COMPLEX PRODUCT MANIFOLDS AND BOUNDS OF CURVATURE

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Abstract. Let $M = X \times Y$ be the product of two complex manifolds of positive dimensions. In this paper, we prove that there is no complete Kähler metric $g$ on $M$ such that: either (i) the holomorphic bisectional curvature of $g$ is bounded by a negative constant and the Ricci curvature is bounded below by $-C(1 + r^2)$ where $r$ is the distance from a fixed point; or (ii) $g$ has nonpositive sectional curvature and the holomorphic bisectional curvature is bounded above by $-B(1 + r^2)^{-\delta}$ and the Ricci curvature is bounded below by $-A(1 + r^2)^{\gamma}$ where $A, B, \gamma, \delta$ are positive constants with $\gamma + 2\delta < 1$. These are generalizations of some previous results, in particular the result of Seshadri and Zheng \cite{8}.

1. Introduction

In \cite{8}, Seshadri and Zheng proved the following result:

\textbf{Theorem 1.1.} Let $M = X \times Y$ be the product of two complex manifolds of positive dimensions. Then $M$ does not admit any complete Hermitian metric with bounded torsion and holomorphic bisectional curvature bounded between two negative constants.

In particular, there is no complete Kähler metric on $M$ with holomorphic bisectional curvature bounded between two negative constants. For earlier results in this direction see \cite{11, 15, 16, 7}. The result of Seshadri-Zheng has been generalized by Tosatti \cite{10} to almost-Hermitian manifolds.

On the other hand, there is an open question whether the assumption on the lower bound of the curvature can be removed. In fact, it is an open question raised by N. Mok (see \cite{8}) whether the bidisc admit a

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complete Kähler metric with bisectional curvature bounded above by -1.

In this work, by using a local version of the generalized Schwartz lemma of Yau [14] and an Omori-Yau type maximum principle of Takegoshi [9], we prove the following:

**Theorem 1.2.** Let \( M = X^m \times Y^n \) be the product of two complex manifolds with positive dimension. Then, there is no complete Kähler metric on \( M \) with Ricci curvature \( \geq -A(1 + r)^2 \) and holomorphic bisectional curvature \( \leq -B \), where \( A \) is some nonnegative constants, \( B \) is some positive constant, and \( r(x, y) = d(o, (x, y)) \) is the distance of \( (x, y) \) and a fixed point \( o \in M \).

On the other hand, Seshadri [7] has constructed a complete Kähler metric on \( \mathbb{C}^n \) with negative curvature. It seems that the assumption on the upper bound of the curvature in Theorem 1.1 is necessary. However, one can also relax the upper bound of the curvature as follows:

**Theorem 1.3.** Let \( M = X^m \times Y^n \) be the product of two complex manifolds with positive dimensions. Then, there is no complete Kähler metric on \( M \) with Ricci curvature \( \geq -A(1 + r)^2 \gamma \), holomorphic bisectional curvature \( \leq -B(1 + r)^2 - \delta \), and nonpositive sectional curvature, where \( \gamma \geq 0 \), \( \delta > 0 \) such that \( \gamma + 2\delta < 1 \), \( A \), \( B \) are some positive constants, and \( r(x, y) = d(o, (x, y)) \) is the distance of \( (x, y) \) and a fixed point \( o \in M \).

Our method relies on a simple observation. Suppose there is a Hermitian metric \( g \) with negative holomorphic bisectional curvature on \( M = X \times Y \). Let \( q \) be any fixed point in \( Y \). Then the holomorphic vector bundle \( V \) over \( X_q := X \times \{q\} \), with fibre \( V_{(x,q)} = T^{1,0}_{x,q}Y \subset T_{(x,q)}M = T_{x,q}^{1,0}X \oplus T_{q}^{1,0}Y \), as a subbundle of \( T^{1,0}M|_{X_q} \), is negative. However, \( V = T^{(1,0)}_{q}Y \times X_q \) is a trivial vector bundle, and hence have nonzero global holomorphic sections. Hence the question reduces to the question on the existence of nontrivial global holomorphic sections on the negative vector bundle \( V \). When \( X \) is compact, then one can conclude easily that this is impossible (See Kobayashi-Wu [6]). Hence \( X \times Y \) does not have a Kähler metric with negative holomorphic bisectional curvature. The result was first proved by Zheng [16] using different method (When \( g \) is Kähler, it is first proved by Yang [12]). Here the metric is not even assumed to be complete.

When \( X \) is noncompact and curvature bounds are relaxed, we can only have vanishing theorem with some restriction on the growth of the global section. Controlling the growth of the global section can be
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The paper is organized as follows: In section 2, we will prove Theorem 1.2 and in section 3 we will prove Theorem 1.3.

2. Proof of Theorem 1.2

Before we prove the theorem, we need several lemmas. First, we have the following local version of the Schwartz lemma by Yau [14]. See also [2, Theorem 2.1].

Lemma 2.1. Let \((M^m, g)\) and \((N^n, h)\) be two complete Kähler manifolds and let \(f\) be a holomorphic map from \(M\) to \(N\). Let \(o \in M\) and let \(R > 0\). Suppose the Ricci curvature of \(B_o(2R)\) is bounded from below by \(-K\) and the holomorphic bisectional curvature at every point in \(f(B_o(2R))\) is bounded above by \(-B\) where \(K\) and \(B\) are positive constants. Then on \(B_o(R)\),

\[
(2.1) \quad f^*\omega_h \leq C \cdot \frac{K + R^{-2} \left(1 + K^{1/2} R \coth(K^{1/2} R)\right)}{B} \omega_g
\]

for some positive constant \(C\) depending only on \(m\).

Proof. Let \(u = \|\partial f\|^2\) which is half the energy density of \(f\). It is clear that \(f^*\omega_h \leq \|\partial f\|^2 \omega_g\). Then \(u\) satisfies the following inequality on \(B(2R)\) (See [3], [14])

\[
(2.2) \quad \frac{1}{2} \Delta u \geq -Ku + Bu^2.
\]

Let \(\eta \geq 0\) be a smooth function on \(\mathbb{R}\) such that (1) \(\eta(t) = 1\) for \(t \leq 1\), (2) \(\eta(t) = 0\) for \(t \geq 2\), (3) \(-C_1 \leq \eta'/\eta^{1/2} \leq 0\) for all \(t \in \mathbb{R}\), and (4) \(|\eta''(t)| \leq C_1\) for all \(t \in \mathbb{R}\) for some absolute constant \(C_1 > 0\). Let \(\phi = \eta(r/R)\).

Suppose \(\phi u\) attains maximum at \(\bar{x} \in B_o(2R)\). We can assume that \(\phi(\bar{x}) > 0\) otherwise \(u(\bar{x}) = 0\) for any \(x \in B_o(R)\) and we are done. Using an argument of Calabi as in [3], we may assume that \(\phi u\) is smooth at \(\bar{x}\). Then, we have (1) \(\nabla(\phi u)(\bar{x}) = 0\) which implies that at
Let $\bar{x}$, $\nabla u = -u\phi^{-1}\nabla\phi$, (2) $\Delta(\phi u)(\bar{x}) \leq 0$. Hence at $\bar{x}$, we have:

$$0 \geq \Delta(\phi u) = \phi \Delta u + 2\langle \nabla \phi, \nabla u \rangle + u\Delta \phi = \phi \Delta u + 2\langle \nabla \phi, -u\phi^{-1}\nabla \phi \rangle + u\Delta \phi = \phi \Delta u - 2uR^{-2} \left( \frac{(|\eta'|)^2}{\eta} + u \left( R^{-1}\eta'\Delta r + R^{-2}\eta'' \right) \right) \geq \phi (-2Ku + 2Bu^2) - C_2 R^{-2} \left( 1 + K \frac{1}{2} R \coth(K \frac{1}{2} R) \right) u.$$ 

where $C_2$ is a positive constant depending only on $m$. Here we have used (2.2), the properties of $\eta$ and the Laplacian comparison [5]. Hence

$$2B(\phi u)^2(\bar{x}) \leq \left[ 2K + C_2 R^{-2} \left( 1 + K \frac{1}{2} R \coth(K \frac{1}{2} R) \right) \right] (\phi u)(\bar{x}).$$

Hence

$$\sup_{B_o(R)} u \leq \sup_{B_o(2R)} (\phi u)(\bar{x}) \leq \frac{2K + C_2 R^{-2} \left( 1 + K \frac{1}{2} R \coth(K \frac{1}{2} R) \right)}{2B}.$$ 

From this the lemma follows.

Before we state the next lemma, let us introduce some notations.

Let $M = X \times Y$ and let $o = (p, q) \in X \times Y$ be a fixed point. For any $x \in X$, let $Y_x = \{x\} \times Y$ with induced metric denoted as $g^x$ and for any $y \in Y$, let $X_y = X \times \{y\}$ with induced metric denoted as $g^y$.

**Lemma 2.2.** Let $M$, $X, Y$ as in Theorem 1.2. Suppose there is a complete Kähler metric on $M$ with Ricci curvature bounded from below by $-A(1+r)^2$ and with holomorphic bisectional curvature bounded from above by $-B < 0$. Let $o = (p, q) \in X \times Y$ be a fixed point. Let $V_p^{X_q}(r)$ be the volume of the geodesic ball of radius $r$ with center at $p$ with respect to the induced metric $g^q$. Then

$$V_p^{X_q}(r) \leq C_1 \exp(C_2 r^2)$$

for some constants $C_1$ and $C_2$ independent of $r$.

**Proof.** Let $x_0 \in X$ be any point. Consider the projection $\pi''_{x_0} : X \times Y \to Y_{x_0}$ such that $\pi''_{x_0}(x, y) = (x_0, y)$. Note that the holomorphic bisectional curvature of $Y_{x_0}$ is still bounded above by $-B$. By Lemma 2.1, there is a constant $C_1$ independent of $x_0$ such that

$$(\pi''_{x_0})^*(g^{x_0})_{|(x,y)} \leq C_1 (1 + r(x, y))^2 g_{|(x,y)}$$

(2.4)
for all \((x, y)\). Similarly, if we choose \(C_1\) large enough, we also have:

\[
(\pi'_{y_0})^*(g_{y_0})|_{(x, y)} \leq C_1 (1 + r(x, y))^2 g|_{(x, y)}
\]

for any \(y_0 \in Y\), and \((x, y) \in M\) where \(\pi'\) is the projection from \(M\) onto \(X_{y_0}\).

Let \(\gamma\) be any smooth curve in \(Y_p\) from \((p, q)\). Then by (2.4), for any \((x, q) \in X_q\) with \(r(x, q) \leq R\), the length \(L(\pi_x \circ \gamma)\) satisfies:

\[
L(\pi_x \circ \gamma) \leq C_1 \frac{1}{2} (1 + r(x, q)) L(\gamma).
\]

Hence there is \(\rho > 0\) such that for \(R > 1\) if \(B_{\rho}(p)\) is the geodesic ball in \(Y_p\) with radius \(\rho\) and center at \((p, q)\), then \(\pi_x(B_{\rho}(p)) \subset B_o(2R)\) for all \((x, q) \in B_o(R)\).

On the other hand, by (2.5), the Jacobian \(J(\pi'_q)\) of \(\pi'_q\) at \((x, y)\) satisfies:

\[
J(\pi'_q)(x, y) \leq C_2 (1 + r(x, y))^{2m}
\]

for some constant \(C_2\) for all \((x, y)\).

Now, let \(R > 1\) be any constant. Let \(dV_x\) be the volume element of \(Y_x\) and \(dV_y\) be the volume element of \(X_y\). By the co-area formula (see [1]), we have

\[
V_o(2R) = \int_M \chi_{B_o(2R)} dV_y
\]

\[
= \int_{X_q} \int_{y \in Y_x} \chi_{B_o(2R)} |J(\pi'_q)|^{-1}(x, y) dV_x dV_q
\]

\[
\geq C_2^{-1} (1 + R)^{-2m} \int_{(x, q) \in B_o(R)} \int_{\pi'_q(B^p(\rho))} dV_x dV_q
\]

\[
\geq C_3 (1 + R)^{-2m} (1 + R)^{-2n} V^p(\rho)V^q(R)
\]

for some constant \(C_3 > 0\) for all \(R\) by (2.4). Here \(V^p(\rho)\) is the volume of \(B^p(\rho)\) in \(Y_p\) and \(V^q(R)\) is the volume of the geodesic ball in \(X_q\) with radius \(R\) and center at \((p, q)\).

By volume comparison, we have \(V_o(2R) \leq \exp(C(1 + R)^2)\) for some constant \(C\). From this and (2.7), the result follows.

We also need the following result of Takegoshi [9, Theorem 1.1]:

**Theorem 2.1.** Let \(M\) be a complete noncompact Riemannian manifold. Suppose there is a smooth function \(f\) such that \(S = \{f > \delta\}\) is nonempty for some \(\delta > 0\) and on \(S\)

\[
\Delta f \geq \frac{C_f f^{1+a}}{(1 + r)^b}
\]

...
for some positive constants $C, a$ and $0 \leq b < 2$ where $r$ is the distance function from some fixed point. Then the volume $V(r)$ of the geodesic ball with radius $r$ satisfies:

$$\liminf_{r \to \infty} \frac{\log V(r)}{r^{2-b}} = \infty.$$ 

Proof of Theorem 1.2. We proceed by contradiction. Let $g$ be a complete Kähler metric on $M$ satisfying the assumptions.

Suppose $o = (p, q) \in M$. Let $u$ be vector in $T^{1,0}_p(M) = T^{1,0}_p(X) \times T^{1,0}_q(Y)$ such that $u \in \{0\} \times T^{1,0}_q(Y)$ and such that $g(u, \bar{u}) = 1$. Let

$$f(x) = f(x, q) = \| (\pi_x)_* u \|^2.$$ 

Then $f$ is a function on $X_q$. Then, by (2.4)

(2.8)  \quad f(x) \leq C_1(1 + r(p, q))^2 g_{u\bar{u}}(p, q) = C_1 g_{u\bar{u}}(p, q).$$

Hence, $f$ is a positive bounded function.

Let $(z^1, z^2, \cdots, z^m, z^{m+1}, \cdots, z^{n+m})$ be a holomorphic coordinate of $M$ at $(x, q)$ such that (1) $(z^1, z^2, \cdots, z^m)$ is a normal coordinate of $X_q$ at $x$ and (2) $g_{ab}(x, q) = \delta_{ab}$, $m+1 \leq a, b \leq m+n$. Then, by identifying $(\pi_x)_* u$ with $u$, we have

$$\Delta_{X_q} f(x) = 2 \sum_{i=1}^{m} \partial_i \partial_{\bar{i}} g_{u\bar{u}}(x, q)$$

$$= 2 \sum_{i=1}^{m} ( -R_{u\bar{u}i\bar{i}} + g^{ba} \partial_i g_{u\bar{b}} \partial_{\bar{i}} g_{u\bar{a}})(x, q)$$

(2.9)

$$= -2 \sum_{i=1}^{m} R_{u\bar{u}i\bar{i}}(x, q) + 2 \sum_{i=1}^{m} \sum_{b=1}^{n+m} |\partial_i g_{u\bar{b}}|^2(x, q)$$

$$\geq 2mBg_{u\bar{u}}(x, q) = 2mf(x).$$

Combining this with (2.8), we have

$$\Delta_{X_q} f \geq \frac{2mB}{C_1} f^2.$$ 

By Lemma 2.2 and Theorem 2.1 we have a contradiction because $f > 0$. 

3. Proof of Theorem 1.3

In order to prove the second main result, we need the following lemma. We will use the notations as in the previous section.
Lemma 3.1. Let $M = X^m \times Y^n$ be the product of two simply connected complex manifolds with positive dimension. Suppose that $g$ is a complete Kähler metric on $M$ with Ricci curvature $\geq -A(1 + r^2)^{\gamma}$, holomorphic bisectional curvature $\leq -B(1 + r^2)^{-\delta}$, and nonpositive sectional curvature, where $\gamma \geq 0$, $\delta > 0$ such that $\gamma + 2\delta < 1$, $A, B$ are some positive constants, and $r = r(x, y)$ is the distance of $(x, y) \in X \times Y$ and a fixed point $o \in M$. Then, there is a positive constant $C$ depending only on $m, n, \gamma, \delta, A$ and $B$, such that for any $x_0 \in X$,

$$\pi''_{x_0}^*(g^{x_0})|_{(x, y)} \leq C(1 + r^2(x, y))^\gamma(1 + r^2(x_0, y))^\delta g|_{(x, y)}$$

for any $(x, y) \in M$ and $y \in Y$.

Proof. For any point $x_0 \in X$, let $u = \|\partial \pi''_{x_0}\|^2$. Then as before, by [4], [14], we have:

$$\Delta u(x, y) \geq -2A(1 + r^2(x, y))^{\gamma}u(x, y) + 2B(1 + r^2(x_0, y))^{-\delta}u(x, y)^2.$$  

Let $v(x, y) = r^2(x_0, y)$. Since $M$ is simply connected and has nonpositive curvature, $r^2(x, y)$ and $v$ are both smooth functions. In the following $\alpha, \beta$ range from $m+1$ to $m+n$. For $(x, y) \in M$, let $z^1, z^2, \ldots, z^m$ be holomorphic coordinates of $x$ in $X$ and $z^{m+1}, \ldots, z^{m+n}$ be holomorphic coordinates of $y$ in $Y$ such that (1) $g_{\alpha\beta}(x, y) = \delta_{\alpha\beta}$, $m+1 \leq \alpha\beta \leq m+n$ and (2) $g_{\alpha\beta}(x_0, y) = \lambda_{\alpha}\delta_{\alpha\beta}$. Here $(z^{m+1}, \ldots, z^{m+n})$ are also considered as holomorphic coordinates of $Y_{x_0}$ because $\pi''_{x_0}$ is a biholomorphism between $Y_x$ and $Y_{x_0}$. Then $u(x, y) = g^{\beta\alpha}(x, y)g_{\alpha\beta}(x_0, y)$. Then

$$\|\nabla v(x, y)\|^2(x, y)$$

$$= 2g^{\beta\alpha}(x, y)v_{\alpha}(x, y)v_{\beta}(x, y)$$

$$= 2g^{\beta\alpha}(x, y)v_{\alpha}(x, y)v_{\beta}(x, y)$$

$$= 2u(x, y)g^{\beta\alpha}(x_0, y)v_{\alpha}(x, y)v_{\beta}(x, y)$$

$$\leq 4u(x, y)v(x, y)$$
where we have used the fact that the $|\nabla r(x, y)| = 1$ and $r(x_0, y) = r(x, y)|_{Y_{x_0}}$. On the other hand, since $r^2$ is convex, we have

$$\Delta v(x, y) = 2g^{\delta\alpha}(x, y)v_{\alpha\delta}(x, y)$$

$$= 2g^{\delta\alpha}(x, y)v_{\alpha\delta}(x, y)$$

$$= 2\sum_\alpha v_{\alpha\delta}(x_0, y)$$

(3.4)

$$\leq 2u(x, y)\sum_\alpha \lambda_\alpha^{-1}v_{\alpha\delta}(x_0, y) \quad \text{(since } v_{\alpha\delta} > 0)$$

$$= u(x, y)\Delta v(x_0, y)$$

$$\leq u(x, y)(\Delta r^2)(x_0, y) \quad \text{(since } r^2 \text{ is convex)}$$

$$\leq C_1 u(x, y)(1 + v(x, y))^{\frac{\gamma + 1}{2}}$$

for some constant $C_1$ by the Laplacian comparison [5] and the assumption on the Ricci curvature of $M$. Here and below $C_i$ will denote constants depending only on $m, n, \gamma, \delta, A, B$. Let

$$w(x, y) = u(x, y)(C_2 + v(x, y))^{-\delta}$$

where $C_2 > 1$ is a constant to be determined later. Then,

(3.5)

$$\Delta w = (C_2 + v)^{-\delta}\Delta u - 2\delta(C_2 + v)^{-1-\delta}\langle \nabla u, \nabla v \rangle$$

$$- u\delta(C_2 + v)^{-1-\delta}\Delta v + u\delta(\delta + 1)(C_2 + v)^{-2-\delta}|\nabla v|^2$$

$$\geq \left(2B - \delta C_1(C_2 + v)^{\frac{\gamma + \delta - 1}{2}}\right)w^2 - 2A(1 + r^2)^\gamma w - 2\delta(C_2 + v)^{-1}\langle \nabla w, \nabla v \rangle$$

$$\geq \left(2B - \delta C_1 C_2^{\frac{\gamma + \delta - 1}{2}}\right)w^2 - 2A(1 + r^2)^\gamma w - 4\delta|\nabla w| |w|^{\frac{2}{2}}$$

where we have used (3.3), (3.4), (3.2) and the that $C_2 > 1$ and $\gamma + 2\delta < 1$. Hence we may choose $C_2 > 0$ large enough, so that

(3.6)

$$\Delta w \geq C_3 w^2 - 2A(1 + r^2)^\gamma w - 4\delta|\nabla w| |w|^{\frac{2}{2}}$$

for some $C_3 > 0$. Then one can proceed as in the proof of Lemma 2.1 to conclude that (3.1) is true.

\[\square\]

Proof of Theorem 1.3. First observe that we may assume $M$ is simply connected because the distance function in the universal cover of $M$ is greater than or equal to the distance function in $M$. Suppose there is complete Kähler metric $g$ on $M$ on $M$ satisfying the curvature assumptions.
Let \( o = (p, q) \). As in the proof of Theorem 1.2, let \( u \) be a vector in \( T^1_{p,0}(M) = T^1_{p,0}(X) \times T^1_{q,0}(Y) \) such that \( u \in \{0\} \times T^1_{q,0}(Y) \) and such that \( g(u, \bar{u}) = 1 \). Let

\[
f(x) = f(x, q) = \|(\pi_x^\ast u)\|^2.
\]

Then \( f(x) \) is a function on \( X_q \). By the same computation as in (2.9),

\[
\Delta_{X_q} f(x) \geq 2mB(1 + r^2(x, q))^{-\delta} f(x).
\]

By Lemma 3.1, we have

\[
0 < f(x) = \|(\pi_x^\ast u)\|^2 = (\pi_x^\ast (g^\ast)) (u, \bar{u}) \leq C_1(1 + r^2(p, q))^\gamma(1 + r^2(x, q))^\delta g(u, \bar{u}) = C_1(1 + r^2(x, q))^\delta,
\]

where \( C_1 \) is a constant independent of \( x \). We may proceed as in the proof of Theorem 1.2 to estimate the volume growth of \( X_q \) and use Theorem 2.1 to finish the proof. However, since the curvature of \( M \) is nonpositive, we may proceed in a more simple way.

Let \( h(x) = \log f(x) - 2\delta \log(C_2 + r^2(x, q)) \) where \( C_2 > 1 \) is some constant to be determined. By (3.8), \( h \) achieves its maximum at some point \((\bar{x}, q)\) \( \in X_q \). Then at \((\bar{x}, q)\)

\[
\nabla_{X_q} \log f(x) = 2\delta \nabla_{X_q} \log(C_2 + r^2(\bar{x}, q)) \quad \text{and} \quad \Delta_{X_q} h \leq 0.
\]

Since \( r^2 \) is convex, we have \( |\nabla_{X_q} r(x, q)| \leq 1 \) and \( \Delta_{X_q} r^2(x, q) \leq \Delta r^2(x, q) \leq C_3(1 + r^2(x, q))^{\frac{1+\gamma}{2}} \) for some constant \( C_3 \) independent of \( x \). Let \( r = r(\bar{x}, q) \), then at \((\bar{x}, q)\), using (3.7) and the fact that \( \gamma + 2\delta < 1 \), we have

\[
0 \geq \Delta_{X_q} h(\bar{x}) = f^{-1} \Delta_{X_q} f - (2\delta)^2 |\nabla_{X_q} \log(C_2 + r^2(\bar{x}, q))|^2 - 2\delta(C_2 + r^2)^{-1} \Delta_{X_q} r^2(\bar{x}, q) + 2\delta |\nabla_{X_q} \log(C_2 + r^2(\bar{x}, q))|^2 \\
\geq 2mB(1 + r^2)^{-\delta} - 2\delta C_3(C_2 + r^2)^{-\frac{1+\gamma}{2}} \\
> 2(C_2 + r^2)^{-\delta} \left( mB - \delta C_3(C_2 + r^2)^{-\frac{1+\gamma+2\delta}{2}} \right) \\
> 2(C_2 + r^2)^{-\delta} \left( mB - \delta C_3 C_2^{-\frac{1+\gamma+2\delta}{2}} \right) \\
> 0,
\]
if we chose $C_2 > 1$ large enough, such that $\delta C_3 C_2^{\frac{1-\gamma-2\delta}{2}} < mB$. This can be done because $\gamma + 2\delta < 1$. Hence we have a contradiction. This completes the proof of the theorem. \hfill \Box

Remark 3.1. Letting $\gamma = 0$ in Theorem 1.3, we know that there is no complete Kähler metric on $X \times Y$ with Ricci curvature bounded from below and sectional curvature $\leq -A(1 + r^2)^{-1}$ for any $\delta < \frac{1}{2}$. We may ask the problem if $\frac{1}{2}$ is the optimal power.

In [5], Greene-Wu proved that if a Hermitian manifold $M$ has holomorphic sectional curvature $\leq -A(1 + r^2)^{-1}$, then $M$ is hyperbolic in the sense of Kobayashi-Royden. Note that $\mathbb{C}^n$ is not hyperbolic in the sense of Kobayashi-Royden. So, there is no Hermitian metric on $\mathbb{C}^n$ with holomorphic sectional curvature $\leq -A(1 + r^2)^{-1}$. On the other hand, the example given by Seshadri [7] has holomorphic bisectional curvature $\leq -A[(1 + r^2) \log(2 + r)]^{-1}$. Therefore the optimal power must be in $[1/2,1]$.

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