Quantum backreaction (Casimir) effect
I. What are admissible idealizations?

Andrzej Herdegen*

Institute of Physics, Jagiellonian University,
Reymonta 4, 30-059 Cracow, Poland

Abstract

Casimir effect, in a broad interpretation which we adopt here, consists in a backreaction of a quantum system to adiabatically changing external conditions. Although the system is usually taken to be a quantum field, we show that this restriction rather blurs than helps to clarify the statement of the problem. We discuss the problem from the point of view of algebraic structure of quantum theory, which is most appropriate in this context. The system in question may be any quantum system, among others both finite as infinite dimensional canonical systems are allowed. A simple finite-dimensional model is discussed. We identify precisely the source of difficulties and infinities in most of traditional treatments of the problem for infinite dimensional systems (such as quantum fields), which is incompatibility of algebras of observables or their representations. We formulate conditions on model idealizations which are acceptable for the discussion of the adiabatic backreaction problem. In the case of quantum field models in that class we find that the normal ordered energy density is a well defined distribution, yielding global energy in the limit of a unit test function. Although we see the “zero point” expressions as inappropriate, we show how they can arise in the quantum field theory context as a result of uncontrollable manipulations.

PACS numbers: 03.70.+k, 03.65.Bz, 11.10.-z

*e-mail: herdegen@th.if.uj.edu.pl
1 Introduction

The Casimir effect, bearing its name from the pioneer work by Casimir [1], has become in recent decades an increasingly popular topic in quantum field theory, with a new review of the subject appearing every few years, see [2, 3, 4, 5, 6]. The effect consists in the response of a quantum field, even in a ground state, to the introduction of external, usually macroscopic, bodies. Initially the effect existed as a theoretical prediction only, and a rather mysterious one, for that matter. However, increasing experimental evidence of its existence (see e.g. [6]) has lead to attempts at better understanding of its theoretical foundation. The problem is, that the theoretical side of the phenomenon has been plagued from the beginning by divergent expressions, as well as conceptual difficulties, which have proved to be surprisingly persistent. This is the more surprising, that models usually considered in this context are linear, so the usual sources of quantum field infinities are absent here.

In an earlier paper [7] I have given a diagnosis of the reasons of this state of affairs and proposed to treat the problem from the algebraic point of view. This is the most natural and fruitful framework in quantum physics, with its beginnings already in the classical book on quantum mechanics by Dirac, and modern developments in quantum field theory and statistical physics described e.g. in monographs [8] and [9]. When viewed from that angle the source of difficulties is rather obvious, and can be briefly termed as uncritical use of the concept of quantum field [7]. More precisely, what we mean is this. The first step to define a quantum theory is to identify a set of quantum observables (we ignore here the question of non-observable variables) together with algebraic relations between them, such as canonical commutation relations. Once we have this, a concrete physical realization of the theory corresponds to a choice of a representation of the algebra of observables. Non-comparable physical situations are realized by non-equivalent representations [8]. Although we want to see the real world as a unity, physics, of course, is about idealizations, and various idealizations need not be compatible (take e.g. an isolated system and a thermodynamic limit system). However, if we want to consider transitions from one physical situation to another and compare values of one and the same observable in various states, all situations taken into account must be describable in one common representation. Now, these scheme is violated in most treatments of the Casimir effect. For a typical situation of a quantum field in a region with movable sharp boundaries the difficulty arises already on the algebraic level: there is no consistent choice of an algebra of observables for all physical situations coming into play. The energy of the “free” field is an observable defined in the vacuum representation of the algebra of the field smeared with Schwartz test functions. For this algebra evolutions imposed on the field by the presence of boundaries cannot be defined. Furthermore, even if one “smooths out” the boundaries so as to make a common choice of an algebra possible, one still has to satisfy rather severe restrictions necessary to ensure the equivalence of representations. These restrictions are typically violated in usual
treatments. For these reasons we have advocated in [7] the view, that the model of sharp boundaries, as well as many other insufficiently regular models, are wrong idealizations in the context of Casimir effect, and we have also proposed (and analyzed) a class of models imitating Dirichlet conditions. Let us stress this: once a model has correctly been chosen, there is no space (nor need) for further *ad hoc* regularizations, and the formalism should yield well-defined answers to legitimate questions. Although views on nonphysical nature of sharp boundary conditions have been also expressed elsewhere (see e.g. [10]), it seems that the conditions for a model to be acceptable in the sense described above have not been analyzed before. For instance, in a series of recent papers Graham et al. [11] investigate a linear model imitating Dirichlet conditions. Being linear, the model should be well-defined without any renormalization (except for a trivial normal ordering for quadratic quantities like energy density; for external potentials without bounded states this is our example (iii)$_1$ at the end of Section 3 below). However, renormalization is *ad hoc* imposed on it by the authors in order to give meaning to a meaningless expression.

The algebraic problems we have described often do not appear if one restricts attention to local quantities in quantum field theory. This fact is connected with what in algebraic formulation is called the local quasiequivalence of representations (see [8]). The point is as follows. In quantum field theory observables are equipped with the property of locality: each local observable carries as a label an open space-time region with compact closure in which it may be measured. As stated above, two representations of the totality of these observables representing two different physical situations may be non-equivalent. However, physically one would expect that even if the two situations are globally non-comparable, one should be able to compare results of local measurements (think of the vacuum representation and a thermodynamic limit representation). Mathematical formulation of this expectation is this: think of states in each of the representations as density operators; restrict attention to an arbitrarily chosen compact region of spacetime; then for each state in one of the representations there is a state in the other which yields the same expectation values for observables localized in the chosen region. If the two representations have this property they are called locally quasiequivalent. As it turns out physically important representations do indeed often have this relative property, and then expectation values of local quantities may be compared. However, in the situation we want to consider in this paper this result falls short of our needs in twofold way. First, we want to calculate expectation values of global quantities, which are limits of local ones for the size of the spacetime region tending to infinity – in this case the global differences of the representations come into play. Second, in situations like fields with imposed boundary conditions, even finite regions which overlap with boundaries are not local in the above sense: for those regions even the scopes of local algebras in presence of boundaries are different than in the vacuum theory.

Another important point we want to stress in our analysis of Casimir effect is the choice of the observable to be compared in various considered states. In our
view the backreaction of a system perturbed by external agents is determined by the expectation value of the energy as defined by the unperturbed system, one and the same (as an operator) in all states to be considered. A more systematic discussion of this point in a wider context will be found in the next section. Here we want to note that some local, in the spirit of the last paragraph, calculations of the Casimir energy do follow similar ideology; in the gravitational context see esp. a paper by Kay [12], and for electromagnetic field with conducting boundaries a paper by Scharf and Wreszinski [13]. However, in many other local calculations, esp. those using “the Green function method”, the situation is somewhat ambiguous: it is often not clear enough what the general viewpoint is, and the result may agree with the above method in some cases, but disagree in others. We shall discuss this point more fully in Section 6 below. For the global energy, as determined by the unperturbed system, to be defined in states of the system influenced by external conditions, as required by the above ideology, we need one common algebra and globally equivalent representations, as explained earlier. This imposes restrictions on the perturbed dynamics, which are usually violated, and the transition from local Casimir energy to global one is then blocked by infinities of physical nature. Any “regularization” thereof is an ad hoc procedure, striving at this late stage to compensate for the wrong idealization in interaction with external conditions. Finally, there is a group of works explicitly comparing the expectation values of different global observables: energy with and without interaction. Here, apparently, is the place of the “zero point” ideology. We shall come back to this point later on, here we only note, that in this case infinities are even more likely to appear. In that method one subtracts expectation values of regularized “bare” energy observables; different energy observables may have different singularities, not cancelling under subtraction.

The present paper is the first of the two in which we develop and describe more fully what was announced in [7] (we use notation slightly changed at some points with respect to that paper). Here we discuss more general results on the admissibility of models for the purpose of investigation of quantum backreaction. In the second paper applications to particular models are discussed. We use rigorous mathematics, and present real proofs. However, we hope that the paper is readable for a wide audience.

In Section 2 we place the quantum field Casimir effects in a wider context of a backreaction of a quantum system to adiabatic changes. This section thoroughly discusses the foundation for the calculation of this backreaction in any quantum system. In Section 3 we discuss quantization of a class of linear systems, which include quantum fields under linear external perturbations. We put stress on less widely known aspects of this otherwise standard procedure which are important in the present context. Section 4 discusses an application to a finite-dimensional system. In Section 5 we treat infinite-dimensional cases, and we formulate conditions for admissibility of a model for the discussion of backreaction effects. More specifically, we consider a quantum field case in Section 6. We show that with a slight strengthening of these conditions not only global energy, but also energy
density may be defined, and in the appropriate limit global energy is recovered. Section 7 contains somewhat more explicit discussion of the points made earlier in this Introduction on the existing calculations of Casimir effect. We also comment there on the “zero point” expressions for the Casimir energy. We try to understand, from the point of view of the formalism presented in the present work, how such expressions may arise. We show how imposing unacceptable idealization of sharp boundaries and doing unjustified manipulations leads from our expression for the energy density to “zero point” expressions for Casimir energy. Appendix gives a simple form to a handful of mathematical facts in Fock space which are needed in the main text. These are known results, but we believe that this summary makes some of them more accessible.

2 A quantum system under external conditions

Trying to put the discussed phenomenon in a broader context we shall adopt the following point of view. The Casimir-type effect consists in the backreaction of a quantum system on the adiabatically changing external conditions under which the system is placed.

The background for this idea is this. We consider a larger closed system consisting of two subsystems $Q$ and $M$. These subsystems interact with each other, but to certain degree (this will be made more precise below) maintain their separate identity. Part $Q$ is our relatively simple quantum system under consideration (say, electromagnetic field), while $M$ is supposed to be of much more complicated nature (say, conductor plates), and to have among its variables some of collective, macroscopic type (separation of the plates). We want to determine the effect of the evolution of the joint system on the collective variables attached to $M$.

Because of the complicated nature of the part $M$ of the system and its interaction with $Q$, to tackle the problem one has to make some simplifying assumptions. There are at least two possibilities, both of them of phenomenological nature. In both cases one simply represents part $M$ of the system by a few collective variables (such as separation of the plates), suppressing all the details of this subsystem, and representing the interaction between $M$ and $Q$ by some simple effective model. The first possibility is to equip the collective variables with a fully quantum nature, and put forward a simple model for the closed system. This approach, when applied to the more specific situation of a quantum field in interaction with macroscopic bodies, is chronologically more recent one in this field, and is called the dynamic Casimir effect (see [6]). Although we admit that this forms an open possibility, we shall not take it up in this article. Firstly, not much can be said with high degree of certainty and mathematical rigor. Secondly, the apparent attractiveness of the approach does not necessarily withstand a closer scrutiny. A macroscopic body undergoes “constant observation”, so effects of decoherence play primary role, which is not taken into account in this approach.
Another possibility, which we take up in this paper, has more restricted aspirations, but admits mathematically rigorous results, as we are going to argue below. We have to admit, however, that there is some confusion at its physical formulation. We hope to contribute to its removal. This second approach consists in approximating the collective quantities, which characterize a macroscopic body as a whole, by classical variables. Moreover, one considers only situations, in which the whole system changes adiabatically. The effect of the evolution on the macroscopic (classical) variables in this context is what we referred to as a Casimir-type effect at the beginning of this section. More specifically, the Casimir effect refers to a quantum field in interaction with macroscopic bodies.

One should be more specific about physical assumptions and approximations involved in the situation implied in the last paragraph. This is, in our opinion, a point not clear enough in many discussions of the Casimir effect. Therefore we shall try to be systematic, even at a risk of being too detailed.

(i) One considers first the isolated quantum system \( Q \) (\( M \) is absent). We give its description in the algebraic formulation of the Heisenberg picture, see e.g. \[9\].

(i)_1 Basic quantum variables at a fixed time form an abstract \(*\)-algebra \( A \), e.g. an algebra of canonical commutation relations (or, more technically, its exponentiation to the Weyl form).

(i)_2 This algebra is represented by operators in a Hilbert space \( \mathcal{H} \):

\[
\pi : A \mapsto \pi(A), \quad A \mapsto \pi(A), \quad \pi(A^*) = \pi(A)^*, \quad \pi(\alpha A + \beta B) = \alpha \pi(A) + \beta \pi(B), \quad \pi(AB) = \pi(A)\pi(B),
\]

where \( \pi(A) \) is a concrete algebra of operators in \( \mathcal{H} \). Vectors in that space, or, more generally, density operators acting in this space, represent states of the system \( Q \). Representation \( \pi \) is assumed to be irreducible; then vectors correspond to pure states.

(i)_3 The intrinsic dynamics of \( Q \) is defined by an automorphism of the algebra \( A \):

\[
\alpha_t : A \mapsto A, \quad A \mapsto \alpha_t A.
\]

This automorphism is implemented by a unitary evolution in the Hilbert space \( \mathcal{H} \):

\[
\pi(\alpha_t A) = U(t)\pi(A)U(t)^*, \quad U(t) = \exp(itH),
\]

where \( H \) has the interpretation of the energy operator of the system. This operator is supposed to have nonnegative spectrum, and usually is assumed to have a ground state, represented by a unit eigenvector to the lowest point in the spectrum. One does not perturb the above relations by adding a multiple of the identity operator to \( H \), so the ground state may be assumed to have zero energy. By irreducibility of \( \pi \) the energy operator \( H \) is then uniquely determined.
One introduces now part $M$ into the system. This part is characterized by classical variables (we shall denote them by $a$), so no new quantum variables are added. Therefore system $Q$ should retain its identity, and changes in its state will influence the classical variables of $M$. Thus various states to be considered must be physically comparable. These assumptions have mathematical consequences.

(ii) Identity of the system $Q$ is formed by the algebra $\mathcal{A}$ ((i) above), so this algebra must remain unaffected by $M$.

(ii) Physical comparability of states demands that also the particular representation $\pi$ of $\mathcal{A}$ ((ii) above) remains unaffected by the introduction of $M$.

We stress the importance of this point as it is both crucial for the scheme, as we see it, and usually overlooked. If various physical situations to be considered demanded different algebras or different (nonequivalent) representations, the approximation would break down, as one could not follow the change in the system $Q$ brought about by the creation of (and changes in) $M$, and its reaction to that occurrence. Further support for this point will be found below. Let us note again, what was discussed in introduction, that the local quasiequivalence of representations if $Q$ is a quantum field system is not enough for our purposes.

(iii) We consider now dynamics in presence of $M$, and assume at first that the variables $a$ are frozen. In this case $Q$ is still a closed system in interaction with conditions created by $M$, and for each fixed $a$ its evolution is again given by an automorphism of the algebra $\mathcal{A}$:

$$\alpha_{at} : \mathcal{A} \mapsto \mathcal{A}, \quad A \mapsto \alpha_{at} A.$$ (2.4)

One assumes implementability of new evolutions in the representation $\pi$: for each $a$ we have

$$\pi(\alpha_{at}A) = U_a(t)\pi(A)U_a(t)^*, \quad U_a(t) = \exp(\imath tH_a).$$ (2.5)

For each $a$ the generator $H_a$ is defined by this up to the addition of a multiple of the identity operator, so we have the freedom

$$H_a \rightarrow H_a + \lambda_a \text{id},$$ (2.6)

where $\lambda_a$ is any real function of parameters $a$.

(iv) One allows now the coupled system $Q - M$ to evolve. Part $Q$ alone is not a closed system any more, so it could be too restrictive to assume that the evolution of its variables would be given at the algebraic level, as an automorphism. However, this evolution should still be describable in terms of unitary
operators in the Hilbert space $\mathcal{H}$ (not forming a one-parameter group, in general); this corresponds to the assumption of conservation of probabilities in the subsystem $Q$. The use of the Schrödinger picture for the quantum part $Q$ will be more convenient in the present context. State of the coupled system $Q - M$ is specified at a given time by a vector in the Hilbert space $\mathcal{H}$ (describing the state of $Q$), and values of $a$ and, possibly, their time derivatives. We formulate the evolution of this system.

(iv) Suppose that $a(t)$ is known as a function of time. We assume that this functional dependence is very slow (system $M$ is “heavy”). It is then a justified approximation to assume that the time-dependent hamiltonian of the evolution of the system $Q$ is given by $H_a(t)$ (with $H_a$ defined in (iii) above). As $a(t)$ is slowly varying we assume the adiabatic approximation to calculate the evolution. One is usually interested in the situations in which the initial state of $Q$ is given by an eigenvector of $H_a$ for the initial value of $a$. Suppose that for each $a$ we have a nondegenerate, normalized eigenvector $\psi_a$ of $H_a$:

$$H_a \psi_a = E_a \psi_a,$$

and the family $\psi_a$ depends continuously on $a$. If at $t = 0$ the state of $Q$ was given by $\psi_{a(0)}$, then at later times in the adiabatic approximation its state is equal to $\psi(t) = e^{i\varphi(t)} \psi_{a(t)}$, where $\varphi(t)$ is a real function depending functionally on $E_a$ and $\psi_a$. If an operator $B$ represents an observable, then the time-dependence of its expectation value is given by

$$\langle B \rangle_t = \langle \psi_{a(t)}, B \psi_{a(t)} \rangle,$$

so it is a function of $a$ in this approximation. It is important to note that the eigenvalues $E_a$ are modified by the addition of $\lambda_a$ under the transformation (2.6), but both the eigenvectors $\psi_a$ and the mean values $\langle B \rangle_t$ remain unchanged.

(iv) Finally, the evolution of the macroscopic variables $a(t)$ must be determined. This is the most controversial part of the problem, but we believe that the foregoing discussion indicates its proper solution. The intrinsic energy stored in the quantum part $Q$ is represented (in the Schrödinger picture) by the operator $H$ ((i) above), which in the coupled system is not a constant of motion any more. Under the assumptions of (iv)1 its expectation value is a function of $a$, depending on the choice of the continuous family of eigenvectors $\psi_a$:

$$\mathcal{E}_a := \langle \psi_a, H \psi_a \rangle,$$

and the time-dependence of this expectation value is through $a(t)$ only. Changes in $\mathcal{E}_a$ correspond to the energy which has been transferred
from $Q$ to the rest of the system, which (with the suppression of all microscopic details of $M$) is described by the variables $a$. Thus $E_a$ plays the role of a potential energy with respect to these variables. We assume that the rest of the total energy of the coupled system is supplied by the kinetic energy of $M$, thus we obtain a potential system, with the generalized force given by

$$F_a = -\frac{\partial E_a}{\partial a}.$$ (2.10)

With a specific form of the kinetic energy for a particular model the motion of $a(t)$ could be determined, and with large inertial parameters (a “heavy” system) the approximation of its slow change should be confirmed.

We have thus spelled out all the assumptions and arrived at the basic formulas (2.9), (2.10). In the following sections we shall take these formulas as a starting point. The derivation of the formulas was not rigorous, as this would demand more information on the underlying microscopic model of the closed system $Q-M$. A detailed analysis of these questions is both outside the usual discussions of Casimir effect, and also beyond the reach of a rigorous calculation at present. However, we believe that the proposed discussion offers more plausibility than most of the statements of the problem to be found in literature. In particular, points made by us in (ii) above are typically ignored; we shall see their consequences when $Q$ is an infinite-dimensional system, e.g. a quantum field. Furthermore, we want to draw a closer attention to the formula (2.9) and contrast it with what one obtains by the generalization of the “zero point” method to the more general context discussed in this section. In the latter case our formulas (2.9) and (2.10) are replaced respectively by

$$E_a^{z-p.} = E_a - E_0,$$

$$F_a^{z-p.} = -\frac{\partial E_a}{\partial a},$$ (2.11)

where $E_a$ is the eigenvalue determined by (2.7), and $E_0$ some reference eigenvalue of $H$. One can object to these formulas on several grounds.

(a) The philosophy behind them seems to be this: the backreaction of $Q$ on $M$ is due to the changes in $H_a$, which may be interpreted as the sum of intrinsic energy $H$ of $Q$ and some interaction energy. However, we think that it is $M$ which absorbs the interaction and transforms it in a phenomenological way into an effect on macroscopic variables $a$, while $Q$ has a rather clear-cut identity.

(b) The energy given by the “zero point” philosophy is not a quantum mechanical average of any clear-cut observable: with changing $a$ one changes the observable $H_a$. Moreover, as already pointed out, $H_a$ and their eigenvalues are subject to the gauge freedom (2.6). The usual argument runs that this is fixed by the quantization of the “proper” classical expression for $H_a$. 

9
We regard this argument as very unreliable. Quantum theory is the more fundamental one, so in case of doubt it should not seek a verdict from the classical theory.

(c) We put forward the following “consistency check”. Suppose that for certain values of parameters $a$ the effect of $M$ on $Q$ vanishes. In this case the backreaction force should vanish as well. The supposition means that for $a = a_0$ the vector $\psi_{a_0}$ is also an eigenvector of $H$, $H\psi_{a_0} = E\psi_{a_0}$ with some eigenvalue $E$. Using this equation one easily shows that our formulas yield

$$F_{a_0} = -E \left. \frac{\partial (\psi_a, \psi_a)}{\partial a} \right|_{a=a_0} = 0,$$

so they pass the check. On the other hand

$$F_{a_0} = -\left. \frac{\partial E_a}{\partial a} \right|_{a=a_0},$$

which, in general, has no reason to vanish.

(d) In Section 4 below we discuss an example of a Casimir-type effect in a canonical system with finite degrees of freedom. In this example the “zero point” method fails dramatically, yielding a completely unphysical result.

How, then, may “zero point” expressions arise? We shall show in Section 7 below how for quantum fields problems “zero point” expressions may be related to ours by unjustified idealizations and manipulations.

## 3 A class of quasi-free systems

We discuss in this section a general quantization scheme for a class of simple models. This class includes linear perturbations of multi-dimensional harmonic oscillators or quantum fields.

Consider first the classical case. Let $\mathcal{R}$ be a real Hilbert space, and denote its scalar product by $(\cdot, \cdot)$. Let $h$ be a selfadjoint, strictly positive (hence invertible, with densely defined inverse $h^{-1}$) operator on $\mathcal{R}$, with the domain $\mathcal{D}_R(h)$. We form the external direct sum $\mathcal{L} = \mathcal{D}_R(h) \oplus \mathcal{R} \subset \mathcal{R} \oplus \mathcal{R}$, and denote its elements by $V = v \oplus u$, $v \in \mathcal{D}_R(h)$, $u \in \mathcal{R}$. With the symplectic form $\sigma$ defined by

$$\sigma(V_1, V_2) = (v_2, u_1) - (v_1, u_2) \quad (3.1)$$

space $\mathcal{L}$ becomes the phase space of a classical model. Let the Hamiltonian function of the model be given by $H(v, u) = \frac{1}{2}\left[ (u, u) + (hv, hv) \right]$ (where all mass parameters have been absorbed by momenta). The evolution determined by this Hamiltonian in $\mathcal{L}$ is given by

$$T_t(v \oplus u) = \left( \cos(ht)v + \sin(ht)h^{-1}u \right) \oplus \left( -\sin(ht)hv + \cos(ht)u \right). \quad (3.2)$$
The differential form of this evolution is actually valid only on a subspace of \( \mathcal{L} \) (dense in \( \mathcal{R} \oplus \mathcal{R} \)), but the evolution itself is properly defined on the whole of \( \mathcal{L} \). Operators \( T_t \) form a one-parameter group of symplectic transformations

\[
T_t T_s = T_{t+s}, \quad \sigma(T_t V_1, T_t V_2) = \sigma(V_1, V_2).
\] (3.3)

Note, also, that

\[
T_{-t} = (\text{id} \oplus -\text{id}) T_t (\text{id} \oplus -\text{id}).
\] (3.4)

Each \( V' \in \mathcal{L} \) may be identified with an element of the dual space by the rule

\[
V' (V) = (v', u) + (u', v) = \sigma(V', (\text{id} \oplus -\text{id}) V).
\] (3.5)

Then using Eqs. (3.3) and (3.4) one easily shows that

\[
(T_t V')(V) = V'(T_t V).
\] (3.6)

The above model may be generalized by considering a more general subspace contained in \( \mathcal{D}_\mathcal{R}(h) \oplus \mathcal{R} \) and invariant under the evolution law (3.2). We use this freedom to choose

\[
\mathcal{L} = \mathcal{D}_\mathcal{R}(h) \oplus \mathcal{D}_R(h^{-1/2})
\] (3.7)

(the invariance under (3.2) is easily checked). The evolution law \( T_t \) may be now expressed as a unitary evolution in a complex Hilbert space. One introduces a complex Hilbert space \( \mathcal{K} \) which is the complexification of \( \mathcal{R} \), \( \mathcal{K} = \mathcal{R} \oplus i\mathcal{R} \), with scalar product (denoted by the same symbol) and complex conjugation defined by

\[
(v_1 + iu_1, v_2 + iu_2) = (v_1, v_2) + (u_1, u_2) + i(v_1, u_2) - i(u_1, v_2),
\] (3.8)

\[
\mathcal{K} \ni x \mapsto Kx \equiv \bar{x} \in \mathcal{K}, \quad K(v + iu) = v - iu.
\] (3.9)

We shall write \( v = \text{Re}(v + iu), \ u = \text{Im}(v + iu) \). The operator \( h \) has a unique extension to a complex-linear operator on \( \mathcal{K} \), denoted by the same symbol, with the domain \( \mathcal{D}(h) = \mathcal{D}_\mathcal{R}(h) \oplus i\mathcal{D}_\mathcal{R}(h) \). This new \( h \) is again a selfadjoint, positive operator, and it commutes with the conjugation. Consider now a real-line operator

\[
j : \mathcal{L} \mapsto \text{Ran} j \subset \mathcal{K}, \quad j(V) = h^{1/2} v - i h^{-1/2} u.
\] (3.10)

Its range \( \text{Ran} j \) is a real-linear subspace of \( \mathcal{K} \), dense in \( \mathcal{K} \), and \( j \) is a bijection of \( \mathcal{L} \) onto \( \text{Ran} j \). Then for all \( V \in \mathcal{L} \):

\[
j(T_t V) = e^{iht} j(V),
\] (3.11)

so \( \text{Ran} j \) is invariant under \( e^{iht} \), and the evolution may be expressed as

\[
T_t V = j^{-1}(e^{iht} j(V)).
\] (3.12)

Space \( \mathcal{K} \), regarded as a real vector space, has a natural symplectic structure introduced with the symplectic form \( \text{Im} (f, g) \). Space \( \text{Ran} j \) is its symplectic subspace. One easily shows that

\[
\sigma(V_1, V_2) = \text{Im} (j(V_1), j(V_2)),
\] (3.13)

so \( j \) is a symplectic transformation of \( \mathcal{L} \) onto \( \text{Ran} j \).
The mapping \( j \), as well known, serves to construct the ground state representation of the quantum version of the model, and the space \( \mathcal{K} \) is then the \(" \text{one-particle space} \) (see below). A natural problem thus arises: to extend the construction of the space \( \mathcal{L} \) to the largest possible space compatible with the symplectic mapping \( \mathfrak{R}_{\mathcal{L}} \), that is to extend \( \mathcal{L} \) and \( j \) so as for \( \text{Ran} \ j \) to cover the whole space \( \mathcal{K} \) (instead of being only dense in \( \mathcal{K} \), as above). One defines on \( \mathcal{D}_R(h^{\pm 1/2}) \) the scalar products

\[
\begin{align*}
(v_1, v_2)_+ &= (h^{1/2}v_1, h^{1/2}v_2), & v_1, v_2 &\in \mathcal{D}_R(h^{1/2}), \\
(u_1, u_2)_- &= (h^{-1/2}u_1, h^{-1/2}u_2), & u_1, u_2 &\in \mathcal{D}_R(h^{-1/2}),
\end{align*}
\]

and denotes by \( \mathcal{R}_+ \) and \( \mathcal{R}_- \) the Hilbert spaces obtained by the completion of \( \mathcal{D}_R(h^{1/2}) \) and \( \mathcal{D}_R(h^{-1/2}) \), respectively, with respect to the norms \( \|v\|_+ = \sqrt{(v, v)_+} \) and \( \|u\|_- = \sqrt{(u, u)_-} \). For \( v \in \mathcal{D}_R(h^{1/2}) \) and \( u \in \mathcal{D}_R(h^{-1/2}) \) we have \( \|h^{1/2}v\| = \|v\|_+ \) and \( \|h^{-1/2}u\| = \|u\|_- \). Therefore operators \( h^{1/2} \) and \( h^{-1/2} \) extend by continuity to bijective isometric operators \( h^{1/2} \) and \( h^{-1/2} \) respectively,

\[
\begin{align*}
h^{1/2} : \mathcal{R}_+ &\rightarrow \mathcal{R}, & \|h^{1/2}v\| &= \|v\|_+, \\
h^{-1/2} : \mathcal{R}_- &\rightarrow \mathcal{R}, & \|h^{-1/2}u\| &= \|u\|_-.
\end{align*}
\]

We note for future use that

\[
\mathcal{R}_+ \cap \mathcal{R}_- = \mathcal{D}_R(h^{\pm 1/2}).
\]

This is easily seen in the spectral representation of \( h \): if \( h \) is a multiplication by a positive, different from zero almost everywhere, function \( f \) in a space \( L^2(M, d\mu) \), then \( \mathcal{R} \) consists of functions \( \psi \) for which \( \int_M |\psi(m)|^2 d\mu(m) < \infty \), \( \mathcal{R}_\pm \) consists of functions for which \( \int_M (f(m))^{\pm 1} |\psi(m)|^2 d\mu(m) < \infty \), and \( \mathcal{D}_R(h^{\pm 1/2}) \) – of those satisfying both conditions.

For \( v \in \mathcal{D}_R(h^{1/2}) \) and \( u \in \mathcal{D}_R(h^{-1/2}) \) one has

\[
|\langle v, u \rangle| = |\langle h^{1/2}v, h^{-1/2}u \rangle| \leq \|v\|_+ \|u\|_-,
\]

thus \( \langle v, u \rangle \) extends to a continuous pairing

\[
\mathcal{R}_+ \times \mathcal{R}_- \ni v, u \mapsto \langle v, u \rangle \in \mathbb{R}, \quad |\langle v, u \rangle| \leq \|v\|_+ \|u\|_-.
\]

Now one can set

\[
\begin{align*}
\tilde{\mathcal{L}} &= \mathcal{R}_+ \oplus \mathcal{R}_-, & \tilde{\sigma}(V_1, V_2) &= \langle v_2, u_1 \rangle - \langle v_1, u_2 \rangle, \\
\tilde{j} : \tilde{\mathcal{L}} &\rightarrow \mathcal{K}, & \tilde{j}(V) &= h^{1/2}v - ih^{-1/2}u, \\
\tilde{T}_t V &= \tilde{j}^{-1}(e^{iht}\tilde{j}(V)).
\end{align*}
\]
As a consequence of (3.18) one has
\[ \hat{L} \cap (R \oplus R) = D_R(h^{1/2}) \oplus D_R(h^{-1/2}). \] (3.24)

It is easy to see that now \( \text{Ran} \hat{j} = \mathcal{K} \), the space given by Eq. (3.24) is dense in \( \hat{L} \) (in its Hilbert space structure norm), and the time evolution on \( \hat{L} \) is the continuous extension of the evolution on the space \( D_R \). Moreover,
\[ \hat{j}(V_1, V_2) = (v_1, v_2)_+ + \langle u_1, u_2 \rangle_- + i\hat{\sigma}(V_1, V_2), \] (3.25)
so, in particular, \( \hat{j} \) is a symplectic mapping of \( (\hat{L}, \hat{\sigma}) \) onto \( (\mathcal{K}, \text{Im}(..)) \). Relations (3.5) and (3.6) are also generalized to
\[ V'(V) = \langle v', u \rangle + \langle u', v \rangle = \hat{\sigma}(V', [\text{id} \oplus (-\text{id})] V), \] (3.26)
\[ (\hat{T}_t V')(V) = V'(\hat{T}_t V). \] (3.27)

Once we have the largest arena consistent with the scheme, particular models are defined by choosing a subspace invariant under the evolution:
\[ \mathcal{L} \subset \hat{L}, \quad \hat{T}_t \mathcal{L} \subset \mathcal{L}. \] (3.28)

The maximal model is invariant under the time reversal, represented by the operator \( \text{id} \oplus (-\text{id}) \) appearing in (3.26). We want to retain this property for the model defined by \( \mathcal{L} \), which is equivalent to the assumption
\[ \mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_-, \quad \mathcal{L}_\pm \subset \mathcal{R}_\pm. \] (3.29)

Examples of particular spaces include the class of spaces
\[ \mathcal{L} = D_R(h^{r+1}) \cap D_R(h^{-s}) \oplus D_R(h^{t}) \cap D_R(h^{-t-\frac{1}{2}}), \]
\[ r, s, t \in (0, \infty), \quad s \leq t + \frac{1}{2}, \quad t \leq s + \frac{3}{2}, \] (3.30)
all of which are contained in (3.24).

The quantum version of the maximal model (with the symplectic space \( \hat{L} \)) is now obtained by standard procedure (see e.g. [9], vol. II). Starting with expression (3.20) one aims at replacing \( V' \) by some “quantum variable” \( \Phi \). In quantum theory a concrete representation of a quantum variable is an operator in a Hilbert space. If the classical variable is real, its quantum counterpart should be represented by a selfadjoint operator. Thus one assumes that a Hilbert space \( \mathcal{H} \) is given, and for each \( V \in \mathcal{L} \) one has a selfadjoint operator \( \Phi(V) \) in that space. The functional dependence of \( \Phi(V) \) on \( V \) is assumed to be linear, and the canonical commutation relations are imposed:
\[ [\Phi(V_1), \Phi(V_2)] = i\hat{\sigma}(V_1, V_2) \text{id}, \] (3.31)
where one still has to clarify the domain problems. If $V = v \oplus u$, then we shall also write $\Phi(V) = \Phi(v, u)$. The element $P(v) \equiv \Phi(v, 0)$ has the interpretation of the quantum momentum for the “test vector” $v$, and $X(u) \equiv \Phi(0, u)$ – of the quantum position variable for the “test vector” $u$. With the linearity of $\Phi(V)$ the above commutation relations are equivalent to those in a more familiar form

$$[X(u_1), X(u_2)] = 0, \quad [P(v_1), P(v_2)] = 0, \quad [P(v), X(u)] = -i(v, u) \text{id}. \quad (3.32)$$

It is well-known that there are many different concrete representations of the above scheme, and this is why it is desirable to formulate the canonical commutation relations in an algebraic way. As there are no bounded operators satisfying these relations, it is usual to take them in an exponentiated variant. This leads to the Weyl form of these relations. The Weyl algebra over the symplectic space $\hat{\mathcal{L}}$ is the unique $C^*$-algebra generated by elements $W(V)$, $V \in \hat{\mathcal{L}}$, and a unit element $1$

by the relations

$$W(V_1)W(V_2) = e^{-\frac{i}{2} \overrightarrow{\sigma}(V_1, V_2)} W(V_1 + V_2), \quad W(V)^{*} = W(-V), \quad W(0) = 1. \quad (3.33)$$

One looks for representations of this algebra by bounded operators in a Hilbert space (which exist for all $C^*$-algebras). Let $\pi$ be such a representation in the Hilbert space $\mathcal{H}$. One says that this representation is regular, if for each $V \in \hat{\mathcal{L}}$ the one-parameter group of unitary operators $R \ni s \mapsto \pi(W(sV)) \quad (3.34)$
is strongly continuous. If this is the case, then there exist, by Stone’s theorem (e.g. [14]), selfadjoint operators $\Phi(V)$ such that

$$\pi(W(V)) = \exp(i\Phi(V)). \quad (3.35)$$

Moreover, one shows that for each finite-dimensional subspace $\mathcal{L}' \subset \hat{\mathcal{L}}$ there exists a dense subspace $\mathcal{D}' \subset \mathcal{H}$ which is contained in the domains of all operators $\Phi(V)$, $V \in \mathcal{L}'$, is an invariant subspace and an essential domain of selfadjointness for all of them, and on which linearity of $\Phi(V)$ in its argument $V \in \mathcal{L}'$ and commutation relations $\Phi(V)$ are satisfied (this follows from the Stone–von Neumann uniqueness theorem, cf. [9], vol.II). While not all canonical systems with these properties arise in this way from regular representations of the corresponding Weyl algebra, most of those needed in physics do, and one usually restricts attention to this class.

The algebra $\mathcal{A}$ of the maximal model is thus the Weyl algebra over $\hat{\mathcal{L}}$. Dynamics of the model is a “quasi-free” evolution obtained by a simple “quantization” of the classical evolution $\hat{T}_t$. Being guided by the replacement $V' \to \Phi$ in Eq. (3.27) and the relation $\Phi(V)$, one defines it on the algebraic level by

$$\alpha_t(W(V)) = W(\hat{T}_t V). \quad (3.36)$$
One looks now for representations $\pi$ of the algebra in which this evolution law may be implemented:

$$\pi(W(\hat{T}_tV)) = U(t) \pi(W(V)) U(t)^* , \quad U(t) = e^{itH}, \quad (3.37)$$

where $H$ is a selfadjoint operator. The ground state representation is obtained if $H$ is a nonnegative operator with zero energy ground state. This representation is constructed in standard way with the use of the Fock space method. Let $W_0(f) = \exp[i\Phi_0(f)], f \in \mathcal{K}$, be the Weyl system of operators in the Fock space $\mathcal{H}$ built on the “one-particle” space $\mathcal{K}$ (see Appendix A.1). We set

$$\pi(W(V)) := W_0(\hat{j}(V)), \quad (3.38)$$

or, which is equivalent,

$$\pi(W(V)) = e^{i\Phi(V)}, \quad \Phi(V) = \Phi_0(\hat{j}(V)). \quad (3.39)$$

Using identity (3.25) and properties of the operators $W_0(f)$ one easily shows that this indeed constitutes a representation of the Weyl algebra (3.33). By the irreducibility of the Weyl system in Fock space this representation is irreducible. Moreover, using Eqs. (3.38) and (3.23) one rewrites the condition (3.37) as

$$W_0(e^{ith}f) = e^{itH}W_0(f)e^{-itH}, \quad f \in \mathcal{H}. \quad (3.40)$$

The discussion of Appendix A.1 shows now that

$$H = d\Gamma(h), \quad (3.41)$$

where $d\Gamma(h)$ is the “second quantization” of $h$ (see Eqs. A.19 – A.21). This energy operator has nonnegative spectrum, and a unique ground state represented by the “Fock vacuum” $\Omega$.

Consider now a restriction of this model defined by a subspace $\mathcal{L}$ invariant under evolution (Eq. (3.28)) and time reflection (Eq. (3.29)). The algebra of the model is the subalgebra of the Weyl algebra (3.33) obtained by restricting the test vectors to $\mathcal{L}$. It is well-known, that the resulting model is not identical with the maximal one if $\mathcal{L} \neq \hat{\mathcal{L}}$ (see [9], vol. II), but one can demand that its ground state representation approximates that of the maximal model. This representation may be constructed as before, but the scope of Weyl operators used in this representation is restricted to $\{W_0(f) \mid f \in \hat{j}(\mathcal{L})\}$ – cf. Eq. (3.38). This set is irreducible in $\mathcal{K}$ if, and only if, the space $\hat{j}(\mathcal{L})$ is dense in $\mathcal{K}$, or, what is the same, $\mathcal{L}$ is dense in $\mathcal{R}_+ \oplus \mathcal{R}_-$. With the time reflection symmetry assumption (3.29) this takes the form

$$\mathcal{L}_+ \text{ is dense in } \mathcal{R}_+, \quad \mathcal{L}_- \text{ is dense in } \mathcal{R}_-. \quad (3.42)$$

We restrict attention to those spaces $\mathcal{L}$ which satisfy this condition. This restriction can be paraphrased by saying that there are no superselection rules in the Fock space of the ground state representation.
Examples

(i) Multidimensional harmonic oscillator

In this case $\mathcal{R}$ is a finite-dimensional Euclidean space, and $h$ is a positive self-adjoint operator defined on the whole of $\mathcal{R}$. We choose $\mathcal{L} = \mathcal{R} \oplus \mathcal{R}$. Space $\mathcal{K}$ is the unitary space obtained by complexification of $\mathcal{R}$, and $\text{Ran } j = \mathcal{K}$. The more familiar simple form of the model is obtained by choosing in $\mathcal{R}$ an orthonormal basis $(e_1, \ldots, e_n)$ of eigenvectors of $h$ and putting $X_i = \Phi(0, e_i)$, $P_i = \Phi(e_i, 0)$. The system is then the set of $n$ independent harmonic oscillators with canonical variables $\{X_i, P_i\}$, unit masses, and frequencies $\omega_i$, where $he_i = \omega_i e_i$.

(ii) Free scalar field

Free quantum fields are usually defined as operator-valued distributions on test functions of all spacetime variables. The evolution equation (Klein-Gordon for the scalar field) is already encoded in this formulation, which is manifestly relativistically covariant. For our purposes the equivalent initial value formulation is preferable – we want to separate evolution law, as far as it is possible, from setting up of the algebra.

Standard identifications for this model are as follows:

$$\mathcal{R} = \mathcal{L}^2(\mathbb{R}^3), \quad \mathcal{K} = \mathcal{L}^2(\mathbb{R}^3),$$

$$h = \sqrt{-\Delta},$$

$$\mathcal{L} = \mathcal{D}(\mathbb{R}^3) \oplus \mathcal{D}(\mathbb{R}^3),$$

where subscript $\mathbb{R}$ denotes the real part of the respective function space, $\mathcal{D}(\mathbb{R}^3)$ is the space of infinitely differentiable complex functions of compact support, and $\Delta$ is the Laplace operator. Standard solution of the initial value problem for the Klein-Gordon equation has the form of Eq. (3.2), and the assumptions (3.28), (3.29) and (3.42) are satisfied.

(iii) Scalar field with external time-independent interaction

Loosely speaking, the choice of $h$ here is the square root of a selfadjoint positive operator of heuristic form “$h^2 = -\Delta + \text{interaction}$”. There are a few possibilities.

(iii) If the interaction is given by an external field $\sigma = \sigma(\vec{x})$ then the choice of spaces $\mathcal{R}$ and $\mathcal{K}$ remains the same as in the free case, while $h^2 = -\Delta + \sigma$ (we assume that $h^2$ is still positive – there are no bound states). Depending on the form of $\sigma$ the choice of $\mathcal{L}$ as in the free case may be admissible (satisfy assumptions (3.28) and (3.42)) or not. A safe choice for $\mathcal{L}$ is supplied by any of the cases given by Eq. (3.30). More generally, $h^2$ may be any positive selfadjoint perturbation of $-\Delta$ in the sense of operators or forms.
The next possibility arises from restricting the region accessible to the field to a proper subset $\Lambda \subset \mathbb{R}^3$. In this case $\mathcal{R} = L^2_\mathbb{R}(\Lambda)$, $\mathcal{K} = L^2(\Lambda)$, and $h^2 = -\Delta_B$, where $\Delta_B$ is a selfadjoint extension of the Laplace operator defined on twice differentiable functions with support inside $\Lambda$, determined by some boundary conditions "$B". Here, of course, the free field choice of $\mathcal{L}$ is not admissible, and a safe choice is again given by the formula (3.30).

Finally, we consider a setting usually assumed for the Casimir effect. The whole physical space $\mathbb{R}^3$ is divided by two-dimensional surfaces into disjoint open regions $\Lambda_1, \ldots, \Lambda_s$. Position of the dividing boundaries is characterized by a set of parameters $a$. One chooses $\mathcal{R}$ and $\mathcal{K}$ as in the free field case. Depending on parameters $a$, a family of positive operators $h_a$ is given by $h^2_a = -\Delta_a$, where $\Delta_a$ is the Laplace operator in $L^2(\mathbb{R}^3)$ determined by the assumed boundary conditions (Dirichlet, Neumann, etc.) at the dividing surfaces with positions given by parameters $a$. In consequence, the choice of the symplectic space $\mathcal{L}$ must be adjusted to the position of the boundaries. A choice of simple possibilities is again given by Eq. (3.30) (with $h_a$ replacing $h$). In the Casimir problem one wants to compare states of the system at different values of $a$. However, spaces $\mathcal{L}$ depend nontrivially on $a$, thus the respective Weyl algebras are also different, and do not define the same quantum system. This constitutes the difficulty of traditional treatments of the Casimir effect which we anticipated in Section 2. Infinities naturally appear then, and are a consequence of an uncritical use of the notion of a quantum field.

In the following sections we discuss Casimir effects for some systems in the category described in the present section. We start with a simple finite-dimensional case.

4 Deformation of a finite-dimensional harmonic oscillator

Our unperturbed quantum system $Q$ is here an $n$-dimensional quantum oscillator described in example (i) of the last section. Thus the algebra of the model is the Weyl algebra based on a finite-dimensional symplectic space $\mathcal{L} = \mathcal{R} \oplus \mathcal{R}$, and its representation is given by $\pi(W(V)) = W_0(j(V))$ in the Fock space $\mathcal{H}$ based on the finite-dimensional one-excitation space $\mathcal{K}$. The energy operator of the model is given by $H = d\Gamma(h)$.

We consider now a combined system $Q - M$, as described in Section 2 and assume that the influence of $M$ on $Q$ for frozen parameters $a$ manifests itself in the change of axes and frequencies of oscillations. Thus the time evolution of the algebra for frozen parameters is given by

$$\alpha_{at}(W(V)) = W(T_{at}),$$

(4.1)
where \( T_a \) has the form (3.2), but with operator \( h \) replaced by an operator \( h_a \) from a family \( \{ h_a \} \). For each \( a \) the irreducible representation of the algebra in which this evolution is implemented by a unitary one-parameter group, with a nonnegative energy operator, is constructed by the same method, as in the free \( Q \) case, in the same Fock space \( \mathcal{H} \). Thus

\[
 j_a(V) = h_a^{1/2} v - i h_a^{-1/2} u, \quad \pi_a(W(V)) = W_a(j_a(V)).
\]  

(4.2)

The Hamilton operator is given by \( d\Gamma(h_a) \), with ground state described by the Fock vacuum \( \Omega \). However, as discussed in Section 2, we want to describe the same physical situation with the use of the representation \( \pi_a \). Therefore for each \( a \) we look for a unitary operator \( U_a \) which by similarity transforms representation \( \pi_a \) onto \( \pi \):

\[
 U_a \pi_a(W(V)) U_a^* = \pi(W(V)), \quad U_a \Phi_a(V) U_a^* = \Phi(V), \quad V \in \mathcal{L}.
\]  

(4.3)

We substitute here the definitions of the representations \( \pi \) and \( \pi_a \), and set \( j_a(V) \equiv f \). This condition then takes the form

\[
 U_a W_a(f) U_a^* = W_0(L_a f), \quad f \in \mathcal{K}, \quad \text{where} \quad L_a := j j_a^{-1}.
\]  

(4.4)

Both \( j \) and \( j_a \) are bijective symplectic mappings, so \( L_a \) is a symplectic transformation of the space \( (\mathcal{K}, \text{Im}(\ldots)) \), and the above condition states that \( U_a \) implements the corresponding Bogoliubov transformation in the Fock space (see Appendix A). As \( \mathcal{K} \) is finite-dimensional, such \( U_a \) exists. The explicit form of transformation \( L_a \) is easily obtained:

\[
 L_a f = h^{1/2} h_a^{-1/2} \text{Re} f + i h^{-1/2} h_a^{1/2} \text{Im} f,
\]  

(4.5)

and then from (A.20) one finds \( L_a = T_a + S_a, \) \( T_a \) complex-linear and \( S_a \) complex-antilinear,

\[
 T_a = \frac{1}{2} (B_a^{-1} + B_a^*), \quad S_a = \frac{1}{2} (B_a^{-1} - B_a^*) K.
\]  

(4.6)

where

\[
 B_a = h_a^{1/2} h_a^{-1/2}, \quad K f = \bar{f}.
\]  

(4.7)

In the representation \( \pi \) the Hamiltonians of the new evolutions are given by

\[
 H_a = U_a d\Gamma(h_a) U_a^*,
\]  

(4.8)

and the ground state of \( H_a \) is given by

\[
 \Omega_a = U_a \Omega.
\]  

(4.9)

Suppose now, that \( a \) is a single real parameter, and under the influence of the external conditions the state of the subsystem \( Q \) changes adiabatically over
the states $\Omega_a$, as discussed in Section 2. The potential for the backreaction force is therefore, in accordance with point (iv) in Section 2, determined by

$$E_a = (\Omega_a, H \Omega_a). \tag{4.10}$$

We take into account that $H = d\Gamma(h)$ and use the expression for a form matrix element of $d\Gamma(h)$ as given in Eqs. (A.24), (A.24):

$$E_a = \sum_i \|a(h^{1/2} f_i)\Omega_a\|^2,$$

where $\{f_i\}$ is an arbitrary orthonormal basis of $\mathcal{K}$. We use Eqs. (A.40), (A.42) and (A.45) to find

$$\|a(h^{1/2} f_i)\Omega_a\|^2 = \|a^*_a (S_a^* h^{1/2} f_i)\Omega_a\|^2 = (f_i, h^{1/2} S_a S_a^* h^{1/2} f_i).$$

Thus we obtain

$$E_a = \text{Tr} \left[ h^{1/2} S_a S_a^* h^{1/2} \right] = \frac{1}{4} \text{Tr} \left[ (h_a - h) h_a^{-1} (h_a - h) \right]. \tag{4.11}$$

Let the eigenvalues and orthonormal eigenvectors of $h$ and $h_a$ be given by

$$he_i = \epsilon_i e_i, \quad h_a e_{ai} = \epsilon_{ai} e_{ai}. \tag{4.12}$$

Then using the spectral representation $h_a^{-1} = \sum_k \epsilon_{ak}^{-1} e_{ak} \langle e_{ak} |$ and employing the basis $e_i$ for the calculation of the trace we find

$$E_a = \sum_{i,k} \frac{(\epsilon_{ak} - \epsilon_i)^2}{4 \epsilon_{ak}} |(e_{ak}, e_i)|^2. \tag{4.13}$$

We consider a simple example. Let $Q$ be a two-dimensional oscillator in physical space, vectors $e_1, e_2$ being its main axes, and let the effect of the external conditions be the rotation of these axes by an angle $\varphi (\equiv a)$, without a change in the frequencies. In this case we have $\epsilon_{\varphi k} = \epsilon_k$, $(\epsilon_{\varphi 1}, \epsilon_2) = -(\epsilon_{\varphi 2}, \epsilon_1) = \sin \varphi$, so

$$E_\varphi = \frac{(\epsilon_2 - \epsilon_1)^2}{4} (\epsilon_1^{-1} + \epsilon_2^{-1}) \sin^2 \varphi. \tag{4.14}$$

The backreaction “force” in this case is a torque

$$F_\varphi = -\frac{(\epsilon_2 - \epsilon_1)^2}{4} (\epsilon_1^{-1} + \epsilon_2^{-1}) \sin 2\varphi. \tag{4.15}$$

For $\epsilon_1 \to 0$ (with $\epsilon_2$ kept constant) the torque tends to infinity. This is what one should expect. This limiting case describes the situation in which the harmonic force in the direction of $e_2$ extends translationally invariant in the direction of $e_1$; any rotation of this picture involves an “infinite” change.

Note, that the “zero point” prescription for the force gives zero in the above example, which is an utterly unphysical prediction.
5  An infinite-dimensional system

Let now $\mathcal{R}$ be an infinite dimensional real Hilbert space. We want to consider a situation analogous to that discussed in the last section: system $Q$ defined by the operator $h$, and its perturbations by a family of operators $h_a$. We need to take into account complications arising from the unboundedness of the operators, as explained in Section 3.

We want to be able to define for our model both evolutions (that determined by $h$ and by $h_a$), and both ground state representations. Thus the model has to fit into structures defined in Section 3 both by $h$ as well as $h_a$. In particular, its symplectic space should in a canonical way be a part of both $\hat{\mathcal{L}}$ and $\hat{\mathcal{L}}_a$.

However, the construction of these spaces is based on different, in general, parts of $\mathcal{R} \oplus \mathcal{R}$, and without some restrictions there is no canonical way of identification of their parts. We assume that $D_{\pm} \equiv D_{\mathcal{R}}(h_{\pm}^{1/2}) \cap D_{\mathcal{R}}(h_a^{1/2})$ is dense in $\mathcal{R}_\pm$ and in $\mathcal{R}_{a,\pm}$, (5.1)

Suppose now that the space of a model contains at least a subspace $\mathcal{L}^0$ such that

$$\mathcal{L}^0 = \mathcal{L}^0_+ \oplus \mathcal{L}^0_-, \quad \mathcal{L}^0_+ \subset D_+, \quad \mathcal{L}^0_\pm \text{ is dense in } \mathcal{R}_\pm \text{ and in } \mathcal{R}_{a,\pm},$$

which is a strengthening of the condition (3.42). Note that then $\tilde{j}(\mathcal{L}^0) = j(\mathcal{L}^0)$, $\tilde{j}_a(\mathcal{L}^0) = j_a(\mathcal{L}^0)$, and both spaces are dense in $\mathcal{K}$. Under these assumptions we show that the following three conditions are equivalent:

(i) The symplectic mapping

$$L_a := jj_a^{-1} : j_a(\mathcal{L}^0) \rightarrow j(\mathcal{L}^0)$$

extends to a bounded operator in $\mathcal{K}$, with a bounded inverse.

(ii) The operators $h$ and $h_a$ satisfy the conditions

$$D_{\mathcal{R}}(h_{\pm}^{1/2}) = D_{\mathcal{R}}(h_a^{1/2});$$

$$B_a \equiv h_a^{1/2}h^{-1/2} \text{ and } B^{-1}_a \text{ extend to bounded operators in } \mathcal{K}.$$  

(iii) There exists a selfadjoint, positive, bounded operator $C_a$ in $\mathcal{R}$, with bounded inverse, and such that

$$h_a = h^{1/2}C_a h^{1/2}$$

in the sense of forms, that is: $h_a$ is the unique selfadjoint operator defined by the closed form $q(v_1, v_2) = (h^{1/2}v_1, C_a h^{1/2}v_2)$ with the form domain $Q(q) = D_{\mathcal{R}}(h^{1/2})$.  

20
If these conditions are satisfied, then \( \hat{\mathcal{L}}_a = \hat{\mathcal{L}} \), so both evolutions are well defined in \( \hat{\mathcal{L}} \).

Note that (i) is a necessary condition for the ground state representations defined by \( h \) and \( h_a \) to be equivalent. The equivalence implies that Eq. (4.5) is satisfied in particular for all \( V \in \mathcal{L}^0 \), or, equivalently, Eq. (4.4) for all \( f \in j_\alpha(\mathcal{L}^0) \). But then, as shown in the Appendix A.3, \( L_a \) extends to a bounded symplectic mapping in \( K \), with a bounded inverse.

Let (i) be satisfied. Then on \( \mathcal{L}^0 \) from Eq. (5.6) we have \( L_a j_a = j \) and \( j_a = L_a^{-1} j \), which implies that for \( w_\pm \in \mathcal{L}^0_\pm \) one has

\[
\| h^{\pm 1/2} w_\pm \| \leq \text{const.} \| h_a^{\pm 1/2} w_\pm \|, \quad \| h_a^{\pm 1/2} w_\pm \| \leq \text{const.} \| h^{\pm 1/2} w_\pm \|. \tag{5.7}
\]

This means that the norms \( \| . \| \) and \( \| . \|_{a\pm} \) are equivalent on \( \mathcal{L}^0_\pm \), so they yield the same completion, on which these inequalities are preserved. But \( \mathcal{L}^0_\pm \) is dense in \( \mathcal{R}_\pm \) and \( \mathcal{R}_{a\pm} \), so \( \mathcal{R}_\pm = \mathcal{R}_{a\pm} \) as sets, hence also \( \hat{\mathcal{L}} = \hat{\mathcal{L}}_a \) as sets, with equivalent norms. Moreover, from Eq. (4.14) we find

\[
\mathcal{D}_\mathcal{R}(h_a^{\pm 1/2}) = \mathcal{R}_{a\pm} \cap \mathcal{R}_\pm = \mathcal{R}_{a\pm} \cap \mathcal{R} = \mathcal{D}_\mathcal{R}(h^{\pm 1/2}).
\]

The boundedness of \( B_a \) and \( B_a^{-1} \) follows now from (5.7), which ends the proof of (ii). Conversely, if (ii) is satisfied, then one easily shows that the formula \( L_a f = B_a^{-1} \text{Re} f + i B_a^* \text{Im} f \) gives the extension needed in (i) (cf. Eq. (4.5)).

The equivalence of (ii) and (iii) follows by polar decomposition of closed operators (e.g., [14]). If we assume (ii), then \( h_a^{1/2} = B_a h^{1/2} \) with the domain \( \mathcal{D}_\mathcal{R}(h^{1/2}) \) is a selfadjoint operator, so \( h_a = h^{1/2} B_a^* B_a h^{1/2} \), and \( C_a = B_a^* B_a \) fulfills the conditions of (iii). Conversely, let \( C_a \) be bounded, positive selfadjoint, with a bounded inverse. Then \( C_a^{1/2} h^{1/2} \) with the domain equal to \( \mathcal{D}_\mathcal{R}(h^{1/2}) \) is a closed operator. Indeed, let \( v_n \in \mathcal{D}_\mathcal{R}(h^{1/2}) \), \( \| v_n - v_m \| \to 0 \) and \( \| C_a^{1/2} h^{1/2} (v_n - v_m) \| \to 0 \). But \( C_a^{1/2} \) is bounded, so also \( \| h^{1/2} (v_n - v_m) \| \to 0 \). As \( h^{1/2} \) is closed, there exists \( v \in \mathcal{D}_\mathcal{R}(h^{1/2}) \) such that \( \| v_n - v \| \to 0 \), \( \| h^{1/2} (v_n - v) \| \to 0 \), and by boundedness of \( C_a^{1/2} \) also \( \| C_a^{1/2} h^{1/2} (v_n - v) \| \to 0 \), which shows that \( C_a^{1/2} h^{1/2} \) is indeed closed. Thus the form \( q \) defined in (iii) is closed, and the condition (5.6) means that \( |C_a^{1/2} h^{1/2}| = h^{1/2}_a \). It follows that there exists an orthogonal operator \( F_a \) such that \( C_a^{1/2} h^{1/2} = F_a h_a^{1/2} \), so \( h_a^{1/2} h^{-1/2} \) extends to a bounded operator \( B_a = F_a^{-1} C_a^{1/2} \). As \( C_a \) has a bounded inverse, so the same is true for \( B_a^{-1} \), which ends the proof of equivalence of (i) – (iii).

These preliminary results show that if the two ground state representations are to be equivalent in our model, we have to assume that (5.6), and then all conditions (i) – (iii), are true. Then the space \( \hat{\mathcal{L}} \) is invariant under both evolutions and forms the widest possible space of the model. Any subspace (if it exists) \( \mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_- \subset \hat{\mathcal{L}} \) which is also invariant under both evolutions and dense in \( \hat{\mathcal{L}} \) can also be taken as the symplectic space of the model.
With these assumptions the symplectic mapping $L_a$ decomposes into the bounded complex-linear and complex-antilinear parts, $L_a = T_a + S_a$. The application of the results described in Appendix A.3 shows that the necessary and sufficient condition for the unitary equivalence of the ground state representations is that $S_a$ is a Hilbert-Schmidt (HS) operator, that is

$$N_a = \text{Tr} \left[ S_a S_a^* \right] < \infty.$$  \hspace{1cm} (5.8)

Going through the steps (4.10 – 4.11) in the present infinite-dimensional context we see that $E_a$ is finite if, and only if, $S_a^* h^{1/2}$ extends to a HS operator, and then

$$E_a = \text{Tr} \left[ h^{1/2} S_a S_a^* h^{1/2} \right].$$  \hspace{1cm} (5.9)

The quantity $N_a$ appearing in (5.8) has a clear-cut physical meaning. The results of the Appendix A.3 show that if (5.8) is satisfied, then $\Omega_a \in \mathcal{D}(N)$, where $N$ is the “particle” (excitation) number operator. A calculation analogous to that carried out for the energy yields

$$(\Omega_a, N \Omega_a) = N_a,$$  \hspace{1cm} (5.10)

so $N_a$ is the mean value of the excitation number in the ground state.

In the rest of this section we obtain the following criterion for admissible perturbations. Let $h_a$ be given by (5.6). The ground state representations are unitarily equivalent ($N_a < \infty$) if, and only if,

$$C_a = \text{id} + \delta_a,$$  \hspace{1cm} (5.11)

where $\delta_a$ is any operator satisfying conditions

$$\delta_a \text{ is a HS operator, } \text{id} + \delta_a > 0.$$  \hspace{1cm} (5.12)

In this case we can write in the sense of forms

$$h_a = h + h^{1/2} \delta_a h^{1/2}.$$  \hspace{1cm} (5.13)

Moreover, if conditions (5.12) are satisfied, then $E_a$ is finite if, and only if,

$$\delta_a h^{1/2} \text{ extends to a HS operator}.$$  \hspace{1cm} (5.14)

With the condition (5.12) satisfied one has

$$N_a = \frac{1}{4} \text{Tr} \left[ \delta_a^2 \right],$$  \hspace{1cm} (5.15)

and if in addition (5.14) holds, then

$$E_a = \frac{1}{4} \text{Tr} \left[ h^{1/2} \frac{\delta_a^2}{\text{id} + \delta_a} h^{1/2} \right].$$  \hspace{1cm} (5.16)
To prove these assertions note first that equations (4.6) remain in force with our assumptions in the present infinite-dimensional context, and then

\[ S_a S_a^* = \frac{1}{4} \frac{(C_a - \text{id})^2}{C_a}. \quad (5.17) \]

If the ground state representations are equivalent, then \( S_a^* \) is a HS operator, so \( C_a^{-1/2} (C_a - \text{id}) \) is HS as well. But \( C_a^{1/2} \) is bounded, therefore also \( \delta_a = C_a - \text{id} \) is HS. The second condition in (5.12) is satisfied by the positivity of \( C_a \). Conversely, suppose that \( \delta_a \) satisfies conditions (5.12). By the first of these conditions \( \delta_a \) has a purely discrete spectrum with no other convergence points than zero, and then by the second \( C_a = \text{id} + \delta_a \geq 0 \), with \( b > 0 \). Hence \( C_a^{-1} \) is bounded, and \( C_a^{-1/2} \delta_a \) is bounded, therefore also \( \delta_a = \frac{1}{4} \text{Tr}(\delta_a^2 (\text{id} + \delta_a)^{-1}) < \infty \).

If the conditions (5.12) are satisfied, then in a completely analogous way one proves the equivalence of the condition (5.14) with the finiteness of \( E_a \), and the equation (5.16).

6 Energy density of quantum field

In this section we consider the case of a quantum field, and for definiteness we take the scalar field (massive or massless) with standard commutation relations and free evolution. Thus here \( \mathcal{R} = L^2(\mathbb{R}^3) \), \( h = \sqrt{m^2 \text{id} - \Delta} \), \( (m \geq 0) \), and we take for the test function space the largest space \( \hat{L} \), as described in the previous section. Perturbations \( h_a \) are assumed to be in the class defined by Eqs. (5.11 - 5.14) (in fact, a slight strengthening of these conditions will be needed).

We show that in this setting the energy density in the ground states \( \Omega_a \) is well defined as a tempered distribution, and for the test function tending to unit function one recovers the energy expectation value \( E_a \).

We assume a slight strengthening of our assumptions and demand that for some \( \alpha \in (0, 1) \) there is:

\[ h^{(1+\alpha)/4} \delta_a h^{(1+\alpha)/4} \text{ is a HS operator}. \quad (6.1) \]

Note that this statement with \( \alpha = 0 \) is a consequence of our earlier assumptions. Indeed, if \( \delta_a \) and \( \delta_a h^{1/2} \) are HS, then

\[
0 \leq \text{Tr}(h^{1/4} \delta_a h^{1/4})^2 = \lim_{n \to \infty} \text{Tr} \left[ P_{(0,n)}(h)(h^{1/4} \delta_a h^{1/4})^2 P_{(0,n)}(h) \right] \\
= \lim_{n \to \infty} \text{Tr} \left[ \delta_a h^{1/2} \delta_a h^{1/2} P_{(0,n)}(h) \right] = \text{Tr}(\delta_a h^{1/2})^2 < \infty ,
\]

where \( \{P_F(h)\}, F \) a Borel set in \( \mathbb{R} \), is the spectral family of \( h \).

Loosely speaking, the energy density operator of the scalar field is determined by the point-splitting procedure and normal ordering with respect to the vacuum.
where

\[ H(\vec{x}) = \lim_{\vec{g} \to \vec{x}} :H_2(\vec{x}, \vec{y}):: \equiv \lim_{\vec{g} \to \vec{x}} \left[ H_2(\vec{x}, \vec{y}) - (\Omega, H_2(\vec{x}, \vec{y})\Omega) \right], \]  

(6.2)

\[ H_2(\vec{x}, \vec{y}) = \frac{1}{2} \left( P(\vec{x}) P(\vec{y}) + \nabla \cdot \nabla X(\vec{y}) \right), \]  

(6.3)

and to get \( X(\vec{x}) \) and \( P(\vec{x}) \) one sets formally \( v \) and \( u \) equal to Dirac delta concentrated at \( \vec{x} \). We are interested in the energy density \( (\Omega_a, H(\vec{x})\Omega_a) \) in the ground state \( \Omega_a \).

We now make this precise. The real Schwartz test function space \( \mathcal{S}_R \) is contained in \( \mathcal{D}_R(h^{1/2}) \cap \mathcal{D}_R(h^{-1/2}) \), so functions from that space may be used for “smearing” both \( X(\vec{x}) \) as \( P(\vec{x}) \). Let \( w_1, w_2 \in \mathcal{S}_R \). The precise meaning of (6.3) is

\[ H_2(w_1, w_2) = \frac{1}{2} (\Phi(0, u) \Phi(w_2, 0) + \Phi(0, \nabla w_1) \cdot \Phi(0, \nabla w_2) + m^2 \Phi(0, w_1) \Phi(0, w_2)). \]  

(6.4)

To find normal-ordered expectation value \( (\Omega_a, H_2(w_1, w_2)) \) we need to know \( (\Omega_a, \Phi(V_1) \Phi(V_2)) \), where we assume that \( V_1 \in \mathcal{S}_R \oplus \mathcal{S}_R \). We recall the definitions of the representations \( \pi \) and \( \pi_a \) and their equivalence relations:

\[ \Phi(V) = \Phi_0(j(V)), \quad \Phi_a(V) = \Phi_0(j_a(V)), \quad \Omega_a = U_a \Omega, \quad U_a \Phi_a(V) U_a^* = \Phi(V) \]  

(Eqs. (6.39), (6.42), (6.40) and (6.43) respectively). Using them one finds

\[ (\Omega, \Phi(V_1) \Phi(V_2) \Omega) = \frac{1}{2} (j(V_1), j(V_2)) \]

(6.5)

\[ = \frac{1}{2} (h^{1/2} v_1, h^{1/2} v_2) + \frac{1}{2} (h^{-1/2} u_1, h^{-1/2} u_2) + \frac{i}{2} \sigma(V_1, V_2), \]

\[ (\Omega_a, \Phi(V_1) \Phi(V_2) \Omega_a) = (\Omega, \Phi_a(V_1) \Phi_a(V_2) \Omega) = \frac{1}{2} (j_a(V_1), j_a(V_2)) \]

(6.6)

\[ = \frac{1}{2} (h_a^{1/2} v_1, h_a^{1/2} v_2) + \frac{1}{2} (h_a^{-1/2} u_1, h_a^{-1/2} u_2) + \frac{i}{2} \sigma(V_1, V_2), \]

\[ (\Omega_a, \Phi(V_1) \Phi(V_2) : \Omega_a) = \frac{1}{2} (h^{1/2} v_1, \delta_a h^{1/2} v_2) - \frac{1}{2} \left( h^{-1/2} u_1, \frac{\delta^a_2}{\delta_1 + \delta} h^{-1/2} u_2 \right), \]

(6.7)

so

\[ (\Omega_a, H_2(w_1, w_2) : \Omega_a) \equiv T_a(w_1, w_2) = T_{a_1}(w_1, w_2) + T_{a_2}(w_1, w_2), \]

(6.8)

where

\[ T_{a_1}(w_1, w_2) = \frac{1}{4} \left( h^{1/2} w_1, \frac{\delta^2_a}{\delta_1 + \delta} h^{1/2} w_2 \right), \]

(6.9)
\[ T_{a2}(w_1, w_2) = \frac{1}{4} \left( h^{1/2} \frac{\delta_a}{\text{id} + \delta_a} h^{1/2} w_2 \right) - \frac{m^2}{4} \left( h^{-1/2} \frac{\delta_a}{\text{id} + \delta_a} h^{-1/2} w_2 \right) - \frac{1}{4} \left( h^{-1/2} \frac{\delta_a}{\text{id} + \delta_a} h^{-1/2} \overrightarrow{w_1} \right) \ \text{.} \]  

(6.10)

We have added the conjugation sign over \( w_1 \) on the r.h. side to make the expression linear rather than antilinear also for complex functions. The \( T_{a1} \) part and the first term in \( T_{a2} \) result from splitting \( \delta_a = \delta_a^2 \text{id} + \delta_a \).  

(6.11)

We show that:

(i) \( T_a(w_1, w_2) \) defines a distribution \( T_a(\vec{x}, \vec{y}) \) on \( S(\mathbb{R}^6) \).

(ii) For each \( \vec{\eta} \in \mathbb{R}^3 \) the expression \( T_a(\vec{\xi} + \vec{\eta}, \vec{\xi} - \vec{\eta}) \) is a distribution on \( S(\mathbb{R}^3) \), and for each test function \( f \) the function

\[ \vec{\eta} \to E_a(\vec{\eta}, f) = \int T_a(\vec{\xi} + \vec{\eta}, \vec{\xi} - \vec{\eta}) f(\vec{\xi}) \, d^3 \xi \]  

is continuous and bounded (we use the “integral” notation of distributions). The energy density according to point-splitting procedure is then the distribution

\[ E_a(f) = E_a(\vec{\eta}, f) \]  

(6.13)

(iii) Let \( f_\varepsilon(\vec{\xi}) = f(\varepsilon \vec{\xi}), f(\vec{0}) = 1, f \in S(\mathbb{R}) \). Then

\[ \lim_{\varepsilon \to 0} E_a(f_\varepsilon) = E_a \]  

(6.14)

Before starting the proof we fix conventions for the Fourier transforms. For \( a, b \in \mathbb{R}^n \) we set

\[ \hat{f}(b) = (2\pi)^{-n/2} \int f(a) e^{-ib \cdot a} d^3 a, \ \ \ \ \hat{f}(a) = \hat{f}(-a) \]  

(6.15)

We consider the \( T_{a1} \) and \( T_{a2} \) parts separately. Expressions (6.12) and (6.13) for \( T_{a1} \) replacing \( T_a \) will be denoted \( E_{a1}(\vec{\eta}, f) \) and \( E_{a1}(f) \) respectively.

As \( \frac{1}{2} \delta_a (\text{id} + \delta_a)^{-1/2} h^{1/2} \) is a HS operator in \( L^2(\mathbb{R}^3) \), it is an integral operator with a kernel \( k_a(\vec{x}, \vec{y}) \in L^2(\mathbb{R}^6) \) (see e.g. [14]). Thus \( T_{a1} \) is obviously a distribution on \( S(\mathbb{R}^6) \), determined by the ordinary function

\[ T_{a1}(\vec{x}, \vec{y}) = \int k_a(\vec{z}, \vec{x}) k_a(\vec{z}, \vec{y}) \, d^3 z \]  

(6.16)
As for each $\tilde{\eta}$ there is $k_a(\tilde{z}, \tilde{\xi} + \tilde{\eta})k_a(\tilde{z}, \tilde{\xi} - \tilde{\eta}) \in L^1(\mathbb{R}^3, d^3 z d^3 \xi)$, the distribution $\mathcal{E}_{a1}(\tilde{\eta}, f)$ is indeed well defined,

$$\mathcal{E}_{a1}(\tilde{\eta}, f) = \int \overline{k_a(\tilde{z}, \tilde{\xi} + \tilde{\eta})}f(\tilde{\xi})k_a(\tilde{z}, \tilde{\xi} - \tilde{\eta}) d^3 z d^3 \xi. \quad (6.17)$$

Now, Fourier-transforming $k_a(\tilde{z}, \tilde{\xi} + \tilde{\eta})$ and $f(\tilde{\xi})k_a(\tilde{z}, \tilde{\xi} - \tilde{\eta})$ with respect to $\tilde{z}$ and $\tilde{\xi}$ one finds

$$\mathcal{E}_{a1}(\tilde{\eta}, f) = \frac{1}{(2\pi)^{3/2}} \int \overline{k_a(\tilde{r}, \tilde{p})}k_a(\tilde{r}, \tilde{q})\hat{f}(\tilde{p} - \tilde{q})e^{-i\tilde{\eta} \cdot (\tilde{p} + \tilde{q})} d^3 r d^3 p d^3 q, \quad (6.18)$$

as the integrand on the r.h. side is absolutely integrable. Therefore $\mathcal{E}_{a1}(\tilde{\eta}, f)$ is continuous in $\tilde{\eta}$. For $\tilde{\eta} = 0$ we get

$$\mathcal{E}_{a1}(f) = \int |k_a(\tilde{z}, \tilde{\xi})|^2 \hat{f}(\tilde{\xi}) d^3 z d^3 \xi. \quad (6.19)$$

As the function $|k_a(\tilde{z}, \tilde{\xi})|^2$ is absolutely integrable, we see immediately that for $f$ as in (iii) there is

$$\lim_{\tilde{\eta} \to 0} \mathcal{E}_{a1}(f_x) = \int |k_a(\tilde{z}, \tilde{\xi})|^2 d^3 z d^3 \xi = \frac{1}{4} \text{Tr} \left[ h^{1/2} \frac{\delta_a^2}{id + \delta_a} h^{1/2} \right] = \mathcal{E}_a. \quad (6.20)$$

We now turn to $T_{a2}$ and take into account our assumption (6.11). Using the identity (6.11) and the fact that $\delta_a$ and $\delta_a h^{1/2}$ are HS, one finds that an equivalent formulation of the assumption is that $h^{(1+\alpha)/4}\delta_a(id + \delta_a)^{-1}h^{(1+\alpha)/4}$ is a HS operator; we denote its kernel in the momentum space by $l_a(-\tilde{p}, \tilde{q})$. We evaluate $T_{a2}(w_1, w_2)$ in momentum space, making use of the identity $\hat{w}_1(\tilde{p}) = \hat{w}_1(-\tilde{p})$:

$$T_{a2}(w_1, w_2) = \frac{1}{4} \int l_a(\tilde{p}, \tilde{q})t(\tilde{p}, \tilde{q})\hat{w}_1(\tilde{p})\hat{w}_2(\tilde{q}) d^3 p d^3 q, \quad (6.21)$$

where

$$t(\tilde{p}, \tilde{q}) = \left[(\tilde{p}^2 + m^2)(\tilde{q}^2 + m^2)\right]^{(1-\alpha)/2}\left(1 + \frac{\tilde{p} \cdot \tilde{q} - m^2}{\sqrt{\tilde{p}^2 + m^2}(\tilde{q}^2 + m^2)}\right). \quad (6.22)$$

As $l_a$ is square integrable, and $t$ polynomially bounded, $T_{a2}$ defines a distribution $T_{a2}(\tilde{s}, \tilde{g})$. Let $f, g \in \mathcal{S}(\mathbb{R}^3)$. Then

$$\int T_{a2}(\tilde{\xi} + \tilde{\eta}, \tilde{\xi} - \tilde{\eta})f(\tilde{\xi})g(\tilde{\eta}) d^3 \xi d^3 \eta$$

$$= 2 \int l_a(\tilde{r} + \tilde{s}, \tilde{r} - \tilde{s}) t(\tilde{r} + \tilde{s}, \tilde{r} - \tilde{s}) \hat{f}(2\tilde{r})\hat{g}(2\tilde{s}) d^3 r d^3 s, \quad (6.23)$$

26
where on the l.h. side the integral notation is symbolic, but on the r.h. side this is the ordinary integration. Now, one shows the following estimate

\[ t(\vec{r} + \vec{s}, \vec{r} - \vec{s}) \leq \frac{4|\vec{r}|^2}{(|\vec{r}|^2 + |\vec{s}|^2 + m^2)^{(3+\alpha)/4}}, \]  

(6.24)

(note that \( t(\vec{p}, \vec{q}) \geq 0 \)). To prove this it is convenient to consider the cases \(|\vec{r}|^2 \geq |\vec{s}|^2 + m^2\) and \(|\vec{r}|^2 < |\vec{s}|^2 + m^2\) separately. In the first region one then uses the obvious bound \( t(\vec{p}, \vec{q}) \leq 2(\vec{p}^2 + m^2)(\vec{q}^2 + m^2)^{(1-\alpha)/8}, \) while in the second one finds that for the given \(|\vec{r}|\) and \(|\vec{s}|\) the function on the l.h. side is the biggest for \( \vec{r} \cdot \vec{s} = 0. \) Using the bound one easily shows that

\[ \int [t(\vec{r} + \vec{s}, \vec{r} - \vec{s})]^2 d^3s \leq \text{const.} |\vec{r}|^{4-\alpha}. \]  

(6.25)

Therefore \( t(\vec{r} + \vec{s}, \vec{r} - \vec{s}) \hat{f}(2\vec{r}) \in L^2(\mathbb{R}^6), \) so

\[ l_a(\vec{r} + \vec{s}, \vec{r} - \vec{s}) t(\vec{r} + \vec{s}, \vec{r} - \vec{s}) \hat{f}(2\vec{r}) \in L^1(\mathbb{R}^6). \]  

(6.26)

Using this fact in (6.23) one finds that

\[ \mathcal{E}_{a2}(\vec{f}, f) = \int l_a(\vec{r} + \vec{s}, \vec{r} - \vec{s}) t(\vec{r} + \vec{s}, \vec{r} - \vec{s}) \hat{f}(2\vec{r}) e^{-i2\vec{\eta} \cdot \vec{s}} d^3r d^3s. \]  

(6.27)

indeed defines a distribution and is continuous in \( \vec{\eta}. \) Thus

\[ \mathcal{E}_{a2}(f) = \frac{1}{\sqrt{2\pi^{3/2}}} \int l_a(\vec{r} + \vec{s}, \vec{r} - \vec{s}) t(\vec{r} + \vec{s}, \vec{r} - \vec{s}) \hat{f}(2\vec{r}) e^{-i2\vec{\eta} \cdot \vec{s}} d^3r d^3s. \]  

(6.28)

Using square-integrability of \( l_a \) and the estimate (6.25) we have

\[ |\mathcal{E}_{a2}(f)|^2 \leq \text{const.} \int |\hat{f}(\vec{r})|^2 |\vec{r}|^{4-\alpha} d^3r. \]  

(6.29)

For \( f \) as in (iii) one easily then finds

\[ \lim_{\epsilon \to 0} \mathcal{E}_{a2}(f_\epsilon) = 0, \]  

(6.30)

which ends the proof of our claims.

In our calculation of the energy density we have used the standard definition of the Wick normal ordering. As an aside, it may be of interest to mention that this definition has been recently improved for the cases where the reference state \( \Omega \) depends on external fields (as e.g. in a fixed curved classical spacetime). The problem with the usual definition in such cases is, that the scalar subtraction
function depends nonlocally on the background. This may be remedied, as it turns out, by an additional subtraction of a smooth function (in “Hadamard states”; see the papers by Hollands and Wald [15], and Brunetti, Fredenhagen and Verch [16]; for an application to external field electrodynamics see also [17]). This has no immediate bearing on the discussion in the present work, but may have applications in related problems with external fields present from the start (as in a curved spacetime).

7 Remarks on relations with some other approaches

In this section we make some remarks on the relation of our approach to other calculations of Casimir energy. We shall discuss a few characteristic examples of local calculations, and next comment on the “zero point” ideology.

First, to make our point on local Casimir energy, we need to consider a general situation briefly sketched in the Introduction, where two representations of local algebras in some open region $M_0$ in spacetime are locally quasiequivalent. Suppose we have two representations $\pi$ and $\tilde{\pi}$ of the algebras of observables in $M_0$, acting in Hilbert spaces $H$ and $\tilde{H}$ respectively. We assume that the representations are locally quasiequivalent, but say nothing on their (global) equivalence. This is the expected state of affairs in many situations typically considered for Casimir problems. For instance, for a scalar field $M_0$ may be the whole spacetime outside some 2-surfaces in 3-space, $\pi$ the vacuum representation of the field, and $\tilde{\pi}$ the representation built on the ground state of the field in presence of the boundary conditions imposed on the boundaries of $M_0$. We choose a state in the representation $\tilde{\pi}$, that is a density operator $\tilde{\rho}$ in $\tilde{H}$. If the representations are not equivalent it makes no sense to ask for a state in the representation $\pi$ which gives the same expectation values as $\tilde{\rho}$ for all observables. However, the local quasiequivalence tells us that if we restrict attention to an open subset $O$ with a compact closure contained in $M_0$, then there exists a density operator $\rho_O$ in $H$ such that

$$\text{Tr}[\tilde{\rho} \tilde{\pi}(A)] = \text{Tr}[\rho_O \pi(A)] \quad \text{for} \quad A \in O.$$ (7.1)

The local energy density is not one of the fundamental local observables $A$, but it may be locally built with the use of them. In the sequel we restrict attention to the scalar field and construct local energy density as in (6.2). Thus given a state $\tilde{\rho}$ the Casimir energy density in $O$ according to the views we follow in this paper is

$$\tilde{E}(\vec{x}) = \text{Tr}[\rho_O H(\vec{x})]$$ (7.2)

(expectation value of a fixed, free field energy density operator). Let $\tilde{\rho}$ be, for simplicity, the projection operator onto the unit vector $\tilde{\Omega}$. Then using (6.2) and (7.1)
we can write for the Casimir energy at a given time $t = 0$:

$$
\tilde{\mathcal{E}}(\vec{x}) = \frac{1}{2} \lim_{\vec{x}' \to \vec{x}} \left\{ \left( \tilde{\Omega}, \left[ \tilde{P}(\vec{x}) \tilde{P}(\vec{x}') + \vec{\nabla} \tilde{X}(\vec{x}) \cdot \vec{\nabla} \tilde{X}(\vec{x}') + m^2 \tilde{X}(\vec{x}) \tilde{X}(\vec{x}') \right] \tilde{\Omega} \right) - \left( \Omega, \left[ P(\vec{x}) P(\vec{x}') + \vec{\nabla} X(\vec{x}) \cdot \vec{\nabla} X(\vec{x}') + m^2 X(\vec{x}) X(\vec{x}') \right] \Omega \right) \right\},
$$

(7.3)

where $X(u) = \Phi(0, u), P(v) = \Phi(v, 0), \tilde{X}(u) = \tilde{\Phi}(0, u), \tilde{P}(v) = \tilde{\Phi}(v, 0)$, and $\Phi$ and $\tilde{\Phi}$ are operators representing the field under $\pi$ and $\tilde{\pi}$ respectively. Recall that in the case discussed in Section 6 there is $\mathcal{H} = \mathcal{H}, \Omega = \Omega, \Phi = \Phi_a$, and one recovers the formula obtained at the beginning of that section. More generally, let in each of the representations a different time evolution be given by unitary operators: free evolution $U(t)$ and evolution influenced by background $\tilde{U}(t)$ respectively, and denote

$$
X_t(\vec{x}) = U(t) X(\vec{x}) U(t)^*, \quad P_t(\vec{x}) = U(t) P(\vec{x}) U(t)^*, \quad \tilde{X}_t(\vec{x}) = U(t) \tilde{X}(\vec{x}) U(t)^*, \quad \tilde{P}_t(\vec{x}) = U(t) \tilde{P}(\vec{x}) U(t)^*.
$$

(7.4)

If for both evolutions there is $P_t(\vec{x}) = \partial \varphi(t, \vec{x})/\partial t, \tilde{P}_t(\vec{x}) = \partial \tilde{\varphi}(t, \vec{x})/\partial t$, then one can write the last formula for $\tilde{\mathcal{E}}(\vec{x})$ at $t = 0$ as

$$
\tilde{\mathcal{E}}(\vec{x}) = \frac{1}{2} \lim_{t, t' \to 0} \left\{ \partial_t \partial_{t'} + \vec{\nabla} \cdot \vec{\nabla}' + m^2 \right\} \left\{ \left( \tilde{\Omega}, \tilde{\varphi}(t, \vec{x}) \tilde{\varphi}(t', \vec{x}') \tilde{\Omega} \right) - \left( \Omega, \varphi(t, \vec{x}) \varphi(t', \vec{x}') \Omega \right) \right\}.
$$

(7.5)

This formula was derived along similar lines by Kay in the context of the free field in a locally flat spacetime with nontrivial topology. There are no boundaries in that case, but the net of local algebras of observables in this spacetime differs from that in the globally flat Minkowski spacetime, so the notion of a global Casimir energy in the sense we use here has no application.

In the context of electromagnetic field bounded by conductors in Minkowski space the opinion similar to ours, that one should compare expectation values of the fixed free field energy density, was expressed by Scharf and Wreszinski. Consider a massless scalar field analogy of the setting. Then $M_0$ is the spacetime region outside boundaries. Put $m = 0$ in the last formula, use the translational symmetry of the two-point functions (which enables the replacement $\vec{\nabla}' \to -\vec{\nabla}$), and the wave equation, which both correlation functions satisfy outside the boundaries. This leaves us with

$$
\tilde{\mathcal{E}}(\vec{x}) = \frac{1}{2} \lim_{t, t' \to 0} \left\{ \partial_t \partial_{t'} + \partial_t^2 \right\} \left\{ \left( \tilde{\Omega}, \tilde{\varphi}(t, \vec{x}) \tilde{\varphi}(t', \vec{x}') \tilde{\Omega} \right) - \left( \Omega, \varphi(t, \vec{x}) \varphi(t', \vec{x}') \Omega \right) \right\},
$$

(7.6)

which is the formula used in [13]. No global energy density may be obtained in this way (if not by an ad hoc regularization of the infinities in the density) due to the algebraic problems explained earlier.
Next, we want to comment on the “Green function” method. In papers following this method one usually states that the (local) Casimir energy is the difference between the energy “in the vacuum state with the barriers present and with them absent” (see e.g. [10]), with no further explanation on what energy is meant. Staying with the scalar field as our example, one then uses with not much comment a formula similar to (7.5), in which, however, the products of fields are replaced by time-ordered products. This brings no change of the result in this simple case, but in general has to be justified. As long as outside the barriers the field follows the same local equation (the distinct time evolutions agree locally), the ambiguity as to what energy is meant does not show up. However, this does not matter only because for sharp boundaries one cannot determine the global energy anyway. And in fact, if the barriers are replaced by external fields one has to make it clear what is being calculated. An example of such calculation is attempted in [2], where one of the sections treats on the quantum Dirac field in an external classical electromagnetic field. The authors’ intention apparently is to compare the energy of the Dirac field itself, so they keep the free field energy expression. However, they take over the form of this expression containing time derivatives, and to eliminate them they use different field equations in the two cases, which spoils the original intention (remember that the Dirac equation is first order, so the time derivative of the field is not an independent initial value variable).

Another example of the external field calculations is to be found in [11]. Here the authors with the intention of finding the global Casimir energy explicitly compare expectation values of two different energy operators: energy of the field with the interaction terms included and the energy as given by the free field theory, in the ground states of the two respective evolutions. In our opinion this is one of the reasons for the appearance of infinities in their expressions, which are eliminated by adding “counterterms” to the model which does not need them (except for trivial normal ordering of quadratic observables). We note, moreover, that it does not follow from the smoothness of the external field alone that the ground state representations, with the external field present or not, are equivalent globally.

In the rest of this section we try to understand, from the point of view of the formalism presented in this paper, how “zero point” expressions may arise in the context of Casimir effect for quantum fields. In our opinion their appearance is a consequence of unjustified manipulations. Accordingly, the equations and transformations to be found below are not to be taken at face value. We indicate this by putting a dot over the equality sign.

The “zero point” expression for Casimir energy has the form
\[
E^{z.p.}_a = \frac{1}{2} \sum_k \omega_{ak} - \frac{1}{2} \sum_k \omega_k ,
\]
(7.7)

where \(\omega_k\) and \(\omega_{ak}\) are appropriately discretized frequencies of free and perturbed field respectively. In our language this would be
\[
E^{z.p.}_a = \frac{1}{2} \text{Tr}(h_a - h) ,
\]
(7.8)
which usually is meaningless, but is then “regularized” to squeeze a finite result. We show how this expression may arise.

We have shown in the previous section that $E_a$ is a limit of the energy density distribution value for the test function tending to one, see Eq. (6.14). Also, it turned out that in this limit only the part $E_a^1(f)$, Eq. (6.19), of the density distribution contributes. Thus we can use Eq. (6.20) to calculate the total energy. Distribution $E_a^1(f)$ is determined by part $T_a^1$, Eq. (6.9), of $(\Omega_a, H_2(w_1, w_2) : \Omega_a)$. If we do not pay due attention to domains we can rewrite Eq. (6.9) by expressing it in terms of $h$ and $h_a$ instead of $h$ and $\delta_a$. The result is

$$T_a^1(w_1, w_2) = \frac{1}{4} \int \langle x | h_a - h + h(h_a^{-1} - h^{-1})h|y \rangle w_1(x)w_2(y) d^3x d^3y, \quad (7.9)$$

which implies

$$E_a = \frac{1}{4} \int \langle x | h_a - h + h(h_a^{-1} - h^{-1})h|x \rangle d^3x$$

$$= \frac{1}{4} \text{Tr} \left[ h_a - h + h(h_a^{-1} - h^{-1})h \right]. \quad (7.10)$$

Let us now, again ignoring difficulties, apply this to the case of sharp boundaries, where $h_a^2 = -\Delta_{B(a)}$ with appropriate boundary conditions $B(a)$. Suppose that the support of $w_1$ and $w_2$ stays outside the boundaries. Then $h_a^2 w_i = h^2 w_i$, and we have

$$(w_1, (h_a - h)w_2) = \frac{1}{2} (w_1, (h_a^{-1} - h^{-1})h^2 w_2) + \frac{1}{2} (w_1, h^2(h_a^{-1} - h^{-1})w_2), \quad (7.11)$$

(written in two terms only for symmetry reasons), or, for $\bar{x}$ and $\bar{y}$ outside the boundary,

$$\langle \bar{x} | h_a - h | \bar{y} \rangle = \frac{1}{2} \langle \bar{x} | (h_a^{-1} - h^{-1})h^2 + h^2(h_a^{-1} - h^{-1}) | \bar{y} \rangle. \quad (7.12)$$

This needs regularization on the boundaries. Assuming some form of it one writes

$$\text{Tr}(h_a - h) = \frac{1}{2} \int \langle \bar{x} | (h_a^{-1} - h^{-1})h^2 + h^2(h_a^{-1} - h^{-1}) | \bar{x} \rangle d^3x$$

$$= \frac{1}{2} \text{Tr} \left[ (h_a^{-1} - h^{-1})h^2 + h^2(h_a^{-1} - h^{-1}) \right]. \quad (7.13)$$

Suppose that the regularization used cuts high momenta, so as to allow one to change the order of operators under the trace sign. Then

$$\text{Tr}(h_a - h) \doteq \text{Tr} \left[ h(h_a^{-1} - h^{-1})h \right]. \quad (7.14)$$

Using this in (7.13) one arrives at (7.8).
A Appendix. Fock space operators and Bogoliubov transformations

In the appendix we give a brief review of some known properties of Fock space operators which are needed in the main text. The main sources of reference for Section A.1 are books [9] (vol. II) and [14]. The content of Sections A.2 and A.3 is a rather common knowledge. Precise original proofs of the criterions of equivalence of representation use rather more advanced and less common techniques [18], so we think a simple proof with the use of creation/annihilation operators is worth presenting in A.4 (The results on equivalence have been later generalized, in the widest form in [19]).

A.1 Weyl system in a Fock space

Let \( \mathcal{H} \) be the symmetric Fock space based on the “one-particle (excitation) space” \( \mathcal{K} \), i.e.

\[
\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n, \quad \mathcal{H}_0 = \mathbb{C}, \quad \mathcal{H}_n = S(\mathcal{K} \otimes \cdots \otimes \mathcal{K}) \quad (n \geq 1),
\]

where \( S \) is the symmetrization projection operator. The scalar product in \( \mathcal{H} \) will be denoted by \( (.,.) \), the “Fock vacuum” vector by \( \Omega \), and the particle (excitation) number operator by \( N \). On the domain \( \mathcal{D}(N^{1/2}) \) the annihilation and creation operators are defined in the usual way: for each \( f \in \mathcal{K} \) and \( \psi, \chi \in \mathcal{D}(N^{1/2}) \) one sets

\[
a^\ast(f)\psi = S(f \otimes \sqrt{N+1}\psi), \quad (\chi, a(f)\psi) = (a^\ast(f)\chi, \psi),
\]

and shows that

\[
\|a^\#(f)\psi\| \leq \|f\| \|(N+1)^{1/2}\psi\|, \quad a^\#(f) = a(f) \text{ or } a^\ast(f),
\]

and for \( \varphi \in \mathcal{D}(N) \)

\[
[a(f), a^\ast(g)]\varphi = (f, g)\varphi.
\]

Operators \( a(f) \) and \( a^\ast(f) \) are respectively antilinear and linear in \( f \).

Let \( \mathcal{H}_f \) be the finite-excitation subspace (dense in \( \mathcal{H} \)), i.e.

\[
\mathcal{H}_f = \bigcup_{k=0}^{\infty} \bigoplus_{n=0}^{k} \mathcal{H}_n.
\]

Operators \( \Phi_0(f) \) are defined in the following way. One initially sets

\[
\Phi_0(f)\psi = \frac{1}{\sqrt{2}}(a(f) + a^\ast(f))\psi \quad \text{for} \quad \psi \in \mathcal{D}(N^{1/2}).
\]
Using the bounds (A.3) one shows that these operators are essentially selfadjoint on $H_f$, so their closures $\Phi_0(f)$ are selfadjoint. For $\psi \in \mathcal{D}(N^{1/2})$, $\varphi \in \mathcal{D}(N)$, $f, g, f_k \in \mathcal{K}$ and real $\alpha, \beta$ one has

$$\Phi_0(\alpha f + \beta g)\psi = \alpha \Phi_0(f)\psi + \beta \Phi_0(g)\psi,$$

(A.7)

$$a(f)\psi = \frac{1}{\sqrt{2}}(\Phi_0(f) + i\Phi_0(if))\psi, \quad a^*(f)\psi = \frac{1}{\sqrt{2}}(\Phi_0(f) - i\Phi_0(if))\psi,$$

(A.8)

if $\|f_k - f\| \to 0$ then $\|\Phi_0(f_k)\psi - \Phi_0(f)\psi\| \to 0 \quad (k \to \infty)$,

(A.9)

$$[\Phi_0(f), \Phi_0(g)]\varphi = i \text{Im}(f, g)\varphi.$$

(A.10)

Using these relations one shows that the Weyl operators defined by

$$W_0(f) = e^{i\Phi_0(f)}$$

(A.11)

have the following properties

$$W_0(f)W_0(g) = e^{-\frac{i}{4}\text{Im}(f, g)}W_0(f + g), \quad W_0(f)^* = W_0(-f), \quad W_0(0) = \text{id};$$

(A.12)

the set $\{W_0(f) \mid f \in \mathcal{K}\}$ is irreducible;

(A.13)

$$\langle \Omega, W_0(f)\Omega \rangle = e^{-\frac{1}{4}\|f\|^2};$$

(A.14)

if $\|f_k - f\| \to 0$ then $\|W_0(f_k)\psi - W_0(f)\psi\| \to 0 \quad (k \to \infty), \quad \psi \in H.$

(A.15)

Let $U$ be a unitary operator in $\mathcal{K}$. One defines a unitary operator $\Gamma(U)$ in $\mathcal{H}$ by

$$\Gamma(U)W_0(f)\Omega = W_0(Uf)\Omega,$$

(A.16)

which implies

$$\Gamma(U)W_0(f)\Gamma(U)^* = W_0(Uf).$$

(A.17)

It is then easy to show that

$$\Gamma(U) : H_n \to H_n, \quad \Gamma(U)\big|_{H_n} = U \otimes \ldots \otimes U.$$  

(A.18)

Let now $h$ be a selfadjoint operator in $\mathcal{K}$. Then $\Gamma(e^{ith})$ is a one-parameter group of unitary operators. The generator of this group, denoted $d\Gamma(h)$, is a selfadjoint operator,

$$\Gamma(e^{ith}) = \exp(itd\Gamma(h)).$$

(A.19)

Let $\mathcal{D}_h$ be any domain of essential selfadjointness of $h$ and denote

$$\mathcal{D}_{d\Gamma(h)} = \bigcup_{k=0}^{\infty} \bigoplus_{n=0}^{k} \mathcal{S}(\mathcal{D}_h \otimes \ldots \otimes \mathcal{D}_h),$$

(A.20)

which means that $\mathcal{D}_{d\Gamma(h)}$ is formed by finite linear combinations of symmetrized products of vectors from $\mathcal{D}_h$. One shows that

$$d\Gamma(h)\big|_{\mathcal{S}(\mathcal{D}_h \otimes \ldots \otimes \mathcal{D}_h)} = h \otimes \text{id} \otimes \ldots \otimes \text{id} + \ldots + \text{id} \otimes \ldots \otimes \text{id} \otimes h,$$

(A.21)

33
and that \(d\Gamma(h)\) is essentially selfadjoint on \(D_{d\Gamma(h)}\). We assume now that \(h\) is a nonnegative operator. Then \(d\Gamma(h)\) is also nonnegative and has the following representation in terms of quadratic forms. As \((a(f))^*\) is densely defined (its domain contains \(D(N^{1/2})\)), the annihilation operator \(a(f)\) is closable, we denote its closure by \(\bar{a}(f)\). Let \(\{f_i\}\) be any orthonormal basis of \(\mathcal{K}\) formed of vectors in \(D(h^{1/2})\), and denote

\[
Q(q) = \{\psi \in \mathcal{H} | \psi \in \bigcap_i D(\bar{a}(h^{1/2}f_i)) \text{ and } \sum_i \|\bar{a}(h^{1/2}f_i)\psi\|^2 < \infty\}. \quad (A.22)
\]

One shows that the following form on \(Q(q)\) is closed

\[
q(\psi, \chi) = \sum_i (\bar{a}(h^{1/2}f_i)\psi, \bar{a}(h^{1/2}f_i)\chi). \quad (A.23)
\]

It is easy to check by direct calculation that the restriction of this form to \(D_{d\Gamma(h)}\) gives

\[
q(\psi, \chi) = (\psi, d\Gamma(h)\chi). \quad (A.24)
\]

As \(D_{d\Gamma(h)}\) is a core of \(d\Gamma(h)\), the unique selfadjoint operator defined by the form \(q\) is identical with \(d\Gamma(h)\). Thus

\[
Q(q) = D(d\Gamma(h)^{1/2}) \quad \text{and} \quad q(\psi, \chi) = (d\Gamma(h)^{1/2}\psi, d\Gamma(h)^{1/2}\chi). \quad (A.25)
\]

In particular, for all \(\psi, \chi \in D(d\Gamma(h))\) identity (A.24) holds.

We note that the particle number operator may be represented as a special case of this construction,

\[
N = d\Gamma(id). \quad (A.26)
\]

### A.2 Symplectic transformations of \((\mathcal{K}, \text{Im}(\ldots))\)

Hilbert space \(\mathcal{K}\), as a real vector space, is a symplectic space with the form \(\text{Im}(\ldots)\). Its real-linear, bijective transformation \(L\) is a symplectic transformation if for all \(f, g \in \mathcal{K}\)

\[
\text{Im}(Lf, Lg) = \text{Im}(f, g). \quad (A.27)
\]

The inverse transformation is then also a symplectic transformation satisfying the same condition. Substituting \(f \to L^{-1}f\) in (A.27) one has

\[
\text{Im}(f, Lg) = \text{Im}(L^{-1}f, g). \quad (A.28)
\]

One defines operators on \(\mathcal{K}\):

\[
T = \frac{1}{2}(L - iL), \quad S = \frac{1}{2}(L + iL), \quad L = T + S, \quad (A.29)
\]

\[
T' = \frac{1}{2}(L^{-1} - iL^{-1}i), \quad S' = \frac{1}{2}(L^{-1} + iL^{-1}i), \quad L^{-1} = T' + S'. \quad (A.30)
\]

Operators \(T\) and \(T'\) are complex-linear, while \(S\) and \(S'\) are complex-antilinear. Using their definitions and the relation (A.28) it is easy to show that operators in
the two pairs $T', T$ and $S', -S$ are mutually adjoint, so $T = T^{**}$ and $S = S^{**}$. Thus both operators are everywhere defined and closed, so they are bounded. Separating the identities $L^{-1}L = \text{id}$ and $LL^{-1} = \text{id}$ into linear and antilinear parts one gets

$$T^*T = S^*S + \text{id}, \quad T^*S = S^*T,$$

(A.31)  

$$TT^* = SS^* + \text{id}, \quad TS^* = ST^*.$$

(A.32)

Conversely, if the operators $T$ and $S$ satisfy the above relations on the whole Hilbert space $\mathcal{K}$, they are bounded and define a symplectic transformation $L = T + S$. Furthermore, if the relations are satisfied, then $T$ is a bijection of $\mathcal{K}$ onto $\mathcal{K}$. Thus if $T = U_T|T|$ is its unique polar decomposition, then $U_T$ is a unitary operator. We set $S = U_T^-R$. It follows then from the first equalities in (A.31) and |(A.32)| that $R^*R = RR^*$. If $R = K|S|$ is the unique polar decomposition of $R$, then this condition is equivalent to $K|S| = |S|K$, so $K$ is a partial antiisometry of $(\text{Ker}|S|)^\perp$ onto itself. From the first relation in (A.31) $|T|^2 = \text{id} + |S|^2$, so $K|T| = |T|K$ as well. The second relation in (A.31) then gives $K^* = K$.

We summarize the results:

$$T = U_T|T|^{1/2}, \quad S = U_T|S|K, \quad |S|, K = 0,$$

(A.33)  

$$|S| \text{ is bounded}, \quad U_T \text{ is unitary}, \quad K \text{ is a conjugation on } (\text{Ker}|S|)^\perp.$$

(A.34)

Conversely, if these conditions are satisfied, then $T$ and $S$ satisfy conditions (A.31) and (A.32), and determine a symplectic transformation by

$$L = T + S, \quad L^{-1} = T^* - S^*.$$

(A.35)

If $|S|$ has no continuous spectrum, then it follows from the above relations that its orthonormal basis of eigenvectors may be chosen such that

$$|S|f_i = \lambda_i f_i, \quad Kf_i = f_i.$$

(A.36)

### A.3 Bogoliubov transformations in a Fock space

With the notation of the foregoing subsections let $L = T + S$ be a symplectic transformation of the space $(\mathcal{K}, \text{Im}(., .))$, and let us denote

$$W_0L(f) = W_0(Lf).$$

(A.37)

It is easy to show that these new operators also satisfy the Weyl relations (A.12). The transformation $W_0(f) \mapsto W_0L(f)$ is called a Bogoliubov transformation. Its equivalent form is

$$\Phi_0(f) \mapsto \Phi_0L(f) = \Phi_0(Lf), \quad W_0L(f) = e^{i\Phi_0L(f)}.$$

(A.38)

For $\psi \in D(\Phi_0(Lf)) \cap D(\Phi_0(Lif))$ one defines

$$a_L(f)\psi = \frac{1}{\sqrt{2}}(\Phi_0L(f) + i\Phi_0L(if))\psi, \quad a^*_L(f)\psi = \frac{1}{\sqrt{2}}(\Phi_0L(f) - i\Phi_0L(if))\psi,$$

(A.39)
and shows by a simple calculation that for $\psi \in \mathcal{D}(N^{1/2})$:

$$a_L(f)\psi = a(Tf)\psi + a^*(Sf)\psi, \quad a^*_L(f)\psi = a^*(Tf)\psi + a(Sf)\psi. \quad (A.40)$$

Then using Eqs. (A.31) one also finds for $\psi \in \mathcal{D}(N^{1/2})$

$$a(f)\psi = a_L(T^*f)\psi - a^*_L(S^*f)\psi, \quad a^*(f)\psi = a^*_L(T^*f)\psi - a_L(S^*f)\psi, \quad (A.41)$$

and for $\varphi \in \mathcal{D}(N)$

$$[a_L(f), a^*_L(g)]\varphi = (f, g)\varphi. \quad (A.42)$$

One says that the Bogoliubov transformation is implementable in $\mathcal{H}$ if there exists a unitary operator $U_L$ such that either of the following (and then both) conditions hold

$$W_0(f) = U_L W_0(f) U_L^*, \quad \Phi_0(f) = U_L \Phi_0(f) U_L^*, \quad f \in \mathcal{K}. \quad (A.43)$$

The necessary and sufficient condition for the implementability of the Bogoliubov transformation is that $S$ be a Hilbert-Schmidt operator, i.e.

$$\text{Tr} \left[ S^*S \right] < \infty. \quad (A.44)$$

If the condition is satisfied, then there exists a unique, up to a phase factor, normalized vector $\Omega_L$ satisfying the conditions

$$a_L(f)\Omega_L = 0, \quad f \in \mathcal{K}. \quad (A.45)$$

Moreover, one has

$$\Omega_L \in \bigcap_{l=1}^\infty \mathcal{D}(N^{1/2}). \quad (A.46)$$

Equations

$$U_L a^*(f_1) \ldots a^*(f_k) \Omega = a^*_L(f_1) \ldots a^*_L(f_k) \Omega_L, \quad k = 0, 1, \ldots, \quad (A.47)$$

with arbitrary test vectors $f_i$, define the unique (up to a phase factor) unitary operator $U_L$ implementing the Bogoliubov transformation. For the completeness we sketch a simple proof of these statements in the next subsection.

A slight generalization of the above results is needed in the main text. Let $\mathcal{J}$ and $\mathcal{J}'$ be real subspaces of $\mathcal{K}$, dense in $\mathcal{K}$, and let $L : \mathcal{J} \mapsto \mathcal{J}'$ be a bijective symplectic transformation (i.e. a real-linear transformation satisfying Eq. (A.27) for $f \in \mathcal{J}$). Suppose that there exists a unitary operator $U_L$ such that

$$W_0(Lf) = U_L W_0(f) U_L^*, \quad f \in \mathcal{J}. \quad (A.48)$$

Then $L$ and $L^{-1}$ extend to bounded symplectic transformations on $\mathcal{K}$, and Eq. (A.43) is satisfied for all $f \in \mathcal{K}$. 

36
Indeed, suppose that (A.48) is fulfilled. Then using Eq. (A.14) one finds
\[ e^{-\frac{1}{4} ||Lf||^2} = (U^*_L \Omega, W_0(f)U^*_L \Omega), \quad f \in \mathcal{J}, \]  
which shows that \( L \) is a continuous transformation on its domain (if \( f_n \to 0 \), then by Eq. (A.15) also \( Lf_n \to 0 \)). Thus \( L \) extends by continuity to a bounded operator on \( \mathcal{K} \). From Eq. (A.48) we have 
\[ U^*_L W_0(f)U_L = W_0(L^{-1}f) \quad \text{for} \quad f \in \mathcal{J}', \]  
thus similar reasoning shows that the extension of \( L \) is a bijective symplectic transformation of \( \mathcal{K} \) onto itself. Equation (A.48) now extends by (A.15) to all \( f \in \mathcal{K} \).

A.4 Proof of the statements (A.44 – A.47)

Let the Bogoliubov transformation be implemented as in Eq. (A.43). For each pair of vectors \( \psi, \varphi \in D(N^{1/2}) \) one has then 
\[ (a^*(f)\psi, U_L \varphi) = (\psi, U_L a(f)\varphi). \]  
We substitute here \( \varphi = \Omega \), \( f = T^{-1}g \), and use Eq. (A.40). This yields
\[ (a^*(g)\psi, \Omega_L) = -(a(ST^{-1}g)\psi, \Omega_L), \]  
where \( \Omega_L = U_L \Omega \). Substituting here for \( \psi \) all vectors of the form \( a^*(g_1)\ldots a^*(g_k)\Omega \) for \( k = 0, 1, \ldots \) recursively, it is easy to see that \( (\Omega, \Omega_L) \) cannot vanish, as otherwise \( \Omega_L \) would be orthogonal to the whole Hilbert space. Let now \( \{f_i\} \) be an orthonormal basis and put in (A.50) \( g = f_i \) and \( \psi = a^*(f_j)\Omega \), which gives
\[ (a^*(f_i)a^*(f_j)\Omega, \Omega_L) = -(ST^{-1}f_i, f_j)(\Omega, \Omega_L). \]  
Take the sum over \( i, j \) of the absolute values squared of both sides of this equation. On the l.h. side one then gets a quantity smaller or equal \( 2 ||\Omega_L||^2 \), so \( ST^{-1} \) is a Hilbert-Schmidt operator. As \( T \) is a bounded operator, the same is true for \( S \).

Conversely, let now \( S \) be a HS operator, so there exists an orthonormal basis \( \{f_i\} \) satisfying (A.36), and denote \( g_i = U_T f_i \), which defines another orthonormal basis. Then
\[ Sf_i = \lambda_i g_i, \quad T f_i = \sqrt{\lambda_i^2 + 1} g_i, \quad \text{so} \quad ST^{-1}g_i = \frac{\lambda_i}{\sqrt{\lambda_i^2 + 1}} g_i, \]  
where
\[ \sum_{i=1}^{\infty} \lambda_i^2 < \infty. \]  
We look for a vector \( \Omega_L \) which lies in the domain of all operators \( a_L(f) \) and satisfies Eq. (A.45). If such vector exists, then it must satisfy Eq. (A.50) for all possible \( g \) and \( \psi \). It is sufficient to substitute for \( g \) all basis vectors \( g_i \) and for \( \psi \) all vectors from the basis of the particle number representation \( \{|n_1, n_2, \ldots\} \) with profiles \( g_1, g_2, \ldots \). This gives the recurrent conditions
\[ \langle n_1 \ldots n_i + 1 \ldots | \Omega_L \rangle = -\frac{\lambda_i}{\sqrt{\lambda_i^2 + 1}} \sqrt{\frac{n_i}{n_i + 1}} \langle n_1 \ldots n_i - 1 \ldots | \Omega_L \rangle, \]  
(A.53)
which are solved for numbers $\langle n_1 n_2 \ldots | \Omega_L \rangle$ uniquely up to a common constant factor $c$ by

$$
\langle n_1 n_2 \ldots | \Omega_L \rangle = 0 \quad \text{if not all } n_i \text{ are even},
$$

$$
\langle 2m_1 2m_2 \ldots | \Omega_L \rangle = c \prod_{i=1}^{\infty} \left( - \frac{\lambda_i}{\sqrt{\lambda_i^2 + 1}} \right)^{m_i} \sqrt{\frac{(2m_i - 1)!!}{(2m_i)!!}}, \quad (A.54)
$$

Using these explicit expressions one finds that for each $l = 0, 1, \ldots$ the following sum converges:

$$
\sum_{n_1, n_2, \ldots} \left( \sum_{i=1}^{\infty} n_i \right)^l |\langle n_1 n_2 \ldots | \Omega_L \rangle|^2 
\leq |c|^{2l} \sum_{m_1, m_2, \ldots} \left( \sum_{i=1}^{\infty} m_i \right)^l \prod_{j=1}^{\infty} \left( \frac{\lambda_j^2}{1 + \lambda_j^2} \right)^{m_j} < \infty. \quad (A.55)
$$

The first inequality is obvious, while the second bound will be shown below. For $l = 0$ the bound shows that the coefficients $\langle n_1 n_2 \ldots | \Omega_L \rangle$ indeed define a vector $\Omega_L$ solving the conditions $A.53$. The bounds for $l \in \mathbb{N}$ show that this vector is in the domain of all operators $N_l/2$. In particular, $\Omega_L$ is in the domain of all operators $a_L(f)$. This completes the proof of the existence and uniqueness (up to a phase) of a normalized vector solving equation $A.45$, and of the property $A.46$. Statements about the operator $U_L$ are now easily proved with the use of commutation relations, and the irreducibility of the Weyl system.

To prove the missing step in Eq. $A.55$ we denote $q_i = \lambda_i^2 (1 + \lambda_i^2)^{-1}$. From the finiteness of the sum $\sum_{i=1}^{\infty} \lambda_i^2$ it follows that also the following expressions converge:

$$
p_l \equiv \sum_{i=1}^{\infty} \left( \frac{q_i}{1 - q_i} \right)^l = \sum_{i=1}^{\infty} \lambda_i^{2l} \quad \text{for all } l \in \mathbb{N},
$$

$$
r \equiv \prod_{i=1}^{\infty} \frac{1}{1 - q_i} = \prod_{i=1}^{\infty} (1 + \lambda_i^2) \leq \exp \left( \sum_{i=1}^{\infty} \lambda_i^2 \right).
$$

One shows by induction with respect to $l$ that

$$
\sum_{m_1, m_2, \ldots} \left( \sum_{i=1}^{\infty} m_i \right)^l \prod_{j=1}^{\infty} q_j^{m_j} = W_l(p_1, \ldots, p_l) r, \quad (A.56)
$$

where $W_l$ are polynomials. For $l = 0$ the l.h. side is an infinite product of geometrical series, so the equality holds with $W_0 = 1$. The step from $l$ to $l+1$ is obtained by the application of the homogeneity operator $\sum_{i=1}^{\infty} q_i \partial/\partial q_i$ to the both sides of the equa-
tion. A direct calculation yields 

\[ \sum_{i=1}^{\infty} q_i \frac{\partial}{\partial q_i} r = p_1 r \] 

and 

\[ \sum_{i=1}^{\infty} q_i \frac{\partial}{\partial q_i} p_k = k(p_k + p_{k+1}), \]

which confirms the inductive claim and completes the proof of the bound (A.55).
References

[1] H.G.B. Casimir, Proc. K. Ned. Akad. Wet. 51 (1948) 793
[2] G. Plunien, B.Müller, W. Greiner, Phys. Rep. 134 (1986) 87
[3] E. Elizalde, A. Romeo, Am. J. Phys. 59 (1991) 711
[4] W. Milonni, The Quantum Vacuum: An Introduction to Quantum Electrodynamics, Academic Press, San Diego, 1994
[5] K.A. Milton, The Casimir Effect: Physical Manifestations of Zero-Point Energy, World Scientific, Singapore, 2001
[6] M. Bordag, U. Mohideen, V.M. Mostepanenko, Phys. Rep. 353 (2001) 1
[7] A. Herdegen, Acta Phys. Pol. B 32 (2001) 55
[8] R. Haag, Local Quantum Physics, Springer, Berlin, 1992
[9] O. Bratteli, D. W. Robinson, Operator Algebras and Quantum Statistical Mechanics, vol. I & II, Springer, Berlin, 1996
[10] D. Deutsch, P. Candelas, Phys. Rev. D, 20 (1979) 3063
[11] N. Graham, R. Jaffe, V. Khemani, M. Quandt, M. Scandurra, H. Weigel, Nucl. Phys. B 645 (2002) 49; Phys.Lett. B 572, (2003) 196; N. Graham, R. Jaffe, V. Khemani, M. Quandt, O. Schroeder, H. Weigel, hep-th/0309130
[12] B.S. Kay, Phys. Rev. D 20 (1979) 3052
[13] G. Scharf, W.F. Wreszinski, Found. Phys. Lett. 5 (1992) 479
[14] M. Reed, B. Simon, Methods of Modern Mathematical Physics, vol. I, Academic Press, London, 1976
[15] S. Hollands, R. M. Wald, Comm. Math. Phys. 223 (2001) 289
[16] R. Brunetti, K. Fredenhagen, R. Verch, Comm. Math. Phys. 237 (2003) 31
[17] P. Marecki, DESY THESIS 2004-002
[18] D. Shale, Trans. Amer. Math. Soc. 103 (1962) 149; A. van Daele, A. Verbeure, Comm. Math. Phys. 20 (1971) 268
[19] H. Araki, S. Yamagami, Publ. RIMS Kyoto Univ. 18 (1982) 283