On the $p$-adic geometry of traces of singular moduli

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The aim of this article is to show that $p$-adic geometry of modular curves is useful in the study of $p$-adic properties of traces of singular moduli. In order to do so, we partly answer a question by Ono ([5, Problem 7.30]). As our goal is just to illustrate how $p$-adic geometry can be used in this context, we focus on a relatively simple case, in the hope that others will try to obtain the strongest and most general results. For example, for $p = 2$, a result stronger than Thm. 2 is proved in [2], and a result on some modular curves of genus zero can be found in [6]. It should be easy to apply our method, because of its local nature, to modular curves of arbitrary level, as well as to Shimura curves.

**Definition 1** For $d$ a positive integer that is congruent to 0 or 3 mod 4, let $O_d$ be the quadratic order of discriminant $-d$. For $f$ in $\mathbb{Z}[j]$ and $E$ an elliptic curve over some ring $R$, let $f(E)$ be the element of $R$ obtained by evaluating the $f$ on the $j$-invariant of $E$. For such $d$ and $f$, let:

$$t_f(d) := \sum_{\text{End}(E) \supset O_d} 2f(E)/\#\text{Aut}(E),$$

where the sum ranges over the set of isomorphism classes of complex elliptic curves whose ring of endomorphisms contains $O_d$. We also define an integer $\alpha(d)$ to be 2 if $\mathbb{Q}(\sqrt{-d}) = \mathbb{Q}(\sqrt{-1})$, 3 if $\mathbb{Q}(\sqrt{-d}) = \mathbb{Q}(\sqrt{-3})$, and 1 otherwise.

For $d$ as above and $m$ a positive integer, the number $t_m(d)$ defined in [5] is obtained by taking $f := (j - 744)|T_0(m)$.

**Theorem 2** Let $d > 0$ be an integer that is congruent to 0 or 3 modulo 4, and let $f$ be in $\mathbb{Z}[j]$. Let $p$ be a prime not dividing $d$ that splits in $\mathbb{Q}(\sqrt{-d})$, and let $n \geq 1$. Then $\alpha(d)t_f(p^{2n}d)$ is an integer, and $\alpha(d)t_f(p^{2n}d) \equiv 0 \mod p^n$.

All that we need from the local moduli theory of ordinary elliptic curves in positive characteristic is summarized in the following proposition. Definitions for the terms occurring in it can be found in [7] and [4].
Proposition 3 Let $p$ be a prime number, $k$ a finite field of characteristic $p$. Let $E_0$ be an ordinary elliptic curve over $k$, and let $A$ be its endomorphism ring. Then $A$ is an order in a quadratic extension of $\mathbb{Q}$, and $A$ is split at $p$: $\mathbb{Z}_p \otimes A$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$; in particular, $A$ is maximal at $p$.

Let $k \rightarrow \overline{k}$ be an algebraic closure, and let $W$ be the ring of Witt vectors of $k$. Let $E/R$ be the universal deformation of $E_0/k$ over $W$-algebras. Let $\alpha: \mathbb{Q}_p/\mathbb{Z}_p \rightarrow E_0(\overline{k})[p^\infty]$ be a trivialisation of the group of torsion points of $p$-power order of $E_0(\overline{k})$. Then $\alpha$ induces a so-called Serre-Tate parameter $q \in R^*$, and $R = W[[q - 1]]$. For $n \geq 0$, let $A_n$ be the subring $\mathbb{Z} + p^n A$ of $A$, i.e., the order of index $p^n$ in $A$. Then the closed subscheme of $\text{Spec} \, R$ over which all elements of $A_n$ lift as endomorphisms of $E$ is the closed subscheme defined by the equation $q^{p^n} = 1$.

Proof. The endomorphism ring $A$ is free of finite rank as a $\mathbb{Z}$-module, and is an integral domain because each non-zero element in it is surjective as a morphism from $E_0$ to itself. The $p$-divisible group $E_0[p^\infty]$ is the direct sum of its local and etale parts $E_0[p^\infty]^0$ and $E_0[p^\infty]^\text{et}$, hence its endomorphism ring is the $\mathbb{Z}_p$-algebra $\mathbb{Z}_p \times \mathbb{Z}_p$. Then $A$ is commutative because it embeds into $\mathbb{Z}_p \times \mathbb{Z}_p$. The image of the Frobenius endomorphism of $E_0$ is of the form $(p^n u, v)$, with $u$ and $v$ in $\mathbb{Z}_p^*$, and $|k| = p^n$. This proves that $A$ is quadratic over $\mathbb{Z}$, and split at $p$.

The construction of $q$ and the statement that $R = W[[q - 1]]$ are in [4 §2]. There it is also shown every $f$ in $A$ determines a closed subscheme $V_f$ of $\text{Spec} \, R$ given by the condition that $f$ can be lifted as an endomorphism of $E_{V_f}$, and universal for that property. This subscheme $V_f$ coincides with the closed subscheme over which $f$ can be lifted as an endomorphism of $E[p^\infty]$. Let $n \geq 0$, and let $V$ be the intersection of the $V_f$ for all $f$ in $A_n$. Then $V$ is defined by the condition that the endomorphism $(p^n, 0)$ of $E_0[p^\infty]$ lifts as an endomorphism of the $p$-divisible group. The proof of Part 4 of [4 Thm 2.1] shows that $V$ is defined by the equation $q^{p^n} = 1$. □

We can now prove Theorem 2. Let $d$, $f$, $p$ and $n$ be as in the statement. For each $E$ with $\text{End}(E) \supset O_{p^{2n}d}$ we have that $\# \text{Aut}(E)/2$ divides $\alpha(d)$, hence $\alpha(d)t_f(p^{2n}d)$ is an integer. Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and choose an embedding of $\overline{\mathbb{Q}}$ into an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. For $E$ an elliptic curve over $\mathbb{C}$ with complex multiplications, let $\overline{E}$ denote its reduction over $\overline{\mathbb{F}}_p$. For each $E$ in the sum in Definition 1 there is a unique $E_0$ such that $\text{End}(E_0) = O_d$ and $\overline{E} \cong \overline{E_0}$. For each such $E_0$, let $\mathcal{E}(E_0)$ denote the set of $E$ with $\text{End}(E) \supset O_d$ and $\overline{E} \cong \overline{E_0}$. Then we have:

$$\alpha(d)t_f(p^{2n}d) = \sum_{\text{End}(E_0) = O_d} \frac{2\alpha(d)}{\# \text{Aut}(E_0)} \cdot \frac{\# \text{Aut}(E_0)}{\# \text{Aut}(E)} \cdot f(E)/\# \text{Aut}(E).$$

By construction, the $2\alpha(d)/\# \text{Aut}(E_0)$ are integers. Fix an $E_0$ as in the sum, and let $q$ be a $q$-parameter of the deformation space of $\overline{E_0}$. Then the relation between deformation spaces and
coarse moduli spaces (see [3, §8.2.1]) and Propositions imply that:

$$\# \text{Aut}(E_0) \sum_{E \in E(E_0)} f(E)/\# \text{Aut}(E) = \sum_{x^{p^n}=1} f(x-1),$$

where we can now view $f$ as an element $\sum_{k \geq 0} f_k t^k$ of $W[[t]]$, with $t = q - 1$ and $W$ the ring of Witt vectors of $\mathbb{F}_p$. The observation that $\sum_{x^{p^n}=1} (x-1)^k$ is in $p^n W$ for all $k \geq 0$ finishes the proof.

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