A note on the Dixmier-Moeglin equivalence in Leavitt path algebras of arbitrary graphs over a field

Kulumani M. Rangaswamy
Department of Mathematics, University of Colorado, Colorado Springs, Colorado, USA

ABSTRACT
The Dixmier-Moeglin Equivalence (for short, the DM-equivalence) is the equivalence of three distinguishing properties of prime ideals in a non-commutative algebra $A$. These properties are of (i) being primitive, (ii) being rational and (iii) being locally closed in the Zariski topology of $\text{Spec}(A)$. The DM-equivalence holds in many interesting algebras over a field. Recently, it was shown that the prime ideals of a Leavitt path algebra of a finite graph satisfy the DM-equivalence. In this note, we investigate the occurrence of the DM-equivalence in a Leavitt path algebra $L$ of an arbitrary directed graph $E$. Our analysis shows that locally closed prime ideals of $L$ satisfy interesting equivalent properties such as being strongly primitive and completely irreducible. This leads to new characterizations of primitive ideals in a Leavitt path algebra of a finite graph. Examples illustrate the results and the constraints.

KEYWORDS
Completely irreducible ideals; Leavitt path algebra; prime ideals; primitive ideals; strongly primitive ideals

2020 MATHEMATICS SUBJECT CLASSIFICATION
16W10; 16D25

1. Introduction

In the study of several classes of non-commutative algebras, an important but often difficult question is to classify the irreducible representations of an algebra and this question is usually reduced to characterizing the primitive ideals which are the kernels of these representations. Toward this goal, it is often necessary to identify the primitive ideals among the prime ideals of these algebras. A fundamental theorem in this connection was proved by J. Dixmier [9] and C. Moeglin [13]: If $U$ is the enveloping algebra of a finite dimensional complex Lie algebra, and $P$ is a prime ideal of $U$, then $P$ is primitive if and only if it is rational, if and only if $P$ is a locally closed point in the prime spectrum of $U$ under the Zariski topology. Here, a prime ideal $P$ of an algebra $A$ over a field $K$ is said to be rational if the center of the Martindale ring of quotients of the prime algebra $A/P$ – called the extended centroid of $A/P$ – is an algebraic extension of $K$. Likewise, an equivalent statement for a prime ideal $P$ to be locally closed in the algebra $A$ is that the intersection $I$ of all the prime ideals of $A$ properly containing $P$ strictly contains $P$. An algebra $A$ over a field $K$ is said to satisfy the Dixmier-Moeglin Equivalence (for short, DM-equivalence) on prime ideals, if for any prime ideal $P$ of $A$, the three properties in the Dixmier-Moeglin Theorem are equivalent. The D-M equivalence seems to hold in a number of interesting classes of algebras. Just to name a few, Goodearl and Letzter [11] established the DM-equivalence in Quantum coordinate rings and also in quantized Weyl algebras, while Bell, Rogalski and Sierra [7] proved the DM-equivalence for prime ideals in the Twisted homogeneous coordinate rings.
Inspired by the above investigations, it was shown in [2] that if \( E \) is a finite graph and \( K \) is a field, then the DM-equivalence for prime ideals always holds in the Leavitt path algebra \( L_K(E) \). The aim of this note is to investigate the occurrence of the DM-equivalence in the case of a Leavitt path algebra \( L := L_K(E) \) of an arbitrary graph \( E \). We begin by obtaining several characterizations of a locally closed prime ideal \( P \) of \( L \) (Theorem 3.9). Such an ideal \( P \), in addition to being primitive, has the interesting property of being completely irreducible, a property of ideals which plays an important role in the multiplicative ideal theory of non-Noetherian commutative rings (see definition below and [10]). This leads to \( P \) possessing a property stronger than being primitive which we call strong primitivity. It is shown that, in the case of a finite graph \( E \), the primitive ideals of \( L_K(E) \) are always strongly primitive, but this does not hold if \( E \) is an infinite graph. We also establish that a locally closed prime ideal \( P \) of \( L \) is always rational. The converse holds if \( P \) is a non-graded prime ideal or a graded prime ideal of the form \( P = \mathcal{I}(H, B_H \{u\}) \). But an example of a countably infinite graph \( E \) shows that the corresponding Leavitt path algebra \( L_{\mathbb{C}}(E) \) over the field \( \mathbb{C} \) of complex numbers possesses many graded prime ideals which are rational but not locally closed. As a consequence of the preceding investigation, we obtain new characterizations of primitive ideals in a Leavitt path algebra of a finite graph (Corollary 3.20).

2. Preliminaries

For the general notation, terminology and results in Leavitt path algebras, we refer to [1]. We give below a short outline of some of the needed basic concepts and results.

A (directed) graph \( E = (E^0, E^1, r, s) \) consists of two sets \( E^0 \) and \( E^1 \) together with maps \( r, s : E^1 \to E^0 \). The elements of \( E^0 \) are called vertices and the elements of \( E^1 \) edges. All the graphs \( E \) that we consider (excepting when specifically stated) are arbitrary in the sense that no restriction is placed either on the number of vertices in \( E \) or on the number of edges emitted by a single vertex.

A vertex \( v \) is called a sink if it emits no edges and a vertex \( v \) is called a regular vertex if it emits a non-empty finite set of edges. An infinite emitter is a vertex which emits infinitely many edges. A graph without infinite emitters is said to be row-finite. For each \( e \in E^1 \), we call \( e^* \) a ghost edge. We let \( r(e^*) \) denote \( s(e) \), and we let \( s(e^*) \) denote \( r(e) \). A path \( \mu \) of length \( n > 0 \) is a finite sequence of edges \( \mu = e_1 e_2 \cdots e_n \) with \( r(e_i) = s(e_{i+1}) \) for all \( i = 1, \ldots, n - 1 \). In this case, \( \mu^* = e_1^* \cdots e_n^* e_1^* \) is the corresponding ghost path. We write \( |\mu| = n \) if \( \mu \) is a path of length \( n \). A vertex is considered a path of length 0. The set of all vertices on the path \( \mu \) is denoted by \( \mu^0 \).

The set of all paths in a graph \( E \) is denoted by \( \text{Path}(E) \).

A path \( \mu = e_1 e_2 \cdots e_n \) in \( E \) is closed if \( r(e_n) = s(e_1) \), in which case \( \mu \) is said to be based at the vertex \( s(e_1) \). A closed path \( \mu \) as above is called simple provided it does not pass through its base more than once, i.e., \( s(e_i) \neq s(e_j) \) for all \( i = 2, \ldots, n \). The closed path \( \mu \) is called a cycle if it does not pass through any of its vertices twice, that is, if \( s(e_i) \neq s(e_j) \) for every \( i \neq j \). An exit for a path \( \mu = e_1 \cdots e_n \) is an edge \( f \) that satisfies \( s(f) = s(e_i) \) for some \( i \) but \( f \neq e_i \). The graph \( E \) is said to satisfy Condition \( (L) \), if every cycle in \( E \) has an exit in \( E \). The graph \( E \) is said to satisfy Condition \( (K) \), if any vertex on a closed path \( \mu \) is also the base for a closed path \( \gamma \) different from \( \mu \).

If there is a path from vertex \( u \) to a vertex \( v \), we write \( u \geq v \). A non-empty subset \( D \) of vertices is said to be downward directed if for any \( u, v \in D \), there exists a \( w \in D \) such that \( u \geq w \) and \( v \geq w \). A subset \( H \) of \( E^0 \) is called hereditary if, whenever \( v \in H \) and \( w \in E^0 \) satisfy \( v \geq w \), then \( w \in H \). A hereditary set is saturated if, for any regular vertex \( v \) in \( E^0 \), \( r(s^{-1}(v)) \subseteq H \) implies \( v \in H \).

**Definition 2.1.** Given an arbitrary graph \( E \) and a field \( K \), the Leavitt path algebra \( L_K(E) \) is defined to be the \( K \)-algebra generated by a set \( \{v : v \in E^0\} \) of pair-wise orthogonal idempotents together with a set of variables \( \{e, e^* : e \in E^1\} \) which satisfy the following conditions:
(1) \( s(e)e = e = er(e) \) for all \( e \in E^1 \).

(2) \( r(e)e^s = e^s = e^s(e) \) for all \( e \in E^1 \).

(3) (The "CK-1 relations") For all \( e, f \in E^1 \), \( e^s e = r(e) \) and \( e^s f = 0 \) if \( e \neq f \).

(4) (The "CK-2 relations") For every regular vertex \( v \in E^0 \),

\[
\nu = \sum_{e \in E^1 : s(e) = v} ee^s.
\]

Every Leavitt path algebra \( L_K(E) \) is a \( \mathbb{Z} \)-graded algebra, namely, \( L_K(E) = \bigoplus_{n \in \mathbb{Z}} L_n \) induced by defining, for all \( v \in E^0 \) and \( e \in E^1 \), \( \deg(v) = 0 \), \( \deg(e) = 1 \), \( \deg(e^s) = -1 \). Here, for each \( n \in \mathbb{Z} \), the homogeneous component \( L_n \) is given by

\[
L_n = \left\{ \sum k_i x_i \beta_i^s \in L : \ |x_i| - |\beta_i| = n \right\}, \text{ where } k_i \in K, \ x_i, \beta_i \in \text{Path}(E).
\]

Further, the \( L_n \) are abelian subgroups satisfying \( L_n L_m \subseteq L_{m+n} \) for all \( m, n \in \mathbb{Z} \). An ideal \( I \) of \( L_K(E) \) is said to be a graded ideal if \( I = \bigoplus_{n \in \mathbb{Z}} (I \cap L_n) \). Equivalently, if \( a \in I \) and \( a = a_1 + \cdots + a_n \) is a graded sum with \( a_k \in L_k \) for all \( k = 1, \cdots, m \), then \( a_k \in I \) for all \( k \).

A breaking vertex of a hereditary saturated subset \( H \) is an infinite emitter \( w \in E^0 \setminus H \) with the property that \( 0 < |s^{-1}(w) \cap r^{-1}(E^0 \setminus H)| < \infty \). The set of all breaking vertices of \( H \) is denoted by \( B_H \). For any \( v \in B_H \), \( v^H \) denotes the element \( v - \sum_{s(e) = v, r(e) \not\in H} ee^s \). Given a hereditary saturated subset \( H \) and a subset \( S \subseteq B_H \), \( (H, S) \) is called an admissible pair. Given an admissible pair \( (H, S) \), the ideal generated by \( H \cup \{v^H : v \in S\} \) is denoted by \( I(H, S) \). It was shown in [16] that the graded ideals of \( L_K(E) \) are precisely the ideals of the form \( I(H, S) \) for some admissible pair \( (H, S) \). Moreover, \( L_K(E)/I(H, S) \cong L_K(E\langle H, S \rangle) \). Here \( E\langle H, S \rangle \) is a quotient graph of \( E \) where

\[
(E\langle H, S \rangle)^0 = (E^0 \setminus H) \cup \{v' : v \in B_H \setminus S\}
\]

and

\[
(E\langle H, S \rangle)^1 = \{e \in E^1 : r(e) \not\in H\} \cup \{e' : e \in E^1 \text{ with } r(e) \in B_H \setminus S\}
\]

and \( r, s \) are extended to \( (E\langle H, S \rangle)^0 \) by setting \( s(e') = s(e) \) and \( r(e') = r(e) \).

The following statement will be useful in our investigation.

**Theorem 2.2.** (Theorem 3.12, [14]) Let \( P \) be an ideal of \( L_K(E) \) with \( I \cap E^0 = H \). Then \( P \) is a prime ideal if and only if \( P \) satisfies one of the following conditions:

(i) \( P = I(H, B_H) \) such that \( (E\langle H, B_H \rangle)^0 \) is downward directed;

(ii) \( P = I(H, B_H) \setminus \{u\} \) where \( u \in B_H \) and \( v \geq u \) for all \( v \in E^0 \setminus H \);

(iii) \( P = I(H, B_H) + < p(e) > \) where \( e \) is a cycle without exits in \( (E\langle H, B_H \rangle)^0 \), \( (E\langle H, B_H \rangle)^0 \) is downward directed and \( p(x) \) is an irreducible polynomial in \( K[x, x^{-1}] \).

**Definition 2.3.** Let \( K \) be a field and let \( A \) be a prime \( K \)-algebra. The Martindale ring of quotients of \( A \), denoted by \( \text{Frac}(A) \) consists of equivalent classes of pairs \( (I, f) \) where \( I \) is a non-zero ideal of \( A \) and \( f \in \text{Hom}_A(I_A, A_A) \). Here two pairs \( (I, f), (J, g) \) are considered equivalent if \( f = g \) on the intersection \( I \cap J \). Addition and multiplication is given by

\[
(I, f) + (J, g) = (I \cap J, f + g) \text{ and } (I, f) \cdot (J, g) = (IJ, f \circ g)
\]

The center of \( \text{Frac}(A) \) is called the extended centroid of \( A \) and is denoted by \( C(A) \). For each \( a \in A \), let \( \lambda_a \) be the map \( A \to A \) given by \( \lambda_a(x) = ax \). Then the map \( a \to \lambda_a \) is an embedding of \( A \) in \( \text{Frac}(A) \).
We will be using an internal characterization the extended centroid $C(A)$ given in [6]: $C(A)$ consists precisely of the equivalence classes $[(I,f)]$, where $f : I \to A$ is a $(A, A)$ bimodule homomorphism. Thus

$$C(A) = \{ (I,f) : 0 \neq I \triangleleft A, f \in \text{Hom}_A(I_A, A_A) \}/ \sim$$

where $\sim$ is the equivalence relation described above.

**3. The results**

In this section, we shall investigate the occurrence of the D-M equivalence for prime ideals in a Leavitt path algebra $L := L_K(E)$ of an arbitrary directed graph $E$ over a field $K$. First, we obtain several characterizations of a locally closed prime ideal $P$ of $L$. In particular, such a prime ideal $P$ actually satisfies a property stronger than primitivity. We call ideals with this property strongly primitive ideals. Strongly primitive ideals in $L_K(E)$ have interesting property of being completely irreducible. We show that, if $E$ is an arbitrary graph, a prime ideal $P$ of $L_K(E)$ is locally closed if and only if $P$ is strongly primitive. We also show that a strongly primitive ideal $P$ of $L_K(E)$ is rational. The converse easily holds if the prime ideal $P$ is either a non-graded ideal or a graded ideal of the form $P = \text{I}(H, B_H \{ u \})$ for some $u \in B_H$, where $H = P \cap E^0$ and $B_H$ is the set of breaking vertices corresponding to $H$. However, by means of two examples, we show that a graded rational prime ideal of $L_K(E)$ need not be strongly primitive and thus not locally closed.

When the graph $E$ is finite, our analysis of the DM-equivalence leads to new characterizations of primitive ideals of $L_K(E)$.

We begin with a definition followed by a description of the primitive ideals of a Leavitt path algebra $L := L_K(E)$ of an arbitrary graph $E$ over a field $K$.

**Definition 3.1.** ([3]) A graph $E$ is said to satisfy the **countable separation property** (for short, the **CSP**), if there is a countable subset $S$ of $E^0$ such that, for every $u \in E^0$ there is a $v \in S$ such that $u \geq v$, that is, there is a path from $u$ to $v$.

**Theorem 3.2.** (Theorem 4.3, [14]) Let $E$ be an arbitrary graph and let $L := L_K(E)$. An ideal $P$ of $L$ with graded part $\text{gr}(P) = \text{I}(H, S)$ is primitive if and only if $P$ satisfies one of the following:

(a) $P$ is a non-graded prime ideal of the form $P = \text{I}(H, B_H) + < p(x) >$ where $p(x) \in K(x)$ is an irreducible polynomial and $c$ is a cycle without exits in $E^0 \setminus H$;

(b) $P$ is a graded prime ideal of the form $P = \text{I}(H, B_H \{ u \})$ for some $u \in B_H$;

(c) $P$ is a graded prime ideal of the form $P = \text{I}(H, B_H)$ and $E \setminus (H, B_H)$ satisfies both Condition (L) and the CSP.

Note: When $P$ satisfies Condition (b), the quotient graph $E \setminus (H, B_H \{ u \})$ contains a sink $u'$ corresponding to $u$ and since it is further downward directed, it is clear that $E \setminus (H, B_H \{ u \})$ satisfies Condition (L) and the CSP with respect to $\{u'\}$. Thus conditions (b) and (c) Theorem 3.2 can be combined by stating that

$(b^*) P = \text{I}(H, S)$ is a graded prime ideal such that $E \setminus (H, S)$ satisfies Condition (L) and the CSP.

Influenced by Theorem 3.2, we introduce the following definition.

**Definition 3.3.** (i) A graph $E$ is said to satisfy the **strong CSP**, if $E^0$ satisfies the CSP with respect to a countable set $S$ such that $S$ is contained in every non-empty hereditary saturated subset of $E^0$;
(ii) Given an arbitrary graph $E$, an ideal $P$ of $L_K(E)$ is said to be strongly primitive if either $P$ is a non-graded prime ideal or a graded prime ideal of the form $I(H, S)$ such that the quotient graph $E \setminus (H, S)$ (is downward directed and) satisfies both Condition $(L)$ and the strong CSP.

Remark 3.4. Thus, not only a non-graded prime ideal is strongly primitive, but also a graded prime ideal of the form $P = I(H, B_H \setminus \{u\})$ for some $u \in B_H$ is strongly primitive. Because, the quotient graph $E \setminus (H, B_H \setminus \{u\})$ contains the sink $u'$ corresponding to $u$ and since it is also downward directed, it satisfies both Condition $(L)$ and the strong CSP with respect to the sink $u'$.

Definition 3.5. (i) A prime ideal $P$ in a ring $R$ is said to be locally closed if $P$ is not the intersection of all the prime ideals of $R$ properly containing $P$.

(ii) A prime ideal $P$ in an algebra $R$ over a field $K$ is said to be rational if the extended centroid of $R/P$ is an algebraic extension of the field $K$.

(iii) ([10]) An ideal $I$ in a ring $R$ is said to be completely irreducible if $I$ is not the intersection of all the ideals of $R$ properly containing $I$.

We wish to show that in a Leavitt path algebra $L$, a prime ideal $P$ is locally closed $< = > P$ is completely irreducible $< = > P$ is strongly primitive. We shall be using the following three results.

Lemma 3.6. (Theorem 2.1, [15]) An arbitrary graph $E$ satisfies Condition $(L)$ if and only if $L := L_K(E)$ is a Zorn ring, that is, every non-zero left/right ideal of $L$ contains a non-zero idempotent.

Lemma 3.7. (Lemma 3.5, [14]) Let $E$ be a downward directed graph and let $L := L_K(E)$. If $A$ is a non-zero ideal of $L$ containing no vertices, then there is a cycle $c$ without exits in $E$ and $A = I(c)$, where $f(x) \in K[x]$ with a non-zero constant term.

Theorem 3.8. (Theorem 5.5, [12]) Let $E$ be an arbitrary graph and $L := L_K(E)$. A proper ideal $I$ of $L$ is completely irreducible if and only if one of the following conditions hold:

(a) $I = I(H, S)$ is a graded ideal which is strongly primitive;
(b) $I = P^n$ for some non-graded prime ideal $P$ and positive integer $n$.

Note: Since $A^2 = A$ for any graded ideal of $L$, it is clear from the definition 3.3 that Theorem 3.8 can be rephrased as follows: An proper ideal $I$ of $L$ is completely irreducible if and only if $I = P^n$ for some strongly primitive ideal $P$ and positive integer $n$.

The next theorem gives a characterization of the locally closed prime ideals in a Leavitt path algebra.

Theorem 3.9. Let $E$ be an arbitrary graph and $L := L_K(E)$. Then the following properties are equivalent for a prime ideal $P$ of $L$:

(a) $P$ is locally closed;
(b) $P$ is completely irreducible;
(c) There is an element $a \in L$ such that $P$ is maximal with respect to the property that $a \not\in P$;
(d) $P$ is strongly primitive.

Proof. (a) $>$(b). Suppose $P$ is a locally closed prime ideal of $L$. Keeping in mind Theorem 2.2, if $P$ is a non-graded prime ideal or is a graded prime ideal of the form $P = I(H, B_H \setminus \{u\})$, then $P$ is completely irreducible by Theorem 3.8 and Remark 3.4. So assume that $P$ is a graded prime ideal of the form $P = I(H, B_H)$. Since the Jacobson radical of $L/P \cong L_K(E \setminus (H, B_H))$ is zero, the...
intersection of all the primitive ideals in \(L/P\) is \([0]\). As \([0]\) is locally closed in \(L/P\), we conclude that \([0]\) is necessarily a primitive ideal of \(L/P \cong L_K(E \setminus (H, B_H))\). In particular, \(E \setminus (H, B_H)\) is downward directed and satisfies Condition (L). Let \(I \neq [0]\) be the intersection of all the non-zero prime ideals of \(L/P \cong L_K(E \setminus (H, B_H))\). Since \(E\) satisfies Condition (L), Lemma 3.6 implies that \(I\) contains a non-zero idempotent, say \(u\). We claim that every non-zero ideal of \(R = L_K(E \setminus (H, B_H))\) contains \(u\). Because, if there is a non-zero ideal \(A\) of \(R\) which does not contain \(u\), then Zorn’s Lemma will give rise to a non-zero ideal \(J\) of \(R\) maximal with respect to the property that \(u \notin J\). We claim that this ideal \(J\) is a prime ideal. To see this, suppose \(a, b\) are elements of \(R\) such that \(a \notin J\) and \(b \notin J\). Since \(u\) belongs to \(J + RaR\) and \(J + RbR\), we have

\[
u = u^2 \in (J + RaR)(J + RbR) = J^2 + JbR + RaJ + RaRbR \subseteq J + R(ab)R.
\]

As \(u \notin J\), \(aRb \notin J\). Thus \(J\) is a prime ideal and so \(J \supseteq I \ni u\), a contradiction. Consequently, the idempotent \(u\) belongs to every non-zero ideal of \(R\). This shows that \([0]\) is completely irreducible in \(R = L_K(E \setminus (H, B_H))\) \(\cong L/P\). Hence \(P\) is completely irreducible in \(L\).

(b) \(\Rightarrow\) (a) Because, a completely irreducible prime ideal is, in particular, locally closed by definition.

Now (b) \(\Rightarrow\) (c) easily follows from the definition of complete irreducibility and (d) \(\Rightarrow\) (b) from Theorem 3.8 and Remark 3.4.

\(\Box\)

Remark 3.10. Since the prime radical of a Leavitt path algebra is zero, it follows from the proof of (a) \(\Rightarrow\) (b) in the above theorem, that if a graded ideal \(I\) of a Leavitt path algebra \(L\) is not the intersection of all the prime ideals properly containing \(I\), then \(I\) itself must be a prime ideal of \(L\).

The following example shows that, in general, a primitive ideal of \(L_K(E)\) need not be strongly primitive.

Example 3.11. Consider the following \("\mathbb{N} \times \mathbb{N}\)-Lattice” graph \(E\) where the vertices in \(E\) are points in the first quadrant of the coordinate plane whose coordinates are integers \(\geq 0\).

Specifically, \(E^0 = \{(m, n) : m, n \in \mathbb{Z} \text{ with } m, n \geq 0\}\). Every vertex \((m, n)\) emits two edges connecting \((m, n)\) with \((m + 1, n)\) and \((m, n + 1)\).

Now \(E^0\) is downward directed and \(E\) contains no cycles and hence trivially satisfies Condition (L). Since \(E^0\) is countable, \(E^0\) satisfies the CSP (for example, with respect to set \(\{(m, n) : m \geq 1, n \geq 1\}\)). Thus the ideal \([0]\) is a primitive ideal. But \([0]\) is not strongly primitive. To justify this, it is enough if we could find a set of hereditary saturated subsets of \(E^0\) whose intersection is the empty set. Indeed, for each \(n \geq 1\), the sets \(H_n = \{(i, j) : i \geq 0, j \geq n\}\) are hereditary saturated subsets of \(E^0\) whose intersection is the empty set.

In contrast to Example 3.11, we show that, if \(E\) is a finite graph (more generally, if \(E^0\) is finite), then the primitive ideals of \(L_K(E)\) are always strongly primitive. In the proof, we will use the following Definition and Lemma.

Definition 3.12. ([1, 8]) A cycle \(c\) in a graph \(E\) is said to be an extreme cycle if \(c\) has exits and, for every path \(x\) starting at a vertex in \(c^0\), there is a path \(\beta\) such that \(s(\beta) = r(x)\) and \(r(x\beta) \in c^0\).
Intuitively, every path that leaves the cycle \( c \) can be extended to a longer path that ends at a vertex on \( c \).

**Lemma 3.13.** (Lemma 3.7.10, [1], Theorem 2.9, [8]) Let \( E \) be a graph with \( E^0 \) is finite. Then every vertex in \( E \) connects to a sink or a no exit cycle or an extreme cycle.

**Proposition 3.14.** Let \( E \) be a finite graph. Then every primitive ideal \( P \) of \( L = L_K(E) \) is strongly primitive.

**Proof.** Let \( P \) be a primitive ideal of \( L \). In view of Remark 3.4, we may assume that \( P \) is a graded primitive ideal of the form \( P = I(H, B_H) \). In fact, it is enough if we assume that \( P = I(H, B_H) \) is a prime ideal such that \( E \) satisfies the strong CSP. If \( E \) satisfies the strong CSP. If \( E \) satisfies Condition (L). We need to show that \( (E \setminus (H, B_H))^0 \) satisfies the strong CSP with respect to \( \{w\} \). Suppose \( (E \setminus (H, B_H))^0 \) does not contain a sink. Since \( E \) satisfies Condition (L) and since \( E \) is finite, it follows from Lemma 3.13 that every vertex in \( (E \setminus (H, B_H))^0 \) connects to an extreme cycle. Observe that, by downward directness of the graph, any two extreme cycles \( c, d \) in \( E \setminus (H, B_H) \) are connected. Consequently, if \( c \) is an extreme cycle in \( E \setminus (H, B_H) \) based at a vertex \( v \), then \( u \) is a vertex \( u \) in \( (E \setminus (H, B_H))^0 \). It is then clear that \( (E \setminus (H, B_H))^0 \) satisfies the strong CSP with respect to \( \{v\} \). This shows that \( P \) is strongly primitive.

The next Proposition shows that a locally closed prime ideal is rational. In the proof, we use the following Lemma.

**Lemma 3.15.** (Proposition 5.1, [2]) Let \( K \) be a field and let \( A \) be a primitive \( K \)-algebra. If \( \dim_K(A) < |K| \), then the extended centroid \( C(A) \) is algebraic over \( K \).

**Proposition 3.16.** Let \( E \) be an arbitrary graph and let \( P \) be a prime ideal of \( L = L_K(E) \). If \( P \) is locally closed, then \( P \) is rational.

**Proof.** The proof is a minor modification of the one given in the proof of (1) implies (2) in Theorem 1.2 in [2]. Suppose that \( P \) is locally closed. By Theorem 3.9, \( P \) is strongly primitive, and in particular, primitive (by Theorem 3.1). Let \( K' \) be an uncountable purely transcendental field extension of \( K \) having transcendence degree \( > \max(|K|, |E|) \) where \( |E| = |E^0| \cup |E^1| \). Note that \( L_K(E) = L_K(E) \otimes_K K' \) and \( P' := P \otimes_K K' \) is a primitive ideal of \( L_K(E) \). Let \( R = L_K(E)/P \) and \( R' = L_K(E)/P' \). We now have an injective homomorphism \( \phi : C(R) \otimes_K K' \to C(R') \) defined as follows: If \( I \) is a nonzero ideal of \( R \) and \( f : I \to R \) is a bimodule homomorphism, then, if \( [(I,f)] \in C(R) \) is the equivalence class associated to \( f \), we define \( \phi([(I,f)]) \) to be the class \( [(I',f')] \) where \( f' = f \otimes id : I \otimes_K K' \to R \otimes_K K' = R' \). Since \( R' \) is primitive and \( \dim_K R' < |K'| \) (Note that \( \dim_K L_K(E) \leq |E|/n_0 \) for any field \( K \)), we conclude that \( C(R') \) is algebraic over \( K' \), by Lemma 3.15. Given \( [(I,f)] \in C(R) \), we see that the bimodule homomorphism \( [(I',f')] \) is algebraic over \( K' \). Thus there exists some non-zero ideal \( J' \subseteq I' \) and some natural number \( n \) such that \( \{ (f \otimes id)^j|_J, \cdots, (f \otimes id)|_J \} \) is linearly dependent over \( K' \). Thus \( \{ f^j|_J, \cdots, f|_J \} \) is linearly dependent over \( K \) where \( J = J' \cap R \). Hence \( [(I,f)] \) is algebraic over \( K \). Consequently, \( C(R) \) is algebraic over \( K \), thus showing \( P \) is a rational prime ideal.

But, in general, a rational prime ideal need not be locally closed as the following two examples show. Our first example is just the Example 3.11 that we considered earlier. The second example
is due to Jason Bell. I am grateful to him for providing such a nice example. In justifying the conclusions, we shall be using the following important Proposition from [5].

**Proposition 3.17.** (Corollary 2.3, [5]) Suppose $R$ is von Neumann regular prime algebra over a field $K$ with $\dim_K(R) < |K|$. Then the extended centroid $C(R)$ of $R$ is an algebraic extension of the field $K$ and thus $\{0\}$ is a rational prime ideal of $R$.

**Example 3.18.** Consider the graph $E$ of Example 3.11. Let $K$ be any uncountable field. Since $E$ is acyclic, $L_K(E)$ is a von Neumann regular algebra ([4]) and since $E^0$ is downward directed, $L_K(E)$ is also a prime algebra. Thus, $\{0\}$ is a prime ideal and, as $E$ is a countable graph, $\aleph_0 = \dim_K(L_K(E)) < |K|$. Then Proposition 3.17 implies that $\{0\}$ is a rational prime ideal of $L_K(E)$. But $\{0\}$ is not locally closed, since $\{0\} = \bigcap\{P_n : n > 0\}$, where, for each $n > 0$, $P_n$ is the ideal generated by the vertex $(n,0)$ and is further a prime ideal as $E^0 \setminus (P_n \cap E^0)$ is downward directed.

**Example 3.19.** Let $E$ be a graph with its vertex set $E^0 = \{(i,j) : i,j \in \Q, \text{ the set of rational numbers}\}$. Given $(i,j),(r,s) \in E^0$, there is an edge $(i,j) \to (r,s)$ whenever $i > r$ or $i = r$ and $j > s$. It is then clear that every vertex in $E$ is an infinite emitter. Let $L := L_K(E)$ where $K$ is an uncountable infinite field. For each rational number $i$, let $H_i = \{(x,y) \in E^0 : x \leq i\}$. It is clear that the set $H_i$ is hereditary. It is also vacuously saturated, as every vertex in $E$ is an infinite emitter. Also, for each $i \in \Q$, $E^0 \setminus H_i$ is downward directed. To see this, suppose $(a,b),(c,d) \in E^0 \setminus H_i$. Clearly, $a > i$ and $c > i$. Without loss of generality, assume $a \leq c$. Then for some rational number $r < b,d$ we have a vertex $(a,r) \in E^0 \setminus H_i$ and edges $(a,b) \to (a,r)$ and $(c,d) \to (a,r)$. This shows that $E^0 \setminus H_i$ is downward directed. Consequently, for each $i \in \Q$, the ideal $P_i = \langle H_i \rangle$ is a graded prime ideal of $L$. Moreover, it is easy to see that $P_i = \bigcap_{i < j \in \Q} P_j$. Thus the prime ideal $P_i$ is not locally closed. We claim $P_i$ is a rational prime ideal. To see this, first note that the graph $E$ is acyclic and so $L$ is von Neumann regular ([4]). Thus $L/P_i$ is a prime von Neumann $K$-algebra and is countable dimensional since the graph $E$ has countable number of vertices and countable number of edges. Since the field $K$ has uncountable cardinality, we appeal to Proposition 3.17 to conclude that $P_i$ is a rational prime ideal.

As a corollary to the preceding analysis, we are able to append additional equivalent properties to the DM-equivalence of a prime ideal in a Leavitt path algebra $L$ of a finite graph, thus extending Theorem 1.2 of [2] and thus leading to new characterizations of primitive ideals of $L$.

**Corollary 3.20.** Let $E$ be a finite graph and $L = L_K(E)$. Then the following properties are equivalent for any prime ideal $P$ of $L$:

(i) $P$ is primitive;
(ii) $P$ is locally closed;
(iii) $P$ is rational;
(iv) $P$ is strongly primitive;
(v) $P$ is completely irreducible;
(vi) There is a non-zero element $a \in L$ such that $P$ is maximal with respect to the property that $a \not\in P$.

**Acknowledgments**

I am thankful to Jason Bell for many helpful suggestions and, in particular, for providing the nice Example 3.19. I also thank Zak Mesyan for suggesting to include Corollary 3.20. I also thank the Referee for a very careful and meticulous reading of this paper.
References

[1] Abrams, G., Ara, P., Siles Molina, M. (2017). *Leavitt Path Algebras*, Lecture Notes in Mathematics, Vol. 2191, London: Springer-Verlag.

[2] Abrams, G., Bell, J., Rangaswamy, K. M. (2012). The Dixmier-Moeglin equivalence for Leavitt path algebras. *Algebr Represent Theor.* 15(3):407–425. DOI: 10.1007/s10468-010-9245-3.

[3] Abrams, G., Bell, J., Rangaswamy, K. M. (2014). On prime non-primitive von Neumann algebras. *Trans. Amer. Math. Soc.* 366(5):2375–2392. DOI: 10.1090/S0002-9947-2014-05878-9.

[4] Abrams, G., Rangaswamy, K. M. (2010). Regularity conditions for arbitrary Leavitt path algebras. *Algebr Represent Theor.* 13(3):319–334. DOI: 10.1007/s10468-008-9125-2.

[5] Ara, P., Bell, J. (2015). Primitivity of Prime countable dimensional regular algebras. *Proc. Amer. Math. Soc.* 143(7):2759–2766. DOI: 10.1090/S0002-9939-2015-12434-3.

[6] Beidar, K. I., Martindale, W. S., III., Mikhalev, A. V. (1996). *Rings with generalized identities, Monographs and Textbooks in Pure and Applied Mathematics*, Vol. 1961. New York: Marcel-Dekker.

[7] Bell, J., Rogalski, D., Sierra, S. (2010). The Dixmier-Moeglin equivalence for twisted homogeneous coordinate rings. *Isr. J. Math.* 180(1):461–507. DOI: 10.1007/s11856-010-0111-0.

[8] Corrales Garcia, M. G., Martin Barquero, D., Martin González, C., Siles Molina, M. (2016). Extreme cycles. The center of a Leavitt path algebra. *Publ. Mat.* 60:235–263.

[9] Dixmier, J. (1996). *Enveloping Algebras*, The 1996 printing of the 1977 English translation, Graduate Studies in Math. Vol. 11, *Amer. Math. Soc.* Providence: American Mathematical Society.

[10] Fuchs, L., Heinzer, W. J., Olberding, B. (2006). Commutative ideal theory without finiteness conditions: Completely irreducible ideals. *Trans. Amer. Math. Soc.* 358(7):3113–3131. DOI: 10.1090/S0002-9947-06-03815-3.

[11] Goodearl, K., Letzter, E. (1999). The Dixmier-Moeglin equivalence in quantum coordinate rings and quantized Weyl algebras. *Trans. Amer. Math. Soc.* 352(3):1381–1403. DOI: 10.1090/S0002-9947-99-02345-4.

[12] Mesyan, Z., Rangaswamy, K. M. (2022). Products and intersections of prime-power ideals in Leavitt path algebras. *J. Algebra Appl.* 22(1):1–26.

[13] Moeglin, C. (1980). Idéaux primitifs des algèbres enveloppantes. *J. Math. Pures Appl.* 59:265–336.

[14] Rangaswamy, K. M. (2013). The theory of prime ideals of Leavitt path algebras over arbitrary graphs. *J. Algebra* 375:73–96. DOI: 10.1016/j.jalgebra.2012.11.004.

[15] Rangaswamy, K. M. (2014). Leavitt path algebras which are Zorn rings. *Contemp. Math.* 609:277–283.

[16] Tomforde, M. (2007). Uniqueness theorems and ideal structure of Leavitt path algebras. *J. Algebra* 318(1):270–299. DOI: 10.1016/j.jalgebra.2007.01.031.