ON THE STRUCTURE OF THUE-MORSE SUBWORDS, WITH
AN APPLICATION TO DYNAMICAL SYSTEMS

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Abstract. We give an in depth analysis of the subwords of the Thue-Morse
sequence. This allows us to prove that there are infinitely many injective pri-
mitive substitutions with Perron-Frobenius eigenvalue 2 that generate a symbolic
dynamical system topologically conjugate to the Thue-Morse dynamical sys-

Key words: Thue-Morse substitution; Thue-Morse factors; Substitution dy-

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1. Introduction

We consider the Thue-Morse sequence $x = 011010011010110 \ldots$ fixed point of
the substitution $\theta$ given by

$$\theta(0) = 01, \quad \theta(1) = 10.$$  

By taking its orbit closure under the shift map, the sequence $x$ generates a dynamical system called the Thue-Morse dynamical system. In the recent paper [3] it is
proved that there are 12 primitive injective substitutions of length 2 that generate
a system topologically conjugate to the Thue-Morse system. A natural question
is: what is the list of all primitive injective substitutions whose incidence matrix
has maximal eigenvalue 2 that generate a system topologically conjugate to the
Thue-Morse system?

The usual way to generate systems topologically conjugate to a given substitu-
dynamical system is to consider the $N$-block substitution associated to the
substitution ([5], [4]). See Section 3 for more details, here we give an example: the
5-block substitution $\theta_5$ associated to the Thue-Morse substitution $\theta$.

There are twelve Thue-Morse subwords of length $N = 5$ (see Example 1 in
Section 2 for the complete list): $w_1 = 00101, \ldots, w_4 = 01011, \ldots, w_{12} = 11010$.

The $\theta_5$-image of a $w_i$ is obtained as the prefix of length 5 of $\theta(w_i)$ followed by
the prefix of length 5 of $\theta(w_i)$ with the first letter discarded. For example, since
$\theta(00101) = 0101100110$, we have $\theta_5(w_1) = w_4w_{10}$, since $w_{10} = 10110$. In this way
one obtains

\[ \begin{align*}
\theta_5(w_1) &= w_4w_{10}, \quad \theta_5(w_4) = w_5w_{11}, \quad \theta_5(w_7) = w_7w_1, \quad \theta_5(w_{10}) = w_8w_2, \\
\theta_5(w_2) &= w_4w_{10}, \quad \theta_5(w_5) = w_6w_{12}, \quad \theta_5(w_8) = w_7w_1, \quad \theta_5(w_{11}) = w_9w_3, \\
\theta_5(w_3) &= w_5w_{11}, \quad \theta_5(w_6) = w_6w_{12}, \quad \theta_5(w_9) = w_8w_2, \quad \theta_5(w_{12}) = w_9w_3.
\end{align*} \]

We go from this substitution, which is not injective, to an injective one by redistributing the four letters in the \( \theta_5 \)-images of words of length 2 with odd indices—which always occur in pairs, i.e., the couples \( w_5w_{11}, \ w_7w_1, \text{ and } w_9w_3 \). Concretely, we define a new substitution \( \zeta_5 \) by keeping \( \zeta_5(w_i) = \theta_5(w_i) \) for all words with an even index, and changing the six others in pairs as, e.g.,

\[ \theta_5(w_7)\theta_5(w_1) = w_7w_1 \quad w_4w_{10} = w_7w_1w_4 \quad w_{10} = \zeta_5(w_7)\zeta_5(w_1). \]

This leads to the substitution given by

\[ \begin{align*}
\zeta_5(w_1) &= w_{10}, \quad \zeta_5(w_4) = w_5w_{11}, \quad \zeta_5(w_7) = w_7w_1w_4, \quad \zeta_5(w_{10}) = w_8w_2, \\
\zeta_5(w_2) &= w_4w_{10}, \quad \zeta_5(w_5) = w_6w_{12}w_9, \quad \zeta_5(w_8) = w_7w_1, \quad \zeta_5(w_{11}) = w_3, \\
\zeta_5(w_3) &= w_{11}, \quad \zeta_5(w_6) = w_6w_{12}, \quad \zeta_5(w_9) = w_8w_2w_5, \quad \zeta_5(w_{12}) = w_9w_3.
\end{align*} \]

Obviously the substitution \( \zeta_5 \) is injective, and it is not hard to see that \( \zeta_5^n(w_6) = \theta_5^n(w_6) \) for all \( n \geq 1 \). Thus, if \( \zeta_5 \) would be a primitive substitution, then \( \zeta_5 \) would generate the same dynamical system as \( \theta_5 \). However, \( \zeta_5 \) is not primitive, since \( \zeta_5^2(w_3) = \zeta_5(w_{11}) = w_3 \).

In Section [4] we will repair this defect by defining a substitution \( \eta_5 \) which generates the same dynamical system as \( \theta_5 \), but is primitive. Actually, we give this construction for all \( \eta_N \), where \( N \) is a power of two plus one. For this we need an explicit expression for \( \theta_N \), which is given in Section [5] based on the combinatorial analysis in Section [2]. Our main result is in Section [6] there exist infinitely many substitutions in the Thue-Morse conjugacy class if we allow also non-constant length substitutions with Perron-Frobenius eigenvalue 2.

### 2. Combinatorics of Thue-Morse subwords

The subwords of the Thue-Morse sequence have been well studied (see, e.g., [11]). We show here that the subwords of length \( N = 2^m + 1 \) have a particularly elegant structure for \( m = 2, 3, \ldots \). Let \( A_m \) be the set of these words. It is well known (and will be reproven here) that the cardinality of \( A_m \) equals \( |A_m| = 3 \cdot 2^m \). We lexicographically order the words in \( A_m \), representing them as

\[ w_1^m < w_2^m < \cdots < w_{|A_m|}^m. \]

Crucial to the following analysis is the partition of \( A_m \) into 4 sets

\[ A_m = Q_1 \cup Q_2 \cup Q_3 \cup Q_4, \]
where each $Q_k$ consists of one quarter of consecutive words from $A_m$. If we want to emphasize the dependence on $m$ we write $Q_k^m$. Let

$$q_k = \min Q_k, \quad \text{for } k = 1, 2, 3, 4.$$  

Thus

$$q_1^m = u_1^m, \quad q_2^m = u_2^m\{A_m\}_{m+1}, \quad q_3^m = u_3^m_{A_m+1}, \quad q_4^m = u_4^m|A_m+1|.$$  

Let $f_0^\infty = 0110\ldots$ and $f_1^\infty = 1001\ldots$ be the two infinite fixed points of $\theta$, and let $f_0 = f_0^n$ and $f_1 = f_1^n$ be the length $2^m+1$ prefixes of $f_0^n$ and $f_1^n$.

**Example 1** The case $m = 2$. The set $A_2$ is given by

$$\{00101, 00110, 01001, 01011, 01100, 01101, 10010, 10011, 10100, 10101, 11001, 11010\}.$$  

Here $q_1 = 00101, q_2 = 01011, q_3 = 10010, q_4 = 10110$, and $f_0 = 01101, f_1 = 10010$.

We use frequently mirror invariance of the Thue-Morse words, i.e., if the mirroring operation is define as the length 1 substitution given by $\tilde{\theta} = 1, \tilde{I} = 0$, then $u$ is a Thue-Morse subword if and only if $\tilde{u}$ is a Thue-Morse subword. This follows directly from $\tilde{\theta}(0) = \theta(1)$.

The Thue-Morse substitution $\theta$ has the following trivial, but important property.

**Lemma 2.1.** If words $u$ and $v$ satisfy $u < v$, then $\theta(u) < \theta(v)$.

The words in $A_{m+1}$ are generated by the words in $A_m$ in a simple way. Each word $u \in A_m$ has two $\delta$-descendants, $\delta(u)$ and $\epsilon(u)$, where, by definition, $\delta(u)$ is the length $2^{m+1} + 1$ prefix, and $\epsilon(u)$ the length $2^{m+1} + 1$ suffix of $\theta(u)$. For example: since $\theta(00101) = 0101100110$, we have

$$\delta(00101) = 0101100111 \in \mathcal{A}_3, \quad \epsilon(00101) = 101100110 \in \mathcal{A}_3.$$  

The next lemma follows from Lemma 2.1.

**Lemma 2.2.** If two words $u$ and $v$ satisfy $u < v$, then $\delta(u) < \delta(v)$. If moreover, $u_1 = v_1$, then $\epsilon(u) < \epsilon(v)$.

In the following we will freely use group notation for words over the alphabet $\{0, 1\}$. For instance $(01)^{-1}0110 = 10$.

**Proposition 2.1.** For all $m$ the smallest words in the $Q_k^m$ can be expressed in $f_1^m$:

1. $q_1 = 1^{-1}f_1 1,$  
2. $q_2 = (10)^{-1}f_1 11,$  
3. $q_3 = f_1,$  
4. $q_4 = (100)^{-1}f_1 110.$

This proposition is tied up with the following one.

**Proposition 2.2.** For all $m = 2, 3, \ldots$

1. $Q_1^{m+1} = \epsilon(Q_3^m \cup Q_4^m)$,  
2. $Q_2^{m+1} = \delta(Q_1^m \cup Q_2^m)$,  
3. $Q_3^{m+1} = \delta(Q_3^m \cup Q_4^m)$,  
4. $Q_4^{m+1} = \epsilon(Q_1^m \cup Q_2^m).$
We first prove (3) of Proposition 2.1. By mirror symmetry of the Thue-Morse words we know that exactly half of the words in \( A^m \) start with 1, so \( q^m_3 \) is the \textit{smallest} word starting with 1 in \( A^m \). From the example above we see that \( q^3_3 = f_2^m \). Then it follows by induction that \( f^m_1 \) is also the smallest word with prefix 1, since \( \delta \) is order preserving, and the other words starting with 1 in \( A^m+1 \) are generated by words with prefix 00 or 01, which under \( \varepsilon \) generate words with prefix 101 or 110, which are both larger than the prefix 100 from \( f^m_1 \).

\textbf{Proof of Proposition 2.2:} We first prove (3). By Proposition 2.1 (3) the smallest symbol in \( Q^m_3 \) is mapped by \( \delta \) to the smallest symbol of \( \delta(Q^m_3 \cup Q^m_4) \). But since \( \delta \) is order preserving, it follows by matching cardinalities that (3) holds.

Since \( Q^m_1 \cup Q^m_2 \) maps under \( \delta \) to words starting with 0, where the largest word is \( \delta(f^m_0) = f^m_0 \), again a cardinality argument shows that its image must be \( Q^m_2 \), so (2) holds.

Since \( \varepsilon \) maps consecutive symbols starting with 0 to consecutive symbols starting with 1, \( \varepsilon(Q^m_1 \cup Q^m_2) \) must be \( Q^m_4 \). Then for \( \varepsilon(Q^m_3 \cup Q^m_4) \) there is only \( Q^m_1 \) left, i.e., (1) holds. \( \square \)

\textbf{Proof of Proposition 2.1:} We already proved (3), i.e., that \( q^m_3 = f^m_1 \). From this it follows that \( 1^{-1} f^m_1 \) is smaller than (or equal to) all words of length \( 2^m \) starting with 0, except maybe those that are not of the form \( 1^{-1} w \) with \( w \in A^m \). But these are of the form \( 0^{-1} w \), where \( w \) starts with 00. This implies that \( 0^{-1} w \) starts with 01, since 000 does not occur in a Thue-Morse word. Conclusion: \( 1^{-1} f^m_1 \) is the smallest of all words of length \( 2^m \) starting with 0. It has a unique right extension to the word \( 1^{-1} f^m_1 1 \), which is still the smallest among all words of length \( 2^m + 1 \), i.e., (1) holds.

To prove (2), note that \( q^{m+1}_2 = \delta(q^m_1) \) by (2) of Proposition 2.2. Also note that \( \theta(q^m_1) \) has suffix 0, and \( \theta(f^m_1) \) has suffix 1 for all \( m \). Applying \( \theta \) to both sides of (1) we obtain

\[ q^{m+1}_2 = \theta(q^m_1) 0^{-1} = (10)^{-1} \theta(f^m_1) 100^{-1} = (10)^{-1} f^{m+1}_1 11. \]

To prove (4), note that \( q^{m+1}_4 = \varepsilon(q^m_1) \), by Proposition 2.2 (4). It follows that

\[ q^{m+1}_4 = 0^{-1} \theta(q^m_1) = 0^{-1} \theta(1^{-1} f^m_1 1) = 0^{-1}(10)^{-1} \theta(f^m_1) 10 = (100)^{-1} f^{m+1}_1 110. \] \( \square \)

We would like to make the following historical remarks. Our Proposition 2.1 (3) is the finite, mirrored, version of Corollaire 4.4 in [2] by Berstel. Our Proposition 2.1 (1) is the finite version of Corollary 2 to Theorem 1 in [1] by Allouche, Curry and Shallitt.

In the next section we will need the following lemma, in which we use some new notation. For a word \( w = w_1 \ldots w_k \), we write \( \text{Pref}_\ell(w) \) for its prefix \( w_1 \ldots w_\ell \) of length \( \ell \leq k \).
Lemma 2.3. For all \( m \geq 1 \) and \( N = 2^m + 1 \) we have

\[
\text{Pref}_N(w_{2i}^{m+1}) = \text{Pref}_N(w_{2i+1}^{m+1}) = w_i^m, \quad \text{for } i = 1, \ldots, |\mathcal{A}_m|.
\]

Proof: Note first that all words \( w_i^m \) have to appear as an \( N \)-prefix of the words \( w_j^{m+1} \), and in lexicographical order. Here \( \leq \) can, and will occur, and the only fact that has to be checked is that there are no words \( w_i^m \) occurring only once.

A quick glance at \( \mathcal{A}_2 \) in the example above shows this is true for \( m = 1 \), since \( \mathcal{A}_1 \) is equal to \{001, 010, 011, 100, 101, 110\}. Suppose it is true for \( m \). Then for all \( i \) the two words \( w_i^{m+1} \) and \( w_{i+1}^{m+1} \) will have two \( \delta \)-descendants that have the same prefix of length \( N + 1 \), and the same holds for the two \( \varepsilon \)-descendants. So there are no words \( w_j^{m+1} \) occurring only once as a \((N + 1)\)-prefix of a word \( w_k^{m+2} \). \( \square \)

Example 2 The case \( m = 3 \). The set \( \mathcal{A}_3 \) has 24 elements given by

\[
\begin{align*}
w_1 &= 001011001, & w_7 &= 010110011, & w_{13} &= 100101100, & w_{19} &= 101100110, \\
w_2 &= 001011100, & w_8 &= 010110100, & w_{14} &= 100110101, & w_{20} &= 110101001, \\
w_3 &= 001100101, & w_9 &= 011001011, & w_{15} &= 100110010, & w_{21} &= 110011011, \\
w_4 &= 001110100, & w_{10} &= 011010100, & w_{16} &= 101100100, & w_{22} &= 110011010, \\
w_5 &= 010011010, & w_{11} &= 011010010, & w_{17} &= 101001011, & w_{23} &= 110100101, \\
w_6 &= 010011000, & w_{12} &= 011011101, & w_{18} &= 101001000, & w_{24} &= 110100110.
\end{align*}
\]

Here \( q_1 = w_1, q_2 = w_7, q_3 = w_{13}, q_4 = w_{19}, \) and \( f_0 = w_{12}, f_1 = w_{13}. \) \( \square \)

3. The Thue-Morse \( N \)-block substitutions \( \theta_N \)

A simple way to produce substitutions that generate dynamical systems topologically conjugate to a given substitution is to construct \( N \)-block substitutions—see Section 4 of [3].

We will describe this construction for a general substitution \( \alpha \) of constant length on an alphabet \( A \). Let the length of \( \alpha \) be \( L \), an integer greater than one. Further, let \( N \) denote any positive integer. Let \( L_\alpha \) be language of \( \alpha \), i.e., the collection of all words occurring in some power \( \alpha^n(a) \), for some \( a \in A \). We define the alphabet \( B = A^N \cap L_\alpha \), and construct a substitution \( \alpha_N \) on \( B \), called the \( N \)-block substitution associated to \( \alpha \). Namely, if \( b = a_1 \ldots a_N \) is an element of \( B \), we apply \( \alpha \) to \( b \), obtaining a word \( v := \alpha(a_1 \ldots a_N) \) of length \( LN \). We then define

\[
\alpha_N(b) = v_1 \ldots v_N, v_2 \ldots v_{N+1}, \ldots, v_L \ldots v_{L+N-1}.
\]

Example 3 Let \( N = 3 \), let \( A = \{0, 1\} \), and let \( \alpha = \theta \), the Thue Morse substitution. Then the words of length \( N \) in the language of \( \theta \) are \( w_1 = 001, \ldots, w_6 = 110 \). Since \( \theta(001) = 010110 \), we have \( \theta_3(w_1) = w_2, w_3, \) and similarly we find

\[
\begin{align*}
\theta_3(w_1) &= w_2w_3, & \theta_3(w_2) &= w_3w_6, & \theta_3(w_3) &= w_3w_6, \\
\theta_3(w_4) &= w_4w_1, & \theta_3(w_5) &= w_4w_1, & \theta_3(w_6) &= w_5w_2. \quad \square
\end{align*}
\]

We give an explicit formula for all \( \theta_N \), where \( N = 2^m + 1 \) for \( m = 2, 3, \ldots \).
It is convenient to define the translation \( \tau \) on \( \{1, \ldots, |\mathcal{A}_m|\} \) by

\[
\tau(i) = (i - 1 + \frac{1}{2}|\mathcal{A}_m|) \mod |\mathcal{A}_m| + 1.
\]

We extend \( \tau \) to \( \tau : \mathcal{A}_m \to \mathcal{A}_m \) by putting \( \tau(w_i) = w_{\tau(i)} \).

**Proposition 3.1.** Let \( \theta \) be the Thue-Morse substitution, and let \( N = 2^m + 1 \), with \( m \geq 2 \). Write \( \theta_N(w_i) = w_F(i)w_{G(i)} \) for \( i \in \mathcal{A}_m \). Then

1. \( F(2i) = F(2i - 1) = \frac{1}{2}|\mathcal{A}_m| + i \) for \( i = 1, \ldots, \frac{1}{2}|\mathcal{A}_m| \).
2. \( G(i) = \tau(F(i)) \) for \( i = 1, \ldots, |\mathcal{A}_m| \).

**Proof:** Note that \( w_F(i) = \text{Pref}_N[\delta(w_i)] \), and \( w_{G(i)} = \text{Pref}_N[\varepsilon(w_i)] \). We first show that \( \theta_N(q_1) = q_2q_4 \). This follows directly from Proposition 2.2 (2) and (4), since by Proposition 2.1 (2) and (4) we have \( \text{Pref}_N[q_2^{m+1}] = q_2^{m+1} \), and \( \text{Pref}_N[q_4^{m+1}] = q_4^m \). So (1) holds for \( i = 1 \). Similarly, we have \( \theta_N(q_3) = q_3q_1 \).

It follows directly from Lemma 2.3 that \( F(2i) = F(2i - 1) \) for all \( i \), and since \( \delta \) is orderpreserving (1) follows from the \( i = 1 \) case.

In the same way (2) follows from the \( i = 1 \) and the \( i = \frac{1}{2}|\mathcal{A}_m| + 1 \) case. \( \square \)

**4. The construction of injective Thue Morse substitutions**

The substitution \( \theta_N \) is exactly 2-to-1. In this section we construct for \( m \geq 3 \) a 1-to-1 substitution \( \eta_N \) on \( \mathcal{A}_m \) which admits one of the fixed points of \( \theta_N \) as a fixed point. The idea for this construction is a sort of converse of a construction in [4]. Notationally it is convenient to introduce the set \( \mathcal{E}_m \) of words with even indices, and the set \( \mathcal{O}_m \) of words with odd indices.

The substitution \( \eta_N \) will be a non-constant length substitution with lengths 1, 2 or 3. It is defined by \( \eta_N(w_i) = \theta_N(w_i) \) for \( w_i \in \mathcal{E}_m \), and

\[
\eta_N(w_i) = \begin{cases} 
  w_{G(i)} & \text{for } w_i \in \mathcal{O}_m \cap \mathcal{Q}_1, \\
  w_F(i) & \text{for } w_i \in \mathcal{O}_m \cap \mathcal{Q}_2, \\
  \theta_N(w_i)w_{F(\tau(i))} & \text{for } w_i \in \mathcal{O}_m \cap \mathcal{Q}_3, \\
  w_{G(\tau(i))}\theta_N(w_i) & \text{for } w_i \in \mathcal{O}_m \cap \mathcal{Q}_4.
\end{cases}
\]

The idea of this definition is that \( \theta_N \) and \( \eta_N \) act in the same way on words of length 2 occurring at even places in the fixed point \( f_0^\omega \) of \( \theta_N \). Suppose for instance that \( w_i \in \mathcal{O}_m \cap \mathcal{Q}_2 \). Then by (1) of Proposition 3.1 there is a unique \( w_j \in \mathcal{O}_m \cap (\mathcal{Q}_1 \cup \mathcal{Q}_2) \) such that \( F(j) = i \). Note that \( w_{G(j)} \in \mathcal{O}_m \cap \mathcal{Q}_4 \), since by (2) of Proposition 3.1 \( G(j) = \tau(F(j)) = \tau(i) \), and \( \tau(Q_2) = \mathcal{Q}_4 \). Therefore for all odd \( j \) with \( w_j \in \mathcal{Q}_1 \cup \mathcal{Q}_2 \) and since \( \tau \) is an involution,

\[
\eta_N(w_{F(j)}w_{G(j)}) = \eta_N(w_iw_{\tau(i)}) = w_{F(i)}w_{G(i)}\theta_N(w_{\tau(i)}) = \theta_N(w_iw_{G(j)}) = \theta_N(w_{F(j)}w_{G(j)}).
\]
Similarly, if \( w_i \in \mathcal{O}_m \cap \mathcal{Q}_4 \), then there is a unique \( w_j \in \mathcal{O}_m \cap (\mathcal{Q}_3 \cup \mathcal{Q}_4) \) such that \( F(j) = i \). Now \( w_{G(j)} \in \mathcal{O}_m \cap \mathcal{Q}_1 \), and we have for all odd \( j \) with \( w_j \in \mathcal{Q}_3 \cup \mathcal{Q}_4 \\
\eta_N(w_{F(j)}w_{G(j)}) = \eta_N(w_{i}w_{r(i)}) = \theta_N(w_{i})w_{F(r(i))}w_{G(r(i))} \\
= \theta_N(w_{i}w_{r(i)}) = \theta_N(w_{F(j)}w_{G(j)}).
Since \( \eta_N(w_i) = \theta_N(w_i) \) for \( w_i \in \mathcal{E}_m \), it follows that for all \( j \in \mathcal{A}_m \\
\eta_N(w_{F(j)}w_{G(j)}) = \theta_N(w_{F(j)}w_{G(j)}).
Since \( w_{F(j)} \) is always followed by \( w_{G(j)} \), it must be that
\( \eta_n^N(f_0^m) = \theta_n^N(f_0^m) \) for \( n = 1, 2, \ldots \).
If we knew that \( \eta_N \) was primitive, i.e., its incidence matrix is primitive, then this
would imply that \( \eta_N \) and \( \theta_N \) generate the same minimal set: \( X_{\eta_N} = X_{\theta_N} \).

5. The problem of primitivity

**Proposition 5.1.** The substitution \( \eta_N \) is primitive.

*Proof:* Let us write \( v \rightarrow w \) for \( v, w \in \mathcal{A}_m \) if there exists an \( n \) such that \( w \) occurs in \( \eta_N^n(v) \). We will prove the following:

\( F \) \( f_0 \rightarrow w, f_1 \rightarrow w \) for all \( w \in \mathcal{A}_m \), (B) either \( v \rightarrow f_0 \), or \( v \rightarrow f_1 \) for all \( v \in \mathcal{A}_m \).

Obviously (B)+(F) implies that \( v \rightarrow w \) for all \( v, w \in \mathcal{A}_m \), i.e., \( \eta_N \) is irreducible.

But primitivity follows easily from this, by observing that \( f_0 \rightarrow f_0 \) and \( f_1 \rightarrow f_1 \) in one step.

For the proof of (F), note that \( \eta_N^n(f_0^m) = \theta_N^n(f_0^m) \) implies that every \( w \) occurs in some \( \eta_N^n(f_0^m) \), since \( \theta_N \) is primitive. For \( \eta_N^n(f_1^m) \) such an equality does not hold, but it is still true that \( \eta_N^n(f_1^m) \) is a prefix of \( \theta_N^n(f_1^m) \), which of course leads to the same conclusion.

The proof of (B) is somewhat more involved, and we will prove something stronger, namely that starting from any \( v \) either \( f_0 \) or \( f_1 \) will occur as first letter of \( \eta_N^n(v) \) for some \( n \geq 1 \). We first study \( \theta_N \), defining the ‘initials’ map \( \phi : \mathcal{A}_m \rightarrow \mathcal{A}_m \) by
\( \phi(w_i) = w_{F(i)}, \) where \( w_{F(i)} = \text{Pref}_1(\theta_N(w_i)) \) for \( i = 1, 2, \ldots, |\mathcal{A}_m| \).

From Proposition 3.1 we obtain
\( \phi(\mathcal{Q}_1) \subseteq \phi(\mathcal{Q}_2), \phi(\mathcal{Q}_2) \subseteq \phi(\mathcal{Q}_3), \phi(\mathcal{Q}_3) \subseteq \phi(\mathcal{Q}_4), \phi(\mathcal{Q}_4) \subseteq \phi(\mathcal{Q}_3). \)

Moreover, \( \phi \) is strictly increasing on \( \mathcal{Q}_2 \setminus \{f_0\} \), which implies that
\( \phi^n(w_i) = f_0 \) for all \( n \geq \frac{1}{2}|\mathcal{A}_m|, w_i \in \mathcal{Q}_1 \cup \mathcal{Q}_2 \).

By mirroring, we have \( \phi^n(w_i) = f_1 \) for all \( n \geq \frac{1}{2}|\mathcal{A}_m| \) and \( w_i \in \mathcal{Q}_3 \cup \mathcal{Q}_4 \).

Next, we define \( \psi : \mathcal{A}_m \rightarrow \mathcal{A}_m \) by
\( \psi(w_i) = w_{H(i)}, \) where \( w_{H(i)} = \text{Pref}_1(\eta_N(w_i)) \) for \( i = 1, 2, \ldots, |\mathcal{A}_m| \).
From the definition of $\eta_N$ we see that $\psi|_{Q_2 \cup Q_3} = \phi|_{Q_2 \cup Q_3}$, which implies that $\psi^n(Q_2 \cup Q_3) \subseteq \{f_0, f_1\}$ for $n \geq \frac{1}{2}|A_m|$. Note that $\psi(Q_1) \subseteq Q_4$, so what remains is to study the behavior of $\psi$ on $Q_4$. First, if $w_i \in Q_4 \cap E_m$, then $\psi(w_i) = \phi(w_i) \in Q_3$, so $\psi^n(w_i) = f_1$ for all large $n$.

Second, if $w_i \in Q_4 \cap O_m$, then $\psi(w_i) > w_i$. So iterating $\psi$ will always result in hitting a letter $w_j$ with $w_j \in Q_4 \cap E_m$, and we are in the first case. \hfill \Box 

6. An infinite Thue-Morse conjugacy list

Our work in the previous sections leads to an answer to the Thue-Morse conjugacy list question.

**Theorem 6.1.** There are infinitely many injective primitive substitutions with Perron-Frobenius eigenvalue 2 that generate a dynamical system topologically conjugate to the Thue-Morse dynamical system.

**Proof:** Infinitely many substitutions are given by the $\eta_N$, where $N = 2^m + 1$ for $m = 2, 3, \ldots$. These generate minimal systems conjugate to the Thue-Morse substitutions because they generate the same systems as the $N$-block substitutions $\theta_N$. From Proposition 3.1 and the defining Equation (3) it follows that the $\eta_N$ are injective, and primitivity is given by Proposition 5.1.

It remains to prove that the Perron-Frobenius eigenvalue of the matrix $M_N$ of $\eta_N$ is equal to 2. This is clear from the definition, but here is a formal proof. Let $e_N$ be the vector of length $|A_m|$ with all ones, and let $d_N$ be the vector of length $|A_m|$ with all zero’s, except for a 1 at the position of $f_0^m$. Let $\ell(v)$ be the length of a word $v$. Then for all $n \geq 1$ one has $d_N^T M_N^n e_N = \ell(\eta_N^n(f_0^m)) = \ell(\theta_N^n(f_0^m)) = 2^n$. Since $M_N$ is a primitive non-negative matrix it follows from the Perron-Frobenius theorem that this implies that the eigenvalue of largest modulus is equal to 2. \hfill \Box 

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