Correlation functions of one-dimensional strongly interacting two-component gases

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We address the problem of calculating the correlation functions of one-dimensional two-component gases with strong repulsive contact interactions. The model considered in this paper describes particles with fractional statistics and in appropriate limits reduces to the Gaudin-Yang model or the spinor Bose gas. In the case of impenetrable particles we derive a Fredholm determinant representation for the temperature-, time-, and space-dependent correlation functions which is very easy to implement numerically and constitute the starting point for the analytical investigation of the asymptotics. Making use of this determinant representation and the solution of an associated Riemann-Hilbert problem we derive the low-energy asymptotics of the correlators in the spin-incoherent regime characterized by near ground-state charge degrees of freedom but a highly thermally disordered spin sector. The asymptotics present features reminiscent of spin-charge separation with the spin part exponentially decaying in space separation and oscillating with a period proportional to the statistics parameter while the charge part presents scaling with anomalous exponents which cannot be described by any unitary conformal field theory. The momentum distribution and the Fourier transform of the dynamical Green’s function are asymmetrical for arbitrary statistics, a direct consequence of the broken space-reversal symmetry. Due to the exponential decay the momentum distribution $n(k)$ at zero temperature does not present algebraic singularities but the tails obey the universal decay $\lim_{k \to \pm \infty} n(k) \sim C/k^4$ with the amplitude $C$ given by Tan’s contact. As a function of the statistics parameter the contact is a monotonic function reaching its minimum for the fermionic system and the maximum for the bosonic system.

I. INTRODUCTION

Many interacting one-dimensional (1D) systems are described at low energy by an effective theory known as Tomonaga-Luttinger liquid (TLL) theory [1][2]. This theory is expressed in terms of many-body collective excitations propagating with different velocities giving rise to an exotic behavior known as spin-charge separation. Tomonaga-Luttinger liquids are characterized by linear dependence on temperature of the specific heat, algebraic decay of correlation functions including the single-particle Green’s function and power-law vanishing of the tunneling density of states as the chemical potential is approached. The properties of the TLL phase can be understood from bosonisation which maps the TLL onto a theory of free massless compactified bosonic fields with the compactified radii playing the role of the phenomenological parameters that define the effective theory [1]. In the case of exactly solvable many-body systems these phenomenological constants can be extracted from the thermodynamic properties of the system which can be computed using the thermodynamic Bethe ansatz. The TLL description is valid when both the charge and spin sectors of the system are at low energy. However, for strongly interacting systems at low densities we can encounter the situation in which the temperature is much larger than the single excitation spin energy but is much smaller than the single excitation for charge. This energy window $E_{\text{spin}} \ll k_B T \ll E_{\text{charge}}$ in which the spin energy is exponentially smaller than the charge energy characterizes the spin-incoherent Tomonaga-Luttinger liquid (SITLL) regime [5][12] which has distinct properties and a higher degree of universality than the TLL. In this paper we are going to investigate the correlation functions of two-component anyonic gases with $\delta$-function interaction in the spin-incoherent regime.

Integrable models play an important role in our understanding of strongly interacting 1D physical systems [13][14]. In principle the knowledge of their wave functions and energy spectrum provide the starting point in computing the correlation functions from first principles without resorting to sometimes hard to justify approximations. In practice, the complexity of the Bethe ansatz wave functions means that such calculations are extremely difficult and new mathematical methods need to be devised. Initial attempts focused on the case of systems equivalent with free fermions [5][8][15][27] but very recently results were obtained for the Lieb-Liniger [28][38] and the XXZ spin-chain [39][51] models away from the free fermion point. Analytical derivations of the low-energy asymptotics of correlation functions are important because they can be compared with the predictions of TLL but they can also provide insight and identify systems not described by TLL theory. An important example is represented by the Gaudin-Yang model [52][53] which describes 1D nonrelativistic fermions with $\delta$-function repulsive interaction (see [1]). At finite coupling strength $c$ and low energies the model is well described by TLL theory [2] and can be understood as the continuum limit of the Hubbard model [14]. The case of impenetrable particles, $c \to \infty$, is particularly interesting due to the infinite spin degeneracy of the ground state. For this reason the TLL description fails and the effect on the correlation
functions on taking this limit is highly nontrivial due to the noncommutativity of $c \to \infty$ and $T \to 0$. Taking the limit of vanishing temperature first, followed by $c \to \infty$, results are obtained which are consistent with the impenetrable limit of the TLL or Conformal Field Theory (CFT) description. The static single-particle Green’s function decays algebraically and the momentum distribution behaves like $n(k) \sim n_{k_F} - \text{const}|k - k_F|^{1/2}\text{sgn}(k - k_F)$ near $k_F$. However, by taking the impenetrable limit first and then $T \to 0$ the static Green’s function is exponentially decaying in space separation and the momentum distribution is no longer singular at $k_F$. The importance of this result derived by Berkovich and Lowenstein \cite{5,6} was overlooked for a long period of time. In 2004 Cheianov and Zvonarev \cite{7,8} computed the low energy asymptotics of both static and dynamic correlators using the Deift-Zhou nonlinear steepest descent method \cite{55,56} and showed that they do not fit the TLL predictions. The asymptotics show signs of spin-charge separation with the spin part exponentially decaying and the charge part presenting scaling behavior with anomalous exponents inconsistent with any unitary conformal theory. These features reveal that the correlators of the Gaudin-Yang model derived by taking the successive limits $c \to \infty$ and $T \to 0$ belong to the spin-incoherent Tomonaga-Luttinger liquid regime \cite{7,12}.

In this paper we investigate the correlation functions of a 1D two-component system with repulsive contact interactions which can be understood as the anyonic generalization of the Gaudin-Yang model. The literature on 1D anyonic models has been growing steadily in the last years \cite{28,83} spurred by several experimental proposals of realizing such systems with ultracold atoms in optical lattices using Raman-assisted tunneling \cite{84,85}, periodically driven lattices \cite{86} or multicolor lattice-depth modulation \cite{87,88}. Much of the attention was focused on single component systems like the anyonic Lieb-Liniger model \cite{89,90,91} for which we know the ground state characterization \cite{90,92}, asymptotic behavior of the correlation functions for homogeneous \cite{26,27,59,53,85} and trapped systems \cite{95,97}, entanglement \cite{88,99}, and nonequilibrium \cite{100,101,102} properties. In the case of multi-component anyonic systems, such as the one considered in this paper, there are fewer known results, especially from the analytical point of view \cite{103,104}. The starting point of our analysis is the derivation of the Fredholm determinant representation for the temperature-, time-, and space-dependent single-particle Green’s function in the strongly interacting regime using the summation of form factors. This representation is obtained by taking the limit $c \to \infty$ first in the wavefunctions which means that the correlators in the zero temperature limit will be in the spin-incoherent regime. This constitutes the anyonic generalization of the fermionic and bosonic result obtained by Izergin and Pronko \cite{106}. The infrared asymptotics of the correlators at zero temperature and zero magnetic field are computed using the solutions of the associated Riemann-Hilbert problems to the static and time generalization of the sine-kernel \cite{107,108}. The spin part of the asymptotics is exponentially decaying (the correlation length is independent on statistics) coupled with an oscillatory component with frequency depending on the statistics parameter while the charge part presents scaling with anomalous exponents. These features (except the statistics dependent oscillatory part which is a defining feature of 1D anyonic systems) show that the system is in the spin-incoherent regime. For the fermionic system not only we recover Cheianov and Zvonarev \cite{7,8} results but we are also able to compute the constants in front of the asymptotics in the time-like regime. As a by-product of our analysis we also derive the zero temperature asymptotics for the correlators of single component anyons (Lieb-Liniger anyons) \cite{89,90,91}. The momentum distribution is not symmetric but unlike the single-component counterpart it does not present a singularity at $(1 - \kappa)k_F$ ($\kappa \in [0, 1]$ is the statistics parameter) due to the exponential decay of the static correlator. The high-momentum tails present the universal $\lim_{k \to 1/\infty} n(k) \sim C/k^4$ behavior of models with contact interactions with the amplitude $C$ called Tan’s contact (it should be pointed out that in the case of penetrable anyons this might not necessarily be true). Similar to the momentum distribution the Fourier transform of the field-field correlator is not symmetric due to the broken space-reversal symmetry.

The plan of the paper is as follows. In Sect. \textsection II we introduce the model, its eigenstates, spectrum, and Bethe ansatz equations. The general form of the determinant representation for the correlators and some particular limits (static, zero temperature, single-component) are presented in Sect. \textsection III The large distance asymptotic behavior of the static correlators at zero temperature and zero magnetic field, momentum distribution and contact are investigated in Sect. \textsection VII The low energy asymptotics of the dynamic correlators in both space-like and time-like regions are calculated in Sect. \textsection VIII and similar results for the single component system are reported in Sect. \textsection IX The derivation of the determinant representation is succinctly presented in Sects. \textsection VII, \textsection VIII, \textsection IX. In Sect. \textsection VII we compute the form factors, in Sect. \textsection VIII we summate them and the thermodynamic limit is taken in Sect. \textsection IX. We conclude in Sect. \textsection X. Some minimal information on Fredholm determinants, their numerical implementation, the asymptotic solution of certain Riemann-Hilbert problems and the rewriting of relevant functions in the thermodynamic limit are presented in several appendices.
II. MODEL AND EIGENSTATES

We consider a one-dimensional two-component system of anyonic particles with spin-independent repulsive contact interactions. For a finite system of length $L$ the Hamiltonian in second quantization is (we use units of $\hbar = 2m = k_B = 1$)

$$\mathcal{H} = \int_0^L dx \left[ (\partial_x \psi^\dagger \partial_x \psi) + c : (\psi^\dagger \psi) : -h(\psi^\dagger \psi) + B(\psi^\dagger \sigma_z \psi) \right],$$

with $c > 0$ the coupling strength, $h$ the chemical potential, $B$ the magnetic field and $: :$ denote normal ordering. In this paper we are interested in the limit of infinite repulsion, $c \to \infty$. It describes the spinor Bose gas which is also integrable \cite{110-114}. For an arbitrary value of $\kappa$ and bosons $\kappa \in [0, 1]$ it describes the spinor Bose gas which is also integrable \cite{110-114}. For an arbitrary value of $\kappa$ and coupling strength $c$ the solution of (1) in principle can be obtained employing the quantum inverse scattering method with anyonic grading \cite{115}. In this paper we are interested in the limit of infinite repulsion, $c \to \infty$, which is simpler and for which the eigenfunctions can be determined in a more easier fashion. An observation is in order. While we have chosen $\kappa \in [0, 1]$ an equally valid choice would have been $\kappa \in [-1, 0]$ with the bosonic system described by $\kappa = -1$.

The time- and space-dependent Green’s functions (field-field correlators) at finite temperature in the presence of a magnetic field are defined as

$$G^{(c)}_{\beta}(x, t | \kappa, T, B, h) \equiv \langle \psi^\dagger_{\beta}(x, t) \psi_{\beta}(0, 0) \rangle_{\kappa, T, B, h} = \frac{\text{Tr} \left[ e^{-\mathcal{H}/T} \psi^\dagger_{\beta}(x, t) \psi_{\beta}(0, 0) \right]}{\text{Tr} \left[ e^{-\mathcal{H}/T} \right]}, \ \beta \in \{1, 2\},$$

$$G^{(s)}_{\beta}(x, t | \kappa, T, B, h) \equiv \langle \psi_{\beta}(x, t) \psi^\dagger_{\beta}(0, 0) \rangle_{\kappa, T, B, h} = \frac{\text{Tr} \left[ e^{-\mathcal{H}/T} \psi_{\beta}(x, t) \psi^\dagger_{\beta}(0, 0) \right]}{\text{Tr} \left[ e^{-\mathcal{H}/T} \right]}, \ \beta \in \{1, 2\},$$

where $\psi^\dagger_{\beta}(x, t) = e^{it\mathcal{H}} \psi^\dagger_{\beta}(x) e^{-it\mathcal{H}}$, $\psi_{\beta}(x, t) = e^{it\mathcal{H}} \psi_{\beta}(x) e^{-it\mathcal{H}}$ and the trace is taken over the Fock space. The correlators of the second type of particles are related to the ones of the first type of particles by changing the sign of the magnetic field

$$G^{(c)}_{1}(x, t | \kappa, T, B, h) = G^{(c)}_{2}(x, t | \kappa, T, -B, h).$$

so in principle it is sufficient to investigate only $G^{(c)}_{1}(x, t)$ (here, and in the following, we dropped the dependence on certain parameters when there is no risk of confusion). In order to study the correlators in the thermodynamic limit we will consider first the finite size system and then will take the limit $L \to \infty$ keeping the densities of both type of particles fixed.

A. Wavefunctions and eigenvectors

We say that a state of the system is in the $(N, M)$-sector if the total number of particles is $N$ of which $M$ are of type 2 ($\alpha = 2$). The Hamiltonian (1) conserves the number of each type of particles separately and an eigenstate in the $(N, M)$-sector can be written as

$$\ket{\Psi_{N, M}(z, \lambda)} = \int_0^L dz_1 \cdots dz_N \sum_{\alpha_1, \cdots, \alpha_N = \{1, 2\}} \chi^\alpha_{N, M}(z | \lambda) \psi^\dagger_{\alpha_1}(z_1) \cdots \psi^\dagger_{\alpha_N}(z_N) \ket{\alpha_1(z_1) \cdots \alpha_N(z_N)}.$$
with \( k = \{k_j\}_{j=1}^N \) and \( \lambda = \{\lambda_j\}_{j=1}^M \) two sets of unequal parameters (note the ordering of the creation operators which helps the subsequent calculations \([93]\)). In the upper limit of the sum appearing in \([0]\) \([N,M]\) means that \( N - M \) of the \( \alpha \)'s are equal to 1 and \( M \) are equal to 2. \([0]\) is the anyonic Fock vacuum which satisfies \((0|0) = 1, \psi_\alpha(z)|0) = 0\) and \(0 = (0|\psi_\alpha^a(z))\).

In order to determine the wavefunctions we are going to use the same method as the one employed by Izergin and Pronko \([106]\) in the case of two-component fermions and bosons. Namely, first we are going to take the limit \( c \to \infty \) in the wavefunctions of the infinite line and then impose periodic boundary conditions \([1]\).

\[
\chi^\alpha_{N,M}(0, z_2, \ldots, z_N|k, \lambda) = \chi^\alpha_{N,M}(L, z_2, \ldots, z_N|k, \lambda).
\]

The first step produces the wavefunctions

\[
\chi^{\alpha_1,\cdots,\alpha_N}_{N,M}(z|k, \lambda) = \frac{1}{N!} \sum_{P \in S_N} \xi^{P(1),\cdots,P(N)}_{N,M}(\lambda) e^{i\frac{\pi}{N} \sum_{1 \leq \alpha < \beta \leq N} \text{sgn}(z_\alpha - z_\beta) \theta(z_{P(1)} < \cdots < z_{P(N)})} \det_N e^{ik_\alpha z_\alpha},
\]

where \( S_N \) is the group of permutations of \( N \) elements, \( \theta(z_{P(1)} < \cdots < z_{P(N)}) \) is a function which is equal to one when \( z_{P(1)} < \cdots < z_{P(N)} \) and zero otherwise, and \( \det_N e^{ik_\alpha z_\alpha} \) is the determinant of the \( N \times N \) matrix with elements \( e^{ik_\alpha z_\alpha} \).

The \( \xi^{\alpha_1,\cdots,\alpha_N}_{N,M} \)'s are components of a \( 2^N \)-dimensional vector \( \xi_{N,M}(\lambda) \) and in principle the components can be chosen at will as long as they produce a complete set of eigenstates \( |\Psi_{N,M}(k, \lambda)\rangle \) with the required symmetry. Following \([106]\) we are going to choose \( \xi_{N,M}(\lambda) \) as the eigenvectors of the XX0 spin chain with periodic boundary conditions and \( N \) spins of which \( M \) are down. Explicitly, the components of \( \xi_{N,M}(\lambda) \) are \([116]\)

\[
\xi^{\alpha_1,\cdots,\alpha_N}_{N,M}(\lambda) = \left( \prod_{1 \leq j < l \leq N} \text{sgn}(n_l - n_j) \right) \det_N e^{i\lambda_n n_b},
\]

where \( n_j \) gives the position of the \( j \)-th particle of type 2 in \( \{\alpha_1, \cdots, \alpha_N\} \) and the rapidities satisfy

\[
e^{i\lambda_n N} = (-1)^{M+1}, \quad l = 1, \ldots, M.
\]

The wavefunctions \([8]\) present the required anyonic symmetry when interchanging two particles of the same type \( (\xi^\alpha_{N,M}) \) are symmetric in \( n \)’s

\[
\chi^\alpha_{N,M}(\cdots, z_i, z_{i+1}, \cdots|k, \lambda) = -e^{i\pi N \text{sgn}(z_i - z_{i+1})} \chi^\alpha_{N,M}(\cdots, z_{i+1}, z_i, \cdots|k, \lambda).
\]

Imposing now periodic boundary conditions on \( \chi^\alpha_{N,M}(z|k, \lambda) \) we find

\[
\xi^{\alpha_1,\cdots,\alpha_N}_{N,M}(\lambda) = \xi^{\alpha_2,\cdots,\alpha_N\alpha_1}_{N,M}(\lambda) e^{i\pi\kappa(N-1)} e^{ik_j L}, \quad j = 1, \ldots, N,
\]

which can be interpreted as the eigenvalue problem for the cyclic shift operator \( C_N \) acting in a \( 2^N \)-dimensional vector space. Eq. \([12]\) yields

\[
e^{ik_j L} = e^{-i\pi\kappa(N-1)} e^{i\sum_{b=1}^M \lambda_b}, \quad j = 1, \ldots, N.
\]

Eqs. \([10]\) and \([13]\) represent the Bethe ansatz equations (BAEs) of the system. The states \([6]\) with wavefunctions \( \chi^\alpha_{N,M}(z|k, \lambda) \) defined in \([8]\) and \( \xi^\alpha_{N,M}(\lambda) \) given by \([6]\) represent a complete set of eigenstates of the Hamiltonian \([1]\) provided that \( k \) and \( \lambda \) satisfy the BAEs. The normalization of the eigenstates is

\[
\langle \Psi_{N,M}(k, \lambda)|\Psi_{N,M'}(k', \lambda')\rangle = |M|^M L^N \delta_{N,N'} \delta_{M,M'} \delta_{k,k'} \delta_{\lambda,\lambda'}.
\]

The allowed values for the quasimomenta \([k_\alpha]_j \in (-\infty, +\infty) \) and \([\lambda_\alpha]_l \in (-\pi, \pi) \) are

\[ [k_\alpha]_j = \frac{2\pi}{L} (j + \delta) + \frac{1}{L} \sum_{b=1}^M \lambda_b, \quad j \in \mathbb{Z}, \quad \delta = \{[-i\pi\kappa(N-1)]\}, \quad (a = 1, \ldots, N) \]

We should point out that imposing periodic boundary conditions in the case of 1D anyonic systems is not completely trivial as it was first pointed out by Averin and Nesterov \([57]\) (see also Appendix A of \([93]\)).
\[ [\lambda_b] = \frac{2\pi}{N} \left( -\frac{N}{2} + \frac{1}{4} - \frac{(-1)^{N-M}}{4} + l \right), \quad l = 1, \ldots, N, \quad (b = 1, \ldots, M), \] (15b)

where we have introduced the notation \( \{ x \} = \gamma \) if \( x = 2\pi \times \text{integer} + 2\pi \gamma \) with \( \gamma \in (-1/2, 1/2] \). The eigenvalues of the Hamiltonian \( \{ H \} \) \( \Psi_{N,M}(k, \lambda) = E_{N,M}(k) \) are

\[ E_{N,M}(k) = \sum_{j=1}^{N} (k_j^2 - h + B) - 2MB. \] (16)

### III. Determinant Representation for the Correlators

Knowledge of the eigenstates and the exact form of the wavefunctions opens the way for the computation of the correlators using a method first introduced by Korepin and Slavnov in the case of the impenetrable Bose gas \[ \text{[11]}. \] We start with a finite system and insert a resolution of identity between the field operators appearing on the right hand side of (3) obtaining a sum over form factors. Fortunately, the form factors can be computed relatively easily and the summation over them can be performed using a procedure which can be called the “insertion of the summation into the determinant” \[ \text{[106] [117]}. \] Taking the thermodynamic limit in \( \text{[3]} \) produces Fredholm determinants which can be easily computed numerically and are particularly suited to asymptotic analysis. The entire derivation is presented in Sections \[ \text{VII} \text{, VIII} \text{ and} \text{IX} \text{ in this paper we present the final results.} \]

The temperature-, time-, and space-dependent single-particle Green’s functions \[ \text{[3]} \] admit the following Fredholm determinant representations (for the definition of the Fredholm determinant see Appendix \[ \text{[A]} \]):

\[ G_1^{(-)}(x, t | \kappa, T, B, h) = e^{-it(h-B)} \int_{-\pi}^{\pi} \frac{dn}{2\pi} F(\gamma, \eta) \left[ \det \left( 1 + \gamma \hat{\Psi}_{T,B,-}^{(\gamma, \eta)} + \hat{R}_{T,B,-}^{(\gamma, \eta)} \right) \right] - \det \left( 1 + \gamma \hat{\Psi}_{T,B,-}^{(\gamma, \eta)} \right) \right], \] \hspace{1cm} (17)

\[ G_1^{(+)}(x, t | \kappa, T, B, h) = e^{it(h-B)} \int_{-\pi}^{\pi} \frac{dn}{2\pi} F(\gamma, \eta) \left[ \det \left( 1 + \gamma \hat{\Psi}_{T,B,+}^{(\gamma, \eta)} \right) - \gamma \hat{R}_{T,B,+}^{(\gamma, \eta)} \right] \] \hspace{1cm} + \left( G(x, t) - 1 \right) \det \left( 1 + \gamma \hat{\Psi}_{T,B,+}^{(\gamma, \eta)} \right) \right], \] \hspace{1cm} (18)

where \( \gamma = 1 + e^{2B/T} \), \( F(\gamma, \eta) = 1 + \sum_{p=1}^{\infty} \gamma^{-p} (e^{i\eta p} + e^{-i\eta p}) \) and \( G(x, t) = \int_{-\infty}^{+\infty} e^{-it k^2 + ikx} dk/2\pi \). The action of the integral operators \( \hat{\Psi}_{T,B,+}^{(\gamma, \eta)} \) and \( \hat{R}_{T,B,+}^{(\gamma, \eta)} \) on an arbitrary function \( f(k) \) is given by

\[ \left( \hat{\Psi}_{T,B,+}^{(\gamma, \eta)} f \right)(k) = \int_{-\infty}^{+\infty} \left( \hat{\Psi}_{T,B,+}^{(\gamma, \eta)} \right) (\eta, \kappa | k, k') f(k') dk', \quad \left( \hat{R}_{T,B,+}^{(\gamma, \eta)} f \right)(k) = \int_{-\infty}^{+\infty} \left( \hat{R}_{T,B,+}^{(\gamma, \eta)} \right) (\eta, \kappa | k, k') f(k') dk', \] \hspace{1cm} (19)

with kernels

\[ \left( \hat{\Psi}_{T,B,+}^{(\gamma, \eta)} \right) (\eta, \kappa | k, k') = 4 \sin^2 \pi \nu(k) \sqrt{\vartheta(k)} \frac{E^{(\pm)}(k)}{2\pi\lambda(k-k')} \sqrt{\vartheta(k')}, \quad \nu(k) = \pm \left( \frac{\eta}{2\pi} - \frac{\kappa}{2} \right), \] \hspace{1cm} (20)

\[ \left( \hat{R}_{T,B,+}^{(\gamma, \eta)} \right) (\eta, \kappa | k, k') = -2 \sin^2 \pi \nu(k) \mean{\vartheta(k)} \frac{E^{(\pm)}(k)}{2\pi\lambda(k-k')} \sqrt{\vartheta(k')}, \] \hspace{1cm} (21)

\[ \left( \hat{R}_{T,B,+}^{(\gamma, \eta)} \right) (\eta, \kappa | k, k') = \sqrt{\vartheta(k)} \frac{e(k)}{2\pi} \sqrt{\vartheta(k')}, \quad \vartheta(k) = \frac{e^{-B/T}}{2 \cosh(B/T) + e^{(k^2-h)/T}}, \] \hspace{1cm} (22)

Here \( \vartheta(k) \) is the Fermi function and

\[ e(x, t | k) = e^{\frac{ixk^2 - i\kappa x}{T}}, \] \hspace{1cm} (23a)

\[ E^{(\pm)}(x, t | k) = ie(k) \left\{ \text{p.v.} \int_{-\infty}^{+\infty} dq \frac{e^{-2(x, t | q)} - \frac{1}{2} \cot \pi \nu(k) e^{-2(x, t | k)}}{q - k} \right\}, \] \hspace{1cm} (23b)

with p.v. denoting the principal value integral. The representations for the correlators \( g_2^{(\pm)}(x, t | \kappa, T, B, h) \) are obtained from \[ \text{[17] and [18] using the symmetry [5]. Other useful symmetries are (the bar stands for complex conjugation)} \]

\[ g_2^{(\pm)}(x, t | \kappa) = g_2^{(\pm)}(-x, -t | \kappa), \quad g_2^{(\pm)}(-x, -t | \kappa) = g_2^{(\pm)}(x, t | -\kappa) \] and \( g_2^{(\pm)}(x, t | -\kappa) = g_2^{(\pm)}(x, t | \kappa) \). \hspace{1cm} (24)
Our results are equivalent with the determinant representations derived by Izergin and Pronko \cite{106} for impenetrable two-component fermions (κ = 0) and two-component bosons (κ = 1) but we should point out a subtle difference. If we would rewrite the results from \cite{106} in our notation their \( E^{(-)}(k) \) function for fermions has a plus sign instead of a minus sign in front of the \( \cot(\eta/2) \) term (see Eq. (23)) for \( \kappa = 0 \). However this minus sign is irrelevant because in the expansion of the determinant it will give contributions of the type \( (- \cot(\eta/2))^p \) integrated over a symmetric interval and therefore only even powers of \( p \) will give a contribution. In the rest of this section we are going to present some particular cases of the general formulae \cite{17} and \cite{18}.

### A. Static correlators

The determinant representations for the correlators simplify considerably in the static limit. For \( t = 0 \) the principal value integral appearing in Eq. (23b) can be computed analytically by closing the contour in the upper half-plane for \( x > 0 \) and in the lower half plane for \( x < 0 \) with the result \( \int_{-\infty}^{+\infty} e^{iqx}/2\pi(q-k) \, dq = \pm \text{sgn}(x) \, e^{ikx}/2 \). The relevant functions \( e(k) \) and \( E^{(\pm)}(k) \) reduce to \( e(x,0 | k) = e^{-ikx} \), \( E^{(\pm)}(x,0 | k) = e^{-ikx} \left[ i \cot \pi \nu^{(\pm)} - \text{sgn}(x) \right] /2 \). Introducing the integral operators \( \psi^{(T,B,\pm)} \) and \( \tau^{(T,B,\pm)} \) acting on the entire real axis with kernels

\[
\psi^{(T,\pm)}(k,k') = -2 \sqrt{\vartheta(k)} \frac{\sin[(k-k')|x|/2]}{\pi(k-k')} \sqrt{\vartheta(k')},
\]

\[
\tau^{(T,\pm)}(k,k') = 2\pi \sqrt{\vartheta(k)} \frac{e^{i\pm k|x/k|/2}}{\sqrt{\vartheta(k')}} \frac{\pi}{2},
\]

then \( \psi^{(T,B,\pm)}(\eta,\kappa | k,k') \overset{t \to 0}{\longrightarrow} \frac{1}{2} \left( 1 - e^{-i \text{sgn}(x) \pi} e^{-i\eta n} \right) \psi^{(T,\pm)}(k,k') \), \( \tau^{(T,B,\pm)}(\eta,\kappa | k,k') \overset{t \to 0}{\longrightarrow} e^{i \text{sgn}(x) (\eta - \pi \kappa)} \tau^{(T,\pm)}(k,k') \) and \( R^{(T,B,-)}(\eta,\kappa | k,k') \overset{t \to 0}{\longrightarrow} r^{(T,-)}(k,k') \). Because the dependence on \( \eta \) is now very simple we can integrate. For example, expanding the first determinant appearing in the right hand side of \cite{17} for \( t = 0 \) and \( x > 0 \) we have

\[
\det \left( 1 + \gamma \psi^{(T,B,-)}(\eta) + R^{(T,B,-)}(\eta) \right) \overset{x \to 0}{\longrightarrow} \sum_{n=0}^{\infty} \left( \frac{\gamma}{2} \right)^n \left( 1 - e^{i \pi \kappa} e^{-i\eta n} \right) \sum_{n=0}^{\infty} \left( \frac{\gamma}{2} - \frac{e^{i \pi \kappa}}{2} \right)^n A_n(k,k'),
\]

where we have use the binomial formula and integrated term by term. Therefore the determinant representation for the static \( G_1^{-}(x,0) \) correlator is given by

\[
G_1^{-}(x,0) = \det \left( 1 + \gamma \psi^{(T,-)} + \tau^{(T,-)} \right) - \det \left( 1 + \overline{\gamma} \psi^{(T,-)} \right), \quad \overline{\gamma} = \left( \gamma - e^{-i \text{sgn}(x) \pi \kappa} \right) /2,
\]

with \( \psi^{(T,-)}(k,k') \) and \( \tau^{(T,-)}(k,k') \) defined in \cite{25} and \cite{26}. In a similar fashion it can be shown that (\( \overline{\gamma} \) is the complex conjugate of \( \gamma \))

\[
G_1^{(+)}(x,0) = \det \left( 1 + \gamma \psi^{(T,+)} - e^{-i \text{sgn}(x) \pi \kappa} \tau^{(T,+)} \right) + (\delta(x) - 1) \det \left( 1 + \overline{\gamma} \psi^{(T,+)} \right).
\]

In contrast with the fermionic (\( \kappa = 0 \)) and bosonic case (\( \kappa = 1 \)) the static correlators are complex and they are different depending on the sign of \( x \). It is easy to see from the representations \cite{28} and \cite{29} that we have

\[
G_1^{(\pm)}(x,0) = \overline{G_1^{(\pm)}}(-x,0).
\]

As we will see below this results in a nonsymmetric momentum distribution. The \( G_2^{(\pm)}(x,0) \) correlators are obtained using \cite{5}. The two types of correlators then satisfy \( G_1^{(+)}(x,0) = \delta(x) - e^{-i \pi \kappa} G_1^{(-)}(-x,0) \).

### B. Correlators at \( T = 0 \) and \( B = 0 \)

At zero temperature the properties of the system are heavily influenced by the value of the magnetic field. This is due to the fact that our model presents a quantum phase transition at \( B = 0 \) with a ground state that is completely polarized (only particles of one type) for \( |B| > 0 \) and a balanced one at \( B = 0 \). The balanced case, including the
asymptotic behavior of the correlators and the momentum distribution, will be studied in detail in Sect. IV and V.

In the zero temperature limit \( \lim_{T \to 0} \gamma \theta(k) = \theta(k_F - k) \) with \( \theta(k) \) the Heaviside function and \( k_F = (h + |B|)^{1/2} \). For the balanced system at zero temperature \( \gamma = 2, k_F = h^{1/2} \) and the correlators of both type of particles are equal \( g_1^{(-)}(x,t) = g_2^{(+)}(x,t) \). Using the identity 1.447 (3) of [118], \( 1 + 2 \sum_{p=1} a_p \cos px = (1 - a^2)/(1 - 2a \cos x + a^2), |a| < 1 \), we find \( F(\gamma = 2, \eta) = 3/(5 - 4 \cos \eta) \) and from [17] and [18] we obtain

\[
\begin{align*}
G_1^{(-)}(x,t) &= e^{-ith} \int_{-\infty}^{\infty} \frac{d\eta}{2\pi} \frac{3}{5 - 4 \cos \eta} \left[ \det \left( 1 + \mathcal{V}^{(0,0,-)}(\eta) + \frac{1}{2} \mathcal{R}^{(0,0,-)}(\eta) \right) - \det \left( 1 + \mathcal{V}^{(0,0,-)}(\eta) \right) \right], \\
G_1^{(+)}(x,t) &= e^{ith} \int_{-\infty}^{\infty} \frac{d\eta}{2\pi} \frac{3}{5 - 4 \cos \eta} \left[ \det \left( 1 + \mathcal{V}^{(0,0,+)}(\eta) - \mathcal{R}^{(0,0,+)}(\eta) \right) \right] + (G(x,t) - 1) \det \left( 1 + \mathcal{V}^{(0,0,+)}(\eta) \right),
\end{align*}
\]

where the integral operators \( \mathcal{V}^{(0,0,\pm)} \) and \( \mathcal{R}^{(0,0,\pm)} \) act on \([-k_F, k_F]\) and have the kernels

\[
\begin{align*}
\mathcal{V}^{(0,0,\pm)}(\eta, \kappa | k, k') &= 4 \sin^2 \pi \nu^{(\pm)} \left( \frac{E^{(\pm)}(k) e(k') - E^{(\pm)}(k') e(k)}{2\pi i (k - k')} \right), \quad \nu^{(\pm)} = \pm \left( \frac{\eta - \kappa}{2\pi} \right), \\
\mathcal{R}^{(0,0,\pm)}(\eta, \kappa | k, k') &= -2 \sin^2 \pi \nu^{(\pm)} \left( \frac{E^{(\pm)}(k) e(k')}{\pi} \right), \quad \mathcal{R}^{(0,0,-)}(\eta, \kappa | k, k') = \frac{e(k)e(k')}{2\pi}.
\end{align*}
\]

The functions \( e(k) \) and \( E^{(\pm)}(k) \) are defined in (23a) and (23b).

C. Correlators at \( T = 0 \) and \( B \neq 0 \)

We consider the case of negative magnetic field, \( B < 0 \), keeping in mind that the correlators at positive magnetic field can be determined using [19]. The ground state is formed by particles of type 1, \( \gamma = 1 \) and \( F(\gamma = 1, \eta) = 2\pi \delta(\eta) \). Therefore, we are effectively dealing with a single component system (see Sect. III D). The integration is now trivial and we find for the correlators of type 1 particles

\[
\begin{align*}
G_1^{(-)}(0,t) &= e^{-it(h-B)} \left[ \det \left( 1 + \mathcal{V}^{(0,B,-)}(0) + \mathcal{R}^{(0,B,-)}(0) \right) - \det \left( 1 + \mathcal{V}^{(0,B,-)}(0) \right) \right], \\
G_1^{(+)}(0,t) &= e^{it(h-B)} \left[ \det \left( 1 + \mathcal{V}^{(0,B,+)}(0) - \mathcal{R}^{(0,B,+)}(0) \right) + (G(x,t) - 1) \det \left( 1 + \mathcal{V}^{(0,B,+)}(0) \right) \right],
\end{align*}
\]

with the integral operators \( \mathcal{V}^{(0,B,\pm)} \) and \( \mathcal{R}^{(0,B,\pm)} \) acting on \([-k_F, k_F]\) with \( k_F = (h + |B|)^{1/2} \). The kernels are

\[
\begin{align*}
\mathcal{V}^{(0,B,\pm)}(0, \kappa | k, k') &= 4 \sin^2 \left( \frac{\pi \kappa}{2} \right) \left( \frac{E^{(\pm)}(k) e(k') - E^{(\pm)}(k') e(k)}{2\pi i (k - k')} \right), \\
\nu^{(\pm)} &= \pm \left( \frac{\kappa}{2\pi} \right), \\
\mathcal{R}^{(0,B,\pm)}(0, \kappa | k, k') &= -2 \sin^2 \left( \frac{\pi \kappa}{2} \right) \left( \frac{E^{(\pm)}(k) e(k')}{\pi} \right), \quad \mathcal{R}^{(0,B,-)}(0, \kappa | k, k') = \frac{e(k)e(k')}{2\pi}.
\end{align*}
\]

Because the ground state is comprised by only particles of type 1 the \( G_2^{(-)}(x,t) \) correlator is zero. However, for the other correlator we get \( G_2^{(+)}(x,t | \kappa, 0, B, h) = G_1^{(+)}(x,t | \kappa, 0, -B, h) \). For \( B > 0 \) we have \( \gamma = \infty \) and \( F(\gamma, \eta) = 1 \), therefore

\[
G_2^{(+)}(x,t) = e^{it(h+B)} \int_{-\infty}^{\infty} \frac{d\eta}{2\pi} \left[ \det \left( 1 + \mathcal{V}^{(0,0,+)}(\eta) - \mathcal{R}^{(0,0,+)}(\eta) \right) + (G(x,t) - 1) \det \left( 1 + \mathcal{V}^{(0,0,+)}(\eta) \right) \right].
\]

The kernels of the integral operators \( \mathcal{V}^{(0,0,+) \pm} \) and \( \mathcal{R}^{(0,0,+) \pm} \) are defined in (32a) and (32b) and they act on \([-k_F, k_F]\) with \( k_F = (h + |B|)^{1/2} \). The \( G_2^{(+)}(x,t) \) correlator describes the situation in which a single particle of type 2 is injected in a sea of particles of type 1 (see also [119, 120]).

D. Single component limit

In the limit \( B \to -\infty, h \to \infty \) such that \( h_1 = h - B \) is finite the energy of the second type of particles becomes infinite \( (h + B \to \infty) \) and they are effectively excluded from the system. In this limit the model reduces to a single
component system with chemical potential $h_1$ and $\gamma = 1$ for which $F(\gamma = 1, \eta) = 2\pi \delta(\eta)$. The Fermi function becomes 
$\vartheta(k) = (1 + e^{(k^2 - h_1)/T})^{-1}$ and the correlators are 
$$\begin{align*}
G_1^{(-)}(x,t) &\Big|_{h_1 = h_B} = e^{-ith_1} \left[ \det \left( 1 + \hat{V}^{(T,-)}(0) + \hat{R}^{(T,-)}(0) \right) - \det \left( 1 + \hat{V}^{(T,-)}(0) \right) \right], \\
G_1^{(+)}(x,t) &\Big|_{h_1 = h_B} = e^{+ith_1} \left[ \det \left( 1 + \hat{V}^{(T,+)}(0) - \hat{R}^{(T,+)}(0) \right) + (G(x,t) - 1) \det \left( 1 + \hat{V}^{(T,+)}(0) \right) \right].
\end{align*}$$

The integral operators act on the real axis and the kernels are 
$$\begin{align*}
\hat{V}^{(T,\pm)}(0,\kappa | k, k') &= 4 \sin^2 \left( \frac{\pi \kappa}{2} \right) \sqrt{\vartheta(k)} \frac{E^{(\pm)}(k) \vartheta(k') - E^{(\pm)}(k') \vartheta(k)}{2\pi i (k - k')}, \ n^{(\pm)} = \mp \frac{\kappa}{2}, \\
\hat{R}^{(T,\pm)}(0,\kappa | k, k') &= -2 \sin^2 \left( \frac{\pi \kappa}{2} \right) \sqrt{\vartheta(k)} \frac{E^{(\pm)}(k) E^{(\pm)}(k')}{\pi} \sqrt{\vartheta(k')}, \ R^{(T,\pm)}(0,\kappa | k, k') = \sqrt{\vartheta(k)} \frac{\vartheta(k')}{2\pi} \sqrt{\vartheta(k')}. 
\end{align*}$$

At $\kappa = 1$ this result reduces to the determinant representation obtained by Korepin and Slavnov for impenetrable bosons and for arbitrary statistics the representation derived for single component anyons in [121] (note that the statistics parameter $\kappa'$ used in [121] is related to ours via $\kappa' = 1 + \kappa$).

IV. LARGE DISTANCE ASYMPTOTICS OF STATIC CORRELATORS AT $T = 0$ AND $B = 0$

The determinant representations presented in the previous section are important for several reasons. Not only that they constitute the starting point in proving that the correlators of quantum integrable systems are governed by classical integrable systems [13, 122, 123] but they are also particularly suited for the derivation of the asymptotic behavior both at small and large spatial separations. We will see that the asymptotic analysis is closely connected with the solution of an associated Riemann-Hilbert problem. Employing this method rigorous results can be derived which then can be compared with the TLL/CFT predictions or, as in the case of the spin-incoherent Tomonaga-Luttinger liquid, they can reveal totally new regimes. In addition to being extremely useful in deriving analytical results the determinant representations can also be efficiently implemented numerically [124] providing accurate information on the correlators. Coupled with the results on the asymptotic behavior the numerical evaluation of the determinants can be used to compute the momentum distribution or the time and space Fourier transform of the dynamical correlators which are both experimentally accessible quantities. In this section we are going to derive the large distance asymptotic behavior of the static correlators at $T = 0$ and $B = 0$. This is a very interesting regime because as we will see, compared with the single component system which is characterized by an algebraic decay of the correlators, in our case we will have an exponential decay even though we are at zero temperature. This is due to the fact that our system is in the spin-incoherent regime. We are also going to numerically evaluate the momentum distribution and show that is nonsymmetric for $\kappa \neq \{0, 1\}$ and that it presents a universal $\gamma$ dependence.

The determinant representation for the static correlators in the ground state of the balanced system can be obtained from (28) and (29) by noticing that if we take the limit $\kappa \to \infty$ we have 
$$\begin{align*}
G_1^{(-)}(x,t) &\Big|_{h_1 = h_B} = e^{-ith_1} \left[ \det \left( 1 + \hat{V}^{(T,-)}(0) + \hat{R}^{(T,-)}(0) \right) - \det \left( 1 + \hat{V}^{(T,-)}(0) \right) \right], \\
G_1^{(+)}(x,t) &\Big|_{h_1 = h_B} = e^{+ith_1} \left[ \det \left( 1 + \hat{V}^{(T,+)}(0) - \hat{R}^{(T,+)}(0) \right) + (G(x,t) - 1) \det \left( 1 + \hat{V}^{(T,+)}(0) \right) \right].
\end{align*}$$

Introducing two functions $e_\pm^s(k)$ these kernels can be written as 
$$\begin{align*}
\vartheta^{(0,-)}(k, k') &= \frac{\xi e_\pm^s(k) e_\pm^s(k') - e_\pm^s(k) e_\pm^s(k')}{k - k'}, \ r^{(0,-)}(k, k') = \frac{e_\pm^s(k) e_\pm^s(k')}{4\pi}, \ e_\pm^s(k) = e^{\pm ikx/2}. 
\end{align*}$$

The particular factorization shown in (33) reveals that the $\hat{V}^{(0,-)}$ operator is of a special type called integrable integral operators [122, 125] for which the resolvent belongs to the same algebra. More precisely, if we define $f_\pm(k)$ as the
solutions of the integral equations

\[ f_+(k) + \int_{-k_F}^{k_F} \sqrt{(0,-)}(k,k') f_+(k') \, dk' = e_+^+(k), \quad f_-(k) + \int_{-k_F}^{k_F} \sqrt{(0,-)}(k,k') f_-(k') \, dk' = e_-^+(k), \tag{44} \]

the resolvent operator which satisfies \((1 - \hat{w}(0,-)) = (1 + \hat{v}(0,-))^{-1} \) has a kernel given by the same formula as the one for \(\sqrt{(0,-)}(k,k')\) given in [43] with the functions \(e_\pm^+ (k)\) replaced by \(f_\pm(k)\). We also point out that \(e_\pm^+(k) e_\pm^-(k) = 0\) which means that the kernel is nonsingular on the diagonal.

We want to rewrite (41) in a form which is more amenable to extracting the asymptotic behavior at large \(x\). Noticing that \(\nu(0,-)(k,k')\) is a rank 1 matrix and using the fact that for a linear function in \(z\) we have \(\frac{\partial}{\partial z} f(z)|_{z=0} = f(1) - f(0)\) we obtain

\[ \mathcal{G}_1^-(x,0) = \frac{\partial}{\partial z} \det \left( 1 + \hat{v}(0,-) + z \hat{r}(0,-) \right) \big|_{z=0} = \text{Tr} \left( \left( 1 + \hat{v}(0,-) \right)^{-1} \hat{r}(0,-) \right) \left( 1 + \hat{v}(0,-) \right), \]

\[ = \frac{1}{4\pi} \int_{-k_F}^{k_F} \int_{-k_F}^{k_F} \left( 1 + \hat{v}(0,-) \right)^{-1} (k,k') e_\pm^+(k) e_\pm^-(k') \, dk \, dk' \det \left( 1 + \hat{v}(0,-) \right), \]

\[ = \frac{1}{4\pi} B_{-\pm} \det \left( 1 + \hat{v}(0,-) \right), \tag{45} \]

where we have used (A5) in the first line, the definition of the \(f_\pm(x)\) function (44) in the second line and introduced the notation \(B_{ab} = \int_{-k_F}^{k_F} e_\pm^a(k) f_b(k) \, dk\) with \(a,b = \pm\). The objects \(B_{ab}\) are called potentials and they play an important role in the asymptotic analysis. We also need to introduce \(\tilde{e}(k)\) and \(\tilde{f}(k)\) defined by \(\tilde{e}(k) = (e_+^+(k),e_-^+(k))^T\) and \(\tilde{f}(k) = (f_+(k),f_-(k))^T\).

### A. Riemann-Hilbert problem for the static correlator

From [45] we see that in order to obtain the large distance asymptotic behaviour of the static correlator we need to analyse the large \(x\) behaviour of both \(B_{-\pm}(x)\) and the Fredholm determinant. Our task is considerably simplified by using a very powerful result of Kitanine, Kozlowski, Maillet, Slavnov and Terras [107] which investigated the more complicated case of the generalized sine-kernel of which our kernel is a particular case. In [107] the authors were mainly interested in the determinant asymptotics but their analysis can also be used to derive the large distance behaviour of \(B_{-\pm}(x)\). The Riemann-Hilbert problem associated with our determinant is the following (Prop. 3.1 of [107]). Let \(\hat{v}(0,-)\) be the integral operator defined in (42) acting on \(L^2([-k_F,k_F])\) and such that \(\det \left( 1 + \hat{v}(0,-) \right) \neq 0\). Then there exists a 2-by-2 matrix \(\chi(k)\) such that \(\tilde{f}(k) = \chi(k) \tilde{e}(k)\) and is the unique solution of the RHP:

- \(\chi(k)\) is analytic on \(\mathbb{C}\setminus[-k_F,k_F]\).
- \(\chi_+(k) G_\chi(k) = \chi_-(k)\) for \(k \in (-k_F,k_F)\) with jump matrix
  \[ G_\chi(k) = \begin{pmatrix} 1 - \xi & \xi |e_\pm^+(k)|^2 \\ -\xi |e_\pm^+(k)|^2 & 1 + \xi \end{pmatrix}. \tag{46} \]

Here \(\chi_\pm(k')\) represent the limiting values of \(\chi_\pm(k)\) when \(k\) tends to the point \(k'\) of \((-k_F,k_F)\) from the left, respectively right, side of the contour.

- \(\chi(k) \rightarrow I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\) and \(\chi(k) = O \left( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \ln |k^2 - k_F^2| \right)\) for \(k \rightarrow \pm k_F\).

- Also
  \[ \chi(k) = I_2 - \frac{\xi}{2\pi i} \int_{-k_F}^{k_F} \frac{1}{k' - k} \begin{pmatrix} -e_+^+(k') f_+(k') & e_-^+(k') f_+(k') \\ -e_-^+(k') f_-(k') & e_+^+(k') f_-(k') \end{pmatrix} \, dk'. \tag{47} \]

The last relation is extremely important because it shows that the potentials \(B_{ab}(k)\) can be obtained from the large-\(k\) expansion of the RHP solution

\[ \lim_{k \rightarrow \infty} \chi(k) = I_2 + \frac{1}{k} \frac{\xi}{2\pi i} \begin{pmatrix} -B_{-+} & B_{++} \\ -B_{+-} & B_{-+} \end{pmatrix} + O \left( \frac{1}{k^2} \right). \tag{48} \]
In general the derivative of the Fredholm determinant with respect to $x$ or $t$ (in the dynamical case) is expressed in terms of the $B_{ab}(k)$ potentials. Therefore, the asymptotic behavior of the determinant can be derived from the large $k$ and $x$ expansion of the RHP solution.

**B. Large distance asymptotic behavior of the static correlator**

The Fredholm determinant appearing in (41) is a particular case of the generalized sine-kernel studied in [107]. Introducing

$$\nu \equiv -\frac{1}{2\pi i} \ln(1 + \xi) = \frac{i}{2} \ln 2 - \frac{\kappa}{2},$$

the asymptotic behavior of the determinant for large $x$ is given by the following theorem:

**Theorem IV.1.** [107] Let $\hat{\psi}^{(0,-)}$ be the integral operator with the kernel defined in (42). For $\kappa \in [0,1)$ and $x \to \infty$ we have

$$\det \left( 1 + \hat{\psi}^{(0,-)} \right) = G_B^2(1,\nu) e^{-2ik_Fvz} \frac{\exp \left\{ - \frac{z(z-1)}{2} + \int_0^z \Psi(t) \, dt \right\}}{x^{2\nu}},$$

(50)

where $G_B(1,z) = G_B(1+z)G_B(1-z)$ and $G_B(z)$ is Barnes $G$-function [109] which has the integral representation

$$G_B(z+1) = (2\pi)^{z/2} \exp \left\{ - \frac{z(z-1)}{2} + \int_0^z \Psi(t) \, dt \right\}, \quad Re(z) > -1, \quad \Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$ (51)

We should point out that the analysis in [107] is valid only for $\kappa \in [0,1)$ so in principle this result cannot be used in the case of two component bosons ($\kappa = 1$). Nevertheless numerical data obtained from the evaluation of the representation [11] agrees with the asymptotic behavior derived by taking the limit $\kappa = 1$ in the previous result.
Thm. [IV.1] together with the expression for $B_{\nu}$ ([CS]) which is derived in Appendix [C] allows us to state the main result of this section, the large distance asymptotic behavior of the static correlators at zero temperature and zero magnetic field is

$$G_1^{(-)}(x, 0) = \frac{\pi e^{C(n)}}{2\xi \sin \pi \nu} e^{-2ik_F\nu x} e^{-(2\nu^2+1)} \left[ \frac{(2k_F\nu x)^{-2\nu}}{\Gamma^2(-\nu)} e^{-ik_Fx} - \frac{(2k_F\nu x)^{2\nu}}{\Gamma^2(\nu)} e^{ik_Fx} \right],$$  \hspace{1cm} (52)

with $\nu$ defined in [40], $\xi$ in [42] and

$$C(\nu) = -2\nu^2 [1 + \ln(2k_F)] + 2\nu \ln \left[ \frac{\Gamma(\nu)}{\Gamma(-\nu)} \right] - 2 \int_0^\nu \ln \left[ \frac{\Gamma(t)}{\Gamma(-t)} \right] dt = \ln \left[ \frac{G_B^2(1, \nu)}{(2k_F)^{2\nu^2}} \right].$$  \hspace{1cm} (53)

This result is valid for large $x > 0$. For $x < 0$ we use $G_1^{(-)}(x, 0) = G_1^{(-)}(-x, 0)$. In Fig. 1 we present the numerical evaluation of the static correlator and our asymptotic formula (52) and we see that already for $x > 5$ the two curves become indistinguishable. The accuracy of the expansion is the largest for $\kappa = 0$ (fermions) and decreases as we increase the statistics parameter but it is still very accurate even in the bosonic case.

In the fermionic case the asymptotic formula (52) was first obtained by Berkovich and Lowenstein [5, 6] (a different proof similar with the one presented here was later derived in [8]). For $\kappa = 0$ [52] can be written as

$$G_1^{(-)}(x, 0) \overset{\kappa=0}{=} C'(0) e^{-\frac{i\pi}{2}k_Fx} x^{-1+\frac{1}{2} \left( \frac{\ln 2}{\pi} \right)^2} \sin (k_Fx - \ln 2 \ln x - \pi - \varphi_0),$$  \hspace{1cm} (54)

with $C'(0) = -4\pi \sqrt{2} \exp[C[-\ln 2/(2\pi)] - 2 \Re \ln \Gamma(i \ln 2/2\pi)]$ and $\varphi_0 = \ln 2 \ln(2k_F)/\pi - 2 \Im \ln \Gamma(i \ln 2/2\pi)$. In the bosonic case ($\kappa = 1$) the first term in the square parenthesis of (52) gives the leading contribution and we find

$$G_1^{(-)}(x, 0) \overset{\kappa=1}{=} 1 - \frac{\pi e^{C(n)}}{3 \sin \pi \nu} \frac{(2k_F)^{-2\nu}}{\Gamma^2(-\nu)} e^{-\frac{i\pi}{2}k_Fx} x^{-\frac{1}{2}+\frac{1}{2} \left( \frac{\ln 2}{\pi} \right)^2}, \ \nu = -\frac{i \ln 2}{2\pi} - \frac{1}{2}.$$  \hspace{1cm} (55)

This result appeared first, without the constant and derivation, in [11] (please note that their result corresponds to $k_F = 2$).

The most striking feature of the asymptotic expansion for the static correlators is the exponential decay even though we are at zero temperature. This is a consequence of the fact that we are considering the correlators in the spin-incoherent regime which is characterized by a highly excited spin sector. A computation of the entropy per length from Takahashi’s formula [114] gives $s = \ln 2$ even at zero temperature which shows that the spin sector is completely disordered. The exponential decay term is the same for all values of the statistics parameter, $e^{-\frac{i\pi}{2}k_Fx}$ and it depends only on the number of components of the system. This is another particularity of the spin-incoherent system [12], for an $N$-component system the exponential decay is given by $e^{-\frac{i\pi}{2}k_Fx}$. This exponential decay is accompanied by an oscillatory term with frequency proportional with the statistics $e^{ik_Fx}$. The algebraic corrections become smaller as we increase $\kappa$. In the vicinity of the fermionic point, $\kappa = 0$, both terms of the expansion in (52) are important while for $\kappa \sim 1$ only one term gives the leading contribution. This mirrors a similar behavior of the correlators of single component impenetrable anyons for which the algebraic decay at zero temperature is given by ($x > 0$) [50]

$$G_1^{(-)}(x, 0) \sim b_0 \frac{e^{i(1-\kappa)k_Fx}}{x^{1-\kappa+\kappa^2/2}} + b_{-1} \frac{e^{-i(1+\kappa)k_Fx}}{x^{1+\kappa+\kappa^2/2}},$$  \hspace{1cm} (56)

with $b_0, b_{-1}$ constants.

C. Momentum distribution and contact

The experimentally relevant momentum distribution is defined as the Fourier transform of the static correlator $n_1(k) = \int e^{-ikx} G_1^{(-)}(x, 0) \, dx$. While in the balanced system we have $n(k) = n_1(k) = n_2(k)$, in the presence of a magnetic field the momentum distributions of each type of particles is different. A simple consequence of $G_1^{(-)}(x, 0) = G_1^{(-)}(-x, 0)$ is that

$$n(k) = 2 \int_0^\infty \cos(kx) \Re G_1^{(-)}(x, 0) \, dx + 2 \int_0^\infty \sin(kx) \Im G_1^{(-)}(x, 0) \, dx.$$  \hspace{1cm} (57)

In the fermionic and bosonic case $\Im G_1^{(-)}(x, 0) = 0$ and therefore the momentum distribution is symmetric, $n(k) = n(-k)$, but for an arbitrary value of the statistics parameter the static correlator has a nonzero imaginary part which
results in a nonsymmetric momentum distribution as it can be seen in the left panel Fig. 2. Compared with the single component case \( n(k) \) does not have any singularities due to the exponential decay. An universal property of models with contact interactions is that the tail of the momentum distribution is given by \( \lim_{|k| \to \infty} n(k) = C/k^4 \) with \( C \) a quantity called contact [126,137]. This is due to the discontinuity of the wavefunction’s derivative at the coinciding points and, as it can be seen in the right panel of Fig. 2 even though the momentum distribution is nonsymmetric the tails exhibit the \( 1/k^4 \) behavior with the amplitude given by the contact. For two-component impenetrable anyons the contact is a monotonic function of the statistics parameter with the minimum obtained for the fermionic system and reaching its maximum for the bosonic system.

V. LONG TIME LARGE DISTANCE ASYMPTOTICS OF DYNAMIC CORRELATORS AT \( T = 0 \) AND \( B = 0 \)

The derivation of the asymptotic behavior of time and space dependent correlators is more involved than the static case but it follows along the same lines. The Fredholm determinants appearing in the representations for the dynamic correlators are also integrable which means that their asymptotics can be derived by solving an associated RHP. Their kernels are particular cases of the time dependent generalization of the sine-kernel for which a comprehensive asymptotic analysis was performed in [10] by Kozlowski. We are interested in the large \( k \) behavior of the dynamic correlators for a fixed value of the saddle point \( k_0 = x/2t \). The results will be different depending on the position of the saddle point with respect to the Fermi points \( \pm k_F \). If \( |x| > 2k_F|t| \) we will say that we are in the space-like regime and for \( |x| < 2k_F|t| \) we are in the time-like regime. Our results are not valid for \( k_0 = \pm k_F \).

At zero temperature and no magnetic field the determinant representation for dynamic correlators is given by (31) with the kernels defined in (32). The operators \( \hat{V}^{(0,0,\pm)} \) are integrable with the kernels of the resolvents which satisfy

\[
\left( 1 - \hat{W}^{(0,0,\pm)} \right) = \left( 1 + \hat{V}^{(0,0,\pm)} \right)^{-1}
\]

where the functions \( F^{(\pm)}(k) \) and \( f(k) \) are solutions of the integral equations

\[
F^{(\pm)}(k) + \int_{-k_F}^{k_F} V^{(0,0,\pm)}(k,k') F^{(\pm)}(k') \, dk' = E^{(\pm)}(k), \quad f(k) + \int_{-k_F}^{k_F} V^{(0,0,\pm)}(k,k') f(k') \, dk' = e(k).
\]

Introducing the potentials \( B_{++} = \int_{-k_F}^{k_F} E^{(\pm)}(k) F^{(\pm)}(k) \, dk, B_{-+} = \int_{-k_F}^{k_F} e(k) f(k) \, dk, B_{+-} = \int_{-k_F}^{k_F} E^{(\pm)}(k) f(k) \, dk, B_{--} = \int_{-k_F}^{k_F} e(k) F^{(\pm)}(k) \, dk \) and

\[
H(\eta) = \frac{3}{5 - 4 \cos \eta} = 1 + \frac{e^{i\eta}}{2 - e^{i\eta}} + \frac{e^{-i\eta}}{2 - e^{-i\eta}}, \quad b_{++} = \frac{2 \sin^2 \pi \nu^{(\pm)} / \pi}{B_{++} + G(x,t)}.
\]
then, analogous with the derivation of (45), we can derive a more suitable representation for the correlators

\begin{align}
G_1^-(x,t) &= e^{-ith} \int_{-\pi}^{\pi} \frac{d\eta}{2\pi} \frac{H(\eta)}{4\pi i z} B_{-\eta}(\eta) \det \left(1 + \tilde{\mathcal{V}}_{(0,0,-)}^\eta(\eta)\right), \\
G_1^+(x,t) &= e^{ith} \int_{-\pi}^{\pi} \frac{d\eta}{2\pi} \frac{H(\eta)}{4\pi i z} b_{++}(\eta) \det \left(1 + \tilde{\mathcal{V}}_{(0,0,+)\eta}(\eta)\right).
\end{align}

(61a)  

(61b)

Making the change of variable \(z = e^{i\eta}\) we obtain integrals over the unit circle with \(H(z) = 1 + z/(2-z) + 1/(2z-1)\),

\begin{align}
G_1^-(x,t) &= e^{-ith} \int_{|z|=1} \frac{dz}{2\pi i} \frac{H(z)}{4i\pi z} B_{-}(z) \det \left(1 + \tilde{\mathcal{V}}_{(0,0,-)}^z(z)\right), \\
G_1^+(x,t) &= e^{ith} \int_{|z|=1} \frac{dz}{2\pi i} \frac{H(z)}{iz} b_{++}(z) \det \left(1 + \tilde{\mathcal{V}}_{(0,0,+)z}(z)\right).
\end{align}

(62a)  

(62b)

The function \(H(z)\) has two poles in the complex plane situated at \(z_1 = 1/2\) and \(z_2 = 2\) with residues \(Res H(z_1) = 1/2\) and \(Res H(z_2) = 2\). Next, we are going to assume that the integrand on the right hand side of (62a) (excluding \(H(z)/z\)) is analytic in the annulus \(1 \leq z < e\) and the integrand on the right hand side of (62b) (also excluding \(H(z)/z\)) is analytic in the annulus \(0 < z \leq 1\). Under these assumptions we can deform the contour on (62a) to a circle of radius \(r^- = 2 + a\) with \(a\) a small positive quantity and the contour in (62b) to a circle of radius \(r^+ = 1/2 - a\) obtaining

\begin{align}
G_1^-(x,t) &= e^{-ith} \frac{1}{4\pi} B_{-}(2) \det \left(1 + \tilde{\mathcal{V}}_{(0,0,-)}(2)\right) + e^{-ith} \int_{|z|=2+a} \frac{dz}{2\pi i} \frac{H(z)}{4i\pi z} B_{-}(z) \det \left(1 + \tilde{\mathcal{V}}_{(0,0,-)}^z(z)\right), \\
G_1^+(x,t) &= e^{ith} b_{++}(1/2) \det \left(1 + \tilde{\mathcal{V}}_{(0,0,+)z}(1/2)\right) + e^{ith} \int_{|z|=1/2-a} \frac{dz}{2\pi i} \frac{H(z)}{iz} b_{++}(z) \det \left(1 + \tilde{\mathcal{V}}_{(0,0,+)z}(z)\right),
\end{align}

(63a)  

(63b)

where the first terms are given by the contribution of the residues of the function \(H(z)\) at \(z_1\) and \(z_2\) (note that the contribution of the residue at \(z_1\) comes with a minus sign due to the fact that the deformed contour surrounds the pole clockwise). We will show below that the second terms give a negligible contribution in the large \(x\) limit so we can focus only on the first terms. Because \(z = e^{i\eta}\) and \(\nu(\pm) = \pm \left(\frac{n}{2\pi} - \frac{\kappa}{2}\right)\) this means that \(\nu^+(z = 2) = \left(\frac{1+b}{2}-\frac{\kappa}{2}\right)\) and \(\nu^-(z = 2) = \left(\frac{1-b}{2}+\frac{\kappa}{2}\right)\).

A. RHP for dynamic correlators

The RHP associated with the Fredholm determinants appearing in the representations for the dynamic correlators is the following [108]:

- \(\chi(k)\) is analytic on \(\mathbb{C}\setminus[-k_F, k_F]\).
- \(\chi_+(k)G_+(k) = \chi_-(k)\) for \(k \in (-k_F, k_F)\) with jump matrix
  \[
  G_+(k) = \begin{pmatrix}
  1 - 4 \sin^2 \pi \nu^{(\pm)}(k) E^{(\pm)}(k) e(k) & 4 \sin^2 \pi \nu^{(\pm)}(k) E^{(\pm)}(k) e(k) \\
  -e^2(k) & 1 + 4 \sin^2 \pi \nu^{(\pm)}(k) E^{(\pm)}(k) e(k)
  \end{pmatrix}.
  \]
- \(\chi(k) \xrightarrow{k \to \infty} I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\) and \(\chi(k) = O\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right) \ln |k^2 - k_F^2|\) for \(k \to \pm k_F\).

Also

\[
\chi(k) = I_2 - \frac{2 \sin^2 \pi \nu^{(\pm)}(k)}{\pi i} \int_{-k_F}^{k_F} \frac{1}{k' - k} \begin{pmatrix}
-e(k') E^{(\pm)}(k') & E^{(\pm)}(k') E^{(\pm)}(k') \\
-e(k') f(k') & E^{(\pm)}(k') f(k')
\end{pmatrix} dk'.
\]

(65)

The last relation shows that

\[
\lim_{k \to \infty} \chi(k) = I_2 + \frac{2 \sin^2 \pi \nu^{(\pm)}(k)}{\pi i} \begin{pmatrix}
-B_{--} & B_{++} \\
-B_{-+} & B_{+-}
\end{pmatrix} + O\left(\frac{1}{k^2}\right).
\]

(66)
FIG. 3. Plots of the dynamic correlator $g^{(+)}(x,t)$ at fixed $t = 2$ and $h = 2$ (black line) and the asymptotic formula (real part blue line, imaginary part green line) for the space-like region.

FIG. 4. Plots of the dynamic correlator $g^{(-)}(x,t)$ at fixed $t = 2$ and $h = 2$ (black line) and the asymptotic formula (real part blue line, imaginary part green line) for the space-like region.
B. Long-time, large-distance asymptotics

The asymptotic solution of the RHP in both space-like and time-like regime can be found in [108] and is briefly presented in Appendices [D] and [E]. In this section we assume $x > 0$ and $t > 0$. The asymptotic behavior of the time dependent Fredholm determinant is given by the following theorem:

**Theorem V.1.** [108] Let $\hat{V}^{(0,0,\pm)}$ be an integral operator with the kernel defined in (62) and acting on $L^2([-k_F,k_F])$. Then for $x \to \infty$ the leading asymptotic behavior is

\[
\det \left( 1 + \hat{V}^{(0,0,\pm)}(\eta) \right) = \frac{G_D^2(1,1,1)}{(2k_F)^{2(\nu(\pm))^2}} e^{2ik_Fx(\nu(\pm))} \frac{1}{(x-2k_Ft+i0^+)^{(\nu(\pm))^2}} \frac{1}{(x+2k_Ft)^{(\nu(\pm))^2}},
\]

with $\nu(\pm) = \pm \left( \frac{n}{2\pi} - \frac{\kappa}{2} \right)$ provided that $|\text{Re} \nu(\pm)| < 1/2$.

The $i0^+$ regularization plays a role only in the time-like regime ($x < 2k_Ft$).

**Results in the space-like regime.** In the case of $\mathcal{G}_1^{(-)}(x,t)$ we are interested in the value of the determinant and the $B_{--}$ potential evaluated at $z = 2$ which corresponds to $\nu = \frac{i\ln 2}{2\pi} + \frac{\kappa}{2}$ (from now on we drop the $\pm$ superscripts when there is no potential for confusion). The result for $B_{--}$ derived in Appendix [D] and Thm. V.1 show that the leading contribution in the second term on the right hand side of (63a) is given by the determinant which on the contour of integration is of the order of $\exp[-2k_Fx \ln(2 + 2\pi)/2\pi]$ and therefore is negligible in the large $x$ limit. Using (63a), [D] and Thm. V.1 we obtain the asymptotic behavior

\[
\mathcal{G}_1^{(-)}(x,t) = e^{\zeta(\nu)} \pi \left( e^{2i\nu} - 1 \right) \sin \pi \nu e^{2ik_Fx} \left[ (2k_F)^{-2\nu} \Gamma^2(-\nu) (x-2k_Ft)^{\nu+1} (x+2k_Ft)^{\nu+1} - \frac{(2k_F)^{2\nu}}{(x-2k_Ft)^{\nu+1}} e^{-ik_Fx} \right] ,
\]

with $\nu = \frac{i\ln 2}{2\pi} + \frac{\kappa}{2}$ and $\zeta(\nu)$ defined in (53). For the $\mathcal{G}_1^{(+)}(x,t)$ correlator we need to evaluate the determinant and the $b_{++}$ potential at $z = 1/2$ which corresponds to $\nu = \frac{i\ln 2}{2\pi} - \frac{\kappa}{2}$. The second term on the right hand side of (63b) is of the order of $\exp[2k_Fx \ln(1/2 - 2\pi)/2\pi]$ and therefore this contribution can be neglected. Collecting the results from (63b), (D) and Thm. V.1 we find

\[
\mathcal{G}_1^{(+)}(x,t) = e^{\zeta(\nu)} e^{2ik_Fx} \left[ \frac{e^{-i\nu t}G(x,t)}{(x-2k_Ft)^{\nu+1} (x+2k_Ft)^{\nu+1}} - \frac{\pi}{\sin \pi \nu} \left( e^{-2i\nu} - 1 \right) \frac{(2k_F)^{2\nu}}{(x-2k_Ft)^{\nu+1}} e^{-ik_Fx} \right] .
\]

with $\nu = \frac{i\ln 2}{2\pi} - \frac{\kappa}{2}$. The asymptotic formulae (68) and (69) for the fermionic system were first derived by Cheianov and Zvonarev [18] who also pointed out the non-trivial, from the conformal point of view, behavior of the correlation functions. The correlators show spin-charge separation with scaling behavior of the charge component and exponential decay of the spin part. The anomalous dimensions of the charge part do not correspond to any unitary conformal field theory. For arbitrary statistics the overall picture remains almost the same: the spin part is exponentially decaying in space separation while the charge part presents scaling. In the fermionic case both terms in the square parenthesis are important but as we increase the statistics parameter one of them become dominant. Similar to the static case, in principle (68) and (69) should not be valid in the bosonic case due to the fact that for $\kappa = 1$ we have $|\text{Re} \nu| = 1/2$ which means Thm. V.1 and the results of (108) cannot be used. However, by taking the $\kappa \to 1$ limit in the above asymptotic expansions we obtain results which agree with the numerics even in the bosonic case but we should point out that the accuracy is decreasing as we approach $\kappa \to 1$. In Figs. 3 and 4 we plot the correlators for $h = 2$ and fixed $t = 2$ and the expansions (68) and (69). For $x > 2k_Ft$ we have almost perfect agreement if we take into consideration that we have considered only the first terms of the expansion.

**Results in the time-like regime.** The solution of the RHP in the time-like regime is sketched in Appendix [E]. For the $\mathcal{G}_1^{(-)}(x,t)$ correlator using the the representation (63a) where the second term is negligible in the large $x$ limit, Thm. V.1 and (E3) we find

\[
\mathcal{G}_1^{(-)}(x,t) = e^{\zeta(\nu)} e^{2ik_Fx} \left[ -(e^{-2i\nu} - 1)^2 \frac{e^{-4i\nu} e^{2i\nu}}{8 \sin^2 \pi \nu} \frac{e^{-i\nu t}G(x,t)}{(2k_Ft+x)^{\nu} (2k_Ft-x)^{\nu}} \left( \frac{2k_Ft+x}{2k_Ft-x} \right)^{2\nu} \right] .
\]
FIG. 5. Plots of the dynamic correlator $G_{\kappa}(x, t)$ at fixed $t = 25$ and $h = 2$ (black line) and the asymptotic formula (70) (real part blue line, imaginary part green line) for the time-like region.

FIG. 6. Plots of the dynamic correlator $G_{\kappa}(x, t)$ at fixed $t = 25$ and $h = 2$ (black line) and the asymptotic formula (71) (real part blue line, imaginary part green line) for the time-like region.
The red curves are obtain for this previously unknown constant and are also valid for all values of the statistics parameter. Plots of the correlators easily derived from (37) and (39) using lim
\[ G_{\omega,k}(t) = \frac{1}{T^2} \left\{ \begin{array}{c}
\frac{(2k_F)^{-2\nu}}{\Gamma^2(-\nu)(2k_F-t-x)^{\nu^2}(2k_F-x+t)^{\nu^2}}
+ \frac{(2k_F)^{2\nu}}{\Gamma^2(\nu)(2k_F-t-x)^{\nu^2}}
\end{array} \right\},
\]
with \( \nu = \frac{i \ln 2}{2\pi} + \frac{2}{\pi} \). In the case of the \( G_{\omega,k}(x,t) \) correlator using the representation (63b), Thm. 1 and (E4) we obtain
\[ G_{\omega,k}(x,t) = -\mathcal{C}(\nu) \pi e^{-i\pi \nu} \left( e^{-2i\pi \nu} - 1 \right)^{-1} \mathcal{E} \left( 2i\nu \right) \left( 2k_F \right)^{2\nu} \frac{e^{-ik_F x}}{\Gamma^2(\nu)(2k_F-t-x)^{\nu^2}}
+ \frac{(2k_F)^{2\nu}}{\Gamma^2(-\nu)(2k_F-t-x)^{\nu^2}}
\]
with \( \nu = \frac{i \ln 2}{2\pi} - \frac{2}{\pi} \). In the fermionic case similar results (modulo a constant) were derived in [8]. Our results contain this previously unknown constant and are also valid for all values of the statistics parameter. Plots of the correlators for \( h = 2 \) and fixed \( t = 25 \) together with the expansions (70) and (71) are presented in Figs. 5 and 6.

VI. ASYMPTOTICS FOR THE SINGLE COMPONENT CASE

The determinant representation for the correlators of the single component system at zero temperature can be easily derived from (37) and (39) using lim \( T \to 0 \) \( \vartheta(k) = \theta(k_F^2 - k) \) with \( k_F = h^{1/2} \) and \( h \) is the chemical potential of the single component system. The derivation of the asymptotics is very similar with the one presented in the previous sections for the two-component case, therefore, we only present the final results. In the case of the static correlators the large distance asymptotic behavior is given by (we drop the subscript)
\[ G^{(-)}(x,0) = \frac{i e^{C(-\frac{x}{2})}}{2\pi} e^{-i k_F x} \left( \begin{array}{c}
\frac{1}{\Gamma^2 \left( 1 - \frac{\kappa}{2} \right)}
\frac{e^{ik_F x}}{x^{1-\kappa}}
\end{array} \right).
\]
which confirms the bosonization result of Calabrese and Mintchev \([56] [59]\) and, in addition, supplies explicit expressions for the amplitudes. In the bosonic limit \((\kappa = 1)\) the first term on the right hand side of \([74]\) is dominant and using \(G_B(1/2) = A^{-3/2}\pi^{-1/4}e^{1/8}\pi^{1/24} [109]\) and \(G(3/2) = \Gamma(1/2)G_B(1/2)\) with \(A = 1.2824\ldots\) Glaisher’s constant we find \(G^{(-)}(x, 0) = k_F \rho_\infty / \pi |k_F x|^{1/2} \), \(\rho_\infty = \pi e^{1/2 - 1/3} A^{-6}\), which reproduces the leading term derived by Vaidya and Tracy \([16] [17]\) for impenetrable bosons (see also \([18] [19]\)). For \(\kappa = 0\) both terms are important and we obtain \(G^{(-)}(x, 0) = \sin(k_F x) / \pi x\) which is in fact the exact result for spinless fermions.

The asymptotic expansions of the dynamic correlators are almost identical with the ones derived for the two component case the main difference appearing in the value of the \(\nu\) parameter. More precisely, in the single component case the asymptotics for \(G^{(-)}(x, t)\) are given by \([68], [70]\) multiplied by 2 with \(\nu^{(-)} = \kappa / 2\) while for \(G^{(+)}(x, t)\) the asymptotics are \([69], [71]\) with the parameter \(\nu^{(+)} = -\kappa / 2\).

VII. FORM FACTORS

In the following sections we present the derivation of the determinant representations \([17]\) and \([18]\). We are going to use the same method employed by the authors of \([106]\) who derived the similar representations for bosons and fermions. Some of the details of the computations are very similar so we will focus mainly on the particular complications induced by the anyonic statistics. The derivation consists of three main steps. In the initial step we are going to consider a finite size system and insert a resolution of identity between the two operators appearing in the definition of the correlators \([5]\). Each term in the trace can be written as a sum over the form factors which are the building blocks of our representation. For example, in the finite system the \(G_\beta^{(-)}(x, t)\) correlators take the form

\[
G_\beta^{(-)}(x, t) = \sum \frac{e^{-EN+1,M(k)/T} \langle \psi_\beta(x,t) \psi_\beta(0,0) \rangle_{N+1,M}}{\sum e^{-EN+1,M(k)/T}}, \quad \beta \in \{1, 2\},
\]

with \(\langle \psi_\beta(x,t) \psi_\beta(0,0) \rangle_{N+1,M}\) the normalized value of the operators \(\psi_\beta(x,t)\) in the state \(|\Psi_{N+1,M}(k, \lambda)\rangle\). Insertion of the identity produces \([90]\) with the form factors defined in \([86]\). The form factors can be expressed as finite size determinants as it will be shown in this section. The second step is the summation of form factors using the “summation under the determinant” trick (see Section VIII). The final step is taking the thermodynamic limit which will be done in Section IX.

A. Determinant representation for the form factors

In this section we are going to obtain the determinant representation for the form factors in the finite system. The form factors are defined as

\[
\mathcal{F}_{N,M}^{(\beta)}(x, t) \equiv \langle \Psi_{N,M}(q, \mu) | \psi_\beta(x, t) | \Psi_{N+1,M}(k, \lambda) \rangle, \quad \beta \in \{1, 2\},
\]

where \(\bar{M} = M\) if \(\beta = 1\) and \(\bar{M} = M - 1\) if \(\beta = 2\). The form factor of the creation operator \(\psi_\beta^\dagger(x, t)\) is given by the complex conjugate of \(\mathcal{F}_{N,M}^{(\beta)}(x, t)\). In \([76]\) the eigenstate in the \((N + 1, M)\)-sector is characterized by the quasimomenta \(k = \{k_a\}_{a=1}^{N+1}, \lambda = \{\lambda_b\}_{b=1}^M\) and the one in the \((N, \bar{M})\)-sector by \(q = \{q_a\}_{a=1}^N, \mu_b = \{\mu_b\}_{b=1}^M\). They satisfy the following set of BAEs:

\[
\begin{align}
(N + 1, M)\text{-sector} & \quad e^{ik_a L} = \omega e^{-i\pi N}, \quad a = 1, \ldots, N + 1, \\
& \quad e^{i\lambda_b (N + 1)} = (-1)^M, \quad b = 1, \ldots, M,
\end{align}
\]

\[
\begin{align}
(N, \bar{M})\text{-sector} & \quad e^{iq_a L} = \zeta e^{-i\pi (N - 1)}, \quad a = 1, \ldots, N, \\
& \quad e^{i\mu_b N} = (-1)^{M-1}, \quad b = 1, \ldots, \bar{M},
\end{align}
\]

where we have introduced

\[
\omega = e^{i \sum_{b=1}^M \lambda_b}, \quad \zeta = e^{i \sum_{a=1}^N q_a}.
\]

Using \(\psi_\beta^\dagger(x, t) = e^{i\mathcal{H}t} \psi_\beta^\dagger(x)e^{-i\mathcal{H}t}\) and the fact that \(|\Psi_{N,M}(q, \mu)\rangle\) and \(|\Psi_{N+1,M}(k, \lambda)\rangle\) are eigenstates of the Hamiltonian with eigenvalues given by \([16]\) we obtain

\[
\mathcal{F}_{N,M}^{(\beta)}(x, t) = \exp \left\{ it \left( \sum_{a=1}^N q_a^2 - \sum_{a=1}^{N+1} k_a^2 + h_\beta \right) \right\} \mathcal{F}_{N,M}^{(\beta)}(x),
\]

where \(h_\beta = \frac{\beta}{\bar{M}}\) and \(\lambda_{N+1} = \lambda_{M+1} = 0\).
where $h_\beta$ is the chemical potential of the particles ($h_1 = h - B, h_2 = h + B$) and we introduced the notation $F_{N,M}^{(\beta)}(x,0) = F_{N,M}^{(\beta)}(x)$. The starting point of our computation is the formula:

$$F_{N,M}^{(\beta)}(x) = (N+1)! \int_0^L dz_1 \cdots dz_N \sum_{\alpha_1, \cdots, \alpha_N = 1}^{[N,M]} \chi^{\alpha_1 \cdots \alpha_N}_{N,M}(z|q, \mu) \chi^{\alpha_1 \cdots \alpha_N \beta}_{N+1,M}(z,x|k, \lambda),$$  

(80)

which is obtained by using the commutation relations to move the $\psi$ to the right in (76) until it hits the vacuum. In the previous relation $z = \{z_1, \cdots, z_N\}$ and the bar denotes complex conjugation.

Introducing a generalization of the sign function

$$\rho \in \sum_{N,M} \{ \cdots \}$$

in (81) vanish). Therefore, (81) becomes

$$\sum_{R \in S_N} \{ \theta (z_{R(1)} < \cdots < z_{R(N)} < x) + \theta (z_{R(1)} < \cdots < z_{R(j)} < x < z_{R(j+1)} < \cdots < z_{R(N)}) e^{i\pi \kappa (N-j)} F_{\beta}(j)$$

$$\times \left( \sum_{Q \in S_N} (-1)^Q e^{-i(q_{21}z_1 + \cdots + q_{2Q}z_Q)} \right) \left( \sum_{P \in S_N} (-1)^P e^{i(k_{P1}z_1 + \cdots + k_{PN}z_N + k_{PN+1}x)} \right) \right),$$  

(81)

with $(-1)^Q$ the signature of the permutation and $F_{\beta}(j) = \sum_{\alpha_1, \cdots, \alpha_N} \xi^{\alpha_1 \cdots \alpha_N}_{N,M} \xi^{\alpha_1 \cdots \alpha_j \beta + 1 \cdots \alpha_{N+1}}_{N+1,M}$. Remembering that $\xi^{\alpha_1 \cdots \alpha_N}_{N,M}$ and $\xi^{\alpha_1 \cdots \alpha_N}_{N+1,M}$ are eigenvectors of the cyclic shift operators $C_N$ and $C_{N+1}$ acting in a $2^N$ or $2^{N+1}$ dimensional space we have $\xi^{\alpha_1 \cdots \alpha_N}_{N,M} = \zeta \xi^{\alpha_1 \cdots \alpha_N}_{N,M}$ and $\xi^{\alpha_1 \cdots \alpha_N}_{N+1,M} = \omega \xi^{\alpha_1 \cdots \alpha_N}_{N+1,M}$ with $\omega$ and $\zeta$ defined in (78) satisfying $\omega^{N+1} = \zeta^N = 1$. Using these relations we obtain $F_{\beta}(j) = (\bar{\omega} \zeta)^{N-j} F_{\beta}(N)$ and the sum over the $R$ permutation can be written as

$$\sum_{R \in S_N} \{ \theta (z_{R(1)} < \cdots < z_{R(N)} < x) + \theta (z_{R(1)} < \cdots < z_{R(j)} < x < z_{R(j+1)} < \cdots < z_{R(N)}) (\bar{\omega} \zeta e^{i\pi \kappa})^{N-j}$$

$$+ \theta (x < z_{R(1)} < \cdots < z_{R(N)}) (\bar{\omega} \zeta e^{i\pi \kappa})^N \} F_{\beta}(N).$$  

(82)

Introducing a generalization of the sign function $\rho(z) = \theta(z) + e^{i\pi \kappa} \bar{\omega} \zeta \theta(z)$ the previous sum over permutation is $\sum_{R \in S_N} \{ \cdots \} = \prod_{j=1}^N \rho(x - z_j)$ (the value of $\rho(0)$ is not important because when two $z$’s are equal the determinants in (81) vanish). Therefore, (81) becomes

$$F_{N,M}^{(\beta)}(x) = \frac{e^{-i\pi \kappa N}}{N!} F_{\beta} \int_0^L dz_1 \cdots dz_N \prod_{j=1}^N \rho(x - z_j) \sum_{P \in S_{N+1}} (-1)^P \prod_{Q \in S_N} \left( \sum_{j=1}^N (-1)^P e^{i(k_{Pj}z_j - q_{Qj})} \right) e^{i(k_{PN+1}x)},$$  

(83)

where we have introduced the notation $F_{\beta} = F_{\beta}(N)$. The integrand is now factorized so the multiple integral can be calculated using the formula (the BAES have to be taken into account)

$$\int_0^L \rho(x - z) e^{i(k-q)z} dz = -i(1 - e^{i\pi \kappa \bar{\omega} \zeta}) e^{i(k-q)x}.$$  

(84)

We find

$$F_{N,M}^{(\beta)}(x) = e^{-i\pi \kappa N} F_{\beta} (-i)^N (1 - e^{i\pi \kappa \bar{\omega} \zeta}) e^{i(\sum_{a=1}^{N+1} k_a - \sum_{a=1}^N q_a)x} \sum_{P \in S_{N+1}} (-1)^P \prod_{j=1}^N \frac{1}{k_{Pj} - q_j},$$  

(85)

2 Note that the wavefunction also satisfies

$$\chi^{\alpha \alpha_1 \cdots \alpha_{N+1} \cdots}(z_1, z_{i+1}, \cdots) = -e^{-i\pi \kappa \text{sgn}(z_1 - z_{i+1})} \chi^{\alpha \alpha_1 \cdots \alpha_{i+1} \cdots}(z_1, z_{i+1}, \cdots).$$
where it only remains to derive the auxiliary lattice form factors \( F_\beta \). The derivation is the same as in the bosonic and fermionic case and can be read directly from \([106]\) (see formulae 3.23 and 3.25). Collecting all the results we can state the main result of this section. The form factors of a system of impenetrable two-component anyons admit the following determinant representation in the finite box

\[
F_{N,M}^{(\beta)}(x) = (-ie^{-i\pi\kappa})^N (1 - e^{i\pi\kappa} \omega \zeta)^N \det_M \det_{N+1} D \exp \left\{ \sum_{a=1}^{N+1} (-i t k_a^2 + i x k_a) - \sum_{j=a}^N (-i t q_j^2 + i x q_j) + i t \beta \right\},
\]

with \( h_1 = h - B, h_2 = h + B \) and \( D \) is a \((N+1) \times (N+1)\) matrix with elements

\[
D_{ab} = \frac{1}{k_a - q_b}, \quad a = 1, \ldots, N + 1, \quad D_{a,N+1} = 1, \quad b = 1, \ldots, N.
\]

The \( B_1 \) and \( B_2 \) are \( M \times M \) matrices with elements

\[
[B_1]_{ab} = \sum_{n=1}^{N} e^{i(\lambda_a - \mu_b) n}, \quad a, b = 1, \ldots, M, \quad (88)
\]

\[
[B_2]_{ab} = \sum_{n=1}^{N} e^{i(\lambda_a - \mu_b) n}, \quad [B_2]_{aM} = 1, \quad a = 1, \ldots, M, \quad b = 1, \ldots, M - 1. \quad (89)
\]

At \( \kappa = 0 \) and \( \kappa = 1 \) \([86]\) reproduces (modulo a phase factor) the results for fermions and bosons derived in \([106]\). For \( M = 0 \) we have \( F_{N,M}^{(2)} = 0, \omega = \zeta = 1 \) and \( \det_M B_1 = 1 \). The form factor \( F_{N,M}^{(1)} \) is in this case equal again modulo a phase with the form factor of single component hardcore anyons \([121]\) (note that the statistics parameter used in \([121]\) is related to ours via \( \kappa' = 1 + \kappa \)). If \( M = N + 1 \) then the form factor \( F_{N,M}^{(1)} \) vanishes and \( \zeta = \omega = 1 \). Now the form factor \( F_{N,M}^{(2)} \) reduces to the result for single component anyons (the \( \det_{M=N+1} B_2 \) factor is related to the normalization of the eigenstates and irrelevant).

### VIII. SUMMATION OF FORM FACTORS

The second step in our derivation of the determinant representations \([17]\) and \([18]\) is represented by the calculation of normalized mean values of bilocal operators by summation of the form factors. From now on we will focus on the correlators of the type 1 particles as the second set of correlators can be obtained using the relation \([5]\). An independent calculation, which we do not present here, using the form factors for the second type of particles confirms the symmetry \([5]\) (for the bosonic and fermionic case see \([106]\)).

### A. Summation of form factors for \( \langle \psi^\dagger_1(x,t) \psi_1(0,0) \rangle_{N+1,M} \)

Inserting a resolution of the identity between the operators of the normalized mean value \( \langle \psi^\dagger_1(x,t) \psi_1(0,0) \rangle \) we find

\[
\langle \psi^\dagger_1(x,t) \psi_1(0,0) \rangle_{N+1,M} = \frac{\langle \Psi_{N+1,M}(k, \lambda) | \psi^\dagger_1(x,t) \psi_1(0,0) | \Psi_{N+1,M}(k, \lambda) \rangle}{\langle \Psi_{N+1,M}(k, \lambda) | \Psi_{N+1,M}(k, \lambda) \rangle},
\]

\[
= \sum_{q, \mu}^{[N,M]} \frac{\langle \Psi_{N+1,M}(k, \lambda) | \psi^\dagger_1(x,t) | \Psi_{N,M}(q, \mu) \rangle \langle \Psi_{N,M}(q, \mu) | \psi_1(0,0) | \Psi_{N+1,M}(k, \lambda) \rangle}{\langle \Psi_{N+1,M}(k, \lambda) | \Psi_{N+1,M}(k, \lambda) \rangle} \frac{\langle \Psi_{N,M}(q, \mu) | \Psi_{N,M}(q, \mu) \rangle}{\langle \Psi_{N+1,M}(k, \lambda) | \Psi_{N+1,M}(k, \lambda) \rangle},
\]

\[
= \sum_{q, \mu}^{[N,M]} \frac{F_{N,M}^{(1)}(x,t) F_{N,M}^{(1)}(0,0)}{L^{2N+1} N^M (N+1)^M},
\]

where we have used the normalization of the eigenstates \([14]\) and the form factors defined in \([86]\). The summation in \([90]\) is over all allowed values of \( q \) and \( \mu \) in the \((N,M)\)-sector. Introducing

\[
\Lambda = \sum_{a=1}^{M} \lambda_a, \quad \Theta = \sum_{a=1}^{M} \mu_a,
\]

where
we see from (95) that the allowed values of the quasimomenta can be written as \( q_a = \tilde{q}_a + \Theta/L \) and \( k_a = \tilde{k}_a + \Lambda/L \) with \( \tilde{q}_a \) not depending on \( \mu \)'s and \( \tilde{k}_a \) on \( \chi \)'s. This means that we can sum over \( \tilde{q}_a \) independently on \( \mu \)'s. The summand in (90) is symmetric in both \( q \)'s and \( \tilde{q} \)'s and also on \( \mu \)'s and it vanishes when two \( \mu \)'s or \( q \)'s coincide (because it involves square moduli of determinants). Therefore, the sum can be written as

\[
\sum_{\mu, \nu} = \sum_{\mu_1 < \cdots < \mu_M} \frac{1}{N!} \sum_{q_1} \cdots \sum_{q_N} \frac{1}{M!} \sum_{\mu_1} \cdots \sum_{\mu_M} \tag{92}
\]

where each sum is independent of the others. For an arbitrary function \( f \) the summation over \( \tilde{q}_a \) or \( \mu_b \) is given by

\[
\sum_{\tilde{q}_a} f(\tilde{q}_a) = \sum_{j \in \mathbb{Z}} f(\tilde{q}_a)_j, \quad \sum_{\mu_b} = \sum_{l=1}^N f(\mu_b)_l, \tag{93}
\]

where the allowed values of the quasimomenta are \( [\tilde{q}_a]_j = 2\pi(j + \delta)/L \) with \( j \in \mathbb{Z} \) and \( \delta = \{-\pi\kappa(N-1)\} \) and \( [\mu_b]_l = 2\pi l/N \) with \( l = 0, \cdots, N-1 \). The summation over \( \tilde{q}_1, \cdots, \tilde{q}_N \) is performed along the same lines as in the case of bosons and fermions. We find

\[
\langle \psi_1^+(x, t) \psi_1(0, 0) \rangle_{N+1, M} = \frac{e^{-i\theta_{1h}}}{NM(N+1)^M} \frac{1}{M!} \sum_{\mu_1} \cdots \sum_{\mu_M} |\det_M B_1|^2 \frac{\partial}{\partial z} \det_{N+1} \left( S^{(–)} + z R^{(–)} \right) \bigg|_{z=0}, \tag{94}
\]

with \( S^{(-)} \) and \( R^{(-)} \) matrices of dimension \( (N+1) \times (N+1) \) with elements

\[
[S^{(-)}]_{ab} = e_{-}(k_a)e_{-}(k_b) - \frac{1 - \cos(\Lambda - \Theta - \pi \kappa)}{L^2} \sum_q e^{-iqx_i + i\omega q} \sum q \frac{1}{k_a - q}(k_b - q), \quad q = \tilde{q} + \Theta/L, \tag{95}
\]

\[
[R^{(-)}]_{ab} = e_{-}(k_a)e_{-}(k_b), \quad e_{-}(k) = e^{itk^22ix_k}. \tag{96}
\]

In the finite system the elements of the matrices \( S^{(-)} \) and \( R^{(-)} \) are well defined due to \( k_a \neq \tilde{q}_b \) which can be seen from the BAEs (13). In the thermodynamic limit \( k \) and \( q \) become arbitrary and \( \frac{1}{T} \sum \tilde{q} \) is replaced by \( \frac{1}{2\pi} \int d\tilde{q} \) which means that poles can appear in the integrands. It is therefore necessary to rewrite the elements of the relevant matrices in a way such that the thermodynamic limit can be taken without problems. The necessary calculations are presented in Appendix F where it is shown that the matrix \( S^{(-)} \) can be written as

\[
S^{(-)} = I + \frac{1 - \cos(\Lambda - \Theta - \pi \kappa)}{2} V^{(-)}_1 - \frac{\sin(\Lambda - \Theta - \pi \kappa)}{2} V^{(-)}_2, \tag{97}
\]

with \( V^{(-)}_{1,2} \) matrices of dimension \( (N+1) \times (N+1) \) \((I \) is the identity matrix of the same dimensionality\) defined by

\[
[V^{(-)}_1]_{ab} = \frac{2}{L} e_{+}(k_a)e_{-}(k_b) - e_{-}(k_a)e_{+}(k_b), \quad [V^{(-)}_2]_{ab} = \frac{2}{L} \frac{e_{-}(k_a)e_{-}(k_b) - e_{-}(k_a)e_{-}(k_b)[e_{-}(k_a)]^{-1}(k_b)}{k_a - k_b}. \tag{98}
\]

The function \( e_{+}^{(-)}(k) \) is defined by

\[
e_{+}^{(-)}(k) = e^{(-)}(k)e_{-}(k), \quad e^{(-)}(k) = 2 \sum_q \frac{e^{-itq^2 + i\omega q} - e^{-itk^2 + i\omega k}}{q - k}, \tag{99}
\]

with \( e_{-}(k) \) given in (96). All the functions appearing in (98) and in (96) for the \( R^{(-)} \) matrix have a well defined thermodynamic limit. Now it only remains to sum over the \( \mu \)'s in (94). This step is identical with the one in (100) and we obtain

\[
\langle \psi_1^+(x, t) \psi_1(0, 0) \rangle_{N+1, M} = \frac{e^{-i\theta_{1h}}}{N} \sum_{n, p=0}^{N-1} e^{-\frac{2\pi n}{N}} \det_M \mathcal{U}_p^{(1, -)} \det_{N+1} \left( S_n^{(-)} + R^{(-)} \right) - \det_{N+1} S_n^{(-)}, \tag{100}
\]

with \( \mathcal{U}_p^{(1, -)} \) a square matrix of dimension \( M \) and elements

\[
[\mathcal{U}_p^{(1, -)}]_{ab} = \frac{1}{N(N+1)} \sum_{n=1}^N \sum_{m=1}^N e^{i(p+m-n)\mu + i\nu \lambda_N - i\mu \lambda_N}, \quad a, b = 1, \cdots, M. \tag{101}
\]

In (100), \( S_n^{(-)} = I + \frac{1 - \cos(\Lambda - 2\pi n/N - \pi \kappa)}{2} V^{(-)}_1 - \frac{\sin(\Lambda - 2\pi n/N - \pi \kappa)}{2} V^{(-)}_2 \) and also one should put \( q = \tilde{q} + \frac{2\pi n}{(LN)} \) in the function \( e^{(-)}(k) \).
B. Summation of form factors for $\langle \psi_1(x,t)\psi_1^+(0,0) \rangle_{N,M}$

After the insertion of the identity we have

$$
\langle \psi_1(x,t)\psi_1^+(0,0) \rangle_{N,M} = \frac{\langle \Psi_{N,M}(q,\mu)\psi_1(x,t)\psi_1^+(0,0)\psi_N,M(q,\mu) \rangle}{\langle \Psi_{N,M}(q,\mu)\psi_N,M(q,\mu) \rangle},
$$

$$
= \frac{[N+1,M]}{\sum_{k,\mu} \langle \Psi_{N,M}(q,\mu)\psi_1(x,t)\psi_N,M(k,\lambda) \rangle \langle \Psi_{N+1,M}(k,\lambda)\psi_1^+(0,0)\psi_N,M(q,\mu) \rangle}{\langle \Psi_{N+1,M}(k,\lambda)\psi_N,M(k,\lambda) \rangle},
$$

$$
= \frac{[N+1,M]}{\sum_{k,\mu} \mathcal{F}_{N,M}(0,0) \mathcal{F}_{N,M}(x,t)} L^{2N+1}N^M(N+1)^M,
$$

(102)

with the summation over all allowed values of $k$ and $\lambda$ in the $(N+1,M)$-sector. Again due to the symmetry of the summand we have $(k = k + \Lambda/L)$

$$
\sum_{k,\lambda} = \sum_{k_1<\cdots<k_{N+1}} \cdots \sum_{k_{N+1}} \rightarrow \frac{1}{N+1} \sum_{k_1} \cdots \sum_{k_{N+1}} \frac{1}{M!} \sum_{\lambda_1} \cdots \sum_{\lambda_M},
$$

(103)

with each summation independent of the others. Summation over $\hat{k}_1,\cdots,\hat{k}_{N+1}$ gives [106]

$$
\langle \psi_1(x,t)\psi_1^+(0,0) \rangle_{N,M} = \frac{e^{ih_1}}{N^M(N+1)^M} \frac{1}{M!} \sum_{\lambda_1} \cdots \sum_{\lambda_M} |\det M B_1|^2 \left[ g(x,t) + \frac{\partial}{\partial x} \right] \det N \left( S^{(+)} - z R^{(+)1} \right) |_{z=0}
$$

(104)

with $g(x,t) = \sum_k e^{-itk^2+ixk}/L$ and $S^{(+)}$ and $R^{(+)}$ are square matrices of dimension $N$ with elements

$$
[S^{(+)1}]_{ab} = e^{-i\kappa\omega(q)^2} \sum_k \frac{e^{-itk^2+ixk}}{(k-qa)(k-qb)},
$$

$$
[S^{(+)2}]_{ab} = \frac{1-e^{-i\kappa\omega(q)^2} e^{-itk^2+ixk}}{L^2 \sum_k} \left( e^{-i\kappa\omega(q)^2} \sum_k \frac{e^{-itk^2+ixk}}{(k-qa)} \right) \left( e^{-i\kappa\omega(q)^2} \sum_k \frac{e^{-itk^2+ixk}}{(k-qb)} \right),
$$

(105)

$$
[R^{(+)1}]_{ab} = \frac{1-e^{-i\kappa\omega(q)^2} e^{-itk^2+ixk}}{L^3 \sum_k} \left( e^{-i\kappa\omega(q)^2} \sum_k \frac{e^{-itk^2+ixk}}{(k-qa)} \right) \left( e^{-i\kappa\omega(q)^2} \sum_k \frac{e^{-itk^2+ixk}}{(k-qb)} \right),
$$

(106)

Similar to the previous case the $S^{(+)}$ matrix can be rewritten in a form which is suitable to taking the thermodynamic limit (see Appendix F). Introducing the function $e^{+}(q)$ defined by

$$
e^{+}(q) = \frac{2}{L} \sum_k \frac{e^{-itk^2+ixk} - e^{-itq^2+iqx}}{k-q},
$$

(107)

the $S^{(+)}$ matrix can be written as

$$
S^{(+)} = I + \frac{1-cos(\Lambda-\Theta-\pi\kappa)}{2} V_1^{(+)} + \frac{sin(\Lambda-\Theta-\pi\kappa)}{2} V_2^{(+)} ,
$$

(108)

with $V_1^{(+)}$ square matrices of dimension $N$ and elements

$$
[V_1^{(+)1}]_{ab} = \frac{2}{L} \frac{e^{+}(q_a) e^{+}(q_b) - e^{-}(q_a) e^{+}(q_b)}{q_a-q_b},
$$

$$
[V_1^{(+)2}]_{ab} = \frac{2}{L} \frac{e^{-}(q_a)-1 e^{-}(q_a) e^{+}(q_a) - e^{-}(q_a) e^{+}(q_a)-1}{q_a-q_b}.
$$

(109)

In the case of the $R^{(+)}$ matrix we obtain

$$
[R^{(+)1}]_{ab} = \frac{1-cos(\Lambda-\Theta-\pi\kappa)}{2} \frac{e^{+}(q_a) e^{+}(q_b)}{L} + \frac{sin(\Lambda-\Theta-\pi\kappa)}{2} \left[ \frac{e^{+}(q_a)}{L e^{-}(q_a)} + \frac{e^{+}(q_b)}{L e^{-}(q_b)} \right]
$$

$$
+ \frac{1+cos(\Lambda-\Theta-\pi\kappa)}{2} \frac{1}{L e^{-}(q_a) e^{+}(q_b)},
$$

(110)
All the functions appearing in \(109\) and \(110\) have a well define thermodynamic limit. The final step is the summation over \(\lambda_1, \ldots, \lambda_M\) with the result

\[
\langle \psi_1(x, t) \psi_1^*(0, 0) \rangle_{N,M} = \frac{e^{ ih_1}}{N+1} \sum_{r,m=0}^N e^{ \frac{2\pi i r m}{N+1}} \det M \ U_{r,+} \left[ g_m(x, t) + \frac{\partial}{\partial z} \right] \det N \left( S^{(+)}_m - z R^{(+)}_m \right) \big|_{z=0},
\]

where the square matrix \(U_{r,+}\) of dimension \(M\) has the elements

\[
[U_{r,+}]_{ab} = \frac{1}{N(N+1)} \sum_{m=1}^N \sum_{n=1}^N e^{-i(r+m-n)\lambda-\imath \mu a + \imath \mu b}, \quad a, b = 1, \ldots, M.
\]

In \(111\) the subscript \(m\) for \(S^{(+)}_m, R^{(+)}_m\) and \(g_m(x, t)\) means that in the definitions of these functions \(\Lambda\) is replaced by \(2\pi m/(N+1)\).

**IX. THERMODYNAMIC LIMIT**

In the thermodynamic limit the allowed values for the quasimomenta fill the entire real axis and the sums over \(\tilde{q}_a\) (or respectively \(\tilde{k}_a\)) should be replaced by integrals

\[
\frac{1}{L} \sum_{\tilde{q}} f(\tilde{q}) \rightarrow \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\tilde{q}) \, d\tilde{q}.
\]

The previous relation relies on the fact that for every value of the statistics parameter \([\tilde{q}_a]_{j+1} - [\tilde{q}_a]_j = 2\pi/L\) and that \([q_a]_j = 2\pi j/L + 2\pi \delta/L + \Theta/L\) where the last two terms can be neglected in the thermodynamic limit. The grand canonical potential which describes the thermodynamics of the system is independent of the statistics and is given by Takahashi’s formula first derived in the context of impenetrable fermions (Gaudin-Yang model) \(106\) \(138\)

\[
\phi(h, B, T) = -\frac{T}{2\pi} \int_{-\infty}^{+\infty} \ln \left( 1 + 2 \cosh(B/T) e^{-(q^2 - h)/T} \right) \, dq.
\]

The thermodynamic limit of \(109\) and \(111\) is performed along the same lines as in the case of the two-component fermions and bosons \(106\) taking into account that \(g(x, t) \rightarrow G(x, t) = \int_{-\infty}^{+\infty} e^{-i k^2 + i x k}/2\pi\) and

\[
e^{(\pm)}(k) \rightarrow \text{p.v.} \int_{-\infty}^{+\infty} \frac{1}{\pi} \frac{e^{-i q^2 + i x q}}{q - k} \, dq,
\]

obtaining \(17\) and \(18\).

**X. CONCLUSIONS**

In this paper we have investigated the correlation functions of a 1D two-component system of anyons with strong repulsive contact interactions. This model is the anyonic generalization of the Gaudin-Yang model and can also be understood as the continuum limit of the two-component anyonic Hubbard model (2AHM) \(87\) \(88\). We have derived determinant representations for the temperature-, time-, and space-dependent correlation functions which can be straightforwardly implemented numerically. The low energy asymptotics of the correlators at zero temperature and zero magnetic field reveal the spin-incoherent nature of the system. The asymptotics present exponential decay modulated by an oscillatory component with frequency proportional with the statistics parameter and an algebraically decaying component with anomalous exponents which do not correspond to any unitary conformal field theory. The momentum distribution and the Fourier transform of the dynamic field-field correlator are not symmetric in momentum due to the broken space-reversal symmetry. The tails of the momentum distribution exhibit the universal \(1/k^4\) behavior with the amplitude given by Tan’s contact which is a monotonic function of the statistics parameter. The natural extension of our work is the consideration of the equivalent lattice system, the 2AHM \(87\) \(88\), in the strong repulsive limit. In the fermionic case, the well known Hubbard model, the determinant representation of the correlators is already known \(139\) \(140\) and the long distance asymptotics of the static correlators has also been determined \(141\). This will be addressed in a future publication.
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Appendix A: Fredholm determinants

Here we present some minimal information on Fredholm determinants (Chap. III of [142]). Consider a Fredholm integral equation of the second kind

\[ f(x) - \lambda \int_a^b K(x, y) f(y) \, dy = g(x) , \tag{A1} \]

with \( f(x), g(x) \) continuous functions on \([a, b]\) and the kernel \( K(x, y) \) continuous, symmetric and bounded. Introducing the \( n \)-th iterated kernel

\[ K^{(n)}(x, y) = \int_a^b K(x, z) K^{(n-1)}(z, y) \, dz , \quad K^{(1)}(x, y) = K(x, y) , \tag{A2} \]

and the trace of the operator and its powers \( \text{Tr} K = \int_a^b K(x, x) \, dx , \ \text{Tr} K^2 = \int_a^b \int_a^b K(x, y) K(y, x) \, dx \, dy \) and so on, the Fredholm determinant of the integral operator \( (1 - \lambda K) \) is given by

\[ \det \left( 1 - \lambda \hat{K} \right) = \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^n}{n!} \int_a^b dx_1 \cdots \int_a^b dx_n \, K_n \left( x_1, \ldots, x_n \middle| y_1, \ldots, y_n \right) , \tag{A3} \]

where

\[ K_n \left( x_1, \ldots, x_n \middle| y_1, \ldots, y_n \right) \equiv \det_{i \leq j, k \leq n} K(x_j, y_k) . \tag{A4} \]

Two useful formulae are

\[ \ln \det \left( 1 - \lambda \hat{K} \right) = -\sum_{n=1}^{\infty} \frac{\lambda^n}{n} \text{Tr} K^n , \quad \text{and} \quad \left( 1 - \lambda \hat{K} \right)^{-1} = 1 + \lambda K^{(1)} + \lambda^2 K^{(2)} + \cdots . \tag{A5} \]

Appendix B: Numerical implementation of Fredholm determinants

Fredholm determinants can be numerically evaluated with relative ease using a method which is based on the classical Nyström method for the solutions of Fredholm integral equations of the second kind [124]. For all the determinants considered in this paper the diagonal elements need to be computed using l’Hôpital rule. In the case of classical Nyström method for the solutions of Fredholm integral equations of the second kind [124]. For all the determinants considered in this paper the diagonal elements need to be computed using l’Hôpital rule. In the case of classical Nyström method for the solutions of Fredholm integral equations of the second kind [124]. For all the determinants considered in this paper the diagonal elements need to be computed using l’Hôpital rule. In the case of classical Nyström method for the solutions of Fredholm integral equations of the second kind [124]. For all the determinants considered in this paper the diagonal elements need to be computed using l’Hôpital rule. In the case of classical Nyström method for the solutions of Fredholm integral equations of the second kind [124]. For all the determinants considered in this paper the diagonal elements need to be computed using l’Hôpital rule. In the case of classical Nyström method for the solutions of Fredholm integral equations of the second kind [124]. For all the determinants considered in this paper the diagonal elements need to be computed using l’Hôpital rule. In the case of classical Nyström method for the solutions of Fredholm integral equations of the second kind [124]. For all the determinants considered in this paper the diagonal elements need to be computed using l’Hôpital rule. In the case of classical Nyström method for the solutions of Fredholm integral equations of the second kind [124]. For all the determinants considered in this paper the diagonal elements need to be computed using l’Hôpital rule. In the case of classical Nyström method for the solutions of Fredholm integral equations of the second kind [124]. For all the determinants considered in this paper the diagonal elements need to be computed using l’Hôpital rule. In the case of classical Nyström method for the solutions of Fredholm integral equations of the second kind [124]. For all the determinants considered in this paper the diagonal elements need to be computed using l’Hôpital rule. In the case of classical Nyström method for the solutions of Fredholm integral equations of the second kind [124]. For all the determinants considered in this paper the diagonal elements need to be computed using l’Hôpital rule. In the case of classical Nyström method for the solutions of Fredholm integral equations of the second kind [124]. For all the determinants considered in this paper the diagonal elements need to be computed using l’Hôpital rule. In the case of classical Nyström method for the solutions of Fredholm integral equations of the second kind [124]. For all the determinants considered in this paper the diagonal elements need to be computed using l’Hôpital rule. In the case of classical Nyström method for the solutions of Fredholm integral equations of the second kind [124]. For all the determinants considered in this paper the diagonal elements need to be computed using l’Hôpital rule. In the case of classical Nyström method for the solutions of Fredholm integral equations of the second kind [124]. For all the determinants considered in this paper the diagonal elements need to be computed using l’Hôpital rule. In the case of classical Nyström method for the solutions of Fredholm integral equations of the second kind [124]. For all the determinants considered in this paper the diagonal elements need to be computed using l’Hôpital rule. In the case of classical Nyström method for the solutions of Fredholm integral equations of the second kind [124]. For all the determinants considered in this paper the diagonal elements need to be computed using l’Hôpital rule. In the case of classical Nyström method for the solutions of Fredholm integral equations of the second kind [124].}

\[ E(x, t | k) = \frac{1}{\pi} \int_{-\infty}^{+\infty} e^{-iyq} \frac{t+iqx}{q-k} \, dq = i \text{erf} (\zeta(x - 2kt)) e^{-ik^2 t + ikx} , \quad t \neq 0 , \]

where \( \zeta = e^{-i \frac{\pi}{4} \text{sgn}(t) / (2\sqrt{|t|})} \). Making use of \( d \text{erf}(x) / dx = 2e^{-x^2} / \sqrt{\pi} \), the derivative of the previous function which appears in the diagonal terms of the determinant can be expressed as

\[ \frac{dE(x, t | k)}{dk} = i e^{-ik^2 t + ikx} \left[ -\frac{4iqt}{\sqrt{\pi}} e^{-\zeta^2 (x - 2kt)^2} - (x - 2kt) \text{erf} (\zeta(x - 2kt)) \right] , \quad t \neq 0 . \]

Another useful formula whose proof is similar with the computation of the Fresnel integrals with the argument tending to infinity is

\[ G(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ik^2 t + ikx} \, dk = \frac{e^{-i \frac{\pi}{4} \text{sgn}(t)}}{2\sqrt{|t|}} e^{\frac{i\zeta^2}{2}} , \quad t \neq 0 . \]
$\Sigma_{\Pi} = \Gamma'_- \cup \Gamma'_+ \cup \mathcal{D}_{-k_F,\delta} \cup \mathcal{D}_{+k_F,\delta}$

**FIG. 8.** Contour for the RHP of $\Pi$ in the static case. The exact definitions of all matrices can be found in [107].

**Appendix C: Asymptotic solution of the RHP in the static case**

In order to obtain an approximate solution of the RHP for $\chi(k)$ in the large $x$ limit several transformations are required which are detailed in [107]. In the final step the following RHP for a matrix denoted by $\Pi(k)$ is considered (Sect. 4.5 of [107]):

- $\Pi(k)$ is analytic in $\mathbb{C}\setminus\Sigma_{\Pi}$ where the contour $\Sigma_{\Pi} = \Gamma'_- \cup \Gamma'_+ \cup \mathcal{D}_{-k_F,\delta} \cup \mathcal{D}_{+k_F,\delta}$ is presented in Fig. 8 and consists of two small circles around $\pm k_F$ of radii $\delta$ and $\Gamma'_\pm$ situated in the upper, respectively, lower half plane.

- $\Pi(k) \xrightarrow[k \to \infty]{} I_2 + O\left(\frac{1}{k}\right)$ uniformly in $k$.

- The jump conditions satisfied by $\Pi(k)$ are:

\[
\begin{align*}
\Pi_+(k)M_+(k) &= \Pi_-(k) \quad \text{for } k \in \Gamma'_-, \\
\Pi_+(k)M_-^{-1}(k) &= \Pi_-(-k) \quad \text{for } k \in \Gamma'_+, \\
\Pi_+(k)\mathcal{P}(k) &= \Pi_-(-k) \quad \text{for } k \in \partial\mathcal{D}_{-k_F,\delta}, \\
\Pi_+(k)\tilde{\mathcal{P}}(k) &= \Pi_-(-k) \quad \text{for } k \in \partial\mathcal{D}_{+k_F,\delta}.
\end{align*}
\]

The exact form of the jump matrices $M_{\pm}(k)$, $\mathcal{P}(k)$, and $\tilde{\mathcal{P}}(k)$ is not important for our analysis (we will however present below the first term in the large $x$ expansion of $\mathcal{P}(k)$, $\tilde{\mathcal{P}}(k)$) but we should point out that $M_{\pm}(k)$ are exponentially close to $I_2$ on $\Gamma'_\pm$ and $\mathcal{P}(k)$ and $\tilde{\mathcal{P}}(k)$ are uniformly $I_2 + O\left(\frac{1}{x^{\nu-1}}\right)$ on $\partial\mathcal{D}_{\pm k_F,\delta}$ with $\nu = 2 \sup_{\partial\mathcal{D}_{\pm k_F,\delta}}|\text{Re}(\nu)|$ where $\nu = -\frac{\ln 2}{2\pi} - \frac{\kappa}{2}$.

Outside the contour $\Sigma_{\Pi}$ we have

\[\Pi(k) = \chi(k)\alpha(k)^{-\sigma_3}, \quad \alpha(k) = \left(\frac{k-k_F}{k+k_F}\right)^{\nu}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.\]

(C1)

Introducing the notation

\[\Pi(k) = I_2 + \frac{1}{k} \begin{pmatrix} \Pi_{11}^{(1)} & \Pi_{12}^{(1)} \\ \Pi_{21}^{(1)} & \Pi_{22}^{(1)} \end{pmatrix} + O\left(\frac{1}{k^2}\right), \quad \chi(k) = I_2 + \frac{1}{k} \begin{pmatrix} \chi_{11}^{(1)} & \chi_{12}^{(1)} \\ \chi_{21}^{(1)} & \chi_{22}^{(1)} \end{pmatrix} + O\left(\frac{1}{k^2}\right),\]

(C2)

expanding (C1) in powers of $1/k$ and equating terms of the same order we obtain

\[\begin{pmatrix} \chi_{11}^{(1)} & \chi_{12}^{(1)} \\ \chi_{21}^{(1)} & \chi_{22}^{(1)} \end{pmatrix} = \begin{pmatrix} \Pi_{11}^{(1)} & \Pi_{12}^{(1)} \\ \Pi_{21}^{(1)} & \Pi_{22}^{(1)} \end{pmatrix} + 2k_F\nu \begin{pmatrix} \Pi_{11}^{(1)} & \Pi_{12}^{(1)} \\ \Pi_{21}^{(1)} & \Pi_{22}^{(1)} \end{pmatrix},\]
which together with [18] shows that $B_{-} = -2\pi i \Pi_{21}^{(i)} / \xi$. Therefore, now we only need to obtain the first terms of the expansion of $\Pi(k)$ in $k$ and in $x$. The following result can be proved in a similar fashion with Prop. 6.2 of [108]

**Proposition 1.** The matrix $\Pi(k)$ admits the asymptotic expansion

$$\Pi(k) = I_2 + \sum_{n=1}^{M} \frac{\Pi^{(n)}(k)}{x^n} + o\left(x^{-M-1-\varepsilon}\right),$$

which is valid uniformly away from $\Sigma_{\Pi}$. For $k$ belonging to any connected component of $\infty$ in $\mathbb{C} \backslash \Sigma_{\Pi}$ the first term of the expansion is given by

$$\Pi^{(1)}(k) = \sum_{\epsilon = \pm} \frac{V^{(x,1)}(\epsilon k_F)}{k - \epsilon k_F},$$

with $((\nu)_n = \Gamma(\nu + n) / \Gamma(\nu))$

$$V^{(-1)}(k) = \begin{pmatrix} -i(\nu)_1^2 & b_{12}(k) \\ b_{21}(k) & i(\nu)_1^2 \end{pmatrix}, \quad V^{(+1)}(k) = \begin{pmatrix} -i(\nu)_1^2 & \tilde{b}_{12}(k) \\ \tilde{b}_{21}(k) & i(\nu)_1^2 \end{pmatrix},$$

$$b_{12}(k) = -i\Gamma^2(1 + \nu) \frac{\sin \pi \nu}{\pi} \frac{e^{-ik_F x}}{x(k_F - k)^{2\nu}}, \quad b_{21}(k) = -i\Gamma^{-2}(\nu) \frac{\pi}{\sin \pi \nu} \frac{e^{ik_F x}}{|x(k_F - k)|^{-2\nu}},$$

$$\tilde{b}_{12}(k) = i\Gamma^2(1 - \nu) \frac{\sin \pi \nu}{\pi} \frac{e^{ik_F x}}{x(k_F + k)^{2\nu}}, \quad \tilde{b}_{21}(k) = -i\Gamma^{-2}(\nu) \frac{\pi}{\sin \pi \nu} \frac{e^{-ik_F x}}{|x(k_F + k)|^{2\nu}}.$$

From (C2) and (C4) we find $\Pi_{21}^{(i)} = \left[V^{(-1)}_{21}(-k_F) + V^{(+1)}_{21}(k_F)\right] / x + o(x^{-2-\varepsilon})$ and, therefore,

$$\frac{1}{4\pi} B_{-} = \frac{\pi}{2x\xi \sin \pi \nu} \left(-\frac{1}{\Gamma^2(\nu)}(2k_F x)^{2\nu} e^{ik_F x} + \frac{1}{\Gamma^2(-\nu)}(2k_F x)^{-2\nu} e^{-ik_F x}\right).$$

3 In Prop. 6.2 of [108] this term has a minus sign. This is due to the fact that the contours on $\partial D_{\pm k_F, \delta}$ have a different orientation compared with the ones used in [108].
Appendix D: Asymptotic solution of the RHP in the dynamic case. The space-like regime

In the space-like regime after a series of transformations Kozlowski [108] obtained the following RHP:

1. $\Pi(k)$ is analytic in $\mathbb{C}\setminus\Sigma_\Pi$ where the contour $\Sigma_\Pi = \Gamma_L^\uparrow \cup \Gamma_L^\downarrow \cup \Gamma_R^\uparrow \cup \Gamma_R^\downarrow \cup \mathcal{D}_{-k_F,\delta}^\uparrow \cup \mathcal{D}_{+k_F,\delta}^\downarrow \cup \mathcal{D}_{k_0,\delta}^\uparrow$ is presented in Fig. 9.\[9\]
2. $\Pi(k) \xrightarrow{k \to \infty} I_2 + O\left(\frac{1}{k}\right)$ uniformly in $k$.
3. $\Pi_+(k)G_\Pi(k) = \Pi_-(k)$ for $k \in \Sigma_\Pi$ with

$$
\begin{cases}
G_\Pi(k) = G_\uparrow(k) & \text{for } k \in \Gamma \equiv \Gamma_L^\uparrow \cup \Gamma_L^\downarrow \cup \Gamma_R^\uparrow \cup \Gamma_R^\downarrow, \\
G_\Pi(k) = \mathcal{P}_{\pm k_F}^{-1}(k) & \text{on } k \in -\partial \mathcal{D}_{\pm k_F,\delta}, \\
G_\Pi(k) = \mathcal{P}_0^{-1}(k) & \text{on } k \in -\partial \mathcal{D}_{\pm k_0,\delta}.
\end{cases}
$$

Again the exact form of the jump matrices is not important for our analysis only the fact that $G_\uparrow(k)$ is exponentially close to the unit matrix on $\Gamma$ for large $x$ and that $\mathcal{P}_{\pm k_F}^{-1}(k)$ and $\mathcal{P}_0^{-1}(k)$ present corrections in powers of $x$ in the same limit on $-\partial \mathcal{D}_{\pm k_F,\delta}$ and $-\partial \mathcal{D}_{\pm k_0,\delta}$. Outside the contour the connection between $\chi(k)$ and $\Pi(k)$ is given by

$$
\chi(k) = \Pi(k)\alpha^{-\sigma_3}(k)
\begin{pmatrix}
I_2 + \sigma^+ \text{p.v.} \int_{-\infty}^{+\infty} e^{-2(k')} dk' \\
\frac{\int_{-\infty}^{+\infty} e^{-2(k')} dk'}{2\pi i}
\end{pmatrix}
\alpha(k) = \left(\frac{k+k_F}{k-k_F}\right)^{\nu(\pm)}
\sigma^+ = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}.
$$

(D1)

Using the same notations as in (C2), expanding in powers of $1/k$ and equating terms of the same order we obtain

$$
\left(\begin{array}{cc}
\chi_1^{(1)} & \chi_2^{(1)} \\
\chi_1^{(2)} & \chi_2^{(2)}
\end{array}\right) = \left(\begin{array}{cc}
\Pi_1^{(1)} + 2k_F\nu(\pm) & \Pi_1^{(12)} + \frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{-2(k')} dk'
\end{array}\right)
\left(\begin{array}{cc}
\Pi_2^{(1)} & \\
\Pi_2^{(12)} - 2k_F\nu(\pm)
\end{array}\right).
$$

Together with (66) the potentials $B_-$ and $b_{++}$ are expressed in terms of elements of $\Pi^{(1)}$ as

$$
B_- = -\frac{i\pi}{2\sin^2 \pi \nu(\pm)} \Pi_2^{(1)} , \quad b_{++} = i\Pi_2^{(1)}.
$$

(D2)

The first terms of the large $x$ expansion of $\Pi$ are given by the following proposition:

**Proposition 2.** (Prop. 6.2. of [108]) The matrix $\Pi$ admits the asymptotic expansion

$$
\Pi(k) = I_2 + \sum_{n \geq 0} \frac{\Pi^{(n,x)}(x)}{x^{\frac{1}{2}}}
$$

(D3)

with the first terms given by

$$
\Pi^{(0,x)}(k) = -\frac{d^{(0)}(k_0)}{k-k_0}\sigma, \quad \Pi^{(1,x)}(k) = -\sum_{\epsilon = \pm} \frac{V^{(\epsilon,0)}(\epsilon k_F)}{k-k_F},
$$

(D4)

when $k$ belongs to any connected component of $\infty$ in $\mathbb{C}\setminus\Sigma_\Pi$. The expansion is valid uniformly away from the contour and $\sigma = \sigma^+$ in the space-like regime and $\sigma = \sigma^-$ in the time like regime.

In the space-like regime we have $u(k) = k - \frac{tk^2}{x}$

$$
\begin{align*}
&d^{(0)}(k) = \alpha^{-2}(k)e^{ixu(k_0)} \frac{\Gamma(1/2)e^{i\pi/4}}{2\pi} \frac{1}{\sqrt{2\pi}}, \\
&V^{(-,0)}(k) = -i \begin{pmatrix} k + k_F \\ u(k) - u(-k_F) \end{pmatrix} \begin{pmatrix} -(-\nu(\pm))^2_1 & -ib_{12}(k) \\ -ib_{21}(k) & (-\nu(\pm))^2_1 \end{pmatrix}, \\
&V^{(+,0)}(k) = -i \begin{pmatrix} k - k_F \\ u(k) - u(k_F) \end{pmatrix} \begin{pmatrix} -(\nu(\pm))^2_1 & -ib_{12}(k) \\ -ib_{21}(k) & (-\nu(\pm))^2_1 \end{pmatrix},
\end{align*}
$$

(D5a)\[D5c\]
with

\[ b_{12}(k) = -i \Gamma^2 \left(1 - \nu(\pm)\right) \frac{\sin \pi \nu(\pm)}{\pi C(L)(k)} \]
\[ b_{21}(k) = -i \frac{\pi C(L)(k)}{\sin \pi \nu(\pm) \Gamma^2 (-\nu(\pm))}, \]  

(D6a)

\[ \tilde{b}_{12}(k) = i \Gamma^2 \left(1 + \nu(\pm)\right) \frac{\sin \pi \nu(\pm)}{\pi C(R)(k)} \]
\[ \tilde{b}_{21}(k) = i \frac{\pi C(R)(k)}{\sin \pi \nu(\pm) \Gamma^2 (\nu(\pm))}, \]  

(D6b)

and

\[ C^{(L)}(k) = - \left(e^{-2i \pi \nu(\pm)} - 1\right) \left(\frac{k + k_F}{u(k) - u(-k_F)}\right)^{2\nu(\pm)} \frac{e^{-iux(-k_F)}}{[x(k_F - k)]^{2\nu(\pm)}}, \]  

(D7a)

\[ C^{(R)}(k) = - \left(e^{-2i \pi \nu(\pm)} - 1\right) \left(\frac{u(k) - u(k_F)}{k - k_F}\right)^{2\nu(\pm)} \frac{e^{-iux(k_F)}}{[x(k_F + k)]^{-2\nu(\pm)}}. \]  

(D7b)

Using (D2), Prop. 3 and (D5), (D6) and (D7) we have

\[ B_{-+} = \frac{i\pi}{2 \sin^2 \pi \nu x} \left(V_{21}^{(-,0)}(-k_F) + V_{21}^{(+,0)}(k_F)\right), \quad \nu = \frac{\kappa}{2} - \frac{\eta}{2\pi}, \]
\[ = e^{ik_F x} \frac{2\nu}{2 \sin^2 \pi \nu} \left(\frac{(2k_F)^{-2\nu}}{\sin \pi \nu \Gamma^2 (-\nu) (x + 2k_F t)^{2\nu + 1}} - \frac{(2k_F)^{2\nu}}{\sin \pi \nu \Gamma^2 (\nu) (x - 2k_F t)^{-2\nu + 1}}\right). \]  

(D8)

In a similar fashion

\[ b_{++} = -\frac{d^{(0)}}{x^{1/2}} - \frac{i}{x} \left(V_{12}^{(-,0)}(-k_F) + V_{12}^{(+,0)}(k_F)\right), \quad \nu = -\frac{\kappa}{2} + \frac{\eta}{2\pi}, \]
\[ = G(x, t) \left(\frac{x - 2k_F t}{x + 2k_F t}\right)^{2\nu} - e^{-ik_F x} \frac{2\nu}{\sin \pi \nu} \left(\frac{(2k_F)^{-2\nu}}{\Gamma^2 (-\nu) (x + 2k_F t)^{2\nu + 1}} - \frac{(2k_F)^{2\nu}}{\Gamma^2 (\nu) (x - 2k_F t)^{-2\nu + 1}}\right), \]  

(D9)

where we have used that \( G(x, t) = e^{ix^2/(4t)} e^{-ix/\sqrt{t}} (2\sqrt{t})^{-1} \) for \( x, t > 0 \).

**Appendix E: Asymptotic solution of the RHP in the dynamic case. The time-like regime**

The RHP for the \( \Pi(k) \) matrix in the time-like case is almost the same as the one presented in the previous appendix for the space-like regime. The contour is presented in Fig. 10 and the only changes are at the level of the parametrices
\(\mathcal{P}_{\pm k_{F}}(k)\) and \(\mathcal{P}_{k_{0}}(k)\). Prop. 2 and \([D2]\) are also true in the time-like regime but \([D5]\) and \([D6]\) are replaced by

\[
d^{(0)}(k) = - \left( e^{-2i\pi\nu(\pm)} - 1 \right)^2 \alpha^2(k) e^{-i\pi u(k_{0})} \Gamma(1/2) e^{\pi/4} \frac{e^{\pi/4}}{2 \pi} \frac{e^{\pi/4}}{2 \pi} e^{1/2},
\]

(E1a)

\[
V^{(-,0)}(k) = -i \left( \frac{k + k_{F}}{u(k) - u(-k_{F})} \right) \left( \frac{-(-\nu(\pm))^2 - ib_{12}(k)}{-ib_{21}(k)} \right),
\]

(E1b)

\[
V^{(+,0)}(k) = -i \left( \frac{k - k_{F}}{u(k) - u(k_{F})} \right) \left( \frac{-\nu(\pm)^2 - ib_{12}(k)}{-ib_{21}(k)} \right),
\]

(E1c)

where

\[
b_{12}(k) = -i \Gamma^2 \left( 1 - \nu(\pm) \right) \sin \frac{\pi \nu(\pm)}{\pi C^{(L)}(k)}, \quad b_{21}(k) = -i \frac{\pi C^{(L)}(k)}{\sin \frac{\pi \nu(\pm)}{\Gamma^2 (\pm)}} \Gamma^2 (\pm),
\]

(E2a)

\[
\tilde{b}_{12}(k) = \frac{i \pi C^{(R)}(k)}{\Gamma (-\nu(\pm))} \Gamma^2 \left( 1 - \nu(\pm) \right) \sin \frac{\pi \nu(\pm)}{\Gamma^2 (\pm)} C^{(R)}(k),
\]

(E2b)

with the \(C^{(L)}(k)\) and \(C^{(R)}(k)\) functions defined in \([D7]\). Like in the space-like case we obtain

\[
B_{--} = \frac{i \pi}{2 \sin^2 \pi \nu} \left( \frac{d^{(0)}(k_{0})}{x^{1/2}} + \frac{V_{12}^{(-,0)}(-k_{F})}{x} + \frac{V_{21}^{(+,0)}(k_{F})}{x} \right), \quad \nu = \frac{\kappa}{2} - \frac{\eta}{2}.
\]

(E3)

and

\[
\tilde{b}_{++} = -i \left( \frac{V_{12}^{(-,0)}(-k_{F})}{x} + \frac{V_{12}^{(+,0)}(k_{F})}{x} \right), \quad \nu = -\frac{\kappa}{2} + \frac{\eta}{2},
\]

(E4)

where we have used \(\Gamma(1 - z)\Gamma(z) = \pi / \sin \pi z\).

**Appendix F: Thermodynamic limit of singular sums**

In this Appendix we are going to rewrite the elements of the \(S^{(\pm)}\) and \(R^{(+)}\) matrices in a form that will allow to take the thermodynamic limit. We are going to need two identities (1.421(3) and 1.422(4) of \([118]\)):

\[
\cot \pi x = \frac{1}{\pi x} + \frac{1}{\pi} \sum_{j=1}^{\infty} \left( \frac{1}{x - j} + \frac{1}{x + j} \right),
\]

(F1a)

\[
\frac{1}{\sin^2 \pi x} = \frac{1}{\pi x} \sum_{j=-\infty}^{\infty} \frac{1}{(x - j)^2}.
\]

(F1b)

1. **Thermodynamic limit of \(S^{(-)}\)**

The quasimomenta appearing in the definition of the \(S^{(-)}\) matrix \([95]\) can be written as

\[
k_{j} = \frac{2\pi}{L} j - \frac{\pi \kappa N}{L} + \frac{\Lambda}{L}, \quad q_{l} = \frac{2\pi}{L} l - \frac{\pi \kappa (N - 1)}{L} + \frac{\Theta}{L}, \quad j, l \in \mathbb{Z},
\]

(F2)
which means that \( q_i - k_j = 2\pi m/L + \pi \kappa/L + (\Theta - \Lambda)/L \) with \( m \in \mathbb{Z} \). Using this relation and the identities (F1a) and (F1b) we obtain two useful formulae (\( q = \tilde{q} + \Theta/L \))

\[
\frac{2}{L} \sum_{q} \frac{1}{q-k} = - \cot \left( \frac{\Lambda - \Theta - \pi \kappa}{2} \right), \quad \frac{1 - e^{i\pi \kappa \tilde{\omega} \zeta}}{L^2} \sum_{q} \frac{1}{(q-k)^2} = 1. \tag{F3}
\]

Using the first formula we can prove the following identity

\[
\frac{1 - e^{i\pi \kappa \tilde{\omega} \zeta}}{L^2} \sum_{q} \frac{e^{-ikt^2+ikx}}{q-k} = \frac{4\sin^2 \left( \frac{\Lambda - \Theta - \pi \kappa}{2} \right)}{L} \left\{ \frac{1}{L^2(k_a - k_b)} \sum_{q} \left( \frac{e^{-ikt^2+ikx}}{q-k} - \frac{e^{-ikt^2+ikx}}{q-k} \right) \right\},
\]

\[
= \left[ 1 - \cos \left( \frac{\Lambda - \Theta - \pi \kappa}{2} \right) \right] \left( e^{-i(k)} - \sin \left( \frac{\Lambda - \Theta - \pi \kappa}{2} \right) [e_{-}(k)]^{-2} \right) \tag{F4}
\]

where we have introduced the function

\[
e_{-}(k) = \frac{2}{L} \sum_{q} \frac{e^{-ikt^2+ikx} - e^{-ikt^2+ikx}}{q-k} \tag{F5}
\]

and \( e_{-}(k) \) is defined in (99). Now we have all the necessary tools to rewrite the \( S^{(-)} \) matrix. We will consider first the off-diagonal elements (\( a \neq b \)). Starting from the definition (95) and successively using the identities \((k_a - q)^{-1}(k_b - q)^{-1} = (k_a - k_b)^{-1}(q - k_a)^{-1} - (q - k_b)^{-1}\) and (F4) we obtain

\[
[S^{(-)}]_{ab} = e_{-}(k_a)e_{-}(k_b) \frac{1 - e^{i\pi \kappa \tilde{\omega} \zeta}}{L^2(k_a - k_b)} \sum_{q} \left( \frac{e^{-ikt^2+ikx}}{q-k_a} - \frac{e^{-ikt^2+ikx}}{q-k_b} \right),
\]

\[
= \frac{2}{L} \left\{ 1 - \cos \left( \frac{\Lambda - \Theta - \pi \kappa}{2} \right) \frac{e_{+}(k_a)e_{-}(k_b) - e_{-}(k_a)e_{+}(k_b)}{k_a - k_b} \right\}
\]

\[
- \sin \left( \frac{\Lambda - \Theta - \pi \kappa}{2} \right) \frac{e_{-}(k_a)^{-1}e_{-}(k_b) - e_{-}(k_a)[e_{-}(k_b)]^{-1}}{k_a - k_b} \right\}, \tag{F6}
\]

with \( e_{+}(k) = e^{(-i)}(k)e_{-}(k) \). What about the diagonal elements? Let us show that in the thermodynamic limit \( [S^{(-)}]_{aa} = 1 + \lim_{k_a \to k_b}[S^{(-)}]_{ab} \) with \( [S^{(-)}]_{ab} \) written as in (F6). Denoting \( k_a \) by \( k \) in order to lighten the notation the diagonal elements of the \( S^{(-)} \) matrix (95) can be written as

\[
[S^{(-)}]_{aa} = e_{-}(k)^2 \left[ \frac{1 - e^{i\pi \kappa \tilde{\omega} \zeta}}{L^2} \sum_{q} \frac{e^{-ikt^2+ikx} - e^{-ikt^2+ikx}}{q-k} + e^{-ikt^2+ikx} \right], \tag{F7}
\]

where we have used the second identity from (F3). From the definition (F5) we have

\[
[e_{-}(k)]' = \frac{2}{L} \sum_{q} \frac{e^{-ikt^2+ikx} - e^{-ikt^2+ikx}}{(q-k)^2} - \frac{2}{L} \sum_{q} \frac{(-2ikt + ix)e^{-ikt^2+ikx}}{q-k}, \tag{F8}
\]

and using the first identity in (F3) it is easy to see that \( e^{-ikt^2+ikx} \sum_{q} \frac{2}{L(q-k)} = -2 \sin(\Lambda - \Theta - \pi \kappa)[e_{-}(k)]^{-2} \). Plugging these intermediary results in (F7) we find

\[
[S^{(-)}]_{aa} = 1 + \frac{1}{L} \left\{ \left( 1 - \cos(\Lambda - \Theta - \pi \kappa)[e_{-}(k)]^2 \right)[e_{-}(k)]' - \sin(\Lambda - \Theta - \pi \kappa)(-2ikt + ix) \right\}, \tag{F9}
\]

which is exactly what we would obtain by taking the limit \( k_a \to k_b \) and using l'Hôpital's rule in (F6).

2. Thermodynamic limit of \( S^{(+)} \) and \( R^{(+)} \)

The functions appearing in the definitions (105), (106) of \( S^{(+)} \) and \( R^{(+)} \) involve sums over \( \tilde{k} \)'s. The general relation for the difference of the quasimomenta (F2) remains the same and the building identities become

\[
\frac{2}{L} \sum_{k} \frac{1}{k-q} = \cot \left( \frac{\Lambda - \Theta - \pi \kappa}{2} \right), \quad \frac{1 - e^{i\pi \kappa \tilde{\omega} \zeta}}{L^2} \sum_{q} \frac{1}{(k-q)^2} = 1. \tag{F10}
\]
Introducing the function $e^{(+)}(q)$ defined by

$$e^{(+)}(q) = \frac{2}{L} \sum_k \frac{e^{-ikt^2+ikx} - e^{-itq^2+iqx}}{k-q}, \quad (F11)$$

and using the first identity in (F10) we obtain (note the sign change in front of the sin term compared with (F4))

$$\frac{1 - e^{i\pi \omega L}}{L} \sum_k \frac{e^{-ikt^2+ikx}}{k-q} = \left[ 1 - \cos(\Lambda - \Theta - \pi \kappa) \right] e^{(+)}(k) + \sin(\Lambda - \Theta - \pi \kappa) \left[ e_{-}(k) \right]^{-2}. \quad (F12)$$

In a manner similar with the previous section we obtain for the off diagonal elements ($q_a \neq q_b$) (again, note the sign change in front of the sin term)

$$[S^{(+)}]_{ab} = \frac{2}{L} \left\{ \frac{1 - \cos(\Lambda - \Theta - \pi \kappa) e^{(+)}(q_a)e_{-}(q_b) - e_{-}(q_a)e^{(+)}(q_b)}{q_a - q_b} \right. $$

$$+ \left. \sin(\Lambda - \Theta - \pi \kappa) \left[ e_{-}(q_a) \right]^{-1} e_{-}(q_b) - e_{-}(q_a) \right\} \right] ^{-1}, \quad (F13)$$

with $e_{+}(k) = e^{(+)}(k)e_{-}(k)$. The diagonal terms are computed using l’Hôpital’s rule. In the case of $R^{(+)}$ using the identity (F12) we find

$$[R^{(+)}]_{ab} = \frac{1 - \cos(\Lambda - \Theta - \pi \kappa)}{2} \frac{e^{(+)}(q_a)e^{(+)}(q_b)}{L} + \frac{\sin(\Lambda - \Theta - \pi \kappa)}{2} \left[ \frac{e^{(+)}(q_a)}{Le_{-}(q_a)} + \frac{e^{(+)}(q_b)}{Le_{-}(q_b)} \right] \right.$$  

$$+ \frac{1 + \cos(\Lambda - \Theta - \pi \kappa)}{2} \frac{1}{Le_{-}(q_a)e_{-}(q_b)}. \quad (F14)$$

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