A Christoffel Darboux formula for multiple orthogonal polynomials

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Abstract

Bleher and Kuijlaars recently showed that the eigenvalue correlations from matrix ensembles with external source can be expressed by means of a kernel built out of special multiple orthogonal polynomials. We derive a Christoffel-Darboux formula for this kernel for general multiple orthogonal polynomials. In addition, we show that the formula can be written in terms of the solution of the Riemann-Hilbert problem for multiple orthogonal polynomials, which will be useful for asymptotic analysis.

1 Introduction

Multiple orthogonal polynomials are polynomials that satisfy orthogonal conditions with respect to a number of weights, or more general with respect to a number of measures. Such polynomials were first introduced by Hermite in his proof of the transcendence of $e$, and were subsequently used in number theory and approximation theory, see [1], [2], and the references cited therein. The motivation for the present work comes from a connection with random matrix theory. In the random matrix model considered in [3] the eigenvalue correlations are expressed in terms of a kernel built out of multiple orthogonal polynomials with respect to two weights

$$w_j(x) = e^{-V(x)+a_jx}, \quad j = 1, 2, \quad a_1 \neq a_2.$$  

(1.1)

A Christoffel-Darboux formula was given in [3] which leads to a description of the kernel in terms of the Riemann-Hilbert problem for multiple orthogonal polynomials [12]. It is the aim of this paper to extend the Christoffel-Darboux formula to multiple orthogonal polynomials with respect to an arbitrary number of weights. We also allow more general weights than those in (1.1).

Let $m \geq 2$ be an integer, and let $w_1, w_2, \ldots, w_m$ be non-negative functions on $\mathbb{R}$ such that all moments $\int_{-\infty}^{\infty} x^k w_j(x)dx$ exist. Let $\vec{n} = (n_1, n_2, \ldots, n_m)$ be a vector of non-negative integers. The (monic) multiple orthogonal polynomial $P_{\vec{n}}$ of type II is a monic polynomial of degree $|\vec{n}|$ satisfying

$$\int P_{\vec{n}}(x) x^k w_j(x)dx = 0 \quad \text{for} \ k = 0, \ldots, n_j - 1, \quad j = 1, \ldots, m.$$  

(1.2)

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Here we define, as usual, $|\vec{n}| = n_1 + n_2 + \cdots + n_m$.

We assume that the system is perfect, i.e., that for every $\vec{n} \in (\mathbb{N} \cup \{0\})^m$, the polynomial $P_{\vec{n}}$ exists and is unique, see [9]. This is for example the case when the weights form an Angelesco system or an AT system, see e.g. [11].

The multiple orthogonal polynomials of type I are polynomials $A_{\vec{n}}^{(k)}$ for $k = 1, \ldots, m$, where $A_{\vec{n}}^{(k)}$ has degree $\leq n_k - 1$, such that the function

$$Q_{\vec{n}}(x) = \sum_{k=1}^{m} A_{\vec{n}}^{(k)}(x)w_k(x)$$

satisfies

$$\int x^j Q_{\vec{n}}(x)dx = \begin{cases} 0 & \text{for } j = 0, \ldots, |\vec{n}| - 2, \\ 1 & \text{for } j = |\vec{n}| - 1. \end{cases}$$

The polynomials $A_{\vec{n}}^{(k)}$ exist, are unique, and they have full degree $\deg A_{\vec{n}}^{(k)} = n_k - 1$, since the system is perfect.

The usual monic orthogonal polynomials $P_n$ on the real line with weight function $w(x)$ satisfy a three term recurrence relation and this gives rise to the basic Christoffel-Darboux formula

$$\sum_{j=0}^{n-1} \frac{1}{h_j} P_j(x)P_j(y) = \frac{1}{h_{n-1}} \frac{P_n(x)P_{n-1}(y) - P_{n-1}(x)P_n(y)}{x-y},$$

where

$$h_j = \int P_j(x)x^j w(x)dx.$$

In order to generalize the formula (1.5) to multiple orthogonal polynomials, we consider a sequence of multi-indices $\vec{n}_0, \vec{n}_1, \ldots, \vec{n}_n$ such that for each $j = 0, 1, \ldots, n$,

$$|\vec{n}_j| = j, \quad \vec{n}_{j+1} \geq \vec{n}_j,$$

where the inequality is taken componentwise. This means that we can go from $\vec{n}_j$ to $\vec{n}_{j+1}$ by increasing one of the components of $\vec{n}_j$ by 1. We view $\vec{n}_0, \vec{n}_1, \ldots, \vec{n}_n$ as a path from $\vec{n}_0 = \vec{0}$ (the all-zero vector) to an arbitrary multi-index $\vec{n} = \vec{n}_n$. This path will be fixed and all notions are related to this fixed path. Given such a path, we define the polynomials $P_j$ and functions $Q_j$ (with single index) as

$$P_j = P_{\vec{n}_j}, \quad Q_j = Q_{\vec{n}_{j+1}}.$$

Our aim is to find a simplified expression for the sum

$$K_n(x,y) = \sum_{j=0}^{n-1} P_j(x)Q_j(y).$$

To do this, we introduce the following notation. We define for every multi-index $\vec{n}$ and every $k = 1, \ldots, m$,

$$h_{\vec{n}}^{(k)} = \int P_{\vec{n}}(x)x^k w_k(x)dx.$$
The numbers \( h^{(k)}_{\bar{n}} \) are non-zero, since the system is perfect. We also use the standard basis vectors
\[
\tilde{e}_k = (0, \ldots, 0, 1, 0, \ldots, 0), \quad \text{where 1 is in the } k\text{th position.} \tag{1.10}
\]
Our result is the following.

**Theorem 1.1** Let \( n \in \mathbb{N} \) and let \( \bar{n}_0, \bar{n}_1, \ldots, \bar{n}_n \) be multi-indices such that (1.6) holds. Let \( P_j \) and \( Q_j \) be as in (1.7). Then we have if \( \bar{n} = \bar{n}_n \),
\[
(x - y) \sum_{j=0}^{n-1} P_j(x)Q_j(y) = P_{\bar{n}}(x)Q_{\bar{n}}(y) - \sum_{k=1}^{m} \frac{h^{(k)}_{\bar{n}}}{h^{(k)}_{\bar{n} - \tilde{e}_k}} P_{\bar{n} - \tilde{e}_k}(x)Q_{\bar{n} + \tilde{e}_k}(y). \tag{1.11}
\]

It is easy to see that (1.11) reduces to the classical Christoffel-Darboux formula (1.5) in case \( m = 1 \). For \( m = 2 \) the formula was proven in [3].

**Remark 1.2** It follows from (1.11) that the kernel (1.8) only depends on the endpoint \( \bar{n} \) of the chosen path from \( \bar{0} \) to \( \bar{n} \) and not on the particular path itself, since clearly the right-hand side of (1.11) only depends on \( \bar{n} \).

This fact can be deduced from the fact that for any multi-index \( \bar{k} \) and for \( i \neq j \), we have
\[
P_{\bar{k}}(x)Q_{\bar{k} + \tilde{e}_i}(y) + P_{\bar{k} + \tilde{e}_i}(x)Q_{\bar{k} + \tilde{e}_i + \tilde{e}_j}(y) = P_{\bar{k}}(x)Q_{\bar{k} + \tilde{e}_j}(y) + P_{\bar{k} + \tilde{e}_j}(x)Q_{\bar{k} + \tilde{e}_i + \tilde{e}_j}(y). \tag{1.12}
\]

The relation (1.12) follows easily from Lemma 3.6 below.

**Remark 1.3** In [10], Sorokin and Van Iseghem proved a Christoffel-Darboux formula for vector polynomials that have matrix orthogonality properties. As a special case this includes the multiple orthogonal polynomials of type I and type II, when one of the vector polynomials has only one component. In this special case, their Christoffel-Darboux formula comes down to the formula
\[
(x - y) \sum_{j=0}^{n-1} P_j(x)Q_j(y) = P_n(x)Q_{n-1}(y) - \sum_{k=n}^{n+m-1} \sum_{j=0}^{n-1} c_{j,k} P_j(x)Q_k(y) \tag{1.13}
\]
where the constants \( c_{j,k} \) are such that
\[
x P_k(x) = \sum_{j=0}^{k+1} c_{j,k} P_j(x),
\]
see also (3.5) below. In the setting of [10] it holds that \( c_{j,k} = 0 \) if \( k \geq j + m + 1 \), so that the right-hand side of (1.13) has \( 1 + \frac{1}{2} m(m+1) \) terms. Note that in our formula (1.11) the right-hand side has only \( 1 + m \) terms.

Another Christoffel-Darboux formula for multiple orthogonal polynomials similar to the one in [10] has been given recently in [6].

**Remark 1.4** As mentioned before, the formula (1.11) is useful in the theory of random matrices. Brézin and Hikami [5] studied a random matrix model with external source given by the probability measure
\[
\frac{1}{Z_n} e^{-\text{Tr}(V(M) - AM)} dM \tag{1.14}
\]
defined on the space of \( n \times n \) Hermitian matrices \( M \). Here we have that \( V(x) = \frac{1}{2} x^2 \), \( A \) is a fixed Hermitian matrix (the external source), and \( Z_n \) is a normalizing constant. For this case,
we can write $M = H + A$, where $H$ is a random matrix from the Gaussian unitary ensemble and $A$ is deterministic. Zinn-Justin \cite{13} considered the case of an arbitrary polynomial $V$.

The $k$-point correlation function $R_k(\lambda_1, \ldots, \lambda_k)$ of the (random) eigenvalues of a matrix from the ensemble \cite{14} can be expressed as a $k \times k$ determinant involving a kernel $K_n(x, y)$

$$R_k(\lambda_1, \ldots, \lambda_k) = \det(K_n(\lambda_i, \lambda_j))_{1 \leq i, j \leq k}, \quad (1.15)$$

see \cite{13}. Suppose that the external source $A$ has $m$ distinct eigenvalues $\alpha_1, \ldots, \alpha_m$ with respective multiplicities $n_1, \ldots, n_m$. Let $\vec{n} = (n_1, \ldots, n_m)$. Then it was shown in \cite{8} that the kernel $K_n$ has the form \cite{8} built out of the multiple orthogonal polynomials associated with the weights

$$w_j(x) = e^{-(V(x) - \alpha_j x)}, \quad j = 1, \ldots, m.$$ 

The Christoffel-Darboux formula \cite{14} gives a compact expression for the kernel.

There is another expression for the kernel \cite{14} in terms of the solution of a Riemann-Hilbert problem. This will be especially useful for the asymptotic analysis of the matrix model \cite{14}. We will discuss this in the next section. The proof of Theorem 1.1 is presented in Section 3.

## 2 Link with the Riemann-Hilbert problem

Van Assche, Geronimo, and Kuijlaars \cite{12} found a Riemann-Hilbert problem that characterizes the multiple orthogonal polynomials. This is an extension of the Riemann-Hilbert problem for orthogonal polynomials due to Fokas, Its, and Kitaev \cite{8}. We seek $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{(m+1) \times (m+1)}$ such that

1. $Y$ is analytic on $\mathbb{C} \setminus \mathbb{R}$,
2. for $x \in \mathbb{R}$, we have $Y_+(x) = Y_-(x)S(x)$, where

$$S(x) = \begin{bmatrix} 1 & w_1(x) & w_2(x) & \cdots & w_m(x) \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \quad (2.1)$$

3. as $z \to \infty$, we have that

$$Y(z) = \left(I + O\left(\frac{1}{z}\right)\right) \begin{bmatrix} z^n & 0 & 0 & \cdots & 0 \\ 0 & z^{-n_1} & 0 & \cdots & 0 \\ 0 & 0 & z^{-n_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & z^{-n_m} \end{bmatrix}, \quad (2.2)$$

where $I$ denotes the $(m+1) \times (m+1)$ identity matrix.

This Riemann-Hilbert problem has a unique solution given by:

$$Y(z) = \begin{bmatrix} P_{\vec{n}}(z) & \vec{R}_{\vec{n}}(z) \\ c_1P_{\vec{n}-\vec{c}_1}(z) & c_1\vec{R}_{\vec{n}-\vec{c}_1}(z) \\ c_2P_{\vec{n}-\vec{c}_2}(z) & c_2\vec{R}_{\vec{n}-\vec{c}_2}(z) \\ \vdots & \vdots \\ c_mP_{\vec{n}-\vec{c}_m}(z) & c_m\vec{R}_{\vec{n}-\vec{c}_m}(z) \end{bmatrix}, \quad (2.3)$$
where $P_n(z)$ is the multiple orthogonal polynomial of type II with respect to the weights $w_1, \ldots, w_m$ and $\vec{R}_n = (R_{n,1}, R_{n,2}, \ldots, R_{n,m})$ is the vector containing the Cauchy transforms

$$R_{n,j}(z) = \frac{1}{2\pi i} \int \frac{P_n(x)w_j(x)}{x-z} \, dx,$$

and

$$c_j = -\frac{2\pi i}{h^{(j)}_{n-e_j}}, \quad j = 1, \ldots, m. \quad (2.4)$$

Van Assche, Geronimo, and Kuijlaars [12] also gave a Riemann-Hilbert problem that characterizes the multiple orthogonal polynomials of type I. Here we seek $X : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{(m+1) \times (m+1)}$ such that

1. $X$ is analytic on $\mathbb{C} \setminus \mathbb{R}$,
2. for $x \in \mathbb{R}$, we have $X_+(x) = X_-(x)U(x)$, where

$$U(x) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -w_1(x) & 1 & 0 & \cdots & 0 \\ -w_2(x) & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -w_m(x) & 0 & 0 & \cdots & 1 \end{bmatrix}, \quad (2.5)$$

3. as $z \to \infty$, we have

$$X(z) = \left( I + O \left( \frac{1}{z} \right) \right) \begin{bmatrix} z^{-n} & 0 & 0 & \cdots & 0 \\ 0 & z^{-n+1} & 0 & \cdots & 0 \\ 0 & 0 & z^{-n+2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & z^{-nm} \end{bmatrix}. \quad (2.6)$$

This Riemann-Hilbert problem also has a unique solution and it is given by

$$X(z) = \begin{bmatrix} \int Q_{n}(x) \frac{dx}{z-x} \\ k_1 \frac{1}{2\pi i} \int Q_{n+\bar{e}_1}(x) \frac{dx}{z-x} \\ k_2 \frac{1}{2\pi i} \int Q_{n+\bar{e}_2}(x) \frac{dx}{z-x} \\ \vdots \\ k_m \frac{1}{2\pi i} \int Q_{n+\bar{e}_m}(x) \frac{dx}{z-x} \\ 2\pi i \bar{A}_{n}(z) \\ k_1 \bar{A}_{n+\bar{e}_1}(z) \\ k_2 \bar{A}_{n+\bar{e}_2}(z) \\ \vdots \\ k_m \bar{A}_{n+\bar{e}_m}(z) \end{bmatrix}, \quad (2.7)$$

where $\bar{A}_n = (A_{n}^{(1)}, A_{n}^{(2)}, \ldots, A_{n}^{(m)})$ is the vector of multiple orthogonal polynomials of type I with respect to $w_1, \ldots, w_m$, $Q_{n}(z) = \sum_{k=1}^{m} A_{n}^{(k)}(x)w_k(x)$ and

$$k_j = h^{(j)}_{n}, \quad j = 1, \ldots, m. \quad (2.8)$$

It is now possible to write the kernel $K_n(x,y)$ in terms of the solutions of the two Riemann-Hilbert problems, see also [3]. First, we observe that $X = Y^{-t}$. If we look at the $j+1,1$-entry of the product $Y^{-1}(y)Y(x) = X^t(y)Y(x)$, where $j = 1, \ldots, m$, then we find by
\[
[Y^{-1}(y)Y(x)]_{j+1,1} = \begin{bmatrix}
2 \pi i A^{(j)}(y) & k_1 A^{(j)}(y) & \cdots & k_m A^{(j)}(y) \\

\end{bmatrix}
\begin{bmatrix}
P_{\bar{n}}(x) \\
c_1 P_{\bar{n}-e_1}(x) \\
c_2 P_{\bar{n}-e_2}(x) \\
\vdots \\
c_m P_{\bar{n}-e_m}(x)
\end{bmatrix}
= 2 \pi i \left( P_{\bar{n}}(x) A^{(j)}(y) - \sum_{k=1}^{m} \frac{h^{(k)}}{h_{\bar{n}-e_k}} P_{\bar{n}-e_k}(x) A^{(j)}(y) \right).
\] (2.9)

where in the last step we used the expressions (2.4) and (2.8) for the constants \(c_j\) and \(k_j\).

Multiplying (2.9) by \(w_j(y)\), dividing by \(2 \pi i\), and summing over \(j = 1, \ldots, m\), we obtain the right-hand side of (1.11). Therefore we see that

\[
(x - y) K_n(x, y) = \frac{1}{2 \pi i} \sum_{j=1}^{m} w_j(y) [Y^{-1}(y)Y(x)]_{j+1,1}
= \frac{1}{2 \pi i} \begin{bmatrix} 0 & \cdots & w_m(y) \end{bmatrix} \begin{bmatrix} 1 \\
0 \\
\vdots \\
0
\end{bmatrix}.
\] (2.10)

It is clear that the right-hand side of (2.10) is 0 for \(x = y\), which is not obvious at all for the right-hand side of (1.11).

In [4] the Riemann-Hilbert problem (2.1)–(2.2) is analyzed in the limit \(n \to \infty\) for the special case of \(m = 2, n_1 = n_2\), and weights

\[
w_1(x) = e^{-n(\frac{1}{2}x^2-ax)}, \quad w_2(x) = e^{-n(\frac{1}{2}x^2+ax)}.
\]

The corresponding multiple orthogonal polynomials are known as multiple Hermite polynomials [2, 11]. The asymptotic analysis of (2.1)–(2.2) is done by the Deift/Zhou steepest descent method for Riemann-Hilbert problems, see [7] and references cited therein.

3 Proof of Theorem 1.1

For the proof we are going to extend the path \(\vec{n}_0, \vec{n}_1, \ldots, \vec{n}_n\) by defining

\[
\vec{n}_{n+k} - \vec{n}_{n+k-1} = \vec{e}_k, \quad k = 1, 2, \ldots, m.
\] (3.1)

We will also extend the definition (1.7) by putting \(P_j = P_{\vec{n}_j}\) and \(Q_{j-1} = Q_{\vec{n}_j}\) for \(j = n + 1, \ldots, n + m\).

3.1 Biorthogonality and recurrence relations

The multiple orthogonal polynomials satisfy a biorthogonality relation.

Lemma 3.1 We have

\[
\int P_k(x) Q_j(x) dx = \delta_{j,k},
\]

where \(\delta_{j,k}\) is the Kronecker delta.
Proof. This is immediate from the definitions \([17]\), the orthogonality conditions \([14]\) of the function \(Q_j\) and \([12]\) of the polynomial \(P_k\) and the fact that \(P_k\) is a monic polynomial. \(\square\)

Because \(xP_k(x)\) is a polynomial of degree \(k + 1\), we can expand \(xP_k(x)\) as

\[
xP_k(x) = \sum_{j=0}^{k+1} c_{j,k} P_j(x). \tag{3.2}
\]

The coefficients can be calculated by Lemma \([3.1]\) by multiplying both sides of \((3.2)\) with \(Q_j(x)\) and integrating over the real line. That gives us

\[
c_{j,k} = \int xP_k(x)Q_j(x)dx. \tag{3.3}
\]

The coefficients \(c_{j,k}\) are 0 if \(j \geq k + 2\).

Because of \((3.1)\) we can write \(yQ_j(y)\) with \(j \leq n - 1\) as a linear combination of \(Q_0, \ldots, Q_{n+m-1}\) and we have by Lemma \([3.1]\)

\[
yQ_j(y) = \sum_{k=0}^{n+m-1} c_{j,k} Q_k(y) \quad \text{for} \quad j = 0, \ldots, n-1. \tag{3.4}
\]

Using the expansions \((3.2)\) and \((3.4)\) for \(xP_k(x)\) and \(yQ_j(y)\) we can write

\[
(x - y) \sum_{k=0}^{n-1} P_k(x)Q_k(y) = \sum_{k=0}^{n-1} xP_k(x)Q_k(y) - \sum_{k=0}^{n-1} P_k(x)yQ_k(y)
= \sum_{k=0}^{n-1} \sum_{j=0}^{k+1} c_{j,k} P_j(x)Q_k(y) - \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} c_{k,j} P_k(x)Q_j(y).
\]

A lot of terms cancel. Since \(c_{j,k} = 0\) for \(j \geq k + 2\), and \(c_{n,n-1} = 1\), what remains is

\[
(x - y) \sum_{k=0}^{n-1} P_k(x)Q_k(y) = P_n(x)Q_n(y) - \sum_{k=n}^{n+m-1} \sum_{j=0}^{n-1} c_{j,k} P_j(x)Q_k(y). \tag{3.5}
\]

We also used the fact that \(P_n = P_{\bar{n}}\) and \(Q_{n-1} = Q_{\bar{n}}\). Note that \((3.5)\) corresponds to the Christoffel-Darboux formula of \([10]\), as mentioned in the introduction.

In the rest of the proof we are going to show that

\[
\sum_{k=n}^{n+m-1} \sum_{j=0}^{n-1} c_{j,k} P_j(x)Q_k(y) = \sum_{k=1}^{m} \frac{h_{\bar{n}}^{(k)}}{h_{\bar{n}-\bar{\epsilon}_k}^{(k)}} P_{\bar{n}-\bar{\epsilon}_k}(x)Q_{\bar{n}+\bar{\epsilon}_k}(y) \tag{3.6}
\]

so that \((3.5)\) then leads to our desired formula \((1.11)\).

3.2 The vector space generated by the polynomials \(P_{\bar{n}-\bar{\epsilon}_1}, \ldots, P_{\bar{n}-\bar{\epsilon}_m}\)

For fixed \(y\), the right-hand side of \((3.5)\) belongs to the vector space spanned by the polynomials of \(P_{\bar{n}-\bar{\epsilon}_1}, \ldots, P_{\bar{n}-\bar{\epsilon}_m}\). In this part of the proof, we characterize this vector space and show that the left-hand side of \((3.6)\) also belongs to this vector space \(V\).

Lemma 3.2 The polynomials \(P_{\bar{n}-\bar{\epsilon}_1}, \ldots, P_{\bar{n}-\bar{\epsilon}_m}\) are a basis of the vector space \(V\) of all polynomials \(\pi\) of degree \(\leq n - 1\) satisfying

\[
\int \pi(x)x^i w_j(x)dx = 0, \quad i = 0, \ldots, n_j - 2, \quad j = 1, \ldots, m. \tag{3.7}
\]
**Proof.** By the orthogonality properties (1.2) of the polynomials \( P_{\vec{n} - \vec{e}_i} \) for \( i = 1, \ldots, m \), it is obvious that they belong to \( V \). We are first going to show that the polynomials \( P_{\vec{n} - \vec{e}_i} \) are linearly independent. Suppose that

\[
a_1 P_{\vec{n} - \vec{e}_1} + a_2 P_{\vec{n} - \vec{e}_2} + \cdots + a_m P_{\vec{n} - \vec{e}_m} = 0
\] (3.8)

for some coefficients \( a_j \). Multiplying (3.8) with \( w_j(x)x^{n_j-1} \), and integrating over the real line, we obtain

\[
a_j h_j^{(j)}(\vec{n} - \vec{e}_j) = 0.\]

Since \( h_j^{(j)}(\vec{n} - \vec{e}_j) \neq 0 \), we get \( a_j = 0 \) for \( j = 1, \ldots, m \), which shows that the polynomials are linearly independent.

Suppose next that \( \pi \) belongs to \( V \). Put

\[
b_j = \frac{1}{h_j^{(j)}(\vec{n} - \vec{e}_j)} \int \pi(x)x^{n_j-1}w_j(x)dx
\]

and define the polynomial \( \pi_1 \) by

\[
\pi_1 = b_1 P_{\vec{n} - \vec{e}_1} + b_2 P_{\vec{n} - \vec{e}_2} + \cdots + b_m P_{\vec{n} - \vec{e}_m}.
\] (3.9)

Then \( \pi_1 - \pi \) belongs to \( V \) and

\[
\int (\pi_1(x) - \pi(x)) x^{n_j-1}w_j(x)dx = 0, \quad j = 1, \ldots, m.
\] (3.10)

This means that \( \pi_1 - \pi \) satisfies the conditions

\[
\int (\pi_1(x) - \pi(x)) x^i w_j(x)dx = 0, \quad i = 0, \ldots, n_j - 1, \quad j = 1, \ldots, m.
\] (3.11)

Because \( \pi_1 - \pi \) is a polynomial of degree \( \leq n - 1 \) and the system is perfect, it follows from (3.11) that \( \pi_1 - \pi = 0 \). Therefore \( \pi = \pi_1 \), and \( \pi \) can be written as a linear combination of the polynomials \( P_{\vec{n} - \vec{e}_1}, \ldots, P_{\vec{n} - \vec{e}_m} \).

The lemma follows.

**Lemma 3.3** For every \( k = n, \ldots, n + m - 1 \), we have that the polynomial

\[
\pi_k(x) = \sum_{j=0}^{n-1} c_{j,k} P_j(x)
\] (3.12)

belongs to the vector space \( V \).

**Proof.** Clearly \( \pi_k \) is a polynomial of degree \( n - 1 \). Using (3.2) we see that

\[
\pi_k(x) = x P_k(x) - \sum_{j=n}^{k+1} c_{j,k} P_j(x).
\] (3.13)

The representation (3.13) of \( \pi_k \) and the orthogonality conditions (1.2) show that \( \pi_k \) satisfies the relations (3.7), so that \( \pi_k \) belongs to \( V \) by Lemma 3.2.

Because of Lemma 3.3 the left-hand side of (3.6) belongs to \( V \) for every \( y \), and so by Lemma 3.2 we can write

\[
\sum_{k=n}^{n+m-1} \sum_{j=0}^{n-1} c_{j,k} P_j(x) Q_k(y) = \sum_{j=1}^{m} \phi_j(y) P_{\vec{n} - \vec{e}_j}(x)
\] (3.14)

for certain functions \( \phi_j(y) \). The next lemma gives an expression for \( \phi_j \). We use the notation \( \vec{s}_0 = \vec{0} \) (all-zero vector) and

\[
\vec{s}_j = \sum_{k=1}^{j} \vec{e}_k, \quad j = 1, \ldots, m.
\]
Lemma 3.4 We have for \( j = 1, \ldots, m \),

\[
h^{(j)}_{\vec{n} - e_j} \phi_j(y) = \sum_{i=1}^{j} h^{(j)}_{\vec{n} + \vec{s}_{i-1}} Q_{\vec{n} + \vec{s}_{i}}(y). \tag{3.15}
\]

Proof. Rewriting the left-hand side of (3.14) using (3.12) and (3.13) we obtain

\[
\sum_{j=1}^{m} \phi_j(y) P_{\vec{n} - e_j}(x) = \sum_{k=n}^{n+m-1} x P_k(x) Q_k(y) - \sum_{k=n}^{n+m-1} \sum_{j=k}^{n-1} c_{j,k} P_j(x) Q_k(y). \tag{3.16}
\]

Now multiply (3.16) with \( x^{n_j-1}w_j(x) \) and integrate with respect to \( x \). Then the left-hand side gives

\[
h^{(j)}_{\vec{n} - e_j} \phi_j(y). \tag{3.17}
\]

The second sum in the right-hand side of (3.16) gives no contribution to the integral because of orthogonality, and the first sum gives

\[
\sum_{k=n}^{n+m-1} \left( \int P_k(x) x^{n_j} w_j(x) dx \right) Q_k(y) = \sum_{i=1}^{m} \left( \int P_{n+i-1}(x) x^{n_j} w_j(x) dx \right) Q_{n+i-1}(y). \tag{3.18}
\]

Because of the choice (3.1) and the definition (1.7) we have

\[
P_{n+i-1} = P_{\vec{n} + \vec{s}_{i-1}}, \quad Q_{n+i-1} = Q_{\vec{n} + \vec{s}_{i}}, \quad i = 1, \ldots, m.
\]

Then we see that the integral in the right-hand side of (3.18) is zero if \( i \geq j + 1 \) and otherwise it is equal to \( h^{(j)}_{\vec{n} + \vec{s}_{i-1}} \). Then (3.15) follows. \( \blacksquare \)

3.3 Completion of the proof of Theorem 1.1

In view of (3.14) and (3.15) it remains to prove that

\[
h^{(j)}_{\vec{n}} Q_{\vec{n} + \vec{e}_j} = \sum_{i=1}^{j} h^{(j)}_{\vec{n} + \vec{s}_{i-1}} Q_{\vec{n} + \vec{s}_{i}}(y) \tag{3.19}
\]

for \( j = 1, \ldots, m \), and then (3.6) follows.

To establish (3.19) we need some properties of the numbers \( h^{(j)}_{\vec{n}} \) and relations between \( Q \)-functions with different multi-indices. We already noted that \( h^{(j)}_{\vec{n}} \neq 0 \). We express the leading coefficients of the polynomials \( A^{(j)}_{\vec{n}} \) in terms of these numbers.

Lemma 3.5 The leading coefficient of \( A^{(j)}_{\vec{n} + \vec{e}_j} \) is equal to \( \frac{1}{h^{(j)}_{\vec{n}}}. \)

Proof. Because of the orthogonality conditions (1.2) and (1.4) we have that

\[
1 = \int P_{\vec{n}}(x) Q_{\vec{n} + \vec{e}_j}(x) dx
\]

\[
= \int P_{\vec{n}}(x) A^{(j)}_{\vec{n} + \vec{e}_j}(x) w_j(x) dx
\]

\[
= \left( \text{leading coefficient of } A^{(j)}_{\vec{n} + \vec{e}_j} \right) \int P_{\vec{n}}(x) x^{n_j} w_j(x) dx
\]

\[
= \left( \text{leading coefficient of } A^{(j)}_{\vec{n} + \vec{e}_j} \right) h^{(j)}_{\vec{n}}
\]

and the lemma follows. \( \blacksquare \)
Lemma 3.6 Let $j \neq k$. Then we have for every multi-index $\vec{n}$ that

$$P_{\vec{n}}(x) = \frac{h^{(k)}_{\vec{n}}}{h^{(k)}_{\vec{n}+\vec{e}_j}} (P_{\vec{n}+\vec{e}_j} - P_{\vec{n}+\vec{e}_k}) = -\frac{h^{(j)}_{\vec{n}}}{h^{(j)}_{\vec{n}+\vec{e}_k}} (P_{\vec{n}+\vec{e}_j} - P_{\vec{n}+\vec{e}_k})$$

(3.20)

and

$$Q_{\vec{n}} = \frac{h^{(k)}_{\vec{n}-\vec{e}_j-\vec{e}_k}}{h^{(k)}_{\vec{n}-\vec{e}_k}} (Q_{\vec{n}-\vec{e}_j} - Q_{\vec{n}-\vec{e}_k}) = -\frac{h^{(j)}_{\vec{n}-\vec{e}_j-\vec{e}_k}}{h^{(j)}_{\vec{n}-\vec{e}_j}} (Q_{\vec{n}-\vec{e}_j} - Q_{\vec{n}-\vec{e}_k}).$$

(3.21)

**Proof.** We know that $P_{\vec{n}}$ is a polynomial of degree $|\vec{n}|$ that satisfies the orthogonality conditions (1.2). It is easy to see that $P_{\vec{n}+\vec{e}_j} - P_{\vec{n}+\vec{e}_k}$ is a polynomial of degree $|\vec{n}|$ that satisfies these same conditions. Because the system is perfect, we then have that

$$\gamma P_{\vec{n}}(x) = P_{\vec{n}+\vec{e}_j}(x) - P_{\vec{n}+\vec{e}_k}(x),$$

(3.22)

for some $\gamma \in \mathbb{R}$. Multiplying (3.22) with $x^n w_k(x)$ and integrating over the real line, we find that

$$\gamma h^{(k)}_{\vec{n}} = h^{(k)}_{\vec{n}+\vec{e}_j} - 0 = h^{(k)}_{\vec{n}+\vec{e}_j}.$$  

This proves the first equality of (3.20). The second equality follows by interchanging $j$ and $k$.

Next we show (3.21). It is easy to see that $Q_{\vec{n}-\vec{e}_j} - Q_{\vec{n}-\vec{e}_k}$ satisfies the same orthogonality conditions (1.4) as $Q_{\vec{n}}$. Since the degrees of the polynomials $A^{(i)}_{\vec{n}-\vec{e}_j} - A^{(i)}_{\vec{n}-\vec{e}_k}$ do not exceed the degrees of $A^{(i)}_{\vec{n}}$ for $i = 1, \ldots, m$, it follows that

$$\gamma Q_{\vec{n}} = Q_{\vec{n}-\vec{e}_j} - Q_{\vec{n}-\vec{e}_k},$$

(3.23)

for some $\gamma \in \mathbb{R}$. To compute $\gamma$, we are going to compare the leading coefficients of the polynomials that come with $w_k(x)$. Using Lemma 3.5, we find that

$$\gamma \frac{1}{h^{(k)}_{\vec{n}-\vec{e}_k}} = \frac{1}{h^{(k)}_{\vec{n}-\vec{e}_j-\vec{e}_k}} = \frac{1}{h^{(k)}_{\vec{n}-\vec{e}_j-\vec{e}_k}}.$$  

This proves the first equality of (3.21). The second equality follows by interchanging $j$ and $k$. $\Box$

Now we are ready to complete the proof of Theorem 1.1.

**Proof of Theorem 1.1**

In view of what was said before, it suffices to prove (3.19). Fix $j = 1, \ldots, m$. We are going to prove by induction that for $k = 0, \ldots, j - 1$,

$$h_{\vec{n}}^{(j)} Q_{\vec{n}+\vec{e}_j} = \sum_{i=1}^{k} h_{\vec{n}+\vec{s}_{i-1}}^{(j)} Q_{\vec{n}+\vec{s}_i} + h_{\vec{n}+\vec{s}_k}^{(j)} Q_{\vec{n}+\vec{s}_{k+1}+\vec{e}_j}.$$   

(3.24)

For $k = 0$, the sum in the right-hand side of (3.24) is an empty sum, and then the equality (3.24) is clear.

Suppose that (3.24) holds for some $k \leq j - 2$. Taking (3.24) with $\vec{n} + \vec{s}_{k+1} + \vec{e}_j$ instead of $\vec{n}$ and $k+1$ instead of $k$, we get

$$Q_{\vec{n}+\vec{s}_{k+1}+\vec{e}_j} = -\frac{h_{\vec{n}+\vec{s}_k}^{(j)}}{h_{\vec{n}+\vec{s}_{k+1}}^{(j)}} (Q_{\vec{n}+\vec{s}_{k+1}+\vec{e}_j} - Q_{\vec{n}+\vec{s}_{k+1}+\vec{e}_j}).$$
Thus
\[ h^{(j)}_{\vec{n}+\vec{s}_h} Q_{\vec{n}+\vec{s}_h+\vec{e}_j} = h^{(j)}_{\vec{n}+\vec{s}_h} Q_{\vec{n}+\vec{s}_h+1} + h^{(j)}_{\vec{n}+\vec{s}_h+1} Q_{\vec{n}+\vec{s}_h+1+\vec{e}_j} \] (3.25)
and using the induction hypothesis (3.24) we obtain (3.24) with \( k \) replaced by \( k + 1 \).

So (3.24) holds for every \( k = 0, 1, \ldots, j - 1 \). Taking \( k = j - 1 \) in (3.24), we obtain (3.19) and this completes the proof of Theorem 1.1. \( \square \)

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