Hyperbolic Flows and the Question of Quantum Chaos*

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To the Memory of Gerard Emch

Abstract

Hyperbolic flows, as formulated by Anosov, are the prototypes of chaotic evolutions in classical dynamical systems. Here we provide a concise updated account of their quantum counterparts originally formulated by Emch, Narnhofer, Thirring and Sewell within the operator algebraic setting of quantum theory; and we discuss their bearing on the question of quantum chaos.

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1. Introduction

Classical hyperbolic flows, as formulated by Anosov [1], are flows over smooth compact connected Riemannian manifolds that admit stable expanding and contracting foliations. Thus they are prototype examples of chaotic dynamical systems, in that orbits stemming from neighbouring points of their phase spaces diverge, generically, exponentially fast from one another.

In view of the fundamental character of both quantum ergodic theory [2]-[4] and quantum chaology [5, 6], it is natural to ask whether a formulation of a quantum counterpart of these flows is feasible. This question was addressed by Emch et al [7] in a treatment that overcame the obstacle imposed by the fact that quantum mechanics does not accommodate the differential geometric structures on which the classical treatment was based [1, 8]. In fact, their treatment was carried out within the framework of operator algebraic quantum theory [9, 10], wherein the observables of a model were represented by the self-adjoint elements of a $W^*$-algebra and the non-commutative differential structure was carried by derivations of that algebra.

The present article is devoted to a concise updated account of the picture of quantum hyperbolic flows presented in Ref. [7]. Its essential content comprises a general formulation of these flows and their chaotic properties, together with concrete examples both of models for which chaos survives quantisation and models for which it does not.

We start, in Section 2, with a brief account of the classical picture of hyperbolic flows. Here the generic model comprises a one-parameter group of diffeomorphisms of a manifold that satisfies a certain hyperbolicity condition. Prototype examples of these flows, which we provide, are the Arnold cat model and the geodesic flow over a compact Riemannian manifold of constant negative curvature.

In Section 3 we recast the classical model into the operator algebraic form given by the Gelfand isomorphism. This enables us to express the hyperbolicity condition in terms of automorphisms of the resultant commutative algebra of observables.

In Section 4 we provide a simple passage from the classical commutative algebraic picture to the quantum non-commutative one, thereby formulating the hyperbolicity condition for the quantum model in terms of automorphisms of its algebra of observables. In particular we show that this condition implies the chaoticity of the quantum model in that the evolutes of neighbouring states, as represented by density matrices, diverge exponentially fast from one another.

In Section 5 we provide an explicit treatment of the quantum version of the Arnold cat model and prove that its hyperbolicity, and thus its chaotic property, survives the quantisation.

Correspondingly, in Section 6 we provide an explicit treatment of the quantum version of the geodesic flow over a compact Riemannian manifold of negative curvature and show that, by contrast with the Arnold cat model, it violates the hyperbolicity condition. In other words, quantisation of its original classical version destroys its hyperbolicity.
In Section 7 we generalise this result to arbitrary finite quantum Hamiltonian systems by showing that they cannot support hyperbolic flows.

We conclude in Section 8 with a brief discussion of the results presented here and their consequences for quantum chaology.

The Appendix is devoted to the proof of a key Proposition involved in the formulation of the classical hyperbolicity condition of Section 2.

### 2. The Classical Picture

The classical model, $\Sigma_{cl}$, is given by a triple $(M, \mu, \phi)$ [8], where $M$ is a smooth, connected, compact Riemannian manifold, $\phi$ is a representation of $\mathbb{R}$ or $\mathbb{Z}$ in the diffeomorphisms of $M$, the representation being continuous in the former case; and $\mu$ is a $\phi$-invariant probability measure on $M$. Thus $\phi$ and $\mu$ represent the dynamics and a stationary state, respectively, of the model. Specifically, for $m \in M$ and $t \in \mathbb{R}$ or $\mathbb{Z}$, $\phi_t m$ is the evolute of $m$ at time $t$; and for measurable regions $A$ of $M$, $\mu(\phi_t A) = \mu(A)$. We denote the tangent space at the point $m$ of $M$ by $T(m)$ and note that, for fixed time $t$, the differential $d\phi_t$ of $\phi_t$ maps $T(m)$ into $T(\phi_t m)$.

In order to formulate the condition for the hyperbolicity of the dynamics of $\Sigma_{cl}$ we first assume that $M$ is equipped with vector fields $V_1, \ldots, V_n$, where $n = \dim(M)$ or $(\dim(M) - 1)$ according to whether the time variable $t$ is discrete or continuous*. It is assumed that at each point $m$ of $M$ these fields are linearly independent and that each $V_j$ has a global integral curve $C_j(m) = \{m_j(s) | s \in \mathbb{R}; m_j(0) = m\}$, given by the unique solution of the equation

$$m_j'(s) = V_j(m_j(s)); \ m_j(0) = m.$$  \hspace{1cm} (2.1)

Thus, the curves $\{C_j(m) | m \in M\}$ are generated by the action on $M$ of a one-parameter group $\{\theta_j(s) | s \in \mathbb{R}\}$ of diffeomorphisms, defined by the formula

$$\theta_j(s)m = m_j(s), \ \forall m \in M, \ s \in \mathbb{R}.$$  \hspace{1cm} (2.2)

The orbits of the $\theta_j$’s are termed horocycles. We note here that the correspondence between the group $\theta_j$ and the vector field $V_j$ is one to one since Eqs. (2.1) and (2.2) may be employed to define $V_j$ in terms of $\theta_j$ by the formula

$$V_j(m) = \theta'_j(0)m \ \forall \ m \in M.$$  \hspace{1cm} (2.3)

To establish consistency, we remark that this equation, together with the group property of $\theta_j$, implies that

$$V_j(\theta_j(s)m) = \theta'_j(0)(\theta_j(s)m) = \frac{\partial}{\partial t}\theta_j(t)s|_{t=0} = \frac{\partial}{\partial t}\theta_j(t+s)s|_{t=0} = \theta'_j(s)m,$$  \hspace{1cm} (2.4)

* The difference between $n$ and $\dim(M)$ in the continuous case corresponds to the one dimensionality of the space generated by the velocity vector.
as demanded by Eqs. (2.1) and (2.2).

**Definition 2.1.** We term the dynamics of the model $\Sigma_{cl}$ hyperbolic if the action of the differential of $\phi_t$ on the vector fields $V_j$ takes the form

$$d\phi_t V_j(m) = V_j(\phi_t m) e^{\lambda_j t};$$

where the $\lambda$'s are real numbers such that, for some positive integer $r$ less than $n$, $\lambda_j$ is positive for $j \in [1, r]$ and negative for $j \in [r+1, n]$. Thus, if $m'$ and $m$ are neighbouring points of $M$ whose difference, as represented on a chart at $m$, is $\sum_{j=1}^{n} a_j V_j(m)$, the hyperbolicity condition signifies that

$$\phi_t m - \phi_t m' \simeq \sum_{j=1}^{n} a_j V_j(\phi_t m) e^{\lambda_j t}.$$  

Hence, defining $T_+(m)$ (resp. $T_-(m)$) to be the subspace of $T(m)$ spanned by the vectors $V_j$ for which $\lambda_j$ is positive (resp. negative), the hyperbolicity condition is that the action of $\phi_t$ on neighbouring points of $M$ serves to expand their separation exponentially fast if their relative displacement on a chart at $m$ lies in $T_+(m)$ and contracts it if that displacement lies in $T_-(m)$. Thus the $\lambda$'s are Lyapunov exponents and, as some of them are positive, the hyperbolicity condition signifies that the flow is chaotic. The following Proposition will be proved in Appendix A.

**Proposition 2.1.** The hyperbolicity condition given by Eq. (2.5) is equivalent to the following one.

$$\phi_t \theta_j(s) \phi_{-t} = \theta_j(se^{\lambda_j t}).$$  

(2.7)

**Example 1. The Arnold Cat.** This is the model $(M, \phi, \theta, \mu)$, where

(i) $M$ is the torus $[0,1) \times [0,1)$ with Euclidean metric;

(ii) the time variable $t$ is discrete, its range being $\mathbb{Z}$, and the dynamical transformations are $\{\phi^n (:= \phi_n)|n \in \mathbb{Z}\}$, where

$$\phi = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix};$$

(2.8)

(iii) $\mu$ is the Lebesgue measure on the torus $M$; and

(iv) denoting the eigenvectors of $\phi$ by $V_1$ and $V_2$ and their respective eigenvalues by $k_1 (> 1)$ and $k_2 (< 1)$, $\theta$ is the pair of one-parameter groups $\theta_1$ and $\theta_2$ defined in terms of $V_1$ and $V_2$ by Eqs. (2.1) and (2.2). Thus

$$\theta_j(s)m = m + V_j s \text{ (mod (1,1))} \forall m \in M, s \in \mathbb{R}, j = 1, 2.$$  

(2.9)

It now follows from these definitions that the model satisfies the hyperbolicity condition (2.7), with $\lambda_j = \ln(k_j)$.

* This model of automorphisms of the torus is often so termed because of Arnold's illustration [8] of their actions on a cat's face placed in the torus.
**Example 2. Geodesic Flow on a Manifold of Negative Curvature** [8]. This is a model of the free dynamics of a particle on a compact region of the Poincaré half plane $\tilde{M} := \{(x, y) | x \in \mathbb{R}, y \in \mathbb{R}_+\}$, whose metric is given by the formula

$$ds^2 = y^{-2}(dx^2 + dy^2).$$  \hfill (2.10)

The points $(x, y)$ of $\tilde{M}$ will sometimes be represented by the complex numbers $z := (x + iy)$.

The manifold $\tilde{M}$ is equipped with the symmetry group $G = SL(2, \mathbb{R})$ [11], which acts transitively on it. The elements $g$ of this group are represented by two-by-two matrices with real-valued entries and unit determinant. Its actions on $\tilde{M}$ are given by the following formula. Denoting $g \in G$ by \[
\begin{pmatrix} a & b \\
 c & d \end{pmatrix},
\]
we have

$$gz = \frac{(az + b)}{(cz + d)}.$$  \hfill (2.11)

We denote by $K$ the subgroup of $G$ whose elements leave the point $i$ invariant. It then follows from the transitivity of $G$ that $G/K$ may be identified with the space $\tilde{M}$. Correspondingly, for a discrete co-compact non-abelian subgroup $\Gamma$, $\Gamma \backslash G/K$ is a compact manifold, $\tilde{M}$, of constant negative curvature. Its unit tangent bundle, $T_1 \tilde{M} := \tilde{M}$ may then be identified with $\Gamma \backslash G$. We take this to be the phase space of the model.

The dynamical group $\phi$ for the free geodesic motion of a particle on $M$ is given by the formula [11, 7]

$$\phi_t m = m \xi(t),$$  \hfill (2.12)

where

$$\xi(t) = \begin{pmatrix} \exp(-t/2) & 0 \\ 0 & \exp(t/2) \end{pmatrix}.$$  \hfill (2.13)

We note that the measure $d\mu := y^{-2}dxdy$ is $\phi$-invariant. Further, the horocyclic actions are given by the formulae

$$\theta_j(s)m = m \xi_j(s) \forall s \in \mathbb{R}, \ j = 1, 2,$$  \hfill (2.14)

where

$$\xi_1(s) = \begin{pmatrix} 1 & s \\ 0 & s \end{pmatrix}$$  \hfill (2.15)

and

$$\xi_2(s) = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}.$$  \hfill (2.16)

It follows directly from these formulae that the model satisfies the hyperbolicity condition (2.7).

3. The Classical Operator Algebraic Picture.
As a first step towards a passage from the above classical picture to a corresponding quantum mechanical one, we now exploit the Gelfand isomorphism, according to which the model \((M, \phi, \mu)\) is equivalent to the \(W^*\) dynamic system \((A_{cl}, \alpha_{cl}, \rho_{cl})\), where \(A_{cl}\) is the abelian \(W^*\) algebra of observables \(L_{\infty}(M, d\mu)\). \(\{\alpha_{cl}(t)| t \in \mathbb{R}\}\) is the one-parameter group of automorphisms of \(A_{cl}\) representing the dynamics of the model and given by the formula

\[
[\alpha_{cl}(t)A](m) = A(\phi_{-t}m) \quad \forall \, A \in A_{cl}, \, m \in M, \, t \in \mathbb{R},
\]

(3.1)

and \(\rho_{cl}\) is the state on \(A_{cl}\) corresponding to the measure \(\mu\), i.e.

\[
\rho_{cl}(A) = \int A \, d\mu.
\]

(3.2)

It follows immediately from these specifications that the \(\phi\)-invariance of \(\mu\) is equivalent to the \(\alpha_{cl}\)-invariance of \(\rho_{cl}\).

Furthermore the diffeomorphism groups \(\theta_j\) correspond to representations \(\sigma_{j,cl}\) of \(\mathbb{R}\) in \(\text{Aut}(A_{cl})\), given by the formula

\[
[\sigma_{j,cl}(s)A](m) = A(\theta_j(-s)m) \quad \forall \, A \in A_{cl}, \, m \in M, \, s \in \mathbb{R}.
\]

(3.3)

The hyperbolicity condition (2.7) is therefore equivalent to the following one.

\[
\alpha_{cl}(t)\sigma_{j,cl}(s)\alpha_{cl}(-t) = \sigma_{j,cl}(se^{\lambda_j t}) \quad \forall \, s \in \mathbb{R}, \, t \in \mathbb{R} \text{ or } \mathbb{Z}, \, j = 1, \ldots, n.
\]

(3.4)

4. The Quantum Picture.

We assume that the generic quantum model corresponds to the algebraic picture of the classical one, but with the difference that the algebra of observables is non-commutative. Thus the quantum model is a triple \((A, \alpha, \rho)\), where \(A\) is a \(W^*\)-algebra, in general non-commutative, \(\rho\) is a normal state on \(A\) and \(\{\alpha_t| t \in \mathbb{R} \text{ or } \mathbb{Z}\}\) is a one-parameter group of automorphisms of \(A\), which is continuous w.r.t. \(t\) in the former case, and \(\rho\) is a normal \(\alpha\)-invariant state on \(A\). Furthermore, we assume that the model is equipped with \(n\) horocyclic actions, given by one-parameter groups \(\{\sigma_j(s)| s \in \mathbb{R}, \, j = 1, \ldots, n\}\) of \(A\) whose infinitesimal generators are linearly independent both of one another and of that of the group \(\alpha\) in the case where the variable \(t\) runs through \(\mathbb{R}\). Accordingly, we take the hyperbolicity condition to be the natural generalisation of Eq. (3.4) for the possibly non-commutative case, i.e.

\[
\alpha_t\sigma_j(s)\alpha_{-t} = \sigma_j(se^{\lambda_j t}) \quad \forall \, s \in \mathbb{R}, \, t \in \mathbb{R} \text{ or } \mathbb{Z}, \, j = 1, \ldots, n,
\]

(4.1)

where again \(\lambda\) is positive for \(j = 1, \ldots, r\) and negative for \(j = r + 1, \ldots, n\). This condition implies the following one for the duals, \(\alpha_t^*\) and \(\sigma_j^*(s)\), of \(\alpha_t\) and \(\sigma_j(s)\), in their actions on the normal states, \(\mathcal{N}(A)\), on \(A\).

\[
\alpha_t^*\sigma_j^*(s)\alpha_t^* = \sigma_j^*(se^{\lambda_j t}).
\]

(4.2)
We denote by $\delta^*_j$ the infinitesimal generator of the group $\sigma^*_j$, in the $\omega^*$ topology. It follows from this formula that its domain, $D(\delta^*)$, is stable under the group $\alpha^*$ and that, if $\rho_1$ and $\rho_2$ are states in this domain, then

$$\|\delta^*\alpha^*_t(\rho_1 - \rho_2)\| = \|\delta^*(\rho_1 - \rho_2)\|e^{\lambda_j t}.$$  (4.3)

Thus, in the quantum context, $\lambda_j$ is a Lyapounov function that provides a measure of the speed at which the evolutes of $\rho_1$ and $\rho_2$ separate along the horocycle $\sigma_j$. Since some of the $\lambda$'s are positive, this represents a chaoticity condition.

We shall show, in the following Sections, that quantisation does not affect the hyperbolic property of the Arnold cat model, but that it destroys that of the geodesic flow over the manifold of negative curvature; and that, in general, it does not admit chaos in finite Hamiltonian systems.

5. The Quantum Arnold Cat.

In order to quantise the classical Arnold cat model, we start by expressing that model in a form readily amenable to quantisation. Thus we first note that it follows from the definition of the classical algebra $A_{cl}$ in Section 2.2 that this algebra is generated by the sinusoidal functions \{\(W_{cl}(\nu)|\nu = (\nu_1, \nu_2)\in \mathbb{Z}^2\}\}, defined by the formula

$$W_{cl}(\nu)[m] = \exp(2\pi i \nu.m) \forall \nu = (\nu_1, \nu_2)\in \mathbb{Z}^2,$$  (5.1)

where the dot denotes the Euclidean scalar product. Correspondingly, since $\mu$ is the Euclidean measure on the torus $M$, it follows from Eqs. (3.2) and (5.1) that

$$\rho_{cl}\left(W_{cl}(\nu)\right) = \delta_{\nu,0},$$  (5.2)

where $\delta$ is the Kronecker delta. Moreover since, by Eq. (2.8), $\phi$ is Hermitean, it follows from Eqs. (3.1) and (5.1) that

$$\alpha_{cl}(t)W_{cl}(\nu) = W_{cl}(\phi_{-t}\nu) \forall t\in \mathbb{Z}, \nu\in \mathbb{Z}^2;$$  (5.3)

while, by Eqs. (2.9), (3.3) and (5.1), the horocyclic actions for the model are given by the formula

$$\sigma_j(s)W_{cl}(\nu) = W_{cl}(\nu)\exp\left(2\pi i \nu.V_j s\right) \forall s\in \mathbb{R}, \nu\in \mathbb{Z}^2.$$  (5.4)

Thus Eqs. (5.1)-(5.4) define the classical model. One may readily check that they satisfy the hyperbolicity condition (3.4), bearing in mind that $V_j$ is the eigenvector of $\phi$ whose eigenvalue is $\exp(\lambda_j)$.

We now quantise the classical model by basing the algebra of observables on Weyl operators instead of the sinusoidal function $W_{cl}$. Thus, in order to construct $A$, we start with an abstract algebra of elements \{\(W(\nu)|\nu\in \mathbb{Z}^2\}\} which satisfy the Weyl condition that

$$W(\nu)W(\nu') = W(\nu + \nu')\exp\left(i\gamma\kappa(\nu, \nu')\right),$$  (5.5)
where $\kappa$ is the simplectic form defined by the formula
\[
\kappa(\nu, \nu') = \nu_1 \nu'_2 - \nu_2 \nu'_1
\] (5.6)
and $\gamma$ is a constant that plays the role of that of Planck. Thus the algebra $A_0$ of the polynomials in the $W(\nu)$'s comprises just the linear combinations of them. We define $\rho$ to be the positive normalised linear form on this algebra given by the precise analogue of the classical state $\rho_{cl}$, as given by Eq. (5.2), i.e.
\[
\rho(W(\nu)) = \delta_{\nu,0}.
\] (5.7)
We define the algebra of observables, $A$, to be the strong closure of the GNS representation of $A_0$ in the state $\rho$ defined by this last equation. We then define the dynamical and horocyclic automorphisms, $\alpha$ and $\sigma_j$, by the canonical counterparts of the classical ones of Eqs. (5.3) and (5.4). Thus
\[
\alpha(t)W(\nu) = W(\phi_{-t}\nu) \forall t \in \mathbb{Z}, \nu \in \mathbb{Z}^2;
\] (5.8)
and
\[
\sigma_j(s)W(\nu) = W(\nu)\exp(2\pi i \nu V_j s) \forall s \in \mathbb{R}, \nu \in \mathbb{Z}^2.
\] (5.9)
It follows from the last two formulae that the model satisfies the hyperbolicity condition (4.1). Thus we have established the following Proposition.

**Proposition 5.1.** *The chaoticity of the flow of the Arnold cat model survives quantisation.*

6. Quantum Geodesic Flow on a Compact Manifold of Negative Curvature.

The model we now consider is the quantised version of that of Example 2 in Section 2, and it may be described as follows [11, 7]. Its $W^{*-}$ algebra of observables is $B(\mathcal{H})$, the set of bounded operators in the Hilbert space $\mathcal{H} := L_2(\hat{M}, d\mu)$, where the measure $d\mu$ is defined following Eq. (2.13). The state space of the model comprises the normal states of $A$ and its Hamiltonian is $-\Delta$, where $\Delta$ is the Laplace-Beltrami operator for the manifold $\hat{M}$. The dynamical automorphisms of the model are thus given by the formula
\[
\alpha_t A = \exp(-i\Delta t)A \exp(i\Delta t) \forall A \in A, t \in \mathbb{R}.
\] (6.1)
Moreover, the spectrum of $\Delta$ is discrete [12]. We denote by $\{f_k|k \in \mathbb{N}\}$ a complete orthonormal set of eigenvectors of this operator and by $\{e_k\}$ the corresponding set of its eigenvalues. We then define the operators $F_{kl}$, with $k, l \in \mathbb{N}$, by the equation
\[
F_{kl}f_i = \delta_{li}f_k \forall k, l, i \in \mathbb{N}.
\] (6.2)
It follows now from Eqs. (6.1) and (6.2) that
\[
\alpha_t F_{kl} = \exp(i\omega_{kl} t)F_{kl} \forall t \in \mathbb{R}, k, l \in \mathbb{N},
\] (6.3)
where
\[ \omega_{kl} = e_l - e_k. \]  
\( (6.4) \)

We denote by \( \mathcal{L}(F) \) the set of finite linear combinations of the \( F_{kl} \)'s. It follows from this definition that \( \mathcal{L}(F) \) is closed with respect to involution and binary addition and multiplication. It is therefore a *-algebra, and it follows from our specifications that its strong closure is \( \mathcal{A} \).

**Proposition 6.1.** Under the above assumptions, the quantum geodesic flow on the manifold cannot be hyperbolic.

We base the proof of this Proposition on Lemmas (6.2) and (6.3) below.

**Lemma 6.2.** Assume that the model satisfies the hyperbolicity condition with respect to horocyclic automorphisms \( \sigma(R) \). Then it follows from the discreteness of the spectrum of \( \Delta \) that any normal stationary state \( \rho \) of the model is \( \sigma \)-invariant.

Assuming the result of this lemma, we denote the GNS triple for the state \( \rho \) by \( (\mathcal{H}_\rho, \pi_\rho, \Phi_\rho) \) and define \( U_\rho \) and \( V_\rho \) to be the continuous unitary representations of \( R \) in \( \mathcal{H}_\rho \) that implement the automorphisms \( \alpha_t \) and \( \sigma(s) \), respectively, according to the standard prescription
\[ U_\rho(t)\pi_\rho(A)\Phi_\rho = \pi_\rho(\alpha_t A)\Phi_\rho \]  
\( (6.5) \)
and
\[ V_\rho(s)\pi_\rho(A)\Phi_\rho = \pi_\rho(\sigma(s) A)\Phi_\rho. \]  
\( (6.6) \)

Hence, by the cyclicity of \( \Phi_\rho \) and the hyperbolicity condition (4.1), as applied to the horocycle \( \sigma \), that
\[ U_\rho(t)V_\rho(s)U_\rho(-t) = V_\rho(se^{\lambda t}) \quad \forall \ s, t \in R \]  
\( (6.7) \)

We define \( H_\rho \) to be the Hamiltonian operator in the GNS space, \( \mathcal{H}_\rho \), according to the formula \( U_\rho(t) = \exp(iH_\rho t) \).

**Lemma 6.3.** Under the assumptions of Lemma 6.2 and with the subsequent definitions, the formula (6.7) implies that the spectrum of \( H_\rho \) is \( R \).

**Proof of Prop. 6.1 assuming Lemmas 6.2 and 6.3.** Our strategy here is to infer from Lemma 6.2 that the assumption of hyperbolicity implies that the spectrum of \( H_\rho \) is discrete. Since this conflicts with Lemma 6.3, we conclude that that assumption is invalid.

We start by noting that, by Eqs. (6.3) and (6.5),
\[ U_\rho(t)\pi_\rho(F_{kl})\Phi_\rho = \pi_\rho(F_{kl})\Phi_\rho \exp(i\omega_{kl} t). \]  
\( (6.8) \)

Since \( \rho \) is a normal stationary state of the model, it follows from the definition of the vectors \( f_k \) that \( \rho \) corresponds to a density matrix of the form \( \sum_{r \in \mathbb{N}} w_r P_r \), where the \( w_r \)'s
are non-negative numbers whose sum is unity and \( P_r ( = F_{rr} ) \) is the projection operator for the vector \( f_r \). Hence
\[
\langle \pi_\rho (A) \Phi_\rho, \pi_\rho (B) \Phi_\rho \rangle = \langle \rho; (A^* B) \rangle = \sum_{r \in \mathbb{N}} w_r (f_r, A^* B f_r) \forall A, B \in \mathcal{A}. \tag{6.9}
\]
It follows from this formula and Eq. (6.2) that
\[
\langle \pi_\rho (F_{kl}) \Phi_\rho, \pi_\rho (F_{k'l'} \Phi_\rho \rangle = w_l \delta_{kk'} \delta_{ll'}. \tag{6.10}
\]
Therefore, defining \( D := \{ (k, l) \in \mathbb{N}^2; w_l \neq 0 \} \) and
\[
\Psi_{kl} = w_l^{-1/2} \pi_\rho (F_{kl}) \Phi_\rho \forall (k, l) \in D, \tag{6.11}
\]
the set of vectors \( \{ \Psi_{kl}(k, l) \in D \} \) is orthonormal. It is also complete for the following reasons. By the definition (6.2) of the operators \( F_{kl} \), the algebra \( \mathcal{A} \) consists of linear combinations of these operators. Therefore, by the normality of the representation \( \pi_\rho \), the algebra \( \pi_\rho (\mathcal{A}) \) consists of linear combinations of the operators \( \pi_\rho (F_{kl}) \). Hence by Eq. (6.11) and the cyclicity of \( \Pi_\rho \) with respect to that algebra, the set \( \{ \Psi_{kl}(k, l) \in D \} \) of orthonormal vectors in \( H_\rho \) is complete.

Now, by Eqs. (6.8) and (6.11),
\[
U_\rho (t) \Psi_{kl} = \Psi_{kl} \exp (i \omega_{kl} t) \forall (k, l) \in D.
\]
and consequently, since \( \{ \Psi_{kl}(k, l) \in D \} \) is an orthonormal basis in \( H_\rho \),
\[
U_\rho (t) = \sum_{(k, l) \in D} \mathcal{P}_{kl} \exp (i \omega_{kl} t),
\]
where \( \mathcal{P}_{kl} \) is the projector for \( \Psi_{kl} \). Hence
\[
H_\rho = \sum_{(k, l) \in D} \omega_{kl} \mathcal{P}_{kl}, \tag{6.12}
\]
and therefore the spectrum of \( H_\rho \) comprises the discrete set \( \omega_{kl}(k, l) \in D \). As this conflicts with Lemma 6.3, which was based on the assumption of a hyperbolic flow, we conclude that the model does not support such a flow.

**Proof of Lemma 6.2.** By the hyperbolicity condition (4.1), as applied to the horocycle \( \sigma \),
\[
\langle \rho; \alpha_t \sigma (s) \alpha_{-t} F_{kl} \rangle = \langle \rho; \sigma (se^{\lambda t}) F_{kl} \rangle. \tag{6.8}
\]
By Eq. (6.3) and the stationarity of \( \rho \), the l.h.s. of this equation is equal to \( \langle \rho : \sigma (s) F_{kl} \rangle \exp (i \omega_{kl} t) \). On the other hand, in the limit where \( \lambda t \to - \infty \), it follows by continuity that the r.h.s. of Eq. (6.8) reduces to \( \langle \rho; F_{kl} \rangle \). Compatibility of these expressions for the two sides of Eq. (6.8) implies that \( \langle \rho; \sigma (s) F_{kl} \rangle \) and \( \langle \rho; F_{kl} \rangle \) are equal to one another if \( \omega_{kl} = 0 \) and are both zero if \( \omega_{kl} \neq 0 \). Hence they are equal in all cases. In view of the normality of \( \rho \) and the strong density of \( \mathcal{L}(F) \), this result implies that \( \rho \) is \( \sigma \)-invariant.
**Proof of Lemma 6.3.** This is achieved in Ref. [7] on the basis of a version of Mackey’s imprimitivity theorem.

### 7. Generic Non-Hyperbolic Flow of Finite Quantum Hamiltonian Systems.

The generic model of a finite quantum Hamiltonian system is not quite the same as the model presented in Section 4. Specifically it consists of a triple \((\mathcal{A}, \alpha, \mathcal{N})\) [10, 13], where \(\mathcal{A}\) is the \(W^*\)-algebra of bounded operators in a separable Hilbert space \(\mathcal{H}\), \(\mathcal{N}\) is the set of normal states on \(\mathcal{A}\) corresponding to the density matrices in \(\mathcal{H}\), and \(\alpha\) is a representation of \(\mathbf{R}\) in the automorphisms of \(\mathcal{A}\) implemented by a unitary group whose infinitesimal generator is \(i\) times a self-adjoint operator \(H\). Thus

\[
\alpha_t A = \exp(iHt/\hbar)A\exp(-iHt/\hbar) \quad \forall \, A \in \mathcal{A}, \; t \in \mathbf{R}.
\]  

Here \(H\) is the Hamiltonian of the model. In general, it is the sum of the kinetic and potential energies of its constituent particles and its spectrum is discrete. Note that these specifications do not include the assumption of a hyperbolicity assumption such as given by Eq. 4.1. In fact, the following proposition establishes the contrary of that assumption for this model.

**Proposition 7.1.** *Finite quantum Hamiltonian systems, as defined above, cannot support hyperbolic flows.*

**Proof.** This follows immediately from the discreteness of the spectrum of \(H\) by the same argument that led from Lemmas 6.2 and 6.3 to Prop. 6.1.

### 8. Conclusions

The general picture of quantum hyperbolic flows, presented in Section 4, is the natural analogue of its algebraically cast classical counterpart and exhibits the chaotic property represented by Eq. (4.3). Moreover, this picture is realised by the quantum Arnold cat model. On the other hand, finite quantum Hamiltonian systems, including the geodesic flow over a compact manifold of constant negative curvature, do not support hyperbolic flows. This accords with a vast body of work on models for which chaos in classical systems is suppressed by quantisation [5, 6]. Since, in those works, the classical chaos leaves its mark on the resultant quantum system in the form of certain scars on its eigenstates, we expect that this is also the case for the quantum Hamiltonian models treated here.

### Appendix A: Proof of Proposition 2.1.

In order to derive Eq. (2.7) from Eq. (2.5), we start by defining

\[
\tilde{m}_{j,t}(s) = \phi_t \theta_j(\exp(-\lambda_j t)s) \phi_{-t} m \quad \forall \, s, t \in \mathbf{R}, \; m \in M, \; j = 1, \ldots, n
\]  

and inferring from this formula that, for fixed \(t\) and \(j\),

\[
\tilde{m}'_{j,t}(s) = \exp(-\lambda_j t)d\phi_t \theta_j'(\exp(-\lambda_j t)s) \phi_{-t} m
\]  

\[(A.2)\]
Hence, by Eq. (2.1),
\[ \tilde{m}_{j,t}'(s) = \exp(-\lambda_j t) \phi_t V(\theta_j(t)(\exp(-\lambda_j t)s)\phi_{-t}m). \]

and therefore, by Eqs. (2.5) and (A.1),
\[ \tilde{m}_{j,t}'(s) = V(\tilde{m}_{j,t}(s)), \] (A.3)
which signifies that \( \tilde{m}_{j,t}(s) \) is the unique solution of Eq. (2.4), i.e. that \( \tilde{m}_{j,t}(s) = \theta_j(s)m. \)

In view of Eq. (A.1), this implies that
\[ \phi_t \theta_j(\exp(-\lambda_j t)s)\phi_{-t} = \theta_j(s), \quad \forall \ s, t \in \mathbb{R}, \ m \in M, \ j = 1, \ldots, n, \]
which is equivalent to Eq. (2.7).

Conversely, in order to derive Eq. (2.5) from Eq. (2.7), we note that, in view of the formula (A.1), the latter equation signifies that \( \tilde{m}_{j,t}(s) = m_j(s). \) Hence, by Eq. (2.1),
\[ \tilde{m}_{j,t}'(s) = V(\tilde{m}_{j,t}(s)). \] (A.4)
Furthermore, by Eq. (A.1), the l.h.s. of this formula is equal to
\[ \frac{\partial}{\partial s} \phi_t \theta_j(\exp(-\lambda_j t)s)\phi_{-t}m = \exp(-\lambda_j t)\phi_t \theta_j'(\exp(-\lambda_j t)s)\phi_{-t}m, \]
which, by Eq. (2.1), is equal to
\[ \exp(-\lambda_j t)\phi_t V(\theta_j(\exp(-\lambda_j t)s)\phi_{-t}m). \]

Hence, by Eq. (A.1), Eq. (A.4) reduces to the form
\[ d\phi_t V(\theta_j(\exp(-\lambda_j t)s)\phi_{-t}m) = \exp(\lambda_j t)V(\tilde{m}_{j,t}(s)), \]
i.e., by Eq. (A.1),
\[ d\phi_t V(\phi_{-t}\tilde{m}_{j,t}(s)) = \exp(\lambda_j t)V(\tilde{m}_{j,t}(s)). \]
Thus, putting
\[ \hat{m} = \phi_{-t}\tilde{m}_{j,t}(s), \]
\[ d\phi_t V(\hat{m}) = V(\phi_t \hat{m}). \]

Since, by Eqs. (A.1) and (A.5), the correspondence between \( m \) and \( \hat{m} \) is one-to-one, this last equation is equivalent to Eq. (2.5).

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