New bounds for the distance Ramsey number*

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Abstract

In this paper we study the distance Ramsey number $R_D(s, t, d)$. The distance Ramsey number $R_D(s, t, d)$ is the minimum number $n$ such that for any graph $G$ on $n$ vertices, either $G$ contains an induced $s$-vertex subgraph isomorphic to a distance graph in $\mathbb{R}^d$ or $\overline{G}$ contains an induced $t$-vertex subgraph isomorphic to the distance graph in $\mathbb{R}^d$. We obtain the upper and lower bounds on $R_D(s, s, d)$, which are similar to the bounds for the classical Ramsey number $R\left(\left\lceil \frac{s}{d/2} \right\rceil, \left\lceil \frac{s}{d/2} \right\rceil\right)$.

1 Introduction

In this paper we analyze properties of distance graphs from the point of view of Ramsey theory (see [9], [16]). Let us remind the notion of distance graph.

Definition 1. A graph $G$ is the (unit) distance graph in $d$-dimensional Euclidean space $\mathbb{R}^d$ if

$V(G) \subseteq \mathbb{R}^d; \quad E(G) \subseteq \{(x; y) \in V^2 : |x - y| = 1\}$.

The study of various properties of finite distance graphs was motivated by Erdős’ work [6], where he stated three fundamental problems of combinatorial geometry. One of the problems is the following: how many can there be unit distances among $n$ points on the plane? In terms of distance graphs this question can be stated as follows. Let $G$ be a distance graph in $\mathbb{R}^2$. What is the maximum value of $|E(G)|$ provided that $|V(G)| = n$?

Another problem that is closely related to properties of distance graphs is the famous Nelson–Hadwiger problem on finding the chromatic number $\chi(\mathbb{R}^d)$ of the space (see [11]). On the one hand, for every distance graph $G$ in $\mathbb{R}^d$ we have $\chi(G) \leq \chi(\mathbb{R}^d)$, where $\chi(G)$ is the usual chromatic number of the graph. On the other hand, Erdős–de Bruijn theorem (see [4]) states that $\chi(\mathbb{R}^d) = \chi(H)$ for some finite distance graph $H$ in $\mathbb{R}^d$.

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These and other well-known problems such as Borsuk’s partition problem (see [12], [13]) give the motivation to analyze different properties of finite distance graphs (various problems concerning distance graphs can be found in [3]).

Another combinatorial field, which lies at the basis of this work, is Ramsey theory. Recall the definition of the Ramsey numbers $R(s, t)$.

**Definition 2.** Given $s, t \in \mathbb{N}$, the Ramsey number $R(s, t)$ is the minimum number $n$ such that for any graph $G$ on $n$ vertices, either $G$ contains an $s$-vertex independent set (i.e., a set without edges) or its complement $\bar{G}$ contains a $t$-vertex independent set.

The main concept in this work is that of distance Ramsey number.

**Definition 3.** The distance Ramsey number $R_D(s, t, d)$ is the minimum number $n$ such that for any graph $G$ on $n$ vertices, either $G$ contains an induced $s$-vertex subgraph isomorphic to the distance graph in $\mathbb{R}^d$ or $\bar{G}$ contains an induced $t$-vertex subgraph isomorphic to the distance graph in $\mathbb{R}^d$.

Since for every $d \geq 1$ an independent set of any finite size can be realized as the distance graph in $\mathbb{R}^d$, we have the following obvious inequality: $R_D(s, t, d) \leq R(s, t)$.

Best known bounds for classical Ramsey numbers are the following:

$$\frac{\sqrt{2}}{e} (1 + o(1)) s 2^{\frac{3}{2}} \leq R(s, s) \leq e^{-\gamma \frac{\ln^2 s}{\ln \ln s}} \cdot 4^s, \quad \gamma > 0.$$  

The lower bound is due to Spencer and can be found in [1], the upper bound is due to Conlon [5].

Conlon’s bound immediately implies the following upper bound on diagonal distance Ramsey numbers:

$$R_D(s, s, d) \leq 4^s e^{-\gamma \frac{\ln^2 s}{\ln \ln s}}, \quad \gamma > 0.$$  

The concept of distance Ramsey number was introduced and studied in the paper [14], in which several asymptotic lower bounds were obtained. Distance Ramsey number was also studied in [10] and [15]. In these papers authors introduced different methods to obtain lower bounds on $R_D(s, t, d)$ for the case of small fixed $d$. The sharpest bounds for $d \in \{2, \ldots, 8\}$ are stated in the following theorems (Theorems 1, 2 see in [15], Theorem 3 see in [10]).

**Theorem 1.** Let $d = 2$. There exists a positive constant $c$ such that

$$R_D(s, s, d) \geq 2^{\frac{3}{2} - \frac{1}{3} \ln s}.$$  

**Theorem 2.** Let $d = 3$. There exists a positive constant $c$ such that

$$R_D(s, s, d) \geq 2^{\frac{5}{2} - c \beta(s) \frac{1}{2} \ln s},$$  

where $\beta(s) = 2^{\alpha(s)}$, and $\alpha(s)$ is inverse Ackermann function.
**Theorem 3.** Let \( d \in \{4, \ldots, 8\} \). We have

\[
R_D(s,s,d) \geq \frac{1}{e \cdot 2^{2d-1-1}}(1 + o(1))k2^{\frac{k}{2}},
\]

where \( k = \lfloor c_ds \rfloor \) and

\[
c_4 = 0.04413, \quad c_5 = 0.01833, \quad c_6 = 0.00806, \quad c_7 = 0.00352, \quad c_8 = 0.00165.
\]

Proofs of these theorems rely on some special properties of distance graphs in small dimensions. In cases \( d = 2, 3 \) the sharpest bound is based on the fact that the number of edges in a distance graph on \( n \) vertices in \( \mathbb{R}^2, \mathbb{R}^3 \) does not exceed \( n^{2-\varepsilon} \) for some \( \varepsilon > 0 \). However, distance graphs do not have this property in spaces \( \mathbb{R}^d, d = 4, \ldots, 8 \). For every \( m \in \mathbb{N} \) we can realize a complete bipartite graph \( K_{m,m} \) as the distance graph in \( \mathbb{R}^4 \). Indeed, consider two circles

\[
C_1 = \{(x_1, x_2, 0, 0) \in \mathbb{R}^4 : x_1^2 + x_2^2 = 1/2\}
\]

and

\[
C_2 = \{(0, 0, x_3, x_4) \in \mathbb{R}^4 : x_3^2 + x_4^2 = 1/2\}.
\]

Then, by Pithagoras’ theorem, the distance between any point of \( C_1 \) and any point of \( C_2 \) equals 1. Hence, we can embed one part of \( K_{m,m} \) into \( C_1 \), and the second part into \( C_2 \). In cases \( d = 4, \ldots, 8 \) the proofs of the bounds are based on the following type of claims: every \( n \)-vertex distance graph in \( \mathbb{R}^d \) contains several non-overlapping independent sets of sufficiently large (depending on \( n \)) total cardinality.

In this paper we describe a method that allows us to obtain much sharper bounds on distance Ramsey number \( R_D(s,s,d) \) for every fixed \( d \geq 4 \). We state the bounds in the following theorem and in proposition 1.

**Theorem 4.** Let \( d \geq 4 \). The following inequality holds:

\[
R_D(s,s,d) \geq 2\left(\frac{1}{2^{d/2}} - o(1)\right)^s.
\]

Theorem 4 significantly strengthens the bounds from Theorem 3. Moreover, Theorem 4 gives essentially the same bounds for \( d \in \{2, 3\} \) as Theorems 1 and 2 do, though these theorems provide an explicit formula for the \( o(1) \) factor in the exponent. As we will see from the proof in general it is difficult to express this factor explicitly using the new method.

For a graph \( G \) let \( Cl(G, r) \) denote the number of \( r \)-cliques in \( G \), and put \( cl(G, r) = |Cl(G, r)|. \)

To prove Theorem 4 we need the following theorem.

**Theorem 5.** For any fixed natural \( d \) there exists \( \varepsilon > 0 \) and there exists \( n_0 \in \mathbb{N} \) such that for every distance graph \( G \) in \( \mathbb{R}^d \) with \( n \geq n_0 \) vertices

\[
cl\left(G, \left\lfloor \frac{d}{2} \right\rfloor + 1\right) \leq n^{\left\lfloor \frac{d}{2} \right\rfloor + 1 - \varepsilon}.
\]

This theorem allows us to generalize the method used to obtain bounds in Theorems 1 and 2. We prove this theorem in Section 2. In Section 3 we present the proof of Theorem 4. Finally, in Section 4 we prove
Proposition 1. For any $1 \leq d \leq s$ we have

$$R_D(s, s, d) \leq 2 \left\lceil \frac{d}{2} \right\rceil R\left(\left\lceil \frac{s}{d/2} \right\rceil, \left\lceil \frac{s}{d/2} \right\rceil \right) \leq 4 \frac{s}{d/2} (1 + o(1)).$$

The proposition significantly strengthens the described above trivial upper bound. Moreover, the estimate for $R_D(s, s, d)$, which is given in Theorem 4 and Proposition 1 turns out to be essentially the same as for the classical Ramsey number $R\left(\left\lceil \frac{s}{d/2} \right\rceil, \left\lceil \frac{s}{d/2} \right\rceil \right)$:

$$\frac{s}{2[d/2]} (1 + \bar{o}(1)) \leq \log R_D(s, s, d) \leq \frac{2s}{[d/2]} (1 + \bar{o}(1)).$$

Therefore, in some sense we solve the problem completely for fixed $d$.

2 Proof of Theorem 5

We use $K_{l_1, \ldots, l_r}$ to denote a complete $r$-partite graph which parts have cardinalities $l_1, \ldots, l_r$.

Theorem 5 follows from Proposition 2 and Corollary 1 of Theorem 6. Let us begin with the proposition.

Proposition 2. If $G$ is a distance graph in $\mathbb{R}^d$, then $G$ does not contain a subgraph isomorphic to $K_{3, \ldots, 3}^{\left\lceil \frac{d}{2} \right\rceil + 1}$.

Proof. The proof uses induction on $d$.

First, we verify the proposition for $d \in \{2, 3\}$. Suppose that the distance graph $G$ in $\mathbb{R}^3$ has a subgraph, isomorphic to $K_{3,3}$. Consider three vertices $v_1, v_2, v_3$ from the first part. The other vertices of the subgraph lie on the line $l$, that is orthogonal to plane $\text{aff} \langle v_1, v_2, v_3 \rangle$ and passes through a circumcenter of the triangle with vertices $v_1, v_2, v_3$. But the line $l$ contains at most two points that lie at unit distance apart from $v_1, v_2, v_3$. Thus, the statement is true for $d \in \{2, 3\}$.

Assume that the proposition holds for $d$. Consider a distance graph $G \subset \mathbb{R}^{d+2}$. Suppose that it has a subgraph isomorphic to $K_{3, \ldots, 3}$ with $\left\lceil \frac{d}{2} \right\rceil + 2$ parts. Again consider vertices $v_1, v_2, v_3$ from the first part. All other vertices of the subgraph lie in the hyperplane that is orthogonal to plane $\text{aff} \langle v_1, v_2, v_3 \rangle$ and passes through a circumcenter of the triangle $v_1v_2v_3$. However, by the induction hypothesis there are no subgraphs in $d$-dimensional space isomorphic to $K_{3, \ldots, 3}$ with $\left\lceil \frac{d}{2} \right\rceil + 1$ parts. This contradiction concludes the proof.

Next we state Theorem 6, which is proven in [7]. We introduce some notation from [7]. Let $K^{(r)}(l_1, \ldots, l_r)$ be a complete $r$-partite $r$-uniform hypergraph which parts have cardinalities $l_1, \ldots, l_r$ (every edge has exactly one vertex from every partite set), and let $f(n; K^{(r)}(l_1, \ldots, l_r))$ be the least natural number such that every $r$-uniform hypergraph with $n$ vertices and $f(n; K^{(r)}(l_1, \ldots, l_r))$ edges has a subhypergraph isomorphic to $K^{(r)}(l_1, \ldots, l_r)$.
Theorem 6. (Erdős, [4] Theorem 1.) Let \( n > n_0(r, l) \), \( l > 1 \). For sufficiently large \( C \) (\( C \) does not depend on \( n, r, l \)) the following inequality holds:

\[
 f(n; K^{(r)}(l, \ldots, l)) \leq n^{r-1}. 
\]

Corollary 1. For given \( l \) and \( r \) there exists \( \varepsilon > 0 \) and \( n_0 \in \mathbb{N} \) such that if \( n \geq n_0 \) and \( n \)-vertex graph \( G \) does not have a subgraph isomorphic to \( K_{l, \ldots, l} \), then

\[
 cl(G, r) \leq n^{r-\varepsilon}. 
\]

Proof. Indeed, consider a graph \( G \) that does not contain a subgraph isomorphic to \( K_{l, \ldots, l} \). Construct a hypergraph \( \widetilde{G} = (V, \tilde{E}) \) with the vertex set that is the same as the vertex set of \( G \) and with the edge set consisting of all the \( r \)-cliques of the graph \( G \). Let \( |\tilde{E}| = m \) and suppose \( m \geq f(n; K^{(r)}(l, \ldots, l)) \). Note that \( m = cl(G, r) \). According to the definition, hypergraph \( \widetilde{G} \) has a subhypergraph isomorphic to \( K^{(r)}(l, \ldots, l) \). Thus \( G \) has a subgraph isomorphic to \( K_{l, \ldots, l} \), which contradicts the assumption.

Hence \( m < f(n; K^{(r)}(l, \ldots, l)) \). By Theorem 6 there exits \( \varepsilon > 0 \), \( \varepsilon = \varepsilon(l, r) \), such that \( m < n^{r-\varepsilon} \).

Proof of Theorem 4. Let \( G \) be a distance graph in \( \mathbb{R}^d \). By Proposition 4 \( G \) does not contain \( K_3, \ldots, 3 \). We apply Corollary 4 with \( r = \lfloor d/2 \rfloor + 1 \) and \( l = 3 \) to \( G \) and get the statement of Theorem 4.

3 Proof of Theorem 4

3.1 How to obtain lower bounds on \( R_D(s, s, d) \)

To obtain a lower bound \( R_D(s, s, d) > n \) for the distance Ramsey number we need to prove that there exists such a graph \( G \) on \( n \) vertices that every induced \( s \)-vertex subgraph of \( G \) and every induced \( s \)-vertex subgraph of \( \overline{G} \) is not isomorphic to a distance graph in \( \mathbb{R}^d \).

Let \( k = \lfloor d/2 \rfloor + 1 \), and let \( \varepsilon = \varepsilon(d) \) be the number from Theorem 5. Theorem 5 states that every graph \( H \) in \( \mathbb{R}^d \) on \( s \) vertices has at most \( s^{k-\varepsilon} \) \( k \)-cliques. We will prove that for a specific natural \( n \) there exists an \( n \)-vertex graph \( G \) such that every induced \( s \)-vertex subgraph of \( G \) and every induced \( s \)-vertex subgraph of its complement \( \overline{G} \) contains more than \( s^{k-\varepsilon} \) cliques of size \( k \). In this case the inequality \( R_D(s, s, d) > n \) takes place. The value \( s \) is supposed to be sufficiently large (see Theorem 5 and Theorem 4).

We use probabilistic method (see, e.g., [1]). For every natural \( n \) consider the classical Erdős – Rényi random graph model \( G(n, 1/2) \) (see, e.g., [1], [2]).
For every subset $S$, $|S| = s$, of the vertex set $V_n$ of a random graph $G \sim G(n, 1/2)$ we define the event $A_S$: the graph $G[S]$ has at most $s^{k-\varepsilon}$ cliques of size $k$. We use $A_S'$ to denote the event that the graph $\bar{G}[S]$ has at most $s^{k-\varepsilon}$ cliques of size $k$.

If we prove that for a certain $n$ there is a positive probability that none of the events $A_S, A'_S$ occur, i.e.

$$P\left(\bigcup_{S \subseteq V_n} (A_S \cup A'_S)\right) > 0,$$

then we obtain the bound $R_D(s, s, d) > n$.

Fix positive $\gamma$. In the case of Theorem 4 we choose $n$ equal to $2^{\left(\frac{1}{2}[d/2]-\gamma\right)s}$. We prove that for any positive $\gamma$ the above described probability is positive, which, in turn, gives us the statement of the theorem. To make the proof more transparent we begin with the case $d \in \{4, 5\}$. In these two cases we want to bound the distance Ramsey number by $2^{(\frac{1}{4}-\gamma)s}$ from below.

In Section 3.2 we deal with the case $d \in \{4, 5\}$. The crucial part of the proof is to bound the probability of each event $A_S, A'_S, S \subset V_n$. First we prove a weaker bound on the probability of single events, which is formulated in Theorem 7. It implies a weaker bound on the distance Ramsey number than the one we are to prove. Next we improve this bound using additional considerations, completing the proof of Theorem 4 for $d \in \{4, 5\}$. In Section 3.3 we discuss the proof of Theorem 4 for $d \geq 6$. This sequence of presentation is intended to clarify the method we use.

### 3.2 Case $d \in \{4, 5\}$

In this case we have $k = 3$, so we deal with triangles.

To bound the probability of each event $A_S, A'_S$ accurately enough we need to prove several propositions. For the sake of simplicity of presentation below we present a simpler method that doesn’t give the sharpest bound. Next we shortly describe how to modify it to obtain a better result.

**Theorem 7.** The following inequalities hold:

$$P(A_S) \leq \mathcal{P}, \quad P(A'_S) \leq \mathcal{P}, \quad \text{where} \quad \mathcal{P} = s! \cdot \left(\frac{7}{8}\right)^{\frac{s^2}{2}(1+o(1))}.$$

We will give the proof of Theorem 7 below. First we state a corollary.

**Corollary 2.** For $d \in \{4, 5\}$ we have the following lower bound for distance Ramsey number:

$$R_D(s, s, d) \geq \left(\frac{8}{7}\right)^{\frac{s^2}{2}(1+o(1))} \approx 2^{0.032107s}.$$

**Proof of corollary 2.** We bound the probability of the union of the events $A_S, A'_S$ by the sum of probabilities:

$$P\left(\bigcup_{S \subseteq V_n} (A_S \cup A'_S)\right) \leq \sum_{S \subseteq V_n} (P(A_S) + P(A'_S)) \leq \left(\frac{n}{s}\right) \cdot s! \cdot \left(\frac{7}{8}\right)^{\frac{s^2}{2}(1+o(1))} \leq n^s \cdot \left(\frac{7}{8}\right)^{\frac{s^2}{2}(1+o(1))}.$$
Therefore, there exists a function $\alpha(s) = 1 + o(1)$ such that if

$$n \leq \left( \frac{8}{7} \right)^{\frac{1}{\alpha(s)}},$$

then the following inequality holds:

$$P \left( \bigcup_{S \subseteq V_n} (A_S \cup A'_S) \right) > 0.$$

□

For the sake of brevity we use the notation $T(G)$ instead of $Cl(G, 3)$ and $t(G)$ instead of $|T(G)|$. To prove Theorem 7 we need the well-known Rödl’s theorem (see [17]).

**Theorem 8.** Let $M$ denote a collection of $l$-sets of $\{1, \ldots, n\}$ such that for all $A, B \in M$ holds $|A \cap B| \leq m - 1$. Put $g(l, m, n) = \max |M|$. For fixed $l, m$ and for $n \to \infty$ holds $g(l, m, n) \sim \left( \frac{n}{m} \right)^l \left( \lim_{n \to \infty} \frac{g(l, m, n)}{\left( \frac{n}{m} \right)^l} = 1 \right).

From now on we say that two graphs are disjoint if they have no edges in common. Fix an arbitrary maximum system of pairwise disjoint triangles in the set $S = \{1, \ldots, s\}$. We use $Tr(S)$ to denote this system.

**Corollary 3.** (from Theorem 8) Let $s \to \infty$. There exists $\psi(s), \psi(s) \to 0$ as $s \to \infty$, such that the following equality holds:

$$|Tr(S)| = \frac{s^2}{6}(1 + \psi(s)).$$

Consider a graph $H = (S, E)$ of order $s$ and a permutation $\sigma$ of its vertex set $S$. Let $\sigma(H)$ denote the graph with edges $\sigma(E) = \{(\sigma(a), \sigma(b)) \mid (a, b) \in E\}$. Consider the value $F(\sigma, H) = |T(\sigma(H)) \cap Tr(S)|$, which is the number of triangles that the sets $T(\sigma(H))$ and $Tr(S)$ have in common.

We choose a random permutation (from the uniform distribution over all permutations) and find the expectation of $F(\sigma, H)$. Define the function $\psi_1$ from the following equation:

$$\frac{s}{(s - 1)(s - 2)}(1 + \psi(s)) = \frac{1}{s}(1 + \psi_1(s)).$$

It is clear that $\psi_1(s) \to 0$ as $s \to \infty$.

**Claim 1.** For every graph $H$ on $s$ vertices the following holds:

$$\mathbb{E}(F(\sigma, H)) = \frac{|T(H)|}{s}(1 + \psi_1(s)).$$
Proof. We have:
\[
\mathbb{E}(F(\sigma, H)) = \sum_{\sigma} (|T(\sigma(H)) \cap Tr(S)|) \cdot \mathbb{P}(\sigma).
\]

The number of common triangles can be calculated as follows. Take a triangle \( \Delta \in T(H) \). Consider the indicator function of the triangle \( \sigma(\Delta) \) being an element of the set \( Tr(S) \):
\[
\mathbb{I}(\sigma(\Delta) \in Tr(S)) = \begin{cases} 
1, & \text{if } \sigma(\Delta) \in Tr(S), \\
0, & \text{if } \sigma(\Delta) \notin Tr(S).
\end{cases}
\]

We have
\[
|T(\sigma(H)) \cap Tr(S)| = \sum_{\Delta \in T(H)} \mathbb{I}(\sigma(\Delta) \in Tr(S)).
\]

Substituting this expression in the formula for the expectation of the number of common triangles we get
\[
\sum_{\sigma} (|T(\sigma(H)) \cap Tr(S)|) \cdot \mathbb{P}(\sigma) = \sum_{\Delta \in T(H)} \sum_{\sigma} \mathbb{I}(\sigma(\Delta) \in Tr(S)) \cdot \mathbb{P}(\sigma) = \\
= \sum_{\Delta \in T(H)} \sum_{\sigma} \mathbb{I}(\sigma(\Delta) \in Tr(S)) \cdot \mathbb{P}(\sigma).
\]

For every pair of triangles \( \Delta, \Delta' \in Tr(S) \) the number of permutations \( \sigma \), such that \( \sigma(\Delta) = \Delta' \), equals \((s-3)! \cdot 3! \) (there are 3! ways to rearrange vertices of the triangle \( \Delta' \), the other vertices are permuted arbitrarily). Thus the number of permutations \( \sigma \) such that \( \sigma(\Delta) \in Tr(S) \), is equal to \((s-3)! \cdot 3! \cdot |Tr(S)| \).

Since \( |Tr(S)| = \frac{s^2}{6}(1 + \psi(s)) \), we have the following chain of equalities:
\[
\sum_{\sigma} \mathbb{I}(\sigma(\Delta) \in Tr(S)) \cdot \mathbb{P}(\sigma) = \frac{(s-3)! \cdot 3! \cdot \frac{s^2}{6}}{s!}(1 + \psi(s)) = \\
= \frac{s^2}{s(s-1)(s-2)}(1 + \psi(s)) = \frac{1}{s}(1 + \psi_1(s)).
\]

This implies
\[
\sum_{\Delta \in T(H)} \sum_{\sigma} \mathbb{I}(\sigma(\Delta) \in Tr(S)) \cdot \mathbb{P}(\sigma) = (1 + \psi_1(s)) \sum_{\Delta \in T(H)} \frac{1}{s} = \frac{|T(H)|}{s}(1 + \psi_1(s)).
\]

Corollary 4. Let \( H \) be a graph on \( s \) vertices. If the inequality \( |T(H)| \leq s^{3-\delta} \) holds for some \( \delta > 0 \), then there exists a permutation \( \sigma \) of the set \( V(H) \) such that \( F(\sigma, H) \leq s^{2-\delta}(1 + \psi_1(s)) \).

Proof of Theorem 7. Let \( G \sim G(n, 1/2) \).

Let \( \delta \) from Corollary 4 be equal to \( \varepsilon \) from Section 3.1. Set \( z = s^{2-\varepsilon}(1 + \psi_1(s)) \).

For any \( s \)-subset \( S \) of the set \( V(G) \) we have
\[
\mathbb{P}(A_S) = \mathbb{P}\left(|T(G[S])| \leq s^{3-\varepsilon}\right) \leq
\]
\begin{align*}
\leq \mathbb{P}\left( \bigcup_{\sigma} (F(\sigma, G[S]) \leq z) \right) & \leq \sum_{\sigma} \sum_{i=0}^{z} \mathbb{P}(F(\sigma, G[S]) = i) = s! \cdot \sum_{i=0}^{z} \mathbb{P}(F(\sigma, G[S]) = i), \\
\end{align*}

where \(\sigma\) is an arbitrary permutation.

Let us bound the sum. Put \(a = \frac{s^2}{6}(1 + \psi(s))\). Taking into account that \(|Tr(S)| = a\) (we also assume that \(s\) is such that \(a/2 > z\)) we obtain:

\begin{align*}
\sum_{i=0}^{z} \mathbb{P}(F(\sigma, G[S]) = i) & = \sum_{i=0}^{z} \binom{a}{i} \cdot \left(\frac{1}{8}\right)^i \cdot \left(\frac{7}{8}\right)^{a-i} \\
& \leq (z+1)a^z \left(\frac{7}{8}\right)^a = 2^o(s^2) \left(\frac{7}{8}\right)^{(1+o(1))}.
\end{align*}

By symmetry, \(\mathbb{P}(A'_S)\) can be bounded analogously.

Next we describe how to improve the obtained bound. Take a graph \(H = (S, E)\) of order \(s\). Instead of \(Tr(S)\) we consider a maximum system of pairwise disjoint graphs isomorphic to \(K_k\) on the set of vertices \(S = \{1, \ldots, s\}\). Let \(Sys(S, k)\) denote such a system. For a fixed \(k\) and for \(s \to \infty\) Rödl’s theorem implies that \(|Sys(S, k)| \sim \frac{s^2}{k(k-1)}\), or, equivalently, \(|Sys(S, k)| = \frac{s^2}{k(k-1)}(1 + \xi_k(s))\).

Let \(\sigma\) be a permutation of the set \(V(H)\). Let \(F_k(\sigma, H)\) denote the number of such triangles from the set \(T(\sigma(H))\) that are subgraphs of one of the complete subgraphs of size \(k\) from \(Sys(S, k)\). Below we indicate the changes in the proof of Theorem \(\mathbb{I}\) Assume \(k \geq 4\).

Let us generalize Claim \(\mathbb{I}\). Before the claim we defined \(\psi_1\). Similarly to how we defined \(\psi_1\) based on \(\psi\) we define \(\xi_k^1\) based on \(\xi_k\).

**Claim 2.** Fix a natural \(k \geq 4\). For every graph \(H\) with \(s\) vertices we have:

\[\mathbb{E}(F_k(\sigma, H)) = \frac{(k-2)|T(H)|}{s} (1 + \xi_k^1(s)).\]

**Proof.** The proof is similar to the proof of Claim \(\mathbb{I}\). We point out several differences in calculations.

Let \(\Delta \in T(H)\). For every \(k\)-clique \(K_k \in Sys(S, k)\) the number of permutations \(\sigma\) such that \(K_k\) contains \(\sigma(\Delta)\) as a subgraph, equals \((s-3)!k(k-1)(k-2)\). Thus, the number of permutations \(\sigma\) such that \(\sigma(\Delta) \in Sys(S, k)\) equals \((s-3)!k(k-1)(k-2)\cdot |Sys(S, k)|\).

This implies

\begin{align*}
\sum_{\sigma} \mathbb{E}(F_k(\sigma, H)) & = \sum_{\Delta \in T(H)} \sum_{\sigma} \mathbb{E}(\sigma(\Delta) \in Sys(S, k)) \cdot \mathbb{P}(\sigma) = \frac{(k-2)|T(H)|}{s} (1 + \xi_k^1(s)).
\end{align*}
Corollary 5. Fix a natural \( k \) greater than 4 and positive \( \delta \). Let \( H \) be a graph on \( s \) vertices. If \( |T(H)| \leq s^{3-\delta} \), then there exists a permutation \( \sigma \) of the set \( V(H) \) such that \( F_k(\sigma, H) \leq (k-2)s^{2-\delta}(1 + \xi_1^H(s)) \).

In the case \( k = 4 \) this corollary gives the following theorem.

Theorem 9.

\[
\mathbb{P}(A_S) \leq \mathcal{P}, \quad \mathbb{P}(A'_S) \leq \mathcal{P}, \text{ where } \mathcal{P} = s! \cdot \left( \frac{41}{64} \right)^{\frac{2}{12}}(1+o(1)).
\]

Proof. The proof is analogous to the proof of Theorem 7. While in that proof we used Corollary 4, here we apply Corollary 5. We use the same notation as in the proof of Theorem 7. That is, let \( \varepsilon \) be the one appeared in Section 3.1. Put \( \delta \) from Corollary 5 to be equal to \( \varepsilon \). We have the following equality:

\[
z = 2s^{2-\varepsilon}(1 + \xi_4^H(s)).
\]

We already know that \( |Sys(S, 4)| = s^2/(1 + \xi_4(s)) \). Hence \( a = s^2/(1 + \xi_4(s)) \). In fact, to complete the proof it remains to prove that

\[
\sum_{i=0}^{z} \mathbb{P}(F_k(\sigma, G[S]) = i) \leq \left( \frac{41}{64} \right)^{\frac{2}{12}}(1+o(1)).
\]

The event \( \{F_k(\sigma, G[S]) = i\} \) implies the following event: at most \( i \) cliques from \( Sys(S, 4) \) contain at least one triangle from the graph \( \sigma(G[S]) \). At the same time the probability of the event that \( G(4, 1/2) \) does not contain any triangles is \( \frac{41}{64} \). Therefore, for large \( s \) we have:

\[
\sum_{i=0}^{z} \mathbb{P}(F_k(\sigma, G[S]) = i) \leq \sum_{i=0}^{z} \sum_{j=0}^{i} \binom{a}{j} \cdot \left( \frac{23}{64} \right)^{j} \cdot \left( \frac{41}{64} \right)^{a-j} \leq \sum_{i=0}^{z} (i+1)a^{i} \left( \frac{41}{64} \right)^{a} \leq (z+1)^{2}a^{z} \left( \frac{41}{64} \right)^{\frac{2}{12}}(1+o(1)) = \left( \frac{41}{64} \right)^{\frac{2}{12}}(1+o(1))^{\frac{2}{12}}(1+o(1)),
\]

which completes the proof.

Analogously to Corollary 2 we obtain

Corollary 6. For \( d \in \{4, 5\} \) the following lower bound holds:

\[
R_D(s, s, d) \geq \left( \frac{64}{41} \right)^{\frac{2}{12}}(1+o(1)) \approx 2^{0.053537s}.
\]

We use \( \mathcal{P}(k) \) to denote the probability that the random graph \( G(k, 1/2) \) does not have subgraphs isomorphic to \( K_t \). One can easily generalize the above described method (Corollaries 2 and 6). Thus, for \( d \in \{4, 5\} \) we obtain the following bound:

\[
R_D(s, s, d) \geq \left( \frac{1}{\mathcal{P}(k, 3)} \right)^{\frac{2}{12}}(1+o(1)).
\]
Let us note that in this bound the value $o(1)$ depends both on $k$ and $s$, so we apply this bound for fixed $k$ and for $s$ that tends to infinity.

It is known that (see a more general claim in the next section)

$$\mathcal{P}(k, 3) = \frac{2^{k^2/4 + f_1(k)}}{2^k} = 2^{-k^2/4 + f_2(k)}, \quad f_1(k) = o(k^2), \quad f_2(k) = o(k^2).$$

Hence

$$\left(\frac{1}{\mathcal{P}(k, 3)}\right)^{\frac{s}{\pi(k-1)}} = 2^{(1/4 - f_3(k))s}, \quad \lim_{k \to \infty} f_3(k) = 0.$$

First we fix large $k$, next choose a sufficiently large $s$. Finally we get:

$$R_D(s, s, d) \geq \left(\frac{1}{\mathcal{P}(k, 3)}\right)^{\frac{s}{\pi(k-1)}} = (2^{(1/4 - f_3(k))s})^{1+o(1)} > 2^{(1/4 - \gamma)s}.$$

This concludes the proof of Theorem 4 for $d \in \{4, 5\}$.

3.3 Cases $d \geq 6$

We generalize the method, described in the previous section, to the case of arbitrary $d$. While there we considered triangles, now we deal with $l$-cliques, where $l = \lfloor d/2 \rfloor + 1$. Instead of $F_k(\sigma, H)$ we consider random variables $F^l_k(\sigma, H)$, where $F^l_k(\sigma, H)$ is the number of such $l$-cliques in $\sigma(H)$ that are contained as a subgraph in one of the $k$-cliques from $\text{Sys}(S, k)$.

Let us give the analogue of Claim 2.

**Claim 3.** Fix natural $k, l$, $l \leq k$. For every graph $H$ with $s$ vertices we have:

$$\mathbb{E}\left(F^l_k(\sigma, H)\right) = \frac{(k - 2) \cdot \ldots \cdot (k - l + 1) \cdot \Omega(H, l)}{s^{l-2}} \left(1 + \zeta^l_k(s)\right).$$

We omit here the proof of the claim, the corollary and further calculations.

It is clear that finally one gets

$$R_D(s, s, d) \geq \left(\frac{1}{\mathcal{P}(k, l)}\right)^{\frac{s}{\pi(k-1)}}(1+o(1)),$$

where for fixed $d$ the value $o(1)$ depends only on $k$ and $s$.

It was shown in the paper [8] that, for fixed natural $l$ greater than 3, the number of graphs with $k$ vertices and without $l$-cliques is

$$2^{k^2(1 - \frac{1}{l+1}) + f(k, l)},$$

where the value of $f(k, l)$ is $o(k^2)$. Further calculations reproduce those from the end of the previous section.
4 Proof of Proposition

Note that every \([d/2]\)-partite graph can be realized as a distance graph in \(\mathbb{R}^d\). Indeed, consider circles \(C_i, i = 1, \ldots, [d/2]\):

\[
C_i = \{(0, \ldots, 0, x_{2i-1}, x_{2i}, 0, \ldots, 0) \in \mathbb{R}^d : x_{2i-1}^2 + x_{2i}^2 = 1/2\}.
\]

Embed the \(i\)th part of the multipartite graph into \(C_i\). By Pithagoras’ theorem, the distance between any two points from \(C_i, C_j\), for distinct \(i\) and \(j\), equals 1.

So, to prove the proposition it is enough to show that for every graph with

\[
m = 2 \left\lfloor \frac{d}{2} \right\rfloor R \left( \left\lfloor \frac{s}{[d/2]} \right\rfloor, \left\lceil \frac{s}{[d/2]} \right\rceil \right)
\]

vertices the following holds: either the graph or its complement has \([d/2]\) independent sets with total cardinality at least \(s\). Take a graph \(G = (V, E)\) on \(m\) vertices. Split its vertex set into \(t = 2[d/2]\) parts so that each part has cardinality

\[
\frac{m}{t} = R \left( \left\lfloor \frac{s}{[d/2]} \right\rfloor, \left\lceil \frac{s}{[d/2]} \right\rceil \right).
\]

Let \(V_1, \ldots, V_t\) denote these parts. Put \(G_i = G[V_i], \ldots, G_t = G[V_t]\). By the definition of the classical Ramsey number for every \(i \in \{1, \ldots, t\}\) either \(G_i\) or \(\overline{G}_i\) has an independent set with cardinality \(y = \left\lceil \frac{s}{[d/2]} \right\rceil\). Assume that (without loss of generality) there are at least \([d/2] = t/2\) indexes \(i\) such that \(G_i\) has an independent set of size \(y\). Take a union of the collection of \(G_i\) over \(t/2\) such indexes \(i\). The union is a subgraph in \(G\), which is realizable as distance graph in \(\mathbb{R}^d\) and and already has at least \(yt/2\) vertices, and \(yt/2 \geq s\). This concludes the proof.
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