Preconditioned SOR Method to Solve Time-Fractional Diffusion Equations

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Abstract. This paper examines the performance of the Preconditioned SOR (PSOR) method together with an unconditionally implicit Caputo’s time–fractional finite difference approximation equation for solving time-fractional partial diffusion equations (TFPDE’s). To do it, the implicit Caputo’s time-fractional approximation equations and preconditioned matrix are used to construct the corresponding preconditioned linear system. In addition to that, formulation and implementation the PSOR iterative method are also presented. Based on numerical results of the proposed iterative method, it can be concluded that the proposed iterative method is superior to the basic iterative methods.

1. Introduction

Based on previous studies in [1,2] many model in mathematical use fractional partial differential equations (FPDEs) to solve fractional problems such as time-fractional partial diffusion equations (TFPDE’s). Following to that, there are several methods used to solve these models. For instance, we have transform method [3], which is used to obtain analytical and/or numerical solutions of the fractional diffusion equations (FDE’s). Other than this method, other researchers have proposed finite difference methods such as explicit and implicit [4,5,6]. Also it is pointed out that the explicit iterative methods are conditionally stable. Therefore, to solve the problems of the TFPDE’s needs to be discretized. By using the implicit finite difference scheme and Caputo fractional operator, a linear system at each time level can be construct through the approximation equations. Therefore, Due to the matrix properties of the linear system, iterative methods are the alternative option for efficient solutions.

As far as iterative methods are concerned, it can be observed that many researchers such as Guang-hui [7], Young [8], Hackbusch [9] then Saad [10] have proposed and discussed several families of iterative methods. Among the existing proposed methods, the preconditioned iterative methods (Zhao [14], Hoang-hao [15], Gunawardena [16], Saad [10]) have been widely accepted to be one of the efficient methods for solving linear systems.

Because of the advantages of these iterative methods, the aim of this study is to develop and investigate the performance of the PSOR method to solve (TFPDE’s) based on the implicit Caputos’s time-fractional finite difference approximation equation. To investigate the performance of
the PSOR, we also applying the Gauss-Seidel (GS) and Preconditioned Gauss-Seidel (PGS) as control methods.

To demonstrate the performance of PSOR, let TFPDE’s be given as
\[ \frac{\partial^{\alpha}Z(y,r)}{\partial^{\alpha}r} = a(y)\frac{\partial^{2}Z(y,r)}{\partial y^{2}} + b(y)\frac{\partial Z(y,r)}{\partial y} + c(y)Z(y,r) \] (1)
where \(a(y), b(y)\) and \(c(y)\) are clear functions or fixed, whereas the parameter \(\alpha\) refers to the fractional order TFPDE’s derivative.

The organization of this paper as follows: Section 2, contained some basic definition for fractional derivative operator, included Caputo’s, then in section 3, contained an explain about approximation for fractional diffusion equations, for section 4 of this paper will discuss about how to construct of the PSOR method. Besides that analysis of numerical problem is mentioned in section 5 and the concluding remarks are given in final section.

2. Preliminaries

Previous to construct the discrete equation of Problem (1), in this section given some definitions can be applied for fractional derivative theory

**Definition 1.**[8] The Riemann-Liouville operator is defined as
\[ J^\alpha f(y) = \frac{1}{\Gamma(\alpha)} \int_{0}^{y} (y-r)^{\alpha-1} f(r)dr, \alpha > 0, y > 0 \] (2)

**Definition 2.**[8] The Caputo’s operator is defined as
\[ D^\alpha f(y) = \frac{1}{\Gamma(m-\alpha)} \int_{0}^{y} f^{(m)}(r) \frac{\Gamma(m)}{(y-r)^{\alpha+m-1}} dr, \alpha > 0 \] (3)
with \(m-1 < \alpha \leq m, m \in \mathbb{N}, y > 0\)

To obtain the numerical solution of Problem (1) with Dirichlet boundary conditions, we get numerical approximations by using the Caputo’s derivative definition and consider the non-local fractional TFPDE’s derivative operator. This approximation equation can be categorized as unconditionally stable scheme. On strength of Problem (1), the solution domain of the problem has been restricted to the finite space domain \(0 \leq y \leq \gamma\) and \(0 < \alpha < 1\), whereas \(\alpha\) refers to the fractional order of TFPDE’s derivative.

. In addition to that, consider Initial and boundary (conditions) of Eq (1) be given as
\[ Z(x,0) = f(y), Z(0,r) = g_0(r), Z(\ell,r) = g_1(r), \]
with \(g_0(r), g_1(r)\) and \(f(y)\) are given clear functions. A discretize approximation to TFPDE’s Eq. (1) by using Caputo’s order-\(\alpha\), is given as \([10,11]\)
\[ \frac{\partial^{\alpha}Z(x,r)}{\partial r^{\alpha}} = \frac{1}{\Gamma(n-1)} \int_{0}^{x} \frac{\partial u(y-s)}{\partial r} (r-s)^{-\alpha} ds, r > 0, 0 < \alpha < 1 \] (4)

3. Approximation For Fractional Diffusion Equation

According to Eq. (4), the formulation of Caputo’s fractional TFPDE’s derivative of the approximation method is given as
\[ D^{\alpha}_t Z_{i,n} \approx \sum_{j=1}^{n} \alpha_0^{(\alpha)} \left(Z_{i,n-j+1} - Z_{i,n-j} \right) \] (5)
where, $\sigma_{\alpha,k} = \frac{1}{(1-\alpha)(1-\alpha)k^{\alpha}}$ and $\omega_{j}^{(\alpha)} = j^{1-\alpha} - (j-1)^{1-\alpha}$.

Before discretizing Eq (1), let the solution domain of the problem be partitioned uniformly. To do this, we consider $m$ and $n$ (positive integers) in which the grid sizes (space and time) directions for the finite difference TFPDE’s algorithm be given as $h = \Delta x = \frac{\gamma - 0}{m}$ and $k = \Delta t = \frac{T}{n}$. Based on these grid sizes, we construct the uniformly grid network of the solution domain with the grid points in the space interval $[0,\gamma)$ are shown as the numbers $x_i = ih, i = 0,1,2,\ldots,m$ and the grid points in the time interval $[0,T]$ are labeled $t_j = jk, j = 0,1,2,\ldots,n$ . Then the values of the function $Z(x,t)$ at the grid points are denoted as $Z_{i,j} = Z(x_i,t_j)$. As mentioned Eq. (5) and the implicit TFPDE’s finite difference discretization scheme , the implicit Caputo’s finite difference approximation equation of Eq. (1) to the grid point centered at $(y_j,r_j) = (ih,nk)$ is defined as

$$\sigma_{\alpha,k} \sum_{j=1}^{n} \omega_{j}^{(\alpha)} (Z_{i,n-1-j} - Z_{i,n-j}) = a_i \frac{1}{h^2} (Z_{i-1,n} - 2Z_{i,n} + Z_{i+1,n}) + b_i \frac{1}{2h} (Z_{i+1,n} - Z_{i-1,n}) + c_i Z_{i,n},$$

with $i=1,2,\ldots,m-1$. Based on Equation (6), this approximation equation TFPDE’s is known as the fully implicit Caputo’s finite difference approximation which is consistent first order accuracy (time) and second order (space). For simplicity, Eq. (6) for $n \geq 2$ can be rewritten as

$$\sigma_{\alpha,k} \sum_{j=1}^{n} \omega_{j}^{(\alpha)} (Z_{i,n-1-j} - Z_{i,n-j}) = p_i Z_{i-1,n} + q_i Z_{i,n} + r_i Z_{i+1,n},$$

with $p_i = a_i \frac{1}{h^2} - \frac{b_i}{2h}, q_i = c_i - \frac{2a_i}{h^2}, r_i = a_i \frac{1}{h^2} + \frac{b_i}{2h}.$

Also, we get for $i=1,2,\ldots,m-1$ and with $\omega_{j}^{(\alpha)} = 1$,

$$p_i Z_{i-1,n} + q_i Z_{i,n} + r_i Z_{i+1,n} = f_{i-1},$$

for $i=1,2,\ldots,m-1$ and with $\omega_{j}^{(\alpha)} = 1, q_{i}^{*} = \sigma_{\alpha,k} - q_{j}, f_{i-1}^{*} = \sigma_{\alpha,k} Z_{i-1}.$

Again Eq. (7.b) can be expressed in a matrix form as

$$AZ = f$$

with

$$A = \begin{bmatrix} -p_2 & q_2 & \cdots & -r_2 \\ -p_3 & q_3 & \cdots & -r_3 \\ \vdots & \vdots & \ddots & \vdots \\ -p_m & q_m & \cdots & -r_m \\ \end{bmatrix},$$

$$Z = \begin{bmatrix} Z_{11} & Z_{21} & \cdots & Z_{m-1,1} \\ Z_{12} & Z_{22} & \cdots & Z_{m-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{1,m-1} & Z_{2,m-1} & \cdots & Z_{m-1,m-1} \\ \end{bmatrix}^T,$$

and $f = \begin{bmatrix} Z_{11} + p_0 Z_{01} \\ Z_{21} + Z_{21} \\ \vdots \\ Z_{m-1,1} + p_{m-1} Z_{m-1,1} \\ \end{bmatrix}^T$.

4. Preconditioned SOR Method

In relation to the linear system in Equation (8), it is clear that the characteristics of its coefficient matrix are large scale and sparse. Basically in first Section, many scientist have discussed various iteration methods such as Zhao [14], Hoang-hao [15], Gunawardena [16], Young [8], Hackbusch [9], Saad [10], Yousif and Evans [13]. To get numerical solving of the tridiagonal linear system (8), we
proposed the Preconditioned Successive Over Relaxation (PSOR) iteration method [14, 15, 16], to solve any linear systems.

Before applying the PSOR iteration method, we need to transform the original linear system (8) into the preconditioned linear system

\[ A^* Z = f^* \]  

where, \( A^* = PAP^T, f^* = Pf, Z = P^Ty \).

Actually, the matrix \( P \) is called a preconditioned matrix and defined as [17]

\[ P = I + S \]  

where

\[
S = \begin{bmatrix}
0 & -r_1 & 0 & 0 & 0 \\
0 & 0 & -r_2 & 0 & 0 \\
0 & 0 & 0 & -r_3 & 0 \\
0 & 0 & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & -r_{m-1} \\
\end{bmatrix}
\]

and the matrix \( I \) is an identical matrix. To develop PSOR iteration method, let consider the coefficient \( A^* \) in equation (8) can be defined as summation of the matrices

\[ A^* = Di - Lo - Va \]  

with \( Di, Lo \) and \( Va \) are diagonal, lower triangular and upper triangular matrices. By using Eq. (9) and (11), the formulation of PSOR iteration method can be rewritten as [14,15,16,17]

\[
y^{(k+1)} = (D - \omega L)^{-1} \left[ (1 - \omega)D + V \omega \right] y^{(k)} + (D - \omega L)^{-1} f^* 
\]

where \( y^{(k+1)} \) represents an unknown vector at \((k+1)\)th iteration. The application of the PSOR iteration method can be explained in this below.

**PSOR method**

i. First \( \tilde{U} \leftarrow 0 \) and \( \varepsilon \leftarrow 10^{-10} \).

ii. Then \( j = 1, 2, \ldots, n \) implement

a. with \( i = 1, 2, \ldots, m-1 \) calculate

\[
y^{(i+1)} = (D - \omega L)^{-1} \left[ (1 - \omega)D + V \omega \right] y^{(i)} + (D - \omega L)^{-1} f^* \\
Z^{(i+1)} = P^T y^{(i+1)}
\]

b. Next to the convergence test. If the convergence criterion i.e. \( \| y^{(i+1)} - y^{(i)} \| \leq \varepsilon = 10^{-10} \) is satisfied, go to Step (iii). If not go back to Step (a).

iii. Display approximate equation solutions.

5. **Evaluation of Numerical Problem**

By using approximation Eq. (7), we consider one problem of the TFPDE’s to test the performance of the Gauss-Seidel (GS), Preconditioned Gauss Seidel (PGS) and Preconditioned Successive Over-Relaxation (PSOR) iteration methods. In order to compare the performance of these proposed iteration methods, three criteria have been considered such as K (number of iterations), Time (in seconds) and Max Error (maximum absolute error) at three values, where value of \( \alpha = 0.25, \alpha = 0.50 \) and \( \alpha = 0.75 \). For application of three iterative schemes, the convergence test considered the tolerance error as \( \varepsilon = 10^{-10} \).

Let us examine the TFPDE’s initial boundary conditions value problem [18]
\[
\frac{\partial^\alpha Z(y,r)}{\partial r^\alpha} = \frac{\partial^2 Z(y,r)}{\partial y^2}, \quad 0 < \alpha \leq 1, 0 \leq y \leq \gamma, r > 0,
\]  
(13)

with the boundary value conditions are \[Z(0,r) = \frac{2kr^\alpha}{\Gamma(\alpha+1)}, \quad Z(t,r) = t^2 + \frac{2kr^\alpha}{\Gamma(\alpha+1)},\]  
(14)

and the initial value condition \[Z(y,0) = y^2.\]  
(15)

Overall results of evolution of numerical problem for equation (13), obtained from application of GS, PGS and PSOR iteration methods are recorded in Table 1, where value of mesh sizes, \(m = 128, 256, 512, 1024,\) and \(2048.\)

6. Conclusion

In order to get the numerical solution of the TFPDE’s problems equation, this study give the derivation of the implicit Caputo’s finite difference approximation equations in which this approximation equation leads a tridiagonal linear system. Via all experimental results by imposing the GS, PGS and PSOR iteration methods, it is clear at \(\alpha = 0.25\) that \(K\) (number of iterations) have declined approximately by 64.27-96.14% conforms to the PSOR iteration method compared with the GS and PGS methods. Then for Time, application of PSOR iteration method are much faster about 25.84-94.18% than the GS and PGS methods. It can be also observed in Table 1 that the PSOR method requires the least amount for \(K\) (number of iterations) and Time at \(\alpha = 0.25\) as compared with Gauss Seidel (GS) and PGS iteration methods. According to the accuracy of both iteration methods, it can be concluded that their numerical solutions of TFPDE’s are in good agreement.

7. References

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**Table 1.** Comparison of K (number iterations), Time ( seconds) and Max Error (maximum errors) for the iteration methods using example at $\alpha = 0.25, 0.50, 0.75$

| M   | Method | $\alpha = 0.25$ |       |       | $\alpha = 0.50$ |       |       | $\alpha = 0.75$ |       |
|-----|--------|----------------|-------|-------|----------------|-------|-------|----------------|-------|
|     |        | K       | Time  | Max Error | K       | Time  | Max Error | K       | Time  | Max Error |
| 128 | GS     | 21017   | 37.73 | 9.97e-05  | 13601   | 5.92  | 9.86e-05  | 6695    | 2.94  | 1.30e-04  |
|     | PGS    | 7292    | 35.86 | 9.96e-05  | 4715    | 2.23  | 9.84e-05  | 2319    | 1.93  | 1.30e-04  |
|     | PSOR   | 281     | 2.24  | 9.95e-05  | 229     | 1.95  | 9.84e-05  | 164     | 1.63  | 1.29e-04  |
| 256 | GS     | 77231   | 343.63| 1.00e-04  | 50095   | 42.17 | 9.90e-05  | 24732   | 20.70 | 1.30e-04  |
|     | PGS    | 26884   | 261.56| 9.98e-05  | 17417   | 16.68 | 9.87e-05  | 8585    | 12.37 | 1.30e-04  |
|     | PSOR   | 1428    | 16.9  | 9.95e05   | 1171    | 12.61 | 9.84e-05  | 814     | 8.90  | 1.29e-04  |
| 512 | GS     | 281598  | 2747.34| 1.02e-04 | 183181  | 339.85| 1.01e-04 | 90783   | 166.75| 1.32e-04 |
|     | PGS    | 98422   | 1916.28| 1.00e-04 | 63298   | 123.01| 9.96e-05 | 31619   | 62.78 | 1.31e-04 |
|     | PSOR   | 5524    | 113.86| 9.96e05   | 4520    | 91.37 | 9.84e-05 | 3137    | 61.98 | 1.30e-04 |
| 1024| GS     | 1017140 | 68285.36| 1.09e-04| 683971  | 2454.53| 1.08e-05| 330622  | 1209.39| 1.40e-04|
|     | PGS    | 357258  | 14064.44| 1.04e-04| 232784  | 1007.47| 1.03e-05| 115617  | 820.93 | 1.35e-04|
|     | PSOR   | 20574   | 817.59 | 9.98e-05 | 18842   | 662.23| 9.87e-05| 11695   | 456.23 | 1.30e-04|
| 2048| GS     | 3631638 | 58914.30| 1.38e-04| 2380896 | 17795.25| 1.38e-04| 1192528 | 8794.26| 1.71e-04|
|     | PGS    | 1183293 | 4104.17 | 1.36e-04| 1150153 | 3239.84 | 134e-05 | 362784  | 1305.50| 1.35e-04|
|     | PSOR   | 7558    | 3043.59| 1.01e-04 | 61941   | 2894.7 | 9.96e-05 | 4307    | 977.10 | 1.32e-04|