GENERALIZED MONGE-AMPÈRE CAPACITIES

E. DI NEZZA AND CHINH H. LU

Abstract. We study various capacities on compact Kähler manifolds which
generalize the Bedford-Taylor Monge-Ampère capacity. We then use these
capacities to study the existence and the regularity of solutions of complex
Monge-Ampère equations.

Contents

1. Introduction 1
2. Generalized Monge-Ampère Capacities 5
  2.1. Energy classes 5
  2.2. The $(\varphi, \psi)$-Capacity 5
  2.3. Proof of Theorem A 14
3. Another proof of the Domination Principle 14
4. Applications to Complex Monge-Ampère equations 16
  4.1. Proof of Theorem B 17
  4.2. (Non) Existence of solutions 18
  4.3. Proof of Theorem C 20
  4.4. Non Integrable densities 21
  4.5. The case of semipositive and big classes 22
  4.6. Critical Integrability 23
References 26

1. Introduction

Let $(X, \omega)$ be a compact Kähler manifold of complex dimension $n$ and let $D$ be
an arbitrary divisor on $X$. Consider the complex Monge-Ampère equation

\begin{equation}
(\omega + dd^c \varphi)^n = f^n,
\end{equation}

where $0 \leq f \in L^1(X)$ is such that $\int_X f^n = \int_X \omega^n$. It follows from \cite{20} and
\cite{19} that equation (1.1) has a unique normalized solution in the finite energy class
$E(X, \omega)$. We say that the solution $\varphi$ is normalized if $\sup_X \varphi = 0$.

If $f$ is strictly positive and smooth on $X$, we know from the seminal paper of
Yau \cite{24} that the solution is also smooth on $X$. Recall that this solves in particular
the Calabi conjecture and allows to construct Ricci flat metrics on $X$ whenever
$c_1(X) = 0$.

\ \ 

Date: February 12, 2014

The authors are partially supported by the French ANR project MACK.
Given \( f \) positive and smooth on \( X \setminus D \), it is natural to investigate the regularity of the solution. In [15] we have proved in many cases that the solution \( \varphi \) is smooth in \( X \setminus D \).

As in the classical case of Yau [24], the most difficult step is to establish an a priori \( C^0 \)-estimate. This estimate is much more difficult in our situation since in general the solution is not globally bounded. A natural idea is to bound the normalized solution from below by a singular quasi plurisubharmonic function (qpsh for short). This is where generalized Monge-Ampère capacities play a crucial role.

We recall the notion of the classical capacity \( \text{Cap}_\omega \) introduced and studied in [22] and [19]:

\[
\text{Cap}_\omega(E) = \sup \left\{ \int_E (\omega + dd^c u)^n \mid u \in \text{PSH}(X, \omega), -1 \leq u \leq 0 \right\}, \quad E \subset X.
\]

A strong comparison between the Lebesgue measure and \( \text{Cap}_\omega \), as is needed in a celebrated method due to Kołodziej [21], does not hold in our setting. We therefore study other capacities to provide an a priori \( C^0 \)-estimate. In dealing with complex Monge-Ampère equations in quasiprojective varieties we were naturally lead to work with generalized capacities of type \( \text{Cap}_{\psi - 1, \psi} \) in [15] (see below for their definition).

In this paper, we make a systematic study of these capacities as well as the more general \( \text{Cap}_{\varphi, \psi} \) capacities: let \( \varphi, \psi \) be two \( \omega \)-plurisubharmonic functions on \( X \) such that \( \varphi < \psi \) on \( X \) modulo possibly a pluripolar set. The \( (\varphi, \psi) \)-Capacity of a Borel subset \( E \subset X \) is defined by

\[
\text{Cap}_{\varphi, \psi}(E) := \sup \left\{ \int_E (\omega + dd^c u)^n \mid u \in \text{PSH}(X, \omega), \varphi \leq u \leq \psi \right\}.
\]

Here, for a \( \omega \)-psh function \( u \), \( (\omega + dd^c u)^n \) is the non-pluripolar Monge-Ampère measure of \( u \). See Section 2 for the definition. When \( \varphi \equiv \psi - 1 \), we drop the index \( \varphi \) and denote the \( (\psi - 1, \psi) \)-Capacity by \( \text{Cap}_\psi \),

\[
\text{Cap}_\psi := \text{Cap}_{\psi - 1, \psi}.
\]

This is exactly the generalized capacity used in our previous paper [15]. If moreover \( \psi \) is constant, \( \psi \equiv C \), we recover the Monge-Ampère capacity defined above

\[
\text{Cap}_C = \text{Cap}_\omega.
\]

Given any subset \( E \subset X \), we define the outer \( (\varphi, \psi) \)-capacity of \( E \) by

\[
\text{Cap}^*_{\varphi, \psi}(E) := \inf \{ \text{Cap}_{\varphi, \psi}(U) \mid U \text{ is an open subset of } X, E \subset U \}.
\]

We say that the \( (\varphi, \psi) \)-capacity characterizes pluripolar sets on \( X \) if for any subset \( E \subset X \), the following holds

\[
\text{Cap}^*_{\varphi, \psi}(E) = 0 \iff E \text{ is a pluripolar subset of } X.
\]

If \( E \subset X \) is a Borel subset we set

\[
h_{\varphi, \psi, E}(x) := \sup \{ u(x) \mid u \in \text{PSH}(X, \omega), u \leq \psi \text{ on } X, u \leq \varphi \text{ q.e. } E \}.
\]

Here, quasi everywhere (q.e. for short) means outside a pluripolar set. Let \( h^*_{\varphi, \psi, E} \) be its upper semicontinuous regularization which we call the \( (\varphi, \psi) \)-extremal function of \( E \). We establish a useful characterization of the \( (\varphi, \psi) \)-capacity in terms of the relative extremal function for any subset.
When \( \varphi \) belong to the finite energy class \( \mathcal{E}(X, \omega) \) we can bound \( \text{Cap}_{\varphi, \psi} \) by \( F(\text{Cap}_\omega) \) for some positive function \( F \) which vanishes at 0. This uniform control turns out to be very useful in studying convergence of the complex Monge-Ampère operator since it allows us to replace quasi-continuous functions by continuous ones without affecting the final result. We also prove that the generalized Monge-Ampère capacity \( \text{Cap}_{\varphi, \psi} \) characterizes pluripolar sets when the lower weight is in \( \mathcal{E}(X, \omega) \):

**Theorem A.** Assume that \( \varphi \in \mathcal{E}(X, \omega) \) and \( \psi \in \text{PSH}(X, \omega) \) such that \( \varphi < \psi \) modulo a pluripolar subset.

(i) Let \( E \subset X \) be a Borel subset of \( X \), and denote by \( h_E \) the \((\varphi, \psi)\)-extremal function of \( E \). The outer \((\varphi, \psi)\)-capacity of \( E \) is given by

\[
\text{Cap}_{\varphi, \psi}^*(E) = \int_{\{h_E < \varphi\}} \text{MA}(h_E) = \int_X \left( \frac{-h_E + \psi}{-\varphi + \psi} \right) \text{MA}(h_E),
\]

where \( h_E := h_{\varphi, \psi, E}^* \) is the \((\varphi, \psi)\)-extremal function of \( E \).

(ii) There exists a function \( F : \mathbb{R}^+ \to \mathbb{R}^+ \) such that \( \lim_{t \to 0^+} F(t) = 0 \) and such that for all Borel subset \( E \),

\[
\text{Cap}_{\varphi, \psi}(E) \leq F(\text{Cap}_\omega(E)).
\]

(iii) \( \text{Cap}_{\varphi, \psi} \) characterizes pluripolar sets.

We stress that the function \( F \) in (ii) is quite explicit (see Theorem 2.9).

As we have underlined, these generalized capacities play an important role in studying complex Monge-Ampère equations on quasi-projective varieties (see [15]). We give in the second part of this paper several other applications.

We consider the following complex Monge-Ampère equation

\[
(\omega + dd^c \varphi)^n = e^{\lambda \varphi} f \omega^n, \quad \lambda \in \mathbb{R}.
\]

Assume that \( 0 < f \in C^\infty(X \setminus D) \) satisfies Condition \( \mathcal{H}_f \), i.e. \( f \) can be written as

\[
f = e^{\psi^+ - \psi^-}, \quad \psi^\pm \text{ are quasi psh functions on } X, \, \psi^- \in L^\infty_{\text{loc}}(X \setminus D).
\]

When \( \lambda = 0 \) and \( f \) satisfies \( \int_X f \omega^n = \int_X \omega^n \), we proved in [15] that there is a unique normalized solution in \( \mathcal{E}(X, \omega) \) which is smooth on \( X \setminus D \). When \( \lambda > 0 \) and \( \int_X f \omega^n < +\infty \) the same result holds since the \( C^0 \) estimate follows easily from the comparison principle.

Consider now the case when \( \lambda < 0 \). In this case solutions do not always exist and when they do, there may be many of them. Our result here says that any solution in \( \mathcal{E}(X, \omega) \) (if exists) is smooth on \( X \setminus D \).

**Theorem B.** Let \( 0 < f \in C^\infty(X \setminus D) \cap L^1(X) \). Assume that \( f \) satisfies Condition \( \mathcal{H}_f \) and \( \varphi \in \mathcal{E}(X, \omega) \) is a solution of

\[
(\omega + dd^c \varphi)^n = e^{\lambda \varphi} f \omega^n, \quad \lambda < 0.
\]

Then \( \varphi \) is smooth on \( X \setminus D \).

Note that when \( \lambda < 0 \) and equation (1.2) has a solution in \( \mathcal{E}(X, \omega) \), the measure \( \mu = f \omega^n \) is dominated by \( \text{MA}(u) \) for some \( u \in \text{PSH}(X, \omega) \cap L^\infty(X) \). In particular, \( f \in L^1(X) \).

We next investigate the case when \( \lambda > 0 \) and \( f \) is not integrable on \( X \). Of course solutions do not always exist. But observe that when \( \varphi \) is singular enough \( e^\varphi f \) will
be integrable on $X$ and it is then reasonable to find a solution. For example, one can look at densities of the type
\[ f \simeq \frac{1}{|s|^2}, \]
which is not integrable. Here $s$ is a holomorphic section of the line bundle associated to $D$. Such densities have been considered by Berman and Guenancia in their study of the compactification of the moduli space of canonically polarized manifolds [5]. They have shown that there exists a unique solution $\varphi \in \mathcal{E}(X, \omega)$ which is smooth in $X \setminus D$. As another application of the generalized Monge-Ampère capacities we show in the following result that in a general context whenever a solution in $\mathcal{E}(X, \omega)$ exists it is smooth outside $D$.

**Theorem C.** Assume $0 < f \in \mathcal{C}^\infty(X \setminus D)$ satisfies Condition $\mathcal{H} f$. If the equation
\[ (\omega + dd^c \varphi)^n = e^{\lambda \varphi} f \omega^n, \quad \lambda > 0 \]
admits a solution $\varphi \in \mathcal{E}(X, \omega)$ then $\varphi$ is smooth on $X \setminus D$.

Let us stress that in Theorem C we do not assume that $\int_X f \omega^n < +\infty$. It turns out that the existence of solutions in $\mathcal{E}(X, \omega)$ is equivalent to the existence of subsolutions in this class, these are easy to construct in concrete situations (see Example 4.7). We also obtain a similar result in the case of semipositive and big classes (see Theorem 4.8 and Example 4.9).

Finally we use generalized capacitites to study the critical integrability of a given $\phi \in \text{PSH}(X, \omega)$.

**Theorem D.** Let $\phi \in \text{PSH}(X, \omega)$ and $\alpha = \alpha(\phi) \in (0, +\infty)$ be the canonical threshold of $\phi$, i.e.
\[ \alpha = \alpha(\phi) := \sup\{t > 0 \mid e^{-t \phi} \in L^1(X)\}. \]
Then there exists $u \in \text{PSH}(X, \omega)$ with zero Lelong number at all points such that $e^{u - \alpha \phi}$ is integrable. Moreover, there exists a unique $\varphi \in \mathcal{E}(X, \omega)$ such that
\[ (\omega + dd^c \varphi)^n = e^{\alpha \phi} \omega^n. \]
It turns out that one can even chose $u = \chi \circ \phi$ in $\mathcal{E}(X, \omega)$, as an explicit function of $\phi$ with attenuated singularities (see Theorem 4.10).

The paper is organized as follows. In section 2 we recall some known facts on energy classes, we introduce generalized capacities on compact Kähler manifolds and prove Theorem A. As an application of the generalized capacities we give another proof of the domination principle in $\mathcal{E}(X, \omega)$ in Section 3. In Section 4 we use generalized capacities to study complex Monge-Ampère equations as (1.2). The proof of Theorem D will be given in Section 4 as well.

**Acknowledgements.** We would like to thank Vincent Guedj and Ahmed Zeriahi for constant help, many suggestions and encouragements. We also thank Robert Berman and Bo Berndtsson for useful discussions. We are indebted to Henri Guenancia for a careful reading and very useful comments on a previous draft version of this paper.
2. Generalized Monge-Ampère Capacities

Let $(X,\omega)$ be a compact Kähler manifold of complex dimension $n$. In this section we prove some basic properties of the $(\varphi,\psi)$-capacity and of the relative $(\varphi,\psi)$-extremal functions.

2.1. Energy classes.

**Definition 2.1.** We let $\text{PSH}(X,\omega)$ denote the class of $\omega$-plurisubharmonic functions ($\omega$-psh for short) on $X$, i.e. the class of functions $\varphi$ such that locally $\varphi = \rho + u$, where $\rho$ is a local potential of $\omega$ and $u$ is a plurisubharmonic function.

Let $\varphi$ be some unbounded $\omega$-psh function on $X$ and consider $\varphi_j := \max(\varphi,-j)$ the ”canonical approximants”. It has been shown in [20] that

$$1_{\{\varphi_j > -j\}}(\omega + \ddc \varphi_j)^n$$

is a non-decreasing sequence of Borel measures. We denote its limit by

$$\text{MA}(\varphi) = (\omega + \ddc \varphi)^n := \lim_{j \to +\infty} 1_{\{\varphi_j > -j\}}(\omega + \ddc \varphi_j)^n.$$

**Definition 2.2.** We denote by $E(X,\omega)$ the set of $\omega$-psh functions having full Monge-Ampère mass:

$$E(X,\omega) := \{ \varphi \in \text{PSH}(X,\omega) | \int_X \text{MA}(\varphi) = \int_X \omega^n \}.$$

Let us stress that $\omega$-psh functions with full Monge-Ampère mass have mild singularities. In particular, any $\varphi \in E(X,\omega)$ has zero Lelong numbers $\nu(\varphi,\cdot) = 0$ (see [20, Corollary 1.8]). We also recall that, for every $\varphi \in E(X,\omega)$ and any $\psi \in \text{PSH}(X,\omega)$, the generalized comparison principle is valid, namely

$$\int_{\{\varphi < \psi\}} (\omega + \ddc \psi)^n \leq \int_{\{\varphi < \psi\}} (\omega + \ddc \varphi)^n.$$

**Definition 2.3.** Let $\chi: \mathbb{R}^- \to \mathbb{R}^-$ be an increasing function such that $\chi(0) = 0$ and $\chi(-\infty) = -\infty$. We denote by $E_{\chi}(X,\omega)$ the class of $\omega$-psh functions having finite $\chi$-energy:

$$E_{\chi}(X,\omega) := \{ \varphi \in E(X,\omega) | \chi(-|\varphi|) \in L^1(\text{MA}(\varphi)) \}.$$

For $p > 0$, we use the notation

$$\mathcal{E}_p(X,\omega) := E_{\chi}(X,\omega), \text{ when } \chi(t) = -(-t)^p.$$

2.2. The $(\varphi,\psi)$-Capacity. In this subsection we always assume that $\varphi,\psi \in \text{PSH}(X,\omega)$ are such that $\varphi < \psi$ quasi everywhere on $X$. The $(\varphi,\psi)$-capacity of a Borel subset $E \subset X$ is defined by

$$\text{Cap}_{\varphi,\psi}(E) := \sup \left\{ \int_E \text{MA}(u) | u \in \text{PSH}(X,\omega), \varphi \leq u \leq \psi \right\}.$$

When $\varphi \equiv \psi - 1$, to simplify the notation we simply denote

$$\text{Cap}_\psi := \text{Cap}_{\psi-1,\psi}.$$

If moreover $\psi \equiv C$ is constant we recover the Monge-Ampère capacity introduced in [2], [22], [19]. The following properties of the $(\varphi,\psi)$-Capacity follow straightforward from the definition.
Proposition 2.4. (i) If \( E_1 \subset E_2 \subset X \) then \( \text{Cap}_{\varphi,\psi}(E_1) \leq \text{Cap}_{\varphi,\psi}(E_2) \).

(ii) If \( E_1, E_2, \cdots \) are Borel subsets of \( X \) then
\[
\text{Cap}_{\varphi,\psi} \left( \bigcup_{j=1}^{\infty} E_j \right) \leq \sum_{j=1}^{+\infty} \text{Cap}_{\varphi,\psi}(E_j).
\]

(iii) If \( E_1 \subset E_2 \subset \cdots \) are Borel subsets of \( X \) then
\[
\text{Cap}_{\varphi,\psi} \left( \bigcup_{j=1}^{\infty} E_j \right) = \lim_{j \to +\infty} \text{Cap}_{\varphi,\psi}(E_j).
\]

The outer \((\varphi,\psi)\)-capacity of \( E \) is defined by
\[
\text{Cap}_{\varphi,\psi}^*(E) := \inf \{ \text{Cap}_{\varphi,\psi}(U) \mid U \text{ is an open subset of } X, E \subset U \}.
\]

We say that the \((\varphi,\psi)\)-capacity characterizes pluripolar sets on \( X \) if for any subset \( E \subset X \), the following holds
\[
\text{Cap}_{\varphi,\psi}^*(E) = 0 \iff E \text{ is a pluripolar subset of } X.
\]

Definition 2.5. If \( E \subset X \) is a Borel subset we set
\[
h_{\varphi,\psi,E} := \sup \{ u \in \text{PSH}(X,\omega), u \leq \varphi \text{ quasi everywhere on } E, u \leq \psi \text{ on } X \},
\]
where "quasi everywhere" means outside a pluripolar set. The upper semicontinuous regularization of \( h_{\varphi,\psi,E} \) is called the relative \((\varphi,\psi)\) extremal function of \( E \).

Proposition 2.6. Let \( E \subset X \).

(i) The function \( h_{\varphi,\psi,E}^* \) is \( \omega \)-psh. It satisfies \( \varphi \leq h_{\varphi,\psi,E}^* \leq \psi \) on \( X \) and \( h_{\varphi,\psi,E}^* = \varphi \) quasi everywhere on \( E \).

(ii) If \( P \subset E \) is pluripolar, then \( h_{\varphi,\psi,E \setminus P}^* = h_{\varphi,\psi,E}^* \); in particular \( h_{\varphi,\psi,P}^* \equiv \psi \).

(iii) If \( (E_j) \) are subsets of \( X \) increasing towards \( E \subset X \), then \( h_{\varphi,\psi,E_j}^* \) decreases towards \( h_{\varphi,\psi,E}^* \).

(iv) If \( h_{\varphi,\psi,E}^* \equiv \psi \) then \( E \) is pluripolar.

Proof. The statement (i) is a standard consequence of Bedford-Taylor’s work [2].

Set \( E_1 := E \setminus P \), and denote by \( h = h_{\varphi,\psi,E}^*, h_1 = h_{\varphi,\psi,E_1}^* \), the corresponding \((\varphi,\psi)\)-extremal functions of \( E, E_1 \). Since \( E_1 \subset E \) it is clear that \( h_1 \geq h \). On the other hand \( h_1 = \varphi \) quasi everywhere on \( E_1 \) hence on \( E \). This yields \( h_1 \leq h \) whence equality.

Let us prove (iii). Since \( (E_j) \) is increasing, \( h_j := h_{\varphi,\psi,E_j}^* \) is decreasing toward \( h \in \text{PSH}(X,\omega) \). It is clear that \( h \geq h_{\varphi,\psi,E}^* \). By definition, for each \( j \in \mathbb{N} \), \( h_j = \varphi \) quasi everywhere on \( E_j \). It then follows that \( h = \varphi \) quasi everywhere on \( E \). We then infer that \( h \leq h_{\varphi,\psi,E}^* \), hence the equality.

To prove (iv) assume that \( h_{\varphi,\psi,E}^* \equiv \psi \). By definition of \( h := h_{\varphi,\psi,E}^* \) and by Choquet’s lemma we can find an increasing sequence \( (u_j) \) such that \( u_j = \varphi \) on \( E \) and \( (\lim_{j \to +\infty} u_j)^* = h \). Note that
\[
E \subset \left\{ \left( \lim_{j \to +\infty} u_j \right)^* < \left( \lim_{j \to +\infty} u_j \right)^* \right\},
\]
modulo a pluripolar set. The latter is also pluripolar, hence \( E \) is pluripolar.

Theorem 2.7. If \( \varphi \in \mathcal{E}(X,\omega) \) and \( E \subset X \) is pluripolar then \( \text{Cap}_{\varphi,\psi}^*(E) = 0 \).
Proof. Assume that \( \varphi \in \mathcal{E}(X, \omega) \) and fix a pluripolar set \( E \subset X \). By translating \( \psi \) and \( \varphi \) by a constant we can assume that \( \psi \leq 0 \). It follows from [20, Proposition 2.2] that \( \varphi \in \mathcal{E}_c(X, \omega) \) for some convex increasing function \( \chi : \mathbb{R}^- \to \mathbb{R}^- \). We can find \( u \in \mathcal{E}_c(X, \omega) \), \( u \leq 0 \) such that \( E \subset \{ u = -\infty \} \). We claim that
\[
\text{Cap}_{\varphi, \psi}(\{ u < -t \}) \leq \frac{2}{\chi(-t)} (E_X(u) + 2^n E_X(\varphi)), \quad \forall t > 0.
\]
Indeed, let \( v \in \text{PSH}(X, \omega) \) such that \( \varphi \leq v \leq \psi \). We obtain immediately that
\[
\int_{\{ u < -t \}} \mathcal{M}A(v) \leq \frac{1}{-\chi(-t)} \int_{\{ u < -t \}} (-\chi \circ u) \mathcal{M}A(v).
\]
From this and [20, Proposition 2.5] we get
\[
\int_{\{ u < -t \}} \mathcal{M}A(v) \leq \frac{2}{\chi(-t)} (E_X(u) + E_X(v)).
\]
This coupled with the fundamental inequality in [20, Lemma 2.3] yield the claim. Since for any \( t > 0 \), \( E \subset \{ u < -t \} \) we obtain
\[
\text{Cap}_{\varphi, \psi}^*(E) \leq \text{Cap}_{\varphi, \psi}(u < -t) \to 0 \quad \text{as} \quad t \to +\infty.
\]
\[\square\]

From now on we fix \( \varphi, \psi \) two functions in \( \mathcal{E}(X, \omega) \) such that \( \varphi < \psi \) quasi everywhere on \( X \).

Given any \( u \in \text{PSH}(X, \omega) \) such that \( u \leq 0 \), it follows from [20, Example 2.14] (see also the Main Theorem in [12]) that \( u_p := -(\nabla u)^p \) belongs to \( \mathcal{E}(X, \omega) \) for any \( 0 < p < 1 \). The same arguments can be applied to get the following result:

**Lemma 2.8.** Let \( \chi : \mathbb{R}^- \to \mathbb{R}^- \) be any measurable function. Assume that there exists \( q > 0 \) such that
\[
\sup_{t \leq -1} |\chi(t)|(-t)^{-q} = C < +\infty.
\]
Then for any \( u \in \text{PSH}(X, \omega) \) such that \( u \leq -1 \) and any \( 0 < p < \frac{1}{q+1} \) we have
\[
\int_X |\chi \circ u_p| \mathcal{M}A(u_p) \leq A,
\]
where \( u_p := -(\nabla u)^p \) and \( A \) is a positive constant depending only on \( C, p, q \).

**Proof.** In the proof we use \( A \) to denote various positive constants which are under control. By considering \( u^j := \max(u, -j) \), the canonical approximants of \( u \), and letting \( j \to +\infty \) it suffices to treat the case when \( u \) is bounded. We compute
\[
\omega + dd^c u_p = \omega + p(1 - p)(-u)^{p-2} du \wedge d^c u + p(-u)^{p-1} dd^c u.
\]
We thus get
\[
0 \leq \omega + dd^c u_p \leq (-u)^{p-1}(\omega + dd^c u) + \omega + (-u)^{p-2} du \wedge d^c u.
\]
We need to verify the following bounds:
\[
\int_X |\chi \circ u_p|(-u)^{p-1}(\omega + dd^c u)^k \wedge \omega^{n-k} \leq A
\]
and
\[
\int_X |\chi \circ u_p|(-u)^{p-2} du \wedge d^c u \wedge (\omega + dd^c u)^k \wedge \omega^{n-1} \leq A,
\]
where $k = 0, 1, \ldots, n$. Let us consider the first one. By assumption we have
\[|\chi \circ u_p|(-u_p)^{-q} \leq C.\]
To bound the first term, it thus suffices to get a bound for
\[\int_X (-u)^{p-1+pq}(\omega + dd^c u)^k \wedge \omega^{n-k},\]
which is easy since $p + pq - 1 < 0$. For the second one it suffices get a bound for
\[\int_X (-u)^{p-2+pq}du \wedge d^c u \wedge (\omega + dd^c u)^{k} \wedge \omega^{n-k-1},\]
which follows easily by the same reason and by integration by parts. □

We know from Theorem 2.7 that $\text{Cap}_{\varphi, \psi}$ vanishes on pluripolar subsets of $X$. This suggests that $\text{Cap}_{\varphi, \psi}$ is dominated by $F(\text{Cap}_\omega)$, where $F$ is some positive function vanishing at 0. The following result gives an explicit formula of $F$.

**Theorem 2.9.** Let $\chi : \mathbb{R} \to \mathbb{R}$ be a convex increasing function and $\varphi \in \mathcal{E}_\chi(X, \omega)$. Let $q > 0$ be a positive real number such that
\[(2.1) \sup_{t \leq -1} |\chi(t)|(-t)^{-q} < +\infty.\]
Then for any $p < \frac{1}{1+q}$ there exists $C > 0$ depending on $p, q, \chi, \varphi$ such that
\[\text{Cap}_{\varphi, 0}(K) \leq C |\chi(-\text{Cap}_\omega(K))|^{\frac{q}{p}}, \quad \forall K \subset X.\]

As a concrete example, when $\varphi \in \mathcal{E}^q(X, \omega)$ for some $q > 0$ and $p < 1/(1+q)$, then we can take $F(s) := s^\frac{q}{p}$ for $s > 0$, getting
\[\text{Cap}_{\varphi, 0}(K) \leq C \text{Cap}_\omega(K)^{\frac{q}{p}}.\]

**Proof.** Fix $p > 0$ such that $p(q+1) < 1$. Let $V_K$ be the extremal $\omega$-plurisubharmonic function of $K$:
\[V_K := \sup\{u \mid u \in \text{PSH}(X, \omega), u \leq 0 \text{ on } K\},\]
and $M_K := \sup_X V_K^*$. It follows from (2.1) and Lemma 2.8 that the function
\[u = -(V_K^* + M_K + 1)^p\]
belong to $\mathcal{E}_\chi(X, \omega)$. Fix $h \in \text{PSH}(X, \omega)$ be such that $\varphi \leq h \leq 0$. It follows from Lemma 2.10 below that
\[\int_X |\chi \circ u|MA(h) \leq C_1,\]
where $C_1 > 0$ only depends on $\chi, p, q$ and $\varphi$. Therefore, using the fact that $V_K^* \equiv 0$ quasi everywhere on $K$ we get
\[\int_K MA(h) \leq \int_X |\chi \circ u|MA(h) \leq C_1 \frac{C_1}{|\chi(-M_K^P)|^{\frac{q}{p}}},\]
It follows from [19] that $M_K \geq C_2 \text{Cap}(K)^{-1/n}$. This coupled with the above yield the result. □
Lemma 2.10. Assume that $\chi$, $p$, $q$ and $\varphi$ are as in Theorem 2.9. Then there exists $C > 0$ depending on $\chi, p, q, \varphi$ such that

$$\int_X |\chi(-(-u)^p)| \operatorname{MA}(v) \leq C, \forall u, v \in \operatorname{PSH}(X, \omega), \sup_{X} u = -1, \varphi \leq v \leq 0.$$ 

Proof. We argue by contradiction, assuming that there are two sequences $(u_j), (v_j)$ of functions in $\operatorname{PSH}(X, \omega)$ such that $\sup_X u_j = -1, \varphi \leq v_j \leq 0$, and

$$\int_X |\chi(-(-u_j)^p)| \operatorname{MA}(v_j) \geq 2^{(n+2)j}, \forall j \in \mathbb{N}.$$ 

Set

$$u := \sum_{j=1}^{+\infty} 2^{-j} u_j, \ v = \sum_{j=1}^{+\infty} 2^{-j} v_j.$$ 

Then $u \in \operatorname{PSH}(X, \omega)$, $u \leq -1$. Moreover, it follows from Lemma 2.8 that

$$u_p := -(-u)^p \in \mathcal{E}_\chi(X, \omega).$$ 

We also have $\varphi \leq v \leq 0$, in particular $v \in \mathcal{E}_\chi(X, \omega)$. But

$$\int_X |\chi \circ u_p| \operatorname{MA}(v) \geq \sum_{j=1}^{+\infty} 2^j = +\infty,$$

which contradicts [20] Proposition 2.5].

Proposition 2.11. Let $E$ be a Borel subset of $X$ and set $h_E := h^*_E$ the relative $(\varphi, \psi)$-extremal function of $E$. Then

$$\operatorname{MA}(h_E) \equiv 0 \text{ on } \{h_E < \psi\} \setminus E.$$ 

Proof. We first assume that $\psi$ is continuous on $X$. Set $h := h_E$ and let $x_0 \in X \setminus \bar{E}$ be such that $(h-\psi)(x_0) < 0$. Let $B := B(x_0, r) \subset X \setminus \bar{E}$ be a small ball around $x_0$ such that $\sup_{\partial B} (h-\psi)(x) = -2\delta < 0$. Let $\rho$ be a local potential of $\omega$ in $B$. Shrinking $B$ a little bit we can assume that $\sup_{\partial B} |\rho| < \delta$ and $\operatorname{osc}_{\partial B} \psi < \delta/2$. By definition of $h$ and by Choquet’s lemma we can find an increasing sequence $(u_j)_j \subset \mathcal{E}(X, \omega)$ such that $u_j = \varphi$ quasi everywhere on $E$, $u_j \leq \psi$ on $X$, and $(\lim_j u_j)^* = h$. For each $j, k \in \mathbb{N}$, we solve the Dirichlet problem to find $v^k_j \in \operatorname{PSH}(X, \omega) \cap L^\infty(X)$ such that $\operatorname{MA}(v^k_j) = 0$ in $B$ and $v^k_j \equiv \max(u_j, -k)$ on $X \setminus B$. Since

$$\rho + v^k_j \leq \rho + h \leq -\delta + \psi \leq \sup_{\partial B} \psi - \delta$$

on $\partial B$, we deduce from the maximum principle that $v^k_j \leq \inf_{\partial B} \psi - \delta/2 - \rho \leq \psi$ on $B$. Furthermore, taking $k$ big enough such that $\psi \geq -k$, we can conclude that $v^k_j \leq \psi$ on $X$. For $j \in \mathbb{N}$ fixed, by the comparison principle $(v^k_j)_k$ decreases to $v_j \in \mathcal{E}(X, \omega)$. Then $u_j \leq v_j \leq h$ since $v_j = u_j = \varphi$ on $E$ and $v_j \leq \psi$ on $X$. It follows from [20] that the sequence of Monge-Ampère measures $\operatorname{MA}(v^k_j)$ converges weakly to $\operatorname{MA}(v_j)$. Thus $\operatorname{MA}(v_j)(B) = 0$. On the other hand, $v_j$ increases almost everywhere to $h$ and these functions belong to $\mathcal{E}(X, \omega)$. The same arguments as in [20] Theorem 2.6 show that $\operatorname{MA}(v_j)$ converges weakly to $\operatorname{MA}(h)$. We infer that $\operatorname{MA}(h)(B) = 0$.

It remains to remove the continuity hypothesis on $\psi$. Let $(\psi_j)$ be a sequence of continuous functions in $\operatorname{PSH}(X, \omega)$ decreasing to $\psi$ on $X$. Let $h_j := h^*_{\varphi, \psi_j, E}$ be the relative $(\varphi, \psi_j)$-extremal function of $K$. Then $h_j$ decreases to $h$, hence $\operatorname{MA}(h_j)$
converges weakly to $MA(h)$. Denote by $V := \{h < \psi\} \setminus \bar{E}$. Now, fix $\varepsilon > 0$ and $U$ an open subset of $X$ such that

$$\text{Cap}_\omega \left[ (U \setminus V) \cup (V \setminus U) \right] \leq \varepsilon.$$  

From the first step we know that $MA(h_j)$ vanishes on $V$. Thus

$$\int_V MA(h) \leq \int_U MA(h) + F(\varepsilon) \leq \liminf_{j \to +\infty} \int_V MA(h_j) + 2F(\varepsilon) = 2F(\varepsilon),$$

It suffices now to let $\varepsilon \to 0$ since $\lim_{\varepsilon \to 0} F(\varepsilon) = 0$ thanks to Theorem 2.9. \hfill \Box

**Lemma 2.12.** Let $E \subset X$ be a Borel subset and $h_E := h_{\varphi, \psi, E}^*$ be its relative $(\varphi, \psi)$-extremal function. Then we have

$$\text{Cap}_{\varphi, \psi}(E) \leq \int_{\{h_E < \psi\}} MA(h_E).$$

**Proof.** Observe first that the $(\varphi, \psi)$-capacity can be equivalently defined by

$$\text{Cap}_{\varphi, \psi}(E) := \sup \left\{ \int_E MA(u) \mid u \in \text{PSH}(X, \omega), \varphi < u \leq \psi \right\}.$$  

For simplicity, set $h := h_E$. Now take any $u \in \text{PSH}(X, \omega)$ such that $\varphi < u \leq \psi$. Then

$$E \subset \{h < u\} \subset \{h < \psi\},$$

where the first inclusion holds modulo a pluripolar set. The comparison principle for functions in $\mathcal{E}(X, \omega)$ (see [20]) yields

$$\int_E MA(u) \leq \int_{\{h < u\}} MA(u) \leq \int_{\{h < \psi\}} MA(h) \leq \int_{\{h < \psi\}} MA(h).$$

By taking the supremum over all candidates $u$, we get the result. \hfill \Box

The following result says that the inequality in Lemma 2.12 is an equality if $E$ is a compact or open subset of $X$.

**Theorem 2.13.** Let $E$ be an open (or compact) subset of $X$ and let $h_E := h_{\varphi, \psi, E}^*$ be the $(\varphi, \psi)$-extremal function of $E$. The $(\varphi, \psi)$-capacity of $E$ is given by

$$\text{Cap}_{\varphi, \psi}(E) = \int_{\{h_E < \psi\}} MA(h_E).$$

**Proof.** From Lemma 2.12 above we get one inequality. We now prove the opposite one. Set $h := h_E$. Assume first that $E$ is a compact subset of $X$. Let $(\psi_j)$ be a sequence of continuous $\omega$-psh functions decreasing to $\psi$. Denote by $h_j := h_{\varphi, \psi_j, E}^*$. It is easy to check that $h_j$ decreases to $h$ and that $\text{Cap}_{\varphi, \psi_j}(E)$ decreases to $\text{Cap}_{\varphi, \psi}(E)$. Since $h_j$ is a candidate defining the $(\varphi, \psi_j)$-capacity of $E$, it follows from Proposition 2.11 and Lemma 2.12 that

$$\text{Cap}_{\varphi, \psi}(E) = \int_{\{h < \psi_j\}} MA(h_j) = \int_E MA(h_j).$$
Fix $j_0 \in \mathbb{N}$. Since $h_j \leq h_{j_0}$ and $\psi \leq \psi_j$, for any $j > j_0$

$$\int_{\{h_j < \psi_j\}} MA(h_j) \geq \int_{\{h_{j_0} < \psi\}} MA(h_{j_0}).$$

Fix $\varepsilon > 0$ and replacing $\psi$ by a continuous function $\tilde{\psi}$ such that $\operatorname{Cap}_\omega(\\{\tilde{\psi} \neq \psi\}) < \varepsilon$. Arguing as in the proof of Proposition 2.11 we get

$$\liminf_{j \to +\infty} \int_{\{h_{j_0} < \psi\}} MA(h_{j_0}) \geq \int_{\{h_j < \psi\}} MA(h).$$

Taking the limit for $j \to +\infty$ in (2.2) we get

$$\operatorname{Cap}_{\phi,\psi}(E) \geq \int_{\{h < \psi\}} MA(h).$$

We now assume that $E \subset X$ is an open set. Let $(K_j)$ be a sequence of compact subsets increasing to $E$. Then clearly $h_j := h_{\phi,\psi,K_j} \uparrow h$ and $\operatorname{Cap}_{\phi,\psi}(K_j) \uparrow \operatorname{Cap}_{\phi,\psi}(E)$. We have already proved that $\operatorname{Cap}_{\phi,\psi}(K_j) \geq \int_{\{h_j < \psi\}} MA(h_j)$. For each fixed $k \in \mathbb{N}$, we have

$$\liminf_{j \to +\infty} \int_{\{h_j < \psi\}} MA(h_j) \geq \liminf_{j \to +\infty} \int_{\{h_k < \psi\}} MA(h_j) \geq \int_{\{h_k < \psi\}} MA(h).$$

Then letting $k \to +\infty$ and using the first part of the proof we get

$$\liminf_{j \to +\infty} \operatorname{Cap}_{\phi,\psi}(K_j) \geq \int_{\{h < \psi\}} MA(h).$$

On the other hand, it is clear that $\lim_{j \to +\infty} \operatorname{Cap}_{\phi,\psi}(K_j) = \operatorname{Cap}_{\phi,\psi}(E)$, and hence

$$\operatorname{Cap}_{\phi,\psi}(E) \geq \int_{\{h < \psi\}} MA(h).$$

\[\square\]

Now we want to give a formula for the outer $(\phi, \psi)$-capacity. Assume that $E$ is a Borel subset of $X$. We introduce an auxiliary function

$$\phi := \phi_{\phi,\psi,E} = \begin{cases} \frac{-h_{\phi,\psi,E} + \psi}{-\phi + \psi} & \text{if } \phi > +\infty \\ 0 & \text{if } \phi = +\infty \end{cases}. $$

Observe that $\phi$ is a quasicontinuous function, $0 \leq \phi \leq 1$ and $\phi = 1$ quasi everywhere on $E$.

**Theorem 2.14.** Let $E \subset X$ be a Borel subset and denote by $h_E := h_{\phi,\psi,E}$ the $(\phi, \psi)$-extremal function of $E$. Then

$$\operatorname{Cap}_{\phi,\psi}^*(E) = \int_{\{h_E < \psi\}} MA(h_E) = \int_X \left( \frac{-h_E + \psi}{-\phi + \psi} \right) MA(h_E).$$

To prove Theorem 2.14 we need the following results.

**Lemma 2.15.** Let $(u_j)$ be a bounded monotone sequence of quasi-continuous functions converging to $u$. Let $\chi$ be a convex weight and $\{\phi_j\} \subset E_\chi(X, \omega)$ be a monotone sequence converging to $\phi \in E_\chi(X, \omega)$. Then

$$\int_X u_j MA(\phi_j) \xrightarrow{j \to +\infty} \int_X u MA(\phi).$$
Proof. Fix $\varepsilon > 0$. Let $U$ be an open subset of $X$ with $\text{Cap}_\omega(U) < \varepsilon$ and $v_j, v$ be continuous functions on $X$ such that $v_j \equiv u_j$ and $v \equiv u$ on $K := X \setminus U$. By Theorem 2.9 (and by letting $\varepsilon \to 0$) it suffices to prove that

$$
\int_X v_j \text{MA}(\varphi_j) \xrightarrow{j \to +\infty} \int_X v \text{MA}(\varphi).
$$

From Dini’s theorem $v_j$ converges uniformly to $v$ on $K$. Thus, using again Theorem 2.9, the problem reduces to proving that

$$
\int_X v \text{MA}(\varphi_j) \xrightarrow{j \to +\infty} \int_X v \text{MA}(\varphi).
$$

But the latter obviously follows since $v$ is continuous on $X$. The proof is thus complete. □

**Proposition 2.16.** Let $E$ be a compact or open subset of $X$ and let $h_E := h^*_\varphi,\psi,E$ denote the $(\varphi,\psi)$-extremal function of $E$. Then

$$
\text{Cap}_{\varphi,\psi}(E) = \int_{\{h_E < \psi\}} \text{MA}(h_E) = \int_X \left( \frac{-h_E + \psi}{-\varphi + \psi} \right) \text{MA}(h_E).
$$

**Proof.** The first equality has been proved in Theorem 2.13. Set $h := h_E$ and $\phi := \phi_{\varphi,\psi,E} = \frac{-h_E + \psi}{-\varphi + \psi}$. Observe that $\{h < \psi\} = \{\phi > 0\}$ modulo a pluripolar set and $\phi \leq 1$. Thus

$$
\int_{\{h < \psi\}} \text{MA}(h) \geq \int_X \phi \text{MA}(h).
$$

Assume that $E$ is compact. By Proposition 2.11 and Theorem 2.13 we have

$$
\text{Cap}_{\varphi,\psi}(E) = \int_E \text{MA}(h).
$$

Since $\phi = 1$ quasi everywhere on $E$ we obtain

$$
\int_E \text{MA}(h) \leq \int_X \phi \text{MA}(h).
$$

We assume now that $E \subset X$ is an open subset. Let $(K_j)$ be a sequence of compact subsets increasing to $E$. Then

$$
\text{Cap}_{\varphi,\psi}(E) = \lim_{j \to +\infty} \text{Cap}_{\varphi,\psi}(K_j) = \lim_{j \to +\infty} \int_X \phi_j \text{MA}(h_j),
$$

where $h_j := h^*_{\varphi,\psi,K_j}$ and $\phi_j := \phi_{\varphi,\psi,K_j}$ is defined by (2.3). Since $\phi_j$ is quasicontinuous for any $j$ and $\phi_j \searrow \phi$, the conclusion follows from Lemma 2.15. □

**Lemma 2.17.** Let $u, v$ be $\omega$-plurisubharmonic functions. Let $G \subset X$ be an open subset. Set $E = \{u < v\} \cap G$ and $h_E := h^*_{\varphi,\psi,E}$. Then

$$
\text{Cap}_{\varphi,\psi}(E) = \text{Cap}_{\varphi,\psi}(E) = \int_{\{h_E < \psi\}} \text{MA}(h_E) = \int_X \left( \frac{-h_E + \psi}{-\varphi + \psi} \right) \text{MA}(h_E).
$$

**Proof.** We start showing the first identity. First, just by definition $\text{Cap}_{\varphi,\psi}(E) \geq \text{Cap}_{\varphi,\psi}(E)$. Fix $\varepsilon > 0$. There exists a function $\tilde{v} \in C(X)$ such that

$$
\text{Cap}_\omega(\{\tilde{v} \neq v\}) < \varepsilon.
$$
Clearly $E \subset \{u < \tilde{v}\} \cap G$ and so, applying Theorem 2.9 we get
\[
\text{Cap}_{\phi, \psi}^*(E) \leq \text{Cap}_{\phi, \psi}((\{u < \tilde{v}\} \cap G) + F(\varepsilon)) \\
\leq \text{Cap}_{\phi, \psi}(E) + 2F(\varepsilon),
\]
where $F(\varepsilon) \to 0$ as $\varepsilon \to 0$. Taking the limit as $\varepsilon \to 0$ we arrive at the first conclusion.

Let now $\{K_j\}$ be a sequence of compact sets increasing to $G$ and $\{u_j\}$ be a sequence of continuous functions decreasing to $u$. Then $E_j = \{u_j + 1/j \leq v\} \cap K_j$ is compact and $E_j \nearrow E$. Set
\[
h := h_{\phi, \psi, E}, \quad \phi := \frac{-h_E + \psi}{-\varphi + \psi}, \quad h_j := h^*_{\phi, \psi, E_j}, \quad \phi_j := \frac{-h_{E_j} + \psi}{-\varphi + \psi}.
\]
Observe that $E_j \searrow h$ and $\phi_j \searrow \phi$. By Proposition 2.16 and Lemma 2.15 we have
\[
\text{Cap}_{\phi, \psi}(E) = \lim_{j \to +\infty} \text{Cap}_{\phi, \psi}(E_j)
\]
\[
= \lim_{j \to +\infty} \int_X \phi_j \text{MA} (h_j)
\]
\[
= \int_X \phi \text{MA} (h) \leq \int_{\{h < \psi\}} \text{MA} (h).
\]
Furthermore, for each fixed $k \in \mathbb{N}$, using Theorem 2.9 we can argue as above to get
\[
\liminf_{j \to +\infty} \int_{\{h_j < \psi\}} \text{MA} (h_j) \geq \liminf_{j \to +\infty} \int_{\{h_k < \psi\}} \text{MA} (h_j) \geq \int_{\{h_k < \psi\}} \text{MA} (h).
\]
Letting $k \to +\infty$ and using Proposition 2.10 again we get
\[
\text{Cap}_{\phi, \psi}(E) \geq \int_{\{h < \psi\}} \text{MA} (h),
\]
which completes the proof. \hfill $\square$

We are now ready to prove Theorem 2.13

**Proof.** As usual, for simplicity, set $h := h_E$. By definition of the outer capacity there is a sequence $(O_j)$ of open sets decreasing to $E$ such that $\text{Cap}_{\phi, \psi}^*(E) = \lim_{j \to +\infty} \text{Cap}_{\phi, \psi}(O_j)$. Furthermore by Choquet’s lemma there exists a sequence $(u_j)$ of $\omega$-psh functions such that $u_j \equiv \psi$ quasi everywhere on $E$, $u_j \leq \psi$ on $X$ and $u_j \nearrow h$. Since $\text{Cap}_{\phi, \psi}^*$ vanishes on pluripolar sets (see Theorem 2.7) we can assume that $u_j \equiv \psi$ on $E$. For any $j$, we set $E_j = O_j \cap \{u_j < \varphi + 1/j\}$ and $h_j := h^*_{\phi, \psi, E_j}$. Then $(E_j)$ is a decreasing sequence of open subsets such that $E \subset E_j \subset O_j$ and $u_j - 1/j \leq h_j \leq h$. Clearly $\text{Cap}_{\phi, \psi}^*(E) = \lim_{j \to +\infty} \text{Cap}_{\phi, \psi}(E_j)$. By Lemma 2.17 and Lemma 2.15 we get
\[
\lim_{j \to +\infty} \text{Cap}_{\phi, \psi}^*(E_j) = \lim_{j \to +\infty} \text{Cap}_{\phi, \psi}(E_j) = \lim_{j \to +\infty} \int_X \phi_j \text{MA} (h_j) = \int_X \phi \text{MA} (h),
\]
where $\phi_j := \phi_{\phi, \psi, E_j}$ is defined by (2.3). \hfill $\square$

**Corollary 2.18.** Let $K \subset X$ be a compact set and $(K_j)$ a sequence of compact subsets decreasing to $K$. Then
\begin{enumerate}
\item $\text{Cap}_{\phi, \psi}^*(K) = \text{Cap}_{\phi, \psi}(K) = \lim_{j \to +\infty} \text{Cap}_{\phi, \psi}(K_j)$,
\item $h^*_{\phi, \psi, K_j} \nearrow h^*_{\phi, \psi, K}$.
\end{enumerate}
Proof. The first equality in statement (i) comes straightforward from Theorem 2.13 and Theorem 2.14. The second one follows from (ii) and Theorem 2.14. It remains to prove (ii). Since \((K_j)\) decreases to \(K\), \(h_j := h_{\phi,\psi,K_j}^*\) increases to some \(h_\infty \in E(X,\omega)\). Clearly \(h_\infty \leq h\). Thus we need to prove that \(h_\infty \geq h\). Since \(\{h_\infty < h\} \subset \{h_\infty < \psi\} \setminus K\) modulo a pluripolar set,

\[
\int_{\{h_\infty < \psi\} \setminus K} MA(h_\infty) \leq \int_{\{h_\infty < \psi\}} MA(h_\infty).
\]

From Proposition 2.11 we know that

\[
\int_{\{h_\infty < \psi\} \setminus K} MA(h_j) = 0, \forall j \in \mathbb{N}.
\]

Fix \(\varepsilon > 0\) and let \(\psi_\varepsilon \in C(X)\) such that \(\text{Cap}_\omega(\{\psi_\varepsilon \neq \psi\}) < \varepsilon\). Then for each fixed \(k \in \mathbb{N}\), we have

\[
\int_{\{h_\infty < \psi_\varepsilon\} \setminus K_k} MA(h_\infty) \leq \int_{\{h_\infty < \psi_\varepsilon\} \setminus K_k} MA(h_\infty) + F(\varepsilon)
\]

\[
\leq \liminf_{j \to +\infty} \int_{\{h_\infty < \psi_\varepsilon\} \setminus K_k} MA(h_j) + F(\varepsilon)
\]

\[
\leq \liminf_{j \to +\infty} \int_{\{h_\infty < \psi_\varepsilon\} \setminus K_k} MA(h_j) + 2F(\varepsilon)
\]

\[
\leq \liminf_{j \to +\infty} \int_{\{h_j < \psi_\varepsilon\} \setminus K_k} MA(h_j) + 2F(\varepsilon)
\]

\[
= 2F(\varepsilon),
\]

where \(F(\varepsilon) \to 0\) as \(\varepsilon \to 0\) thanks to Theorem 2.9. Thus, letting \(\varepsilon \to 0\) then \(k \to +\infty\) and using the domination principle below (Proposition 3.1) we can conclude that \(h_\infty \geq h\). \(\square\)

2.3. Proof of Theorem A. Let us briefly resume the proof of Theorem A. Statements (i) and (ii) have been proved in Theorem 2.14 and Theorem 2.9 respectively. One direction of the last statement has been proved in Theorem 2.7. Now, if \(E\) is a Borel subset of \(X\) such that \(\text{Cap}^*_{\phi,\psi}(E) = 0\) then it follows from Theorem 2.14 that

\[
\int_{\{h_{\phi,\psi,K}^* < \psi\}} MA(h_{\phi,\psi,K}^*) = 0.
\]

We then can apply the domination principle (see [7] or Proposition 3.1 below for a proof) to conclude.

3. Another proof of the Domination Principle

The following domination principle was proved by Dinew using his uniqueness result [16], [7]. As an application of the \((\phi, \psi)\)-Capacity we propose here an alternative proof.

Proposition 3.1. If \(u, v \in \mathcal{E}(X, \psi)\) such that \(u \leq v\) \(MA(v)\)-almost everywhere then \(u \leq v\) on \(X\).
Proof. We first claim that for every \( \varphi \in \mathcal{E}(X, \omega) \) such that \( 0 \leq \varphi - u \leq C \) for some constant \( C > 0 \) and for any \( s > 0 \) one has
\[
\int_{\{v < u - s\}} MA(\varphi) = 0.
\]
Indeed, fix \( s > 0 \) and let \( \varphi \) be such a function. Let \( C > 0 \) be a constant such that \( \varphi - u \leq C \) on \( X \). Choose \( \delta \in (0, 1) \) such that \( \delta C < s \).

Now, by using the comparison principle and the fact that \( 0 \leq \varphi - u \leq C \) we get
\[
\delta^n \int_{\{v < u - s\}} MA(\varphi) = \int_{\{v < u - s\}} (\delta \omega + dd^c \delta \varphi)^n 
\leq \int_{\{v < \delta \varphi + (1 - \delta)u - s\}} MA(\delta \varphi + (1 - \delta)u) 
\leq \int_{\{v < \delta \varphi + (1 - \delta)u - s\}} MA(v) 
\leq \int_{\{v < u\}} MA(v) = 0.
\]
Thus, the claim is proved. Now for each \( t > 0 \) let \( h_t \) denote the \((\varphi, 0)\)-extremal function of the open set \( G_t := \{u < -t\} \). It is clear that for every \( t > 0, h_t \in \mathcal{E}(X, \omega) \) and \( \sup_X (h_t - u) < +\infty \). The previous step yields
\[
\int_{\{v < u - s\}} MA(h_t) = 0, \forall s > 0.
\]
Fix \( \varepsilon > 0 \). Let \( \tilde{u} \) be a continuous function on \( X \) such that \( \text{Cap}_\omega(\{u \neq \tilde{u}\}) < \varepsilon \).

Since \( h_t \) increases to \( 0 \) as \( t \) increases to \( +\infty \) (see Lemma 2.2 below), we infer that
\[
\int_{\{v < \tilde{u} - s\}} \omega^n \leq \lim inf_{t \to +\infty} \int_{\{v < u - s\}} MA(h_t) + \text{Cap}_{u, 0}(\{u \neq \tilde{u}\}).
\]
Repeating this argument we get
\[
\int_{\{v < u - s\}} \omega^n \leq \varepsilon + \text{Cap}_{u, 0}(\{u \neq \tilde{u}\}).
\]
Letting \( \varepsilon \to 0 \) and using Theorem 2.9 we get \( \text{Vol}(\{v < u - s\}) = 0 \), for any \( s > 0 \) which implies that \( u \leq v \) on \( X \) as desired. \( \square \)

Lemma 3.2. Let \( v \in \text{PSH}(X, \omega) \). For each \( t > 0 \), set \( G_t := \{v < -t\} \). Denote by \( h_t \) the \((\varphi, 0)\)-extremal function of \( G_t \). Then \( h_t \) increases quasi everywhere on \( X \) to \( 0 \) when \( t \) increases to \( +\infty \).

Proof. We know that \( h_t \) increases quasi everywhere to \( h \in \mathcal{E}(X, \omega) \) and that \( h \leq 0 \). By Theorem 2.7 (up to consider \( -(v)^p \) with \( p \in (0, 1) \) instead of \( v \)), we get
\[
\lim_{t \to +\infty} \text{Cap}_{\varphi, 0}(G_t) = 0.
\]
It follows from Theorem 2.13 that for each \( t > 0 \),
\[
\int_{\{h < 0\}} MA(h_t) \leq \int_{\{h_t < 0\}} MA(h_t) = \text{Cap}_{\varphi, 0}(G_t).
\]
We thus get
\[
\int_{\{h < 0\}} MA(h) \leq \lim inf_{t \to +\infty} \int_{\{h < 0\}} MA(h_t) = 0.
\]
The comparison principle yields $\text{Vol}\{h < 0\} = 0$ which completes the proof. □

**Remark 3.3.** Lemma 3.2 is stated and proved in the case $\psi \equiv 0$. Observe that it also holds for any $\psi \in E(X, \omega)$ such that $\varphi < \psi$. To see this we can follow the same arguments of above but for the final step where we get $\psi \leq h \ MA(h)$-almost everywhere. We then conclude using the domination principle.

### 4. Applications to Complex Monge-Ampère equations

In this section (in the same spirit of [15]) we prove Theorem B by using $\text{Cap}_\psi := \text{Cap}_{\psi - 1, \psi}$. Let us recall the setting. Let $X$ be a compact Kähler manifold of dimension $n$ and let $\omega$ be a Kähler form on $X$. Let $D$ be an arbitrary divisor on $X$. Consider the complex Monge-Ampère equations

\[(\omega + dd^c \varphi)^n = e^{\lambda \varphi} f^\omega n, \quad \lambda \in \mathbb{R}.
\]

We say that $f$ satisfies Condition $H_f$ if

\[f = e^{\psi^+ - \psi^-}, \quad \psi^\pm \text{ are quasi psh functions on } X, \quad \psi^- \in L^\infty_{\text{loc}}(X \setminus D).
\]

We have already treated the case when $\lambda = 0$ in [15]. If $\lambda > 0$ and $f$ is integrable then the same arguments can be applied. More precisely, $C^0$-estimates follow from comparison principle while the $C^2$ estimate follows exactly the same way as in [15].

The case when $\lambda < 0$ is known to be much more difficult. We need the following observation where we make use of the generalized capacity $\text{Cap}_\psi$:

**Lemma 4.1.** Let $\varphi \in E(X, \omega)$ be normalized by $\sup_X \varphi = 0$. Assume that there exist a positive constant $A$ and $\psi \in \text{PSH}(X, \omega/2)$ such that $\text{MA}(\varphi) \leq e^{-A\psi} \omega^n$. Then there exists $C > 0$ depending only on $\int_X e^{-2A\varphi} \omega^n$ such that

\[\varphi \geq \psi - C.
\]

Observe that for all $A > 0$ and $\varphi \in E(X,\omega)$, $e^{-A\varphi} \omega^n \in L^1(X)$ as follows from Skoda integrability theorem [23, 24], since functions in $E(X, \omega)$ have zero Lelong number at all points [20].

**Proof.** Set

\[H(t) = \left[\text{Cap}_\psi(\{\varphi < \psi - t\})\right]^{1/n}, \quad t > 0.
\]

Observe that $H(t)$ is right-continuous and $H(+\infty) = 0$ (see [15 Lemma 2.6]). It follows from [15] Lemma 2.7 that $\text{Cap}_\psi \leq 2^n \text{Cap}_\psi$. By a strong volume-capacity domination in [19] we also have

\[\text{Vol}_\omega \leq \exp\left(\frac{-C_1}{\text{Cap}_\omega^{1/n}}\right),
\]
where $C_1$ depends only on $(X, \omega)$. Thus using [15, Proposition 2.8] and the assumption on the measure $\text{MA} (\varphi)$, we get
\[
\begin{aligned}
\text{Cap}_\psi (\{ \varphi < \psi - t \}) & \leq \int_{\{ \varphi < \psi - t \}} \text{MA} (\varphi) \\
& \leq \int_{\{ \varphi < \psi - t \}} e^{-A\varphi} e^{A\psi} \text{MA} (\varphi) \\
& \leq \left[ \int_X e^{-2A\varphi} \omega^n \right]^{1/2} \left[ \int_{\{ \varphi < \psi - t \}} \omega^n \right]^{1/2} \\
& \leq C_2 \left[ \text{Cap}_\psi (\{ \varphi < \psi - t \}) \right]^2,
\end{aligned}
\]
which $C_2$ depends on $\int_X e^{-2A\varphi} \omega^n$. We then get
\[
sH(t+s) \leq C_2^{1/n} H(t)^2, \ \forall t > 0, \forall s \in [0,1].
\]
Then by [17, Lemma 2.4] we get $\varphi \geq \psi - C_3$, where $C_3$ only depends on $\int_X e^{-2A\varphi} \omega^n$. \hfill \hfill \Box

Now, we are ready to prove Theorem B.

4.1. Proof of Theorem B. It suffices to treat the case when $\lambda = -1$. Since $f$ satisfies Condition $\mathcal{H}_f$ we can write $\log f = \psi^+ - \psi^-$, where $\psi^\pm$ are psh functions on $X$, $\psi^-$ is locally bounded on $X \setminus D$ and there exists a uniform constant $C > 0$ such that
\[
\frac{dd^c \psi^+}{\omega} \geq -C \omega, \ \sup_X \psi^+ \leq C.
\]
We apply the smoothing kernel $\rho_* \in$ Demailly regularization theorem [13] to the functions $\varphi$ and $\psi^\pm$. For $\varepsilon$ small enough, we get
\[
\frac{dd^c \rho_*(\varphi + \psi^-)}{\omega} \geq -C_1 \omega, \ \frac{dd^c \rho_*(\psi^+)}{\omega} \geq -C_1 \omega, \ \sup_X \rho_*(\psi^+) \leq C_1,
\]
where $C_1$ depends on $C$ and the Lelong numbers of the currents $C\omega + dd^c \psi^\pm$. By the classical result of Yau [24], for each $\varepsilon$, there exists a unique smooth $\omega$-psh function $\phi_\varepsilon$ satisfying
\[
\text{MA} (\phi_\varepsilon) = e^{c_\varepsilon + \rho_*(\psi^+)} - \rho_*(\psi^+) \omega^n = g_\varepsilon \omega^n, \ \sup_X \phi_\varepsilon = 0,
\]
where $c_\varepsilon$ is a normalization constant such that
\[
\int_X g_\varepsilon \omega^n = \int_X e^{-\varphi} f \omega^n = \int_X \omega^n.
\]
Since by Jensen’s inequality $e^{\rho_*(\varphi + \log f)} \leq \rho_*(e^{-\varphi + \log f})$ and $e^{\rho_*(\varphi + \log f)}$ converges point-wise to $e^{-\varphi} f$ on $X$, it follows from the general Lebesgue dominated convergence theorem that $e^{\rho_*(\varphi + \log f)}$ converges to $e^{-\varphi} f$ in $L^1(X)$ when $\varepsilon \downarrow 0$. This means that $c_\varepsilon$ converges to zero when $\varepsilon \to 0$. It then follows from [15, Lemma 3.4] that $\phi_\varepsilon$ converges in $L^1(X)$ to $\varphi - \sup_X \varphi$. We now apply the $C^2$ estimate in [15, Theorem 3.2] to get
\[
n + \Delta \phi_\varepsilon \leq C_3 e^{-2\rho_*(\psi^+)} \leq C_4 e^{-2(\psi^+ - \psi^-)},
\]
where $C_3, C_4$ are uniform constants (do not depend on $\varepsilon$). Now, we need to bound $\varphi$ from below. By the assumption on $f$ we have

\[ MA(\varphi) = e^{\psi^+-(\varphi+\psi^-)}\omega^n \leq e^{-(\varphi+\psi^-)}C\omega^n. \]

Consider $\psi := \frac{1}{2}(\varphi + \psi^-)$. Since this function belongs to $\text{PSH}(X, \omega/2)$ we can apply Lemma 4.1 to get

\[ \varphi - \sup_X \varphi \geq \psi - C_7 \]

This gives $\varphi \geq C_6 \psi - C_7$. Applying again this argument to $\phi_\varepsilon$ and noting that $c_\varepsilon$ converges to 0, and hence under control, we get

\[ \phi_\varepsilon \geq \rho_\varepsilon (\varphi + \psi^-) - C_9 \geq C_{10}. \]

We can now conclude using the same arguments in [15, Section 3.3].

4.2. (Non) Existence of solutions. In the previous subsection, no regularity assumption on $D$ has been done. We now discuss about the existence of solutions in concrete examples, assuming more information on $D, f$.

Let $D = \sum_{j=1}^{N} D_j$ be a simple normal crossing divisor on $X$. Recall that ”simple normal crossing” means that around each intersection point of $k$ components $D_{j_1}, \ldots, D_{j_k}$ ($k \leq N$), we can find complex coordinates $z_1, \ldots, z_n$ such that for each $l = 1, \ldots, k$ the hypersurface $D_{j_l}$ is locally given by $z_l = 0$.

For each $j$, let $L_j$ be the holomorphic line bundle defined by $D_j$. Let $s_j$ be a holomorphic section of $L_j$ defining $D_j$, i.e $D_j = \{ s_j = 0 \}$. We fix a hermitian metric $h_j$ on $L_j$ such that $|s_j|_{h_j} \leq 1/e$.

We assume that $f$ has the following particular form:

\[ f = \frac{h}{\prod_{j=1}^{N} |s_j|^2(-\log |s_j|)^{1+\alpha}}, \quad \alpha > 0, \]

where $h$ is a bounded function: $0 < 1/B \leq h \leq B$, $B > 0$.

**In this subsection we always assume that** $\lambda < 0$.

**Proposition 4.2.** Assume that $f$ satisfies (4.2) with $0 < \alpha \leq 1$. Then there is no solution in $\mathcal{E}(X, \omega)$ to equation

\[ (\omega + dd^c\varphi)^n = e^{\lambda \varphi} f\omega^n. \]

**Proof.** We can assume (up to normalization) that $\lambda = -1$. Then observe that if there exists $\varphi \in \mathcal{E}(X, \omega)$ such that $(\omega + dd^c\varphi)^n = e^{-\varphi} \mu$, where $\mu$ is a positive measure, then we can find $A > 0$ such that

\[ \mu \leq A (\omega + dd^c u)^n, \]

where $u := e^{(\varphi - \sup_X \varphi)/n}$ is a bounded $\omega$-psh function. Indeed, $u$ is a $\omega$-psh function and

\[ \omega + dd^c u \geq \omega + \frac{u}{n} dd^c \varphi \geq \frac{u}{n} (\omega + dd^c \varphi) \geq 0. \]

This coupled with [15] Proposition 4.4 and 4.5 yields the conclusion. 

The above analysis shows that there is no solution if the density has singularities of Poincaré type or worse. We next show that when $f$ is less singular than the Poincaré type density (i.e. $\alpha > 1$), equation (4.1) has a bounded solution provided
\( \lambda = -\varepsilon \) with \( \varepsilon > 0 \) very small. We say that \( f \) satisfies Condition \( S(B, \alpha) \) for some \( B > 0, \alpha > 0 \)

\[
f \leq \frac{B}{\prod_{j=1}^{N} |s_j|^2(-\log |s_j|)^{1+\alpha}}.
\]

**Theorem 4.3.** Assume that \( f \) satisfies Condition \( S(B, \alpha) \) with \( \alpha > 1 \). We also normalize \( f \) so that \( \int_X f \omega^n = \int_X \omega^n \). Then for \( \lambda = -\varepsilon \) with \( \varepsilon > 0 \) small enough depending only on \( C, \alpha, \omega \), there exists a bounded solution \( \varphi \) to (4.1). The solution is automatically continuous on \( X \). In particular, it is also smooth on \( X \setminus D \) if \( f \) is smooth there.

**Proof.** The last statement follows easily from our previous analysis. Let us prove the existence. We use the Schauder Fixed Point Theorem. Let \( C = C(2B, \alpha) \) be the constant in Lemma 4.4 below. Choose \( \varepsilon > 0 \) very small such that \( e^{\varepsilon C} \leq 2 \).

Consider the following compact convex set in \( L^1(X) \):

\[
C := \{ u \in \text{PSH}(X, \omega) \mid -C \leq u \leq 0 \}.
\]

Let \( \psi \in C \) and \( c_\psi \) be a constant such that

\[
\int_X e^{-\varepsilon \psi + c_\psi} f \omega^n = \int_X \omega^n.
\]

Since \(-C \leq \psi \leq 0\), it is clear that \(-C \varepsilon \leq c_\psi \leq 0\). Let \( \varphi \) be the unique bounded \( \omega \)-psh function such that \( \sup_X \varphi = 0 \) and

\[
(\omega + dd^c \varphi)^n = e^{-\varepsilon \psi + c_\psi} f \omega^n.
\]

The density on the right-hand side satisfies Condition \( S(B, \alpha) \) since \( c_\psi \leq 0 \) and since \( e^{\varepsilon C} \leq 2 \). We thus get from Lemma 4.4 below that \( \varphi \geq -C \). Thus we have defined a mapping from \( C \) to itself

\[
\Phi : C \to C, \quad \Phi(\psi) := \varphi.
\]

Let us prove that \( \Phi \) is continuous on \( C \). Let \( \psi_j \) be a sequence in \( C \) which converges to \( \psi \) in \( L^1(X) \). Denote by

\[
c_j := c_{\psi_j}, \quad c := c_\psi, \quad \Phi(\psi_j) = \varphi_j, \quad \Phi(\psi) = \varphi.
\]

It is enough to prove that any cluster point of the sequence \( (\varphi_j) \) is equal to \( \varphi \). Therefore, we can assume that \( \varphi_j \) converges to \( \varphi_0 \) in \( L^1(X) \) and up to extracting a subsequence that \( \psi_j \) converges almost everywhere to \( \psi \) on \( X \) and also that \( c_j \) converges to \( c_0 \in [-C \varepsilon, 0] \). Since \( e^{-\varepsilon \psi_j + c_j} f \) converges in \( L^1(X) \) to \( e^{-\varepsilon \psi + c_0} f \) in \( L^1(X) \) and almost everywhere, it follows from [8, Lemma 3.4] that

\[
(\omega + dd^c \varphi_0)^n = e^{-\varepsilon \psi + c_0} f \omega^n.
\]

It is clear that \( c_0 = c \) and it follows from Hartogs’ lemma that \( \sup_X \varphi_0 = 0 \). Thus \( \varphi_0 = \varphi \). This concludes the continuity of \( \Phi \).

Now, since \( C \) is compact and convex in \( L^1(X) \) and since \( \Phi \) is continuous on \( C \), by Schauder Fixed Point Theorem there exists a fixed point of \( \Phi \), say \( \varphi \). Then \( \varphi - c_\varphi / \varepsilon \) is the desired solution.

We refer the reader to [15, Section 4.2] for the proof of the following lemma.
Lemma 4.4. Assume that $f$ satisfies Condition $S(B, \alpha)$ with $\alpha > 1$, $B > 0$. Let $\varphi \in \mathcal{E}(X, \omega)$ be the unique function such that

$$(\omega + dd^c \varphi)^n = f \omega^n, \quad \sup_X \varphi = 0.$$  

Then $\varphi \geq -C$ with $C = C(B, \alpha) > 0$.

4.3. Proof of Theorem C. Assume that $\varphi \in \mathcal{E}(X, \omega)$ satisfies

$$(\omega + dd^c \varphi)^n = e^{\lambda \varphi} f \omega^n, \quad \lambda > 0.$$  

Up to rescaling $\omega$ it suffices to treat the case when $\lambda = 1$. The proof of Theorem C is quite similar to that of Theorem B. The difference here is that $f$ is not integrable. For convenience of the reader we rewrite the arguments here. Since $f$ satisfies Condition $H_f$ we can write $\log f = \psi^+ - \psi^-$, where $\psi^\pm$ are psh functions on $X$, $\psi^-$ is locally bounded on $X \setminus D$ and there exists a uniform constant $C > 0$ such that

$$dd^c \psi^\pm \geq -C \omega, \quad \sup_X \psi^+ \leq C.$$  

We apply the smoothing kernel $\rho_\varepsilon$ in Demailly regularization theorem [13] to the functions $\varphi$ and $\psi^\pm$. For $\varepsilon$ small enough, we get

$$dd^c \rho_\varepsilon(\psi^-) \geq -C_1 \omega, \quad dd^c \rho_\varepsilon(\varphi + \psi^+) \geq -C_1 \omega, \quad \sup_X \rho_\varepsilon(\varphi + \psi^+) \leq C_1,$$  

where $C_1$ depends on $C$, the Lelong numbers of the currents $C \omega + dd^c \psi^\pm$ and $\sup_X \varphi$. By the classical result of Yau [24], for each $\varepsilon$, there exists a unique smooth $\omega$-psh function $\phi_\varepsilon$ satisfying

$$\text{MA}(\phi_\varepsilon) = e^{c_\varepsilon + \rho_\varepsilon(\varphi + \psi^+) - \rho_\varepsilon(\psi^-)} \omega^n = g_\varepsilon \omega^n, \quad \sup_X \phi_\varepsilon = 0,$$  

where $c_\varepsilon$ is a normalization constant such that

$$\int_X g_\varepsilon \omega^n = \int_X e^{\varphi} f \omega^n = \int_X \omega^n.$$  

Since by Jensen’s inequality $e^{\rho_\varepsilon(\varphi + \log f)} \leq \rho_\varepsilon(e^{\varphi + \log f})$ and $e^{\rho_\varepsilon(\varphi + \log f)}$ converges point-wise to $e^\varphi f$ on $X$, it follows from the general Lebesgue dominated convergence theorem that $e^{\rho_\varepsilon(\varphi + \log f)}$ converges to $e^\varphi f$ in $L^1(X)$ when $\varepsilon \downarrow 0$. This means that $c_\varepsilon$ converges to zero when $\varepsilon \to 0$. It then follows from Lemma 3.4 in [13] that $\phi_\varepsilon$ converges in $L^1(X)$ to $\varphi - \sup_X \varphi$. We now apply the $C^2$ estimate in Theorem 3.2 in [15] to get

$$n + \Delta \phi_\varepsilon \leq C_3 e^{-2\rho_\varepsilon(\psi^-)} \leq C_4 e^{-2\psi^-},$$  

where $C_3, C_4$ are uniform constants (do not depend on $\varepsilon$). Now, we need to bound $\varphi$ from below. By the assumption on $f$ we have

$$\text{MA}(\varphi) = e^{\varphi + \psi^+ - \psi^-} \omega^n \leq e^{-(\psi^- - C_1)} \omega^n.$$  

Consider $\psi := \frac{1}{\omega} \psi^-$. Since this function belongs to $\text{PSH}(X, \omega/2)$ we can apply Lemma 4.11 to get

$$\varphi - \sup_X \varphi \geq \psi - C_5.$$  

Now the remaining part of the proof follows by exactly the same way as we have done in [15] Section 3.3].
4.4. Non Integrable densities. When $0 \leq f \notin L^1(X)$ it is not clear that we can find a solution $\varphi \in \mathcal{E}(X, \omega)$ of equation
\[(\omega + dd^c \varphi)^n = e^{\varphi} f \omega^n.\]
We show in the following that it suffices to find a subsolution. Another similar result has been proved by Berman and Guenancia in [5] using the variational approach. We provide here a simple proof using our generalized Monge-Ampère capacities.

**Theorem 4.5.** Let $0 \leq f$ be a measurable function such that $\int_X f \omega^n = +\infty$. If there exists $u \in \mathcal{E}(X, \omega)$ such that $\text{MA}(u) \geq e^u f \omega^n$ then there is a unique $\varphi \in \mathcal{E}(X, \omega)$ such that
\[
\text{MA}(\varphi) = e^{\varphi} f \omega^n.
\]

**Proof.** The uniqueness follows easily from the comparison principle. Indeed, one can find a proof in [6, Proposition 3.1]. We now establish the existence. For each $j \in \mathbb{N}$ we can find $\varphi_j \in \text{PSH}(X, \omega) \cap L^\infty(X)$ such that
\[
(\omega + dd^c \varphi_j)^n = e^{\varphi_j} \min(f, j) \omega^n.
\]
It follows from the comparison principle that $\varphi_j$ is non-increasing and $\varphi_j \geq u$. Then $\varphi_j \downarrow \varphi \in \mathcal{E}(X, \omega)$ and by continuity of the complex Monge-Ampère operator along decreasing sequence in $\mathcal{E}(X, \omega)$ we get
\[
\text{MA}(\varphi) = e^{\varphi} f \omega^n.
\]
Indeed, since $\text{MA}(\varphi_j)$ converges weakly to $\text{MA}(\varphi)$, from Fatou’s lemma we get
\[
\text{MA}(\varphi) \geq e^{\varphi} f \omega^n
\]
in the sense of positive Borel measures. To get the reverse inequality we need to show that the right-hand side has full mass, i.e.
\[
\int_X e^{\varphi} f \omega^n = \int_X \omega^n.
\]
Fix $\varepsilon > 0$. Since $\varphi$ is $\omega$-psh, in particular quasi-continuous, we find $U$ an open subset of $X$ such that Cap$_\omega(U) < \varepsilon$ and $\varphi$ is continuous on $K := X \setminus U$. Then $\varphi$ is bounded on $K$ and hence $f$ must be integrable on $K$. We thus can apply the Lebesgue Dominated Convergence Theorem on $K$ to get
\[
\lim_{j \to +\infty} \int_K \text{MA}(\varphi_j) = \int_K e^{\varepsilon j} \min(f, j) \omega^n = \int_K e^{\varepsilon} f \omega^n.
\]
We can assume that $\varphi_j \leq 0$. It follows from Theorem 2.3 that
\[
\int_U \text{MA}(\varphi_j) \leq \text{Cap}_{u,0}(U) \leq F(\varepsilon) \to 0 \quad \text{as } \varepsilon \downarrow 0.
\]
This implies that
\[
\int_X e^{\varphi} f \omega^n \geq \int_K e^{\varphi} f \omega^n = \lim_{j \to +\infty} \int_K \text{MA}(\varphi_j) = \int_X \text{MA}(\varphi) - \lim_{j \to +\infty} \int_U \text{MA}(\varphi_j) \geq \int_X \omega^n - F(\varepsilon).
\]
By letting $\varepsilon \to 0$ we get $\int_X e^{\varphi} f \omega^n = \int_X \omega^n$, which completes the proof. \qed

**Remark 4.6.** Theorem 4.5 also holds if $\omega$ is merely semipositive and big.
Example 4.7. Let $D = \sum_{j=1}^{N} D_j$ be a simple normal crossing divisor on $X$. Assume that the $D_j$ are defined by $s_j = 0$, where $s_j$ are holomorphic sections such that $|s_j| < 1/e$. Consider the following density

$$f = \frac{1}{\prod_{j=1}^{N} |s_j|^2}.$$ 

Then for suitable positive constants $C_1, C_2$ the following function

$$\varphi := -2 \sum_{j=1}^{N} \log(-\log|s_j| + C_1) - C_2$$

is a subsolution of $\text{MA}(\varphi) = e^{\varphi} f^n$. In fact, it suffices to find a function $u \in \mathcal{E}(X, \omega/2)$ such that $e^u f$ is integrable (see Example 4.9).

4.5. The case of semipositive and big classes. In this section we try to extend our result in Theorem C to the case of semipositive and big classes. Let $\theta$ be a smooth closed semipositive $(1, 1)$-form on $X$ such that $\int_X \theta^n > 0$. Assume that $E = \sum_{j=1}^{M} a_j E_j$ is an effective simple normal crossing divisor on $X$ such that $\{\theta\} - c_1(E)$ is ample. Let $0 \leq f$ is a non-negative measurable function on $X$. Consider the following degenerate complex Monge-Ampère equation

$$\text{(4.3)}$$

$$(\theta + dd^c \varphi)^n = e^{\varphi} f^n.$$ 

As in Theorem C we obtain here a similar regularity for solutions in $\mathcal{E}(X, \omega)$:

Theorem 4.8. Assume that $0 < f \in C^\infty(X \setminus D)$ satisfies Condition $\mathcal{H}_f$. Let $\theta$ and $E$ be as above. If there is a solution in $\mathcal{E}(X, \omega)$ of equation (4.3) then this solution is also smooth on $X \setminus (D \cup E)$.

Note that in Theorem 4.8 we do not assume that $f$ is integrable on $X$. We also stress that there is at most one solution in $\mathcal{E}(X, \theta)$ (see [5]).

Proof. We adapt the proof of Theorem 3 in [15] where we followed essentially the ideas in [8]. Assume that $\varphi \in \mathcal{E}(X, \theta)$ is a solution to equation (4.3). By assumption on $f$ we can find a uniform constant $C > 0$ such that

$$f = e^{\psi^+ - \psi^-}, \quad dd^c \psi^\pm \geq -C \omega^n, \quad \sup_X \psi^+ \leq C, \quad \sup_X \psi^- \leq C, \quad \psi^-, \psi^+ \in L^\infty_{\text{loc}}(X \setminus D).$$

We regularize $\varphi$ and $\psi^\pm$ by using the smoothing kernel $\rho_\varepsilon$ in Demailly’s work [13]. Then for $\varepsilon > 0$ small enough we have

$$dd^c \rho_\varepsilon (\psi^-) \geq -C_1 \omega, \quad dd^c \rho_\varepsilon (\varphi + \psi^+) \geq -C_1 \omega, \quad \sup_X \rho_\varepsilon (\varphi + \psi^+) \leq C_1,$$

where $C_1$ depends on $C$ and the Lelong numbers of the currents $C \omega + dd^c \psi^\pm$. For each $\varepsilon > 0$ by the famous result of Yau [24] there exits a unique smooth $\phi_\varepsilon \in \text{PSH}(X, \theta + \varepsilon \omega)$ normalized by $\sup_X \phi_\varepsilon = 0$ such that

$$(\theta + \varepsilon \omega + dd^c \phi_\varepsilon)^n = e^{\psi^+ + \phi_\varepsilon} - \psi^- - \phi_\varepsilon \omega^n = g_\varepsilon \omega^n,$$

where $c_\varepsilon$ is a normalized constant. As in the proof of Theorem 3 in [15] we can prove that $c_\varepsilon$ converges to 0 as $\varepsilon \downarrow 0$. We then can show that $\phi_\varepsilon$ converges in $L^1$ to $\varphi - \sup_X \varphi$. Now, we can apply Theorem 5.1 and Theorem 5.2 in [15] to get uniform bound on $\phi_\varepsilon$ and $\Delta_\omega \phi_\varepsilon$ on every compact subset of $X \setminus (D \cup E)$. From this we can get the smoothness of $\varphi$ on $X \setminus (D \cup E)$ as in [15]. \qed
It follows from Theorem [13] (which is also valid in the case of semipositive and big classes) that to solve the equation it suffices to find a subsolution in \( \mathcal{E}(X, \theta) \). We show in the following example that in some cases it is easy to find a subsolution in \( \mathcal{E}(X, \theta) \).

**Example 4.9.** We consider the density given in Example [4.7]. Assume that \( \theta \) satisfies \( \{ \theta \} - c_1(E) > 0 \), where \( E = \sum_{j=1}^{M} a_j E_j \) is an effective simple normal crossing divisor on \( X \). Assume that \( E_j \) is defined by the zero locus of a holomorphic section \( \sigma_j \) such that \( |\sigma_j| < 1/e \). Then for some constants \( p \in (0, 1) \) and \( a > 0 \), \( A \in \mathbb{R} \) the following function

\[
u := -\left( -a \sum_{j=1}^{N} \log |s_j| - \frac{1}{2} \sum_{j=1}^{M} a_j \log |\sigma_j| \right)^p - A
\]

belongs to \( \mathcal{E}(X, \theta/2) \) and verifies \( \int_X e^\nu f \omega^n = 2^{-n} \int_X \nu^n \). It follows from [3] that there exists \( v \in \mathcal{E}(X, \theta/2) \) such that \( v \leq 0 \) and

\[(\theta/2 + dd^c v)^n = e^\nu f \omega^n.\]

It is easy to see that \( \nu := u + v \in \mathcal{E}(X, \theta) \) is a subsolution of [4.3].

**4.6. Critical Integrability.** Recently, Berndtsson [6] solved the openness conjecture of Demailly and Kollár [13] which says that given \( \phi \in \text{PSH}(X, \omega) \) and

\[\alpha(\phi) = \sup \{ t > 0 \mid e^{-t \phi} \in L^1(X) \} < +\infty,\]

then one has \( e^{-\alpha\phi} \notin L^1(X) \) (a stronger version of the openness conjecture has been quite recently obtained by Guan and Zhou [18]).

In the following result, we use the generalized capacity to show that \( e^{-\alpha\phi} \) is however not far to be integrable in the following sense:

**Theorem 4.10.** Let \( \phi \in \text{PSH}(X, \omega) \) and \( \alpha = \alpha(\phi) \in (0, +\infty) \) be the canonical threshold of \( \phi \). Then we can find \( \varphi \in \text{PSH}(X, \omega) \) having zero Lelong number at all points of \( X \) such that

\[\int_X e^{\varphi - \alpha\phi} \omega^n < +\infty.\]

One can moreover chose \( \varphi = \chi \circ \phi \in \mathcal{E}(X, \omega) \) for some \( \chi \) increasing convex function. We thank S. Boucksom and H. Guenancia for indicating this.

**Proof.** Let \( \alpha_j \) be an increasing sequence of positive numbers which converges to \( \alpha \). By assumption we have \( e^{-\alpha_j \phi} \) is integrable on \( X \). We can assume that \( \phi \leq 0 \). We solve the complex Monge-Ampère equation

\[(\omega + dd^c \varphi_j)^n = e^{2\varphi_j - \alpha_j \phi} \omega^n.\]

For each \( j \), since \( e^{-\alpha_j \phi} \) belongs to \( L^p \) for some \( p_j > 1 \), it follows from the classical result of Kołodziej [21] that \( \varphi_j \) is bounded. Moreover, the comparison principle reveals that \( \varphi_j \) is non-increasing. Now, we need to bound \( \varphi_j \) uniformly from below by some singular quasi-psh function.

Let \( 1/2 > \varepsilon > 0 \) be a very small positive number. By assumption we know that

\[e^{(\varepsilon - \alpha)\phi} \in L^p(X), \quad p = p_\varepsilon := \frac{\alpha - \varepsilon/2}{\alpha - \varepsilon} > 1.\]
Set $\psi := \varepsilon \phi \in \operatorname{PSH}(X, \omega/2)$ and consider the function

$$H(t) := \left[ \operatorname{Cap}_\psi(\varphi_j < \psi - t) \right]^{1/n}, \quad t > 0.$$ 

It follows from [15, Lemma 2.7] that $\operatorname{Cap}_\omega \leq 2^n \operatorname{Cap}_\psi$. By a strong volume-capacity domination in [19, Remark 5.10] we also have

$$\text{Vol}_\omega \leq \exp \left( -C_1 \left[ \operatorname{Cap}_\psi(\varphi_j < \psi - t) \right]^{1/n} \right),$$

where $C_1$ depends only on $(X, \omega)$. Fix $t > 0$, $s \in [0, 1]$. Using [15, Proposition 2.8] and Hölder inequality we get

$$s^n \operatorname{Cap}_\psi(\{ \varphi_j < \psi - t - s \}) \leq \int_{\{ \varphi_j < \psi - t \}} \text{MA}(\varphi_j) \leq \int_{\{ \varphi_j < \psi - t \}} e^{-\varphi_j} e^{\psi} \text{MA}(\varphi_j) \leq \int_{\{ \varphi_j < \psi - t \}} e^{(\varepsilon - \alpha)\phi_n} \leq \left[ \int_X e^{(\varepsilon/2 - \alpha)\phi_n} \right]^{1/p} \left[ \int_{\{ \varphi_j < \psi - t \}} \omega_n \right]^{1/q} \leq C_2 \left[ \operatorname{Cap}_\psi(\{ \varphi_j < \psi - t \}) \right]^{2},$$

where $p = p_\varepsilon := \frac{-\varepsilon/2}{\alpha - \varepsilon} > 1$ and $q > 1$ is the exponent conjugate of $p$. The constant $C_2 > 0$ depends on $\varepsilon$ and also on $\int_X e^{(\varepsilon/2 - \alpha)\phi_n}$. We then get

$$sH(t + s) \leq C_2^{1/n} H(t)^2, \quad \forall t > 0, \forall s \in [0, 1].$$

Then by [17, Lemma 2.4] we get

$$\varphi_j \geq \varepsilon \phi - C_\varepsilon,$$

where $C_\varepsilon$ only depends on $\varepsilon$ and $\int_X e^{(\varepsilon/2 - \alpha)\phi_n}$. Then we see that $\varphi_j$ decreases to $\varphi \in \operatorname{PSH}(X, \omega)$ and $\varphi$ satisfies

$$\varphi \geq \varepsilon \phi - C_\varepsilon.$$

Since $\varepsilon$ is arbitrarily small we conclude that $\varphi$ has zero Lelong number everywhere on $X$. Finally, it follows from Fatou’s lemma that $e^{\varphi - \alpha \phi}$ is integrable on $X$.

We now show that $\varphi$ can be chosen to be in $\mathcal{E}(X, \omega)$, more precisely $\varphi = \chi \circ \phi$,

$$\int_X e^{\chi \circ \phi - \alpha \phi} \omega_n < +\infty,$$

for some $\chi : \mathbb{R}^- \to \mathbb{R}^-$ increasing convex function such that $\chi(-\infty) = -\infty$ and $\chi'(-\infty) = 0$. Note that $\chi \circ \phi \in \mathcal{E}(X, \omega)$ thanks to [12]. We are grateful to H. Guenancia for the following constructive proof.

We can always assume that $\phi \leq -1$. For each $k \in \mathbb{N}$ let

$$a_k := \log \int_X e^{-(\alpha - 2^{-k - 1})\phi} \omega_n < +\infty.$$ 

Define the sequence $(c_k)$ inductively by

$$c_1 = a_1, \quad c_{k+1} := \max(c_k + 4k, a_{k+1}), \quad \forall k \geq 1.$$
Define another sequence \((t_k)\) by
\[
(4.6) \quad t_1 := 1, \quad t_{k+1} := 2^{k+1}(c_{k+1} - c_k), \quad \forall k \geq 1.
\]
Define \(\chi : (-\infty, -1] \to \mathbb{R}^-\) by
\[
\chi(t) := -2^{-k}t - c_k \quad \text{if} \quad t \in [t_k, t_{k+1}], \quad \forall k \geq 1.
\]
It follows from (4.4) that
\[
e^{(\alpha - 2^{-k-1})t} \Vol(\phi < -t) \leq \int_X e^{-((\alpha - 2^{-k-1})t)} \phi \omega^n \leq e^{c_k}.
\]
Thus using (4.5), (4.6) and the above inequality we get
\[
\int_X e^{\chi(\phi) - \alpha \phi} \omega^n \leq e^{\chi(-1) + \alpha} + \int_{\mathbb{R}^n} e^{\alpha t + \chi(-1)} \Vol(\phi < -t) dt
\]
\[
\leq C + \alpha \sum_{k=1}^{+\infty} \int_{t_k}^{t_{k+1}} e^{\alpha t + \chi(-1)} \Vol(\phi < -t) dt
\]
\[
\leq C + \alpha \sum_{k=1}^{+\infty} \int_{t_k}^{t_{k+1}} e^{c_k + 2^{-k-1}t - 2^{-k-1}t} dt
\]
\[
\leq C + \alpha \sum_{k=1}^{+\infty} e^{-2^{-k-1}t} dt
\]
\[
\leq C + \alpha \sum_{k=1}^{+\infty} 2^{k+1} e^{-2^{-k-1}t_k}
\]
\[
\leq C + \alpha \sum_{k=1}^{+\infty} 2^{k+1} e^{-2^{-k-1}(c_k - c_{k-1})}
\]
\[
\leq C + \alpha \sum_{k=1}^{+\infty} 2^{k+1} e^{-2(k-1)}
\]
\[
\leq C + 4\alpha.
\]

The above result is quite optimal as the following example shows:

**Example 4.11.** Let \((X, \omega)\) be a compact Kähler manifold and \(D\) be a smooth complex hypersurface on \(X\) defined by a holomorphic section \(s\) such that \(|s| \leq 1/e\).

Consider
\[
(4.7) \quad \phi = 2 \log |s| - (-\log |s|)^p, \quad p \in (0, 1).
\]

By rescaling \(\omega\) we can assume that \(\phi \in \text{PSH}(X, \omega)\). Then for any \(q > 0\)
\[
\int_X e^{-\phi} \omega^n = +\infty.
\]

The example above has been given in [4] in the case of one complex variable which is locally similar to our setting. Assume now that \(\phi\) is given by (4.7). It
follows from Theorem 4.10 that we can find \( \varphi \in \text{PSH}(X, \omega) \) having zero Lelong number everywhere such that
\[
\int_X e^{\varphi - \phi} \omega^n < +\infty.
\]
In this concrete example one such function \( \varphi \) can be given explicitly by
\[
\varphi = -(\log|s|)^p - (1 + \varepsilon) \log(\log|s|), \quad \varepsilon > 0.
\]

**Proof of Theorem D.** It follows from the above proof of Theorem 4.10 that there exists \( u \in \mathcal{E}(X, \omega/2) \) such that \( e^{u-\alpha \phi} \) is integrable. We then can argue as in Example 4.9 to find a subsolution which also yields a solution thanks to Theorem 4.5. The uniqueness follows from the comparison principle (see [5]). \( \square \)

**REFERENCES**

[1] P. Ahag, U. Cegrell, S. Kolodziej, H. H. Pham, A. Zeriahi, *Partial pluricomplex energy and integrability exponents of plurisubharmonic functions*, Advances Math. 222 (2009), 2036–2058.

[2] E. Bedford, B. A. Taylor, *A new capacity for plurisubharmonic functions*, Acta Math. 149 (1982), no. 1-2, 1-40.

[3] S. Benelkouchi, V. Guedj, A. Zeriahi, *Plurisubharmonic functions with weak singularities*, Acta Univ. Upsaliensis Skr. Uppsala Univ. C Organ. Hist. Vol. 86 (2009), 57-74.

[4] R. J. Berman, S. Boucksom, V. Guedj, A. Zeriahi, *A variational approach to complex Monge-Ampère equations*, Publ. Math. Inst. Hautes Études Sci. 117 (2013), 179-245.

[5] R. J. Berman, H. Guenancia, *Kähler-Einstein metrics on stable varieties and log canonical pairs*, arXiv:1304.2087.

[6] B. Berndtsson, *The openness conjecture for plurisubharmonic functions*, arXiv:1305.5781.

[7] T. Bloom, N. Levenberg, *Pluripotential energy*, Potential Analysis, Volume 36, Issue 1 (2012), 155-176.

[8] S. Boucksom, P. Eyssidieux, V. Guedj, A. Zeriahi, *Monge-Ampère equations in big cohomology classes*, Acta Math. 205 (2010), no. 2, 199-262.

[9] U. Cegrell, *Pluricomplex energy*, Acta Math. 180 (1998), no. 2, 187-217.

[10] U. Cegrell, *The general definition of the complex Monge-Ampère operator*, Ann. Inst. Fourier (Grenoble) 54 (2004), no. 1, 159-179.

[11] U. Cegrell, S. Kołodziej, A. Zeriahi, *Subextension of plurisubharmonic functions with weak singularities*, Math. Z. 250 (2005), no. 1, 7-22.

[12] D. Coman, V. Guedj, A. Zeriahi, *Domains of definitions of Monge-Ampère operators on compact Kähler manifolds*, Math. Z. 259 (2008), no. 2, 393-418.

[13] J. P. Demailly, *Regularization of closed positive currents and intersection theory*, J. Alg. Geom. 1 (1992), no. 3, 361-409.

[14] J. P. Demailly, J. Kollár, *Semicontinuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds*, Ann. Sci. École Norm. Sup. (4) 34 (2001), no. 4, 525-556.

[15] E. Di Nezza, H. C. Lu, *Complex Monge-Ampère equations on quasi-projective varieties*, arXiv:1401.6898.

[16] S. Dinew, *Uniqueness in \( \mathcal{E}(X, \omega) \)*, J. Funct. Anal. 256 (2009), no. 7, 2113-2122.

[17] P. Eyssidieux, V. Guedj, A. Zeriahi, *Singular Kähler Einstein metrics*, Journal of the American Mathematical Society, Volume 22, Number 3, (2009), 607-639.

[18] Q. Guan, X. Zhou, *Strong openness conjecture for plurisubharmonic functions*, arXiv:1311.3781.

[19] V. Guedj, A. Zeriahi, *Intrinsic capacities on compact Kähler manifolds*, J. Geom. Anal. 15 (2005), no. 4, 607-639.

[20] V. Guedj, A. Zeriahi, *The weighted Monge-Ampère energy of quasiplurisubharmonic functions*, J. Funct. Anal. 250 (2007), no. 2, 442-482.

[21] S. Kołodziej, *The complex Monge-Ampère equation*, Acta Math. 180 (1998) 69-117.

[22] S. Kołodziej, *The complex Monge-Ampère equation on compact Kähler manifolds*, Indiana Univ. Math. J. 52 (2003), no. 3, 667-686.
[23] H. Skoda, *Sous-ensembles analytiques d’ordre fini ou infini dans $\mathbb{C}^n$*, Bull. Soc. Math. de France 100 (1972), 353-408. J. Amer. Math. Soc. 3 (1990), no. 3, 579-609.

[24] S. T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation*, Comm. Pure Appl. Math. 31 (1978), no. 3, 339-411.

[25] A. Zeriahi, *Volume and capacity of sublevel sets of a Lelong class of plurisubharmonic functions*, Indiana Univ. Math. J., 50 (2001), 671-703.

Institut Mathématiques de Toulouse, Université Paul Sabatier, 31062 Toulouse, France

E-mail address: eleonora.dinezza@math.univ-toulouse.fr

Chalmers University of Technology, Mathematical Sciences, 412 96 Gothenburg, Sweden

E-mail address: chinh@chalmers.se