Massless scalar fields in 1+1 dimensions and Krein spaces

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Abstract

We consider the Krein realization of the Hilbert space for a massless scalar field in 1+1 dimensions. We find convergence criteria and the completion of the space of test functions $S$ with the topology induced by the Krein scalar product. Finally, we show that the interpretation for the Fourier components as probability amplitudes for the momentum operator is lost in this case.
1 Introduction

The massless scalar field in 1 + 1 dimensions is affected by the problem of indefiniteness of the metric [1]. One would like to represent the algebra of the scalar massless field on a Hilbert space by staying as close as possible to Wightman’s axioms. However, if the metric is not positive definite, one cannot construct the Hilbert space by completing the vector space obtained from the smeared field.

In this note we would like to focus on some of the difficulties one encounters in defining this apparently simple field theory.

Let us assume that a vacuum exists such that

\[ \langle 0 | \phi(f) | 0 \rangle = 0 \] (1)

and

\[ \langle g, f \rangle \equiv \langle 0 | \phi(g) \phi(f) | 0 \rangle = \frac{1}{4\pi} \int_{\mathcal{R}} \frac{dp}{|p|} \left[ g^*(|p|, p) f(|p|, p) - g^*(0, 0) f(0, 0) \theta(1 - |p|) \right] , \] (2)

where \( f, g \in S(\mathcal{R}^2) \). The two-point function gives the sesquilinear form, which in a good theory is the scalar product. Here, due to the subtraction, the induced metric is not positive definite.

This goal is not only a challenging problem: it is also related to some important issues of field theory. One is whether the negative norm allows an evasion of Coleman’s theorem and thus there is a spontaneous breakdown of the symmetry

\[ \phi(g) \to \phi(g) + \text{constant} \cdot g(0) \] (3)

(solid state physics has an equivalent theorem by Mermin and Wagner). Moreover during the discussions a few surprising facts appear. We hope that they are not paradigms for more realistic cases.

It has been suggested [2, 3] that a natural solution of this problem is in the use of the Krein metric [4]. We have reconsidered this problem and reached conclusions that are somewhat different from those of previous authors. We find that Stone’s theorem is evaded for the translation operator. In particular one loses the interpretation for the Fourier components as probability amplitudes for the momentum operator. Our conclusion is that one needs a new strategy.
2 Krein space

In this section we consider the vector space $F$ of functions $f(|p|, p), f \in \mathcal{S}(\mathbb{R}^2)$ where a sesquilinear form is given in eq. (2). Please notice that continuous functions cannot be used (counter example: take a function which behaves like $(\ln |p|)^{-1}$ near the origin, then the integral does not exists. The reason is that the sesquilinear form is a well defined distribution, but not a Radon measure. This fact is the origin of all the troubles which we will find).

Let us recall the definition of Krein space [4]. A vector space $F$ with a sesquilinear Hermitian form $Q$ defined on it is said to have a $Q$-metric

$$Q(f, g) \equiv \langle f, g \rangle, \quad f, g \in F,$$  \hspace{1cm} (4)

in general indefinite.

If this space furthermore admits a canonical decomposition

$$F = F^{(+)} [+] F^{(-)}$$  \hspace{1cm} (5)

where $F^{(+)}$ is orthogonal in the $Q$-metric to $F^{(-)}$ and both $F^{(+)}$ and $F^{(-)}$ are Hilbert spaces with norms

$$\|f\|^2 \equiv \langle f, f \rangle, \quad f \in F^{(+)}$$
$$\|f\|^2 \equiv -\langle f, f \rangle, \quad f \in F^{(-)}$$  \hspace{1cm} (6)

then $F$ is a Krein space.

**Canonical decomposition**

This section is devoted to finding the canonical decomposition of eq. (3). We state a few simple lemmas:

**Lemma 1**

If $f(0) = 0 \implies \langle f, f \rangle \geq 0$, where the case $\langle f, f \rangle = 0$ implies $f = 0$.

From this follows

**Lemma 2**

If $f \neq 0$ and $\langle f, f \rangle = 0 \implies f(0) \neq 0$.

**Lemma 3**

Let $f$ and $g$ be any two linearly independent functions. If $\langle f, f \rangle \leq 0$ and $\langle g, g \rangle \leq 0$, then $\langle f, g \rangle \neq 0$. 
Proof. Since $f \neq 0$ and $g \neq 0$ and their sesquilinear form is not positive, we get from Lemma 1 $f(0) \neq 0$ and $g(0) \neq 0$. Define

$$h \equiv g(0)f - f(0)g$$

then $h(0) = 0$ but $h \neq 0$. From Lemma 1 it follows that $\langle h, h \rangle > 0$. This conclusion is compatible only with $\langle f, g \rangle \neq 0$, otherwise

$$\langle h, h \rangle = |g(0)|^2\langle f, f \rangle + |f(0)|^2\langle g, g \rangle \leq 0.$$  

(8)

Now we can prove the following theorem (see Ref. [2])

Theorem 1

The vector space admits a canonical decomposition

$$\mathcal{F} = \mathcal{F}^{(+)}[+]\mathcal{F}^{(-)}.$$  

(9)

and $\mathcal{F}^{(-)}$ contains only one element.

Proof. First we prove that every vector subspace $\mathcal{F}^{(-)}$ contains only one linear independent element. In fact suppose that it contains more than one. Let $f, g \in \mathcal{F}^{(-)}$. Then from Lemma 1 we get $f(0) \neq 0$ and $g(0) \neq 0$. We define

$$h \equiv f(0)g - g(0)f,$$

then $h(0) = 0$, it follows that $\langle h, h \rangle > 0$ and therefore $h \notin \mathcal{F}^{(-)}$.

Consider an element of $\chi \in \mathcal{F}$ with

$$\langle \chi, \chi \rangle = -1$$

(11)

and define

$$f_+ = f + \langle \chi, f \rangle \chi$$

$$f_- = -\langle \chi, f \rangle \chi.$$  

(12)

Then, by construction,

$$\langle f_+, f_- \rangle = 0.$$  

(13)

Lemma 4

$$\langle f_+, f_+ \rangle \geq 0 \ (\langle f_+, f_+ \rangle = 0 \text{ only if } f_+ = 0).$$

Proof. Since $\langle \chi, f_+ \rangle = 0$ and $\langle \chi, \chi \rangle = -1$ it cannot be that $\langle f_+, f_+ \rangle \leq 0$. In fact if $\langle f_+, f_+ \rangle \leq 0$ one can apply Lemma 2 and get $\langle f_+, \chi \rangle \neq 0$ which is contrary to the assumption.
We prove now a kind of anti-Schwartz inequality, which is valid for any pair of functions with sesquilinear form less than or equal to zero (consequently they have to be non-zero in zero, otherwise the norm is strictly positive).

Lemma 5

Let \( f \) and \( g \) be two non-zero linearly independent continuous functions with non-positive sesquilinear form:

\[
\langle f, f \rangle \leq 0 \quad \text{and} \quad \langle g, g \rangle \leq 0
\]  

(14)

then

\[
|\langle f, g \rangle|^2 > \langle f, f \rangle \langle g, g \rangle.
\]

(15)

Proof. Let us first consider the case

\[
\langle f, f \rangle = \langle g, g \rangle = 0.
\]

(16)

Define

\[
h \equiv f(0)g - g(0)f.
\]

(17)

Since, by construction \( h(0) = 0 \), but \( h \neq 0 \) (\( f \) and \( g \) are linearly independent), then

\[
\langle h, h \rangle > 0
\]

(18)

and therefore

\[
- f^*(0)g(0)\langle g, f \rangle - f(0)g^*(0)\langle f, g \rangle > 0.
\]

(19)

This is possible only if

\[
|\langle f, g \rangle|^2 > 0.
\]

(20)

Now let \( \langle g, g \rangle < 0 \). We define

\[
k \equiv f - g \frac{\langle g, f \rangle}{\langle g, g \rangle}.
\]

(21)

By construction

\[
\langle g, k \rangle = 0 \quad \text{and} \quad k \neq 0.
\]

(22)

This is possible only if \( \langle k, k \rangle > 0 \) (see Lemma 2). Therefore

\[
\langle k, k \rangle = \langle f, f \rangle + \frac{|\langle g, f \rangle|^2}{\langle g, g \rangle} - 2\frac{|\langle f, g \rangle|^2}{\langle g, g \rangle} > 0.
\]

(23)
Finally
\[ |\langle f, g \rangle|^2 > \langle f, f \rangle \langle g, g \rangle. \quad (24) \]

**Krein metric**

We can now introduce a positive metric (Krein metric) by using the canonical decomposition provided in eq. (12).

\[ (f, g) \equiv \langle f_+, g_+ \rangle + \langle f, \chi \rangle \langle \chi, g \rangle. \quad (25) \]

Notice the following alternative form. Use

\[ f_+ = f + \langle \chi, f \rangle \chi \quad (26) \]

then

\[ \langle f_+, g_+ \rangle = \langle f, g \rangle + \langle \chi, g \rangle \langle f, \chi \rangle \quad (27) \]

and finally

\[ (f, g) = \langle f, g \rangle + 2 \langle f, \chi \rangle \langle \chi, g \rangle. \quad (28) \]

From the last two equations we have important inequalities:

\[ \langle f, f \rangle \geq -\langle f, \chi \rangle \langle \chi, f \rangle \quad (29) \]

and

\[ \langle f, f \rangle \geq -2 \langle f, \chi \rangle \langle \chi, f \rangle. \quad (30) \]

The norm induced by Krein’s scalar product also satisfies the triangle inequality.

The positive metric introduced in Ref. [3] differs from ours. See [4].

### 3 Convergence criteria

The problem is now to consider all possible Cauchy sequences in the Krein norm. We prove in this section that the Krein metric is not necessary for this goal. Let us introduce the notion of strong and weak convergence associated to the sesquilinear form:

**Weak convergence**: \( f_n \) is said to converge weakly to \( f \) if for any \( g \)

\[ \lim_{n \to \infty} \langle g, (f_n - f) \rangle = 0 \quad (31) \]
Strong convergence. \( f_n \) is said to converge strongly to \( f \) if

\[
\lim_{n \to \infty} \langle (f_n - f), (f_n - f) \rangle = 0. \tag{32}
\]

The surprising result is:

**Theorem 2**

Convergence in the Krein metric is equivalent to convergence in the sesquilinear form (if both weak and strong).

**Proof.** 1) Let \( f_n \) a Cauchy sequence in the Krein metric. We use the relation

\[
\langle f, g \rangle = (f, g) - 2\langle \chi, g \rangle \langle f, \chi \rangle. \tag{33}
\]

For \( g = \chi \) we get

\[
-\langle f, \chi \rangle = (f, \chi) \tag{34}
\]

and therefore by using the Schwartz inequality for the positive scalar product \((\cdot, \cdot)\) we find

\[
|\langle f, \chi \rangle| = |(f, \chi)| \leq \|f\|. \tag{35}
\]

Finally from eq. (33) we get

\[
|\langle f, g \rangle| \leq |(f, g)| + 2|\langle \chi, g \rangle| |\langle f, \chi \rangle| \leq 3\|f\|\|g\|. \tag{36}
\]

The last equation implies both weak and strong convergence in the sesquilinear form.

2) Let \( f_n \) be a Cauchy sequence in the weak and strong sense in the sesquilinear form. We use eq. (33)

\[
\langle [f_n - f_n'], [f_n - f_{n'}] \rangle = \langle [f_n - f_n'], [f_n - f_{n'}] \rangle + 2|\langle \chi, [f_n - f_{n'}] \rangle|^2
\]

\[
\leq |\langle [f_n - f_{n'}], [f_n - f_{n'}] \rangle| + 2|\langle \chi, [f_n - f_{n'}] \rangle|^2 \tag{37}
\]

Then \( f_n \) is a Cauchy sequence also in the Krein metric.

To conclude the section we stress that the completion of \( \mathcal{F} \) can be performed by using the indefinite metric.
4 Hilbert space

Now we consider all the Cauchy sequences in order to complete the space. It is useful to choose
\[ \chi(0) = 1 \]  
(38)
and to decompose
\[ \hat{f} \equiv f - f(0) \chi. \]
(39)

In Ref. [3] a sequence is proposed with very interesting properties. Here we use a similar one (its support is in \( p_0 > 0 \))
\[ v_n = \frac{4\pi}{\ln n} \tilde{\theta}(-1 + 2np_0)\chi(p) \]
(40)
where
\[ \tilde{\theta} \in S \]
\[ 0 \leq \tilde{\theta} \leq 1 \]
\[ \tilde{\theta}(x) = 0 \quad \forall x \leq 0 \]
\[ \tilde{\theta}(x) = 1 \quad \forall x \geq 1. \]
(41)

We denote by the same symbol \( v_n \) the element of \( \mathcal{F} \) given by \( v_n(|p|, p) \).

Lemma 6
\( v_n \) is a Cauchy sequence in \( \mathcal{F} \).

Moreover

Lemma 7
For any \( f \in \mathcal{F} \)
\[ \lim_{n \to \infty} \langle v_n, f \rangle = f(0). \]
(42)

Let us denote by \( v \) the formal element of the Hilbert space
\[ v \equiv \lim_{n \to \infty} v_n. \]
(43)

The existence of the Cauchy sequence \( v_n \) allows us to prove

Lemma 8
If \( \{f_n = \hat{f}_n + f_n(0)\chi\} \) is a Cauchy sequence then \( f_0 \equiv \lim_{n \to \infty} f_n(0) \) exists

Proof. Since both \( f_n \) and \( v_n \) are Cauchy sequences then also \( \langle v_n, f_n \rangle = f_n(0) \) converges.
Finally we have to characterize the limit of $\hat{f}_n$. Notice that

$$\langle \hat{f}_n, \hat{f}_n \rangle = \frac{1}{4\pi} \int_{\mathcal{R}} \frac{dp}{|p|} |\hat{f}_n|^2(p).$$

(44)

Therefore the limit of $\hat{f}_n$ is any function in $L_2(dp/p, \mathcal{R})$ for which the integral

$$\frac{1}{4\pi} \int_{\mathcal{R}} \frac{dp}{|p|} \chi^*(p) \hat{f}(p)$$

(45)

exists, i.e. $\hat{f} \in L(dp/p, K)$ where $K$ is some compact interval containing the point $p = 0$.

No other state should exists with support at $p = 0$ only, in fact the elements of $\mathcal{F}$ are not differentiable at this point.

5 Problems with the translations

There is a disturbing feature with the state $v$ given in eq. (43). Notice that at every point

$$\lim_{n \to \infty} v_n(p) = 0.$$  (46)

Thus, it is disturbing to find that the limit of $v_n$ in the Krein metric is non zero. Our uneasiness comes from the prejudice that the Fourier component at $p$ is (proportional to) the probability amplitude for a state to have momentum $p$. Thus if all components are zero we expect the vector to be zero. The paradox is resolved if we look once again at the distribution which defines the metric in eq. (2). Notice that

$$\lim_{n \to \infty} v_n(p) = 0$$  (47)

in the norm

$$||f|| = \sup_{x \in K} |f|$$

(48)

where $K \subset \mathcal{R}$ is compact. A Radon measure is the dual space of continuous functions in $\mathcal{R}$ with the above norm [6]. But eq. (12) tells us that the limit is non zero. Thus we have proven the following theorem

Theorem 3 The distribution

$$\frac{1}{4\pi} \int_{\mathcal{R}} \frac{dp}{|p|} \left[ e^{ipx} - \theta(1 - |p|) \right]$$

(49)
is not a Radon measure.

The theorem above excludes also the possibility of a difference of positive measures. On the other hand we expect the translation operator to be written in terms of projection operators on states of definite momentum (Stone’s theorem)

\[ U(a) = \int_{\mathbb{R}} e^{ipa} dE(p) \]  

and that therefore the Fourier transform of the two-point function should be a measure. Stone’s theorem is evaded here because \( U(a) \) is unitary only in the indefinite metric (in the Krein metric it is not unitary, see eq. (28)).

Finally we recall that the state \( v \) is invariant under translations \([3]\). This is an immediate consequence of eq. (12). Moreover the procedure can be repeated on the vector space of any number of particles and thus an infinite number of states invariant under translations can be constructed. In particular, since \( v_n \) has support only in the region \( p_0 > 0 \), one can define

\[ |v; k\rangle \equiv \lim_{n \to \infty} \frac{1}{\sqrt{k!}} [\phi(v_n)]^k |0\rangle. \]  

For \( k \neq 0 \), one gets easily

\[ \langle v; k, v; k \rangle = 0 \]  

and their Krein norm (we have assumed \( \chi(0) = 1 \))

\[ (v; k, |v; k\rangle = (2|\chi, v\rangle|^2)^k = (2|\chi(0)|^2)^k = 2^k. \]  

Finally one can construct (for any complex number \( \alpha \))

\[ |0_{\alpha}\rangle \equiv \lim_{N \to \infty} \sum_{k=0}^{N} \frac{i\alpha^k}{\sqrt{k!}} |v; k\rangle. \]  

with the properties

\[ \langle 0_{\beta}, 0_{\alpha}\rangle = 1 \]  

\[ (0_{\beta}, 0_{\alpha}) = e^{2\beta^* \alpha}. \]  

Since they are invariant under translations, we prefer to call them vacuums.
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