Transport Synthetic Acceleration for the Solution of the One-Speed Nonclassical Spectral $S_N$ Equations in Slab Geometry

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Abstract

The nonclassical transport equation models particle transport processes in which the particle flux does not decrease as an exponential function of the particle’s free-path. Recently, a spectral approach was developed to generate nonclassical spectral $S_N$ equations, which can be numerically solved in a deterministic fashion using classical numerical techniques. This paper introduces a transport synthetic acceleration procedure to speed up the iteration scheme for the solution of the monoenergetic slab-geometry nonclassical spectral $S_N$ equations. We present numerical results that confirm the benefit of the acceleration procedure for this class of problems.

Keywords: Nonclassical transport; Spectral method; Discrete ordinates; Synthetic Acceleration; Slab geometry

1. Introduction

In the classical theory of linear particle transport, the incremental probability that a particle at position $\mathbf{x} = (x, y, z)$ will experience a collision while traveling an incremental distance $ds$ in the background material is given by

$$dp = \sigma_t(\mathbf{x})ds,$$

where $\sigma_t$ represents the macroscopic total cross section [1]. The implicit assumption is that $\sigma_t$ is independent of the particle’s direction-of-flight $\Omega$ and of the particle’s free-path $s$, defined as the distance traveled by the particle since its previous interaction (birth or scattering). This assumption leads to the particle flux being exponentially attenuated (Beer-Lambert law). We remark that extending the results discussed here to include energy- or frequency-dependence is straightforward.

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The theory of nonclassical particle transport employs a generalized form of the linear Boltzmann equation to model processes in which the particle flux is not attenuated exponentially. This area has been significantly researched in recent years [2–15]. Originally introduced to describe photon transport in the Earth’s cloudy atmosphere [16–21], it has found its way to other applications, including nuclear engineering [22–26] and computer graphics [27–29]. Furthermore, an analogous theory yielding a similar kinetic equation has been independently derived for the periodic Lorentz gas by Marklof and Strömbergsson [30–33] and by Golse (cf. [34]).

The nonclassical transport equation allows the nonclassical macroscopic total cross section $\Sigma_t$ to be a function of the particle’s free-path, and is defined in an extended phase space that includes $s$ as an independent variable. If we define

$$P(x, \Omega, s)ds = \begin{cases} \text{the probability that a particle released at position } x \text{ in the direction } \Omega \text{ will experience its next collision while traveling} \\ \text{an incremental interval between } s \text{ and } s + ds \end{cases}, \quad (1.2)$$

then we can define the ensemble average

$$p(s) = \langle P(x, \Omega, s) \rangle_{(x, \Omega, R)} \quad (1.3)$$

over all “release positions” $x$ in a realization of the system, all directions $\Omega$, and all possible realizations $R$. In this case, $p(s)$ represents the free-path distribution function, and the nonclassical cross section $\Sigma_t(s)$ satisfies

$$p(s) = \Sigma_t(s)e^{-\int_0^s \Sigma_t(s')ds'} \quad (1.4)$$

It is possible to extend this definition to include angular-dependent free-path distributions and cross sections [5], but in this paper we will restrict ourselves to the case given by Eq. (1.4).

The steady-state, one-speed nonclassical transport equation with isotropic scattering can be written as [3]

$$\frac{\partial}{\partial s}\Psi(x, \Omega, s) + \Omega \cdot \nabla \Psi(x, \Omega, s) + \Sigma_t(s)\Psi(x, \Omega, s) = \quad (1.5a)$$

$$\delta(s) \left[ \frac{c}{4\pi} \int_0^\infty \Sigma_t(s')\Psi(x, \Omega', s')ds'd\Omega' + \frac{Q(x)}{4\pi} \right], \quad x \in V, \ \Omega \in 4\pi, \ 0 < s,$$

where $\Psi$ is the nonclassical angular flux, $c$ is the scattering ratio, and $Q$ is an isotropic internal source. The Dirac delta function $\delta(s)$ on the right-hand side of Eq. (1.5a) represents the fact that a particle that has just undergone scattering or been born will have its free-path value (distance since previous interaction) set to $s = 0$. If we consider vacuum boundaries, Eq. (1.5a) is subject to the boundary condition

$$\Psi(x, \Omega, s) = 0, \quad x \in \partial V, \ n \cdot \Omega < 0, \ 0 < s. \quad (1.5b)$$

We remark that, if $\Sigma_t(s)$ is assumed to be independent of $s$, then $\Sigma_t(s) = \sigma_t$ and the free-path distribution in Eq. (1.4) reduces to the exponential

$$p(s) = \sigma_t e^{-\sigma_ts} \quad (1.6)$$
In this case, Eq. (1.5a) can be shown to reduce to the corresponding classical linear Boltzmann equation

\[ \mathbf{\Omega} \cdot \nabla \Psi_c(x, \Omega) + \sigma_t \Psi_c(x, \Omega) = \frac{c}{4\pi} \int_{4\pi} \sigma_t \Psi_c(x, \Omega')d\Omega' + \frac{Q(x)}{4\pi}, \quad x \in V, \ \Omega \in 4\pi, \quad (1.7a) \]

with vacuum boundary condition given by

\[ \Psi_c(x, \Omega) = 0, \quad x \in \partial V, \ n \cdot \Omega < 0. \quad (1.7b) \]

Here, the classical angular flux \( \Psi_c \) is given by

\[ \Psi_c(x, \Omega) = \int_0^\infty \Psi(x, \Omega, s)ds. \quad (1.8) \]

Recently, a spectral approach was developed to represent the nonclassical flux as a series of Laguerre polynomials in the variable \( s \) \[35\]. The resulting equation has the form of a classical transport equation that can be solved in a deterministic fashion using traditional methods. Specifically, the nonclassical solution was obtained using the conventional discrete ordinates (S\(_N\)) formulation \[36\] and a source iteration (SI) scheme \[37\]. However, for highly scattering systems the spectral radius of the transport problem can get arbitrarily close to unity \[36\], and numerical acceleration becomes important.

The goal of this paper is to introduce transport synthetic acceleration techniques—namely, S\(_2\) synthetic acceleration (S\(_2\)SA)—to speed up the solution of the nonclassical spectral S\(_N\) equations. We also present numerical results that confirm the benefit of using this approach; to our knowledge, this is the first time such acceleration methods are applied to this class of nonclassical spectral problems.

The remainder of the paper is organized as follows. In Section 2 we present the nonclassical spectral S\(_N\) equations for slab geometry. We discuss transport synthetic acceleration in Section 3 and present an iterative method to efficiently solve the nonclassical problem. Numerical results are given in Section 4 for problems with both exponential (Section 4.1) and nonexponential (Section 4.2) choices of \( \rho(s) \). We conclude with a brief discussion in Section 5.

### 2. Nonclassical Spectral S\(_N\) Equations in Slab Geometry

In this section we briefly sketch out the derivation of the one-speed nonclassical spectral S\(_N\) equations in slab geometry. For a detailed derivation, we direct the reader to the work presented in \[35\].

In slab geometry, Eqs. (1.5) can be written as

\[ \frac{\partial}{\partial s} \Psi(x, \mu, s) + \mu \frac{\partial}{\partial x} \Psi(x, \mu, s) + \Sigma_t(s)\Psi(x, \mu, s) = \delta(s) \left[ \frac{c}{2} \int_{-1}^{1} \int_0^{\infty} \Sigma_t(s')\Psi(x, \mu', s')ds'd\mu' + \frac{Q(x)}{2} \right], \quad 0 < x < X, \ -1 < \mu < 1, \ 0 < s, \quad (2.1a) \]

\[ \Psi(0, \mu, s) = 0, \quad 0 < \mu \leq 1, \ 0 < s, \quad (2.1b) \]

\[ \Psi(X, \mu, s) = 0, \quad -1 \leq \mu < 0, \ 0 < s, \quad (2.1c) \]
where $\mu$ is the cosine of the scattering angle. Equation (2.1a) can be written in an equivalent “initial value” form:

$$\frac{\partial}{\partial s} \Psi(x, \mu, s) + \mu \frac{\partial}{\partial x} \Psi(x, \mu, s) + \Sigma_t(s) \Psi(x, \mu, s) = 0,$$

$$\Psi(x, \mu, 0) = \frac{c}{2} \int_{-1}^{1} \int_{0}^{\infty} \Sigma_t(s') \Psi(x, \mu', s') ds' d\mu' + \frac{Q(x)}{2}. \quad (2.2a)$$

Note that, due to scattering and internal source being isotropic, the right-hand side of Eq. (2.2b) does not depend on $\mu$.

Defining $\psi$ such that

$$\Psi(x, \mu, s) \equiv \psi(x, \mu, s) e^{-\int_{0}^{s} \Sigma_t(s') ds'}, \quad (2.3)$$

we can rewrite the nonclassical problem as

$$\frac{\partial}{\partial s} \psi(x, \mu, s) + \mu \frac{\partial}{\partial x} \psi(x, \mu, s) = 0, \quad (2.4a)$$

$$\psi(x, \mu, 0) = \frac{c}{2} \int_{-1}^{1} \int_{0}^{\infty} p(s') \psi(x, \mu', s') ds' d\mu' + \frac{Q(x)}{2}, \quad (2.4b)$$

where $p(s)$ is given by Eq. (1.4). This problem has the vacuum boundary conditions

$$\psi(0, \mu, s) = 0, \quad 0 < \mu \leq 1, 0 < s, \quad (2.4c)$$

$$\psi(X, \mu, s) = 0, \quad -1 \leq \mu < 0, 0 < s. \quad (2.4d)$$

Next, we write $\psi$ as a truncated series of Laguerre polynomials in $s$:

$$\psi(x, \mu, s) = \sum_{m=0}^{M} \psi_m(x, \mu) L_m(s), \quad (2.5)$$

where $L_m(s)$ is the Laguerre polynomial of order $m$ and $M$ is the expansion (truncation) order. The Laguerre polynomials $\{L_m(s)\}$ are orthogonal with respect to the weight function $e^{-s}$ and satisfy $\frac{d}{ds} L_m(s) = (\frac{d}{ds} - 1) L_{m-1}(s)$ for $m > 0$ [38]. We introduce this expansion in the nonclassical problem and perform the following steps [35]: (i) multiply Eq. (2.4a) by $e^{-\int_{0}^{s} \Sigma_t(s') ds'}$; (ii) integrate from 0 to $\infty$ with respect to $s$; and (iii) use the properties of the Laguerre polynomials to simplify the result. This procedure returns the following nonclassical spectral problem:

$$\mu \frac{\partial}{\partial x} \psi_m(x, \mu) + \psi_m(x, \mu) = S(x) + \frac{Q(x)}{2} - \sum_{j=0}^{m-1} \psi_j(x, \mu), \quad m = 0, 1, ..., M, \quad (2.6a)$$

$$\psi_m(0, \mu) = 0, \quad 0 < \mu \leq 1, m = 0, 1, ..., M, \quad (2.6b)$$

$$\psi_m(X, \mu) = 0, \quad -1 \leq \mu < 0, m = 0, 1, ..., M, \quad (2.6c)$$

where the in-scattering term $S(x)$ (the scattering source) is given by

$$S(x) = \frac{c}{2} \int_{-1}^{1} \sum_{k=0}^{M} \psi_k(x, \mu') \left[ \int_{0}^{\infty} p(s) L_k(s) ds \right] d\mu'. \quad (2.6d)$$
The nonclassical angular flux $\Psi$ is recovered from Eqs. (2.3) and (2.5). The classical angular flux $\Psi_c$ is obtained using Eq. (1.8), such that

$$
\Psi_c(x, \mu) = \int_{0}^{\infty} \Psi(x, \mu, s) ds = \sum_{m=0}^{M} \psi_m(x, \mu) \int_{0}^{\infty} L_m(s) e^{-\int_{0}^{s} \Sigma(t) ds} ds.
$$

Finally, using the discrete ordinates formulation [36], we can write the nonclassical spectral $S_N$ equations

$$
\mu_n \frac{d}{dx} \psi_{m,n}(x) + \psi_{m,n}(x) = S(x) + \frac{Q(x)}{2} - \sum_{j=0}^{m-1} \psi_{j,n}(x), \quad m = 0, 1, \ldots, M, n = 1, 2, \ldots, N,
$$

(3.1a)

$$
\psi_{m,n}(0) = 0, \quad m = 0, 1, \ldots, M, n = 1, 2, \ldots, \frac{N}{2},
$$

(3.1b)

$$
\psi_{m,n}(X) = 0, \quad m = 0, 1, \ldots, M, n = \frac{N}{2} + 1, \ldots, N,
$$

(3.1c)

$$
S(x) = \frac{c}{2} \sum_{n=1}^{N} \omega_n \sum_{k=0}^{M} \psi_{k,n}(x) \left[ \int_{0}^{\infty} p(s) L_k(s) ds \right],
$$

(3.1d)

$$
\Psi_{c_n}(x) = \sum_{m=0}^{M} \psi_{m,n}(x) \int_{0}^{\infty} L_m(s) e^{-\int_{0}^{s} \Sigma(t) ds} ds, \quad n = 1, 2, \ldots, N.
$$

(3.1e)

Here, the cosine of the scattering angle $\mu$ has been discretized in $N$ discrete values $\mu_n$. Thus, $\psi_{m,n}(x) = \psi_m(x, \mu_n)$, $\Psi_{c_n}(x) = \Psi_c(x, \mu_n)$, and the angular integral has been approximated by the angular quadrature formula with weights $\omega_n$.

3. Source Iteration and Synthetic Acceleration

To solve the nonclassical spectral $S_N$ equations using standard source iteration [37], we lag the scattering source on the right-hand side of Eq. (2.8a):

$$
\mu_n \frac{d}{dx} \psi_{m,n}^{i+1}(x) + \psi_{m,n}^{i+1}(x) = S_i(x) + \frac{Q(x)}{2} - \sum_{j=0}^{m-1} \psi_{j,n}^{i+1}(x),
$$

(3.1a)

where $i$ is the iteration index and

$$
S_i(x) = \frac{c}{2} \sum_{n=1}^{N} \omega_n \sum_{k=0}^{M} \psi_{k,n}^{i}(x) \left[ \int_{0}^{\infty} p(s) L_k(s) ds \right],
$$

In order to accelerate the convergence of this approach, the iterative scheme is broken into multiple stages.

Standard synthetic acceleration methods consist of two stages. The first stage is a single transport sweep. The second stage is error-correction, which uses an approximation of the error equation to estimate the error at each iteration. Our synthetic acceleration scheme has the following steps:
1. **Determine the new “half iterate”** $\psi^{i+\frac{1}{2}}$ (solution estimate) using one transport sweep.

   This is done by solving
   \[
   \mu_n \frac{d}{dx} \psi^{i+\frac{1}{2}}_{m,n}(x) + \psi^{i+\frac{1}{2}}_{m,n}(x) = S^i(x) + \frac{Q(x)}{2} - \sum_{j=0}^{m-1} \psi^{i+\frac{1}{2}}_{j,n}(x). \tag{3.2}
   \]

2. **Approximate the error** $\epsilon^{i+1}$ in this half iterate using an approximation to the error equation (error estimate).

   To do that, we first subtract Eq. (3.2) from the exact equation Eq. (2.8a), then add and subtract $\psi^{i+\frac{1}{2}}_k(x)$ to the in-scattering term on the right-hand side. This yields
   \[
   \mu_n \frac{d}{dx} \left( \psi_{m,n}(x) - \psi^{i+\frac{1}{2}}_{m,n}(x) \right) + \left( \psi_{m,n}(x) - \psi^{i+\frac{1}{2}}_{m,n}(x) \right) =
   (S(x) - S^i(x)) - \sum_{j=0}^{m-1} \left( \psi_{j,n}(x) - \psi^{i+\frac{1}{2}}_{j,n}(x) \right), \tag{3.3a}
   \]

   where
   \[
   S(x) - S^i(x) = \frac{c}{2} \sum_{n=1}^{N} \omega_n \sum_{k=0}^{M} \left( \psi_{k,n}(x) - \psi^{i+\frac{1}{2}}_{k,n}(x) + \psi^{i+\frac{1}{2}}_{k,n}(x) - \psi^{i}_{k,n}(x) \right) \int_0^\infty p(s)L_k(s)ds. \tag{3.3b}
   \]

   Defining the error $\epsilon^{i+1}_{m,n}$ as
   \[
   \epsilon^{i+1}_{m,n}(x) \equiv \psi_{m,n}(x) - \psi^{i+\frac{1}{2}}_{m,n}(x), \tag{3.4}
   \]

   we rewrite Eqs. (3.3) as
   \[
   \mu_n \frac{d}{dx} \epsilon^{i+1}_{m,n}(x) + \epsilon^{i+1}_{m,n}(x) - S^{i+1,\epsilon}(x) = \left( S^{i+\frac{1}{2}}(x) - S^i(x) \right) - \sum_{j=0}^{m-1} \epsilon^{i+1}_{j,n}(x), \tag{3.5a}
   \]

   with
   \[
   S^{i+1,\epsilon}(x) = \frac{c}{2} \sum_{n=1}^{N} \omega_n \sum_{k=0}^{M} \epsilon^{i+1}_{k,n}(x) \int_0^\infty p(s)L_k(s)ds. \tag{3.5b}
   \]

   We solve Eqs. (3.5) and obtain the error estimate $\epsilon^{i+1}$.

3. **Correct the solution estimate using the error estimate.**

   The corrected solution estimate $\psi^{i+1}$ is given by
   \[
   \psi^{i+1}_{m,n}(x) = \psi^{i+\frac{1}{2}}_{m,n}(x) + \epsilon^{i+1}_{m,n}(x). \tag{3.6}
   \]

4. **Check for convergence and loop back if necessary.**

   We remark that this transport synthetic acceleration procedure accelerates each one of the $M$ Laguerre moments of the angular flux. In this paper, we have chosen to approximate the error estimate in Eqs. (3.5) by setting $N = 2$, thus applying $S_2$ synthetic acceleration ($S_2$SA).
4. Numerical Results

In this section we provide numerical results that confirm the benefit of using transport synthetic acceleration for the iterative numerical solution of the nonclassical spectral $S_N$ equations \((2.8)\). For validation purposes, we first apply this nonclassical approach to solve a transport problem with an exponential $p(s)$, which leads to classical transport. Then, we proceed to solve a nonclassical transport problem that mimics diffusion, with a nonexponential $p(s)$.

For all numerical experiments in this section we use the Gauss-Legendre angular quadrature \([39]\) with \(N = 16\) for Eqs. \((2.8)\) and \(N = 2\) for Eqs. \((3.5)\), thus solving the nonclassical spectral $S_{16}$ equations using $S_2$ synthetic acceleration. We discretize the spatial variable into 200 elements and use the linear discontinuous Galerkin finite element method \([40]\). Furthermore, the improper integrals $\int_0^\infty (\cdot)ds$ in these equations are calculated numerically in the same fashion as in \([35]\): the upper limit is truncated to 1.5 times the length of the slab, and a Gauss-Legendre quadrature is used to solve them. Here, we set the order of this quadrature to $M$, the same order as the Laguerre expansion.

The stopping criterion adopted is that the relative deviations between two consecutive estimates of the classical scalar flux

\[
\Phi(x) = \sum_{n=1}^{N} \omega_n \Psi_{c_n}(x) \tag{4.1}
\]

in each point of the spatial discretization grid need to be smaller than or equal to a prescribed positive constant $\xi$. For all our calculations we fix $\xi = 10^{-6}$, such that the stopping criterion is given by

\[
\frac{||\Phi^{i+1}(x) - \Phi^i(x)||}{||\Phi^i(x)||} \leq \xi. \tag{4.2}
\]

4.1. Exponential $p(s)$

To validate the approach, we use the nonclassical method to solve a transport problem in which $p(s)$ is given by the exponential function provided in Eq. \((1.6)\). This yields \([3]\)

\[
\Sigma_t(s) = \frac{p(s)}{\int_s^\infty p(s')ds'} = \frac{\sigma_t e^{-\sigma ts}}{\int_s^\infty \sigma_t e^{-\sigma ts'}ds'} = \sigma_t \text{ (independent of } s). \tag{4.3}
\]

In this case, the flux $\Psi_{c_n}$ given by Eq. \((2.8e)\) should match the one obtained by solving the corresponding classical $S_N$ transport problem

\[
\mu_n \frac{d}{dx} \Psi_{c_n}(x) + \sigma_t \Psi_{c_n}(x) = \frac{c}{2} \sigma_t \sum_{n=1}^{N} \omega_n \Psi_{c_n}(x) + \frac{Q(x)}{2}, \quad 0 < x < X, \quad n = 1, 2, \ldots, N, \tag{4.4a}
\]

\[
\Psi_{c_n}(0) = 0, \quad n = 1, 2, \ldots, N/2, \tag{4.4b}
\]

\[
\Psi_{c_n}(X) = 0, \quad n = \frac{N}{2} + 1, \ldots, N. \tag{4.4c}
\]
Let us consider a slab of length $X = 20$, total cross section $\sigma_t = 1.0$, scattering ratio $c = 0.999$, and internal source $Q(x) = 1.0$, and let us assume the truncation order of the Laguerre expansion to be $M = 10$. Figure 1 depicts the scalar flux obtained when solving the nonclassical (2.8) and classical (4.4) problems. As expected, the solutions match each other.

Table 1: Convergence Data for Nonclassical Transport with Exponential $p(s)$

| $c$  | Number of Iterations | Spectral Radius |
|------|----------------------|-----------------|
|      | SI       | $S_2$SA | SI   | $S_2$SA |
| 0.8  | 56       | 6       | 0.7997 | 0.1328 |
| 0.9  | 110      | 6       | 0.8997 | 0.1565 |
| 0.99 | 906      | 6       | 0.9899 | 0.1748 |
| 0.999| 6439     | 6       | 0.9989 | 0.1685 |
Next, we compare the iteration count and spectral radius for stand-alone source iteration (SI) and transport synthetic acceleration (S2SA) for the nonclassical method. Once again, we set \( Q(x) = 1.0 \), and \( p(s) \) and \( \Sigma_t(s) \) are given respectively by Eqs. (1.6) and (4.3), with \( \sigma_t = 1.0 \). However, this time we increase the domain size to \( X = 200 \). We assume the truncation order of the Laguerre expansion to be \( M = 50 \), and vary the scattering ratio \( c \) from 0.8 to 0.999. Table 1 presents the number of iterations and the spectral radius for each case. As expected, we observe a significant reduction in the spectral radius and iteration count, with the number of iterations decreasing 3 orders of magnitude for the highest scattering ratio of \( c = 0.999 \).

### 4.2. Nonexponential \( p(s) \)

Let us consider the diffusion equation in a homogeneous slab

\[
-\frac{1}{3\sigma_t} \frac{d}{dx} \phi(x) + (1 - c)\sigma_t \phi(x) = Q(x),
\]

with Marshak boundary conditions [11]

\[
\phi(0) - \frac{2}{3\sigma_t} \frac{d}{dx} \phi(0) = 0,
\]

\[
\phi(X) + \frac{2}{3\sigma_t} \frac{d}{dx} \phi(X) = 0.
\]

Here, \( \phi \) is the (diffusion) scalar flux.

If the free-path distribution \( p(s) \) is given by the nonexponential function

\[
p(s) = 3\sigma_t^2 e^{-\sqrt{3} \sigma_t s},
\]

it has been shown that the collision-rate density \( \sigma_t \phi(x) \) of the diffusion problem, given by Eqs. (4.5), will match the nonclassical collision-rate density [7, 8]

\[
f(x) = \int_0^\infty \Sigma_t(s) \int_{-1}^1 \Psi(x, \mu, s) d\mu ds,
\]

where \( \Psi(x, \mu, s) \) is the solution of the nonclassical problem given by Eqs. (2.1), and

\[
\Sigma_t(s) = \frac{\int_s^\infty \frac{p(s)}{p(s') ds'}}{\int_s^\infty \frac{3\sigma_t^2 e^{-\sqrt{3} \sigma_t s} ds'}{3\sigma_t^2 e^{-\sqrt{3} \sigma_t s'} ds'}} = \frac{3\sigma_t^2 s}{1 + \sqrt{3} \sigma_t s}.
\]

Once again, let us consider a slab of length \( X = 20 \), \( \sigma_t = 1.0 \), \( c = 0.999 \), and \( Q(x) = 1.0 \). Figure 2 shows a comparison between the collision-rate densities of the diffusion problem (Eqs. (4.5)) and the nonclassical spectral S\(_N\) method (Eqs. (2.8)), with the latter being given by

\[
f(x) = \sum_{n=1}^N \omega_n \sum_{m=0}^M \psi_{m,n}(x) \int_0^\infty p(s) L_m(s) ds,
\]
where $M = 10$. As in the previous case, the solutions match as expected.

At this point, we compare the iteration count and spectral radius for stand-alone source iteration (SI) and transport synthetic acceleration ($S_2SA$). We set $Q(x) = 1.0$, and $p(s)$ and $\Sigma_t(s)$ are given respectively by Eqs. (4.6) and (4.8), with $\sigma_t = 1.0$. Once more, we increase the domain size to $X = 200$, and assume the truncation order of the Laguerre expansion to

Table 2: Convergence Data for Nonclassical Transport with Nonexponential $p(s)$

| $c$  | Number of Iterations | Spectral Radius |
|------|-----------------------|-----------------|
|      | SI     | $S_2SA$ | SI        | $S_2SA$   |
| 0.8  | 56     | 6       | 0.7997    | 0.1538    |
| 0.9  | 110    | 7       | 0.8997    | 0.1811    |
| 0.99 | 906    | 6       | 0.9989    | 0.1885    |
| 0.999| 6443   | 6       | 0.9989    | 0.1802    |
be $M = 50$. Table 2 presents the number of iterations and the spectral radius for different choices of the scattering ratio $c$. Similar to the previous case, there is a reduction in both the spectral radius and iteration count, with a decrease of 3 orders of magnitude in the iterations for the highest scattering ratio.

5. Discussion

We have introduced a transport synthetic acceleration procedure that speeds up the source iteration scheme for the solution of the one-speed nonclassical spectral $S_N$ equations in slab geometry. Specifically, we used $S_2$ synthetic acceleration to solve nonclassical spectral $S_{16}$ equations for problems involving exponential and nonexponential free-path distributions. The numerical results successfully confirm the advantage of the method; to our knowledge, this is the first time a numerical acceleration approach is used in this class of nonclassical spectral problems. Moreover, although we assumed for simplicity monoenergetic transport and isotropic scattering, extending the method to include energy-dependence and anisotropic scattering shall not lead to significant additional theoretical difficulties.

When compared to stand-alone SI, $S_2$SA yields a significant reduction in number of iterations (up to three orders of magnitude) and spectral radii. The values of the spectral radius for stand-alone SI remain virtually unchanged for the exponential and nonexponential cases for a fixed value of the scattering ratio $c$. However, all spectral radii for $S_2$SA are larger in the nonexponential case than in the exponential case for the same value of $c$, increasing from 6.5% (when $c = 0.999$) to 13.6% (when $c = 0.8$).

In fact, we do not see spectral radius values that are exactly consistent with those found when applying corresponding techniques to the classical $S_N$ transport equation. This can be attributed to the fact that the nonclassical equation contains an altogether different scattering term, which depends on the free-path $s$. Although a full convergence analysis is beyond the scope of this paper, we shall perform it in a future work in order to investigate this feature.

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