Tree Projections and Structural Decomposition Methods: Minimality and Game-Theoretic Characterization

Gianluigi Greco and Francesco Scarcello

University of Calabria, 87036, Rende, Italy
{ggreco}@mat.unical.it, {scarcello}@deis.unical.it

Abstract

Tree projections provide a mathematical framework that encompasses all the various (purely) structural decomposition methods that have been proposed in the literature to single out classes of nearly-acyclic (hyper)graphs, such as the tree decomposition method, which is the most powerful decomposition method on graphs, and the (generalized) hypertree decomposition method, which is its natural counterpart on arbitrary hypergraphs.

The paper analyzes this framework, by focusing in particular on “minimal” tree projections, that is, on tree projections without useless redundancies. First, it is shown that minimal tree projections enjoy a number of properties that are usually required for normal form decompositions in various structural decomposition methods. In particular, they enjoy the same kind of connection properties as (minimal) tree decompositions of graphs, with the result being tight in the light of the negative answer that is provided to the open question about whether they enjoy a slightly stronger notion of connection property, defined to speed-up the computation of hypertree decompositions. Second, it is shown that tree projections admit a natural game-theoretic characterization in terms of the Captain and Robber game. In this game, as for the Robber and Cops game characterizing tree decompositions, the existence of winning strategies implies the existence of monotone ones. As a special case, the Captain and Robber game can be used to characterize the generalized hypertree decomposition method, where such a game-theoretic characterization was missing and asked for. Besides their theoretical interest, these results have immediate algorithmic applications both for the general setting and for structural decomposition methods that can be recast in terms of tree projections.

1 Introduction

1.1 Structural Decomposition Methods and Open Questions

Many NP-hard problems in different application areas, ranging, e.g., from AI [13] to Database Theory [6], are known to be efficiently solvable when restricted to instances whose underlying structures can be modeled via acyclic graphs or hypergraphs. Indeed, on these kinds of instances, solutions can usually be computed via dynamic programming, by incrementally processing the acyclic (hyper)graph, according to some of its topological orderings. However, structures arising from real applications are hardly precisely acyclic. Yet, they are often not very intricate and, in fact, tend to exhibit some limited degree of cyclicity, which suffices to retain most of the nice properties of acyclic ones. Therefore,
several efforts have been spent to investigate invariants that are best suited to identify nearly-acyclic graph/hypergraphs, leading to the definition of a number of so-called structural decomposition methods, such as the (generalized) hypertree [14], fractional hypertree [23], spread-cut [8], and component hypertree [16] decompositions. These methods aim at transforming a given cyclic hypergraph into an acyclic one, by organizing its edges (or its nodes) into a polynomial number of clusters, and by suitably arranging these clusters as a tree, called decomposition tree. The original problem instance can then be evaluated over such a tree of subproblems, with a cost that is exponential in the cardinality of the largest cluster, also called width of the decomposition, and polynomial if this width is bounded by some constant.

Despite their different technical definitions, there is a simple mathematical framework that encompasses all purely structural decomposition methods, which is the framework of the tree projections [18]. Roughly, given a pair of hypergraphs \( (\mathcal{H}_1, \mathcal{H}_2) \), a tree projection of \( \mathcal{H}_1 \) w.r.t. \( \mathcal{H}_2 \) is an acyclic hypergraph \( \mathcal{H}_a \) such that each hyperedge of \( \mathcal{H}_1 \) is contained in some hyperedge of \( \mathcal{H}_a \), that is in its turn contained in a hyperedge of \( \mathcal{H}_2 \), which is called the resource hypergraph—see Figure 1 for an illustration.

Therefore, in the tree projection framework, the resource hypergraph \( \mathcal{H}_2 \) is arbitrary. Whenever it is instead computed with some specific technique from the hypergraph \( \mathcal{H}_1 \), we obtain as special cases the so-called purely structural decomposition methods. Consider, for instance, the tree decomposition method [9, 12], based on the notion of treewidth [26], which is the most general decomposition method over classes of graphs (see, e.g., [13, 22]). Let \( k \) be a fixed natural number, and consider any (hyper)graph \( \mathcal{H}_1 \) over a set \( V \) of nodes. Let \( \mathcal{H}_1^{tk} \) be the hypergraph associated whose hyperedges are all possible sets of at most \( k + 1 \) variables. Then, a hypergraf \( \mathcal{H}_1 \) has treewidth bounded by \( k \) if, and only if, there is a tree projection of \( \mathcal{H}_1 \) w.r.t. \( \mathcal{H}_1^{tk} \) (see, e.g., [19, 20]).

In fact, our current understating of structural decompositions for binary (graph) instances is fairly complete. The situation pertaining decompositions methods for arbitrary (hypergraphs) instances is much more muddled instead. In particular, the following two questions have been posed in the literature for the general tree projection framework as well as for structural decomposition methods specifically

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\(^{1}\)For the sake of completeness, observe that the only known structural technique that does not fit the general framework of tree projections is the one based on the submodular width [25], which is not purely structural. Indeed, this method, which is specifically tailored to solve constraint satisfaction problem (or conjunctive query) instances, identify a number of decompositions on the basis of both the given constraint hypergraph and the associated constraint relations.
tailored to deal with classes of queries without a fixed arity bound. Such questions were in particular open for the generalized hypertree decomposition method, which on classes of unbounded-arity queries is a natural counterpart of the tree decomposition method.

(Q2) Is there a natural notion of normal-form for tree projections? Whenever some tree projection of a pair \((\mathcal{H}_1, \mathcal{H}_2)\) exists, in general there are also many tree projections with useless redundancies. Having a suitable notion of minimality may allow us to identify the most desirable tree projections. In fact, for several structural decomposition methods, normal forms have been defined to restrict the search space of decomposition trees, without loosing any useful decomposition.

Such a nice feature is however missing for the general case of tree projections. As a consequence, consider for instance the basic problem of deciding whether a tree projection of a hypergraph \(\mathcal{H}_1\) with respect to a hypergraph \(\mathcal{H}_2\) exists or not. Because every subset of any hyperedge of \(\mathcal{H}_2\) may belong to the tree projection \(\mathcal{H}_a\) we are looking for, this latter hypergraph might in principle consists of an exponential number of hyperedges (w.r.t. to the size of \(\mathcal{H}_1\) and \(\mathcal{H}_2\)). Therefore, even proving that the existence problem is feasible in NP is not easy, without a notion of minimality that allows us to get rid of redundant hyperedges.

Furthermore, in the case of tree decompositions, it is known that we can focus, w.l.o.g., on connected ones \([11]\), that is, basically, on tree decompositions such that, for each set of connected vertices, the sub-hypergraph induced by the nodes covered in such vertices is connected in its turn. Again, connected decompositions provide us with a “normal form” for decomposition trees, which can be exploited to restrict the search space of the possible decompositions and, thus, to speed-up their computation \([11]\). However, no systematic study about connection properties of tree projections (and of decomposition methods other than tree decomposition) appeared in the literature. Algorithms have been implemented limiting the search space to a kind of connected (generalized) hypertree decompositions \([29]\), but it was left open whether the resulting method is a heuristic one or it does give an exact solution.

(Q2) Is there a natural game-theoretic characterization for tree projections? Tree decompositions have a nice game-theoretic characterization in terms of the Robber and Cops game \([28]\): A hypergraph \(\mathcal{H}\) has treewidth bounded by \(k\) if, and only if, \(k + 1\) Cops can capture a Robber that can run at great speed along the hyperedges of \(\mathcal{H}\), while being not permitted to run trough a node that is controlled by a Cop. In particular, the Cops can move over the nodes, and while they move, the Robber is fast and can run trough those nodes that are left or not yet occupied before the move is completed. An important property of this game is that there is no restriction on the strategy used by the Cops to capture the Robber. In particular, the Cops are not constrained to play monotone strategies, that is, to shrink the Robber’s escape space in a monotonically decreasing way. More precisely, playing non-monotone strategies gives no more power to the Cops \([28]\). In many results about treewidth (e.g., \([5]\)), this property turns out to be very useful, because good strategies for the Robber may be easily characterized as those strategies that allow the Robber to run forever.

Hypertree decomposition is an efficiently recognizable structural method \([15]\), which provides a 3-approximation for generalized hypertree decompositions \([4]\). This method is also known to have a nice game-theoretic characterization, in terms of the (monotone) Robber and Marshals game \([15]\), which can be viewed as a natural generalization of the Robber and Cops games. The game is the same as the one characterizing acyclicity, but with \(k\) Marshals acting simultaneously to capture the Robber: A hypergraph \(\mathcal{H}\) has hypertree width bounded by \(k\) if, and only if, \(k\) Marshals, each one with the ability of controlling a hyperedge of \(\mathcal{H}\), can capture a Robber that can run at great speed along the hyperedges, while being not permitted to run trough a node that belongs to a hyperedge controlled by a Marshal. Note that Marshals are more powerful than the Cops of the Robber and Cops game characterizing
treewidth, in that they can move over whole hyperedges. However, Marshals are now required to play monotonically, because non-monotone strategies give some extra-power that does not correspond to valid decompositions \[1\].

Despite the similarities between hypertree and generalized hypertree decompositions as they are apparent from the original definitions by Gottlob et al. \[15\], game theoretic characterizations for generalized hypertree width were still missing. In \[1\], it is raised the question about whether there is a (natural) game theoretic characterization for generalized hypertree width, where non-monotonicity does not represent a source of additional power. Such a characterization is missing for the tree projection setting, too.

### 1.2 Contributions

In this paper, we provide useful properties and characterizations of tree projections (and structural decomposition methods), by answering the two questions illustrated above. In particular,

- We define and investigate minimal tree projections, where the minimal possible subsets of any view are employed. Intuitively, such tree projections typically correspond to more efficient decompositions. We show that some properties required for “normal form” decompositions in various notions of structural decomposition methods (see, e.g., \[14\]) are a consequence of minimality. In particular, minimal tree projections enjoy the same kind of connection property as tree decompositions.

- We define a normal form for (minimal) tree projections. In particular, it turns out that, given any pair of hypergraphs \((H_1, H_2)\), there always exists a tree projection of \(H_1\) w.r.t. \(H_2\) in normal form having polynomial size with respect to the size of the given hypergraphs. An immediate consequence of this result is that checking whether a tree projection exists or not is feasible in NP. In fact, this property has already been exploited in the NP-completeness proof of tree projections \[16\].

- We give a negative answer to the question raised in \[29\] for (generalized) hypertree decomposition. We observe that the notion of connected decomposition proposed there differs from the one defined for tree decompositions and mentioned above. In particular, we show a hypergraph where this restriction leads to worse tree projections, more precisely, where all such connected (generalized) hypertree decompositions have width higher than the hypertree width of the considered hypergraph. Hence, the algorithm proposed in \[29\] for connected hypertree decompositions is not complete, as far as the computation of unrestricted decompositions is considered.

- We define the Captain and Robber game to be played on pairs of hypergraphs, and we show that in this game the Captain has a winning strategy if, and only if, she has a monotone winning one. Then, we show that tree projections and thus, e.g., generalized hypertree decompositions, may be characterized in terms of the Captain and Robber game. Hence, these notions have now a natural game characterization where monotone and non-monotone strategies have the same power.

**Organization.** The rest of the paper is organized as follows. Section 2 illustrates some basic notions and concepts. The setting of minimal tree projections is discussed in Section 3. The game-theoretic characterization is illustrated in Section 4. A few final remarks and some further results are discussed in Section 5 by exploiting the properties of minimal tree projections and the game-theoretic characterization.
2 Preliminaries

Hypergraphs and Acyclicity. A hypergraph $H$ is a pair $(V, H)$, where $V$ is a finite set of nodes and $H$ is a set of hyperedges such that, for each $h \in H$, $h \subseteq V$. If $|h| = 2$ for each (hyper)edge $h \in H$, then $H$ is a graph. For the sake of simplicity, we always denote $V$ and $H$ by $\text{nodes}(H)$ and $\text{edges}(H)$, respectively.

A hypergraph $H$ is acyclic (more precisely, $\alpha$-acyclic\cite{10}) if, and only if, it has a join tree\cite{6}. A join tree $J^H$ for a hypergraph $H$ is a tree whose vertices are the hyperedges of $H$ such that, whenever a node $X \in V$ occurs in two hyperedges $h_1$ and $h_2$ of $H$, then $h_1$ and $h_2$ are connected in $J^H$, and $X$ occurs in each vertex on the unique path linking $h_1$ and $h_2$ (see Figure 1 for an illustration). In words, the set of vertices in which $X$ occurs induces a (connected) subtree of $J^H$. We will refer to this condition as the connectedness condition of join trees.

Tree Decompositions. A tree decomposition\cite{26} of a graph $G$ is a pair $(T, \chi)$, where $T = (N, E)$ is a tree, and $\chi$ is a labeling function assigning to each vertex $v \in N$ a set of vertices $\chi(v) \subseteq \text{nodes}(G)$, such that the following conditions are satisfied: (1) for each node $Y \in \text{nodes}(G)$, there exists $p \in N$ such that $Y \in \chi(p)$; (2) for each edge $\{X, Y\} \in \text{edges}(G)$, there exists $p \in N$ such that $\{X, Y\} \subseteq \chi(p)$; and (3) for each node $Y \in \text{nodes}(G)$, the set $\{p \in N \mid Y \in \chi(p)\}$ induces a (connected) subtree of $T$. The width of $(T, \chi)$ is the number $\max_{p \in N}(|\chi(p)| - 1)$.

The Gaifman graph of a hypergraph $H$ is defined over the set $\text{nodes}(H)$ of the nodes of $H$, and contains an edge $\{X, Y\}$ if, and only if, $\{X, Y\} \subseteq h$ holds, for some hyperedge $h \in \text{edges}(H)$. The treewidth of $H$ is the minimum width over all the tree decompositions of its Gaifman graph. Deciding whether a given hypergraph has treewidth bounded by a fixed natural number $k$ is known to be feasible in linear time\cite{7}.

(Generalized) Hypertree Decompositions. A hypertree for a hypergraph $H$ is a triple $(T, \chi, \lambda)$, where $T = (N, E)$ is a rooted tree, and $\chi$ and $\lambda$ are labeling functions which associate each vertex $p \in N$ with two sets $\chi(p) \subseteq \text{nodes}(H)$ and $\lambda(p) \subseteq \text{edges}(H)$. If $T' = (N', E')$ is a subtree of $T$, we define $\chi(T') = \bigcup_{v \in N'} \chi(v)$. In the following, for any rooted tree $T$, we denote the set of vertices $N$ of $T$ by $\text{vertices}(T)$, and the root of $T$ by $\text{root}(T)$. Moreover, for any $p \in N$, $T_p$ denotes the subtree of $T$ rooted at $p$.

A generalized hypertree decomposition\cite{15} of a hypergraph $H$ is a hypertree $HD = (T, \chi, \lambda)$ for $H$ such that: (1) for each hyperedge $h \in \text{edges}(H)$, there exists $p \in \text{vertices}(T)$ such that $h \subseteq \chi(p)$; (2) for each node $Y \in \text{nodes}(H)$, the set $\{p \in \text{vertices}(T) \mid Y \in \chi(p)\}$ induces a (connected) subtree of $T$; and (3) for each $p \in \text{vertices}(T)$, $\lambda(p) \subseteq \text{nodes}(\lambda(p))$. The width of a generalized hypertree decomposition $(T, \chi, \lambda)$ is $\max_{p \in \text{vertices}(T)} |\lambda(p)|$. The generalized hypertree width $\text{ghw}(H)$ of $H$ is the minimum width over all its generalized hypertree decompositions.

A hypertree decomposition\cite{14} of $H$ is a generalized hypertree decomposition $HD = (T, \chi, \lambda)$ where: (4) for each $p \in \text{vertices}(T)$, $\lambda(p) \cap \chi(T_p) \subseteq \chi(p)$. Note that the inclusion in the above condition is actually an equality, because Condition (3) implies the reverse inclusion. The hypertree width $\text{hw}(H)$ of $H$ is the minimum width over all its hypertree decompositions. Note that, for any hypergraph $H$, it is the case that $\text{ghw}(H) \leq \text{hw}(H) \leq 3 \times \text{ghw}(H) + 1$\cite{4}. Moreover, for any fixed natural number $k > 0$, deciding whether $\text{hw}(H) \leq k$ is feasible in polynomial time (and, actually, it is highly-parallelizable)\cite{14}, while deciding whether $\text{ghw}(H) \leq k$ is NP-complete\cite{16}.

Tree Projections. For two hypergraphs $H_1$ and $H_2$, we write $H_1 \preceq H_2$ if, and only if, each hyperedge of $H_1$ is contained in at least one hyperedge of $H_2$. Let $H_1 \preceq H_2$; then, a tree projection of $H_1$ with
respect to \( H_2 \) is an acyclic hypergraph \( H_a \) such that \( H_1 \leq H_a \leq H_2 \). Whenever such a hypergraph \( H_a \) exists, we say that the pair of hypergraphs \((H_1, H_2)\) has a tree projection.

Note that the notion of tree projection is more general than the above mentioned (hyper)graph based notions. For instance, consider the generalized hypertree decomposition approach. Given a hypergraph \( H \) and a natural number \( k > 0 \), let \( H^k \) denote the hypergraph over the same set of nodes as \( H \), and whose set of hyperedges is given by all possible unions of \( k \) edges in \( H \), i.e., \( \text{edges}(H^k) = \{ h_1 \cup h_2 \cup \cdots \cup h_k \mid \{h_1, h_2, \ldots, h_k\} \subseteq \text{edges}(H) \} \). Then, it is well known and easy to see that \( H \) has generalized hypertree width at most \( k \) if, and only if, there is a tree projection for \((H, H^k)\).

Similarly, for tree decompositions, let \( H^{tk} \) be the hypergraph over the same set of nodes as \( H \), and whose set of hyperedges is given by all possible clusters \( B \subseteq \text{nodes}(H) \) of nodes such that \( |B| \leq k + 1 \). Then, \( H \) has treewidth at most \( k \) if, and only if, there is a tree projection for \((H, H^{tk})\).

### 3 Minimal Tree Projections

In this section, a partial ordering of tree projections is defined. It is shown that minimal tree projections have nice properties with both theoretical and practical interest.

Let \( H \) and \( H' \) be two hypergraphs. We say that \( H \) is contained in \( H' \), denoted by \( H \subseteq H' \), if for each hyperedge \( h \in \text{edges}(H) \)−\( \text{edges}(H') \), there is a hyperedge \( h' \in \text{edges}(H') \)−\( \text{edges}(H) \) with \( h \subseteq h' \) (and hence \( h \subset h' \)). Moreover, we say that \( H \) is properly contained in \( H' \), denoted by \( H \subset H' \), if \( H \subseteq H' \) and \( H \neq H' \).

Note that \( \text{edges}(H) \subseteq \text{edges}(H') \) entails \( H \subseteq H' \) (and hence \( \text{edges}(H) \subset \text{edges}(H') \) entails \( H \subset H' \)). Moreover, \( H \subseteq H' \) implies \( H \leq H' \), but the converse is not true. For example, if \( \text{edges}(H) = \{h_1, h_2\} \) with \( h_2 \subset h_1 \) and \( \text{edges}(H') = \{h_1\} \), then \( H \subseteq H' \) and \( H' \leq H \) hold, as \( \text{edges}(H') \subset \text{edges}(H) \). Moreover, \( H \leq H' \) holds too, but \( H \) is not contained in \( H' \) as there is no hyperedge \( h' \in \text{edges}(H') \)−\( \text{edges}(H) \) such that \( h_2 \subset h' \).

**Definition 3.1** Let \( H_1 \) and \( H_2 \) be two hypergraphs. Then, a tree projection \( H_a \) for \((H_1, H_2)\) is minimal if there is no tree projection \( H_a' \) of \( H_1 \) wrt. \( H_2 \) with \( H_a' \subset H_a \).

### 3.1 Basic Facts

We first point out a number of basic important properties of tree projections of a given pair of hypergraphs \((H_1, H_2)\).

**Fact 3.2** The relationship \( \subseteq \) of Definition 3.1 induces a partial ordering over the tree projections of \( H_1 \) wrt. \( H_2 \).

**Proof.** Observe first that the relation ‘\( \subseteq \)’ over hypergraphs is reflexive. We next show that it is anti-symmetric, too. Let \( H_1 \) and \( H_2 \) be two hypergraphs such that \( H_1 \subseteq H_2 \) and \( H_2 \subseteq H_1 \), and assume by contradiction that \( H_1 \neq H_2 \). Thus, \( \text{edges}(H_2) \neq \text{edges}(H_1) \). Moreover, \( \text{edges}(H_2) \not\supseteq \text{edges}(H_1) \) holds, for otherwise it is trivially impossible that \( H_2 \subseteq H_1 \). Then, let \( h_1 \) be the largest hyperedge (with the maximum number of nodes) in \( \text{edges}(H_1) \)−\( \text{edges}(H_2) \). Since \( H_1 \subseteq H_2 \), it is the case that there is a hyperedge \( h_2 \in \text{edges}(H_2) \)−\( \text{edges}(H_1) \) with \( h_1 \subset h_2 \). But we also know that \( H_2 \subseteq H_1 \) holds, and hence there is a hyperedge \( h'_1 \in \text{edges}(H_1) \)−\( \text{edges}(H_2) \) with \( h_2 \subset h'_1 \). Thus, \( h_1 \subset h_2 \subset h'_1 \), which is impossible due to the maximality of \( h_1 \).
Eventually, we show that the relation ‘⊆’ over hyperedges is transitive. Indeed, assume $H_1 \subseteq H_2$ and $H_2 \subseteq H_3$. Let $h_1$ be a hyperedge in $\text{edges}(H_1) \setminus \text{edges}(H_3)$. We distinguish two cases. If $h_1 \in \text{edges}(H_2)$, and hence $h_1 \in \text{edges}(H_2) \setminus \text{edges}(H_3)$, then there is a hyperedge $h_3 \in \text{edges}(H_3) \setminus \text{edges}(H_2)$ such that $h_3 \subseteq h_1$. Otherwise, i.e., if $h_1 \notin \text{edges}(H_2)$, and hence $h_1 \in \text{edges}(H_1) \setminus \text{edges}(H_2)$, then there is a hyperedge $h_2 \in \text{edges}(H_2) \setminus \text{edges}(H_1)$ such that $h_1 \subseteq h_2$. Then, we have to consider two subcases. If $h_2 \in \text{edges}(H_3)$, then we have that $h_2$ is actually a hyperedge in $\text{edges}(H_3) \setminus \text{edges}(H_1)$ such that $h_1 \subseteq h_2$. Instead, if $h_2 \notin \text{edges}(H_3)$, and hence $h_2 \in \text{edges}(H_2) \setminus \text{edges}(H_3)$, then there is a hyperedge $h'_3 \in \text{edges}(H_3) \setminus \text{edges}(H_2)$ with $h_2 \subseteq h'_3$. It follows that $h_1 \subseteq h_2 \subseteq h'_3$. Putting it all together, we have shown that in all the possible cases, for each hyperedge $h_1 \in \text{edges}(H_1) \setminus \text{edges}(H_3)$ there is a hyperedge $h' \in \text{edges}(H_3) \setminus \text{edges}(H_1)$ such that $h_1 \subseteq h'$. It follows that $H_1 \subseteq H_3$ holds.

By the above properties, ‘⊆’ is a partial order, and ‘<’ is a strict partial order over hypergraphs. □

Hence, minimal tree projections always exist, as long as a tree projection exists.

**Fact 3.3** The pair $(H_1, H_2)$ has a tree projection if, and only if, it has a minimal tree projection.

A further property (again rather intuitive) is that minimal tree projections are reduced hypergraphs. Recall that a hypergraph $H_a$ is reduced if $\text{edges}(H_a)$ does not contain two hyperedges $h_a$ and $\bar{h}_a$ such that $h_a \subseteq \bar{h}_a$.

**Fact 3.4** Every minimal tree projection is reduced.

**Proof.** Assume for the sake of contradiction that $H_a$ is a minimal tree projection of $H_1$ w.r.t. $H_2$ such that $H_a$ is not reduced. Let $h_a$ and $\bar{h}_a$ be two hyperedges of $H_a$ such that $h_a \subseteq \bar{h}_a$. Consider the tree projection $H'_a \neq H_a$ obtained by removing $h_a$ from $H_a$, and notice that $H'_a \subseteq H_a \subseteq H_2$ and $H_1 \leq H'_a$. Thus, $H'_a$ is a tree projection of $H_1$ w.r.t. $H_2$. However, we have that $\text{edges}(H'_a) \subset \text{edges}(H_a)$, which entails that $H'_a \subseteq H_a$ holds, thereby contradicting the minimality of $H_a$. □

The last basic fact is rather trivial: minimal tree projections do not contain nodes that do not occur in $H_1$.

**Fact 3.5** Let $H_a$ be a minimal tree projection of $H_1$ w.r.t. $H_2$. Then, $\text{nodes}(H_a) = \text{nodes}(H_1)$.

**Proof.** Let $H_a$ be a minimal tree projection of $H_1$ w.r.t. $H_2$. Of course, $\text{nodes}(H_a) \supseteq \text{nodes}(H_1)$ clearly holds. On the other hand, if $\text{nodes}(H_a) \supseteq \text{nodes}(H_1)$, the hypergraphs $H'_a$ obtained by deleting from every hyperedge each node in $\text{nodes}(H_a) \setminus \text{nodes}(H_1)$ is still an acyclic hypergraph, and $H_1 \leq H'_a \leq H_a$ holds. Moreover, it is straightforward to check that $H'_a \subset H_a$, which contradicts the minimality of $H_a$.

□

### 3.2 Component trees

We now generalize to the setting of tree projections some properties of join trees that are required for efficiently computable decompositions in various notions of structural decomposition methods (see, e.g., [14]). To formalize these properties, we need to introduce some additional definitions, which will be intensively used in the following.

Assume that a hypergraph $H$ is given. Let $V, W$, and $\{X, Y\}$ be sets of nodes. Then, $X$ is said $[V]$-adjacent (in $H$) to $Y$ if there exists a hyperedge $h \in \text{edges}(H)$ such that $\{X, Y\} \subseteq (h - V)$. A
[V]-path from X to Y is a sequence X = X₀, . . . , Xₖ = Y of nodes such that Xᵢ is [V]-adjacent to Xᵢ₊₁, for each i ∈ [0...k-1]. We say that X [V]--touches Y if X is [0]-adjacent to Z ∈ nodes(H), and there is a [V]-path from Z to Y; similarly, X [V]-.touches the set W if X [V]-touches some node Y ∈ W. We say that W is [V]-connected if ∀X, Y ∈ W there is a [V]-path from X to Y. A [V]-component (of H) is a maximal [V]-connected non-empty set of nodes W ⊆ (nodes(H) − V).

For any [V]-component C, let edges(C) = {h ∈ edges(H) | h ∩ C = ∅}, and for a set of hyperedges H ⊆ edges(H), let nodes(H) denote the set of nodes occurring in H, that is nodes(H) = ∪h∈H h. For any component C of H, we denote by Fr(C, H) the frontier of C (in H), i.e., the set nodes(edges(C))

Moreover, ∂(C, H) denote the border of C (in H), i.e., the set Fr(C, H) \ C. Note that C₁ ⊆ C₂ entails Fr(C₁, H) ⊆ Fr(C₂, H). We write simply Fr(C) or ∂C, whenever H is clear from the context.

We find often convenient to think at join trees as rooted trees: For each hyperedge h ∈ edges(H), the tree obtained by rooting JT at vertex h is denoted by JT[h] (if it is necessary to point out its root). Moreover, for each hyperedge h' ∈ edges(H) with h' ≠ h, let JT[h]h' denote the subtree of JT[h] rooted at h', and let nodes(JT[h]h') be the set of all nodes of H occurring in the vertices of JT[h]h'.

**Definition 3.6**  Let H₁ and Hₐ be two hypergraphs with the same set of nodes such that H₁ ≤ Hₐ and Hₐ is acyclic. A join tree JT of Hₐ, rooted at some vertex root ∈ edges(Hₐ), is said an H₁-component tree if the following conditions hold for each vertex hᵣ ∈ edges(Hₐ) in JT:

1. **SUBTREES→COMPONENTS.** For each child hₛ of hᵣ in JT, there is exactly one [hₛ]-component of H₁, denoted by Cᵣ(hₛ), such that nodes(JT[hₛ]) = Cᵣ(hₛ) ∪ (hₛ ∩ hᵣ). Moreover, hₛ ∩ Cᵣ(hₛ) ≠ ∅ and hₛ ⊆ Fr(Cᵣ(hₛ), H₁) hold.

2. **COMPONENTS→SUBTREES.** For each [hₛ]-component Cᵣ of H₁ such that Cᵣ ⊆ Cᵣ(hₛ), with Cᵣ(root) being conventionally defined as nodes(H₁), there is exactly one child hₛ of hᵣ in JT such that Cᵣ = Cᵣ(hₛ).

Interestingly, any reduced acyclic hypergraph Hₐ has such an Hₐ-component tree (i.e., H₁ = Hₐ, here), as pointed out in the result below.

**Theorem 3.7**  Let Hₐ be a reduced acyclic hypergraph (e.g., any minimal tree projection). For any hyperedge h ∈ edges(Hₐ), there exists a join tree JT rooted at h that is an Hₐ-component tree.

**Proof.** Let Hₐ be any reduced acyclic hypergraph and let h ∈ edges(Hₐ) be any of its hyperedges, and consider Definition 3.6 with its two parts: SUBTREES→COMPONENTS and COMPONENTS→SUBTREES.

**SUBTREES→COMPONENTS.** We first show that there is a join tree JT for Hₐ such that, for each pair hᵣ, hₛ ∈ edges(Hₐ) where hₛ is a child of hᵣ in JT[h],

1. there is exactly one [hₛ]-component Cᵣ of Hₐ, denoted by Cᵣ(hₛ), such that nodes(JT[hₛ]) = Cᵣ(hₛ) ∪ (hₛ ∩ hᵣ);

2. hₛ ∩ Cᵣ(hₛ) ≠ ∅;

3. hₛ ⊆ Fr(Cᵣ(hₛ), Hₐ).

The choice of the term “frontier” to name the union of a component with its outer border is due to the role that this notion plays in hypergraph games, such as the one described in the subsequent section.
Since $H_a$ is a reduced acyclic hypergraph, the hypertree width of $H_a$ is 1. In particular, from the results in [14] (in particular, from Theorem 5.4 in [14]) it follows that, for each hyperedge $h \in \text{edges}(H_a)$, there is a width-1 hypertree decomposition $HD = (T, \chi, \lambda)$ for $H_a$, where $T$ is root at a vertex $\text{root}(T)$ such that $\lambda(\text{root}(T)) = \{h\}$ and, for each vertex $r \in \text{vertices}(T)$ and for each child $s$ of $r$, the following conditions hold: (1) there is (exactly) one $[\chi(r)]$-component $C_r$ of $H_a$ such that $\chi(T_s) = C_r \cup (\chi(s) \cap \chi(r));$ (2) $\chi(s) \cap C_r \neq \emptyset$, where $C_r$ is the $[\chi(r)]$-component of $H_a$ satisfying Condition (1); and (3) $h_s \cap \text{Fr}(C_r, H_a) \neq \emptyset$ holds, where $\{h_s\} = \lambda(s)$ and $C_r$ is the $[\chi(r)]$-component of $H_a$ satisfying Condition (1).

Let us now denote by $h_p$ the unique (as the width is 1) hyperedge contained in $\lambda(p)$, for each vertex $p$ of $T$. Recall that $h = h_{\text{root}(T)}$ is the hyperedge associated with the root of $T$. Let $JT[h]$ be the tree rooted at $h$ obtained from $T$ by replacing each vertex $p$ with the corresponding hyperedge $h_p$. Then, for each vertex $r \in \text{vertices}(T)$ and for each child $s$ of $r$, the three conditions above that hold on $HD$ can be rewritten as follows: (1) there is (exactly) one $[\chi(r)]$-component $C_r$ of $H_a$ such that $\text{nodes}(JT[h], h_s) = C_r \cup (h_s \cap C_r);$ (2) $h_s \cap C_r \neq \emptyset$, where $C_r$ is the $[\chi(r)]$-component of $H_a$ satisfying Condition (1); and (3) $h_s \subseteq \text{Fr}(C_r, H_a)$, where $C_r$ is the $[\chi(r)]$-component of $H_a$ satisfying Condition (1).

It remains to show that $JT$ is actually a join tree for $H_a$. To this end, we claim that the following two properties hold on $HD$.

**Property P1:** $\forall p \in \text{vertices}(T), \chi(p) = \text{nodes}(\lambda(p))$.

**Proof.** Recall that for each vertex $r \in \text{vertices}(T)$ and for each child $s$ of $r$, the following conditions hold on the hypertree decomposition $HD = (T, \chi, \lambda)$ for $H_a$: (1) there is (exactly) one $[\chi(r)]$-component $C_r$ of $H_a$ such that $\chi(T_s) = C_r \cup (\chi(s) \cap \chi(r));$ (2) $\chi(s) \cap C_r \neq \emptyset$, where $C_r$ is the $[\chi(r)]$-component of $H_a$ satisfying Condition (1); and (3) $h_s \cap \text{Fr}(C_r, H_a) \neq \emptyset$ holds, where $\{h_s\} = \lambda(s)$ and $C_r$ is the $[\chi(r)]$-component of $H_a$ satisfying Condition (1). In fact, $\chi(s) \not\subseteq \chi(r)$ holds, as $\chi(r) \cap C_r = \emptyset$ while $\chi(s) \cap C_r \neq \emptyset$. Now, from Condition (4) in the definition of hypertree decompositions it follows that, for each vertex $p \in \text{vertices}(T)$, $\chi(p) = \text{nodes}(\lambda(p)) \cap \chi(T_p)$. Thus, for each node $Y \in \text{nodes}(H_a)$, the vertex $\bar{p}$ with $Y \in \chi(\bar{p})$ that is the closest to the root of $T$ is such that $\chi(\bar{p}) = \text{nodes}(\lambda(\bar{p}))$. Indeed, each node $X \in h_p$, where $\lambda(\bar{p}) = \{h_p\},$ must occur in the $\chi$-labeling of some vertex in the subtree rooted at $\bar{p}$ together with $Y$ in order to satisfy Condition (1) in the definition of hypertree decomposition. Therefore, $X \in \chi(T_{\bar{p}})$. Hence, for the vertex $\text{root}(T)$, it is trivially the case that $\chi(\text{root}(T)) = \text{nodes}(\lambda(\text{root}(T)))$. Consider now an arbitrary vertex $r \in \text{vertices}(T)$ and let $s$ be a child of $r$. Thus, $\{h_s\} = \lambda(s)$, for some hyperedge $h_s$. Recall that $\chi(s) \not\subseteq \chi(r)$, and take any node $Y \in h_s$ such that $Y \in \chi(s) \setminus \chi(r)$. Because of Condition (2) in the definition of hypertree decomposition, $Y$ cannot occur in the $\chi$-labeling of any vertex in path connecting $\text{root}(T)$ and $r$ in $T$. Thus, $s$ is the vertex closest to the root where $Y$ occurs. Hence, $\chi(s) = \text{nodes}(\lambda(s))$.

**Property P2:** $\forall p_1, p_2 \in \text{vertices}(T), \lambda(p_1) \neq \lambda(p_2)$.

**Proof.** Assume for the sake of contradiction that there are two vertices $p_1$ and $p_2$ such that $\lambda(p_1) = \lambda(p_2)$. Because of Property P1, $\text{nodes}(\lambda(p_1)) = \text{nodes}(\lambda(p_2)) = \lambda(p_1) = \lambda(p_2)$. Then, by Condition (2) in the definition of hypertree decomposition, each vertex $p$ in the path between $p_1$ and $p_2$ is such that $\text{nodes}(\lambda(p)) = \chi(p) = \chi(p_1) = \chi(p_2)$ (because the hypergraph is reduced). In particular, this property holds for one vertex $r \in \text{vertices}(T)$ and for one child $s$ of $r$. However, $\chi(r) = \chi(s)$ is impossible as we have observed in the proof of Property P1.
Lemma 3.8

Now, we show that hyperedges of $\mathcal{H}_a$ one-to-one correspond to vertices of $JT$, and that the connectedness condition holds on $JT$.

For the first property, note that each vertex $p$ of $T$ corresponds to the hyperedge $h_p$, by construction. Moreover, by Property $P_2$, each vertex of $JT$ is mapped to a distinct hyperedge. Thus, it remains to show that for each hyperedge $h \in \text{edges}(\mathcal{H}_a)$, there is a vertex $p$ of $T$ such that $h = h_p$. Indeed, note that by Condition (1) of hypertree decompositions, for each hyperedge $h \in \text{edges}(\mathcal{H}_a)$, there is a vertex $p$ in $T$ such that $h \subseteq \chi(p)$. By Property $P_1$ above, this entails that there is a hyperedge $h_p \in \text{edges}(\mathcal{H}_a)$ such that $h_p \subseteq \chi(p)$ and $h \subseteq h_p$. However, since $\mathcal{H}_a$ is reduced, $h = h_p$ holds.

We eventually observe that the connectedness condition holds on $JT$. Indeed, if a node $Y \in \text{nodes}(\mathcal{H}_a)$ occurs in a vertex $h_p$ of $JT$, i.e., $Y \subseteq h_p$, we have that $Y \in \chi(p)$ holds by Property $P_1$. By Condition (2) of hypertree decompositions, the set $\{p \in \text{vertices}(T) \mid Y \subseteq \chi(p)\}$ induces a (connected) subtree of $T$. It follows that the set $\{h_p \in \text{edges}(\mathcal{H}_a) \mid Y \subseteq h_p\}$ induces a connected subtree of $JT$.

Components$\rightarrow$Subtrees. Let us now complete the proof by showing that the join tree $JT$ also satisfies the part Components$\rightarrow$Subtrees in Definition 3.6. Recall that $C_r(h)$ is defined as $\text{nodes}(\mathcal{H}_a)$ for the root $h$, and that $C_r(h_r)$ is the unique $[h_r]$-component with $\text{nodes}(JT[h_r]) = C_r(h_r) \cup (h_r \cap h_r)$, where $h_r$ is a child of $h$ in $JT[h]$. In fact, to conclude the proof, we next show that, for each vertex $h_r$ in $JT[h]$, and for each $[h_r]$-component $C_r$ of $\mathcal{H}_a$ such that $C_r \subseteq C_r(h_r)$, there is exactly one child $h_s$ of $h_r$ such that $C_r = C_r(h_s)$.

Let $C_r$ be an $[h_r]$-component such that $C_r \subseteq C_r(h_r)$. Assume, first, that $h_r$ is the child of a vertex $h_p \in \text{edges}(\mathcal{H}_a)$ of $JT[h]$, i.e., $h_r$ is distinct from the root $h$ of $JT[h]$. Then, because of the part Subtrees$\rightarrow$Components above, we have that $\text{nodes}(JT[h_p]) \supseteq C_r(h_r)$. In particular, this entails that $\text{nodes}(JT[h_p]) \supseteq C_r(h_r)$. Thus, $\text{nodes}(JT[h_p]) \supseteq C_r$. Then, since $h_r \cap C_r = \emptyset$, we have that for each node $X \in C_r$, $X$ occurs in some vertex of a subtree of $JT[h_p]$ rooted at a child $h_{s_r}(X)$ of $h_r$, with $X \subseteq h_{s_r}(X)$. In particular, because of the connectedness condition of join trees, there is precisely one such subtree, since $X \not\subseteq h_r$. Now, we can apply the part Subtrees$\rightarrow$Components above on $h_{s_r}(X)$ to observe that there is exactly one $[h_r]$-component $C_r(h_{s_r})$ of $\mathcal{H}_a$ such that $\text{nodes}(JT[h_{s_r}(X)]) = C_r(h_{s_r}) \cup (h_r \cap h_{s_r}(X))$. However, since $X \not\subseteq h_r$, $X \subseteq C_r(h_{s_r})$ holds. Hence, $C_r = C_r(h_{s_r})$.

Finally, consider now the case where $h_r$ is the root of $JT[h]$, i.e., $h_r = h$. Then, let $C_r$ be an $[h_r]$-component and let $X \in C_r$. Let $h_X$ be the hyperedge that is the closest to the root of $JT[h]$ and such that $X \subseteq h_X$. Note that because of the connectedness condition, there is precisely one such hyperedge $h_X$. By using the same line of reasoning as above, it follows that the child $h_{s_r}(X)$ of $h$ such that $X \subseteq h_{s_r}(X)$ is the only one satisfying the condition in the statement. □

3.3 Preservation of Components

In the light of Theorem 3.7, the connectivity of an arbitrary tree projection $\mathcal{H}_a$ for $\mathcal{H}_1$ (with respect to some hypergraph $\mathcal{H}_2$) is characterized in terms of its components. We next show that it can be also characterized in terms of the components of the original hypergraph $\mathcal{H}_1$. This is formalized in the following two lemmas.

Lemma 3.8 Let $\mathcal{H}_1$ and $\mathcal{H}_a$ be two hypergraphs with the same set of nodes such that $\mathcal{H}_1 \subseteq \mathcal{H}_a$. Then, for each $h \in \text{edges}(\mathcal{H}_a)$ and $[h]$-component $C_1$ in $\mathcal{H}_1$, there is an $[h]$-component $C_a$ of $\mathcal{H}_a$ such that $C_1 \subseteq C_a$. 

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Proof. Since $H_1 \leq H_a$, for each hyperedge $h' \in \text{edges}(H_1)$, there is a hyperedge $h_a \in \text{edges}(H_a)$ such that $h_1 \subseteq h_a$. Then, for any set of nodes $h$ and any $[h]$-component $C_1$ of $H_1$, it follows that $C_1$ is also $[h]$-connected in $H_a$. Hence, there is an $[h]$-component $C_a$ of $H_a$ such that $C_1 \subseteq C_a$. \qed

Lemma 3.9 Let $H_1$ and $H_a$ be two hypergraphs with the same set of nodes such that $H_1 \leq H_a$. Then, for each $h \in \text{edges}(H_a)$ and $[h]$-component $C_a$ in $H_a$, there are $C_{1a}, \ldots, C_{na}$ $[h]$-components of $H_1$ such that $C_a = \bigcup_{i=1}^{n} C_{1a}$.\ 

Proof. After Lemma 3.8 the result follows from the fact that $H_1$ and $H_a$ are defined over the same set of nodes. Indeed, let $X$ be a node in $C_a$. Then, since $X \notin h$, $X$ belongs to an $[h]$-component $C(X)$ of $H_1$, and because of Lemma 3.8 $C(X) \subseteq C_a$ holds. Thus, $C_a = \bigcup_{X \in C_a} C(X)$. \qed

At a first sight, however, since each hyperedge in $H_1$ is contained in a hyperedge of $H_a$, one may naturally be inclined at thinking that such a “bigger” hypergraph $H_a$ is characterized by a higher connectivity, because some nodes that are not (directly) connected by any edge in $H_1$ may be included together in some edge of $H_a$. Indeed, in general, for any given set of nodes $h$, evaluating $[h]$-components of $H_1$ gives proper subsets of the analogous components evaluated in $H_a$. Next, we show that this is not the case if minimal tree projections are considered.

Theorem 3.10 Let $H_a$ be a minimal tree projection of $H_1$ w.r.t. $H_2$. Then, for each hyperedge $h \in \text{edges}(H_a)$, $C$ is an $[h]$-component of $H_a$ $\iff$ $C$ is an $[h]$-component of $H_1$.

Proof. Let $H_a$ be a minimal tree projection of $H_1$ with respect to $H_2$. Let $h$ be in $\text{edges}(H_a)$, and assume, by contradiction, that: $C$ is an $[h]$-component of $H_a$ $\iff$ $C$ is an $[h]$-component of $H_1$. From Lemma 3.8 and Lemma 3.9 it follows that there is an $[h]$-component $C_a$ in $H_a$, and $n > 1$ $[h]$-components $C_{1a}, \ldots, C_{na}$ of $H_1$ such that $C_a = \bigcup_{i=1}^{n} C_{1a}$. See Figure 2 for an illustration.
Let $H$ be the set of all hyperedges of $\mathcal{H}_a$ that intersect $C_a$, i.e., $H = \{ h_a \mid h_a \in \text{edges}(\mathcal{H}_a) \land h_a \cap C_a \neq \emptyset \}$, and consider the hypergraph $\mathcal{H}_a'$ defined over the same set of nodes of $\mathcal{H}_a$ and such that:

$$\text{edges}(\mathcal{H}_a') = (\text{edges}(\mathcal{H}_a) - H) \cup \{ h_a \cap (C_i^1 \cup h) \mid h_a \in H, i \in \{1, \ldots, n\} \}.$$ 

Note that, since $C_a = \bigcup_{i=1}^n C_i^1$ with $n > 1$, there is at least a hyperedge $h_a \in \text{edges}(\mathcal{H}_a)$ such that $h_a \cap (C_i^1 \cup h) \subset h_a$, for some [h]-component $C_i^1$. Thus, $\mathcal{H}_a' \neq \mathcal{H}_a$. Let in fact $h_1$ be any hyperedge in $\text{edges}(\mathcal{H}_a') \setminus \text{edges}(\mathcal{H}_a)$. Then, $h_1 \in \{ h_a \cap (C_i^1 \cup h) \mid h_a \in H, i \in \{1, \ldots, n\} \}$. That is, $h_1 = h_a \cap (C_i^1 \cup h)$ for some hyperedge $h_a \in H$ and [h]-component $C_i^1$ of $\mathcal{H}_1$. In particular, note that the case where $h_1 = h_a = h_a \cap (C_i^1 \cup h)$ is impossible, for otherwise we would have $h_1 \in \text{edges}(\mathcal{H}_a)$. Thus, $h_1 = h_a \cap (C_i^1 \cup h) \subset h_a$, which in turn entails that $h_a \cap (C_i^1 \cup h) \subset h_a$, for each [h]-component $C_i^1$. This property suffices to show that $h_a \notin \text{edges}(\mathcal{H}_a')$. Indeed, assume by contradiction that $h_a \in \text{edges}(\mathcal{H}_a')$. As $h_a \in H$, there is a hyperedge $h_a' \in H$ such that $h_a = h_a' \cap (C_i^1 \cup h)$ for some [h]-component $C_i^1$, and therefore such that $h_a \subseteq h_a'$. However, since $h_a \cap (C_i^1 \cup h) \subset h_a$, we conclude that $h_a \neq h_a'$, and hence, $h_a \subset h_a'$. This is impossible since $\mathcal{H}_a$ is a minimal tree projection, and thus a reduced hypergraph by Fact 3.4. It follows that $\mathcal{H}_a' \subset \mathcal{H}_a$, because for (the generic) hyperedge $h_1 \in \{ \text{edges}(\mathcal{H}_a') \setminus \text{edges}(\mathcal{H}_a) \}$ there exists $h_a \in \{ \text{edges}(\mathcal{H}_a) \setminus \text{edges}(\mathcal{H}_a') \}$ such that $h_1 \subset h_a$.

We now claim that the following three properties hold on $\mathcal{H}_a'$.

**Property P1:** $\mathcal{H}_a' \leq \mathcal{H}_2$.

**Proof.** We have to show that for each hyperedge $h_a' \in \text{edges}(\mathcal{H}_a')$, there is a hyperedge $h_2 \in \text{edges}(\mathcal{H}_2)$ such that $h_a' \subseteq h_2$. To this end, observe that for each hyperedge $h_a' \in \text{edges}(\mathcal{H}_a')$, there is by definition of $\text{edges}(\mathcal{H}_a')$ a hyperedge $h_a \in \text{edges}(\mathcal{H}_a)$ such that $h_a' \subseteq h_a$. Then, since $\mathcal{H}_a$ is a tree projection of $\mathcal{H}_1$ w.r.t. $\mathcal{H}_2$, there is in turn a hyperedge $h_2 \in \text{edges}(\mathcal{H}_2)$ such that $h_a \subseteq h_2$. That is, $h_a' \subseteq h_2$, for some $h_2 \in \text{edges}(\mathcal{H}_2)$. $
$

**Property P2:** $\mathcal{H}_1 \leq \mathcal{H}_a'$.

**Proof.** We have to show that for each hyperedge $h_1 \in \text{edges}(\mathcal{H}_1)$, there is a hyperedge $h_a' \in \text{edges}(\mathcal{H}_a')$ such that $h_1 \subseteq h_a'$. Let $h_1$ be a hyperedge of $\mathcal{H}_1$. Since $\mathcal{H}_a$ is a tree projection of $\mathcal{H}_1$, we have that there is a hyperedge $h_a \in \text{edges}(\mathcal{H}_a)$ such that $h_1 \subseteq h_a$. In the case where $h_1 \cap C_a = \emptyset$, we distinguish two subcases. Either $h_1 \subseteq h$, or $h_1 \setminus h \neq \emptyset$. In the former scenario, we have just to observe that $h$ occurs in $\text{edges}(\mathcal{H}_a')$, as $h \cap C_a = \emptyset$, and hence $h = h_a$. In the latter scenario, $h_a \cap C_a$ must be empty, as $h_a$ is [h]-connected in $\mathcal{H}_a$ and $h_1 \subseteq h_a$. Again, we have that $h_a$ occurs in $\text{edges}(\mathcal{H}_a')$. Consider now the case where $h_1 \cap C_a \neq \emptyset$, and let $X \in h_1 \cap C_a$. Because of Lemma 3.9 $X$ must belong to an [h]-component $C_i^1$ in $\mathcal{H}_1$. Then, $\text{edges}(\mathcal{H}_a')$ contains, by definition, the hyperedge $h_a' = h_a \cap (C_i^1 \cup h)$. In fact, since $h_1 \subseteq h_a$, we also have $h_1 \cap (C_i^1 \cup h) \subseteq h_a'$. In order to conclude that $h_1 \subseteq h_a'$, it remains to observe that all the vertices in $h_1 \setminus h$ are contained in $C_i^1$ since $h_1 \setminus h$ is [h]-connected in $\mathcal{H}_1$ and $X \in h_1 \cap C_i^1$. $
$

**Property P3:** $\mathcal{H}_a'$ is acyclic.

**Proof.** The proof of this property is rather technical, and hence we find convenient to illustrate its main ideas here, as they shed some light on the connectivity of minimal tree projections. From Theorem 3.7, we know that $\mathcal{H}_a$ has an $\mathcal{H}_a$-component tree rooted at $h$, say $JT[h]$. For such a join tree, there is a one-to-one correspondence between components of $\mathcal{H}_a$ and subtrees of $JT[h]$. Accordingly, any such a component $C$, denote by $JT[h]_C$ the subtree rooted at the child $h_0$ of $h$ such that $C = C(h_0)$. Then, the line of the proof is to apply a normalization procedure
over the subtree $JT'[h]_{C_a}$ which is in charge of decomposing $C_a$, in order to build the subtrees $JT'[h]_{C_1}, \ldots, JT'[h]_{C_m}$, each one being in charge of decomposing an $[h]$-component in $H_1$. An illustration is reported in Figure 2. The resulting tree $JT'[h]$ can be shown to be a join tree for $H'_a$, thus witnessing that $H'_a$ is acyclic.

Let us now prove formally the result. Recall that $H_a$ is reduced because of Fact 3.4. From Theorem 3.7, we know that $H_a$ has an $H_a$-component tree rooted at $h$, say $JT[h]$. For such a join tree, there is a one-to-one correspondence between components of $H_a$ and subtrees of $JT[h]$. Accordingly, for any such a component $C$, denote by $JT[h]_{C_i}$ the subtree rooted at the child $h_a$ of $h$ such that $C = C_i(h_a)$.

Let $C_a, C_a^1, \ldots, C_a^m$ be the $[h]$-components of $H_a$, where $C_a$ is the component such that $C_a = \bigcup_{i=1}^n C_i^1$, with $n > 1$ and $C_1^1, \ldots, C_m^1$ are $[h]$-components of $H_1$. Based on $JT[h]$, we shall build a tree $JT'[h]$ whose vertices are the hyperedges of $H'_a$. In particular, $JT'[h]$ is a built as follows:

- The root of $JT'[h]$ is the hyperedge $h$.
- Each subtree $JT[h]_{C_a}$ occurs in $JT'[h]$ as a subtree of $h$.
- For each $[h]$-component $C_1^1 \subseteq C_a$ in $H_1$, $JT'[h]$ contains, as a subtree of $h$, the subtree $JT[h]_{C_1^1}$ that is built from $JT[h]_{C_a}$ by replacing each hyperedge $h_a$ with the hyperedge $h_a \cap (C_1^1 \cup h)$.
- No further vertices are in $JT'[h]$.

Next, we show that $JT'[h]$ is a join tree. Actually, $JT'[h]$ may contain two vertices associated to the same hyperedge of $H'_a$ (because of different original hyperedges that may lead to the same intersections). Thus, formally $JT'[h]$ cannot be precisely a join tree, and we shall rather show that it is a hypertree decomposition of width 1 where $\chi(p) = \text{nodes}(\lambda(p))$, for each vertex $p$, which of course entails the acyclicity of the considered hypergraph. However, for the sake of presentation, we keep the notation of join trees, avoiding the use of the $\chi$ and $\lambda$-labelings, and we allow that $JT'[h]$ contains two vertices associated with the same hyperedge of $H'_a$.

(i) For each vertex $h'$ in $JT'[h]$, $h'$ is in $\text{edges}(H'_a)$. Let $h'$ be in $JT'[h]$. In the case where $h' = h$, or $h'$ occurs in a subtree of the form $JT'[h]_{C_a}$, then $h'$ precisely coincides with a hyperedge of $H_a$ such that $h' \not\in H$. Thus, $h'$ also belongs to $\text{edges}(H'_a)$, by definition. If $h'$ occurs in a subtree of the form $JT'[h]_{C_1^1}$, then $h' = h_a \cap (C_1^1 \cup h)$, by constuction of $JT'[h]$, for some hyperedge $h_a$ in $JT[h]_{C_a}$ which, in particular, belongs to $H_a \setminus h \subseteq C_a$. The case $h_a \subseteq h$ (actually, $h_a \subset h$) is impossible, since $h_a$ and $h$ are both hyperedges of $H'_a$, which is minimal and hence reduced by Fact 3.4. Thus, $h_a \cap C_a \neq \emptyset$ and hence $h_a \in H$. Then, the hyperedge $h' = h_a \cap (C_1^1 \cup h)$ is in $\text{edges}(H'_a)$.

(ii) For each hyperedge $h'$ in $\text{edges}(H'_a)$, $h'$ is in $JT'[h]$. Let $h' \neq h$ be a hyperedge of $H'_a$; indeed, for $h' = h$ the property trivially holds. If $h'$ is also a hyperedge of $H_a$, then either $h' \not\in H$, or $h' \in H$ and there is an $[h]$-component $C_a^i$ with $h' = h' \cap (C_a^i \cup h)$, i.e., with $h' \subseteq (C_a^i \cup h)$. If $h' \not\in H$, then $h' \cap C_a = \emptyset$. Then, we have that $h' \cap C_a \neq \emptyset$ for some $[h]$-component $C_a^i \neq C_a$. Hence, due to Theorem 3.7, $h'$ occurs in $JT[h]_{C_a^i}$. The result then follows since $JT[h]_{C_a^i}$ also occurs as a subtree of $JT'[h]$. Consider now the case where $h' \in H$ and there is an $[h]$-component $C_a^i$ with $h' = h' \cap (C_a^i \cup h)$, i.e., with $h' \subseteq (C_a^i \cup h)$. Since $h' \not\subseteq h$, it holds that $h' \cap C_a^i \neq \emptyset$ and hence, due to Lemma 3.9, $h' \cap C_a \neq \emptyset$. Then, $h'$ occurs in $JT[h]_{C_a}$. 

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because of Theorem 3.7 and, by construction, \( h' \) occurs in \( JT'[h]_{C_1} \). Finally, assume that \( h' \) is not a hyperedge of \( H_a \). Thus, \( h' = h_a \cap (C_1^a \cup h) \), for some hyperedge \( h_a \in \text{edges}(H_a) \) and \([h]-\text{component} C_1^a \) with \( h_a \cap C_a \neq \emptyset \) and \( h_a \not\subseteq (C_1^a \cup h) \). Due to Theorem 3.7, \( h_a \) occurs in \( JT[h]_{C_1} \). Then, by construction, \( h' \) occurs in \( JT'[h]_{C_1} \).

(iii) The connectedness condition holds on \( JT'[h] \). Let \( h'_{a_1} \) and \( h'_{a_2} \) be two hyperedges in \( H'_a \) such that \( h'_{a_2} \) occurs in the subtree of \( JT'[h] \) rooted at \( h'_{a_1} \). Since subtrees of the form \( JT'[h]_{C_1} \) are not altered in the transformation, we can focus on the case where \( h'_{a_2} \) occurs in some subtree of the form \( JT'[h]_{C_1^a} \) and where either \( h'_{a_1} = h \) or \( h'_{a_2} \) occurs in the same subtree. In fact, \( h'_{a_2} \) (resp., \( h'_{a_1} \)) belonging to \( JT'[h]_{C_1^a} \) entails that \( h'_{a_2} = h_a \cap (C_1^a \cup h) \) (resp., \( h'_{a_1} = h_a \cap (C_1^a \cup h) \)), for some hyperedge \( h_a \in \text{edges}(H_a) \) (resp., \( h_a \in \text{edges}(H_a) \)). Note that to deal uniformly with the two cases above, if \( h'_{a_1} = h \), then we can just set \( h_a = h_a \). Now, let \( Y \) be a node in \( h'_{a_1} \cap h'_{a_2} \). Then, \( Y \) belongs to \( h'_{a_2} \cap h_{a_1} \). Consider a hyperedge \( h'_a \) in the path between \( h'_{a_1} \) and \( h'_{a_2} \). Again, \( h'_a \) belonging to \( JT'[h]_{C_1^a} \) entails that \( h' = h_a \cap (C_1^a \cup h) \), where \( h_a \) is an edge occurring in the path between \( h_{a_1} \) and \( h_{a_2} \) in \( JT[h] \). Since \( JT[h] \) is a join tree, \( Y \) also occurs in \( h_a \), and hence \( Y \) is in \( h'_a \).

In the light of the above properties, \( H'_a \) is a tree projection of \( H_1 \) w.r.t. \( H_2 \) such that \( H'_a \subset H_a \), thereby contradicting the fact that \( H_a \) is a minimal tree projection.

\[ \square \]

### 3.4 Connected Tree Projections

We next present another interesting property of minimal tree projections: they always admit join trees in a desirable form that we call connected. Such a form is based on the well known notion of connected decomposition defined for the treewidth (see, e.g., [11]). Let \( (T, \chi) \) be a tree decomposition of a graph \( G \). For any pair of adjacent vertices \( p_r \) and \( p_s \) of \( T \), let \( T_r \) and \( T_s \) be the two connected subtrees obtained from \( T \) by removing the edge connecting \( p_r \) and \( p_s \). Then, \( (T, \chi) \) is connected if the subgraphs induced by the nodes covered be the \( \chi \)-labeling in \( T_r \) and \( T_s \), respectively, are connected, for each pair of vertices \( p_r \) and \( p_s \).

Next, we define a natural extension of this notion to the more general framework of tree projections of hypergraph pairs.

**Definition 3.11** A tree projection \( H_a \) of \( H_1 \) w.r.t. \( H_2 \) is connected if it has an \( H_1 \)-connected join tree, i.e., a join tree \( JT \) with the following property: For each pair of adjacent vertices \( h_r, h_s \) of \( JT \), the sub-hypergraph of \( H_1 \) induced by the nodes in \( \text{nodes}(JT[h_r,h_s]) \) is \([0]\)-connected.

Note that the novel notion coincides with the original one whenever we considers the treewidth method, that is, whenever we look for tree projections of pairs of the form \((H, H^{tk})\), for any fixed natural number \( k > 0 \), where \( H^{tk} \) is the hypergraph whose hyperedges are all possible sets of at most \( k + 1 \) nodes in \( H \).

**Example 3.12** The tree projection \( H_a \) of \( H_1 \) w.r.t. \( H_2 \) reported in Figure 1 is not connected, because it has no \( H_1 \)-connected join trees.

For instance, consider the join tree \( JT_a \) depicted in the same figure, and let \( h_r = \{E, F, G, H, I, J, K\} \) and \( h_s = \{A, D, E, F, J, K\} \). Then, the sub-hypergraph of \( H_1 \) induced by \( \text{nodes}(JT_a[h_r,h_s]) \) consists of the hyperedges \( \{D, F, E\}, \{K, J\}, \{A, B, C\}, \{C, D\}, \) and \( \{A, F\} \).
and thus \{K, J\} is clearly disconnected from the others. On the other hand, note that the join tree \(JT'_a\) for the minimal tree projection \(\mathcal{H}'_a\) reported in Figure 3 is \(H_1\)-connected.

We next show that such a connected join tree always exists for any minimal tree projection, as in the special case of the tree decomposition method. In order to establish the result, we shall exploit an algorithm, called make-it-connected, that has been described in [11] and that enjoys the following properties.

**Proposition 3.13 (cf. [11])** Let \(\langle T, \chi \rangle\) be a width-\(k\) tree decomposition of a graph \(G\). Then, Algorithm make-it-connected builds in polynomial time a connected width-\(k'\) tree decomposition \(\langle T', \chi' \rangle\) of \(G\), with \(k' \leq k\), such that: (1) for each vertex \(p'\) of \(T'\), there is a vertex \(p\) of \(T\) such that \(\chi'(p') \subseteq \chi(p)\); and (2) if \(\langle T, \chi \rangle\) is not connected, then there is a vertex \(\bar{p}\) of \(T\) such that \(\chi(\bar{p}) \neq \chi'(p')\), for each vertex \(p'\) of \(T'\).

**Theorem 3.14** If \(\mathcal{H}_a\) is a minimal tree projection of \(H_1\) w.r.t. \(H_2\), then any join tree \(JT\) for \(\mathcal{H}_a\) is \(H_1\)-connected.

**Proof.** Assume that \(\mathcal{H}_a\) is a minimal tree projection of \(H_1\) w.r.t. \(H_2\), hence from Fact 3.5 \(\text{nodes}(\mathcal{H}_1) = \text{nodes}(\mathcal{H}_a)\). Let \(JT\) be a join tree for \(\mathcal{H}_a\), and let \(\langle T, \chi \rangle\) be a labeled tree whose vertices one-to-one correspond with the vertices of \(JT\). In particular, for each hyperedge \(h \in \text{edges}(\mathcal{H}_a)\), \(T\) contains the vertex \(p_h\), which is moreover such that \(\chi(p_h) = h\). From the connectedness property of join trees and the fact that \(H_1 \leq \mathcal{H}_a\), it immediately follows that \(\langle T, \chi \rangle\) is a tree decomposition of (the Gaifman graph of) \(H_1\). Assume now, for the sake of contradiction, that \(JT\) is not \(H_1\)-connected. Then, \(\langle T, \chi \rangle\) is not connected too. Thus, we can apply algorithm make-it-connected on \(\langle T, \chi \rangle\), which produces the connected tree decomposition \(\langle T', \chi' \rangle\), with \(T' = \langle N', E' \rangle\) of (the Gaifman graph of) \(\mathcal{H}_1\).

Let \(\mathcal{H}'_a\) be the acyclic hypergraph such that \(\text{nodes}(\mathcal{H}'_a) = \text{nodes}(\mathcal{H}_a)\) and \(\text{edges}(\mathcal{H}'_a) = \{\chi'(p') \mid p' \in N'\}\), and let \(\mathcal{H}''_a\) be the reduced hypergraph obtained from \(\mathcal{H}'_a\) by removing its hyperedges that are proper subsets of some hyperedge in \(\mathcal{H}'_a\). Therefore, we have \(\text{edges}(\mathcal{H}''_a) \subseteq \text{edges}(\mathcal{H}'_a)\) and \(\mathcal{H}_a \supseteq \mathcal{H}''_a\). Of course, \(\mathcal{H}''_a\) is acyclic too. Moreover, we claim that \(\mathcal{H}''_a \subseteq \mathcal{H}_a\). Indeed, for each hyperedge \(\chi'(p') \in \text{edges}(\mathcal{H}'_a)\), by Proposition 3.13(1), there is a hyperedge \(\chi(p) \in \text{edges}(\mathcal{H}_a)\) such that \(\chi'(p') \subseteq \chi(p)\). Moreover, \(\chi(p)\) cannot occur in \(\text{edges}(\mathcal{H}''_a)\), as \(\mathcal{H}''_a\) is reduced. Hence, \(\chi(p)\) is
in \( \text{edges}(\mathcal{H}_a) \setminus \text{edges}(\mathcal{H}_a') \), and we actually have \( \chi'(p') \subset \chi(p) \). That is, \( \mathcal{H}_a'' \) is an acyclic hypergraph with \( \mathcal{H}_a'' \subseteq \mathcal{H}_a \).

Now, observe that since \( (T', \chi') \) is a tree decomposition of (the Gaifman graph of) \( \mathcal{H}_1 \) and since \( \mathcal{H}_a'' \subseteq \mathcal{H}_a \), we have \( \mathcal{H}_1 \leq \mathcal{H}_a'' \leq \mathcal{H}_a \leq \mathcal{H}_2 \). Thus, \( \mathcal{H}_a'' \) is a tree projection of \( \mathcal{H}_1 \) w.r.t. \( \mathcal{H}_2 \). However, by Proposition 3.13.(2), there is a vertex \( \bar{p} \) of \( T \) such that \( \chi(\bar{p}) \neq \chi'(p') \), for each vertex \( p' \) of \( T' \). Thus, \( \mathcal{H}_a'' \neq \mathcal{H}_a \). Hence, \( \mathcal{H}_a'' \) is a tree projection for \( (\mathcal{H}_1, \mathcal{H}_2) \) such that \( \mathcal{H}_a'' \subset \mathcal{H}_a \), which contradicts the minimality of \( \mathcal{H}_a \). \( \square \)

Eventually, by exploiting Fact 3.3 we get the following corollary.

**Corollary 3.15** \( (\mathcal{H}_1, \mathcal{H}_2) \) has a tree projection if, and only if, \( (\mathcal{H}_1, \mathcal{H}_2) \) has a connected tree projection.

**Remark 3.16** A different notion of connected decomposition has been introduced in [29] for the special case of (generalized) hypertree decompositions, in order to speed-up their computation. According to [29], a (generalized) hypertree decomposition \( \text{HD} = (T, \chi, \lambda) \) is connected if the root \( r \) of \( T \) is such that \( |\lambda(r)| = 1 \), and for each pair of nodes \( p \) and \( s \), with \( s \) child of \( p \) in \( T \), and for each \( h \in \lambda(s) \), \( h \cap \chi(s) \cap \chi(p) \neq \emptyset \). The connected (generalized) hypertree width \( c\text{hw} \) is the minimum width over all the possible connected (generalized) hypertree decompositions. Whether or not \( c\text{hw}(\mathcal{H}) = \text{hw}(\mathcal{H}) \) for every hypergraph \( \mathcal{H} \) was an open question [29].

Next, we give a negative answer to this question by showing that the latter notion of connectedness gives a structural method that is weaker than the unrestricted (generalized) hypertree decomposition, even on graphs. Consider the graph \( G_{\text{hex}} \) in Figure 4. As shown in the same figure, there is a hypertree decomposition \( \text{HD}_{\text{hex}} = (T, \chi, \lambda) \) of this (hyper)graph having width 3, and thus \( \text{hw}(G_{\text{hex}}) \leq 3 \). In \( \text{HD}_{\text{hex}} \), for each vertex \( p \) of \( T \), \( \chi(p) = \text{nodes}(\lambda(p)) \) holds, and thus Figure 4 shows only the \( \lambda \)-labeling of each vertex. Moreover, only the left branch is detailed, showing how to deal with the single cluster of hexagons. The other subtrees are of the same form, and thus are not reported, for the sake of simplicity. Note that \( \{0\} \) and its child \( \{0, 21, 42\} \) violate the required connectedness property. In fact, it turns out that the only way to attack such hexagons is by using, at some vertex \( s \) of the decomposition tree, some nodes that are not directly connected to (the nodes occurring in) the parent vertex of \( s \). Indeed, the reader can check there is neither a hypertree decomposition nor a generalized hypertree decomposition of \( G_{\text{hex}} \) that is connected according to [29] and has width 3. Thus, the following holds.

**Fact 3.17** There is a graph \( G_{\text{hex}} \) such that \( c\text{ghw}(G_{\text{hex}}) > \text{hw}(G_{\text{hex}}) \).

### 3.5 Tree Projections in Normal Form

We next show the main result of this section, where all the above ingredients are exploited together: all minimal tree projections have join trees in a suitable normal form. This normal form is of theoretical interest, since it can be exploited to establish further results on the setting (as its game-theoretic characterization discussed in Section 4). Moreover, it is of practical interest, since it can be used to prune the search space in solution approaches aimed at computing tree projections.

**Definition 3.18** A join tree of a tree projection of \( \mathcal{H}_1 \) w.r.t. \( \mathcal{H}_2 \) is said in **normal form** if it is \( \mathcal{H}_1 \)-connected and it is an \( \mathcal{H}_1 \)-component tree.

**Example 3.19** Consider again the tree projection \( \mathcal{H}_a \) and its join tree \( JT_a \) illustrated in Figure 1. Consider the vertices \( h_r = \{ E, F, G, H, I, J, K \} \) and \( h_s = \{ A, D, E, F, J, K \} \) in
\[JT_a[\{E, F, G, H, I, J, K\} \}, \text{and note that there is exactly one } [h_r]-\text{component } C_r(h_s) = \{A, B, C, D\}\] of \(H_1\) such that \(\text{nodes}(JT[h], h_s) = \{A, B, C, D, E, F, J, K\} \) \(\subset C_r(h_s) \cup (h_s \cap h_r)\). However, \(\text{Fr}(C_r(h_s), H_1) = \{A, B, C, D, E, F\}\) and hence \(h_s \not\subset \text{Fr}(C_r(h_s), H_1)\). Thus, one condition in the (SUBTREES\(\rightarrow\)COMPONENTS)\(-\)part of Theorem 3.20 is violated.

Indeed, the tree projection \(H_a\) is not minimal. This is witnessed by the tree projection \(H'_a\) for \((H_1, H_2)\) that is reported on the left of Figure 5 and that is properly contained in \(H_a\). A join tree \(JT'_a\) for \(H'_a\) is reported on the right of the same figure. The careful reader may check that \(JT'_a\) satisfies all conditions in Theorem 3.20.

**Theorem 3.20 (Normal Form)** Let \(H_a\) be a minimal tree projection of \(H_1\) w.r.t. \(H_2\). For any hyperedge \(h \in \text{edges}(H_a)\), there is a join tree for \(H_a\) in normal form rooted at \(h\).

**Proof.** Let \(h \in \text{edges}(H_a)\) be any hyperedge of the tree projection \(H_a\). Since minimal tree projections are reduced, Theorem 3.7 entails that \(H_a\) has a join tree \(JT\) that is an \(H_a\)-component tree rooted at \(h\). Again from minimality and Theorem 3.10 we get that \([h']\)-components in \(H_a\) and \([h']\)-components in \(H_1\) do coincide, for each \(h' \in \text{edges}(H_a)\). Thus, \(JT\) has the following properties:

**SUBTREES\(\rightarrow\)COMPONENTS.** For each vertex \(h_r\) of \(JT[h]\) and each child \(h_s\) of \(h_r\), there is exactly one \([h_r]\)-component \(C_r(h_s)\) of \(H_1\) such that \(\text{nodes}(JT[h], h_s) = C_r(h_s) \cup (h_s \cap h_r)\). Moreover, \(h_s \cap C_r(h_s) \neq \emptyset\) holds.

**COMPONENTS\(\rightarrow\)SUBTREES.** For each vertex \(h_r\) of \(JT[h]\) and each \([h_r]\)-component \(C_r\) of \(H_1\) such that \(C_r \subset C_r(h_s)\), there is exactly one child \(h_s\) of \(h_r\) such that \(C_r = C_r(h_s)\).

Hence, in order to prove that \(JT\) is an \(H_1\)-component tree, it remains to show that, for each vertex \(h_r\) of \(JT[h]\) and each child \(h_s\) of \(h_r\), \(h_s \subset \text{Fr}(C_r(h_s), H_1)\) holds. Assume, for the sake of contradiction, that there is a vertex \(h_r\) and a child \(h_s\) of \(h_r\) such that \(h_s \not\subset \text{Fr}(C_r(h_s), H_1)\) does not hold. From Theorem 3.7 we know that \(h_s \not\subset \text{Fr}(C_r(h_s), H_a)\). It follows that there exists a non-empty set \(W \subset h_s \setminus C_r(h_s)\) of nodes such that \(X \not\subset \text{Fr}(C_r(h_s), H_1)\) for each \(X \subset W\). Moreover, as \(\text{nodes}(JT[h], h_s) = C_r(h_s) \cup (h_s \cap h_r)\), we have that \(W \subset h_s \setminus h_r\). Consider the hypergraph \(H'_a\) obtained from \(H_a\) by replacing each hyperedge \(h\) occurring in \(JT[h]\) with \(h \setminus W\), and note that \(H'_a \subset H_a\). Of course, the tree \(JT'\) obtained from \(JT\) by replacing any such \(h\) with \(h \setminus W\) is a join...
tree for $H'$. Finally, $H'_a$ is again a tree projection for $(H_1, H_2)$ because every hyperedge of $H_1$ is still covered by some vertex in $JT'$. Indeed, there is no hyperedge $h \in \text{edges}(H_1)$ such that both $h \cap W \neq \emptyset$ and $h \cap C(h_x) \neq \emptyset$, by construction of $W$. This contradicts the fact that $H_a$ is a minimal tree projection for $(H_1, H_2)$.

Finally, from Theorem 3.14, $JT$ is $H_1$-connected.

4 Game-Theoretic Characterization

The Robber and Captain game is played on a pair of hypergraphs $(H_1, H_2)$ by a Robber and a Captain controlling some squads of cops, in charge of the surveillance of a number of strategic targets. The Robber stands on a node and can run at great speed along the edges of $H_1$; however, she is not permitted to run through a node that is controlled by a cop. Each move of the Captain involves one squad of cops, which is encoded as a hyperedge $h \in \text{edges}(H_2)$. The Captain may ask any cops in the squad $h$ to run in action, as long as they occupy nodes that are currently reachable by the Robber, thereby blocking an escape path for the Robber. Thus, “second-lines” cops cannot be activated by the Captain. Note that the Robber is fast and may see cops that are entering in action. Therefore, while cops move, the Robber may run through those positions that are left by cops or not yet occupied. The goal of the Captain is to place a cop on the node occupied by the Robber, while the Robber tries to avoid her capture.

For a comparison, observe that this game is somehow in the middle between the Robber and Marshals game of [15], where the marshals occupy a full hyperedge at each move, and the Robber and Cops game of [28], where each cop stands on a vertex and thus, if there are enough cops, any subset of any edge can be blocked at each move. Instead, the Captain cannot employ “second-lines” cops, but only cops whose positions are under possible Robber attacks.

Example 4.1 Consider the Robber and Captain game played on the pair $(H_1, H_2)$ of hypergraphs depicted in Figure 1 and the sequences of moves illustrated in Figure 5.

Initially, the Robber stands on the node $K$, and each other node is reachable. The Captain selects the squad $\{E, F, G, H, I, J, K\}$ and uses the three cops blocking $E$, $F$, and $G$. The Robber sees the cops and, while they enter in action, is fast enough to run on $A$. Note that, when the Robber is on $A$ and nodes $E$, $F$, and $G$ are blocked by the Captain, the Robber can move over $\{A, B, C, D\}$, while $\{E, F\}$ are also under possible Robber attacks because they are adjacent to her escape space. All other nodes are no longer reachable by the Robber and no longer depicted. Hence, the Captain might ask cops to occupy some of the nodes in $\{A, B, C, D, E, F\}$, provided they are covered by some hyperedge. In fact, the strategy of the Captain is to select the hyperedge/squad $\{A, D, E, F, J, K\}$, and then to use those cops in this squad that block nodes $A$, $D$, $E$, and $F$. During this move of the cops, the potential escape door $\{E, F\}$ for the Robber is still blocked, and hence its available space shrinks. Indeed, during the move of the Captain, the Robber can just move either on $B$ or on $C$. Finally, the Captain uses the squad $\{A, B, C, D, H\}$ and order its cops to move to $A$, $B$, $C$, and $D$, thereby capturing the Robber, as its potential escape door $\{A, D\}$ remains blocked by the cops.

In the rest of the section, we formalize and analyze the game. To this end, we intensively use the notions and the notations given in the previous section, by implicitly applying them to the hypergraph $H_1$, unless stated otherwise.

Definition 4.2 (R&C Game) Let $H_1$ and $H_2$ be two hypergraphs. The Robber and Captain game on $(H_1, H_2)$ (short: R&C($H_1, H_2$) game) is formalized as follows. A position for the Captain is a set $M$
of vertices where the cops stand such that $M \subseteq h_2$, for some hyperedge (squad) $h_2 \in \text{edges}(H_2)$. A **configuration** is a pair $(M, v)$, where $M$ is a position for the Captain, and $v \in \text{nodes}(H_1)$ is the node where the Robber stands. The initial configuration is $(\{\}, v_0)$, where $v_0$ is a node arbitrarily picked by the Robber.

Let $(M_i, v_i)$ be the configuration at step $i$. This is a capture configuration, where the Captain wins, if $v_i \in M_i$. Otherwise, the Captain activates the cops in a novel position $M_{i+1}$ such that: $\forall X \in M_{i+1}$, $X \ [M_i]$-touches $v_i$ (in $H_1$). Then, the Robber selects some available node $v_{i+1}$ (if any) such that there is a $[M_i \cap M_{i+1}]$-path from $v_i$ to $v_{i+1}$ (in $H_1$). If the game continues forever, the Robber wins. \[ \square \]

Note that it does not make sense for the Captain to assume that the Robber is on a particular node, given the ability of the Robber of changing positions before the cops land. Thus, given a configuration $(M_i, v_i)$, we may assume w.l.o.g. that the next Captain’s move is only determined by the $[M_i]$-component (of $H_1$) that contains $v_i$, rather than by $v_i$ itself. And, accordingly, positions can equivalently be written as $(M_i, C_i)$, where $C_i$ is an $[M_i]$-component. In this case, capture configurations have the form $(M, \{\})$, and the initial configuration has the form $(\{\}, \text{nodes}(H_1))$. 

![Figure 5: The Robber and Captain game on $(H_1, H_2)$](image-url)
In the following, assume that a R&C($H_1, H_2$) game is given. Moreover recall that, for any component $C$ (of $H_1$), $Fr(C, H_1)$ is the frontier of $C$, i.e., (by omitting hereafter $H_1$, which is understood) the set $Fr(C) = \text{nodes}(edges(C)) = C \cup \{Z \mid Z \cap X \in C, h \in edges(H_1) \text{ s.t. } \{X, Z\} \subseteq h\}$. Then, observe that the moves of the Captain are confined in the frontier of the current component where the Robber stands.

**Fact 4.3** Let $M_i$ and $M_{i+1}$ be positions for the Captain and let $C_i$ be an $[M_i]$-component. Then, $\forall X \in M_{i+1}$, $X [M_i]$-.touches $C_i$ if, and only if, $M_{i+1} \subseteq Fr(C_i)$.

**Proof.** In fact, a node $X$ is in $Fr(C_i)$ if, and only if, either $X \in C_i$, or $X \in M_i$ and there is a node $Z \in C_i$ with $\{X, Z\} \subseteq h$, for some edge $h$ in $edges(H_1)$. But, this condition precisely coincides with the definition that $X [M_i]$-touches $C_i$. \hfill $\square$

**Definition 4.4 (Strategies)** A strategy $\sigma$ for R&C($H_1, H_2$) is a function that encodes the moves of the Captain, i.e., given a configuration $(M_i, C_i)$, with $C_i \neq \emptyset$, $\sigma$ returns a position $M_{i+1}$ such that $M_{i+1} \subseteq Fr(C_i)$.

A game-tree for $\sigma$ is a rooted tree $T(\sigma)$ defined over configurations as follows. Its root is the configuration $(\emptyset, \text{nodes}(H_1))$. Let $(M_i, C_i)$ be a vertex in $T(\sigma)$ and let $M_{i+1} = \sigma(M_i, C_i)$. Then, $(M_i, C_i)$ has exactly one child $(M_{i+1}, C_{i+1})$, for each $[M_i]$-component $C_{i+1}$ such that $C_i \cup C_{i+1}$ is $[M_i \cap M_{i+1}]$-connected; we call such a $C_{i+1}$ an $[\{M_i, C_i\}, M_{i+1}]$-option for the Robber. If there is no $[(M_i, C_i), M_{i+1}]$-option, then $(M_i, C_i)$ has exactly one child $(M_{i+1}, \{\})$. No further edge or vertex is in $T(\sigma)$.

Then, $\sigma$ is a winning strategy if $T(\sigma)$ is a finite tree. Moreover, define a position $M_{i+1}$ to be a monotone move of the Captain in $(M_i, C_i)$, if for each $[\{M_i, C_i\}, M_{i+1}]$-option $C_{i+1}$, $C_{i+1} \subseteq C_i$. We say that $\sigma$ is a monotone strategy if, for each edge from $(M_i, C_i)$ to $(M_{i+1}, C_{i+1})$, it holds that $M_{i+1}$ is a monotone move in $(M_i, C_i)$.

**Example 4.5** Consider again the exemplification of the Robber and Captain in Figure 5. In particular, the bottom-right part of the figure depicts a game-tree associated with a winning strategy: The Captain initially moves on $\{E, F, G\}$, and there are two connected components available to the Robber, namely $\{A, B, C, D\}$ and $\{H, I, J, K\}$. The left branch of the tree illustrates the strategy when the Robber goes into the component $\{A, B, C, D\}$. Note that this branch precisely corresponds to the moves that are discussed in Example 4.1. The right branch addresses the case where the Robber goes into the component $\{H, I, J, K\}$. In both cases, the Captain will eventually capture the Robber. Observe that this winning strategy is monotone.

### 4.1 Monotone vs Non-monotone Strategies

In this section, we show that there is no incentive for the Captain to play a strategy $\sigma$ that is not monotone, since it is always possible to construct and play a monotone strategy $\sigma'$ that is equivalent to $\sigma$, i.e., such that $\sigma'$ is winning if, and only if, $\sigma$ is winning. This crucial property conceptually relates our game with the *Robber and Cops game* characterizing the treewidth [28], and differentiates it from most of the hypergraph-based games in the literature, in particular, from the *Robber and Marshals game*, whose monotone strategies characterize hypertree decompositions [15], while non-monotone strategies do not correspond to valid decompositions [11].

\footnote{Note the little abuse of notation: $\sigma(M_i, C_i)$ instead of $\sigma((M_i, C_i))$.}
We point out that the proof below does not apply to the traditional Robber and Cops game, because in our setting cops can be placed just on the positions that are reachable by the Robber. As a matter of fact, our techniques are substantially different from those used to show that non-monotonic moves provide no extra-power in the Robber and Cops game. We start by illustrating some properties of the novel game.

In the following, assume that $\sigma$ and $T(\sigma)$ are a strategy and a game tree for it, respectively. Moreover recall that, for any component $C$ (of $\mathcal{H}_1$), $\partial(C)$ denotes the border of $C$ (in $\mathcal{H}_1$), i.e., the set $\text{Fr}(C) \setminus C$.

Then, let the escape-door of the Robber in $v_i = (M_i, C_i)$ when attacked with $M_{i+1}$ be defined as $\text{ED}(v_i, M_{i+1}) = \partial(C_i) \setminus M_{i+1}$. Note that this is equivalent to state that $\text{ED}(v_i, M_{i+1}) = M_{i+1} \cap \text{Fr}(C_i) \setminus M_{i+1}$, because $C_i$ is an $[M_i]$-component. Consider for instance Example 4.1 at the configuration $v_i = (\{E,F,G\}, \{A,B,C,D\})$, when the Robber is attacked by the Captain with the cops $\{A,D,E,F\}$. In this case, the frontier is $\text{Fr}(\{A,B,C,D\}) = \{A,B,C,D,E,F\}$, hence the escape-door is $\{E,F\} \setminus \{A,D,E,F\} = \emptyset$.

In the following lemma, we show that this set precisely characterizes those vertices through which the Robber may escape from the current component $C_i$, when the Captain changes her position from $M_i$ to $M_{i+1}$.

**Lemma 4.6** Let $M_i$ and $M_{i+1}$ be positions for the Captain, let $C_i$ be an $[M_i]$-component, and let $v_i = (M_i, C_i)$. Then, $C_{i+1}$ is a $[v_i, M_{i+1}]$-option if, and only if, $C_{i+1}$ is an $[M_{i+1}]$-component with $C_{i+1} \cap (C_i \cup \text{ED}(v_i, M_{i+1})) \neq \emptyset$.

**Proof.** Recall from Definition 4.3 that $C_{i+1}$ is an $[(M_i, C_i), M_{i+1}]$-option if $C_{i+1}$ is an $[M_{i+1}]$-component such that $C_{i+1} \cup C_i$ is $[M_{i+1} \cap M_i]$-connected.

(*if-part*) Assume that $C_{i+1}$ is an $[M_{i+1}]$-component with $C_{i+1} \cap (C_i \cup \text{ED}(v_i, M_{i+1})) \neq \emptyset$. Since $C_{i+1}$ (resp., $C_i$) is an $[M_{i+1}]$-component (resp., $[M_i]$-component), we have that $C_{i+1}$ (resp., $C_i$) is contained in an $[M_{i+1} \cap M_i]$-component, say $C_{i+1}'$ (resp., $C_i'$). Therefore, if $C_{i+1} \cap C_i \neq \emptyset$, we immediately can conclude that $C_{i+1} \cup C_i$ is $[M_{i+1} \cap M_i]$-connected, with $C_{i+1}' = C_{i}'$. Thus, let us consider the case where $C_{i+1} \cap C_i = \emptyset$ and, hence, $C_{i+1} \cap \text{ED}(v_i, M_{i+1}) \neq \emptyset$. Consider now $p_{i+1} \in C_{i+1} \cap \text{ED}(v_i, M_{i+1})$. By definition of $\text{ED}(v_i, M_{i+1})$, $p_{i+1}$ belongs in particular to $M_{i+1} \cap \text{Fr}(C_i) \setminus M_i$.

Thus, $p_{i+1} \notin M_i \cap M_{i+1}$. However, $p_{i+1} \in C_{i+1}$ and $p_{i+1} \in \text{Fr}(C_i)$. From the latter, we have that there is a node $Z \in C_i$ and a hyperedge $h_1 \in \text{edges}(\mathcal{H}_1)$ such that $\{p_{i+1}, Z\} \subseteq h_1$. It follows that there is an $[M_{i+1} \cap M_i]$-path from $p_{i+1} \in C_{i+1}$ to $Z \in C_i$. Thus, $C_{i+1} \cup C_i$ is $[M_{i+1} \cap M_i]$-connected.

(*only-if-part*) Assume that $C_{i+1}$ is an $[M_{i+1}]$-component such that $C_{i+1} \cup C_i$ is $[M_{i+1} \cap M_i]$-connected. Consider the case where $C_{i+1} \cap C_i = \emptyset$. Then, there is a node $p_{i+1} \in M_{i+1} \cap \text{Fr}(C_i)$ such that $p_{i+1} \in C_{i+1}$. Thus, $p_{i+1} \notin M_i$. It follows that $p_{i+1} \in M_i \cap \text{Fr}(C_i) \setminus M_{i+1}$, and hence $C_{i+1} \cap \text{ED}(v_i, M_{i+1}) \neq \emptyset$. $\square$

Moreover, we next characterize monotone moves based on escape-doors.

**Lemma 4.7** Let $M_i$ and $M_{i+1}$ be positions for the Captain, let $C_i$ be an $[M_i]$-component, and let $v_i = (M_i, C_i)$. Then, $\text{ED}(v_i, M_{i+1}) = \emptyset$ if, and only if, for each $[v_i, M_{i+1}]$-option $C_{i+1}$, $C_{i+1} \subseteq C_i$.

**Proof.** (*if-part*) Assume that $\forall [v_i, M_{i+1}]$-option $C_{i+1}$, $C_{i+1} \subseteq C_i$. Moreover, assume for the sake of contradiction that $\text{ED}(v_i, M_{i+1}) \neq \emptyset$, and let $X \in \text{ED}(v_i, M_{i+1}) = M_i \cap \text{Fr}(C_i) \setminus M_{i+1}$. In particular note that $X \notin M_{i+1}$, from which we conclude that there must be an $[M_{i+1}]$-component $C_{i+1}$ such that $X \in C_{i+1}$. Thus, $X \in C_{i+1} \cap \text{ED}(r, M_{i+1})$ and hence we can apply Lemma 4.6 to conclude that $C_{i+1}$ is a $[v_i, M_{i+1}]$-option. However, $X$ is not in $C_i$, since $X$ belongs to $M_i$ (and $C_i$ is an $[M_i]$-component). Thus, $C_{i+1} \nsubseteq C_i$, which is impossible. Therefore, $\text{ED}(v_i, M_{i+1}) = \emptyset$. $\square$
(only-if-part) Assume that \( ED(v_i, M_{i+1}) = \emptyset \), and for the sake of contradiction that \( C_{i+1} \) is a \([v_i, M_{i+1}]\)-option such that \( C_{i+1} \not\subseteq C_i \). Let \( Y \) be a node in \( C_{i+1} \setminus C_i \), and observe that there must be a node \( X \in C_{i+1} \cap C_i \), because of Lemma 4.6.

Consider now an \([M_{i+1}]\)-path from \( Y \) to \( X \) and let \( Z_1, Z_2 \) be two nodes in this path such that \( Z_1 \in C_{i+1} \cap C_i \), \( Z_2 \in C_{i+1} \setminus C_i \), and \( \{Z_1, Z_2\} \subseteq h \) for some hyperedge \( h \in \text{edges}(H_1) \). Note that these two nodes exist because of the properties of the endpoints \( Y \) and \( X \). Now, it must be the case that \( Z_2 \) is in \((\text{Fr}(C_i) \setminus C_i) \cap C_{i+1}\). Since \( M_i \supseteq \text{Fr}(C_i) \setminus C_i \), the latter entails that \( Z_2 \in M_i \cap \text{Fr}(C_i) \cap C_{i+1} \). Finally, since \( C_{i+1} \) is an \([M_{i+1}]\)-component, we conclude that \( Z_2 \in M_i \cap \text{Fr}(C_i) \setminus M_{i+1} \), i.e., \( Z_2 \in ED(v_i, M_{i+1}) \) which is impossible. \( \square \)

The lemma above easily leads us to characterize monotone strategies as those ones for which there are no escape-doors.

**Corollary 4.8** The strategy \( \sigma \) is monotone if, and only if, for each vertex \( v_i = (M_i, C_i) \) in \( T(\sigma) \), and for each child \((M_{i+1}, C_{i+1})\) of \( v_i \), \( ED(v_i, M_{i+1}) = \emptyset \).

Assume now that \( \sigma \) is a non-monotone winning strategy. Armed with the above notions and results, we shall show how \( \sigma \) can be transformed into a monotone winning strategy, by “removing” the various escape-doors.

Let \( p = (M_p, C_p) \) be a configuration reached in \( T(\sigma) \) from \((\emptyset, \text{nodes}(H_1))\) by a (possibly empty) succession of moves \( \pi \). Assume that \( M_r \) is the move of the Captain in \( p \) and that this move is monotone, i.e., for each \([p, M_r]\)-option \( C, C \subseteq C_r \) (note that any move in the initial configuration is monotone). Let \( r = (M_r, C_r) \) be a child of \( p \) in \( T(\sigma) \), and let \( s = (M_s, C_s) \) be a child of \( r \) such that \( C_s \not\subseteq C_r \), i.e., such that \( ED(r, M_s) \neq \emptyset \) (by Corollary 4.8). This witnesses that \( M_s \) is a non-monotone move—see Figure 6.

Let \( M_r' = M_r \setminus ED(r, M_s) \subset M_r \), and consider the function \( \sigma' \) built as follows:

\[
\sigma'(M, C) = \begin{cases} 
M_r' & \text{if } (M, C) = (M_p, C_p) \\
\sigma(M, C) & \text{otherwise.}
\end{cases}
\]
Intuitively, we are removing from \( r \) the source of non-monotonicity that was suddenly evidenced while moving to \( s \), i.e., the fact that \( \text{ED}(r, M_s) \neq \emptyset \). Next, we show that this modification does not affect the final outcome of the game.

**Lemma 4.9** \( \sigma' \) is a winning strategy.

**Proof.** By definition, \( \sigma' \) leaves unchanged the configurations of \( \sigma \) encoded in those subtrees of \( T(\sigma) \) that are not rooted below \( p = (M_p, C_p) \)—see Figure 6. Therefore, throughout this proof we have only to take care of what happens in the subtree of \( T(\sigma) \) rooted at \((M_p, C_p)\). Indeed, the first crucial difference between \( \sigma' \) and \( \sigma \) occurs when the Captain plays \( \sigma'(M_p, C_p) = M'_r \). Beforehand, we note that this is a “valid” move, since \( M'_r \subset M_r \subseteq \text{Fr}(C_p) \) (cf. Fact 4.3).

Let \( C'_r \) be the \( [M'_r]\)-component with \( C_r \cup \text{ED}(r, M_s) \subseteq C'_r \). Note that such a component exists since \( \text{ED}(r, M_s) \subseteq \text{Fr}(C_r) \) and \( M'_r \subseteq M_r \). In addition, we claim that \( C'_r \) is a \([p, M'_r]\)-option. Indeed, \( C_r \) is a \([p, M_r]\)-option, for which therefore \( C_r \cap (C_p \cup \text{ED}(p, M'_r)) \neq \emptyset \) holds, by Lemma 4.6. Moreover, since \( M'_r \subset M_r \), \( \text{ED}(p, M_r) \subseteq \text{ED}(p, M'_r) \) holds. Hence, given that \( C_r \subseteq C'_r \), it follows that \( C'_r \cap (C_p \cup \text{ED}(p, M'_r)) \neq \emptyset \), and again by Lemma 4.6 that \( C'_r \) is a \([p, M'_r]\)-option. Thus, there is an edge from \( p \) to \((M'_r, C'_r)\) in \( T(\sigma') \). In order to complete the picture, we need the following two properties.

**Property P1:** For each \([p, M_r]\)-option \( C \), either \( C \subseteq C'_r \) or \( C \) is a \([p, M'_r]\)-option.

**Proof.** We distinguish two cases depending on whether \( \text{Fr}(C) \cap \text{ED}(r, M_s) \) is empty or not. In the case where \( \text{Fr}(C) \cap \text{ED}(r, M_s) = \emptyset \), then \( C \) is an \([M'_r]\)-component given that \( M'_r = M_r \setminus \text{ED}(r, M_s) \). Moreover, by Lemma 4.6, \( C \) is an \([M_r]\)-component such that \( C \cap (C_p \cup \text{ED}(p, M_r)) = \emptyset \). Hence, it trivially holds that \( C \cap (C_p \cup \text{ED}(p, M'_r)) = \emptyset \), because \( \text{ED}(p, M_r) \subseteq \text{ED}(p, M'_r) \). Thus, by Lemma 4.6, \( C \) is a \([p, M'_r]\)-option. Eventually, consider the case where \( \text{Fr}(C) \cap \text{ED}(r, M_s) \neq \emptyset \), and recall that \( C'_r \) is the \([M'_r]\)-component with \( C_r \cup \text{ED}(r, M_s) \subseteq C'_r \). Since \( M'_r = M_r \setminus \text{ED}(r, M_s) \), we then have that \( C \subseteq C'_r \).

**Property P2:** For each \([p, M'_r]\)-option \( C' \neq C'_r \), \( C' \) is a \([p, M_r]\)-option.

**Proof.** Let \( C' \neq C'_r \) be a \([p, M'_r]\)-option, hence in particular an \([M'_r]\)-component. Since \( \text{ED}(r, M_s) \subseteq C'_r \) and \( C'_r \) is also an \([M'_r]\)-component, we have that \( C' \cap \text{ED}(r, M_s) = \emptyset \). Moreover, \( C' \cap M'_r = \emptyset \), with \( M'_r = M_r \setminus \text{ED}(r, M_s) \). Thus, \( C' \cap M_r = \emptyset \) and hence, \( C' \) is also an \([M_r]\)-component. Then, in the light of Lemma 4.6 to conclude the proof, it suffices to show that \( C' \cap (C_p \cup \text{ED}(p, M_r)) \neq \emptyset \). In the case where \( C' \cap C_p = \emptyset \), we have concluded. Therefore, consider the case where \( C' \cap C_p = \emptyset \). In this case, as \( C' \) is a \([p, M'_r]\)-option and hence, \( C' \cap (C_p \cup \text{ED}(p, M'_r)) = \emptyset \) because of Lemma 4.6, we have \( C' \cap \text{ED}(p, M'_r) = \emptyset \). Recall now that \( \text{ED}(p, M_r) = M_p \cap \text{Fr}(C_p) \setminus M_r \) and \( \text{ED}(p, M'_r) = M_p \cap \text{Fr}(C_p) \setminus M'_r \). Thus, \( \text{ED}(p, M'_r) \subseteq (M_r \setminus M'_r) \cup \text{ED}(p, M_r) \), and then \( \text{ED}(p, M'_r) \subseteq \text{ED}(r, M_s) \cup \text{ED}(p, M_r) \) holds, as \( M'_r = M_r \setminus \text{ED}(r, M_s) \) by definition of the strategy \( \sigma' \). Given that \( C' \cap \text{ED}(r, M_s) = \emptyset \), we immediately can conclude that \( C' \cap \text{ED}(p, M'_r) \subseteq C' \cap \text{ED}(p, M_r) \). However, \( \text{ED}(p, M_r) \subseteq \text{ED}(p, M'_r) \), because \( M'_r \subset M_r \), and hence, \( C' \cap \text{ED}(p, M'_r) = C' \cap \text{ED}(p, M_r) \) actually holds. Given that \( C' \cap \text{ED}(p, M'_r) \neq \emptyset \), we have therefore that \( C' \cap \text{ED}(p, M_r) \neq \emptyset \). Thus, \( C' \cap (C_p \cup \text{ED}(p, M_r)) \neq \emptyset \).

Note that in the light of the two results above, the function \( \sigma' \) encodes a winning strategy when attacking each \([p, M'_r]\)-option \( C' \neq C'_r \), since these components remain completely unchanged when
changing $\sigma$ with $\sigma'$. Thus, as illustrated in Figure 6, all subtrees of $T(\sigma)$ rooted at the children of $(M_p, C_p)$ and attacking options outside $C'_r$ are preserved in the game-tree $T(\sigma')$. Hence, we have only to take care of how $\sigma'$ attacks the remaining component $C'_r$.

By definition of $\sigma'$, $\sigma'(M'_r, C'_r) = M_s$, which is a valid position since $M_s \subseteq \text{Fr}(C'_r)$ because of the facts that $\sigma$ is a strategy (and, hence, we can apply Fact 4.3) to conclude that $M_r \subseteq \text{Fr}(C_r)$ and $C'_r \supseteq C_r$, so that $\text{Fr}(C_r) \subseteq \text{Fr}(C'_r)$. Moreover, the following property holds, which guarantees that the novel move is actually monotone.

Property $P_3$: Assume by contradiction that $\text{ED}(r', M_s) = \partial C'_r \setminus M_s \neq \emptyset$, and let $X \in \text{ED}(r', M_s)$. From $\partial C'_r \subseteq M'_r = \text{Fr}(r, M_s)$, we get that $X \in M_r \setminus \text{Fr}(C_r)$. However, this is impossible because $M_r \subseteq \text{Fr}(C_r)$, by definition of the Robber and Captain game.

We next show that applying $M_s$ to $C'_r$ leads exactly to the same strategy obtained when attacking $C_r$ with the same move $M_s$. Let $r'$ be the position $(M'_r, C'_r)$.

Property $P_4$: For each $[r, M_s]$-option $C$, $C$ is an $[r', M_s]$-option.

Proof. Let $C$ be an $(r, M_s)$-option and, hence, an $[M_s]$-component such that $C \cap (C_r \cup \text{ED}(r, M_s)) \neq \emptyset$, by Lemma 4.6. By definition, $C_r \cup \text{ED}(r, M_s) \subseteq C'_r$. Hence, $C \cap C'_r \neq \emptyset$. Again by Lemma 4.6 we conclude that $C$ is a $[r', M_s]$-option.

Property $P_5$: For each $[r', M_s]$-option $C$, $C$ is an $[r, M_s]$-option.

Proof. Let $C$ be an $(r', M_s)$-option and, hence, an $[M_s]$-component such that $C \cap (C'_r \cup \text{ED}(r', M_s)) \neq \emptyset$, by Lemma 4.6. Our goal is to show that $C$ is also such that $C \cap (C_r \cup \text{ED}(r, M_s)) \neq \emptyset$, so that $C$ is also an $[r, M_s]$-option (again by Lemma 4.6). By Property $P_3$, $\text{ED}(r', M_s) = \emptyset$ and, hence, $C \cap C'_r \neq \emptyset$. Let $Y$ be any vertex in $C \cap C'_r$ and assume, for the sake of contradiction, that $C \cap (C_r \cup \text{ED}(r, M_s)) = \emptyset$. Then, since $C$ is an $[M_s]$-component, $M_s$ separates $Y$ from the vertices in $C_r \cup \text{ED}(r, M_s)$, i.e., each path connecting $Y$ with some vertex in $C_r \cup \text{ED}(r, M_s)$ must include a vertex belonging to $M_s$. 5 Let $M_s \subseteq Y\cup M_s$ be the set of all such vertices blocking the paths from $Y$ to $C_r \cup \text{ED}(r, M_s)$. Then, consider the set $\{Y\} \cup C_r \cup \text{ED}(r, M_s)$ which is contained in $C'_r$. Hence, $\{Y\} \cup C_r \cup \text{ED}(r, M_s)$ is an $[M'_r]$-component, because $C'_r$ is an $[M'_r]$-component. Therefore, the separator $M_s$ cannot be included in $M'_r$. It follows that there is a node $p \in M_s \setminus M'_r$. Recall, now, that $M'_r = M_r \setminus \text{ED}(r, M_s)$. Thus, $p \notin M_r \setminus \text{ED}(r, M_s)$ holds. In addition, since $\text{ED}(r, M_s) = M_r \cap \text{Fr}(C_r) \setminus M_s$, we have that $M_s \cap \text{ED}(r, M_s) = \emptyset$. So, given that $p \in M_s \subseteq M_s$, we have that $p \notin \text{ED}(r, M_s)$.

It follows that $p \notin (M_r \setminus \text{ED}(r, M_s)) \cup \text{ED}(r, M_s) = M_r$. Now observe that $M_s \subseteq \text{Fr}(C_r)$ (because of Fact 4.3), while $\text{Fr}(C_r) \setminus C_r \subseteq M_r$. Then, since $p \notin M_r$ and $p \in M_s$, we conclude that $p$ is in $C_r$. Thus, we can assume w.l.o.g that $Y$ is in $\text{Fr}(C_r)$. Since $Y \notin C_r \cup \text{ED}(r, M_s)$, because of the assumption that $C \cap (C_r \cup \text{ED}(r, M_s)) = \emptyset$ and the fact that $Y \in C$, we conclude that $Y \in \text{Fr}(C_r) \setminus C_r$, i.e., $Y \in M_r$. And, actually, $Y \in M'_r = M_r \setminus \text{ED}(r, M_s)$, given that $Y \notin \text{ED}(r, M_s)$. But, this is impossible, since $Y \subseteq C'_r$ and $C'_r$ is an $[M'_r]$-component.

It follows that after the move $M'_r = \sigma'(M_p, C_p)$, the set of options available to the Robber is precisely the set that the Robber would have obtained with the move $M_s$. Then, because $\sigma'$ attacks these options precisely as $\sigma$, we conclude that $\sigma'$ is still a winning strategy. ⊓⊔
Example 4.10 For an example application of the above lemma, consider Figure \[7\]. The figure reports two hyperedges \(H_1\) and \(H_2\), plus the game-tree for a winning strategy \(\sigma\). In particular, note that \(\sigma\) is non-monotone, because the Robber is allowed to return on \(A\) and \(G\), after that these nodes have been previously occupied by the Captain with the move \(M_r = \{A, C, D, E, G\}\). In fact, the figure also reports the strategy \(\sigma'\) that is obtained from \(\sigma\), by turning the non-monotone move of the Captain (in the left branch of the tree) into a monotone one according to the construction of Lemma 4.9.

Note that the novel move of the Captain is \(M_r' = \{C, D, E, G\}\), with \(\{A\} = \text{ED}(r', B, C)\) being the escape door for the Robber. In fact, this novel move does not affect the (winning) strategy of the Captain in the right branch.

Now that the transformation from \(\sigma\) to \(\sigma'\) has been clarified, we can state the main result of this section, which is based on the fact that minimal winning strategies have to be monotone. For a strategy \(\sigma\), let \(||\sigma||\) be the size of \(\sigma\) measured as total number of cops used over all the vertices of \(T(\sigma)\), i.e., \(||\sigma|| = \sum_{(M,C) \in T(\sigma)} |M|\). Let \(\sigma_1\) and \(\sigma_2\) be two winning strategies. We write \(\sigma_1 \prec \sigma_2\) iff \(||\sigma_1|| < ||\sigma_2||\). We say that a winning strategy \(\sigma\) is minimal, if there is no winning strategy \(\bar{\sigma}\) such that \(\bar{\sigma} \prec \sigma\).

Note that the existence of a winning strategy always entails the existence of a minimal winning one.

Theorem 4.11 On the R&C(\(H_1, H_2\)) game, the existence of a winning strategy implies the existences of a monotone winning strategy.

Proof. We claim that minimal winning strategies must be monotone. Indeed, let \(\sigma\) be a non-monotone winning strategy and assume, for the sake of contradiction, that \(\sigma\) is minimal. Consider the transfor-
mation from $\sigma$ to $\sigma'$ discussed in the proof of Lemma 4.9 by recalling that $\sigma'$ is a winning strategy. Then, by definition of $\sigma'$ and the properties pointed out in that proof, we have that $\sigma' \prec \sigma$, which is impossible.

As a remark, the transformation in the proof of Lemma 4.9 entails the existence of a constructive method to build a monotone strategy from a non-monotone one.

### 4.2 Tree Projections and the R&C Game

In this section, we prove that the Robber and Captain game precisely characterizes the tree projection problem, in the sense that a winning strategy for R&C($H_1, H_2$) exists if and only if ($H_1, H_2$) has a tree projection. Hence, any decomposition technique that can be restated in terms of tree projections is in turn characterized by R&C games. In particular, if we consider pairs of the form ($H_1, H_2^1$), we get a game characterization for the notion of $k$-width generalized hypertree decomposition, for which such a characterization was still missing in the literature.

For an exemplification of the results below, the reader may consider the game-tree illustrated in the bottom-right part of Figure 5 and the tree projection in Figure 4.

**Theorem 4.12** If there is a winning strategy in R&C($H_1, H_2$), then ($H_1, H_2$) has a tree projection.

**Proof.** From Theorem 4.11 if there is a winning strategy in R&C($H_1, H_2$), there exists a monotone winning strategy, say $\sigma$, for this game. Based on $\sigma$ we build a hypergraph $H_a(\sigma)$ where, for each vertex $(M, C)$ in $T(\sigma)$ with $M \neq \emptyset$, $\text{edges}(H_a(\sigma))$ contains the hyperedge $M$; and, no further hyperedge is in $\text{edges}(H_a(\sigma))$. Note that, by construction, $H_a(\sigma) \subseteq H_2$, since each position $M$ is such that $M \subseteq h_2$ for some hyperedge $h_2 \in \text{edges}(H_2)$. Let $h_1$ be a hyperedge in $\text{edges}(H_1)$. Since $\sigma$ is a winning strategy, we trivially conclude that the Captain has necessarily covered in a complete form $h_1$ in some position. Thus, $H_1 \leq H_a(\sigma)$. Eventually, in order to show that $H_a(\sigma)$ is a tree projection, it remains to check that $H_a(\sigma)$ is acyclic.

To this end, we build a tree $JT$ by exploiting the strategy $T(\sigma)$. $JT$ contains all the hyperedges in $\text{edges}(H_a)$ and, for each pair of adjacent configurations $(M_s, C_s)$ and $(M_r, C_r)$ in $T(\sigma)$, the vertices $M_s$ and $M_r$ of $JT$ are connected with an edge in $JT$. We claim that $JT$ is a join tree for $H_a(\sigma)$. In the following, assume (for the sake of exposition) that $JT$ is rooted at the hyperedge encoding the first move of the Captain (e.g., $\{E, F, G\}$ in the game-tree depicted in Figure 5).

Note first that by construction of $JT$ and since $\sigma$ is monotone, each vertex of $JT$ has exactly one parent, but the root. Thus, $JT$ is in fact an acyclic graph, and we have just to focus on showing that the connectedness condition is satisfied.

Let $h_1$ and $h_2$ be two distinct vertices in $JT$ and let $X \in h_1 \cap h_2$. Let $h$ be the vertex in the shortest path between $h_1$ and $h_2$ that is the closest to the root of $JT$. Assume, w.l.o.g., that $h_2 \neq h$ and assume that $h_2$ is a child of $h'$. Because of Fact 4.3, $h_2 \subseteq \text{Fr}(C_{h'})$, where $C_{h'}$ is the $[h']$-component such that $(h, C_{h'})$ is in $T(\sigma)$ and where $\sigma(h, C_{h'}) = h_2$. Thus, $X \in \text{Fr}(C_{h'})$. Assume, for the sake of contradiction, that $X$ does not occur in a hyperedge in the path between $h_1$ and $h_2$, i.e., that the connectedness condition is violated. In particular, w.l.o.g., we may just focus on the case where $X \notin h'$. Then, since $h' \supseteq \text{Fr}(C_{h'}) \setminus C_{h'}$, $X \in \text{Fr}(C_{h'})$ and $X \notin h'$ immediately entail that $X$ is in $C_{h'}$. Then, because of the monotonicity of $\sigma$, $X$ also occurs in an $[h]$-component $C_h$ such that $(h, C_h) \in T(\sigma)$. Now, observe that the scenario $h_1 = h$ is impossible. Indeed, $h_1$ contains $X$ that would be also contained in an $[h_1]$-component, which is impossible by the monotonicity of the strategy. Thus, assume that $h_1$ is the child of an edge $h'$. By using the same line of reasoning as above, we conclude
that $X$ occurs in an $[h]$-component $\bar{C}_h$ such that $(h, \bar{C}_h) \in T(\sigma)$. Thus, $C_h = \bar{C}_h$. By definition of $JT'$, this means that $h_2$ and $h_1$ occur in a subtree rooted at some child of $h$, which is impossible since $h$ is in the shortest path between $h_1$ and $h_2$. 

We now complete the picture by showing the converse result.

**Theorem 4.13** If $(\mathcal{H}_1, \mathcal{H}_2)$ has a tree projection, then there is a winning strategy in $R\&C(\mathcal{H}_1, \mathcal{H}_2)$.

**Proof.** Assume that $(\mathcal{H}_1, \mathcal{H}_2)$ has a tree projection. From Fact 3.3 it has a minimal tree projection, say $\mathcal{H}_a$. Let $JT'$ be a join tree for $\mathcal{H}_a$ in normal form (cf. Theorem 3.20), and let $h$ be any hyperedge in $edges(\mathcal{H}_a)$.

Based on $JT'$, we build a strategy $\sigma$ as follows. Let $h_0 = \emptyset$ and $C_0 = nodes(\mathcal{H}_1)$. The first move of the Captain is $h$. Recall from Section 3 that $C_\tau(h_s)$ is the unique $[h_s]$-component with $nodes(JT'[h]_s) = C_\tau(h_s) \cup (h_s \cap h_r)$, where $h_s$ is a child of $h_r$ in $JT'[h]$ (with $C_\tau(h)$ be defined as $nodes(\mathcal{H}_a)$). Given the current position $(h_p, C_p)$ and the current move $h_r$, assume that the following inclusion relationship holds: for each $[(h_p, C_p), h_r]$-option $C_r, C_r \subseteq C_\tau(h_s)$. Then, $\sigma(h_r, C_r)$ is defined as the hyperedge $h_s$ that is the child of $h_r$ in $JT'[h]$ and that is such that $C_r = C_\tau(h_s)$. It follows that the strategy $\sigma$ is well-defined under this assumption, because such a hyperedge exists by Theorem 3.20.

Now note that we can set $C_p = C_\tau(h_r)$ (for the first move, just recall that $C_\tau(h) = nodes(\mathcal{H}_1)$).

We now show that the above inclusion relationship actually holds, that is, for each vertex $h_r$ of $JT'[h]$ and for each child $h_s$ of $h_r$, we have $C_\tau(h_s) \subseteq C_\tau(h_r)$. To see this is true, recall again by Theorem 3.20 that $nodes(JT'[h]_s) = C_\tau(h_s) \cup (h_s \cap h_p)$ and $nodes(JT'[h]_r) = C_\tau(h_s) \cup (h_s \cap h_r)$. Assume, for the sake of contradiction, that $C_\tau(h_s) \not\subseteq C_\tau(h_r)$ and let $X \in C_\tau(h_s) \setminus C_\tau(h_r)$. Since, $nodes(JT'[h]_s) \subseteq nodes(JT'[h]_r)$, it follows that $X \in h_s \cap h_p$. This is impossible, since $C_\tau(h_s)$ is a $[h_s]$-component, with $X \in C_\tau(h_s)$.

Finally, to complete the proof just notice that the above also entails that $\sigma$ is a monotone strategy, eventually covering all the nodes in $\mathcal{H}_1$, hence it is a winning strategy.

## 5 Applications and Conclusion

In this paper, we have analyzed structural decomposition methods to identify nearly-acyclic hypergraphs by focusing on the general concept of tree projections.

We defined and studied a natural notion of minimality for tree-projections of pairs of hypergraphs. It turns out that minimal tree-projections always exist (whenever some tree-projection exists), and that they enjoy some useful properties, such as the existence of join trees in a suitable normal form that is crucial for algorithmic applications. In particular, such join trees have polynomial size with respect to the given pair of hypergraphs. As an immediate consequence of these properties, we get that deciding whether a tree projection of a pair of hypergraphs $(\mathcal{H}_1, \mathcal{H}_2)$ exists is an NP-problem. Note that this result is expected but not trivial, because in general a tree projection may employ any subset of every hyperedge of $\mathcal{H}_2$. In fact, the results proved in detail in the present paper have been (explicitly) used by [16] in the membership part of the proof that deciding the existence of a tree projection is an NP-complete problem, which closed the long-standing open question about its computational complexity [17] [18] [27] [24].

Moreover, we provided a natural game-theoretic characterization of tree projections in terms of the Captain and Robber game, which was missing and asked for even in the special case of generalized hypertree decompositions. In this game, monotone strategies have the same power as non-monotone
strategies. Even this result is not just of theoretical interest. Indeed, by exploiting the power of non-monotonicity for some easy-to-compute strategies in the game, called greedy strategies, larger islands of tractability for the homomorphism problem (hence, for the constraint satisfaction problem and for the problem of evaluating conjunctive queries, and so on) have been identified in [21]. In particular, for the special case of generalized hypertree decompositions, these strategies lead to the definition of a new tractable approximation, called greedy hypertree decomposition, which is strictly more powerful than the (standard) notion of hypertree decomposition.

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