A note on separation conditions of resonance sets in the instability analysis for high-frequency oscillations in geometric optics

Jiaojiao Pan*

Abstract

In this paper, we study the instability of highly-oscillating solutions to semi-linear hyperbolic systems. An instability criterion was given in [9] under rather strong separation conditions of resonance sets: coupled resonance sets are pairwise disjoint. Here we show that such separation conditions in [9] can be relaxed: one of the coupled non-transparent resonance sets is allowed to intersect with at most two others. We obtain the same instability criterion as in [9]. Finally, we give some applications to coupled Klein-Gordon systems with equal masses and nonlinear terms specified particularly, where on the intersections of resonance sets, the related interaction coefficients are non-transparent.

Keywords. High-frequency oscillations; separation conditions; WKB solution; Klein-Gordon systems.

Mathematics Subject Classification. 35B35, 35Q60, 35L03.

1 Introduction

1.1 Problem formulation

We study the instability of highly-oscillating solutions to semilinear systems of the form

$$
\partial_t u + \frac{1}{\varepsilon} A_0 u + \sum_{1 \leq j \leq d} A_j \partial_{x_j} u = \frac{1}{\sqrt{\varepsilon}} B(u, u),
$$

(1.1)

in the small wavelength limit \( \varepsilon \to 0 \). Here the unknown \( u = u(t, x) = (u_1, u_2, \cdots, u_N) \in \mathbb{R}^N \), with time variable \( t \in \mathbb{R}_+ \), space variable \( x \in \mathbb{R}^d \). Furthermore, \( A_0 \in \mathbb{R}^{N \times N} \) is skew-symmetric, \( A_j \in \mathbb{R}^{N \times N} \) is symmetric, the source term \( B(u, u) \) is symmetric bilinear, whose specific form can be derived from a phenomenological description of nonlinear interactions [16].

The problem mainly tells the propagation of light with relevant initial highly oscillating condition of the form

$$
u(\varepsilon, 0, x) = a(x)e^{ikx/\varepsilon} + a(x)^*e^{-ikx/\varepsilon} + O(\sqrt{\varepsilon}),
$$

(1.2)

where \( a \) is of high Sobolev regularity, \( k \) is a given wave-number in \( \mathbb{R}^d \), and \( a^* \) denotes the complex conjugation of \( a \). For fixed \( \varepsilon > 0 \), the classical theory on symmetric hyperbolic systems gives the existence
and uniqueness of local-in-time solutions to (1.1)-(1.2) in Sobolev spaces $H^s$ with $s > d/2$ and the a priori existence time is $O(\varepsilon^{1/2})$.

Under certain compatibility condition, an approximate solution for (1.1)-(1.2) over time interval $[0, T]$ with $T > 0$ independent of $\varepsilon$ can be constructed via WKB expansion. A main concern is the stability of such WKB solutions. We say a WKB approximate solution $u_a$ is stable if the real solution $u$ stays close to $u_a$ in some time interval $[0, T]$ with $T$ independent of $\varepsilon$, and a WKB solution $u_a$ is unstable provided in some short interval $[0, T_\varepsilon]$ with $T_\varepsilon \to 0$, $|u - u_a|$ is much amplified compared to the initial differential.

Transparency is introduced in by Joly, Métévier, Rauch (see [4], see also Dumas’s survey [2]) which ensures the stability of WKB solution and is analogous to the null conditions which imply global existence for nonlinear wave equations, see Klainerman’s classical work [7]. The link between transparency and null forms can be seen in Lannes’ Bourbaki review [8].

Joly, Métévier and Ranch [4] considered Maxwell-Bloch systems in the critical regime of geometric optics and the transparency conditions are verified. Consequently, the existence and the stability of WKB solutions to Maxwell-Bloch equations in a supercritical regime are obtained. Following Joly, Métévier and Rauch, it was verified by Texier [18, 19] that the Euler-Maxwell equations satisfy a form of transparency, and by Lu that the Maxwell-Landau-Lifschitz equations also are transparent in one dimensional setting [10]. Cheverry, Guès and Métévier [1] showed that for systems of conservation laws, linear degeneracy of a field implies transparency. Jeanne [6] showed that the Yang-Mills equations provide another example of a physical system exhibiting transparency properties.

A further question is that: if the transparency conditions are not satisfied, will the WKB approximate solutions become unstable or not? In [9], Texier and Lu gave a rather complete description on the stability of WKB solutions to a class of semi-linear hyperbolic systems issued from highly-oscillating initial data with large amplitudes. They found that the stability of WKB solution is determined by the sign of some stability character index $\Gamma$ determined by the linear operator of the equation and nonlinear form of the equation. Roughly speaking, if $\Gamma > 0$ which ensures the symbol of the linearized operator around a WKB approximate solution admits an eigenvalue with positive real part, the solution of the linearized system has certain exponential growth and thus is unstable; if $\Gamma < 0$ which ensures the eigenvalues of the symbol of the linearized operator are pure imaginary, the solution of the linearized system stay bounded and is stable. While, if $\Gamma = 0$ and the transparency conditions are not satisfied, the stability analysis of the WKB solutions is more delicate, and it seems both stability and instability are possible. For example, in [12], Lu considered a system of two coupled Klein-Gordon equations with different velocities and different masses and instability of the WKB solutions is discovered. With a proper choice of nonlinear source terms, it was shown in [12] that even the equations linearized around the leading WKB terms are initially stable (transparency conditions are satisfied), while the resonances associated with the higher-order harmonics of the WKB solutions generated by the nonlinearities can also generate instantaneous instabilities. In particular, such higher-order harmonics are not present in the data. These studies are rated to the work [13] and [14] where the instability phenomena with loss of hyperbolicity are studied. Later in [11], Zhang and Lu considered the non relativistic limit of the Klein-Gordon equations with quadratic nonlinearities. It is introduced compatible conditions weaker than the strong transparency conditions and a singular localization method to prove the stability of WKB solutions over long time intervals in [11]. In particular, such weaker compatible conditions ensure $\Gamma = 0$.

In [9], a key assumption is the resonance sets are essentially pairwise disjoint: related interaction coefficients are transparent in corresponding intersections of the resonance sets (see Assumption 1.4).
However, such reparation conditions of resonance sets may not hold in practice. For example in Section 5.2 in [9] concerning a coupled Klein-Gordon system with equal masses and different velocities, the separation conditions are satisfied only in one dimensional setting. Actually, the resonance sets often intersect in physical models. Our goal in this paper is to offer certain weaker separation conditions which still ensure the main instability results in [9] hold. This will make the abstract instability criterion theory in [9] applicable to more physical models.

This paper is organized as follows. In Section 1.1 to Section 1.3, we mainly introduce the formulation of the problem, the existing instability criterion (Theorem 1.6) and estimates of symbolic flows related to 2×2 block matrices for non-transparent resonance in [9]. We give new relaxed separation conditions related to 2-coupled non-transparent resonances and 3-non-transparent resonances coupled in pairs and present new instability theorem in Section 1.4. In Section 1.5, main ideas of the proof for Theorem 1.11 are presented. What’s more, we show the the estimates of symbolic flows related to 3×3 block matrices for 2-coupled non-transparent resonances and 3-non-transparent resonances coupled in pairs respectively in Section 2 and Section 3, and obtain new instability criterion in Theorem 1.11. Then we give an application in Section 4 for new relaxed instability criterion which contains coupled Klein-Gordon systems with equal masses whose nonlinear terms are specified particularly. Finally, we show a more complete result about the instability for coupled Klein-Gordon systems with equal masses given in [9].

1.2 Assumptions

Here we recall some basic assumptions introduced in [9], where the stability character is given.

**Assumption 1.1.** Assume that the matrix $A_0$ is real skew-symmetric, the matrices $A_j (1 \leq j \leq d)$ are real symmetric, and the hermitian matrices $\{A_0/i + \sum_{1 \leq j \leq d} \xi_j A_j\}_{\xi \in \mathbb{R}^d}$ have the spectral decomposition

$$A_0/i + \sum_{1 \leq j \leq d} \xi_j A_j = \sum_{1 \leq j \leq J} \lambda_j(\xi) \Pi_j(\xi), \quad (1.3)$$

where $\lambda_j(\xi)$ are smooth eigenvalues and $\Pi_j(\xi)$ are smooth eigenprojectors satisfying the following bounds

$$|\partial_\xi^\alpha \lambda_j(\xi)| \leq C_\beta (1 + |\xi|^2)^{(1 - |\beta|)/2}, \quad |\partial_\xi^\alpha \Pi_j(\xi)| \leq C_\beta (1 + |\xi|^2)^{(-|\beta|)/2}, \quad \forall \beta \in \mathbb{N}^d, \quad (1.4)$$

**i.e.** $\lambda_j(\cdot) \in S^1$ and $\Pi_j(\cdot) \in S^0$. Here $S^m, \quad m \in \mathbb{R}$ denotes the set of matrix-valued symbols $a \in C^\infty(\mathbb{R}^d; C^\infty(\mathbb{R}^d))$ satisfying that for $\forall \alpha, \beta \in \mathbb{N}^d$ with $|\alpha| \leq \bar{s}$ and some $C_{\alpha \beta} > 0$, for $\forall (x, \xi), \quad \bar{s}$

$$|\partial_\xi^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha \beta} (\xi)^{|m-|\beta|}, \quad \langle \xi \rangle := (1 + |\xi|^2)^{\frac{1}{2}}. \quad (1.5)$$

**Assumption 1.2.** For some $K_a \in \mathbb{N}$, some $T_a > 0$, there exists an approximate solution $u_a$ to (1.1) in $[0, T_a]$ satisfying

$$\partial_t u_a + \frac{1}{\varepsilon} A_0 u_a + \sum_{1 \leq j \leq d} A_j \partial_{x_j} u_a = \frac{1}{\sqrt{\varepsilon}} B(u_a, u_a) + \varepsilon^{K_a} v_a. \quad (1.6)$$

The approximate solution has the form of a WKB expansion

$$u_a(\varepsilon, t, x) = e^{-i(k \cdot x - wt)/\varepsilon} u_{0, -1}(t, x) + e^{i(k \cdot x - wt)/\varepsilon} u_{0, 1}(t, x) + \sqrt{\varepsilon} v_a(\varepsilon, t, x) \in \mathbb{R}^N,$$
where the phase \((w, k)\) is a characteristic for the hyperbolic operator satisfying:

\[
(\partial_\tau + \frac{1}{\varepsilon} A_0 + \sum_{1 \leq j \leq d} A_j \partial_{x_j})(e^{\pm i(k \cdot x - wt)/\varepsilon} \vec{e}_{1}) = 0, \quad \vec{e}_{-1} = (\vec{e}_{1})^* \tag{1.7}
\]

with \(\vec{e}_{1}\) and \(\vec{e}_{-1}\) are fixed unit vectors in \(\mathbb{C}^N\), and \(u_{0, \pm 1}\) are leading amplitudes polarized along \(\vec{e}_{\pm 1}\) i.e.

\[
u_{0,1}(t,x) = g(t,x)\vec{e}_{1}, \quad u_{0,-1}(t,x) = g(t,x)^*\vec{e}_{-1}, \quad g \in C^1([0,T_a], H^s_\alpha(\mathbb{R}^d)).
\]

At the same time, there holds \(v_a, r_a^\varepsilon \in C^0([0,T_a], H^s_\alpha(\mathbb{R}^d))\) and

\[
\sup_{\varepsilon > 0} \sup_{|\alpha| \leq s_\alpha} (|\varepsilon \partial_\alpha (v_a, r_a^\varepsilon)|_{L^\infty[0,T],L^2} + |(\mathcal{F}(v_a, r_a^\varepsilon))|_{L^\infty[0,T_a],L^1}) < \infty.
\]

With certain compatibility conditions (weak transparency conditions), Assumption 1.2 holds true, see [4] or [9]. In particular, \(v_a, r_a^\varepsilon\) are trigonometric polynomials in \(\theta\) with \(\theta := (k \cdot x - wt)/\varepsilon\). Given the perturbation unknown \(\hat{u}\) as

\[
u = v_a + \hat{u},
\]

then we have the perturbed system

\[
\partial_\tau \hat{u} + \frac{1}{\varepsilon} A_0 \hat{u} + \sum_{1 \leq j \leq d} A_j \partial_{x_j} \hat{u} = \frac{1}{\sqrt{\varepsilon}} B(u_a) \hat{u} + \frac{1}{\sqrt{\varepsilon}} B(u, \hat{u}) - \varepsilon^K_a r_a^\varepsilon. \tag{1.8}
\]

where \(B(\vec{u})v := B(\vec{u}, v) + B(v, \vec{u})\) is a bilinear

**Definition 1.3.** Given \(i, j \in \{1, \cdots, J\}\), with \(J\) defined in Assumption 1.1, the \((i,j)\)-resonance set is defined as follows

\[
\mathcal{R}_{ij} := \{\xi \in \mathbb{R}^d, \omega = \lambda_i(\xi + k) - \lambda_j(\xi)\}. \tag{1.9}
\]

For \(\xi \in \mathbb{R}^d\), the matrices

\[
\Pi_i(\xi + k)B(\vec{e}_{1})\Pi_j(\xi) \in \mathbb{C}^{N \times N}, \quad \Pi_j(\xi)B(\vec{e}_{1})\Pi_i(\xi + k) \in \mathbb{C}^{N \times N}
\]

are called \((i,j)\)-interaction coefficients, and \((\lambda_i(\xi + k) - \lambda_j(\xi) - \omega)\) \((1 \leq i, j \leq J)\) are called \((i,j)\)-resonant phase. Furthermore, interaction coefficients \(\Pi_i(\xi + k)B(\vec{e}_{1})\Pi_j(\xi)\) or \(\Pi_j(\xi)B(\vec{e}_{-1})\Pi_i(\xi + k)\) is said to be transparent if for some \(C > 0\), there holds for all \(\xi \in \mathbb{R}^d\),

\[
|\Pi_i(\xi + k)B(\vec{e}_{1})\Pi_j(\xi)| \leq C|\lambda_i(\xi + k) - \lambda_j(\xi) - \omega|,
\]

or

\[
|\Pi_j(\xi)B(\vec{e}_{-1})\Pi_i(\xi + k)| \leq C|\lambda_i(\xi + k) - \lambda_j(\xi) - \omega|.
\]

If both \((i,j)\)-interaction coefficients are transparent, then \((i,j)\)-resonance is said to be transparent. Denote \(\mathcal{R} = \{(i,j), \mathcal{R}_{ij} \neq \emptyset\}\) the set containing all the non-empty resonance indices and \(\mathcal{R}_0 \subset \mathcal{R}\) the set of indices of which at least one of the interaction coefficients are non-transparent, i.e. \(\mathcal{R}_0 = \{(i,j), \mathcal{R}_{ij} \neq \emptyset \text{ and } (i,j)\text{-resonance is non-transparent}\}\).
Denote $\Gamma_{ij}$ the trace of the product of the $(i,j)$-interaction coefficients:

$$
\Gamma_{ij}(\xi) := \text{tr} \Pi_i(\xi + k) B(\xi_1) \Pi_j(\xi) B(\xi_{-1}) \Pi_i(\xi + k). \quad (1.10)
$$

The stability index is given as follows

$$
\Gamma = \begin{cases}
-1, & \text{if } \max_{(i,j)\in \mathcal{R}_0} \sup_{\xi\in \mathcal{R}_{ij}} \text{Re} \Gamma_{ij}(\xi) < 0 \text{ and } \max_{(i,j)\in \mathcal{R}_0} \sup_{\xi\in \mathcal{R}_{ij}} |\text{Im} \Gamma_{ij}(\xi)| = 0, \\
1, & \text{if } \max_{(i,j)\in \mathcal{R}_0} \sup_{\xi\in \mathcal{R}_{ij}} \text{Re} \Gamma_{ij}(\xi) > 0 \text{ or } \max_{(i,j)\in \mathcal{R}_0} \sup_{\xi\in \mathcal{R}_{ij}} |\text{Im} \Gamma_{ij}(\xi)| \neq 0.
\end{cases} \quad (1.11)
$$

The main results in [9] say that if $\Gamma = 1$, then with proper choice of initial data, the WKB solutions are unstable in a short time interval of order $O(\sqrt{\varepsilon} \ln \varepsilon)$, and if $\Gamma = -1$, the WKB solutions are stable in time interval of order $O(1)$.

And furthermore, we give several necessary notations

$$
\gamma_{ij} := |\max_{\xi \in \mathcal{R}_{ij}} \text{Re}(\Gamma_{ij}(\xi) \frac{1}{2})|,
$$

$$
\gamma_{ij}^+ := |a|_{L^\infty} |\max_{\xi \in \mathcal{R}_{ij}^h} \text{Re}(\Gamma_{ij}(\xi) \frac{1}{2})|,
$$

$$
\gamma^+ := \max_{(i,j)\in \mathcal{R}_0} \gamma_{ij}^+.
$$

where $\mathcal{R}_{ij}^h$ is a neighborhood $\mathcal{R}_{ij}$ given as

$$
\mathcal{R}_{ij}^h := \{\xi \in \mathbb{R}^d, |\lambda_1(\xi + k) - \lambda_1(\xi) - \omega| \leq h\}. \quad (1.14)
$$

Given a frequency cut-off function $\chi_{ij}(\xi) \in C_c^\infty(\mathbb{R}^d)$ satisfying $0 \leq \chi_{ij}(\xi) \leq 1$, $\chi_{ij}(\xi) \equiv 1$ on the neighborhood $\mathcal{R}_{ij}^h$ of resonant set $\mathcal{R}_{ij}$, we use $\chi_{ij}^{\#}(\xi)$ to denote an extension of $\chi_{ij}(\xi)$ in the sense that $(1 - \chi_{ij}^{\#}(\xi))\chi_{ij}(\xi) = 0$. In a similar way, we can define $\varphi_{ij}^{\#}(x)$ an extension of $\varphi_{ij}(x) \in C_c^\infty(\mathbb{R})$ satisfying $(1 - \varphi_{ij}^{\#}(x))\varphi_{ij}(x) = 0$ with $\varphi_{ij}(x) \equiv 1$ on a neighborhood of $x_0$.

Here we recall $S^m$ the space of classical symbols of order $m$. Assuming $a \in S^m$, we denote $\text{op}_\varepsilon(a)$ the pseudo-differential operators in semi-classical quantization who act on functions or distributions $u(x)$ with

$$
\text{op}_\varepsilon(a)u = \int e^{ix\xi}a(x,\varepsilon\xi)\hat{u}(\xi)d\xi, \quad \varepsilon > 0,
$$

and semi-classical norm $\| \cdot \|_{\varepsilon,s}$ with

$$
\|u\|_{\varepsilon,s}^2 = \int (1 + |\varepsilon\xi|^2)^s|\hat{u}(\xi)|^2d\xi.
$$

It is rather clear that there holds the identity $\text{op}_\varepsilon(\sigma)(e^{ip\theta}v) = e^{ip\theta}\text{op}_\varepsilon(\sigma_\varepsilon)(v)$ with $\sigma_\varepsilon(x,\xi) := \sigma(x,\xi + pk)$. For the case of coupled non-transparent resonances, we recall the following separation conditions.
Assumption 1.4. Given \((i, j) \in \mathbb{R} \setminus \mathbb{R}_0\), the \((i, j)\)-resonance is transparent. Given \((i, j) \in \mathbb{R}_0\), the \((i, j)\)-interaction coefficients are transparent on a neighborhood of

\[
\mathcal{R}_{ij} \cap ((\mathcal{R}_{i'i} - k) \cup (\mathcal{R}_{j'j} + k))
\]

for all \(i', j'\) with \((i', i) \in \mathbb{R}_0\), \((j', j) \in \mathbb{R}_0\), and on a neighborhood of

\[
\mathcal{R}_{ij} \cap (\mathcal{R}_{ii'} \cup \mathcal{R}_{jj'})
\]

for all \(i' \neq j', j' \neq i\), with \((i, i') \in \mathbb{R}_0\), \((j', j) \in \mathbb{R}_0\).

Assumption 1.5. Suppose

- (Boundedness) The resonant set in \(\mathbb{R}\) is bounded.
- (Rank-one coefficients) For all \((i, j) \in \mathbb{R}_0\), for all \(\xi\) in an open set containing \(\mathcal{R}_{ij}\), the ranks of the \((i, j)\)-interaction coefficients are at most 1; except for \((i, j) \in \mathbb{R}_0\) such that one interaction coefficient is identically equal to zero, in which case we make no assumptions on the rank of the other coefficient.

From the form of the stability index \(\Gamma\) in (1.11), it is much more likely that \(\Gamma = 1\) which corresponds to the instability result. Here we will focus on the instability case and we recall Theorem 2.13 in [9] concerning the instability of the WKB solutions.

In the context of Assumption 1.4, for \((i, j)\) ranging over the set \(\mathbb{R}_0\) of non-transparent resonances, the maximum \(\gamma\) of coefficients \(\gamma_{ij}\) is attained at \((i_0, j_0)\). Consider the datum as

\[
u(0, x) := u_a(0, x) + \varepsilon^k e^{ix(\xi_0 + k)/\varepsilon} \varphi_{i_0j_0}(x) \bar{c}_{i_0j_0},
\]

where

- \(\xi_0\) is such that \(\gamma := |\max_{\xi \in \mathcal{R}_{i_0j_0}} \text{Re}(\Gamma_{i_0j_0}(\xi)^{1/2})|\) is attained at \(\xi_0\);
- \(x_0\) is such that \(|a|_{L^\infty}\) is attained at \(x_0\);
- \(\varphi_{i_0j_0} \in C_0^\infty(\mathbb{R}^d)\) is a scalar spatial truncation on a neighborhood of \(x_0\);
- \(\bar{c}_{i_0j_0}\) is some fixed constant eigenvector corresponding to the eigenvalue of \(M\) which generates exponential grow \(O(e^{s \gamma^{1/2}})\).

Theorem 1.6. Let \(T_\infty := T_{\infty}/\gamma_{|a|_{L^\infty}}\), under Assumptions 1.1, 1.2, 1.4 and 1.5, in the unstable case \(\Gamma > 0\), for any \(K > 0\), if \(K_a + \frac{1}{3} \geq K\), for \(d/2 < s \leq s\alpha\):

- either for some \(T < T_\infty\), for any \(\varepsilon\) small enough, the initial-value problem (1.1)-(1.15) does not have a solution \(u \in \mathcal{C}^0([0, T \sqrt{\varepsilon}] \ln \varepsilon], H^s(\mathbb{R}^d))\);
- or for some \(T < T_\infty\), for any \(\varepsilon_0 > 0\), the solution \(u\) to (1.1)-(1.15) satisfies

\[
\sup_{0 < \varepsilon < \varepsilon_0} \sup_{0 \leq t \leq T \sqrt{\varepsilon} \ln \varepsilon} |u(t, \cdot)|_{L^\infty} = \infty;
\]

- or for any \(K' > 0\), for some \(T < T_\infty\), there holds the deviation estimate

\[
\sup_{0 < \varepsilon < \varepsilon_0} \sup_{0 \leq t \leq T \sqrt{\varepsilon} \ln \varepsilon} \varepsilon^{-K'} |(u - u_a)(t)|_{L^2} = \infty,
\]

for some \(x_0 \in \mathbb{R}^d\), some \(\beta > 0\), some \(\varepsilon_0 > 0\).
1.3 Estimates of the symbolic flow

The proof of the instability in [9] of the WKB solutions when \( \Gamma > 0 \) relies on a short-time Duhamel representation formula for solutions of zeroth-order pseudo-differential equations. To achieve such Duhamel representation formula, it is crucial to obtain the optimal upper bounds for the flows of the symbols of the pseudo-differential operators. After frequency localizations near \((i,j)\)-resonance set and normal form reductions, the separation conditions in Assumption 1.4 ensure that except the \((i,j)\)-interaction coefficients, all others will be eliminated. As a result, the study of the symbolic flow reduces to the local equations in \(2 \times 2\) block matrices such as the following \(M_{ij}\) in (1.18). We recall the following lemma in [9] which concludes the main results about the upper bounds for the symbolic flows of \(2 \times 2\) block matrices.

**Lemma 1.7.** Under Assumptions 1.1, 1.2, 1.4 and 1.5, for all \(T > 0\) and all \(0 \leq \tau \leq t \leq T|\ln \varepsilon|, \alpha \in \mathbb{N}^d\), the solution \(S_0\) to

\[
\partial_t S_0 + \frac{1}{\sqrt{\varepsilon}} M_{ij} S_0 = 0, \quad S_0(\tau, \tau) = \text{Id}
\]

satisfies the bound

\[
|\partial^\alpha_x S_0(\tau, t)| \lesssim |\ln \varepsilon|^* \exp(t\gamma_{ij}^+),
\]

with the block matrices of non-transparent coefficients \(\{M_{ij}\}_{(i,j)\in \mathbb{N}_0}\) defined as

\[
M_{ij}(\varepsilon, t, x, \xi) = \chi_{ij}^\# \begin{pmatrix} i(\lambda_{i+1} - \omega) & -\sqrt{\varepsilon} \varphi_{ij}^\# \chi_{ij} b_{ij}^+ \\ -\sqrt{\varepsilon} \varphi_{ij}^\# \chi_{ij} b_{ji}^- & i\lambda_j \end{pmatrix},
\]

where

\[
b_{ij}^\pm = g(\sqrt{\varepsilon} t, x)\Pi_{i+1} B_1 \Pi_j, \quad b_{ji}^- = g^*(\sqrt{\varepsilon} t, x)\Pi_j B_{-1} \Pi_{i+1}
\]

and \(1 \leq i, j \leq J, i \neq j\). If the \(M_{ij}\) in (1.18) is replaced by

\[
M_{ij}(\varepsilon, t, x, \xi) = \chi_{ij}^\# \begin{pmatrix} i\lambda_{i+1} & -\sqrt{\varepsilon} \varphi_{ij}^\# \chi_{ij} b_{ij}^+ \\ -\sqrt{\varepsilon} \varphi_{ij}^\# \chi_{ij} b_{ji}^- & i(\lambda_j + \omega) \end{pmatrix},
\]

the proof in [9] shows the same bound for symbolic flows of \(2 \times 2\) block matrices in (1.17). Furthermore, \(|\ln \varepsilon|^*\) denotes \(|\ln \varepsilon|^{N^*}\) for some \(N^* > 0\) independent of \((\varepsilon, \tau, t, x, \xi)\) and the support of \(\chi_{ij}^\#\) is a small neighborhood of \(\mathcal{R}_{ij}\). We specifically suppose \(\chi_{ij}^\#\) be a smooth cut-off function in frequency space with \(0 \leq \chi_{ij}^\# \leq 1, \chi_{ij}^\# = 1\) on a neighborhood of \(\mathcal{R}_{ij}^h\) and \(\chi_{ij}^\# = 0\) away from \(\mathcal{R}_{ij}^h\), such as \(\mathbb{R}^d \setminus \mathcal{R}_{ij}^{2h}\).

Assumption 1.4 concerning the separation conditions of the resonance sets is the key to reduce the massive matrix of interaction coefficients into some \(2 \times 2\) block matrices. After different normal form reduction and space-frequency localization, we now consider the following \(3 \times 3\) matrices cases

\[
M = \begin{pmatrix} i\mu_1 & -\sqrt{\varepsilon} b_{12} & -\sqrt{\varepsilon} b_{13} \\ -\sqrt{\varepsilon} b_{21} & i\mu_2 & -\sqrt{\varepsilon} b_{23} \\ -\sqrt{\varepsilon} b_{31} & -\sqrt{\varepsilon} b_{32} & i\mu_3 \end{pmatrix}.
\]

Here we can not deal with the cases where all the interaction coefficients in \(3 \times 3\) block matrices are non-transparent, while if some of the interaction coefficients in \(3 \times 3\) block matrices are transparent, the estimate
of the symbolic flow can be obtained. Furthermore, we give relaxed separation conditions and reasonable conditions in Assumption 1.8 to show those specific $3 \times 3$ block matrices for which we can also obtain the estimate of symbolic flow finally.

1.4 Main results

For the sake of simplicity, we give the following reasonable conditions.

**Assumption 1.8.** *(Reasonable conditions)* Assume $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_J$, $R_{ij} \cap (R_{ij} + k) = \emptyset$ and $\forall (i, j) \in R_0 \Rightarrow i < j$.

Combining the property that the nonzero eigenvalue $\lambda_i(\xi) \to \infty$ as $|\xi| \to \infty$, we acquiescently choose $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_J$ in this paper. We select a characteristic temporal frequency $\omega \in \mathbb{R}$ associated with initial wavenumber $k \in \mathbb{R}^d$ satisfying that the phase $\beta = (\omega, k) = (\lambda_1(k), k)$ belongs to the fastest positive branch on the variety. Then it is reasonable to assume the resonance occurs only when $i < j$ for $R_{ij}$ (i.e. $\lambda_i \geq \lambda_j$ for $R_{ij}$). Thus under reasonable conditions in Assumption 1.8, we can conclude following three cases for 2-coupled non-transparent resonances: $(i, j), (j, j^{'})$-resonances, $(i, j), (i, j^{'})$-resonances, and $(i, j^{'}, (j, j^{'})$-resonances, where $i < j < j^{'}$ and $\{(i, j), (i, j^{'}, (j, j^{'})\} \subset R_0$. Combined with these considerations, we can reduce Assumption 1.4 to the following relaxed separation conditions Assumption 1.9.

**Assumption 1.9.** *(Relaxed separation conditions)* Let $R_0 \subset \mathbb{R}$ be the collection of non-transparent pairs, i.e. if $(i, j) \in \mathbb{R} \setminus R_0$, $(i, j)$-resonance is transparent. Assume that at least one of the following statements holds:

(1). For any $i < j < j'$ such that $(i, j), (j, j') \in \mathbb{R}_0$, at least one of $b_{ij}^{+}, b_{jj}^{-}$ is transparent on $R_{jj'} \cap (R_{ij} - k)$, or at least one of $b_{ij}^{-}, b_{jj}^{+}$ is transparent on $R_{ij} \cap (R_{jj'} + k)$;

(2). For any $i < j < j'$ such that $(i, j), (i, j') \in \mathbb{R}_0$, at least one of $b_{ij}^{+}, b_{ji}^{-}$ is transparent on $R_{ij} \cap R_{ij'}$, or at least one of $b_{ij}^{-}, b_{ji}^{+}$ is transparent on $R_{ij} \cap R_{ij'}$;

(3). For any $i < j < j'$ such that $(i, j'), (j, j') \in \mathbb{R}_0$, at least one of $b_{jj}^{+}, b_{jj'}^{-}$ is transparent on $R_{ij'} \cap R_{jj'}$, or at least one of $b_{jj}^{-}, b_{jj'}^{+}$ is transparent on $R_{ij'} \cap R_{jj'}$.

Furthermore, we considered the 3-non-transparent resonances coupled in pairs which can be concluded as $R_1 = \{(i, j), (i, j'), (j, j')\} \subset \mathbb{R}_0$ and give the following assumption.

**Assumption 1.10.** For non-transparent resonances in $R_1$, anyone of the following statements (separation conditions) holds:

(1). If $R_{ij} \cap R_{ij'} \cap R_{jj'} \neq \emptyset$, $b_{jj}^{+}$ (or $b_{jj}^{-}$) is transparent in $R_{jj'}$ (or $R_{ij}$) and anyone of interaction coefficients in $R_1 \setminus (j, j')$ (or $R_1 \setminus (i, j)$) is transparent on corresponding resonance set;

(2). If $R_{ij} \cap R_{ij'} \cap (R_{jj} + k) \neq \emptyset$, $b_{ij}^{+}$ (or $b_{ij}^{-}$) is transparent in $R_{jj'}$ (or $R_{ij'}$) and anyone of interaction coefficients in $R_1 \setminus (j, j')$ (or $R_1 \setminus (i, j')$) is transparent on corresponding resonance set;
(3). If $\mathcal{R}_{ij} \cap \mathcal{R}_{ij'} + k \cap \mathcal{R}_{jj'} + k \neq \emptyset$, $b_{ij}^+$ (or $b_{ij}^-$) is transparent in $\mathcal{R}_{ij'}$ (or $\mathcal{R}_{ij}$) and anyone of interaction coefficients in $\mathcal{R}_1 \setminus (i, j')$ (or $\mathcal{R}_1 \setminus (i, j)$) is transparent on corresponding resonance set.

Usually, we call $\mathcal{R}_{ij}$ the corresponding resonance set of interaction coefficients $b_{ij}^+$ and $b_{ji}^-$.

**Theorem 1.11.** Under reasonable conditions in Assumption 1.8, we have the following results:

(i). Theorem 1.6 holds if Assumption 1.4 in Theorem 1.6 is substituted by Assumption 1.9;

(ii). Theorem 1.6 holds if Assumption 1.4 in Theorem 1.6 is substituted by Assumption 1.10.

1.5 Ideas of the proof

This paper generalizes the main results in [9] and allows non-separation of resonance sets. To overcome extra difficulties caused by the intersection of resonance sets, there are two main new ideas compared to [9]:

1. Careful choices of frequency shit are given to delete oscillations $e^{ip\theta}$ in $M_{ijj'}$. Since the oscillations $e^{ip\theta}$ with interaction coefficients appear in the perturbed system, appropriate frequency shift is needed here to eliminate the oscillations. For example, for any $i < j < j'$ such that $(i, j), (j, j') \in \mathbb{R}_0$, we give the frequency shift on $U_i, U_j$ and $U_{j'}$ as $U_i = e^{-i\theta} \text{op}_{\epsilon}(\Pi_i) \dot{u}, U_j = \text{op}_{\epsilon}(\Pi_j) \dot{u}$ and $U_{j'} = e^{i\theta} \text{op}_{\epsilon}(\Pi_{j'}) \dot{u}$. Then the perturbed system (1.8) can be diagonalized and furthermore, using normal form reduction and under a $O(\sqrt{\epsilon})$ remainder, the oscillations $e^{ip\theta}$ are eliminated with interaction coefficients left supported in corresponding resonance sets. For this case, any other frequency shifts are not ready to delete oscillations $e^{ip\theta}$, that is the reason why we need to choose suitable frequency shift carefully. While for 3-non-transparent resonances coupled in pairs, frequency shift and normal form reduction can not delete all the oscillations. Under such conditions, we give several reasonable conditions to help remove the oscillations and finally it is reduced to one of 2-coupled non-transparent resonances. What’s more, we find that different frequency shifts used in 2-coupled non-transparent resonances are also applicable for 3-non-transparent resonances coupled in pairs if we similarly change separation conditions, which offer us two more possibilities to give the final instability of WKB approximate solution $u_a$.

2. Estimates for the symbolic flow related to $3 \times 3$ block matrices $M_{ijj'}$ are established. After normal form reduction, non-transparent interaction coefficients supported in corresponding resonance sets are left without oscillations, which provides us the convenience for the following space-frequency localization. Utilizing the estimates for symbolic flow related to $2 \times 2$ block matrices $\{M_{ij}\}_{(i,j) \in \mathbb{R}_0}$, which is given in Lemma 1.7, the key point is the classified discussion in different non-transparent sets and their intersections. Especially in their intersections, separation conditions will help reduce one of the non-transparent interaction coefficients and successfully give the estimates for the symbolic flow. Whether in cases of 2-coupled non-transparent resonances or 3-non-transparent resonances coupled in pairs, classified discussion combined with separation conditions and reasonable conditions show the estimates for the symbolic flow and finally give the instability by the same method of Duhamel representation in [9].

**Remark 1.12.** For 3-non-transparent resonances coupled in pairs in $\mathbb{R}_1$ in Assumption 1.10, we choose suitable reasonable condition $\mathcal{R}_{ij} \cap (\mathcal{R}_{ij} + k) = \emptyset$ and separation conditions to wipe out the oscillation
items and reduce the case to one of the cases talked above in Assumption 1.9 describing 2-coupled non-transparent resonances. From Theorem 1.6 and Theorem 1.11, it can be observed that if non-transparent resonance set $\mathcal{R}_0$ for the highly-oscillating solutions to semi-linear systems is composed of $\mathcal{R}_1$ satisfying conditions in Assumption 1.10, 2-coupled non-transparent resonances satisfying relaxed separation conditions in Assumption 1.9 and 1-single non-transparent resonance, we can also verify the WKB approximate solution $u_n$ satisfies the instability result in Theorem 1.6.

2 Cases of two coupled resonances

To prove the first result in Theorem 1.11, there is no influence if we consider the following specific assumptions of separation conditions. Given $(i, j, j') = (1, 2, 3)$, Assumption 1.9 becomes:

1. Assume $(i, j) = (1, 2), (j, j') = (2, 3) \in \mathcal{R}_0$, at least one of $b_{23}^+, b_{32}^-$ is transparent on $\mathcal{R}_{23} \cap (\mathcal{R}_{12} - k)$, or at least one of $b_{12}^+, b_{21}^-$ is transparent on $\mathcal{R}_{12} \cap (\mathcal{R}_{23} + k)$;

2. Assume $(i, j) = (1, 2), (i, j') = (1, 3) \in \mathcal{R}_0$, at least one of $b_{13}^+, b_{31}^-$ is transparent on $\mathcal{R}_{12} \cap \mathcal{R}_{13}$, or at least one of $b_{12}^+, b_{21}^-$ is transparent on $\mathcal{R}_{12} \cap \mathcal{R}_{13}$;

3. Assume $(i, j') = (1, 3), (j, j') = (2, 3) \in \mathcal{R}_0$, at least one of $b_{23}^+, b_{32}^-$ is transparent on $\mathcal{R}_{13} \cap \mathcal{R}_{23}$, or at least one of $b_{13}^+, b_{31}^-$ is transparent on $\mathcal{R}_{13} \cap \mathcal{R}_{23}$.

Now we need to show the instability in Theorem 1.6 holds under Assumption 1.1, Assumption 1.2, Assumption 1.5 and furthermore, any one of above three cases of assumptions.

2.1 Case 1

We start by considering the first case where $(1, 2), (2, 3) \in \mathcal{R}_0$, at most one of $b_{23}^+, b_{32}^-$ is non-transparent on $\mathcal{R}_{23} \cap (\mathcal{R}_{12} - k)$, or at most one of $b_{12}^+, b_{21}^-$ is non-transparent on $\mathcal{R}_{12} \cap (\mathcal{R}_{23} + k)$.

2.1.1 Diagonalization and shift of frequency

By the eigenmodes of the hyperbolic operator, we decompose $\dot{u}$ and shift the component related to $\Pi_1$ and $\Pi_3$. Then we make the following frequency shift

$$U_1 = e^{-i\theta} \text{op}_\varepsilon(\Pi_1) \dot{u}, U_2 = \text{op}_\varepsilon(\Pi_2) \dot{u}, U_3 = e^{i\theta} \text{op}_\varepsilon(\Pi_3) \dot{u}, U_4 = \text{op}_\varepsilon(\Pi_4) \dot{u}, \ldots, U_J = \text{op}_\varepsilon(\Pi_J) \dot{u},$$

where the projectors $\Pi_j (j = 1, 2, 3, \ldots, J)$ are eigenprojectors of $A(i\xi) + A_0$ and $\text{op}_\varepsilon(\Pi_j) (j = 1, 2, 3, \ldots, J)$ are defined as above. The perturbation unknown $\dot{u}$ is decomposed by

$$\dot{u} = e^{i\theta} U_1 + U_2 + e^{-i\theta} U_3 + U_4 + \cdots + U_J.$$

Then we find that $U = (U_1, U_2, U_3, \ldots, U_J)$ solves

$$\partial_t U + \frac{1}{\varepsilon} \text{op}_\varepsilon(iA) U = \frac{1}{\sqrt{\varepsilon}} \text{op}_\varepsilon(iB) U + F.$$  (2.1)
The symbol of the propagator is

$$A = \text{diag}(\lambda_{1,+} - w, \lambda_{2,} \lambda_{3,-} + w, \lambda_{4}, \cdots, \lambda_{J}),$$

(2.2)

and the symbol of singular source term is

$$B = \begin{pmatrix} B_{[1,3]} & B_{[1,3],J} \\ B_{[J,1,3]} & B_{[J,J]} \end{pmatrix}$$

(2.3)

where the top left block is

$$B_{[1,3]} = \sum_{p=\pm 1} \begin{pmatrix} e^{ip\phi} \Pi_{1,+}(p+1) B_{p} B_{1} & \cdots & e^{ip\phi} \Pi_{1,+}(p+1) B_{p} B_{J} \\ e^{ip\phi} \Pi_{2,+}(p+1) B_{p} B_{2} & \cdots & e^{ip\phi} \Pi_{2,+}(p+1) B_{p} B_{J} \\ e^{ip\phi} \Pi_{3,+}(p+1) B_{p} B_{3} & \cdots & e^{ip\phi} \Pi_{3,+}(p+1) B_{p} B_{J} \end{pmatrix} \in \mathbb{R}^{3N \times (J-3)N},$$

with \(\Pi_{j,+q}(\xi) = \Pi(\xi + qk)\) for \(q \in \mathbb{Z}\) and \(B_{p} = B(u_{0,p})\), \(p \in \{-1, 1\}\), where \(u_{0,\pm 1}\) are the leading amplitudes in the WKB approximate solution. The other blocks are

$$B_{[1,3],J} = \sum_{p=\pm 1} \begin{pmatrix} e^{ip\phi} \Pi_{1,+}(p+1) B_{p} B_{1} & \cdots & e^{ip\phi} \Pi_{1,+}(p+1) B_{p} B_{J} \\ e^{ip\phi} \Pi_{2,+}(p+1) B_{p} B_{2} & \cdots & e^{ip\phi} \Pi_{2,+}(p+1) B_{p} B_{J} \\ e^{ip\phi} \Pi_{3,+}(p+1) B_{p} B_{3} & \cdots & e^{ip\phi} \Pi_{3,+}(p+1) B_{p} B_{J} \end{pmatrix} \in \mathbb{R}^{(J-3)N \times 3N},$$

and

$$B_{[J,1,3]} = \sum_{p=\pm 1} \begin{pmatrix} e^{ip\phi} \Pi_{4,+}(p+1) B_{p} B_{4} & \cdots & e^{ip\phi} \Pi_{4,+}(p+1) B_{p} B_{J} \\ \vdots & \vdots & \vdots \\ e^{ip\phi} \Pi_{J,+}(p+1) B_{p} B_{J} & \cdots & e^{ip\phi} \Pi_{J,+}(p+1) B_{p} B_{J} \end{pmatrix} \in \mathbb{R}^{(J-3)N \times (J-3)N}.$$
Assumptions in Theorem 1.6 ensure the interaction coefficients in $B^{or}$, $B_{[1,3,J]}$, $B_{[J,1,3]}$ and $B_{[J,J]}$ are transparent with oscillations and the interaction coefficients in $\tilde{B}$ are supported away from corresponding resonance sets. In a skillful way, we will delete them using the following normal form reduction, which means we can delete $\mathcal{D}$ in the system with a $O(\sqrt{\varepsilon})$ remainder.

### 2.1.2 Normal form reduction

Using normal form reduction with $\mu_1 = \lambda_{1,1} - \omega$, $\mu_2 = \lambda_2$, $\mu_3 = \lambda_{3,1} + \omega$, $\mu_j = \lambda_j$ for $4 \leq j \leq J$, it is sufficient to solve

$$i(-p\omega + \mu_{i+p} - \mu_{j})(Q_p)_{(i,j)} = (\mathcal{D}_p)_{(i,j)}, \quad 1 \leq i, j \leq J,$$

where we rewrite the above $\mathcal{D}$ as $\mathcal{D} = \sum_{|p| \leq 3} e^{ip\theta} \mathcal{D}_p$.

When $p = 0$, we have

$$i(\mu_1 - \mu_2)(Q_0)_{(1,2)} = (1 - \chi_{12})\Pi_{1,1}B_1\Pi_2 \Rightarrow i(\lambda_{1,1} - \lambda_2 - \omega)(Q_0)_{(1,2)} = (1 - \chi_{12})b_{12}^+;$$

$$i(\mu_2 - \mu_1)(Q_0)_{(2,1)} = (1 - \chi_{12})\Pi_2B_1\Pi_{1,1} \Rightarrow -i(\lambda_{1,1} - \lambda_2 - \omega)(Q_0)_{(2,1)} = (1 - \chi_{12})b_{21}^-;$$

$$i(\mu_2 - \mu_3)(Q_0)_{(1,2)} = (1 - \chi_{23,1})\Pi_2B_1\Pi_{3,1} \Rightarrow i(\lambda_{2,1} - \lambda_3 - \omega)_{-1}(Q_0)_{(2,3)} = (1 - \chi_{23,1})(b_{23}^+)_1;$$

$$i(\mu_3 - \mu_2)(Q_0)_{(1,2)} = (1 - \chi_{23,1})\Pi_{3,1}B_1\Pi_2 \Rightarrow -i(\lambda_{2,1} - \lambda_3 - \omega)_{-1}(Q_0)_{(3,2)} = (1 - \chi_{23,1})(b_{32}^+)_1;$$

$$i(\mu_1 - \mu_j)(Q_0)_{(1,j)} = b_{j}^+ \Rightarrow i(\lambda_{1,1} - \lambda_j - \omega)(Q_0)_{(1,j)} = b_{j}^+, \quad (4 \leq j \leq J);$$

$$i(\mu_j - \mu_3)(Q_0)_{(j,3)} = (b_{3j}^-)_1 \Rightarrow i(\lambda_{j,1} - \lambda_3 - \omega)_{-1}(Q_0)_{(j,3)} = (b_{3j}^-)_1, \quad (4 \leq j \leq J);$$

In $\text{supp}(1 - \chi_{12})$, $[(\lambda_{1,1} - \lambda_2 - \omega)]$ has a maximum positive lower bound and in $\text{supp}(1 - \chi_{23,1})$, $[(\lambda_{2,1} - \lambda_3 - \omega)_{-1}]$ has a maximum positive lower bound too. Then we can divide the right-hand side by the phase to define $(Q_0)_{(1,2)}$, $(Q_0)_{(2,1)}$, $(Q_0)_{(2,3)}$ and $(Q_0)_{(3,2)}$ in the above cases. Relaxed separation conditions in Assumption 1.9 ensure the phases factor out in the right-hand sides, then we can solve $(Q_0)_{(1,j)}$, $(Q_0)_{(3,j)}$, $(Q_0)_{(j,1)}$ and $(Q_0)_{(j,3)}$ for $4 \leq j \leq J$ in $S^0$.

When $|p| = 1$, we have

$$i(\lambda_{1,1} - \lambda_1 - \omega)_{+1}(Q_1)_{(1,1)} = \Pi_{1,2}B_1\Pi_{1,1} = (b_{11})_{+1},$$

$$-i(\lambda_{1,1} - \lambda_1 - \omega)(Q_1)_{(1,1)} = \Pi_1B_{1,1} = b_{11}^-;$$

$$i(\lambda_{1,1} - \lambda_3 - \omega)_{-1}(Q_1)_{(1,3)} = \Pi_1B_{3,1} = (b_{13})_{-1};$$

$$i(\lambda_{2,1} - \lambda_2 - \omega)(Q_1)_{(2,2)} = \Pi_{2,1}B_1\Pi_2 = b_{22}^+;$$

$$-i(\lambda_{2,1} - \lambda_2 - \omega)_{-1}(Q_1)_{(2,2)} = \Pi_{2,1}B_{1,1} = (b_{22})_{-1};$$

$$-i(\lambda_{3,1} - \lambda_3 - \omega)(Q_1)_{(3,1)} = \Pi_3B_{1,1} = b_{31}^+;$$

$$i(\lambda_{3,1} - \lambda_3 - \omega)_{-1}(Q_1)_{(3,3)} = \Pi_3B_{3,1} = (b_{33})_{-1},$$

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\[-i(\lambda_{3,+1} - \lambda_3 - \omega)_{-2}(Q_{-1})_{(3,3)} = \Pi_{3,-2}B_{-1}\Pi_{3,-1} = (b_{33}^-)_{-2},\]

\[i(\lambda_{i,+1} - \lambda_j - \omega)(Q_{1})_{(i,j)} = \Pi_{i,+1}B_1\Pi_j = b_{ij}^+, \quad (4 \leq i, j \leq J),\]

\[-i(\lambda_{j,+1} - \lambda_i - \omega)_{-1}(Q_{-1})_{(i,j)} = \Pi_{i,-1}B_{-1}\Pi_j = (b_{ij}^-)_{-1}, \quad (4 \leq i, j \leq J).\]

When \(|p| = 2\), we have

\[-i(\lambda_{2,+1} - \lambda_1 - \omega)_{-1}(Q_{-2})_{(1,2)} = \Pi_{1,-1}B_{-1}\Pi_2 = (b_{12}^-)_{-1},\]

\[i(\lambda_{2,+1} - \lambda_1 - \omega)_{+1}(Q_{2})_{(2,1)} = \Pi_{2,+2}B_1\Pi_{1,+1} = (b_{21}^+)_{+1},\]

\[-i(\lambda_{3,+1} - \lambda_2 - \omega)_{-2}(Q_{-2})_{(2,3)} = \Pi_{2,-2}B_{-1}\Pi_{3,-1} = (b_{23}^-)_{-2},\]

\[i(\lambda_{3,+1} - \lambda_2 - \omega)(Q_{2})_{(3,2)} = \Pi_{3,-2}B_1\Pi_2 = b_{32}^+,\]

\[-i(\lambda_{j,+1} - \lambda_1 - \omega)_{-1}(Q_{-2})_{(i,1)} = \Pi_{1,-1}B_{-1}\Pi_j = (b_{ij}^-)_{-1}, \quad (4 \leq j \leq J),\]

\[i(\lambda_{3,+1} - \lambda_j - \omega)_{-2}(Q_{-2})_{(j,3)} = \Pi_{j,-2}B_{-1}\Pi_{3,-1} = (b_{j3}^-)_{-2}, \quad (4 \leq j \leq J),\]

\[i(\lambda_{3,+1} - \lambda_j - \omega)(Q_{2})_{(3,j)} = \Pi_{3,+1}B_1\Pi_j = b_{3j}^+, \quad (4 \leq j \leq J),\]

\[i(\lambda_{j,+1} - \lambda_1 - \omega)_{+1}(Q_{2})_{(j,1)} = \Pi_{j,+2}B_1\Pi_{1,+1} = (b_{j1}^+)_{+1}, \quad (4 \leq j \leq J).\]

When \(|p| = 3\), we have

\[-i(\lambda_{3,+1} - \lambda_1 - \omega)_{-2}(Q_{-3})_{(1,3)} = \Pi_{1,-2}B_{-1}\Pi_{3,-1} = (b_{13}^-)_{-2},\]

\[i(\lambda_{3,+1} - \lambda_1 - \omega)_{+1}(Q_{3})_{(3,1)} = \Pi_{3,+2}B_1\Pi_{1,+1} = (b_{31}^+)_{+1}.\]

Relaxed separation conditions in Assumption 1.9 are used again to solve for \(Q\) in \(S^0\) in above cases of \(|p| = 1, 2, 3\), and combining them. We solve the equations (2.6) and obtain \((Q_p)_{(i,j)} \in S^0\) for \(1 \leq i, j \leq J\) and \(|p| = 0, 1, 2, 3\) in corresponding support sets. Given the change of variable \(\bar{U}(t) = (\text{Id} + \sqrt{e}\text{op}_x(Q(\sqrt{e}t)))^{-1}U(\sqrt{e}t)\) and combining the above normal form reduction similar to Corollary 3.5 of [9], we find equation in \(\bar{U}(t)\):

\[\partial_t \bar{U} + \frac{1}{\sqrt{e}}\text{op}_x(iA)\bar{U} = \text{op}_x(\hat{B})\bar{U} + \sqrt{e}\bar{F}, \quad (2.7)\]

where the symbol \(\hat{B}\) is as

\[
\hat{B} = \begin{pmatrix}
0 & \chi_{12}\Pi_{1,+1}B_1\Pi_2 & 0 \\
\chi_{12}\Pi_{2}B_{-1}\Pi_{1,+1} & 0 & \chi_{12}^{\#}B_{1}\Pi_{2}B_{-1}\Pi_{3,-1} \\
0 & \chi_{23}\Pi_{3,-1}B_{-1}\Pi_{2} & 0 \\
\chi_{23}^{\#}B_{23} & 0 & 0
\end{pmatrix}
\quad (2.8)
\]

\[
\hat{B} := \begin{pmatrix}
0 & \chi_{12}b_{12}^+ & 0 \\
\chi_{12}b_{21}^+ & 0 & \chi_{23}^{\#}b_{23}^+ \\
0 & \chi_{23}\Pi_{3,-1}(b_{32}^-)_{-1} & 0
\end{pmatrix}.
\]

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2.1.3 Space-frequency localization

Furthermore, we let

\[ V = \text{op}_\varepsilon(\chi)(\varphi\tilde{U}), \quad W = (W_1, W_2)(\text{op}_\varepsilon(\chi)((1 - \varphi)\tilde{U}), (1 - \text{op}_\varepsilon(\chi))\tilde{U}) \]  

(2.9)
satisfying

\[ \tilde{U} = V + W_1 + W_2. \]

Then we have

\[ \partial_t V + \frac{1}{\sqrt{\varepsilon}}\text{op}_\varepsilon(M)V = \sqrt{\varepsilon}F_V, \]  

(2.10)

\[ \partial_t W + \frac{1}{\sqrt{\varepsilon}}\text{op}_\varepsilon(iA)W = \text{op}_\varepsilon(D)W + \sqrt{\varepsilon}F_W \]  

(2.11)

with symbol

\[ M = i\chi^#A - \sqrt{\varepsilon}\varphi^#\tilde{B}, \quad A = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad D = (1 - \varphi)\chi^#\tilde{B}. \]  

(2.12)

\( \tilde{F}, F_W \) and \( F_V \) satisfy the same estimates in Lemma 3.1 of [9]. While in different frequency sets \( \text{supp}\chi(\xi) \) and spatial truncations \( \text{supp}\varphi(x) \), small differences in the symbols \( M, \tilde{B} \) and \( (V, W) \) occur, which will be specified in following discussions.

2.1.4 Estimates of symbolic flow

Now, we are trying to obtain the bounds for symbolic flow in the following equations

\[ \partial_t S_0 + \frac{1}{\sqrt{\varepsilon}}MS_0 = 0, \quad S_0(\tau, \tau) = \text{Id}. \]  

(2.13)

In other words, we need to give estimate for \( |S_0(\tau, t)| = |\exp(-\frac{t}{\sqrt{\varepsilon}}M)| \) and most importantly, we want to obtain

\[ |S_0(\tau, t)| \lesssim |\ln \varepsilon|^n \exp(t\gamma^+) \]  

(2.14)

for frequency \( \xi \) in any non-transparent resonance sets or not. As discussed in [9], if \( \xi \) is in transparent resonance sets or no resonance occurs, the normal form reduction will eliminate corresponding effects in (2.10) and (2.11) with two small remainders on the right side. Hence we need to discuss when frequency \( \xi \) in different non-transparent sets, under certain separation conditions (i.e. separation conditions in Assumption 1.9), whether the bound for symbolic flow in (2.14) holds or not.

If \( \mathcal{R}_{12} \cap (\mathcal{R}_{23} + k) = \emptyset \) and \( \xi \in \mathcal{R}_{12} \), let \( \chi = \chi_{12} \) and \( \varphi = \varphi_{12} \) in (2.9) and (2.12), then we have

\[ \tilde{B} = \chi_{12} \begin{pmatrix} 0 & b_{12}^+ & 0 \\ b_{21}^- & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]
and

\[ M = \imath \chi_{12}^# A - \sqrt{\varepsilon} \varphi_{12}^# \hat{B} \]

\[
= \begin{pmatrix}
\imath \chi_{12}^# (\lambda_{1,1} - \omega) & -\sqrt{\varepsilon} \varphi_{12}^# \chi_{12} b_{12}^+ & 0 \\
-\sqrt{\varepsilon} \varphi_{12}^# \chi_{12} b_{21}^- & \imath \chi_{12}^# \lambda_2 & 0 \\
0 & 0 & \imath \chi_{12}^# (\lambda_{3,-1} + \omega)
\end{pmatrix},
\]

(2.15)

then (2.13) can be solved directly using Lemma 1.7 with \( |S_0(\tau, t)| \lesssim |\ln \varepsilon|^* \exp(t \gamma_{12}^+)) \leq |\ln \varepsilon|^* \exp(t \gamma^+) \) for \( \xi \in \mathcal{R}_{12} \).

And for \( \xi \in (\mathcal{R}_{23} + k) \), let \( \chi = \chi_{23,-1}, \varphi = \varphi_{23} \) in (2.9) and (2.12), then we have

\[ \hat{B} = \chi_{23,-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & (b_{23}^+)_1 \\ 0 & (b_{32}^+)_1 & 0 \end{pmatrix} := \chi_{23}(\xi - k) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b_{23}^+(\xi - k) \\ 0 & b_{32}^- (\xi - k) & 0 \end{pmatrix}. \]

(2.16)

Let \( \xi' = \xi - k \), then \( \xi' \in \mathcal{R}_{23} \) and we have

\[ \hat{B} = \chi_{23}(\xi') \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b_{23}^+(\xi') \\ 0 & b_{32}^- (\xi') & 0 \end{pmatrix}. \]

For the equation

\[ \partial_t S_0 + \frac{1}{\sqrt{\varepsilon}} M(\xi') S_0 = 0, \quad S_0(\tau, \tau) = \text{Id} \]

with

\[ M(\xi') = \imath \chi_{23,-1}^# A - \sqrt{\varepsilon} \varphi_{23}^# \hat{B} \]

\[
= \begin{pmatrix}
\imath \chi_{23}^# (\lambda_1 (\xi' + 2k) - \omega) & 0 & 0 \\
0 & \lambda_2 (\xi' + k) & 0 \\
0 & 0 & \lambda_3 (\xi' + \omega)
\end{pmatrix} - \sqrt{\varepsilon} \varphi_{23}^# \chi_{23} (\xi') \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b_{23}^+(\xi') \\ 0 & b_{32}^- (\xi') & 0 \end{pmatrix}
\]

\[
= \begin{pmatrix}
\imath \chi_{23}^# (\lambda_1 (\xi' + 2k) - \omega) & 0 & 0 \\
0 & \imath \chi_{23}^# (\xi') \lambda_2 (\xi' + k) & 0 \\
0 & 0 & \imath \chi_{23}^# (\xi') (\lambda_3 (\xi') + \omega)
\end{pmatrix}.
\]

using the Lemma 1.7, we obtain

\[ |S_0(\tau, t)| \lesssim |\ln \varepsilon|^* \exp(t \gamma_{23}^+ (\xi')) = |\ln \varepsilon|^* \exp(t \gamma_{23}^+). \]

(2.17)

where

\[ \gamma_{23}^+ := |a|_{L^\infty} \max_{\xi \in \mathcal{R}_{23}} \text{Re}(\Gamma_{23}(\xi)^{\frac{1}{2}}), \]

\[ \gamma_{23}^+(\xi') := |a|_{L^\infty} \max_{\xi' \in \mathcal{R}_{23}} \text{Re}(\Gamma_{23}(\xi')^{\frac{1}{2}}) = |a|_{L^\infty} \max_{\xi \in \mathcal{R}_{23}} \text{Re}(\Gamma_{23}(\xi)^{\frac{1}{2}}) = \gamma_{23}^+ \]

Then by Lemma 1.7, we get the estimate \( |S_0(\tau, t)| \lesssim |\ln \varepsilon|^* \exp(t \gamma_{23}^+)) \leq |\ln \varepsilon|^* \exp(t \gamma^+) \) for \( \xi \in (\mathcal{R}_{23} + k) \), where \( \mathcal{R}_{12} \cap (\mathcal{R}_{23} + k) = \emptyset \).

From (2.16) to (2.17), the following Remark can be concluded.
Remark 2.1. Variable substitution in (2.16) to (2.17) shows the same upper bound for the symbolic flow. Similarly, we can conclude that if there is a uniform Variable substitution in frequency parameter i.e. for $\lambda_i$ and $\lambda_j$, we have

$$M_{ij}(\varepsilon, t, x, \xi) = \chi_{ij}^# \left( \begin{array}{cc} i\lambda_i & -\sqrt{\varepsilon} \varphi_{ij}^# \chi_{ij}(b_{ij}^+)^{-1} \\ -\sqrt{\varepsilon} \varphi_{ij}^# \chi_{ij}(b_{ij}^-)^{-1} & i(\lambda_{ij-1} + \omega) \end{array} \right),$$

corresponding the phase to $\lambda_i - (\lambda_{ij-1} + \omega) = (\lambda_{ij-1} + \lambda_j + \omega)_{ij-1}$ and following the proof in [9], we can obtain (2.14) for (2.13) too.

If $\mathcal{R}_{12} \cap (\mathcal{R}_{23} + k) \neq \emptyset$, for $\xi \in \mathcal{R}_{12} \setminus (\mathcal{R}_{23} + k)$, let $\chi = \chi_{\mathcal{R}_{12} \setminus (\mathcal{R}_{23} + k)}$ and $\varphi = \varphi_{12}$ in (2.9) and (2.12), then we have

$$\bar{B} = \chi_{\mathcal{R}_{12} \setminus (\mathcal{R}_{23} + k)} \left( \begin{array}{ccc} 0 & b_{12}^+ & 0 \\ b^{-}_{21} & 0 & 0 \\ 0 & 0 & 0 \end{array} \right),$$

and for $\xi \in (\mathcal{R}_{23} + k) \setminus \mathcal{R}_{12}$, let $\chi = \chi_{(\mathcal{R}_{23} + k) \setminus \mathcal{R}_{12}}$ and $\varphi = \varphi_{23}$ in (2.9) and (2.12), then we have

$$\bar{B} = \chi_{(\mathcal{R}_{23} + k) \setminus \mathcal{R}_{12}} \left( \begin{array}{ccc} 0 & 0 & b_{23}^- \\ 0 & 0 & 0 \\ b_{32}(\xi - k) & 0 & 0 \end{array} \right).$$

Let $\xi' = \xi - k$, then $\xi' \in \mathcal{R}_{23} \setminus (\mathcal{R}_{12} - k)$ and we have

$$\bar{B} = \chi_{\mathcal{R}_{23} \setminus (\mathcal{R}_{12} - k)}(\xi') \left( \begin{array}{ccc} 0 & 0 & b_{23}^- \\ 0 & 0 & 0 \\ b_{32}(\xi') & 0 & 0 \end{array} \right),$$

Combining Lemma 1.7 and Remark 2.1, we can obtain (2.14) for $\xi \in \mathcal{R}_{12} \setminus (\mathcal{R}_{23} + k)$ and $\xi \in (\mathcal{R}_{23} + k) \setminus \mathcal{R}_{12}$ directly.

While in $\mathcal{R}_{12} \cap (\mathcal{R}_{23} + k)$, two non-transparent resonances $(1, 2)$ and $(2, 3)$ are coupled together. By Assumption 1.9, if at least one of $b_{32}^-$, $b_{23}^+$ is transparent in $(\mathcal{R}_{12} - k) \cap \mathcal{R}_{23}$, then at least one of $(b_{32}^-)_1$, $(b_{23}^+)_1$ is transparent in $\mathcal{R}_{12} \cap (\mathcal{R}_{23} + k)$. That is to say one of $(b_{32}^-)_1$, $(b_{23}^+)_1$ is transparent in $\mathcal{R}_{12} \cap (\mathcal{R}_{23} + k)$ or both $(b_{32}^-)_1$, $(b_{23}^+)_1$ are transparent in $\mathcal{R}_{12} \cap (\mathcal{R}_{23} + k)$, then we can eliminate transparent term in $\mathcal{R}_{12} \cap (\mathcal{R}_{23} + k)$ and similarly non-transparent properties of interaction coefficients $b_{12}^+$, $b_{21}^-$ in $\mathcal{R}_{12} \cap (\mathcal{R}_{23} + k)$ are considered. Under different separation conditions, (2.13) can be divided into different equations.

Firstly, if at least one of $b_{32}^-$, $b_{23}^+$ is transparent in $(\mathcal{R}_{12} - k) \cap \mathcal{R}_{23}$ and there is only $b_{23}^+$ transparent in $(\mathcal{R}_{12} - k) \cap \mathcal{R}_{23}$. Using normal form reduction, we have

$$\bar{B} = \chi_{12} \chi_{23} \left( \begin{array}{ccc} 0 & b_{12}^+ & 0 \\ b_{21}^- & 0 & 0 \\ 0 & (b_{23}^-)_{-1} & 0 \end{array} \right).$$

Let

$$V = \text{op}_\varepsilon(\chi_{12} \chi_{23} \chi_{-1})(\varphi_{12} \varphi_{23} U),$$

$$W = (W_1, W_2) = (\text{op}_\varepsilon(\chi_{12} \chi_{23} \chi_{-1})(1 - \varphi_{12} \varphi_{23} U), (1 - \text{op}_\varepsilon(\chi_{12} \chi_{23} \chi_{-1})) U).$$

(2.20) (2.21)
\[ \bar{U} = V + W_1 + W_2. \]

Then we obtain the following equations:

\[ \partial_t V + \frac{1}{\sqrt{\varepsilon}} \text{op}_\varepsilon (M)V = \sqrt{\varepsilon} F_V, \]  
\[ (2.22) \]

\[ \partial_t W + \frac{1}{\sqrt{\varepsilon}} \text{op}_\varepsilon (iA)W = \text{op}_\varepsilon (D)W + \sqrt{\varepsilon} F_W \]  
\[ (2.23) \]

with symbol

\[ M = i(\chi_{12} \chi_{23,-1})\# A - \sqrt{\varepsilon} (\varphi_{12} \varphi_{23})\# \hat{B}, \quad A = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad D = (1 - (\varphi_{12} \varphi_{23})\#)(\chi_{12} \chi_{23,-1})\# \hat{B}. \]  
\[ (2.24) \]

Based on the given \( M \), we now consider the equation

\[ \partial_t S_0 + \frac{1}{\sqrt{\varepsilon}} MS_0 = 0, \quad S_0(\tau, \tau) = \text{Id}. \]  
\[ (2.25) \]

Then combine (2.24) to obtain

\[ M = \begin{pmatrix} i(\chi_{12} \chi_{23,-1})\# (\lambda_{1,1} - \omega) & -\sqrt{\varepsilon} (\varphi_{12} \varphi_{23})\# \chi_{12} \chi_{23,-1} b_{12}^+ \\ -\sqrt{\varepsilon} (\varphi_{12} \varphi_{23})\# \chi_{12} \chi_{23,-1} b_{12}^- & i(\chi_{12} \chi_{23,-1})\# \lambda_2 \\ 0 & 0 \end{pmatrix} \]  
\[ =: \begin{pmatrix} i\mu_1' & -\sqrt{\varepsilon} (b_{12})' \\ -\sqrt{\varepsilon} (b_{21})' & i\mu_2' \\ 0 & 0 \end{pmatrix}. \]  
\[ (2.26) \]

We denote \( S_0(\tau, t) = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}(t) \), then (2.25) can be write into

\[ \partial_t \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \frac{1}{\sqrt{\varepsilon}} M_{12} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0, \]  
\[ (2.27) \]

\[ \partial_t y_3 - (b_{32})'_{-1} y_2 + \frac{1}{\sqrt{\varepsilon}} i\mu_3' y_3 = 0, \]  
\[ (2.28) \]

with

\[ M_{12} = \begin{pmatrix} i\mu_1' & -\sqrt{\varepsilon} (b_{12})' \\ -\sqrt{\varepsilon} (b_{21})' & i\mu_2' \end{pmatrix}. \]

For the case of (2.27)-(2.28), by Lemma 1.7, we have

\[ | \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(t) | = |e^{\frac{1}{\sqrt{\varepsilon}} M_{12} \tau} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(\tau) | \lesssim \exp((t - \tau) \gamma_{12}^+). \]
Then for \(|y_3|\), we have
\[
|y_3(t)| = |e^{(\tau - t) \mu_3'} y_3(\tau) + \int_\tau^t e^{(t' - t) \mu_3'} (b_{32})_{-1} y_2(t') dt' |
\approx |e^{(\tau - t) \mu_3'}| \cdot |y_3(\tau)| + \int_\tau^t |e^{(t' - t) \mu_3'}| \cdot |(b_{32})_{-1}| \cdot |y_2(t')| dt' \\
\lesssim 1 + CT |\ln |exp((t - \tau) \gamma_{12}) \lesssim exp(t \gamma^{+}').
\]

Finally, we obtain
\[
|S_0(\tau, t)| = \left| \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}(t) \right| \approx \exp(t \gamma^{+}) \lesssim \exp(t \gamma^{+}). \tag{2.29}
\]

Secondly, if at least one of \(b_{32}, b_{23}^{+}\) is transparent in \((\mathcal{R}_{12} - k) \cap \mathcal{R}_{23}\) and there is only \(b_{32}^{-}\) transparent in \((\mathcal{R}_{12} - k) \cap \mathcal{R}_{23}\). After normal form reduction, we have
\[
\tilde{B} = \chi_{12} \chi_{23}^{23,-1} \begin{pmatrix} 0 & b_{12}^{+} & 0 \\ b_{21}^{-} & 0 & (b_{23}^{+})^{-1} \\ 0 & 0 & 0 \end{pmatrix}.
\]

and
\[
M = \begin{pmatrix} i(\chi_{12} \chi_{23}^{23,-1}) \# (\lambda_{11} + \omega) & i(\chi_{12} \chi_{23}^{23,-1}) \# \chi_{12} \chi_{23}^{23,-1} b_{12}^{+} & 0 \\ -\sqrt{\varepsilon}(\varphi_{12} \varphi_{23}) \# \chi_{12} \chi_{23}^{23,-1} b_{21}^{-} & 0 & i(\chi_{12} \chi_{23}^{23,-1}) \# \chi_{12} \chi_{23}^{23,-1} (b_{23}^{+})^{-1} \# (\lambda_{31} + \omega) \\ 0 & 0 & 0 \end{pmatrix}.
\]

Then (2.25) can be write into
\[
\partial^t_\tau \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{\sqrt{\varepsilon}} M_{12} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ (b_{23}^{+})^{-1} y_3 \end{pmatrix}, \tag{2.30}
\]
\[
\partial_t y_3 + \frac{1}{\sqrt{\varepsilon}} i \mu_3' y_3 = 0. \tag{2.31}
\]

For (2.30)-(2.31), we have
\[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(t) = e^{\frac{\tau - t}{\sqrt{\varepsilon}} M_{12}} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(\tau) + \int_\tau^t e^{(t' - t) \frac{\tau - t}{\sqrt{\varepsilon}} M_{12}} \begin{pmatrix} 0 \\ (b_{23}^{+})^{-1} y_3 \end{pmatrix}(t') dt',
\]
\[
y_3(t) = e^{\frac{\tau - t}{\sqrt{\varepsilon}} i \mu_3'}. 
\]

It is obvious that
\[
|y_3| \lesssim 1
\]
and by Lemma 1.7, we have
\[ \left| \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}(t) \right| \lesssim |e^{\frac{\tau - t}{\sqrt{\epsilon}}M_{12}}| \cdot \left| \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}(\tau) \right| + \int_{\tau}^{t} |e^{(t' - t')\frac{\tau - t'}{\sqrt{\epsilon}}M_{12}}| \cdot \left| \begin{pmatrix} 0 \\ (b_{12}^{+})_{-1}y_3 \end{pmatrix}(t') \right| dt'. \]
\[ \lesssim \exp((t - \tau)\gamma_{12}) \cdot \exp(C\epsilon|\exp(t\gamma_{12}^+)\lesssim \exp(t\gamma^+). \]

Thus we have,
\[ |S_0(\tau, t)| = \left| \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right| \lesssim \exp(t\gamma_{12}^+) \lesssim \exp(t\gamma^+). \quad (2.32) \]

Similarly, if at least one of \( b_{21}^+ \), \( b_{12}^+ \) is transparent in \( \mathcal{R}_{12} \cap (\mathcal{R}_{23} + k) \), we can also obtain the same upper bounds for \( S_0 \). Finally, if both \( (b_{32}^-)_{-1}, (b_{23}^+)_1 \) are transparent in \( \mathcal{R}_{12} \cap (\mathcal{R}_{23} + k) \), after normal form reduction, \( M \) in (2.25) can be reduced to \( M_{12} \) or if both \( b_{12}^+, b_{21}^- \) are transparent in \( \mathcal{R}_{12} \cap (\mathcal{R}_{23} + k) \), \( M \) in (2.25) can be reduced to \( M_{23} \). Thus we can solve (2.25) respectively with \( |S_0(\tau, t)| \lesssim \exp(t\gamma_{12}^+) \) and \( |S_0(\tau, t)| \lesssim \exp(t\gamma_{23}^+) \) directly using Lemma 1.7.

### 2.2 Case 2.

Assume \((1, 2, 3) \in \mathcal{R}_0 \), at least one of \( b_{13}^+, b_{31}^- \) is transparent on \( \mathcal{R}_{12} \cap \mathcal{R}_{13} \), or at least one of \( b_{12}^+, b_{21}^- \) is transparent on \( \mathcal{R}_{12} \cap \mathcal{R}_{13} \). By the eigenmodes of the hyperbolic operator, we decompose \( \dot{u} \) and shift the component related to \( \Pi_1 \). Then we define
\[ U_1 = e^{-i\theta}\mathop{\text{op}}_{\epsilon}(\Pi_1)\dot{u}, \quad U_2 = \mathop{\text{op}}_{\epsilon}(\Pi_2)\dot{u}, \quad U_3 = \mathop{\text{op}}_{\epsilon}(\Pi_3)\dot{u}, \quad U_4 = \mathop{\text{op}}_{\epsilon}(\Pi_4)\dot{u}, \quad \cdots, \quad U_J = \mathop{\text{op}}_{\epsilon}(\Pi_J)\dot{u}. \]

The perturbation unknown \( \dot{u} \) is decomposed by
\[ \dot{u} = e^{i\theta}U_1 + U_2 + U_3 + U_4 + \cdots + U_J, \]
and
\[ A = \text{diag}(\lambda_{1,1} - w, \lambda_2, \lambda_3, \cdots, \lambda_J). \]

Similar to Case 1, we utilize normal form reduction to have
\[ \tilde{B} = \left( \begin{array}{ccc} 0 & \chi_{12}b_{12}^+ & \chi_{13}b_{13}^+ \\ \chi_{12}b_{21}^- & 0 & 0 \\ \chi_{13}b_{31}^- & 0 & 0 \end{array} \right), \]
and when \( \xi \in \mathcal{R}_{12} \cap \mathcal{R}_{13} \) for \( \mathcal{R}_{12} \cap \mathcal{R}_{13} \neq \emptyset \), we have
\[ M = i(\chi_{12} \chi_{13})^{#}A - \sqrt{\epsilon}(\varphi_{12} \varphi_{13})^{#}\tilde{B} \]
\[ = \left( \begin{array}{ccc} i(\chi_{12} \chi_{13})^{#}(\lambda_{1,1} - w) & -\sqrt{\epsilon}(\varphi_{12} \varphi_{13})^{#}\chi_{12} \chi_{13} b_{12}^+ & -\sqrt{\epsilon}(\varphi_{12} \varphi_{13})^{#}\chi_{12} \chi_{13} b_{13}^+ \\ -\sqrt{\epsilon}(\varphi_{12} \varphi_{13})^{#}\chi_{12} \chi_{13} b_{21}^- & i(\chi_{12} \chi_{13})^{#}\lambda_2 & 0 \\ -\sqrt{\epsilon}(\varphi_{12} \varphi_{13})^{#}\chi_{12} \chi_{13} b_{31}^- & 0 & i(\chi_{12} \chi_{13})^{#}\lambda_3 \end{array} \right). \quad (2.33) \]

Here the method of variable substitution and the classified discussion on intersections of resonance sets occupy an essential position.
We now consider the solutions of $\partial_t S_0 + \frac{1}{\varepsilon} MS_0 = 0$, $S_0(\tau, \tau) = \text{Id}$ in different non-transparent resonance sets $R_{12}$ and $R_{13}$. When $\xi \in R_{12}$ for $R_{12} \cap R_{13} = \emptyset$, $\xi \in R_{12} \setminus R_{13}$ for $R_{12} \cap R_{13} = \emptyset$ and $\xi \in R_{12} \cap R_{13}$ for $R_{12} \cap R_{13} \neq \emptyset$ with the condition that at least one of $b_{13}^+$, $b_{13}^-$ is transparent in $R_{12} \cap R_{13}$ and utilizing normal form reduction to eliminate the transparent terms, we can obtain the following bound for the symbolic flow

$$|S_0(\tau, t)| \lesssim |\ln \varepsilon|^* \exp(t\gamma_{12}) \lesssim |\ln \varepsilon|^* \exp(t\gamma^+) \tag{2.34}$$

Moreover, when $\xi \in R_{13}$ for $R_{12} \cap R_{13} = \emptyset$, $\xi \in R_{13} \setminus R_{12}$ for $R_{12} \cap R_{13} = \emptyset$ and $\xi \in R_{12} \cap R_{13}$ for $R_{12} \cap R_{13} \neq \emptyset$ with the condition that at least one of $b_{12}^+$, $b_{23}^-$ is transparent on $R_{12} \cap R_{13}$, we similarly have

$$|S_0(\tau, t)| \lesssim |\ln \varepsilon|^* \exp(t\gamma_{13}) \lesssim |\ln \varepsilon|^* \exp(t\gamma^+) \tag{2.35}$$

### 2.3 Case 3

In Case 3, for resonances $(1,3), (2,3) \in \mathbb{R}_0$, at least one of $b_{23}^+, b_{32}^-$ is transparent on $R_{13} \cap R_{23}$, or at least one of $b_{13}^+, b_{31}^-$ is transparent on $R_{13} \cap R_{23}$, we decompose $\hat{u}$ and shift the component related to $\Pi_3$ and define

$$U_1 = \text{op}_\varepsilon(\Pi_1) \hat{u}, \quad U_2 = \text{op}_\varepsilon(\Pi_2) \hat{u}, \quad U_3 = e^{i\theta} \text{op}_\varepsilon(\Pi_3) \hat{u}, \quad U_4 = \text{op}_\varepsilon(\Pi_4) \hat{u}, \quad \ldots, \quad U_J = \text{op}_\varepsilon(\Pi_J) \hat{u},$$

and define $A = \text{diag}(\lambda_1, \lambda_2, \lambda_{3,-1} + w, \lambda_4, \ldots, \lambda_J)$.

Then, similar to Case 1 and Case 2, we have

$$\tilde{B} = \begin{pmatrix} 0 & 0 & \chi_{13,-1}(b_{13}^-) & 0 \\ 0 & 0 & \chi_{23,-1}(b_{23}^+) & 0 \\ \chi_{13,-1}(b_{31}^-) & \chi_{23,-1}(b_{32}^-) & 0 & 0 \\ \chi_{13,-1}(b_{13}^-) & \chi_{23,-1}(b_{23}^+) & 0 & 0 \\ \chi_{32}(b_{13}^-) & \chi_{32}(b_{23}^+) & 0 & 0 \\ \chi_{32}(b_{31}^-) & \chi_{32}(b_{32}^-) & 0 & 0 \\ \chi_{32}(b_{13}^-) & \chi_{32}(b_{23}^+) & 0 & 0 \\ \chi_{32}(b_{31}^-) & \chi_{32}(b_{32}^-) & 0 & 0 \end{pmatrix} (\xi),$$

and when $\xi \in (R_{13} + k) \cap (R_{23} + k)$ for $(R_{13} + k) \cap (R_{23} + k) \neq \emptyset$, we have

$$M = i(\chi_{13,-1}\chi_{23,-1}) A - \sqrt{\varepsilon}(\varphi_{13}\varphi_{23}) \tilde{B}$$

Combining Remark 2.1, when $\xi \in (R_{13} + k)$ for $(R_{13} + k) \cap (R_{23} + k) = \emptyset$, $\xi \in (R_{13} + k) \setminus (R_{23} + k)$ for $(R_{13} + k) \cap (R_{23} + k) \neq \emptyset$ and $\xi \in (R_{13} + k) \cap (R_{23} + k)$ for $(R_{13} + k) \cap (R_{23} + k) \neq \emptyset$ with the condition that at least one of $b_{23}^+, b_{32}^-$ is transparent in $R_{12} \cap R_{13}$, we can obtain following bound for the symbolic flow

$$|S_0(\tau, t)| \lesssim |\ln \varepsilon|^* \exp(t\gamma_{13}) \lesssim |\ln \varepsilon|^* \exp(t\gamma^+) \tag{2.37}$$
Moreover, when \( \xi \in (\mathcal{R}_{23}+k) \) for \((\mathcal{R}_{13}+k)\cap(\mathcal{R}_{23}+k) = \emptyset \), \( \xi \in (\mathcal{R}_{23}+k) \) for \((\mathcal{R}_{13}+k)\cap(\mathcal{R}_{23}+k) \neq \emptyset \) and \( \xi \in (\mathcal{R}_{13}+k) \cap (\mathcal{R}_{23}+k) \) for \((\mathcal{R}_{13}+k)\cap(\mathcal{R}_{23}+k) \neq \emptyset \) with the condition that at least one of \( b_{i\beta}^+ \), \( b_{\beta j}^- \) is transparent in \( \mathcal{R}_{13} \cap \mathcal{R}_{23} \), we similarly have

\[
|S_0(\tau, t)| \lesssim \ln |\varepsilon|^* \exp(t\gamma_{23}^+) \lesssim \ln |\varepsilon|^* \exp(t\gamma^+). \tag{2.38}
\]

### 2.4 Instability of WKB solution

Combining Case 1 to Case 3, for all \( T > 0 \), \( 0 \leq \tau \leq t \leq T|\ln \varepsilon| \), and under the given Assumptions in Theorem 1.11, we can conclude that the solution \( S_0 \) to

\[
\partial_t S_0 + \frac{1}{\sqrt{\varepsilon}} M_{ij\gamma} S_0 = 0, \quad S_0(\tau, \tau) = \text{Id}
\]

always satisfies the upper bounds

\[
|S_0(\tau, t)| \lesssim |\ln \varepsilon|^* \exp(t\gamma^+),
\]

where \( \{M_{ij\gamma}\}_{1 \leq i, j, \gamma \leq J} \) are \( 3 \times 3 \) block matrices of non-transparent resonance terms after normal form reductions. Similar to analysis of general cases in Section 6.3.2 of [9], we have the following estimate:

\[
|\partial^\ell_x S_0(\tau, t)| \lesssim |\ln \varepsilon|^* \exp(t\gamma^+).
\]

Combining [9], we prove the first result in Theorem 1.11 using the same method of Duhamel representation for the instability.

### 3 Three coupled resonances

For sake of simplicity, it takes no influences if we directly consider non-transparent resonances set \( \mathcal{R}_1 = \{(1, 2), (2, 3), (1, 3)\} \subset \mathcal{R}_0 \). We have the freedom to choose different frequency shift and we will see that we can do some flexible extensions for the separation conditions in the arguments. By the eigenmodes of the hyperbolic operator, we decompose \( \hat{u} \) and shift the component related to \( \Pi_1 \) as follows

\[
U_1 = e^{-i\vartheta} \text{op}_{\varepsilon}(\Pi_1) \hat{u}, \quad U_2 = \text{op}_{\varepsilon}(\Pi_2) \hat{u}, \quad U_3 = \text{op}_{\varepsilon}(\Pi_3) \hat{u}, \quad U_4 = \text{op}_{\varepsilon}(\Pi_4) \hat{u}, \quad \ldots, \quad U_J = \text{op}_{\varepsilon}(\Pi_J) \hat{u}.
\]

The perturbation unknown \( \hat{u} \) is decomposed by

\[
\hat{u} = e^{i\vartheta} U_1 + U_2 + U_3 + \cdots + U_J
\]

and

\[
A = \text{diag}(\lambda_{1, +1} - w, \lambda_2, \lambda_3, \ldots, \lambda_J).
\]

Then, similar to Case 1 to Case 3 in Section 2, we have

\[
B = \sum_{p=\pm 1} \left( \begin{array}{cccc}
   e^{i(p-1)\vartheta} \Pi_{1,+(p+1)} B_p \Pi_{1, +1} & e^{i(p-1)\vartheta} \Pi_{1,+(p+1)} B_p \Pi_2 & \cdots & e^{i(p-1)\vartheta} \Pi_{1,+(p+1)} B_p \Pi_J \\
   e^{i(p+1)\vartheta} \Pi_{2,+(p+1)} B_p \Pi_{1, +1} & e^{i(p+1)\vartheta} \Pi_{2,+(p+1)} B_p \Pi_2 & \cdots & e^{i(p+1)\vartheta} \Pi_{2,+(p+1)} B_p \Pi_J \\
   \vdots & \vdots & \ddots & \vdots \\
   e^{i(p+1)\vartheta} \Pi_{J,+(p+1)} B_p \Pi_{1, +1} & e^{i(p+1)\vartheta} \Pi_{J,+(p+1)} B_p \Pi_2 & \cdots & e^{i(p+1)\vartheta} \Pi_{J,+(p+1)} B_p \Pi_J
\end{array} \right).
\]
Since we only have \((1, 2), (1, 3), (2, 3) \in \mathcal{R}_0\), which means \((2, 3) \notin \mathcal{R}_0\). Then after normal form reduction, we have the following \(\hat{B}\) with oscillation terms:

\[
\hat{B} = \begin{pmatrix}
0 & \chi_{12}b_{12}^+ & \chi_{13}b_{13}^+ \\
\chi_{12}b_{21}^- & 0 & e^{-i\theta} \chi_{23,-1}(b_{32}^-)^{-1} \\
\chi_{13}b_{31}^- & e^{i\theta} \chi_{23}b_{23}^- & 0
\end{pmatrix}. \tag{3.1}
\]

Since normal form reduction can help us to eliminate transparent interaction coefficients in above \(B\), here we assume \(\mathcal{R}_{12} \cap \mathcal{R}_{13} \cap \mathcal{R}_{23} \neq \emptyset\) and \(b_{23}^+\) is transparent in \(\mathcal{R}_{23}\), then \(\mathcal{R}_{12} \cap \mathcal{R}_{13} \cap (\mathcal{R}_{23} + k) = \emptyset\). Or, we assume \(\mathcal{R}_{12} \cap \mathcal{R}_{13} \cap (\mathcal{R}_{23} + k) \neq \emptyset\) and \(b_{32}^-\) is transparent in \(\mathcal{R}_{23}\), then \(\mathcal{R}_{12} \cap \mathcal{R}_{13} \cap \mathcal{R}_{23} = \emptyset\). Thus we can eliminate the oscillation terms \(e^{i\theta} \chi_{23}b_{23}^+\) and \(e^{-i\theta} \chi_{23,-1}(b_{32}^-)^{-1}\) in \(\hat{B}\). And combining the assumption that one of \(b_{13}^+, b_{31}^-\) is transparent in \(\mathcal{R}_{13}\), or one of \(b_{12}^+, b_{21}^-\) is transparent in \(\mathcal{R}_{12}\), we eliminate one of \(\chi_{13}b_{13}^+, \chi_{13}b_{51}^+, \chi_{12}b_{12}^+, \chi_{12}b_{21}^-\) in \(\hat{B}\) in normal form reduction and we can obtain the coupled case of \(\hat{B}\) as follows:

\[
\hat{B}_1 = \begin{pmatrix}
0 & 0 & \chi_{13}b_{13}^+ \\
\chi_{12}b_{21}^- & 0 & 0 \\
\chi_{13}b_{31}^- & 0 & 0
\end{pmatrix}, \quad \hat{B}_2 = \begin{pmatrix}
0 & \chi_{12}b_{12}^- & 0 \\
\chi_{12}b_{21}^- & 0 & 0 \\
\chi_{13}b_{31}^- & 0 & 0
\end{pmatrix},
\]

\[
\hat{B}_3 = \begin{pmatrix}
0 & \chi_{12}b_{12}^- & \chi_{13}b_{13}^+ \\
0 & \chi_{12}b_{12}^- & \chi_{13}b_{13}^+ \\
\chi_{13}b_{31}^- & 0 & 0
\end{pmatrix}, \quad \hat{B}_4 = \begin{pmatrix}
0 & \chi_{12}b_{12}^- & \chi_{13}b_{13}^+ \\
0 & \chi_{12}b_{12}^- & \chi_{13}b_{13}^+ \\
0 & 0 & 0
\end{pmatrix}.
\]

Now we use the combine Case 2 in Section 2, we obtain the bound of symmetric flow: \(|S_0(\tau, t)| \lesssim \exp(\tau \gamma^+)\).

Following the way to Case 1 in Section 2 and by the eigenmodes of hyperbolic operator, we decompose \(\dot{u}\) and shift the component related to \(\Pi_1\) and \(\Pi_3\). Then we can make the following frequency shift:

\[
U_1 = e^{-i\theta} \mathfrak{o}_e(\Pi_1)\dot{u}, \quad U_2 = \mathfrak{o}_e(\Pi_2)\dot{u}, \quad U_3 = e^{i\theta} \mathfrak{o}_e(\Pi_3)\dot{u}, U_4 = \mathfrak{o}_e(\Pi_4)\dot{u}, \quad \ldots, \quad U_J = \mathfrak{o}_e(\Pi_J)\dot{u}.
\]

Then the perturbation unknown \(\dot{u}\) is decomposed by

\[
\dot{u} = e^{i\theta} U_1 + U_2 + e^{-i\theta} U_3 + U_4 + \cdots + U_J,
\]

and the symbol of the propagator is

\[
A = \text{diag}(\lambda_{1,+1} - w, \lambda_2, \lambda_{3,-1} + \omega, \cdots, \lambda_J).
\]

The symbol of singular source term is

\[
B = \sum_{p = \pm 1} \begin{pmatrix}
e^{i\theta} \Pi_{1,+(p+1)}B_p \Pi_{1,+1} & e^{i(p-1)\theta} \Pi_{1,+(p+1)}B_p \Pi_{3,-1} & \cdots & e^{i(p-1)\theta} \Pi_{1,+(p+1)}B_p \Pi_{J} \\
e^{i(p+1)\theta} \Pi_{2,+(p+1)}B_p \Pi_{1,+1} & e^{i(p-1)\theta} \Pi_{2,+(p+1)}B_p \Pi_{3,-1} & \cdots & e^{i(p-1)\theta} \Pi_{2,+(p+1)}B_p \Pi_{J} \\
e^{i(p+2)\theta} \Pi_{3,+(p+1)}B_p \Pi_{1,+1} & e^{i(p+1)\theta} \Pi_{3,+(p+1)}B_p \Pi_{3,-1} & \cdots & e^{i(p+1)\theta} \Pi_{3,+(p+1)}B_p \Pi_{J} \\
\vdots & \vdots & \vdots & \vdots \\
e^{i(p+1)\theta} \Pi_{J,+(p+1)}B_p \Pi_{1,+(p+1)} & e^{i(p+1)\theta} \Pi_{J,+(p+1)}B_p \Pi_{2,+(p-1)} & \cdots & e^{i(p+1)\theta} \Pi_{J,+(p+1)}B_p \Pi_{J}
\end{pmatrix}
\]

with \(\Pi_{j,+q}(\xi) = \Pi(\xi + qk)\) for \(q \in \mathbb{Z}\) and

\[
B_p = B(u_{0,p}), \quad p \in \{-1, 1\},
\]

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where \( u_{0, \pm 1} \) are the leading amplitudes in the WKB solution. Combining the separation conditions in Assumption 1.9, after normal form reduction we have

\[
\tilde{B} = \begin{pmatrix}
\chi_{12} \Pi_1, +1 B_1 \Pi_2 & \chi_{13} e^{-i\delta} \Pi_1, +1 B_1 \Pi_3, -1 \\
\chi_{13} e^{i\delta} \Pi_3 B_1 \Pi_2, +1 B_1 \Pi_2 & \chi_{23} e^{-i\delta} \Pi_3, -1 B_1 \Pi_3, -1 \\
\chi_{15} e^{-i\delta} b_1^+, B_1 \Pi_2, -1 \Pi_2 & \chi_{16} e^{i\delta} b_1^- B_1 \Pi_2, -1 \Pi_2 \end{pmatrix}
\]

Assume \( \mathcal{R}_{12} \cap (\mathcal{R}_{13} + k) \cap (\mathcal{R}_{23} + k) \neq \emptyset \) and \( b_1^+ \) is transparent in \( \mathcal{R}_{13} \), then \( \mathcal{R}_{12} \cap \mathcal{R}_{13} \cap (\mathcal{R}_{23} + k) = \emptyset \) shows the oscillation terms \( \chi_{13} e^{-i\delta} (b_1^+)^{-1} \), \( \chi_{13} e^{i\delta} b_3^- \) will disappear in normal form reduction. Or, we assume \( \mathcal{R}_{12} \cap \mathcal{R}_{13} \cap (\mathcal{R}_{23} + k) \neq \emptyset \) and \( b_3^- \) is transparent in \( \mathcal{R}_{13} \), then \( \mathcal{R}_{12} \cap (\mathcal{R}_{13} + k) \cap (\mathcal{R}_{23} + k) = \emptyset \) will also eliminate the oscillation terms in normal form reduction. Combining the assumption that one of \( b_1^+ \), \( b_3^- \) is transparent in \( \mathcal{R}_{13} \), or one of \( b_2^+ \), \( b_3^- \) is transparent in \( \mathcal{R}_{23} \), the coupled case of \( \tilde{B} \) could be reduced to one of the following cases

\[
\tilde{B} = \begin{pmatrix}
0 & \chi_{12} b_1^+ & 0 \\
\chi_{12} b_3^- & 0 & \chi_{23} (b_2^+) \\
0 & \chi_{23} (b_3^-) & 0 \\
\end{pmatrix}, \quad \tilde{B} = \begin{pmatrix}
0 & \chi_{12} b_1^+ & 0 \\
\chi_{12} b_3^- & 0 & \chi_{23} (b_2^+) \\
0 & \chi_{23} (b_3^-) & 0 \\
\end{pmatrix},
\]

Combine the cases 1 in Section 2, we can obtain the bound of symmetric flows: \( |S_0(\tau, t)| \lesssim \exp(t \gamma^+) \).

From another point of view, we can decompose \( \dot{u} \) and shift the component related to \( \Pi_3 \). Then we define

\[
U_1 = \text{op}_\varepsilon(\Pi_1) \dot{u}, \quad U_2 = \text{op}_\varepsilon(\Pi_2) \dot{u}, \quad U_3 = e^{i\theta} \text{op}_\varepsilon(\Pi_3) \dot{u}, \quad U_4 = \text{op}_\varepsilon(\Pi_4) \dot{u}, \quad \cdots, \quad U_J = \text{op}_\varepsilon(\Pi_J) \dot{u}.
\]

Then the perturbation unknown \( \dot{u} \) is decomposed by

\[
\dot{u} = U_1 + U_2 + e^{-i\delta} U_3 + \cdots + U_J
\]

and

\[
A = \text{diag}(\lambda_1, \lambda_2, \lambda_3, -1 + \omega, \cdots, \lambda_J).
\]

Similar to Case 1 and Case 2 in Section 2, we have

\[
B = \sum_{p=\pm 1} \begin{pmatrix}
e^{i\theta} \Pi_{1, +p} B_p \Pi_1 & e^{i\theta} \Pi_{1, +p} B_p \Pi_2 & \cdots & e^{i\theta} \Pi_{1, +p} B_p \Pi_J \\
e^{i\theta} \Pi_{2, +p} B_p \Pi_1 & e^{i\theta} \Pi_{2, -p} B_p \Pi_2 & \cdots & e^{i\theta} \Pi_{2, -p} B_p \Pi_J \\
e^{i(p+1)\theta} \Pi_3, +p B_p \Pi_1 & e^{i(p+1)\theta} \Pi_3, -p B_p \Pi_2 & \cdots & e^{i(p+1)\theta} \Pi_3, -p B_p \Pi_J \\
\vdots & \vdots & \ddots & \vdots \\
e^{i\theta} \Pi_{J, +p} B_p \Pi_1 & e^{i\theta} \Pi_{J, +p} B_p \Pi_2 & \cdots & e^{i\theta} \Pi_{J, +p} B_p \Pi_J \\
e^{i(p-1)\theta} \Pi_{J, +p} B_p \Pi_1 & e^{i(p-1)\theta} \Pi_{J, -p} B_p \Pi_2 & \cdots & e^{i(p-1)\theta} \Pi_{J, -p} B_p \Pi_J \\
\end{pmatrix}
\]
Since we only have $(1, 2), (1, 3), (2, 3) \in \mathcal{R}_0$ which means $(2, 1) \notin \mathcal{R}_0$. Then after normal form reduction, we have

$$
\tilde{B} = \begin{pmatrix}
0 & \chi_{12} e^{i\theta} b_{12}^+ & \chi_{13} e^{i\theta} b_{13}^+ \\
\chi_{12} e^{-i\theta} b_{12}^- & 0 & \chi_{13} e^{-i\theta} b_{13}^- \\
\chi_{13} e^{i\theta} b_{13}^- & \chi_{23} e^{i\theta} b_{23}^- & 0
\end{pmatrix}.
$$

(3.3)

Assume $\mathcal{R}_{12} \cap (\mathcal{R}_{13} + k) \cap (\mathcal{R}_{23} + k) \neq \emptyset$ and $b_{12}^+$ is transparent in $\mathcal{R}_{12}$, or we assume $(\mathcal{R}_{12} + k) \cap (\mathcal{R}_{13} + k) \cap (\mathcal{R}_{23} + k) \neq \emptyset$ and $b_{21}^-$ is transparent in $\mathcal{R}_{12}$, then the oscillation terms $\chi_{12} e^{i\theta} b_{12}^+ e^{-i\theta} \chi_{13} e^{-i\theta} b_{13}^-$ will be eliminated in normal form reduction. Now we use the assumption one of $b_{13}^+, b_{31}^-$ is transparent in $\mathcal{R}_{13}$, or one of $b_{23}^+, b_{32}^-$ is transparent in $\mathcal{R}_{23}$, the coupled case of $\tilde{B}$ could be to one of the following cases

$$
\tilde{B}_1 = \begin{pmatrix}
0 & 0 & \chi_{13} e^{-i\theta} b_{13}^- \\
0 & 0 & \chi_{23} e^{-i\theta} b_{23}^- \\
\chi_{13} e^{i\theta} b_{13}^- & \chi_{23} e^{i\theta} b_{23}^- & 0
\end{pmatrix}, \quad
\tilde{B}_2 = \begin{pmatrix}
0 & 0 & \chi_{13} e^{i\theta} b_{13}^+ \\
0 & 0 & \chi_{23} e^{i\theta} b_{23}^+ \\
\chi_{13} e^{-i\theta} b_{13}^+ & \chi_{23} e^{-i\theta} b_{23}^+ & 0
\end{pmatrix},
$$

$$
\tilde{B}_3 = \begin{pmatrix}
0 & 0 & \chi_{13} e^{-i\theta} b_{13}^- \\
0 & 0 & \chi_{23} e^{-i\theta} b_{23}^- \\
0 & \chi_{23} e^{i\theta} b_{23}^- & 0
\end{pmatrix}, \quad
\tilde{B}_4 = \begin{pmatrix}
0 & 0 & \chi_{13} e^{i\theta} b_{13}^+ \\
0 & 0 & \chi_{23} e^{i\theta} b_{23}^+ \\
0 & \chi_{23} e^{-i\theta} b_{23}^+ & 0
\end{pmatrix}.
$$

Combine Remark 2.1 and the same method as Case 3 in Section 2 to obtain the bound of symmetric flows: $|S_0(\tau, t)| \leq \exp(\gamma_\tau^+)$ Similar to Section 2.4, we finally obtain the second result in Theorem 1.11 by the same method of Duhamel representation to establish the instability.

4 Applications

In this section, we give examples comprising coupled Klein-Gordon systems in $\mathbb{R}^d$ with equal masses and different velocities to show the application of our theorem. We have the following Klein-Gordon operators

$$
\partial_t + A_1(\partial_x) + \frac{1}{\varepsilon} L_0, \quad \partial_t + A_1(\theta_0 \partial_x) + \frac{1}{\varepsilon} L_0
$$

with $0 < \theta < 1$ implying different velocities and

$$
A_1(\partial_x) = \begin{pmatrix}
0 & \partial_x & 0 \\
\partial_x & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad
L_0 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \omega_0 \\
0 & -\omega_0 & 0
\end{pmatrix},
$$

(4.1)

where $\omega_0 > 0$ and $x \in \mathbb{R}^d$. For $U = (u, v) = (u_1, u_2, u_3, v_1, v_2, v_3) \in \mathbb{R}^{2(d+2)}$ with $u_1 \in \mathbb{R}^d, u_2 \in \mathbb{R}, u_3 \in \mathbb{R}$, $v_1 \in \mathbb{R}^d, v_2 \in \mathbb{R}, v_3 \in \mathbb{R}$, we give the coupled systems as follows

$$
\begin{cases}
(\partial_t + A_1(\partial_x) + L_0)u = \frac{1}{\sqrt{2}} B^1(U, U), \\
(\partial_t + A_1(\theta_0 \partial_x) + L_0)v = \frac{1}{\sqrt{2}} B^2(U, U),
\end{cases}
$$

(4.2)

where $B^1$ and $B^2$ are bilinear $\mathbb{R}^{2(d+2)} \times \mathbb{R}^{2(d+2)} \to \mathbb{R}^{d+2}$ specified as

$$
B^1(U, U') = \frac{1}{2} \begin{pmatrix}
0 & u_2 v_2' + u_2' v_2 + u_2 u_2' \\
0 & 0
\end{pmatrix}, \quad
B^2(U, U') = \frac{1}{2} \begin{pmatrix}
u_3 v_3' + u_3' v_3 + u_3 u_3' \\
0 & 0
\end{pmatrix}
$$

(4.3)
with $U = (u_1, u_2, u_3, v_1, v_2, v_3)$ and $U' = (u'_1, u'_2, u'_3, v'_1, v'_2, v'_3)$. We denote
\[
A(\partial_x) = \begin{pmatrix} A_1(\partial_x) & 0 \\ 0 & A_1(\theta_0 \partial_x) \end{pmatrix}, \quad A_0 = \begin{pmatrix} L_0 & 0 \\ 0 & L_0 \end{pmatrix}, \quad B(U, U') = \begin{pmatrix} B^1(U, U') \\ B^2(U, U') \end{pmatrix}.
\]

Then, we obtain the 1-multiplicity eigenvalues for matrix $A_1(\xi) + L_0/i$
\[
\lambda_1(\xi) = \sqrt{\omega_0^2 + |\xi|^2} = -\lambda_4(\xi), \quad \lambda_2(\xi) = \sqrt{\omega_0^2 + \theta_0^2|\xi|^2} = -\lambda_3(\xi)
\]
and 2d-multiplicity eigenvalues $\lambda_5(\xi) = 0$. As discussed in [9], corresponding smooth spectral decomposition in Assumption 1.1 and (1.4) is satisfied for $A_1(\partial_x) + L_0/i$ and $A_1(\theta_0 \partial_x) + L_0/i$.

### 4.1 Verification of Assumption 1.2: WKB solution

We select the phase $\beta = (\omega, k)$ belongs to fast Klein-Gordon branch on the variety (i.e. $\omega = \sqrt{\omega_0^2 + |k|^2}$) satisfying

- The only harmonic of $\beta$ on the fast branches are $p \in \{-1, 1\}$, $p^2\omega^2 = \omega_0^2 + p^2|k|^2 \Rightarrow p \in \{-1, 1\}$;
- No harmonic of $\beta$ belongs to the slow branches i.e. $p^2\omega^2 \neq \omega_0^2 + p^2\theta_0^2|k|^2$;
- No auto-resonances occur i.e. for $\xi \in \mathbb{R}^d$, $\lambda_1(\xi + k) = \pm \omega + \lambda_1(\xi)$ and $\lambda_2(\xi + k) = \pm \omega + \lambda_2(\xi)$ have no solution;

where $\omega \in \mathbb{R}$ is a characteristic temporal frequency and $k \in \mathbb{R}^d$ is the initial wavenumber. By [9], it suffices to check the weak transparency condition (6.48) in [9] to verify Assumption 1.2. Denoting $\Pi(p \beta)$ the orthogonal projector onto $\ker(-ip \omega + A(ip k) + A_0)$, for $|p| = 1$, the $\ker(-ip \omega + A(ip k) + A_0)$ are 1-dimensional and generated by $\vec{e}_1$ and $\vec{e}_{-1} = (\vec{e}_1)^*$. Then we obtain
\[
\vec{e}_1 = \frac{1}{\sqrt{2} \omega} \left( \frac{k}{\omega}, 1, \frac{i\omega_0}{\omega}, 0 \right)_{\mathbb{C}^{d+2}} \in \mathbb{C}^{2(d+2)}
\]
and
\[
\Pi(\beta)U = (U, \vec{e}_1)\vec{e}_1, \quad \Pi(-\beta)U = (U, \vec{e}_{-1})\vec{e}_{-1},
\]
where $(\cdot, \cdot)$ represents the Hermitian scalar product in $\mathbb{C}^{2(d+2)}$. For $p = 0$, the kernels are generated by
\[
\vec{e}_0 = \frac{1}{\sqrt{|u_1|^2 + |v_1|^2}} (u_1, 0, 0, v_1, 0, 0),
\]
and the orthogonal projector $\Pi(0)$ onto the kernel of $A_0 + A_0/i$ is
\[
\Pi(0)U = (U, \vec{e}_0)\vec{e}_0 = (u_1, 0, 0, v_1, 0, 0).
\]

Thus by definition of $B$ in (4.4), for any $U, U'$, we have
\[
\Pi(0)B(U, U') = 0, B(\Pi(0)U, U') = 0, B(U, \Pi(0)U') = 0.
\]
Then we get
\[ \Pi(0)B(\Pi(0)U, \Pi(0)U') = 0, \] for \( p = 0 \).

and for \( |p| = 1 \):
\[ 
\Pi(\beta) \sum_{p_1 + p_2 = 1} B(\Pi(p_1 \beta)U, \Pi(p_2 \beta)U') = \Pi(\beta)B(\Pi(\beta)U, \Pi(0 \cdot \beta)U') + \Pi(\beta)B(\Pi(0 \cdot \beta)U, \Pi(\beta)U') = 0, \\
\Pi(-\beta) \sum_{p_1 + p_2 = -1} B(\Pi(p_1 \beta)U, \Pi(p_2 \beta)U') = \Pi(-\beta)B(\Pi(-\beta)U, \Pi(0 \cdot \beta)U') + \Pi(-\beta)B(\Pi(0 \cdot \beta)U, \Pi(-\beta)U') = 0. 
\]

Then we verified (6.48) in [9] and furthermore, we obtain the existence of WKB approximate solution.

4.2 Verification of Assumption 1.9: Resonances and transparency

Combining the form of the characteristic variety and the \( \beta \) chosen in Section 4.1, the resonant pairs are
\[ \mathcal{R} = \{(1, 2), (1, 5), (2, 5), (3, 4), (5, 3), (5, 4)\}. \] (4.8)

Then we have
\[ B(\vec{e}_1)U = B(\vec{e}_1, U) + B(U, \vec{e}_1) = \frac{1}{\sqrt{2}}(0_d, u_2 + v_2, 0, 0_d, \frac{i\omega}{\omega}(u_3 + v_3), 0), \] (4.9)
\[ B(\vec{e}_{-1})U = B(\vec{e}_{-1}, U) + B(U, \vec{e}_{-1}) = \frac{1}{\sqrt{2}}(0_d, u_2 + v_2, 0, 0_d, -\frac{i\omega}{\omega}(u_3 + v_3), 0). \] (4.10)

We denote \( U_5(\xi) \) a element in the image of \( \Pi_5(\xi) \) and \( U_5(\xi) \) satisfies the following form
\[ U_5(\xi) = (u_1, 0, \frac{-i\xi \cdot u_1}{\omega_0}, v_1, 0, \frac{-i\theta_0 \xi \cdot u_1}{\omega_0}), \quad (u_1, v_1) \in \mathbb{C}^d \times \mathbb{C}^d. \] (4.11)

After a brief observation, we have
\[ \Pi_5(\cdot)B(\vec{e}_{\pm 1}) = 0, \] (4.12)
which means for any \( U \in \mathbb{C}^{2(d+2)} \), \( B(\vec{e}_{\pm 1})U \) belongs to the orthogonal of the range of \( \Pi_5(\xi) \).

The other projectors are
\[ \Pi_j(\xi)U = (U, \Omega_j(\xi))\Omega_j(\xi), \quad j = 1, 2, 3, 4, \]
where
\[ \Omega_j(\xi) = \frac{1}{\sqrt{2}}(\frac{\xi}{\lambda_j}, 1, \frac{i\omega_0}{\lambda_j}, 0_{\mathbb{C}^{d+2}}), \quad j = 1, 4; \]
\[ \Omega_{j'}(\xi) = \frac{1}{\sqrt{2}}(0_{\mathbb{C}^{d+2}}, \frac{\theta_0 \xi}{\lambda_{j'}}, 1, \frac{i\omega_0}{\lambda_{j'}}), \quad j' = 2, 3. \]

Then we have
\[ (\Omega_1(\xi + k), B(\vec{e}_1)U_5(\xi)) = 0, \] (4.13)
\[ (\Omega_4(\xi), B(\vec{e}_{-1})U_5(\xi + k)) = 0. \] (4.14)
Combining $\Pi_5(\cdot)B(\bar{\epsilon}_{\pm 1}) = 0$, it shows that the resources $(1, 5), (5, 4)$ are transparent and furthermore, we have

$$(\Omega_1(\xi + k), B(\bar{\epsilon}_1)\Omega_2(\xi)) = \frac{1}{2\sqrt{2}},$$

$$(\Omega_2(\xi), B(\bar{\epsilon}_{-1})\Omega_1(\xi + k)) = \frac{1}{2\sqrt{2}} \cdot \frac{\omega_0^2}{\omega \lambda_1(\xi + k)},$$

$$(\Omega_3(\xi + k), B(\bar{\epsilon}_1)\Omega_4(\xi)) = -\frac{1}{2\sqrt{2}} \cdot \frac{\omega_0^2}{\omega \lambda_4(\xi)},$$

$$(\Omega_4(\xi), B(\bar{\epsilon}_{-1})\Omega_3(\xi + k)) = \frac{1}{2\sqrt{2}},$$

$$(\Omega_2(\xi + k), B(\bar{\epsilon}_1)U_5(\xi)) = \frac{\xi \cdot u_1 + \theta_0 \xi \cdot v_1}{2\omega},$$

$$(\Omega_5(\xi), B(\bar{\epsilon}_{-1})\Omega_5(\xi + k)) = -\frac{(\xi + k)(\xi \cdot u_1 + \theta_0 \xi \cdot v_1)}{2\omega}.$$  

Thus we get

$$R_0 = \{(1, 2), (2, 5), (3, 4), (5, 3)\},$$

(4.15)

in which set no auto-resonances are contained. Furthermore, we consider the following corresponding resonant sets:

$$R_{12} = \{\xi : \sqrt{\omega_0^2 + |\xi + k|^2} = \omega + \sqrt{\omega_0^2 + \theta_0^2|\xi|^2}\},$$

$$R_{25} = \{\xi : \theta_0|\xi + k| = |k|\},$$

$$R_{34} = \{\xi : \sqrt{\omega_0^2 + |\xi|^2} - \sqrt{\omega_0^2 + \theta_0^2|\xi + k|^2} = \omega\},$$

$$R_{53} = \{\xi : \theta_0|\xi| = |k|\},$$

where $\omega = \sqrt{\omega_0^2 + |k|^2}$. Now, we are trying to verify separation conditions in Assumption 1.9.

Firstly, we consider $(1, 2)$-non-transparent resonance and its coupled $(2, 5)$-resonance in $R_0$ and give the following frequency shift:

$$U_1 = e^{i\theta}\text{op}_\varepsilon(\Pi_1)\dot{u}, \quad U_2 = \text{op}_\varepsilon(\Pi_2)\dot{u}, \quad U_3 = \text{op}_\varepsilon(\Pi_3)\dot{u}, \quad U_4 = \text{op}_\varepsilon(\Pi_4)\dot{u}, \quad U_5 = e^{-i\theta}\text{op}_\varepsilon(\Pi_5)\dot{u}.$$  

Obviously (4.12) shows that $b_{52}^-$ is transparent in $R_{25}$ i.e. $(b_{52}^-)_{-1}$ is transparent in $(R_{25} + k)$. After normal form reduction, we have

$$\bar{B} = \begin{pmatrix}
0 & \chi_{12}b_{12}^+ & 0 \\
\chi_{12}b_{12}^- & 0 & \chi_{25,-1}(b_{25}^-)_{-1} \\
0 & 0 & 0
\end{pmatrix},$$

(4.16)

and

$$R_{12} \cap (R_{25} + k) = \{\xi : |\xi + k| = \sqrt{3\omega_0^2 + 4|k|^2}, \theta_0|\xi| = |k|\}.$$  

Since $\omega_0$ and $|k|$ are fixed, bounded constants and by inequality

$$\left(\frac{1}{\theta_0} - 1\right)|k| = ||\xi| - |k|| \lesssim |\xi + k| = \sqrt{3\omega_0^2 + 4|k|^2} = C(\omega, k) \lesssim ||\xi| + |k|| = \left(\frac{1}{\theta_0} + 1\right)|k|,$$

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we can easily draw a conclusion that when $\theta_0$ small enough, it gives

$$\mathcal{R}_{12} \cap (\mathcal{R}_{25} + k) \neq \emptyset.$$  

Then in $\mathcal{R}_{12} \cap (\mathcal{R}_{25} + k)$, (4.16) can be reduced to

$$\hat{B} = \chi_{12}\chi_{25,-1} \begin{pmatrix} 0 & b_{12}^+ & 0 \\ b_{21}^- & 0 & (b_{25}^+)_1 \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.17)$$

By Case 1, we get $|S_0(\tau, t)| \lesssim \exp(t \gamma_{12}^+)$ for corresponding symbolic flow. When $\xi \in \mathcal{R}_{12}(\mathcal{R}_{25} + k)$ and $\xi \in (\mathcal{R}_{25} + k) \setminus \mathcal{R}_{12}$, we will obtain $|S_0(\tau, t)| \lesssim \exp(t \gamma_{12}^+)$ and $|S_0(\tau, t)| \lesssim 1$ respectively. When $\theta_0$ does not satisfy the smallness and $\mathcal{R}_{12} \cap (\mathcal{R}_{25} + k) = \emptyset$, we can obtain the same bounds for symbolic flow both $\xi \in \mathcal{R}_{12}$ and $\xi \in (\mathcal{R}_{25} + k)$.

Then, we consider (2, 5)-non-transparent resonance and its coupled (1, 2)-resonance and (5, 3)-resonance in $\mathcal{R}_0$. We make the frequency shift as

$$U_1 = e^{-2i\theta \text{op}_x(P_1)}\hat{u}, \quad U_2 = e^{-i\theta \text{op}_x(P_2)}\hat{u}, \quad U_3 = e^{i\theta \text{op}_x(P_3)}\hat{u}, \quad U_4 = \text{op}_x(P_4)\hat{u}, \quad U_5 = \text{op}_x(P_5)\hat{u},$$

where $\theta = (k \cdot x - wt)/\varepsilon$. Combining (3.12), we know that $b_{52}^-$ and $(b_{53}^+)_1$ are transparent in $\mathcal{R}_{25}$ and $(\mathcal{R}_{53} + k)$ respectively, then we have

$$\hat{B} = \chi_{12,+1}(b_{12}^+)_{+1} 0 0 \\ \chi_{12,+1}(b_{21}^-)+1 0 0 \\ 0 0 0 \chi_{53,+1}(b_{35}^+)_{-1}. \quad (4.18)$$

Since

$$\mathcal{R}_{25} \cap (\mathcal{R}_{53} + k) = \{ \xi : |\xi| = \sqrt{1 - \theta_0^2|k|} \},$$

$$\mathcal{R}_{25} \cap (\mathcal{R}_{12} - k) = \{ \xi : |\xi + 2k| = \sqrt{3\omega_0^2 + 4|k|^2}, \theta_0|\xi + k| = |k| \},$$

they show that

$$\mathcal{R}_{25} \cap (\mathcal{R}_{53} + k) \cap (\mathcal{R}_{12} - k) = \{ \xi : |\xi| = \sqrt{1 - \theta_0^2|k|}, |\xi|^2 = 3\omega_0^2 \}.$$

If the fixed $\xi$ and $\omega_0$ satisfy $(1 - \theta_0^2)|k| = 3\omega_0^2$, then $\mathcal{R}_{25} \cap (\mathcal{R}_{53} + k) \cap (\mathcal{R}_{12} - k) \neq \emptyset$, in which intersection we have

$$\hat{B} = \chi_{25}\chi_{53,-1}\chi_{12,+1} \begin{pmatrix} 0 & (b_{12}^+)_{+1} & 0 \\ (b_{21}^-)+1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.19)$$

Then using similar estimates in Case 1 in Section 2, we have $|S_0(\tau, t)| \lesssim \exp(t \gamma_{12}^+)$. 

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For the fixed $\xi$ and $\omega_0$, if $(1 - \theta_0^2)|k| \neq 3\omega_0^2$, then $\mathcal{R}_{25} \cap (\mathcal{R}_{53} + k) \cap (\mathcal{R}_{12} - k) = \emptyset$, then we only need to discuss the case of intersection $\mathcal{R}_{25} \cap (\mathcal{R}_{53} + k)$ and $\mathcal{R}_{25} \cap (\mathcal{R}_{12} - k)$, in those intersections we have

$$
\tilde{B}_1 = \chi_{25\lambda_{53}, 1} \begin{pmatrix} 0 & 0 & b_{25}^+ \\ 0 & 0 & (b_{35}^-)_{-1} \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{B}_2 = \chi_{25\lambda_{12}, +1} \begin{pmatrix} 0 & (b_{12}^+)_{+1} & 0 \\ (b_{12}^-)_{+1} & 0 & b_{25}^- \\ 0 & 0 & 0 \end{pmatrix}.
$$

(4.20)

Then we can give the bounds $|S_0(\tau, t)| \lesssim 1$ and $|S_0(\tau, t)| \lesssim \exp(t\gamma_{12}^+)$ respectively for symbolic flow directly by same procedures in Case 1 of Section 2.

Furthermore, we consider (3, 4)-non-transparent resonance and its coupled (5, 3)-resonance in $\mathcal{R}_0$. We make the frequency shift as

$$U_1 = \text{op}_\varepsilon(\Pi_1)\hat{u}, \quad U_2 = \text{op}_\varepsilon(\Pi_2)\hat{u}, \quad U_3 = \text{op}_\varepsilon(\Pi_3)\hat{u}, \quad U_4 = e^{i\theta} \text{op}_\varepsilon(\Pi_4)\hat{u}, \quad U_5 = e^{-i\theta} \text{op}_\varepsilon(\Pi_5)\hat{u},$$

and

$$(\mathcal{R}_{34} + k) \cap \mathcal{R}_{53} = \{ \xi : |\xi - k| = \sqrt{3\omega_0^2 + 4|k|^2}, \theta_0|\xi| = |k| \},$$

then combining (3.12), we know that $b_{53}^+$ are transparent in $\mathcal{R}_{53}$, we have

$$\tilde{B} = \chi_{34,-1}\chi_{53} \begin{pmatrix} 0 & (b_{34}^+)_{-1} & b_{35}^- \\ (b_{34}^-)_{+1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

Then we can directly use the theory in Case 1 in Section 2 to give the discussion for the bound of symbolic flow: $|S_0(\tau, t)| \lesssim \exp(t\gamma_{34}^+)$.

Finally, we consider (5, 3)-non-transparent resonance and its coupled (2, 5)-resonance and (3, 4)-resonance in $\mathcal{R}_0$. We make the frequency shift as

$$U_1 = \text{op}_\varepsilon(\Pi_1)\hat{u}, \quad U_2 = e^{-2i\theta} \text{op}_\varepsilon(\Pi_2)\hat{u}, \quad U_3 = \text{op}_\varepsilon(\Pi_3)\hat{u}, \quad U_4 = e^{i\theta} \text{op}_\varepsilon(\Pi_4)\hat{u}, \quad U_5 = e^{-i\theta} \text{op}_\varepsilon(\Pi_5)\hat{u},$$

We know that $(b_{52}^-)_{+1}$ and $b_{53}^+$ are transparent in $(\mathcal{R}_{25} - k)$ and $\mathcal{R}_{53}$ respectively, thus we have

$$\tilde{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \chi_{34,-1}(b_{34}^+)_{-1} \chi_{34,-1}(b_{34}^-)_{+1} \\ 0 & \chi_{34,-1}(b_{34}^-)_{-1} & \chi_{34,-1} \chi_{53} \end{pmatrix}.$$ 

(4.21)

We need to consider the following intersections

$$\mathcal{R}_{53} \cap (\mathcal{R}_{25} - k) = \{ \xi : |\xi + k| = \sqrt{1 - \theta_0^2|k|} \},$$

$$\mathcal{R}_{53} \cap (\mathcal{R}_{34} + k) = \{ \xi : |\xi - k| = \sqrt{3\omega_0^2 + 4|k|^2}, \theta_0|\xi| = |k| \}.$$

Moreover, we have

$$\mathcal{R}_{53} \cap (\mathcal{R}_{25} - k) \cap (\mathcal{R}_{34} + k) = \{ \xi : |\xi| = \frac{1}{\theta_0}|k|, |\xi|^2 = 3\omega_0^2 + |k|^2 \},$$

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If the fixed $\xi$, $\omega_0$ and $\theta_0$ satisfy $\frac{1}{\theta_0^2} |k|^2 = 3\omega_0^2 + |k|^2$, then $R_{53} \cap (R_{25} - k) \cap (R_{34} + k) \neq \emptyset$, in which intersection we have

$$\vec{B} = \chi_{53}\chi_{25,+1}\chi_{34,-1} \begin{pmatrix} 0 & 0 & (b_{25}^+)^{+1} \\ 0 & 0 & (b_{34}^-)^{-1} \\ 0 & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (4.22)$$

then we have $|S_0(\tau, t)| \lesssim \exp(t_{\gamma}^+)$.

Thus we obtain $|S_0(\tau, t)| \lesssim \exp(t_{\gamma}^+)$ and we can apply Theorem 1.11 to coupled Klein-Gordon systems with equal masses whose nonlinear terms specified as (4.3) for $x, \xi \in \mathbb{R}^d (d \geq 1)$. After a short observation, we know: $\Gamma_{12}(\xi) > 0 \Rightarrow \Gamma = 0$ and combining Theorem 1.11 (the relaxed instability criterion), we obtain the instability of WKB solutions $u_a$.

**Remark 4.1.** In Section 5.2 of [9], under the Assumption 2.8 (Assumption 1.4 in this paper), the example 'coupled Klein-Gordon systems with equal masses' with different nonlinear term $B(U, V)$ defined in [9] in (5.39) shows the instability criterion holds only in 1-dimensional space. Now we are trying to verify the relaxed instability criterion in Theorem 1.11 to improve the results i.e. to turn the restriction 1-dimensional space into $d$ dimensional space with any $d \geq 1$.

For $R_0 = \{(1, 2), (1, 5), (3, 4), (5, 4)\}$, we already have

- $R_{12} \cap R_{15} = \emptyset$ for (1, 2)-resonance and its coupled (1, 5)-resonance in $R_0$;
- $R_{15} \cap R_{12} \cap (R_{54} + k) = \emptyset$ for (1, 5)-resonance and its coupled (1, 2), (5, 4)-resonance in $R_0$;
- $R_{34} \cap R_{54} = \emptyset$ for (3, 4)-resonance and its coupled (5, 4)-resonance in $R_0$;
- $R_{54} \cap (R_{15} - k) \cap R_{34} = \emptyset$ for (5, 4)-resonance and its coupled (1, 5), (3, 4)-resonance in $R_0$.

Combining (5.41) in [9]: $\Pi_5(\cdot)B(\vec{\xi} \pm 1) = 0$, we know that $b_{51}^- \cdot (b_{34}^-)^{-1}$ and its coupled (5, 4)-resonance in $R_{54}$ respectively. Thus we only need to consider $R_{15} \cap (R_{54} + k) = \{\xi \in \mathbb{R}^d : |\xi| = 0\}$, in which intersection by the method of frequency shift and normal form reduction, we have

$$\vec{B} = \chi_{15}\chi_{54,-1} \begin{pmatrix} 0 & 0 & b_{15}^+ \\ 0 & 0 & (b_{45}^-)^{-1} \\ 0 & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (4.24)$$

Then we obtain $|S_0(\tau, t)| \lesssim 1$ directly by Case 1 in Section 2. Hence Theorem 1.11 (the relaxed instability criterion) is applicable for this case. Combining the calculation result: $\Gamma_{12}(\xi) > 0 \Rightarrow \Gamma = 0$, it shows instability for coupled Klein-Gordon systems with equal masses in $d$ dimensional space whose nonlinear terms specified as (5.39) in [9].

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