NEW NOTES ON PLANAR SEMIMODULAR LATTICES. I.
TWO REMARKS ON SLIM RECTANGULAR LATTICES

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Abstract. Let \( L \) be a slim, planar, semimodular lattice (slim means that it does not contain an \( M_3 \)-sublattice). We call the interval \( I = [o,i] \) of \( L \) rectangular, if there are \( u_l, u_r \in [o,i] \setminus \{o,i\} \) such that \( i = u_l \lor u_r \) and \( o = u_l \land u_r \) where \( u_l \) is to the left of \( u_r \).

The first remark: a rectangular interval of a rectangular lattice is a rectangular lattice. As an application, we get a recent result of G. Czédli. In a 2017 paper, G. Czédli introduced a very powerful diagram type for slim, planar, semimodular lattices, the \( C_1 \)-diagrams.

We revisit the concept of natural diagrams, which I introduced with E. Knapp about a dozen years ago. Given a slim rectangular lattice \( L \), we construct its natural diagram in one simple step. The second remark shows that for a slim rectangular lattice, a natural diagram is the same as a \( C_1 \)-diagram. Therefore, natural diagrams have all the nice properties of \( C_1 \)-diagrams.

1. Introduction

In 2006, we started studying planar, semimodular lattices in my papers with E. Knapp [9]–[13]. More than four dozen publications have been devoted to this topic since; see G. Czédli’s list
http://www.math.u-szeged.hu/~czedli/m/listak/publ-psml.pdf

An SPS lattice \( L \) is a planar semimodular lattice that is also slim (it does not contain an \( M_3 \)-sublattice).

Following my paper with E. Knapp [12], a planar semimodular lattice \( L \) is rectangular, if its left boundary chain has exactly one doubly-irreducible element other than the bounds (the left corner) and its right boundary chain has exactly one doubly-irreducible element other than the bounds (the right corner) and the two corners are complementary.

Rectangular lattices are easier to work with than planar semimodular lattices, because they have much more structure. Moreover, a planar semimodular lattice has a (congruence-preserving) extension to a rectangular lattice, so we can prove many result for planar semimodular lattices by verifying them for rectangular lattices (G. Grätzer and E. Knapp [12]). It turns out that there is another way to go to slim rectangular lattices from SPS lattices.

Before we state it, we need a definition. Let \( L \) be a planar lattice. We call the interval \( I = [o,i] \) of \( L \) rectangular, if there are \( u_l, u_r \in [o,i] \setminus \{o,i\} \) such that \( i = u_l \lor u_r \) and \( o = u_l \land u_r \), where \( u_l \) is to the left of \( u_r \).

Now we state of first remark.

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2.2 Fork extensions

We discuss in Section 4.3 of CFL2 a result of G. Czédli and E. T. Schmidt \[5\]: for an SPS lattice $L$ and covering square $C$ in $L$, we can insert a fork in $L$ at $C$ to obtain the lattice extension $L[C]$, which is also an SPS lattice, see Figure 1.

![Figure 1. Inserting a fork into $L$ at $C$](image-url)
Lemma 1. Let $L$ be an SPS lattice and let $S = \{o, m, b_l, b_r, t\}$ be a minimal covering $S_7$ in $L$. Then $L$ has a sublattice $L^-$ with 4-cell $C = S - \{m, b_l, b_r\}$ such that $L = L^- [C]$.

The lattice $L^-$ is the lattice $L$ with the fork deleted.

The structure of slim rectangular lattices is described as follows.

Theorem 2 (G. Czédli and E. T. Schmidt [5]). $K$ is a slim rectangular lattice iff it can be obtained from a grid by inserting forks ($n$-times).

We thus associate a natural number $n$ with a slim rectangular lattice $K$; we call it the rank of $K$, and denote it by $\text{Rank}(K)$. It is easy to see that the $\text{Rank}(K)$ is well defined. For instance, it is the length of the lower left boundary of $K$ minus the length of the lower left boundary of $G$.

There is a slightly stronger version of this result, implicit in G. Czédli and E. T. Schmidt [5]. We present it with a short proof.

Theorem 3 (Structure Theorem). For every slim rectangular lattice $K$, there is a grid $G$, the natural number $n = \text{Rank}(K)$, and sequences

\begin{align*}
(1) & 
G = K_1, K_2, \ldots, K_{n-1}, K_n = K \\
(2) & 
C_1 = \{o_1, c_1, d_1, i_1\}, C_2 = \{o_2, c_2, d_2, i_2\}, \ldots, C_{n-1} = \{o_{n-1}, c_{n-1}, d_{n-1}, i_{n-1}\}
\end{align*}

of 4-cells in the appropriate lattices such that

\begin{align*}
(3) & 
G = K_1, K_1[C_1] = K_2, \ldots, K_{n-1}[C_{n-1}] = K_n = K.
\end{align*}

Moreover, the principal ideals $\downarrow c_{n-1}$ and $\downarrow d_{n-1}$ are distributive.

Proof. We prove this result by induction on $n$. If $n = 0$, then $K$ is distributive by G. Grätzer and E. Knapp [12], so the statement is trivial. Now let us assume that the statement holds for $n-1$. Let $K$ be a slim rectangular lattice with $n$ covering $S_7$-s. As in Lemma 1, we take $S$, a minimal covering $S_7$ in $K$. Then we form the sublattice $K^-$
by deleting the fork at $S$. So we get a 4-cell $C = C_{n-1} = \{o_{n-1}, c_{n-1}, d_{n-1}, i_{n-1}\}$ of $K^-$ such that $K = K^- | C$. Since $K^-$ has $n - 1$ covering $S_T$-s, we get the sequence

$$G = K_1, K_1 | C_1 = K_2, \ldots, K_{n-2} | C_{n-2} = K_{n-1} = K^-,$$

which, along with $K = K^- | C$, proving the statement for $K$.

By the minimality of $S$, the principal ideals $\downarrow c_{n-1}$ and $\downarrow d_{n-1}$ are distributive. □

3. Proving Remark \[\[\]

Remark\[\] obviously holds for grids.

Otherwise, we can assume that the slim rectangular lattice $K$ is not a grid. Let $K^-$ be the lattice defined in the proof of the Structure Theorem. Let

$$C_{n-1} = \{o_{n-1}, c_{n-1}, d_{n-1}, i_{n-1}\}$$

be the covering square in $K^-$, with which we obtain $K$ from $K^-$ by inserting a fork in $C_{n-1}$. We add the element $m$ in the middle of $C_{n-1}$, and add the sequences of elements $x_1, \ldots$ on the left going down and $y_1, \ldots$ on the right going down as in Figure 1.

Let $I = [o, i]_K$ be a rectangular interval in $K$ with bounds $o, i$ and corners $u_l, u_r$. We want to prove that $I$ is a slim rectangular lattice. Of course, the lattice $I$ is slim.

We induct on $n = \text{Rank}(K)$.

There are three types of subcases.

Case 1. $I$ has no element internal to $\downarrow i_{n-1}$. For instance, $I \cap \downarrow i_{n-1} = \emptyset$. Then $[o, i]_{K^-} = I$. By induction, $[o, i]_{K^-}$ is rectangular, therefore, so is $I$.

Case 2. $m$ is an internal element of $I$. For instance, $u_l$ is $c_{n-1}$ or it is to the left of $c_{n-1}$ and symmetrically. In this case, $C$ is a covering square in $[o, i]_{K^-}$ and we obtain $[o, i]_K$ by adding a fork to $C$ in $[o, i]_{K^-}$. A fork extension of a slim rectangular lattice is also slim rectangular, so $I$ is slim rectangular.

Case 3. $m$ is not an internal element of $I$ but some $x_l$ or $y_l$ is. For instance, $x_2$ is an internal element of $I$. Then we obtain $I$ from $[o, i]_{K^-}$ by replacing a cover preserving $C_m \times C_2$ by $C_m \times C_3$, and so it is rectangular.

4. Applications of Remark \[\[\]

The next statement follows directly from Remark \[\[\].

Corollary 4. Let $L$ be an SPS lattice and let $I$ be a rectangular interval of $L$. Let $(P)$ be any property of slim rectangular lattices. Then $(P)$ holds for the lattice $I$.

For instance, let $(P)$ be the property: the intervals $[o, u_l]$ and $[o, u_r]$ are chains and all elements of the lower boundary of $I$ except for $u_l, u_r$ are meet-reducible. Then we get the main result of G. Czédli [3]:

Corollary 5. Let $L$ be an SPS lattice and let $I$ be a rectangular interval of $L$, then $[o, u_l]$ and $[o, u_r]$ are chains and all elements of the lower boundary of $I$ except for $u_l, u_r$ are meet-reducible.

Another nice application is the following.

Corollary 6. Let $L$ be an SPS lattice and let $I$ be a rectangular interval of $L$ with corners $u_l, u_r$. Then for any $x \in I$, the following equation holds:

$$x = (x \land u_l) \lor (x \land u_r).$$
There is a more elegant way to formulate the last result.

**Corollary 7.** Let $L$ be an SPS lattice and let $a, b, c$ be pairwise incomparable elements of $L$. If $a$ is to the left of $b$, and $b$ is to the left of $c$, then

$$b = (b \land a) \lor (b \land c).$$

See Figure 3.

![Figure 3. Illustrating Corollary 7](image)

5. **Background for planar diagrams**

**Planar diagrams.** In a planar ordered set $P$, an $X$-configuration (see Figure 4) is formed by two edges $E$ and $F$ of $P$ satisfying the following properties:

(i) $0_E$ is to the left of $0_F$;
(ii) $1_E$ is to the right of $1_F$.

![Figure 4. The X-configuration and an example](image)

**Lemma 8.** A diagram of a bounded planar ordered set $P$ is the diagram of a planar lattice iff it does not have an $X$-configuration.

This is a useful result, even though it is almost a tautology. The following result is an easy consequence of Lemma 8.

**Corollary 9.** A sublattice of a planar lattice is also planar.
**C1-diagrams.** This research tool, introduced by G. Czédli, has been playing an important role in some recent papers, see G. Czédli [1, 3], G. Czédli and G. Grätzer [4], and G. Grätzer [7]; for the definition, see G. Czédli [1] and G. Grätzer [7].

In the diagram of an slim rectangular $K$, a normal edge (line) has a slope of $45^\circ$ or $135^\circ$. Any edge (line) of slope strictly between $45^\circ$ and $135^\circ$ is steep.

Figure 5 depicts the lattice $S_7$. A peak sublattice $S_7$ (peak sublattice, for short) of a lattice $L$ is a sublattice isomorphic to $S_7$ such that three edges at the top are covers in the lattice $L$.

**Definition 10.** A diagram of a slim rectangular $L$ is a C1-diagram, if the middle edge of a steep sublattice is steep and all other edges are normal.

**Theorem 11.** Every slim rectangular lattice $L$ has a C1-diagram.

This was proved in G. Czédli [1, Theorem 5.5]. My note [8] presents a short and direct proof.

### 6. Natural diagrams

Slim rectangular lattices have some particularly nice diagrams such as the natural diagrams of my paper with E. Knapp [13], discovered about a dozen years ago and forgotten.

For a slim rectangular lattice $L$, let $C_l(L)$ be the lower left and $C_r(L)$ the lower right boundary chain of $L$, respectively, and let $l_L$ be the left and $r_L$ the right corner of $L$, respectively.

We regard $G = C_l(L) \times C_r(L)$ as a planar lattice, with $C_l(L) = C_l(G)$ and $C_r(L) = C_r(G)$. Then the map

$$\psi: x \mapsto (x \wedge l_L, x \wedge r_L)$$

is a meet-embedding of $L$ into $G$; the map $\psi$ also preserves the bounds. By Corollary [9] the image of $L$ under $\psi$ in $G$ is a diagram of $L$, we call it the natural diagram representing $L$. For instance, the second diagram of Figure 5 shows the natural diagram representing $S_7$.

The following statement is the crucial step in proving Theorem [11]

**Lemma 12.** Let $L$ be a slim rectangular lattice, and let us represent $L$ in the form $L = K[C]$, where $K$ is a slim rectangular lattice and $C$ is a distributive 4-cell of $K$. Let $\mathcal{D}$ be a diagram of $K$ which is both natural and $C_1$. Then the diagram $\mathcal{D}[C]$ of $L$ is also a natural diagram and a $C_1$-diagram.
Proof. As illustrated in Figure 1, the diagram $D[C]$ is natural because of the choice of $u$ and $v$ and the process in Step 2 made possible by the distributivity of $C$.

The diagram $D[C]$ is $C_1$ because all the new edges are normal (by the distributivity of $C$) except for $M$. □

7. The second remark

Now we can state the second remark.

**Remark 2** (natural = $C_1$). Let $L$ be a slim rectangular lattice. Then the natural diagram of $L$ is a $C_1$-diagram. Conversely, every $C_1$-diagram is natural.

Proof. Let us assume that the slim rectangular lattice $L$ can be obtained from a grid $G$ by adding forks $n$-times, where $n = \text{Rank}(L)$. We induct on $n$. The case $n = 0$ is trivial because then $L$ is a grid. So let us assume that the theorem holds for $n - 1$.

By the Structure Theorem, there is a slim rectangular lattice $K$ and a distributive 4-cell $C = \{o, a, b, i\}$ of $K$ such that $K$ can be obtained from the grid $G$ by adding forks $(n - 1)$-times and also $L = K[C]$ holds.

Now form the natural diagram $D$ of $K$. By induction, it is a $C_1$-diagram. By Lemma 12, the diagram $D[C]$ is both natural and $C_1$.

We prove the converse the same way. □

8. Applications of Remark 2

We use Remark 2 to prove two results of G. Czédli [1].

**Theorem 13.** Let $L$ be a slim rectangular lattice. Then $L$ has a $C_2$-diagram.

Proof. Let $C_l$ and $C_r$ be chains of the same length as $C_l(L)$ and $C_r(L)$, respectively. Then $C_l(L) \times C_r(L)$ and $C_l \times C_r$ are isomorphic, so we can regard the map $\psi$, see [1], as a map from $L$ into $C_l \times C_r$, a bounded and meet-preserving map. So the natural diagram it defines is the diagram of the lattice $L$.

If we choose $C_l$ and $C_r$ so that the edges are of the same (geometric) size, we obtain a $C_2$-diagram of the slim rectangular lattice $L$. □

Natural diagrams have a left-right symmetry. The symmetric diagram is obtained with the map

$$\tilde{\psi}: x \mapsto (x \wedge \text{rc}(L), x \wedge \text{lc}(L))$$

replacing (4).

**Theorem 14** (Uniqueness Theorem). Let $L$ be a slim rectangular lattice. Then the $C_1$-diagram of $L$ is unique up to left-right symmetry.

Statements and declarations

Data availability statement. Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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