MANIFOLD COVERED BY TWO GEODESIC BALLS

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ABSTRACT. In this paper we prove a covering type sphere theorem: If a compact Riemannian manifold with nonnegative sectional curvature can be covered by two open geodesic balls with suitable radius (depending on the diameter and injectivity radius), then it is homeomorphic to a sphere. This can be seen as a geometric analogue of the Brown theorem in topology.

1. INTRODUCTION

The sphere is one of the most fundamental models in Riemannian geometry. An important topic in Riemannian geometry is to characterize the sphere. That is so called sphere type theorem, i.e. seeking suitable geometric conditions (curvature, diameter, volume etc.) ensure that the manifold is homeomorphic to a sphere. The classical result is Rauch-Berger-Klingenberg 1/4-pinching theorem. It claims that a simply connected compact Riemannian manifold with $\frac{1}{4} < \text{sec}_M \leq 1$ is homeomorphic to a sphere. After Rauch-Berger-Klingenberg’s work, there are many notable results on sphere type theorem. These include Grove-Shiohama diameter sphere theorem [3], Micallef-Moore’s positive isotropic curvature sphere theorem [5], Perelman’s almost maximal volume sphere theorem [8] and Brendle-Schoen’s differential sphere theorem [2] etc.. In this paper we consider a covering type sphere theorem: A compact Riemannian manifold covered by two open geodesic balls with suitable radius is homeomorphic to a sphere. The motivation comes from a simple fact: (Theorem 2.1) If a compact Riemannian manifold is covered exactly by two closed geodesic balls, then it is homeomorphic to a sphere. It will be discussed detailedly in next section. On the other hand, considering a compact Riemannian manifold covered by two geodesic balls also appeared naturally in the original proof of 1/4-pinching theorem (the heart part is to show that the manifold can covered by two disks. c.f. [7] page 347).

Let $M^n$ be an $n$-dimensional compact Riemannian manifold. Let $B_p(r) = \{ x \in M | d(p, x) < r \}$ be the open geodesic ball of center at $p$ and radius $r$. Let $p, q \in M^n$ such that $d = d(p, q)$ is equal to the diameter of $M^n$. Denote the injectivity radius of $M^n$ by $i = i(M^n)$. The first result of this paper is

**Theorem 1.1.** Let $M^n$ be an $n$-dimensional compact Riemannian manifold with nonnegative sectional curvature. Then there is a constant $\epsilon(d, i)$ only depending on $d$ and $i$ such that if

$$M^n = B_p\left(\frac{d}{2} + \epsilon\right) \cup B_q\left(\frac{d}{2} + \epsilon\right),$$

where $\epsilon < \epsilon(d, i)$, then $M^n$ is homeomorphic to $n$-sphere $S^n$.

Assume that the sectional curvature is positive, we have

Date: October 18, 2016.

2010 Mathematics Subject Classification. Primary 53C20; Secondary 53C23.

The author was supported by National Natural Science Foundation of China N0.11301416.
Theorem 1.2. Let $M^n$ be an orientable even dimensional compact Riemannian manifold. The sectional curvature satisfies $0 < \sec_M \leq K$. Then there is a constant $\epsilon(d,K)$ only depending on $d$ and $K$ such that if

$$M^n = B_p\left(\frac{d}{2} + \epsilon\right) \cup B_q\left(\frac{d}{2} + \epsilon\right),$$

where $\epsilon < \epsilon(d,K)$, then $M^n$ is homeomorphic to $n$-sphere $S^n$.

Particularly, if we rescale the Riemannian metric such that the upper bound of sectional curvature $K = 1$, then $\epsilon$ is only depending on the diameter $d$.

Remark 1.3. The constants $\epsilon(d,i)$ and $\epsilon(d,K)$ are computable. In our proofs we can choose $\epsilon(d,i) = -\frac{d - \sqrt{d^2 + 4}}{2}$ and $\epsilon(d,K) = -\frac{d - \sqrt{d^2 + \pi}}{2}$. But in this paper we do not discuss the sharp values of $\epsilon(d,i)$ and $\epsilon(d,K)$.

We recall the well-known Brown theorem (c.f. [10] page 158, [9] and [1]):

Theorem 1.4. If a compact manifold $M^n$ is covered by two open sets which are both homeomorphic to Euclid ball, then $M^n$ is homeomorphic to $n$-sphere $S^n$.

Theorem 1.4 was conjectured by Alexander in dimension 3 and proved by Moise [6] in that case. In general case it follows from Brown’s proof of generalized Schoenflies theorem [1] (c.f. [9] theorem 3 and its explanation). Theorem 1.1 can be regarded as a geometric version of Brown theorem.

This paper is organized as follows. In section 2 we will discuss the main motivation of this paper. In section 3 we give an account of preliminary knowledge. Some key estimates on the length of geodesics are given in section 4. The main theorems will be proved in section 5. Section 6 contains a general account of the main theorems and related discusses.

2. motivation

Recall that an $n$-sphere can be obtained (topologically) by gluing two closed $n$-Euclid balls along their boundaries. The following simple but astonishing observation shows that the geometric version of this fact is also true.

Theorem 2.1. Let $M^n$ be a compact Riemannian manifold. If there exists two points $p, q \in M$ such that

$$M^n = \overline{B}_p(r) \cup \overline{B}_q(d - r)$$

for some $0 < r < d$, where $d = d(p,q)$, then $M^n$ is homeomorphic to $n$-sphere $S^n$.

The author believes that this result may be known by many experts.

Proof. The condition implies that the boundary $\partial \overline{B}_p(r) = \partial \overline{B}_q(d - r)$. We denote it by $\partial B$.

We will show that for any $x \in \overline{B}_p(r) \setminus \{p\}$, there is a unique minimal geodesic from $p$ to $q$ passing through $x$.

Let $\gamma_1$ be the minimal geodesic connecting $x$ and $q$ and passing through $y \in \partial B$. Let $\gamma_2$ be the minimal geodesic from $y$ to $p$. Denote the length of $\gamma_1$ from $q$ to $y$ by $l_1$ and the length of $\gamma_2$ by $l_2$. Then $l_1 = d - r$ and $l_2 = r$. The triangle inequality

$$d = l_1 + l_2 \geq d = d(p,q)$$

ensures that $\gamma_2$ agrees with $\gamma_1$. This shows that $x$ lies on a minimal geodesic from $p$ to $q$.

We write this geodesic by $\gamma$.

1 I thank professor Morton Brown for pointing out the theorem appeared in [6].
If there is another minimal geodesic $\tilde{\gamma}$ from $p$ to $q$ passing through $x$, then the angle between $\gamma'$ and $-\tilde{\gamma}$ at $x$ is less than $\pi$. Since $l(\tilde{\gamma}) = l(\gamma) = d$, the triangle inequality
\[ d = l(\gamma_{pq}) + l(\gamma_{xq}) > d \]
leads to a contradiction.

So for very small $\epsilon$, the distance function $d(p, x)$ is smooth and has no critical point in $M \setminus B_p(\epsilon) \cup B_q(\epsilon)$. By the isotopy lemma in Morse theory [4], we know that $M^n$ is homeomorphic to $n$-sphere $S^n$. Or more directly, $d(p, x)$ has no critical point in the sense of Grove-Shiohama [3] except $p$ and $q$ (theorem 3.3 below). This also leads that $M^n$ is homeomorphic to $S^n$.

Note that $\overline{B}_p(r)$ may have arbitrary topological type when $r$ is very large. So the theorem is not directly though the proof is simple! We also should note that the theorem have no curvature assumption.

The key argument in above proof is that $\overline{B}_p(r)$ and $\overline{B}_q(d - r)$ have same boundary. This leads that all the triangle inequalities become equalities. We can ask a natural question: Can we relax the covering condition in theorem 2.1 such that $M^n = B_p(r + \epsilon) \cup B_q(d - r + \epsilon)$, $\epsilon$ is very small (obviously $\epsilon$ can not be very large), is $M^n$ homeomorphic to $S^n$? In this paper we would answer this question positively. In this case we need to control the triangle inequalities in suitable range. So naturally we need some curvature assumptions.

We have interesting consequences of theorem 2.1.

**Corollary 2.2.** If the boundary of $\overline{B}_p(r)$ is isometric to an $n - 1$-Euclid sphere, then $B_p(r)$ is homeomorphic to an Euclid ball.

**Proof.** Since $\partial \overline{B}_p(r)$ is isometric to an $n - 1$-Euclid sphere, we can glue $\overline{B}_p(r)$ to an $n$-Euclid disk along their boundaries and obtain a compact manifold $M^n$. Applying theorem 2.1 to $M^n$, we get the result. □

**Corollary 2.3.** Let $V_p(r)$ denote the volume of $B_p(r)$. If $V(M) = V_p(r) + V_q(d - r)$ for some $0 < r < d$, then $M^n$ is homeomorphic to $n$-sphere $S^n$.

**Proof.** The condition $V(M) = V_p(r) + V_q(d - r)$ implies $M^n = \overline{B}_p(r) \cup \overline{B}_q(d - r)$. □

### 3. Toponogov comparison theorem and critical point theory for distance functions

In this section we collect some necessary knowledge for our purpose. All these are standard and can be found in many literatures (for example see [7]).

#### 3.1. Toponogov triangle comparison

Let $M^n$ be a complete Riemannian manifold with $\sec_M \geq k$. Given any three points $p, q_1, q_2 \in M^n$, there is a minimal geodesic $\sigma_1$ (resp. $\sigma_2$) from $p$ to $q_1$ (resp. to $q_2$). We denote the length of $\sigma_1$ (resp. $\sigma_2$) by $l_1$ (resp. $l_2$). $\sigma_1$ and $\sigma_2$ form an angle $\alpha$ at $p$. Let $\overline{\sigma_1}, \overline{\sigma_2}, \overline{\sigma_3} \in S^k$ be three points in the model space $S^k$ ($k < 0$ Hyperbolic space, $k = 0$ Euclid space, $k > 0$ sphere space). Let $\overline{\sigma_1}$ (resp. $\overline{\sigma_2}$) be the minimal geodesic from $\overline{\sigma_1}$ to $\overline{\sigma_2}$ (resp. to $\overline{\sigma_3}$). The lengths denote respectively by $\overline{l}_1$ and $\overline{l}_2$. $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ form an angle $\overline{\alpha}$ at $\overline{\sigma}$. A form of Toponogov comparison theorem is

**Theorem 3.1.** If $l_1 = \overline{l}_1, l_2 = \overline{l}_2, \alpha = \overline{\alpha}$, then $d(q_1, q_2) \leq d(\overline{\sigma_1}, \overline{\sigma_2})$. 

3.2. **Critical point of distance functions.** The critical point theory for distance functions is a partial analogue of Morse theory. It was developed by Grove and Shiohama in their famous work on diameter sphere theorem \[3\]. Now it becomes an important tool in many geometric problems.

Let \( M^n \) be a complete Riemannian manifold. Fix a point \( p \in M^n \). We say that \( x \in M^n \) is a critical point for distance function \( d(p, x) \), if for any tangent vector \( v \in T_xM^n \) there exists a minimal geodesic \( \sigma \) from \( x \) to \( p \) such that the angle \( \angle(v, \sigma'(0)) \leq \pi/2 \). The importance of this concept is the following isotopy lemma.

**Lemma 3.2.** If \( d(p, x) \) has no critical point in \( B_p(r_2) \setminus B_p(r_1) \) \((r_2 > r_1)\), then \( B_p(r_2) \) is homeomorphic to \( B_p(r_1) \). Moreover, \( B_p(r_2) \) deformation retracts onto \( B_p(r_1) \).

As a directly consequence, we have

**Theorem 3.3.** 1) If \( d(p, x) \) has no critical point in \( B_p(r) \) except \( p \), then \( B_p(r) \) is homeomorphic to an Euclid ball.

2) If \( M^n \) is compact and \( d(p, x) \) has only two critical points, then \( M^n \) is homeomorphic to an \( n \)-sphere.

4. **LENGTH ESTIMATES**

In this section we establish some necessary length estimates.

Let \( x \neq p \) be a point in \( B_p(\frac{d}{2} + \epsilon) \).

**Notations.**

\( \gamma_1 \): the minimal geodesic connecting \( x \) and \( p \).
\( \gamma_2 \): the minimal geodesic connecting \( x \) and \( q \) and passing through \( y \in \partial B_p(\frac{d}{2} + \epsilon) \).
\( \gamma_3 \): the minimal geodesic connecting \( y \) and \( p \).
\( \theta \): the angle between \( \gamma_1 \) and \( \gamma_2 \) at \( x \).
\( \alpha \): the angle between \( \gamma_2 \) and \( \gamma_3 \) at \( y \).
\( l_1 = \text{length}(\gamma_1) = d(x, p); l_2 = \text{length}(\gamma_2) = d(x, y); l_3 = \text{length}(\gamma_3) = d(y, p); l_4 = \text{length}(\gamma_2) \).

**Basic properties.**

\[ l_2 < \frac{d}{2} + \epsilon. \]
\[ l_3 = \frac{d}{2} + \epsilon. \]
\[ \frac{d}{2} - \epsilon < l_4 < \frac{d}{2} + \epsilon. \]

**Proof of basic properties.** Since \( y \in \partial B_p(\frac{d}{2} + \epsilon), \) we have \( l_1 = \frac{d}{2} + \epsilon \). By the covering condition, we know that \( y \in B_q(\frac{d}{2} + \epsilon) \). Hence \( l_4 < \frac{d}{2} + \epsilon \). The triangle inequality implies that \( l_4 > d - l_3 = \frac{d}{2} - \epsilon \). Because \( l_2 + l_4 = d(x, q) \leq d \), we have \( l_2 \leq d - l_4 < \frac{d}{2} + \epsilon \).

Applying the Toponogov triangle comparison to \( \Delta pyq \): \[ d^2 \leq l_1^2 + l_4^2 - 2l_1l_4 \cos \alpha. \]

Then \[ \cos \alpha \leq \frac{l_1^2 + l_4^2 - d^2}{2l_1l_4}. \]

Since \( d > l_3 \), \( \frac{l_1^2 + l_4^2 - d^2}{2l_1l_4} \) is an increasing function on \( l_4 \). One has \[ \frac{l_1^2 + l_4^2 - d^2}{2l_1l_4} < \frac{(\frac{d}{2} + \epsilon)^2 + (\frac{d}{2} + \epsilon)^2 - d^2}{2(\frac{d}{2} + \epsilon)^2} = 1 - \frac{d^2}{2(\frac{d}{2} + \epsilon)^2}. \]
So we obtain

\[ \cos \alpha < 1 - \frac{d^2}{2(\frac{d}{2} + \epsilon)^2}. \]

Applying also the Toponogov triangle comparison to \( \Delta px \):

\[ \hat{\ell}_1^2 \leq \hat{\ell}_2^2 + \hat{\ell}_3^2 - 2\hat{l}_2\hat{l}_3 \cos(\pi - \alpha) = \hat{\ell}_2^2 + \hat{\ell}_3^2 + 2\hat{l}_2\hat{l}_3 \cos \alpha. \]

If \( x \) is a critical point of \( d(p, x) \), then there exists a \( \gamma_1 \) such that \( \theta \leq \frac{\pi}{2} \). So we have

\[ \hat{\ell}_3 \leq \hat{\ell}_1^2 + \hat{\ell}_2^2. \]

and

\[ \hat{l}_1 \geq \hat{i}. \]

We have the following two important estimates on \( \hat{l}_2 \).

**Lemma 4.1.** \( \hat{l}_2 < \frac{d}{2} + \epsilon - \sqrt{\hat{l}_2^2 - 4\epsilon(d + \epsilon)}. \)

**Proof.** By 4.2, we have

\[ \hat{\ell}_1^2 \leq (l_3 - l_2)^2 + 2l_2l_3(\cos \alpha + 1). \]

From 4.1, we get

\[ l_2l_3(\cos \alpha + 1) < (\frac{d}{2} + \epsilon)^2[2 - \frac{d^2}{2(\frac{d}{2} + \epsilon)^2}] = 2\epsilon(d + \epsilon). \]

Thus

\[ \hat{l}_2 \leq \hat{l}_1^2 < (l_3 - l_2)^2 + 4\epsilon(d + \epsilon). \]

So

\[ \hat{l}_2 < l_3 - \sqrt{l_2^2 - 4\epsilon(d + \epsilon)} = \frac{d}{2} + \epsilon - \sqrt{l_2^2 - 4\epsilon(d + \epsilon)}. \]

\[ \square \]

**Lemma 4.2.** \( \hat{l}_2 > (\frac{d}{2} + \epsilon)[\frac{d^2}{2(\frac{d}{2} + \epsilon)^2} - 1]. \)

**Proof.** Combining 4.2 with 4.3, we obtain

\[ 0 \leq 2l_2^2 + 2l_2l_3 \cos \alpha. \]

Substituting 4.1 into it, we have

\[ 0 < l_2 + (\frac{d}{2} + \epsilon)[1 - \frac{d^2}{2(\frac{d}{2} + \epsilon)^2}]. \]

We conclude that

\[ \hat{l}_2 > (\frac{d}{2} + \epsilon)[\frac{d^2}{2(\frac{d}{2} + \epsilon)^2} - 1]. \]

\[ \square \]
5. PROOF OF THE MAIN THEOREMS

The main strategy is to show that the distance function \( d(p, x) \) has no critical point in \( B_p(\frac{d}{2} + \epsilon) \) except \( p \). We observe that when \( \epsilon \) tends to 0, from lemma 4.1 and 4.2, we have \( \frac{d}{2} < l_2 < \frac{d}{2} - i \). This contradiction leads to

**Lemma 5.1.** There is a constant \( \epsilon(d, i) \) such that \( d(p, x) \) has no critical point in \( B_p(\frac{d}{2} + \epsilon) \) except \( p \) when \( \epsilon < \epsilon(i, d) \).

**Proof.** Following lemma 4.1 and 4.2, we consider the inequality

\[
\frac{d}{2} + \epsilon \left[ \frac{d^2}{2(\frac{d}{2} + \epsilon)^2} - 1 \right] \geq \frac{d}{2} + \epsilon - \sqrt{i^2 - 4\epsilon(d + \epsilon)}.
\]

This is equivalent to

\[
8(d\epsilon + \epsilon^2) - i^2 + \frac{d^4}{4(\frac{d}{2} + \epsilon)^2} - d^2 \leq 0.
\]

Because we always have \( -\frac{d^4}{4(\frac{d}{2} + \epsilon)^2} - d^2 < 0 \), the solution of

\[
8(d\epsilon + \epsilon^2) - i^2 \leq 0,
\]

namely

\[
0 < \epsilon \leq -d + \sqrt{d^2 + \frac{d^2}{2}}
\]

also satisfies inequality 5.2. So we can choose

\[
\epsilon(d, i) = \frac{-d + \sqrt{d^2 + \frac{d^2}{2}}}{2}.
\]

When \( \epsilon < \epsilon(d, i) \), there is a contradiction between lemma 4.1 and 4.2. Hence \( d(p, x) \) has no critical point in \( B_p(\frac{d}{2} + \epsilon) \) except \( p \). \( \square \)

From formula 4.1, it is also obviously that \( d(p, x) \) has no critical point on \( \partial B_p(\frac{d}{2} + \epsilon) \). So when \( \epsilon < \epsilon(d, i) \), \( d(p, x) \) has no critical point in \( B_p(\frac{d}{2} + \epsilon) \) except \( p \). Similarly \( d(q, x) \) has no critical point in \( B_q(\frac{d}{2} + \epsilon) \) except \( q \). So both \( B_p(\frac{d}{2} + \epsilon) \) and \( B_q(\frac{d}{2} + \epsilon) \) are homeomorphic to Euclid ball. By the Brown theorem, we have that \( M^n \) is homeomorphic to \( S^n \).

If the dimension is even and \( 0 < \text{sec}_M \leq K \), by Klingenberg theorem (c.f. [7] page 178), the injectivity radius \( i \geq \pi / \sqrt{K} \). In the proof of lemma 4.1, \( l_1 \geq i \) can be replaced by \( l_1 \geq \sqrt{K} \). So we can choose

\[
\epsilon(d, K) = \frac{-d + \sqrt{d^2 + \frac{\epsilon^2}{2K}}}{2}
\]

A little more patience we also can solve directly. In fact 5.2 is equivalent to

\[
32(d\epsilon + \epsilon^2) + 4(d^2 - i^2)(d\epsilon + \epsilon^2) - d^2i^2 \leq 0.
\]

Hence

\[
0 \leq d\epsilon + \epsilon^2 \leq \Delta = \frac{-(d^2 - i^2) + \sqrt{(d^2 - i^2)^2 + 8d^2i^2}}{16}.
\]

One gets

\[
0 < \epsilon \leq \frac{-d + \sqrt{d^2 + 4\Delta}}{2}.
\]
in theorem 1.2.

Remark 5.2. The Brown theorem is essential for our proof. Since the injectivity radius may be very small, we can not construct the homeomorphism directly via exponential map like $1/4$-pinching theorem.

6. REMARKS

We also can consider manifold covered by two geodesic balls with different radius.

**Theorem 6.1.** Let $M^n$ be an $n$-dimensional compact Riemannian manifold with nonnegative sectional curvature. Then there is a constant $\epsilon(d, i, s)$ depending on $d$, $i$ and $s$ such that if

$$M^n = B_p(s + \epsilon) \cup B_q(d - s + \epsilon),$$

where $\epsilon < \epsilon(d, i, s)$, then $M^n$ is homeomorphic to $n$-sphere $S^n$.

**Theorem 6.2.** Let $M^n$ be an even dimensional compact Riemannian manifold. The sectional curvature satisfies $0 < \sec M \leq K$. Then there is a constant $\epsilon(d, K, s)$ depending on $d$, $K$ and $s$ such that if

$$M^n = B_p(s + \epsilon) \cup B_q(d - s + \epsilon),$$

where $\epsilon < \epsilon(d, K, s)$, then $M^n$ is homeomorphic to $n$-sphere $S^n$.

They can be proved in the same way as shown before. In fact, the basic properties in section 4 would become $l_2 < s + \epsilon; \ l_3 = s + \epsilon; \ d - s - \epsilon < l_4 < d - s + \epsilon$. Then all the remanent procedures are routine.

A problem is whether the injectivity radius restriction in $\epsilon(d, i)$ can be removed. At least in present proof, it is difficult to do so. We can consider two examples (c.f. page 179): 1) $M_k = S^3/\mathbb{Z}_k$, constant curvature 3-sphere acted isometrically by $k$-order cyclic group. $\sec M_k = 1$ and $d(M_k) = \pi/2$. As $k \to \infty$, $\text{Volume}(M_k) \to 0$. Hence the injectivity radius $i(M_k) \to 0$; 2) Berger sphere $(S^3, g_\epsilon)$, $\epsilon^2 \leq \sec \leq 4 - 3\epsilon^2$. The Hopf fiber is a closed geodesic of length $2\pi \epsilon$. The injectivity radius $i \to 0$ as $\epsilon \to 0$.

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