Constructive – non-Constructive Approximation and Maximum Independent Set Problem

Marc Demange1,2 Vangelis Th. Paschos1

1 LAMSADE, Université Paris-Dauphine
Place du Maréchal De Lattre de Tassigny, 75775 Paris Cedex 16, France
e-mails: {demange,paschos}@lamsade.dauphine.fr
2 CERMSEM, Université Paris I, 90, rue de Tolbiac, 75634 Paris Cedex 13

Abstract. We apply in the case of the maximum independent set, a general thought process consisting in integrating an information on the optimal objective value in its instance. This thought process for the study of the relative hardness between determining solutions of combinatorial optimization problems and computing (approximately or exactly) their optimal values, allows us to define classes of independent set problems the approximability of which is particularly interesting.

1 Introduction

In theoretical computer science, problems (a fortiori the NP-complete ones) are originally defined as decisions about the existence of solutions verifying some properties. So in the decision framework (denoted by D in what follows) a problem is a question whose objective is to answer by yes or no.

However, the polynomial approximation theory needs a different framework, called optimization framework, defined below, simultaneously establishing and using the notion of the feasible solution and the one of the objective value; both notions are crucial for defining the concept of the approximate solution and for characterizing its quality.

So, in the optimization framework (denoted by O), the instance of a problem \( \Pi \) is expressed in terms of an optimization program of the form

\[
\begin{cases}
\text{opt } f(x) \\
x \in C
\end{cases}
\]

where \( \text{opt} \in \{\min, \max\} \), \( f \) is the objective function and \( C \) the set of feasible solutions. Then, \( \Pi \) consists of determining an optimal solution \( x^* \) and an algorithm determining it is called constructive. An approximation algorithm dealing with this framework (i.e., determining a feasible solution and guaranteeing a nearness of its value to the value of an optimal one) will be called constructive approximation algorithm. The approximation point of view of O will be denoted by O-ap.

There exists also another framework, the one of the optimal value (denoted by V), where the instances are the same as in O but the goal is now to determine the optimal value and an algorithm doing this will be called non-constructive. In this context, one can also define a concept of approximation
(V-ap) and an algorithm determining a feasible value and guaranteeing a nearness of this value to the optimal one is called non-constructive approximation algorithm.

A broad discussion about these several computation frameworks is performed in [2].

In this paper we consider the relative hardness between the constructive and the non-constructive approximations in the framework of maximum independent set problem.

We propose in what follows a general thought process consisting in integrating in the instances of the problems an information carrying over their optimal value, in such a way that the new (modified) problem is a priori V-easier (easier when dealing with V) than the original one. Let us note that in this case, the recognition of the instances of the derived problems is not necessarily polynomial\(^3\). So, via this transformation, work in framework O allows to better understand the links between constructiveness and non-constructiveness and a pertinent question is: in what an information about the optimal value of a problem helps in determining either the optimal or a good approximate solution for it?

We study the above question in the case of one of the most famous NP-complete problems, the maximum independent set.

Let \( G = (V, E) \) be a graph of order \( n \); an independent set is a subset of \( V' \subseteq V \) such that whenever \( \{v_i, v_j\} \subseteq V' \), \( v_i v_j \notin E \), and the maximum independent set problem (IS) is to find an independent set of maximum size. A natural generalization of IS is the one where positive integer weights are associated with the vertices of \( G \); then, the objective becomes to maximize the sum of weights of the vertices of an independent set; we denote this problem by WIS; we suppose also that the (integer) weights are universal constants (independent of \( n \)).

We consider a type of information addressed to (restricted) classes of instances of WIS, classes defined using this information. This thought process can be rich enough, since it allows us to apprehend the boundaries between constructiveness and non-constructiveness. We consider graphs defined by means of an information on the optimal value of the stability problem to be solved, and we give results concerning the constructive solution (exact or approximate) of IS on such graphs.

Let us remark that these results can be presented alternatively as autonomous independent set results on restricted classes of graphs, or as results in a "constructive - non-constructive framework".

Let \( G = (V, E) \) be a graph of order \( n \). We denote by \( a(G) \) the stability number (cardinality of a maximum independent set) of \( G \) and (for WIS) by \( w_v \) the weight of the vertex \( v \), \( i = 1, \ldots, n \) (by \( w \), we denote the vector of the weights); as usual, \( \Gamma(v) \), \( v \in V \), denotes the neighbourhood of \( v \). For a set \( V' \subseteq V \), we denote by \( G[V'] \) the subgraph of \( G \) induced by \( V' \); for a set \( A \) of edges, we denote by \( T[A] \) the set of vertices, endpoints of the edges of \( A \).

Moreover, given an instance \( I \) of a problem \( \Pi \), we denote by \( v(I) \) its optimal (objective) value. Finally, whenever vector 1 is indexed by a set of vertices,

\(^3\) This type of problems are, eventually, not NPO in the sense of [2].