1, \ldots, x_{n+1})\) in \(C_{k+1+m}^{n+1}\). If \(x_1 = 0\), then it belongs to the image of the bottom map. If \(x_1 \neq 0\), then by the property \((*)\), we can never have \(x_j = 0\) for any other coordinate and so \((x_1, \ldots, x_{n+1}) = (y_1 + 1, \ldots, y_{n+1} + 1)\) for some tuple \((y_1, \ldots, y_{n+1})\) in \(C_{k+m}^{n+1}\). We need to check that \((y_1, \ldots, y_{n+1})\) satisfies \((*)\). Let \(y_{i+1} \in \{0, \ldots, k - 1\}\). Then \(y_i + 1 \leq y_{i+1} + 1\). This proves \((y_1, \ldots, y_{n+1}) \in C_{k+m}^{n+1}\), showing that every element in the bottom right lattice falls in the image of one of the maps going into it. Complete the proof that the square is a lax pushout, it remains to show that \((0, z_1, \ldots, z_n) \leq (y_1 + 1, \ldots, y_{n+1} + 1)\) for two \((n+1)\)-tuples in \(C_{k+1+m}^{n+1}\) if and only if

\[
(x_1, \ldots, x_{n+1}) \leq (y_1 + 1, \ldots, y_{n+1} + 1) \text{ and } (0, x_1, \ldots, x_n) \leq (y_1, \ldots, y_{n+1})
\]

for some \(n\)-tuple \((x_1, \ldots, x_n)\) in \(C_{k+m}^{n}\). The ‘if’ part is easy to see (it follows from the fact that the square is lax commuting). To show the ‘only if’ part, assume \((0, z_1, \ldots, z_n) \leq (y_1 + 1, \ldots, y_{n+1} + 1)\) in \(C_{k+1+m}^{n+1}\). Then the \((n+1)\)-tuple \((y_1, y_2, \ldots, y_{n+1})\) belongs to \(C_{k+m}^{n+1}\), which easily implies that the \(n\)-tuple \((y_2, \ldots, y_n)\) belongs to \(C_{k+m}^{n}\). This is the desired \(n\)-tuple, since

\[
(z_1, \ldots, z_n) \leq (y_2 + 1, \ldots, y_n + 1) \text{ and } (0, y_2, \ldots, y_n) \leq (y_1, \ldots, y_{n+1}).
\]

This completes the proof.

\[\square\]

5 The Rows of Fig. 1

In this section we briefly consider the ‘orthogonal’ situation, i.e., iterated stacking of rows of Fig. 1 instead of columns. Theorem 3 has the following analogue in this case.

Theorem 11 The poset \(\Sigma_m^k C_n^m\) is isomorphic to the sublattice \(C_{m+k}^{n+k}\) of \(C_{m+k}\) consisting of those elements that satisfy

\[\text{(*) for each } i \in \{1, \ldots, n\}, x_i \leq x_{n+1} \leq x_{n+2} \leq \cdots \leq x_{n+k}.\]

The proof of this theorem can be established along the lines of the proof of Theorem 3, where the lax pushout from Lemma 10 would now be replaced with the following lax pushout:

\[
\begin{array}{c}
(x_1, \ldots, x_{n+k}) \downarrow \quad (x_1, \ldots, x_{n+k})
\end{array}
\]

\[
\begin{array}{ccc}
C_{m+k}^n & \longrightarrow & C_{m+k}^{n+k} & \longrightarrow & (y_1, \ldots, y_{n+k})
\end{array}
\]

\[
\begin{array}{c}
(x_1, \ldots, x_{n+k}, m) \downarrow \quad C_{m+k}^{n+k} \downarrow \quad C_{m+k+1}^{n+k+1} & \longrightarrow & (y_1, \ldots, y_{n+k}, m + 1)
\end{array}
\]

\[
\begin{array}{c}
(z_1, \ldots, z_{n+k+1}) \downarrow \quad (z_1, \ldots, z_{n+k+1})
\end{array}
\]
This representation recovers pictures from the Introduction in the case when \( k = 1 \) and \( n = 1, 2 \) — see Figs. 6 and 7.

Maximal chains in the poset \( \Sigma_m C_m^2 \) correspond to ‘lattice walks’ on a cartesian plane of length \( 3m \) starting and ending at \((0, 0)\), remaining in the first quadrant and using only the NE, W, and S steps, that is, \((1, 1)\), \((-1, 0)\) and \((0, -1)\) steps — this is the Kreweras situation mentioned in the Introduction (see [3, 10]). More generally, we have:

**Theorem 12** There is a bijection between the set of maximal chains in the poset \( \Sigma_m C_m^n \) and the set of walks of length \((n + 1)m\) in \( C_m^n \), starting and ending at \((0, \ldots, 0)\), and using only \((1, 1, \ldots, 1)\), \((-1, 0, \ldots, 0)\), \((0, -1, \ldots, 0)\), \(\ldots\), and \((0, 0, \ldots, -1)\) steps.

**Fig. 6** Lattices \( \Sigma_m^k C_m^1 \) for \( k \in \{0, 1, 2\} \) and \( m \in \{0, 1, 2, 3, 4\} \)
Fig. 7  Lattices $\Sigma^k_m C^2$ for $k \in \{0, 1, 2\}$ and $m \in \{0, 1, 2, 3, 4\}$
Proof (sketch) Using the representation $\Sigma_m C_m^n \approx C_m^{n+1^*}$ of Theorem 7, the desired bijection is given by:

- for $1 \leq i \leq n$, associating an increase in the $i$-th coordinate of a maximal chain in $C_m^{n+1^*}$ with the step in a walk given by $-1$ in the $i$-th position,
- and associating an increase in the $(n+1)$-th coordinate with the diagonal step $(1, 1, \ldots, 1)$ in a walk.

The first few terms of the sequence of maximal chain numbers for the stacking of lattices along the rows of Fig. 1 are shown in Table 2.

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