On Weakly Complete Group Algebras
of Compact Groups

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Abstract. A topological vector space over the real or complex field $\mathbb{K}$ is weakly complete if it is isomorphic to a power $\mathbb{K}^J$. For each topological group $G$ there is a weakly complete topological group Hopf algebra $\mathbb{K}[G]$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, for which three insights are contributed: Firstly, there is a comprehensive structure theorem saying that the topological algebra $\mathbb{K}[G]$ is the cartesian product of its finite dimensional minimal ideals whose structure is clarified. Secondly, for a compact abelian group $G$ and its character group $\hat{G}$, the weakly complete complex Hopf algebra $\mathbb{C}[G]$ is the product algebra $\mathbb{C}[\hat{G}] = \mathbb{C}[\{\xi, \eta\}]$ while the vector subspace of primitive elements is $\text{Hom}(\hat{G}, (\mathbb{C}, +))$. This forces the group $G(\mathbb{R}[G]) \subseteq \mathbb{G}(\mathbb{C}[G])$ to be $\text{Hom}(\hat{G}, S^1) = \mathbb{G} = G$ with the complex circle group $S^1$. While the relation $G(\mathbb{R}[G]) \cong G$ remains true for any compact group, $\mathbb{G}(\mathbb{C}[G]) \cong G$ holds for a compact abelian group $G$ if and only if it is profinite. Thirdly, for each pro-Lie algebra $L$ a weakly complete universal enveloping Hopf algebra $U_{\mathbb{K}}(L)$ over $\mathbb{K}$ exists such that for each connected compact group $G$ the weakly complete real group Hopf algebra $\mathbb{R}[G]$ is a quotient Hopf algebra of $U_{\mathbb{R}}(\mathcal{L}(G))$ with the (pro-)Lie algebra $\mathcal{L}(G)$ of $G$. The group $\mathbb{G}(U_{\mathbb{R}}(\mathcal{L}(G)))$ of grouplike elements of the weakly complete enveloping algebra of $\mathcal{L}(G)$ maps onto $G(\mathbb{R}[G]) \cong G$ and is therefore nontrivial in contrast to the case of the discrete classical enveloping Hopf algebra of an abstract Lie algebra.

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1. Introduction

For much of the material surrounding a theory of group Hopf algebras in the category of weakly complete real or complex vector spaces we refer to [3] and the forthcoming fourth edition of [6]. Some additional facts are presented here. A topological vector space over a locally compact field $\mathbb{K}$ is called weakly complete if it is isomorphic to $\mathbb{K}^J$ for some set $J$. This text considers $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ only and deals

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with the categories $\mathcal{G}$ of topological groups, respectively, $\mathcal{WA}$ of weakly complete topological algebras. Inside each $\mathcal{WA}$-object $A$ we have the subset $A^{-1}$ of all units (i.e., invertible elements) which turns out to be a $\mathcal{G}$-object in the subspace topology. In [3] it was shown that the functor $A \to A^{-1} : \mathcal{WA} \to \mathcal{G}$ has a left adjoint functor $G \mapsto \mathbb{K}[G] : \mathcal{G} \to \mathcal{WA}$. Automatically, $\mathbb{K}[G]$ is a topological Hopf algebra. Then for each topological group $G$ there is a $\mathcal{G}$-morphism $\eta_G : G \to \mathbb{K}[G]^{-1}$ such that for each $\mathcal{WA}$-object $A$ and $\mathcal{G}$-morphism $f : G \to A^{-1}$ there is a unique $\mathcal{WA}$-morphism $f' : \mathbb{K}[G] \to A$ such that $f(g) = f'(\eta_G(g))$. The weakly complete algebra $A = \mathbb{K}[G]$ is seen to have a comultiplication $c : A \to A \otimes A$ making it into a symmetric Hopf algebra. The subset

$$\mathbb{G}(A) = \{a \in A^{-1} : c(a) = a \otimes a\}$$

is a subgroup of $A^{-1}$ and its elements are called grouplike. If $G$ is a discrete group, then $\mathbb{K}[G]$ is the traditional group algebra over $\mathbb{K}$ and $\eta_G$ is an embedding and its image is $\mathbb{G}(\mathbb{K}[G])$. In [3] it was shown that for a compact group $G$ the map $\eta_G$ is an embedding and for $\mathbb{K} = \mathbb{R}$ induces an isomorphism onto $\mathbb{G}(\mathbb{R}[G])$. We shall see in this text that this fails over the complex ground field even for all compact abelian groups which are not profinite. The assignment $G \mapsto \mathbb{G}(\mathbb{C}[G])$ may be viewed as ‘complexification’ of the compact group $G$. This viewpoint becomes important if one considers the module category of $\mathbb{C}[G]$ in $\mathcal{W}$, i.e. the representations of $G$ on weakly complete complex vector spaces, but we will not pursue this in the present note.

Since we shall deal with compact groups throughout this paper, we shall always consider $G$ as a subgroup of the group $\mathbb{K}[G]^{-1}$ of units of $\mathbb{K}[G]$. In particular, this means $G \subseteq \mathbb{R}[G] \subseteq \mathbb{C}[G]$.

The article [3] and the 4th Edition of the book [6] identify a category $\mathcal{H}$ of weakly complete topological Hopf algebras for which the functor $G \to \mathbb{K}[G]$ implements an equivalence of the category of compact groups and the category $\mathcal{H}$. The vector space continuous dual of $\mathbb{K}[G]$ turns out to be the traditional representation algebra $R(G, \mathbb{R})$. This approach yields a new access to the Tannaka–Hochschild duality of the categories of compact groups and “reduced” real Hopf algebras.

In all of this, the precise nature of the weakly complete Hopf algebras $\mathbb{K}[G]$ even for compact groups remained somewhat obscure in the nondiscrete case. The present paper will present a precise piece of information on the topological algebra structure of $\mathbb{K}[G]$ in terms of a direct product of its finite dimensional minimal ideals whose precise structure links this presentation with the classical information of finite dimensional $G$-modules (cf. e.g. [6], Chapters 3 and 4). Certain complications have to be overcome on that level if one insists on an explicit identification of the algebra and ideal structure of $\mathbb{R}[G]$ as the case $\mathbb{C}[G]$ is easier.

For abelian compact groups $G$ we shall present a very direct access to the structure of the weakly complete Hopf algebra of $\mathbb{C}[G]$ by identifying an isomorphism between $\mathbb{C}[G]$ and $\mathbb{C}^G$ and by further identifying the group of grouplike elements to be isomorphic to $(\mathcal{L}(G), +) \oplus G$ with the pro-Lie algebra $\mathcal{L}(G)$ of $G$ (see [6]). The subgroup $G$ of $\mathbb{C}[G]^{-1}$ therefore agrees with the group of grouplike elements of $\mathbb{C}[G]$ if and only if $\mathcal{L}(G) = \{0\}$ if and only if $G$ is totally disconnected.

The presence of weakly complete Lie algebras over $\mathbb{K}$ in the group $\mathbb{K}$-Hopf algebra of a compact group motivates a proof of the existence of a weakly complete universal
enveloping algebra over $\mathbb{K}$ for $\mathcal{W}A$ Lie algebras over $\mathbb{K}$. In contrast to the case of enveloping algebras on the purely algebraic side, the $\mathcal{W}A$ enveloping Hopf algebras will sometimes have grouplike elements. The universal property of the $\mathcal{W}A$ enveloping Hopf algebras will show that each $\mathcal{W}A$ Hopf-group algebra $k[G]$ of a compact connected group $G$ is a quotient algebra of the $\mathcal{W}A$ enveloping algebra of $\mathcal{L}(G)$. So $\mathcal{W}A$ enveloping algebras have a tendency of being larger than $\mathcal{W}A$ group algebras.

For a special class of profinite dimensional Lie algebras a similar but different approach to appropriate enveloping algebras is considered in [4].

A preprint of the present material appeared in the Series of Preprints of the Mathematical Research Institute of Oberwolfach [5].

2. Weakly Complete Hopf Algebras

For the basic theory of weakly complete Hopf algebras we may safely refer to [3] and [6], 4th Edition. For the present discussion we need a reminder of some basic concepts.

**Definition 2.1.** Let $A$ be a weakly complete symmetric Hopf algebra, i.e. a group object in the monoidal category $(\mathcal{W}, \otimes)$ of weakly complete vector spaces (see [6], Appendix 7 and Definition A3.62), with comultiplication $c: A \rightarrow A \otimes A$ and coidentity $k: A \rightarrow \mathbb{K}$.

An element $a \in A$ is called **grouplike** if $c(a) = a \otimes a$ and $k(a) = 1$. The subgroup of grouplike elements in the group of units $A^{-1}$ will be denoted $\mathbb{G}(A)$.

An element $a \in A$ is called **primitive**, if $c(a) = a \otimes 1 + 1 \otimes a$. The Lie algebra of primitive elements of $A_{\text{Lie}}$, i.e. the weakly complete Lie algebra obtained by endowing the weakly complete vector space underlying $A$ with the Lie bracket obtained by $[a, b] = ab - ba$, will be denoted $\mathcal{P}(A)$.

Any weakly complete symmetric Hopf algebra $A$ has an exponential function $\exp_A: A \rightarrow A^{-1}$ as explained in [3], Theorem 3.12 or in [6], 4th Edition, A7.41.

**Theorem 2.2.** Let $A$ be a weakly complete symmetric Hopf algebra. Then the following statements hold:

(i) The set $\mathbb{G}(A)$ of grouplike elements of a weakly complete symmetric Hopf algebra $A$ is a closed subgroup of $(A, \cdot)$ and therefore is a pro-Lie group.

(ii) The set $\mathcal{P}(A)$ of primitive elements of $A$ is a closed Lie subalgebra of $A_{\text{Lie}}$ and therefore is a pro-Lie algebra.

(iii) $\mathcal{P}(A) \cong \mathcal{L}(\mathbb{G}(A))$ and the exponential function $\exp_A$ of $A$ induces the exponential function $\exp_{\mathbb{G}(A)}: \mathcal{P}(A) \rightarrow \mathbb{G}(A)$ of the pro-Lie group $\mathbb{G}(A)$.

For a proof see e.g. [3], Theorem 6.15.

**Definition 2.3.** For an arbitrary topological group $G$ we define $R(G, \mathbb{K}) \subseteq C(G, \mathbb{K})$ to be that set of continuous functions $f: G \rightarrow \mathbb{K}$ for which the linear
span of the set of translations \( g f, g f(h) = f(hg) \), is a finite dimensional vector subspace of \( C(G, \mathbb{K}) \). The functions in \( R(G, \mathbb{K}) \) are called representative functions.

Clearly \( R(G, \mathbb{K}) \) is a subalgebra of \( C(G, \mathbb{K}) \) also known as the representation algebra of \( G \). In [3], Theorem 7.7(a) the following duality result was shown.

**Theorem 2.4.** (The Dual of a Weakly Complete Group Algebra \( \mathbb{K}[G] \))

For an arbitrary topological group \( G \), the function

\[
F_G : \mathbb{K}[G]^\prime \to \mathcal{R}(G, \mathbb{K}), \quad F_G(\omega) = \omega \circ \eta_G
\]

is a natural isomorphism of Hopf algebras.

This applies, of course, to compact groups, in which case the Hopf algebra \( R(G, \mathbb{K}) \) is a well-known object.

For easy reference we record the following facts in the case of a compact group \( G \) for which we recall \( G \subseteq \mathbb{K}[G] \):

**Theorem 2.5.** For any compact topological group \( G \), the following statements hold:

(i) We have \( G \subseteq \mathbb{G}(\mathbb{K}(G)) \subseteq \mathbb{K}[G]^{-1} \)

(ii) In the case of \( \mathbb{K} = \mathbb{R} \) the equality \( G = \mathbb{G}(\mathbb{R}[G]) \) holds.

For (1) see [3], 5.4, and for (ii) see [3], 8.7.

For the complex case, we shall see later in this paper that in many cases, a compact group \( G \) is a proper subgroup of \( \mathbb{G}(\mathbb{C}[G]) \).

### 3. Some Preservation Properties of \( \mathbb{K}[-] \)

Let us explicitly formulate and prove some preservation properties of our functor \( \mathbb{K}[-] \). Left adjoint functors preserve epics. A morphism of compact groups is an epimorphism if and only if it is surjective (see [6], RA3.17). Therefore the following lemma is to be expected.

**Lemma 3.1.** For every surjective morphism \( f : G \to H \) of compact groups the morphism \( \mathbb{K}[f] : \mathbb{K}[G] \to \mathbb{K}[H] \) of weakly complete \( \mathbb{K} \)-Hopf algebras is surjective.

**Proof.** From the surjectivity of \( f : G \to H \) we conclude that

\[
f(\text{span}(G)) = \text{span}(f(G)) = \text{span}(H)
\]

is dense in \( \mathbb{K}[H] \) by Proposition 5.3 of [3], and likewise \( \mathbb{K}(f)(\mathbb{K}[G]) \) is dense in \( \mathbb{K}[H] \). But \( \mathbb{K}[f] \) is, in particular, a \( \mathcal{W} \)-morphism, that is, a morphism of weakly complete vector spaces. Every such has a closed image by [6], Theorem 7.30(iv). Hence \( \mathbb{K}(f)(\mathbb{K}[G]) = \mathbb{K}[H] \).

This particular left adjoint functor \( \mathbb{K}[-] \), however, also preserves the injectivity of morphisms:
Theorem 3.2. If \( G \) is a closed subgroup of the compact group \( H \), then \( \mathbb{K}[G] \subseteq \mathbb{K}[H] \) (up to natural isomorphism).

Proof. From the injectivity of a morphism of compact groups \( j: G \to H \) we derive the surjectivity of \( C(j, \mathbb{K}): C(H, \mathbb{K}) \to C'(G, \mathbb{K}) \) by the Tietze Extension Theorem. Now we set \( M := C(j, \mathbb{K})(R(H, \mathbb{K})) \subseteq R(G, \mathbb{K}) \). Since \( R(H, \mathbb{K}) \) is dense in \( C'(H, \mathbb{K}) \) in the norm topology, \( M \) is dense in \( R(G, \mathbb{K}) \) in the norm topology. Then it is dense in \( L^2(G, \mathbb{K}) \) in the \( L^2 \)-topology, and \( M \) is a \( G \)-module. In the case of \( \mathbb{K} = \mathbb{R} \) we can now apply Lemma 8.11 of [3] and conclude that \( M = R(G, \mathbb{R}) \). Thus \( R(G, j): R(H, \mathbb{R}) \to R(G, \mathbb{R}) \) is surjective. By Theorem 7.7 of [3] this implies that \( \mathbb{R}[j']: \mathbb{R}[H'] \to \mathbb{R}[G]' \) is surjective. The duality between \( \mathbb{K} \)-vector spaces and weakly complete \( \mathbb{K} \)-vector spaces shows that \( \mathbb{R}[j]: \mathbb{R}[G] \to \mathbb{R}[H] \) is injective. This proves the theorem for \( \mathbb{K} = \mathbb{R} \). But then the commuting diagram

\[
\begin{array}{ccc}
\mathbb{C} \otimes \mathbb{R}[G] & \xrightarrow{C \otimes \mathbb{R}[j]} & \mathbb{C} \otimes \mathbb{R}[H] \\
\cong & & \cong \\
\mathbb{C}[G] & \xrightarrow{C[j]} & \mathbb{C}[H]
\end{array}
\]

shows that \( \mathbb{C}[j] \) is also injective.

In the category of weakly complete vector spaces every injective morphism is an embedding by duality since every surjective morphism of vector spaces is a coretraction.

Corollary 3.3. Let \( G_0 \) denote the identity component of the compact group \( G \). Then

(i) The Hopf algebra \( \mathbb{K}[G_0] \) is a Hopf subalgebra of \( \mathbb{K}[G] \).

(ii) \( \mathbb{R}[G_0] \) is algebraically and topologically generated by \( \mathcal{P}(\mathbb{R}[G]) \cong \mathcal{L}(G) \).

Proof. (i) is a consequence of Theorem 3.2.

(ii) The compact group \( G_0 \) is algebraically and topologically generated by \( \exp_G(\mathcal{L}(G)) \) (cf. [7], Corollary 4.22, p. 191), and \( \text{span}(G_0) = \mathbb{R}[G_0] \) by [3], Corollary 5.3.

4. A Principal Structure Theorem of \( \mathbb{K}[G] \)

Let \( G \) be a compact group and let \( E \) be a finite dimensional vector space over \( \mathbb{K} \in \{ \mathbb{R}, \mathbb{C} \} \). We recall that the character of a representation \( \rho: G \to \text{End}_{\mathbb{K}}(E) \) is the continuous map \( g \mapsto \text{tr}_{\mathbb{K}}(\rho(g)) \). We also say that \( \chi \) is the character of the \( G \)-module \( E \). A representation, respectively, a \( G \) module, is determined by its character up to isomorphism. A character is called irreducible if the corresponding representation is irreducible (over the ground field \( \mathbb{K} \)), equivalently, the corresponding \( G \)-module is simple. We denote the set of all irreducible characters of \( G \) over \( \mathbb{K} \) by \( \widehat{G}_{\mathbb{K}} \). For each character \( \chi \) of \( G \), we select a finite dimensional \( G \)-module \( E_{\chi,\mathbb{K}} \) having \( \chi \) as its character. If \( \varepsilon \) is an irreducible character, then the ring \( \mathbb{L}_{\varepsilon,\mathbb{K}} = \text{End}_{G}(E_{\varepsilon,\mathbb{K}}) \) of all
\( K \)-linear endomorphisms of \( E_{\varepsilon, K} \) which commute with the \( G \)-action is, by Schur’s Lemma, a finite dimensional division ring over \( K \). Hence

\[
L_{\varepsilon, K} = \mathbb{C} \quad \text{if} \quad K = \mathbb{C},
\]
\[
L_{\varepsilon, K} \in \{ \mathbb{R}, \mathbb{C}, \mathbb{H} \} \quad \text{if} \quad K = \mathbb{R},
\]

where \( \mathbb{H} \), as is usual, denotes the skew-field of quaternions. We view \( E_{\varepsilon, K} \) as a right module over \( L_{\varepsilon, K} \). We denote the corresponding representation by \( \rho_{\varepsilon, K} : G \to \text{End}_{\mathbb{K}}(E_{\varepsilon, K}) \subseteq \text{End}_{K}(E_{\varepsilon, K}) \).

Before we enter the presentation of the principal theorem on the weakly complete group algebra \( K[G] \) of a compact group we elaborate on some basic ideas of finite dimensional representation theory, indeed extending some of the presentation such as it can be found e.g. in Chapter 3 of [6]. The first lemma extends the details of Proposition 3.21 of [6] and the comments which precede it.

**Lemma 4.1.** Let \( E \) be a finite dimensional vector space over \( K \) and \( \rho : G \to \text{End}_{K}(E) \) an irreducible representation of a group \( G \). Let \( A \) denote the \( K \)-span of the set \( \{ \rho(g) \mid g \in G \} \). Then \( A = \text{End}_{L}(E) \), where \( L = \text{End}_{A}(E) = \text{End}_{G}(E) \), \( L \in \{ \mathbb{R}, \mathbb{C}, \mathbb{H} \} \).

**Proof.** First of all we note that \( A \) is a \( K \)-algebra containing \( \text{id}_{E} \). Hence every \( A \)-submodule of the additive group \( E \) is a \( G \)-invariant linear subspace, and vice versa. Therefore \( E \) is a simple \( A \)-module and so Jacobson’s Density Theorem applies, which, for the sake of completeness, we cite here in its entirety (see e.g. [2]).

**Jacobson’s Density Theorem.** Let \( M \neq 0 \) be an (additive) abelian group, let \( A \subseteq \text{End}(M) \) be a subring and suppose that \( M \) is simple as a left \( A \)-module. Put \( L = \text{End}_{A}(M) \). Then \( L \) is a division ring and \( M \) is a right \( L \)-module in a natural way. For every \( 2k \)-tuple \( (x_1, \ldots, x_k, y_1, \ldots, y_k) \in M^{2k} \), such that the elements \( x_1, \ldots, x_k \) are linearly independent, there exists \( a \in A \) such that \( a(x_i) = y_i \) holds for all \( i = 1, \ldots, k \).

Now the division ring \( L \) is a finite dimensional \( K \)-algebra over \( K \), and hence is isomorphic to \( \mathbb{R}, \mathbb{C}, \) or \( \mathbb{H} \). Moreover, \( A \subseteq \text{End}_{L}(E) \). Let \( x_1, \ldots, x_m \) be a \( L \)-basis for \( E \), and let \( \phi \in \text{End}_{L}(E) \) be arbitrary. Then there exists an element \( a \in A \) such that \( a(x_i) = \phi(x_i) \) holds for all \( i = 1, \ldots, m \). Therefore \( \phi = a \) and thus \( \text{End}_{L}(E) = A \).

The following result now extends [6], Lemma 3.14.

**Lemma 4.2.** Let \( E \) and \( F \) be finite dimensional vector spaces over \( K \) and suppose that \( \rho : G \to \text{End}_{K}(E) \) and \( \sigma : H \to \text{End}_{K}(F) \) are irreducible representations of groups \( G, H \). Suppose also that \( \text{End}_{G}(E) = L = \text{End}_{H}(F) \). Then \( \text{Hom}_{L}(F, E) \) is an irreducible \( G \times H \)-module over \( K \), where \( (g, h)(f) = \rho(g) \circ f \circ \sigma(h^{-1}) \).

**Proof.** We define \( A \subseteq \text{End}_{K}(E) \) and \( B \subseteq \text{End}_{K}(F) \) as in Corollary 4.1. Then \( A = \text{End}_{L}(E) \) and \( B = \text{End}_{L}(F) \). The \( K \)-vector space \( \text{Hom}(F, E) \) is in a natural way a right \( A \)-module and a left \( B \)-module. For every nonzero \( f \in \text{Hom}_{L}(F, E) \) we have \( AfB = \text{Hom}_{L}(F, E) \). Therefore \( \text{Hom}_{L}(F, E) \) is simple as \( G \times H \)-module over \( K \).
We are now ready to prove a principal structure theorem for the weakly complete group algebra $\mathbb{K}[G]$ of a compact group $G$ for either $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. For each $\varepsilon \in \hat{G}_\mathbb{K}$ we have the $G$-module $E_\varepsilon$ and the corresponding irreducible representation $\rho_\varepsilon : G \to \text{End}_\varepsilon (E_\varepsilon)$ into the group of units of the concrete matrix ring $M_\varepsilon := \text{End}_\varepsilon (E_\varepsilon)$ over $\mathbb{L} = L_\varepsilon$ of $\mathbb{L}$-dimension $(\text{dim}_L E_\varepsilon)^2$. Accordingly there is a unique function $\rho_G : G \to \prod_{\varepsilon \in \hat{G}_\mathbb{K}} M_\varepsilon$ which is an injective group morphism into the multiplicative group of units of the product defined by the universal property of the product such that

$$
\begin{array}{ccc}
G & \xrightarrow{\rho_G} & \prod_{\varepsilon \in \hat{G}_\mathbb{K}} M_\varepsilon \\
\downarrow & & \downarrow \text{pr}_\chi \\
G & \xrightarrow{\rho_\chi} & M_\chi
\end{array}
$$

commutes for all $\chi \in \hat{G}_\mathbb{K}$.

**Theorem 4.3.** For any compact group $G$ the weakly complete symmetric Hopf algebra $\mathbb{K}[G]$ is a direct product

$$
\mathbb{K}[G] = \prod_{\varepsilon \in \hat{G}_\mathbb{K}} \mathbb{K}_\varepsilon[G]
$$

of finite dimensional minimal two-sided ideals $\mathbb{K}_\varepsilon[G]$ such that for each $\varepsilon \in \hat{G}_\mathbb{K}$ there is a $\mathbb{K}$-algebra isomorphism

$$
\mathbb{K}_\varepsilon[G] \cong M_\varepsilon = \text{End}_{L_\varepsilon}(E_\varepsilon).
$$

In particular, each of these two-sided ideals $\mathbb{K}_\varepsilon[G]$ is a two sided simple $G \times G$-module and as an algebra is isomorphic to a full matrix ring over $\mathbb{L}$.

**Remark 4.4.** The diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\eta_G} & \mathbb{K}[G] \\
\downarrow & & \downarrow \cong \\
G & \xrightarrow{\rho_G} & \prod_{\varepsilon \in \hat{G}_\mathbb{K}} M_\varepsilon \\
\downarrow & & \downarrow \text{pr}_\chi \\
G & \xrightarrow{\rho_\chi} & M_\chi
\end{array}
$$

commutes for all $\chi \in \hat{G}_\mathbb{K}$.

**Proof.** By Theorem 2.4 and [6] Theorem 3.28, the topological dual $\mathbb{K}[G]' \cong R(G, \mathbb{K})$ is the direct sum of the finite dimensional two-sided $G$-submodules $R_\varepsilon(G, \mathbb{K})$ as $\varepsilon$ ranges through the set of irreducible characters in $\hat{G}_\mathbb{K}$. The $G \times G$-module $R_\varepsilon(G, \mathbb{K})$ is defined in [6] as the image of the linear map

$$
\phi : E_\varepsilon' \otimes \mathbb{K} E_\varepsilon \to R(G, \mathbb{K}),
$$

where

$$
\phi(u \otimes v)(g) = \langle u, \rho_\varepsilon(g)v \rangle.
$$
If we put $\psi(f)(g) = \text{tr}_K(\rho_e(\chi)(f))$, for $f \in \text{End}_K(E,e)\chi$ and $g \in G$, then the diagram

$$
\begin{array}{c}
E'_e \otimes_K E_e \xrightarrow{\phi} R_e(G, \mathbb{K}) \\
s \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
\text{End}_K(E_e) \xrightarrow{\psi} R_e(G, \mathbb{K})
\end{array}
$$

commutes, where $s(u \otimes v) = [w \mapsto v(u, w)]$. We recall that group $G \times G$ acts on $R_e(G, \mathbb{K})$ via $(a, b)(\lambda) = [g \mapsto \lambda(a^{-1}gb)]$. If we put $(a, b)(u \otimes v) = (u \circ \rho_e(a^{-1})) \otimes \rho_e(b)v$ and $(a, b)(f) = \rho_e(a^{-1}) f \circ \rho_e(a^{-1})$, then all maps in this diagram are $G \times G$-equivariant.

Suppose that $\mathbb{K} = \mathbb{L}$. Then $\text{End}_K(E_e, \mathbb{K}) = \text{End}_L(E_e, \mathbb{K})$ is simple as a $G \times G$-module by Lemma 4.2 and thus $\psi$ is an isomorphism.

Suppose next that $\mathbb{K} \subset \mathbb{L}$. Then $\mathbb{K} = \mathbb{R}$ and $\mathbb{L} = \mathbb{C}$ or $\mathbb{L} = \mathbb{H}$. By the averaging process in [6] Lemma 2.15 there exists a $G$-invariant positive definite $\mathbb{L}$-hermitian form $(\cdot|\cdot)$ on $E$, semilinear in the first argument and linear in the second argument. This allows us to rewrite $R_e(G, \mathbb{K})$ as the span of the maps $g \mapsto \text{Re}(u|gv)$, for $u, v \in E$. The $G$-invariance of $(\cdot|\cdot)$ yields that $\text{Re}(au|gbv) = \text{Re}(u|a^{-1}gbv)$. If we consider the algebra inclusion

$$
\psi : \text{End}_L(E_e, \mathbb{K}) \to \text{End}_K(E_e, \mathbb{K}),
$$

then $\text{Re}(u|v) = \text{tr}_K[w \mapsto v(u, w)]$ holds for the trace map of $\text{End}_K(E_e, \mathbb{K})$. It follows that the map $\psi \circ j$ in the diagram

$$
\begin{array}{c}
E'_e \otimes_K E_e \xrightarrow{\phi} R_e(G, \mathbb{K}) \\
s \downarrow \quad \quad \quad \downarrow \\
\text{End}_K(E_e) \xrightarrow{\psi} R_e(G, \mathbb{K})
\end{array}
$$

is surjective and $G \times G$-equivariant. Since $\text{End}_L(E_e, \mathbb{K})$ is a simple $G \times G$-module over $\mathbb{K}$ by Lemma 4.2, the map $\psi \circ j$ is an isomorphism.

For the remaining part of the proof we apply standard duality theory. We put

$$
R^\chi = \bigoplus_{\chi \neq e \in G_K} R_e(G, \mathbb{K})
$$

and define $\mathbb{K}_\chi[G]$ as the annihilator of $R^\chi$. The annihilator mechanism supplies us with the diagram

$$
\begin{array}{c}
\mathbb{K}[G] \leftrightarrow \{0\} \\
\mathbb{K}_\chi[G] \leftrightarrow R^\chi \xrightarrow{\cong} \{R_e(G, \mathbb{K})\} \\
\{0\} \leftrightarrow R(G, \mathbb{K}).
\end{array}
$$

By the duality of $\mathcal{V}_\mathbb{K}$ and $\mathcal{W}_\mathbb{K}$ it follows that $\mathbb{K}[G] \cong \prod_{\chi \in \hat{G}_K} \mathbb{K}_\chi[G]$ with

$$
\mathbb{K}_e[G] \cong R_e(G, \mathbb{K})'.
$$
Now, if any closed vector subspace $J$ of $\mathbb{K}[G]$ satisfies $G \cdot J \subseteq J$ and $J \cdot G \subseteq J$, then we also have $\text{span}(G) \cdot J \subseteq J$ and $J \cdot \text{span}(G) \subseteq J$ (where we view $G$ as a subset of $\mathbb{K}[G]$). Then Proposition 5.3 of [3] says that $\text{span}(G) = \mathbb{K}[G]$, and so $\mathbb{K}[G] \cdot J \subseteq J$ and $J \cdot \mathbb{K}[G] \subseteq J$. That is, $J$ is a closed two-sided ideal of $\mathbb{K}[G]$. Therefore each $\mathbb{K}_\varepsilon[G]$ is a two-sided ideal in $\mathbb{K}[G]$. It remains to clarify the multiplicative structure of the ideals $\mathbb{K}_\varepsilon[G]$. If we consider $\varepsilon \in \hat{G}_\mathbb{K}$ and the representation $\rho_{\varepsilon,\mathbb{K}}$, then the map

$$G \xrightarrow{\rho_{\varepsilon,\mathbb{K}}} \text{Gl}_{\mathbb{L}_\varepsilon,\mathbb{K}}(E_{\varepsilon,\mathbb{K}}) \xrightarrow{\text{inc}} \text{End}_{\mathbb{L}_\varepsilon,\mathbb{K}}(E_{\varepsilon,\mathbb{K}})$$

and the universal property of $\mathbb{K}[G]$ described in the Weakly Complete Group Algebra Theorem 5.1 of [3] provides a morphism of weakly complete algebras

$$\pi_\varepsilon : \mathbb{K}[G] \rightarrow \text{End}_{\mathbb{L}_\varepsilon,\mathbb{K}}(E_{\varepsilon,\mathbb{K}})$$

extending $\rho_{\varepsilon,\mathbb{K}}$. We also have the product projection of weakly complete algebras $\text{pr}_\varepsilon : \mathbb{K}[G] \rightarrow \mathbb{K}_\varepsilon[G]$. Both maps $\pi_\varepsilon$ and $\text{pr}_\varepsilon$ have the same kernel $\prod_{\varepsilon \neq \varepsilon' \in \hat{G}_\mathbb{K}} A_{\varepsilon'}$. So there is an injective morphism

$$\alpha : \mathbb{K}_\varepsilon[G] \rightarrow \text{End}_{\mathbb{L}_\varepsilon,\mathbb{K}}(E_{\varepsilon,\mathbb{K}})$$

such that $\pi_\varepsilon = \alpha \circ \text{pr}_\varepsilon$. Since both algebras have the same dimension, $\alpha$ is an isomorphism of $\mathbb{K}$-algebras. 

**Corollary 4.5.** There is an isomorphism of $G \times G$-modules

$$R(G, \mathbb{K}) = \bigoplus_{\varepsilon \in \hat{G}_\mathbb{K}} \text{End}_{\mathbb{L}_\varepsilon,\mathbb{K}}(E_{\varepsilon,\mathbb{K}}).$$

Thus the multiplicity $m$ of $E_{\varepsilon,\mathbb{K}}$ as a $G$-module in $R(G, \mathbb{K})$ is

$$m = \dim_{\mathbb{L}_\varepsilon,\mathbb{K}}(E_{\varepsilon,\mathbb{K}}) = \frac{\dim_{\mathbb{K}} E_{\varepsilon,\mathbb{K}}}{\dim_{\mathbb{K}} \mathbb{L}_\varepsilon}.$$ 

This conclusion is well-known for $\mathbb{K} = \mathbb{C}$ (see e.g. Theorems 3.22 and 3.28 in [6]) but we could not readily find a reference for $\mathbb{K} = \mathbb{R}$. While the algebra structure of the weakly complete symmetric Hopf algebra $\mathbb{K}[G]$ is satisfactorily elucidated in Theorem 4.3, the comultiplication seems to be not easily accessible due to complications of the way how the representation theory of $G \times G$ reduces to that of $G$ in general. In the case of commutative compact groups $G$ and the complex ground field $\mathbb{C}$ these complications go away, and so we shall clarify the situation in these circumstances in the subsequent section.

### 5. The Weakly Complete Group Algebras of Compact Abelian Groups: An Alternative View

We have seen the usefulness of the concept of a weakly complete group algebra $\mathbb{K}[G]$ over the real or complex numbers. We obtained its existence from the Adjoint Functor Existence Theorem. This is rather remote from a concrete construction. It
may therefore be helpful to see the whole apparatus in a much more concrete way
at least for a substantial subcategory of the category of compact groups, namely,
the category of compact abelian groups for which we already have a familiar duality
theory due to Pontryagin and Van Kampen (see e.g. [6], Chapter 7).

In this section let $G$ be a compact abelian group and $\widehat{G} = \mathcal{CAB}(G, \mathbb{T})$ (with
the category $\mathcal{CAB}$ of compact abelian groups and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$) its discrete character
group. These groups are written additively. For $\mathbb{K} = \mathbb{C}$ there is a natural bijection
$G \to \widehat{G}_\mathbb{C}$ from the character group to the set of equivalence classes of complex simple
$G$-modules (cf. [6], Lemma 2.30 (p.43), Exercise E3.10 (p.66)), and also Proposition
3.56 (p.87) for some information on $\widehat{G}_\mathbb{R}$). This bijection associates with a character
$\chi \in \widehat{G} = \text{Hom}(G, \mathbb{T})$ the class of the module $E_\chi = \mathbb{C}$, $\chi \cdot c = e^{2\pi i c}$. Accordingly,
[6] Theorem 3.28 (12) reads $R(G, \mathbb{C}) = \sum_{\chi \in \widehat{G}} \mathbb{C} \cdot f_\chi$, for a suitable basis $f_\chi$, $\chi \in \widehat{G}$,
$f_\chi(g) = e^{2\pi i \chi(g)}$. In other words, as a $G$-module, $R(G, \mathbb{C}) \cong \mathbb{C}^{\widehat{G}}$. Accordingly, we
expect $\mathbb{C}[G]$ to be uncomplicated. Our Theorem 4.3 makes this clear:

The complex algebra $\mathbb{C}[G]$ may be naturally identified with the componentwise algebra
$\mathbb{C}^{\widehat{G}}$.

In the abelian case, our understanding of the comultiplication of $\mathbb{C}[G] = \mathbb{C}^{\widehat{G}}$
is much more explicit than in the general situation of Theorem 4.3. Each character
$\chi : G \to \mathbb{T}$ determines a morphism $f_\chi : G \to \mathbb{C}^{-1} = \mathbb{C}^\times$, $f_\chi(g) = e^{2\pi i \chi(g)}$, $g \in G \subseteq \mathbb{C}^{\widehat{G}}$. By the universal property of $\mathbb{C}[G] = \mathbb{C}^{\widehat{G}}$, this value agrees with the $\chi$-th
projection of $g \in G \subseteq \mathbb{C}^{\widehat{G}}$. Hence

$$(\forall g \in G, \chi \in \widehat{G}) \eta_\chi(g)(\chi) = e^{2\pi i (\chi,g)}.$$ 

Accordingly, if we write $S^1 = \{z \in \mathbb{C}; |z| = 1\}$, then $g \in \text{Hom}(\widehat{G}, S^1) \cong \widehat{G} \cong G$.
Then in view of $G \subseteq \mathbb{R}[G] \subseteq \mathbb{C}[G]$ we have

$\text{Hom}(\widehat{G}, S^1) \subseteq \text{span}_\mathbb{R}(\text{Hom}(\widehat{G}, S^1)) = \mathbb{R}[G] \subseteq \mathbb{C}[G] = \mathbb{C}^{\widehat{G}}$.

Recall from [3], Theorem 5.5 that we have an isomorphism

$\alpha_G : \mathbb{C}[G \times G] \to \mathbb{C}[G] \otimes_{\mathbb{W}} \mathbb{C}[G]$,

and from [3] Lemma 5.12 we recall the comultiplication $\gamma_G : \mathbb{C}[G] \to \mathbb{C}[G] \otimes_{\mathbb{W}} \mathbb{C}[G]$ to be the composition

$$\begin{array}{c}
\mathbb{C}[G] \xrightarrow{\delta_G} \mathbb{C}[G \times G] \xrightarrow{\alpha_G} \mathbb{C}[G] \otimes_{\mathbb{W}} \mathbb{C}[G].
\end{array}$$

Now for a compact abelian group $G$, the diagonal morphism $\delta_G : G \to G \times G$ has
the group operation of $\widehat{G}$ as its dual, namely:

$\widehat{\delta_G} : \widehat{G} \times \widehat{G} \to \widehat{G}$. $\widehat{\delta_G}(\chi_1, \chi_2) = \chi_1 + \chi_2$,

as we write abelian group operations additively in general. If now we also write
$\mathbb{C}[G] \otimes_{\mathbb{W}} \mathbb{C}[G] = \mathbb{C}^{\widehat{G} \times \widehat{G}}$ (identifying $\phi \otimes \psi$ with $(\chi_1, \chi_2) \mapsto \phi(\chi_1)\psi(\chi_2)$), then we have

$\gamma_G = \mathbb{C}^{\widehat{\delta_G}} : \mathbb{C}^{\widehat{G}} \to \mathbb{C}^{\widehat{G} \times \widehat{G}}$, i.e., $(\forall \phi \in \mathbb{C}^{\widehat{G}})$, $\gamma_G(\phi)(\chi_1, \chi_2) = \phi(\chi_1 + \chi_2)$. 

This allows us to determine explicitly the elements of the group \( \mathcal{G}(\mathbb{C}^\hat{G}) \) of all grouplike elements:

Indeed a nonzero element \( \phi \in \mathbb{C}^\hat{G} \) is in \( \mathcal{G}(\mathbb{C}^\hat{G}) \) if and only if

\[
\gamma_G(\phi) = \phi \otimes \phi \quad \text{in} \quad \mathbb{C}^\hat{G} \otimes_W \mathbb{C}^\hat{G} = \mathbb{C}^\hat{G} \times \hat{G},
\]

where \((\phi \otimes \phi)(\chi_1, \chi_2) = \phi(\chi_1)\phi(\chi_2)\). This is the case if and only if

\[
(\forall \phi_1, \phi_2 \in \hat{G}) \phi(\chi_1 + \chi_2) = \gamma_G(\phi)(\chi_1, \chi_2) = (\phi \otimes \phi)(\chi_1, \chi_2) = \phi(\chi_1)\phi(\chi_2),
\]

that is, if and only if \( \phi \) is a morphism of groups from \( \hat{G} \) to \( \mathbb{C}^\times = (\mathbb{C} \setminus \{0\}, \cdot) \).

Similarly, an element \( \phi \in \mathbb{C}^\hat{G} \) is primitive if and only if \( \phi(\chi_1 + \chi_2) = \gamma_G(\phi)(\chi_1, \chi_2) = (\phi \otimes 1) + 1 \otimes \phi \) if and only if \( \phi: \hat{G} \to (\mathbb{C}, +) \) is a morphism of topological groups.

Let us summarize this discourse:

**Theorem 5.1.** (The Group Hopf Algebra \( \mathbb{C}[G] \) for Compact Abelian \( G \)) Let \( G \) be a compact abelian group and \( A \) its weakly complete commutative symmetric group Hopf algebra \( \mathbb{C}[G] \) and let \( \hat{G} = \text{Hom}(G, \mathbb{T}) \) be its character.

(i) Then \( A \) may be identified with \( \mathbb{C}^\hat{G} \) such that \( g: \hat{G} \to A^{-1} \) is defined by

\[
(\forall \chi \in \hat{G}) g(\chi) = e^{2\pi i \langle \chi, g \rangle} \in \mathbb{S}^1,
\]

where \( \mathbb{S}^1 = \{ z \in \mathbb{C} : |z| = 1 \} \subseteq \mathbb{C}^\times \). The natural image of \( G \) in \( A^{-1} \) is

\[
G = \text{Hom}(\hat{G}, \mathbb{S}^1) \cong \hat{G},
\]

and

\[
G = \text{Hom}(\hat{G}, \mathbb{S}^1) \subseteq \mathbb{R}[G] \subseteq \mathbb{C}[G] = \mathbb{C}^\hat{G}.
\]

(ii) If, as is possible in the category of weakly complete vector spaces, the weakly complete vector spaces \( A \otimes_W A \) and \( \mathbb{C}^\hat{G} \times \hat{G} \) are identified, then the comultiplication \( \gamma_G: A \to A \otimes_W A \) of \( A \) is given by

\[
(\forall \phi: \hat{G} \to \mathbb{C}, \chi_1, \chi_2 \in \hat{G}) \quad \gamma_G(\phi)(\chi_1, \chi_2) = \phi(\chi_1 + \chi_2) \in \mathbb{C}.
\]

(iii) The group of grouplike elements of \( A \) is

\[
\mathbb{G}(A) = \text{Hom}(\hat{G}, \mathbb{S}^\times) \subseteq \mathbb{C}^\hat{G}.
\]

(iv) The weakly complete Lie algebra of primitive elements of \( A \) is

\[
\mathbb{P}(A) = \text{Hom}(\hat{G}, \mathbb{C}) \subseteq \mathbb{C}^\hat{G}.
\]

We write \( \mathbb{R}^\times_+ \) for the multiplicative subgroup \( \{ z \in \mathbb{C} : 0 < z \in \mathbb{R} \subseteq \mathbb{C} \} \) of \( \mathbb{C}^\times \).
Corollary 5.2. For a compact abelian group $G$ and the weakly complete commutative unital algebra $A := \mathbb{C}[G]$ we have a commutative diagram

\[
\begin{array}{ccc}
\mathbb{P}(A) = \text{Hom}(\hat{G}, \mathbb{R}) + \text{Hom}(\hat{G}, i\mathbb{R}) & \xrightarrow{\cong} & \mathcal{L}(G) \times \mathcal{L}(G) \\
\exp_A & & \downarrow \text{id}_{\mathcal{L}(G) \times \exp G} \\
\mathcal{G}(A) = \text{Hom}(\hat{G}, \mathbb{R}_+^\times) \cdot \text{Hom}(\hat{G}, \mathbb{S}^1) & \xrightarrow{\cong} & \mathcal{L}(G) \times G.
\end{array}
\]

The unique maximal compact subgroup of $\mathcal{G}(A)$ is $G = \text{Hom}(\hat{G}, \mathbb{S}^1)$.

**Proof.** There is an elementary isomorphism of topological groups

\[(r, t + \mathbb{Z}) \mapsto e^r e^{2\pi it} = e^{r+2\pi it} : \mathbb{R} \times \mathbb{T} \to \mathbb{C}^\times.\]

Accordingly, $\mathcal{G}(A) = \text{Hom}(\hat{G}, \mathbb{C}^\times) \cong \text{Hom}(\hat{G}, \mathbb{R}) \oplus \text{Hom}(\hat{G}, \mathbb{T})$.

Now $\text{Hom}(\hat{G}, \mathbb{R}) \cong \text{Hom}(\mathbb{R}, G)$ (cf. [6], Proposition 7.11(iii)), $\text{Hom}(\mathbb{R}, G) = \mathcal{L}(G)$ by [6], Definition 5.7 (cf. Proposition 7.36ff., Theorem 7.66) and $\text{Hom}(\hat{G}, \mathbb{T}) = \hat{G} \cong G$ by [6], Theorem 2.32. For the exponential function $\exp_A$ of a weakly complete unital symmetric Hopf algebra is treated in Theorem 2.2 above. ■

A compact abelian group is totally disconnected (i.e. profinite) if and only if $\mathcal{L}(G) = \{0\}$ (cf. [6], Corollary 7.72).

**Remark 5.3.** For a compact abelian group $G$ the equality $G = \mathcal{G}(\mathbb{C}[G])$ holds if and only if $G$ is totally disconnected (i.e. profinite).

**Proof.** By Theorem 5.1, the equality holds if and only if $L(G) = \{0\}$ if and only if $\text{Hom}(\hat{G}, \mathbb{R}) = \{0\}$ if and only if $\hat{G}$ is a torsion group (cf. e.g. [6], Propositions A1.33, A1.39) if and only if $G$ is totally disconnected (see Corollary 8.5). ■

In particular, e.g., $\mathbb{T} \neq \mathcal{G}(\mathbb{C}[\mathbb{T}])$.

Now we understand $\mathbb{C}[G] = \mathbb{C}^\hat{G}$ rather explicitly, but $\mathbb{R}[G]$ only rather implicitly. However, Theorem 4.3 applies with $\mathbb{K} = \mathbb{R}$ in order to shed some light on its intrinsic structure.

We define the function $\sigma_G : \mathbb{C}[G] \to \mathbb{C}[G]$ as follows: For $\chi \in \hat{G}$ we set $\hat{\chi}(g) = \chi(-g) = -\chi(g)$. Then we define

\[(\forall \phi \in \mathbb{C}^\hat{G}) \sigma(\phi)(\chi) = \overline{\phi(\chi)}.\]

**Exercise 5.4.** For a compact abelian group $G$, the function $\sigma_G$ is an involution of weakly complete real algebras of $\mathbb{C}[G]$ whose precise fixed point algebra is $\mathbb{R}[G]$. Accordingly, $\mathbb{C}[G] = \mathbb{R}[G] \oplus i\mathbb{R}[G]$.

**Remark 5.5.** If $\aleph$ is any cardinal and $\hat{G}$ is any abelian group with torsion free rank $\aleph$, then $\text{Hom}(\hat{G}, \mathbb{R}) \cong \mathbb{R}^\aleph$. 
Proof. In [6], Theorem 8.20, pp.387ff, it is discussed that $G$ contains totally disconnected compact subgroups $\Delta$ such that the annihilator in the character group of $G$, say, $\Delta^\perp \subseteq \hat{G}$ is free, and $\hat{G}/\Delta^\perp$ is a torsion group. This means that $G/\Delta$ is a torus. We note that the inclusion $\Delta^\perp \to \hat{G}$ induces an isomorphism $K \otimes_{\mathbb{Z}} \Delta^\perp \to K \otimes_{\mathbb{Z}} \hat{G}$ and the (torsion free) rank of $\hat{G}$ is rank $\Delta^\perp$. If $\Delta^\perp \cong \mathbb{Z}^{|X|}$ for a set $X$ of cardinality rank $\Delta^\perp$, then $\text{Hom}(\hat{G}, K) \cong K^X$.

5.1. The exponential function of $\mathbb{C}[G] = \mathbb{C}^\hat{G}$

We recall from Theorem 2.2 that every weakly complete associative unital algebra $W$ has an exponential function, which is immediate in the case of $W = \mathbb{C}^\hat{G}$ as it is calculated componentwise. If the weakly complete algebra $W$ is even a Hopf algebra, such as $\mathbb{C}^\hat{G}$, then the group $G(W)$ of grouplike elements is a pro-Lie group and $\mathbb{P}(W)$ is the (real!) Lie algebra of the pro-Lie group $G(W)$ ([3], Theorem 6.15). If indeed $W = \mathbb{C}^\hat{G} = \mathbb{C}[G]$ for a compact abelian group $G$, then the exponential function $\exp: L(G(W)) \to G(W)$ of $G(W)$ is the restriction of the (componentwise!) exponential function $\exp: \mathbb{C}^\hat{G} \to (\mathbb{C}^\hat{G})^{-1} = ((\mathbb{C}^X,.,))^{\hat{G}}$ to $L(G(W)) = \text{Hom}(\hat{G}, \mathbb{C})$.

6. The Weakly Complete Enveloping Algebra of a Weakly Compact Lie Algebra

We have observed that for compact groups $G$ the weakly complete real group algebra $\mathbb{R}[G]$ contains a substantial volume of different materials: the pro-Lie group $G$ itself, its Lie algebra $\mathfrak{L}(G)$, the exponential function between them and, as was discussed in detail in [3], a substantial portion of the Radon measure theory of $G$. The topological Hopf algebra $\mathbb{K}[G]$ is, in a sense, universally generated by $G$. So it seems natural to ask the question whether $\mathfrak{L}(G)$ generates $\mathbb{K}[G]$ in a universal way—perhaps in some fashion that would resemble the universal enveloping algebra of a Lie algebra such as it is presented in the famous Poincaré-Birkhoff-Witt-Theorem (see e.g. [1], §2, n° 7, Théorème 1., p.30). This is not exactly the case, but a few aspects should be discussed.

So we let $\mathbb{K}$ again denote one of the topological fields $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{WA}$ denote the category of weakly complete associative unital algebras over $\mathbb{K}$ and and $\mathcal{WL}$ the category of weakly complete Lie algebras over $\mathbb{K}$. The functor $A \mapsto A_{\text{Lie}}$ which associates with a weakly complete associative algebra $A$ the weakly complete Lie algebra obtained by considering on the weakly complete vector space $A$ the Lie algebra obtained with respect to the Lie bracket $[x,y] = xy - yx$ is called the underlying Lie algebra functor. Since $A$ is a strict projective limit of finite dimensional $\mathbb{K}$-algebras by [3], Theorem 3.2, then $A_{\text{Lie}}$ is a strict projective limit of finite dimensional $\mathbb{K}$-Lie algebras, briefly called pro-Lie algebras. Every pro-Lie algebra is weakly complete. (Caution: A comment following Theorem 3.12 of [3] exhibits an example of a weakly complete $\mathbb{K}$-Lie algebra which is not a pro-Lie algebra!)

Lemma 6.1. The underlying Lie algebra functor $A \mapsto A_{\text{Lie}}$ from $\mathcal{WA}$ to $\mathcal{WL}$ has a left adjoint $U: \mathcal{WL} \to \mathcal{WA}$.
Proof. The category $\mathcal{WL}$ is complete. (Exercise. Cf. Theorem A3.48 of [6], p. 781.) The “Solution Set Condition” (of Definition A3.59 in [6], p. 786) holds. (Exercise: Cf. the proof of [3], Section 5.1 “The solution set condition”.) Hence $U$ exists by the Adjoint Functor Existence Theorem (i.e., Theorem A3.60 of [6], p. 786).

In other words, for each weakly complete Lie algebra $L$ there is a natural morphism $\lambda_L: L \to U(L)$ such that for each continuous Lie algebra morphism $f: L \to A_{\text{Lie}}$ for a weakly complete associative unital algebra $A$ there is a unique $\mathcal{WA}$-morphism $f': U(L) \to A$ such that $f = f'_{\text{Lie}} \circ \lambda_L$.

\[
\begin{array}{ccc}
\mathcal{WL} & \mathcal{WA} \\
L \downarrow \lambda_L & U(L)_{\text{Lie}} & U(L) \\
\forall f & f'_{\text{Lie}} & \cong f' \\
A_{\text{Lie}} \downarrow \text{id} & A_{\text{Lie}} & A.
\end{array}
\]

If necessary we shall write $U_K$ instead of $U$ whenever the ground field should be emphasized. We shall call $U_K(L)$ the weakly complete enveloping algebra of $L$ (over $K$).

Example 6.2. Let $L = K$, the smallest possible nonzero Lie algebra over $K$. Then $U(L) = K\langle X \rangle$ (see [3], Definition following Corollary 3.3), and define $\lambda_L: L \to U(L)_{\text{Lie}}$ by $\lambda_L(t) = t \cdot X$. Indeed the universal property is satisfied by [3], Corollary 3.4. Namely, let $f: K \to A_{\text{Lie}}$ be a morphism of weakly complete Lie algebras. Then there is a unique morphism $f': U(L) \to A$ such that $f'(X) = f(1)$ by [3], Corollary 3.4. Then $f'(t \cdot X) = t \cdot f'(X) = t \cdot f(1) = f(t)$.

Thus by Lemma 3.5 of [3] and the subsequent remarks we have:

\[\text{The weakly complete enveloping algebra } U_C(\mathbb{C}) \text{ over } \mathbb{C} \text{ of the smallest nonzero complex Lie algebra is isomorphic to the weakly complete commutative algebra } \mathbb{C}[[X]]^\mathbb{C} \text{ with the complex power series algebra } \mathbb{C}[[X]] \cong \mathbb{C}^{\mathbb{N}_0}, \mathbb{N}_0 = \{0, 1, 2, \ldots \}.\]

The size of the weakly complete enveloping algebras therefore is considerable.

Proposition 6.3. The universal enveloping functor $U$ is multiplicative, that is, there is a natural isomorphism $\alpha_{L_1, L_2}: U(L_1 \times L_2) \to U(L_1) \otimes_W U(L_2)$.

Proof. We have a natural bilinear inclusion map of weakly complete vector spaces $j: U(L_1) \times U(L_2) \to U(L_1) \otimes_W U(L_2)$ yielding

\[
L_1 \times L_2 \xrightarrow{\lambda_{L_1} \times \lambda_{L_2}} U(L_1)_{\text{Lie}} \times U(L_2)_{\text{Lie}} \xrightarrow{j} U(L_1)_{\text{Lie}} \otimes_W U(L_2)_{\text{Lie}}
\]

and

\[
U(L_1)_{\text{Lie}} \otimes_W U(L_2)_{\text{Lie}} = (U(L_1) \otimes_W U(L_2))_{\text{Lie}}.
\]
the composition $\alpha_0$ of which is a morphism of weakly complete Lie algebras. Hence the universal property yields a morphism of weakly complete associative algebras
\begin{equation}
\alpha: \mathbf{U}(L_1 \times L_2) \to \mathbf{U}(L_1) \otimes_W \mathbf{U}(L_2)
\end{equation}
such that $\alpha_0 = \alpha_{\text{Lie}} \circ \lambda_{L_1} \otimes \lambda_{L_2}$.

The functorial property of $\mathbf{U}$ allows us to argue that each of $\mathbf{U}(L_m)$, $m = 1, 2$ is a retract of $\mathbf{U}(L_1 \times L_2)$ so that we may assume $\mathbf{U}(L_m) \subseteq \mathbf{U}(L_1 \times L_2)$, $m = 1, 2$. Now the multiplication in $\mathbf{U}(L_1 \times L_2)$ gives rise to a continuous bilinear map $\mathbf{U}(L_1) \times \mathbf{U}(L_2) \to \mathbf{U}(L_1 \times L_2)$, and then the universal property of the tensor product of weakly complete associative algebras yields the morphism
\begin{equation}
\beta: \mathbf{U}(L_1) \otimes_W \mathbf{U}(L_2) \to \mathbf{U}(L_1 \times L_2).
\end{equation}
Similarly to the proof of [3], Theorem 5.5 (preceding the statement of the theorem) we argue that $\alpha$ and $\beta$ are inverses of each other, and so $\alpha$ of (1) is the desired isomorphism $\alpha_{\lambda_1 \lambda_2}$.

**Lemma 6.4.** For any weakly complete unital algebra $A$, the vector space morphism $\Delta_A: A \to A \otimes_W A$, $\Delta_A(a) = a \otimes 1 + 1 \otimes a$ is a morphism of weakly complete Lie algebras $A_{\text{Lie}} \to (A \otimes_W A)_{\text{Lie}}$.

**Proof.** Since the functions $a \mapsto a \otimes 1$, $1 \otimes a$ are morphisms of topological vector spaces, so is $\Delta_A$ For $y_1, y_2 \in A$, write $z_j := \Delta_A(y_j) = y_j \otimes 1 + 1 \otimes y_j$ Then just as in the classical case, we calculate $[\Delta_A(y_1), \Delta_A(y_2)] = [z_1, z_2] = z_1 z_2 - z_2 z_1 = (y_1 \otimes 1 + 1 \otimes y_1)(y_2 \otimes 1 + 1 \otimes y_2) - (y_2 \otimes 1 + 1 \otimes y_2)(y_1 \otimes 1 + 1 \otimes y_1)$
\begin{align*}
&= (y_1 y_2 \otimes 1 + y_1 \otimes y_2 + y_2 \otimes y_1 + 1 \otimes y_1 y_2) - (y_2 y_1 \otimes 1 + y_2 \otimes y_1 + y_1 \otimes y_2 + 1 \otimes y_2 y_1) \\
&= [y_1, y_2] \otimes 1 + 1 \otimes [y_1, y_2] = \Delta_A[y_1, y_2].
\end{align*}
Thus $\Delta_A$ is a morphism of Lie algebras as asserted.

Now we consider a weakly complete Lie algebra $L$ and recall that $\lambda_L: L \to \mathbf{U}(L)_{\text{Lie}}$ is a morphism of weakly complete Lie algebras. Thus by Lemma 6.4,
\[ p_L = \Delta_{\mathbf{U}(L)} \circ \lambda_L: L \to (\mathbf{U}(L) \otimes \mathbf{U}(L))_{\text{Lie}} \]
is a morphism of weakly complete Lie algebras. Now by the universal property of $\mathbf{U}$, $p_L$ induces a unique natural morphism of weakly complete associative unital algebras $\gamma_L: \mathbf{U}(L) \to \mathbf{U}(L) \otimes_W \mathbf{U}(L)$ such that $p_L = (\gamma_L)_{\text{Lie}} \circ \lambda_L$. This yields the following insight, where $k_L: L \to \{0\}$ denotes the constant morphism.

**Corollary 6.5.** Each weakly complete enveloping algebra $\mathbf{U}(L)$ is a weakly complete Hopf algebra with the comultiplication $\gamma_L$ and the coidentity $\mathbf{U}(k_L): \mathbf{U}(L) \to \mathbf{K}$.

**Proof.** Observe that $\gamma_L$ is a morphism of weakly complete unital algebras satisfying $\gamma_L(y) = y \otimes 1 + 1 \otimes y$ for $y = \lambda(x), x \in L$. The associativity of this comultiplication is readily checked as in the case of abstract enveloping algebras. The constant morphism of weakly complete Lie algebras $L \to \{0\}$ yields a morphism of weakly complete unital algebras $\mathbf{U}(L) \to \mathbf{U}(\{0\}) = \mathbf{K}$ which is the coidentity of the Hopf algebra.
Our results from [3] regarding weakly complete associative unital algebras and Hopf algebras over $\mathbb{K}$ apply to the present situation.

**Theorem 6.6.** (The Weakly Complete Enveloping Algebra) Let $L$ be a weakly complete Lie algebra. Then the following statements hold:

(i) $U(L)$ is a strict projective limit of finite-dimensional associative unital algebras and the group of units $U(L)^{-1}$ is dense in $U(L)$. It is an almost connected pro-Lie group (which is connected in the case of $\mathbb{K} = \mathbb{C}$). The algebra $U(L)$ has an exponential function $\exp: U(L)_{\text{Lie}} \to U(L)^{-1}$.

(ii) The pro-Lie algebra $P(U(L))$ of primitive elements of $U(L)$ contains $\lambda_L(L)$.

(iii) The subalgebra generated by $\lambda_L(L)$ in $U(L)$ is dense in $U(L)$.

(iv) The pro-Lie algebra $P(U(L))$ is the Lie algebra of the pro-Lie group $G(U(L))$ of grouplike elements of $U(L)$. We use the abbreviation $G := G(U(L))$ and note that the exponential function $\exp_G: \mathfrak{L}(G) \to G$ is the restriction and corestriction of the exponential function $\exp$ of $U(L)$ to $P(U(L))$, respectively, $G$. The image $\exp(\mathfrak{L}(G))$ generates algebraically and topologically the identity component $G_0$ of $G$.

**Proof.** (i) See [3], Theorems 3.2, 3.11, 3.12, 4.1.

(ii) The very definition of the comultiplication $\gamma_L$ for Corollary 6.5 shows that for any $y \in \lambda_L(L)$, the image under the comultiplication $\gamma_L$ is $y \otimes 1 + 1 \otimes y$, which means that $y$ is primitive.

(iii) An argument analogous to that in the proof of Proposition 5.3 of [3] showing that, for the case of any topological group $T$, the subset $\eta_G(T)$ of the weakly complete group algebra $\mathbb{K}[T]$ spans a dense subalgebra, shows here that the closed subalgebra $S$ generated in $U(L)$ by $\lambda_L(L)$ has the universal property of $U(L)$ and therefore agrees with $U(L)$.

(iv) See Theorem 2.2 and [3], Theorem 6.15.

**Remark 6.7.** We note right away that for any weakly complete Lie algebra $L$ which has at least one nonzero finite dimensional $\mathbb{K}$-linear representation, the morphism $\lambda_L: L \to U(L)_{\text{Lie}}$ is nonzero. By Ado’s Theorem, this applies, in particular, to any Lie algebra which has a nontrivial finite dimensional quotient and therefore is true for the Lie algebra $\mathfrak{L}(P)$ of any pro-Lie group $P$.

**Corollary 6.8.** (i) The weakly complete enveloping algebra $U(L)$ of a weakly complete Lie algebra $L$ with a nontrivial finite dimensional quotient has nontrivial grouplike elements.

(ii) If $L$ is a pro-Lie algebra, then $\lambda_L: L \to U(L)_{\text{Lie}}$ maps $L$ isomorphically onto a closed Lie subalgebra of the pro-Lie algebra $P(U(L))$ of primitive elements.

**Proof.** (i) If $P(U(L))$ is nonzero, then $U(L)$ has nontrivial grouplike elements by Theorem 6.6 (iii), and by part (ii) of 6.6, this is the case if $\lambda_L$ is nonzero which is
the case for all $L$ satisfying the hypothesis of the Corollary by the remark preceding it.

(ii) Since each finite dimensional quotient of $L$ has a faithful representation by the Theorem of Ado, and since the finite dimensional quotients separate the points of $L$, the morphism $\lambda_L$ is injective. However, injective morphisms of weakly complete vector spaces are open onto their images.

It follows that for pro-Lie algebras $L$ we may assume that $L$ is in fact a closed Lie subalgebra of primitive elements of $U(L)$ which generates $U(L)$ algebraically and topologically as a weakly complete algebra.

It remains an open question under which circumstances we then have in fact $L = P(U(L))$. In the classical setting of the discrete enveloping Hopf algebra in characteristic 0 this is the case: see e.g. [8], Theorem 5.4.

One application of the functor $U$ is of present interest to us. Recall that for a compact group we naturally identify $G$ with the group of grouplike elements of $\mathbb{R}[G]$ (cf. [3], Theorems 8.7, 8.9 and 8.12), and that $L(G)$ may be identified with the pro-Lie algebra $P(\mathbb{R}[G])$ of primitive elements. (Cf. also Theorem 2.2 above.) We may also assume that $L(G)$ is contained the set $P(U(L(G)))$ of primitive elements of $U(\mathbb{R}(L(G)))$.

**Theorem 6.9.** (i) Let $G$ be a compact group. Then there is a natural morphism of weakly complete algebras $\omega_G: U_{\mathbb{R}}(L(G)) \rightarrow \mathbb{R}[G]$ fixing the elements of $L(G)$ elementwise.

(ii) The image of $\omega_G$ is the closed subalgebra $\mathbb{R}[G_0]$ of $\mathbb{R}[G]$.

(iii) The pro-Lie group $G(U_{\mathbb{R}}(L(G)))$ is mapped onto $G_0 = G(\mathbb{R}[G_0]) \subseteq \mathbb{R}[G]$. The connected pro-Lie group $G(U_{\mathbb{R}}(L(G)))_0$ maps surjectively onto $G_0$ and $P(U_{\mathbb{R}}(L(G)))$ onto $P(\mathbb{R}[G])$.

**Proof.** (i) follows at once from the universal property of $U$.

(ii) As a morphism of weakly complete Hopf algebras, $\omega_G$ has a closed image which is generated as a weakly complete subalgebra by $L(G)$ which is $\mathbb{R}[G_0]$ by Corollary 3.3 (ii).

(iii) The morphism $\omega_G$ of weakly complete Hopf algebras maps grouplike elements to grouplike elements, whence we have the commutative diagram

$$
\begin{array}{c}
\mathbb{L}(G) \subseteq P(U_{\mathbb{R}}(L(G))) \xrightarrow{\exp_G(U_{\mathbb{R}}(L(G)))} P(\mathbb{R}(G))=L(G) \\
\xrightarrow{\omega_G(U_{\mathbb{R}}(L(G)))} G(\mathbb{R}(L(G))) \xrightarrow{G(\omega_G)} G(\mathbb{R}[G]) = G.
\end{array}
$$

Since $P(\omega)$ is a retraction and the image of $\exp_G$ topologically generates $G_0$, the image of $G(\omega_G) \circ \exp_{U_{\mathbb{R}}(L(G))}$ topologically generates $G_0$. Since the image of the exponential function of the pro-Lie group $G(U_{\mathbb{R}}(L(G)))$ generates topologically its identity component, $G(\omega_G)$ maps this identity component onto $G_0$.

Since $L(G) \subseteq P(U_{\mathbb{R}}(L(G)))$, and since also any morphism of Hopf algebras maps a primitive element onto a primitive element we know $\omega_G(P(U_{\mathbb{R}}(L(G)))) = P(\mathbb{R}[G])$. ■
It remains an open question whether $G(U_R(L(G)))$ is in fact connected. An overview of the situation may be helpful:

$$
\begin{align*}
U_R(L(G)) & \xrightarrow{\omega_{G,onto}} \mathbb{R}[G_0] \\
G(U_R(G)) & \xrightarrow{onto} G(\mathbb{R}[G]) = G \\
G(U_R(L(G)))_0 & \xrightarrow{onto} G(\mathbb{R}[G_0]) = G_0 \\
\exp_{U_R(L(G))} & = \exp_{\mathbb{R}[G]} \\
P(U_R(L(G))) & = P(\mathbb{R}[G_0]) = P(\mathbb{R}[G]) = L(G).
\end{align*}
$$

A noteworthy consequence of the preceding results is the insight that for any nonzero weakly complete real Lie algebra of the kind $L = \mathbb{R}^X \times \prod_{j \in J} L_j$ for any set $X$ and any family of compact finite dimensional simple Lie algebras $L_j$, the weakly complete enveloping algebra $U(L)$ has grouplike elements. In the discrete situation, the enveloping algebra $U(L)$ of a Lie algebra $L$ for characteristic zero has no grouplike elements.

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