Backwards Two-Particle Dispersion in a Turbulent Flow

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We derive an exact equation governing two-particle backwards mean-squared dispersion for both deterministic and stochastic tracer particles. For the deterministic trajectories, we probe consequences of our formula for short time and arrive at approximate expressions for the mean squared dispersion which involve second order structure functions of the velocity and acceleration fields. For the stochastic trajectories, we analytically compute an exact \(t^3\) contribution to the squared separation of stochastic paths. We argue that this contribution appears also for deterministic paths at long times and present direct numerical simulation (DNS) results for incompressible Navier-Stokes flows to support this claim. We also numerically compute the probability distribution of particle separations for the deterministic paths and the stochastic paths and show their strong self-similar nature.

Dispersions of particles in a turbulent flow is of great importance in a wide range of natural phenomena - from environmental problems such as the spread of a cloud of contaminants to astrophysical problems of understanding solar flares.\(^3\) In 1926, Richardson\(^1\) argued that in a turbulent flow the separation between a pair of Lagrangian tracer particles \(\delta x(t) = x_1(t) - x_2(t)\) separate like \(\langle (\delta x(t))^2 \rangle \sim \varepsilon t^3\) where \(\varepsilon\) is the so-called Richardson constant and \(\varepsilon\) is the space-average energy dissipation. This prediction is now known as the Richardson-Obukhov scaling law\(^2\) and has been a subject of controversy and inquiry since \(^2\).\(^3\)\(^4\). In three dimensional homogeneous isotropic turbulence, numerous studies have verified that particle-pair separations grows like \(t^3\) at long time however the physical mechanisms behind this growth are still not well understood. Experimental and numerical studies of two-particle dispersion are challenging as they require setting wide range separation between the dissipation scale and the integral scale of the flow\(^12\).\(^13\). Backward dispersion is particularly difficult because it involves integrating the Navier-Stokes equation forward in time and tracking the tracers backward in time.

In this letter we will investigate properties of backwards separation for both deterministic and stochastic trajectories. It has been observed that the features characterizing the backward and forward dispersion are quite different\(^2\)\(^10\).\(^11\). The measured Richardson constant for example is observed to be later backwards in time. We study deterministic tracers to make contact with previous work and to understand what aspects of fluid flow contribute to the observed scalings. In parallel, backwards stochastic trajectories are particularly important in physics since they provide a means to calculate the evolution of passively transported diffusive quantities such as dye, heat, or magnetic fields\(^6\).\(^8\).\(^11\). A better understanding of stochastic dispersion may help to understand how these field behave when advected by a turbulent flow. See Sawford et al.\(^14\) for a detailed discussion of this connection.

\(\int \delta u(L) \cdot \delta u(t)\) is the second order velocity structure function, \(\delta u(r; t)\) is the Eulerian increment of the velocity field at the final time, \(\delta a_L(r; s)\) is the Lagrangian acceleration increment and \(\langle \rangle_x\) denotes integration over the final particle positions \(x_f\). This formula is exact - no assumptions were needed to arrive at equation (4). The \(\frac{1}{2}t^3\) term appearing in equation (4) is the so-called Batchelor regime\(^1\) where the particles undergo ballistic motion and is present for all time. Note that in the forward case, the last term would have an opposite sign and \(\{u_f, r_f, t_f\}\) would be

A. Deterministic Tracers

Consider a passive tracer with position \(x(t)\) advected by the velocity field \(u\), which in turn evolves due to the action of a acceleration field \(a\). For a Navier-Stokes velocity \(u\), the \(a\) is given by: \(a \equiv f_{ext} - \nabla p + \nu \Delta u\). The tracer is labelled at the final time \(t_f\) and travels backwards for times \(t < t_f\). The dynamical relations are

\[
\frac{d}{dt} x(t) = u(x(t), t), \quad x(t_f) = x_f, \quad (1)
\]

\[
\frac{d}{dt} u(x(t), t) = a(x(t), t). \quad (2)
\]

Defining \(\tau(s) \equiv s - t\) and \(\tau(t) \equiv \tau(t_f)\), it follows from equations (1) and (2) that

\[
x(t) = x_f - \tau(x(t_f), t_f) + \int_t^{t_f} \tau(s) a(x(s), s) ds. \quad (3)
\]

The backward separation of particle pairs separated by \(r_f\) at the final time is \(r(\tau) \equiv x(t; x_f + r_f) - x(t; x_f)\). The space-averaged squared separation is easily shown to satisfy:

\[
\langle (r(\tau) - r_f)^2 \rangle_x = \frac{\tau^2 S_{uu}^2(r_f)}{x_f} + \left(\int_s^{t_f} \tau(s) \delta a_L(r_f; s) ds \right)^2 \frac{1}{x_f} - 2\tau \int_s^{t_f} \tau(s) \langle \delta u(r_f; s) \delta a_L(r_f; s) \rangle_x ds, \quad (4)
\]

where \(S_{uu}^2(r_f) = \langle (\delta u(r_f; t_f))^2 \rangle_x\) is the second order velocity structure function, \(\delta u(x_f + r_f, t_f) - u(x_f, t_f)\) is the Eulerian increment of the velocity field at the final time, \(\delta a_L(r; s) \equiv a(x(s; x_f + r_f), s) - a(x(s; x_f), s)\) is the Lagrangian acceleration increment and \(\langle \rangle_x\) denotes integration over the final particle positions \(x_f\). This formula is exact - no assumptions were needed to arrive at equation (4). The \(\frac{1}{2}t^3\) term appearing in equation (4) is the so-called Batchelor regime\(^1\).
replaced by $\{u_0, r_0, t_0\}$. If we consider small $\tau$, then a first order Taylor expansion of equation (4) yields:

$$\langle [r(t) - r_f]^2 \rangle_{x_f} \approx S^2(\tau) + 2(\varepsilon) x^3 + \frac{1}{4} S^4(\tau) \tau^4$$

(5)

where $S^2(\tau)$ is the second order acceleration structure function at the final time. We have also assumed that the final separation $r_f$ is in the inertial range so as to use the Ott-Mann relation – a Lagrangian analogue of the $4/5$th law – expressing $\langle \delta u(r_f; t_f) \cdot \delta a(r_f; t_f) \rangle_{x_f} \approx -2(\varepsilon)x$ where $\varepsilon$ is the viscous energy dissipation [13,19].

The $\tau^3$ term appearing in (5) has been derived in [14] for the forward case with an opposite sign and highlighted with DNS studies for the statistics of velocity differences in [15]. In [15], the equivalent term to our $\tau^4$ term has been observed for the case of the velocity difference statistics. They described it as the initial abrupt variation of the velocity difference and here we further understand it as a Bachelor-type-range (in a sense of first order expansion) for the velocity separation.

B. Stochastic Tracers

Consider now the following backward stochastic equation governing the flow of passive tracer particles:

$$d\tilde{x}(t) = u(\tilde{x}(t), t)dt + \sqrt{2\kappa}dW_t, \quad \tilde{x}(t_f) = x_f.$$  

(6)

Here $\kappa$ is the molecular diffusivity, $W_t$ is standard Brownian motion and $\tilde{d}$ is the backwards Itô differential [21]. Note that if the viscosity of the fluid is fixed and $\kappa \to 0$, we recover the deterministic equation (1). Along a path defined by equation (6), the backward Itô lemma [24,30] can be used to show that the Navier-Stokes velocity satisfies

$$du(\tilde{x}(t), t) = a^n|\tilde{x}(t)|dt + \sqrt{2\kappa}\nabla u|\tilde{x}(t)| \cdot dW_t,$$

(7)

where $a^n \equiv f_{ext} - \nabla p + (\nu - \kappa)\Delta u$. This is a stochastic generalization of equation (2). The quadratic variation term $-\kappa\Delta u$ appearing in the backward Itô Lemma has the opposite sign than the forward case which would yield a similar expression but with $a^n_{forw} \equiv f_{ext} - \nabla p + (\nu + \kappa)\Delta u$. By integrating (7), we obtain

$$\tilde{x}(t) = x_f - \tau u(\tilde{x}(t), t) + \int_t^{t_f} \tau(s)a^n(\tilde{x}(s), s)ds$$

$$+ \sqrt{2\kappa} \int_t^{t_f} \tau(s)\nabla u(\tilde{x}(s), s) \cdot dW_s - \sqrt{2\kappa}W_t.$$  

(8)

This equation is the analogue of equation (3) but for paths with intrinsic additive white noise. Note that setting $\kappa = \nu$ leads to a dramatic simplification where the laplacian term in the acceleration vanishes: $a^n = f_{ext} - \nabla p$. Further, Eyink in [17] has shown strong evidence that the backward dispersion is independent of $\kappa$ at long times. Motivated by its simplicity and the result of [17], for the remainder of this letter – in both the theoretical and numerical results – we make the choice of $\kappa = \nu$.

Consider two trajectories $\tilde{x}^1(t)$ and $\tilde{x}^2(t)$ ending at the same point $x_f$ and satisfying equation (6) with independent Brownian motions $W^1$ and $W^2$ respectively. The natural object of study in this setting is the spaced averaged mean (in the sense of averaging over the Brownian motions) squared distance $\langle \tilde{r}(\tau) \rangle \equiv \langle (\tilde{x}^1(\tau) - \tilde{x}^2(\tau)) \rangle$ between the trajectories advected by independent Brownian motions. In the deterministic case, in order to study the squared separation of two particles it was necessary to consider particles which (at the final time) were separated by a positive distance $|r_f|$. In the stochastic setting this is no longer necessary and starting particles at the same point effectively removes the dependence on the final separation and final velocity difference.

If the turbulence is homogeneous and isotropic or, more generally, the flow is ergodic, then the space averaged energy dissipation $\langle \varepsilon \rangle_x$ is constant in time. Assuming this, at arbitrary time the space averaged mean dispersion can be proven to satisfy:

$$E_{1,2} \langle |r(\tau)| + \sqrt{2}\delta W_t \rangle_{x_f} = \frac{4}{3} \langle \varepsilon \rangle_x r^3 + \varepsilon(\tau)$$

(9)

where the difference of the Brownian motions is denoted $\delta W_t = W^1_t - W^2_t$ and where

$$\varepsilon(\tau) = E_{1,2} \langle \delta A(\tau) \rangle_{x_f}^2 + E_{1,2} \langle \delta A(\tau) \delta U(\tau) \rangle_{x_f}.$$  

(10)

The terms $A(\tau)$ and $U(\tau)$ are defined to be the integral terms involving $a^n$ and $\nabla u$ respectively in equation (6). The $\delta$ refers to the difference in a quantity evaluated on each path. The expectations are taken over realizations of the two independent Brownian motions. The $\tau^3$ term is $E_{1,2} \langle \delta U(\tau) \rangle^2_{x_f}$ after using Itô isometry and the fact that $\langle \varepsilon \rangle_x = \nu \langle \nabla u^2 \rangle_x$. If we instead decided to study forward stochastic tracers, then equation (9) still holds but with minor differences in the error term including that integral $A(\tau)$ involve $a^n_{forw}$ (defined earlier) instead of $a^n$. Saffman in 1956 studied the effect of molecular diffusivity on forward particle dispersion using (6) as a model [22]. There he computed short-time $t^3$ deviation of the stochastic dispersion to the deterministic for tracers in a homogeneous isotropic turbulent flow. The methods he employed are different from our own and, though his results have a similar form, they are not directly related to the long-time $4/3 \langle \varepsilon \rangle_x r^3$ behavior we calculate.

C. Numerical Results

In order to understand the behavior of the different terms in equation (10), we evaluate them using turbulence data from direct numerical simulations. We use the JHU Turbulence Database Cluster [31,32], which provides online DNS data over an entire large-eddy turnover time for isotropic and homogeneous turbulence at Taylor-scale Reynolds number $Re_\lambda = 433$. The integration of particle trajectories is performed inside the database using the getPosition functionality [33] and a backward fourth-order Runge-Kutta integration scheme.
Figure 1 shows our results for the different terms in (9) compensated by $4/3\langle\varepsilon\rangle_x\tau^3$. Error bars are calculated by the maximum difference between two subensembles of $N/2$ samples and $N = 5 \times 10^8$. The terms $E_{1,2} \langle [\delta A(\tau)]^2 \rangle_{x_f}$ and $-E_{1,2} \langle \delta A(\tau) \delta U(\tau) \rangle_{x_f}$ behave in a very similar fashion. They grow as $\tau^3$ for short time and seem to reach an asymptotic $\tau^3$ for $\tau \gg \tau_v$, $\tau_v$ being the Kolmogorov time-scale. The difference $\mathcal{E}(\tau)$ between these two terms however never exceeds 16% of $4/3\langle\varepsilon\rangle_x\tau^3$ (29% including the error bars) at all points in time. See inset of Figure 1. As a consequence of $\mathcal{E}(t)$ being small relative to $4/3\langle\varepsilon\rangle_x\tau^3$, we observe the dispersion is

$$E_{1,2} \langle \left\{ \sqrt{2\nu} \delta \mathbf{W}_t \rangle_{x_f} \rangle_{x_f} \approx \frac{4}{3} \langle\varepsilon\rangle_x\tau^3$$

for almost three decades.

We now take a moment to speculate on the behavior of particle trajectories for very high Reynolds number turbulence. Note that, by the zeroth law of turbulence $\langle\varepsilon\rangle_x$ tends to a non-zero constant in the inviscid limit. Therefore the $\frac{4}{3} \langle\varepsilon\rangle_x\tau^3$ term will remain for arbitrarily small viscosity. Figure 1 shows that, in the inertial range, the terms $\mathcal{E}(t)$ is small compared to $\frac{4}{3} \langle\varepsilon\rangle_x\tau^3$. In the inviscid limit, the inertial range expands to all scales and if $\mathcal{E}(t)$ is small in this range then there is good evidence for spontaneous separation of particle trajectories emanating from a single point in the zero viscosity limit. This is the hallmark of the phenomenon of spontaneous stochasticity 17, 21, 24, 28.

We now compare the stochastic particle dispersion of section 13 to the deterministic dispersion of section A measured for finite final separations. We study the statistics of $\langle \rho(\tau; r_f) \rangle = r_f + \tau \delta \mathbf{u}(r_f; t_f)$. By doing so, we effectively “remove” the effect of the Bachelor regime and expose longer range of $\tau^3$ scaling. It is equivalent to consider only the integral term over $\delta \mathbf{a}(r_f; s)$ in (10). It was numerically verified that the quantity $\langle |\rho(\tau; r_f)|^2 \rangle_{x_f}$ has the asymptotic behavior

$$\langle |\rho(\tau; r_f)|^2 \rangle_{x_f} \approx \left\{ \begin{array}{ll} \langle |\rho(\tau)|^2 \rangle_{x_f} & \tau \gg \tau_v \\ \frac{4}{3} S_2^3(r_f) & \tau \ll \tau_v \end{array} \right.$$  (12)

In Figure 2 we highlight the limiting cases given by (12). The inset shows the $\frac{4}{3} S_2^3(r_f)$ scaling law up to $\tau \approx \tau_v$ as predicted analytically. In the main plot, results rescaled by $\frac{4}{3} \langle\varepsilon\rangle_x\tau^3$ are shown for eight different final separations $r_f \in [\eta, 20\eta]$ where $\eta$ is the Kolmogorov length scale and $T_L$ is the large scale turn-over time. The dispersion of the particles advected by the Brownian motion is in good agreement with the deterministic separations and all the curves tend to converge toward to the stochastic dispersion. Define $g_{r_f} = \langle |\rho(T_L; r_f)|^2 \rangle_{x_f}/\langle\varepsilon\rangle_x T^3_L$ and we measure the values in Table I. The longest $\tau^3$ scaling (one decade) observed for deterministic dispersion is for $|r_f| = 6\eta$ and $\eta_{stat} = 4/3$ within numerical error. Note that in backwards setting, our measured Richardson constant is larger than forward results ($g \approx 0.5, 0.52$ 13–15) and is in good agreement with the measurements in 13, 16, 17.

We have also numerically calculated the probability distribution functions (PDF) $P(\rho, \tau)$, $P(\tilde{\rho}, \tau)$ for the deterministic $\rho$ and stochastic $\tilde{\rho}$ where $\rho \equiv P(\rho, \tau)$.

| $|r_f|/\eta$ | 4 | 6 | 8 |
|----------------|---|---|---|
| $g_{r_f}/(4/3)$ | 0.86 ± 0.10 | 0.99 ± 0.08 | 1.17 ± 0.02 |

| $|r_f|/\eta$ | 1 | 3 | 10 | 20 |
|----------------|---|---|---|---|
| $g_{r_f}/(4/3)$ | 0.50 ± 0.07 | 0.76 ± 0.01 | 1.24 ± 0.07 | 1.71 ± 0.05 |

TABLE I. Values of $g_{r_f}$
\[ \sqrt{\langle \rho(\tau; r_f)^2 \rangle_{x_f}} \] and \( \bar{\rho} = \sqrt{\mathcal{E}_{\tau} \langle |\bar{r}(\tau)| + \sqrt{2 \bar{\rho} \delta W_i^2} \rangle_{x_f}} \). Figure 3 plots for \( \tau = T_L \) the probability distributions with similarity scaling. The straight dashed line is the Richardson PDF:

\[ P(\rho, t) = \frac{B}{\langle \rho(\tau)^2 \rangle} \exp \left[ -A \left( \frac{\rho}{\langle \rho(\tau)^2 \rangle^{1/2}} \right)^{2/3} \right]. \] (13)

All the curves are in good agreement with each others and Richardson for 0.4 \( \lesssim \langle \rho/\sqrt{\langle \rho(\tau)^2 \rangle} \rangle^{2/3} \lesssim 2 \). Note that \( L \), the integral scale, we have \( (L/\sqrt{4/3\langle \varepsilon \rangle_{x_f} T_L^3}) \approx 1.24 \) in our case. Particle separations beyond that limit are outside the inertial range. Notice that the PDFs tend to spike above the Richardson PDF at small \( \rho \) indicating for strong intermittency at small scales.

![Figure 3](image3.png)

FIG. 3. Pair separation PDF for eight different initial separations in pure DNS data compared to the dispersion for the stochastic advection model with similarity scaling. Infinite Reynolds self-similar PDFs are shown for Richardson (straight dashed line).

Figure 4 plots the PDF for the stochastic advection model at seven different times \( \tau \in [0.023 \tau_r, 44.1 \tau_r] \). The probability distribution exhibits a clear self-similarity behavior for 0 \( \lesssim \langle \rho/\sqrt{\langle \rho(\tau)^2 \rangle} \rangle^{2/3} \lesssim 2 \) at all times. Note that even though we effectively removed the dependency of the dispersion on the final separation, the PDF is still not well described by Richardson at large scales (greater than 2 in similarity units).

In this letter we investigate properties of mean squared dispersion for both deterministic and stochastic particle passive tracers in a turbulent flow. We analytically predicted small time behavior for deterministic backwards dispersion and investigated the \( \tau^3 \)-scaling law numerically. We also looked at the convergence towards a long-time \( \tau^3 \) scaling law of the backwards dispersion for different final separations. In addition, we developed a mathematical formalism for studying backwards stochastic particle trajectories with additive white noise. Using this formalism we derived a formula for the mean dispersion and showed that the main contribution to this

![Figure 4](image4.png)

FIG. 4. Pair separation PDF for seven different times for the stochastic advection case with similarity scaling. The straight dashed line is the Infinite Reynolds self-similar PDF.

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