EXTREMAL CROSS-POLYTOPES AND GAUSSIAN VECTORS

GERGELY AMBRUS

Abstract. Let $C = C_n(\lambda_1, \ldots, \lambda_n)$ be the $n$-dimensional orthogonal cross-polytope whose axes are of length $\lambda_1, \ldots, \lambda_n$. Subject to the condition $\sum \lambda_i^2 = 1$, the mean width of $C$ is minimised when $\lambda_i = 1/\sqrt{n}$ for every $i$, and it is maximised when $C$ is at most two dimensional. As a corollary, a lower bound on the mean width of a general convex body $K$ is derived in terms of the successive inner radii of $K$. A more general result is presented for Gaussian random vectors.

1. Context and motivation

Let $K$ be a convex body in $\mathbb{R}^n$. A natural way to measure how close $K$ is to a ball is to relate its volume to that of the largest ball inscribed in $K$, or to the smallest ball circumscribed about $K$. Equivalently, one may as well relate $\text{Vol}(K)$ to the inradius of $K$ and to the circumradius of $K$. However, the best possible estimates in this case are the trivial ones.

More interesting inequalities can be obtained by taking into account the successive inner and outer radii of $K$. These are defined as follows. Let $A^n_i$ denote the set of $i$-dimensional affine subspaces of $\mathbb{R}^n$, and for a subspace $L \in A^n_i$, denote by $K|L$ the orthogonal projection of $K$ onto $L$. The successive inner and outer radii of $K$ for $1 \leq i \leq n$ are given by

$$r_i(K) = \max_{L \in A^n_i} r(K \cap L) \quad \text{and} \quad R_i(K) = \min_{L \in A^n_i} R(K|L).$$

Note that $r_n(K) = r(K)$, $R_n(K) = R(K)$, $2r_1(K)$ is the minimum width of $K$, and $2R_1(K)$ is the diameter of $K$.

We also introduce the intrinsic volumes of $K$ for $0 \leq i \leq n$ by

$$(1) \quad V_i(K) = \binom{n}{i} \frac{\kappa_n}{\kappa_i \kappa_{n-i}} \int_{\mathcal{L}_i^n} \text{Vol}_i(K|L) \, d\mu_i(L),$$

where $\mathcal{L}_i^n$ is the Grassmannian of all $i$-dimensional linear subspaces of $\mathbb{R}^n$, equipped with the unique Haar probability measure $\mu_i$, and $\kappa_n$ is the volume of $B^n$, the unit ball of dimension $n$:

$$\kappa_n = \frac{\pi^{n/2}}{\Gamma(1 + \frac{n}{2})}.$$
Alternatively, $V_i(K)$ may be expressed as the coefficients in Steiner’s formula

$$\text{Vol}(K + \lambda B^n) = \sum_{i=0}^{d} \lambda^{n-i} \kappa_{n-i} V_i(K).$$

The most well-known special cases are: $V_n(K) = \text{Vol}(K)$; $2V_{n-1}(K)$ is the surface area of $K$; $2\kappa_{n-1}/(n\kappa_n)V_1(K)$ is the mean width of $K$; and $V_0(K) = 1$ is the Euler characteristic. For further references, see Gruber [4] or Schneider [6].

Since the intrinsic volumes are the average volumes of projections of $K$ onto lower dimensional subspaces, it is natural to expect a relationship between these and the successive radii of $K$. This link was established by M. Henk and M. Hernández Cifre [5] who proved that the following inequalities hold:

$$V_i(K) \leq 2^i s_i(R_1(K), \ldots, R_n(K)) \text{ for every } 0 \leq i \leq n;$$

$$V_{n-1}(K) \geq \frac{2^{n-1}}{(n-1)!} \sqrt{s_{n-1}(r_1(K)^2, \ldots, r_n(K)^2)};$$

$$V_{n-2}(K) \geq \frac{2\sqrt{2}}{\pi} \frac{2^{n-2}}{(n-2)!} \sqrt{s_{n-2}(r_1(K)^2, \ldots, r_n(K)^2)},$$

where $s_i$ stands for the $i$th elementary symmetric polynomial:

$$s_i(\lambda_1, \ldots, \lambda_n) = \sum_{1 \leq k_1 < \cdots < k_i \leq n} \lambda_{k_1} \cdots \lambda_{k_i}.$$

The upper bound (2) is sharp. In order to derive the lower bounds on $V_i(K)$, the authors of [5] apply a sequence of Steiner symmetrisations to $K$, leading to a convex body $\tilde{K}$ which contains the orthogonal cross-polytope spanned by $\pm r_1(K)e_1, \ldots, \pm r_n(K)e_n$, where $(e_i)_n$ is the standard basis of $\mathbb{R}^n$. Using that Steiner symmetrisations do not increase the intrinsic volumes (see e.g. [4]), one arrives at the following bound:

$$V_i(K) \geq V_i(C_n(r_1(K), \ldots, r_n(K))),$$

where $$C_n(\lambda_1, \ldots, \lambda_n) = \text{conv}(\pm \lambda_i e_i : i = 1, \ldots, n).$$

Thus, in order to derive the best bounds provided by this method, one faces the following question:

**Problem 1.** For $1 \leq i \leq n$, determine the vectors $u = (u_1, \ldots, u_n) \in \mathbb{R}_+^n$ minimising

$$V_i(C_n(u_1, \ldots, u_n)) \sqrt{s_i(u_1^2, \ldots, u_n^2)}.$$

We note that the quantity (4) is invariant under scaling (see e.g. Corollary 2.1. of [4]), thus we may assume that $u \in S^{n-1}$. It is proved in [5] that for $i = n-1$ and $i = n-2$, the minimum of (4) is attained when $u$ is a multiple of $(1, \ldots, 1)$, that is, when the cross-polytope is regular. The authors also conjecture that the same statement should hold for every $i$.

In the present note, we settle the $i = 1$ case of Problem 1, showing that the minimum is attained in the regular case, whereas the maximum
is attained when the cross-polytope is at most 2 dimensional. Perhaps it is more convenient to formulate an equivalent question about Gaussian vectors.

**Problem 2.** Let \( n \geq 1 \), and \( \xi_1, \ldots, \xi_n \) be independent, identically distributed standard normal variables. Determine the unit vector \( u = (u_1, \ldots, u_n) \in \mathbb{R}^n \) with non-negative coordinates which minimises

\[
\mathbb{E} \max_{1 \leq i \leq n} \|u_i \xi_i\|.
\]

(5) Denoting the random vector \( (\xi_1, \ldots, \xi_n) \) by \( \xi \) and introducing the Hadamard product of \( v, w \in \mathbb{R}^n \) by

\[
(v \odot w)_i = v_iw_i,
\]

the quantity in (5) becomes \( \mathbb{E}\|u \odot \xi\|_\infty \). Note that \( u \odot \xi \) is an \( n \)-dimensional Gaussian random vector with independent coordinates, whose covariance matrix has trace 1.

The equivalence of Problem 2 and the \( i = 1 \) case of Problem 1 follows by the following standard transformation. For a convex body \( K \in \mathbb{R}^n \) and \( x \in S^{n-1} \), let \( h_K(x) \) denote the support function and \( \rho_K(x) \) the radial function of \( K \). If \( K \) is symmetric, \( \|\cdot\|_K \) denotes the assigned norm with unit ball \( K \). \( K^* \) stands for the polar body of \( K \). Furthermore, \( \sigma(x) = \sigma_{K^*}(x) \) denotes the \((k - 1)\)-dimensional surface (Lebesgue) measure on \( S^{k-1} \); note that \( \sigma \) is a scaled copy of the rotationally invariant probability measure on \( S^{k-1} \), with total mass \( k\kappa_k \). With these conventions, using (1),

\[
\kappa_{n-1}V_1(K) = \int_{S^{n-1}} h_K(x)d\sigma(x) = \int_{S^{n-1}} \frac{1}{\rho_{K^*}(x)}d\sigma(x) = \int_{S^{n-1}} \|x\|_{K^*}d\sigma(x).
\]

Since \( C_n^*(u_1, \ldots, u_n) \) is a rectangular box of half-axes \( e_1/u_1, \ldots, e_n/u_n \),

\[
V_1(C_n(u_1, \ldots, u_n)) = \kappa_{n-1} \int_{S^{n-1}} \|u \odot x\|_\infty d\sigma(x)
\]

\[
= \frac{1}{(2\pi)^{(n-1)/2}} \int_{S^{n-1}} \|u \odot x\|_\infty \int_0^\infty e^{-r^2/2} r^{n-1}dr d\sigma(x)
\]

\[
= \frac{\sqrt{2\pi}}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-|x|^2/2} \|u \odot x\|_\infty dx
\]

\[
= \sqrt{2\pi} \mathbb{E}\|u \odot \xi\|_\infty.
\]

Relaxing the independence condition of Problem 2 we can ask the following, more general question. We call a multivariate random variable symmetric, if the mean values of its coordinate variables are 0.

**Problem 3.** Among the \( n \)-dimensional symmetric Gaussian random vectors \( X \) satisfying \( \text{tr \, Cov} \, X = 1 \), which ones minimise and maximise \( \mathbb{E}\|X\|_\infty \)?

Problem 2 is a special case of the above one when \( \text{Cov} \, X \) is assumed to be diagonal.

We answer Problems 2 and 3 formulated above.

**Theorem 1.** Let \( \xi = (\xi_i)_{i=1}^n \) be a \( n \)-dimensional standard normal vector with i.i.d. coordinate variables, and let \( u \in \mathbb{R}^n \) be a unit vector. For \( n = 2 \), the expectation \( \mathbb{E}\|u \odot \xi\|_\infty \) is independent of the choice of \( u \). For \( n \geq 3 \), the expectation is maximised when at most two coordinates of \( u \) are non-zero, and it is minimised when \( u = (\pm 1/\sqrt{n}, \ldots, \pm 1/\sqrt{n}) \).
Thus, the regular cross-polytope is the minimiser for Problem [1].

**Theorem 2.** Among the $n$-dimensional symmetric Gaussian random vectors $X$ satisfying $\text{tr} \text{Cov} X = 1$, $\mathbb{E} \|X\|_\infty$ is maximal when $\text{Cov} X$ is diagonal with at most two non-zero entries, and minimal when the absolute values of the coordinates of $X$ are identical almost everywhere.

We note that among the affine equivalence classes of convex bodies $K$ in $\mathbb{R}^n$, $\int_{S^{n-1}} \|x\|_K d\sigma(x)$ is minimal for the class of the cube [1]. Using the Brascamp-Lieb inequality, it also follows that the cube has minimal mean width among its affine images of the same volume (see Ball [3]). This, however, does not imply the above results, as the cube has the smallest volume among the rectangular boxes to be considered in the present problem.

In order to derive a lower estimate for $V_1(K)$ from Theorem [1], we follow the proof of Dvoretzky’s theorem presented by Ball [1]. Let $\mu$ be the median of $\|x\|_\infty$ on $\mathbb{R}^n$ with respect to the standard Gaussian measure. On the one hand, (6) implies that

$$V_1\left(C_n\left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right)\right) = \sqrt{\frac{2\pi}{n}} \cdot \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\|x\|_\infty^2} dx > \sqrt{\frac{\pi}{2n}} \mu.$$  

On the other hand, $\mu$ satisfies that

$$\frac{1}{2} = \frac{1}{(2\pi)^{n/2}} \int_{[-\mu,\mu]^n} e^{-|x|^2/2} dx = \left(\frac{2}{\pi} \int_0^\mu e^{-s^2/2} ds\right)^n \approx (1 - e^{-n^2/2})^n,$$

thus, from $2^{-1/n} \approx 1 - (\log 2)/n$ we deduce that $\mu \approx \sqrt{2\log n}$, and

$$V_1\left(C_n\left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right)\right) \approx \sqrt{\pi} \frac{\log n}{n}.$$

Taking (3) into account, we arrive at the following estimate.

**Corollary 1.** There exists an absolute constant $c$, so that for any convex body $K \subset \mathbb{R}^n$,

$$V_1(K) \geq c \sqrt{\frac{\log n}{n}} \sqrt{r_1(K)^2 + \cdots + r_n(K)^2}.$$

Numerical calculations show that the value of $c$ can be chosen to be 1.74.

This estimate, however, is not optimal; see the relevant remarks in [5].

2. Proofs

We start with a technical lemma.

**Lemma 1.** For any $0 < q \leq 2$, the function

$$F(x) = \frac{e^{x^2/2}}{x^{q-1}} \int_0^x e^{-t^2/2} dt$$

is strictly monotone increasing for $x > 0$.

**Proof.** It is easy to obtain that

$$F'(x) = \frac{1}{x^q} \left(x + (x^2 - q + 1)e^{x^2/2} \int_0^x e^{-t^2/2} dt\right) =: \frac{1}{x^q} f(x).$$
Here $f(0) = 0$ and
$$f'(x) = x^2 + 2 - q + x(x^2 + 3 - q) e^{x^2/2} \int_0^x e^{-t^2/2} dt$$
which is positive for $x > 0$. Thus, $F'(x) > 0$ for every $x > 0$. □

**Proof of Theorem 1.** We may and do assume that $u_i \geq 0$ for every $i = 1, \ldots, n$. Using that for a non-negative random variable $X$
$$E X = \int_0^\infty P(X > t) dt = \int_0^\infty 1 - P(X \leq t) dt,$$
we can express the expectation in question as
$$E \|u \odot \xi\|_\infty = \int_0^\infty 1 - \prod_{i=1}^n P(|u_i \xi_i| \leq t) dt$$
(8)
$$= \int_0^\infty 1 - \left( \frac{2}{\pi} \right)^{n/2} \int_0^{t/u_1} e^{-s^2/2} ds \ldots \int_0^{t/u_n} e^{-s^2/2} ds dt$$
$$= \int_0^\infty 1 - \phi(t/u_1) \ldots \phi(t/u_n) dt,$$
where
$$\phi(a) = \text{Erf} \left( \frac{a}{\sqrt{2}} \right) = \sqrt{\frac{2}{\pi}} \int_0^a e^{-s^2/2} ds,$$
also using the convention that $c/0 = \infty$ for $c \geq 0$.

Thus, if $u_1 \neq 0$, integrating by parts leads to
$$\frac{\partial E \|u \odot \xi\|_\infty}{\partial u_1} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{t e^{-t^2/2u_1^2}}{u_1^2} \phi(t/u_2) \ldots \phi(t/u_n) dt$$
(9)
$$= \left[ -\sqrt{\frac{2}{\pi}} e^{-t^2/2u_1^2} \phi(t/u_2) \ldots \phi(t/u_n) \right]_0^\infty +$$
$$+ \frac{2}{\pi} \int_0^\infty e^{-t^2/2u_1^2} \sum_{i=2}^n \left( \frac{e^{-t^2/2u_i^2}}{u_i} \prod_{j=2 \atop j \neq i}^n \phi(t/u_j) \right) dt.$$

Note that on the right hand side, the first term vanishes. Now, the Lagrange multiplier method implies that for the critical points, $\nabla E \|u \odot \xi\|_\infty = \lambda u$ for some $\lambda \neq 0$. In particular, assuming that $u_1 \geq u_2 > 0$, we obtain that
$$\frac{1}{u_1} \frac{\partial E \|u \odot \xi\|_\infty}{\partial u_1} = \frac{1}{u_2} \frac{\partial E \|u \odot \xi\|_\infty}{\partial u_2},$$
which, by (9), implies that

\[
\int_0^\infty \frac{e^{-t^2/2u_1^2}}{u_1} \sum_{i=3}^n \left( \frac{e^{-t^2/2u_i^2}}{u_i} \prod_{j=2, j\neq i}^n \phi(t/u_j) \right) dt
\]

(10)

\[
= \int_0^\infty \frac{e^{-t^2/2u_2^2}}{u_2} \sum_{i=3}^n \left( \frac{e^{-t^2/2u_i^2}}{u_i} \prod_{j=3, j\neq i}^n \phi(t/u_j) \right) dt.
\]

The quotient of the above integrands is

\[
\frac{u_2 e^{-t^2/2u_2^2} \phi(t/u_2)}{u_1 e^{-t^2/2u_1^2} \phi(t/u_1)} = \frac{s}{\phi(s)} e^{-s^2/2} \frac{e(\mu s)^2/2}{\mu s} \phi(\mu s),
\]

where \( s = t/u_1 \) and \( \mu = u_1/u_2 \). Setting \( q = 2 \) in Lemma 1 implies that for any fixed \( s > 0 \), this is a strictly monotone increasing function of \( \mu \). In particular, \( \mu > 1 \) would imply that the quotient is strictly greater than 1 for every \( s > 0 \). Thus, equality in (10) can hold only if \( \mu = 1 \), that is, \( u_1 = u_2 \).

Therefore, all the non-zero coordinates of the extremal vectors \( u \) must be equal, and in order to find the minimum and the maximum values of \( E\|u \circ \xi\|_\infty \), it suffices to compute the expectations for the set of points

\[
u^k = \left( \frac{1}{\sqrt{k}}, \ldots, \frac{1}{\sqrt{k}}, 0, \ldots, 0 \right), \quad k = 1, \ldots, n.
\]

Introducing the notation

\[
E_k = \frac{1}{\sqrt{k}} \mathbb{E}\|\xi_1, \ldots, \xi_k\|_\infty,
\]

our goal is to show that

\[
E_1 = E_2 > E_3 > \cdots > E_n.
\]

Note that (6) and (7) imply that the above inequality is asymptotically true, as \( E_n \approx \sqrt{\log n}/2n \).

Fix \( n \geq 1 \), and for \( 0 \leq \rho \leq 1/\sqrt{n+1} \), introduce

\[
u(\rho) = \left( \sqrt{\frac{1-\rho^2}{n}}, \ldots, \sqrt{\frac{1-\rho^2}{n}}, \rho \right) \in \mathbb{R}^{n+1}.
\]

Then, by (8),

\[
\mathbb{E}\|u(\rho) \circ \xi\|_\infty = \int_0^\infty 1 - \phi \left( \frac{t}{\rho} \right) \phi \left( t\sqrt{\frac{n}{1-\rho^2}} \right)^n dt =: R(\rho).
\]

This leads to

\[
R'(\rho) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{t\rho}{\rho^2} e^{-t^2/2\rho^2} \phi \left( t\sqrt{\frac{n}{1-\rho^2}} \right)^n dt
\]

\[
- \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{t\rho^{3/2}}{(1-\rho^2)^{3/2}} e^{-n^2/2(1-\rho^2)} \phi \left( \frac{t}{\rho} \right) \phi \left( t\sqrt{\frac{n}{1-\rho^2}} \right)^{n-1} dt.
\]
Proof of Theorem 2. May and do assume that
\[ u(t) = t \frac{n}{\sqrt{1 - \rho^2}} \left( \sqrt{n - \frac{n}{1 - \rho^2}} \right)^n \]
where the first term on the right hand side vanishes. Similarly,
\[ \int_0^\infty t \rho n^{3/2} e^{-nt^2/2(1-\rho^2)} \phi \left( t \rho \right) \phi \left( t \sqrt{n - \frac{n}{1 - \rho^2}} \right) \]
\[ + \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-t^2/2\rho^2} \rho n^{-1} \sqrt{n - \frac{n}{1 - \rho^2}} e^{-nt^2/2(1-\rho^2)} \phi \left( t \sqrt{n - \frac{n}{1 - \rho^2}} \right) \]

The above equations lead to
\[ R'(\rho) = \frac{2}{\pi} \int_0^\infty e^{-nt^2/2(1-\rho^2)} (n - 1) \sqrt{n - \frac{n}{1 - \rho^2}} \phi \left( t \sqrt{n - \frac{n}{1 - \rho^2}} \right) \]
\[ \cdot \left[ e^{-t^2/2\rho^2} \phi \left( t \sqrt{n - \frac{n}{1 - \rho^2}} \right) - e^{-nt^2/2(1-\rho^2)} \rho \sqrt{n - \frac{n}{1 - \rho^2}} \phi \left( t \rho \right) \right] dt. \]

Since \( 1/\rho \geq \sqrt{n/(1-\rho^2)} \), Lemma (1) \((q = 2)\) implies that for every \( t > 0 \)
\[ \frac{1}{t \sqrt{n/(1-\rho^2)}} e^{nt^2/2(1-\rho^2)} \phi \left( t \sqrt{n - \frac{n}{1 - \rho^2}} \right) \leq \frac{\rho t}{t} e^{t^2/2\rho^2} \phi \left( t \rho \right), \]
which, by (11), shows that for any \( t > 0 \), the function \( R'(\rho) \) is monotone decreasing on the interval \([0,1/\sqrt{n} + 1]\). Hence, for every \( n \geq 1 \),
\[ E_n = R(0) \geq R \left( \frac{1}{\sqrt{n} + 1} \right) = E_{n+1}. \]
Furthermore, (11) shows that equality holds above if and only if \( n = 1 \). \( \square \)

Proof of Theorem 3. Let \( u_i = \sqrt{\text{Cov}X_{ij}} \). For the minimum inequality, we may and do assume that \( u_1^2 \) is the largest diagonal entry of CovX, hence \( u_1 \geq 1/\sqrt{n} \). Since \( ||X||_\infty \geq |X_1| \), and \( E|X_1| = \sqrt{2/\pi} u_1 \), we obtain the lower bound \( E||X||_\infty \geq \sqrt{2/(n\pi)} \). This bound is sharp only if for all \( i \), \( u_i = 1/\sqrt{n} \) and \( ||X||_\infty = |X_1| \) almost everywhere, which yields that for every \( i \) and \( j \), \( |X_i| = |X_j| \) almost everywhere.

For the upper bound, let \( \xi_1, \ldots, \xi_n \) be i.i.d standard normal variables. By a theorem of Šidák (7), (8), which is a relative of Slepian’s lemma,
\[ \mathbb{P}(||X||_\infty \leq t) = \mathbb{P}(|X_1| \leq t, \ldots, |X_n| \leq t) \]
\[ \geq \prod_{i=1}^n \mathbb{P}(|X_i| \leq t) = \prod_{i=1}^n \mathbb{P}(|u_i\xi_i| \leq t). \]
Thus, the question reduces to Problem 2, and the upper bound provided by Theorem 1 is sharp if the coordinate variables of $X$ are independent.

3. Further remarks

Theorem 1 may also be proved by induction on $n$; the inductive statement asserts that for any constant $C \geq 0$, the quantity

$$\int_{\mathbb{R}^n} \max\{C, \|u \odot x\|_{\infty}\} d\gamma(x),$$

where $\gamma$ is the standard $n$-variate Gaussian distribution, is maximal for $u = (1, 0, \ldots, 0)$, and minimal for $u = (1/\sqrt{n}, \ldots, 1/\sqrt{n})$. The initial step is the $n = 2$ case. After determining the possible extremum places using Lagrange multipliers, the remaining statement amounts to the following.

**Lemma 2.** For any $c \geq 0$, the following inequality holds:

$$c + \int_{c}^{\infty} 1 - (\phi(\sqrt{2}t))^2 dt \leq c + \int_{c}^{\infty} 1 - \phi(t) dt = \sqrt{2} \frac{c}{\pi} e^{-c^2/2} + c \phi(c).$$

The proof is somewhat technical, thus we leave it to the interested reader.

There are two natural directions to generalise the above results. First, Problem 1 is open for $2 \leq i \leq n - 3$. A method similar to the one presented here may be applied to these cases as well; however, when computing the mixed volumes of cross-polytopes, one faces a formula (see e.g. Corollary 2.1 of [5]) which is too complicated to carry out the necessary analysis. It may be possible to express the mixed volumes in a more suitable way; in that respect, it is illustrative that (6) differs from the $i = 1$ case of the above cited formula.

The other direction is to generalise Problem 2 the following way.

**Problem 4.** Let $n \geq 1$, $p, q \in (1, \infty]$, and let $\xi$ be an $n$-variate Gaussian vector with i.i.d. standard normal coordinate variables. Determine the maximum and minimum of $E\|u \odot \xi\|_p$ subject to the condition $\|u\|_q = 1$.

Clearly, in the $p = q = 1$ case the expectation is independent of the choice of $u$, whereas Theorem 1 provides the answer to the $q = 2, p = \infty$ case. We now extend this for $0 < q < 2$ as well.

**Theorem 3.** For $p = \infty$ and $0 < q \leq 2$, the answer for Problem 4 is given as follows: $E\|u \odot \xi\|_\infty$ is maximised by $u = (1, 0, \ldots, 0)$, and it is minimal for $u = (n^{-1/q}, \ldots, n^{-1/q})$.

**Proof.** The argument applied in the course of the proof of Theorem 1 together with the general case of Lemma 1 imply that the extremal vectors are among the $u^k_q$ given by

$$u^k_q = (k^{-1/q}, \ldots, k^{-1/q}, 0, \ldots, 0), \quad k = 1, \ldots, n.$$

Introducing

$$E_{k,q} = k^{-1/q} E\|(\xi_1, \ldots, \xi_k)\|_\infty,$$

the chain of inequalities

$$E_{1,q} > E_{2,q} > \cdots > E_{n,q}$$
easily follows by
\[
\left(\frac{k+1}{k}\right)^{1/q} > \left(\frac{k+1}{k}\right)^{1/2} \geq \frac{\mathbb{E}\|\xi_1, \ldots, \xi_{k+1}\|_\infty}{\mathbb{E}\|\xi_1, \ldots, \xi_k\|_\infty}.
\]
\[\Box\]

It would be natural to expect that for every \( p \) and \( q \), the behaviour of the extremal vectors is similar to the above case. However, this is very far from being true. We illustrate this phenomenon by the \( n = 2 \) and \( p = 2 \) case, where by elementary but tedious calculations one can show the following.

Let \( q_L = 3/2 \), \( q_M = \log 2 / (\log \pi - \log 2) \approx 1.53 \) and \( q_U = 2 \). For \( 1 \leq q \leq q_L \), the expectation is maximal when \( u = (1,0) \), and there is exactly one local (and global) minimum at \((2^{-1/q}, 2^{-1/q})\). For \( q_L < q < q_U \), there is a local maximum at both of these directions, and there are two further local (and global) minimum places. When \( q = q_M \), these two maxima are equal; for \( q < q_M \), the global maximum is at \((1,0)\), whereas for \( q > q_M \), the global maximum is at \((2^{-1/q}, 2^{-1/q})\). For \( q \geq q_U \), the vector \((1,0)\) becomes a global minimum place, and the only local extremum places are \((1,0)\), \((0,1)\) and \((2^{-1/q}, 2^{-1/q})\). For general \( p \), a similar pattern holds with \( q_U = 2 \), and \( q_L \to 2 \) as \( p \to \infty \).

The same situation holds in higher dimensions, with \( q_U \) depending on \( n \). Thus, in particular, it is not true in general that the extremal vectors \( u \) are necessarily of the form \((\alpha, \ldots, \alpha, 0, \ldots, 0)\), up to a permutation of coordinates.

For \( p = \infty \) and \( q > 2 \), the distribution of the minimum and maximum places is unclear. Theorem 1 and the discussion above show that for \( q \leq 2 \), the maximum is achieved at \((1,0, \ldots, 0)\), whereas the minimum is taken when \( u \) is parallel to \((1, \ldots, 1)\). However, for \( q = \infty \), the role of these two directions is clearly swapped. Thus, there must be a transition phase as \( q \to \infty \), and it is plausible to expect that the behaviour of the extremal places depends heavily on the dimension as well. That for any \( q > 2 \), whether all the extremal places have equal non-zero coordinates (up to sign), remains an open question.

4. Acknowledgements

I would like to thank F. Fodor and V. Víg for communicating the problem to me and for the fruitful discussions and ideas, and K. Ball, S. Litvak, and G. Pisier for the useful advices. I am grateful for the hospitality of the Mathematical Sciences Research Institute, Berkeley, CA.

References

[1] K. Ball, An elementary introduction to convex geometry. In: Flavors of geometry, ed. S. Levy, Cambridge University Press, Cambridge, 1997.
[2] U. Betke, M. Henk, Intrinsic volumes and lattice points of crosspolytopes. Monatsch. Math. 115 (1993), no. 1–2, 27–33.
[3] A. A. Giannopoulos, V. D. Milman and M. Rudelson, Convex bodies with minimal mean width. GAFA, Lecture Notes in Mathematics, 1745 (2000), 81–93.
[4] P. Gruber, Convex and Discrete Geometry. Springer-Verlag, 2007.
[5] M. Henk, M. Hernández Cifre, Intrinsic volumes and successive radii. J. Math.Analysis Appl. 343 (2008), no. 2, 733-742.
[6] R. Schneider, Convex bodies: the Brunn-Minkowski theory. Cambridge University Press, 1993.
[7] Z. Šidák, Rectangular Confidence Regions for the Means of Multivariate Normal Distributions. J. Amer. Stat. Association, 62 (1967), no. 318, 626–633.

[8] Z. Šidák, On Multivariate Normal Probabilities of Rectangles: Their Dependence on Correlations. Ann. Math. Stat., 39 (1968), no. 5, 1425–1434.

E-mail address, G. Ambrus: ambrus@renyi.hu

Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Réáltaanoda u. 13-15, 1053 Budapest, Hungary