This paper considers an initial market model, specified by the pair \((S, \mathcal{F})\) where \(S\) is its discounted assets’ price process and \(\mathcal{F}\) its flow of information, and an arbitrary random time \(\tau\). This random time can represent the death time of an agent or the default time of a firm, and in both cases \(\tau\) can not be seen before it occurs. Thus, the progressive enlargement of \(\mathcal{F}\) with \(\tau\), denoted by \(\mathcal{G}\), sounds tailor-fit for modelling the new flow of information that incorporates both \(\mathcal{F}\) and \(\tau\). For the stopped model \((S^\tau, \mathcal{G})\), we describe in different manners and as explicit as possible the numéraire portfolio, the log-optimal portfolio, the log-optimal deflator (which is the dual of the log-optimal portfolio), and we elaborate their duality without any further assumption.

1 Introduction

Since the seminal papers of Merton [44, 45], the theory of utility maximization and optimal portfolio has been developed successfully in many directions and in different frameworks. These achievements can be found in the works of Karatzas et al. [37], Kramkov and Schachermayer [40], Cvitanic, Schachermayer and Wang [22], Karatzas and Zitkovic [38], and the references therein to cite few. In these works, the authors considered a fixed investment horizon and practically neglected the impact of a random horizon on the optimal selection portfolio and/or investor’s behaviour. The economic problem of how a random horizon will impact an investment is old and can be traced back to Fisher [26]. In mathematical context, this problem is very difficult and only recently there were some advances. This problem of random horizon in finance can be viewed as a general setting for many other financial and economics frameworks. Among these, we cite the example of credit risk theory where the random horizon is the default time of a firm, and life insurance with its challenging mortality and/or longevity risks where the random time is the death time of an agent.

This paper considers an initial market model represented by the pair \((S, \mathcal{F})\), where \(S\) represents the discounted stock prices for \(d\)-stocks, and \(\mathcal{F}\) is the “public” information that is available to all agents. To this initial market model, we add a random time \(\tau\) that might not be seen through \(\mathcal{F}\) when it occurs (mathematically speaking this means that \(\tau\) might not be an \(\mathcal{F}\)-stopping time). In this context, we adopt the progressive enlargement of filtration to catch the information from both \(\mathcal{F}\) and \(\tau\). This modelling of the new informational system, that we denote by \((S^\tau, \mathcal{G})\), allows us to keep in mind credit risk theory and life insurance as potential applications of our results. In fact the death time of an agent can not be seen with certainty before its occurrence. Similarly for the default of a firm, up to our knowledge, there is no single financial literature that models the information in \(\tau\) as fully seen from the beginning as in the case of insider trading. For this new informational financial model,
our ultimate goal lies in measuring the impact of \( \tau \) on the optimal portfolios, mainly the log-optimal portfolio and the numéraire portfolio.

Our mathematical and financial achievements reside in describing the log-optimal portfolio, the log-optimal deflator, and the numéraire portfolio for \((S^\tau, \mathcal{G})\) in different manners. One way lies in proving that the random horizon induces randomness in agent’s utility. It is worth mentioning that random utilities appeared first in economics within the random utility model theory due to the psychometric literature that provided empirical evidence about stochastic choice behaviour. For details about this theme, we refer the reader to [48, 43, 21, 20] and the references therein to cite few. A random field utility represents the preference of an agent (or the agent’s impatience as called in Fisher [26]), which is updated at each instant using the available aggregate flow of public information about the market.

Our results in this paper relies on two important results. The first result is detailed and deeply discussed in [10] - consists of explicit description of the set of all deflators for the model \((S^\tau, \mathcal{G})\). The second result, stated in [11], characterizes the log-optimal portfolio and establishes its duality for general semimartingale models without the No-Free-Lunch-with-Vanishing-Risk assumption on the market model.

This paper contains six sections including the current one. Section 2 deals with the preliminaries, where we recall the important results on which our paper is based on besides the mathematical model and notation. Section 3 addresses the dual problem of the log-utility maximization problem for the model \((S^\tau, \mathcal{G})\), while Section 4 addresses the log-utility maximization problem itself in details. Section 4 illustrates the results of Sections 3-4 on the popular model of jump-diffusion model for \((S, \mathcal{F})\), while letting \( \tau \) to follow a quite arbitrary model. Finally, Section 4 discusses the impact of random horizon on the numéraire portfolio. The paper contains an appendix where some proofs are relegated and some useful technical (new and existing) results are detailed.

2 Preliminaries

In this section, we recall some important results on which this paper relies on, and define mathematically the economic model that we investigate. This section has three subsections. The first subsection outlines the model and its ingredients. The second subsection recalls an important result on the set of deflators for the setting considered herein, while the last subsection states a general result on the log-optimal portfolio that we recently proved.

2.1 The mathematical model and Notation

This subsection contains basically two important parts. The first part, the next paragraph below, consists of universal notations that will be used throughout the whole paper in a way or another. The second part describes the mathematical models and its ingredients.

Throughout the paper, by \( \mathbb{H} \) we denote an arbitrary filtration that satisfies the usual conditions of completeness and right continuity. For any process \( X \), the \( \mathbb{H} \)-optional projection and dual \( \mathbb{H} \)-optional projection of \( X \), when they exist, will be denote by \( X^{o, \mathbb{H}} \) and \( X^{o, \mathbb{H}} \) respectively. Similarly, we denote by \( X^{p, \mathbb{H}} \) and \( X^{p, \mathbb{H}} \) the \( \mathbb{H} \)-predictable projection and dual predictable projection of \( X \) when they exist. The set \( \mathcal{M}(\mathbb{H}, Q) \) denotes the set of all \( \mathbb{H} \)-martingales under \( Q \), while \( \mathcal{A}(\mathbb{H}, Q) \) denotes the set of all optional processes with integrable variation under \( Q \). When \( Q = P \), we simply omit the probability for the sake of simple notations. For an \( \mathbb{H} \)-semimartingale \( X \), by \( L(X, \mathbb{H}) \) we denote the set of \( \mathbb{H} \)-predictable processes that are \( X \)-integrable in the semimartingale sense. For \( \varphi \in L(X, \mathbb{H}) \), the resulting integral of \( \varphi \) with respect to \( X \) is denoted by \( \varphi \cdot X \). For \( \mathbb{H} \)-local martingale \( M \), we denote by \( L^{1, \text{loc}}(M, \mathbb{H}) \) the set \( \mathbb{H} \)-predictable processes \( \varphi \) that are \( X \)-integrable and the resulting integral \( \varphi \cdot M \) is an \( \mathbb{H} \)-local martingale. If \( C(\mathbb{H}) \) is the set of processes that are adapted to \( \mathbb{H} \in \{ \mathcal{F}, \mathcal{G} \} \), then \( C_{\text{loc}}(\mathbb{H}) \) is
the set of processes, \( X \), for which there exists a sequence of \( \mathbb{H} \)-stopping times, \((T_n)_{n \geq 1} \), that increases to infinity and \( X^{T_n} \) belongs to \( \mathcal{C}(\mathbb{H}) \), for each \( n \geq 1 \). For any \( \mathbb{H} \)-semimartingale, \( L \), we denote by \( \mathcal{E}(L) \) the Doleans-Dade (stochastic) exponential, it is the unique solution to the stochastic differential equation \( dX = X \cdot dL, \quad X_0 = 1 \), given by

\[
\mathcal{E}_t(L) = \exp(L_t - \frac{1}{2}(L^c)_t) \prod_{0 \leq s \leq t} (1 + \Delta L_s)e^{-\Delta L_s}.
\]

Our mathematical model starts with a filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}, P)\). Here the filtration \( \mathbb{F} := (\mathcal{F}_t)_{t \geq 0} \), which represents the “public” flow of information available to all agent over time, satisfies the usual conditions of right continuity and completeness. On this stochastic basis, we suppose given a \( d \)-dimensional locally bounded and quasi-left-continuous \( \mathbb{F} \)-semimartingale, \( S \), that models the discounted price process of \( d \) risky assets. In addition to this initial market model, \((S, \mathbb{F}, P)\), we consider a random time \( \tau \), that might represent the death time of an agent or the default time of a firm, and hence it might not be an \( \mathbb{F} \)-stopping time in general. To this random time, we associate the following associated non-decreasing process \( D \) and the filtration \( \mathbb{G} := (\mathcal{G}_t)_{t \geq 0} \) given by

\[
D := I_{[\tau, +\infty]}, \quad \mathcal{G}_t := \mathcal{G}^0_t := \mathcal{F}_t \lor \sigma(D_s, s \leq t).
\]

It is clear \( \mathbb{G} \) makes \( \tau \) a stopping time. In fact, it is the smallest filtration, satisfying the usual conditions, that makes \( \tau \) a stopping time and contains \( \mathbb{F} \). It is the progressive enlargement of \( \mathbb{F} \) with \( \tau \). Besides \( D \) and \( \mathbb{G} \), other \( \mathbb{F} \)-adapted processes intimately related to \( \tau \) play central roles in our analysis. Among these, the following survival probabilities called Azéma supermartingales in the literature, and are given by

\[
G_t := I_{[0, \tau]} = P(\tau > t| \mathcal{F}_t) \quad \text{and} \quad \tilde{G}_t := I_{[0, \tau]} = P(\tau \geq t| \mathcal{F}_t),
\]

while the process

\[
m := G + D^o, \quad \mathbb{F}
\]

is an \( \mathbb{F} \)-martingale. Then thanks to [4], the process

\[
\mathcal{T}(M) := M^\tau - \tilde{G}^{-1} I_{[0, \tau]} \cdot [M, m] + I_{[0, \tau]} \cdot \left( \sum \Delta MI_{\{G = 0 \land G_-\}} \right)^{p, \mathbb{F}},
\]

is a \( \mathbb{G} \)-local martingale for any \( M \in \mathcal{M}_{loc}(\mathbb{F}) \). In [12], the authors introduced

\[
N^G := D - \tilde{G}^{-1} I_{[0, \tau]} \cdot D^o, \quad \mathbb{F}
\]

which is a \( \mathbb{G} \)-martingale with integrable variation such that \( H \cdot N^G \) is a \( \mathbb{G} \)-local martingale with locally integrable variation for any \( H \) belonging to

\[
T^o_{loc}(N^G, \mathbb{G}) := \left\{ K \in \mathcal{O}(\mathbb{F}) \mid |K|G^{-1}I_{\{G > 0\}} \cdot D \in A^+_{loc}(\mathbb{G}) \right\}.
\]

For \( p \in [1, +\infty) \) and a \( \sigma \)-algebra \( \mathcal{H} \) on \( \Omega \times [0, +\infty] \), we define \( L^p_{loc}(\mathcal{H}, \mathbb{P} \otimes \mathcal{D}) \) as the set of all processes \( X \) for which there exists a sequence of \( \mathbb{F} \)-stopping times \((T_n)_{n \geq 1} \) that increases to infinity almost surely and \( X^{T_n} \) belongs to \( L^p(\mathcal{H}, \mathbb{P} \otimes \mathcal{D}) \) given by

\[
L^p(\mathcal{H}, \mathbb{P} \otimes \mathcal{D}) := \left\{ X \mathcal{H}\text{-measurable} \mid \mathbb{E}[|X_T|^p | I_{(\tau, +\infty)}] < +\infty \right\}.
\]

The remaining two subsections of this section are central in our analysis of the log-optimal portfolio and its dual for \((S^*, \mathbb{G})\). In fact, the next subsection describes explicitly the set of all deflators for the model \((S^*, \mathbb{G})\). This allows us to undertake an optimization problem over this set.
2.2 The set of all deflators for \((S^r, G)\)

We start this subsection by recalling the two definitions of deflators.

**Definition 2.1.** Let \(X\) be a \(\mathbb{H}\)-semimartingale and \(Z\) be a process.

(a) We call \(Z\) an \(\mathbb{H}\)-local martingale deflator for \(X\) (or a local martingale deflator for \((X, \mathbb{H})\), or also called a \(\sigma\)-martingale density) if \(Z > 0\) and there exists a real-valued and \(\mathbb{H}\)-predictable process \(\varphi\) such that \(0 < \varphi \leq 1\) and both processes \(Z\) and \(Z(\varphi \cdot X)\) are \(\mathbb{H}\)-local martingales. Throughout the paper, the set of all local martingale deflators for \((X, \mathbb{H})\) will be denoted by \(Z_{\text{loc}}(X, \mathbb{H})\).

(b) We call \(Z\) an \(\mathbb{H}\)-deflator for \(X\) (or a deflator for \((X, \mathbb{H})\)) if \(Z > 0\) and \(ZE(\varphi \cdot X)\) is an \(\mathbb{H}\)-supermartingale, for any \(\varphi \in L(X, \mathbb{H})\) such that \(\varphi \Delta X \geq -1\). In the rest of the paper, the set of all deflators for \((X, \mathbb{H})\) is denoted by \(D(X, \mathbb{H})\).

**Lemma 2.2.** Let \(\sigma\) be an \(\mathbb{H}\)-stopping time. \(Z\) is a deflator for \((X^\sigma, \mathbb{H})\) if and only if there exists unique pair of processes \((K_1, K_2)\) such that \(K_1 = (K_1)^\sigma\), \(\mathcal{E}(K_1)\) is also a deflator for \((X^\sigma, \mathbb{H})\), \(K_2\) is any \(\mathbb{H}\)-local supermartingale satisfying \((K_2)^\sigma \equiv 0\), \(\Delta K_2 > -1\), and \(Z = \mathcal{E}(K_1 + K_2) = \mathcal{E}(K_1)\mathcal{E}(K_2)\).

The proof of this lemma is straightforward and will be omitted. This lemma shows in a way or another that when dealing with the stopped model \((X^\sigma, \mathbb{H})\), there is no loss of generality in assuming that deflators for this model are also stopped at \(\sigma\). This assumption will be considered throughout the rest of the paper. Throughout the rest of the paper, we adopt the convention \(1/0^+ = +\infty\).

**Theorem 2.3.** Suppose \(G > 0\), and let \(Z^G\) be a \(G\)-semimartingale. Then the following are equivalent.

(a) \(Z^G\) is a deflator for \((S^r, G)\) (i.e. \(Z^G \in D(S^r, G)\)).

(b) There exists a unique \((K^F, V^F, \varphi^{(o)}, \varphi^{(pr)})\) such that \(K^F \in \mathcal{M}_{\text{loc}}(\mathbb{F})\), \(V^F\) is an \(\mathbb{F}\)-predictable and nondecreasing process, \(\varphi^{(o)} \in \mathcal{I}^{\text{loc}}(N^G, G), \varphi^{(pr)}\) belongs to \(L^{\text{loc}}(\text{Prog}(\mathbb{F}), P \otimes D)\) such that \(\mathcal{E}(K^F) \exp(-V^F) \in \mathcal{D}(S, \mathbb{F})\),

\[
\varphi^{(pr)} > -\left[G_-(1 + \Delta K^F) + \varphi^{(o)} G\right]/G, \quad P \otimes D - \text{a.e.},
\]

\[
\frac{G^-}{G}(1 + \Delta K^F) \varphi^{(o)} < \frac{(1 + \Delta K^F)G^-}{\Delta G^F}, \quad P \otimes D_{\mathbb{F}} - \text{a.e.}
\]

\[
Z^G = \mathcal{E}(K^G) \exp(-(V^F)^r), \quad K^G = T(K^F - \frac{1}{G^-} \cdot m) + \varphi^{(o)} \cdot N^G + \varphi^{(pr)} \cdot D.
\]

(c) There exists unique \((Z^F, \varphi^{(o)}, \varphi^{(pr)})\) such that \(Z^F \in \mathcal{D}(S, \mathbb{F})\), \((\varphi^{(o)}, \varphi^{(pr)})\) belongs to \(\mathcal{I}^{\text{loc}}(N^G, G), \in L^{\text{loc}}(\text{Prog}(\mathbb{F}), P \otimes D)\),

\[
\varphi^{(pr)} > -1, \quad P \otimes D - \text{a.e.}, \quad \frac{G^-}{G} < \varphi^{(o)} < \frac{G^-}{G - G}, \quad P \otimes D_{\mathbb{F}} - \text{a.e.},
\]

and

\[
Z^G = \frac{(Z^F)^r}{\mathcal{E}(G^- \cdot m)^r} \mathcal{E}(\varphi^{(o)} \cdot N^G) \mathcal{E}(\varphi^{(pr)} \cdot D).
\]

2.3 A result on log-optimal portfolio without NFLVR

The this last subsection recalls an important result on the log-optimal portfolio and its dual (i.e. the log-optimal deflator) for a general model \((\Omega, X, \mathbb{H}, P)\) without the no-free-lunch-with-vanishing-risk assumption, where \(X\) is a \(d\)-dimensional \(\mathbb{H}\)-semimartingale that is locally bounded and quasi-left-continuous (i.e. \(X\) does not jump at predictable stopping times). This result uses the powerful
techniques of predictable characteristics for semimartingales that we start recalling first. For the filtration \( \mathbb{H} \), we denote

\[
\tilde{O}(\mathbb{H}) := O(\mathbb{H}) \otimes B(\mathbb{R}^d), \quad \tilde{P}(\mathbb{H}) := P(\mathbb{H}) \otimes B(\mathbb{R}^d),
\]

where \( B(\mathbb{R}^d) \) is the Borel \( \sigma \)-field on \( \mathbb{R}^d \), the \( \mathbb{H} \)-optional and \( \mathbb{H} \)-predictable \( \sigma \)-fields respectively on the \( \Omega \times [0, +\infty) \times \mathbb{R}^d \). With a càdlàg \( \mathbb{H} \)-adapted process \( X \), we associate the optional random measure \( \mu_X \) defined by

\[
\mu_X(dt, dx) := \sum_{u>0} I_{\{\Delta X_u \neq 0\}} \delta_{(u, \Delta X_u)}(dt, dx).
\]

For a product-measurable functional \( W \geq 0 \) on \( \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \), we denote \( W \star \mu_X \) (or sometimes, with abuse of notation, \( (x) \star \mu_X \)) the process

\[
(W \star \mu_X)_t := \int_0^t \int_{\mathbb{R}^d} W(u, x) \mu_X(du, dx) = \sum_{0<u\leq t} W(u, \Delta X_u)I_{\{\Delta X_u \neq 0\}}.
\]

We define, on \( \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \), the measure \( M^P_{\mu_X} := P \otimes \mu_X \) by

\[
M^P_{\mu_X} (W) := \int W dM^P_{\mu_X} := E [(W \star \mu_X)_\infty],
\]

(when the expectation is well defined). The conditional expectation given \( \tilde{P}(\mathbb{H}) \) of a product-measurable functional \( W \), denoted by \( M^P_{\mu_X} (W | \tilde{P}(\mathbb{H})) \), is the unique \( \tilde{P}(\mathbb{H}) \)-measurable functional \( \tilde{W} \) satisfying

\[
E [(W I_\Sigma \star \mu_X)_\infty] = E [(\tilde{W} I_\Sigma \star \mu_X)_\infty] \quad \text{for all } \Sigma \in \tilde{P}(\mathbb{H}).
\]

For the reader’s convenience, we recall the canonical decomposition of \( X \) (for more related details, we refer the reader to [31, Theorem 2.34, Section II.2])

\[
X = X_0 + X^c + h \star (\mu_X - \nu_X) + b \cdot A^X + (x - h) \star \mu_X,
\]

where \( h \), defined as \( h(x) := xI_{\{|x|\leq 1\}} \), is the truncation function, and \( h \star (\mu_X - \nu_X) \) is the unique pure jump \( \mathbb{H} \)-local martingale with jumps given by \( h(\Delta X)I_{\{\Delta X \neq 0\}} \). For the matrix \( C^X \) with entries \( C^{ij} := \langle X^c, X^c \rangle \), and \( \nu_X \), we can find a version satisfying

\[
C^X = c^X \cdot A^X, \quad \nu_X(dt, dx) = dA^X_t F^X_t(dx), \quad F^X_t(\emptyset) = 0, \quad \int (|x|^2 + 1) F^X_t(dx) \leq 1.
\]

Here \( A^X \) is increasing and continuous due to the quasi-left-continuity of \( X \), \( b^X \) and \( c^X \) are predictable processes, \( F^X_t(dx) \) is a predictable kernel, \( b^X_t(\omega) \) is a vector in \( \mathbb{R}^d \) and \( c^X_t(\omega) \) is a symmetric \( d \times d \)-matrix, for all \((\omega, t) \in \Omega \times \mathbb{R}_+\). The quadruplet

\[
(b^X, c^X, F^X, A^X)
\]

are the predictable characteristics of \((X, \mathbb{H})\).

For the sake of simplicity, throughout the rest of this subsection, as there is no risk of confusion, we denote by \((b, c, F, A) := (b^X, c^X, F^X, A^X)\). For more details about the predictable characteristics and other related issues, we refer the reader to [31, Section II.2]. Now, we are in the stage of stating the following.
Theorem 2.4. Let $X$ be an $\mathbb{H}$-quasi-left-continuous semimartingale with predictable characteristics $(b, c, F, A)$. We define

$$K_{\log}(y) := \frac{-y}{1+y} + \ln(1+y) \text{ for any } y > -1. \quad (14)$$

If $(X, \mathbb{H})$ is $\sigma$-special, then the following assertions are equivalent.

(a) The set $\mathcal{D}_{\log}(X, \mathbb{H})$, given by

$$\mathcal{D}_{\log}(X, \mathbb{H}) := \{ Z \in \mathcal{D}(X, \mathbb{H}) \mid E[-\ln(Z_T)] < +\infty \}, \quad (15)$$

is not empty (i.e. $\mathcal{D}_{\log}(X, \mathbb{H}) \neq \emptyset$).

(b) There exists a $\mathbb{H}$-predictable process $\tilde{\varphi} \in \mathcal{L}(X, \mathbb{H})$ such that, for any $\varphi$ belonging to $\mathcal{L}(X, \mathbb{H})$, the following hold

$$E \left[ V_T^X + \frac{1}{2}(\varphi^{tr} c\tilde{\varphi})_T + \int K_{\log}(\tilde{\varphi}^{tr} x) F(dx) \cdot A \right] < +\infty, \quad (16)$$

$$V^{\varphi} := \left| \tilde{\varphi}^{tr} (b - c\tilde{\varphi}) + \int \left[ \frac{\tilde{\varphi}^{tr} x}{1 + \tilde{\varphi}^{tr} x} \right] F(dx) \cdot A, \quad (17)$$

$$\langle \varphi - \tilde{\varphi} \rangle^{tr} (b - c\tilde{\varphi}) + \int \left( \frac{(\varphi - \tilde{\varphi})^{tr} x}{1 + \tilde{\varphi}^{tr} x} - \langle \varphi - \tilde{\varphi} \rangle^{tr} h(x) \right) F(dx) \leq 0. \quad (18)$$

(c) There exists a unique $\tilde{Z} \in \mathcal{D}(X, \mathbb{H})$ such that

$$\inf_{Z \in \mathcal{D}(X, \mathbb{H})} E[-\ln(Z_T)] = E[-\ln(\tilde{Z}_T)] < +\infty. \quad (19)$$

(d) There exists a unique $\tilde{\theta} \in \Theta(X, \mathbb{H})$ such that

$$\sup_{\theta \in \Theta(X, \mathbb{H})} E[\ln(1 + (\theta \cdot X)_T)] = E[\ln(1 + (\tilde{\theta} \cdot X)_T)] < +\infty. \quad (20)$$

Furthermore, $\tilde{\theta}(1 + (\tilde{\theta} \cdot X)_-)^{-1}$ and $\tilde{\varphi}$ coincide $P \otimes A$-a.e., and

$$\tilde{\varphi} \in L(X^c, \mathbb{H}) \cap \mathcal{L}(X, \mathbb{H}), \quad \sqrt{(1 + \tilde{\varphi}^{tr} x)^{-1} - 1} \ast \mu \in \mathcal{A}^+(\mathbb{H}), \quad (21)$$

$$\frac{1}{Z} = \mathcal{E}(\tilde{\varphi} \cdot X), \quad \tilde{Z} := \mathcal{E}(K^X - V^{\varphi}), \quad K^X := \tilde{\varphi} \cdot X^c + \frac{-\tilde{\varphi}^{tr} x}{1 + \tilde{\varphi}^{tr} x} \ast (\mu - \nu). \quad (22)$$

3 Optimal deflator for $(S^r, \mathbb{G})$: Existence and explicit description

This section describes, in different manners and as explicit as possible using $F$-adapted processes only, the log-optimal deflator for the model $(S^r, \mathbb{G})$. In other words, we analyze deeply the following minimization problem

$$\min_{Z \in \mathcal{D}_{\log}(S^r, \mathbb{G})} E[-\ln(Z_T)], \quad (23)$$

where $\mathcal{D}_{\log}(S^r, \mathbb{G}) := \{ Z \in \mathcal{D}(S^r, \mathbb{G}) \mid E[-\ln(Z_T)] < +\infty \}$.

and $\mathcal{D}(S^r, \mathbb{G})$ is the set of all deflectors for the model $(S^r, \mathbb{G})$ as defined in Definition 2.1. Our results about the solution to (23) is delivered through two main theorems. The second theorem provides the full, complete characterizations of the solution in different manners, while the first theorem allows us to simplify the optimization problem in order to apply Subsection 2.3. This latter theorem requires the following notations and definitions, that were initially introduced in [18].
Definition 3.1. Let \( N \) be an \( \mathbb{H} \)-local martingale such that \( 1 + \Delta N > 0 \).
1) We call a Hellinger process of order zero for \( N \), denoted by \( h^{(0)}(N, \mathbb{H}) \), the process \( h^{(0)}(N, \mathbb{H}) := (H^{(0)}(N, \mathbb{H}))_p^\mathbb{H} \) when this projection exists, where
\[
H^{(0)}(N, \mathbb{H}) := \frac{1}{2} (N^c)^\mathbb{H} + \sum (\Delta N - \ln(1 + \Delta N)).
\] (24)
2) We call an entropy-Hellinger process for \( N \), denoted by \( h^{(E)}(N, \mathbb{H}) \), the process \( h^{(E)}(N, \mathbb{H}) := (H^{(E)}(N, \mathbb{H}))_p^\mathbb{H} \) when this projection exists, where
\[
H^{(E)}(N, \mathbb{H}) := \frac{1}{2} (N^c)^\mathbb{H} + \sum ((1 + \Delta N) \ln(1 + \Delta N) - \Delta N).
\] (25)

Below, we elaborate our first result of this subsection.

Theorem 3.2. The following holds.
\[
\inf_{Z^G \in \mathcal{D}(S', \mathcal{G})} E[- \ln(Z_T^G)] = \inf_{Z \in \mathcal{D}(S, \mathcal{F})} E[- \ln(Z_{T \wedge T}/\mathcal{E}_{T \wedge T}(G_{-1}^1 \cdot m))].
\] (26)

Proof. Thanks to Theorem 2.3.(c), we deduce that \( Z^T/\mathcal{E}(G_{-1}^1 \cdot m)^\tau \) belongs to \( \mathcal{D}(S', \mathcal{G}) \) for any \( Z \in \mathcal{D}(S, \mathcal{F}) \), and the following inequality holds
\[
\inf_{Z^G \in \mathcal{D}(S', \mathcal{G})} E[- \ln(Z_T^G)] \leq \inf_{Z \in \mathcal{D}(S, \mathcal{F})} E[- \ln(Z_{T \wedge T}/\mathcal{E}_{T \wedge T}(G_{-1}^1 \cdot m))].
\]

To prove the reverse inequality, we consider \( Z^G \in \mathcal{D}(S', \mathcal{G}) \), and apply Theorem 2.3. This implies the existence of a triplet \((Z^F, \varphi^{(o)}, \varphi^{(pr)})\) that belongs to \( \mathcal{D}(S, \mathcal{F}) \times \mathcal{I}_loc^o(N^G, \mathcal{G}) \times L^1_loc(\text{Prog}(\mathcal{F}), \mathcal{P} \otimes \mathcal{D}) \) and satisfies
\[
\varphi^{(pr)} > -1, \quad P \otimes \mathcal{D} - a.e., \quad \frac{\tilde{G}}{G} < \varphi^{(o)} \leq \frac{\tilde{G}}{G - \tilde{G}}, \quad P \otimes \mathcal{D}^{0, \mathcal{F}} - a.e.,
\]
and
\[
Z^G = \frac{(Z^F)^\tau}{\mathcal{E}(G_{-1}^1 \cdot m)^\tau} \mathcal{E}(\varphi^{(o)} \cdot N^G) \mathcal{E}(\varphi^{(pr)} \cdot D).
\]

As a result, we get
\[
- \ln(Z^G) = - \ln \left( (Z^F)^\tau/\mathcal{E}(G_{-1}^1 \cdot m)^\tau \right) - \ln(\mathcal{E}(\varphi^{(o)} \cdot N^G)) - \ln(\mathcal{E}(\varphi^{(pr)} \cdot D)).
\]

Thus, in virtue of Proposition 3.2, the process \(- \ln(Z^G)\) is uniformly integrable iff \(- \ln((Z^F)^\tau/\mathcal{E}(G_{-1}^1 \cdot m)^\tau)\), \(- \ln(\mathcal{E}(\varphi^{(o)} \cdot N^G))\) and \(- \ln(\mathcal{E}(\varphi^{(pr)} \cdot D))\) are uniformly integrable, and hence
\[
E[- \ln(Z_T^G)] \geq E \left[ - \ln \left( Z_{T \wedge T}^F/\mathcal{E}_{T \wedge T}(G_{-1}^1 \cdot m) \right) \right]
\]
\[
\geq \inf_{Z \in \mathcal{D}(S, \mathcal{F})} E \left[ - \ln(Z_{T \wedge T}/\mathcal{E}_{T \wedge T}(G_{-1}^1 \cdot m)) \right].
\]

The first inequality is due to the fact that both \( E[- \ln(\mathcal{E}(\varphi^{(o)} \cdot N^G))] \) and \( E[- \ln(\mathcal{E}(\varphi^{(pr)} \cdot D))] \) are nonnegative. Therefore, the proof of the theorem follows immediately. \(\square\)
We end this section by describing explicitly the optimal deflator for the model \((S^r, \mathbb{G}, \ln)\). This requires the predictable characteristics of \((S, \mathbb{F})\) and/or that of \((S^r, \mathbb{G})\). Thus, throughout the rest of the paper, for the sake of simplicity, the random measure \(\mu_S\) associated with the jumps of \(S\) will be denoted for simplicity by \(\mu\), while \(S^c\) denotes the continuous \(\mathbb{F}\)-local martingale part of \(S\), and the quadruplet

\[
(b, c, F, A) \text{ are the predictable characteristics of } (S, \mathbb{F}).
\]

Or equivalently the canonical decomposition of \(S\) (see Theorem 2.34, Section II.2 of [31] for details) is given by

\[
S = S_0 + S^c + h \ast (\mu - \nu) + b \ast A + (x - h) \ast \mu, \quad h(x) := xI_{|x| \leq 1}.
\]  

(27)

Throughout the rest of this section, we consider Jacod’s decomposition for the \(\mathbb{F}\)-martingale \(G^{-1} \ast m\) and the space \(\mathcal{L}(S, \mathbb{F})\) given by

\[
G^{-1} \ast m = \beta_m \ast S^c + (f_m - 1) \ast (\mu - \nu) + g_m \ast \mu + m_-, \quad \mathcal{L}(S, \mathbb{F}) := \left\{ \theta \in \mathcal{P}(\mathbb{F}) \mid 1 + x^{tr} \theta_1(\omega) > 0 \quad P\otimes F_t \otimes dA_t \text{-a.e}\right\}.
\]  

(29)

**Theorem 3.3.** Let \(K_{\log}(\cdot)\) be given by (14), and suppose \(S\) is quasi-left-continuous and \(\sigma\)-special, and \(G > 0\). Then the following assertions are equivalent.

(a) There exist \(K \in \mathcal{M}_b,\text{loc}(\mathbb{F})\) and a nondecreasing and \(\mathbb{F}\)-predictable process \(V\) such that \(\mathcal{E}(K)\exp(-V) \in \mathcal{D}(S, \mathbb{F})\) and the nondecreasing process

\[
G_\ast V + G_\ast h^F(G^{-1} \ast m, P) - (K, m)^F - \left(\mathcal{G} \ast H^{(0)}(K, P)\right)^{p,F},
\]

is integrable.

(b) The set \(\mathcal{D}_{\log}(S^r, \mathbb{G})\) is not empty.

(c) There exists a unique \(\mathcal{E}^G \in \mathcal{D}_{\log}(S^r, \mathbb{G})\) such that

\[
\min_{Z \in \mathcal{D}_{\log}(S^r, \mathbb{G})} E[-\ln(Z_T)] = E[-\ln(\tilde{Z}^G_T)].
\]  

(31)

(d) There exists \(\tilde{\lambda} \in \mathcal{L}(S, \mathbb{F})\) such that, for any \(\theta \in \mathcal{L}(S, \mathbb{F})\), the following hold

\[
E \left[(G_\ast \tilde{V})_T + G_\ast \left(\int f_m(x)K_{\log}(\tilde{\lambda}^{tr}x)F(dx) + \tilde{\lambda}^{tr}c\tilde{\lambda}\right) \cdot A_T\right] < +\infty,
\]

(32)

\[
(\theta - \tilde{\lambda})^{tr} \left[b + c(\beta_m - \tilde{\lambda}) + \int_{\mathbb{R}^d \setminus \{0\}} \frac{f_m(x)}{1 + \tilde{\lambda}^{tr}x} x - h(x) F(dx)\right] \leq 0,
\]

(33)

\[
\tilde{V} := \tilde{\lambda}^{tr} \left[b + c(\beta_m - \tilde{\lambda}) + \int_{\mathbb{R}^d \setminus \{0\}} \frac{f_m(x)}{1 + \tilde{\lambda}^{tr}x} x - h(x) F(dx)\right] \cdot A.
\]

(34)

If furthermore one of the above assertions holds, then \(\tilde{Z}^G\) solution to (31) and the process \(\tilde{\lambda}\) of assertion (d) are related via

\[
\tilde{Z}^G = \mathcal{E}(\tilde{K}^G)\exp(-\tilde{V}^r), \quad \text{where } \tilde{K}^G := \mathcal{T}(\tilde{K}^G) - G^{-1} \ast \mathcal{T}(m), \quad \text{and}
\]

\[
K^F := (\beta_m - \tilde{\lambda}) \ast S^c + \frac{f_m - 1 - \tilde{\lambda}^{tr}x}{1 + \tilde{\lambda}^{tr}x} \ast (\mu - \nu) + \frac{g_m}{1 + \tilde{\lambda}^{tr}x} \ast \mu + m_-.
\]  

(35)

(36)
Proof. The proof of (b)⇐⇒(c) is a direct application of Theorem 2.4 for the model \((X, \mathbb{H}) = (S^r, G)\). Thus, the remaining part of the proof will be achieved in three steps. The first step proves the equivalence \((a)⇐⇒(b)\), the second step proves \((b)⇐⇒(d)\) and the last step proves \((35)-(36)\).

**Step 1.** Here we prove \((a)⇐⇒(b)\). Let \(Z^G \in \mathcal{D}(S^r, G)\). Thus, thanks to Theorem 2.4 there exist two \(G\)-local martingales \(\mathcal{E}(\varphi^{(0)} \cdot N^G)\) and \(\mathcal{E}(\varphi^{(pr)} \cdot D)\) and \(Z^G \equiv \mathcal{E}(K) \exp(-V) \in \mathcal{D}(S, F)\), where \(K \in \mathcal{M}_{0,loc}(F)\) and \(V\) is an nondecreasing and \(F\)-predictable process, such that

\[
Z^G = \frac{(Z^F)^r}{\mathcal{E}(G^{-1} \cdot m)^r} \mathcal{E}(\varphi^{(0)} \cdot N^G) \mathcal{E}(\varphi^{(pr)} \cdot D).
\]

As a result, we obtain

\[
-\ln(Z^G) = -\ln((Z^F)^r/\mathcal{E}(G^{-1} \cdot m)^r) - \ln(\mathcal{E}(\varphi^{(0)} \cdot N^G)) - \ln(\mathcal{E}(\varphi^{(pr)} \cdot D)).
\]

Thanks to Proposition B.2, we deduce that \(Z^G \in \mathcal{D}_{log}(S^r, G)\) if and only if the three \(G\)-local martingale \((Z^F)^r/\mathcal{E}(G^{-1} \cdot m)^r\), \(\mathcal{E}(\varphi^{(0)} \cdot N^G)\) and \(\mathcal{E}(\varphi^{(pr)} \cdot D)\) belong to \(\mathcal{D}_{log}(S^r, G)\). Then by combining this with

\[
-\ln((Z^F)^r/\mathcal{E}(G^{-1} \cdot m)^r) = G\text{-local mart.} - \frac{I_{[0, \tau]} \cdot \langle K, m \rangle^F}{G^{-1}} + H^{(0)}(K, P)^\tau
\]

we conclude that the process in the RHS term is nondecreasing and \(G\)-integrable, or equivalently its \(F\)-predictable dual projection (\(F\)-compensator) is a nondecreasing and integrable process. This resulting predictable process coincides with the process defined in (30) due to

\[
\left(\frac{I_{[0, \tau]} \cdot \langle m \rangle^F}{G^{-1}} - H^{(0)}(G^{-1} \cdot m, P)^\tau\right)^{p,F} = \frac{1}{G^{-1}} \cdot \langle m \rangle^F - \left(\tilde{G} \cdot H^{(0)}(G^{-1} \cdot m, P)^\tau\right)^{p,F}
\]

\[
= \frac{1}{2G^{-1}} \cdot \langle m \rangle^F + G^{-1} \cdot \left(\sum \left(\frac{\Delta m}{G^{-1}}\right)^2\right)^{p,F} - \left(\sum \tilde{G} \cdot \left(\Delta m - \ln(1 + \frac{\Delta m}{G^{-1}})\right)\right)^{p,F}
\]

\[
= \frac{1}{2G^{-1}} \cdot \langle m \rangle^F + \left(\sum (\Delta m + G^{-1} \ln(1 + \frac{\Delta m}{G^{-1}})) - \Delta m\right)^{p,F}
\]

\[
= G^{-1} \cdot h^F(G^{-1} \cdot m, P).
\]

This ends the proof of the equivalence between assertions (a) and (b).

**Step 2.** Here we prove \((b)⇐⇒(d)\) using Theorem 2.4. To this end, we start deriving the predictable characteristics of \((S^r, G)\), denoted by \((b^G, c^G, F^G, A^G)\) and are given by

\[
b^G := b + c \beta_m + \int h(x) f_m(x) - 1) F(dx), \quad \mu^G := \int_{[0, \tau]} \mu, \quad c^G := c
\]

\[
\nu^G := \int_{[0, \tau]} f_m * \nu, \quad F^G(dx) := \int_{[0, \tau]} f_m(x) F(dx), \quad A^G := A^r.
\]

Thus, by directly applying Theorem 2.4 to the model \((S^r, G)\), we deduce \(\mathcal{D}_{log}(S^r, G) \neq \emptyset\) is equivalent to the existence of a \(G\)-predictable process \(\varphi \in \mathcal{L}(S^r, G)\) satisfying

\[
E \left[ V^G_T + \frac{1}{2} \mathcal{E}(\varphi^{(r)} \cdot \varphi \cdot A^G) \right] < +\infty,
\]

\[
V^G := |\varphi^{(r)} b^G - \varphi^{(r)} c^G | + \int f_m(x) - \varphi^{(r)} h(x) F^G(dx) \cdot A^G,
\]

\[
(\theta - \varphi)^{tr} (b^G - c^G) + \int f_m(x) - \varphi^{(r)} h(x) F^G(dx) \leq 0.
\]
for any bounded $\theta \in \mathcal{L}(S^T, \mathbb{G})$. Thanks to Lemma A.1, we deduce the existence of $\tilde{\lambda} \in \mathcal{L}(S, \mathbb{F})$ such that $\varphi I_{[0,\tau]} = \tilde{\lambda} I_{[0,\tau]} P \otimes A$-a.e.. Thus, by inserting this in (38)-(39)-(40), we conclude that $V^G = \tilde{V}^\tau \in \mathcal{A}_{loc}^\tau(\mathbb{G})$, which is equivalent to (31) due to Lemma A.1 and

$$
E \left[ (G - \tilde{V})_T + \frac{1}{2} (G - \tilde{\lambda}^r c^{\tilde{\lambda}} \cdot A)_T + \int \mathcal{K}_{log}(\tilde{\lambda}^r x) f_m(x) F(dx) G - A T \right] < +\infty,
$$

$$
(\theta - \tilde{\lambda})^r (b + c(\beta_m - \tilde{\lambda})) + \int \left( \frac{(\theta - \tilde{\lambda})^r x}{1 + \lambda^r x} f_m(x) - (\theta - \tilde{\lambda})^r h(x) \right) F(dx) \leq 0,
$$

$P \otimes A$-a.e. on $[0,\tau]$ for any bounded $\theta \in \mathcal{L}(S, \mathbb{F})$. The above first inequality is obviously (32), while (33) follows immediately from combining the second inequality above and Lemma A.1 again. This proves (b)$\Rightarrow$(d), while the converse follows from the fact that assertion (d) implies (38)-(39)-(40) with $\varphi = \tilde{\lambda} I_{[0,\tau]}$. This latter fact is obviously equivalent to assertion (b) due to Theorem 2.4 as stated above. This ends the second step, and the proof of the theorem is complete.

4 Optimal portfolio for $(S^\tau, \mathbb{G}, \ln)$: The general framework

This section addresses the optimal portfolio for the economic model $(S^\tau, \mathbb{G}, \ln)$ when $S$ is a general d-dimensional $\mathbb{F}$-semimartingale that is locally bounded and quasi-left-continuous. This is for the sake of simplifying the notation and avoiding the technicalities only. These two assumption can be definitely removed at the expenses of some technicalities that we are avoiding herein. Below, we elaborate the main result of this section that characterizes, in different manners using the processes under $\mathbb{F}$ only, the existence of the optimal portfolio for the model $(\ln, S^\tau, \mathbb{G})$.

**Theorem 4.1.** Let $\mathcal{K}_{log}(\cdot)$ be given by (14). Suppose $G > 0$, $S$ is quasi-left-continuous and $\sigma$-special, and $\mathcal{D}(S, \mathbb{F}) \neq \emptyset$. Then the following are equivalent.

(a) There exists $\tilde{\theta}^G \in \Theta(S^\tau, \mathbb{G})$ such that

$$
\max_{\theta \in \Theta(S^\tau, \mathbb{G})} E \left[ \ln (1 + (\theta \cdot S^T)_T) \right] = E \left[ \ln \left( 1 + (\tilde{\theta}^G \cdot S^T)_T \right) \right] < +\infty. \quad (41)
$$

(b) There exist $K \in \mathcal{M}_{loc}(\mathbb{F})$ and a nondecreasing and $\mathbb{F}$-predictable process $V$ such that $\mathcal{E}(K) \exp(-V) \in \mathcal{D}(S, \mathbb{F})$, and

$$
E \left[ (G - V)_T + (\tilde{G} \cdot H^{(0)}(K, P))_{\mathbb{F}T}^T + (G - h E \left( \frac{1}{G - m} \cdot m, P \right))_T + (K, m)_{\mathbb{F}T}^T \right] < +\infty.
$$

(c) There exists $\tilde{\lambda} \in \mathcal{L}(S, \mathbb{F})$ such that, for any $\theta \in \mathcal{L}(S, \mathbb{F})$, the following hold

$$
E \left[ (G - \tilde{V})_T + (G - \tilde{\lambda}^r c^{\tilde{\lambda}} \cdot A)_T + (G - \mathcal{K}_{log}(\tilde{\lambda}^r x) * \nu)_T \right] < +\infty, \quad (42)
$$

$$
(\theta - \tilde{\lambda})^r (b + c(\beta_m - \tilde{\lambda})) + \int (\theta - \tilde{\lambda})^r \left[ \frac{f_m(x)}{1 + \lambda^r x} x - h(x) \right] F(dx) \leq 0, \quad (43)
$$

$$
\tilde{V} := \tilde{\lambda}^r b + \tilde{\lambda}^r c(\beta_m - \tilde{\lambda}) + \int \tilde{\lambda}^r \left[ \frac{f_m(x)}{1 + \lambda^r x} x - h(x) \right] F(dx) \cdot A. \quad (44)
$$

Furthermore, we have $\tilde{\theta}^G (1 + (\tilde{\theta}^G \cdot S^T)_-)^{-1} = \tilde{\lambda}$ on $[0,\tau]$. 

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The proof of this theorem follows immediately from combining Theorem 2.4 and Theorem 3.3 and hence it will be omitted herein.

The following is a consequence of the above theorem, and naturally connects the optimal portfolio for \((S^\tau, \mathbb{G}, \ln)\) with the optimal portfolio for \((S, \mathbb{F}, U)\) where \(U\) is a random field utility that will be specified.

**Theorem 4.2.** Suppose \(G > 0\) and \(\mathcal{E}(G^{-1} \cdot m)\) is a martingale, and consider \(Q := \mathcal{E}_T(G^{-1} \cdot m) \cdot P\). Then the following are equivalent.

(a) The optimal portfolio for \((\ln, S^\tau, \mathbb{G})\) exists
(b) \(\tilde{U}(t,x) := \mathcal{E}(G^{-1} \cdot m) \ln(x)\) for any \(x > 0\).
(c) The model \((\ln, S, Q, \mathbb{F})\) admits the optimal portfolio.

Furthermore, the three portfolios coincide on \([0,T]\) when they exist.

**Proof.** It is clear that (b) \(\iff\) (c) is obvious. Thus the remaining part of this proof focuses on proving (a) \(\iff\) (c). This follows as a direct application of Theorems 2.4-4.1 as follows. To this end, we start by noticing that

\[
\nu^Q(dt, dx) = f_m(x)\nu(dt, dx),
\]

and

\[
S = S_0 + (S^c \cdot (S^\tau, G^{-1} \cdot m)^F) + (h \star (\mu - \nu) - (h \star (\mu - \nu), G^{-1} \cdot m)^F) + c\beta_m \cdot A + \int (f_m(x) - 1)h(x)F(dx) + c\beta_m \cdot A
\]

Then the predictable characteristics of \((S, \mathbb{F})\) under \(Q\), will be denoted by \((b^Q, c^Q, F^Q, A^Q)\), and are given by

\[
b^Q := b + \int (f_m(x) - 1)h(x)F(dx) + c\beta_m, \quad c^Q := c, \quad F^Q := f_m \cdot F, \quad A^Q := A.
\]

Therefore, using these characteristics and applying Theorem 2.4, we deduce that \((S, Q, \mathbb{F}, \ln)\) admits the optimal portfolio if and only if assertion (d) of Theorem 3.3 (or assertion (c) of Theorem 4.1) holds, which is equivalent to the existence of the optimal portfolio for \((S^\tau, \mathbb{G}, \ln)\). This ends the proof of the theorem.

**Theorem 4.3.** Suppose \(G > 0\), \(S\) is quasi-left-continuous and \(\sigma\)-special, and \((\ln, S, \mathbb{F})\) admits the optimal portfolio. Then the optimal portfolio for the model \((\ln, S^\tau, \mathbb{G})\) exists if and only

\[
E \left[ (G_\tau \cdot h^F(G^{-1} \cdot m, P)_T) \right] < +\infty.
\]

Here \(h^F(N, P)\) is given by Definition 3.1, for any \(N \in \mathcal{M}_{0,\text{loc}}(\mathbb{F})\) such that \(1 + \Delta N \geq 0\).

**Proof.** Due to Theorem 2.4 \((\ln, S, \mathbb{F})\) admits the optimal portfolio if and only if there exists a deflator \(Z := \mathcal{E}(K) \exp(-V)\), where \(K \in \mathcal{M}_{0,\text{loc}}(\mathbb{F})\) and \(V\) is a nondecreasing and \(\mathbb{F}\)-predictable process, such that

\[
E[-\ln(Z_T)] = E[V_T + H^0(K, P)_T] < +\infty.
\]

Thus, due to Lemma 3.1 we conclude that \(\sqrt{K^2} \cdot K\) is an integrable process (or equivalently \(K\) is a martingale such that \(\sup_{0 \leq t \leq T} |K_t| \in L^1(P)\)), and hence the process \((K, m)^F\) has integrable variation as \(m\) is a BMO martingale. Therefore, a combination of these with Theorem 3.3 we deduce that the optimal portfolio for the model \((\ln, S^\tau, \mathbb{G})\) exists if and only if (45) holds. This ends the proof of the theorem.

\[\Box\]
Theorem 5.1. Suppose $G > 0$, $S$ is quasi-left-continuous and $\sigma$-special, and
\[
\mathbb{E} \left[ (G_\cdot \cdot h^E(G_\cdot^{-1} \cdot m, P))_T \right] < +\infty.
\]
Then the optimal portfolio for $(\ln, S^\tau, G)$ exists iff there exist $K \in M_{0,loc}(\mathbb{F})$ and a nondecreasing and $\mathbb{F}$-predictable $V$ such that $\mathcal{E}(K) \exp(-V) \in \mathcal{D}(S, \mathbb{F})$ and $G_\cdot \cdot V + \langle K, m \rangle^\mathbb{F} + G_\cdot \cdot H^0(0, K, P)$ has integrable variation.

The proof follows immediately from Theorem 4.1.

Corollary 4.4. Suppose that $G > 0$, and $S$ is $\sigma$-special and quasi-left-continuous. Then the following conditions are all sufficient for the fact that the optimal portfolio for $(S^\tau, G, \ln)$ exists if and only if the optimal portfolio for $(S, \mathbb{F}, \ln)$ does also, and both portfolios coincide on $[0, \tau]$ when they exist.

(a) $\tau$ is a pseudo-stopping time (i.e. $\mathbb{E}[M_t] = \mathbb{E}[M_0]$ for any bounded $\mathbb{F}$-martingale)
(b) $\tau$ is independent of $\mathbb{F}$.
(c) Every $\mathbb{F}$-martingale is a $G$-local martingale (i.e. immersion holds).

Proof. Under condition (a) of the corollary, the proof of the claim follows immediately from combining Theorem 4.2 with the fact that $\mathcal{E}(G_\cdot^{-1} \cdot m) \equiv 1$ or equivalently $\tilde{U}(t, x) \equiv \ln(x)$). This latter fact can be found in [46]. Finally, it is easy to see that condition (a) is implied by either conditions (b) or (c). This ends the proof of the corollary.

5. The case when $S$ follows a jump-diffusion model

This section illustrates Sections 3-4 on the case where the uncertainties in the initial model $(S, \mathbb{F})$ is a one-dimensional process generated by Poisson process and a Brownian motion. Precisely, we suppose that a standard Brownian motion $W$ and a Poisson process $N$ with intensity $\lambda > 0$ are defined on the probability space $(\Omega, \mathcal{F}, P)$, the filtration $\mathbb{F}$ is the completed and right continuous filtration generated by $W$ and $N$, and the stock price process is given by the following dynamics.

\[
S_t := S_0 \mathcal{E}(X)_t, \quad X_t := \int_0^t \sigma_s dW_s + \int_0^t \xi_s dN^\mathbb{F}_s + \int_0^t \mu_s ds, \quad N^\mathbb{F}_t := N_t - \lambda t,
\]

and there exists a constant $\delta \in (0, +\infty)$ such that $\mu, \sigma$ and $\xi$ are bounded adapted processes satisfying the following
\[
\sigma > 0, \quad \xi > -1, \quad \sigma + |\xi| \geq \delta, \quad P \otimes dt\text{-a.e.}. \tag{47}
\]

Since $m$ is an $\mathbb{F}$-martingale, then there exists two $\mathbb{F}$-predictable processes $\varphi^{(m)}$ and $\psi^{(m)}$ such that $\int_0^t \left( (\varphi^{(m)})^2 + |\psi^{(m)}|^2 \right) ds < +\infty$ $P$-a.s., and
\[
G_\cdot^{-1} \cdot m = \varphi^{(m)} \cdot W + (\psi^{(m)} - 1) \cdot N^\mathbb{F}. \tag{48}
\]

Theorem 5.1. Suppose $G > 0$ and $S$ and $X$ are given by (46)-(47). Then the following $\mathbb{F}$-predictable process
\[
\tilde{\theta} := \xi + \text{sign}(\xi) \sqrt{\xi^2 + 4 \lambda \psi^{(m)}} \over 2\sigma - \frac{1}{\xi}, \quad \text{where} \quad \xi := \frac{\mu - \lambda \xi}{\sigma} + \varphi^{(m)} + \frac{\sigma}{\xi}, \tag{49}
\]
is $S$-integrable satisfying $1 + \tilde{\theta} \xi > 0$ $P \otimes dt$-a.e., and the following hold.

(a) The solution to
\[
\min_{Z \in \mathcal{D}(S^\tau, G)} \mathbb{E} \left[ -\ln(Z_T) \right] = \mathbb{E} \left[ -\ln(\tilde{Z}_T^\mathbb{F}) \right] < +\infty, \tag{50}
\]
exists and is given by \(\tilde{Z}^G := \mathcal{E}(\tilde{K}^G)\) where

\[
\tilde{K}^G := -\sigma \theta \cdot T(W) - \frac{\psi(m) \zeta \tilde{\theta}}{1 + \theta \zeta} \cdot T(N^F).
\] (51)

(b) It holds that

\[
\max_{\theta \in \Theta(S^\ast, \mathcal{G})} \mathbb{E}[\ln(\mathcal{E}_T(\theta \cdot X^T))] = \mathbb{E}[\ln(\mathcal{E}_T(\tilde{\theta} \cdot X^T))].
\] (52)

(c) It holds that

\[
\max_{\theta \in \Theta(\mathcal{S}, \mathcal{F})} \mathbb{E}[\ln(\mathcal{E}_T(\theta \cdot \overline{X}))] = \mathbb{E} \left[ \ln \left( \mathcal{E}_T \left( \frac{\tilde{\theta}}{\psi(m)} \cdot \overline{X} \right) \right) \right] < +\infty,
\] (53)

where \(\mathcal{S} := S_0 \mathcal{E}(\overline{X})\), \(\overline{X}_0 = 0\), and

\[
d\overline{X} := \sqrt{\psi(m)} \sigma dW + \psi(m) \zeta dN^F + \left[ \lambda \zeta (\psi(m) - 1) + \mu + \sigma \varphi(m) (1 - \sqrt{\psi(m)}) \right] ds.
\] (54)

**Proof.** This proof is achieved in three steps. The first step proves assertions (a) and (b), while step 2 proves assertion (c).

**Step 1.** For the model (46)-(47), the predictable characteristics of Section 3 can be derived as follows. Let \(\delta_a(dx)\) be the Dirac mass at the point \(a\). Then in this case we have \(d = 1\) and

\[
\mu(dt, dx) = \delta_G S_{\perp}(dx) dN_t, \quad \nu(dt, dx) = \delta_G S_{\perp}(dx) \lambda dt, \quad F_t(dx) = \lambda \delta_G S_{\perp}(dx),
\]

\[
A_t = t, \quad c = (S_\cdot \sigma)^2, \quad b = (\mu - \lambda \zeta I_{\{\psi > 1\}}) S_\cdot, \quad (\beta_{m,G}, g_m, m^+) = \left( \frac{\varphi(m)}{S_\cdot \sigma}, 0, 0 \right).
\]

As a result, the set

\[
\mathcal{L}_{(\omega, t)}(S, \mathcal{F}) := \{ \varphi \in \mathbb{R} \mid \varphi x > -1 \} = \{ \varphi \in \mathbb{R} \mid \varphi_{S_\cdot \zeta} > -1 \} = (-1/(S_\cdot \zeta)^+, +\infty) \cap (-\infty, 1/(S_\cdot \zeta)^-)
\]

is an open set in \(\mathbb{R}\) (with the convention \(1/0^+ = +\infty\)). Then the condition (43), characterizing the optimal portfolio \(\varphi\), becomes an equation as follows.

\[
0 = \mu - \lambda \zeta I_{\{\psi > 1\}} S_{-\sigma^2} \left( \frac{\varphi(m)}{S_{-\sigma}} \cdot \theta \right) + \lambda \frac{\psi(m) \zeta}{1 + S_{-\sigma} \theta \zeta} - \lambda \zeta I_{\{|\psi| \leq 1\}}
\]

\[
= \mu - \lambda \zeta + \sigma \varphi(m) - S_{-\sigma^2} \theta + \frac{\psi(m) \lambda \zeta}{1 + \theta S_{-\zeta}}.
\] (55)

By changing the variable and putting \(\varphi := 1 + \theta S_{-\zeta} > 0\), the above equation is equivalent to

\[
0 = -\frac{\sigma^2}{\zeta} \varphi^2 + [\mu - \lambda \zeta + \sigma \varphi(m) + \frac{\sigma^2}{\zeta}] \varphi + \psi(m) \lambda \zeta.
\]

This equation has always (since \(\psi(m) > 0\)) a unique positive solution

\[
\check{\varphi} := \frac{\Gamma \zeta + |\psi| \sqrt{T^2 + 4 \sigma^2 \lambda \psi(m)}}{2 \sigma^2}, \quad \Gamma := \mu - \lambda \zeta + \sigma \varphi(m) + \frac{\sigma^2}{\zeta}.
\]
Hence, we deduce that $\tilde{\lambda} := \frac{\tilde{\theta}}{S}$, where $\tilde{\theta}$ is given by (49), coincides with $(\tilde{\varphi} - 1)/(S - \zeta)$, satisfies $1 + \zeta \tilde{\theta} > 0$, and hence it is the unique solution to (55). It is also clear that $\tilde{\theta}$ is $S$-integrable (or equivalently $\lambda$ is $S$-integrable) due to the assumptions in (47)-(48). As a result, the optimal wealth process is $\mathcal{E}(\lambda \cdot S^r) = \mathcal{E}(\tilde{\theta} \cdot X^r)$ and hence $\tilde{\theta}$ is the solution to (55) and assertions (a) and (b) follow immediately using the above analysis and Theorems 4.1.

**Step 2.** Herein, we prove assertion (c) using Theorem 2.4. Tom this end, we calculate the random measure jumps $\mathfrak{p}$ and its compensator $\mathfrak{p}$, and the predictable characteristics $(\mathfrak{b}, \mathfrak{r}, \mathfrak{F}, \mathfrak{A})$ for the model $(\widetilde{S}, \mathfrak{F})$ as follows.

$$
\mathfrak{p}(dt, dx) := \mu_S(dt, dx) = \delta_{\zeta \psi_{\tilde{\varphi}}(m)_{S_{t^-}}}(dx) d\mathcal{N}_t, \quad \mathfrak{r}(dt, dx) = \delta_{\zeta \psi_{\tilde{\varphi}}(m)_{S_{t^-}}}(dx) \lambda dt
$$

$$
\mathfrak{b} = \mathfrak{S}_{-}(\mu - \lambda \zeta + \lambda \psi(m) \zeta I_{\{\zeta S_{-} \psi(m) \leq 1 \}} + \sigma \varphi(m)(1 - \sqrt{\psi(m))}), \quad \mathfrak{A}_t = t,
$$

$$
\mathfrak{F}_t(dx) = \lambda \delta_{\zeta \psi_{\tilde{\varphi}}(m)_{S_{t^-}}}(dx), \quad \mathfrak{r} = \psi(m)(\mathfrak{S}_{-}\sigma)^2, \quad \beta_m = \frac{\varphi(m)}{\mathfrak{S}_{-}\sigma \sqrt{\psi(m)}}, \quad m \perp \equiv 0.
$$

Then similarly as in the first step, we deduce that the set $\mathcal{L}_{(\omega, t)}(\mathfrak{S}, \mathfrak{F})$ is an open real set (since $\mathcal{L}_{(\omega, t)}(\mathfrak{S}, \mathfrak{F}) = (-\langle \mathfrak{S}_{-} \psi(m) \zeta^{-} \rangle^{-1}, +\infty) \cap (-\infty, (\mathfrak{S}_{-} \psi(m) \zeta^{-} \rangle^{-1})$) and hence the equation (18) becomes

$$
0 = \mathfrak{b} + \mathfrak{r}(\beta_m - \varphi) + \int \frac{x}{1 + \varphi x} - h(x))\mathfrak{F}(dx).
$$

This is equivalent to

$$
0 = \mu - \lambda \zeta + \sigma \varphi(m) - \sigma^2 \psi(m) \mathfrak{S}_{-}\varphi + \frac{\varphi(m) \lambda \zeta}{1 + \zeta \psi(m) \mathfrak{S}_{-}\varphi}.
$$

Thus, by comparing this equation to (55), we deduce that the optimal strategy for the problem (55), that we denote by $\tilde{\theta}$ satisfies

$$
\tilde{\theta} = \mathfrak{S}_{-}\varphi = S_{-}\tilde{\varphi} = \frac{\tilde{\theta}}{\psi(m)}.
$$

(Where $\varphi$ is the root of the above equation). This ends the proof of assertion (c), and the proof of the theorem as well.

For other related discussions on the quantification of the impact of $\tau$ on the log-optimal portfolio in this setting of jump-diffusion model, and/or more particular models and examples, such as discrete-time, general Lévy models, and volatility models, we refer the reader to [51].

### 6 Numéraire portfolio under random horizon

This section addresses the impact of $\tau$ on the numéraire portfolio. To this end, we start by the mathematical definition of this financial concept.

**Definition 6.1.** Let $(X, \mathbb{H}, P)$ be a model and $Q$ be a probability measure such that $Q \ll P$. We call the numéraire portfolio, for the model $(X, \mathbb{H}, Q)$ when it exists, the unique $\mathbb{H}$-predictable process $\tilde{\phi} \in L(X, \mathbb{H})$ such that $\mathcal{E}(\tilde{\phi} \cdot X) > 0$, and $\mathcal{E}(\phi \cdot X)/\mathcal{E}(\tilde{\phi} \cdot X)$ is a supermartingale under $Q$, for any $\phi \in L(X, \mathbb{H})$ satisfying $\mathcal{E}(\phi \cdot X) \geq 0$. When $Q = P$, we simply say numéraire portfolio for $(X, \mathbb{H})$. 


By comparing Definitions 2.1 and 6.1 it is clear that if the numéraire portfolio \( \tilde{\phi} \) for \((X, \mathbb{H})\) exists, then \( Z := 1/\mathcal{E}(\tilde{\phi} \cdot X) \) belongs to \( \mathcal{D}(X, \mathbb{H}) \).

It is known that this numéraire portfolio, that was initially introduced in [41], is intimately related to the notion of deflator (or local martingale deflator) in a way or another. The connection of the existence of numéraire portfolio to deflectors was first established by [35], see also [7, 15, 34] and the references therein for different proofs and/or related topics.

By taking into account a possible change of probability and or even a density, a natural extension of the above definition will be as follows.

**Definition 6.2.** Consider \((X, \mathbb{H}, P)\), and let \(Z\) be a positive \(\mathbb{H}\)-local martingale. We call numéraire portfolio for \((X, \mathbb{H}, Z)\), when it exists, the unique \(\tilde{\theta} \in L(X, \mathbb{H})\) such that \(\mathcal{E}(\tilde{\theta} \cdot X) > 0\), and the process \(Z \mathcal{E}(\phi \cdot X)/\mathcal{E}(\tilde{\theta} \cdot X)\) is a supermartingale, for any \(\phi \in L(X, \mathbb{H})\) satisfying \(\mathcal{E}(\phi \cdot X) \geq 0\).

**Remark 6.3.** In the definition above, it is enough to consider the test processes \(\phi \in L(X, \mathbb{H})\) such that \(\mathcal{E}(\phi \cdot X) > 0\). In fact, we consider \(\phi_0 \in L(X, \mathbb{H})\) such that \(\mathcal{E}(\phi_0 \cdot X) > 0\). Then for any \(\phi \in L(X, \mathbb{H})\) satisfying \(\mathcal{E}(\phi \cdot X) \geq 0\) and any \(\epsilon \in (0, 1]\) we have \(\phi_\epsilon := \epsilon \phi_0 + (1 - \epsilon)\phi\) belongs to \(L(X, \mathbb{H})\) satisfying \(\mathcal{E}(\phi_\epsilon \cdot X) > 0\) and \(\mathcal{E}(\phi_\epsilon \cdot X)\) converges to \(\mathcal{E}(\phi_0 \cdot X)\) when \(\epsilon\) goes to zero.

Below, we elaborate the principal result of this section.

**Theorem 6.4.** Let \(Z^{(m)} := \mathcal{E}(G^{-1}_- \cdot m)\). Then the numéraire portfolio for \((S^\tau, \mathbb{G}, P)\), denoted by \(\tilde{\theta}^{(G)}\), exists if and only if the numéraire portfolio for \((S, \mathbb{F}, Z^{(m)})\), denoted by \(\tilde{\theta}^{(F)}\), does exist also. Furthermore,

\[
\tilde{\theta}^{(G)} = \tilde{\theta}^{(F)} I_{[0, \tau]}.
\]

**Proof.** The proof is achieved in two parts, where we prove \((a) \implies (b)\) and its converse respectively.

**Part 1.** Suppose that the numéraire portfolio \(\tilde{\theta}^{(G)}\), for the model \((S^\tau, \mathbb{G}, P)\), exists. Then on the one hand, there exists an \(\mathbb{F}\)-predictable process \(\tilde{\theta}^{(F)}\) such that \(\tilde{\theta}^{(G)} I_{[0, \tau]} = \tilde{\theta}^{(F)} I_{[0, \tau]}\). On the other hand, for \(\theta \in L(S, \mathbb{F})\) such that \(\mathcal{E}(\theta \cdot S) > 0\), the process \(X := \mathcal{E}(\theta \cdot S^\tau)/\mathcal{E}(\tilde{\theta} \cdot S^\tau)\) is a positive supermartingale. Or equivalently

\[
\frac{\mathcal{E}(\theta \cdot S^\tau)}{\mathcal{E}(\tilde{\theta} \cdot S^\tau)} = \mathcal{E}(\theta \cdot S^\tau)\mathcal{E}(\tilde{\theta} \cdot S^\tau)^{-1} = \mathcal{E}(\theta \cdot S^\tau)\mathcal{E}
\left(-\tilde{\theta} \cdot S^\tau + \frac{1}{1 + \theta \Delta S^\tau} \cdot [\tilde{\theta} \cdot S^\tau]\right)

= \mathcal{E}
\left((\theta - \tilde{\theta}) \cdot S^\tau + \frac{\tilde{\theta} - \theta}{1 + \theta \Delta S^\tau} \cdot [S^\tau, \tilde{\theta} \cdot S^\tau]\right)

:= \mathcal{E}(L),
\]

and \(L\) is a \(\mathbb{G}\)-local supermartingale. Consider the decomposition for \(S\) given by \(S = S_0 + M + A + \sum \Delta S I_{\{\Delta S > 1\}}\), where \(M\) is a \(\mathbb{G}\)-local martingale with bounded jumps and \(A\) is a finite variation and predictable process. Therefore, we derive

\[
L = (\theta - \tilde{\theta}) \cdot (M^\tau + A^\tau) + \sum (\theta - \tilde{\theta}) \Delta S^\tau I_{\{\Delta S > 1\}} - \frac{\theta - \tilde{\theta}}{1 + \theta \Delta S^\tau} \cdot [S^\tau, \tilde{\theta} \cdot S^\tau]

= \mathbb{G}\)-local martingale + \(\theta - \tilde{\theta} I_{[0, \tau]} \cdot (M, m)^\mathbb{F} + W^\theta,
\]
where $W^\theta$ is given by

$$W^\theta := (\theta - \tilde{\theta}) \cdot \left\{ A^\tau + \sum \Delta S^\tau I_{\{\Delta S > 1\}} - \frac{1}{1 + \tilde{\theta} \Delta S^\tau} \cdot [S^\tau, \tilde{\theta} \cdot S^\tau] \right\}.$$ 

Thus, the process $L$ is a $\mathcal{G}$-local supermartingale if and only if $W^\theta \in \mathcal{A}_{loc}(\mathcal{G})$ and

$$\left( \frac{\langle \theta - \tilde{\theta} \rangle_{[0,\tau]} \cdot \langle M, m \rangle^F + W^\theta}{\mathcal{E}(\theta \cdot S)} \right)^{\mathcal{P},\mathcal{F}} \leq 0.$$ 

This is equivalent to

$$(\theta - \tilde{\theta}) \cdot \left\{ \langle M, m \rangle^F + G_- \cdot A \right\} +$$

$$(\theta - \tilde{\theta}) \cdot \left( \sum \tilde{G} \Delta S I_{\{\Delta S > 1\}} - \frac{\tilde{G}}{1 + \tilde{\theta} \Delta S} \cdot [S, \tilde{\theta} \cdot S] \right) \leq 0.$$  

(57)

Now, we derive

$$X := \mathcal{E}(G_-^{-1} \cdot m) \frac{\mathcal{E}(\theta \cdot S)}{\mathcal{E}(\theta \cdot S)} = \mathcal{E}(G_-^{-1} \cdot m) \mathcal{E} \left( \frac{\langle \theta - \tilde{\theta} \rangle \cdot S + \frac{\tilde{G}}{G_- - 1 + \tilde{\theta} \Delta S} \cdot [S, \tilde{\theta} \cdot S]}{\mathcal{E}(\theta \cdot S)} \right)$$

$$= \mathcal{E}(L_1),$$

where

$$L_1 := G_-^{-1} \cdot m + (\theta - \tilde{\theta}) \cdot S + \frac{\theta - \tilde{\theta}}{G_-} \cdot [m, S] + \frac{G_-}{G_- - 1 + \tilde{\theta} \Delta S} \cdot [S, \tilde{\theta} \cdot S]$$

$$= \mathcal{G}$-local martingale + $\frac{\theta - \tilde{\theta}}{G_-} \cdot (\langle M, m \rangle^F + G_- \cdot A)$$

$$+ (\theta - \tilde{\theta}) \cdot \left( \sum \tilde{G} \Delta S I_{\{\Delta S > 1\}} - \frac{\tilde{G}}{1 + \tilde{\theta} \Delta S} \cdot [S, \tilde{\theta} \cdot S] \right)^{\mathcal{P},\mathcal{F}}.$$ 

Thanks to (57), we deduce that $L_1$ is an $\mathcal{F}$-local supermartingale, and hence $X$ is a nonnegative $\mathcal{F}$-supermartingale. This proves assertion (b).

**Part 2.** Here we prove that assertion (b) implies assertion (a). Suppose that the numéraire portfolio for $(X, \mathcal{F}, Z)$ exists that we denote by $\tilde{\theta}(\mathcal{F})$. Then for any $\phi \in L(X, \mathcal{H})$ satisfying $\mathcal{E}(\phi \cdot X) > 0$, $X := Z \mathcal{E} \left( \mathcal{E}(G_-^{-1} \cdot m) \mathcal{E}(\phi \cdot S) / \mathcal{E}(\theta \cdot S) \right)$ is a positive supermartingale. On the one hand, it is known that there exists a local martingale $M$ and a nondecreasing and $\mathcal{F}$-predictable process $V$ such that $X = \mathcal{E}(M) \exp(-V)$. On the other hand, we have

$$\frac{\mathcal{E}(M)^\tau}{\mathcal{E}(G_-^{-1} \cdot m)^\tau} = \mathcal{E}(\mathcal{T}(M) - G_-^{-1} \cdot \mathcal{T}(m))$$

is a nonnegative $\mathcal{G}$-local martingale.

Therefore we easily conclude that $X^\tau / \mathcal{E}(G_-^{-1} \cdot m)^\tau$ is a nonnegative $\mathcal{G}$-supermartingale, or equivalently the process

$$\frac{\mathcal{E}(\phi \cdot S)^\tau}{\mathcal{E}(\theta \cdot S)^\tau} = \frac{\mathcal{E}(\phi \cdot S)^\tau}{\mathcal{E}(\theta \cdot S)^\tau} = \frac{X^\tau / \mathcal{E}(G_-^{-1} \cdot m)^\tau}{\mathcal{E}(\theta \cdot S)^\tau},$$

is a nonnegative $\mathcal{G}$-supermartingale. This ends the proof of theorem. \qed
Lemma B.1. and they are general and not technical at all. The results of this section are new and are very useful, especially the first lemma and the proposition, and hence \( E(G^{-1} \cdot m) \equiv 1 \). This ends the proof of the corollary. □

APPENDIX

A Some \( \mathcal{G} \)-properties versus those in \( \mathbb{F} \)

Lemma A.1. The following assertions hold.
(a) For any \( \theta \in \mathcal{L}(S^T, \mathcal{G}) \), there exists \( \varphi \in \mathcal{L}(S, \mathbb{F}) \) that coincides with \( \theta \) on \( [0, \tau] \).
(b) Let \( \nu \) be an \( \mathbb{F} \)-predictable process. If \( \nu I_{[0, \tau]} \leq 0 \) \( \mathbb{P} \)-a.e., then \( \nu \leq 0 \) \( \mathbb{P} \)-a.e.
(c) Let \( \varphi \) be a nonnegative and \( \mathbb{F} \)-predictable process. If \( \varphi < +\infty \) \( \mathbb{P} \)-a.e. on \( [0, \tau] \), then \( \varphi < +\infty \) \( \mathbb{P} \)-a.e.
(d) Let \( V \) be an \( \mathbb{F} \)-predictable and nondecreasing process that takes values in \( [0, +\infty] \). If \( V^\tau \) is \( \mathbb{G} \)-locally integrable, then \( V \) is \( \mathbb{F} \)-locally integrable.
(e) For any \( \mathbb{G} \)-predictable process \( \varphi^G \), there exists an \( \mathbb{F} \)-predictable process \( \varphi^F \) such that \( \varphi^G = \varphi^F I_{[0, \tau]} \). Furthermore, if \( \varphi^G > 0 \) (respectively \( \varphi^G \leq 1 \)), then \( \varphi^F > 0 \) (respectively \( \varphi^F \leq 1 \)).

The following lemma recalls the \( \mathbb{G} \)-compensator of any \( \mathbb{F} \)-optional process stopped at \( \tau \).

Lemma A.2. Let \( V \in \mathcal{A}_{loc}(\mathbb{F}) \), then we have
\[
(V^\tau)^{p, \mathcal{G}} = I_{[0, \tau]} G^{-1} \cdot (\widetilde{G} \cdot V)^{p, \mathbb{F}}.
\]

For the proof of this lemma and other related results, we refer to \[1\] \[2\] \[3\].

B Some useful integrability properties

The results of this section are new and are very useful, especially the first lemma and the proposition, and they are general and not technical at all.

Lemma B.1. Consider \( K \in \mathcal{M}_{0,loc}(\mathbb{H}) \) with 1 + \( \Delta K > 0 \), and let \( H^{(0)}(K, P) \) be given by Definition \[5.7\] If \( E[H^{(0)}(K, P)] < +\infty \), then \( E[\sqrt{[K, K]_T}] < +\infty \) or equivalently \( E[\sup_{0\leq t\leq T} |K_t|] < +\infty \).

Proposition B.2. Let \( Z \) be a positive supermartingale such that \( Z_0 = 1 \). Then the following hold.
(a) \(-\ln(Z)\) is a uniformly integrable submartingale if and only if there exists a local martingale \( N \) and a nondecreasing and predictable process \( V \) such that \( \Delta N > -1 \), \( Z = \mathcal{E}(N) \exp(-V) \) and
\[
E\left[V_T + H^{(0)}_T(N, \mathbb{H})\right] < +\infty.
\] (58)
(b) Suppose that there exist a finite sequence of positive supermartingale \((Z^{(i)})_{i=1,...,n}\) such that the product \(Z := \prod_{i=1}^{n} Z^{(i)}\). Then \(-\ln(Z)\) is uniformly integrable submartingale if and only if all \(-\ln(Z^{(i)})\), \(i = 1,...,n\), are uniformly integrable submartingales.

Proof. The proof of this proposition is achieved in two parts, where we prove assertions (a) and (b) respectively. 

**Part 1.** It is clear that there exist unique local martingale \(N\) and a nondecreasing and predictable process \(V\) such that \(N_0 = V_0 = 0, \Delta N > -1,\) and hence \(\Delta N = -\ln(\mathcal{E}(N) \exp(-V))\).

Thus, we derive

\[-\ln(Z) = -\ln(\mathcal{E}(N)) + V = -N + H^{(0)}(N, \mathbb{H}) + V, \tag{59}\]

where both processes \(V\) and \(H^{(0)}(N, \mathbb{H})\) are nondecreasing.

Suppose that \(-\ln(Z)\) is a uniformly integrable submartingale, and let \(\tau_n\) be a sequence of stopping times that increases to infinity and \(N^{\tau_n}\) is a martingale. Then on the one hand, by stopping (59) with \(\tau_n\), and taking expectation afterwards we get

\[E[-\ln(Z_{\tau_n \wedge T})] = E\left[V_{\tau_n \wedge T} + H^{(0)}(N, \mathbb{H})\right].\]

On the other hand, since \(\{-\ln(Z_{\tau_n \wedge T}), \ n \geq 0\}\) is uniformly integrable and the RHS term of the above equality is increasing, by letting \(n\) goes to infinity in this equality, (59) follows immediately. Now suppose that (60) holds. As a consequence \(E[H^{(0)}(N, \mathbb{H})] < +\infty\), and by combining this with Lemma B.1 and (59), we deduce that \(-\ln(Z)\) is a uniformly integrable submartingale.

**Part 2.** Here we prove assertion (b). From the proof of assertion (a), it is clear that there exists a sequence of local martingale \(N^{(i)}\) and a sequence of nondecreasing and predictable processes \(V^{(i)}\) such that for all \(i = 1,...,n, \Delta N^{(i)} > -1,\) and \(Z^{(i)} = \mathcal{E}(N^{(i)}) \exp(-V^{(i)}).

Furthermore, we derive

\[-\ln(Z) = -\sum_{i=1}^{n} N^{(i)} + \sum_{i=1}^{n} H^{(0)}(N^{(i)}, \mathbb{H}) + \sum_{i=1}^{n} V^{(i)}.\]

If \(-\ln(Z)\) is a uniformly integrable submartingale, then thanks to assertion (a), we deduce that

\[\sum_{i=1}^{n} H^{(0)}(N^{(i)}, \mathbb{H}) + \sum_{i=1}^{n} V^{(i)}\]

is integrable. Hence, thanks again to assertion (a), we deduce that \((\ln(Z^{(i)}))_{i=1,...,n}\) are uniformly integrable submartingales. Now suppose that \((\ln(Z^{(i)}))_{i=1,...,n}\) are uniformly integrable submartingales, then in virtue of assertion (a) and Lemma B.1, \((N^{(i)})_{i=1,...,n}\) are uniformly integrable martingales, and hence \(-\ln(Z)\) is a uniformly integrable submartingale. This ends the proof of the proposition. □

**Lemma B.3.** Let \(\lambda \in \mathcal{L}(X, \mathbb{H}),\) and \(\delta \in (0,1)\) such that

\[
\frac{\lambda^{tr}x}{1 + \lambda^{tr}x} I_{[\lambda^{tr}x > \delta]} \ast \mu_X + \left(\frac{\lambda^{tr}x}{1 + \lambda^{tr}x}\right)^2 I_{[\lambda^{tr}x \leq \delta]} \ast \mu_X \in \mathcal{A}_{loc}^+(\mathbb{H}). \tag{61}
\]

Then \(\sqrt{(1 + \lambda^{tr}x)^{-1} - 1} \ast \mu_X \in \mathcal{A}_{loc}^+(\mathbb{H}).\)
Proof. By using \( \sqrt{\sum_i x_i^2} \leq \sum_i |x_i| \), we derive
\[
\sqrt{((1 + \lambda^r x)^{-1} - 1)^2 \star \mu_X} = \sqrt{\sum \left( \frac{\lambda^r \Delta X}{1 + \lambda^r \Delta X} \right)^2} \leq \sqrt{\sum \frac{(\lambda^r \Delta X)^2}{(1 + \lambda^r \Delta X)^2} I_{(|\lambda^r \Delta X| \leq \delta)} + \sum \frac{|\lambda^r \Delta X|}{1 + \lambda^r \Delta X} I_{(|\lambda^r \Delta X| > \delta)}.}
\]
Thus, the lemma follows immediately from the latter inequality. \qed

C Martingales and deflators via predictable characteristics

For the following representation theorem, we refer to [30] Theorem 3.75] and to [31] Lemma 4.24).

**Theorem C.1.** Consider the model \((X, \mathbb{H}, Q)\) as defined above with its predictable characteristics \((b, c, F, A)\) such that \(X\) is quasi-left-continuous. Let \(N \in \mathcal{M}_{0, loc}(\mathbb{H})\). Then, there exist \(\phi \in L(X^c, \mathbb{H})\), \(N' \in \mathcal{M}_{0, loc}(\mathbb{H})\) with \([N', S] = 0\) and functionals \(f \in \bar{\mathcal{P}}(\mathbb{H})\) and \(g \in \bar{\mathcal{O}}(\mathbb{F})\) such that
(a) Both \(\sum_{s=0}^t (f(s, \Delta S_s) - 1)^2 I_{\{\Delta S_s \neq 0\}} \) and \(\sum_{s=0}^t g(s, \Delta S_s)^2 I_{\{\Delta S_s \neq 0\}} \) are locally integrable.
(b) \(M^f_P(g | \bar{\mathcal{P}}) = 0\), \(P \otimes \mu\).-a.e.
(c) The process \(N\) is given by
\[
N = \phi \cdot X^c + (f - 1) \cdot (\mu - \nu) + g \cdot \mu + N'.
\]
The quadruplet \((\beta, f, g, N')\) is called throughout the paper by Jacod’s components of \(N\) (under \(P\)).

The following theorem describes how general deflators can be characterized using the predictable characteristics. A version of this theorem can be found in [24].

**Theorem C.2.** Suppose \(X\) is quasi-left-continuous. \(Z \in \mathcal{D}_\log(X, \mathbb{H})\) if and only if there exists a triplet \((\beta, f, V)\) such that \(\beta \in L(X^c, \mathbb{H})\), \(f\) is \(\bar{P}(\mathbb{H})\)-measurable, positive and \(\sqrt{(f - 1)^2 \star \mu} \) belongs to \(\mathcal{A}^+_{loc}(\mathbb{H})\), \(V\) is an \(\mathbb{H}\)-predictable and nondecreasing process, and the following hold
\[
Z = \mathcal{E} \left( \beta \cdot X^c + (f - 1) \cdot (\mu - \nu) \right) \exp(-V),
\]
\[
E \left[ V_T + \left( \frac{1}{2} \theta^c X + \int (f(x) - 1 - \ln(f(x))) F^X(dx) \right) \cdot A_T^X \right] \leq E[-\ln(Z_T)] < +\infty, \quad (64)
\]
\[
\left( \int |f(x) \theta^x - \theta^x h(x)| F^X(dx) \right) \cdot A_T^X < +\infty \quad P\text{-a.s.}\]
(65)
\[
\left( \theta^b X + \theta^c X \right) \beta + \int |f(x) \theta^x x - \theta^x h(x)| F^X(dx) \right) \cdot A_T^X \leq V, \quad (66)
\]
P-a.s. for any bounded process \(\theta \in \mathcal{L}(X, \mathbb{H})\).

Proof. Let \(Z \in \mathcal{D}_\log(X, \mathbb{H})\), then \(Z^{-1} \cdot Z\) a local supermartingale. Hence, there exists an \(\mathbb{H}\)-local martingale \(N\) and a nondecreasing and \(\mathbb{H}\)-predictable process \(V\) such that \(Z = \mathcal{E}(N) \exp(-V)\). Then we derive
\[
-\ln(Z) = -N + V + \frac{1}{2} \left( N^c \right) + \sum (\Delta N - \ln(1 + \Delta N)).
\]

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Thus $Z \in \mathcal{D}_{\log}(X, \mathbb{H})$ if and only if $V + \frac{1}{2} \langle N^c \rangle + \sum (\Delta N - \ln(1 + \Delta N))$ is integrable. Then there exists a positive and $\mathcal{P}(\mathbb{H})$-measurable functional $f$ such that $\sqrt{(f-1)^2 * \mu}$ is locally integrable, $\beta \in L(X^c, \mathbb{H})$, and nonnegative and $\mathbb{H}$-predictable process $v$ such that $N$ can be chosen to be $N := \beta \cdot X^c + (f-1) * (\mu X - \nu X)$ and $V = v \cdot A^X$. Then $Z \in \mathcal{D}_{\log}(X, \mathbb{H})$ if and only if $V + \frac{1}{2} \beta^T c^X \beta \cdot A^X + (f - 1 - \ln(f)) \nu^X \in A^+(\mathbb{H})$ and $Z\mathcal{E}(\theta \cdot X)$ is a supermartingale, for any locally bounded $\mathbb{H}$-predictable process $\theta$ such that $1 + \theta^T x > 0 \ P \otimes A^X$-a.e.. Here $(b^X, c^X, \nu^X := F^X \otimes A^X)$ is the predictable characteristics of $(X, \mathbb{H})$.

On the one hand, we have $Z\mathcal{E}(\theta \cdot X) = \mathcal{E}(N - v \cdot A^X + \theta \cdot X + [\theta \cdot X, N])$ is a positive supermartingale and hence $N - v \cdot A^X + \theta \cdot X + [\theta \cdot X, N]$ is a local supermartingale. This is equivalent, (after simplification and transformation), to the conditions (65)-(66). This ends the proof of theorem.

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