INTERACTING QUARK MATTER EQUATION OF STATE FOR COMPACT STARS

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\textbf{ABSTRACT}

Lattice quantum chromodynamics (QCD) studies of the thermodynamics of hot quark–gluon plasma demonstrate the importance of accounting for the interactions of quarks and gluons if one wants to investigate the phase structure of strongly interacting matter. Motivated by this observation and using state-of-the-art results from perturbative QCD, we construct a simple, effective equation of state (EOS) for cold quark matter that consistently incorporates the effects of interactions and furthermore includes a built-in estimate of the inherent systematic uncertainties. This goes beyond the MIT bag model description in a crucial way, yet leads to an EOS that is equally straightforward to use. We also demonstrate that, at moderate densities, our EOS can be made to smoothly connect to hadronic EOSs, with the two exhibiting very similar behavior near the matching region. The resulting hybrid stars are seen to have masses similar to those predicted by the purely nucleonic EOSs.

\textit{Key words:} dense matter – equation of state – stars: neutron

\textbf{Online-only material:} color figures

1. INTRODUCTION

The structure of a stellar object is determined by competition between its internal pressure and gravity, with the former given by the equation of state (EOS) of the matter composing the star (Glendenning 1997). For compact stars, such as (hybrid) neutron or quark stars, what is required is the EOS of strongly interacting matter at high density and low temperature. Such information is, in principle, available from the underlying theory, quantum chromodynamics (QCD), which, however, has thus far not admitted a non-perturbative first-principles solution at non-zero quark (or baryon) density. The reason for this is the well-known sign problem of lattice QCD (de Forcrand 2010).

At low densities, the physics determining the stellar EOS is that of low-energy nuclear interactions, which can be treated successfully through relativistic or non-relativistic effective theories (see Lattimer 2012 and references therein). In the cores of stars, the density of the matter is, however, likely to significantly exceed that of nuclear saturation density. This may lead to the presence of matter better described by deconfined quark than hadronic degrees of freedom. In this region, a low-energy description of the thermodynamics clearly breaks down, leaving one with very limited tools for the determination of the EOS. In the absence of more realistic competitors, the compact star community has widely adopted the simplest alternative, the MIT bag model EOS (Farhi & Jaffe 1984), in which all effects from interactions are contained in one additive “bag” constant arising from the energy difference between the perturbative ground state and the physical, chiral symmetry breaking vacuum of the theory. This model was used in the very first works on quark stars (Itoh 1970; Witten 1984; Alcock et al. 1986; Haensel et al. 1986), and has since been employed in most investigations of hybrid and quark stars as well as core collapse supernovae (Berezhiani et al. 2003; Alford et al. 2005, 2007; Drago et al. 2007, 2013; Mintz et al. 2010; Fischer et al. 2011; Ozel et al. 2010; Weissenborn et al. 2011; Zdunik & Haensel 2012). For some exceptions, using different low-energy models or the quasiparticle approach, see, e.g., Dexheimer et al. (2013), Klahn et al. (2013), Orsaria et al. (2013), and Peshier et al. (2002).

In addition to its simplicity, the MIT bag model owes its popularity to its success in describing the vacuum properties of hadrons—despite only incorporating confinement in the crudest possible fashion. In the context of high temperatures, however, it has long been known that the bag model fails badly in the description of thermodynamics: it assumes that interactions can be altogether neglected above the phase transition region, a claim that has been convincingly disproven by lattice QCD simulations. Indeed, at any phenomenologically relevant temperature, the strong coupling constant $\alpha_s(T) \sim 1/\log(T/\Lambda_{QCD})$ is not vanishingly small, and perturbative corrections proportional to its powers (and logarithms) are numerically significant.

In the regime of high density and low temperature, common wisdom based on the MIT bag model EOS states that the masses of pure quark stars are unlikely to reach $2 M_\odot$, and hybrid stars containing quark matter cores are generically less massive than their purely nucleonic counterparts (Weissenborn et al. 2011). This conclusion was, however, challenged more than 10 yr ago in Fraga et al. (2001, 2002), where a study of the perturbative EOS of high-density quark matter was seen to lead to quark stars with masses above $2 M_\odot$.\textsuperscript{6} Somewhat later, the three-loop perturbative EOS used in these works was further extended by including the effects of the strange quark mass, leading to quark stars with masses in excess of $2.5 M_\odot$ (Kurkela et al. 2010a; see Fraga & Romatschke 2004 for a discussion of quark mass effects at two loops). At that time, these masses were considered unrealistically high, but since then the discoveries of the pulsars PSR J1614–2230, with $M = 1.97 \pm 0.04 M_\odot$ (Demorest et al. 2010), and PSR

\textsuperscript{6} A comprehensive phenomenological study using this parameterization of the EOS, including effects from color superconductivity and matching onto different low-density nuclear EOSs, was later performed by Alford et al. (2005).
J0348+0432, with $M = 2.01 \pm 0.04 M_\odot$ (Antoniadis et al. 2013), have dramatically changed the situation.

In this Letter, our aim is to package the state-of-the-art perturbative QCD EOS of Kurkela et al. (2010a) in a simple, easy-to-use form that can be used in modeling cold quark matter in compact stars. Although the diagrammatic computation behind this result is very involved, we have observed that it is possible to condense the EOS into a simple analytic fitting function that provides an accurate description of the pressure and its two first derivatives with respect to the baryon number chemical potential. As we will explicitly demonstrate in the following, this result is immediately amenable to the determination of the structure of quark stars, and can in addition be matched to various nuclear EOSs with considerable ease.

As our result constitutes a purely diagrammatic correction to the pressure of free quarks, it does not incorporate any fundamentally non-perturbative contributions to the EOS. While we believe that the breakdown of the perturbative series around nuclear matter saturation density signals the presence of non-perturbative physics, a calculation of this type cannot provide a quantitative description of these effects. Nevertheless, it is still possible to use our result to improve the perturbative part of the EOS in calculations including non-perturbative physics, such as a bag constant\(^7\) or the color superconducting gap.

While being as convenient to use as the MIT bag model EOS, the result we provide improves it in two important ways. First, as demonstrated in the case of high temperature and vanishing baryon density, Here, one is faced with the task of connecting a low-energy hadron resonance gas EOS to the high-temperature limit, where a perturbative description of the thermodynamics is expected to be feasible. At intermediate temperatures, one has the options of using resummed perturbation theory, the MIT bag model, or other, more elaborate model calculations to estimate the behavior of the EOS. The matching of these EOSs to the hadron resonance gas results can, in addition, be used to determine the approximate location of the crossover transition, where the degrees of freedom used to describe the system effectively change from hadronic to partonic ones.

Despite the obvious similarities of the two systems, there is one crucial difference between the description of QCD matter at high density and high temperature: for hot and dilute quark–gluon plasma (QGP), lattice QCD provides a reliable non-perturbative first-principles method to evaluate bulk thermodynamic quantities, and thus check the accuracy of the perturbative and model predictions. This is illustrated in Figure 1, where we plot the pressure of hot QGP at vanishing baryon number density, normalized to that of a free system. The figure exhibits a perturbative band obtained from the so-called “dimensional reduction resummation” (see Blaizot et al. 2003; Laine & Schroder 2006) of the highest complete weak-coupling result of order $\alpha_s^2$ (Zhai & Kastening 1995; Kajantie et al. 2002),\(^8\) as well as a curve corresponding to the prediction of the MIT bag model, in which we have used the typical value of $(150 \text{ MeV})^4$ for the bag constant. The two predictions are compared to state-of-the-art lattice data from Borsanyi et al. (2010, 2013), corresponding to two massless and one massive flavor of dynamical quarks.

What one sees in Figure 1 is quite striking: whereas the resummed perturbative EOS clearly provides a successful description of the zero density EOS (and even more so at small but non-zero baryon number chemical potential $\mu_B$; Mogliacci et al. 2013), the MIT bag model prediction fails badly in the same task. Inspecting the figure, one quickly sees that this can be attributed to the extremely sharp rise of the bag curve toward the non-interacting Stefan–Boltzmann limit, proportional to $1/T^4$. At the same time, the logarithmic dependence of the perturbative EOS on the temperature, originating from the running of the strong coupling constant, is in excellent agreement with the slow rising of the lattice data.

As noted above, at non-zero baryon density there is at the moment still no benchmark from lattice QCD to which one could compare perturbative and model results. At the same time, there is, however, also no physical reason to expect interactions to play any smaller a role in this regime, or that the bag model would provide a better description of the EOS. Therefore, if one wishes to allow for the presence of quark matter inside compact stars, it is crucial to use an EOS that correctly accounts for the

\(^7\) In the MIT bag model, confinement is modeled by adding a “bag constant” to the free pressure, corresponding to the free energy difference of the perturbative and non-perturbative chiral symmetry breaking vacua. It is not clear that this physical effect can be included in the EOS in the form of an additive constant, but it is also no less consistent to add such a term to our EOS than to the non-interacting one.

\(^8\) Note that at high temperatures the strong infrared $(T/p_T)$-tail of the gluonic Bose–Einstein distribution renders the EOS sensitive to the dispersion relation of soft gluons (with $p \sim \sqrt{s}/T$), creating a need for performing resummations. This complication is largely absent in the case of cold fermionic matter.
interactions. As we will argue in the following section, such a result is in fact already available, and is provided by state-of-the-art perturbative QCD.

3. THE HIGH-DENSITY EOS: BEYOND THE BAG

Within perturbative thermal field theory, the EOS of a given system is obtained by expanding the path integral representation of the partition function in terms of zero-point Feynman diagrams. The expansion is, however, somewhat complicated by the fact that diagrams with any number of loops can contribute at the same order in $\alpha_s$. This is seen explicitly in the fact that at order $\alpha_s^2$ the pressure of zero-temperature QCD obtains contributions from an infinite set of so-called plasmon or ring diagrams (Kraemmer & Rebhan 2004).

Having determined the weak coupling expansion to a given order in $\alpha_s$, we observe that the result has become a function of an unphysical auxiliary parameter, the renormalization scale $\Lambda$. As long as the perturbative expansion converges, this dependence is, however, guaranteed to decrease order by order, and thus the sensitivity of our result on the parameter can be interpreted as reflecting the systematic error introduced by the truncation of the series. This error is commonly estimated by choosing a physically reasonable fiducial scale and varying the renormalization scale around it by a factor of two; below, we too follow this procedure, choosing as the central scale the commonly used value $\Lambda = (2/3)\mu_B$.

For the pressure of QCD at zero-density, the weak coupling expansion has so far been determined to $O(\alpha_s^3)\ln \alpha_s$ at temperatures $T \gtrsim \mu/\sqrt{\pi}$ (Vuoerinen 2003; Ipp et al. 2006), to $O(\alpha_s^2)$ at $T = 0$ (Freedman & McLerran 1977; Baluni 1978; Blaizot et al. 2001; Fraga et al. 2001; Kurkela et al. 2010c) and to $O(\alpha_s^2 \ln \alpha_s)$ between these two limits (Toimela 1985; see also Andersen & Strickland 2002; Gerhold et al. 2004). The calculation relevant for compact star physics is clearly the $O(\alpha_s^2)$ zero-temperature work of Kurkela et al. (2010c), which most importantly also takes into account the non-zero value of the strange quark mass. It is exactly this EOS, applied to the special case of electrically neutral and $\beta$-stable quark matter, that we will analyze in this Letter. It is a function of the baryon chemical potential $\mu_B$ and parameterized by the strong coupling constant and strange quark mass, which are taken at arbitrary reference scales, $\alpha_s(1.5\text{ GeV}) = 0.326$ and $m_s(2\text{ GeV}) = 0.938\text{ GeV}$ (Bazavov et al. 2012; Aoki et al. 2013), and then evolved as functions of the MS scale $\Lambda$.

We find that the EOS and its first and second derivatives are to a very high accuracy described by the compact fitting function

$$P_{\text{QCD}}(\mu_B, X) = P_{\text{SB}}(\mu_B) \left( c_1 - \frac{a(X)}{(\mu_B/\text{GeV}) - b(X)} \right),$$

$$a(X) = d_1 X^{-\nu_1}, \quad b(X) = d_2 X^{-\nu_2},$$

where we have denoted the pressure of three massless non-interacting quark flavors (at $N_c = 3$) by

$$P_{\text{SB}}(\mu_B) = \frac{3}{4\pi^2} (\mu_B/3)^3.$$  

9 There is some freedom involved in the choice of the thermodynamical potential that one chooses to truncate at a given order in $\alpha_s$, while other functions are derived from it demanding thermodynamic consistency. Unlike in Kurkela et al. (2010a), we have for simplicity chosen to truncate here the pressure as a function of $\mu_B$.

The dependence of the result on the renormalization scale is contained in the functions $a(X)$ and $b(X)$, which depend on a dimensionless parameter proportional to the scale parameter, $X \equiv 3\Lambda/\mu_B$, that is allowed to vary from 1 to 4.

The values of the constants $\{ c_1, d_1, d_2, \nu_1, \nu_2 \}$ are fixed by minimizing the value of the following merit function:

$$\chi^2 = [\Delta P(\mu_B, X)]^2 + [\Delta N(\mu_B, X)]^2 + [\Delta c_s^2(\mu_B, X)]^2,$$

where $\Delta P$, $\Delta N$, and $\Delta c_s^2$ are the differences between the values of the pressure, quark number density, and speed of sound squared obtained from the fit and from the corresponding full perturbative expressions of Kurkela et al. (2010a), normalized to the corresponding Stefan–Boltzmann values. For our best fit values

$$c_1 = 0.9008, \quad d_1 = 0.5034, \quad d_2 = 1.452,$$

$$\nu_1 = 0.3553, \quad \nu_2 = 0.9101,$$

we obtain a good fit ($\chi^2/\nu < 0.03$) in the region defined by the conditions $\mu_B < 2\text{ GeV}, P(\mu_B) > 0$, and $X \in [1, 4]$, with the resulting EOS displayed in Figure 2. We have checked that all relevant observables depending on the pressure and its first and second derivatives (such as the energy density as a function of pressure) are faithfully described by the fit.

To investigate the dependence of our result on the strange quark mass, we have also performed fits at unphysical quark masses, varying $m_s(2\text{ GeV})$ between 0 and 140 MeV. The authors of Alford et al. (2005) argue that the inclusion of a non-zero strange quark mass should generate a term in the EOS resulting EOS displayed in Figure 2. We have checked that all relevant observables depending on the pressure and its first and second derivatives (such as the energy density as a function of pressure) are faithfully described by the fit.

Figure 2. Same as in Figure 1, but for the pressure of zero-temperature quark matter in $\beta$ equilibrium as a function of the baryon chemical potential. (A color version of this figure is available in the online journal.)

we then the simple relation $P(m_1, \mu_B)/P_{\text{SB}}(\mu_B) = P(m_2, \mu_B)/P_{\text{SB}}(\mu_B)$ is seen to hold within the accuracy of the fitting function in the entire range of parameters considered

$$2(X m_2/\text{GeV}) \mu_B' = (2 - X m_1/\text{GeV}) \mu_B,$$

then the simple relation $P(m_1, \mu_B)/P_{\text{SB}}(\mu_B) = P(m_2, \mu_B)/P_{\text{SB}}(\mu_B)$ is seen to hold within the accuracy of the fitting function in the entire range of parameters considered
two points in Figure 3. The red and blue dots on the hybrid curve represent the cases where the maximal density inside the star reaches the corresponding value in Figure 3, respectively. The red and blue dots on the hybrid curve represent the cases where the maximal density inside the star reaches the corresponding value in Figure 3, respectively. The red dot.

(A color version of this figure is available in the online journal.)

Figure 3. Pressure resulting from the matching of our EOS with that of Akmal et al. (1998). Note that the behaviors of the APR and quark matter EOSs are strikingly similar in the region near the matching point, marked here by the red dot.

Figure 4. Mass–radius diagram for pure quark and hybrid quark–neutron stars generated using the equation of state of Equation (1) as well as that displayed in Figure 3, respectively. The red and blue dots on the hybrid curve represent the cases where the maximal density inside the star reaches the corresponding two points in Figure 3.

(A color version of this figure is available in the online journal.)

above. Here, the $m_i$ again stand for the strange quark masses measured at the scale 2 GeV.

Finally, it should be noted that during the past 15 yr there have been several attempts to describe various perturbative EOSs with fitting functions of increasing sophistication (see, e.g., Fraga et al. 2001; Alford et al. 2005). For the sake of comparison, we have also analyzed these ansätze, and have found that our fitting function consistently provides $\chi^2$-values at least 40 times smaller than any of its competitors.

To demonstrate the uses of our effective EOS, we next determine the mass–radius relations of pure quark stars as well as a hybrid star obtained by joining our EOS to the standard nucleonic EOS of Akmal, Pandharipande, and Ravenhall (APR: Akmal et al. 1998) via the Maxwell construction (see Figure 3 for the resulting pressure). The latter calculation is performed in the extremal case, where the latent heat vanishes and the pressure is continuous at the transition, and is meant only as an illustration of the fact that such matchings are highly straightforward to implement. The resulting mass–radius curves, obtained by solving the Tolman–Oppenheimer–Volkov equations (Glendenning 1997), are displayed in Figure 4. From here, we see that the maximal masses of the quark stars depend strongly on the value of the renormalization scale, and that it is not difficult to reach masses in excess of 2 $M_\odot$.

4. CONCLUSIONS

In spite of its rudimentary nature, the MIT bag model has for decades been the framework of choice in studies of quark matter in compact stars. Primary questions, such as the possible existence of deconfined matter in the cores of neutron stars, have thus essentially been addressed neglecting all interactions between the constituents of the system. In light of recent important advances in perturbative QCD at high density, it is clear that the state-of-the-art weak coupling EOSs for cold quark matter should become the new standard in these investigations.

For the above reasons, this Letter has been aimed at providing a simple parameterized EOS for cold quark matter, which significantly improves the picture provided by the bag model (and other low-energy effective models). Most importantly, it is based on a first-principles calculation within the fundamental microscopic theory (Kurkela et al. 2010a), and automatically contains a realistic estimate for the magnitude of its inherent theoretical uncertainties. At the same time, it is, however, as simple to use as the MIT bag model EOS, having been condensed to a single-line analytic formula in Equation (1). To demonstrate the impact this result has on compact star physics, we determined the mass–radius relation of pure quark stars based on the new EOS. In glaring contrast to studies carried out with the bag model EOS, this showed that it is in fact easy to construct pure quark stars with masses in excess of 2 $M_\odot$, consistent with current observational bounds.

Finally, one should note that in this work we only briefly experimented with the matching of our EOS to its low-density hadronic counterparts. A naive matching to the APR result demonstrated a striking similarity between the two EOSs near the transition point, and led to stars almost as massive as purely nucleonic ones. It is certainly possible to carry out this exercise in a much more systematic way, which we in fact aim to do in a separate publication (see also Kurkela et al. 2010b). There, one may additionally consider including in the EOS various non-perturbative contributions originating, e.g., from quark pairing or non-trivial vacuum properties (see our discussion of the bag constant in footnote 7). After all, our current result is simply the perturbative EOS of unpaired quark matter, aimed at replacing the free quark matter part in the MIT bag model EOS.

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