$t$–analogs of $q$–characters of quantum affine algebras of type $A_n$, $D_n$

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Dedicated to Professor Ryoshi Hotta on his sixtieth birthday

Abstract. We give a tableaux sum expression of $t$–analog of $q$–characters of finite dimensional representations (standard modules) of quantum affine algebras $U_q(Lg)$ when $g$ is of type $A_n$, $D_n$.

1. Introduction

Let $g$ be a simple Lie algebra of type $ADE$ over $\mathbb{C}$, $Lg = g \otimes \mathbb{C}[z, z^{-1}]$ be its loop algebra, and $U_q(Lg)$ be its quantum universal enveloping algebra, or the quantum loop algebra for short. It is a subquotient of the quantum affine algebra $U_q(\hat{g})$, i.e., without central extension and degree operator. It is customary to define $U_q(Lg)$ as an algebra over $\mathbb{Q}(q)$, but here we consider $q$ as a nonzero complex number which is not a root of unity, for simplicity.

By Drinfeld [2], Chari-Pressley [1], simple $U_q(Lg)$-modules are parametrized by $I$-tuples of polynomials $P = (P_i(u))_{i \in I}$ with normalization $P_i(0) = 1$. They are called Drinfeld polynomials. Let us denote by $L(P)$ the simple module with Drinfeld polynomial $P$. It gives a basis $\{L(P)\}_P$ of the Grothendieck group $\text{Rep} U_q(Lg)$ of the category of finite dimensional representations of $U_q(Lg)$.

In [18] the author introduced another set of $U_q(Lg)$-modules $M(P)$, called standard modules, parametrized also by Drinfeld polynomials. It gives us another base of $\text{Rep} U_q(Lg)$. Then the author [18] showed that the multiplicity $[M(P) : L(Q)]$ is equal to a specialization of a polynomial $Z_{PQ}(t) \in \mathbb{Z}[t, t^{-1}]$ at $t = 1$. And the polynomial $Z_{PQ}(t)$ is defined as Poincaré polynomials of intersection cohomology of graded quiver varieties, which are fixed point sets of $\mathbb{C}^*$-actions on quiver varieties, introduced earlier by the author [15, 17].

The polynomials $Z_{PQ}(t)$ can be considered as an analog of Kazhdan-Lusztig polynomials which are Poincaré polynomials of intersection cohomology of Schubert varieties. As Kazhdan-Lusztig polynomials are defined via an involution on the Hecke algebra, our $Z_{PQ}(t)$ are determined by means of a bar involution on

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the \( t \)-analog of the Grothendieck ring \( R_t \) defined as \( \text{Rep} \mathbf{U}_q(L_\mathbb{G}) \otimes \mathbb{Z}[t, t^{-1}] \). In order to compute the bar involution (and hence \( \overline{Z}_{PQ}(t) \)), the author introduced \( t \)-analogs of \( q \)-characters \( \chi_{q,t} \) and gave a combinatorial algorithm to compute \( \tilde{\chi}_{q,t}(M(P)) \) \cite{19, 21}. The original \( q \)-characters \( \chi_q \) had been introduced and studied by Knight, Frenkel-Reshetikhin, Frenkel-Mukhin \cite{13, 4, 5}. In summary, the multiplicity \([M(P) : L(Q)]\) can be given by a purely combinatorial algorithm.

In this paper, we shall give an explicit expression of \( \tilde{\chi}_{q,t}(M(P)) \) when \( g \) is of type \( A_n, D_n \) in terms of Young tableaux or their variants. Such expressions had been known for the original \( q \)-characters \( \chi_q \) by Kuniba-Suzuki \cite{14} and Nazarov-Tarasov \cite{22}. (They did not use the terminology of \( q \)-characters. But their calculation can be translated to \( q \)-characters. See \cite{3, 4} §11 and reference therein.) In fact, the author finds this expression via a certain relation between \( q \)-characters and Kashiwara’s crystal base (see \cite{3, 5}), where expression in terms of tableaux was given by Kashiwara-Nakashima \cite{11}. (See also \cite{6}.) It seems that the relation between crystals and \( \chi_q \) has not been known. The author is also motivated by his earlier work \cite{16} on an expression of Poincaré polynomials of original quiver varieties of type \( A_n \) in terms of Young tableaux. This work was motivated by works of Shimomura, Hotta-Shimomura \cite{23, 7} in turn.

In this paper we use the following notation: \((P)\) be 1 if a statement \( P \) is true and 0 otherwise.

### 2. \( t \)-analogs of \( q \)-characters

We shall not discuss the definition of quantum loop algebras, nor their finite dimensional representations in this paper. (See \cite{19} for a survey.) We just review properties of \( \chi_{q,t} \), as axiomized in \cite{21}.

Let \( g \) be a simple Lie algebra of type \( ADE \), let \( I \) be the index set of simple roots. Let \( L(P) \) (resp. \( M(P) \)) be the simple (resp. standard) \( \mathbf{U}_q(L_\mathbb{G}) \)-module with Drinfeld polynomial \( P \). A simple module \( L(P) \) is called an \( l \)-fundamental representation when \( P_t(u) = (1 - au)^{\delta_{IN}} \) for some \( a \in \mathbb{C}^\ast \) and \( N \in I \). Since it depends only on \( N \) and \( a \), we denote it by \( L(A_N)_a \). This will play an important role later.

Let \( Y_t \equiv \mathbb{Z}[t, t^{-1}, Y_{i,a}, Y_{i,a}^{-1}]_{i \in I, a \in \mathbb{C}^\ast} \) be a Laurent polynomial ring of uncountably many variables \( Y_{i,a} \)'s with coefficients in \( \mathbb{Z}[t, t^{-1}] \). A monomial in \( Y_t \) means a monomial only in \( Y_{i,a} \)'s, containing no \( t \)'s. Let

\[
A_{i,a} \equiv Y_{i,a}Y_{i,a}^{-1} \prod_{j : j \neq i} Y_{j,a}^{c_{ij}},
\]

where \( c_{ij} \) is the \((i,j)\)-entry of the Cartan matrix. Let \( \mathcal{M} \) be the set of monomials in \( Y_t \).

**Definition 2.1.** (1) For a monomial \( m \in \mathcal{M} \), we define \( u_{i,a}(m) \in \mathbb{Z} \) be the degree in \( Y_{i,a} \), i.e.,

\[
m = \prod_{i,a} Y_{i,a}^{u_{i,a}(m)}.
\]

(2) A monomial \( m \in \mathcal{M} \) is said \( i \)-dominant if \( u_{i,a}(m) \geq 0 \) for all \( a \). It is said \( l \)-dominant if it is \( i \)-dominant for all \( i \).

(3) Let \( m, m' \) be monomials in \( \mathcal{M} \). We say \( m \leq m' \) if \( m/m' \) is a monomial in \( A_{i,a}^{-1} \) \((i \in I, a \in \mathbb{C}^\ast)\). Here a monomial in \( A_{i,a}^{-1} \) means a product of nonnegative
powers of $A_{i,a}^{-1}$. It does not contain any factors $A_{i,a}$. In such a case we define $v_{i,a}(m,m') \in \mathbb{Z}_{\geq 0}$ by

$$m = m' \prod_{i,a} A_{i,a}^{-v_{i,a}(m,m')}.$$ 

This is well-defined since the $q$-analog of the Cartan matrix is invertible. We say $m < m'$ if $m \leq m'$ and $m \neq m'$.

(4) For an $i$-dominant monomial $m \in \mathcal{M}$ we define

$$E_i(m) \defeq m \prod_{a} u_{i,a} \sum_{r_a = 0} A_{i,a}^{-r_a}$$

where $\binom{n}{t}$ is the $t$-binomial coefficient.

Suppose that $l$-dominant monomials $m_{P_1}, m_{P_2}$ and monomials $m_1 \leq m_{P_1}, m_2 \leq m_{P_2}$ are given. We define an integer $d(m_1, m_{P_1}; m_2, m_{P_2})$ by

$$d(m_1, m_{P_1}; m_2, m_{P_2}) \defeq \sum_{i,a} (v_{i,a}(m_1, m_{P_1}) u_{i,a}(m_2) + v_{i,a}(m_{P_1}) v_{i,a}(m_2, m_{P_2})).$$

For an $I$-tuple of rational functions $Q/R = (Q_i(u)/R_i(u))_{i \in I}$ with $Q_i(0) = R_i(0) = 1$, we set

$$m_{Q/R} \defeq \prod_{i \in I} \prod_{\alpha \beta} Y_{i,\alpha} Y_{i,\beta}^{-1},$$

where $\alpha$ (resp. $\beta$) runs roots of $Q_i(1/u) = 0$ (resp. $R_i(1/u) = 0$), i.e., $Q_i(u) = \prod_{\alpha} (1 - au)$ (resp. $R_i(u) = \prod_{\beta} (1 - \beta u)$). As a special case, an $I$-tuple of polynomials $P = (P_i(u))_{i \in I}$ defines $m_P = m_{P_{i=1}}$. In this way, the set $\mathcal{M}$ of monomials are identified with the set of $I$-tuple of rational functions, and the set of $l$-dominant monomials are identified with the set of $I$-tuple of polynomials.

The $t$-analog of the Grothendieck ring $\mathcal{R}_t$ is a free $\mathbb{Z}[t, t^{-1}]$-module with base $\{M(P)\}$ where $P = (P_i(u))_{i \in I}$ is the Drinfeld polynomial. (We do not recall the definition of standard modules $M(P)$ here, but the reader safely consider them as formal variables.)

The $t$-analog of the $q$-character homomorphism is a $\mathbb{Z}[t, t^{-1}]$-linear homomorphism $\widetilde{\chi}_{q,t}: \mathcal{R}_t \to \mathcal{Y}_t$. It is defined as the generating function of Poincaré polynomials of graded quiver varieties, and will not be reviewed in this paper. But the following is known.

**Theorem 2.3.** (1) The $\widetilde{\chi}_{q,t}$ of a standard module $M(P)$ has a form

$$\widetilde{\chi}_{q,t}(M(P)) = m_P + \sum a_m(t)m,$$

where the summation runs over monomials $m < m_P$.

(2) For each $i \in I$, $\widetilde{\chi}_{q,t}(M(P))$ can be expressed as a linear combination (over $\mathbb{Z}[t, t^{-1}]$) of $E_i(m)$ with $i$-dominant monomials $m$.

(3) Suppose that two $I$-tuples of polynomials $P^1 = (P^1_i), P^2 = (P^2_i)$ satisfy the following condition:

$$a/b \notin \{q^n \mid n \in \mathbb{Z}, n \geq 2\} \text{ for any pair } a, b \text{ with } P^1_i(1/a) = 0, P^2_j(1/b) = 0 (i, j \in I).$$
Then we have
\[ \widetilde{\chi}_{q,t}(M(P^1P^2)) = \sum_{m_1,m_2} t^{2d(m_1,m_1:m^2,m_2)} a_{m_1}(t) a_{m_2}(t) m_1 m_2, \]
where \( \widetilde{\chi}_{q,t}(M(P^a)) = \sum_{m_a} a_{m_a}(t) m_a \) with \( a = 1, 2 \).
Moreover, properties (1), (2), (3) uniquely determine \( \widetilde{\chi}_{q,t} \).

Apart from the existence problem, one can consider the above properties (1), (2), (3) as the definition of \( \widetilde{\chi}_{q,t} \) (an axiomatic definition). We only use the above properties, and the reader can safely forget the original definition.

Remark 2.5. It is more suitable to consider a slightly modified version \( \chi_{q,t} \) in stead of \( \widetilde{\chi}_{q,t} \) for computing the bar operation. Therefore \( \chi_{q,t} \) was mainly used in [19]. Anyhow, they are simply related as
\[ \text{if } \chi_{q,t}(M(P)) = \sum_{m} a_{m}(t) m, \quad \chi_{q,t}(M(P)) = \sum_{m} t^{-d(m_m^1,m_m^2,m_m^3)} a_{m}(t) m. \]
See [19, 5.1.3].

Let us explain briefly why the properties (1), (2), (3) determine \( \chi_{q,t} \). First consider the case \( M(P) \) is an \( l \)-fundamental representation. (We have \( M(P) = L(P) \) in this case.) Then one can determine \( \chi_{q,t}(M(P)) \) starting from \( m_P \) and using the property (2) inductively. (The idea can be seen in the examples below.) For general \( P \), write it as \( P = P^1P^2P^3 \cdots \) so that each \( M(P^a) \) is an \( l \)-fundamental representation, and the condition [2, 4] is met with respect to the ordering. Then we apply (3) successively to get \( \chi_{q,t}(M(P)) \) from \( \chi_{q,t}(M(P^a)) \) with \( a = 1, 2, \cdots \).

We attach to each standard module \( M(P) \), an oriented colored graph \( \Gamma_P \) as follows. (It is a slight modification of the graph in [4, 5.3].) The vertices are monomials in \( \chi_{q,t}(M(P)) \). We draw a colored edge \( \overrightarrow{i_{m_1}} \) from \( m_1 \) to \( m_2 \) if \( m_2 = m_1 A_{i_{m_1}} \). We also write the coefficients of the monomials in \( \chi_{q,t}(M(P)) \). In fact, edges are determined from monomials on vertices.

Here are examples.

Example 2.6. Let \( g \) be of type \( A_2 \). We put a numbering \( I = \{1, 2\} \).
(1) The graph of \( \chi_{q,t}(M(P)) \) with \( m_P = Y_{1,1}Y_{2,q} \) is the following:
\[
\begin{align*}
Y_{1,1}Y_{2,q} \xrightarrow{2q^2} Y_{1,1}Y_{1,1}q^{-2}Y_{2,q}^{-1} \xrightarrow{1q^3} Y_{1,1}Y_{1,1}q^{-1} \xrightarrow{1q} Y_{1,1}Y_{1,1}q^{-1}Y_{2,q} \xrightarrow{1q} Y_{1,1}Y_{1,1}q^{-1}Y_{2,q} \xrightarrow{2q} \end{align*}
\]
(2) The graph of \( \chi_{q,t}(M(P)) \) with \( m_P = Y_{1,1}^2 \) is the following:
\[
\begin{align*}
Y_{1,1}^2 \xrightarrow{1q} (1 + q^2)Y_{1,1}q^{-2}Y_{2,q} \xrightarrow{1q} Y_{1,1}^2Y_{2,q} \xrightarrow{1q} Y_{1,1}^2Y_{2,q} \xrightarrow{2q} \end{align*}
\]

(1 + t^2)Y_{1,1}Y_{2,q}^{-1} \xrightarrow{1q} (1 + t^2)Y_{1,1}Y_{1,1}q^{-2}Y_{2,q} \xrightarrow{2q} Y_{2,q}^{-2}
(3) The graph of \( \widetilde{\chi}_{q,t}(M(P)) \) with \( mp = Y_{1,1}Y_{1,q^2} \) is the following:

\[
\begin{array}{ccc}
Y_{1,1}Y_{1,q^2} & \xrightarrow{1q} & Y_{2,q} \\
\downarrow{1,q^3} & & \downarrow{1,q^3} \\
Y_{1,1}Y_{1,q^4}Y_{2,q^3} & \xrightarrow{1q} & Y_{1,q^2}^2Y_{1,q^4}Y_{2,q^3} \\
\downarrow{2,q^4} & & \downarrow{2,q^4} \\
Y_{1,1}Y_{1,q^4}^{-1} & \xrightarrow{1q} & Y_{1,q^2}Y_{1,q^4}^{-1} \\
\downarrow{2,q^4} & & \downarrow{2,q^4} \\
Y_{1,1}Y_{1,q^4}^{-1} & \xrightarrow{1q} & Y_{1,q^2}Y_{1,q^4}^{-1} \\
\end{array}
\]

In the first two examples, \( \widetilde{\chi}_{q,t}(M(P)) \) does not contain \( l \)-dominant monomials other than \( mp \). Therefore the graphs are determined from Theorem 2.3(1), (2), as we mentioned above. (It is instructive to check that (2) holds in these examples.) Other examples can be found in [19]. (Caution: (1) In [loc. cit.], \( \chi_{q,t} \) in stead of \( \widetilde{\chi}_{q,t} \) was used. (2) There are mistakes in Example 5.3.3 in [loc. cit.])

3. A monomial realization of crystal bases

In this section, we give a realization of crystal bases of highest weight modules, called a monomial realization. We can avoid the usage of this material in later sections, but it will give us a natural motivation for our tableaux sum expression of \( q \)-characters. Moreover, this section can be read independently from the other sections. See also Kashiwara’s article [10] in this volume.

In this section, \( \mathfrak{g} \) is an arbitrary symmetrizable Kac-Moody Lie algebra. Let \( I \) be the index set of simple roots. Let \( \{ \alpha_i \}_{i \in I}, \{ h_i \}_{i \in I} \) be the sets of simple roots and simple coroots. Let \( P \) be a weight lattice, and \( P^+ \) be the set of dominant weights. We assume that there exists \( \Lambda \in P \) such that \( \langle h_i, \Lambda \rangle = \delta_{ij} \) (fundamental weights).

We shall not recall here the notion of crystals. See e.g., [3]. We denote by \( B(\Lambda) \) the crystal of the highest weight module with highest weight \( \lambda \in P^+ \).

Let \( \mathcal{M}^e \) be the set of monomials in \( Y = \mathbb{Z}[Y_{i,a}]_{i \in I, a \in \mathbb{C}^*} \) such that \( a \) is a power of \( q \):

\[
\mathcal{M}^e \overset{\text{def.}}{=} \left\{ m = \prod_{i,n} Y_{i,a}^{u_{i,a}n(m)} \biggm| u_{i,a}n(m) \in \mathbb{Z} \text{ is zero except for finitely many} \ (i,n) \right\}.
\]

We set

\[
\begin{align*}
\varepsilon_{i,n}(m) & \overset{\text{def.}}{=} - \sum_{k : k \geq n} u_{i,q^k}(m), & \varphi_{i,n}(m) & \overset{\text{def.}}{=} \sum_{k : k \leq n} u_{i,q^k}(m), \\
\varepsilon_i(m) & \overset{\text{def.}}{=} \max_n \varepsilon_{i,n}(m), & \varphi_i(m) & \overset{\text{def.}}{=} \max_n \varphi_{i,n}(m), \\
p_i(m) & \overset{\text{def.}}{=} \max\{ n \mid \varepsilon_{i,n}(m) = \varepsilon_i(m) \}, & q_i(m) & \overset{\text{def.}}{=} \min\{ n \mid \varphi_{i,n}(m) = \varphi_i(m) \}, \\
\text{wt}(m) & \overset{\text{def.}}{=} \sum_{i,n} u_{i,q^n}(m) \Lambda_i.
\end{align*}
\]

If \( n \) is sufficiently large, we have \( \varepsilon_{i,n}(m) = 0 \). If \( n \) is sufficiently small, \( \varepsilon_{i,n}(m) \) is a fixed integer independent of \( n \). Therefore \( \varepsilon_i(m) \) is a nonnegative integer. If
\[ \varepsilon_i(m) = 0, \text{ we understand } p_i(m) = \infty. \text{ Similarly, we set } q_i(m) = -\infty \text{ if } \varphi_i(m) = 0. \]

We define operators \( \tilde{e}_i, \tilde{f}_i \) by

\[
\tilde{e}_i(m) \overset{\text{def}}{=} \begin{cases} 
\prod_{j=0}^{\varepsilon_i(m)-1} A_{i,q_i(m)-j} & \text{if } \varepsilon_i(m) > 0, \\
0 & \text{if } \varepsilon_i(m) = 0,
\end{cases}
\]

\[
\tilde{f}_i(m) \overset{\text{def}}{=} \begin{cases} 
\prod_{j=0}^{\varphi_i(m)-1} A_{i,q_i(m)+j} & \text{if } \varphi_i(m) > 0, \\
0 & \text{if } \varphi_i(m) = 0.
\end{cases}
\]

Unfortunately this does not give us a crystal in general. Therefore we need an extra assumption or modification. Here we assume that \( \mathfrak{g} \) is \textit{without odd cycles}, i.e., there exists a function \( I \in i \mapsto a_i \in \{0,1\} \) such that \( a_i + a_j = 1 \) whenever \( a_{ij} < 0 \) here. This condition is satisfied when \( \mathfrak{g} \) is finite-dimensional (so enough for our present purpose), or of affine type other than \( A_n^{(1)} \). (See [20], [8] or [10] for other modifications of the rule to have a crystal for arbitrary \( \mathfrak{g} \).)

Let

\[
\mathcal{M}' = \left\{ m = \prod_{i,n} Y_{i,q_i}^{u_i,q_i(m)} \in \mathcal{M}^2 \mid \text{if } n \equiv a_i \text{ mod } 2 \right\}.
\]

It is clear that \( \mathcal{M}' \) is invariant under \( \tilde{e}_i, \tilde{f}_i \).

**Theorem 3.1.** (1) The set \( \mathcal{M}' \) together with maps \( \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i \) satisfies the axioms of a crystal in the sense of [8].

(2) The crystal generated by an \( l \)-dominant monomial \( m \in \mathcal{M}' \) (i.e., \( u_i,q_i(m) \geq 0 \) for all \( i,n \)) is isomorphic to the crystal \( \mathcal{B}(\text{wt}(m)) \) of the highest weight module.

**Warning:** The crystals, we constructed, are for \( \mathfrak{g} \), not for \( \mathfrak{g} \) or \( \mathfrak{L}_q \).

Here are examples.

**Example 3.2** (Compare with Example [20]). Let \( \mathfrak{g} \) be of type \( A_2 \).

(1) The crystal graph of \( \mathcal{M}' \) starting from \( Y_{1,1}Y_{2,q} \) is the following:

\[
Y_{1,1}Y_{2,q} \xrightarrow{2} Y_{1,1}Y_{1,q}Y_{2,q}^{-1} \xrightarrow{1} Y_{1,1}Y_{1,q}^{-1} \xrightarrow{1} Y_{1,q}^{-1}Y_{1,q}Y_{2,q} \xrightarrow{2} Y_{1,q}^{-1}Y_{2,q}^{-1}.
\]

(2) The crystal graph of \( \mathcal{M}' \) starting from \( Y_{2,1}^2 \) is the following:

\[
Y_{1,1}^2 \xrightarrow{1} Y_{1,1}Y_{1,q}^{-1}Y_{2,q} \xrightarrow{1} Y_{1,q}^{-2}Y_{2,q} \xrightarrow{2} Y_{2,q}^{-2}.
\]
(3) The crystal graph of $\mathcal{M}'$ starting from $Y_{1,1}Y_{1,q^2}$ is the following:

```
\begin{align*}
& Y_{1,1}Y_{1,q^2} \\
& \downarrow^1 \\
& Y_{1,1}^{-1}Y_{1,q^2}Y_{2,q^3} \xrightarrow{1} Y_{1,q^2}Y_{1,q^4}Y_{2,q}Y_{2,q^3} \\
& \downarrow^2 \\
& Y_{1,1}^{-1}Y_{2,q}Y_{2,q^5} \xrightarrow{2} Y_{2,q^3}Y_{2,q^5}
\end{align*}
```

Note that (2) and (3) are different realizations of the same crystal $B(2\Lambda_1)$. Comparing with Example 2.6, we find that vertices and edges are subsets of those of graphs for $\tilde{\chi}_{q,t}$. This is the case for any $l$-dominant monomials in $\mathcal{M}'$, as we shall explain below.

There are two proofs of this theorem. The author’s proof is based on [20, 8.6], in particular depends on the theory of quiver varieties. There is more direct proof due to Kashiwara [12]. We explain the author’s proof here since it is related closely to $q$-characters, although we shall not explain quiver varieties. (See e.g., [19, §8] for a summary of the theory of quiver varieties.)

The $t$-analogues of $q$-characters of standard modules are the generating function of Poincaré polynomials of graded quiver varieties, which are fixed points of the quiver variety $\mathcal{M}$ with respect to a $C^*$-action. It is expressed schematically as

$$\tilde{\chi}_{q,t}(M(P)) = \sum_{m} P_t(\mathcal{M}(m))m,$$

where $\mathcal{M}(m)$ is the connected component corresponding to a monomial $m$, and $P_t(\mathcal{M}(m))$ is its Poincaré polynomial. The choice of the $C^*$-action corresponds to a choice of the Drinfeld polynomial $P$. The $l$-dominant monomial $m_P$ corresponding to $P$ is a certain distinguished component, which is a single point (and hence $P_t(\mathcal{M}(m_P)) = 1$). Corresponding to each monomial $m$, we consider the following locally-closed subvariety of $\mathcal{M}$:

$$\mathcal{Z}(m) \overset{\text{def}}{=} \left\{ x \in \mathcal{M} \middle| \lim_{t \to \infty} t \circ x \in \mathcal{M}(m) \right\},$$

where $\circ$ denotes the $C^*$-action. If $m_P \in \mathcal{M}'$, then all monomials appearing in $\tilde{\chi}_{q,t}(M(P))$ is contained in $\mathcal{M}'$ by Theorem 2.3. Moreover, it can be shown that the above $\mathcal{Z}(m)$ is a lagrangian subvariety of $\mathcal{M}$. Moving all $m$, the union of all $\mathcal{Z}(m)$ forms a closed lagrangian subvariety $\mathcal{Z}$ of $\mathcal{M}$. So the irreducible components of $\mathcal{Z}$ can be identified with monomials. In [20], a crystal structure is defined on the set of irreducible components of $\mathcal{Z}$, following the work of Kashiwara-Saito [12]. The crystal structure is isomorphic to a direct sum of the crystals of highest weight modules. We translate this result in terms of monomials, and get the above theorem.

Applying the above result to $q$-characters, we get the following

1In fact, when the author obtained [20, 8.6], he thought it a new result on crystals. But afterwards, Kashiwara informed him that it follows directly from the main result of [8] together with the formula $T_{\lambda} \otimes B_l = T_{\lambda,\lambda} \otimes B_l$ (see [8] for the notation). The last sentence of the abstract in [20] should be corrected as ‘This result is equivalent to Kashiwara’s combinatorial description given by his embedding theorem’.

2The class of $C^*$-actions used for the $q$-characters is different from that for crystals in [20, §8]. The condition ensures that the action in the former class is also in the latter class. The absence of odd cycles is used here.
Theorem 3.3. Suppose that \( \mathfrak{g} \) is of type ADE. Let \( M(P) \) be a standard module, and let \( M(\mathcal{P}) \) be the set of monomials appearing in \( \tilde{\chi}_{q,t}(M(P)) \). Suppose that the monomial \( m_P \) corresponding to the \( l \)-highest weight vector is contained in \( \mathcal{M}' \). Then \( \mathcal{M}(P) \) has a structure of a crystal (with respect to \( \mathfrak{g} \)), which is isomorphic to a direct sum of the crystals of highest weight modules. Moreover, the crystal graph is obtained from the graph \( \Gamma_P \) by forgetting the multiplicities of monomials and erasing some arrows.

For example, the crystal of 2.6(3) is \( B(2\Lambda_1) \oplus B(\Lambda_2) \) while those of (1), (2) are crystals of simple modules.

Since the original \( \chi_q \)-character for non-simply-laced case has a slightly different \( A_i,a \) from one used in this paper, the proof (ours or Kashiwara’s) of the above theorem does not apply. But the statement seems to be true.

In general, it is not easy to determine the crystal structure on \( \mathcal{M}(P) \). But we can do it for a special case. Choose and fix orientations of edges in the Dynkin diagram. We define integer \( m(i) \) for each vertex \( i \) so that \( m(i) - m(j) = 1 \) if we have an oriented edge from \( i \) to \( j \), i.e., \( i \to j \). Then we define \( P \) by

\[
P_i(u) = (1 - u a q^{m(i)})^{w_i}
\]

for \( w_i \in \mathbb{Z}_{\geq 0} \), \( a \in \mathbb{C}^* \). A special case is when \( M(P) \) is an \( l \)-fundamental representation, i.e., \( w_i = 0 \) except for one vertex \( i \).

Proposition 3.4. Suppose that \( \mathfrak{g} \) is of type ADE and \( P \) as above. Then the above \( \mathcal{M}(P) \) is isomorphic to the crystal \( B(\sum_i w_i \Lambda_i) \) of the highest weight \( \mathfrak{g} \)-module.

This is not true for non-simply-laced case, even if we get the crystal structure, as conjectured above.

This can be proved by showing that the above lagrangian subvariety \( \mathfrak{3} \) is isomorphic to another lagrangian subvariety \( \Sigma \), whose irreducible components are known to be obtained from the highest weight vector by applying Kashiwara’s operators. (The detail depends on the theory of quiver varieties. So it is not given here.) The first two examples of 2.6 satisfy the assumption of Proposition 3.4.

Proposition 3.4 means that all the monomials appearing in \( \tilde{\chi}_{q,t}(M(P)) \) can be determined from the crystal \( B(\sum_i w_i \Lambda_i) \) of the highest weight \( \mathfrak{g} \)-module. So far, the relation between the coefficients of monomials and the theory of crystal bases is unclear. For example, \( l \)-fundamental representations are known to have crystal bases \( \mathfrak{3} \), but their relation to \( q \)-characters are not known.

4. type \( A_n \)

We number the vertex of the Dynkin graph of type \( A_n \) as follows:

\[
\begin{array}{cccccc}
1 & 2 & \cdots & n-1 & n \\
\end{array}
\]

We have

\[
A_{i,a} = Y_{i,a} q^{-\frac{i}{2}} Y_{i,a}^{-1} Y_{i-1,a}^{-1} Y_{i+1,a}^{-1},
\]

where we understand \( Y_{i-1,a} = 1 \) and \( Y_{i+1,a} = 1 \) if \( i = 1 \) and \( i = n \) respectively.

We first consider the pullback of the vector representation by the evaluation homomorphism. It is the \( l \)-fundamental representation \( L(\Lambda_1)_a \). Then Proposition 3.4
implies that the vertex of the graph of $\tilde{\chi}_{q,t}(L(\Lambda_1)_{\alpha})$ is the same as that of crystal $B(\Lambda_1)$ and all coefficients are 1. Therefore we get

$$Y_{1,a} \xrightarrow{1,aq} Y_{1,aq^2} Y_{2,aq^3} \xrightarrow{2,aq^2} \ldots \xrightarrow{n-1,aq^{n-1}} Y_{n-1,aq^n} Y_{n,aq^{n-1}} \xrightarrow{n,aq^n} Y_{n,aq^{n+1}}$$

Now it becomes clear that there are no extra arrows. So the graph $\Gamma_P$ is exactly the same as the crystal graph.

We introduce the symbol $\begin{bmatrix} i \end{bmatrix}$ by the above equations. In fact, this can be easily shown from Theorem 2.3 without any knowledge about the representation theory.

Let $B = \{1, \ldots, n + 1\}$. We give the usual ordering $<$ on $B$.

DEFINITION 4.1. (1) A column tableau $T$ is a map

$$T: \{aq^{N-1}, aq^{N-3}, \ldots, aq^{1-N}\} \to B,$$

for $a \in \mathbb{C}^*$, $1 \leq N \leq n$. We call $N$ the length and $a$ the center of $T$ respectively. We associate a monomial $m_T$ to $T$ by

$$m_T = \prod_{p=1}^{N} a^{i_p N + 1 - 2p},$$

where $i_p = T(aq^{N+1-2p})$. We write this graphically as

$$T = \begin{bmatrix} i_1 \\ \vdots \\ i_N \end{bmatrix} a^{1-N}.$$

For $p \neq N$ the suffix of $i_p$, which is $aq^{N+1-2p}$, is omitted since it can be determined from that of $i_N$. The same graphical notation will be used for $m_T$. We extend $T$ to a map from $\mathbb{C}^* \to B \sqcup \{0\}$ by setting

$$T(b) = 0 \quad \text{if } b \neq aq^{N-1}, \ldots, aq^{1-N}.$$

In this case, $\{aq^{N-1}, aq^{1-N}\}$ will be called the support of $T$.

(2) A tableau $T$ is a finite sequence of column tableaux $T = (T_1, T_2, \ldots, T_L)$. Its shape is the sequence of lengths and centers of columns: $(N_1, a_1, N_2, a_2, \ldots, N_L, a_L)$. We write $T$ graphically as

$$T = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_L \end{bmatrix} a^{1-N_L},$$

where $T_\alpha$’s are placed so that $T_1(b), T_2(b), \ldots, T_L(b)$ appear in a row for each $b \in \mathbb{C}^*$. As above suffixes for $T_\alpha$ ($\alpha \leq L - 1$) are omitted since they are determined from the suffix of $T_L$ and the positions of $T_\alpha$. Since we assume that supports of $T_\alpha$ are contained in $aq^{2L}$, all rows are matched.
The associated monomial $m_T$ is given by

$$m_T = \prod_{\alpha=1}^{L} m_{T_{\alpha}}.$$  

(3) We define $s_{\alpha,\beta}$ by

$$\frac{a_{\alpha}q^{N_{\alpha}}}{a_{\beta}q^{N_{\beta}}} = q^{2s_{\alpha,\beta}}.$$  

If the left hand side is not in $q^{2\mathbb{Z}}$, we simply set $s_{\alpha,\beta} = -\infty$. This is the number of boxes in $T_{\alpha}$ which is located upper than the top of $T_{\beta}$. Then we set

$$d(T_{\alpha}, T_{\beta}) \overset{\text{def.}}{=} \sum_{b \in \mathbb{C}^*} (T_{\alpha}(aq^{-2}) < T_{\beta}(b) < T_{\alpha}(b))$$  

$$- (N_{\alpha} < T_{\beta}(a_{\alpha}q^{-1-N_{\alpha}}) \leq T_{\alpha}(a_{\alpha}q^{1-N_{\alpha}}))$$  

$$+ (N_{\alpha} - s_{\alpha,\beta} < N_{\alpha} < T_{\beta}(a_{\alpha}q^{-1-N_{\alpha}}))$$

$$d(T) \overset{\text{def.}}{=} \sum_{\alpha < \beta} d(T_{\alpha}, T_{\beta}).$$

Note that $T_{\alpha}(a_{\alpha}q^{1-N_{\alpha}})$ is the entry of the bottom of $T_{\alpha}$ and $T_{\beta}(a_{\alpha}q^{-1-N_{\alpha}})$ is the entry of $T_{\beta}$ of one row below. From the definition, it is clear that $d(T_{\alpha}, T_{\beta}) = 0$ unless both of their supports are contained in $aq^{2\mathbb{Z}}$ for some $a \in \mathbb{C}^*$.

(4) A tableau $T$ is said to be column increasing if the entries in each column strictly increase from top to bottom.

We have

$$m_T = \prod_{a} \prod_{i=1}^{n} Y_{i,a}^{\# \{ \alpha \mid T_{\alpha}(a) = i \}},$$

where $\# \{ \alpha \mid T_{\alpha}(a) = i \} = \# \{ \alpha \mid T'_{\alpha}(a) = i \}$.

**Definition 4.3.** Two tableaux $T$ and $T'$ are equivalent if $\# \{ \alpha \mid T_{\alpha}(a) = i \} = \# \{ \alpha \mid T'_{\alpha}(a) = i \}$ for all $i = 1, \ldots, n+1$, $a \in \mathbb{C}^*$. Namely $T'$ is obtained from $T$ by permuting boxes in the same rows.

It is clear that monomials $m_T$ and $m_{T'}$ are equal if $T$ and $T'$ are equivalent. The converse is not true, but we can determine when $m_T = m_{T'}$.

**Lemma 4.4.** Let $T$ and $T'$ be tableaux. Then the corresponding monomials $m_T$ and $m_{T'}$ are equal if and only if $T$ and $T'$ become equivalent after we add several columns of the form

$$\begin{array}{c}
1 \\
2 \\
\vdots \\
+1
\end{array}$$

to $T$ and $T'$.

**Proof.** Let

$$\# \{ \alpha \mid T_{\alpha}(a) = i \} = \# \{ \alpha \mid T'_{\alpha}(a) = i \} - \# \{ \alpha \mid T'_{\alpha}(a) = i \}.$$
By (4.2) $m_T = m_{T'}$ if and only if
$$\# \begin{array}{c} aq^{-1} \\ i \\ \vdots \\ \end{array} = \# \begin{array}{c} aq^{-1} \\ i \\ \vdots \\ \end{array}
$$
for any $a, i$. Replacing $a$ by $aq^{2n+1-i}$, we get
$$\# \begin{array}{c} aq^{2n+2-2i} \\ i \\ \vdots \\ \end{array} = \# \begin{array}{c} aq^{2n-2i} \\ i \\ \vdots \\ \end{array}
$$
Namely $m_T$ and $m_{T'}$ are equal if and only if
$$\# \begin{array}{c} aq^{2n+2-2i} \\ i \\ \vdots \\ \end{array}
$$
is independent of $i = 1, \ldots, n+1$ for any $a \in \mathbb{C}^*$. Let $d_a$ be this integer. If $d_a > 0$, we add $(d_a)$-columns of the above form to $T$. If $d_a < 0$, we add $(-d_a)$-columns to $T$. Then the resulting tableaux are equivalent.

**Lemma 4.5.** Let $T$ be a tableau. The corresponding monomial $m_T$ is $l$–dominant if and only if $T$ is equivalent to a tableau $T'$ such that every column is of the form
$$\begin{array}{c} 1 \\ 2 \\ \vdots \\ \end{array}$$
for some $a \in \mathbb{C}^*, i \in \{1, 2, \ldots, n+1\}$.

**Proof.** For $i = 1, \ldots, n$ we have
$$Y_{i,a} = \begin{array}{c} 1 \\ 2 \\ \vdots \\ \end{array}$$
Thus an $l$–dominant monomial is given by a tableau $T''$ whose column is of the form as above with $a \in \mathbb{C}^*, i \in \{1, \ldots, n\}$. On the other hand, Lemma 4.4 means that $m_T = m_{T''}$ if and only if $T$ is equivalent to a tableau $T'$ which is obtained by adding columns of the above type with $i = n+1$ to $T''$.

Now we give a tableaux sum expression of $t$–analogs of $q$–characters. We start with $l$–fundamental representations. Let
$$\mathcal{B}(\Lambda_N)_a = \left\{ T = \begin{array}{c} i_1 \\ \vdots \\ i_N \end{array} \mid i_p \in \mathbb{B}, i_1 < i_2 < \cdots < i_N \right\}.
$$

**Proposition 4.6.** For $1 \leq N \leq n$, we have
$$\tilde{\chi}_{q,t}(L(\Lambda_N)_a) = \sum_{T \in \mathcal{B}(\Lambda_N)_a} m_T.
$$

**Proof.** We check the conditions 2.3(1)(2). By Lemma 4.5 it is clear that the only $l$–dominant monomial in the right hand side of (4.7) is the $l$–highest weight vector, i.e., $i_1 = 1, \ldots, i_N = N$.

Let $T \in \mathcal{B}(\Lambda_N)_a$ as above. The exponent of $Y_{i,aq^{i+1-p}}$ is positive if and only if $i_p = i, i_{p+1} \neq i+1$. In this case the exponent is equal to 1, and the exponent
of other $Y_{i,b}$'s ($b \neq aq^{2(i+1-p)}$) are all 0. Let $T'$ be the tableau obtained from $T$ by changing $i$ to $i+1$. It is in $B(A_N)_a$ since $i_{p+1} \neq i+1$ and we have

$$m_T + m_{T'} = Y_{i,aq^{2(i+1-p)}} \left(1 + A_{i,aq^{2(i-p)}+1}^{-1}\right) M,$$

where $M$ does not contain the factor $Y_{i,b}^\pm$ for any $b \in \mathbb{C}^*$. This shows that the right hand side of (4.7) satisfies the condition 2.3(1).

Our next task is to compute $d(m_T, m_P; m_{T'}, m_{P'})$ for two column tableaux $T$, $T'$ with corresponding $\ell$-dominant monomials $m_P$, $m_{P'}$. We represent them graphically as

$$T = \begin{array}{c} \vdots \\ \vdots \\ i_N \end{array}_{aq^{N-1}} \quad , \quad T' = \begin{array}{c} \vdots \\ \vdots \\ j_M \end{array}_{aq^{N+1-2M}}.$$

Note that we fix the entries of $T'$ so that $j_P$ and $i_p$ have the same vertical coordinate if we write $T$ and $T'$ graphically by the rule Definition [4.4](2). We set $i_p = j_{p'} = 0$ if $p \neq 1, \ldots, N$, $p' \neq s + 1, \ldots, M$. The corresponding $\ell$-dominant monomials $m_P$ and $m_{P'}$ are given by $m_P = Y_{N,a}$, $m_{P'} = Y_{M-s,a}q^{N-M-s}$.

**Lemma 4.8.**

$$d(m_T, m_P; m_{T'}, m_{P'}) = \sum_{p=1}^{N} (i_{p-1} < j_p < i_p) - (N < j_{N+1} \leq i_N) + (N - s < N < j_{N+1}).$$

**Proof.** We have

$$m_T = m_P \prod_{p=1}^{N} \prod_{i=p}^{i_{p-1}} A_{i,aq^{N+1-2p+i}}^{-1},$$

$$m_{T'} = m_{P'} \prod_{p=s+1}^{M} \prod_{i=p-s}^{j_{p-1}} A_{i,aq^{N+1-2p+i}}^{-1}.$$

Hence we have

$$v_{i,aq^{N+1-2p+i}}(m_T, m_P) = (p \leq i \leq i_p - 1).$$

On the other hand, we have

$$u_{i,aq^{N-2p+i}}(m_{T'}) = (j_p = i) - (j_{p+1} = i + 1).$$

Thus we get

$$\sum_{i} v_{i,aq^{N-2p+i}}(m_T, m_P) u_{i,aq^{N-2p+i}}(m_{T'}) = (p \leq j_p \leq i_p - 1) - (p + 1 \leq j_{p+1} \leq i_p).$$
Summing up with respect to \( p \), we get

\[
\sum_{p=1}^{N} \sum_{i} v_{i,aq^{N-2p+i+1}}(m_T,m_P)u_{i,aq^{N-2p+i}}(m_{T'})
\]

\[
= \sum_{p=1}^{N} (p \leq j_p \leq i_p - 1) - (p + 1 \leq j_{p+1} \leq i_p)
\]

\[
= \sum_{p=1}^{N} [(p \leq j_p \leq i_p - 1) - (p \leq j_p \leq i_{p}-1)] - (N + 1 \leq j_{N+1} \leq i_N)
\]

Here we have used that \( p \leq j_p \leq i_p - 1 \) never hold if \( p = 1 \) in the last equality. Note that \( i_{p-1} < i_p \) for \( p = 1, \ldots, N \). Thus \( p \leq j_p \leq i_{p-1} \) implies \( p \leq j_p \leq i_p - 1 \). Hence each term of the above summation is

\[(i_{p-1} < j_p < i_p).
\]

Note that \( p \leq j_p \) holds automatically since \( p - 1 \leq i_{p-1} \).

We have \( u_{N,a}(m_P) = 1 \) and other \( u_{i,b}(m_P) \)'s are all 0. By (4.9) we have

\[
v_{N,aq^{-1}}(m_{T'},m_P) = (N - s \leq N < j_{N+1}).
\]

Combining all together, we get the assertion.

Now let \( M(P) \) be an arbitrary standard module. We decompose \( P = P^1 P^2 P^3 \ldots \) so that each \( M(P^\alpha) \) is an \( l \)-fundamental representation and the condition 2.4 is met with respect to the ordering. (There might be several orderings satisfying 2.4. In that case, we just fix one such ordering.) Let \((N_\alpha,a_\alpha)\) be the shape of a column tableau corresponding to \( M(P^\alpha) \) by Proposition 4.6. Then let \( B(P) \) be the set of column increasing tableaux with shape \((N_1,a_1,N_2,a_2,\ldots,N_L,a_L)\). We apply Theorem 2.3(3) successively to get the following:

**Theorem 4.10.**

\[
\tilde{\chi}_{q,t}(M(P)) = \sum_{T \in B(P)} t^{2d(T)} m_T.
\]

**Example 4.11.** Let \( g \) be of type \( A_2 \). We give only Young tableaux. The corresponding \( m_T \) and \( d(T) \) are given in Example 2.6.

1. \( \tilde{\chi}_{q,t}(M(P)) \) with \( m_P = Y_{1,1} Y_{2,q} \) is given by

\[
\begin{array}{c}
1 \\
1,2 \\
1,2,3 \\
1,2,3,4
\end{array}
\begin{array}{c}
2, q^2 \\
1, q \\
1, q \\
1, q \\
1, q
\end{array}
\begin{array}{c}
1, q \\
1, q \\
1, q \\
1, q \\
1, q
\end{array}
\begin{array}{c}
2 \\
2, q^2 \\
2, q^2 \\
2, q^2 \\
2, q^2
\end{array}
\begin{array}{c}
1, q \\
1, q \\
1, q \\
1, q \\
1, q
\end{array}
\begin{array}{c}
2 \\
2, q^2 \\
2, q^2 \\
2, q^2 \\
2, q^2
\end{array}
\begin{array}{c}
1, q^3 \\
1, q^3 \\
1, q^3 \\
1, q^3 \\
1, q^3
\end{array}
\begin{array}{c}
2 \\
2, q^2 \\
2, q^2 \\
2, q^2 \\
2, q^2
\end{array}
\begin{array}{c}
1, q^3 \\
1, q^3 \\
1, q^3 \\
1, q^3 \\
1, q^3
\end{array}
\begin{array}{c}
2 \\
2, q^2 \\
2, q^2 \\
2, q^2 \\
2, q^2
\end{array}
\begin{array}{c}
1, q^3 \\
1, q^3 \\
1, q^3 \\
1, q^3 \\
1, q^3
\end{array}
\begin{array}{c}
2 \\
2, q^2 \\
2, q^2 \\
2, q^2 \\
2, q^2
\end{array}
\end{array}
\]
(2) \( \bar{\chi}_{q,t}(M(P)) \) with \( m_P = Y_{1,1}^2 \) is given by

\[
\begin{array}{c}
1 & 1 \\
1 & 2 \\
2 & 2 \\
\end{array}
\]
\[
\begin{array}{c}
1 & 1 \\
1 & 3 \\
3 & 3 \\
\end{array}
\]

(3) \( \bar{\chi}_{q,t}(M(P)) \) with \( m_P = Y_{1,1}Y_{1,q^2} \) is given by

\[
\begin{array}{c}
1 & 1 \\
1 & 2 \\
2 & 1 \\
\end{array}
\]
\[
\begin{array}{c}
1 & 1 \\
1 & 2 \\
2 & 2 \\
\end{array}
\]

Remark 4.12. (1) In [22] a different convention for tableaux was used. Each column is located so that \( T_1(b), T_2(bq^2), \ldots \) appear in the same row.

(2) When the shapes of tableaux are those of ordinary Young tableaux, i.e., the tops of columns are the same and the lengths are nonincreasing, \( d(T) \) is equal to \( C - l(T^t) \) where \( T^t \) is the transpose of \( T \), \( l(T^t) \) was defined in [16], and \( C \) is a constant depending only the shape of \( T \) and numbers of figures. (In fact, \( C \) is equal to \( \frac{1}{2} \dim_{\mathbb{C}} \mathfrak{M}(v, w) \), where \( \mathfrak{M}(v, w) \) is the quiver variety containing the point corresponding to \( T \).) Moreover, the assumption of Proposition 3.4 is satisfied in this case. (See also [19] §8.5.)

5. type \( D_n \)

We number the vertices as follows.

\[
\begin{array}{c}
1 & 2 & \ldots & n-2 \\
1 & 2 & \ldots & n-1 \\
n & \underline{n} & \underline{n} & \underline{n} \\
\end{array}
\]

The vector representation of \( g \) is known to be lifted to a \( \mathbf{U}_q(\mathbf{L}_g) \)-module. It is an \( \ell \)-fundamental representation \( \mathbf{L}(\Lambda_1)_\ell \). The graph of its \( q \)-character is the same as that of crystal \( \mathcal{B}(\Lambda_1) \) as in the case of \( A_n \). It is
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where

\[
\begin{align*}
\pi_i &= Y^{i-1}_{i-1,aq} Y_{i,aq}^{i-1} \quad (1 \leq i \leq n - 2) \\
\pi_{n-1} &= Y^{n-1}_{n-2,aq}^{-1} Y_{n-1,aq} Y_{n,aq}^{n-2} \\
\pi_n &= Y^{n-1}_{n-1,aq} Y_{n,aq}^{-1} \\
\pi_{n-2} &= Y_{n-2,aq}^{-1} Y_{n-1,aq}^{n-1} Y_{n,aq}^{-1} \\
\pi_{n-3} &= Y_{i-1,aq}^{n-2-i} Y_{i,aq}^{n-1-i} \quad (1 \leq i \leq n - 2).
\end{align*}
\]

Here $Y_{0,b}$ is understood as 1. This is also easily shown by Theorem 2.3. (The notation is borrowed from [11].)

Let $B = \{1, \ldots, n, n, \ldots, 1\}$. We give the ordering $\prec$ on the set $B$ by

\[
1 \prec 2 \prec \cdots \prec n - 1 \prec \pi \prec \pi^{-1} \prec 2 \prec 1.
\]

Remark that there is no order between $n$ and $\pi$.

We define a tableau $T$ and its associated monomial exactly as in the type $A_n$ case. We just replace $B$. (So far we do not include column corresponding to spin representations.) The following is an analog of Lemma 4.4. (In fact, it will not be used later.)

**Lemma 5.1.** Let $T$ and $T'$ be tableaux. Then the corresponding monomials $m_T$ and $m_{T'}$ are equal if and only if $T$ and $T'$ become equivalent after we add several pairs of columns

\[
\begin{array}{cccc}
\pi & \cdots & \pi & \cdots & \pi \\
1 & \cdots & 1 & \cdots & 1 \\
\vdots & \cdots & \vdots & \cdots & \vdots \\
i & \cdots & i
\end{array}
\]

for some $i \in \{1, \ldots, n\}$, $a \in \mathbb{C}^*$ to $T$ and $T'$.

**Proof.** Let $\#^i_{\pi^i}$ be the number of boxes with entry $i$ in the row corresponding to $a$ of $T$ minus that of $T'$. Then $m_T = m_{T'}$ if and only if

\[
\begin{align*}
\#^i_{\pi^i} Y_{aq}^{i+1} - \#^i_{n-aq} &= \#^i_{\pi^{-1}aq} Y_{aq}^{i+1} - \#^i_{\pi^{-1}aq} Y_{aq}^{i+1} \quad \text{for } 1 \leq i \leq n - 2 \\
\#^i_{n-aq} Y_{aq}^{i+2} - \#^i_{\pi^{-1}aq} &= \#^i_{\pi^{-1}aq} Y_{aq}^{i+2} - \#^i_{\pi^{-1}aq} Y_{aq}^{i+2} \\
\#^i_{\pi^{-1}aq} Y_{aq}^{i+3} + \#^i_{n-aq} Y_{aq}^{i+1} &= \#^i_{\pi^{-1}aq} Y_{aq}^{i+3} + \#^i_{\pi^{-1}aq} Y_{aq}^{i+1}.
\end{align*}
\]

From the second and third equations we get

\[
\#^i_{n-aq} Y_{aq}^{i+2} + \#^i_{\pi^{-1}aq} = \#^i_{\pi^{-1}aq} Y_{aq}^{i+2} + \#^i_{\pi^{-1}aq}.
\]

Moving $a \in \mathbb{C}^*$, we get

\[
\#^i_{\pi^i} = \#^i_{\pi^{-1}}.
\]

Set this number $d_{n,a}$. Substituting this back to the second equation, we get

\[
\#^i_{n-aq} Y_{aq}^{i+3} - d_{n,a} = \#^i_{\pi^{-1}aq} - d_{n,aq}.
\]
Set this number $d_{n-1,aq^2}$. Using the first equation, we define $d_{i,aq^2(n-i)}$ inductively by

$$d_{i,aq^2(n-i)} \overset{\text{def}}{=} \# \bigg\{ j \in \mathbb{N} \setminus \{0\} \bigg| d_{j,aq^2(n-j)} = \# \bigg\{ j \in \mathbb{N} \setminus \{0\} \bigg| d_{j,aq^2(n-j)} \bigg\} - \sum_{j:i+1 \leq j \leq n} d_{j,aq^2(n-j)} \bigg\}.$$ 

For each $i \in \{1, \ldots, n\}$, we add $(d_{i,a})$-pairs of columns (as in the statement) to $T'$ if $d_{i,a} > 0$ and we add $(-d_{i,a})$-pairs to $T$ if $d_{i,a} < 0$. The resulting tableaux are equivalent.

We also have an obvious analog of Lemma 4.5 for $D_n$. Let

$$B(A_N)_a = \left\{ T = \begin{array}{c} \vdots \\ i_1 \\ \vdots \\ i_N \end{array} \bigg| a \in B, i_1 \not< i_2 \not< \cdots \not< i_N \right\}.$$ 

We define the associated degree by

$$l(T) = \#\{ p \mid i_p = i, i_{p+n-1-i} = \text{7} \text{ for } i = 1, \ldots, n-2 \}.$$ 

For $i = 1, \ldots, n$ and an integer $s$, we define $p(i,s)$ and $p'(i,s)$ so that

$$\begin{align*}
&\begin{cases}
    s = N - 2p(i,s) + i = N - 2p(i,s) - 2 + 2n - i & \text{if } 1 \leq i \leq n-1,
    s = N - 2p(n,s) + n - 1 = N - 2p(n,s) + n + 1 & \text{if } i = n.
\end{cases}
\end{align*}$$

If such $p(i,s), p'(i,s)$ do not exist, they are undefined. We have

$$u_{i,aq^r}(mT) = \begin{cases}
    \begin{array}{l}
    (i_{p(i,s)} = i) - (i_{p(i,s)+1} = i + 1) \\
    + (i_{p'(i,s)} = \overline{i+1}) - (i_{p'(i,s)+1} = \overline{i})
    \end{array} & \text{if } 1 \leq i \leq n-1, \\
    \begin{array}{l}
    (i_{p(n,s)} = n - 1) + (i_{p(n,s)} = n) \\
    - (i_{p'(n,s)} = \overline{n}) - (i_{p'(n,s)} = n - 1)
    \end{array} & \text{if } i = n.
\end{cases}$$

If $p(i,s), p'(i,s)$ are undefined, the right hand side is understood as 0.

We also have

$$v_{i,aq^{r+1}}(mT,Y_{N,a}) = \begin{cases}
    \begin{array}{l}
    (p(i,s) \leq i < i_{p(i,s)}) + (\overline{i} \leq i_{p'(i,s)})
    \end{array} & \text{if } 1 \leq i \leq n-2, \\
    \begin{array}{l}
    (n \leq i_{p(n-1,s)}) \\
    (\overline{n} \leq i_{p(n,s)})
    \end{array} & \text{if } i = n-1, \\
    \begin{array}{l}
    (\overline{i} \leq i_{p(n,s)})
    \end{array} & \text{if } i = n.
\end{cases}$$

**PROPOSITION 5.5.** For $1 \leq N \leq n-2$, we have

$$\sum_{T \in B(A_N)_a} i^{2l(T)}m_T.$$ 

**PROOF.** Let $m$ be a monomial in $Y_{i,aq^r}$ for fixed $a$. Following [5], we say $m$ is right negative if the factor $Y_{i,aq^r}$ appearing in $m$, for which $s$ is maximal, have negative powers. The product of right negative monomials is right negative. An $l$-dominant monomial is not right negative.

Let us prove that if $m_T$ is not right negative, then $i_1 = 1, \ldots, i_N = N$ by induction. Since $m_T$ is not right negative, there exists $p_0$ such that $i_{p_0} \not< i_{p_0+1}$ is not right negative, that is $i_{p_0} = 1$. By the rule $i_p \not< i_{p+1}$, we have $p_0 = 1$, i.e., $i_1 = 1$. This proves the first step of the induction.
Suppose that we know $i_p = p$ for $p = 1, \ldots, k$. Consider

$$m' = m_p Y_{k,aq^{N-k}}^{-1} = \prod_{s=0}^{k} Y_{aq^{N-1-2s}}^{-1} Y_{aq^{N-3-2k}}^{-1} \cdots Y_{aq^{N-2k}}^{-1}.$$ 

This is right negative since all $Y_{aq^p}$ appearing in $m'$, for which $s$ is maximal, must be equal to $Y_{k,aq^{-k}}^{-1}$ since $m_T$ is not right negative. By (5.3), $Y_{aq^{N-k}}^{-1}$ appears only when $i_p(k,N-k)+1 = k+1$ or $i_p'(k,N-k)+1 = k$. We have $p(k,N-k) = k$, $p'(k,N-k) = n-1$. So the latter case does not occur since $p'(k,N-k) \leq N \leq n-2$. Thus we have $i_p = p$ with $p = k+1$. This completes the induction step. In particular, the only $l$–dominant term in (5.6) is the $l$–highest weight term $Y_{N,a}$.

Next we show that the right hand side of (5.6) satisfies the condition 2.3(1) for $i = 1, \ldots, n$. We only give the proof for the case $i \neq n$. The case $i = n$ can be checked in a similar way.

Let $T$ be as above. We consider the following statement for $T$:

1. $i$ occurs, but $i + 1$ does not occur.
2. $i$ does not occur, but $i + 1$ occurs.
3. Both $i$ and $i + 1$ occur (hence consecutively), or neither occurs.
4. $\overline{i+1}$ occurs, but $\overline{i}$ does not occur.
5. $\overline{i+1}$ does not occur, but $\overline{i}$ occurs.
6. Both $\overline{i+1}$ and $\overline{i}$ occur (hence consecutively), or neither occurs.

Tableaux of type (1) and (2) appear in pairs, i.e., they are obtained by the replacement of $i$ and $i + 1$. If $T$ and $T'$ are such a pair, we have

$$m_T + m_{T'} = Y_{i,aq^t}(1 + A_{i,aq^{t+1}}^{-1})M,$$

for some $s$. Here $M$ is the contribution from the other terms. Similarly sequences of type (4) and (5) appear in pairs, and we have (5.7) for the pair $(T, T')$. Monomials of sequences of other types (i.e., (3) and (6)) do not contain $Y_{i,aq^t}$. This proves the assertion when $t = 1$.

Our remaining task is to study the exponent of $t$. First consider the case when $T$ satisfies both (1) and (4). So $i_p = i, i_{p'} = \overline{i+1}$ for some $p, p'$. Then $T$ is a member of a quadruplet $(T, T', T'', T''')$, where other members are obtained from $T$ by replacing $i, \overline{i+1}$ by $i+1, i$. First consider the case $p' = p + n - 1 - i$. We have

$$t^{2l(T)}m_T + t^{2l(T')}m_{T'} + t^{2l(T'')}m_{T''} + t^{2l(T''')}m_{T'''}$$

$$= Y_{i,aq^{N-2p+i}}^2 + (t^2 + 1)A_{i,aq^{N-2p+i+1}}^{-1} + A_{i,aq^{N-2p+i+1}}^{-2}$$

$$= E_i(Y_{i,aq^{N-2p+i}}^2 M)$$

by the definition of $l(T)$ in (5.2). Here $M$ does not contain the factor $Y_{i,b}^\pm$ for any $b \in \mathbb{C}^*$. This contribution satisfies the condition 2.3(1).
Next consider the case \( p' = p + n - 2 - i \). We have

\[
\begin{align*}
t^{2(\mathcal{T})}m_T + t^{2(\mathcal{T}')}m_{T'} + t^{2(\mathcal{T}'')}m_{T''} + t^{2(\mathcal{T}'')}m_{T'''} & = Y_{i,\alpha q^{-2p+i}} Y_{i,\alpha q^{-2p+i+2k}} \\
& \times \left( 1 + i^2A_i^{-1}A_i^{-1}A_i^{-1} + A_{i,\alpha q^{-2p+i+1}} + A_{i,\alpha q^{-2p+i+3}} + A_{i,\alpha q^{-2p+i+1}}^{-1}A_{i,\alpha q^{-2p+i+3}}^{-1} \right) M \\
& = Y_{i,\alpha q^{-2p+i}} Y_{i,\alpha q^{-2p+i}} \left( 1 + A_{i,\alpha q^{-2p+i+1}}^{-1}A_{i,\alpha q^{-2p+i+1}}^{-1} \right) \left( 1 + A_{i,\alpha q^{-2p+i-1}}^{-1}A_{i,\alpha q^{-2p+i-1}}^{-1} \right) M \\
& + (t^{2} - 1)Y_{i,\alpha q^{-2p+i}} Y_{i,\alpha q^{-2p+i}} A_{i,\alpha q^{-2p+i+1}}^{-1}M.
\end{align*}
\]

Since

\[
Y_{i,\alpha q^{-2p+i}} Y_{i,\alpha q^{-2p+i+2k}} A_{i,\alpha q^{-2p+i+1}}^{-1} \in \mathbb{Z}[Y_{j,b}^{\pm} j \neq i],
\]

this contribution also satisfies 2.3(1).

In the remaining case \( p' \neq p + n - 1 - i, p + n - 2 - i \), we have

\[
\begin{align*}
t^{2(\mathcal{T})}m_T + t^{2(\mathcal{T}')}m_{T'} + t^{2(\mathcal{T}'')}m_{T''} + t^{2(\mathcal{T}'')}m_{T'''} & = Y_{i,\alpha q^{-2p+i}} Y_{i,\alpha q^{-2p+i+2k}} \\
& \times \left( 1 + A_{i,\alpha q^{-2p+i+1}}^{-1}A_{i,\alpha q^{-2p+i+1}}^{-1} \right) \left( 1 + A_{i,\alpha q^{-2p+i-1}}^{-1}A_{i,\alpha q^{-2p+i-1}}^{-1} \right) M
\end{align*}
\]

This also satisfies 2.3(1).

Next consider the case when \( T \) satisfies both (1) and (6). We have \( i_p = i \) for some \( p \). Then \( T \) appears in a pair \( (T, T') \) as above. The exponents \( l(T) \) and \( l(T') \) are possibly different only if \( i_{p+n-1-i} = \overline{7} \) or \( i_{p+n-2-i} = \overline{7} + 1 \). But the condition (6) implies \( i_{p+n-2-i} = \overline{7} + 1 \) and \( i_{p+n-1-i} = \overline{7} \) in either cases. Thus the exponents \( l(T) \) and \( l(T') \) are the same even in this case. Then we have

\[
t^{2(\mathcal{T})}m_T + t^{2(\mathcal{T}')}m_{T'} = Y_{i,\alpha q^{-2p+i}} (1 + A_{i,\alpha q^{-2p+i+1}}^{-1})M.
\]

This satisfies 2.3(1). In the case \( T \) satisfies (3) and (4), we similarly have 2.3(1). In the remaining case when \( T \) satisfies (3) and (6), \( m_T \) does not contain the factor \( Y_{i,b} \) for any \( b \). Thus it satisfies 2.3(1). Hence the right hand side of (5.6) satisfies 2.3(1). \( \square \)

**Example 5.8.** Let \( \mathfrak{g} = D_4 \) and \( M(P) = L(\Lambda_2) \). The graph \( \Gamma_P \) is Figure 11. The same example was appeared in [19, 5.3.2]. The subscripts \( q^{-1} \) are not written. A careful reader finds that the tableaux appearing here are slightly different from those in Figure 11.

**Remark 5.9.** Let \( \text{Res} \) be the functor sending \( \mathbf{U}_q(L\mathfrak{g}) \)-modules to \( \mathbf{U}_q(\mathfrak{g}) \)-modules by restriction. As was shown in [14], Proposition 5.5 implies that the restriction \( \text{Res} L(\Lambda_N)_a \) decomposes as

\[
\text{Res} L(\Lambda_N)_a = \begin{cases} V(\Lambda_N) \oplus V(\Lambda_{N-2}) \oplus \cdots \oplus V(\Lambda_4) \oplus V(\Lambda_1) & \text{if } N \text{ is odd}, \\ V(\Lambda_N) \oplus V(\Lambda_{N-2}) \oplus \cdots \oplus V(\Lambda_2) \oplus V(0) & \text{if } N \text{ is even}, \end{cases}
\]

where \( V(\lambda) \) is the irreducible highest weight \( \mathbf{U}_q(\mathfrak{g}) \)-module with highest weight \( \lambda \). This follows from the observation that the (ordinary) character of \( V(\Lambda_N) \) is also described by tableaux sum, but with an extra condition that \( (i_p, i_{p+1}) \neq (7, n) \).
Figure 1. The graph for $L(\Lambda_2)_1$
5.1. Spin representations. It is known that spin representations of $\mathfrak{g}$ can be lifted to a $U_q(\mathfrak{Lg})$-module. As in above cases, the vertex of the graph of $\chi_{q,t}$ is the same as that of crystal and all coefficients are 1. Following (11), we introduce the half size numbered box:

\[ I = \begin{cases} Y_{i-1,aq^{i-1}}^{-1} Y_{i,aq^{i-2}} & \text{if } 1 \leq i \leq n - 2, \\ Y_{n-1,aq^{n-2}}^{-1} & \text{if } i = n - 1, \\ Y_{n,aq} & \text{if } i = n. \end{cases} \]

\[ I = \begin{cases} Y_{n-1,aq^{n+1}}^{-1} Y_{n,aq^{n+1}}^{-1} & \text{if } 1 \leq i \leq n - 2, \\ Y_{n-1,aq^{n+1}}^{-1} & \text{if } i = n. \end{cases} \]

Let $E_{s,p}^+$ (resp. $E_{s,p}^-$) be

\[ E_{s,p}^+ = \left\{ T \mid i_1 < i_2 < \cdots < i_n \right\}. \]

We define $m_T$ as before. We have

\[ u_{i,aq^{n-2p+1}}(m_T) = (i_p = i) - (i_{p+1} = i + 1) \quad \text{if } 1 \leq i \leq n - 2, \quad (5.10) \]

\[ u_{n-1,aq^{n-2p}}(m_T) = (i_p = n - 1) - (i_{p+1} = \overline{n}), \]

\[ u_{n,aq^{n-2p}}(m_T) = - (i_p = n - 1) + (i_{p+1} = n). \]

The proof of the following is left to the reader as an exercise.

**Proposition 5.11.**

\[ \tilde{\chi}_{q,t}(L(\Lambda_{n-1})_a) = \sum_{T \in E_{s,p}^+} m_T, \quad \tilde{\chi}_{q,t}(L(\Lambda_n)_a) = \sum_{T \in E_{s,p}^+} m_T. \]

In particular, we find that $u_{i,aq^p}(m_T)$ is at most 1 and if $u_{i,aq^p}(m_T) = 1$, then $u_{i,b}(m_T) = 0$ for other $b$. This implies that the graph $\Gamma_P$ is the same as the crystal graph. More precisely, edges are given by

\[ aq^{n+1-2p} \xrightarrow{i,aq^{n-2p+1}} \cdots \xrightarrow{i,aq^{n+1-2p'}} (1 \leq i \leq n - 1), \]

\[ \cdots \xrightarrow{i,aq^{n+1-2p}} aq^{n+1-2p} \]

We have

\[ v_{i,aq^{n-2p+1}}(m_T, m_P) = \begin{cases} (p \leq i < i_p) & \text{if } 1 \leq i \leq n - 2, \\ (p \neq n \leq i_p) & \text{if } i = n - 1, \\ (\overline{p} \neq \overline{n} \leq i_p) & \text{if } i = n. \end{cases} \]

Suppose that a Drinfeld polynomial $P$ is given. We define the set of column increasing tableaux $B(P)$ as in the type $A_n$ case. For a tableau $T = (T_1, T_2, \ldots, T_l) \in B(P)$, we define $d(T_\alpha, T_\beta) \overset{\text{def}}{=} d(m_{T_\alpha}, m_{T_\beta}; m_{T_\beta}, m_{T_\alpha})$, substituting (5.3, 5.4, 5.10) into (2.3). (We do not try to simplify the expression as in the type $A_n$ case.) Then we set $d(T) = \sum_{\alpha \neq \beta} d(T_\alpha, T_\beta)$ as before. We also set $l(T) = \sum_{\alpha} l(T_\alpha)$, where $l(T_\alpha)$ was defined in (5.3). We get
Theorem 5.13.

\[ \tilde{\chi}_{q,t}(M(P)) = \sum_{T \in B(P)} t^{2d(T)+2l(T)} m_T. \]

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