Binary Mixture of quasi one dimensional dipolar Bose Einstein Condensates with tilted dipoles

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Abstract

We consider a $^{168}$Er-$^{164}$Dy dipolar mixture, trapped by a cigar shaped harmonic potential. We derive the quasi-1D inter-species effective potential exhibiting the tilting angles and show that it is a quite natural generalization of the situation of a single dipolar gas. By solving the coupled Gross-Pitaevskii equations, we observe a transition from miscible to immiscible mixture as the orientations of the magnetic moments are varied. The atom numbers are also shown to lead to noticeable effects on the mixture.

1 Introduction

The nature of inter particle interactions in ultracold gases at low-dimensions has been extensively studied theoretically and experimentally over the last years[1, 2]. In one and two dimensions, physics is qualitatively (and quantitatively) different and new phenomena are observed [3, 4, 5, 6].

The properties of quantum gases are crucially determined by the leading role of interactions, which are in turn affected by a constrained geometry. This can be bestly seen when the interactions are anistropic and/or long ranged[7, 8, 9, 10, 11]. For instance, for dipolar gases, not only the amplitude but also the sign of the interactions are strongly affected by the trapping geometry. Indeed, the stability diagram contains the trap anisotropy as a crucial parameter[6, 12, 13, 14, 15, 16, 17].
Binary mixtures of cold atoms or cold molecules show a richer phase diagram than a single species gas owing to the existence of many different interactions. Indeed, in addition to the intra-species interactions, the inter-species forces have been predicted and then shown experimentally to play a prominent role in the dynamics of the mixture [18, 19].

A relevant characteristic of binary mixtures is their ability to mix and demix depending on various parameters, such as the intra and inter-species interactions, the number of particles as well as the trapping geometries. In this context, much have been done in the full three dimensional as well as in the quasi-2D and quasi-1D cases with dipolar gases [20, 21, 22, 23] with the predictions of structure formation due essentially to the dipolar interactions. This mixing-demixing behavior is bestly described by the overlap of the macroscopic wave functions and has recently attracted attention [22, 24].

For dipolar binary mixtures, another tuning parameter that governs the miscibility-immiscibility transition as well as the stability regions is the orientation of the magnetic moment. Furthermore, it may also be used to control the anisotropy by leading to a roton-like Bogoliubov spectrum [10] and may drive the condensate to a phonon instability [25].

Except from few works on single dipolar gases, such as [25] who have considered 2D bright solitons in a dipolar BEC, or [20] who discussed a quasi-2D BEC with tilted dipoles, the studies of dipolar bose mixtures under various orientations of the magnetic moments for strong anisotropies are, to our knowledge, just at their beginning. As a first example, we can cite in particular the work of [27] where one of the key results is that ”the long-range repulsion tends to suppress the spatial structure induced by the immiscibility, while the nonlocal attraction helps to enhance it”. We will show in our numerical treatment that it is indeed the case, namely that changing the orientations of the dipoles and therefore changing the interactions from attractive to repulsive, induces a clear tendency to mixing.

Moreover, when studying a $^{168}$Er-$^{164}$Dy mixture, the authors of [22] have found that the mixture is partially miscible (as compared to for instance a $^{164}$Dy-$^{162}$Dy mixture which is completely miscible), and this was attributed to the inter-species dipolar strength. The stability window, in terms of the fraction of atom numbers, was also found to be very narrow for pancake traps. In our case, we will show that, not only the stability window can be wider for cigare-shaped traps, accommodating higher numbers of atoms in each species and therefore almost reaching the Thomas-Fermi regime, but also that the mixture can be made completely miscible by just changing the orientations of the magnetic moments of the two species while keeping fixed all the other parameters. We will also demonstrate, based on energy arguments, that it is the less energetic species, with an attractive dipolar component, that will occupy the center of the trap. By contrast, it is noted in [23] that ”the species containing the larger number of atoms stay at the center
and the species containing the smaller number of atoms breaks into two equal parts and stays symmetrically on two sides with a minimum of interspecies overlap. This claim is however parameter and model dependent. Indeed, in the previous reference, the author considers first that the two species have different scattering lengths and then gradually lowers the axial trap to get an axially free gas.

In order to extend the discussions started in the preceding quoted papers, we aim in this work at examining more deeply how a dipolar binary mixture is affected by the changes of the orientations of the magnetic moments. More specifically, owing to the actual experimental studies and to the high magnetic moments of the atoms involved, we choose a mixture composed of $^{168}$Er and $^{164}$Dy gases.

We will focus on a quasi-1D geometry since it provides a strong anisotropic problem which will help us to apprehend better the role of the tilting angles. Indeed, varying the orientations of the magnetic moments may affect not only the sign of the dipole-dipole interactions (DDI), changing them from repulsive to attractive, but also the anisotropy of the system even if everything else is kept fixed. This was shown by Gligoric et al. in the two extreme limits (side by side and head to tail). Furthermore, the DDI may drastically change the properties of the system in quasi-1D even if the short range interactions dominate the physics in the full 3D geometry.

To this end, we construct first the quasi-1D effective potential for both the intra-species and the inter-species interactions. While the former has been derived elsewhere, we show that the latter nicely generalizes the expression derived in to the case of mixture of different species and depending on both orientations of the magnetic moments, expression which was not a priori foreseeable since it depends on both the relative angle and the total angle.

The paper is organized as follows. In the next section, we present the zero temperature model for a trapped binary mixture of dipolar gases in the mean field approach and obtain its quasi-one dimensional reduction when the transverse degrees of freedom are frozen.

Section 3 is devoted to the numerical resolution of the two coupled Gross-Pitaevskii equations. The long range character of the DDI makes the numerical problem (being an integro-differential one) quite challenging and we build a special algorithm for it. We then study the behavior of the densities of each species in terms of the orientations of the magnetic moments. We also examine their individual energies and show that the species with lower energy and an attractive dipolar component does indeed occupy the center of the trap. We then observe how the mixing and demixing occurs as one varies the tilting angles. An interesting parameter is a measure of the overlap between the wave functions which we define and plot as a function of the tilting. As it can easily be understood, this
parameter is seen to be minimum for a head-to-tail configuration while it is maximum for a side by side configuration. Energy arguments provide simple explanations of these behaviors. But since energy depends also on the number of atoms in each species, it is quite natural to pursue an analogous analysis along the same preceding lines by examining the role of the number of atoms in the mixing-demixing process.

The last section summarises our conclusions and provides some perspectives for forthcoming works.

2 The model

In the following, we consider a dipolar Bose gas composed of two-species, with inter and intra-species dipolar and contact interactions. The system is enclosed in a cigar-shaped harmonic potential, which is an axially-symmetric trap with strong transverse confinement, generated by a deep optical lattice. The confinement is ensured by two harmonic potentials $V_{ex,i} = \frac{m_i}{2}(\omega_i^2 x^2 + \omega_{i\perp}^2 \rho^2)$ $(i = 1, 2)$, $\omega_i$ and $\omega_{i\perp}$ being the trap frequencies of the axial ($\vec{x}$) and transverse ($\vec{\rho}$) directions respectively, with $\omega_{i\perp} >> \omega_i$ in order to freeze the transverse degrees of freedom. In the mean-field approximation, the Hamiltonian of the system writes

$$H = \sum_{i=1}^{2} \int dr \Psi_{i}^{\dagger}(r,t)(-\frac{\hbar^2}{2m_i} \nabla^2 + V_{ex,i}(r))\Psi_{i}(r,t) + \sum_{i=1}^{2} \int dr dr' \Psi_{i}^{\dagger}(r,t)\Psi_{i}^{\dagger}(r',t)U_{i}(r,r')\Psi_{i}(r',t)\Psi_{i}(r,t) + \int dr dr' \Psi_{1}^{\dagger}(r,t)\Psi_{2}^{\dagger}(r',t)U_{12}(r,r')\Psi_{1}(r',t)\Psi_{2}(r,t)$$

(2.1)

where $\Psi_{i}^{\dagger}(r,t)$ and $\Psi_{i}(r,t)$ are the boson creation and annihilation operators for the species $i$, $U_{i}(r,r')$ are the intra-species potentials

$$U_{i}(r,r') = \frac{4\pi\hbar^2 a_{i}}{m_i} \delta(r-r') + \frac{\mu_0 \mu_i^2}{4\pi} U_{i}^{(3D)}(r-r'),$$

(2.2)

and $U_{12}(r,r') = U_{21}(r',r)$ the inter-species potential

$$U_{12}(r,r') = \frac{2\pi\hbar^2 a_{12}}{m_r} \delta(r-r') + \frac{\mu_0 \mu_1 \mu_2}{4\pi} U_{12}^{(3D)}(r-r'),$$

(2.3)

where we have considered the most general contact+dipolar interactions. In the previous expressions, $m_i$, $a_i$, and $\mu_i$ are the masses, the s-wave scattering lengths and the magnetic moments of the two species, $m_r = \frac{m_1 m_2}{m_1 + m_2}$ the reduced mass, $a_{12}$ the s-wave scattering length corresponding to the binary interaction. $\mu_0$ is the permeability of free space.

The dipolar potentials $U_{i}^{(3D)}_{dd}$ and $U_{i}^{(3D)}_{dd_{12}}$ are defined as usual by

$$U_{i}^{(3D)}_{dd}(r-r') = \frac{1 - 3 \cos^2 \theta_i}{|r-r'|^3},$$

$$U_{i}^{(3D)}_{dd_{12}}(r-r') = \frac{\cos(\theta_{1}-\theta_{2}) - 3 \cos \theta_1 \cos \theta_2}{|r-r'|^3},$$

(2.4)
where \( \theta_i \) are the angles formed by the vector \( \mathbf{r} - \mathbf{r}' \) and the magnetic moments \( \mathbf{\mu}_i \) (see appendix).

Upon introducing the aspect ratios \( \lambda_i = \frac{m_i \omega_i}{\omega_{1\perp}} \) and using the notations of [21], the coupled Gross-Pitaevskii equations (GPE) write as

\[
\begin{align*}
\frac{\partial \Psi_1(x,t)}{\partial t} &= \left\{ -\frac{1}{2} \nabla_x^2 + \frac{1}{2} \left( \lambda_1^2 \lambda_2^2 + \rho^2 \right) + g_1^{(3D)} |\Psi_1(x,t)|^2 + g_{dd}^{(1)} \int U_{dd}^{(3D)}(x-x') |\Psi_1(x',t)|^2 dx' \\
&+ g_{12}^{(3D)} |\Psi_2(x,t)|^2 + g_{dd}^{(12)} \int U_{dd}^{(3D)}(x-x') |\Psi_2(x',t)|^2 dx' \right\} \Psi_1(x,t), \\
\frac{\partial \Psi_2(x,t)}{\partial t} &= \left\{ -\frac{1}{2} \nabla_x^2 + \frac{m_\omega}{m_{12} \omega_{1\perp}} \left( \lambda_2^2 \lambda_2^2 + \rho^2 \right) + g_2^{(3D)} |\Psi_2(x,t)|^2 + g_{dd}^{(2)} \int U_{dd}^{(3D)}(x-x') |\Psi_2(x',t)|^2 dx' \\
&+ g_{21}^{(3D)} |\Psi_1(x,t)|^2 + g_{dd}^{(21)} \int U_{dd}^{(3D)}(x-x') |\Psi_1(x',t)|^2 dx' \right\} \Psi_2(x,t),
\end{align*}
\]

(2.5)

where \( m_{12} = \frac{m_1}{m_2}, m_\omega = \frac{\omega_1}{\omega_{1\perp}}, g_1^{(3D)} = \frac{4\pi a_1 N_1}{m_r}, g_2^{(3D)} = \frac{4\pi a_2 N_2}{m_r}, g_{dd}^{(1)} = 3N_1 a_{dd}^{(1)}, g_{dd}^{(2)} = 3N_2 a_{dd}^{(2)}, g_{dd}^{(12)} = 3N_1 a_{dd}^{(12)}, g_{21}^{(21)} = 3N_1 a_{dd}^{(12)}. \)

The atom numbers \( N_1 \) and \( N_2 \) have been introduced in order for the wave functions to be normalized to unity.

In the system (2.5-2.6), lengths are expressed in units of the oscillator length \( l_1 = \sqrt{\frac{\hbar}{m_1 \omega_{1\perp}}} \) and densities in units of \( l_1^{-3} \). We also express energy and time in units of \( \hbar \omega_{1\perp} \) and \( \omega_{1\perp} \), respectively. For the sake of clarity, we have also introduced length scales \( a_{dd}^{(i)} \) and \( a_{dd}^{(12)} \), corresponding to the DDI defined by \( \frac{\mu_{0\perp}^2}{4\pi} = \frac{3\hbar^2}{m_i a_{dd}^{(i)}} \) and \( \frac{\mu_{0\perp}^2 a_{dd}^{(i)}}{4\pi} = \frac{3\hbar^2}{m_i a_{dd}^{(12)}} \).

For a cigar-shaped trap, the transverse degrees of freedom are frozen. The kinematics of the system can be considered as quasi-one dimensional. In this situation, a good approximation is to split the individual wave functions into products of ground state functions of the harmonic oscillator (in the \( \rho \) direction) and functions of \( x, t \) alone. We may therefore take

\[
\Psi_i(x,t) = \frac{1}{\sqrt{\pi l_i^2}} \exp(-\rho^2/2l_i^2) \psi_i(x,t).
\]

(2.7)

Inserting the ansatz (2.7) into the coupled GPE (2.5-2.6), multiplying both sides by \( \exp(-\rho^2/2l_i^2) \) and integrating over \( \rho \), we get the final quasi-1D coupled GPE:

\[
\begin{align*}
\frac{\partial \psi_1(x,t)}{\partial t} &= \left\{ -\frac{1}{2} \nabla_x^2 + \frac{1}{2} \lambda_1^2 \lambda_2^2 + g_1^{(3D)} \int U_{dd}^{(1D)}(x-x') |\psi_1(x',t)|^2 dx' \\
&+ g_{12}^{(3D)} |\psi_2(x,t)|^2 + g_{dd}^{(12)} \int U_{dd}^{(1D)}(x-x') |\psi_2(x',t)|^2 dx' \right\} \psi_1(x,t), \\
\frac{\partial \psi_2(x,t)}{\partial t} &= \left\{ -\frac{1}{2} \nabla_x^2 + \frac{1}{2} \lambda_2^2 \lambda_2^2 + g_2^{(3D)} \int U_{dd}^{(1D)}(x-x') |\psi_2(x',t)|^2 dx' \\
&+ g_{21}^{(3D)} |\psi_1(x,t)|^2 + g_{dd}^{(21)} \int U_{dd}^{(1D)}(x-x') |\psi_1(x',t)|^2 dx' \right\} \psi_2(x,t),
\end{align*}
\]

(2.8-2.9)
In these equations, $U^{(1D)}_{dd_i}$ are the quasi-1D intra-species DDI given by [2, 5, 7, 8, 30, 31]

$$U^{(1D)}_{dd_i}(v_i) = \frac{1 + 3 \cos 2\alpha_i}{4l_i^3} \left( \frac{4}{3} \delta(v_i) + 2\sqrt{v_i} - \sqrt{\pi}(1 + 2v_i)e^{v_i}\text{erfc}(\sqrt{v_i}) \right),$$

(2.10)

where $v_i = (x - x')^2/l_i^2$, $\alpha_i$ are the angles formed by the magnetic moments and the $x$ axis and erfc is the complementary error function. Moreover, $U^{(1D)}_{dd_12}$ is the quasi-1D inter-species DDI which can be shown to write as (see appendix for more details)

$$U^{(1D)}_{dd_12}(v_r) = \cos(\alpha_1 - \alpha_2) + \frac{3 \cos(\alpha_1 + \alpha_2)}{2(l_1^2 + l_2^2)^{3/2}} \left( \frac{4}{3} \delta(v_r) + 2\sqrt{v_r} - \sqrt{\pi}(1 + 2v_r)e^{v_r}\text{erfc}(\sqrt{v_r}) \right),$$

(2.11)

where $v_r = (x - x')^2/(l_1^2 + l_2^2)$. One notices that the expressions (2.10) and (2.11) are symmetric around $\alpha_i = \pi/2$ and become equivalent when the tilting angles are equal. Furthermore, the three dipolar potentials are attractive when $\alpha_i \leq 0.5 \arccos(-1/3)$ and repulsive otherwise. This will have great implications in the following.

### 3 Numerical results

In order to solve the coupled GPE (2.8-2.9), we use imaginary-time propagation with Crank Nicolson method as depicted in [20, 23, 30, 32]. This provides the ground state solutions of the mixture. The DDI are evaluated by means of a fast Fourier transform [33].

One may notice on (2.8-2.9) that the static properties depend on a large number of control parameters, including the number of particles in each species, the strengths of the two types of interactions (contact and DDI), the trap geometry as well as the orientations of the dipoles. Since the former parameters have been considered elsewhere [21, 22, 23], we will focus in the following on the effects of the tilting angles. Moreover, in order to be as close as possible to the experimental situations, and since it is quite difficult if not impossible to experimentally vary the angles independently [34], we will for simplicity take $\alpha_1 = \alpha_2 = \alpha$ which will then be varied from 0 up to $\pi/2$.

In the binary mixture, $^{168}\text{Er}$ is the species labeled by 1, with $\mu_1 = 7\mu_B$ ($\mu_B$ the Bohr magneton), $a_{dd}^{(1)} = 66a_0$ ($a_0$ the Bohr radius). For $^{164}\text{Dy}$, we take $\mu_2 = 10\mu_B$ and $a_{dd}^{(2)} = 131a_0$ and this gives $a_{dd}^{(12)} = 25a_0$ [35]. In order to apprehend better the effects of the dipolar interactions on the miscibility-immiscibility transition, we deliberately choose repulsive and equal intra-species contact interactions: $a_{1}^{(3D)} = a_{2}^{(3D)} = 200a_0$. The intra-species contact interactions are also repulsive $a_{12}^{(3D)} = 140a_0$. Have we chosen $a_{12}^{(3D)} < 0$, the net total interaction for $\alpha = 0$ would have been attractive and consequently we will only get mixed configurations. Moreover, the traps have equal parameters: $\omega_{1\perp} = \omega_{2\perp} = 480\pi$, $\lambda_1 = \lambda_2 = 0.2$. 


We begin by fixing \( N_1 = 2000 \) and plot the densities (measured in units of \( 1/l_1 \)) and the energies of the two species \( |\psi_i(x)|^2 \) for \( N_2 = 500 \) as the tilting angle is varied from \( \alpha = 0 \) where the DDI are attractive, up to \( \alpha = \pi/2 \) where they become repulsive. The DDI change their signs for the "magic" angle \( \alpha^* = \frac{1}{2} \arccos (-1/3) \) which is around \( \pi/3 \).

Figure 1 depicts the densities of the two species for various titling angles. For \( \alpha = 0 \), although the DDI are attractive, they are not strong enough to balance the repulsive contact forces. The net result is a lower energy for the Dy as shown in figure 2. The Er density partially splits into two symmetrical parts around the center of the trap, which leads to a quasi-demixed configuration. However, as soon as the tilting angle approaches and then goes beyond \( \alpha = \pi/6 \), this splitting becomes less acute witnessing a tendency to mixing. Indeed, when \( \alpha \) goes beyond the magic angle \( \alpha^* \), the DDI vanish and then turn repulsive, being maximally repulsive for \( \alpha = \pi/2 \). This leads to an increasing positive energy for both species and therefore to an almost total mixing due to the spreading and flattening of the Dy.

In order to see how the mixing-demixing transition is affected by the number of atoms, the calculations are repeated for \( N_2 = 2000 \) (figures 3 and 4) and \( N_2 = 5000 \) (figures 5 and 6). In these situations, the intra and inter-species interactions become comparable in absolute values. In the figures 3 and 5, a partial demixing is occurring at \( \alpha = 0 \) and \( \pi/6 \), but now the Er density falls to zero around the center which means that the Dy atoms are occupying the center of the trap as their energy is lower, and are surrounded by the Er atoms which are then expelled at the peripheries. This partial demixing is more acute for growing Dy atom numbers, since, unlike the figure 3, in the case \( N_2 = 5000 \), there is a clear phase separation as the two peaks of the Er density become more distant and the region where there is no Er atoms becomes wider.

For repulsive DDI, \((\alpha = \pi/3, \pi/2)\), as the energies of the two species become closer (see figures 4 and 6), one observes a tendency to mixing leading to an almost total mixing for \( \alpha = \pi/2 \). This argument is clearly the most physical one. The argument of masses, reported in [22] can no longer be viable since it is most natural to compare the \( a_{dd} \)'s of the two species.

Furthermore, one observes on figure 6 a remarkable variation of the Dy energy around \( \alpha = \pi/4 \). This brutal variation is not evident on the density profiles shown in figure 5. For these orientations of the magnetic moments, the attractive and repulsive parts of the DDI are equal. Since there is no net attractive component, the interaction becomes suddenly repulsive due to the high number of Dy atoms.

This behavior may be well illustrated by the overlap (or miscibility) parameter \( \eta = \int_x \sqrt{\psi_1(x)^2|\psi_2(x)|^2} \) as defined in [22]. This quantity is best suited to the inhomogenous case instead of the parameter \( \Delta \) (see [22] Eq.9).
Figure 1: Density profiles for $^{168}\text{Er}$ ($N_1 = 2000$, purple-continuous) and $^{164}\text{Dy}$ ($N_2 = 500$, green-dashed).

Figure 2: Individual energies for $^{168}\text{Er}$ ($N_1 = 2000$, purple-continuous) and $^{164}\text{Dy}$ ($N_2 = 500$, green-dashed).
Figure 3: Density profiles for $^{168}\text{Er} (N_1 = 2000, \text{purple-continuous})$ and $^{164}\text{Dy} (N_2 = 2000, \text{green-dashed})$.

Figure 4: Individual energies for $^{168}\text{Er} (N_1 = 2000, \text{purple-continuous})$ and $^{164}\text{Dy} (N_2 = 2000, \text{green-dashed})$. 


Figure 5: Density profiles for $^{168}$Er ($N_1 = 2000$, purple-continuous) and $^{164}$Dy ($N_2 = 5000$, green-dashed).

Figure 6: Individual energies for $^{168}$Er ($N_1 = 2000$, purple-continuous) and $^{164}$Dy ($N_2 = 5000$, green-dashed).
The figure 7 depicts $\eta$ as a function of Dy atom number for different tilting angles $\alpha$. This quantity clearly varies from very small values for $\alpha = 0$, where the DDI are maximally attractive leading to a partially demixed system, up to almost 1 (for $\alpha = \pi/2$), where the DDI are maximally repulsive and the system is totally mixed.

![Figure 7](image.png)

Figure 7: Overlap parameter for $N_1 = 2000$ atoms of $^{168}\text{Er}$ versus the $^{164}\text{Dy}$ atom number $N_2$ for various titling angles.

In the figure 8, we represent $\eta$ as a function of $\alpha$ for different Dy atom numbers. On this figure, we observe a strong partial demixing for small tilting angles, which we are tempted to call an immiscible configuration. For growing angles, the overlap nearly reaches its maximum value (depending on $N_2$) in the strongly repulsive DDI case ($\alpha = \pi/2$).

A noticeable feature is the fixed point (around ($\alpha = \pi/5$) which indicates that the overlap crosses a constant value whatever the number of Dy atoms. This result comes from the very definition of $\eta$ since $\psi_2$ is normalized to unity.
4 Concluding remarks

In the present work, we have studied a zero temperature dipolar bose mixture, namely $^{168}$Er-$^{164}$Dy, trapped in quasi-1D geometry in the mean field approximation. We have been particularly concerned with the effect of the orientation of the magnetic moments on the transition from miscibility to immiscibility.

Upon solving numerically the coupled Gross-Pitaevskii equations, we have explored various situations where one changes the tilting angles as well as the number of atoms. The outcomes of our calculations show that a transition from demixed to mixed configuration can be driven by changing the orientations of the dipoles and therefore changing the DDI from attractive to repulsive. This behavior is particularly marked for growing atom numbers, where we observe that the less energetic species occupy the center of the trap while expelling the more energetic one at the peripheries. Indeed, the density of the latter shows two symmetric peaks around the center separated by a void region which becomes wider as the number of atoms of the former species grows up.

Moreover, upon examining the overlap of the macroscopic wave functions of the two species, we notice a clear tendency to maximum mixing due to the repulsive DDI. The demixing is however partial since within the parameter space that we have considered, the immiscibility conditions are not fully reached.
Our results are not only consistent with previous works\cite{22,23} but also extend their claims. Indeed, we found that the mixture can be made completely miscible (the overlap almost reaching its maximum value) by just changing the orientations of the magnetic moments of the two species while keeping fixed all the other parameters. Based on energy arguments, we conclude that it is the less energetic species, with an attractive dipolar component, that will occupy the center of the trap. The argument of masses invoked in\cite{22} is no longer viable. Moreover, in the quasi-1D geometry, the stability window, in terms of the fraction of atom numbers or trap anisotropy, is found to be wider accommodating for higher numbers of atoms in each species and therefore almost reaching the Thomas-Fermi regime at very low aspect ratios. This result can be helpful for future experimental setups.

As an interesting perspective, it would be quite natural to examine the excitation spectrum of this quasi-1D mixture. Indeed, the emergence of an anisotropic roton mode, with very special properties as a function of the polarization angle has been predicted in\cite{36} for a quasi-2D traps. The question is to what extent such a mode may survive in reduced geometries. This and other related questions will be addressed in a forthcoming paper.

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**Appendix**

In this appendix, we provide the details for the computation of the quasi-1D effective potential (2.11) which appears in the Eqs. (2.8-2.9). It is given by the integral expression:

\[
U^{(1D)}_{dd_{12}}(x_1 - x_2) = \frac{1}{\pi^2 l_1^2 l_2^2} \int d^2 \rho_1 d^2 \rho_2 dx_2 U^{(3D)}_{dd_{12}}(r_1 - r_2) \exp(-\rho_1^2/l_1^2) \exp(-\rho_2^2/l_2^2),
\]

(1)

where, up to a factor \(\mu_0 \mu_1 \mu_2/4\pi\), the 3D potential is given by\cite{6}

\[
U^{(3D)}_{dd_{12}}(r) = \frac{(\mathbf{e}_1 \cdot \mathbf{e}_2)r^2 - 3(\mathbf{e}_1 \cdot \mathbf{r})(\mathbf{e}_2 \cdot \mathbf{r})}{r^5}.
\]

(2)

\(\mathbf{e}_1\) and \(\mathbf{e}_2\) are the unit vectors along the dipole moments directions with \(\mathbf{e}_i \cdot \mathbf{r} = \cos \theta_i\) and \(\mathbf{e}_1 \cdot \mathbf{e}_2 = \cos(\theta_1 - \theta_2)\) which leads to (2.4).

In order to compute the integrals appearing in (1), we introduce the relative \((y, z)\) and the center of mass (CM) cartesian coordinates \((Y, Z)\) in the \(y-z\) plane:

\[
y_{1,2} = Y \pm \frac{m_2}{m_1 + m_2} y,
\]

\[
z_{1,2} = Z \pm \frac{m_2}{m_1 + m_2} z.
\]

(3)

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It is now straightforward to notice that the CM coordinates can be integrated out to yield an expression depending solely on the relative coordinates. Noting \( x = x_1 - x_2 \), we get

\[
U_{dd12}^{(1D)}(x) = \frac{1}{\pi(l_1^2 + l_2^2)} \int dydz U_{dd12}^{(3D)}(x, y, z) \exp \left( -\frac{y^2 + z^2}{l_1^2 + l_2^2} \right). \tag{4}
\]

Now assuming that the dipoles are in the \( x - z \) plane, \( e_i = (\cos \alpha_i, 0, \sin \alpha_i) \), we may change back to polar coordinates \( (y = \rho \cos \phi, z = \rho \sin \phi) \). One obtains the simple relations

\[
\cos(\theta_1 - \theta_2) = \cos(\alpha_1 - \alpha_2), \quad \cos \theta_i = \frac{x \cos \alpha_i + \rho \sin \phi \sin \alpha_i}{(x^2 + \rho^2)^{1/2}}, \tag{5}
\]

which allow us to write (4) in the simpler form

\[
U_{dd12}^{(1D)}(x) = \frac{\cos(\alpha_1 - \alpha_2) + 3 \cos(\alpha_1 + \alpha_2)}{2(l_1^2 + l_2^2)} \int_0^\infty d\rho \rho^2 \frac{\rho^2 - 2x^2}{(x^2 + \rho^2)^{5/2}} \exp \left( -\frac{\rho^2}{l_1^2 + l_2^2} \right), \tag{6}
\]

and directly yields the result (2.11).

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