Real-complex transition driven by quasiperiodicity: a new universality class beyond $\mathcal{PT}$ symmetric one

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We study a one-dimensional lattice model subject to non-Hermitian quasiperiodic potentials. Firstly, we strictly demonstrate that there exists an interesting dual mapping relation between $|a| < 1$ and $|a| > 1$ with regard to the potential tuning parameter $a$. The localization property of $|a| < 1$ can be directly mapping to that of $|a| > 1$, the analytical expression of the mobility edge of $|a| > 1$ is therefore obtained through spectral properties of $|a| < 1$. More impressively, we prove rigorously that even if the phase $\theta \neq 0$ in quasiperiodic potentials, the model becomes non-$\mathcal{PT}$ symmetric, however, there still exists a new type of real-complex transition driven by non-Hermitian disorder, which is a new universality class beyond $\mathcal{PT}$ symmetric class.

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I. INTRODUCTION

In quantum physics, the conservation of energy and probability demands that a closed system exhibits real energies, which leads to all physical observables must be represented by Hermitian operators. However, when an open quantum system couples to a surrounding environment, the Hamiltonian of the system becomes non-Hermitian and the physical process can be effectively described through a complex eigenenergy. Recently, non-Hermitian systems have been attracted growing interest motivated by experimental realization of photonic lattices and ultracold atomic gases. A celebrated paradigm in the non-Hermitian physics was parity-time ($\mathcal{PT}$) symmetric class, namely, a large class of non-Hermitian Hamiltonians, discovered by Bender and Boettcher, can exhibit entirely real spectra as long as they commute with the $\mathcal{PT}$ operator. From an intuitive point of view, a necessary (but not sufficient) condition for $\mathcal{PT}$ symmetry to preserve is that the involved complex potential should satisfy $V(n) = V^*(−n)$ in the discrete lattice, where $n$ is the lattice site number. Although the impact of $\mathcal{PT}$ symmetry in real quantum systems is still debated, broad research interests have been put into the study of $\mathcal{PT}$ symmetry-breaking transitions between real and complex eigenenergies in various non-Hermitian systems.

Here a natural question arises: can other class of non-Hermitian systems host entirely real spectra beyond $\mathcal{PT}$ symmetric class? Disordered systems provide a potential candidate. A paradigm to understand the Anderson localization in one dimension (1D) is the famous Aubry-André (AA) model, one of the most studied example displaying a localization-delocalization transition. Another interesting aspect of AA-like models is the presence of mobility edges separating extended from localized states. The combination of non-Hermitian properties and disordered systems is an interesting and active research topic. A seminal work dealing with disorder in non-Hermitian systems is the Hatano-Nelson model, in which an asymmetric hopping caused by an imaginary gauge field results in a localization-delocalization transition. Other non-Hermitian models with either random or quasiperiodic disorder have been investigated, in which the localization properties of systems are the main concerns and contents.

With regard to the real-complex transition of spectra, in a pioneering study, Hamazaki et al. numerically found that a real-complex transition occurred in non-Hermitian interacting systems without $\mathcal{PT}$ symmetry, the real energy region was located in the localized phase of the model with time-reversal symmetry. In the further research, Liu et al. numerically demonstrated that the real energy region was located in the extended phase of a quasiperiodic $p$-wave superconductor chain without $\mathcal{PT}$ symmetry. However, these available studies are limited to numerical simulations, there are very few rigorous results to clarify the underlying physics of the real energy. In this work, we are devoted to introduce an exactly solvable model and analytically demonstrate that there exists a new type of real-complex transitions driven by non-Hermitian disorder, which is different from $\mathcal{PT}$ symmetric class.

II. MODEL AND THE DUAL MAPPING

We consider one-dimensional non-Hermitian quasiperiodic model, described by the following eigenvalue equation,

$$E\psi_n = \psi_{n+1} + \psi_{n-1} + \frac{V}{1 - ae^{i(2\pi \alpha n + \theta)}} \psi_n,$$

where $V$ is the complex potential strength, $\alpha$ is irrational, and $\psi_n$ is the amplitude of wave function at the
Thus, spectral properties of the counterpart of the energy spectrum, which is unconventional and different from the real-complex transition in the energy spectrum, can be derived by the equation

\[ -V \frac{1}{1 - a e^{-i\theta}} + V, \]

thus, spectral properties of \( V \frac{1}{1 - a e^{-i\theta}} \) can be dual mapping to these of the potential \( V \frac{1}{1 - a e^{-i\theta}} + V \) (\(|a| > 1\)). Consequently, utilizing the mobility edge energy \( E_m = a + 1/a \) (\(|a| < 1\)), with regard to \(|a| > 1\) we can directly obtain the exact explicit form of the mobility edge:

\[ E_m = V - a - 1/a. \]

To support the analytical result above, we now present detailed numerical analysis of Eq. (1). In the disordered system, the localization property of wave functions can be measured by the inverse participation ratio (IPR)\(^\text{47}\). For any given normalized wave function, the corresponding IPR is defined as IPR = \( \sum_{n=1}^{N} |\psi_n|^4 \), which measures the inverse of the number of sites being occupied by particles. It is well known that the IPR of an extended state scales like \( L^{-\alpha} \) which approaches zero in the thermodynamic limit. However, for a localized state, since only finite number of sites are occupied, the IPR is finite even in the thermodynamic limit. In Fig. 1 we show the numerical IPR diagram in the \( \text{Re}(E), V \) plane, different colours of the eigenvalue curves indicate different magnitudes of the IPR of the corresponding wave functions. The black eigenvalue curves denote the extended states, and the bright yellow eigenvalue curves denote the localized states. It is clearly demonstrating a mobility edge separating localized from extended states along the blue line defined by Eq. (4).

III. REAL-COMPLEX TRANSITION

Besides the localization, the most novel discovery of this work is that there exists a new type of real-complex transitions in Eq. (1) driven by non-Hermitian disorder, rather than \( \mathcal{PT} \) symmetry.

For the sake of simplicity, here we focus on the mathematical proof of the \(|a| < 1\) case. First let us multiply both sides of Eq. (1) by \( e^{i\alpha k} \) and sum over \( n \). After setting \( f(k) = \sum_{n} e^{i\alpha k} \psi_n \), one obtains

\[ a[E - 2 \cos(k + 2\pi \alpha)]f(k + 2\pi \alpha) = [E - V - 2 \cos(k)]f(k), \]

According to Sarnak’s method\(^\text{18}\), the spectrum of Eq. (5) is govern by a characeristic function defined as

\[ G(E) = \frac{1}{2\pi} \int_0^{2\pi} \log |E - V - 2 \cos(k)| \, dk \]

Here, we don’t show more complicated mathematical proofs, but directly quote Sarnak’s Lemma\(^\text{48}\):

(i) There is no energy spectrum when \( G(E) < \log |a| \).

(ii) When \( G(E) > \log |a| \), Eq. (5) has dense spectrum, the corresponding wave functions are localized.

(iii) When \( G(E) > \log |a| \), wave functions of Eq. (5) are extended. However, the spectrum of the system must be within the energy interval \( U_E = [V - 2, V + 2] \), which is derived by the equation \( E - V - 2 \cos(k) = 0 \).
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and \( \Sigma_2 \) represents the minimum distance of two sets \( \Sigma_1 \) and \( \Sigma_2 \).
Thus, we obtain
\[
h(\Sigma_1, \Sigma_2) \leq \frac{2\pi V}{(1-a)^2} |\theta_1 - \theta_2 - n\omega_{R/Z}| \tag{9}
\]
where \( h \) represents the minimum distance of two sets \( \Sigma_1 \) and \( \Sigma_2 \). Because the frequency \( \alpha \) is an irrational number, we can take a list of \( n_k \) to make \( |\theta_1 - \theta_2 - n_k\omega_{R/Z}| \) approach zero, which means \( h = 0 \). Consequently, we strictly prove \( \Sigma_1 = \Sigma_2 \), irrelevant to \( \theta_1 \) and \( \theta_2 \). Thus even if \( \theta \neq 0 \), i.e., it doesn’t exist the \( \mathcal{PT} \) symmetry, the real spectrum of the system still stays the same, these results strongly demonstrate that the real-complex transition driven by \( \exp(iz) \)-like non-Hermitian disorder belongs to a unique class, rather than the \( \mathcal{PT} \) symmetric class.

In order to verify the theoretical analysis, we implement numerical calculations of eigenvalues. As shown in Fig. 2, even if the phase \( \theta \) is different, the eigenvalues of the system with the same \( V \) are same, and the mobility edge \( E_c = a + 1/a \) exactly separates the real from complex spectrum, thus numerical results are in excellent agreement with the analytical results. We have also checked other combinations of parameters and get the

![Figure 2](image-url)

Figure 2. (Color online) The eigenvalues of Eq. (1) with the parameter \( a = 0.5 \) under open boundary conditions. The total number of sites is set to be \( L = 500 \). (a) and (c) use the same potential strength \( V = 2 \) but different phase \( \theta \) for comparison. Similarly, (b) and (d) use the same potential strength \( V = 3 \) but different phase \( \theta \) for comparison. The blue solid lines represent the boundary between the real and complex energy spectrum, it is clearly shown that the real spectrum region of four cases is restricted in \( E_{\text{min}} = V - 2 \) and \( E_c = a + 1/a \).

![Figure 3](image-url)

Figure 3. (Color online) The eigenvalues of Eq. (1) with the parameter \( a = 0.5 \) under open boundary conditions, \( \alpha \) is set to a rational number 1/4. The total number of sites is set to be \( L = 500 \). In this situation, the \( \theta \neq 0 \) case has different spectrum with the \( \theta = 0 \) case, and the real-complex transition is within the logical framework of the \( \mathcal{PT} \) symmetry broken.

Hamiltonian as \( H_{\theta_1} \) and \( H_{\theta_2} \) when two phases are \( \theta_1 \) and \( \theta_2 \), respectively. All eigenvalues of \( H_{\theta_1} \) and \( H_{\theta_2} \) are denoted as \( \Sigma_1 \) and \( \Sigma_2 \), consequently.

Then for any integer \( n \), there exists the following estimate
\[
\|H_{\theta_1} - H_{\theta_2}\| \leq \frac{2V}{(1-a)^2} |\sin(\theta_1 - \theta_2 - nw)\pi| \leq \frac{2\pi V}{(1-a)^2} |\theta_1 - \theta_2 - n\omega_{R/Z}|, \tag{8}
\]
thus, we obtain
\[
h(\Sigma_1, \Sigma_2) \leq \frac{2\pi V}{(1-a)^2} |\theta_1 - \theta_2 - n\omega_{R/Z}|. \tag{9}
\]
same results as expected. In addition, there exist several deviated points in Fig. 2, this is because, not like finite periodic systems, disordered systems are statistical and based on large samples. The proportion of deviated points will decrease sharply and be almost negligible with the increase of sample size.

A possible reason of being classified as the $\mathcal{PT}$ symmetric class in previous works is to ignore the difference of the rational and irrational of the parameter $\alpha$. For $\alpha$ being rational, the system is perfectly periodic, in this situation the real-complex transition is indeed driven by the $\mathcal{PT}$ symmetry, see Fig. 3. When the phase $\theta = 0$, the system preserves the $\mathcal{PT}$ symmetry, the eigenvalues are real, while the phase $\theta \neq 0$, the system doesn’t preserve the $\mathcal{PT}$ symmetry, the eigenvalues are complex. For $\alpha$ being irrational, as we proved above, this real-complex transition violates the logical framework of the $\mathcal{PT}$ symmetry broken, hence can’t be classified as the $\mathcal{PT}$ symmetric class.

IV. SUMMARY

In summary, we have shown an interesting dual mapping between $|\alpha| < 1$ and $|\alpha| > 1$ in a non-Hermitian quasiperiodic model, the spectral properties of the two can be mapping to each other. More impressively, we mathematically prove that the eigenenergies in the extended phase are fully real, and an unconventional real-complex transition occurs accompanied by the metal-insulator transition. The meaning of unconventional is that even if our model doesn’t preserve the conventional $\mathcal{PT}$ symmetry, the real-complex transition in the spectrum also appears exactly as our theoretical predictions. Our findings act in cooperation with the previous numerical results of non-Hermitian disorder, and establish that the real-complex transition driven by non-Hermitian disorder is a new universality class, which is different from $\mathcal{PT}$ symmetric class.

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