Twisted Morita–Mumford classes on braid groups

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Evaluating the twisted Morita–Mumford classes $\bar{h}_p$ (Kawazumi [12]) on the Artin braid group $B_n$, we give the stable algebraic independence of the $\bar{h}_p$’s on the automorphism group of the free group, $\text{Aut}(F_n)$. This is sharper than the results obtained by restricting them to the mapping class group (Kawazumi [9]).

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Introduction

In the cohomological study of the mapping class group for a surface, the Morita–Mumford classes, $e_i = (-1)^{i+1} \kappa_i$, $i \geq 1$, [19, 17] play some important roles. As was proved by Miller [16] and Morita [17] independently, they are algebraically independent in the stable range $\ast < \frac{3g}{2}$. Madsen and Weiss [15] proved that the rational stable cohomology algebra of the mapping class groups, $H^{\ast}(M_{\infty}; \mathbb{Q})$, is generated by the Morita–Mumford classes. The Morita–Mumford classes have twisted variants, $m_{i,j} \in H^{2i+j-2}(M_{g,1}; \bigwedge^\ast H)$, $i,j \geq 0$, introduced by the author [11]. Here we denote by $\Sigma_{g,1}$ a 2–dimensional oriented compact connected $C^\infty$ manifold of genus $g$ with 1 boundary component, $M_{g,1}$ its mapping class group, $M_{g,1} := \pi_0 \text{Diff}(\Sigma_{g,1}, \text{id on } \partial \Sigma_{g,1})$, and $H$ the integral first homology group of the surface $\Sigma_{g,1}$. The mapping class group $M_{g,1}$ acts on $H$ in an obvious way. The twisted variants also satisfy the algebraic independence. More precisely, the algebra $H^{\ast}(M_{g,1}; \bigwedge^\ast H) \otimes \mathbb{Q}$ is the polynomial algebra in the set $\{m_{ij}; i \geq 0, j \geq 1, \text{ and } i+j \geq 2\}$ over the algebra $H^{\ast}(M_{g,1}; \mathbb{Q})$ in the range where the total degree $\leq \frac{3g}{2}$ (Kawazumi [9, Theorem 1.C]). Hence, from the theorem of Madsen and Weiss [15] stated above, the algebra $H^{\ast}(M_{g,1}; \bigwedge^\ast H) \otimes \mathbb{Q}$ is stably isomorphic to the polynomial algebra in the set $\{m_{ij}; i \geq 0, j \geq 0, \text{ and } i+j \geq 2\}$ over $\mathbb{Q}$. Similar results hold for any other symplectic coefficients (Kawazumi [9, Theorem 1.B]). Furthermore all the cohomology classes on the mapping class group obtained by contracting the coefficients of the twisted ones using the intersection pairing $H^{\ast} \otimes \mathbb{Z}$ are exactly the algebra generated by the (original) Morita–Mumford classes $e_i$’s (Morita [18], Kawazumi and Morita [13]).

Some of the twisted ones have the advantage over the original ones of being defined on the automorphism group of a free group, which has the mapping class group and the
braid group as proper subgroups. Let \( n \geq 2 \) be an integer, \( F_n \) a free group of rank \( n \) with free basis \( x_1, x_2, \ldots, x_n \)

\[
F_n = \langle x_1, x_2, \ldots, x_n \rangle,
\]

and \( \text{Aut}(F_n) \) the automorphism group of the group \( F_n \). The Dehn–Nielsen theorem tells us the natural action of the group \( M_{g,1} \) on the free group \( \pi_1(\Sigma_{g,1}) \) of rank \( 2g \) induces an injective homomorphism \( \mathcal{M}_{g,1} \rightarrow \text{Aut}(F_{2g}) \). In view of a theorem of Artin [2] the braid group \( B_n \) of \( n \) strings is embedded into the group \( \text{Aut}(F_n) \).

Now we denote by \( H \) and \( H^* \) the first integral homology and cohomology groups of the group \( F_n \)

\[
H := H_1(F_n; \mathbb{Z}) = F_n/\text{abel} = F_n/[F_n,F_n] \quad \text{and} \quad H^* := H^1(F_n; \mathbb{Z}) = \text{Hom}(H, \mathbb{Z}),
\]

respectively, on which the automorphism group \( \text{Aut}(F_n) \) acts in an obvious way. We write \([\gamma] := \gamma \mod [F_n,F_n] \in H \) for \( \gamma \in F_n \), and \( X_i := [x_i] \in H \) for \( i, 1 \leq i \leq n \). In [12] we introduced cohomology classes

\[
h_p \in H^p(\text{Aut}(F_n); H^* \otimes H^{\otimes (p+1)}) \quad \text{and} \quad \overline{h}_p \in H^p(\text{Aut}(F_n); H^{\otimes p})
\]

for \( p \geq 1 \). Restricted to the mapping class group \( \mathcal{M}_{g,1} \) they coincide with the twisted Morita–Mumford classes

\[
(p + 2)! h_p|_{\mathcal{M}_{g,1}} = m_{0,p+2} \in H^p(\mathcal{M}_{g,1}; H^{\otimes (p+2)}), \quad \text{and}

p! \overline{h}_p|_{\mathcal{M}_{g,1}} = -m_{1,p} \in H^p(\mathcal{M}_{g,1}; H^{\otimes p}).
\]

Here \( H \) and \( H^* \) are isomorphic to each other as \( \mathcal{M}_{g,1} \) modules because of the intersection pairing of the surface \( \Sigma_{g,1} \). The class \( p! \overline{h}_p \) can be regarded as an element in \( H^p(\text{Aut}(F_n); \wedge^p H) \).

In this note we confine ourselves to studying the behavior of \( \overline{h}_p \)’s restricted to the braid group \( B_n \), and consider the rational coefficients

\[
H_Q := H \otimes \mathbb{Z} Q \quad \text{and} \quad H_Q^* := H^* \otimes \mathbb{Z} Q.
\]

In this paper we prove the following result:

**Theorem 1** The cohomology classes \( \overline{h}_p \)’s are algebraically independent in the algebra \( H^*(B_n; \wedge^* H_Q) \) in the range where the total degree \( \leq n \).

Here the total degree of \( \overline{h}_p \) is defined to be \( 2p \). **Theorem 1** implies the algebraic independence on the automorphism group \( \text{Aut}(F_n) \). This is sharper than that obtained by restricting them to the mapping class group \( \mathcal{M}_{g,1} \) [9, Theorem 1.C], where the range is given by the inequality the total degree \( \leq \frac{2}{3}g = \frac{1}{3}n \).
Theorem 1 was announced in [10]. Its proof given in Section 3 is based on some kind of primitiveness of the $\tilde{h}_p$’s (Proposition 1.2) and the evaluation of $\tilde{h}_{n-1}$ on the pure braid group of $n$ strings, $P_n$ (Lemma 2.4). In Section 4 we will give some remarks on the cohomology of the automorphism group $\text{Aut}(F_n)$.

1 Twisted Morita–Mumford classes on the automorphism group $\text{Aut}(F_n)$

Throughout this paper we denote by $C^*(G; M)$ the normalized standard complex of a group $G$ with values in a $G$–module $M$, and use the Alexander–Whitney cup product $\cup: C^*(G; M_1) \otimes C^*(G; M_2) \to C^*(G; M_1 \otimes M_2)$. Moreover we denote by $Z_p^*(G; M)$, $p \geq 0$, the $p$–cocycles in the cochain complex $C^*(G; M)$.

Now we recall the definition of the twisted cohomology classes $h_p$ and $\tilde{h}_p$ on the automorphism group $\text{Aut}(F_n)$ for $p \geq 1$. The semi-direct product $A_n := F_n \rtimes \text{Aut}(F_n)$ admits an extension of groups

\begin{equation}
F_n \xrightarrow{i} \overline{A}_n \xrightarrow{\pi} \text{Aut}(F_n)
\end{equation}

given by $i(\gamma) = (\gamma, 1)$ and $\pi(\gamma, \varphi) = \varphi$ for $\gamma \in F_n$ and $\varphi \in \text{Aut}(F_n)$. The map $k_0 : \overline{A}_n \to H$, $(\gamma, \varphi) \mapsto [\gamma]$, satisfies the cocycle condition. We write also $k_0$ for the cohomology class $[k_0] \in H^1(\overline{A}_n; H)$. For each $p \geq 1$ we define $h_p$ by the image of the $(p+1)$-st power of the cohomology class $k_0$ under the Gysin map of the extension (1–1)

\begin{equation}
h_p := \pi^*(k_0^{\otimes (p+1)}) \in H^p(\text{Aut}(F_n); H^* \otimes H^{\otimes (p+1)})
\end{equation}

[12]. Contracting the coefficients by the $\text{GL}(H)$–homomorphism

\begin{equation}
r_p : H^* \otimes H^{\otimes (p+1)} \to H^{\otimes p}, \quad f \otimes v_0 \otimes v_1 \otimes \cdots \otimes v_p \mapsto f(v_0)v_1 \otimes \cdots \otimes v_p,
\end{equation}

we define

\begin{equation}
\overline{h}_p := r_p^*(h_p) \in H^p(\text{Aut}(F_n); H^{\otimes p}).
\end{equation}

The $p$-th exterior power $k_0^{p \otimes} = p!k_0^{\otimes p}$ can be regarded as a cohomology class with coefficients in $\bigwedge^p H$. Hence, if we consider the rational coefficients $H \mathbb{Q}$, we may regard $\overline{h}_p$ as a cohomology class in $H^p(\text{Aut}(F_n); \bigwedge^p H \mathbb{Q})$.

A Magnus expansion $\theta$ of the free group $F_n$ gives an explicit cocycle representing the class $h_p$. The completed tensor algebra generated by $H$, $\widetilde{T} = \widetilde{T}(H) := \prod_{m=0}^\infty \hat{H}^{\otimes m}$, has a decreasing filtration of two-sided ideals $\widetilde{T}_p := \prod_{m \geq p} \hat{H}^{\otimes m}$, $p \geq 1$. It should
be remarked that the subset $1 + \mathcal{T}_1$ is a subgroup of the multiplicative group of the algebra $\mathcal{T}$. We call a map $\theta : F_n \to 1 + \mathcal{T}_1$ a Magnus expansion of the free group $F_n$, if $\theta : F_n \to 1 + \mathcal{T}_1$ is a group homomorphism, and if $\theta(\gamma) \equiv 1 + [\gamma] \pmod{\mathcal{T}_2}$ for any $\gamma \in F_n$. We write $\theta(\gamma) = \sum_{m=0}^{\infty} \theta_m(\gamma), \theta_m(\gamma) \in H^{\otimes m}$. The $m$-th component $\theta_m : F_n \to H^{\otimes m}$ is a map, but not a group homomorphism. A Magnus expansion std: $F_n \to 1 + \mathcal{T}_1$ is defined by std$(x) := 1 + X_i, 1 \leq i \leq n$. Here we denote $X_i := [x_i] \in H$, the homology class of the generator $x_i$. We call it the standard Magnus expansion. As is described in classical references, the value std$(\gamma)$ for any word $\gamma \in F_n$ is explicitly computed by means of Fox' free differentials. All the results of this paper can be derived from the expansion std.

We define a map $\tau_1^\theta : \text{Aut}(F_n) \to H^* \otimes H^{\otimes 2}$ by

$$(1–5) \quad \tau_1^\theta(\varphi)[\gamma] = \theta_2(\gamma) - |\varphi| \otimes 2 \theta_2(\varphi^{-1}(\gamma)) \in H^{\otimes 2}$$

for $\gamma \in F_n$ and $\varphi \in \text{Aut}(F_n)$. Here $|\varphi| \in \text{GL}(H)$ is the automorphism of $H = F_n^\text{abel}$ induced by $\varphi$. This map $\tau_1^\theta$ satisfies the cocycle condition [12, Lemma 2.1]. Now we introduce a $\text{GL}(H)$–homomorphism

$$\varsigma_p : (H^* \otimes H^{\otimes 2})^\otimes p = \text{Hom}(H, H^{\otimes 2})^\otimes p \to \text{Hom}(H, H^{\otimes (p+1)}) = H^* \otimes H^{\otimes (p+1)}$$

for each $p \geq 1$. If $p \geq 2$, we define

$$(1–6) \quad \varsigma_p(u_1 \otimes u_2 \otimes \cdots \otimes u_{(p-1)} \otimes u_{(p)}) := (u_1 \otimes 1_H^{\otimes (p-1)}) \circ (u_2 \otimes 1_H^{\otimes (p-2)}) \circ \cdots \circ (u_{(p-1)} \otimes 1_H) \circ u_{(p)},$$

where $u_{(i)} \in \text{Hom}(H, H^{\otimes 2}) = H^* \otimes H^{\otimes 2}, 1 \leq i \leq p$. In the case $p = 1$, we define $\varsigma_1 := 1_{H^* \otimes H^{\otimes 2}}$. Then we have:

**Theorem 1.1** [12, Theorem 4.1]

$$h_p = \varsigma_p(\tau_1^\theta)^\otimes p \in H^0(\text{Aut}(F_n); H^* \otimes H^{\otimes (p+1)})$$

for any Magnus expansion $\theta$ and each $p \geq 1$. In the case $p = 1$ we have $[\tau_1^\theta] = h_1 \in H^1(\text{Aut}(F_n); H^* \otimes H^{\otimes 2})$.

Some kind of primitiveness of the cohomology classes $h_p$ and $\bar{h}_p$, follows from the theorem. We write simply $A_n := \text{Aut}(F_n)$ for the remainder of the section. Suppose $n_1 + n_2 \leq n$. Let $A_n$ act on the words in the letters $x_{n_1+1}, x_{n_1+2}, \ldots, x_{n_1+n_2}$ in an obvious way. Then we have a natural homomorphism

$$\iota = \iota_{n_1,n_2} : A_{n_1} \times A_{n_2} \to A_n.$$

We denote by $\omega_1 : A_{n_1} \times A_{n_2} \to A_{n_1}$ and $\omega_2 : A_{n_1} \times A_{n_2} \to A_{n_2}$ the first and the second projections of the product $A_{n_1} \times A_{n_2}$, respectively, and by $H_{(n_1)}, H_{(n_2)}$ and

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we deduce (2). This completes the proof of the proposition.

The open subset

\[ Y_n := \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n; z_i \neq z_j \text{ for } i \neq j\} \]

Proof of Proposition 1.2

1. \( t^*h_p = \varphi_1^* h_p + \varphi_2^* h_p \in H^p(A_{n_1} \times A_{n_2}; H^* \otimes H^\otimes(p+1)), \)
2. \( t^* h_p = \varphi_1^* h_p + \varphi_2^* h_p \in H^p(A_{n_1} \times A_{n_2}; H^* \otimes H^\otimes(p+1)). \)

Proof Using the standard expansion \( \text{std} \), we write simply

\[ \tau^{(k)} := \varphi_1^{* \text{std}} \in \mathbb{Z}^1(A_{n_1} \times A_{n_2}; H^* \otimes H^\otimes2). \]

Clearly we have \( \text{std}(\gamma_1) \in \prod_{p=0}^\infty H_{(n_1)}^\otimes p \subset \hat{T} \) for any word \( \gamma_1 \) in the letters \( x_1, \ldots, x_{n_1} \). Similar conditions hold for any word \( \gamma_2 \) in the letters \( x_{n_1+1}, \ldots, x_{n_1+n_2} \) and any \( \gamma_3 \) in \( x_{n_1+n_2+1}, \ldots, x_n \). Hence, from the definition of \( \tau_{\gamma_1}^\otimes (1–5) \), we have

\[ t^* \tau_{\gamma_1}^{\text{std}} = \tau^{(1)} + \tau^{(2)} \in \mathbb{Z}^1(A_{n_1} \times A_{n_2}; H^* \otimes H^\otimes2). \]

If we use the GL(H)–homomorphism \( \varsigma_2 : (H^* \otimes H^\otimes2)^\otimes2 \to H^* \otimes H^\otimes3 \) in (1–6), then we have

\[ \varsigma_2^* (\tau^{(1)} \tau^{(2)}) = \varsigma_2^* (\tau^{(2)} \tau^{(1)}) = 0 \in \mathbb{Z}^2(A_{n_1} \times A_{n_2}; H^* \otimes H^\otimes3). \]

In fact, \( f(u) = 0 \) for any \( f \in H_{(n_1)}^* \) and \( u \in H_{(n_2)}^* \) and vice versa. From Theorem 1.1 follows

\[ t^* h_p = \varsigma_p^* (t^* \tau_{\gamma_1}^{\text{std}} \otimes p) = \varsigma_p^* ((\tau^{(1)} + \tau^{(2)}) \otimes p) \]

\[ = \varsigma_p^* (\tau^{(1)} \otimes p) + \varsigma_p^* (\tau^{(2)} \otimes p) = \varphi_1^* h_p + \varphi_2^* h_p. \]

Here \( \varsigma_p^* \) of each mixed term in \( \tau^{(1)} \) and \( \tau^{(2)} \) vanishes by (1–7). Applying \( r_{p*} \) to (1), we deduce (2). This completes the proof of the proposition.

2 Evaluation on the Artin braid groups

The \( n \)-th symmetric group \( \mathfrak{S}_n \) acts on the space \( \mathbb{C}^n \) by permuting the components. The open subset

\[ Y_n := \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n; z_i \neq z_j \text{ for } i \neq j\} \]
is stable under the action of the group $\mathfrak{S}_n$. By definition, the Artin braid group of $n$ strings, $B_n$, is the fundamental group of the quotient space $Y_n/\mathfrak{S}_n$, $B_n := \pi_1(Y_n/\mathfrak{S}_n)$. As was shown by Artin [2], the group $B_n$ admits a presentation

\begin{align*}
\text{generators:} & \quad \sigma_i, \quad 1 \leq i \leq n - 1, \\
\text{relations:} & \quad \sigma_i \sigma_j = \sigma_j \sigma_i, \quad \text{if } |i - j| \geq 2, \\
& \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \text{for } 1 \leq i \leq n - 2.
\end{align*}

(2–1)

The pure braid group of $n$ strings, $P_n$, is defined to be the fundamental group of the space $Y_n$, $P_n := \pi_1(Y_n)$. We have a natural extension of groups

$$P_n \to B_n \to \mathfrak{S}_n.$$ 

As is known, $A_{i,j}$, $1 \leq i < j \leq n$, given by

$$A_{i,j} := \sigma_{j-1} \sigma_{j-2} \cdots \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1}^{-1} \cdots \sigma_{j-2}^{-1} \sigma_{j-1}^{-1}$$

can serve as a generating system of the group $P_n$. For details, see Birman [3].

The braid group $B_n$ admits a natural homomorphism into the group $\text{Aut}(F_n)$, $\xi : B_n \to \text{Aut}(F_n)$. To recall how to construct it, we consider an action of the group $\mathfrak{S}_n$ on the space $Y_{n+1} \subset \mathbb{C}^{n+1} = \mathbb{C}^n \times \mathbb{C}$ given by

$$\rho(z_1, \ldots, z_n, z_{n+1}) = (z_{\rho^{-1}(1)}, \ldots, z_{\rho^{-1}(n)}, z_{n+1})$$

for $\rho \in \mathfrak{S}_n$. We denote by $\hat{B}_n$ the fundamental group of the quotient space $Y_{n+1}/\mathfrak{S}_n$, $\hat{B}_n := \pi_1(Y_{n+1}/\mathfrak{S}_n)$.

The forgetful map $Y_{n+1} \to Y_n$, $(z_1, \ldots, z_n, z_{n+1}) \mapsto (z_1, \ldots, z_n)$, induces a fibration

$$\mathbb{C} \setminus \{n \text{ points}\} \to Y_{n+1}/\mathfrak{S}_n \to Y_n/\mathfrak{S}_n$$

with a section $s : Y_n/\mathfrak{S}_n \to Y_{n+1}/\mathfrak{S}_n$ given by $(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_n, \frac{1}{n} \sum_{i=1}^n z_i + \frac{1}{n} \sum_{j=1}^n |z_j - \frac{1}{n} \sum_{i=1}^n z_i|)$ (Arnol’d [1]). This fibration with the section $s$ induces an extension of groups

(2–2)

$$F_n \xrightarrow{s} \hat{B}_n \xrightarrow{\pi} B_n$$

with a split homomorphism $s : B_n \to \hat{B}_n$. Thus we obtain a morphism of extensions of groups

(2–3)

$$\begin{array}{ccc}
F_n & \xrightarrow{\xi} & \hat{B}_n & \xrightarrow{\pi} & B_n \\
\| & & \| & & \| \\
F_n & \xrightarrow{\xi} & \hat{B}_n & \xrightarrow{\pi} & \text{Aut}(F_n).
\end{array}$$

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We now evaluate the cohomology classes \(P\) and \(\hat{\xi}\) are explicitly given by
\[
\iota(\xi(x)(\gamma)) = s(x)\gamma s(x)^{-1} \\
\hat{\iota}(\iota(x)s(x)) = (\gamma, \xi(x)) \in F_n \times \text{Aut}(F_n) = \overline{A_n}
\]
for \(x \in B_n\) and \(\gamma \in F_n\). The group \(\hat{B}_n\) is embedded into \(B_{n+1}\) in an obvious way. Then the homomorphisms \(s\) and \(\iota\) are described as
\[
(2-4) \quad s(\sigma_i) = \sigma_i \quad \text{for} \quad 1 \leq i \leq n - 1, \\
\iota(x) = \sigma_n \sigma_n^{-1} \cdots \sigma_{j+1}^2 \sigma_{j+1}^{-1} \cdots \sigma_{n-1}^{-1} \sigma_{n}^{-1} \\
= A_{j,n+1} \quad \text{for} \quad 1 \leq j \leq n
\]
in terms of the presentation (2-1). So the homomorphism \(\xi\) is explicitly given by
\[
(2-5) \quad \xi(\sigma_i)(x_j) = \begin{cases} 
  x_{i+1}, & \text{if} \ j = i, \\
  x_{i+1}^{-1}x_{i+1}, & \text{if} \ j = i + 1, \\
  x_j, & \text{otherwise.}
\end{cases}
\]

We now evaluate the cohomology classes \(h_1\) and \(\overline{\tau}_{n-1}\) on the braid group \(B_n\). Here we use the standard Magnus expansion \(\text{std} : F_n \rightarrow 1 + T_1\) introduced in Section 1. For the rest of this section we write simply \(k_0\), \(\tau_1\), \(h_p\) and \(\overline{\tau}_p\) for \(\hat{\xi}^* k_0\), \(\xi^* \tau_1\), \(\xi^* h_p\) and \(\xi^* \overline{\tau}_p\), respectively. Let \(\{l_i\}_{i=1}^{n} \subset H^*\) denote the dual basis of \(\{x_i\}_{i=1}^{n} = \{ [x_i] \}_{i=1}^{n} \subset H\).

**Lemma 2.1**
\[
\tau_1(\sigma_i) = l_i \otimes (X_i \otimes X_{i+1} - X_{i+1} \otimes X_i) \in H^* \otimes H^{\otimes 2}
\]

**Proof** From (1-5)
\[
\tau_1(\sigma_i) = \sum_{j=1}^{n} l_j \otimes (\text{std}_2(x_j) - | \sigma_i |^{\otimes 2} \text{std}_2(\sigma_i^{-1}(x_j))) \\
= -l_i \otimes | \sigma_i |^{\otimes 2} \text{std}_2(\sigma_i^{-1}(x_i)) - l_{i+1} \otimes | \sigma_i |^{\otimes 2} \text{std}_2(\sigma_i^{-1}(x_{i+1})) \\
= -l_i \otimes | \sigma_i |^{\otimes 2} \text{std}_2(x_i x_{i+1} x_{i+1}^{-1}) - l_{i+1} \otimes | \sigma_i |^{\otimes 2} \text{std}_2(x_i) \\
= -l_i \otimes | \sigma_i |^{\otimes 2} \text{std}_2(x_i x_{i+1} x_{i+1}^{-1}).
\]

On the other hand, we have
\[
\text{std}_2(x_i x_{i+1} x_{i+1}^{-1}) = X_i \otimes X_{i+1} - X_{i+1} \otimes X_i.
\]

In fact, \(X_i \otimes X_{i+1} = \text{std}_2(x_i x_{i+1}) = \text{std}_2(x_i x_{i+1} x_{i+1}^{-1}) = \text{std}_2(x_i x_{i+1} x_{i+1}^{-1}) + \text{std}_2(x_i) + X_{i+1} \otimes X_i = \text{std}_2(x_i x_{i+1} x_{i+1}^{-1}) + X_{i+1} \otimes X_i\). Therefore we obtain \(\tau_1(\sigma_i) = -l_i \otimes | \sigma_i |^{\otimes 2}(X_i \otimes X_{i+1} - X_{i+1} \otimes X_i) = -l_i \otimes (X_{i+1} \otimes X_i - X_i \otimes X_{i+1})\), as was to be shown.

The pure braid group \(P_n\) acts on the homology \(H\) trivially. Hence, from [12, Theorem 3.1], the restriction of \(\tau_1\) to \(P_n\) does not depend on the choice of Magnus expansions.
Lemma 2.2
\[ \tau_1(A_{ij}) = (l_i - l_j) \otimes (X_i \otimes X_j - X_j \otimes X_i) \]

Proof Recall the map \( \tau_1 \) satisfies the cocycle condition on the automorphism group \( \text{Aut}(F_n) \). When we set \( \gamma := \sigma_{j-1} \sigma_{j-2} \cdots \sigma_{i+1} \), we have \( A_{ij} = \gamma \sigma_i^2 \gamma^{-1} \), so that
\[
\begin{align*}
\tau_1(A_{ij}) &= \tau_1(\gamma \sigma_i^2 \gamma^{-1}) = \tau_1(\gamma) + \gamma \tau_1(\sigma_i^2) + \gamma \sigma_i^2 \tau_1(\gamma^{-1}) \\
&= \tau_1(\gamma) + \gamma \tau_1(\sigma_i^2) + \gamma \tau_1(\gamma^{-1}) = \tau_1(1) + \gamma \tau_1(\sigma_i^2) = \gamma \tau_1(\sigma_i^2) \\
&= \gamma(l_i \otimes (X_i \otimes X_{i+1} - X_{i+1} \otimes X_i)) + \gamma \sigma_i(l_i \otimes (X_i \otimes X_{i+1} - X_{i+1} \otimes X_i)) \\
&= \gamma((l_i - l_{i+1}) \otimes (X_i \otimes X_{i+1} - X_{i+1} \otimes X_i)) \\
&= (l_i - l_j) \otimes (X_i \otimes X_j - X_j \otimes X_i),
\end{align*}
\]
as was to be shown. \( \square \)

To prove the nontriviality of \( T_{n-1} \) on the group \( B_n \), we recall some basic facts on the cohomology of the pure braid group \( P_n \). The space \( Y_n \) is an Eilenberg–MacLane space of type \( (P_n, 1) \). The subspace \( Y_n \cap \{ z_1 + \cdots + z_n = 0 \} \) is a deformation retract of the space \( Y_n \) and a Stein manifold of complex dimension \( n - 1 \). Hence the cohomological dimension of the group \( P_n \), \( cdP_n \), is not greater than \( n - 1 \). Let \( A^*(Y_n) \) be the algebra of all the complex-valued differential forms on the space \( Y_n \). As was shown by Arnol’d [1], the \( \mathbb{Z} \)-subalgebra generated by the 1–forms
\[ \omega_{i,j} := \frac{1}{2\pi i} \frac{dz_i - dz_j}{z_i - z_j}, \quad 1 \leq i < j \leq n, \]
is isomorphic to the cohomology algebra \( H^*(Y_n; \mathbb{Z}) = H^*(P_n; \mathbb{Z}) \). Especially in the case \( \ast = 1 \), \( \{ [\omega_{i,j}] \}_{1 \leq i < j \leq n} \) is a \( \mathbb{Z} \)-basis of \( H^1(P_n; \mathbb{Z}) \), so that \( \{ [A_{ij}] \}_{1 \leq i < j \leq n} \) is a \( \mathbb{Z} \)-free basis of \( H^1(P_n; \mathbb{Z}) = P_n \) abel.

Lemma 2.3
\[ k^0_n \neq 0 \in H^n(Y_{n+1}; \bigwedge^n H_Q), \text{ where } P_{n+1} = \pi_1(Y_{n+1}) \text{ is regarded as a subgroup of } B_n = \pi_1(Y_{n+1}/\mathbb{H}_n). \]
\[ h_{n-1} \neq 0 \in H^{n-1}(P_n; H^*_Q \otimes \bigwedge^n H_Q). \]

Proof (1) From (2–3) and (2–4) we have
\[ k_0(A_{ij}) = \begin{cases} 0, & \text{if } i < j \leq n, \\ X_i, & \text{if } i < j = n + 1. \end{cases} \]
that is
\[ k_0 = \sum_{i=1}^{n} \omega_{i,n+1} \otimes X_i \in H^1(Y_{n+1}; H). \]
If we restrict the \( n \)-form
\[ \omega_{1,n+1} \omega_{2,n+1} \cdots \omega_{n,n+1} = (1/2\pi \sqrt{-1})^n \prod_{i=1}^{n} (dz_i - dz_{n+1})/(z_i - z_{n+1}) \]
to the subspace \( Y_{n+1} \cap \{z_{n+1} = 0\} \), then we obtain the non-zero \( n \)-form
\[ (1/2\pi \sqrt{-1})^n \prod_{i=1}^{n} (dz_i - dz_{n+1})/(z_i - z_{n+1}) \]
does not vanish, as was to be shown.

(2) Since \( \text{cd} P_n \leq n - 1 \), the Gysin map of the extension
\[ F_n \rightarrow P_{n+1} \pi \rightarrow P_n \]
gives an isomorphism
\[ \pi_* : H^n(P_{n+1}; M) \cong H^{n-1}(P_n; H^* \otimes M) \]
for any \( P_n \)-module \( M \). Hence \( h_{n-1} = \pi_* k_0^n \neq 0 \) by (1).

The map \( r_n : HQ^* \otimes \bigwedge^n HQ \rightarrow \bigwedge^{n-1} HQ \) is an isomorphism because \( \dim_Q H_Q = n \).
Hence we obtain:

**Lemma 2.4**
\[ \overline{\alpha}_{n-1} \neq 0 \in H^{n-1}(P_n; \bigwedge^{n-1} HQ). \]

### 3 Proof of Theorem 1

Our proof of Theorem 1 is based on Proposition 1.2 and Lemma 2.4. For \( q \leq n \) we denote by \( P_{n-q}(q) \) the set of all the non-negative partitions \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-q} \geq 0) \) of \( q \) into \( n - q \) parts. For \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-q} \geq 0) \in P_{n-q}(q) \) we introduce a cohomology class \( \overline{\alpha}_\lambda \) and a subgroup \( P_\lambda \subset P_n \) by
\[ \overline{\alpha}_\lambda := \overline{\alpha}_{\lambda_1} \overline{\alpha}_{\lambda_2} \cdots \overline{\alpha}_{\lambda_{n-q}} \in H^q(B_n; \bigwedge^q HQ) \subset H^q(P_n; \bigwedge^q HQ), \quad \text{and} \]
\[ P_\lambda := P_{\lambda_1+1} \times P_{\lambda_2+1} \times \cdots \times P_{\lambda_{n-q}+1} \subset P_n, \]
respectively. Here \( P_{0+1} = P_1 \) is the trivial group \( \{1\} \). Denote by \( \iota_\lambda : P_\lambda \hookrightarrow P_n \) the obvious inclusion map and \( \varpi_\lambda : P_\lambda \twoheadrightarrow P_{\lambda+1} \) the obvious projection. Theorem 1 follows from:

\[ \text{Geometry & Topology Monographs 13 (2008)} \]
Theorem 3.1 The cohomology classes \( \{h_\lambda; \lambda \in \mathcal{P}_{n-q}(q)\} \) are linearly independent in \( H^q(P_n; \wedge^q H_Q) \).

In fact, when \( q \leq n/2 \), the set of all the non-negative partitions of \( q \) into \( n - q \) parts does not depend on \( n \).

Endow the partitions \( \mathcal{P}_{n-q}(q) \) with the lexicographic order. For example, \( (q \geq 0 \geq \cdots \geq 0) \) is the maximal partition. Theorem 3.1 is reduced to the following

**Assertions** For any \( \lambda \) and \( \mu \in \mathcal{P}_{n-q}(q) \) we have:

(A) \( \iota_\lambda^* h_\lambda \neq 0 \in H^q(P_\lambda; \wedge^q H_Q) \)

(B) If \( \mu \geq \lambda \), then \( \iota_\lambda^* h_\mu = 0 \in H^q(P_\lambda; \wedge^q H_Q) \).

In fact, assume we have a nontrivial linear relation

\[
\sum_{\lambda \in \mathcal{P}_{n-q}(q)} c_\lambda h_\lambda = 0 \in H^q(P_n; \wedge^q H_Q).
\]

Choose the minimum \( \lambda \) satisfying \( c_\lambda \neq 0 \). Applying \( \iota_\lambda^* \) to the relation, we obtain \( c_{\lambda \land \lambda}^* h_\lambda = 0 \) from Assertion B. Assertion A implies \( c_\lambda = 0 \), which contradicts the choice of \( \lambda \).

**Proof of Assertion A** Let \( b_1 \geq b_2 \geq \cdots \geq b_{\lambda_1} > b_{\lambda_1+1} = 0 \) be the dual partition of \( \lambda \). The number of \( \lambda_k \)'s equal to \( p \) is \( b_p - b_{p+1} \). We abbreviate \( \overline{h}_{p,k} := \omega_k^* h_p \). Since \( cd P_{\lambda_k+1} \leq \lambda_k \), we have \( \overline{h}_{p,k} = 0 \) if \( p > \lambda_k \), or equivalently, \( k > b_p \). Moreover we have \( \overline{h}_{\lambda_k,k} \overline{h}_{p,k} = 0 \) for any \( p \geq 1 \) since \( H^{\lambda_k+p}(P_{\lambda_k+1}; \wedge^{\lambda_k+p} H_Q) = 0 \). From Proposition 1.2 we have

\[
\iota_\lambda^* h_p = \sum_{k=1}^{n-q} \overline{h}_{p,k} \in H^p(P_\lambda; \wedge^p H),
\]
so that
\[ t_h^* \mathcal{H}_\lambda = \prod_{k=1}^{n-q} t_h^* \mathcal{H}_{\lambda_k} = \prod_{p=1}^{\lambda_1} (t_h^* \mathcal{H}_p)^{b_p-b_{p+1}} \]
\[ = \prod_{p=1}^{\lambda_1} (\mathcal{H}_{p,1} + \mathcal{H}_{p,2} + \cdots + \mathcal{H}_{p,n-q})^{b_p-b_{p+1}} \]
\[ = \prod_{p=1}^{\lambda_1} (\mathcal{H}_{p,1} + \mathcal{H}_{p,2} + \cdots + \mathcal{H}_{p,n})^{b_p-b_{p+1}} = \prod_{p=1}^{\lambda_1} (\mathcal{H}_{p,b_{p+1}+1} + \cdots + \mathcal{H}_{p,b_p})^{b_p-b_{p+1}} \]
\[ = \prod_{p=1}^{\lambda_1} (b_p - b_{p+1})! \mathcal{H}_{p,b_{p+1}+1} \cdots \mathcal{H}_{p,b_p} \]
\[ = \left( \prod_{p=1}^{\lambda_1} (b_p - b_{p+1})! \right) \mathcal{H}_{\lambda_1,1} \mathcal{H}_{\lambda_2,2} \cdots \mathcal{H}_{\lambda_{n-q},n-q}. \]

Here the fifth equal sign comes from the equation \( \mathcal{H}_{\lambda_1,k} \mathcal{H}_{p,k} = 0 \). Clearly \( r_h := \prod_{p=1}^{\lambda_1} (b_p - b_{p+1})! \) is a positive integer. From Lemma 2.4 and the Künneth formula \( \mathcal{H}_{\lambda_1,1} \mathcal{H}_{\lambda_2,2} \cdots \mathcal{H}_{\lambda_{n-q},n-q} \neq 0 \in H^q(P_n; \mathbb{Q}). \) This proves Assertion A. \( \square \)

**Proof of Assertion B** Suppose \( \mu > \lambda \) with respect to the lexicographic order, namely, \( \mu_1 = \lambda_1 \geq \mu_2 = \lambda_2 \geq \cdots \geq \mu_h = \lambda_h \geq \mu_{h+1} > \lambda_{h+1} \) for some \( h, 0 \leq h < n-q \).

Let \( \nu := (\nu_1 \geq \nu_2 \geq \cdots \geq \nu_h) \) be the (truncated) partition of \( q' := \lambda_1 + \lambda_2 + \cdots + \lambda_h \) defined by \( \nu_k := \lambda_k = \mu_k, k \leq h. \) From Assertion A
\[ t_h^* (\mathcal{H}_{\mu_1} \mathcal{H}_{\mu_2} \cdots \mathcal{H}_{\mu_h}) = r_{\nu_1} \mathcal{H}_{\mu_1,1} \mathcal{H}_{\mu_2,2} \cdots \mathcal{H}_{\mu_h,h} \in H^q(P_n; \mathbb{Q}). \]

In fact, from \( \mu_h > \lambda_{h+1} \), we have \( \mathcal{H}_{\mu_i,j} = 0 \) if \( i < j \). Since \( \mu_{h+1} \geq \lambda_k \) for any \( k \geq h+1, \) we have
\[ t_h^* (\mathcal{H}_{\mu_1} \cdots \mathcal{H}_{\mu_h}) = r_{\nu_1} \mathcal{H}_{\mu_1,1} \cdots \mathcal{H}_{\mu_h,h}(\mathcal{H}_{\mu_{h+1},1} + \cdots + \mathcal{H}_{\mu_{h+1},h}) = 0 \]
Hence \( t_h^* (\mathcal{H}_{\mu}) = 0, \) as was to be shown. \( \square \)

This completes the proof of Theorem 3.1 and Theorem 1.

## 4 Concluding remarks

We conclude this note by giving some remarks on the twisted cohomology of the automorphism group \( \text{Aut}(F_n) \) and the braid group \( B_n. \)

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The IA–automorphism group $IA_n$ is defined to be the kernel of the action of the group $\text{Aut}(F_n)$ on the homology group $H = F_n^{\text{abel}}$. We have an extension of groups $IA_n \to \text{Aut}(F_n) \to \text{GL}(H)$. The map $\tau_1^0$ restricted to $IA_n$ gives an isomorphism of the abelianization of the group $IA_n$ onto the module $H^* \otimes \bigwedge^2 H$

$$\tau_1 : IA_n^{\text{abel}} \cong H^* \otimes \bigwedge^2 H$$

(Chen and Pakianathan [5], Farb [6], Kawazumi [12]). Here we embed $\bigwedge^2 H$ into $H^\otimes 2$ by $X_i \wedge X_j \mapsto X_i \otimes X_j - X_j \otimes X_i$ for $1 \leq i, j \leq n$. Lemma 2.2 implies $\xi^* : H^1(IA_n; \mathbb{Z}) \to H^1(P_n; \mathbb{Z})$ is surjective. From the result of Arnol’d [1] quoted in Section 2, the cohomology algebra $H^*(P_n; \mathbb{Z})$ is generated by the first cohomology classes. Hence we obtain:

**Corollary 4.1** The algebra homomorphism

$$\xi^* : H^*(IA_n; \mathbb{Z}) \to H^*(P_n; \mathbb{Z})$$

induced by the homomorphism $\xi : P_n \to IA_n$ is surjective.

It should be remarked that it does not imply that the map $\xi^* : H^*(\text{Aut}(F_n); M) \to H^*(B_n; M)$ is surjective for a $\mathbb{Q}[\text{GL}(H)]$–module $M$. In fact, the quotient groups $\text{Aut}(F_n)/IA_n = \text{GL}(H)$ and $B_n/P_n = \mathbb{G}_n$ differ from each other.

Fred Cohen [4, Lemma 7.2, page 261] described the action of the symmetric group $\mathfrak{S}_n$ on the integral cohomology of the group $P_n$, $H^*(P_n; \mathbb{Z})$. Later Lehrer and Solomon [14] gave another explicit description of the $\mathbb{Q}[\mathfrak{S}_n]$–module $H^*(P_n; \mathbb{Q})$. Moreover Cohen [4, Theorem 3.1, page 225] computed the twisted cohomology $H^*(B_n; H^\otimes m \otimes \mathbb{F})$ for any field $\mathbb{F}$ and any $m \geq 0$. It would be interesting if one could describe the submodule of $H^*(B_n; M)$ generated by all the possible algebraic combinations coming from the twisted Morita–Mumford classes $h_p$’s in an explicit manner. Here we should remark the $\mathfrak{S}_n$–invariant inner product $\cdot : H \otimes H \to \mathbb{Z}$ defined by $X_i \cdot X_j = \delta_{ij}$, $1 \leq i, j \leq n$, gives a $B_n$–isomorphism $H \cong H^*$.

As was stated in Introduction, the algebra $H^*(\mathcal{M}_{g,1}; \bigwedge^* HQ)$ is stably isomorphic to the polynomial algebra in the twisted Morita–Mumford classes $m_{ij}$’s. The intersection pairing of the surface $\Sigma_{g,1}$, $H^\otimes 2 \to \mathbb{Z}$, gives an isomorphism $H \cong H^*$ of $\mathcal{M}_{g,1}$–modules, so that the cocycle $\tau_1^0$ restricted to $\mathcal{M}_{g,1}$ can be regarded as a cocycle $\tau_1^0 : \mathcal{M}_{g,1} \to H^\otimes 3$. As was proved by Kawazumi and Morita in [13], for any twisted Morita–Mumford class $m_{ij}$ we have an $\mathcal{M}_{g,1}$–homomorphism $C : (H^\otimes 3)^{\otimes (2i+j-2)} \to \mathbb{Z}$ obtained from the intersection pairing such that $C_s[\tau_1^0]^{2i+j-2} = m_{ij}$. In other words, the natural map

$$((\bigwedge^* H^1(\mathcal{I}_{g,1}; \mathbb{Q})) \otimes M)^{\text{Sp}(H)} \to H^*(\mathcal{M}_{g,1}; M)$$

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is stably surjective for any finite dimensional \( \mathbb{Q}[\text{Sp}(H)] \)–module \( M \). Here \( \mathcal{I}_{g,1} \) is the Torelli group, i.e, the kernel of the action of \( \mathcal{M}_{g,1} \) on the homology \( H \).

Recently Galatius [7] proved the rational reduced cohomology \( \tilde{H}^*(\text{Aut}(F_n); \mathbb{Q}) \) vanishes in a stable range. It would be very interesting to know whether a similar result holds also for twisted coefficients.

**Expectation 4.2** For a finite dimensional \( \mathbb{Q}[\text{GL}(H)] \)–module \( M \), the natural map

\[
((\bigwedge^* H^1(IA_n; \mathbb{Q})) \otimes M)^{\text{GL}(H)} \rightarrow H^*(\text{Aut}(F_n); M)
\]

is surjective in some stable range.

In the case \( M \) is the trivial module \( \mathbb{Q} \), this expectation is exactly the fact that \( \tilde{H}^*(\text{Aut}(F_n); \mathbb{Q}) \) vanishes in some stable range, which Galatius [7] proved. A result of Hatcher and Wahl [8] tells us it holds also for \( M = (H^*)^\otimes m \) for any \( m \geq 1 \).

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