**Abstract.** We survey and analyze different ways in which bornologies, coarse structures and uniformities on a group agree with the group operations.

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1. **Bornological groups**

1.1. **Bornological spaces.** A family \( \mathcal{I} \) of subsets of a set \( X \) is called an *ideal* (in the Boolean algebra \( \mathcal{P}_X \) of all subsets of \( X \)) if \( \mathcal{I} \) is closed under formation of finite unions and subsets. If \( \bigcup \mathcal{I} = X \) then \( \mathcal{I} \) is called a *bornology*, so a bornology is an ideal containing the ideal \( \mathcal{F}_X \) of all finite subsets.

A bornology \( \mathcal{B} \) on \( X \) is called *tall* if any infinite subset \( Y \) of \( X \) contains an infinite subset \( Z \) such that \( Z \in \mathcal{B} \).

A bornology \( \mathcal{B} \) on \( X \) is called *antitall* if, for any \( Y \subseteq X \), \( Y \notin \mathcal{B} \), there exists \( Z \subseteq Y \) such that \( Z' \notin \mathcal{B} \) for each infinite subset \( Z' \) of \( Z \).

By [1, Proposition 1], every bornology is the meet of some tall and antitall bornologies.

A family \( \mathfrak{F} \subseteq \mathcal{B} \) is called a *base* of \( \mathcal{B} \) if, for any \( A \in \mathcal{B} \), there exists \( B \in \mathfrak{F} \) such that \( A \subseteq B \). Every bornology with a countable base is antitall. In particular, a bornology of all bounded subsets of a metric space is antitall. We note also that, for every bornology \( \mathcal{B} \) on \( X \) with a countable base, there exists a metric \( d \) on \( X \) such that \( \mathcal{B} \) is a bornology of all bounded subsets of \((X, d)\).

A set \( X \) endowed with a bornology \( \mathcal{B} \) is called a *bornological space*, and is denoted by \((X, \mathcal{B})\). Each subset \( Y \in \mathcal{B} \) is called *bounded*. If \( \mathcal{B} = \mathcal{P}_X \) then \((X, \mathcal{B})\) is *bounded*.

Let \((X, \mathcal{B})\) be a bornological space. Every subset \( Y \subseteq X \) defines the bornological *subspace* \((Y, \mathcal{B}_Y), \mathcal{B}_Y = \{ Y \cap A : A \in \mathcal{B} \}\).

The product of a family of bornological spaces is the Cartesian product of its supports endowed with the Cartesian product of its bornologies.

A mapping \( f : (X, \mathcal{B}) \to (X', \mathcal{B}') \) is called *bornologous* if \( f(A) \in \mathcal{B}' \) for each \( A \in \mathcal{B} \).

A *variety* is a class of bornological spaces closed under formation of subspaces, products and bornological images.
We denote by $\mathcal{M}_{\text{single}}$ the variety of all singletons, $\mathcal{M}_{\text{bound}}$ the variety of all bounded spaces, $\mathcal{M}_\kappa$ the variety of all $\kappa$-bounded spaces. For an infinite cardinal $\kappa$, a space $(X, \mathcal{B})$ is called $\kappa$-bounded if $[X]^{<\kappa} \subseteq \mathcal{B}$, $[X]^{<\kappa} = \{Y \subseteq X : |Y| < \kappa\}$.

Every variety of bornological spaces lies in the chain
\[
\mathcal{M}_{\text{single}} \subset \mathcal{M}_{\text{bound}} \subset \ldots \subset \mathcal{M}_\kappa \subset \ldots \subset \mathcal{M}_\omega,
\]
see Proof of Theorem 2 in [2].

1.2. Bornologies on groups. A bornology on a group $G$ is called right (left) invariant if $B g \subseteq B(gB \subseteq B)$ for each $g \in G$. A group $G$ endowed with a right (left) invariant bornology is called a right (left) bornological group.

We say that a group $G$ endowed with a bornology $B$ is a bornological group if the group multiplication and inversion are bornologous mappings. In this case, $B$ is called a group bornology. We note that $B$ is a group bornology if and only if, for any $A, B \in B$, we have $AB^{-1} \in B$.

1.3. Duality. Now we endow $G$ with the discrete topology and identify the Stone-Čech compactification $\beta G$ of $G$ with the set of all ultrafilters on $G$ and denote $G^* = \beta G \setminus G$, the set of all free ultrafilters on $G$. Then the family $\{\overline{A} : A \subseteq G\}$, where $\overline{A} = \{p \in \beta G : A \in p\}$, forms the base for the topology of $\beta G$. Given a filter $\varphi$ on $G$, we denote $\overline{\varphi} = \cap \{\overline{A} : A \in \varphi\}$, so $\varphi$ defines the closed subset $\overline{\varphi}$ of $\beta G$, and each closed subset $K$ of $\beta G$ can be obtained in this way: $K = \overline{\varphi}$, where $\varphi = \{A \subseteq G : K \subseteq A\}$.

We use the standard extension [3, Section 4.1] of the multiplication on $G$ to the semigroup multiplication on $\beta G$ such that, for each $p \in \beta G$, the mapping $x \mapsto xp$, $x \in \beta G$ is continuous, and for each $g \in G$, the mapping $x \mapsto gx$, $x \in \beta G$ is continuous. Given two ultrafilters $p, q \in \beta G$, we choose $P \in p$ and, for each $x \in P$, pick $Q_x \in q$. Then $\bigcup_{x \in P} x Q_x \in pq$ and the family of all these subsets forms a local basis for the product $pq$.

It follows directly from the definition of the multiplication in $\beta G$ that $G^*$, $\overline{G^*}$ are ideals in the semigroup $\beta G$, and $G^*$ is the unique maximal closed ideal in $G$. By Theorem 4.44 from [3], the closure $\overline{K(\beta G)}$ of the minimal ideal $K(G)$ of $\beta G$ is an ideal, so $\overline{K(\beta G)}$ is the smallest closed ideal in $\beta G$. For the structure of $\overline{K(\beta G)}$ and some other ideals in $\beta G$ see [3, Sections 4,6].

For an ideal $\mathcal{I}$ in $\mathcal{P}_G$ and a closed subset $K$ of $\beta G$, we put
\[
\mathcal{I}^\wedge = \{p \in \beta G : G \setminus A \in p \text{ for each } A \in \mathcal{I}\}, \quad K^\vee = \{G \setminus A : A \in \varphi, \overline{\varphi} = K\},
\]
and get the following statements:

- $\mathcal{I}$ is left translation invariant if and only if $\mathcal{I}^\wedge$ is a left ideal of the semigroup $\beta G$;
- $\mathcal{I}$ is right translation invariant if and only if $(\mathcal{I}^\wedge)G \subseteq \mathcal{I}^\wedge$;
- $(\mathcal{I}^\wedge)^\vee = \mathcal{I}$;
- $\mathcal{I}$ is a bornology if and only if $\mathcal{I}^\wedge \subseteq G^*$.
Thus, we have the duality between left invariant bornologies on $G$ and closed left ideals of $\beta G$ containing in $G^*$. 

We say that a subset $A$ of a group $G$ is

- **large** if $G = FA$ for some $F \in \mathcal{F}_G$;
- **small** if $L \setminus A$ is large for every large subset $L$ of $G$;
- **sparse** if, for every infinite subset $X$ of $G$, there exists a finite subset $F \subset X$ such that $\bigcap_{g \in F} gA$ is finite.

**Theorem 1.1.** For every infinite group $G$, the family $Sm_G$ of all small subsets of $G$ is a left and right invariant bornology and $Sm_G^\wedge = \overline{K(\beta G)}$.

This is Theorem 4.40 from [3] in the form given in [4, Theorem 12.5].

**Theorem 1.2.** For every infinite group $G$, the family $Sp_G$ of all sparse subsets of $G$ is a left and right invariant bornology and $Sp_G^\wedge = G^*G^*$.

This is Theorem 10 from [5].

More applications of this duality can be find in [6].

Let $B$ be a group bornology on $G$. By [7], $B^\wedge$ is an ideal in $\beta G$ but the converse statement does not hold: $Sm_G^\wedge$ is an ideal but $Sm_G$ is not a group bornology.

1.4. Plenty of bornologies. We say that a left invariant bornology $B$ of $G$ is **maximal** if $B \neq \mathcal{P}_G$ and, for every left invariant bornology $B'$ on $G$, $B \subset B'$ implies $B' = \mathcal{P}_G$.

**Theorem 1.3.** For every infinite group $G$, of cardinality $\kappa$, there are $2^{2^\kappa}$ distinct maximal left invariant bornologies on $G$.

**Proof.** By Theorem 6.30 from [3], there is a family $\mathcal{F}$, $|\mathcal{F}| = 2^{2^\kappa}$ of pairwise disjoint closed left ideals in $\beta G$. Each $I \in \mathcal{F}$ contains some minimal closed left ideal $L_I$. Then $\{L_I^\vee : I \in \mathcal{F}\}$ is the desired family of maximal left invariant bornologies. \[\square\]

**Theorem 1.4.** [8]. For every countable group $G$, there are $2^{2^{\aleph_0}}$ distinct group bornologies on $G$.

Let $G$ be an Abelian group of cardinality $\kappa$. By Theorem 6.3.3 from [9], there are $2^{2^\kappa}$ distinct group bornologies of $G$. 
**Question 1.1.** How many distinct group bornologies on an infinite group \( G \) of cardinality \( \kappa \)? Is this number \( 2^{2^\kappa} \)?

**Theorem 1.4.** [7]. For every infinite group \( G \), the following statements hold:

(i) if \( \mathcal{B} \) is a left invariant bornology on \( G \) and \( \mathcal{B} \neq \mathfrak{F}_G \) then there is a left invariant bornology \( \mathcal{B}' \) on \( G \) such that \( \mathfrak{F}_G \subset \mathcal{B}' \subset \mathcal{B} \);

(ii) if \( G \) is either countable or Abelian and \( \mathcal{B} \) is a left and right invariant bornology on \( G \) such that \( \mathcal{B} \neq \mathfrak{F}_G \) then there is a left and right invariant bornology \( \mathcal{B}' \) on \( G \) such that \( \mathfrak{F}_G \subset \mathcal{B}' \subset \mathcal{B} \);

(iii) if \( G \) is the group \( S_\kappa \) of all permutations of an infinite cardinal \( \kappa \) then there exists a left and right invariant bornology \( \mathcal{B} \) on \( G \) such that \( \mathfrak{F}_G \subset \mathcal{B} \) and there are no left and right invariant bornologies on \( G \) between \( \mathfrak{F}_G \) and \( \mathcal{B} \).

**Theorem 1.5.** Let \( G \) be an infinite group and let \( \mathcal{B} \) be a group bornology of \( G \) such that \( \mathcal{B} \neq \mathfrak{F}_G \). If \( G \) is either countable or Abelian then there is a group bornology \( \mathcal{B}' \) such that \( \mathfrak{F}_G \subset \mathcal{B}' \subset \mathcal{B} \).

This is Theorem 6.4.1. from [9]. We do not know (Question 6.4.1. in [9]) if Theorem 1.5. true for every infinite group \( G \), in particular, for \( G = S_\kappa \).

By [4, Theorem 12.9], every infinite group can be partitioned into countably many small subsets.

**Question 1.2.** Given an infinite group \( G \), do there exist a small subset \( S \) and a countable subset \( A \) of \( G \) such that \( \{aS : a \in A \} \) is a covering (partition) of \( G \)?

This is so if \( G \) is amenable or \( G \) has a subgroup of countable index.

2. **Coarse groups**

2.1. **Coarse spaces.** Following [10], we say that a family \( \mathcal{E} \) of subsets of \( X \times X \) is a coarse structure on a set \( X \) if

- each \( \varepsilon \in \mathcal{E} \) contains the diagonal \( \Delta_X, \Delta_X = \{(x, x) : x \in X \} \);
- if \( \varepsilon, \delta \in \mathcal{E} \) then \( \varepsilon \circ \delta \in \mathcal{E} \) and \( \varepsilon^{-1} \in \mathcal{E} \) where \( \varepsilon \circ \delta = \{(x, y) : \exists z ((x, z) \in \varepsilon, (z, y) \in \delta) \}, \varepsilon^{-1} = \{(y, x) : (x, y) \in \varepsilon \};\)
- if \( \varepsilon \in \mathcal{E} \) and \( \Delta_X \subseteq \varepsilon' \subseteq \varepsilon \) then \( \varepsilon' \in \mathcal{E} \);
- for any \( x, y \in X \), there exists \( \varepsilon \in \mathcal{E} \) such that \( (x, y) \in \varepsilon \).
Each $\varepsilon \in \mathcal{E}$ is called an entourage of the diagonal. A subset $\mathcal{E}' \subseteq \mathcal{E}$ is called a base for $\mathcal{E}$ if, for every $\varepsilon \in \mathcal{E}$ there exists $\varepsilon' \in \mathcal{E}'$ such that $\varepsilon \subseteq \varepsilon'$.

The pair $(X, \mathcal{E})$ is called a coarse space. For $x \in X$ and $\varepsilon \in \mathcal{E}$, we denote $B(x, \varepsilon) = \{ y \in X : (x, y) \in \varepsilon \}$ and say that $B(x, \varepsilon)$ is the ball of radius $\varepsilon$ around $x$. We note that a coarse space can be considered as an asymptotic counterpart of a uniform topological space and could be defined in terms of balls, see [4], [9]. In this case a coarse space is called a ballean.

A subset $Y$ of $X$ is called bounded if there exist $x \in X$ and $\varepsilon \in \mathcal{E}$ such that $Y \subseteq B(x, \varepsilon)$. The coarse structure $\mathcal{E} = \{ \varepsilon \in X \times X : \Delta_X \subseteq \varepsilon \}$ is the unique coarse structure such that $(X, \mathcal{E})$ bounded.

Given a coarse space $(X, \mathcal{E})$, each subset $Y \subseteq X$ has the natural coarse structure $\mathcal{E}|_Y = \{ \varepsilon \cap (Y \times Y) : \varepsilon \in \mathcal{E} \}$, $(Y, \mathcal{E}|_Y)$ is called a subspace of $(X, \mathcal{E})$. A subset $Y$ of $X$ is called large (or coarsely dense) if there exists $\varepsilon \in \mathcal{E}$ such that $X = B(Y, \varepsilon)$ where $B(Y, \varepsilon) = \bigcup_{y \in Y} B(y, \varepsilon)$.

Let $(X, \mathcal{E})$, $(X', \mathcal{E}')$ be coarse spaces. A mapping $f : X \rightarrow X'$ is called coarse if, for every $\varepsilon \in \mathcal{E}$ there exists $\varepsilon' \in \mathcal{E}$ such that, for every $x \in X$, we have $f(B(x, \varepsilon)) \subseteq B(f(x), \varepsilon')$. If $f$ is surjective and coarse then $(X', \mathcal{E}')$ is called a coarse image of $(X, \mathcal{E})$. If $f$ is a bijection such that $f$ and $f^{-1}$ are coarse mappings then $f$ is called an isomorphism. The coarse spaces $(X, \mathcal{E})$, $(X', \mathcal{E}')$ are called coarsely equivalent if there exist large subsets $Y \subseteq X$, $Y' \subseteq X'$ such that $(Y, \mathcal{E}|_Y)$ and $(Y', \mathcal{E}'|_{Y'})$ are isomorphic.

To conclude the coarse vocabulary, we take a family $\{(X_\alpha, \mathcal{E}_\alpha) : \alpha < \kappa \}$ of coarse spaces and define the product $P_{\alpha<\kappa}(X_\alpha, \mathcal{E}_\alpha)$ as the Cartesian product $P_{\alpha<\kappa}X_\alpha$ endowed with the coarse structure with the base $P_{\alpha<\kappa}\mathcal{E}_\alpha$. If $\varepsilon_\alpha \in \mathcal{E}_\alpha$, $\alpha < \kappa$ and $x, y \in P_{\alpha<\kappa}X_\alpha$, $x = (x_\alpha)_{\alpha<\kappa}$, $y = (y_\alpha)_{\alpha<\kappa}$ then $(x, y) \in (\varepsilon_\alpha)_{\alpha<\kappa}$ if and only if $(x_\alpha, y_\alpha) \in \varepsilon_\alpha$ for every $\alpha < \kappa$.

For lattices of coarse structures and varieties of coarse spaces, see [11] and [2].

For every coarse space $(X, \mathcal{E})$, the family of all bounded subsets of $X$ is a bornology. On the other hand, for every bornology $\mathcal{B}$ on $X$, there is the smallest by inclusion coarse structure $\mathcal{E}_\mathcal{B}$ on $X$ such that $\mathcal{B}$ is a bornology of all bounded subsets of $(X, \mathcal{E}_\mathcal{B})$. A coarse $\mathcal{E}$ on $X$ is of the form $\mathcal{E}_\mathcal{B}$ if and only if $(X, \mathcal{E})$ is thin: for every $\varepsilon \in \mathcal{E}$, there exists a bounded subset $A$ of $(X, \mathcal{E})$ such that $B(x, \varepsilon) = \{ x \}$ for all $x \in X \setminus A$.

### 2.2. Coarse structures on groups.

Let $G$ be a group with the identity $e$.

We remind that a family $\mathcal{I}$ of subsets of $G$ is a group bornology if $\mathcal{I}$ is closed under formation of subsets and finite unions, $[G]^{<\omega} \subseteq \mathcal{I}$ and $AB^{-1} \in \mathcal{I}$ for all $A, B \in \mathcal{I}$. A group bornology $\mathcal{I}$ is called invariant if $\cup_{g \in G} g^{-1}Ag \in \mathcal{I}$ for each $A \in \mathcal{I}$.

Let $X$ be a $G$-space with the action $G \times X \rightarrow X$, $(g, x) \mapsto gx$. We assume that $G$ acts on $X$ transitively, take a group bornology $\mathcal{I}$ on $G$ and consider the coarse structure $\mathcal{E}(G, \mathcal{I}, X)$ on $X$ with the base $\{ \varepsilon_A : A \in \mathcal{I}, e \in A \}$, $\varepsilon_A = \{ (x, gx) : x \in X, g \in A \}$. Then $(x, y) \in \varepsilon_A$ if and only if $yx^{-1} \in A$ so $B(x, \varepsilon) = Ax$, $Ax = \{ gx : g \in A \}$.
By [12, Theorem 1], for every coarse structure $\mathcal{E}$ on $X$, there exist a group $G$ of permutations of $X$ and a group ideal $\mathcal{I}$ on $G$ such that $\mathcal{E} = \mathcal{E}(G, \mathcal{I}, X)$.

Now let $X = G$ and $G$ acts on $X$ by the left shifts. We denote $\mathcal{E}_G = \mathcal{E}(G, \mathcal{I}, G)$. Thus, every group bornology $\mathcal{I}$ on $G$ turns $G$ into the coarse space $(G, \mathcal{E}_G)$. We note that a subset $A$ of $G$ is bounded in $(G, \mathcal{E}_G)$ if and only if $A \in \mathcal{I}$.

A group $G$ endowed with a coarse structure $\mathcal{E}$ is called left (right) coarse group if, for every $\varepsilon \in \mathcal{E}$, there exists $\varepsilon' \in \mathcal{E}$ such that $gB(x, \varepsilon) \subseteq B(gx, \varepsilon')$ ($B(x, \varepsilon)g \subseteq B(xg, \varepsilon')$) for all $x, g \in G$. Equivalently, $(G, \mathcal{E})$ is a left (right) coarse group if $\mathcal{E}$ has a base consisting of left (right) invariant entourages. An entourage $\varepsilon$ is left (right) invariant if $g\varepsilon = \varepsilon (\varepsilon g = g)$ for each $g \in G$, $g\varepsilon = \{(gx, gy) : (x, y) \in \varepsilon\}$. For finitely generated groups, the right coarse groups $(G, \mathcal{E}_{\lbrack G \rbrack}_{<\kappa})$ in metric form take a great part of Geometrical Group Theory, see [13, Chapter 4].

A group $G$ endowed with a coarse structure $\mathcal{E}$ is called a coarse group if the group multiplication $(G, \mathcal{E}) \times (G, \mathcal{E}) \rightarrow (G, \mathcal{E})$, $(x, y) \mapsto xy$ and the inversion $(G, \mathcal{E}) \rightarrow (G, \mathcal{E})$, $x \mapsto x^{-1}$ are coarse mappings. In this case, $\mathcal{E}$ is called a group coarse structure.

The following two statements are from [14], or see also [9, Chapter 6].

**Proposition 2.1.** A group $G$ endowed with a coarse structure $\mathcal{E}$ is a right coarse group if and only if there exists a group bornology $\mathcal{I}$ on $G$ such that $\mathcal{E} = \mathcal{E}_\mathcal{I}$.

**Proposition 2.2.** For a group $G$ endowed with a coarse structure $\mathcal{E}$, the following conditions are equivalent:

(i) $(G, \mathcal{E})$ is a coarse group;

(ii) $(G, \mathcal{E})$ is left and right coarse group;

(iii) there exists an invariant group bornology $\mathcal{I}$ on $G$ such that $\mathcal{E} = \mathcal{E}_\mathcal{I}$.

Applying Theorem 1.4, we get $2^{2^{\aleph_0}}$ distinct right coarse structures on any countable group. For every infinite group $G$ and any infinite cardinal $\kappa$, $\kappa \leq | G |$, the bornology $[G]^{<\kappa}$ defines an unbounded right coarse structure on $G$. But if $G$ has only two conjugated classes then there is only one, bounded, group coarse structure on $G$.

### 2.3 Asymorphisms

For an infinite cardinal $\kappa$, we say that two groups $G$ and $H$ are $\kappa$-asymorphic ($\kappa$-coarsely equivalent) if the right coarse structures on $G$ and $H$ defined by the bornologies $[G]^{<\kappa}$ and $[H]^{<\kappa}$ are asymorphic (coarsely equivalent). In the case $\kappa = \aleph_0$, $G$ and $H$ are called finitarily asymorphic and finitarily coarsely equivalent respectively.

A classification of countable locally finite groups (each finite subset generates finite subgroup) up to finitary asymorphisms is obtained in [15] (cf. [4, p. 103]).

**Theorem 2.1.** Two countable locally finite groups $G_1$ and $G_2$ are finitarily asymorphic if and only if the following conditions hold:
(i) for every finite subgroup \( F \subset G_1 \), there exists a finite subgroup \( H \) of \( G_2 \) such that \(|F|\) is a divisor of \(|H|\);

(ii) for every finite subgroup \( H \) of \( G_2 \), there exists a finite subgroup \( F \) of \( G_1 \) such that \(|F|\) is a divisor of \(|H|\).

It follows that there are continuum many distinct types of countable locally finite groups and each group is finitarily asymorphic to some direct sum of finite cyclic groups.

The following coarse classification of countable Abelian groups is obtained in [16].

**Theorem 2.2.** Two countable \( G \) and \( H \) are groups are finitarily coarsely equivalent if and only if the torsion-free ranks of \( G \) and \( H \) coincide and \( G, H \) are either both finitely generated or infinitely generated.

In particular, any two countable torsion Abelian groups are finitarily coarsely equivalent.

**Theorem 2.3.** [17] Let \( G \) be an Abelian group of cardinality \( \gamma, \gamma > \aleph_0 \) and let \( \kappa \) be a cardinal such that \( \aleph_0 < \kappa \leq \gamma \). Then \( G \) is \( \kappa \)-asymorphic to the free Abelian group of rank \( \gamma \).

**Theorem 2.4.** [17] Let \( G \) be an Abelian group of cardinality \( \gamma, \gamma > \aleph_0 \). Then \( G \) is not \( \kappa \)-coarsely equivalent to the free group \( F_\gamma \) of rank \( \gamma \) provided that either \( \aleph_0 < \kappa < \gamma \) or \( \kappa = \gamma \) and \( \gamma \) is a singular cardinal. In particular, \( G \) and \( F_\gamma \) are not \( \kappa \)-asymorphic.

**Theorem 2.5.** [18] For every countable group \( G \) there are \( 2^{2^{\aleph_0}} \) distinct classes of finitarily coarsely equivalent subsets of \( G \).

### 2.4. Free coarse groups.

A class \( \mathfrak{M} \) of groups is a *variety* if \( \mathfrak{M} \) is closed under subgroups, products and homomorphic images. We assume that \( \mathfrak{M} \) is non-trivial (i.e. there exists \( G \in \mathfrak{M} \) such that \(|G| > 1\)) and recall that the free group \( F_\mathfrak{M}(X) \) is defined by the following conditions: \( F_\mathfrak{M}(X) \in \mathfrak{M}, X \subset F_\mathfrak{M}(X), X \) generates \( F_\mathfrak{M}(X) \) and every mapping \( X \longrightarrow G, G \in \mathfrak{M} \) can be extended to homomorphism \( F_\mathfrak{M}(X) \longrightarrow G \).

For a coarse space \((X, \mathcal{E})\), the *free coarse group* \( F_\mathfrak{M}(X, \mathcal{E}) \) is defined as a coarse group \((F_\mathfrak{M}(X), \mathcal{E}')\) such that \((X, \mathcal{E})\) is a subspace of \((F_\mathfrak{M}(X), \mathcal{E}')\) and every coarse mapping \((X, \mathcal{E}) \longrightarrow (G, \mathcal{E}''), G \in \mathfrak{M}, (G, \mathcal{E}'') \) is a coarse group, can be extended to coarse homomorphism \((F_\mathfrak{M}(X), \mathcal{E}') \longrightarrow (G, \mathcal{E}'')\).

The following theorem is proved in [19] with explicit description of the coarse structure of \( F_\mathfrak{M}(X, \mathcal{E}) \).
Theorem 2.6. For every coarse space \((X, \mathcal{E})\) and every non-trivial variety \(\mathcal{M}\) of groups, there exists the free coarse structure \(F_{\mathcal{M}}(X, \mathcal{E})\).

2.5. Maximality. A topological space \(X\) with no isolated points is called maximal if \(X\) has an isolated point in any stronger topology. A topological groups \(G\) is called maximal if \(G\) is maximal as a topological space. Every maximal topological group has an open countable Boolean subgroup, and can be constructed with usage of Martin Axiom. On the other hand, the existence of a maximal topological group implies a \(P\)-point in \(\omega^\ast\), for references see [20].

An unbounded coarse space \((X, \mathcal{E})\) is called maximal if \(X\) is bounded in every coarse structure \(\mathcal{E}'\) such that \(\mathcal{E} \subset \mathcal{E}'\). A coarse group \(G\) is called maximal if \(G\) is maximal as a coarse space. If a coarse group \((G, I)\) is maximal then \(\{g^2 : g \in G\}\) is bounded in \((G, I)\), and under \(CH\) a maximal coarse Boolean group is constructed in [21], (see also [9, Chapter 10]), but the following question remains open.

Question 2.1. Does there exists a maximal coarse group in ZFC?

2.6. Normality. We say that the subsets \(Y, Z\) of a coarse space \((X, \mathcal{E})\) are asymptotically disjoint if, for every \(\varepsilon \in \mathcal{E}\) there exists a bounded subset \(\mathcal{V}_\varepsilon\) of \(X\) such that

\[
B(Y \setminus \mathcal{V}_\varepsilon, \varepsilon) \cup B(Z \setminus \mathcal{V}_\varepsilon, \varepsilon) = \emptyset.
\]

We say that \(Y, Z\) are asymptotically separated if, there exists a family \(\{\mathcal{V}_\varepsilon : \varepsilon \in \mathcal{E}\}\) of bounded subsets of \(X\) such that

\[
B(Y \setminus \mathcal{V}_\varepsilon, \varepsilon) \cup B(Z \setminus \mathcal{V}_\gamma, \gamma) = \emptyset
\]

for all \(\varepsilon, \gamma \in \mathcal{E}\). By [9, Theorem 4.3.1], if \(Y, Z\) are disjoint and asymptotically disjoint then there exists a slowly oscillating function \(f : (X, \mathcal{E}) \to [0, 1]\) such that \(f|_{Y} = 0\) and \(f|_{Z} = 1\).

A coarse space \((X, \mathcal{E})\) is called normal if any two asymptotically disjoint subsets of \(X\) are asymptotically separated. By [9, Theorem 4.3.2], \((X, \mathcal{E})\) is normal if and only if, for each subspace \(Y\) of \(X\), every bounded slowly oscillating function \(f : Y \to \mathbb{R}\) can be extended to a bounded slowly oscillating function on \(X\).

Every metrizable coarse space is normal. In particular, the finitary space of a countable group is normal. If \(G\) is an uncountable Abelian group then the finitary space of \(G\) is not normal [9, Example 4.3.2].

Question 2.2. Does there exist an uncountable group \(G\) with normal finitary space?

2.7. Coarse structures on topological groups. A subset \(A\) of a topological group \(G\) (all topological groups are supposed to be Hausdorff) is called totally bounded if, for every neighborhood \(U\) of the identity, there exists \(F \in \mathcal{F}_G\) such that \(A \subseteq FU, A \subseteq UF\). The group bornology \(\mathcal{B}_\varepsilon\) of all totally bounded subsets...
of \((G, \tau)\) defines two coarse structures \(E_l\) and \(E_r\) with the bases
\[
\{(x, y) : x^{-1}y \in B\}, \ B \in B_r\}, \ \{(x, y) : xy^{-1} \in B\}, \ B \in B_r\}.
\]

The following questions are from [22].

**Question 2.3** Given a group bornology \(B\) on a group \(G\), how one can detect whether there exists a group topology \(\tau\) on \(G\) such that \(B = B_\tau\)?

Let \((G, \tau)\) be a topological group. We denote by \(\tau^\#\) the strongest group topology on \(G\) such that \(B_\tau^\# = B_\tau\), and say that \((G, \tau)\) is \(b\)-determined if and only if \(\tau^\# = \tau\). Clearly, every discrete group is \(b\)-determined. A totally bounded group \(G\) is \(b\)-determined if and only if \(\tau\) is the maximal totally bounded topology on \(G\).

**Question 2.4.** Given a topological group \(G\), how one can detect whether \(G\) is \(b\)-determined?

For the coarse structures \(E_l\), \(E_r\) and slowly oscillating functions on locally compact groups, see [23].

3. Uniform Groups

We recall that a family \(\mathcal{U}\) of subsets of \(X \times X\) is a uniformity on a set \(X\) if
- \(\triangle_X \subseteq u\) for each \(u \in \mathcal{U}\);
- if \(u, v \in \mathcal{U}\) then \(u \cap v \in \mathcal{U}\);
- if \(u \in \mathcal{U}\) and \(u \subseteq v\) then \(v \in \mathcal{U}\).
- for every \(u \in \mathcal{U}\), there exists \(v \in \mathcal{U}\) such that \(v \circ v^{-1} \subseteq u\).

A family \(\mathcal{F} \subseteq \mathcal{U}\) is called a base of \(\mathcal{U}\) if, for every \(u \in \mathcal{U}\), there exists \(v \in \mathcal{F}\) such that \(v \subseteq u\). A set \(X\), endowed with a uniformity \(\mathcal{U}\), is called a uniform spaces.

We say that a uniformity \(\mathcal{U}\) on \(X\) is left (right) invariant if \(\mathcal{U}\) has a base consisting of left (right) translation invariant entourages (cf. 2.2).

A filter \(\varphi\) on a group \(G\) is called a group filter if, for every \(A \in \varphi\), there exists \(B \in \varphi\) such that \(BB^{-1} \in \varphi\). Clearly, \(e \in A\) for each \(A \in \varphi\), and \(\varphi\) is called principal if \(\{e\} \in \varphi\).

Every group filter \(\varphi\) defines two uniformities \(\mathcal{L}_\varphi\) and \(\mathcal{R}_\varphi\) on \(G\) with the bases
\[
\{(x, y) : x^{-1}y \in A\}, \ A \in \varphi\}, \ \{(x, y) : xy^{-1} \in A\}, \ A \in \varphi\}.
\]

**Proposition 3.1.** A uniformity \(\mathcal{U}\) on a group \(G\) is right invariant if and only if \(\mathcal{U} = \mathcal{R}_\varphi\) for some group filter \(\varphi\) on \(G\).

We recall that a topological group \(G\) is balanced (= SIN) if the left and right uniformities on \(G\) coincide (= \(G\) has a base of invariant neighborhoods of the identity).
Proposition 3.2. A uniformity $U$ on a group $G$ is left and right invariant if and only if $(G, U)$ is a balanced topological group.

Example 3.1. Let $H$ be a topological group and let $f$ be an arbitrary automorphism of $H$. Then the semidirect product $G = H \rtimes < f >$ is a right uniform group defined by the filter of neighbourhoods of $e$ in $H$. If $f$ is discontinuous then $G$ is not left uniform. Now let $H$ be the Cartesian product of infinitely many copies of $\mathbb{Z}_p$. It is easy to find a discontinuous automorphism of order 2 of $H$. Hence, the right uniform group $G$ contains a compact topological group of index 2 but $G$ is not a topological group.

We say that a group $G$ is uniformizable if there is a non-principal group filter $\varphi$ on $G$ such that $\bigcap \varphi = \{e\}$.

We recall that a group $G$ is topologizable if $G$ admits a non-discrete (Hausdorff) group topology. Clearly, every topologizable group is uniformizable, but the class of uniformizable groups is much wider than the class of topologizable groups. By [24], every group can be embedded into some non-topologizable group, and if a group $G$ contains a uniformizable subgroup then $G$ is uniformizable.

Does there exists a non-uniformizable group? Yes, once upon a time, Alexander Ol’shansky used the following example to demonstrate a countable non-topologizable group.

Example 3.2. Let $n, m$ be a natural number, $n \geq 2$, $m$ is odd and $m \geq 665$. We consider the Adian group $A(n, m)$, see [25]. This group is generated by $n$ elements and has the following properties:

(a) $A(n, m)$ is torsion free;

(b) The center $C$ of $A(n, m)$ is an infinite cyclic group, $C = \langle c \rangle$;

(c) $A(n, m)/C$ is an infinite group of period $m$.

We put $G = A(n, m)/\langle c^m \rangle$, denote $f : A(n, m) \rightarrow G$ the quotient map and observe that if $g \in G \setminus \{e\}$ then either $g \in \{f(c), f(c^2), \ldots, f(c^{m-1})\}$ or $g^m \in \{f(c), f(c^2), \ldots, f(c^{m-1})\}$.

It follows that if $\varphi$ is a group filter on $G$ and $\bigcap \varphi = e$ then $\varphi = \{e\}$, so $G$ is non-uniformizable.

Question 3.1. Does every non-uniformizable group contain a non-topologizable subgroup?

Question 3.2. Can one find a criterion of uniformizability of a countable group in spirit of Markov criterion of topologizability?
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CONTACT INFORMATION
I. Protasov:
Faculty of Computer Science and Cybernetics, Kyiv University,
Academic Glushkov pr. 4d, 03680 Kyiv, Ukraine
i.v.protasov@gmail.com