STRING DUALITY AND ENUMERATION OF CURVES BY JACOBI FORMS

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For a Calabi-Yau threefold admitting both a $K3$ fibration and an elliptic fibration (with some extra conditions) we discuss candidate asymptotic expressions of the genus 0 and 1 Gromov-Witten potentials in the limit (possibly corresponding to the perturbative regime of a heterotic string) where the area of the base of the $K3$ fibration is very large. The expressions are constructed by lifting procedures using nearly holomorphic Weyl-invariant Jacobi forms. The method we use is similar to the one introduced by Borcherds for the constructions of automorphic forms on type IV domains as infinite products and employs in an essential way the elliptic polylogarithms of Beilinson and Levin. In particular, if we take a further limit where the base of the elliptic fibration decompactifies, the Gromov-Witten potentials are expressed simply by these elliptic polylogarithms. The theta correspondence considered by Harvey and Moore which they used to extract the expression for the perturbative prepotential is closely related to the Eisenstein-Kronecker double series and hence the real versions of elliptic polylogarithms introduced by Zagier.

1 Introduction

Topological sigma models with target Calabi-Yau threefolds have been extensively investigated in recent years. The study of the so-called A-model [Wit] starts with the following setup.

Let $X$ be a Calabi-Yau threefold. The non-trivial Hodge numbers $h^{11}$ and $h^{12}$ of $X$ are related to the Euler characteristic of $X$ by $\chi = 2(h^{11} - h^{12})$. We assume that $H_2(X, \mathbb{Z})$ is torsion-free. We identify $H_2(X, \mathbb{Z})$ with $H_4(X, \mathbb{Z})$ and interchangeably use both viewpoints. Let $D_1, \ldots, D_l$ be the divisors generating $H_2(X, \mathbb{Z}) \cong H_4(X, \mathbb{Z})$ where $l := h^{11}$. Introduce the topological invariants

$$\kappa_{ijk} := D_i \cdot D_j \cdot D_k,$$

$$\rho_i := c_2(X) \cdot D_i,$$ 

(1.1)

where $c_2(X)$ is the second Chern class of $X$. In what follows we assume that the divisors $D_1, \ldots, D_l$ are nef, i.e. $D_i \cdot C \geq 0$ ($i = 1, \ldots, l$) for any irreducible algebraic curve $C$ in $X$. Hence $\rho_i \geq 0$ ($i = 1, \ldots, l$) by [Mi].

For a fixed complex structure on $X$, the A-model moduli parameters are conveniently expressed by the complexified Kähler form $J_C$ belonging to some open subset of the complexified Kähler cone $\{J_C = B + \sqrt{-1}T \mid B \in \}$
$H^2(X, \mathbb{R})/H^2(X, \mathbb{Z})$, $J \in \mathcal{K}$} where $\mathcal{K}$ is the Kähler cone of $X$ and $B$ is associated with the ‘$B$-field’ in the sigma model language. Near the large-radius limit $J_C$ can be parametrized as

$$J_C = \sum_{i=1}^{l} \left( \frac{1}{2\pi \sqrt{-1}} \log q_i \right) D_i, \quad |q_i| \ll 1, \quad (1.2)$$

where $\frac{1}{2\pi \sqrt{-1}} \log q_i (i = 1, \ldots, l)$ are the so-called flat coordinates. They are multi-valued and can be shifted by arbitrary integers.

The polylogarithm functions are defined by

$$\text{Li}_r(\xi) = \sum_{n=1}^{\infty} \frac{\xi^n}{n^r}, \quad |\xi| < 1, \quad r \in \mathbb{N}. \quad (1.3)$$

The prepotential or the genus 0 Gromov-Witten potential of $X$ is known to have the following world-sheet instanton expansion:

$$F_0(q_*) = \frac{1}{6} \sum_{i,j,k} \kappa_{ijk} (\log q_i)(\log q_j)(\log q_k) - \frac{\chi}{2} \zeta(3) + \sum_{d \in \mathcal{S}} N_0(d) \text{Li}_3(q^d), \quad (1.4)$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ (Re $s > 1$) is the Riemann zeta function and $q^d := q_1^{d_1} \cdots q_l^{d_l}$ for $d = (d_1, \ldots, d_l) \in \mathcal{S} := \mathbb{N}_0^l \setminus \{0\}$. The (virtual) numbers of rational curves on $X$ are counted by $N_0(d)$’s which are related to the Gromov-Witten invariants for genus 0 curves with at least 3 marked points [AM, KM]. (The Gromov-Witten invariants are defined for genus $g$ curves with $n$ marked points satisfying $2 - 2g - n < 0$.)

On the other hand, the genus 1 Gromov-Witten potential of $X$ is believed to have the following expansion [BCOV):

$$F_1(q_*) = -\frac{1}{2} \sum_i \rho_i \log q_i + \sum_{d \in \mathcal{S}} N_{1,0}(d) \text{Li}_1(q^d), \quad (1.5)$$

where

$$N_{1,0}(d) = N_0(d) + 12 \sum_{d' \leq d} N_1(d'), \quad (1.6)$$

$^a$For a careful treatment of the A-model moduli space see [Mo1, Mo2].

$^b$Here we are taking an unconventional overall normalization and the undetermined constant has been set to zero.
and we have considered $S$ as a poset by the partial order: $d' \leq d$ ($d, d' \in S$) iff $d'_i | d_i$ ($i \in I$). The $N_1(d)$’s predict the numbers of elliptic curves on $X$ and are related to the Gromov-Witten invariants for genus 1 curves with at least 1 marked points.

After the pioneering work [CdGP] there have been an abundance of works on the computations of $F_0$ and $F_1$ for various Calabi-Yau threefolds in which they reduced the tasks to B-model calculations using mirror symmetry. See for instance [Mo1] and references therein. However, not much is known about the instanton expansions of the genus $g$ Gromov-Witten potentials $F_g(q_*)$ when $g \geq 2$.

In this work we are concerned with $F_0$ and $F_1$ when $X$ admits a $K3$ fibration as well as an elliptic fibration (with some extra conditions). $K3$ fibered Calabi-Yau threefolds have been of great interest in recent developments of string theory, due to their relevance to heterotic-type IIA string duality conjectures [KV]. Namely, type IIA string compactified on a $K3$ fibered Calabi-Yau threefold is a potential candidate for the dual of a heterotic string compactified on $K3 \times T^2$ [KLM, AL, A]. If $D_1$ corresponds to the generic fiber of the $K3$ fibration, the flat coordinate $\frac{1}{2\pi \sqrt{-1}} \log q_1$ is linearly related to the dilaton-axion of heterotic string and the limit $q_1 \to 0$ where the area of the base of the $K3$ fibration is very large corresponds to the perturbative limit of heterotic string. Thus it is important to understand the behaviors of $F_0$ and $F_1$ in this limit in order to test duality conjectures and there have already been many works on the subject, the list of which includes [KLT, HaM, CCLM, Kaw1, CCL, HeM, C]. The additional assumption about the existence of an elliptic fibration is also important in string theory in the context of $F$-theory [V][MV]. It is related to the existence of the decompactification limit of the $T^2$ factor of the $K3 \times T^2$ on which heterotic string was compactified. We refer to [MV, A] for more details.

The purpose of this paper is, with the above assumptions on $X$, to discuss certain expressions which are conjecturally suitable for the descriptions of $F_0$ and $F_1$ near the limit $q_1 \to 0$. To give such expressions we use the lifting procedures for some nearly holomorphic Weyl-invariant Jacobi forms similar to the one introduced by Borcherds [Bo1] in his construction of automorphic forms on type IV domains as infinite products. These operations involve the Hecke operators for Jacobi forms and, more interestingly, the elliptic polylogarithms of Beilinson and Levin [BeL] in an essential way. In particular if we consider the limit where the base of the elliptic fibration decompactifies, $F_0$ and $F_1$ can be simply expressed in terms of these elliptic polylogarithms.

In an important paper [HaM], by which the present work is much influenced, Harvey and Moore introduced a kind of theta correspondence and extracted the perturbative prepotential of heterotic string from the computa-
tions of some modular integrals. The relation between the two approaches, one by lifting and the other by modular integrals, to investigate the limiting behaviors of $F_0$ and $F_1$ may be partially explained by the following fact. In [BeL], they introduced the notion of the elliptic polylogarithm in their effort to give a motive theoretic interpretation on the work of Zagier [Z] where the real versions of elliptic polylogarithms were introduced as the elliptic extensions of certain real combinations of polylogarithms. As shown in [Z] Zagier’s elliptic polylogarithms are equal to certain double series (which we call the Eisenstein-Kronecker double series). These double series are generalizations of the ones appearing in the classical Kronecker limit formulas. However, as we will remark in this paper, Zagier’s real combinations of polylogarithms and the Eisenstein-Kronecker double series naturally appear in the evaluation of some modular integrals.

The organization of this paper is as follows. In §2, we summarize several properties of (nearly holomorphic) Weyl-invariant Jacobi forms which will be frequently used in later sections. In §3 we consider the theta correspondence associated with a nearly holomorphic Weyl-invariant Jacobi form following the approach of [HaM] and discuss its relation to the Eisenstein-Kronecker double series and the real version of elliptic polylogarithms [Z]. In §4 we discuss the relation between the Borcherds lifting [Bo1] and the elliptic polylogarithms introduced in [BeL]. §5 is devoted to applications to the problem described above providing several explicit (albeit conjectural) examples. There are many things yet to be clarified. We shall discuss some of them in §6.

**Notation.** $\mathbb{N}$ is the set of positive integers. $\mathbb{N}_0$ is the set of non-negative integers. $e[x]$ stands for $\exp(2\pi \sqrt{-1}x)$.

## 2 Weyl-invariant Jacobi Forms

Let $\mathbb{H}$ be the complex upper half plane $\{\tau \in \mathbb{C} \mid \text{Im} \tau > 0\}$. The normalized Eisenstein series of weight $k (\in 2\mathbb{N})$ is defined by

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \quad \tau \in \mathbb{H}, \quad (2.1)$$

where $q = e[\tau]$ and $\sigma_m(n)$ is the sum of the $m^{th}$ powers of all the positive divisors of $n$. The Bernoulli numbers $B_n \ (n \in \mathbb{N}_0)$ are defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad (2.2)$$
Thus, $B_{2n+1} = 0 \ (n \geq 1)$ and

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30} \ldots \quad (2.3)$$

Let $\mathfrak{g}$ be a simple Lie algebra of rank $s$ with a fixed Cartan subalgebra $\mathfrak{h}$. We identify $\mathfrak{h}$ and $\mathfrak{h}^*$ using the Killing form $(\ , \ )$. We extend $(\ , \ )$ by $\mathbb{C}$-linearity. We normalize the highest root $\theta$ as $(\theta, \theta) = 2$. Let $P^\vee$ be the coroot lattice of $\mathfrak{g}$. Then $Q^\vee$ is a positive definite even integral lattice of rank $s$. Let $P$ be the weight lattice of $\mathfrak{g}$ so that $P^\vee = Q^\vee$. The Weyl group of $\mathfrak{g}$ is denoted by $W$.

A holomorphic function $\phi_{k,m} : \mathbb{H} \times \mathfrak{h}_C \to \mathbb{C} \ (k \in \mathbb{Z}, m \in \mathbb{N})$ is called a Weyl-invariant Jacobi form of weight $k$ and index $m$ if it satisfies the following conditions [Wir]:

1. For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$,
   $$\phi_{k,m}\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^k e^{\left[\frac{mc(z, z)}{2(c\tau + d)}\right]} \phi_{k,m}(\tau, z). \quad (2.4)$$

2. For any $\lambda, \mu \in Q^\vee$,
   $$\phi_{k,m}(\tau, z + \lambda\tau + \mu) = e^{-m\left(\frac{1}{2}(\lambda, \lambda)\tau + (\lambda, z)\right)} \phi_{k,m}(\tau, z). \quad (2.5)$$

3. For any $w \in W$,
   $$\phi_{k,m}(\tau, wz) = \phi_{k,m}(\tau, z). \quad (2.6)$$

4. $\phi_{k,m}$ can be expanded as:
   $$\phi_{k,m}(\tau, z) = \sum_{n \in \mathbb{N}_0} c(n, \gamma) q^n \zeta^\gamma. \quad (2.7)$$

Here $\zeta^\gamma = e[(\gamma, z)]$.

The coefficients $c(n, \gamma)$ depend only on $mn - \frac{(\gamma, \gamma)}{2}$ and $\gamma \mod mQ^\vee$. They are zero except for a finite number of $\gamma \in P$ for a fixed $n$. It follows from (2.4) that

$$\phi_{k,m}(\tau, -z) = (-1)^k \phi_{k,m}(\tau, z). \quad (2.8)$$
Hence $k$ must be an even integer if $\epsilon \in W$ where $\epsilon$ acts linearly on $\mathfrak{h}_C$ as $\epsilon(z) = -z$, ($z \in \mathfrak{h}_C$). This is the case if $g$ is none of: $A_n$ ($n \geq 2$), $D_{2n+1}$ ($n \geq 2$), $E_6$.

If we allow $\phi_{k,m}$ to have poles at the cusps so that a finite number of negative powers of $q$ are allowed in the sum in (2.7), it is called a nearly holomorphic Weyl-invariant Jacobi form. The vector space of (nearly holomorphic) Weyl-invariant Jacobi forms of weight $k$ and index $m$ is denoted as $J_{k,m}(g)$ ($J_{nh,k,m}(g)$).

Let $\alpha_1^\vee, \ldots, \alpha_s^\vee$ be the simple coroots of $g$ and let $a_0^\vee, \ldots, a_s^\vee$ be the positive integers such that $a_0^\vee = 1$ and $\theta^\vee = \sum_{i=1}^s a_i^\vee \alpha_i^\vee$ where $\theta^\vee$ is the highest coroot of $g$. It was shown in [Wir] that if $g \neq E_8$ the bigraded ring $\oplus_{k,m} J_{k,m}(g)$ is a polynomial ring over $\mathbb{C}[E_4, E_6]$ freely generated by $\varphi_0, \ldots, \varphi_s$ with the weight and index of $\varphi_i$ given respectively by $-d_i$ and $a_i^\vee$. Here $d_0 = 0$ and $d_i = m_i + 1$ ($i \neq 0$) with $m_1, \ldots, m_s$ being the exponents of $g$. See also the earlier works [L,BS,Sai]. For $g = A_1$, this structure theorem reduces to Th 9.3 in [EZ] (see also [FF]).

Let us fix

$$\phi_{k,m}(\tau, z) = \sum_{n, \gamma} c(n, \gamma)q^n \zeta^\gamma \in J_{k,m}^{nh}(g), \quad (2.9)$$

for an even integer $k$. We now list up miscellaneous identities satisfied by the coefficients $c(n, \gamma)$ for later use. The set $P_0 := \{\gamma \in P \mid c(0, \gamma) \neq 0\}$ is especially important for our purpose. By (2.6) and (2.8) we have apparently

$$c(0, w\gamma) = c(0, \gamma), \quad \forall w \in W, \quad (2.10)$$

and

$$c(0, -\gamma) = c(0, \gamma). \quad (2.11)$$

Let us introduce

$$I_0 := \frac{1}{2} \sum_{\gamma} c(0, \gamma), \quad I_{2n} := \sum_{\gamma > 0} c(0, \gamma)(\gamma, \gamma)^n, \quad (n \geq 1). \quad (2.12)$$

We expand $E_2 \phi_{k,m}$ as

$$(E_2 \phi_{k,m})(\tau, z) = \sum_{n, \gamma} \tilde{c}(n, \gamma)q^n \zeta^\gamma, \quad (2.13)$$

and define $\tilde{P}_0$, $\tilde{I}_0$ and $\tilde{I}_{2n}$ ($n \in \mathbb{N}$) with the obvious replacement of $c(0, \gamma)$ by $\tilde{c}(0, \gamma)$. 6
Since $W$ acts irreducibly on $\mathfrak{h}_R$, an argument similar to [Bo1, Lemma 6.1] implies that for any $v \in \mathfrak{h}_R$, $\sum_{\gamma > 0} c(0, \gamma)(\gamma, v)^{2n}$ ($n = 1, 2, \ldots$) is equal to $(v, v)^n$ up to a multiplicative constant. This should also be true for $v \in \mathfrak{h}_C$. Thus the distribution of $P_0$ is ‘rotationally symmetric’. Concretely we will use

$$\sum_{\gamma > 0} c(0, \gamma)(\gamma, v)^2 = \frac{I_2}{s} (v, v),$$

(2.14)

$$\sum_{\gamma > 0} c(0, \gamma)(\gamma, v)^4 = \frac{3I_4}{s(s + 2)} (v, v)^2,$$

for any $v \in \mathfrak{h}_C$. Similarly we have

$$\sum_{\gamma > 0} \tilde{c}(0, \gamma)(\gamma, v)^2 = \frac{\tilde{I}_2}{s} (v, v), \quad \forall v \in \mathfrak{h}_C.$$

(2.15)

Let $z_1, \ldots, z_s$ be the components of $z \in \mathfrak{h}_C$ when expressed in terms of the standard orthonormal basis. Set $\zeta_i = e[z_i]$ ($i = 1, 2, \ldots, s$). As a little calculation shows, the differential operator

$$D_{k,m} := q \frac{\partial}{\partial q} - \frac{1}{2m} \sum_{i=1}^{s} \left( \zeta_i \frac{\partial}{\partial \zeta_i} \right)^2 + \frac{s - 2k}{24} E_2(\tau),$$

(2.16)

increases the modular weight by two, i.e.

$$D_{k,m} \left( \phi_{j_{k,m}}(g) \right) \subset J_{k+2,m}^h(g).$$

(2.17)

A simple but useful observation [Bo1, Lemma 9.2] is that the constant term in the $q$-expansion of a nearly holomorphic modular form of weight 2 vanishes. Hence if we let the symbol $|_{q^0}$ mean picking up the constant term in the $q$-expansion, we have $(D_{0,m} \phi_{0,m})(\tau, 0)|_{q^0} = 0$, from which it follows that

$$\frac{1}{24}(E_2 \phi_{0,m})(\tau, 0)|_{q^0} = \frac{I_2}{ms}, \quad (k = 0),$$

(2.18)

or equivalently,

$$\sum_{n > 0} \sigma_1(n)c(-n, \gamma) = \frac{I_0}{12} - \frac{I_2}{ms}, \quad (k = 0).$$

(2.19)

Similarly $(E_4 \phi_{-2,m})(\tau, 0)|_{q^0} = 0$ implies

$$\sum_{n > 0} \sigma_3(n)c(-n, \gamma) = -\frac{I_0}{120}, \quad (k = -2).$$

(2.20)
It is also not difficult to see

\[ \frac{1}{24}(E_2^2 \phi_{-2,m})(\tau, 0)|_{q^0} = \frac{1}{2} \left( \frac{dE_2}{dq} \phi_{-2,m} \right)(\tau, 0)|_{q^0} \]

\[ = -12 \sum_{n>0} n\sigma_1(n)c(-n, \gamma), \quad (k = -2), \]

(2.21)

where we used

\[ \frac{dE_2}{dq} = \frac{E_2^2 - E_4}{12}. \]

(2.22)

On the other hand, \((\mathcal{D}_{0,m}\mathcal{D}_{-2,m}\phi_{-2,m})(\tau, 0)|_{q^0} = 0\) leads after a small algebra to

\[ \frac{1}{24}(E_2^2 \phi_{-2,m})(\tau, 0)|_{q^0} = \frac{\tilde{I}_2}{ms} - \frac{12I_4}{m^2s(s+2)}, \quad (k = -2). \]

(2.23)

Therefore combining (2.21) and (2.23) we obtain

\[ \sum_{n>0} n\sigma_1(n)c(-n, \gamma) = -\frac{\tilde{I}_2}{6ms} + \frac{I_4}{m^2s(s+2)}, \quad (k = -2), \]

(2.24)

or equivalently,

\[ \sum_{n>0} (msn - 2(\gamma, \gamma))\sigma_1(n)c(-n, \gamma) = -\frac{\tilde{I}_2}{6} + \frac{I_4}{m(s+2)}, \quad (k = -2). \]

(2.25)

3 Theta correspondence, real polylogarithms and the Eisenstein-Kronecker double series

Any nearly holomorphic Weyl-invariant Jacobi form \(\phi_{k,m} \in J_{k,m}^{nh}(\mathfrak{g})\) can be expanded as

\[ \phi_{k,m}(\tau, z) = \sum_{\alpha \in P/mQ^\vee} h_\alpha(\tau) \theta_{\alpha,m}(\tau, z), \]

(3.1)

where

\[ \theta_{\alpha,m}(\tau, z) := \sum_{\gamma \in \alpha + mQ^\vee} q^{\frac{1}{m}(\gamma,\gamma)} \zeta^\gamma, \]

(3.2)
are the theta functions associated with the integrable representations of the affine Lie algebra [Kac]. In this section, we fix $\phi_k,m \in J_{k,m}^h(\mathfrak{g})$ for a simple Lie algebra $\mathfrak{g}$ of rank $s$, together with its expansion [3.3]. We assume that $k$ is a non-positive even integer.

Let $L$ and $M$ be the indefinite lattices defined by

$$L := H(-m)_2 \oplus P, \quad M := H(-m)_1 \oplus L,$$

where $H(-m)_1 = \mathbb{Z}e_1 + \mathbb{Z}e_2$ and $H(-m)_2 = \mathbb{Z}f_1 + \mathbb{Z}f_2$ are mutually isomorphic indefinite lattices of rank two with $((e_i, e_j)) = ((f_i, f_j)) = \left( \begin{array}{cc} 0 & -m \\ -m & 0 \end{array} \right)$.

Then $M$ can be decomposed as

$$M = \bigcup_{\alpha \in P/mQ^\vee} M_\alpha, \quad M_\alpha := H(-m)_1 \oplus H(-m)_2 \oplus (\alpha + mQ^\vee).$$

Let $\mathbb{D}$ be one of the two connected components of

$$\{ [\omega] \in \mathbb{P}(M_\mathbb{C}) \mid (\omega, \omega) = 0 \text{ and } (\omega, \bar{\omega}) < 0 \},$$

where $[\omega]$ is a line through $\omega \in M_\mathbb{C} \setminus \{0\}$.

The cone $\{ x \in L_\mathbb{R} \mid (x, x) < 0 \}$ consists of two connected components. We fix one of them (the future light-cone) and denote it by $C^+(L_\mathbb{R})$. Then the tube domain

$$\mathcal{H} := L_\mathbb{R} + \sqrt{-1}C^+(L_\mathbb{R}) \subset L_\mathbb{C},$$

gives a realization of the type IV domain $\mathbb{D}$ with the isomorphism $\mathcal{H} \sim \mathbb{D}$ given by

$$y \in \mathcal{H} \longrightarrow [\omega(y)] \in \mathbb{D}, \quad \text{with} \quad \omega(y) = e_1 + \frac{(y, y)}{2m} e_2 + y.$$

We will use this isomorphism freely in the following.

We also always parametrize the elements of $\mathcal{H}$ as

$$\mathcal{H} \ni y = \sigma f_1 + \tau f_2 + mz, \quad z \in K_\mathbb{C}.$$

Thus if we set

$$Y := \text{Im } \sigma \text{ Im } \tau - \frac{m}{2} (\text{Im } z, \text{Im } z),$$

we find that

$$(\omega, \bar{\omega}) = 2(\text{Im } y, \text{Im } y) = -4mY < 0.$$
Each element $\omega \in D$ determines a negative definite two dimensional vector subspace of $M_{\mathbb{R}}$ with an oriented basis. Given a lattice element $\lambda \in M$ we denote by $\lambda^{[\omega]}$ the projection of $\lambda$ onto this vector subspace and set $\lambda^+ := \lambda - \lambda^{[\omega]}$. Therefore we have a relation:

$$\frac{1}{2} \langle \lambda^{[\omega]}, \lambda^{[\omega]} \rangle = \frac{|(\lambda, \omega)|^2}{(\omega, \bar{\omega})}. \quad (3.11)$$

Let $F$ be the fundamental domain of the modular group $SL_2(\mathbb{Z})$ in $\mathbb{H}$. We put $v_1 := \text{Re} \rho$, $v_2 := \text{Im} \rho$ and $q_{\rho} := e^{i \rho}$ for $\rho \in \mathbb{H}$. Now we consider the following theta correspondence:

$$\mathcal{V} = \frac{1}{4} \int_{\mathcal{F}} \frac{dv_1}{v_2} \sum_{\alpha \in P/mQ^\vee} \left( E_2(\rho) - \frac{3}{\pi v_2} \right)^{-k/2} h_{\alpha}(\rho) \Theta_{\alpha,m}(\rho, \bar{\rho}, [\omega]), \quad (3.12)$$

where the indefinite theta functions $\Theta_{\alpha,m}$ are defined by

$$\Theta_{\alpha,m}(\rho, \bar{\rho}, [\omega]) := \sum_{\lambda \in M_{\alpha}} q_{\rho}^{\frac{1}{2} \langle \lambda^{[\omega]}, \lambda^{[\omega]} \rangle - \frac{1}{2} \langle \lambda^{[\omega]}, \lambda^{[\omega]} \rangle}. \quad (3.13)$$

This is a generalization of [HaM] and a special case of [Bo2]. The modular integral (3.12) is divergent and it must be regularized appropriately. The evaluation of (3.12) is now a standard task, however here we wish to emphasize that the final result can be neatly expressed in terms of the real analytic polylogarithms introduced in the work of Zagier [Z]. Besides aesthetic, this gives a motivation for the construction to be explained in the next section.

Thus we first collect relevant materials from [Z] with a slight change of the notation. For a positive odd integer $r$ we define

$$\mathbb{L}_r(\xi) := (-1)^{r-1} \sum_{j=a}^{r} 2^{1-j} \binom{r-j}{a-1} \left( \frac{-\log|\xi|}{r-j} \right)^{r-j} \text{Re} \left( \text{Li}_j(\xi) \right), \quad (3.14)$$

where $r = 2a - 1$. This differs from $D_{a,a}$ in [Z] only by a multiplicative constant. $\text{Li}_r$ is a single-valued real analytic function on $\mathbb{C} \setminus [1, \infty)$ satisfying the inversion relation

$$\text{Li}_r(\xi^{-1}) = \text{Li}_r(\xi) + \frac{(\log|\xi|)^r}{r!}. \quad (3.15)$$

For instance, we have

$$\text{Li}_1(\xi) = \text{Re} \left( \text{Li}_1(\xi) \right), \quad (3.16)$$

$$\text{Li}_3(\xi) = \frac{1}{2} \text{Re} \left[ \log|\xi| \cdot \text{Li}_2(\xi) - \text{Li}_3(\xi) \right]. \quad (3.17)$$
The Bernoulli polynomials $B_n(x)$ ($n \in \mathbb{N}_0$) are monic polynomials of degree $n$ defined through
\[
\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}.
\] (3.18)

We need the first five of them and they are given explicitly by
\[
\begin{align*}
B_0(x) &= 1, \\
B_1(x) &= x - \frac{1}{2}, \\
B_2(x) &= x^2 - x + \frac{1}{6}, \\
B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \\
B_4(x) &= x^4 - 2x^3 + x^2 - \frac{1}{30}.
\end{align*}
\] (3.19)

The Bernoulli polynomials are related to the Bernoulli numbers by
\[
B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k},
\] (3.20)

so that
\[
B_{2n}(x) - B_{2n}(-x) = -2nx^{2n-1}.
\] (3.21)

Zagier [Z] introduced the following elliptic extension of $\text{Li}_r(x)$:
\[
\text{Li}_r(q, \xi) := \sum_{n=0}^{\infty} \text{Li}_r(q^n \xi) + \sum_{n=1}^{\infty} \text{Li}_r(q^n \xi^{-1}) - \mathcal{X}_r(q, \xi),
\] (3.22)

where
\[
\mathcal{X}_r(q, \xi) := \frac{(\log|q|)^r}{(r+1)!} B_{r+1}\left(\frac{\log|\xi|}{\log|q|}\right)
\]
\[
= \sum_{j=-1}^{r} \frac{B_{j+1}}{(r-j)! (j+1)!} (\log|\xi|)^{r-j} (\log|q|)^j.
\] (3.23)

Note that $\mathcal{X}_r(q, \xi)$ would be a polynomial in $\log|\xi|$ and $\log|q|$ if the summation started from $j = 0$. 

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To evaluate $V$ we have to divide it into three parts [DKL]
\[
V = V_0 + V_1 + V_2,
\]
where $V_0$, $V_1$ and $V_2$ correspond respectively to the zero orbit, the degenerate orbits and the non-degenerate orbits of the modular group. The Eisenstein-Kronecker double series $E_a(q, \xi)$ (Re $t > 1$) is defined as
\[
E_a(q, \xi) := \sum_{w \in \mathbb{Z} + \tau \mathbb{Z}, \ w \neq 0} \frac{\chi_w(\tau, x)}{|w|^2 t},
\]
where $q = e[\tau]$, $\xi = e[x]$ and
\[
\chi_w(\tau, x) := e\left[\frac{\text{Im}(\bar{w}x)}{\text{Im} \tau}\right].
\]
The Eisenstein-Kronecker double series appear quite naturally in the evaluation of $V_1$. See for instance [Kaw2]. Interestingly Zagier found that $L_i(r)(q, \xi)$ coincides modulo some factor with $E_a(q, \xi)$. The case $r = 1 = a$ reduces to the classical Kronecker limit formula and as usual we have to make an analytic continuation and discard the divergent part. Consequently $V_1$ can be expressed in terms of the real elliptic polylogarithms $\text{Li}_r(q, \xi)$ where $1 \leq r \leq 1 - k$.

A more detailed explanation is as follows. Suppose $(\ell, n, \gamma) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{P}$.

Using the relations between $E_a(q, \xi)$ and $\text{Li}_r(q, \xi)$, $V_1$ is a linear combination of \( \frac{1}{2}\sum_{\gamma} c_r(0, \gamma) \text{Li}_r(q, \zeta^\gamma) \) \( (1 \leq r \leq 1 - k) \) where the coefficients $c_r(0, \gamma)$ are determined through the expansions
\[
(E_2, \frac{1}{2}) \phi_{k, m}) = \sum_{n, \gamma} c_r(n, \gamma) q^n \zeta^\gamma.
\]

By applying (3.13) and (3.21) it is straightforward to show
\[
\frac{1}{2} \sum_{\gamma} c_r(0, \gamma) \text{Li}_r(q, \zeta^\gamma) = \frac{c_r(0, 0)}{2} \text{Li}_r(1) + \sum_{(0, n, \gamma) > 0} c_r(0, \gamma) \text{Li}_r(q^n \zeta^\gamma)
\]
\[
- \frac{c_r(0, 0)}{2} \frac{B_{r+1}}{(r+1)!} (\log|q|)^r - \sum_{\gamma > 0} c_r(0, \gamma) \chi_r(q, \zeta^\gamma).
\]
As mentioned we have to be careful about the case \( r = 1 \) since \( \text{Li}_1(1) \) is divergent and a suitable regularization is needed. We will choose a regularization scheme such that \( \text{Li}_1(1) \) is formally replaced by \(-\frac{1}{2} \log \hat{Y}\) where

\[
\hat{Y} := \log|p| \log|q| - \frac{m}{2} (\log|\zeta|, \log|\zeta|)
= (2\pi)^2 Y,
\]

(3.29)

with \( \log|\zeta| := -2\pi \text{Im } z \).

If we include the contribution from \( \mathcal{V}_2, \sum_{(\ell,n,\gamma)>0} c_\ell(0,\gamma) \text{Li}_r(q^n\zeta^\gamma) \) in (3.28) is replaced by \( \sum_{(\ell,n,\gamma)>0} c_\ell(\ell n,\gamma) \text{Li}_r(p^n q^n \zeta^\gamma) \). Therefore the final result of the evaluation of \( \mathcal{V} \) can be written in terms of \( \text{Li}_r \) (\( 1 \leq r \leq 1 - k \)).

In the following we summarize the results of the calculations for \( k = 0 \) and \( k = -2 \) since these cases are of direct interest in later sections. However, in principle one may work out for other cases. To reach the results we have frequently used the various identities given in §2. In particular the terms in \( \mathcal{V}_1 \) stemming from the summand with \( j = -1 \) in (3.23) are cancelled against \( \mathcal{V}_0 \). (In [Bo2] it was shown that such cancellations must always occur by quite a general argument.)

**\( k = 0 \)**

\[
\mathcal{V}_0 = -\hat{Y} \frac{1}{24 \log|q|} (E_2^{\phi_0, m})(\tau, 0)|_{q^0}
\]

(3.30)

\[
\mathcal{V}_1 = \frac{1}{2} \sum_{\gamma} c(0,\gamma) \text{Li}_1(q,\zeta^\gamma)
\]

(3.31)

\[
\mathcal{V} = -\frac{T_2}{ms} \log|p| - \frac{T_0}{12} \log|q| + \frac{1}{2} \sum_{\gamma>0} c(0,\gamma)(\gamma,\log|\zeta|) - \frac{c(0,0)}{4} \log \hat{Y} + \sum_{(\ell,n,\gamma)>0} c(\ell n,\gamma) \text{Li}_1(p^n q^n \zeta^\gamma)
\]

(3.32)

**\( k = -2 \)**

\[
\mathcal{V}_0 = -\hat{Y} \frac{1}{48 \log|q|} (E_2^{\phi_{-2,m}})(\tau, 0)|_{q^0}
\]

(3.33)
\[ V_1 = \frac{1}{2} \sum_{\gamma} \hat{c}(0, \gamma) \text{Li}_1(q, \zeta^\gamma) + \frac{12}{Y} \cdot \frac{1}{2} \sum_{\gamma} c(0, \gamma) \text{Li}_3(q, \zeta^\gamma) \]  

\[ V = \left( -\frac{\tilde{T}_2}{ms} + \frac{6T_4}{m^2 s(s+2)} \right) \log|p| - \frac{\tilde{T}_0}{12} \log|q| + \frac{12}{Y} \sum_{\gamma > 0} \hat{c}(0, \gamma) (\gamma, \log|\zeta|) \]

\[ + \frac{\tilde{T}_0}{4} \log|Y| + \sum_{(\ell, n, \gamma) > 0} \hat{c}(\ell, n, \gamma) \text{Li}_1(p^\ell q^n \zeta^\gamma) \]

\[ + \frac{12}{Y} \cdot \frac{T_4}{4ms(s+2)} \log|p| \cdot (\log|\zeta|, \log|\zeta|) + \frac{T_0}{720} (\log|q|)^3 \]

\[ - \frac{T_0}{24s} \log|q| \cdot (\log|\zeta|, \log|\zeta|) + \frac{1}{12} \sum_{\gamma > 0} c(0, \gamma) (\gamma, \log|\zeta|)^3 \]

\[ - \frac{c(0, 0)}{4} \zeta(3) + \sum_{(\ell, n, \gamma) > 0} c(\ell, n, \gamma) \text{Li}_3(p^\ell q^n \zeta^\gamma) \]  

(3.35)

4 The Borcherds lifting and the Beilinson-Levin elliptic polylogarithms

The main purpose of this section is to introduce a lifting procedure for a nearly holomorphic Weyl-invariant Jacobi form whose weight is a non-positive even integer. This is a generalization of the lifting which was introduced by Borcherds [Bo1] for the case of weight 0 and was applied for the construction of automorphic forms on type IV domains as infinite products [Bo1,GN1,GN2].

(To avoid a confusion, we should remark in advance that we will not take the “exponential” unlike [Bo1,GN1,GN2].) As mentioned in Introduction, our lifting procedure involves the elliptic polylogarithms of Beilinson and Levin [BeL]. Thus we begin by reviewing these functions. For a more precise formulation we refer to [BeL] and especially to §4.4 of that paper.

The polylogarithms defined by (1.3) satisfy

\[ \xi \frac{\partial}{\partial \xi} \text{Li}_r(\xi) = \text{Li}_{r-1}(\xi), \quad r \in \mathbb{N}, \]

with \( \text{Li}_0(\xi) = \xi/(1 - \xi) \). In other words we have

\[ \text{Li}_r(\xi) = \int_{\xi}^\xi \text{Li}_{r-1}(\xi) \frac{d\xi'}{\xi'}, \quad r \in \mathbb{N}. \]

(4.2)
Using this relation inductively, each polylogarithm can be analytically continued to a multi-valued holomorphic function on \( \mathbb{P}^1 \setminus \{0, 1, \infty\} = \mathbb{C} \setminus \{0, 1\} \).

The monodromy of the polylogarithms has been studied in [R]. See also [H] for instance. Consider the \((r+1)\)th order differential equation

\[
\frac{d}{d\xi} \left[ \frac{1 - \xi}{\xi} \left( \frac{d}{d\xi} \right)^r f(\xi) \right] = 0, \tag{4.3}
\]

whose “local” solution space is spanned by \( \text{Li}_r, S_{r-1}, S_{r-2}, \ldots, S_0 \). Here we have set \( S_{r-j} = (2\pi \sqrt{-1})^j (\log \xi)^{r-j}, \) \((1 \leq j \leq r)\). Under the monodromy transformations corresponding to the elements of \( \pi_1(\mathbb{C} \setminus \{0, 1\}, *) \) where * is a base point, \( \text{Li}_r(\xi) \) generally shifts by \( \mathbb{Q}\)-linear combinations of \( S_{r-1}, S_{r-2}, \ldots, S_0 \). Thus, in order to kill off the monodromy and attain the effective single-valuedness we are led to define \( \text{Li}_r(\xi) \) as \( \text{Li}_r(\xi) \) modulo any \( \mathbb{Q}\)-linear combinations of \( S_{r-1}, S_{r-2}, \ldots, S_0 \).

In the following we frequently write down equations involving \( \text{Li}_r \) with various different arguments. It should be understood that these are valid only modulo \( \mathbb{Q}\)-linear combinations of \( S_{r-1}, S_{r-2}, \ldots, S_0 \) where \( S_{r-1}, S_{r-2}, \ldots, S_0 \) are determined according to the argument of \( \text{Li}_r \).

For instance we have \( \text{Li}_r(1) = \zeta(r) \) \((r > 1)\) since \( \text{Li}_r(1) \) \((r > 1)\) coincides with \( \zeta(r) \) modulo \( \frac{(2\pi \sqrt{-1})^r}{(r-1)!} \mathbb{Z} \). (See [H].) Actually \( \text{Li}_r(1) = 0 \) if \( r \) is even, since \( \zeta(r) = -\frac{(2\pi \sqrt{-1})^r}{(2\pi \sqrt{-1})^r} B_r \in (2\pi \sqrt{-1})^r \mathbb{Q} \) in that case.

By an induction argument using (4.1) one can show that the ordinary polylogarithms satisfy the inversion formula

\[
(-1)^{r-1} \text{Li}_r(\xi^{-1}) = \text{Li}_r(\xi) + \sum_{j=0}^{r} B_j (2\pi \sqrt{-1})^j (\log \xi)^{r-j}, \tag{4.4}
\]

where we have chosen the branch of \( \text{Li}_r \) so that \( \lim_{\xi \to 1} \text{Li}_r(\xi^{\pm 1}) = \zeta(r) \). Then it is easy to see that

\[
(-1)^{r-1} \text{Li}_r(\xi^{-1}) = \text{Li}_r(\xi) + \frac{(\log \xi)^r}{r!}. \tag{4.5}
\]

More directly one can show (4.3) by induction since (4.1) is true if we replace \( \text{Li} \) by \( \text{Li}_r \).

Notice that if \( r \) is odd, (4.3) is the same type of transformation law as (3.15) for the real analytic single-valued polylogarithm \( \text{Li}_r \). Henceforth we almost always assume that \( r \) is a positive odd integer to render the formulas in this section in parallel with those in the previous section.
In the course of their motivic interpretation of Zagier’s results [Z], Beilinson and Levin [BeL] introduced the elliptic polylogarithm:

\[ \mathcal{L}_r(q, \xi) := \sum_{n=0}^{\infty} \mathcal{L}_r(q^n\xi) + \sum_{n=1}^{\infty} \mathcal{L}_r(q^n\xi^{-1}) - \mathcal{X}_r(q, \xi) , \]  

where

\[ \mathcal{X}_r(q, \xi) := \frac{(\log q)^r}{(r + 1)!} \tilde{B}_{r+1} \left( \frac{\log \xi}{\log q} \right) \]

\[ = \sum_{j=0}^{r} \frac{B_{j+1}}{(r-j)!(j+1)!} (\log \xi)^{r-j} (\log q)^j , \]  

with

\[ \tilde{B}_n(x) := B_n(x) - x^n . \]

The RHS of (4.6) can be obtained from the formal sum \( \sum_{n \in \mathbb{Z}} \mathcal{L}_r(q^n\xi) \) by using (4.5) and then applying the \( \zeta \) function regularization. Obviously \( \mathcal{L}_r(q, \xi) \) is an analog of \( \mathcal{L}_r(q, \xi) \). However, notice that this time \( \mathcal{X}_r(q, \xi) \) is a polynomial of \( \log \xi \) and \( \log q \).

The key devices in the lifting procedure are the Hecke operators. Given a \( \phi_{k,m}(\tau, z) = \sum_{n, \gamma} c(n, \gamma)q^n\zeta^\gamma \in J^{nh}_{k,m}(g) \), the actions of the Hecke operators \( V_\ell \) (\( \ell \in \mathbb{N} \)) are defined by [EZ]

\[ (\phi_{k,m}|V_\ell) (\tau, z) := \ell^{k-1} \sum_{a=0}^{d-1} \sum_{b=0}^{d-1} d^{-k} \phi_{k,m} \left( \frac{a\tau + b}{d} , az \right) \in J^{nh}_{k,\ell m}(g) , \]

where the first sum runs over all the positive integers \( a, d \) such that \( ad = \ell \).

We can formally extend the definition of the polylogarithm \( \mathcal{L}_r(\xi) \) for all \( r \in \mathbb{Z} \) by repeatedly using (4.5). Then we see that \( \mathcal{L}_r(\xi) \) (\( r \leq 0 \)) are rational functions with poles at \( \xi = 1 \) and satisfy

\[ (-1)^{r-1} \mathcal{L}_r(\xi^{-1}) = \mathcal{L}_r(\xi) , \quad r \leq 0 . \]

A crucial identity for us is

\[ \sum_{\ell=1}^{\infty} p^\ell (\phi_{k,m}|V_\ell) (\tau, z) = \sum_{\ell, n, \gamma} c(\ell n, \gamma) \mathcal{L}_{1-k}(p^\ell q^n\zeta^\gamma) , \]

with

\[ \tilde{B}_n(x) := B_n(x) - x^n . \]
which can be shown by a little calculation and is valid for all \( k \in \mathbb{Z} \). Thus the lifting procedure is directly related to polylogarithms. Eq. (1.11) with \( k = 1 - r \) roughly corresponds to the non-degenerate part \( V_2 \) in §3.

The part which is independent of \( p \) (corresponding to \( V_1 \)) may be constructed by considering, in analogy to (3.28), the weighted sum of elliptic polylogarithms over \( P_0 \):

\[
\frac{1}{2} \sum_{\gamma} c(0, \gamma) \mathcal{L}_{ir}(q, \zeta^\gamma) = \frac{c(0, 0)}{2} \zeta(r) + \sum_{(0, n, \gamma) > 0} c(0, \gamma) \mathcal{L}_{ir}(q^n \zeta^\gamma)
\]

\[
- \frac{c(0, 0)}{2} \frac{B_{r+1}}{(r+1)!} \log q)^r - \sum_{\gamma > 0} c(0, \gamma) \mathcal{X}_r(q, \zeta^\gamma),
\]

where \( k = 1 - r \) and we used again the inversion relation (4.5). This prompts us to define the Hecke operator \( V_0 \) by

\[
(\phi_{k,m} | V_0)(\tau, z) := \frac{c(0, 0)}{2} \zeta^*(1 - k) + \sum_{(0, n, \gamma) > 0} c(0, \gamma) \mathcal{L}_{1-k}(q^n \zeta^\gamma),
\]

where \( \zeta^*(1) = 0 \) and \( \zeta^*(n) = \zeta(n) \) if \( n \) is a nonzero integer.

Thus we are led to define

\[
\tilde{\mathcal{L}}_{ir}^\phi(p, q, \zeta) := \sum_{\ell=0}^{\infty} p^{\ell} (\phi_{1-r, m} | V_\ell)(\tau, z)^\ell
\]

\[
- \frac{c(0, 0)}{2} \frac{B_{r+1}}{(r+1)!} \log q)^r - \sum_{\gamma > 0} c(0, \gamma) \mathcal{X}_r(q, \zeta^\gamma),
\]

where the symbol ! means that we should replace \( \mathcal{L}_{ir} \) by \( \mathcal{L}_{ir} \) when applying (4.11).

We wish to define \( \mathcal{L}_{ir}^\phi(p, q, \zeta) \) by adding to \( \tilde{\mathcal{L}}_{ir}^\phi(p, q, \zeta) \) the part which will consist of terms depending on \( \log p \). For \( r = 1 \) and \( r = 3 \) the following definitions seem to be working:

\[
\mathcal{L}_{1}^{\phi}(p, q, \zeta) := -\frac{T_2}{ms} \log p + \tilde{\mathcal{L}}_{1}^{\phi}(p, q, \zeta),
\]

and

\[
\mathcal{L}_{3}^{\phi}(p, q, \zeta) := -\frac{T_4}{4ms(s+2)} \log p \cdot (\log \zeta, \log \zeta) + \tilde{\mathcal{L}}_{3}^{\phi}(p, q, \zeta).
\]
More explicit expressions are given by

\[ \mathcal{L}^\phi_1(p, q, \zeta) = -\frac{T_2}{ms} \log p - \frac{T_0}{12} \log q + \frac{1}{2} \sum_{\gamma > 0} c(0, \gamma)(\gamma, \log \zeta) \]
\[ + \sum_{(\ell, n, \gamma) > 0} c(\ell n, \gamma) \mathcal{L}^\phi_1(p^\ell q^n \zeta^\gamma), \tag{4.17} \]

and

\[ \mathcal{L}^\phi_3(p, q, \zeta) = -\frac{T_4}{4ms(s+2)} \log p \cdot (\log \zeta, \log \zeta) + \frac{T_0}{720} (\log q)^3 \]
\[ - \frac{T_2}{24s} \log q \cdot (\log \zeta, \log \zeta) + \frac{1}{12} \sum_{\gamma > 0} c(0, \gamma)(\gamma, \log \zeta)^3 \]
\[ + \frac{c(0, 0)}{2} \zeta(3) + \sum_{(\ell, n, \gamma) > 0} c(\ell n, \gamma) \mathcal{L}^\phi_3(p^\ell q^n \zeta^\gamma), \tag{4.18} \]

where \( \log \zeta := 2\pi \sqrt{-1z} \). It should be kept in mind that the \( c(n, \gamma) \)'s are the coefficients of \( \phi_{0,m} \) in (4.17) while they are the coefficients of \( \phi_{-2,m} \) in (4.18).

The choices of \( \log p \) dependent terms are apparently motivated by the calculations in the previous section and may look ad hoc within the logic of this section. However we can show that with these choices \( \mathcal{L}^\phi_1(p, q, \zeta) \) and \( \mathcal{L}^\phi_3(p, q, \zeta) \) have nice wall-crossing behaviors under the exchange of \( p \) and \( q \) which determine part of the (quasi-)automorphic properties of \( \mathcal{L}^\phi_1(p, q, \zeta) \) and \( \mathcal{L}^\phi_3(p, q, \zeta) \). In order to know these behaviors we must use various identities derived in §2. For \( \mathcal{L}^\phi_1(p, q, \zeta) \) we have

\[ \mathcal{L}^\phi_1(p, q, \zeta) = \mathcal{L}^\phi_1(q, p, \zeta), \tag{4.19} \]

by using

\[ \sum_{\ell > 0 \atop n < 0} c(\ell n, \gamma) \log(p^\ell q^n \zeta^\gamma) = -\sum_{n > 0} \sigma_3(n)c(-n, \gamma)(\log p - \log q) \]
\[ = \left( -\frac{T_0}{12} + \frac{T_2}{ms} \right) (\log p - \log q). \tag{4.20} \]
On the other hand we have

\[
\mathcal{L}_3^\phi(p, q, \zeta) - \mathcal{L}_3^\phi(q, p, \zeta) = \left( - \frac{\mathcal{I}_2}{12ms} + \frac{\mathcal{I}_4}{2m^2 s(s + 2)} \right) (\log p - \log q) \left\{ \log p \log q - \frac{m}{2} (\log \zeta, \log \zeta) \right\}.
\]

This can be proved by using

\[
- \sum_{n > 0, \gamma < 0} c(\ell n, \gamma) \frac{(\log(p^\nu q^\mu \zeta^\gamma))^3}{3!} = -\frac{1}{6} \sum_{n > 0} \sigma_3(n) c(-n, \gamma)((\log p)^3 - (\log q)^3)
\]

\[
- \frac{1}{2s} \sum_{n > 0} \sigma_1(n, \gamma) c(-n, \gamma) (\log p - \log q)(\log \zeta, \log \zeta)
\]

\[
+ \frac{1}{2} \sum_{n > 0} n \sigma_1(n)(-n, \gamma)(\log p - \log q) \log p \log q,
\]

and noting that the RHS can be rewritten as

\[
\frac{\mathcal{I}_0}{720}((\log p)^3 - (\log q)^3))
\]

\[
+ \left( - \frac{\mathcal{I}_2}{24s} + \frac{\mathcal{I}_4}{4ms(s + 2)} \right) (\log p - \log q)(\log \zeta, \log \zeta)
\]

\[
+ \left( - \frac{\mathcal{I}_2}{12ms} + \frac{\mathcal{I}_4}{2m^2 s(s + 2)} \right) (\log p - \log q) \left\{ \log p \log q - \frac{m}{2} (\log \zeta, \log \zeta) \right\}.
\]

The functions \( \mathcal{L}_3^\phi(p, q, \zeta) \) and \( \mathcal{L}_3^\phi(p, q, \zeta) \) introduced in the above and their properties (4.19) and (4.21) will play important roles in the next section.

5 String duality and enumeration of curves

Now we can turn to the problem mentioned in Introduction. We are concerned with Calabi-Yau threefolds admitting fibrations. The existence of an algebraic fibration structure on \( X \) is related to that of a nef effective divisor \( D \) on \( X \).
and the allowed types of fibrations are classified [O] by the numerical dimension \( \nu(X, D) := \max\{n \in \mathbb{N} \mid D^n \neq 0\} \) where \( \equiv \) stands for the numerical equivalence relation and the value of \( c_2(X) \cdot D \). (For a nef divisor \( D \) we have \( c_2(X) \cdot D \geq 0 \) [Mi].)

We first assume that \( \nu(X, D_1) = 1 \) and \( \rho_1 > 0 \) which means that \( X \) allows a type I_+ fibration [O]. In other words, there exists a fibration \( \pi_1 : X \to W_1 \) with \( W_1 \cong \mathbb{P}^1 \) such that a generic non-singular fiber is an algebraic K3 surface. We have \( \kappa_{11i} = 0 \) (\( \forall i \)) by \( \nu(X, D_1) = 1 \) and actually \( \rho_1 = 24 \). The divisor \( D_1 \) may be identified with a generic fiber K3 of this fibration.

Another assumption we make upon \( X \) is that \( \nu(X, D_2) = 2 \) and \( \rho_2 > 0 \). This is the case of type II_+ fibration and there exists an elliptic fibration \( \pi_2 : X \to W_2 \). We assume that this fibration has a section \( \sigma_2 : W_2 \to X \). The condition \( \nu(X, D_2) = 2 \) can be rephrased as: \( \kappa_{222} = 0 \) but \( \kappa_{22i} \neq 0 \) for some \( i \).

Due to the above assumptions, \( W_2 \) itself is a \( \mathbb{P}^1 \) fibration over \( \mathbb{P}^1 \), \( \pi_{21} : W_2 \to W_1 \). We further assume that this fibration is smooth. In other words, \( W_2 \) is a ruled surface over \( \mathbb{P}^1 \), hence is one of the Hirzebruch surfaces, say \( \Sigma_a \) (\( 0 \leq a \leq 12 \)). There are two basic divisors of \( \Sigma_a \), namely \( C_0 \) (a section of \( \pi_{21} \)) and \( f \) (the fiber of \( \pi_{21} \)) with \( C_0 \cdot C_0 = -a \), \( f \cdot f = 0 \) and \( C_0 \cdot f = 1 \). There is another section \( C_{\infty} := C_0 + af \) satisfying \( C_{\infty} \cdot C_{\infty} = a \), \( C_0 \cdot C_{\infty} = 0 \) and \( C_{\infty} \cdot f = 1 \).

The divisor \( D_1 \) may be identified with the restriction of \( \pi_2 : X \to W_2 \) on \( f \). Thus \( D_1 \) is an elliptic K3. On the other hand, the divisor \( D_2 \) may be identified with the restriction of \( \pi_2 : X \to W_2 \) on \( C_{\infty} \). To put differently, \( q_1 \) and \( q_2 \) are the parameters counting how many times the images of the instanton holomorphic maps wind respectively around \( C_0 \) and \( f \). If we represent \( X \) in the Weierstrass form the discriminant divisor \( \Delta \subset W_2 \) is given by (see for instance [A])

\[
\Delta = 24C_0 + (24 + 12a)f. \tag{5.1}
\]

Hence \( \Delta \cdot f = 24 \) and \( \Delta \cdot C_{\infty} = 24 + 12a \). This gives immediately that \( \chi(D_1) = 24 \) and \( \chi(D_2) = 24 + 12a \) assuming that the singular fibers are of Kodaira type I_1. On the other hand, \( \rho_1 = c_2(D_1) \) and \( \rho_2 = c_2(D_2) \) in view of \( D_1 \cdot D_1 = 0 \) and \( D_2 \cdot D_2 = 0 \). Therefore we have

\[
\rho_1 = 24, \quad \rho_2 = 24 + 12a. \tag{5.2}
\]

As the divisor \( D_3 \) we may choose \( \sigma_2(W_2) \). It turns out that

\[
\rho_3 = 92. \tag{5.3}
\]

\[c\text{ See the footnote 6 of [BJPS].}\]
We make one more important assumption. We assume that the singular fibers of \( \pi_1 : X \to W_1 \) are irreducible. This ensures that there are no extra contributions to \( H_4(X, \mathbb{Z}) \) or equivalently \( H^2(X, \mathbb{Z}) \) from the singular fibers \([A]\).

Our goal in this section is to study the behaviors of \( F_0 \) and \( F_1 \) in the limit \( q_1 \to 0 \) using the results obtained in the previous sections. The main reason why we are interested in this limit lies in its possible connection to perturbative heterotic string compactified on \( K3 \times T^2 \) \([KLM,AL,A]\). However, even if we fail to find such a connection, the asymptotic behaviors we propose may well be true.

Before presenting our proposal, we introduce several auxiliary functions. Suppose we have a simple Lie algebra \( g \) of rank \( s \) and \( \phi_{-2,m}(\tau, z) \in J^{nh}_{-2,m}(g) \) together with the expansion \( \phi_{-2,m}(\tau, z) = \sum_{n, \gamma} c(n, \gamma) q^n \zeta^\gamma \). Then we define \( \phi_{0,m} := \frac{24}{s+4} D_{-2,m} \phi_{-2,m} \in J^{nh}_{0,m}(g) \).

Using these \( \phi_{0,m} \) and \( \phi_{-2,m} \) one can construct \( \mathcal{L}_1^\psi(p, q, \zeta) \) and \( \mathcal{L}_3^\psi(p, q, \zeta) \) as explained in the previous section. Then we can introduce

\[
\mathcal{F}_0(p, q, \zeta) := \mathcal{L}_1^\psi(p, q, \zeta),
\]

and

\[
\mathcal{F}_{0,1}(p, q, \zeta) := \mathcal{L}_3^\psi(p, q, \zeta)
\]

\[
- \frac{24}{s+4} \left\{ \frac{p \partial}{\partial p} \frac{q \partial}{\partial q} - \frac{1}{2m} \sum_{i=1}^{s} \left( \zeta_i \frac{\partial}{\partial \zeta_i} \right)^2 \right\} \mathcal{F}_0(p, q, \zeta) .
\]

The explicit expression of \( \mathcal{F}_{0,1} \) is given by

\[
\mathcal{F}_{0,1} = \left( \frac{\hat{T}_2}{m s} + \frac{6T_1}{m^2 s(s+2)} \right) \log p - \frac{\hat{T}_0}{12} \log q + \frac{1}{2} \sum_{\gamma > 0} \bar{c}(0, \gamma)(\gamma, \log \zeta)
\]

\[
+ \sum_{\ell, n, \gamma > 0} \bar{c}(\ell n, \gamma) \mathcal{L}_1(p^{2\ell} q^n \zeta^\gamma),
\]

which should be compared with (3.37).

The main conjecture of this paper is that by imposing an extra condition on \( X \) to be mentioned below and by selecting a suitable Lie algebra \( g \) of rank \( s \) and \( \phi_{-2,m} \in J^{nh}_{-2,m}(g) \) with \( l = h^{11} = s + 3 \), the Gromov-Witten potentials
$F_0$ and $F_1$ have the following asymptotic behaviors:

$$F_0(q_*) = \log u \left\{ \log p \log q - \frac{m}{2} (\log \zeta, \log \zeta) \right\} + F_0(p, q, \zeta) + O(q_1), \quad (5.8)$$

$$F_1(q_*) = -\frac{1}{2} \cdot 24 \cdot \log u + F_0(p, q, \zeta) + O(q_1). \quad (5.9)$$

Here by setting

$$t_i = \log q_i, \quad (i = 1, \ldots, l), \quad (5.10)$$

we should have

$$t_1 = \log u - \log q - \frac{a}{2} (\log p - \log q), \quad t_2 = \log p - \log q, \quad (5.11)$$

and $t_3, \ldots, t_l$ are expressed in terms of $\log q$ and $\log \zeta$. This determines the behaviors of $F_0$ and $F_1$ in the limit where the size of $W_1$ is very large. If we take a further limit where $q_2 \to 0$ or $p \to 0$ in addition to $q_1 \to 0$, then it is obvious that $F_0$ and $F_1$ are simply described by the elliptic polylogarithms of Beilinson and Levin. This is the limit where $W_2$ decompactifies, namely both the fiber and the base of $W_2$ are becoming very large.

A short calculation using (4.19) and (4.21) shows that the relation

$$F_g(u, p, q, \zeta) = F_g(u', q, p, \zeta), \quad (5.12)$$

holds in the limit $q_1 \to 0$, where $g = 0, 1$ and

$$\log u' = \log u + \left( -\frac{\tilde{I}_2}{12ms} + \frac{\tilde{I}_4}{2m^2s(s + 2)} \right) (\log p - \log q). \quad (5.13)$$

It is tempting to conjecture that (5.12) continues to be valid to all orders in $q_1$ and for all genera $g$.

We can view $F_0$ as the enumerating function of rational curves residing entirely in the fibers of $\pi_1 : X \to W_1$. Let us fix a generic fiber of $\pi_1 : X \to W_1$ and denote it by $Z$. Then $\pi : Z \to B$ with $B \cong \mathbb{P}^1$ is an elliptic K3 surface with a section. Here $\pi$ is obtained by restricting $\pi_2$ to the fiber $f$ of $W_2$. Let us temporarily assume that the Mordell-Weil group (= the group of sections) of $\pi : Z \to B$ is trivial so that the section of $\pi : Z \to B$ is unique. Let $S$ and $F$ be respectively the section and a generic fiber of $\pi : Z \to B$. Thus we have $S^2 = -2$, $F^2 = 0$ and $S \cdot F = 1$. Besides $S$, rational curves in $Z$ arise from the singular fibers of $\pi : Z \to B$. The singular fibers may be either irreducible or reducible. In the irreducible cases they must be of Kodaira
type $I_1$ (a rational curve with a node) or of Kodaira type $II$ (a rational curve with a cusp). A reducible singular fiber consists of $\mathbb{P}^1$’s intersecting with one another according to the pattern of the extended Dynkin diagram of one of the simply-laced Lie algebras. Let us assume that we have a single reducible singular fiber and the rest of the singular fibers are irreducible. The finite dimensional simply-laced Lie algebra of rank $s_0$ associated with the reducible fiber is denoted by $g_0$. Let $\tilde{g}_0$ be the untwisted affine Lie algebra corresponding to $g_0$ and let $\Pi = \{ \alpha_0, \ldots, \alpha_{s_0} \}$ be the set of the simple roots of $\tilde{g}_0$. Thus the $\mathbb{P}^1$’s in the reducible fiber correspond to $\Pi$ and their intersection matrix is equal to the negative of the Cartan matrix of $(\tilde{g}_0, \Pi)$. Let $\delta$ and $\Lambda_0$ be as in the standard theory of affine Lie algebras [Kac]. Thus we have $(\delta, \delta) = (\Lambda_0, \Lambda_0) = 0$ and $(\delta, \Lambda_0) = 1$. $\delta$ is orthogonal to $\Pi$ and $\Lambda_0$ is orthogonal to $\alpha_1, \ldots, \alpha_{s_0}$, however $(\Lambda_0, \alpha_0) = 1$. In addition we have the relation $\alpha_0 = \delta - \theta$. We can identify $F$ with $\delta$ and $S$ with $-\Lambda_0 - \delta$ with sign flips understood when considering intersections. When we have a reducible fiber the relative sizes of the $\mathbb{P}^1$’s give extra contributions to $\text{Pic}(Z)$. Thus the cohomology classes $[\Lambda_0], [\alpha_0], [\alpha_1], \ldots, [\alpha_{s_0}]$ will span $\text{Pic}(Z)$.

The complexified Kähler moduli space of $X$ will be given by the complexified Kähler cone divided by some discrete group. This group will arise from the monodromy of the fibrations of $X$. Since we have assumed that there are no reducible singular fibers in the $K3$ fibration $\pi_1 : X \to W_1$, the monodromy-invariant part of $\text{Pic}(Z)$ and $D_1$ will span $H^2(X, \mathbb{Z})$. It has been argued that from the monodromy of $\pi_1 : X \to W_1$ an outer automorphism of $g_0$ can appear [AG,BIKMSV] in addition to the inner automorphisms described by the Weyl group. In such a case we divide $g_0$ by the outer automorphism and the resulting simple Lie algebra is denoted by $g$. Otherwise we simply set $g = g_0$. Let $s$ be the rank of $g$. It seems natural to assume that the monodromy-invariant part of $\text{Pic}(Z)$ has signature $(1, s + 1)$ and contains $Q^\vee(-1)$ where $Q^\vee$ is the coroot lattice of $g$. We conjecture that $g$ coincides with the one we have been using for the description of (nearly holomorphic) Weyl-invariant Jacobi forms. If the Mordell-Weil group of $\pi : Z \to B$ is non-trivial or if there are more than one reducible singular fibers in $\pi : Z \to B$, the above picture should be modified appropriately.

What we have conjectured implies that the complexified Kähler moduli space contains (a suitable compactification of) the space

$$\frac{(\mathbb{H} \times \mathbb{C}^s)}{(SL_2(\mathbb{Z}) \times \tilde{W})},$$

(5.14)

which is coordinatized by $\tau$ and $z$ as a codimension two submoduli space. Here $\tilde{W} = \tilde{W} \times Q^\vee$ with $\tilde{W} = W \times Q^\vee$ being the affine Weyl group of $g$. If we go
to a further boundary we will have
\[ \mathbb{C}^* / \tilde{W}, \] (5.15)
which is coordinatized by \( z \). The last space also arises from possible connections to heterotic string theory on \( K3 \times T^2 \). Let \( G \) be a compact simple simply-connected Lie group associated with \( \mathfrak{g} \). We fix a maximal torus \( T \subset G \). Let \( E \) be an elliptic curve. By fixing a base point we identify \( E \) with the \( T^2 \) of \( K3 \times T^2 \) on which heterotic string is compactified. The Wilson lines we can put on \( E \) is described by the set of flat \( G \)-bundles on \( E \) up to isomorphism.

There is a natural bijection \([FMW,D]\) between such set and \( \text{Hom}(\pi_1(E), T)/W \cong (E \otimes_\mathbb{Z} \mathbb{Q}_\vee)/W \sim C_s / \tilde{W} \)\.(5.16)

However, apparently we have \((E \otimes_\mathbb{Z} \mathbb{Q}_\vee)/W \sim C_s / \tilde{W}\). It is known \([L,BS,FMW]\) that \((E \otimes_\mathbb{Z} \mathbb{Q}_\vee)/W\) is a weighted projective space with weights \( a_0^\vee, \ldots, a_s^\vee \).

We should also note that the generators \( \varphi_0, \ldots, \varphi_s \) of the ring \( \oplus_{k,m} J_{k,m}(\mathfrak{g}) \) define (part of) the (inverse) mirror maps (or the solutions to the Jacobi inversion problems). Actually this was one of the original motivations for the study of Weyl-invariant Jacobi forms \([Sai]\). See also \([Sat]\).

Further restrictions on \( \phi_{-2,m} \) may arise from the following consideration. Let \( Z \) be as before an elliptic \( K3 \) surface with a section but assume for simplicity that all the singular fibers are of Kodaira type \( I_1 \). Hence there are 24 of them. Let \( S \) be a section and let \( F \) be a generic fiber as before. As argued in many works \([YZ,Be,Go,BrL]\), the number of rational curves with \( n \) nodes corresponding to the class \( S + nF \) is given by \( p_{24}(n) \) where
\[ \frac{q}{\Delta(\tau)} = \sum_{n=0}^{\infty} p_{24}(n) q^n = 1 + 24q + \cdots, \quad \Delta(\tau) := q \prod_{n=1}^{\infty} (1 - q^n)^{24}. \] (5.17)
Notice that we have a piece
\[ p(\phi_{-2,m}|V_1)(\tau, z) = p\phi_{-2,m}(\tau, z), \] (5.18)
in \( \mathcal{F}_0 \). Thus it might be tempting to relate \( \phi_{-2,m}(\tau, z) \) to \( 1/\Delta(\tau) \). However, \( 1/\Delta(\tau) \) does not have modular weight \(-2\). Presumably the source of the discrepancy partly lies in that we are not considering a single elliptic \( K3 \) but a family of elliptic \( K3 \) surfaces parametrized by \( W_1 \cong \mathbb{P}_1 \). Thus the rational curves in \( D_1 \) will come in continuous families. For instance, if there are smooth rational curves in a Calabi-Yau threefold parametrized by a smooth parameter space \( \mathcal{B} \) the virtual number of rational curves is given by \((-1)^{\dim B} \chi(\mathcal{B}) \) \([CdFKM,CFKM,Kat]\). Since we have \( X \supset W_2 \cong \Sigma_a \) and
the parameter associated with $f \cong \mathbb{P}^1$ is given by $q_2 = p q^{-1}$, we should have $c(-1, 0) = (-1)(2 - 2 \cdot 0) = -2$ and except for this case $c(-n, \gamma) = 0$ ($n \in \mathbb{N}$). Thus $-2$ replaces the naïve expectation of 1. (The point $t_2 = 0$ in the Kähler moduli space of $X$ is the place where $f \cong \mathbb{P}^1$ shrinks. This point is the place where the U(1) gauge symmetry is enhanced to SU(2) producing two extra massless vector multiplets explaining the physical meaning of $-2$.) By equating the terms proportional to $\zeta(3)$ we should have

$$c(0, 0) = -\chi. \quad (5.19)$$

This tells us that the naïve expectation 24 should be replaced by $-\chi$. Actually there should be infinitely many cases where the virtual number of rational curves are given by $-\chi$ considering the expression of $\mathcal{F}_0$. The very appearance of $-\chi$ as the virtual number of rational curves has been observed and proved for certain Calabi-Yau threefolds with elliptic fibrations. See §4.1 of [BKK] and [KMV].

We also note that expressions similar to (5.18) appeared in different contexts [KMV,HSS].

Taking into account the above considerations we assume that $\phi_{-2,m}(\tau, z)$ can be expressed as

$$\phi_{-2,m}(\tau, z) = -\frac{2 \Phi_{10,m}(\tau, z)}{\Delta(\tau)}, \quad (5.20)$$

where $\Phi_{10,m}(\tau, z)$ belongs to $J_{10,m}(g)$ with $\Phi_{10,m}(\tau, z) = 1 + O(g)$. Note that since $\Phi_{10,m}(\tau, 0)$ is a modular form of weight ten, we should have $\Phi_{10,m}(\tau, 0) = E_4(\tau) E_6(\tau)$. Of course $\Phi_{10,m}$ can be expanded (over $\mathbb{C}$) in the basis of the structure theorem [Wir] mentioned in §2, however we generally expect that the Fourier coefficients of $\phi_{-2,m}$ are integers.

With the assumption (5.20) it is easy to see that

$$\mathcal{I}_0 = 240, \quad \tilde{\mathcal{I}}_0 = 264, \quad \tilde{\mathcal{I}}_2 = \mathcal{I}_2, \quad (5.21)$$

and

$$\frac{\mathcal{I}_4}{2 m^2 s (s + 2)} = \frac{\mathcal{I}_2}{12 ms} - 1, \quad (5.22)$$

25
by (5.23). Hence we have simplified expressions for $F_0$ and $F_{0,1}$:

$$F_0 = \left( \frac{m}{2} - \frac{T_2}{24s} \right) \log p \cdot (\log \zeta, \log \zeta) + \frac{1}{3} (\log q)^3 - \frac{T_2}{24s} \log q \cdot (\log \zeta, \log \zeta)$$

$$+ \frac{1}{12} \sum_{\gamma > 0} c(0, \gamma)(\gamma, \log \zeta)^3 + \frac{c(0, 0)}{2} \zeta(3) + \sum_{(\ell, n, \gamma) > 0} c(\ell n, \gamma) \mathcal{L}_3(p^\ell q^n \zeta^\gamma),$$

(5.23)

and

$$F_{0,1} = -12 \log p - 22 \log q + \frac{1}{2} \sum_{\gamma > 0} c(0, \gamma)(\gamma, \log \zeta)$$

$$+ \sum_{(\ell, n, \gamma) > 0} \check{c}(\ell n, \gamma) \mathcal{L}_1(p^\ell q^n \zeta^\gamma).$$

(5.24)

Also (5.13) is simplified as

$$\log u' = \log u - (\log p - \log q).$$

(5.25)

It is convenient to introduce the classical parts of $F_0$ and $F_1$ arising from constant instanton maps:

$$F_0^c := \frac{1}{6} \sum_{i,j, k=1}^l \kappa_{ijk} t_i t_j t_k, \quad F_1^c := -\frac{1}{2} \sum_{i=1}^l \rho_i t_i.$$  

(5.26)

We have to express $t_i$ in terms of the logarithms of $p, q$ and $\zeta$ so that we can take the limits $q_i \to 0$ in $F_0 - F_0^c$ and $F_1 - F_1^c$. We expect that in addition to (5.11) we should have

$$t_3 = \log q - (\gamma_0, \log \zeta),$$

(5.27)

$$t_{i+3} = (\Lambda_i, \log \zeta), \quad (i = 1, \ldots, s),$$

(5.28)

where $\Lambda_i (i = 1, \ldots, s)$ are the fundamental weights of $g$ and $\gamma_0$ is some positive weight. Then (5.12) turns into

$$F_g(q_1, q_2, q_3, q_4, \ldots, q_l) = F_g\left( q_1 q_2 a^{-2}, q_2^{-1}, q_2^2 q_3, q_4, \ldots, q_l \right).$$

(5.29)

In the rest of this section we present several conjectural examples as applications of the general theory developed so far. They are not particularly new since they have more or less appeared in the literature. We hope we can report on other examples in the near future.
Let $X(w_1, \ldots, w_5)_{h_11, h_{12}}$ symbolically denote a Calabi-Yau threefold, with the subscripts denoting its Hodge numbers, obtained from the hypersurface of degree $w_1 + \cdots + w_5$ in the weighted projective space $\mathbb{P}(w_1, \ldots, w_5)$.

**Example 1.** Take $g = A_1$ and $m = 1$. Thus $Q = Q^c = \sqrt{2}\mathbb{Z}$ and $P = \frac{1}{\sqrt{2}}\mathbb{Z}$. Set $x = \frac{1}{\sqrt{2}}z$ and $\xi = e[x]$. Let $E_{k, m}(\tau, x)$ be the Eisenstein-Jacobi series of weight $k$ and index $m$ as defined in [EZ]. Consider [Kaw1,CCL]

$$
\phi_{-2, 1}(\tau, z) = \frac{-(12 - a)E_6(\tau)E_{4, 1}(\tau, x) + (12 + a)E_4(\tau)E_{6, 1}(\tau, x)}{12\Delta(\tau)}
$$

$$
= -\frac{2}{q} - 2\xi^2 + (32 + 12a)\xi + 420 - 24a + \frac{32 + 12a}{\xi} - \frac{2}{\xi^2}
$$

$$
+ \left((420 - 24a)\xi^2 + (53760 + 96a)\xi + 174528 - 144a\right)\xi + O(q^2),
$$

(5.30)

where $a = 0, \ldots, 12$. Since $I_2 = 12 + 6a$, we have predictions

$$
F_{0}^{cl} = \log u \left(\log p \log q - (\log \xi)^2\right) - \frac{a}{2} \log p(\log \xi)^2
$$

$$
+ \frac{(\log q)^3}{3} - \left(\frac{a}{2} + 1\right)\log q(\log \xi)^2 + \left(a + \frac{4}{3}\right)(\log \xi)^3,
$$

(5.31)

and

$$
F_{1}^{cl} = -12 \log u - 12 \log p - 22 \log q + (6a + 14) \log \xi.
$$

(5.32)

Choosing $\gamma_0 = \sqrt{2}$ we set

$$
t_3 = \log q - 2 \log \xi, \quad t_4 = \log \xi.
$$

(5.33)

Therefore we find

$$
\rho_4 = 156 - 12a,
$$

(5.34)

and

$$
F_{0}^{cl} = t_1 \left(t_2 t_3 + t_3^2 + 2t_2 t_4 + 4 t_3 t_4 + 3 t_4^2\right)
$$

$$
+ \frac{a}{2} t_2^2 t_3 + \left(1 + \frac{a}{2}\right) t_2 t_3^2 + \frac{4}{3} t_3^3 + a t_2^2 t_4 + 2(a + 2) t_2 t_3 t_4
$$

$$
+ (4 + a) t_2 t_4^2 + 8 t_3^2 t_4 + (14 - a) t_3 t_4^2 + (8 - a) t_4^3.
$$

(5.35)

\[\text{This choice is not surprising. Due to the structure theorem mentioned in §2, } \Phi_{10, 1} \text{ can always be expressed as } \Phi_{10, 1} = cE_6 E_{4, 1} + (1 - c)E_4 E_{6, 1} \text{ for some number } c.\]
Note that this is consistent with $\nu(X, D_1) = 1$ and $\nu(X, D_2) = 2$.

The would-be Calabi-Yau threefold $X$ must satisfy $\chi = -420 + 24a$. For $a = 2$ [CCL] and $a = 12$ [Kaw1], there are known candidates for $X$. They are respectively given by $X(10, 6, 2, 1, 1)_{4,190}$ and $X(10, 3, 3, 2, 2)_{4,70}$. The conjecture for the case $a = 2$ was tested against the B-model calculation of $F_0$ in [CCL] by using the result of [BKKM]. The test of the case $a = 12$, which was not attempted in [Kaw1], has now been done up to some orders to find a good agreement with the B-model calculation of $F_0$. We thank Hosono for kindly providing the relevant data of the B-model calculation for $X(10, 3, 3, 2, 2)_{4,70}$.

The $K_3$ fibers $D_1$ of $X(10, 6, 2, 1, 1)_{4,190}$ and $X(10, 3, 3, 2, 2)_{4,70}$ coincide with the degree 10 hypersurface in $P(5, 3, 1, 1)$.

**Example 2.** Take $g = A_1$ and $m = 2$. Consider

$$
\phi_{-2,2}(\tau, z) = -\frac{5E_6(\tau)E_{4,2}(\tau, x) + 7E_4(\tau)E_{6,2}(\tau, x)}{6\Delta(\tau)}
- \frac{2}{q} + 96\xi + 288 + \frac{96}{\xi} + \left(-2\xi^4 + 96\xi^3 + 10192\xi^2 + 69280\xi + 123756 + \frac{69280}{\xi} + \frac{10192}{\xi^2} + \frac{96}{\xi^3} - \frac{2}{\xi^4}\right)q + O(q^2).
$$

(5.36)

Since $\mathfrak{I}_2 = 48$, we should have

$$
F_0^{\text{cl}} = \log u \left( \log p \log q - 2(\log \xi)^2 \right) - 2 \log p (\log \xi)^2 + \frac{(\log q)^3}{3} - 4 \log q (\log \xi)^2 + 8(\log \xi)^3,
$$

(5.37)

and

$$
F_1^{\text{cl}} = -12 \log u - 12 \log p - 22 \log q + 48 \log \xi.
$$

(5.38)

Now assume that $a = 2$. We take $t_3$ and $t_4$ as in the previous example. Then we find

$$
\rho_4 = 88,
$$

(5.39)

and

$$
F_0^{\text{cl}} = t_1 \left( t_2 t_3 + t_3^2 + 2 t_2 t_4 + 4 t_3 t_4 + 2 t_4^2 \right)
+ t_2^2 t_4 + 2 t_2 t_3^2 + \frac{4}{3} t_3^3 + 2 t_4^2 t_4 + 8 t_2 t_3 t_4
+ 4 t_2 t_4^2 + 8 t_3^2 t_4 + 8 t_3 t_4^2 + \frac{8}{3} t_4^3.
$$

(5.40)
A candidate Calabi-Yau threefold is $X(8, 4, 2, 1, 1)_{4,148}$ whose $K3$ fiber $D_1$ is the degree 8 hypersurface in $\mathbb{P}(4, 2, 1, 1)$. This example should definitely be related to the proposal in [C]. The nearly holomorphic Weyl-invariant Jacobi form $\phi_{0,2}$ for this model coincides with the one considered in [GN2].

**Example 3.** Take $g = E_8$ and $m = 1$. Hence $P = Q = Q^\vee$. We consider the case $a = 12$ and take

$$
\phi_{-2,1}(\tau, z) = \frac{2E_6(\tau)\theta_{0,1}(\tau, z)}{\Delta(\tau)} = -\frac{2}{q} + 960 - 2 \sum_{(\gamma, \gamma) = 2} \zeta^\gamma + O(q),
$$

(5.41)

where $\theta_{0,1}(\tau, z) = \sum_{\gamma \in Q} q^{(\gamma, \gamma)} \zeta^\gamma$. This choice corresponds to the case studied in [HaM]. Since $I_2 = -480$, we can predict that

$$
\begin{align*}
F_{0}^{cl} &= \log u \left( \log p \log q - \frac{1}{2} (\log \zeta, \log \zeta) \right) \\
&+ 3 \log p \cdot (\log \zeta, \log \zeta) + \frac{1}{3} (\log q)^3 + \frac{5}{2} \log q \cdot (\log \zeta, \log \zeta) \\
&- \frac{1}{6} \sum_{(\gamma, \log \zeta) = 2} (\gamma, \log \zeta)^3,
\end{align*}
$$

(5.42)

and

$$
F_{1}^{cl} = -12 \log u - 12 \log p - 22 \log q - 2(\rho, \log \zeta),
$$

(5.43)

where $\rho$ is the Weyl vector of $E_8$. Since $P = Q$, we can take [HaM]

$$
t_3 = \log q - (\theta, \log \zeta), \quad t_{i+3} = (\alpha_i, \log \zeta), \quad (i = 1, \ldots, 8).
$$

(5.44)

With this choice we obtain that

$$
(\rho_4, \ldots, \rho_{11}) = (368, 548, 732, 1092, 900, 704, 504, 300).
$$

(5.45)

A candidate Calabi-Yau threefold is $X(42, 28, 12, 1, 1)_{11,493}$ as found in [HaM]. We remark that there are cousins of this threefold: $X(30, 20, 8, 1, 1)_{10,376}$ and $X(24, 16, 6, 1, 1)_{9,321}$. They are related respectively to $E_7$ and $E_6$. It would be interesting to find out the appropriate nearly holomorphic Weyl-invariant Jacobi forms for these Calabi-Yau threefolds.
6 Discussions

In this paper we have proposed the asymptotic behaviors of \( F_0 \) and \( F_1 \) for a class of Calabi-Yau threefolds simultaneously admitting a \( K3 \) fibration and an elliptic fibration. The key ingredients in this proposal were the functions \( \mathcal{L}^0_n(p, q, \zeta) \) and \( \mathcal{L}^1_n(p, q, \zeta) \) which were constructed by liftings. Notably in the limit where the base of the elliptic fibration decompactifies, \( F_0 \) and \( F_1 \) are described by the weighted sums of the elliptic polylogarithms of Beilinson and Levin. This should generally be true if the Calabi-Yau threefold is elliptic regardless of the existence of a \( K3 \) fibration structure. String duality is in a sense the search of a conformal field theory spectrum in some asymptotic region of the moduli space of one string theory. In this sense elliptic fibrations seem to be more inherent to string duality phenomena.

What about the asymptotic behavior of \( F_g \) \((g \geq 2)\)? It seems natural to presume that an essential piece of \( F_g \) \((g \geq 2)\) in the limit \( q_1 \to 0 \) is captured by the automorphic form of weight \( 2g - 2 \) [Gr,Bo1]:

\[
\mathcal{F}_g(p, q, \zeta) := \sum_{\ell=0}^{\infty} p^\ell (\phi_{2g-2,m}(V_\ell) (\tau, z), \quad (g \geq 2),
\]

where \( \phi_{2g-2,m} \in J^\text{nh}_{2g-2,m}(g) \) will be constructed from \( \phi_{-2,m} \) by appropriate applications of the differential operators \( D_{k,m} \) and multiplications of \( E_4 \) and \( E_6 \) so that the resulting expression has weight \( 2g - 2 \). This \( \phi_{2g-2,m} \) will have a single pole at \( q = 0 \) and (1.11) tells us that (6.1) scales as \( t_2^{2-2g} \) when \( t_2 \sim 0 \). In general (6.1) has poles of order \( 2g - 2 \) along rational quadratic divisors [Bo1,Th9.3]. This is an expected behavior of \( F_g \) [BCOV]. Note also that in contrast to the cases \( g = 0 \) and \( g = 1 \) there do not appear any powers of logarithms. This is consistent since there is no restriction on the number of marked points when considering the genus \( g \geq 2 \) Gromov-Witten invariants of Calabi-Yau threefolds. Using (1.10) it is easy to see that \( \mathcal{F}_g(p, q, \zeta) = \mathcal{F}_g(q, p, \zeta) \) for \( g \geq 2 \) which is part of the automorphic properties of \( \mathcal{F}_g(p, q, \zeta) \) and is consistent with the conjecture (5.12). However, we do not know how precisely \( \mathcal{F}_g \) is related to \( F_g \) at the moment. More detailed properties of \( \mathcal{F}_g \) are currently under study.

A most urgent problem would be to explain the appearance of the nearly holomorphic Weyl-invariant Jacobi form purely from the geometry of the Calabi-Yau threefold \( X \). A modest starting point would be to establish the part (5.18) in \( F_0 \).

We studied \( \mathcal{L}^r_{n}(p, q, \zeta) \) only for \( r = 1 \) and \( r = 3 \) since these are of direct relevance to the main object of this paper. However, the procedure should be general and in principle one may consider \( \mathcal{L}^r_{n}(p, q, \zeta) \) for larger values of \( r \).
This may be of independent interest though the connection to string theory is not so clear.

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