On certain subclass of univalent functions with finitely many fixed coefficients defined by Bessel function

R. Ezhilarasi and T.V. Sudharsan

Abstract
In this present investigation, we study a new class of functions that are analytic and Univalent with finitely many fixed coefficients defined by modified Hadamard product involving Bessel function. Further, we also establish coefficient condition, radii of starlikeness and convexity, extreme points and integral operators applied to functions in this class.

Keywords
Analytic; Starlike; Convex; Bessel Function.

AMS Subject Classification
30C45, 30C80.

1. Introduction
Let \( S \) denote the subclass of \( A \) consisting of functions which are analytic and univalent in the open disc
\[
D = \{ z : |z| < 1 \}.
\]
Let \( S^* \) denote the subclass of \( A \) where in addition the functions in \( S^* \) are also univalent in \( D \). The class \( SD(\alpha) \) was introduced in [8] and was recently considered in [12] that consists of functions of the form (1.1) satisfying the criteria
\[
\Re \left\{ \frac{f(z)}{z} \right\} \geq \alpha \left| f'(z) - \frac{f(z)}{z} \right|, \alpha \geq 0. \tag{1.2}
\]
New subclasses of \( S^* \) by fixing a finite number of coefficients of functions has been considered earlier by many authors (see [4, 5] for details). We also denote by \( T \) a subclass of \( S^* \) introduced and studied by Silverman [9], consisting of functions of the form
\[
f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n > 0; z \in D. \tag{1.3}
\]
Robertson [7] introduced the subclasses of \( A \), given by \( T^*(\beta) \) and \( C(\beta) \) respectively called as starlike functions of order \( \beta \) and convex functions of order \( \beta \) consisting of functions which satisfy the following inequalities:
\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \beta \quad \text{and} \quad \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \beta.
\]

The generalized Bessel function \( \omega_{u,b,c}(z) \) of the first kind of order \( u \) in terms of Euler gamma function is given by the representation
\[
\omega_{u,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n! \Gamma(u+n+\frac{b+1}{2})} \left( \frac{z}{2} \right)^{2n+u}, z \in \mathbb{C}. \tag{1.4}
\]
The function \( \varphi_{u,b,c}(z) \) defined by the transformation
\[
\varphi_{u,b,c}(z) = 2^u \Gamma(u+\frac{b+1}{2}) z^{1-\frac{c}{2}} \omega_{u,b,c}(\sqrt{z}), \tag{1.5}
\]
using the generalised Bessel function \( \omega_{u,b,c}(z) \) is studied by many researchers [2, 3]. Ramachandran et al.[6] obtained the...
following series representation for the function \( \varphi_{u,b,c}(z) \) given by (1.5)

\[
\varphi_{u,b,c}(z) = z + \sum_{n=1}^{\infty} \frac{(-c)^n}{4^n (\kappa)_n} z^{n+1}, z \in \mathbb{C}
\]  

(1.6)

where \( \kappa = u + \frac{b+1}{2} \) \( \notin \mathbb{Z}_0 \), \( N = \{1, 2, \cdots \} \), \( \mathbb{Z}_0 = \{0, -1, -2, \cdots \} \), and \( (\kappa)_n \) is the Pochhammer symbol given by

\[
(\kappa)_n = \begin{cases} 
1, & n = 0 \\
\kappa(\kappa+1)(\kappa+2) \cdots (\kappa+n-1), n \in \mathbb{N} 
\end{cases}
\]  

(1.7)

The Hadamard product or convolution of two functions \( f \) and \( g \) given by (1.1) and \( g \) defined as \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \) is defined by

\[
(f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.
\]  

(1.8)

For convenience, \( \varphi_{u,b,c}(z) \) is replaced by \( \varphi_{b,c}(z) \).

Ramachandran et al. [6] introduced an operator \( B^\ast_c : \mathcal{S} \to \mathcal{S} \) which is defined by the convolution

\[
B^\ast_c f(z) = \varphi_{b,c}(z) * f(z)
\]

\[
= z + \sum_{n=2}^{\infty} \frac{(-c)^{n-1}}{4^{n-1} (\kappa)_{n-1} (n-1)!} z^n
\]

\[
= z + \sum_{n=1}^{\infty} \frac{(-c)^{n-1}}{4^{n-1} (\kappa)_n (n-1)!} z^n
\]

(1.9)

The function \( B^\ast_c \) of the form (1.9) which is nothing but a transformation of the generalized hypergeometric function.

Also \( B^\ast_c f(z) = z_0 F_1 (\kappa, c \frac{z^2}{4} ; f(z)) \) and

\[
\varphi(\kappa, c \frac{z^2}{4}) = z_0 F_1 (\kappa, z)
\]

A class \( UB(\lambda, \eta, k, c) \) has been recently studied by Shanmugam et al. [10] and [11]. A function \( f \) of the form (1.3) is said to be in the class \( UB(\lambda, \eta, k, c) \) if

\[
\text{Re} \left\{ \frac{G''(z)}{G'(z)} \right\} > k \left| \frac{z G''(z)}{G'(z)} - 1 \right| + \eta,
\]

(1.10)

where \( c > 1, 0 \leq \lambda < 1, k \geq 0, 0 \leq \eta < 1, z \in \mathbb{D} \) and

\[
G(z) = (1 - \lambda)(B^\ast_c f(z)) + \lambda z (B^\ast_c f(z))'.
\]

Motivated by the work of Ramachandran et al. [6], here we consider and study the subclass \( BSD(\alpha, \lambda, c) \) with fixed finitely many coefficients defined by modified Hadamard product with Bessel function.

We begin with the definition of the class \( BSD(\alpha, \lambda, c) \).

**Definition 1.1.** Let \( c > 1, 0 \leq \lambda < 1, \alpha \geq 0 \). A function \( f \in \mathcal{S} \) is in \( BSD(\alpha, \lambda, c) \), if it satisfies the following inequality

\[
\text{Re} \left\{ \frac{G(z)}{z} \right\} \geq \alpha \left| \frac{G'(z) - G(z)}{z} \right|,
\]

(1.11)

where \( G(z) = (1 - \lambda)(B^\ast_c f(z)) + \lambda z (B^\ast_c f(z))' \).

Further, let \( T BSD(\alpha, \lambda, c) = \mathcal{S} \cap BSD(\alpha, \lambda, c) \).

**Theorem 1.2.** Let the function \( f \) be of the form (1.3). Then, \( f \in T BSD(\alpha, \lambda, c) \) if and only if \( \alpha \geq 0 \)

\[
\sum_{n=2}^{\infty} (1 + (n-1)\lambda)(1 + (n-1)\alpha) \delta(c, k, n) a_n |z^n| \leq 1.
\]

(1.12)

**Proof.** Let \( f \) of the form (1.3) satisfies (1.12). Then we have

\[
\text{Re} \left\{ \frac{G(z)}{z} \right\} - \alpha \left| \frac{G'(z) - G(z)}{z} \right|
\]

\[
= \left\{ \text{Re} \left\{ \frac{G(z)}{z} \right\} - 1 \right\} + \alpha \left| \frac{G'(z) - G(z)}{z} \right|
\]

\[
\geq 1 - \left| \frac{G(z)}{z} \right| - \alpha \left| \frac{G'(z) - G(z)}{z} \right|
\]

\[
= 1 - \sum_{n=2}^{\infty} (1 + (n-1)\lambda) \delta(c, k, n) a_n z^{n-1}
\]

\[
- \alpha \sum_{n=2}^{\infty} (1 + (n-1)\lambda) \delta(c, k, n) a_n z^{n-1}
\]

\[
= 1 - \sum_{n=2}^{\infty} (1 + (n-1)\lambda) D(c, k, n) a_n
\]

\[
- \alpha \sum_{n=2}^{\infty} (1 + (n-1)\lambda) D(c, k, n) a_n
\]

\[
\geq 0.
\]

Therefore, \( f \in T BSD(\alpha, \lambda, c) \). Conversely, let

\[
\text{Re} \left\{ \frac{G(z)}{z} \right\} - \alpha \left| \frac{G'(z) - G(z)}{z} \right| > 0.
\]

This implies,

\[
\text{Re} \left\{ 1 - \sum_{n=2}^{\infty} (1 + (n-1)\lambda) \delta(c, k, n) a_n z^{n-1} \right\}
\]

\[
- \alpha \sum_{n=2}^{\infty} (1 + (n-1)\lambda) \delta(c, k, n) a_n z^{n-1} > 0.
\]

If we allow \( z \) to take real values and as \( z \to 1 \), we get

\[
1 - \sum_{n=2}^{\infty} (1 + (n-1)\lambda) \delta(c, k, n) a_n
\]

\[
- \alpha \sum_{n=2}^{\infty} (1 + (n-1)\lambda) \delta(c, k, n) a_n \geq 0
\]
or

\[
\sum_{n=2}^{\infty} (1 + (n-1)\alpha)(1 + (n-1)\lambda)\delta(c, \kappa, n)|a_n| \leq 1.
\]

\[\square\]

Corollary 1.3. Let \( f \in TBSD(\alpha, \lambda, c) \). Then,

\[
a_n \leq \frac{1}{(1 + (n-1)\alpha)(1 + (n-1)\lambda)\delta(c, \kappa, n)}, n \geq 2. \tag{1.13}
\]

The subclass \( TBSD(\alpha, \lambda, c, p_k) \) of \( TBSD(\alpha, \lambda, c) \) consists of functions

\[
f(z) = z - \sum_{i=2}^{k} \frac{p_i}{(1 + (i-1)\alpha)(1 + (i-1)\lambda)\delta(c, \kappa, n)^i} z^i - \sum_{n=k+1}^{\infty} a_n z^n \tag{1.14}
\]

where \( \alpha \geq 0, 0 \leq p_i \leq 1 \) and \( 0 \leq \sum_{i=2}^{k} p_i \leq 1 \).

2. Main Results

We start with obtaining the coefficient bounds for the class \( TBSD(\alpha, \lambda, c, p_k) \) for functions \( f \) of the form (1.14).

Theorem 2.1. A function of the form (1.14) belongs to the class \( TBSD(\alpha, \lambda, c, p_k) \) if and only if

\[
\sum_{n=k+1}^{\infty} (1 + (n-1)\lambda)(1 + (n-1)\alpha)\delta(c, \kappa, n)a_n \leq 1 - \sum_{i=2}^{k} p_i \tag{2.1}
\]

where \( \alpha \geq 0, 0 \leq p_i \leq 1 \) and \( 0 \leq \sum_{i=2}^{k} p_i \leq 1 \). The result is sharp.

Proof. From (1.14), we have for \( i = 2, 3, \cdots , k \),

\[
a_i = \frac{p_i}{(1 + (i-1)\alpha)(1 + (i-1)\lambda)\delta(c, \kappa, i)}, \tag{2.2}
\]

\[0 \leq p_i \leq 1, \quad 0 \leq \sum_{i=2}^{k} p_i \leq 1.\]

By Theorem 1.2,

\[
\sum_{n=2}^{\infty} (1 + (n-1)\alpha)(1 + (n-1)\lambda)\delta(c, \kappa, n)a_n = \sum_{i=2}^{k} (1 + (i-1)\alpha)(1 + (i-1)\lambda)\delta(c, \kappa, i)a_i
\]

\[+ \sum_{n=k+1}^{\infty} (1 + (n-1)\alpha)(1 + (n-1)\lambda)\delta(c, \kappa, n)a_n \]

\[= \sum_{i=2}^{k} p_i + \sum_{n=k+1}^{\infty} (1 + (n-1)\alpha)(1 + (n-1)\lambda)\delta(c, \kappa, n)a_n \]

\[\leq 1. \]

Conversely,

\[
\Re \left\{ \frac{G(z)}{z} \right\} - \alpha \left| G'(z) - \frac{G(z)}{z} \right| \geq 1 - \left| \frac{G(z)}{z} - 1 - \alpha \left| G'(z) - \frac{G(z)}{z} \right| \right|
\]

\[= 1 - \sum_{n=2}^{\infty} (1 + (n-1)\alpha)(1 + (n-1)\lambda)\delta(c, \kappa, n)|a_n| - \alpha \sum_{n=2}^{\infty} (1 + (n-1)\alpha)(1 + (n-1)\lambda)\delta(c, \kappa, n)|a_n| \]

\[= 1 - \sum_{n=2}^{\infty} (1 + (n-1)\alpha)(1 + (n-1)\lambda)\delta(c, \kappa, n)|a_n| + \sum_{i=2}^{k} p_i - \sum_{n=k+1}^{\infty} (1 + (n-1)\alpha)(1 + (n-1)\lambda)\delta(c, \kappa, n)|a_n| \]

\[\geq 0. \]

Hence \( f \in TBSD(\alpha, \lambda, c, p_k) \). \( \square \)

Finally, it is observed that the inequality (2.1) of Theorem 2.1 is sharp and for \( n \geq 1 \), the extremal function is given by

\[
f(z) = z - \sum_{i=2}^{k} \frac{p_i}{(1 + (i-1)\alpha)(1 + (i-1)\lambda)\delta(c, \kappa, i)^i} z^i - \sum_{n=k+1}^{\infty} a_n z^n \tag{2.3}
\]

\[\text{Corollary 2.2. Let } f \in TBSD(\alpha, \lambda, c, p_k). \text{ Then,}
\]

\[\text{for } n \geq k+1,
\]

\[
a_n \leq \frac{1 - \sum_{i=2}^{k} p_i}{(1 + (n-1)\alpha)(1 + (n-1)\lambda)\delta(c, \kappa, n)^n}.
\]
Theorem 2.3. The class $T_{BSD}(\alpha, \lambda, c, p_k)$ is convex.

Proof. Let $f, g \in T_{BSD}(\alpha, \lambda, c, p_k)$. Then,

$$f(z) = z - \sum_{i=2}^{k} \frac{p_i}{(1 + (i - 1)\alpha)(1 + i\lambda)}(c, \kappa, \iota)z^i - \sum_{n=k+1}^{\infty} a_n z^n,$$

and

$$g(z) = z - \sum_{i=2}^{k} \frac{p_i}{(1 + (i - 1)\lambda)(1 + i\alpha)}(c, \kappa, \iota)z^i - \sum_{n=k+1}^{\infty} b_n z^n,$$

with $0 \leq p_i \leq 1$, $0 \leq \sum_{i=2}^{k} p_i \leq 1$.

Let us assume that $h(z) = \mu f(z) + (1 - \mu)g(z)$. Hence,

$$h(z) = z - \sum_{i=2}^{k} \frac{p_i}{(1 + (i - 1)\alpha)(1 + i\lambda)}(c, \kappa, \iota)z^i - \sum_{n=k+1}^{\infty} (\mu a_n + (1 - \mu)b_n)z^n.$$

Consider,

$$\sum_{n=k+1}^{\infty} (1 + (n - 1)\alpha)(1 + n\lambda)(c, \kappa, \iota)(\mu a_n + (1 - \mu)b_n)$$

$$= \mu \sum_{n=k+1}^{\infty} (1 + (n - 1)\alpha)(1 + n\lambda)(c, \kappa, \iota)a_n$$

$$+ (1 - \mu) \sum_{n=k+1}^{\infty} (1 + (n - 1)\alpha)(1 + n\lambda)(c, \kappa, \iota)b_n$$

$$\leq \mu \left(1 - \sum_{i=2}^{k} p_i\right) + (1 - \mu) \left(1 - \sum_{i=2}^{k} p_i\right)$$

$$= 1 - \sum_{i=2}^{k} p_i.$$

Therefore, $h(z) \in T_{BSD}(\alpha, \lambda, c, p_k)$.

Theorem 2.4. Let

$$f_k(z) = z - \sum_{i=2}^{k} \frac{p_i}{(1 + (i - 1)\lambda)(1 + i\alpha)}(c, \kappa, \iota)z^i,$$

and for $n \geq k + 1$, let

$$f_n(z) = z - \sum_{i=2}^{k} \frac{p_i}{(1 + (i - 1)\lambda)(1 + i\alpha)}(c, \kappa, \iota)z^i - \frac{\left(1 - \sum_{i=2}^{k} p_i\right)}{(1 + (n - 1)\lambda)(1 + (n - 1)\alpha)}(c, \kappa, \iota)z^n.$$

Then $f \in T_{BSD}(\alpha, \lambda, c, p_k)$ if and only if the function $f$ can be represented in the form

$$f(z) = \sum_{n=k}^{\infty} \lambda_n f_n(z),$$

where $\lambda_n \geq 0$, $(n \geq k)$ and $\sum_{n=k}^{\infty} \lambda_n = 1$.

Proof. Let $f \in T$ can be expressed in the form (2.6). Then

$$f(z) = z - \sum_{i=2}^{k} \frac{p_i}{(1 + (i - 1)\alpha)(1 + i\lambda)}(c, \kappa, \iota)z^i - \sum_{n=k+1}^{\infty} \frac{\lambda_n}{(1 + (n - 1)\lambda)(1 + (n - 1)\alpha)}(c, \kappa, \iota)z^n.$$

Now,

$$\sum_{n=k+1}^{\infty} (1 + (n - 1)\alpha)(1 + n\lambda)(c, \kappa, \iota)\lambda_n$$

$$\times \frac{\left(1 - \sum_{i=2}^{k} p_i\right)}{(1 + (n - 1)\alpha)(1 + n\lambda)(c, \kappa, \iota)}$$

$$= \sum_{n=k+1}^{\infty} \lambda_n \left(1 - \sum_{i=2}^{k} p_i\right)$$

$$= \left(1 - \sum_{i=2}^{k} p_i\right) \sum_{n=k+1}^{\infty} \lambda_n$$

$$= \left(1 - \sum_{i=2}^{k} p_i\right) (1 - \lambda_k)$$

$$\leq 1 - \sum_{i=2}^{k} p_i,$$

which implies $f \in T_{BSD}(\alpha, \lambda, c, p_k)$.

Conversely, for $n \geq k + 1$, let

$$\lambda_n = \frac{(1 + (n - 1)\alpha)(1 + n\lambda)(c, \kappa, \iota)}{\left(1 - \sum_{i=2}^{k} p_i\right)},$$

and $\lambda_k = 1 - \sum_{n=k+1}^{\infty} \lambda_n$.

Thus $f$ can be expressed as $f(z) = \sum_{n=k}^{\infty} \lambda_n f_n(z)$. 

Corollary 2.5. The extreme points of the class $T_{BSD}(\alpha, \lambda, c, p_k)$ are the functions $f_n$, $(n \geq k)$ given by (2.4) and (2.5).
maps the class of starlike functions onto the class of close to convex functions and is defined as

\[ I(f) = \int_0^z \frac{f(t)}{t} \, dt. \]

**Theorem 2.6.** Let \( f \) of the form (1.14) be in the class \( TBSD(\alpha, \lambda, c, p_k) \). Then \( I(f) \in TBSD(\alpha, \lambda, c, p_k) \) where \( q_k = \frac{p_k}{k} \).

**Proof.** First of all,

\[
I(f) = z - \sum_{i=2}^k \frac{q_i}{(1 + (i - 1)\alpha)(1 + (i - 1)\lambda)\beta(c, \kappa, i)z^i} - \sum_{n=k+1}^\infty \frac{a_n}{n} z^n.
\]

Now,

\[
\sum_{n=k+1}^\infty \frac{1}{(1 + (n - 1)\alpha)(1 + (n - 1)\lambda)\beta(c, \kappa, n)\frac{a_n}{n}} \leq \frac{1}{k+1} \sum_{n=k+1}^k \frac{1}{(1 + (n - 1)\alpha)(1 + (n - 1)\lambda)\beta(c, \kappa, n)a_n} \\
\leq \frac{1}{k+1} \left( 1 - \sum_{n=k+1}^k p_n \right) \\
= \frac{1}{k+1} - \frac{\sum_{n=k+1}^k p_n}{k+1} \\
\leq 1 - \frac{p}{k+1}.
\]

Hence \( I(f) \in TBSD(\alpha, \lambda, c, p_k) \).

Next, we obtain the radii results for the function in the class \( TBSD(\alpha, \lambda, c, p_k) \) to be starlike or convex of order \( \beta \). These results are stated in next two theorems.

**Theorem 2.7.** Let the function \( f \) given by (1.14) belongs to the class \( TBSD(\alpha, \lambda, c, p_k) \). Then \( f \in S^*(\beta) \) in the disk \( |z| < r_1 \), where \( r_1 \) is the largest value that satisfies

\[
\sum_{i=2}^k \left( \frac{(2-i)-\beta}{(1+(i-1)\alpha)(1+(i-1)\lambda)\beta(c, \kappa, i)}p_i i^{i-1} \right) \\
\left( (2-n)-\beta \right) \left( 1 - \sum_{i=2}^k p_i \right) \\
\leq 1 - \beta.
\]

**Proof.** To show the theorem, it is enough to establish that

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \beta \text{ for } |z| < r_1.
\]

Now, Upon simple computations, we have

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \beta \text{ for } |z| < r, \text{ if and only if}
\]

\[
\sum_{i=2}^k \left( \frac{(2-i)-\beta}{(1+(i-1)\alpha)(1+(i-1)\lambda)\beta(c, \kappa, i)}p_i i^{i-1} \right) \\
\left( (2-n)-\beta \right) \left( 1 - \sum_{i=2}^k p_i \right) \\
\leq 1 - \beta.
\]

By using Corollary 1.3, we get

\[
a_n = \frac{1 - \sum_{i=2}^k p_i}{(1 + (n - 1)\alpha)(1 + (n - 1)\lambda)\beta(c, \kappa, n)\lambda_n},
\]

where \( \lambda_n \geq 0, n \geq k + 1 \) and \( \sum_{n=k+1}^\infty \lambda_n \leq 1 \).

For each fixed \( r \), choosing an integer \( n_0 = n_0(r) \), for which

\[
(2-n) - \beta \left( 1 - \sum_{i=2}^k p_i \right) \\
\leq \frac{1}{(1 + (n_0 - 1)\lambda)(1 + (n_0 - 1)\alpha)D(c, \kappa, n_0)} \leq \beta.
\]

Hence \( f \) is starlike of order \( \beta \) in \( |z| < r_1 \), provided

\[
\sum_{i=2}^k \left( \frac{(2-i)-\beta}{(1+(i-1)\alpha)(1+(i-1)\lambda)\beta(c, \kappa, i)}p_i i^{i-1} \right) \\
\left( (2-n_0)-\beta \right) \left( 1 - \sum_{i=2}^k p_i \right) \\
\leq 1 - \beta.
\]

We find the value of \( r_0 \) and corresponding \( n_0(r_0) \), so that

\[
\sum_{i=2}^k \left( \frac{(2-i)-\beta}{(1+(i-1)\alpha)(1+(i-1)\lambda)\beta(c, \kappa, i)}p_i i^{i-1} \right) \\
\left( (2-n_0)-\beta \right) \left( 1 - \sum_{i=2}^k p_i \right) \\
\leq 1 - \beta.
\]

This is the radius of starlikeness of order \( \beta \) for functions in the class \( TBSD(\alpha, \lambda, c, p_k) \).
The radius of convexity for functions in the class $TBSD(\alpha, \lambda, c, p_k)$ is given in the next theorem.

**Theorem 2.8.** Let the function $f$ given by (1.14) belong to the class $TBSD(\alpha, \lambda, c, p_k)$. Then $f \in C(\beta)$ in $|z| < r_2$, where $r_2$ is the largest value that satisfies

$$
\sum_{i=2}^{\infty} \frac{i(i - \beta)p_i r^{i-1}}{(1 + (i - 1)\alpha)(1 + (i - 1)\lambda) \delta(c, \kappa, i)} + n(n - \beta) \left(1 - \sum_{i=2}^{k} p_i\right) r^{n-1} \leq 1 - \beta.
$$

(2.15)

**Proof.** Upon simple computations, for $|z| < r$,

$$\left|\frac{zf''(z)}{f(z)}\right| \leq 1 - \beta$$

if and only if

$$
\sum_{i=2}^{\infty} \frac{i(i - \beta)r^{i-1}}{(1 + (i - 1)\alpha)(1 + (i - 1)\lambda)} + \sum_{n=k+1}^{\infty} n(n - \beta) a_n r^{n-1} \leq 1 - \beta.
$$

(2.16)

By virtue of Corollary 1.3 and for each fixed $r$, choosing an integer $n_0 = n_0(r)$ for which $n_0(n_0 - \beta)r_0^{n_0-1} \leq (1 + (n_0 - 1)\alpha)(1 + (n_0 - 1)\lambda) \delta(c, \kappa, n_0)$ is maximum, we get

$$
\sum_{n=k+1}^{\infty} n(n - \beta) a_n r^{n-1} \leq \frac{n_0(n_0 - \beta) \left(1 - \sum_{i=2}^{k} p_i\right) r_0^{n_0-1}}{(1 + (n_0 - 1)\alpha)(1 + (n_0 - 1)\lambda) \delta(c, \kappa, n_0)}.
$$

(2.17)

Therefore, $f$ is convex of order $\beta$ in $|z| < r_2$, provided

$$
\sum_{i=2}^{\infty} \frac{i(i - \beta)r^{i-1}}{(1 + (i - 1)\alpha)(1 + (i - 1)\lambda) \delta(c, \kappa, i)} + n_0(n_0 - \beta) \left(1 - \sum_{i=2}^{k} p_i\right) r_0^{n_0-1} \leq 1 - \beta.
$$

(2.18)

We find the value of $n_0$ and corresponding $n_0(r_0)$, so that

$$
\sum_{i=2}^{\infty} \frac{i(i - \beta)r_0^{i-1}}{(1 + (i - 1)\alpha)(1 + (i - 1)\lambda) \delta(c, \kappa, i)} + n_0(n_0 - \beta) \left(1 - \sum_{i=2}^{k} p_i\right) r_0^{n_0-1} \leq 1 - \beta.
$$

This gives the radius of convexity of order $\beta$ for the functions $f$ in $TBSD(\alpha, \lambda, c, p_k)$.

\[\Box\]

**Acknowledgements**

The work of third-named author is supported by a grant from the Science and Engineering Research Board, Government of India under Mathematical Research Impact Centric Support of Department of Science and Technology (DST)(vide ref: MTR/2017/000607).

**References**

[1] J. W. Alexander, Functions which map the interior of the unit circle upon simple regions, *Ann. of Math.*, (2)17(1)(1915), 12–22.
[2] E. Deniz, Convexity of integral operators involving generalized Bessel functions, *Integral Transforms Spec. Funct.*, 24(3)(2013), 201–216.
[3] E. Deniz, H. Orhan and H. M. Srivastava, Some sufficient conditions for univalence of certain families of integral operators involving generalized Bessel functions, *Taiwanese J. Math.*, 15(2)(2011), 883–917.
[4] K. K. Dixit and I. B. Misra, A class of uniformly convex functions of order $\alpha$ with negative and fixed finitely many coefficients, *Indian J. Pure Appl. Math.*, 32(5)(2001), 711–716.
[5] S. Owa and H. M. Srivastava, A class of analytic functions with fixed finitely many coefficients, *J. Fac. Sci. Tech. Kinki Univ.*, 23(1987), 1–10.
[6] C. Ramachandran, K. Dhanalakshmi and L. Vanitha, Certain aspects of univalent functions with negative coefficients defined by Bessel function, *International Journal of Brazilian Archive of Biology and Technology*, Vol 59, e16161044, Jan - Dec - 2016.
[7] M. I. S. Robertson, On the theory of univalent functions, *Ann. of Math.*, (2)37(1936), 374–408.
[8] T. Rosy, *Studies on Subclasses of Starlike and Convex Functions*, Ph.D. Thesis, submitted to University of Madras, Chennai, India, 2001.
[9] H. Silverman, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.*, 51(1975), 109–116.
[10] T. N. Shanmugam, C. Ramachandran and R. A. Prabhu, Certain aspects of univalent functions with negative coefficients defined by Rafid operator, *Int. J. Math. Anal.*, 7(9-12)(2013), 499–509.
[11] H. M. Srivastava et al., A new subclass of $k$-uniformly convex functions with negative coefficients, *J. Inequal. Pure Appl. Math.*, 8(2)(2007), Article ID 43, 14 pp.
[12] S. Sunil Varma and T. Rosy, Certain properties of a sub-
On certain subclass of univalent functions with finitely many fixed coefficients defined by Bessel function —

class of univalent functions with finitely many fixed coefficients, *Khayyam J. Math.*, 3(1)(2017), 25–32.

**********
ISSN(P): 2319 – 3786
Malaya Journal of Matematik
ISSN(O): 2321 – 5666
**********