A GENERAL SOLUTION OF THE MASTER EQUATION
FOR A CLASS OF FIRST ORDER SYSTEMS

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Inspired by the formulation of the Batalin-Vilkovisky method of quantization in terms of “odd time”, we show that for a class of gauge theories which are first order in the derivatives, the kinetic term is bilinear in the fields, and the interaction part satisfies some properties, it is possible to give the solution of the master equation in a very simple way. To clarify the general procedure we discuss its application to Yang-Mills theory, massive (abelian) theory in the Stueckelberg formalism, relativistic particle and to the self-interacting antisymmetric tensor field.
The formulation of the Batalin-Vilkovisky (BV) method of quantization in terms of a Grassmann odd parameter behaving as time (“odd time”) was helpful to derive systematically the “ad hoc” definitions of Batalin and Vilkovisky. In Ref. existence of an appropriate Lagrangian for the odd time formulation (“odd time Lagrangian”) was assumed. Although it is possible to write an odd time Lagrangian by using the Hamiltonian formalism of the BV method, it does not give new hints about finding a solution of the master equation of the Lagrangian formalism. In the Hamiltonian formalism a general solution of the master equation is available in terms of the Becchi-Rouet-Stora-Tyutin (BRST)-charge which gives a vanishing generalized Poisson bracket with itself.

Recently, the extended form method developed in Ref. was utilized to write actions for topological quantum field theories which lead to solution of the master equation.

Inspired by these, we show that for a class of gauge theories which are first order in the derivatives, the kinetic part is bilinear in the fields (variables), and the interaction terms possessing some properties, it is possible to write an action which simply leads to the solution of the master equation in the minimal ghost sector. In fact, the actions of Ref. can be obtained from this general method.

First we recall the basic concepts of the odd time formulation of the BV method and discuss the related Lagrangian. This discussion is not essential for the rest of the paper, but it is useful to understand how the solution of the master equation is inspired.

We give the rules of constructing an action in terms of the minimal ghost fields, the antifields and the initial gauge theory action satisfying some conditions, and prove that it is the desired solution of the master equation. We illustrate the method by its application to Yang-Mills theory, massive (abelian) theory in Stueckelberg formalism, the relativistic particle and the self-interacting antisymmetric tensor field.

When we deal with a gauge theory we can introduce the odd time \( \tau_0 \) (a parameter possessing odd Grassmann parity), such that the change of a function \( f \) by the BRST-charge \( \Omega_{BRST} \), is written symbolically as

\[
\Omega_{BRST} f = \frac{\partial f(\tau_0)}{\partial \tau_0}.
\]

We assume that there exists an odd time Lagrangian \( L(\Phi(\tau_0), \dot{\Phi}(\tau_0)) \), which
carries information about the BRST transformations. $\Phi$ includes the original fields of the original gauge theory, and the related ghost fields; $\dot{\Phi}(\tau_0) \equiv \partial \Phi(\tau_0)/\partial \tau_0$. The “odd time canonical momentum” which results from this Lagrangian is

$$\Pi(\tau_0) = \frac{\partial L(\Phi(\tau_0), \dot{\Phi}(\tau_0))}{\partial \dot{\Phi}(\tau_0)}. \quad (1)$$

On the cotangent bundle of a supermanifold an odd canonical two form is known to exist when the cotangent bundle has an equal number of odd and even coordinates [7]. Thus we can define an “odd Poisson bracket” (antibracket)

$$(f, g) \equiv \partial_r f \frac{\partial g}{\partial \Phi} - \partial_l f \frac{\partial g}{\partial \Pi} \frac{\partial \Pi}{\partial \Phi}, \quad (2)$$

where $\partial_r$ and $\partial_l$ indicate the right and the left derivatives. In this phase space odd time evolution is given by the Grassmann-even Hamiltonian $S$:

$$\frac{\partial f}{\partial \tau_0} = (S, f). \quad (3)$$

Thus $S$ must satisfy

$$\frac{\partial S}{\partial \tau_0} = (S, S) = 0. \quad (4)$$

This is the master equation of Batalin and Vilkovisky.

The easiest way of defining an odd time Lagrangian is to take it independent of the velocities. Then

$$S(\Phi) = -L(\Phi),$$

and the master equation is automatically satisfied. Of course, we should give the conditions to construct $L(\Phi)$, such that $S(\Phi)$ is the action of the BV method of quantization.

To gather the original fields and the ghosts one can extend the ordinary differential forms to include also the ghost number. This can be achieved by generalizing the exterior derivative as [4]

$$d \to \tilde{d} \equiv d + \delta, \quad (5)$$

where $\delta$ denotes the BRST transformation. In order to utilize this generalization of $d$ to find the solution of the master equation we follow the following procedure.
i) If the original gauge theory is not already first order in $d$ and the terms containing $d$ are not bilinear in fields, one should find an equivalent formulation of it possessing these properties.

$ii$) The minimal ghost content of the theory should be found by analyzing the related gauge invariance and the proper solution condition of Batalin and Vilkovisky. Generalize the original fields to include also the ghosts and antifields which possess the same grading with the original ones in terms of $\tilde{d}$. Now, substitute the original fields $\phi$, with the generalized ones $\Phi \equiv (\tilde{A}, \tilde{B})$, in the Lagrangian. The resulting action is the one which can be used in the BV method if it is in the form

$$S(\Phi) = \tilde{B}d\tilde{A} + \alpha\tilde{B}\tilde{B} + \beta\tilde{A}\tilde{A} + \gamma\tilde{A}\tilde{A}\tilde{B}, \quad (6)$$

where $\alpha$ or $\beta$ vanishes and the BRST transformation of the fields

$$\delta\tilde{A} = \frac{\partial S}{\partial \tilde{B}}, \quad \delta\tilde{B} = -\frac{\partial S}{\partial \tilde{A}}, \quad (7)$$

can be written in terms of $\tilde{D} = d + \tilde{A}$ and the related curvature $\tilde{F}$, when $\beta = 0$ as

$$\delta\tilde{A} = \tilde{F} - \tilde{B}, \quad \delta\tilde{B} = -\tilde{D}\tilde{B}, \quad (8)$$

and when $\alpha = 0$ as

$$\delta\tilde{A} = \tilde{F}, \quad \delta\tilde{B} = -\tilde{D}\tilde{B} + \tilde{A}. \quad (9)$$

In (6) multiplication is defined such that $S$ is a scalar possessing zero ghost number. In (7)-(9) the components of the right hand side are restricted to possess the same grading and one more ghost number of the components of the left hand side.

It is possible to choose the signs of the components of $\tilde{A}$ and $\tilde{B}$ such that

$$\delta_1 S \equiv \frac{\partial S}{\partial \tilde{B}} \delta \tilde{B} = \frac{1}{2}(S, S),$$

$$\delta_2 S \equiv \frac{\partial S}{\partial \tilde{A}} \delta \tilde{A} = \frac{1}{2}(S, S),$$

where $(S, S)$ is the appropriate master equation. By taking the derivative of these with respect to $\tilde{A}$ and $\tilde{B}$, one can show that $S$ satisfies the master equation $(S, S) = constant \neq 0$ would lead to the non-consistency of equations of motion) if

$$\delta^2 \tilde{A} = 0, \quad \delta^2 \tilde{B} = 0,$$

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where for an arbitrary functional $f$, BRST transformation is defined as

$$\delta f = \frac{\partial_r f}{\partial A} \delta A + \frac{\partial_r f}{\partial B} \delta B.$$  

In the case given in (8), formally we have

$$\delta^2 A = \dot{\mathcal{D}} \cdot (\dot{\mathcal{F}} - \dot{\mathcal{B}}) + \dot{\mathcal{D}} \dot{\mathcal{B}},$$

$$\delta^2 B = -\dot{\mathcal{D}} \cdot (\dot{\mathcal{D}} \dot{\mathcal{B}} + (\dot{\mathcal{F}} - \dot{\mathcal{B}}) \cdot \dot{\mathcal{B}}.$$ 

These vanish due to the Bianchi identities $\dot{\mathcal{D}} \cdot \dot{\mathcal{F}} = 0$, the definition of the curvature $\dot{\mathcal{F}} = \dot{\mathcal{D}} \cdot \dot{\mathcal{D}}$, and because when the related conditions permit that $\dot{\mathcal{B}}^2$ exists we have $B \cdot \dot{B} = B_i \dot{B}_j - (-1)^{\epsilon(B_i \epsilon(B_j)} \dot{B}_j \dot{B}_i = 0$, where $\epsilon$ indicates the Grassmann parity.

When we deal with the case given in (9)

$$\delta^2 A = \dot{\mathcal{D}} \cdot \dot{\mathcal{F}},$$

$$\delta^2 B = -\dot{\mathcal{D}} \cdot (\dot{\mathcal{D}} \dot{\mathcal{B}} + \dot{A}) - \dot{\mathcal{F}} \dot{B} + \dot{\mathcal{F}},$$

which vanish due to the Bianchi identities and the definition of curvature.

In the case $\dot{A} = \dot{B}$ (Chern-Simons type) we have both $\alpha = 0$ and $\beta = 0$ in (9) and $\delta \dot{A} = \dot{\mathcal{F}}$, so that $\delta^2 \dot{A} = 0$ follows from the Bianchi identities.

By construction $S(\Phi)$ possesses the correct classical limit. Moreover, $\Phi$ is found by using the proper solution condition, so that it includes all the fields of the minimal sector and because of the form of $S$ (6),

$$\text{rank} \left| \frac{\partial^2 S}{\partial \Phi \partial \Phi} \right| = N,$$

where $N$ is the number of the components of $\dot{A}$ or $\dot{B}$. Hence, we conclude that $S(\Phi)$ is the desired action. It seems that one could relax the conditions on the interaction part of the action (9), but a general proof of $(S, S) = 0$ is lacking.

To clarify the procedure outlined above, let us see some applications of it.

1) Yang-Mills Theory
It is defined in terms of the second order action (we suppress $Tr$)

\[ L_0 = -\frac{1}{2} \int d^4x \, F_{\mu\nu} F^{\mu\nu}, \tag{10} \]

where $F = d \wedge A + A \wedge A$. The theory given by

\[ L = -\frac{1}{2} \int d^4x \, (B_{\mu\nu} F^{\mu\nu} - \frac{1}{2} B_{\mu\nu} B^{\mu\nu}), \tag{11} \]

is equivalent to (10) on mass-shell, and moreover it is first order in $d$. (11) is invariant under the infinitesimal gauge transformations

\[ \delta A_\mu = D_\mu \Lambda, \quad \delta B_{\mu\nu} = [B_{\mu\nu}, \Lambda], \]

where $D = d + [A, \cdot]$ is the covariant derivative. They are irreducible, so that for the covariant quantization we need only (in the minimal sector) the ghost field $\eta$, which possesses ghost number 1.

Generalize the fields of (11) according to (5) as

\[ A \rightarrow \tilde{A}, \quad B \rightarrow \tilde{B}, \]

so that one obtains ($\partial_\mu A_\nu - \partial_\nu A_\mu \rightarrow 2d\tilde{A}$)

\[ S = -\frac{1}{2} \int d^4x \, [\tilde{B}(2d\tilde{A} + \tilde{A}\tilde{A}) - \frac{1}{2} \tilde{B}\tilde{B}], \tag{12} \]

which is defined to possess 0 ghost number. Grading of the extended forms $\tilde{A}$ and $\tilde{B}$, respectively, are 1 and 2, and their first components are $A_\mu$ and $B_{\mu\nu}$.

By using the fact that

\[ N_{gh}(\phi) + N_{gh}(\phi^*) = -1, \]

where $N_{gh}$ denotes the ghost number, we write the generalized fields as

\[ \tilde{A} = A_{(1+0)} + \eta_{(0+1)} - B^*_{(2-1)}, \]
\[ \tilde{B} = B_{(2+0)} + A^*_{(3-1)} + \eta^*_{(4-2)}, \]

where the first number in parenthesis is the order of d-forms and the second is the ghost number. Here "*" indicates the antifields as well as the Hodge-map.
Substitution of these in (11) and using the property of the multiplication that the product is different from zero only when its ghost number vanishes, we get

\[ S = - \int d^4x \left( \frac{1}{2} B_{\mu\nu} F^{\mu\nu} - B^{\mu\nu} [\eta, B^*_{\mu\nu}] + A^*_\mu D^\mu \eta + \frac{1}{2} \eta^*[\eta, \eta] - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \right). \] (13)

We may perform a partial gauge fixing \( B^* = 0 \), and then use the equations of motion with respect to \( B_{\mu\nu} \) to obtain

\[ S \rightarrow \tilde{S} = - \int d^4x \left( \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + A^*_\mu D^\mu \eta + \frac{1}{2} \eta^*[\eta, \eta] \right), \]

which is the minimal solution of the master equation for Yang-Mills theory.

2) Massive Abelian Theory in Stueckelberg Formalism

It is defined in terms of the second order Lagrangian

\[ L_0 = - \int d^4x \left[ \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + m^2 (A_\mu - m^{-1} \partial_\mu v) (A^\mu - m^{-1} \partial^\mu v) \right], \] (14)

where \( F = d \wedge A \). The action linear in \( d \) and possessing a kinetic term bilinear in the fields,

\[ L = - \int d^4x \left[ \frac{1}{2} B_{\mu\nu} (d \wedge A)^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + m (A_\mu - m^{-1} \partial_\mu v) K^\mu - \frac{1}{2} K^K_{\mu} K^\mu \right], \] (15)

is equivalent to (14) on mass-shell. It is invariant under the gauge transformations

\[ \delta A_\mu = \partial_\mu \Lambda, \quad \delta B_{\mu\nu} = 0, \]
\[ \delta v = m \Lambda, \quad \delta K_\mu = 0, \]

which are irreducible, so that in the minimal sector there is only one ghost: \( \eta \). By substituting the original fields with the generalized ones we get

\[ S = \int d^4x \left[ - \tilde{B} d \tilde{A} + \tilde{B} \tilde{B} + m (\tilde{A} - m^{-1} d \tilde{v}) \tilde{K} + \frac{1}{2} \tilde{K} \tilde{K} \right]. \] (16)

The generalized fields are

\[ \tilde{A} = A_{(1+0)} + \eta_{(0+1)} - B^*_{(2-1)}, \]
\[ \tilde{B} = B_{(2+0)} + A^*_{(3-1)} + \eta^*_{(4-2)}, \]
\[ \tilde{v} = v_{(0+0)} - K^*_{(1-1)}, \]
\[ \tilde{K} = K_{(3+0)} + v^*_{(4-1)}. \]
By respecting the rules of multiplication one finds in components

\[ S = L - \int d^4x \left[ A^*_\mu \partial^\mu \eta - m\eta v^* \right]. \]

We can eliminate \( B \) and \( K \) by using their equations of motion to obtain

\[ S \to \tilde{S} = \int d^4x \left[ -\frac{1}{2} F^2_{\mu\nu} - m^2 (A_\mu - m^{-1} \partial_\mu v)^2 - A^*_\mu \partial^\mu \eta + m\eta v^* \right]. \]

Indeed, this is the minimal solution of the master equation for the theory given by (14).

3) Relativistic Particle

In terms of the canonical variables satisfying the Poisson bracket relation \( \{ p_\mu, q^\nu \} = \delta^\nu_\mu \), the relativistic particle is given by

\[ L_0 = \int (p \cdot dq - \frac{1}{2} ep \cdot p), \quad (17) \]

where \( dq^\mu = \partial_t q^\mu dt \). A variable possesses two different gradings: one of them is due to the one dimensional manifold of \( t \) and the other one is related to the space-time manifold. (17) is invariant under

\[ \delta q^\mu = p^\mu \Lambda, \; \delta p = 0, \; \delta e = \partial_t \Lambda. \]

Now, we generalize the fields as

\[ [q, e] \to \tilde{q}; \; p \to \tilde{p}. \]

\( q \) and \( e \) are treated on the same footing due to the fact that there is no \( de \) term in (17). Hence the odd time Hamiltonian is

\[ S = \int [\tilde{p}d\tilde{q} - \frac{1}{2} \tilde{q} \tilde{p}^2], \quad (18) \]

where

\[ \tilde{q} = q^\mu_{(1+0+0)} + e_{(0+1+0)} + \eta_{(0+0+1)} - p^\mu_{(1+1-1)}, \]

\[ \tilde{p} = p^{\mu(d-1+0+0)} + q^\mu_{(d-1+1-1)} + e^\mu_{(d+0-1)} + \eta^\mu_{(d+1-2)}. \]
The numbers in the parenthesis indicate, respectively, grading due to space-time, grading due to 1-dimensional manifold and ghost number.

By calculating the product in components one can show that

\[ S = \int dt \left[ p \cdot \partial_t q + e^* \partial_t \eta - \frac{1}{2} ep^2 + q^* \cdot p \eta \right], \]

which is the minimal solution of the master equation for the relativistic particle.

4) The Self-interacting Antisymmetric Tensor Field

The action \[ L_0 = -\int d^4x \left[ B_{\mu \nu} (d \wedge A + A \wedge A)^{\mu \nu} - \frac{1}{2} A_\mu A^\mu \right], \] (19)
is invariant under the transformations

\[ \delta B_{\mu \nu} = \epsilon_{\mu \nu \rho \sigma} D^\rho A^\sigma, \quad \delta A_\mu = 0, \]

and is analysed in terms of the BRST methods in Ref. \[ 10 \]. If we set \( \Lambda_\mu = D_\mu \alpha \), the gauge transformation vanishes on shell \( \delta B|_{F=0} = 0 \). This is a first-stage reducible theory, hence we need to introduce the ghost fields \( C^\mu_0, C_1; \text{N}_{gh}(C^\mu_0) = 1, \text{N}_{gh}(C_1) = 2. \)

By following the general procedure we find

\[ S = -\int d^4x \left[ \tilde{B} (2d \tilde{A} + \tilde{A} \tilde{A}) - \frac{1}{2} \tilde{A} \tilde{A} \right], \] (20)

where the generalized fields are

\[ \tilde{A} = A_{(1+0)} + B^{*}_{(2-1)} + C^*_{0(3-2)} + C^*_{1(4-3)}, \]
\[ \tilde{B} = B_{(2+0)} - A^{*}_{(3-1)} + C_{0(1+1)} + C_{1(0+2)}. \]

In terms of the components (20) reads

\[ H = -\int d^4x \left\{ B_{\mu \nu} F^{\mu \nu} + 2 \epsilon_{\mu \nu \rho \sigma} C^\mu_0 D^\nu B^{* \rho \sigma} + 2 C_1 D^\mu C^{\mu *}_0 + \epsilon^{\mu \nu \rho \sigma} C_1 [B^*_{\mu \nu}, B^*_{\rho \sigma}] - \frac{1}{2} A_\mu A^\mu \right\}, \]

(21)

This is the minimal solution of the master equation of the theory defined by \( 19 \).
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