AVERAGE OF CLASS NUMBERS OF SOME FAMILY OF ARTIN-SCHREIER EXTENSIONS OF RATIONAL FUNCTION FIELDS

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Abstract. In this paper we obtain average of class numbers of some family of Artin-Schreier extensions of rational function field $F_q(t)$, where $q$ is a power of 3.

1. Introduction

The average of class numbers of a family of global fields has been studied by many authors. This problem was initiated by Gauss who made two famous conjectures on average values of class numbers of orders in quadratic fields. These conjectures were proved by Lipschitz in imaginary quadratic fields case and by Siegel [8] in real quadratic fields case. Takhtadzjan and Vinogradov [9] gave an average formula for class numbers of quadratic fields with prime discriminants. Let $k = F_q(t)$ be a rational function field over the finite field $F_q$, where $q$ is a power of a prime number $p$, and $A = F_q[t]$ be the polynomial ring. Assume that $q$ is odd. In [3], Hoffstein and Rosen gave an average of class numbers of orders of quadratic extensions of $k$ and also an average value of class numbers of maximal orders of quadratic extensions of $k$. When $q \equiv 1 \mod 3$, Rosen [7] gave an average of class numbers of orders of Kummer extensions of $k$ and Jung [5] obtained an average of class numbers of maximal orders of Kummer extensions $K = k(\sqrt[3]{P})$ of $k$. 

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where \( P \) runs over monic irreducible polynomials. In [2], when \( q \) is even, Chen obtained an average of class numbers of orders of quadratic extensions of \( k \). In [1], Bae, Jung and Kang extended Chen’s result to any Artin-Schreier extensions of \( k \). Let \( K_u = k(\alpha_u) \) be the Artin-Schreier extension of \( k \) generated by a root \( \alpha_u \) of \( x^p - x = u \), where \( u = \frac{B}{A} \in k \) is normalized (see §2.1). Then \( G(K_u) = A \) which is a monic polynomial in \( A \) to any Artin-Schreier extensions of \( k \). Let \( k \) extensions of \( k \), \( 2 \). Preliminaries

Let \( k = \mathbb{F}_q(t) \) and \( \mathbb{A} = \mathbb{F}_q[t] \), where \( q \) is a power of a prime \( p \). Let \( \infty_k = (1/t) \) be the infinite prime of \( k \). We denote by \( \mathbb{A}^+ \) the set of monic polynomials in \( \mathbb{A} \) and by \( \mathcal{P} \) the set of monic irreducible polynomials in \( \mathbb{A} \). Write \( \mathbb{A}_n = \{ N \in \mathbb{A} : \deg(N) = n \} \), \( \mathbb{A}_n^+ = \mathbb{A}^+ \cap \mathbb{A}_n \) and \( \mathcal{P}_n = \mathcal{P} \cap \mathbb{A}_n \). For any \( 0 \neq N \in \mathbb{A} \), let \( |N| = \#(\mathbb{A}/NA) = q^{\deg(N)}, \Phi(N) = \#(\mathbb{A}/NA)^\times \), where \( \#X \) denotes the cardinality of a set \( X \), and \( \text{sgn}(N) \) denote the leading coefficient of \( N \). Let \( \zeta_A(s) = \frac{1}{1-q^s} \) be the zeta function of \( \mathbb{A} \).

### 2.1. Artin-Schreier extensions

Let \( \varphi(x) = x^p - x \) be the Artin-Schreier operator. For \( u = \frac{B}{A} \in k \) with \( A \in \mathbb{A}^+, B \in \mathbb{A} \) and \( \gcd(A, B) = 1 \), we say that \( u \) is normalized if it satisfies the following conditions: (1) if \( A = \prod_{i=1}^r P_i^{e_i} \), then \( p \nmid e_i \) for each \( 1 \leq i \leq r \), (2) if \( \deg(B) > \deg(A) \), then \( p \nmid (\deg(B) - \deg(A)) \), (3) if \( \deg(B) = \deg(A) \), then \( \text{sgn}(B) \notin \varphi(\mathbb{F}_q) \). Let \( K_u = k(\alpha_u) \) be the Artin-Schreier extension of \( k \) generated by a root \( \alpha_u \) of \( \varphi(x) = u \). Let \( \mathcal{O}_u \) be the integral closure of \( \mathbb{A} \) in \( K_u \). If we write \( u = f(t) + \frac{B}{A} \) with \( f(t) \in \mathbb{A} \) and \( \deg(B) < \deg(A) \), then \( f(t) \) and \( A \) are uniquely determined by the field \( K_u \). Also, if \( K \) is a \( \mathbb{Z}/p\mathbb{Z} \)-extension of \( k \), then there exists such a normalized \( u \in k \) such that \( K = K_u \).
Let $G(K) = A$ be the denominator of $u$ as above. The discriminant $d_u$ of $\mathcal{O}_u$ over $\mathbb{A}$ is $(A \cdot \text{rad}(A))^{p-1}$, where $\text{rad}(A)$ denotes the product of the distinct monic irreducible divisors of $A$ (see [1, Corollary 2.7]). The local discriminant $d_{\infty_k}$ at $\infty_k$ is $\infty_k^{(p-1)(\deg(f)+1)}$ if $\deg(f) > 0$ and 1 otherwise. The discriminant $d_{K_u}$ of $K_u$ is defined to be $d_u \cdot d_{\infty_k}$. We say that the Artin-Schreier extension $K/k$ is real, inert imaginary or ramified imaginary according as $\infty_k$ splits completely, is inert or ramifies in $K$. Then, the extension $K_u/k$ is real, inert imaginary or ramified imaginary according as $\deg(B) < \deg(A)$, $\deg(A) = \deg(B)$ or $\deg(A) < \deg(B)$. (See [4] for details.)

2.2. $L$-functions and class number formulas. Fix an isomorphism $\psi : \mathbb{F}_p \to \mu_p$ sending 1 to a primitive $p$-th root $\zeta_p$ of unity, where $\mu_p$ is the group of $p$-th roots of unity in $\mathbb{C}$. For $u \in k$ and $P \in \mathcal{P}$ which is prime to the denominator of $u$, define $[u, P) \in \mathbb{F}_p$ by $(P, K_u/k)(\alpha_u) = \alpha_u + [u, P)$, where $(P, K_u/k)$ is the Artin automorphism at $P$. Extend this to $N \in \mathbb{A}_+^*$, which is prime to the denominator of $u$, by multiplicativity. For any $N \in \mathbb{A}_+^*$, define \( \{ \frac{u}{N} \} = \psi([u, N]) \) if $N$ is prime to the denominator of $u$ and \( \{ \frac{u}{N} \} = 0 \) otherwise. Let $\chi_u$ be the character defined by $\chi_u(N) = \{ \frac{u}{N} \}$. For $0 \leq i \leq p - 1$, the $L$-function $L(s, \chi_u^i)$ associated to $\chi_u^i$ is defined by

$$L(s, \chi_u^i) = \sum_{N \in \mathbb{A}_+^*} \frac{\chi_u^i(N)}{|N|^s} = \sum_{n=0}^{\infty} \sigma_n^{(i)}(u)q^{-ns} \quad \text{with} \quad \sigma_n^{(i)}(u) = \sum_{N \in \mathbb{A}_+^*} \chi_u^i(N).$$

It is well known that $L(s, \chi_u^i)$ is a polynomial in $q^{-s}$ of degree $\deg(\text{rad}(A)) + \deg(B) - 1$ or $\deg(A) + \deg(\text{rad}(A)) - 1$ according as $\infty_k$ ramifies in $K_u$ or otherwise for $1 \leq i \leq p - 1$.

Let $u = \frac{B}{p} \in k$ be a normalized one with $B \in \mathbb{A}$ and $P \in \mathcal{P}_m$ and $K_u$ be the associated Artin-Schreier extension of $k$. Let $h_u$ and $R_u$ be the ideal class number and regulator of $\mathcal{O}_u$, respectively. Since $d_u = p^{2(p-1)}$ and $d_{\infty_k}^{(p-1)(c+1)}$ if $c = \deg(B) - \deg(P) > 0$ and 1 otherwise, by [1, Proposition 5.1], we have

$$\prod_{i=1}^{p-1} L(1, \chi_u^i) = \begin{cases} (q - 1)^{p-1}q^{(1-p)m}h_uR_u & \text{if } \infty_k \text{ splits,} \\ \frac{q^2-1}q p^{-1}q^{(1-p)m}h_u & \text{if } \infty_k \text{ is inert,} \\ q^{(p-1)(p-1)(c+1)-2(p-1)m}h_u & \text{if } \infty_k \text{ is ramified.} \end{cases}$$

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3. Average value of ideal class numbers of Artin-Schreier extensions

In this section, when \( p = 3 \), we study the averages of class numbers of maximal orders of real/inert imaginary/ramified imaginary Artin-Schreier extensions \( K_u \) of \( k \) with monic irreducible \( G(K_u) \), respectively.

3.1. Real Artin-Schreier extension. For \( P \in \mathcal{P} \), let \( \mathfrak{F}_P = \{ B \in \mathbb{A} : B \neq 0, \deg(B) < \deg(P) \} \) and \( \mathcal{F}_P \) be the set of real Artin-Schreier extensions \( K \) of \( k \) with \( G(K) = P \). It is easy to show that for any \( B_1, B_2 \in \mathfrak{F}_P \), \( \frac{K_{B_1}}{P} = \frac{K_{B_2}}{P} \) if and only if \( B_1 = B_2 \). Hence, the map \( B \mapsto \frac{K_B}{P} \) is a bijection from \( \mathfrak{F}_P \) onto \( \mathcal{F}_P \).

Theorem 3.1. Assume that \( p = 3 \). Then we have

\[
\sum_{P \in \mathcal{P}} \sum_{B \in \mathfrak{F}_P} \frac{h_{B/P} R_{B/P}}{(q^m - 1)\# \mathcal{P}_m} = \frac{\zeta_{a}(2)3 \zeta_{a}(3)^2}{\zeta_{a}(6)} q^{2(m-1)} + O\left(16^mq^{3m}\right).
\]

Proof. Let

\[
F_m(s) = \sum_{P \in \mathcal{P}_m} \sum_{B \in \mathfrak{F}_P} \frac{\prod_{i=1}^{2} L(s, \chi_{B/P})}{(q^m - 1)\# \mathcal{P}_m}.
\]

Since \( L(s, \chi_{B/P}) \) is a polynomial in \( q^{-s} \) of degree \( 2m - 1 \), we have

\[
\prod_{i=1}^{2} L(s, \chi_{B/P}) = \sum_{n=0}^{4m-2} \sum_{n_1+n_2=n} \sum_{N_1 \in \mathbb{A}_{n_1}^+, N_2 \in \mathbb{A}_{n_2}^+} \chi_{B/P}(N_1N_2^2)q^{-ns}.
\]

Then,

\[
F_m(s) = \sum_{n=0}^{4m-2} \frac{\sum_{P \in \mathcal{P}_m} \sum_{B \in \mathfrak{F}_P} a_n(B/P)q^{-ns}}{(q^m - 1)\# \mathcal{P}_m}
\]

with

\[
a_n(B/P) = \sum_{n_1+n_2=n} \chi_{B/P}(N_1N_2^2).
\]

Following the arguments of [7, §2], we have \( |a_n(B/P)| \leq \binom{4m-2}{n} q^{2} \) and

\[
\left| \sum_{n=m}^{4m-2} a_n(B/P)q^{-ns} \right| \leq 2^{4m-2} q^{m\frac{1}{2}-\sigma} \frac{1}{1 - q^{1-\sigma}}.
\]
for \( s \in \mathbb{C} \) with \( \sigma = \text{Re}(s) > \frac{1}{2} \). Thus, we have

\[
\left| \frac{\sum_{n=m}^{4m-2} \sum_{P \in \mathcal{P}} \sum_{B \in \mathcal{B}_P} a_n(B/P)q^{-ns}}{(q^m - 1)\#\mathcal{P}_m} \right| \leq \frac{2^{4m-2}q^{m(\frac{1}{2} - \sigma)}}{1 - q^{\frac{1}{2} - \sigma}}.
\]

In particular, when \( s = 1 \), the summation is less than or equal to \( 16^m q^{-\frac{m}{2}} \).

Now, we consider

\[
f_m(s) = \sum_{n=0}^{m-1} \sum_{P \in \mathcal{P}_m} \sum_{B \in \mathcal{B}_P} a_n(B/P)q^{-ns} \frac{q^{-ns}}{(q^m - 1)\#\mathcal{P}_m}
\]

and

\[
f_m^{(1)}(s) = \sum_{n=0}^{m-1} \sum_{n_1+n_2=n} \sum_{B \in \mathcal{B}_P} \chi_{B/P}(N_1N_2)q^{-ns}
\]

Note that \( P \nmid N_1N_2 \) for any \( N_1 \in \mathbb{A}^+_m, N_2 \in \mathbb{A}^+_m \) with \( n_1 + n_2 = n \) since \( n < m \). By definition, if \( N_1N_2^2 \) is cube, we have \( \chi_{B/P}(N_1N_2^2) = 1 \). If \( N_1N_2^2 \) is not cube, by [6, Corollary 2.2], we have

\[
\sum_{B \in \mathcal{B}_P} \chi_{B/P}(N_1N_2^2) = -1.
\]

Hence, we have \( f_m(s) = f_m^{(1)}(s) + f_m^{(2)}(s) \) with

\[
f_m^{(1)}(s) = \sum_{n=0}^{m-1} \sum_{n_1+n_2=n} q^{-ns}
\]

and

\[
f_m^{(2)}(s) = -\frac{1}{q^m - 1} \sum_{n=0}^{m-1} \sum_{n_1+n_2=n} q^{-ns}.
\]

For \( s \in \mathbb{C} \) with \( \sigma = \text{Re}(s) > \frac{1}{3} \), as \( m \to \infty \), we have

\[
|f_m^{(2)}(s)| \leq \frac{1}{q^m - 1} \sum_{n=0}^{m-1} \sum_{n_1+n_2=n} q^{-ns}
\]

\[
(3.2) \quad = \frac{1}{1 - q^{-m}} \left( \frac{q^{-m} - q^{-m\sigma}}{(1 - q^{1-\sigma})^2} - \frac{mq^{-m\sigma}}{1 - q^{1-\sigma}} \right) \to 0.
\]
Let \( L(s) = \sum_{(N_1,N_2)} |N_1|^{-s} |N_2|^{-s} \), where \((N_1,N_2)\) runs over monic polynomials \(N_1, N_2 \in \mathbb{H}^+\) such that \( P \nmid N_1N_2 \) and \( N_1N_2^2 \) is a cube. Then, as in [7, \S 2], we have

\[
(3.3) \quad \left| f_m^{(1)}(s) - L(s) \right| \leq Cm^2 q^{\frac{m}{2}(1-3\sigma)} 
\]

for some constant \( C \) and \( \sigma = \Re(s) > \frac{1}{3} \). Hence, by (3.2) and (3.3), for \( s \in \mathbb{C} \) with \( \sigma = \Re(s) > \frac{1}{3} \), we have

\[
(3.4) \quad f_m(s) = \frac{\zeta_{\mathbb{A}}(2s)\zeta_{\mathbb{A}}(3s)^2}{\zeta_{\mathbb{A}}(6s)} + O\left(m^2 q^{\frac{m}{2}(1-3\sigma)}\right). 
\]

Finally, since \( Cm^2 q^{-2m} = o(16^m q^{-\frac{m}{2}}) \), by (3.1) and (3.4), we have

\[
(3.5) \quad F_m(1) = \frac{\zeta_{\mathbb{A}}(2)\zeta_{\mathbb{A}}(3)^2}{\zeta_{\mathbb{A}}(6)} + O\left(16^m q^{-\frac{m}{2}}\right). 
\]

Hence, by (2.1) and (3.5), we have

\[
\sum_{P \in \mathcal{P}_m} \sum_{B \in \Phi_P} h_{B/P} R_{B/P} \left( q^m - 1 \right) \# \mathcal{P}_m = \frac{\zeta_{\mathbb{A}}(2)\zeta_{\mathbb{A}}(3s)^2}{(q-1)^2 \zeta_{\mathbb{A}}(6)} q^{2m} + O(16^m q^{3m}). 
\]

Since \( \frac{1}{q^1} = \zeta_{\mathbb{A}}(2)q^{-1} \), we get the result. \( \square \)

**3.2. Inert imaginary Artin-Schreier extension.** Let \( \{0, \xi_1, \ldots, \xi_{p-1}\} \) be a set of representatives of \( \mathbb{F}_q/\mathfrak{o}(\mathbb{F}_q) \). For \( P \in \mathcal{P} \), let \( \Phi_P = \{ \xi_aP + B : B \in \Phi_P, 1 \leq a \leq p - 1 \} \) and \( \mathcal{G}_P \) be the set of inert imaginary Artin-Schreier extensions \( K \) of \( k \) with \( G(K) = P \). For any \( B_1, B_2 \in \Phi_P \) and \( 1 \leq a, b \leq p - 1 \), \( K_{\xi_a + B_1/P} = K_{\xi_b + B_2/P} \) if and only if \( a = b \) and \( B_1 = B_2 \). Thus, the map \( \xi_aP + B \mapsto K_{\xi_a + B/P} \) is a bijection from \( \Phi_P \) onto \( \mathcal{G}_P \).

**Theorem 3.2.** Assume that \( p = 3 \). Then we have

\[
\sum_{P \in \mathcal{P}_m} \sum_{B \in \Phi_P} \sum_{a=1}^{2} h_{\xi_a + B/P} \frac{3\zeta_{\mathbb{A}}(3)\zeta_{\mathbb{A}}(4)}{\zeta_{\mathbb{A}}(6)} q^{2(m-1)} + O(16^m q^{3m}). 
\]

**Proof.** For a positive integer \( m \) and \( a \in \{1, 2\} \), let

\[
G_{m,a}(s) = \frac{\sum_{P \in \mathcal{P}_m} \sum_{B \in \Phi_P} \prod_{i=1}^{2} L(s, \chi_{\xi_a + B/P}^{i})}{2(q^m - 1) \# \mathcal{P}_m}, 
\]
and $G_m(s) = G_{m,1}(s) + G_{m,2}(s)$. Since $L(s, \chi_{\xi_a+B/P}^i)$ is a polynomial in $q^{-s}$ of degree $2m-1$, we have

$$\prod_{i=1}^{2} L(s, \chi_{\xi_a+B/P}^i) = \sum_{n=0}^{4m-2} \sum_{n_1+n_2=n} \frac{\chi_{\xi_a+B/P}(N_1N_2^2)q^{-ns}}{2(q^m-1)\#P_m}.$$

Then,

$$G_{m,a}(s) = \sum_{n=0}^{4m-2} \sum_{P \in \mathcal{P}_m} \sum_{B \in \mathfrak{B}_P} a_n(\xi_a + B/P)q^{-ns} \frac{\chi_{\xi_a+B/P}(N_1N_2^2)}{2(q^m-1)\#P_m}$$

with

$$a_n(\xi_a + B/P) = \sum_{n_1+n_2=n} \chi_{\xi_a+B/P}(N_1N_2^2).$$

Following the arguments of [7, §2], we have $|a_n(\xi_a + B/P)| \leq (4m-2)q^{\frac{n}{2}}$ and

$$\left| \sum_{n=m}^{4m-2} a_n(\xi_a + B/P)q^{-ns} \right| \leq \frac{2^{4m-2}q^{m(\frac{1}{2} - \sigma)}}{1 - q^{\frac{1}{2}-\sigma}}.$$

for $\sigma = \text{Re}(s) > \frac{1}{2}$. Thus, we have

$$\sum_{n=0}^{4m-2} \sum_{P \in \mathcal{P}_m} \sum_{B \in \mathfrak{B}_P} a_n(\xi_a + B/P)q^{-ns} \leq \frac{2^{4m-2}q^{m(\frac{1}{2} - \sigma)}}{1 - q^{\frac{1}{2}-\sigma}}.$$

In particular, when $s = 1$, the summation is less than or equal to $16^m q^{-\frac{m}{2}}$.

Now, we consider

$$g_{m,a}(s) = \sum_{n=0}^{m-1} \sum_{P \in \mathcal{P}_m} \sum_{B \in \mathfrak{B}_P} a_n(\xi_a + B/P)q^{-ns} \frac{\chi_{\xi_a+B/P}(N_1N_2^2)}{2(q^m-1)\#P_m}$$

$$= \sum_{n=0}^{m-1} \sum_{N_1 \in \mathbb{A}_{n_1}^+, N_2 \in \mathbb{A}_{n_2}^+} \sum_{P \in \mathcal{P}_m} \sum_{B \in \mathfrak{B}_P} \chi_{\xi_a+B/P}(N_1N_2^2)q^{-ns} \frac{\chi_{\xi_a+B/P}(N_1N_2^2)}{2(q^m-1)\#P_m}.$$

Note that $P \nmid N_1N_2$ for any $N_1 \in \mathbb{A}_{n_1}^+, N_2 \in \mathbb{A}_{n_2}^+$, with $n_1 + n_2 = n$ since $n < m$. By definition, if $N_1N_2^2$ is not cube, we have $\chi_{\xi_a+B/P}(N_1N_2^2) = 1$. If $N_1N_2^2$ is not cube, by [6, Corollary 2.2], we have

$$\sum_{B \in \mathfrak{B}_P} \chi_{\xi_a+B/P}(N_1N_2^2) = \left\{ \frac{\xi_a}{N_1N_2^2} \right\} \sum_{B \in \mathfrak{B}_P} \chi_{B/P}(N_1N_2^2) = -\left\{ \frac{\xi_a}{N_1N_2^2} \right\}.$$
Finally, since $C_m g (3.8)$ $B$ are equivalent, denoted by $c$ and positive integer $3.3$. Ramified imaginary Artin-Schreier extension. For $\sigma = \text{Re}(s) > \frac{1}{3}$, as $m \to \infty$, we have

$$g_{m,a}(s) = \sum_{n=0}^{m-1} \sum_{n_1+n_2=n} \left\{ \frac{s}{N_1 N_2^2} \right\} q^{-ns}.$$  

For $s \in \mathbb{C}$ with $\sigma = \text{Re}(s) > \frac{1}{3}$, as $m \to \infty$, we have

$$\left| g_{m,a}(s) \right| \leq \frac{1}{q^m - 1} \sum_{n=0}^{m-1} \sum_{n_1+n_2=n} q^{-n\sigma} \to 0$$

as in (3.2). Let $g_m(s) = g_{m,1}(s) + g_{m,2}(s) = f_m^{(1)}(s) + g_{m,1}^{(2)}(s) + g_{m,2}^{(2)}(s)$. Then, by (3.4) and (3.7), for $s \in \mathbb{C}$ with $\sigma = \text{Re}(s) > \frac{1}{3}$, we have

$$g_m(s) = \frac{\zeta_A(2)\zeta_A(3)^2}{\zeta_A(6)} + O \left( m^2 q^{\frac{m}{2}(1-3\sigma)} \right).$$

Finally, since $C m^2 q^{-\frac{2m}{3}} = o(16m q^{-\frac{m}{2}})$, by (3.6) and (3.8), we have

$$G_m(1) = \frac{\zeta_A(2)\zeta_A(3)^2}{\zeta_A(6)} + O \left( 16m q^{-\frac{m}{2}} \right).$$

Hence, by (2.1) and (3.9), we have

$$\sum_{P \in \mathcal{P}_m} \sum_{B \in \mathfrak{B}_P} \sum_{n=1}^{2} h_{B,B/P} = \frac{3(q-1) \zeta_A(2) \zeta_A(3)^2}{q^3 - 1} q^{2m} + O(16m q^{-\frac{m}{2}}).$$

Since $q - 1 = \frac{q}{\zeta_A(2)}$ and $1 = \frac{\zeta_A(4)}{q^3}$, we get the result. 

3.3. Ramified imaginary Artin-Schreier extension. For $P \in \mathcal{P}$ and positive integer $c$ with $p \nmid c$, let $\mathfrak{B}_{P,c} = \{ B \in \mathbb{A} : P \nmid B, \deg(B) = \deg(P) + c \}$ and $\mathcal{H}_{P,c}$ be the set of ramified imaginary Artin-Schreier extensions $K$ of $k$ with $G(K) = P$ and whose discriminant $d_K$ is $P^{2(p-1)} \cdot \infty^{(p-1)(c+1)}$. For any $B, B' \in \mathfrak{B}_{P,c}$, we have that $K_B/P = K_{B'/P}$ if and only if $B' = B + P(D^p - D)$ for some $D \in \mathbb{A}$. We say that $B, B' \in \mathfrak{B}_{P,c}$ are equivalent, denoted by $B \sim B'$, if $B' = B + P(D^p - D)$ for some $D \in \mathbb{A}$. Let $[B]$ be the equivalence class of $B \in \mathfrak{B}_{P,c}$ with respect to $\sim$, and $\tilde{\mathfrak{B}}_{P,c} = \{ [B] : B \in \mathfrak{B}_{P,c} \}$. Then, the map $[B] \mapsto K_B/P$ is a bijection from $\tilde{\mathfrak{B}}_{P,c}$ onto $\mathcal{H}_{P,c}$. For $B \in \mathfrak{B}_{P,c}$, we have a surjective map

$$D \in \mathbb{A} : \deg(D) \leq \lfloor c/p \rfloor \to [B], \quad D \mapsto B + P(D^p - D).$$
For $D, E \in \mathbb{A}$ with $\deg(D), \deg(E) \leq \lfloor c/p \rfloor$, we have that $B + P(D^p - D) = B + P(E^p - E)$ if and only if $D - E \in \mathbb{F}_p$. Hence, the map in (3.10) is $p$ to 1, so we have $\#|B| = \frac{q^{\lfloor c/p \rfloor}}{p}$. Since $\#\tilde{H}_{P,c} = \#\mathbb{A}_{\deg(P) + c} - \#\mathbb{A}_c = q^c (q - 1)(q^{\deg(P)} - 1)$, we have

$$\#\tilde{H}_{P,c} = pq^{c - \lfloor c/p \rfloor} (q - 1)(q^{\deg(P)} - 1).$$

**Theorem 3.3.** Assume that $p = 3$. Then we have

$$\sum_{P \in \mathbb{P}_m} \sum_{|B| \in \tilde{H}_{P,c}} h_{B/P} I_q(m, c) = \frac{\zeta_\mathbb{A}(2) \zeta_\mathbb{A}(3)^2}{\zeta_\mathbb{A}(6)} q^{2m + 2c} + O \left( 4^{2m + c} \frac{q^{m + 2c}}{q^2} \right),$$

where $I_q(m, c) = 3q^{c - \lfloor c/3 \rfloor} (q - 1)(q^m - 1)\#\mathbb{P}_m$.

**Proof.** For a positive integers $m$ and $c$ with $3 \nmid c$, let

$$H_{m,c}(s) = \frac{\sum_{P \in \mathbb{P}_m} \sum_{|B| \in \tilde{H}_{P,c}} \prod_{i=1}^2 L(s, \chi^i_{B/P})}{I_q(m, c)}.$$

Since $L(s, \chi^i_{B/P})$ is a polynomial in $q^{-s}$ of degree $2m + c - 1$, we have

$$\prod_{i=1}^2 L(s, \chi^i_{B/P}) = \sum_{n=0}^{4m+2c-2} \sum_{N_1 \in \mathbb{A}_1^+, N_2 \in \mathbb{A}_2^+} \chi_{B/P}(N_1 N_2^2) q^{-ns}.$$

Then,

$$H_{m,c}(s) = \frac{3 \sum_{n=0}^{4m+2c-2} \sum_{P \in \mathbb{P}_m} \sum_{B \in \tilde{H}_{P,c}} a_n(B/P) q^{-ns}}{q^{c/3} I_q(m, c)},$$

with

$$a_n(B/P) = \sum_{N_1 \in \mathbb{A}_1^+, N_2 \in \mathbb{A}_2^+} \chi_{B/P}(N_1 N_2^2).$$

Following the arguments of [7, §2], we have $|a_n(B/P)| \leq \left( \frac{4m+2c-2}{n} \right) q^{n/2}$ and

$$\left| \sum_{n=m}^{4m+2c-2} a_n(B/P) q^{-ns} \right| \leq 2^{4m+2c-2} q^{m(\frac{1}{2} - \sigma)} (1 - q^{1/2 - \sigma})^{-1}.$$
for $\sigma = \text{Re}(s) > \frac{1}{3}$. Thus, we have

\[
(3.11) \quad \left| 3 \sum_{n=m}^{n=m+2c-2} \sum_{p \in \mathcal{P}_m} \sum_{B \in \mathcal{B}_{p,c}} a_n(B/P)q^{-ns} \right| \leq \frac{24m+2c-2q^{m(\frac{1}{3}-\sigma)}}{1-q^{2-\sigma}}.
\]

In particular, when $s = 1$, the summation is less than or equal to $4^{2m+c}q^{-m/2}$.

Now, we consider

\[
h_{m,c}(s) = \frac{3 \sum_{n=0}^{m-1} \sum_{p \in \mathcal{P}_m} \sum_{B \in \mathcal{B}_{p,c}} a_n(B/P)q^{-ns}}{q^{c/3}I_q(m,c)}
\]

\[
= \frac{3 \sum_{n=0}^{m-1} \sum_{n_1+n_2=n} \sum_{p \in \mathcal{P}_m} \sum_{B \in \mathcal{B}_{p,c}} \chi_{B/P}(N_1N_2^2)q^{-ns}}{q^{c/3}I_q(m,c)}.
\]

Note that $P \nmid N_1N_2$ for any $N_1 \in \mathbb{A}_{n_1}^+, N_2 \in \mathbb{A}_{n_2}^+$ with $n_1 + n_2 = n$ since $n < m$. By definition, if $N_1N_2^2$ is cube, we have $\chi_{B/P}(N_1N_2^2) = 1$. If $N_1N_2^2$ is not cube, by [6, Corollary 2.4], we have

\[
\sum_{B \in \mathcal{B}_{p,c}} \chi_{B/P}(N_1N_2^2) = -\sum_{C \in \mathcal{A}_c} \left\{ \frac{C}{N_1N_2^2} \right\}.
\]

Hence, we have $h_{m,c}(s) = f_m^{(1)}(s) + h_{m,c}^{(2)}(s)$ with

\[
h_{m,c}^{(2)}(s) = -\frac{3}{q^{c/3}I_q(m,c)} \sum_{n=0}^{m-1} \sum_{n_1+n_2=n} \sum_{p \in \mathcal{P}_m} \sum_{C \in \mathcal{A}_c} \left\{ \frac{C}{N_1N_2^2} \right\} q^{-ns}.
\]

For $\sigma = \text{Re}(s) > \frac{1}{2}$, by using the fact that $\sum_{C \in \mathcal{A}_c} \left| \left\{ \frac{C}{N_1N_2^2} \right\} \right| \leq \# \mathcal{A}_c = (q-1)q^e$, we have

\[
|h_{m,c}^{(2)}(s)| \leq \frac{3}{q^{c/3}I_q(m,c)} \sum_{n=0}^{m-1} \sum_{n_1+n_2=n} \sum_{p \in \mathcal{P}_m} \sum_{C \in \mathcal{A}_c} \left| \left\{ \frac{C}{N_1N_2^2} \right\} \right| q^{-ns}
\]

\[
\leq \frac{1}{q^{m-1}} \sum_{n=0}^{m-1} q^{-ns} \sum_{n_1+n_2=n} \sum_{N_1 \in \mathbb{A}_{n_1}^+, N_2 \in \mathbb{A}_{n_2}^+} 1 = \frac{1}{q^{m-1}} \sum_{n=0}^{m-1} (n+1)q^{(1-\sigma)n}.
\]
If \( \sigma = 1 \), we have
\[
\frac{1}{q^m - 1} \sum_{n=0}^{m-1} (n + 1)q^{(1-\sigma)n} = \frac{m(m+1)}{2(q^m - 1)} \to 0
\]
and if \( \sigma \neq 1 \), we have
\[
\frac{1}{q^m - 1} \sum_{n=0}^{m-1} (n + 1)q^{(1-\sigma)n} = \frac{1}{1 - q^{-m}} \left( \frac{q^{-m} - q^{-\sigma m}}{(1-q^{-1})^2} - \frac{mq^{-\sigma m}}{1-q^{-1}} \right) \to 0
\]
as \( m \to \infty \). Hence, for \( \sigma = \text{Re}(s) > \frac{1}{2} \), we have
\[
(3.12) \quad h^{(2)}_{m,c}(s) \to 0
\]
as \( m \to \infty \). Hence, by (3.4) and (3.12), for \( s \in \mathbb{C} \) with \( \sigma = \text{Re}(s) > \frac{1}{5} \), we have
\[
(3.13) \quad h_{m,c}(s) = \frac{\zeta_{\mathbb{A}}(2s)\zeta_{\mathbb{A}}(3s)^2}{\zeta_{\mathbb{A}}(6s)} + O \left( m^2q^{\frac{m}{3}(1-3\sigma)} \right) .
\]
Finally, since \( C m^2 q^{-\frac{2m}{3}} = o(4^{2m+c}q^{-\frac{m}{2}}) \), we have
\[
(3.14) \quad H_{m,c}(1) = \frac{\zeta_{\mathbb{A}}(2)\zeta_{\mathbb{A}}(3)^2}{\zeta_{\mathbb{A}}(6)} + O(4^{2m+c}q^{-\frac{m}{2}}).
\]
Hence, by (2.1) and (3.14), we have
\[
\sum_{P \in P_m} \sum_{[\beta] \in \delta_{P,c}} \frac{h_{\beta/P}}{I_q(m,c)} = \frac{\zeta_{\mathbb{A}}(2)\zeta_{\mathbb{A}}(3)^2}{\zeta_{\mathbb{A}}(6)} q^{2m+2c} + O \left( 4^{2m+c}q^{\frac{3m}{2}+2c} \right).
\]

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