The Drinfeld associator of $\mathfrak{gl}(1|1)$

Jens Lieberum

Mathematisches Institut, Rheinsprung 21, CH-4051 Basel, lieberum@math.unibas.ch

Abstract

We determine explicitly a rational even Drinfeld associator $\Phi$ in a completion of the universal enveloping algebra of the Lie superalgebra $\mathfrak{gl}(1|1)^{\otimes 3}$. More generally, we define a new algebra of trivalent diagrams that has a unique even horizontal group-like Drinfeld associator $\Phi$. The associator $\Phi$ is mapped to $\overline{\Phi}$ by a weight system. As a related result of independent interest, we show how O. Viro’s generalization $\Delta^1$ of the multi-variable Alexander polynomial can be obtained from the universal Vassiliev invariant of trivalent graphs. We determine $\Phi$ by using the invariant $\Delta^1(\otimes)$ of a planar tetrahedron $\otimes$.

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Introduction

Building on concepts of Mac Lane and Kohno, Drinfeld introduced associators in 1989/90 in [Dr1] in context with a weakened version of the coassociativity axiom of Hopf algebras and quasitriangular Hopf algebras. Around the same time, the development of a systematic approach to knot theory started with the concept of Vassiliev invariants ([Vas]). The relation between Vassiliev invariants and Drinfelds work was established by Kontsevichs analytic construction of a universal Vassiliev invariant $Z$ and its algebraic description that requires the existence of a Drinfeld associator. The most useful and most convenient associators for topological applications are horizontal even group-like Drinfeld associators.

In the main result of this paper we determine explicitly an even horizontal group-like associator $\overline{\Phi}$ in a completion of $U(\mathfrak{gl}(1|1))^{\otimes 3}$. More generally, we define an algebra $\mathcal{C}(n)$ over $\Lambda(n) = \mathbb{Q}[d_1^{\pm 1}, \ldots, d_n^{\pm 1}]$ that is generated by trivalent diagrams on $n$
strings modulo some graphical relations. In a completion \( \hat{\mathcal{C}}^0(3) \) of a \( \mathbb{Q} \)-subalgebra of \( \mathcal{C}(3) \) there exists a unique Drinfeld associator of the form

\[
\Phi = \exp \left( F \cdot \left[ t^{12}, t^{23} \right] \right) \in \hat{\mathcal{C}}^0(3),
\]

where \( F \in \mathbb{Q}[[C, D, E]] \subset \mathbb{Q}[[d_1, d_2, d_3]] \) is a formal power series that starts with

\[
F = \frac{1}{24} - \frac{C + 4D}{5760} + \frac{4C^2 + 36CD + 48D^2 - 31E}{2903040} - \frac{6C^3 + 96C^2D + 240CD^2 + 192D^3 + 13CE - 184DE}{464486400} + \ldots
\]

and \( C = d_1d_3 - d_2(d_1 + d_2 + d_3) \), \( D = (d_1 + d_2)^2 + (d_2 + d_3)^2 \), \( E = (d_1 + d_2)^2(d_2 + d_3)^2 \).

The main result of this paper is the following theorem.

**Theorem 1** The series \( F \) is the unique solution of the equation

\[
\cosh(Fk) + d_1d_3 \frac{\sinh(Fk)}{k} = \sqrt{\frac{\varphi(d_2)\varphi(d_1 + d_2 + d_3)}{\varphi(d_1 + d_2)\varphi(d_2 + d_3)}}
\]

where \( \varphi(x) = 2 \sinh(x/2)/x \) and \( k^2 = -d_1d_2d_3(d_1 + d_2 + d_3) \).

Due to a computation of P. Vogel the solution \( F \) is given explicitly by

\[
F = X \Psi(d_1d_3 X, -d_2(d_1 + d_2 + d_3) X), \quad \text{where}
\]
\[
\Psi(u, v) = \sum_{p, q=0}^{\infty} \left( \frac{-1/2}{p} \right) \left( \frac{-1/2}{q} \right) \frac{u^p v^q (u + 2)^{p+q+1}}{2(p+q+1)}
\]

and

\[
X = d_1^{-1}d_3^{-1} \left( \frac{\varphi(d_2)\varphi(d_1 + d_2 + d_3)}{\varphi(d_1 + d_2)\varphi(d_2 + d_3)} - 1 \right).
\]
R-matrices and 6j-symbols ([Tu2]). Although this construction neither extends in its full generality to quantum supergroups nor to versions of Vassiliev invariants for 3-manifolds the value $\Delta^1(\bigcirc)$ and its relation to the Kontsevich integral $Z$ turned out to be useful to determine $\Phi$ explicitly. The translation between $\Delta^1(\bigcirc)$ and $\Phi$ is in the spirit of a general connection between well-behaved invariants of trivalent graphs and associators that is investigated by D. Bar-Natan and D. Thurston. We extend the relation between $\Delta^1$ and the Kontsevich integral $Z$ from $\bigcirc$ to arbitrary trivalent graphs (Theorem 31). This is a result of independent interest. It generalizes an unpublished proof of A. Vaintrob who related the multi-variable Alexander polynomial of links to the Kontsevich integral. With the standard definition the Kontsevich integral of trivalent graphs ([MuO]) Theorem 31 would only hold up to a factor that depends on the colored graph but not on its embedding. In order to avoid this factor, to simplify computations, and to emphasize the roles played by cyclic orientations of vertices and half-framings we introduce a different normalization of the Kontsevich integral of oriented trivalent graphs (Theorem 20).

The paper is organized as follows. In Section 1 we recall definitions and properties of a module $\mathcal{A}(\Gamma)$ of trivalent diagrams on an oriented unitrivalent graph $\Gamma$ and of Drinfeld associators in a completion $\hat{\mathcal{A}}(3)$ of $\mathcal{A}(\Gamma_3)$ where $\Gamma_n$ consists of $n$ intervals. In Section 2 we introduce the module $\mathcal{C}(\Gamma)$ and investigate its structure for $\Gamma = \Gamma_n$ (Theorem 8 and Corollary 9). Section 3 contains results about Drinfeld associators in $\hat{\mathcal{C}}^0(3)$ and in a quotient $\hat{\mathcal{C}}^1(3)$ of $\hat{\mathcal{C}}^0(3)$ (Theorem 13). In particular, we establish the existence and uniqueness of the series $F$ and deduce equations (2) to (4) from Theorem 1. In Sections 4 to 9 we prepare the proof of Theorem 1 that will be given in Section 10. Sections 4 to 8 are also used in Section 11 where we relate the Kontsevich integral $Z$ to Viro’s Alexander invariant $\Delta^1$ (Theorem 31).

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1 Drinfeld associators in $\hat{\mathcal{A}}_k(3)$

A graph is called unitrivalent if all of its vertices have valency one or three. Let $\Gamma$ be a unitrivalent graph with oriented edges and cyclically oriented vertices. As an exception, we allow circles as connected components of $\Gamma$ that we consider as a single oriented edge without vertex. The graph $\Gamma$ may have multiple edges between vertices. When $\Gamma$ has no univalent vertex, we call it a trivalent graph. Let $V(G)$
(resp. $E(G)$) be the set of vertices (resp. edges) of a unitrivalent graph $G$. A trivalent diagram $D$ with skeleton $\Gamma$ is a unitrivalent graph $G$ whose univalent vertices are glued to $\Gamma \setminus V(\Gamma)$ by an injective gluing map. The unitrivalent graph $G$ has the following properties: trivalent vertices of $G$ are cyclically oriented, but in contrast to $\Gamma$, edges of $G$ are not oriented. In addition, we require that each connected component of $G$ has at least one univalent vertex.

We represent a trivalent diagram graphically by a generic picture of $\Gamma \cup G$ in the plane. We use thicker lines to draw $\Gamma$ than we use for $G$. We assume that cyclic orientations of oriented vertices are always counterclockwise. When it is of importance we indicate orientations of edges of $\Gamma$ by arrows, and we include the names $i,j,\ldots$ of edges (resp. vertices) in our graphical representation of trivalent diagrams by writing them close to the corresponding edges (resp. vertices). Homeomorphisms $h : \Gamma \cup G \longrightarrow \Gamma \cup G'$ between trivalent diagrams on $\Gamma$ have to respect orientations of edges and vertices and induce a homeomorphism of $\Gamma$ that is homotopic to the identity. By abuse of language we call the homeomorphism class of a trivalent diagram simply trivalent diagram. For a vertex $v \in V(\Gamma)$ and an edge $i \in E(\Gamma)$ that is incident to $v$, we define $s_{v,i} \in \{\pm 1\}$ by $s_{v,i} = 1$ if the edge $i$ is oriented towards $v$ and $s_{v,i} = -1$ otherwise. We specify relations between trivalent diagrams by using graphical representations of the part where these diagrams differ.

**Definition 2** Let $\mathcal{A}(\Gamma)$ be the $\mathbb{Q}$-vector space generated by trivalent diagrams on $\Gamma$ modulo the relations $(STU)$ and $(InvV)$.

\[
(STU) \quad \begin{array}{c}
\includegraphics[width=0.2\textwidth]{STU.png}
\end{array} = \begin{array}{c}
\includegraphics[width=0.2\textwidth]{STU2.png}
\end{array} - \begin{array}{c}
\includegraphics[width=0.2\textwidth]{STU3.png}
\end{array},
\]

\[
(InvV) \quad s_{v,i} + s_{v,j} + s_{v,k} = 0.
\]

The signs on the right side of relation $(STU)$ depend on the cyclic order of the trivalent vertex and on the orientation of the shown part of $\Gamma$ in this relation. The degree of a trivalent diagram $D = \Gamma \cup G$ is defined by $\deg(D) = (1/2)\#V(G)$. This definition induces a grading on $\mathcal{A}(\Gamma)$. It is well-known (see [BN1]) that the relations $(IHX)$ and $(AS)$ below are consequences of the $(STU)$-relation.

\[
(IHX) \quad \begin{array}{c}
\includegraphics[width=0.2\textwidth]{IHX.png}
\end{array} = \begin{array}{c}
\includegraphics[width=0.2\textwidth]{IHX2.png}
\end{array} - \begin{array}{c}
\includegraphics[width=0.2\textwidth]{IHX3.png}
\end{array}, \quad (AS) \quad \begin{array}{c}
\includegraphics[width=0.2\textwidth]{AS.png}
\end{array} = - \begin{array}{c}
\includegraphics[width=0.2\textwidth]{AS2.png}
\end{array}.
\]

A basis of $\mathcal{A}(\Gamma)$ is not known explicitly. Let $\Gamma_n = \bigsqcup_{i=1}^n I_i$ be the disjoint union of oriented intervals $I_i$. The bijection between $\{1,\ldots,n\}$ and the intervals of $\Gamma_n$ is a part of the definition of $\Gamma_n$. The vector spaces $\mathcal{A}(n) = \mathcal{A}(\Gamma_n)$ are particularly interesting for several reasons. One reason is that for connected trivalent graphs $\Gamma$
there exist isomorphisms $\mathcal{A}(\Gamma) \cong \mathcal{A}(b_1(\Gamma))$ where $b_1$ is the first Betti number. A second reason is that $\mathcal{A}(n)$ is an algebra in the following way. We represent trivalent diagrams on $\Gamma_n$ graphically in the strip $\mathbb{R} \times [0, 1]$ such that $\partial I_i = \{i\} \times \{0, 1\}$ and these intervals direct from $(i, 1)$ to $(i, 0)$. We say that $\{1\} \times \{1, \ldots, n\}$ (resp. $\{0\} \times \{1, \ldots, n\}$) is the upper (resp. lower) boundary of a trivalent diagram on $\Gamma_n$.

The product $ab$ of trivalent diagrams $a, b$ is induced by gluing the lower boundary points of $a$ to the corresponding upper boundary points of $b$. The 1-element of $\mathcal{A}(n)$ is the diagram $\Gamma_n$.

For a commutative $\mathbb{Q}$-algebra $k$ we define the $\mathbb{N}$-graded $k$-module

$$\mathcal{A}_k(\Gamma) := k \otimes_{\mathbb{Q}} \mathcal{A}(\Gamma)$$

and denote its homogeneous components $k \otimes_{\mathbb{Q}} \mathcal{A}(\Gamma)_i$ by $\mathcal{A}_k(\Gamma)_i$. The case $k \neq \mathbb{Q}$ will only be important in this section and in Section 3. When $k = \mathbb{Q}$ we omit the symbol $k$. We define the completion $\hat{\mathcal{A}}_k(\Gamma)$ by

$$\hat{\mathcal{A}}_k(\Gamma) = \prod_{i=0}^{\infty} \mathcal{A}_k(\Gamma)_i.$$  

We represent elements of completions of graded vector spaces by formal power series $\sum_{i=k}^{\infty} a_i$ with homogeneous elements $a_i$ of degree $i$. We consider completions as metric spaces (and in particular as topological spaces) by

$$d \left( \sum_{i=k}^{\infty} a_i, \sum_{i=k}^{\infty} b_j \right) = \sum_{i=k}^{\infty} \delta_{a_i, b_i} 2^{-i}$$

where $\delta_{a_i, b_i} \in \{0, 1\}$ is 0 iff $a_i = b_i$. The algebra structure on $\mathcal{A}_k(n) := \mathcal{A}_k(\Gamma_n)$ extends in a unique way to a topological algebra structure on $\hat{\mathcal{A}}_k(n) := \hat{\mathcal{A}}_k(\Gamma_n)$.

Define continuous $k$-linear maps $\Delta_i : \hat{\mathcal{A}}_k(n) \rightarrow \hat{\mathcal{A}}_k(n+1)$ ($i = 0, \ldots, n+1$), where $\Delta_i(D)$ ($i = 1, \ldots, n$) is obtained from the trivalent diagram $D$ by replacing the $i$-th interval of $D$ by two copies and by summing over all ways of lifting the univalent vertices that are glued to the $i$-th interval to the copies of that interval. This sum has $2^\ell$ terms when $\ell$ univalent vertices are glued to the $i$-th interval of $D$. We label the new intervals in $\Delta_i(D)$ by $i$ and $i+1$ and replace labels $j > i$ by $j+1$. Define $\Delta_{n+1}(D)$ as the union of $D$ with a new skeleton component labeled $n+1$, and define $\Delta_0(D)$ by first adding to $D$ a new skeleton component labeled 0 followed by replacing all labels $i$ of the skeleton components by $i+1$.

Define $\epsilon_i : \hat{\mathcal{A}}_k(n) \rightarrow \hat{\mathcal{A}}_k(n-1)$ by $\epsilon_i(D) = 0$ if the $i$-th interval of $D$ contains a univalent vertex of $D \setminus \Gamma$, and by deleting the $i$-th interval and by replacing labels $j > i$ by $j-1$ otherwise.
Let $f : \{1, \ldots, m\} \to \{1, \ldots, n\}$ be an injective map. For $a \in \hat{A}(m)$ we denote by $a f^{(1)} \ldots f^{(m)} \in \hat{A}(n)$ the element obtained by applying $f$ to the labels of the skeleton components of trivalent diagrams in $a$ and by adding intervals with labels $i \in \{1, \ldots, n\} \setminus f(\{1, \ldots, m\})$ to the skeleton of these diagrams. Let $t_{ij}$ be the unique trivalent diagram on $\Gamma_n$ of degree 1 where the $i$-th edge of $\Gamma_n$ is connected to the $j$-th edge of $\Gamma_n$ by a single edge. Define $R \in \hat{A}_k(2)$ by $R = \exp(t_{12}/2)$. Now we are ready to define a Drinfeld associator.

**Definition 3** A Drinfeld associator $\Phi$ in $\hat{A}_k(3)$ is a solution of equations (DA1)-(DA4) in $\hat{A}_k(n)$ ($n = 4, 3, 3, 2$).

\[
\begin{align*}
(DA1) & \quad \Delta_0(\Phi) \circ \Delta_2(\Phi) \circ \Delta_4(\Phi) = \Delta_3(\Phi) \circ \Delta_1(\Phi), \\
(DA2) & \quad \Delta_1(R_{12}) = \Phi^{312} \circ R_{13} \circ (\Phi^{132})^{-1} \circ R_{23} \circ \Phi \\
(DA3) & \quad \Phi \circ \Phi^{321} = 1 \\
(DA4) & \quad \epsilon_1(\Phi) = \epsilon_2(\Phi) = \epsilon_3(\Phi) = 1
\end{align*}
\]

Equation (DA4) implies that there exists a unique $P \in \hat{A}_k(3)$ such that $\Phi = \exp(P)$. Let $P(\hat{A}_k(n))$ be the closed $k$-submodule of $\hat{A}_k(n)$ consisting of series that involve only trivalent diagrams $D$ with the property that $D \setminus \Gamma_n$ is non-empty and connected. When $P \in P(\hat{A}_k(n))$ we say that $\exp(P)$ is group-like. A Drinfeld associator is called horizontal, if it lies in the closed subalgebra of $\hat{A}_k(3)$ generated by $t_{12}$ and $t_{23}$. A horizontal Drinfeld associator uniquely determines a formal series $S \in k\langle\langle A, B \rangle\rangle$ in non-commutative, associative indeterminates $A$ and $B$ such that $\Phi = S(t_{12}, t_{23})$ (see [BN3], Fact 9 and Corollary 4.4). When $\Phi$ is group-like the series $S$ satisfies

\[
(DA5) \quad S = \exp(p) \text{ for a Lie series } p \text{ in } A \text{ and } B \text{ over } k.
\]

We say that a Drinfeld associator $\Phi$ is even, if $\Phi = \sum_{i=0}^{\infty} a_{2i} \in A_k(3)_{2i}$. For the following fact see [Dr2], Theorem A” and [BN4], Corollary 4.2.

**Fact 4** There exists an even horizontal group-like Drinfeld associator in $\hat{A}(3)$.

Associators in $\hat{A}_k(3)$ are not unique, but for two associators $\Phi_1$, $\Phi_2$ there exists an element $T \in \hat{A}_k(2)$ satisfying $\epsilon_1(T) = \epsilon_2(T) = 1$ and $T^{21} = T$ such that

\[
\Phi_1 = \Delta_0(T)\Delta_2(T)\Phi_2\Delta_1(T^{-1})\Delta_3(T^{-1})
\]
(see Theorem 8 of [LM1]). We say that $\Phi_1$ and $\Phi_2$ are related by a twist $T$. Group-like (resp. even) associators are related by group-like (resp. even) twists. The relation between horizontal associators is more involved: the proofs of Fact 4 rely on an action of the formal Grothendieck-Teichmüller group on horizontal group-like Drinfeld associators and on the existence of a horizontal group-like Drinfeld associator $\Phi_{KZ} \in \widehat{A}_C(3)$.

\section{The $\Lambda(n)$-algebras $C(n)$ and $D(n)$}

Let $\Lambda(\Gamma)$ be the commutative $\mathbb{Q}$-algebra generated by elements $d_i, d_i^{-1}$ ($i \in E(\Gamma)$) modulo relations $d_i d_i^{-1} = 1$ for each $i \in E(\Gamma)$ and $s_{v,i}d_i + s_{v,j}d_j + s_{v,k}d_k = 0$ for each trivalent vertex $v$ of $\Gamma$ that is incident to the three edges $i, j, k \in E(\Gamma)$. Let us investigate the structure of $\Lambda(\Gamma)$. For $e \in E(\Gamma)$ let $\alpha_e \in H^1(\Gamma, \partial \Gamma, \mathbb{Z}) =: H$ be given by the 1-cocycle that evaluates on a 1-chain $c$ to the coefficient of $e$ in $c$. Then there exists a homomorphism from $H$ into the additive group of $\Lambda(\Gamma)$ that sends $\alpha_e$ to $d_e$. It is easy to see that $\Lambda(\Gamma) = \{0\}$ iff $\alpha_e = 0$ for some edge $e$ of $\Gamma$ (or more explicitly, $\Lambda(\Gamma) = \{0\}$ iff $\Gamma$ has an edge $e$ such that the number of connected components of $\Gamma$ increases when we cut $\Gamma$ at a point $p$ in the interior of $e$, and at least one of the new connected components contains no univalent vertex besides $p$). When $\Lambda(\Gamma) \neq \{0\}$ then $\Lambda(\Gamma)$ is isomorphic to a localization of a $\mathbb{Q}$-algebra of polynomials in $\text{rank}(H)$ indeterminates.

\textbf{Definition 5} Let $C(\Gamma)$ be the quotient of $\Lambda(\Gamma) \otimes_{\mathbb{Q}} A(\Gamma)$ by the relations $(CL1A)$ and $(CL2A)$.

\begin{align*}
(CL1A) & \quad \begin{array}{c}
\begin{array}{c}
\text{\includegraphics{CL1A}}
\end{array}
\end{array} = 2d_i^2, \\
(CL2A) & \quad \begin{array}{c}
\begin{array}{c}
\text{\includegraphics{CL2A}}
\end{array}
\end{array}
\end{align*}

Both relations in the definition of $C(\Gamma)$ relate ‘coefficients’ to ‘legs’ of trivalent diagrams what explains the letters $C$ and $L$ in the names of the relations. The following relations are consequences of the definition of $C(\Gamma)$.

\begin{align*}
(LS) & \quad \begin{array}{c}
\begin{array}{c}
\text{\includegraphics{LS}}
\end{array}
\end{array} \quad \text{and} \quad \begin{array}{c}
\begin{array}{c}
\text{\includegraphics{LS}}
\end{array}
\end{array} = 0, \\
(IntV) & \quad \begin{array}{c}
\begin{array}{c}
\text{\includegraphics{IntV}}
\end{array}
\end{array} = 0.
\end{align*}
Relation \((LS)\) (‘leg slide’) is implied by the invertibility of the elements \(d_i\) and by relation \((CL2A)\). In context with relation \((STU)\), relation \((LS)\) is equivalent to relations \((IntV)\) (‘internal vertex’).

The definition of \(\text{deg}(D)\) for a trivalent diagram \(D\) and \(\text{deg}(e) = 1\) \((e \in E(\Gamma))\) induce a \(\mathbb{Z}\)-grading of \(\mathcal{C}(\Gamma)\). Consider the \(\mathbb{N}\)-graded \(\mathbb{Q}\)-subalgebra \(\Lambda^+(\Gamma)\) of \(\Lambda(\Gamma)\) generated by elements \(d_e\) \((e \in E(\Gamma))\). Define

\[
\mathcal{C}^+(\Gamma) = \tau(\Lambda^+(\Gamma) \otimes \mathcal{A}(\Gamma)) \quad \text{and} \quad \mathcal{C}^0(\Gamma) = \tau(\mathcal{A}(\Gamma))
\]

(6)

where \(\tau : \Lambda(\Gamma) \otimes \mathcal{A}(\Gamma) \rightarrow \mathcal{C}(\Gamma)\) denotes the canonical projection. Let \(\mathcal{C}(n) = \mathcal{C}(\Gamma_n)\) and \(\Lambda(n) = \Lambda(\Gamma_n)\). We have \(\Lambda(n) = \mathbb{Q}[d_1^{\pm 1}, \ldots, d_n^{\pm 1}]\). There exists a unique structure of a \(\Lambda(n)\)-algebra on \(\mathcal{C}(n)\) such that the map \(\tau\) is a homomorphism of rings. The space \(\mathcal{C}^+(n) := \mathcal{C}^+(\Gamma_n)\) (resp. \(\mathcal{C}^0(n) := \mathcal{C}^0(\Gamma_n)\)) is a \(\Lambda^+(n) := \Lambda^+(\Gamma_n)\)-subalgebra (resp. \(\mathbb{Q}\)-subalgebra).

A unitrivalent diagram on a set \(\mathcal{M}\) is a unitrivalent graph with oriented trivalent vertices and at least one univalent vertex on each connected component together with an assignment of a label in \(\mathcal{M}\) to each univalent vertex. Examples of unitrivalent diagrams can be obtained from trivalent diagrams \(D\) on \(\Gamma_n\) by labeling the univalent vertices of \(D \setminus \Gamma_n\) according to the connected components of \(\Gamma_n\).

**Definition 6** Let \(\mathcal{D}(n)\) be the \(\Lambda(n)\)-module generated by unitrivalent diagrams on the set \(\{1, \ldots, n\}\) modulo the relations \((IHX)\), \((AS)\), \((IntV)\), \((CL1B)\), \((CL2B)\).

\[
(CL1B) \quad \begin{array}{c|c}
& i & i \\
\hline
j & & k
\end{array} = d_i^2 \quad \begin{array}{c|c}
& j \\
\hline
& k
\end{array}
\]

\[
(CL2B) \quad \begin{array}{c|c}
& i \\
\hline
j & \ \ \ \ \ \ \ j
\end{array} = d_id_j
\]

Relation \((CL1B)\) concerns connected components of unitrivalent diagrams because all univalent vertices of the diagrams in the relation have labels. Relation \((CL2B)\) can be applied to parts of connected components of unitrivalent diagrams.

Let \(D\) be a unitrivalent diagram on \(\{1, \ldots, n\}\). Let \(W_i\) be the set of all linear orders on the set of univalent vertices of \(D\) labeled \(i\) with the following property: for all connected components \(C\) of \(D\) and all univalent vertices \(l_1 < l_2\) of \(C\) labeled \(i\), a univalent vertex \(l_3\) labeled \(i\) with \(l_1 < l_3 < l_2\) also belongs to \(C\). We define \(W = W_1 \times \ldots \times W_n\) and

\[
\chi(D) = \frac{1}{\#W} \sum_{w \in W} D_w,
\]

8
where $D_w$ is the trivalent diagram on $\Gamma_n$ obtained by gluing the univalent vertices labeled $i$ of $D$ to the $i$-th interval of $\Gamma_n$ according to the order $w$.

**Proposition 7** The definition of $\chi$ induces an isomorphism of $\mathbb{Z}$-graded $\Lambda(n)$-modules $D(n) \rightarrow C(n)$.

**Proof:** First verify that $\chi$ is well-defined: it is clear that $\chi$ is compatible with relations $(AS), (IHX), (IntV)$ and it is easy to see that $\chi$ is compatible with relation $(CL2B)$. By relation $(LS)$ the second vertex on the interval $i$ in relation $(CL1A)$ can be moved freely on that interval. We use this to see that $\chi$ is compatible with relation $(CL1B)$ in the case $j \neq i \neq k$. The compatibility of $\chi$ with relation $(AS)$ implies compatibility with relation $(CL1B)$ in the cases $i = j$ and $i = k$. When $D$ and $D'$ are trivalent diagrams on $\Gamma_n$ that differ only by the order of their univalent vertices on $\Gamma_n$, then by relation $(STU)$ the element $D - D' \in C(n)$ can be expressed in terms of diagrams $D''$ such that $D'' \setminus \Gamma_n$ has more trivalent vertices than $D \setminus \Gamma_n$. This implies that $\chi$ is surjective. The proof of the injectivity of $\chi$ is more difficult and similar to the proof of Theorem 8 in [BN1]. $\square$

Let

\[ G(n) = \left\{ \begin{array}{c} \begin{array}{c} \begin{array}{c} \begin{array}{c} a \end{array} \begin{array}{c} b \end{array} \end{array} \begin{array}{c} \begin{array}{c} c \end{array} \begin{array}{c} d \end{array} \end{array} \begin{array}{c} \begin{array}{c} e \end{array} \begin{array}{c} f \end{array} \end{array} \end{array} \\ 1 \leq a \leq b \leq n, \ 2 \leq c < d \leq n, \ 2 \leq e \leq f \leq n \end{array} \right\}. \]

The disjoint union of diagrams turns $D(n)$ into a commutative $\Lambda(n)$-algebra whose 1-element is the empty diagram.

**Theorem 8** The commutative $\Lambda(n)$-algebra $D(n)$ is freely generated by $G(n)$.

The proof of Theorem 8 will occupy the rest of this section. We obtain the following corollary of Theorem 8 by using Proposition 7 and the ascending filtration of $C(n)$ defined for trivalent diagrams $D$ by the number of connected components of $D \setminus \Gamma_n$.

**Corollary 9** The $\Lambda(n)$-module $C(n)$ is free. For any order on $G(n)$ a basis of $C(n)$ is given by ordered monomials in $\chi(G(n))$.

\footnote{Notice that our definition of $\chi$ is slightly different from the standard definition of $\chi$ because the standard definition is not compatible with relation $(CL1B)$.}
By Corollary 9 the $\Lambda^+(n)$-module $C^+(n)$ is torsion-free. This can be seen directly by using that the $\mathbb{Q}$-vector space $C^+(n)$ admits an $\mathbb{N}^n$-grading where the elements $d_i$ act by isomorphisms of degree $(0, \ldots, 0, 1, 0, \ldots, 0)$ where the 1 is at the $i$-th place. Using the same grading and Corollary 4 one sees that for $n > 1$ the $\Lambda^+(n)$-module $C^+(n)$ is not free.

We call a unitrivalent diagram $D$ a comb, if it is a tree whose unique spanning tree for the set of trivalent vertices is empty or a point or homeomorphic to an interval. We say that a unitrivalent diagram $D$ is a wheel when it contains a circle whose complement is a disjoint union of intervals and each of these intervals has exactly one labeled univalent vertex. The first step in the proof of Theorem 8 is the following lemma.

**Lemma 10** The $\Lambda(n)$-algebra $D(n)$ is generated by $G(n)$.

**Proof:** By relation (IntV) the algebra $D(n)$ is generated by trees and wheels. By relation (AS) wheels of odd degree whose univalent vertices have the same label are trivial in $D(n)$. Relation (CL2B) then implies that wheels of odd degree are trivial in $D(n)$ and wheels of even degree are related to $1 - 1$ by a monomial in $d_i^{\pm 1}$ ($i = 1, \ldots, n$).

Now we consider trees. By relation (IHX) it is sufficient to consider combs. By relation (CL2B) we only have to consider combs of degrees 1, 2, 3, and 4. All combs of degree 1 appear in our list of generators, so there is nothing to do. Now we treat combs of degree 3. In the computation below we use relations (CL2B), (IHX) and (AS), and (CL2B) and (AS) to write such a comb as a linear combination of combs that have two univalent vertices labeled by 1.

\[
    d^2_i \bigg| \begin{array}{c|c}
        j & k \\
        \hline
        \ell & \ell \\
    \end{array}igg| = \bigg| \begin{array}{c|c}
        i & j \\
        \hline
        k & k \\
    \end{array}igg| = \bigg| \begin{array}{c|c}
        i & j \\
        \hline
        k & k \\
    \end{array}igg| - \bigg| \begin{array}{c|c}
        i & j \\
        \hline
        k & \ell \\
    \end{array}igg| - \bigg| \begin{array}{c|c}
        i & j \\
        \hline
        \ell & k \\
    \end{array}igg| + \bigg| \begin{array}{c|c}
        i & j \\
        \hline
        k & \ell \\
    \end{array}igg| = d_k d_{\ell} - d_j d_k - d_{i} d_{\ell} + d_{i} d_j. \tag{7}
\]

By relation (AS), the symmetry of the comb of degree 3, and by invertibility of $d_i$ we see that we can write a comb of degree 3 as a linear combination of elements of $G(n)$.

We continue with combs of degree 4. The proof of the first equality below is similar to the computation in equation (7), and the second equality follows from relations (CL1B) and (AS).
By equation (8) and relation \((AS)\) we can write combs of degree 4 as linear combinations of elements of \(G(n)\). We apply equation (8) to combs of degree 2 as follows:

\[
d_1 1 \sim \ell_k = d_1d_2^{−1} j \sim \ell_j = d_k 1 \sim \ell_j + d_\ell 1 \sim \ell_j + d_j 1 \sim \ell_k.
\]

(9)

This completes the proof. \(\square\)

In the proof of the following lemma we use the Lie superalgebra \(\text{gl}(1|1)\) that we will treat later in more detail.

**Lemma 11** The elements of \(G(n)\) are \(\Lambda(n)\)-linearly independent in \(\mathcal{D}(n)\).

**Proof:** The algebra \(\text{Sym}(\text{gl}(1|1)\oplus n)\) is generated by \(H_i, D_i, E_i, F_i (i = 1, \ldots, n)\), where \(X_i\) denotes the element \(X\) of the \(i\)-th copy of \(\text{gl}(1|1)\) (see equation (22)). We consider \(\text{Sym}(\text{gl}(1|1)\oplus n)\) as a module over \(Q[d_1, \ldots, d_n]\) where \(d_i\) acts by multiplication with \(D_i\). A well-known construction using the element \(\omega\) (see equation (44)) and equation (14) shows that there exist morphisms of \(\Lambda(n)\)-algebras

\[
u_n: \mathcal{D}(n) \longrightarrow \Lambda(n) \otimes_{Q[d_1, \ldots, d_n]} \text{Sym}(\text{gl}(1|1)\oplus n)\text{gl}(1|1)
\]

satisfying

\[
u_n(i \sim j) = (1/2)(d_jH_i + d_iH_j) + F_iE_j - E_iF_j,
\]

(10)

\[
u_n(1 \sim i j) = d_1(E_jF_i + F_jE_i) + d_i(E_1F_j + F_1E_j) + d_j(E_iF_1 + F_iE_1),
\]

(11)

\[
u_n(1 \sim \ell j) = d_1^2(F_iE_j + F_jE_i) + d_1d_i(E_1F_j + E_jF_1) + d_1d_j(E_iF_1 + E_1F_i) + 2d_id_je,F_1E_1,
\]

(12)

\[
u_n(1 \sim \ell) = -2d_1^2.
\]

(13)

The \(\Lambda(n)\)-algebra \(\mathcal{D}(n)\) has an \(\mathbb{N}^3\)-grading given for unitrivalent diagrams \(D\) by \(\partial(D) = (\partial_1(D), \partial_2(D), \partial_1(D))\), where \(\partial_1(D)\) is the number of connected components of \(D\) of degree 1, \(\partial_2(D)\) is the number of connected components in \(D\) of even degree,
and \( \partial_3(D) \) is the number of connected components in \( D \) of odd degree \( \geq 3 \). The elements of \( G(n) \) are homogeneous with respect to \( \partial \). The formulas for \( u_n(D) \) \( (D \in G(n)) \) from above imply that elements of \( G(n) \) of the same degree are \( \Lambda(n) \)-linearly independent. □

It follows from computations of [FKV] that the maps \( u_n \) from the proof of Lemma [1] are compatible with relations \((IntV)\) and satisfy

\[
u_n(i \bigcirc j) = -2d_id_j, \quad \text{and} \quad u_n \left( \bigotimes \right) = \frac{1}{2} u_n \left( \bigotimes + \bigotimes - \bigotimes - \bigotimes \right). \quad (14)
\]

Equation \((14)\) or direct computations imply that \( u_n \) is compatible with relations \((CL1B)\) and \((CL2B)\). The maps \( u_n \) are not injective because in contrast to Theorem [8] the elements in equations \((11)\) and \((12)\) are nilpotent.

**Proof:** Let \( Q_n \) be the quotient field of \( \Lambda(n) \). Consider the commutative \( Q_n \)-algebra \( D_n = Q_n \otimes_{\Lambda(n)} D(n) \). Let the degree of a unitrivalent diagram be the number of its components. Then \( D_n \) admits the structure of a connected primitively generated graded Hopf algebra of finite type over \( Q_n \) whose space of primitive elements \( P_n \) is the homogeneous part of degree 1 of \( D_n \) and whose counit is the augmentation map. By [M1M] the algebra \( Q_n \) is freely generated by \( P_n \). By Lemmas [10] and [11] the map \( D(n) \to D_n \) induced by \( \Lambda(n) \subset Q_n \) is injective and maps \( G(n) \) to a basis of \( P_n \). This completes the proof. □

3 Drinfeld associators in \( \hat{C}_k^m(3) \)

Let \( C^1(n) \) be the quotient of \( C^0(n) \) by the ideal \( I_1 \) generated by trivalent diagrams \( D \) with \( b_1(D \setminus \Gamma_n) > 0 \). Define \( \hat{\Lambda}_k^+(\Gamma) \) (resp. \( \hat{C}_k^0(\Gamma), \hat{\Lambda}_k^+(n), \hat{C}_k^0(n), \hat{C}_k^1(n) \)) by extending coefficients of \( \Lambda^+(\Gamma) \) (resp. \( C^0(\Gamma), \Lambda^+(n), C^+(n), C^0(n), C^1(n) \)) from \( \mathbb{Q} \) to the commutative \( \mathbb{Q} \)-algebra \( k \) followed by completion. Define continuous morphisms of \( k \)-algebras

\[
\Delta_i : \hat{\Lambda}_k^+(n) \to \hat{\Lambda}_k^+(n+1) \quad (i = 0, \ldots, n+1),
\]

\[
\Delta_i(d_j) = d_j \quad \text{for} \quad j < i, \quad \Delta_i(d_j) = d_{j+1} \quad \text{for} \quad j > i, \quad \Delta_i(d_i) = d_i + d_{i+1}, \quad \text{and}
\]

\[
\delta_i : \hat{\Lambda}_k^+(n) \to \hat{\Lambda}_k^+(n-1) \quad (i = 1, \ldots, n),
\]

\[
\delta_i(d_j) = d_j \quad \text{for} \quad j < i, \quad \delta_i(d_j) = d_{j+1} \quad \text{for} \quad j > i, \quad \delta_i(d_i) = 0.
\]

Then the maps \( \Delta_i \otimes \Delta_i : \hat{\Lambda}_k^+(n) \otimes \hat{\Lambda}(n) \to \hat{\Lambda}_k^+(n+1) \otimes \hat{\Lambda}_k(n+1) \) and \( \delta_i \otimes \delta_i : \hat{\Lambda}_k^+(n) \otimes \hat{\Lambda}(n) \to \hat{\Lambda}_k^+(n-1) \otimes \hat{\Lambda}(n-1) \) induce continuous linear maps.

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\[ \Delta_i : \hat{C}_k^m(n) \rightarrow \hat{C}_k^m(n + 1) \quad \text{and} \quad \epsilon_i : \hat{C}_k^m(n) \rightarrow \hat{C}_k^m(n - 1) \quad (m = ' + ', 0, 1). \tag{15} \]

The proof that \( \epsilon_i \) in equation (15) is well-defined uses an \( \mathbb{N}^n \)-grading of \( \hat{C}_k^m(n) \). A direct proof that \( \Delta_i \) is well-defined uses that \( \hat{C}_k^m(n) \) is a torsion-free \( \Lambda^+ \)-(n)-module and requires a computation to ensure compatibility with relation (CL1A).

**Definition 12** A Drinfeld associator \( \Phi \) in \( \hat{A}_k^+(3) \) (resp. \( \hat{C}_k^m(3) \) with \( m = 0, 1 \)) is a solution of equations (DA1)-(DA4) in \( \hat{C}_k^m(n) \). (resp. \( \hat{C}_k^m(n) \)).

Since the canonical map \( \tau_m \) from \( \hat{A}_k(n) \) to \( \hat{C}_k^m(n) \) commutes with \( \Delta_i \) and \( \epsilon_i \) there exist even horizontal group-like Drinfeld associators in \( \hat{C}_k^m(3) \) by Fact [4]. We use the notion of a twist \( T \in \hat{C}_k^m(2) \) (\( m = ' + ', 0, 1 \)) as defined in Section [4]. For example,

\[
\exp(\mathbf{d}_1^2 \mathbf{d}_2^2) \quad \text{and} \quad \exp \left( \mathbf{d}_1 \mathbf{d}_2 (\mathbf{d}_1 + \mathbf{d}_2) \right)
\]

are non-trivial twists in \( \hat{C}_k^m(2) \). With the definition of even, horizontal, and group-like Drinfeld associators in \( \hat{C}_k^m(3) \) (\( m = 0, 1 \)) as in Section [4], we have the following theorem.

**Theorem 13** (1a) There exists exactly one even horizontal group-like Drinfeld associator in \( \hat{A}_k(3) \).

(1b) There exists exactly one even group-like Drinfeld associator in \( \hat{C}_k^1(3) \).

(2) The unique Drinfeld associator in (1a) and (1b) is equal to

\[
\exp \left( F \cdot [t^{12}, t^{23}] \right) \in \hat{C}_k^m(3) \subset \hat{C}_k(3) \quad (m = 0, 1)
\]

for a unique \( F \in \mathbb{Q}[[C, D, E]] \subset \hat{A}(3) \) where the inclusion is given by

\[
C = (d_1 + d_2)^2 + (d_2 + d_3)^2, \quad D = (d_1 + d_2)^2(d_2 + d_3)^2, \quad E = d_1d_3 - d_2(d_1 + d_2 + d_3).
\]

**Proof:** Existence of \( F \): Let \( \Phi \) be an even horizontal group-like Drinfeld associator in \( \hat{A}(3) \). By (DA5) we have \( \Phi = \exp(p(t^{12}, t^{23})) \) for a Lie series \( p \) in \( A \) and \( B \) that involves only terms of even degrees. The free Lie algebra with two generators \( A \) and \( B \) is spanned linearly by a set \( K \) that is defined recursively by \( A, B \in K \) and by \( \text{ad}_A(c), \text{ad}_B(c) \in K \) for \( c \in K \). Therefore Lemma [4] below implies that

\[
\Phi_m = \tau_m(\Phi) = \exp(F \cdot [t^{12}, t^{23}])
\]
for some $F \in \mathbb{Q}[[ (d_1 + d_2)^2, (d_2 + d_3)^2, E ]]$. Equation (DA3) implies that $p$ satisfies $p(t^{23}, t^{12}) = p(t^{12}, t^{23}) \in \hat{C}(3)$. As a consequence, $F$ is invariant under the permutation of $(d_1 + d_2)^2$ and $(d_2 + d_3)^2$. This implies $F \in \mathbb{Q}[[ C, D, E ]]$.

**Uniqueness of $\Phi_1$**: The proofs of Theorems 8 and 9 of [LM1] can be adapted to see that any even group-like associator in $\hat{C}_k^1(3)$ is related to $\Phi_1$ by an even group-like twist $T \in \hat{C}_k^1(2)$ (see also Lemma 4.17 of [BN2]). Corollary 9 implies $T = 1$. Therefore $\Phi_1$ is unique.

**Uniqueness of $F$**: Corollary 3 implies that the $\Lambda^+(n)$-module

$$M = \mathcal{C}^+(n)/\Lambda^+(n)I_1 \supset \mathcal{C}^1(n)$$

is torsion-free and $0 \neq [t^{12}, t^{23}] \in M$. This implies that $F \in \Lambda^+(3)$ is uniquely determined by $\Phi_1$. It is easy to see that $C, D, E$ are algebraically independent by using that $d_1 + d_2, d_2 + d_3, E$ are algebraically independent. Therefore $F$ is uniquely determined as a formal series in $C, D, E$.

**Uniqueness of $\Phi_0$**: Except for the uniqueness of the Lie series $p$ equation (DA5) holds also for horizontal group-like Drinfeld associators in $\hat{C}_k^0(3)$. As in the first part of the proof we see that an even horizontal group-like Drinfeld associator $\Psi$ in $\hat{C}_k^0(3)$ can be expressed as $\Psi = \exp(F' \cdot [t^{12}, t^{23}])$ for some $F' \in k[[C, D, E]]$. Let $\pi : \hat{C}_k^0(n) \rightarrow \hat{C}_k^1(n)$ be the canonical projection. Then

$$\pi(\Psi) = \exp(F' \cdot [t^{12}, t^{23}]) = \Phi_1 = \exp(F \cdot [t^{12}, t^{23}])$$

by the uniqueness of $\Phi_1$ which implies $F = F'$ by the uniqueness of $F$. This completes the proof. □

It follows from Theorem 13 that any even group-like associator in $\hat{A}_k(3)$ is mapped to an associator with rational coefficients in $\hat{C}_k^1(3)$ by the canonical projection. The next lemma was used in the proof of Theorem 13.

**Lemma 14** The following identities hold in $\mathcal{C}^0(3)$:

\[
[t^{12}, [t^{12}, t^{23}]] = (d_1 + d_2)^2 t^{12} t^{23}, \quad [t^{23}, [t^{12}, t^{23}]] = (d_2 + d_3)^2 t^{12} t^{23}, \quad (17)
\]
\[
[t^{12}, [t^{23}, t^{12}]] = (d_1 d_3 - d_2 (d_1 + d_2 + d_3)) [t^{12}, t^{23}] = [t^{23}, [t^{12}, t^{23}]]. \quad (18)
\]

**Proof**: In the computation in $\mathcal{C}^0(\Gamma)$ below, the first equality follows from relations $(LS)$ and $(STU)$. For the second equality we apply relations $(LS)$, $(AS)$, and $(CL2A)$. The third equality follows from relation $(CL1A)$. 

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$i \downarrow ^k j \uparrow = \frac{1}{2} \left( i \downarrow ^k j \uparrow + i \downarrow ^k j \uparrow + i \downarrow ^k j \uparrow \right) \tag{19}$

Now we prove equation (20) by the computation in $C^0(3)$ below. The three equalities in this computation follow by applying relations $(CL2A)$ and $(AS)$, equation (19), relations $(STU)$, $(AS)$, and $(CL2A)$, respectively.

\[
\begin{align*}
\begin{array}{c}
\end{array}
\end{align*}
\]

In the computation below, we apply relations $(STU)$ and $(AS)$ to obtain the first three equalities. The fourth equality follows from equations (19) and (20) and from relation $(CL2A)$ and $(AS)$. The fifth equality is implied by equation (19).

\[
\begin{align*}
\begin{array}{c}
\end{array}
\end{align*}
\]

This proves the first part of equation (17). The second part of equation (17) follows by applying the $\mathbb{Q}$-algebra automorphism of $C^0(3)$ induced by interchanging the intervals 1 and 3 of the skeleton. The proof of equation (18) is similar to the proof of equation (17). $\square$
Lemma 14 and Theorem 13 imply that the denominator of the homogeneous part of degree 2n in \( \exp(F \cdot x) \in \mathbb{Q}[[C, D, E, x]] \) (\( \deg(x) = 2 \)) is a lower bound for this denominator in any even group-like associator in \( \hat{A}(3) \). Now we come to P. Vogel’s proof of the formula for \( F \) using Theorem 1.

**Proof:** [of equations (2) to (4)] There exists a unique solution \( \Psi \in \mathbb{Q}[[u, v]] \) of

\[
1 + u = \cosh(\Psi \sqrt{uv}) + \sqrt{u/v} \sinh(\Psi \sqrt{uv}) = 1 + u\Psi + (1/2)uv\Psi^2 + \ldots \quad (21)
\]

By Theorem 1 the series \( \Psi \) is related to \( F \) by equations (2) and (4). Equation (21) implies the following equation in \( R = \mathbb{Q}[[u, v]]((uv)^{-1/2}) \):

\[
\left(1 + \frac{1}{\sqrt{u/v}}\right) e^{2\Psi \sqrt{uv}} - 2(1 + u)e^{\Psi \sqrt{uv}} + 1 - \frac{1}{\sqrt{u/v}} = 0.
\]

The solutions in \( R[\sqrt{v}, (\sqrt{u} + \sqrt{v})^{-1}] \) of this quadratic equation are

\[
e^{\Psi \sqrt{uv}} = \frac{1 + u \pm \sqrt{(1 + u)^2 - 1 + u/v}}{1 + \sqrt{u/v}} = \frac{\sqrt{v}(1 + u) \pm \sqrt{u} \sqrt{1 + v(u + 2)}\sqrt{u + \sqrt{v}}}{\sqrt{u + \sqrt{v}}}
\]

where we can determine the sign \( \pm \) because \( \Psi \in \mathbb{Q}[[u, v]] \). Let \( H \in R[[w]] \) be the unique solution of

\[
e^{H \sqrt{uv}} = \frac{\sqrt{v} \sqrt{1 + uw} + \sqrt{u} \sqrt{1 + vw}}{\sqrt{u + \sqrt{v}}} \in R[[w]].
\]

Then \( \Psi = H(u + 2) \). Let \( H' \) be the partial derivative of \( H \) with respect to \( w \). Then

\[
H' = \frac{1}{2(\sqrt{u} + \sqrt{v})} \left( \frac{\sqrt{uv}}{\sqrt{1 + uw}} + \frac{\sqrt{uv}}{\sqrt{1 + vw}} \right)
\]

\[
= \frac{\sqrt{uv}}{2(\sqrt{u} + \sqrt{v})} \frac{\sqrt{v} \sqrt{1 + uw} + \sqrt{u} \sqrt{1 + vw}}{\sqrt{(1 + uw)(1 + vw)}}.
\]

Therefore

\[
H' = \frac{1}{2\sqrt{(1 + uw)(1 + vw)}} = \sum_{p, q \geq 0} \left( \frac{-1/2}{p} \right) \left( \frac{-1/2}{q} \right) \frac{u^p v^q w^{p+q}}{2}
\]

which implies

\[
H = \sum_{p, q \geq 0} \left( \frac{-1/2}{p} \right) \left( \frac{-1/2}{q} \right) \frac{u^p v^q w^{p+q+1}}{2(p + q + 1)}
\]

because \( H \) is 0 for \( w = 0 \). This implies equation (3) by substituting \( w = u + 2 \). \( \square \)
By solving equation (21) iteratively one sees that the series $\Psi$ in equation (3) satisfies $\Psi \in Q[u][v] \subset Q[u, v]$ and that the coefficient of $v^q$ of $\Psi$ is a polynomial in $u$ of degree $q$.

It would be interesting if the associator of Theorem 13 could be used to investigate the coefficients of the image of $\Phi_{KZ} \in \hat{A}_C(3)$ in $\hat{C}_m^l(3)$ ($m = 0, 1$) by the canonical projection. The simplest relation holds for $m = 1$:

**Remark 15** For every group-like Drinfeld associator $\Phi \in \hat{C}_k^1(3)$ there exists a unique series $G(\Phi) \in k[[d_1 + d_2, d_1 d_2]] \subset \Lambda_k^+(2)$ such that $\Phi$ is related to the associator $\exp(F \cdot [t^{12}, t^{23}]) \in \hat{C}_k^1(3)$ of Theorem 13 by the twist $\exp(G(\Phi) \cdot \Box)$.

The existence of the twist in Remark 13 follows as in the proof of Theorem 13 by using the structure of $\mathcal{C}_k^1(2)$. The uniqueness of the twist uses the structure of $\mathcal{C}_k^1(1)$ and the triviality of $H^2(K^\bullet)$ where the cochain complex $K^\bullet$ consists of the spaces $K^n = \bigcap_{i=1}^n \text{Ker}(\epsilon_i) \subset \mathcal{C}_k^1(n)$ and the coboundary maps $\delta^n = \sum_{i=0}^{n+1} (-1)^i \Delta_i$ (compare Proposition 4.4 and Remark 4.9 of [BN2]).

### 4 Tensor products and duality of $\mathfrak{gl}(1|1)$-modules

Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}/(2)$-graded vector space. The algebra $\text{End}(V)$ has a natural $\mathbb{Z}/(2)$-grading with homogeneous components $\text{End}(V)_0 = \text{End}(V_0) \oplus \text{End}(V_1)$ and $\text{End}(V)_1 = \text{Hom}(V_0, V_1) \oplus \text{Hom}(V_1, V_0)$. The map $\text{str} : \text{End}(V) \to \mathbb{Q}$ defined by

$$\text{str}(\varphi) = \text{tr}(\varphi|_{\text{End}(V)_0}) - \text{tr}(\varphi|_{\text{End}(V)_1})$$

is called supertrace. The number $\text{sdim}(V) = \dim(V_0) - \dim(V_1) = \text{str}(\text{id}_V)$ is called the superdimension of $V$. We consider $\text{End}(V)$ as a Lie superalgebra $\mathfrak{gl}(V)$ with bracket $[\cdot, \cdot]$ induced by

$$[A, B] = AB - (-1)^{\deg(A) \deg(B)} BA$$

for homogeneous elements $A, B \in \text{End}(V) = \mathfrak{gl}(V)$. For $\dim V_0 = \dim V_1 = 1$ the Lie superalgebra $\mathfrak{gl}(V)$ is isomorphic to $\mathfrak{gl}(1|1)$ which has a homogeneous basis given by the following four matrices:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$  \hspace{1cm} (22)

The non-vanishing brackets of basis elements are given by
\[ [H, E] = -[E, H] = 2E, \quad [F, H] = -[H, F] = 2F, \quad [E, F] = [F, E] = D. \]

For \( t = (\lambda, \mu, \sigma) \in \mathbb{Q}^* \times \mathbb{Q} \times \mathbb{Z}/(2) \) there exists a unique 2-dimensional \( \text{gl}(1|1) \)-module \( V_t \) with \( \text{sdim}(V_t) = 0 \) such that for all \( v \in V_t \) with \( \text{deg}(v) = \sigma \) we have \( H \cdot v = (\mu + 1)v, D \cdot v = \lambda v \) and \( E \cdot v = 0 \).

For each triple \( t = (\lambda, \mu, \sigma) \in \mathbb{Q}^* \times \mathbb{Q} \times \mathbb{Z}/(2) \) we fix a choice of a vector \( 0 \neq v_t \in V_t \) with \( \text{deg}(v_t) = \sigma \). Denote \( F \cdot v_t \) by \( w_t \). Then \( H \cdot w_t = (\mu - 1)w_t, D \cdot w_t = \lambda w_t, F \cdot w_t = 0 \) and \( E \cdot w_t = \lambda v_t \). In particular, the vectors \( v_t \) and \( w_t \) form a basis of \( V_t \).

It is easy to see that the modules \( V_t \) are simple.

For triples \( t_i = (\lambda_i, \mu_i, \sigma_i) \in \mathbb{Q}^* \times \mathbb{Q} \times \mathbb{Z}/(2) \) with \( \lambda_1 + \lambda_2 \neq 0 \) and for \( e_2 = (0, 1, 0), e_3 = (0, 0, 1) \in \mathbb{Q}^* \times \mathbb{Q} \times \mathbb{Z}/(2) \) we define \( \text{gl}(1|1) \)-linear maps

\[
Y_{t_1, t_2} : V_{t_1+t_2+e_2} \rightarrow V_{t_1} \otimes V_{t_2} \quad \text{by} \quad Y_{t_1, t_2}(v_{t_1+t_2+e_2}) = v_{t_1} \otimes v_{t_2}, \quad \text{and} \quad (23)
\]

\[
Y_{t_1, t_2} : V_{t_1+t_2-e_2+e_3} \rightarrow V_{t_1} \otimes V_{t_2} \quad \text{by} \quad Y_{t_1, t_2}(w_{t_1+t_2-e_2+e_3}) = w_{t_1} \otimes w_{t_2}. \quad (24)
\]

Since \( Y_{t_1, t_2} \) and \( Y_{t_1, t_2} \) are non-trivial, well-defined, and have non-isomorphic simple images, equation \( [23] \) holds for reasons of dimension.

\[
V_{t_1} \otimes V_{t_2} \cong V_{t_1+t_2+e_2} \oplus V_{t_1+t_2-e_2+e_3} \quad \text{if} \quad \lambda_1 + \lambda_2 \neq 0. \quad (25)
\]

The \( \mathbb{Z}/(2) \)-graded space \( (V_t)^* = \text{Hom}_\mathbb{Q}(V_t, \mathbb{Q}) \) becomes a \( \text{gl}(1|1) \)-module by

\[
(a \cdot \beta)(v) = (-1)^{\text{deg}(a) \text{deg}(\beta)} \beta(-a \cdot v)
\]

for all \( v \in V_t \) and homogeneous elements \( a \in \text{gl}(1|1), \beta \in (V_t)^* \). Let \( \alpha \in (V_t)^* \) be given by \( \alpha(w_t) = 1, \alpha(v_t) = 0 \). We have \( H \cdot \alpha = (-\mu + 1)\alpha, D \cdot \alpha = -\lambda \alpha, E \cdot \alpha = 0, \) and \( \text{deg}(\alpha) = \sigma + 1 \). This implies

\[
(V_t)^* \cong V_{t^*} \quad \text{where} \quad t^* = (-\lambda, -\mu, \sigma + 1). \quad (26)
\]

For \( (\mu, \sigma) \in \mathbb{Q} \times \mathbb{Z}/(2) \) there exists a unique 1-dimensional representation \( I_\mu^\sigma = \mathbb{Q} \) of \( \text{gl}(1|1) \) with \( \text{sdim}(I_\mu^\sigma) = (-1)^\sigma \) and \( H \cdot v = \mu v \) for all \( v \in I_\mu^\sigma \). The formulas

\[
D \cdot v = E \cdot v = F \cdot v = 0
\]

hold for all \( v \in I_\mu^\sigma \). The modules \( I_\mu^\sigma \) and \( V_t \) form a complete set of isomorphism types of simple \( \text{gl}(1|1) \)-modules up to isomorphisms of degree 0. Define a \( \text{gl}(1|1) \)-linear map \( \bigwedge : V_{t^*} \otimes V_t \rightarrow I_0^0 = \mathbb{Q} \) by

\[
\bigwedge(v_{t^*} \otimes w_t) = 1 \quad \text{and} \quad \bigwedge(w_{t^*} \otimes v_t) = (-1)^\sigma. \quad (27)
\]
The map \( \cap_t \) can be obtained as the composition of \( \iota \otimes \text{id} \) with the evaluation map, where \( \iota \) is the isomorphism that we used to prove equation (26). We define \( \cup_t : I^0_t \rightarrow V_t \otimes V_t \) by \( \cup(1) = w_{t^*} \otimes v_t - (-1)^{\sigma_t} v_{t^*} \otimes w_t \). Then we have

\[
(\cap_{t^*} \otimes 1_t) \circ (1_t \otimes \cup_t) = I_t = (1_t \otimes \cap_t) \circ (\cup_t \otimes 1_t),
\]

where \( I_t = \text{id}_{V_t} \). There exist unique \( \text{gl}(1|1) \)-linear maps \( \mathcal{A}_{t_1,t_2} \), \( \mathcal{A}_{t_1,t_2} \) satisfying

\[
\mathcal{A}_{t_1,t_2} \circ Y_{t_1,t_2} = I_{t_1+t_2+e_2} \quad \text{and} \quad \mathcal{A}_{t_1,t_2} \circ Y_{t_1,t_2} = I_{t_1+t_2-e_2+e_3}.
\]

We have \( \mathcal{A}_{t_1,t_2} \circ Y_{t_1,t_2} = 0 \) and \( \mathcal{A}_{t_1,t_2} \circ Y_{t_1,t_2} = 0 \) because these maps are homomorphisms between non-isomorphic simple modules. By equation (25) and Schurs lemma (see [Kac]) the elements

\[
Y_{t_1,t_2} \circ \mathcal{A}_{t_1,t_2}, \quad Y_{t_1,t_2} \circ \mathcal{A}_{t_1,t_2}
\]

are a basis of the vector space \( \text{End}_{\text{gl}(1|1)}(V_{t_1} \otimes V_{t_2}) \) of \( \text{gl}(1|1) \)-linear endomorphisms of \( V_{t_1} \otimes V_{t_2} \) of degree 0. In the following lemma we present some relations between the morphisms introduced in this section.

**Lemma 16** For \( t_i = (\lambda_i, \mu_i, \sigma_i) \in \mathbb{Q}^+ \times \mathbb{Q} \times \mathbb{Z}/(2) \) \((i = 1, \ldots, 4)\) with \( t_3 = t_1 + t_2 + e_2 \) and \( t_4 = t_1 + t_2 - e_2 + e_3 \) we have

\[
\mathcal{A}_{t_1,t_2} = (1_{t_1} \otimes \cap_{t_2}) \circ (Y_{t_1,t_2} \otimes 1_t) = (\cap_{t_1} \otimes 1_{t_2}) \circ (1_{t_1} \otimes Y_{t_1,t_2}),
\]

\[
\mathcal{A}_{t_1,t_2} = (-1)^{\sigma_2} \frac{\lambda_1}{\lambda_3} (1_{t_3} \otimes \cap_{t_2}) \circ (Y_{t_1,t_2} \otimes 1_t)
\]

\[
= (-1)^{\sigma_1} \frac{\lambda_2}{\lambda_3} (\cap_{t_1} \otimes 1_{t_3}) \circ (1_{t_1} \otimes Y_{t_1,t_3}).
\]

**Proof:** We compare the three elements of \( \text{Hom}_{\text{gl}(1|1)}(V_{t_1} \otimes V_{t_2}, V_{t_4}) \) from the first equation by comparing the images of \( v_{t_1} \otimes v_{t_2} \) and \( w_{t_1} \otimes w_{t_2} \) which determine the morphisms. We obtain

\[
(1_{t_1} \otimes \cap_{t_2}) \circ (Y_{t_1,t_2} \otimes 1_t)(v_{t_1} \otimes v_{t_2}) = (1_{t_1} \otimes \cap_{t_2})(v_{t_1} \otimes v_{t_2} \otimes v_{t_2}) = 0,
\]

\[
\mathcal{A}_{t_1,t_2}(v_{t_1} \otimes v_{t_2}) = 0,
\]

\[
(\cap_{t_1} \otimes 1_{t_2}) \circ (1_{t_1} \otimes Y_{t_1,t_4})(v_{t_1} \otimes v_{t_2}) = (\cap_{t_1} \otimes 1_{t_2})(v_{t_1} \otimes v_{t_2} \otimes v_{t_4}) = 0
\]

as the value of \( v_{t_1} \otimes v_{t_2} \), and
(\mathbb{I}_t \otimes \mathbb{I}_t)(w_{t_1} \otimes w_{t_2}) = (\mathbb{I}_t \otimes \mathbb{I}_t)(F \cdot (v_{t_1} \otimes v_{t_2}) \otimes w_{t_2}) \\
= (\mathbb{I}_t \otimes \mathbb{I}_t)(w_{t_1} \otimes v_{t_2} \otimes w_{t_2} + (-1)^{\sigma_1} v_{t_1} \otimes w_{t_2} \otimes w_{t_2}) = w_{t_1}, \\
\mathbb{A}_{\lambda_1, \lambda_2}(w_{t_1} \otimes w_{t_2}) = w_{t_1}, \\
(\mathbb{I}_t \otimes \mathbb{I}_t)(w_{t_1} \otimes v_{t_2} \otimes v_{t_2}) = (\mathbb{I}_t \otimes \mathbb{I}_t)(w_{t_1} \otimes F \cdot (v_{t_1} \otimes v_{t_2})) \\
= (\mathbb{I}_t \otimes \mathbb{I}_t)(w_{t_1} \otimes v_{t_1} \otimes v_{t_2} - (-1)^{\sigma_1} w_{t_1} \otimes v_{t_1} \otimes w_{t_2}) = w_{t_1}.

We illustrate a small difference in the proof of the second equation by a sample computation.

(\mathbb{I}_t \otimes \mathbb{I}_t)(w_{t_1} \otimes w_{t_2}) = \frac{1}{\lambda_1}(\mathbb{I}_t \otimes \mathbb{I}_t)(E \cdot (w_{t_3} \otimes w_{t_2}) \otimes v_{t_2}) \\
= (\mathbb{I}_t \otimes \mathbb{I}_t)\left(\frac{\lambda_3}{\lambda_1} v_{t_3} \otimes w_{t_2} \otimes v_{t_2} + (-1)^{\sigma_3} \frac{\lambda_2}{\lambda_1} w_{t_3} \otimes v_{t_2} \otimes v_{t_2}\right) = (-1)^{\sigma_2} \frac{\lambda_3}{\lambda_1} v_{t_3}.

The rest of the proof is straightforward. \( \square \)

5 The tensor functor \( W_0 \)

Let \( \Gamma \) be a unitrivalent graph with oriented edges and vertices whose edges \( i \in E(\Gamma) \) are colored by pairs \( (\lambda_i, \mu_i) \in \mathbb{Q}^* \times \mathbb{Q} \). We say that the coloring of \( \Gamma \) is \textit{admissible}, if the following two conditions hold at every trivalent vertex \( v \) of \( \Gamma \) that is incident to \( i, j, k \in E(\Gamma) \):

\[
\begin{align*}
\sigma_{v,i} \lambda_i + \sigma_{v,j} \lambda_j + \sigma_{v,k} \lambda_k &= 0, \\
\sigma_{v,i} \mu_i + \sigma_{v,j} \mu_j + \sigma_{v,k} \mu_k &= \sigma_{v,i} \sigma_{v,j} \sigma_{v,k}.
\end{align*}
\]

When \( \Gamma \) is admissibly colored then \( \Lambda(\Gamma) \neq 0 \) by equation (31) because \( \lambda_i \neq 0 \) for all \( i \in E(\Gamma) \). Let us define a category \( \mathcal{C}_0 \). Objects of \( \mathcal{C}_0 \) are finite sequences \( (c_1, \ldots, c_k) \) of triples

\[
c_i = (\lambda_i, \mu_i, \sigma_i) \in \mathbb{Q}^* \times \mathbb{Q} \times \mathbb{Z}/(2).
\]

Morphisms from \( c^0 = (c_1^0, \ldots, c_k^0) \) to \( c^1 = (c_1^1, \ldots, c_k^1) \) with \( c_j^i = (\lambda_j^i, \mu_j^i, \sigma_j^i) \) consist of unitrivalent graphs \( \Gamma \) with admissible coloring, where the univalent vertices of \( \Gamma \) are related to \( c^0 \) and \( c^1 \) as follows: the boundary of \( \Gamma \) is decomposed into two disjoint subsets \( \partial \Gamma = \partial_0 \Gamma \cup \partial_1 \Gamma \), \( \partial_i \Gamma \) is in bijection with \( \{1, \ldots, k_i\} \), the \( j \)-th vertex
of \( \partial_i \Gamma \) is incident to an edge of \( \Gamma \) that is colored by \((\lambda^i_j, \mu^i_j)\), and that edge is directed towards the boundary iff \( \sigma^i_j + i \equiv 0 \mod 2 \). We represent morphisms of \( C_0 \) graphically by generic pictures of \( \Gamma \) in \( \mathbb{R} \times [0,1] \) where the \( i \)-th boundary point of \( \partial_j \Gamma \) has coordinates \((i, j)\). The composition \( b \circ a \in \text{Hom}_{C_0}(e^0, c^2) \) of morphisms \( a \in \text{Hom}_{C_0}(e^0, e^1), b \in \text{Hom}_{C_0}(e^1, c^2) \) is defined graphically by placing \( b \) onto the top of \( a \) and by shrinking the result to \( \mathbb{R} \times [0,1] \). We define a tensor product of objects of \( C_0 \) by concatenation of sequences. For \( a \in \text{Hom}_{C_0}(e^0, e^1), b \in \text{Hom}_{C_0}(e^2, c^3) \) we define \( a \otimes b \in \text{Hom}_{C_0}(e^0 \otimes e^2, e^1 \otimes c^3) \) graphically by placing \( a \) to the left of \( b \).

To an object \( c = (c_1, \ldots, c_k) \) of \( C_0 \) with \( c_i = (\lambda_i, \mu_i, \sigma_i) \) we assign the \( gl(1|1) \)-module

\[
W_0(c) = V_{i_1} \otimes \ldots \otimes V_{i_k} \quad \text{with} \quad t_i = ((-1)^{\sigma_i} \lambda_i, (-1)^{\sigma_i} \mu_i, \sigma_i).
\]  

Let \( t = (\lambda, \mu, 0) \). We do not specify colors or orientations of edges in pictures, when they are uniquely determined by the context or when the formulas hold for arbitrary orientations. Assume that the graphs in equations (34) to (36) are colored by \((\lambda, \mu)\). Then we define

\[
W_0 \left( \begin{array}{c}
\downarrow
\end{array} \right) = 1_t, \quad W_0 \left( \begin{array}{c}
\uparrow
\end{array} \right) = 1_{t^*},
\]

\[
W_0 \left( \begin{array}{c}
\bigcirc
\end{array} \right) = \bigcap_{t}, \quad W_0 \left( \begin{array}{c}
\bigcirc
\end{array} \right) = \bigcup_{t^*},
\]

\[
W_0 \left( \begin{array}{c}
\cup
\end{array} \right) = \bigcup_{t}, \quad W_0 \left( \begin{array}{c}
\bigcup
\end{array} \right) = \bigcap_{t^*}.
\]

Let \( c_i = (\lambda_i, \mu_i, \sigma_i) \). Consider orientations and colors of the graphs \( \bigcirc \) and \( \bigcup \) such that

\[
\bigcirc \in \text{Hom}_{C_0}(c_1, (c_2, c_3)) \quad \text{and} \quad \bigcup \in \text{Hom}_{C_0}((c_1, c_2), c_3).
\]

Then we define \( t_i = ((-1)^{\sigma_i} \lambda_i, (-1)^{\sigma_i} \mu_i, \sigma_i) \) and

\[
W_0 \left( \begin{array}{c}
\bigcirc
\end{array} \right) = \left\{ \begin{array}{ll}
Y_{t_2,t_3} & \text{if } \sigma_1 + \sigma_2 + \sigma_3 \equiv 0 \mod 2, \\
\lambda_1 Y_{t_2,t_3} & \text{if } \sigma_1 + \sigma_2 + \sigma_3 \equiv 1 \mod 2,
\end{array} \right.
\]

\[
W_0 \left( \begin{array}{c}
\bigcup
\end{array} \right) = \left\{ \begin{array}{ll}
\lambda_3 Y_{t_1,t_2} & \text{if } \sigma_1 + \sigma_2 + \sigma_3 \equiv 0 \mod 2, \\
\lambda_2 Y_{t_1,t_2} & \text{if } \sigma_1 + \sigma_2 + \sigma_3 \equiv 1 \mod 2.
\end{array} \right.
\]

Let \( X_{t_1,t_2} \in \text{Hom}_{gl(1|1)}(W_0((c_1, c_2)), W_0((c_2, c_1))) \) be the superpermutation of tensor factors induced by \( X_{t_1,t_2}(v \otimes w) = (-1)^{\deg(v) \deg(w)} w \otimes v \). We define

\[
W_0 \left( \begin{array}{c}
\bigtimes
\end{array} \right) = X_{t_1,t_2} \quad \text{for} \quad \bigtimes \in \text{Hom}_{C_0}((c_1, c_2), (c_2, c_1)).
\]
Lemma 17 The map $W_0$ induces a tensor functor from $C_0$ to $\mathfrak{gl}(1|1)$-modules.

Proof: The tensor category $C_0$ has a presentation by admissibly colored generators\(^2\)

\[
\downarrow, \uparrow, \bigcirc, \bigcirc, \bigcirc, \bigcirc, \bigcirc, \bigcirc, \bigcirc, \bigcirc, \bigcirc, \bigcirc
\]

(where for the last three graphs all orientations are possible) modulo the relations

\[
\begin{align*}
\bigcirc = \bigcirc = \bigcirc = \bigcirc = \bigcirc = \bigcirc, \\
\bigcirc = \bigcirc = \bigcirc = \bigcirc = \bigcirc = \bigcirc
\end{align*}
\]

(41)

(42)

where the pictures in equations (41), (42) represent words in the generators and $\otimes, \circ$ such that all compositions are defined. The value of $W_0$ on objects and generators of $C_0$ has been defined in equations (33) to (40). It remains to verify the compatibility of $W_0$ with the defining relations of $C_0$.

The relation on the left side of equation (41) follows from equation (28). Assume that the graphs on the right side of equation (41) are colored and oriented such that they are mapped by $W_0$ to elements of $\text{Hom}_{\mathfrak{gl}(1|1)}(V_{t_1} \otimes V_{t_2}, V_{t_3})$ for certain parameters $t_i = (\lambda_i, \mu_i, \sigma_i) \in \mathbb{Q}^* \times \mathbb{Q} \times \mathbb{Z}/(2)$. We will distinguish two cases. For $\sigma_1 + \sigma_2 + \sigma_3 \equiv 1 \text{ mod } 2$ we compute

\[
W_0 \left( \bigcirc \right) = L_{t_1,t_2} = (I_{t_4} \otimes \bigcirc_{t_2}) \circ (Y_{t_1,t_2} \otimes I_{t_2}) = W_0 \left( \bigcirc \right),
\]

where we used the first equation of Lemma 16 and equations (33) to (39). For $\sigma_1 + \sigma_2 + \sigma_3 \equiv 0 \text{ mod } 2$ we compute

\[
W_0 \left( \bigcirc \right) = (-1)^{\sigma_3} \lambda_3 L_{t_1,t_2} = (-1)^{\sigma_2 + \sigma_3} \lambda_1 (I_{t_4} \otimes \bigcirc_{t_2}) \circ (Y_{t_1,t_2} \otimes I_{t_2}) = W_0 \left( \bigcirc \right),
\]

(43)

where we used the second equation of Lemma 16 this time. The equation

\[
W_0 \left( \bigcirc \right) = W_0 \left( \bigcirc \right)
\]

Some of these generators are superfluous, but facilitate the statement of the defining relations of $C_0$. 

\[^2\text{Some of these generators are superfluous, but facilitate the statement of the defining relations of } C_0.\]
is verified similarly. For the verification of the compatibility of $W_0$ with the relations in equation (42) we use that $W_0$ maps graphs to morphisms of degree 0 together with simple properties of the superpermutation $X_{t_1,t_2}$ and the map $\bigcap t$. This completes the proof. □

The morphisms $\Upsilon_{t_1,t_2}$ verify $\Upsilon_{t_1,t_2} = (-1)^{\sigma_1\sigma_2}X_{t_2,t_1}Y_{t_2,t_1}$ whereas for $\Upsilon_{t_1,t_2}$ we have $\Upsilon_{t_1,t_2} = (-1)^{\sigma_1+1}(\sigma_2+1)X_{t_2,t_1} \circ \Upsilon_{t_2,t_1}$. Using equation (38) this translates as follows to trivalent vertices $v$ of colored trivalent graphs $\Gamma$ that are incident to $i,j,k \in E(\Gamma)$. We define $s_v \in \{\pm 1\}$ by $s_v = 1$ iff $|s_{v,i} + s_{v,j} + s_{v,k}| = 1$. Then we have

$$W_0 \left( \begin{array}{c} \Upsilon \\ \end{array} \right) = s_v W_0 \left( \begin{array}{c} \bigcirc \\ \end{array} \right). \quad (43)$$

When $s_v = -1$ we distinguish the cases $s_{v,i} + s_{v,j} + s_{v,k} = -3$ where $v$ is called a source, and $s_{v,i} + s_{v,j} + s_{v,k} = 3$ where $v$ is called a sink.

6 $\Lambda(\Gamma)$-linear weight systems

Maps from trivalent diagrams to modules are called weight systems. In this section we will combine a well-known construction of weight systems related to $\text{gl}(1|1)$ with the results of the previous section. Define a category $\mathcal{C}$ that has the same objects as $\mathcal{C}_0$. The set $\text{Hom}_\mathcal{C}(c,c')$ is a direct sum of modules $\mathcal{C}(\Gamma)$ where $\Gamma \in \text{Hom}_{\mathcal{C}_0}(c,c')$. The graphical definition of the composition of morphisms in $\mathcal{C}_0$ extends to trivalent diagrams and induces $\mathbb{Q}$-linear maps

$$\beta : \Lambda(\Gamma_1) \otimes \mathbb{Q} \Lambda(\Gamma_2) \longrightarrow \Lambda(\Gamma_1 \circ \Gamma_2), \text{ and } \alpha : \mathcal{C}(\Gamma_1) \otimes \mathbb{Q} \mathcal{C}(\Gamma_2) \longrightarrow \mathcal{C}(\Gamma_1 \circ \Gamma_2)$$

with the property that for all $p_i \in \Lambda(\Gamma_i)$, $D_i \in \mathcal{C}(\Gamma_i) (i = 1, 2)$ we have

$$\alpha((p_1D_1) \otimes (p_2D_2)) = \beta(p_1 \otimes p_2)\alpha(D_1 \otimes D_2).$$

We consider $\mathcal{C}_0$ as a subcategory of $\mathcal{C}$ in the obvious way. For objects $c$ of $\mathcal{C}$ we define a functor $W$ by $W(c) = W_0(c)$. Now we extend $W_0$ to morphisms of $\mathcal{C}$. Define $\omega \in \text{gl}(1|1) \otimes \mathbb{Q}$ by

$$\omega = (1/2)(H \otimes D + D \otimes H) + F \otimes E - E \otimes F. \quad (44)$$

Sometimes we use the notation $\omega = \sum_v a_v \otimes b_v$ for $\omega$. The elements $\omega$ and $D \otimes D$ are a basis of the space of invariants in $\text{gl}(1|1) \otimes \text{gl}(1|1)$ by the adjoint representation.
Consider an object \( c = (c_1, \ldots, c_k) \) of \( \mathcal{C} \) with \( c_i = (\lambda_i, \mu_i, \sigma_i) \). Let \( \Gamma = \text{id} \in \mathcal{C}_0 \). Define \( T_c \in \text{End}_c(c) \) as the trivalent diagram of degree 1 on \( \Gamma \) that connects the first interval of \( \Gamma \) with the second interval. We define \( W(T_c) \in \text{End}_{gl(1|1)}(W(c)) \) by

\[
W(T_c)(v_1 \otimes \ldots \otimes v_k) = \sum_{\nu} (-1)^{\deg(v_1) + \deg(b_{\nu}) + \sigma_1 + \sigma_2} a_{\nu} \cdot v_1 \otimes b_{\nu} \cdot v_2 \otimes v_3 \otimes \ldots \otimes v_k, \tag{45}
\]

where \( v_i \in W(c_i) \). For morphisms \( D \) of \( \mathcal{C}_0 \) we define \( W(D) = W_0(D) \). For any unitrivalent graph \( \Gamma \) with admissible coloring, we consider \( gl(1|1) \)-modules as a \( \Lambda(\Gamma) \)-module, where \( d_i \) acts by multiplication with \( \lambda_i \) when \( (\lambda_i, \mu_i) \) is the color of the edge \( i \). Equation (41) implies that this definition is compatible with the defining relations of \( \Lambda(\Gamma) \). We have the following lemma.

**Lemma 18** The definition of \( W \) extends uniquely to a functor from \( \mathcal{C} \) to \( gl(1|1) \)-modules that is \( \Lambda(\Gamma) \)-linear on \( \mathcal{C}(\Gamma) \) and \( \mathbb{Q} \)-linear on general morphisms.

**Proof:** Using relation \((STU)\) we see that the modules \( \mathcal{C}(\Gamma) \) are generated by diagrams \( D \) such that \( D \setminus \Gamma \) contains no trivalent vertex. For the diagrams \( D \) one can choose pictures such that horizontal stripes around intervals in \( D \setminus \Gamma \) are equal to \( T_c^{12} \) for various objects \( c \) of \( \mathcal{C} \). Linearity of \( W \) now implies that the definition of \( W \) determines the value of all morphisms of \( \mathcal{C} \). We have to verify that \( W \) is independent of the choices from above and compatible with the defining relations of \( \mathcal{C}(\Gamma) \).

It follows from Lemma 17 and general properties of \( \omega \) (\( \omega \) is invariant, supersymmetric, and of degree 0) that the definition of \( W \) induces well-defined \( \mathbb{Q} \)-linear maps \( \tilde{W}_\Gamma \) from \( \mathcal{A}(\Gamma) \) to \( \mathbb{Q} \) for all admissibly colored unitrivalent graphs \( \Gamma \). It remains to show that the \( \Lambda(\Gamma) \)-linear maps \( W_\Gamma : \mathcal{C}(\Gamma) \longrightarrow \mathbb{Q} \) defined by \( W_\Gamma \circ \tau = \tilde{W}_\Gamma \) are well-defined, where \( \tau \) denotes the canonical map from \( \mathcal{A}(\Gamma) \) to \( \mathcal{C}(\Gamma) \). As in the proof of Lemma 11 this follows from particular properties of \( gl(1|1) \) and \( \omega : \) equations (46) and (47) (see [FKV]) imply the compatibility of \( W_\Gamma \) with relations \((CL1A)\) and \((CL2A)\).

\[
W_\Gamma \left( \begin{array}{c} \lambda \end{array} \right) = -2d_i d_j W_\Gamma \left( \begin{array}{c} i \end{array} \right), \quad W_\Gamma \left( \begin{array}{c} \chi \end{array} \right) = 0, \quad W_\Gamma \left( \begin{array}{c} \varphi \end{array} \right) = 0, \tag{46}
\]
\[
W_\Gamma \left( \begin{array}{c} \chi \end{array} \right) = \frac{1}{2} W_\Gamma \left( \chi + \chi - \chi - \chi \right). \tag{47}
\]

Now consider a trivalent diagram \( D \) on \( \Gamma \), where \( \Gamma \) is an admissibly colored trivalent graph. Let \( p \in \Gamma \setminus V(D) \) be a point on an edge with color \( (\lambda_p, \mu_p) \). By cutting \( D \) at \( p \), we obtain a diagram \( D_p \in \text{End}_c((\lambda_p, \mu_p, 0)) \).

\(^3\)The explicit appearance of the factor \((-1)^{\sigma_1 + \sigma_2} \) in equation (45) is sometimes avoided by introducing an antisymmetry relation for univalent vertices that are glued to \( \Gamma \).
Lemma 19 There exists a unique linear map $W^\circ : \text{End}_C(\emptyset) \to \mathbb{Q}$ such that for a trivalent diagram $D$ on an admissibly colored trivalent graph $\Gamma$ and any $p \in \Gamma \setminus V(D)$ on an edge with color $(\lambda_p, \mu_p)$ we have

$$W(D_p) = \lambda_p W^\circ(D) t_p,$$

where $t_p = (\lambda_p, \mu_p, 0)$.

The map $W^\circ$ is $\Lambda(\Gamma)$-linear on $C(\Gamma) \subset \text{End}_C(\emptyset)$.

Proof: We only have to show that the definition of $W^\circ(D)$ does not depend on the choice of $p$, because then Lemma 18 implies the compatibility of $W^\circ$ with the defining relations of $C(\Gamma) \subset \text{End}_C(\emptyset)$ and the $\Lambda(\Gamma)$-linearity. Consider two points $p_1, p_2 \in \Gamma \setminus V(D)$ on edges with colors $(\lambda_i, \mu_i)$ $(i = 1, 2)$. We treat the case $\lambda_1 \neq \lambda_2$ in detail. Let $t_i = (\lambda_i, \mu_i, 0)$. Pictorial representations of the diagrams $D$, $D_{p_1}$, and $D_{p_2}$ are shown in equation (48), where the box labeled $x$ represents a morphism $x \in \text{End}_C((t_1, t_2))$.

\[ D = p_1 \begin{array}{c} x \end{array} p_2, \quad D_{p_1} = \begin{array}{c} x \end{array} p_2, \quad D_{p_2} = p_1 \begin{array}{c} x \end{array}. \]  

(48)

It follows from equations (34), (38), (39) and definitions that there exist $a, b \in \mathbb{Q}$ such that

$$W(x) = a Y_{t_1, t_2} \circ \lambda_{t_1, t_2} + b Y_{t_1, t_2} \circ \lambda_{t_1, t_2} = \frac{a}{\lambda_1 - \lambda_2} W \left( \begin{array}{c} \circ \lambda \end{array} \right) + \frac{b}{\lambda_1 - \lambda_2} W \left( \begin{array}{c} \circ \lambda \end{array} \right)$$

which implies

$$W(D_{p_i}) = \frac{a}{\lambda_1 - \lambda_2} W \left( \begin{array}{c} \circ \lambda \end{array} \right) + \frac{b}{\lambda_1 - \lambda_2} W \left( \begin{array}{c} \circ \lambda \end{array} \right) = \lambda_i \frac{a+b}{\lambda_1 - \lambda_2} t_i. \quad (49)$$

The value $\frac{a+b}{\lambda_1 - \lambda_2}$ in equation (49) does not depend on $p_i$ $(i \in \{1, 2\})$. It follows that $W^\circ(D)$ is well-defined for diagrams $D$ whose colored skeleton has two edges colored by $(\lambda_i', \mu_i')$ $(i = 1, 2)$ with $\lambda_i' \neq \lambda_2'$ because then we can treat the case $\lambda_1 = \lambda_2$ by a two-fold application of the argument above.

The remaining case is well-known from computations concerning the 1-variable Alexander polynomial of links. Alternatively, there is a proof for $\lambda_1 = \lambda_2$ (or, more generally, $\lambda_1 \neq -\lambda_2$) similar to the proof above, where a picture of $D$ is chosen as in equation (48), but with different orientations and with $x \in \text{End}_C((t_1, t_2))$. $\square$
Recall from [LM1] the definition of the Kontsevich integral $Z$ of framed $q$-tangles. We assume that $Z$ is defined using an even group-like horizontal Drinfeld associator in $\hat{A}(3)$. For a $q$-tangle $T$ we have $Z(T) \in \hat{A}(\Gamma(T))$ where $\Gamma(T)$ is the underlying 1-manifold of $T$. In this section we will study an extension of $Z$ to graphs (compare [MuO]). We consider a category $G^{na}$ whose morphisms are isotopy classes of oriented (half-)framed unitrivalent graphs $G$ with cyclically oriented vertices. By definition, $G$ is properly embedded into $\mathbb{R} \times [0,1] \times \mathbb{R}$ and we have $G \cap \mathbb{R} \times \{i\} \times \mathbb{R} = \{1, \ldots, n_i\} \times \{i\} \times \{0\}$ for $i = 0,1$ and for certain $n_0, n_1 \geq 0$. We represent a strand of $G$ with a right-handed half twist of the framing graphically by $\bigcirc$. The objects of $G^{na}$ are non-associative words in the symbols $+$ and $\pm$. For example, $(- (+ -))$ is an object of $G^{na}$. Unitrivalent graphs $G \in \text{Hom}_{G^{na}}(w_0, w_1)$ are related to $w_i$ as follows: the $i$-th symbol $a \in \{+, -\}$ of $w_0$ (resp. $w_1$) corresponds to the $i$-th lower (resp. upper) boundary point $p = (i, 0, 0)$ (resp. $p = (i, 1, 0)$) of $G$ where $a = +$ (resp. $a = -$) means that the graph $G$ must be oriented downwards (resp. upwards) at $p$.

The category $\hat{A}^{na}$ has the same objects as $G^{na}$. The set $\text{Hom}_{\hat{A}^{na}}(w_0, w_1)$ is the direct sum of all modules $\hat{A}(\Gamma)$ where $\Gamma$ is a unitrivalent graph whose boundary is partitioned into two ordered sets called lower and upper boundary as in Section 5, and $a \in \{+, -\}$ in $w_0$ and $w_1$ is related to the boundary points in a graphical representation of $\Gamma$ in the same way as above. We consider the invariant $Z$ of $q$-tangles from $\hat{A}^{na}$ as a tensor functor from the subcategory $T^{na} \subset G^{na}$ of $q$-tangles to $\hat{A}^{na}$.

Let us recall how $Z$ depends on the orientations of edges of $q$-tangles. Let $\Gamma$ be an oriented 1-dimensional manifold with boundary and let $\Gamma' \subset \Gamma$ be a set of connected components of $\Gamma$. Let $D$ be a trivalent diagram on $\Gamma$. Define $S_{\Gamma'}(D)$ by inverting the orientation of all components of $\Gamma'$ and by multiplying the result by $(-1)^m$ where $m$ is the number of univalent vertices of $D \setminus \Gamma$ that are glued to $\Gamma'$. Let $T$ be a $q$-tangle and $T' \subset T$ be a set of connected components of $T$. Define $S_{T'}(T)$ by inverting the orientation of all components of $T'$ in $T$. With this notation we have

$$Z(S_{T'}(T)) = S_{\Gamma(T')} (Z(T)).$$

We omit the index of $S$ when all components of a unitrivalent graph (resp. of the skeleton of a trivalent diagram) are concerned. For $a \in \text{End}_{\hat{A}}(+)$ we define

$$S^\ast(a) = (\bigcap \otimes \uparrow) \circ (\uparrow \otimes a \otimes \uparrow) \circ (\uparrow \otimes \bigcup).$$
It is unknown if there exist elements \( a \in \hat{\mathcal{A}}(1) \subseteq \text{End}_{\hat{\mathcal{A}}}(+) \) with \( S(a) \neq S^*(a) \).

In diagrams of morphisms \( G \) of \( \mathcal{G}^{na} \) we use projections of generic representatives of \( G \) to the first two coordinates such that the cyclic order at trivalent vertices is counterclockwise in the projection. For a unitrivalent graph \( \Gamma \) we define \( \Gamma^{\text{op}} \) by inverting the cyclic order of all trivalent vertices of \( \Gamma \). This induces maps

\[
\hat{\mathcal{A}}(\Gamma) \longrightarrow \hat{\mathcal{A}}(\Gamma^{\text{op}}), \ a \mapsto a^{\text{op}} \quad \text{and} \quad \text{Hom}_{\hat{\mathcal{A}}^{na}}(w_0, w_1) \longrightarrow \text{Hom}_{\hat{\mathcal{A}}^{na}}(w_0, w_1), \ G \mapsto G^{\text{op}}.
\]

We denote the Kontsevich integral of the trivial knot by \( \nu \) and regard it as an element of \( \hat{\mathcal{A}}(1) \). Since \( \nu \) is equal to 1 in degree 0 the element \( \nu \) is invertible and there exist unique roots of \( \nu^n \ (n \in \mathbb{Z}) \) that are equal to 1 in degree 0. The elements \( \nu^k \ (k \in \mathbb{Q}) \) satisfy \( S(\nu^k) = S^*(\nu^k) \). In the formulas below the box labeled \( \nu^k \) represents the element \( \nu^k \) or \( S(\nu^k) \) according to the orientation of the interval with the box.

**Theorem 20** (1) For any choice of

\[
a, b \in \text{End}_{\hat{\mathcal{A}}^{na}}(+) \quad , \quad c \in \text{Hom}_{\hat{\mathcal{A}}^{na}}(-, (++) \) \quad , \quad \text{and} \quad d \in \text{Hom}_{\hat{\mathcal{A}}^{na}}(+, (--) \)

there exists a unique extension of \( Z \) to a tensor functor from \( \mathcal{G}^{na} \) to \( \hat{\mathcal{A}}^{na} \) satisfying

\[
Z \left( \begin{array}{c} \circ \\
\downarrow \end{array} \right) = \begin{array}{c} a \\
S'(a) \end{array}, \\
Z \left( \begin{array}{c} \circ \\
\downarrow \end{array} \right) = \begin{array}{c} b \\
S'(b) \end{array}, \\
Z \left( \begin{array}{c} \circ \\
\downarrow \end{array} \right) = c, \\
Z \left( \begin{array}{c} \circ \\
\downarrow \end{array} \right) = d,
\]

and

\[
Z \left( \begin{array}{c} \circ \\
\downarrow \end{array} \right) = \exp \left( \frac{\int}{4} \right).
\]

(2) When \( \bigotimes \circ c = c^{\text{op}} \) and \( \bigotimes \circ d = d^{\text{op}} \) holds then we have \( Z(G^{\text{op}}) = Z(G)^{\text{op}} \) for all morphisms \( G \) of \( \mathcal{G}^{na} \).

**Sketch of proof:** First we consider oriented graphs whose vertices are oriented boxes with a distinguished lower boundary (called **coupons**). A category \( \mathcal{G}^{na} \) is defined in the same way as \( \mathcal{G}^{na} \) except that morphisms are embedded framed graphs with coupons

\[
C_a = \begin{array}{c} \cdots \\
\downarrow \\
a \end{array} \in \text{Hom}_{\mathcal{G}^{na}}(s, t)
\]

27
that are colored by elements \( a \in \text{Hom}_{\tilde{\mathcal{A}}^{na}}(s,t) \). It is easy to see that there exists a unique extension of \( Z \) to a tensor functor from \( \mathcal{G}^{na} \) to \( \tilde{\mathcal{A}}^{na} \) that verifies \( Z(C_a) = a \) for all coupons \( C_a \) as above. This general construction implies that part (1) of the theorem holds iff

\[
Z \left( \begin{array}{c}
\cdot \\
\cdot
\end{array} \right) = Z \left( \begin{array}{c}
\cdot \\
\cdot
\end{array} \right) \quad \text{and} \quad Z \left( \begin{array}{c}
\cdot \\
\cdot
\end{array} \right) = Z \left( \begin{array}{c}
\cdot \\
\cdot
\end{array} \right).
\]

These two equations follow from a well-known identity in \( \text{Hom}_{\tilde{\mathcal{A}}}((++-),+) \):

\[
Z \left( \begin{array}{c}
\cdot \\
\cdot
\end{array} \right) = \frac{\Delta_1}{\Delta_1^{1/2}} \frac{-1/2}{\nu_0}. \tag{52}
\]

(2) The isotopy invariance of \( Z \) implies

\[
Z \left( \begin{array}{c}
\cdot \\
\cdot
\end{array} \right) = Z \left( \begin{array}{c}
\cdot \\
\cdot
\end{array} \right).
\]

When the upper two strands at a trivalent vertex are both oriented downwards or both oriented upwards then the equation

\[
\begin{array}{c}
\cdot \\
\cdot
\end{array} = \frac{1}{2} \left( \begin{array}{c}
\cdot \\
\cdot
\end{array} - \begin{array}{c}
\cdot \\
\cdot
\end{array} - \begin{array}{c}
\cdot \\
\cdot
\end{array} \right)
\]

and the symmetry properties of \( c \) and \( d \) imply

\[
Z \left( \begin{array}{c}
\cdot \\
\cdot
\end{array} \right) = Z \left( \begin{array}{c}
\cdot \\
\cdot
\end{array} \right).
\]

Part (2) of the theorem now follows from part (1) and the equations above. \( \Box \)

From now on we will fix the choices below in the definition of the extension of \( Z \) to unitrivalent graphs.

\[
\begin{align*}
Z \left( \begin{array}{c}
\cdot \\
\cdot
\end{array} \right) &= \begin{array}{c}
\cdot \\
\cdot
\end{array} \quad & Z \left( \begin{array}{c}
\cdot \\
\cdot
\end{array} \right) &= \begin{array}{c}
\cdot \\
\cdot
\end{array} \tag{53} \\
Z \left( \begin{array}{c}
\cdot \\
\cdot
\end{array} \right) &= \begin{array}{c}
\cdot \\
\cdot
\end{array} & Z \left( \begin{array}{c}
\cdot \\
\cdot
\end{array} \right) &= \begin{array}{c}
\cdot \\
\cdot
\end{array} \tag{54}
\end{align*}
\]

With this definition we have \( Z(G) \in \tilde{\mathcal{A}}(\Gamma(G)) \) where \( \Gamma(G) \) is the underlying graph of \( G \). For explicit computations in the following sections we list more values of \( Z \).
The computation of these values can be simplified by first generalizing symmetry properties of $Z$ from $q$-tangles (Proposition 3.1 of [LM2]) to trivalent graphs.

8 The Alexander series of a tetrahedron

For a trivalent diagram $D$ on an admissibly colored trivalent graph $\Gamma$ we define

$$\hat{W}^\circ(D) = W^\circ(D) h^{(1/2)\#V(D)-1} \in h^{-1}\mathbb{Q}[[h]] \subset \mathbb{Q}[[h]][h^{-1}]$$

This definition induces a continuous linear map $\hat{W}^\circ : \hat{C}^0(\Gamma) \rightarrow h^{-1}\mathbb{Q}[[h]]$. Notice that $(1/2)\#V(D) > \deg(D)$ when $\Gamma \subset D$ has trivalent vertices. For an admissibly colored framed unitrivalent graph $G \subset \mathbb{R}^2 \times I$ we define $Z(G) = \tau(Z(G))$ where the continuous linear map $\tau : \hat{A}(\Gamma(G)) \rightarrow \hat{C}^0(\Gamma(G))$ is defined on trivalent diagrams $D$ by $\tau(D) = D$ and the skeleton $\Gamma(G)$ of $D$ is colored according to $G$.

Definition 21 The invariant $\hat{\nabla}(G) = \hat{W}^\circ(Z(G))$ of an admissibly colored framed trivalent graph $G$ is called the Alexander series of $G$.

In the following three sections we will compute the Alexander series of a trivially embedded colored tetrahedron $T$ in two different ways. With the first computation we will determine the value of the Alexander series $\hat{\nabla}(T)$, and with the second computation we will use this value to prove Theorem [I]. Let $\mathcal{O}$ be the trivial knot with color $c = (\lambda, \mu)$ and with 0-framing. It follows from Section 2.2 of [BNC] that for $c = (1,0)$ we have $\hat{\nabla}(\mathcal{O}) = 1/(e^{h/2} - e^{-h/2})$. For $c = (\lambda, \mu)$ we obtain from this value the more general formula

$$\hat{\nabla}(\mathcal{O}) = \frac{1}{e^{\lambda h/2} - e^{-\lambda h/2}}.$$  

(57)
by using that the left side of equation (57) suffices to compute \( \hat{W}(\mathcal{Z}(\mathcal{O})) \) because \( \mathcal{O} \) is 0-framed and we know the structure of \( \mathcal{C}(1) \) (see [Thu] for a more general result). For a trivalent diagram \( D \in \text{Hom}_{\mathcal{C}_0}(c, c') \) the definition

\[
\hat{W}(D) = W(D) h^{\text{deg}D} \in \text{Hom}_{\mathfrak{gl}(1|1)}(W(c), W(c'))[[h]]
\]

(58) induces a continuous linear map \( \hat{W} : \mathcal{C}_0^0(\Gamma) \rightarrow \text{Hom}_{\mathfrak{gl}(1|1)}(W(c), W(c'))[[h]] \). Recall the definition \( \varphi(x) = (e^{x/2} - e^{-x/2})/x = 2 \sinh(x/2)/x \) from the introduction. Equation (57) and Lemma 19 imply that for the trivalent diagram \( \nu \) on a skeleton \( I_\nu \in \text{End}_{\mathcal{C}_0}(t) \) with \( t = (\lambda, \mu, \sigma) \) we have

\[
\hat{W}(\nu) = (1/\varphi(\lambda h)) I_\nu.
\]

(59)

Let \( G \in \text{End}_{\mathcal{G}^{\text{ass}}}(+) \). Then \( H = \bigcap \circ (\text{id}_{\mathcal{L}} \otimes \mathcal{G}) \circ \bigcup \in \text{End}_{\mathcal{G}^{\text{ass}}}(\emptyset) \) is called the closure (or trace) of \( G \). When \( H \) is admissibly colored and the upper (or lower) edge of \( G \) has color \( (\lambda, \mu) \) then the definition of \( Z \), Lemma 13, and equations (60) and (59) imply that for \( t = (\lambda, \mu, 0) \) we have

\[
\hat{W}(H)_{I_\nu} = \frac{h^{(1/2)\#V(\Gamma(H))}}{\lambda h \varphi(\lambda h)} \hat{W}(\mathcal{Z}(G)).
\]

(60)

For \( i = 1, 2, 3 \) let \( t_i = (\lambda_i, \mu_i, \sigma_i) \in \mathbb{Q}^* \times \mathbb{Q} \times \mathbb{Z}/(2) \) with

\[
\lambda_1 + \lambda_2 + \lambda_3 \, , \, \lambda_1 + \lambda_2 + \lambda_3 \in \mathbb{Q}^*.
\]

(61)

We will use in the following sections the triples \( t_i = (\lambda_i, \mu_i, \sigma_i) \in \mathbb{Q}^* \times \mathbb{Q} \times \mathbb{Z}/(2) \) \((i = 4, \ldots, 11)\) defined by

\[
t_4 = t_1 + t_2 + e_2 \, , \, t_5 = t_4 + t_3 + e_2 \, , \, t_6 = t_2 + t_3 + e_2 \, , \, t_7 = t_1 - 3 - 2e_2 + e_3 \, (i \geq 7)
\]

(62)

and the colors \( c_i = (((-1)^\sigma, \lambda_i, (-1)^\sigma \mu_i) \,(i = 1, \ldots, 11) \). Equation (25) implies that the tensor product of three modules decomposes as

\[
V_{t_1} \otimes V_{t_2} \otimes V_{t_3} \cong V_{t_5} \oplus V_{t_8} \oplus V_{t_{11}}.
\]

(63)

The following lemma concerns simple modules of multiplicity 1 in equation (53).

**Lemma 22** With \( t_i \) as in equation (62) the following formulas hold.

\[
(\mathcal{I}_{t_1} \otimes Y_{t_2, t_3}) \circ Y_{t_1, t_6} = (Y_{t_1, t_2} \otimes \mathcal{I}_{t_3}) \circ Y_{t_4, t_5},
\]

\[
(\mathcal{I}_{t_1} \otimes Y_{t_2, t_3}) \circ Y_{t_1, t_6} = (Y_{t_1, t_2} \otimes \mathcal{I}_{t_3}) \circ Y_{t_7, t_9},
\]

\[
\lambda_{t_1, t_6} \circ (\mathcal{I}_{t_1} \otimes \lambda_{t_2, t_3}) = \lambda_{t_4, t_5} \circ (\lambda_{t_1, t_2} \otimes \mathcal{I}_{t_3}).
\]

\[
\lambda_{t_1, t_9} \circ (\mathcal{I}_{t_1} \otimes \lambda_{t_2, t_3}) = \lambda_{t_7, t_9} \circ (Y_{t_1, t_2} \otimes \mathcal{I}_{t_3}).
\]
Consider the three diagrams $T_k$ ($k = 1, 2, 3$) shown in Figure 1.

![Diagram](image)

Figure 1: Three colored diagrams whose closures are planar tetrahedra

Fix a choice of $k \in \{1, 2, 3\}$. Define $\sigma_i = \sigma_i(k)$ by $\sigma_i = 1$ iff the edge of $T_k$ labeled by $c_i$ points downwards. Then equations (61) and (62) ensure that the coloring of $T_k$ is admissible. Let $b_i = ((-1)^{\alpha_i} \lambda_i, (-1)^{\beta_i} \mu_i, \sigma_i)$. Define $b = b_5$ if $k = 1$, $b = b_{11}$ if $k = 2$, and $b = b_8$ if $k = 3$. Then $T_k = U_k \circ A_k \circ L_k$ where $U_k \in \text{Hom}_G((b_1 (b_2 b_3)), b)$ consists of the upper half of $T_k$, $L_k \in \text{Hom}_G(b, ((b_1 b_2) b_3))$ consists of the lower half of $T_k$, and $A_k \in \text{Hom}_G(((b_1 b_2) b_3), (b_1 (b_2 b_3)))$ consists of three vertical colored strands.

It follows from equations (53), (58), and Lemma 22) that the graph $L_1$ is mapped by $\widehat{W} \circ \overline{Z}$ to the following morphism from $V_{t_5}[h]$ to $(V_{t_1} \otimes V_{t_2} \otimes V_{t_3})[h]$:  

$$\widehat{W}(Z(L_1)) = W(\Gamma(L_1)) = (Y_{t_1,t_2} \otimes I_{t_3}) \circ Y_{t_4,t_3} = (I_{t_1} \otimes Y_{t_2,t_3}) \circ Y_{t_1,t_6}. \quad (64)$$

The graph $U_1$ is mapped by $\widehat{W} \circ \overline{Z}$ to the following morphism from $(V_{t_1} \otimes V_{t_2} \otimes V_{t_3})[h]$ to $V_{t_5}[h]$ (see equations (53), (59), (63)):

$$\widehat{W}(Z(U_1)) = \varphi(\lambda_5 h) \varphi(\lambda_6 h) W(\Gamma(U_1)) \quad \lambda_5 \lambda_6 \varphi(\lambda_5 h) \varphi(\lambda_6 h) \lambda_{t_1,t_6} \circ (I_{t_1} \otimes \lambda_{t_2,t_3}). \quad (65)$$

The associator $\Phi$ is a series with constant term 1, and all higher order terms of $\Phi$ involve a commutator (see (DA5)). Therefore, by equation (63) and Schurs lemma, the action of $\Phi$ on $(V_{t_1} \otimes V_{t_2} \otimes V_{t_3})[h]$ restricts to

$$I_{t_5} \in \text{End}_{gl(1|1)}(V_{t_5}) \subset \text{End}_{gl(1|1)}(V_{t_1} \otimes V_{t_2} \otimes V_{t_3})[h]. \quad (66)$$
This implies

\[ \lambda_{t_1,t_6} \circ (I_{t_1} \otimes I_{t_2,t_3}) \circ \hat{W}(Z(A_1)) \circ (I_{t_1} \circ Y_{t_2,t_3}) \circ Y_{t_1,t_6} = I_{t_5}. \]  

(67)

Let \( S_1 \) be the closure of \( T_1 \). Equations (60), (64), (65), and (67) allow to compute \( \hat{\nabla}(S_1) \) without knowing \( \Phi \):

\[ \hat{\nabla}(S_1)I_{t_5} = \frac{h}{\lambda_5 \varphi(\lambda_5 h)} \hat{W}(Z(T_1)) = \lambda_6 h \varphi(\lambda_6 h)I_{t_5} = (e^{\lambda_6 h/2} - e^{-\lambda_6 h/2})I_{t_5}. \]  

(68)

In general, the Alexander series of a planar tetrahedron is given by the following lemma.

**Lemma 23** Let \( T \) be a planar tetrahedron with admissible coloring and blackboard framing. There exists a unique edge \( e \) of \( T \) such that by reversing the orientation of \( e \) we obtain a tetrahedron with one source and without a sink. Let \((\lambda, \mu)\) be the color of \( e \). Then we have

\[ \hat{\nabla}(T) = e^{\lambda h/2} - e^{-\lambda h/2}. \]

**Proof:** By Lemma 7.2.A of [Vir] there are four isotopy classes of planar oriented tetrahedra with blackboard framing. By Lemma 7.2.B of [Vir] two of these tetrahedra do not have an admissible coloring. By Lemma 7.2.C of [Vir] the remaining two oriented tetrahedra have a unique edge \( e \) as in the lemma. For one admissibly colored tetrahedron \( S_1 \) we have computed \( \hat{\nabla}(S_1) \) in equation (68). The second tetrahedron is the closure \( S_2 \) of \( T_2 \) (see Figure 1). The computation of \( \hat{\nabla}(S_2) \) proceeds along the same lines as the computation of \( \hat{\nabla}(S_1) \).  

\[ \blacksquare \]

9 **Associativity and \( gl(1|1) \)-modules**

For the diagram \( T_3 \) in Figure 1 we will see in Section 10 that the contribution of the associator \( \Phi \) in the computation of \( \hat{W}(Z(T_3)) \) is non-trivial. We will use Lemma 24 and Corollary 25 below for a similar purpose in this computation as we used Lemma 22 to deduce equation (64).

**Lemma 24** For \( t_i = (\lambda_i, \mu_i, \sigma_i) \) as in equations (67), (62) and \( \kappa = \frac{\lambda_2(\lambda_1 + \lambda_2 + \lambda_3)}{(\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)} \) we have two bases of \( \text{Hom}_{gl(1|1)}(V_{t_8}, V_{t_1} \otimes V_{t_2} \otimes V_{t_3}) \) that are related by

\[
\begin{pmatrix}
(I_{t_1} \otimes Y_{t_2,t_3}) \circ Y_{t_1,t_8} \\
(I_{t_1} \otimes Y_{t_2,t_3}) \circ Y_{t_1,t_6}
\end{pmatrix}
= \begin{pmatrix}
\begin{pmatrix}
(\lambda_2 \lambda_3) \\
-\lambda_2 - \lambda_3
\end{pmatrix}^{-1} \\
1
\end{pmatrix}
\begin{pmatrix}
\frac{\kappa}{\lambda_2 \lambda_3} \\
-\lambda_2 - \lambda_3
\end{pmatrix}
\begin{pmatrix}
(Y_{t_1,t_2} \otimes I_{t_3}) \circ Y_{t_1,t_6} \\
(Y_{t_1,t_2} \otimes I_{t_3}) \circ Y_{t_1,t_3}
\end{pmatrix}.
\]

(69)

\[ \blacksquare \]
Proof: We compute

\[
(I_1 \otimes Y_{t_2,t_3})(Y_{t_1,t_9}(w_{t_9})) = (I_1 \otimes Y_{t_2,t_3})(F \cdot (v_1 \otimes v_{t_9}))
\]
\[
= w_{t_1} \otimes Y_{t_2,t_3}(v_{t_9}) + (-1)^{\sigma_1} v_{t_1} \otimes Y_{t_2,t_3}(w_{t_9})
\]
\[
= \frac{1}{\lambda_2 + \lambda_3} w_{t_1} \otimes E : (w_{t_2} \otimes w_{t_3}) + (-1)^{\sigma_1} v_{t_1} \otimes w_{t_2} \otimes w_{t_3}
\]
\[
= \frac{\lambda_2}{\lambda_2 + \lambda_3} w_{t_1} \otimes v_{t_2} \otimes w_{t_3} - \frac{(-1)^{\sigma_2} \lambda_3}{\lambda_2 + \lambda_3} w_{t_1} \otimes w_{t_2} \otimes v_{t_3} + (-1)^{\sigma_1} v_{t_1} \otimes w_{t_2} \otimes w_{t_3}
\]

and similarly (or simpler)

\[
(I_1 \otimes Y_{t_2,t_3})(Y_{t_1,t_6}(w_{t_8})) = (-1)^{\sigma_2} w_{t_1} \otimes v_{t_2} \otimes w_{t_3} + w_{t_1} \otimes w_{t_2} \otimes v_{t_3},
\]
\[
(Y_{t_1,t_2} \otimes I_3)(Y_{t_7,t_3}(w_{t_8})) = \frac{(-1)^{\sigma_2} \lambda_2}{\lambda_1 + \lambda_2} w_{t_1} \otimes v_{t_2} \otimes w_{t_3} + w_{t_1} \otimes w_{t_2} \otimes v_{t_3}
\]
\[
- \frac{(-1)^{\sigma_1 + \sigma_2} \lambda_1}{\lambda_1 + \lambda_2} v_{t_1} \otimes w_{t_2} \otimes w_{t_3},
\]
\[
(Y_{t_1,t_2} \otimes I_3)(Y_{t_4,t_3}(w_{t_8})) = w_{t_1} \otimes v_{t_2} \otimes w_{t_3} + (-1)^{\sigma_1} v_{t_1} \otimes w_{t_2} \otimes w_{t_3}.
\]

We see that the vectors \((I_1 \otimes Y_{t_2,t_3})(Y_{t_1,t_9}(w_{t_9}))\) and \((I_1 \otimes Y_{t_2,t_3})(Y_{t_1,t_6}(w_{t_8}))\) as well as \((Y_{t_1,t_2} \otimes I_3)(Y_{t_7,t_3}(w_{t_8}))\) and \((Y_{t_1,t_2} \otimes I_3)(Y_{t_4,t_3}(w_{t_8}))\) are linearly independent. Equation (63) implies that \(\text{Hom}_{gl(1|1)}(V_{t_8}, V_{t_1} \otimes V_{t_2} \otimes V_{t_3})\) is two-dimensional, so we have found two bases of that space. Verify that equation (69) holds when we evaluate the morphisms in this equation on \(w_{t_8}\). This implies the lemma because \(w_{t_8}\) generates \(V_{t_8}\). \(\square\)

Dually to Lemma 24 we have the following corollary.

**Corollary 25** With \(t_i\) and \(\kappa\) as in Lemma 24 two bases of

\[
\text{Hom}_{gl(1|1)}(V_{t_1} \otimes V_{t_2} \otimes V_{t_3}, V_{t_8})
\]

are related by

\[
\left(\begin{array}{c}
\lambda_{t_1,t_9} \circ (I_1 \otimes A_{t_2,t_3}) \\
\lambda_{t_1,t_6} \circ (I_1 \otimes A_{t_2,t_3})
\end{array}\right) = \left(\begin{array}{c}
\frac{(-1)^{\sigma_2} \lambda_1}{\lambda_1 - \lambda_2} \\
\frac{1}{\lambda_2 + \lambda_3}
\end{array}\right)
\left(\begin{array}{c}
\lambda_{t_7,t_3} \circ (A_{t_1,t_2} \otimes I_3) \\
\lambda_{t_4,t_3} \circ (A_{t_1,t_2} \otimes I_3)
\end{array}\right).
\]

**Proof:** By Lemma 24 linear maps \(f_i, g_i\) are related by \((f_1, f_2)^T = A(g_1, g_2)^T\) for a certain matrix \(A\). Therefore, maps \(f'_i, g'_i\) satisfying \(f'_i \circ f_j = \delta_{ij} I_{t_8}\) and \(g'_i \circ g_j = \delta_{ij} I_{t_8}\) are related by \((f'_1, f'_2)^T = (A^T)^{-1}(g'_1, g'_2)^T\). This implies the corollary. \(\square\)
In the rest of this section we prepare the analogue of equation (67) for the computation in Section 10. By the definition of the \( \text{gl}(1|1) \)-module structure of \( V_{t_1} \otimes V_{t_2} \) we have

\[
F \cdot (v_{t_1} \otimes v_{t_2}) = w_{t_1} \otimes v_{t_2} + (-1)^{\sigma_1} v_{t_1} \otimes w_{t_2} \quad \text{and} \quad (70)
\]

\[
E \cdot (w_{t_1} \otimes w_{t_2}) = -(-1)^{\sigma_1} \lambda_2 w_{t_1} \otimes v_{t_2} + \lambda_1 v_{t_1} \otimes w_{t_2} . \quad (71)
\]

For \( \lambda_1 \neq -\lambda_2 \) equations (70) and (71) are formulas for a change of bases in the two dimensional eigenspace of \( H \) on \( V_{t_1} \otimes V_{t_2} \) and imply

\[
(\lambda_1 + \lambda_2) w_{t_1} \otimes v_{t_2} = \lambda_1 F \cdot (v_{t_1} \otimes v_{t_2}) - (-1)^{\sigma_1} E \cdot (w_{t_1} \otimes w_{t_2}), \quad (72)
\]

\[
(\lambda_1 + \lambda_2) v_{t_1} \otimes w_{t_2} = -(-1)^{\sigma_1} \lambda_2 F \cdot (v_{t_1} \otimes v_{t_2}) + E \cdot (w_{t_1} \otimes w_{t_2}). \quad (73)
\]

**Lemma 26** With \( t_i \) and \( \kappa \) as in Lemma 24 the following formulas hold:

\[
(\lambda_{t_1,t_2} \otimes I_3) \circ (I_t \otimes Y_{t_2,t_3}) = Y_{t_4,t_3} \circ \lambda_{t_1,t_6} + \frac{(-1)^{\sigma_2} \lambda_1}{\lambda_1 + \lambda_2} Y_{t_4,t_3} \circ \lambda_{t_1,t_6},
\]

\[
(\lambda_{t_1,t_2} \otimes I_3) \circ (I_t \otimes Y_{t_2,t_3}) = Y_{t_7,t_3} \circ \lambda_{t_1,t_9} - \frac{(-1)^{\sigma_2} \lambda_2}{\lambda_2 + \lambda_3} Y_{t_7,t_3} \circ \lambda_{t_1,t_9},
\]

\[
(\lambda_{t_1,t_2} \otimes I_3) \circ (I_t \otimes Y_{t_2,t_3}) = Y_{t_7,t_3} \circ \lambda_{t_1,t_6},
\]

\[
(\lambda_{t_1,t_2} \otimes I_3) \circ (I_t \otimes Y_{t_2,t_3}) = \kappa Y_{t_4,t_3} \circ \lambda_{t_1,t_9}.
\]

**Proof:** We have

\[
(\lambda_{t_1,t_2} \otimes I_3) \circ (I_t \otimes Y_{t_2,t_3})(v_{t_1} \otimes v_{t_6}) = (\lambda_{t_1,t_2} \otimes I_3)(v_{t_1} \otimes v_{t_2} \otimes v_{t_3}) = v_{t_4} \otimes v_{t_3}
\]

\[
= \left( Y_{t_4,t_3} \circ \lambda_{t_1,t_6} + \frac{(-1)^{\sigma_2} \lambda_1}{\lambda_1 + \lambda_2} Y_{t_4,t_3} \circ \lambda_{t_1,t_6} \right) (v_{t_1} \otimes v_{t_6}).
\]

Using equation (72) we see that

\[
(\lambda_{t_1,t_2} \otimes I_3) \circ (I_t \otimes Y_{t_2,t_3})(w_{t_1} \otimes w_{t_6}) = (\lambda_{t_1,t_6} \otimes I_3)(w_{t_1} \otimes F \cdot (v_{t_2} \otimes v_{t_3}))
\]

\[
= (\lambda_{t_1,t_6} \otimes I_3)(w_{t_1} \otimes w_{t_2} \otimes v_{t_3} + (-1)^{\sigma_2} w_{t_1} \otimes v_{t_2} \otimes w_{t_3})
\]
\[
(x_1, t_2) \otimes I_3) \left( \frac{(-1)^{\sigma_2} \lambda_1}{\lambda_1 + \lambda_2} F \cdot (v_{t_1} \otimes v_{t_2}) \otimes w_{t_3} - \frac{(-1)^{\sigma_1 + \sigma_2}}{\lambda_1 + \lambda_2} E \cdot (w_{t_1} \otimes w_{t_2}) \otimes w_{t_3} \right) \\
= \frac{(-1)^{\sigma_2} \lambda_1}{\lambda_1 + \lambda_2} w_{t_4} \otimes w_{t_3} \\
= \left( \mathcal{Y}_{t_4, t_3} \circ \mathcal{J}_{t_4, t_3} + \frac{(-1)^{\sigma_2} \lambda_1}{\lambda_1 + \lambda_2} \mathcal{Y}_{t_4, t_3} \circ \mathcal{J}_{t_4, t_3} \right) (w_{t_1} \otimes w_{t_6}).
\]

Since an element of \( \text{Hom}_{gl(1|1)}(V_{t_1} \otimes V_{t_6}, V_{t_4} \otimes V_{t_3}) \) is determined by the images of \( v_{t_1} \otimes v_{t_6} \) and \( w_{t_1} \otimes w_{t_6} \) this implies the first equation of the lemma. The remaining three equations are proved similarly. \( \square \)

The following corollary holds for reasons of symmetry.

**Corollary 27** With \( t_i \) and \( \kappa \) as in Lemma 24 the following formulas hold:

\[
(I_1 \otimes \mathcal{J}_{t_2, t_3}) \circ (Y_{t_1, t_2} \otimes I_3) = Y_{t_1, t_6} \circ \mathcal{J}_{t_4, t_3} + \frac{(-1)^{\sigma_2} \lambda_3}{\lambda_2 + \lambda_3} Y_{t_1, t_6} \circ \mathcal{J}_{t_4, t_3},
\]

\[
(I_1 \otimes \mathcal{J}_{t_2, t_3}) \circ (Y_{t_1, t_2} \otimes I_3) = Y_{t_1, t_6} \circ \mathcal{J}_{t_4, t_3} - \frac{(-1)^{\sigma_2} \lambda_1}{\lambda_1 + \lambda_2} Y_{t_1, t_6} \circ \mathcal{J}_{t_4, t_3},
\]

\[
(I_1 \otimes \mathcal{J}_{t_2, t_3}) \circ (Y_{t_1, t_2} \otimes I_3) = Y_{t_1, t_6} \circ \mathcal{J}_{t_4, t_3},
\]

\[
(I_1 \otimes \mathcal{J}_{t_2, t_3}) \circ (Y_{t_1, t_2} \otimes I_3) = \kappa Y_{t_1, t_6} \circ \mathcal{J}_{t_4, t_3}.
\]

**Proof:** Let \( \tau_{V, W} \in \text{Hom}(V \otimes W, W \otimes V) \) be the linear map induced by the permutation of tensor factors. When we interchange the labels \( t_{3n-2} \) and \( t_{3n} \) \( (n = 1, 2, 3) \) in the equations of Lemma 24, replace the equations \( X = Y \in \text{Hom}(V \otimes W, V' \otimes W') \) by \( \tau_{V', W} \cdot X \tau_{V, W} = \tau_{V', W} \cdot Y \tau_{V, W} \), and apply the properties preceding equation 43 and similar equations for \( \mathcal{J} \) and \( \mathcal{J}' \), then we obtain the equations of the corollary. \( \square \)

By equation 23 commutators of elements of \( \text{End}_{gl(1|1)}(V_{t_1} \otimes V_{t_2} \otimes V_{t_3}) \) lie in the subspace \( \text{End}_{gl(1|1)}(V_{t_3}) \) of \( \text{End}_{gl(1|1)}(V_{t_1} \otimes V_{t_2} \otimes V_{t_3}) \). We make some explicit computations.

**Lemma 28** With \( t_i \) and \( \kappa \) as in Lemma 24 we have

\[
[([Y_{t_1, t_2} \circ \mathcal{J}_{t_1, t_2}] \otimes I_3), I_1 \otimes ([Y_{t_2, t_3} \circ \mathcal{J}_{t_2, t_3}]]) \\
= [([Y_{t_1, t_2} \circ \mathcal{J}_{t_1, t_2}] \otimes I_3), I_1 \otimes ([Y_{t_2, t_3} \circ \mathcal{J}_{t_2, t_3}]]) \\
= [I_1 \otimes (Y_{t_2, t_3} \circ \mathcal{J}_{t_2, t_3}), (Y_{t_1, t_2} \circ \mathcal{J}_{t_1, t_2} \otimes I_3)] \\
= [I_1 \otimes (Y_{t_2, t_3} \circ \mathcal{J}_{t_2, t_3}), (Y_{t_1, t_2} \circ \mathcal{J}_{t_1, t_2} \otimes I_3)]
\]

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Lemma 29. The remaining equations can be proven similarly.

Proof: For the first commutator we compute

\[
\begin{align*}
= & \frac{(-1)^{\sigma_2} \lambda_1 \kappa}{\lambda_1 + \lambda_2} (Y_{t_1,t_2} \otimes I_{t_3}) \circ Y_{t_4,t_3} \circ \lambda_{d_7,t_3} \circ (\lambda_{d_{1,t_2} \otimes I_{t_3}}) \\
- & \frac{(-1)^{\sigma_2} \lambda_3}{\lambda_2 + \lambda_3} (Y_{t_1,t_2} \otimes I_{t_3}) \circ Y_{t_7,t_3} \circ \lambda_{d_{4,t_3}} \circ (\lambda_{d_{1,t_2} \otimes I_{t_3}})
\end{align*}
\]

where the first equality follows from Lemma \( \boxed{22} \) and Corollary \( \boxed{27} \), the second equality is a consequence of Lemma \( \boxed{22} \) and the third equality is implied by Corollary \( \boxed{27} \) and Lemma \( \boxed{22} \). The remaining equations can be proven similarly. \( \square \)

The following lemma will be used to express the action of \( \Phi \) on \( (V_{t_1} \otimes V_{t_2} \otimes V_{t_3})[[h]] \) in terms of the commutators of our basis elements from Lemma \( \boxed{28} \).

**Lemma 29** With \( t_i = (\lambda_i, \mu_i, \sigma_i) \) as in equations \( \boxed{61}, \boxed{62} \), and

\[
\begin{align*}
\end{align*}
\]

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\[ Y = (-1)^{\sigma_1 + \sigma_2 + \sigma_3} \in \text{End}_C((b_1, b_2, b_3)) \quad (b_i = ((-1)^{\sigma_i} \lambda_i, (-1)^{\sigma_i} \mu_i, \sigma_i)) \]

we have

\[ W(Y) = (\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3) \left[ (Y_{t_1, t_2} \circ \lambda_{t_1, t_2}) \otimes I_{t_3}, I_{t_1} \otimes (Y_{t_2, t_3} \circ \lambda_{t_2, t_3}) \right]. \]

**Proof:** With \( \omega = \sum_\nu a_\nu \otimes b_\nu \) as in equation (44) we compute

\[ 2 \sum_\nu (-1)^{\deg(v_1)} \deg(b_\nu) a_\nu \cdot v_{t_1} \otimes b_\nu \cdot v_{t_2} = (\lambda_1 (\mu_2 + 1) + \lambda_2 (\mu_1 + 1)) v_{t_1} \otimes v_{t_2}, \]
\[ 2 \sum_\nu (-1)^{\deg(w_1)} \deg(b_\nu) a_\nu \cdot w_{t_1} \otimes b_\nu \cdot w_{t_2} = (\lambda_1 (\mu_2 - 1) + \lambda_2 (\mu_1 - 1)) w_{t_1} \otimes w_{t_2}. \]

Equation (45) implies

\[ 2W\left((-1)^{\sigma_1 + \sigma_2} \left| \begin{array}{c} \end{array} \right| \right) = (a + b)(Y_{t_1, t_2} \circ \lambda_{t_1, t_2}) \otimes I_{t_3} + (a - b)(Y_{t_1, t_2} \circ \lambda_{t_1, t_2}) \otimes I_{t_3}, \]
\[ 2W\left((-1)^{\sigma_2 + \sigma_3} \left| \begin{array}{c} \end{array} \right| \right) = (c + d)I_{t_1} \otimes (Y_{t_2, t_3} \circ \lambda_{t_2, t_3}) + (c - d)I_{t_1} \otimes (Y_{t_2, t_3} \circ \lambda_{t_2, t_3}), \]

where \( a = \lambda_1 \mu_2 + \lambda_2 \mu_1, b = \lambda_1 + \lambda_2, c = \lambda_2 \mu_3 + \lambda_3 \mu_2, \) and \( d = \lambda_2 + \lambda_3. \) By Lemma 28 we have

\[ W(Y) = \left[ W\left((-1)^{\sigma_1 + \sigma_2} \left| \begin{array}{c} \end{array} \right| \right), W\left((-1)^{\sigma_2 + \sigma_3} \left| \begin{array}{c} \end{array} \right| \right) \right] \]
\[ = (1/4) ((a + b)(c + d) + (a - b)(c - d) - (a + b)(c - d) - (a - b)(c + d)) \]
\[ = bd[(Y_{t_1, t_2} \circ \lambda_{t_1, t_2}) \otimes I_{t_3}, I_{t_1} \otimes (Y_{t_2, t_3} \circ \lambda_{t_2, t_3})]. \]

This completes the proof. \( \Box \)

### 10 Proof of Theorem 4

Let \( T_3 = U_3 \circ A_3 \circ L_3 \) be the colored graph in Figure 9. The upper half \( U_3 \) of \( T_3 \) is mapped by \( \hat{\mathcal{W}} \circ \hat{Z} \) to the following morphism from \( (V_{t_1} \otimes V_{t_2} \otimes V_{t_3})[[h]] \) to \( V_s[[h]] \) (see equations (89), (90), (94), (98)):

\[ \text{(89), (90), (94), (98)} \]
\[ \widehat{W}(Z(U_3)) = \varphi(\lambda_8 h)^{1/2} \varphi(\lambda_1 h)^{-1/2} \varphi(\lambda_6 h)^{1/2} \varphi(\lambda_2 h)^{1/2} W(\Gamma(U_3)) \]
\[ = -\lambda_6 \varphi(\lambda_8 h)^{1/2} \varphi(\lambda_1 h)^{-1/2} \varphi(\lambda_6 h)^{1/2} \varphi(\lambda_2 h)^{1/2} \tilde{J}_{t_1, t_6} \circ (I_{t_1} \circ \tilde{J}_{2, t_3}). \]

For similar reasons the lower half \( L_3 \) of \( T_3 \) is mapped by \( \widehat{W} \circ Z \) to the following morphism from \( V_6[[h]] \) to \( (V_{t_1} \otimes V_{t_2} \otimes V_{t_3})[[h]] \):

\[ \widehat{W}(Z(L_3)) = \varphi(\lambda_1 h)^{1/2} \varphi(\lambda_7 h)^{1/2} W(\Gamma(L_3)) = \lambda_7 \varphi(\lambda_1 h)^{1/2} \varphi(\lambda_7 h)^{1/2} (Y_{t_1, t_2}) \otimes I_{t_3} \circ Y_{t_7, t_4}. \]

Since \( \text{End}_{gl(1)}(V_6) = \mathbb{Q} \mathbb{I}_6 \) there exists a formal power series \( x \in \mathbb{Q}[[h]] \) satisfying

\[ x \mathbb{I}_6 = \tilde{J}_{2, t_1} \circ (I_{t_1} \otimes \tilde{J}_{2, t_3}) \circ \widehat{W}(Z(A_3)) \circ (Y_{t_1, t_2} \otimes I_{t_3}) \circ Y_{t_7, t_4}. \]

We will determine \( x \) in two different ways. Using equations (74), (75), and (76) we compute

\[ \widehat{\nabla}(S_3)_{I_{t_1}} = \frac{h}{\lambda_8 \varphi(\lambda_8 h)} \widehat{W}(Z(T_3)) = -\frac{x \lambda_6 \lambda_7 \varphi(\lambda_6 h)^{1/2} \varphi(\lambda_2 h)^{1/2} \varphi(\lambda_7 h)^{1/2} h}{\lambda_8 \varphi(\lambda_8 h)^{1/2}} \mathbb{I}_4. \]

By Lemma 23 we have \( \widehat{\nabla}(S_3) = e^{-\lambda_2 h/2} - e^{\lambda_2 h/2} = -\lambda_2 h \varphi(\lambda_2 h) \) for the closure \( S_3 \) of \( T_3 \). Equation (77) implies

\[ x = \frac{\lambda_2 \lambda_8}{\lambda_6 \lambda_7} \sqrt{\frac{\varphi(\lambda_2 h) \varphi(\lambda_8 h)}{\varphi(\lambda_6 h) \varphi(\lambda_7 h)}} = \kappa \sqrt{\frac{\varphi(v) \varphi(u + v + w)}{\varphi(u + v) \varphi(v + w)}}, \]

with \( \kappa = \lambda_2 \lambda_8 / (\lambda_6 \lambda_7) \) as in Lemma 24 and \( u = \lambda_1 h, v = \lambda_2 h, w = \lambda_3 h \). Now we use equation (76) directly to derive an equation for \( x \) that depends on a Drinfeld associator. We start with a general remark. Let \( R \) be a commutative ring with 1. Let \( M_2(R[[h]]) \) be the algebra of \( 2 \times 2 \)-matrices over \( R[[h]] \). Let \( a, b \in hR[[h]] \) be elements of the augmentation ideal of \( R[[h]] \). Then we have

\[ \exp \left( \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \right) = \begin{pmatrix} \cosh(c) & a \sinh(c)/c \\ b \sinh(c)/c & \cosh(c) \end{pmatrix} \in M_2(R[[h]]), \]

where \( c^2 = ab \) (the result does not depend on the choice of a (formal) root \( c \) of \( c^2 \) because \( \cosh(c) \) and \( \sinh(c)/c \) are even power series in \( c \)). Notice that the bases of Lemma 24 and Corollary 25 establish isomorphisms.
Let $F \in \mathbb{Q}[[C, D, E]] \subset \mathbb{Q}[[d_1, d_2, d_3]]$ be the formal power series of Theorem 13. Let $Y$ be as in Lemma 29 with $\sigma_2 = 1$. Lemmas 28 and 29 imply

\[
F(u, v, w)\hat{W}(Y) = a (Y_{t_1, t_2} \otimes I_{t_3}) \circ Y_{t_3, t_3} \circ (\hat{\Delta}_{t_1, t_2} \otimes I_{t_3}) + b (Y_{t_1, t_2} \otimes I_{t_3}) \circ Y_{t_4, t_3} \circ (\hat{\Delta}_{t_1, t_2} \otimes I_{t_3}),
\]

where $a = (u + v) w F(u, v, w)$, \quad (81)

and $b = -u(v + w) \kappa F(u, v, w)$. \quad (82)

By equations (76), (50), Corollary 25, and equations (79) to (83) with $c_2 = ab = -uvw (u + v + w)^2$, we have

\[
e^2 = ab = -uvw (u + v + w)^2 F(u, v, w)^2 \quad (84)
\]

we have

\[
x I_{t_8} = \hat{\Delta}_{d_1, t_6} \circ (I_{t_1} \otimes \hat{\Delta}_{t_2, t_3}) \circ \hat{W}(\exp(F(d_1, -d_2, d_3) \cdot Y)) \circ (Y_{t_1, t_2} \otimes I_{t_3}) \circ Y_{t_7, t_3}
\]

\[
= (\kappa \hat{\Delta}_{t_7, t_3} \circ (\hat{\Delta}_{d_1, t_2} \otimes I_{t_3}) - \lambda_3/(\lambda_2 + \lambda_3) \hat{\Delta}_{t_4, t_3} \circ (\hat{\Delta}_{d_1, t_2} \otimes I_{t_3}))
\]

\[
\circ \exp(F(u, v, w)\hat{W}(Y)) \circ (Y_{t_1, t_2} \otimes I_{t_3}) \circ Y_{t_7, t_3}
\]

\[
= (\kappa \cosh(c) - (\lambda_3 b/(\lambda_2 + \lambda_3)) \sinh(c)/c) I_{t_8}
\]

\[
= \kappa (\cosh(c) + uw F(u, v, w) \sinh(c)/c) I_{t_8}. \quad (85)
\]

Equations (78) and (83) imply

\[
\cosh(c) + uw \sinh(c)/(c/F(u, v, w)) = \sqrt{\varphi(v)\varphi(u + v + w)/\varphi(u + v)\varphi(v + w)}. \quad (86)
\]

Since this formula holds for arbitrary values of $u, v, w \in \mathbb{Q}'h$ with $u + v \neq 0$, $v + w \neq 0$, and $u + v + w \neq 0$ we have proven the equation between formal power series stated in Theorem 1.

11 $\hat{\nabla}$ and Viro’s Alexander invariant

In this section we relate the Alexander series $\hat{\nabla}$ to Viro’s Alexander invariant $\Delta^1$. Proofs that are direct translations of proofs from [Vir] will only be sketched in what follows. We start with deriving relations between values of $\hat{\nabla}$ on different trivalent framed graphs with admissible coloring. When the colors and orientations of the
lower three edges of $\hat{\phi}$ are fixed, then by equation (31) there are two possible colors of the upper edge in an admissible coloring of a graph. In the first case the upper and lower edge point into the same direction, and we compute
\[
\hat{\nabla}(\phi) = \hat{\nabla}(\mathcal{O})^{-1}\hat{\nabla}(l)
\]
by using equations (53), (55), (57), (59), and Lemma 19. In the second case $W \circ Z_{\hat{\phi}}$ maps $\phi$ to $x\psi$ where $x \in \mathbb{Q}[[h]]$ and $\psi$ is a morphism between non-isomorphic simple $\text{gl}(1|1)$-modules. Therefore, we obtain
\[
\hat{\nabla}(\phi^c) = 0 \quad \text{if } c \neq d.
\]
Consider two parallel strands colored by $a = (\lambda, \mu), b = (\lambda', \mu')$ and define $s \in \{\pm 1\}$ (resp. $s' \in \{\pm 1\}$) iff the left (resp. right) strand points downwards. Then we have
\[
\hat{\nabla}(\phi) = \sum_c \hat{\nabla}(\mathcal{O}) \hat{\nabla}(x^c_{a \lambda}) \quad \text{if } s\lambda + s'\lambda' \neq 0,
\]
where the sum runs over the two colors $c$ such that the coloring of $x^c_{a \lambda}$ is admissible. In proofs of equation (89) and equation (90) below, we use equation (25) to show that the left sides of these equations are equal to linear combinations
\[
\sum_p x_p \hat{\nabla}(\chi_p),
\]
for certain $x_p \in \mathbb{Q}[[h]]$ and we use equation (87) to determine the coefficients $x_p$ (see the proof of 9.2.A in [Vir] for more details).

Now consider $a^b_{d \lambda} |_{c \mu}$ with colors $a, b, c, d, e \in \mathbb{Q}^\times \times \mathbb{Q}$. Let $a = (\lambda, \mu), b = (\lambda', \mu')$, and define $s, s' \in \{\pm 1\}$ as above. We assume that $s\lambda + s'\lambda' \neq 0$. Then
\[
\hat{\nabla}(a^b_{d \lambda}) = \sum_f \hat{\nabla}(a^b_{d \lambda}) \hat{\nabla}(\mathcal{O}) \hat{\nabla}(a^c_{e \mu}),
\]
where the sum runs over the one or two colors $f$ such that the coloring of $a^c_{e \mu}$ is admissible and $a^b_{d \lambda} \subset a^c_{e \mu}$ coincides with $a^c_{e \mu}$ as oriented colored graph. Let $i$ be the edge colored $c$ in $a^b_{d \lambda}$. When the restriction $s\lambda + s'\lambda' \neq 0$ is satisfied (resp. violated) we say that equation (90) can (resp. cannot) be applied to the edge $i$. 

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For a strand colored by $c = (\lambda, \mu)$ with a right-handed half-twist, it follows by a direct computation that

$$
\hat{\nabla}\left(\begin{smallmatrix} 1 \\ \lambda \\ \mu \\
\end{smallmatrix}\right) = e^{\lambda\mu h/4} \hat{\nabla}\left(\begin{smallmatrix} 1 \\
\end{smallmatrix}\right).$$

(91)

At a trivalent vertex $v$ of $G$ we obtain from part (2) of Theorem 20 and equation (43) that

$$
\hat{\nabla}\left(\begin{smallmatrix} \gamma \\ \overrightarrow{\alpha} \\
\end{smallmatrix}\right) = s_v \hat{\nabla}\left(\begin{smallmatrix} \gamma \\
\end{smallmatrix}\right).$$

(92)

We have collected all properties of $\hat{\nabla}$ that are needed to state the following proposition.

**Proposition 30** The invariant $\hat{\nabla}$ of embedded colored framed trivalent graphs $G$ is uniquely determined by its value on the trivial knot in equation (54), its values on planar tetrahedra in Lemma 22, and by the skein relations in equations (87) to (92).

**Proof:** Consider a diagram of $G$. We use equations (89) and (92) to replace each crossing in that diagram by a planar graph with two trivalent vertices (notice that despite the restriction in equation (89) this is always possible). By equation (91), we may assume that the resulting planar graph has blackboard framing.

Let $n$ be the number of connected components plus the number of trivalent vertices of a planar trivalent graph $G$. The connected components $F$ of $\mathbb{R}^2 \setminus G$ are called the faces of the diagram, and the trivalent vertices (resp. the edges) in the closure of a face $F$ are called the vertices (resp. edges) of that face. Let $\ell$ be the minimal number of vertices of a face of $G$.

We will prove the proposition by induction on the pairs $(n, \ell) \in \mathbb{N} \times \mathbb{N}_0$ with lexicographical order. For $\ell = 0$ and $n = 1$ the graph $G$ is a trivial knot. For $\ell = 0$ and $n > 1$, we use equation (89), the property $\sum_c \hat{\nabla}(\mathcal{O}) = 0$ where the sum runs over the same values of $c$ as the sum in equation (89), and equation (87) to show that $\hat{\nabla}(G) = 0$ in this case. For $\ell = 1$ the graph $G$ cannot have an admissible coloring, so we do not need to consider that case. In the case $\ell = 2$ we can apply equation (87) which will reduce $n$ by 2, or we can apply equation (88) to show that $\hat{\nabla}(G) = 0$. Now let $\ell \geq 3$.

**Case 1:** Assume that equation (90) can be applied to an edge of $F$. Then we can reduce $\ell$ by one while preserving $n$.

\footnote{For reasons of Euler characteristic we have $\ell \leq 5$ (see [BeS]), but we do not need this in the proof.}
Case 2: When equation (90) cannot be applied to an edge of $F$ we choose a trivalent vertex $v$ of $G$ that is connected to a vertex of the face $F$ by an edge $e$, and that is not itself a vertex of $F$. Such a vertex $v$ exists because $\ell > 2$ was minimal.

Case 2a: We assume that equation (90) can be applied to $e$. Then we proceed as shown schematically in Figure 2. Equation (90) can be applied to the two edges $i$, $j$ in the second step in Figure 2 because equation (90) could not be applied to an edge of $F$ in the first picture. This way we again decrease $\ell$ by 1 while preserving $n$.

![Figure 2: Reducing the number of vertices of a face](image)

Case 2b: When equation (90) cannot be applied to $e$, we proceed as follows. We first use equation (87) to add a bubble to one edge of $F$. This will increase the number of edges and vertices of $F$ by two. Let $e$ be the new edge of $F$ that belongs to the bubble, let $(\lambda, 0)$ be the color of $e$, and let $v$ be one vertex of $e$. As shown in Figure 3, our plan is to apply equation (90) $\ell$ times to push $v$ around $F$, and then to use equation (87) again to remove the bubble.

![Figure 3: Changing the colors of the edges of a face](image)

This will express $\widehat{\nabla}(G)$ as a linear combination of values $\widehat{\nabla}(G')$ on diagrams $G'$ of the same shape as $G$, but where orientations of the edges of the face $F$ may have changed and where the colors of these edges have changed by additive constants $(\pm \lambda, \pm 1)$. There are infinitely many possible choices of $\lambda \in \mathbb{Q}^*$ such that equation (90) can be applied $\ell$ times as needed above. For any such choice of $\lambda$, we can apply case 2a to all diagrams $G'$ as above. This completes the proof. □

Let $B = \mathbb{Q}[[h]]/\mathbb{Q}[[h^{-1}]]$ be the quotient field of $\mathbb{Q}[[h]]$, $M = \exp(\mathbb{Q}h) \subset B^*$, and $W = \mathbb{Q}$. Define $\beta : M \times W \longrightarrow M$ by $\beta(m, w) = m^w$. Let $G$ be a trivalent graph with admissible coloring. Define the colored graph $q(G)$ by replacing all colors $(\lambda, \mu)$ by $(\exp(\lambda h/4), \mu) \in M \times W$. Then the colors of $q(G)$ verify condition 2.8.A of [Vir] for the 1-palette $P = (B, M, W, \beta)$. Viro’s Alexander invariant $\Delta^1$ (see Section 6.3 of [Vir]), considered as a map $G \mapsto \widehat{\Delta}^1(q(G))$, verifies the same equations as $\widehat{\nabla}$ in Proposition 30 (see 7.2.D and Section 9 of [Vir]). This implies the following theorem.

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5In that case $\ell$ must be $\geq 4$ and even.
Theorem 31  For an admissibly colored framed trivalent graph $G \subset S^3$ we have
\[ \widehat{\nabla}(G) = \Delta^1(q(G)). \]

More generally, the invariants $\widehat{\nabla}(G)$ for all admissible colorings of $G$ determine Viro’s Alexander invariant of graphs with ‘universal’ colors as in Section 7.4 of [Vir]. Theorem 31 and 7.7.G of [Vir] imply the following relation between $\widehat{\nabla}(L)$ and the multi-variable Alexander polynomial $\nabla_L$.

Corollary 32  For a link $L$ with $n$ components colored by $(\lambda_i, 0)$ we have
\[ \nabla_L(e^{\lambda_1 h/2}, \ldots, e^{\lambda_n h/2}) = \widehat{\nabla}(L). \]

Corollary 32 can also be proven directly by using the characterization of $\nabla$ by axioms from [Tu1]. Except for some technical details this direct proof is a translation of the proof given in [Vir]. I learned about Corollary 32 from A. Vaintrob, but he never published his proof. Since $\widehat{\nabla}$ was defined using the universal Vassiliev invariant $Z$, Corollary 32 implies that the coefficient of $h^k$ in $\nabla_L(e^{\lambda_1 h/2}, \ldots, e^{\lambda_n h/2})$ is a Vassiliev invariant of degree $k + 1$ (this can also be proven directly using [Har] or [Tu1]).

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