SOME PROPERTIES OF SOLUTIONS TO THE WEIGHTED HARDY-LITTLEWOOD-SOBOLEV TYPE INTEGRAL SYSTEM

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Abstract. This paper is concerned with the properties of solutions for the weighted Hardy-Littlewood-Sobolev type integral system
\[
\begin{align*}
  u(x) &= \frac{1}{|x|^\alpha} \int_{\mathbb{R}^n} \frac{v^q(y)}{|y|^\beta |x-y|^\lambda} dy, \\
  v(x) &= \frac{1}{|x|^\beta} \int_{\mathbb{R}^n} \frac{u^p(y)}{|y|^\alpha |x-y|^\lambda} dy
\end{align*}
\]
and the fractional order partial differential system
\[
\begin{align*}
  (-\Delta)^{\frac{n+\lambda}{2}} (|x|^{-\alpha}u(x)) &= |x|^{-\beta}v^q(x), \\
  (-\Delta)^{\frac{n+\lambda}{2}} (|x|^{-\beta}v(x)) &= |x|^{-\alpha}u^p(x).
\end{align*}
\]
Here $x \in \mathbb{R}^n \setminus \{0\}$. Due to $0 < p, q < \infty$, we need more complicated analytical techniques to handle the case $0 < p < 1$ or $0 < q < 1$. We first establish the equivalence of integral system (1) and fractional order partial differential system (2). For integral system (1), we prove that the integrable solutions are locally bounded. In addition, we also show that the positive locally bounded solutions are symmetric and decreasing about some axis by means of the method of moving planes in integral forms introduced by Chen-Li-Ou. Thus, the equivalence implies the positive solutions of the PDE system, also have the corresponding properties. This paper extends previous results obtained by other authors to the general case.

1. Introduction. We consider the weighted Hardy-Littlewood-Sobolev type integral system
\[
\begin{align*}
  u(x) &= \frac{1}{|x|^\alpha} \int_{\mathbb{R}^n} \frac{v^q(y)}{|y|^\beta |x-y|^\lambda} dy, \\
  v(x) &= \frac{1}{|x|^\beta} \int_{\mathbb{R}^n} \frac{u^p(y)}{|y|^\alpha |x-y|^\lambda} dy
\end{align*}
\]
where
\[
\begin{align*}
  \alpha, \beta &\geq 0, \quad 0 < p, q < \infty, \quad 0 < \lambda < n, \quad \alpha + \beta + \lambda < n, \\
  \frac{\alpha}{n} &< \frac{1}{p+1}, \quad \frac{\beta}{n} < \frac{1}{q+1}, \quad \frac{1}{p+1} + \frac{1}{q+1} = \frac{\lambda+\alpha+\beta}{n}.
\end{align*}
\]
This integral system is related with the fractional order partial differential system
\begin{equation}
\begin{cases}
(-\Delta)^{\frac{\alpha}{n}}(|x|^\alpha u(x)) = |x|^{-\beta}v^q(x), \\
(-\Delta)^{\frac{\beta}{n}}(|x|^\beta v(x)) = |x|^{-\alpha}u^p(x),
\end{cases}
\end{equation}
(1.3)

Chen and Li [1] pointed out that (1.1) is related with the best constant of the following weighted Hardy-Littlewood-Sobolev (WHLS) inequality introduced by Stein and Weiss [13],
\begin{equation}
\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^\alpha |x-y|^\lambda |y|^{\beta}} dx dy \right| \leq C_{\alpha,\beta,s,\lambda,n} \|f\|_r \|g\|_s
\end{equation}
where
\begin{equation}
1 - \frac{1}{r} - \frac{\lambda}{n} < \frac{\alpha}{n} < 1 - \frac{1}{r} \quad \text{and} \quad \frac{1}{r} + \frac{1}{s} + \frac{\lambda + \alpha + \beta}{n} = 2.
\end{equation}
(1.4)

When \(\lambda = n - 2k\), \(\alpha = \beta = 0\), (1.1) is reduced to integral system involving the Riesz potentials
\begin{equation}
\begin{cases}
u(x) = \int_{\mathbb{R}^n} \frac{v^q(y)}{|x-y|^\lambda} dy \\
v(x) = \int_{\mathbb{R}^n} \frac{u^p(y)}{|x-y|^\lambda} dy.
\end{cases}
\end{equation}
(1.6)

and (1.3) is reduced to the 2k-order Lane-Emden type system
\begin{equation}
\begin{cases}
(-\Delta)^k u = v^q \quad \text{in } \mathbb{R}^n, \\
(-\Delta)^k v = u^p \quad \text{in } \mathbb{R}^n,
\end{cases}
\end{equation}
(1.7)

where \(k \geq 1\) is an integer.

Chen and Li [2] proved recently that the HLS type integral system (1.6) is equivalent to (1.7) when \(\lambda = n - 2k\), where \(k\) is an integer. Thus, the radial symmetry of solutions of (1.6) implies the symmetry result of solutions of (1.7). When \(n-\lambda\) is not an integer, they directly defined the PDE solutions of (1.7) is in fact a solution of the integral system (1.6). Following their definition, we define a solution \((u, v)\) of (1.3) in the sense of distribution when \(n-\lambda\) is not an integer.

For the integrable positive solutions of (1.1), when \(p, q \geq 1, pq \neq 1\), Jin and Li [6, 7] proved the radial symmetry, the monotonicity and their optimal integrability. Li, Lim, Lei, and Ma estimated the asymptotic rates when \(|x| \to 0\) and \(|x| \to \infty\), respectively (cf. [8, 10, 11]).

Lei and Lü [9] first establish (1.1) is equivalent to (1.3), then prove that the integrable solutions of (1.1) are locally bounded. In addition, they also show that the positive locally bounded solutions are symmetric and decreasing about some axis by means of the method of moving planes in integral forms introduced by Chen-Li-Ou.

But when \(p \in (0, 1)\) or \(q \in (0, 1)\), there is few study of the qualitative properties of positive solutions. When \(\frac{n}{2} < p, q < \infty\), Hang [5] proved the regularity and radial symmetry of nonnegative integrable solutions of the system (1.1) as \(\alpha = \beta = 0\).

For the integrable positive solutions of (1.1), when \(0 < p, q < \infty\), Onodera [12] proved the radial symmetry, the monotonicity, the optimal integrability and the asymptotic rates when \(|x| \to 0\) and \(|x| \to \infty\).

**Proposition 1.1.** Let \((u, v) \in L^{p+1}(\mathbb{R}^n) \times L^{s+1}(\mathbb{R}^n)\) be a pair of nonnegative solutions of system (1.1) with (1.2). Then \(u\) and \(v\) are smooth away from the origin, radially symmetric and strictly decreasing in the radial direction. Moreover, the center of the symmetry must be the origin unless \(\alpha = \beta = 0\).
Since \((f, g) \in L^r(R^n) \times L^s(R^n)\) in (1.4), the assumption \((u, v) \in L^{r+1}(R^n) \times L^{s+1}(R^n)\) is natural. The integrable solutions \((u, v) \in L^{r+1}(R^n) \times L^{s+1}(R^n)\) are called the finite energy solutions.

**Proposition 1.2.** Under the same assumptions of Proposition 1.1, then

\[ u \in L^r(R^n) \quad \text{and} \quad v \in L^s(R^n) \]

hold for \(r, s\) satisfying

\[
\max\left\{ \frac{\alpha}{n}, \frac{\beta q + \bar{\lambda} - n}{n} \right\} < \frac{1}{r} < \min\left\{ \frac{\lambda + \alpha}{n}, \frac{q(\lambda + \beta) + \bar{\lambda} - n}{n} \right\}, \tag{1.8}
\]

\[
\max\left\{ \frac{\beta}{n}, \frac{\alpha p + \bar{\lambda} - n}{n} \right\} < \frac{1}{s} < \min\left\{ \frac{\lambda + \beta}{n}, \frac{p(\lambda + \alpha) + \bar{\lambda} - n}{n} \right\}. \tag{1.9}
\]

Here, \(\bar{\lambda} = \lambda + \alpha + \beta\).

**Proposition 1.3.** Under the same assumptions of Proposition 1.1, \(u\) and \(v\) have the following profiles.

(i) Around the origin. If \(\lambda + \beta(q + 1) < n\), then it holds that

\[ u(x) \simeq \frac{A_0}{|x|^{\alpha}}, \tag{1.10} \]

and

\[ v(x) \simeq \begin{cases} 
\frac{A_1}{|x|^{\alpha}}, & \text{if } \lambda + \alpha(p + 1) < n \\
\frac{A_2}{|\ln |x||}, & \text{if } \lambda + \alpha(p + 1) = n \\
\frac{A_3}{|x|^{\alpha(p+1) + \beta + \lambda - n}}, & \text{if } \lambda + \alpha(p + 1) > n
\end{cases} \tag{1.11} \]

where \(A_0 = \int_{R^n} \frac{\nu^s(y)}{|y|^{\alpha + n}} dy, A_1 = \int_{R^n} \frac{\nu^s(y)}{|y|^{\lambda + n}} dy, A_2 = |S^{n-1}|(\int_{R^n} \frac{\nu^s(y)}{|y|^{\alpha + n}} dy)^p\) and

\[ A_3 = \left( \int_{R^n} \frac{\nu^s(y)}{|y|^{\alpha + n}} dy \right)^p \int_{R^n} \frac{dz}{|z|^{\alpha + \lambda + (p+1)\beta - n}}, \] Here, \(e\) is a unit vector in \(R^n\) and \(|S^{n-1}|\) is the surface area of the unit sphere.

(ii) Around the infinity. If \(\lambda q + \beta(q + 1) > n\), then it holds that

\[ u(x) \simeq \frac{B_0}{|x|^{\lambda + \alpha}}, \tag{1.12} \]

and

\[ v(x) \simeq \begin{cases} 
\frac{B_1}{|x|^{\lambda + \beta}}, & \text{if } \lambda p + \alpha(p + 1) > n \\
\frac{B_2}{|\ln |x||}, & \text{if } \lambda p + \alpha(p + 1) = n \\
\frac{B_3}{|x|^{\alpha + \lambda + (p+1)\beta - n}}, & \text{if } \lambda p + \alpha(p + 1) < n
\end{cases} \tag{1.13} \]

where \(B_0 = \int_{R^n} \frac{\nu^s(y)}{|y|^{\alpha + n}} dy, B_1 = \int_{R^n} \frac{\nu^s(y)}{|y|^{\lambda + n}} dy, B_2 = |S^{n-1}|(\int_{R^n} \frac{\nu^s(y)}{|y|^{\alpha + n}} dy)^p\) and

\[ B_3 = \left( \int_{R^n} \frac{\nu^s(y)}{|y|^{\alpha + n}} dy \right)^p \int_{R^n} \frac{dz}{|z|^{2n - (\alpha + \lambda)(p+1)\beta - n}|e - z|^\lambda}. \]

Clearly, Propositions 1.1-1.3 are helpful for understanding the shape of the positive solutions of the system (1.1).

In this paper, we consider the properties of solutions of (1.1) for \(0 < p, q < \infty\). We first establish the equivalence of PDEs system (1.3) and the integral system.
Let $(u, v)$ be a pair of positive solutions of the PDE system (1.3) with (1.2) satisfying $(u, v) \in L^{p+1}(\mathbb{R}^n) \times L^{q+1}(\mathbb{R}^n)$, if $pq \neq 1$ and $\max\{\alpha(p + 1), \beta(q + 1), \alpha \frac{2p+1}{p}, \beta \frac{2q+1}{q}\} \leq \min\{n - \lambda, \frac{n-\lambda+1}{2}, 2\}$, then $(u, v)$ is also solutions of the integral system (1.1) if we omit the constants.

**Theorem 1.2.** Solution of the integral system (1.1) is also the solution of the PDE system (1.3) if we omit the constants.

**Theorem 1.3.** If $(u, v)$ is a pair of the finite energy solutions of (1.1) with (1.2), then $u, v \in L^{\infty}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$.

**Theorem 1.4.** Let $u, v \in L^{\infty}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ be positive solutions of (1.1) with (1.2). If

$$(\lambda + \beta)(q + 1) \leq 2n, \quad (\lambda + \alpha)(p + 1) \leq 2n,$$

then $u, v$ are symmetric and decreasing about the origin.

**Remark.** (1) On the regularities of the finite energy solutions, Proposition 1.2 gives the optimal integrability. Here, Theorem 1.3 shows that the finite energy solutions must be locally bounded.

(2) Proposition 1.1 shows the finite energy solutions have radial symmetry property. Here, Theorem 1.4 shows the axisymmetry for the locally bounded solutions.

(3) According to Theorem 1.1, the result of Theorems 1.3 are still true for the positive solutions of the fractional order partial differential system (1.3).

(4) Difficulties in proving Theorem 1.1 and Theorem 1.4 are the case $0 < p < 1$ or $0 < q < 1$. To deal with the first difficulty in proving Theorem 1.1, we need to use the integrable condition and more complicated analytical techniques. To overcome the second difficulty in proving Theorem 1.4, we need to use the method in [12].

2. Equivalence. In this section, we will prove Theorem 1.1 and 1.2.

To prove Theorem 1.1, we first give the following result.

**Theorem 2.1.** If $(u, v)$ is a pair of positive solutions of

$$(\Delta)^n(|x|^\alpha u) = |x|^{-\beta}v^\alpha, \quad (\Delta)^n(|x|^\beta v) = |x|^{-\alpha}u^\beta,$$

with (1.2) satisfying $(u, v) \in L^{p+1}(\mathbb{R}^n) \times L^{q+1}(\mathbb{R}^n)$, where $m = \frac{n-\lambda}{2}$ is a positive integer, if $pq \neq 1$, and $\max\{\alpha(p + 1), \beta(q + 1)\} < 2$ and $\max\{\alpha \frac{2p+1}{p}, \beta \frac{2q+1}{q}\} \leq m + \frac{3}{2}$, then for $1 \leq j \leq m - 1$,

$$(\Delta)^j(|x|^\alpha u) > 0, \quad (\Delta)^j(|x|^\beta v) > 0.$$ 

**Remark.** The condition $\max\{\alpha(p + 1), \beta(q + 1)\} < 2$ is to ensure that the average of $u, v$ are well-defined.

**Proof.** Here we only prove the case $0 < p < 1, q > 1$. The other cases similarly can be proved.

For $\alpha \geq 0$ and $\beta \geq 0$, we use the ideas introduced by Wei-Xu (cf. [14]), Lei-Lü (cf. [9]). Write

$$u_j = (\Delta)^j(|x|^\alpha u), \quad v_j = (\Delta)^j(|x|^\beta v).$$

We need to prove $u_j, v_j > 0$ for $j = 1, 2, \cdots, m - 1$.

**Step 1.** We claim $u_{m-1} \geq 0$. 
Otherwise, there exists \( x_1 \) such that
\[
    u_{m-1}(x_1) < 0. 
\] (2.2)

We will deduce a contradiction by four substeps.

(i) Define the average of \( u, v \) as
\[
    \bar{u}(r) := \frac{1}{|\partial B_r(x_1)|} \int_{\partial B_r(x_1)} |x|^\alpha u(x) \, ds, \quad \bar{v}(r) := \frac{1}{|\partial B_r(x_1)|} \int_{\partial B_r(x_1)} |x|^\beta v(x) \, ds.
\]

Write
\[
    \bar{u}_j(r) := \frac{1}{|\partial B_r(x_1)|} \int_{\partial B_r(x_1)} u_j(x) \, ds, \quad \bar{v}_j(r) := \frac{1}{|\partial B_r(x_1)|} \int_{\partial B_r(x_1)} v_j(x) \, ds.
\]

Here \( j = 1, 2, \ldots, m - 1 \). Then for \( r > 0 \), we have
\[
-\Delta \bar{u}_m - 1 = \bar{u}_1, \quad -\Delta \bar{v}_m - 1 = \bar{v}_1, \
 basics, \quad \bar{v}_m = \bar{v}_{m-1}, \quad -\Delta \bar{v}_{m-1} > 0, \quad -\Delta \bar{v}_{m-1} > 0. \] (2.3)

We claim that
\[
    -\Delta \bar{u}_{m-1} \geq c(r + |x_1|)^{-\beta(p+1)} \bar{u}^\alpha, \
    -\Delta \bar{v}_{m-1} \geq c(r + |x_1|)^{-\alpha(1 + \frac{s}{s-p})} \bar{v}^\alpha \frac{p(n-1)N}{r^{p(p+1)-N}}, \] (2.4)

for any \( r > 0 \). Where positive real number \( s > 1 \) and positive integers \( M, N \) satisfy \( \frac{s}{p} = p, (1 - p^2)M \leq N \leq (1 - \frac{p^2}{p+2})M \). Here \( c \) is an absolute constant (independent of \( r \)).

In fact, by Hölder inequality and \( q > 1 \), we easily obtain the first inequality in (2.4). We mainly prove the second inequality in (2.4).

By (2.3), we have
\[
-\Delta \bar{v}_{m-1} = \frac{1}{|\partial B_r(x_1)|} \int_{\partial B_r(x_1)} (-\Delta)^m (|x|^\beta v(x)) \, ds \
= \frac{1}{|\partial B_r(x_1)|} \int_{\partial B_r(x_1)} |x|^{-\alpha} u^p(x) \, ds \
\geq (r + |x_1|)^{-\alpha(1 + \frac{s}{s-p})} \frac{p(n-1)N}{r^{p(p+1)-N}} \int_{\partial B_r(x_1)} |x|^\beta v(x) \, ds. \] (2.5)

On the other hand, for \( p < 1 \), let \( s > 1 \) be positive real number and \( M, N \) be positive integers such that \( \frac{s}{p} = p, (1 - p^2)M \leq N \leq (1 - \frac{p^2}{p+2})M \), then we have \( \frac{p}{p+1} \leq \frac{M-N}{s-p} < p \). Using the generalized Hölder inequality with \( \frac{M-N}{s} + \frac{s-M+N}{p+1} = 1 \) and \( u \in L^{p+1} \),
\[
\bar{u}^p = \frac{1}{|\partial B_r(x_1)|^p} \left( \int_{\partial B_r(x_1)} |x|^\alpha u^p \, ds \right)^{\frac{M-N}{M}} \left( \int_{\partial B_r(x_1)} |x|^\alpha u^p \, ds \right)^{\frac{Np}{p(p+1)}} \
\leq \frac{1}{|\partial B_r(x_1)|^p} \left( \int_{\partial B_r(x_1)} |x|^\alpha u^p \, ds \right)^{\frac{M-N}{M}} \left( \int_{\partial B_r(x_1)} u^{p+1} \, ds \right)^{\frac{Np}{p(p+1)}} \
\times \left( \int_{\partial B_r(x_1)} |x|^\alpha \frac{1}{|x|^\frac{M-N}{p+1}} \, ds \right)^{\frac{Np}{p(p+1)-N}} \
\leq C(r + |x_1|)^{\frac{Np}{p}} \left( \frac{1}{|\partial B_r(x_1)|} \int_{\partial B_r(x_1)} |x|^\alpha u^p \, ds \right)^{\frac{M-N}{M}} \left( \frac{1}{|\partial B_r(x_1)|} \int_{\partial B_r(x_1)} |x|^\beta v(x) \, ds \right)^{\frac{Np}{r^{p(p+1)-N}}},\]
then we have
\[
\frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} (|x|^a u)^p ds \geq C (r + |x_1|)^{\frac{nN}{M-N} - \frac{\alpha}{M}} \bar{u}^{\frac{\alpha(N-1)}{M}} r^{\frac{\alpha(N-1)}{M}},
\]  
(2.6)

Combining with (2.5), we get the second inequality in (2.4).

Eq. (2.3) shows
\[ -r^{1-n} (r^{n-1} \bar{u}'_{m-1})' > 0, \quad \forall r > 0. \]
Thus, \( \bar{u}'_{m-1} < 0 \). In view of (2.2),
\[ \bar{u}_{m-1}(r) \leq \bar{u}_{m-1}(0) = u_{m-1}(x_1) < 0, \quad \forall r \geq 0. \]
(2.7)

By this and (2.3), we get
\[ -r^{1-n} (r^{n-1} \bar{u}'_{m-2}(r))' = \bar{u}_{m-1}(r) < \bar{u}_{m-1}(0) := -c. \]
After integrating, we obtain \( \bar{u}'_{m-2} > c_1 r \) and hence
\[ \bar{u}_{m-2}(r) \geq \bar{u}_{m-2}(0) + c_2 r^2, \quad \forall r \geq 0. \]
(2.8)

So we can find a suitably large \( r_1 > 0 \) such that \( \bar{u}_{m-2}(r) > c_2 r^2 \) for \( r \geq r_1 \).

(ii) In general, by the same argument above, we can see that
\[ (-1)^j \bar{u}_{m-j}(r) \geq \tau_j r^{2(j-1)}, \quad \text{for } r \geq \bar{r}_j, j = 1, 2, \ldots, m. \]
(2.9)

By an analogous process, we can also deduce that
\[ (-1)^j \bar{u}_{m-j}(r) \geq \tau_{j} r^{2(j-1)}, \quad \text{for } r \geq \bar{r}_j, j = 1, 2, \ldots, m. \]
(2.10)

(iii) We claim that \( m \) must be even.
If \( m \) is odd, we have a contradiction with the fact that \( u > 0 \). So \( m \) must be even.

(iv) From (ii) and (iii), we have
\[
\bar{u}(r) \geq c_0 r^{\sigma_0}, \quad \sigma_0 = 2(m-1)
\]  
(2.11)

for \( r > \bar{r}_0 > 0 \).

Setting \( A = n - \alpha - \frac{\alpha}{M-N} + 2(m-1) \frac{s}{M-N} + 2m \), and Suppose now that
\[ \bar{u}(r) \geq c_0 \left( \frac{\bar{s}}{M-N} \right)^k r^{\sigma_k} A^{\frac{M-N}{\bar{s}}}, \quad \text{for } r \geq r_k. \]
(2.12)

Combining with (2.4), we have
\[
r^{n-1} \bar{v}'_{m-1}(r) \leq r_k^{n-1} \bar{v}'_{m-1}(r_k) - C \int_{r_k}^r t^{n-1} \frac{t^{n-1} - \alpha - \frac{\alpha}{M-N} + \sigma_k \frac{s}{M-N}}{A^{\frac{M-N}{\bar{s}}}} \frac{(\frac{\bar{s}}{M-N})^{k+1}}{A^{\frac{M-N}{\bar{s}}}} dt.
\]
\[ \bar{v}'_{m-1}(r) \leq -c_0 \left( \frac{\bar{s}}{M-N} \right)^{k+1} \frac{r^{1-\alpha - \frac{\alpha}{M-N} + \sigma_k \frac{s}{M-N}} - r_k^{1-\alpha - \frac{\alpha}{M-N} + \sigma_k \frac{s}{M-N}}}{n - \alpha - \frac{\alpha}{M-N} + \sigma_k \frac{s}{M-N}}. \]

Hence, by \( \alpha \frac{2p+1}{p} \leq m + \frac{1}{2} \) and \( \frac{p}{p+1} \leq \frac{M-N}{s} < p \),
\[ \bar{v}'_{m-1}(r) \leq -c_0 \left( \frac{\bar{s}}{M-N} \right)^{k+1} \frac{r^{2-\alpha - \frac{\alpha}{M-N} + \sigma_k \frac{s}{M-N}}}{2A^{\frac{M-N}{\bar{s}}}} \frac{n - \alpha - \frac{\alpha}{M-N} + \sigma_k \frac{s}{M-N}}{n - \alpha - \frac{\alpha}{M-N} + \sigma_k \frac{s}{M-N}} \]
for \( r \geq \frac{1}{n-\alpha - \frac{\alpha}{M-N} + \sigma_k \frac{s}{M-N}} r_k \).

Similarly
\[ \bar{v}_{m-1}(r) \leq -c_0 \left( \frac{\bar{s}}{M-N} \right)^{k+1} \frac{r^{2-\alpha - \frac{\alpha}{M-N} + \sigma_k \frac{s}{M-N}}}{4A^{\frac{M-N}{\bar{s}}}} \left( n - \alpha - \frac{\alpha}{M-N} + \sigma_k \frac{s}{M-N} \right)^2 \]
Then we have

\[(−1)^{k} \tilde{v}_{m−i}(r) ≥ c_{0} \frac{(\alpha s/\beta N)k+1}{4^{m}A^{k+1}^{\alpha s/\beta N}} \left(\frac{1}{n−\alpha−\sigma k\beta N^}\right)^{2k} \]

for \(r ≥ 2^{\frac{1}{\alpha−\sigma s}+\sigma k\beta N^} r_k\).

By induction, we have

\[\tilde{v}(r) \geq c_{0} \frac{(\alpha s/\beta N)k+1}{4^{m}A^{k+1}^{\alpha s/\beta N}} \left(\frac{1}{n−\alpha−\sigma k\beta N^} + 2m2k\right)^{2k+1},\]

(2.13)

for \(r ≥ 2^{\frac{1}{\alpha−\sigma s}+\sigma k\beta N^} r_k\).

Hence

\[\tilde{v}(r) \geq c_{0} \frac{(\alpha s/\beta N)^{k+1}}{4^{m}A^{k+1}^{\alpha s/\beta N}} \left(\frac{1}{n−\alpha−\sigma k\beta N^} + 2m2k\right)^{2k+1},\]

(2.14)

for \(r ≥ 2^{\frac{1}{\alpha−\sigma s}+\sigma k\beta N^} r_k\).

Set

\[\sigma_{0} = 2(m−1), r_{0} = \tilde{r}_{0} \]

\[\sigma_{k+1} = 2m−\alpha−\sigma k\beta N^ \]

\[r_{k+1} = 2^{\frac{1}{\alpha−\sigma s}+\sigma k\beta N^} r_k,\]

By induction, it is easy to see that

\[n−\alpha−\sigma k\beta N^ + 2m ≤ A^{k+1}\]

by noticing that

\[n−\alpha−\sigma k\beta N^ + 2m ≤ A(n−\alpha−\sigma k−1\beta N^ + 2m).\]

We set

\[b_{0} = 0, b_{k+1} = \frac{s}{\beta N^}b_{k} + 2m(k+1).\]

Then we have

\[\tilde{v}(r) \geq c_{0} \frac{(\alpha s/\beta N)^{k+1}}{4^{m}A^{k+1}^{\alpha s/\beta N}},\]

for \(r ≥ r_{k+1}^{−1}\).

By using the iteration formulas above, we have

\[\sigma_{k} = 2(m−1)\left(\frac{s}{\beta N^}\right)^{k} + (2m−\alpha−\sigma k\beta N^)\left(\frac{s}{\beta N^}\right)^{k−1} \]

\[b_{k} = 2m\left(\frac{s}{\beta N^}\right)^{k−1} − (k+1)\frac{s}{\beta N^} + k\left(\frac{s}{\beta N^} − 1\right)^{2}\]

Notice that

\[r_{k+1} ≤ 2^{\frac{1}{\alpha−\sigma s}+\sigma k\beta N^} r_{0}.\]

Hence, if we take \(M ≥ 1\) is large enough so that

\[MA^{\frac{3}{\alpha−\sigma s}+\sigma k\beta N^} ≥ 2^{1+\frac{1}{\alpha−\sigma s}+\sigma k\beta N^} r_{0} \quad \text{if} \quad c_{0} ≥ 1\]

and

\[MA^{\frac{3}{\alpha−\sigma s}+\sigma k\beta N^} c_{0}^{−1} ≥ 2^{1+\frac{1}{\alpha−\sigma s}+\sigma k\beta N^} r_{0} \quad \text{if} \quad c_{0} < 1,\]
and then we take \( r_1 = MA^{\frac{3}{m-n}} \) if \( c_0 \geq 1 \) and \( r_0 = MA^{\frac{3}{m-n}}c_0^{-1} \) if \( c_0 < 1 \), then by \( m > 1 \), we have

\[
\varepsilon(r_1) \geq \frac{c}{4m} A \frac{(4m-6)(\frac{3}{m-n})^{k+2} + (6-3n-\frac{3n}{m-n}) \frac{3}{m-n} + \frac{3n}{m-n}}{(\frac{3}{m-n}-1)^2} \rightarrow \infty \text{ as } k \rightarrow \infty.
\]

Since \( r_1 \) is independent of \( k \), this is impossible.

**Step 2.** Moreover, \( u_{m-1} > 0 \).

Otherwise, there exists \( x_0 \neq 0 \) such that \( u_{m-1}(x_0) = 0 \) and \( u_{m-1}(x) \geq 0 \). This shows \( x_0 \) is a local minimizer and hence

\[
0 \leq \Delta u_{m-1}(x_0) = -|x_0|^{-\beta}v^q(x_0) < 0.
\]

It is impossible. Similarly, \( v_{m-1} > 0 \) is also true.

**Step 3.** \( u_{m-j}, v_{m-j} > 0 \) for \( j = 2, 3, \ldots, m-1 \).

Suppose \( x_0 \) satisfying \( u_{m-j}(x_0) < 0 \). Similar to the argument in Step 1, we see that the signs of \( \bar{u}_j \) are alternating (cf. (2.9)). Combining (2.3) we can deduce by finite steps that \(-\Delta \bar{u} < 0\). Therefore, by integrating twice, we get

\[
\bar{u}(r) \geq \bar{u}(0) = c > 0.
\]

Thus, similar to the argument in Step 1, from (2.4), we can choose \( R^* \) sufficiently large such that \( \bar{u}_{m-1}(R^*) < 0 \). Step 2 shows that it is impossible. \( \square \)

**Proof of Theorem 1.1.** **Case I.** \( n - \lambda \) is even. Write \( m = \frac{n-\lambda}{2} \).

Fix \( x_0 \in \mathbb{R}^n \setminus \{0\} \). Denote \( B_r(x_0) \) by \( B \). Let \( \phi(x) \) solve

\[
(-\Delta)^m \phi(x) = \delta(x), \quad x \in B_r(x_0)
\]

\[
\phi = \Delta \phi = \cdots = \Delta^{m-1} \phi = 0, \quad \text{on } \partial B_r(x_0).
\]

(2.15)

By the results in section 4 of [3], \( \phi \) satisfies

\[
\phi(x) \rightarrow C|x-x_0|^{2m-n}, \quad \text{when } r \rightarrow \infty.
\]

(2.16)

In addition, on \( \partial B \) for \( j = 0, 1, \ldots, m-1 \), there hold

\[
\partial_{r^j}((-\Delta)^j \phi) \leq 0
\]

(2.17)

and

\[
|\partial_{r^j}((-\Delta)^j \phi)| \leq C r^{2m-n-1-2j}.
\]

(2.18)

Multiply the PDE

\[
(-\Delta)^m(|x|^\alpha u(x)) = |x|^{-\beta}v^q(x)
\]

by \( \phi(x) \) and integrate by parts. We have

\[
\int_B |x|^{-\beta}v^q \phi dx = \int_B (-\Delta)^m(|x|^\alpha u) \phi dx
\]

\[
= |x_0|^\alpha u(x_0) + \sum_{j=0}^{m-1} \int_{\partial B} (-\Delta)^j(|x|^\alpha u) \partial_{r^j}((-\Delta)^m-1-j \phi)|ds.
\]

(2.19)

By using (2.17) and Theorem 2.1, we can deduce from (2.19) that

\[
\int_B |x|^{-\beta}v^q \phi dx \leq |x_0|^\alpha u(x_0), \quad \forall r > 0.
\]

Letting \( r \rightarrow \infty \) and using (2.16), we see

\[
\int_{\mathbb{R}^n} |x|^{-\beta}v^q |x-x_0|^{2m-n} dx \leq C |x_0|^\alpha u(x_0) < \infty.
\]

(2.20)
Similarly,
\[ \int_{\mathbb{R}^n} |x|^{-\alpha} u^p |x - x_0|^{2m-n} dx < \infty. \]  
(2.21)

Thus, we can find \( r_1 \to \infty \) such that
\[ \int_{\partial B(x_0, r_1)} |x|^{-\alpha} u^p |x - x_0|^{2m-n+1} ds \to 0. \]  
(2.22)

When \( p > 1 \), using the Hölder inequality, we get
\[
\int_{\partial B(x_0, r_1)} |x|^{\alpha} u(x) |x - x_0|^{-n+1} ds 
\leq C r_1^{\frac{\alpha(p+1)-2n}{p}} \left( \int_{\partial B(x_0, r_1)} |x|^{-\alpha} u^p |x - x_0|^{2m-n+1} ds \right)^{1/p}.
\]

When \( p < 1 \), let \( s > 1 \) be positive real number and let \( M, N \) be positive integers such that \( \frac{s}{M} = p, (1 - p^2)M \leq N \leq (1 - \frac{s^2}{p+1})M \), then we have \( \frac{p}{p+1} \leq \frac{M-N}{s} < p \).

Using the generalized Hölder inequality with \( \frac{M-N}{s} + \frac{s-M+N}{s+1} = 1 \), Hölder inequality with \( \frac{pN}{(p+1)(s-M+N)} + 1 - \frac{pN}{(p+1)(s-M+N)} = 1 \) and the condition \( u \in L^{p+1}(\mathbb{R}^n) \), we get
\[
\int_{\partial B(x_0, r_1)} |x|^{\alpha} u(x) |x - x_0|^{-n+1} ds 
\leq C r_1^{\frac{\alpha}{M-N} + \alpha - 2m} \left( \int_{\partial B(x_0, r_1)} |x|^{-\alpha} u^p |x - x_0|^{2m-n+1} ds \right)^{1/(M-N)/s}.
\]

Noting \( \alpha \cdot \frac{s}{M-N} + \alpha - 2m \leq 0 \), from \( \max\{\alpha \frac{2n+1}{p}, \beta \frac{2n+1}{q}\} \leq n - \lambda = 2m \) and \( \frac{p}{p+1} \leq \frac{M-N}{s} < p \). Then from (2.22) we deduce that as \( r_1 \to \infty \),
\[ \int_{\partial B(x_0, r_1)} |x|^{\alpha} u(x) |x - x_0|^{-n+1} ds \to 0. \]  
(2.23)

When \( j \geq 1 \), replacing \( |x|^{-\beta} u^q \) by \( (-\Delta)^j (|x|^\alpha u) \) in (2.19), we also have
\[ \int_B (-\Delta)^j (|x|^\alpha u) \phi dx \leq |x_0|^\alpha u(x_0), \quad \forall r > 0. \]

Letting \( r \to \infty \), we obtain
\[ \int_{\mathbb{R}^n} (-\Delta)^j (|x|^\alpha u) |x - x_0|^{2j-n} dx < \infty. \]

Summing \( j \) from 1 to \( m - 1 \) yields
\[ \int_{\mathbb{R}^n} \sum_{j=1}^{m-1} (-\Delta)^j (|x|^\alpha u) |x - x_0|^{n-2j} dx < \infty. \]

Therefore, we can find a subsequence of \( r_1 \) (denoted by itself) such that as \( r_1 \to \infty \),
\[ \int_{\partial B(x_0, r_1)} \sum_{j=1}^{m-1} (-\Delta)^j (|x|^\alpha u) |x - x_0|^{n-2j} ds \to 0. \]  
(2.24)

Combining (2.18), (2.23) and (2.24) and letting \( r_1 \to \infty \), we obtain
\[ \int_{\partial B(x_0, r_1)} \sum_{j=0}^{m-1} (-\Delta)^j (|x|^\alpha u) \partial_r ((-\Delta)^{m-1-j} \phi) ds \to 0. \]  
(2.25)
Similarly, combining two cases I and II together, we complete the proof of Theorem 1.1.

\[ \int_{B_{r_t}} |y|^{-\beta} v^\eta(y) \phi(y) dy \to \int_{\mathbb{R}^n} \frac{cv^\eta(y)dy}{|y - x_0|^{n-2m}|y|^{\beta}} , \]

when \( r_t \to \infty \). Inserting this result and (2.25) into (2.19), we get

\[ |x_0|^\alpha u(x_0) = c \int_{\mathbb{R}^n} \frac{v^\eta(y)dy}{|y - x_0|^{n-2m}|y|^{\beta}}. \]

Similarly,

\[ |x_0|^\beta v(x_0) = c \int_{\mathbb{R}^n} \frac{u^p(y)dy}{|y - x_0|^{n-2m}|y|^{\alpha}}. \]

**Case II.** \( n - \lambda \) is not an integer, this follows from our definition. The following is an intuitive idea behind this definition. For any \( \phi \in C_0^\infty(\mathbb{R}^n) \), let \( \psi \in H^{n-\lambda}(\mathbb{R}^n) \) satisfy

\[ (-\Delta)^{(n-\lambda)/2}\psi = \phi. \]

Namely, there exists a constant \( c \) such that

\[ \psi(x) = c \int_{\mathbb{R}^n} \frac{\phi(y)dy}{|x - y|^{\lambda}}. \]

Clearly, \( \psi \in H^{(n-\lambda)/2}(\mathbb{R}^n) \). Since \( u, v \) solve (2), we have

\[ \int_{\mathbb{R}^n} (-\Delta)^{\frac{n-\lambda}{4}} (|x|^\alpha u(x)) (-\Delta)^{\frac{n-\lambda}{4}} \psi(x) dx = \int_{\mathbb{R}^n} |x|^{-\beta} v^\eta(x) \psi(x) dx. \] (2.26)

Integrating by parts, we see the left hand side of (2.26) becomes

\[ \int_{\mathbb{R}^n} (-\Delta)^{\frac{n-\lambda}{4}} (|x|^\alpha u(x)) (-\Delta)^{\frac{n-\lambda}{4}} \psi(x) dx \]

\[ = \int_{\mathbb{R}^n} |x|^\alpha u(x) (-\Delta)^{\frac{n-\lambda}{2}} \psi(x) dx = \int_{\mathbb{R}^n} |x|^\alpha u(x) \phi(x) dx. \]

Exchanging the order of the integral variables, we see the right hand side of (2.26) becomes

\[ \int_{\mathbb{R}^n} |x|^{-\beta} v^\eta(x) \psi(x) dx = c \int_{\mathbb{R}^n} |x|^{-\beta} v^\eta(x) \int_{\mathbb{R}^n} \frac{\phi(y)dy}{|x - y|^{\lambda}} dx \]

\[ = c \int_{\mathbb{R}^n} \phi(x) \int_{\mathbb{R}^n} \frac{v^\eta(y)dy}{|x - y|^{\lambda}|y|^{\beta}} dx. \]

Thus, from (2.26) it follows

\[ |x|^\alpha u(x) = c \int_{\mathbb{R}^n} \frac{v^\eta(y)dy}{|x - y|^{\lambda}|y|^{\beta}}. \]

Similarly,

\[ |x|^\beta v(x) = c \int_{\mathbb{R}^n} \frac{u^p(y)dy}{|x - y|^{\lambda}|y|^{\alpha}}. \]

Combining two cases I and II together, we complete the proof of Theorem 1.1. \( \square \)

**Proof of Theorem 1.2.** **Case I.** \( n - \lambda \) is even. Write \( m = \frac{n-\lambda}{2} \). Clearly, from (1.1) we obtain

\[ |x|^\alpha u(x) = \int_{\mathbb{R}^n} \frac{|y|^{-\beta} v^\eta(y)dy}{|y - x|^{n-2m}}, \quad |x|^\beta v(x) = \int_{\mathbb{R}^n} \frac{|y|^{-\alpha} u^p(y)dy}{|y - x|^{n-2m}}. \] (2.27)
We assume the solutions $u, v$ are smooth except for the origin. In fact, the assumption can be relaxed to $u, v \in L^\infty_{loc}(R^n \setminus \{0\})$. Based on this local boundedness, the regularity of $u, v$ can be lifted to $u, v \in C^\infty_{loc}(R^n \setminus \{0\})$ via an analogous potential estimates as in Chapter 4 in [4]. Differentiate (2.27) with respect to $x$, since $c|x|^{2m-n}$ is the basic solution of $(-\Delta)^m u = 0$, using the convolution properties of the delta function, we can see (1.3).

**Case II.** $n - \lambda$ is a positive real number.

Assume $u, v$ satisfy

$$|x|^\alpha u(x) = \int_{R^n} \frac{v^q(y)dy}{|x-y|^\lambda|y|^\beta}.$$ 

Using the Fourier transform, we have

$$[|x|^\alpha u(x)]^\wedge(\xi) = [|x|^{-\beta} v^q(x) * |x|^{-\lambda}]^\wedge(\xi)
= c(|x|^{-\lambda}|x|^{-\beta} v^q(x))^\wedge(\xi) = c|\xi|^{\lambda-n}|x|^{-\lambda}v^q(x)^\wedge(\xi).$$

By this result and the Parseval identity, it follows

$$\int_{R^n} (-\Delta)^{n+\lambda}(|x|^\alpha u(x))(-\Delta)^{n+\lambda} \phi(x)dx
= c \int_{R^n} |\xi|^{n+\lambda} |x|^\alpha u(x)^\wedge(\xi) \phi^\wedge(\xi) d\xi
= c \int_{R^n} [|x|^{-\beta} v^q(x)]^\wedge(\xi) \phi^\wedge(\xi) d\xi
= \int_{R^n} |x|^{-\beta} v^q(x) \phi(x) dx.$$ 

Similarly,

$$\int_{R^n} (-\Delta)^{n+\lambda}(|x|^\beta v(x))(-\Delta)^{n+\lambda} \phi(x) dx = \int_{R^n} |x|^{-\alpha} u^p(x) \phi(x) dx.$$

These results show that $u, v$ are weak solutions of PDE system.

Combining two cases I and II together, we complete the proof of Theorem 1.2. \qed

3. **Local boundedness.** Proof of Theorem 1.3. Assume $(u, v) \in L^{p+1}(R^n) \times L^{q+1}(R^n)$ is a pair of positive solutions with the critical condition

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{\lambda}{n}.$$ 

(3.1)

We prove $u, v \in L^\infty_{loc}(R^n \setminus \{0\})$.

Such a critical condition implies that $\lambda + (p+1)\alpha \geq n$ and $\lambda + (q+1)\beta \geq n$ can not hold at the same time. Without loss of generality, we assume

$$\lambda + (q+1)\beta < n.$$ 

(3.2)

**Step 1.** $u \in L^\infty_{loc}(R^n \setminus \{0\})$.

According to (3.1) in [12], we have

$$A_* := \int_{R^n \setminus B_1(0)} \frac{v^q(y)dy}{|y|^{\lambda+\beta}} < \infty.$$ 

We can prove that

$$B_* := \int_{B_1(0)} \frac{v^q(y)dy}{|y|^\beta} < \infty.$$
In fact, if \( x \in B_2 \setminus B_1 \) and \( y \in B_1 \), then \(|x - y| \leq 3\) and hence \(|x| > \frac{|x - y|}{3}\). Thus, when \( x \in B_2 \setminus B_1 \), there holds
\[
\int_{B_1(0)} \frac{v^q(y)dy}{|y|^\beta} < C \int_{B_1(0)} \frac{|x|^\lambda v^q(y)dy}{|x - y|^\lambda |y|^\beta}.
\]
Integrating in \( B_2 \setminus B_1 \) yields
\[
|B_2 \setminus B_1| \int_{B_1(0)} \frac{v^q(y)dy}{|y|^\beta} < C \int_{B_2 \setminus B_1} \int_{B_1(0)} \frac{|x|^\lambda v^q(y)dy}{|x - y|^\lambda |y|^\beta} dx.
\]
\[
\leq C \int_{B_2 \setminus B_1} |x|^\lambda u(x)dx \leq C \|u\|_{p+1} < \infty.
\]
It is easy to see \( \int_{B_1(0)} \frac{v^q(y)dy}{|y|^\beta} < \infty \). Similarly, we can also get that \( \int_{B_1(0)} \frac{v^q(y)dy}{|y|^\alpha} < \infty \).

Let \( x_0 \in \mathbb{R}^n \setminus \{0\} \). Clearly, the singularities of the following improper integral
\[
|x_0|^\alpha u(x_0) = \int_{\mathbb{R}^n} \frac{v^q(y)dy}{|x_0 - y|^\beta}
\]
are 0, \( x_0, \infty \). If \( y \) is close to the origin 0, then there exists \( 1 > \delta > 0 \) such that \(|x_0 - y| > |x_0|/2\) for \( y \in B_3(0) \). Thus
\[
\int_{B_3(0)} \frac{v^q(y)dy}{|x_0 - y|^\beta} \leq \frac{C}{|x_0|^\lambda} \int_{B_3(0)} \frac{v^q(y)dy}{|y|^\beta} \leq C B_* < \infty.
\]
If \( y \) converges to infinity, then there exists \( R > 1 \) such that \(|y - x_0| > |y|/2\) for \( y \in \mathbb{R}^n \setminus B_R(0) \). Thus,
\[
\int_{\mathbb{R}^n \setminus B_R(0)} \frac{v^q(y)dy}{|x_0 - y|^\beta} \leq C \int_{\mathbb{R}^n \setminus B_R(0)} \frac{v^q(y)dy}{|y|^\beta + \lambda} \leq C A_* < \infty.
\]
If \( y \) is close to \( x_0 \), then there exists \( \frac{|x_0|}{2} > \delta > 0 \) such that \(|y| > |x_0|/2\) for \( y \in B_3(x_0) \). Thus
\[
\int_{B_3(x_0)} \frac{v^q(y)dy}{x_0 - y|^\beta} \leq \frac{C}{|x_0|^\beta} \int_{B_3(x_0)} \frac{v^q(y)dy}{x_0 - y|^\lambda} = C |x_0|^{-\beta} I.
\]
It is sufficient to prove \( I < \infty \).

Take \( s \) satisfying
\[
\frac{1}{qs} = \frac{1}{n} \max\{ \beta, p\alpha + \lambda - n \} + \epsilon,
\]
with sufficiently small \( \epsilon \in (0, h) \), where
\[
h = \min\{ \frac{n - \lambda - q\beta}{2nq}, \frac{n - \lambda - q(p\alpha + \lambda - n)}{2nq}, \frac{\lambda}{n}, \frac{\lambda}{n}, \frac{\lambda}{n}, \frac{n(p + 1)(\alpha + \lambda) - n}{n}, \frac{n - (p + 1)\alpha}{n} \}.
\]
Then \( \|v^q\|_s \leq \infty \) by Theorem 1.1 in [12]. By the Hölder inequality, we deduce that
\[
I \leq C \|v^q\|_s \left( \int_0^\delta r^{n-\lambda} \frac{dr}{r} \right)^{1/t}.
\]
Here \( \frac{1}{t} + \frac{1}{s} = 1 \).

When \( \frac{1}{qs} = \frac{\beta}{n} + \epsilon \), from (3.5) it follows
\[
\frac{1}{t} = \frac{n - q\beta - nq\epsilon}{n} > \frac{\lambda}{n},
\]
which leads to $I < \infty$.

When $\frac{1}{q} = \frac{p\alpha + \lambda - n}{n} + \epsilon$. Clearly, $\frac{1}{p+1} > \frac{2}{n}$ implies

$$\frac{n - \tilde{\lambda}}{p + 1} > \frac{n - \tilde{\lambda}}{n} - \frac{\beta}{(p + 1)(q + 1)}.$$ Multiplying by $(p + 1)(q + 1)$ and using the critical condition (3.1), we obtain

$$(q + 1)(n - \tilde{\lambda}) > (pq - 1)\alpha - \beta.$$ Thus,

$$\frac{1}{t} = \frac{n - q(p\alpha + \tilde{\lambda} - n) - q\epsilon}{n} > \frac{\lambda}{n}.$$ and hence $I < \infty$.

Combining the results above, we can see that $|x_0|^\alpha u(x_0) < \infty$. Thus, $u \in L^\infty_{loc}(\mathbb{R}^n \setminus \{0\})$.

**Step 2.** $v \in L^\infty_{loc}(\mathbb{R}^n \setminus \{0\})$.

In view of (1.2), there hold

$$n - \alpha(p + 1) > 0.$$ Using (1.10), we can find $\delta \in (0, |x_0|/2)$ sufficiently small such that

$$u(y) \leq C|y|^{-\alpha} \text{ for } y \in B_\delta(0).$$ Therefore,

$$\int_{B_\delta(0)} \frac{u^p(y)dy}{|x_0 - y|^\lambda |y|^\alpha} \leq C \int_{B_\delta(0)} \frac{dy}{|x_0 - y|^\lambda |y|^n(p + 1)}$$

$$\leq \frac{C}{|x_0|^\lambda} \int_0^\delta r^{n-\alpha(p+1)} \frac{dr}{r} < \infty. \quad (3.3)$$

Let $\tilde{\alpha} = \frac{2n}{p+1} - \alpha - \lambda$, $\tilde{\beta} = \frac{2n}{q+1} - \beta - \lambda$, then from [12] and [8], we know that $\tilde{\alpha}, \tilde{\beta}, p, q, \lambda$ also satisfy (1.9). In addition,

$$\lambda + (q + 1)\beta < n. \quad (3.4)$$ is equivalent to

$$\lambda q + \beta(q + 1) > n. \quad (3.5)$$ Then, by Theorem 1.3(ii) in [12], we can also find $R > 2|x_0|$ such that

$$u(y) \leq C|y|^{-\frac{2n}{p+1} + \alpha} \text{ for } y \in \mathbb{R}^n \setminus B_R(0).$$ Thus,

$$\int_{\mathbb{R}^n \setminus B_R(0)} \frac{u^p(y)dy}{|x_0 - y|^\lambda |y|^\alpha} \leq C \int_{\mathbb{R}^n \setminus B_R(0)} \frac{dy}{|y|^{2n-\alpha(p+1)}}$$

$$\leq C \int_R^\infty r^{\alpha(p + 1) - n} \frac{dr}{r} < \infty. \quad (3.6)$$ Finally, by Step 1 we can find $\delta > 0$ sufficiently small, such that

$$\int_{B_\delta(x_0)} \frac{u^p(y)dy}{|x_0 - y|^\lambda |y|^\alpha} \leq \frac{C}{|x_0|^\alpha} \int_0^\delta r^{n-\lambda} \frac{dr}{r} < \infty. \quad (3.7)$$ Combining (3.3)-(3.7), we obtain $|x_0|^\beta v(x_0) < \infty$.

Theorem 1.3 is proved. \qed
4. Axisymmetry of locally bounded solutions. Assume \( u, v \in L^\infty_{loc}(R^n \setminus \{0\}) \) solve the following system

\[
\begin{align*}
  u(x) &= \int_{R^n} \frac{v^q(y)}{|x|^{\alpha}|x-y|^{\lambda}|y|^\beta} dy, \\
  v(x) &= \int_{R^n} \frac{u^p(y)}{|x|^{\beta}|x-y|^{\lambda}|y|^\alpha} dy.
\end{align*}
\] (4.1)

Not as the finite energy solutions, the singularity of the locally bounded solutions is hard to handle when we move the planes. To avoid this difficulty, we use the Kelvin type transform with translation.

By the transformation

\[
\tilde{u}(x) = |x|^\alpha u(x), \quad \tilde{v}(x) = |x|^\beta v(x),
\]
we get

\[
\begin{align*}
  \tilde{u}(x) &= \int_{R^n} \frac{v^q(y)}{|x-y|^{\lambda}|y|^{\beta(q+1)}|y-z_0|^{\alpha(q+1)}} dy, \quad x \in R^n, \\
  \tilde{v}(x) &= \int_{R^n} \frac{u^p(y)}{|x-y|^{\lambda}|y|^{\alpha(p+1)}|y-z_0|^{\beta(p+1)}} dy, \quad x \in R^n.
\end{align*}
\] (4.2)

Let \( x_0 \in R^n \setminus \{0\} \). Then the Kelvin-type transforms of \( \tilde{u}(x), \tilde{v}(x) \),

\[
\begin{align*}
  \tilde{u}(x) &= \frac{1}{|x|^\lambda} \tilde{u}(\frac{x}{|x|^2} + x_0), \quad x \in R^n \setminus \{0\}, \\
  \tilde{v}(x) &= \frac{1}{|x|^\lambda} \tilde{v}(\frac{x}{|x|^2} + x_0), \quad x \in R^n \setminus \{0\},
\end{align*}
\] (4.3)

satisfy the following system

\[
\begin{align*}
  u(x) &= C_1 \int_{R^n} \frac{v^q(y)}{|x-y|^{\lambda}|y|^{2n-(q+1)(\lambda+\beta)}|y-z_0|^{\alpha(q+1)}} dy, \\
  v(x) &= C_2 \int_{R^n} \frac{u^p(y)}{|x-y|^{\lambda}|y|^{2n-(p+1)(\lambda+\alpha)}|y-z_0|^{\beta(p+1)}} dy,
\end{align*}
\] (4.4)

where \( x \in R^n \setminus \{0\}, z_0 = -\frac{x_0}{|x_0|^2} \), and \( C_1, C_2 > 0 \). By virtue of

\[
\begin{align*}
  u(x) &= \int_{R^n} \frac{v^q(y)}{|x|^{\alpha}|x-y|^{\lambda}|y|^\beta} dy \geq \int_{B_1(0)} \frac{v^q(y)}{|x|^{\alpha}|x-y|^{\lambda}|y|^\beta} dy \geq \frac{C}{|x|^{\alpha(1 + |x|^\lambda)}},
\end{align*}
\]

we obtain that, for suitably large \( |x| \),

\[
\begin{align*}
  \tilde{u}(x) &= \frac{1}{|x|^\lambda} \tilde{u}(\frac{x}{|x|^2} + x_0) = \frac{1}{|x|^\lambda} \frac{x}{|x|^2} + x_0 |^\alpha u(\frac{x}{|x|^2} + x_0) \\
  &\geq \frac{1}{|x|^\lambda(1 + \frac{1}{|x|^2} + x_0 |^\lambda)} \geq \frac{C}{|x|^{\lambda}}.
\end{align*}
\] (4.5)

From the result that \( u \in L^\infty_{loc}(R^n \setminus \{0\}) \), we obtain that

\[
\tilde{u}(x) = \frac{1}{|x|^\lambda} \tilde{u}(\frac{x}{|x|^2} + x_0) = \frac{1}{|x|^\lambda} \frac{x}{|x|^2} + x_0 |^\alpha u(\frac{x}{|x|^2} + x_0) \\
\leq \frac{C}{|x|^{\lambda}} \frac{x}{|x|^2} + x_0 |^\alpha \leq \frac{C}{|x|^{\lambda}} (1 + x_0 |^\alpha) = \frac{C}{|x|^{\lambda}}
\] (4.6)

when \( |x| \) is large. Combining (4.5) and (4.6), we have

\[
\frac{c}{|x|^{\lambda}} \leq \tilde{u}(x) \leq \frac{C}{|x|^{\lambda}}, \quad \text{for } |x| \text{ large}.
\] (4.7)
Another important observation is that if \( u \in L^\infty_{loc}(R^n \setminus \{0\}) \), then
\[
\bar{u} \in L^\infty_{loc}(R^n \setminus \{0, z_0\}).
\]  
(4.8)

Similarly, \( \bar{v} \) has the same properties of (4.7) and (4.8).

We will prove that the solutions \( \bar{u}, \bar{v} \) of (4.4) are axis symmetric by the method of moving planes introduced in [3]. Without loss of generality, let \( x_0 = (0, \cdots, 0, -1) \), then \( z_0 = (0, \cdots, 0, 1) \). We move the planes along all the directions except for the axis involving \( \overline{0z_0} \) to derive the symmetry of \( \bar{u} \) and \( \bar{v} \) where 0 represents the origin. Therefore, \( u \) and \( v \) are symmetric about the axis involving \( \overline{0z_0} \). Thus, Theorem 1.4 is true once the following theorem is proved.

**Theorem 4.1.** Let \( \bar{u}, \bar{v} \in L^\infty_{loc}(R^n \setminus \{0, z_0\}) \) be positive solutions of (4.4) with (1.2). If
\[
2n - (q + 1)(\lambda + \beta) \geq 0, \quad 2n - (p + 1)(\lambda + \alpha) \geq 0.
\]  
(4.9)

Then \( \bar{u}, \bar{v} \) are axially symmetric and decreasing about the axis involving the line segment \( \overline{0z_0} \).

**Proof.** We may assume \( q > p \) without loss of generality. Then by (1.2), we have
\[
q > \frac{1}{p}. \quad \text{Let us choose } \rho > 1 \text{ so that } 1/p < \rho < q.
\]

For \( k \in R \), define
\[
H_k = \{ x = (x_1, x_2, \cdots, x_n) | x_1 \geq k \},
\]
\[
x^k = (2k - x_1, x_2, \cdots, x_n), \quad \bar{u}_k(x) = \overline{u(x^k)}, \quad \bar{v}_k(x) = \overline{v(x^k)}.
\]

To prove Theorem 4.1, we compare \( \overline{u(x)} \) with \( \overline{u_k(x)} \) and \( \overline{v(x)} \) with \( \overline{v_k(x)} \) on \( H_k \), respectively. The proof consists of three steps. In step 1, we show there exists a real number \( R < 0 \) such that for \( k \leq R \) and \( x \in H_k \), we have
\[
\overline{u_k(x)} \leq \overline{u(x)}, \quad \overline{v_k(x)} \leq \overline{v(x)} \text{ a.e.} \tag{4.10}
\]

Thus, we can start moving the plane from \( k \leq R \) to the right as long as (4.10) holds. In step 2, we show that if the plane stops at \( x_1 = k_0 \) for some \( k_0 < 0 \), then \( \overline{u}, \overline{v} \) must be symmetric and monotone about the plane \( x_1 = k_0 \); otherwise, we can move the plane continuously. In step 3, we prove that we can move the plane all the way to \( x_1 = 0 \). Since the direction of \( x_1 \) can be chosen arbitrarily except for the axis involving \( \overline{0z_0} \), we deduce that \( \overline{u}, \overline{v} \) must be symmetric and decreasing about this axis.

**Step 1.** Define
\[
B^\overline{u}_k = \{ x \in H_k : \overline{u(x)} < \overline{u_k(x)} \}, \quad B^\overline{v}_k = \{ x \in H_k : \overline{v(x)} < \overline{v_k(x)} \}.
\]

To use the method of moving planes, we need the following formulas, which are obtained by change of variables:
\[
\overline{u}(x) = C_1 \int_{R^n} \frac{\overline{u}(y)}{|x - y|^\lambda |y|^{2n - (q + 1)(\lambda + \beta)}|y - z_0|^\beta(q + 1)} dy
\]
\[
= C_1 \int_{H_k} \frac{\overline{u_k}(y)}{|x - y|^\lambda |y|^{2n - (q + 1)(\lambda + \beta)}|y - z_0|^\beta(q + 1)} dy
\]
\[
+ C_1 \int_{H_k} \frac{\overline{u}_k(y)}{|x^k - y|^\lambda |y|^{2n - (q + 1)(\lambda + \beta)}|y^k - z_0|^\beta(q + 1)} dy.
\]
Similarly, we have
\[
\mathcal{V}_k(x) = C_1 \int_{H_k} \frac{\mathcal{V}^p(y)}{|x - y|^\lambda |y|^{2n-(p+1)(\lambda + \alpha)} |y - z_0|^\alpha (p+1)} dy + C_1 \int_{H_k} \frac{\mathcal{V}^p(y)}{|x - y|^\lambda |y|^{2n-(p+1)(\lambda + \alpha)} |y - z_0|^\alpha (p+1)} dy.
\]

Then we have
\[
\mathcal{V}(x) = C_2 \int_{H_k} \frac{\mathcal{V}^p(y)}{|x - y|^\lambda |y|^{2n-(p+1)(\lambda + \alpha)} |y - z_0|^\alpha (p+1)} dy + C_2 \int_{H_k} \frac{\mathcal{V}^p(y)}{|x - y|^\lambda |y|^{2n-(p+1)(\lambda + \alpha)} |y - z_0|^\alpha (p+1)} dy.
\]

Since for \(x, y \in H_k\) with \(k < 0\),
\[
|x - y| < |x^k - y|, \ |y^k - z_0| > |y - z_0|, \ |y^k| > |y|,
\]
then we have
\[
0 \leq \mathcal{V}_k(x) - \mathcal{V}(x) = C_2 \int_{H_k} \left( \frac{1}{|x - y|^\lambda} - \frac{1}{|x^k - y|^\lambda} \right) \mathcal{V}^p(y) dy + C_2 \int_{H_k} \left( \frac{1}{|x - y|^\lambda} - \frac{1}{|y|^{2n-(p+1)(\lambda + \alpha)} |y - z_0|^\alpha (p+1)} \right) \mathcal{V}^p(y) dy
\]
\[
\leq C_2 \int_{B^r_k} \left( \frac{1}{|x - y|^\lambda} - \frac{1}{|y|^{2n-(p+1)(\lambda + \alpha)} |y - z_0|^\alpha (p+1)} \right) \mathcal{V}^p(y) dy
\]
\[
\leq C_2 \int_{B^r_k} \left( \frac{(\mathcal{V}^{1/\rho}(y))^{pp} - (\mathcal{V}^{1/\rho}(y))^{pp}}{(\mathcal{V}^{1/\rho}(y))^{pp} - (\mathcal{V}^{1/\rho}(y))^{pp}} \right) \mathcal{V}^{p-1/\rho}(y) dy
\]
\[
\leq C_2 \int_{B^r_k} \left( \frac{1}{|x - y|^\lambda} - \frac{1}{|y|^{2n-(p+1)(\lambda + \alpha)} |y - z_0|^\alpha (p+1)} \right) \mathcal{V}^p(y) dy.
\]

Next, we can find \(s > \max\{\frac{\alpha}{n}, \frac{n(\alpha - 1)}{n - \lambda}\} > 0\), such that for any \(\epsilon > 0\),
\[
\overline{u} \in L^s(R^n \setminus [B_r(0) \cup B_s(z_0)]).
\]

In fact, from (4.7), we see that there exists \(R > 0\) such that when \(|x| > R\), \(\overline{u} \leq C|x|^{-\lambda}\). Hence, by (4.8), we get
\[
\int_{R^n \setminus [B_r(0) \cup B_s(z_0)]} \overline{u}^s dx \leq \int_{R^n \setminus [B_r(0) \cup B_s(z_0)]} \overline{u}^s dx + \int_{R^n \setminus B_R} \overline{u}^s dx
\]
\[
\leq C + C \int_{R} r^{n-1-s\lambda} dr < \infty.
\]

Similarly, \(\overline{u}\) has the same property.
On the other hand, $s > \frac{n}{\lambda}$ implies that $\frac{n}{(n-\lambda)s} > 1$. Then we can use the Hardy-Littlewood-Sobolev inequality and the Hölder inequality to obtain

$$
\left\| \nabla_k(x) - \nabla(x) \right\|_{L^p(B^\gamma_0)} \\
\leq C \left\| \frac{1}{y^{2n-(p+1)(\lambda+\beta)}(y) |y_{-0}|^{p+(p+1)}} \right\|_{L^{\frac{n}{n-s(n-\lambda s)}}(B^\gamma_0)} \\
\leq \frac{1}{y^{2n-(p+1)(\lambda+\beta)}(y) |y_{-0}|^{p+(p+1)}} \left\| \nabla_k(y) \right\|_{L^p(B^\gamma_0)}^{-1/p} \\
\times \left\| \nabla_k^1(y) - \nabla^1(y) \right\|_{L^{\infty}(B^\gamma_0)}.
$$

Now let us estimate $\left\| \nabla_k^1(y) - \nabla^1(y) \right\|_{L^{\infty}(B^\gamma_0)}$. For $k < 0$ and $x \in B^\gamma_0$, note that in $H_k \setminus B^\gamma_0$,

$$
\left\| \nabla_k^1(y) - \nabla^1(y) \right\|_{L^{\infty}(B^\gamma_0)} \geq 0,
$$

and that in $B^\gamma_0$,

$$
\left\| \nabla_k^1(y) - \nabla^1(y) \right\|_{L^{\infty}(B^\gamma_0)} \geq 0,
$$

then for $y \in H_k \setminus B^\gamma_0$,

$$
\left\| \nabla_k^1(y) - \nabla^1(y) \right\|_{L^{\infty}(B^\gamma_0)} \geq 0,
$$

for $y \in B^\gamma_0$,

$$
\left\| \nabla_k^1(y) - \nabla^1(y) \right\|_{L^{\infty}(B^\gamma_0)} \geq 0.
$$

And hence

$$
\pi(x) \geq C_1 \int_{B^\gamma_0} \frac{\nabla_k^1(y)}{|x - y|^{\lambda} |y^{2n-(p+1)(\lambda+\beta)}|y_{-0}|^{p+(p+1)}}
\frac{\nabla^1(y)}{|x - y|^{\lambda} |y^{2n-(p+1)(\lambda+\beta)}|y_{-0}|^{p+(p+1)}}
\frac{dy}{|x - y|^{\lambda} |y^{2n-(p+1)(\lambda+\beta)}|y_{-0}|^{p+(p+1)}}
\frac{dy}{|x - y|^{\lambda} |y^{2n-(p+1)(\lambda+\beta)}|y_{-0}|^{p+(p+1)}}
\frac{dy}{|x - y|^{\lambda} |y^{2n-(p+1)(\lambda+\beta)}|y_{-0}|^{p+(p+1)}}
\frac{dy}{|x - y|^{\lambda} |y^{2n-(p+1)(\lambda+\beta)}|y_{-0}|^{p+(p+1)}}
\frac{dy}{|x - y|^{\lambda} |y^{2n-(p+1)(\lambda+\beta)}|y_{-0}|^{p+(p+1)}}
$$
\[ \mathcal{V}_k(x) \leq C_1 \int_{B_k^q} \frac{\pi^q(y)}{|x - y|^\lambda |y|^{2n-(q+1)(\lambda + \beta)}|y^k - z_0|^{\beta(q+1)}} dy \\
+ C_1 \int_{B_k^q} \frac{\pi^q_k(y)}{|x - y|^\lambda |y|^{2n-(q+1)(\lambda + \beta)}|y^k - z_0|^{\beta(q+1)}} dy \\
+ C_1 \int_{H_k \setminus B_k^q} \frac{\pi^q(y)}{|x - y|^\lambda |y|^{2n-(q+1)(\lambda + \beta)}|y^k - z_0|^{\beta(q+1)}} dy \\
+ C_1 \int_{H_k \setminus B_k^q} \frac{\pi^q_k(y)}{|x - y|^\lambda |y|^{2n-(q+1)(\lambda + \beta)}|y^k - z_0|^{\beta(q+1)}} dy. \]

Therefore from the inequality \((a+c)^{1/p} - (b+c)^{1/p} \leq a^{1/p} - b^{1/p}\) for \(a \geq b \geq 0, c \geq 0\) and the Minkowski inequality, it follows that

\[ 0 \leq \mathcal{V}_k^{1/p}(x) - \pi^{1/p}(x) \leq (C_1 \int_{B_k^q} \frac{\pi^q(y)}{|x - y|^\lambda |y|^{2n-(q+1)(\lambda + \beta)}|y^k - z_0|^{\beta(q+1)}} dy \\
+ C_1 \int_{B_k^q} \frac{\pi^q_k(y)}{|x - y|^\lambda |y|^{2n-(q+1)(\lambda + \beta)}|y^k - z_0|^{\beta(q+1)}} dy)^{1/p} \\
- (C_1 \int_{B_k^q} \frac{\pi^q(y)}{|x - y|^\lambda |y|^{2n-(q+1)(\lambda + \beta)}|y^k - z_0|^{\beta(q+1)}} dy \\
+ C_1 \int_{B_k^q} \frac{\pi^q_k(y)}{|x - y|^\lambda |y|^{2n-(q+1)(\lambda + \beta)}|y^k - z_0|^{\beta(q+1)}} dy)^{1/p})^p dy \\
\leq (2C_1)^{1/p} (\int_{B_k^q} \frac{(\pi^q_k(y) - \pi^{1/p}(y))^p}{|x - y|^\lambda |y|^{2n-(q+1)(\lambda + \beta)}|y^k - z_0|^{\beta(q+1)}} dy)^{1/p} \\
\leq (2C_1)^{1/p} \left( \int_{B_k^q} \frac{(\pi^q_k(y) - \pi^{1/p}(y))^p}{|x - y|^\lambda |y|^{2n-(q+1)(\lambda + \beta)}|y^k - z_0|^{\beta(q+1)}} dy \right)^{1/p} \]
Since \( \overline{u}, \overline{v} \in L^s(R^n \setminus [B_{r}(0) \cup B_{t}(z_{0})]) \), by the assumption (4.9), we can choose a sufficiently large \(|R|\) with \( R < 0 \), such that for \( k \leq R < 0 \),

\[
C \left\| \frac{1}{|y|^{2n-(p+1)(\lambda+\alpha)|\alpha(p+1)|} \frac{\lambda}{L^{\frac{n}{(\alpha(p+1)+\lambda)}}(B_{r}^\ast) \frac{\lambda}{L^{\frac{n}{(\alpha(p+1)+\lambda)}}(B_{r}^\ast)} \right\| \leq 1 \frac{1}{L^{\frac{n}{(\alpha(p+1)+\lambda)}}(B_{r}^\ast)} \leq \frac{1}{2}.
\]

Substituting this into (4.12) yields

\[
\|\overline{u}_k(x) - \overline{u}(x)\|_{L^s(B_{r}^\ast)} \leq \frac{1}{2} \|\overline{u}_k(x) - \overline{u}(x)\|_{L^s(B_{r}^\ast)}.
\]

Similarly,

\[
\|\overline{v}_k(x) - \overline{v}(x)\|_{L^s(B_{r}^\ast)} \leq \frac{1}{2} \|\overline{v}_k(x) - \overline{v}(x)\|_{L^s(B_{r}^\ast)}.
\]

These imply that \( \|\overline{u}_k(x) - \overline{u}(x)\|_{L^s(B_{r}^\ast)} = 0 \), \( \|\overline{v}_k(x) - \overline{v}(x)\|_{L^s(B_{r}^\ast)} = 0 \). Thus, (4.10) is proved.

**Step 2.** We now move \( x_1 = k \) to the right as long as (4.10) holds. Suppose that at a point \( k_0 < 0 \), we have on \( H_{k_0} \),

\[
\overline{u}(x) \geq \overline{u}_k(x), \quad \overline{v}(x) \geq \overline{v}_k(x), \quad \text{but} \quad \overline{u}(x) \not\equiv \overline{u}_k(x) \quad \text{or} \quad \overline{v}(x) \not\equiv \overline{v}_k(x).
\]

Then the plane can be moved further to the right. More precisely, there exists an \( \epsilon \) such that

\[
\overline{u}(x) \geq \overline{u}_k(x), \quad \overline{v}(x) \geq \overline{v}_k(x), \quad \text{on} \quad H_k \quad \text{for} \quad \text{all} \quad k \in \{k_0, k_0 + \epsilon\}.
\]

In the case \( \overline{v}(x) \not\equiv \overline{v}_k(x) \) on \( H_{k_0} \), we have in fact \( \overline{u}(x) > \overline{u}_k(x) \) in the interior of \( H_{k_0} \). Let

\[
\Phi_{k_0}^\ast = \{x \in H_{k_0} : \overline{u}(x) \geq \overline{u}_k(x)\}, \quad \Phi_{k_0}^\ast = \{x \in H_{k_0} : \overline{v}(x) \geq \overline{v}_k(x)\}.
\]

Then, obviously \( \Phi_{k_0}^\ast \) has measure zero and \( \limsup_{k \to k_0} B_{k}^\ast \subset \Phi_{k_0}^\ast \). The same is true for that of \( \overline{v} \).

Similar to the derivation of (4.12),

\[
\|\overline{u}_k(x) - \overline{u}(x)\|_{L^s(B_{r}^\ast)} \leq C \left\| \frac{1}{|y|^{2n-(p+1)(\lambda+\alpha)|\alpha(p+1)|} \frac{\lambda}{L^{\frac{n}{(\alpha(p+1)+\lambda)}}(B_{r}^\ast) \frac{\lambda}{L^{\frac{n}{(\alpha(p+1)+\lambda)}}(B_{r}^\ast)} \right\| \leq \frac{1}{2} \|\overline{u}_k(x) - \overline{u}(x)\|_{L^s(B_{r}^\ast)}.
\]

Then \( \overline{u}, \overline{v} \in L^s(R^n \setminus [B_{r}(0) \cup B_{t}(z_{0})]) \) ensure that one can choose \( \epsilon \) sufficiently small, so that for all \( k \in \{k_0, k_0 + \epsilon\} \),

\[
C \left\| \frac{1}{|y|^{2n-(p+1)(\lambda+\alpha)|\alpha(p+1)|} \frac{\lambda}{L^{\frac{n}{(\alpha(p+1)+\lambda)}}(B_{r}^\ast) \frac{\lambda}{L^{\frac{n}{(\alpha(p+1)+\lambda)}}(B_{r}^\ast)} \right\| \leq \frac{1}{2} \|\overline{u}_k(x) - \overline{u}(x)\|_{L^s(B_{r}^\ast)} \leq \frac{1}{2}.
\]

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Here $D^*$ is the reflection of the set $D$ about the plane $x_1 = k$. So,

$$\|\overset{\sim}{u}_k(x) - \overset{\sim}{u}(x)\|_{L^\infty(B^*_k)} \leq \frac{1}{2} \|\overset{\sim}{u}_k(x) - \overset{\sim}{u}(x)\|_{L^\infty(B^*_k)}.$$

Similarly,

$$\|u_k(x) - u(x)\|_{L^\infty(B^*_k)} \leq \frac{1}{2} \|u_k(x) - u(x)\|_{L^\infty(B^*_k)}.$$

These imply that $\|\overset{\sim}{u}_k(x) - \overset{\sim}{u}(x)\|_{L^\infty(B^*_k)} = 0$, $\|u_k(x) - u(x)\|_{L^\infty(B^*_k)}$ for all $k$ in $[k_0, k_0 + \epsilon)$. This verifies (4.13).

**Step 3.** Steps 1 and 2 show that we can start moving the plane continuously to the right as long as (4.10) holds. We claim that we can move the plane to $x_1 = 0$. Otherwise, if the plane stops at $x_1 = k_0$ with $k_0 < 0$, then (4.11) implies

$$0 = \bar{u}_{k_0}(x) - \bar{u}(x) = C_1 \int_{H_{k_0}} \frac{1}{|x - y|^{\lambda}} - \frac{1}{|x - y|^{\lambda}} |\nabla u(y)|
\times (\frac{1}{|y|^{2n-(q+1)(\lambda+\beta)}|y - z_0|^{\beta(q+1)}} - \frac{1}{|y|^{2n-(q+1)(\lambda+\beta)}|y - z_0|^{\beta(q+1)}})dy \neq 0.$$

It is impossible. Since the direction of $x_1$ is arbitrary except for the axis involving $\overline{s_0}$, we see that $\bar{u}$ and $\bar{v}$ are symmetric and decreasing about this axis.

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