Study of the 2d Ising Model with Mixed Perturbation

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Abstract

We study the thermodynamical observables of the 2d Ising model in the neighborhood of the magnetic axis by means of numerical diagonalization of the transfer matrix. In particular, we estimate the leading order corrections to the Zamolodchikov mass spectrum and find evidence of non-vanishing contributions due to the stress-energy tensor.

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Introduction

In the past the 2d Ising model has been the subject of both analytical and numerical study. After the original solution due to Onsager [1] much work has been done to improve our knowledge of the model.

It has been shown that at criticality the model can be described by a Minimal Unitary Conformal Field Theory [2]. Both magnetic and thermal perturbations of this model have been investigated and it has been found that these are the only two integrable perturbations of the model. The integrability of the model with a magnetic perturbation has been exploited by Zamolodchikov in order to obtain the exact S-matrix and mass spectrum of the theory [3]. Further, the exact knowledge of the S-matrix has been utilized to calculate the first few terms in the spectral expansion of the two point correlation functions via the exact calculation of the form factors [4].

On the other hand, an infrared safe short distance expansion (IRS) for the correlators has been proposed in [5, 6] (see [7] for a comparison of the expansions to Monte Carlo simulations). An interesting feature of the latter approach is that, in principle, it does not require the integrability of the model.

Apart from the analytic results there exists an extensive numerical investigation of the model. In particular, the transfer matrix technique has been employed in a large set of investigations: the study of the magnetic perturbation of the model [8], even at finite temperature [9], the critical equation of state [10] and a classification of irrelevant operators which enters in the observables of the theory [11, 12].

However, the knowledge of the mixed perturbation of the model is still limited. The most important contributions to the study of this regime are due to McCoy and Wu [13], Delfino et al. [14] and to Zamolodchikov and Fonseca [15].

This work explores this latter regime, in fact, our purpose is to investigate the mixed perturbation model in the neighborhood of the magnetic axis. We apply the method developed by [8] to this case. There are two key points of this method: the high precision numerical determination of the eigenvalues of the transfer matrix as a function of the couplings and the analysis of the data by means of scaling functions obtained from a CFT approach.

The original contributions of this work can be summarized in three points. Firstly, we verify that the CFT inspired scaling functions are in perfect agreement with our data, moreover we are able to estimate the leading order corrections induced by the presence of the thermal perturbation.

Secondly, we have been able to predict the existence of a term due to the stress-energy tensor in the free energy and determine its amplitude. This is an interesting result because the contributions of the stress-energy tensor have been claimed to be zero if only integrable perturbations are involved. However, this is issue is still a matter of study.

Finally we give an estimate of the corrections to the Zamolodchikov mass spectrum and we find perfect agreement (up to some acceptable error) with Delfino et al. [14].

This work is organized as follows. In Section 1 we review the standard definitions used in the Ising model and we define the normalizations and conventions we use. Section 2 is devoted to the explanation of the transfer matrix technique. The computation of the scaling functions is illustrated in Section 3. Our results are presented in Section 4 and
finally, we draw our conclusions in Section 5. In Appendix A we summarize all of the known results on the amplitudes of the scaling functions, in Appendix B we present an example of the obtained scaling functions.

1 The Ising Model

In this section we review the existing results of the 2d Ising model. In particular, we shall present the model both in its lattice form and in its continuum formulation as a field theory. We also define the observables that we shall use in the following work.

1.1 The Lattice Model

The Ising model in a magnetic field at an arbitrary temperature is defined by the partition function

$$Z(\beta, h_\ell) = \sum_{\sigma_i=\pm1} e^{\beta \sum_{\langle n, m \rangle} \sigma_n \sigma_m + h_\ell \sum_n \sigma_n},$$

where the spin variable $\sigma_n$ takes the values $\pm 1$; the notation $\langle n, m \rangle$ represents nearest neighbor sites on the lattice; the sites are labeled by $n = (n_o, n_1)$ and the two sizes of the square lattice are $L_0$ and $L_1$ (they are taken to be different because our transfer matrix calculations will treat the two directions asymmetrically); the total number of sites of the lattice will be denoted as $N = L_0 L_1$.

The coupling $\beta$ is the inverse of the temperature, while the magnetic perturbation is introduced by the coupling $h_\ell \equiv H \beta$, where $H$ is the magnetic coupling. This model undergoes a second order phase transition when $h_\ell = 0$ and $\beta$ reaches its critical value $\beta_c$

$$\beta_c \equiv \frac{1}{2} \log(\sqrt{2} + 1) = 0.4406868 \ldots .$$

Now we define the observables which we shall consider in the following work.

- The Magnetization per site is defined as

$$M(\beta, h_\ell) = \frac{1}{N} \frac{\partial}{\partial h_\ell} \log Z(\beta, h_\ell) = \frac{1}{N} \langle \sum_i \sigma_i \rangle.$$  

- The Magnetic Susceptibility is defined as

$$\chi(\beta, h_\ell) = \frac{\partial M(\beta, h_\ell)}{\partial h_\ell}.$$ 

- The Free Energy is defined as

$$f(\beta, h_\ell) = \frac{1}{N} \log Z(\beta, h_\ell).$$
The free energy is composed of a “bulk” term $f_b(\beta, h_\ell)$, which is an analytic function of $h_\ell$ and $\beta$, and a “singular” term encoding the relevant information about the theory in the neighborhood of the critical point. From the exact solution of the lattice model at $h_\ell = 0$ and $\beta = \beta_c$ we can compute the value of $f_b(0, 0)$

$$f_b = \frac{2G}{\pi} + \frac{1}{2} \log 2 = 0.9296953982\ldots,$$

where $G$ is the Catalan constant.

- The *Internal Energy* density is defined as

$$\hat{E}(\beta, h_\ell) = \frac{1}{2N} \langle \sum_{(n,m)} \sigma_n \sigma_m \rangle.$$  

As for the free energy, we have both a bulk and a singular part. If we define $E_b(\beta, h_\ell)$ as the bulk contribution, we can obtain the value of $\epsilon_b = E_b(0, 0) = \frac{1}{\sqrt{2}}$ (by using Kramers-Wannier duality or CFT techniques) and we can also define (for future convenience)

$$E(\beta, h_\ell) = \hat{E}(\beta, h_\ell) - \epsilon_b.$$  

- *Time slice correlation functions.*

We can define the zero momentum projections of the two-point correlation functions $\langle \sigma(r) \sigma(0) \rangle$ and $\langle \epsilon(r) \epsilon(0) \rangle$ (they are also called time slice correlators). The magnetization of a row of the lattice (time slice) is given by

$$S_{n_0} = \frac{1}{L_1} \sum_{n_1} \sigma_{(n_0,n_1)}$$

hence, the correlation function between time slices is given by

$$G^0_{\sigma\sigma}(n_\tau) = \sum_{n_0} \left[ \langle S_{n_0} S_{n_0+n_\tau} \rangle - \langle S_{n_0} \rangle^2 \right]$$

where label 0 indicates that it is the zero momentum projection of the original correlator.

## 1.2 The Ising Field Theory

The $2d$ Ising model near the phase transition can be described via a minimal unitary Conformal Field Theory (with central charge $c = 1/2$) perturbed with the energy and magnetization densities $\epsilon(x)$ and $\sigma(x)$

$$\mathcal{A} = \mathcal{A}_{CFT} + h \int d^2x \, \sigma(x) + \tau \int d^2x \, \epsilon(x).$$

Where $\mathcal{A}_{CFT}$ is the action of the model at the critical temperature without external magnetic field.
The coupling constant $h$ represents the magnetic perturbation of the model and, as stated before, it is the continuum version of the previously defined coupling $h_\ell$. The other coupling constant $\tau$ represents the thermal perturbation and, near criticality, it is proportional to the reduced temperature $t$

$$t = \frac{\beta_c - \beta}{\beta_c}$$

which we use in the rest of the paper.

In the following Section we give a brief report of the known results on the field theory description of the model.

### 1.2.1 The Critical Theory  $(t = 0, h_\ell = 0)$

The Ising model is the lowest model of the so-called “Minimal Unitary” series of conformal theories whose central charge is given by

$$c_p = 1 - \frac{6}{p(p+1)}, \quad p = 3, \ldots.$$  

The peculiarity of these models is that they possess a finite set of primary fields; as a consequence the whole space of local operators of the theory can be built by applying the generators of the Virasoro algebra to the primary fields. The operators obtained in this way are called secondary fields or descendants. Following this route, one is led to organize the operator content of the theory in conformal families, i.e. the sets of descendants of each primary field.

The operator spectrum of the Ising model consists of three primary operators

- Identity $\Rightarrow \Delta_I = 0$
- Magnetization $\Rightarrow \Delta_\sigma = 1/16$
- Energy $\Rightarrow \Delta_\epsilon = 1/2$

where $X_i = 2\Delta_i$ is the scaling dimension of each operator.

We assume for the field $\sigma(x)$ and $\epsilon(x)$ the usual CFT normalization

$$\langle \sigma(x)\sigma(0) \rangle = |x|^{-1/4}, \quad \langle \epsilon(x)\epsilon(0) \rangle = |x|^{-2}$$

so they have scaling dimensions 1/16 and 1/2 respectively; the parameters $h$ and $t$ have dimensions 15/8 and 1.

Hence there are three conformal families that descend from these operators and the general expression for the descendants $O_{[\phi]}$ is

$$O_{[\phi]} = L_{-k_s} \ldots L_{-k_1} T_{-m_p} \ldots T_{-m_1} \phi$$

where

$$\sum_{i=1}^{s} k_i = n; \quad \sum_{i=1}^{p} m_i = n,$$  

4
the operator $\phi$ is one of the primary fields of the theory.

The scaling dimension and the conformal spin of the operators are given by

$$X_O = \Delta_O + \bar{\Delta}_O = n + \pi + \Delta_\phi + \bar{\Delta}_\phi$$
$$s_O = n - \pi.$$  \hspace{1cm} (17)

Among secondary fields, the quasi-primary fields play a special role. A descendant field $Q$ is called quasi-primary when

- $L_1|Q\rangle = 0$
- $|Q\rangle$ is not a null vector.

It will be shown in Section 3 that quasi-primary fields can be used as building blocks for the construction of an effective Hamiltonian for the lattice model near criticality.

### 1.2.2 The Ising Field Theory with Magnetic Perturbation ($t = 0, h_\ell \neq 0$)

If $\beta$ is fixed to its critical value $\beta_c = \frac{1}{4} \log(1 + \sqrt{2})$ and the magnetic field is switched on, the field theory is still integrable, i.e. it possesses an infinite number of integrals of motion. This implies the exact knowledge of the $S$-matrix and the mass spectrum. The latter result, due to Zamolodchikov [16], suggests that the model could be described by a scattering theory with a spectrum of eight self-conjugated particles with masses

$$m_2 = 2m_1 \cos \frac{\pi}{5} = (1.6180339887..) m_1,$$
$$m_3 = 2m_1 \cos \frac{\pi}{30} = (1.9890437907..) m_1,$$
$$m_4 = 2m_2 \cos \frac{7\pi}{30} = (2.4048671724..) m_1,$$
$$m_5 = 2m_2 \cos \frac{2\pi}{15} = (2.9562952015..) m_1,$$
$$m_6 = 2m_2 \cos \frac{\pi}{30} = (3.2183404585..) m_1,$$
$$m_7 = 4m_2 \cos \frac{\pi}{5} \cos \frac{7\pi}{30} = (3.8911568233..) m_1,$$
$$m_8 = 4m_2 \cos \frac{\pi}{5} \cos \frac{2\pi}{15} = (4.7833861168..) m_1,$$

where $m_1 \equiv m_1(h)$ is the fundamental mass of the theory, and is given by

$$m_1(h) = Ch^{\frac{1}{15}}.$$  \hspace{1cm} (19)

The numerical value of $C$ was computed by Fateev [17]

$$C = \frac{4\sin \frac{\pi}{5} \Gamma \left( \frac{1}{5} \right)}{\Gamma \left( \frac{2}{5} \right) \Gamma \left( \frac{3}{5} \right)} \left[ 4\pi^2 \Gamma \left( \frac{2}{5} \right) \Gamma^2 \left( \frac{3}{16} \right) \right]^{\frac{1}{2}} = 4.40490858... .$$  \hspace{1cm} (20)

The vacuum expectation values of energy and magnetization can be parametrized as

$$\langle \epsilon \rangle = A_\epsilon \ h^{8/15}, \quad \langle \sigma \rangle = A_\sigma \ h^{1/15}$$  \hspace{1cm} (21)

where the amplitudes $A_\epsilon$ and $A_\sigma$ can be computed exactly [18]

$$A_\epsilon = 2.00314... , \quad A_\sigma = 1.27758227... .$$  \hspace{1cm} (22)
1.2.3 Non-integrable Perturbation of Ising Model \((t \neq 0, h_\ell \neq 0)\)

The non-integrable perturbation of the 2\(d\) Ising model\(^1\) was treated in \([14]\) in the framework of the Form Factors Perturbation theory.

The starting point of FFPT is to consider the mixed perturbation as the perturbation of an integrable QFT.

The action (Eq. 11) describes a one parameter family of theories labeled by the adimensional scaling variable \(\xi\)

\[
\xi \equiv \frac{t}{h^{8/15}}.
\] (23)

In this framework it is possible to calculate (at first order) the corrections to the mass spectrum and vacuum energy of the integrable theory. Two limits have been investigated in \([14]\): the magnetic perturbation of a free massive Majorana fermion \((\xi \to \infty)\) and the thermal perturbation of the integrable magnetic perturbation of the critical Ising model \((\xi \to 0)\).

The detailed discussion of \([14]\) shows that in the former case \((\xi \to \infty)\), a straightforward application of the method is not an easy task. In fact, if we are in the high temperature regime of the model, the corrections at first order vanish. A second order calculation is required but is a non-trivial computation. However, in the low temperature regime, the divergence remains at first order, and is the signal kink’s confinement.

The latter case \((\xi \to 0)\), however, is more tractable and gives rise to some quantitative predictions of the mass spectrum of the model. The results are listed below

\[
\frac{\delta \mathcal{E}_{\text{vac}}}{\delta m_1} \simeq -0.0558\ldots m_1^0,
\]
\[
\frac{\delta m_2}{\delta m_1} \simeq 0.8616\ldots,
\]
\[
\frac{\delta m_3}{\delta m_1} \simeq 1.5082\ldots.
\] (24)

The non integrable perturbation gives rise to an important quantitative difference on the masses above the threshold. In fact, the integrability of the pure magnetic perturbation prevents the creation of new particles at energies above the threshold (and from any other inelastic process). The explicit breakdown of integrability implies that particles above threshold become unstable and are expected to decay. This new feature manifests itself in the corrections to the masses, namely they develop an imaginary part. This effect has been seen explicitly in the case of the first mass above the threshold \(m_4\). The first order correction is real and given by

\[
\frac{\delta m_4}{\delta m_1} \simeq 1.1460\ldots
\] (25)

while, at second order, one expects a non-zero value of \(\text{Im } m_4^2\) (it was not computed in \([14]\) because the authors’ analysis only covers the first order contribution).

\(^1\)See, e.g. \([19]\) for a detailed discussion of the model with thermal perturbation only \((t \neq 0, h_\ell = 0)\).
1.3 Relation Between Lattice and Continuum Operators

It is useful to define, for future convenience, the relations between the lattice and continuum definitions of the energy and magnetization operators. Near the critical point, the simplest choices for the lattice operators are

- **Spin operator**

  \[ \sigma_l(x) = \sigma_x \]  

  where the index \( l \) indicates that it is a lattice discretization of the continuum operator. The magnetization per site is defined as

  \[ \sigma = \frac{1}{N} \sum_x \sigma_l(x). \]  

- **Energy operator**

  \[ \epsilon_l = \frac{1}{4} \sigma_x \left[ \sum_{y \in \{n.n.x\}} \sigma_y \right] - \epsilon_b \]  

  where the sum runs over the nearest neighbor sites of \( x \) and \( \epsilon_b \) is the bulk term. The energy per site is defined as

  \[ \epsilon = \frac{1}{N} \sum_x \epsilon_l(x). \]  

Off-critical corrections to scaling of these operators will be discussed later.

1.4 Converting between Lattice and Continuum Units

In order to fix the conversions between lattice and continuum units, we shall follow the careful discussion of [8]. One can write the lattice versions of \( \sigma \) and \( \epsilon \) as

\[ \begin{align*}
  \sigma_l &= f^\sigma_0(t, h) \sigma + f_i(t, h) \phi_i \\
  \epsilon_l &= g^\epsilon_0(t, h) \epsilon + g_i(t, h) \phi_i
\end{align*} \]  

(30)

where \( f^\sigma_0, f_i, g^\epsilon_0, g_i \) are suitable functions of \( t \) and \( h \) (which also depend on the parity properties of the operator), while the operators \( \phi_i \) are all other fields (relevant and irrelevant) of the theory respecting the symmetries of the lattice. We also have

\[ h_\ell = b_0(t, h) h \]  

(31)

which is the relation between the lattice coupling constant \( h_\ell \) and the continuum magnetic field \( h \); \( b_0(t, h) \) is an even function of \( h \).

At first order in \( t \) and \( h \), i.e. near criticality when \( t \to 0 \) and \( h \to 0 \), we have

\[ \begin{align*}
  \sigma_l &= R_\sigma \sigma, \\
  \epsilon_l &= R_\epsilon \epsilon, \\
  h_\ell &= R_h h
\end{align*} \]  

(32)
where the constants $R_\sigma$, $R_\epsilon$, $R_h$ are defined as

\begin{align*}
R_\sigma &= \lim_{t,h \to 0} f_0^\sigma(t,h) \\
R_\epsilon &= \lim_{t,h \to 0} f_0^\epsilon(t,h) \\
R_h &= \lim_{t,h \to 0} b_0(t,h).
\end{align*}

The previous normalizations were fixed in [8] by comparison with the explicit expression of the spin-spin and energy-energy critical correlation functions on the lattice. The numerical results are

\begin{align*}
R_\sigma &= 0.83868 \ldots \\
R_\epsilon &= \frac{1}{\pi} = 0.31831 \ldots \\
R_h &= R_\sigma^{-1} = 1.1923 \ldots.
\end{align*}

## 2 Transfer Matrix Technique

We face the problem of computing mass spectrum and observables by numerical diagonalization of the transfer matrix. This technique, introduced in 1941 by Kramers and Wannier [20], was extensively used by Baxter to obtain analytic solutions of certain statistical mechanical models [21]. For further details and discussions about numerical implementations of the transfer matrix see, e.g. [22].

The basic idea is to rewrite the Boltzmann weight by means of the transfer matrix $T(u_i, u_j)$

\begin{equation}
T(u_{n_0}, u_{n_0+1}) = V(u_{n_0})^{1/2} U(u_{n_0}, u_{n_0+1}) V(u_{n_0+1})^{1/2}
\end{equation}

with

\begin{align*}
U(u_{n_0}, u_{n_0+1}) &= \exp \left( \beta \sum_{n_1=1}^{L_1} \sigma_{(n_0,n_1)} \sigma_{(n_0+1,n_1)} \right) \\
V(u_{n_0}) &= \exp \left( \beta \sum_{n_1=1}^{L_1} \sigma_{(n_0,n_1)} \sigma_{(n_0,n_1+1)} + h_\ell \sum_{n_1=1}^{L_1} \sigma_{(n_0,n_1)} \right)
\end{align*}

where $u_{n_0} = (\sigma_{(n_0,1)}, \sigma_{(n_0,2)}, \ldots, \sigma_{(n_0,L_1)})$ is the spin configuration at the row (time-slice) $n_0$. The previous position implies that the partition function becomes

\begin{align*}
Z(\beta, h_\ell) &= \sum_{\sigma_i = \pm 1} e^{\beta \sum_{(n,m)} \sigma_n \sigma_m + h_\ell \sum_n \sigma_n} \\
&= \sum_{\sigma_i = \pm 1} T(u_1, u_2) T(u_2, u_3) \cdots T(u_{L_0}, u_1) \\
&= \text{tr } T_{L_0} = \sum_i \lambda_i^{L_0}
\end{align*}

where $T$ is a positive and symmetric $2^{L_1} \times 2^{L_1}$ matrix, whose (real) eigenvalues are the $\lambda_i$. 
2.1 Observables in Transfer Matrix Formalism

The numerical computation of eigenvalues and eigenvectors of the transfer matrix enables us to compute all the observables we need, provided that we specify the values of $h_\ell$, $t$ and $L_1$. We can derive all the observables from the partition function $Z(\beta, h_\ell)$ \(^\text{(37)}\). For the derivation see \[8\].

- **Free energy**

  \[ f(\beta, h_\ell) = \frac{1}{L_0 L_1} \log Z(\beta, h_\ell). \]  \hspace{1cm} (38)

  This expression simplifies in the limit $L_0 \to \infty$, in fact the leading contribution is due to the maximum eigenvalue $\lambda_{\text{max}}$

  \[ f(\beta, h_\ell) \sim \frac{1}{L_1} \log \lambda_{\text{max}}. \]  \hspace{1cm} (39)

- **Magnetization**

  \[ \langle \sigma_i \rangle = \frac{\text{tr} \ S \ T^{L_0}}{\text{tr} \ T^{L_0}} \]  \hspace{1cm} (40)

  where $S(u_{n_0}, u_{n_1}) = \delta(u_{n_0}, u_{n_1})\sigma_{(n_0,1)}$. In the limit $L_0 \to \infty$ we have

  \[ \langle \sigma_i \rangle = \langle 0 | S | 0 \rangle \]  \hspace{1cm} (41)

  where $|0\rangle$ is the eigenvector associated to the $\lambda_{\text{max}}$ eigenvalue.

- **Energy**

  The case of the internal energy is analogous similar to that of magnetization. In the limit $L_0 \to \infty$ we have

  \[ \langle \epsilon_i \rangle = \langle 0 | E | 0 \rangle \]  \hspace{1cm} (42)

  where the matrix $E(u_{n_0}, u_{n_1})$ is given by

  \[ E(u_{n_0}, u_{n_1}) = \delta(u_{n_0}, u_{n_1})\sigma_{(n_0,1)}\sigma_{(n_0,2)}. \]  \hspace{1cm} (43)

- **Correlation functions and mass spectrum**

  In the limit $L_0 \to \infty$ the time slice correlation function is defined as

  \[ \langle S_0 \ S_i \rangle = \sum_i \exp(-m_i \ |t|) \langle 0 | \tilde{S} | i \rangle \langle i | \tilde{S} | 0 \rangle \]  \hspace{1cm} (44)

  with $\tilde{S} = \frac{1}{L_1} \delta(u_{n_0}, u_{n_1}) \sum_{n_1} \sigma_{(n_0,n_1)}$. The mass spectrum $m_i$ is given by

  \[ m_i = - \log \left( \frac{\lambda_i}{\lambda_0} \right) \]  \hspace{1cm} (45)

  where the eigenvalues are organized in decreasing order of magnitude $\lambda_{\text{max}} \equiv \lambda_0 > \lambda_1 > \cdots > \lambda_i > \cdots$ and $|i\rangle$ are the normalized eigenvectors of $T$. 

9
3 Scaling Functions

In order to study the collected data obtained from the TM, we need to know the behavior of the measured operator as a function of the perturbation variables. The fundamental ingredient of this construction is the knowledge of the whole spectrum of operators of the theory, including the OPE between them.

Hence, the operator content of the 2d Ising model at the critical point previously discussed, enables us to build an effective Hamiltonian for the perturbed model. As discussed in detail in [8], the aim of this effective Hamiltonian is not to describe the model at a scale comparable with the lattice spacing; instead it has to be considered as the Hamiltonian describing the model after a suitable number of Renormalization Group transformations, i.e. at a scale that is larger with respect to the lattice spacing.

3.1 Lattice Construction of the Ising Model Via CFT Operators

The main idea then is to use the whole spectrum of conformal operators, defined on the continuum, to describe the corrections to scaling (due to the lattice) in the observables of the model. In order to build this Hamiltonian explicitly, we have to take into account, in principle, the following ingredients:

- **Symmetries of the model**
  Unlike the case of the model at the critical point, which exhibits two exact symmetries (\(\mathbb{Z}_2\) and duality), the presence of the magnetic field explicitly breaks all of them. Hence, in this case, there are no constraints coming from symmetries (it is useful to remember that this argument, in the critical case, selects only the fields belonging to the conformal family of the identity).

- **Symmetries of the lattice**
  It is crucial to define the geometry of lattice we are using in the transfer matrix calculation. In the following we consider a square lattice. This means that the rotational symmetry of the CFT is broken down to the dihedral subgroup \(D_4\) and also operators with spin are allowed. Hence, the residual symmetry group (rotations of integer multiples of \(\pi/2\)) implies that only operators with spin \(j = 4k, k \in \mathbb{N}\), can appear on the lattice (see [8] for a detailed discussion).

- **Lattice ↔ continuum relations**
  Lattice operators are defined in terms of continuum operators as follows

  \[
  \sigma_i = f_0^\ell(h, t)\sigma + h_\ell f_0^\ell(h, t)\epsilon + h_\ell f_1^\ell(h, t)\epsilon_i + h_\ell f_1^\ell \eta_i \\
  \epsilon_\ell = g_0^\ell(h, t)\epsilon + h_\ell g_0^\ell(h, t)\sigma + h_\ell g_1^\ell(h, t)\sigma_i + g_1^\ell(h, t)\epsilon_i + g_1^\ell \eta_i
  \]

  (46)

  where \(f\) and \(g\) are functions of the reduced temperature \(t\), and they are even functions of the magnetization \(h\). Furthermore, the operators appearing in the previous expressions have to be compatible with the described symmetries.

Now we are able to write the following lattice (effective) Hamiltonian

\[
H_{\text{lat}} = H_{\text{CFT}} + h_\ell \sigma + t \epsilon + u_i \Psi_i
\]

(47)
where $\Psi_i$ are the quasi-primary fields belonging to the whole set of conformal families of the theory with spin $j = 4k$, $k \in \mathbb{N}$.

The least irrelevant fields which enter the expression of $H_{\text{lat}}$ are built starting from the following quasi-primary fields of the family of the identity

$$
Q^1_2 = L_{-2} 1
$$
$$
Q^1_4 = (L_{-2}^2 - \frac{3}{5} L_{-4}) 1
$$

where the notation $Q^\eta_n$ is used to denote with $\eta$ the conformal family and with $n$ the level of descent. The same can be done for all the other families (for a list of low-lying quasi-primary states, up to level 10, see [11]).

All the results are reported in Table 1.

| | Spin-0 Sector | Spin-4 Sector | RG Eigenvalue |
|---|---|---|---|
| Identity | $Q^1_2 \bar{Q}^1_2 \equiv TT$ | $Q^1_4 + \bar{Q}^1_4 \equiv T^2 + T^2$ | $-2$ |
| Energy | $Q^\sigma_6 \bar{Q}^\sigma_6$ | $Q^\sigma_4 + \bar{Q}^\sigma_4$ | $-7$ |
| Spin | $Q^\sigma_5 \bar{Q}^\sigma_5$ | $Q^\sigma_7 + Q^\sigma_7 \bar{Q}^\sigma_3$ | $-4 - \frac{1}{8}$ |

Table 1: Low-lying quasi-primary operators. $T$ is the stress-energy tensor.

### 3.2 Computation of the Scaling Functions

We are now in a position to compute the singular part of the scaling functions of thermodynamic observables, e.g. the free energy, making use of the lattice Hamiltonian we have constructed. This is achieved by starting with the partition function $Z(h_\ell, \beta)$ of the lattice Hamiltonian $H_{\text{lat}}$. Expanding this expression, it is possible to write down a formal series expansion in the variables $h_\ell$ and $\xi \to 0$. Hence we are able to obtain the scaling functions expressions \[ for the non-scaling corrections to the observables of Section 1.1. To obtain these results, some remarks are in order:

- In the (numerical) transfer matrix analysis we are interested in the limit $\xi \to 0$ (the thermal perturbation is smaller than the magnetic one), so we can consider the following expressions for the VEV of the operators

$$
\langle O \rangle = \bar{A}_O \left[ \frac{\dim \Omega}{\dim h} \right] q_O(\xi, h_\ell)
$$

\[ We report in appendix B the expression of some of the scaling functions.
where \( q_\xi \) is an analytic function of its arguments. In the general case this ansatz is not correct because the presence of resonances induce the appearance of logarithms in the expression of VEVs.

- In the formal expansion of the partition function \( Z(h_\ell, \beta) \), we find that there also appear products of the conformal fields contained in \( H_{\text{lat}} \). The correct way to deal with these is to use the fusion rules of the conformal theory

\[
\begin{align*}
[\epsilon][\epsilon] & \sim [1] \\
[\sigma][\epsilon] & \sim [\sigma] \\
[\sigma][\sigma] & \sim [1] + [\epsilon]
\end{align*}
\]

where the notation \([\ldots]\) means that we are referring to the whole conformal family.

### 3.2.1 A Peculiar Case: The Free Energy

In this Section we show how to determine the scaling functions of the model in the region of interest \((\xi \to 0)\). We have computed the scaling functions by means of two different methods: the renormalization group, and CFT approach.

If we consider the free energy as an example, in the renormalization group approach, we can write it as the sum of three contributions

\[
f(t, h_\ell) = f_b(t, h_\ell) + f_{\text{sing}}(t, h_\ell) + f_{\text{log}}(t, h_\ell)
\]

The bulk term takes into account analytic contributions in the variables \( t \) and \( h_\ell \) due to non-critical behavior

\[
f_b(t, h_\ell) = f_{b,0}^b + f_{b,2}^b h_\ell^2 + f_{b,4}^b h_\ell^4 + \left( f_{b,1,0}^b + f_{b,1,2}^b h_\ell^2 + f_{b,1,4}^b h_\ell^4 \right) t + \\
+ \left( f_{b,2,0}^b + f_{b,2,2}^b h_\ell^2 + f_{b,2,4}^b h_\ell^4 \right) t^2 + \left( f_{b,3,0}^b + f_{b,3,2}^b h_\ell^2 + f_{b,3,4}^b h_\ell^4 \right) t^3 + O(h_\ell^6, t^4).
\]

Only even powers of \( h_\ell \) appear because the free energy is even under \( \mathbb{Z}_2 \) transformations. The non-analytic contribution is given by the master equation of the RG

\[
f_{\text{sing}}(t, h_\ell) = g_h^{2/\Delta} Y \left( \frac{g_t}{g_h^{1/\Delta}}, \left\{ g_u g_h^{\frac{y_u}{\Delta}} \right\} \right)
\]

where the scaling variables \( g_h(t, h_\ell), g_t(t, h_\ell), g_u(t, h_\ell) \) are defined in the usual way \[10\]

\[
\begin{align*}
g_t(t, h_\ell) & = t + b_t h_\ell^2 + c_t t^2 + d_t t^3 + e_t h_\ell^2 + f_t t^4 + g_t h_\ell^4 + h_t t^2 h_\ell^2 + \ldots \\
g_h(t, h_\ell) & = h_\ell(1 + c_h t + d_h t^2 + e_h h_\ell^2 + f_h t^3 + g_h h_\ell^2) + \ldots \\
g_u(t, h_\ell) & = u + a_u t + b_u h_\ell^2 + c_u t^2 + d_u t^3 + e_u h_\ell^2 + f_u h_\ell^4 + l_u t^2 h_\ell^2 + \ldots
\end{align*}
\]

and some of the coefficients are known either exactly or numerically (see appendix \[3\]). It has been known for a long time \[23\] that, in order to take into account the logarithmic divergence of the specific heat of the Onsager solution \[1\], the term \( f_{\text{log}} \) \[51\] must have the following form

\[
f_{\text{log}} = g_t^2(t, h_\ell) \log(|g_t(t, h_\ell)|^{-1}) \tilde{Y} \left( \frac{g_h}{g_t^{1/\Delta}}, \left\{ g_u g_h^{\frac{y_u}{\Delta}} \right\} \right).
\]

\[53\]
The function $\tilde{Y}(\cdot, \cdot)$ can be considered as a constant in our analysis, whose exact value is $\tilde{Y}(\cdot, \cdot) \equiv A_f = -\frac{4\beta^2}{\pi}$.

The previous expression needs some clarification in order to extend it around the magnetic axis. Following the discussion of [23], we point out that if we want $f(t, h_\ell)$ to be an analytic function of $t$ at fixed $h_\ell$, a logarithmic term should appear in the expansion of $Y(\cdot, \cdot)$. In order to make the $\log |g_\ell|$ term disappear, leaving us with the correct $\log |g_h|$, we have to extract a term like

$$x^2 |g_h|^\frac{1}{2} \log |x|^{-1} A_f$$

from the expansion of $Y(\cdot, \cdot)$ when $x \to 0$, where $x$ is the adimensional variable $\frac{g_\ell}{g_h^{1/2}}$.

Hence the logarithmic contribution has the following form

$$f_{log}(t, h_\ell) = g_\ell^2 \log |g_h|^{-1} \frac{A_f}{\Delta}. \quad (57)$$

This gives the correct divergence of the specific heat $\frac{d}{dt} \sim 2 \frac{A_f}{\Delta} \log |h_\ell| + \ldots \quad (58)$

With regard to the CFT approach one remark is in order: the CFT derivation can generate only the singular term ($f_{sing}$), while the bulk and the logarithmic contributions must be added separately.

The comparison of the two different approaches (CFT and Renormalization Group) enables us to understand the relation between the operators and the corrections they give rise to (for a careful discussion about this crucial point see [3]). A relevant example is given by the stress-energy tensor: it is the only responsible for terms like $h^{32/15}$, $t h^{32/15}$, $t^2 h^{32/15}$. This will have fundamental implications in discussing the numerical data.

4 Analysis of Results

The analysis of the data obtained by transfer matrix technique is done as follows:

- To analyze the large amount of data, due to the large number of required terms in the expansion of the scaling functions, and in order to obtain sensible results in the fits, we fix as many terms as possible, resorting to both exact and high precision numerical results already known in the literature;

- Fixing all known parameters in the scaling function and then fitting the data enables us to check their correctness in this particular regime ($\xi \to 0$, i.e. the neighborhood of the magnetic axis);

- The high precision of our data enables us to conjecture the presence or the absence of contributions due to well identified sources, i.e. terms like $(At + Bt^2) h^{32/15}$, which are entirely due to the stress-energy tensor (the irrelevant fields $T\bar{T}$, $T^2$ and $\bar{T}$).

This Section is devoted to developing such an analysis and to discuss our results.

---

3In principle it is possible to include higher logarithmic powers in the free energy scaling function, However the leading order of the expansion of such terms are zero with a precision of $10^{-6}$ in our fits.


|   |   |   |
|---|---|---|
| $t$ | $h_\ell$ | $L$ |
| 0.0130481 | 0.01 | 9 |
| 0.0119726 | 0.02 | 10 |
| 0.0108971 | 0.03 | 11 |
| 0.0087367 | 0.04 | 12 |
| 0.0076518 | :   | :  |
| 0.0065668 | 0.17 | 18 |
| 0.0043875 | 0.18 | 19 |
| 0.0010992 | 0.19 | 20 |
| 0.0   | 0.20 | 21 |

Table 2: Parameters value.

4.1 Outline of the Numerical Computations

We perform our numerical computations extracting the four larger eigenvalues of the transfer matrix at given $t$, $h_\ell$, and $L$, the size of the transverse direction of the lattice. The diagonalization method utilizes an iterative algorithm, due to [24], that evaluates the highest-lying eigenvalues of the transfer matrix with arbitrary high precision. The choice of an iterative algorithm lies in the reason that the typical size of the reduced transfer matrix is of order $10^5$ and the exact diagonalization, even if partial, is an unobtainable computationally.

Our programs we run on 10 PCs equipped with Pentium III processors and 256 Mb of RAM for a total CPU time of about 2 months.

We performed our numerical computations for each available choice of the three parameters $t$, $h_\ell$ and $L$ that we collected in table 2 for a total of 2600 different runs.

4.2 Infinite Volume Extrapolations

After obtaining the values of the thermodynamical quantities at fixed $L$, we would like to extrapolate them, taking the thermodynamic limit. In order to do this, we follow the method of [10], which we outline briefly for sake of completeness.

The basic idea is that the behavior of any thermodynamic quantity in a massive QFT will show an exponential decay as a function of $L$. The task we would like to achieve is to find the asymptotic value of the observables; in order to reach this result, we must be able to subtract all the possible exponential behaviors from our data. To achieve we iteratively subtract the exponential behavior from our data, removing both leading and subleading corrections. We define

$$b^i(L_1 - 2) = c \exp(-x(L_1 - 2)) + b^{i+1}(L_1)$$

$$b^i(L_1 - 1) = c \exp(-x(L_1 - 1)) + b^{i+1}(L_1)$$

$$b^i(L_1) = c \exp(-x(L_1)) + b^{i+1}(L_1)$$  \hspace{1cm} (59)
where $b^0(L_1)$ is the transfer matrix quantity. A step of iteration is defined by solving the previous system with respect to $b^{i+1}(L_1)$, $c$ and $x$. The iteration chain stops when the predicted value becomes numerically unstable. The taken result is the last stable prediction and its error is evaluated from the variation with respect to previous step.

### 4.3 The Fitting Procedure

In order to analyze our data we fit them with the scaling function previously obtained for a generic operator $\mathcal{O}$.

The fit is performed in three steps:

1. We start fitting the data at fixed $h_\ell$ with a polynomial in the reduced temperature

\[
\mathcal{O}(t)q|_{h_\ell} - \mathcal{O}(0)|_{h_\ell} = A_1^O(h_\ell) t + A_2^O(h_\ell) t^2 + A_3^O(h_\ell) t^3 + A_4^O(h_\ell) t^4 + \ldots \tag{60}
\]

where we found convenient to subtract the value of the observable at the critical temperature; with this subtraction we are a position to discard all the terms in the expansion that depend on $h_\ell$ only. The results of the fit are collected in a table displaying the values of $A_1^O(h_\ell)$ and $A_2^O(h_\ell)$ as a functions of $h_\ell$.

2. By means of the expansion for the scaling function, we can fit the functions $A_1^O(h_\ell)$ and $A_2^O(h_\ell)$ against the magnetic field $h_\ell$. In this way we were able to obtain the amplitudes of the corrections to the observable $\mathcal{O}$ at first and second order in the reduced temperature $t$.

3. We use all predicted values of the parameters of the fit to calculate the $\chi^2$ with a unique fitting function for all the data (the coefficients $A_3^O(h_\ell)$ and $A_4^O(h_\ell)$ of step 1 are only useful to the determination of the $\chi^2$ of the entire set of data).

The list of requirements a fit must fulfill to be accepted is the following:

- The reduced $\chi^2$ of the fit had to be of order unity, i.e. we required it to have a confidence level larger than 30%.

- In order to be included in the fitting function, the subleading terms must have an amplitude larger than the corresponding error.

- The number of degrees of freedom must be larger than 5.

The previous requirements are very hard to fulfill at the same time; this way we also take into account the systematic errors due to extrapolations.

### 4.4 Numerical Results

#### 4.4.1 Determination of $e_h$

The least precise of the constants collected in the appendix is $e_h$. The high precision of our data enables us to make an attempt to improve this estimate. Following the same
procedure of \[8\], and fixing all the known terms in the scaling function we are able to give the following result

\[ e_h = -0.007298(3) \] (61)

where the data we used as input are collected in appendix A.

### 4.4.2 Free Energy

The analysis of the free energy can be performed in four steps.

Firstly, we fit the term proportional to \( t \) with the following function

\[
-0.623226 - 0.511645 h_\ell^{\frac{8}{3}} + 0.329993 h_\ell^{\frac{16}{3}} - 0.0427280 h_\ell^2 + \\
-0.0107535 h_\ell^2 \log|h_\ell| + h_\ell^{\frac{32}{3}} K_0 + 0.0184589 h_\ell^{\frac{32}{3}} + h_\ell^{\frac{32}{3}} K_1 + \\
+ 0.0001605 h_\ell^{\frac{64}{3}} + h_\ell^{\frac{64}{3}} K_2 + h_\ell^{\frac{64}{3}} K_3 + \ldots. \] (62)

As we explain in the appendix A we set \( Y^{(2,1)}(0,0) = 0 \) according to the results of [25] and [26], moreover we can found \( Y^{(1,1)}(0,0) \) compatible with zero within the error in all our fits.

The fit shows a non-zero correction due to the stress-energy tensor, in fact we are able to estimate the value of \( K_0 \)

\[-0.05828 < K_0 < -0.05820. \] (63)

where the expression of \( K_0 \) in terms of the scaling fields is given by

\[ K_0 = (a_u + 0.664773u) Y^{(0,1)}(0,0). \] (64)

This is quite an interesting result, in fact it is known that in both the integrable perturbation of the critical Ising model, the amplitude of this term is compatible with zero.

Secondly, we perform the fit of \( t^2 \) term with the following function

\[
K_4 - 0.131877 \log|h_\ell| - 0.244467 h_\ell^{\frac{8}{3}} + 0.299064 h_\ell^{\frac{16}{3}} + \\
+ K_5 h_\ell^{\frac{32}{3}} - 0.0318458 h_\ell^2 + 0.0100527 h_\ell^2 \log|h_\ell| + h_\ell^{\frac{32}{3}} K_6 + \ldots. \] (65)

As previously stated we put \( Y^{(2,1)}(0,0) = Y^{(1,1)}(0,0) = 0 \), and we found that leading correction is

\[ 0.03655 < K_4 < 0.03657. \] (66)

The expression of \( K_4 \) in terms of the scaling fields is:

\[ K_4 = 0.481290866 + f_{2,0}^b \] (67)
in this way we are able to give an estimate for the bulk term

\[-0.44474 < f_{2,0}^b < -0.44472. \tag{68}\]

Furthermore, we can clearly see that the term due to the stress-energy tensor \(K_6\) (which comes with the power \(h^{32/15}_\ell\)) is different from zero, as for the previous case. As discussed in \[8,9\] it is difficult to have reliable estimates for the amplitudes of the subleading terms due to large systematic deviations induced by the uncertainties in the leading corrections, nevertheless we can still assert that they are different from zero.

Thirdly, we check that within the precision of our computation, the function \(\tilde{Y}(\cdot, \cdot)\) is a constant. In fact, if we expand it in Taylor series, we find that the first contribution to the scaling function comes in the \(t^2\) term and has the following form

\[\tilde{Y}^{(1,0)}(0, 0) \log |h_\ell| h^{22/15}_\ell. \tag{70}\]

If we perform the fit with this new contribution, we find that its amplitude is compatible with zero. Hence, for our purposes it is safe to consider \(\tilde{Y}(\cdot, \cdot)\) as a constant.

Fourthly, we compute the \(\chi^2\) with the global scaling function (Eq. 60) both on \(t\) and \(h_\ell\) utilizing the value of the constants predicted in the previous steps, in order to verify the correctness of the fitting procedure.

It is important to mention that in the fitting functions we use the known values of the constants of Appendix \[A\]. It is a non-trivial test on the validity of both our results, and the well known values we used.

### 4.4.3 Magnetization

Firstly, we write down the correction proportional to the \(t\) term

\[
-0.2728775 \frac{h^{17}_\ell}{h^{17}_\ell} + 0.3519930 \frac{1}{h^{17}_\ell} - 0.0747026 h_\ell + 0.0215069 h_\ell \log |h_\ell| + h^{12}_\ell \mathcal{H}_0 + 0.0467625 h^{24}_\ell + h^{5}_\ell \mathcal{H}_1 - 0.0004923 h^{31}_\ell + h^{11}_\ell \mathcal{H}_2 + h^{37}_\ell \mathcal{H}_3 + h^{41}_\ell \mathcal{H}_4 + \ldots \tag{71}\]

where all the known quantities are taken into account. In particular we also impose the constraints we found in the analysis of the free energy, i.e. \(Y^{(2,1)}(0, 0) = \)

\[As discussed in \[8\], notwithstanding systematic errors, one can give also a rough estimate of the subleading amplitudes. Following the same route we are able to find

\[-0.721 < K_5 < -0.717. \tag{69}\]
$Y^{(1,1)}(0,0) = 0$, and we find perfect agreement also in this case. The leading correction is due to the stress-energy tensor, as we expected from the analysis of the free energy, and its amplitude is given by

$$-0.124 < H_0 < -0.122$$

which is compatible with $H_0 \equiv \frac{32}{15} K_0$ within our errors.

Secondly, the contribution due to $t^2$ is given by

$$-\frac{0.1318769}{h_{\ell}} - \frac{0.1303825}{h_{\ell}^{15}} + 0.3190016 \ h_{\ell}^{15} + \frac{22}{15} h_{\ell}^{15} H_5 - 0.0536389 \ h_{\ell} +$$

$$+ 0.0201054 \ h_{\ell} \ \log |h_{\ell}| + \ h_{\ell}^{15} H_6 + \ldots$$

and is obtained following the same strategy as before. This order also shows a non-zero contribution due to stress-energy tensor, i.e. $H_6$.

The amplitude of leading order correction is given by

$$-0.723 < H_5 < -0.718.$$ 

We can check that it is in reasonable agreement with the subleading amplitude of the free energy $H_5 \equiv K_5$ (see (72)).

### 4.4.4 Internal Energy

Firstly, the scaling function for the $t$ term is

$$W_0 + 0.2992532 \ \log |h_{\ell}| + 0.5547414 \ h_{\ell}^{15} + h_{\ell}^{15} W_1 + h_{\ell}^{15} W_2 - 0.0918396 \ h_{\ell}^2 +$$

$$+ h_{\ell}^{15} W_3 + h_{\ell}^{15} W_4 - 0.0228115 \ h_{\ell}^2 \ \log |h_{\ell}| + \ldots$$

the predicted value for the leading contribution is

$$-0.1659 < W_0 < -0.1657$$

which expressed in term of scaling function is $W_0 \equiv (-2.1842763 + 4.5383706 \ f_{2,0}^b)$, the value of $f_{2,0}^b$ is consistent with the previous estimate (68). As before the fit is compatible with the constraint $Y^{(2,1)}(0,0) = Y^{(1,1)}(0,0) = 0$.

Secondly, the scaling function for the $t^2$ term is

$$\frac{W_5}{h_{\ell}^{15}} + W_6 - 0.2797532 \ \log |h_{\ell}| + h_{\ell}^{15} \ W_7 + h_{\ell}^{15} \ W_6 + h_{\ell}^{15} \ W_8 +$$

$$- 0.0727482 \ h_{\ell}^{15} W_5 + \ldots$$

the predicted value for the leading contribution is

$$-0.414 < W_5 < -0.408$$

where $W_5 \equiv 0.5672963 \ Y^{(3,0)}(0,0)$, and again the fit is compatible with the constraint $Y^{(2,1)}(0,0) = Y^{(1,1)}(0,0) = 0$. 

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4.4.5 Susceptibility

The form of the scaling function is in total agreement with the computed values of the this observable. However no new predictions are available.

4.4.6 Mass spectrum

In order to check the results of the FFPT proposed by [14], we performed the fit on the square of the first three masses of the theory, and computed the following ratio

$$\frac{\delta m^2_i}{\delta m^2_1} = \frac{m(t)^2_i - m(0)^2_i}{m(t)^2_1 - m(0)^2_1}$$  \hspace{1cm} (79)

where $m(t)^2_i$ are the perturbed masses, and $m(0)^2_i$ the unperturbed ones.

The results of our analysis are reported below:

$$\frac{\delta m^2_2}{\delta m^2_1} = 1.393(17)$$  \hspace{1cm} (80)

$$\frac{\delta m^2_3}{\delta m^2_1} = 3.16(30).$$

These results are in perfect agreement with the theoretical prediction. In fact, since the deviation from the integrable model is small, it is correct to write $\delta m^2_i = 2m(0)_i \delta m_i$, and finally we have

$$\frac{\delta m^2_2}{\delta m^2_1} = 0.86(1)$$  \hspace{1cm} (81)

$$\frac{\delta m^2_3}{\delta m^2_1} = 1.58(15).$$

4.4.7 Magnetization Overlap

The magnetic overlap $|F^\sigma_1|^2$ can also be analyzed as before. We are able to find the leading order corrections of terms proportional to $t$

$$\frac{\mathcal{R}_0}{h^\frac{15}{16}} + h^\frac{2}{16}\mathcal{R}_1 + h^\frac{3}{16}\mathcal{R}_2 + h^\frac{4}{16}\mathcal{R}_3 + h^\frac{5}{16}\mathcal{R}_4 + h^\frac{6}{16}\mathcal{R}_5 + h^\frac{7}{16}\mathcal{R}_6 + h^\frac{8}{16}\mathcal{R}_7 + h^2\mathcal{R}_8 + \ldots$$  \hspace{1cm} (82)

and $t^2$

$$\frac{\mathcal{R}_9}{h^\frac{16}{15}} + \frac{\mathcal{R}_{10}}{h^\frac{17}{15}} + \frac{\mathcal{R}_{11}}{h^\frac{18}{15}} + h^\frac{3}{16}\mathcal{R}_{12} + h^\frac{5}{16}\mathcal{R}_{13} + h^\frac{7}{16}\mathcal{R}_{14} + \ldots$$  \hspace{1cm} (83)

The numerical values are

$$0.628 < \mathcal{R}_0 < 0.631$$

$$0.661 < \mathcal{R}_9 < 0.664$$  \hspace{1cm} (84)

It would be interesting to calculate the same corrections on theoretical grounds (at least for the $t$ correction) in order to make a comparison.
5 Discussion and Conclusions

In this paper we studied the effect of a mixed relevant perturbation on the Ising model using the Transfer Matrix technique. We consider the neighborhood of the magnetic axis in the limit $\xi \ll 1$, where $\xi$ is the adimensional parameter $t/h^{8/15}$.

We have concentrated our efforts on the following areas

- We calculated the scaling functions (appendix B) applying a CFT approach to the problem.
- We used all known predictions about the behavior of scaling functions (see appendix A), and verified that they all agree with our data.
- Being able to identify the contribution of secondary fields to the scaling function, we predicted that there is a non zero contribution due to the stress-energy tensor, and we evaluate it. In our opinion this a quite interesting result, because it is known that the contribution of these particular secondary fields is zero if we study the model with only one relevant perturbation.
- We obtained estimates of several amplitudes never predicted by any other analytical method before.
- We calculated the correction to the Zamolodchikov mass spectrum of the Ising model with a magnetic field, and we found perfect agreement with Delfino et al. [14].

There are two possible developments of this work: We can use our data to improve the knowledge of the equation of state of the Ising model, in order to map all the possible regime of perturbations. We can extend the analysis of [9] to study the effect of mixed perturbation on the finite temperature results.

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A Known Numbers

It is known that it is possible to write the free energy of the model in terms of nonlinear scaling fields [27].

The scaling fields are analytic functions of $t$ and $h_\ell$ respecting the $\mathbb{Z}_2$ parity of $h_\ell$. Their Taylor expansions are expected to be

\begin{align*}
g_t(t, h_\ell) &= t + b_t h_\ell^2 + c_t t^2 + d_t t^3 + e_t t h_\ell^2 + f_t t^4 + g_t h_\ell^4 + h_t t^2 h_\ell^2 + \ldots \\
g_h(t, h_\ell) &= h_\ell (1 + c_h t + d_h t^2 + e_h h_\ell^2 + f_h t^3 + g_h t^2) + \\
g_u(t, h_\ell) &= u + a_u t + b_u h_\ell^2 + c_u t^2 + d_u t^3 + e_u t h_\ell^2 + f_u h_\ell^4 + l_u t^2 h_\ell^2 + \ldots
\end{align*}

Here we report all the analytically known coefficients [26, 28, 29, 30]

\begin{align*}
c_h &= \frac{\beta_c}{\sqrt{2}}, & d_h &= \frac{23\beta_c^2}{16}, & f_h &= \frac{191\beta_c^3}{48\sqrt{2}}, \\
c_t &= \frac{\beta_c}{\sqrt{2}}, & d_t &= \frac{7\beta_c^2}{6}, & f_t &= \frac{17\beta_c^3}{6\sqrt{2}}, \\
e_t &= b_t \beta_c \sqrt{2}, & b_t &= \frac{E_0 \pi}{16\beta_c^2}
\end{align*}

where

\begin{equation}
E_0 = 0.0403255003\ldots
\end{equation}

From the analysis of the model at the critical temperature and $h_\ell \neq 0$, we obtain a new estimate for $e_h$

\begin{equation}
e_h = -0.007298(3).
\end{equation}

For the free energy, we use also the coefficient reported in the following

\begin{align*}
Y(0, 0) &= 0.99279949\ldots \\
Y^{(1,0)}(0, 0) &= A_{E,2}^l/b_t = -0.511645336\ldots \\
A_{E,2}^l &= \frac{A_{E,2}^l \pi}{Y(0, 0) 8\beta_c} = 0.0208602\ldots
\end{align*}

where $G$ is the Catalan constant and the $A$'s are defined in [8].

A high precision study of the thermal perturbation have been performed by Orrick et al. in [26], from this work we are able to extract a set of parameters for our scaling functions. In particular we are able to observe that the known coefficients for both thermal and magnetic perturbations are exactly consistent, and we can extend the knowledge about the thermal coefficients to gain predictions about the unknown magnetic terms.
To achieve these results we compare the scaling function of the susceptibility along the thermal axis with the high precision estimates of \[26\] and \[25\], in this way we obtained

\[
y^{(2,0)}(0,0) = 0.9625817322\ldots
\]
\[
f_{0,2}^b \equiv A_{f,b}^l Y(0,0) = -0.05206662255\ldots
\]
\[
f_{1,2}^b = -0.00348278\ldots
\]
\[
f_{2,2}^b = 0.000528775\ldots.
\]

These results are obtained for the thermal perturbation in the regime \(t \neq 0, h_\ell = 0\), we observe that these results are valid also in our regime of interest, because the analytic continuation of Section 3.2.1 do not affect similar kinds of terms, in total agreement with our fits.

Furthermore we obtain also the relations:

\[
y^{(2,1)}(0,0)(a_u + 0.701128u) = 0
\]
\[
y^{(2,1)}(0,0)u = 0
\]

In order to fulfill the above requirements, we set \(Y^{(2,1)}(0,0) = 0\) because the other choice \(u = 0, a_u = 0\) was not consistent with our fits.

Finally, from Onsager’s exact solution \[1\] we shall extract

\[
f_{1,0}^b = -0.623226\ldots
\]

### B Scaling Functions

Here we report some of the scaling functions we obtained. We remark that the functions have been evaluated to higher order of the expansion reported here \(O(h_\ell^5)O(t^5))\).

**Free Energy:**

\[
f(t)|_{h_\ell} - f(0)|_{h_\ell} = \mathcal{A}^l_1(h_\ell) t + \mathcal{A}^l_2(h_\ell) t^2 + \ldots
\]

\[
\mathcal{A}^l_1(h_\ell) = f_{1,0}^b + Y^{(1,0)}(0,0) h_\ell^\frac{2}{5} + \frac{16}{15} c_h Y(0,0) h_\ell^{\frac{16}{5}} + u Y^{(1,1)}(0,0) h_\ell^\frac{8}{5} + \left(-\frac{16}{15} A_f \log |h_\ell| b_t + f_{1,2}^b + b_t Y^{(2,1)}(0,0)\right) h_\ell^2 + \ldots
\]
\[ A_2^f(h_\ell) = \left( -\frac{8}{15} A_f \log |h_\ell| + f_{0,2}^b + \frac{1}{2} Y^{(2,0)}(0,0) \right) + \left( \frac{8}{15} c_h Y^{(1,0)}(0,0) + c_t Y^{(1,0)}(0,0) \right) h_\ell^{\frac{8}{15}} + \left( \frac{8}{225} c_h^2 Y(0,0) + \frac{16}{15} d_h Y(0,0) + \frac{1}{2} u Y^{(2,1)}(0,0) \right) h_\ell^{\frac{8}{15}} + \]
\[ + \frac{1}{2} b_t Y^{(3,0)}(0,0) h_\ell^{\frac{8}{15}} + \left( a_u Y^{(1,1)}(0,0) + \frac{8}{5} u c_h Y^{(1,1)}(0,0) + u c_t Y^{(1,1)}(0,0) \right) h_\ell^{\frac{8}{15}} + \]
\[ + \left( -\frac{16}{15} A_f b_t c_h - \frac{16}{15} A_f \log |h_\ell| b_t c_t - \frac{8}{15} A_f e_h - \frac{16}{15} A_f \log |h_\ell| e_t + f_{2,2}^b + b_t c_t Y^{(2,0)}(0,0) + e_t Y^{(2,0)}(0,0) \right) h_\ell^{\frac{8}{15}} + \ldots \]

(95)

Internal Energy:
\[ E(t)|_{h_\ell} - E(0)|_{h_\ell} = A_1^F(h_\ell) t + A_2^E(h_\ell) t^2 + \ldots \] (96)

\[ A_1^F(h_\ell) = \left( -\frac{16}{15} A_f \log |h_\ell| T_c + 2 f_{2,0}^b T_c + T_c Y^{(2,0)}(0,0) \right) + \]
\[ + \left( \frac{16}{15} c_h T_c Y^{(1,0)}(0,0) + 2 c_t T_c Y^{(1,0)}(0,0) \right) h_\ell^{\frac{8}{15}} + \]
\[ + \left( \frac{16}{225} c_h^2 T_c Y(0,0) + \frac{32}{15} d_h T_c Y(0,0) + u T_c Y^{(2,1)}(0,0) \right) h_\ell^{\frac{8}{15}} + \]
\[ + b_t T_c Y^{(3,0)}(0,0) h_\ell^{\frac{8}{15}} + \left( 2 a_u T_c Y^{(1,1)}(0,0) \right) h_\ell^{\frac{8}{15}} + \]
\[ + \frac{16}{5} u c_h T_c Y^{(1,1)}(0,0) + 2 u c_t T_c Y^{(1,1)}(0,0) \right) h_\ell^{\frac{8}{15}} + \ldots \]

(97)
\[ A_{\tilde{e}}^\ell (h_\ell) = \frac{T_c}{2 h_{\ell}^{\frac{5}{2}}} Y^{(3,0)} (0, 0) + \left( -\frac{8}{5} A_f c_h T_c - \frac{16}{5} A_f \log |h_\ell| c_t T_c + 3 f_{3,0}^b T_c + \right. \\
+ 3 c_t T_c Y^{(2,0)} (0, 0) \bigg) \left. \left( -\frac{28}{75} c_h^2 T_c Y^{(1,0)} (0, 0) + \frac{8}{5} c_h c_t T_c Y^{(1,0)} (0, 0) + \right. \\
+ \frac{8}{5} d_h T_c Y^{(1,0)} (0, 0) + 3 d_t T_c Y^{(1,0)} (0, 0) + \frac{1}{2} u T_c Y^{(3,1)} (0, 0) \right) h_{\ell}^{\frac{2}{5}} + \\
+ \frac{1}{2} b_t T_c Y^{(4,0)} (0, 0) h_{\ell}^{\frac{16}{5}} + \left( -\frac{112}{3375} c_h^3 T_c Y (0, 0) + \frac{16}{75} c_h d_h T_c Y (0, 0) + \right. \\
+ \frac{16}{5} f_h T_c Y (0, 0) + \frac{3}{2} a_u T_c Y^{(2,1)} (0, 0) + \frac{8}{5} u c_h T_c Y^{(2,1)} (0, 0) + \\
+ \frac{3}{2} u c_t T_c Y^{(2,1)} (0, 0) \bigg) h_{\ell}^{\frac{12}{5}} + \left( -\frac{4}{5} b_t c_h T_c Y^{(3,0)} (0, 0) + 3 b_t c_t T_c Y^{(3,0)} (0, 0) + \right. \\
- \frac{4}{15} e_h T_c Y^{(3,0)} (0, 0) + \frac{3}{2} c_t Y^{(3,0)} (0, 0) \bigg) h_{\ell}^{\frac{22}{5}} + \left( \frac{24}{5} a_u c_h T_c Y^{(1,1)} (0, 0) + \right. \\
+ \frac{36}{25} u c_h^2 T_c Y^{(1,1)} (0, 0) + 3 a_u c_t T_c Y^{(1,1)} (0, 0) + \frac{24}{5} u c_h c_t Y^{(1,1)} (0, 0) + \\
+ \frac{3}{2} c_u T_c Y^{(1,1)} (0, 0) + \frac{24}{5} u d_h T_c Y^{(1,1)} (0, 0) + 3 u d_t T_c Y^{(1,1)} (0, 0) + \\
+ \frac{1}{4} u^2 T_c Y^{(3,2)} (0, 0) \bigg) h_{\ell}^{\frac{14}{5}} + \ldots \right]

(98)

C Example of numerical data

Here we report, for sake of completeness, an example of the data we used to fit. We refer to value of $\beta = 0.4373$.  

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| $h_\ell$ | $E(t, h_\ell)$ | $f(t, h_\ell)$ | $M(t, h_\ell)$ | $\chi(t, h_\ell)$ |
|-------|----------------|----------------|----------------|-----------------|
| 0.01  | 0.035901(3)   | 0.931944971(5) | 0.76120(4)    |                 |
| 0.02  | 0.0569877(2)  | 0.939788377(1) | 0.802572(1)   |                 |
| 0.03  | 0.07237192(2) | 0.94794151643(2) | 0.82640939(1) |                 |
| 0.04  | 0.084838424(2) | 0.95629353642(1) | 0.84316904(1) | 1.45013(2)     |
| 0.05  | 0.0954482541(1) | 0.96479217393(1) | 0.856062081(1) | 1.150734(1)    |
| 0.06  | 0.1047420407(1) | 0.97340669611(1) | 0.8665107103(1) | 0.95108(1)     |
| 0.07  | 0.11303997732(1) | 0.98211679357(1) | 0.8752714873(1) | 0.808372(1)    |
| 0.08  | 0.12055040575(1) | 0.99090802119(1) | 0.8827957916(1) | 0.701236(1)    |
| 0.09  | 0.12741797596(1) | 0.999769568339(1) | 0.8937481546(1) | 0.6178284(1)   |
| 0.10  | 0.13374801551(1) | 1.008693035843(1) | 0.89520749914(1) | 0.5510433(1)   |
| 0.11  | 0.13962007086(2) | 1.017671707919(1) | 0.90043585646(1) | 0.4963581(1)   |
| 0.12  | 0.14509597932(2) | 1.026700091153(1) | 0.90516485833(3) | 0.4507573(1)   |
| 0.13  | 0.150224955365(1) | 1.035773608172(1) | 0.909474255678(2) | 0.4121515(1)   |
| 0.14  | 0.155046932292(2) | 1.044888386026(1) | 0.913426172207(2) | 0.3790476(1)   |
| 0.15  | 0.159594851224(1) | 1.054041105248(1) | 0.917069865355(1) | 0.3503494(1)   |
| 0.16  | 0.16389626622(1) | 1.06322888925(1) | 0.920445095734(1) | 0.32523603(1)  |
| 0.17  | 0.167974505628(1) | 1.072449221416(1) | 0.923584148311(1) | 0.30076426(1)  |
| 0.18  | 0.17184952388(1) | 1.081699881718(1) | 0.926514827477(1) | 0.28381176(1)  |
| 0.19  | 0.17553575445(1) | 1.090978897417(1) | 0.92925895744(1) | 0.2657631501(1) |
| 0.20  | 0.179056649653(1) | 1.100284504134(1) | 0.931835974906(1) | 0.2499123945(1) |

| $h_\ell$ | $1/m_1(t, h_\ell)$ | $1/m_2(t, h_\ell)$ | $1/m_3(t, h_\ell)$ | $|F_1^2(t, h_\ell)|^2$ |
|-------|-----------------|-----------------|-----------------|-----------------|
| 0.03  | 1.66609(4)      |                 |                 |                 |
| 0.04  | 1.42446(2)      | 0.86(2)         |                 |                 |
| 0.05  | 1.262217(1)     | 0.775(2)        |                 |                 |
| 0.06  | 1.1438966(2)    | 0.70476(2)      |                 |                 |
| 0.07  | 1.0528411(3)    | 0.64979(3)      | 0.5334(3)       | 0.3859807(2)    |
| 0.08  | 0.980049866(1)  | 0.60587(2)      | 0.4973(3)       | 0.37813145(1)   |
| 0.09  | 0.920186381(1)  | 0.56978(5)      | 0.4676(3)       | 0.3706210923(3) |
| 0.10  | 0.8698623782(3) | 0.539467(1)     | 0.4417(3)       | 0.363856671(3)  |
| 0.11  | 0.82681092694(1) | 0.5135465(5) | 0.4203(2)       | 0.356382554(1) |
| 0.12  | 0.789451203443(1) | 0.4910712(1) | 0.4019(1)       | 0.3495820201(7) |
| 0.13  | 0.756643166957(1) | 0.47135216(1) | 0.3859(7)       | 0.34296250455(5) |
| 0.14  | 0.727541458508(3) | 0.453877992(1) | 0.37178(2)      | 0.33650783473(4) |
| 0.15  | 0.701504369616(1) | 0.4382602533(3) | 0.35921(1)      | 0.3302055233(2) |
| 0.16  | 0.678034397921(1) | 0.424197853(2) | 0.347936(2)     | 0.324045673287(8) |
| 0.17  | 0.656741586699(1) | 0.411453290663(3) | 0.33773872(2) | 0.31802057612(3) |
| 0.18  | 0.637311086114(2) | 0.399836374051(1) | 0.32846313(3) | 0.3121262996(1) |
| 0.19  | 0.619489526443(1) | 0.389192731029(1) | 0.31998(1)   | 0.306347184601(2) |
| 0.20  | 0.603068643116(1) | 0.379395558407(1) | 0.312184(1) | 0.300689114577(1) |
References

[1] L. Onsager, Phys. Rev. 65 (1944) 117.
[2] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. B241(1984)333.
[3] A.B. Zamolodchikov, Al B. Zamolodchikov, Ann. Phys. 120 (1979), 253. For a review see for instance: G. Mussardo, Phys. Rep. 218 (1992), 215.
[4] G. Delfino and G. Mussardo, Nucl. Phys. B 455 (1995) 724 [arXiv:hep-th/9507010]; G. Delfino and P. Simonetti, Phys. Lett. B 383 (1996) 450 [arXiv:hep-th/9605065].
[5] R. Guida and N. Magnoli, Nucl. Phys. B 471 (1996) 361 [arXiv:hep-th/9511209].
[6] R. Guida and N. Magnoli, Nucl. Phys. B 483 (1997) 563 [arXiv:hep-th/9606072].
[7] M. Caselle, P. Grinza and N. Magnoli, Nucl. Phys. B 579 (2000) 635 [arXiv:hep-th/9909065].
[8] M. Caselle and M. Hasenbusch, Nucl. Phys. B 579 (2000) 667 [arXiv:hep-th/9911216].
[9] M. Caselle and M. Hasenbusch, arXiv:hep-th/0204088.
[10] M. Caselle, M. Hasenbusch, A. Pelissetto and E. Vicari, J. Phys. A 34 (2001) 2923 [arXiv:cond-mat/0011305].
[11] M. Caselle, M. Hasenbusch, A. Pelissetto and E. Vicari, J. Phys. A 35 (2002) 4861 [arXiv:cond-mat/0106372].
[12] M. Caselle, P. Grinza and N. Magnoli, J. Phys. A 34 (2001) 8733 [arXiv:hep-th/0103263].
[13] B. M. McCoy and T. T. Wu, Phys. Rev. D 18 (1978) 1259.
[14] G. Delfino, G. Mussardo and P. Simonetti, Nucl. Phys. B 473 (1996) 469 [arXiv:hep-th/9603011].
[15] P. Fonseca and A. Zamolodchikov, arXiv:hep-th/0112167.
[16] A. B. Zamolodchikov, Int. J. Mod. Phys. A 4 (1989) 4235.
[17] V. A. Fateev, Phys. Lett. B 324 (1994) 45.
[18] V. Fateev, S. Lukyanov, A. B. Zamolodchikov and A. B. Zamolodchikov, Nucl. Phys. B 516 (1998) 652 [arXiv:hep-th/9709034].
[19] V. P. Yurov and A. B. Zamolodchikov, Int. J. Mod. Phys. A 6 (1991) 3419.
[20] H. A. Kramers, G. H. Wannier, Phys. Rev. 60 (1941) 252.
[21] R. J. Baxter *Exactly Solvable Models in Statistical Mechanics*, (Academic Press, London, 1982).

[22] M. P. Nightingale, in *Finite Size Scaling and Numerical Simulation of Statistical Systems*, ed. V. Privman, World Scientific 1990.

[23] A. Aharony and M. E. Fisher, Phys. Rev. B 27 (1983) 4394.

[24] H. L. Richards, M. A. Novotny and P. A. Rikvold, Phys. Rev. B48 (1993) 14584.

[25] S. Gartenhaus and W. S. McCullough, Phys. Rev. B 38 (1988) 11688.

[26] W. P. Orrick, B. Nickel, A. J. Guttmann, J. H. H. Perk, J. Statist. Phys. 102 (2001) 795-841; and W. P. Orrick, B. G. Nickel, A. J. Guttmann, J. H. H. Perk Phys. Rev. Lett. 86 (2001) 4120-4123.

[27] F. J. Wegner, in *Phase transitions and critical phenomena, Vol. 6*, eds. C. Domb and M. Green (New York, Academic Press, 1976), p. 7.

[28] M. Caselle, M. Hasenbusch, A. Pelissetto and E. Vicari, J. Phys. A 33 (2000) 8171 [arXiv:hep-th/0003049].

[29] B. Nickel, J. Phys. A 32 (1999) 3889; 33 (2000) 1693.

[30] J. Salas and A. D. Sokal, J. Statist. Phys. 98 (2000) 551 [arXiv:cond-mat/9904038].