Tests of flavour independence in heavy quark potential models

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Abstract

We review some rigorous consequences of flavour independence on the spectrum and properties of hadrons in potential models, with emphasis on hadrons with two heavy quarks, such as \((b\bar{c})\) mesons and \((QQq)\) baryons.

1 Introduction

Potential models are rather successful in hadron spectroscopy. Once the spectrum of known states is reproduced with a few parameters, some predictions can be made. For instance, the location of the P-states of charmonium or bottomonium has been guessed from the lowest S-states and their radial excitations; the mass of the \(\Lambda_b\) (bud) baryon was predicted near 5.6 GeV in any reasonable potential model, while a much larger mass was sometimes computed in other approaches; a recent survey of realistic potentials gives the mass of the lowest \((b\bar{c})\) state with an uncertainty as small as \(\pm 20\text{ MeV}\); etc.

In the light-quark sector, the success of potential models is a little accidental, though one can argue that the Schrödinger equation generates a spectrum which is a very regular function of the constituent masses and thus mimics the regularities of the genuine QCD spectrum in flavour space.

For heavy quarkonia \((Q\overline{Q'})\) or heavy baryons \((QQ'q)\) with two heavy quarks, there is a deeper reason. The true picture results in a potential for describing the relative motion of the two heavy quarks, once the gluon and light-quark degrees of freedom are integrated out. This is similar to the Born–Oppenheimer treatment of the inter-nuclear motion in molecular physics. The potential between the heavy quarks has first been guessed in phenomenological works, and then derived from more fundamental studies.

A key property of QCD is flavour independence. The gluons are coupled to the colour of the quark independently of its isospin, hypercharge, or charm. This means
the potential is the same whatever quarks experience it, at least before any relativistic correction is included. This has motivated studies on how the 2-body or 3-body Schrödinger bound states evolve when the constituent masses vary in a given potential. Some results will be summarized below. Amazingly, these studies have found applications in atomic physics, where we have a similar situation, namely the very same Coulomb potential binding masses as different as $e^+, \mu^+$ or $\pi$.

2 Some early results on flavour independence

2.1 Gap to the OZI-allowed threshold

In the simplest model, mesons are given by the Hamiltonian

$$H(\alpha) = \alpha p^2 + V(r),$$

where $\alpha$ is half the inverse reduced mass, and $V$ is universal, i.e., independent of $\alpha$. The flavoured mesons ($c\bar{q}$) and ($b\bar{q}$) have nearly the same $\alpha \simeq 1/(2m_q)$, and thus the same energy. On the other hand,

$$\alpha(b\bar{b}) < \alpha(b\bar{c}) < \alpha(c\bar{c}),$$

and since every level of $H(\alpha)$ has an energy which increases with $\alpha$, we obtain the hierarchy [1, 2]

$$G(b\bar{b}) < G(b\bar{c}) < G(c\bar{c}),$$

where

$$G(Q\bar{Q}) = (Q\bar{q}) + (\bar{Q}q) - (Q\bar{Q})$$

is the gap between the lowest quarkonium state ($Q\bar{Q}$) and its Zweig-allowed threshold. Experimentally, $G(c\bar{c}) \simeq 0.7$ GeV, and $G(b\bar{b}) \simeq 1.1$ GeV.

2.2 Quark-mass differences

The above considerations deal with the ground state of various quark-antiquark configurations. Explicit models have shown that the excitation spectrum of both ($c\bar{c}$) and ($b\bar{b}$) can be simultaneously reproduced by the same potential. This is a convincing illustration of flavour independence. In such studies, the quark masses are free parameters. There is some freedom in fixing their value, but it turns out that quark-mass differences such as ($m_b - m_c$) are rather well constrained by the data. Shifting both $m_b$ and $m_c$ up or down mostly results in changing the size of the wave functions. Leptonic or radiative widths can thus be used to choose an optimal set of values. Rigourous bounds on ($m_b - m_c$) from the spectrum have been studied by Bertlmann and Martin [3].
2.3 $SU(3)_F$ breaking for baryon masses

If one dares at applying non-relativistic models to baryons consisting of light quarks, the flavour-independent character of the central potential provides convexity relations such as

\[(gqq) + (qss) < 2(qqs)\]
\[(s qq) + ( s q s) < 2(sqs)\] (5)

Hyperfine corrections contain $1/(m_i m_j)$ factors which makes them larger for $q$ than for $s$ quarks. This compensates the above inequalities in specific combinations, and one eventually obtains a simple understanding of the famous Gell-Mann–Okubo formula and “equal-spacing” rule of decuplet [4]

\[2(N + \Xi) \simeq 3\Lambda + \Sigma\]
\[\Omega - \Xi^* \simeq \Xi^* - \Sigma^* \simeq \Sigma^* - \Delta\] (6)

2.4 Mass of $\Lambda_b(bdu)$

Consider the Hamiltonian

\[H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{p_3^2}{2m_3} + V(r_1, r_2, r_3),\] (7)

where $V$ is a given operator, not necessarily pairwise (one could even accept here some relativized form of the kinetic energy for the quarks 2 and 3). The lowest binding energy is an increasing and concave function of the inverse mass $m_i^{-1}$. This leads to an upper bound on the binding of $\Lambda_b(bdu)$, when extrapolated from $\Lambda(sud)$ and $\Lambda_c(cud)$. The inequality involves quark masses, which are not observable, but cannot be varied beyond a limited range. The study can be refined to accommodate spin–spin forces, and supplemented by a lower bound on $\Lambda_b$ in terms of meson masses, if one assumes a certain relation between the quarkonium and the baryon potentials, on which more shortly. The mass of $\Lambda_b$ is eventually constrained in a rather narrow window near 5.6 GeV, and explicit estimates are indeed, clustered around this value [5].

3 Applications to $(b\bar{c})$

At first sight, one expects the lowest $(b\bar{c})$ meson approximately half between $J/\Psi$ and $\Upsilon$. In a flavour-independent potential, this is in fact a lower bound [3], i.e., we have

\[2(b\bar{c}) \geq (c\bar{c}) + (b\bar{b}).\] (8)

If one knows the excitation spectrum of $(c\bar{c})$ and $(b\bar{b})$, one can extract model-independent bounds on the average kinetic energy in the ground state, which governs the evolution
of the ground-state energy when the reduced mass varies. This leads to an upper bound on the lowest \((b\bar{c})\) state \([6]\),

\[
(b\bar{c}) \leq \frac{(c\bar{c}) + (b\bar{b})}{2} - \frac{9}{8} \delta E(c\bar{c}) \left[ 1 - \left( \frac{m_b + m_c}{2m_b} \right)^{1/3} \right] + 2\delta E(b\bar{b}) \left[ \left( \frac{m_b + m_c}{2m_c} \right)^{1/3} - 1 \right],
\]

(9)

where \(\delta E\) denotes the orbital excitation energy, \(E(\ell = 1) - E(\ell = 0)\). In summary,

\[
6.26 \leq (b\bar{c}) \leq 6.43 \text{ GeV/}c^2,
\]

(10)

for the spin-averaged ground state, and, indeed, all predictions of realistic potentials cluster near 6.3 GeV/\(c^2\) \([7]\), in between the lower and the upper bounds provided by flavour independence. With hyperfine corrections, one obtains typically 6.26 GeV/\(c^2\) for the pseudoscalar, and 6.33 GeV/\(c^2\) for the vector \([7]\).

4 Baryons with two heavy quarks

Regularity patterns similar to those of mesons are expected in the baryon sector (the mathematics of the 3-body problem is of course more delicate than that of the 2-body one, and sometimes requires some mild conditions on the shape of the confining potential, which are satisfied by all current models \([8]\)). For instance, one expects an analogue of \((8)\)

\[
2(cqq) \geq (ccq) + (qqq)
\]

(11)

which leads to an upper bound \((ccq) \leq 3.7\text{ GeV}\) for the centre of gravity of the ground-state multiplet of \((ccq)\). A upper bound can also be derived for \((ccs)\). On the other hand, the convexity relation

\[
2(bcq) \geq (ccq) + (bbq),
\]

(12)

cannot be tested immediately, as well as the even more exotic-looking \([9]\)

\[
3(bcq) \geq (bbb) + (ccc) + (qqq),
\]

(13)

and its analogue with \(q \to s\). Of more immediate use is the relation

\[
(bcq) \geq (bqq) + (cqq) - (qqq),
\]

(14)

which leads to a rough lower bound \((bcq) \geq 6.9\text{ GeV/}c^2\), if one inputs the following rounded and spin-averaged values: \((bqq) = 5.6, (cqq) = 2.4\), and \((qqq) = 1.1\text{ GeV/}c^2\).

To derive these inequalities, one uses the Schrödinger equation, even for the light quarks. Very likely, the regularities exhibited by flavour-independent potentials also
hold in more rigorous QCD calculations and in the experimental spectrum. Any failure of the above inequalities would be very intriguing.

Sometimes, one can be more precise, and derive inequalities that include spin–spin corrections, for instance relations between $J^P = (1/2)^+$ baryons with different flavour content. See [8] for details.

Another mathematical game triggered by potential models consists of writing inequalities among meson and baryon masses. The basic relation is [8]

$$2(q_1q_2q_3) \geq (q_1 \bar{q}_2) + (q_2 \bar{q}_3) + (q_3 \bar{q}_1), \quad (15)$$

obtained by assuming that the potential energy operators fulfill the following inequality

$$2V_{qqq}(r_1, r_2, r_3) \geq \sum_{i<j} V_{qq}(|r_i - r_j|), \quad (16)$$

which holds (with equality) for a colour-octet exchange, in particular one-gluon exchange, and for the simple model

$$V_{qq}(r) = \lambda r, \quad V_{qqq} = \lambda \min_j (d_1 + d_2 + d_3) \quad (17)$$

where $d_i$ is the distance from the $i$-th quark to a junction $J$ whose location is adjusted to minimize $V_{qqq}$ [10]. We already mentioned possible applications to $\Lambda$. In the double-charm sector, we obtain [11] $(ccq) \geq 3.45$ GeV/$c^2$ for the $(1/2)^+$ state. This is rather crude, not surprisingly. Years ago, Hall and Post [12] pointed out in a different context that the pairs are not at rest in a 3-body bound state, and that their collective kinetic energy is neglected in inequalities of type (15).

Computing the $(QQq)$ energies in a given potential model does not raise any particular difficulty. The 3-body problem is routinely solved by means of the Faddeev equations or variational methods. On the other hand, successful approximations often shed some light on the dynamics. In particular, the Born–Oppenheimer method works very well for large ratios ($M/m$) of the quark masses. At fixed $QQ$ separation $R$, one solves the 2-centre problem for the light quark $q$. The energy of $q$ is added to the direct $QQ$ interaction to generate the effective potential $V_{QQ}(R)$ in which the heavy quarks evolve. One then computes the $QQ$ energy and wave function. Note that one can remove the centre-of-mass motion exactly, and also estimate the hyperfine corrections.

The physics behind the Born–Oppenheimer approximation is rather simple. As the heavy quarks move slowly, the light degrees of freedom readjust themselves to their lowest configuration (or stay in the same $n$-th excitation, more generally). At this point, there is no basic difference with quarkonium. The $QQ$ potential does not represent an elementary process. It can be viewed as the effective interaction generated by the gluon field being in its ground-state, for a given $Q\bar{Q}$ separation.

The results shown in Table 1 come from the simple potential

$$V = \frac{1}{2} \sum_{i<j} \left[ A + Br_{ij}^\beta + \frac{C}{m_i m_j} \sigma_i \cdot \sigma_j \delta^{(3)}(r_{ij}) \right], \quad (18)$$
with parameters $\beta = 0.1$, $A = -8.337$, $B = 6.9923$, $C = 2.572$, in units of appropriate powers of GeV. The quark masses are $m_q = 0.300$, $m_s = 0.600$, $m_c = 1.905$ and $m_b = 5.290$ GeV. The $1/2$ factor is a pure convention, although reminiscent from the discussion of inequalities (15) and (16). The smooth central term can be seen as a handy interpolation between the short-range Coulomb regime modified by asymptotic-freedom corrections and an elusive linear regime screened by pair-creation effects. The spin-spin term is treated at first order to estimate $M_0$. This model fits all known ground-state baryons with at most one heavy quark.

Table 1: Masses, in GeV, of $(QQq)$ baryons in a simple potential model. We show the spin-averaged mass $\overline{M}$, and the mass $M_0$ of the lowest state with $J^P = (1/2)^+$. 

| State | ccq | ccs | bcq | bcs | bbq | bbs |
|-------|-----|-----|-----|-----|-----|-----|
| $\overline{M}$ | 3.70 | 3.80 | 6.99 | 7.07 | 10.24 | 10.30 |
| $M_0$ | 3.63 | 3.72 | 6.93 | 7.00 | 10.21 | 10.27 |

A more conventional Coulomb-plus-linear potential was used in Ref. [11], with similar results. One remains, however, far from the large number of models available for $(b\bar{c})$ [11], and the non-relativistic treatment of the light quark might induce systematic errors. The uncertainty is then conservatively estimated to be $\pm 50$ MeV, as compared to $\pm 20$ MeV for $(b\bar{c})$. Note also that the $b$-quark mass $m_b$ is tuned to reproduce the experimental mass of $\Lambda_b$ at 5.62 GeV/$c^2$, and this latter value is not firmly established.

The Born–Oppenheimer framework leaves room for improvements. A relativistic treatment of the light quark was attempted in [11], using the bag model. For any given $QQ$ separation, a bag is constructed in which the light quark moves. The shape of the bag is adjusted to minimize the energy. In practice, a spherical approximation is used, so that the radius is the only varying quantity. The energy of the bag and light quark is interpreted as the effective $QQ$ potential. Unlike the rigid MIT cavity, we have a self-adjusting bag, which follows the $QQ$ motion. Again, this is very similar to the bag model picture of charmonium [13].

Unfortunately, there are variants in the bag model, with different values of the parameters, and with or without corrections for the centre-of-mass motion. These variants lead to rather different values for the $(ccq)$ masses [11]. This contrasts with the clustered shoots of potentials models, and deprives the bag model of predictive power in this sector of hadron spectroscopy.

It is hoped that the $QQ$ potential will be calculated by lattice or sum-rule methods.

The excitation spectrum of $(QQq)$ baryons has never been calculated in great detail, at least to our knowledge. In Ref. [11], an estimate is provided for the spin excitation (ground state with $J^P = (3/2)^+$), the lowest negative-parity level, and the radial excitation of the ground state.

The spin excitation is typically 100 MeV above the ground state, and thus should decay radiatively, with an $M1$ transition. The orbital and radial excitations of $(ccq)$
are unstable, since they can emit a pion. The radial excitation of \((ccs)\) can decay into \((ccq) + K\), but the orbital excitation cannot, and thus should be rather narrow, since restricted to \((ccs) + \gamma\), or to the isospin-violating \((ccs) + \pi^0\).

5 Exotic hadrons?

There are several types of multiquarks in the literature. Jaffe’s \(H\) dibaryon with strangeness \(S = -2\), \((ssuudd)\), or the “Pentaquark”, \(((csuud, for instance)\) are tentatively bound by chromomagnetic forces, while the “Tetraquark” uses a combination of flavour-independent chromoelectric forces, and Yukawa-type of long range forces.

This latter contribution was pointed out by Törnqvist [14] and Manohar and Wise[15], who studied pion-exchange between heavy mesons, and stressed that, among others, some \(DD^*\) and \(BB^*\) configurations experience attractive long-range forces. By itself, this Yukawa potential seems unlikely to bind \(DD^*\), but might succeed for the heavier \(BB^*\) system.

Years ago, Ader et al. [16] showed that \((QQ\bar{q}\bar{q})\) should become stable for very large quark-mass ratio \((M/m)\), a consequence of the flavour independence of chromoelectric forces. The conclusion was confirmed in subsequent studies [17].

In the limit of large \((M/m)\), \((QQ\bar{q}\bar{q})\) bound states exhibit a simple structure. There is a localized \(QQ\) diquark with colour 3, and this diquark forms a colour singlet together with the two \(\bar{q}\), as in every flavoured antibaryon. In other words, this multiquark uses well-experimented colour coupling, unlike speculative mock-baryonia or other states proposed in “colour chemistry” [18], which contain clusters with colour 6 or 8.

The stability of \((QQ\bar{q}\bar{q})\) in flavour-independent potentials is analogous to that of the hydrogen molecule [19]. If one measures the binding in units of the threshold energy, i.e., the energy of two atoms, one notices that the positronium molecule \((e^+e^+e^-e^-)\) with equal masses is bound by only 3%, while the very asymmetric hydrogen reaches 17%. This can be understood by writing the molecular Hamiltonian as

\[
H = H_S + H_A \\
= \left(\frac{1}{4M} + \frac{1}{4m}\right) \left(p_1^2 + p_2^2 + p_3^2 + p_4^2\right) + V \\
+ \left(\frac{1}{4M} - \frac{1}{4m}\right) \left(p_1^2 + p_2^2 - p_3^2 - p_4^2\right) \\
\] (19)

The Hamiltonian \(H_S\), which is symmetric under charge conjugation, has the same threshold as \(H\), since only the inverse reduced mass \((M^{-1} + m^{-1})\) enters the energy of the \((M^+m^-)\) atoms. Since \(H_S\) is nothing but a rescaled version of the Hamiltonian of the positronium molecule, it gives 3% binding below the threshold. Then the antisymmetric part \(H_A\) lowers the ground-state energy of \(H\), a simple consequence of the variational principle.

In simple quark models without spin forces, we have a similar situation. The equal mass case is found unbound, and \((QQ\bar{q}\bar{q})\) becomes stable, and more and more stable,
as \((M/m)\) increases. One typically needs \((bb\bar{q}\bar{q})\), with \(q = u\) or \(d\), to achieve binding with the nice diquark clustering we mentioned. However, if one combines this quark attraction with the long-range Yukawa forces, one presumably gets binding for \((cc\bar{q}\bar{q})\) with \(DD^*\) quantum numbers. A more detailed study is presently under way \(^{20}\).

The experimental signature of tetraquark heavily depends on its exact mass. Above \(DD^*\), we have a resonance, seen as a peak in the \(DD^*\) mass spectrum. Below \(DD^*\), one should look at \(DD\gamma\) decay of tetraquark. If it lies below \(DD\), then it is stable, and decays via weak interactions, with a lifetime comparable to that of other charmed particles.

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