On some class of homogeneous polynomials and explicit form of integrable hierarchies of differential–difference equations

A K Svinin

Institute for System Dynamics and Control Theory, Siberian Branch of Russian Academy of Sciences, PO Box 292, 664033 Irkutsk, Russia

E-mail: svinin@icc.ru

Received 19 January 2011, in final form 3 March 2011
Published 29 March 2011
Online at stacks.iop.org/JPhysA/44/165206

Abstract

We introduce two classes of homogeneous polynomials and show their role in construction of integrable hierarchies for some integrable lattices.

PACS number: 02.30.Ik

1. Introduction

Our main goal in this paper is to introduce two classes of homogeneous polynomials \( T^l_s \) and \( S^l_s \) of many variables and to show their applicability in the theory of integrable differential–difference equations (lattices). More exactly, we construct in terms of these polynomials explicit form of some integrable hierarchies. We base our studies on the relationship of integrable lattices with KP hierarchy. To this aim, we consider bi-infinite sequences of KP hierarchies in the form of local differential–difference conservation laws [10] expressed in terms of generating relations. Our approach, in particular, goes back to [2, 4, 10].

Let us give a sketch of our approach for investigation of some integrable lattices and their hierarchies. A presentation of the sequence of KP hierarchy in the form of two generating equations

\[
\partial_s h(i, z) = \partial H^{(s)}(i, z) \quad \text{and} \quad \partial_s a(i, z) = a(i, z)(H^{(s)}(i + 1, z) - H^{(s)}(i, z)), \quad i \in \mathbb{Z},
\]

where \( \partial_s \) stands for a derivative with respect to the evolution parameter \( t_s \), can be found in [4]. Taking this representation as a basis, we have shown in [6–8] that many integrable lattices can be obtained as reductions of this system. Our study in these papers was based on two theorems which select an infinite number of invariant submanifolds \( S^j_{l-1} \). In particular, the restriction of this sequence of KP hierarchies on \( S^j_0 \) gives the totality of reduction equations

\[
\partial_i a_k(i) = a_k(i), \quad i \in \mathbb{Z},
\]

where \( \partial_i \) stands for a derivative with respect to the evolution parameter \( t_i \). We call this set of equations the \( n \)th discrete KP hierarchy. We denote by \( \mathcal{M} \) the corresponding phase space. Further we
consider intersections of the form $S^n_0 \cap S^{p-1}$ and show that this corresponds to restriction of the $n$th discrete KP hierarchy on some submanifold $M_{n,p,l} \subset M$ restricting the latter to some differential–difference system of equations on finite number of fields. The restriction on $M_{n,p,l}$ is given by an infinite set of algebraic equations $J_{k}^{(n,p,l)}[a_1, \ldots, a_{k+1}] = 0$ for $k \geq 1$. 

In this paper, we consider only a class of one-field lattices corresponding to submanifolds $M_{n,1}$ with $n \geq 1$ and $p > n$. The particular case when $p = n + 1$, is given by Itoh–Narita–Bogoyavlenskii (INB) lattice [1, 3, 5] or extended Volterra equation in terminology of [5]. For the general case of $M_{n,p,l}$, we construct corresponding integrable hierarchy in explicit form. In [9], we have already shown the explicit form of integrable hierarchy of INB lattice equations in terms of homogeneous polynomials $S^l$. In this paper, we construct a more general class of such polynomials and corresponding integrable hierarchies. 

The rest of the paper is organized as follows. In the next two sections, we give an explanation of the relationship of the KP hierarchy with some integrable lattices. In section 4, we define two classes of homogeneous polynomials $T^l$ and $S^l$ of many variables and show some identities which directly follow from their definition. In section 5, we consider the restriction of $n$th discrete KP hierarchy on submanifolds $M_{n,1}$ corresponding to one-component lattices and prove there some technical lemmas. In particular, we write down an infinite number of linear equations on KP wavefunction $\Psi = \{\psi_i\}$ which corresponds to the sequence of invariant submanifolds inclusions. We show the explicit form of evolution equations of some class of integrable hierarchies. In subsections 5.5 and 5.6, we provide the reader with examples and show the explicit form of some integrable hierarchies of one-component lattices.

2. The KP hierarchy

2.1. The KP hierarchy

It is known that there exists a close relationship of the KP hierarchy with some integrable lattices and their hierarchies. In this and the next section, we briefly describe this relationship based on our approach developed in [6–8] considering a free bi-infinite chain of KP hierarchies and its suitable reductions.

We write evolution equations of the KP hierarchy itself in the form of local conservation laws [10] 
\[ \partial_t h(z) = \partial H^{(\ell)}(z), \] 
where formal Laurent series $h(z) = z + \sum_{j \geq 2} h_j z^{-j+1}$ and $H^{(\ell)}(z) = z^\ell + \sum_{j \geq 1} H_j^\ell z^{-j}$ are related with formal KP wavefunction as $h(z) = \partial\psi/\psi$ and $H^{(\ell)}(z) = \partial_\ell \psi/\psi$, respectively. Each coefficient $H^\ell_j$, in fact, is calculated to be some differential polynomial $H^\ell_j = H^\ell_j[h_2, \ldots, h_{\ell+1}]$. More exactly, the Laurent series $H^{(\ell)}(z)$ is calculated as projection of $z^\ell$ onto the linear space $\mathcal{H}_\ell = \langle 1, h, h^2, \ldots \rangle$ spanned by Faa di Bruno iterates $h^{(k)} \equiv (\partial + h)^k(1)$. It can be shown that dynamics of coefficients $H^\ell_j$ in virtue of KP flows (1) is given by the invariance relation $(\partial_t + H^{(\ell)})(\mathcal{H}_\ell \subset \mathcal{H}_\ell$ which can be written explicitly as  
\[ \partial_t H^{(k)} = H^{(k+1)} - H^{(k)}H^{(\ell)} + \sum_{j=1}^{k} H^j_k H^{(\ell-j)} + \sum_{j=1}^{\ell} H^j_k H^{(k-j)}. \] 
Conversely, one can start from these equations, which constitute an infinite number of commuting flows including the first flow corresponding to evolution parameter $t_1 = x$ on the phase space whose points are parameterized by the semi-infinite matrix $(H_j^\ell)$. System (2) known as the Central System is equivalent in fact to the KP hierarchy and the latter can be obtained by choosing $t_1$ as the spatial variable [2].
2.2. The system describing infinite chain of KP hierarchies

Let us consider bi-infinite sequence of KP hierarchies \( \{ h(i, z) : i \in \mathbb{Z} \} \). To our aim, it is convenient to introduce another Laurent series \( a(i, z) = z + \sum_{j \geq 1} a_j(i)z^{-j+1} \equiv z\psi_{i+1}/\psi_i \), which as can be checked to satisfy the equation

\[
\partial_s a(i, z) = a(i, z)(H_s(i+1, z) - H_s(i, z)).
\]

The latter in turn can be rewritten in the form of differential–difference conservation laws:

\[
\partial_s \xi(i, z) = H_s(i+1, z) - H_s(i, z)
\]

with

\[
\xi(i, z) = \ln a(i, z) = \ln z - \sum_{j \geq 1} \left( - \sum_{k \geq 1} a_k(i)z^{-k} \right)^j = \ln z + \sum_{j \geq 1} \xi_j(i)z^{-j}.
\]

Thus, as a starting point we consider following an infinite set of evolution equations:

\[
\partial_s h_k(i) = \partial_s H_k(i), \quad \partial_s \xi_k(i) = H_s^k(i+1) - H_s^k(i),
\]

which as is shown below to admit a broad class of reductions yielding an infinite number of differential–difference equations. Let us remember that the KP hierarchy is equivalent to the Central System (2). Therefore, we can assume that the point of our phase space is defined by infinite number of functions \( \{ H_j^s(i), a_k(i) \} \).

3. The \( n \)th discrete KP hierarchy

3.1. Reductions of equations (4)

We are going to show in this and the next section how some integrable lattices can be obtained as a result of special reductions of equations (4). To this aim, we need the following two theorems.

**Theorem 1** ([6]). The submanifold \( S_{n-1}^n \) defined by the condition

\[
z^{l-n}a^n(i, z) \in H_s(i), \quad \forall i \in \mathbb{Z}
\]

is invariant with respect to flows given by (4).

**Theorem 2** ([7]). The following chain of invariant submanifolds inclusions:

\[
S_{l-1}^n \subset S_{2l-1}^n \subset S_{3l-1}^n \subset \cdots
\]

is valid.

Let us spend some time clarifying certain details. In theorem 1, by definition,

\[
a^{(r)}(i, z) = z^r \frac{\psi_{i+r}}{\psi_i} = \begin{cases} 
\prod_{j=1}^r a(i + j - 1, z), & r \geq 1, \\
1, & r = 0,
\prod_{j=1}^{-r} a^{-1}(i - j, z), & r \leq -1.
\end{cases}
\]

Thus, the coefficients of the Laurent series \( a^{(r)}(i, z) = z^r + \sum_{j \geq 1} a_j^{(r)}(i)z^{-j+r} \) are some quasi-homogeneous polynomials \( a_j^{(r)}[a_1, \ldots, a_j] \). In what follows, we use obvious identity

\[
a^{(r+r_2)}(i) = a^{(r)}(i)a^{(r_2)}(i + r_1) = a^{(r)}(i)a^{(r)}(i + r_2)
\]
Clearly, in terms of the wavefunction \( \Psi \), shifts with respect to this variable. For simplicity, we sometimes do not indicate dependence on discrete variable \( r \) for any \( r_1, r_2 \in \mathbb{Z} \), which yields an infinite set of identities

\[
a_k^{[r_1+r_2]}(i) = a_k^{[r_1]}(i) + \sum_{j=1}^{k-1} a_j^{[r_1]}(i) a_{k-j}^{[r_2]}(i + r_1) + a_k^{[r_2]}(i + r_1)
\]

\[
= a_k^{[r_2]}(i) + \sum_{j=1}^{k-1} a_j^{[r_2]}(i) a_{k-j}^{[r_1]}(i + r_2) + a_k^{[r_1]}(i + r_2).
\]

(7)

It is worth remarking that theorem 1 was proven in [4] in the case \( n = 1 \). It is useful to define another set of quasi-homogeneous polynomials \( \{q_j^{(n,r)}\} \) with the help of the generating relation

\[
z^r = a^{[r]} + \sum_{j \geq 1} q_j^{(n,r)} z^{[n-1]} a^{[r-j]}
\]

(8)

Clearly, in terms of the wavefunction \( \Psi = \{\psi_i\} \) this relation takes the form

\[
z^r \psi_i = z^r \psi_{i+r} + \sum_{j \geq 1} z^r \psi_i q_j^{(n,r)}(i) \psi_{i+r-j}.
\]

(9)

Relation (8) generates the triangular infinite system

\[
a_k^{[r]} + \sum_{j=1}^{k-1} a_k^{[r-j]} q_j^{(n,r)} + q_k^{(n,r)} = 0, \quad k \geq 1.
\]

(10)

One can check that a more general relation than (10), namely

\[
a_k^{[m]}(i) = a_k^{[r]}(i) + \sum_{j=1}^{k-1} a_k^{[r-j]}(i) q_j^{(n,r-m)}(i + m) + q_k^{(n,r-m)}(i + m)
\]

(11)

with any \( r, m \in \mathbb{Z} \), is valid [8]. Resolving the latter in favor of \( q_k^{(n,r-p)}(i + p) \) yields

\[
q_k^{(n,r-m)}(i + m) = a_k^{[m]}(i) + \sum_{j=1}^{k-1} q_j^{(n,r-(k-j)m)}(i) a_k^{[m]}(i) + q_k^{(n,r)}(i).
\]

It should be noted that polynomials \( q_j^{(n,r)} \) satisfy the following identities:

\[
q_k^{(n,r)}(i) + \sum_{j=1}^{k-1} q_j^{(n,r)}(i) q_{k-j}^{(n,r)}(i + r_1 - j n) + q_k^{(n,r)}(i + r_1)
\]

\[
= q_k^{(n,r)}(i) + \sum_{j=1}^{k-1} q_j^{(n,r)}(i) q_{k-j}^{(n,r)}(i + r_2 - j n) + q_k^{(n,r)}(i + r_2)
\]

(12)

for any \( r_1, r_2 \in \mathbb{Z} \). These identities can be obtained from (9) if we rewrite it in the operator form: \( z^r \Psi = Q_{(n,r)}(\Psi) \). Then we derive the set of identities (12) from the relation

\[
z^{r_1+r_2} \Psi = Q_{(n,r_1)}(Q_{(n,r_2)}(\Psi) = Q_{(n,r_2)}(Q_{(n,r_1)}(\Psi).
\]

Let us remark that condition (5) is equivalent to the relation

\[
z^{l-n} a^{[n]} = H^{(l)} + \sum_{k=1}^{l} a_k^{[n]} H^{(l-k)}.
\]

1 For simplicity, we sometimes do not indicate dependence on discrete variable \( i \in \mathbb{Z} \) in formulae which contain no shifts with respect to this variable.
3.2. The nth discrete KP hierarchy

Let us consider now the restriction of equations (4) on $S^n$ defined by simple relation
\[ z^{1-n} a^{[n]} = h + a^{[n]} \], from which we obtained \( h_k = a^{[n]}_k \) first of all. Therefore, we know that on $S^n$ one has \( H_k^1 = h_{k+1} = a^{[n]}_{k+1} \). To obtain explicit expressions for all $H_k^s$ as quasi-homogeneous polynomials of $a_k$ we need theorem 2. As a result we have [8]
\[ H_k^s = F^{(n,s)}_k [a_1, \ldots, a_{k+s}] \equiv a^{[n]}_k + \sum_{j=1}^{s-1} q^{(n,s)}_j a^{[s-j]n}_{k+j} \].

This totality of relations can be written down as a whole with the help of one generating relation
\[ H(s) = F^{(n,s)} = z^s - sn (a^{[sn]} + \sum_{j=1}^s z^{(n-1)s} q^{(n,s)}_j a^{[s-j]n}) \],
(13)
where $F^{(n,s)} \equiv z^s + \sum_{j \geq 1} F^{(n,s)}_j z^{-j}$. Thus, the restriction of dynamics given by (4) on $S^n$ leads to evolution equations in the form of differential–difference conservation laws
\[ \partial_s \xi_k(i) = F^{(n,s)}_k (i + 1) - F^{(n,s)}_k (i) \],
(14)
on infinite number of fields $\{a_k = a_k(i)\}$. The corresponding phase space we denote by $\mathcal{M}$. The hierarchy of the flows on $\mathcal{M}$ given by evolution equations (14) we call nth discrete KP hierarchy. These equations also admit a rich family of reductions.

The following remark is in order. A reformulation of the generating relation (13) in terms of the wavefunction $\Psi = \{\psi_i\}$ is
\[ \partial_s \psi_i = z^s \psi_{i+sn} + \sum_{j=1}^s z^{s-j} q^{(n,s)}_j (i) \psi_{i+(s-j)n} \].
(15)

Checking the compatibility of (15) with (9) we obtain
\[ \partial_s q^{(n,r)}_k (i) = \sum_{j=1}^s \sum_{r=1}^s \frac{1}{r} q^{(n,s)}_j (i) \psi_{i+(s-j)n} - q^{(n,r)}_k (i) - \sum_{j=1}^s q^{(n,s)}_j (i + r - (k + s - j)n) q^{(n,r)}_k (i). \]
(16)

3.3. ...and its reductions

Obviously, that restriction of dynamics given by (4) on intersection $S^n \cap S^{n-1}$ is equivalent to restriction of nth discrete KP hierarchy on some submanifold $\mathcal{M}_{n,p,l} \subset \mathcal{M}$. We can easily write down the equations defining $\mathcal{M}_{n,p,l}$. They evidently follow from the generating relation
\[ z^{1-n} a^{[p]} = F^{(n,l)} + \sum_{j=1}^l a^{[p]}_j F^{(n,l-j)} \].

From here, we obtain that $\mathcal{M}_{n,p,l}$ is defined by infinite number of equations
\[ J_k^{(n,p,l)} [a_1, \ldots, a_{k+l}] = 0, \quad \forall k \geq 1 \]
(17)
with

\[ J^{(n,p,l)}_k = a_k^{[p]} - F_k^{(n,l)} - \sum_{j=1}^{l-1} a_j^{[p]} F^{(n,l-j)}_k \]

\[ = a_k^{[p]}(i) - a_k^{[ln]}(i) - \sum_{j=1}^{l-1} q_j^{(n,ln-p)}(i + p) a_k^{[l-j][n]}(i). \]

As a consequence of theorem 3, we have the following.

**Theorem 3.** The following chain of invariant submanifolds inclusions:

\[ M_{n,p,l} \subset M_{n,2p,2l} \subset M_{n,3p,3l} \subset \cdots \] (18)

is valid.

3.4. Linear equations on KP wavefunction corresponding to submanifold \( M_{n,p,l} \)

Some remarks about linear equations on KP formal wavefunction \( \{ \psi_i \} \) which follow as a result of restriction on \( M_{n,p,l} \) are in order. Let \( J^{(n,p,l)}(z) \equiv \sum_{j=1}^{l} J^{(n,p,l)}_j z^{-j} \). We observe that an infinite number of equations (17) can be presented by a single generating relation

\[ J^{(n,p,l)}(i, z) = z^l \psi_i + \sum_{j=1}^{l} z^{l-j} q_j^{(n,ln-p)}(i + p) \psi_{i+j}[n] = 0. \]

Clearly, in terms of the KP wavefunction, we can rewrite the latter relation as

\[ J^{(n,p,l)}(i, z) = z^l \psi_{i+p} - \sum_{j=1}^{l} z^{l-j} q_j^{(n,ln-p)}(i + p) \psi_{i+j}[n] = 0. \]

Thus, in terms of KP wavefunction, the restriction of \( n \)th discrete KP hierarchy on \( M_{n,p,l} \) is given by the linear equation

\[ z^l \psi_{i+p} + \sum_{j=1}^{l} z^{l-j} q_j^{(n,ln-p)}(i + p) \psi_{i+j}[n] = z^l \psi_{i+p}. \] (19)

The second linear equation which we should have in mind is (15).

When considering restriction of \( n \)th discrete KP hierarchy on \( M_{n,p,l} \), one can find many examples of integrable lattices known from the literature (some examples can be found in the paper [8]) and construct ‘new’ ones. Classical examples are Volterra and Toda lattices. In what follows, we restrict ourselves by considering only a class of one-field lattices corresponding to \( M_{n,p,1} \).

4. The polynomials \( S_d^x \) and \( T_d^x \)

In the next section, we consider restriction of \( n \)th discrete KP hierarchy on the submanifold \( M_{n,p,1} \), with \( n \geq 1 \) and \( p > n \). For further use, let us prepare in this section two classes of polynomials through

\[ S_d^x(y_1|y_2,\ldots,y_{d-1}) = \prod_{j=1}^{d} (y_{n+j} - y_{n+j+1}) \] (20)

We use perhaps unusual but quite convenient notation writing ‘first’ and ‘last’ argument through the vertical bar.
and
\[ T_s^j \left( y_1 \right) |_{yp-(s-l+1)n+1} = \sum_{1 \leq \lambda_1 \leq \cdots \leq \lambda_s \leq s+1} \left( \prod_{j=1}^l y_{(p-1)n+1}^{\lambda_j} j_{n+1} \right) \] (21)
for \( s \geq 0 \).\(^3\) Let us show some identities for these polynomials. Firstly consider \( S_s^l \). A partition of the set
\[ B_{l,s} = \{ \lambda_j : 1 \leq \lambda_1 \leq \cdots \leq \lambda_s \leq s + 1 \} \]
into a pair of disjoint subsets \( B_{l,s} = B_l^1 \cup B_{l,s-1} \) with
\[ B_l^1 = \{ \lambda_j : 1 \leq \lambda_1 = s + 1, 1 \leq \lambda_1 \leq \cdots \leq \lambda_2 \leq s + 1 \} \]
and
\[ B_{l,s-1} = \{ \lambda_j : 1 \leq \lambda_1 \leq \cdots \leq \lambda_s \leq s \}, \]
as can be checked, leads to the identity
\[ S_s^l (y_1) |_{yp-(s-l+1)n+1} = S_s^l_{l-1} (y_1) |_{yp-(s-l)n+1} + y_{(p-1)n+1} S_s^{l-1} (y_{(n+1)}) |_{yp-(s-l+2)n+1}. \] (22)
Let
\[ B_k^l = \{ \lambda_j : 1 \leq \lambda_1 = s - k + 2, 1 \leq \lambda_1 \leq \cdots \leq \lambda_2 \leq s - k + 2 \} \]
and
\[ B_{l,s-k+1} = \{ \lambda_j : 1 \leq \lambda_1 \leq \cdots \leq \lambda_s \leq s - k + 2 \} \]
for \( k = 1, \ldots, s + 1 \). Clearly, \( B_{l,s-k+1} = B_k^l \cup B_{l,s-k} \) and \( B_{l,0} = B_{s+1}^l \). Thus, we have the following:
\[ B_{l,s} = \bigcup_{j=1}^{s+1} B_j^l. \]
This decomposition of \( B_{l,s} \) in turn yields the identity
\[ S_s^l (y_1) |_{yp-(s-l+1)n+1} = \sum_{j=1}^{s+1} y_{(p-n+1)} \cdot S_s^{l-1} \left( y_{(n+1)} \right) |_{yp-(s-l-j+2)n+1}. \] (23)
On the other hand, a partition of \( B_{l,s} \) into
\[ B_1^l = \{ \lambda_j : \lambda_1 = 1, 1 \leq \lambda_1 \leq \cdots \leq \lambda_s \leq s + 1 \} \]
and
\[ B_{l,s-1} = \{ \lambda_j : 2 \leq \lambda_1 \leq \cdots \leq \lambda_s \leq s + 1 \} \]
gives the following identity:
\[ S_s^l (y_1) |_{yp-(s-l+1)n+1} = S_s^l_{s-1} (y_{p-n+1}) |_{yp-(s-l)n+1} + y_{(p-1)n+1} S_s^{l-1} (y_1) |_{yp-(s-l+2)n+1}. \] (24)
Making use of the partition \( B_{l,s-k+1} = B_k^l \cup B_{l,s-k} \) and \( B_{l,0} = B_{s+1}^l \) with
\[ B_k^l = \{ \lambda_j : \lambda_1 = k, k \leq \lambda_1 \leq \cdots \leq \lambda_s \leq s + 1 \} \]
and
\[ B_{l,s-k+1} = \{ \lambda_j : k \leq \lambda_1 \leq \cdots \leq \lambda_s \leq s + 1 \} \]
\(^3\) It is convenient to think that \( T_s^l = 0 \) for \( s = 0, \ldots, l - 2 \).
we are led to the partition
\[ B_{l,s} = \bigcup_{j=1}^{s+1} B_j^{l,s} \]
and corresponding identity
\[ S_j^l(y_1 | y_{p-(l+1)n+1}) = \sum_{j=1}^{s+1} y_{(j-1)p-(j-l)n+1} S_{j-1}^{l-1} (y_{(j-1)(p-n)+1} | y_{p-(l+1)n+1}). \]  
(25)

It is worth remarking that identities (23) and (25) being in nature recurrence relations both uniquely define polynomials \( S_j^l \) starting from \( S_0^0 \equiv 1 \).

Consider now polynomials \( T_j^l \). A partition of the set \( D_{l,s} \) into two subsets
\[ D_l^1 = \{ \lambda_j : \lambda_1 = 1, 2 \leq \lambda_2 < \cdots < \lambda_l \leq s \} \]
and
\[ D_{l,s-1} = \{ \lambda_j : 2 \leq \lambda_1 < \cdots < \lambda_l \leq s \} \]
leads to the identity
\[ T_l^j (y_1 | y_{p-(l+1)n+1}) = T_{l-1}^j (y_{p-n+1} | y_{p-(l+1)n+1}) + y_1 T_{l-1}^{j-1} (y_{p+1} | y_{p-(l+1)n+1}). \]  
(26)

Let
\[ D_l^j = \{ \lambda_j : \lambda_1 = k, k + 1 \leq \lambda_2 < \cdots < \lambda_l \leq s \} \]
and
\[ D_{l,s-k+1} = \{ \lambda_j : k \leq \lambda_1 < \cdots < \lambda_l \leq s \} \]
for \( k = 1, \ldots, s-l+2 \). Taking into account that \( D_{l,s-k+1} = D_k^l \cup D_{l,s-k} \) and \( D_{l,s-1} = D_{s-l+2}^l \) we are led to the partition
\[ D_{l,s} = \bigcup_{j=1}^{s-l+2} D_j^l \]
and corresponding identity
\[ T_j^l (y_1 | y_{p-(l+1)n+1}) = \sum_{j=1}^{s-l+2} y_{(j-1)p-(j-l)n+1} T_{j-1}^{l-1} (y_{(j-1)(p-n)+1} | y_{p-(l+1)n+1}). \]  
(27)

Finally, consider the partition \( D_{l,s} = D_l^1 \cup D_{l,s-1} \) with
\[ D_l^1 = \{ \lambda_j : \lambda_l = s+1, 1 \leq \lambda_1 < \cdots < \lambda_{l-1} \leq s \} \]
and
\[ D_{l,s-1} = \{ \lambda_j : 1 \leq \lambda_1 < \cdots < \lambda_l \leq s \} \]
and corresponding identity
\[ T_j^l (y_1 | y_{p-(l+1)n+1}) = T_{l-1}^j (y_1 | y_{p-l+1}) + y_{p-(l+1)n+1} T_{l-1}^{j-1} (y_1 | y_{p-l+1}). \]  
(28)

Let
\[ D_k^l = \{ \lambda_j : \lambda_l = s-k+2, 1 \leq \lambda_1 < \cdots < \lambda_{l-1} \leq s-k+1 \} \]
and
\[ D_{l,s-k+1} = \{ \lambda_j : 1 \leq \lambda_1 < \cdots < \lambda_j \leq s - k + 1 \}. \]

We have \( D_{l,s-k+1} = D_k^l \cup D_{l,s-k} \) and \( D_{l,l-1} = D_{s-l+2}^l \). The partition
\[ D_{l,s} = \bigcup_{j=1}^{s-l+2} \tilde{D}_j^l, \]
leads to the identity
\[ T_{l,s}^l (y_1 | y_{p-s-l+2} n+1) = \bar{S}_{l,s}^l (i) + r_i \bar{T}_{l-1}^l (i). \]

Let us identify \( y_k = r_i k - 1 \) for \( k = 1, \ldots, sp - (s - l + 1)n + 1 \), where \( r = r(i) \equiv r_i \) is some unknown function of discrete variable \( i \in \mathbb{Z} \). We define discrete polynomial functions \( S_i^l[r] \) and \( T_i^l[r] \) by
\[ S_i^l(i) \equiv S_{i-1}^l(i) + r_i S_{i+1}^{l-1}(i + n) \]
and
\[ T_i^l(i) \equiv T_{i-1}^l(i + p - n) + r_i T_{i+1}^{l-1}(i). \]

Since \( T_i^l \equiv 0 \) for \( s = 0, \ldots, l - 2 \), then we also can write
\[ T_i^l(i) = \sum_{j=1}^s r_i (j + 1) T_{i+1}^{l-1}(i + j - 1) n. \]

(29)
5. The manifold $\mathcal{M}_{n,p,1}$

5.1. Technical lemmas

Our aim is to show how polynomials $S_{l}^s[r]$ and $T_{l}^s[r]$ constructed in the previous section appear when constructing integrable hierarchies for some differential–difference equations in its explicit form.

Let us consider the submanifold $\mathcal{M}_{n,p,1} \subset \mathcal{M}$ with $n \geq 1$ and $p > n$ defined by equations $J_{k}^{(n,p,1)} = a_{k+1}^{[p]} - a_{k+1}^{[n]} \equiv 0$ for $k \geq 1$ which, using (7) can be rewritten equivalently as

$$\sum_{j=1}^{k} a_{k-j+1}^{[n]}(i)a_{j}^{[p-n]}(i+n) + a_{k+1}^{[p-n]}(i+n) = 0.$$ 

Making use again of identity (7) we see that solution of this equation is given by

$$a_{k}^{[p-n]}(i) = a_{k-1}^{[n]}(i) r_i \quad \text{with} \quad r_i \equiv a_i^{[p-n]}(i)$$

for all $k \geq 2$.

Lemma 1. On $\mathcal{M}_{n,p,1}$

$$a_{k}^{(p-n)s}(i) = \sum_{j=1}^{s} a_{k-j}^{[-j+n+(j-1)p]}(i) r_i^{(j-1)(p-n)},$$

and

$$a_{k}^{(n-p)s}(i) = -\sum_{j=1}^{s} a_{k-j}^{[-j+p+(j-1)n]}(i) r_i^{j(n-p)}$$

for all integers $s \geq 1$.

Proof. By virtue of (7) and (31), we have

$$a_{k}^{(p-n)s}(i) = a_{k}^{(p-n)(s-1)}(i) + \sum_{j=1}^{k-1} a_{k-j}^{(p-n)(s-1)}(i) a_{j}^{(p-n)}(i+n) - a_{k}^{(p-n)(s-1)}(i)$$

$$+ \left\{ a_{k}^{(p-n)(s-1)}(i) + \sum_{j=1}^{k-2} a_{k-j}^{(p-n)(s-1)}(i) a_{j}^{(p-n)}(i+n) \right\} r_i^{p-(p-n)(s-1)}$$

$$+ a_{k-1}^{[-n]}(i+n)(s-1))$$

Making use of this recurrence relation we immediately obtain (32) and (33). \qed

Lemma 2. On $\mathcal{M}_{n,p,1}$

$$q_{k}^{*(p-n)s}(i) = (-1)^{k} S_{k}^{*(k-1)n},$$

$$q_{k}^{*(n-p)s}(i) = T_{k}^{*(k-1)n+(n-p)s}$$

for $s \geq 1$.  

10
Let us remark that this lemma, in particular, says that on $\mathcal{M}_{n,p}$ we have $q^{(n-lp)}_k \equiv 0$ for $s = 1, \ldots, k - 1$.

**Proof of lemma 2.** In the particular case $r = 1$, identity (11) is specified as

\[
a_k^{[n]}(i) = \sum_{j=1}^{k-1} a_{k-j}^{[-jn]}(i)q_j^{(n-m)}(i + m) + q_k^{(n-m)}(i + m).
\]

Let $m = (p - n)s$. We observe that by virtue of the latter identity, (34) is equivalent to the relation

\[
a_k^{(p-n)s}(i) = \sum_{j=1}^{k-1} a_{k-j}^{[-jn]}(i)T_{s-1}^{k-j}(i - (j - 1)n) + T_{s-1}^{k}(i - (k - 1)n).
\]

(36)

So, let us prove (36) instead of (35). For $k = 1$, the latter becomes $a_1^{(p-n)s}(i) = T_{s-1}^{1}(i)$, which is evident, because

\[
a_1^{(p-n)s}(i) = \sum_{j=1}^{s} a_{j}^{[-jn]}(i)T_{s-1}^{k-j}(i - (j - 1)n) = T_{s-1}^{1}(i)
\]

simply by definition and without reference to any invariant submanifold. Further, we proceed by induction. Suppose we have proved (36) for $k = 1, \ldots, k_0$, where $k_0 \geq 1$. Then it is easy to see, making use of (7), that for these values of $k$ the relations of the form

\[
a_k^{[m+(p-n)s]}(i) - a_k^{[n]}(i) = T_{s-1}^{k}(i + m - (k - 1)n) + \sum_{j=1}^{k-1} a_j^{[m-(k-j)n]}(i)T_{s-1}^{k-j}(i - (j - 1)n)
\]

hold for any $m \in \mathbb{Z}$. In particular, let $m = -n$; then

\[
a_k^{[-n+(p-n)s]}(i) - a_k^{[-n]}(i) = T_{s-1}^{k}(i - kn) + \sum_{j=1}^{k-1} a_j^{[-(k-j+1)n]}(i)T_{s-1}^{k-j}(i - (k - j)n).
\]

(38)

With the help of (30), (32), (37) and (38) we have the following:

\[
a_{k+1}^{(p-n)s}(i) - a_k^{[-n]}(i)T_{s-1}^{1}(i) = \sum_{j=1}^{s} \{a_k^{[-(j+1)n+p]}(i) - a_k^{[-n]}(i)\}r_{j+p}(p-n)
\]

\[
= \sum_{j=1}^{s-1} \left\{T_{j-1}^{k}(i - kn) + \sum_{j=1}^{k-1} a_j^{[-(k-j+1)n]}(i)T_{j-1}^{k-j}(i - (j - 1)n)\right\}r_{j+p+n}^{(p-n)}
\]

\[
= \sum_{j=1}^{s-1} T_{j-1}^{k}(i - kn)r_{j+p}(p-n) + \sum_{j=1}^{k-1} a_j^{[-(k-j+1)n]}(i)\left(\sum_{j=1}^{s-1} T_{j-1}^{k-j}(i - (j - 1)n)r_{j+p}(p-n)\right)
\]

\[
= T_{s-1}^{k}(i - kn) + \sum_{j=1}^{k-1} a_j^{[-(k-j+1)n]}(i)T_{s-1}^{k-j}(i - (k - j)n).
\]

\[\text{Here } a_1 = a_1(i).\]
Thus, we have proved that if (36) is valid for \( k = 1, \ldots, k_0 \) then it is valid for \( k = k_0 + 1 \), and therefore we can now use induction with respect to \( k \). Relation (34) is proved using similar reasoning.

5.2. Additional identities for \( S_l^i[r] \) and \( T_l^i[r] \)

Making use of lemma 2 and identities (12) with \( r_1 = (p - n)(s_1 + 1) \) and \( r_2 = (p - n)(s_2 + 1) \), we are able to obtain following identities:

\[
S_l^i(i) + \sum_{j=1}^{l-1} (-1)^j S_{s_1}^{l-j}(i)T_j^i(i + (l-j)n) + (-1)^j T_j^i(i) = S_l^i(i + (s_2 + 1)(p - n)) + \sum_{j=1}^{l-1} (-1)^j S_{s_1}^{l-j}(i + s_2 + 1)(p - n)) + (-1)^j T_j^i(i + (s_1 + 1)(p - n)).
\]

In particular, let \( s_1 = s_2 = s \). Then we have

\[
S_l^i(i) + \sum_{j=1}^{l-1} (-1)^j S_{s_1}^{l-j}(i)T_j^i(i + (l-j)n) + (-1)^j T_j^i(i) = 0
\]

and

\[
S_l^i(i) + \sum_{j=1}^{l-1} (-1)^j S_{s_1}^{l-j}(i + jn)T_j^i(i) + (-1)^j T_j^i(i) = 0.
\]

5.3. Linear equations on KP wavefunction and its compatibility

Let us discuss linear equations on KP wavefunction \( \Psi = \{\psi_i\} \). On \( \mathcal{M}_{n,p,1} \) we have the linear equation

\[
z \psi_{i+n} + T_0^i(i+n)\psi_i = z \psi_{i+p}
\]

with \( T_0^i(i) = r_i \) which is a specification of (19). On the other hand, from theorem 3 we have

\[
\mathcal{M}_{n,p,1} \subset \mathcal{M}_{n,2p,2} \subset \mathcal{M}_{n,3p,3} \subset \cdots
\]

and the corresponding infinite set of linear equations

\[
z^k \psi_{i+kn} + \sum_{j=1}^{k} T_{k-1}^j(i + (k - j + 1)n)z^{k-j} \psi_{i+(k-j)n} = z^k \psi_{i+kp}
\]

for \( k \geq 2 \), which can be obtained also as a consequence of linear equation (40). Remark that we obtain coefficients of (41) making use of lemma (2). Let us remember that \( T_l^i[r]'s \) in (41) are discrete polynomials defined for some fixed \( n \geq 1 \) and \( p > n \) via (21). Let us consider linear evolution equation

\[
\psi'_i = z \psi_{i+n} - s_i \psi_i \quad \text{with} \quad s_i \equiv a_i^{(n)}(i)
\]

for the first flow of \( n \)th discrete KP hierarchy. By straightforward calculations we check that the compatibility of the latter equation with (41) is guaranteed when the relation

\[
\partial T_k^i(i) = T_k^{i+1}(i) - T_k^{i+1}(i - n) + T_k^i(i)(s_i - s_{i+(k+1)p -(k+2)n})
\]

holds. In particular,

\[
\partial T_k^i(i) = \partial r_i = r_i(s_i - s_{i+p-n})
\]
or
\[ \sum_{j=1}^{p-n} a'_{i+j-1} = \sum_{j=1}^{p-n} d_{i+j-1} \left( \sum_{j=1}^{n} a_{i-j} - \sum_{j=p-n}^{p-1} a_{i+j} \right). \]

### 5.4. Integrable hierarchies of differential–difference equations associated with \( \mathcal{M}_{n,p,1} \)

In this subsection we are going to show a class of integrable hierarchies corresponding to \( \mathcal{M}_{n,p,1} \). Making use of lemma 2 we obtain
\[ \partial^*_s \psi_i \equiv \partial \left( p-n \right)^s \psi_i = z \left( p-n \right)^s \psi_i + \left( p-n \right)^s \left( p-n \right)^{s-1} \sum_{j=1}^{n} \left( -1 \right)^j \left( p-n \right)^{s-1} S_{i-n-1}^j (i - (j-1)n) \psi_{i+(p-n)sn-jn}. \] (42)

The condition of compatibility of (42) with (40) yields
\[ \partial^*_s T^0_k (i) = \partial^*_s r_i = \left( -1 \right)^{(p-n)s} \left( p-n \right)^s \psi_{i+(p-n)sn} + \sum_{j=1}^{p-n} \left( -1 \right)^j \left( p-n \right)^{s-1} S_{i-n-1}^j (i - (j-1)n) \psi_{i+(p-n)sn-jn}, \] (43)
while when checking compatibility of (42) with (41) we obtain
\[ \partial^*_s T^k_l (i) = T^k_l \left( p-n \right)^s (i) + \sum_{j=1}^{p-n} \left( -1 \right)^j S_{i-n-1}^j (i + (p-n)(k+1) + (l-j)n) T^k_l \left( p-n \right)^{s-j} (i) \]
\[ - T^k_l \left( p-n \right)^s (i - (p-n)sn) - \sum_{j=1}^{p-n} \left( -1 \right)^j S_{i-n-1}^j (i - (p-n)sn) \]
\[ \times T^k_l \left( p-n \right)^{s-j} (i - (p-n)sn + jn). \] (44)

We remark that all other relations obtained as a result of checking compatibility of (42) with (41) are just identities of the form (39). We remark also that equations (44) could be obtained in an easier way by using (16) with lemma 2. Following along this line, in addition to (44), we obtain the following:
\[ \partial^*_s S^k_l (i) = \left( -1 \right)^{(p-n)s} \left\{ S^k_l \left( p-n \right)^s (i) + \sum_{j=1}^{p-n} S_{i-n-1}^j (i + (k-j)n) \right\} \]
\[ - S^k_l \left( p-n \right)^s (i - (p-n)sn) \]
\[ - \sum_{j=1}^{p-n} S_{i-n-1}^j (i + (p-n)(k-sn+1)) S^k_l \left( p-n \right)^{s-j} (i - (p-n)sn + jn) \}, \] (45)

### 5.5. The case \( \mathcal{M}_{n,n+1,1} \)

Let us consider the submanifold \( \mathcal{M}_{n,n+1,1} \). Since in this case \( p-n = 1 \), then \( r_i = a_i \). Linear equation (40) in this case becomes
\[ z \psi_{i+sn} + a_{i+sn} \psi_i = z \psi_{i+sn+1}, \] (46)
while evolution equation (42) takes the form
\[ \partial^*_s \psi_i = z^s \psi_{i+sn} + \sum_{j=1}^{s} z^{s-j} \left( -1 \right)^j S^j_{i-n-1} (i - (j-1)n) \psi_{i+(s-j)n} \]
and (43) is specified as
\[ \partial_s a_i = (-1)^i a_i \left\{ S_{s-1}^i (i - (s - 1)n + 1) - S_{s-1}^i (i - sn) \right\}. \] (47)

The first flow in this hierarchy is given by differential–difference equation
\[ a'_i = -a_i \left( S_{s-1}^i (i + 1) - S_{s-1}^i (i - n) \right) \]
\[ = a_i \left( \sum_{j=1}^{n} a_{i-j} - \sum_{j=1}^{n} a_{i+j} \right). \] (48)

This equation with quadratic nonlinearity is nothing but the INB lattice mentioned in the introduction. It is known from [1] that the INB lattice, being in a sense integrable generalization of the Volterra lattice \[ a'_i = a_i (a_i - 1 - a_i + 1) \], for any \( n \geq 1 \), gives integrable discretization of the Korteweg–de Vries (KdV) equation.

Therefore, as a particular case we constructed integrable hierarchy for the INB equation (48) in its explicit form [9]. Remark that polynomials \( S_l^i \) and \( T_l^i \) in this case are specified as
\[ S_l^i (y_1 | y_2 \ldots | y_s + (l - 1)n + 1) = \sum_{1 \leq \lambda_1 \leq \ldots \leq \lambda_l \leq s + 1} \left( \prod_{j=1}^{l} y_{\lambda_j + n(j-1)} \right) \]
and
\[ T_l^i (y_1 | y_2 \ldots | y_s + (l - 1)n + 1) = \sum_{1 \leq \lambda_1 < \ldots < \lambda_l \leq s + 1} \left( \prod_{j=1}^{l} y_{\lambda_j + n(j-1)} \right). \]

These polynomials were introduced in [9].

As a consequence of linear equation (46) corresponding to the chain of inclusions \[ M_{n,0} \subset M_{n,2} \subset M_{n,3} \subset \cdots \], we have an infinite number of linear equations
\[ z^k \psi_{i+k} + \sum_{j=1}^{k} T_{k+1}^j (i + (k - j)n) z^{k-j} \psi_{i+(k-j)n} = z^k \psi_{i+k+n \in \mathbb{N}} \]
for \( k \geq 2 \). Finally, (44) and (45) in this case become
\[ \partial_s T_k^i (i) = T_k^{is} (i) + \sum_{j=1}^{s} (-1)^j S_{s-1}^j (i + (l - j)n + k + 1) T_k^{is-j} (i) \]
\[ - T_k^{is} (i - sn) - \sum_{j=1}^{s} (-1)^j S_{s-1}^j (i - sn) T_k^{is-j} (i - (s + j)n) \]
and
\[ \partial_s S_k^i (i) = (-1)^i \left\{ S_k^{is} (i) + \sum_{j=1}^{s} S_{s-1}^j (i + (k - j)n) S_k^{is-j} (i) \right\} \]
\[ - S_k^{is} (i - sn) - \sum_{j=1}^{s} S_{s-1}^j (i + k - sn + 1) S_k^{is-j} (i - (s + j)n) \right\}, \]
respectively. In the paper [9], we have proved the following.
Theorem 4. Each of the constraints
\[ T^{s+1}_i (i + 1) = T^{s+1}_i (i), \quad s \geq l \]
and
\[ S^{s+1}_i (i + 1) = S^{s+1}_i (i), \quad s \geq 0 \]
is consistent with the INB lattice hierarchy (47).

This theorem gives an infinite number of constraints for the INB lattice hierarchy including periodicity and stationary conditions.

5.6. The case \( M_{1,g+1,1} \)

As a second example, let us consider the submanifold \( M_{1,g+1,1} \). In this case \( p - n = g \) and consequently \( r_i = a_i^{(g)}(i) \). Linear equation (40) and its consequences (41) are specified as
\[ z\psi_{i+1} + r_{i+1}\psi_i = z\psi_{1+g+1} \]
and
\[ z^k\psi_{i+k} + \sum_{j=1}^{k} T_{k-1}^j(i + k - j + 1)z^{k-j}\psi_{i+k-j} = z^k\psi_{1+g+k}, \]
respectively. Linear evolution equation (42) becomes
\[ \partial^*_{s}\psi_i \equiv \partial_{gs}\psi_i = z^{gs}\psi_{1+gs} + \sum_{j=1}^{gs}(-1)^jz^{gs-j}S_{s-1}^j(i - j + 1)\psi_{1+gs-j}. \]
The corresponding hierarchy of differential–difference equations is given by
\[ \partial^*_{s}r_i = (-1)^{gs}r_i \left\{ S_{s-1}^g(i + gs + g + 1) - S_{s-1}^g(i - gs) \right\}. \]
In particular, the first flow of this hierarchy is managed by
\[ \partial^*_{s}r_i = (-1)^{gs}r_i \left( \prod_{j=1}^{g} r_{i+j} - \prod_{j=1}^{g} r_{i-j} \right). \]
This equation is known from [1]. It, as well as the INB equation, represents, for any \( g \geq 1 \), integrable discretization of the KdV equation. It is a simple exercise to check that in virtue of the latter (cf [1])
\[ \partial^*_{s}S_0^g(i) = (-1)^{gs}S_0^g(i) \left( \sum_{j=1}^{g} S_0^g(i + j) - \sum_{j=1}^{g} S_0^g(i - j) \right), \]
i.e., \( S_0^g \) satisfies the INB equation. Finally, let us write below (44) and (45) in this case. We have
\[ \partial^*_{s}T_k^l(i) = T_k^{gs+g}(i) + \sum_{j=1}^{gs}(-1)^jS_{s-1}^j(i + g(k + 1) + l - j)T_k^{gs+j}(i) - T_k^{gs+j}(i + gs) \]
\[ - \sum_{j=1}^{gs}(-1)^jS_{s-1}^j(i + gs)T_k^{gs+j}(i + gs + j). \]
and

\[ \partial^x_y S_k(i) = (-1)^{gs} \left\{ S_k^{i+gs}(i) + \sum_{j=1}^{gs} S_{i-k-1}^j(i+j)S_k^{i+gs-j}(i) \right. \]

\[ \left. - S_k^{i+gs}(i-gs) - \sum_{j=1}^{gs} S_{i-k-1}^j(i+g(k-s+1))S_k^{i+gs-j}(i-gs+j) \right\}. \]

6. Conclusion

Our main goal in the paper was to construct explicit form for integrable hierarchies for some class of differential–difference equations. Perhaps the most important case in this class of equations is given by Volterra lattice. What one learned from this presentation is that evolution equations of integrable hierarchies from this class are essentially formulated with the help of special homogeneous polynomials which we present by explicit formulas (20) and (21). In forthcoming papers, we will show how this information can be exploited for investigation of some problems concerning equations from this class. Remark that in this paper we consider only one-component lattices. It might be interesting to extend these results to multi-component ones.

Acknowledgments

This work has been supported by Russian Foundation for Basic Research grant no 09-01-00192-a.

References

[1] Bogoyavlenskii O I 1988 Some constructions of integrable dynamical systems Math. USSR Izv. 31 47–76
[2] Casati P, Falqui G, Magri F and Pedroni M 1996 The KP theory revisited I, II, III, IV SISSA preprints ref. SISSA/2–5/96/FM
[3] Itoh Y 1975 An H-theorem for a system of competing species Proc. Japan Acad. 51 374–9
[4] Magri F, Pedroni M and Zubelli J P 1997 On the geometry of Darboux transformations for the KP hierarchy and its connection with the discrete KP hierarchy Commun. Math. Phys. 188 305–25
[5] Narita K 1982 Soliton solutions to extended Volterra equation J. Phys. Soc. Japan 51 1682–5
[6] Svinin A K 2002 Extended discrete KP hierarchy and its reductions from a geometric viewpoint Lett. Math. Phys. 61 231–9
[7] Svinin A K 2004 Invariant submanifolds of the Darboux–Kadomtsev–Petviashvili chain and an extension of the discrete Kadomtsev–Petviashvili hierarchy Theor. Math. Phys. 144 1542–61
[8] Svinin A K 2008 Reductions of integrable lattices J. Phys. A: Math. Theor. 41 315205
[9] Svinin A K 2009 On some class of reductions for the Itoh–Narita–Bogoyavlenskii lattice J. Phys. A: Math. Theor. 42 454021
[10] Wilson G 1981 On two constructions of conservation laws for Lax equations Q. J. Math. Oxford 32 491–512