Dynamics and steady states of a tracer particle in a confined critical fluid

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The dynamics and the steady states of a point-like tracer particle immersed in a confined critical fluid are studied. The fluid is modeled field-theoretically in terms of an order parameter (concentration or density field) obeying dissipative or conservative equilibrium dynamics and (non-)symmetry-breaking boundary conditions. The tracer, which represents, e.g., a colloidal particle, interacts with the fluid by locally modifying its chemical potential or its correlations. The coupling between tracer and fluid gives rise to a nonlinear and non-Markovian tracer dynamics, which is investigated here analytically and via numerical simulations for a one-dimensional system. From the coupled Langevin equations for the tracer-fluid system we derive an effective Fokker-Planck equation for the tracer by means of adiabatic elimination as well as perturbation theory within a weak-coupling approximation. The effective tracer dynamics is found to be governed by a fluctuation-induced (Casimir) potential, a spatially dependent mobility, and a spatially dependent (multiplicative) noise, the characteristics of which depend on the interaction and the boundary conditions. The steady-state distribution of the tracer is typically inhomogeneous. Notably, when detailed balance is broken, the driving of the temporally correlated noise can induce an effective attraction of the tracer towards a boundary.

I. INTRODUCTION

Fluids near critical points are characterized by fluctuations with large correlation length and long relaxation time, which give rise to a plethora of intriguing phenomena. A particular example is the critical Casimir force (CCF), which acts on objects immersed in a near-critical medium \cite{1-4}. CCFs have been utilized to externally control the behavior of colloidal particles in critical solvents by, e.g., altering the solvent temperature or tuning the surface properties of the particles (see Ref. \cite{6} for a review). Except in \( d = 2 \) spatial dimensions, where exact methods exist \cite{7-10}, the interaction between two colloidal particles or between a colloid and a solid wall is typically analyzed separately in the near-distance (Derjaguin) and the far-distance limit. In the former, an approximate parallel plate geometry is realized, allowing one to invoke known results for the CCF in a thin film \cite{11-13}. By contrast, in the far-distance limit, where the ratio between the particle radius and the distance is small, the CCF can be obtained from a “small-sphere expansion” of the Boltzmann weight \cite{11,14-16}. Recently, the non-equilibrium dynamics of colloidal particles in critical media has received increased attention \cite{17,18}, examples including studies of drag forces \cite{18-23}, aggregation \cite{18,24}, diffusion \cite{25-31}, shear flow \cite{32,33}, solvent coarsening \cite{34-36}, and interplay between criticality and activity \cite{37,38}.

Here, we study the behavior of a point-like tracer particle in a confined critical fluid within a dynamical field theory (see Fig. 1). The fluid medium is modeled in terms of a scalar order-parameter (OP) field \( \phi(\mathbf{r}) \) governed by non-conserved or conserved equilibrium dynamics within the Gaussian approximation \cite{39,40}. In a single-component fluid, the OP represents the deviation of the local fluid density from its critical value \( \phi \propto n - n_c \), whereas, in a binary mixture, \( \phi \) correspondingly represents the concentration deviation of a certain species \cite{41}. We distinguish between a reactive and a passive tracer: a reactive tracer interacts with the fluid in a way that preserves detailed balance and that thus renders a steady state in accordance with equilibrium statistical mechanics. A passive tracer, by contrast, is affected by the fluid but does not act back on it and hence represents a non-equilibrium system. In the present case, the passive tracer is driven by a temporally correlated noise and thus can, in fact, be regarded as a special type of “active” particle \cite{42-45}. Following Refs. \cite{25,27}, fluid and tracer are coupled such that, in the reactive case, one obtains either a locally enhanced mean OP or a locally suppressed OP variance (and reduced correlation length) at the tracer location. This resembles the typical behavior of a colloid in a critical fluid \cite{1,12} or of a magnetic point defect in the Ising model \cite{47,48}. A reactive tracer can thus be regarded as a simplified model for a colloidal particle. We analyze theoretically and via simulations the resulting dynamics and long-time steady states for various symmetry- and non-symmetry-breaking boundary conditions (BCs) imposed on the OP by the confinement. Specifically, we focus on a one-dimensional system, i.e., an interval of length \( L \). However, we also calculate equilibrium distributions in three dimensions, which turn out to be qualitatively similar to the one-dimensional case. By comparing with the

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FIG. 1. Situation considered in the present study: a point-like tracer at location $R(t)$ (black dot) is immersed in a fluctuating medium described by a scalar field $\phi(z,t)$ (order parameter). The system is confined by boundaries at $z = 0, L$, which exert reflective BCs on the tracer and either periodic, Neumann, Dirichlet, or capillary BCs on the field $\phi$ [see Eq. (2.3)].

literature on CCFs, we show that the equilibrium distributions obtained here encode the static behavior of a colloid in a half-space geometry in the far-distance limit [11, 14, 15]. The point-like representation of the tracer thus retains the essential character of the interaction with the medium, while it facilitates analytical and numerical approaches.

The present study is structured as follows: In Secs. II and III, we introduce the model and provide the necessary preliminaries. In Secs. IV and V, the statics and dynamics of a passive and a reactive tracer, respectively, are analyzed. Results of numerical simulations are presented in Sec. VI, together with a physical interpretation of the observed phenomena and a discussion in the context of existing literature. Sec. VII provides a summary of our study. Technical details are collected in Appendices A to H.

II. MODEL

Before specializing to the one-dimensional case, we introduce the model for arbitrary spatial dimension $d$. We consider a system consisting of a tracer at position $\mathbf{R}(t)$ and a fluctuating OP field $\phi(\mathbf{r}, t)$. The system is finite in at least one direction, which we take to be the $z$-direction, having length $L$. In thermal equilibrium, the system is described by the joint steady-state probability distribution

$$P_s(\mathbf{R}, [\phi]) = \frac{1}{Z_0} \exp \left( -\frac{1}{T} \mathcal{H}(\mathbf{R}, [\phi]) \right)$$

with the Hamiltonian

$$\mathcal{H}(\mathbf{R}, [\phi]) = \mathcal{H}_\phi[\phi] + \mathcal{H}_R(\mathbf{R}),$$

$$\mathcal{H}_\phi[\phi] \equiv \int_V d^d r \left\{ \frac{1}{2} |\nabla \phi(\mathbf{r})|^2 - h_1 \phi(\mathbf{r})[\delta(z) + \delta(L-z)] \right\}, \quad \mathcal{H}_R(\mathbf{R}) \equiv -h^\phi(\mathbf{R}) + \frac{1}{2} c^\phi(\mathbf{R})^2,$$  \hspace{1cm} (2.2)

where $Z_0 = \int_V d^d R \int D\phi \exp(-\mathcal{H}(\mathbf{R}, [\phi])/T)$ is a normalization factor and $V$ is the system volume. The OP fulfills one of the following BCs:

- periodic: $\phi(\{\mathbf{r}_\parallel, z\}) = \phi(\{\mathbf{r}_\parallel, z + L\})$,
- Dirichlet: $\phi(\{\mathbf{r}_\parallel, z \in \{0, L\}\}) = 0$,
- Neumann: $\partial_z \phi(\{\mathbf{r}_\parallel, z\})|_{z \in \{0, L\}} = 0$,
- capillary BCs: $\partial_z \langle \phi(\{\mathbf{r}_\parallel, z\}) \rangle|_{z \in \{0, L\}} = -h_1$,

with the decomposition $\mathbf{r} = \{\mathbf{r}_\parallel, z \equiv r_z\}$. Capillary BCs are imposed onto the (mean) OP by the action of boundary fields of equal strength $h_1$ [see Eq. (3.5) below] [39]. Tracer and OP field are coupled either linearly (with strength $h$) or quadratically (with strength $c$) [25,27]. The coupling $h$ corresponds to a local bulk field (chemical potential), which leads to an excess OP around the tracer, resembling critical adsorption on a colloid [1,12]. A quadratic coupling $c > 0$ describes a locally reduced correlation length and results in a suppression of the OP fluctuations near the tracer. The dynamics of the system is described in terms of coupled Langevin equations [25,27]:

$$\dot{\mathbf{R}}(t) = -\gamma_R \nabla_R \mathcal{H} + \sqrt{\gamma_R} \eta(t),$$  \hspace{1cm} (2.4a)

$$\dot{\phi}(\mathbf{r}, t) = -\gamma_\phi (-\nabla^2)^a \frac{\delta \mathcal{H}}{\delta \phi(\mathbf{r})} + \sqrt{\gamma_\phi} \xi(\mathbf{r}, t),$$  \hspace{1cm} (2.4b)

where $\gamma_R$ and $\gamma_\phi$ are kinetic coefficients and the Gaussian white noises $\eta$ and $\xi$ have zero mean and correlations

$$\langle \eta_\alpha(t) \eta_\beta(t') \rangle = 2 T_R \delta_\alpha_\beta \delta(t-t'),$$  \hspace{1cm} (2.5a)

$$\langle \xi(\mathbf{r}, t) \xi(\mathbf{r}', t') \rangle = 2 T_\phi (-\nabla^2)^a \delta(\mathbf{r} - \mathbf{r}') \delta(t-t').$$  \hspace{1cm} (2.5b)
Dissipative dynamics (model A \[69\]) is realized for \(a = 0\) and conserved dynamics (model B) for \(a = 1\). The tracer is subject to reflective BCs at \(z \in \{0, L\}\). In general, we allow different heat bath temperatures for the tracer (\(T_R\)) and the OP (\(T_\phi\)). The case \(T_R \neq T_\phi\) describes an energy transfer between tracer and OP, which can be regarded as a simplified means to capture varying levels of non-equilibrium activity in the system \([50–54]\). The Fokker-Planck equation (FPE) associated with Eq. (2.4) is given by \[P \equiv P(\mathbf{R}, \phi, t)\] \[\text{cq} \[55, 56\]. The Fokker-Planck equation (FPE) associated with Eq. (2.4) is given by \[ \partial_t P = \int_V d^4 \mathbf{r} \frac{\delta}{\delta \phi(\mathbf{r})} \left[ \gamma_\phi (-\nabla^2)^a \frac{\delta \mathcal{H}}{\delta \phi(\mathbf{r})} + T_\phi \gamma_\phi (-\nabla^2)^a \frac{\delta}{\delta \phi(\mathbf{r})} \right] P + \gamma_R \nabla \mathbf{R} \cdot (\nabla \mathcal{H}) P + T_R \gamma_R \nabla^2 P, \] (2.6)

which, in the equilibrium case, is solved by Eq. (2.1). We consider in the following also the distribution \(P_s(\mathbf{R}, \phi)\) in thermal equilibrium with the OP field (describing, e.g., a colloidal particle) is realized for \(T = T_\phi\) reduces to \(\text{balanced by dissipation and therefore represents a driven, “active” particle [42, 51–53, 57] [58]. We henceforth focus on a confined system of size \(L\) in \(d = 1\) spatial dimensions (see Fig. 1), such that Eq. (2.8) reduces to \[ \dot{\phi}(\mathbf{r}, t) = \chi^{-1}(-\nabla^2)^a \{ \nabla_\mathbf{r}^2 \phi(\mathbf{r}, t) + \zeta [h - c\phi(\mathbf{R}(t), t)] \delta(\mathbf{r} - \mathbf{R}(t)) + h_1 [\delta(z) + \delta(L - z)] \} + \chi^{-1/2}(\zeta, t), \] (2.8b)

where, for convenience, we dropped all tildes. Furthermore, we have introduced the “adiabaticity” parameter \[ \chi \equiv \gamma_R/\gamma_\phi, \] (2.9)
as well as the control parameter \(\zeta\), which takes the values \(\zeta = 0\) for a passive and \(\zeta = 1\) for a reactive tracer. A tracer in thermal equilibrium with the OP field (describing, e.g., a colloidal particle) is realized for \(T_R = T_\phi = T, \zeta = 1\), and its steady state obeys Eq. (2.1). A passive tracer, or one for which \(T_R \neq T_\phi\), receives energy input that is not balanced by dissipation and therefore represents a driven, “active” particle [42, 51–53, 57] [58].

We henceforth focus on a confined system of size \(L\) in \(d = 1\) spatial dimensions (see Fig. 1), such that Eq. (2.8) reduces to \[ \dot{\mathbf{R}}(t) = (\nabla \mathbf{R}) \mathbf{R}(t) [h - c\phi(\mathbf{R}(t), t)] + \eta(t), \] (2.10a)
\[ \dot{\phi}(z, t) = \chi^{-1}(-\partial_z^2)^a \{ \partial_z^2 \phi(z, t) + \zeta [h - c\phi(\mathbf{R}(t), t)] \delta(z - R(t)) + h_1 [\delta(z) + \delta(L - z)] \} + \chi^{-1/2}(\zeta, t). \] (2.10b)

Since \(L\) is the only OP-specific length scale in this system, we define the dimensionless counterpart of \(\chi\) in Eq. (2.9) as \[ \tilde{\chi} \equiv T_R L^{2a} \chi. \] (2.11)

In the adiabatic limit, \(\tilde{\chi} \ll 1\), the dynamics of the OP field is much faster than the one of the tracer [59]. This will be utilized in Secs. [4] and [5] below to derive effective Markovian descriptions.

The time-dependent statistics of \(R\) and \(\phi\) is described by a probability distribution \(P(R, t) = P(R, R_0, \phi, t, t_0)\), where \(R_0 \equiv R(t_0)\) denotes the tracer position at the initial time \(t_0\). Integrating \(P\) over the fluctuations of the OP field \(\phi\) renders the marginal probability distribution \[ \tilde{P}(R, t) \equiv \int D\phi P(R, \phi, t), \] (2.12)

which, due to global conservation of probability, fulfills \[ \int_0^L dR \tilde{P}(R, t) = 1. \] (2.13)

One aim of the present study is to determine the time-dependence of \(\tilde{P}(R, t)\) by deriving effective Fokker-Planck equations. Importantly, we assume the OP to always remain in thermal equilibrium. Accordingly, in the presence of a reactive tracer, the OP distribution \(P_\phi(R, \phi)\) is time independent and given by \(P_\phi\) in Eq. (2.1). In the passive case, instead, one has [see Eq. (2.2)] \[ P_\phi(\phi) = \hat{Z}_0^{-1} \exp(-\mathcal{H}_\phi(\phi)/T_\phi), \] (2.14)
which is independent of $R$. The initial joint probability distribution is given by

$$P(R, R_0, [\phi], t = t_0, t_0) = \delta(R - R_0)P_\phi(R_0, [\phi]).$$

(2.15)

Note that, at times $t > t_0$, the joint distribution does not generally factorize into $R$- and $\phi$-independent parts. For simplicity, we set $t_0 = 0$ and usually suppress the dependence of $P$ on $R_0$ and $t_0$. We will frequently express the tracer probability distribution $\tilde{P}(R, t)$ in terms of the dimensionless coordinate $\rho \equiv R/L$, which formally renders a new distribution $\tilde{b}(\rho, t) = L\tilde{P}(\rho L, t)$. However, for notational simplicity we will use $\rho$ also as a shorthand notation in the expressions for $\tilde{P}$.

### III. PRELIMINARIES

#### A. Mode expansion and boundary conditions of the OP

We begin by introducing a set of eigenfunctions $\sigma_n(z)$ and eigenvalues $k_n^2$ of the operator $-\partial_z^2$ for the various BCs [see Eq. (2.3)] considered in this study:

$$\sigma_n^{(p)}(z) = \frac{1}{\sqrt{L}} \exp \left( i k_n^{(p)} z \right), \quad k_n^{(p)} = \frac{2\pi n}{L}, \quad n = 0, \pm 1, \pm 2, \ldots,$$

(periodic BCs),

$$\sigma_n^{(D)}(z) = \sqrt{\frac{2}{L}} \sin \left( k_n^{(D)} z \right), \quad k_n^{(D)} = \frac{\pi n}{L}, \quad n = 1, 2, \ldots,$$

(Dirichlet BCs),

$$\sigma_n^{(N)}(z) = \sqrt{\frac{2 - \delta n,0}{L}} \cos \left( k_n^{(N)} z \right), \quad k_n^{(N)} = \frac{\pi n}{L}, \quad n = 0, 1, 2, \ldots,$$

(Neumann BCs).

The eigenfunctions are orthonormal:

$$\int_0^L dz \sigma_m(z)\sigma_n^{*}(z) = \delta_{m,n},$$

(3.2)

and complete:

$$\sum_n \sigma_n(z)\sigma_n^{*}(z') = \delta(z - z').$$

(3.3)

Note that complex conjugation is only relevant for periodic BCs. The eigenfunction expansions of the OP field and the noise take the form:

$$\phi(z, t) = \sum_n \sigma_n(z)\phi_n(t), \quad \xi(z, t) = \sum_n \sigma_n(z)\xi_n(t),$$

(3.4)

which imply the inverse relations [see Eq. (3.2)]:

$$\phi_n(t) = \int_0^L dz \sigma_n^{*}(z)\phi(z, t), \quad \xi_n(t) = \int_0^L dz \sigma_n^{*}(z)\xi(z, t).$$

(3.5)

For a real-valued function such as $\phi(z, t)$, one has $\phi_{-n}(t) = \phi_n^{*}(t)$. According to Eq. (2.5b), the correlations of the noise modes are given by

$$\langle \xi_n(t)\xi_{n'}^{*}(t') \rangle = 2T_\rho k_m^{2n}\delta_{m,n}\delta(t - t').$$

(3.6)

Upon inserting Eq. (3.4) into Eq. (2.10b), one obtains the dynamic equation of the OP modes:

$$\partial_t \phi_n = \chi^{-1}k_n^{2a} \left\{ -k_n^2 \phi_n + \frac{1}{2} \left[ h_1 \sigma_n^{*}(R(t)) + h_2 \sigma_n^{*}(L(t)) \right] \right\} + \chi^{-1/2}\xi_n.$$ 

(3.7)

We assume the initial condition $\phi(z, t = t_i) = 0$ to apply in the infinite past ($t_i = -\infty$), such that at $t = t_0 = 0$ the OP is equilibrated [see Eq. (2.15)] and its initial condition plays no role anymore. The zero mode $\phi_{n=0}$, which occurs (except for Dirichlet BCs) in the case of dissipative dynamics ($a = 0$), has to be treated separately. For all BCs except standard Dirichlet ones [Eq. (2.3b)], Eq. (2.10) with $a = 1$ conserves the OP globally, i.e., $\int_0^L dz \phi(z, t) = \text{const.}$ Standard Dirichlet BCs generally entail a non-zero flux through the boundaries, requiring to use suitable “no-flux”
basis functions [60] instead in order to ensure global OP conservation. However, since this is technically involved, we do not consider conserved dynamics in conjunction with Dirichlet BCs in the following.

Due to the equilibrium assumption, the mean OP profile $\langle \phi(z) \rangle$ is time-independent and, for the BCs specified in Eq. (3.1), in fact, vanishes, $\langle \phi(z) \rangle = 0$. A spatially inhomogeneous profile can arise here either due to a local bulk field ($h \neq 0$), representing a linearly coupled reactive tracer, or due to boundary fields ($h_1 \neq 0$). For a system with $h = 0$ and boundary fields of identical strength at $z \in \{0, L\}$, the mean OP profile is given by [see Appendix B1]

\[
\langle \phi(z) \rangle_{h_1} = h_1 L \left[ \left( \frac{1}{2} - \frac{z}{L} \right)^2 - \frac{1}{12} \right],
\]

which fulfills standard capillary BCs [see Eq. (2.34) as well as, e.g., Ref. [61] and references therein]:

\[
\partial_z \langle \phi(z) \rangle_{h_1} \big|_{z \in \{0, L\}} = \mp h_1.
\]

Capillary BCs are applicable only in the absence of a zero mode, as otherwise $\langle \phi(z) \rangle_{h_1}$ would be divergent [see Appendix B1]. A system without a zero mode is naturally realized for conserved dynamics.

### B. Solution of the OP dynamics

We next present exact as well as perturbative solutions of Eq. (3.7), which will be used in the course of this work.

1. **Solution of Eq. (3.7) for $c = 0$ or $\zeta = 0$**

For $c = 0$ and arbitrary $\zeta$, as well as generally for $\zeta = 0$ (passive tracer), the solution of Eq. (3.7) is given by:

\[
\phi_n(t) = \int_{-\infty}^{t} ds \ e^{-k_n^{2+2a}(t-s)/\chi} \left\{ \chi^{-1} k_n^{2a} \left[ \zeta \chi_0^*(R(s)) + h_1 \tau_n^* \right] + \chi^{-1/2} \zeta_n(s) \right\}, \quad (c = 0) \tag{3.10a}
\]

\[
\phi_0(t) = (t - t_i) \chi^{-1} \delta_{a,0} \left( \zeta h \sigma_0 + h_1 \tau_0 \right) + \chi^{-1/2} \int_{t_i}^{t} ds \xi_0(s), \quad (3.10b)
\]

where $\tau_n \equiv \sigma_n(0) + \sigma_n(L)$ and $\sigma_0 = 1/\sqrt{L}$. The zero mode $\phi_0$ performs a diffusive motion superimposed on a linear growth. Since this growth depends on the initial time $t_i$, the latter has to be kept finite in Eq. (3.10b), whereas in Eq. (3.10a), it is unproblematic to set $t_i = -\infty$ in the lower integration boundary. In the adiabatic limit $\tilde{\chi} \ll 1$, Eq. (3.10a) reduces, for $\zeta = 0$ or $h = 0$, to [see Eq. (C2)]

\[
\phi_n^{(0)}(t) \approx h_1 \frac{\tau_n^*}{k_n^{2+2a}} + \sqrt{\chi} \frac{\zeta_n(t)}{k_n^{2+2a}}, \quad (n \neq 0, \quad \tilde{\chi} \ll 1) \tag{3.11}
\]

while the zero mode [Eq. (3.10b)] does not admit an expansion for small $\tilde{\chi}$. We conclude that BCs involving a zero mode are not compatible with the adiabatic limit. This is expected, since the relaxation time of a zero mode is infinite, which violates the assumption of a fast OP dynamics. Besides standard periodic or Neumann BCs, we shall thus also consider modified variants thereof for which the zero mode is explicitly removed [see Eq. (3.19) below]. Note that the adiabatic approximation can in general not be used to calculate equilibrium variances of $\phi$ (cf. Sec. III C 3).

2. **Solution of Eq. (3.7) for $c \neq 0$ and $h = 0$**

For $c \neq 0$, a perturbative solution of Eq. (3.7) in orders of $c$ can be constructed. To this end, we formally expand the OP as $\phi(z, t) = \phi^{(0)} + \phi^{(1)} + \ldots$, where $\phi^{(1)} \sim O(c^1)$, and assume the noise $\zeta \sim O(c^0)$ [62]. Inserting this expansion in Eq. (3.7) (with $h = 0$) and grouping terms of the same order in $c$, we obtain

\[
\phi_n^{(1)}(t) = -\zeta \chi^{-1} c k_n^{2a} \int_{-\infty}^{t} ds e^{-k_n^{2+2a}(t-s)/\chi} \phi^{(0)}(R(s), s) \sigma_n^*(R(s)), \quad (n \neq 0) \tag{3.12a}
\]

\[
\phi_0^{(1)}(t) = -\zeta \chi^{-1} c \delta_{a,0} (t - t_i) \phi^{(0)}(R(t), t) \sigma_0^*(R(t)), \quad (3.12b)
\]
while the $O(e^0)$ solution $\phi_n^{(0)}(t)$ is given by Eq. (3.10) with $h = 0$. Note that the only dependence on $h_1$ enters through $\phi_n^{(0)}$. Determining, analogously to Appendix C the adiabatic limit $\tilde{\chi} \ll 1$ of Eq. (3.12a), gives

$$\phi_n^{(1)}(t) \simeq \frac{\zeta c}{k_n} \phi_n^{(0)}(R(t), t) \sigma_n^{*}(R(t)). \quad (n \neq 0, \ \tilde{\chi} \ll 1) \quad (3.13)$$

C. Order-parameter correlation functions

The (connected) equilibrium OP correlation function is defined as usual as

$$C_\phi(z, z', t, t') = \langle \delta \phi(z, t) \delta \phi(z', t') \rangle = \langle \phi(z, t) \phi(z') \rangle - \langle \phi(z) \rangle \langle \phi(z') \rangle, \quad \delta \phi(z, t) \equiv \phi(z, t) - \langle \phi(z) \rangle, \quad (3.14)$$

where $\delta \phi$ denotes the fluctuating part of $\phi$. We provide in the following expressions for $C_\phi$ within the Gaussian model and evaluate them in the adiabatic limit. Technical details are deferred to Appendix B.

1. Time-dependent correlation functions

The time-dependent correlation function follows straightforwardly from the solution of the Langevin equation [Eq. (3.7)] discussed above. In the case of a passive tracer ($\zeta = 0$), Eqs. (3.6) and (3.10) render the equilibrium two-time correlator

$$C_\phi(z, z', t, t')|_{\zeta=0} = T_\phi \sum_n \sigma_n(z) \sigma_n^{*}(z') e^{-k_n^2 |t-t'|/\chi} + \frac{2T_\phi}{L\chi} \min(t-t_i, t'-t_i) \delta_{a,0}, \quad (3.15)$$

where the prime indicates a sum excluding $n = 0$ and we used Eqs. (3.8) and (B6) to identify the mean part $\langle \phi(z) \rangle_{h_1}$, as required by Eq. (3.14). The last term in Eq. (3.15) stems from the zero mode and exists only for periodic or Neumann BCs [see Eq. (3.11)] and non-conserved dynamics ($a = 0$). In the following, we will also need the correlation function of $\partial_z \phi$:

$$C_{\partial \phi}(z, z', t-t')|_{\zeta=0} = \langle \partial_z \delta \phi(z, t) \partial_z \delta \phi(z', t') \rangle|_{\zeta=0} = \partial_z \partial_z C_\phi(z, z', t, t')|_{\zeta=0}$$

where

$$\bar{\sigma}_n(z) \equiv \frac{1}{k_n} \partial_z \sigma_n(z) = \begin{cases} \text{i} \sigma_n^{(p)}(z), & n = \pm 1, \ldots, (p) \\ \frac{1}{\sqrt{2}} \cos(k_n^{(D)} z), & n = 1, 2, \ldots, (D) \\ -\frac{1}{\sqrt{2}} \sin(k_n^{(N)} z), & n = 1, 2, \ldots, (N) \end{cases} \quad (3.17)$$

and $\bar{\sigma}_0^{(p)}(z) = \bar{\sigma}_0^{(N)}(z) = 0$. The notation in the right column indicates the BCs of the OP [see Eq. (2.3)]. Since the zero mode does not contribute to $C_{\partial \phi}$, the latter is solely a function of the time difference. The correlation function $C_{\partial \phi}^{(p)}$ for periodic BCs can be expressed in terms of the corresponding one for Neumann and Dirichlet BCs [63]:

$$C_{\partial \phi}^{(p)}(z-z', t-t') = \frac{1}{2} \left[ C_{\partial \phi}^{(N)}(z-z', t-t') + C_{\partial \phi}^{(D)}(z-z', t-t') \right]_{L/2}, \quad (3.18)$$

where the r.h.s. is to be evaluated for a system of size $L/2$ instead of $L$.

2. Static correlation functions

In equilibrium for $t = t'$, the (connected) OP correlation function [see Eq. (3.11)] can be directly determined from Eq. (2.7):

$$C_\phi(x, y) \equiv \langle \delta \phi(x) \delta \phi(y) \rangle|_{\zeta=0} = T_\phi \sum_n \sigma_n(x) \sigma_n^{*}(y) \frac{1}{k_n^2} = \begin{cases} \frac{LT_\phi}{\frac{3}{2} - \frac{1}{2} \left( |x-y| + \frac{1}{2} (x-y)^2 \right)}, & (p^*) \\ \frac{LT_\phi}{\frac{1}{2}} \min(x, y) - \frac{1}{2} xy, & (D) \\ \frac{LT_\phi}{\frac{3}{2} - \frac{1}{2} \max(x, y) + \frac{1}{12} (x^2 + y^2)} \left[ \frac{1}{2} \text{max}(x, y) + \frac{1}{2} (x^2 + y^2) \right], & (N^*, \pm) \\ \frac{T_\phi}{\frac{3}{2} - \frac{1}{2} \text{max}(x, y) + \frac{1}{12} (x^2 + y^2)} + C_{\partial \phi}^{(p^*, N^*)}(x, y), & (p, N) \end{cases} \quad (3.19)$$
The corresponding expression for Neumann BCs with a vanishing mean OP indicates that the zero mode is removed from the set of modes in Eq. (3.1). Physically, the vanishing of the zero mode can be ensured within conserved dynamics if a vanishing mean OP is imposed as initial condition [see Eq. (3.7)]. However, in order to elucidate the effect of the conservation law, we will use (p* and (N*) BCs also in conjunction with dissipative dynamics, if this is required to obtain a well-defined model (e.g., in the adiabatic limit, see Sec. III C 3 below). Note that, for capillary BCs, the OP correlation function takes a Neumann form (see Appendix B 2). As a characteristic of the Gaussian model, Eq. (3.19) holds independently from the presence of boundary or bulk fields (see Appendix B 1); in particular, it applies to an OP coupled linearly to a passive or reactive tracer. If the tracer is coupled quadratically, by contrast, the static correlation function differs in the passive and the reactive case (see Appendix B 2).

For $x = y = R$, Eq. (3.19) reduces to the variance ($\rho \equiv R/L$)

$$V_\phi(R) \equiv C_\phi(R, R) = \langle \delta\phi(R)^2 \rangle|_{\zeta=0} = T_\rho \sum_n \frac{\sigma_n(R)^2}{k_n^2} = \begin{cases} T_\rho L \times \frac{1}{12}, & (p*) \\
T_\rho L \left(\rho - \rho^2\right), & (D) \\
T_\rho L \left(\frac{1}{4} - \rho + \rho^2\right), & (N*, \pm) \end{cases}$$

which turns out to be a central quantity for the tracer dynamics and is illustrated in Fig. 2(a). It is useful to remark that $V_\phi(R) = \langle \phi^{(0)}(R)^2 \rangle - \langle \phi \rangle_\rho^2$, which follows from Eqs. (3.10) and (B6) (with $\zeta = 0$). Note that we defined $V_\phi$ only for BCs without a zero mode, as the variance is infinite otherwise.

3. Adiabatic Approximation

The adiabatic limit of $C_\phi$ can be obtained by inserting Eq. (3.11) into Eq. (3.14):

$$C_\phi^{ad}(z, z', t - t')|_{\zeta=0} = 2T_\phi \chi \sum_n \frac{\sigma_n(z)\sigma_n^*(z')}{k_n^{4+2a}} \delta(t - t') = L^3 \tilde{\chi}_\phi C(z/L, z'/L) \delta(t - t'),$$

with [cf. Eq. (2.11)]

$$\tilde{\chi}_\phi \equiv T_\phi L^{2a} \chi$$

and

$$C(\tilde{z}, \tilde{z}') = \begin{cases} \frac{1}{6} \left[|\tilde{z} - \tilde{z}'|^3 - |\tilde{z} + \tilde{z}'|^3 + 2\tilde{z} \tilde{z}' (\tilde{z}^2 + \tilde{z}'^2 + 2)\right], & (D) \\
\frac{1}{6} \left[|\tilde{z} - \tilde{z}'|^3 + |\tilde{z} + \tilde{z}'|^3 - 3\tilde{z}^2 \tilde{z}'^2 - \frac{1}{2} \tilde{z}^4 - 2 \tilde{z}^2 - \frac{1}{2} \tilde{z}'^4 - 2 \tilde{z}'^2\right] + \frac{1}{245}, \quad (N^*, a = 0) \end{cases}$$

The corresponding expression for Neumann BCs with $a = 1$ is of similar polynomial form, but rather lengthy and not stated here. The expression for periodic BCs can be constructed by means of Eq. (3.18). Since the relaxation time of a zero mode is divergent [see also Eq. (3.15)], the actual limit $\tilde{\chi}_\phi \to 0$ does not exist for standard periodic or Neumann BCs. We remark that $\int_{-\infty}^{\infty} dt C_\phi(z, z', t - t') = \int_{-\infty}^{\infty} dt C_\phi^{ad}(z, z', t - t')$, i.e., the term on the right-hand side of Eq. (3.21) multiplying the $\delta$-function corresponds to the time integral of the correlation function.

FIG. 2. (a) Static variance $V_\phi(R) = \langle \delta\phi(R)^2 \rangle$ [Eq. (3.20)] of an OP field $\phi$ obeying Dirichlet (D) or Neumann BCs without a zero mode ($N^*$), in the absence of a (reactive) tracer ($\zeta = 0$). (b) Amplitude $m(R)$ of the correlation function $\langle [\partial_R \delta\phi(R, t)][\partial_R \delta\phi(R, t')] \rangle$ in the adiabatic limit [see Eq. (3.26)].
Using Eq. (3.21), one obtains
\[ C_{\partial\phi}(z, z', t - t') \big|_{\zeta = 0} = 2T_0 \chi \sum_n \frac{\tilde{\sigma}_n(z) \tilde{\sigma}_n(z')}{k_n^{2+2\alpha}} \delta(t - t') \]
\[ = L \tilde{\chi}_0 \delta(t - t') \times \begin{cases} -\hat{z} - \hat{z}' - |\hat{z} - \hat{z}'| + \frac{2}{3} \hat{z}', & (D) \\ \hat{z} + \hat{z}' - |\hat{z} - \hat{z}'| - 2\hat{z}', & (N^*, a = 0) \\ \frac{1}{3} \hat{z}'(2 + \hat{z}^2 + \hat{z}'^2) + \frac{1}{3} |\hat{z} - \hat{z}'|^3 - \frac{1}{3} (\hat{z} + \hat{z}')^3, & (N^*, a = 1) \end{cases} \] (3.24)
for the expression of \( C_{\partial\phi} \) in the adiabatic limit. We introduce a function \( m(R) \) by writing
\[ C_{\partial\phi}^\text{rad}(R, R, t - t') \big|_{\zeta = 0} = 2L \tilde{\chi}_0 m(R) \delta(t - t'), \quad m(R) = \frac{1}{L^{1+2\alpha}} \sum_n \frac{|\tilde{\sigma}_n(R)|^2}{k_n^{2+2\alpha}}. \] (3.25)
For the various BCs, one has (see Fig. 2(b)):
\[ m(R) \big|_{a = 0} = \begin{cases} \frac{1}{3}, & (p^*) \\ \frac{1}{3} - \rho + \rho^2, & (D) \\ \rho - \rho^2, & (N^*, \pm) \end{cases} \] (3.26a)
and
\[ m(R) \big|_{a = 1} = \begin{cases} \frac{1}{25}, & \left( p \right) \\ \frac{5}{2} \rho^2 - \frac{2}{3} \rho^3 + \frac{1}{3} \rho^4, & (N^*, \pm) \end{cases} \] (3.26b)
Note that \( m \) is dimensionless and we generally suppress the dependence on \( L \). It turns out that \( m \) encodes the modification of the tracer mobility due to the OP fluctuations.

D. Effective Langevin equation for the tracer

The Langevin equation in Eq. (2.10a) can be rewritten in a more compact form:
\[ \dot{h}(t) = \Xi_h(R(t), t) + \eta(t), \quad \Xi_h(z, t) \equiv h \partial_z \phi(z, t), \quad \Xi_e(z, t) \equiv -c \phi(z, t) \partial_z \phi(z, t) = -\frac{c}{2} \partial_z \phi^2(z, t), \] (3.27)
wherein \( \Xi_h \) and \( \Xi_e \) represent effective forcing terms. In the case of a passive tracer, the only source of inhomogeneity for the OP are boundary fields \( h_1 \neq 0 \), which allows us to decompose the OP as \( \phi(z) = \langle \phi(z) \rangle_{h_1} + \delta \phi(z) \), with the
mean equilibrium profile \( \langle \phi(z) \rangle_{h_1} \) stated in Eq. \( \text{[3.8]} \) and \( \langle \delta \phi(z) \rangle = 0 \). Accordingly, the mean parts of \( \Xi_h \) and \( \Xi_c \) give rise to (time-independent) effective potentials \( U_h \) and \( U_c \):

\[
\begin{align*}
\langle \Xi_h(z) \rangle &= -\partial_z U_h(z), \quad U_h(z) \equiv -h \langle \phi(z) \rangle_{h_1}, \\
\langle \Xi_c(z) \rangle &= -\partial_z U_c(z), \quad U_c(z) \equiv \frac{c}{2} [V_\phi(z) + \langle \phi(z) \rangle_{h_1}^2],
\end{align*}
\]

where \( V_\phi \) is the fluctuation variance stated in Eq. \( \text{[3.20]} \). It turns out that the effective potentials \( U_{hc} \) (see Fig. \[3\]) determine the dynamics and the steady state of a passive tracer [see Eqs. \( \text{[4.1]} \) and \( \text{[4.31]} \) below], as well as of a reactive tracer for weak couplings \( h, c \) [see Eqs. \( \text{[5.20]} \) and \( \text{[5.26]} \) below]. In particular, the expression for \( U_c \) in Eq. \( \text{[3.28b]} \) also applies if \( h_1 = 0 \), in which case \( V_\phi \) is given by Eq. \( \text{[3.20]} \) for the various BCs. Accordingly, a linearly coupled tracer is attracted to (repelled from) the boundary if \( h \) and \( h_1 \) have the same (opposite) sign. A quadratically coupled tracer, instead, is generally repelled from the boundaries with \( h_1 \neq 0 \), which, interestingly, can lead to the emergence of two distinct minima in \( U_c \) provided \( |h_1| \) is large. This behavior reflects the critical Casimir interactions between wall and tracer and is further discussed in Sec. \[VI.B.2\].

By subtracting the mean parts from \( \Xi_h \) and \( \Xi_c \), we obtain new noises

\[
\begin{align*}
\Pi_h(z,t) &= \Xi_h(z,t) - \langle \Xi_h(z) \rangle, \\
\Pi_c(z,t) &= \Xi_c(z,t) - \langle \Xi_c(z) \rangle
\end{align*}
\]

having vanishing means and correlations are given by [see Eqs. \( \text{[3.15]} \) and \( \text{[3.16]} \)]

\[
\begin{align}
\langle \Pi_h(z,t)\Pi_h(z',t') \rangle &= h^2 C_\phi(z,z',t-t'), \\
\langle \Pi_c(z,t)\Pi_c(z',t') \rangle &= c^2 \left\{ \partial_z \partial_{z'} \langle \phi(z) \rangle_{h_1} \langle \phi(z') \rangle_{h_1} C_\phi(z,z',t-t') \\
&\quad + [\partial_z C_\phi(z,z',t-t')] [\partial_{z'} C_\phi(z,z',t-t')] + C_\phi(z,z',t-t') \right\},
\end{align}
\]

while \( \langle \Pi_h \Pi_c(R(t),t) \Pi_c(R(t),t) \rangle = 0 \). We emphasize that \( \Pi_c \), which is essentially the square of the Gaussian process \( \phi \), is itself not Gaussian \[65, 66\]. A Gaussian approximation can be made by assuming a small coupling \( c \) such that higher-order cumulants of \( \Pi_c \) become negligible.

Due to the non-linear and non-Markovian nature of Eq. \( \text{[3.27]} \), its time-dependent solution can in general not be determined exactly. In the following, we thus solve Eq. \( \text{[3.27]} \) perturbatively, either by assuming a weak coupling or by applying an adiabatic approximation. The latter approach amounts to replacing the noises \( \Pi_h \) and \( \Pi_c \) by suitable Markovian approximations.

### IV. PASSIVE TRACER

We begin by discussing a passive tracer, which is described by Eq. \( \text{[2.10]} \) with \( \zeta = 0 \). Since the forcing term [first term on the r.h.s. of Eq. \( \text{[2.10]} \)] is not balanced by a corresponding dissipation term in Eq. \( \text{[2.10]} \), a passive tracer represents a driven, non-equilibrium system. We separately analyze the cases of a linear \( (h \neq 0, c = 0) \) and a quadratic coupling \( (h = 0, c \neq 0) \) between tracer and OP field.

#### A. Linear tracer-field coupling

1. **Adiabatic approximation**

We apply the adiabatic elimination procedure described in Refs. \[67, 68\] in order to integrate out the fast dynamics of \( \phi \) from Eq. \( \text{[2.10]} \), which results in the following FPE for the effective tracer probability distribution \( \bar{P} \) (see Appendix \[E\]):

\[
\partial_t \bar{P}(R,t) = -\partial_R \left[ \mu(R) \bar{P}(R,t) \right] + \partial^2_R \left[ D(R) \bar{P}(R,t) \right],
\]

with drift and diffusion coefficients given by [see Eq. \( \text{[E5]} \) with \( \zeta = 0 \)]

\[
\begin{align}
\mu(R) &\equiv -U'_h(R) + \frac{1}{2} D'(R), \\
D(R) &\equiv T_R [1 + \tilde{\chi}_{kh} m(R)],
\end{align}
\]
where the prime denotes a derivative, \( \tilde{\chi} \), \( m(R) \), and \( U_h(R) \propto h_1 \) are stated in Eqs. (2.11), (3.26), and (3.28a), respectively, and we defined the effective dimensionless coupling constant

\[
\kappa_h \equiv \frac{L h^2 T_\phi}{T_R^2}.
\]

(4.3)

The term \( D'(R)/2 \) in Eq. (4.2a) represents a spurious drift \cite{55}, while the prefactor of \( U'_h(R) \) represents a mobility, which is unity here, i.e., in physical units [see Eq. (2.4a)], the effective and bare mobilities of a passive tracer are identical. As revealed by a comparison of the characteristic relaxation rates of the OP and the tracer, the adiabatic approximation requires not only \( \tilde{\chi} \ll 1 \), but, in fact, \( \tilde{\chi}_h \ll 1 \) [see Eq. (4.10)].

The steady state solution of Eq. (4.1) (with vanishing flux at the boundaries) is given by [see also Eq. (D3)]

\[
\hat{P}_s(R) = \frac{1}{LZ} \sqrt{1 + \tilde{\chi} \kappa_h m(R)} \exp \left[ - \int_0^R \frac{d z}{D(z)} U'_h(z) \right].
\]

(4.4)

The term \( \varepsilon \) is absent for \( h_1 = 0 \). In this case, the normalization constant \( Z \) evaluates for \( a = 0 \) (non-conserved OP dynamics) to \( Z^{(p)} = 1/\sqrt{1 + \tilde{\chi} \kappa_h m^{(p)}} \), \( Z^{(D)} = \ln \left( \frac{12 + 7 \tilde{\chi} \kappa_h + 4 \sqrt{3 \tilde{\chi} \kappa_h (3 + \tilde{\chi} \kappa_h)}}{12 + \tilde{\chi} \kappa_h} \right) / \sqrt{\tilde{\chi} \kappa_h} \), and \( Z^{(N^*)} = 2 \arccot \left( \frac{2}{\sqrt{\tilde{\chi} \kappa_h}} \right) / \sqrt{\tilde{\chi} \kappa_h} \); for \( a = 1 \) and generally for \( h_1 \neq 0 \), \( Z \) has to be calculated numerically. We recall that standard Dirichlet BCs are not compatible with global probability conservation (\( a = 1 \)). In the extreme adiabatic limit \( \tilde{\chi} = 0 \), Eq. (4.4) reduces to

\[
\hat{P}_s(R) = \tilde{Z}^{-1} \exp \left( -U_h(R)/T_R \right), \quad (\tilde{\chi} = 0)
\]

(4.5)

with \( \tilde{Z} = \exp \left( -h_1/(12T_R) \right) \sqrt{\pi T_R h_1} \text{erf} \left( \sqrt{h_1/T_R} \right) \). Equation (4.5) is what one obtain by neglecting the noise \( \Pi_h \) [see Eq. (3.29a)] in Eq. (3.27). Note that \( \hat{P}_s \) in Eq. (4.5) is non-trivial only for capillary BCs, as otherwise \( \hat{P}_s(R) = 1/L \) for \( \tilde{\chi} = 0 \).

It is interesting to contrast the above results to a “naive” derivation of the adiabatic limit of Eq. (3.27): by using Eq. (3.11) in the expression for \( \Pi_h \) in Eq. (3.29a), we obtain, to leading order in \( \chi \), a Gaussian Markovian (white) noise

\[
\Pi_h(R(t), t) \simeq h \sqrt{\chi} \sum_n \frac{1}{\kappa_n + 2 \zeta} \tilde{\sigma}_n(R(t)) \xi(t),
\]

(4.6)

with correlations given by Eqs. (3.25) and (3.30a), i.e.,

\[
(\Pi_h(R(t), t) \Pi_h(R(t'), t')) = 2 T_R \tilde{\chi} \kappa_h m(R) \delta(t - t'),
\]

(4.7)

which are consistent with Eq. (4.2b). However, \( \Pi_h \) is also a multiplicative noise and thus requires specifying a stochastic integration rule \cite{55, 69}. The form of the spurious drift in Eq. (4.2a) suggests a Stratonovich convention, such that the Langevin equation associated with Eq. (4.1) takes the form (see Appendix D)

\[
\partial_t R = -U'_h(R) + \sqrt{D(R)} \tilde{\sigma} \theta, \quad \langle \theta(t) \theta(t') \rangle = 2 \delta(t - t'),
\]

(4.8)

where \( \theta \) is a Gaussian white noise of zero mean.

2. Weak-coupling approximation

In a complementary approach to the adiabatic approximation [Sec. IV A 1], we determine here a perturbative solution in terms of \( h \) for \( \hat{P}(R, t) [\text{Eq. (2.12)}] \). As it turns out, one can thereby capture certain non-Markovian effects neglected in the adiabatic limit. To this end, we regard the first term on the r.h.s. of Eq. (2.10a) as a generic time-dependent force, which allows us to set up the following FPE for the “reduced” probability density \( \hat{P}(R, t) = P(R, R_0, t, t_0) \) of the tracer \cite{25, 27, 55}:

\[
\partial_t \hat{P}(R, t) = -\partial_R \hat{J}(R, t),
\]

(4.9)

with the flux

\[
\hat{J}(R, t) \equiv -T_R \partial_R \hat{P}(R, t) + [h - c \phi(R, t)] [\partial_R \phi(R, t)] \hat{P}(R, t).
\]

(4.10)
Due to the reflective BCs for the tracer, one has

$$\hat{J}_{R \in \{0, L\}} = 0, \quad (4.11)$$

consistent with Eq. (2.13a). Note that the reduced distribution \( \hat{P} \) is distinct from \( \hat{P} \) in Eq. (2.12), since both \( \hat{P} \) and \( \hat{J} \) depend implicitly on the OP field \( \phi(z, t) \), which will be specified below in terms of its correlations [see Sec. III C].

Averaging over the OP fluctuations described by Eq. (2.14), relevant for the passive case, yields an approximation for \( \hat{P} \):

$$\hat{P}(R, t) \simeq \langle \hat{P}(R, t) \rangle_\phi. \quad (4.12)$$

We now take \( c = 0 \) and \( h_1 = 0 \) and formally decompose the reduced distribution as \( \hat{P} = \hat{P}_0 + \hat{P}_1 + \hat{P}_2 + \ldots \), assuming \( \hat{P}_1 \sim \mathcal{O}(h^1) \). (Note that the actual dimensionless control parameter is \( \kappa_h \) [Eq. (4.3)], which can be made explicit by appropriate rescaling.) Due to vanishing boundary fields, we have \( \langle \phi \rangle = 0 \). Accordingly, since Eq. (4.9) (with \( c = 0 \)) is linear in \( \phi \), the first non-trivial influence of \( \phi \) arises at \( \mathcal{O}(h^2) \). Inserting the expansion of \( \hat{P} \) into Eq. (4.9) and collecting terms of the same order in \( h \), renders

$$\begin{align*}
& (\partial_t - T_R \partial_R^2) \hat{P}_0 = \delta(R - R_0) \delta(t - t_0), \\
& (\partial_t - T_R \partial_R^2) \hat{P}_1 = -h \partial_R \left[ (\partial_R \phi(R, t)) \hat{P}_0(R, t) \right], \\
& (\partial_t - T_R \partial_R^2) \hat{P}_2 = -h \partial_R \left[ (\partial_R \phi(R, t)) \hat{P}_1(R, t) \right].
\end{align*} \quad (4.13)$$

The solution of Eq. (4.13a) fulfilling the initial and the no-flux conditions of Eqs. (2.15) and (4.11) can be expressed in terms of Neumann modes [see Eq. III C14] as

$$\begin{align*}
\hat{P}_0(R, R_0, t, t_0) &= \frac{1}{L} + \frac{2}{L} \sum_{n=1}^{\infty} \cos(k_n^{(N)} R) \cos(k_n^{(N)} R_0) \exp(-T_R(k_n^{(N)})^2(t - t_0)) \\
&= \frac{1}{L} \sum_{n=-\infty}^{\infty} \cos(k_n^{(N)} R) \cos(k_n^{(N)} R_0) \exp(-T_R(k_n^{(N)})^2(t - t_0)).
\end{align*} \quad (4.14)$$

Since \( \hat{P} = \hat{P}_0 \) [see Eq. (4.12)] already validates Eq. (2.13) at all \( t \), we require

$$\int_0^L dR \hat{P}_{i \geq 1}(R, t) = 0. \quad (4.15)$$

The solution of the inhomogeneous Fokker-Planck equation in Eq. (4.13b) can be stated in terms of the associated Green’s function \( G \) (determined below) as

$$\hat{P}_1(R, R_0, t, t_0) = -h \int_0^L dy \int_{t_0}^{t} ds \ G(R, y, t - s) \partial_y \left[ (\partial_y \phi(y, s)) \hat{P}_0(y, R_0, s, t_0) \right]. \quad (4.16)$$

We assume no-flux BCs [Eq. (4.11)] to apply also to \( \hat{P}_1 \). Together with Eq. (4.15), this implies that the Green’s function takes the same form as \( \hat{P}_0 \) in Eq. (4.14) except for the absence of the term with \( n = 0 \):

$$G(R, R_0, t - t_0) = \frac{2}{L} \sum_{n=1}^{\infty} \cos(k_n^{(N)} R) \cos(k_n^{(N)} R_0) \exp \left[ -T_R(k_n^{(N)})^2(t - t_0) \right]. \quad (4.17)$$

We remark that the completeness relation in Eq. (3.3) implies

$$G(R, R_0, t \rightarrow t_0) = \delta(R - R_0) - \frac{1}{L}. \quad (4.18)$$

The solution of Eq. (4.13c) follows in an analogous way:

$$\hat{P}_2(R, R_0, t, t_0) = h^2 \int_0^L dy \int_0^L dz \int_{t_0}^{t} du \int_{t_0}^{u} dv \ G(R, y, t - s) \partial_y \left[ G(y, z, s - w) \partial_z \left[ (\partial_y \phi(y, s)) (\partial_z \phi(z, u)) \hat{P}_0(z, R_0, u, t_0) \right] \right]. \quad (4.19)$$
Upon averaging over the fluctuations of $\phi$ [see Eq. (4.12)], we obtain the distributions

$$
\bar{P}_1 = 0,
$$

(4.20a)

$$
\bar{P}_2(R, R_0, t, t_0) = \hbar^2 \int_0^L dy \int_0^L dz \int_{t_0}^t ds \int_{t_0}^s dt \ G(R, y, t - s) \partial_y \{G(y, z, s - u) \partial_z [C_{\phi}(y, z, s - u) P_0(z, R_0, u, t_0)]\},
$$

(4.20b)

with $C_{\phi}$ defined in Eq. (3.16). By means of Eq. (3.18), the expression of Eq. (4.20b) for periodic BCs on $\phi$ can be obtained from the corresponding expressions for Neumann and Dirichlet BCs as

$$
\bar{P}_2^{(p)}(R, R_0, t, t_0) = \frac{1}{2} \left[ \bar{P}_2^{(N)}(R, R_0, t, t_0) + \bar{P}_2^{(D)}(R, R_0, t, t_0) \right]_{L/2}.
$$

(4.21)

In order to evaluate Eq. (4.20b), we use Eqs. (4.14) and (4.17) and subsequently perform the spatial and temporal integrations. For Neumann BCs, the resulting expression is convergent and simplifies to three infinite sums over the mode indices, which can be efficiently evaluated numerically owing to the exponential term. For Dirichlet BCs, instead, one is left with four infinite sums [which can be reduced to three sums in the late-time limit, see Eq. (4.26)], and $\bar{P}_2$ grows logarithmically with the summation cutoff. This divergence appears to be an artifact of the perturbative approach, as it turns out that the shape of $\bar{P}$ still qualitatively describes the numerical results (see Sec. [VI]).

For simplicity, we henceforth set $t_0 = 0$ [see Eq. (2.15)]. In order to determine the late time limit of $\bar{P}_2$, we reorder the time integration variables in Eq. (4.20b) to obtain

$$
\bar{P}_2(R, R_0, t) = \hbar^2 \int_0^L dy \int_0^L dz \int_0^t dv \ G(R, y, v) \partial_y \{G(y, z, w) \partial_z [C_{\phi}(y, z, w) P_0(z, R_0, t - v - w)]\}.
$$

(4.22)

Owing to their exponential time dependence, the functions $G$ [Eq. (4.17)] and $C_{\phi}$ [Eq. (3.16)] are essentially nonzero only for times $v, w \ll k_{\text{min}}^{-2} \sim L^{2+2a}$, where $k_{\text{min}}$ is the minimal wavenumber in the system and we recall that $a = 0 \ (a = 1)$ for dissipative (conserved) OP dynamics. When considering the limit $t \to \infty$, the integrand thus gives substantial contributions only for $v \ll t$ and $w \ll t - v$. Accordingly, we may replace $P_0$ [Eq. (4.14)] by its late-time limit,

$$
\bar{P}_0_{t \to \infty} = \frac{1}{L},
$$

(4.23)

which is independent of position. This allows us to extend the upper limit of the $w$ integral from $t - v$ to $t$, resulting in

$$
\bar{P}_2(R, R_0, t \to \infty) = \hbar^2 \int_0^L dy \int_0^L dz \int_0^\infty dv \ G(R, y, v) \partial_y \{G(y, z, w) \partial_z [C_{\phi}(y, z, w)]\}.
$$

(4.24)

The time and space integrals can be performed explicitly, rendering, for Neumann BCs on the OP:

$$
\bar{P}_2^{(N)}(R, R_0, t \to \infty) = \frac{\kappa_h}{2 L \pi^2} \sum_{l=1}^\infty \frac{\cos(2\pi l R/L)}{l^2(1 + \frac{\pi^2 l^2}{4 \kappa_h^2})},
$$

(4.25)

with the effective dimensionless coupling constant $\kappa_h$ defined in Eq. (4.13). The weak-coupling approximation applies to the regime $\kappa_h \ll 1$. For Dirichlet BCs (which are relevant only for $a = 0$), instead, we obtain

$$
\bar{P}_2^{(D)}(R, R_0, t \to \infty) = \frac{\kappa_h}{L \pi^2} \sum_{l=1}^N \sum_{m=1}^N \sum_{k=1}^N \left[ \frac{k - m}{k - m + \mathcal{J}_{k-m}} + \frac{k - m}{k + m} \right] \cos\frac{\pi l R}{L} \frac{\cos(\pi l R/L)}{l^2(m^2 + k^2/(\chi T_R))},
$$

(4.26)

where the function $\mathcal{J}_n$ arises from the spatial integrals and is given by

$$
\mathcal{J}_n = \begin{cases} 1, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}
$$

(4.27)
3. Discussion

Figure 4 illustrates the steady-state distribution $\bar{P}_s(R)$ for a linearly coupled passive tracer obtained within the adiabatic [Eq. (4.4), panel (a)] and the weak-coupling approximation [Eq. (4.28), panel (b)]. Here and in the following, we use a notation such as (D) for Neumann BCs of the OP. The time evolution of $\bar{P}_s$ is found to be slightly faster for dissipative (a) than for conserved dynamics (b). The dissipative dynamics for Dirichlet BCs follows analogously, but does not exhibit any new features and is thus not shown.

If periodic BCs are imposed on the OP, Eq. (4.2) reduces to $\mu = 0$ and $D = T_R$, such that Eq. (4.4) describes a simple diffusion process subject to reflective BCs, which has $\bar{P}_s = 1/L$ as steady-state solution. However, numerical simulations (not shown) reveal this to be an artifact of the present order of the adiabatic approximation. In fact,
within the weak-coupling approximation, Eq. (4.21) implies that the behavior of the tracer is similar for periodic, Neumann or Dirichlet BCs. The adiabatic and the weak-coupling approximations need not necessarily agree in the limit \( \tilde{\chi} \ll 1 \), because, in the former approach the field \( \phi \) is interpreted as a noise, whereas in the latter, it is formally regarded as a time-dependent potential.

**B. Quadratic tracer-field coupling**

1. **Adiabatic approximation**

In the case of a quadratically coupled passive tracer, the Langevin equation in Eq. (3.27) (with \( h = 0 \)) reduces to

\[
\dot{R}(t) = -\partial_R U_c(R(t)) + \Pi_c(R(t), t) + \eta(t),
\]

where the effective potential \( U_c \) and the noise \( \Pi_c \) are reported in Eqs. (3.28b) and (3.29b), respectively. We apply the adiabatic elimination procedure described in Refs. \[55, 72\] in order to obtain an effective FPE for the tracer distribution \( \bar{P}(R, t) \). We furthermore assume the coupling \( c \) to be small, such that \( \Pi_c \) can be approximated as a Gaussian Markovian white noise with correlation (see Appendix F)

\[
\langle \Pi_c(R(t), t) \Pi_c(R(t'), t') \rangle \simeq 2\mathcal{P}(R)\delta(t - t'),
\]

where the amplitude \( \mathcal{P} \sim \mathcal{O}(c^2\chi) \) is specified in Eq. (F3). Within these approximations, the FPE associated with Eq. (4.29) follows as (see §4.8 in Ref. \[72\])

\[
\partial_t \bar{P}(R, t) = -\partial_R \left[ \mu(R) \bar{P}(R, t) \right] + \partial^2_R \left[ D(R) \bar{P}(R, t) \right],
\]

with the effective drift and diffusion coefficients

\[
\mu(R) \equiv -\partial_R U_c(R) + \frac{1}{2} \partial_R \mathcal{P}(R),
\]

\[
D(R) \equiv T_R + \mathcal{P}(R).
\]

The spurious drift term \((1/2)\mathcal{P}'(R)\) in Eq. (4.32a) indicates a Stratonovich character of the noise \( \Pi_c \) in Eq. (4.29) \[55\]. The steady-state distribution resulting from Eq. (4.31) is given by [see also Eq. (D3)]

\[
\bar{P}_s(R) = \frac{1}{Z} e^{-\mathcal{V}(R)}, \quad \mathcal{V}(R) \equiv \ln D(R) - \int_0^R dz \frac{\mu(z)}{D(z)} \simeq \frac{c}{2T_R} \left[ V_\phi(R) + (\phi(R))^2 \right] + \mathcal{O}(c^2\chi) = \frac{U_c(R)}{T_R},
\]

where the amplitude \( U_c \) is specified in Eq. (D3).
where we expanded the potential \( V \) up to \( \mathcal{O}(\epsilon^0) \) and disregarded any \( R \)-independent constants. Specializing \( \tilde{P}_s \) to the various BCs results in \( (\rho \equiv R/L) \)

\[
\tilde{P}_s(R) = \begin{cases} \\
\frac{1}{2} \kappa \sqrt{\pi} \exp \left[ \frac{\kappa}{2} \left( \frac{1}{2} - \rho \right)^2 \right], & (p^*) \\
\frac{1}{2} \kappa \sqrt{\pi} \exp \left[ - \frac{\kappa}{2} \left( \frac{1}{2} - \rho \right)^2 \right], & (D) \\
\frac{1}{2} \kappa \sqrt{\pi} \exp \left[ - \frac{\kappa}{2} \left( \frac{1}{6} - \rho \right)^2 \right], & (N^*) \\
\frac{1}{2} \kappa \sqrt{\pi} \exp \left[ - \frac{\kappa}{2} \left( \frac{1}{6} - \rho + \rho^2 \right) \right], & (\pm) 
\end{cases}
\]

(4.34)

with the dimensionless effective coupling

\[
\kappa_c \equiv \frac{c L T_\phi}{T_R}.
\]

(4.35)

The above assumption of a weak coupling implies that \( \tilde{P}_s \) in Eq. (4.34) is correct only to \( \mathcal{O}(\kappa_c) \) [see also Eq. (4.40) below]. Note that Eq. (4.34) is independent of the parameter \( \kappa \) [see Eq. (2.4b)], as the effect of the conservation law enters at \( \mathcal{O}(\kappa_c^2) \) [see Eq. (1.33)].

2. Weak-coupling approximation

Here, analogously to Sec. [IV A 2], we develop a perturbative solution of the tracer dynamics by assuming a weak coupling \( c \) [the actual dimensionless control parameter being \( \kappa_c \), see Eq. (4.35)]. We thus consider Eq. (4.9) with \( h = 0 \) and formally expand the reduced tracer distribution as \( \tilde{P} = \tilde{P}_0 + \tilde{P}_1 + \tilde{P}_2 + \ldots \), with \( \tilde{P}_i \sim \mathcal{O}(c^i) \). Upon inserting this expansion into Eq. (4.9), one obtains the hierarchy

\[
\begin{align*}
(\partial_t - T_R \partial_t^2) \tilde{P}_0 &= \delta(R - R_0) \delta(t - t_0), \\
(\partial_t - T_R \partial_t^2) \tilde{P}_1 &= c \partial_R \left[ \phi(R, t) (\partial_R \phi(R, t)) \tilde{P}_0(R, t) \right], \\
(\partial_t - T_R \partial_t^2) \tilde{P}_2 &= c \partial_R \left[ \phi(R, t) (\partial_R \phi(R, t)) \tilde{P}_1(R, t) \right].
\end{align*}
\]

(4.36a–c)

The solution for \( \tilde{P}_0 \) is given in Eq. (4.14). For the leading correction, we obtain, analogously to Eq. (4.16):

\[
\tilde{P}_1(R, R_0, t, t_0) = c \int_0^L dy \int_{t_0}^t ds G(R, y, t - s) \partial_y \left[ \phi(y, s) (\partial_y \phi(y, s)) \tilde{P}_0(y, R_0, s, t_0) \right],
\]

(4.37)

with the Green function \( G \) reported in Eq. (4.17). The \( \mathcal{O}(c^2) \) contribution to \( \tilde{P} \) follows along the same lines from Eq. (4.36c). Averaging over the fluctuations of \( \phi \) [see Eq. (4.12)] and using their time-translation invariance, renders

\[
\begin{align*}
\tilde{P}_1(R, R_0, t, t_0) &= \frac{1}{c} \int_0^L dy \int_{t_0}^t ds G(R, y, t - s) \partial_y \left[ P_0(y, R_0, s, t_0) \partial_y U_c(y) \right], \\
\tilde{P}_2(R, R_0, t, t_0) &= \int_0^L dy \int_0^L dz \int_{t_0}^t ds \int_{t_0}^t du G(R, y, t - s) \partial_y \left\{ G(y, z, s - u) \partial_z \left[ \langle \Xi_c(y, s) \Xi_c(z, u) \rangle P_0(z, R_0, u, t_0) \right] \right\},
\end{align*}
\]

(4.38a–b)

with the effective potential \( U_c \) stated in Eq. (3.28b) and \( \langle \Xi_c(y, s) \Xi_c(z, u) \rangle = \langle \Pi_c(y, s) \Pi_c(z, u) \rangle + \langle \Xi_c(y) \Xi_c(z) \rangle \) [see Eqs. (3.29b) and (3.30b)].

While Eq. (4.38) has to be evaluated numerically in the general case, the steady-state distribution \( \tilde{P}_s \) up to \( \mathcal{O}(c) \) can be readily determined from Eqs. (4.36a) and (4.36b): the former yields \( \tilde{P}_{s,0} = 1/L \), whereas the latter reduces, after averaging over the field, to [see Eq. (3.28b)]

\[
T_R \partial_R^2 \tilde{P}_{s,1} = - \frac{1}{L} \partial_R^2 U_c(R).
\]

(4.39)

Upon imposing the no-flux condition [Eq. (4.11)], we obtain

\[
\tilde{P}_s = \frac{1}{L} \left[ \alpha - \frac{U_c(R)}{T_R} \right] = \begin{cases} \\
\frac{1}{L} \left[ 1 + \frac{\kappa}{2} \left( \frac{1}{6} - \rho + \rho^2 \right) \right], & (p) \\
\frac{1}{L} \left[ 1 - \frac{\kappa}{2} \left( \frac{1}{6} - \rho + \rho^2 \right) \right], & (D) \\
\frac{1}{L} \left[ 1 + \frac{\kappa}{2} \left( \frac{1}{3} - \rho + \rho^2 \right) \right], & (N^*) \\
\frac{1}{L} \left[ 1 + \frac{\kappa}{2} \left( \frac{1}{3} - \rho + \rho^2 \right) \right], & (\pm)
\end{cases}
\]

(4.40)
FIG. 6. Time evolution of the probability distribution $\bar{P}(R,R_0 = L/2,t,t_0 = 0)$ [solid curves, corresponding to various times] at $\mathcal{O}(\epsilon\chi^3)$ of a passive tracer in the adiabatic limit coupled quadratically to an OP with Dirichlet BCs. The distributions are obtained by numerically solving the FPE in Eq. (4.31) with an initial condition $P(R,R_0,t_0,t_0) = \delta(R - R_0)$ and using an effective coupling $\kappa_c = 1$ [Eq. (4.35)]. The thick black line represents the steady-state distribution $\bar{P}_s(D)(R)$ given in Eq. (4.34), which to $\mathcal{O}(\kappa_c^2)$ coincides with the weak-coupling expression in Eq. (4.40). By contrast, if the OP fulfills Neumann BCs, the steady-state probability is enhanced in the center of the system and reduced at the boundaries [see Eq. (4.33) and Fig. 3(b)].

where the integration constant $\alpha$ is fixed via Eq. (4.15) and we introduced the dimensionless coupling

$$H_1 \equiv h_1\sqrt{L/T\phi}.$$ (4.41)

These expressions agree with the ones obtained from the long-time limit of Eq. (4.38a) (not shown) and from the expansion of Eq. (4.33) to $\mathcal{O}(\kappa_c)$. A dependence of $\bar{P}_s$ on $\chi$ and $a$, i.e., the conservation law [see Eq. (2.4b)], is introduced at $\mathcal{O}(\kappa_c^2)$ via the correlation function of $\Xi_c$ [Eq. (3.30b)] present in Eq. (4.38b).

3. Discussion

If non-symmetry-breaking BCs are imposed on the OP, in both the adiabatic [Eq. (4.31)] and the weak-coupling approximation [Eq. (4.36b)] the tracer follows, to leading order in $\kappa_c$ and $\tilde{\chi}$, a Brownian motion in a quadratic potential $U_c(R)$ subject to reflective BCs [see Eqs. (3.20) and (3.27) and Fig. 2(a)]. For Neumann BCs, this is, in fact, a confined Ornstein-Uhlenbeck process. While the time-dependent solution for such processes can in principle be determined analytically [73], the resulting expression is rather involved and we instead present in Fig. 6 a numerical solution of Eq. (4.31). Since the behavior of $(N_cN)$ and $(D_cD)$ is qualitatively similar, only the latter case is shown in the figure.

According to Eqs. (3.20) and (4.33), at late times the occupation probability of the tracer is enhanced in the center (at the boundaries) of the system if the OP is subject to Neumann (Dirichlet) BCs. Notably, the dynamics of $\bar{P}$ in the case $(D_cD)$ is similar to the one for $(N_cN)$ [see Fig. 5(a)], analogously, $(N_cN)$ is similar to $(D_cD)$. This is a consequence of the formal similarity between the expressions for $V_c$ [Eq. (3.20)] and $m(R)$ [Eq. (3.25)], which enter the FPEs in Eq. (4.1) and Eq. (4.31), respectively. We remark that, up to the order considered in Eq. (4.39), effects of the interaction between the tracer and the field at different times are neglected. In Eq. (4.31), this is a direct consequence of the Markovian assumption for $\Pi_c$. The uniform distribution for periodic BCs predicted by both the adiabatic and the weak-coupling approach [see Eqs. (4.34) and (4.40)] are considered to be an artifact of the low order of the approximations used here, as numerical simulations (not shown) reveal a non-uniform steady-state distribution.

For capillary BCs, the tracer behaves to leading order as a Brownian particle in an effective potential $U_c$ [see Eq. (3.28b) and Fig. 3]. The latter is a fourth-order polynomial and shows a crossover from a unimodal shape ($|H_1| \ll 1$) with a minimum at the center of the system, to a bimodal shape ($|H_1| \gg 1$) with minima at

$${R_{\pm}} = \frac{L}{2} \pm \frac{\sqrt{1 - 6/H_1^2}}{2\sqrt{3}}, \quad (|H_1| \gg 1)$$ (4.42)

V. REACTIVE TRACER

We now turn to the discussion of a reactive tracer, which is described by Eq. (2.10) with $\zeta = 1$. Since Eq. (2.10) satisfies in this case a fluctuation-dissipation relation for all degrees of freedom [10, 11, 74], the resulting joint stea-
state distribution for $\phi$ and $R$ is provided by Eqs. (2.1) and (2.2). A reactive tracer represents a simplified model for a colloid in a fluid. While previous studies of the equilibrium behavior of a single confined colloid considered spatially resolved particles in a half-space [11][12] or in a slit geometry [75][77], we focus here on a point-like particle under strong confinement. The confinement induces long-ranged interactions between the colloid and both boundaries (walls). Further connections to previous studies are discussed in Sec. VI B 2.

In Sec. V A, we determine the equilibrium distribution of the tracer for various couplings and OP BCs. Besides being interesting in its own right, the equilibrium distributions provide a means to independently check the steady-state solutions obtained from the effective FPEs derived in Sec. V B. While we focus here on a one-dimensional system, equilibrium distributions of a tracer in a three-dimensional slit are presented in Appendix G.

A. Equilibrium tracer distribution

The equilibrium (steady-state) probability distribution for the tracer follows by marginalizing the joint distribution in Eq. (2.7) over the OP field $\phi$ [see Eq. (2.12)]:

$$\bar{P}_s(R) = \frac{1}{Z} \int \mathcal{D}\phi P_s(R, \phi) = \frac{1}{Z} \int \mathcal{D}\phi \exp \left\{ -\frac{1}{2T} \int_0^L d\xi \left( \frac{1}{2} (\partial_\xi \phi)^2 - h_1 (\phi(0) + \phi(L)) \right) \right\},$$

where the normalization factor $Z$ is determined by the requirement $\int_0^L dR \bar{P}_s(R) = 1$ [see Eq. (2.13)]. For generality, we keep here different temperatures for the tracer ($T_h$) and the field ($T_\phi$), although they essentially only rescale the coupling parameters. Boundary fields $h_1$ give rise to an inhomogeneous mean profile $\langle \phi(z) \rangle$ of the OP field [see Eq. (3.8)].

Owing to the Gaussian nature of Eq. (5.1), we can integrate out the field degrees-of-freedom by inserting the orthonormal transformation in Eq. (3.4), which induces a unit Jacobian. The actual integration is performed over the modes $\phi_n$ with the aid of the following standard result for multidimensional Gaussian integration [78][79]:

$$\int \mathcal{D}\phi \exp \left\{ -\frac{1}{2} \sum_{ij} \phi_i \Gamma_{i,j} \phi_j^* + \sum_i K_i^* \phi_i \right\} = \frac{(2\pi)^{N/2}}{\det \Gamma^{1/2}} \exp \left( \frac{1}{2} \sum_{i,j} K_i \Gamma_{i,j}^{-1} K_j^* \right),$$

where $\Gamma$ is a $N \times N$ matrix and $K_i$ is a given field. Complex conjugation is relevant only for periodic BCs, in which case the modes $\phi_n$ and $K_n$ are complex-valued with $\phi_n^* = \phi_{-n}$ (analogously for $K_n$) and the integration measure has to be suitably chosen [78][80]. A possible zero mode, occurring for periodic and Neumann BCs, can be regularized by replacing the vanishing eigenvalue $k_0$ [see Eq. (3.1)] by a nonzero parameter $\varepsilon [81]$, which is set to zero in the end of the calculation [see also Appendix G].

1. System without boundary fields

We consider first a system without boundary fields, i.e., $h_1 = 0$. In the case of a linear coupling between OP and tracer (i.e., $h \neq 0$, $c = 0$), Eq. (5.1) evaluates to

$$\bar{P}_s(R)|_{c=0} = \frac{1}{Z} \exp \left[ \frac{h^2}{2TR} V_\phi(R) \right] = \frac{1/L}{\sqrt{\kappa_h}} \exp \left[ \frac{\kappa_h}{2} \left( \frac{1}{2} - \rho \right)^2 \right],$$

where $V_\phi(R) = \langle \phi^2 \rangle$ is reported in Eq. (3.20) and $\kappa_h$ in Eq. (4.3). The zero mode of the OP provides an $R$-dependent constant in the exponential and thus plays no role for the distribution [this is different if $c \neq 0$, see Eq. (5.5) below]. In the limit $\kappa_h \to 0$, Eq. (5.3) reduces to $\bar{P}_s(R) \simeq 1/L$, as expected. In the limit $\kappa_h \to \infty$, instead, using the asymptotic behavior $\text{erf}(x \to \infty) \simeq \exp(x^2)/\sqrt{\pi x}$, the tracer becomes highly localized for Neumann or Dirichlet BCs on the OP:

$$\bar{P}_s(R)|_{c=0, \kappa_h \to \infty} = \frac{1/L}{\delta(R - L/2)}.$$

In the linear coupling case, we keep here different temperatures for the tracer ($T_h$) and the field ($T_\phi$), although they essentially only rescale the coupling parameters. Boundary fields $h_1$ give rise to an inhomogeneous mean profile $\langle \phi(z) \rangle$ of the OP field [see Eq. (3.8)].

Owing to the Gaussian nature of Eq. (5.1), we can integrate out the field degrees-of-freedom by inserting the orthonormal transformation in Eq. (3.4), which induces a unit Jacobian. The actual integration is performed over the modes $\phi_n$ with the aid of the following standard result for multidimensional Gaussian integration [78][79]:

$$\int \mathcal{D}\phi \exp \left\{ -\frac{1}{2} \sum_{ij} \phi_i \Gamma_{i,j} \phi_j^* + \sum_i K_i^* \phi_i \right\} = \frac{(2\pi)^{N/2}}{\det \Gamma^{1/2}} \exp \left( \frac{1}{2} \sum_{i,j} K_i \Gamma_{i,j}^{-1} K_j^* \right),$$

where $\Gamma$ is a $N \times N$ matrix and $K_i$ is a given field. Complex conjugation is relevant only for periodic BCs, in which case the modes $\phi_n$ and $K_n$ are complex-valued with $\phi_n^* = \phi_{-n}$ (analogously for $K_n$) and the integration measure has to be suitably chosen [78][80]. A possible zero mode, occurring for periodic and Neumann BCs, can be regularized by replacing the vanishing eigenvalue $k_0$ [see Eq. (3.1)] by a nonzero parameter $\varepsilon [81]$, which is set to zero in the end of the calculation [see also Appendix G].

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where $V_\phi(R) = \langle \phi^2 \rangle$ is reported in Eq. (3.20) and $\kappa_h$ in Eq. (4.3). The zero mode of the OP provides an $R$-dependent constant in the exponential and thus plays no role for the distribution [this is different if $c \neq 0$, see Eq. (5.5) below]. In the limit $\kappa_h \to 0$, Eq. (5.3) reduces to $\bar{P}_s(R) \simeq 1/L$, as expected. In the limit $\kappa_h \to \infty$, instead, using the asymptotic behavior $\text{erf}(x \to \infty) \simeq \exp(x^2)/\sqrt{\pi x}$, the tracer becomes highly localized for Neumann or Dirichlet BCs on the OP:

$$\bar{P}_s(R)|_{c=0, \kappa_h \to \infty} = \frac{1/L}{\delta(R - L/2)}.$$
Equation (5.3) is illustrated in Fig. 7(a). If the OP is subject to Dirichlet BCs, the probability of the tracer is enhanced at the center of the system and reduced at the boundaries, while the trends are opposite in the case of Neumann BCs. These behaviors are qualitatively similar to a passive tracer in the adiabatic limit (see Fig. 4(a)).

This can be intuitively explained based on the dynamical coupling \( \partial_t \phi(R) \) [see Eq. (2.10a)], which acts as an effective noise [see Eq. (3.27)] in the Langevin equation: for Neumann (Dirichlet) BCs, this noise is strongest at the center (boundaries) and thus drives the tracer towards the boundaries (center). Note, however, that this effect can be overwhelmed by non-Markovian contributions, responsible for the generally strong attraction of a passive tracer towards the boundaries (see discussion in Sec. VI).

For a tracer quadratically coupled to the OP (i.e., \( c \neq 0, h = 0 \)), Eq. (5.1) evaluates to

\[
\bar{P}_s(R) \big|_{h=0} = \frac{1}{Z} \left[ \det \Gamma(R) \right]^{-1/2} = \begin{cases} \frac{1}{L'}, & (p), (N) \\ \frac{1}{Z} \left[ 1 + \frac{c}{T_R} V_\phi(R) \right]^{1/2}, & (D), (N^*). \end{cases}
\] (5.5)

where the matrix \( \Gamma \) is given by

\[
\Gamma_{n,m} = \frac{1}{T_\phi} k_n^2 \delta_{n,m} + \frac{c}{T_R} \sigma_n(R) \sigma^*_m(R).
\] (5.6)

and the continuum limit of its determinant is calculated in Appendix E. Upon normalization, Eq. (5.5) results in

\[
\bar{P}_s(R) \big|_{h=0} = \begin{cases} \frac{1}{L'}, & (p), (N) \\ \frac{1}{2L \arctan(\sqrt{\kappa_c}/2) \sqrt{\kappa_c^{-1} + \rho - \rho^2}}, & (D) \\ \frac{L \sqrt{\kappa_c^{-1} + 1/3 - \rho + \rho^2 \ln \left( \frac{12 + 7 \kappa_c + 4 \sqrt{3 \kappa_c (3 + \kappa_c)}}{12 + \kappa_c} \right)}}{2L \arctan(\sqrt{\kappa_c}/2) \sqrt{\kappa_c^{-1} + \rho - \rho^2}}, & (N^*) \end{cases}
\] (5.7)

with the dimensionless coupling constant \( \kappa_c \) in Eq. (4.35). In the limit \( \kappa_c \to \infty \), the tracer effectively imposes Dirichlet BCs on the OP field, such that Eq. (5.7), becoming independent of \( \kappa_c \), reduces to

\[
\bar{P}_s(R) \big|_{h=0, \kappa_c \to \infty} = \begin{cases} \frac{1}{L'}, & (p), (N) \\ \frac{1}{\pi L \sqrt{\rho (1 - \rho)}} \ln \left( \frac{12 + 7 \kappa_c + 4 \sqrt{3 \kappa_c (3 + \kappa_c)}}{12 + \kappa_c} \right), & (D) \\ \frac{\arccosh(7)}{2L \sqrt{1/3 - \rho + \rho^2}}, & (N^*) \end{cases}
\] (5.8)

Accordingly, if the OP is subject to Dirichlet (Neumann) BCs, a quadratically coupled tracer is most likely to be found at the boundaries (center) of the system, as illustrated in Fig. 7(b). This behavior is similar to the passive case (see Eq. 4.34 and Fig. 6)
2. System with boundary fields

We assume now boundary fields of equal strength \( h_1 \) to act on the OP. Upon performing in Eq. (5.1) the Gaussian integration over \( \phi \) and omitting all \( R \)-independent constants, one obtains the following equilibrium distribution for a linearly coupled tracer \( (c=0) \):

\[
\bar{P}_s(R)|_{c=0} = \frac{1}{2} \exp \left\{ \frac{T_\phi}{2} \sum_{n=1}^{\infty} \frac{1}{k_n^2} \left[ \tilde{h}^2 \sigma_n^2(R) + 2\tilde{h}_1 \sigma_n(R)(\sigma_n(0) + \sigma_n(L)) \right] \right\}
\]

\[
= \frac{1}{2} \exp \left\{ \frac{1}{2} \tilde{h}^2 V_{\phi}^{(N)}(R) + \tilde{h}_1 \left[ C_\phi^{(N)}(0,R) + C_\phi^{(N)}(L,R) \right] \right\}
\]

with \( \tilde{h} \equiv h/T, \tilde{h}_1 \equiv h_1/T, \) and \( C_\phi^{(N)} \) and \( V_{\phi}^{(N)} \) reported in Eqs. (3.19) and (3.20), respectively. Note that the term in the last square brackets essentially simplifies to \( \langle \phi(R) \rangle_{\tilde{h}_1} \) [see Eq. (3.11)], such that this part of the distribution resembles Eq. (4.5) of the passive case. The normalized tracer probability distribution is given by (with \( \rho \equiv R/L \))

\[
\bar{P}_s(R) = \frac{\sqrt{k_{h_1}}}{L \sqrt{2\pi} \text{erf} \left( \frac{\sqrt{k_{h_1}}}{2 \sqrt{2}} \right)} \exp \left[ \frac{k_{h_1}}{2} \left( \frac{1}{2} - \rho \right)^2 \right], \quad (++)\.,(---) \quad (5.10)
\]

with the dimensionless effective coupling

\[
k_{h_1} \equiv T_\phi L \tilde{h}(\tilde{h} + \tilde{h}_1), \quad (5.11)
\]

which generalizes Eq. (4.3). We use a notation such as \( (+++) \) to indicate the presence of two boundary fields with \( h_1 > 0 \) and a tracer field \( h < 0 \). When \( \tilde{h} \) and \( \tilde{h}_1 \) are such that \( k_{h_1} > 0 \), \( \bar{P}_s(R) \) is largest at the boundaries [see Fig. 8(a)].

This attractive interaction between wall and tracer is, in fact, characteristic for an effective \((++)\) boundary condition between wall and tracer (see Sec. VI B 2). For large values of \( k_{h_1} \), one has \( \bar{P}_s(R) \sim \exp \left( -T_\phi \tilde{L} \tilde{h}\tilde{h}_1(1-R/L)R/L \right) \), which implies, in particular, that \( \bar{P}_s \approx \frac{1}{2} \delta(R) + \delta(L-R) \) in the limit \( h_1 \to \infty \).

The case \( k_{h_1} < 0 \) corresponds to an effective \((-+)\) boundary condition between wall and tracer and applies if \( h \) and \( h_1 \) have opposite signs and fulfill \( |h| < 2|h_1| \). In this case, one obtains

\[
\bar{P}_s(R) = \frac{\sqrt{k'_{h_1}}}{L \sqrt{2\pi} \text{erf} \left( \frac{\sqrt{k'_{h_1}}}{2 \sqrt{2}} \right)} \exp \left[ \frac{-k'_{h_1}}{2} \left( \frac{1}{2} - \rho \right)^2 \right], \quad (--)\.,(++) \quad (5.12)
\]

with \( k'_{h_1} \equiv T_\phi L |\tilde{h}|(2|\tilde{h}_1| - |\tilde{h}|) \). This distribution has its maximum at \( L/2 \), indicating a repulsion between wall and tracer [see Fig. 8(a)]. In fact, in the limit \( k'_{h_1} \to \infty \), the tracer is strongly localized at the center, \( \bar{P}_s(R) \approx \delta(R-L/2) \).
If the two boundary fields have opposite signs, we obtain asymmetric distributions, which, for sufficiently large $|h_1|$, are monotonous and have a maximum at one boundary.

In the case of a quadratically coupled tracer with non-vanishing boundary fields (i.e., $h = 0$, $h_1 \neq 0$), Eq. (5.1) yields

$$
\mathcal{P}_s(R) \big|_{h=0} = \frac{1}{Z} \left[ \det \mathbf{\Gamma}(R) \right]^{-1/2} \exp \left\{ \frac{h^2}{2T_\phi} \sum_{n,m} \left[ \sigma_n(0) + \sigma_n(L) \right] \Gamma(R)_{n,m} \left[ \sigma_m(0) + \sigma_m(L) \right] \right\},
$$

(5.13)

where the determinant and the exponential of the matrix $\mathbf{\Gamma}$ [Eq. (5.6)] are calculated in Appendix G2 [see Eqs. (G6) and (G10)]. Taking, as required in this case, $\sigma_n$ to be Neumann modes without a zero mode, one obtains

$$
\mathcal{P}_s(R) \big|_{h=0} = \frac{1}{Z} \frac{1}{L \sqrt{1 + \kappa_c \left( \frac{1}{3} - \rho + \rho^2 \right)}} \exp \left\{ -\frac{1}{2} H_1^2 \kappa_c \left( \frac{1}{3} - \rho + \rho^2 \right)^2 \right\},
$$

(5.14)

where the dimensionless couplings $\kappa_c$ and $H_1$ are reported in Eqs. (4.35) and (4.41), respectively, and the normalization factor $Z = \bar{Z}/L$ has to be calculated numerically. If a zero mode is present, one obtains a flat distribution, $\mathcal{P}_s(R) |_{h=0} = 1/L$, instead [see Eq. (G11)]. In the limit $\kappa_c \to \infty$, the OP obeys Dirichlet BCs at the tracer location and the distribution in Eq. (5.14) becomes independent of $\kappa_c$, reducing to

$$
\mathcal{P}_s(R) \big|_{h=0, \kappa_c \to \infty} = \frac{1}{Z} \frac{1}{L \sqrt{1 + \rho + \rho^2}} \exp \left\{ -\frac{1}{2} H_1^2 \left( \frac{1}{3} - \rho + \rho^2 \right)^2 \right\},
$$

(5.15)

Upon increasing the parameter $H_1$, the distributions in Eqs. (5.14) and (5.15) show a cross-over [see Fig. 8b)] from a single- to a double-peaked shape with peaks located at

$$
\frac{R_+}{L} = \frac{1}{2} \pm \frac{1}{2\sqrt{3}}. \quad (|H_1| \gg 1)
$$

(5.16)

In the limit $H_1 \to \infty$, Eqs. (4.42) and (5.16) coincide and $\mathcal{P}_s$ reduces to a sum of two Dirac-$\delta$ functions at $R_{\pm}$.

3. Many tracers

The above results can be straightforwardly generalized to $N_p > 1$ tracers in the system. To this end, the terms in the square brackets in Eq. (5.1) are replaced according to $h\phi(R) \to \sum_{j=1}^{N_p} \tilde{h}_j \phi(R_j)$ and $c\phi^2(R) \to \sum_{j=1}^{N_p} c_j \phi^2(R_j)$, where $\tilde{h}_j$ and $c_j$ denote the coupling constants pertaining to the $j$th tracer. As before, boundary fields of equal strength $\tilde{h}_1 = h_1/T_\phi$ act on the OP $\phi$. In the case of a linear coupling between each tracer and the OP, Eq. (5.9) generalizes to

$$
\mathcal{P}_s(R_1, R_2) \big|_{c=0} = \frac{1}{Z} \exp \left\{ \frac{T_\phi}{2} \sum_{n=1}^{N_p} \left[ \sum_{j=1}^{N_p} \tilde{h}_j \sigma_n(R_j) + \tilde{h}_1 (\sigma_n(0) + \sigma_n(L)) \right]^2 \right\}
$$

(5.17)

$$
= \frac{1}{Z} \exp \left\{ \frac{1}{2} \sum_{i,j=1}^{N_p} \tilde{\sigma}_i \tilde{\sigma}_j C_\phi(R_i, R_j) + \tilde{\sigma}_1 \sum_{j=1}^{N_p} \tilde{h}_j \left[ C_\phi(R_j, 0) + C_\phi(R_j, L) \right] \right\},
$$

with $\tilde{\sigma}_i \equiv \tilde{h}_i/T_R$. In the final result, we omitted all terms independent of $R_k$, as they are canceled by the normalization. While the sums in Eq. (5.17) can be calculated for any number of tracers, we focus in the following on $N_p = 2$ and couplings of equal magnitude $|\tilde{h}_j| = \tilde{h}$. In the case of Dirichlet BCs on $\phi$, the contributions associated with the boundary field $h_1$ identically vanish and one is left with

$$
\mathcal{P}_s(R_1, R_2) \big|_{c=0} = \begin{cases} 
\frac{1}{Z} \exp \left\{ -\frac{1}{2} T_\phi \tilde{h}^2 \left[ (\rho_1 + \rho_2)^2 + |\rho_1 - \rho_2| \right] \right\}, & (D++D) \\
\frac{1}{Z} \exp \left\{ -\frac{1}{2} T_\phi \tilde{h}^2 \left[ (\rho_1 - \rho_2)^2 - |\rho_1 - \rho_2| \right] \right\}, & (D+-D)
\end{cases}
$$

(5.18)
FIG. 9. Joint equilibrium probability distribution \( \bar{p}_s(\rho_1, \rho_2) = L^2 \bar{P}_s(R_1 = \rho_1 L, R_2 = \rho_2 L) \) [Eqs. (5.18) and (5.19)] of two reactive tracers (with positions \( R_1, R_2 \)) confined to an interval \([0, L]\) and coupled linearly to a critical OP field. Up to a sign (indicated in the captions), the value of the coupling \( h \) is the same for the two tracers. The OP field obeys (a,b) Dirichlet, (c,d) Neumann, or (e,f) capillary BCs [see Eq. (3.9)].

with \( \rho_k \equiv R_k/L, \tilde{h} = h/T_R \), and a normalization factor \( Z \) that can be calculated analytically. In the case of capillary BCs on \( \phi \) (which, as before, requires taking the \( \sigma_n \) to be Neumann modes), the joint probability distribution of the two tracers is given by

\[
\bar{P}_s(R_1, R_2) |_{c=0} = \begin{cases} 
\frac{1}{2} \exp \left\{ T_0 L \left[ \tilde{h} \tilde{h}_1 \left( -\rho_1 + \rho_1^2 - \rho_2 + \rho_2^2 \right) + \tilde{h}^2 \left( -\rho_1 + \rho_1^2 - \rho_2 + \rho_2^2 - \frac{1}{2} |\rho_1 - \rho_2| \right) \right] \right\}, & (++++) \\
\frac{1}{2} \exp \left\{ T_0 L \left[ \tilde{h} \tilde{h}_1 \left( -\rho_1 + \rho_1^2 + \rho_2 - \rho_2^2 \right) + \frac{1}{2} \tilde{h}^2 |\rho_1 - \rho_2| \right] \right\}, & (+--+) 
\end{cases}
\]

When the OP obeys Neumann BCs, the distribution follows by setting \( h_1 = 0 \) in this expression.

Figure 9 illustrates the probability distribution of two linearly coupled reactive tracers in confinement. The essential features can be readily understood based on the behavior of a single tracer [see Figs. 7(a) and 8(a)]. To this end, we first note that two tracers having couplings of the same (opposite) sign attract (repel) each other. This behavior directly reflects the interactions of a single tracer with a boundary field [see Fig. 8(a)] and is well-known in the context of critical Casimir forces (see Sec. VI B 2). Accordingly, two tracers with (++)-type couplings tend to occupy the same region in the system, which, in the case of Dirichlet BCs, is in the center [Fig. 9(a)], or, in the case of Neumann or capillary BCs of the same sign, at the boundaries [Fig. 9(c,e)]. For the latter two BCs, due to the attractive interactions between tracer and boundary, the tracers also have a nonzero probability to be located at opposite walls [in the case (N++N), the effect is suppressed except for small \( h \), and is hence not visible in Fig. 9(c)]. By contrast, two tracers coupled linearly with opposite signs [(+-)] to an OP subject to Neumann BCs likely reside at different boundaries of the system [Fig. 9(d)]. If the interactions between boundary and tracer are (partly) repulsive [as for (D+), see Fig. 7(a)], one tracer tends to be located near the center of the system, while the other resides at a boundary [see Fig. 9(b,f)]. The above calculations can be readily extended to tracers with quadratic or mixed couplings.

### B. Dynamics

We discuss in the following the effective dynamics of a reactive tracer in the adiabatic regime, i.e., assuming the OP field \( \phi \) to be a fast variable. This regime is typically realized in experiments on colloidal particles in critical
solvents [24].

1. Linear tracer-field coupling

We first consider a tracer linearly coupled to a critical fluctuating OP field. This situation approximately describes a colloidal particle subject to critical adsorption. In order to perform the adiabatic elimination of \( \phi \) in the Langevin equations in Eq. (2.10), we apply the method of Refs. [67, 68], which renders the following FPE for the effective tracer distribution \( \bar{P} \) (see Appendix [E]):

\[
\partial_t \bar{P}(R, t) = -\partial_R \left[ \mu(R) \bar{P}(R, t) \right] + \partial^2_{R^2} \left[ D(R) \bar{P}(R, t) \right],
\]

with the drift and diffusion coefficients given by [see Eq. (E3)]

\[
\mu(R) \equiv \mu_0(R) + D'(R), \quad \mu_0(R) \equiv -[1 - \tilde{\chi} \kappa_h m(R)] \mathcal{U}'(R),
\]

\[
D(R) \equiv T \left[ 1 - \tilde{\chi} \kappa_h m(R) \right],
\]

and \( T = T_R = T_\phi \). Furthermore, the coupling constants \( \tilde{\chi} \) and \( \kappa_h \) are defined in Eqs. (2.11) and (4.3), \( m(R) \) is reported in Eq. (3.25), and

\[
\mathcal{U}(R) \equiv - \left[ \frac{\hbar^2}{2T} V_\phi(R) + \hbar \langle \phi(R) \rangle \kappa_1 \right]
\]

is an effective potential [see Eqs. (3.8) and (3.20)]. The contribution \(-\hbar^2 V_\phi/(2T)\) in \( \mathcal{U} \) stems from the polarization of the OP by the tracer (representing critical adsorption) and is not present in the passive case [see Eq. (4.2a)].

The term \( 1 - \tilde{\chi} \kappa_h m(R) \) in Eq. (5.21a) represents an effective mobility [in units of the bare mobility \( \gamma_R \); see Eq. (2.4a)], which is generally reduced compared to an uncoupled or a passive tracer [see Eq. (4.2)]. Relative to the boundaries, the effective mobility is reduced (enhanced) in the center of the system for Neumann (Dirichlet) BCs. A positive effective mobility requires \( \tilde{\chi} \kappa_h \ll 1 \), which also defines the regime of validity of the adiabatic approximation [see Eq. (E10)].

As one easily checks, the steady-state solution \( \bar{P}_s \) of Eq. (5.20) is given by

\[
\bar{P}_s(R) = \frac{1}{Z} \exp(-\mathcal{U}(R)/T),
\]

which coincides with the equilibrium distribution for non-symmetry breaking and capillary BCs, Eqs. (5.3) and (5.9), respectively. For periodic BCs, Eq. (5.21) reduces to \( \mu = 0 \) and \( D = T \), such that the tracer behaves as a simple Brownian particle in confinement, for which \( \bar{P}_s = 1/L \) [see Eq. (5.3)].

The Langevin equation associated with Eq. (5.20) takes its simplest form in the “isothermal” convention (also known as anti-Ito or Hänggi-Klimontovich convention [89, 83, 85], see Appendix [D]):

\[
\partial_t R = \mu_0(R) + \sqrt{D(R)} \psi_0, \quad \langle \psi(t) \psi(t') \rangle = 2\delta(t - t'),
\]

where \( \psi \) is a Gaussian white noise. We remark that, using any other convention for the noise requires adding a spurious drift term to the Langevin equation in Eq. (5.24) in order to recover the correct form of the drift in Eq. (5.21a) (see Appendix [D]). It has been previously noted that the isothermal convention is indeed a natural choice for the Langevin description of particles in an equilibrium system [83, 80]. Here, we have rigorously derived the underlying FPE from a system of stochastic differential equations with additive noise [Eq. (2.10)], for which there is no ambiguity in its interpretation.

2. Quadratic tracer-field coupling

We now determine the effective dynamics of a point-like colloidal particle quadratically coupled to a critical medium. In the strong coupling limit, \( c \to \infty \), the particle imposes Dirichlet BCs on the OP at \( R \). However, in the present approach we are concerned with the opposite limit of a weak coupling. We focus on the dynamics to \( O(c) \), which is obtained (analogously to the passive case, see Sec. [IV B 1]) by inserting the adiabatic weak-coupling solutions for \( \phi \) reported in Eq. (3.13) into the Langevin equation in Eq. (3.27). Applying the adiabatic elimination procedure of
Refs. 55, 72, 87, using the \(\mathcal{O}(\phi^2)\)-expression \(\langle \phi(0) \rangle^2 = V_0(R) + \langle \phi(R) \rangle^2\) [see Eqs. (3.8) and (3.20)], renders the linear Langevin equation

\[
\dot{R}(t) = -\partial_R U_c(R(t)) + \eta(t),
\]

with the potential \(U_c\) reported in Eq. (3.28b). The associated FPE is given by

\[
\partial_t \tilde{P}(R,t) = \partial_R \left[ U_c'(R) \tilde{P}(R,t) \right] + \partial_R^2 \left[ T_R \tilde{P}(R,t) \right],
\]

which coincides with the one for a passive tracer [see Eq. (4.32)] at \(O(e)\). Accordingly, the associated steady-state distribution is given by Eq. (4.33), i.e.,

\[
\tilde{P}_s(R) = \frac{1}{Z} \exp \left[ -U_c(R)/T_R \right],
\]

which agrees at \(O(e)\) with Eq. (4.40) as well as with the exact equilibrium distributions in Eqs. (5.5) and (5.14). Numerical simulations (see Sec. VII) indeed confirm that, for a quadratically coupled tracer, steady-state distributions in the passive and reactive cases are similar 88.

VI. SIMULATION RESULTS & DISCUSSION

In the next subsection, we present results for the stationary distribution and the mean squared displacement of the tracer obtained from numerical Langevin simulations of Eq. (2.10). Furthermore, we discuss the generic mechanism underlying the observed behaviors (Sec. VI B 1) and place our results in the general context of boundary critical phenomena (Sec. VI B 2).

A. Simulations

Numerical results for the tracer position are generated by solving the Langevin Eq. (2.10a) in real space (for a system size of \(L = 100\) in simulation units) and Eq. (2.10b) in mode space using Eq. (3.7) (with a total number of 100 modes) 89. For both equations a standard Euler forward integration scheme is employed 55 with time steps \(\Delta t \sim O(0.1)\) in the non-conserved \([a = 0\) in Eq. (2.10b)] and \(\Delta t \sim O(0.01)\) in the conserved \((a = 1)\) case. All other parameters are set to unity in the simulations, unless otherwise indicated. In particular, the effective coupling constants [see Eqs. (4.3) and (4.35)] take the values \(\kappa_h = h^2 LT\phi/T_R^2 = 100\) and \(\kappa_c = cL\phi/T_R = 100\), while the adiabaticity parameter [see Eq. (2.1)] \(^\chi = 1\) (except if \(T_R/T\phi \neq 1\) and in parts of Fig. 12). We remark that, since standard Dirichlet BCs violate global OP conservation (see discussion in Sec. III A), we consider in this case only non-conserved dynamics. For all other BCs, we typically study both conserved as well as non-conserved dynamics. Simulation data is recorded after an initial transient period [of approximate duration \(\chi(L/\pi)^{2+2a}\)] required to equilibrate the OP field.

1. Statics

In Fig. 10 the numerically determined steady-state distributions \(\tilde{P}_s(R)\) of a tracer are summarized, focusing on OP fields obeying Dirichlet or Neumann BCs. A passive tracer (first row) is genuinely out of thermal equilibrium and thus \(\tilde{P}_s\) depends, in principle, on the specific dynamics and the conservation law. However, for Neumann BCs, simulations reveal only a marginal difference between conserved (dashed curves) and dissipative dynamics (thin curves with filling) in the considered parameter regime. Since, for a quadratically coupled tracer, the presence of a zero mode would lead to a uniform distribution \(\tilde{P}_s = 1/L\) [see Eq. (5.7)], we consider in that case only Neumann BCs without a zero mode \((N^*eN^*,\text{ last column})\). While this is easily arranged in a simulation with dissipative dynamics, it requires conserved dynamics in an actual physical setup.

In the case of a reactive tracer (second row), the equilibrium distributions in Eqs. (5.3) and (5.7) match the numerical results essentially exactly, both for dissipative and conserved dynamics. For a passive tracer, the distributions obtained within the weak-coupling approximation in Eqs. (4.25) and (4.40) [see also Figs. 4 and 6] still capture the simulations qualitatively – in particular, the attraction of the tracer towards a boundary or the center of the system. This is remarkable insofar as the numerical results are obtained using effective couplings of significantly larger magnitude than typically permitted in a perturbative solution. This points to the generic character of the phenomenon (see Sec. VI B 1 below). Note that, if Dirichlet or Neumann BCs act on the OP, the sign of the coupling \(h\) is irrelevant, since \(\tilde{P}_s\) depends on \(h^2\) [see Eq. (4.3)].
The local maximum of $\bar{P}_s$ at $R = L/2$ observed in Fig. 10 in the case of a passive tracer for (D±D) is predicted by the adiabatic approximation [see Eq. (4.4) and Fig. 4(a)]. This is detailed in Fig. 11(a), which shows $\bar{P}_s$ determined from simulations in the adiabatic regime for (D±D), non-conserved OP dynamics, and various values of $\tilde{\chi}\kappa_h$. As the latter parameter is decreased, the central maximum of $\bar{P}_s$ grows at the expense of the maxima at the boundaries. Figure 11(b) illustrates the corresponding behavior of $\bar{P}_s$ in the case (N±N) for conserved and non-conserved OP.

FIG. 11. Steady-state distribution $\bar{P}_s(R)$ of a tracer confined to the interval $[0, L]$, numerically determined from simulations of Eq. (2.10) for $h_1 = 0$. The tracer is either coupled linearly (coupling $h$, denoted as ±) or quadratically (coupling $c$) to a fluctuating OP field $\phi$, the latter obeying either Dirichlet or Neumann BCs ($N^*$ denotes Neumann BCs without a zero mode). The dimensionless couplings [see Eqs. (4.3) and (4.35)] take the values $\kappa_h = \kappa_c = 100$, while the adiabaticity parameter [see Eq. (2.11)] $\tilde{\chi} = 1$ (except for the last row). The first two rows show $\bar{P}_s$ for a passive ($\zeta = 0$) and a reactive tracer ($\zeta = 1$), while the thin curves (with adjacent filling) pertain to non-conserved dynamics and the dashed curves to conserved dynamics. The numerically determined distributions of a reactive tracer (second row) are accurately captured by the analytical equilibrium distributions in Eqs. (5.3) and (5.7) (thick black curves). The last row illustrates the steady-state distributions obtained for vanishing tracer noise $T_R = 0$, both in the reactive and passive cases, with the gray bars at the boundaries representing Dirac-$\delta$ distributions. In the last row, the dashed curves in the case (D±D) pertain to $T_R/T_\phi = 1$, $10^{-1}$, $10^{-4}$ (center bottom to top). The horizontal lines represent $\bar{P}_s = 1$. The local maximum of $\bar{P}_s$ at $R = L/2$ observed in Fig. 10 in the case of a passive tracer for (D±D) is predicted by the adiabatic approximation [see Eq. (4.4) and Fig. 4(a)]. This is detailed in Fig. 11(a), which shows $\bar{P}_s$ determined from simulations in the adiabatic regime for (D±D), non-conserved OP dynamics, and various values of $\tilde{\chi}\kappa_h$. As the latter parameter is decreased, the central maximum of $\bar{P}_s$ grows at the expense of the maxima at the boundaries. Figure 11(b) illustrates the corresponding behavior of $\bar{P}_s$ in the case (N±N) for conserved and non-conserved OP.

FIG. 11. Steady-state distribution $\bar{P}_s = L\bar{P}_r$ of a passive tracer in the adiabatic regime ($\tilde{\chi} \ll 1$), linearly coupled to an OP field subject to (a) Dirichlet and (b) Neumann BCs. The various solid curves (with adjacent filling) represent simulation results obtained for (a) $\tilde{\chi}\kappa_h = 10, 100, 1000$ (in the direction of the arrow) and (b) $\tilde{\chi}\kappa_h = 10$ (50) in the non-conserved (conserved) case. The dashed-dotted (dashed) curves represent the expression in Eq. (4.4) for $\tilde{\chi}\kappa_h = 10$ [$\tilde{\chi}\kappa_h = 50$ in the conserved case in (b)] and $h_1 = 0$. [Diagram not shown in text]
In order to assess the tracer dynamics, we consider the mean-squared displacement (MSD) of the tracer location,

$$\langle \Delta R(t)^2 \rangle \equiv \langle |R(t) - R(0)|^2 \rangle = \int_0^L dR (R - R_0)^2 \bar{P}(R, R_0, t),$$  

(6.1)

where we take $R_0 \equiv R(0) = L/2$ as initial position. Figure 13 illustrates the MSD of a tracer coupled linearly to a (non-conserved) OP field with Dirichlet BCs in the adiabatic regime. In simulation, the average in Eq. (6.1) is obtained over multiple stochastic realizations of the noise, whereas the theoretical prediction is determined by inserting in Eq. (6.1) for $\bar{P}$ the (numerically calculated) solution of the FPEs reported in Eqs. (4.1) and (5.20). Since these FPEs describe a Markovian process with time-independent drift and diffusion coefficients, the resulting MSD

Notably, for $\tilde{\chi}\kappa h \lesssim O(1)$, the adiabatic result in Eq. (4.4) (broken curves in Fig. 11) agrees well with the numerically determined steady-state distributions. The adiabatic approximation breaks down for larger values of $\tilde{\chi}\kappa h$ [see Eq. (E10)], where non-Markovian effects instead dominate the behavior of a passive tracer, leading to an enhancement of the probability near the boundaries (see Sec. VI B1 below).

The last row in Fig. 10 shows the steady-state distributions resulting for vanishing tracer noise intensity [$T_R = 0$, see Eq. (2.5a)], in which case the only source of noise stems from the coupling to the OP [see Eq. (3.27)]. In the case (D±D), the resulting distributions (thin curves with filling) are close to the ones obtained for $T_R = T_\phi$ (dashed curves). For all other couplings and BCs, one obtains two Dirac-δ-like distributions located at the boundaries (represented by thin vertical bars in the plot). This behavior can be readily understood by noting that, in Eq. (3.27) the coupling terms $\partial_R\phi$ and $\phi$ vanish at the boundaries for Neumann and Dirichlet BCs, respectively. Interestingly, in the case (N*cN∗), the transition from $T_R = T_\phi$ to $T_R = 0$ proceeds by a growth of the central peak of $\bar{P}$ and a rather sharp increase of the probability at the boundaries.

Figure 12 illustrates the steady-state distribution of a tracer in the presence of non-vanishing boundary fields $h_1$. In the passive case (top row), $\bar{P}$ is dominated by the deterministic potentials $U_b$ and $U_c$ [see Eqs. (4.5) and (4.34)] and is thus inhomogeneous even for $\tilde{\chi} = 0$, in contrast to the case with non-symmetry-breaking BCs ($h_T = 0$). We find that, for $\tilde{\chi} \ll 1$, the simulation results (thin curves with filling) are accurately captured by the expressions obtained based on the adiabatic approximation in Eqs. (4.4) and (4.34) (thick curves). As was the case for $h_1 = 0$, the analytically determined equilibrium distributions for a reactive tracer in Eqs. (5.10), (5.12), and (5.14) (thick curves, bottom row) exactly match the numerical results.

2. Dynamics
A short-time solution of the FPE in Eqs. (4.1) and (5.20) yields \[ \langle \Delta R^2(t) \rangle \simeq 2D(R_0)t + \mathcal{O}(t^2), \quad (t \ll t_R) \] with the diffusivity \( D(R) \) given in Eqs. (4.2b) and (5.21b). For a passive tracer within the adiabatic regime, the OP acts as an additional Markovian noise source [cf. Eq. (3.27)], causing the diffusivity to surpass the free one \( t_R \). In contrast, a reactive tracer polarizes the surrounding medium, which hinders displacement and consequently reduces the diffusivity relative to the free one. These behaviors are consistent with the results of Ref. [25] and we emphasize that the trends observed when varying \( \tilde{\chi} \) apply only to \( \tilde{\chi} \ll 1 \). Remarkably, already for \( \tilde{\chi}\kappa_h \lesssim \mathcal{O}(10) \), our simulation results (solid curves) are accurately captured by the analytical predictions (dashed curves). At shorter times as well as for larger \( \tilde{\chi}\kappa_h \), the non-Markovian character of the OP fluctuations becomes prominent, causing the simulations to increasingly deviate from the adiabatic approximation [see Eq. (E10)]. In fact, for \( t \to 0 \), the MSD obtained from the simulations approaches the one of a free (\( h = 0 \)) tracer, \( \langle \Delta R(t) \rangle^2 = 2tR_0^2 \) [dashed-dotted curve, see inset in Fig. 13(a)]. We remark that, even in the Markovian regime, the spatially heterogeneous character of the diffusivity \( D(R) \) can lead to non-Brownian diffusion [83, 92, 93]. A more detailed analysis of the diffusivity will be performed in a separate study.

B. Discussion

1. Passive tracer

The steady-state distributions of a passive tracer (first row in Figs. 10 and 12) can be understood by inspecting the forcing terms \( \Xi_{h,c} \) in the Langevin equation in Eq. (3.27). In the presence of boundary fields \( h_1 \neq 0 \), the tracer dynamics is essentially controlled by the non-vanishing mean OP profile \( \langle \phi(z) \rangle \), which gives rise to the deterministic potentials \( U_{c,h}(z) \) [see Eqs. (3.28), (4.4), and (4.33)]. For \( h_1 = 0 \), instead, the OP fluctuations, as encoded in the effective noises \( \Pi_{h,c} \) [Eq. (3.29)], determine the behavior of the tracer. In the linearly coupled case, the correlations of the effective noise \( \Pi_{h,c} \) are proportional to \( C_{\phi s}(R,R',\Delta t) \) [see Eqs. (3.16) and (3.30a)], which is illustrated in Fig. 14 as a function of \( R \) and \( \Delta t \) around some fixed location \( R' \simeq 0 \) near one boundary [24]. Note first that \( C_{\phi s}(R,R',\Delta t) \) is positive if \( R \) is near \( R' \). Assume now that the tracer is located near a boundary and receives a “kick” from the noise which, in the absence of a boundary, would move the tracer beyond it. Due to the boundary condition, however, the tracer is reflected back to a position, where, owing to the positive temporal correlation of the noise, it is likely to again be kicked towards the boundary in the next time step. As a consequence, a passive tracer linearly coupled
to a uniform OP field has an enhanced probability to reside near a boundary, as observed in Fig. 10. This effect is, in fact, generically expected for a confined stochastic process driven by a temporally correlated noise [95, 96]. The dynamics of a quadratically coupled tracer, by contrast, is dominated by the non-vanishing mean of the forcing term $\Xi_c$ [Eq. (3.28b)], as described by the deterministic potential $U_c(R)$ [see Fig. 3 and Eqs. (4.33) and (5.27)].

2. Reactive tracer

The equilibrium distribution $\bar{P}_c$ of a reactive tracer (second row in Fig. 10 and Fig. 12) encodes the fluctuation-induced interactions between inclusions in a critical medium [2, 5, 10]. Accordingly, it is informative to connect the present results to previous studies of the critical Casimir force (CCF) acting on a spatially extended spherical particle [11, 14, 15]. To this end, the Hamiltonian coupling of the point-like tracer [see Eq. (2.2)] must be mapped to a boundary condition for the OP $\phi$ at the particle surface: a linear coupling ($h \neq 0$, $c = 0$) corresponds to a $+$ (or $-$) boundary condition, while a quadratic coupling ($c \neq 0$, $h = 0$) corresponds to a Dirichlet boundary condition (provided that $|h|$ and $|c|$ are sufficiently large). In the limit where the particle-wall distance $R$ is large compared to the particle radius $a$, a small-sphere expansion renders the Casimir (excess) free energy [11, 14, 15]

$$F_{\text{cas}} \simeq -T_c \frac{A_{\psi} A_{\psi}}{B_{\psi}} \left( \frac{\theta}{2R} \right)^{x_\psi}.$$  \hspace{1cm} (6.3)

Here, $\psi = \phi$ and $x_\psi = \beta/\nu$, if both the wall and the particle impose symmetry-breaking BCs [i.e., $(\pm \pm)$], whereas $\psi = \phi^2$, $x_\psi = d - \nu^{-1}$, if neither particle nor wall, or only one, impose symmetry-breaking BCs [corresponding to the cases $(\pm D/N)$, $(D/D/N)$, $(NN)$]. (Here, $\beta$ and $\nu$ denote the standard bulk critical exponents.) The amplitudes $A_{\psi}$ and $B_{\psi}$ and the exponent $x_{\psi}$ are defined via the associated bulk correlation function, $\langle \psi(r)\psi(r') \rangle_{\text{bulk}} = B_{\psi} r^{-2x_{\psi}}$, and the profile in the half-space with boundary condition $a$, $\langle \psi(r) \rangle_{\text{half-space}} = A_{\psi} (2r_a)^{-x_{\psi}}$. The excess free energy contributes to the Casimir potential $U(R) = F_{\text{cas}}(R) + U_{\text{add}}(R)$, which enters a Boltzmann-like probability distribution for the tracer [6],

$$\bar{P}_c(R) \propto \exp(-U(R)/T).$$  \hspace{1cm} (6.4)

The potential $U_{\text{add}}(R)$ accounts for additional interactions, such as those stemming from van der Waals forces, which are relevant at short distances $R$ and regularize a possible divergence of $F_{\text{cas}}$ for $R \to 0$ [13, 97]. We do not consider these here. The CCF acting between particle and wall follows from Eq. (6.3) as $K = -dF_{\text{cas}}/dR = x_{\psi} F_{\text{cas}}/R$.

Since $F_{\text{cas}}$ in Eq. (6.3) is singular for a point-like particle ($q \to 0$) and, moreover, does not include the effect of the second, distant wall, we do not expect quantitative agreement with the results obtained in our study. We thus focus instead on the sign of the CCF and asymptotic behavior of $F_{\text{cas}}$: for the BCs considered here, Eq. (6.3) implies an attractive CCF for the combinations $(a,b) = (D,D)$, $(N,+)$, $(+,+)$, and a repulsive one for $(D,N)$, $(D,+)$, $(+,-)$ (see also Ref. [15]). These predictions are consistent with the behaviors observed in Figs. 10 and 12. For the Gaussian model in $d = 1$ dimensions, the exponent $x_{\psi}$ takes the value $x_{\psi} = -1$ if either particle or wall (or both) have non-symmetry breaking BCs. This implies that $F_{\text{cas}} \propto R$ as $R \to 0$, in agreement with the asymptotic behavior.
resulting from (the negative logarithm of) the expressions in Eqs. (5.3), (5.7), and (5.14) [98]. In $d = 3$ dimensions (see Appendix H), we have $x_\phi = x_\varphi = 1$ within the Gaussian model, which agrees in the cases (D/N, $\pm$, D/N) and ($\pm$,$\pm$) with the asymptotics reported in Eq. (H11).

VII. SUMMARY

We have investigated in this study the behavior of a confined point-like tracer particle coupled to a fluctuating order parameter (OP) field $\phi$ within the Gaussian approximation. The OP field represents a critical fluid medium in equilibrium and follows either dissipative or conservative dynamics (model A/B [39]). The tracer is governed by a Langevin equation [see Eq. (2.8)] and is subject to reflective BCs. We have considered passive as well as reactive types of tracers. The former is out of equilibrium, since the coupling to the fluctuating fluid represents an energy source that is unbalanced by dissipation. In this sense, a passive tracer bears resemblance to an “active” particle [43, 44, 57, 99]. By contrast, a reactive tracer interacts with the fluid in accordance with the fluctuation-dissipation theorem, such that its steady state obeys equilibrium statistical mechanics. A reactive tracer can be viewed as a simplified model of a colloidal particle in a critical fluid. The action of a reactive tracer on the fluid is described either by a local chemical potential (linear coupling) or by a locally altered correlation length (quadratic coupling) [see Eq. (2.7)]. A linear coupling enhances the OP around the tracer, as is typically observed for colloids [13, 104]. We have also considered non-vanishing boundary fields ($h_1 \neq 0$). These induce a non-uniform OP profile, which manifests as a deterministic force acting on the tracer [see Eq. (3.28)].

The central quantity in our study is the probability distribution $\tilde{P}(R, t)$ of the tracer position $R$. While previous studies of tracers in fluctuating media focused mostly on bulk systems [25–27], we have addressed here the effect of the zero mode, such that the otherwise diverging relaxation time of a critical fluid [41] is cut off. One of our main results is given by Eq. (5.20), which describes the effective (adiabatic) equilibrium distributions of a point-like colloidal particle in a confined critical medium in the presence of critical adsorption, i.e., a local enhancement of the OP around the particle.

We have validated our analytical results by numerically solving the associated Langevin equations [Eq. (2.10)] to obtain the steady-state distribution [see Figs. 10 to 12] as well as the mean-squared displacement [see Fig. 13]. While we focused a one-dimensional system, analytical calculations of the equilibrium distribution of a reactive tracer in three dimensions reveal that they are qualitatively similar to the one-dimensional case (see Appendix H). A reactive tracer typically obeys a Boltzmann-like equilibrium distribution, $P_\text{eq}(R, t) \sim \exp(-U(R)/T)$, with an effective potential $U(R)$. The latter is a consequence of the coupling to the OP field [see Sec. VII B] and encodes the critical Casimir interactions between tracer and boundaries [see Sec. VII C]. In a homogeneous medium ($h_1 = 0$) and for a linearly coupled tracer (coupling $h$), one has $U(R) \propto -h^2 \langle \phi(R)^2 \rangle$ [see Eqs. (5.3) and (5.23)], while $U(R) \propto c \langle \phi(R)^2 \rangle$ in the case of a quadratic coupling ($c$) [see Eq. (5.27)], where $\langle \phi(R)^2 \rangle = V_\phi(R)$ is the variance of $\phi$ [see Eq. (3.20)].

For a quadratically coupled tracer, these expressions apply only in the weak-coupling approximation, as the exact equilibrium distribution is given in Eq. (5.51). Non-uniform OP profiles $\langle \phi(R) \rangle$, caused by boundary fields $h_1 \neq 0$ [see Eq. (3.8)], render contributions to $U$ of the form $-h\langle \phi(R) \rangle$ and $c\langle \phi(R)^2 \rangle$, respectively.

The effective potential $U$ determines also the dynamics of the tracer at the leading order in the adiabatic and the weak-coupling approximations [see Eqs. (4.1), (4.31), (5.20), and (5.26)]. Since, within the Gaussian model considered here, $\langle \delta \phi(R)^2 \rangle$ and $\langle \phi(R) \rangle$ are quadratic functions of $R$, the tracer can in certain cases be effectively described by a confined Ornstein-Uhlenbeck process [101]. Beyond leading order, deviations from this simple behavior arise because the mobility acquires a spatial dependence [see discussion after Eqs. (4.1) and (5.20)]. Note that the adiabatic dynamics of a quadratically coupled reactive tracer has been considered here only to $O(e)$ and the inclusion of higher-order corrections is reserved for a future study. In the adiabatic regime, the mean-squared displacement of the tracer grows linearly in time, with an effective diffusivity that increases (decreases) with the adiabaticity parameter $\chi\kappa_\text{h}$ [see Eq. (E10)] in the passive (reactive) case (see Fig. 13).

A linearly coupled passive tracer has a higher probability to be located near the boundaries than in the center of
the system. This is a generic effect resulting from the interplay between confinement and a temporally correlated noise [95,96]. It also arises in the case of active matter, where it gives rise to the accumulation of active particles at surfaces and plays a role for motility-induced phase separation [43,44,99]. The behavior of a quadratically coupled (passive or reactive) tracer, instead, is dominated by the effective potential \( \propto c\langle \phi(R)\rangle^2 \), stemming from the nonzero average of the OP-related noise \( \sim \phi^2 \) [see Eq. (3.27) and Fig. 2(a)]. Interestingly, in the parameter regimes considered here, the steady-state distributions of a passive and a reactive tracer are qualitatively similar (see Figs. 10 and 12). An exception is a tracer coupled linearly to an OP field subject to Dirichlet BCs (see Fig. 10), in which case the steady-state distribution exhibits a crossover behavior controlled by the non-Markovian character of the dynamics [see Fig. 11(a)]. We finally remark that the steady-state distribution of a passive tracer is similar for dissipative and conservative OP dynamics.

The present study opens up various possibilities to investigate colloidal dynamics in a critical medium within an analytical approach. In particular, our model can be readily extended to more than one tracer (see Sec. V A 3), which could be utilized to address many-body critical Casimir interactions [75,102,106]. Furthermore, while we assumed the OP to remain in equilibrium at all times, non-equilibrium scenarios such as OP quenches [33,63,107,108] appear to be a rewarding topic. It is also pertinent to extend the present work towards two and three spatial dimensions, which are the relevant cases for membranes [28,30,109,110], interfaces [111–113], and colloidal suspensions [6]. Moreover, effects of off-criticality as well as hydrodynamics [114,115] could be taken into account in the future. The Gaussian weak-coupling approximations employed here provide the leading order contributions of a perturbation expansion of a \( \phi^d \)-theory [46]. In fact, the couplings are expected to flow under a renormalization group and attain fixed-point values, depending on the dimension, are not necessarily small. This should be addressed in a future study. Further attention should also be devoted to non-Markovian effects in the dynamics, which become relevant at large coupling strengths.

Appendix A: Dimensional considerations

From the fact that the argument of the exponential in Eq. (2.7) must be dimensionless, one infers the following dimensions of the field and the static couplings: [\( \phi = [L]^{1-d/2}[\phi]^{1/2} \), [\( h = [L]^{d/2-1}[T_R][T_\phi]^{-1/2} \), [\( c = [L]^{d-2}[T_R]/[T_\phi] \), and [\( h_1 = [L]^{-d/2}[T_\phi]^{1/2} \). The dimensions of the dynamical couplings follow from Eqs. (2.4a) and (2.4b): [\( [\gamma_\phi] = [L]^{d+2\epsilon+1}[t]^{-1} \), [\( [\gamma_R] = [L]^{2}[T_R]^{-1}[t]^{-1} \), [\( [\chi] = [L]^{-2\epsilon}[T_R]^{-1} \), where [\( [R_\alpha] = [L] \). Although [\( [T_R] = [T_\phi] \), we distinguish here the two temperatures for clarity. Note that \( t \) refers to the unrescaled time [see Eq. (2.8)].

Appendix B: Equilibrium correlation functions of the OP

Here, we determine the equilibrium OP profile and correlation function within the Gaussian model as defined by Eqs. (2.2) and (2.7).

1. System with a linearly coupled tracer or boundary fields

We consider an OP field \( \phi \), which may be subject to boundary fields \( h_1 \) as well as to a bulk field \( h \) (at location \( R \), representing a linearly coupled tracer). Connected correlation functions of \( \phi \) can be determined in the usual fashion [79] from the generating functional

\[
\mathcal{Z}[J] \equiv \int \mathcal{D}\phi \, e^{-\frac{1}{\beta} \hat{H}[\phi,J]}, \quad (B1)
\]

which is obtained by introducing a position-dependent auxiliary field \( J(z) \) into the Hamiltonian in Eq. (2.2):

\[
\hat{H}[\phi,J] \equiv \int dz \left[ \frac{1}{2} (\partial_z \phi)^2 - J(z)\phi(z) \right] - h_1 \left[ \phi(0) + \phi(L) \right] - h\phi(R). \quad (B2)
\]

We first consider the averaged profile

\[
\langle \phi(z) \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\phi \, \phi(z) \, e^{-\frac{1}{\beta} \hat{H}[\phi,J=0]} = T_\phi \frac{\delta}{\delta J(z)} \ln \mathcal{Z}[J] \bigg|_{J=0}. \quad (B3)
\]
Switching to mode space, we write \( J(z) = \sum_n \sigma_n(z) J_n \) and use Eqs. (3.2) and (3.4) to bring the Hamiltonian into the form

\[
\hat{\mathcal{H}} = \sum_n \left[ \frac{1}{2} k_n^2 |\phi_n|^2 - J_n^* \phi_n - \bar{\tau}_n \phi_n \right],
\] (B4)

with \( \bar{\tau}_n \equiv h_1 [\sigma_n(0) + \sigma_n(L)] + h \sigma_n(R) \). In order to regularize a possible zero mode \( k_0 = 0 \), we set \( k_0 = \epsilon > 0 \) and perform the limit \( \epsilon \to 0 \) at the end of the calculation. After a Gaussian integration using Eq. (5.2), the generating functional takes the form

\[
Z[J] = \int D\phi_n e^{-\frac{1}{2} \sum_n J_n^* \tau_n J_n} \exp \left( \frac{1}{2T_\phi} \sum_n |J_n^* + \bar{\tau}_n|^2 \right),
\] (B5)

where we omitted an unimportant normalization factor. The mode-space expression for the averaged profile resulting from Eq. (B3) follows as

\[
\langle \phi(z) \rangle = \sum_n \sigma_n(z) \langle \phi_n \rangle = T_\phi \sum_n \sigma_n(z) \frac{\partial \ln Z[J]}{\partial J_n} \bigg|_{J=0} = \sum_n \sigma_n(z) \frac{\bar{\tau}_n^*}{k_n^2}.
\] (B6)

The zero mode \( k_0 = \epsilon \to 0 \) renders a divergent mean profile in equilibrium if \( h \neq 0 \) or \( h_1 \neq 0 \). In the absence of a zero mode (which applies, in particular, to conserved dynamics), we take Neumann modes for \( \sigma_n \) [see Eq. (3.1c) [117]] and use standard Fourier-series relations [118] to evaluate Eq. (B6) with \( h = 0 \):

\[
\langle \phi(z) \rangle_{h_1} \equiv \langle \phi(z) \rangle_{h=0} = h_1 L \left[ \frac{1}{6} - \frac{z}{L} + \left( \frac{z}{L} \right)^2 \right].
\] (B7)

This profile agrees with the one obtained within linear MFT [see Eq. (38) in [61]] and accordingly fulfills, in an averaged sense, \((++)\) capillary BCs of critical adsorption,

\[
\partial_z \langle \phi(z) \rangle_{h_1} \bigg|_{z \in \{0,L\}} = \mp h_1,
\] (B8)

as well as \( \int_0^L dz \langle \phi(z) \rangle = 0 \). If, instead, \( h_1 = 0 \) and the OP field is only subject to a bulk-like field \( h \neq 0 \) at location \( R \), Eq. (B6) renders

\[
\langle \phi(z) \rangle_{h_1=0} = \begin{cases} hL \left[ \frac{1}{L} \min(R,z) - \frac{zR}{L^2} \right], & \text{if } \min(R,z) < z < R, \\ hL \left[ \frac{1}{3} - \frac{1}{L} \max(R,z) + \frac{1}{2L^2} \left( R^2 + z^2 \right) \right], & \text{if } z < \min(R,z) \end{cases}
\] (B9)

In an analogous way, we obtain from Eq. (B5) the connected static correlation function [see Eqs. (3.14) and (3.15)]:

\[
C_\phi(x,y) = \langle \delta \phi(x) \delta \phi(y) \rangle = \sum_{n,m} \sigma_n(x) \sigma_m(y) \langle \delta \phi_n \delta \phi_m \rangle = T_\phi^2 \sum_{n,m} \sigma_n(x) \sigma_m(y) \frac{\partial^2 \ln Z}{\partial J_n \partial J_m} \bigg|_{J=0} = T_\phi \sum_n \frac{\sigma_n(x) \sigma_n^*(y)}{k_n^2},
\] (B10)

where \( \delta \phi \equiv \phi - \langle \phi \rangle \) denotes the fluctuation part of \( \phi \) and we used \( J_n^* = J_{-n} \) (which follows from \( J(z) \) being real-valued) and \( [\sigma_n^{(b)}(z)]^* = \sigma_{-n}^{(b)}(z) \). Note that \( C_\phi \) is independent of \( h \) and \( h_1 \) and thus applies irrespective of the presence of boundary fields or linearly coupled tracers. Explicit expressions for \( C_\phi \) are provided in Eq. (3.19). It is useful to note that the profile in Eq. (B7) can be written as

\[
\langle \phi(z) \rangle_{h_1} = \frac{h_1}{T_\phi} \left[ C_\phi^{(N)}(0,z) + C_\phi^{(N)}(L,z) \right],
\] (B11)

which readily follows from Eqs. (B6) and (B10).

2. System with a quadratically coupled tracer

In the case of a quadratically coupled tracer, correlations can be determined analogously to Eq. (B10) by defining \( Z \) [Eq. (B11)] in terms of the Hamiltonian \( \hat{\mathcal{H}}[\phi,J] = \int dz \left[ \frac{1}{2} (\partial_z \phi)^2 - J(z) \phi(z) \right] + \frac{1}{2} \phi(R)^2 = \frac{1}{2} \sum_{n,m} \phi_n \Gamma_{nm} \phi_m - \frac{1}{2} \sum_{n,m} \phi_n \Gamma_{nm} \phi_m \).
\[ \sum_n J_n^\star \phi_n, \text{ with } \Gamma \text{ given in Eq. (5.6) [see also Eq. (G3)]. We focus here on the variance } \langle \phi(R)^2 \rangle \text{ of the field at the location of the tracer, which can be obtained from Eq. (5.1) as} \]

\[ \langle \phi(R)^2 \rangle = -2TR \partial_t \ln Q(R), \quad Q(R) \equiv \int_{V} D\phi \exp \left\{ - \int_V \frac{1}{2T\phi} (\partial_x \phi)^2 - \frac{c}{2TR} \phi(R)^2 \right\} = [\det \Gamma(R)]^{-1/2}, \quad (B12) \]

where we have used Eqs. (5.2) and (5.5) and omitted an (infinite) numerical prefactor which cancels out in Eq. (B12). (The same result follows from \( \delta^2 Z/\delta J(R)^2 \).) Upon using the expression for \( \det \Gamma \) stated in Eqs. (G6) and (G7), one obtains (see Fig. 15)

\[ \langle \phi(R)^2 \rangle = \left\{ \begin{array}{l} \frac{\kappa_c}{T\phi L} + \frac{1}{V\phi} \quad (p^\star) \\ \frac{T_{\phi}L}{\kappa_c + \tau_2^\star} \quad (pN) \\ \frac{T_{\phi}L}{\kappa_c + (\rho - \rho^\star)^2} \quad (D) \\ \frac{T_{\phi}L}{\kappa_c + (\frac{1}{2} - \rho + \rho^2)} \quad (N^\star) \end{array} \right. \quad (B13) \]

with \( \rho \equiv R/L, \kappa_c \) given in Eq. (4.35) and \( V\phi \) in Eq. (3.20). In the absence of a coupling to the tracer, i.e., for \( \kappa_c = 0 \), Eq. (B13) reduces to the expression in the passive case, Eq. (3.20). In the limit \( \kappa_c \to \infty \), corresponding to Dirichlet BCs at \( R \), \( \langle \phi(R)^2 \rangle \) vanishes.

**Appendix C: Adiabatic limit of the OP dynamics**

In order to determine an approximation to the solution \( \phi(z,t) \) [Eq. (3.10)] in the adiabatic limit (\( \chi \ll 1 \)), we substitute in Eq. (3.10a) the integration variable \( s = t - u\chi/(k^2 k_2^\alpha) \) and obtain

\[ \phi_n(t) = \frac{\chi}{k_2^2 k_2^\alpha} \int_0^\infty du e^{-u} \left\{ \chi^{-1} k_2^\alpha \left[ \zeta h_\sigma^\star_n \left( R \left( t - \frac{u \chi}{k_2^2 k_2^\alpha} \right) \right) \right] + h_1^\star \right\]. \quad (n \neq 0) \quad (C1) \]

The dependence on \( \chi \) can be made explicit by rescaling time accordingly, see the discussion around Eq. (2.8). Due to the exponential, the integrand gives substantial contributions only if \( u \lesssim \mathcal{O}(1) \). Accordingly, for \( \chi \ll 1 \) it is justified to Taylor expand the terms in the square bracket in Eq. (C1) up to first order in \( u\chi/(k^2 k_2^\alpha) \), rendering

\[ \phi_n(t) \approx \int_0^\infty du e^{-u} \left\{ \frac{\zeta h_\sigma^\star_n(R(t))}{k_2^2} + \frac{h_1^\star}{k_2^2} \tau_2^\star + \frac{\zeta h_\sigma^\star_n(R(t))}{k_2^2 k_2^\alpha} u \dot{R}(t) \partial_R \sigma_\phi^\star(R(t)) + \frac{\sqrt{\chi}}{k_2^2 k_2^\alpha} \xi_n(t) \right\]. \quad (n \neq 0) \quad (C2) \]

It is not feasible here to expand beyond \( \mathcal{O}(u) \) or, correspondingly \( \mathcal{O}(\chi) \), since this would generate derivatives of the noise \( \xi_n(t) \). We remark that, alternatively to deriving an equation of motion, the adiabatic approximation can also be applied on the level of the correlation function \( C_{\phi} \) in Eq. (8.15).
Given the FPE

$$\partial_t P(R, t) = -\partial_R [\mu(R) P(R, t)] + \partial^2_R [D(R) P(R, t)], \quad \text{(D1)}$$

the form of the associated Langevin equation depends on the chosen integration rule for the noise \[53\]. Possible rules can be parametrized by a quantity $\alpha$, with $0 \leq \alpha \leq 1$, such that Eq. (D1) maps to \[69, 83\]

$$\partial_t R = \mu(R) - \alpha D'(R) + \sqrt{D(R)} \theta, \quad \langle \theta(t') \theta(t') \rangle = 2 \delta(t - t'), \quad \text{(D2)}$$

with a Gaussian white noise $\theta$. The parameter $\alpha$, in fact, determines the point $t^*_i = t_{i-1} + \alpha(t_i - t_{i-1})$ in the interval $[t_i, t_{i-1}]$ at which the noise is evaluated in a discretization of Eq. (D2). Common choices in the literature are $\alpha = 0$ (Ito convention), $\alpha = 1/2$ (Stratonovich convention), and $\alpha = 1$ (called anti-Ito, isothermal, or H"anggi-Klimontovich convention) \[69, 83\]. Conversely, when starting from a Langevin equation of the form $\partial_t R = \bar{\mu}(R) + \sqrt{D(R)} \theta$, the drift term $\mu(R)$ in the FPE in Eq. (D1) would be replaced by $\bar{\mu}(R) + \alpha D'(R)$. The contribution $\alpha D'(R)$ denotes a "spurious" drift and appears in either the Langevin or the FPE when $\alpha \neq 0$.

The steady-state solution $P_s(R)$ of Eq. (D1) is given by

$$P_s(R) = \frac{1}{Z} \frac{1}{D(R)} \exp \left[ \int_0^R dz \frac{\mu(z)}{D(z)} \right] \quad \text{(D3)}$$

with a normalization constant $Z$. The drift typically takes the following generic form:

$$\mu(z) = -\frac{D(z)}{T} U'(z) + \alpha D'(z), \quad \text{(D4)}$$

where $U(z)$ is a potential, $D(z)/T$ represents a mobility and $\alpha D'(z)$ is the "spurious" drift. Using Eq. (D4) in Eq. (D3) renders

$$P_s(R) = \frac{1}{Z} \frac{1}{D(R)^{1-\alpha}} \exp \left[ -U(R) \right]. \quad \text{(D5)}$$

Accordingly, in order to recover the standard Boltzmann equilibrium distribution from the FPE in Eq. (D1), the drift in Eq. (D4) must involve a "spurious" contribution with $\alpha = 1$ (isothermal convention) \[83\].

FIG. 16. Assessment of the predictions for the steady-state distribution $\bar{p}_s = LP_s$ of a passive tracer ($\zeta = 0$) determined from the two adiabatic elimination schemes in Eqs. (E3) and (E5). The tracer is linearly coupled to an OP field subject to capillary BCs with (a) $hh_1 > 0$ and (b) $hh_1 < 0$. The solid curves (with filling) represent simulation results (see Sec. VI) obtained for $\kappa h_1 \simeq 25$ and (a) $\chi = 0.5$, (b) $\chi = 0.1$. The dashed-dotted curve represents $\bar{p}_s$ [Eq. (D3)] obtained from Eq. (E5), which accurately describes the simulation results even for large values of $\chi \kappa h_1$ [see Eq. (E10)]. In comparison, the dotted curve shows $\bar{p}_s$ determined from Eq. (E5) (with $\zeta = 0$).
Appendix E: Adiabatic elimination for a linearly coupled tracer

We focus on a linearly coupled tracer and recall that Eqs. (2.10a) and (3.7) form a system of Langevin equations:

\[
\dot{R}(t) = \hbar \sum_{n} k_n \hat{\sigma}_n(R(t)) \phi_n(t) + \eta(t),
\]

\[
\partial_t \phi_n = \chi^{-1} k^2_n \left\{ -k^2_n \partial_n \phi + \zeta h \sigma^*_n(R(t)) + h_1 \tau_n^* \right\} + \chi^{-1/2} \xi_n,
\]

where \(\tau_n = \sigma_n(0) + \sigma_n(L)\). We seek to determine an effective Markovian Langevin equation for the tracer position \(R\), assuming the field \(\phi\) to be a fast variable, i.e., \(\chi \ll 1\). In the literature, a number of adiabatic elimination schemes for multidimensional Langevin equations (linear in the fast variable) have been developed, see, e.g., Refs. [55, 67, 72, 119–122]. For the specific system in Eq. (E1), the Chapman-Enskog based approach of Ref. [67, 68] results in a FPE which renders a steady-state distribution in accord with equilibrium statistical mechanics [see Eqs. (5.3) and (5.9)].

Equation (E1) can be matched onto the formalism in Refs. [67, 68] by identifying \(c_1 = R, b_n = \phi_n, \alpha_{1i} = -\hbar k_i \hat{\sigma}(R), \gamma_{ij} = \chi^{-1} k^2_i k^2_j, \beta_i = \chi^{-1} h^2 k^2_i \sigma^*_i(R) + \chi^{-1} h_1 k^2_i \tau_i, V_{ij} = (T_\alpha/k^2_i) \delta_{ij}\), where \(V_{ij} = \langle \delta \phi_i \delta \phi_j \rangle\) is the equilibrium correlation function of the field modes, which can be inferred from Eq. (B10). In the case of periodic BCs, one has \(V_{ij} = (T_\phi/k^2_i) \delta_{i,-j}\), instead.

A reactive tracer (\(\zeta = 1\)) affects the local value of \(\phi\), implying that the scheme stated in Eq. (2.9) of Ref. [68] is appropriate. This renders the following FPE for the tracer distribution \(\bar{P}\):

\[
\partial_t \bar{P}(R, t) = -\partial_R \left[ (\mu(R) \bar{P}(R, t)) + \partial_R^2 [D(R) \bar{P}(R, t)] \right],
\]

with the drift and the diffusion coefficient

\[
\mu(R) = A(R) \left[ 1 - \chi h^2 L^{1+2a} m(R) \right] - \left( \frac{3}{2} \xi T_R - \frac{1}{2} T_\phi \right) \chi h^2 L^{1+2a} m'(R),
\]

\[
D(R) = T_R - \chi h^2 L^{1+2a} m(R) (2\xi T_R - T_\phi).
\]

Here, \(m(R)\) is defined in Eq. (3.25) and

\[
A(R) \equiv \sum_{n} \frac{1}{k_n} \left[ \chi h^2 \sigma_n(R) \sigma^*_n(R) + \chi h_1 \sigma_n(R) \tau_n^* \right] = \frac{1}{T_\phi} \partial_R \left\{ \frac{1}{2} \chi h^2 V_{\phi}(R) + \chi h_1 \left[ C_\phi(0, R) + C_\phi(L, R) \right] \right\}
\]

is an effective forcing term [recall Eq. (B11)], which is, in fact, independent of \(T_\phi\). For clarity, we have made the presence of \(\zeta\) explicit. Note that the steady-state solution of Eq. (E2), which is given by Eq. (D3) with \(A(R) = U'(R)\), attains the correct Boltzmann-like equilibrium form [see Eqs. (5.3), (5.9), and (D5)] only for \(T_\phi = T_R\).

By contrast, if the tracer is passive (\(\zeta = 0\)), the appropriate scheme is given by Eq. (2.10) of Ref. [68], which yields the FPE in Eq. (E2) with the following drift and diffusion coefficients:

\[
\mu(R) = A(R) + \frac{1}{2} T_\phi \chi h^2 L^{1+2a} m'(R),
\]

\[
D(R) = T_R + T_\phi \chi h^2 L^{1+2a} m(R).
\]

For \(\zeta = 0\), these expressions are identical to the ones in Eq. (E3), except for the prefactor of \(A\) (mobility), which is unity in Eq. (E5a).

In order to demonstrate that the drift in Eq. (E5a) rather than the one in Eq. (E3a) (for \(\zeta = 0\)) provides the correct description of a passive tracer, we compare in Fig. 16 the steady-state distributions \(\bar{P}\) resulting from Eq. (D3) to the ones obtained from simulation (see Sec. VI). (Note that this requires us to use \(h_1 \neq 0\), as otherwise \(A\) vanishes.) As illustrated in Fig. 16, the expressions in Eq. (E5) indeed provide a more accurate description of \(\bar{P}\) than the ones in Eq. (E3), in particular also for larger values of the adiabaticity parameter \(\chi\).

It is useful to compare the above result to a “naive” adiabatic elimination based on the adiabatic approximation of the OP \(\phi\) given in Eq. (C2). Inserting this into Eq. (2.10a) renders

\[
\mathcal{M}(R(t)) \dot{R}(t) = \mathcal{A}(R(t)) + \Pi_h(R(t), t) + \eta(t),
\]

with the noise \(\Pi_h\) reported in Eq. (4.6). Notably, the coupling to the OP (\(\zeta = 1\)) has generated an effective damping term

\[
\mathcal{M}(R) \equiv 1 + \chi h^2 \sum_{n} \frac{C_\phi(0, R) \sigma_n(R)^2}{k^2_n h^2} = 1 + \frac{\zeta \chi h^2 L}{T_\phi} m(R).
\]
Upon dividing Eq. (E6) by $\mathcal{M}$ and expanding to $\mathcal{O}(\chi)$, the associated FPE takes, for $\zeta = 1$, the form given in Eqs. (E2) and (E3) and, for $\zeta = 0$, the form in Eqs. (E2) and (E5), except for the spurious drift term. The latter cannot be determined from the Langevin approach, as it leaves the stochastic interpretation of the multiplicative noise unclear.

In order to quantify the regime of validity of the adiabatic approximation, we estimate [noting that $\mathcal{A}(R)$ in Eq. (E4) is linear in $R$] the characteristic relaxation time $t_R$ of the process described by Eq. (E2) at $\mathcal{O}(\chi^0)$ by the one of an Ornstein-Uhlenbeck process [55]:

$$ t_R \sim (\zeta h^2 + 2hh_1)^{-1}. $$

(E8)

Note that, for small $|h|$ and $|h_1|$, the relaxation time is bounded by the one of a free (uncoupled, $h = 0$) tracer in confinement, $t_{R,\text{free}} \sim L^2/T_R$. On the other hand, the typical OP relaxation time is given by $t_\phi \sim \chi L^{2+2\alpha}/\pi^{2+2\alpha}$ [see Eq. (3.15)]. The adiabatic approximation holds if the ratio of the characteristic relaxation times of the OP and the tracer is small, i.e.,

$$ t_\phi/t_R \ll 1. $$

(E9)

Using Eqs. (2.11), (4.3), and (5.11) (with $T_R = T_\phi$), this implies, in particular,

$$ \hat{\chi}_R \ll 1, \quad \hat{\chi}_R h_1 \ll 1. $$

(E10)

**Appendix F: Noise contribution in the adiabatic limit for a quadratically coupled passive tracer**

Since $C_\phi$ and its derivatives appear quadratically in the correlation function in Eq. (3.30b), one cannot simply use Eqs. (3.21) and (3.24) [or, equivalently, Eq. (3.11)] in order to obtain the adiabatic limit, since this gives rise to terms $\propto \delta^2(t-t')$ (see also Refs. [55, 65] for related discussions). In the standard adiabatic elimination procedure [55, 72], $\Pi_\sigma$ is approximated by a zero-mean white noise $\tilde{\Pi}_\sigma$ with correlations

$$ \langle \tilde{\Pi}_\sigma(z,t)\tilde{\Pi}_\sigma(z,t') \rangle = 2\mathcal{P}(z)\delta(t-t'), $$

(F1)

where $\tilde{\Pi}_\sigma$ and $\Pi_\sigma$ have the same fluctuation amplitude,

$$ \int_{-\infty}^{\infty} dt \langle \tilde{\Pi}_\sigma(z,t)\tilde{\Pi}_\sigma(z,t') \rangle = \int_{-\infty}^{\infty} dt \langle \tilde{\Pi}_\sigma(z,t)\tilde{\Pi}_\sigma(z,t') \rangle = \mathcal{P}(z), $$

(F2)

assuming time translation invariance. In order to calculate the amplitude $\mathcal{P}(z)$, we insert the mode representations stated in Eqs. (3.15) and (3.16) into Eq. (3.30b) and obtain (in the case $h_1 = 0$)

$$ \mathcal{P}(z) = \chi^2 T_\phi^2 \sum_{n,m} \left[ \frac{\sigma_n(z)\sigma_m(z)}{k_n^2(k_n^2 + 2\alpha m + \gamma_n^2)} + \frac{\sigma_n(z)\sigma_m(z)}{k_n^2(k_n^2 + 2\alpha m + \gamma_n^2)} \right]. $$

(F3)

Note that, due to the adiabatic limit, $n = 0$ or $m = 0$ are excluded from the sum.

**Appendix G: Equilibrium distributions and matrix calculations**

In order to obtain the equilibrium distributions in Eqs. (5.5) and (5.13), the determinant and the inverse of the matrix $\Gamma$ [Eq. (5.6)] entering the multivariate Gaussian integral in Eq. (5.2) must be evaluated. To this end, we define a diagonal matrix $A$ and a vector $u$,

$$ A_{nm} = \frac{k_n^2}{T_\phi} \delta_{n,m}, \quad u_n = \sqrt{\frac{c}{T_\phi}} \sigma_n(R), $$

(G1)

where the eigenfunctions $\sigma_n$ and indices for the various BCs are specified in Eq. (3.1). For the zero mode $n = m = 0$, we replace the vanishing entry $A_{00}$ with

$$ A_{00} = \frac{c}{T_\phi} > 0, $$

(G2)

where $\varepsilon$ is regularization parameter which set to zero at the end of the calculation. According to Eqs. (G1) and (G2), we can write Eq. (5.6) as

$$ \Gamma = A + uu^\dagger, $$

(G3)

where $^\dagger$ denotes transposition and complex conjugation (the latter being relevant only in the case of periodic BCs).
1. Determinant

The form of Eq. (G3) allows us to apply the matrix determinant lemma:

$$\det \Gamma = (1 + u^\dagger A^{-1}u) \det(A). \quad \text{(G4)}$$

For the purpose of regularization, we keep the mode number $M$ finite and let $M \to \infty$ only at the end of the calculation. Considering first BCs without a zero mode, it follows from the expression for the variance $V_\phi(R)$ in Eq. (3.20) that

$$u^\dagger A^{-1}u \overset{M \to \infty}{=} \frac{c}{T_R} V_\phi(R). \quad \text{(G5)}$$

Accordingly, Eq. (G4) evaluates to

$$\det \Gamma = \mathcal{N} \left[ 1 + \frac{c}{T_R} V_\phi(R) \right] = \begin{cases} \mathcal{N}^{(p)} \left[ 1 + \frac{1}{T_R} \kappa_c \right], & \text{(p)} \\ \mathcal{N}^{(D)} \left[ 1 + \kappa_c (\rho - \rho^2) \right], & \text{(D)} \\ \mathcal{N}^{(N)} \left[ 1 + \kappa_c \left( \frac{1}{4} - \rho + \rho^2 \right) \right], & \text{(N)} \end{cases} \quad \text{(G6)}$$

where $\rho \equiv R/L$ and $\kappa_c = cLT_\phi/T_R$ [see Eq. (4.35)]. While the quantity $\mathcal{N} \equiv \det(A)$ formally diverges for $M \to \infty$, it is independent of $R$ and $c$ and thus canceled by the normalization of the distribution [see, e.g., Eq. (5.5)]. We obtain

$$\mathcal{N}^{(D,N)} = \frac{\pi^2 M}{T_R^2 L^{2M}} \prod_{n=1}^M \sigma_n^2, \quad \text{(D,N)}$$

$$\mathcal{N}^{(p)} = \left( \frac{2}{2^2} \frac{\pi^2 M}{T_R^2 L^{2M}} \right)^2 \prod_{n=1}^M \sigma_n^2, \quad \text{(p)}$$

where the last equation follows from the fact that $A$ is a real symmetric matrix. Upon introducing the static correlation function $C_\phi$ via Eq. (3.19), one obtains:

$$\sum_{n,m} \sigma_n(x) \Gamma^{-1}_{nm}(R) \sigma_m^*(y) = C_\phi(x,y) - \frac{c}{T_R} \frac{C_\phi(x,R) C_\phi(R,y)}{1 + \frac{c}{T_R} V_\phi(R)}. \quad \text{(G10)}$$

In the specific case of Neumann BCs with a zero mode we also need the following expression:

$$\sum_{n,m=0}^\infty \sigma_n(0) \Gamma^{-1}_{nm}(R) \Gamma^{-1}_{nm}(L) = LT_\phi + 4 \frac{T_R}{c}, \quad \text{(N)} \quad \text{(G11)}$$

which is obtained by inserting $C_\phi^{(N)}$ [Eq. (3.19)] into Eq. (G10) and performing the limit $\varepsilon \to 0$.

2. Inverse

According to the Sherman-Morrison formula [89], the inverse of $\Gamma$ in Eq. (G3) is given by

$$\Gamma^{-1} = A^{-1} - \frac{G}{1 + u^\dagger A^{-1}u}. \quad \text{(G8)}$$

with $(uu^\dagger)_{nm} = u_n u_m^*$ and

$$G_{nm} \equiv (A^{-1} uu^\dagger A^{-1})_{nm} = (A^{-1} u)_n (A^{-1} u)_m^*, \quad \text{(G9)}$$

where the last equation follows from the fact that $A$ is a real symmetric matrix. Upon introducing the static correlation function $C_\phi$ via Eq. (3.19), one obtains:

$$\sum_{n,m} \sigma_n(x) \Gamma^{-1}_{nm}(R) \sigma_m^*(y) = C_\phi(x,y) - \frac{c}{T_R} \frac{C_\phi(x,R) C_\phi(R,y)}{1 + \frac{c}{T_R} V_\phi(R)}. \quad \text{(G10)}$$

In the specific case of Neumann BCs with a zero mode we also need the following expression:

$$\sum_{n,m=0}^\infty \sigma_n(0) \Gamma^{-1}_{nm}(R) \Gamma^{-1}_{nm}(L) = LT_\phi + 4 \frac{T_R}{c}, \quad \text{(N)} \quad \text{(G11)}$$

which is obtained by inserting $C_\phi^{(N)}$ [Eq. (3.19)] into Eq. (G10) and performing the limit $\varepsilon \to 0$.

Appendix H: Equilibrium distributions of a reactive tracer in $d = 3$ dimensions

The calculation of the equilibrium distributions for $d = 1$ in Sec. VA is extended here to a film geometry in $d > 1$ dimensions. Where necessary, we specialize the calculation to $d = 3$. The film is assumed to have boundaries at
z = 0, L and is macroscopically extended (transverse area A) in the other directions. For the purpose of regularization, we first consider a finite A and perform the thin film limit A → ∞ at the end of the calculation.

Inserting the standard eigenmode expansion of the OP,

$$\phi(r\|, z) = \frac{1}{\sqrt{A}} \sum_{n,p} \sigma_n(z) e^{i p r_i} \phi_n(p),$$  \hspace{1cm} (H1)

into the Hamiltonian in Eq. (2.2) renders

$$\mathcal{H} = \sum_{n,p} \left\{ \frac{1}{2} \sum_{m,q} \phi_n(p) \tilde{\Gamma}_{n,p,m,q} \phi_m(q) - \frac{\hbar}{T_R} \sigma_n(R_z) e^{i p R_{\|}} \phi_n(p) - \frac{\sqrt{A_h}}{T_\phi} [\sigma_n(0) + \sigma_n(L)] \delta_p,0 \phi_n(p) \right\},$$ \hspace{1cm} (H2)

with

$$\tilde{\Gamma}_{n,p,m,q}(R_z) = \frac{1}{T_\phi} (k_n^2 + p^2) \delta_{n,m} \delta_{p,q} + \frac{c}{AT_R} e^{i p R_{\|}} \sigma_n(R_z) e^{-i q R_{\|}} \sigma_{-m}(R_z),$$ \hspace{1cm} (H3)

wherein R_z denotes the z-coordinate of the tracer position. In writing the last term in \(\tilde{\Gamma}\), we used the fact that \(\phi_{-n}(-p) = \phi_n^*(p)\) for a real-valued \(\phi(r)\), such that one recovers the expected contribution \(c^{-2} \sum_{n,\pm} e^{i p R_{\|}} \sigma_n(R_z) \phi_n(p) \phi_m(q)\) in the Hamiltonian. We now separately discuss the case of a linearly and a quadratically coupled tracer.

### 1. Linearly coupled tracer

We assume \(\hbar \neq 0, h_1 = c = 0\) and perform the Gaussian integration over the OP field [see Eqs. (2.12) and (5.2)], which yields the equilibrium distribution of \(R_z\):

$$\bar{P}_s(R_z)|_{c=0} = \frac{1}{Z} \exp[\Phi(R_z)], \quad \Phi(R_z) = \frac{\hbar^2 T_\phi K_{d-1}}{2 T_R^2} \int_0^\infty dp \, p^{d-2} \sum_n |\sigma_n(R_z)|^2,$$ \hspace{1cm} (H4)

with

$$K_d \equiv \Omega_d/(2\pi)^d, \quad \Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)},$$ \hspace{1cm} (H5)

where \(\Omega_d\) is the surface area of the d-dimensional unit sphere. In Eq. (H4), we have performed the thin film limit \(A \to \infty\) by applying the rule

$$\frac{1}{A} \sum_{p} \to \frac{1}{(2\pi)^{(d-1)}} \int_\lambda^\Lambda \, d^{d-1} p,$$ \hspace{1cm} (H6)

where \(\lambda\) and \(\Lambda\) are low- and high-wavenumber cutoffs introduced for the purpose of regularization. Note that the expression for \(\Phi\) in Eq. (H4) essentially corresponds to the variance of the OP \(\phi(R_{\|}, R_z)\) in d dimensions, i.e., \(\Phi(R_z) = \frac{\hbar^2}{2 T_\phi} V_{\phi}(R_{\|} = 0, R_z)\) [cf. Eq. (3.20)], rendering Eq. (H4) fully analogous to Eq. (5.3). In order to further evaluate Eq. (H4), we make use of the fact that [see Eq. (3.1)]

$$|\sigma_n(R_z)|^2 = \begin{cases} \frac{1}{T}, & (p) \\ \frac{1}{L} (1 - \cos(2n \pi R_z/L)), & (D) \\ \frac{1}{L} (1 + \cos(2n \pi R_z/L)), & (N, n \geq 1) \\ \frac{1}{L}, & (N, n = 0) \end{cases}$$ \hspace{1cm} (H7)

In the case of periodic BCs, we accordingly obtain a spatially uniform distribution \(\bar{P}_s\). For the other BCs, we get

$$\Phi(R_z) = \frac{\hbar^2 T_\phi K_{d-1}}{2 T_R^2 L} \sum_{n=1}^\infty \int_\lambda^\Lambda \, dp \, p^{d-2} \frac{1 \mp \cos(2n \pi R_z/L)}{(\pi n/L)^2 + p^2},$$ \hspace{1cm} (H8)
where the $- (+)$ sign applies to Dirichlet (Neumann) BCs. In order to calculate Eq. (H8) in $d = 3$, we first determine the sum over $n$ (see §1.4.5 in [118]) and subsequently integrate over $p$. The result can be written in the form

$$\Phi(R_z) = \hat{\Phi}(R_z) + \Phi_0,$$

where $\Phi_0$ is a $R_z$-independent term that diverges $\sim \lambda^{d-3}$ ($\sim \log \lambda$ in $d = 3$) and $\sim \Lambda^{d-1}$, while the $R_z$-dependent part $\hat{\Phi}(R_z)$ is independent of the cutoffs. Since all $R_z$-independent parts are canceled by the normalization in Eq. (H4), one may omit them in $\hat{\Phi}$ and accordingly obtain

$$\hat{\Phi}(R_z) = \pm \frac{\kappa}{8} \left[ \psi(1 - R_z/L) + \psi(R_z/L) \right],$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function [123] and

$$\kappa = \frac{h^2 T_\phi K_2}{T_R^2 L},$$

is a dimensionless coupling [cf. Eq. (4.3)]. The normalization factor $Z$ in Eq. (H4) has to be computed numerically. The function $\hat{\Phi}$ diverges at the boundaries:

$$\hat{\Phi}(R_z \to 0) \approx \mp \frac{\kappa}{8} \frac{L}{R_z},$$

(analogously for $R_z \to L$), rendering $\tilde{P}_s$ non-normalizable for Neumann BCs. This divergence of $\hat{\Phi}$ is cut off if $\Lambda$ is finite. In a physical system, van der Waals interactions regularize the singular Casimir potential near the boundary [3]. Instead of $\tilde{P}_s$, we thus show in Fig. 17 the quantity $\Phi$, which essentially corresponds to the negative Casimir potential defined in Eq. (6.4), $\mathcal{U}(R_z) = -T\Phi(R_z)$ (setting $T = T_R = T_\phi$).

For $h_1 \neq 0$ (but still $c = 0$), we take Neumann modes for $\sigma_n$ and obtain from Eq. (H2):

$$\Phi(R_z) = \frac{T_\phi}{2} \sum_{n,p} \left| A^{1/2} \tilde{h} \sigma_n(R_z) e^{i\mathbf{R}_p}\mathbf{p} + A^{1/2} \tilde{h}_1 [\sigma_n(0) + \sigma_n(L)] \delta_p \right|^2$$

$$= \frac{T_\phi}{2} \sum_{n} \left\{ K_{d-1} \tilde{h}^2 \int dp p^{d-2} \frac{\sigma_n^2(R_z)}{k_n^2 + \mathbf{p}^2} + 2 \tilde{h}_1 \frac{\sigma_n(0) + \sigma_n(L)}{k_n^2} \right\},$$

with $\tilde{h} \equiv h/T_R$, $\tilde{h}_1 \equiv h_1/T_\phi$ and where, in the last step, we performed the continuum limit $A \to \infty$. As before, we generally omit any constants independent of $R_z$. The required expressions have already been determined in Eqs. (5.10) and (H9), resulting (for $d = 3$) in the exponent

$$\hat{\Phi}(R_z) = -\frac{\kappa}{8} \left[ \psi(1 - R_z/L) + \psi(R_z/L) \right] + LT_\phi \tilde{h}^2 \frac{1}{6} \left[ \left( \frac{R_z}{L} \right) + \left( \frac{R_z}{L} \right) \right]^2,$$

which is illustrated in Fig. 17a,b.

### 2. Quadratic coupling

For $c \neq 0$ and $h = h_1 = 0$, a Gaussian integration over the field modes using Eq. (H2) yields [cf. Eq. (5.5)]

$$\tilde{P}_s(R_z)|_{h=0} = \frac{1}{Z} \left[ \det \tilde{\Gamma}(R_z) \right]^{-1/2}.$$  

(14)

Upon expressing $\tilde{\Gamma}$ in terms of $(u)_{np} = \sqrt{c/(AT_R)} e^{ipR_1 \sigma_n(R_z)}$ and $A_{np,mq} = \delta_{n,m} \delta_{p,q} (k_n^2 + \mathbf{p}^2)/T_\phi$, the determinant can be calculated using the formalism in Appendix [G]. In the thin film limit, this results in [see Eq. (G4)]

$$u^\dagger A^{-1} u = \frac{c T_\phi}{(2\pi)^{d-1} T_R} \sum_n \int d^{d-1} p \frac{\sigma_n(R_z)^2}{k_n^2 + \mathbf{p}^2},$$

(15)

which has essentially been calculated after Eq. (H4). Accordingly, for periodic BCs, it is independent of $R_z$, whereas for Neumann or Dirichlet BCs, the analysis around Eq. (H8) implies that it is dominated by an $R_z$-independent
contribution which diverges $\propto \Lambda^{d-1}$ in the continuum limit. Specifically in $d = 3$, Eq. (H14) can be expressed as

$$P_s = Z^{-1}[1 + c \alpha \Lambda + cf(R_z)]^{-1/2},$$

where $\alpha$ is a constant and $f$ is a function describing the $R_z$-dependent part [in fact, $f \propto \hat{\Phi}$, see Eq. (H9)]. In the limit $\Lambda \to \infty$, one obtains $P_s \approx [1 + f(R_z)/(\alpha \Lambda)])/L \to 1/L$.

Finally, in the case $c \neq 0$ and $h_1 \neq 0$ (but $h = 0$), one finds (for finite $A$)

$$P_s(R_z)\big|_{h_0=0} = \frac{1}{Z} \frac{1}{\det \tilde{\Gamma}(R_z)}^{1/2} \exp \left\{ \frac{1}{2} \tilde{A} \hat{h}_z^2 \sum_{n,m} \left[ \sigma_n(0) + \sigma_n(L) \right] \tilde{\Gamma}(R_z)^{-1}_{nm} \right\}.$$  \hspace{1cm} (H16)

As discussed above [see Eq. (H14)], in the continuum limit $\Lambda \to \infty$, the contribution of the determinant is dominated by a divergent term and is thus canceled by the normalization factor $Z$. The exponent in Eq. (H16) coincides (apart from the prefactor $A$) with its one-dimensional counterpart in Eq. (5.13) and, taking Neumann modes for $\sigma_n$ and using Appendix G, we obtain (with $\rho \equiv R_z/L$)

$$P_s(R_z)\big|_{h_0=0} = \frac{1}{Z} \exp \left\{ -\frac{1}{2} H_1^2 \kappa_c \left[ \frac{\delta - \rho + \rho^2}{1 + \kappa_c} \right]^2 \right\}, \hspace{1cm} (\pm c\pm)$$ \hspace{1cm} (H17)

The (dimensionless) effective couplings $H_1 \equiv h_1 \sqrt{L^d/T_R}$ and $\kappa_c \equiv cL^{2-d}T_0/T_R$ generalize the ones in Eqs. (4.35) and (4.41) to $d$ dimensions. The expression in Eq. (H17) differs from Eq. (5.14) by the absence of the $R_z$-dependent prefactor stemming from the determinant. In the limit $\kappa_c \to \infty$, the probability distribution is given by

$$P_s(R_z)\big|_{h_0=0,\kappa_c\to\infty} = \frac{1}{Z} \exp \left\{ -\frac{1}{2} H_1^2 \left[ \frac{z^2 - \rho + \rho^2}{3} \right]^2 \right\}, \hspace{1cm} (\pm \delta \pm)$$ \hspace{1cm} (H18)

which is independent of $\kappa_c$. The normalization factors $Z$ in Eqs. (H17) and (H18) have to be determined numerically. Analogously to the one-dimensional case [see Eq. (5.15)], the most likely position of the tracer is at a certain distance from the nearest boundary. In the limit $H_1 \to \infty$, Eq. (H18) reduces to two $\delta$-functions:

$$P_s(R_z)\big|_{h_0=0,\kappa_c\to\infty} = \frac{1}{Z} \left[ \delta (R_z - R_-) + \delta (R_z - R_+) \right], \hspace{1cm} (H_1 \to \infty)$$ \hspace{1cm} (H19)

with $R_{\pm} = L \left( \frac{1}{2} \pm \frac{1}{2\sqrt{3}} \right)$, identically to Eq. (5.16). For Neumann BCs with a zero mode, Eq. (G11) implies $P_s(R_z)\big|_{h_0=0} = 1/L$. 

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[46] In Refs. [25, 27], a tracer with \( \zeta = 1 \) is called “active”. However, in order to avoid confusion with other sources of activity, we prefer here the term “reactive”.

[47] As shown in Appendix E, the validity of the adiabatic approximation not only requires \( \chi \), but instead the product \( \frac{L h^2}{T_0} \chi \) (in the linearly coupled case) to be small.

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[50] Note that the reduced distribution \( \hat{P} \) is also different from the full distribution \( P \) introduced in Eq. (2.6). More precisely, we require \( v, w \ll \min(k_{\text{min}}^{-2}/T_R, k_{\text{min}}^{-2+2\zeta}) \).

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M. Gross, A. Gambassi, and S. Dietrich, “Surface-induced nonequilibrium dynamics and critical Casimir forces,” J. Phys.: Condens. Matter 30, 414002 (2018)

O. A. Vasilyev, M. Labbé-Laurent, S. Dietrich, and S. Kondrat, “Probing interface localization–delocalization transitions by colloids,” J. Chem. Phys. 153, 014901 (2020)

J. J. Binney, N. J. Dowrick, A. J. Fisher, and M. E. J. Newman, The Theory of Critical Phenomena (Oxford University Press, 1992).

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D. I. Uzunov, Introduction to the Theory of Critical Phenomena: Mean Field, Fluctuations and Renormalization (World Scientific, Singapore, 1993).

In the context of a Landau-Ginzburg model, a natural regularization is provided by a finite correlation length $\xi \sim \varepsilon^{-1/2}$, which can be achieved by adding the term $\varepsilon \phi^2$ to the integrand in Eq. (2.2).

The complex conjugation is only relevant for periodic BCs and follows by noting that $\sum_{n=-\infty}^{\infty} \sigma_n \phi_n = \sum_{n=-\infty}^{\infty} \sigma_{-n} \phi_{-n} = \sum_{n=0}^{\infty} \sigma_n \phi_n$, see Eq. (3.1a).

A. W. C. Lau and T. C. Lubensky, “State-dependent diffusion: Thermodynamic consistency and its path integral formulation,” Phys. Rev. E 76, 011123 (2007)

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Y. L. Klimontovich, “Ito, Stratonovich and kinetic forms of stochastic equations,” Physica A 163, 515 (1990)

G. Volpe, L. Helden, T. Brettschneider, J. Wehr, and C. Bechinger, “Influence of Noise on Force Measurements,” Phys. Rev. Lett. 104, 170602 (2010)

We remark that the adiabatic elimination procedure presented in Refs. 61, 62 applies only to a linear tracer-OP coupling.

An exception occurs for OP fields subject to periodic BCs, in which case $P_s = 1/L$ for a reactive tracer, while $P_s$ is non-uniform for a passive tracer (see also the discussion in Sec. IVB3).

W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, Numerical Recipes 3rd Edition: The Art of Scientific Computing, 3rd ed. (Cambridge University Press, 2007).

In order to increase the statistical accuracy of the data for conserved dynamics shown in Fig. 11(b), we have taken advantage of the mirror symmetry of $P_s(R)$ around $R = L/2$.

H. Risken, The Fokker-Planck Equation, 2nd ed. (Springer, 1989).

A. G. Cherstvy, A. V. Chechkin, and R. Metzler, “Anomalous diffusion and ergodicity breaking in heterogeneous diffusion processes,” New. J. Phys. 15, 083039 (2013)

N. Leibovich and E. Barkai, “Infinite ergodic theory for heterogeneous diffusion processes,” Phys. Rev. E 99, 042138 (2019).

The behavior is, in fact, similar in the bulk, where $C_{\phi \phi}$ as a function of $R$ is symmetric around $R'$. We remark that the adiabatic elimination procedure presented in Refs. 61, 62 applies only to a linear tracer-OP coupling.

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This strictly applies only when the quadratic potential has its minimum in the center of the system (see Fig. 3). We recall furthermore that a tracer coupled quadratically to an OP subject to boundary fields $h_i$ experiences a quartic potential.

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[116] In contrast to standard field theoretical approaches (see Ref. [124] and references therein), we do not separate here the mean profile $\langle \phi(z) \rangle$ from the fluctuating part beforehand.

[117] The contribution of boundary fields vanishes when representing $\phi$ in terms of Dirichlet modes, which can be readily seen by using $\sigma_n(0) = 0 = \sigma_n(L)$ in the expressions in Eqs. (5.9) and (5.13).

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