DIHEDRAL LINKING INVARIANTS

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Abstract. A Fox p-colored knot $K$ in $S^3$ gives rise to a corresponding $p$-fold branched cover $M$ of $S^3$ along $K$. The pre-image of the knot $K$ under the covering map is a $p+1$-component link $L$ in $M$, and the set of pairwise linking numbers of the components of $L$ is an invariant of $K$. This powerful invariant played a key role in the development of early knot tables, and appears in formulas for many other important knot and manifold invariants. We give an algorithm for computing this invariant for all odd $p$, generalizing an algorithm of Perko, and tabulate the invariant for thousands of $p$-colorable knots.

1. Introduction

Linking numbers of curves in branched covers of the 3-sphere along a knot $K$ are a rich source of invariants of $K$. One such invariant is the $p$-dihedral linking invariant, which is defined for Fox $p$-colorable knots, where $p$ is odd. A Fox $p$-coloring of $K$ gives rise to a $p$-fold branched covering $f : M \to S^3$ along $K$. The knot $K$ lifts to a $(p+1)/2$-component link in $M$, and the set of pairwise linking numbers of its components is the dihedral linking invariant of the $p$-colored knot $K$.

We refer the reader to [17] for details on the rich history of non-cyclic branched covering spaces in knot theory and their associated linking invariants, particularly the key role they played in the development of early knot tables. For example, using computations of Bankwitz and Schumann [1], Reidemeister used the dihedral linking invariant distinguish two knots with the same Alexander polynomial [18].

Linking invariants derived from branched covers, and dihedral linking invariants in particular, also appear in formulas for many other knot and manifold invariants, for example: Cappell and Shaneson’s formula for the Rokhlin $\mu$-invariant [7], Litherland’s formula for the Casson-Gordon invariants [12], and Kjuchukova’s $\Xi_p$-invariant [11, 6], which gives an obstruction to a knot being homotopy-ribbon, as well as bounds on 4-genera associated to a $p$-colored knot [9, 4, 5].

Perko gave an explicit geometric algorithm for computing the dihedral linking invariant for all 3-colorable knots in his 1964 thesis, and tabulated the invariant for all such knots up to 11 crossings [15]. The primary challenge in generalizing Perko’s algorithm to $p > 3$ is to find a simple way to combinatorially encode a cell structure on the covering space, which also lends itself to fast computation. We develop such a method and give a fully general geometric algorithm for computing the dihedral linking invariant for all $p$-colorable knots. We then tabulate the invariant for thousands of knots. Code and additional data can be found at [3].

A number of methods have been introduced for computing the dihedral linking invariant for particular knots and knot families. For example, in [16, Theorem 4], Perko gave an elegant method for computing linking numbers for 2-bridge knots from the Schubert normal form. Perko also introduced a visually appealing way of computing the linking invariant from a Perko surface, a Seifert-like surface constructed from the diagram of a $p$-colored knot [17]; see Figure 1. However, such surfaces may not exist for a given diagram of the knot, and it is not known whether every
A colorable knot admits a diagram with a Perko surface. In another direction, Hartley and Murasugi gave general, algebraic formulas for linking numbers of branch curves via the Reidemeister-Schreier algorithm, contrasting them with Perko’s method in [15], which they say requires “considerable geometric intuition” [10].

We now review the necessary background information. Consider a diagram $D_K$ of $K$, and let $p$ be an odd. A Fox $p$-coloring of $D_K$ is an assignment of the values $\{0, 1, \ldots, p-1\}$ to the arcs of $D_K$ such that at each crossing, $a + b \equiv 2c \mod p$, where $a$ and $c$ are the values on the overstrands and $b$ is the value on the understrand. Equivalently, a Fox $p$-coloring is described by a surjection $\rho : \pi_1(S^3 - K) \to D_p$, where $D_p$ denotes the dihedral group of order $2p$. Label the vertices of a regular $p$-gon $\{0, \ldots, p-1\}$. Denote by $\text{Ref}_n \in D_p$ the reflection over a line through the $n^{th}$ vertex of a regular $p$-gon, and denote by $\text{Ref}_n(x) \in \{0, \ldots, p-1\}$ the image of vertex $x$ under this reflection. The standard meridian of an arc in the diagram $D_K$ colored $n$ is mapped to the reflection $\text{Ref}_n$, and a crossing with overstrand colored $c$ and understand colored $a$ and $b$ satisfies $\text{Ref}_c(a) = b$.

Given a $p$-coloring $\rho : \pi_1(S^3 - K) \to D_p$, the corresponding \textit{irregular $p$-fold dihedral cover of $S^3$}, denoted $M_\rho$, is the branched cover of $S^3$ along $K$ corresponding to a subgroup $\rho^{-1}(\mathbb{Z}_2)$ of $\pi_1(S^3 - K)$. The knot $K$ has one lift $K^0 \subset M_\rho$ of branching index 1, and $(p-1)/2$ lifts $K^1, \ldots, K^{(p-1)/2} \subset M_\rho$ of branching index 2.

The linking number $\text{lk}(\alpha, \beta) \in \mathbb{Q}$ of two knots $\alpha, \beta$ in a closed, connected, oriented 3-manifold $M$ is defined whenever both $\alpha$ and $\beta$ are rationally null-homologous in $M$. Suppose $\Sigma_\alpha$ is a 2-chain such that $\partial \Sigma_\alpha = k \cdot \alpha$, where $k \in \mathbb{Z}$. Then $\text{lk}(\alpha, \beta) = \frac{1}{k} (\beta \cdot \Sigma_\alpha)$, where $\cdot$ denotes the algebraic intersection number. This linking number is well-defined and symmetric [2]. If either $\alpha$ or $\beta$ represent a nonzero class in $H_1(M; \mathbb{Q})$, we set $\text{lk}(\alpha, \beta) = \infty$. The \textit{$p$-dihedral linking invariant} of $K$ together with a choice of $p$-coloring $\rho$ of $K$ is the multiset

$$DLN(K, \rho) = \{\text{lk}(K^i, K^j)|i \neq j \in \{0, 1, \ldots, (p-1)/2\}\}.$$

An outline of our algorithm is as follows. Neuwirth gave a combinatorial method for constructing a branched cover of a 3-manifold $M$ along a 1-subcomplex $K$ from a choice of \textit{splitting complex} for $K$, and a permutation representation of the knot group $\pi_1(M - K) \to S_n$ corresponding to the desired branched cover [14]. We carry out this construction where $M = S^3$, the splitting complex $C$ is the cone on $K$, and the permutation representation is the coloring $\rho : \pi_1(S^3 - K) \to D_p \leq S_p$. The resulting construction gives the irregular $p$-fold cover $M_\rho$ of $S^3$ along $K$, equipped with a cell

![Figure 1. Perko’s surfaces bounding the branch curves of the Figure-8 knot, which has 5-dihedral linking invariant $\{0, 2, -2\}$ [17].](image-url)
structure determined by the lift \( \tilde{C} \) of \( C \) to \( M_\rho \). As noted in [17], this cell structure on \( M \) actually dates back to Wirtinger.

We then find a rational 2-chain \( \Sigma^j \) bounding each connected component \( K^j \) of the singular set, by solving a \( qn \times qn \) system of linear equations, where \( n \) is the number of crossings in a chosen diagram of \( K \) and \( q = \frac{p-1}{2} \). If a solution exists, we conclude \( K^j \) is rationally null-homologous, and \( \text{lk}(K^i, K^j) \) exists provided \( K^i \) is rationally null-homologous as well. If no solution exists, \( K^j \) is not rationally null-homologous. To compute \( \text{lk}(K^i, K^j) \) we choose a push-off of \( K^i \) transverse to the 2-chain \( \Sigma^j \), and compute the signed intersection number \( K^i \cdot \Sigma^j \).

In Section 2 we carry out this algorithm for the Figure-8 knot. We then turn to computing the invariant for an arbitrary \( p \)-colored knot \((K, \rho)\). In Section 3 we introduce configuration diagrams, a fully combinatorial method for encoding the cell structure on the corresponding branched cover \( M_\rho \). In Section 4, we find a rational 2-chain \( \Sigma^j \) with boundary \( K^j \) using the configuration diagram of \((K, \rho)\), and in Section 5 we compute the linking number of \( K^j \) and \( K^i \) using the configuration diagram of \((K, \rho)\). This allows us to tabulate the dihedral linking invariant. Additional worked examples are in Section 6. In the final section, we include a tabulation of the dihedral linking invariant for hundreds of 3-, 5-, and 7-colorable knots. We do not include the full tabulation here due to space considerations; the code used to produce the tabulation and additional data are available at [3]. Note that the tabulation for 3-colorable knots up to 11 crossings agrees with that of Perko [15].

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### 2. Extended Example

In this section, let \( K \) be the Figure-8 knot, with diagram \( D_K \) and 5-coloring \( \rho \) as shown in Figure 2. We will carry out our algorithm on \( K \).

The 3-manifold \( M_\rho \) can be constructed combinatorially from the diagram \( D_K \) by means of a splitting complex for \( K \), namely the cone on \( D_K \) [14]. First, the cone on \( D_K \) gives rise to an associated cell structure on \( S^3 \), as shown in Figure 2 with cone point at infinity.

![Figure 2. Cell structure on \( S^3 \) determined by the cone on \( D_K \).](image_url)
We label the cells as follows:

1. “horizontal” 1-cells $k_0, k_1, k_2, k_3$, which are arcs in the diagram $D_K$ of $K$;
2. “vertical” 1-cells $a_0, a_1, a_2, a_3$, below each crossing of $D_K$;
3. “vertical” 2-cells $A_0, A_1, A_2, A_3$, below each arc of $D_K$
4. one 3-cell, $E$, the complement of the cone
5. one 0-cell $e$ at the cone point
6. a 0-cell at the head of each arc $k_i$

Denote the $k$-skeleton of this cell structure by $C^k$.

Next we lift this cell structure to the dihedral cover $M_\rho$ of $S^3$, branched along $K$ determined by the 5-coloring $\rho$ of $K$ shown in Figure 2. Consider the open cover of the 2-skeleton $C^2$ which consists of: a small open 3-ball $U_i$ centered at $e_i$ and containing a segment of the over-arc at crossing $i$; a small tubular neighborhood $V_j$ of each arc $k_i$ of $D_K$ disjoint from $D_K - k_i$, and a (topological) 3-ball neighborhood $W$ of $e$ disjoint from $D_K$ and containing the remainder of the 2-skeleton $S^2$; see Figure 3. Together with the 3-cell $E$, this forms an open cover of $S^3$. It suffices to understand the lift of the cell structure on its intersection with each of the $U_i, V_j, W,$ and $E$.

Since $M_\rho$ is a 5-fold dihedral cover of $(S^3, K)$, $K$ has one index-1 preimage $K_0$ and two index-2 preimages $K_1$ and $K_2$. Denote the three corresponding lifts of each 1-cell $k_i$ by $k_0^i, k_1^i,$ and $k_2^i$.

There are five lifts of the 3-cell $E$, denoted $E^0 \ldots E^4$. We label them as follows. Let $a$ and $b$ be two points in $E$, and $g : [0, 1] \to S^3$ a path from $a$ to $b$ which intersects exactly one of the two-cells $A_i$ once, transversely, where $A_i$ is colored $n$. Denote by $\tilde{g}_m$ the lift of $g$ such that $\tilde{g}_m(0) \in E^n$. Then $\tilde{g}_m(1) \in E^{Ref_n(m)}$. This labelling is well-defined since the monodromy homomorphism $\rho$ is trivial on the complement $E$ of the cone.

The neighborhood $W$ of the 0-cell $e$ has 5 lifts to $M$. The pair $(W, C^2)$ is homeomorphic to the cone on $(S^2, G_K)$ where $G_K$ is the 4-valent graph corresponding to the knot diagram $D_K$. Its five lifts intersect the 3-cells $E_i$ as shown in Figure 4. The subscripts on the $E_i$ are labelled by the 5 Dehn colorings of $K$ corresponding to the Fox coloring $\rho$, as defined in [8].
Figure 4. Position of the lifts $E^j$ of $E$ in a neighborhood of each lift $e_k$ of $e$. The position of $E^j$ is marked by $j$.

Figure 5. Lifts of a tubular neighborhood of the arc $k_i$ to the 5-fold dihedral cover.

The other cells $a_i$ and $A_i$ above each have 5 lifts to $M_\rho$. We introduce a systematic way of labelling these cells.

Since $D_k$ has an even number of crossings, the vector field $-\partial/\partial z$ along $K$ has two lifts along each of the index-2 curves $K_1$ and $K_2$. We choose one such lift $V^1$ along $K^1$ and one lift $V^2$ along $K^2$. Now denote by $R_i^1$ and $L_i^1$ the two lifts of $A_i$ adjacent to $k_i^1$, with $V^1$ tangent to $R_i^1$. Similarly denote by $R_i^2$ and $L_i^2$ the two lifts of $A_i$ adjacent to $k_i^2$, with $V^2$ tangent to $R_i^2$. See Figure 5 which also serves to label the remaining lifts of the cells $A_i$ and $a_i$. In particular, the lift of $A_i$ adjacent to $k_i^0$ is denoted $B_i$. The five lifts of $a_i$ are $b_i$, $r_i^1$, $l_i^1$, $r_i^2$, and $l_i^2$, as shown in Figure 5.

To fill in Figure 6, which depicts the lift of the cell structure in $U_i$ with all cells labelled, we carry out the following steps:
(1) Lift the 2-skeleton $C^2 \cap U_i$ in a neighborhood of each crossing, and label the positions of the 3-cells $E_j$ according to the coloring.

(2) At crossing 0, arbitrarily label one index-2 lift of $k_0 \: k_0^1$, and label the other $k_0^2$.

(3) At crossing 0, arbitrarily label one of the 2-cells adjacent to $k_0^1 \: R_0$, and label the other $L_0^1$.

(4) At crossing 0, arbitrarily label one of the 2-cells adjacent to $k_0^2 \: R_0^2$, and label the other $L_0^2$.

Next we will find a two-chain $\Sigma^j$ such that $\partial K_j = \Sigma^j$ for $j = 0, 1, 2$. A priori, these two-chains take the following forms:

$$\Sigma^0 = 3 \sum_{i=0}^{3} B_i + 3 \sum_{i=0}^{3} x_i^1(R_i^1 - L_i^1) + 3 \sum_{i=0}^{3} x_i^2(R_i^2 - L_i^2)$$

$$\Sigma^1 = 3 \sum_{i=0}^{3} y_i^1 R_i^1 + (1 - y_i^1)L_i^1 + 3 \sum_{i=0}^{3} y_i^2(R_i^2 - L_i^2)$$

$$\Sigma^2 = 3 \sum_{i=0}^{3} z_i^1(R_i^1 - L_i^1) + 3 \sum_{i=0}^{3} z_i^2 R_i^2 + (1 - z_i^2)L_i^2$$

Indeed, each $\Sigma^j$ is a linear combination of the 2-cells $B_i$, $R_i^k$, and $L_i^k$, and each $k_j^i$ must appear exactly once as a summand in $\partial \Sigma^j$. The linear combinations take the form above if and only if:

$$\partial \Sigma^0 - K^0 = 3 \sum_{i=0}^{2} \sum_{k=1}^{2} \alpha_i^k (r_i^k - l_i^k),$$

$$\partial \Sigma^1 - K^1 = 3 \sum_{i=0}^{2} \sum_{k=1}^{2} \beta_i^k (r_i^k - l_i^k),$$

$$\partial \Sigma^2 - K^2 = 3 \sum_{i=0}^{2} \sum_{k=1}^{2} \gamma_i^k (r_i^k - l_i^k).$$

Since $\partial \Sigma^j = K^j$, we determine each $\alpha_i^k$, $\beta_i^k$, and $\gamma_i^k$ and set them equal to 0. This results in three linear systems in the variables $x_i^j$, $y_i^j$, and $z_i^j$. If the linear system corresponding to $\Sigma^j$ has a solution, it determines $\Sigma^j$ explicitly; if not, $K^j$ cannot be rationally nullhomologous.

To compute $\alpha_i^k$, $\beta_i^k$, and $\gamma_i^k$, it suffices to look at the lift of the cell structure in $f^{-1}(U_i)$, where $U_i$ is the neighborhood of crossing $i$ of $D_K$ described above.

To compute $\alpha_0^k$, $\beta_0^k$, and $\gamma_0^k$, we compute $\partial \Sigma^j$, but only record the boundaries of the 2-cells incident to $r_0^1$ or $l_0^1$, and only list the 1-cells which intersect $f^{-1}(U_0)$. Referring to Figure 6 at Crossing 0, this gives:
Figure 6. Cell structure on the 5-fold branched cover of $S^3$ along the Figure-8 knot.
A choice of solution for the first system is $x_j^0$ in the direction of the orientation of $K^0$. The 2-chain $R^0_j$ is on the right; see Figure 5. Then push $K^j$ into the 3-cells $E^{h_j}$, and denote this push-off by $(K^j)'$.
Now
\[
(K_1')^i \cdot \Sigma^0 = (K_1')^i \cdot \sum_{i=0}^{3} B_i + (K_1')^i \cdot (R_1^1 - L_1^1 - R_3^1 + L_3^1) = (0 + 0 - 1 + 0) + (1 - 1 - 1 + 0) = -2
\]
and
\[
(K_2')^i \cdot \Sigma^0 = (K_2')^i \cdot \sum_{i=0}^{3} B_i + (K_2')^i \cdot (R_1^1 - L_1^1 - R_3^1 + L_3^1) = (0 + 1 + 0 + 0) + (0 + 0 + 0 + 1) = 2
\]
Similarly, a solution to the second system is \( y_1^1 = 1, y_3^1 = 1 \), and all other \( y_i^j = 0 \). This corresponds to the 2-chain
\[
\Sigma_1 = L_0^1 + L_1^1 + R_2^1 + R_3^1.
\]
This gives \( (K_1')^i \cdot \Sigma^1 = -2 \) and \( (K_2')^i \cdot \Sigma^1 = 0 \).

Finally, a solution to the third system is \( z_1^1 = -1 \), and all other \( z_i^j = 0 \). We take
\[
\Sigma_2 = -R_2^1 + L_2^1 + L_0^2 + L_1^2 + L_2^2 + L_3^2
\]
This gives \( (K_1')^i \cdot \Sigma^2 = 2 \) and \( (K_2')^i \cdot \Sigma^2 = 0 \).

The dihedral linking invariant of the Figure-8 is therefore
\[
\text{DLN}(K, \rho) = \{ \text{lk}(K^0, K^1), \text{lk}(K^0, K^2), \text{lk}(K^1, K^2) \} = \{-2, 2, 0\}.
\]
Note that the algorithm double-checks itself, by computing both \( \text{lk}(K^i, K^j) \) and \( \text{lk}(K^j, K^i) \) and confirming that they agree.

3. General Setup

Let \( K \) be a knot and \( \rho : \pi_1(S^3 - K) \to D_p \) a \( p \)-coloring of \( K \). Let \( M_\rho \) denote the corresponding \( p \)-fold irregular branched cover of \((S^3, K)\). Let \( D_K \) be a diagram of \( K \) with an even number of crossings \( n \); we can always arrange this by adding a crossing with a first Reidemeister move.

As in Section 2, we consider a cell structure on \( S^3 \) determined by the cone on \( D_K \).

1. “horizontal” 1-cells \( k_0, k_1, \ldots, k_{n-1} \), which are arcs in the diagram \( D_K \) of \( K \);
2. “vertical” 1-cells \( a_0, a_1, \ldots, a_{n-1} \), below each crossing of \( D_K \);
3. “vertical” 2-cells \( A_0, A_1, \ldots, A_{n-1} \), below each arc of \( D_K \)
4. one 3-cell, \( E \), the complement of the cone
5. one 0-cell \( e \) at the cone point
6. one 0-cell at the head of each arc \( k_i \)

Denote the \( k \)-skeleton by \( C_k \).

Next we lift this cell structure to the dihedral cover \( M_\rho \) of \( S^3 \), branched along \( K \) determined by the \( p \)-coloring \( \rho \) of \( K \). We let \( U_i \) and \( V_i \) represent neighborhoods of crossing \( i \) and arc \( k_i \) as before; see Figure 3.

Since \( M_\rho \) is a \( p \)-fold dihedral cover of \((S^3, K)\), \( K \) has one index-1 preimage \( K^0 \), and \( q = \frac{p-1}{2} \) index-2 preimages \( K^1, K^2, \ldots, K^q \). Denote the corresponding lifts of each 1-cell \( k_i \) by \( k_i^0, k_i^1, \ldots, k_i^q \).
There are \( p \) lifts of the 3-cell \( E \), denoted \( E^0 \ldots E^p \). We again label them by the following rule: Let \( a \) and \( b \) be two points in \( E \), and \( g : [0,1] \to S^3 \) a path from \( a \) to \( b \) which intersects exactly one of the two-cells \( A_i \) once, transversely, where \( k_i \) is colored \( n \). Denote by \( \tilde{g}_m \) the lift of \( g \) such that \( \tilde{g}_m(0) \in E^m \). Then \( \tilde{g}_m(1) \in E^{Ref_m(m)} \).

The other cells \( a_i \) and \( A_i \) above each have \( p \) lifts to \( M_\rho \). Let \( B_i \) denote the lift of \( A_i \) incident to \( k_0^i \).

There are two lifts of \( A_i \) incident to each \( k_j^i \) for \( 1 \leq j \leq q \), which we denote by \( R_j^i \) and \( L_j^i \), labelled as in Section 2. First, arbitrarily label the two lifts of \( A_0 \) incident to \( k_0^0 \), \( R_0^0 \) and \( L_0^0 \). Then lift the vertical vector field \( -\partial/\partial z \) along \( K \) (tangent to \( A_i \)) to \( M_\rho \). Because the number of crossings of \( D_K \) is even, \( -\partial/\partial z \) has two lifts to \( M_\rho \). Let \( V^j \) denote the lift of \( V \) tangent to \( R_0^j \). Then for \( i \geq 1 \), let \( R_i^j \) be the lift of \( A_i \) incident to \( k_i^j \) and tangent to \( V^j \), and let \( L_i^j \) be the other lift of \( A_i \) incident to \( k_i^j \). Finally, let \( r_i^j \) denote the lift of \( a_i \) on the boundary of \( R_i^j \), let \( i_i^j \) denote the lift of \( a_i \) on the boundary of \( L_i^j \), and let \( b_i \) be the lift of \( a_i \) on the boundary of \( B_i \). See Figure 5 for an example when \( p = 5 \).

It will be useful to record the positions of the 2-cells \( R_i^j \) and \( L_i^j \) relative to the 3-cells \( E^0, \ldots, E^{p-1} \). The 2-cells \( R_i^j \) and \( L_i^j \) are incident to two 3-cells. Informally, let \( h = h_i^j \) denote the superscript of the 3-cell \( E^k \) such that, if one stands on the arc \( k_i^j \) facing in the direction of its orientation, \( R_i^j \) is on the right and \( L_i^j \) is on the left. Let \( t = t_i^j \) denote the superscript of the 3-cell \( E^t \) such that, if one stands on the arc \( k_i^j \) facing in the direction of its orientation, \( L_i^j \) is on the right and \( R_i^j \) is on the left.

**Definition 1.** The configuration diagram for arc \( i \) is a set of \( q = \frac{p-1}{2} \) arrows \( arr_1^i, \ldots, arr_q^i \) between vertices \( \{0, \ldots, p-1\} \) of a regular \( p \)-gon (necessarily with disjoint endpoints), such that the head and tail of \( arr_r^i \) are \( h_r^i \) and \( t_r^i \) \( \in \{0, \ldots, p-1\} \), respectively. Note that if \( c(i) \) is the color of the arc \( k_i \), none of the arrows \( arr_1^i, \ldots, arr_q^i \) have an endpoint on vertex \( c(i) \). We let \( arr_0^i \) be an arrow with head and tail \( h_0^i = t_0^i = c(i) \), but omit this arrow from the diagram. See Figure 7.

Observe that the configuration diagram for arc \( i + 1 \) is obtained from the configuration diagram for arc \( i \) by a reflection over the vertex \( c(o(i)) \), where \( s = o(i) \) is the subscript of the arc \( k_s \) passing over \( k_i \) at its head. The configuration diagrams for each crossing of the Figure-8 knot are shown in Figure 6 together with the lifts of the relevant 2-cells.

### 4. Constructing 2-chains bounded by the singular set

In this section, we construct a rational 2-chain \( \Sigma^j \) bounding each \( K^j \), for \( j \in \{0, \ldots, q\} \) or determine no such 2-chain exists. A priori, these 2-chains take the following forms:

\[
\Sigma^0 = \sum_{i=0}^{n-1} B_i + \sum_{j=1}^q \sum_{i=0}^{n-1} x_i^{0,j} (R_i^j - L_i^j)
\]

\[
\Sigma^k = \sum_{i=0}^{n-1} x_i^{k,k} R_i^k + (1 - x_i^{k,k}) L_i^k + \sum_{j \in \{0, \ldots, q\} \setminus \{k\}} \sum_{i=0}^{n-1} x_i^{k,j} (R_i^j - L_i^j)
\]

Now for all \( k \in \{0, \ldots, q\} \),
\[ \partial \Sigma^k - K^k = \sum_{j=1}^{q} \sum_{i=0}^{n-1} \alpha_{i,j}^{k,j} (r_i^j - l_i^j). \]

We compute each \( \alpha_{i,j}^{k,j} \) using the configuration diagram at crossing \( i \) and set the result equal to 0, giving us a system of equations for which a solution determines the 2-chain \( \Sigma^k \).

Figure 7 shows the 8 possible configurations of 2-cells incident to \( r_i^j - l_i^j \) when the local writhe number of the crossing is positive. We rotate the picture so that \( R_i^j \) is to the right of \( k_i^j \) and \( L_i^j \) is to the left of \( k_i^j \). Informally, let \( a(i,j) \) and \( b(i,j) \in \{0, 1, \ldots, q\} \) be the index-2 lifts of the over arc \( k_{o(i)} \) which sit above and below \( r_i^j - l_i^j \) in the picture, respectively. Note that if \( a(i,j) = 0 \), \( B_{o(i)} \) is incident to \( r_i^j - l_i^j \) from above, and if \( b(i,j) = 0 \), \( B_{o(i)} \) is incident to \( r_i^j - l_i^j \) from below. A formal definition, which also works for the case \( j = 0 \), is as follows.

**Definition 2.** Let \( i \in \{0, \ldots, n-1\} \) and let \( j \in \{0, \ldots, q\} \) where \( q = \frac{p-1}{2} \). Then set

\[ a(i,j) = s \text{ if } h_i^j = h_{o(i)}^s \text{ or } h_i^j = t_{o(i)}^s, \]

and set

\[ b(i,j) = s \text{ if } t_i^j = h_{o(i)}^s \text{ or } t_i^j = t_{o(i)}^s. \]

**Example 1.** We can read off the values of \( a(i,j) \) and \( b(i,j) \) from the configuration diagrams at crossings \( i \) and \( o(i) \). First observe from Figure 2 that
variables \( x \) and \( \epsilon \). Together with the local writhe number \( h \), let

\[
\begin{align*}
\text{Definition 3.} & \quad \text{Then set} \\
& \text{The complete set of values of } a(i, j) \text{ and } b(i, j) \text{ are below.} \\
& \begin{array}{|c|c|c|c|}
\hline
a(i, j) & j = 0 & j = 1 & j = 2 \\
\hline
i = 0 & 1 & 0 & 1 \\
i = 1 & 2 & 2 & 1 \\
i = 2 & 1 & 2 & 2 \\
i = 3 & 2 & 1 & 0 \\
\hline
\end{array}
\begin{array}{|c|c|c|c|}
\hline
b(i, j) & j = 0 & j = 1 & j = 2 \\
\hline
i = 0 & 1 & 2 & 2 \\
i = 1 & 2 & 1 & 0 \\
i = 2 & 1 & 0 & 1 \\
i = 3 & 2 & 2 & 1 \\
\hline
\end{array}
\end{align*}
\]

Next, observe \( a(i, j) \) is the arrow in configuration diagram \( \alpha(i) \) such that its head or tail is incident to \( h_i^2 \); that is, either \( h_i^2 = h_a^{o(i)} \) or \( h_i^2 = t_a^{o(i)} \). Similarly \( b(i, j) \) is the arrow in configuration diagram \( \alpha(i) \) such that its head or tail is incident to \( t_j^2 \); that is, either \( t_j^2 = h_b^{o(i)} \) or \( t_j^2 = t_b^{o(i)} \).

Since \( a(0) = 2 \), we can read off each \( a(0, j) \) and \( b(0, j) \) from the first and third diagrams in Figure 8. For example, \( h_0^0 = 1 = h_0^2 = t_0^0 \), and \( t_0^0 = 4 = t_2^2 \), so \( a(0, 1) = 0 \) and \( b(0, 1) = 2 \). Similarly, \( h_0^2 = 2 = h_2^1 \) and \( t_0^2 = 3 = h_3^1 \), so \( a(0, 2) = 1 \) and \( b(0, 2) = 2 \). We can read off each \( a(1, j) \) and \( b(1, j) \) from the first and third diagrams, and so on.

The configuration diagrams for the Figure-8 knot.

![Figure 8](image-url)

We record these configurations combinatorially using two \( \{0, 1, -1\} \)-valued functions, \( \epsilon_a \) and \( \epsilon_b \). Together with the local writhe number \( \epsilon(i) \) at crossing \( i \), \( \epsilon_a \) and \( \epsilon_b \) determine the coefficients of the variables \( x_{o(i)}^{a(i)} \) and \( x_{o(i)}^{b(i)} \) in \( \alpha_i^{k,j} \).

**Definition 3.** Let \( i \in \{0, \ldots, n-1\} \), \( j \in \{0, \ldots, q\} \) with \( a(i, j), b(i, j), \) and \( \alpha(i) \) as defined above. Then set

\[
\epsilon_a(i, j) = \begin{cases} 
1 & h_i^2 = h_a^{o(i)} \text{ and } a(i, j) \neq 0 \\
-1 & h_i^2 = t_a^{o(i)} \text{ and } a(i, j) \neq 0 \\
0 & a(i, j) = 0 
\end{cases}
\]

and

\[
\epsilon_b(i, j) = \begin{cases} 
1 & t_i^2 = t_b^{o(i)} \text{ and } b(i, j) \neq 0 \\
-1 & t_i^2 = h_b^{o(i)} \text{ and } b(i, j) \neq 0 \\
0 & b(i, j) = 0 
\end{cases}
\]

Note that if \( j = 0 \), \( a(i, j) = b(i, j) \), so \( \epsilon_a(i, j) = \epsilon_b(i, j) \).

**Example 2.** We list the \( \epsilon_a(i, j) \) and \( \epsilon_b(i, j) \) for the Figure-8 in Section 2 in the table below.
We write \( C \) of \( B \) by considering the 8 possible configurations of cells in Figures 9 and 10, we see that \( \alpha_i = \epsilon_a(i, j) = x_i^j - x_i^{j+1} - \epsilon_a(i, j)x_{a(i)}^{k(i,j)} - \epsilon_b(i, j)x_{b(i)}^{k(i,j)} + C(i, j, k) \), where \( C(i, j, k) \) is a constant. This constant \( C(i, j, k) \) is the coefficient of \( r_i^j - l_i^j \) in the boundary of \( \sum_{i=0}^{n-1} B_i \) when \( k = 0 \), and in the boundary of \( \sum_{i=0}^{n-1} L_i \) when \( k \in \{1, \ldots, q\} \).

We write \( C(i, j, k) = C_a(i, j, k) + C_b(i, j, k) \) with \( C_a \) and \( C_b \) defined as follows. Note that \( C_a \) is nonzero precisely when \( L_i^k \) (when \( k \neq 0 \)) or \( B_i \) (when \( k = 0 \)) is incident to \( r_i^j - l_i^j \) from above. Similarly, \( C_b \) is nonzero when \( L_i^k \) (when \( k \neq 0 \)) or \( B_i \) (when \( k = 0 \)) is incident to \( r_i^j - l_i^j \) from below.

**Definition 4.** Let \( i \in \{0, \ldots, n-1\} \), \( j \in \{1, \ldots, q\} \), and \( k \in \{0, \ldots, q\} \). Then set

\[
C_a(i, j, k) = \begin{cases} \epsilon(i) a(i, j) = k, k \neq 0, \text{ and } \epsilon(i) \epsilon_a(i, j) = -1 \\ -\epsilon(i) a(i, j) = k \text{ and } k = 0 \\ 0 \text{ else} \end{cases}
\]

\[
C_b(i, j, k) = \begin{cases} \epsilon(i) b(i, j) = k, k \neq 0, \text{ and } \epsilon(i) \epsilon_b(i, j) = 1 \\ -\epsilon(i) b(i, j) = k \text{ and } k = 0 \\ 0 \text{ else} \end{cases}
\]

**Example 3.** The values of \( C_a \) and \( C_b \) for the Figure-8 knot in Section 2 are as follows:

\[
\begin{array}{|c|c|c|c|}
\hline
C_a(i, j, 0) & j = 0 & j = 1 & j = 2 \\
\hline
i = 0 & 0 & 0 & 0 \\
i = 1 & 0 & 0 & 0 \\
i = 2 & 0 & 0 & 0 \\
i = 3 & 0 & 0 & -1 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|}
\hline
C_a(i, j, 1) & j = 0 & j = 1 & j = 2 \\
\hline
i = 0 & 0 & 0 & 0 \\
i = 1 & 0 & 0 & -1 \\
i = 2 & 0 & 0 & 0 \\
i = 3 & 0 & 0 & -1 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
C_a(i, j, 2) & j = 0 & j = 1 \\
\hline
i = 0 & 0 & 0 \\
i = 1 & 0 & -1 \\
i = 2 & 0 & 0 \\
i = 3 & -1 & 0 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|}
\hline
C_b(i, j, 0) & j = 0 & j = 1 & j = 2 \\
\hline
i = 0 & 0 & 0 & 0 \\
i = 1 & 0 & 0 & 1 \\
i = 2 & 0 & -1 & 0 \\
i = 3 & 0 & 0 & 0 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|}
\hline
C_b(i, j, 1) & j = 0 & j = 1 & j = 2 \\
\hline
i = 0 & 0 & 0 & 0 \\
i = 1 & 0 & 0 & 0 \\
i = 2 & -1 & 0 & 0 \\
i = 3 & 0 & 0 & 0 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
C_b(i, j, 2) & j = 0 \\
\hline
i = 0 & 0 \\
i = 1 & -1 \\
i = 2 & 0 \\
i = 3 & 0 \\
\hline
\end{array}
\]
\[
\begin{array}{c|c|c|c}
C_b(i, j, 2) & j = 0 & j = 1 & j = 2 \\
\hline
i = 0 & 0 & 0 & -1 \\
i = 1 & 0 & 0 & 0 \\
i = 2 & 0 & 0 & 0 \\
i = 3 & 1 & 0 & 0 \\
\end{array}
\]

**Example 4.** We can now compute \( \alpha_{0,1} \) for the Figure-8 knot in Section 2 (where it was denoted \( \alpha_0 \)) directly from the combinatorial formula

\[\alpha_i^{k,j} = x_i^{k,j} - x_i^{k,j+1} - \epsilon_a(i, j) x_i^{k,a(i,j)} - \epsilon_b(i, j) x_i^{k,b(i,j)} + C(i, j, k).\]

Set \( i = 0, j = 1 \), and \( k = 0 \). Clearly \( x_i^{k,j} - x_i^{k,j+1} = x_{i,0}^{0,1} - x_{i,1}^{0,1} \).

Recall that \( o(0) = 2 \).

We found \( a(0, 1) = 0 \) and \( b(0, 1) = 2 \) in Example 3.

Next, \( \epsilon_a(0, 1) = 0 \) since \( a(0, 1) = 0 \) by Definition 3. Also using Definition 3, and by considering Figure 8, \( \epsilon_b(0, 1) = 1 \) since \( t_0^1 = 4 = t_2^2 \).

Since crossing 0 has negative local writhe number, \( \epsilon(i) = -1 \).

Now \( C_a(0, 1, 0) = -(-1) = 1 \) by Definition 4 since \( a(0, 1) = 0 = k \). Similarly, \( C_b(0, 1, 0) = 0 \) since \( 0 \neq b(0, 1) \).

Therefore \( \alpha_{0,1}^{0,1} = x_{0,1}^{1} - x_{1,1}^{0,2} + 1 \), which agrees with the direct computation in Section 2.

**Theorem 5.** Let \( q = \frac{p - 1}{2} \) with \( p \) odd. Let \( \rho: \pi_1(S^3 \to K) \) be a \( p \)-coloring of \( K \), with corresponding irregular \( p \)-fold dihedral cover \( f: M_\rho \to S^3 \). Let \( K^0 \) be the index-1 connected component of \( f^{-1}(K) \) and let \( K^1, K^2, \ldots, K^q \) be the index-2 connected components of \( f^{-1}(K) \). Then \( K^k \) is rationally null-homologous in \( M_\rho \) if and only if the system of equations

\[x_i^{k,j} - x_i^{k,j+1} - \epsilon_a(i, j) x_i^{k,a(i,j)} - \epsilon_b(i, j) x_i^{k,b(i,j)} + C(i, j, k) = 0\]

has a solution \((x_0^{k,j}, \ldots, x_{n-1}^{k,j})^q_{j=1} \in \mathbb{Q}^{nq} \).

Furthermore, if \( k = 0 \), the index-1 lift \( K^0 \) of \( K \) bounds the 2-chain

\[\Sigma^0 = \sum_{i=0}^{n-1} B_i + \sum_{j=1}^{q} \sum_{i=0}^{n-1} x_i^{0,j}(R_i^j - L_i^j),\]

and if \( k \neq 0 \), the index-2 lift \( K^k \) of \( K \) bounds the 2-chain

\[\Sigma^k = \sum_{i=0}^{n-1} x_i^{k,k} R_i^k + (1 - x_i^{k,k}) L_i^k + \sum_{j \in \{0, \ldots, q\} \setminus \{k\}} \sum_{i=0}^{n-1} x_i^{k,j}(R_i^j - L_i^j)\]

**Proof.** We systematically check each possible configuration of 2-cells incident to \( r_i^j - l_i^j \), and compute the total number of times \( r_i^j - l_i^j \) appears in the boundary of

\[\Sigma^k = \sum_{i=0}^{n-1} x_i^{k,k} R_i^k + (1 - x_i^{k,k}) L_i^k + \sum_{j \in \{0, \ldots, q\} \setminus \{k\}} \sum_{i=0}^{n-1} x_i^{k,j}(R_i^j - L_i^j)\]

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if \( k \neq 0 \),

\[
\Sigma^0 = \sum_{i=0}^{n-1} B_i + \sum_{j=1}^{q} \sum_{i=0}^{n-1} x_i^{0,j}(R_i^j - L_i^j),
\]

if \( k = 0 \).

There are 16 possible configurations. The 8 configurations corresponding to a crossing with positive local writhe number are shown in Figure 9 and the 8 configurations corresponding to a crossing with negative local writhe number are shown in Figure 10. In each case, the coefficients of \( x_{i}^{k,j} \), \( x_{i+1}^{k,j} \), \( x_{\alpha(i)}^{k,a(i,j)} \), and \( x_{\alpha(i)}^{k,b(i,j)} \) in \( \alpha_{i}^{k,j} \) are shown, as are the values of \( \epsilon_a \) and \( \epsilon_b \). We observe that the coefficients of \( x_{i}^{k,j} \) and \( x_{i+1}^{k,j} \) coming from the boundaries of the “horizontal” 2-cells \( \partial(x_i^j(R_i^j - L_i^j) + x_{i+1}^j(R_{i+1}^j - L_{i+1}^j)) \) are 1 and -1 respectively. We then check that the coefficients of \( x_{\alpha(i)}^{k,a(i,j)} \) and \( x_{\alpha(i)}^{k,b(i,j)} \) coming from the boundaries of the two “vertical” 2-cells are \( -\epsilon_a \) and \( -\epsilon_b \), respectively.

There are either 0 or 1 additional 2-cells incident to \( r_i^j - l_i^j \), depending on the value of \( k \). This contributes the constant term \( C = C_a + C_b \alpha_{i}^{k,j} \), where \( C = 0 \) or \( \pm 1 \). If \( k = 0 \), then \( C = \pm 1 \) if and only if \( B_i \) is incident to \( r_i^j - l_i^j \). If \( k \neq 0 \), then \( C = \pm 1 \) if and only if \( a(i,j) = k \) or \( b(i,j) = k \).

The sign depends on the local writhe number of the crossing. In each of the configurations in Figures 9 and 10, we verify that \( C = C_a(i,j,k) + C_b(i,j,k) \) is the coefficient of \( r_i^j - l_i^j \) in the relevant additional 2-cell, completing the proof.

\[ \square \]

### 5. Computing Linking Numbers

The linking number of \( K^j \) and \( K^k \) is the algebraic intersection number of \( K^j \) with a rational 2-chain bounding \( K^k \). Theorem 6 determines when such a 2-chain exists and, when it does, describes one such 2-chain \( \Sigma^k \) explicitly. Now in Theorem 6 we give a formula for the intersection number of \( K^j \) with \( \Sigma^k \). By iterating over all \( j, k \in \{0, \ldots, q\} \), Theorem 6 allows us to compute the dihedral linking invariant of \((K, \rho)\). In the statement of the theorem, we allow \( j = k \), in which case the result can be interpreted as a self-linking number of \( K^j \) with respect to a lift of the blackboard framing of the diagram \( D_K \) of \( K \) that determined the cell structure on \( S^3 \).

**Theorem 6.** Let \( q = \frac{p-1}{2} \) with \( p \) odd. Let \( \rho : \pi_1(S^3 \to K) \) be a \( p \)-coloring of \( K \), with corresponding irregular \( p \)-fold dihedral cover \( f : M_\rho \to S^3 \). Suppose that \( K^j \) and \( K^k \), \( j, k \in \{0, \ldots, q\} \), are rationally null-homologous connected components of the singular set \( f^{-1}(K) \). Let \( \Sigma^k \) be the rational 2-chain given in Theorem 5. Then the linking number of \( K^j \) with \( K^k \) is

\[
\text{lk}(K^j, K^k) = \sum_{i=0}^{n-1} \text{Int}(i, j, k)
\]

where \( \text{Int}(i, j, k) = \epsilon_a(i, j) x_{\alpha(i)}^{k,a(i,j)} - C_a(i, j, k) \)

Note that \( \epsilon_a(i, j) = 0 \) when \( a(i, j) = 0 \), so we treat the first term as 0 in this case even though \( x_{i}^{k,0} \) is not defined.
Figure 9. Configurations of cells incident to $r^j_i - l^j_i$ at a positive crossing.
Figure 10. Configurations of cells incident to $r^j_l - l^j_l$ at a negative crossing.
Proof. Let \((K^j)'\) be a push-off of \(K^j\), obtained by pushing \(K^j\) into the 3-cells \(E^k\) such that \((K^j)'\) intersects the 2-skeleton transversely. Let \((k^i)\)' denote the corresponding push-off of \(k^i\). Let \(\text{Int}(i,j,k)\) denote the signed intersection number of \((k^i)'\cup(k^j)'\) with \(\Sigma^k\).

We now show \(\text{Int}(i,j,k) = \epsilon_a(i,j)x_{o(i)}^{k,a(i,j)} - C_a(i,j,k)\).

Depending on the values of \(\epsilon(i)\) and \(\epsilon_a(i,j)\), \((K^j)'\) meets one of the 2-cells \(R_{o(i)}^{a(i,j)}\), \(L_{o(i)}^{a(i,j)}\), or \(B_{o(i)}\) transversely, as shown in Figure 11 when \(j \neq 0\), and in Figure 12 when \(j \neq 0\).

Recall that
\[
\Sigma^0 = \sum_{i=0}^{n-1} B_i + \sum_{j=1}^{q} \sum_{i=0}^{n-1} x_i^{0,j}(R_i^j - L_i^j),
\]
and if \(k \neq 0\),
\[
\Sigma^k = \sum_{i=0}^{n-1} x_i^{k,k}R_i^k + (1 - x_i^{k,k})L_i^k + \sum_{j \in \{0, \ldots, q\} - \{k\}} \sum_{i=0}^{n-1} x_i^{k,j}(R_i^j - L_i^j),
\]
where the \(x_i^{k,j}\) are a solution to the system of equations in Theorem 5.

**Case 1:** \(k = 0\).

At crossing \(i\), \((K^j)'\) meets \(R_{o(i)}^{a(i,j)}\), \(L_{o(i)}^{a(i,j)}\), or \(B_{o(i)}\), as shown in Figure 11 if \(j \neq 0\) or Figure 12 if \(j = 0\).

**Case 1a:** Suppose \((K^j)'\) meets \(R_{o(i)}^{a(i,j)}\). If \(\epsilon(i) = 1\), the intersection number of \((K^j)'\) with \(R_{o(i)}^{a(i,j)}\) is positive, and the coefficient of \(R_{o(i)}^{a(i,j)}\) in \(\Sigma^k\) is \(x_{o(i)}^{k,a(i,j)}\). Then \(\text{Int}(i,j,k) = x_{o(i)}^{k,a(i,j)}\). In this case, \(\epsilon_a(i,j) = 1\) and \(C_a(i,j,k) = 0\). Therefore \(\text{Int}(i,j,k) = \epsilon_a(i,j)x_{o(i)}^{k,a(i,j)} - C_a(i,j,k)\) as desired. If instead \(\epsilon(i) = -1\), the intersection number of \((K^j)'\) with \(R_{o(i)}^{a(i,j)}\) is negative, but the coefficient of \(R_{o(i)}^{a(i,j)}\) in \(\Sigma^k\) is still \(x_{o(i)}^{k,a(i,j)}\). Therefore \(\text{Int}(i,j,k) = -x_{o(i)}^{k,a(i,j)}\). In this case, \(\epsilon(i,j) = -1\) and \(C_a(i,j,k) = 0\). Therefore \(\text{Int}(i,j,k) = \epsilon_a(i,j)x_{o(i)}^{k,a(i,j)} - C_a(i,j,k)\) as well.

**Case 1b:** Suppose instead that \((K^j)'\) meets \(L_{o(i)}^{a(i,j)}\). If \(\epsilon(i) = 1\), the intersection number of \((K^j)'\) with \(L_{o(i)}^{a(i,j)}\) is positive, and the coefficient of \(L_{o(i)}^{a(i,j)}\) in \(\Sigma^k\) is \(-x_{o(i)}^{k,a(i,j)}\). Then \(\text{Int}(i,j,k) = -x_{o(i)}^{k,a(i,j)}\). In this case, \(\epsilon_a(i,j) = -1\) and \(C_a(i,j,k) = 0\). Therefore \(\text{Int}(i,j,k) = \epsilon_a(i,j)x_{o(i)}^{k,a(i,j)} - C_a(i,j,k)\) as desired. If instead \(\epsilon(i) = -1\), the intersection number of \((K^j)'\) with \(L_{o(i)}^{a(i,j)}\) is negative, but the coefficient of \(L_{o(i)}^{a(i,j)}\) in \(\Sigma^k\) is still \(-x_{o(i)}^{k,a(i,j)}\). Therefore \(\text{Int}(i,j,k) = x_{o(i)}^{k,a(i,j)}\). In this case, \(\epsilon(i,j) = 1\) and \(C_a(i,j,k) = 0\). Therefore \(\text{Int}(i,j,k) = \epsilon_a(i,j)x_{o(i)}^{k,a(i,j)} - C_a(i,j,k)\) as well.

**Case 1c:** Finally, suppose that \((K^j)'\) meets \(B_{o(i)}\). Since \(k = 0\), the coefficient of \(B_{o(i)}\) in \(\Sigma^k\) is 1. The intersection number of \((K^j)'\) with \(B_{o(i)}\) is simply \(\text{Int}(i,j,k) = \epsilon(i)\). In this case, \(\epsilon_a(i,j) = 0\) and \(C_a(i,j,k) = -\epsilon(i)\). Therefore \(\text{Int}(i,j,k) = \epsilon_a(i,j)x_{o(i)}^{k,a(i,j)} - C_a(i,j,k)\).

**Case 2:** \(k \neq 0\).
At crossing $i$, $(K^j)'$ meets $R_{o(i)}^{a(i,j)}$, $L_{o(i)}^{a(i,j)}$, or $B_{o(i)}$, as shown in Figure 11 if $j \neq 0$ or Figure 12 if $j = 0$.

**Case 2a:** Suppose $(K^j)'$ meets $R_{o(i)}^{a(i,j)}$. This is identical to Case 1a.

**Case 2b:** Suppose $(K^j)'$ meets $L_{o(i)}^{a(i,j)}$ and $a(i,j) \neq k$. This is identical to Case 1b.

**Case 2c:** Suppose $(K^j)'$ meets $L_{o(i)}^{a(i,j)}$ and $a(i,j) = k$. If $\epsilon(i) = 1$, the intersection number of $(K^j)'$ with $L_{o(i)}^{a(i,j)}$ is positive, and the coefficient of $L_{o(i)}^{a(i,j)}$ in $\Sigma^k$ is $1 - x_{o(i)}^{k,a(i,j)}$. Then $\text{Int}(i,j,k) = 1 - x_{o(i)}^{k,a(i,j)}$. In this case, $\epsilon_a(i,j) = -1$ and $C_a(i,j,k) = -1$. Therefore $\text{Int}(i,j,k) = \epsilon_a(i,j)x_{o(i)}^{k,a(i,j)} - C_a(i,j,k)$ as desired. If instead $\epsilon(i) = -1$, the intersection number of $(K^j)'$ with $L_{o(i)}^{a(i,j)}$ is negative, but the coefficient of $L_{o(i)}^{a(i,j)}$ in $\Sigma^k$ is still $1 - x_{o(i)}^{k,a(i,j)}$. Therefore $\text{Int}(i,j,k) = x_{o(i)}^{k,a(i,j)} - 1$. In this case, $\epsilon(i,j) = 1$ and $C_a(i,j,k) = 1$. Therefore $\text{Int}(i,j,k) = \epsilon_a(i,j)x_{o(i)}^{k,a(i,j)} - C_a(i,j,k)$.

**Case 2d:** Suppose that $(K^j)'$ meets $B_{o(i)}$. Since $k \neq 0$, the coefficient of $B_{o(i)}$ in $\Sigma^k$ is 0. Therefore $\text{Int}(i,j,k) = 0$. In this case, $\epsilon_a(i,j) = 0$ and $C_a(i,j,k) = 0$. Therefore $\text{Int}(i,j,k) = \epsilon_a(i,j)x_{o(i)}^{k,a(i,j)} - C_a(i,j,k)$.
\[ \varepsilon_{i,j} = \begin{cases} 1 & \text{if } \text{Int}(i,j,k) = x_0 \text{ or } C_f(i,j,k) \\ -1 & \text{if } \text{Int}(i,j,k) = -x_0 \text{ or } C_f(i,j,k) \\ 0 & \text{otherwise} \end{cases} \]

**Figure 11.**
\[ a(i,0) = 1 \]
\[ \text{Int}(i,0,k) = x_{o(i)}^{k,a(i,0)} + C_a(i,0,k) \]

\[ a(i,0) = -1 \]
\[ \text{Int}(i,0,k) = -x_{o(i)}^{k,a(i,0)} + C_a(i,0,k) \]

\[ a(i,0) = 0 \]
\[ \text{Int}(i,0,k) = C_a(i,0,k) \]

**Figure 12.**
6. Additional Examples

In this section, we carry out our algorithm in full detail on the trefoil. We then include a few additional examples to illustrate the use of the code at [3].

6.1. The trefoil. Consider the trefoil knot with 3-coloring and configuration diagram shown in Figure 13.

From the configuration diagram, we compute the values of \( a(i, j) \) and \( b(i, j) \), shown in the tables below.

| \( a(i, j) \) | \( j = 0 \) | \( j = 1 \) |
|--------------|-----------|-----------|
| \( i = 0 \)  | 1         | 1         |
| \( i = 1 \)  | 1         | 0         |
| \( i = 2 \)  | 1         | 1         |
| \( i = 3 \)  | 1         | 0         |

| \( b(i, j) \) | \( j = 0 \) | \( j = 1 \) |
|--------------|-----------|-----------|
| \( i = 0 \)  | 1         | 0         |
| \( i = 1 \)  | 1         | 1         |
| \( i = 2 \)  | 1         | 0         |
| \( i = 3 \)  | 0         | 1         |

We then compute the values of \( \epsilon_a(i, j) \) and \( \epsilon_b(i, j) \), shown in the tables below.

| \( \epsilon_a(i, j) \) | \( j = 0 \) | \( j = 1 \) |
|------------------------|-----------|-----------|
| \( i = 0 \)            | 1         | -1        |
| \( i = 1 \)            | 1         | 0         |
| \( i = 2 \)            | -1        | 1         |
| \( i = 3 \)            | 0         | 1         |

| \( \epsilon_b(i, j) \) | \( j = 0 \) | \( j = 1 \) |
|------------------------|-----------|-----------|
| \( i = 0 \)            | -1        | 0         |
| \( i = 1 \)            | -1        | 1         |
| \( i = 2 \)            | 1         | 0         |
| \( i = 3 \)            | 0         | 1         |

Last, we compute the values of \( C_a(i, j, k) \) and \( C_b(i, j, k) \).

| \( C_a(i, j, 0) \) | \( j = 0 \) | \( j = 1 \) |
|---------------------|-----------|-----------|
| \( i = 0 \)        | 0         | 0         |
| \( i = 1 \)        | 0         | -1        |
| \( i = 2 \)        | 0         | 0         |
| \( i = 3 \)        | -1        | 0         |

| \( C_b(i, j, 0) \) | \( j = 0 \) | \( j = 1 \) |
|---------------------|-----------|-----------|
| \( i = 0 \)        | 0         | 1         |
| \( i = 1 \)        | 0         | 0         |
| \( i = 2 \)        | 0         | 1         |
| \( i = 3 \)        | 1         | 0         |
Now we determine the $\alpha_{i,j}^k$ using Theorem 5. Each table gives the augmented matrix corresponding to the system of equations $\alpha_{i,j}^k = 0$ for $i \in \{0, 1, 2, 3\}$, $j \in \{0, 1\}$, and fixed $k \in \{0, 1\}$.

One choice of solution for each system is:

Finally, we determine the linking numbers of the $K^j$ using Theorem 6.

Therefore, the dihedral linking invariant of the trefoil is $\{\text{lk}(K^0, K^1)\} = \{2\}$.

6.2. The knot $8_{16}$. The knot $8_{16}$ has determinant 35, so is both 5- and 7-colorable. We now illustrate how to use the code available at [3] to determine the 5- and 7-dihedral linking invariants.

The list of over-arc subscripts $[o(0), \ldots, o(7)]$ is

overstrands=$[6, 4, 0, 7, 2, 3, 1, 5]$.

The list of local writhe numbers $[\epsilon(0), \ldots, \epsilon(7)]$ is

signs=$[1, 1, 1, -1, 1, -1, 1, -1]$.

The unique 5-coloring $\rho_5 [c(0), \ldots, c(7)]$ of $8_{16}$ up to equivalence is

coloring5=$[2, 3, 2, 2, 0, 4, 0, 1]$.

The function DLNmatrix(5, overstrands, signs, coloring5) returns the 3x3 matrix below, whose $jk$-entry is the linking number of $K^j$ and $K^k$. Note that the diagonal entries are self-linking numbers and are not part of the dihedral linking invariant.
\[
\begin{pmatrix}
-22 & 18 & -6 \\
18 & -14 & 6 \\
-6 & 6 & -2 \\
\end{pmatrix}
\]

Therefore \( DLN(8_{16}, \rho_5) = \{18, 6, -6\} \).

The unique 8-coloring \( \rho_7 \) of \( 8_{16} \) up to equivalence is

\[
\text{coloring}_7 = [3, 4, 5, 1, 1, 2, 0, 1].
\]

The function \( DLN\text{matrix}(7, \text{overstrands}, \text{signs}, \text{coloring}_7) \) returns the 4x4 matrix below, whose \( jk \)-entry is the linking number of \( K^j \) and \( K^k \). Again note that the diagonal entries are self-linking numbers and are not part of the dihedral linking invariant.

\[
\begin{pmatrix}
22 & -6 & -22 & 18 \\
-6 & 0 & 6 & -2 \\
-22 & 6 & 20 & -14 \\
18 & -2 & -14 & 8 \\
\end{pmatrix}
\]

Therefore \( DLN(8_{16}, \rho_7) = \{-6, -22, 18, 6, -2, -14\} \).
In this section we tabulate the dihedral linking invariant for 3, 5, and 7-colorable knots with up to 12 crossings. Due to space considerations, we list the linking numbers over all \( p \)-colorings of a given knot for fixed \( p \), \textit{without duplicates}. An extension of this tabulation, which includes the tabulation for additional 12-crossing knots, and also gives the complete set of dihedral linking numbers for each \( p \)-coloring of \( K \) as a multiset, is available at [3]. The tabulation of the linking invariant for 3-colorable knots up to 11 crossings is due to Perko [15], and the data produced here agrees with that of [15], except possibly for the sign, which corresponds to exchanging \( K \) with its mirror.

To carry out the tabulation, we extract a list of knots, their braid words, and determinants from Knot Info [13]. We produce a list of all \( p \)-colorable knots by taking those knots whose determinants are divisible by \( p \). From the braid word, we determine the overstrand and sign lists. We solve a system of linear equations mod \( p \) to determine all \( p \)-colorings of each knot, and then use the function DLNmatrix in dihedrallinking.py, available at [3], to produce a matrix of linking numbers between the branch curves.
7.1. Tabulation of the Dihedral Linking Invariant for 3-Colorable Knots.
For knots of up to 11 crossings, see also [15].

|    |     |     |     |
|----|-----|-----|-----|
| 3 - 1 | -2  | 10 - 42 | 2   |
| 6 - 1 | 2   | 10 - 59 | -2  |
| 7 - 4 | 2   | 10 - 61 | -4  |
| 7 - 7 | 6   | 10 - 62 | 0   |
| 8 - 5 | -4  | 10 - 63 | -4  |
| 8 - 10 | 0   | 10 - 64 | -4  |
| 8 - 11 | 6   | 10 - 65 | 0   |
| 8 - 15 | -4  | 10 - 66 | -4  |
| 9 - 1  | -6  | 10 - 67 | 14/5|
| 9 - 4  | -2  | 10 - 68 | -6/5|
| 9 - 9  | 6   | 10 - 72 | 0   |
| 9 - 15 | 6   | 10 - 76 | 0   |
| 9 - 17 | 6   | 10 - 77 | 0   |
| 9 - 23 | -4  | 10 - 80 | 0   |
| 9 - 29 | 10  | 10 - 81 | 0   |
| 9 - 34 | -4  | 10 - 82 | 0   |
| 9 - 37 | -4  | 10 - 83 | 0   |
| 9 - 38 | -4  | 10 - 84 | 0   |
| 9 - 41 | -4  | 10 - 85 | 0   |
| 9 - 46 | -4  | 10 - 86 | 0   |
| 9 - 47 | -4  | 10 - 87 | 0   |
| 9 - 48 | -4  | 10 - 88 | 0   |
| 10 - 4 | 2   | 10 - 89 | 2   |
| 10 - 5 | 2   | 10 - 90 | 2   |
| 10 - 9 | 2   | 10 - 91 | 2   |
| 10 - 10 | -2 | 10 - 92 | 2   |
| 10 - 14 | 2   | 10 - 93 | 2   |
| 10 - 19 | -6 | 10 - 94 | 2   |
| 10 - 21 | -6 | 10 - 95 | 2   |
| 10 - 29 | -6 | 10 - 96 | 2   |
| 10 - 31 | -6 | 10 - 97 | 2   |
| 10 - 32 | -6 | 10 - 98 | 2   |
| 10 - 36 | -6 | 10 - 99 | 2   |
| 10 - 40 | -6 | 10 - 100 | 2   |
| 11a - 137 | 8 | 11a - 263 | -6, -4 | 11n - 14 | 10 |
| 11a - 139 | 0 | 11a - 264 | 0 | 11n - 18 | 10 |
| 11a - 146 | 10 | 11a - 271 | 6 | 11n - 28 | 0 |
| 11a - 152 | 0 | 11a - 273 | 4 | 11n - 29 | -4 |
| 11a - 155 | ∞, 14/3, 10/3 | 11a - 274 | 6 | 11n - 30 | 0 |
| 11a - 157 | 0 | 11a - 275 | -2 | 11n - 31 | 0 |
| 11a - 159 | -6 | 11a - 277 | ∞, 26/9, 2/3 | 11n - 32 | -4 |
| 11a - 161 | 4 | 11a - 278 | 30/11 | 11n - 33 | 0 |
| 11a - 152 | 0 | 11a - 280 | -8 | 11n - 38 | -4 |
| 11a - 171 | -2 | 11a - 283 | 2 | 11n - 62 | -4 |
| 11a - 173 | -14/3, ∞, 34/3 | 11a - 286 | -2 | 11n - 63 | 0 |
| 11a - 175 | -2 | 11a - 290 | 2 | 11n - 64 | -4 |
| 11a - 176 | -2 | 11a - 291 | ∞, -22/3, -26/3 | 11n - 65 | 2/7 |
| 11a - 178 | 2 | 11a - 292 | -30 | 11n - 66 | -26/7 |
| 11a - 179 | 14 | 11a - 293 | ∞, 14/3 | 11n - 67 | 2/7 |
| 11a - 181 | ∞, 26/3, 22/3 | 11a - 294 | 26 | 11n - 68 | -14 |
| 11a - 184 | 10 | 11a - 296 | 22 | 11n - 69 | -10 |
| 11a - 187 | -26/5 | 11a - 300 | -2 | 11n - 71 | 0, -2, -4 |
| 11a - 194 | 4 | 11a - 304 | -10 | 11n - 72 | -6, -4 |
| 11a - 196 | 2 | 11a - 305 | 0 | 11n - 73 | 0, -2, -4 |
| 11a - 199 | 22/7 | 11a - 306 | 2 | 11n - 74 | 0, -2, -4 |
| 11a - 201 | 18/5 | 11a - 309 | 2 | 11n - 75 | 0, 4, 2 |
| 11a - 202 | 34/7 | 11a - 314 | ∞, 26/9, -2/3 | 11n - 76 | 0, 4, 2 |
| 11a - 203 | 6 | 11a - 318 | -6 | 11n - 77 | -6, -4 |
| 11a - 208 | 6 | 11a - 324 | 12 | 11n - 78 | 0, -2, -4 |
| 11a - 209 | -8 | 11a - 332 | 0, 2, -4 | 11n - 79 | 0 |
| 11a - 212 | 6 | 11a - 335 | 2 | 11n - 80 | 4 |
| 11a - 214 | 22/5 | 11a - 338 | -4 | 11n - 81 | -6, -4 |
| 11a - 216 | -2 | 11a - 340 | -4 | 11n - 85 | 0 |
| 11a - 219 | -10/7 | 11a - 344 | 2 | 11n - 86 | -2 |
| 11a - 223 | -4 | 11a - 346 | -14/5 | 11n - 87 | 4 |
| 11a - 230 | -2 | 11a - 347 | 26 | 11n - 92 | 6 |
| 11a - 231 | 0, 4, 2 | 11a - 351 | -4 | 11n - 94 | 2 |
| 11a - 232 | 4 | 11a - 352 | -6, -2, -4 | 11n - 95 | -6/5 |
| 11a - 236 | -6 | 11a - 353 | -34/5 | 11n - 97 | -2/5 |
| 11a - 237 | 14/5 | 11a - 354 | 22 | 11n - 98 | 22/5 |
| 11a - 239 | 34/11 | 11a - 355 | -6 | 11n - 99 | -2/5 |
| 11a - 243 | 2 | 11a - 360 | -2 | 11n - 100 | 10/7 |
| 11a - 244 | -4 | 11a - 361 | 6/7 | 11n - 101 | 18/7 |
| 11a - 245 | -4 | 11a - 362 | 2/5 | 11n - 102 | 10/7 |
| 11a - 248 | -34/11 | 11a - 365 | 10 | 11n - 104 | -4 |
| 11a - 249 | -22/9, 2/3, -2/3 | 11a - 366 | ∞, 14/3 | 11n - 105 | -4 |
| 11a - 258 | -4 | 11n - 1 | 10 | 11n - 106 | 0 |
| 11a - 260 | 0 | 11n - 2 | -2 | 11n - 107 | -4 |
| 11a - 261 | 4 | 11n - 13 | -2 | 11n - 109 | 2 |
| 11n - 118 | -2 | 12a - 36 | -4 | 12a - 181 | 6 |
| 11n - 119 | 5 | 12a - 48 | 4 | 12a - 182 | 0 |
| 11n - 121 | 8 | 12a - 49 | -50 | 12a - 189 | 18 |
| 11n - 122 | -6/7 | 12a - 51 | -4 | 12a - 193 | 6 |
| 11n - 123 | 22 | 12a - 57 | -4 | 12a - 195 | 4 |
| 11n - 125 | 6 | 12a - 60 | 4 | 12a - 197 | 2 |
| 11n - 126 | ∞, -22/3 | 12a - 62 | 0 | 12a - 203 | 8 |
| 11n - 136 | -6 | 12a - 66 | -54 | 12a - 206 | 6 |
| 11n - 137 | 22/7 | 12a - 67 | 8 | 12a - 207 | 42/5 |
| 11n - 138 | 6/7 | 12a - 71 | 2 | 12a - 208 | 46/7 |
| 11n - 139 | 2/5 | 12a - 74 | 6 | 12a - 209 | 54/19 |
| 11n - 140 | 18/5 | 12a - 77 | 4 | 12a - 210 | -6 |
| 11n - 141 | 2/5 | 12a - 79 | -54/7 | 12a - 212 | -2 |
| 11n - 142 | 12 | 12a - 81 | -4 | 12a - 215 | 10 |
| 11n - 143 | -6/5 | 12a - 86 | -4 | 12a - 223 | -4 |
| 11n - 145 | 2 | 12a - 88 | 66/19 | 12a - 224 | 0 |
| 11n - 146 | 1 | 12a - 95 | -8 | 12a - 229 | 62/13 |
| 11n - 148 | -2/7 | 12a - 97 | 0 | 12a - 231 | -54/5 |
| 11n - 149 | 2/5 | 12a - 100 | -4 | 12a - 233 | 4 |
| 11n - 150 | -10/7 | 12a - 103 | 66/13 | 12a - 234 | -2 |
| 11n - 153 | -6/7 | 12a - 113 | -4 | 12a - 236 | -6 |
| 11n - 155 | -2/5 | 12a - 114 | -4 | 12a - 239 | -2 |
| 11n - 158 | 6/7 | 12a - 116 | 0 | 12a - 244 | ∞, 58/3, 10/3 |
| 11n - 161 | 2/5 | 12a - 117 | -4 | 12a - 245 | ∞, 46/3, 2/3 |
| 11n - 164 | -6, -2, -4 | 12a - 119 | -6, -4 | 12a - 246 | 4 |
| 11n - 167 | ∞, -10/3, -14/3 | 12a - 121 | -4 | 12a - 251 | 2 |
| 11n - 168 | -18/7 | 12a - 122 | 0 | 12a - 256 | -18 |
| 11n - 170 | -8 | 12a - 127 | 0 | 12a - 260 | -34/5 |
| 11n - 173 | 1 | 12a - 130 | 2 | 12a - 265 | 14/3, 62/21, 10/3 |
| 11n - 176 | -6/7 | 12a - 136 | 8 | 12a - 269 | 34/11 |
| 11n - 181 | -4 | 12a - 150 | -26/7 | 12a - 270 | ∞, 2/3, 22/9, -2/3 |
| 11n - 182 | 6/7 | 12a - 151 | 0 | 12a - 280 | -4 |
| 11n - 183 | -4 | 12a - 155 | -38/13 | 12a - 281 | -4 |
| 11n - 184 | -4 | 12a - 156 | 14/13 | 12a - 287 | 12 |
| 11n - 185 | -4 | 12a - 157 | 0 | 12a - 288 | 74/13 |
| 12a - 1 | 8 | 12a - 163 | -8 | 12a - 290 | 0 |
| 12a - 2 | 4 | 12a - 164 | 0, -2, -4 | 12a - 291 | 0 |
| 12a - 10 | 4 | 12a - 165 | -4 | 12a - 292 | 2 |
| 12a - 11 | 8 | 12a - 166 | 0, -2, -4 | 12a - 295 | -8, -6, -4 |
| 12a - 16 | -6 | 12a - 167 | 0, 4, 2 | 12a - 296 | -16 |
| 12a - 23 | 42/11 | 12a - 168 | 0 | 12a - 297 | 0, 6, 4, 2 |
| 12a - 29 | 4 | 12a - 172 | -2 | 12a - 298 | 0, 6, 4, 2 |
| 12a - 30 | 0 | 12a - 175 | 6 | 12a - 302 | 2 |
| 12a - 33 | 0 | 12a - 177 | ∞, 14/3, -38/9, 10/3 | 12a - 303 | 2 |
| 12a - 35 | 6 | 12a - 179 | 2 | 12a - 306 | 2 |
| 12a - 308 | -16 | 12a - 426 | 9/2 | 12a - 549 | -6 |
| 12a - 311 | ∞, -46/3, 2/3 | 12a - 427 | 0, -2, 2 | 12a - 554 | ∞, 14/3, 6 |
| 12a - 312 | 26/7 | 12a - 428 | -4 | 12a - 561 | -6 |
| 12a - 313 | -34/5 | 12a - 429 | -4 | 12a - 562 | 38/11 |
| 12a - 319 | -10 | 12a - 433 | -8, -6, -4 | 12a - 563 | ∞, 26/9, 2/3, -2/3 |
| 12a - 321 | -6 | 12a - 434 | 10 | 12a - 564 | 30/11 |
| 12a - 325 | 6/5 | 12a - 435 | 0, 4, -4 | 12a - 567 | -46/11 |
| 12a - 329 | 0 | 12a - 439 | 14/5 | 12a - 569 | 46/9, ∞, 22/3 |
| 12a - 331 | 54/11 | 12a - 444 | -4 | 12a - 570 | 58/11 |
| 12a - 332 | ∞, 50/9, 22/3 | 12a - 445 | 0 | 12a - 571 | 0 |
| 12a - 334 | 70/11 | 12a - 446 | -4 | 12a - 574 | -8, -6, -4 |
| 12a - 336 | -26/5 | 12a - 448 | 10 | 12a - 575 | -4 |
| 12a - 342 | -6 | 12a - 450 | 30 | 12a - 576 | 0, 6, 4, 2 |
| 12a - 345 | -10 | 12a - 451 | 6 | 12a - 577 | 0 |
| 12a - 346 | -46/11 | 12a - 456 | 2 | 12a - 578 | 0 |
| 12a - 347 | -58/17 | 12a - 459 | -22 | 12a - 579 | -2 |
| 12a - 348 | 10 | 12a - 463 | -4 | 12a - 580 | 2 |
| 12a - 358 | -50/19 | 12a - 464 | -2 | 12a - 588 | -2 |
| 12a - 360 | 2 | 12a - 472 | 9/2 | 12a - 594 | 0, -2, 2 |
| 12a - 362 | 34/5 | 12a - 475 | -4 | 12a - 596 | -2 |
| 12a - 373 | -4 | 12a - 482 | 6 | 12a - 597 | 6 |
| 12a - 374 | 0 | 12a - 485 | 2 | 12a - 611 | -14 |
| 12a - 375 | 0 | 12a - 486 | -2 | 12a - 615 | -6 |
| 12a - 376 | 0 | 12a - 490 | -2 | 12a - 617 | -2 |
| 12a - 377 | 0 | 12a - 491 | 46/5 | 12a - 623 | 42 |
| 12a - 386 | 26/3, 22/3 | 12a - 493 | ∞, -34/3, 10/3 | 12a - 626 | -2 |
| 12a - 389 | 8 | 12a - 494 | 54/7 | 12a - 628 | -50/11 |
| 12a - 390 | 0 | 12a - 496 | 2 | 12a - 629 | 2 |
| 12a - 392 | -20 | 12a - 498 | 6 | 12a - 631 | 2 |
| 12a - 393 | -6 | 12a - 503 | ∞, 34/3 | 12a - 633 | -6 |
| 12a - 395 | 6 | 12a - 504 | 34/5 | 12a - 634 | 0, -2 |
| 12a - 396 | 0, -6, -4 | 12a - 505 | 50/7 | 12a - 635 | -6 |
| 12a - 401 | -50/11 | 12a - 508 | -2 | 12a - 636 | -6/11 |
| 12a - 404 | 6 | 12a - 509 | -2 | 12a - 637 | 10 |
| 12a - 408 | -26/5 | 12a - 515 | -46/5 | 12a - 643 | -2 |
| 12a - 409 | -26/5 | 12a - 519 | 6 | 12a - 647 | -8, -4 |
| 12a - 411 | 34/5 | 12a - 523 | -2 | 12a - 648 | -4 |
| 12a - 413 | ∞, -58/21, -14/3 | 12a - 524 | 38/7 | 12a - 650 | -6 |
| 12a - 414 | 26/5 | 12a - 525 | 38/5 | 12a - 654 | -14 |
| 12a - 415 | 14/5 | 12a - 528 | 2 | 12a - 655 | 10 |
| 12a - 416 | 2 | 12a - 529 | -42/13 | 12a - 665 | -16 |
| 12a - 419 | 2 | 12a - 540 | 2 | 12a - 666 | 6 |
| 12a - 420 | -10 | 12a - 541 | 6 | 12a - 668 | 0 |
| 12a - 422 | 2 | 12a - 542 | 10/13 | 12a - 670 | -4 |
| 12a - 425 | -2 | 12a - 546 | -8 | 12a - 672 | 0 |
| 12a - 679   | -2, -4       | 12a - 793   | 46/17       | 12a - 913   | -2       |
|-------------|-------------|-------------|-------------|-------------|---------|
| 12a - 683   | 0, 2, -4    | 12a - 796   | -2          | 12a - 914   | 6       |
| 12a - 684   | 2           | 12a - 798   | -4          | 12a - 916   | 2       |
| 12a - 685   | -14         | 12a - 800   | -4          | 12a - 919   | -2      |
| 12a - 689   | 0           | 12a - 801   | -6, -4      | 12a - 923   | -30/7   |
| 12a - 692   | 0, 4, 2     | 12a - 803   | 2           | 12a - 924   | -42/5   |
| 12a - 693   | 4           | 12a - 804   | -54/17      | 12a - 927   | 8       |
| 12a - 694   | -4          | 12a - 805   | -2          | 12a - 928   | 22      |
| 12a - 699   | -2          | 12a - 806   | -2          | 12a - 930   | -4      |
| 12a - 701   | 4, 2        | 12a - 807   | -2          | 12a - 931   | -4      |
| 12a - 702   | 4           | 12a - 808   | -2          | 12a - 936   | -6      |
| 12a - 703   | 4           | 12a - 809   | -46/13      | 12a - 939   | 34      |
| 12a - 705   | 54/19       | 12a - 810   | -58/15, 14/3, 10/3 | 12a - 940 | -38 |
| 12a - 706   | 46          | 12a - 812   | -46         | 12a - 941   | 0       |
| 12a - 708   | -34/7       | 12a - 814   | 0           | 12a - 944   | -10     |
| 12a - 711   | 6/5         | 12a - 818   | -42         | 12a - 949   | 0       |
| 12a - 712   | 10, 6, 4    | 12a - 823   | 0           | 12a - 952   | -14     |
| 12a - 718   | 2           | 12a - 827   | -4          | 12a - 964   | -10     |
| 12a - 719   | -26/5       | 12a - 828   | 0           | 12a - 965   | -2      |
| 12a - 721   | -6          | 12a - 841   | 22/5        | 12a - 970   | 4       |
| 12a - 723   | 14          | 12a - 845   | -10/11      | 12a - 972   | -14/5   |
| 12a - 725   | ∞, 26/3, 22/3 | 12a - 847   | 4           | 12a - 973   | ∞, -44/9, -22/3 |
| 12a - 730   | 34/7        | 12a - 852   | 0           | 12a - 974   | -36/7   |
| 12a - 731   | -6          | 12a - 853   | 22/13       | 12a - 975   | -4      |
| 12a - 735   | 22/5        | 12a - 855   | 2           | 12a - 976   | -22     |
| 12a - 736   | -2          | 12a - 858   | -6/5        | 12a - 979   | 0       |
| 12a - 737   | 4           | 12a - 861   | 4           | 12a - 981   | 0       |
| 12a - 739   | -4          | 12a - 862   | -26/11      | 12a - 982   | 0       |
| 12a - 741   | -62/7       | 12a - 864   | 4           | 12a - 986   | 2       |
| 12a - 742   | 0, 4, -2    | 12a - 873   | -10/3, ∞, 38/3 | 12a - 987 | 4, 2 |
| 12a - 750   | ∞, 2, 10/3  | 12a - 876   | -10         | 12a - 990   | 0, -2, 2 |
| 12a - 760   | -10         | 12a - 878   | 2           | 12a - 999   | 0       |
| 12a - 762   | -2          | 12a - 882   | 2           | 12a - 1000  | 0       |
| 12a - 766   | -2          | 12a - 886   | ∞, 50/3, -2/3 | 12a - 1001 | 2 |
| 12a - 767   | 18/7        | 12a - 888   | -10         | 12a - 1002  | 2       |
| 12a - 768   | -10         | 12a - 893   | -6          | 12a - 1010  | 14      |
| 12a - 769   | -14/3, -10/3, 34/3 | 12a - 894 | -10         | 12a - 1016  | -4      |
| 12a - 771   | 50          | 12a - 895   | ∞, -10/3    | 12a - 1018  | 0       |
| 12a - 772   | -4          | 12a - 902   | -30         | 12a - 1021  | -26/5   |
| 12a - 775   | 2           | 12a - 903   | -4          | 12a - 1022  | -26/3, 22/3 |
| 12a - 777   | -6          | 12a - 904   | 2           | 12a - 1025  | 50/13   |
| 12a - 780   | -6          | 12a - 905   | 6, 4, 2     | 12a - 1026  | 38/7    |
| 12a - 785   | -4          | 12a - 907   | -34/5       | 12a - 1029  | -2      |
| 12a - 787   | -14/9, ∞, -14/3 | 12a - 909   | -2          | 12a - 1036  | -14     |
| 12a - 791   | 2           | 12a - 912   | 2           | 12a - 1043  | -6/5    |
7.2. Tabulation of the Dihedral Linking Invariant for 5-Colorable Knots.

| 4 - 1 | 0, -2, 2 | 10 - 136 | 0, 4, -4 |
|-------|----------|----------|----------|
| 5 - 1 | -2       | 10 - 137 | 0        |
| 7 - 4 | -6, -2, 2| 10 - 138 | 0, 4, -4 |
| 8 - 8 | -6, -2, 2| 10 - 142 | -14, -6, 2 |
| 8 - 9 | 0, -2, 2 | 10 - 155 | -22/5, 18/5, -2/5 |
| 8 - 16| -6, 6, 18| 10 - 156 | -18, 10, -4 |
| 8 - 18| 0, 4, -4 | 10 - 158 | 30/11, -50/11, -10/11 |
| 8 - 21| -10, 2, -4| 10 - 161 | -16/11, -14/11, -18/11 |
| 9 - 2 | 0, -2, 2 | 10 - 162 | 54/11, -2/11, 26/11 |
| 9 - 12| -10, 2, -4| 10 - 164 | 6/11, 10/11, 14/11 |
| 9 - 23| 10, -6, 2 | 11a - 3 | 0, 4, 2 |
| 9 - 24| 0, -2, 2 | 11a - 5 | 0 |
| 9 - 31| -14, -6, 2 | 11a - 6 | 0, 4, -4 |
| 9 - 37| 0, -2, 2 | 11a - 7 | 0, 4, 2 |
| 9 - 39| 6, 4, 2 | 11a - 9 | 0, -2, -4 |
| 9 - 40| 0, 6, 2, -2, -6 | 11a - 16 | -10, 6, -2 |
| 9 - 49| 2/5, -8/5, -18/5 | 11a - 19 | -6 |
| 10 - 3 | 6, -2, 2 | 11a - 21 | -6, -2, 2 |
| 10 - 10 | -10, 2, -4 | 11a - 25 | -6 |
| 10 - 12 | -6, -2, 2 | 11a - 29 | 0, 4, -4 |
| 10 - 20 | -6, -2, 2 | 11a - 31 | -6, -2, 2 |
| 10 - 21 | -2 | 11a - 33 | -10, 6, -2 |
| 10 - 24 | -2 | 11a - 41 | -6, -2, -4 |
| 10 - 25 | -6, -2, 2 | 11a - 43 | -6, -10, -2 |
| 10 - 33 | -6, 0, 6 | 11a - 49 | 0, -2, -4 |
| 10 - 40 | -6, -18, 6 | 11a - 51 | 0, 4, -4 |
| 10 - 56 | 0, -2, 2 | 11a - 60 | 0, -2, -4 |
| 10 - 58 | 0, 4, -4 | 11a - 62 | 0, 4, 2 |
| 10 - 59 | 0 | 11a - 67 | 0 |
| 10 - 60 | 0, 4, -4 | 11a - 76 | -10, 6, -2 |
| 10 - 62 | 2 | 11a - 82 | -10, 2, -4 |
| 10 - 66 | -6, -10, -2 | 11a - 99 | 66/31, -242/31, -88/31 |
| 10 - 81 | 0, -2, 2 | 11a - 104 | -10/3, 10, 10/3 |
| 10 - 86 | 10, -6, 2 | 11a - 112 | 50/11, 10/11, -30/11 |
| 10 - 100 | 50/11, 20/11, -10/11 | 11a - 124 | -14, 10, -38 |
| 10 - 101 | -16/11, 30/11, -62/11 | 11a - 125 | -6, -10, -2 |
| 10 - 103 | -98/5, 22/5, 2/5, 42/5, 12/5, -18/5, -28/5 | 11a - 126 | -14, 10, 34 |
| 10 - 106 | -6, -18, 6 | 11a - 132 | 0, -2, 2 |
| 10 - 109 | 0, 4, -4 | 11a - 140 | -6, -2, -4 |
| 10 - 116 | 28/11, 30/11, 26/11 | 11a - 148 | 70/31, -174/31, -52/31 |
| 10 - 120 | -16/11, -14/11, -18/11 | 11a - 150 | 6, -2, 2 |
| 10 - 121 | 22/19, 106/19, -62/19 | 11a - 157 | 0, -2, 2 |
| 10 - 122 | 0, 4, -4 | 11a - 160 | 130, -104, -338 |
| 10 - 129 | 0, -2, 2 | 11a - 168 | -40, 50, -130 |
| 10 - 132 | -14, 6, -4 | 11a - 170 | -362/19, 158/19, -102/19 |
| Code  | Values                   |
|-------|--------------------------|
| 11n - 171 | -74/19, 6/19, -34/19     |
| 11n - 173 | -34/11, -6/11, -20/11    |
| 11n - 175 | 86/19, -6/19, -98/19     |
| 11n - 178 | -34/3, -4, 10/3          |
| 11n - 180 | 18/19, -26/19, -70/19    |
| 11n - 181 | -34/19, 2/19, -70/19     |
| 11n - 185 | -56/29, -150/29, 38/29   |
| 12a - 11  | 0, 4, 2                  |
| 12a - 25  | 0, 4, -4                 |
| 12a - 26  | 0, 4, 2                  |
| 12a - 28  | 0                        |
| 12a - 34  | 0, -2, -4                |
| 12a - 41  | -6, 0, 6                 |
| 12a - 53  | -6, -2, -4               |
| 12a - 55  | -6, -2, -4               |
| 12a - 58  | 0, 10, -10               |
| 12a - 68  | 0, 4, -4                 |
| 12a - 74  | 62, -18, 22              |
| 12a - 75  | 0, -2, -4                |
| 12a - 77  | -6, -2, 2                |
| 12a - 83  | -2                       |
| 12a - 85  | 0, -6, 6                 |
| 12a - 87  | -6, -2, 2                |
| 12a - 95  | -2                       |
| 12a - 96  | -6, -2, -4               |
| 12a - 100 | 318/55, -2/5, 68/55, -182/55, 18/5, -22/5 |
| 12a - 106 | 0, -6, 6                 |
| 12a - 108 | -10, 4, 18               |
| 12a - 120 | -10, 4, 18               |
| 12a - 129 | 0, -2, -4                |
| 12a - 135 | 0                        |
| 12a - 137 | 0, 4, -4                 |
| 12a - 142 | -6, -2, 2                |
| 12a - 143 | -6, -2, -4               |
| 12a - 146 | -6, 0, 6                 |
| 12a - 165 | -6, -2, -4               |
| 12a - 179 | 0, -2, -4                |
| 12a - 181 | 0, 4, -4                 |
| 12a - 189 | 0                        |
| 12a - 193 | 0, 4, 2                  |
| 12a - 199 | 18, -22, 58              |
| 12a - 200 | 0, 4, 2                  |
| 12a - 202 | 0, 4, -4                 |
| 12a - 206 | -6, 0, 6                 |
| 12a - 208 | -6, 0, 6                 |
| 12a - 215 | 0, 4, -4 | 12a - 456 | 6, -2, 2 |
| 12a - 216 | 0 | 12a - 464 | -6, -2, 2 |
| 12a - 218 | 0 | 12a - 471 | -6, 0, 6 |
| 12a - 227 | -2 | 12a - 475 | -14, 10, -38 |
| 12a - 230 | 54, -38, -130 | 12a - 478 | 228/11, 722/11, -266/11 |
| 12a - 232 | -22/3, 14, 10/3 | 12a - 483 | 450/71, 34/71, -382/71 |
| 12a - 235 | 10, 4, -2 | 12a - 485 | 302/11, -762/11, -230/11 |
| 12a - 245 | -8, 10, -26 | 12a - 488 | 6, -2, 2 |
| 12a - 248 | 0, 4, -4 | 12a - 495 | -278/61, -182/61, -86/61 |
| 12a - 259 | -6, 0, 6 | 12a - 503 | -6, 0, 6 |
| 12a - 264 | 0, -2, 2 | 12a - 506 | -6, 0, 6 |
| 12a - 271 | 4/3, 2/3, 2 | 12a - 507 | 14, -6, 4 |
| 12a - 273 | 0, -6, 6 | 12a - 511 | 10, -10, -30 |
| 12a - 280 | -6, -2, -4 | 12a - 513 | -14, -14/3, 14/3 |
| 12a - 288 | 6, 4, 2 | 12a - 516 | -18, -22/3, 10/3 |
| 12a - 300 | -6, -2, -4 | 12a - 517 | -6, -2, -4 |
| 12a - 309 | 0, -2, 2 | 12a - 532 | 10, 4, -2 |
| 12a - 310 | 0, -2, 2 | 12a - 535 | -14, 6, -4 |
| 12a - 322 | 18/31, -190/31, -86/31 | 12a - 539 | -2 |
| 12a - 326 | -6, -2, 2 | 12a - 540 | -6, -2, 2 |
| 12a - 327 | ∞, -38/5, -6/5, 14/5, -42/5, 26/5, -14/5 | 12a - 553 | -204/59, -578/59, 170/59 |
| 12a - 330 | -6, 0, 6 | 12a - 556 | -2, -4/3, -2/3 |
| 12a - 339 | -2 | 12a - 561 | ∞, 6/5, -26/5, -58/5 |
| 12a - 345 | -10, 4, 18 | 12a - 564 | -6, -2, 2 |
| 12a - 348 | 0, 2, -2, -4, 4 | 12a - 566 | -10/3, 14, 16/3 |
| 12a - 349 | 2 | 12a - 570 | 2 |
| 12a - 358 | -12, 4, -4 | 12a - 572 | -70/31, 14/31, 98/31 |
| 12a - 359 | 0, -2, 2 | 12a - 587 | -10/3, -6, -14/3 |
| 12a - 360 | 0 | 12a - 588 | -14, 4, 22 |
| 12a - 366 | 2 | 12a - 589 | 6, 2/3, 10/3 |
| 12a - 376 | 6, -2, 2 | 12a - 603 | 6, 4, 2 |
| 12a - 377 | 0, 10, -10 | 12a - 605 | -6, 22, 8 |
| 12a - 381 | 0, 4, -4 | 12a - 612 | 10, -12, -34 |
| 12a - 387 | 0, -2, 2 | 12a - 613 | 38, 12, -14 |
| 12a - 388 | 0, -2, 2 | 12a - 614 | -10, 2, 4 |
| 12a - 389 | 294/41, -302/41, -4/41 | 12a - 616 | -342/59, -204/59, -66/59 |
| 12a - 397 | -66/31, -106/31, -86/31 | 12a - 619 | -10/3, 6, 4/3 |
| 12a - 398 | -6, -2, -4 | 12a - 621 | -6, 6, 18 |
| 12a - 408 | -166/33, -14/11, 82/33 | 12a - 628 | 0 |
| 12a - 427 | 0, 2, -2, -4, 4 | 12a - 631 | 0, -2, 2 |
| 12a - 428 | 6, -2, 2 | 12a - 634 | 6, -2, 2 |
| 12a - 431 | -14, -2, -26 | 12a - 640 | 6, 4, 2 |
| 12a - 434 | -10, 6, 22 | 12a - 642 | -6, 6, 18 |
| 12a - 435 | 0, 10, -10 | 12a - 650 | 6, -10, -2 |
| 12a - 443 | -6, -10, -2 | 12a - 652 | -2 |
| 12a - 448 | 0, 4, -4 | 12a - 653 | -334/59, -104/59, 126/59 |
| 12a - 654 | 0, -6, 6 |
|-----------|---------|
| 12a - 656 | 32/59, -106/59, 170/59 |
| 12a - 663 | -8, 10, -26 |
| 12a - 665 | -70/31, 14/31, 98/31 |
| 12a - 677 | -98/11, 84/11, 266/11 |
| 12a - 681 | -2 |
| 12a - 684 | -6, -2, 2 |
| 12a - 695 | -14, 6, -4 |
| 12a - 701 | -6, 6, 18 |
| 12a - 702 | 10, 4, -2 |
| 12a - 708 | 0, -2, 2 |
| 12a - 712 | 650/121, 84/121, -482/121 |
| 12a - 731 | 26, -10, 8 |
| 12a - 732 | -8, 6, -22 |
| 12a - 737 | 0, -2, 2 |
| 12a - 742 | -6, 22, 8 |
| 12a - 750 | -6, -2, 2 |
| 12a - 752 | 8/3, 10, -14/3 |
| 12a - 753 | -8/3, -6, 2/3 |
| 12a - 776 | 10, -6, -22 |
| 12a - 778 | 0, -2, 2 |
| 12a - 779 | -6, -2, 2 |
| 12a - 780 | -48/5, 2/5, 42/5, -138/5, -8/5, -18/5 |
| 12a - 785 | -330/41, 482/41, 76/41 |
| 12a - 792 | 2 |
| 12a - 793 | 0, 4, -4 |
| 12a - 799 | -14, 6, -4 |
| 12a - 801 | -14, -6, 2 |
| 12a - 811 | -4 |
| 12a - 813 | -4 |
| 12a - 815 | 0 |
| 12a - 817 | -4 |
| 12a - 818 | 10, -6, 2 |
| 12a - 828 | -8/19, -126/19, 110/19 |
| 12a - 830 | 0, -2, 2 |
| 12a - 831 | 0, -2, 2 |
| 12a - 836 | 0, -2, 2 |
| 12a - 837 | 0, -2, 2 |
| 12a - 843 | 6, -2, 2 |
| 12a - 858 | -6, -2, -4 |
| 12a - 866 | 10/3, 4/3, -2/3 |
| 12a - 876 | -4 |
| 12a - 878 | 0 |
| 12a - 883 | 28/11, -14/11, 70/11 |
| 12a - 890 | -8, 0, 8 |
|   |   |   |
|---|---|---|
| 12a - 906 | 0 |   |
| 12a - 907 | -34/5, 118/15, 28/15, -4/5, 6/5, -14/5, -62/15 |   |
| 12a - 908 | 92/109, 498/109, -314/109 |   |
| 12a - 909 | -10, 2, -4 |   |
| 12a - 914 | 6/11, 20/11, 34/11 |   |
| 12a - 920 | 10/11, -50/11, -20/11 |   |
| 12a - 921 | -2/5, 18/5, -22/5, 28/5, -42/5, 98/5, -12/5 |   |
| 12a - 923 | -2 |   |
| 12a - 925 | 6, 1, -4 |   |
| 12a - 927 | -46/11, 4/11, 54/11 |   |
| 12a - 934 | 18/19, -26/19, -70/19 |   |
| 12a - 940 | 10, -6, -22 |   |
| 12a - 941 | 0, -2, -4 |   |
| 12a - 944 | 0, -2, 2 |   |
| 12a - 951 | -12, -2, 8 |   |
| 12a - 953 | 0, -2, -4 |   |
| 12a - 959 | -58/11, 82/11, 12/11 |   |
| 12a - 964 | 28/11, 30/11, 26/11 |   |
| 12a - 970 | 0, 4, 2 |   |
| 12a - 975 | -2/5, 3/5, 38/5, 28/5, -22/5, -42/5 |   |
| 12a - 977 | -48/11, -54/11, -42/11 |   |
| 12a - 979 | -2, -2/3, 2/3 |   |
| 12a - 981 | 2 |   |
| 12a - 986 | 0 |   |
| 12a - 987 | 0, -6, 6 |   |
| 12a - 988 | 30/19, -50/19, 110/19 |   |
| 12a - 989 | 238/19, 70/19, -98/19 |   |
| 12a - 990 | 0 |   |
| 12a - 992 | -10/19, 14/19, -34/19 |   |
| 12a - 997 | -38/31, -58/31, -18/31 |   |
| 12a - 1006 | 18/19, 50/19, 82/19 |   |
| 12a - 1010 | -6, -2, 2 |   |
| 12a - 1025 | 0, 4, -4 |   |
| 12a - 1026 | 0, 4, -4 |   |
| 12a - 1035 | 90/29, -38/29, -166/29 |   |
| 12a - 1037 | 154/61, -94/61, -342/61 |   |
| 12a - 1040 | 6, -10, -2 |   |
| 12a - 1041 | -326/79, 150/79, 626/79 |   |
| 12a - 1044 | 38/71, 258/71, -182/71 |   |
| 12a - 1049 | 2 |   |
| 12a - 1050 | 0, -6, 6 |   |
| 12a - 1067 | -8, 4, -2 |   |
| 12a - 1068 | -26/11, -28/11, -30/11 |   |
| 12a - 1069 | 12/29, -62/29, 86/29 |   |
| 12a - 1082 | -12/11, 50/11, -74/11 |   |
Tabulation of the Dihedral Linking Invariant for 7-Colorable Knots.

| Knot | Linking Invariant |
|------|-------------------|
| 5 - 2 | 0, -2, 2         |
| 7 - 1 | -2               |
| 7 - 7 | -6, -2, 2        |
| 8 - 5 | 0, 10, 6, 2, -4, -10 |
| 8 - 16 | 6, -2, -6, -22, -14, 18 |
| 9 - 4 | 0, 2, -2, -4, -6   |
| 9 - 12 | -6, -2, 2        |
| 9 - 27 | 0, -2, 2         |
| 9 - 41 | -16/7, -30/7, -2/7, 12/7, 26/7 |
| 9 - 42 | 10, -2, -8, -14, 4 |
| 10 - 20 | 0, 6, 2, -2, -6, 4 |
| 10 - 22 | 10, 6, 2, -2, -6 |
| 10 - 29 | 6, 2, -2, -4, -8, -14 |
| 10 - 35 | 6, 2, -2, -6, -10 |
| 10 - 48 | 0, -2, 2         |
| 10 - 65 | 0, -2, 2         |
| 10 - 67 | 0, -2, 2         |
| 10 - 71 | 0, -2, 2         |
| 10 - 74 | 0, -2, 2         |
| 10 - 77 | 0, -2, 2         |
| 10 - 82 | 6, 2, -2, -4, -8, -14 |
| 10 - 90 | 0, 10, 6, 8, -2, -10 |
| 10 - 95 | 6, -2, 2         |
| 10 - 104 | 142/41, 92/41, -84/41, -210/41, 42/41, -34/41 |
| 10 - 105 | -42/43, -190/43, 106/43, 80/43, 54/43, -68/43 |
| 10 - 108 | -74/29, -218/29, -30/29, -124/29, 20/29, 70/29 |
| 10 - 111 | -64/13, -2/13, -86/13, -42/13, 20/13, 82/13 |
| 10 - 120 | 0, 4, -4         |
| 10 - 122 | 0, -2, 2         |
| 10 - 138 | -18, 10, -2, -4, -10, 4 |
| 10 - 141 | 8, 2, -4, 14, -10 |
| 10 - 156 | 46/13, 38/13, -54/13, -8/13, -4/13, 42/13 |
| 10 - 157 | 30/7, 16/7, 2/7, -26/7 |
| 10 - 160 | -2/13, 6/13, -10/13, -22/13, -34/13, -14/13 |
| 10 - 162 | 46/13, -2/13, 16/13, -8/13, 22/13, -14/13 |
| 11a - 14 | -6, -2, 2       |
| 11a - 16 | 0, 2, -2, -4, -6 |
| 11a - 34 | 0, 2, -2, -4, -6, -10 |
| 11a - 49 | 6, 2, -2, -4, -8, -14 |
| 11a - 66 | 6, -2, -4, 14, -6, 4 |
| 11a - 86 | -18, 10, -2, -4, -10, 4 |
| 11a - 89 | 0, 2, -2, -4, -6 |
| 11a - 98 | 0, 6, 2, -2, -4, -6 |
| 11a - 119 | 10, 6, 2, -2, 14, -10 |
| 11a - 121 | 6, -2, 14, -6, -14, -26 |
|   |   |   |
|---|---|---|
|11a - 125 | 2/13, -12/13, -6/13, -4/13, -18/13, 10/13 |
|11a - 138 | 0, 6, 2, -2, -6, 4 |
|11a - 156 | 10, 6, 2, -2, -6 |
|11a - 158 | -368/127, 432/127, 874/127, -726/127, 74/127, -10/127 |
|11a - 163 | 2/13, -38/13, -70/13, -6/13, -30/13, 10/13 |
|11a - 175 | 6, -2, 2 |
|11a - 196 | 520/203, 2/7, -824/203, -26/7, -54/7, -12/7, 86/203, -390/203, 1430/203, 30/7, -1258/203, 16/7 |
|11a - 203 | 0, 2, -2, -4, -6 |
|11a - 205 | 6, 12, -2, -10, 4, 18 |
|11a - 208 | 6, -2, -4, 14, -22, -12 |
|11a - 215 | -952/83, 858/83, -330/83, -1574/83, -358/83, 264/83 |
|11a - 216 | -90/13, 6/13, 54/13, 30/13, -18/13, -42/13 |
|11a - 228 | 250/197, 364/197, 478/197, -344/197, -230/197, -938/197 |
|11a - 244 | 0, 2, 8, 14, -6, -14 |
|11a - 251 | -10, 4, 18 |
|11a - 253 | -10, 4, 18 |
|11a - 256 | 46/13, 28/13, -12/13, -70/13, -30/13, 10/13 |
|11a - 276 | -12/13, -58/13, -4/13, 34/13, -42/13, -50/13 |
|11a - 279 | 146/71, 14/71, 28/71, -38/71, -90/71, 80/71 |
|11a - 280 | 326/13, -282/13, -462/13, -102/13, 112/13, -68/13 |
|11a - 283 | -54/13, -12/13, 6/13, 30/13, -18/13, -36/13 |
|11a - 285 | 202/29, -466/29, 98/29, -132/29, 432/29, 662/29 |
|11a - 286 | -102/83, -108/83, -114/83, -12/83, 78/83, -18/83 |
|11a - 297 | -98/29, 186/29, 96/29, -202/29, 6/29, -8/29 |
|11a - 302 | 6, 2, -4, -6, -14, 4 |
|11a - 306 | 6, -2, 14, 22, -10 |
|11a - 312 | -54/13, 6/13, -70/13, -62/13, 14/13, 82/13 |
|11a - 313 | 130/71, 262/71, -14/71, -290/71, -146/71, -2/71 |
|11a - 319 | 12/13, 6/13, -28/13, -62/13, -22/13, 18/13 |
|11a - 332 | 48/13, 38/13, -28/13, 58/13, -18/13, -94/13 |
|11a - 334 | -6, -2, -4 |
|11a - 345 | 6/13, -40/13, 24/13, -86/13, 42/13, -22/13 |
|11a - 354 | 0, -2, 2 |
|11a - 357 | 10, 2, -6, -14, 18 |
|11a - 363 | 6, 2, -2, -6, -10 |
|11n - 4 | -16, 6, -4, -6, -26, 18 |
|11n - 15 | 0, 2, -2, -4, -6 |
|11n - 21 | 10, 6, 2, -2, -6 |
|11n - 28 | 6, 2, 12, -4, -14, 18 |
|11n - 56 | 0, 10, 6, 8, -2, -10 |
|11n - 57 | 0, 2, -2, -4, -6 |
|11n - 58 | 6, -4, 14, 22, -14, 4 |
|11n - 64 | 0, -2, 2 |
|11n - 68 | 0, -2, 2 |
| 11n - 71 | 0, -2, 2 |
| 11n - 75 | 0, -2, 2 |
| 11n - 83 | 0, -2, 2 |
| 11n - 84 | 0, 6, 2, -2, -6, 4 |
| 11n - 96 | -38/13, -2/13, -6/13, 30/13, -34/13, -42/13 |
| 11n - 107 | 6, 2, -2, 14, -6, -10 |
| 11n - 111 | -26/29, 2/29, 10/29, -22/29, 6/29, -54/29 |
| 11n - 113 | 6, -4, 14, 22, -14, 4 |
| 11n - 115 | 76/13, 58/13, 230/13, -96/13, 402/13, -250/13 |
| 11n - 117 | -174/41, -86/41, 14/41, 102/41, 202/41, 2/41 |
| 11n - 118 | -24/13, -12/13, -6/13, -30/13, -18/13, -42/13 |
| 11n - 125 | -106/43, -70/43, 210/43, 70/43, -142/43, 34/43 |
| 11n - 136 | 10, 2, -2, -4, -6, 4 |
| 11n - 141 | 6, 2, -2, -4, -10, 4 |
| 11n - 146 | -30/43, -70/43, -34/43, -66/43, -102/43, -38/43 |
| 11n - 156 | -162/29, 30/29, -66/29, -78/29, 6/29, 18/29 |
| 11n - 161 | -70/43, 30/43, -16/43, -20/43, 38/43, 34/43 |
| 11n - 163 | 46/13, -38/13, -24/13, -10/13, 4/13, 18/13 |
| 11n - 167 | 0, 2, -2, -4, 4 |
| 11n - 169 | -16, 14, -6, 4, -26 |
| 11n - 170 | 102/43, 210/43, -20/43, -142/43, 34/43, 156/43 |
| 11n - 172 | -24/29, -2/29, -26/29, -22/29, -4/29, 18/29 |
| 11n - 176 | 0, 4, 2 |
| 11n - 179 | 40/29, -70/29, -118/29, 16/29, -94/29, 150/29 |
| 11n - 183 | 6/13, -14/13, -28/13, -4/13, -18/13, -42/13 |
| 11n - 185 | 0, -2, 2 |
| 12a - 6 | 6, 2, -2, -4, -8, -14 |
| 12a - 9 | -4, 14, -6, -22, -14, 4 |
| 12a - 23 | -6, -2, 2 |
| 12a - 24 | -16, 14, -6, 4, -26 |
| 12a - 28 | 0, -2, 2 |
| 12a - 30 | -6, -2, 2 |
| 12a - 33 | -6, -2, 2 |
| 12a - 37 | 0, 2, -2, -4, -6 |
| 12a - 43 | 0, -2, -4 |
| 12a - 44 | 0, 4, -4 |
| 12a - 46 | 0, 2, -2, -4, 4 |
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| 12a - 53 | 0, -2, -4 |
| 12a - 59 | -10, 2, -4 |
| 12a - 63 | -10, 2, -4 |
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| 12a - 87 | 10, -2, -6, -22, -12, 4 |
| 12a - 99 | 0, -2, 2 |
| 12a - 119 | 0, 2, 8, 14, -6, -14 |
| 12a - 135 | 10, 2, -4, -10, 4, 18 |
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| 12a - 187 | 0, 6, 2, -2, -4, -6 |
| 12a - 190 | 6, 2, 12, -4, 22, -10 |
| 12a - 193 | 0, 10, -2, -6, -10, 4 |
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| 12a - 209 | 0, 4, -4 |
| 12a - 210 | 0, 4, -4 |
| 12a - 212 | 0 |
| 12a - 218 | 0, -2, 2 |
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| 12a - 224 | 6, 2, -8, 26, -22, 16 |
| 12a - 227 | -10, 2, -4 |
| 12a - 235 | 0, 2, -2, -4, 4 |
| 12a - 237 | 0, -2, 4 |
| 12a - 243 | -6, 0, 6 |
| 12a - 253 | 0, 2, -2, -4, -6 |
| 12a - 263 | 0, 2, -2, -4, -6 |
| 12a - 284 | -16, 10, 74, 48, 22, -54 |
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| 12a - 299 | 10, 2, -2, -4, -6, 4 |
| 12a - 302 | 0, 6, 2, -2, -4, -6 |
| 12a - 306 | -18, 6, -6, -30, 18 |
| 12a - 309 | 120/307, 70/307, 200/307, 250/307, -10/307, 150/307 |
| 12a - 310 | 350/13, -70/13, 230/13, 110/13, -250/13, 50/13 |
| 12a - 311 | -18, -2, 14, -6, -12, 4 |
| 12a - 314 | -18, 6, 2, 14, -6, 22 |
| 12a - 317 | 78/41, 266/41, 54/41, 172/41, -40/41, -158/41 |
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| 12a - 337 | 0, -2, 4 |
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| 12a - 361 | 138/29, 56/29, -26/29, -274/29, -150/29, -68/29 |
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| 12a - 385 | 6, 2, -4, -6, -14, 4 |
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| 12a - 388 | -70/43, 210/43, 30/43, 70/43, -20/43, 120/43 |
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| 12a - 409 | 0, 2, -2, -4, -6 |
| 12a - 414 | 0, 6, 2, -2, 4 |
| 12a - 421 | 0, -2, -4 |
| 12a - 423 | 0, 2, -2, -4, 4 |
| 12a - 426 | 142/41, 92/41, -84/41, -210/41, -34/41, 42/41 |
| 12a - 433 | -18, 2, -8, 14, 26, 4 |
| 12a - 446 | 2, 12, -6, 22, -14, 4 |
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| 12a - 469 | -46/29, 98/29, -234/29, -94/29, 150/29, -42/29 |
| 12a - 474 | -130/43, 92/43, 118/43, -326/43, 66/43, -104/43 |
| 12a - 476 | 1280/377, -1324/377, 2798/377, -2410/377, -238/377, 194/377 |
| 12a - 480 | 2/13, -38/13, -6/13, 42/13, -22/13, 18/13 |
| 12a - 481 | 6, 2, -4, -6, -14, 4 |
| 12a - 486 | 0, 6, 2, -2, -4, -6 |
| 12a - 488 | -110/43, -116/43, 8/43, 2/43, 14/43, -234/43 |
| 12a - 495 | -38/13, -28/13, 30/13, -86/13, 10/13, 20/13 |
| 12a - 502 | 0, 2, -2, -4, -6 |
| 12a - 513 | 0, -2, 2 |
| 12a - 520 | -2 |
| 12a - 529 | -6, -2, 2 |
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| 12a - 535 | 0, 10, 6, 2, 8, -6 |
| 12a - 546 | 350/13, 204/13, 58/13, -96/13, -250/13, 50/13 |
| 12a - 554 | -6, -2, 2 |
| 12a - 572 | 0, -2, 2 |
| 12a - 576 | 2 |
| 12a - 581 | 6, -2, 2 |
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| 12a - 592 | -138/41, -64/41, 62/41, 10/41, 114/41, -12/41 |
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| 12a - 697 | 0, -2, -4 |
| 12a - 704 | 18/43, -106/43, -44/43, 118/43, 6/43, 68/43 |
| 12a - 709 | -18/29, 142/29, 24/29, -94/29, -56/29, 62/29 |
| 12a - 712 | -6, -2, 2 |
| 12a - 719 | 0, 6, 2, -2, -6, 4 |
| 12a - 723 | -2 |
| 12a - 728 | 6, -2, 2 |
| 12a - 731 | -6, -2, 2 |
| 12a - 738 | 6, 2, -2, -4, -10, 4 |
| 12a - 739 | 10, 6, 2, -2, -6, -14 |
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| 12a - 751 | 28/13, 38/13, 86/13, -30/13, -10/13, -20/13 |
| 12a - 764 | -18, 6, 26, -6, 16, 4 |
| 12a - 769 | 10, -2, -8, -14, 4 |
| 12a - 773 | -6, -2, 2 |
| 12a - 791 | 6, -2, 2 |
| 12a - 794 | 6, 2, -2, -4, -10, 4 |
| 12a - 799 | 686/503, 1186/503, -908/503, -2502/503, -658/503, 936/503 |
| 12a - 803 | 0, -2, 2 |
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| 12a - 822 | 0, 10, -2, -6, -10, 4 |
| 12a - 842 | 0, 6, 2, -2, -6, 4 |
| 12a - 844 | -24/13, -6/13, -4/13, 34/13, 14/13, -42/13 |
| 12a - 846 | -24/13, -6/13, -4/13, 34/13, 14/13, -42/13 |
| 12a - 847 | -74/29, 250/29, -70/29, -66/29, 92/29, 88/29 |
| 12a - 848 | 8, -2, -22, -12, 18 |
| 12a - 854 | -14, -6, 2 |
| 12a - 869 | 154/43, 130/43, 106/43, 80/43, 54/43, 104/43 |
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| 12a - 876 | -6, -2, -4 |
| 12a - 880 | 0, 4, -4 |
| 12a - 882 | 0 |
| 12a - 885 | 14, -10, 2 |
| 12a - 894 | 2, 4/3, -2, 14/3, -4/3, -2/3 |
| 12a - 896 | 0, -2, 2 |
| 12a - 897 | 0, -2, 2 |
| 12a - 905 | 46/41, 72/41, -6/41, -110/41, -32/41, 98/41 |
| 12a - 906 | 0, 6, 2, -2, -4, -6, 4 |
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