ON THE HOLOMONY OF KALUZA-KLEIN METRICS

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Abstract. We investigate Kaluza-Klein metrics with a recurrent light-
like vector field over a pseudo-Riemannian manifold \((B, g)\).

1. Introduction

Principal \(S^1\)-bundles have been used by physicists to unify Einsteins gen-
eral relativity theory with electrodynamics. This theory, by their main con-
tributors known as Kaluza-Klein theory, has been later on generalized to
bundles where the fiber is no longer an abelian Lie group (Yang-Mills the-
ory). We refer to [Ba] for a general picture of the mathematical backgrounds
of Gauge theory.

We focus here in particular on the construction of pseudo-Riemannian
manifolds with the so called Kaluza-Klein-metric. Given a pseudo-
Riemannian manifold \((B, g)\), we consider a principal \(S^1\)-bundle \(P\) over \(B\)
together with a connection form \(A\) on it. For the Kaluza-Klein-metric a
fundamental vector corresponding to the \(S^1\)-action has length \(\pm 1\) whereas
the horizontal spaces, corresponding to the connection form \(A\), inherit the
metric from the base manifold. Finally any horizontal vector is orthogonal
to any vertical vector \((i.e.\ \text{which is tangent to the fiber})\). The well known
Boothby-Wang fibration now gives examples of such principal \(S^1\)-bundles
over a projective symplectic manifold and it is known that if the base \(B\) is a
projective Kähler manifold, \(P\) will carry a natural Sasakian structure. We
can fix transverse holonomy of the Sasakian manifold by fixing the holonomy
of the base manifold. We refer to [Bl] and [BG2] for a detailed treatment of
Sasakian geometry.

It is now known that Lorentzian connections on a \(n+1\)-dimensional man-
ifold either admit full \(so(1,n)\)-holonomy or there is a holonomy-invariant di-
rection, case in which we speak of special Lorentzian holonomy. By the work
of L. Bérard Bergery, A. Ikemakhen ([BBI]) and T. Leistner ([L1]) indecom-
posable special holonomy has been classified. In section 4 we explore more
generally \((i.e.\ \text{in the pseudo-Riemannian case})\) principal \(S^1\)-bundles with a
Kaluza-Klein-metric admitting a holonomy-invariant direction or similarly
a recurrent vector field. Under the condition that the recurrent vector field
projects fiberwise on a fixed direction, it turns out that the recurrent vector
field has to be parallel and projects onto a Killing field on the base mani-
fold and that \(P\) has to admit a flat connection. Inversely we can construct
examples of pseudo-Riemannian manifolds with a recurrent vector field by

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considering $S^1$-bundles with a flat connection over a base manifold with a Killing field and form a Kaluza-Klein-metric. In particular we can consider base manifolds to be K-contact or Sasakian.

Finally we can iterate the construction: A first Boothby-Wang construction over a projective symplectic manifold yields a K-contact manifold, and a second $S^1$-bundle-construction yields a manifold on which there is a Kaluza-Klein-metric with a parallel vector field. In case the starting manifold was projective Kähler the special Lorentzian holonomy (of type II) is completely known.

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2. Transverse connections

Let $(B, g)$ be a pseudo-Riemannian manifold with Levi-Civita covariant derivative $\nabla$.

Let $\xi$ be a vector field on $(B, g)$ such that $g(\xi, \xi) = \sigma$ with $\sigma = \pm 1$.

Note $D = D_\xi$ the sub-bundle $\xi^\perp$. Note $p(X)$ the orthogonal projection of the vector field $X$ on $B$ onto $D$. Note $\nabla^D$ the covariant derivative on the bundle $D$ defined by: $\nabla^D_\xi Y = p([\xi, Y])$, $\nabla^D_X Y = p(\nabla_X Y)$ for $Y$ a section of $D$ and $X$ orthogonal to $\xi$.

It can be readily checked that $\nabla^D$ is really a covariant derivative and that $\nabla^D$ is torsion-free in the sense: $\nabla^D_X Y - \nabla^D_Y X = p([X, Y])$ for $X, Y$ sections of $D$. Furthermore $\nabla^D$ preserves the bundle metric on $D$ obtained by restriction of $g$.

Note $\text{Hol}^D_o$ the holonomy group of the transverse connection in the point $o$ and $\mathfrak{hol}^D_o$ its Lie algebra.

3. Kaluza-Klein metrics

3.1. Definitions and basic results. Given a pseudo-Riemannian manifold $(B, g)$, let $\nabla$ be its Levi-Civita covariant derivative. Let $\mathfrak{g}$ be the Lie algebra of the group $G = U(1)$. $S^1$ is the underlying manifold of the group $G$. $\mathfrak{g}$ identifies to $i\mathbb{R}$. We consider $(P, \pi, B)$ a principal $S^1$-bundle over the manifold $B$. Let $A : TP \to \mathfrak{g}$ be a connection form on the bundle $P$, i.e. for $g \in G$ and $X$ a vector field on $P$, $A(X \cdot g) = Ad(g^{-1})A(X)$ (= $A(X)$ in this case), and for $a \in \mathfrak{g}, \xi \in P$, $A(\frac{d}{dt}_{t=0}(\xi \cdot \exp(ta))) = a$.

Let $v$ be the vector field on $P$ such that $T\pi(v) = 0$ and $A(v)$ is constant equal to $i$.

For $X$ a vector field on $B$, we call horizontal lift of $X$ and note $X^*$ the unique vector field on $P$ such that $A(X^*) = 0$ and $T\pi(X^*) = X$.

Recall that the curvature form $\Omega$ in $\Gamma((T^*B \wedge T^*B) \otimes \mathfrak{g})$ of the connection form $A$ verifies $\Omega(X, Y) := -A([X^*, Y^*])$. As $S^1$ is abelian, $\Omega$ is closed. We have:

**Proposition 1.** $[X^*, v] = 0, [X^*, Y^*] = [X, Y]^* + i\Omega(X, Y)v.$

**Proof.** Classic results. □
Fix $\sigma = \pm 1$. Note $\tilde{g}$ the pseudo-Riemannian metric $\pi^*g + \sigma A \otimes A$ on $P$, called in the following Kaluza-Klein-metric. Let $D$ be the Levi-Civita covariant derivative corresponding to $\tilde{g}$.

Note $\phi$ the section of $\text{End}_B(TB)$ such that $g(\phi X, Y) = \sigma \frac{i}{2} \Omega(X, Y)$.

**Proposition 2.** $\sigma \frac{i}{2} (\nabla \Omega)(X, Y, Z) = g(\langle \nabla_X \phi \rangle Y, Z)$

**Proposition 3.** $D_v v = 0,$

$D_v X^* = D_X^* v = (\phi X)^*$

$D_X Y^* = (\nabla_X Y)^* + \frac{i}{2} \Omega(X, Y) v$.

**Proof.** Use the Koszul formula. $\square$

**Lemma 4.** $v$ is a Killing vector field s.t. $\tilde{g}(v, v) = -\sigma$.

**Proposition 5.** Let $D$ be the transverse bundle $v^\perp$. The transverse connection $D^D$ verifies: $D^D v = 0$, $D^D (\nabla v) = (\nabla_X Y)^*$.

The holonomy group in the point $o$ of the connection $D^D$ verifies $\text{Hol}^D_\sigma = \pi^*(\text{Hol}_{\tilde{g}(\nabla)})$.

### 3.2. Boothby-Wang fibration

Consider the $S^1$-bundle $P$ as in the preceding section. It is known (see chapter 2 in [Bl], resp. [K]) that $c_1(P) = \frac{1}{2\pi} \Omega$ is an integral second De Rham cohomology class i.e. which lies in $H^2(B, \mathbb{Z})_b \subset H^2(B, \mathbb{R})$ ($b$ stands for Betti-part).

Inversely suppose that $\Omega$ is a closed $i\mathbb{R}$-valued 2-form such that $\frac{1}{2\pi} \Omega \in H^2(B, \mathbb{Z})_b$. As the principal $S^1$-bundles over $B$ are classified by $H^2(B, \mathbb{Z})$, we can consider a bundle $P$ corresponding to $\frac{1}{2\pi} \Omega$. If $B$ is simply connected, then the bundle $P$ is uniquely defined. There is a (generally not unique) connection form $A$ on the bundle $P$ such that $\Omega$ is the curvature form of $A$.

### 3.3. Examples of Kähler base manifolds

If $(B, g, \Omega)$ is a Kähler manifold, the condition $\frac{1}{2\pi} \Omega \in H^2(B, \mathbb{Z})_b$ means, by the Kodaira embedding theorem, that $B$ is projective, i.e. holomorphically embeds into some projective space $\mathbb{C}P^n$. The construction of section 2.1 gives then examples of Sasakian manifolds (see 1.5.2) with holonomy of the transverse bundle given by the holonomy of $(B, g)$: By a theorem of Y. Hatakeyama, the total space of the principal $S^1$-bundle is normal contact if and only if the base manifold is Kähler and projective (see [II]).

By the Berger theorem it is known that the list of holonomy algebras of irreducible, non locally-symmetric Kähler manifolds is restricted to $\mathfrak{u}(n)$, $\mathfrak{su}(n)$ (Calabi-Yau), $\mathfrak{sp}(n)$ (hyperKähler).

#### 3.3.1. Projective Kähler symmetric spaces

Irreducible Kähler symmetric spaces are automatically Einstein. Furthermore if they are compact and simply-connected, they are projective (see [Be], 8.89, 8.2, 8.99). See also ([Be], 10.K) for the list of compact irreducible Kähler symmetric spaces.

#### 3.3.2. Projective base manifold with full $U(n)$-holonomy

Consider $(X, g)$ a compact homogeneous Kähler manifold. If $X$ is simply-connected, it is known (see [Be], 8.2, 8.98-99) that $X$ admits a Kähler-Einstein metric $g$ with positive scalar curvature, and $X$ is projective. If $(X, g)$ is not a symmetric space and irreducible, it admits necessarily then full $U(n)$-holonomy as Kähler-manifolds whose holonomy is included in $SU(n)$ are Ricci-flat.
3.3.3. **Projective Calabi-Yau base manifolds.** It is known that manifolds with full $SU(n)$-holonomy are automatically projective as soon as their complex dimension is bigger than 3.

3.3.4. **Projective hyperKähler base manifolds.** Very little examples are known by today of projective hyperKähler manifolds. One series of such can be obtained in the following way: Take $X$ a projective K3-surface e.g. the Fermat quartic:

$$
\{(x_0 : x_1 : x_2 : x_3) | x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0 \} \subset \mathbb{C}P^3
$$

$X$ itself and the Hilbert scheme $\text{Hilb}^n(X)$ of dimension $2n$ are then projective and hyperKähler. See ([GHJ], 21.1-2 and 26, and [Be] 14.C) for more information about these examples and projective hyperKähler manifolds in general.

4. **Kaluza-Klein metrics with a recurrent vector field**

4.1. **Recurrent vector fields.** We say that the non-vanishing vector field $\tilde{R}$ on $P$ is *recurrent* if there is a scalar-valued 1-form $\omega$ on $P$ such that $D\tilde{R} = \omega \otimes \tilde{R}$.

Note that if $\tilde{R}$ is a recurrent vector field, then $\langle \tilde{R} \rangle$ and $\tilde{R}^\perp$ are parallel distributions. In case the connection is torsion-free a parallel distribution is automatically integrable and tangent to a foliation of the manifold. In case $\tilde{R}$ is light-like, $\langle \tilde{R} \rangle$ is a sub-distribution of $\tilde{R}^\perp$ and the leaves corresponding to $\langle \tilde{R} \rangle$ are contained in the leaves corresponding to $\tilde{R}^\perp$.

4.2. **Screen bundle.** The quotient-bundle $S := \tilde{R}^\perp / \langle \tilde{R} \rangle$ is called the *screen bundle* corresponding to the recurrent vector field $\tilde{R}$. Note $q$ the corresponding canonical projection $\tilde{R}^\perp \to S$. $S$ carries a natural covariant derivative $D_S$:

$$
D_X^S(qY) := q(D_XY).
$$

**Lemma 6.** $D_S$ is well defined.

$D_S$ restricts to any leaf $L$ of the foliation $\mathcal{F}$ determined by $\tilde{R}^\perp$.

4.3. **Kaluza-Klein metrics with a recurrent vector field.**

**Proposition 7.** Given a pseudo-Riemannian manifold $(B, g)$, and let $\sigma$ be $\pm 1$.

(i) Let $(P, \pi, B)$ a principal $S^1$-bundle over the manifold $B$ and connection form $A$. Let $\tilde{g} := \pi^* g + \sigma A \otimes A$ be a metric on $P$.

Suppose there is a light-like recurrent vector field $\tilde{R}$ on $P$ of the form $f\xi^* + hv$, where $\xi$ is a vector field on $B$ such that $g(\xi, \xi) = \sigma$ and $f$ and $h$ are in $C^\infty(P)$.

Then $\varepsilon = \text{sign}(fh)$ is locally constant $\pm 1$, and we have: $\xi$ is a Killing vector field, $D(\xi^* + \varepsilon v) = 0$, $\phi \xi = 0$, $\Omega(\xi, \cdot) = 0$, $\phi = -\varepsilon \nabla \xi$,

Furthermore $\Omega = -i\sigma \varepsilon \eta$ for the 1-form $\eta := g(\xi, \cdot)$. The connection form $A_0 := A + i\sigma \varepsilon \pi^* \eta$ on the bundle $\pi : P \to B$ is flat.
(ii) Reciprocally suppose \((B, g)\) admits a Killing vector field \(\xi\) such that \(g(\xi, \xi) = \sigma\). Suppose the principal \(S^1\)-bundle \(\pi : P \to B\) admits a flat connection \(A_0\). Let \(\tilde{g}\) be the metric \(\pi^*g + \sigma A \otimes A\) for the connection form \(A := A_0 - i\varepsilon\pi^*\eta\) with \(\eta := g(\xi, \cdot)\) and \(\varepsilon = \pm 1\). \(P\) admits then the parallel light-like vector field \(\xi^* + \varepsilon v\) for the Levi-Civita connection associated to \(\tilde{g}\).

**Notation:** We will note \(r\) the parallel vector field \(\xi^* + \varepsilon v\), and \(\tilde{r}\) the complementary field \(\xi^* - \varepsilon v\).

**Proof.**

1) The condition that \(\tilde{R}\) is light-like writes \(\tilde{g}(\tilde{R}, \tilde{R}) = 0\), giving \(f^2 \cdot \sigma - h^2 \cdot \sigma = 0\), say \(f = \varepsilon h\) with \(\varepsilon = \pm 1\).

\[\tilde{R} = f \cdot (\xi^* + \varepsilon v)\]

As a consequence \(f\) is non vanishing.

2) \(D\tilde{R} = \omega \otimes \tilde{R}\),

a) \(Dv\tilde{R} = \omega(v)\tilde{R}\)

\[\Rightarrow v(f)(\xi^* + \varepsilon v) + f \cdot (Dv\xi^* + \varepsilon Dv) = \omega(v)f \cdot (\xi^* + \varepsilon v)\]

\[\Rightarrow v(f)(\xi^* + \varepsilon v) + f \cdot (\phi\xi) = \omega(v)f \cdot (\xi^* + \varepsilon v)\]

From this follow the two equations:

\[v(f)v = \omega(v)fv\]  \(\text{(1)}\)

\[v(f)\xi^* + f \cdot (\phi\xi) = \omega(v)f\xi^*\]  \(\text{(2)}\)

Equation \(\text{[1]}\) gives

\[\omega(v) = v(\ln f)\]  \(\text{(3)}\)

From equation \(\text{[2]}\) follows

\[\phi\xi = 0\]  \(\text{(4)}\)

otherwise stated

\[\Omega(\xi, \cdot) = 0\]  \(\text{(5)}\)

b) \(DX \cdot \tilde{R} = \omega(X^*)\tilde{R}\)

\[X^*(f)(\xi^* + \varepsilon v) + f DX^*\xi^* + \varepsilon f DX^*v = \omega(X^*)f\xi^* + \varepsilon\omega(X^*)fv\]

From this follow the two equations:

\[X^*(f)v = \omega(X^*)fv\]

\[X^*(f)\xi^* + f \cdot (\nabla_X\xi)^* + \varepsilon f \cdot (\phi X)^* = \omega(X^*)f\xi^*\]

then

\[X^*(f) = \omega(X^*)f\]  \(\text{(6)}\)

\[\nabla_X\xi)^* = -\varepsilon(\phi X)^*\]  \(\text{(7)}\)

The two equations \(\text{[3]}\) and \(\text{[4]}\) give

\[\omega = d(\ln f)\]  \(\text{(8)}\)

From equation \(\text{[8]}\) follows in particular that the vector field \(\tilde{R}_0 := \xi^* + \varepsilon v\) is parallel.

Equation \(\text{[7]}\) rewrites:

\[\phi X = -\varepsilon\nabla_X\xi\]  \(\text{(9)}\)
Using the definition of $\phi$ and the antisymmetry of $\Omega$, one sees from this equation that $\xi$ is a Killing vector field. The relation $\Omega = -i\varepsilon\sigma d\eta$ follows from the fact: $d\eta(X,Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y]) = 2g(-\nabla_X \xi, Y)$, true if $\xi$ is a Killing vector field, and consequently $d\eta(X,Y) = 2\varepsilon g(\phi X, Y) = \sigma \varepsilon i \Omega(X,Y)$.

The reverse implication is immediate after the preceding discussion. □

Note that under the conditions of the proposition $r^\perp$ is exactly the horizontal distribution of the connection form $A_0$.

Proposition 8. Under the conditions of proposition 7, we have: $L_\xi \eta = 0$, $L_\xi \phi = 0$, $\nabla_\xi \phi = 0$.

Proof. Since $L_\xi = d \circ \iota_\xi + \iota_\xi \circ d$, the two first statements follow from $\eta(\xi) = \sigma$ and $d\eta(\xi, \cdot) = 0$.

For the third note

$$0 = -\frac{\sigma \varepsilon}{2} (L_\xi d\eta)(X,Y) = \xi g(\phi X, Y) - g(\phi [\xi, X], Y) - g(\phi X, [\xi, Y])$$

$$= (L_\xi g)(\phi X, Y) + g((L_\xi \phi)X, Y).$$

From this follows $L_\xi \phi = 0$.

In particular it implies: $\nabla_\xi (\phi X) - \nabla_{\phi X} \xi = \phi (\nabla_\xi X - \nabla_X \xi)$ and so $(\nabla_\xi \phi)X = \nabla_{\phi X} \xi - \phi (\nabla_X \xi) = -\varepsilon \phi^2 X + \varepsilon \phi^2 X = 0$, proving the last statement. □

Lemma 9. Under the conditions of proposition 7 we have: $(L_{\phi X} \eta)(Y) = d\eta(\phi X, Y)$.

Proof. This follows from the definition of $L$ and from $\eta(\phi X) = 0$. □

4.4. Transverse connection on $\xi^\perp$ in case $D$ admits a recurrent vector field. Note $D = D_\xi$ the sub-bundle $\xi^\perp$ and let $\nabla^D$ be the transverse connection corresponding to $\xi$ on the bundle $D$.

Proposition 10. Let $D$ be as in proposition 7. Let $X$ and $Y$ be sections of $D$.

$$Dr = 0,$$

$$D_r Y^* = (\nabla^D_X Y^*)^*,$$

$$D_r r = 0,$$

$$D_X * Y^* = (\nabla^D_X Y^*)^* + \varepsilon \frac{i}{2} \Omega(X,Y)r,$$

$$D_{\phi X} * r = -2\varepsilon (\phi X)^*,$$

$$D_{\phi Y} * = (\phi Y)^*,$$

$$D_{\phi Y} r = 0.$$
Proposition 11. Let $D$ be as in proposition 4. Let $X, Y, Z$ be sections of $\mathcal{D}$. The curvature of the covariant derivative $\tilde{D}$ expresses by:

\[
\begin{align*}
R(r, \cdot) &= 0, \\
R(r, X^*)Y^* &= (\tilde{R}^D(\xi, X)Y)^*, \\
R(r, X^*)\tilde{r} &= 0, \\
R(X^*, Y^*)Z^* &= (\tilde{R}^D(X, Y)Z)^* + \sigma \Xi((\nabla^D_X\phi)Y - (\nabla^D_Y\phi)X, Z), \\
R(X^*, Y^*)\tilde{r} &= -2\Xi((\nabla^D_X\phi)Y - (\nabla^D_Y\phi)X)^*, \\
R(r, v) &= 0, \\
R(X^*, v)Y^* &= (\nabla^D_X\phi)^* + \varepsilon g(\phi X, \phi Y)r, \\
R(X^*, v)\tilde{r} &= 2\varepsilon(\phi^2 X)^*.
\end{align*}
\]

Let $S := r^\perp/\langle r \rangle$ be the screen bundle corresponding to $r$. Note $g$ the corresponding canonical projection.

As we saw before the covariant derivative $D^S$ restricts to any leaf $\mathcal{L}$ of the foliation $\mathcal{F}$ associated to $D$. Note $\mathfrak{h}ol^L_S$ its holonomy algebra.

Call $\iota$ the $C^\infty(B)$-linear mapping $\Gamma(TB) \to \Gamma(TP)$ defined by $\iota(\xi) = r$, $\iota(X) = X^*$, for $X$ a section of $\mathcal{D}$. Note that $\iota$ is simply the horizontal lift associated to the connection form $A_0$. The mapping $l^S$ (or simply $l$) shall be defined as follows: For $p \in P$ and $v \in \mathcal{D}_{\pi(p)}$, let $l_p(v) := \iota(\iota(X))_p$ for $X$ a section of $\mathcal{D}$ such that $X_{\pi(p)} = v$. $l_p$ is an isomorphism between $\mathcal{D}_{\pi(p)}$ and $S_p$.

**Proposition 12.** For $X$ a vector field on $B$ and $Y$ a section of $\mathcal{D}$, $D^L_{\iota \iota}q(\iota Y) = q(\iota(\nabla^D_X Y))$.

For $\iota \in \mathcal{L}$, $l^S_{\iota \iota}(\mathfrak{h}ol^L_{\pi(\iota)}) = \mathfrak{h}ol^L_{\iota}$.

4.5. Examples of base manifolds.

4.5.1. K-contact manifolds. $(B^{2n+1}, \phi, \xi, \eta, g)$ is a K-contact manifold, meaning that:

1. $(B^{2n+1}, \phi, \xi, \eta, g)$ is an almost contact metric manifold, i.e. $\phi$, $\xi$, $\eta$ are respectively a $(1,1)$-tensor, a vector field and a 1-form on the manifold $B$, $g$ is a Riemannian metric on $B$, $\eta(\xi) = 1$, $d\eta(\xi, \cdot) = 0$, $\phi^2 = -I + \eta \otimes \xi$, $g(X, Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y)$,
2. $g(\phi X, Y) = -d\eta(X, Y)$
3. $\xi$ is a Killing vector field

$(B^{2n+1}, \phi, \xi, \eta, g)$ verifies then the conditions of proposition 11(ii) with $\sigma = 1$ and $\varepsilon = 1$.

4.5.2. Sasakian manifolds. $(B^{2n+1}, \phi, \xi, \eta, g)$ is a Sasakian manifold, meaning that:

1. $(B^{2n+1}, \phi, \xi, \eta, g)$ is a K-contact manifold
2. $\phi$ satisfies $(\nabla_X\phi)(Y) = g(X, Y)\xi - \eta(Y)\chi$.

In this case we have $\nabla^D\phi = 0$, and as a consequence $\mathfrak{hol}^D_{\iota} \subset U(\mathcal{D}_{\iota})$. Additionally we see from proposition 11 that:
Proposition 13. Suppose \((B^{2n+1}, \phi, \xi, \eta, g)\) is a Sasakian manifold. Let \(X, Y, Z\) be sections of \(\mathcal{D} = \xi^\perp\). The curvature of the covariant derivative \(D\) on the manifold \(P\) constructed as before expresses by:

\[
\begin{align*}
R(\cdot, \cdot)r &= 0, \\
R(r, \cdot) &= 0, \\
R(X^*, Y^*)Z^* &= (R^D(X, Y)Z)^*, \\
R(X^*, Y^*)\tilde{r} &= 0, \\
R(X^*, v)Y^* &= g(X, Y)r, \\
R(X^*, v)\tilde{r} &= -2X^*.
\end{align*}
\]

Note that \(\phi\) is parallel along any path \(\gamma\). For \(X, Y\) sections of \(\mathcal{D}\):

\[R(X^*, v)Y^* = g(X, Y)r,\] as a consequence we obtain:

Proposition 14. If \((B^{2n+1}, \phi, \xi, \eta, g)\) is a Sasakian manifold the holonomy algebra of the Levi-Civita covariant derivative \(D\) of \(g\) on the manifold \(P\) is of the form: For the decomposition \(T_o P = \langle r_o \rangle \oplus \langle iD_o \rangle \oplus \langle \tilde{r}_o \rangle\).

\[\mathfrak{ho}_o^D = \left\{ \begin{pmatrix} 0 & u^* & 0 \\ 0 & A & -u \\ 0 & 0 & 0 \end{pmatrix} \bigg| A \in \mathfrak{g}, u \in \text{Hom}(\tilde{\pi}_o, (iD)_o) \right\},\]

where \(\mathfrak{g} = \mathfrak{ho}_{\pi(o)}\) is a sub-Lie-algebra of \(\mathfrak{u}(iD)_o\).

Note that in the theorem appears \(\mathfrak{ho}_{\pi(o)}\) and not \(\mathfrak{ho}_{\pi(o)}^{C,D}\) like in proposition 12. In the proof, the fact that \(\phi\) commutes with parallel transport on the transverse bundle is crucial.

5. Double bundles

The base manifold can be obtained by a preliminary circle bundle construction. We discuss this possibility in the following: Let \((B, g)\) be a pseudo-Riemannian manifold with Levi-Civita connection \(\nabla\). Let \(\pi_1 : P_1 \rightarrow B\) be a first principal \(S^1\)-bundle over \(B\) with connection form \(A_1\) and metric \(\tilde{g}_1 = \pi_1^*g + \sigma A_1 \otimes A_1\).

We have seen that the corresponding fundamental field \(\xi = v\) is then a Killing field such that \(\tilde{g}_1(v, v) = -\sigma\).

Let \(\mathcal{D}_1\) be the transverse bundle \(\xi^\perp\) equipped with the transverse covariant derivative \(D_1^\perp\). Let \(t_p\) be the mapping \(T_{\pi_1(p)}B \rightarrow \mathcal{D}_1, t_p\) defined by \(t_p(X_p) = X_p^*\) for \(X\) a vector field on \(B\). \(t_p\) is an isomorphism.

Consider a second principal \(S^1\)-bundle \(\pi_2 : P_2 \rightarrow P_1\) admitting a flat connection form \(A_{2,0}\). Fix \(\varepsilon_2 := \pm 1\). Let \(\eta_2 := \tilde{g}_1(\xi, \cdot)\) be a 1-form on \(P_1\). Define the connection form \(A_2\) by \(A_2 := A_{2,0} + i\varepsilon_2 \pi_2^*\eta_2\). Let \(\tilde{g}_2\) be the metric \(\pi_2^*\tilde{g}_1 - \sigma A_2 \otimes A_2\). By proposition 7 follows that \(P_2\) admits the parallel vector field \(r = \xi^* + \varepsilon_2 v_2\) for the Levi-Civita connection associated to \(\tilde{g}_2\).

Let \(\mathcal{C}\) be a leaf of the foliation determined by \(\mathcal{D}_2 := r^\perp\) and \(o\) be a point contained in it. Let \(\mathcal{S}\) be the screen bundle determined by \(r\).

Proposition 15. \(\mathfrak{ho}_o^{\mathcal{S}}\) is isomorphic to \(\mathfrak{ho}_{\pi_1 \pi_2(o)}^{\nabla}\)

Proof. By propositions 5 and 12 the holonomy algebra \(\mathfrak{ho}_o^{\mathcal{S}}\) verifies:

\[\mathfrak{ho}_o^{\mathcal{S}} = \mathfrak{ho}_{\pi_1 \pi_2(o)}(\mathfrak{ho}_o^{\nabla}(\pi_1 \pi_2(o)))\]

\(\square\)
References

[Ba] H. Baum, *Eichfeldtheorie*, in preparation.

[Be] A. L. Besse, *Einstein manifolds*, Springer-Verlag, 1986.

[BBI] L. Bérard Bergery, A. Ikemakhen, *On the Holonomy of Lorentzian Manifolds*, Proceeding of symposia in pure math., volume 54, 1993, pp. 27-40.

[Bl] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Birkhäuser, 2002.

[BG1] C. P. Boyer, K. Galicki, *Sasakian Geometry, Holonomy and Supersymmetry*, arXiv: math/0703231v2.

[BG2] C. P. Boyer, K. Galicki, *Sasakian Geometry*, Oxford University Press, 2007.

[GHJ] M. Gross, D. Huybrechts, D. Joyce, *Calabi-Yau Manifolds and Related Geometries*, Springer-Verlag, 2003.

[H] Y. Hatakeyama, *Some notes on differentiable manifolds with almost contact structures*, Tohoku Math. J. (2) 15 (1963), pp. 176-181.

[K] S. Kobayashi, *Principal fibre bundles with the 1-dimensional toroidal group*, Tohoku Math. J. (2) 8 (1956), pp. 29-45.

[L1] T. Leistner, *PhD-Thesis (Dissertation)*, Humboldt-Universität Berlin, 2003.

[L2] T. Leistner, *Berger algebras, weak-Berger algebras and Lorentzian holonomy*, sfb 288 preprint no. 567, 2002.

[L3] T. Leistner, *Towards a classification of Lorentzian holonomy groups*, arXiv:math.DG/0305139, 2003.

[L4] T. Leistner, *Towards a classification of Lorentzian holonomy groups. Part II: Semisimple, non-simple weak-Berger algebras*, arXiv:math.DG/0309274, 2003.