Surface angular momentum of light beams

Marco Ornigotti\textsuperscript{1} and Andrea Aiello\textsuperscript{2,3,4}\textsuperscript{*}

\textsuperscript{1}Institute of Applied Physics, Friedrich-Schiller University, Jena, Max-Wien Platz 1, 07743 Jena, Germany
\textsuperscript{2}Max Planck Institute for the Science of Light, Günther-Scharowsky-Strasse 1/Bau24, 91058 Erlangen, Germany
\textsuperscript{3}Institute for Optics, Information and Photonics, University of Erlangen-Nuernberg, Staudtstrasse 7/B2, 91058 Erlangen, Germany

\textsuperscript{*} andrea.aiello@mpl.mpg.de

Abstract:
Traditionally, the angular momentum of light is calculated for “bullet-like” electromagnetic wave packets, although in actual optical experiments “pencil-like” beams of light are more commonly used. The fact that a wave packet is bounded transversely and longitudinally while a beam has, in principle, an infinite extent along the direction of propagation, renders incomplete the textbook calculation of the spin/orbital separation of the angular momentum of a light beam. In this work we demonstrate that a novel, extra surface part must be added in order to preserve the gauge invariance of the optical angular momentum per unit length. The impact of this extra term is quantified by means of two examples: a Laguerre-Gaussian and a Bessel beam, both circularly polarized.

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In his *Treatise of Electricity and Magnetism* [11], Maxwell pointed out that the electromagnetic field carries both energy and momentum, and that the momentum can have both linear and angular contributions. Today it is common knowledge that the angular momentum (AM) of...
light can be thought of having two contributions: a spin part associated to polarization (firstly theoretically investigated by Poynting [2] and then experimentally demonstrated by Beth [3]), and an orbital part associated with the spatial distribution of the field, as first recognised by Darwin [4]. About sixty years later, thanks to the seminal work of Allen and Woerdman [5], the topic of AM of light experienced a new renaissance, producing a considerable amount of literature on the subject, including theoretical discussions about the separation of AM in spin and orbital parts for optical beams [6–14], the spin-Hall effect of light [15–18], the geometric spin-Hall effect of light [19,20], and a huge variety of experimental applications such as optical tweezers, spatial light modulators and vortex beams. A satisfactory review of the development of the field of optical AM of light can be found in Refs. [21–23].

The AM of light has been frequently separated, by many authors, into a spin part and an orbital part (see, e.g., [24] for a didactic presentation of the subject). As emphasized by Birula&Birula [14] such separation can be either gauge dependent or gauge invariant. The latter result is achieved, according to Darwin [4], by expressing the optical AM in terms of the Fourier transform of the electromagnetic fields. On the contrary, the gauge dependent separation is obtained, traditionally, when in the expression of the optical AM

$$J = \varepsilon_0 \int r \times (E \times B) \, d^3r,$$

(1)

the magnetic field $B$ is replaced by the curl of the vector potential $A$, namely $B = \nabla \times A$. After partial integration Eq. (1) takes the split form [25,26]

$$J = \varepsilon_0 \int E \times A \, d^3r + \varepsilon_0 \int \sum_\xi E_\xi (r \times \nabla) A_\xi \, d^3r - \varepsilon_0 \int \sum_\xi \nabla_\xi (E_\xi r \times A) \, d^3r,$$

(2)

where $\xi \in \{x,y,z\}$ and the first and the second integral represent the spin part and the orbital part, respectively, of the optical AM. The third integrals is usually omitted in the textbook expressions of $J$ because, as consequence of the Gauss’ theorem, it will be identically zero if the fields vanish sufficiently quickly as $|r| \to \infty$. This requirement, as pointed out by Crichton&Marston [27] and Nieminen et al. [28], is often understood while its fulfillment should be carefully checked case by case. For example, the electric and magnetic fields of an optical “bullet-like” wave packet with a finite transverse and longitudinal (with respect to the direction of propagation) extent, by definition vanish as $|r| \to \infty$ and the third term in Eq. (2) can be safely neglected. However, this is no longer true for a “pencil-like” beam of light whose span along the direction of propagation is virtually infinite. This simple but important fact was already noticed by Barnett&Allen [6] (see Eqs. (3.15-18) in their paper) who, nevertheless, simply classified the third integral in Eq. (2) as a nonparaxial contribution to the AM per unit length of the beam and did not investigate its physical content.

In this paper we thoroughly investigate the physical nature of the latter term in Eq. (2) that we call the “Surface Angular Momentum” (SuAM) of light. At the fundamental level, we demonstrate that even for an optical beam with a small angular spread around the direction of propagation (paraxial beams), the SuAM term must be retained in order to guarantee the gauge invariance of the theory. From an experimental point of view, we show that SuAM arises whenever the intensity of an optical beam is recorded by a planar detector which, in practice, performs a two dimensional integration over a cross section of the beam perpendicular to its direction of propagation. As this is the case occurring in the majority of experimental optical setups, we believe that the impact of our new results may be significant. All our conclusions are valid for light beams propagating in free space.

The structure of this Article is as follows. In Sec. II we first calculate the total AM density $j(r)$ of a monochromatic beam of light described in terms of the vector potential $A$. Then, by
following the standard procedure, we decompose \( J(r) \) into the sum of three terms: the well-known orbital and spin AM parts and the third term, leading to the SuAM. Next, we integrate homogeneous plane waves solely (namely, without contributions from evanescent waves) both

\[
J = \int j(r) \, d^2r, \quad u(r) = \int u(r) \, d^2r, \quad E = i/2 \times B, \quad \mathbf{E} = \mathbf{E}^\dagger, \quad \mathbf{B} = \mathbf{B}^\dagger.
\]

2. Theory of SuAM

To begin with, consider a monochromatic beam of light with angular frequency \( \omega \) whose real electric and magnetic fields \( \mathbf{E}(r, t) \) and \( \mathbf{B}(r, t) \) are expressed in terms of the complex field amplitudes \( \mathbf{E}(r) \) and \( \mathbf{B}(r) \) as \( \mathbf{E}(r, t) = \text{Re}[\mathbf{E}(r) \exp(-i\omega t)] \) and \( \mathbf{B}(r, t) = \text{Re}[\mathbf{B}(r) \exp(-i\omega t)] \). Following van Enk and Nienhuis [8] we write the cycle-averaged linear momentum, angular momentum and energy densities, respectively,

\[
\begin{align*}
\mathbf{p}(r) &= \frac{\epsilon_0 \hbar}{2} \text{Re}(\mathbf{E}^\dagger \times \mathbf{B}), \\
\mathbf{j}(r) &= \frac{\epsilon_0 \hbar}{2} \mathbf{r} \times \text{Re}(\mathbf{E}^\dagger \times \mathbf{B}), \\
u(r) &= \frac{\epsilon_0}{4} \left( \mathbf{E} \cdot \mathbf{E}^\dagger + c^2 \mathbf{B} \cdot \mathbf{B}^\dagger \right),
\end{align*}
\]

where \( \mathbf{E} \) and \( \mathbf{B} \) are written in terms of the transverse vector potential \( \mathbf{A}(r) \) as

\[
\begin{align*}
\mathbf{E} &= i\omega \mathbf{A} \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A} \quad \text{with} \quad \nabla \cdot \mathbf{A} = 0. \quad (4)
\end{align*}
\]

By using Eqs. (3) we can express the AM densities \( \mathbf{j}(r) \) as [27, 29]:

\[
\mathbf{j}(r) = \frac{\omega \epsilon_0}{2} \text{Re} \left\{ A^\dagger \cdot (i\mathbf{r} \times \nabla) \mathbf{A} - iA^\dagger \times \mathbf{A} + \frac{1}{2} \sum_{\xi \in \{x,y,z\}} \frac{\partial}{\partial \xi} \left[ A^\dagger \mathbf{z} \times (\mathbf{r} \times \mathbf{A}) \right] \right\}.
\]

According to [27], the first term in Eq. (5) represents the orbital part of the AM and the second term gives the spin AM of light. The third term is a three-divergence that vanishes when integrated over a volume \( V \) whose surface boundary \( S \) is far enough from the field sources to give \( \mathbf{A}(r)|_S \approx 0 \). However, as noticed in [27], the latter relation should be checked whenever the decomposition (5) is used. In the remainder of this Article, we will show some consequences of the failure of the relation \( \mathbf{A}(r)|_S \approx 0 \).

Haus and Pan [24] pointed out that for a monochromatic beam of light of angular frequency \( \omega \) propagating in the \( z \) direction, the classical optics analogous of the helicity \( \pm \hbar \) of a photon, namely the projection of the spin AM along the direction of the linear momentum, is given by the ratio \( \hat{\epsilon}_z \cdot \mathbf{J}/U = \pm 1/\omega \), where \( \mathbf{J} \) and \( U \) denote the cycle-averaged AM and energy per unit length:

\[
\mathbf{J} = \int j(r) \, d^2r, \quad U = \int u(r) \, d^2r,
\]

where \( d^2r = dx\,dy \) and the integrations extend over all the plane \( xy \) at \( z = \text{const} \). It should be noticed that for a monochromatic optical beam that can be represented by a superposition of homogeneous plane waves solely (namely, without contributions from evanescent waves) both
$J$ and $U$ do not depend upon the longitudinal coordinate $z$ and, therefore, are conserved during free propagation. Henceforth, we shall use over line symbol to denote normalization according to the rule

$$
\mathcal{J} = \frac{J}{U/\omega}.
$$

This definition of helicity is consistent with the physical picture of a beam of light as observed on a laboratory bench, where it travels across plates, lenses and other optical devices before eventually shining the surface of a planar detector. Last but not least, it should be noticed that the calculations by Haus and Pan \cite{33}, leading to small longitudinal fields components, are accounted for.

As a consequence of the surface integration, a beam-like field “feels” the contribution given by the $z$-part of the latter term in Eq. \cite{34}, namely $\partial_z[A^*_z(r \times A)]$, because such a term is not integrated when one calculates $J$. In practice, since

$$
\int \partial_z[A^*_z(r \times A)] \, d^3r = \partial_z \int A^*_z(r \times A) \, d^3r,
$$

we can write the SuAM contribution in the following form:

$$
J_{\text{surf}} = \frac{\partial}{\partial z} \int \text{Re} \left\{ iA^*_z(r \times A) \right\} \, d^3r,
$$

where $\text{Re} \left\{ iA^*_z(r \times A) \right\} = i[z(A^* \times A) - r \cdot (A^* \times A) \hat{e}_z]/2$. From Eq. \cite{34} it appears that the $J_{\text{surf}}$ depends both on the potential vector $A$ and the position vector $r$, thus revealing a seemingly hybrid spin/orbital nature. Now, we can calculate $J_{\text{surf}}$ from Eq. \cite{35} by writing the vector potential as a superposition of homogeneous plane waves

$$
A(r) = \frac{1}{2\pi} \int \tilde{A}(k_{\perp}) \exp (i r_{\perp} \cdot k_{\perp} + i z k_z) \, d^2k,
$$

where $\tilde{A}(k_{\perp}) : k \cdot \tilde{A}(k_{\perp}) = 0$ is the so-called angular spectrum of the field evaluated at $z = 0$ \cite{35} and $k = (k_x, k_y, k_z)$ is the wave vector with $k = |k|$ and $\omega = kc$. For the sake of consistency, henceforth we shall use the following notation: $r_{\perp} = (x, y, 0)$, $k_{\perp} = (k_x, k_y, 0)$, $r_{\perp} \cdot k_{\perp} = x k_x + y k_y$, $k_{\perp} = |k_{\perp}|$ and $d^2k = dk_x \, dk_y$. By definition, $k_z$ is not an independent variable and can be fixed to the non-negative value $k_z = + (k^2 - k_{\perp}^2)^{1/2} \geq 0$, which is appropriate for the forward homogeneous fields considered here. For a circularly polarized optical beam, the angular spectrum can be chosen of the form

$$
\tilde{A}(k_{\perp}) = \tilde{A}(k_{\perp}) \mathcal{E}(\sigma, k_{\perp}),
$$

where the real-valued amplitude $|\tilde{A}(k_{\perp})|$ and phase $\tilde{\alpha}(k_{\perp})$ determines the spatial profile of the beam and $\mathcal{E}(\sigma, k_{\perp})$ is the complex-valued polarization vector defined as:

$$
\mathcal{E}(\sigma, k_{\perp}) = n_{\sigma} - \frac{k \cdot (k \cdot n_{\sigma})}{k^2}
$$

$$
=- k \times (k \times n_{\sigma}) / k^2.
$$

In the equation above $n_{\sigma} = (\hat{e}_z + i \sigma \hat{e}_x) / \sqrt{2}$ with $\sigma = \pm 1$, represents the wave vector-independent configuration-space helicity assigned in the (global) laboratory reference frame.
\{\hat{e}_x, \hat{e}_y, \hat{e}_z\}. By definition \(\mathbf{e}_\perp(\sigma, \mathbf{k}_\perp)\) is not normalized; however, since its norm
\[
\mathbf{e}_\perp^*(\sigma, \mathbf{k}_\perp) \cdot \mathbf{e}_\perp(\sigma, \mathbf{k}_\perp) = 1 - |\mathbf{k} \cdot \mathbf{n}_\sigma|^2 / k^2
\]
\[
= 1 - \frac{k_x^2}{k^2}
\]
\[
\equiv 1 - \vartheta^2,
\]  
(12)
is polarization-independent, we can assume that the normalization factor is tacitly contained in the expression of \(A(\mathbf{k}_\perp)\). Here, the dimensionless parameter \(0 \leq \vartheta < 1 / \sqrt{2}\):
\[
\vartheta^2 = \frac{k_x^2 + k_y^2}{2k^2},
\]  
(13)
defines the angular spread of the beam around the direction of propagation \(z\), and the paraxial regime is characterized by angular spectra such that the condition \(\vartheta \ll 1\) is satisfied. At this point, by substituting Eq. (9) into Eq. (5) and using Eqs. (10-11) one obtains after a straightforward calculation the orbital part
\[
J_{x,\text{orb}} = -\frac{\varepsilon_0 \omega^2}{2} \int |\tilde{A}|^2 \left[ \sigma x k_z \frac{k_x k_z}{k^2 + k_z^2} + k_z \frac{\partial \alpha}{\partial k_y} \right] d^2 k,
\]  
(14a)
\[
J_{y,\text{orb}} = -\frac{\varepsilon_0 \omega^2}{2} \int |\tilde{A}|^2 \left[ \sigma k_y k_z \frac{k_x^2}{k^2 + k_z^2} - k_z \frac{\partial \alpha}{\partial k_y} \right] d^2 k,
\]  
(14b)
\[
J_{z,\text{orb}} = \frac{\varepsilon_0 \omega^2}{2} \int |\tilde{A}|^2 \left[ \sigma - \frac{\vartheta^2}{1 - \vartheta^2} \left( k_x \frac{\partial \alpha}{\partial k_x} - k_y \frac{\partial \alpha}{\partial k_y} \right) \right] d^2 k,
\]  
(14c)
the spin part
\[
J_{x,\text{spin}} = \frac{\varepsilon_0 \omega^2}{2} 2\sigma \int |\tilde{A}|^2 \frac{k_x k_z}{k^2 + k_z^2} d^2 k,
\]  
(15a)
\[
J_{y,\text{spin}} = \frac{\varepsilon_0 \omega^2}{2} 2\sigma \int |\tilde{A}|^2 \frac{k_y k_z}{k^2 + k_z^2} d^2 k,
\]  
(15b)
\[
J_{z,\text{spin}} = \frac{\varepsilon_0 \omega^2}{2} \sigma \int |\tilde{A}|^2 \frac{1 - 2\vartheta^2}{1 - \vartheta^2} d^2 k,
\]  
(15c)
and the surface part of the optical AM per unit of length:
\[
J_{x,\text{surf}} = -\frac{\varepsilon_0 \omega^2}{2} \sigma \int |\tilde{A}|^2 \frac{k_x k_z}{k^2 + k_z^2} d^2 k,
\]  
(16a)
\[
J_{y,\text{surf}} = -\frac{\varepsilon_0 \omega^2}{2} \sigma \int |\tilde{A}|^2 \frac{k_y k_z}{k^2 + k_z^2} d^2 k,
\]  
(16b)
\[
J_{z,\text{surf}} = \frac{\varepsilon_0 \omega^2}{2} \sigma \int |\tilde{A}|^2 \frac{\vartheta^2}{1 - \vartheta^2} d^2 k,
\]  
(16c)
In addition, the energy per unit length is given by
\[
\frac{U}{\omega} = \frac{\varepsilon_0 \omega^2}{2} \int |\tilde{A}|^2 d^2 k.
\]  
(17)
Note that all the quantities (14-17) are \(z\)-independent, so they cannot vanish if one would perform an additional integration along the axis \(z\) to obtain the AMs in a given, finite, volume.
The explicit form for $J_{surf}$ is the first main result of this Article. What does it represent? First of all, it should be noticed that at this stage the angular spectrum $\mathbf{A}(k_z \, t)$ is perfectly general and may represent, for example, an electromagnetic beam with arbitrary orbital AM. The $z$-component of the integrand in Eq. (16c) is proportional to $\vartheta^2/(1 - \vartheta^2)$ as noticed already in [6]. This contribution is $O(\vartheta^2)$ and therefore disappears in the strict paraxial limit where such terms are not retained [24]. In order to understand better how $J_{surf}$ affects the total AM per unit length we sum the three contribution and obtain

$$J_{z}^{eel} + J_{z}^{spin} + J_{z}^{surf} = \frac{e_0 \omega}{2} \int |\mathbf{A}|^2 \left[ \frac{\sigma}{1 - \vartheta^2} + \left( k_\perp \frac{\partial \tilde{\alpha}}{\partial k_\perp} - k_z \frac{\partial \tilde{\alpha}}{\partial k_z} \right) \right] d^2k. \quad (18)$$

It is instructive to compare Eq. (18) with Eq. (38) in [9] where Barnett calculates the ratio between the cycle-averaged angular momentum density flux $\mathcal{M}_{zz}$ and the energy density flux $\mathcal{F}$ through the xy plane, namely $\mathcal{M}_{zz}/\mathcal{F}$. In our case, for a circularly polarized beam with helicity $\sigma$ and by using Eq. (9), it follows that

$$\mathcal{M}_{zz} = \frac{e_0 c^2}{2} \int |\mathbf{A}|^2 k_z \left[ \sigma + \left( k_\perp \frac{\partial \tilde{\alpha}}{\partial k_\perp} - k_z \frac{\partial \tilde{\alpha}}{\partial k_z} \right) \right] d^2k, \quad (19)$$

and

$$\mathcal{F} = \frac{c^2 e_0 \omega}{2} \int |\mathbf{A}|^2 k_z d^2k. \quad (20)$$

The main difference between Eq. (18) and Eq. (19) resides, apart from the trivial factor $c^2/\omega$, in the extra multiplicative term $k_z = k(1 - 2\vartheta^2)^{1/2} \simeq k(1 - \vartheta^2)$ in the integrand. Of course, in the paraxial limit $k_z \to k$ and the two expressions coincides. There is, however, a profound conceptual and practical difference between the energy density $U$ and the energy density flux $\mathcal{F}$ (Poynting vector flux). The physical quantity that is actually measured by common detection devices as CCD detectors, photographic plates, photoresists, etc., is the time-averaged value of the scalar energy density (namely, loosely speaking, the number of photons in the unit volume) integrated over the detector surface, rather than the Poynting vector flux [32–34]. This distinction between scalar and vector quantities becomes crucial when measuring spin-dependent non-paraxial optical phenomena as, e.g., the geometric spin Hall effect of light [20]. In fact, the standard theory of photo-detection (see, e.g., sec. 4.11 of [32] and chap. III of [25]) shows that the observable intensity of the electromagnetic field is proportional to the probability of observing a photoionization in a phototube detector which is given by the expectation value of the electric-field energy density operator $\hat{\mathbf{E}}^(-)(r,t) \cdot \hat{\mathbf{E}}^+(r,t)$. As shown in [32], only when a light beam is at least approximately paraxial the observable intensity coincides with the flux across the detector surface of the Poynting vector operator $\hat{\mathcal{I}}(r,t) = e_0 c^2 \left\{ \hat{\mathbf{E}}^-(r,t) \times \hat{\mathbf{B}}^+(r,t) - \hat{\mathbf{B}}^-(r,t) \times \hat{\mathbf{E}}^+(r,t) \right\}$. Therefore, in general, is the electromagnetic energy density and not the Poynting vector flux that matters in ordinary photodetection.

It should be noticed that both $J_{surf}$ and $J_{spin}$ are proportional to the helicity $\sigma$ of the incident beam. However, this should not be interpreted as a signature of the spin nature of the SuAM, since such a dependence also appears in spin-to-orbit conversion phenomena [55]. From Eqs. (15) (16) it follows that

$$J_{z}^{spin} + J_{z}^{surf} = \frac{e_0 \omega}{2} \sigma \int |\mathbf{A}|^2 d^2k, \quad (21)$$
from which, by using Eqs. (7) and (17), we obtain
\[ J_{z}^{\text{spin}} + J_{z}^{\text{surf}} = \sigma. \] (22)

Equation (22) shows that adding the z-component of the SuAM term to the z-component of the spin AM per unit length ensures that the total “spin” AM per unit length along the propagation axis z is exactly equal to \( \sigma \) times the energy per unit length of the beam, the latter being completely arbitrary. This our second main result.

2.1. Examples

In the remainder of this section, we calculate explicitly the SuAM for Laguerre-Gaussian and Bessel beams to investigate how orbital AM affects distribution can be neglected for |\( \ell | \) comparable with the standard helicity value \( |\sigma| \approx 1 \). In this case, for \( \theta_0 \ll 1 \), the small angle approximation furnishes \( J_{z}^{\text{spin}} / \sigma \approx 1 - \theta_0^2 / 4 \), in agreement with (23).

Next, we consider the case of a \( \ell \)-th-order Bessel beam that, in the cylindrical polar coordinates \((r, \phi, z)\), takes the form \( E(r, \phi, z) = J_\ell(rk \sin \theta_0) \exp(i\ell \phi) \exp(izk \cos \theta_0) \), where \( \theta_0 \) is the angular aperture of the characteristic Bessel cone. A straightforward calculation shows that Eqs. (23) are still valid providing that \( \beta = (\theta_0/2)^2(2p + |\ell| + 1) \). Although \( \beta \) depends on \(|\ell|\) such a dependence is not a consequence of the orbital AM of the beam (this would lead to a dependence from the signed quantity \( \ell \)), but it comes from the radial dependence of the beam profile. For a fundamental Gaussian beam \( \ell = 0 \) and \( \beta \rightarrow \theta_0^2 / 4 \). In this case, for \( \theta_0 \ll 1 \), the small angle approximation furnishes \( J_{z}^{\text{spin}} / \sigma \approx 1 - \theta_0^2 / 4 \), in agreement with (28).

3. Gauge invariant calculation

In this section we calculate again the AM per unit length for a monochromatic beam of light. However, this time we will not use the vector potential and, thus, our calculation will be manifestly gauge invariant. Having this goal in mind, we write the electromagnetic fields in their (homogeneous) angular spectrum form as

\[ E(r) = \frac{1}{2\pi} \int \vec{E}(k_\perp) \exp(i\vec{r}_\perp \cdot k_\perp + ikz) \, d^2k, \] (24a)

\[ B(r) = \frac{1}{2\pi} \int \vec{B}(k_\perp) \exp(i\vec{r}_\perp \cdot k_\perp + ikz) \, d^2k, \] (24b)

where \( \vec{E} \) and \( \vec{B} \) fulfill the following relations stemming from Maxwell equations:

\[ k \cdot \vec{E} = 0, \] (25a)

\[ k \cdot \vec{B} = 0, \] (25b)

\[ \epsilon \vec{B} = \frac{k}{k} \times \vec{E}. \] (25c)
Fig. 1. SuAM for a circularly polarized fundamental Gaussian beam (green line) and Bessel beam (blue line). At $\theta_0 = \theta_c \equiv \sqrt{2}$ rad $\approx 81^\circ$ the tails of the Gaussian angular spectrum distribution $|A(k_\perp)|^2 \propto \exp[-2(k_x^2 + k_y^2)/\theta_0^2]$ becomes no longer negligible for $k_x^2 + k_y^2 \geq k^2$ where evanescent waves occur. The angular spectrum representation Eq. (2) with $k_z = \pm (k_x^2 - k_y^2 - k_z^2)^{1/2} \in \mathbb{R}$ does not account for evanescent waves, therefore it breaks down for $\theta_0 > \theta_c$. This critical value is marked by a dashed vertical line. Such a problem does not occur for a Bessel beam because in this case the angular spectrum does not possess tails but is sharply peaked about $\theta_0$.

As before, for a circularly polarized beam of light we can choose
\[
\vec{E}(k_\perp) = \vec{E}(k_\perp) e_\perp(\sigma, k_\perp) = |\vec{E}(k_\perp)| \exp[i\vec{\theta}(k_\perp)] e_\perp(\sigma, k_\perp)
\]
and from Eq. (25) it follows that $c \vec{B}(k_\perp) = \vec{E}(k_\perp) \vec{\beta}_\perp(\sigma, k_\perp)$, where
\[
\vec{\beta}_\perp(\sigma, k_\perp) = k \times e_\perp(\sigma, k_\perp)/k = (k \times n_\sigma)/k.
\]

Once again, we assume that the normalization factor for the polarization vector $e_\perp(\sigma, k_\perp)$ is contained in the complex-valued amplitude $\vec{E}(k_\perp)$.

By substituting Eqs. (24-27) into Eqs. (3) we obtain, after a straightforward calculation, the following expressions for the energy per unit length
\[
\frac{U}{\omega} = \frac{\epsilon_0}{2\omega} \int |\vec{E}|^2 d^2k,
\]
for the Poynting vector
\[
S = c^2 \frac{\epsilon_0}{2\omega} \int |\vec{E}|^2 k d^2k,
\]
and for the AM per unit length:

$$J_x = -\frac{\epsilon_0}{2\omega} \int |\mathbf{E}|^2 k_z \frac{\partial \tilde{\theta}}{\partial k_y} dk^2,$$

$$J_y = \frac{\epsilon_0}{2\omega} \int |\mathbf{E}|^2 k_z \frac{\partial \tilde{\theta}}{\partial k_x} dk^2,$$

$$J_z = \frac{\epsilon_0}{2\omega} \int |\mathbf{E}|^2 \left[ \frac{\sigma}{1-\tilde{\theta}^2} + \left( k_x \frac{\partial \tilde{\theta}}{\partial k_y} - k_y \frac{\partial \tilde{\theta}}{\partial k_x} \right) \right] dk^2. \quad (30a)$$

$$J_y = \frac{\epsilon_0}{2\omega} \int |\mathbf{E}|^2 k_z \frac{\partial \tilde{\theta}}{\partial k_x} dk^2,$$

$$J_z = \frac{\epsilon_0}{2\omega} \int |\mathbf{E}|^2 \left[ \frac{\sigma}{1-\tilde{\theta}^2} + \left( k_x \frac{\partial \tilde{\theta}}{\partial k_y} - k_y \frac{\partial \tilde{\theta}}{\partial k_x} \right) \right] dk^2. \quad (30b)$$

$$J_z = \frac{\epsilon_0}{2\omega} \int |\mathbf{E}|^2 \left[ \frac{\sigma}{1-\tilde{\theta}^2} + \left( k_x \frac{\partial \tilde{\theta}}{\partial k_y} - k_y \frac{\partial \tilde{\theta}}{\partial k_x} \right) \right] dk^2. \quad (30c)$$

By comparing Eq. (30c) with the expression of the AM flux below

$$M_{zz} = c^2 \frac{\epsilon_0}{2\omega^2} \int |\mathbf{A}|^2 k_z \left[ \sigma + \left( k_x \frac{\partial \tilde{\alpha}}{\partial k_y} - k_y \frac{\partial \tilde{\alpha}}{\partial k_x} \right) \right] dk^2, \quad (31)$$

one may be tempted to identify the term proportional to $\sigma$ in Eq. (30c) with the spin part of the AM per unit length. However, such a term differs from the expression of $J_z^{\text{spin}}$ given by Eq. (15). Moreover, from Eqs. (14-15) it follows that

$$J_z^{\text{orb}} + J_z^{\text{spin}} = \frac{\epsilon_0}{2\omega} \int |\mathbf{A}|^2 \left[ \frac{\sigma}{1-\tilde{\theta}^2} + \left( k_x \frac{\partial \tilde{\alpha}}{\partial k_y} - k_y \frac{\partial \tilde{\alpha}}{\partial k_x} \right) \right] dk^2, \quad (32)$$

which does not coincide with Eq. (30c). However, if we add $J_z^{\text{surf}}$ to Eq. (32) we obtain the result given by Eq. (18) that we rewrite below:

$$J_z^{\text{orb}} + J_z^{\text{spin}} + J_z^{\text{surf}} = \frac{\epsilon_0}{2\omega} \int |\mathbf{A}|^2 \left[ \frac{\sigma}{1-\tilde{\theta}^2} + \left( k_x \frac{\partial \tilde{\alpha}}{\partial k_y} - k_y \frac{\partial \tilde{\alpha}}{\partial k_x} \right) \right] dk^2. \quad (33)$$

Finally, the two expressions given by Eq. (30c) and Eq. (33) coincide (apart from the trivial factor $\omega$ linking the electric field and the vector potential amplitudes $\mathbf{E}$ and $\mathbf{A}$, respectively). An analogous result is found for the other two components $J_x$ and $J_y$. It should be noted that Eq. (33) is independent from $z$, thus indicating that the total (spin plus orbital plus surface) AM per unit of length is conserved during the free propagation of the beam.

In summary, we can recollect the results obtained in this section as follows: By using the angular spectrum representation for the electromagnetic fields, we have calculated the AM per unit length $J$ for an arbitrary beam of light, either paraxial or non paraxial. Then, we compared this result given by Eqs. (30) with the expressions (14-16) of $J^{\text{orb}}$, $J^{\text{spin}}$, and $J^{\text{surf}}$ and we found that

$$J = J^{\text{orb}} + J^{\text{spin}} + J^{\text{surf}}. \quad (34)$$

The equation above unambiguously shows that the gauge invariant expression of $J$ on the left side can be obtained via the gauge dependent calculation on the right side, only when the SuAM is accounted for, irrespective of the paraxial or non paraxial nature of the beam of light.

4. Conclusions

One of the key aspects of this work, is the illustration of the interplay between paraxial approximation, AM of light and gauge invariance. The actual situation can be schematically illustrated as follows. In the simplest electromagnetic field, the plane wave, the electric and magnetic fields are strictly perpendicular to the direction of propagation of the wave. Differently, for a beam of
finite waist, a nonzero component of the electric field $E_z$ parallel to the direction of propagation $z$ of the beam, is unavoidable \cite{33}. This component, although small with respect to $E_x$ and $E_y$ in collimated beams, is responsible for the difference between A) linear and angular momentum fluxes from one side, and B) energy density and AM per unit of length from the other side. In fact, as shown in \cite{9}, both fluxes depends only upon the transverse fields components $E_x$, $E_y$ and $B_x$, $B_y$. On the opposite, it can be shown that the energy density and the AM per unit of length, receive a contribution from $E_z$ and $B_z$, as well \cite{20}. What quantities are more physically meaningful, A) or B)? There is not a definite answer to this question. In principle, the fluxes obey more general conservation laws and therefore, from a theoretical point of view, are more appealing. On the other hand, the quantities more commonly measured in standard laboratories, are the energy density and the AM per unit of length. In practice, it will be the experimental configuration at hand to set the choice between A) and B), because spin, orbital and total AM of light may be measured by means of different techniques \cite{23}. Many detection apparatuses yield the ratio between the angular momentum per unit length and the energy density per unit length. Typical examples thereof are Stokes parameter measurements in paraxial polarization optics (the third Stokes parameter, usually denotes $s_3$, gives a measure of $\sigma$; see, e.g., \cite{27} and Sec. 6.2 of \cite{31}) and interferometric methods in single photon AM detection \cite{39}. In all these cases, the energy density of light per unit length is measured by means of commonplace photodetectors as CCD cameras, photographic plates, photoresists, etc. and the quantities calculated in this work are the relevant ones.

How gauge invariance enters this discussion? In the paraxial approximations the quantities A) and B) coincide and the spin/orbital separation of the AM is manifestly gauge-invariant in both cases. However, beyond paraxial approximation only the fluxes A) remain explicitly gauge-invariant, as shown by Barnett \cite{9}. Oppositely, gauge-invariance is no longer guaranteed for the AM per unit of length outside paraxial approximation. In fact, in this paper we have demonstrated that the textbook expression for the angular momentum per unit length of a beam of light, containing a spin and an orbital part solely, is not complete and that a third part must be included in order to preserve the gauge invariance of the theory. We call this new term the \textit{surface} AM (SuAM). This quantity is derived considering the virtually infinite extent of the beam along the direction of propagation. We believe that our results may have a relevant conceptual impact upon the very lively and timely research field about AM of light \cite{40,41}.

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