DIFFERENTIABILITY OF THERMODYNAMICAL QUANTITIES IN NON-UNIFORM EXPANDING DYNAMICS

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Abstract. In this paper we study the ergodic theory of a robust non-uniformly expanding maps where no Markov assumption is required. We prove that the topological pressure is differentiable as a function of the dynamics and analytic with respect to the potential. Moreover we not only prove the continuity of the equilibrium states and their metric entropy as well as the differentiability of the maximal entropy measure and extremal Lyapunov exponents with respect to the dynamics. Moreover, we prove a local large deviations principle and central limit theorem and show that the rate function, mean and variance vary continuously with respect to observables, potentials and dynamics. Finally, we also prove that the correlation function associated to the maximal entropy measure is differentiable with respect to the dynamics and $C^1$-convergent to zero. In addition, precise formulas for the derivatives of thermodynamical quantities are given.

1. Introduction

The thermodynamical formalism was brought from statistical mechanics to dynamical systems by the pioneering works of Sinai, Ruelle and Bowen [Sin72, Bow75, BR75] in the mid seventies. Indeed, the correspondence between one-dimensional lattices and uniformly hyperbolic maps, via Markov partitions, allowed to translate and introduce several notions of Gibbs measures and equilibrium states in the realm of dynamical systems. Nevertheless, although uniformly hyperbolic dynamics arise in physical systems (see e.g. [HM03]) they do not include some relevant classes of systems including the Manneville-Pomeau transformation (phenomena of intermittency), Hénon maps and billiards with convex scatterers. We note that all the previous systems present some non-uniformly hyperbolic behavior and its relevant measure exhibits a weak Gibbs property. More recently there have been established many evidences that non-uniformly hyperbolic dynamical systems admit countable and generating Markov partitions. This is now parallel to the development of a thermodynamical formalism of gases with infinitely many states. We refer the reader to [Sat99, Pin11, PV13] for some recent progress in this direction.

A cornerstone of the theory that has driven the recent attention of many authors both in the physics and mathematics literature concerns the differentiability of thermodynamical quantities as the topological pressure, SRB measures or equilibrium states with respect to the underlying dynamical system. For natural reasons these have been referred as linear response formulas (see e.g. [Rue09]). This has proved to be a hard subject not yet completely understood. In fact,
this has been studied mostly for uniformly hyperbolic diffeomorphisms and flows in [KKPW89, Rue97, BL07, Ji12], for the SRB measure of some partially hyperbolic diffeomorphisms in [Dol04] and for one-dimensional piecewise expanding and quadratic maps in [Rue05, BS08, BS09, BS10, BBS13], and a general picture is still far from complete. In this paper we address these questions for robust classes of non-uniformly expanding maps.

If, on the one hand, the study of finer statistical properties of thermodynamical quantities as equilibrium states, mixing properties, large deviation and limit theorems, stability under deterministic perturbations or regularity of the topological pressure is usually associated to good spectral properties of the Ruelle-Perron-Frobenius operator, on the other hand neither the stability of the equilibrium states or differentiability results for on thermodynamical quantities could follow directly from the spectral gap property since transfer operators acting on the space of Hölder continuous potentials may not even vary continuously with the dynamical system (see e.g. Subsection 6.1.1 for an example). Revealing to be of fundamental importance, the functional analytic approach to thermodynamical formalism has gained special interest in the last few years and produced new and interesting results even in both uniformly and non-uniformly hyperbolic setting (see e.g. [KL99, BKL01, Cas02, Cas04, GL06, BT07, DL08, BG09, BG10, Ru10, CV13, DZ13]).

In this article we address the study of linear response formulas, continuity and differentiability of several thermodynamical quantities and limit theorems in a robust nonuniformly expanding setting of [VV10, CV13] that admit the coexistence of expanding and contracting behavior and need not admit any Markov partition. Such classes of maps include important classes of examples as bifurcation of expanding homeomorphisms, subshifts of finite type or intermittency phenomena as in the class of Maneville-Pommeau maps, as illustrated in the Section 6. Hence, one of the difficulties is that in this open set of non-uniformly expanding maps the dynamical systems are not topologically conjugate.

So, in opposition to many of the known strategies that deal with topological conjugacy classes and make use of the conjugacy to deduce some regularity results for the topological pressure or SRB measures, our strategy builds on the Birkhoff’s method of projective cones. In fact, in [CV13] this method was applied to the Ruelle-Perron-Frobenius operator acting on good Banach spaces to establish that the Ruelle-Perron-Frobenius transfer operator has a spectral gap in the Banach spaces of both Hölder continuous and smooth observables, to obtain good statistical properties of equilibrium states and continuous dependence of the topological pressure with respect to the dynamics and the potential.

Our starting point here is to prove that the Ruelle-Perron-Frobenius operator $\mathcal{L}_{f,\phi}$ associated to local diffeomorphisms is always differentiable with respect to the dynamics and potential as an a linear operator in $L(C^r(M, \mathbb{R}), C^{r-1}(M, \mathbb{R}))$ for $r \geq 1$. In fact we also deduce a chain-rule like formula for the derivative of the transfer operators $\mathcal{L}^n_{f,\phi}$ in the space of linear operators as above. Then we prove that topological pressure $P_{\text{top}}(f, \phi)$ and equilibrium states $\mu_{f,\phi}$ can be obtained as limits involving $\mathcal{L}^n_{f,\phi}(1)$ of the $(\mathcal{L}^n_{f,\phi})^*\eta$, where $\mathcal{L}^*_{f,\phi}$ stands for the dual of the transfer operator and $\eta$ is a fixed probability measure. Therefore the program to prove differentiability of many thermodynamical quantities has been carried out with success by proving that these are limit of differentiable objects with uniform derivatives. To the best of our knowledge these are the first differentiability
formulas for the topological pressure and equilibrium states for multidimensional non-uniformly expanding maps.

The second main purpose in this article is to obtain some stability of the limit laws with respect to the dynamics and potential. On the one hand, the spectral gap property and the differentiability of the topological pressure is well known to imply a central limit theorem and a local large deviations principle. On the other hand, we prove that the correlation function associated to the maximal entropy measure is smooth with respect to the dynamical system $f$ and convergent to zero. In consequence, we deduce that the mean and variance in the central limit theorem vary smoothly with respect to $f$. Continuity results are obtained for more general equilibrium states. In addition, we use a continuous inverse mapping theorem for fibered maps to deduce that the large deviations rate function vary continuously with respect to the dynamical system and potential. To the best of our knowledge the previous results are new even in the uniformly hyperbolic setting.

This paper is organized as follows. In Section 2, we describe the setting of our results and state our main results on the regularity of the thermodynamical quantities and the applications to the stability of the limit laws. Some preliminary results are given in Section 3 while our main results concerning the differentiability of topological pressure, conformal measures and equilibrium states are proven in Section 4. In Sections 5.1 and 5.2 we prove that the correlation function is convergent to zero in the $C^1$-topology and obtain the differentiability of mean and variance in the central limit theorem. A local large deviations principle and the regularity of the rate function is discussed in Section 5.3. Finally, some applications and examples are discussed in Section 6.

2. Statement of the main results

2.1. Setting. In this section we introduce some definitions and establish the setting. Let $M$ be compact and connected Riemannian manifold of dimension $m$ with distance $d$. Let $f : M \rightarrow M$ be a local homeomorphism and assume that there exists a function $x \mapsto L(x)$ such that, for every $x \in M$ there is a neighborhood $U_x$ of $x$ so that $f_x : U_x \rightarrow f(U_x)$ is invertible and

$$d(f_x^{-1}(y), f_x^{-1}(z)) \leq L(x) d(y, z), \quad \forall y, z \in f(U_x).$$

In particular every point has the same finite number of preimages $\deg(f)$ which coincides with the degree of $f$.

For all our results we assume that $f$ satisfies conditions (H1) and (H2) below. Assume there are constants $\sigma > 1$ and $L \geq 1$, and an open region $\mathcal{A} \subset M$ such that

(H1) $L(x) \leq L$ for every $x \in \mathcal{A}$ and $L(x) < \sigma^{-1}$ for all $x \notin \mathcal{A}$, and $L$ is close to 1: the precise condition is given in (3.1) and (3.2).

(H2) There exists a finite covering $\mathcal{U}$ of $M$ by open domains of injectivity for $f$ such that $\mathcal{A}$ can be covered by $q < \deg(f)$ elements of $\mathcal{U}$.

The first condition means that we allow expanding and contracting behavior to coexist in $M$: $f$ is uniformly expanding outside $\mathcal{A}$ and not too contracting inside $\mathcal{A}$. In the case that $\mathcal{A}$ is empty then $f$ is uniformly expanding. The second condition requires that every point has at least one preimage in the expanding region.
An observable \( g : M \to \mathbb{R} \) is \( \alpha \)-Hölder continuous if the Hölder constant

\[
|g|_\alpha = \sup_{x \neq y} \frac{|g(x) - g(y)|}{d(x, y)^\alpha}
\]
is finite. As usual, given \( r \in \mathbb{N}_0 \) and \( \alpha \in (0, 1) \) we endow the space \( C^{r+\alpha}(M, \mathbb{R}) \) of \( C^r \) observables \( g \) such that \( D^r g \) is \( \alpha \)-Hölder continuous with the norm \( \| \cdot \|_{r, \alpha} = \| \cdot \|_r + | \cdot |_\alpha \). We write for simplicity \( \| \cdot \|_\alpha \) for the case that \( r = 0 \). Throughout, we let \( \phi : M \to \mathbb{R} \) denote a potential at least Hölder continuous and satisfying either

\[
(P) \sup \phi - \inf \phi < \varepsilon_\phi \quad \text{and} \quad |e^{\phi}|_{\alpha} < \varepsilon_\phi \ e^{\inf \phi}
\]
provided that \( \phi \) is \( \alpha \)-Hölder continuous, or

\[
(P') \sup \phi - \inf \phi < \varepsilon_\phi \quad \text{and} \quad \max_{s \leq r} \|D^s \phi\|_0 < \varepsilon_\phi
\]
if \( \phi \) is \( C^r \), where \( \varepsilon_\phi > 0 \) depends only on \( L, \sigma, q, \deg(f) \), \( r \), a positive integer \( m \) and small \( \delta > 0 \) stated precisely in \([CV13]\) (see equations (3.1) and (3.2) below).

These are open conditions on the set of potentials, satisfied by constant potentials. In particular we can consider measures of maximal entropy and equilibrium states associated to potentials \( \beta \phi \) with \( \phi \) at least Hölder continuous and \( \beta \) small, which in the physics literature is known as the high temperature setting.

Throughout the paper we shall denote by \( \mathcal{F} \) an open set of local homeomorphisms with Lipschitz inverse and \( \mathbb{W} \) be some family of Hölder continuous potentials satisfying (H1), (H2) and (P) with uniform constants. Moreover, we shall denote by \( \mathcal{F}^{r+\alpha} \) an open set of \( C^{r+\alpha} \) local diffeomorphisms such that (H1) and (H2) hold with uniform constants and their inverse branches are \( C^{r+\alpha} \), and \( \mathbb{W}^{r+\alpha} \) to denote an open set of \( C^{r+\alpha} \) potentials such that (P) or (P') holds. We notice that the higher regularity of the dynamics is required for the inverse branches which are related with the Perron-Frobenius operator.

We shall always use the term differentiable to mean \( C^1 \)-differentiable.

### 2.2. Strong statistical properties of equilibrium states.

Let us first introduce some necessary definitions and collect from \([VV10]\) [CV13] some of the known results on the existence and statistical properties of equilibrium states for this robust class of transformations. Given a continuous map \( f : M \to M \) and a potential \( \phi : M \to \mathbb{R} \), the variational principle for the pressure asserts that

\[
P_{\text{top}}(f, \phi) = \sup \left\{ h_\mu(f) + \int \phi \, d\mu : \mu \text{ is } f \text{-invariant} \right\}
\]
where \( P_{\text{top}}(f, \phi) \) denotes the topological pressure of \( f \) with respect to \( \phi \) and \( h_\mu(f) \) denotes the metric entropy. An equilibrium state for \( f \) with respect to \( \phi \) is an invariant measure that attains the supremum in the right hand side above.

In our setting equilibrium states arise as invariant measures absolutely continuous with respect to an expanding, conformal and non-lacunary Gibbs measure \( \nu \). Since we will not use these notions here we shall refer the reader to \([VV10]\) for precise definitions and details. Many important properties arise from the study of transfer operators. We consider the Ruelle-Perron-Frobenius transfer operator \( \mathcal{L}_{f, \phi} \) associated to \( f : M \to M \) and \( \phi : M \to \mathbb{R} \) as the linear operator defined on a Banach space \( X \subset C^0(M, \mathbb{R}) \) of continuous functions \( g : M \to \mathbb{R} \) and given by

\[
\mathcal{L}_{f, \phi}(g)(x) = \sum_{f(y) = x} e^{\phi(y)} g(y).
\]
Since $f$ is a local homeomorphism it is clear that $L_{f,\phi}g$ is continuous for every continuous $g$ and, furthermore, $L_{f,\phi}$ is indeed a bounded operator relative to the norm of uniform convergence in $C^0(M,\mathbb{R})$ because $\|L_{f,\phi}\| \leq \deg(f) e^{\sup|\phi|}$. Analogously, $L_{f,\phi}$ preserves the Banach space $C^{r+\alpha}(M,\mathbb{R})$, with $r+\alpha > 0$, provided that $\phi$ is $C^{r+\alpha}$. Moreover, it is not hard to check that $L_{f,\phi}$ is a bounded linear operator in the Banach space $C^r(M,\mathbb{R}) \subset C^0(M,\mathbb{R})$ ($r \geq 1$) endowed with the norm $\| \cdot \|_r$
 whenever $f$ is a $C^r$-local diffeomorphism and $\phi \in C^r(M,\mathbb{R})$.

We say that the Ruelle-Perron-Frobenius operator $L_{f,\phi}$ acting on a Banach space $X$ has the spectral gap property if there exists a decomposition of its spectrum $\sigma(L_{f,\phi}) \subset \mathbb{C}$ as follows: $\sigma(L_{f,\phi}) = \{\lambda_1\} \cup \Sigma_1$ where $\lambda_1$ is a leading eigenvalue for $L_{f,\phi}$ with one-dimensional associated eigenspace and there exists $0 < \lambda_0 < \lambda_1$ such that $\Sigma_1 \subset \{z \in \mathbb{C} : |z| < \lambda_0\}$. When no confusion is possible, for notational simplicity we omit the dependence of the Perron-Frobenius operator on $f$ or $\phi$. We build over the following result which is a consequence of the results in [VV10] [CV13].

**Theorem 2.1.** Let $f : M \to M$ be a local homeomorphism with Lipschitz continuous inverse satisfying (H1) and (H2), and let $\phi : M \to \mathbb{R}$ be a Hölder continuous potential such that (P) holds. Then

1. there exists a unique equilibrium state $\mu_{f,\phi}$ for $f$ with respect to $\phi$, it is expanding, exact and absolutely continuous with respect to some conformal, non-lacunary Gibbs measure $\nu_{f,\phi}$;
2. the Ruelle-Perron-Frobenius has a spectral gap property in the space of Hölder continuous observables and the density $d\mu_{f,\phi}/d\nu_{f,\phi}$ is Hölder;
3. $P_{\text{top}}(f, \phi) = \log \lambda_{f,\phi}$, where $\lambda_{f,\phi}$ is the spectral radius of the Ruelle-Perron-Frobenius;
4. the topological pressure function $F \times W \ni (f, \phi) \mapsto P_{\text{top}}(f, \phi)$ is continuous;
5. the invariant density function $F \times W \to C^\alpha(M,\mathbb{R})$ given by $(f, \phi) \mapsto d\mu_{f,\phi}/d\nu_{f,\phi}$ is continuous whenever $C^\alpha(M,\mathbb{R})$ is endowed with the $C^0$ topology.

If, in addition, the potential $\phi : M \to \mathbb{R}$ is $C^r$-differentiable and satisfies (P') then

1. the topological pressure $F^r \times W^r \ni (f, \phi) \mapsto P_{\text{top}}(f, \phi)$ and the invariant density function $F^r \times W^r \to C^r(M,\mathbb{R})$ given by $(f, \phi) \mapsto d\mu_{f,\phi}/d\nu_{f,\phi}$ vary continuously in the $C^r$ topology;
2. the conformal measure function $F^r \times W^r \to \mathcal{M}(M)$ given by $(f, \phi) \mapsto \nu_{f,\phi}$ is continuous in the weak$^*$ topology. In consequence, the equilibrium measure $\mu_{f,\phi}$ varies continuously in the weak$^*$ topology;

Let us mention that condition (1) above holds more generally for all Hölder continuous potentials such that $\sup \phi - \inf \phi < \log \deg(f) - \log q$ (see [VV10] Theorem A). The aforementioned results lead to the natural questions about the regularity of some thermodynamical quantities as the topological pressure, equilibrium states, Lyapunov exponents, entropy, the central limit theorem, the large deviation rate function or the correlation function when one perturbs the dynamics $f$ or the potential $\phi$. Our purpose in the present paper is to address these questions in this non-uniformly expanding context.

### 2.3. Statement of the main results.

#### 2.3.1. Spectral theory of transfer operators.
Our first result adresses the problem of the regularity of the transfer operators with respect to the dynamical system given
by a local diffeomorphism. As discussed before, in general the Koopman operator
when acting in the space of $C^{r+\alpha}$-observables is not differentiable with respect
to the dynamics in the operator norm topology. This implies also on the lack of
differentiablility for the transfer operators. Nevertheless, we get the following:

**Theorem A.** (Differentiability of transfer operator) Let $M$ be a compact connected
Riemannian manifold and $\phi \in C^r(M, \mathbb{R})$ be any fixed potential, with $r \geq 1$. Then
the map
$$\text{Diff}_\text{loc}(M) \rightarrow L(C^r(M, \mathbb{R}), C^{r-1}(M, \mathbb{R}))$$
is differentiable.

In general we can only expect pointwise continuity of the transfer operators
acting on the space of $C^r$-observables. More precisely, given a fixed observable
$g \in C^r(M, \mathbb{R})$ the map $f \mapsto \mathcal{L}_{f,\phi}(g)$ is continuous. Moreover, we refer the reader to
Section 6 for an explicit example where the transfer operator in not even pointwise
continuous when acting on the space of Hölder continuous observables.

For the time being let us focus on the regularity of the transfer operators as
$\mathcal{L}_{f,\phi} : C^{r+\alpha}(M, \mathbb{R}) \rightarrow C^{r+\alpha}(M, \mathbb{R})$ on the potential $\phi$ and deduce the analyticity of
spectral radius, leading eigenfunction and eigenmeasure, and the equilibrium state
when the dynamics $f$ is fixed.

**Theorem B.** Assume $r + \alpha > 0$. Let $f : M \rightarrow M$ be a local homeomorphism with
$C^{r+\alpha}$ inverse branches satisfying (H1) and (H2) and let $\mathcal{W}^{r+\alpha} \subset C^{r+\alpha}(M, \mathbb{R})$ be
an open subset of Hölder continuous potentials $\phi : M \rightarrow \mathbb{R}$ such that either (P)
holds (in the case $r = 0$) or (P') holds (in the case $r > 0$) with uniform constants.
Then the following functions are analytic:

1. The Ruelle-Perron-Frobenius operator $C^{r+\alpha}(M, \mathbb{R}) \ni \phi \mapsto \lambda_\phi \in L(C^{r+\alpha}(M, \mathbb{R}))$;
2. The spectral radius function $\mathcal{W}^{r+\alpha} \ni \phi \mapsto \lambda_\phi = \exp(P_{\text{top}}(f, \phi))$;
3. The invariant density function $\mathcal{W}^{r+\alpha} \ni \phi \mapsto h_\phi \in C^{r+\alpha}(M, \mathbb{R})$;
4. The conformal measure function $\mathcal{W}^{r+\alpha} \ni \phi \mapsto \mu_{f,\phi} \in (C^{r+\alpha})^*$. In particular,
   for any fixed $g \in C^{r+\alpha}(M, \mathbb{R})$ the map $\phi \mapsto \int g \, d\mu_{f,\phi}$ is analytic;
5. The equilibrium state function $\mathcal{W}^{r+\alpha} \ni \phi \mapsto \mu_{f,\phi} = h_\phi \mu_{f,\phi} \in (C^{r+\alpha})^*$. In particular,
   for any fixed $g \in C^{r+\alpha}(M, \mathbb{R})$ the map $\phi \mapsto \int g \, d\mu_{f,\phi}$ is analytic.

First, let us mention that the previous differentiability results hold more generally
when there is a spectral gap for the transfer operator, by the operator perturbation
theory using the continuity of the transfer operators in terms of the potential $\phi$.
In addition, we obtain precise formulas for the first derivatives of the expressions
(ii)-(v) in the previous theorem along Proposition 4.4 to Proposition 5.7 and Corollary 4.8.
Since the topological pressure is given by the logarithm of the spectral
radius, obtain that for all $H \in C^{r+\alpha}(M, \mathbb{R})$
$$D_\phi P_{\text{top}}(f, \phi) \big|_{\phi_0} \cdot H = \int h_{f,\phi_0} \cdot H \, d\nu_{f,\phi_0} = \int H \, d\mu_{f,\phi_0}.$$
(i) The pressure function $P_{\text{top}}(\cdot, \phi) : \mathcal{F}^2 \to \mathbb{R}$ given by $f \mapsto P_{\text{top}}(f, \phi)$ is differentiable;

(ii) If $\phi \equiv 0$ then the maximal entropy measure function $\mathcal{F}^2 \ni f \mapsto \mu_f \in (C^2(M,\mathbb{R}))^*$ is differentiable. In particular, the map $\mathcal{F}^2 \ni f \mapsto \int g \, d\mu_f$ is differentiable for any fixed $g \in C^2(M,\mathbb{R})$.

(iii) If $\phi \equiv 0$ and $\mathcal{F}^2 \ni f \mapsto g_f \in C^2(M,\mathbb{R})$ is differentiable at $f_0$ then the map $\mathcal{F}^2 \ni f \mapsto \int g_f \, d\mu_f$ is differentiable at $f_0$.

In fact, our proof of the differentiability of the topological pressure with respect to the dynamics involves the analysis of the iterations of the transfer operator at the constant function one. For that reason it was necessary not only to prove differentiability as to obtain precise formulas for the first derivatives of the expressions above. Given $g \in C^1(M,\mathbb{R})$ fixed, we obtain an expression for the first derivative of $f \mapsto L_{f,\phi}(g)$, an important ingredient as the chain rule

$$D_f L^n_{f,\phi}(g)|_{f_0} \cdot H = \sum_{i=1}^{n} L_{f_0,\phi}^{-1}(D_f L_{f,\phi}(L_{f_0,\phi}^{-1}(g))|_{f_0}) \cdot H,$$

was obtained even without the differentiability of the transfer operator in the strong norm topology, and the expression for the derivative of $\mathcal{F}^2 \ni f \mapsto \int g \, d\mu_f$ acting in $H \in C^2(M,M)$ and given by

$$D_f \mu_f(g)|_{f_0} \cdot H = \sum_{i=0}^{\infty} \int D_f \hat{\mathcal{L}}_f(P_0(g)) \cdot H \, d\mu_0.$$ 

We omit the potential $\phi \equiv 0$ above for notational simplicity. Furthermore, since partial derivatives are continuous then the function $(f,\phi) \mapsto P_{\text{top}}(f,\phi)$ is differentiable. We refer the reader to Proposition 4.10, Lemma 4.14 and Theorem 4.15 for more details.

2.3.2. Applications: Stability and differentiability in dynamical systems. In this subsection we derive some interesting consequences on the stability of the robust class of non-uniformly expanding maps considered. The following is a consequence of Theorem B and Theorem C.

**Corollary A.** Given $r \geq 2$, let $\mathcal{F}^r$ and $\mathcal{W}^r$ be open sets of local diffeomorphisms and potentials as above. If $f \mapsto \phi_f \in \mathcal{W}^2$ is differentiable then the pressure function $\mathcal{F}^2 \ni f \mapsto P_{\text{top}}(f,\phi_f)$ is differentiable. In particular, if $\delta > 0$ is small then the pressure functions $\mathcal{F}^r \times (-\delta,\delta) \ni (f,t) \mapsto P_{\text{top}}(f,-t \log \|Df\| \pm 1)$ are differentiable.

As a byproduct of Theorems B and C we also obtain the regularity of the measure theoretical entropy, extremal Lyapunov exponents and sum of the positive Lyapunov exponents associated to the equilibrium states.

**Corollary B.** Assume that $r \geq 1$ and $\alpha > 0$. Then

$$\mathcal{F}^{r+\alpha} \times \mathcal{W}^{1+\alpha} \ni (f,\phi) \mapsto h_{\mu_f,\phi}(f) = P_{\text{top}}(f,\phi) - \int \phi \, d\mu_f,\phi$$

and the Lyapunov exponent functions

$$\mathcal{F}^{r+\alpha} \ni f \mapsto \int \log \|Df(x)\| \, d\mu_f,\phi \quad \text{and} \quad \mathcal{F}^{r+\alpha} \ni f \mapsto \int \log \|Df(x)^{-1}\|^{-1} \, d\mu_f,\phi$$
and

\[ F^{r+\alpha} \ni f \mapsto \int \log |\det Df(x)| \, d\mu_{f,\phi} \]

are continuous. Furthermore, if \( \phi \equiv 0 \) and \( r \geq 3 \) and \( \alpha \geq 0 \) then the previous functions vary differentiably with respect to the dynamics \( f \).

Other application of our results include a strong stability of the statistical laws. In [CV13] we deduced that this class of maps has exponential decay of correlations, which is well known to imply a Central Limit Theorem. To prove the stability of this limit theorem our first step is to prove that time-\( n \) correlation function with respect to the maximal entropy measure is differentiable with respect to \( f \) and its derivative converges to zero in the \( C^1 \)-topology. More precisely,

**Corollary C.** Given \( F^2 \) an open set of local diffeomorphisms and \( W^2 \) an open set of potentials as above, consider the correlation function

\[ C_{\varphi, \psi}(f, \phi, n) = \int (\varphi \circ f^n)\psi \, d\mu_{f,\phi} - \int \varphi \, d\mu_{f,\phi} \int \psi \, d\mu_{f,\phi} \]

defined for \( f \in F^2, \, \phi \in W^2 \), observables \( \varphi, \psi \in C^\alpha(M, \mathbb{R}) \) and \( n \in \mathbb{N} \). Then

i) The map \((f, \phi) \mapsto C_{\varphi, \psi}(f, \phi, n)\) is analytic in \( \phi \) and continuous in \( f \), and

ii) The map \( f \mapsto C_{\varphi, \psi}(f, 0, n) \) is differentiable, \( \partial_f C_{\varphi, \psi}(f, 0, n) \) is convergent to zero as \( n \to \infty \), and the convergence can be taken uniform in a small neighborhood of \( f \).

One should mention that property (i) above holds more generally, namely when we consider \( f \in F^{1+\alpha} \) and \( \phi \in W^{1+\alpha} \). The regularity of the correlation function also allow us to establish the regularity of the quantities involved in the central limit theorem with respect to the dynamics and potential. More precisely,

**Theorem D.** Let \( \phi \in W^2 \) and \( f \in F^2 \) be given. If \( \psi \in C^\alpha(M, \mathbb{R}) \) then:

i. either \( \psi = u \circ f - u + \int \psi \, d\mu_{f,\phi} \) for some \( u \in L^2(M, F, \mu_{f,\phi}) \) (we say \( \psi \) is cohomologous to a constant)

ii. or the convergence in distribution

\[ \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \psi \circ f^j \xrightarrow{\mathcal{D}} \frac{d}{n \to +\infty} N(m, \sigma^2) \]

holds with mean \( m = m_{f,\phi}(\psi) = \int \psi \, d\mu_{f,\phi} \) and variance \( \sigma^2 \) given by

\[ \sigma^2 = \sigma_{f,\phi}^2(\psi) = \int \tilde{\psi}^2 \, d\mu_{f,\phi} + 2 \sum_{n=1}^{\infty} \int \tilde{\psi}(\tilde{\psi} \circ f^n) \, d\mu_{f,\phi} > 0, \]

where \( \tilde{\psi} = \psi - \int \psi \, d\mu_{f,\phi} \) is a mean zero function depending on \((f, \phi)\).

Moreover, both functions \((f, \phi, \psi) \mapsto m_{f,\phi,\psi} \) and \((f, \phi, \psi) \mapsto \sigma_{f,\phi}^2(\psi) \) are analytic on \( \phi, \psi \) and continuous on \( f \). Finally, if \( \psi \in C^2(M, \mathbb{R}) \) and \( \phi \equiv 0 \) then \( (f, \psi) \mapsto m_f(\psi) \) and \((f, \psi) \mapsto \sigma_f^2(\psi) \) are differentiable.

Let us make some comments on the last result. Using the continuity of the variance \( \sigma_{f,\phi}^2(\psi) \) with respect to the dynamics \( f \), potential \( \phi \) and observable \( \psi \), and since the first case in the theorem above corresponds to the case that \( \sigma_{f,\phi}^2(\psi) = 0 \) then we obtain the following consequences of our results for the cohomological equation.
Corollary D. Let \( F^2 \) and \( W^2 \) be as above. Then, if \( \psi \) is not cohomologous to a constant for \( (f, \phi) \) then the same property holds for all close \((f, \bar{\phi})\). In consequence, the sets \( \{ (f, \phi) \in F^2 \times W^2 : \psi \) is cohomologous to \( \int \psi \, d\mu_{f, \phi} \} \) and \( \{ \psi \in C^2(M, \mathbb{R}) : \psi \) is cohomologous to \( \int \psi \, d\mu_{f, \phi} \} \) are closed.

Therefore, a particularly interesting open question is to understand if the sets defined above have empty interior, meaning that open and densely on the dynamical system and the potential any Hölder continuous observable would not be cohomologous to a constant.

Other consequence of the differentiability of the pressure function is related to a local large deviations principle. First we recall some notions. Given an observable \( \psi : M \to \mathbb{R} \) and \( t \in \mathbb{R} \) the free energy \( E_{f, \phi, \psi} \) is given by

\[
E_{f, \phi, \psi}(t) = \lim_{n \to \infty} \frac{1}{n} \log \int e^{t S_n} \, d\mu_{f, \phi},
\]

where \( S_n := \sum_{j=0}^{n-1} \psi \circ f^j \) is the usual Birkhoff sum. In our setting we will prove that the limit above does exist for all Hölder continuous \( \psi \) and \( |t| \leq t_{\phi, \psi} \), for some small \( t_{\phi, \psi} > 0 \). Moreover we study its regularity in the parameters \( t, \phi, \psi \) and \( f \).

Theorem E. Let \( \alpha > 0, f \in F^{1+\alpha} \) and \( \phi \in C^\alpha(M, \mathbb{R}) \) satisfy (H1), (H2) and (P). Then for any Hölder continuous observable \( \psi : M \to \mathbb{R} \) there exists \( t_{\phi, \psi} > 0 \) such that for all \( |t| \leq t_{\phi, \psi} \) the following limit exists

\[
E_{f, \phi, \psi}(t) := \lim_{n \to \infty} \frac{1}{n} \log \int e^{t S_n} \, d\mu_{f, \phi} = P_{\text{top}}(f, \phi + t\psi) - P_{\text{top}}(f, \phi).
\]

In consequence, \( E_{f, \phi}(t) \) is analytic in \( t, \phi \) and \( \psi \). Moreover, if \( \psi \) is cohomologous to a constant then \( t \mapsto E_{f, \phi, \psi}(t) \) is affine and, otherwise, \( t \mapsto E_{f, \phi, \psi}(t) \) is real analytic and strictly convex in \([-t_{\phi, \psi}, t_{\phi, \psi}]\). Furthermore, if \( \psi \in C^2(M, \mathbb{R}) \) then for every fixed \( |t| \leq t_{\phi, \psi} \) the function \( E_{f, \phi, \psi}(t) \) is continuous and if \( \phi \in W^2 \) we have that \( F^2 \ni f \mapsto E_{f, \phi, \psi}(t) \) is differentiable.

So, if \( \psi \) is not cohomologous to a constant then the function \([-t_{\phi, \psi}, t_{\phi, \psi}] \ni t \mapsto E_{f, \phi, \psi}(t) \) is strictly convex it is well defined the “local” Legendre transform \( I_{f, \phi, \psi} \) given by

\[
I_{f, \phi, \psi}(s) = \sup_{-t_{\phi, \psi} \leq t \leq t_{\phi, \psi}} \left\{ st - E_{f, \phi, \psi}(t) \right\}.
\]

Let us mention that local rate functions have also been used in [RY08] and let us refer the reader to Section 5.3 for more details. In fact, using differentiability of the pressure function we obtain a level-1 large deviation principle and deduce that stability of the rate function with the dynamical system. More precisely,

Theorem F. Let \( V \) be a compact metric space and \((f_v)_{v \in V}\) be a parametrized and injective family of maps in \( F^2 \) and let \( \phi \in C^2(M, \mathbb{R}) \) be a potential so that \( (P') \) holds. If the observable \( \psi \in C^2(M, \mathbb{R}) \) is not cohomologous to a constant then there exists an interval \( J \subset \mathbb{R} \) such that for all \( v \in V \) and \([a, b] \subset J\)

\[
\lim_{n \to \infty} \frac{1}{n} \log \mu_{f_v, \phi} \left( x \in M : \frac{1}{n} S_n \psi(x) \in [a, b] \right) \leq - \inf_{s \in [a, b]} I_{f_v, \phi, \psi}(s)
\]

and

\[
\liminf_{n \to \infty} \frac{1}{n} \log \mu_{f_v, \phi} \left( x \in M : \frac{1}{n} S_n \psi(x) \in (a, b) \right) \geq - \inf_{s \in (a, b)} I_{f_v, \phi, \psi}(s)
\]
If in addition $\psi \in C^2(M,\mathbb{R})$ then the rate function $(s,v) \mapsto I_{f,\phi,\psi}(s)$ is continuous on $J \times V$ in the $C^0$-topology.

![Figure 1. Continuity of the rate functions](image)

Let us mention that some upper and lower large deviation bounds for a larger class of transformations and the class of $C^0$-observables were obtained previously in [AP06, Va12]. The previous provides sharper results for Hölder continuous observables.

3. Preliminaries

In this section we provide some preparatory results needed for the proof of the main results. Namely, we recall some properties of the transfer operators.

3.1. Spectral radius of Ruelle-Perron-Frobenius operators and conformal measures. Let $\mathcal{L}_{f,\phi} : C^0(M,\mathbb{R}) \to C^0(M,\mathbb{R})$ be the Ruelle-Perron-Frobenius transfer operator associated to $f : M \to M$ and $\phi : M \to \mathbb{R}$ previously defined by

$$\mathcal{L}_{f,\phi}g(x) = \sum_{f(y) = x} e^{\phi(y)} g(y).$$

for every $g \in C^0(M,\mathbb{R})$. We consider also the dual operator $\mathcal{L}_{f,\phi}^* : \mathcal{M}(M) \to \mathcal{M}(M)$ acting on the space $\mathcal{M}(M)$ of Borel measures in $M$ by

$$\int g \, d(\mathcal{L}_{f,\phi}^* \eta) = \int (\mathcal{L}_{f,\phi} g) \, d\eta$$

for every $g \in C^0(M,\mathbb{R})$. Let $r(\mathcal{L}_{f,\phi})$ be the spectral radius of $\mathcal{L}_{f,\phi}$. In our context conformal measures associated to the spectral radius always exist. More precisely,

**Proposition 3.1.** Assume that $f$ satisfies assumptions (H1), (H2). If $\phi$ satisfies $\sup \phi - \inf \phi < \log \deg(f) - \log q$ then there exists a conformal measure $\nu = \nu_{f,\phi}$ such that $\mathcal{L}_{f,\phi}^* \nu = \lambda \nu$, where $\lambda = r(\mathcal{L}_{f,\phi})$. Moreover, $\nu$ is a non-lacunary Gibbs measure and $P_{top}(f,\phi) = \log \lambda$.

**Proof.** See Theorem B, Theorem 4.1 and Proposition 6.1 in [VV10].
3.2. Spectral gap for the transfer operator in $C^\alpha(M, \mathbb{R})$. Recall that the Hölder constant of $\varphi \in C^\alpha(M, \mathbb{R})$ is the least constant $C > 0$ such that $|\varphi(x) - \varphi(y)| \leq Cd(x, y)^\alpha$ for all points $x \neq y$. For any $\delta > 0$, the local Hölder constant $|\varphi|_{\alpha, \delta}$ is the corresponding notion for points $x, y$ such that $d(x, y) < \delta$. If $\delta$ is small then there exists a positive integer $m$ such that every $(C, \alpha)$-Hölder continuous map in balls of radius $\delta$ is globally $(Cm, \alpha)$-Hölder continuous (see [CV13, Lemma 3.5]). This put us in a position to state the precise relation on the constant $L, \sigma, q$ and $\varepsilon_\varphi$ on the hypothesis (H1), (P) and (P'). We assume:

$$e^{\varepsilon_\varphi} \cdot \left( \frac{(\deg(f) - q)\sigma^{-\alpha} + qL^\alpha[1 + (L - 1)^\alpha]}{\deg(f)} \right) + \varepsilon_\varphi 2mL^\alpha \text{diam}(M)^\alpha < 1$$

(3.1) and

$$[1 + \varepsilon_\varphi] \cdot e^{\varepsilon_\varphi} \cdot \left( \frac{(\deg(f) - q)\sigma^{-\alpha} + qL^\alpha[1 + (L - 1)^\alpha]}{\deg(f)} \right) < 1$$

(3.2)

This choice was taken to obtain the following cone invariance.

**Theorem 3.2.** Assume that $f$ satisfies (H1), (H2) and that $\varphi$ satisfies (P). Then there exists $0 < \hat{\lambda} < 1$ such that $\mathcal{L}_{f, \varphi}(\Lambda_{\kappa, \delta}) \subset \Lambda_{\lambda_{\kappa, \delta}}$ for every large positive constant $\kappa$, where

$$\Lambda_{\kappa, \delta} = \{ \varphi \in C^0(M, \mathbb{R}) : \varphi > 0 \text{ and } |\varphi|_{\alpha, \delta} \leq \kappa \inf \varphi \}.$$

is a cone of locally Hölder continuous observables. Moreover, given $0 < \hat{\lambda} < 1$, the cone $\Lambda_{\lambda_{\kappa, \delta}}$ has finite $\lambda_{\kappa, \delta}$-diameter in the projective metric $\Theta_{\kappa}$. Furthermore, if $\varphi \in \Lambda_{\kappa, \delta}$ satisfies $\int \varphi d\nu_{f, \varphi} = 1$ and $h_{f, \varphi}$ denotes the $\Theta_{\kappa}$-limit of $\varphi_n = L^\alpha_{\varphi}(\varphi)$ then, $\varphi_n$ converges exponentially fast to $h_{f, \varphi}$ in the Hölder norm.

**Proof.** See Theorem 4.1, Proposition 4.3 and Corollary 4.5 in [CV13].

Hence the normalized operator $\hat{\mathcal{L}}_{f, \varphi} = \lambda_{f, \varphi}^{-1}\mathcal{L}_{f, \varphi}$ has the spectral gap property.

**Theorem 3.3.** There exists $0 < r_0 < 1$ such that the operator $\hat{\mathcal{L}}_{f, \varphi}$ acting on the space $C^0(M, \mathbb{R})$ admits a decomposition of its spectrum given by $\Sigma = \{ 1 \} \cup \Sigma_0$, where $\Sigma_0$ contained in a ball $B(0, r_0)$. Furthermore, there exists $C > 0$ and $\tau \in (0, 1)$ such that $\| L^\alpha_{\varphi} - h_{f, \varphi} \|_\alpha \leq C\tau^n \| \varphi \|_\alpha$ for all $n \geq 1$ and $\varphi \in C^0(M, \mathbb{R})$, where $h_{f, \varphi} \in C^\alpha(M, \mathbb{R})$ is the unique fixed point for $\hat{\mathcal{L}}_{f, \varphi}$.

**Proof.** See Theorem 4.6, Proposition 4.3 and Corollary 4.5 in [CV13].

As a consequence of the previous results it follows that the density of the equilibrium state with respect to the corresponding conformal measure vary continuously in the $C^0$-norm. We recall the precise statement and the proof of the result since some estimates in the proof will be needed later on.

**Proposition 3.4.** Let $\mathcal{F}$ be a family of local homeomorphisms with inverse Lipschitz and $\mathcal{W}$ be a family of Hölder potentials as above. Then the topological pressure $\mathcal{F} \times \mathcal{W} \ni (f, \varphi) \mapsto \log \lambda_{f, \varphi} = P_{\text{top}}(f, \varphi)$ and the density function

$$\mathcal{F} \times \mathcal{W} \to (C^\alpha(M, \mathbb{R}), \| \cdot \|_0)$$

$$(f, \varphi) \mapsto \frac{d\nu_{f, \varphi}}{d\nu_{f, \varphi}}$$

are continuous. Moreover, $h_{f, \varphi} = \lim \lambda_{f, \varphi}^{-n} L_{f, \varphi}^n 1$ and the convergence is uniform in a neighborhood of $(f, \varphi)$. 

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Proof. Recall that $P_{\text{top}}(f, \phi) = \log \lambda_{f, \phi}$ where $\lambda_{f, \phi}$ is the spectral radius of the operator $L_{f, \phi}$. Moreover, it follows from the proof of Theorem 3.2 that for any $\phi \in \Lambda_{N, \delta}$ satisfying $\int \phi \, d\nu_{f, \phi} = 1$ one has in particular
\[
\left\| \frac{d\mu_{f, \phi}}{d\nu_{f, \phi}} - \frac{d\mu_{\tilde{f}, \phi}}{d\nu_{f, \phi}} \right\|_0 \leq C \tau^n
\]
for all $n \geq 1$. Notice the previous reasoning applies to $\phi \equiv 1 \in \Lambda_{N, \delta}$. Moreover, since the spectral gap property estimates depend only on the constants $L, \sigma$ and $\deg(f)$ it follows that all transfer operators $L_{f, \phi}$ preserve the cone $\Lambda_{N, \delta}$ for all pairs $(f, \phi)$ and that the constants $R_1$ and $\Delta$ can be taken uniform in a small neighborhood $U$ of $(f, \phi)$. Furthermore, one has that $\int \lambda_{f, \phi}^{-1} L_{f, \phi} \, d\nu_{f, \phi} = 1$ and so the convergence
\[
\lim_{n \to +\infty} \frac{1}{n} \log \| L_{f, \phi}^{n}(1) \|_0 = \lim_{n \to +\infty} \frac{1}{n} \log \| \lambda_{f, \phi}^{-n} L_{f, \phi}^{n}(1) \|_0 = 0
\]
given by Theorem 3.2 can be taken uniform in $U$. This is the key ingredient to obtain the continuity of the topological pressure and density function. Indeed, let $\varepsilon > 0$ be fixed and take $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} \log \| L_{f, \phi}^{n_0}(1) \|_0 - \log(\lambda_{f, \phi}) < \varepsilon$. This proves that $\frac{1}{n_0} \log \| L_{f, \phi}^{n_0}(1) \|_0 - \log(\lambda_{f, \phi}) < \frac{\varepsilon}{3}$ for all $\tilde{f} \in U$. Moreover, using $P_{\text{top}}(f, \phi) = \log \lambda_{f, \phi}$ by triangular inequality we get
\[
\left| P_{\text{top}}(f, \phi) - P_{\text{top}}(\tilde{f}, \phi) \right| \leq \left| \frac{1}{n_0} \log \| L_{f, \phi}^{n_0}(1) \|_0 - \log(\lambda_{f, \phi}) \right|
+ \left| \frac{1}{n_0} \log \| L_{f, \phi}^{n_0}(1) \|_0 - \log(\lambda_{f, \phi}) \right|
+ \left| \frac{1}{n_0} \log \| L_{f, \phi}^{n_0}(1) \|_0 - \frac{1}{n_0} \log \| L_{\tilde{f}, \phi}^{n_0}(1) \|_0 \right|
\]
Now, it is not hard to check that, for $n_0$ fixed, the function $U \to C^0(M, \mathbb{R})$
\[
\tilde{f} \mapsto L_{\tilde{f}, \phi}^{n_0}(1) = \sum_{\tilde{f}^n(y) = x} e^{S_{\nu_{\tilde{f}, \phi}}(y)}
\]
is continuous. Consequently, there exists a neighborhood $V \subset U$ of $f$ such that $\left| \frac{1}{n_0} \log \| L_{f, \phi}^{n_0}(1) \|_0 - \frac{1}{n_0} \log \| L_{\tilde{f}, \phi}^{n_0}(1) \|_0 \right| < \frac{\varepsilon}{3}$ for every $\tilde{f} \in V$. Altogether this proves that $\left| P_{\text{top}}(f, \phi) - P_{\text{top}}(\tilde{f}, \phi) \right| < \varepsilon$ for all $\tilde{f} \in V$. Since $\varepsilon$ was chosen arbitrary we obtain that both the leading eigenvalue and topological pressure functions vary continuously with the dynamics $f$. Finally, by equation (3.3) above applied to $\phi \equiv 1$ and triangular inequality we obtain that
\[
\left\| \frac{d\mu_{f, \phi}}{d\nu_{f, \phi}} - \frac{d\mu_{\tilde{f}, \phi}}{d\nu_{f, \phi}} \right\|_0 \leq 2C \tau^n + \left\| \lambda_{f, \phi}^{-n} L_{f, \phi}^{n} - \lambda_{\tilde{f}, \phi}^{-n} L_{\tilde{f}, \phi}^{n} \right\|_0
\]
for all $n$. Hence, proceeding as before one can make the right hand side above as close to zero as desired provided that $\tilde{f}$ is sufficiently close to $f$. This proves the continuity of the density function and finishes the proof of the proposition. \( \square \)

3.3. Spectral gap for the transfer operator in $C^r(M, \mathbb{R})$. Here we recall the analogous results for the action of the transfer operator in the space of smooth observables. In particular we have the corresponding spectral gap property for the action of Ruelle-Perron-Frobenius transfer operators in the space of smooth observables whose proof can be found in [CV13, Section 5].
Theorem 3.5. There exists $0 < r_0 < 1$ such that the operator $\tilde{L}_{f,\phi}$ acting on the space $C^r(M, \mathbb{R})$ ($r \geq 1$) admits a decomposition of its spectrum given by $\Sigma = \{1\} \cup \Sigma_0$, where $\Sigma_0$ contained in a ball $B(0, r_0)$. In consequence, there exists $C > 0$ and $\tau \in (0, 1)$ such that $\|\tilde{L}_{f,\phi}^n \varphi - h_{f,\phi} \int \varphi \, d\nu_f,\phi\|_r \leq C r^n \|\varphi\|_r$ for all $n \geq 1$ and $\varphi \in C^r(M, \mathbb{R})$, where $h_{f,\phi} \in C^r(M, \mathbb{R})$ is the unique fixed point for $\tilde{L}_{f,\phi}$.

Let us mention also that by Proposition 5.4 in [CV13] one has that

$$P_{\text{top}}(f, \phi) = \lim_{n \to \infty} \frac{1}{n} \log \|L_{f,\phi}^n\|_r$$

and that the limit can be taken uniform in a $C^r$ neighborhood of $(f, \phi)$. This is the key fact that will be used later on to prove the differentiability of the topological pressure.

4. DIFFERENTIABILITY RESULTS

In this section we address the regularity of the Perron-Frobenius operator, spectral radius and corresponding eigenmeasure and eigenfunction. For simplicity we address first the dependence on the potential and later on the dynamical system. In particular, Theorems [A] and [C] will be proven on Subsection 4.2 and 4.3 while Theorem [B] is proved along Subsection 4.1 below.

4.1. Differentiation with respect to the potential. First we fix $f$ and will focus on the differentiability questions with respect to the potential $\phi$. Let $\mathcal{W}$ be an open set of potentials in $C^\alpha(M, \mathbb{R})$, $\alpha > 0$, satisfying the condition (P), endowed with the $C^\alpha$-topology. For notational simplicity, when no confusion is possible, we write simply $\mathcal{L}_{f,\phi}$, $\lambda_{f,\phi}$ and $h_{\phi}$ omitting the dependence on $f$. The following results hold generally when the transfer operator has the spectral gap property.

Proposition 4.1. Assume that $r + \alpha > 0$. The map $C^{r+\alpha} \ni \phi \mapsto \mathcal{L}_{f,\phi}^n \in \mathcal{L}(C^{r+\alpha})$ is analytic, hence $C^\infty$. Moreover, for every vectors $g, H \in C^{r+\alpha}$ and for every $n \geq 1$, the first derivative acting in $H$ is given by

$$(D_\phi L_{f,\phi}^n(g))|_{\phi_0}(H) = \sum_{i=1}^n \mathcal{L}_{f,\phi_0}(H \cdot \mathcal{L}_{f,\phi_0}^{n-i}(g)).$$

Proof. Note that

$$\mathcal{L}_{f,\phi}(H) = \mathcal{L}_{f,\phi}(e^H g) = \sum_{i=0}^\infty \mathcal{L}_{f,\phi} \left( \frac{1}{i!} H^i g \right) = \mathcal{L}_{f,\phi}(g) + \sum_{i=1}^\infty \frac{1}{i!} \mathcal{L}_{f,\phi}(H^i g).$$

Let us denote by $\mathcal{L}_i^1(C^{r+\alpha}, C^{r+\alpha})$ the space of symmetric $i$-linear maps with domain in $[C^{r+\alpha}]^i$ into $C^{r+\alpha}$. Note also that the maps

$$C^{r+\alpha} \ni \phi \mapsto \left( H \mapsto \mathcal{L}_{f,\phi}(H^i) \right) \in \mathcal{L}_i^1(C^{r+\alpha})$$

are continuous for every $i \in \mathbb{N}$, and that the product between functions is also continuous in $C^{r+\alpha}$. Therefore for $k \in \mathbb{N}$, it follows that

$$\sup_{\|g\|_{r+\alpha} = 1} \|\mathcal{L}_{f,\phi+H}(g) - \mathcal{L}_{f,\phi}(g) - \sum_{i=1}^k \frac{1}{i!} \mathcal{L}_{f,\phi}(H^i g)\|_{r+\alpha} \leq \sum_{i=k+1}^{\infty} \sup_{\|g\|_{r+\alpha} = 1} \frac{\|\frac{1}{i!} \mathcal{L}_{f,\phi}(H^i g)\|_{r+\alpha}}{\|g\|_{r+\alpha} \|H\|_{r+\alpha}}$$

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which converges to zero as \( H \) tends to zero. By Theorem 1.4 in [Fr79], this implies that \( \phi \mapsto \mathcal{L}_{f,\phi} \) is \( C^k \), for any \( k \in \mathbb{N} \), and its \( k \)-th derivative applied in \( H \) is \( \mathcal{L}_{f,\phi}(H^t) \). Note that this also implies that \( \phi \mapsto \mathcal{L}_{f,\phi} \) is analytic. By applying the chain rule to the composition \( \phi \mapsto \mathcal{L}_{f,\phi}^n(g) \) we finish the proof of the proposition. □

Remark 4.2. Our previous argument implies in particular that for any given \( g \in C^\alpha(M, \mathbb{R}) \) the map \( t \mapsto \mathcal{L}_{f,t\phi} \) is real analytic since the previous argument shows that \( \mathcal{L}_{f,t\phi}(g) \) is also equal to the power series

\[
\sum_{i=0}^{\infty} \mathcal{L}_{f,\phi} \left( \frac{1}{i!} [(1-t)\phi]^i g \right) = \mathcal{L}_{f,\phi}(g) + \mathcal{L}_{f,\phi}((1-t)\phi g) + \sum_{i=2}^{\infty} \mathcal{L}_{f,\phi} \left( \frac{1}{i!} [(1-t)\phi]^i g \right)
\]

and the series is convergent.

Let us mention that [VV10] proved that the sequence \( \frac{1}{n} \sum_{j=0}^{n-1} f_j^* \nu_{f,\phi} \) converges to the unique equilibrium state \( \mu \). Here we deduce much stronger properties fundamental to the proof that the spectral radius of the Ruelle-Perron-Frobenius operator varies differentiably with respect to the potential \( \phi \). We show that \( (\mathcal{L}_{f,\phi}^n)^* \xi \) converge exponentially fast to \( \nu_{f,\phi} \) for any probability measure \( \xi \in \mathcal{M}(M) \). More precisely,

Proposition 4.3. There exists \( C > 0 \) and \( \tau \in (0,1) \) such that for every \( \varphi \in C^\alpha(M, \mathbb{R}) \) and every probability measure \( \xi \in \mathcal{M}(M) \) it holds that

\[
\left| \int \varphi \, d(\mathcal{L}_{f,\phi}^n)^* \xi - \int h_{f,\phi} \, \varphi \, d\xi \right| \leq C \tau^n \| \varphi \|_\alpha.
\]

Proof. The proof is a simple consequence of the spectral gap property. In fact,

\[
\left| \int \varphi \, d(\mathcal{L}_{f,\phi}^n)^* \xi - \int h_{f,\phi} \, d\xi \right| \leq \int \mathcal{L}_{f,\phi}^n(\varphi) - h_{f,\phi} \, d\xi
\]

\[
\leq \left\| \mathcal{L}_{f,\phi}^n(\varphi) - h_{f,\phi} \int \varphi \, d\nu_{f,\phi} \right\|_0 \leq C \tau^n \| \varphi \|_\alpha,
\]

where \( C \) and \( \tau \) are given by Theorem 3.3. This proves our proposition. □

Proposition 4.4. The spectral radius map \( \mathcal{W} \ni \phi \mapsto \lambda_{f,\phi} \) is analytic. Furthermore, given \( \phi_0 \in \mathcal{W} \) and \( H \in C^\alpha(M, \mathbb{R}) \) we have:

\[
D_{\phi} \lambda_{f,\phi} \big|_{\phi_0} \cdot H = \lambda_{f,\phi_0} \cdot \int h_{f,\phi_0} \cdot H \, d\nu_{f,\phi_0}.
\]

Proof. Note that it is immediate that \( \phi \mapsto \lambda_{f,\phi} \) is analytic, since \( \phi \mapsto \mathcal{L}_{f,\phi} \) is analytic (in the norm operator topology), and since the spectral radius of \( \mathcal{L}_{f,\phi} \) coincides with an isolated eigenvalue of \( \mathcal{L}_{f,\phi} \) with multiplicity one. Let us calculate explicitly the derivative of \( \phi \mapsto \lambda_{f,\phi} \).

Let \( \phi_0 \in \mathcal{W} \) be fixed. It follows from the \( C^0 \)-statistical stability statement in Proposition 3.3 that \( \lambda_{f,\phi_0}^{-n} \mathcal{L}_{f,\phi_0}^n(1) \to h_{f,\phi} \) and that the limit is uniform in a small neighborhood \( \mathcal{W} \) of \( \phi_0 \). Moreover, since \( \mathcal{L}_{f,\phi}^n(1)(x) \leq K \) for some constant \( K \) that can be taken uniform in \( \mathcal{W} \) it follows that \( h_{f,\phi} \) can be taken uniformly bounded from above for all \( \phi \in \mathcal{W} \). Since the sequence \( \mathcal{L}_{f,\phi}^n(1) \) is Cauchy in the projective metric it also follows that \( h_{f,\phi} \) can be taken uniformly bounded away from zero for all \( \phi \in \mathcal{W} \). In consequence, \( \lim_{n \to \infty} \frac{1}{n} \log \int \mathcal{L}_{f,\phi}^n \, d\nu_{f,\phi_0} = \log \lambda_{f,\phi} \) uniformly with
Furthermore, 

\[ \text{verges to } \phi \text{ which is uniformly convergent to zero with respect to } \phi \text{ and this convergence is uniform with respect to } \phi \text{ where the convergence is uniform with respect to } \phi \text{ which are well defined and converge to the constant } \log \lambda. \]

Taking into account Proposition 4.3 it follows that

\[ F_n(\phi) = \frac{1}{n} \log \int L_{f,\phi}^n d\nu_{f,\phi}, \]

which are well defined and converge to the constant \( \log \lambda_{f,\phi} \), and prove that the derivatives of \( F_n \) converge uniformly as \( n \) tends to infinity. Write \( DF_n(\phi) \cdot H \) as

\[ \frac{1}{n} \int D_{\phi} L_{f,\phi}^n(1)_{f,\phi} \cdot H d\nu_{f,\phi} = \int \frac{1}{n} \sum_{i=1}^{n} L_{f,\phi}^{n-i}(1) \cdot H d\nu_{f,\phi} \]

where \( A_n \), that uses the normalized operators \( \hat{L}_{f,\phi} = \lambda_{f,\phi}^{-1} L_{f,\phi} \), is given by

\[ A_n(\phi) \cdot H = \frac{1}{n} \sum_{i=1}^{n} \hat{L}_{f,\phi}^{n-i}(1) \cdot H d\nu_{f,\phi}. \]

Taking into account Proposition 4.3 it follows that

\[ |A_n(\phi) \cdot H - \frac{1}{n} \sum_{i=0}^{n-1} \int \hat{L}_{f,\phi}^{i}(1) \cdot H d\nu_{f,\phi} \cdot \int h_{f,\phi} d\nu_{f,\phi}| \]

\[ \leq \frac{1}{n} \sum_{i=1}^{n} \left| \int \hat{L}_{f,\phi}^{n-i}(1) \cdot H d\nu_{f,\phi} - \int \hat{L}_{f,\phi}^{n-i}(1) \cdot H d\nu_{f,\phi} \right| \]

\[ \leq \frac{1}{n} \sum_{i=1}^{n} C \tau^i \cdot \| \hat{L}_{f,\phi}^{n-i}(1) \cdot H \|_{\alpha} \leq \frac{1}{n} \sum_{i=1}^{n} 4C \tau^i \cdot (C \tau^{n-i} + \| h_{f,\phi} \|_{\alpha}) \cdot \| H \|_{\alpha}, \]

which is uniformly convergent to zero with respect to \( \phi \) and unitary \( H \in C^\alpha(M, \mathbb{R}) \). Furthermore,

\[ \frac{1}{n} \sum_{i=0}^{n-1} \int \hat{L}_{f,\phi}^{i}(1) \cdot H d\nu_{f,\phi} \rightarrow \int h_{f,\phi} \cdot H d\nu_{f,\phi} \text{ uniformly with respect to } \phi, \]

and this convergence is uniform with respect to \( \phi \in H \). Since \( \int \hat{L}_{f,\phi}^{n}(1) d\nu_{f,\phi} \) converges to \( \int h_{f,\phi} d\nu_{f,\phi} \) uniformly with respect to \( \phi \), we obtain that

\[ DF_n(\phi) \cdot H = \frac{A_n(\phi) \cdot H}{\int \hat{L}_{f,\phi}^{n}(1) d\nu_{f,\phi}} \rightarrow \int h_{f,\phi} \cdot H d\nu_{f,\phi}, \]

where the convergence is uniform with respect to \( \phi \) and \( H \in C^\alpha(M, \mathbb{R}) \) satisfying \( \| H \|_{\alpha} = 1 \). Now, just observe that \( e^{F_n(\phi)} \) is differentiable and uniformly convergent to \( \lambda_{f,\phi} \). Thus, as a consequence of the chain rule it follows that

\[ D_{\phi} \lambda_{f,\phi} |_{\phi_0} \cdot H = \lambda_{f,\phi_0} \cdot \int h_{f,\phi_0} \cdot H d\nu_{f,\phi_0}. \]

This finishes the proof of the proposition. \( \square \)

Since in our context it follows that \( P_{\text{top}}(f, \phi) = \log \lambda_{f,\phi} \) and the arguments in the later proof lead to the following immediate consequence:

**Corollary 4.5.** The map \( \mathcal{W} \ni \phi \mapsto P_{\text{top}}(f, \phi) \) is differentiable. Furthermore, given \( \phi_0 \in \mathcal{W} \) and \( H \in C^\alpha(M, \mathbb{R}) \) we have:

\[ D_{\phi} P_{\text{top}}(f, \phi) |_{\phi_0} \cdot H = \int h_{f,\phi_0} \cdot H d\nu_{f,\phi_0} = \int H d\mu_{f,\phi_0}. \]
From Proposition 3.3 the invariant density is Hölder continuous function and varies continuously in the $C^0$-topology. Here we show that it varies differentiably with respect to the potential.

**Proposition 4.6.** The map $W \ni \phi \mapsto h_{f,\phi} \in C^{r+\alpha}(M,\mathbb{R})$ is differentiable. Furthermore, given $\phi_0 \in W$ and $H \in C^{\alpha}(M,\mathbb{R})$ we have:

$$D_\phi h_{f,\phi} |_{\phi_0} \cdot H = h_{f,\phi_0} \cdot \int [(I - \hat{L}_{f,\phi_0}|_{E_0})^{-1}(1 - h_{f,\phi_0})] \cdot H \, d\nu_{f,\phi_0}. $$

**Proof.** Let $\phi_0 \in W$ be fixed. By the $C^0$-statistical stability Theorem we have that $\lambda_{f,\phi}^{-n}L_{f,\phi}^n \cdot 1$ converges uniformly to $h_{f,\phi}$ with respect to $\phi$ in some sufficiently small neighborhood $W$ of $\phi_0$. Hence, consider as before a family of functions $F_n : W \rightarrow C^0(M,\mathbb{R})$ given by $F_n(\phi) = \lambda_{f,\phi}^{-n}L_{f,\phi}^n \cdot 1$. We claim that the derivatives of $F_n$ are uniformly convergent. Indeed, taking into account Proposition 4.4 for the derivative of the spectral radius $\lambda_{f,\phi}$ one can write

$$DF_n(\phi) \cdot H = \lambda_{f,\phi}^{-n}D\phi L_{f,\phi}^n \cdot H - n\lambda_{f,\phi}^{-1}L_{f,\phi}^n \cdot D\phi \lambda_{f,\phi} \cdot H$$

$$= \left[ \sum_{i=1}^{n} \hat{L}_{f,\phi}^{i-1}(1) \cdot h_{f,\phi} \cdot H \, d\nu_{f,\phi} \right]. \quad (4.2)$$

We now prove that the later expression is uniformly convergent (with respect to $\phi$ and $\{H \in C^{\alpha}(M,\mathbb{R}) : \|H\|_{\alpha} \leq 1\}$) to the series $h_{f,\phi} \cdot \sum_{i=0}^{+\infty} \int (\hat{L}_{f,\phi}^{i-1}(1) \cdot H - h_{f,\phi} \cdot H) \, d\nu_{f,\phi}$. On the one hand,

$$h_{f,\phi} \cdot \sum_{i=0}^{n-1} \int (\hat{L}_{f,\phi}^{i-1}(1) \cdot H - h_{f,\phi} \cdot H) \, d\nu_{f,\phi} = h_{f,\phi} \cdot \int \sum_{i=0}^{n-1} \hat{L}_{f,\phi}^{i-1}(1 - h_{\phi}) \cdot H \, d\nu_{f,\phi}$$

which converges to $h_{f,\phi} \cdot \int [(I - \hat{L}_{f,\phi}|_{E_0})^{-1}(1 - h_{f,\phi})] \cdot H \, d\nu_{f,\phi}$ as $n$ tends to infinite. On the other hand, using (4.2) one can write

$$DF_n(\phi) \cdot H - h_{f,\phi} \cdot \sum_{i=0}^{n-1} \int [L_{f,\phi}^{i-1}(1) \cdot H - h_{f,\phi} \cdot H] \, d\nu_{f,\phi}$$

$$= \sum_{i=1}^{n} \hat{L}_{f,\phi}^{i-1}(1)H - \hat{L}_{f,\phi}^{n-i}(1) \int h_{f,\phi}H \, d\nu_{f,\phi} - h_{f,\phi} \int (\hat{L}_{f,\phi}^{n-i}(1)H - h_{f,\phi}H) \, d\nu_{f,\phi}$$

and is equal to a sum $\sum_{i=1}^{n} \hat{L}_{f,\phi}^{i}(\xi_n)$ where the functions $\xi_n$ have zero mean average with respect to the conformal measure $\nu_\phi$. Therefore it follows from Theorem 3.3

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that there exists $C > 0$ and $0 < \tau < 1$ such that
\[
\|DF_n(\phi) \cdot H - h_{f,\phi} \cdot \sum_{i=0}^{n-1} \int [\hat{L}^i_{f,\phi}(1) \cdot H - h_{f,\phi} \cdot H] \, d\nu_{f,\phi}\|_{\alpha}
\]
\[
\leq \sum_{i=1}^{n} C^i \|\hat{L}^{-i}(1)H - \hat{L}^{-i}(1)H - h_{f,\phi} \cdot \int (\hat{L}^{-i}(1)H - h_{f,\phi}H) \, d\nu_{f,\phi}\|_{\alpha}
\]
\[
\leq \sum_{i=1}^{n} C^i \|\hat{L}^{-i}(H \circ f^{-i}) - h_{f,\phi} \cdot \int \hat{L}^{-i}(H \circ f^{-i}) \, d\nu_{f,\phi}\|_{\alpha}
\]
\[
+ \sum_{i=1}^{n} C^i \|\hat{L}^{-i}(1) \cdot \int h_{f,\phi} \cdot H \, d\nu_{f,\phi} - h_{f,\phi} \cdot \int h_{f,\phi} \cdot H \, d\nu_{f,\phi}\|_{\alpha}
\]
\[
\leq \sum_{i=1}^{n} 4C^2 \tau^n \cdot \|H\|_{\alpha} \cdot (1 + \|h_{f,\phi}\|_{\alpha})^2,
\]
which is convergent to zero. Since all convergences above are uniform with respect to $\phi$ and $H \in C^\alpha(M,\mathbb{R})$ with $\|H\|_{\alpha} \leq 1$ the previous estimates prove our claim.

Now we can finish the proof of the proposition by estimating
\[
DF_n(\phi) \cdot H \to \int [(I - \hat{L}_{f,\phi})^{-1}(1 - h_{f,\phi})] \cdot H \, d\nu_{f,\phi} \cdot h_{f,\phi},
\]
uniformly with respect to $\phi$ and $H \in C^\alpha(M,\mathbb{R}), \|H\|_{\alpha} = 1$. Thus we deduce that
\[
D_{\phi} h_{\phi,\phi_0} \cdot H = h_{\phi,\phi_0} \cdot \int [(I - \hat{L}_{f,\phi_0})^{-1}(1 - h_{\phi,\phi_0})] \cdot H \, d\nu_{f,\phi_0},
\]
which finishes the proof of the proposition. 

Now we use the previous information to deduce that the conformal measures $\nu_{\phi}$ are differentiable. The precise statement is as follows:

**Proposition 4.7.** The map $\mathcal{W} \ni \phi \mapsto \nu_{f,\phi} \in (C^{r+\alpha})^*$ is differentiable. In particular, the map $\mathcal{W} \ni \phi \mapsto \int g \, d\nu_{f,\phi}$ is differentiable for any fixed $g \in C^\alpha(M,\mathbb{R})$.

Furthermore, given $\phi_0 \in \mathcal{W}$ and $H \in C^\alpha(M,\mathbb{R})$
\[
D_{\phi} \int g \, d\nu_{f,\phi,\phi_0} \cdot H = \int [(I - \hat{L}_{f,\phi_0})^{-1}(1 - h_{f,\phi_0})] \cdot H \, d\nu_{f,\phi_0}.
\]

**Proof.** Fix any $g \in C^\alpha(M,\mathbb{R})$ and $\phi_0 \in \mathcal{W}$. Since we deal with differentiability conditions it is enough to consider the sequence of functionals $F_n$ defined in a small neighborhood $W$ of $\phi_0$ by $F_n(\phi) := \int \hat{L}^n_{f,\phi} g \, d\nu_{f,\phi_0}$. If the neighborhood $W$ is small enough then we have that $F_n(\phi)$ converges uniformly to $\int g \, d\nu_{f,\phi}$, with respect the $\phi$ and $g$. Moreover, for any $H \in C^\alpha(M,\mathbb{R})$, it is not hard to check that
\[
DF_n(\phi) \cdot H = \frac{A_n(\phi)}{h_{f,\phi} d\nu_{f,\phi_0}} - B_n(\phi)
\]
where the formal series $A_n$ and $B_n$ are defined as
\[
A_n(\phi) \cdot H := \int \left[ \sum_{i=1}^{n} \hat{L}^i_{f,\phi}(\hat{L}^{-i}_{f,\phi}(g) \cdot H) - n \int h_{f,\phi} \cdot H \, d\nu_{f,\phi} \cdot \hat{L}^n_{f,\phi} g \right] d\nu_{f,\phi_0}.
\]
and
\[ B_n(\phi) \cdot H := \frac{\int (D\phi h_{f,\phi} |_\phi \cdot H) dv_{f,\phi_0} \cdot \int \hat{L}_{f,\phi}^n (g) dv_{f,\phi_0}}{(\int h_{f,\phi} dv_{f,\phi_0})^2} \]
respectively. We proceed to establish the uniform convergence of these formal series. First we note that as before one can write
\[ A_n(\phi) \cdot H - \sum_{i = 0}^{n-1} \left( \hat{L}_{f,\phi}^i (g) \cdot H - \int gdv_{f,\phi} \cdot h_{f,\phi} \cdot H \right) dv_{f,\phi} \cdot \int h_{f,\phi} dv_{f,\phi_0} \]
\[ = \sum_{i = 1}^{n} \int \hat{L}_{f,\phi}^i \left( \hat{L}_{f,\phi}^{n-i} (g) H - \hat{L}_{f,\phi}^{n-i} (g) \int h_{f,\phi} H \right) dv_{f,\phi} \cdot \int h_{f,\phi} dv_{f,\phi_0}. \]

Thus, proceeding as in the proof of Proposition 4.6 we get that
\[ \left| A_n(\phi) \cdot H - \sum_{i = 0}^{n-1} \left( \hat{L}_{f,\phi}^i (g) \cdot H - \int gdv_{f,\phi} \cdot h_{f,\phi} \cdot H \right) dv_{f,\phi} \cdot \int h_{f,\phi} dv_{f,\phi_0} \right| \]
is bounded from above by
\[ \sum_{i = 1}^{n} C_\tau \| \hat{L}_{f,\phi}^{n-i} (g) H - \hat{L}_{f,\phi}^{n-i} (g) \int h_{f,\phi} H \| dv_{f,\phi} \]
\[ - \int (\hat{L}_{f,\phi}^{n-i} (g) H - \int gdv_{f,\phi} h_{f,\phi} H) dv_{f,\phi} h_{f,\phi} \|_\alpha \]
\[ \leq \sum_{i = 1}^{n} C_\tau \| \hat{L}_{f,\phi}^{n-i} (g H \circ f^{n-i}) - h_{\phi} \int \hat{L}_{f,\phi}^{n-i} (g H \circ f^{n-i}) dv_{f,\phi} \|_\alpha \]
\[ + \sum_{i = 1}^{n} C_\tau \| \hat{L}_{f,\phi}^{n-i} (g) \int h_{f,\phi} H \| dv_{f,\phi} \| - \int gdv_{f,\phi} \int h_{f,\phi} H \ dv_{f,\phi} h_{f,\phi} \|_\alpha \]
\[ \leq \sum_{i = 1}^{n} 8C_\tau^2 \tau^n \| g \|_\alpha \cdot \| H \|_\alpha \cdot (1 + \| h_{f,\phi} \|_\alpha)^2, \]
which is uniformly convergent to zero with respect to \( \phi \) and unitary vectors \( H \in C^\alpha (M, \mathbb{R}) \) and \( g \in C^\alpha \). Thus we get that \( A_n(\phi) \cdot H \) is uniformly convergent to
\[ \sum_{i = 0}^{+\infty} \int \left( \hat{L}_{f,\phi}^i (g) \cdot H - \int gdv_{f,\phi} \cdot h_{f,\phi} \cdot H \right) dv_{f,\phi} \cdot \int h_{f,\phi} dv_{f,\phi_0}. \quad (4.4) \]

As for \( B_n \) it is not hard to check that \( B_n(\phi) \cdot H \) is uniformly convergent to
\[ \frac{\int (D\phi h_{f,\phi} |_\phi \cdot H) dv_{f,\phi_0} \cdot \int gdv_{f,\phi}}{\int h_{f,\phi} dv_{f,\phi_0}} = \sum_{i = 0}^{+\infty} \int (\hat{L}_{f,\phi}^i (1) \cdot H - \int h_{f,\phi} \cdot H) dv_{f,\phi} \cdot \int gdv_{f,\phi} \quad (4.5) \]
uniformly with respect to \( \phi \) and unitary vectors \( H \in C^\alpha (M, \mathbb{R}) \) and \( g \in C^\alpha \). Therefore the result follows by combining equality (4.3) above with the limit series (4.4) and (4.5) since \( DF_n(\phi) \cdot H \) is uniformly convergent with respect to \( \phi \) and
unitary vectors $H \in C^\alpha(M, \mathbb{R})$ and $g \in C^\alpha(M, \mathbb{R})$ to the expression
\[
\sum_{i=0}^{+\infty} \int \left[ \hat{L}_{f,\phi}(g) \cdot \hat{L}_{f,\phi}(f) \cdot H \right] d
\]
\[
= \int \left[ (I - \hat{L}_{f,\phi}) (g - \int g df_{f,\phi}) \right] \cdot H df_{f,\phi}.
\]
Thus we get as claimed
\[
D_{f,\phi} \int g df_{f,\phi} \cdot H = \int [(I - \hat{L}_{f,\phi})^{-1}(g - \int g df_{f,\phi})] \cdot H df_{f,\phi}.
\]

We will now deduce the differentiability of the equilibrium states $\mu_\phi$ with respect to the potential $\phi$. In fact, using that $\mu_{f,\phi} = h_{f,\phi} \nu_{f,\phi}$ the following consequence is immediate from our previous two differentiability results.

**Corollary 4.8.** The map $W \ni \phi \mapsto \mu_{f,\phi} \in (C^{r+\alpha})^*$ is differentiable and, consequently, $W \ni \phi \mapsto \int g d\mu_{f,\phi}$ is differentiable for any fixed $g \in C^\alpha(M, \mathbb{R})$. Furthermore, given $\phi_0 \in W$ and $H \in C^\alpha(M, \mathbb{R})$
\[
D_{f,\phi} \int g df_{f,\phi,\phi_0} \cdot H = \int [(I - \hat{L}_{f,\phi})^{-1}(g - \int g df_{f,\phi,\phi_0})] \cdot H df_{f,\phi,\phi_0}.
\]

Now, notice that it follows from our previous results that $h_{f,\phi,\nu_{f,\phi}}$ and $\mu_{f,\phi}$ vary analytically with respect to the potential $\phi$ by the explicit power series formulas obtained in the previous results. This finishes the proof of Theorem 4.7.

**4.2. Differentiability of the topological pressure with respect to the dynamics.** In this subsection we prove the differentiability of the topological pressure using the differentiability of inverse branches for the dynamics. More precisely,

**Lemma 4.9.** (Local Differentiability of inverse branches) Let $r \geq 1$, $0 \leq k \leq r$ and $f : M \to M$ be a $C^r$-local diffeomorphism on a compact connected manifold $M$. Let $B = B(x, \delta) \subset M$ some ball such that the inverse branches $f_1, \ldots, f_s : B \to M$ are well defined diffeomorphisms onto their images. Then $C^r(M, M) \ni f \mapsto (f_1, \ldots, f_s) \in C^{r-k}$ is a $C^k$ map.

**Proof.** Let $F : C^r(M, M) \times [C^{r-k}(B, M)]^s \to [C^{r-k}(B; M)]^s$ given by
\[
F(h, h_1, \ldots, h_s) = (h \circ h_1, \ldots, h \circ h_s).
\]

Note that $F$ is $C^k$. In fact, on one hand, $\partial_h F$ is $C^\infty$ (in suitable charts, we see it as a continuous linear map in $h$). By Theorem 4.2 and Corollary 4.2 in [FR79], the composition $h \mapsto h \circ h_i$ is differentiable and a precise expression for the derivative is given. In our setting, for an increment $H = (H_1, \ldots, H_s) \in T(C^{r-k}(B; M))^s$, we obtain that $\partial_{h_i} F \cdot H_i = h' \circ h_i \cdot H_i$, which is clearly a $C^k$ map.

Note that $F(f, f_1, \ldots, f_s) = (id, \ldots, id)$. For the point $(f, f_1, \ldots, f_s)$, we have that $\partial_{h_i} F(f, f_1, \ldots, f_s) \cdot H = (f' \circ h_1 \cdot H_1, \ldots, f' \circ h_s \cdot H_s)$, is an isomorphism, since $\tilde{f}$ is a local diffeomorphism and so $[f' \circ f_j(x)]$ is invertible, for any $x \in M$. Therefore, by Implicit Function Theorem, we obtain that the map $G : C^r(M, M) \to...
Proposition 4.10. (Differentiability of transfer operator) Let \( r \geq 1, f : M \rightarrow M \) be a \( C^r \) local diffeomorphism on a compact connected manifold \( M \) and \( \phi \in C^r(M, \mathbb{R}) \) be any fixed potential. The map

\[
\text{Diff}_{loc}(M) \quad \rightarrow \quad L(C^r(M, \mathbb{R}), C^{r-1}(M, \mathbb{R}))
\]

is differentiable.

Proof. Let \( \{ \varphi_j : j = 1, \ldots, l \} \) be a \( C^\infty \) partition of unity associated to some finite covering \( B_1, \ldots, B_l \) of \( M \) by balls with radius smaller or equal to \( \delta > 0 \) and define the auxiliar operators \( \mathcal{L}_j = \mathcal{L}_{j, f, \phi} := \mathcal{L}_{f, \phi} \cdot \varphi_j \). In particular it holds that \( \mathcal{L}_{f, \phi} = \sum_{j=1}^l \mathcal{L}_j \). Therefore, all we need to prove is that any auxiliar operator \( \mathcal{L}_j \) is differentiable.

We claim first that \( f \mapsto \mathcal{L}_j(g) \) is differentiable, for any fixed \( g \in C^r(M, \mathbb{R}) \). Recall that \( \varphi_j \) vanishes outside the ball \( B_j \). We write \( f_1, \ldots, f_s \) for the inverse branches of \( f \) in \( B_j \), and also \( T_i = \partial_{f_i} f_1 \), for \( i = 1, \ldots, s \). Therefore, by a slight abuse of notation, since \( \mathcal{L}_j(g)|_{\partial B_j} = 0 \) we have

\[
\mathcal{L}_j(g) = \sum_{i=1}^s g(f_i) \cdot e^\varphi(f_i) \varphi_j,
\]

which implies that \( \partial_{f_i} \mathcal{L}_j(g) \cdot H = \sum_{i=1}^s (g \cdot e^\varphi)' \circ f_i \cdot [T_i \cdot H] \cdot \varphi_j \) does exist. Since \( C^r(M, \mathbb{R}) \ni g \mapsto \partial_{f_i} \mathcal{L}_j(g) \in C^{r-1}(M, \mathbb{R}) \) is linear, continuous and the rest of differentiability depends continuously of \( g \), this finishes the proof of the lemma.

Proposition 4.11. Let \( r \geq 1 \) and \( \phi, g \in C^r(M, \mathbb{R}) \) be fixed. Then, the map \( \text{Diff}_{loc} \ni f \mapsto \mathcal{L}^n_{f, \phi} \in L(C^r(M, \mathbb{R}), C^{r-1}(M, \mathbb{R})) \) is differentiable. Furthermore, given \( H \in C^r(M, M) \), \( g_1, g_2 \in C^r(M, \mathbb{R}) \) and \( t \in \mathbb{R} \) it holds

\begin{align*}
\text{i)} & \quad D_f \mathcal{L}^n_{f, \phi}(g)|_{f_0} \cdot H = \sum_{i=1}^n \mathcal{L}^{i-1}_{f, \phi}(D_f \mathcal{L}^{i-1}_{f, \phi}(g)|_{f_0} \cdot H); \\
\text{ii)} & \quad \text{there exists } c_{f, \phi} > 0 \text{ so that } \|D_f \mathcal{L}_{f, \phi}(g)|_{f_0} \cdot H\| \leq c_{f, \phi} \|g\|_1 \|H\|; \\
\text{iii)} & \quad D_f \mathcal{L}^n_{f, \phi}(g_1 + t g_2)|_{f_0} \cdot H = D_f \mathcal{L}^n_{f, \phi}(g_1)|_{f_0} \cdot H + t D_f \mathcal{L}^n_{f, \phi}(g_2)|_{f_0} \cdot H; \\
\text{iv)} & \quad \text{if } \phi = 0, \text{ then } D_f \mathcal{L}^n_{f}(1)|_{f_0} \cdot H = 0.
\end{align*}

Proof. Property i) is obtained by induction as follows. The case \( n = 1 \) follows from the previous proposition. Now suppose the formula is valid for \( n \). Using the induction assumption and the fact that \( \mathcal{L}^n_{f_0 + H, \phi}(g) \) also belongs in \( C^r(M, \mathbb{R}) \) we
obtain for all fixed diffeomorphism \( f_0 \) and \( H \in C^r(M, M) \) that
\[
\mathcal{L}_{f_0 + H, \phi}^{n+1}(g) = \mathcal{L}_{f_0, \phi}(\mathcal{L}_{f_0 + H, \phi}^n(g)) + D_f \mathcal{L}_{f, \phi}|_{f_0} (\mathcal{L}_{f_0 + H, \phi}^n(g)) \cdot H + o(H)
\]
\[
= \mathcal{L}_{f_0, \phi} \left( \mathcal{L}_{f_0}^n(g) + \sum_{i=1}^{n} \mathcal{L}_{f_0, \phi}^{i-1} (D_f \mathcal{L}_{f, \phi}(\mathcal{L}_{f_0, \phi}^i(g))|_{f_0}) \cdot H + \delta(H) \right)
\]
\[
+ D_f \mathcal{L}_{f, \phi}|_{f_0} (\mathcal{L}_{f_0, \phi}^n(g)) \cdot H + o(H)
\]
\[
= \mathcal{L}_{f_0, \phi} \left( \mathcal{L}_{f_0}^n(g) + \sum_{i=1}^{n} \mathcal{L}_{f_0, \phi}^{i-1} (D_f \mathcal{L}_{f, \phi}(\mathcal{L}_{f_0, \phi}^i(g))|_{f_0}) \cdot H + \delta(H) \right)
\]
\[
+ D_f \mathcal{L}_{f, \phi}|_{f_0} (\mathcal{L}_{f_0, \phi}^n(g)) \cdot H + (D_f \mathcal{L}_{f, \phi}|_{f_0} (D_f (\mathcal{L}_{f_0, \phi}^n(g))|_{f_0}) \cdot H + o(H)) \cdot H
\]
\[
= \mathcal{L}_{f_0, \phi} \left( \mathcal{L}_{f_0}^n(g) + \sum_{i=1}^{n+1} \mathcal{L}_{f_0, \phi}^{i-1} (D_f \mathcal{L}_{f, \phi}(\mathcal{L}_{f_0, \phi}^{i-1}(g))|_{f_0}) \cdot H + \delta(H),
\]
where \( o(H), \delta(H), \delta(H) \) are terms converging to zero faster than \( \|H\| \). This finishes the proof of i). Part ii) is obtained by straightforward computation using the explicit formula from the previous proposition. Part iii) follows using that
\[
\mathcal{L}_{f_0, \phi}(g_1 + tg_2) = \mathcal{L}_{f_0, \phi}(g_1) + t \mathcal{L}_{f_0, \phi}(g_2)
\]
for \( r_1(H), r_2(H) \) that tend to zero as \( H \) approaches zero. Finally, part iv) follows immediately from the fact that \( \mathcal{L}_{f_0, \phi}(1) = \deg(f)^n \) and that \( \deg(f) \) is locally constant. This finishes the proof. \( \square \)

Throughout, \( f \) will denote a local diffeomorphism satisfying (H1) and (H2) and \( \phi \) a Hölder potential such that (P) holds.

**Lemma 4.12.** For any probability measure \( \eta \), the topological pressure \( P_{\text{top}}(f, \phi) \) is given by
\[
P_{\text{top}}(f, \phi) = \lim_{n \to +\infty} \frac{1}{n} \log \left[ \int \mathcal{L}_{f, \phi}^n(1) \, d\eta \right].
\]
In particular, for any given \( x \in M \)
\[
P_{\text{top}}(f, \phi) = \lim_{n \to +\infty} \frac{1}{n} \log[\mathcal{L}_{f, \phi}^n(1)(x)].
\]

**Proof.** Since the second assertion above is a direct consequence of the first one with \( \eta = \delta_x \) (the Dirac measure at \( x \)) it is enough to prove the first one. Let \( \eta \) be any fixed probability measure. Recall that the topological pressure is the logarithm of the spectral radius of the transfer operator \( \mathcal{L}_{f, \phi} \), that is, \( P_{\text{top}}(f, \phi) = \log \lambda_{f, \phi} \). Moreover, since \( \mathcal{L}_{f, \phi} \) is a positive operator then the spectral radius can be computed as
\[
\lambda_{f, \phi} = \lim_{n \to +\infty} \sqrt[n]{\|\mathcal{L}_{f, \phi}^n\|} = \lim_{n \to +\infty} \sqrt[n]{\|\mathcal{L}_{f, \phi}^n(1)\|_u}
\]
Using that the functions \( \lambda_{f, \phi}^n \mathcal{L}_{f, \phi}^n(1) \) are uniformly convergent to the eigenfunction \( h_{f, \phi} \) which is bounded away from zero and infinity one has that there exists \( K > 0 \) and \( n_0 \geq 1 \) such that \( K^{-1} \leq \lambda_{f, \phi}^n \mathcal{L}_{f, \phi}^n(1) \leq K \) for all \( n \geq n_0 \). In consequence, we get
\[
\lim_{n \to +\infty} \frac{1}{n} \log \int \lambda_{f, \phi}^n \mathcal{L}_{f, \phi}^n(1) \, d\eta = 0,
\]
which proves the lemma.

The next lemma will be fundamental to study the differentiability of equilibrium states. In fact we show that the topological pressure is differentiable once that one requires smooth potentials.

Lemma 4.13 (Differentiability of Topological Pressure with respect to dynamics). Let \( \phi \) be a fixed \( C^2 \) potential on \( M \) satisfying (P'). Then the topological pressure function \( P_\phi : \mathcal{F}^2(M) \to \mathbb{R} \) given by \( P_\phi(f) = P_{\text{top}}(f, \phi) \) is differentiable with respect to \( f \).

Proof. By the last lemma we are reduced to prove the differentiability of the function

\[
C^2 \ni f \mapsto P(f, \phi) = \lim_{n \to +\infty} \frac{1}{n} \log \int \mathcal{L}_f^n(1) \, d\nu_{f, \phi}
\]

for some fixed \( \hat{f} \). We will use derivation of sequence \( P_n(f) = \frac{1}{n} \log \int \mathcal{L}_f^n(1) \, d\nu_{f, \phi} \), which converge to the topological pressure of \( f \) uniformly in a small neighborhood of \( \hat{f} \). By the chain rule, the derivative of \( P_n \) with respect to \( f \) is given by

\[
D_f P_n(f) = \frac{\nu_{f, \phi}(\frac{d}{d\nu_{f, \phi}} \mathcal{L}_f^n(1)(\cdot))}{n \nu_{f, \phi}(\mathcal{L}_f^n(1)(\cdot))}.
\]

This yields that

\[
D_f P_n(\hat{f}) \cdot (H) = \frac{\int D_f \mathcal{L}_{\hat{f}, \phi}^n(1) |f \cdot (H) d\nu_{f, \phi}}{n \cdot \int \mathcal{L}_{\hat{f}, \phi}^n(1) d\nu_{f, \phi}}
= \frac{\int \sum_{i=1}^n \mathcal{L}_{\hat{f}, \phi}^{i-1}(D_f \mathcal{L}_{\hat{f}, \phi}(\mathcal{L}_{\hat{f}, \phi}^{n-i}(1)) |f \cdot (H) d\nu_{f, \phi}}{n \cdot \int \mathcal{L}_{\hat{f}, \phi}^n(1) d\nu_{f, \phi}}.
\]

In fact the later can be written also as the sum

\[
\int \sum_{i=1}^n \mathcal{L}_{\hat{f}, \phi}^{i-1}(\sum_{j=1}^{\deg(\hat{f})} e^{\phi(\hat{f}_j(\cdot))} \cdot D \mathcal{L}_{\hat{f}, \phi}^{n-i}(1) |f \cdot (H) d\nu_{f, \phi})
+ \int \sum_{i=1}^n \mathcal{L}_{\hat{f}, \phi}^{i-1}(\sum_{j=1}^{\deg(\hat{f})} e^{\phi(\hat{f}_j(\cdot))} \cdot D \mathcal{L}_{\hat{f}, \phi}^{n-i}(1) |f \cdot (H) d\nu_{f, \phi})
\]

where \( \hat{f}_j \) denote the inverse branches of the map \( \hat{f} \). To analyze the previous expressions we consider the three sums below

\[
B_n(\hat{f}) \cdot H = \frac{1}{n} \int \sum_{i=1}^n \mathcal{L}_{\hat{f}, \phi}^{i-1}(\sum_{j=1}^{\deg(\hat{f})} e^{\phi(\hat{f}_j(\cdot))} \cdot D \mathcal{L}_{\hat{f}, \phi}^{n-i}(1) |f \cdot (H) d\nu_{f, \phi})
\]

and

\[
C_n(\hat{f}) \cdot H = \frac{1}{n} \int \sum_{i=1}^n \mathcal{L}_{\hat{f}, \phi}^{i-1}(\sum_{j=1}^{\deg(\hat{f})} e^{\phi(\hat{f}_j(\cdot))} \mathcal{L}_{\hat{f}, \phi}^{n-i}(1) |f \cdot (H) d\nu_{f, \phi})
\]

To establish our result we will use the following:
Claim 1: $B_n(\hat{f}) \cdot H$ is uniformly convergent on $(\hat{f}, \phi)$ and $H \in C^2(M, M)$ with $\|H\|_2 \leq 1$ to the expression

$$\|C\|$$

and the sum $\hat{f}$ to the expression $\|H\|_2 \leq 1$.

As for Claim 1, observe that the following uniform convergence holds

$$\lim_{n \to \infty} \int \sum_{j=1}^{\deg(\hat{f})} e^{\phi(f_j(\cdot))} \cdot Df_{\hat{f},\phi}(f_{\hat{f},\phi}) \cdot [(T_{\hat{f},f} \cdot H)(\cdot)] d\nu_{\hat{f},\phi}.$$
Moreover, one also has that

\[
B_n(\hat{f}, \hat{\phi}) \cdot H - \frac{1}{n} \sum_{i=0}^{n-1} \int \sum_{j=1}^{\deg(\hat{f})} e^{\phi(\hat{f}_j(\cdot))} D\hat{\phi}^{i}(\cdot) \int_{T_{j}|f \cdot H}(\cdot) \, d\nu_{f,\phi} \int h_{f,\phi} \, d\nu_{f,\phi} \\
\leq \frac{1}{n} \sum_{i=1}^{n} \left| \int \sum_{j=1}^{\deg(\hat{f})} e^{\phi(\hat{f}_j(\cdot))} D\hat{\phi}^{i}(\cdot) \int_{T_{j}|f \cdot H}(\cdot) \, d\nu_{f,\phi} \int h_{f,\phi} \, d\nu_{f,\phi} \right|
\]

\[
- \int \sum_{j=1}^{\deg(\hat{f})} e^{\phi(\hat{f}_j(\cdot))} D\hat{\phi}^{n-i}(\cdot) \int_{T_{j}|f \cdot H}(\cdot) \, d\nu_{f,\phi} \int h_{f,\phi} \, d\nu_{f,\phi} \\
\leq \frac{1}{n} \sum_{i=1}^{n} \left| \int \sum_{j=1}^{\deg(\hat{f})} e^{\phi(\hat{f}_j(\cdot))} D\hat{\phi}^{i}(\cdot) \int_{T_{j}|f \cdot H}(\cdot) \, d\nu_{f,\phi} \int h_{f,\phi} \, d\nu_{f,\phi} \right|
\]

which is uniformly convergent to zero with respect to \((\hat{f}, \hat{\phi})\) and all \(H \in C^2(M, M)\) with \(\|H\|_2 \leq 1\). This proves Claim 1. We now proceed to prove Claim 2.

\[
\frac{1}{n} \sum_{i=0}^{n-1} \int \sum_{j=1}^{\deg(\hat{f})} e^{\phi(\hat{f}_j(\cdot))} \hat{\phi}_j(\cdot) \int_{T_{j}|f \cdot H}(\cdot) \, d\nu_{f,\phi} \int h_{f,\phi} \, d\nu_{f,\phi} \\
\rightarrow_{n \to +\infty} \int \sum_{j=1}^{\deg(\hat{f})} e^{\phi(\hat{f}_j(\cdot))} h_{f,\phi}(\hat{f}_j(\cdot)) \int_{T_{j}|f \cdot H}(\cdot) \, d\nu_{f,\phi} \int h_{f,\phi} \, d\nu_{f,\phi}
\]

uniformly with respect to \((\hat{f}, \hat{\phi})\) and \(H\). Then, the difference between \(C_n(\hat{f}) \cdot H\) and

\[
\frac{1}{n} \sum_{i=0}^{n-1} \int \sum_{j=1}^{\deg(\hat{f})} e^{\phi(\hat{f}_j(\cdot))} \hat{\phi}_j(\cdot) \int_{T_{j}|f \cdot H}(\cdot) \, d\nu_{f,\phi} \int h_{f,\phi} \, d\nu_{f,\phi}
\]

is bounded from above (in absolute value) by

\[
\frac{1}{n} \sum_{i=1}^{n} \left| \int \sum_{j=1}^{\deg(\hat{f})} e^{\phi(\hat{f}_j(\cdot))} \hat{\phi}_j(\cdot) \int_{T_{j}|f \cdot H}(\cdot) \, d\nu_{f,\phi} \int h_{f,\phi} \, d\nu_{f,\phi} \right|
\]

\[
- \int \sum_{j=1}^{\deg(\hat{f})} e^{\phi(\hat{f}_j(\cdot))} \hat{\phi}_j(\cdot) \int_{T_{j}|f \cdot H}(\cdot) \, d\nu_{f,\phi} \int h_{f,\phi} \, d\nu_{f,\phi} \\
\leq \frac{1}{n} \sum_{i=1}^{n} \left| \int \sum_{j=1}^{\deg(\hat{f})} e^{\phi(\hat{f}_j(\cdot))} \hat{\phi}_j(\cdot) \int_{T_{j}|f \cdot H}(\cdot) \, d\nu_{f,\phi} \int h_{f,\phi} \, d\nu_{f,\phi} \right|
\]

\[
- \int \sum_{j=1}^{\deg(\hat{f})} e^{\phi(\hat{f}_j(\cdot))} \hat{\phi}_j(\cdot) \int_{T_{j}|f \cdot H}(\cdot) \, d\nu_{f,\phi} \int h_{f,\phi} \, d\nu_{f,\phi} \\
\leq \frac{1}{n} \sum_{i=1}^{n} \left| \int \sum_{j=1}^{\deg(\hat{f})} e^{\phi(\hat{f}_j(\cdot))} \hat{\phi}_j(\cdot) \int_{T_{j}|f \cdot H}(\cdot) \, d\nu_{f,\phi} \int h_{f,\phi} \, d\nu_{f,\phi} \right|
\]

where \(\hat{C} = \|\phi\|_1 \cdot \max_{j=1, \ldots, \deg(\hat{f})} \{\|T_{j}|f \cdot H(\hat{f}_j(\cdot))\|_1\}\). Since the later expression is uniformly convergent to zero with respect to \((\hat{f}, \hat{\phi})\) and \(H \in C^2(M, M)\) such that \(\|H\|_2 \leq 1\) this proves Claim 2 and finishes the proof of the lemma.
Corollary 4.14. The topological pressure $P_{\text{top}} : \mathcal{F}^2 \times \mathcal{W}^2 \to \mathbb{R}$ is differentiable.

Proof. Just note that the derivatives calculated in Corollary 4.13 and in the Lemma above are partial derivatives for the function $P_{\text{top}}(f, \phi)$, and jointly continuous with respect to both variables $f$ and $\phi$.

\[ \square \]

4.3. Differentiability of maximal entropy measure with respect to dynamics. Through this section we deal with maximal entropy measures and henceforth we fix the potential $\phi \equiv 0$ and fix $f_0$ local diffeomorphism satisfying (H1) and (H2). For that reason we shall omit the dependence on $\phi$. Recall that for every $C^1$ local diffeomorphism $f$ satisfying (H1) and (H2) we have maximal eigenvalue $\lambda_f = \deg(f)$, eigenfunction $h_f = \frac{d\nu_f}{d\mu_f} = 1$ and conformal measure $\nu_f = \mu_f$ for the Perron-Frobenius operator. In particular, the topological entropy $h_{\text{top}}(f) = \log \deg(f)$ is constant.

Let $r \in \mathbb{N}_0$ and $\alpha \in (0, 1)$ be such that $r + \alpha > 0$ and $f \in \mathcal{F}_r^+ \alpha$ be given. It follows from Theorems 3.13 and 3.15 that all transfer operators $\hat{L}_f : C^{k+\alpha}(M, \mathbb{R}) \to C^{k+\alpha}(M, \mathbb{R})$ have the spectral gap property for $k + \alpha \in \{\alpha, 1 + \alpha, \ldots, r + \alpha\} \cap \mathbb{R}^*_+$, provided that $f$ is sufficiently $C^{r+\alpha}$-close to $f_0$. In consequence, it is not hard to check that if $E^{k+\alpha}_{0,f} = \{ \hat{g} \in C^{k+\alpha}(M, \mathbb{R}) : \int \hat{g} d\nu_f = 0 \}$ then $C^{k+\alpha}(M, \mathbb{R}) = \{ \ell h_f : \ell \in \mathbb{R} \} \oplus E^{k+\alpha}_{0,f}$ is a $\hat{L}_f, \phi$-invariant decomposition in $C^{k+\alpha}(M, \mathbb{R})$. Furthermore there are constants $C_{f,k+\alpha} > 0$ and $\tau_{f,k+\alpha} \in (0, 1)$ such that for all $\hat{g} \in E^{k+\alpha}_{0,f}$ it follows:

\[
\| \hat{L}_f^n \hat{g} \|_k \leq C_{f,k+\alpha} \tau_{f,k+\alpha} \| \hat{g} \|_k, \quad \text{for all } n \geq 1.
\]

Set $C_f = \max\{C_{f,k+\alpha} : k \leq r\}$, $\tau_f = \max\{\tau_{f,k+\alpha} : k \leq r\}$. We also set $c_f$ to be a bound for the norm of of $D_f \hat{L}_f$. Notice that these constants can be taken uniform in a neighborhood of $f_0$. For that reason, we shall omit the dependence of $C_f, c_f$, and $\tau_f$ on $f$. Consider also the spectral projections $P^{k+\alpha}_{0,f} : C^{k+\alpha}(M, \mathbb{R}) \to E^{k+\alpha}_{0,f}$ given by $P^{k+\alpha}_{0,f}(g) = g - \int g d\nu_f$. In what follows, when no confusion is possible we shall omit the dependence on $f$ in the corresponding subspaces and spectral projections.

Theorem 4.15. The map $\mathcal{F}^2 \ni f \mapsto \mu_f \in (C^2(M, \mathbb{R}))^*$ is differentiable. In particular, for any $g \in C^2(M, \mathbb{R})$ the map $\mathcal{F}^2 \ni f \mapsto \int g \, d\mu_f$ is $C^1$-differentiable and its derivative acting in $H \in C^2(M, M)$ is given by

\[
D_f \mu_f (g) |_{f_0} : H = \sum_{i=0}^{\infty} \int D_f \hat{L}_f^i (\hat{L}_f^i (P_0 (g))) : H \, d\mu_{f_0}.
\]

Proof. Let $g \in C^2(M, \mathbb{R})$ and $f_0 \in \mathcal{F}^2$ be fixed. We define a sequence of maps $F_n : \mathcal{F}^2 \to \mathbb{R}$ given by $F_n(f) = \int \hat{L}_f^n (g) \, d\mu_{f_0}$ and notice that $F_n(f)$ is convergent to $\int g \, d\mu_f$, whereas the convergence is uniform in a sufficiently small neighborhood.
of \( f_0 \) and for \( g \) in the unit sphere of \( C^2(M, \mathbb{R}) \). Moreover, if \( H \in C^2(M, M) \) then

\[
DF_n(f) \cdot H = \sum_{i=1}^{n-1} \int \tilde{L}_f^{-1}(D_f \tilde{L}_f(\tilde{L}_f^{-i}(g))) \cdot H \ d\mu_0
\]

\[
= \sum_{i=1}^{n-1} \int \tilde{L}_f^{-1}(D_f \tilde{L}_f \left( \sum g \ d\mu_f + \tilde{L}_f^{-i}(P_0(g)) \right)) \cdot H \ d\mu_0
\]

\[
= \sum_{i=1}^{n-1} \int \tilde{L}_f^{-1}(D_f \tilde{L}_f(\tilde{L}_f^{-i}(P_0(g)))) \cdot H \ d\mu_0.
\]

On the other hand, since we assumed \( \phi \equiv 0 \) then \( \mu_f = \nu_f \) and \( \tilde{L}_f^* \mu_f = \mu_f \). Thus,

\[
\sum_{i=1}^{n-1} \int D_f \tilde{L}_f(\tilde{L}_f^{-i}(P_0(g))) \cdot H \ d\mu_f = \sum_{i=1}^{n-1} \int D_f \tilde{L}_f(\tilde{L}_f^{-i}(P_0(g))) \cdot H \ d\mu_f
\]

and

\[
\sum_{i=0}^{n-1} \left| \int D_f \tilde{L}_f(\tilde{L}_f^{-i}(P_0,f(g))) \cdot H \ d\mu_f \right| \leq \sum_{i=0}^{n-1} \| D_f \tilde{L}_f(\tilde{L}_f^{-i}(P_0,f(g))) \|_f \cdot \| H \|_0
\]

\[
\leq \sum_{i=0}^{n-1} \| \tilde{L}_f^{-i}(P_0,f(g)) \|_1 \cdot \| H \|_1 \cdot c
\]

\[
\leq \sum_{i=0}^{n-1} C \tau^i \cdot \| g \|_1 \cdot c \cdot \| H \|_1,
\]

that is bounded from above by \( \frac{C}{1-\tau} \cdot 2\| g \|_1 c \| H \|_1 \). So the previous upper bound is uniform for \( g \) in the unit sphere of \( C^2(M, \mathbb{R}) \) and \( H \) in the unit sphere of \( C^2(M, M) \). This implies that the limit

\[
\lim_{n \to +\infty} \sum_{i=1}^{n-1} \int D_f \tilde{L}_f(\tilde{L}_f^{-i}(P_0,g)) \cdot H \ d\mu_f,
\]

does exist and is uniform with respect to the dynamics, \( g \) in the unit sphere of \( C^2(M, \mathbb{R}) \) and \( H \) in the unit sphere of \( C^2(M, M) \). We proceed and estimate

\[
|DF_n(f) \cdot H - \sum_{i=1}^{n-1} \int D_f \tilde{L}_f(\tilde{L}_f^{-i}(P_0,g)) \cdot H \ d\mu_f|
\]

\[
\leq \sum_{i=1}^{n-1} \left| \int D_f \tilde{L}_f(\tilde{L}_f^{-i}(P_0,g)) \cdot H \ d\mu_f - \int D_f \tilde{L}_f(\tilde{L}_f^{-i}(P_0,g)) \cdot H \ d\mu_f \right|
\]

\[
= \sum_{i=1}^{n-1} \left| \int D_f \tilde{L}_f(\tilde{L}_f^{-i}(P_0,g)) \cdot H \ d\mu_f - \int D_f \tilde{L}_f(\tilde{L}_f^{-i}(P_0,g)) \cdot H \ d\mu_0 \right|
\]

\[
\leq \sum_{i=1}^{n-1} C \tau^i \cdot 2 \| D_f \tilde{L}_f(\tilde{L}_f^{-i}(P_0,g)) \|_1 \| H \|_2 \leq \sum_{i=1}^{n-1} C \tau^i \cdot 2 \| \tilde{L}_f^{-i}(P_0,g) \|_2 c \| H \|_2
\]

\[
\leq \sum_{i=1}^{n-1} C \tau^i \cdot 2 \cdot C \tau^{n-1} \cdot 2 \cdot \| g \|_2 \cdot c \| H \|_2 \leq 4c C^2 (n-1) \tau^{n-1} \cdot \| g \|_2 \cdot \| H \|_2
\]

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which converges to zero. Thus \( \lim DF_n(f) \cdot H = \sum_{i=1}^{\infty} \int D_f \hat{L}_f(\hat{L}^{-i}_{f_0}(P_0(g))) \cdot H \, d\mu_f \) uniformly with respect to the dynamics \( f \), and \( g \) in the unit sphere of \( C^2(M, \mathbb{R}) \) and \( H \) in the unit sphere of \( C^2(M, M) \). One can deduce that for all \( f \) close to \( f_0 \) the sequence \( F_n(f) \) converges uniformly to \( \int g \, d\mu_f \) and the sequence \( DF_n \) is also uniformly convergent to the linear continuous functional defined above. We conclude that

\[
D_f \mu_f(g)|_{f_0} \cdot H = \sum_{i=0}^{\infty} \int D_f \hat{L}_f(\hat{L}^i_{f_0}(P_0(g))) \cdot H \, d\mu_{f_0}.
\]

This finishes the proof of the theorem. \( \square \)

Since previous results contain the statements of Theorems \( \text{B} \) and \( \text{C} \), their proofs are now complete.

5. Stability and differentiability in dynamical systems

In this section we prove that many dynamical objects vary continuously or differentiably with respect to perturbations of the potential and dynamics. We organize this into subsections for the readers convenience.

5.1. Smoothness of the correlation function. Our purpose here is to prove Corollary \( \text{C} \) on the smoothness of the correlation function \( (f, \phi) \mapsto C_{\varphi, \psi}(f, \phi, n) \) and its asymptotic behaviour when \( n \) tends to infinite. Let \( \mathcal{F}^2 \) an open set of local diffeomorphisms, \( \mathcal{W}^a \) be an open set of potentials as introduced before. Consider the observables \( \varphi, \psi \in C^a(M, \mathbb{R}) \) and \( n \in \mathbb{N} \). First we will prove that the map \( (f, \phi) \mapsto C_{\varphi, \psi}(f, \phi, n) \) is analytic in \( \phi \) and differentiable in \( f \) whenever \( \phi \equiv 0 \). Recall one can write

\[
C_{\varphi, \psi}(f, \phi, n) = \int \varphi \left[ \hat{L}^n_{f_0}(\psi h_{f, \phi}) - h_{f, \phi} \int \psi \, d\mu_{f, \phi} \right] \, d\nu_{f, \phi}.
\]

This expression varies analytically with \( \phi \) since it is composition of analytic functions. As for the differentiability of the correlation function with respect to \( f \), if \( \phi \equiv 0 \) then clearly \( \mu_f = \nu_f, \, h_f = 1 \). Moreover, for \( f_0 \in \mathcal{F}^2 \) and \( H \in C^2(M, M) \)

\[
D_f C_{\varphi, \psi}(f, 0, n)|_{f_0} \cdot H = [D_f \mu_f|_{f_0} \cdot H] \left( \varphi \hat{L}^n_{f_0}(\psi) - \int \psi \, d\mu_{f_0} \right)
\]

\[
+ \int \varphi \left( D_f \hat{L}^n_{f_0}(\psi)|_{f_0} \cdot H - [D_f \mu_f|_{f_0} \cdot H](\psi) \right) \, d\mu_{f_0}
\]

\[
= \sum_{i=0}^{\infty} \int D_f \hat{L}_f \left( \hat{L}^{i}_{f_0}(P_0(\varphi(\hat{L}^n_{f_0}(\psi) - \int \psi \, d\mu_{f_0}))) \right)|_{f_0} \cdot H \, d\mu_{f_0}
\]

\[
+ \sum_{i=1}^{\infty} \int \varphi \hat{L}^{i-1}_{f_0}(D_f \hat{L}_f(\hat{L}^{n-i}_{f_0}(\psi))|_{f_0} \cdot H) \, d\mu_{f_0}
\]

\[- \sum_{i=0}^{\infty} \int D_f \hat{L}_f \left( \hat{L}^i_{f_0}(P_0(\psi)) \right)|_{f_0} \cdot H \, d\mu_{f_0} \cdot \int \varphi \, d\mu_{f_0}.\]
Hence we deduce
\[ D_f C_{\varphi, \psi}(f, 0, n)_{|f_0} \cdot H = \sum_{i=0}^{n-1} \int D_f \tilde{\mathcal{L}}_f \left( \tilde{\mathcal{L}}^{i}_{f_0} \left( P_0(\varphi(\tilde{\mathcal{L}}^{n}_{f_0}(\psi) - \int \psi d\mu_{f_0})) \right) \right)_{|f_0} \cdot H d\mu_{f_0} \]
\[ + \sum_{i=0}^{n-1} \int \varphi \tilde{\mathcal{L}}^{n-i}_{f_0} \left( D_f \tilde{\mathcal{L}}_f (\tilde{\mathcal{L}}^i_{f_0} \psi)_{f_0} \cdot H \right) d\mu_{f_0} \]
\[ - \sum_{i=0}^{\infty} \int D_f \tilde{\mathcal{L}}_f \left( \tilde{\mathcal{L}}^i_{f_0} (P_0(\psi)) \right)_{f_0} \cdot H d\mu_{f_0} \cdot \int \varphi d\mu_{f_0}. \]

Consider the series
\[ A_n(f_0, H) = \sum_{i=0}^{\infty} \int D_f \tilde{\mathcal{L}}_f \left( \tilde{\mathcal{L}}^{i}_{f_0} \left( P_0(\varphi(\tilde{\mathcal{L}}^{n}_{f_0}(\psi) - \int \psi d\mu_{f_0})) \right) \right)_{|f_0} \cdot H d\mu_{f_0} \]
and
\[ B_n(f_0, H) = \sum_{i=0}^{n-1} \int \varphi \tilde{\mathcal{L}}^{n-i-1}_{f_0} \left( D_f \tilde{\mathcal{L}}_f (\tilde{\mathcal{L}}^i_{f_0} \psi)_{f_0} \cdot H \right) d\mu_{f_0} \]
\[ - \sum_{i=0}^{n-1} \int D_f \tilde{\mathcal{L}}_f \left( \tilde{\mathcal{L}}^i_{f_0} (P_0(\psi)) \right)_{f_0} \cdot H d\mu_{f_0} \cdot \int \varphi d\mu_{f_0}. \]

We will prove that both expressions \( A_n(f_0, H) \) and \( B_n(f_0, H) \) converge uniformly to zero in \( \{ H \in C^2(M, M) : \| H \|_2 \leq 1 \} \) and for all \( f \) close enough to \( f_0 \). In fact, on the one hand
\[ |A_n(f_0, H)| \leq \sum_{i=0}^{\infty} c \cdot \| H \|_2 \cdot \| \tilde{\mathcal{L}}^{i}_{f_0} (P_0(\varphi(\tilde{\mathcal{L}}^{n}_{f_0}(\psi) - \int \psi d\mu_{f_0})) \) \|_2 \]
\[ \leq \sum_{i=0}^{\infty} c \cdot \| H \|_2 C \tau^i \cdot \| P_0 \|_2 \cdot \| \varphi \|_2 C \tau^n \| \psi \| - \int \psi d\mu_{f_0} \|_2 \]
which is uniformly convergent to zero for \( \{ H \in C^2(M, M) : \| H \|_2 \leq 1 \} \) and \( f \) close to \( f_0 \). On the other hand, \( |B_n(f_0, H)| \) is bounded from above by
\[ \sum_{i=0}^{n-1} \| \varphi \|_2 \cdot \| \tilde{\mathcal{L}}^{n-i-1}_{f_0} (D_f \tilde{\mathcal{L}}_f (\tilde{\mathcal{L}}^i_{f_0} \psi)_{f_0} \cdot H) - \int D_f \tilde{\mathcal{L}}_f \left( \tilde{\mathcal{L}}^i_{f_0} (P_0(\psi)) \right)_{f_0} \cdot H d\mu_{f_0} \|_0 \]
\[ \leq \sum_{i=0}^{n-1} C \tau^{n-i} \| \varphi \|_2 \cdot \| D_f \tilde{\mathcal{L}}_f (\tilde{\mathcal{L}}^i_{f_0} \psi)_{f_0} \cdot H - \int D_f \tilde{\mathcal{L}}_f \left( \tilde{\mathcal{L}}^i_{f_0} (P_0(\psi)) \right)_{f_0} \cdot H d\mu_{f_0} \|_2 \]
\[ \leq \sum_{i=0}^{n-1} \| \varphi \|_2 \cdot C^2 \tau^{n-1} \cdot c \cdot \| \psi \|_2 \| H \|_2 \]
which is again uniformly convergent to zero as \( n \) goes to infinity. This proves not only that the correlation function for the maximal entropy measure is differentiable in \( f \) but also that \( D_f C_{\varphi, \psi}(f, 0, n)_{|f_0} \) is uniformly convergent to zero as \( n \) goes to infinity. This finishes the proof of Corollary C.
5.2. Stability of the Central Limit Theorem. Our purpose here is to prove Theorem \[ \text{[D]} \] Let \( \mathcal{W} \) be an open set of \( C^2 \) potentials and \( \mathcal{F} \) an open set of \( C^2 \) local diffeomorphisms satisfying the conditions (H1), (H2) and (P') with uniform constants as before. For any \( \psi \in C^\infty(M, \mathbb{R}) \) consider the mean and variance given, respectively, by

\[
\begin{align*}
    m_{f,\phi} &= \int \psi \, d\mu_{f,\phi} \quad \text{and} \quad \sigma^2_{f,\phi} = \int \dot{\psi}^2 \, d\mu_{f,\phi} + 2 \sum_{j=1}^{\infty} \int \dot{\psi}(\dot{\psi} \circ f^j) \, d\mu_{f,\phi},
\end{align*}
\]

where \( \dot{\psi} = \psi - m_{f,\phi} \). We omit the dependence of \( m_{f,\phi} \) and \( \sigma^2_{f,\phi} \) on \( \psi \) for notational simplicity. By invariance of the measure \( \mu_{f,\phi} \) we can also write

\[
\sigma^2_{f,\phi} = \lim_{n \to \infty} \frac{1}{n} \int \left( \sum_{j=0}^{n-1} \dot{\psi} \circ f^j \right)^2 \, d\mu_{f,\phi} \geq 0.
\]

Moreover, it follows from the exponential decay of correlations that the Central Limit Theorem holds (see e.g. [CV13, Corollary 2]). So, either \( \sigma^2_{f,\phi} = 0 \) and consequently \( \psi = u \circ f - u + \int \psi \, d\mu_{f,\phi} \) for some \( u \in L^2(X, \mathcal{F}, \mu_{f,\phi}) \) or the random variables \( \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \psi \circ f^j \) converge in distribution to the Gaussian \( N(m_{f,\phi}, \sigma^2_{f,\phi}) \).

First we will prove that the functions \((f, \phi) \mapsto m_{f,\phi}\) and \((f, \phi) \mapsto \sigma^2_{f,\phi}\) are analytic on \( \phi \) and that are differentiable on \( f \) in the case of the maximal entropy measure, provided that \( \psi \) is smooth enough. Since \( \psi \mapsto \mathcal{L}_{f,\phi} \) is analytic, if \( T(z) = \frac{\log |\lambda_f|}{\log \lambda_f} \) it follows from the classical perturbation theory and Nagaev’s method that \( \sigma^2_{f,\phi}(\psi) = -D^2T(z)|_{z=0} \) (see e.g. [Sar12] for details) and the dependence is analytic in \( \phi \). In fact one can check

\[
\begin{align*}
    \sigma^2_{f,\phi}(\psi) &= \left( \int \psi \, d\mu_{f,\phi} \right)^2 + \int (I - \mathcal{L}_{f,\phi}|_{E_0})^{-1} (\psi h_{f,\phi} - h_{f,\phi}) \int \psi \, d\mu_{f,\phi} \psi \, d\nu_{f,\phi} \\
    &\quad + \int \psi \, d\mu_{f,\phi} \int (I - \mathcal{L}_{f,\phi}|_{E_0})^{-1} (1 - h_{f,\phi}) \psi \, d\nu_{f,\phi} \\
    &= \left( \int \psi \, d\mu_{f,\phi} \right)^2 + \int \sum_{k=0}^{\infty} \mathcal{L}_{f,\phi}|_{E_0} \left( \psi h_{f,\phi} - h_{f,\phi} \right) \int \psi \, d\mu_{f,\phi} \psi \, d\nu_{f,\phi} \\
    &\quad + \int \psi \, d\mu_{f,\phi} \int \sum_{k=0}^{\infty} \mathcal{L}_{f,\phi}|_{E_0} (1 - h_{f,\phi}) \psi \, d\nu_{f,\phi}
\end{align*}
\]

Hence, for any fixed \( \psi \in C^{1+\alpha}(M, \mathbb{R}) \) the variance map \((f, \phi) \mapsto \sigma^2_{f,\phi}(\psi)\) is continuous since it is obtained as composition of continuous functions and \( \mathcal{L}_{f,\phi}(1 - h_{f,\phi}) \) is uniformly convergent to zero in a neighborhood of \((f, \phi)\). We obtain further regularity in the case of maximal entropy measures. More precisely,

**Lemma 5.1.** Let \( \phi \equiv 0 \) and \( \psi \in C^2(M, \mathbb{R}) \) be given. Then the variance map \( \mathcal{F} \ni f \mapsto \sigma^2_{f,0}(\psi) \) is differentiable.

**Proof.** Notice that we want to study the variance

\[
\sigma^2_{f,0}(\psi) = \int \dot{\psi}^2 \, d\mu_f + 2 \sum_{n=1}^{\infty} C_{\psi,\dot{\psi}}(f,0,n)
\]

where \( \dot{\psi} = \dot{\psi}(f) = \psi - \int \psi \, d\mu_f \), that is differentiable in \( f \). So, to prove the differentiability of the previous expression, using the chain rule, we are reduced to
prove the differentiability of the map \( f \mapsto \sum_{n=1}^{\infty} C_{\tilde{\psi},\tilde{\psi}}(f, 0, n) \) assuming that the observable \( \tilde{\psi} \) is fixed and independent of \( f \). Fix \( \hat{f} \in \mathcal{F}^2 \) and proceed to consider the sequence of functions

\[
F_k(f) = \sum_{n=0}^{k} C_{\tilde{\psi},\tilde{\psi}}(f, 0, n),
\]

which is differentiable and uniformly convergent to \( F(f) = \sum_{n=0}^{\infty} C_{\tilde{\psi},\tilde{\psi}}(f, 0, n) \) in a small neighborhood of \( \hat{f} \). We claim that the derivatives of \( F_k \) are uniformly convergent. In fact, \( D_f F_k(f) \cdot H = \sum_{n=0}^{k} D_f C_{\tilde{\psi},\tilde{\psi}}(f, 0, n) \cdot f \cdot H \) and so, under the notations of Subsection 5.1

\[
D_f F_k(f) |_f \cdot H = 2 \sum_{n=1}^{k} \left[ A_n(f, H) + B_n(f, H) \right]
\]

\[
+ \sum_{i=0}^{n-1} \int D_f \hat{\mathcal{L}}_f \left( \hat{\mathcal{L}}_f(P_0(\tilde{\psi})) \right)_f \cdot H d\mu_f \cdot \int \tilde{\psi} d\mu_f
\]

\[
- \sum_{i=0}^{\infty} \int D_f \hat{\mathcal{L}}_f \left( \hat{\mathcal{L}}_f(P_0(\tilde{\psi})) \right)_f \cdot H d\mu_f \cdot \int \tilde{\psi} d\mu_f.
\]

On the one hand, \( \sum_{n=1}^{n} |A_n(f, H) + B_n(f, H)| \) is bounded from above by

\[
\sum_{n=1}^{\infty} \left| c \cdot \| H \|_2 C^1 \cdot \| P_0 \|_2 \| \tilde{\psi} \|_2 C^1 \cdot \tilde{\psi} - \int \tilde{\psi} d\mu_f \|_2
\]

\[
+ \sum_{i=0}^{n-1} \| \tilde{\psi} \|_2 \cdot C^2 \tau^{n-1} \cdot c \cdot \| \tilde{\psi} \|_2 \| H \|_2,
\]

that is summable. Then it is well defined the limit \( \sum_{n=0}^{\infty} A_n(f, H) + B_n(f, H) \) and the convergence is uniform for \( f \) in a small neighborhood of \( \hat{f} \) and \( \| H \|_2 = 1 \). On the other hand, in view of the differentiability of the maximal entropy measure,

\[
\sum_{j=1}^{n} \left| \sum_{i=0}^{j-1} \int D_f \hat{\mathcal{L}}_f \left( \hat{\mathcal{L}}_f(P_0(\tilde{\psi})) \right)_f \cdot H d\mu_f \cdot \int \tilde{\psi} d\mu_f
\]

\[
- \sum_{i=0}^{\infty} \int D_f \hat{\mathcal{L}}_f \left( \hat{\mathcal{L}}_f(P_0(\tilde{\psi})) \right)_f \cdot H d\mu_f \cdot \int \tilde{\psi} d\mu_f.
\]

and so, we get the convergence of the series

\[
\lim_{n \to \infty} \sum_{j=0}^{n} \sum_{i=0}^{j-1} \int D_f \hat{\mathcal{L}}_f \left( \hat{\mathcal{L}}_f(P_0(\tilde{\psi})) \right)_f \cdot H d\mu_f \cdot \int \tilde{\psi} d\mu_f
\]

\[
- \sum_{i=0}^{\infty} \int D_f \hat{\mathcal{L}}_f \left( \hat{\mathcal{L}}_f(P_0(\tilde{\psi})) \right)_f \cdot H d\mu_f \cdot \int \tilde{\psi} d\mu_f
\]
also uniform in a neighborhood of \( \hat{f} \) and with \( \|H\|_2 = 1 \). This proves that \( D_f F_k(f), H \) is uniformly convergent, proving that \( F \) is differentiable. This finishes the proof of the lemma.

Finally, if \( \sigma_{f,\phi}^2 > 0 \) one can use the continuity of the function \( (f, \phi) \mapsto \sigma_{f,\phi}^2 \) to obtain \( U \subset \mathcal{F}^2 \times \mathcal{W}^2 \) open such that for every \( (\hat{f}, \hat{\phi}) \in U \) it holds that \( \sigma_{\hat{f},\hat{\phi}}^2 > 0 \). In consequence, if \( \psi \) is not a coboundary in \( L^2(\mu_{f,\phi}) \) then the same property holds for all close \( \hat{f} \) and \( \hat{\phi} \). This finishes the proof of Theorem [D] and Corollary [D].

5.3. Differentiability of the free energy and stability of large deviations.

In this section we prove the differentiability of the free energy function and deduce some further properties for large deviations corresponding to Theorems [E] and [F].

5.3.1. Free energy function. First we establish some properties of the free energy function as consequence of the spectral gap property. Recall that an observable \( \psi : M \to \mathbb{R} \) is cohomologous to a constant if there exists \( A \in \mathbb{R} \) and an observable \( \psi : M \to \mathbb{R} \) such that \( \psi = \psi \circ f - \psi + A \). Now we prove the following:

**Proposition 5.2.** Let \( f \) and \( \phi \) be as above and satisfy assumptions (H1), (H2) and (P). Then for any Hölder continuous observable \( \psi : M \to \mathbb{R} \) there exists \( t_{\phi,\psi} > 0 \) such that for all \( |t| \leq t_{\phi,\psi} \) the following limit exists

\[
\mathcal{E}_{f,\phi,\psi}(t) := \lim_{n \to \infty} \frac{1}{n} \log \int e^{t S_n \psi} \, d\mu_{f,\phi} = P_{\text{top}}(f, \phi + t \psi) - P_{\text{top}}(f, \phi).
\]

Moreover, if \( \psi \) is cohomologous to a constant then \( t \mapsto \mathcal{E}_{f,\phi,\psi}(t) \) is affine and otherwise \( t \mapsto \mathcal{E}_{f,\phi,\psi}(t) \) is real analytic, strictly convex. Furthermore, if \( (f, \phi) \in \mathcal{F}^2 \times \mathcal{W}^2 \) then for every \( t \in (-t_{\phi,\psi}, t_{\phi,\psi}) \) the function \( \mathcal{F}^2 \ni f \mapsto \mathcal{E}_{f,\phi,\psi}(t) \) is differentiable and \( \mathcal{F}^2 \ni f \mapsto \mathcal{E}_{f,\phi,\psi}(t) \) is continuous.

**Proof.** The first part of the proof goes along some well known arguments that we include here for completeness. Observe first that for all \( n \in \mathbb{N} \)

\[
\int e^{t S_n \psi} \, d\mu_{f,\phi} = \left( \frac{\lambda_{f,\phi}^{-n}}{\lambda_f^{-\phi}} \right)^n \int \mathcal{L}_{f,\phi}^{n} \mathcal{L}_{f,\phi}^{n} (h_{f,\phi} e^{t S_n \psi}) \, d\nu_{f,\phi},
\]

Since (P) is an open condition, then for every \( |t| \leq t_{\phi,\psi} \) the potential \( \phi + t\psi \) satisfies (P) provided that \( t_{\phi,\psi} \) is small enough.

Since \( h_{f,\phi} \) is positive and bounded away from zero and infinity this implies that \( \lambda_{f,\phi}^{-n} \mathcal{L}_{f,\phi}^{n} (h_{f,\phi} e^{t S_n \psi}) \) is uniformly convergent to \( h_{f,\phi+t\psi} \mathcal{L}_{f,\phi}^{n} (h_{f,\phi} e^{t S_n \psi}) \), thus uniformly bounded from zero and infinity for all large \( n \). Therefore using the dominated convergence theorem

\[
\lim_{n \to \infty} \frac{1}{n} \log \int e^{t S_n \psi} \, d\mu_{f,\phi} = \log \lambda_{f,\phi+t\psi} - \log \lambda_{f,\phi} = P_{\text{top}}(f, \phi + t \psi) - P_{\text{top}}(f, \phi),
\]

proving the first assertion of the proposition. Now, assume first that there exists \( A \in \mathbb{R} \) and a potential \( \hat{\psi} : M \to \mathbb{R} \) such that \( \hat{\psi} = \hat{\psi} \circ f - \hat{\psi} + A \). Then it follows
from the variational principle and invariance that

\[ P_{\text{top}}(f, \phi + t\psi) = \sup_{\mu \in \mathcal{M}_1(f)} \left\{ h_\mu(f) + \int (\phi + t\psi) \, d\mu \right\} \]

\[ = tA + \sup_{\mu \in \mathcal{M}_1(f)} \left\{ h_\mu(f) + \int \phi \, d\mu \right\} \]

\[ = tA + P_{\text{top}}(f, \phi) \]

and, consequently, \( \mathcal{E}_{f,\phi,\psi}(t) = tA \) is affine.

Now, we proceed to prove that if \( \psi \) is not cohomologous to a constant then the free energy function is strictly convex. Since \( t \mapsto P_{\text{top}}(f, \phi + t\psi) \) is real analytic (recall Remark 4.2) then to prove that \( t \mapsto \mathcal{E}_{f,\phi,\psi}(t) \) is strictly convex it is enough to show that \( \mathcal{E}_{f,\phi,\psi}''(t) > 0 \) for all \( t \). Assume that there exists \( t \) such that \( \mathcal{E}_{f,\phi,\psi}''(t) = 0 \).

Up to replace \( \phi \) by the potential \( \tilde{\phi} = \phi + t\psi \) we may assume without loss of generality that \( t = 0 \), that is, \( \mathcal{E}_{f,\phi,\psi}''(0) = 0 \). Hence, using Corollary 4.5 and differentiation under the sign of integral we obtain

\[ \mathcal{E}_{f,\phi,\psi}'(t) = \frac{dP_{\text{top}}(f, \phi + t\psi)}{dt} \bigg|_{t=0} = \int \psi \, d\mu_{f,\phi + t\psi} = \lim_{n \to \infty} \frac{1}{n} \int \frac{\mathcal{S}_n(\psi)}{e^{t\mathcal{S}_n(\psi)} \, d\mu_{f,\phi}} \]

(hence \( \mathcal{E}_{f,\phi,\psi}'(0) = \int \psi \, d\mu_{f,\phi} \)). Using Theorem B and differentiating again with respect to \( t \) under the sign of integral it follows that

\[ \mathcal{E}_{f,\phi,\psi}'''(t) = \lim_{n \to \infty} \frac{1}{n} \left[ \frac{\mathcal{S}_n(\psi)^2}{e^{t\mathcal{S}_n(\psi)} \, d\mu_{f,\phi}} - \left( \frac{\mathcal{S}_n(\psi) e^{t\mathcal{S}_n(\psi)} \, d\mu_{f,\phi}}{e^{t\mathcal{S}_n(\psi)} \, d\mu_{f,\phi}} \right)^2 \right] \geq 0 \]

(hence \( \mathcal{E}_{f,\phi,\psi}'''(0) = \lim_{n \to \infty} \frac{1}{n} [\mathcal{S}_n(\psi)^2 \, d\mu_{f,\phi} - (\mathcal{S}_n(\psi) \, d\mu_{f,\phi})^2] > 0 \)) because, if \( \mu_n = e^{t\mathcal{S}_n(\psi)} \, d\mu_{f,\phi} \), the inequality is equivalent to \( \int \mathcal{S}_n(\psi) \, d\mu_n \leq (\int \mathcal{S}_n(\psi) \, d\mu_n)^{\frac{1}{2}} (\int 1 \, d\mu_n)^{\frac{1}{2}} \) that holds by Hölder’s inequality. In particular, \( \mathcal{E}_{f,\phi,\psi}''(t) = 0 \) if and only if \( \psi \) is cohomologous to a constant. Thus we conclude that \( \mathcal{E}_{f,\phi,\psi} \) is a strictly convex function. Finally, using that the topological pressure is differentiable with respect to the dynamics the proof of the proposition is now complete. \( \square \)
The following result illustrates some characteristics of the behaviour on the free energy function.

**Corollary 5.3.** For any Hölder continuous potential $\psi$ so that $\int \psi \, dm_{f,\phi} = 0$ the free energy function $[-t_{f,\phi}, t_{f,\phi}] \ni t \mapsto \mathcal{E}_{f,\phi}(t)$ satisfies:

1. $\mathcal{E}_{f,\phi}(0) = 0$ and $\mathcal{E}_{f,\phi}(t) \geq 0$ for all $t \in (-t_{f,\phi}, t_{f,\phi})$;
2. $t \inf \psi \leq \mathcal{E}_{f,\phi}(t) \leq t \sup \psi$ for all $t \in (0, t_{f,\phi})$;
3. $t \sup \psi \leq \mathcal{E}_{f,\phi}(t) \leq t \inf \psi$ for all $t \in [-t_{f,\phi}, 0)$.

**Proof.** It follows from the first part of Proposition 5.2 that $\mathcal{E}_{f,\phi}(0) = 0$. Now, since $\mathcal{E}'_{f,\phi,\psi}(t) > 0$ then $\mathcal{E}'_{f,\phi,\psi}$ is strictly increasing. Therefore, using $\mathcal{E}'_{f,\phi,\psi}(0) = \int \psi \, dm_{f,\phi} = 0$ it follows that $\mathcal{E}_{f,\phi,\psi}$ is strictly increasing for $t \in (0, t_{f,\phi})$ and strictly decreasing for $t \in (-t_{f,\phi}, 0)$. This proves item (1) above. Finally, (2) and (3) is a simple consequence of the mean value theorem and, using $\mathcal{E}'_{f,\phi,\psi}(t) = \int \psi \, dm_{f,\phi} + t \psi$, the fact that $\inf \psi \leq \mathcal{E}'_{f,\phi,\psi}(t) \leq \sup \psi$. This finishes the proof of the corollary. □

In what follows assume that $\psi$ is not cohomologous to a constant and that $m_{f,\phi} = \int \psi \, dm_{f,\phi} = 0$. Therefore, since the function $[-t_{f,\phi}, t_{f,\phi}] \ni t \mapsto \mathcal{E}_{f,\phi}(t)$ is strictly convex it is well defined the “local” Legendre transform $I_{f,\phi,\psi}$ given by

$$I_{f,\phi,\psi}(s) = \sup_{-t_{f,\phi} \leq t \leq t_{f,\phi}} \{ st - \mathcal{E}_{f,\phi}(t) \}.$$ 

**Remark 5.4.** This is a convex function since it is supremum of affine functions and, using that $\mathcal{E}_{f,\phi,\psi}$ is strictly convex and non-negative, $I_{f,\phi,\psi} \geq 0$. Moreover, notice that since $\mathcal{E}_{f,\phi,\psi} + ct = \mathcal{E}_{f,\phi,\psi}(t) + ct$ then we also get that $I_{f,\phi,\psi + c}(t) = I_{f,\phi,\psi}(t - c)$ for every $c, t \in \mathbb{R}$.

When the free energy function is differentiable it is not hard to check the variational property $I_{f,\phi,\psi}(\mathcal{E}'_{f,\phi,\psi}(t)) = t \mathcal{E}'_{f,\phi,\psi}(t) - \mathcal{E}_{f,\phi,\psi}(t)$ and the domain of $I_{f,\phi,\psi}$ contains the interval $[\mathcal{E}'_{f,\phi,\psi}(-t_{f,\phi}), \mathcal{E}'_{f,\phi,\psi}(t_{f,\phi})]$. Moreover, $I_{f,\phi,\psi}(s) = 0$ if and only if $s = m_{f,\phi}$ belongs to the domain of $I_{f,\phi,\psi}$. It is also well known that the strict convexity of $\mathcal{E}_{f,\phi,\psi}$ together with differentiability of $\mathcal{E}_{f,\phi,\psi}$ yields that $[-t_{f,\phi}, t_{f,\phi}] \ni t \mapsto I_{f,\phi,\psi}(t)$ is strictly convex and differentiable. Using the previous remark we collect all of these statements in the following:

**Corollary 5.5.** Let $f \in \mathcal{F}$ be arbitrary and let $\psi$ be an Hölder continuous observable. Then the rate function $I_{f,\phi,\psi}$ satisfies:

1. The domain $[\mathcal{E}'_{f,\phi,\psi}(-t_{f,\phi}), \mathcal{E}'_{f,\phi,\psi}(t_{f,\phi})]$ contains $m_{f,\phi} = \int \psi \, dm_{f,\phi}$;
2. $I_{f,\phi,\psi} \geq 0$ is strictly convex and $I_{f,\phi,\psi}(s) = 0$ if and only $s = \int \psi \, dm_{f,\phi}$;
3. $s \mapsto I_{f,\phi,\psi}(s)$ is real analytic.

5.3.2. Estimating deviations. Now we use the previous free energy function to obtain a “local” large deviation results. In fact, the following results hold from Gartner-Ellis theorem (see e.g. [DZ98, RY08]) as a consequence of the differentiability of the free energy function.

**Theorem 5.6.** Let $f$ be a local diffeomorphism so that (H1) and (H2) holds and let $\phi$ be a Hölder continuous potential such that (P) holds. Given any interval $[a, b] \subseteq [\mathcal{E}'_{f,\phi,\psi}(-t_{f,\phi}), \mathcal{E}'_{f,\phi,\psi}(t_{f,\phi})]$ it holds that

$$\limsup_{n \to \infty} \frac{1}{n} \log \mu_{f,\phi} \left( x \in M : \frac{1}{n} S_n \psi(x) \in [a, b] \right) \leq - \inf_{s \in [a, b]} I_{f,\phi,\psi}(s)$$
and
\[ \liminf_{n \to \infty} \frac{1}{n} \log \mu_{f,\phi}(x \in M : \frac{1}{n} S_n \psi(x) \in (a, b)) \geq - \inf_{s \in (a,b)} I_{f,\phi,\psi}(s) \]

Finally, to finish this section we deduce a regular dependence of the large deviations rate function with respect to the dynamics and potential. To the best of our knowledge these results are new even in the uniformly expanding setting.

**Proposition 5.7.** Let \( \psi \) be a Hölder continuous observable. There exists an interval \( J \subset \mathbb{R} \) containing \( m f, \phi \) such that for all \( [a, b] \subset J \) and \( f \in F^{1+\alpha} \)
\[ \limsup_{n \to \infty} \frac{1}{n} \log \mu_{f,\phi}(x \in M : \frac{1}{n} S_n \psi(x) \in [a, b]) \leq - \inf_{s \in [a,b]} I_{f,\phi,\psi}(s) \]
and
\[ \liminf_{n \to \infty} \frac{1}{n} \log \mu_{f,\phi}(x \in M : \frac{1}{n} S_n \psi(x) \in (a, b)) \geq - \inf_{s \in (a,b)} I_{f,\phi,\psi}(s) \]

Moreover, if \( V \) is a compact metric space and \( V \ni v \mapsto f_v \in F^2 \) is a continuous and injective map then the rate function \( (s, v) \mapsto I_{f_v,\phi,\psi}(s) \) is continuous on \( J \times V \).

**Proof.** Fix \( f_0 \in F \). We obtain a large deviation principle for Birkhoff averages on subintervals of a given interval \( [\mathcal{E}_{f_0,\phi,\psi}(-t,\phi,\psi), \mathcal{E}_{f_0,\phi,\psi}(t,\phi,\psi)] \) given by Theorem 5.6. Observe that the interval \( [\mathcal{E}_{f,\phi,\psi}(-t,\phi,\psi), \mathcal{E}_{f,\phi,\psi}(t,\phi,\psi)] \) is non-degenerate and varies continuously with \( f \) and \( \psi \). Hence, we may take a non-degenerate interval \( J \) contained in all intervals \( [\mathcal{E}_{f,\phi,\psi}(-t,\phi,\psi), \mathcal{E}_{f,\phi,\psi}(t,\phi,\psi)] \) for all \( f \in F \) sufficiently close to \( f_0 \). This proves the first assertion above.

Finally, from the variational relation using the Legendre transform and the convexity of the free energy function (that is, \( \mathcal{E}_{f,\phi,\psi}'(s) > 0 \) for all \( s \)) we get that for any \( s \in J \) there exists a unique \( t = t(s, v) \) such that \( s = \mathcal{E}_{f,\phi,\psi}'(t) \)
\[ I_{f,\phi,\psi}(s) = s \cdot t(s, v) - \mathcal{E}_{f,\phi,\psi}(t(s, v)). \] (5.1)

Now, we consider the continuous skew-product
\[ F : V \times J \to V \times \mathbb{R} \]
\[ (v, t) \mapsto (v, \mathcal{E}_{f,\phi,\psi}'(t)) \]
and notice that it is injective because it is strictly increasing along the fibers. Since \( V \times J \) is a compact metric space then \( F \) is a homeomorphism onto its image \( F(V \times J) \). In particular this shows that for every \( (v, s) \in F(V \times J) \) there exists a unique \( t = t(v, s) \) varying continuously with \( (v, s) \) such that \( F(v, t(v, s)) = (v, s) \) and \( s = \mathcal{E}_{f,\phi,\psi}'(t) \). Finally, relation (5.1) above yields that \( (s, v) \mapsto I_{f,\phi,\psi}(s) \) is continuous on \( J \times V \). This finishes the proof of the corollary.

It is not hard to check that the rate function is real analytic with respect to the potential. However, since the proof is much simpler than the previous one we shall omit it and leave as an exercise to the reader.

6. SOME EXAMPLES

In this section we provide some applications where we discuss mainly the smooth or continuous variation relevant dynamical quantities for non-uniformly expanding maps obtained through bifurcation theory. In particular, although our results apply for uniformly expanding dynamics we discuss some robust examples that can be far from being expanding.
6.1. One-dimensional examples.

6.1.1. Discontinuity of Perron-Frobenius operator for circle expanding maps. In order to illustrate the discontinuity of the Perron-Frobenius operator when acting on the space of functions with low regularity. We provide a one-dimensional example just for simplicity.

We claim that the transfer operator \( \mathcal{L}_{f, \varphi} : C^\alpha(S^1, \mathbb{R}) \to C^\alpha(S^1, \mathbb{R}) \) associated to the doubling circle map \( f : S^1 \to S^1 \) is discontinuous both in the operator norm as well as pointwise. In fact, up to consider the metric \( \tilde{d}(x, y) = d(x, y)^\alpha \) we are reduced to prove the discontinuity of the transfer operators acting on the space of Lipschitz observables. The key idea is that the composition operator \( \varphi \to \varphi \circ g \) acting in the space of Lipschitz functions does not vary continuously with \( g \), as we now detail.

Let \( S^1 \simeq \mathbb{R}/[-1/2, 1/2] \) be the circle and consider the expanding maps of the circle \( f_n(x) = 2(x + \frac{1}{10n}) \mod 1 \), \( n \geq 1 \). It is clear that the sequence \( (f_n)_n \) is convergent to the doubling map \( f(x) = 2x \mod 1 \) in the \( C^\infty \)-topology.

Now, pick a Lipschitz function \( \varphi \) in the circle so that \( \varphi(x) = |x| \) for \( |x| \leq 1/8 \) and \( \varphi(x) = 0 \) for \( 1/2 \geq |x| \geq 1/5 \), and consider the potential \( \phi \equiv 0 \). In this way, if \( 0 < x_n < y_n < 1/10n \), we obtain that

\[
\text{Lip}((\mathcal{L}_{f_n, \varphi} - \mathcal{L}_{f, \varphi})(\varphi)) \geq \frac{|\mathcal{L}_{f_n, \varphi}(\varphi)(y_n) - \mathcal{L}_{f_n, \varphi}(\varphi)(x_n) + \mathcal{L}(\varphi)(x_n) - \mathcal{L}(\varphi)(y_n)|}{y_n - x_n}
\]

\[
= \frac{|y_n/2 - 1/10n| - |x_n/2 - 1/10n| + |x_n/2| - |y_n/2|}{y_n - x_n}
\]

\[
= \frac{|-y_n - x_n|}{y_n - x_n} = 1 = \text{Lip}(\varphi)
\]

for all \( n \in \mathbb{N} \). Thus the sequence of transfer operators \( (\mathcal{L}_{f_n, \varphi})_n \) does not converge to \( \mathcal{L}_{f, \varphi} \) even in the strong operator topology. Nevertheless we have that

\[
(f, \varphi) \mapsto \mathcal{L}_{f_n, \varphi}1 = \sum_{f(y)=x} \varphi(y)
\]

is indeed continuous, which was enough for us to prove the differentiability of the topological pressure function.

6.1.2. Manneville-Pomeau maps. Given \( \alpha > 0 \), let \( f_\alpha : [0, 1] \to [0, 1] \) be the local diffeomorphism given by

\[
f_\alpha(x) = \begin{cases} 
x(1 + 2^\alpha x^\alpha) & \text{if } 0 \leq x \leq \frac{1}{2} \\
2x - 1 & \text{if } \frac{1}{2} < x \leq 1
\end{cases}
\]

(6.1)

and the family of potentials \( \varphi_{\alpha, t} = -t \log |Df_\alpha| \). Note that \( f \) is a \( C^{r+\beta} \)-local diffeomorphism, where \( r = 1 + [\alpha] \) and \( \beta = \alpha - [\alpha] \). In consequence \( \varphi_{\alpha, t} \in C^{r-1+\beta}(M, \mathbb{R}) \). Since it is not hard to check that a similar construction of an expanding map with an indifferent fixed point can be realized as a circle local diffeomorphism, we will deal with this family for simplicity. Moreover, conditions (H1) and (H2) are clearly verified for all \( f_\alpha \). It is well known that if \( \alpha \in (0, 1) \) then an intermittency phenomenon occurs for the potentials \( \varphi_{\alpha, t} \) at \( t = 1 \). However no phase transitions occur at high temperature, as we now discuss.
Assume first $\alpha \in (0, 2)$. The family $\varphi_{\alpha,t}$ of $C^{r-1+\beta}$-potentials do satisfy condition (P) for all $|t| \leq t_0$ small, that can be taken not depending on $\alpha$ since $\alpha < 2$ and

$$|\varphi_{\alpha,t}(x) - \varphi_{\alpha,t}(y)| = |t \log |Df_{\alpha}(x)| - t \log |Df_{\alpha}(y)|| = |t| \log \frac{|Df_{\alpha}(x)|}{|Df_{\alpha}(y)|} \leq |t| \log (2+\alpha).$$

In fact, if on the one hand, $\sup \varphi_{\alpha,t} - \inf \varphi_{\alpha,t} \leq t_0 \log 4$, on the other hand the Hölder constant of $\varphi_{\alpha,t}$ can be made small provided that $t$ is small. Hence, it follows from Theorem 2.1 that for all $|t| \leq t_0$ there exists a unique equilibrium state $\mu_{\alpha,t}$ for $f_{\alpha}$ with respect to $\varphi_{\alpha,t}$, it has exponential decay of correlations in the space of Hölder observables, and that the pressure $t \mapsto P_{\text{top}}(f_{\alpha}, -t \log |Df_{\alpha}|)$ and the equilibrium state $t \mapsto \mu_{\alpha,t}$ are continuous in the interval $(-t_0, t_0)$. In consequence, the Lyapunov exponent function $t \mapsto \lambda(\mu_{\alpha,t}) = \int \log |f'_{\alpha}| \, d\mu_{\alpha,t}$ is also continuous.

Now we will discuss the case that $\alpha \in [2, +\infty)$. In this case one can say that $f_{\alpha}$ is at least $C^3$ and the potentials $\varphi_{\alpha,t}$ are at least $C^2$. Moreover, $|\varphi_{\alpha,t}(x)| \leq |t|2^{\alpha} (1 + \alpha)|x|^{\alpha-1}$ can be taken uniformly small, thus satisfying (P'), provided that $|t| \leq t_a$ small. Therefore our results imply that no transition occurs once one considers the order of contact $\alpha$ of the indifferent fixed point to increase. In fact, not only the maximal entropy measure varies differentiably with the contact order $\alpha$ of the indifferent fixed point as we deduce from Corollary 5A that the topological pressure

$$(1, +\infty) \times [-t_a, t_a] \to \mathbb{R}$$

$$(\alpha, t) \mapsto P_{\text{top}}(f_{\alpha}, -t \log |Df_{\alpha}|)$$

and the Lyapunov exponent function

$$(1, +\infty) \times [-t_a, t_a] \to \mathbb{R}$$

$$(\alpha, t) \mapsto \int \log |Df_{\alpha}| \, d\mu_{\alpha,t}$$

are differentiable.

6.1.3. Bifurcations of circle expanding maps. Let $f$ be a $C^{r+\alpha}$-expanding map of the circle $S^1$ with degree $d$ and let $p \in S^1$ be a fixed point for $f$, with $r \geq 2$ and $\alpha \geq 0$. Assume that $(f_t)_{t \in [0, 1]}$ is a one-parameter family of $C^{r+\alpha}$-local diffeomorphisms of the circle such that $f_0 = f$, all maps $f_t$ satisfy hypothesis (H1) and (H2) with uniform constants and $f_1$ exhibits a periodic attractor at $p$.

Then, Theorems 3B and 3C yield that there exists a $C^3$-neighborhood $\mathcal{F}$ of the curve $(f_t)_{t \in [0, 1]}$ and a $C^\alpha$ neighborhood $\mathcal{W}$ of the constant zero potential such that the pressure function $\mathcal{F} \times \mathcal{W} \ni (\hat{f}, \hat{\phi}) \mapsto P_{\text{top}}(\hat{f}, \hat{\phi})$ is analytic on $\hat{\phi}$ and differentiable in $\hat{f}$. Moreover, both the maximal entropy measure function $\hat{f} \mapsto \mu_{\hat{f}}$ and the Lyapunov exponent function $\hat{f} \mapsto \int \log |D\hat{f}| \, d\mu_{\hat{f}}$ varies differentiably. In particular, since

$$t \mapsto \dim_H(\mu_{f_t}) = \frac{h_{f_{t_1}}(f_t)}{\int \log |Df_{t_1}| \, d\mu_{f_t}} = \frac{\log d}{\int \log |Df_{t_1}| \, d\mu_{f_t}}$$

then the Hausdorff dimension of the maximal entropy measure is smooth on $t$ along the bifurcation.

6.2. Higher dimensional examples.
6.2.1. Derived from expanding maps. Let \( f_0 : \mathbb{T}^d \to \mathbb{T}^d \) be a linear expanding map. Fix some covering \( \mathcal{U} \) by domains of injectivity for \( f_0 \) and some \( U_0 \in \mathcal{U} \) containing a fixed (or periodic) point \( p \). Then deform \( f_0 \) on a small neighborhood of \( p \) inside \( U_0 \) by a pitchfork bifurcation in such a way that \( p \) becomes a saddle for the perturbed local diffeomorphism \( f \). In particular, such perturbation can be done in the \( C^r \)-topology, for every \( r > 0 \). By construction, \( f \) coincides with \( f_0 \) in the complement of \( P_1 \), where uniform expansion holds. Observe that we may take the deformation in such a way that \( f \) is never too contracting in \( P_1 \), which guarantees that conditions (H1) and (H2) hold. Since the later are open conditions let \( \mathcal{V}^2 \) be a small open neighborhood of \( f \) by \( C^2 \) local diffeomorphisms satisfying (H1) and (H2). Since condition (P') is clearly satisfied by \( \phi \equiv 0 \) one can take \( \mathcal{V}^2 \) to be an open set of \( C^2 \)-potentials close to zero and satisfying (P') with uniform constants. It follows from [VV10, CV13] that there exists a unique equilibrium state for \( f \) with respect to \( \phi \), it has full support, is has exponential decay of correlations in the space of Hölder observables and that equilibrium states and topological pressure vary continuously with the dynamics.

Concerning higher regularity of these functions it follows from Theorems [3] and [4] that the pressure function \( (f, \phi) \mapsto P(f, \phi) \) is analytical in \( \phi \) and differentiable with respect to \( f \), the invariant density function \( (f, \phi) \mapsto h_{f, \phi} \) and the equilibrium state function \( (f, \phi) \mapsto \mu_{f, \phi} \) are analytical in \( \phi \). Furthermore, if one considers perturbations in the \( C^3 \)-topology then the largest, smallest and sum of Lyapunov exponents and the metric entropy of the equilibrium states \( \mu_{f, \phi} \) vary continuously with respect to \( f \) and \( \phi \); the largest, smallest and sum of Lyapunov exponents of the maximum entropy \( \mu_{f, 0} \) vary differentially with respect to \( f \). Finally, the unique measure of maximal entropy \( \mu_f \) is differentiable with respect to \( f \).

**Remark 6.1.** Let us mention an easy modification of the previous example allows to consider multidimensional expanding maps with indifferent periodic points. In consequence all the results discussed above hold also in this context.

6.2.2. Non-uniformly expanding repellers through Hopf bifurcations. Hopf bifurcations constitute an important class of bifurcations and arise in many physical phenomena as e.g. the Selkov model of glycolysis or the Belousov-Zhabotinsky reaction. We obtain applications also to these class of examples.

Let \( f_0 \) be a linear endomorphism on the 2-dimensional torus \( M = \mathbb{T}^2 \) with two complex conjugate eigenvalues \( \sigma e^{i\gamma} \) with \( \sigma > 3 \) and \( \gamma \in \mathbb{R} \) satisfying the non-resonance condition \( k \gamma \notin 2\pi \mathbb{Z} \) for \( k \in \{1, 2, 3, 4\} \). Following [HV05] we consider a one parameter family \( (f_t)_{t} \) of \( C^5 \)-local diffeomorphisms going through a Hopf bifurcation at \( t = 0 \) in a small neighborhood \( V \) of the fixed point corresponding to the origin. More precisely, in local cylindrical coordinates \( (r, \theta) \) in a neighborhood of zero the map \( f_0 \) can be expressed as \( f_0(r, \theta) = (\sigma r, \theta + \zeta) \). So, proceeding as in [HV05] we can obtain a one-parameter family

\[
\tilde{f}_t(r, \theta) = (g(t, r^2, r, \theta + \zeta)
\]

with \( g \) being a real valued \( C^\infty \) map on \([-1, 1]^2 \) and constants \( C, \delta > 0 \) such that \( g(t, 0) = 1 - t \leq g(t, s) \) for all \( s \geq 0 \), that \( g(t, s) = \sigma \) whenever \( s \geq \delta_0 \), that \( \partial_s g(t, s) \in (0, C/\delta] \) for all \( 0 \leq s < \delta \), and also there exists \( 1 < \sigma_1 < \sigma \) and \( 0 < \delta_1 < \delta \) satisfying \( g(t, s) > \sigma_1 \) for all \( s > \sigma_1 \) and \( \partial_s g(t, s) > 0 \) for \( s \geq 0, \delta_1 ] \). Using the non-resonance condition, for any family \( (f_t)_{t} \) that is \( C^5 \)-close to \( (\tilde{f}_t)_{t} \) there exists a curve of fixed points \( (p_t)_{t} \) close to the origin that also go
through a Hopf bifurcation at some parameter $t_*$ (depending continuously of the family) close to zero.

The complement $\Lambda_t$ of the basin of attraction of the periodic attractor $p_t$ is a repeller. Moreover, since $\Lambda_t$ contains an invariant circle obtained as the boundary of the immediate basin of attraction of $p_t$ then cannot be a uniformly expanding repeller. Nevertheless, if $t_0$ is not much larger than $t_*$ then the curve $(f_t)_{t \in [0, t_0]}$ can be assumed to satisfy conditions (H1) and (H2) with uniform constants. In particular, for any small Hölder continuous potential $\phi$

$$(t, \phi) \mapsto P_{\top}(f_t, \phi)$$

is differentiable and the equilibrium state $(t, \phi) \mapsto \mu_{f_t, \phi}$ varies continuously. Moreover, if $\psi$ measures. Moreover, if $\Lambda_t$ is the repeller for $t \in (t_*, t_0]$. In fact, $t \mapsto \mu_t$ varies differentiably along the Hopf bifurcation.

6.2.3. Large deviations for (non)-uniformly expanding maps. Assume that $f$ is $C^2$ local diffeomorphism and $\Lambda \subset M$ be a transitive and $f$-invariant set such that $f|\Lambda$ uniformly expanding. In [You90], Young obtained a large deviations principle for the unique SRB measure which in our setting generalizes as follows: if $\phi$ is a Hölder continuous potential then for every $\psi : M \to \mathbb{R}$ continuous

$$\limsup_{n \to \infty} \frac{1}{n} \log \nu_{f, \phi} \left( x \in M : \frac{1}{n} S_n \psi(x) \in [a, b] \right) \leq - \inf_{s \in [a, b]} K_{f, \phi}(s) \tag{6.2}$$

and

$$\liminf_{n \to \infty} \frac{1}{n} \log \nu_{f, \phi} \left( x \in M : \frac{1}{n} S_n \psi(x) \in (a, b) \right) \geq - \inf_{s \in (a, b)} K_{f, \phi}(s) \tag{6.3}$$

where $K_{f, \psi}(s) = - \sup \left\{ -P_{\top}(f, \phi) + h_\eta(f) + \int \phi d\eta : \int \psi d\eta = s \right\}$. We refer the reader to [Va12] for a proof of the previous assertions and extension for weak Gibbs measures. Moreover, if $\psi$ is Hölder continuous then Theorem [F] yields a large deviation principle where the rate function $K_{f, \psi}$ in (6.2) and (6.3) is replaced by $I_{f, \phi, \psi}$, where $I_{f, \phi, \psi}(s) = \sup_{-t_0, \phi, \psi, t \in [t_0, t]} \left\{ st - \mathcal{E}_{f, \phi, \psi}(t) \right\}$ is the Legendre transform of the free energy function varies continuously. In particular this proves that the two rate functions above do coincide in the interval $(\mathcal{E}_{f, \phi, \psi}(t_0), \mathcal{E}_{f, \phi, \psi}(t_0))$.

Now, take $T_- = \min \left\{ \int \psi d\eta \right\}$ and $T_+ = \max \left\{ \int \psi d\eta \right\}$ where the minimum and maximum are taken over all $f$-invariant measures (we omit the dependence on $f$, $\phi$ and $\psi$ for notational simplicity). Then for any fixed $t \in (T_-, T_+)$

$$(f, \phi) \mapsto \sup \left\{ P_{\top}(f, \phi) - h_\eta(f) - \int \phi d\eta : \eta \in \mathcal{M}_1(f) \text{ and } \int \psi d\eta = t \right\}$$

is continuous, provided that $\psi$ is Hölder continuous. This illustrates the space of invariant probability measures is rich for uniformly expanding dynamical systems.

Remark 6.2. Our large deviation results apply also to the robust class of multidimensional local diffeomorphisms $\mathcal{F}^2$ obtained by bifurcations of expanding maps as in Subsections 6.2.1 and 6.2.2 above, and it yields that for any Hölder continuous observable $\psi$ not cohomologous to a constant there exists an interval $J \subset \mathbb{R}$ such that

$$\limsup_{n \to \infty} \frac{1}{n} \log \nu_{f, \phi} \left( x \in M : \frac{1}{n} S_n \psi(x) \in [a, b] \right) \leq - \inf_{s \in [a, b]} I_{f, \phi, \psi}(s)$$
and
\[
\liminf_{n \to \infty} \frac{1}{n} \log \nu_{f, \phi}\left( x \in M : \frac{1}{n} S_n \psi(x) \in (a, b) \right) \geq - \inf_{s \in (a, b)} I_{f, \phi, \psi}(s).
\]
for all \( f \in F \) and \([a, b] \subset J\). Previous to this local large deviations principle some upper and lower bounds were obtained in [Va12]. In addition, for any injectively parametrized family \( V \ni v \to f_v \), as in the Hopf bifurcation construction, the rate function \((s, v) \mapsto I_{f_v, \psi}(s)\) varies continuously with the dynamics and the potential.

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