Numerical General Relativistic MHD with Magnetically Polarized Matter

Oscar M. Pimentel, F. D. Lora-Clavijo, and Guillermo A. González
Grupo de Investigación en Relatividad y Gravitación, Escuela de Física, Universidad Industrial de Santander, A. A. 678, Bucaramanga 680002, Colombia

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Abstract

Magnetically polarized matter in astrophysical systems may be relevant in some magnetically dominated regions, for instance, in the funnel that is generated in some highly magnetized disk configurations where relativistic jets are thought to spread, or in pulsars where the fluids are subject to very intense magnetic fields. With the aim of dealing with magnetic media in the astrophysical context, we present for the first time the conservative form of the ideal general relativistic magnetohydrodynamics (GRMHD) equations with a non-zero magnetic polarization vector $m^\mu$. Then, we follow the Anile method to compute the eigenvalue structure in the case where the magnetic polarization is parallel to the magnetic field, and it is parameterized by the magnetic susceptibility $\chi_m$. This approximation allows us to describe diamagnetic fluids, for which $\chi_m < 0$, and paramagnetic fluids, for which $\chi_m > 0$. The theoretical results were implemented in the CAFE code to study the role of magnetic polarization in several one-dimensional Riemann problems. We found that independent of the initial condition, the first waves that appear in the numerical solutions are faster in diamagnetic materials than in paramagnetic ones. Moreover, the constant states between the waves change notably for different magnetic susceptibilities. All of these effects are more appreciable if the magnetic pressure is much higher than the fluid pressure. Additionally, with the aim of analyzing magnetic media in a strong gravitational field, we carry out for the first time a test of the magnetized Michel accretion of a magnetically polarized fluid. With this test, we found that the numerical solution is effectively maintained over time ($t > 4000$), and that the global convergence of the code is $\geq 2$ for $\chi_m \lesssim 0.005$ for all magnetic field strengths $\beta$ we considered. Finally, when $\chi_m = 0.008$ and $\beta \geq 10$, the global convergence of the code is reduced to a value between the first and second orders.

Key words: magnetohydrodynamics (MHD) – methods: numerical – relativity

1. Introduction

Nowadays, it is widely known that strong magnetic and gravitational fields are fundamental for giving a correct description of some systems involving compact objects. For example, it is believed that fluid accreting onto a black hole is the main power source of active galactic nuclei and microquasars (Frank et al. 2002). In those systems, highly collimated radio jets can be observed, suggesting the existence of strong magnetic fields (Blandford & Znajek 1977; De Villiers et al. 2005; Beckwith et al. 2008). Additionally, the interaction between the magnetic field and the differential rotation of the disk produces turbulence via magnetorotational instability (MRI; Balbus & Hawley 1991). This turbulence acts on the disk as an effective viscosity that transports angular momentum and dissipates energy. Such a mechanism is fundamental for explaining how the accretion process begins and is maintained over time. The first observations of the structure of those magnetic fields near the event horizon have been reported recently in Johnson et al. (2015), who resolved the linearly polarized emission of Sagittarius A*. On the other hand, some models try to explain the central engine of the gamma-ray burst include a highly magnetized millisecond pulsar or a magnetar, whose magnetic field is about $\sim 10^{14}$ G (Zhang & Mészáros 2001).

Now, as it was pointed out by Font (2008), some high-energy astrophysical processes involving the dynamics of a fluid around a compact object can be successfully modeled by general relativistic ideal magnetohydrodynamics (GRMHD) in the test fluid approximation. In this theoretical context, the gravitational field of the fluid is negligible in comparison with that of the compact object, and the fluid is considered to be a perfect conductor (Anile 1989). Nevertheless, since GRMHD is described by a nonlinear, time-dependent, and multidimensional system of equations, it is very difficult to find analytic solutions (Gammie et al. 2003). Indeed, the only solution with a magnetic field in the context of relativistic accretion disk theory was obtained by Komissarov (2006), and it describes the stationary state of a non-self-gravitating disk-like fluid with a toroidal magnetic field in Kerr spacetime. Fortunately, nowadays, we can study realistic astrophysical systems by numerically solving the GRMHD equations. For example, Gammie et al. (2003) simulated the accretion of a toroidal disk with an ad hoc poloidal magnetic field and found that the MRI increases the magnetic strength sufficiently to distort the disk and drop the fluid into the black hole. In Siegel & Metzger (2017), the authors carried out three-dimensional GRMHD simulations of toroidal accretion disks, which resulted from the merger of neutron stars. They found that in the state where the heating due to MRI turbulence is balanced by neutrino cooling, the disk outflows launch more mass than those simulations with hydrodynamic $\alpha$-viscosity. These results strongly suggest that the outflows of a post-merger disk are possible sites of the nucleosynthesis of heavy elements.

Another interesting astrophysical system that can be analyzed in the context of GRMHD is the accretion disk around a black hole, whose symmetry plane is misaligned with the midplane of the disk. This feature is observed, for example, in the active galactic nucleus NGC 4258 (Caproni et al. 2007). The first simulations of these kind of systems (Fragile et al. 2007) exhibit remarkable differences with the accretion of an aligned disk. For instance, the innermost stable circular orbit is located at a larger radius than in the untilted disks, and the disk precesses uniformly...
with a frequency that could explain some observed quasiperiodic oscillations.

In a slightly different context, GRMHD is useful for studying the acceleration and collimation of relativistic jets. Currently, it is widely believed that the jets are accelerated and self-collimated by a large-scale magnetic field via the Blandford–Znajek (Blandford & Znajek 1977) or the Blandford–Payne (Blandford & Payne 1982) mechanism. Those models have been tested by observations (Narayan & McClintock 2012) and by numerical GRMHD simulations (Komissarov 2005; McKinney 2005; Lii et al. 2012). Additionally, Beckwith et al. (2008) have shown, through magnetized disk simulations, that jet launching strongly depends on the initial configuration of the magnetic field. Furthermore, as suggested in Tchekhovskoy et al. (2011), it is possible to explain the apparent observed efficiency in generating energy in jets, within some active galactic nuclei, by magnetically arrested accretion.

In all of the works mentioned above, the magnetic fields are due only to the motion of free charges. Nevertheless, at a microscopic level, all substances contain spinning electrons that move around orbits, forming small dipoles. Those dipoles can generate a magnetic field either due to a net alignment among themselves or as a response to an external magnetic field by means of magnetic torques (paramagnetism) or induced magnetic moments (diamagnetism; Lorrain & Corson 1970). Therefore, in order to study the response of the fluid to an applied magnetic field, it is necessary to include magnetic polarization terms in the energy–momentum tensor of the fluid. Fortunately, this tensor has already been computed by Maugin (1978) using the relativistic invariant Lorentz force in a particle aggregate. Recently, Chatterjee et al. (2015) obtained the same tensor but starting from the Lagrangian density of a fermion system in the presence of a magnetic field.

The magnetic polarization of an astrophysical fluid has already been considered in studies of the equilibrium structure of neutron stars. For example, Blandford & Hernquist (1982) computed the magnetic susceptibility of the degenerate free electrons in the star crust and concluded that the magnetization does not significantly contribute to the surface properties, but it probably may be coupled indirectly to observable effects. In fact, in Suh & Mathews (2010), the authors investigated the possibility that soft gamma-ray repeaters and anomalous X-ray pulsars might be observational evidences for magnetic domain formation in magnetars. However, as far as we know, in the ideal GRMHD context, there are no previous works dealing with the evolution of a magnetically polarized test fluid with a magnetic field in the vicinity of a relativistic astrophysical object. Therefore, the aim of this paper is to present the theoretical and numerical background for the study of the evolution of a magnetic medium in the fixed gravitational field of a compact object. With this background, we can analyze the role of magnetic polarization using some standard tests in the literature.

This paper is organized as follows: we first present in Section 2 the GRMHD equations for a magnetically polarized fluid as a conservative system in order to obtain numerical solutions through Godunov-type shock-capturing schemes (Toro 2009). In Section 3, we obtain the eigenvalue structure of the system of equations by following the Anile method (Anile 1989). The eigenvalues are computed in the linear approximation, where the magnetic polarization vector is in the same direction as the applied magnetic field. These eigenvalues are physically relevant not only because they are fundamental in the Godunov-type schemes, but also because they provide important information about the wave propagation speeds. Next, in Section 4, we give a brief summary of the numerical methods implemented in the CAFE code (Lora-Clavijo et al. 2015a), where we incorporate the new theoretical results presented in previous sections. In Section 5, we present several magnetized Riemann problems with magnetic polarization in order to analyze the structure of the solutions when we consider the fluid as either a diamagnetic or a paramagnetic medium. Additionally, we use the stationary magnetized Michel solution to estimate the values of $\chi_m$ for which the code converges to second order, in the case where a strong gravitational field is present in the system. Finally, in Section 6 we summarize the main results of this work.

Throughout the paper, the Greek indices run from 0 to 3, while the Latin ones run from 1 to 3. Additionally, the equations are written in units where the gravitational constant $G$ and the speed of light $c$ are equal to one.

2. The GRMHD Equations for a Magnetically Polarized Fluid as a Conservative System

The evolution of a magnetically polarized test fluid in the presence of a strong gravitational field and a magnetic field can be obtained from the local conservation laws, i.e., from the conservation of the baryon number and the conservation of the energy and momentum,

$$\nabla_\mu (\rho u^\mu) = 0, \quad (1)$$

$$\nabla_\mu T^{\mu\nu} = 0, \quad (2)$$

and the homogeneous Maxwell equation,

$$\nabla_\mu F_{\mu\nu} = 0, \quad (3)$$

where $\nabla_\mu$ is the covariant derivative, $u^\mu$ is the four-velocity vector of the fluid, $\rho$ is the rest-mass density, $T^{\mu\nu}$ is the energy–momentum tensor, and $F_{\mu\nu}$ is the Faraday tensor.

Now, within the ideal GRMHD approximation, where the fluid is assumed to be a perfect conductor, the Faraday tensor takes the form

$$F_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta} b^\alpha u^\beta, \quad (4)$$

where $b^\alpha$ is the magnetic field measured by an observer comoving with the fluid, and $\epsilon_{\mu\nu\alpha\beta}$ is the Levi–Civita tensorial density,

$$\epsilon_{\mu\nu\alpha\beta} = \frac{1}{\sqrt{-g}} \eta_{\mu\nu\alpha\beta}, \quad (5)$$

with $g$ the determinant of the metric tensor $g_{\mu\nu}$ and $\eta_{\mu\nu\alpha\beta}$ the Levi–Civita tensor. Substituting Equation (4) into Equation (3) and multiplying the resulting equation by $\eta_{\mu\nu\alpha\beta}$, one obtains

$$\nabla_\mu (u^\alpha b^\nu - b^\alpha u^\nu) = 0, \quad (6)$$

which is known as the relevant Maxwell equation (Anile 1989).

On the other hand, the energy–momentum tensor for a magnetically polarized fluid in a magnetic field can be written as the following superposition

$$T^{\mu\nu} = T^{\mu\nu}_f + T^{\mu\nu}_{em} \quad (7)$$

where $T^{\mu\nu}_f$ and $T^{\mu\nu}_{em}$ are the fluid and electromagnetic field energy–momentum tensors, respectively.
Huang et al. (2010) and Chatterjee et al. (2015), with the aim of giving a first approximation of the fluid description, we will consider the thermodynamic pressure to be isotropic and the heat flux to be zero. It is worth mentioning that the total thermodynamic pressure is actually the sum of the kinetic pressure and an additional contribution due to the Lorentz force density related to the magnetization currents (Potekhin & Yakovlev 2012). Thus, the kinetic pressure is anisotropic in a magnetically polarized fluid, but the total thermodynamic pressure is isotropic. These assumptions enable us to treat the fluid as a perfect one. Thus, the energy–momentum tensor of the fluid takes the form,

$$T^\mu_\nu = phu^\mu u^\nu + pg^\mu_\nu, \quad (8)$$

where $p$ is the thermodynamic pressure and $h = (e + p)/\rho$ is the specific enthalpy, $e$ being the energy density. On the other hand, the energy and momentum associated with the electromagnetic field and their interaction with matter can be described, in the ideal GRMHD approximation, through the tensor (Maugin 1978; Chatterjee et al. 2015)

$$T^\mu_\nu = \left( u^\mu u^\nu + \frac{1}{2}g^\mu_\nu \right) b^2 - \frac{1}{\mu_0} b^\mu b^\nu - \left( u^\mu u^\nu + g^\mu_\nu \right) b^\mu m_\nu + m^\mu b^\nu, \quad (9)$$

where $b^2 = b^\mu b_\mu$, $\mu_0$ is the vacuum permeability, and $m^\mu$ is the magnetic polarization four-vector in the comoving observer. It is important to mention that since the electric field in the comoving frame is zero for a perfect conductor, then it is reasonable to assume that the electric polarization of the matter is also zero in the same frame. With these two contributions, $T^\mu_\nu$ takes the form (Pimentel et al. 2016, 2017)

$$T^\mu_\nu = \rho h u^\mu u^\nu + p^* g^\mu_\nu - \Pi^\mu_\nu, \quad (10)$$

with

$$\rho h = \rho + \frac{b^2}{\mu_0} - b^\mu m_\mu, \quad (11)$$

$$p^* = p + \frac{b^2}{2\mu_0} + b^\mu m_\mu, \quad (12)$$

$$\Pi^\mu_\nu = \frac{1}{\mu_0} b^\mu b^\nu - m^\mu b^\nu. \quad (13)$$

This is the energy–momentum tensor of a magnetically polarized perfect fluid endowed with a magnetic field.

Once we have defined the total energy–momentum tensor, the next step is to write Equations (1), (2), and (6) as a conservative system of equations, in which the time derivatives are separated from the space ones, and so numerical methods can be applied to compute the evolution of the system from an initial state. To do this, we use the well known 3 + 1 formalism (Poisson 2004), in which the spacetime is foliated with spacelike hypersurfaces of t-constant, $\Sigma_t$. These hypersurfaces have a normalized timelike normal vector, $n^\mu$, and a set of three spacelike tangent vectors, $e^\mu$, which define the induced metric $\gamma_{ij} = g_{\mu\nu} e^\mu_i e^\nu_j$ on $\Sigma_t$. Now, taking $x^i$ as a set of three spatial coordinates on the hypersurface, it is possible to use the coordinate transformations $x^\mu = x^i(t, x^i)$ to write the line element as

$$ds^2 = -(\alpha^2 - \beta^i\beta_i)dt^2 + 2\beta^idx^idt + \gamma_{ij}dx^idx^j, \quad (14)$$

where $\alpha$ is the lapse function and $\beta^i$ is the shift vector.

In addition to the comoving observer, it is convenient to introduce another observer, called the Eulerian observer, that moves perpendicular to $\Sigma_t$ with a four-velocity $n^\mu = (-\alpha, 0, 0, 0)$. The spatial velocity of the fluid $v^i$, measured by this observer, is tangent to the hypersurfaces and satisfies the equation

$$v^i = \frac{u^i}{W} + \frac{\beta^i}{\alpha}, \quad (15)$$

where $W = -u^\mu n_\mu = \alpha u^0$ is the Lorentz factor between the Eulerian and comoving observers. Now, the GRMHD equations with magnetic polarization, Equations (1), (2), and (6), can be written as a conservative system of the form

$$\partial_0(\sqrt{\gamma} U) + \partial_\nu(\sqrt{\gamma} F^\nu) = \sqrt{\gamma} S, \quad (16)$$

$$\partial_0(\sqrt{\gamma} B^i) = 0. \quad (17)$$

Equation (17) is the divergence-free constraint for the magnetic field, $\gamma$ is the determinant of the induced metric, and $U$ is the state vector

$$U = [D, S_j, \tau, B^k]^T, \quad (18)$$

whose components are the mass density $D = -\rho u^\mu n_\mu$, the energy flux vector $S_j = -T^\mu_\nu n_\mu e^\nu_j$, the energy density $-D$, $\tau = T^\mu_\nu n_\mu n_\nu - D$, and the magnetic field $B^k$, all of them measured in the Eulerian observer frame. By doing the projections along the basis $\{n^\mu, e^\mu_i\}$, we obtain

$$D = \rho W, \quad (19)$$

$$S_j = \rho h W^2 v_j - \frac{\alpha}{\mu_0} b_j b^0 + \frac{\alpha}{2}(m^0 b_j + b^0 m_j), \quad (20)$$

$$\tau = \rho h W^2 - p^* - \frac{\alpha^2}{\mu_0}(b^0)^2 + \alpha^2 m^0 b^0 - D. \quad (21)$$

Now, it is possible to show that when the matter is magnetically polarized, the flux vector $F^i$ takes the form

$$F^i = \left[ \begin{array}{c} S_j v^j \vspace{0.1cm} + p^* \delta_j^i - \frac{b_j B^i}{\mu_0 W} + \frac{1}{2W}(m^i B_j + b^0 M^i) \vspace{0.1cm} \tau v^i \vspace{0.1cm} + p^* v^i - \frac{\alpha}{\mu_0 W} b^0 B^i + \frac{\alpha}{2W}(m^i B_j + b^0 M^i) \vspace{0.1cm} \bar{v}^i B^k \vspace{0.1cm} - \bar{v}^k B^i \end{array} \right]. \quad (22)$$

with $M^i$ the magnetic polarization vector in the Eulerian frame and $\bar{v}^i = v^i - \beta^i/\alpha$. It is worth mentioning that $b^\mu$ and $m^\mu$ are related to the same quantities in the Eulerian frame through the equations

$$b^i = \frac{B^i}{W} + \frac{\alpha b^0 u^i}{W}, \quad b^0 = \frac{W}{\alpha} B^i v_i, \quad (23)$$

$$m^i = \frac{M^i}{W} + \frac{\alpha m^0 u^i}{W}, \quad m^0 = \frac{W}{\alpha} M^i v_i. \quad (24)$$
Finally, the terms that do not contain derivatives of the state variables are included in the source vector,

\[
S = \begin{bmatrix}
0 \\
T^{\mu\nu}(\partial_\mu g_{\rho\nu} - g_{\rho\nu} \Gamma^\zeta_{\mu\nu}) \\
\frac{1}{\rho} T^{\rho\nu} \partial_\nu - \alpha T^{\mu\nu} \Gamma^\zeta_{\mu\nu} \\
\end{bmatrix},
\tag{25}
\]

The conservative system, Equation (16), becomes a closed system once we introduce an equation of state, \(p = p(\rho, e)\), and a constitutive relation between the magnetic polarization vector and the magnetic field, \(m^\mu = m^\mu(b^\nu)\).

### 3. The Eigenvalue Structure

The system of Equations (16)–(17) does not have a general exact solution, so it is necessary to solve them numerically. Some current numerical methods devoted to solving the system of evolution equations in conservative form are based on the eigenvalue structure, i.e., on the propagation speeds of the signals in the fluid. Therefore, with the aim of computing the eigenvalues of Equation (16), we follow a similar procedure to the one described in Anile (1989), whose starting point is to write Equations (1), (2), and (5) in the covariant form

\[
A_{pK}^\mu \nabla_\mu V^P = 0,
\tag{26}
\]

where \(A_{pK}^\mu\) are \(N \times N\) coefficient matrices, \(V^P\) is a column vector whose \(N\) components are state variables, and the indices \(P, K = 1, 2, \ldots, N\). We recall that the eigenvalue structure we want to compute in this paper differs from the usual GRMHD eigenvalues in that we are including the magnetic polarization of the fluid.

To write the equations in the covariant form (Equation (26)), we project Equation (6) along \(u_\nu\) and \(b_\nu\) to obtain

\[
\nabla_\mu b^\mu + u^\mu u_\mu \nabla_\mu b_\nu = 0,
\tag{27}
\]

\[
b^2 \nabla_\mu u^\mu + \frac{1}{2} u^\mu \nabla_\mu (b^2) - b^\mu b^\nu \nabla_\mu u_\nu = 0,
\tag{28}
\]

respectively. Now, we obtain the energy conservation equation by projecting Equation (2) along \(u_\nu\), and by using Equation (27) in the resulting expression; so, we get

\[
u^\mu \nabla_\mu e + (e + p - b^\mu m_\mu) \nabla_\mu u^\mu + m^\mu b^\nu \nabla_\mu u_\nu = 0.
\tag{29}
\]

Nevertheless, from an equation of state \(e = e(\rho, s)\), with \(s\) being the specific entropy, and \(e\) being the energy conservation takes the following form,

\[
e'_\mu u^\nu \nabla_\nu e + e'_\mu u^\nu \nabla_\nu s + K \nabla_\mu u^\mu + m^\mu b^\nu \nabla_\mu u_\nu = 0,
\tag{30}
\]

where \(K = e + p - b^\mu m_\mu, e'_\mu = \partial e / \partial p,\) and \(e'_\mu = \partial e / \partial s\).

The second equation in the Anile method is the one that describes entropy production. This equation is computed from the first law of thermodynamics for a magnetically polarized material (Groot & Suttorp 1972),

\[
T \nabla_\mu s = \nabla_\mu e - \frac{p}{\rho^2} \nabla_\mu p + \frac{1}{\rho} m_\mu \nabla_\mu b^\nu,
\tag{31}
\]

\(T\) being the temperature. Now, from the conservation of the baryon number (Equation (1)) and the conservation of energy (Equation (29)), we can rewrite Equation (31) as

\[
\rho T u^\mu \nabla_\mu s = b^\nu m_\mu \nabla_\mu u^\nu + u^\nu m_\mu \nabla_\mu b^\nu
- m^\mu b^\nu \nabla_\mu u_\nu.
\tag{32}
\]

Note that unlike in the GRMHD case, where the adiabaticity condition \(u^\nu \nabla_\nu s = 0\) is satisfied, when the magnetic polarization of the matter is considered, the right-hand side of the last equation does not vanish in general.

Another relevant equation in the Anile method is the conservation of momentum. We also compute this equation in the case of a magnetically polarized fluid; first, we contract Equation (2) with \(b_\nu\) in order to obtain the auxiliary relation

\[
\left( e + p - \frac{1}{2} b^\mu m_\mu \right) \nabla_\mu b^\nu + b^\nu \nabla_\mu p - b^\mu m_\nu \nabla_\nu b_\nu
- \frac{1}{2} b^\mu b^\nu \nabla_\mu m_\nu + \frac{1}{2} m^\mu b^\nu \nabla_\mu b_\nu + \frac{1}{2} b^2 \nabla_\mu m^\nu = 0.
\tag{33}
\]

Then, we project Equation (2) on the hypersurface \(\Sigma_t\) by contracting it with the projector tensor \(h_\mu_\nu = g_\mu_\nu + u_\mu u_\nu\). In the resulting expression, we replace \(\nabla_\mu u^\mu\) and \(\nabla_\mu b^\nu\), which can be obtained from Equations (30) and (33), respectively, to get the conservation of momentum equation,

\[
A^\mu \nabla_\mu p + B^\mu \nabla_\mu b^\nu + C^\mu \nabla_\mu u^\nu
+ D^\mu \nabla_\mu m^\nu + E^\mu \nabla_\mu s = 0,
\tag{34}
\]

where

\[
A^\mu = h^\mu - \frac{b^\mu}{\mu_0} \frac{1}{2} K^{-1} \epsilon_{\rho e} u^\rho u^e - \zeta \epsilon^\mu b^\nu,
\]

\[
B^\mu = \frac{1}{\mu_0} \left( (h^\mu + u^\mu) b_\nu + \zeta \epsilon^\mu \left( b^\nu m_\rho - \frac{1}{2} m^\nu b_\rho \right) \right) - h^\mu m_\nu + \eta \epsilon^\mu b_\nu,
\]

\[
C^\mu = \left( K + \frac{b^2}{\mu_0} \right) \left( u^\mu \epsilon^\nu - K^{-1} u^\nu b^\mu b^\rho g_{\rho\nu} \right),
\]

\[
D^\mu = \frac{1}{2} \frac{\zeta}{\mu_0} \epsilon^\nu b^\nu \epsilon^\mu - \frac{h^\nu b_\nu}{\mu_0} - \frac{1}{2} \frac{\zeta}{\mu_0} \epsilon^\nu \epsilon^\nu + \frac{1}{2} \epsilon^\mu \epsilon^\nu + \frac{1}{2} b^\mu \epsilon^\nu,
\]

\[
E^\mu = - \frac{b^2}{\mu_0} K^{-1} \epsilon^\nu u^\nu u^\nu,
\]

with \(\zeta = \left( K + \frac{1}{2} b^\mu m_\mu \right)^{-1}\) and \(\eta = \frac{1}{2} m^\mu - \frac{1}{2} b^\mu m_\mu\).

Finally, the relevant Maxwell Equation (6) can be rewritten as

\[
F^\mu \nabla_\mu p + G^\mu \nabla_\mu b^\nu + H^\mu \nabla_\mu u^\nu
+ I^\mu \nabla_\mu m^\nu + J^\mu \nabla_\mu s = 0,
\tag{35}
\]

where

\[
F^\mu = \zeta b^\mu u^\nu - K^{-1} \epsilon^\nu u^\nu b^\mu,
\]

\[
G^\mu = \frac{u^\mu \epsilon^\nu - u^\nu \epsilon^\mu}{\mu_0} \left( b^\nu m_\rho - \frac{1}{2} m^\nu b_\rho \right),
\]

\[
H^\mu = - b^\nu \epsilon^\mu - K^{-1} g_{\rho\nu} m^\rho b^\mu,
\]

\[
I^\mu = \frac{1}{2} \frac{b^2}{\mu_0} \epsilon^\nu u^\nu - b^\nu u^\nu b_\nu \zeta,
\]

\[
J^\mu = - K^{-1} \epsilon^\nu b^\nu u^\nu.
\]
It is important to mention that when $m^\mu = 0$, the conservation of energy Equation (30), the entropy production (Equation (32)), the conservation of momentum Equation (34), and the relevant Maxwell Equation (35) reduce to the equations of the ideal GRMHD as expected (cf. Anile 1989).

On the other hand, as was pointed out at the end of Section 2, it is necessary to introduce a constitutive relation between $m^\mu$ and $b^\mu$ to form a closed set of equations. In this work, we concentrate our attention on the special case in which the magnetic polarization is always in the same direction as the magnetic field, so we can relate them through the equation

$$m^\mu = \frac{\chi}{\mu_0} b^\mu,$$

where $\chi = \chi_m/(1 + \chi_m)$, and $\chi_m$ is the magnetic susceptibility of the fluid. This last physical parameter is negative for diamagnets and positive for paramagnets. As argued by Felderhof & Kroh (1999), this particular relation is useful to describe polar fluids in equilibrium or systems in which the constituent particles do not have permanent magnetic dipole moments. When a polar fluid is not in equilibrium, it is necessary to introduce a phenomenological relaxation equation that describes the evolution of $m^\mu$ in the presence of an applied magnetic field (Schumacher et al. 2010).

For the particular choice of $m^\mu$ presented in Equation (36), it is possible to see that the adiabaticity condition, which implies zero heat transfer between two adjacent elements of the fluid, is satisfied. With this condition, all of the terms proportional to $u^\mu \nabla_\mu$ in the equations vanish. On the other hand, when using the constitutive relation (36) to reduce the number of variables, and therefore to close the system of equations, it is important to take into account that $b^\mu$ has to satisfy Equations (27) and (28).

With these considerations, the GRMHD equations with magnetization can be set in the covariant form, Equation (26), with $V = (u^\mu, b^\mu, \rho, s)^T$ and

$$f^\mu_{\nu} = -\eta^{-1}(-e^\prime_\nu u^\mu + u^\nu b^\mu),$$

with the aim of reducing the expressions.

Now, to compute the characteristic structure of the system of Equation (26), we introduce a hypersurface given by $\phi(x^\nu) = 0$ with normal vector $\phi_\nu = \nabla_\nu \phi$. This hypersurface is said to be characteristic if $\det(A^\mu_{\nu} \phi_\nu) = 0$. Therefore, by contracting $A^\mu_{\nu}$ with the vector $\phi_\nu$, we obtain the characteristic matrix

$$A^\mu_{\nu} = \begin{bmatrix} \eta + (1 - \chi)b^2 & a\delta^\nu_\mu & -a\delta^\nu_\mu + \eta^{-1}b_\nu(Bb^\nu + ab^\nu) & f^\nu_{\eta} & 0^\mu_{\eta} \\ B\delta^\nu_\mu & 0 & 0 & 0 & 0 \\ \eta\phi_\nu + \chi(Bb_\nu - b^2\phi_\nu) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $a = u^\mu \phi_\nu$, $B = b^\mu \phi_\nu$, and

$$m^\nu_{\nu} = \tilde{p}^\nu_{\mu} \phi_\mu b_\nu - (1 - \chi)B\delta^\nu_\mu,$$

$$f^\nu = f^\nu_{\eta} \phi_\nu = -\eta^{-1}(-e^\prime_\nu au^\nu + Bu^\nu),$$

$$\tilde{\nu}_{\nu} = \tilde{\nu}^\nu_{\mu} \phi_\mu,$$

$$= \phi^\nu + au^\nu + \eta^{-1}(1 - \chi)(Bb^\nu - e^\prime_\nu b^2au^\nu).$$

Hence, the characteristic equation takes the simple form

$$\det(A^\mu_{\nu}) \phi_\nu = -\eta^{-1}u^\nu \tilde{A}^2 \tilde{N}_\nu Q = 0,$$

with

$$\tilde{A} = [\eta + (1 - \chi)b^2]a^2 - (1 - \chi)B^2,$$

$$\tilde{N}_\nu = f_1 a^4 - f_2 a^2G - f_3 a^2B^2 + f_4 B^2G,$$

$$Q = b^2\chi^2 - b^2(b^2 + 2\eta)\chi + \eta(\eta + b^2),$$

where

$$f_1 = (e^\prime_\nu - 1)\eta + (e^\prime_\nu + 1)\chi b^2,$$

$$f_2 = \eta + (1 - 2\chi)b^2e^\prime_\nu - \chi b^2,$$

$$f_3 = (e^\prime_\nu + 1)\chi,$$

$$f_4 = 1 - \chi,$$

and $G = \phi_\nu \phi^\nu$ are the functions that define $\tilde{N}_\nu$. The characteristic Equation (45) depends on the vector $\phi_\mu$, which represents the local velocity of the travelling wave (Ruggeri & Tommaso 1981). The characteristic wave speeds can be computed by setting zero the functions $a$, $\tilde{A}$, and $\tilde{N}_\nu$ to zero, since $\eta^{-1}$ and $Q$ do not depend on $\phi_\nu$. Now, a wave that propagates with speed $\lambda$ in the first spatial direction (let us call
this \( x \) has a vector \( \phi_x = (-\lambda, 1, 0, 0) \). Then, the eigenvalue that results from the equation \( a = 0 \) is
\[
\lambda_e = \alpha v^x - \beta^x
\]
and corresponds to the propagation speed of disturbances in the mass density (material waves). \( \lambda_e \) is commonly known as the entropic eigenvalue and appears as a double root in the system of equations (Komissarov 1999).

On the other hand, the Alfvén wave speeds are the roots of the equation \( \tilde{A} = 0 \). These waves are produced from the interaction of a conducting fluid and the magnetic field. When one considers the matter to be magnetically polarized fluid, the Alfvén eigenvalues become dependent on the magnetic susceptibility of the material, with the form
\[
\lambda_a = \frac{b^s \pm \sqrt{H a^s}}{b^0 \pm \sqrt{H a^0}},
\]
where \( H = b^2 + \eta/(1 - \lambda) \). Now, in order to preserve the real character of the Alfvén eigenvalues and therefore to ensure the hyperbolicity of the system of equations, we need to have \( H = \eta(1 + \chi_m) + b^2 \geq 0 \). This condition constrains the magnetic susceptibility to the interval
\[
\chi_m \geq -\left(1 + \frac{b^2}{\eta}\right).
\]

Additionally, we can see from Equation (54) that when \( \chi = 1 \), the Alfvén waves travel at the same speed as the entropic waves, but with a non-zero magnetic field. Nevertheless, this value for \( \chi \) corresponds to an infinite magnetic susceptibility.

The remaining eigenvalues are solutions to the quartic equation \( \tilde{N}_t = 0 \) and correspond to magnetosonic waves. Unfortunately, the analytic expressions for these eigenvalues are not simple, so one of the best options is to compute them numerically with a Newton–Raphson method. Nevertheless, as pointed out by Leismann et al. (2005), this solution makes the solver unstable for high Lorentz factors. Hence, the authors implemented in their code the analytical expressions for the fast magnetosonic waves with \( B'_{\|} = 0 \), in the degenerate case \( B^s = 0 \), and achieved very similar results to those obtained by the Newton–Raphson algorithm. In our paper, we followed the same method presented by Leismann et al. to obtain the boundary values for the magnetosonic speeds. These values have the following form:
\[
\lambda_{bmax} = a - \frac{\alpha \sqrt{b^2 + cd}}{c},
\]
where
\[
a = b^0 b^s q_s \alpha^2 - H [q_1 \chi \beta^s + q_2 W^2 (\alpha v^s - \beta^s)] + q_1 H [W^2 (\alpha v^s - \beta^s) + \beta^s] \Omega^2,
\]
\[
b = \alpha b^0 q_s (b^s + b^0 \beta^s) - H v^s W^2 (q_2 - \Omega^2 q_1),
\]
\[
c = (b^0)^2 \alpha^2 q_s + H [q_1 + q_1 W^2 - 1] \Omega^2 - W^2 q_2,
\]
\[
d = H [q_2 (v^s)^2 W^2 + q_3 \chi \gamma_{ax} - q_4 \Omega^2 ((v^s)^2 W^2 + \gamma_{xx})] - q_4 (b^s + b^0 \beta^s)^2,
\]
and the \( q \) functions are defined in terms of the sound speed and the magnetic susceptibility as
\[
q_1 = C_i^2 - 1 + 2x,
\]
\[
q_2 = q_1 + \chi q_4,
\]
\[
q_3 = C_i^2 (1 + C_i^2),
\]
\[
q_4 = C_i^4 - 1.
\]

With this eigenvalue structure, we can solve the equations by using a numerical method that only requires information about the wave speeds, such as the HLL Riemann solver. In the next section we will describe briefly the numerical methods that we use for solving the magnetically polarized GRMHD equations.

4. Numerical Methods

The conservative system of GRMHD equations with magnetically polarized matter (Equation (16)) will be solved numerically with a new extension of the CAFE code, where we will implement the theoretical formalism presented in Sections 2 and 3. CAFE was developed by Lora-Clavijo et al. (2015a), and its first version was designed to solve the equations of the ideal RMHD in three dimensions. The hydrodynamic version of this code has been used to study the shock cone that is produced when a relativistic ideal fluid is accreted onto a rotating Kerr black hole (Cruz-Osorio et al. 2012) and when the accretion is produced in a fixed Schwarzschild spacetime (Lora-Clavijo & Guzmán 2013). Additionally, in Lora-Clavijo et al. (2013, 2014) the authors used a new version of CAFE, which is capable of solving the hydrodynamic equations coupled with the Einstein equations, to analyze the horizon growth of black hole seeds due to the accretion of radiation fluids and collisional dark matter. More recently, CAFE has been used to model the Bondi–Hoyle accretion of an ideal supersonic gas with velocity gradients (Cruz-Osorio & Lora-Clavijo 2016) and with density gradients (Lora-Clavijo et al. 2015b) onto a Kerr black hole. Finally, Cruz-Osorio et al. (2017) also studied this kind of accretion in the Schwarzschild background, when some rigid and small bodies are randomly distributed around the black hole.

CAFE uses the method of lines with a Runge–Kutta algorithm to solve the discretized version of the system of equations (16), which is obtained with a finite volume scheme. The evolution of the state vector is computed with a high-resolution shock-capturing scheme (HRSC), together with a spatial reconstructor. In this work, we use the approximate Riemann solver proposed by Harten, Lax, van Leer, and Einfeldt or HLL (Harten et al. 1997; Einfeldt 1988), because it is a robust method that only requires information about the eigenvalue structure of the system of equations. More exactly, it uses the maximum and minimum wave speeds to estimate an approximate value for the fluxes in the middle of the mesh points (intercells). Finally, we use the MINMOD linear piecewise reconstructor to compute the state vector on both sides of the intercells.

On the other hand, the code guarantees the constraint of the magnetic field, given in Equation (17), with the flux constraint transport method (Yee 1966), which was first implemented by Evans & Hawley (1988) in an artificial-viscosity scheme and then modified by Balsara & Spicer (1999) in order to use the flux formulas from the HRSC schemes. With this method, the divergence-free condition for the magnetic field is satisfied to machine precision by evolving the integral form of the induction equation. Hence, if \( \nabla \cdot B = 0 \) at the initial time, then this value will remain to machine precision order in the subsequent time steps.
and the set of primitive variables \( \mathbb{U} \) and the set of primitive variables \( \mathbb{V} = [\rho, v^i, p, B^i]^T \); so, we need to find, at each time step, the primitive variables in terms of the conservative ones. Unfortunately, from Equations (19)–(21), we can see that it is not possible to find explicitly the functions \( \mathbb{W}(U) \), so the code uses a Newton–Raphson algorithm to recover the primitive variables in all points of the spacetime grid for an ideal gas equation of state. The mathematical procedure to compute \( \mathbb{W} \) when the magnetic polarization vector is different from zero is described in more detail in the Appendix.

### 5. Numerical Experiments

Once we have obtained the GRMHD equations with magnetization in conservative form and its corresponding eigenvalue structure, we can implement these results into the CAFE code to carry out some physically relevant numerical tests, which will be presented in the next two subsections. The fluids that we are going to study in all problems obey the ideal equation of state,

\[
p = \rho e (\Gamma - 1),
\]

where \( e \) is the specific internal energy and \( \Gamma \) is the adiabatic index.

#### 5.1. One-dimensional Magnetized Riemann Problems with Magnetic Polarization

In this subsection, we analyze the consequences of considering the magnetic polarization of matter in some standard shock-tube problems in the Minkowski spacetime. In each simulation, we assume the magnetic susceptibility of the fluids to be constant in all spacetime domains. These 1D tests represent the physical analogs of the Riemann problems because the initial configurations consist of two different constant states, \( \mathbb{W}_L \) and \( \mathbb{W}_R \), separated by a diaphragm or interface at the position \( x = 0 \). Here, the subscripts \( L \) and \( R \) denote the left \( (x \leq 0) \) and right \( (x > 0) \) states, respectively. The first two tests that we present here are the shock tube and the collision of two fluids, both proposed by Komissarov (1999). We carry out these simulations in the domain \( x \in [-2, 2] \), with a spatial resolution \( \Delta x = 1/800 \), and a Courant factor of 0.25. Then, we present the problems proposed by van Putten (1993) and Balsara (2001), for which we obtain the numerical solution in the domain \( x \in [-0.5, 0.5] \), with a spatial resolution \( \Delta x = 1/1600 \), and a Courant factor of 0.25. Finally, we present the generic Alfvén wave test, for which we use the domain \( x \in [-0.5, 0.5] \), a spatial resolution \( \Delta x = 1/1600 \), and a Courant factor of 0.25. The parameters used in these tests are given in Table 1.

Each one of the numerical tests mentioned in the last paragraph was carried out for some constant values of magnetic susceptibility, so we can determine the differences in the evolution of a diamagnetic \( (\chi_m < 0) \) and a paramagnetic \( (\chi_m > 0) \) fluid. Now, we also evolve the initial states with a zero magnetic susceptibility in order to reproduce the original RMHD solutions (Giacomazzo & Rezzolla 2006) and compare them with our results. In this respect, we anticipate that although the wave structure of the solutions turns out to be the same as in the RMHD case (with \( \chi_m = 0 \)), there are remarkable differences in the evolution when the magnetic polarization of the fluid is considered.

| Test       | State | \( \Gamma \) | \( \rho_0 \) | \( p \) | \( v^i \) | \( v^j \) | \( v^k \) | \( B^i \) | \( B^j \) | \( B^k \) |
|------------|-------|--------------|-------------|---------|---------|---------|---------|---------|---------|---------|
| Shock Tube 1 | Left  | 4/3         | 1.0         | 1000.0  | 0.0     | 0.0     | 0.0     | 1.0     | 0.0     | 0.0     |
|             | Right | 4/3         | 0.1         | 1.0     | 0.0     | 0.0     | 0.0     | 1.0     | 0.0     | 0.0     |
| Collision   | Left  | 4/3         | 1.0         | 1.0     | 5/\sqrt{26} | 0.0 | 0.0 | 10.0 | 10.0 | 0.0 |
|             | Right | 4/3         | 1.0         | 1.0     | -5/\sqrt{26} | 0.0 | 0.0 | 10.0 | -10.0 | 0.0 |
| Balsara 1   | Left  | 2           | 1.0         | 1.0     | 0.0     | 0.0     | 0.0     | 0.5     | 1.0     | 0.0     |
|             | Right | 2           | 0.125       | 1.0     | 0.0     | 0.0     | 0.0     | 0.5     | -1.0    | 0.0     |
| Balsara 2   | Left  | 5/3         | 1.0         | 30.0    | 0.0     | 0.0     | 0.0     | 5.0     | 0.0     | 6.0     |
|             | Right | 5/3         | 1.0         | 1.0     | 0.0     | 0.0     | 0.0     | 5.0     | 0.0     | 6.0     |
| Balsara 3   | Left  | 5/3         | 1.0         | 1000.0  | 0.0     | 0.0     | 0.0     | 10.0    | 0.0     | 7.0     |
|             | Right | 5/3         | 1.0         | 1.0     | 0.0     | 0.0     | 0.0     | 10.0    | 0.0     | 7.0     |
| Balsara 4   | Left  | 5/3         | 1.0         | 1.0     | 0.1     | 0.0     | 0.0     | 10.0    | 0.0     | 7.0     |
|             | Right | 5/3         | 1.0         | 1.0     | 0.1     | -0.999 | 0.0     | 10.0    | -7.0    | -7.0    |
| Balsara 5   | Left  | 5/3         | 1.08        | 0.95    | 0.4     | 0.3     | 0.2     | 2.0     | 0.3     | 0.3     |
|             | Right | 5/3         | 1.0         | 1.0     | -0.45   | -0.2    | 0.2     | 2.0     | -0.7    | 0.5     |
| Generic Alfvén Wave | Left | 5/3         | 1.0         | 5.0     | 0.0     | 0.3     | 0.4     | 1.0     | 6.0     | 2.0     |
|             | Right | 5/3         | 0.9         | 5.3     | 0.0     | 0.0     | 0.0     | 1.0     | 5.0     | 2.0     |
5.1.1. Komissarov Shock-tube Test

The first shock-tube test consists of a fluid with $\Gamma = 4/3$, initially at rest, and with a constant magnetic field, which is normal to the initial discontinuity. The rest-mass density and pressure on the left state are different from those on the right. However, the main feature of this test is that the pressure on the left is $10^3$ times that on the right one. Such a gradient produces a thin shell in the density that moves with a relativistic velocity of approximately 0.9 times the speed of light. This behavior is showed in Figure 1 at time $t = 0.8$ for the magnetic susceptibilities $\chi_m = -0.20, -0.10, -0.05, 0.00, 0.05, 0.10, 0.20$. We can also distinguish a contact discontinuity and a shock wave, both moving to the right, and a rarefaction zone in the central region of the domain. This test is very interesting because the magnetic field is normal to the initial discontinuity and therefore, as pointed out in Komissarov (1999), the solution is purely hydrodynamical, i.e., the magnetic field does not have a dynamical effect on the evolution. As a consequence of this fact, the solution showed in Figure 1 does not change with the magnetic susceptibility.

5.1.2. Komissarov Collision Test

The second test describes the head-on one-dimensional collision of two fluids with the same rest-mass density and pressure. Both fluids, characterized by an adiabatic index $\Gamma = 4/3$, collide with each other with a relativistic velocity $v^x = 0.98058$ at the position $x = 0$. In this test, the tangential components of the magnetic field $B^Y$ on both states have

Figure 1. Komissarov shock-tube test at time $t = 0.8$. We use a spatial resolution of $\Delta x = 1/800$ and a Courant factor of 0.25.
opposite directions. The numerical solution of this initial value problem at time $t = 0.8$ is presented in Figure 2, for the magnetic susceptibilities $\chi_m = -0.30, -0.20, -0.10, -0.05, 0.00, 0.05, \text{ and } 0.10$. We can observe two fast shock waves moving in opposite directions, with two slow shocks behind each of them. The solutions for different magnetic susceptibilities show that the fast shocks move faster in diamagnetic materials and the slow shocks move slightly faster in paramagnetic ones.

Due to the symmetry of the collision, the tangential magnetic field and the normal velocity vanishes between the slow shocks (Komissarov 1999), independently of $\chi_m$. Nevertheless, when the magnetic polarization in the fluid changes, the constant states between the waves take different values. For instance, the maximum value of the rest-mass density increases with $\chi_m$, the transverse component of the magnetic field in the constant states between the slow and fast shocks changes considerably from the usual RMHD case ($\chi_m = 0$), and the total pressure gradient through the slow shocks is reversed from a particular positive value of $\chi_m$. Finally, it is worth mentioning that in the paramagnetic materials considered here, a transverse relativistic flow developed. The relativistic character of this flow is significantly reduced in diamagnetic fluids.

5.1.3. Balsara 1

This test is the relativistic generalization of the Brio–Wu problem in Newtonian MHD (Brio & Wu 1988). The initial
data consist of a two-state fluid with $\Gamma = 2$. On both sides of the diaphragm, the fluid is initially at rest, but with different mass densities and pressures. Moreover, the component of the magnetic field that is normal to the interface is the same in both states, while the tangential component has the opposite direction. Figure 3 shows the structure of the solution at $t = 0.4$ for the values of magnetic susceptibility $\chi_m = -0.20, -0.10, -0.05, 0.00, 0.05, 0.10, \text{and } 0.20$. For all of these values, we obtain solutions that consist of a leftward fast rarefaction, a leftward compound wave, a contact discontinuity, a rightward slow shock, and a rightward fast rarefaction. These waves are the same as those obtained in the usual relativistic MHD simulations (Giacomazzo & Rezzolla 2007; Lora-Clavijo et al. 2015a). Nevertheless, the rarefaction waves move considerably faster for diamagnetic materials than for materials with paramagnetic properties, while the compound, contact, and shock waves move slower in diamagnetic fluids than in paramagnetic ones.

On the other hand, the rest-mass density gradient through the shock becomes greater when the diamagnetic character of the fluid increases. The solution for the total pressure shows that the magnetic polarization tends to increment the gradients of this state variable through the waves. In particular, we obtain an interesting behavior across the slow shock, because after a certain positive value of $\chi_m$, the total pressure gradient changes sign, hence the dynamics of the fluid becomes different. For
example, when $\chi_m \leq 0$, the fluid moves in opposite directions across the shock, but in the paramagnetic cases, all of the fluid moves to the right. Furthermore, it is interesting to see that, between the shock and the rightward rarefaction, any magnetic polarization tends to generate a transverse flow in the direction opposite to the rest of the fluid. Finally, the magnetic field strength increases in paramagnetic fluids, specially in the constant state between the shock and the rightward rarefaction.

5.1.4. Balsara 2

The second test of Balsara consists of a fluid with $\Gamma = 5/3$ initially at rest, in which the left state pressure is 30 times the pressure on the right, and the transverse magnetic field is discontinuous at $x = 0$. The numerical solution to this problem at time $t = 0.4$ is presented in Figure 4 for the magnetic susceptibilities $\chi_m = -0.20, -0.10, -0.05, 0.00, 0.05, 0.10,$ and $0.15$. The different plots show a fast rarefaction wave moving to the left, a slow rarefaction wave moving to the right, and a rightward contact discontinuity that moves along with the two shock waves. As we can see, all of the waves in this solution move faster in diamagnetic materials than in paramagnetic ones, but the rarefaction waves present the most significant differences.

In this test, it is important to mention that the rarefaction zones are reduced when we consider diamagnetic fluids and that the most important changes in the state variables are obtained between the rarefaction waves and between the shocks. In particular, the relativistic character of the flow in
the $x$-direction increases when we consider fluids that are increasingly diamagnetic. Conversely, a transverse relativistic flow is generated between the shocks for the paramagnetic case with $\chi_m = 0.15$. We can also observe a transverse flow that moves in the opposite direction to the rest of the fluid in the solution with $\chi_m = -0.2$. As a final comment, we note that it is more difficult for the code to deal with paramagnetic materials, because the solution for $\chi_m = 0.15$ shows oscillations, especially between the shocks.

5.1.5. Balsara 3

The initial state of Balsara 3 is quite similar to that of Balsara 2, with the main difference that now, the left pressure is 10,000 times the pressure on the right. With this $\Delta p$, the range of values of $\chi_m$ that we can treat with the code is reduced. In particular, we use the magnetic susceptibilities $\chi_m = -0.20, -0.10, -0.05, 0.00, 0.05, 0.08,$ and $0.10$. The wave structure of the solution, shown in Figure 5 at time $t = 0.4$, is the same as in the previous test. Nevertheless, since the pressure gradient is so big, it is difficult to capture the true constant state between the shocks (Giacomazzo & Rezzolla 2007) with the resolution that we utilized. Another consequence of the high pressure in the left state is that the magnetic susceptibility does not produce significant changes in the solution as compared with Balsara 2. The main change is that the slow rarefaction wave zone increases when we consider diamagnetic materials.
5.1.6. Balsara 4

The fourth test of Balsara is a head-on collision of two fluids, with $\Gamma = 5/3$ and a transverse magnetic field, in opposite directions on both sides of the interface. This test is similar to the Komissarov collision but with a different adiabatic index and with the fluids moving against each other with a normal velocity of 0.999 times the speed of light. In Figure 6, we show a snapshot of the numerical solution at time $t = 0.4$ for the values $\chi_m = -0.20, -0.10, -0.05, 0.00, 0.05, 0.10,$ and 0.20. We observe in this numerical solution the same shock waves as in the Komissarov collision, and we notice again that the fast shocks travel faster in diamagnetic fluids. However, the slow shocks show a different behavior because they move slower in materials with positive $\chi_m$. Finally, the maximum value of the rest-mass density does not change appreciably when we vary the magnetic susceptibility.

5.1.7. Balsara 5

The last test of Balsara is a collision of two fluids with an adiabatic index of $\Gamma = 5/3$, but in this case, the transverse components of the velocity and the magnetic field are discontinuous in the plane $x = 0$ (see Table 1). The numerical solution to this problem for the values $\chi_m = -0.20, -0.10, -0.05, 0.00, 0.05, 0.10,$ and 0.20, which is shown in Figure 7 at time $t = 0.55$, is composed of four waves moving to the left: a fast shock, an Alfvén discontinuity, a slow rarefaction, and a...
contact discontinuity; and three waves moving to the right: a slow shock, an Alfvén discontinuity, and a fast shock (Giacomazzo & Rezzolla 2006).

The numerical solutions with different magnetic susceptibilities show that the fast shocks, the fast rarefaction, and the Alfvén discontinuities move faster in diamagnetic fluids. Additionally, the maximum values of the rest-mass density and the total pressure increase when we reduce the magnetic susceptibility. Finally, as in previous cases, we observe that from a positive value of $\chi_m$, the total pressure gradient across the rarefaction wave and the slow rightward shock change signs.

5.1.8. Generic Alfvén Wave

In this last test, we have an initial state similar to that in Balsara 5: the transverse components of the velocity and the magnetic field are discontinuous at the interface. Nevertheless, in this case, the fluid on the right is initially at rest, while on the left, there is a purely transverse flow. The discontinuity in the magnetic field (magnitude and direction) is in turn higher in this test than in the previous one. The solution, which is shown in Figure 8 at time $t = 0.4$ for $\chi_m = -0.20, -0.10, -0.05, 0.00, 0.05, 0.10, \text{and} 0.20$, consists of three leftward waves—a fast rarefaction, an Alfvén discontinuity, and a slow shock—
and four rightward waves—a contact discontinuity, a slow shock, an Alfvén discontinuity, and a fast shock.

The solutions for different values of magnetic susceptibility exhibit similar behavior to those of the previous test. However, in this case, the Alfvén waves seem to travel at the same speed regardless of $\chi_m$, and the constant state between the leftward Alfvén discontinuity and the leftward slow shock tends to occupy more $x$ domains in diamagnetic fluids. As a final comment about these last two tests, we notice that it is more difficult to capture numerically the Alfvén discontinuities in paramagnetic materials than in those with diamagnetic properties.

5.2. Magnetized Spherical Accretion of a Magnetically Polarized Perfect Fluid

In this section, we will analyze the ability of the code to maintain the stationary solution that describes the spherical accretion of a magnetically polarized perfect fluid onto a Schwarzschild black hole in the presence of a radial magnetic field. Previously, in De Villiers & Hawley (2003), it has been shown that the hydrodynamical solution obtained by Michel (1972) remains the same when a radial magnetic field is added. This magnetized solution does not represent any real physical system (Antón et al. 2006), but it is a useful nontrivial test in numerical GRMHD. Now, following De
Villiers & Hawley (2003), it is possible to show that when we consider the constitutive relation for a linear medium, Equation (36), and maintain the radial character of the magnetic field, the Michel solution is not affected. To prove this result, we compute the component

\[ T_r' = (\rho + b^2(1 - \chi))u^r u_t - b c H b(1 - \chi), \]

which is the only one that is involved in the conservation of the energy flux \( \nabla_r T_r' = 0 \). Now, from Equation (23), it is possible to show that \( b c H b = b u^r u_t \), so the terms with a magnetic field cancel each other. Therefore, we can use this test, but now in the context of GRMHD with magnetically polarized matter to estimate the order of the global convergence of the code with different magnetic susceptibilities.

The initial data for this test are the set of primitive variables \( \mathbf{W} = [\rho, v^r, p, B^r]^T \), where the rest-mass density and the pressure are related by a polytropic equation of state with an adiabatic index of \( \Gamma = 4/3 \). The first three variables are given by the Michel solution, while the radial component of the magnetic field is chosen to satisfy the divergence-free condition. The hydrodynamic solution is completely determined by fixing the critical radius \( r_c \) and the critical rest-mass density \( \rho_c = \rho(r_c) \). To carry out our simulations, we use the same parameters as in Giacomazzo & Rezzolla (2007), for which \( r_c = 8 \) (we take \( M_{\text{BH}} = 1 \) and \( \rho_c = 6.25 \times 10^{-2} \)). Now, we use Eddington–Finkelstein coordinates, so we can choose a spatial domain \( r \in [1.9, 20.9] \) with \( N = 50 \) radial zones. On the other hand, the component \( B^r \) at \( t = 0 \) is obtained through the equation \( B^r = C_{\text{IH}} \rho_{\text{IH}} \rho_{\text{IH}} \), where \( C_{\text{IH}} = 4 \rho_{\text{IH}} \beta_{\text{IH}} \). So, we only need to define the magnetic field strength, \( \beta_{\text{IH}} = b^2 / \rho \), and the rest-mass density, \( \rho_{\text{IH}} \), at the event horizon \( \rho_{\text{EH}} = 2 \).

In the simulations, we use the following positive values of magnetic susceptibility, \( \chi_m = 0.000, 0.001, 0.005, \) and \( 0.008 \), since we noticed in the last section that it is more difficult to deal with paramagnetic materials. For each of these values, we compute the error,

\[ L_1 = \frac{\sum (\rho(r) - \rho_{\text{exact}}(r))}{\sum \rho_{\text{exact}}(r)}, \]

in the rest-mass density with different values of \( \beta_{\text{IH}} \). In particular, we take \( \beta_{\text{IH}} = 1, 10, \) and \( 25 \), which is equivalent to a ratio between the magnetic pressure and the gas pressure \( p_m / p = 3.88, 38.80, \) and \( 97.00 \), respectively. We found that, although the numerical solution is effectively maintained over time \( t > 4000 \), the stationary state of the radial velocity deviates from the analytic solution when we increase the values of \( \beta_{\text{IH}} \) and \( \chi_m \). This behavior has already been reported by Shibata & Sekiguchi (2005), but only for the case where the fluid is not magnetically polarized.

Now, with the aim of estimating the order of convergence, we present in Figure 9 the \( L_1 \) norm of the error for the rest-mass density, at time \( t = 200 \), as a function of the number of radial zones \( N \). Figure 9(a) corresponds to the GRMHD evolution for the three values of \( \beta_{\text{IH}} \), without magnetic polarization, i.e with \( \chi_m = 0 \). We notice from this plot that the convergence of the code is slightly greater than 2 for the magnetic field strengths that we considered. A similar result is obtained when we evolve the magnetized Michel solution with a magnetic susceptibility of \( \chi_m = 0.001 \) (see Figure 9(b)). Nevertheless, we notice in Figure 9(c) that when \( \chi_m \) is incremented to 0.005, the convergence is reduced to second order for all values of \( \beta_{\text{IH}} \) used in the simulations. The plots of Figure 9(d) show that when the magnetic susceptibility of the fluid is \( \chi_m = 0.008 \), the order of convergence is considerably reduced in the cases with \( \beta_{\text{IH}} = 10 \) and \( \beta_{\text{IH}} = 25 \). We can say that the ability of the code to deal with magnetically polarized fluids in strong magnetic and
The gravitational fields depends not only on the ratio between the magnetic and gas pressures, but also on the magnetic susceptibility. The global convergence of the code is ≥2 for $\chi_m \lesssim 0.005$ for all magnetic field strengths $\beta$ considered here. Nevertheless, when $\chi_m = 0.008$, the global convergence of the code is between the first and second orders for $\beta = 10$ and $\beta = 25$. These results are interesting because a magnetic field strength of $\beta \approx 4$ corresponds to a large magnetic field of $\approx 10^{19}$ G (Giacomazzo & Rezzolla 2007), and the typical values for the magnetic susceptibility of the materials are around $10^{-5}$. Therefore, if we assume the strongest magnetic field in the universe, $\approx 10^{15}$ G, which could correspond to a magnetar, we can study with CAFE the magnetic polarization of realistic magnetic media in a strong gravitational field with high precision and second order of convergence.

As a final comment, as we mention in Section 4, CAFE preserves the free divergence constraint of the magnetic field to machine precision in all simulations that we present in this work. We show in Figure 10 the evolution over time of the maximum value of $\partial_t(\sqrt{\gamma}B^t)$ in the hardest spherical accretion test, in which the fluid has a magnetic susceptibility of $\chi_m = 0.008$ and the magnetic field is such that $\beta_B = 25$. In this figure, we note that after the first time step, the magnetic field divergence oscillates between $7 \times 10^{-15}$ and $1 \times 10^{-14}$.

6. Concluding Remarks

In this paper, we have presented for the first time the conservative form of the ideal GRMHD equations for a magnetically polarized fluid endowed with a magnetic field and around a strong gravitational field. With the aim of solving numerically the last system of equations, we have also computed its eigenvalue structure by following the Anile procedure. We have found that in the particular case where the magnetic polarization vector is in the same direction as the magnetic field, the speed of the material waves is independent of the magnetic susceptibility $\chi_m$ and is the same as in the usual GRMHD case. Nevertheless, the Alfvén eigenvalues and the bounds on the magnetosonic speeds depend on the magnetic susceptibility. The constitutive relation (36), which we use to compute the eigenvalues, allows us to study the role of the diamagnetic and paramagnetic fluids in astrophysical scenarios. Now, in order to compute the primitive variables from the conservative ones, we generalized the primitive variable recovery proposed by Mignone & Bodo (2006) to include the magnetic polarization of the material.

With the new numerical and theoretical results implemented in the CAFE code, we have carried out the first one-dimensional shock-tube simulations with magnetically polarized fluids in the Minkowski spacetime. When the magnetic susceptibility is zero, the numerical solutions are the same as those obtained in the usual GRMHD tests (Giacomazzo & Rezzolla 2006; Lora-Clavijo et al. 2015a), but when $\chi_m = 0$, we obtained significant differences. For instance, we found that the propagation speed of the fastest waves in the solutions is greater in diamagnetic materials than in paramagnetic ones. This behavior is interesting because it is independent of the initial configuration of the problem. Another remarkable result is that the magnetic susceptibility considerably increases the relativistic character of the flows in some problems, such as the Komissarov collision or Balsara 2. Moreover, in the Balsara 2 and Balsara 3 tests, we noticed that magnetic polarization can reverse the direction of the flows compared with the solutions where $\chi_m = 0$. Additionally, we also found that the total pressure gradient across the waves can be reversed due to the paramagnetic character of the fluids. This behavior clearly appears, for instance, in the Balsara 4 and the generic Alfvén tests. All of these differences between the cases with $\chi_m = 0$ and the cases with $\chi_m \neq 0$ are more evident when the magnetic pressure dominates over the gas pressure. Additionally, it is important to mention that the ability of the code to deal with magnetically polarized fluids is reduced with increasing $|\chi_m|$; in particular, it is more difficult to numerically evolve paramagnetic fluids.

On the other hand, with the aim of testing the code in the strong gravitational field regime, we have presented for the first time the magnetized Michel accretion of a magnetically polarized fluid. We have shown that with the constitutive relation (36), the Michel solution is not affected, so we can use it to test the code. Now, we have only considered paramagnetic fluids because it is more difficult to maintain the stationary solution over time when $\chi_m > 0$. From the simulations, we have found that the solution is effectively maintained over time ($t > 4000$), and that the global convergence of the code is 2 for $\chi_m \lesssim 0.005$ and for all magnetic field strengths $\beta$ that we considered. Nevertheless, when $\chi_m = 0.008$ and $\beta \geq 10$, the global convergence of the code is reduced to a value between the first and second orders. These results are interesting because even the strong magnetic fields in the magnetars are within $\beta \leq 4$ (Zhang & Mészáros 2001), and the typical values for the magnetic susceptibility of the materials are around $10^{-5}$.

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Appendix

Primitive Variable Recovery

In order to write down the primitive variables $\rho$, $v^i$, $p$, and $B^k$, in terms of the conservative ones, $D$, $S_\parallel$, $\tau$, and $B^k$, we need
to solve the $5 \times 5$ algebraic system, Equations (19)–(21). Unfortunately, the GRMHD with magnetically polarized matter shares the same feature as GRMHD: it is not possible to find $W(U)$ in a closed form (Noble et al. 2006). In this paper, we follow the method proposed by Mignone & Bodo (2006) to reduce the five nonlinear equations to only one by setting $Z = \rho \, h W^2$. Considering the constitutive relation (36), we compute the scalar $S^2 = \gamma^2 S_i S^i$ from Equation (20). The resulting expression takes the form

$$S^2 = (Z + \tilde{\chi} B^2)^2(1 - W^{-2}) - \tilde{\chi}(2Z + \tilde{\chi} B^2) \left( \frac{B \cdot S}{Z} \right)^2,$$

where we have defined $\tilde{\chi} = 1 - \chi$. The conservative variable $\tau$ can also be written in terms of $Z$ as

$$\tau = Z + \tilde{\chi} B^2 - p - D + \frac{1 - 2 \tilde{\chi}}{2 W^2} B^2 + \frac{1 - 2 \tilde{\chi}}{2} \left( \frac{B \cdot S}{Z} \right)^2.$$

(69)

In the last two expressions, we have used the fact that $\alpha^2(b^2) = W^2 (B \cdot S)^2 / S^2$. Additionally, with the ideal gas equation of state, $p = \rho c (\Gamma - 1)$, we can replace the thermodynamic pressure in Equation (69) by

$$p = \frac{Z - WD}{W} \, \Gamma - 1 \quad \frac{1}{\Gamma}. $$

(70)

Therefore, we have reduced the $5 \times 5$ system of equations to a pair of equations for unknown $Z$ and $W$. Nevertheless, from Equation (68), we can find the following expression for the Lorentz factor in terms of $Z$ and the conservative variables

$$W(Z) = \left[ 1 - \left( \frac{B \cdot S}{Z} \right)^2 \left( \frac{2 \tilde{\chi} B^2 + S^2 Z^2}{Z^2} \right)^{-1/2} \right].$$

(71)

In this way, Equation (69), with $p$ and $W$ given in Equations (70) and (71), respectively, becomes a nonlinear equation for the unknown $Z$. To solve this equation, CAFE uses a Newton–Raphson algorithm together with the bisection method for finding the roots of the function

$$f(Z) = Z + \tilde{\chi} B^2 - (D + \tau) - p + \frac{1 - 2 \tilde{\chi}}{2 W^2} B^2 + \frac{1 - 2 \tilde{\chi}}{2} \left( \frac{B \cdot S}{Z} \right)^2 = 0. $$

(72)

Those roots correspond to the values of $Z$.

Once $Z$ has been obtained, it is possible to compute the Lorentz factor with Equation (71), and therefore the rest-mass density is obtained as $\rho = D / W$. Finally, the thermodynamic pressure is computed from the equation of state (70), and the components of the spatial velocity $\nu^i$ through the equation

$$\nu^i = \frac{S^i + \tilde{\chi}(B \cdot S)B^i / Z}{Z + \tilde{\chi} B^2},$$

which is obtained from Equation (20). Note that when the magnetic susceptibility is zero, i.e., when $\tilde{\chi} = 1$, all expressions in this appendix reduce to those of the GRMHD (Giacomazzo & Rezzolla 2007).
Poisson, E. 2004, A Relativist’s Toolkit: The Mathematics of Black-Hole Mechanics (Cambridge: Cambridge Univ. Press)
Potekhin, A. Y., & Yakovlev, D. G. 2012, PhRvC, 85, 039801
Ruggeri, & Tommaso, S. A. 1981, Annales de l’I.H.P. Physique théorique, 34, 65
Schumacher, K. R., Riley, J. J., & Finlayson, B. A. 2010, PhRvE, 81, 016317
Shibata, M., & Sekiguchi, Y.-I. 2005, PhRvD, 72, 044014
Siegel, D. M., & Metzger, B. D. 2017, PhRvL, 119, 231102
Suh, I.-S., & Mathews, G. J. 2010, ApJ, 717, 843
Tchekhovskoy, A., Narayan, R., & McKinney, J. C. 2011, MNRAS, 418, L79
Toro, E. 2009, Riemann Solvers and Numerical Methods for Fluid Dynamics (Berlin: Springer)
van Putten, M. H. P. M. 1993, JCoPh, 105, 339
Yee, K. 1966, ITAP, 14, 302
Zhang, B., & Mészáros, P. 2001, ApJL, 552, L35

Suh, I.-S., & Mathews, G. J. 2010, ApJ, 717, 843
Tchekhovskoy, A., Narayan, R., & McKinney, J. C. 2011, MNRAS, 418, L79
Toro, E. 2009, Riemann Solvers and Numerical Methods for Fluid Dynamics (Berlin: Springer)
van Putten, M. H. P. M. 1993, JCoPh, 105, 339
Yee, K. 1966, ITAP, 14, 302
Zhang, B., & Mészáros, P. 2001, ApJL, 552, L35