Chapter 1

Positive energy theorems in General Relativity

1.1 Introduction

The aim of this chapter is to present an introduction and also an overview of some of the most relevant results concerning positivity energy theorems in General Relativity. These theorems provide the answer to a long standing problem that has been proved remarkably difficult to solve. They constitute one of the major results in classical General Relativity and they uncover a deep self-consistency of the theory.

In this introductory section I would like to present the theorems in a complete form but with the least possible amount of technical details, in such a way that the reader can have a rough idea of the basic ingredients. The examples that illustrate the hypothesis of the theorems are discussed in the following sections.

An isolated system is an idealization in physics that assumes that the sources are confined to a finite region and the fields are weak far away from the sources. This kind of systems are expected to have finite total energy. In General Relativity there are several ways of defining isolated systems. For our purpose the most appropriate definition is through initial conditions for Einstein equations. The reasons for that are twofold. First, the notion of total energy has been discovered and formulated using a Hamiltonian formulation of the theory which involves the study of initial conditions. We refer the reader to the chapter of Domenico Giulini for this topic. Second, the proofs of the positive mass theorem are mainly given in terms of initial conditions. For a discussion of the initial value formulation of Einstein equations we refer to the chapter of James Isenberg.

Initial conditions for Einstein equations are characterized by initial data set given by \((S, h_{ij}, K_{ij}, \mu, j^i)\) where \(S\) is a connected 3-dimensional manifold, \(h_{ij}\) a (positive definite) Riemannian metric, \(K_{ij}\) a symmetric tensor field, \(\mu\) a scalar
field and $j^i$ a vector field on $S$, such that the constraint equations

$$D_j K^{ij} - D^i K = -8\pi j^i,$$  \hspace{1cm} (1.1)

$$R - K_{ij} K^{ij} + K^2 = 16\pi \mu,$$  \hspace{1cm} (1.2)

are satisfied on $S$. Here $D$ and $R$ are the Levi-Civita connection and scalar curvature associated with $h_{ij}$, and $K = K_{ij} h^{ij}$. In these equations the indices $i, k, \ldots$ are 3-dimensional indices, they are raised and lowered with the metric $h_{ij}$ and its inverse $h^{ij}$. The matter fields are assumed to satisfy the dominant energy condition

$$\mu \geq \sqrt{j^i j_i}.$$

(1.3)

The initial data model an isolated system if the fields are weak far away from sources. This physical idea is captured in the following definition of asymptotically flat initial data set. Let $B_R$ be a ball of finite radius $R$ in $\mathbb{R}^3$. The exterior region $U = \mathbb{R}^3 \setminus B_R$ is called an end. On $U$ we consider Cartesian coordinates $x^i$ with their associated euclidean radius $r = \left( \sum_{i=1}^3 (x^i)^2 \right)^{1/2}$ and let $\delta_{ij}$ be the euclidean metric components with respect to $x^i$. A 3-dimensional manifold $S$ is called Euclidean at infinity, if there exists a compact subset $\mathcal{K}$ of $S$ such that $S \setminus \mathcal{K}$ is the disjoint union of a finite number of ends $U_k$. The initial data set $(S, h_{ij}, K_{ij}, \mu, j^i)$ is called asymptotically flat if $S$ is Euclidean at infinity and at every end the metric $h_{ij}$ and the tensor $K_{ij}$ satisfy the following fall off conditions

$$h_{ij} = \delta_{ij} + \gamma_{ij}, \quad K_{ij} = O(r^{-2}),$$

where $\gamma_{ij} = O(r^{-1})$, $\partial_k \gamma_{ij} = O(r^{-2})$, $\partial_i \partial_k \gamma_{ij} = O(r^{-3})$ and $\partial_k K_{ij} = O(r^{-3})$. These conditions are written in terms of Cartesian coordinates $x^i$ attached at every end $U_k$. Here $\partial_i$ denotes partial derivatives with respect to these coordinates.

At first sight it could appear that the notion of asymptotically flat manifold with “multiple ends” $U_k$ is a bit artificial. Certainly, the most important case is when $S = \mathbb{R}^3$, for which this definition trivializes with $\mathcal{K} = B_R$ and only one end $U = \mathbb{R}^3 \setminus B_R$. Initial data for standard configurations of matter like stars or galaxies are modeled with $S = \mathbb{R}^3$. Also, gravitational collapse can be described with this kind of data. However, initial conditions with multiple ends and non-trivial interior $\mathcal{K}$ appear naturally in black hole initial data as we will see. In particular, the initial data for the Schwarzschild black hole has two asymptotic ends. On the other hand, this generalization does not imply any essential difficulty in the proofs of the theorems.

Only conditions on $h_{ij}$ and $K_{ij}$ are imposed in (1.4) and not on the matter fields $\mu$ and $j^i$, however since they are coupled by the constraint equations (1.1)–(1.2) the fall off conditions (1.4) impose also fall off conditions on $(\mu, j^i)$.

The fall off conditions (1.4) are far from being the minimal requirements for the validity of the theorem. This is a rather delicate issue that have important consequences in the definition of the energy. We will discuss this point in section
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We have chosen these particular fall off conditions because they are simple to present and they encompass a rich family of physical models.

For asymptotically flat initial data the total energy and linear momentum of the spacetime are defined as integrals over 2-spheres at infinity at every end by the following expressions

\[ E = \frac{1}{16\pi} \lim_{r \to \infty} \oint_{S_r} (\partial_j h_{ij} - \partial_i h_{jj}) s^i d\sigma, \]  
\[ P_i = \frac{1}{8\pi} \lim_{r \to \infty} \oint_{S_r} (K_{ik} - K h_{ik}) s^k d\sigma, \]

where \( s^i \) is its exterior unit normal and \( d\sigma \) is the surface element of the 2-sphere with respect to the euclidean metric. We emphasize that for every end \( U_k \) we have a corresponding energy and linear momentum \( E(k), P_i(k) \), which can have different values. We will discuss examples of that in section 1.2.

The quantities \( E \) and \( P_i \) are defined on the asymptotic ends and they depend only on the asymptotic behaviour of the fields \( h_{ij} \) and \( K_{ij} \). However, since \( h_{ij} \) and \( K_{ij} \) satisfy the constraint equations (1.1)–(1.2) and the dominant energy condition (1.3) holds these quantities carry in fact information of the whole initial conditions.

The energy \( E \) and the linear momentum \( P_i \) are components of a 4-vector \( P^a = (E, P_i) \) (indices \( a, b, c, \ldots \) are 4-dimensional). We will discuss this in section 1.3. The total mass of the spacetime is defined by

\[ M = \sqrt{E^2 - P_i P_j \delta^{ij}}. \]  

We have all the ingredients to present the positive energy theorem.

**Theorem 1.1.1 (Positive energy theorem).** Let \( (S, h_{ij}, K_{ij}, \mu, j^i) \) be an asymptotically flat (with possible many asymptotic ends), complete, initial data set, such that the dominant energy condition (1.3) holds. Then the energy and linear momentum \( (E, P_i) \) defined by (1.5)–(1.6) satisfies

\[ E \geq \sqrt{P_i P_j \delta^{ij}} \geq 0. \]  

at every end. Moreover, \( E = 0 \) at any end if and only if the initial data correspond to the Minkowski space-time.

The word “complete” means that \( (S, h_{ij}) \) as Riemannian manifold is complete. That is, no singularities are present on the initial conditions. But the space-time can be singular since singularities can developed from regular initial conditions, for example in the gravitational collapse. We will discuss that in more detail in section 1.2.

Note that Theorem 1.1.1 allows the vector \( P^a \) to be null and non trivial. However, it has been shown in [2] that if the energy momentum vector \( P^a \) is null then it vanishes identically.
One remarkable aspect of this theorem is that it is non-trivial even in the case where $S = \mathbb{R}^3$ and no matter fields $\mu = j^i = 0$ are present. This correspond to the positivity of the energy of the pure vacuum gravitational waves. We present explicit examples of this in section 1.2.

For spacetimes with black holes there are spacelike surfaces that touch the singularity. For that kind of initial conditions theorem 1.1.1 does not apply. Physically it is expected that it should be possible to prove a positivity energy theorem for black holes without assuming anything about what happens inside the black hole. That is, it should be possible to prove an extension of the positive energy theorem for initial conditions with inner boundaries if the boundary represents a black hole horizon. The following theorem deals precisely with that problem.

**Theorem 1.1.2** (Positive energy theorem with black hole inner boundaries).

Let $(S, h_{ij}, K_{ij})$ be an asymptotically flat, complete, initial data set, with $S = \mathbb{R}^3 \setminus B$, where $B$ is a ball. Assume that the dominant energy condition (1.3) holds and and that $\partial B$ is a black hole boundary. Then the energy momentum $E, P^i$ defined by (1.5)–(1.6) satisfies

$$E \geq \sqrt{P^i P_i} \geq 0.$$  \hspace{1cm} (1.9)

Moreover, $E = 0$ if and only if the initial data correspond to the Minkowski space-time.

We will explain what are black hole inner boundary conditions in section 1.2.

The plan of the chapter is the following. In section 1.2 we discuss the concept of the energy $E$ and we present examples that illustrate the hypothesis of the positive energy theorem. In section 1.3 we analyze the linear momentum $P_i$ and describe its transformation properties. In section 1.4 we review the main steps of the proof of theorems 1.1.1 and 1.1.2. Finally in section 1.5 other recent related results are discussed and the relevant current open problems are presented.

### 1.2 Energy

A remarkable feature of the asymptotic conditions (1.5) is that they imply that the total energy can be expressed exclusively in terms of the Riemannian metric $h_{ij}$ of the initial data (and the linear momentum in terms of $h_{ij}$ and the second fundamental form $K_{ij}$). Hence the notion of energy can be discussed in a pure Riemannian setting, without mention the second fundamental form. Moreover, as we will see, there is a natural corollary of the positive energy theorem for Riemannian manifolds. This corollary is relevant for several reasons. First, it provides a simpler and relevant setting to prove the positive energy theorem. Second, and more important, it has surprising applications in other areas of mathematics. Finally, to deal first with the Riemannian metric and then, in the next section, with the second fundamental form to incorporate
the linear momentum, reveal the different mathematical structures behind the energy concept.

In the previous section we have introduced the notion of an end $U$, the energy is defined in terms of Riemannian metrics on $U$. To emphasize this important point we isolate the notion of energy defined in the introduction in the following definition.

**Definition 1.2.1 (Energy).** Let $h_{ij}$ be a Riemannian metric on an end $U$ given in the coordinate system $x^i$ associated with $U$. The energy is defined by

$$E = \frac{1}{16\pi} \lim_{r \to \infty} \oint_{S_r} (\partial_j h_{ij} - \partial_i h_{jj}) s^i ds_0. \quad (1.10)$$

Note that in this definition there is no mention to the constraint equations (1.1)–(1.2). Also, the definition only involve an end $U$, there is no assumptions on the interior of the manifold.

In the literature it is custom to call $E$ the total mass and denote it by $m$ or $M$. In this article, in order to emphasize that $E$ is in fact the zero component of a four vector we prefer to call it energy and reserve the name mass to the quantity $M$ defined by (1.7). When the linear momentum is zero, both quantities coincides.

The definition of the total energy has three main ingredients: the end $U$, the coordinate system $x^i$ and the Riemannian metric $h_{ij}$. The metric is always assumed to be smooth on $U$, we will deal with singular metrics but these singularities will be in the interior region of the manifold and not on $U$.

There exists two potentials problems with the definition 1.2.1. The first one is that the integral (1.10) could be infinite. The second, and more subtle, problem is that the mass seems to depend on the particular coordinate system $x^i$. Both problems are related with fall off conditions for the metric. In the previous section we have introduced in equation (1.4) an example of this kind of conditions. These conditions are probably sufficient to model most physically relevant initial data. However, it is interesting to study the optimal fall off conditions that are necessary to have a well defined notion of energy and such that the energy is independent of the coordinate system.

To study this problem, we introduce first a general class of fall off conditions as follows. Given an end $U$ with coordinates $x^i$, and an arbitrary real number $\alpha$, we say that the metric $h_{ij}$ on $U$ is asymptotically flat of degree $\alpha$ if the components of the metric with respect to these coordinates have the following fall off in $U$ as $r \to \infty$

$$h_{ij} = \delta_{ij} + \gamma_{ij}, \quad (1.11)$$

with $\gamma_{ij} = O(r^{-\alpha})$, $\partial_k \gamma_{ij} = O(r^{-\alpha-1})$. The subtle point is to determine the appropriate $\alpha$ decay. To understand the meaning of this coefficient let us discuss the following relevant example given in [19] (see also [6]). Take the euclidean metric $\delta_{ij}$ in Cartesian coordinates $x^i$ and consider coordinates $y^i$ defined by

$$y^i = \frac{\rho}{r} x^i, \quad (1.12)$$
where $\rho$ is defined by
\[ r = \rho + c \rho^{1-\alpha}, \] (1.13)
for some constants $c$ and $\alpha$. Note that $\rho = \left( \sum_{i=1}^{3} (y^i)^2 \right)^{1/2}$. The components $g'_{ij}$ of the euclidean metric in coordinates $y^i$ have the following form
\[ g'_{ij} = \delta_{ij} + \gamma_{ij}, \] (1.14)
where $\gamma_{ij}$ satisfies the decay conditions (1.11) with the arbitrary $\alpha$ prescribed in the coordinate definition (1.13). That is, the metric in the new coordinate system $y^i$ is asymptotically flat of degree $\alpha$.

We calculate the energy in the coordinates $y^i$ using the definition (1.10). We obtain
\[ E = \begin{cases} 
\infty, & \alpha < 1/2, \\
\frac{c^2}{8}, & \alpha = 1/2, \\
0, & \alpha > 1/2.
\end{cases} \] (1.15)

Of course, we expect that the energy of the euclidean metric should be zero in any coordinate system. The interesting point of this example is the limit case $\alpha = 1/2$, the example shows that if the energy has any chance to be coordinate independent, then we should impose $\alpha > 1/2$. The following theorem, proved in [3] and [12], says that this condition is also sufficient.

**Theorem 1.2.2.** Let $U$ be an end with a Riemannian metric $h_{ij}$ such that it satisfies the fall off conditions (1.11) with $\alpha > 1/2$. Assume also that the scalar curvature $R$ is integrable in $U$, that is
\[ \int_U |R| \, dv < \infty. \] (1.16)

Then the energy defined by (1.10) is unique and it is finite.

In this theorem unique means if we calculate the energy in any coordinate system for which the metric satisfies the decay conditions (1.11) with $\alpha > 1/2$ we obtain the same result. This theorem ensure that the energy is a geometrical invariant of the Riemannian metric in the end $U$. Historically, this theorem was proved after the positive energy theorems. In the original proofs of the positive energy theorems different decay conditions for the metric have been used. The decay conditions are usually formulated in terms of integrals of derivatives (i.e. Sobolev spaces) (see [3]) which are more flexible for many applications. This particular formulation (which is simpler to present) of theorem 1.2.2 was taken from [14]. The decay conditions with $\alpha > 1/2$ together with the condition (1.16) on the scalar curvature are called mass decay conditions. The freedom in the coordinates $x^i$ is only a rigid motion at infinity (see [3]).

Theorem 1.2.2 completes the geometric characterization of the energy at the end $U$. We turn now to positivity. It is clear that the energy can have any sign.
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on $U$. The model example is given by the initial data for the Schwarzschild black hole, with metric on $U$ given by

$$h_{ij} = \psi^4 \delta_{ij},$$

(1.17)

where $\psi$ is the following function

$$\psi = 1 + \frac{C}{2r},$$

(1.18)

with $C$ an arbitrary constant. Computing the energy for this metric we obtain $E = C$. The constant $C$ can of course have any sign. It is however important to emphasize that theorem 1.2.2 asserts that the energy is well defined and it is an invariant of the geometry of the end even when it is negative.

To ensure the positivity of the energy we need to impose two important conditions. One is a local condition: the positivity of the local energy given by the dominant energy condition (1.3). The other is a global condition on the manifold: the manifold should be complete or should have black hole boundaries.

Initial conditions with

$$K_{ij} = 0,$$

(1.19)

are called time symmetric initial data. That is, time symmetric initial data are characterized only by a Riemannian metric $h_{ij}$. Conversely, any Riemannian metric can be interpreted as a time symmetric initial data. However, an arbitrary metric will not satisfy the dominant energy condition (1.3). In effect, inserting condition (1.19) in the constraint equation (1.2) and using the dominant energy condition (1.18) we obtain

$$R \geq 0.$$  

(1.20)

Only metrics that satisfy (1.20) can be interpreted as time symmetric initial data for which the dominant energy condition holds. But then, any metric such that (1.20) holds satisfies the dominant energy condition and it is a good candidate for the positive energy theorem. And hence we obtain the following corollary of theorem 1.1.1.

**Corollary 1.2.3** (Riemannian positive mass theorem). Let $(S, h_{ij})$ be a complete, asymptotically flat, Riemannian manifold. Assume that the scalar curvature is non-negative (i.e. condition (1.20)). Then the energy is non-negative at every end and it is zero at one end if and only if the metric is flat.

This corollary was proved with the optimal decay conditions for the metric in [3] and [26].

The interesting mathematical aspect of this corollary is that there is no mention to the constraint equations, the second fundamental form or the matter fields. This theorem is a result in pure Riemannian geometry. It has surprising applications in the solution of the Yamabe problem (see the review article [26] and reference therein).
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Note that it is not necessary to impose that the whole second fundamental form is zero to have (1.20), from equation (1.2) it is clear that is enough to have $K = 0$. This class of initial data are called maximal and they have important properties (see the chapter by J. Isenberg). In particular, positive energy theorems for this kind of data are easier to prove (mainly because condition (1.20) holds) than for general initial data.

Let us discuss some examples of corollary 1.2.3. We begin with the case with one asymptotic end and trivial topology, namely $S = \mathbb{R}^3$. For arbitrary functions $\psi$, metrics of the form (1.17) are called conformally flat, they provide a very rich family of initial conditions which have many interesting applications (for example, initial data for black hole collisions, see the review article [15]).

The scalar curvature for this class of metrics is given by

$$R = -8\psi^{-5}\Delta\psi,$$

(1.21)

where $\Delta$ is the euclidean Laplacian. If $\psi$ satisfies the fall off conditions

$$\psi = 1 + u, \quad u = O(r^{-1}), \quad \partial_k u = O(r^{-2}),$$

(1.22)

then the energy for this class of metric is given by

$$E = -\frac{1}{2\pi}\lim_{r \to \infty} \oint_{S_r} \partial_r \psi \, ds_0.$$

(1.23)

For $\psi$ given by (1.18) we obtain $R = 0$, and then the metric satisfies the local condition (1.20) for any choice of the constant $C$. However, this metric can not be extended to $\mathbb{R}^3$ since the function $\psi$ is singular at $r = 0$ and hence, as expected, corollary 1.2.3 does not apply to this case. Let us try to prescribe a function with the same decay (and hence identical energy) but such that it is regular at $r = 0$. For example

$$\psi = 1 + \frac{C}{2\sqrt{r^2 + C^2}}.$$  \hspace{1cm} (1.24)

Using (1.23) we obtain again that $E = C$. For any value of $C$ the function $\psi$ is strictly positive and bounded on $\mathbb{R}^3$ and hence the metric is smooth on $\mathbb{R}^3$. That is, it satisfies the completeness assumption in corollary 1.2.3. Using (1.21) we compute the scalar curvature

$$R = 12\psi^{-5}\frac{C^3}{(r^2 + C^2)^{5/2}}.$$  \hspace{1cm} (1.25)

We have $R \geq 0$ if and only if $C \geq 0$. Also, in this example the mass is zero if and only if the metric is flat.

Other interesting examples can be constructed with conformally flat metrics as follows. Let $\psi$ be a solution of the Poisson equation

$$\Delta\psi = -2\pi\tilde{\mu},$$

(1.26)
that satisfies the decay conditions (1.22), where \( \tilde{\mu} \) is a non-negative function of compact support in \( \mathbb{R}^3 \). Solution of (1.26) can be easily constructed using the Green function of the Laplacian. By equation (1.21), the scalar curvature of the associated conformal metric (1.17) will be non-negative and the function \( \tilde{\mu} \) is related to the matter density \( \mu \) by

\[
\mu = \frac{R}{16\pi} = \tilde{\mu}\psi^{-5}.
\]  

Note that we can not prescribe, in this example, exactly the matter density \( \mu \), we prescribe a conformal rescaling of \( \mu \). However, it is enough to control the support of \( \mu \). The support of \( \mu \) represents the localization of the matter sources. Outside the matter sources the scalar curvature (for time symmetric data) is zero.

For conformally flat metrics in \( \mathbb{R}^3 \) there is a very simple proof of corollary 1.2.3. We write equation (1.21) as

\[
\frac{R}{8} = -\partial^i \left( \frac{\partial_i \psi}{\psi^{5/2}} \right) - \frac{5}{\psi^6} |\partial \psi|^2.
\]  

Integrating this equation in \( \mathbb{R}^3 \), using for the first term in the right-hand side the Gauss theorem, the condition \( \psi \to 1 \) as \( r \to \infty \) and the expression (1.23) for the energy we finally obtain

\[
E = \frac{1}{2\pi} \int_{\mathbb{R}^3} \left( \frac{R}{8} + \frac{5}{\psi^6} |\partial \psi|^2 \right) dv_0,
\]  

where \( dv_0 \) is the flat volume element. This formula proves that for metric of the form (1.17) we have \( E \geq 0 \) if \( R \geq 0 \) and \( E = 0 \) if and only if \( h_{ij} = \delta_{ij} \). This proof easily generalize for conformally flat maximal initial data.

Asymptotically flat initial conditions in \( \mathbb{R}^3 \) with no matter sources (i.e. \( \mu = j^i = 0 \)) represent pure gravitational waves. They are conceptually important because they describe the dynamic of pure vacuum, independent of any matter model. Note that in that case the energy condition (1.3) is trivially satisfied.

In the previous examples the only solution with pure vacuum \( R = 0 \) in \( \mathbb{R}^3 \) is the flat metric, because by equation (1.21) we obtain \( \Delta \psi = 0 \) and the decay condition (1.22) implies \( \psi = 1 \). In order to construct pure waves initial data we allow for more general kind of conformal metrics, let \( h_{ij} \) be given by

\[
h = e^{\sigma} \left[ e^{-2q}(d\rho^2 + dz^2) + \rho^2 d\varphi^2 \right],
\]  

where \( (\rho, z, \varphi) \) are cylindrical coordinates in \( \mathbb{R}^3 \) and the functions \( q \) and \( \sigma \) depend only on \( (\rho, z) \). That is, the metric \( h_{ij} \) given by (1.30) is axially symmetric.

The scalar curvature of the metric (1.30) is given by

\[
-\frac{1}{8} Re^{\sigma-2q} = \frac{1}{4} \Delta \sigma + \frac{1}{16} |\partial \sigma|^2 - \frac{1}{4} \Delta_2 q,
\]  

where \( Re^{\sigma-2q} \) is the Ricci scalar of the metric (1.30) and \( \Delta \) is the Laplacian.
where $\Delta$, as before, is the 3-dimensional flat Laplacian and $\Delta_2$ is the 2-dimensional Laplacian in cylindrical coordinates given by

$$\Delta_2 q = \partial_{\rho}^2 q + \partial_z^2 q. \quad (1.32)$$

If we impose $R = 0$, equation (1.31) reduce to

$$\Delta \psi - \frac{1}{4} \Delta_2 q = 0, \quad (1.33)$$

where $\psi^4 = e^\sigma$. To construct metrics of the form (1.30) that satisfies $R = 0$ a function $q$ is prescribed and then the linear equation (1.33) is solved for $\psi$. The function $q$ can not be arbitrary, it should satisfy a global condition (which is related with the Yamabe problem mentioned above), see \[11\] for details. This kind of metric are called Brill waves, they have been used by D. Brill in one of the first proofs of the positive energy theorem \[8\]. Let us discuss this proof.

In order to be smooth at the axis the metric (1.30) should satisfies $q = 0$ at $\rho = 0$. For simplicity we also impose a strong fall off condition on $q$ at infinity, namely $q = O(r^{-2})$, $\partial_\rho q = O(r^{-2})$. For $\sigma$ we impose $\sigma = O(r^{-1})$ and $\partial_\rho \sigma = O(r^{-2})$. Using these decay assumptions is straightforward to check that the energy of the metric (1.30) is given by

$$E = -\frac{1}{8\pi} \lim_{r \to \infty} \oint_{S_r} \partial_\rho \sigma \, ds. \quad (1.34)$$

By Gauss theorem, using that $q = 0$ at the axis and the fall off condition of $q$ at infinity we obtain that

$$\int_{\mathbb{R}^3} \Delta_2 q \, dv_0 = 0. \quad (1.35)$$

Integrating equation (1.31) in $\mathbb{R}^3$, using (1.35) and using the expression (1.34) for the energy we obtain

$$E = \frac{1}{8\pi} \int_{\mathbb{R}^3} \left( \frac{1}{2} |\partial \sigma|^2 + Re^{-2q} \right) \, dv_0. \quad (1.36)$$

That is, $R \geq 0$ implies $E \geq 0$. In particular for vacuum $R = 0$, we have

$$E = \frac{1}{16\pi} \int_{\mathbb{R}^3} |\partial \sigma|^2 \, dv_0. \quad (1.37)$$

This positivity proof can be extended in many ways, in particular it has applications for the inequality between energy and angular momentum discussed in section 1.5 (see the review article \[17\] and the lectures notes \[13\], \[14\], and reference therein).

We turn now to manifolds with many asymptotic flat ends and interior $\mathcal{K}$ with non-trivial topology defined in section 1.1. Let us present some basic example of the definition of asymptotic euclidean manifold, without mention the metric.
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Take out a point in $\mathbb{R}^3$, the manifold $S = \mathbb{R}^3 \setminus \{0\}$ is asymptotic Euclidean with two ends, which we denote by $U_0$ and $U_1$. In effect, let $B_2$ and $B_1$ be two balls centered at the origin with radius 2 and 1 respectively. Define $K$ be the annulus centered at the origin $B_2 \setminus B_1$. Then $S \setminus K$ has two components $U_0$ and $U_1$, where $U_0 = \mathbb{R}^3 \setminus B_2$ and $U_1 = B_1 \setminus \{0\}$. The set $U_0$ is clearly an end. The set $U_1$ is also an end since the a ball minus a point is diffeomorphic to $\mathbb{R}^3$ minus a ball. This can be explicitly seen using Cartesian coordinates centered at the origin $x^i$, then the map given by the inversion

$$y^i = r^{-2}x^i,$$  \hspace{1cm} (1.38)

provide the diffeomorphism between $\mathbb{R}^3 \setminus B_1$ and $B_1 \setminus \{0\}$.

In the same way $\mathbb{R}^3$ minus a finite number $N$ of points $i_k$ is a Euclidean manifold with $N + 1$ ends. For each $i_k$ take a small ball $B_k$ of radius $r_{i(k)}$, centered at $i_k$, where $r_{i(k)}$ is small enough such that $B_k$ does not contain any other $i_{k'}$ with $k' \neq k$. Take $B_R$, with large $R$, such that $B_R$ contains all points $i_k$. The compact set $K$ is given by $K = B_R \setminus \sum_{k=1}^{N} B_k$ and the open sets $U_k$ are given by $B_k \setminus i_k$, for $1 \leq k \leq N$, and $U_0$ is given by $\mathbb{R}^3 \setminus B_R$.

Another example is a torus $T^3$ minus a point $i_0$. Take a small ball $B$ centered at $i_0$. Then the manifold is asymptotic euclidean with $K = T^3 \setminus B$ and only one end $U = B \setminus i_0$. This is an example of an Euclidean manifold with one asymptotic end but non-trivial $K$. More generally, given any compact manifold, if we subtract a finite number of points we get an asymptotically Euclidean manifold with multiple ends. Note that the topology of the compact core $K$ can be very complicated.

Let us consider now Riemannian metrics on these asymptotic euclidian manifolds. Consider the manifold $S = \mathbb{R}^3 \setminus \{0\}$ and the metric given by (1.17) and (1.18). The function $\psi$ is smooth on $S$ for any value of the constant $C$, however if $C < 0$ then $\psi$ vanished at $r = -2/C$ and hence the metric is not defined at those points. That is, the metric $h_{ij}$ is smooth on $S$ only when $C \geq 0$. We have seen that $S$ has two asymptotic ends, let us check that the metric $h_{ij}$ is asymptotically flat (i.e. it satisfies the decay conditions (1.4)) at both ends $U_0$ and $U_1$. On $U_0$, the metric in the coordinates $x^i$ is clearly asymptotically flat. But note that in this coordinates the metric is not asymptotically flat at the end $U_1$ (which, in these coordinates is represented by a neighborhood of $r = 0$), in fact the components of the metric are singular at $r = 0$. However, using a coordinate an inversion of coordinates like (1.38) is straightforward to prove that the metric is asymptotically flat also at $r = 0$. More precisely, consider the coordinate transformation

$$y^i = \left(\frac{C}{2}\right)^2 \frac{1}{r^2}x^i, \quad \rho = \left(\frac{C}{2}\right)^2 \frac{1}{r}. \hspace{1cm} (1.39)$$

In terms of this coordinates the metric has the form

$$h'_{ij} = \left(1 + \frac{C}{2\rho}\right)^4 \delta_{ij}. \hspace{1cm} (1.40)$$
We have chosen the constant factor in the coordinate transformation (1.40) in such a way that the transformation is in fact the well known isometry of this metric, this choice is however not essential. The metric (1.40) is clearly asymptotically flat at the $U_1$. Note that we have two energies, one for each end, the two are equal and given by the constant $C$. In this example the positivity of the mass is enforced purely by the global requirement of completeness of the metric (the energy condition is satisfied for arbitrary $C$). It is this condition that fails when $C < 0$. In that case the metric is defined on a manifold with boundary $S = \mathbb{R}^3 \setminus B_{-2/C}$, and the metric vanished at the boundary $\partial B_{-2/C}$. In particular, the 2-surface $\partial B_{-2/C}$ has zero area. This motivated the concept of “zero area singularities” introduced in [7], where interesting results are presented concerning negative energy defined on this class of singular metrics.

In the previous example the energies at the different ends are equal. It is straightforward to construct an example for which the two energies are different. Consider the following function

$$\psi = 1 + \frac{C}{2r} + g,$$  \hspace{1cm} (1.41)

where $g$ is a smooth function on $\mathbb{R}^3$ such that $g = O(r^{-2})$ as $r \to \infty$ and $g(0) = a$. Making the same calculation we get that the energy at one end is $E_0 = C$ (here we use the decay conditions on $g$, otherwise the function $g$ will contribute to the energy at that end). But at the other end the components of the metric in the coordinates $y^i$ are given by

$$h'_{ij} = \left(1 + \frac{C(1+g)}{2r}\right)^4 \delta_{ij},$$  \hspace{1cm} (1.42)

and hence we have that

$$E_1 = C(1+a).$$  \hspace{1cm} (1.43)

Note that in order to satisfy the energy condition (1.20) $g$ (and hence $a$) can not be arbitrary, we must impose the following condition on $g$

$$\Delta g \leq 0.$$  \hspace{1cm} (1.44)

Using (1.44), the decay assumption on $g$ and the maximum principle for the Laplacian (see, for example, the version of the maximum principle in the appendix of [13]) it is easy to prove that $g \geq 0$ and then $a \geq 0$.

Consider the manifold $S = \mathbb{R}^3 \setminus \{i_1\}, \{i_2\}$ with three asymptotic ends. And consider the function given by (this nice example was constructed in [10])

$$\psi = 1 + \frac{C_1}{2r_1} + \frac{C_2}{2r_2},$$  \hspace{1cm} (1.45)

where $r_1$ and $r_2$ are the euclidean radius centered at the points $i_1$ and $i_2$ respectively, and $C_1$ and $C_2$ are constant. Note that $\Delta \psi = 0$ and hence the metric defined by (1.17) has $R = 0$. As before, only when $C_1, C_2 \geq 0$ the metric is
smooth on $S$. Also, using a similar calculation as in the case of two ends it is not difficult to check that the metric is asymptotically flat on the three ends. Moreover, the energies of the different ends are given by

$$
E_0 = C_1 + C_2, \quad E_1 = C_1 + \frac{C_1 C_2}{L}, \quad E_2 = C_2 + \frac{C_1 C_2}{L},
$$

(1.46)

where $L$ be the euclidean distance between $i_1$ and $i_2$. We see that they are all positive and, in general, different. These initial conditions model a head on collision of two black holes and they have been extensively used in numerical simulations of black hole collisions (see for example [1] and reference therein). We analyze the case of the wormhole (see [29]), which is an example of a $K$ with more complicated topology. Consider the metric on the compact manifold $S = S^1 \times S^2$ given by

$$
\gamma = d\mu^2 + (d\theta^2 + \sin^2 \theta d\varphi^2),
$$

(1.47)

where the coordinates ranges are $-\pi < \mu \leq \pi$, and the sections $\mu = \text{const}$ are 2-spheres. Let $h_{ij}$ be given by

$$
h_{ij} = \psi^4 \gamma_{ij}
$$

(1.48)

where the function $\psi$ is

$$
\psi = \sum_{n=-\infty}^{\infty} \left[ \cosh(\mu + 2n\pi) \right]^{-1/2}.
$$

(1.49)

This function blows up at $\mu = 0$. Hence, the metric $h_{ij}$ is defined on $S$ minus the point $\mu = 0$. We have seen that this is an asymptotic euclidean manifold with one asymptotic end. It can be proved that the metric is asymptotically flat at that end (see [29] for details). Also, the function $\psi$ is chosen in such a way that the scalar of $h_{ij}$ curvature vanished. Moreover, the energy is given by

$$
E = 4 \sum_{n=1}^{\infty} (\sinh(n\pi))^{-1},
$$

(1.50)

which is positive.

Finally, consider the initial data for the Reissner-Nördstrom black hole given by a metric of the form (1.17) with $\psi$ given by

$$
\psi = \frac{1}{2r} \sqrt{(q + 2r + C)(-q + 2r + C)},
$$

(1.51)

where $C$ and $q$ are constant. The scalar curvature of this metric is given by

$$
R = \frac{2q^2}{\psi^6 r^4}.
$$

(1.52)

Which is non-negative for any value of the constants. When $C > |q|$, then the metric is asymptotically flat with two ends $U_0$ and $U_1$ as in the example (1.18).
The energy on both ends is given by \( E = C \). The positive energy theorem applies to this case. If \( C < |q| \) then the metric is singular, there is only one end \( U_0 \) and the energy on that end is given by \( C \). Note that in this case it is still possible to have positive energy \( 0 < C < |q| \), but the positive energy theorem does not apply because is a singular metric. The borderline case \( C = |q| \) represent the extreme black hole. The manifold is \( \mathbb{R}^3 \) minus a point and the metric is smooth on that manifold. However the metric is asymptotically flat only at the end \( U_0 \), on the other end is asymptotically cylindrical. And hence this version of the positive energy theorem does not apply for these data. The asymptotically cylindrical end is a feature of all extreme black holes. For discussions on this kind of geometry see [17] and reference therein.

So far, we have discussed complete manifolds without boundaries or manifolds with boundaries in which the metric is singular at the boundaries. We analyze now the important case of black hole boundaries.

Black hole boundaries are defined in terms of marginally trapped surfaces. A marginally trapped surface is a closed 2-surface such that the outgoing null expansion \( \Theta^+ \) vanishes (more details on this important concept can be seen in [40]). If such surface is embedded on a space-like 3-dimensional surface, then the expansion \( \Theta^+ \) can be written in terms of the initial conditions as follows

\[
\Theta^+ = H - K_{ij} s^i s^j + K, \tag{1.53}
\]

where

\[
H = D_i s^i, \tag{1.54}
\]

is the mean curvature of the surface. Here \( s^i \) is the unit normal vector to the surface. For time symmetric initial data, condition \( \Theta^+ = 0 \) reduces to

\[
H = 0. \tag{1.55}
\]

Surfaces that satisfies condition (1.55) are called minimal surfaces, because (1.55) is satisfied if and only if the first variation of the area of the surface vanishes. These kind of surfaces have been extensively studied in Riemannian geometry (see the book [31] for an introduction to the subject). We have seen that a marginally trapped surface on a time symmetric initial data is a minimal surface. That is, black hole boundaries translate, for these kind of data, into a pure Riemannian boundary condition. Then, we have the following corollary of theorem 1.1.2.

**Corollary 1.2.4 (Black holes in Riemannian geometry).** Let \( (S, h_{ij}) \) be a complete, asymptotically flat, Riemannian manifold with compact boundary. Assume that the scalar curvature is non-negative (i.e. condition (1.20)) and that the boundary is a minimal surface (i.e. it satisfies (1.55)). Then the energy is non-negative and it is zero at one end if an only if the metric is flat.

Let us give a very simple example that illustrate this theorem. Consider the function \( \psi \) given by (1.18). It is well known that the surface \( r = C/2 \) is a minimal surface (it represents the intersection of the Schwarzschild black
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hole event horizon with the spacelike surface $t = constant$ in Schwarzschild coordinates. To verify, that we compute $H$ for the 2-surfaces $r = constant$ for the metric (1.17). The unit normal vector is given by

$$s^i = \psi^{-2} \left( \frac{\partial}{\partial r} \right)^i.$$  \hspace{1cm} (1.56)

Then we have

$$H = D_i s^i = \frac{4}{\psi^3} \left( \partial_r \psi + \frac{\psi}{2r} \right).$$  \hspace{1cm} (1.57)

Then, condition (1.55) is equivalent to

$$0 = \partial_r \psi + \frac{\psi}{2r} = \frac{1}{2r} - \frac{C}{4r^2},$$  \hspace{1cm} (1.58)

and hence for $r = C/2$ we have a minimal surface. Note that $C$ must be positive in order to have a minimal surface. Previously we have discussed this example in the complete manifold, without boundaries, $\mathbb{R}^3 \setminus \{0\}$. In that case corollary 1.2.3 applies. We can also consider the same metric but in the manifold with boundary $R \setminus B_{C/2}$. Since we have seen that $\partial B_{C/2}$ is a minimal surface, then corollary 1.2.4 applies to that case. To emphasize the scope of this corollary, we slightly extend this example in the following form. Consider $\psi$ given by

$$\psi = (1 + \frac{C}{2r}) \chi(r),$$  \hspace{1cm} (1.59)

where $\chi(r)$ is a function such that it is $\chi = 1$ for $r > C/2$ and arbitrary for $r < C/2$. Corollary 1.2.4 applies to this case since again the boundary is a minimal surface. Note that inside the minimal surface the function $\chi$ is arbitrary, in particular it can blows up and it does not need to satisfies the energy condition. The corollary 1.2.3 certainly does not apply to this case.

1.3 Linear momentum

The total mass $M$ defined by (1.7) in terms of the energy and linear momentum (1.5)–(1.6) represents the total amount of energy of the space-time. The first basic question we need to address is in what sense $M$ is independent of the choice of initial conditions that describe the same space-time. That is, given a fixed space-time we can take different space-like surfaces on it, on each surface we can calculate the initial data set and hence we have a corresponding $M$, do we get the same result? We will see that the answer of that question strongly depend on the fall off conditions (1.4).

To illustrate that, let us consider the Schwarzschild space-time. We recall that in the following examples the space-time is fixed and we only chose different space-like surfaces on it. The space-time metric is given in Schwarzschild coordinates $(t, r_s, \theta, \phi)$ by

$$ds^2 = - \left( 1 - \frac{2C}{r_s} \right) dt^2 + \left( 1 - \frac{2C}{r_s} \right)^{-1} dr^2_s + r^2_s (d\theta^2 + \sin^2 \theta d\phi^2).$$  \hspace{1cm} (1.60)
These coordinates are singular at \( r_s = 2C \) and hence they do not reveal the global structure of the surfaces \( t = \text{constant} \). The most direct way to see that these surfaces are complete 3-dimensional manifolds is using the isotropical radius \( r_s \) defined by

\[
    r_s = r \left( 1 + \frac{C}{2r} \right)^2 .
\]  

(1.61)

In isotropic coordinates the line element is given by

\[
    ds^2 = \left( \frac{1 - \frac{C}{2r}}{1 + \frac{C}{2r}} \right)^2 dt^2 + \left( 1 + \frac{C}{2r} \right)^4 \left( dr^2 + d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right).
\]  

(1.62)

The initial data on the slice \( t = \text{constant} \) are given by

\[
    h_{ij} = \left( 1 + \frac{C}{2r} \right)^4 \delta_{ij}, \quad K_{ij} = 0.
\]  

(1.63)

These are the time symmetric initial data studied in section 1.2. The linear momentum of these data is obviously zero, then the total mass \( M \) is equal to the energy \( E \) calculated in the previous section and we obtain the expected result \( M = C \).

We take another foliation of space-like surfaces. We write the metric (1.62) in the Gullstrand – Painlevé coordinates \((t_{gp}, r_s, \theta, \phi)\) (see [28] and reference therein). We obtain

\[
    ds^2 = -\left( 1 - \frac{2C}{r_s} \right) dt_{gp}^2 + 2 \sqrt{2C} dr_{gp} dr_s + dr_s^2 + r_s^2 d\theta^2 + r_s^2 \sin^2 \theta d\phi^2.
\]  

(1.64)

The slices \( t_{gp} = \text{constant} \) in these coordinates have the following initial data

\[
    h_{ij} = \delta_{ij}, \quad K_{ij} = \frac{\sqrt{2m}}{r_s^{3/2}} \left( \delta_{ij} - \frac{3}{2} s_i s_j \right),
\]  

(1.65)

where \( s^i \) is the radial unit normal vector with respect to the flat metric \( \delta_{ij} \). We see that the intrinsic metric is flat and hence the energy \( E \) is clearly zero. The linear momentum is also zero, because if we calculate the integral (1.6) at an sphere of finite radius (note that the limit is in danger to diverge because the radial dependence of \( K_{ij} \) in (1.65) the angular variables integrate to zero. And hence we obtain that for these surfaces the total mass \( M \) is zero. What happens is that the second fundamental form (1.65) does not satisfy the decay condition (1.4) since it falls off like \( O(r^{-3/2}) \). It can be proved that any initial conditions that satisfy (1.4) in the same space-time give the same total mass \( M \).

We consider another foliation which reveal the Lorentz transformation properties of \((E, P^i)\). Let \((x, y, z)\) be the associated Cartesian coordinates of the isotropical coordinates \((r, \theta, \phi)\), that is

\[
    x = r \cos \phi \sin \theta, \quad y = r \sin \phi \cos \theta, \quad z = r \cos \theta.
\]  

(1.66)
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We consider the line element (1.62) written in terms of the coordinates \((t, x, y, z)\) and we perform the following change of coordinates which represents a boost in the \(z\) direction

\[
\hat{t} = \gamma^{-1}(t - vz), \quad (1.67)
\]

\[
\hat{z} = \gamma^{-1}(-vt + \hat{z}), \quad (1.68)
\]

\[
\hat{x} = x, \quad (1.69)
\]

\[
\hat{y} = y, \quad (1.70)
\]

where \(v\) is a constant and \(\gamma = \sqrt{1 - v^2}\). Consider a surface \(\hat{t} = \text{constant}\) in these coordinates. The intrinsic metric is given by

\[
h = \psi^4(d\hat{x}^2 + d\hat{y}^2) + \gamma^{-2}(-N^2v^2 + \psi^4)d\hat{z}^2, \quad (1.71)
\]

where

\[
\psi = 1 + \frac{C}{2r}, \quad N = \frac{1 - \frac{C}{2r}}{1 + \frac{C}{2r}}. \quad (1.72)
\]

The radius \(r\) can be written in terms of the hat coordinates as follows

\[
r = \sqrt{x^2 + y^2 + z^2} = \sqrt{\hat{x}^2 + \hat{y}^2 + \gamma^{-2}(v\hat{t} + \hat{z})^2}. \quad (1.73)
\]

One can check that the metric \(h\) given by (1.71) is asymptotically flat in the coordinates \((\hat{x}, \hat{y}, \hat{z})\). Then, we can compute the energy of this metric and we obtain

\[
E = \gamma^{-1}C. \quad (1.74)
\]

To obtain the linear momentum we need to compute the second fundamental form of the slice. The calculations are long (see, for example, [20] for details), the final result is the following

\[
P_x = 0, \quad P_y = 0, \quad P_z = vC\gamma^{-1}. \quad (1.75)
\]

Using (1.74) and (1.75) we obtain

\[
M = \sqrt{E^2 - P^iP^j\delta_{ij}} = C. \quad (1.76)
\]

That is, the quantities \(E, P^i\) transform like a 4-vector under asymptotic Lorentz transformations of coordinates.

1.4 Proof

In section 1.2 we have presented two proofs of the positive energy theorem for two particular cases, for other proofs that applies to other relevant particular cases (like spherical symmetry and weak field limit) see [9], [14], [13] and references therein.
The first general proof of the positive energy theorem was done by Schoen and Yau [37]. Shortly after it was followed by a proof by Witten [41] using completely different methods. The proof of the Penrose inequality done by Huisken and Illmanen in [25] (we briefly discuss this work in the following section 1.5) also provide a new proof of the positive energy theorem (which is based on an idea of Geroch [21]).

The simpler of all these proofs is, by far, Witten’s one. Also it resembles other positivity proofs is physics: the total energy is written as a positive definite integral in the space. In this section we review this proof. The aim is to present all the relevant steps in the most elementary way.

This proof uses, in an essential way, spinors. We refer the reader to the chapter in this book by Robert Geroch for an introduction to this subject. We will follow the notation of that chapter in this section.

There exists various reformulations of the original proof by Witten, in this section we essentially follow references [34] [39] [24] [36].

The proof uses only spinors defined on the spacelike surface, however it is more transparent to begin with spinor fields in the spacetime and then, at the very end, to restrict them to the spacelike surface. Also, this way of constructing the proof easily generalize to the proof of the positivity of the energy at null infinity (Bondi mass) (see [36]).

Let \((M, g_{ab})\) be a four dimensional Lorenzian manifold with connection \(\nabla_a\). In this section we use the signature \((+−−−)\) to be consistent with the literature on spinors. Unfortunately this signature gives a negative sign to the Riemannian metrics on spacelike surfaces used in the previous sections.

Let \(\lambda_A\) be an spinor field in the spacetime, the spin connection is denoted by \(\nabla_{AA'}\), and we use the standard notation \(a = AA'\) to identify spinor indices with tensorial indices.

The proof of the positive energy theorem is based on the remarkable properties of a 2-form \(\Omega\) called the Nester-Witten form [41] [30], defined as follows. The computations of this section involve integration on different kind of surfaces and hence it is convenient to use differential forms instead of ordinary tensors. We will denote them with boldface and no indices (for an introduction to forms see, for example, appendix B in [40], we will follow the notation and convention of this reference).

Consider the following complex tensor

\[
\Omega_{ab} = -i\bar{\lambda}_B \nabla_{AA'} \lambda_{B'}.
\]  

(1.77)

From this tensor we construct the complex 2-form \(\Omega\) by

\[
\Omega = \Omega_{[ab]}.
\]  

(1.78)

Explicitly we have

\[
\Omega = \frac{i}{2} (\bar{\lambda}_{A'} \nabla_{BB'} \lambda_{A} - \bar{\lambda}_{B'} \nabla_{AA'} \lambda_{B}).
\]  

(1.79)

The forms used in the following are always tensor fields (usually complex) but they are constructed out of spinors, as in the case of \(\Omega\). In order to define
these forms we need to antisymmetrize tensorial indices, to avoid a complicated notation we will always define first the tensor field in terms of spinors (as in equation \((1.77)\)) and then define the form antisymmetrizing the tensor indices (as in \((1.78)\)). When there are more than two tensorial indices the explicit expression of the differential form (like \((1.79)\)) can be lengthy and it is not usually needed. The spinor \(\lambda^A\) has an associated (future directed) null vector \(\xi_a\) given by

\[
\xi^a = \lambda^A \bar{\lambda}^A'.
\] (1.80)

Note that the \(\Omega\) can not be written in terms of derivatives of pure tensors fields like \(\xi_a\) and \(\nabla_a\).

The strategy of the proof is the following. Consider the exterior derivative \(d\Omega\) (which is a 3-form) and integrate it on a spacelike, asymptotically flat, 3-surface \(S\). Using Stoke’s theorem we obtain

\[
\sum_k \lim_{r \to \infty} \oint_{S_r} \Omega = \int_S d\Omega.
\] (1.81)

We are assuming that \(S\) is an asymptotically euclidean manifold with \(k\) asymptotic ends \(U_k\). The 2-form \(\Omega\) has two important properties. The first one is that the left hand side of \((1.81)\) gives is the total energy-momentum of a prescribed asymptotic end. The second is that the integrand of the right hand side is non-negative. Both properties depend on the way in which the spinor field \(\lambda^A\) is prescribed.

We begin with the first property. Note that the integrand in the left hand side of \((1.81)\) is complex. But the imaginary part of \(\Omega\) is given by

\[
\Omega - \bar{\Omega} = i \nabla_{[a} \xi_{b]} = i d\xi,
\] (1.82)

where, to be consistent with our notation, we write \(\xi\) for the the 1-form \(\xi_a\). That is, the imaginary part is the exterior derivative of a 1-form and hence its integral over a closed 2-surface is zero. Hence the boundary integral is always real, for arbitrary spinors \(\lambda^A\).

To prove the desired property, we need to impose fall off conditions on the spinor \(\lambda^A\). Fix one arbitrary end \(k\) (from now on we will always work on that end, and hence we suppress the label \(k\)). Let \(\lambda^A\) be an arbitrary \textit{constant} spinor, we require that the spinor \(\lambda^A\) satisfies on that end

\[
\lambda^A = \hat{\lambda}^A + \gamma^A, \quad \gamma^A = O(r^{-1}).
\] (1.83)

We also assume that the partial derivatives of \(\gamma^A\) are \(O(r^{-2})\) and we require that \(\lambda^A\) \textit{decays to zero at every other end}.

The idea is to prove that at the chosen end we have

\[
P_a \dot{\xi}^a = \frac{1}{8\pi} \lim_{r \to \infty} \oint_{S_r} \Omega,
\] (1.84)
where \( P_a = (E, P_i) \), with \( E \) and \( P_i \) defined by \((1.5)-\(1.6)\), and \( \xi^a \) is the constant null vector determined by the constant spinor \( \lambda^A \) by

\[
\xi^a = \hat{\lambda}^A \hat{\lambda}'^A. \tag{1.85}
\]

Note that the boundary integral in the right hand side of \((1.84)\) determines both the energy and the linear momentum of the end.

To prove \((1.84)\) the most important step is to prove that the value of the integral depends only on the constant spinor \( \lambda^A \) and not on \( \gamma^A \). We emphasize, as we will see, that a naive counting of the fall behaviour of the different terms in \( \Omega \), under the assumption \((1.83)\), does not prove this result. Using the decomposition \((1.83)\) we write \( \Omega \) as

\[
\Omega = \hat{\Omega} + \Gamma, \tag{1.86}
\]

where

\[
\hat{\Omega}_{ab} = -i \hat{\lambda}_B' \nabla_{AA'} \hat{\lambda}_B, \quad \hat{\Omega} = \hat{\Omega}_{|ab|}, \tag{1.87}
\]

and

\[
\Gamma_{ab} = -i \left( \hat{\lambda}_B' \nabla_{AA'} \gamma_B + \tilde{\gamma}_{B'} \nabla_{AA'} \hat{\lambda}_B + \tilde{\gamma}_{B'} \nabla_{AA'} \gamma_B \right), \quad \Gamma = \Gamma_{|ab|}. \tag{1.88}
\]

That is, \( \hat{\Omega} \) depends only on \( \hat{\lambda}^A \).

We would like to prove that \( \Gamma = O(r^{-3}) \) and hence it does not contribute to the integral at infinity \((1.84)\). Consider the third term in \((1.88)\). The covariant derivative \( \nabla_{AA'} \gamma_B \) has two terms, the first one contains partial derivatives of \( \gamma_B \) which, by assumption, are \( O(r^{-2}) \). The second term contains products of \( \gamma_B \) and the connections coefficients of the space-time metric \( g_{ab} \) evaluated at the asymptotic end of the spacelike surface \( S \). These coefficients are first derivatives of \( g_{ab} \), they can be written as first derivatives of the intrinsic Riemannian metric and the second fundamental form of the surfaces and hence, by assumption (recall that \( S \) is asymptotically flat and hence we have the fall off conditions \((1.4)\)) they are \( O(r^{-2}) \). We conclude that \( \nabla_{AA'} \gamma_B = O(r^{-2}) \) and hence \( \tilde{\gamma}_{B'} \nabla_{AA'} \gamma_B = O(r^{-3}) \). We proceed in a similar way for the second term: since \( \hat{\lambda}^A \) is constant the covariant derivative \( \nabla_{AA'} \hat{\lambda}_B \) contains connection coefficients times constants and hence we have \( \nabla_{AA'} \hat{\lambda}_B = O(r^{-2}) \), and then \( \tilde{\gamma}_{B'} \nabla_{AA'} \hat{\lambda}_B = O(r^{-3}) \). But using the same argument we obtain that the first term in \((1.88)\) is \( O(r^{-2}) \) and then it can contribute to the integral. But we can re-write \( \Gamma_{ab} \) as follows

\[
\Gamma_{ab} = -i \left( \nabla_{BB'} (\gamma_A \hat{\lambda}_A') - \gamma_A \nabla_{BB'} \hat{\lambda}_A + \tilde{\gamma}_{B'} \nabla_{AA'} \hat{\lambda}_B + \tilde{\gamma}_{B'} \nabla_{AA'} \gamma_B \right). \tag{1.89}
\]

The first term in \((1.89)\), which is the problematic one, contribute to \( \Gamma \) with the derivative of a 1-form, and hence it integrate to zero over a closed 2-surface. The new second term in \((1.89)\) is clearly \( O(r^{-3}) \). We have proved that

\[
\lim_{r \to \infty} \oint_{S_r} \Omega = \lim_{r \to \infty} \oint_{S_r} \hat{\Omega}. \tag{1.90}
\]
Note that \( \tilde{\Omega} \) is \( O(r^{-2}) \) and hence the integral converges. Also, the asymptotic value of \( \tilde{\Omega} \) at infinity contain a combination of first derivative of the intrinsic metric and the second fundamental form of the surface \( S \) multiplied by the constants \( \tilde{\lambda}^A \). It can be proved, essentially by an explicit calculation, that this combination is precisely \( P_a \xi^a \) (see [41], [30] and also [4]).

We turn to the second property of \( \Omega \). Recall that the exterior derivative of a \( p \)-form is given by

\[
d\Omega = (p + 1) \nabla_{[a} \Omega_{b_1 \ldots b_p]}.
\]

We have

\[
d\Omega = \alpha + \beta,
\]

where \( \alpha \) and \( \beta \) are the following 3-forms

\[
\alpha_{abc} = -i \tilde{\lambda}_C \nabla_a \nabla_b \lambda_C, \quad \alpha = \alpha_{[abc]},
\]

and

\[
\beta_{abc} = -i \nabla_a \tilde{\lambda}_C \nabla_b \lambda_C, \quad \beta = \beta_{[abc]}.
\]

That is, \( \alpha \) has second derivatives of the spinor \( \lambda_A \) and \( \beta \) has squares of first derivatives of \( \lambda_A \).

We compute first \( \alpha \). Observe that there is a commutator of covariant derivatives and hence we can replace it by the curvature tensor. However, what is surprising is that precisely the Einstein tensor appears. To see this, is easier to work with the dual of \( \alpha \) defined by

\[
\ast \alpha = \frac{1}{3!} \epsilon_{abcd} \alpha^{abc}.
\]

We use the commutator relations

\[
2 \nabla_{[a} \nabla_b] \lambda_C = -\epsilon_{A'B'C} X_{ABC} E \lambda_E - \epsilon_{AB} \Phi_{A'B'C} E \lambda_E,
\]

where \( X_{ABCD} \) and \( \Phi_{A'B'C'D} \) are the curvature spinors. These spinors are defined in terms of the Riemann tensor \( R_{abcd} = R_{AA'BB'CC'DD'} \) by

\[
X_{ABCD} = \frac{1}{4} R_{AX'B'} Y^C Y^D, \quad \Phi_{ABCD} = \frac{1}{4} R_{AX'B'} Y^C Y^D.
\]

See [33] for further details on the curvature spinors. The Einstein tensor is given by

\[
G_{ab} = -6 \Lambda g_{ab} - \Phi_{ab},
\]

where \( \Lambda \) is given by

\[
\Lambda = \frac{1}{6} X_{AB} A^B.
\]

We also use the identities

\[
X_{ABC} B = 3 \Lambda \epsilon_{AC},
\]

and

\[
\epsilon_{abcd} = i (\epsilon_{AC} \epsilon_{BD} \epsilon_{A'D'} \epsilon_{B'C'} - \epsilon_{AD} \epsilon_{BC} \epsilon_{A'C'} \epsilon_{B'D'}). \quad (1.101)
\]
And then we obtain
\[ *\alpha = -\frac{1}{2 \cdot 3!} \xi^e G^{ef}, \]
and hence
\[ \alpha = -\frac{1}{2 \cdot 3!} \xi^e G^{ef} \epsilon_{fabc}. \]

The expressions (1.102) and (1.103) are pure tensorial expressions.

To compute \( \beta \) we proceed in a similar form. We work first with the dual
\[ *\beta = \frac{1}{3!} \epsilon_{abcd} \beta^{bcd}. \]

It is important (we see later why) to split the covariant derivative \( \nabla_a \) into its temporal and spatial component. Let \( t^a \) denote the unit timelike normal to the surface \( S \) and \( h_{ab} \) is the intrinsic metric of the surface. We define the spatial \( D_a \) derivative as
\[ D_a = h^{ab} \nabla_b. \]

Note that \( D_a \) is not the covariant derivative \( D \) of the intrinsic metric \( h \) used in the previous sections, they are related by the equation
\[ D_{AB} \lambda_C = D_{AB} \lambda_C + \frac{1}{\sqrt{2}} \pi_{ABC} \lambda_D, \]
where \( \pi_{ABCD} = \pi_{(AB)(CD)} \) is the spinor representation of the second fundamental form of the surface.

From equation (1.105) we obtain
\[ \nabla_a = D_a - t_a t^b \nabla_b. \]

We replace the derivative \( \nabla_a \) by (1.107) in the definition of \( \beta \) given by (1.94) and we compute the dual defined by (1.104) to obtain
\[ *\beta = -\frac{1}{3!} \epsilon_{abcd} \left( D^b \lambda^C' D^d \lambda^C \right) + W_a, \]
where
\[ W_a = \frac{1}{3!} \epsilon_{abcd} \left( t^b t^f \nabla_f \lambda^C' D^d \lambda^C + t^d t^f \nabla_f \lambda^C D^b \lambda^{C'} \right). \]

Note that \( W_a \) satisfies
\[ t^a W_a = 0. \]

Using the identity (1.111) we further decompose the first term in the right hand side of (1.108)
\[ -i \epsilon_{abcd} D^b \lambda^C' D^d \lambda^C = D^b \lambda^C' D_{BA'} \lambda_A - D^b \lambda_A' D_{AB'} \lambda_B. \]
where in the second line we have used the spinorial identity
\[ \epsilon_{AB}\epsilon_{CD} + \epsilon_{BC}\epsilon_{AD} + \epsilon_{CA}\epsilon_{BD} = 0. \]

Combining (1.108) and (1.111) we finally obtain
\[ *\beta = D_{C'B'}\lambda^{C'}D^{B'}\bar{\lambda}_{A'} - D_b\lambda_A D^b\bar{\lambda}_{A'} + W_a. \] (1.114)

We are in position now to perform the integral over \( S \) of \( d\Omega \). Using (1.92), (1.103) and (1.114) we obtain
\[ \int_S d\Omega = \int_S \left( 4\pi T_{ab}\xi^b + D_{C'B'}\lambda^{C'}D^{B'}\bar{\lambda}_{A'} - D_b\lambda_A D^b\bar{\lambda}_{A'} \right) t^a dv, \] (1.115)

where we have used Einstein equations
\[ G_{ab} = 8\pi T_{ab}, \] (1.116)

to replace the Einstein tensor by the energy-momentum tensor in the expression (1.103) for \( \alpha \). Note that the term \( W_a \) in (1.114) does not appear in the integral because it is orthogonal to \( t^a \) (c.f. equation (1.110)).

Assume that the spinor \( \lambda^A \) has the fall-off behaviour (1.83), then the identity holds, using the Stoke’s theorem (1.81) (note that by (1.83) all the other boundary integrals vanish) we finally obtain the famous Witten identity
\[ P_a\dot{\xi}^a = \frac{1}{8\pi} \int_S \left( 4\pi T_{ab}\xi^b + D_{C'B'}\lambda^{C'}D^{B'}\bar{\lambda}_{A'} - D_b\lambda_A D^b\bar{\lambda}_{A'} \right) t^a dv, \] (1.117)

If we assume that the energy-momentum tensor \( T_{ab} \) satisfies the dominant energy condition then we have
\[ T_{ab}\xi^a \xi^b \geq 0, \] (1.118)

and hence the first term in the integrand of (1.117) is non-negative. The last term in (1.117) is also non-negative since it involves the contraction with the Riemannian metric (which is negative definite) and the timelike vector \( t^{AA'} \). To handle the second and third term we impose on \( \lambda^A \) the following equation which is called the Sen-Witten equation [38] [41]
\[ D_{AB}\lambda^A = 0. \] (1.119)

Let us assume for the moment that there is a solution of this equation with the fall-off behaviour (1.83). Then, from (1.117) we obtain
\[ P_a\dot{\xi}^a \geq 0. \] (1.120)

But the constant null vector \( \dot{\xi}^a \) is arbitrary, hence it follows that \( P_a \) should satisfy (1.8). To prove the rigidity part of theorem 1.1.1 the key ingredient is that \( E = 0 \) implies, again by the identity (1.117), that the spinor satisfies the equation
\[ D_{AB}\lambda_C = 0, \] (1.121)
that is, it is covariant constant in the whole manifold. From this equation it can be deduced that the initial data on the surface correspond to the Minkowski space-time (see [32] for the details of this argument).

It remains to discuss the solutions of equation (1.119). The existence of solution of these equations under the required fall-off conditions (1.83) has been proved in [35] [36] [32]. The main point is that equation (1.119) constitute an elliptic system of first order for the two complex components of the spinor (this can be easily seen using the standard definition of ellipticity for systems, see, for example, [16] where this specific example is discussed). And hence this equation can, essentially, be handled as a Poisson equation. Solutions under weak decay conditions on the data of equation (1.119) has been proved in [4].

Finally, let us discuss the proof of theorem 1.1.2. This was done in [36] [22]. Remarkably, the proof is very similar, the only extra ingredient is that in the Stoke’s theorem we need to include an extra internal boundary term. This term has the form (see [36])

$$\int_{\partial B} \Omega = \int_{\partial B} \left( \Theta_+ \lambda^0 \bar{\lambda}^0 - \rho' \lambda^1 \bar{\lambda}^1 + \lambda^2 \partial \lambda^0 - \bar{\lambda}^0 \partial \lambda^1 \right) ds. \quad (1.122)$$

In this equation $\Theta_+$ is the null expansion defined previously by (1.53). The coefficient $\rho'$ represent the ingoing null expansion on the surface, it is not important for our purposes. The functions $\lambda^0$ and $\lambda^1$ are the component of $\lambda^A$ in an appropriated spinorial diad adapted to the 2-surface $\partial B$. Finally, $\partial$ is a tangential differential operator to the 2-surface. It can be shown that the appropriate inner Dirichlet boundary for equation (1.119) is to prescribe one of the component $\lambda^0$ or $\lambda^1$ (but not both) (this is a consequence of the elliptic character of this equation, see, for example [16] for an elementary treatment of this). If we prescribe $\lambda^1 = 0$ on $\partial B$ and use that, by hypothesis this surface satisfies $\Theta_+ = 0$, then the boundary term (1.122) vanished and we can proceed in the same way as above to prove the positivity of the energy. Note that without the condition $\Theta_+ = 0$ it is not possible to make the boundary term zero.

1.5 Further results and open problems

In this article we have discussed only the positive energy theorem in 3 space dimensions. The spinorial proof presented in section 1.4 works in any dimensions (see [32]), however in higher dimensions the existence of an spin structure involves restrictions on the topology of the manifold $S$. In section 1.4 we have used Weyl spinors which are well adapted to 4 spacetime dimensions. For higher dimensions Dirac spinors are usually used. The other proofs currently available [37] and [25] do not work in arbitrary high dimensions. To prove the positive energy theorem in all dimensions is one of the relevant open problem in this area.

The positive energy theorem can be refined to incorporate other physically relevant parameters. For example, using a similar argument as in Witten’s proof
it is possible to prove \cite{22, 23} that the total mass $M$ satisfies

$$M \geq |q|, \tag{1.123}$$

where $q$ is the electric charge and the non-electromagnetic part of the energy momentum tensor must satisfy appropriate conditions.

Recently, for axially symmetric black holes the following inequality

$$M \geq \sqrt{|J|}, \tag{1.124}$$

has been proved. Here $J$ is the angular momentum of the black hole (see the review article \cite{17} and reference therein). The equality in (1.124) is achieved only for the extreme Kerr black hole. This inequality is proved for one black hole, a relevant open problem is to prove it for multiple black holes.

Another important extension is the Penrose inequality for black holes. The Riemannian black hole positivity theorem 1.2.4 can be generalized to include the area of the minimal surface, namely

$$M \geq \sqrt{\frac{A}{16\pi}}, \tag{1.125}$$

with equality only for the Schwarzschild black hole. This result was proved by \cite{25} and \cite{5}. The general case remains open, see the review article \cite{27}.

Finally, we have discussed the concept of total energy and linear momentum of an isolated system. It would be very desirable to have a quantity that measures the energy of a finite region of the spacetime. These kind of quantities are called quasi-local mass. For a comprehensive review on this important open problem see \cite{39}. The following related, pure quasi-local, inequality for axially symmetric black holes has been recently proved

$$A \geq 8\pi|J|, \tag{1.126}$$

where $A$ is the area and $J$ is the quasi-local angular momentum of the black hole (see the review article \cite{17} and reference therein). The equality in (1.126) is achieved if and only if the local geometry of the black hole is equal to the extreme Kerr black hole local geometry. For non-axially symmetric black holes it is difficult to define the quasi-local angular momentum $J$ (see \cite{39}). An important open problem is to generalize the inequality (1.126) for non-axially symmetric black holes (or to find suitable counter examples).
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