A basic homogenization problem for the $p$-Laplacian in $\mathbb{R}^d$ perforated along a sphere: $L^\infty$ estimates

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Abstract

We consider a boundary value problem for the $p$-Laplacian, posed in the exterior of small cavities that all have the same $p$-capacity and are anchored to the unit sphere in $\mathbb{R}^d$, where $1 < p < d$. We assume that the distance between anchoring points is at least $\varepsilon$ and the characteristic diameter of cavities is $\alpha \varepsilon$, where $\alpha = \alpha(\varepsilon)$ tends to 0 with $\varepsilon$. We also assume that anchoring points are asymptotically uniformly distributed as $\varepsilon \downarrow 0$, and their number is asymptotic to a positive constant times $\varepsilon^{1-d/p}$. The solution $u^{\varepsilon}$ is required to be 1 on all cavities and decay to 0 at infinity. Our goal is to describe the behavior of solutions for small $\varepsilon > 0$. We show that the problem possesses a critical window characterized by $\tau := \lim_{\varepsilon \to 0} \alpha(\varepsilon) / \varepsilon$, where $\alpha(\varepsilon) = \varepsilon^{1/p} - \varepsilon^{d/p}$. We prove that outside the unit sphere, as $\varepsilon \downarrow 0$, the solution converges to $A. U$ for some constant $A$, where $U(x) = \min\{1, |x|^{-\tau}\}$ is the radial $p$-harmonic function outside the unit ball. Here the constant $A_*$ equals 0 if $\tau = 0$, while $A_* = 1$ if $\tau = \infty$. In the critical window where $\tau$ is positive and finite, $A_* \in (0, 1)$ is explicitly computed in terms of the parameters of the problem. We also evaluate the limiting $p$-capacity in all three cases mentioned above. Our key new tool is the construction of an explicit ansatz function $u^{\varepsilon}_{A_*}$ that approximates the solution $u^{\varepsilon}$ in $L^\infty(\mathbb{R}^d)$ and satisfies $\|\nabla u^{\varepsilon} - \nabla u^{\varepsilon}_{A_*}\|_{L^p(\mathbb{R}^d)} \to 0$ as $\varepsilon \to 0$.

Keywords: $p$-Laplacian, $p$-capacity, homogenization, $L^\infty$ estimates.

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1 Introduction

In this paper we consider a boundary value problem for the $p$-Laplacian in a domain obtained from $\mathbb{R}^d$ by perforating it (i.e., removing small cavities) along the unit sphere. The problem is formulated as follows. Given $\varepsilon > 0$, let $S = S(\varepsilon)$ be a finite set of points on the unit sphere $S^{d-1} \subset \mathbb{R}^d$ such that the Euclidean distance between any two points in $S$ is at least $\varepsilon$. Points in $S$ will be referred to as anchors. For each anchor $s$, let $K_s$ be a compact subset of the closed unit ball $\bar{B}(0,1) \subset \mathbb{R}^d$. Let $0 < \alpha = \alpha(\varepsilon) \leq \frac{1}{8\varepsilon}$ satisfy $\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$. Define

$$\Gamma = \Gamma_\varepsilon := \bigcup_{s \in S} (s + \alpha \varepsilon K_s).$$

(1.1)

Two examples of such sets are depicted in Figure 1.

Assume that $1 < p < d$ and let $u = u^{\varepsilon}$ be the Perron solution of the boundary value problem

$$\begin{cases}
\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u) = 0 & \text{in } \mathbb{R}^d \setminus \Gamma, \\
\quad u = 1 & \text{on } \Gamma, \\
\quad u(x) \to 0 & \text{as } |x| \to \infty.
\end{cases}$$

(1.2)

(We recall the definition of Perron solutions in Section 2.) We are interested in the asymptotic behavior of $u^{\varepsilon}$ as $\varepsilon \to 0$ under the following two key hypotheses:

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(H1) The anchors are asymptotically equidistributed, that is
\[ \varepsilon^{d-1} \sum_{s \in S(\varepsilon)} \delta_s \rightharpoonup \sigma \mu \quad \text{as} \quad \varepsilon \to 0, \] (1.3)
where \( \mu \) is the uniform probability measure on the unit sphere \( S^{d-1} \) and \( \sigma > 0 \).

(H2) All the sets \( K_s \) have the same p-capacity, i.e., there is some compact set \( K \subseteq B(0,1) \subset \mathbb{R}^d \) such that \( \text{cap}_p(K_s) = \text{cap}_p(K) > 0 \) for all \( \varepsilon > 0 \) and \( s \in S(\varepsilon) \), where
\[ \text{cap}_p(K) := \inf \left\{ \int_{\mathbb{R}^d} |\nabla \psi|^p : \psi \in C_0^\infty(\mathbb{R}^d), \psi \geq 1 \text{ on } K \right\}, \] (1.4)
see [9, Chapter 2]. A special case to keep in mind is when all the \( K_s \) are rotated copies of \( K \), as in the left part of Figure 1.

The Perron solution \( u \) of (1.2) is related to the p-capacity of \( \Gamma \). Specifically, the following identity holds [9, Theorems 9.33 and 9.35]:
\[ \text{cap}_p(\Gamma) = \int_{\mathbb{R}^d} |\nabla u|^p. \] (1.5)

In this context, \( u \) is called the p-equilibrium potential of \( \Gamma \). In previous works in similar setups [7, 10, 11] (summarized at the end of this section), asymptotics of \( u^\varepsilon \) in the Sobolev space \( H^{1,p} \) were studied. The methods of the present paper enable us to obtain precise \( L^\infty \) asymptotics.

To motivate our results, note that since the p-capacity is sub-additive [9, Theorem 2.2], we have:
\[ \text{cap}_p(\Gamma) \leq \sum_{s \in S} \text{cap}_p(s + \alpha \varepsilon K_s) = |S|((\alpha \varepsilon)^{d-p}\text{cap}_p(K) = \frac{(\alpha \varepsilon)^{d-p}}{\varepsilon^{d-1}}(e^{d-1}|S|)\text{cap}_p(K), \] (1.6)
where we used the easily checked scaling relation
\[ \text{cap}_p(aK) = a^{d-p} \text{cap}_p(K), \quad \forall a > 0. \] (1.7)

Taking into account that \( \varepsilon^{d-1}|S| \to \sigma \) as \( \varepsilon \to 0 \), we obtain that
\[ \text{cap}_p(\Gamma) \leq C \left( \frac{\alpha}{\alpha_c} \right)^{d-p}, \] (1.8)

where \( C \) is some constant independent of \( \varepsilon \),
\[ \alpha_c := \varepsilon^{\frac{1}{\gamma}}, \] (1.9)
and
\[ \gamma := \frac{d-p}{p-1}. \] (1.10)

On the other hand, since \( p \)-capacity is monotone and \( \Gamma \subset B(0, 1 + \varepsilon) \), we have
\[ \text{cap}_p(\Gamma) \leq (1 + \varepsilon)^{d-p} \text{cap}_p(B(0, 1)). \] (1.11)

Thus, when the limiting value (as \( \varepsilon \to 0 \)) of \( \frac{\alpha}{\alpha_c} \) increases from 0 to \( \infty \), the natural upper bounds (1.8) and (1.11) for \( \text{cap}_p(\Gamma) \) cross each other. It turns out that in the same regime, the solution \( u = u^\varepsilon \) of (1.2) (outside of \( S^{d-1} \)) gradually transitions from 0 to the \( p \)-equilibrium potential \( U \) of the unit ball, defined as the solution of
\[
\begin{cases}
\Delta_p U = 0 & \text{in } \mathbb{R}^d \setminus B(0, 1), \\
U = 1 & \text{on } B(0, 1), \\
U(x) \to 0 & \text{as } |x| \to \infty,
\end{cases}
\] (1.12)

and given by
\[ U(x) := \begin{cases} 
1 & 0 \leq |x| \leq 1, \\
|x|^{-\gamma} & |x| > 1.
\end{cases} \] (1.13)

The discussion above motivates our third hypothesis:
\[ (H_3) \lim_{\varepsilon \to 0} \frac{\alpha(\varepsilon)}{\alpha_c} = \tau \in [0, \infty]. \]

Most of the paper will be devoted to the analysis of the critical window, where \( 0 < \tau < \infty \). Our first result describes the asymptotics of the capacity \( \text{cap}_p(\Gamma_\varepsilon) \) and of the potential \( u = u^\varepsilon \) away from the unit sphere.

**Theorem 1.1.** Suppose that hypotheses \((H_1), (H_2), (H_3)\) hold. Then, as \( \varepsilon \to 0 \),
\[
0 \leq |x| \leq 1, \quad |x| \to \infty,
\] (1.14)

uniformly on compact subsets of \( \mathbb{R}^d \setminus S^{d-1} \), where for \( \tau \in [0, \infty) \),
\[ A_* = A_*(\tau) = \frac{\left( \sigma \tau^{d-p} \text{cap}_p(K) \right)^{\frac{1}{\tau d-p}}}{\left( \sigma \tau^{d-p} \text{cap}_p(K) \right)^{\frac{1}{\tau d-p}} + \left( \text{cap}_p(B(0, 1)) \right)^{\frac{1}{\tau d-p}}} \] (1.15)

Furthermore,
\[
\text{cap}_p(\Gamma_\varepsilon) \to \begin{cases} 
0 & \text{if } \tau = 0, \\
A_*^p \text{cap}_p(B(0, 1)) + (1 - A_*)^p \text{cap}_p(K) \sigma \tau^{d-p} & \text{if } \tau \in (0, \infty), \\
\text{cap}_p(B(0, 1)) & \text{if } \tau = \infty.
\end{cases}
\] (1.16)
It will be convenient to define $A_* = 1$ if $\tau = \infty$. If we assume that $\alpha = C\varepsilon^\zeta$ for $\zeta > 0$, then this theorem reveals a phase transition when the exponent $\zeta$ crosses $1/\gamma$.

**Example.** To illustrate the Theorem, consider the simple special case where $d = 3$, $p = 2$ (so $\gamma = 1$ and $\alpha_c = \varepsilon$) and $K_s = B(0, 1)$ for all $s$. Suppose that $\alpha = \tau\alpha_c$. In this case, $U(x) = \min\{1, |x|^{-1}\}$ and $w^\varepsilon(x)$ can be interpreted as the probability that a Brownian motion started from $x$ ever hits $\Gamma = \bigcup_{s \in S} B(s, \tau \varepsilon^2)$. The expression for $A_*$ simplifies to $A_* = \sigma\tau/(1 + \sigma\tau)$.

For $\tau \in (0, \infty)$, the expression \eqref{eq:1.14} suggests that $\text{cap}_p(\Gamma_\varepsilon)$ should approach $A_p^* \text{cap}_p(B(0, 1))$ as $\varepsilon \to 0$. This, however, contradicts the actual limiting value of the capacity given by \eqref{eq:1.16}. Indeed, the limiting equilibrium potential only captures the first summand in \eqref{eq:1.16}, which accounts for the contribution from the bulk. The second term in \eqref{eq:1.16}, which accounts for the contribution of the equilibrium potential near the cavities, completely disappears when looking at \eqref{eq:1.14} alone. This is similar to the “term coming from nowhere” discussed in [4].

The preceding observation raises the question: How does the solution of \eqref{eq:1.2} behave near the unit sphere? To answer this question, we will introduce the $p$-potential of $K_*$ in a ball $B(0, R)$, namely, the Perron solution $V_R^s : \mathbb{R}^d \to [0, 1]$ of

\[
\begin{align*}
\Delta_p V_R^s &= 0 \quad \text{in} \quad B(0, R) \setminus K_s, \\
V_R^s &= 1 \quad \text{on} \quad K_s, \\
V_R^s(x) &= 0 \quad \text{for} \quad |x| \geq R.
\end{align*}
\]
We also define $V_{\infty}^s$ by replacing the last requirement in (1.17) by $V_{\infty}^s(x) \to 0$ as $|x| \to \infty$. With $U$ defined by (1.13) and $U_{1+\varepsilon}(x) := U\left(\frac{x}{1+\varepsilon}\right)$, we define, for each $A \in [0,1]$, the ansatz function

$$u_A(x) := AU_{1+\varepsilon}(x) + (1 - A) \sum_{s \in S} V_s \frac{x - s}{\alpha \varepsilon}.$$  \hspace{1cm} (1.18)

For $\tau < \infty$, the convergence $u^\tau(x) \to A_i U(x)$ as $\varepsilon \to 0$ in Theorem 1.1 does not hold uniformly in $\mathbb{R}^d \setminus \mathbb{S}^{d-1}$ because $\sup_{x \in \mathbb{R}^d \setminus \mathbb{S}^{d-1}} u^\tau(x) = 1$. The following theorem states that $\nabla u_A$ approximates $\nabla u$ in $L^p(\mathbb{R}^d)$ and $u_A$ approximates $u$ in $L^\infty(\mathbb{R}^d)$ as $\varepsilon \to 0$. See Figures 2 and 3 for a depiction of $u_A$, and of how it changes when $\tau$ increases.

**Theorem 1.2.** Suppose that hypotheses $(H_1)$, $(H_2)$, $(H_4)$ hold. Then as $\varepsilon \to 0$, we have

$$\|\nabla u - \nabla u_A\|_{L^p(\mathbb{R}^d)} \to 0,$$  \hspace{1cm} (1.19)

and

$$\|u - u_A\|_{L^\infty(\mathbb{R}^d)} \to 0.$$  \hspace{1cm} (1.20)

**Remark 1.1.** Several technical challenges arise because there are no regularity assumptions on the compact sets $K_s$. First, the solution $u$ of (1.2) depends monotonically on $\alpha(\varepsilon)$ only if $K_s$ are star-shaped. Moreover, in general $u$ need not be continuous on the boundary of $\Gamma$.

**Related work.** There is an extensive literature devoted to the $p$-Laplacian and nonlinear potential theory, see, e.g., the books [2,9,12,14] and the review paper [15]. Chapter 3 of [13] discusses homogenization problems for the $p$-Laplacian. Most works on this topic focus on the derivation of effective limiting equations in domains that are perforated in the bulk; see, e.g., [7,11] and references therein. In the papers [7,10,11], which are most closely related to our study, the authors considered homogenization for the $p$-Laplacian in domains where the perforation takes place only near a $(d - 1)$ dimensional surface. In [10] the authors assume that $p < 1 + d/2$ and the cavities are obtained by intersecting a periodic structure with a hyperplane; the critical scaling $\alpha$ was already identified there under a mild regularity assumption on the cavities. In [11], the hyperplane is replaced by a convex surface and $p < 1 + d/4$. In [7] the cavities are obtained by intersecting a periodic collection of small balls with the $\varepsilon$-neighborhood of a smooth surface. The latter paper also contains a detailed review of earlier literature. In the works mentioned above, the authors determined the asymptotics of the $p$-capacity of the obstacle, and the limiting behavior of the solution in Sobolev space in the weak topology; the behavior of the solution near the perforated surface was not described. That is our main goal here.

**Roadmap.** While it is not so hard to show that the solution $u^\tau$ is generally well-behaved away from the cavities $s + \alpha \varepsilon K_s$, with rare regions of high local energy, the main obstacle to obtaining our results (especially the $L^\infty$ estimates in Theorem 1.2) is ruling out such exceptional regions.

The rest of the paper is organized as follows. Section 2 presents some necessary background and preliminaries.

## 2 Preliminaries

### 2.1 Notation

Given an open set $\Omega \subset \mathbb{R}^d$, let $H^{1,p}(\Omega)$ denote the Sobolev space of $L^p(\Omega)$ functions with weak gradient in $L^p(\Omega)$, and let $H_{0}^{1,p}(\Omega)$ denote the closure in $H^{1,p}(\Omega)$ of $C_c^\infty(\Omega)$. Write $H_{0}^{1,p}(\Omega)$ for the space of functions on $\Omega$ which, when restricted to every subdomain $\Omega_1$ compactly contained in $\Omega$, are in $H^{1,p}(\Omega_1)$. 

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Definition 2.1. [9] Chapters 6, 7] Given an open set \( \Omega \subset \mathbb{R}^d \), a continuous function \( w \in H^{1,p}_{\text{loc}}(\Omega) \) is called \( p \)-harmonic if \( \Delta_p w = 0 \) weakly, i.e., if
\[
\int_{\Omega} |\nabla w|^{p-2} \nabla w \cdot \nabla \eta = 0, \quad \forall \eta \in C_0^\infty(\Omega).
\]
A function \( \psi : \Omega \to (-\infty, \infty) \) is \( p \)-superharmonic if it is lower semi-continuous, not identically \( \infty \) on any connected component of \( \Omega \), and satisfies the following comparison inequality on compactly contained subdomains \( D \subset \Omega \): If \( \psi \geq w \) on \( \partial D \) and \( w \in C(\overline{D}) \) is \( p \)-harmonic in \( D \), then \( \psi \geq w \) in \( D \).

The following principle, a special case of [9, Proposition 7.6], will be used several times.

Lemma 2.1 (Comparison principle). Let \( v, w \) be bounded \( p \)-harmonic functions in an open set \( \Omega \subset \mathbb{R}^d \). If
\[
\limsup_{x \to y} v(x) \leq \liminf_{x \to y} w(x)
\]
for all \( y \in \partial \Omega \) (including \( y = \infty \) if \( \Omega \) is unbounded) then \( v \leq w \) in \( \Omega \).

Next we give the definition of Perron solutions from [9, Chapter 9].

Definition 2.2. Given a domain \( \Omega \subset \mathbb{R}^d \) and boundary values \( f : M \to \mathbb{R} \), where \( \partial \Omega \subset M \subset \Omega^c \), the upper class \( U^f_\Omega \) of \( f \) consists of \( p \)-superharmonic functions \( \psi : \Omega \to (-\infty, \infty] \) that are bounded below and satisfy
\[
\liminf_{x \to y} \psi(x) \geq f(y), \quad \forall y \in \partial \Omega.
\]
(2.1)
(Recall that if \( \Omega \) is unbounded, then we include \( \infty \) in \( \partial \Omega \).) The upper Perron solution
\[
h = \overline{H}^f_\Omega : \Omega \cup M \to [-\infty, \infty]
\]
of the boundary value problem
\[
\begin{aligned}
\Delta_p h &= 0 \quad \text{in} \quad \Omega, \\
h(y) &= f(y) \quad \text{for} \quad y \in M
\end{aligned}
\]
(2.2)
is defined in \( \Omega \) by
\[
\overline{H}^f_\Omega(x) := \inf\{\psi(x) : \psi \in U^f_\Omega\}
\]
(2.3)
and extended to agree with \( f \) in \( M \). The lower Perron solution of (2.2) is defined by \( \underline{H}^f_\Omega := -\overline{H}^f_{-f} \).

We say that \( f \) is resolutive in \( \Omega \) if the upper and lower Perron solutions of (2.2) coincide; in this case we refer to both of these simply as the Perron solution.

Theorem 9.25 in [9] ensures that if \( f \) is continuous on \( \partial \Omega \) and \( (\mathbb{R}^d \setminus \Omega) \) contains a compact set of positive \( p \)-capacity, then \( f \) is resolutive.

Following [9], we define the Dirichlet space
\[
L^{1,p}(\Omega) := \{ v \in H^{1,p}_{\text{loc}}(\Omega) : \nabla v \in L^p(\Omega) \},
\]
(2.4)
and let \( L^{1,p}_{0}(\Omega) \) be the closure of \( C_0^\infty(\Omega) \) in \( L^{1,p}(\Omega) \) with respect to the semi-norm \( \left( \int_{\Omega} |\nabla v|^p \right)^{\frac{1}{p}} \), see [9] Chapter 1.9, p. 13.

Definition 2.3. For a function \( v \in L^{1,p}(\Omega) \), we define its energy in a Borel set \( B \subset \Omega \) by
\[
E(v, B) = \int_B |\nabla v|^p.
\]
(2.5)
We write \( E(v, \mathbb{R}^d) \) simply as \( E(v) \).
We now define the \( p \)-capacity (with respect to a domain \( \Omega \)) of a compact set \( K \subset \Omega \) by
\[
\text{cap}_p(K, \Omega) := \inf \{ E(\psi, \Omega), \, \psi \geq 1 \text{ on } K, \, \psi \in C^\infty_0(\Omega) \}.
\]
This definition readily implies that
\[
\text{cap}_p(K, B(0, R)) \downarrow \text{cap}_p(K) \text{ as } R \uparrow \infty,
\]
where \( \text{cap}_p(K) = \text{cap}_p(K, \mathbb{R}^d) \). In \cite[27-28]{9} it is shown that
\[
\text{cap}_p(K, \Omega) = \inf \{ E(\psi, \Omega), \, \psi \geq 1 \text{ on } K, \, \psi \in H^1_0(\Omega) \cap C(\Omega) \}.
\]
Fix \( \phi \in C^\infty_0(\Omega) \) such that \( \phi = 1 \) on \( K \), and define the admissible class
\[
\mathcal{A}(K, \Omega, \phi) := \{ \psi \in L^1_0(\Omega) \text{ such that } \psi - \phi \in L^1_0(\Omega \setminus K) \}.
\]
Then we claim that
\[
\text{cap}_p(K, \Omega) \leq E(\psi, \Omega) \quad \text{for } \psi \in \mathcal{A}(K, \Omega, \phi).
\]
Indeed, suppose that \( \psi \in L^1_0(\Omega) \) and \( h_n \in C^\infty_0(\Omega \setminus K) \) satisfy \( \nabla(\psi - \phi - h_n) \to 0 \) in \( L^p(\Omega \setminus K) \). Then \( \nabla(\phi + h_n) = \nabla \phi = 0 \) a.e. in \( K \). Therefore, (2.8) implies that
\[
\text{cap}_p(K, \Omega) \leq E(\phi + h_n, \Omega) = E(\phi + h_n, \Omega \setminus K) \to E(\psi, \Omega \setminus K)
\]
as \( n \to \infty \).

Next, let \( \psi^K \) be the Perron solution of
\[
\begin{aligned}
\Delta_p \psi^K &= 0 \quad \text{in } \Omega \setminus K, \\
\psi^K &= 1 \quad \text{on } K, \\
\psi^K &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
where by convention, \( \infty \in \partial \Omega \) if \( \Omega \) is unbounded. By \cite[Theorem 9.33]{9}, \( \psi^K \in \mathcal{A}(K, \Omega, \phi) \), and it coincides with the \( p \)-potential of \( K \) defined in \cite[Chapter 6]{9}. Moreover, \( \psi^K \) is continuous and \( p \)-harmonic on \( \Omega \setminus K \), and it satisfies \( \psi^K \equiv 1 \) on \( K \). It follows from Corollary 1.21 in \cite{9} that \( \nabla \psi^K(x) = 0 \) for almost every \( x \in K \). By \cite[Theorem 9.35]{9}, \( E(\psi^K, \Omega) = \text{cap}_p(K, \Omega) \). This yields an alternative definition of \( p \)-capacity: If \( \phi \in C^\infty_0(\Omega) \) satisfies \( \phi = 1 \) on \( K \), then
\[
\text{cap}_p(K, \Omega) := \min \{ E(\psi, \Omega) : \psi \in \mathcal{A}(K, \Omega, \phi) \}.
\]
By the strict convexity of the \( L^p \) norm, \( \psi^K \) is the unique (up to translation by a constant) minimizer of the extremal problem \( (2.12) \).

Thus the Perron solutions \( u, V^p_\phi \) and \( U \), defined in \((1.2),(1.17)\) and \((1.12)\), have the following properties:
\[
\int_{\mathbb{R}^d} |\nabla u|^p = \text{cap}_p(\Gamma), \quad \int_{B(0,R)} |\nabla V^p_\phi|^p = \text{cap}_p(K_\phi, B(0,R)), \quad \int_{\mathbb{R}^d} |\nabla U|^p = \text{cap}_p(B(0,1)) = \gamma^{p-1} \omega_{d-1},
\]
where \( \omega_{d-1} \) is the surface area of the unit ball in \( \mathbb{R}^d \), and the last equality is from \cite[Section 2.11]{9}.

**Remark 2.1.** Fix \( \phi \in C^\infty_0(\mathbb{R}^d) \) such that \( \phi = 1 \) on \( \bar{B}(0,1) \) and \( \phi = 0 \) outside \( B(0,2) \), and suppose that \( R > 2 \). Since \( V^p_{\phi^K} \) coincides with the \( p \)-potential of \( K_\phi \) in \( B(0,R) \), \cite[Lemma 8.5]{9} yields that \( V^p_{\phi^K} - \phi \in H^1_0(B(0,R) \setminus K_\phi) \), which readily implies that for any domain \( \Omega \) that contains \( B(0,R) \), we have \( V^p_{\phi^K} \in H^1_0(\Omega \setminus K_\phi) \) and \( E(V^p_{\phi^K}, \Omega) = E(V_{\phi^K}, B(0,R)) \). While the classical gradient of \( V^p_{\phi^K} \) on \( \partial B(0,R) \) need not exist, the distributional gradient may be taken to be zero on this boundary, and in any case, does not affect the energy.

We will need an estimate on the rate of convergence in \( (2.7) \).

**Lemma 2.2.** Suppose that \( K \subset \bar{B}(0,1) \subset \mathbb{R}^d \) is compact. Then
\[
\text{cap}_p(K) \leq \text{cap}_p(K, B(0,R)) \leq (1 - R^{-\gamma})^{-p} \text{cap}_p(K).
\]

Proof. The first inequality in (2.14) was already noted in (2.7), so we focus on the second. Let \( K_n \) be the union of the closed dyadic cubes of side length \( 2^{-n} \) that intersect \( K \). Denote by \( \psi_n \) the \( p \)-potential of \( K_n \) in \( \mathbb{R}^d \) and let \( \delta > 0 \). Since \( K = \bigcap_{n=1}^{\infty} K_n \), Theorem 2.2(iv) in [9] implies that there exists \( n \) such that \( K_n \subset B(0, 1 + \delta) \) and

\[
\text{cap}_p(K_n) \leq \text{cap}_p(K) + \delta.
\]

Note that \( \psi_n \) is continuous in \( \mathbb{R}^d \) since \( \mathbb{R}^d \setminus K_n \) is regular by the corkscrew condition (See [9] Theorems 6.27 and 6.31).) Recall that \( U \) given by (1.13) is the \( p \)-potential of the unit ball in \( \mathbb{R}^d \). Then \( \psi_n(x) \leq U\left(\frac{x}{1 + \delta}\right) \) for all \( x \) by the definition of upper Perron solutions, so the function

\[
\psi(x) = \max \left\{ 0, \frac{\psi_n(x) - (1 + \delta)^{-\gamma} R^{-\gamma}}{1 - (1 + \delta)^{-\gamma} R^{-\gamma}} \right\}
\]

is in \( H^{1,p}_0(B(0,R)) \) by [9] Lemmas 1.23 and 1.26. Thus, using \( \psi \) in (2.8) yields

\[
\text{cap}_p(K, B(0,R)) \leq E(\psi, B(0,R)) \leq (1 - (1 + \delta)^{-\gamma} R^{-\gamma})^{-p} \text{cap}_p(K_n),
\]

which implies the second inequality of (2.14) since \( \delta > 0 \) is arbitrary. \( \square \)

2.2 Energy estimate on truncated cones

Definition 2.4. Let \( y \in \mathbb{S}^{d-1} \) and let \( Q_\delta(y) := B(y, \delta) \cap \mathbb{S}^{d-1} \) be the open spherical cap of Euclidean radius \( \delta \) centered at \( y \). We define the spherical cone

\[
\Lambda_\delta(y) := \{ rq : q \in Q_\delta(y) \text{ and } r > 0 \}.
\]

Lemma 2.3. Let \( Q \) be an open set on the unit sphere, and for \( R > 1 \) set

\[
\Lambda(Q, R) := \{ rq : q \in Q \text{ and } r > R \}.
\]

Suppose that a nonnegative function \( v \in L^{1,p}(\Lambda(Q, R)) \cap C(\Lambda(Q, R)) \) satisfies

\[
v(x) \to 0 \quad \text{as} \quad |x| \to \infty,
\]

and

\[
v(Ry) \geq \tilde{\Lambda} > 0, \quad \forall y \in Q.
\]

Then,

\[
E(v, \Lambda(Q,R)) \geq E(v_{\tilde{\Lambda}}, \Lambda(Q,R)) ,
\]

where \( v_{\tilde{\Lambda}}(x) = \tilde{\Lambda} R^{-1} U(x) = \tilde{\Lambda} U(x/R) \) for all \( x \in \Lambda(Q,R) \).

Proof. Assume (2.17) and (2.18) hold. Since \( v \in L^{1,p}(\Lambda(Q, R)) \), for \( \mu \) almost every \( y \in Q \), the gradient \( \nabla v(ry) \) exists and is finite for a.e. \( r \in (R, \infty) \). Furthermore, [6] Theorem 4.9.2 implies that for \( \mu \) almost every \( y \in Q \), the function \( r \mapsto v(ry) \) is absolutely continuous in \( (R, \infty) \). Thus, for \( \mu \) almost every \( y \in Q \), Hölder’s inequality yields that

\[
\tilde{\Lambda} \leq \int_{R}^{\infty} |\nabla v(ry)| \, dr = \int_{R}^{\infty} |\nabla v(ry)| \, r^{\frac{d-1}{p}} R^{-\frac{d}{p}} \, dr \leq \left( \int_{R}^{\infty} |\nabla v(ry)|^p \, r^{d-1} \, dr \right)^{\frac{1}{p}} \left( \int_{R}^{\infty} r^{-\frac{d}{p}} \, dr \right)^{\frac{1}{1-\frac{d}{p}}}
\]

\[
= \left( \frac{1}{\gamma R^{\gamma}} \right) \left( \int_{R}^{\infty} |\nabla v(ry)|^p \, r^{d-1} \, dr \right)^{\frac{1}{p}}
\]

Taking \( p \)-th power and rearranging terms, we obtain

\[
\tilde{\Lambda}^p R^{d-\gamma} r^{p-1} \leq \int_{R}^{\infty} |v(ry)|^p \, r^{d-1} \, dr .
\]

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Let $\omega_{d-1}$ denote the area of the unit sphere. Integrating (2.21) over $Q$, we obtain
\[
\bar{A}^p R^{d-p} \omega_{d-1} \mu(Q) \leq \omega_{d-1} \int_{\mathbb{R}} |\nabla v(y)|^p \rho^{d-1} \, dr \, d\mu(y) = \int_{\Lambda(Q,R)} |\nabla v(x)|^p \, dx = E(v, \Lambda(Q,R)) \, . \tag{2.22}
\]

By the condition for equality in Hölder’s inequality, the second inequality in (2.20) is an equality if for some constant $C$,
\[
\left[|\nabla v(y)|^p \right]^{\frac{1}{p-1}} \leq Cr^{\frac{1-d}{p}} \text{ for a.e. } r \geq R \, . \tag{2.23}
\]
The function $v_A(x) = \bar{A} R^d U(x)$ satisfies (2.23) and the boundary condition $v_A(Ry) = \bar{A}$ for all $y \in Q$. Since
\[
|\nabla v_A(y)| = -\frac{d}{dr} v_A(ry)
\]
for $y \in \partial B(0,1)$, the first inequality in (2.20) is also an equality if $v = v_A$. Therefore,
\[
E(v_A, \Lambda(Q,R)) = \bar{A}^p R^{d-p} \omega_{d-1} \mu(Q) \, . \tag{2.24}
\]
Combining (2.22) and (2.24), we obtain (2.19).

### 2.3 Energy of an ansatz function

Recall from (1.18) the definition
\[
u_A(x) = AU_{1+\varepsilon}(x) + (1 - A) \sum_{s \in S} V_s^{\alpha \varepsilon} \left(\frac{x - s}{\alpha \varepsilon}\right) \, .
\]
Since the gradients of the summands on the right-hand side have disjoint supports, we have
\[
E(u_A) = A^p E(U_{1+\varepsilon}) + (1 - A)^p \sum_{s \in S} E(x \mapsto V_s^{\alpha \varepsilon} \left(\frac{x - s}{\alpha \varepsilon}\right)) \, . \tag{2.25}
\]
(By Remark 2.1 we can compute the energy over all of $\mathbb{R}^d$ in the summands on the right-hand side.) For any function $\psi \in L^p(\mathbb{R}^d)$, vector $s \in \mathbb{R}^d$ and scalar $r > 0$, scaling arguments yield that
\[
E\left(x \mapsto \psi\left(\frac{x - s}{r}\right)\right) = r^{d-p} E(\psi) \, . \tag{2.26}
\]
Therefore,
\[
E(u_A) = A^p (1 + \varepsilon)^{d-p} E(U) + (1 - A)^p \sum_{s \in S} (\alpha \varepsilon)^{d-p} E(V_s^{\alpha \varepsilon}) \, . \tag{2.27}
\]
Using the fact that $E(U) = \text{cap}_p(\bar{B}(0,1))$, we obtain
\[
E(u_A) = (1 + \varepsilon)^{d-p} A^p \text{cap}_p(\bar{B}(0,1)) + (1 - A)^p \sum_{s \in S} \kappa_s \, , \tag{2.28}
\]
where
\[
\kappa_s := \text{cap}_p\left(\alpha \varepsilon K_s, B\left(s, \frac{\varepsilon}{10}\right)\right) \, . \tag{2.29}
\]
By Lemma 2.2 applied to $K_s$ and hypothesis (H2) on equality of capacities, we infer that
\[
\left|\text{cap}_p\left(K_s, B\left(0, \frac{1}{10\alpha}\right)\right) - \text{cap}_p(K)\right| \leq J(\alpha) \text{cap}_p(K) \, , \tag{2.30}
\]
where
\[
J(\alpha) := (1 - (10\alpha)\gamma)^{-p} - 1 \to 0 \text{ as } \varepsilon \to 0 \, . \tag{2.31}
\]
Therefore,
\[
|E(u_A) - (1 + \varepsilon)^{d-p}A\Gamma_p(\bar{B}(0,1)) - (1 - A)^p(\alpha\varepsilon)^{d-p}|S|\Gamma_p(K)\rangle \leq (1 - A)^p(\alpha\varepsilon)^{d-p}|S|J(\alpha)\Gamma_p(K).
\]  
(2.32)

Next, as \( \varepsilon \to 0 \), we have by (1.5) and the definition of \( \tau \) in (H\(_3\)) that
\[
(\alpha\varepsilon)^{d-p}|S| = \left(\frac{\tau}{\alpha\varepsilon}\right)^{d-p}\varepsilon^{d-1}|S| \to \tau^{d-p}\sigma.
\]  
(2.33)

In conjunction with (2.32), this gives
\[
E(u_A) \to \varphi_\tau(A) \text{ as } \varepsilon \to 0,
\]  
(2.34)

where
\[
\varphi_\tau(A) = A\Gamma_p(\bar{B}(0,1)) + (1 - A)^p\Gamma_p(K)\sigma^{d-p}.
\]  
(2.35)

If \( \tau \in [0, \infty) \), then the convergence in (2.34) is uniform in \( A \in [0, 1] \). Note that \( \varphi_\infty(1) = \Gamma_p(\bar{B}(0,1)) \) and \( \varphi_\infty(A) = \infty \) for all \( A \in (0, 1) \), and in the latter case, the convergence in (2.34) means that the left-hand side tends to \( \infty \). For \( \tau \in (0, \infty) \), the function \( \varphi_\tau \) is a continuous strictly convex function on \([0, 1]\). Differentiation shows that it attains its minimum at \( A_\ast = A_\ast(\tau) \) given by (1.15), with
\[
\varphi_\tau(A_\ast) = \left(\frac{(\sigma^{d-p}\Gamma_p(K)\Gamma_p(\bar{B}(0,1)))}{(\sigma^{d-p}\Gamma_p(K)) + (\Gamma_p(\bar{B}(0,1)))}\right)^{p-1}.
\]  
(2.36)

Clearly, \( \varphi_\tau(A_\ast) \leq \varphi_\tau(1) = \Gamma_p(\bar{B}(0,1)) \) for all \( \tau \in [0, \infty) \).

**Remark 2.2.** Let \( \eta : \mathbb{R}^d \to [0, 1] \) be a \( C^\infty \) cutoff function such that \( \eta \equiv 1 \) on \( B(0,1) \) and \( \eta \equiv 0 \) outside \( B(0,2) \). Then for \( R > 2 \), we have that \( V_\ast - \eta \in L_0^1(B(0, R) \setminus K_s) \) for each \( s \in S \). Thus, for every anchor \( s \),
\[
V_\ast\left(\frac{x - s}{\alpha\varepsilon}\right) - \eta\left(\frac{x - s}{\alpha\varepsilon}\right) \in L_0^1(B(s, \varepsilon/10) \setminus (s + \alpha\varepsilon K_s)).
\]

Consequently, for all \( A \in [0, 1] \), the ansatz function \( u_A \) given by (1.18) belongs to the admissible class \( \mathcal{A}(\Gamma, \mathbb{R}^d, \phi) \) with
\[
\phi(x) := A\eta\left(\frac{x}{1 + \varepsilon}\right) + (1 - A)\sum_{s \in \mathcal{S}} \eta\left(\frac{x - s}{\alpha\varepsilon}\right).
\]

By (2.10), this implies that \( E(u) \leq E(u_A) \).

**Corollary 2.1.** For every \( y \) on the unit sphere, \( \delta > 0 \) and \( \tau \in [0, \infty] \), we have
\[
E(u_A, \Lambda_\delta(y)) \to \mu(\Lambda_\delta)\varphi_\tau(A) \text{ as } \varepsilon \to 0,
\]  
(2.37)

and the convergence is uniform in \( A \in [0, 1] \) and in \( y \in S^{d-1} \), provided that \( \tau < \infty \). Moreover, for every \( \delta \in (0, 1/2) \) and sufficiently small \( \varepsilon \) (that may depend on \( \delta \)),
\[
\forall A \in [0, 1], \quad E(u_A, \Lambda_\delta(y)) \geq (1 - \delta)^{d-p}E(u_A, \Lambda_\delta(y)) \text{ for } y \in S^{d-1}.
\]  
(2.38)

**Proof.** By the definition of \( u_A \) and (2.13), we have
\[
(1 - A)^p\sum_{s \in S \cap \mathcal{Q}_{\delta, \varepsilon}(y)} \kappa_s \leq E(u_A, \Lambda_\delta(y)) - A^p(1 + \varepsilon)^{d-p}E(u, \Lambda_\delta(y)) \leq (1 - A)^p\sum_{s \in S \cap \mathcal{Q}_{\delta, \varepsilon}(y)} \kappa_s,
\]  
(2.39)

where \( \kappa_s \) was defined in (2.29). Since \( U \) is radial,
\[
E(U, \Lambda_\delta(y)) = \mu(\Lambda_\delta)\Gamma_p(\bar{B}(0,1)).
\]

The rest of the proof of (2.37) proceeds exactly like the proof of (2.34), since the left-hand and right-hand sides of (2.39) have the same asymptotics:
\[
\varepsilon^{d-1}|S \cap \mathcal{Q}_{\delta, \varepsilon}(y)| \to \sigma\mu(\Lambda_\delta) \text{ as } \varepsilon \to 0.
\]
It remains to verify (2.38). If $0 < \tau < \infty$, then this follows from the uniform convergence (in the parameter $A$) in (2.37). If $\tau = 0$, then $A_\ast = 0$. In this case, (2.38) is obvious if $A \leq \delta$. On the other hand, if $\tau = 0$ and $A > \delta$, then the left-hand side of (2.38) is at least $\delta^p E(u, A, \Lambda^\delta(y)) > 0$ which does not depend on $\varepsilon$, while the right-hand side of (2.38) tends to 0 as $\varepsilon \downarrow 0$; Thus, (2.38) holds in this case as well provided $\varepsilon$ is small enough.

Finally, if $\tau = \infty$, then $A_\ast = 1$ and $u_{A_\ast} = U_{1+\varepsilon}$; in this case, (2.38) is obvious if $A > 1 - \delta$, while the lower bound
\[ E(u, A, \Lambda^\delta(y)) \geq (1 - A)^p \sum_{s \in S \cap Q_{\delta, \varepsilon}(x)} \kappa_s, \]
implies that
\[ \lim_{\varepsilon \to 0} \min_{A \in [0, 1-\delta]} E(u, A, \Lambda^\delta(y)) = \infty, \]
so (2.38) holds if $\tau = \infty$ and $A \leq 1 - \delta$ as well.

3 Bounding oscillation and energy of $u$ in cones

In this section, the positive constants $C, C_1, C_2, \ldots$ depend only on $d, p$.

**Lemma 3.1.** Suppose that for some $r > 0$ and $z \in \mathbb{R}^d$ we have
\[ B(z, r) \subset B(z, 5r/4) \subset \mathbb{R}^d \setminus \Gamma \] (3.1)
and for some $\lambda > 0, \beta \geq 0$, the solution $u$ of (1.2) satisfies
\[ E(u, B(z, 5r/4)) \leq \lambda^p r^{d-1-\beta}. \] (3.2)

Then,
\[ \text{osc}_{B(z,r)} u \leq C_1 \lambda r^{\frac{p-1-\beta}{p}}, \] (3.3)
where
\[ \text{osc}_D u := \sup_D u - \inf_D u \] (3.4)
stands for the oscillation over the set $D$.

**Proof.** By Poincaré’s inequality, there exists a real $t$ such that
\[ J_p = \int_{B(z, 5r/4)} |u-t|^p \leq C_2 r^p \int_{B(z, 5r/4)} |\nabla u|^p = C_2 r^p E(u, B(z, 5r/4)). \] (3.5)
The hypothesis (3.2) and the inequality (3.5) give
\[ J_p \leq C_2 \lambda^p r^{p+d-1-\beta}. \] (3.6)
Let
\[ u_1(x) = \max\{u(x) - t, 0\}, \quad u_2(x) = \max\{t - u(x), 0\}. \] (3.7)
By [12] Lemma 3.6] we have
\[ \|u_i\|_{L^\infty(B(z,r))} \leq C_3 \left(\frac{J_p}{r^d}\right)^{\frac{1}{p}} \leq C_4 \lambda r^{\frac{p-1-\beta}{p}}, \quad i = 1, 2, \] (3.8)
which implies (3.3) with $C_1 = 2C_4$. 

\[ \square \]
Figure 4: Geometric objects used in the proof of Lemma 3.2. The top figure indicates the point $x_0$ and the line segment $L(x_0)$ in the cored wedge $\Lambda^*_\varepsilon(s)$ defined by (3.10). The bottom figure shows the points $x_k$ and $\xi$ on $L(x_0)$ and the corresponding overlapping balls centered at these points.

**Definition 3.1.** Fix $\beta = \frac{p^{-1}}{2p}$. Given $\delta < 1/20$ and $\varepsilon < \delta/20$, an anchor $s \in S$ will be called a **good anchor** if

$$E(u, \Lambda_\zeta(s)) \leq \zeta^{d-1-\beta}, \quad \forall \zeta \in [\varepsilon, \delta].$$

(3.9)

Otherwise, $s$ will be called a **bad anchor**.
Lemma 3.2. Fix $\varepsilon, \delta, \beta$ as in the definition above. Suppose $s$ is a good anchor and let
\[
\Lambda^*_s(s) := \left\{ x \in \Lambda_{s/2}(s) \setminus B\left( s, \frac{\varepsilon}{10} \right) : 1 - \delta \leq |x| \leq 1 + \delta \right\}.
\]
Then there exists $C_5$, such that
\[
\text{osc}_{\Lambda^*_s(s)} u \leq C_5\delta^{\frac{2-1-\beta}{p}}.
\]

Proof. Take $x_0 \in \Lambda^*_s(s)$ and connect $x_0$ radially by a line segment $L(x_0)$ to the cap $\Lambda^*_s(s) \cap \{|x| = 1 + \delta\}$ or the cap $\Lambda^*_s(s) \cap \{|x| = 1 - \delta\}$, depending on whether $|x_0| > 1$ or not. (Observe that the oscillation of $u$ on each of these caps is at most $C_1\delta^{\frac{3}{p}-\beta}$ by Lemma 3.1.) Assume first that $1 \leq |x_0| \leq 1 + \delta/4$. We then define a sequence of points $\{x_k\}_{k \geq 0}$ along $L(x_0)$ and corresponding radii $r_k$ as follows (See Figure 4):

For $k = 0, 1, \ldots, 20$, let $|x_k| = |x_0| + k\varepsilon/20$ and $r_k = \varepsilon/20$. For $k > 20$, let $r_k = 2^{k-22}\varepsilon/5$ and $|x_k| = |x_{k-1}| + r_k$. Denote by $k_s$ the last $k$ such that $|x_k| \leq 1 + 3\delta/4$. Then for all $k \in [0, k_s]$, we have $|x_k - s| \geq 3r_k/2$ and $3r_k/2 \leq |x_k| - 1 \leq 3\delta/4$, whence $r_k \leq \delta/2$. Note that for $k \leq k_s$, we have
\[
B(x_k, 5r_k/4) \subset \Lambda_{\max}(s) \setminus \Gamma.
\]
We also have $|x_k| > 1 + r_k$, and $1 + 3\delta/4 \leq |x_{k+1}| = |x_k| + 2r_k$, so
\[
|x_k| - 1 + r_k > (3|x_k| - 3 + 3r_k, \ (|x_k| - 1 - r_k))/3 = 2(|x_k| + 2r_k - 1)/3 \geq \delta/2.
\]
Let $\xi$ be the point on $L(x_0)$ that satisfies $|\xi| = 1 + 3\delta/4$. Observe that $B(\xi, 5\delta/16) \subset \Lambda_{\beta}(s) \setminus \Gamma$. Applying the preceding lemma to the overlapping balls $B(x_k, r_k)_{k \leq k}$ and $B(\xi, \delta/4)$ yields, for some $C_0$, that
\[
\forall k \leq k_s, \quad \text{osc}_{B(x_k, r_k)} u \leq C_0 r_k^{\frac{2-1-\beta}{p}}, \quad \text{osc}_{B(\xi, \delta/4)} u \leq C_3 \delta^{\frac{2-1-\beta}{p}}.
\]

Summing the resulting (almost geometric) series concludes the proof for the oscillation over the line segment $L(x_0)$. Considering the union of these line segments over all $\{x_0 \in \Lambda^*_s(s) : 1 \leq |x_0| \leq 1 + \delta/4\}$ allows us to bound the oscillation over $\{x \in \Lambda^*_s(s) : |x| \geq 1\}$. A similar argument applies to $\{x \in \Lambda^*_s(s) : |x| \leq 1\}$. Since these two sets intersect on the unit sphere, the proof is complete. \(\square\)

Lemma 3.3. Fix $\varepsilon, \delta, \beta$ as in Definition 3.1 and let $y$ be a point on the unit sphere $S^{d-1}$. Suppose that
\[
E(u, \Lambda_{2\delta}(y)) \leq M\delta^{d-1},
\]
for some $M > 0$. Then the set $S_\beta$ of bad anchors satisfies $|S_\beta \cap Q_\delta(y)| \leq C_7 M \delta^{d}(\delta/\varepsilon)^{d-1}$, where $C_7 = C(d, p)$.

The proof is a variant of the classical proof of the Hardy-Littlewood maximal inequality.

Proof. By the definition of bad anchors, for each $s \in S_\beta \cap Q_\delta(y)$ there is a $\zeta = \zeta(s) \in [\varepsilon, \delta]$ such that
\[
E(u, \Lambda_{\zeta}(s)) > \zeta^{d-1-\beta} \geq \delta^{-\beta} \zeta^{d-1}.
\]

The caps $Q_{\zeta(s)}(s)$ with $s \in S_\beta \cap Q_\delta(y)$ form a Besicovitch covering of $S_\beta \cap Q_\delta(y)$. By the Besicovitch Covering Theorem, we can extract a finite subcover $\{Q_{\zeta_j(s_j)}\}_{j=1}^N$, where $\zeta_j = \zeta(s_j)$ for each $j$, so that every point on the unit sphere belongs to at most $C_8 = C_8(d)$ elements of this subcover. Therefore,
\[
\sum_{j=1}^N 1_{\Lambda_{\zeta_j}(s_j)} \leq C_8 1_{\Lambda_{2\delta}(y)}.
\]

For each $s \in S_\beta \cap Q_\delta(y)$, there is some $j \leq N$ such that $s \in Q_{\zeta_j}(s_j)$, whence the cap $Q_{\zeta/2}(s)$ is contained in $Q_{2\zeta_j}(s_j)$. Since the $\varepsilon/2$ neighborhoods $Q_{\varepsilon/2}(s)$ of anchors are disjoint, for some $C_9 > 0$ we have
\[
C_9 |S_\beta \cap Q_\delta(y)| \varepsilon^{d-1} \leq \mu\left(\bigcup_{s \in S_\beta \cap Q_\delta(y)} Q_{\varepsilon/2}(s)\right) \leq \mu\left(\bigcup_{j=1}^N Q_{2\zeta_j}(s_j)\right) \leq C_9 \sum_{j=1}^N \mu(Q_{2\zeta_j}(s_j)).
\]

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Now by \(3.14\), for each \(j\) we have
\[
\mu(Q_{2c_j}(s_j)) \leq C_{10}d_j^d \leq C_{10}\delta^\beta E(u, \Lambda_{c_j}(s_j)).
\] (3.17)

Thus by (3.16) and (3.17),
\[
|S_\delta \cap Q_\delta(y)| \leq \frac{C_{10}\delta^\beta}{C_9} \sum_{j=1}^N E(u, \Lambda_{c_j}(s_j)) = C_{11}\delta^\beta \int_{\mathbb{R}^d} \left( |\nabla u|^p \sum_{j=1}^N 1_{\Lambda_{c_j}(s_j)} \right) \leq C_{11}\delta^\beta C_9 E(u, \Lambda_{2\delta}(y)),
\] (3.18)
where we have used (3.15) in the last step. Combining (3.13) and (3.18) concludes the proof. \(\square\)

**Lemma 3.4.** Let \(\beta := \frac{p-1}{2p}\). Then there exist \(\delta_0 > 0\) (which may depend on \(d, p, \sigma\)) and \(C_{12}, C_{13} > 0\) (which depend on \(d, p\)) such that for \(\delta \in (0, \delta_0)\), there exists \(\varepsilon_0 > 0\) with the following property. For each \(\varepsilon \in (0, \varepsilon_0)\) and \(y \in S^{d-1}\) such that \(u = u^\varepsilon\) satisfies
\[
E(u, \Lambda_{2\delta}(y)) \leq \delta^{-\beta/2}\delta^{d-1},
\] (3.19)
there exists some \(A = A(y, \varepsilon)\), such that
\[
|u(x) - A| \leq C_{12}\delta^\beta, \quad \forall x \in \Lambda_{\delta}(y) : |x| = 1 \pm \delta,
\] (3.20)
and
\[
E(u, \Lambda_{\delta}(y)) \geq (1 - 2\delta^{\beta/3})E(u_A, \Lambda_{\delta}(y)).
\] (3.21)

**Proof.** Since \(\beta < (p-1)/2\), we have \((p-1-\beta)/p > (p-1)/2p = \beta\). We will use this repeatedly below. Also, recall the following consequence of the definition (1.18) of \(u_A\) and (2.13), that was already noted in (2.39):
\[
E(u_A, \Lambda_{\delta}(y)) \leq A^p(1 + \varepsilon)^{d-p}E(U, \Lambda_{\delta}(y)) + (1 - A)^p \sum_{s \in S^{d-1}} \kappa_s,
\] (3.22)
where
\[
\kappa_s := \text{cap}_K \left( \alpha \varepsilon K_s, B \left( s, \frac{\varepsilon}{10} \right) \right).
\] (3.23)

First observe that under the assumption (3.19), Lemma 3.1 gives that
\[
\text{osc}_{\Lambda_{\delta}(y) \cap \partial B(0,1+\delta)} u \leq C_1\delta^{\beta - \frac{1-p}{2}} \leq C_1\delta^\beta.
\] (3.24)

Similarly,
\[
\text{osc}_{\Lambda_{\delta}(y) \cap \partial B(0,1-\delta)} u \leq C_1\delta^\beta.
\] (3.25)

Hence,
\[
u(x) = A_+ + O(\delta^\beta), \quad \forall x \in \Lambda_{\delta}(y) : |x| = 1 \pm \delta,
\] (3.26)
where \(A_+ = A_+(y)\) and \(A_- = A_-(y)\) are two constants. In view of Lemma 3.3 if \(\delta_0 > 0\) is small enough and \(\delta \in (0, \delta_0)\), then for sufficiently small \(\varepsilon > 0\), there exists a good anchor in \(Q_\delta(y)\), and therefore by Lemma 3.2 and (3.26), we have \(|A_+ - A_-| \leq C_{13}\delta^\beta\). Thus
\[
|u(x) - A_+| \leq C_{14}\delta^\beta, \quad \forall x \in \Lambda_{\delta}(y) : |x| = 1 \pm \delta.
\] (3.27)

Denote by \(Q^\varepsilon_{\delta-\varepsilon}(y) = (S \setminus S_\delta) \cap Q_{\delta-\varepsilon}(y)\) the set of good anchors in \(Q_{\delta-\varepsilon}(y)\). Lemma 3.2 and (3.20) imply that there exists \(C_{15}\) such that for each \(s \in Q^\varepsilon_{\delta-\varepsilon}(y)\),
\[
u(x) \leq A_1 := \min \{1, A_+ + C_{15}\delta^\beta\}, \quad \forall x \in \partial B \left( s, \frac{\varepsilon}{10} \right).
\] (3.28)
Define
\[ A := A_1 \text{ if } A_+ > 1/2 \text{ and } A := \max\{0, A_+ - C_14\delta^\beta\} \text{ if } A_+ \leq 1/2. \]

We assume that \( \delta \) is small enough to ensure that \( C_{15}\delta^\beta < 1/4 \), so \( A = 1 \) iff \( A_1 = 1 \). Also, note that \( A_1 \leq A + (C_{14} + C_{15})\delta^\beta, \) so \( 1 - A_1 \geq (1 - A)(1 - 2(C_{14} + C_{15})\delta^\beta) \) if \( A_+ \leq 1/2 \). Thus the inequality
\[ (1 - A_1)p \geq (1 - \delta^{\beta/3})(1 - A)p \quad (3.29) \]
holds for all values of \( A_+ \), provided that \( \delta \) is small enough.

Now that \( A \) has been chosen, our next goal will be to prove that if \( \delta \) is small enough, then the inequality
\[ E(u, \Lambda_\delta(y) \cap B(0,1 + \delta)) \geq (1 - 2\delta^{\beta/3})(1 - A)p \sum_{s \in S \cap \mathbb{Q}_{\delta+\epsilon}(y)} \kappa_s \quad (3.30) \]
holds for sufficiently small \( \epsilon \). Note that if \( A_1 = 1 \) then also \( A = 1 \), so the right-hand side of (3.30) vanishes, and the inequality certainly holds; thus we need only prove (3.30) when \( A_1 < 1 \).

Given \( s \in \mathbb{Q}_{\delta+\epsilon}(y) \), fix a \( C^\infty \) cutoff function \( \phi_s : B\left(s, \frac{\epsilon}{10}\right) \to [0,1] \) that is identically 1 in \( B(s,\alpha\epsilon) \) and vanishes outside \( B(s,2\alpha\epsilon) \). Since \( A_1 < 1 \) and (3.28) holds, the function \( u_s : B(s,\epsilon/10) \to [0,1] \) given by
\[ u_s(x) := \max\{0, \frac{u(x) - A_1}{1 - A_1}\}, \]
is in the admissible class
\[ \mathcal{A}(s + \alpha\epsilon K_s, B(s,\epsilon/10), \phi_s) \]
defined in (2.9), by basic properties of \( H_{\theta}^{p} \) (See [9, Lemmas 1.23 and 1.26]). Thus, by (2.10),
\[ \forall s \in \mathbb{Q}_{\delta+\epsilon}(y), \quad \kappa_s \leq E\left(u_s, B\left(s, \frac{\epsilon}{10}\right)\right) \leq (1 - A_1)^{-p}E\left(u, B\left(s, \frac{\epsilon}{10}\right)\right). \]

Therefore,
\[ E(u, \Lambda_\delta(y) \cap B(0,1 + \delta)) \geq \sum_{s \in \mathbb{Q}_{\delta+\epsilon}(y)} E\left(u, B\left(s, \frac{\epsilon}{10}\right)\right) \geq (1 - A_1)p \sum_{s \in \mathbb{Q}_{\delta+\epsilon}(y)} \kappa_s. \quad (3.31) \]

Next, we will compare the right-hand-sides of (3.31) and (3.30). The equidistribution hypothesis (H1) implies that for \( \epsilon \) sufficiently small,
\[ |S \cap Q_\delta(y)| \geq (1 - \delta)\sigma\epsilon^{1-d}\mu(Q_\delta) \geq C_{17}\sigma(\delta/\epsilon)^{d-1}. \quad (3.32) \]

Invoking Lemma 3.3 with \( M = \delta^{-\beta/2} \) yields that
\[ |S_\beta \cap Q_\delta(y)| \leq C_7\delta^{\beta/2}(\delta/\epsilon)^{d-1}. \quad (3.33) \]

The \( \mu \)-measure of the shell \( Q_{\delta+2\epsilon}(y) \setminus Q_{\delta-2\epsilon}(y) \) is \( O(\epsilon^d\delta^{-d}) \), so the number of anchors in the smaller shell \( Q_{\delta+\epsilon}(y) \setminus Q_{\delta-\epsilon}(y) \) is
\[ O(\delta/\epsilon)^{d-2} \leq C_7\delta^{\beta/2}(\delta/\epsilon)^{d-1} \]
(since caps of radius \( \epsilon/2 \) around these anchors are pairwise disjoint, and contained in the larger shell.) Thus for small enough \( \epsilon \),
\[ \left|\left(S \cap Q_{\delta+\epsilon}(y)\right) \setminus Q_{\delta-\epsilon}(y)\right| \leq |S_\beta \cap Q_\delta(y)| + |S \cap (Q_{\delta+\epsilon}(y) \setminus Q_{\delta-\epsilon}(y))| \leq 2C_7\delta^{\beta/2}(\delta/\epsilon)^{d-1} \leq \frac{C_{17}}{2}\sigma\delta^{\beta/3}(\delta/\epsilon)^{d-1}, \quad (3.34) \]
where the rightmost inequality assumes that \( \delta \) is small enough so that \( 2C_7\delta^{\beta/2} \leq \frac{C_{17}}{2}\sigma \).

Lemma 2.2 (see also (2.30)) implies that if \( \epsilon > 0 \) is small enough, then for every anchor \( s \),
\[ \text{cap}_p(\alpha\epsilon K) \leq \kappa_s \leq 2\text{cap}_p(\alpha\epsilon K). \quad (3.35) \]
By comparing (3.34) to (3.32), and using (3.35), we obtain that
\[
\sum \left\{ \kappa_s : s \in \left( S \cap Q_{\delta+\varepsilon}(y) \right) \right\} \leq \delta^{2/3} \sum \left\{ \kappa_s : s \in S \cap Q_{\delta+\varepsilon}(y) \right\},
\]
or equivalently,
\[
\sum \left\{ \kappa_s : s \in Q_{\delta+\varepsilon}(y) \right\} \geq \left( 1 - \delta^{2/3} \right) \sum \left\{ \kappa_s : s \in S \cap Q_{\delta+\varepsilon}(y) \right\}.
\]
Combining this inequality with (3.31) and (3.29), we have established that (3.30) holds.

Recall from (1.2) that \( u(x) \to 0 \) as \(|x| \to \infty \) and recall that \( U_R(x) = U(x/R) \) is identically 1 for \( x \in B(0, R) \).

By considering separately the two cases \( A_+ > 1/2 \) and \( A_+ \leq 1/2 \), we infer from (3.27) that
\[
\forall x \in \Lambda_\delta(y) \cap \partial B(0, 1 + \delta), \quad u(x) \geq \tilde{A} := (1 - C_{20}\delta^3)A.
\]

Thus, Lemma 2.3 implies that
\[
E(u, \{ x \in \Lambda_\delta(y) : |x| \geq 1 + \delta \}) \geq \tilde{A}^p E(U_{1+\delta}, \Lambda_\delta(y)) = (1 + \delta)^{d-p} \tilde{A}^p E(U, \Lambda_\delta(y)).
\]

For sufficiently small \( \delta \), we have \( (1 - C_{20}\delta^3)^p \geq 1 - 2\delta^{2/3} \), so if \( \epsilon \) is small enough, then
\[
E(u, \{ x \in \Lambda_\delta(y) : |x| \geq 1 + \delta \}) \geq (1 - \delta^{2/3}) \tilde{A}^p E(U, \Lambda_\delta(y)).
\]
Combining (3.30) and (3.39), we obtain by (3.22) that (3.21) holds. This completes the proof.

4 Asymptotics for \( u^\varepsilon \) in the bulk: Proof of Theorem 1.1

In this section, we first use the lower bound on the energy of \( u \) in cones, obtained in Lemma 3.4, in conjunction with \( u \) minimizing energy globally, to deduce an upper bound for the energy of \( u \) in all cones. This will imply that \( u \) is close to \( u_{A_+} \) on \( \partial B(0, 1 + \delta) \), from which Theorem 1.1 will follow easily.

**Lemma 4.1.** Let \( \beta = \frac{\delta d}{4p} \) as in Definition 3.1. There exists \( \delta_0 > 0 \) (that may depend on \( d, p, \sigma \)) and \( C_{21} > 0 \) (that may depend on \( d, p \)) such that for all \( \delta \in (0, \delta_0/2) \), points \( z \in S^{d-1} \) and \( m \geq 1 \), we have
\[
E(u, \Lambda_\delta(z)) \leq E(u_{A_+}, \Lambda_\delta(z)) + C_{21} \delta^{2m+3},
\]
provided that \( \epsilon \) is sufficiently small. Consequently, there exists \( \theta = \theta(d, p) > 0 \) such that for \( \delta \in (0, \delta_0/2) \) and \( \epsilon \) sufficiently small,
\[
|u(x) - A_+| \leq C_\# \delta^\theta \quad \text{for all} \quad x \in \mathbb{R}^d \quad \text{such that} \quad |x| = 1 \pm \delta,
\]
where \( C_\# \) does not depend on \( \epsilon, \delta \) (but may depend on \( d, p, \tau, \sigma, \text{cap}_p(K) \)).

As in the previous section, the constants \( C_i \) in the proof only depend on \( d, p \).

**Proof.** Let \( \Omega = \mathbb{R}^d \setminus \Gamma \). Observe that for \( x \in \mathbb{R}^d \) and \( y \in S^{d-1} \), we have
\[
x \in \Lambda_\delta(y) \Leftrightarrow y \in Q_{2\delta}(x/|x|).
\]
Therefore, by Fubini,
\[
\int_{\mathbb{R}^{d-1}} E(u, \Lambda_\delta(y)) \, d\mu(y) = \int_{\mathbb{R}^{d-1}} \int_{\Omega \cap \Lambda_\delta(y)} |\nabla u(x)|^p \, dx \, d\mu(y) = \int_{\Omega} \int_{Q_{2\delta}(x/|x|)} |\nabla u(x)|^p \, d\mu(y) \, dx = \mu(Q_{2\delta}) E(u).
\]
Define
\[
Y_\delta = \left\{ y \in S^{d-1} : E(u, \Lambda_\delta(y)) > \delta^{-\beta/2} \delta^{d-1} \right\}.
\]
Since $E(u) = \text{cap}_p(\Gamma) \leq \text{cap}_p(B(0,2)) = C_{22}$ and $\mu(Q_{2\delta}) = O(\delta^{d-1})$, equation (4.3) implies that

$$\mu(Y_\delta) \leq C_{23}\delta^{d/2}.\quad (4.5)$$

Denote $\chi_m(y) = E(u_{A_\delta}, \Lambda_{\delta^m}(y))$ and observe that if $\delta^m < \delta_0(d, p, \sigma)$, then Lemma 3.4 and (2.38) (applied to $\delta^m$ in place of $\delta$) imply that for sufficiently small $\varepsilon$, 

$$E(u, \Lambda_{\delta^m}(y)) \geq (1 - 2\delta^{m\beta/3})\chi_m(y) \quad \text{for } y \in S^{d-1} \setminus Y_{\delta^m}.\quad (4.6)$$

Next, note that for $z \in S^{d-1}$ and $y \in S^{d-1} \setminus Q_{\delta + \delta^m}(z)$, we have

$$x \in \Lambda_{\delta^m}(y) \Rightarrow \left\{ x \in \mathbb{R}^d \setminus \Lambda_{\delta}(z) \text{ and } y \in Q_{\delta^m}(x/|x|) \right\}. \quad (4.7)$$

Therefore, by Fubini

$$\int_{S^{d-1} \setminus Q_{\delta + \delta^m}(z)} E(u, \Lambda_{\delta^m}(y)) \, d\mu(y) = \int_{S^{d-1} \setminus Q_{\delta + \delta^m}(z)} \int_{\Omega \setminus \Lambda_{\delta}(z)} |\nabla u(x)|^p \, dx \, d\mu(y) \leq \int_{\Omega \setminus \Lambda_{\delta}(z)} \int_{S^{d-1} \setminus (Q_{\delta + \delta^m}(z) \cup Y_{\delta^m})} |\nabla u(x)|^p \, d\mu(y) \, dx. \quad (4.8)$$

The right-hand side factors as $\mu(Q_{\delta^m})E(u, \Omega \setminus \Lambda_{\delta}(z))$, so reversing the order of expressions gives

$$\mu(Q_{\delta^m})E(u, \Omega \setminus \Lambda_{\delta}(z)) \geq \int_{S^{d-1} \setminus (Q_{\delta + \delta^m}(z) \cup Y_{\delta^m})} E(u, \Lambda_{\delta^m}(y)) \, d\mu(y) \geq \int_{S^{d-1} \setminus Q_{\delta + \delta^m}(z)} (1 - 2\delta^{m\beta/3})\chi_m(y) \, d\mu(y), \quad (4.9)$$

where we have used (2.38) in the last step. By Fubini,

$$\int_{S^{d-1}} \chi_m(y) \, d\mu(y) = \mu(Q_{\delta^m})E(u_{A_\delta}) \quad (4.10)$$

and

$$\int_{Q_{\delta - \delta^m}(z)} \chi_m(y) \, d\mu(y) \leq \mu(Q_{\delta^m})E(u_{A_\delta}, \Lambda_{\delta}(z)). \quad (4.11)$$

Write $Y^+ = Y_{\delta^m} \cup \left( Q_{\delta + \delta^m}(z) \setminus Q_{\delta - \delta^m}(z) \right)$, so that for small $\delta$,

$$\mu(Y^+) \leq C_{23}\delta^{m\beta/2} + C_{24}\delta^{d-2+m} \leq 2C_{23}\delta^{m\beta/2}. \quad (4.12)$$

Now (2.37) implies the bound $\chi_m(y) \leq C_{25}\mu(Q_{\delta^m})$ for $\varepsilon$ small enough; together with (4.11), it gives

$$\int_{Y^+} \chi_m(y) \, d\mu(y) \leq C_{26}\mu(Q_{\delta^m})\delta^{m\beta/2}. \quad (4.13)$$

Subtracting (4.10) and (4.12) from (4.9) yields

$$\int_{S^{d-1} \setminus (Q_{\delta + \delta^m}(z) \cup Y_{\delta^m})} \chi_m(y) \, d\mu(y) \geq \mu(Q_{\delta^m}) \left[ E(u_{A_\delta}, \mathbb{R}^d \setminus \Lambda_{\delta}(z)) - C_{26}\delta^{m\beta/2} \right], \quad (4.14)$$

whence by (4.8),

$$E(u, \Omega \setminus \Lambda_{\delta}(z)) \geq (1 - 2\delta^{m\beta/3}) \left[ E(u_{A_\delta}, \mathbb{R}^d \setminus \Lambda_{\delta}(z)) - C_{26}\delta^{m\beta/2} \right]. \quad (4.15)$$

Thus,

$$E(u_{A_\delta}) \geq E(u) = E(u, \Lambda_{\delta}(z)) + E(u, \Omega \setminus \Lambda_{\delta}(z)) \geq E(u, \Lambda_{\delta}(z)) + (1 - 2\delta^{m\beta/3}) \left[ E(u_{A_\delta}, \Omega \setminus \Lambda_{\delta}(z)) - C_{26}\delta^{m\beta/2} \right].$$
Rearranging terms and using that there exists $C_{27}$ such that if $\varepsilon$ is small, then $E(u_{A_*}) \leq C_{27}$, we conclude that

$$E(u, \Lambda_\delta(z)) \leq E(u_{A_*}, \Lambda_\delta(z)) + 2C_{27}\delta^{3/3},$$

(4.16)

provided $\delta < \delta_0$ and $\varepsilon < \varepsilon_0(\delta)$ are small enough. Applying this inequality with $2\delta$ in place of $\delta$, the hypothesis of Lemma 3.4 is satisfied, provided $m$ is chosen to satisfy $m\beta/3 > d$. Therefore, by (3.20), we have that for sufficiently small $\varepsilon$,

$$|u(x) - A| \leq C_{12}\delta^3$$

for all $x \in A_\delta(z)$ such that $|x| = 1 \pm \delta$, (4.17)

where $A = A(z, \varepsilon)$. By (3.21) and (4.16),

$$(1 - 2\delta^{3/3})E(u_{A_*}, \Lambda_\delta(z)) \leq E(u_{A_*}, \Lambda_\delta(z)) + 2C_{27}\delta^d,$$

(4.18)

since $m\beta/3 > d$. Now we separate cases.

**Case 1.** If $\tau \in (0, \infty)$, then (4.18) and Corollary 2.1 yield, for sufficiently small $\varepsilon > 0$, that

$$(1 - 3\delta^{3/3})\mu(Q_\delta)\varphi_\tau(A) \leq \mu(Q_\delta)\varphi_\tau(A_*) + 2C_{27}\delta^d.$$  

(4.19)

Thus, since $\varphi_\tau(A_*) \leq C$, we infer that

$$C_{28}\delta^{3/3} \geq \varphi_\tau(A) - \varphi_\tau(A_*) \geq c_\#|A - A_*|^{2\mu},$$

where in the right-hand inequality, $c_\# = c_\#(d, p, \tau, \sigma, \text{cap}_p(K)) > 0$. Therefore,

$$|A - A_*| \leq (C_{28}/c_\#)^{1/(2\mu)} \cdot \delta^{3/(6\mu)}.$$  

In conjunction with (4.17), this yields the final claim of the lemma.

**Case 2.** If $\tau = 0$, then $A_* = 0$, so $E(u_{A_*}) \to 0$ as $\varepsilon \to 0$ by (2.34). Since $E(u_{A_*}, \Lambda_\delta(z)) \geq C_{29}A_\tau^\delta$ by (2.24), we infer from (4.18) that $|A - A_*| = A = O(\delta^{3/\mu})$ for small $\varepsilon$.

**Case 3.** If $\tau = \infty$, then $A_* = 1$ and

$$E(u_{A_*}, \Lambda_\delta(z)) = E((1 + \varepsilon)^\gamma U, \Lambda_\delta(z)) \leq C_{30}\delta^{d-1}$$

(4.20)

by (2.24). On the other hand, for $A < 1$, by (2.39) and the definition of $\kappa_\delta$, for any $\varepsilon > 0$ small enough we have

$$\frac{E(u_{A_*}, \Lambda_\delta(z))}{(1 - A)^p} \geq \sum_{s \in S^nQ_{\varepsilon\tau}(y)} \kappa_s \geq C_{31}\sigma(\delta/\varepsilon)^{d-1}(\alpha\varepsilon)^{d-p}\text{cap}_p(K) \geq c_0\delta^{d-1}(\alpha/\alpha_\varepsilon)^{d-p}$$

(4.21)

for some $c_0 = c_0(d, p, K, \sigma) > 0$. The right-hand side of (4.21) tends to $\infty$ as $\varepsilon \downarrow 0$, so by (4.18), we have $|A - A_*| = 1 - A \leq \delta$ for small enough $\varepsilon$.  

**Proof of Theorem 1.1.** By Lemma 4.1, if $\delta < \delta_0(d, p, \sigma)$, then for sufficiently small $\varepsilon$, we have

$$\forall x \in \partial B(0, 1 + \delta) \cup \partial B(0, 1 - \delta), \quad A_* - C_\#\delta^0 \leq u(x) \leq A_* + C_\#\delta^0,$$

where $\theta, C_\#$ do not depend on $\varepsilon, \delta$. The comparison principle (Lemma 2.1) then implies that for sufficiently small $\varepsilon$,

$$\forall x \in \mathbb{R}^d \setminus B(0, 1 + \delta), \quad (A_* - C_\#\delta^0)U_{1+\delta}(x) \leq u(x) \leq (A_* + C_\#\delta^0)U_{1+\delta}(x).$$

(4.22)

and

$$\forall x \in B(0, 1 - \delta), \quad A_* - C_\#\delta^0 \leq u(x) \leq A_* + C_\#\delta^0.$$  

(4.23)

(For the latter conclusion, the maximum principle suffices.) Since $\delta > 0$ can be chosen arbitrarily small, this concludes the proof of uniform convergence on compact subsets of $\mathbb{R}^d \setminus \partial B(0, 1)$.  

5 Separation theorem

The oscillation and energy estimates we used in the previous section to prove Theorem 1.1 are not sufficient to establish Theorem 1.2. For this purpose, we will need to bound \( u \) closer to the cavities, which is the goal of this section.

Recall that \( \gamma = \frac{d-p}{p-1} \). In this section we prove some estimates that are valid for all parameters \( \delta < \min\{1/2, 1/(2\gamma)\} \) and \( 0 < \varepsilon < \delta/10 \). In particular, we do not assume the asymptotic equidistribution hypothesis \((H_1)\) in [1.3], and instead of the asymptotic relation \((H_3)\) \( \alpha(\varepsilon)\varepsilon^{-1/\gamma} \to \tau \) we just assume that for some \( \tau_1 \in (0, \infty) \), we have \( \alpha < \min\{1/8, \tau_1 \varepsilon^{1/\gamma}\} \).

Let \( S \subset \partial B(0,1) \) be a set of anchors, with Euclidean distance at least \( \varepsilon \) between any two anchors. The next theorem ensures that for \( \alpha \) in the critical window and below it, the bumps in the equilibrium potential are separated.

**Theorem 5.1.** For \( \zeta > 0 \), write \( \Omega_{\zeta} = B(0,1+\delta) \setminus \bigcup_{s \in S} \bar{B}(s, \zeta \varepsilon) \). Let \( w : \bar{B}(0,1+\delta) \to [0,1] \) be the Perron solution of the boundary value problem

\[
\begin{cases}
\Delta_p w = 0 & \text{in } \Omega_{\alpha}, \\
 w = 0 & \text{on } \partial B(0,1+\delta), \\
 w = 1 & \text{on } \bigcup_{s \in S} \partial B(0, \alpha \varepsilon).
\end{cases}
\]

(5.1)

Then for some \( C = C(d,p) \), we have

\[
\sup_{z \in \Omega_{\alpha/10}} |w(z)| \leq C \tau_1^\gamma \delta.
\]

(5.2)

The proof of Theorem 5.1 is based on the lemma below which requires the following notation:

\[
\begin{align*}
D & := \sup \left\{ w(x) \mid x \in \bigcup_{s \in S} \partial B(s, \varepsilon/20) \right\}, \\
F & := \sup \left\{ w(x) \mid x \in \bigcup_{s \in S} \partial B(s, \varepsilon/10) \right\}, \\
G & := \sup \left\{ w(x) \mid x \in \partial B(0,1+\varepsilon/5) \right\}.
\end{align*}
\]

We note here that by the maximum principle,

\[
\sup_{z \in \Omega_{\alpha/10}} w(z) \leq F.
\]

(5.3)

**Lemma 5.1.** There exist constants \( c_1, c_2 > 0 \) and \( c_3 \in (0,1) \) that only depend on \( d, p \), such that

\[
\begin{align*}
(a) \quad & D \leq F + c_1 \tau_1^\gamma \varepsilon (1 - F), \\
(b) \quad & G \leq (1 - c_2 \varepsilon/\delta) F, \\
(c) \quad & F \leq (1 - c_3) D + c_3 G.
\end{align*}
\]

(5.4) (5.5) (5.6)

**Proof.** Let \( 0 < r < R \) and set

\[
h_{r,R}(x) = \frac{|x|^{-\gamma} - R^{-\gamma}}{r^{-\gamma} - R^{-\gamma}} \quad \text{for } r < |x| < R.
\]

(5.7)

Observe that \( h_{r,R} \) is \( p \)-harmonic in \( B(0,R) \setminus \bar{B}(0,r) \) and takes values 0 and 1 on \( \partial B(0,R) \) and \( \partial B(0,r) \) respectively.

(a) Given the definition of \( F \), the comparison principle (Lemma 2.1) implies that for each \( s \in S \), we have

\[
w(x) \leq F + (1 - F) \nu_1(x-s), \quad \forall x \in B(s, \varepsilon/10) \setminus B(s, \alpha \varepsilon),
\]

(5.8)

where \( \nu_1(x) = h_{r,R}(x) \) with \( r = \alpha \varepsilon, R = \varepsilon/10 \). Since

\[
|x| = \varepsilon/20 \implies \nu_1(x) = \frac{(\varepsilon/20)^{-\gamma} - (\varepsilon/10)^{-\gamma}}{(\alpha \varepsilon)^{-\gamma} - (\varepsilon/10)^{-\gamma}} \leq \frac{(\varepsilon/20)^{-\gamma}}{(\alpha \varepsilon)^{-\gamma}} \leq c_1 \tau_1^\gamma \varepsilon,
\]

(5.9)
we infer from (5.8) that
\[
    w(x) \leq F + (1 - F) c_1 \tau_1^\gamma \varepsilon \quad \forall x \in \partial B \left( s, \frac{\varepsilon}{20} \right).
\]  
(5.10)

This gives (5.4).

(b) Observe that by (5.3) we have \( w \leq F \) on \( \partial B(0, 1 + \varepsilon/10) \). Hence, by the comparison principle, for all \( s \in S \) we have
\[
    w(x) \leq F \nu_2(x), \quad \forall x \in B(0, 1 + \delta) \setminus B\left(0, 1 + \frac{\varepsilon}{10}\right),
\]  
(5.11)

where \( \nu_2(x) = h_{r,R}(x) \) with \( r = 1 + \varepsilon/10 \) and \( R = 1 + \delta \).

Since the derivative of \( t \mapsto t^{-\gamma} \) is bounded above and below by positive constants for \( t \in [1, 2] \), the mean value theorem gives
\[
    1 - \nu_2 \left( 1 + \frac{\varepsilon}{5} \right) = \frac{(1 + \varepsilon/10)^{-\gamma} - (1 + \varepsilon/5)^{-\gamma}}{(1 + \varepsilon/10)^{-\gamma} - (1 + \delta)^{-\gamma}} \geq c_2 \frac{\varepsilon}{\delta}. 
\]  
(5.12)

Combining (5.11) and (5.12), we have that
\[
    w(x) \leq \left( 1 - c_2 \frac{\varepsilon}{\delta} \right) F, \quad \forall x \in \partial B \left( 0, 1 + \frac{\varepsilon}{5} \right).
\]  
(5.13)

This gives (5.5).
Figure 6: The cored half-ball $H$ obtained by removing a unit ball from the middle of a half-ball of radius 8.

(c) Denote by $\xi$ the standard basis vector $(1, 0, \ldots, 0)$ in $\mathbb{R}^d$, and consider the open cored half-ball

$$H := \{ x = (x_1, \ldots, x_d) \in B(0, 8) \setminus \bar{B}(4\xi, 1) : x_1 > 0\}. \quad (5.14)$$

Let $\psi$ be the Perron solution of $\Delta_p \psi = 0$ in $H$, with boundary values $\psi = 0$ on the flat part $\{ x \in \partial H : x_1 = 0 \}$ of $\partial H$, and $\psi = 1$ on the curved part $\{ x \in \partial B(0, 8) : x_1 > 0 \} \cup \partial B(4\xi, 1)$. (By Proposition 9.31 in [9], the upper and lower Perron solutions coincide since the boundary conditions are lower semi-continuous and $H$ is regular.) By the strong maximum principle (see [12], Corollary 2.22), we have $\psi(x) < 1$ for all $x \in H$. By the continuity of $\psi$ in $H$, for some $c_3 > 0$, we have

$$\sup \{ \psi(x) : x \in \partial B(4\xi, 2) \} = 1 - c_3. \quad (5.15)$$

Fix $s \in S$ and let $s_* = (1 + \varepsilon/5)s$. The cored half-ball

$$H_* := \{ x \in B(s_*, 2\varepsilon/5) \setminus \bar{B}(s, \varepsilon/20) : \langle x, s \rangle < 1 + \varepsilon/5 \} \quad (5.16)$$

is a rotated and translated copy of $H$, scaled by $\varepsilon/20$. The function $w$ is continuous on $\bar{H}_*$, and $p$-harmonic in $H_*$, with boundary values $w \leq G$ on the flat part of $\partial H_*$ and $w \leq D$ on the curved part. Therefore, by the comparison principle and (5.15), we have

$$\sup \{ w(x) : x \in \partial B(s, \varepsilon/10) \} \leq G + (1 - c_3)(D - G) = (1 - c_3)D + c_3G. \quad (5.17)$$

This proves (5.6).

**Proof of Theorem 5.1.** Combining (5.4)-(5.6), we have

$$F \leq (1 - c_3) \left[ F + c_1 \tau_1 \varepsilon (1 - F) \right] + c_3 \left[ 1 - c_2 \varepsilon / \delta \right] F. \quad (5.18)$$

This gives

$$0 \leq (1 - c_3)c_1 \tau_1 \varepsilon (1 - F) - c_3 c_2 \varepsilon / \delta F, \quad (5.19)$$

which in turn implies that

$$c_3 c_2 \varepsilon / \delta F \leq c_1 \tau_1 \varepsilon, \quad (5.20)$$

whence

$$F \leq \left( \frac{c_1}{c_3 c_2} \right) \tau_1 \varepsilon. \quad (5.21)$$

Thus there exists a constant $C_{32} > 0$ such that $F \leq C_{32} \tau_1 \varepsilon$. By (5.3), this completes the proof. \qed
6 Asymptotics for \( u^\varepsilon \) near the unit sphere: Proof of Theorem 1.2

We will need Clarkson’s inequalities, in the slightly more general form given in [3] for functions \( f, g \) taking values in \( \mathbb{R}^d \):

\[
p \geq 2 \Rightarrow \left\| \frac{f + g}{2} \right\|_p^p + \left\| \frac{f - g}{2} \right\|_p^p \leq \left( \frac{1}{2} \right) \left( \|f\|_p^p + \|g\|_p^p \right),
\]

and

\[
1 < p \leq 2 \Rightarrow \left\| \frac{f + g}{2} \right\|_p^q + \left\| \frac{f - g}{2} \right\|_p^q \leq \left( \frac{1}{2} \right) \left( \|f\|_p^q + \frac{1}{2} \|g\|_p^q \right)^{\frac{q}{p}},
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Proof of Theorem 1.2.** Recall that \( \beta = (p - 1)/(2p) \). As before, we let \( C_i \) denote constants that depend only on \( d, p \). Pick a small \( \delta > 0 \). We invoke Lemma 4.1 with \( m \) chosen to satisfy \( m\beta/3 \geq d \), and infer from (2.37) that

\[
\forall y \in \mathbb{S}^{d-1}, \quad E(u, \Lambda_\delta(y)) \leq E(u_{A_\delta}, \Lambda_\delta(y)) + 2^d C_1 \delta^d \leq C_2 \delta^{d - 1},
\]

provided \( \varepsilon \) is small enough. Thus the hypothesis of Lemma 3.4 holds for every \( y \in \mathbb{S}^{d-1} \), so that lemma yields

\[
E(u, \Lambda_\delta(y)) \geq (1 - 2\delta^{\beta/3})E(u_{A_\delta}, \Lambda_\delta(y)).
\]

(6.3)

for some \( A \) that may depend on all parameters of the problem, including \( \delta \) and \( \varepsilon \). In conjunction with (2.38), this yields that for small enough \( \delta \),

\[
E(u, \Lambda_\delta(y)) \geq (1 - 3\delta^{\beta/3})E(u_{A_\delta}, \Lambda_\delta(y)).
\]

(6.4)

By averaging this inequality over \( y \in \mathbb{S}^{d-1} \), Fubini’s Theorem yields (as in the proof of (1.3)) that for sufficiently small \( \varepsilon \),

\[
E(u)\mu(Q_\delta) \geq (1 - 3\delta^{\beta/3})E(u_{A_\delta})\mu(Q_\delta).
\]

(6.5)

We deduce that for \( \varepsilon \) small enough,

\[
0 \leq E(u_{A_\delta}) - E(u) \leq C\delta^{\beta/3}.
\]

(6.6)

Let \( \phi \) be the cutoff function defined in Remark 2.2. Then \( u, u_{A_\delta} \) are both in the convex admissible class \( \mathcal{A}(\Gamma, \mathbb{R}^d, \phi) \) defined in (2.9), and \( u \) minimizes the energy in this class, so the inequality

\[
E\left( \frac{u + u_{A_\delta}}{2} \right) \geq E(u)
\]

holds. In conjunction with (6.6) and Clarkson’s inequalities, this implies that

\[
\|\nabla u - \nabla u_{A_\delta}\|_p = O(\delta^\varepsilon)
\]

for some constant \( c > 0 \) and all sufficiently small \( \varepsilon \). Since \( \delta \) can be arbitrarily small, this proves the first statement of the theorem.

It only remains to prove the \( L^\infty \) convergence.

Recall that we have already established in (4.22) that for small \( \delta \) and (given \( \delta \)) sufficiently small \( \varepsilon \), we have

\[
\forall x \in \mathbb{R}^d \setminus B(0, 1 + \delta), \quad (A_\ast - C_\# \delta^\theta)U_{1+\delta}(x) \leq u(x) \leq (A_\ast + C_\# \delta^\theta)U_{1+\delta}(x).
\]

(6.7)

for some \( 0 < \theta < 1 \).

If \( \tau = \infty \), then \( A_\ast = 1 \) and the first inequality in (6.7), together with the minimum principle, imply that \( u \geq 1 - C_\# \delta^\theta \) in \( B(0, 1 + \delta) \), completing the proof in this case. Thus, we may assume that \( \tau < \infty \).

Next, recall the continuous p-harmonic function \( w \) from Theorem 5.1 defined in the regular domain

\[
\Omega_\alpha = B(0, 1 + \delta) \setminus \bigcup_{s \in \mathcal{S}} B(s, \alpha \varepsilon)
\]

holding. In conjunction with (6.6) and Clarkson’s inequalities, this implies that

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Next, recall the continuous p-harmonic function \( w \) from Theorem 5.1 defined in the regular domain

\[
\Omega_\alpha = B(0, 1 + \delta) \setminus \bigcup_{s \in \mathcal{S}} B(s, \alpha \varepsilon)
\]
considered there. Theorem 5.1 implies that \( w \leq C_{34}(2\tau)^\gamma \delta \) in the closure of the domain

\[
\Omega_{1/10} = B(0,1 + \delta) \setminus \left[ \cup_{s \in S} B(s,\varepsilon/10) \right].
\]

The inequality \( u(x) - A_* - C_\# \delta^\theta \leq w \) holds on \( \partial \Omega_{s,\varepsilon} \), so it must holds in \( \Omega_{s,\varepsilon} \) as well, by the comparison principle (Lemma 2.1). In particular, for \( \delta < \delta_0 \) small enough (that may depend on \( d,p,\sigma,\tau \)), and \( \epsilon \) small enough given \( \delta \), we have

\[
\forall x \in \partial B(s,\varepsilon/10), \quad u \leq A_* + C_\# \delta^\theta + C_{34}(2\tau)^\gamma \delta \leq A_* + 2C_\# \delta^\theta.
\]

On the other hand, \( u \geq A_* - C_\# \delta^\theta \) in \( B(0,1 + \delta) \) by (6.7) and the minimum principle (or by Lemma 2.1) Thus

\[
|u - u_{A_*}| \leq 2C_\# \delta^\theta \quad \text{in} \quad \Omega_{1/10}.
\]

Finally, fix an anchor \( s \in S \) and let \( D_s := B(s,\varepsilon/10) \setminus (s + \alpha \varepsilon K_s) \). Observe that \( u_{A_*} \), restricted to \( D_s \), is the Perron solution of the \( p \)-Laplace equation in \( D_s \), with the boundary condition \( f_s \) which is identically 1 on \( s + \alpha \varepsilon K_s \) and equals \( A_* \) on \( \partial B(s,\varepsilon/10) \). The inequality

\[
u_{A_*}(x) - 2C_\# \delta^\theta \leq u(x) \leq u_{A_*}(x) + 2C_\# \delta^\theta
\]

holds for \( x \in \partial D_s \).

Let \( \psi \) be a superharmonic function in the upper class \( U_f^\Omega \) (see Definition 2.2) corresponding to the boundary conditions

\[
f \equiv 1 \quad \text{on} \quad \Gamma, \quad f(\infty) = 0
\]

in the definition (1.2) of \( u \) as a Perron solution in \( \Omega = \mathbb{R}^d \setminus \Gamma \). Then \( \psi + 2C_\# \delta^\theta \), restricted to \( D_s \), is in the upper class \( U_{f_s}^D \), by (6.9), so \( \psi + 2C_\# \delta^\theta \geq u_{A_*} \) in \( D_s \). Taking an infimum over all such \( \psi \in U_f^\Omega \), we infer that

\[
u + 2C_\# \delta^\theta \geq u_{A_*} \quad \text{in} \quad D_s.
\]

Similarly, let \( \psi_1 \) be a superharmonic function in the upper class \( U_{-f}^\Omega \). Then \( \psi_1 + 2C_\# \delta^\theta \), restricted to \( D_s \), is in the upper class \( U_{-f_s}^D \), by (6.9), so \( \psi_1 + 2C_\# \delta^\theta \geq -u_{A_*} \) in \( D_s \). Taking an infimum over all such \( \psi_1 \in U_{-f}^\Omega \), we infer that

\[-\nu + 2C_\# \delta^\theta \geq -u_{A_*} \quad \text{in} \quad D_s.
\]

Combining this with (6.10) concludes the proof, since \( \delta \) can be arbitrarily small. \( \square \)

7 Concluding remarks.

1. One method to obtain an asymptotically equidistributed set of anchors is to choose \( S(\varepsilon) \) as any \( \varepsilon \)-separated set on the sphere of maximal cardinality. The asymptotic equidistribution

\[
\frac{1}{|S(\varepsilon)|} \sum_{s \in S(\varepsilon)} \delta_s \rightarrow \mu \quad \text{as} \quad \varepsilon \downarrow 0,
\]

then follows from a classical argument of Maak, as presented, e.g., in [5, Chapter 12], while the existence of the limit \( \sigma = \lim_{\varepsilon \downarrow 0} \varepsilon^{d-1}|S(\varepsilon)| \) is established in [8].

2. The main results of this paper directly extend to problem (1.2) considered in a ball \( B(0,R) \subset \mathbb{R}^d \) with \( R > 1 \). Indeed, let \( 1 < p < d, \ R > 1 \) and let \( w = w^\gamma \) be the Perron solution of the following problem

\[
\begin{aligned}
\Delta_p w &= 0 \quad \text{in} \quad B(0,R) \setminus \Gamma, \\
w &= 1 \quad \text{on} \quad \Gamma, \\
w &= 0 \quad \text{on} \quad \partial B(0,R).
\end{aligned}
\]

Define an ansatz

\[
w^\gamma_A(x) := AW_{f,R}(x) + (1 - A) \sum_{s \in S} V_x \frac{x - s}{\alpha \varepsilon},
\]

(7.3)
where \( W_{\varepsilon,R} \) is the \( p \)-equilibrium potential of the ball \( \bar{B}(0, 1 + \varepsilon) \) relative to the ball \( B(0, R) \), given by

\[
W_{\varepsilon,R}(x) := \begin{cases} 
1 & \text{if } 0 \leq |x| \leq 1 + \varepsilon, \\
\frac{|x|^{-\gamma} - R^{-\gamma}}{(1 + \varepsilon)^{-\gamma} - R^{-\gamma}} & \text{if } 1 + \varepsilon < |x| \leq R,
\end{cases}
\]

and \( V^\tau \) are as in (1.17).

The following corollary holds:

**Corollary 7.1.** Suppose that hypotheses \((H_1), (H_2), (H_3)\) hold. Then, as \( \varepsilon \to 0 \),

\[
w_\tau(x) \to \begin{cases} 
0 & \text{if } \tau = 0, \\
A_RW_{0,R}(x) & \text{if } \tau \in (0, \infty), \\
W_{0,R}(x) & \text{if } \tau = \infty
\end{cases}
\]

uniformly on compact subsets of \( \bar{B}(0, R) \setminus \mathbb{S}^{d-1} \), where for \( \tau \in [0, \infty) \),

\[
A_R = A_R(\tau) = \frac{(\sigma\tau^{d-p}\text{cap}_p(K))^{\frac{1}{p-1}}}{(\sigma\tau^{d-p}\text{cap}_p(K))^{\frac{1}{p-1}} + (\text{cap}_p(\bar{B}(0,1)\cap B(0,R)))^{\frac{1}{p-1}}}.
\]

Furthermore, as \( \varepsilon \to 0 \),

\[
\text{cap}_p(\Gamma_\varepsilon, B(0,R)) \to \begin{cases} 
0 & \text{if } \tau = 0, \\
A^\tau_p\text{cap}_p(\bar{B}(0,1), B(0,R)) + (1 - A^\tau_p)\text{cap}_p(B(0,1), B(0,R)) & \text{if } \tau \in (0, \infty), \\
\text{cap}_p(\bar{B}(0,1), B(0,R)) & \text{if } \tau = \infty.
\end{cases}
\]

Moreover,

\[
\|\nabla w - \nabla w^\varepsilon\|_{L^p(B(0,R))} \to 0,
\]

and

\[
\|w - w_{A_R}\|_{L^\infty(B(0, R))} \to 0.
\]

The proof of this corollary is a line by line adaptation of the arguments in the preceding sections.

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