EXISTENCE RESULTS FOR THE HIGHER-ORDER $Q$-CURVATURE EQUATION

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Abstract. We obtain existence results for the $Q$-curvature equation of order $2k$ on a closed Riemannian manifold of dimension $n \geq 2k+1$, where $k \geq 1$ is an integer. We obtain these results under the assumptions that the Yamabe invariant of order $2k$ is positive and the Green’s function of the corresponding operator is positive, which are satisfied in particular when the manifold is Einstein with positive scalar curvature. In the case where $2k+1 \leq n \leq 2k+3$ or the manifold is locally conformally flat, we assume moreover that the operator has positive mass. In the case where $n \geq 2k+4$ and the manifold is not locally conformally flat, the results essentially reduce to the determination of the sign of a complicated constant depending only on $n$ and $k$.

1. Introduction and main results

Given an integer $k \geq 1$, a smooth, closed Riemannian manifold $(M,g)$ of dimension $n > 2k$ and a smooth positive function $f$ in $M$, we consider the equation

$$P_{2k}u = f |u|^{2^*_k-2} u \text{ in } M,$$

where $P_{2k}$ is the GJMS operator with leading part $\Delta^k$, $\Delta := \delta d$ is the Laplace–Beltrami operator with nonnegative eigenvalues and $2^*_k := 2n/(n-2k)$ is the critical Sobolev exponent. The so-called GJMS operators were discovered by Graham, Jenne, Mason and Sparling [18] by using a construction based on the Fefferman–Graham ambient metric [14,15]. They provide a natural extension to higher orders of the Yamabe operator [42] ($k = 1$) and the Paneitz–Branson operator [4,32] ($k = 2$). When $u$ is positive, (1.1) arises in the problem of prescribing Branson’s $Q$-curvature of order $2k$ in a given conformal class (see Branson [5]). More precisely, the positive solutions $u$ to the equation (1.1) correspond to the conformal metrics $u^{4/(n-2k)}g$ with $Q$-curvature of order $2k$ equal to $\frac{2}{n-2k}f$.

Let $Y_{2k}$ be the conformal invariant defined by

$$Y_{2k} := \inf_{\tilde{g} \in [g]} \left( \frac{\text{Vol}_{\tilde{g}}(M) \int_M Q_{2k,\tilde{g}} \, dv_{\tilde{g}}}{\int_M u^{2^*_k} \, dv_g} \right) = \inf_{u \in C^\infty(M) \atop u > 0 \text{ in } M} \frac{\int_M u P_{2k} u \, dv_g}{\left( \int_M u^{2^*_k} \, dv_g \right)^{\frac{n-2k}{n}}}.$$
where \(|g|\) is the conformal class of \(g\), and \(\text{Vol}_3(M), dv_3\) and \(Q_{2k,\tilde{g}}\) are the volume, volume element and \(Q\)-curvature of order \(2k\), respectively, of \((M, \tilde{g})\). Throughout this paper, we assume that \(Y_{2k} > 0\). As is easily seen, this is equivalent to the coercivity of the operator \(P_{2k}\), which is also equivalent to \(\lambda_1(P_{2k}) > 0\), where \(\lambda_1(P_{2k})\) is the first eigenvalue of \(P_{2k}\).

In the case where \(k = 1\), it is well-known that there exists at least one positive solution to the equation \((1.1)\) with \(f \equiv 1\) if and only if \(Y_2 > 0\) (see the historic work of Aubin [2], Schoen [36], Trudinger [41] and Yamabe [42]). In the case where \(k = 2\), the existence of at least one positive solution to this problem has been obtained under positivity assumptions on the scalar curvature and \(Q\)-curvature of order 4 (see Gursky and Malchiodi [21]) and later extended to the cases where \(Y_2 > 0\) and \(Y_4 > 0\) in dimension \(n \geq 6\) (see Gursky, Hang and Lin [20]) and the case where \(Y_2 > 0\) and \(Q_4 > 0\) in dimension \(n \geq 5\) (see Hang and Yang [22, 23]). This question has also been solved by Qing and Raske [33] in the locally conformally flat case for all orders \(k \geq 2\), under a topological assumption on the Poincaré exponent of the holonomy representation of the fundamental group, using an approach introduced by Schoen [37] for \(k = 1\). More general existence results have also been obtained in the case where \(f \not\equiv 1\) (see among others Aubin [2], Escobar and Schoen [12], Hebey [24] and Hebey and Vaugon [25] for \(k = 1\), Djadli, Hebey and Ledoux [10], Esposito and Robert [13] and Robert [34] for \(k = 2\), Chen and Hou [9] for \(k = 3\) and Robert [35] for higher orders).

We let \(W\) be the Weyl tensor of \((M, g)\) and \(|W|\) be the norm of \(W\) with respect to \(g\). In the case where \(2k + 1 \leq n \leq 2k + 3\) or \((M, g)\) is locally conformally flat, assuming that \(Y_{2k} > 0\), for every point \(\xi \in M\), we let \(m(\xi)\) be the mass of \(P_{2k}\) at \(\xi\) (see \(3.2\) for the definition of the mass). Our main result is the following:

**Theorem 1.1.** Let \(k \geq 1\) be an integer, \((M, g)\) be a smooth, closed Riemannian manifold of dimension \(n \geq 2k + 1\) and \(f\) be a smooth positive function in \(M\). Assume that \(Y_{2k} > 0\) and there exists a maximal point \(\xi\) of \(f\) such that

\[
\Delta f(\xi) = 0 \quad \text{if} \quad n \geq 2k + 2
\]

and

\[
\begin{cases}
|W(\xi)|^2 f(\xi) + c(n,k) \Delta^2 f(\xi) > 0 & \text{if} \quad n \geq 2k + 5 \\
W(\xi) \neq 0 & \text{if} \quad n = 2k + 4 \\
m(\xi) > 0 & \text{if} \quad 2k + 1 \leq n \leq 2k + 3,
\end{cases}
\]

where \(c(n,k)\) is a positive constant depending only on \(n\) and \(k\) (see \(2.65\) for the value of \(c(n,k)\)). Then there exists a nontrivial solution \(u \in C^{2k}(M)\) to the equation \((1.1)\), which minimizes the energy functional \((2.1)\). If moreover the Green’s function of the operator \(P_{2k}\) is positive, then \(u\) is positive, which implies that the \(Q\)-curvature of order \(2k\) of the metric \(u^{4/(n-2k)}g\) is equal to \(\frac{2}{n-2k}f\).

In particular, Theorem 1.1 extends to all orders previous results obtained by Aubin [2] for \(k = 1\) (in this case, the positivity of the Green’s function is not an issue), Esposito and Robert [13] for \(k = 2\) and Chen and Hou [9] for \(k = 3\).

In the case where \(f\) is constant, we obtain the following:

**Theorem 1.2.** Let \(k \geq 1\) be an integer and \((M, g)\) be a smooth, closed Riemannian manifold of dimension \(n \geq 2k + 1\). Assume that \(Y_{2k} > 0\) and its Green’s function is positive. Assume moreover that if \(2k + 1 \leq n \leq 2k + 3\) or \((M, g)\) is locally
conformally flat, then \( m(\xi) > 0 \) for some point \( \xi \in M \). Then there exists a conformal metric to \( g \) with constant \( Q \)-curvature of order \( 2k \).

Notice that Theorem 1.2 is a direct consequence of Theorem 1.1 in the case where \((M, g)\) is not locally conformally flat of dimension \( n \geq 2k + 4 \). A more general result about the locally conformally flat case will be stated in Section 3.

When \((M, g)\) is Einstein, Fefferman and Graham [15, Proposition 7.9] (see also Gover [17] for a proof based on tractors) established the formula

\[
P_{2k} = \prod_{j=1}^{k} \left( \Delta + \frac{(n + 2j - 2)(n - 2j)}{4n(n-1)} S \right),
\]

where \( S \) is the Scalar curvature of \((M, g)\). In this case, it is easy to see that if \( S \) is positive, then \( P_{2k} \) is coercive, and so \( Y_{2k} > 0 \). Furthermore, successive applications of the maximum principles yield that the Green’s function of the operator \( P_{2k} \) is positive. Therefore, we obtain the following corollary of Theorem 1.1:

**Corollary 1.1.** Let \( k \geq 1 \) be an integer and \((M, g)\) be a smooth, closed Einstein manifold of positive scalar curvature and dimension \( n \geq 2k + 1 \). Let \( f \) be a smooth positive function in \( M \) such that there exists a maximal point \( \xi \) of \( f \) satisfying (1.2) and (1.3). Then there exists a conformal metric to \( g \) with \( Q \)-curvature of order \( 2k \) equal to \( \frac{2}{n-2k} f \).

The positivity of the Green’s function of the operator \( P_4 \) has been shown to be true by Gursky and Malchiodi [21] and Hang and Yang [22, 23] under positivity assumptions on the \( Q \)-curvature of order 4 and the scalar curvature or the Yamabe invariant of the manifold. Positivity results for the mass of \( P_4 \) have also been obtained by Gursky and Malchiodi [21], Hang and Yang [22], Humbert and Raulot [26] and Michel [31], thus extending the positive mass theorem obtained by Schoen and Yau [38–40] for \( k = 1 \). As far as the authors know, no such results have yet been obtained for higher orders. As regards the case where \( n = 2k \), we point out that the problem of prescribing the \( Q \)-curvature involves a different equation than (1.1) which contains an exponential non-linearity. Some references in this case are Chang and Yang [8], Djadli and Malchiodi [11] and Li, Li and Liu [29] for \( k = 2 \) and Baird, Fardoun and Regbaoui [3] for higher orders.

The proofs of Theorems 1.1 and 1.2 are based on the approach introduced by Aubin [2] and Schoen [36] in the case where \( k = 1 \). This approach consists in deriving an asymptotic expansion for the energy functional associated with the equation (1.1), which we apply to a suitable family of test functions depending on a real parameter (see (2.1) for the energy functional; see (2.5) and (3.4) for the definitions of our families of test functions). To simplify the calculations of curvature terms, we use the conformal normal coordinates introduced by Lee and Parker [28] and later improved by Cao [7] and Günther [19]. Our proof also crucially relies on the derivation of an expression for the highest-order terms of the GJMS operators (see (2.7)), which we obtain by using Juhl’s formulae [27]. In the case where \( n \geq 2k + 4 \), the proof essentially reduces to determining the sign of a constant \( C(n, k) \), which appears in the energy expansion (see (2.6)). In particular, we recover the values found in [9, 13] for \( C(n, k) \) with \( k \in \{2, 3\} \). We then conclude the proof by using a minimization result in the spirit of Aubin [2] (see Mazumdar [30, Theorem 3]). When the Green’s function of the operator \( P_{2k} \) is positive, by an
application of the Green’s representation formula, we obtain moreover that the minimizing solution is positive (see the argument in [30, end of Section 3]). We point out that at one place in the proof, namely in the very last computation to determine the sign of $C(n,k)$ (see (2.63)), we have used the computation software Maple to expand a complicated polynomial with integer coefficients.

The paper is organized as follows. In Section 2, we prove Theorem 1.1 in the case where $n \geq 2k + 4$. In Section 3, we complete the proof of Theorems 1.1 in the remaining case where $2k + 1 \leq n \leq 2k + 3$ and we state and prove a more general result in the case where $g$ is conformally flat in some open subset of the manifold. Theorem 1.2 then directly follows from this new result together with Theorem 1.1.

2. Proof of Theorem 1.1 in the case where $n \geq 2k + 4$

Given an integer $k \geq 1$ and a smooth positive function $f$ in $M$, we let $I_{k,f}$ be the energy functional defined as

$$I_{k,f,g}(u) := \frac{\int_M u P_{2k} u \, dv_g}{\left( \int_M f \left| u \right|^2 k \, dv_g \right)^{n-2k/n}}$$

for all functions $u \in C^{2k}(M)$ such that $u \neq 0$. We fix a point $\xi \in M$. By applying a conformal change of metric (see Cao [7], Günther [19] and Lee and Parker [28]), we may assume that

$$\det g(x) = 1 \quad \forall x \in \Omega$$

for some neighborhood $\Omega$ of the point $\xi$, where $\det g$ is the determinant of $g$ in geodesic normal coordinates at $\xi$. In particular (see [28]), it follows from (2.2) that

$$\text{Ric}(\xi) = \text{Sym} \nabla \text{Ric}(\xi) = \text{Sym} \left( \text{Ric}_{ab,cd}(\xi) + \frac{2}{9} W_{eabf}(\xi) W_{cd}^f(\xi) \right) = 0,$$

where Sym stands for the symmetric part, Ric is the Ricci tensor, and $\text{Ric}_{ab,cd}$ and $W_{eabf}$ are the coordinates of $\nabla^2 \text{Ric}$ and $W$, respectively, with the standard convention on raising and lowering indices. By taking traces in (2.3) and using Bianchi’s identities, we obtain

$$S(\xi) = \left| \nabla S(\xi) \right| = 0, \quad \Delta S(\xi) = \frac{1}{6} \left| W(\xi) \right|^2 \quad \text{and} \quad \text{Ric}_{ab,cd}(\xi) = -\frac{1}{12} \left| W(\xi) \right|^2$$

Let $r_0 > 0$ be such that the injectivity radius of the metric $g$ at the point $\xi$ is greater than $3r_0$ and $B(\xi, 3r_0) \subset \Omega$, where $B(\xi, r_0)$ is the ball of center $\xi$ and radius $3r_0$ with respect to $g$. We then let $\chi$ be a smooth cutoff function in $[0, \infty)$ such that $\chi \equiv 1$ in $[0, r_0]$, $0 \leq \chi \leq 1$ in $(r_0, 2r_0)$ and $\chi \equiv 0$ in $[2r_0, \infty)$. For every $\mu > 0$, we then define our test functions as

$$U_\mu(x) := \chi \left( d_g(x, \xi) \right) \mu^{\frac{2k-n}{n}} U\left( \mu^{-1} \exp_\xi^{-1} x \right) \quad \forall x \in M,$$

where $d_g$ is the geodesic distance with respect to $g$, $\exp_\xi$ is the exponential map with respect to $g$ at the point $\xi$ and $U$ is the function in $\mathbb{R}^n$ (we identify $T_\xi M$ with $\mathbb{R}^n$) defined as

$$U(x) := \left( 1 + |x|^2 \right)^{-\frac{n-2k}{2}} \quad \forall x \in \mathbb{R}^n.$$
It is easy to verify that $U$ is a solution of the equation
\[ \Delta_0^k U = \left[ \prod_{j=-k}^{k-1} (n + 2j) \right] U^{2k-1} \quad \text{in } \mathbb{R}^n, \]
where $\Delta_0$ is the Euclidean Laplacian.

**Proposition 2.1.** Let $k \geq 1$ be an integer, $(M, g)$ be a smooth, closed Riemannian manifold of dimension $n \geq 2k + 4$ and $f$ be a smooth positive function in $M$. Assume that $g$ satisfies \((2.2)\) for some point $\xi \in M$. Let $I_{k,f,g}$ be as in \((2.1)\) and $U_\mu$ be as in \((2.5)\). Then there exists a positive constant $C(n,k)$ depending only on $n$ and $k$ (see \((2.62)\) for the value of $C(n,k)$) such that as $\mu \to 0$,
\[ I_{k,f,g}(U_\mu) = \omega_n^2 f(\xi) \frac{\Delta f(\xi)}{\Delta f(\xi) \mu^2} \left( 2(k-1)! B \left( \frac{n}{2} - k, 2k \right) \right)^{-1} \]
\[ \times \left[ 1 + \frac{(n-2k)\Delta f(\xi)}{2(n-2)\Delta f(\xi) \mu^2} - \frac{(n-2k)\Delta^2 f(\xi)}{4n(n-2)f(\xi)} \left( \frac{\Delta f(\xi)}{n(n-2)f(\xi)} \right)^2 \mu^4 \right] \]
\[ - C(n,k) \mu^4 \left\{ \begin{array}{ll} |W(\xi)|^2 \ln(1/\mu) + O(1) & \text{if } n = 2k + 4 \\ |W(\xi)|^2 + o(1) & \text{if } n > 2k + 4 \end{array} \right\}, \quad (2.6) \]
where $\omega_n$ is the volume of the standard $n$-dimensional sphere and $B$ is the beta function defined as
\[ B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad \forall a,b > 0. \]

**Proof of Proposition 2.1.** We let $P$ be the Schouten tensor defined as
\[ P := \frac{1}{n-2} \left( \text{Ric} - \frac{S}{2(n-1)} g \right) \]
and $B$ be the Bach tensor whose coordinates are given by
\[ B_{ij} := P_{ab} W_{ij}^{a b} + P_{ij}^{a} - P_{ia:j}^{a}, \]
where $W_{iajb}$, $P_{ab}$ and $P_{ij}^{a}$ are the coordinates of $W$, $P$ and $\nabla^2 P$, respectively. We let $\langle \cdot, \cdot \rangle$ be the multiple inner product induced by the metric $g$ for the tensors of same rank, i.e. such that $\langle S, T \rangle = S^{i_{1} \ldots i_{l}}T_{i_{1} \ldots i_{l}}$ for all tensors $S$ and $T$ of rank $l \in \mathbb{N}$. The first step in the proof of Proposition 2.1 is as follows:

**Step 2.1.** For every $k \in \mathbb{N}$ such that $n \geq 2k + 1$, we have
\[ P_{2k} = \Delta^k + k \Delta^{k-1}(J_1) + k(k-1)\Delta^{k-2}(J_2) \]
\[ + k(k-1)(k-2)\Delta^{k-3}(J_3) \]
\[ + k(k-1)(k-2)(k-3)\Delta^{k-4}(J_4) + Z, \quad (2.7) \]
where $Z$ is a smooth linear operator of order less than $2k - 4$ if $k \geq 3$, $Z := 0$ if $k \leq 2$, the functions $J_1$ and $J_2$ are defined as
\[ J_1 := \frac{n-2}{4(n-1)} S \]
and

\[ J_2 := \frac{1}{6} \left( \frac{3n^2 - 12n - 4k + 8}{16(n-1)^2} S^2 - (k+1)(n-4)|P|^2 - \frac{3n+2k-4}{4(n-1)} \Delta S \right), \]

and the tensors \( T_1, T_2, T_3, T_4 \) and \( T_5 \) are defined as

\[ T_1 := \frac{n-2}{4(n-1)} \nabla S - \frac{2}{3} (k+1) \delta P, \]
\[ T_2 := \frac{2}{3} (k+1) P, \]
\[ T_3 := \frac{n-2}{6(n-1)} \nabla^2 S + \frac{(k+1)(n-2)}{6(n-1)} S P - \frac{k+1}{3} (\delta \nabla P + 2\nabla \delta P + 2 R \ast P) \]
\[ - \frac{2}{15} (k+1)(k+2) \left( 3 P^# P + \frac{B}{n-4} \right), \]
\[ T_4 := \frac{2}{3} (k+1) \nabla P \]

and

\[ T_5 := \frac{2}{5} (k+1) \left( \frac{5k+7}{9} P \otimes P + \nabla^2 P \right), \]

where \# stands for the musical isomorphism with respect to \( g \) (i.e. \( P^# := g^{-1} P \)), and \( \delta \nabla P, \nabla \delta P \) and \( R \ast P \) stand for the covariant tensors whose coordinates are given by

\[ (\delta \nabla P)_{ij} := -P_{ij;a} \cdot (\nabla \delta P)_{ij} := -P_{i;a j} \quad \text{and} \quad (R \ast P)_{ij} := R_{ia b} P_j b + R_{ij ab} P^{ab}, \]

(2.8)

where \( R_{ij ab} \), \( P_{ab} \) and \( P_{ij;ab} \) are the coordinates of the Riemann tensor, \( P \) and \( \nabla^2 P \), respectively.

**Proof of Step 2.1.** Throughout this proof, for every integer \( l \), \( o^l \) stands for a linear operator of order less than \( l \) if \( l > 0 \) and \( o^0 := 0 \) if \( l \leq 0 \). Juhl’s formulæ [27] (see also Fefferman and Graham [16]) give

\[ P_{2k} = M_2^k - \sum_{j=1}^{k-1} j (k-j) M_2^{j-1} M_4 M_2^{k-j-1} \]
\[ + \frac{1}{4} \sum_{j=1}^{k-2} j (j+1) (k-j) (k-j-1) M_2^{j-1} M_6 M_2^{k-j-2} \]
\[ + \sum_{j=2}^{k-2} (j+1)(j-1) \sum_{i=1}^{j-1} i (k-i) M_2^{j-1} M_4 M_2^{j-i-1} M_4 M_2^{k-j-2} + o^{2k-5}, \]

(2.9)

where the operators \( M_2, M_4 \) and \( M_6 \) are defined as

\[ M_2 := \Delta + \mu_2, \quad M_4 := 4 \delta P^# d + \mu_4 \quad \text{and} \quad M_6 := \delta A^# d + \mu_6, \]

where \( \mu_6 \) is a smooth function in \( M \) which we do not need explicitly, \( \mu_2 \) and \( \mu_4 \) are the functions defined as

\[ \mu_2 := \frac{n-2}{4(n-1)} S \quad \text{and} \quad \mu_4 := \frac{\Delta S}{2(n-1)} + \frac{S^2}{4(n-1)^2} + (n-4)|P|^2, \]
and $A_6$ is the tensor defined as

$$A_6 := 48 P^# P + \frac{16}{n-4} B.$$  

We point out that throughout this paper, we use the same sign convention for the Riemann tensor as in the paper of Lee and Parker [28], which is the opposite of the convention used by Juhl [27]. Straightforward expansions yield

\[
M_k^2 = \Delta^k + \frac{n-2}{4(n-1)} \sum_{j=1}^{k} \Delta^{j-1} (S \Delta^{k-j}) \\
+ \frac{(n-2)^2}{16(n-1)^2} \sum_{j=2}^{k} \sum_{i=1}^{j-1} \Delta^{i-1} (S \Delta^{j-i-1} (S \Delta^{k-j})) + o^{2k-5} \\
= \Delta^k + \frac{n-2}{4(n-1)} \sum_{j=1}^{k} \Delta^{j-1} (S \Delta^{k-j}) + \frac{(n-2)^2}{16(n-1)^2} \sum_{j=2}^{k} \sum_{i=1}^{j-1} \Delta^{i-1} (S \Delta^{j-i-1} (S \Delta^{k-j})) + o^{2k-4} \\
= \Delta^k + \frac{n-2}{4(n-1)} \sum_{j=1}^{k} \Delta^{j-1} (S \Delta^{k-j}) + \frac{k(k-1)(n-2)^2}{32(n-1)^2} \Delta^{k-2} (S \Delta^{k-j}) + o^{2k-4} \\
\tag{2.10}
\]

and

\[
M_j^{-1} M_k^{k-j-1} = 4 \Delta^j \delta P^# d\Delta^{k-j-1} + \Delta^{j-1} (\mu_4 \Delta^{k-j-1}) \\
+ \frac{n-2}{n-1} \sum_{i=1}^{j-1} \Delta^{i-1} (S \Delta^{j-i-1} \delta P^# d\Delta^{k-j-1}) \\
+ \frac{n-2}{n-1} \sum_{i=j+1}^{k} \Delta^{i-1} \delta P^# d\Delta^{i-j-1} (S \Delta^{k-i-1}) + o^{2k-5} \\
= 4 \Delta^j \delta P^# d\Delta^{k-j-1} + \mu_4 \Delta^{k-j-2} - \frac{(k-2)(n-2)}{n-1} \Delta^{k-3} (S \Delta^{k-j}) + o^{2k-4} \\
\tag{2.11}
\]

and

\[
M_j^{-1} M_6 M_k^{k-j-2} = \Delta^j \delta A_6^# d\Delta^{k-j-2} + o^{2k-5} = -\Delta^{k-3} (A_6, \nabla^2) + o^{2k-4} \tag{2.12}
\]

and

\[
M_j^{-1} M_4 M_k^{j-i-1} M_4 M_k^{j-2} = 16 \Delta^i \delta P^# d\Delta^{j-i-1} \delta P^# d\Delta^{k-j-2} + o^{2k-5} \\
= 16 \Delta^{k-4} (P \otimes P, \nabla^4) + o^{2k-4}. \tag{2.13}
\]

Furthermore, by induction, one can check that

\[
S \Delta^j = \Delta^j (S \cdot) - j (\Delta S) \Delta^j + 2j \Delta^j (\nabla S, \nabla) + 2j (j-1) \Delta^{j-2} (\nabla^2 S, \nabla^2) + o^{2j-2} \tag{2.14}
\]

and

\[
\delta P^# d\Delta^j = \Delta^j ((\delta P, \nabla) - (P, \nabla^2)) + j \Delta^j (\delta \nabla P + 2 \nabla \delta P + 2 R \ast P, \nabla^2) \\
- 2 (\nabla P, \nabla^3) - 2j (j-1) \Delta^{j-2} (\nabla^2 P, \nabla^4) + o^{2j}. \tag{2.15}
\]
where $\delta\nabla P$, $\nabla\delta P$ and $R\ast P$ are as in (2.8). The proof of (2.15) relies on the commutation formula

$$u_{abcd} = u_{cdab} + R^{e}_{c_{ad}} u_{eb} + R^{e}_{ab_{d}} u_{ce} + R^{e}_{bc_{a}} u_{de} + o^{2} u,$$

which gives

$$\delta P^{#} \ast du (\nabla) - \Delta \delta P^{#} \ast du = (P^{bc} u_{ac_{b}})_{;c} - (P^{bc} u_{bc})_{;ca}^{a}$$

$$= P^{bc} (u_{bc} - u_{bc}) - P^{bc} a u_{bc} - 2 P^{bc} a u_{bc}^{a} - 2 P^{bc} a u_{bc} + o^{2} u$$

$$= 2 P^{bc} (R^{a_{d}} u_{ad} + R^{d_{a}} u_{bd}) - P^{bc} a u_{bc} - 2 P^{bc} a u_{bc}^{a} - 2 P^{bc} a u_{bc} + o^{2} u$$

$$= (\delta \nabla P + 2 \nabla\delta P + 2 R \ast P, \nabla^{3} u) - 2 (\nabla P, \nabla^{3} u) + o^{2} u.$$

By combining Faulhaber’s formulae with (2.11)–(2.15), we obtain

$$\sum_{j=1}^{k} \Delta^{j-1} (S \Delta^{k-j}) = k \Delta^{k-1} (S \cdot) - \frac{k(k-1)}{2} \Delta^{k-2} ((\Delta S) \cdot)$$

$$+ k (k-1) \Delta^{k-2} (\nabla S, \nabla) + \frac{2k(k-1)(k-2)}{3} \Delta^{k-3} (\nabla^{2} S, \nabla^{2}) + o^{2k-4} \quad (2.16)$$

and

$$\sum_{j=1}^{k-1} j(k-j) M^{j-1}_{2} M_{2}^{k-j-1} = k(k-1)(k+1) \left( \frac{2}{3} \Delta^{k-2} ((\delta P, \nabla) - (P, \nabla^{2})) \right)$$

$$+ \frac{k-2}{3} \Delta^{k-3} ((\delta \nabla P + 2 \nabla\delta P + 2 R \ast P, \nabla^{2}) - 2 (\nabla P, \nabla^{3})) + \frac{1}{6} \Delta^{k-2} (\mu_{\gamma})$$

$$- \frac{2(k-2)(k-3)}{5} \Delta^{k-4} (\nabla^{2} P, \nabla^{2}) - \frac{(k-2)(n-2)}{6(n-1)} \Delta^{k-3} (S P, \nabla^{2}) \right) + o^{2k-4} \quad (2.17)$$

and

$$\sum_{j=1}^{k-2} j(k+1)(k-j)(k-j-1) M_{2}^{j-1} M_{6} M_{2}^{k-j-2} = - \frac{k(k-1)(k-2)(k+1)(k+2)}{30} \Delta^{k-3} (A_{6}, \nabla^{2}) + o^{2k-4} \quad (2.18)$$

and

$$\sum_{j=1}^{k-2} (j+1)(k-j-1) \sum_{i=1}^{j-1} i(k-i) M_{2}^{i-1} M_{4} M_{2}^{j-i-1} M_{4} M_{2}^{k-j-2}$$

$$= \frac{2k(k-1)(k-2)(k-3)(k+1)(5k+7)}{45} \Delta^{k-4} (P \otimes P, \nabla^{4}) + o^{2k-4} \quad (2.19)$$

Finally, (2.7) follows by putting together (2.9), (2.10) and (2.16)–(2.19). This ends the proof of Step 2.1.

The next step is as follows:

**Step 2.2.** Assume that $n \geq 2k + 4$ and $k \geq 3$. Then for every smooth linear operator $Z$ of order less than $2k - 4$, as $\mu \to 0$,

$$\int_{\mathcal{M}} U_{\mu} Z U_{\mu} \, dv_{g} = \begin{cases} 0 (\mu^{4}) & \text{if } n = 2k + 4 \\ 0 (\mu^{4}) & \text{if } n > 2k + 4. \end{cases} \quad (2.20)$$
Proof of Step 2.2. By rewriting the integral in geodesic normal coordinates, we obtain

$$
\int_M U_\mu Z U_\mu \, dv_g = \int_{B(0, 2r_0)} U_\mu Z U_\mu \, dx = \sum_{|\alpha| < 2k} \int_{B(0, 2r_0)} z_\alpha U_\mu \partial^\alpha U_\mu \, dx, \quad (2.21)
$$

where

$$
\tilde{U}_\mu (x) := \mu^{2k - |\alpha|} U (x/\mu) \text{ and } \tilde{Z} (x) := \sum_{|\alpha| < 2k} z_\alpha (x) \partial^{(\alpha)} \quad \forall x \in B (0, 2r_0) \quad (2.22)
$$

for some smooth functions $z_\alpha$ in $B (0, 2r_0)$, where $\alpha$ is a multi-index. A straightforward change of variable then gives

$$
\int_{B (0, 2r_0)} z_\alpha U_\mu \partial^{(\alpha)} U_\mu \, dx = \mu^{2k - |\alpha|} \int_{B(0, 2r_0/\mu)} z_\alpha (\mu x) U (x) \partial^{(\alpha)} U (x) \, dx. \quad (2.23)
$$

An easy induction yields that for every multi-index $\alpha$, there exists a constant $C_\alpha$ such that

$$
|\partial^{(\alpha)} U (x)| \leq C_\alpha (1 + |x|^2)^{-\frac{n - 2k + |\alpha|}{2}} \quad \forall x \in \mathbb{R}^n \quad (2.24)
$$

It follows from (2.23) and (2.24) that

$$
\int_{B (0, 2r_0)} \sum \left\{ \begin{array}{ll}
O \left( \mu^{2k - |\alpha|} \right) & \text{if } |\alpha| > 4k - n \\
O \left( \mu^{n-2k} \ln (1/\mu) \right) & \text{if } |\alpha| = 4k - n \\
O \left( \mu^{n-2k} \right) & \text{if } |\alpha| < 4k - n.
\end{array} \right. \quad (2.25)
$$

Finally, (2.20) follows from (2.21) and (2.25).

We then prove the following:

Step 2.3. Assume that $n \geq 2k + 4$ and $g$ satisfies (2.2) for some point $\xi \in M$. Then, as $\mu \to 0$,

$$
\int_M U_\mu \Delta^k U_\mu \, dv_g = 2^{2k-n} (2k - 1)! \omega_n \left( \frac{n}{2} - k, 2k \right)^{-1} + O \left( \mu^{n-2k} \right). \quad (2.26)
$$

If $k \geq 2$, then for every smooth function $f$ in $M$,

$$
\int_M f U_\mu \Delta^{k-2} U_\mu \, dv_g = \frac{2^{2k-n-1} (n-1)! (k-2)! \omega_n}{(n-2)(n-4)(n-2k-2)} f (\xi) \mu^4 \times 
\sum_{l=k-2}^{2k-4} \frac{l!}{(l-k+2)! (2k-l-4)! (n+l-2k-1)!} \mu \left( \frac{n}{2} - k - 1, l + 1 \right)^{-1} \times
\begin{cases}
2 \ln (1/\mu) & \text{if } n = 2k + 4 \text{ and } l = k - 2 \\
B \left( \frac{n}{2} + l - 2k, 2k - l - 2 \right) & \text{otherwise}
\end{cases}
+ O \left( \mu^4 \right) \quad \text{if } n = 2k + 4 \quad (2.27)
$$

\text{if } n > 2k + 4,
for every smooth, covariant tensor $T$ of rank 1,

$$
\int_M (T, \nabla U_\mu) \Delta^{k-2} U_\mu \, dv_g = - \frac{2^{2k-n-2} (n-2k) (n-1)! (k-2)! \omega_n T^{a}_{;a} (\xi) \mu^4}{(n-2) (n-4) (n-2k-2)}
\times \sum_{l=k-2}^{2k-4} \frac{l!}{(l-k+2)! (2k-l-4)! (n+l-2k)!} B \left( \frac{n}{2} - k - 1, l + 1 \right)^{-1}
\times \begin{cases}
2 \ln (1/\mu) & \text{if } n = 2k + 4 \text{ and } l = k - 2 \\
B \left( \frac{n}{2} + l - 2k, 2k - l - 2 \right) & \text{otherwise}
\end{cases}
+ \begin{cases}
O (\mu^4) & \text{if } n = 2k + 4 \\
o (\mu^4) & \text{if } n > 2k + 4
\end{cases}
$$

(2.28)

and for every smooth, covariant tensor $T$ of rank 2,

$$
\int_M (T, \nabla^2 U_\mu) \Delta^{k-2} U_\mu \, dv_g = \frac{2^{2k-n-4} (n-2k) (n-1)! (k-2)! \omega_n T^{a}_{;a} (\xi) \mu^4}{(n-2) (n-4) (n-2k-2)}
\times \sum_{l=k-2}^{2k-4} \frac{l!}{(l-k+2)! (2k-l-4)! (n+l-2k+1)!} B \left( \frac{n}{2} - k - 1, l + 1 \right)^{-1}
\times \begin{cases}
-2 (n-4) (n+2l-2k) B \left( \frac{n}{2} - 2k + l + 1, 2k - l - 2 \right) T^{a}_{;a} (\xi) \mu^2 \\
(n-2k+2) (T^{ab}_{;ab} (\xi) + T^{ab}_{;ba} (\xi)) - (n+2l-2k) T^{a}_{;b} (\xi) \mu^4
\end{cases}
\times \begin{cases}
2 \ln (1/\mu) & \text{if } n = 2k + 4 \text{ and } l = k - 2 \\
B \left( \frac{n}{2} + l - 2k, 2k - l - 2 \right) & \text{otherwise}
\end{cases}
+ \begin{cases}
O (\mu^4) & \text{if } n = 2k + 4 \\
o (\mu^4) & \text{if } n > 2k + 4
\end{cases}
$$

(2.29)

If $k \geq 3$, then for every smooth, covariant tensor $T$ of rank 2,

$$
\int_M (T, \nabla^2 U_\mu) \Delta^{k-3} U_\mu \, dv_g = - \frac{2^{2k-n-5} (n-2k) (n-1)! (k-3)! \omega_n T^{a}_{;a} (\xi) \mu^4}{(n-2) (n-4) (n-2k-2)}
\times \sum_{l=k-3}^{2k-6} \frac{(n+2l-2k)!}{(l-k+3)! (2k-l-6)! (n+l-2k+1)!} B \left( \frac{n}{2} - k - 1, l + 1 \right)^{-1}
\times \begin{cases}
2 \ln (1/\mu) & \text{if } n = 2k + 4 \text{ and } l = k - 3 \\
B \left( \frac{n}{2} + l - 2k + 1, 2k - l - 3 \right) & \text{otherwise}
\end{cases}
+ \begin{cases}
O (\mu^4) & \text{if } n = 2k + 4 \\
o (\mu^4) & \text{if } n > 2k + 4
\end{cases}
$$

(2.30)

and for every smooth, covariant tensor $T$ of rank 3,

$$
\int_M (T, \nabla^3 U_\mu) \Delta^{k-3} U_\mu \, dv_g = \frac{2^{2k-n-6} (n-2k) (n-2k+2) (n-1)! (k-3)! \omega_n}{(n-2) (n-4) (n-2k-2)}
\times (T^{ab}_{;ab} (\xi) + T^{ba}_{;ba} (\xi) + T^{a}_{;b} b (\xi) + T^{a}_{;b} b (\xi)) \mu^4
$$
\[ \sum_{l=k-3}^{2k-6} \frac{(n+2l-2k)!}{(l-k+3)!(2k-l-6)!(n+l-2k)!} \cdot \frac{2\ln(1/\mu)}{B\left(\frac{n}{2} - k - 1, l + 1\right)}^{-1} \]
\[ \times \begin{cases} 2 \ln(1/\mu) & \text{if } n = 2k + 4 \text{ and } l = k - 3 \\ B\left(\frac{n}{2} + l - 2k + 1, 2k - l - 3\right) & \text{otherwise} \end{cases} \]
\[ + \begin{cases} O(\mu^4) & \text{if } n = 2k + 4 \\ o(\mu^4) & \text{if } n > 2k + 4 \end{cases} \] (2.31)

If \( k \geq 4 \), then for every smooth, covariant tensor \( T \) of rank 4,
\[ \int_M (T, \nabla^4 U_\mu) \Delta^4 U_\mu \, dv_g = \frac{2^{2k-8} \omega_n}{3 \omega_n} \omega_n \]
\[ + \sum_{l=k-4}^{2k-8} \frac{B\left(\frac{n}{2} - k - 1, l + 1\right)}{(l-k+4)!(2k-l-8)!(n+l-2k+3)!} \cdot \frac{2\ln(1/\mu)}{B\left(\frac{n}{2} - k - 1, l + 1\right)}^{-1} \]
\[ \times \begin{cases} 2 \ln(1/\mu) & \text{if } n = 2k + 4 \text{ and } l = k - 4 \\ B\left(\frac{n}{2} + l - 2k + 2, 2k - l - 4\right) & \text{otherwise} \end{cases} \]
\[ + \begin{cases} O(\mu^4) & \text{if } n = 2k + 4 \\ o(\mu^4) & \text{if } n > 2k + 4 \end{cases} \] (2.32)

**Proof of Step 2.3.** We let \( j \) and \( l \) be two integers such that
\[
\max(2(k-l-2), 0) \leq j \leq k-l \quad \text{and} \quad \max(k-4,0) \leq l \leq k \]
and \( T \) be a smooth, covariant tensor of rank \( j \). By using geodesic normal coordinates, we obtain
\[ \int_M (T, \nabla^j U_\mu) \Delta^j U_\mu \, dv_g - \int_{B(\xi, r_0)} (T, \nabla^j U_\mu) \Delta^j U_\mu \, dv_g \]
\[ = \int_{B(0,2r_0) \setminus B(0,r_0)} \tilde{Z}_1 \tilde{U}_{\mu} \tilde{Z}_2 \tilde{U}_{\mu} \, dx \]
\[ = \sum_{|\alpha| \leq l} \sum_{|\beta| \leq 2l} \int_{B(0,2r_0) \setminus B(0,r_0)} z_{1,\alpha} z_{2,\alpha} (\partial^{\alpha_2} \tilde{U}_{\mu}) (\partial^{\alpha_1} \tilde{U}_{\mu}) \, dx, \] (2.33)
where \( \tilde{U}_{\mu} \) is as in (2.22) and
\[ \tilde{Z}_1 (x) := \sum_{|\alpha| \leq j} z_{1,\alpha} (x) \partial^{\alpha} \quad \text{and} \quad \tilde{Z}_2 (x) := \sum_{|\alpha| \leq 2l} z_{2,\alpha} (x) \partial^{\alpha} \quad \forall x \in B(0,2r_0) \]
for some smooth functions \( z_{1,\alpha} \) and \( z_{2,\alpha} \) in \( B(0,2r_0) \). By proceeding as in (2.23)–(2.25), we obtain
\[ \int_{B(0,2r_0) \setminus B(0,r_0)} z_{1,\alpha} z_{2,\alpha} (\partial^{\alpha_2} \tilde{U}_{\mu}) (\partial^{\alpha_1} \tilde{U}_{\mu}) \, dx = O(\mu^{n-2k}). \] (2.34)
It follows from (2.33) and (2.34) that
\[ \int_M (T, \nabla^j U_\mu) \Delta^j U_\mu \, dv_g = \int_{B(\xi, r_0)} (T, \nabla^j U_\mu) \Delta^j U_\mu \, dv_g + O(\mu^{n-2k}). \] (2.35)
On the other hand, by using (2.24) and rewriting the integral in the right-hand side of (2.35) in geodesic normal coordinates, we obtain

\[
\int_{B(\xi,r_0)} (T, \nabla^j U_\mu) \Delta^l U_\mu \, dv_g = \sum_{j'=0}^j \int_{B(0,r_0)} \hat{T}^{i_1 \ldots i_{j'}} \circ \exp_\xi U_{\mu,i_{j''} \ldots i_j} \Delta^l_0 U_\mu \, dx, \quad (2.36)
\]

where \( U_{\mu,i_{j''} \ldots i_j} := \partial^{(i_{j''} \ldots i_j)} (U_\mu \circ \exp_\xi) \) and the tensor \( \hat{T} \) is defined as

\[
\hat{T}^{i_1 \ldots i_{j'}} := \begin{cases}
T^{i_1 \ldots i_{j'}} & \text{if } j' = j \\
- \Gamma^{i_1 \ldots i_{j'}}_{e_{i_1} \ldots e_{i_j}} T^{e_{i_1} \ldots e_{i_j}} & \text{if } j' < j
\end{cases}
\]

where \( \Gamma^{i_1 \ldots i_{j'}}_{e_{i_1} \ldots e_{i_j}} \) is the generalized Christoffel symbol such that \( \Gamma^{i_1 \ldots i_{j'}}_{e_{i_1} \ldots e_{i_j}} \) is symmetric in \( i_1, \ldots, i_{j'} \) and

\[
u_{e_{i_1} \ldots e_{i_j}} = \nu_{e_{i_1} \ldots e_{i_j}} - \sum_{j'=0}^{j-1} \Gamma^{i_{j'} \ldots i_{j'}}_{e_{i_1} \ldots e_{i_j}} u_{i_{j'} \ldots i_{j'}}
\]

in geodesic normal coordinates. By using (2.36) together with a straightforward change of variable and a Taylor expansion, we then obtain

\[
\begin{aligned}
\int_{B(\xi,r_0)} (T, \nabla^j U_\mu) \Delta^l U_\mu \, dv_g &= \sum_{j'=0}^j \mu^{2k-2l-j'} \int_{B(0,r_0)/\mu} \hat{T}^{i_1 \ldots i_{j'}} (\exp_\xi (\mu x)) U_{i_{j''} \ldots i_j} (x) \Delta^l_0 U (x) \, dx \\
&= \sum_{j'=\max(2(k-l-2),0)}^j \sum_{j'=0}^{j'+2l-2k+4} \frac{\mu^{2k-2l-j'+j''}}{j''!} \hat{T}^{i_{1 \ldots i_{j''}} \ldots i_{j'+l} \ldots i_j} U \, dx + O \left( \sum_{j'=0}^j \mu^{\max(5,2k-2l-j')} \right) \\
&\times \int_{B(0,r_0)/\mu} |x|^{\max(j'+2(l-2k+5),0)} |U_{i_{j''} \ldots i_j} \Delta^l_0 U| \, dx \Bigg) . \quad (2.37)
\end{aligned}
\]

On the other hand, by using (2.24), we obtain

\[
\mu^{\max(5,2k-2l-j')} \int_{B(0,r_0)/\mu} |x|^{\max(j'+2l-2k+5,0)} |U_{i_{j''} \ldots i_j} \Delta^l_0 U| \, dx \\
= O \left( \mu^{\max(5,2k-2l-j')} \right) \int_{B(0,r_0)/\mu} |x|^{\max(j'+2l-2k+5,0)} (1 + |x|^2)^{-\frac{2n+j'+2l-4k}{2}} \, dx \\
= \begin{cases}
O (\mu^4) & \text{if } n = 2k + 4 \\
o (\mu^4) & \text{otherwise.} \quad (2.38)
\end{cases}
\]
It follows from (2.35), (2.37) and (2.38) that
\[
\int_M (T, \nabla^j U_\mu) \Delta^j U_\mu \, dv_g = \sum_{j'=\max(2(k-l-2),0)}^j \sum_{j''=0}^{j'+2l-2k+4} \frac{\mu^{2k-2l-j'+j''}}{j''!} \nonumber
\]
\[
\times \int^{i_1 \cdots i_l \cdots i_{2l} \cdots i_{j'+j''}} (\xi) \int_{B(0,ro/\mu)} U_{i_1 \cdots i_l \cdots i_{j'+j''}} \Delta^l U \, dx + \begin{cases} O(\mu^4) & \text{if } n = 2k + 4 \\ o(\mu^4) & \text{if } n > 2k + 4 \end{cases} \quad (2.39) \nonumber
\]
An easy induction gives
\[
U_{i_1 \cdots i_l}(x) = \sum_{m=0}^{\lfloor j/2 \rfloor} \frac{2l-2m}{m!(j-2m)!} \partial^l U (r) \nonumber
\]
\[
\times \sum_{\sigma \in \mathcal{S}(j)} \delta_{i_1(r(1)) \ldots i_l(r(j))} \cdots \delta_{i_{2l}(r(2l-1)) \cdots i_{(2m+1)}(r(2m+1)) \cdots i_{(2l)}(r(l))} \forall x \in \mathbb{R}^n, \quad (2.40) \nonumber
\]
where \( r := |x|^2 \), \( U(r) := U(x) = (1 + r)^{(2k-n)/2} \), \( \mathcal{S}(j) \) is the set of all permutations of \( (1, \ldots, j) \) and \( \delta_{i_1(r(1)), \ldots, i_{2l}(r(2l))} \) stand for the Kronecker symbols. Furthermore, it is easy to see that
\[
\partial^l U (r) = (-1)^j 2^{-j} (n - 2k)(n - 2k + 2) \cdots (n - 2k + 2j - 2)(1 + r)^{-\frac{n^2 - 2k + 2j}{4}} = \frac{2(-1)^j j!}{(n - 2k - 2)} B \left( \frac{n}{2} - k - 1, j + 1 \right)^{-1} (1 + r)^{-\frac{n^2 - 2k + 2j}{4}}. \quad (2.41) \nonumber
\]
Another induction yields
\[
\Delta^l U (x) = \begin{cases} \frac{2^{2l+1} l!}{(n - 2k - 2)(k - l - 1)!} \sum_{l'=l}^{2l} \frac{l!(k + l - l' - 1)!}{(l' - l)!(2l - l')!} \\ \times B \left( \frac{n}{2} - k - 1, l' + 1 \right)^{-1} (1 + r)^{-\frac{n^2 + 2l - 2k}{4}} & \text{if } l < k \\ 2^{2k} (2k - 1)! B \left( \frac{n}{2} - k, 2k \right)^{-1} (1 + r)^{-\frac{n^2 + 2k}{4}} & \text{if } l = k \end{cases} \quad (2.42) \nonumber
\]
for all \( x \in \mathbb{R}^n \). In the case where \( j = 0, l = k \) and \( T \equiv 1 \), it follows from (2.42) that
\[
\int_{B(0,ro/\mu)} U \Delta^k U \, dx = 2^{2k-1} (2k - 1)! \omega_{n-1} B \left( \frac{n}{2} - k, 2k \right)^{-1} \int_0^{(ro/\mu)^2} \frac{r^{n/2}}{(1 + r)^n} \, dr, \quad (2.43) \nonumber
\]
where \( \omega_{n-1} = \operatorname{Vol}(S^{n-1}, g_0) \) is the volume of the standard \((n - 1)\)-dimensional sphere. On the other hand, in the case where \( l < k \), by putting together (2.40)–(2.42) , we obtain
\[
\int_{B(0,ro/\mu)} U_{i_1 \cdots i_l} x_{i_{j'+1}} \cdots x_{i_{j'+j''}} \Delta^l U \, dx = \frac{2^{2l+1} l!}{(n - 2k - 2)(k - l - 1)!} \sum_{l'=l}^{2l} \sum_{m=0}^{\lfloor j'/2 \rfloor} \frac{2^{2l+1} l!}{(l' - l)!(2l - l')! m!(j' - m)!} B \left( \frac{n}{2} - k - 1, l' + 1 \right)^{-1} \quad (2.44) \nonumber
\]
\[
\times B \left( \frac{n}{2} - k - 1, j' - m + 1 \right)^{-1} \int_{0}^{(r_0/\mu)^2} \frac{r^\alpha}{(1 + r)^\beta} \, dr \sum_{\sigma \in \mathcal{S}(j')} \delta_{\sigma(1) \sigma(2)} \delta_{\sigma(1) \sigma(2)} \delta_{\sigma(1) \sigma(2)} \int_{S_{n-1}} y_{i_1} \cdots y_{i_{j'}} y_{i_{j'+1}} \cdots y_{i_{j'+m}} \, dv_{g_0}(y) \cdot (2.44)
\]

A standard computation gives
\[
\int_{0}^{(r_0/\mu)^2} \frac{r^{n-1}}{(1 + r)^b} \, dr = \begin{cases} 
2 \ln (1/\mu) + O(1) & \text{if } b = a \\
B(a, b - a) + O(\mu^2(b-a)) & \text{if } b > a.
\end{cases} \tag{2.45}
\]

On the other hand, by using the fact (see for example Brendle [6, Proposition 28]) that for every homogeneous polynomial \( \Phi \) of degree \( j \geq 2 \),
\[
\int_{S_{n-1}} \Phi(y) \, dv_{g_0}(y) = -\frac{1}{j(n+j-2)} \int_{S_{n-1}} \Delta_0 \Phi(y) \, dv_{g_0}(y),
\]

another induction yields that when \( j \) is even,
\[
\int_{S_{n-1}} y_{i_1} \cdots y_{i_{j}} \, dv_{g_0}(y) = \frac{(n-2) \omega_{n-1}}{2^{j+1}(j/2)!^2} B \left( \frac{n-2}{2}, \frac{j+2}{2} \right) \times \sum_{\sigma \in \mathcal{S}(j)} \delta_{\sigma(1) \sigma(2)} \cdots \delta_{\sigma(j-1) \sigma(j)} \cdot (2.46)
\]

The integral in (2.46) vanishes when \( j \) is odd. By observing that
\[
\omega_n = 2^{n-1} B \left( \frac{n}{2}, \frac{n}{2} \right) \omega_{n-1}, \tag{2.47}
\]

we obtain that for even \( j \),
\[
B \left( \frac{n-2}{2}, \frac{j+2}{2} \right) = \frac{2^{2n-2} (n-1)! (j/2)! \omega_n}{(n-2)(n+j/2-1)! \omega_{n-1}} \left( \frac{n}{2}, \frac{n+j}{2} \right)^{-1}. \tag{2.48}
\]

By using (2.45)–(2.48) together with the identity
\[
B \left( \frac{n}{2}, \frac{n+j''-j'' - 2m}{2} \right)^{-1} B \left( \frac{n+j'+j''-2m}{2}, \frac{n+j'-j''+2l'-4k}{2} \right) = \left( \frac{j'+j''}{2} + n - m - 1 \right)! \times B \left( \frac{n+j'-j''+2l'-4k}{2}, \frac{j''-j'+4k-2l'}{2} \right),
\]

\[
= \frac{(n+j'-m+2k-1)! \left( \frac{j''-j'+4k-2l'}{2} \right)!}{(n+j'-m+l'-2k-1)! \left( \frac{j''-j'+2k-2l'}{2} \right)!} \times B \left( \frac{n+j'+j''-2m}{2}, \frac{n+j'-j''+2l'-4k}{2} \right),
\]
we obtain that if \(j' + j''\) is even, then

\[
\int_0^{(\tau_0/\mu)^2} \frac{r^{n+j'+j''-m-2}}{(1+r)^{n+j'+j''-m+l'}-2k} \int_{\mathcal{S}_{n-1}} \left. \sum_{i} y_{\mathcal{S}_{(n+1)}} \cdots y_{\mathcal{S}_{j'}} y_{j'+1} \cdots y_{j'+j''} \right|_{\mathcal{S}_{y_0}(y)} \, dr \, dy_0(y) = \frac{1}{(n+j'-m+l'-2k-1)!} \left( \frac{j''-l'}{2} + 2k - l' - 1 \right)! \left( \frac{j''+j'}{2} - m \right)!
\]

\[
\left\{ \begin{array}{ll}
2 \ln (1/\mu) + O(1) & \text{if } n + j' + j'' + 2l' - 4k = 0 \\
B \left( \frac{n + j' - j'' + 2l' - 4k}{2}, \frac{j'' - j' + 4k}{2} \right) + O(\mu^{n+j'-j''+2l'-4k}) & \text{if } 0 < n + j' - j'' + 2l' - 4k < n
\end{array} \right.
\]

\[
\times \sum_{\sigma' \in \mathcal{G}(S_{j',j'',m,s})} \delta_{i_\sigma'(\sigma(2m+1)), \sigma'(\sigma(2m+2)), \cdots, \delta_{i_\sigma'(\sigma(j'-1)), \sigma'(\sigma(j'))}} \\
\times \delta_{i_{\sigma'(j'+1)}^2 \sigma'(\sigma(j'+2)), \cdots, \delta_{i_{\sigma'(j'+j''-1)}^2 \sigma'(\sigma(j'))}}
\]

(2.49)

where

\[S_{j',j'',m,s} := (\sigma(2m+1), \cdots, \sigma(j'), j'+1, \cdots, j'+j'')\]

and \(\mathcal{G}(S_{j',j'',m,s})\) stands for the set of all permutations of \(S_{j',j'',m,s}\). In the case where \(j = 0, l = k\) and \(T = 1\), (2.26) follows from (2.35), (2.36), (2.43), (2.45) and (2.47). On the other hand, in the case where \(l < k\), by combining (2.39), (2.44) and (2.49) (and replacing \(j''\) by \(j''-2m'+2l'-2k+4\) for \(m' \in \{0, \cdots, [j'/2] + l - k + 2\}\) so that \(j' + j''\) is even and \(0 \leq j'' \leq j' + 2l - 2k + 4\), we obtain

\[
\int_M (T, \nabla' U_\mu) \Delta' U_\mu \, dv_\mu = \frac{2^{2k-n-2} (n-1)! \omega_n}{(n-2k-2)^2} \left( \sum_{l'=l'=\max(2k-l-2,0)}^{2l} \sum_{m=0}^{\lceil j'/2 \rceil} \sum_{m'=0}^{1-k+2} \frac{2^{2m'-j'\prime} l'! (k + l - l')! c(n,k,j',l',m,m') \mu^{4-2m'}}{(l'-l')! (2l-l')! (k+l-l'-m'+1)! (j''-2m'+2l-2k+4)!} \times B \left( \frac{n}{2} - k, 1, l' + 1 \right)^{-1} T^{m+1} \cdots T^{j'+1} \cdot \tilde{T}^{\alpha(m-1,2m'+1-k+2)} (\xi) \right)
\]

\[
\times \sum_{\sigma \in \mathcal{G}(j')} \sum_{\sigma' \in \mathcal{G}(S_{j',j''-2m'+2l-2k+4,s})} \delta_{i_{\sigma(1)}^2 \sigma(2)} \cdots \delta_{i_{\sigma(2m-1)}^2 \sigma(2m)} \\
\times \delta_{i_{\sigma'(j'+2m+1)}^2 \sigma'(\sigma(2m+2)), \cdots, \delta_{i_{\sigma'(\sigma(j'-1))}^2 \sigma'(\sigma(j'))}} \\
\times \delta_{i_{\sigma'(j'+1)}^2 \sigma'(\sigma(j'+2)), \cdots, \delta_{i_{\sigma'(\sigma(j'-j''-1))}^2 \sigma'(\sigma(j'))}} \\
\times \left\{ \begin{array}{ll}
2 \ln (1/\mu) + O(1) & \text{if } n = 2k + 4, l' = l \text{ and } m' = 0 \\
B \left( \frac{n}{2} + m' + l' - l - k - 2, k + l - l' - m' + 2 \right) + O(\mu^{2m'}) & \text{otherwise}
\end{array} \right.
\]

(2.50)
where
\[
c(n, k, k', l, l', m, m') := (-1)^{j'-m} B \left( \frac{n}{2} - k - 1, j' - m + 1 \right)^{-1} 
\times \frac{(j' - m)!}{m!(j' - 2m)!(n + j' - m + l' - 2k - 1)!(j' - m - m' + l - k + 2)!}.
\]

Straightforward computations yield
\[
\sum_{\sigma \in \mathcal{S}(j')} \sum_{\sigma' \in \mathcal{S}(j', j')}
\sum_{m,m'} \delta_{i_{\sigma(1)}i_{\sigma(2)}} \cdots \delta_{i_{\sigma(2m-1)}i_{\sigma(2m)}} \delta_{i_{\sigma'(2m+1)}i_{\sigma'(2m+2)}} 
\cdots \delta_{i_{\sigma'(j'-1)}i_{\sigma'(j')}} \delta_{i_{\sigma'(j'+1)}i_{\sigma'(j'+2)}} \cdots \delta_{i_{\sigma'(j'+m'-1)}i_{\sigma'(j'+m')}}
\begin{cases}
1 & \text{if } j' = j'' = m = 0 \\
2 \delta_{i_{1i_2}} & \text{if } j' = j'' = 1 \text{ and } m = 0 \\
2 (2 - m) \delta_{i_{1i_2}} & \text{if } j' = 2, j'' = 0 \text{ and } m \leq 1 \\
16 \delta_{i_{1i_2}} \delta_{i_{3i_4}} + \delta_{i_{1i_3}} \delta_{i_{2i_4}} + \delta_{i_{1i_4}} \delta_{i_{2i_3}} & \text{if } j' = j'' = 2 \text{ and } m = 0 \\
4 \delta_{i_{1i_2}} \delta_{i_{3i_4}} & \text{if } j' = j'' = 2 \text{ and } m = 1 \\
2 (4 - 2m) \delta_{i_{1i_2}} \delta_{i_{3i_4}} + \delta_{i_{1i_3}} \delta_{i_{2i_4}} + \delta_{i_{1i_4}} \delta_{i_{2i_3}} & \text{if } j' = 3, j'' = 1 \text{ and } m \leq 1 \\
8 (4 - 2m) \delta_{i_{1i_2}} \delta_{i_{3i_4}} + \delta_{i_{1i_3}} \delta_{i_{2i_4}} + \delta_{i_{1i_4}} \delta_{i_{2i_3}} & \text{if } j' = 4, j'' = 0 \text{ and } m \leq 2.
\end{cases}
\]

On the other hand, by using (2.3) and the fact that for all \(a, b, c, d, e \in \{1, \ldots, n\}\), in geodesic normal coordinates,
\[
g_{ab} (\xi) = \delta_{ab}, \quad g_{ab, c} (\xi) = 0 \quad \text{and} \quad g_{ab, cd} (\xi) = \frac{1}{3} (R_{acdb} (\xi) + R_{adbc} (\xi)),
\]
and
\[
\Gamma^a_{bc} (\xi) = 0, \quad \Gamma^a_{bc, d} (\xi) = \frac{1}{3} (R_{abcd} (\xi) + R_{acdb} (\xi)) \quad \text{and} \quad \Gamma^a_{cde} (\xi) = 0,
\]
we obtain
\[
\begin{align*}
\tilde{T} (\xi) &= T (\xi) & \text{if } j = 0 \\
\tilde{T}^a_{\cdot ; a} (\xi) &= T^a_{\cdot ; a} (\xi) & \text{if } j = 1 \\
\left\{ \begin{array}{l}
\tilde{T}^a_{\cdot ; a} (\xi) = 0, \\
\tilde{T}^a_{\cdot ; a} (\xi) = T^a_{\cdot ; a} (\xi), \\
\tilde{T}^a_{\cdot ; b} (\xi) = T^a_{\cdot ; b} (\xi)
\end{array} \right. & \text{if } j = 2 \\
\left\{ \begin{array}{l}
\tilde{T}^a_{\cdot ; b} (\xi) + \tilde{T}^a_{\cdot ; b} (\xi) + \tilde{T}^a_{\cdot ; a} (\xi)
\end{array} \right. & \text{if } j = 3 \\
\left\{ \begin{array}{l}
\tilde{T}^a_{\cdot ; b} (\xi) + \tilde{T}^a_{\cdot ; b} (\xi) + \tilde{T}^a_{\cdot ; b} (\xi)
\end{array} \right. & \text{if } j = 4.
\end{align*}
\]

We then obtain (2.27) by putting together (2.50), (2.51) and (2.52) and using the identities
\[
c(n, k, 0, k - 2, l', 0, 0) = \frac{n - 2k - 2}{2(n + l' - 2k - 1)!}
\]
Then, for every smooth function $f$ and $k$, Robert [13] in the case where $\rho = \mu = 0$

**Proof of Step 2.4.**

Steps 2.1–2.4:

By using the identities

We can now end the proof of Proposition 2.1 by putting together the results of

As regards the integral in the denominator of $I_{k,f,g}(u)$, we obtain the following:

**Step 2.4.** Assume that $n \geq 2k + 1$ and $g$ satisfies (2.2) for some point $\xi \in M$. Then, for every smooth function $f$ in $M$, as $\mu \to 0$,

$$
\int_M f U^2_{\mu} \, dv_g = \frac{\omega_n}{2^n} f(\xi) - \frac{\omega_n \Delta f(\xi) \mu^2}{2^{n+1}(n-2)} + \frac{\omega_n \Delta^2 f(\xi) \mu^4}{2^{n+3}(n-2)(n-4)} + o(\mu^4). \quad (2.53)
$$

**Proof of Step 2.4.** By observing that $U^2_{\mu}$ does not depend on $k$ in $B(0,r_0)$, we obtain that (2.53) is in fact identical to an estimate obtained by Esposito and Robert [13] in the case where $k = 2$ (note that in our case, $\text{Ric}(\xi) = 0$ and $\nabla S(\xi) = 0$ since we are working with conformal normal coordinates, see (2.3) and (2.4)).

We can now end the proof of Proposition 2.1 by putting together the results of Steps 2.1–2.4:
End of proof of Proposition 2.1. We assume that $k \geq 2$ and refer to Aubin [2] for the case where $k = 1$. By using (2.53), we obtain

$$
\left( \int_M f U_{\mu}^2 \, dv_g \right)^{\frac{n-2k}{n}} = \left( \frac{\omega_n}{2^n} f(\xi) \right)^{\frac{n-2k}{n}} \left[ 1 + \frac{(n-2k) \Delta f(\xi) \mu^2}{2n (n-2) f(\xi)} \right].
$$

We let $J_1, J_2, T_1, T_2, T_3, T_4, T_5$ and $Z$ be as in Step 2.1. Since $k \geq 1$, by integrating by parts, we obtain

$$
\int_M U_\mu \Delta^{k-1} (J_1 U_\mu) \, dv_g = \int_M (\Delta (J_1 U_\mu)) \Delta^{k-2} U_\mu \, dv_g
$$

$$
= \int_M (U_\mu \Delta J_1 - 2 (\nabla J_1, \nabla U_\mu) + J_1 \Delta U_\mu) \Delta^{k-2} U_\mu \, dv_g. \tag{2.55}
$$

By integrating by parts again, it follows from (2.7) and (2.55) that

$$
\int_M U_\mu P_{2k} U_\mu \, dv_g = \int_M U_\mu \Delta^{k} U_\mu \, dv_g + k \int_M \left( ((k-1) \, J_2 + \Delta J_1) U_\mu + ((k-1) \, T_1 - 2 \nabla J_1, \nabla U_\mu) + ((k-1) \, T_2 - J_1 g, \nabla^2 U_\mu) \right) \Delta^{k-2} U_\mu \, dv_g
$$

$$
+ k \, (k-1) \, (k-2) \int_M \left( (T_3, \nabla^2 U_\mu) + (T_4, \nabla^3 U_\mu) \right) \Delta^{k-3} U_\mu \, dv_g
$$

$$
+ k \, (k-1) \, (k-2) \, (k-3) \int_M (T_5, \nabla^4 U_\mu) \Delta^{k-4} U_\mu \, dv_g + \int_M U_\mu Z U_\mu \, dv_g. \tag{2.56}
$$

By using (2.4), we obtain

$$
P_{a \, b}^{\; a \, b}(\xi) = P_{b, a}^{\; b, a}(\xi) = P_{a \, b}^{\; b, a}(\xi) = -\frac{|W(\xi)|^2}{12(n-1)}. \tag{2.57}
$$

By using (2.3), (2.4) and (2.57) together with straightforward computations, we obtain

$$(T_3)_a^{\; a}(\xi) = -\frac{n + 3k + 1}{36(n-1)} |W(\xi)|^2 = -\frac{n + 3k + 1}{36n (n-1)} |W(\xi)|^2 g_a^{\; a}$$

and

$$(T_4)_a^{\; ab}(\xi) + (T_4)_a^{\; ba}(\xi) + (T_4)_a^{\; b, a}(\xi) = -\frac{k + 1}{6(n-1)} |W(\xi)|^2
$$

$$
= \frac{k + 1}{(n-1)(n+2)} \left( (\nabla S \otimes g)_a^{\; ab} + (\nabla S \otimes g)_a^{\; ba} \right) (\xi) + (\nabla S \otimes g)_a^{\; b, a}(\xi)
$$

and

$$(T_5)_a^{\; ab}(\xi) + (T_5)_a^{\; ba}(\xi) + (T_5)_a^{\; b, a}(\xi) = -\frac{k + 1}{10(n-1)} |W(\xi)|^2
$$

$$
= -\frac{(k+1) |W(\xi)|^2}{10n (n-1)(n+2)} \left( (g \otimes g)_a^{\; ab}(\xi) + (g \otimes g)_a^{\; ba}(\xi) + (g \otimes g)_a^{\; b, a}(\xi) \right).
$$

By using these identities together with (2.30)–(2.32) and observing that

$$(\nabla S \otimes g, \nabla^3 U_\mu) = -\Delta (\nabla S, \nabla U_\mu) - 2 (\nabla^2 S, \nabla^2 U_\mu) - (\nabla^3 S, \nabla U_\mu \otimes g),$$
and

\[ (\nabla^2 S)_a^a (\xi) = -\frac{1}{6} |W (\xi)|^2 = -\frac{1}{6n} |W (\xi)|^2 g_a^a (\xi), \]

we obtain that for \( k \geq 3 \),

\[
\int_M (T_3, \nabla^2 U_\mu) \Delta^{k-3} U_\mu \, dv_g = \frac{n + 3k + 1}{36n (n - 1)} |W(\xi)|^2 \int_M (\Delta U_\mu) \Delta^{k-3} U_\mu \, dv_g + \begin{cases} O(\mu^4) & \text{if } n = 2k + 4 \\ o(\mu^4) & \text{if } n > 2k + 4 \end{cases}
\]

\[
= \frac{n + 3k + 1}{36n (n - 1)} |W(\xi)|^2 \int_M U_\mu \Delta^{k-2} U_\mu \, dv_g + \begin{cases} O(\mu^4) & \text{if } n = 2k + 4 \\ o(\mu^4) & \text{if } n > 2k + 4 \end{cases} \quad (2.58)
\]

and

\[
\int_M (T_4, \nabla^3 U_\mu) \Delta^{k-3} U_\mu \, dv_g = -\frac{k + 1}{(n - 1) (n + 2)} \int_M (\Delta (\nabla S, \nabla U_\mu)) + \frac{1}{3n} |W(\xi)|^2 \Delta U_\mu + (\nabla^3 S, \nabla U_\mu \otimes g) \Delta^{k-3} U_\mu \, dv_g + \begin{cases} O(\mu^4) & \text{if } n = 2k + 4 \\ o(\mu^4) & \text{if } n > 2k + 4 \end{cases}
\]

\[
= \frac{k + 1}{3n (n - 1) (n + 2)} \int_M (|W(\xi)|^2 U_\mu + 3n (\nabla S, \nabla U_\mu)) \Delta^{k-2} U_\mu \, dv_g
\]

\[
- \frac{k + 1}{(n - 1) (n + 2)} \int_M U_\mu \Delta^{k-3} (\nabla^3 S, \nabla U_\mu \otimes g) \, dv_g + \begin{cases} O(\mu^4) & \text{if } n = 2k + 4 \\ o(\mu^4) & \text{if } n > 2k + 4 \end{cases}
\]

\[
(2.59)
\]

and for \( k \geq 4 \),

\[
\int_M (T_5, \nabla^4 U_\mu) \Delta^{k-4} U_\mu \, dv_g
\]

\[
= -\frac{(k + 1) |W(\xi)|^2}{10n (n - 1) (n + 2)} \int_M (\Delta^2 U_\mu) \Delta^{k-4} U_\mu \, dv_g + \begin{cases} O(\mu^4) & \text{if } n = 2k + 4 \\ o(\mu^4) & \text{if } n > 2k + 4 \end{cases}
\]

\[
= -\frac{(k + 1) |W(\xi)|^2}{10n (n - 1) (n + 2)} \int_M U_\mu \Delta^{k-2} U_\mu \, dv_g + \begin{cases} O(\mu^4) & \text{if } n = 2k + 4 \\ o(\mu^4) & \text{if } n > 2k + 4 \end{cases}
\]

\[
(2.60)
\]

It follows from (2.56) and (2.58)–(2.60) that

\[
\int_M U_\mu P_2 U_\mu \, dv_g = \int_M U_\mu \Delta^k U_\mu \, dv_g + k \int_M \left( (k - 1) J_2 + \Delta J_3 \right) + \left( \frac{(k - 1) (k - 2) (n + 3k + 1)}{36n (n - 1)} - \frac{(k - 1) (k - 2) (k + 1) (3k + 1)}{30n (n - 1) (n + 2)} \right) |W(\xi)|^2 \int_M U_\mu
\]

\[
+ \left( (k - 1) T_1 - 2 \nabla J_1 - \frac{(k - 1) (k - 1) (k - 2)}{(n - 1) (n + 2)} \nabla S, \nabla U_\mu \right)
\]

\[
+ \left( (k - 1) T_2 - J_1 g, \nabla^2 U_\mu \right) \Delta^{k-2} U_\mu \, dv_g
\]
Straightforward computations together with (2.3), (2.4) and (2.57) give

and

By using these identities together with (2.20), (2.27)–(2.29), (2.54) and (2.61), we obtain that (2.6) holds true with

and

By using these identities together with (2.20), (2.27)–(2.29), (2.54) and (2.61), we obtain that (2.6) holds true with $C(n, k)$ defined as

$$C(n, k) := \frac{(n-3)(n-5)!k!}{16(n-2)!}$$

$$\times \sum_{l=k-2}^{2k-4} \frac{l!}{(l-k+2)!(2k-l-4)!(n+l-2k+1)!} \left( \frac{8(n+l-2k)(n+l-2k+1)}{36n} + \frac{8(n-2k)(n+l-2k+1)}{6(n+2)} \right)$$

$$\times \left( \frac{(k-1)(3n+2k-4)}{144} - \frac{n-2}{24} - \frac{k+1}{18(n-1)} \right)^2$$

$$\times B \left( \frac{n}{2} - k - 1, l + 1 \right) \left\{ \begin{array}{ll}
2\chi_{l=k-2} & \text{if } n = 2k + 4 \\
B \left( \frac{n}{2} + l - 2k, 2k - l - 2 \right) & \text{otherwise}
\end{array} \right.$$

$$= \frac{(n-3)(n-5)!k!}{5760n(n+2)(k-1)(n-2k-2)}$$

$$\times \sum_{l=k-2}^{2k-4} \frac{l!c(n, k, l)}{(l-k+2)!(2k-l-4)!(n+l-2k+1)!} \times B \left( \frac{n}{2} - k - 1, l + 1 \right) \left\{ \begin{array}{ll}
2\chi_{l=k-2} & \text{if } n = 2k + 4 \\
B \left( \frac{n}{2} + l - 2k, 2k - l - 2 \right) & \text{otherwise}
\end{array} \right.$$

$$= \sum_{l=2}^{n} \left\{ \begin{array}{ll}
\frac{n}{2} - k - 1 & \text{if } n = 2k + 4 \\
\frac{n}{2} + l - 2k, 2k - l - 2 & \text{otherwise}
\end{array} \right.$$

$$= \sum_{l=2}^{n} \left\{ \begin{array}{ll}
\frac{n}{2} - k - 1 & \text{if } n = 2k + 4 \\
\frac{n}{2} + l - 2k, 2k - l - 2 & \text{otherwise}
\end{array} \right.$$
where
\[ c(n, k, l) := 4(n + l - 2k) (n + l - 2k + 1) (5n (n + 2) (k - 1) (3n + 2k - 4) - 30n(n + 2)(n - 2) - 20(n + 2)(k - 1)(k - 2)(n + 3k + 1) + 24 (k - 1)(k - 2)(k + 1)(3k + 1)) + 20n(n - 2k)(n + l - 2k + 1) \times (6(n + 2)(n - 2) - (n + 2)(k - 1)(3n + 4k - 2) + 12(k + 1)(k - 1)(k - 2)) + 5n(n + 2)(n - 2k)(4(k + 1)(k - 1)(n - 2k - 2l + 4) - 3(n - 2)(2(n - 2k + 2) - n(n + 2l - 2k))].

By letting \( k := 3 + a, n := 2k + 4 + b \) and \( l := k - 2 + c \) and using the software Maple to expand the expression of \( c(n, k, l) \), we then obtain
\[ c(n, k, l) = 4(15ab^3 + 1200a^2b + 3880ab + 1920 + 10656a + 480b + 4528a^2 + 624a^3 + 40b^2 + 450ab^2 + 80a^3b + 80a^2b^2 + 32a^4)c^2 + 2(71552a^2b + 414912a + 500a^2b^3 + 247984ab + 31840a^3 + 53660ab^2 + 3200a^4 + 640a^3b^2 + 11020b^3 + 150ab^4 + 128a^5 + 9056a^3b + 660a^4 + 161440a^2 + 448a^4b + 15b^4 + 311040b + 10520a^2b^2 + 4830ab^3 + 426240 + 85840b^2)c + 128a^6 + 576a^5b + 1088a^4b^2 + 1020a^3b^3 + 560a^2b^4 + 150ab^5 + 15b^6 + 3904a^5b + 14720a^4b + 21896a^3b^2 + 15940a^2b^3 + 5640ab^4 + 720b^5 + 49408a^4 + 149280a^3b + 167032a^2b^2 + 81120ab^3 + 13780b^4 + 332096a^3 + 754720a^2b + 563824ab^2 + 134240b^3 + 1250304a^2 + 1900224ab + 704640b^2 + 2499840a + 1900800b + 2073600.
\]

Since all the coefficients in this expression are positive, it follows that \( C(n, k) \) is positive whenever \( k \geq 3, n \geq 2k + 4 \) and \( l \geq k - 2 \). Furthermore, in the case where \( k = 2 \) and \( l = 0 \), we find
\[ c(n, 2, 0) = 5n(n + 2)(n - 4)^2(n^2 - 4n - 4) > 0 \quad \forall n \geq 8.
\]

Therefore, in all cases, we find that \( C(n, k) \) is positive. This ends the proof of Proposition \( 2.1 \). \( \square \)

We can now prove Theorem \( 1.1 \) by using Proposition \( 2.1 \).

Proof of Theorem \( 1.1 \) in the case where \( n \geq 2k + 4 \). Let \( \xi \in M \) be a maximal point of \( f \) and \( \bar{g} = \varphi^{4/(n-2)}g \) be a conformal metric to \( g \) such that \( \varphi(\xi) = 1 \) and \( \det \bar{g}(x) = 1 \) for all \( x \) in a neighborhood of the point \( \xi \). Notice that since \( \xi \) is a maximal point of \( f \), if \( \Delta_{\bar{g}} f(\xi) = 0 \), then \( \nabla^j f = 0 \) for all \( j \in \{1, 2, 3\} \). In particular, since \( \varphi(\xi) = 1 \), it follows that
\[ \Delta_{\bar{g}} f(\xi) = 0 \quad \text{and} \quad \Delta_{\bar{g}}^2 f(\xi) = \Delta_{\bar{g}}^2 f(\xi), \]
where \( \Delta_{\bar{g}} \) and \( \Delta_{\bar{g}}^2 \) are the Laplace–Beltrami operators with respect to the metrics \( g \) and \( \bar{g} \), respectively, and the covariant derivatives, the Ricci tensor and the multiple inner product in the right-hand side of the second identity are with respect to the metric \( g \). Let \( c(n, k) \) be the constant defined as
\[ c(n, k) := \begin{cases} 0 & \text{if } n = 2k + 4 \\ \frac{(n - 2k)(2k - 1)!}{8n(n - 2)(n - 4)C(n, k)} & \text{if } n > 2k + 4, \end{cases} \]
(2.65)
where $C(n, k)$ is as in (2.6) (see also (2.62)). By applying Proposition 2.1 together with (2.64) and the fact that $|W|$ is conformally invariant, we then obtain that if (1.2) and (1.3) hold true, then

$$\inf_{u \in C^{2k}(M) \setminus \{0\}} I_{k,f,g}(u) < \omega_n^{\frac{2k}{n}} (2k - 1)! B \left( \frac{n}{2} - k, 2k \right)^{-1} \left( \max_{x \in M} f(x) \right)^{-\frac{n-2k}{n}}. \quad (2.66)$$

On the other hand, by conformal invariance of the operator $P_{2k}$, we obtain

$$\inf_{u \in C^{2k}(M) \setminus \{0\}} I_{k,f,g}(u) = \inf_{u \in C^{2k}(M) \setminus \{0\}} I_{k,f,g}(u). \quad (2.67)$$

By putting together (2.66) and (2.67) and applying Theorem 3 of Mazumdar [30], we then obtain that the conclusions of Theorem 1.1 hold true.

3. The remaining cases

This section is devoted to the proof of Theorem 1.1 in the remaining case where $2k + 1 \leq n \leq 2k + 3$ together with the following result in the case where $g$ is conformally flat in some open subset of the manifold:

**Theorem 3.1.** Let $k \geq 1$ be an integer, $(M,g)$ be a smooth, closed Riemannian manifold of dimension $n \geq 2k + 1$ and $f$ be a smooth positive function in $M$. Assume that $Y_{2k} > 0$ and there exists a maximal point $\xi$ of $f$ such that $m(\xi) > 0$ (see (3.2) for the definition of the mass), $\nabla^j f(\xi) = 0$ for all $j \in \{1, \ldots, n - 2k\}$ and $g$ is conformally flat in some neighborhood of the point $\xi$. Then there exists a nontrivial solution $u \in C^{2k}(M)$ to the equation (1.1), which minimizes the energy functional (2.1). If moreover the Green’s function of the operator $P_{2k}$ is positive, then $u$ is positive, which implies that the $Q$-curvature of order $2k$ of the metric $u^{4/(n-2k)}g$ is equal to $\frac{2}{n-2k} f$.

Notice that Theorem 1.2 is now a direct consequence of Theorems 1.1 and 3.1.

Throughout this section, we fix a point $\xi \in M$ and assume that $2k + 1 \leq n \leq 2k + 3$ or $g$ is conformally flat in some neighborhood of $\xi$. In these cases, our proofs are based on the method of Schoen [36] for the resolution of the remaining cases of the Yamabe problem, which has been extended to the $k = 2$ case by Gursky and Malchiodi [21] and Hang and Yang [22,23]. We consider a family of global test functions involving the Green’s function and derive an expression for the energy functional $I_{k,f,g}$ (see (2.1)) associated with the equation (1.1). Then, analogously as in the case $n \geq 2k + 4$, by using the expansion obtained in Proposition 2.1, we obtain the existence of a nontrivial solution to the equation (1.1) under a positivity assumption on the mass of the operator $P_{2k}$.

We now discuss the definition of the mass. By applying a conformal change of metric, we may assume that

$$\begin{cases} g \text{ satisfies (2.2) in some neighborhood } \Omega \text{ of } \xi \text{ if } 2k + 1 \leq n \leq 2k + 3 \\ g \text{ is flat in some neighborhood } \Omega \text{ of } \xi \text{ if } n \geq 2k + 4. \end{cases} \quad (3.1)$$

Then, in the geodesic normal coordinates at $\xi$ determined by $g$, the Green’s function $G_{2k}(x) := G_{2k}(x, \xi)$ of the operator $P_{2k}$ has the expansion

$$G_{2k}(x) = b_{n,k} d_g(x, \xi)^{2k-n} + m(\xi) + o(1) \quad (3.2)$$
as $x \to \xi$ (see Lee and Parker [28] for $k = 1$ and Michel [31] for $k \geq 2$), where $m(\xi) \in M$ is called the mass of the operator $P_{2k}$ at the point $\xi$ and the constant $b_{n,k}$ is defined as

$$b_{n,k}^{-1} := 2^{k-1} (k-1)! (n-2) (n-4) \cdots (n-2k) \omega_{n-1}.$$ 

It is important to point out that the sign of $m(\xi)$ does not depend on our choice of conformal metric (see Michel [31, Théorème 3.1]).

Now that the mass is defined, we consider the regular part of the Green’s function, which plays a crucial role in the proofs of our theorems. It follows from (3.2) that there exists a continuous function $h_{2k}$ in $M$ such that $h_{2k}(\xi) = m(\xi)$ and

$$G_{2k}(x) = b_{n,k} d_g(x, \xi)^{2k-n} + h_{2k}(x) \quad \forall x \in M \setminus \{\xi\}.$$ 

(3.3)

Furthermore, we have that $h_{2k} \in C^\infty(\Omega)$ in the case where $g$ is flat in $\Omega$ and $h_{2k} \in W^{2k,p}(\Omega)$ for all $p \in [1, n/(n-4))$ if $n \geq 5$ and $p \in [1, \infty)$ if $n \in \{3,4\}$ in the case where $2k + 1 \leq n \leq 2k+3$ and $g$ satisfies (2.2) in $\Omega$. This follows from classical elliptic regularity theory (see Agmon, Douglis and Nirenberg [1]) together with the fact that $P_{2k} h_{2k} = \Delta^k h_{2k} = 0$ in $\Omega$ in the case where $g$ is flat in $\Omega$ and $P_{2k} h_{2k} = O(d_g(\cdot, \xi)^{4-n})$ in $\Omega$ in the case where $g$ satisfies (2.2) in $\Omega$ (see [31, Lemme 2.2]).

For every $\mu > 0$, letting $\chi$ and $U_\mu$ be as in Section 2, we consider the test functions $V_\mu$ defined as

$$V_\mu(x) := U_\mu(x) + b_{n,k}^{-1} \mu^{n-2k} (\chi(d_g(x, \xi)) h_{2k}(x) + (1 - \chi(d_g(x, \xi))) G_{2k}(x))$$ 

(3.4)

for all $x \in M$. Note that $V_\mu \in W^{2k,2n/(n+2k)}(M)$ so that in particular the integral $\int_M V_\mu P_{2k} V_\mu d_g$ is well defined. We then obtain the following:

**Proposition 3.1.** Let $k \geq 1$ be an integer, $(M, g)$ be a smooth, closed Riemannian manifold of dimension $n \geq 2k+1$, $f$ be a smooth positive function in $M$ and $\xi$ be a point in $M$ such that $\nabla^j f(\xi) = 0$ for all $j \in \{1, \ldots, n-2k\}$ and (3.1) holds true. Let $I_{k,f,g}$ be as in (2.1) and $V_\mu$ be as in (3.4). Then, as $\mu \to 0$,

$$I_{k,f,g}(V_\mu) = \omega_n^{2k} (2k-1)! B\left(\frac{n}{2} - k, 2k\right)^{-1} f(\xi)^{-\frac{n-2k}{n}}$$

$$\times \left(1 - b_{n,k}^{-1} B\left(\frac{n}{2}, n - 2k\right) \mu^{n-2k} + o\left(\mu^{n-2k}\right)\right).$$ 

(3.5)

**Proof of Proposition 3.1.** The first step in the proof is as follows:

**Step 3.1.** Assume that $g$ satisfies (3.1) for some point $\xi \in M$. Then, as $\mu \to 0$,

$$\int_M V_\mu P_{2k} V_\mu d_g = 2^{2k-n} \omega_n (2k-1)! B\left(\frac{n}{2} - k, 2k\right)^{-1}$$

$$\times \left(1 + b_{n,k}^{-1} B\left(\frac{n}{2}, n - 2k\right) \mu^{n-2k} + o\left(\mu^{n-2k}\right)\right).$$ 

(3.6)

**Proof of Step 3.1.** We write

$$V_\mu(x) = b_{n,k}^{-1} \mu^{\frac{n-2k}{n}} G_{2k}(x) + \left(U_\mu(x) - \mu^{\frac{n-2k}{n}} \chi(d_g(x, \xi)) d_g(x, \xi)^{2k-n}\right)_{W_\mu(x)}$$

where $W_\mu(x)$...
for all \( x \in M \setminus \{ \xi \} \). Straightforward estimates give

\[
\int_M V_\mu P_{2k} V_\mu \, dv_g = \int_{B(\xi, r_0)} V_\mu P_{2k} W_\mu \, dv_g
\]

\[
= \int_{B(\xi, r_0)} \left( U_\mu + b_{n,k}^{-1} V_\mu^{1-2k} h_{2k} \right) P_{2k} W_\mu \, dv_g
\]

\[
+ O \left( \mu^{n-2k} \int_{B(\xi, r_0) \setminus B(\xi, r_0)} |P_{2k} W_\mu| \, dv_g \right)
\]

\[
= \int_{B(\xi, r_0)} U_\mu P_{2k} W_\mu \, dv_g + b_{n,k}^{-1} \mu^{n-2k} \int_{B(\xi, r_0)} h_{2k} P_{2k} W_\mu \, dv_g
\]

\[
+ O \left( \mu^{n-2k} \sum_{|\alpha| \leq 2k} \int_{B(0, r_0) \setminus B(0, r_0)} \left| \partial^\alpha \left( \mu^2 + |x|^2 \right)^{2k-n} - |x|^{2k-n} \right| \, dx \right)
\]

\[
= \int_{B(\xi, r_0)} U_\mu P_{2k} W_\mu \, dv_g + b_{n,k}^{-1} \mu^{n-2k} \int_{B(\xi, r_0)} h_{2k} P_{2k} W_\mu \, dv_g + o(\mu^{n-2k}). \quad (3.7)
\]

We claim that

\[
|P_{2k} W_\mu - \Delta^k U_\mu| \leq C \mu^{n-2(k-2)} d_g(x, \xi)^{2-n} \quad \forall x \in B(\xi, r_0) \setminus \{ \xi \} \quad (3.8)
\]

for some constant \( C \) independent of \( x, \mu \) and \( \xi \). Assuming (3.8) and proceeding as in Step 2.3, it then follows from (3.7) that

\[
\int_M V_\mu P_{2k} V_\mu \, dv_g = \int_{B(\xi, r_0)} U_\mu \Delta^k U_\mu \, dv_g + b_{n,k}^{-1} \mu^{n-2k} \int_{B(\xi, r_0)} h_{2k} \Delta^k U_\mu \, dv_g
\]

\[
+ O \left( \mu^{n-2(k-1)} \int_{B(0, r_0)} |x|^{2-n} \left( \mu^2 + |x|^2 \right)^{2k-n} \, dx \right) + o(\mu^{n-2k})
\]

\[
= \int_{B(0, r_0)} \tilde{U}_\mu \Delta^k \tilde{U}_\mu \, dx + b_{n,k}^{-1} \mu^{n-2k} \int_{B(0, r_0)} h_{2k} \left( \exp_{\xi} x \right) \Delta^k \tilde{U}_\mu \, dx + o(\mu^{n-2k})
\]

\[
= 2^{2k-1} (2k - 1)! \omega_{n-1} B \left( \frac{n}{2} - k, 2k \right)^{-1} \left( \int_0^{(r_0/\mu)^2} r^{n-2} \left( 1 + r \right)^{\frac{n-2}{2}} \, dr \right)
\]

\[
+ b_{n,k}^{-1} \omega_n B \left( \frac{n}{2} - k, 2k \right)^{-1} \left( \int_0^{(r_0/\mu)^2} r^{n-2} \left( 1 + r \right)^{\frac{n-2}{2}} \, dr \right) + o(\mu^{n-2k})
\]

\[
= 2^{2k-n} (2k - 1)! \omega_n B \left( \frac{n}{2} - k, 2k \right)^{-1}
\]

\[
\times \left( 1 + b_{n,k}^{-1} B \left( \frac{n}{2}, \frac{n}{2} \right)^{-1} B \left( \frac{n}{2}, k \right) M(\xi) \mu^{n-2k} + o(\mu^{n-2k}) \right).
\]

Therefore, it remains to prove (3.8) to complete the proof of Step 3.2. Notice that (3.8) is clearly satisfied with \( C = 0 \) in the case where \( n \geq 2k + 4 \) and \( g \) is flat in \( \Omega \). Therefore, we may assume in what follows that we are in the case where \( 2k + 1 \leq n \leq 2k + 3 \) and \( g \) satisfies (2.2) in \( \Omega \). By using (2.7), we obtain

\[
P_{2k} = \Delta^k + k \Delta^{k-1} (J_1) + k (k - 1) \Delta^{k-2} ((T_1, \nabla) + (T_2, \nabla^2)) + k (k - 1) (k - 2) \Delta^{k-3} (T_4, \nabla^2) + Z, \quad (3.9)
\]
where $Z$ is a smooth linear operator of order less than $2k - 3$ if $k \geq 2$, $Z := 0$ if $k = 1$. By induction, one can check that

\[
\begin{align*}
\Delta^{k-1} (J_1, \cdot) & = J_1 \Delta^{k-1} - 2 (k - 1) (\nabla J_1, \nabla \Delta^{k-2}) + o^{2k-3}, \\
\Delta^{k-2} (T_1, \nabla) & = (T_1, \nabla \Delta^{k-2}) + o^{2k-3}, \\
\Delta^{k-2} (T_2, \nabla^2) & = (T_2, \nabla^2 \Delta^{k-2}) - 2 (k - 2) (\nabla T_2, \nabla^3 \Delta^{k-3}) + o^{2k-3}
\end{align*}
\] (3.10)–(3.12)

and

\[
\Delta^{k-3} (T_4, \nabla^3) = (T_4, \nabla^3 \Delta^{k-3}) + o^{2k-3},
\] (3.13)

where $o^{2k-3}$ is as in the proof of Step 2.1. It follows from (2.3), (2.4) and (3.9)–(3.11) that

\[
P_{2k} W_\mu = \Delta^k W_\mu + \frac{2k (k - 1) (k + 1)}{3 (n - 2)} \left( (\text{Ric}, \nabla^2 \Delta^{k-2} W_\mu) - (\delta \text{Ric}, \nabla \Delta^{k-2} W_\mu) \right) - (k - 2) (\nabla \text{Ric}, \nabla^3 \Delta^{k-2} W_\mu) + O \left( d_g (\cdot, \xi)^2 |\nabla^{2k-2} W_\mu| + \sum_{j=0}^{2k-4} |\nabla^j W_\mu| \right)
\] (3.14)

in $M \setminus \{\xi\}$, uniformly with respect to $\mu$ and $\xi$. By using geodesic normal coordinates together with (2.3) and a Taylor expansion, it follows from (3.14) that

\[
(P_{2k} W_\mu - \Delta^k U_\mu) (\exp_\xi x) = (P_{2k} W_\mu - \Delta^k W_\mu) (\exp_\xi x)
= O \left( |x|^2 |\nabla^{2k-2} \widetilde{W}_\mu (x)| + |x| |\nabla^{2k-3} \widetilde{W}_\mu (x)| + \sum_{j=0}^{2k-4} |\nabla^j \widetilde{W}_\mu (x)| \right)
\] (3.15)

uniformly with respect to $x \in B (0, r_0) \setminus \{0\}$, $\mu$ and $\xi$, where

\[
\widetilde{W}_\mu (x) := \mu^{\frac{2k-n}{2}} U (x/\mu) - \mu^{\frac{n-2k}{2}} |x|^{2k-n}.
\]

Similarly as in (2.40), we obtain that for every $j \in \mathbb{N}$, there exists a constant $C_j$ independent of $x$ and $\mu$ such that

\[
|\nabla^j \widetilde{W}_\mu (x)| = \mu^{\frac{2k-n-2j}{2}} |\nabla^j W (x/\mu)|
\leq C_j \sum_{m=0}^{[j/2]} \mu^{\frac{2k-n-j+4m}{2}} r^{\frac{j-2m}{2}} |\partial_r^{j-m} W (r/\mu^2)|
\] (3.16)

for all $x \in B (0, r_0) \setminus \{0\}$, where $r := |x|^2$ and

\[
W (x) = W (r) := (1 + r)^{(2k-n)/2} - r^{(2k-n)/2}.
\]

Furthermore, it is easy to see that

\[
|\partial_r^{j} W (r)| \leq C_j' r^{\frac{2k-n-2j-2}{2}}
\] (3.17)

for some constant $C_j'$ independent of $r$. We then obtain (3.8) by putting together (3.15)–(3.17). This completes the proof of Step 3.1. $\square$
**Step 3.2.** Assume that $g$ satisfies (3.1) for some point $\xi \in M$. Let $f$ be a smooth function $f$ in $M$ such that $\nabla^j f(\xi) = 0$ for all $j \in \{1, \ldots, n-2k\}$. Then, as $\mu \to 0$,

$$
\int_M f \left| V_\mu \right|^2 \, dv_g = \frac{\omega_n}{2^n} \left( f(\xi) + 2_k b_{n,k} \left( \frac{n}{2}, \frac{n}{2} \right)^{-1} \int_B \left( \frac{n}{2} - k \right) f(\xi) m(\xi) \mu^{n-2k} + o(\mu^{n-2k}) \right). \tag{3.18}
$$

**Proof of Step 3.2.** By using a Taylor expansion together with straightforward estimates, we obtain

\[
\int_M f \left| V_\mu \right|^2 \, dv_g = \int_{B(\xi, 0)} f \left| U_\mu + b_{n,k} \mu^{\frac{n-2k}{2}} h_{2k} \right|^2 \, dv_g + O(\mu^n) \\
= f(\xi) \int_{B(\xi, 0)} U_\mu h_{2k} \, dv_g + 2_k b_{n,k} \int_{B(\xi, 0)} f h_{2k} U_\mu \, dv_g \\
+ O \left( \int_{B(0, 0)} \left[ |x|^{n-2k+1} \left( \frac{\mu}{\mu^2 + |x|^2} \right)^n + \mu^{n-2k} \left( \frac{\mu}{\mu^2 + |x|^2} \right)^{2k} \right] dx + \mu^n \right) \\
= \frac{\omega_n}{2^n} f(\xi) + \frac{n \omega_{n-1}}{n - 2k} b_{n,k} f(\xi) m(\xi) \mu^{n-2k} \int_0^{(ro/\mu)^2} r \frac{n-2k}{n-2k-2} dr + o(\mu^{n-2k}). \tag{3.19}
\]

Then (3.18) follows from (2.45), (2.47) and (3.19).

We can now end the proofs of Proposition 3.1 and Theorems 1.1 and 1.2.

**End of proof of Proposition 3.1.** We obtain (3.5) by putting together (3.6) and (3.18).

**Proofs of Theorem 1.1 in the case where $2k + 1 \leq n \leq 2k + 3$ and of Theorem 3.1.** Let $\xi \in M$ be a maximal point of $f$ such that $\nabla^j f(\xi) = 0$ for all $j \in \{1, \ldots, n-2k\}$ (notice that for a maximal point, this is equivalent to (1.2) in the case where $2k + 1 \leq n \leq 2k + 3$). By applying Proposition 3.1 together with a conformal change of metric, we then obtain that if $m(\xi) > 0$, then there exists a function $V \in W^{2k,2n/(n+2k)}(M) \setminus \{0\}$ such that

$$
I_{k,f,g}(V) < \omega_n^2 (2k - 1)! B \left( \frac{n}{2} - k, 2k \right)^{-1} \left( \max_{x \in M} f(x) \right)^{-2k/n}.
$$

Notice that $W^{2k,2n/(n+2k)}(M) \hookrightarrow L^{2k}(M)$ so that a density argument gives

$$
\inf_{u \in C^2(M) \setminus \{0\}} I_{k,f,g}(u) < \omega_n^2 (2k - 1)! B \left( \frac{n}{2} - k, 2k \right)^{-1} \left( \max_{x \in M} f(x) \right)^{-2k/n}.
$$

We can then conclude the proofs of Theorems 1.1 and 3.1 by applying Theorem 3 of Mazumdar [30].

**References**

[1] S. Agmon, A. Douglas, and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I.*, Comm. Pure Appl. Math. 12 (1959), 623–727.

[2] Th. Aubin, *Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire.*, J. Math. Pures Appl. (9) 55 (1976), no. 3, 269–296 (French).

[3] P. Baird, A. Fardoun, and R. Regbaoui, *Prescribed Q-curvature on manifolds of even dimension*, J. Geom. Phys. 59 (2009), no. 2, 221–233.
[4] T. P. Branson, Differential operators canonically associated to a conformal structure, Math. Scand. 57 (1985), no. 2, 293–345.
[5] , Sharp inequalities, the functional determinant, and the complementary series, Trans. Amer. Math. Soc. 347 (1995), no. 10, 3671–3742.
[6] S. Brendle, Blow-up phenomena for the Yamabe equation, J. Amer. Math. Soc. 21 (2008), no. 4, 951–979.
[7] J. G. Cao, The existence of generalized isothermal coordinates for higher-dimensional Riemannian manifolds, Trans. Amer. Math. Soc. 324 (1991), no. 2, 901–920.
[8] S.-Y. A. Chang and Paul C. Yang, Extremal metrics of zeta function determinants on 4-manifolds, Ann. of Math. (2) 142 (1995), no. 1, 171–212.
[9] X. Chen and F. Hou, Remarks on GJMS operator of order six, Pacific J. Math. 289 (2017), no. 1, 35–70.
[10] Z. Djadli, E. Hebey, and M. Ledoux, Paneitz-type operators and applications, Duke Math. J. 104 (2000), no. 1, 129–169.
[11] Z. Djadli and A. Malchiodi, Existence of conformal metrics with constant Q-curvature, Ann. of Math. (2) 168 (2008), no. 3, 813–858.
[12] J. F. Escobar and R. M. Schoen, Conformal metrics with prescribed scalar curvature, Invent. Math. 86 (1986), no. 2, 243–254.
[13] P. Esposito and F. Robert, Mountain pass critical points for Paneitz-Branson operators, Calc. Var. Partial Differential Equations 15 (2002), no. 4, 493–517.
[14] C. Fefferman and C. R. Graham, Conformal invariants, ´Elie Cartan et les mathématiques d’aujourd’hui - Lyon, 25-29 juin 1984, Astérisque S131 (1985), 95–116.
[15] , The ambient metric, Annals of Mathematics Studies, vol. 178, Princeton University Press, Princeton, NJ, 2012.
[16] , Juhl’s formulae for GJMS operators and Q-curvatures, J. Amer. Math. Soc. 26 (2013), no. 4, 1191–1207.
[17] A. R. Gover, Laplacian operators and Q-curvature on conformally Einstein manifolds, Math. Ann. 336 (2006), no. 2, 311–334.
[18] C. R. Graham, R. Jenne, L. J. Mason, and G. A. J. Sparling, Conformally invariant powers of the Laplacian. I. Existence, J. London Math. Soc. (2) 46 (1992), no. 3, 557–565.
[19] M. G¨unther, Conformal normal coordinates, Ann. Global Anal. Geom. 11 (1993), no. 2, 173–184.
[20] M. J. Gursky, F. Hang, and Y.-J. Lin, Riemannian manifolds with positive Yamabe invariant and Paneitz operator, Int. Math. Res. Not. IMRN 5 (2016), 1348–1367.
[21] M. J. Gursky and A. Malchiodi, A strong maximum principle for the Paneitz operator and a non-local flow for the Q-curvature, J. Eur. Math. Soc. (JEMS) 17 (2015), no. 9, 2137–2173.
[22] F. Hang and P. C. Yang, Sign of Green’s function of Paneitz operators and the Q-curvature, Int. Math. Res. Not. IMRN 19 (2015), 9775–9791.
[23] , Q-curvature on a class of manifolds with dimension at least 5, Comm. Pure Appl. Math. 69 (2016), no. 8, 1452–1491.
[24] E. Hebey, Changements de m´etriques conformes sur la sph`ere. Le probl`eme de Nirenberg, Bull. Sci. Math. 114 (1990), no. 2, 215–242 (French, with English summary).
[25] E. Hebey and M. Vaugon, Courbure scalaire prescrite pour des variétés non conformément difféomorphes à la sphère, C. R. Acad. Sci. Paris Sér. I Math. 316 (1993), no. 3, 281–282 (French).
[26] E. Humbert and S. Raulot, Positive mass theorem for the Paneitz-Branson operator, Calc. Var. Partial Differential Equations 36 (2009), no. 4, 525–531.
[27] A. Juhl, Explicit formulas for GJMS-operators and Q-curvatures, Geom. Funct. Anal. 23 (2013), no. 4, 1278–1370.
[28] J. M. Lee and T. H. Parker, The Yamabe problem, Bull. Amer. Math. Soc. (N.S.) 17 (1987), no. 1, 37–91.
[29] J. Li, Y. Li, and P. Liu, The Q-curvature on a 4-dimensional Riemannian manifold (M,g) with \( \int_M Q dV_g = 8\pi^2 \), Adv. Math. 231 (2012), no. 3-4, 2194–2223.
[30] S. Mazumdar, GJMS-type operators on a compact Riemannian manifold: best constants and Coron-type solutions, J. Differential Equations 261 (2016), no. 9, 4907–5034.
[31] B. Michel, Masse des opérateurs GJMS, arXiv:1012.4414 (2010) (French).
S. M. Paneitz, *A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds (summary)*, SIGMA Symmetry Integrability Geom. Methods Appl. **4** (2008), no. 36.

J. Qing and D. Raske, *On positive solutions to semilinear conformally invariant equations on locally conformally flat manifolds*, Int. Math. Res. Not. **2006** (2006), no. 94172.

F. Robert, *Positive solutions for a fourth order equation invariant under isometries*, Proc. Amer. Math. Soc. **131** (2003), no. 5, 1423–1431.

F. Robert, *Admissible $Q$-curvatures under isometries for the conformal GJMS operators*, Nonlinear elliptic partial differential equations, Contemp. Math., vol. 540, Amer. Math. Soc., Providence, RI, 2011, pp. 241–259.

R. M. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geom. **20** (1984), no. 2, 479–495.

R. M. Schoen, *On the number of constant scalar curvature metrics in a conformal class*, Differential geometry, Pitman Monogr. Surveys Pure Appl. Math., vol. 52, Longman Sci. Tech., Harlow, 1991, pp. 311–320.

R. Schoen and S.-T. Yau, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. **65** (1979), no. 1, 45–76.

R. Schoen and S.-T. Yau, *Proof of the Positive-Action Conjecture in Quantum Relativity*, Phys. Rev. Lett. **42** (1979), no. 9, 547–548.

R. Schoen and S.-T. Yau, *Conformally flat manifolds, Kleinian groups and scalar curvature*, Invent. Math. **92** (1988), no. 1, 47–71.

N. S. Trudinger, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. Pisa (3) **22** (1968), 265–274.

H. Yamabe, *On a deformation of Riemannian structures on compact manifolds*, Osaka Math. J. **12** (1960), 21–37.