Two Loop Renormalization of Massive (p,q) Supersymmetric Sigma Models

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ABSTRACT

We calculate the $\beta$-functions of the general massive (p,q) supersymmetric sigma model to two loop order using (1,0) superfields. The conditions for finiteness are discussed in relation to (p,q) supersymmetry. We also calculate the effective potential using component fields to one loop order and consider the possibility of perturbative breaking of supersymmetry. The effect of one loop finite local counter terms and the ultra-violet behaviour of the off-shell (p,q) models to all orders in perturbation theory are also addressed.
1. Introduction

In the past massless supersymmetric sigma models have been extensively studied in connection with superstring theory (see [1,2,3,4] and the references therein) and also for their relation to differential geometry and topology. More recently there has been interest in models which include a potential for the bosonic fields (so called massive sigma models) [5,6,7,8]. Although originally studied in connection with the need to eliminate the infrared divergences occurring in the massless models [1,9], the existence of a potential for the bosonic fields allows for interesting nonperturbative effects, such as soliton solutions interpolating between the different vacua of the theory [10]. The phenomena which occur in these models are analogous to those appearing in 4 dimensional Yang-Mills-Higgs theories. In addition, certain classes of these theories have interesting renormalization properties, which have attracted attention due to their relationship to (2,0) Landau-Ginzburg models and the moduli space of superstring vacua. In particular relatively simply massive linear sigma models can flow under the renormalization group to highly non trivial conformally invariant field theories in the infrared limit [8,11].

In the literature there exist many two loop calculations of generalized sigma models discussing several issues. A brief but certainly not complete listing is as follows. Reference [1] calculated the two loop $\beta$-functions of the purely metric bosonic and (1,1) supersymmetric sigma model. The $\beta$-function for bosonic model with with torsion was calculated in [12]. These were followed by other calculations in the bosonic and (1,1) supersymmetric models by several authors (including discussions of the ambiguities associated with dimensional regularization). In particular [13,14] used component fields while [15,16] used (1,1) superfields. The (1,0) supersymmetric sigma model $\beta$-functions have been calculated to three loops for a vanishing antisymmetric tensor [17,18] using component fields, and for a purely gauge field background by [19] with (1,0) superfields. To the best of our knowledge, there exists no two loop calculation using (1,0) superfields where all the background fields are non vanishing.
In this paper we study, to two loop order, the renormalization and finiteness of the general off-shell \((p,q)\) supersymmetric massive sigma model, on flat two dimensional Minkowski Space. We will be particularly interested in the addition of mass terms, not considered in previous supersymmetric calculations. We use \((1,0)\) superspace perturbation theory and no assumptions are made about any of the background fields. It should be noted that even in the case of \((1,1)\) supersymmetry the most general form of the potential, where a central charge appears in the supersymmetry algebra, cannot be expressed with (standard) \((1,1)\) superfields, but can with \((1,0)\) superfields [5]. Our work therefore applies to the most general massive \((p,q)\) supersymmetric sigma model.

After introducing the model in the next section we discuss the background field quantization method in section 3. Then, in section 4, this method is used to calculate the one loop contributions to the \(\beta\)-functions and the effective potential. We discuss the conditions for finiteness in relation to \((2,0)\) and \((1,1)\) supersymmetry and the possibility of the perturbative breaking of supersymmetry. In section 5 the \(\beta\)-functions are calculated to two loop order and the effect of one loop finite local counter terms discussed. Finally, in section 6 we discuss the ultra-violet behaviour of the general \((p,q)\) supersymmetric model to all orders of perturbation theory. Here we show that there are no mass renormalizations to any order of perturbation theory for the \((2,0)\) supersymmetric models. We conclude with a brief discussion of the conditions for conformal invariance and the relevance of massive sigma models to string theory.
2. The (1,0) Supersymmetric Massive Sigma Model

The massive supersymmetric sigma model is defined by maps from flat (1,0) superspace $\Sigma^{(1,0)}$ into a rank n vector bundle $\Xi$ over a d dimensional Riemannian target manifold $\mathcal{M}$. The field content consists of d even scalar superfields $\Phi^i(x, \theta^+)$ ($i = 1, \ldots, d$) mapping $\Sigma^{(1,0)}$ into $\mathcal{M}$ and n odd right handed spinor superfields $\Psi^a_-(x, \theta^+)$ ($a = 1, \ldots, n$) which map $\Sigma^{(1,0)}$ into the pull back of $\Xi$ by $\Phi$ (ie. they map into the fibre above a point $\Phi^i(x, \theta^+)$ in $\mathcal{M}$). The $\Phi^i$ can be thought of as the coordinates on the manifold $\mathcal{M}$ while $\Psi^a_-$ is a section of $S_- \otimes \Phi^*\Xi$, where $S_-$ is the (right handed) spin bundle over $\Sigma^{(1,0)}$.

In order to define the (1,0) supersymmetry algebra we introduce light cone coordinates $(x_\not=, x_=)$ defined as

$$
  x_\not= = \frac{1}{\sqrt{2}}(x^0 + x^1), \quad x_= = \frac{1}{\sqrt{2}}(x^0 - x^1);
$$

$$
  \partial_\not= = \frac{1}{\sqrt{2}}(\partial_0 + \partial_1), \quad \partial_= = \frac{1}{\sqrt{2}}(\partial_0 - \partial_1).
$$

Hence $\Box = \partial^\mu\partial_\mu = 2\partial_\not=\partial_=$. The ”stacked” subscripts count the Lorentz weight of a variable (ie. their transformation properties under the Lorentz group SO(1,1)) as in [20]. Vector components have Lorentz weight $\pm 2$ while those of spinors have Lorentz weight $\pm 1$. The (1,0) superspace covariant derivative $D_+$ is defined as,

$$
  D_+ = \frac{\partial}{\partial \theta^+} + i\theta^+ \partial_\not= , \quad (2.1)
$$

so that $D_+^2 = i\theta^+ \partial_\not=$. Throughout this paper a hat over a derivative refers to the covariant derivative induced by an $\Xi$ connection $A^a_{i \ b}$, on $\mathcal{M}$. For example

$$
  \hat{D}_+ \Psi^-_a = D_+ \Psi^-_a + D_+ \Phi^i A^a_{i \ b} \Psi^-_b , \quad (2.2)
$$

where here we have pulled back the covariant derivative to $\Sigma^{(1,0)}$. 


Given these definitions the action for the massive supersymmetric sigma model is written as

\[
S = -i \int d^2 x d\theta^+ \left\{ (g_{ij} + b_{ij}) D_+ \Phi^i \Phi^j + i h_{ab} \Psi^a_+ \bar{D}_+ \Psi^b_- + im s_a \Psi^a_- \right\}, \quad (2.3)
\]

where \( g_{ij} \) and \( b_{ij} \) are metric and antisymmetric tensor fields on \( \mathcal{M} \) respectively. \( h_{ab} \) is a fibre metric which, following [5,6], we can assume without loss of generality to be covariantly constant with respect to the fibre covariant derivative \( \nabla_i \). The parameter \( m \) is a constant of mass dimension one and \( s^a \) is an arbitrary section of \( \mathcal{E} \). We denote by \( S_{\Phi}, S_{\Psi} \) and \( S_m \) the three terms in (2.3) respectively. Throughout most this paper the fibre metric assumed implicitly. Repeated vector bundle indices will still be summed over using \( h_{ab} \).

When expanded in terms of the component fields, \( \Phi^i = \phi^i + \theta^+ \eta^i_+ \) and \( \Psi^a_- = \psi^a_- + \theta^+ F^a \), the action (2.3) can be reduced to an integral over two dimensional Minkowski space by integrating over \( \theta^+ \). After eliminating the auxiliary fields \( F^a \) by their equations of motion, one recovers the standard supersymmetric sigma model but with a potential for the bosonic fields [5,6]

\[
S = \int d^2 x \left\{ (g_{ij} + b_{ij}) \partial_\pm \phi^i \partial_\pm \phi^j + ig_{ij} \eta^i_+ \nabla^{(+)\eta^j_+} - i h_{ab} \psi^a_- \bar{\nabla}_- \psi^b_- - \frac{1}{2} \psi^a_- \psi^b_- F_{abij} \eta^i_+ \eta^j_+ + m \tilde{\nabla}_i s_a \eta^a_+ \psi^a_- - \frac{1}{4} m^2 h_{ab} s_a s_b \right\}, \quad (2.4)
\]

where the covariant derivatives in (2.4) are defined in section 3 below. Furthermore, supersymmetry requires that \( h_{ab} \) is positive definite so that the potential \( V(\phi) = \frac{1}{4} h_{ab} s_a s_b \), and hence the energy, is positive. A generic section will have isolated zeros. The constant values of the scalar fields at these points constitute the classical vacuum configurations. In the case of (1,1) supersymmetry at least one of these vacua must survive as the quantum zero energy vacuum unless the Witten index vanishes, which is possible only for target space with vanishing Euler number. In these special cases, which include models with a group manifold for the target space, the the section \( s_a \) may have several or no zeros, depending on the choice of parameters defining the potential [10].
3. Background Field Quantization

In order to perform the quantization and renormalization of the \((1,0)\) supersymmetric sigma model it is most convenient to use background field method \([21]\) together with \((1,0)\) superspace perturbation theory \([3,4,19,22,23]\). \((1,0)\) superfield methods allows for the general \((p,q)\) massive supersymmetric model to be considered and therefore is the most useful for our purposes. All previous calculations of sigma model \(\beta\)-functions have only used either component field or, in the case of \((1,1)\) supersymmetry, \((1,1)\) superfields (with the exception of \([19]\) which also uses \((1,0)\) superfields but only includes background gauge fields).

The background field method entails splitting up the fields into "background" and "quantum" parts and integrating over the quantum fields in the generating functional. This is achieved by summing over all graphs with no external quantum legs. To renormalize the theory to two loop order the action is expanded to fourth order in the quantum fields so that the relevant vertices can be determined. By far the simplest way to expand the action is to use the algorithm introduced by Mukhi \([24]\), exploiting the manifest tensorial structure of the action. Here we will generalize this algorithm so as to include the torsion implicitly into the terms in the expansion which greatly simplifies the calculation.

First, we wish to spilt up the fields \(\Phi^i_{\text{total}}\) and \(\Psi^a_{\text{-total}}\), which appear in the action \((2.3)\), into background and quantum fields. In order to construct a manifestly covariant expansion of the action, the quantum fields must transform covariantly under coordinate and gauge transformations. If we naively expand the action \((2.3)\) around fixed background fields \(\Phi^i\) and \(\Psi^a\) to

\[
S(\Phi^i_{\text{total}}, \Psi^a_{\text{-total}}) = S(\Phi^i + \pi^i, \Psi^a + \chi^a_{-})
\]

we immediately see that the fields \(\pi^i\) and \(\chi^a_{-}\) are not vectors under coordinate transformations. Thus, in order to perform an expansion which is manifestly coordinate invariant, we must find another definition for the quantum fields. There
is a well known procedure for doing this [1]. For any background field $\Phi^i$ and total field $\Phi^i_{total}$, sufficiently close to it, there exists a unique geodesic of unit length, passing through each point. We define the quantum field $\xi^i$ as the tangent to this geodesic at the point $\Phi^i$. Specifically, if $\Phi^i(s)$ is the geodesic with $\Phi^i(0) = \Phi^i$, the background field, and $\Phi^i(1) = \Phi^i_{total}$ the total field appearing in the action. We then have

$$\frac{d^2 \Phi^i(s)}{ds^2} + \Gamma^i_{jk} \frac{d \Phi^j(s)}{ds} \frac{d \Phi^k(s)}{ds} = 0 .$$

Then we define the quantum field as $\xi^i = \frac{d \Phi^i(s)}{ds} |_{s=0}$. Similarly, $\zeta^a$ is the tangent to the geodesic $\Psi^a(s)$, joining $\Psi^a_{-total}$ to the total field appearing in the action $\Psi^a_{-total}$. The geodesic equation for $\Psi^a(s)$ is

$$\frac{d^2 \Psi^a(s)}{ds^2} + A^a_{ib} \Psi^b(s) \frac{d \Phi^i(s)}{ds} = 0 ,$$

and the quantum field is defined to be $\zeta^a = \frac{d \Psi^a(s)}{ds} |_{s=0}$. This non trivial definition will in general lead to a non linear renormalization of the quantum fields at higher loops [21] but this will not affect the calculation here.

The advantage of using the background field method is that, due to a shift symmetry in the choice of the background/quantum split, the symmetries of the action are preserved under quantization [21]. The action (2.3) is invariant under general coordinate transformations of the target space $\mathcal{M}$ and gauge transformations of the vector bundle $\Xi$. Although it should be noted that these symmetries do not give rise to associated Noether charges as the spacetime fields $g_{ij}$, $b_{ij}$, $A^a_{ib}$, and $s^a$ (i.e., coupling constants) must be varied in addition to the worldsheet fields $\Phi^i$ and $\Psi^a$. These symmetries are also subject to anomalies in the quantum theory which can be removed as we will discuss below. By using the background field method this tensorial structure of the action can be maintained in perturbation theory. This allows us to use the methods of standard tensor analysis and choose to work in a frame where the expressions are relatively simple, such as normal coordinates, and
then transform back to a general frame for the final answer. In normal coordinates, one finds $\xi^i = \pi^i$ and $\zeta^a = \chi^a$ [1], and the expansion can be readily performed.

While this method is certainly sufficient, it requires rather lengthy calculations. To avoid such tedious algebra we use a further simplification which allows one to write down the (n+1)th term in the expansion, in covariant form, directly from the nth term. This method utilizes the algorithm first worked out by Mukhi [24]. In Mukhi's algorithm the various geometric objects (tensors and covariant derivatives) appearing in the nth term of the expansion combine simply to give the (n+1)th term without resorting to the use of normal coordinates.

It is useful at this point to introduce some additional geometric structures on the target space bundle. First, as the quantum fields $\xi^i$ are target manifold vectors, we need to replace the derivative operators by the pull back of $\nabla$ on $\mathcal{M}$ to $\Sigma^{(1,0)}$ when acting on $\xi^i$

$$
\nabla_\pm \xi^i = \partial_\pm \xi^i + \Gamma^i_{jk} \partial_\pm \Phi^j \xi^k \\
\nabla = \xi^i = \partial = \xi^i + \Gamma^i_{jk} D_+ \Phi^j \xi^k,
$$

(3.1)

where $\Gamma^i_{jk}$ are the connection coefficients of $\nabla$.

Under the transformation $b_{ij} \to b_{ij} + \partial_i \lambda_j$ where $\lambda_i$ is an arbitrary 1-form, the action (2.3) changes by a surface term. Thus there is a gauge freedom in the choice of $b_{ij}$ and hence we define the (gauge invariant) field strength as

$$
H_{ijk} = \frac{3}{2} \partial_{[i} b_{jk]}.
$$

(3.2)

In the action (2.4) the antisymmetric field strength $H_{ijk}$ acts as a torsion for the covariant derivative. We therefore define the connection coefficients and corresponding Riemann tensor with torsion as

$$
\Gamma^{(+)}^{ij}_{jk} = \Gamma^{ij}_{jk} + H^{ij}_{jk} \\
R^{(+)}^{ij}_{jkl} = R^{ij}_{jkl} - \nabla_i H^{ij}_{kj} + \nabla_k H^{ij}_{lj} + H_{mlj} H^{mi}_{k} - H_{mkj} H^{mi}_{l},
$$

(3.3)

and similarly for $\Gamma^{(-)}^{ij}_{jk}$ and $R^{(-)}^{ij}_{jkl}$ with $H_{ijk}$ replaced by $-H_{ijk}$. It follows im-
mediately from \((3.3)\) that
\[
R^{\pm}_{ij} = R_{ij} \mp \nabla^k H_{kij} - H_{ikl} H_{j}^{\ kl} ,
\]
(3.4)
and hence
\[
R^{\pm}_{(ij)} = R_{ij} - H_{ikl} H_{j}^{\ kl} , \quad R^{\pm}_{[ij]} = \mp \nabla^k H_{kij} .
\]
(3.5)
The standard symmetries of the Riemann tensor (not associated with the Levi-Civita connection) are for \(R^{\pm}_{jkl}\)
\[
R^{\pm}_{ijkl} = \mp \frac{2}{3} \nabla^l H_{ijk} ,
R^{\pm}_{(ij)kl} = R^{\pm}_{ij(kl)} = 0 .
\]
(3.6)
While the Bianchi identity is
\[
\nabla^{(\pm)}_i R^{(\pm)}_{[mnjk]} = \mp 2H_{[ij}^{\ p} R^{(\pm)}_{mpnk]} .
\]
(3.7)
As a result of their antisymmetry, \(H_{ijk}\) and \(b_{ij}\) compose the components of differential forms \(H = H_{ijk} d\Phi^i \wedge d\Phi^j \wedge d\Phi^k\) and \(b = b_{ij} d\Phi^i \wedge d\Phi^j\). Hence it follows from \((3.2)\) that \(dH = \nabla_i H_{jk} = 0\). This identity for \(H_{ijk}\), along with the above symmetries of the Riemann tensor \(R_{ijkl}\), yield the following useful relation
\[
R^{(+)}_{ijkl} = R^{(-)}_{klij} .
\]
(3.8)
It will be convenient to define two different derivative operators with torsion for the left and right moving \(\xi^i\) fields. These covariant derivatives are defined from the derivatives in \((3.1)\) and connections in \((3.3)\) as
\[
\nabla^{(+)}_i \xi^i = \nabla_i \xi^i + H_{jk}^{\ i} \partial_i \Phi^j \xi^k , \\
\nabla^{(-)}_i \xi^i = \nabla_i \xi^i - H_{jk}^{\ i} D_i \Phi^j \xi^k .
\]
(3.9)
Mukhi’s method [24] is based on the observation that the nth term \(L^n\) in the expansion of the action can be calculated from \(L^{n-1}\) by acting with the operator
\[ \xi^i \hat{D}^1_i, \]
\[ \mathcal{L}^n(x_2, \theta_2^+ ) = \frac{1}{n} \int d^2 x_1 d\theta_1^+ \left\{ \xi^i(x_1, \theta_1^+) \hat{D}^1_i \mathcal{L}^{n-1}(x_2, \theta_2^+) \right\} \]
\[ \equiv \frac{1}{n} \mathcal{A}(\mathcal{L}^{n-1}), \]

where \( \hat{D}^1_i \) is the covariant functional derivative with respect to \( \xi^i(x_1, \theta_1^+) \), defined on tensors \( T_{klm...}^{abc...} \) by
\[ \hat{D}^1_i T_{klm...}^{abc...}(x_2, \theta_2^+) = \hat{\nabla}^i T_{klm...}^{abc...}(x_1, \theta_1^+) \delta^2(x_1 - x_2) \delta(\theta_1^+ - \theta_2^+) . \]

With the geometrical structures defined above, it is not much trouble to prove the following generalization of Mukhi’s algorithm [24] for the sigma model on (1,0) superspace with torsion
\[ \mathcal{A}(\xi^i) = 0 \]
\[ \mathcal{A}(\zeta^a) = 0 \]
\[ \mathcal{A}(\Phi^i) = \xi^i \]
\[ \mathcal{A}(\Psi^a) = \zeta^a \]
\[ \mathcal{A}(D_+ \Phi^i) = \nabla_+^{(-)} \xi^i + H^i_{jk} D_+ \Phi^j \xi^k \]
\[ \mathcal{A}(\partial_- \Phi^i) = \nabla_-^{(+)} \xi^i - H^i_{jk} \partial_- \Phi^j \xi^k \]
\[ \mathcal{A}(\hat{\nabla}_+ \Psi^a) = \hat{\nabla}_+ \zeta^a - F^{ab}_{ij} D_+ \Phi^i \Psi^b \xi^j \]
\[ \mathcal{A}(\nabla_+^{(-)} \xi^i) = R^{(-)}_{jk} D_+ \Phi^j \xi^k \xi^l - H^i_{jk} \nabla_+^{(-)} \xi^j \xi^k \]
\[ \mathcal{A}(\nabla_-^{(+)} \xi^i) = R^{(+)}_{jk} \partial_- \Phi^j \xi^k \xi^l + H^i_{jk} \nabla_-^{(+)} \xi^j \xi^k \]
\[ \mathcal{A}(\hat{\nabla}_- \zeta^a) = - F^{ab}_{ij} D_- \Phi^i \xi^j \xi^b \]
\[ \mathcal{A}(T_{a_1...a_m}^{i_1...i_n}) = \hat{\nabla}_k T_{a_1...a_m}^{i_1...i_n} \xi^k . \]

In the above rules we have incorporated the torsion implicitly into the geometrical terms in the algorithm developed by Mukhi [24], as applied to (1,0) superspace. This will greatly simplify the algebra.
4. One Loop Renormalization

The action (2.3), expanded to second order in the quantum fields using the algorithm described in the last section is

\[ S^{(2)} = -i \int d^2x \, d\theta^+ \left\{ g_{ij} \nabla^+(\xi) \nabla^-(\xi) + i \xi^a \nabla_+ \xi^a + im \nabla_i s_a \xi^i \xi^a + R^{(+)}_{i(kl)} D_+ \Phi^i \partial_- \Phi^j \xi^k \xi^l + \frac{1}{2} im \nabla_i \nabla_- s_a \Psi^a \xi^i \xi^j - 2i \Psi^- F^{ab} D_+ \Phi^i \xi^j \xi^b \right\} \]  

The first two terms yield the propagator for the \( \xi^i \) and \( \zeta^a \) fields. However, before the propagators can be read off, \( g_{ij} \) and \( h_{ab} \) must be absorbed into a redefinition of the quantum fields. This is done by referring all the fields to a vielbein frame

\[ g_{ij} = e^I_i e^J_j \delta_{IJ} , \]
\[ h_{ab} = \hat{e}^A_a \hat{e}^B_b \delta_{AB} , \]

where the fields are then \( \xi^i = e^I_i \xi^I \), \( \zeta^a = \hat{e}^A_a \zeta^A \). We will consider the quantum fields to be in vielbein frames implicitly and write all tensorial expressions in terms of coordinate frames.

To regulate the ultra-violet divergences we use dimensional regularization and then renormalize with modified minimal subtraction. However, the (1,0) super-symmetry algebra can not be defined in \( 2 + \epsilon \) dimensions [2] as there is no clear distinction between left and right movers there. To enable us to use dimensional regularization we follow the methods introduced in [4] whereby we perform the \( D_+ \) algebra first and integrate over all but one of the Grassmannian coordinates, in order to obtain integrals which can be written in a manifestly Lorentz invariant manner in \( 2 + \epsilon \) dimensions and then regularized. As in [19] we will not assume Lorentz invariant integration [4] so that some non Lorentz scalar momentum integrals may be non vanishing. Otherwise we would find no contributions to the
\( \beta \)-functions at two loops. We will use the conventions of [12] with \( c = 0 \) and define

\[
e^\mu_\lambda e^\lambda_\nu = -\delta^\mu_\nu + O(\epsilon^2) \tag{4.2}
\]

in \( D = 2 + \epsilon \) dimensions. Other choices of regularization scheme are equivalent to a field redefinition [13].

Furthermore, in the vielbein frame the spin connection terms appearing in the covariant derivatives, \( \omega^{(+)}_{IJ} = \omega^{(+)k}_{IJ} \partial_\pm \Phi^k \) and \( \omega^{(-)}_{IJ} = \omega^{(-)k}_{IJ} D_\pm \Phi^k \), transform as \( SO(d) \) gauge potentials over \( \Sigma^{(1,0)} \). This is possible only in two spacetime dimensions where we may assign different connections to the left and right moving modes in a Lorentz and gauge invariant manner. The only gauge and Lorentz invariant expressions of the connection are the square of the field strength tensor and covariant derivatives terms. The field strength tensor has mass dimension \( \frac{3}{2} \) and so its square can not be the coefficient of any divergence in (1,0) superspace [1]. This also assumes that we use the convention (4.2) [13]. Therefore we may replace covariant derivatives by flat space derivatives while performing the loop integrations as the connection terms will not contribute any Lorentz and gauge invariant divergences. There is an important caveat here. If a diagram has a positive superficial degree of divergence the connection terms may well contribute to the divergences. This problem occurs at the 2 loop level. In such cases the connection terms in \( \nabla^{(-)}_+ \) covariant derivatives must be pulled back to the background fields when performing the \( D_+ \) algebra. This simplification greatly reduces the number of graphs which need to be considered.

As is well known there are potential sigma model anomalies arising from the chiral structure of \((p,0)\) models which spoil the gauge and Lorentz invariance of the effective action [2,3,25]. This causes the connection terms to appear in non invariant forms in the one loop effective action. It is, however, possible to remove the anomalies to all orders in perturbation theory in a manner consistent with (1,0) supersymmetry, by modifying the definition of \( H_{ijk} \) through the addition of the
Chern-Simons terms

\[ H_{ijk} \rightarrow H_{ijk} + \frac{3\alpha'}{4} \Omega_3(A) - \frac{3\alpha'}{4} \Omega_3(\omega(-)) , \]

\[ \Omega_3(A)_{ijk} = A_{[i}^{AB} \partial_j A_{k]}^{AB} + \frac{2}{3} A_{[i}^{AB} A_{j}^{BC} A_{k]}^{CB} , \]

(4.3)

\[ \Omega_3(\omega(-))_{ijk} = \omega^{(-)IJ}_{[i} \partial_j \omega^{(-)JK}_{k]} + \frac{2}{3} \omega^{(-)IJ}_{[i} \omega^{(-)JK}_{j} \omega^{(-)KI}_{k]} , \]

where \( \omega(-) \) is the spin connection associated with positive torsion [3,25]

\[ \omega^{(-)JK}_{i} = \omega^{JK}_{i} + H_{ijk} e^j e^k . \]

In this case we no longer have \( dH = 0 \) but rather

\[ dH = \frac{3}{4} \alpha' (F \wedge F - R^{(+)} \wedge R^{(+)} ) . \]

(4.4)

Thus the identity (3.8) is no longer valid, and furthermore, the connection terms in the covariant derivatives will contribute Lorentz and gauge variant divergences. However, the effect of the anomalies on the two loop \( \beta \)-function has been discussed elsewhere for the massless (1,0) sigma model [26, 18]. In [26] it was conjectured that the effect of the anomalies on the higher loop \( \beta \)-functions was simply to replace \( H_{ijk} \) by the modification (4.3) and this was shown to hold at the two loop level [29,30].

The additional mass term considered in this paper, when expanded to arbitrary orders in the quantum fields, does not contain any derivative operators. Hence the anomaly structure of the massive (1,0) supersymmetric sigma model is unchanged from the massless model and thus we may use the analysis of [18]. Therefore for our purposes we may ignore the contributions of the connection coefficients in (4.4) and continue to use (3.8). After the computation is performed the effect of the one loop anomalies can be accounted for by making the replacement (4.3) in the two loop \( \beta \)-functions.

While the presence of the potential term in (2.3) appears to act as a mass term for the fields, it in fact only does so, in the general \((p,q)\) supersymmetric case,
for the $\phi^i$ and $\eta^i_+$ fields. We therefore treat the potential term in (2.3) as a pure interaction term in the perturbative expansion. In order to regulate the infrared divergences that occur we add a mass $M$ into the propagator when evaluating the divergent integrals. The ultra-violet divergences can then be isolated from the infrared ones. This modification is a purely formal device achieved by simply using a massive propagator, in place of a massless one, in the momentum integrals. The infrared divergent terms can then be ignored for the purpose of calculating the $\beta$-functions. The massive propagators used are

\[
< 0 | T \{ \xi^I(x, \theta_1^+) \xi^J(y, \theta_2^+) \} | 0 > = \delta^{IJ} D_+(x, \theta_1^+) \delta(\theta_1^+ - \theta_2^+) \Delta(x - y)
\]

\[
< 0 | T \{ \zeta^A(x_1, \theta_1^+) \zeta^B(x_2, \theta_2^+) \} | 0 > = i \delta^{AB} \partial^x D_+(x_1, \theta_1^+) \delta(\theta_1^+ - \theta_2^+) \Delta(x_1 - x_2)
\]

where $\Delta(x_1 - x_2)$ is the Feynman propagator for a free bosonic field of mass $M$

\[
(\Box + M^2) \Delta(x_1 - x_2) = \delta(x_1 - x_2)
\]

The possible divergent one loop graphs are shown in figure 1. Note that triple lines represent background fields, solid lines $\xi^i$ propagators and dashed lines $\zeta^a -$ propagators. A slash on a propagator represents the insertion of a momentum factor. However, only figure 1a yields a divergent integral after the $D_+$ algebra is performed. Thus the $\frac{1}{\epsilon}$ one loop divergences come only from the tadpole graphs and are

\[
\Gamma^{(1,1)}_{Div} = -i I \int d^2 x d\theta^+ \{ -R^{(+)}_{ij} D_+ \Phi^i \partial_+ \Phi^j + \frac{1}{2} i \Psi^a \hat{\nabla}^{(+)} k F^{ab} D_+ \Phi^i \Psi^b - \frac{1}{2} i m \hat{\nabla}_i \hat{\nabla}^i s_a \Psi^a \} .
\]

The divergent tadpole integral $I$ is given in the appendix.

Thus the requirements for one loop finiteness are

\[
R^{(+)}_{ij} = 0 \quad \hat{\nabla}^{(+)} k F^{ab}_{ki} = 0 , \quad \hat{\nabla}^2 s_a = 0 ,
\]

The first two equations in (4.6) are the well known one loop finiteness equations for the heterotic sigma model, which may be rewritten as the Einstein and Yang-Mills
field equations with the antisymmetric field strength as a source [26]. The last equation is the new contribution due to the potential $s_a$, not discussed in other supersymmetric calculations. The presence of similar massive terms has, however, been discussed in the bosonic case in connection to the tachyon of bosonic string theory [27,28].

In order to renormalize the theory we define the renormalized fields $g_{ij}^r$, $b_{ij}^r$, $A_i^a b$ and $s_a^r$ in terms of the bare ones $g_{ij}^0$, $b_{ij}^0$, $A_i^a b$ and $s_a^0$ with the divergences from each level of loop diagrams subtracted off. In $2 + \epsilon$ dimensions $g_{ij}^0$, $b_{ij}^0$ and $A_i^a b$ all have mass dimension $\epsilon$ while $ms_a^0$ has dimension $1 + \epsilon$. Hence we write [1]

$$
\begin{align*}
g_{ij}^0 &= \mu^\epsilon \left( g_{ij}^r - \sum_{\nu=1}^{\infty} \epsilon^{-\nu} g_{ij}^\nu \right) , \\
b_{ij}^0 &= \mu^\epsilon \left( b_{ij}^r - \sum_{\nu=1}^{\infty} \epsilon^{-\nu} b_{ij}^\nu \right) , \\
A_i^a b &= \mu^\epsilon \left( A_i^a b - \sum_{\nu=1}^{\infty} \epsilon^{-\nu} A_i^a b^\nu \right) , \\
s_a^0 &= \mu^{1+\epsilon} \left( s_a^r - \sum_{\nu=1}^{\infty} \epsilon^{-\nu} s_a^\nu \right) ,
\end{align*}
$$

(4.7)

where $g_{ij}^\nu$, $b_{ij}^\nu$, $A_i^a b^\nu$ and $s_a^\nu$ are the $\frac{1}{\epsilon^n}$ divergent contributions to $g_{ij}$, $b_{ij}$, $A_i^a b$ and $s_a$ respectively, calculated from all levels of loop diagrams. In (4.7) we have included the mass with $s_a$ since we wish to include the classical contribution to the conformal anomaly. By demanding that the bare fields do not depend on the (arbitrary) renormalization scale $\mu$, it is well known that the $\beta$-functions can be derived from the $\nu = 1$ terms only in (4.7) [1]. The higher order poles being calculable from the $\frac{1}{\epsilon}$ terms via t’Hooft’s pole equations.

Following this procedure the one loop divergences in (4.5) give rise to the $\beta$-
functions

\[ \beta^{(g)}_{ij} = \mu \frac{dg_r^{ij}}{d\mu} = \alpha' (R_{ij} - H_{imn} H_{jm}^{mn}) + O(\alpha'^2) , \]

\[ \beta^{(b)}_{ij} = \mu \frac{db_r^{ij}}{d\mu} = -\alpha' \nabla^k H_{kij} + O(\alpha'^2) , \]

\[ \beta^{(A)ab}_{i} = \mu \frac{dA_r^{iab}}{d\mu} = -\frac{\alpha'}{2} (\nabla^k F_{ki}^{ab} + H_i^{kj} F_{kj}^{ab}) + O(\alpha'^2) , \]

\[ \beta^{(s)}_{a} = \mu \frac{ds_r^{a}}{d\mu} = s_a - \frac{\alpha'}{2} \nabla^2 s_a + O(\alpha'^2) , \]

(4.8)

The factors of \( \alpha' \) have been inserted to enable us keep track of the different loop contributions.

### (2,0) Supersymmetry

In [5] and [6] the conditions for the model to admit an additional off-shell and on-shell left handed supersymmetry respectively, were found. The requirements in the case of zero potential [5,20] are that \( \mathcal{M} \) is a complex manifold with complex structure \( I \), \( g \) Hermitian with respect to \( I \). In addition, the holonomy of the connection \( \Gamma^{(+)} \) must be a subgroup of \( U(d/2) \). Off-shell supersymmetry further requires that \( (\Xi, \hat{I}, h) \) is also an Hermitian manifold. Off-shell (2,0) supersymmetry of the mass terms requires that, in the complex coordinates \( (\mu = 1, ..., d/2) \), \( (\alpha, ..., n/2) \) associated with the complex structures \( (I, \hat{I}) \), the potential satisfies [5]

\[ s^\alpha = \frac{1}{2} \hat{I}_r^{\alpha \beta} (M_r^\beta - L_r^\beta) , \]

(4.9)

where \( r = 1, 2 \), \( \nabla \hat{I}_r^{\alpha \beta} = 0 \), \( \nabla \hat{\mu} L_r^\alpha = 0 \) and \( \nabla \mu M_r^\alpha = 0 \). From these conditions it clearly follows that

\[ \hat{\nabla}^2 M_r^\alpha = g^{\mu \mu} \hat{\nabla} \hat{\mu} \hat{\nabla} \mu M_r^\alpha = 0 , \]

\[ \hat{\nabla}^2 L_r^\alpha = g^{\mu \mu} \hat{\nabla} \hat{\mu} \hat{\nabla} \mu L_r^\alpha = 0 , \]

and hence

\[ \hat{\nabla}^2 s^\alpha = 0 . \]

Thus one sees that the mass term is one loop finite in the case of off-shell (2,0)
supersymmetry. In the case of on-shell supersymmetry, however, there are many fewer restrictions on the target bundle [6]. In particular Ξ need not be complex or even dimensional. Thus we are unable to conclude that the mass terms are in general one loop finite.

We will show by power counting arguments in section 6 below, that for off-shell (2,0) models the mass term contributes no logarithmic divergences at any order. The β-function \( \beta(s) = s_a \) is then exact to all orders of perturbation theory.

(1,1) Supersymmetry

We now turn our attention to the case of (1,1) supersymmetry. Here the anomaly (4.3) vanishes and, as will be shown in section 5, the β-functions (4.8) receive no two loop corrections. For the model to admit a right handed supersymmetry [5], we must identify Ξ with the tangent bundle \( T\mathcal{M} \) and \( \hat{\nabla}_i \) with \( \nabla_i^{(-)} \).

We introduce a vielbein frame \( E_a^i \), with inverse \( E^a_i \), such that

\[
h_{ab} = E_a^i E_b^j g_{ij}, \quad \nabla_j^{(-)} E_a^i = 0. \tag{4.10}
\]

The curvature and field strength are now simply related through

\[
F_{ij}^{ab} = E^a_m E^b_n R_{ijmn}^{(+)}. \tag{4.11}
\]

With this identification the condition \( R_{ij}^{(+)} = 0 \) along with the Bianchi identity (3.7) implies

\[
\hat{\nabla}^{(+)} F_{ki} = 0,
\]

so we need only ensure \( R_{ij}^{(+)} = 0 \) for the massless sectors. In this case \( S_\Phi \) and \( S_\Psi \) can be combined using the (1,1) superfield

\[
X^i = \phi^i + \theta^+ \eta_+^i + \theta^- \psi_-^i + \theta^\theta^+ F^i
\]

\[
= \Phi^i + \theta^- \Psi_-^i \tag{4.12}
\]

where \( \psi_-^i = E_a^i \psi_-^a \), \( F^i = E_a^i F^a \) and \( \Psi_-^i = E_a^i \Psi_-^a \). \( \eta_+^i \) and \( \psi_-^i \) can now be thought of as the left and right handed components of a single spinor. In (4.12) \( \theta^- \).
is another anticommuting spinorial coordinate similar to $\theta^+$ with the corresponding superderivative

$$D_- = \frac{\partial}{\partial \theta^-} + i \theta^- \partial_-, \quad (4.13)$$

which anticommutes with $D_+$. The first two terms of the action (2.3) can then be written as

$$S_{\Phi + \Psi} = \int d^2 x d\theta^+ d\theta^- (g_{ij}(\mathcal{X}) + b_{ij}(\mathcal{X})) D_+ \mathcal{X}^i D_- \mathcal{X}^j. \quad (4.14)$$

The mass term $S_m$, however, cannot always be written with (1,1) superfields.

The potentials consistent with (1,1) supersymmetry are given by [5]

$$s_a = E_a^i s_i, \quad s_i = u_i - X_i, \quad (4.15)$$

where $X_i$ is a (possibly vanishing) Killing vector of $\mathcal{M}$; $\nabla_{(i} X_{j)} = 0$ and $u_i$ is a one form satisfying

$$X^k H_{ijk} = \nabla_{[i} u_{j]} , \quad (4.16)$$

and the restriction [5,33]

$$X^i u_i = 0 . \quad (4.17)$$

In the general case $X^i \neq 0$. The (1,1) supersymmetry algebra then contains a central charge and the massive terms in (2.3) cannot be written in (1,1) superfield form [5].

As can be seen from the definition of $u_i$ (4.16), there are an infinite number of $u_i$ for a given $X_i$ in the definition of $s_i$ which is reminiscent of the gauge freedom in Electromagnetism with (4.17) as a gauge fixing condition. However, different choices of the function $u_i$ will lead to physically distinct potentials $s_i$. From (4.10),
(4.15) and (4.16) one can derive
\[
\hat{\nabla}^2 s_a = E_a^i \nabla(-)^2 s_i \\
= E_a^k (\nabla_k \nabla_i u_i + R_{jk}^{(+)} u^j + R_{jk}^{(+)} X^j) \\
= E_a^k \nabla_k \nabla_i u_i ,
\]
when \( R_{ij}^{(+)} = 0 \).

For a given \( s_i \) we can find another solution \( u'_i \) to (4.16) so the theory defined by \( s'_i = u'_i - X_i \) is one loop finite. We simply take \( u_i \) to be \( u'_i = u_i + \nabla_i \lambda \) where, for an arbitrary constant \( \sigma \),
\[
\nabla^2 \lambda = \sigma - \nabla^i u_i .
\]
It follows that
\[
\nabla^(-)^2 s'_i = 0 .
\]
However, we must satisfy the restriction (4.17) as well. This can be done by requiring \( \lambda \) to satisfy the boundary condition \( X^i \nabla_i \lambda = -X^i u_i \).

To see that this is possible, we can choose coordinates \((t, x^m)\) such that \( X = \frac{\partial}{\partial t} \) then \( \lambda \) satisfies \( \nabla^2 \lambda = \sigma - \nabla^i u_i \) and \( \frac{\partial \lambda}{\partial t} = -u_t \). This is analogous to choosing the Coulomb gauge in Electromagnetism. Furthermore this uniquely determines the function \( u_i \), provided \( X^i \neq 0 \).

If \( \sigma \neq 0 \) then we may rescale \( m \rightarrow \sigma m \) and \( s_i \rightarrow \sigma^{-1} s_i \) to absorb \( \sigma \) into \( m \) and \( s_i \). Thus for \( X^i \neq 0 \) there exist only two physically distinct potentials \( s_a^{(\sigma)} \) which are one loop finite, provided \( R_{ij}^{(+)} = 0 \). If we assume \( \mathcal{M} \) to be asymptotically flat and \( s_a \) to vanish at infinity, we are forced to take \( \sigma = 0 \) and we are left with a unique choice for \( u_i \). Since \( X_i \) is Killing, \( \nabla^i X_i = 0 \). Thus it follows from (4.18) that the (1,1) supersymmetric sigma model with potential \( s_a \) is finite at one loop if and only if \( \nabla^i s_i = \sigma \) and \( R_{ij}^{(+)} = 0 \).
Given any target space $\mathcal{M}$ we can choose $X = 0$ so that (4.17) is trivially satisfied. Only in this case may we write $S_m$ in (1,1) superfield form as

$$S_m = \int d^2x d\theta^+ d\theta^- m\lambda(X).$$

Then, $s_i$ is given by

$$s_i = u_i = \nabla_i \lambda$$

and is finite at one loop precisely when

$$\nabla^2 \lambda = \sigma.$$  

(4.20)

Solutions to (4.20), with $\sigma = 0$, can be found by choosing $\lambda$ as the real part of a holomorphic function. In this case there are clearly an infinite number of physically distinct potentials which are one loop finite. The classical vacua of the theory are given by the critical points of $\lambda$. This is consistent with the general requirements of the $N = 2$ nonrenormalization theorems (in the special case that the model possess (2,2) supersymmetry), where $\lambda$ is interpreted as the superpotential in the F-term, thus providing a check on our calculations.

Lastly, it we briefly outline what happens in models with (2,2) supersymmetry. Since these models are special cases of (1,1) sigma models, we know that (4.18) holds. However, as there also exists a second left handed supersymmetry we know that $\nabla^2 s_\alpha = 0$ automatically. Thus for (2,2) supersymmetry we must have $\nabla^i u_i = 0$. Since $u_i$ is holomorphic [3] this is easily seen to be the case, because in complex coordinates $\nabla_\bar{\mu} u_\mu = 0$ hence $\nabla^\mu u_\mu = g^{\bar{\mu} \mu} \nabla_{\bar{\mu}} u_\mu = 0$.

The One Loop effective Potential

To complete our discussion of the one loop quantization of the massive supersymmetric sigma model we now calculate the effective potential. For this it is sufficient to set $\eta^i_\perp = \psi_\alpha = 0$ and fix $\phi^i$ to be constant. While the sigma model is
most readily defined by the superspace action (2.3), in this simple case we choose to calculate the effective potential using ordinary component fields rather than superfields. To this end we expand the action (2.4) without integrating over the auxiliary field $F^a$ (i.e. with $V = -m s_a F^a - F^a F^a$), to second order in the quantum component fields. This yields, after integrating out the quantum auxiliary fields,

$$S^{(2)} = \int d^2x \left\{ \delta_{IJ} \partial_+ A^I \partial_+ A^J + i \delta_{IJ} \omega_+^I \partial_+ \omega_+^J - i \delta_{AB} \chi_+^A \partial_+ \chi_+^B 
+ m \hat{\nabla} _{I} s_A \omega_+^I \chi_+^A - \frac{1}{4} (m^2 \hat{\nabla} _{I} s_A \hat{\nabla} _{J} s_A - 2m F^A \hat{\nabla} _{(I} \hat{\nabla} _{J)} s_A ) A^I A^J \right\},$$

(4.21)

where we have referred all fields to the vielbein frames and $A^I$, $\omega_+^I$ and $\chi_+^A$ are the quantum fields for $\phi^I$, $\eta_+^I$ and $\psi_+^A$ respectively.

It is useful to define the matrices

$$M_IA = m \hat{\nabla} _{I} s_A ,$$

$$M_{IJ}^2 = m^2 \hat{\nabla} _{I} s_A \hat{\nabla} _{J} s_A ,$$

$$K^F_{IJ} = \frac{1}{2} (m^2 \hat{\nabla} _{I} s_A \hat{\nabla} _{J} s_A - 2m F^A \hat{\nabla} _{(I} \hat{\nabla} _{J)} s_A ) .$$

(4.22)

$M_IA$ and $K_{IJ}$ appear in the action (4.22) as mass matrices for the (left handed) fermions and bosons respectively. Only in the case of (1,1) supersymmetry do they provide a mass for the right handed fermions.

From (4.21) the propagators for the component quantum fields can be read off. We do not add a mass term for these fields as we did above since in general the infrared divergences cancel in the effective potential - although we will discuss an exception to this below. In (4.21) there are boson-boson and fermion-fermion interactions. The effective potential can be found by summing over all one loop diagrams with zero external momentum. There are two types of graph to consider, the purely bosonic loop 2a and the purely fermionic loop 2b. It is a straightforward
calculation to determine their contributions to be (in Euclidean space)

\[ 5a) = \frac{\alpha'}{4\pi} \int d^2p \; \text{Tr} \ln \left( \delta_{IJ} + \frac{K_{IJ}}{p^2} \right) \]

\[ 5b) = -\frac{\alpha'}{4\pi} \int d^2p \; \text{Tr} \ln \left( \delta_{IJ} + \frac{M_{IJ}^2}{2p^2} \right). \]

where the trace is over the manifold indices. We must also include the counter term graphs in (4.23). However these are just the graphs in figure 5 with only one and two vertices respectively. The momentum integral in (4.23) can then be performed and we arrive at the one loop correction to the potential

\[ V_{\text{eff}} = -m s_A F^A - h_{AB} F^A F^B \]

\[ + \frac{\alpha'}{4} \text{Tr} \left[ K_{IJ}^F - K_{IJ} F^F \text{ln} \left( \frac{K_{IJ}^F}{\mu^2} \right) - \frac{1}{2} M_{IJ}^2 + \frac{1}{2} M_{IJ}^2 \text{ln} \left( \frac{M_{IJ}^2}{2\mu^2} \right) \right]. \] (4.24)

We now find the equation of motion of \( F^A \) to be

\[ F^A = -\frac{1}{2} m s_A + \frac{m\alpha'}{8} \text{Tr} \left[ \hat{\nabla}(I \hat{\nabla} J) s^A \text{ln} \frac{K_{IJ}}{\mu^2} \right] + \mathcal{O}(h^2), \] (4.25)

where

\[ K_{IJ} = \frac{m^2}{2}(\hat{\nabla} I s_A \hat{\nabla} J s^A + s^A \hat{\nabla}(I \hat{\nabla} J) s_A). \] (4.26)

Substituting this into (4.24) yields

\[ V_{\text{eff}} = \frac{1}{4} m^2 s_A s^A + \frac{\alpha'}{4} \text{Tr} \left[ K_{IJ} - K_{IJ} F^F \text{ln} \left( \frac{K_{IJ}}{\mu^2} \right) - \frac{1}{2} M_{IJ}^2 + \frac{1}{2} M_{IJ}^2 \text{ln} \left( \frac{M_{IJ}^2}{2\mu^2} \right) \right] \]

\[ + \frac{m^2 \alpha'^2}{64} \left( \text{Tr} \left[ \hat{\nabla}(I \hat{\nabla} J) s^A \text{ln} \frac{K_{IJ}}{\mu^2} \right] \right)^2. \] (4.27)

We have kept the last term in (4.27) since, although it is of order \( \alpha'^2 \), it is determined by one loop corrections and is needed to ensure that the effective potential is positive.
Suppose that \( \hat{\nabla}_I s^A \) is invertible. We may then expand (4.27) near the classical vacuum \( \phi_{cl} \) as, 
\[
y^I = O(\alpha'), \n\]
\[
V_{eff}(\phi_{cl} + y) = \frac{m^2}{4} \left( \hat{\nabla}_I s^A(\phi_{cl}) y^I - \frac{\alpha'}{4} \text{Tr} \left[ \hat{\nabla}_I \hat{\nabla}_J s^A(\phi_{cl}) \ln \frac{M^2_{IJ}(\phi_{cl})}{2\mu^2} \right] \right)^2 + \alpha'^2 V_2(\phi_{cl}) + O(\alpha'^3),
\]
where \( V_2 \) represents the higher loop contributions to \( V_{eff} \). Since \( \hat{\nabla}_I s^A \) is invertible, we can solve for \( y^I \) so that the first term in (4.28) vanishes and \( V_{eff} \) is minimized. The classical vacuum is then shifted by \( y^I \), of order \( \alpha' \) and the vacuum energy is \( \alpha'^2 V_2(\phi_{cl}) + O(\alpha'^3) \). Since supersymmetry must be preserved in this case (see below) we conclude that \( V_2(\phi_{cl}) = 0 \), justifying the inclusion of the \( O(\alpha'^2) \) term in (4.27).

If \( \hat{\nabla}_I s^A \) is degenerate at \( \phi_{cl} \), so that the fermion mass matrix \( M^2_{IJ} \) has zero modes then the effective potential diverges logarithmically there. It is tempting to view the presence of massless fermions as an indication that the theory dynamically breaks supersymmetry, with the massless fermions interpreted as Goldstone modes. It is sometimes incorrectly stated that supersymmetry can not be broken perturbatively due to the nonrenormalization theorems. In fact what actually prevents perturbative corrections to the vacuum energy is a non vanishing Witten index, which counts the number of bosonic minus the number of fermionic zero energy states. A non vanishing index therefore implies the existence of a supersymmetric (zero energy) vacuum state. As is well known the Witten index is a topological invariant, equal to the Euler number of \( \mathcal{M} \) in the \((1,1)\) supersymmetry case, and therefore cannot receive any quantum corrections, including non perturbative effects. Vacuum states where \( M^2_{IJ} \) has a zero mode, however, do not contribute to the Witten index (Euler number) and so may in principle be removed by quantum effects. This issue has been raised and discussed some time ago where it was concluded that no such breaking of supersymmetry occurred (see \[34\] and the references therein). However, the models discussed there claimed to break supersymmetry even in cases where the Witten index was nonzero. What
we are describing here actually corresponds to the case \( a = 0 \) of [34] which, to the best of our knowledge, has not been discussed.

Unfortunately it is precisely the existence of the massless fermions which causes the logarithmic infrared divergence in the effective potential. This renders the loop expansion invalid near the classical vacuum. Thus, despite the fact that a straightforward analysis shows the effective potential to be non-vanishing at the vacuum, we cannot conclude that supersymmetry has been perturbatively broken. If we add a mass term in the propagator as an infrared regulator, as we have done above when calculating the \( \beta \)-functions, then we find that the effective potential is well behaved and the vacuum energy is indeed lifted above zero. However, this is not surprising as the addition of the mass term explicitly breaks supersymmetry. In order to substantiate the claim that supersymmetry is perturbatively broken we must find a reliable approximation in which to work. Although in a weaker sense supersymmetry is broken at 1 loop as standard perturbation theory becomes untenable about such a vacuum state. In the non-perturbative regime, dynamical breaking of supersymmetry can be seen using the \( 1/N \) expansion in the large \( N \) limit [35] for similar types of two dimensional fields theories.

5. Two Loop Renormalization

In this section we proceed to calculate the two loop contributions to the gauge and Lorentz invariant parts of the \( \beta \)-functions, bearing in mind the discussion in the previous section regarding the effect of the one loop anomalies. In order to calculate the \( \beta \)-functions to two loop order we must first expand the action (2.3) to fourth order in the quantum fields. This calculation is greatly simplified by using the algorithm we developed in section 3. To third and fourth order, the action is

\[
S_{\Phi}^{(3)} = \frac{-i}{3} \int d^2 x d\theta^+ \left\{ \left( R_{i[kl]}^{(+)} + \Phi^i \partial_+ \Phi^j \xi^k \xi^l \right) \xi^m + 2 R_{i[kl]}^{(+)} \nabla^+_+ \xi^i \partial_+ \Phi^j \xi^k \xi^l \\
+ 2 R_{i[kl]}^{(-)} \nabla^+_+ \xi^i \nabla^+_+ \xi^j \xi^k \xi^l + 2 H_{ijk} \nabla^+_+ \xi^i \nabla^+_+ \xi^j \xi^k \xi^l \right\},
\]

24
\[ S^{(3)}_{\Psi} = -\frac{i}{6} \int d^2 x d\theta^+ \left\{ i \psi^a \psi^b (\nabla_+^{(a)} \nabla_-^{(b)} F^{abc}) + R^{(+)p}_{i(mn} R^{(+)q}_{klj]} + \nabla_+^{(a)} \nabla_-^{(b)} F^{abc} \right\} D_+ \Phi^i \xi^j \xi^k \xi^l \\
- 2i \Phi^a \nabla_+^{(a)} F^{abc} \nabla_+^{(b)} \xi^j \xi^k \xi^l - 6i \Phi^a F^{abc} \nabla_+^{(a)} \xi^j \xi^k \xi^l \\
- 6i \Phi^a \nabla_+^{(a)} F^{abc} D_+ \Phi^i \xi^j \xi^k \xi^l + 6i F^{abc} D_+ \Phi^i \xi^j \xi^k \xi^l \right\}, \]

\[ S^{(3)}_m = -\frac{i}{6} \int d^2 x d\theta^+ \left\{ i m \nabla_+(i \nabla_j \nabla_k) s_a \psi^a \xi^j \xi^k + 3 m \nabla_+(i \nabla_j) s_a \xi^a \xi^j \right\}. \]

\[ S^{(4)}_{\Phi} = -\frac{i}{12} \int d^2 x d\theta^+ \left\{ (4 R^{(+)p}_{i(mn} R^{(+)q}_{klj]} + \nabla_+^{(a)} \nabla_-^{(b)} F^{abc} \right\} D_+ \Phi^i \partial_- \Phi^j \xi^k \xi^l \xi^m \xi^n \\
+ (6 \nabla_+(i H_k)_{ij} + 4 R_{ijkl} \nabla_+^{(a)} \xi^j \xi^k \xi^l \\
+ 3 \nabla_-^{(a)} R^{(+)b}_{ijkl,j} D_+ \Phi^i \nabla_-^{(b)} \xi^j \xi^k \xi^l \xi^m \\
+ 3 \nabla_-^{(a)} \partial_- \Phi^i \nabla_-^{(b)} \xi^j \xi^k \xi^l \xi^m \right\}. \]

\[ S^{(4)}_{\psi} = -\frac{i}{24} \int d^2 x d\theta^+ \left\{ 12 i \nabla_+^{(a)} F^{abc} D_+ \Phi^i \xi^j \xi^k \xi^l \xi^m \\
- 8i \psi^a (\nabla_+^{(a)} \nabla_-^{(b)} F^{abc} + R^{(+)p}_{i(mn} R^{(+)q}_{klj]} + \nabla_+^{(a)} \nabla_-^{(b)} F^{abc} \right\} D_+ \Phi^i \partial_- \Phi^j \xi^k \xi^l \xi^m \xi^n \\
+ 16 i \psi^a \nabla_+^{(a)} F^{abc} \nabla_+^{(b)} \xi^j \xi^k \xi^l \xi^m + 12 i \nabla_+^{(a)} \nabla_-^{(b)} \nabla_+^{(c)} \nabla_-^{(d)} F^{abcd} \\
+ i \psi^a \nabla_-^{(a)} \nabla_+(i \nabla_k F^{abc}) + 4 H_{ijkl} F^{abc} \nabla_+^{(a)} \nabla_-^{(b)} \nabla_+^{(c)} \nabla_-^{(d)} F^{abcd} \\
+ i \psi^a \nabla_-^{(a)} \nabla_+(i \nabla_k F^{abc}) + 2 H_{ijkl} F^{abc} \nabla_+^{(a)} \nabla_-^{(b)} \nabla_+^{(c)} \nabla_-^{(d)} F^{abcd} \\
+ F^{abcd} \nabla_+^{(a)} \nabla_-^{(b)} \nabla_+^{(c)} \nabla_-^{(d)} F^{abcd} \right\}. \]

\[ S^{(4)}_m = -\frac{i}{24} \int d^2 x d\theta^+ \left\{ i m \nabla_+(i \nabla_j \nabla_k) s_a \psi^a \xi^j \xi^k \xi^l + 4 i m \nabla_+(i \nabla_j \nabla_k) s_a \xi^a \xi^j \xi^k \xi^l \right\}. \]

In the above expansions we have introduced a "twisted" covariant derivative \( \nabla^{(\pm)} \) defined on target manifold bundles as

\[ \nabla_+^{(a)} T^{a_1 \ldots a_m} = \nabla_+^{(a)} T^{a_1 \ldots a_m} - H_{k_1}^{a_1 \ldots a_m} - H_{k_2}^{a_1 \ldots a_m} + H_{k_3}^{a_1 \ldots a_m} \ldots + (-1)^n H_{k_n}^{a_1 \ldots a_m} T^{a_1 \ldots a_m}, \]

and similarly for \( \nabla^{(+)} \) with \( H_{ijk} \) replaced by \( -H_{ijk} \).
The counter terms, derived from (4.5) and expanded to second order in the quantum fields are of the form

\[
S_{D}^{(2)} = \frac{-i}{2\pi \epsilon} \int d^2 x d\theta^+ \left\{ \frac{C_{-ij}^2}{\xi^i \xi^j} + \frac{C_{-ij}^2}{\xi^i \xi^j} + \frac{C_{+jk}^2}{\xi^j \xi^k} + \frac{C_{+jk}^2}{\xi^j \xi^k} \right. \\
+ \frac{C_{+ij}^2}{\xi^i \xi^j} + \frac{C_{+ij}^2}{\xi^i \xi^j} + \frac{C_{+ij}^2}{\xi^i \xi^j} + \frac{C_{+ij}^2}{\xi^i \xi^j} \\
+ \left. \frac{C_{+jk}^2}{\xi^j \xi^k} + \frac{C_{+jk}^2}{\xi^j \xi^k} + \frac{C_{+jk}^2}{\xi^j \xi^k} + \frac{C_{+jk}^2}{\xi^j \xi^k} \right\},
\]

(5.2)

However, as will be seen below we only need to know the \(C_{+ij}^2\), \(C_{+jk}^2\), \(C_{+ij}^2\), \(D_{\neq ai}^2\) and \(D_{-ai}^2\) coefficients in order to calculate the contributions of (5.2) to the two loop \(\beta\)-functions. These coefficients are

\[
C_{+ij}^2 = -R_{ij}^{(+)} , \\
C_{+jk}^2 = -(\nabla_k R_{ij}^{(+)} + H_{ik}^m R_{mj}^{(+)} ) D_+ \Phi^i , \\
C_{+jk}^2 = -(\nabla_k R_{ij}^{(-)} - H_{ik}^m R_{mj}^{(-)} ) \partial_- \Phi^i , \\
D_{+ai}^2 = i \Psi^a \nabla^{(+)} k F_{ki}^{ab} , \\
D_{\neq ai}^2 = 0 .
\]

Simple power counting shows that the only divergent two loop diagrams have at most 3 vertices. By dimensional analysis, it can be seen that no vertices coming from the expansion of the mass term \(S_m\) in (2.3) can contribute to the renormalization of the \(g_{ij}\), \(b_{ij}\) and \(A_{\alpha_{ij}}^a\) fields. This is because any such diagram would necessarily involve terms with \(mD_+ \Phi^i \partial_- \Phi^j\) or \(m \Psi^a \Psi^b D_+ \Phi^i\) respectively. However, both of these terms have mass dimension 5/2 and hence the corresponding graphs must have a negative superficial degree of divergence. Thus any divergences must come from divergent subgraphs, but these are canceled, according to Hepp’s theorem, by the counter term graphs.

\(S_\Psi\) Renormalization

Since no vertices coming from the expansion of \(S_m\) and only one graph from \(S_\Psi\) contribute to the \(g_{ij}\) and \(b_{ij}\) renormalization, we have only the graphs in figure
where the divergent integrals $I$, $J$, $K$ and $L$ are given in the appendix. Only one of the three possible combinations of curvature terms appears in graph 3(c), the others are either finite or lead to $\frac{1}{\epsilon}$ contributions only. The $D_+$ algebra, however, reduces the momentum integral to one that is not a Lorentz scalar. To evaluate this integral we contract the internal momentum with the external $\partial^i \Phi$ field and express everything in terms of $\eta_{\mu \nu}$ and $\epsilon_{\mu \nu}$. The integration then yields the extra factor of $\epsilon$ appearing at the front of 3(c), 3(d) and 3(e) and also the $\partial \Phi^i$ field. In this way we obtain non vanishing $\frac{1}{\epsilon}$ poles. If we had assumed symmetric Lorentz integration then these graphs would vanish.

It is not hard to see, using (3.5), that the $\frac{1}{\epsilon}$ pole terms from graphs (3a,3b,3z) cancel each other. We now find, after some algebraic manipulations, that the two loop $\frac{1}{\epsilon}$ pole divergences of (3c,3d,3e) are

\[
\Gamma^{(1,2)}_{\text{Div} \Phi} = \frac{-i}{64\pi^2\epsilon} \int d^2 x d\theta^+ \left\{ \mathcal{R}^{(+)}_{iklm} \mathcal{R}^{(+)}_{jklm} - F^{ab}_{ik} F^{abk}_{ij} \right\} D_+ \Phi^i \partial \Phi^j . \tag{5.3}
\]

We may now proceed, as in section 4, to calculate the two loop $\beta$-functions $\beta_{ij}^{(g)}$.
and $\beta^{(b)}_{ij}$. They are easily seen to be

$$
\beta^{(q)}_{ij} = \alpha' R^{(+)}_{ij} + \frac{\alpha'^2}{8} \left( R^{(+)}_{iklm} R^{(+)}_{kij} - F^{ab} F^{ab} \right) + O(\alpha'^3),
$$

$$
\beta^{(b)}_{ij} = \alpha' R^{(+)}_{ij} + O(\alpha'^3).
$$

$S_\Psi$ Renormalization

The vertices from $S_\Phi$ and $S_\Psi$ both contribute to the renormalization of $S_\Psi$. The nonvanishing graphs are given in figure 4. Following a similar analysis to the $S_\Phi$ renormalization we obtain the contributions

$$
4a) = \frac{2i}{9} K \int d^2 x d\theta^+ \left\{ i \Psi^a H^{mn} H^{jmn} \nabla^{(+)}_{k} F^{ab} D_i \Phi^i \Psi_b^b \right\},
$$

$$
4b) = -\frac{2i}{9} K \int d^2 x d\theta^+ \left\{ i \Psi^a H^{mn} R^{(+)}_{ikmn} F^{abk} D_+ \Phi^i \Psi_b^b \right\},
$$

$$
4c) = -\frac{i}{2} K' \int d^2 x d\theta^+ \left\{ i \Psi^a H^{kmn} F^{ac} F^{cb} D_+ \Phi^i \Psi_b^b \right\},
$$

$$
4d) = -i IJ \int d^2 x d\theta^+ \left\{ i \Psi^a F^{acj} \nabla^{+} F^{eb} D_+ \Phi^i \Psi_b^b \right\},
$$

$$
4e) = \frac{3i}{8} IJ \int d^2 x d\theta^+ \left\{ i \Psi^a g^{mn} \nabla^{(+)}_{k} R^{(+)}_{ijmn} F^{abk} D_+ \Phi^i \Psi_b^b \right\},
$$

$$
4f) = -\frac{i}{2} IJ \int d^2 x d\theta^+ \left\{ i \Psi^a R^{kj} \nabla^{(+)}_{k} F^{ab} D_+ \Phi^i \Psi_b^b \right\},
$$

$$
4g) = -\frac{i}{8} IJ \int d^2 x d\theta^+ \left\{ i \Psi^a \nabla_i H^{lkj} \nabla^{(-)}_{i} F^{abk} D_+ \Phi^i \Psi_b^b \right\},
$$

$$
4w) = -\frac{i}{4\pi \epsilon} J \int d^2 x d\theta^+ \left\{ i \Psi^a R^{(jk)} \nabla_{k} F^{ac} F^{i} D_+ \Phi^i \Psi_b^b \right\},
$$

$$
4x) = -\frac{i}{4\pi \epsilon} J \int d^2 x d\theta^+ \left\{ i \Psi^a (\nabla_k R^{(ijk)} + H^{m}_{ik} R^{(m)}_{n} F^{abk} D_+ \Phi^i \Psi_b^b \right\},
$$

$$
4y) = -\frac{i}{2\pi \epsilon} J \int d^2 x d\theta^+ \left\{ i \Psi^a \nabla^{(+)}_{k} F^{ac} F^{eb} D_+ \Phi^i \Psi_b^b \right\},
$$

$$
4z) = \frac{i}{16\pi \epsilon} J \int d^2 x d\theta^+ \left\{ i \Psi^a (R^{(jk)} \nabla_{k} F^{ab} + \nabla^{(-)}_{i} R^{(ijk)} F^{ab} D_+ \Phi^i \Psi_b^b \right\}.
$$

From these graphs, the integrals given in the appendix and the identities (3.5), (3.6) and (3.8) we find that the $\frac{1}{t}$ poles of graphs (4a,4f,4w) and (4c,4d,4y) have
completely canceled with each other. Furthermore after some tedious algebra, the $\frac{1}{\epsilon}$ poles from the graphs (4b,4e,4x) and the second term in 4z also cancel, while the first term in 4z cancels with 4g. Thus we find

$$\Gamma^{(1,2)}_{Div} \Psi = 0$$

and arrive at the Yang-Mills two loop $\beta$-function

$$\beta_i^{(A)ab} = -\frac{\alpha'}{2} \hat{\nabla}^{(+)} k F_{ki}^{ab} + O(\alpha'^3) . \quad (5.5)$$

**Sm Renormalization**

Finally we consider the renormalization of the mass term $S_m$. The only diagrams contributing to the mass renormalization at two loops are given in figure 5. They can be calculated to be

$$5a) = \frac{i}{2} J I \int d^2 x d\theta^+ \left\{ i m R_{ij} \hat{\nabla}^i \hat{\nabla}^j s_a \Psi_a \right\} ,$$

$$5b) = -\frac{2i}{9} K \int d^2 x d\theta^+ \left\{ i m H_{ikl} H_j^{kl} \hat{\nabla}^i \hat{\nabla}^j s_a \Psi_a \right\} ,$$

$$5c) = -\frac{i}{4} K' \int d^2 x d\theta^+ \left\{ i m H_i^{jk} F_{jk}^{ab} \hat{\nabla}^i s_a \Psi_a \right\} ,$$

$$5d) = \frac{i}{2} J I \int d^2 x d\theta^+ \left\{ i m \hat{\nabla}^k F_{ki}^{ab} \hat{\nabla}^i s_a \Psi_a \right\} ,$$

$$5y) = \frac{i}{4\pi\epsilon} J \int d^2 x d\theta^+ \left\{ i m R^{(+)}(ij) \hat{\nabla}^i \hat{\nabla}^j s_a \Psi_a \right\} ,$$

$$5z) = \frac{i}{4\pi\epsilon} J \int d^2 x d\theta^+ \left\{ i m \hat{\nabla}^{(+)} k F_{ki}^{ab} \hat{\nabla}^i s_a \Psi_a \right\} .$$

These divergences can now be added up, using expressions for the divergent integrals in the appendix and (3.5) to give the total two loop contribution to the $\frac{1}{\epsilon}$ pole divergence. One finds that the $\frac{1}{\epsilon}$ terms in (5a,5b,5y) and (5c,5d,5z) completely
cancel leaving only $\frac{1}{e}$ poles. Hence

$$\Gamma^{(1,2)}_{\text{Div } m} = 0$$

and so we find the $\beta$-function $\beta^{(s)}_a$, calculated to two loop order, to be

$$\beta^{(s)}_a = s_a - \frac{\alpha'}{2} \hat{\nabla}^2 s_a + O(\alpha'^3).$$

Field Redefinitions

There is an inherent quantum mechanical ambiguity in the above calculation caused by the possible introduction of $O(\alpha')$ finite local counter terms, equivalent to a change in renormalization scheme. These terms have no effect at the one loop level, however the one loop diagrams constructed from them will alter the two loop $\beta$-functions. We will therefore end our discussion by considering the effect that the addition of such terms to the action (2.3) has on the $\beta$-functions found above. As discussed above the addition of finite local counter terms is need for the effect of the sigma model anomaly to be included in them case of chiral $(p,0)$ supersymmetry. In addition they are also need to preserve $(4,0)$ supersymmetry in perturbation theory and ensure that the off-shell $(4,0)$ supersymmetric sigma models are ultraviolet finite [38].

The addition of finite local counter terms is tantamount to making a redefinition of $g_{ij}, b_{ij}, A^a_{i \ b}$ and $s_a$ that appear in (2.3) to

$$g_{ij} \rightarrow g_{ij} + \alpha' \bar{g}_{ij},$$

$$b_{ij} \rightarrow b_{ij} + \alpha' \bar{b}_{ij},$$

$$A^a_{i \ b} \rightarrow A^a_{i \ b} + \alpha' A^a_{i \ b},$$

$$s_a \rightarrow s_a + \alpha' s_a.$$  \hspace{1cm} (5.6)

(5.6)

As the potential terms do not effect the other beta functions, it is of little interest here to consider $s_a$ redefinitions. Let us suppose then that only $g_{ij}, b_{ij}$ and $A^a_{i \ b}$
have been redefined as in (5.6). The connection $\Gamma^{(-)}_{jk}$ is shifted to

$$
\Gamma^{(-)i}_{jk} \rightarrow \Gamma^{(-)i}_{jk} + \alpha' \bar{\Gamma}^{i}_{jk} + O(\alpha'^2),
$$

where

$$
\bar{\Gamma}^{i}_{jk} = \frac{1}{2}(\nabla_j \bar{g}_{ik} + \nabla_k \bar{g}_{ij} - \nabla_i \bar{g}_{jk}) - \frac{3}{2} \nabla_{[k} \bar{b}_{ij]} ,
$$

and we use the original metric $g_{ij}$ to raise and lower indices. The Yang-Mills field strength $F_{ij}^{ab}$ is shifted to

$$
F_{ij}^{ab} \rightarrow F_{ij}^{ab} + \alpha' \hat{\nabla}_{[i} \bar{A}^{ab}_{j]} + O(\alpha'^2).
$$

A straightforward calculation shows that the two loop $\beta$-functions become (to $O(\alpha'^2)$)

$$
\beta^{(g)}_{ij} = \beta^{(g)}_{ij} + \alpha'^2 (\hat{\nabla}^{(-)}^{k} \Gamma_{k(ij)} - \nabla^{(-)}_{(i} \nabla^{(-)}_{j)} \bar{g}^{k}_{ij}) + \alpha'^2 \nabla^{(-)}_{(i} v_{j)},
$$

$$
\beta^{(b)}_{ij} = \beta^{(b)}_{ij} + \alpha'^2 (\hat{\nabla}^{(-)}^{k} \Gamma_{k[ij]} - \nabla^{(-)}_{[i} \nabla^{(-)}_{j]} \bar{g}^{k}_{ij}) + \alpha'^2 \nabla^{(-)}_{[i} v_{j]},
$$

$$
\beta^{(A)ab}_{i} = \beta^{(A)ab}_{i} + \frac{\alpha'^2}{2} \bar{\Gamma}_{ki} F^{ab} \bar{\Gamma}_{kij} - \frac{\alpha'^2}{2} (\bar{A}^{ka}_{c} F_{ij}^{cb} + \bar{A}^{kb}_{c} F_{ij}^{ac} + \hat{\nabla}^{k} \hat{\nabla}_{[k} \bar{A}^{ab}_{i]} )
$$

$$
+ \frac{\alpha'^2}{2} v^{k} \bar{F}^{ab}_{ki},
$$

$$
\beta^{(s)}_{a} = \beta^{(s)}_{a} + \frac{\alpha'^2}{2} (\hat{\nabla}^{k} \bar{A}^{ab}_{k} + \bar{A}^{b}_{ka} \hat{\nabla}^{k} s_{b}) + \frac{\alpha'^2}{2} v^{k} \hat{\nabla}^{k} s_{a},
$$

where $v_{k} = g^{ij} \Gamma_{kij}$ and $\hat{\nabla}^{(-)}$ is defined in (5.1).

We are also free to redefine the background fields $\Phi^{i}$ and $\Psi^{a}$. Indeed if we consider the diffeomorphism generated by the vector $v^{i}$, $\Phi^{i} \rightarrow \Phi^{i} + \alpha'^2 v^{i}$ accompanied by a gauge transformation with parameter $u^{a} = -\alpha'^2 v^{i} A^{a}_{i}$ then the $\beta$-functions

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are (to $O(\alpha'^2)$)

\[ \beta_{ij}^{(g)} = \beta_{ij}^{(g)} + \alpha'^2 (\nabla(-)^k \bar{\Gamma}_{k(ij)} - \nabla(-)^k \nabla^k \tilde{g}_{k(ij)}) , \]
\[ \beta_{ij}^{(b)} = \beta_{ij}^{(b)} + \alpha'^2 (\nabla(-)^k \bar{\Gamma}_{k[ij]} - \nabla(-)^k \nabla^k \tilde{g}_{k[ij]} ) , \]
\[ \beta_{i}^{(A)ab} = \beta_{i}^{(A)ab} + \frac{\alpha'^2}{2} \tilde{\Gamma}_{ki} F^{abjk} - \frac{\alpha'^2}{2} \left( \tilde{A}^{ka} c F^{cb}_{ij} + \tilde{A}^{kb} c F^{ac}_{ij} + \tilde{\nabla}^k \tilde{\nabla}_{k}[\tilde{A}^{ab}_{ij}] \right) , \]
\[ \beta_{a}^{(s)} = \beta_{a}^{(s)} + \frac{\alpha'^2}{2} (\tilde{\nabla}^k \tilde{A}^{ab}_{k} + \tilde{A}^{ka}_{k} b \tilde{\nabla}^k s_b) . \]

It is by the above procedure that the effect of the sigma model anomalies can be included by setting

\[ \bar{\Gamma}_{ijk} = \frac{3}{4} \Omega_{3ijk}(\omega(-)) - \frac{3}{4} \Omega_{3ijk}(A) , \]

where $\Omega_3$ is defined in (4.3).

6. Concluding Remarks

In this paper we calculated the two loop $\beta$-functions of the general massive $(p,q)$ supersymmetric sigma model using $(1,0)$ superfields. At two loops the $\beta$-functions are (without taking account of any potential sigma model anomalies)

\[ \beta_{ij}^{(g)} = \alpha' (R_{ij} - H_{ikl} H^{kl}_{j}) + \frac{\alpha'^2}{8} \left( R_{iklm}^{(+)} R_{j}^{(+)} R^{klm}_{j} - F^{ab}_{ik} F^{ab}_{kj} \right) , \]
\[ \beta_{ij}^{(b)} = -\alpha' \nabla^k H_{kj} , \]
\[ \beta_{i}^{(A)ab} = -\frac{\alpha'}{2} (\tilde{\nabla}^k F^{ab}_{ki} + H^{kij} F^{ab}_{kj}) , \]
\[ \beta_{a}^{(s)} = s_a - \frac{\alpha'}{2} \tilde{\nabla}^2 s_a . \]

These results are in agreement with previous calculations in special cases where various background fields vanish [17,18,19].

As is well known [1,13,14,15,16], when we restrict the model so that it possess (1,1) supersymmetry, the two loop divergences from $R_{ijkl}^{(+)}$ and $F_{ij}^{ab}$ in (6.1) cancel.
This can be seen by setting $A_i^{ab} = \omega_{i}^{(+)}ab$, so that the anomaly (4.3) vanishes (in fact it can be arranged, by the addition of finite local counter terms, that any spin connection differing from $\omega^{(+)}$ by covariant term may be used [37]). Furthermore, we have $F_{ij}^{ab} = E^{am}E^{bn}R_{ijmn}^{(+)}$ and can identify $(\eta^i_{+}, E^i_{a}\psi^a_{+})$ as the left and right handed components of a single spinor. It then clearly follows that the $\alpha'^2$ terms in (6.1) vanish. There will in general be higher loop divergences [36].

Ultra-violet Behaviour At All Orders

Here we would like here to discuss ultra-violet behaviour to all orders for the general off shell $(p,q)$ supersymmetric massive sigma model. As is well known theories with $N = 4$ supersymmetry are often finite as this places very strict conditions on the possible counter terms. Following the power counting argument of [20], we now show that off-shell $(4,q)$ supersymmetric massive sigma models are perturbatively ultra-violet finite to all orders, for $q \leq 4$. We will also see that there are no mass renormalizations to all orders of perturbation theory for the off-shell $(2,q)$ models.

For an off shell $(p,q)$ supersymmetric sigma model in two dimensions the superspace measure is

$$d^2x d^p\theta^+ d^q\theta^-.$$

This has Lorentz weight $p - q$ and mass dimension $\frac{1}{2}(p + q) - 2$. Thus any logarithmically divergent counter term must have mass dimension $2 - \frac{1}{2}(p + q)$ and Lorentz weight $q - p$. The possible divergences are of the form

$$\Gamma_{Div} \sim m^a\mathcal{O}(\partial, D_-, D_+)F(\Phi)(\Psi^+)^r(\Psi^-)^s,$$

where $\mathcal{O}$ is a differential operator and $F$ a scalar function. $F$ may contain derivatives of $\Phi$, so long as they are in in scalar combinations. Hence we may choose $F$ in such a way that it contains the maximum number of derivatives. In this case $\mathcal{O}$ is either of the form $\mathcal{O} \sim (D_+)^a$ or $\mathcal{O} \sim (D_-)^b$. 
Now the counter term $\Gamma_{D_{\text{iv}}}$ has Lorentz weight

$$[\Gamma_{D_{\text{iv}}}]_l = q - p = [O]_l + r - s ,$$

and mass dimension,

$$[\Gamma_{D_{\text{iv}}}]_m = 2 - \frac{1}{2}(p + q) = \alpha + [O]_m + [F]_m + \frac{1}{2}(r + s) .$$

However, all of $O$’s Lorentz weight must come from worldsheet derivatives, and our choice of $F$ implies that

$$[O]_m = \frac{1}{2}[O]_l .$$

Hence, the mass dimension of $F$ must satisfy

$$[F]_m = 2 - \alpha - \frac{1}{2}(p + q) - \frac{1}{2}(r + s) - [O]_m$$

$$= 2 - \alpha - \frac{1}{2}(p + q) - \frac{1}{2}(r + s) - \frac{1}{2}((p + r) - (q + s)) |$$

$$\leq 2 - \alpha - r - p ,$$

where we have used the inequality $|a - b| \geq |a| - |b|$. As $\Gamma_{D_{\text{iv}}}$ has a non negative degree of divergence and there are no negative mass dimensional constants, $[F]_m \geq 0$. Thus, for $p > 2$ there are no possible logarithmically divergent counter terms.

As $p = 3$ implies $p = 4$, we see that off-shell (4,q) supersymmetric sigma models are perturbatively ultra-violet finite to all orders.

A quick look at (6.1) appears to show a contradiction with this claim. There is a two loop contribution to the metric $\beta$-function which does not in general vanish in the case of chiral (4,0) supersymmetry. This problem has been recognized before and is resolved by the observation that (4,0) supersymmetry is broken in perturbation theory [38]. To remedy this requires the addition of finite local terms (ie. field redefinitions) at each order of perturbation theory to restore (4,0) supersymmetry. The metric and antisymmetric tensor fields of the finite theory are then not those that appear in (2.3) but differ by terms of higher order in $\alpha'$. In other words in an appropriate regularization scheme the (4,0) models are ultra-violet finite.
The inequality (6.2) has a further application to the case of massive (2,q) models. Here it follows that \( \alpha = 0 \) for any counter term and therefore there are no possible \( S_m \) counter terms. Thus there are no mass renormalizations to all orders of perturbation theory for the off-shell (2,q) models.

Conformal Invariance and String Theory

Finally we would like to make some comments concerning the relationship of massive supersymmetric sigma model to string theory. As is easily seen from the action (2.3) the presence of the mass term \( m s_a \Psi^a \) breaks conformal invariance at the classical level and this appears as the first term in \( \beta_a^{(s)} \). From the point of view of string theory the sigma model \( \beta \)-functions become equations of motion for the (bosonic) spacetime fields \( g_{ij}, b_{ij}, A^a_{\ i\ b} \) and \( \varphi \) [26]. Here \( \varphi \) is the dilaton which is absent from our calculations as it does not couple to a flat worldsheet and hence does not appear action (2.3). If we likewise consider \( s_a \) as a spacetime field the minus sign in \( \beta_a^{(s)} \) implies that it is a tachyon with \( (\text{mass})^2 \sim -1/\alpha' \). Although, since a tachyon does not appear in the spectrum of the superstring, it’s interpretation as a spacetime field is problematic. Usually in order to render the theory spacetime supersymmetric the GSO projection is performed which then removes the tachyon from the spectrum of the superstring. However, one may wish to consider string theories with worldsheet supersymmetry but no spacetime supersymmetry, in which case the tachyon would be in the physical spectrum.

The vanishing of the \( \beta \)-functions (4.8) only ensures that rigid scale invariance is preserved. As is discussed in [27,31,28] full conformal invariance of the sigma model occurs when the complete \( \beta \)-functions, including any additional contributions from the dilaton, vanish. The central charge is then given by the dilaton \( \beta \)-function, which is a constant by virtue of the Curci-Paffuti relation [27,31,28]. Furthermore the Curci-Paffuti relation enables the dilaton \( \beta \)-function, and it’s contributions to the other \( \beta \)-functions, to be calculated by flat worldsheet techniques [28,32]. Only then can the complete conditions for conformal invariance found and the central charge be made to vanish. Thus, even though the action (2.3) is not classically
conformally invariant it may be possible that the quantum theory is.

Any statement regarding a nontrivial solution of $\beta^{(s)}_a = 0$ (or its generalization to include the dilaton) must be made carefully as we are comparing terms of different order in $\alpha'$ and therefore we cannot assume that the loop expansion is meaningful. A special case is offered by massive linear sigma models for which $F_{ij}^{ab} = H_{ijk} = R_{ijkl} = \varphi = 0$. $\beta^{(s)}_a$ is then the only nonvanishing $\beta$-function. Since the superspace measure $d^2x d\theta^+$ has mass dimension $-\frac{3}{2}$ and each vertex carries a factor of $m$, the only divergent contributions to the effective action come from graphs with a single vertex. Of these however, only the one loop tadpole graph has a $\frac{1}{\epsilon}$ pole which contributes to the $\beta$-function. Thus $\beta^{(s)}_a$ in (4.8) is exact at order $\alpha'$ to all orders of perturbation theory and receives no other (perturbative) contributions.

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* It would be interesting to know if this were still true had we only assumed $F_{ij}^{ab} = R_{ijkl}^{(+)} = \varphi = 0$. Here an extension of the arguments in [39] based on the algorithm (3.10) again show that only $\beta^{(s)}_a$ is nonzero.
7. Appendix

In order to regulate the divergent integrals we use the conventions of [4,12]. The divergent integrals used above, after taking the limit $M^2 \to 0$ and ignoring any infrared divergences, are found by dimensional regularization to be

$$I = \int \frac{d^Dk}{i(2\pi)^D} \frac{1}{-k^2 + M^2} = -\frac{1}{2\pi\epsilon} + O(\epsilon) ,$$

$$J = \int \frac{d^Dk}{i(2\pi)^D} \frac{k_{\neq}k_{=}}{-k^2 + M^2} \frac{1}{-(p-k)^2 + M^2} = \frac{1}{4\pi\epsilon} + \frac{1}{4\pi} + O(\epsilon) ,$$

$$K = -K_1 + K_2 - K_3 + K_4 - K_5 + K_6 = \frac{1}{16\pi^2\epsilon^2} - \frac{9}{32\pi^2\epsilon} + O(\epsilon^0) ,$$

$$K' = K_3 - K_6 ,$$

$$L = \int \frac{d^Dq}{i(2\pi)^D} \int \frac{d^Dk}{i(2\pi)^D} \frac{k_{\neq}k_{=}}{-k^2 + M^2} \frac{1}{-q^2 + M^2} \frac{1}{-(p-k-q)^2 + M^2} = -\frac{1}{8\pi^2\epsilon^2} + O(\epsilon^0) ,$$

where
\[ K_1 = \int \frac{d^D q}{i(2\pi)^D} \int \frac{d^D k}{i(2\pi)^D} \left\{ \frac{k_{\neq}}{[-k^2 + M^2]} \frac{q_{=}}{[-q^2 + M^2]} \left( \frac{1}{[-(p - k - q)^2 + M^2]} \right) \right\} \]
\[ = \frac{1}{32\pi^2 \epsilon} + O(\epsilon^0) , \]
\[ K_2 = \int \frac{d^D q}{i(2\pi)^D} \int \frac{d^D k}{i(2\pi)^D} \left\{ \frac{k_{\neq}}{[-k^2 + M^2]} \frac{q_{=}}{[-q^2 + M^2]} \left( \frac{1}{[-(p - k - q)^2 + M^2]} \right) \right\} \]
\[ = \frac{1}{16\pi^2 \epsilon^2} + \frac{1}{32\pi^2 \epsilon} + O(\epsilon^0) , \]
\[ K_3 = \int \frac{d^D q}{i(2\pi)^D} \int \frac{d^D k}{i(2\pi)^D} \left\{ \frac{k_{\neq}}{[-k^2 + M^2]} \frac{q_{\neq}}{[-q^2 + M^2]} \left( \frac{1}{[-(p - k - q)^2 + M^2]} \right) \right\} \]
\[ = \frac{1}{16\pi^2 \epsilon^2} + \frac{1}{8\pi^2 \epsilon} + O(\epsilon^0) , \]
\[ K_4 = \int \frac{d^D q}{i(2\pi)^D} \int \frac{d^D k}{i(2\pi)^D} \left\{ \frac{k_{\neq}}{[-k^2 + M^2]} \frac{q_{\neq}}{[-q^2 + M^2]} \left( \frac{1}{[-(p - k - q)^2 + M^2]} \right) \right\} \]
\[ = O(\epsilon^0) , \]
\[ K_5 = \int \frac{d^D q}{i(2\pi)^D} \int \frac{d^D k}{i(2\pi)^D} \left\{ \frac{k_{\neq}}{[-k^2 + M^2]} \frac{q_{=}}{[-q^2 + M^2]} \left( \frac{1}{[-(p - k - q)^2 + M^2]} \right) \right\} \]
\[ = \frac{1}{32\pi^2 \epsilon} + O(\epsilon^0) , \]
\[ K_6 = \int \frac{d^D q}{i(2\pi)^D} \int \frac{d^D k}{i(2\pi)^D} \left\{ \frac{k_{\neq}}{[-k^2 + M^2]} \frac{q_{=}}{[-q^2 + M^2]} \left( \frac{1}{[-(p - k - q)^2 + M^2]} \right) \right\} \]
\[ = -\frac{1}{16\pi^2 \epsilon^2} + \frac{1}{8\pi^2 \epsilon} + O(\epsilon^0) . \]
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Figure 1: contributions to $\Gamma_{Div}^{(1,1)}$

Figure 2: 1 loop contributions to $V_{eff}$

Figure 3: contributions to $\Gamma_{Div}^{(1,2)}$
Figure 4: contributions to $\Gamma_{\text{Div}}^{(1,2)}$
Figure 5: contributions to $\Gamma_{\text{Div} \ m}^{(1,2)}$