ON DEFORMED PREPROJECTIVE ALGEBRAS

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Abstract. Deformed preprojective algebras are generalizations of the usual preprojective algebras introduced by Crawley-Boevey and Holland, which have applications to Kleinian singularities, the Deligne-Simpson problem, integrable systems and noncommutative geometry. In this paper we offer three contributions to the study of such algebras: (1) the 2-Calabi-Yau property; (2) the unification of the reflection functors of Crawley-Boevey and Holland with reflection functors for the usual preprojective algebras; and (3) the classification of tilting ideals in 2-Calabi-Yau algebras, and especially in deformed preprojective algebras for extended Dynkin quivers.

1. Introduction

Deformed preprojective algebras were introduced in [12] in order to study noncommutative deformations of Kleinian singularities; they have been studied further in [10], and used in [11] to solve an additive analogue of the Deligne-Simpson problem. They also have a role in integrable systems, see e.g. [4], and noncommutative geometry, see e.g. [1]. In this paper we further develop the theory of these algebras.

The definition is as follows. Let $K$ be a field and let $Q$ be a quiver with vertex set $Q_0$ and arrow set $Q_1$. We assume always that $Q$ is finite, and write $h(a)$ and $t(a)$ for the head and tail vertices of an arrow $a$. The double $\overline{Q}$ of $Q$ is obtained by adjoining a reverse arrow $a^* : j \to i$ for each arrow $a : i \to j$ in $Q$. We use the notation $(a^*)^* = a$, and for an arrow $a \in Q_1$, we set $\varepsilon(a) = 1$ if $a \in Q_1$ and $\varepsilon(a) = -1$ otherwise. Let $I = Q_0$ and fix a weight $\lambda \in K I$. The corresponding deformed preprojective algebra is

$$\Pi^\lambda(Q) := K\overline{Q}/(\rho_{\lambda})$$

where $\rho = \sum_{a \in Q_1} \varepsilon(a)aa^*$ and $\rho_{\lambda} = \rho - \sum_{i \in I} \lambda_i e_i$. Here $K\overline{Q}$ denotes the path algebra of $\overline{Q}$. We use the convention that the path $ba$ exists if $h(a) = t(b)$, and the trivial path at vertex $i$ is denoted $e_i$. The usual (undeformed) preprojective algebra is $\Pi^0(Q)$.

Given a ring $R$, we usually consider left $R$-modules, and write $\text{Mod } R$ for the corresponding category. Given an algebra $A$, we consider $A \otimes_K A$ as an $A$-bimodule using the outer $A$-actions, so $b(a \otimes a')c = ba \otimes a'c$. If $M$ is an $A$-bimodule, the space $\text{Hom}_{AA}(M, A \otimes_K A)$ of $A$-bimodule homomorphisms becomes an $A$-bimodule using the inner $A$-actions, that is, for $f \in \text{Hom}(M, A \otimes_K A)$ we have $(bfc)(m) = \sum a_\lambda a \otimes ba'_\lambda$ where $f(m) = \sum a_\lambda a \Lambda$. The algebra $A$ is said to be homologically smooth if $A$ has a finite projective resolution.

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by finitely generated $\mathcal{A}$-bimodules, so $\mathcal{A}$ is isomorphic to a perfect complex in the derived category of $\mathcal{A}$-bimodules, and it is said to be $d$-Calabi-Yau, for a natural number $d$, if it is homologically smooth and $\mathbf{R}\mathrm{Hom}_{\mathcal{A}\mathcal{A}}(\mathcal{A},\mathcal{A} \otimes K \mathcal{A}) \cong \mathcal{A}[-d]$ in that same derived category [17, Definition 3.2.3], [28, Definition 7.2]. Our first result may already be known to specialists, but we didn’t find a reference. After releasing our work as a preprint, Travis Schedler explained to us a different proof, based on the methods of [20].

**Theorem 1.1.** If $Q$ is a connected non-Dynkin quiver, then $\Pi^{\lambda}(Q)$ is $2$-Calabi-Yau.

This is proved in section 2.3. If $Q$ is a Dynkin quiver, then $\Pi^{0}(Q)$ is a finite-dimensional self-injective algebra, so not in general $2$-Calabi-Yau. If $Q$ is connected and non-Dynkin, it is known that $\Pi^{0}(Q)$ is $2$-Calabi-Yau, see [6, 9, 7, 16].

The dimension vector of a finite-dimensional $\Pi^{\lambda}(Q)$-module $M$ is $\dim M \in \mathbb{Z}^I$. We write $\varepsilon_i \in \mathbb{Z}^I$ for the coordinate vector of a vertex $i$. The Ringel form of $Q$ is the bilinear form $\langle -, - \rangle : \mathbb{Z}^I \times \mathbb{Z}^I \to \mathbb{Z}$ given by

$$\langle \alpha, \beta \rangle := \sum_{i \in I} \alpha_i \beta_i - \sum_{\alpha \in Q_1} \alpha_{i(\alpha)} \beta_{h(\alpha)}.$$  

(1.1)

We call $q(\alpha) := \langle \alpha, \alpha \rangle$ the quadratic form of $Q$, and have a symmetric bilinear form $\langle \alpha, \beta \rangle := \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$.

We say that a vertex $i$ in $Q$ is loop-free if there are no arrows with both head and tail at $i$, or equivalently $q(\varepsilon_i) = 1$. For such a vertex $i$, the corresponding reflections $s_i \in \mathrm{Aut}(\mathbb{Z}^I)$ and $r_i \in \mathrm{Aut}(K^I)$ are defined by

$$s_i \alpha := \alpha - \langle \alpha, \varepsilon_i \rangle \varepsilon_i, \quad (r_i \lambda)_j = \lambda_j - \langle \varepsilon_i, \varepsilon_j \rangle \lambda_j.$$  

For $\lambda \in K^I$ and $\alpha \in \mathbb{Z}^I$, let $\lambda \cdot \alpha = \sum \lambda_i \alpha_i$. Then it is easy to see that $r_i(\lambda) \cdot \alpha = \lambda \cdot s_i(\alpha)$ for $\lambda, \alpha \in K^I$. The Weyl group $W$ is the group of automorphisms of $\mathbb{Z}^I$ generated by the simple reflections for loop-free vertices. Then $W$ acts on $\mathbb{Z}^I$ and $K^I$ with $(w \lambda) \cdot \alpha = \lambda \cdot (w \cdot \alpha)$ for $w \in W$, $\alpha \in \mathbb{Z}^I$ and $\lambda \in K^I$. According to [12], deformed preprojective algebras come equipped with reflection functors

$$E_i : \text{Mod} \Pi^{\lambda}(Q) \to \text{Mod} \Pi^{r_i(\lambda)}(Q)$$

which exist when $i$ is a loop-free vertex and $\lambda_i \neq 0$. Moreover, if $M$ is finite-dimensional, then

$$\dim E_i(M) = s_i(\dim M).$$

The condition $\lambda_i \neq 0$ means that these reflection functors do not exist for $\Pi^{0}(Q)$. Instead there is a theory of tilting modules and functors, see [10, 3, 2, 9]. Our next results unify these two situations. Using cokernels and kernels, for any loop-free vertex $i$ we construct functors

$$C_i^\lambda : \text{Mod} \Pi^{\lambda}(Q) \to \text{Mod} \Pi^{r_i(\lambda)}(Q)$$

in section 3.1. Since $r_i(r_i \lambda) = \lambda$, it follows that $C_i^r(\lambda)$ and $K_i^{r_i(\lambda)}$ are functors in the reverse direction. Note that if $\lambda_i \neq 0$, then also $(r_i \lambda)_i \neq 0$. If $\lambda_i = 0$, then $r_i \lambda = \lambda$, so $C_i^\lambda$ and $K_i^{r_i(\lambda)}$ are functors from $\text{Mod} \Pi^{\lambda}(Q)$ to itself.

If $i$ is a loop-free vertex and $\lambda_i = 0$, there is a unique trivial simple module $S_i$ for $\Pi^{\lambda}(Q)$ with dimension vector the coordinate vector for $i$. We denote its annihilator by $I_i$. This is a 2-sided ideal in $\Pi^{\lambda}(Q)$ with $\Pi^{\lambda}(Q)/I_i \cong K$. Also $I_i = \Pi^{\lambda}(Q)(1 - \varepsilon_i)\Pi^{\lambda}(Q)$, from which it follows that $I_i I_i = I_i$.

**Theorem 1.2.** If $i$ is a loop-free vertex, then

(i) $C_i^\lambda$ is left adjoint to $K_i^{r_i(\lambda)}$.

(ii) If $\lambda_i \neq 0$, then $C_i^\lambda \cong K_i^{r_i(\lambda)} \cong E_i$, and $C_i^\lambda$ is an equivalence of categories which satisfies $C_i^r(\lambda) C_i^\lambda \cong \text{id}_{\text{Mod} \Pi^{\lambda}(Q)}$.

(iii) If $\lambda_i = 0$, then $C_i^\lambda \cong I_i \otimes_{\Pi^{\lambda}(Q)} (-)$ and $K_i^{r_i(\lambda)} \cong \text{Hom}_{\Pi^{\lambda}(Q)}(I_i, -)$.

This is proved in section 3.1. In future we drop the superscript $\lambda$ and just write $C_i$ and $K_i$, leaving it to the reader to interpret appropriately. In section 3.2 we prove the following.

**Theorem 1.3.** If $i$ and $j$ are loop-free vertices, then

(i) $C_i C_j \cong C_j C_i$ and $K_i K_j \cong K_j K_i$ if there is no arrow between $i$ and $j$ in $Q$.

(ii) $C_i C_j C_i = C_j C_i C_j$ and $K_i K_j K_i \cong K_j K_i K_j$ if there is exactly one arrow between $i$ and $j$ in $Q$. 
We remark that part (ii) of the theorem is known in the case where \( \lambda = 0 \) by [2]. (It can also be shown by using relations on tilting ideals as in [3].) On the other hand, our statement (ii) does not assume that \( \lambda = 0 \).

Let \( A \) be a \( K \)-algebra. Given a left \( A \)-module \( M \), one writes \( \text{add} M \) for the full subcategory of \( \text{Mod} A \) consisting of the modules isomorphic to a direct summand of a finite direct sum of copies of \( M \), so \( \text{add} A \) is the category of finitely generated projective left \( A \)-modules. Let \( n \geq 1 \). Recall that an \( A \)-module \( T \) is an \( n \)-tilting \( A \)-module if it satisfies the following three conditions.

1. There is an exact sequence \( 0 \to P_n \to \cdots \to P_1 \to P_0 \to T \to 0 \) with \( P_0, P_1, \ldots, P_n \in \text{add} A \).
2. \( \text{Ext}^i_A(T, T) = 0 \) for all \( i > 0 \).
3. There is an exact sequence \( 0 \to A \to T_0 \to T_1 \to \cdots \to T_n \to 0 \) with \( T_0, T_1, \ldots, T_n \in \text{add} T \).

In this paper, by a tilting module we always mean a 1-tilting module. Note that by (1), any tilting module is finitely generated. One says that \( T \) is partial tilting if \( T \) satisfies (1) (with \( n = 1 \)) and (2). One says that an ideal \( I \) of \( A \) is a tilting ideal if \( I \) is a tilting module as both left and right \( A \)-modules.

Tilting ideals for 2-Calabi-Yau algebras were studied in [8, 9] under the assumption that the algebras are complete. In this paper we study such ideals for arbitrary 2-Calabi-Yau algebras.

For an \( A \)-module \( M \), let \( \text{Ann}_A(M) = \{ a \in A \mid aM = 0 \} \) be the annihilator ideal of \( M \) in \( A \). We say that a simple \( A \)-module \( S \) is rigid if \( \text{Ext}^1_A(S, S) = 0 \). Given a finite-dimensional rigid simple \( A \)-module \( S \), we write \( I_S \) for its annihilator ideal in \( A \). In section 4.1 we prove the following.

**Proposition 1.6.** If \( A \) is 2-Calabi-Yau and \( S \) is a finite-dimensional rigid simple \( A \)-module, then \( I_S \) is a tilting ideal in \( A \), it has finite codimension in \( A \), and there is an isomorphism \( \text{End}_A(I_S) \cong A^{op} \), under which \( a \in A \) corresponds to the homothety of right multiplication by \( a \).

Let \( S \) be a set of finite-dimensional rigid simple \( A \)-modules. In the category of finite-dimensional \( A \)-modules, we write \( \mathcal{E}(S) \) for the Serre subcategory generated by \( S \), so \( \mathcal{E}(S) \) consists of the finite-dimensional modules whose composition factors belong to \( S \). For a finite sequence \( S_1, S_2, \ldots, S_r \) of modules in \( S \), we consider the ideal \( I_{S_1S_2\cdots S_r} = I_{S_1}I_{S_2}\cdots I_{S_r} \) in \( A \). For the empty sequence we define \( I_\emptyset = A \). We denote by \( \mathcal{I}(S) \) the set of all ideals of this form. The following result, proved in section 1.2, is an analogue of [8, Theorem III.1.6].

**Theorem 1.5.** Suppose that \( A \) is 2-Calabi-Yau. Any element \( I \in \mathcal{I}(S) \) is a tilting ideal with \( A/I \in \mathcal{E}(S) \) and \( \text{End}_A(I) = A \). Conversely any partial tilting left ideal \( I \) in \( A \) with \( A/I \in \mathcal{E}(S) \) is in \( \mathcal{I}(S) \). If \( I, I' \in \mathcal{I}(S) \) are isomorphic as left modules, they are equal.

**Proposition 1.6.** Suppose that \( A \) is 2-Calabi-Yau and \( S, T \in \mathcal{S} \).

1. \( I_SI_S = I_S \).
2. \( I_SI_T = I_TI_S \) if \( \text{Ext}^1_A(S, T) = 0 \).
3. \( I_SI_TI_S = I_TI_SI_T \) if \( \text{Ext}^1_A(S, T) \) is 1-dimensional as a right \( \text{End}_A(S) \)-module and as a left \( \text{End}_A(T) \)-module.

This and the next theorem are proved in section 1.3. For simplicity (to avoid valued quivers, and because it is sufficient for \( \Pi^1(Q) \)) we consider the case that \( S \) is split, by which we mean that \( \text{End}_A(S) = K \) for all \( S \in \mathcal{S} \). In this case, the Ext-quiver \( Q(S) \) has as vertices the isomorphism classes of elements \( S \in \mathcal{S} \), and with \( \dim_K \text{Ext}^1_A(S, T) \) arrows from \( S \) to \( T \). For a 2-Calabi-Yau algebra this is the double of an acyclic quiver. The associated Coxeter group \( W(S) \) is generated by elements \( \sigma_{S_1} \), one for each \( S \in \mathcal{S} \) up to isomorphism, subject to the relations that \( \sigma_S^2 = 1 \) for any \( S \), \( \sigma_S\sigma_T = \sigma_T\sigma_S \) if there are no arrows from \( S \) to \( T \) and \( \sigma_S\sigma_T\sigma_S = \sigma_T\sigma_S\sigma_T \) if there is exactly one arrow from \( S \) to \( T \).

**Theorem 1.7.** Suppose that \( A \) is 2-Calabi-Yau. If \( S \) is a set of finite-dimensional rigid simple \( A \)-modules which is split, then there is a bijection \( W(S) \to \mathcal{I}(S) \) given by \( w \mapsto I_{S_1S_2\cdots S_r} \) for a reduced expression \( \sigma_{S_1}\sigma_{S_2}\cdots\sigma_{S_r} \) for \( w \).

The previous results apply in particular to the algebra \( A = \Pi^1(Q) \) where \( Q \) is a connected non-Dynkin quiver. Below we give the classification of finite-dimensional rigid simple modules in this case.
Recall that the simple roots for $Q$ are the coordinate vectors $\varepsilon_i$ of loop-free vertices $i$. An element of $\mathbb{Z}^l$ is a real root if it is the image of a simple root under the action of the Weyl group. The fundamental region $F$ is the set of vectors $\alpha$ in $\mathbb{N}^l$ such that $\alpha \neq 0$, the support of $\alpha$ is connected and $(\alpha, \varepsilon_i) \leq 0$ for any $i \in I$. An imaginary root is an element of $\mathbb{Z}^l$ of the form $w\beta$ or $-w\beta$ for some $w \in W$ and $\beta \in F$. A root is a real or imaginary root. It is standard that any root $\alpha$ is either positive, meaning that it belongs to $\mathbb{N}^l$, or negative, meaning that it belongs to $(-\mathbb{N})^l$. It is easy to see that $q(s_i, \alpha) = q(\alpha)$ holds. Therefore $q(\alpha) = 1$ if $\alpha$ is a real root, and $q(\alpha) = 0$ if $\alpha$ is an imaginary root.

For $\lambda \in K^l$, we define $\Sigma^+_{\lambda}$ to be the set of positive real roots $\alpha$ with $\lambda \cdot \alpha = 0$ and such that there is no decomposition $\alpha = \beta + \gamma + \ldots$ as a sum of two or more positive roots with $\lambda \cdot \beta = \lambda \cdot \gamma = \ldots$. It is the intersection of the set of real roots and the set $\Sigma_\lambda$ of $[10]$. The following result is clear from $[10]$ Theorem 1.2 in case the base field $K$ is algebraically closed (see also $[11]$ Theorem 2)), or for general $K$ by the argument of $[13]$ Theorems 1.8, 1.9—we omit the details.

**Proposition 1.8.** The map sending a module to its dimension vector gives a 1:1 correspondence between the isomorphism classes of finite-dimensional rigid simple $\Pi^A(Q)$-modules and the elements of $\Sigma^\text{re}_\lambda$. The endomorphism algebra of any finite-dimensional rigid simple module is isomorphic to $K$. The dimension vector of any finite-dimensional non-rigid simple module is a positive imaginary root.

We can say more in case $Q$ is an extended Dynkin quiver, see $[9]$ Theorem III.1.6. Let $R$ be the set of all finite-dimensional rigid simple $\Pi^A(Q)$-modules. The last two theorems are proved in section 4.4.

**Theorem 1.9.** If $Q$ is an extended Dynkin quiver and $A = \Pi^A(Q)$, then there are only finitely many isomorphism classes of finite-dimensional rigid simple $A$-modules, and $I(R)$ is the set of all tilting ideals of finite codimension in $\Pi^A(Q)$.

In order to understand the structure of $Q(R)$ we have the following result. Here $\delta$ is the minimal positive imaginary root for an extended Dynkin quiver $Q$. The notation is as in $[12]$.

**Theorem 1.10.** If $Q$ is extended Dynkin quiver and $\lambda \cdot \delta = 0$, then $Q(R)$ is the double of a disjoint union of extended Dynkin quivers. If in addition $K$ is an algebraically closed field of characteristic zero, then there is a bijection between the connected components of $Q(R)$ and the singular points of the affine quotient variety $\text{Rep}(\Pi^A(Q), \delta) \sslash \text{GL}(\delta)$.

2. The PBW and 2-Calabi-Yau properties

2.1. Filtrations and the PBW property. Let $A$ be a $K$-algebra. A filtration of $A$ is a family $\mathcal{F} = \{A_\leq i \mid i \in \mathbb{Z}\}$ of $K$-subspaces $A_\leq i$ of $A$ satisfying $A = \bigcup_{i \geq 0} A_\leq i$, $A_\leq i \subseteq A_{\leq i+1}$ and $A_\leq i A_\leq j \subseteq A_{\leq i+j}$ for any $i, j \geq 0$. If $A$ admits a filtration, then we say that $A = \bigcup_{i \geq 0} A_\leq i$ is a filtered $K$-algebra. In this case, the associated graded $K$-algebra $\text{gr} A$ is defined as follows:

$$\text{gr} \mathcal{F} A = \text{gr} A := \bigoplus_{i \geq 0} A_\leq i / A_{\leq i-1},$$

where $A_{\leq -1} = 0$. The algebra $\text{gr} A$ is a $(\mathbb{N})$-graded $K$-algebra such that the $i$-th component is $A_{\leq i} / A_{\leq i-1}$.

If $A = \bigoplus_{i \geq 0} A_i$ is a graded $K$-algebra then $A_\leq i = \bigoplus_{j \leq i} A_j$ defines a filtration of $A$, and the associated graded algebra is isomorphic to $A$ as graded $K$-algebras. More precisely, if $x = \sum_{i=0}^l x_i \in A$ with $x_i \in A_i$, then the isomorphism is described as follows:

$$A \to \text{gr} A, \quad x \mapsto (x_i + A_{\leq i-1})_i.$$

Let $A$ be a filtered $K$-algebra and let $M$ be an $A$-module. A filtration of $M$ is a family $\mathcal{M} = \{M_{\leq i} \mid i \geq 0\}$ of $K$-subspaces $M_{\leq i}$ of $M$ satisfying $M = \bigcup_{i \geq 0} M_{\leq i}$, $M_{\leq i} \subseteq M_{\leq i+1}$ and $A_\leq i M_{\leq j} \subseteq M_{\leq i+j}$. For filtered $A$-modules $M = \bigcup_{i \geq 0} M_{\leq i}$ and $N = \bigcup_{i \geq 0} N_{\leq i}$, a morphism of filtered modules of degree $j \in \mathbb{Z}$ is a morphism of $A$-modules $f : M \to N$ such that $f(M_{\leq i}) \subseteq N_{\leq i+j}$ holds for any $i \geq 0$. In this case, the morphism $f$ induces a morphism

$$\text{gr} f : \text{gr} M \to \text{gr} N$$

of graded $(\text{gr} A)$-modules of degree $j$. We have two well-known and straightforward lemmas.
Lemma 2.1. Let $M, N$ be filtered $A$-modules and $f : M \to N$ be a morphism of filtered modules of degree $j \geq 0$. If $\text{gr} f$ is injective, then so is $f$.

Proof. Let $x \in M$ such that $f(x) = 0$. There exists $i \geq 0$ such that $x \in M_{\leq i}$. Because $f(x) = 0$, we have $(\text{gr} f)(\mathfrak{p}) = 0$ in $N_{\leq i+j}/N_{\leq i+j-1}$, where $\mathfrak{p} \in M_{\leq i}/M_{\leq i-1}$. Since $\text{gr} f$ is injective, we have $\mathfrak{p} = 0$, that is, $x \in M_{\leq i-1}$. By using this argument inductively, we have $x \in M_{\leq i-1} = 0$.

Let $A$ and $B$ be two filtered $K$-algebras. For a filtered left $A$-module $M$ and a filtered right $B$-module $N$, we have the following filtration of $M \otimes_K N$:

$$(M \otimes_K N)_{\leq i} := \sum_{i=j+k} M_{j} \otimes_K N_{\leq k}.$$  \hspace{1cm} (2.2)

In particular this induces a filtration on the $K$-algebra $A \otimes_K B^{\text{op}}$, and then $M \otimes_K N$ is a filtered $A$-$B$-bimodule, by which we mean that it is a filtered $A \otimes_K B^{\text{op}}$-module.

Lemma 2.2. Let $A$ and $B$ be two filtered $K$-algebras. Let $M$ be a filtered left $A$-module and $N$ be a filtered right $B$-module. Then $\text{gr}(M \otimes_K N)$ is isomorphic to $\text{gr} M \otimes_K \text{gr} N$ as a $\text{gr} A$-$\text{gr} B$-bimodule. If $M = A$ and $N = B$, then this is an isomorphism of graded algebras.

Proof. A morphism $\phi$ from $\text{gr}(M \otimes_K N)$ to $\text{gr} M \otimes_K \text{gr} N$ is given as follows: for $z = x \otimes y + (M \otimes_K N)_{\leq i-1} \in \text{gr}(M \otimes_K N)$, with $x \in M_{\leq j}, y \in N_{\leq k}$ for $j + k = i$, let $\phi(z) = (x + M_{\leq j-1}) \otimes (y + N_{\leq k-1}) \in (M \otimes_K N)_{i}$. Then this assignment induces a morphism of $\text{gr} A$-$\text{gr} B$-bimodules. By comparing $K$-bases of $M_{\leq i}/M_{\leq i-1}, N_{\leq i}/N_{\leq i-1}$ and their images under $\phi$, one can show that $\phi$ is an isomorphism.

The deformed preprojective algebra $\Pi^A(Q)$ has a filtration $0 = \Pi_{\leq -1} \subseteq \Pi_{\leq 0} \subseteq \Pi_{\leq 1} \subseteq \cdots \subseteq \Pi^A(Q)$, where $\Pi_{\leq n}$ is spanned by the paths in $\overline{Q}$ of length at most $n$. There is a natural map $\Pi^0(Q) \to \Pi^A(Q)$ and it is surjective [12, Lemma 2.3].

Lemma 2.3. If $Q$ is connected and non-Dynkin, then the natural map $\Pi^0(Q) \to \text{gr} \Pi^A(Q)$ is an isomorphism.

In other terminology, this says that $\Pi^A(Q)$ is a PBW deformation of $\Pi^0(Q)$.

Proof. If $Q$ is connected and non-Dynkin, it is known that $\Pi^0(Q)$ is Koszul, see [13] (and [26, 14, 24] for special cases). The lemma thus follows from [13, Theorem A].

2.2. Projective resolution and Calabi-Yau property. We begin with results about a partial projective resolution of $\Pi^A(Q)$. The results are already known for the undeformed preprojective algebra [9, Lemma 1], and although not written anywhere, it has long been known to the first author that they generalize to $\Pi^A(Q)$, since they were further adapted to multiplicative preprojective algebras in [13, section 3].

Let $\Pi = \Pi^A(Q)$. Let $P_0$ and $P_1$ be the following projective $\Pi$-bimodules,

$$P_0 = \bigoplus_{i \in I} \Pi e_i \otimes_K e_i \Pi \quad \text{and} \quad P_1 = \bigoplus_{a \in \Pi Q_1} \Pi e_{h(a)} \otimes_K e_{t(a)} \Pi.$$

For any $i \in I$, we write $\eta_i$ for the element $e_i \otimes e_i$ in the $i$th summand of $P_0$, and for any arrow $a \in \Pi Q_1$, we write $\eta_a$ for the element $e_{h(a)} \otimes e_{t(a)}$ in the $a$th summand of $P_1$.

Proposition 2.4. There is an exact sequence $P_0 \xrightarrow{f} P_1 \xrightarrow{g} P_0 \xrightarrow{h} \Pi \to 0$ of $\Pi$-bimodules, where the maps are given by

$$f(\eta_i) = \sum_{a \in \Pi Q_1} \varepsilon(a)(\eta_a a^* + an_a^*), \quad g(\eta_a) = an_{t(a)} - \eta_{h(a)} a, \quad h(\eta_i) = e_i.$$

Proof. We have $\Pi = K\overline{Q}/I$, where $I$ is the ideal in $K\overline{Q}$ generated by the elements $\rho_i - \lambda_i e_i$. Let $S = K\overline{Q}_0 = \bigoplus_{i \in I} K e_i$. Combining the exact sequence [26, Theorem 10.1] for $\Omega_S(\Pi)$ with [26, Theorem 10.3], gives an exact sequence

$$I/I^2 \xrightarrow{\alpha} \Pi \otimes_K \Omega_S(K\overline{Q}) \otimes_K \Pi \xrightarrow{\beta} \Pi \otimes_S \Pi \xrightarrow{\gamma} \Pi \to 0.$$
Here \( \alpha(I^2 + x) = 1 \otimes (x \otimes 1 - 1 \otimes x) \otimes 1 \) for \( x \in I \), \( \beta(1 \otimes \omega \otimes 1) \) is the image of \( \omega \in \Omega_S(KQ) \subseteq KQ \otimes_S KQ \) in \( \Pi \otimes \Pi \), and \( \gamma \) is the multiplication map.

We identify \( \Pi \otimes_S \Pi \) with \( P_0 \), and then \( \gamma \) is identified with the map \( h \). Let \( V = KQ_1 \), the vector space spanned by the arrows in \( Q \), which is naturally an \( S \)-bimodule. By [25, Theorem 10.5], there is an isomorphism

\[
\Pi Q \otimes_S V \otimes_S KQ \to \Omega_S(KQ)
\]

sending \( 1 \otimes a \otimes 1 \) to \( a \otimes 1 - 1 \otimes a \). Thus we can identify

\[
P_1 \cong \Pi \otimes_S V \otimes_S \Pi \cong \Pi \otimes KQ \Omega_S(KQ) \otimes KQ \Pi
\]

with \( \eta_a \) corresponding to \( 1 \otimes (a \otimes 1 - 1 \otimes a) \otimes 1 \). Now under the identification of \( \Pi \otimes_S \Pi \) with \( P_0 \), the element \( a \otimes 1 - 1 \otimes a \) corresponds to

\[
a \otimes e_{i(a)} - e_{h(a)} \otimes a = a\eta_{i(a)} - \eta_{h(a)}a.
\]

It follows that \( \beta \) corresponds to the map \( g \). Finally \( I \) is generated as an ideal by the elements \( \rho_i - \lambda_i e_i \), so there is a surjective map \( \phi : P_0 \to I/I^2 \) sending \( \eta_i \) to \( I^2 + \rho_i - \lambda_i e_i \). Its composition with \( \alpha \) sends \( \eta_i \) to \( 1 \otimes (\rho_i - 1 \otimes \rho_i) \otimes 1 \) since \( \lambda_i e_i \otimes 1 = \lambda_i e_i \otimes e_i = 1 \otimes \lambda_i e_i \). Now

\[
\rho_i \otimes 1 - 1 \otimes \rho_i = \sum_{a \in Q_i, \quad h(a) = i} \varepsilon(a) (aa^* \otimes 1 - 1 \otimes aa^*)
\]

\[
= \sum_{a \in Q_i, \quad h(a) = i} \varepsilon(a) (a(a^* \otimes 1 - 1 \otimes a^*) + (a \otimes 1 - 1 \otimes a) a^*)
\]

which corresponds in \( P_1 \) to the same element as \( f(\eta_i) \). Thus \( \alpha \phi \) corresponds to \( f \), giving the result. \( \square \)

**Proposition 2.5.** The complexes \( P_1 \xrightarrow{\phi} P_1 \xrightarrow{g} P_0 \) and

\[
\text{Hom}_{\Pi}(P_0, \Pi \otimes \Pi) \xrightarrow{g^*} \text{Hom}_{\Pi}(P_1, \Pi \otimes \Pi) \xrightarrow{\phi^*} \text{Hom}_{\Pi}(P_0, \Pi \otimes \Pi)
\]

are isomorphic as complexes of \( \Pi \)-bimodules.

Note that the change of sign for \( g^* \) is irrelevant for the truth of the proposition; it is introduced because it arises in the definition of \( R \text{Hom}_{\Pi}(\Pi, \Pi \otimes \Pi) \cong \text{Hom}_{\Pi}(P^*, \Pi \otimes \Pi) \).

**Proof.** Clearly we have \( \text{Hom}_{\Pi}(\Pi e_i \otimes \Pi e_j, \Pi \otimes \Pi) \cong e_i \Pi \otimes \Pi e_j \). Thus there are isomorphisms

\[
\alpha : \text{Hom}_{\Pi}(P_0, \Pi \otimes \Pi) \to P_0, \quad \alpha(\phi) = \sum_{i, \lambda} p'_{i\lambda} \eta_{i\lambda},
\]

where \( \phi(\eta_i) = \sum_{\lambda} p_{i\lambda} \otimes p'_{i\lambda} \in e_i \Pi \otimes \Pi e_i \), and

\[
\beta : \text{Hom}_{\Pi}(P_1, \Pi \otimes \Pi) \to P_1, \quad \beta(\psi) = \sum_{a \in Q_i} \varepsilon(a)p'_{a\lambda} \eta_{a\lambda}, \quad p_{a\lambda} \in e_{h(a)} \Pi \otimes \Pi e_{h(a)}.
\]

Now we have a commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_{\Pi}(P_0, \Pi \otimes \Pi) & \xrightarrow{-g^*} & \text{Hom}_{\Pi}(P_1, \Pi \otimes \Pi) \\
\alpha \downarrow & & \beta \downarrow \\
P_0 & \xrightarrow{f} & P_1 \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{Hom}_{\Pi}(P_0, \Pi \otimes \Pi) & \xrightarrow{f^*} & \text{Hom}_{\Pi}(P_0, \Pi \otimes \Pi) \\
\alpha \downarrow & & \alpha \downarrow \\
P_0 & \xrightarrow{g} & P_0 \\
\end{array}
\]

\( \square \)

Let \( D \) be the duality \( \text{Hom}_K(-, K) \).

**Proposition 2.6.** For finite dimensional \( \Pi \)-modules \( M, N \), we have \( \text{Ext}^1_{\Pi}(N, M) \cong D \text{Ext}^1_{\Pi}(M, N) \) and

\[
\dim_K \text{Ext}^1_{\Pi}(M, N) = \dim_K \text{Hom}_{\Pi}(M, N) + \dim_K \text{Hom}_{\Pi}(N, M) - (\dim M, \dim N).
\]

\[ (2.3) \]
Proof. Since the sequence in Proposition 2.4 is split as a sequence of right \( \Pi \)-modules, it induces an exact sequence
\[
P_0 \otimes \Pi M \to P_1 \otimes \Pi M \to P_0 \otimes \Pi M \to M \to 0
\]
which is the start of a projective resolution of \( M \). Applying \( \text{Hom}_\Pi(-, N) \) gives a complex
\[
0 \to \text{Hom}_\Pi(P_0 \otimes \Pi M, N) \to \text{Hom}_\Pi(P_1 \otimes \Pi M, N) \to \text{Hom}_\Pi(P_0 \otimes \Pi M, N) \to 0. \tag{2.4}
\]
Since \( \text{Hom}_\Pi(\Pi e_\lambda \otimes_K e_\lambda \Pi \otimes \Pi M, N) \cong \text{Hom}_K(e_\lambda M, e_\lambda N) \), we see that the spaces in the complex are finite dimensional, and the alternating sum of their dimensions is \((\dim M, \dim N)\). Now the cohomology in the first two places is \( \text{Hom}_\Pi(M, N) \) and \( \text{Ext}_\Pi^1(M, N) \). For \( P \) a f.g. projective \( \Pi \)-bimodule, we have natural isomorphisms
\[
D \text{Hom}_\Pi(P \otimes \Pi M, N) \cong D \text{Hom}_\Pi(P, \text{Hom}_K(M, N)) \cong D (D M \otimes \Pi \text{Hom}_\Pi(P, \Pi \otimes_K \Pi \otimes \Pi M, N))
\]
\[
\cong \text{Hom}_\Pi(\text{Hom}_\Pi(P, \Pi \otimes_K \Pi) \otimes \Pi N, M).
\]
Using Proposition 2.5, we see that the dual of the complex (2.4) is isomorphic to the same complex, but with \( M \) and \( N \) exchanged. Thus the cohomology in the second two places of the complex (2.4) is isomorphic to \( D \text{Ext}_\Pi^1(N, M) \) and \( D \text{Hom}_\Pi(N, M) \). The result follows. \( \square \)

**Theorem 2.7.** If \( Q \) is a connected non-Dynkin quiver, then the map \( f \) in Proposition 2.4 is injective, so the exact sequence given there is a projective resolution of \( \Pi \) as a \( \Pi \)-bimodule, and the global dimension of \( \Pi \) is at most 2.

**Proof.** For \( \lambda = 0 \) this follows from the fact, mentioned in the proof of Lemma 2.3, that \( \Pi^0(Q) \) is Koszul with known Hilbert series. In general, to show the dependence on \( \lambda \), we write the map \( f \) in Proposition 2.4 as \( f^\lambda : P_0^\lambda \to P_1^\lambda \). Now the filtration on \( \Pi \) induces filtrations on the bimodules \( P_0^\lambda \) and \( P_1^\lambda \), and there are natural maps \( P_i^\lambda \to \text{gr} P_i^\lambda \), which are isomorphisms thanks to Lemmas 2.2 and 2.3. We have a commutative square
\[
P_0^\lambda \xrightarrow{f^0} P_1^\lambda
\]
and since \( f^0 \) is injective, Lemma 2.4 implies that \( f^\lambda \) is injective. Tensoring the bimodule projective resolution of \( \Pi \) with a \( \Pi \)-module \( M \), the resulting sequence remains exact (since the projective resolution of \( \Pi \) is split as a sequence of one-sided \( \Pi \)-modules), so shows that \( M \) has a projective resolution of length at most 2. \( \square \)

**Proof of Theorem 1.7** By Theorem 2.7, in the derived category of \( \Pi \)-bimodules, \( \Pi \) is isomorphic to the complex \( P_0 \to P_1 \to P_0 \), so the Calabi-Yau property follows from Proposition 2.5. \( \square \)

3. Reflection functors

In this section, we define reflection functors on the module categories of deformed preprojective algebras and observe relations on these functors. Since these functors changes weights, contrary to section 2, we denote by \( \Pi^\lambda = \Pi^\lambda(Q) \) the deformed preprojective algebra of a quiver \( Q \) with a weight \( \lambda \).

### 3.1. Reflection functors via cokernels and kernels.

In this subsection, we induce and study reflection functors via cokernels and kernels on modules over deformed preprojective algebras.

A representation \( V \) of \( \Pi^\lambda = \Pi^\lambda(Q) \) is a set \( V = (V_i, V_a) \) of \( K \)-vector spaces \( V_i \) \( (i \in I) \) and morphisms of \( K \)-vector spaces \( V_a : V_{h(a)} \to V_{t(a)} \) \((a \in \overline{Q}_1)\) satisfying the following relations for each \( i \in I \):
\[
\sum_{\beta(a)} \varepsilon(a) V_a V_{a^\ast} = \lambda_i \text{id}_{V_i}.
\]
Here the sum is over all arrows \( a \) in \( \overline{Q} \) with head at \( i \). A morphism between two representations \( U, V \) is a set \( f = (f_i)_{i \in I} \) of \( K \)-linear morphisms \( f_i : U_i \to V_i \) satisfying \( f_{h(a)} U_a = V_a f_{t(a)} \) for any \( a \in \overline{Q}_1 \). We denote by \( \text{Rep} \Pi^\lambda \) the category of representations of \( \Pi^\lambda \). Then it is well-known that there is an equivalence
For each loop-free vertex \( i \in I \), a representation \( V = (V_j, V_a) \) of \( \Pi^\lambda \), let
\[
V_\oplus = \bigoplus_{h(a)=i} V_{i(a)},
\]
and let \( \mu^V_a \) and \( \pi^V_a \) be a canonical injection and a canonical surjection between \( V_{i(a)} \) and \( V_\oplus \). Let
\[
\mu^V = \sum_{h(a)=i} \mu^V_a \cdot : V_i \rightarrow V_\oplus
\]
\[
\pi^V = \sum_{h(a)=i} \varepsilon(a) V_a \pi^V_a : V_\oplus \rightarrow V_i.
\]
So we have a sequence
\[
V_i \xrightarrow{\mu^V} V_\oplus \xrightarrow{\pi^V} V_i.
\]
(3.1)

It is easy to see that \( \pi^V \mu^V = \lambda_i \text{id}_{V_i} \). If \( V \) is clear from the context, then we write \( \pi^V = \pi \), \( \mu^V = \mu \), etc. For \( b, c \in Q_1 \) with \( h(b) = h(c) = i \), we have
\[
\pi_{c} \mu \pi_{b} = \varepsilon(b) V_{c} \pi_{b}.
\]
(3.2)

We define a representation \( C_i(V) = (C_i(V)_j, C_i(V)_a) \) of \( \Pi_{i}^{\lambda} \) as follows. For \( j \in I \), let
\[
C_i(V)_j = \begin{cases} V_j & (j \neq i), \\ \text{Coker}(\mu) & (j = i). \end{cases}
\]
We have an exact sequence \( V_i \xrightarrow{\mu} V_\oplus \xrightarrow{\varepsilon} C_i(V)_i \rightarrow 0 \). Since \( \pi \mu = \lambda_i \text{id}_{V_i} \), there is a morphism \( \gamma : C_i(V)_i \rightarrow V_\oplus \) which makes the following diagram commutative with an exact row:
\[
\begin{array}{ccc}
V_i & \xrightarrow{\mu} & V_\oplus \\
\mu \pi - \lambda_i \text{id}_{V_\oplus} & \downarrow \gamma & \xrightarrow{\varepsilon} C_i(V)_i \\
V_\oplus & \xrightarrow{\varepsilon} & 0
\end{array}
\]
(3.3)

Then for an arrow \( a \in Q_1 \), let
\[
C_i(V)_a = \begin{cases} V_a & (h(a) \neq i \neq t(a)), \\ \varepsilon(a) c \mu_a & (h(a) = i), \\ \pi_a \gamma & (t(a) = i). \end{cases}
\]
(3.4)

For a morphism \( f : U \rightarrow V \) of representations of \( \Pi^\lambda \), let \( C_i(f)_j = f_j \) if \( j \neq i \) and \( C_i(f)_i \) be a map induced from the following commutative diagram:
\[
\begin{array}{ccc}
U_i & \xrightarrow{\mu^U} & U_\oplus \\
\mu^U f_i & \downarrow \phi & \xrightarrow{\varepsilon^U} C_i(U)_i \\
V_i & \xrightarrow{\mu^V} & V_\oplus \\
\mu^V f_i & \downarrow \phi & \xrightarrow{\varepsilon^V} C_i(V)_i
\end{array}
\]
(3.5)

where \( \oplus_b f_{i(b)} := \oplus_{h(a)=i} f_{i(a)} \).

We see that \( C_i \) is actually a functor. It is known that \( C_i \) is a functor in the case where \( \lambda = 0 \) [2]. If \( \lambda \neq 0 \), then this was stated in [12] without a detailed proof. Therefore we write the proof here.

**Proposition 3.1.** For each loop-free vertex \( i \in I \), the above \( C_i \) gives a covariant functor
\[
C_i : \text{Mod} \Pi^\lambda \rightarrow \text{Mod} \Pi_{i}^{\lambda}.
\]
Proof. We first see that \( V' = C_1(V) \) is a representation of \( \Pi^\cdot \lambda(\mathcal{Q}) \). We show that \( \sum_{h(a)=j} c(a)V_aV_{a'} \) holds for each vertex \( j \). Assume that \( j = i \), then we have

\[
\sum_{h(a)=i} c(a)V_aV_{a'} \quad \sum_{h(a)=i} c(a)ep_{a} \pi_{a} \gamma c = \sum_{h(a)=i} c\mu_{a} \pi_{a}(\mu\pi - \lambda_{i} id_{V_{a'}}) = c(\mu\pi - \lambda_{i} id_{V_{a'}}) = -\lambda_{i} c.
\]

Since \( c \) is surjective, we have the desired equality.

Assume that \( j \neq i \) and there are \( m \) arrows between \( i \) and \( j \) in \( Q \). We have \( (r_{i} \lambda)_{j} = \lambda_{j} + m\lambda_{i} \) and

\[
\sum_{h(a)=j, t(a) \neq i} c(a)V_aV_{a'} + \sum_{h(a)=j, t(a) = i} c(a)V_aV_{a'} = \sum_{h(a)=j, t(a) \neq i} c(a)V_aV_{a'} + \sum_{h(a)=j, t(a) = i} c(a)\pi_{a} \gamma c = \sum_{h(a)=j, t(a) \neq i} c(a)V_aV_{a'} - \sum_{h(a)=j, t(a) = i} \pi_{a} \mu_{a} \mu_{a'} + \lambda_{i} \sum_{h(a)=j, t(a) = i} \pi_{a} \mu_{a} \mu_{a'}
\]

\[\sum_{h(a)=j, t(a) \neq i} c(a)V_aV_{a'} \quad \sum_{h(a)=j, t(a) \neq i} c(a)\pi_{a} \gamma c = (\lambda_{j} + m\lambda_{i}) id_{V_{j}}.\]

Therefore \( V' = C_1(V) \) is a representation of \( \Pi^\cdot \lambda(\mathcal{Q}) \).

Let \( f : U \to V \) be a morphism of representations of \( \Pi^{\cdot}\lambda \). We check that \( f' = C_1(f) \) is a morphism of representations of \( \Pi^{\cdot}\lambda \). Let \( b \in \overline{Q}_{1} \). If \( h(b) = i \), then

\[
f'_{b}U_{b} = f'_{b}(c(b)\mu_{b})U_{b} = f_{b}(c(b)\mu_{b})U_{b} = \varepsilon(b)c^{\cdot}(\varepsilon_{b} f_{b}(a) \mu_{b})U_{b} = \varepsilon(b)c^{\cdot}(\varepsilon_{b} \mu_{b})V_{b} f_{b}(b)
\]

holds, where \( \varepsilon_{b}(a) = \varepsilon(h(a)) = f_{b}(a) \).

Assume that \( t(b) = i \). For any arrow \( c \) of \( \overline{Q} \) with \( h(c) = i \), we have

\[
\pi^{V_{b}}_{b} \mu^{V}_{b}(\varepsilon_{b} f_{b}(a) \mu_{b})U_{c} = \pi^{V_{b}}_{b} \mu^{V}_{b} \mu^{V}_{c} U_{c} = \varepsilon(c) f_{b}(b) \mu_{b} \mu^{U}_{c} U_{c} = \varepsilon(c) f_{b}(b) \mu_{b} \mu^{U}_{c} U_{c} = \pi^{V}_{b} \mu^{U}_{b} \mu^{U}_{c} U_{c}.
\]

Since \( \varepsilon^{U} \) is surjective, we have the desired equality. If \( h(b) \neq i \neq t(b) \), then it is easy to see that \( V'_{h(b)} f'_{b} = f'_{b} V'_{b} \) holds. Thus \( f' \) is a morphism.

Finally, \( C_{1}(gf) = C_{1}(g) C_{1}(f) \) holds by the diagram (3.5), which completes the proof. \( \square \)

For a loop-free vertex \( i \in I \), since \( r_{i} r_{j} = id \), by the same construction, we also have a covariant functor from \( \text{Mod} \Pi^{\cdot} \lambda \) to \( \text{Mod} \Pi^{\cdot} \lambda \). Therefore, to distinguish directions of functors, we use the superscript \( \lambda \),

\[
C_{1}^{\lambda} = C_{1} : \text{Mod} \Pi^{\cdot} \lambda \to \text{Mod} \Pi^{\cdot} \lambda.
\]

Next we define reflection functors via kernels. For a representation \( V = (V_{i}, \lambda) \) of \( \Pi^{\cdot} \lambda \) and a loop-free vertex \( i \in I \), we define a representation \( K_{i}^{\lambda}(V) = (K_{i}^{\lambda}(V_{j}), K_{i}^{\lambda}(V_{a})) \) as follows: For \( j \in I \), let

\[
K_{i}^{\lambda}(V_{j}) = \begin{cases} \lambda_{j} & (j \neq i), \\ \text{Ker}(\pi) & (j = i). \end{cases}
\]

We have a short exact sequence \( 0 \to K_{i}^{\lambda}(V_{i}) \to V_{\oplus} \to V_{i} \). Since \( \pi \mu = \lambda_{i} id_{V_{i}} \), there is a morphism \( \gamma : V_{\oplus} \to K_{i}^{\lambda}(V_{i}) \), which makes the following diagram commutative:

\[
\begin{array}{c}
\begin{array}{c}
0 \to K_{i}^{\lambda}(V_{i}) \to V_{\oplus} \to V_{i} \\
\gamma \quad \beta \end{array}
\end{array}
\]

\[\begin{array}{c}
\begin{array}{c}
\mu \gamma \to V_{i} \\
\mu \lambda - \lambda_{i} \text{id}_{V_{i}} \end{array}
\end{array}\]

(3.6)
Recall that $V_{i(a)} \xrightarrow{\mu_a} V_{i(a)} \xrightarrow{\pi_a} V_{i(a)}$ are the canonical inclusion and surjection. Then for an arrow $a \in Q_1$, let

$$K_i^\lambda(V)_a = \begin{cases} V_a & (h(a) \neq i \neq t(a)), \\ \epsilon(a) \gamma_i \mu_a & (h(a) = i), \\ \pi_a & (t(a) = i). \end{cases}$$

**Proposition 3.2.** For each loop-free vertex $i \in I$, we have a covariant functor $K_i^\lambda : \text{Mod} \Pi^\lambda \to \text{Mod} \Pi_i^{\lambda}$. Proof. The proof is similar for Proposition 3.1. □

**Proof of Theorem 3.2** (i) Let $V = (V_i, V_a)$ be a representation of $\Pi^\lambda$ and $W = (W_i, W_a)$ be a representation of $\Pi_i^{\lambda}$. Let $f : C_i^\lambda(V) \to W$ be a morphism of $\Pi_i^{\lambda}$-modules. Then we have the following commutative diagram with exact rows and an induced morphism $g_i$:

$$
\begin{array}{cccccc}
V_i & \xrightarrow{\mu} & V_{i(a)} & \xrightarrow{c} & C_i^\lambda(V)_i & \rightarrow 0 \\
\downarrow{g_i} & & \downarrow{\otimes f(a)} & & \downarrow{f_i} & \\
0 & \rightarrow & K_i^{\lambda}(W)_i & \xrightarrow{c} & W_{i(a)} & \xrightarrow{\pi} W_i
\end{array}
$$

Let $g : V \to K_i^{\lambda}(W)$ be a map given by $g_j = f_j$ if $j \neq i$ and $g_i$ as in the diagram. Then this $g$ is a morphism of $\Pi^\lambda$-modules. We have a map $\text{Hom}_{\Pi_i^{\lambda}}(C_i^\lambda(V), W) \to \text{Hom}_{\Pi^\lambda}(V, K_i^{\lambda}(W))$ by $f \mapsto g$. The dual argument induces an inverse direction map and they are inverse each other. Therefore $C_i^\lambda$ is left adjoint to $K_i^{\lambda}$.

(ii) Assume that $\lambda_i \neq 0$. Then the sequence splits, so the functors $C_i^\lambda$ and $K_i^{\lambda}$ are naturally isomorphic. By the definition of $E_i$, $E_i$ and $K_i^{\lambda}$ are isomorphic as functors, see [12]. Therefore $C_i^\lambda \cong K_i^{\lambda} \cong E_i$ is an equivalence of categories.

(iii) Use the same proof as in [3, section 5]; see also [2, section 2]. □

3.2. Braid relations on reflection functors. In this subsection we prove Theorem 1.3. We use the following technical lemma.

**Lemma 3.3.** Assume that there is the following exact sequence

$$V \oplus W \xrightarrow{A} V \oplus X \oplus Y \xrightarrow{(\rho, \sigma, \tau)} Z \rightarrow 0,$$

Then it induces the following exact sequence

$$W \xrightarrow{B} X \oplus Y \xrightarrow{(\sigma, \tau)} Z \rightarrow 0,$$

where $A = \begin{pmatrix} 0 & f \\ -g & 0 \\ 0 & h \end{pmatrix}$.

**Proof.** This follows from a direct calculation. □

The following theorem is a more precise statement of Theorem 1.3 for the functors $C_i$.

**Theorem 3.4.** Let $i, j \in I$ be loop-free vertices. We have the following isomorphisms of functors.

(a) $C_i^{\tau_i \lambda} C_j^{\lambda} \cong C_{j}^{\tau_i \lambda} C_i^{\lambda}$ if there is no arrow between $i$ and $j$ in $Q$.

(b) $C_i^{\tau_i \lambda} C_j^{\tau_j \lambda} C_i^{\lambda} \cong C_j^{\tau_i \lambda} C_i^{\tau_j \lambda} C_i^{\lambda}$ if there is exactly one arrow between $i$ and $j$ in $Q$.

Note that since $r_i$ is the dual of $s_i$, for loop-free vertices $i, j$ of $Q$, if there is no arrow between $i$ and $j$ in $Q$, then $r_i r_j = r_j r_i$, and if there is exactly one arrow between $i$ and $j$ in $Q$, then $r_i r_j r_i = r_j r_i r_j$.

**Proof.** For simplicity, we write $C_i^{\tau_{i_2} \tau_{i_3} \lambda} C_i^{\tau_{i_2} \tau_{i_3} \lambda}$ for loop-free vertices $i_1, i_2, i_3$ of $Q$. The isomorphism (a) is clear by the definition of $C_i$. We show the statement (b). Assume that there is exactly one arrow $\alpha$ from $j$ to $i$ in $Q$. Let $V = (V_i, V_a)$ be a representation of $\Pi^\lambda$. We use the following notations of vector spaces:

$$V^i := \bigoplus_{h(a) = i} V_{t(a)}, \quad V^j := \bigoplus_{h(a) = j} V_{t(a)}.$$
Then the sequence \( \Phi \) for \( i \) and \( j \) can be described as follows:

\[
V_i \xrightarrow{(X^i - V_0)} V^i \oplus V_j \xrightarrow{(Y^j - V_0)} V_i, \quad V_j \xrightarrow{(X^j - V_0)} V^j \oplus V_i \xrightarrow{(Y^j - V_0)} V_j.
\]

By the definitions of \( C_j(V) \) and \( C_i(V) \), we have the following commutative diagrams with exact rows:

\[
\begin{array}{ccc}
V_j & \xrightarrow[t(X^j - V_0)]{t(X^j - V_0)} & V^j \oplus V_i \xrightarrow{(e_1c_1')} \ C_j(V)_j \xrightarrow{0} & V_i & \xrightarrow[t(X^i - V_0)]{t(X^i - V_0)} & V^i \oplus V_j \xrightarrow{(e_2c_2')} \ C_i(V)_i \xrightarrow{0} \\
\phi & \downarrow & \phi' & \downarrow & \phi'' & \downarrow & \phi'''
\end{array}
\]

where \( \Phi = t(X^j V_0)(Y^j - V_0^*) - \lambda_j \) id and \( \Psi = t(X^i V_0^*)(Y^i - V_0) - \lambda_i \) id.

By the definition of \( C_{ij}(V) \) and \( C_{ji}(V) \), we have the following commutative diagrams with exact rows:

\[
\begin{array}{ccc}
V_i & \xrightarrow[t(X^i - c_1')]{t(X^i - c_1')} & V^i \oplus C_j(V)_j \xrightarrow{(d_1d'_1)} \ C_{ij}(V)_i \xrightarrow{0} & V_j & \xrightarrow[t(X^j - c_2')]{t(X^j - c_2')} & V^j \oplus C_i(V)_i \xrightarrow{(d_2d'_2)} \ C_{ji}(V)_j \xrightarrow{0} \\
\phi & \downarrow & \phi' & \downarrow & \phi'' & \downarrow & \phi'''
\end{array}
\]

where \( \Phi' = t(X^i - c_1')(Y^j - \gamma_1') - (r_j \lambda)_i \) id and \( \Psi' = t(X^j c_2')(Y^j - \gamma_2') - (r_i \lambda)_j \) id.

By the definitions of \( C_{iji}(V) \) and \( C_{ijji}(V) \), we have the following commutative diagrams with exact rows:

\[
\begin{array}{ccc}
C_j(V)_j & \xrightarrow[t(X^i d_1')]{t(X^i d_1')} & V^j \oplus C_i(V)_i \xrightarrow{(e_1e_1')} \ C_{ijj}(V)_j \xrightarrow{0} & C_i(V)_i & \xrightarrow[t(X^j d_2')]{t(X^j d_2')} & V^j \oplus C_i(V)_i \xrightarrow{(e_2e_2')} \ C_{ijji}(V)_j \xrightarrow{0} \\
\phi'' & \downarrow & \phi''' & \downarrow & \phi'' & \downarrow & \phi'''
\end{array}
\]

where \( \Phi'' = t(\gamma_1 d_1')(c_1 - \delta_1') - (r_i r_j \lambda)_j \) id and \( \Psi'' = t(\gamma_2 d_2')(c_2 - \delta_2') - (r_j r_i \lambda)_i \) id.

We show that \( C_{iji}(V) \cong C_{ijji}(V) \) as left \( \Pi^N \)-modules, where \( \lambda' = r_j r_i \lambda = r_j r_i \lambda \). By combining the exact rows of the left hand diagrams in (3.7) and (3.8), we have the following exact sequence:

\[
V_i \oplus V_j \xrightarrow{A} V_i \oplus V^i \oplus V^j \xrightarrow{(d_1d'_1d'_2c_1)} C_{ijj}(V)_j \xrightarrow{0}, \quad A = \begin{pmatrix} -\id & V_0 \\ X^i & 0 \\ 0 & X^j \end{pmatrix}.
\]

By Lemma 3.3 we have

\[
V_j \xrightarrow{t(X^j V_0, X^i)} V^j \oplus V_i \xrightarrow{(d_1d'_1)} C_{ijj}(V)_j \xrightarrow{0}.
\]

By combining the exact rows of the right hand diagrams in (3.8) and (3.9), and by applying Lemma 3.3, we have the following exact sequence:

\[
V_j \xrightarrow{t(X^j V_0, X^i)} V^j \oplus V_i \xrightarrow{(e_2e_2'd_2)} C_{ijji}(V)_i \xrightarrow{0}.
\]

By the right diagram of (3.7), we have \( X^i V_0 = \gamma_2 c_2' \). Thus we have \( C_{ij}(V)_i = C_{ijji}(V)_i \) and \( (e_2 e_2'd_2) = (d_1 d'_1'c_1') \).

Similarly, \( C_{ijji}(V)_j = C_{ijji}(V)_j \) holds, where this vector space is given by a cokernel of \( t(-X^j V_0^*; X^i) : V_i \rightarrow V^j \oplus V^i \). We have \( (e_1c_1'd_1) = (d_2 d'_2c_2) \).

By the above argument, we have \( C_{ijji}(V) = C_{ijji}(V) \) as vector spaces. We next show that this equation is compatible with an action of \( \Pi^N \). Let \( b \in \mathcal{Q}_1 \). We show that \( C_{ijji}(V)_b = C_{ijji}(V)_b \) holds. If \( h(b), t(b) \notin \{ i, j \} \), then \( C_{ijji}(V)_b = V_b = C_{ijji}(V)_b \).

If \( h(b) = i \) and \( t(b) \neq j \), recall that \( d_1 = e_2 \). Thus we have \( C_{ijji}(V)_b = \varepsilon(b)e_2 \mu_b = \varepsilon(b)d_1 \mu_b = C_{ijji}(V)_b = C_{ijji}(V)_b \). In a similar way, the equation holds if \( h(b) = j \) and \( t(b) \neq i \).
Assume that $t(b) = i$ and $h(b) \neq j$. Then $C_{ij}(V)_b = \pi_b \cdot \varepsilon_2$ and $C_{ji}(V)_b = \pi_b \cdot \delta_1$ hold. We have
\[\varepsilon_2 \varepsilon_2 = \gamma_2 \varepsilon_2 - (r_j r_i \lambda)_i \text{id} = X_i Y_i - \lambda_i \text{id} \quad \text{and} \quad \varepsilon_2 \varepsilon_2' d_2 = \gamma_2 \varepsilon_2' d_2 = X_i Y_i - (r_j r_i \lambda)_i \text{id} = \delta_1 d_1,\]
Therefore $\varepsilon_2 \varepsilon_2' d_2 = \delta_1 (d_1' d_1)$. Since $(\varepsilon_2 \varepsilon_2' d_2) = (d_1' d_1)$ is surjective, we have $\varepsilon_2 = \delta_1$. Thus $C_{ij}(V)_b = \pi_b \cdot \varepsilon_2 = \pi_b \delta_1 = C_{ji}(V)_b$. In a similar way, the equation holds if $t(b) = j$ and $h(b) \neq i$.

Assume that $b = \alpha^*$. Then $C_{ij}(V)_b = \varepsilon_2'$ and $C_{ji}(V)_b = -\varepsilon_1'$. Since $(e_1' d_1) = (d_2' d_2 c_2)$, we have
\[\varepsilon_2' \varepsilon_2' d_2 = -d_2' c_2 = -e_1' d_1\]
\[\varepsilon_2' \varepsilon_2' d_2 = (d_2' \delta_2) - (r_j r_i \lambda)_i \text{id}) d_2 = -d_2' \varepsilon_2' d_2 = (X_i Y_i - \lambda_i \text{id} - e_1 (X_i Y_i - \lambda_i \text{id}) = e_1' (X_i Y_i - \lambda_i \text{id}) = -(e_1' d_1' c_1).\]

Namely, $\varepsilon_2' (e_1' d_1) = -e_1' (d_2' d_2 c_2)$. Therefore $\varepsilon_2' = -\varepsilon_1'$. In a similar way, the equation holds if $b = \alpha$. This completes the proof that $C_{ij}(V) = C_{ji}(V)$ as $\Pi^\alpha$-modules.

Finally we show that this identification is a morphism of functors, that is, if $f : U \to V$ is a morphism of $\Pi^\alpha$-modules, then $C_{ij}(f) = C_{ji}(f)$. Let $k$ be a vertex of $Q$. If $k \neq i, j$, then $C_{ij}(f)_k = f_k = C_{ji}(f)_k$. If $k = i$, then $C_{ij}(f)_i = C_{ji}(f)_i$. Since $C_{ij}(f)_i$ is induced from (3.11) and $C_{ji}(f)_i$ is induced from (3.12), we have $C_{ij}(f)_i = C_{ji}(f)_i$. In a similar way, we have $C_{ij}(f)_j = C_{ji}(f)_j$. This completes the proof of the theorem. \(\square\)

**Proof of Theorem 4.3** The relations on the $C_i$ are shown in the previous theorem. The relations on the $K_i$ follow, since $K_i$ is right adjoint to $C_i$. \(\square\)

### 4. Tilting ideals for 2-Calabi-Yau algebras

#### 4.1. Preliminaries

We denote by $D_{td}(\mathrm{Mod} A)$ the triangulated subcategory of the derived category $D(\mathrm{Mod} A)$ which consists of the complexes whose total homology is a finite-dimensional $A$-module.

**Proposition 4.1.** If $A$ is a d-Calabi-Yau algebra, then

(a) There is a functorial isomorphism $D \mathrm{Hom}_{D(\mathrm{Mod} A)}(M, N) \cong \mathrm{Hom}_{D(\mathrm{Mod} A)}(N, M[d])$ for $M$ in $D_{td}(\mathrm{Mod} A)$ and $N$ in $D(\mathrm{Mod} A)$.

(b) In particular $D \mathrm{Ext}^1_A(M, N) \cong \mathrm{Ext}^{d-1}_A(N, M)$ for $M$ a finite-dimensional $A$-module and $N$ any $A$-module.

(c) $\mathrm{Ext}^1_A(M, A) = 0$ if $i \neq d$ and $D \mathrm{Ext}^1_A(M, A) \cong M$ for a finite-dimensional $A$-module $M$.

**Proof.** For part (a) use [21] Lemma 4.1]. Parts (b) and (c) are special cases. \(\square\)

Proposition 4.1 is a special case of the following result.

**Proposition 4.2.** Let $A$ be a $d$-Calabi-Yau algebra with $d \geq 2$, let $S$ be a finite-dimensional simple left $A$-module and let $I = \text{Ann}_A(S)$.

(i) As a left $A$-module, $A/I$ is isomorphic to a finite direct sum of copies of $S$, and as a right $A$-module it is isomorphic to a finite direct sum of copies of $D(S)$.

(ii) If $\text{Ext}^1_A(S, S) = 0$, then $I$ is a faithfully balanced $A$-bimodule, that is, the natural maps $A \to \text{End}(I_A)$ and $A^{op} \to \text{End}(A_I)$ given by the homotheties of left and right multiplication, are isomorphisms.

(iii) If $\text{Ext}^1_A(S, S) = 0$ for $i = 1, 2, \ldots, d - 1$, then as a left $A$-module or as a right $A$-module, $I$ is a $(d - 1)$-tilting module.

**Proof.** (i) Use that $A/I$ embeds in $\text{End}_K(S) \cong S \otimes_K D(S)$.

(ii) By the $d$-Calabi-Yau property, we have $D \text{Ext}^1_A(A/I, A) \cong \text{Ext}^{d-1}_A(A, A/I) = 0$ for $i < d$. It follows that the restriction map $\text{Hom}_A(A, A) \to \text{Hom}_A(I, A)$ is an isomorphism. Since $\text{Ext}^1(S, S) = 0$, the restriction map $\text{Hom}_A(A, A/I) \to \text{Hom}_A(I, A/I)$ is surjective, but since the natural map $\text{Hom}_A(A/I, A/I) \to \text{Hom}_A(A, A/I)$
is onto, we have $\text{Hom}_A(I, A/I) = 0$. This implies that the natural map $\text{Hom}_A(I, I) \to \text{Hom}_A(I, A)$ is an isomorphism. This gives an isomorphism $A^{\text{op}} \to \text{End}(A/I)$. The other isomorphism follows by symmetry.

(iii) Since $A$ is $d$-Calabi-Yau, the global dimension of $A$ is $d$. Therefore $I$ has a projective dimension at most $d - 1$ on both sides. It is equal to $d - 1$ since

$$\text{Ext}_A^{d-1}(I, S) \cong \text{Ext}_A^d(A/I, S) \cong D\text{Hom}_A(S, A/I) \neq 0.$$  

We show that $A/I$ admits a projective resolution with finitely generated projective $A$-modules. Since $A$ is homologically smooth, it has a finite projective resolution by finitely generated $A$-bimodules. By applying $(-) \otimes_A A/I$ to a bimodule resolution of $A$, $A/I$ has a finite projective resolution by finitely generated projective $A$-modules. This implies that $A$I has a projective resolution with finitely generated projective $A$-modules by Schanuel’s lemma, using, for example [22 Lemma 2.5]. By symmetry also $I_A$ has a projective resolution with finitely generated projective right $A$-modules.

Next we show that $\text{Ext}_A^i(A, I) = 0$ for any $i > 0$. For $i = 1, \ldots, d - 1$, we have

$$D\text{Ext}_A^i(A, I) \cong D\text{Ext}_A^{i+1}(A/I, I) \cong \text{Ext}_A^{d-i-1}(I, A/I) \cong \text{Ext}_A^{d-i}(A/I, A/I) = 0$$

where the second isomorphism comes from the $d$-Calabi-Yau property and the last equality comes from the condition that $\text{Ext}_A^i(S, S) = 0$ for $i = 1, 2, \ldots, d - 1$. Since the projective dimension of $I$ is at most $d - 1$, $\text{Ext}_A^i(A, I) = 0$ for any $i > 0$. By symmetry also $\text{Ext}_A^i(I, A) = 0$ for $i > 0$.

Finally, we show that $A$ admits a finite coresolution by modules in $\text{add}(A/I)$. Let $0 \to P_{d-1} \to \cdots \to P_0 \to I \to 0$ be a resolution of $I$ by finitely generated projective right $A$-modules. Applying the functor $\text{Hom}(-, I_A)$ to this resolution, using that $\text{End}(I_A) \cong A$, that $\text{Ext}_A^i(I_A, I_A) = 0$ for $i > 0$, and $\text{Hom}(P_i, I_A) \in \text{add}(A/I)$, we obtain the desired exact sequence. By symmetry also $A$ admits a finite coresolution by modules in $\text{add}(I_A)$.

The following lemma is standard. See for example [8 Lemma III.1.1] in the Krull-Schmidt case and [27 Lemma 2.8] in the tilting module case.

**Lemma 4.3.** If $T$ is a partial tilting $A$-module and $S$ is a finite-dimensional simple right $A$-module, then at least one of $S \otimes_A T = 0$ or $\text{Tor}_1^A(S, T) = 0$ holds.

**Proof.** Since $D(S)$ is a simple left module, either $\text{Hom}_A(T, D(S)) = 0$ or there is a surjection $T \to D(S)$. In the second case, applying $\text{Hom}_A(T, -)$ and using that $T$ is partial tilting, one obtains that $\text{Ext}_A^1(T, D(S)) = 0$. Rewriting in terms of tensor products and Tor gives the result. \hfill \Box

### 4.2. Tilting ideals

For an $A$-module $S$, let $I_S = \text{Ann}_A(S)$. In the proof of the following proposition, we refer the proof of [8 Proposition III.1.5].

**Proposition 4.4.** Let $A$ be 2-Calabi-Yau, let $T$ be a tilting module for $A$ and let $S$ be a finite-dimensional rigid simple $A$-module.

(a) We have $I_S \otimes_A T \cong I_S \otimes_A T$ in the derived category of $A$-modules.

(b) If $\text{Tor}_1^A(D(S), T) = 0$, then the natural map $I_S \otimes_A T \to I_ST$ is an isomorphism.

(c) $I_ST$ is a tilting module for $A$ and $\text{End}_A(I_ST) \cong \text{End}_A(T)$.

**Proof.** (a) We have a short exact sequence

$$0 \to I_S \to A \to A/I_S \to 0.$$  

Applying $- \otimes_A T$ to this short exact sequence, we have $\text{Tor}_1^A(I_S, T) \cong \text{Tor}_2^A(A/I_S, T) = 0$, giving the result.

(b) We have an exact sequence

$$\text{Tor}_1^A(A/I_S, T) \to I_S \otimes_A T \to T \to (A/I_S) \otimes_A T \to 0,$$

and $A/I_S$ is isomorphic to a finite direct sum of copies of $D(S)$ as a right $A$-module, so if $\text{Tor}_1^A(D(S), T) = 0$, then the map $I_S \otimes_A T \to I_ST$ is an isomorphism.

(c) By Lemma 4.3 we have $D(S) \otimes_A T = 0$ or $\text{Tor}_1^A(D(S), T) = 0$. In the first case, $I_ST = T$ and the assertion is trivial. Thus we may assume that $\text{Tor}_1^A(D(S), T) = 0$. Now the projective dimension of $T$ is at most one and the projective dimension of $T/I_ST$ is at most two, since the global dimension of $A$ is two.
Thus the projective dimension of \( I_S T \) is at most one. By (a) and (b) we have \( I_S T \cong I_S \otimes_A^L T \). Since \( I_S \) is a tilting ideal and \( T \) is a tilting module, this is tilting complex; see [3] Lemma III.1.2(a)]. Now the condition on projective dimension implies that \( I_S T \) is a tilting module.

**Proof of Theorem 4.5.** We show by induction on \( r \) that \( I_{S_1, S_2, ... , S_r} = I_{S_1} I_{S_2} \cdots I_{S_r} \) is a tilting ideal of finite codimension in \( A \). This is clear for \( r = 1 \). If we know it for \( r - 1 \), then \( I_{S_1, S_2, ... , S_r} \) is a tilting ideal of finite codimension.

Now \( I_S / I_{S_1} I_{S_2} \cdots I_{S_r} \cong (A/I_{S_1}) \otimes_A I_{S_2} \cdots I_{S_r} \)

which is finite dimensional since \( A/I_{S_1} \) is finite dimensional and \( I_{S_2} \cdots I_{S_r} \) is a finitely generated left \( A \)-module.

Also \( I_{S_1, S_2} = I_{S_1} I_{S_2} \) is a tilting module by Proposition 4.4. Similarly, \( I_{S_1, \ldots, S_r} = I_{S_1, \ldots, S_{r-1}} I_{S_r} \) is tilting as a right \( A \)-module.

Now let \( I \) be a partial tilting left ideal in \( A \) with \( A/I \in \mathcal{E}(S) \). We show by induction on \( \dim_K (A/I) \) that \( I \in \mathcal{I}(S) \). If \( I \not\cong A \), choose a simple submodule \( S \) of \( A/I \).

We know \( D \mathrm{Hom}_A(S, A) \cong \mathrm{Ext}_A^2(A, S) = 0 \), so \( \mathrm{Hom}_A(S, A) = 0 \). It follows that \( \mathrm{Ext}^1(S, I) \not\cong 0 \). Thus \( \mathrm{Tor}^1(D(S), I) \neq 0 \). Thus by Lemma 1.3 \( D(S) \otimes_A I = 0 \). Thus \( (A/I_S) \otimes_A I = 0 \), so \( I_S I = I \).

Let \( U = \{a \in A \mid S \in \mathcal{I}(S) \} \). It is a left ideal with \( I_S U \subseteq I \subseteq U \). Since \( I_S I = I \) it follows that \( I_S U = I \).

Since the natural map \( A \to \mathrm{Hom}_A(I_S, S) \) is an isomorphism, we have \( U \cong \mathrm{Hom}_A(I_S, I) \). Since \( \mathrm{Ext}_A^2(I_S, I) \cong \mathrm{Ext}_A^2(A/I_S, I) \cong D \mathrm{Hom}_A(I_S I, A/I_S) = 0 \), we have \( U \cong \mathrm{Hom}_A(I_S, I) \cong \mathcal{R} \mathrm{Hom}_A(I_S, I) \). This implies that \( U \) is a partial tilting module. Since there is a surjection from \( A/I_S U \) to \( A/U \), \( A/U \) belongs to \( \mathcal{E}(S) \). By induction, \( U \) is in \( \mathcal{I}(S) \) and so is \( I = I_S U \).

Finally, if \( I, I' \in \mathcal{I}(S) \) are isomorphic as \( A \)-modules, then \( I = I' \) by the argument of [8] Theorem III.1.6(d)]. Namely, by the 2-Calabi-Yau property \( \mathrm{Ext}_A^1(A/I, A) = 0 \), so an isomorphism \( f : I \to I' \) lifts to a map \( f' : A \to A \). Thus there is an element \( a \in A \) such that \( f' \) is right multiplication by \( a \). Then \( I' = \mathrm{Im} f = Ia \subseteq I \), and by symmetry \( I \subseteq I' \).

**4.3. Relations.**

**Lemma 4.5.** Let \( A \) be 2-Calabi-Yau and let \( S, T \) be finite-dimensional rigid simple \( A \)-modules such that \( \mathrm{Ext}_A^1(S, T) \) is 1-dimensional as a right \( \mathrm{End}_A(S) \)-module and as a left \( \mathrm{End}_A(T) \)-module. Let \( E_S^T \) (resp. \( E_T^S \)) be the unique \( A \)-module of length two such that the top is \( T \) (resp. \( S \)) and the socle is \( S \) (resp. \( T \)). We denote by \( \mathcal{E}(\{S, T\}) \) the Serre subcategory of the category of finite-dimensional \( A \)-modules generated by \( S \) and \( T \). Then the following statements hold.

(a) \( \mathcal{E}(\{S, T\}) \) is a uniserial category, that is, any indecomposable module in \( \mathcal{E} \) has a unique composition series.

(b) If \( L \in \mathcal{E}(\{S, T\}) \) is indecomposable, then \( L \) is isomorphic to one of \( S, T, E_S^T \) and \( E_T^S \).

**Proof.** (a) By assumption \( \mathrm{Ext}_A^1(S, T) \) is 1-dimensional as a right \( \mathrm{End}_A(S) \)-module and as a left \( \mathrm{End}_A(T) \)-module. By the 2-Calabi-Yau property, also \( \mathrm{Ext}_A^1(T, S) \) is 1-dimensional as a right \( \mathrm{End}_A(T) \)-module and as a left \( \mathrm{End}_A(S) \)-module. The claim then follows from [15] subsection 8.3].

(b) Let \( M \) be an indecomposable module in \( \mathcal{E} \) of length \( n \). Without loss of generality its top is \( S \). If \( n \geq 3 \), since \( \mathcal{E}(\{S, T\}) \) is uniserial, there is an indecomposable factor module \( N \) of \( M \) of length three. Then there is a non-split short exact sequence \( 0 \to S \to N \to E_T^S \to 0 \). But applying \( \mathrm{Hom}_A(S, \_ \_) \) to this sequence and using that \( \mathrm{Hom}_A(S, E_T^S) = 0 \), \( \mathrm{Ext}_A^1(S, S) = 0 \) and that \( \mathrm{Ext}_A^1(S, T) \) is 1-dimensional as a right \( \mathrm{End}_A(S) \)-module, we see that \( \mathrm{Ext}_A^1(S, E_T^S) = 0 \). Thus by the 2-Calabi-Yau property \( \mathrm{Ext}_A^1(E_T^S, S) = 0 \), a contradiction. Thus any indecomposable module in \( \mathcal{E} \) has length at most two, and the assertion holds.

For a subcategories \( \mathcal{B} \) and \( \mathcal{C} \) of an abelian category \( \mathcal{A} \), we denote by \( \mathcal{B} \ast \mathcal{C} \) the subcategory of \( \mathcal{A} \) consisting of objects \( A \in \mathcal{A} \) admitting a short exact sequence \( 0 \to B \to A \to C \to 0 \) in \( \mathcal{A} \) with \( B \in \mathcal{B} \) and \( C \in \mathcal{C} \).

**Proposition 4.6.** Let \( A \) be 2-Calabi-Yau. For finite-dimensional rigid simple \( A \)-modules \( S, T \), the following hold.

(a) \( \mathrm{add}(S) \ast \mathrm{add}(S) = \mathrm{add}(S) \).

(b) \( \mathrm{add}(S) \ast \mathrm{add}(T) = \mathrm{add}(T) \ast \mathrm{add}(S) \) if \( \mathrm{Ext}_A^1(S, T) = 0 \).
(c) \( \text{add}(S) \ast \text{add}(T) \ast \text{add}(S) = \text{add}(T) \ast \text{add}(S) \ast \text{add}(T) \) if \( \text{Ext}^1_A(S,T) \) is 1-dimensional as a right \( \text{End}_A(S) \)-module and as a left \( \text{End}_A(T) \)-module.

Proof. Part (a) holds since \( \text{Ext}^1_A(S,S) = 0 \), and (b) since \( \text{Ext}^1_A(S,T) = \text{Ext}^1_A(T,S) = 0 \). Part (c) follows from Lemma 4.5. □

Proof of Proposition 4.6. For a finite-dimensional \( A \)-module \( M \) and a sequence \( S_1, S_2, \ldots, S_r \) of rigid finite dimensional simple modules, we have that \( I_{S_1 \cdots S_r}M = 0 \) if and only if \( M \in \text{add} S_1 \ast \cdots \ast \text{add} S_r \). Now parts (a)-(c) follow from the corresponding parts of Proposition 4.6. For example in case (c), the module \( M = A/I_{STS} \) is finite-dimensional by Theorem 4.5, and annihilated by \( I_{STS} \), so is in \( \text{add}(S) \ast \text{add}(T) \ast \text{add}(S) \). Thus \( M \) is in \( \text{add}(T) \ast \text{add}(S) \ast \text{add}(T) \) by Proposition 4.6(c), so \( I_{TST}M = 0 \), and hence \( I_{TST} \subseteq I_{STS} \). Similarly \( I_{STS} \subseteq I_{TST} \), giving equality. □

Now let \( S \) be a set of pairwise non-isomorphic finite-dimensional rigid simple \( A \)-modules. For simplicity we assume that \( S \) is split, meaning that \( \text{End}_A(S) = K \) for all \( S \in S \). We denote by \( \mathcal{E}(S) \) the Serre subcategory of the category of finite-dimensional \( A \)-modules generated by \( S \). Moreover we denote by \( D_{\mathcal{E}(S)}(\text{Mod} A) \) the triangulated subcategory of \( D(\text{Mod} A) \) consisting of the objects \( X \) such that the total cohomology of \( X \) belongs to \( \mathcal{E}(S) \), that is, \( \bigoplus_{n \in \mathbb{Z}} H^n(X) \in \mathcal{E}(S) \).

Let \( Z = \bigoplus_{S \in S} \mathbb{Z}e_S \) be the free \( \mathbb{Z} \)-module with basis elements \( e_S \), where \( S \) runs through the elements of \( S \), up to isomorphism, and let \( (-,-) \) be the bilinear form on \( Z \) given by

\[
(e_S,e_T) = \sum_{i \geq 0} (-1)^i \dim_K \text{Ext}^i_A(S,T).
\]

It is defined and symmetric for \( A \) 2-Calabi-Yau. For \( M \in \mathcal{E}(S) \), let

\[
[M] = \sum_{S \in S} (M : S)e_S \in Z,
\]

where \((M : S)\) is the multiplicity of \( S \) as a composition factor of \( M \). For \( X \) in \( D_{\mathcal{E}(S)}(\text{Mod} A) \), we define

\[
F(X) = \sum_{n \in \mathbb{Z}} (-1)^n[H^nX] \in Z.
\]

Let \( W(S) \) be the Coxeter group, as in the introduction. We define an action of \( W(S) \) on \( Z \) by \( \sigma_S(x) := x - (e_S,x)e_S \).

**Proposition 4.7.** Let \( S,T \in S \) and let \( m = \dim_K \text{Ext}^1_A(S,T) \).

(a) \( \text{Tor}^1_A(I_S,S) \cong S \) as left \( A \)-modules.

(b) If \( S \not\cong T \), then the \( A \)-module \( I_S \otimes_A T \) has exactly two composition factors \( S \) and \( T \), with multiplicities \( m \) and 1, respectively.

(c) We have

\[
I_S \otimes_A^L T \cong \begin{cases} I_S \otimes_A T & (S \not\cong T), \\ \text{Tor}^1_A(I_S,T)[1] & (S \cong T). \end{cases}
\]

(d) The functor \( I_S \otimes_A^L \) sends \( D_{\mathcal{E}(S)}(\text{Mod} A) \) to itself, and \( F(I_S \otimes_A^L X) = \sigma_S(F(X)) \) for \( X \) in \( D_{\mathcal{E}(S)}(\text{Mod} A) \).

Proof. By assumption \( \text{End}_A(S) = K \), so by the Jacobson Density Theorem, \( A/I_S \cong \text{End}_K(S) \cong S \otimes_K D(S) \) as \( A \)-bimodules. Thus also \( D(A/I_S) \cong S \otimes_K D(S) \). For \( i \geq 0 \), we have

\[
\text{Tor}^1_A(A/I_S,T) \cong D \text{Ext}^1_A(T,D(A/I_S)) \cong D \text{Ext}^1_A(T,S \otimes_K D(S))
\]

as left \( A \)-modules. Thus, tensoring the exact sequence \((4.1)\) with \( T \), gives an exact sequence

\[
0 \to S \otimes_K D \text{Ext}^1_A(T,S) \to I_S \otimes_A T \to T \to S \otimes_K D \text{Hom}_A(T,S) \to 0,
\]

as well as \( \text{Tor}^1_A(I_S,T) \cong \text{Tor}^1_A(A/I_S,T) \cong S \otimes_K D \text{Ext}^2_A(T,S) \).

(a) Taking \( S = T \), by the 2-Calabi-Yau property, \( \text{Ext}^1_A(S,S) \) is one dimensional over \( K \), so \( \text{Tor}^1_A(I_S,S) \) is isomorphic to \( S \).
(b) Follows from the exact sequence \( \text{[1.2]} \).
(c) Follows from (a), (b) and Lemma \( \text{[4.3]} \).
(d) Follows from (c).

The following proposition is the Hom version of Proposition \( \text{[4.7]} \). We omit the proof since it is essentially the same as the tensor version.

**Proposition 4.8.** Let \( S, T \in S \) and let \( m = \dim_K \text{Ext}^1_A(S, T) \).

(a) \( \text{Ext}^1_A(I_S, S) \cong S \) as left \( A \)-modules.
(b) If \( S \not\cong T \), then the \( A \)-module \( \text{Hom}_A(I_S, T) \) has exactly two composition factors \( S \) and \( T \), with multiplicities \( m \) and 1, respectively.
(c) We have
\[
\text{RHom}_A(I_S, T) \cong \begin{cases} 
\text{Hom}_A(I_S, T) & (S \not\cong T), \\
\text{Ext}^1_A(I_S, T)[1] & (S \cong T).
\end{cases}
\]
(d) The functor \( \text{RHom}_A(I_S, -) \) sends \( \text{D}_{\mathcal{E}(S)}(\text{Mod} \, A) \) to itself, and \( F(\text{RHom}_A(I_S, X)) = \sigma_S(F(X)) \) for \( X \) in \( \text{D}_{\mathcal{E}(S)}(\text{Mod} \, A) \).

**Proof of Theorem 4.7.** We adapt the argument of \( \text{[8]} \) Theorem III.1.9. By \( \text{[3]} \) Theorem 3.3.1(ii) one can pass between any two reduced expressions for \( w \) by the operations of (i) replacing \( \sigma_S \sigma_T \) with \( \sigma_T \sigma_S \), if there is no arrow in \( Q(S) \) from \( S \) to \( T \), and (ii) replacing \( \sigma_S \sigma_T \sigma_S \) with \( \sigma_T \sigma_S \sigma_T \) if there is a unique arrow in \( Q(S) \) from \( S \) to \( T \). Thus by Proposition \( \text{[1.6]}(i) \) the mapping is well-defined.

To show the mapping is surjective, let \( I \in \mathcal{I}(S) \), take an expression \( I = I_{S_1}I_{S_2} \ldots I_{S_k} \) with \( k \) minimal and let \( w = \sigma_{S_1} \ldots \sigma_{S_k} \). By \( \text{[3]} \) Theorem 3.3.1(i), one can pass to a reduced expression for \( w \) by the operations (i) and (ii), as above, and (iii) remove \( \sigma_S \sigma_S \). By Proposition \( \text{[1.6]}(i) \) and the minimality of \( k \), operation (iii) doesn’t occur, so the expression \( w = \sigma_{S_1} \ldots \sigma_{S_k} \) is reduced.

To prove that the mapping is injective, we may assume that \( S \) contains only finitely many isomorphism classes of rigid simple modules, for if two Coxeter group elements \( w, w' \) with reduced expressions involving the generators \( \sigma_{S_1}, \ldots, \sigma_{S_k} \) are sent to the same ideal, then they are equal in \( W(\{S_1, \ldots, S_k\}) \), so also equal in \( W(S) \). Now the action of \( W(S) \) on \( Z \) is the ‘geometric representation’ considered in \( \text{[8]} \) section 4.2.

We claim that if \( w = \sigma_{S_1} \ldots \sigma_{S_k} \) is a reduced expression, then \( I = I_{S_1}I_{S_2} \ldots I_{S_k} \cong I_{S_1} \circ \sigma_{S_2} \circ \ldots \circ \sigma_{S_k} \). Then \( F(I \circ \sigma_{S_2} \cdots \sigma_{S_k} X) = w(F(X)) \) for \( X \in \text{D}_{\mathcal{E}(S)}(\text{Mod} \, A) \) by Proposition \( \text{[4.7]}(d) \), and since the geometric representation is faithful by \( \text{[3]} \) Theorem 4.2.7, it follows that \( I \) determines \( w \).

We prove the claim by induction on \( k \), so let \( w' = \sigma_{S_1} \ldots \sigma_{S_{k-1}} \) and \( I' = I_{S_1} \ldots I_{S_{k-1}} \). By Proposition \( \text{[4.3]} \) for the opposite algebra, and with \( S = D(S_k) \), it suffices to show that \( \text{Tor}^A_1(I', S_k) = 0 \). By Lemma \( \text{[4.8]} \) for the opposite algebra, if this fails, then \( I' \circ \sigma_{S_k} = 0 \). Thus it suffices to show that \( F(I' \circ \sigma_{S_k}) \) is positive, but by the induction this is \( w'(F(S_k)) \), and this is positive by \( \text{[3]} \) Proposition 4.2.5(i)].}

### 4.4. Extended Dynkin quivers

In this subsection \( Q \) is an extended Dynkin quiver and \( \delta \) is the minimal positive imaginary root.

**Proof of Theorem 4.9.** Let \( X \) be the set of roots for \( Q \) together with zero. If \( \alpha \in X \) then the orbit \( \{ \alpha + n\delta \mid n \in \mathbb{Z} \} \) under addition of \( \delta \) is a subset of \( X \), and since any orbit contains a root for the Dynkin quiver, the set of orbits is finite. But each orbit can contain at most one element of \( \Sigma^\mathfrak{r} \), for if \( \alpha \) and \( \beta = \alpha + m\delta \) belong to \( \Sigma^\mathfrak{r} \), with \( m > 0 \), then the decomposition \( \beta = \alpha + \gamma \) with \( \gamma = m\delta \) contradicts that \( \beta \in \Sigma^\mathfrak{r} \).

Suppose \( I \) is a tilting ideal in \( A = \Pi^\mathfrak{h}(Q) \) with \( A/I \) finite dimensional. If \( S \) is a finite-dimensional rigid simple module for \( A = \Pi^\mathfrak{h}(Q) \), and \( T \) is a finite-dimensional simple module which is not rigid, then \( \dim T \) is an imaginary root, so a multiple of \( \delta \), so \( (\dim S, \dim T) = 0 \). Thus \( \text{Ext}^1_A(S, T) = \text{Ext}^1_A(T, S) = 0 \) by Proposition \( \text{[2.6]} \). It follows that \( A/I \) decomposes as a direct sum \( Y \oplus Z \) where \( \dim Y \) is a multiple of \( \delta \) and \( Z \) has composition factors in \( \mathcal{R} \). Now if \( Y \neq 0 \), then \( \text{End}_A(Y) \neq 0 \), so \( \text{Ext}^1_A(Y, Y) \neq 0 \) by Proposition \( \text{[2.6]} \). Thus \( \text{Ext}^1_A(A/I, A/I) \neq 0 \). On the other hand, since \( \text{Ext}^1_A(I, I) = 0 \) we have a commutative diagram with
Lemma 4.11. This assertion directly follows from Proposition 4.9.

Proof. For $d = (d_i) \in \mathbb{Z}^k$, we have

$$2q_Q(\sum_{i=1}^k d_i \alpha_i) = \sum_{i,j} (d_i \alpha_i, d_j \alpha_j)_Q = \sum_{i,j} d_i d_j (\alpha_i, \alpha_j)_Q = 2q_{\Gamma}(d),$$

where $(-,-)_Q$ is the symmetric bilinear form of $Q$. Thus we have the assertion.

By (a), $q_{\Gamma}$ is positive semi-definite, since so is $q_Q$. Therefore $\Gamma$ is a disjoint union of Dynkin quivers and extended Dynkin quivers. We show that the connected components of $\Gamma$ are not Dynkin. For $\alpha_j \in \Sigma^r_\lambda$, the element $\beta = \delta - \alpha_j$ is a positive real root and $\lambda \cdot \beta = 0$. Consider a decomposition $\beta = \gamma_1 + \cdots + \gamma_r$ with the $\gamma_i$ positive (and necessarily real) roots with $\lambda \cdot \gamma_i = 0$, and with $r$ as large as possible. The maximality implies that $\gamma_i \in \Sigma^r_\lambda$, so each $\gamma_i$ is an $\alpha_{\ell}$ for some $\ell$. Collecting terms, this implies that for any vertex $j$ of $\Gamma$, there is a dimension vector $d = (d_i) \in \mathbb{N}^k$ such that $d_j > 0$ and $\delta = \sum_{i=1}^k d_i \alpha_i$. By (a), we have $q_{\Gamma}(d) = q_Q(\delta) = 0$. Thus connected component of $\Gamma$ containing vertex $j$ cannot be Dynkin.

Lemma 4.10. For $d \in \mathbb{Z}^k$, $\sum_{i=1}^k d_i \alpha_i$ is a multiple of $\delta$ if and only if the restriction of $d$ to each connected component of $\Gamma$ is a multiple of the radical vector for that component.

Proof. This assertion directly follows from Proposition 4.9.

Lemma 4.11. Let $d = (d_i) \in \mathbb{N}^k$ such that $\sum_{i=1}^k d_i \alpha_i = \delta$ with $d_1 > 0$. Let $\Gamma'$ be the connected component of $\Gamma$ containing the vertex 1. We denote by $\Gamma'_0 = \{1, \ldots, s\}$ the set of vertices of $\Gamma'$.

(a) We have $d_i = 0$ for $i > s$.

(b) The vector $(d_1, d_2, \ldots, d_s)$ is the minimal positive imaginary root $\delta'$ for $\Gamma'$.

Proof. By Lemma 4.10, a vector $(d_1, d_2, \ldots, d_s)$ is a multiple of $\delta'$, so write $(d_1, d_2, \ldots, d_s) = \ell \delta'$ for an integer $\ell > 0$ Again by Lemma 4.10, $\sum_{i=1}^s d_i \alpha_i$ is a multiple of $\delta$. Since $\delta$ is minimal, we have $\delta = \sum_{i=1}^s d_i \alpha_i$ and $d_i = 0$ for $i > s$. Moreover by Lemma 4.10, $\sum_{i=1}^s d_i \alpha_i$ is a multiple of $\delta$, so $m \delta$ for an integer $m > 0$. Then $\delta = \ell' \sum_{i=1}^s d_i \alpha_i = \ell \delta'$. Thus $\ell = m = 1$ and the assertion holds.

Proof of Theorem 1.10. Part is already proved in Proposition 4.9. Let $\Gamma'$ be a connected component of $\Gamma$ such that $\Gamma'_0 = \{1, \ldots, s\}$ with minimal imaginary positive root $\delta'$. By Lemma 4.11 the semisimple $\Pi^\lambda(Q)$-module $\bigoplus_{i=1}^s S^{|\delta'|}$ has dimension vector $\delta$. Recall that, for $K$ algebraically closed of characteristic zero, the elements of $\text{Rep}(\Pi^\lambda(Q), \alpha) / \text{GL}(\alpha)$ are in 1:1 correspondence with the isomorphism classes of semisimple $\Pi^\lambda(Q)$-modules of dimension vector $\delta$, and by Theorem 3.2, the non-singular points correspond to the simple modules. Thus this semisimple module represents a singular point of $\text{Rep}(\Pi^\lambda(Q), \delta) / \text{GL}(\delta)$. 

(\vspace{1em})

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This argument defines a map from the set of connected components of $\Gamma$ to the set of singular points of $\text{Rep}(\Pi^\lambda(Q), \delta)/\text{GL}(\delta)$. We denote this map by $\Phi$.

Conversely, let $x \in \text{Rep}(\Pi^\lambda(Q), \delta)/\text{GL}(\delta)$ be a singular point. Then $x$ corresponds to a semisimple $\Pi^\lambda(Q)$-module $M_x = \bigoplus_{i=1}^k \mathbb{C}^\otimes d_i$ with $\sum_{i=1}^kd_i\alpha_i = \delta$. We may assume that $d_1 > 0$. Let $\Gamma'$ be a connected component of $\Gamma$ containing the vertex 1 with $\Gamma'_0 = \{1, \ldots, s\}$. By Lemma 4.11, we have $d_i = 0$ for $i > s$. Namely, $x$ determines a unique connected component $\Gamma'$. We denote this map by $\Psi$.

It is easy to see that $\Psi$ and $\Phi$ are mutually inverse. \qed

References

[1] V. Baranovsky, V. Ginzburg, and A. Kuznetsov, Quiver varieties and a noncommutative $\mathbb{P}^2$, *Compositio Math.*, 134(3):283–318, 2002.

[2] P. Baumann and J. Kamnitzer, Preprojective algebras and MV polytopes, *Represent. Theory*, 16:152–188, 2012.

[3] P. Baumann, J. Kamnitzer, and P. Tingley, Affine Mirković-Vilonen polytopes, *Publ. Math. Inst. Hautes Études Sci.*, 120:113–205, 2014.

[4] Y. Berest, O. Chalykh, and F. Eshmatov, Recollement of deformed preprojective algebras and the Calogero-Moser correspondence, *Mosc. Math. J.*, 8(1):21–37, 2008.

[5] P. Baumann and J. Kamnitzer, Preprojective algebras and MV polytopes, *Combinatorics of Coxeter groups*, volume 231 of *Graduate Texts in Mathematics*, Springer, New York, 2005.

[6] R. Bocklandt, Graded Calabi Yau algebras of dimension 3, *J. Pure Appl. Algebra*, 212(1):14–32, 2008.

[7] S. Brenner, C. M. R. Butler, and A. D. King, Periodic algebras which are almost Koszul, *Algebr. Represent. Theory*, 5(4):331–367, 2002.

[8] A. Buan, O. Iyama, I. Reiten, and J. Scott, Cluster structures for 2-Calabi-Yau categories and unipotent groups, *Compos. Math.*, 145(4):1035–1079, 2009.

[9] W. Crawley-Boevey, On the exceptional fibres of Kleinian singularities, *Amer. J. Math.*, 122(5):1027–1037, 2000.

[10] W. Crawley-Boevey, On matrices in prescribed conjugacy classes with no common invariant subspace and sum zero, *Duke Math. J.*, 118(2):339–352, 2003.

[11] W. Crawley-Boevey and M. P. Holland, Noncommutative deformations of Kleinian singularities, *Duke Math. J.*, 92(3):605–635, 1998.

[12] W. Crawley-Boevey and P. Shaw, Multiplicative preprojective algebras, middle convolution and the Deligne–Simpson problem, *Advances in Mathematics*, 201(1):180 – 208, 2006.

[13] P. Etingof and C.-H. Eu, Koszulity and the Hilbert series of preprojective algebras, *Compos. Math.*, 145(4):1035–1079, 2009.

[14] P. Gabriel, Indecomposable representations. II, In *Symposia Mathematica, Vol. XI (Convegno di Algebra Commutativa, INDAM, Rome, 1971)*, pages 81–104. Academic Press, London-New York, 1973.

[15] C. Geiss, B. Leclerc, and J. Schröer, Semicanonical bases and preprojective algebras. II. A multiplication formula, *Compos. Math.*, 143(5):1313–1334, 2007.

[16] V. Ginzburg, Calabi-Yau algebras, *arXiv:math/0612139*, 2006.

[17] J.-W. He, F. Van Oystaeyen, and Y. Zhang, PBW deformations of Koszul algebras over a nonsemisimple ring, *Math. Z.*, 270(1-2):185–210, 2015.

[18] O. Iyama and I. Reiten, Fomin-Zelevinsky mutation and tilting modules over Calabi-Yau algebras, *Amer. J. Math.*, 130(4):1087–1149, 2008.

[19] D. Kaplan and T. Schedler, Multiplicative preprojective algebras are 2-Calabi-Yau, *arXiv:1905.12925*, 2021.

[20] B. Keller, Calabi-Yau triangulated categories, In *Trends in representation theory of algebras and related topics, EMS Ser. Congr. Rep.*, pages 467–489. Eur. Math. Soc., Zürich, 2008.

[21] Y. Kimura, Singularity categories of derived categories of hereditary algebras are derived categories, *J. Pure Appl. Algebra*, 224(2):836–859, 2020.

[22] L. Le Bruyn, Noncommutative smoothness and coadjoint orbits, *J. Algebra*, 258(1):60–70, 2002.

[23] A. Malkin, V. Ostrik, and M. Vybornov, Quiver varieties and Lusztig’s algebra, *Adv. Math.*, 203(2):514–536, 2006.

[24] R. Martínez-Villa, Applications of Koszul algebras: the preprojective algebra, In *Representation theory of algebras (Cocoyoc, 1994)*, volume 18 of *CMS Conf. Proc.*, pages 487–504. Amer. Math. Soc., Providence, RI, 1996.

[25] A. H. Schofield. *Representation of rings over skew fields*, volume 92 of *London Mathematical Society Lecture Note Series*, Cambridge University Press, Cambridge, 1986.

[26] M. Van den Bergh, Calabi-Yau algebras and superpotentials, *Selecta Math. (N.S.)*, 21(2):555–603, 2015.