Toward computational algorithm for time-fractional Fokker–Planck models

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Abstract
This article describes an efficient algorithm based on residual power series to approximate the solution of a class of partial differential equations of time-fractional Fokker–Planck model. The fractional derivative is assumed in the Caputo sense. The proposed algorithm gives the solution in a form of rapidly convergent fractional power series with easily computable coefficients. It does not require linearization, discretization, or small perturbation. To test simplicity, potentiality, and practical usefulness of the proposed algorithm, illustrative examples are provided. The approximate solutions of time-fractional Fokker–Planck equations are obtained by the residual power series method are compared with those obtained by other existing methods. The present results and graphics reveal the ability of residual power series method to deal with a wide range of partial fractional differential equations emerging in the modeling of physical phenomena of science and engineering.

Keywords
Caputo fractional derivatives, Fokker–Planck equation, residual power series, numerical algorithms

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Introduction
Recent decades have witnessed great attention toward fractional calculus, which can be considered as a generalization of classical integer-order integration and differentiation. Many definitions have been suggested for fractional derivatives such as Riesz, Riemann-Liouville, Grunwald-Letnikov, Caputo, and conformable fractional definitions.\textsuperscript{1–4} Such differential and integral operators of non-integer order include all historical states of the function in a weighted form called the memory effect. Anyhow, a large number of physical systems are modeled using fractional differential equations (FDEs), particularly fractional partial differential equations (FPDEs). The FPDEs have achieved significance and publicity due to their tremendous use in different fields such as electrochemistry, electrical circuits, theoretical biology, and quantum mechanics.\textsuperscript{5–8} Furthermore, the most significant feature for using the FPDEs in such and other applications is the non-local property, while the differential operator of integer order is local. In this light, since the next state of a fractional system depends not only on the current state but also on its entire historical states. This leads to deep consistency of the mathematical model components in dynamic systems and physical processes. Nevertheless, solving those FDEs is a challenge, especially for numerical calculations. Thus, an effective, reliable, and

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appropriate numerical methods are needed in handling the partial differential equations (PDEs) with a fractional order of physical interest.

The Fokker–Planck equation is one of the classical widely used equations of statistical physics, which was first presented by Fokker and Planck for describing the Brownian motion for particles and the change of probability of a random function in space and time. In addition, chemical Fokker–Planck equation can be derived from an uncontrolled, second-order truncation of the Kramers–Moyal expansion of the chemical master equation. This equation turns out to be more accurate than the linear-noise approximation of the rate equation. This equation is derived from an uncontrolled, second-order truncation of the Kramers–Moyal expansion of the chemical master equation. Anyhow, Fokker–Planck equation arises in the modeling of many natural science phenomena, including quantum optics, electron relaxation, polymer dynamics, solid-state system, probability flux, and other theoretical and practical models.

The concern of this analysis is to consider the numerical approximate solutions of the Fokker–Planck PDE with time-fractional derivative of the following form

\[ D^\beta_t v(x, t) = \left[ -\frac{\partial}{\partial x} f_1(x, t) + \frac{\partial^2}{\partial x^2} f_2(t, x) \right] v(x, t), \]

along with the initial condition

\[ v(x, 0) = v_0(x) \]

where \( D^\beta_t \) is the Caputo time-fractional derivatives, \( v_0(x) \) is the given analytic function on \( \mathbb{R} \), and \( v(x, t) \) is suggested to be a causal function of space and time, which means that \( v(x, t) \) is vanishing for \( x < 0 \) and \( t < 0 \). The smooth functions \( f_1(x, t) \) and \( f_2(t, x) \) are the diffusion and drift coefficients, respectively. Under such assumptions, we assume that the initial value problem (IVP) (1) and (2) has a unique analytical solution on the domain of interest.

The fractional Fokker–Planck equation (F-FPE) has been successfully used in biological molecules, chemical physics, energy consumption, and engineering. Indeed, fractional diffusion, a specific type of F-FPE, has been also applied to several situations such as frequency-dependent damping behavior of materials, viscoelasticity, and diffusion processes. Unfortunately, it is not easy to obtain the exact solution for FDEs in general. So, many numerical and analytical techniques are employed to approximate these solutions. The multi-step reduced differential transform method, the predictor–corrector approach, the Laplace transform method, the variational iteration method (VIM), and Adomain decomposition method (ADM) are some of the advanced numerical and approximate methods that have been applied for F-FPEs.

In this article, the residual power series (RPS) method is implemented for solving IVPs (1) and (2). The RPS technique was developed to approximate solutions for certain class of fuzzy differential equations. Later, it was applied in solving different types of differential equations due to its simplicity, accuracy, and efficiency. The residual power series method (RPSM) has many advantages; first, it is easy to construct a power series solution for handling both linear and non-linear equations without the terms of linearization, discretization, or perturbation. Second, the present method provides the solutions in Taylor expansions; therefore, the exact solutions will be available when the solutions are polynomials. This technique is a direct way to ensure the rate of convergence for series solution, as it depends on minimizing the residual error related. Third, the solutions along with their derivatives can be applied for each arbitrary point in the given interval. Fourth, the RPSM does not require modifications while converting from lower to higher order. Consequently, it has to be easily applied to the proposed system by selecting an appropriate value for the initial guesses approximations. Fifth, the RPS technique needs minor computational requirements with less time and more accuracy. In addition, the presented method is not affected by round-off errors, since it gives the solution in a form of fractional power series (FPS), and substituting values for the solution variables happens as a final step. Finally, it is of global nature in terms of the solutions obtained as well as its ability to solve various types of mathematical, physical, and engineering problems.

The rest of this work is organized as follows. In section “Preliminaries and notations,” some essential definitions and primary results relating to fractional calculus are given. In sections “Description of the RPS algorithm,” the RPS algorithm is presented for solving time-fractional Fokker–Planck equations (TF-FPEs). Numerical and analytical results for some illustrative examples using the RPS algorithm are introduced in section “Numerical experiments.” Meanwhile, numeric comparison between the proposed method and those available in the literature is discussed. Concluding remarks are given in the last section.

Preliminaries and notations

In this section, we revisit some essential definitions and basic properties of popular fractional operators, Riemann–Liouville fractional integral and Caputo fractional derivative. Then, we survey the most important results of the FPS representation. Throughout this analysis, the set of real numbers and the set of natural numbers are denoted by \( \mathbb{R} \) and \( \mathbb{N} \), respectively, while the gamma function is denoted by \( \Gamma \).
Definition 2.1. The integral operator for Riemann–Liouville of order \( \beta \geq 0 \) is given by\(^2\)

\[
\mathcal{J}_t^\beta v(x, t) = \begin{cases} 
\frac{1}{\Gamma(n-\beta)} \int_0^t \frac{v(x, \eta)}{(t-\eta)^{n-\beta}} d\eta, & x \in I, 0 < \eta(t, \beta)0 \\
v(x, t) & \beta = 0
\end{cases}
\]

(3)

where \( I \) is the domain of interest for \( x \).

Next, we present the Caputo fractional derivative\(^3\) of order \( \beta \), which is an alternative operator to the Riemann–Liouville fractional operator as follows

\[
\mathcal{D}_t^\beta u(x) = \frac{1}{\Gamma(n-\beta)} \int_0^t \frac{u^{(n)}(s)}{(t-s)^{n-\beta}} ds, \quad n - 1 < \beta < n \quad \text{for} \quad n \in \mathbb{N}
\]

(4)

In fact, Caputo fractional derivative allows us to include the classical initial and boundary conditions in the formulation of the model, whereas the derivative of a constant is 0. For such reasons, the Caputo sense is considered in this analysis to handle the Fokker–Planck equation.

Definition 2.2. For \( n - 1 < \beta < n, \quad n \in \mathbb{N} \). The Caputo time-fractional derivative operator of order \( \beta \) is defined by\(^1\)

\[
\mathcal{D}_t^\beta v(x, t) = \begin{cases} 
\frac{1}{\Gamma(n-\beta)} \int_0^t \frac{\partial^n v(x, \xi)}{\partial \xi^n} (t-\xi)^{n-\beta-1} d\xi, & n - 1 < \beta < n \\
\frac{\partial^n v(x, t)}{\partial \xi^n}, & \beta = n \in \mathbb{N}
\end{cases}
\]

(5)

Similarly, the Caputo space-fractional derivative operator of order \( \beta \) is defined by

\[
\mathcal{D}_x^\beta v(x, t) = \begin{cases} 
\frac{1}{\Gamma(n-\beta)} \int_0^t \frac{\partial^n v(z, t)}{\partial z^n} (x-z)^{\alpha-\beta-1} d\zeta, & n - 1 < \beta < n \\
\frac{\partial^n v(z, t)}{\partial z^n}, & \beta = n \in \mathbb{N}
\end{cases}
\]

(6)

Theorem 2.1. If \( n - 1 < \alpha \leq n \) and \( n \in \mathbb{N} \), then \( \mathcal{D}_t^\beta \mathcal{J}_t^\beta v(x, t) = v(x, t) \) and \( \mathcal{J}_t^\beta \mathcal{D}_t^\beta v(x, t) = v(x, t) - \sum_{j=0}^{n-1} (\partial v(x, 0^+)) / (\partial t^j / \Gamma(j + 1)) \), where \( x \in I, t > 0 \).

The following are some properties of the operators \( \mathcal{D}_x^\beta \) and \( \mathcal{D}_t^\beta \) for \( \beta > 0, \beta > -1 \), that can be found in Podlubny\(^1\)

\[
\mathcal{D}_t^\beta t^\beta = \frac{\Gamma(q + 1)}{\Gamma(q - \beta + 1)} t^{\beta - q}
\]

(7)

\[
\mathcal{J}_t^\beta t^\beta = \frac{\Gamma(q + 1)}{\Gamma(q + \beta + 1)} t^{\beta + q}
\]

(8)

Definition 2.3. An FPS representation at \( t_0 \) has the following form\(^19\)

\[
\sum_{m=0}^{\infty} v_m(x)(t-t_0)^{m\beta} = v_0(x) + v_1(x)(t-t_0)^\beta + \ldots
\]

(9)

where \( 0 \leq n - 1 < \beta \leq n \), \( x \in I \) and \( t \geq t_0 \) is called multiple fractional power series (MFPS) about \( t_0 \).

Theorem 2.2. Suppose that \( v(x, t) \) has the following MFPS representation at \( t = t_0 \):

\[
v(x, t) = \sum_{m=0}^{\infty} v_m(x)(t-t_0)^{m\beta},
\]

(10)

then the coefficients \( v_m(x) \) will be in the form

\[
v_m(x) = (D_{t_0}^{(m\beta)} v(x, t))/(\Gamma(m\beta + 1)),
\]

such that \( D_{t_0}^{(m\beta)} = D_{t_0}^\beta \cdot D_{t_0}^\beta \cdot \ldots \cdot D_{t_0}^\beta \), where \( \rho \) is a positive real number, called the radius of convergence for the MFPS.

Description of the RPS algorithm

The main goal of this section is to present the methodology of the RPS technique in obtaining the MFPS approximation of the time-fractional Fokker–Planck model based on the formula of generalized Taylor in Caputo sense by providing a fractional recursion formula to obtain the coefficients of the MFPS depending on minimizing the residual function. To do this, let us assume the solution \( v(x, t) \) of TF-FPES (1) and (2) has the following MFPS expansion about \( t_0 = 0 \)

\[
v(x, t) = \sum_{m=0}^{\infty} v_m(x) t^{m\beta} = v_0(x) + \sum_{m=1}^{\infty} v_m(x) t^{m\beta},
\]

(11)

By starting with the initial guess approximation \( v(x, 0) = v_0(x) \), the series solution of equation (11) can be rewritten as

\[
v(x, t) = v_0(x) + \sum_{n=1}^{\infty} v_n(x) t^{n\beta}
\]

(12)

To obtain the MFPS approximate solution, let \( v(x, t) \) indicate the \( n \)-th-truncated MFPS of \( v(x, t) \) such that
\[ v_k(x, t) = v_0(x) + \sum_{n=1}^{k} v_n(x) \frac{\rho^n}{\Gamma(n\rho + 1)} \]  
\[ \text{where } n \in \mathbb{N}, \rho > 0, \text{and } \Gamma(z) \text{ is the gamma function.} \]

Now, define the \( k \)-th residual function as follows

\[ \text{Res}_k(x, t) = D_t^\rho v_k(x, t) + \left[ \frac{\partial}{\partial x} f_i(x, t) - \frac{\partial^2}{\partial x^2} f_2(t, x) \right] v(x, t) \]

where the residual function can be given in the form

\[ \text{Res}_v(x, t) = \lim_{k \to \infty} \text{Res}_k(x, t) \]

Evidently, \( \text{Res}_v(x, t) = 0 \) and \( \lim_{k \to \infty} \text{Res}_k(x, t) = 0 \) for each \( x \in \mathbb{R} \) and \( 0 < t < \rho \), where \( \rho \) is the radius of convergence for the MFPS (11). According to the RPSM,25–27,30 it can be noted that \( D_t^\rho \text{Res}_v(x, t) = 0 \). In addition, \( D_t^{(k-1)\rho} \text{Res}_k(x, t) \to 0 = D_t^{(j-1)\rho} \text{Res}_j(x, t) \) for each \( j = 1, 2, \ldots, k \). Therefore, the following relations can hold

\[ D_t^{(k-1)\rho} \text{Res}_k(x, t) = 0, \quad k = 1, 2, 3, \ldots \]

These relations help us to determine the values of the coefficients \( v_n(x), n = 1, 2, \ldots, k \). And so, the approximate solution for TF-FPEs (1) and (2) has been completely constructed.

Anyhow, the next algorithm clarifies the procedure in obtaining the unknown coefficients of equation (13).

**Algorithm 3.1.** To determine the required coefficients of \( v_k(x, t) \), do the following steps:

1. **Step 1:** The initial condition \( v(x, 0) = v_0(x) \), which is the zeroth FPS approximate solution of \( v(t) \).
2. **Step 2:** The \( k \)-th truncated MFPS \( v_k(x, t) = v_0(x) + \sum_{n=1}^{k} v_n(x) (\rho^n / \Gamma(n\rho + 1)) \) into the \( k \)-th residual function \( \text{Res}_k(x, t) \) of equation (14).
3. **Step 3:** \( D_t^{(k-1)\rho} \text{Res}_k(x, t) \) for \( k = 1, 2, \ldots, N \).
4. **Step 4:** The resulting fractional equations \( D_t^{(k-1)\rho} \text{Res}_k(x, t) = 0 \) for \( v_k(x) \).
5. **Step 5:** The obtained coefficients, for \( k = 1, 2, \ldots, N \), back into equation (13).

**Numerical experiments**

The purpose of this section is to show the high degree of accuracy, efficiency, and applicability of this algorithm. The approximate analytical solutions of TF-FPEs are constructed in a rapidly convergent FPS form. Numeric comparisons of the results obtained by the proposed method, ADM13 and VIM13 are provided. The tabular and graphical results reveal that the RPS approach is easy to implement and accurate when applied to the TF-FPEs, as well as it introduces a promising tool for solving many fractional PDEs. The present computations are performed using Mathematica 10 (Wolfram Mathematica) software package.

**Example 4.1.** Consider the following TF-FPE

\[ D_t^\rho v(x, t) = \left[ \frac{\partial}{\partial x} x^2 + \frac{\partial^2}{\partial x^2} x^2 / 2 \right] v(x, t), \quad x \in [0, 2], t > 0, 0 < \beta \leq 1 \]

with the initial conditions

\[ v(x, 0) = x \]

The exact solution of IVPs (17) and (18) for standard case at \( \beta = 1 \) is given by \( v(x, t) = x e^t \).

Using the last description of RPS algorithm, the solution of IVPs (17) and (18) is

\[ v(x, t) = x + \sum_{n=1}^{\infty} v_n(x) (\rho^n / \Gamma(n\rho + 1)) \]

and \( k \)-th-residual function as follows

\[ \text{Res}_k(x, t) = D_t^\rho v_k(x, t) + \frac{\partial}{\partial x} (x^2 v_k(x, t)) - \frac{\partial^2}{\partial x^2} \left( \frac{x^2}{2} \right) v_k(x, t) \]

For \( k = 1 \), the first residual function is

\[ \text{Res}_1(x, t) = D_t^\rho \left( v_0(x) + \sum_{n=1}^{1} v_n(x) (\rho^n / \Gamma(n\rho + 1)) \right) \]

Depending on equation (16), the first unknown coefficient of MFPS expansion is \( v_1(x) = x \). Hence, the first FPS approximate solution is

\[ v_1(x) = x - x \frac{\rho^1}{\Gamma(\beta + 1)} \]
\[ v_1(x,t) = x + x \frac{t^\beta}{\Gamma(\beta + 1)} \] (21)

As the former, to determine the second coefficient \( v_2(x) \), consider \( k = 2 \) in the MFPS (13) and then substitute \( v_2(x,t) \) into the second residual function \( \text{Res}_2(x,t) \) of equation (19) to get

\[
\text{Res}_2^2(x,t) = (2v_2(x) - 2x) \frac{t^\beta}{\Gamma(\beta + 1)} - x\left(2v_2'(x) + xv_2''(x)\right) \frac{t^\beta}{\Gamma(2\beta + 1)}
\] (22)

By applying \( D_\beta^t \) on both sides of equation (22), it follows that

\[
D_\beta^t \text{Res}_2^2(x,t) = (2v_2(x) - 2x) \frac{t^\beta}{\Gamma(\beta + 1)} - \left(2v_2'(x) + xv_2''(x)\right) \frac{t^\beta}{\Gamma(2\beta + 1)}
\] (23)

Using the fact that \( D_\beta^t \text{Res}_2^2(x,0) = 0 \), the second unknown coefficient in the MFPS is \( v_2(x) = x \). Therefore, the second FPS approximate solution is

\[
v_2(x,t) = x + x \frac{t^\beta}{\Gamma(\beta + 1)} + x \frac{t^\beta}{\Gamma(2\beta + 1)}
\] (24)

Applying similar argument for \( k = 3 \), the third unknown coefficient in the MFPS (13) will be \( v_3(x) = x \). Moreover, the third FPS approximate solution can be written as

\[
v_3(x,t) = x + x \frac{t^\beta}{\Gamma(\beta + 1)} + x \frac{t^\beta}{\Gamma(2\beta + 1)} + x \frac{t^\beta}{\Gamma(3\beta + 1)}
\] (25)

In the same manner, the process can be repeated till the arbitrary order and then the coefficients of the MFPS solution (13) can be obtained. Consequently, we have \( v_k(x) = x \) for \( k \geq 1 \). Furthermore, if we collect all the last results, the solution \( v(x,t) \) can be given as follows

\[
v(x,t) = x \left(1 + \frac{t^\beta}{\Gamma(\beta + 1)} + \frac{t^\beta}{\Gamma(2\beta + 1)} + \frac{t^\beta}{\Gamma(3\beta + 1)} + \ldots + \frac{t^\beta}{\Gamma(k\beta + 1)} + \ldots\right)
\] (26)

where \( E_\beta(t) = \sum_{n=0}^{\infty} (t^n / \Gamma(n\beta + 1)) \) is the Mittag-Leffler function. By setting \( \beta = 1 \), equation (26) can be reduced to \( v(x,t) = xe^t \), which is the exact solution of the classical form of IVPs (17) and (18).

In view of the obtained previous results and without loss of generality, the geometric behavior of the 10th FPS approximate solution of IVPs (17) and (18) has been studied by drawing the three-dimensional (3D) space figures at different values of \( \beta \) for \( x \in [0,2] \) and \( t \in [0,10] \). Figure 1 represents a comparison between the 10th FPS approximate solution for \( \beta \in \{1.0, 0.95, 0.75, 0.5\} \) and the exact solution. Regarding these figures, it is worth mentioning that running the RPS algorithm via Mathematica 10.0 to get such graphs took 0.66, 80.77, 88.41, and 90.22 times in seconds for \( \beta \in \{1.0, 0.95, 0.75, 0.5\} \), respectively. Moreover, to show the efficiency and accuracy of the present algorithm, the numerical results at \( x = 1 \) with some selected grid points \( t \) with step size 0.1 on \([0,1]\) are given in Table 1. The results in Table 1 show that the FPS approximate solutions are in good agreement with exact solutions.

On the contrary, by applying the ADM, we have the following iteration

\[
v_0(x,t) = x
\]

and

\[
v_{k+1}(x,t) = J_\beta^t \left[-\frac{\partial}{\partial x} x + \frac{\partial^2 x^2}{\partial x^2} \right] v_k(x,t)
\]

Using the property of equation (8), it follows that

\[
v_1(x,t) = x \frac{t^\beta}{\Gamma(\beta + 1)}
\]

\[
v_2(x,t) = x \frac{t^\beta}{\Gamma(2\beta + 1)}
\]

\[
v_3(x,t) = x \frac{t^\beta}{\Gamma(3\beta + 1)}
\]

Continuing this process, the \( n \)th approximate solution is \( v_n(x,t) = x(t^\beta / \Gamma(n\beta + 1)) \).

According to the ADM, the \( k \)th ADM solution of IVPs (17) and (18) is given by

\[
v_k(x,t) = x \left(1 + \frac{t^\beta}{\Gamma(\beta + 1)} + \frac{t^\beta}{\Gamma(2\beta + 1)} + \frac{t^\beta}{\Gamma(3\beta + 1)} + \ldots + \frac{t^\beta}{\Gamma(k\beta + 1)} + \ldots\right)
\]

Hence, the solution is \( v(x,t) = \lim_{k \to \infty} v_k(x,t) = x \sum_{n=0}^{\infty} (t^\beta / \Gamma(n\beta + 1)) = xe^t \).

Obviously, the RPSM produced an identical analytical solution of the ADM solution for this example. Anyhow, to see the effect of the fractional derivative to
Fokker–Planck equation, the tabulated and graphical results for the approximate solutions at different values of fractional order \( \beta \) using the RPSM and (ADM) with \( k = 20 \) are summarized and listed in Table 2 and Figure 2. It is obvious from the current results that the RPS algorithm is of good agreements with earlier literature works, and the RPS solution of TF-FPE approaches to the solution of the classical case as soon as \( \beta \) approaches to 1.

Table 1. Numerical results of the 10th FPS approximate solution for Example 4.1 at \( \beta = 1 \).

| \( t \) | Exact solution | Approximation solution | Absolute error | Relative error |
|-------|----------------|------------------------|----------------|----------------|
| 0.1   | 1.1051709180756477 | 1.1051709180756475 | 2.22045 \times 10^{-16} | 2.00914 \times 10^{-16} |
| 0.2   | 1.2214027581601699  | 1.2214027581601692  | 6.66134 \times 10^{-16} | 5.45384 \times 10^{-16} |
| 0.3   | 1.3498588075760032  | 1.3498588075759777  | 4.55919 \times 10^{-14} | 3.37214 \times 10^{-14} |
| 0.4   | 1.4918246976412703  | 1.4918246976401834  | 1.08691 \times 10^{-12} | 7.28576 \times 10^{-12} |
| 0.5   | 1.6487212707001282  | 1.6487212706873657  | 1.76250 \times 10^{-11} | 7.74082 \times 10^{-11} |
| 0.6   | 1.8221188003905090  | 1.8221188002948574  | 9.56517 \times 10^{-11} | 5.24948 \times 10^{-11} |
| 0.7   | 2.0137527074704766  | 2.013752706945813   | 5.25895 \times 10^{-10} | 2.61152 \times 10^{-10} |
| 0.8   | 2.2255409284924680  | 2.2255409261876826  | 2.30479 \times 10^{-9}  | 1.03561 \times 10^{-9}  |
| 0.9   | 2.459603111569500   | 2.4596031026621000  | 8.49485 \times 10^{-9}  | 3.45375 \times 10^{-9}  |

FPS: fractional power series.

**Example 4.2.** Consider the following TF-FPE

\[
D^\beta_t v(x, t) = \left[ -\frac{\partial}{\partial x} x^2 + \frac{\partial^2}{\partial x^2} x^2 \right] v(x, t),
\]

for \( x \in [0, 2], \ t \geq 0, \ 0 < \beta \leq 1 \) with the initial conditions

\[
v(x, 0) = x^2
\]
The exact solution of IVPs (27) and (28) for standard case at β = 1 is given by \( v(x, t) = x^2 e^t \)

In view of the RPS technique, by starting with \( v_0(x) = x^2 \) as the initial approximation, the \( k \)-th residual function of IVPs (27) and (28) can be written as

\[
\text{Res}_k^\beta(v, t) = D^\beta_t v_k(x, t) + \frac{\partial}{\partial x} \left( \frac{x}{6} v_k(x, t) \right) - \frac{\partial^2}{\partial x^2} \left( \frac{x^2}{12} v_k(x, t) \right)
\]

(29)

where \( v_k(x, t) \) is the \( k \)-th-truncated MFPS given by equation (13). Thus, for \( k = 1 \), the first residual function is given by

\[
\text{Res}_1^\beta(v, t) = v_1(x)
\]

\[
+ \frac{1}{12} x \left( -6x - (2v'(1) + xv''(1)) \right) \frac{t^\beta}{\Gamma(\beta + 1)}
\]

(30)

Based on the result of equation (16), it yields that \( v_1(x) = x^2/2 \). Therefore, the first FPS approximate solution is

\[
v_1(x, t) = x^2 + \frac{x^2}{2 \Gamma(\beta + 1)} \frac{t^\beta}{\Gamma(\beta + 1)}
\]

(31)

To determine the second coefficient, let \( k = 2 \) in the \( k \)-th-truncated MFPS (13), and substitute \( v_2(x, t) \) into the \( \text{Res}_2^\beta(v, t) \) of equation (30) to get

\[
\text{Res}_2^\beta(v, t) = \frac{1}{12} (12v_1(x) - 6x^2)
\]

\[
+ (12v_2(x) - x(2v'_1(x) + xv''_1(x))) \frac{t^\beta}{\Gamma(\beta + 1)}
\]

\[
- x(2v'_2(x) + xv''_2(x)) \frac{t^\beta}{\Gamma(2\beta + 1)}
\]

(32)

By considering the fact of equation (16) and solving \( D^\beta_t \text{Res}_2^\beta(v, 0) = 0 \) for \( v_2(x) \), then it can obtain that \( v_2(x) = x^3/4 \). Hence, the second FPS approximate solution can be written as

\[
v_2(x, t) = x^2 + \frac{x^2}{2 \Gamma(\beta + 1)} \frac{t^\beta}{\Gamma(\beta + 1)} + \frac{x^2}{4 \Gamma(2\beta + 1)} \frac{t^\beta}{\Gamma(2\beta + 1)}
\]

(33)

For the third unknown coefficient, substitute \( v_3(x, t) \) into \( \text{Res}_3^\beta(v, t) \) of equation (30) to get

\[
\text{Res}_3^\beta(v, t) = \frac{1}{12} (12v_1(x) - 6x^2)
\]

\[
+ (12v_2(x) - x(2v'_1(x) + xv''_1(x))) \frac{t^\beta}{\Gamma(\beta + 1)}
\]

\[
+ (12v_3(x) - x(2v'_2(x) + xv''_2(x))) \frac{t^\beta}{\Gamma(2\beta + 1)}
\]

\[
+ x(2v'_3(x) + xv''_3(x)) \frac{t^\beta}{\Gamma(3\beta + 1)}
\]

(34)
Now, compute $D_t^{2\beta}R_3(x,t)$ for equation (34) and use $D_t^{\beta}R_3(x,0) = 0$ to get that $v_3(x) = \frac{x^2}{8}$. Therefore, the third FPS approximate solution is given by

$$v_3(x,t) = x^2 + \frac{x^2}{2} \frac{t^{\beta}}{\Gamma(\beta + 1)} + \frac{x^2}{4} \frac{t^{2\beta}}{\Gamma(2\beta + 1)} + \frac{x^2}{8} \frac{t^{3\beta}}{\Gamma(3\beta + 1)}$$

(35)

Using the same process for $k \geq 4$, the $k$th unknown coefficient $v_k(x)$ can be obtained. Consequently, the solution $v(x,t)$ of IVPs (27) and (28) can be expressed in the form of an infinite series given by

$$v(x,t) = x^2 \left( 1 + \frac{t^{\beta}}{2\Gamma(\beta + 1)} + \frac{t^{2\beta}}{2\Gamma(2\beta + 1)} + \frac{t^{3\beta}}{2\Gamma(3\beta + 1)} + \ldots \right) = x^2 \sum_{n=0}^{\infty} \frac{t^{n\beta}}{2^n \Gamma(n\beta + 1)}$$

(36)

For $\beta = 1$, the result in equation (36) can be reduced to $v(x,t) = x^2 e^t$, which represents the exact solution for classical form of TF-FPEs.

In view of the previous discussion, the geometric behavior of the 10th FPS approximate solution of IVPs (27) and (28) has been constructed and presented in Figure 3 by drawing the 3D space graphs at different values of $\beta$ for each $x \in [0, 2]$ and $t \in [0, 10]$. Furthermore, Figure 3 represents a comparison between the 10th FPS approximate solution for $\beta = \{0.95, 0.75, 0.5\}$ and the solution when $\beta = 1$. Regarding these figures, it is worth mentioning that running the RPS algorithm via Mathematica 10.0 to get such graphs took 0.75, 79.66, 79.84, and 93.72 times in seconds for $\beta \in \{1.0, 0.95, 0.75, 0.5\}$, respectively. It should be noted that the FPS approximate solutions are in good agreement with each other.

To illustrate the efficiency and accuracy of the fractional residual power series (FRPS) algorithm, some numerical results at fixed value of $x = 1$ and some selected grid points $t$ with step size 0.1 on $[0, 1]$ and $k = 10$ are given in Table 3.

On the contrary, by applying the ADM, we have the following iteration...
Table 3. Numerical results of the 10th FPS approximate solution for Example 4.2 at $\beta = 1$.

| $t$ | Exact solution | Approximation solution | Absolute error | Relative error |
|-----|----------------|------------------------|----------------|---------------|
| 0.1 | 1.0512710963760241 | 1.0512710963760241 | 0.000000 | 0.000000 |
| 0.2 | 1.1051709180756477 | 1.1051709180756475 | 2.22045 × 10^{-16} | 2.00914 × 10^{-16} |
| 0.3 | 1.161834247282830 | 1.161834247282830 | 0.000000 | 0.000000 |
| 0.4 | 1.221407851601699 | 1.221407851601699 | 6.66134 × 10^{-16} | 5.43584 × 10^{-16} |
| 0.5 | 1.2840254166877354 | 1.2840254166877354 | 5.99520 × 10^{-15} | 4.69907 × 10^{-15} |
| 0.6 | 1.3498588075760032 | 1.3498588075760037 | 4.55191 × 10^{-14} | 3.37214 × 10^{-14} |
| 0.7 | 1.4190675485932573 | 1.4190675485930082 | 2.49134 × 10^{-13} | 1.75562 × 10^{-13} |
| 0.8 | 1.4918246976412703 | 1.491824697641834 | 1.08691 × 10^{-12} | 7.28576 × 10^{-13} |
| 0.9 | 1.5683121854916900 | 1.5683121854861810 | 3.98792 × 10^{-12} | 2.54281 × 10^{-12} |

FPS: fractional power series.

Table 4. The FPS approximate solutions for Example 4.2 at $x = 1$ and $k = 10$.

| $t$ | $\beta = 1$ | $\beta = 0.9$ | $\beta = 0.8$ | $\beta = 0.7$ | $\beta = 0.6$ | $\beta = 0.5$ |
|-----|-------------|-------------|-------------|-------------|-------------|-------------|
| 0.1 | 1.0512710963760241 | 1.067873047 | 1.089647419 | 1.118280169 | 1.156149183 | 1.206731021 |
| 0.2 | 1.1051709180756477 | 1.130764608 | 1.162383287 | 1.201599744 | 1.250543786 | 1.312165429 |
| 0.3 | 1.161834247282830 | 1.194217917 | 1.232886661 | 1.279239821 | 1.335069142 | 1.402828108 |
| 0.4 | 1.221407851601699 | 1.259271717 | 1.303403550 | 1.354966929 | 1.415525830 | 1.486763385 |
| 0.5 | 1.2840254166877354 | 1.326444650 | 1.374962402 | 1.430442617 | 1.494100595 | 1.567059193 |
| 0.6 | 1.3498588075760032 | 1.396085151 | 1.448036274 | 1.506438647 | 1.572012045 | 1.645293775 |
| 0.7 | 1.4190675485932573 | 1.468464873 | 1.523137711 | 1.583537184 | 1.649908028 | 1.722406977 |
| 0.8 | 1.4918246976412703 | 1.543818988 | 1.600527385 | 1.662132979 | 1.728533408 | 1.799016640 |
| 0.9 | 1.5683121854916900 | 1.622364270 | 1.680474811 | 1.742531667 | 1.808003639 | 1.875558788 |
| 1.0 | 1.648721271 | 1.704308722 | 1.763203672 | 1.824985031 | 1.888644747 | 1.952358374 |

FPS: fractional power series.

$$v_0(x, t) = x^2$$ and $$v_{k+1}(x, t)$$

$$= J^\beta_t \left[ -\frac{\partial}{\partial x} x + \frac{\partial^2}{\partial x^2} \frac{x^2}{12} \right] v_k(x, t)$$

Using the property of equation (8), we get

$$v_1(x, t) = x^2 \frac{t^\beta}{2\Gamma(\beta + 1)}$$

$$v_2(x, t) = J^\beta_t \left[ -\frac{\partial}{\partial x} x + \frac{\partial^2}{\partial x^2} \frac{x^2}{2} \right]$$

$$\left( x^2 \frac{t^\beta}{2\Gamma(\beta + 1)} \right) = x^2 \frac{t^\beta}{4\Gamma(2\beta + 1)}$$

$$v_3(x, t) = J^\beta_t \left[ -\frac{\partial}{\partial x} x + \frac{\partial^2}{\partial x^2} \frac{x^2}{2} \right]$$

$$\left( x^2 \frac{t^\beta}{4\Gamma(2\beta + 1)} \right) = x^2 \frac{t^\beta}{8\Gamma(3\beta + 1)}$$

and so on; therefore, in this manner, the rest of terms can be obtained. So, the ADM solution of IVPs (27) and (28) is

$$v(x, t) = \lim_{k \to \infty} v_k(x, t) = x^2 + \lim_{k \to \infty} \sum_{n=0}^{k} \frac{t^{\beta n}}{2^n \Gamma(n^\beta + 1)}$$

The previous result is exactly in agreement with the result obtained by the RPSM. To see the effect of the fractional derivative to Fokker–Planck equation, the tabulated and graphical results for the approximate solutions at different values of fractional order $\beta$ by using the RPSM and ADM$^{13}$ with $k = 10$ are given in Table 4 and Figure 4. It is obvious from these results that the RPS solution of TF-FPE approaches to the solution of the classical case as soon as $\beta$ approaches to 1.

In addition, the VIM$^{13}$ can be used to solve the IVPs (27) and (28) with the following iterations

$$v_0(x, t) = x^2$$ and $$v_{k+1}(x, t) = v_k(x, t) - \int_0^t D^\beta_t v_k(x, \tau)$$

$$\left[ -\frac{\partial}{\partial x} x + \frac{\partial^2}{\partial x^2} \frac{x^2}{2} \right] v_k(x, \tau) d\tau$$

Using this iteration with the help of property of equation (7), it follows that

$$v_0(x, t) = x^2$$

$$v_1(x, t) = x^2 \left( 1 + \frac{1}{2} \right)$$

$$v_2(x, t) = x \left( 1 + t + \frac{t^2}{8} - \frac{t^{2-\beta}}{2\Gamma(3-\beta)} \right)$$
Planck PDEs of fractional order with fitted initial conditions. The RPS algorithm has been applied directly to obtain the solution in rapidly convergent MFPS without being linearized, discretized, or exposed to perturbation. Graphs and numerical results show that the proposed method is complete reliability and performance with great potential for use in many scientific applications. The present results show that the RPS technique is a simple and quite powerful tool in finding the approximate solutions for different kinds of fractional PDEs. A comparison between the RPSM and those available in the literature are carried out through numerical examples. High agreements of numerical results are clear and remarkable.

### Concluding remarks

Developing analytical and numerical solutions for fractional mathematical models of physical and chemical phenomena are very essential in science. In this work, an analytic-approximate method, so-called RPS, has been employed effectively to solve a class of Fokker–Planck PDEs of fractional order with fitted initial conditions. The RPS algorithm has been applied directly to obtain the solution in rapidly convergent MFPS without being linearized, discretized, or exposed to perturbation. Graphs and numerical results show that the proposed method is complete reliability and performance with great potential for use in many scientific applications. The present results show that the RPS technique is a simple and quite powerful tool in finding the approximate solutions for different kinds of fractional PDEs. A comparison between the RPSM and those available in the literature are carried out through numerical examples. High agreements of numerical results are clear and remarkable.

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The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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**Table 5.** A comparison of the RPS, ADM, and VIM of Example 4.2 for different values of $\beta$.

| $t$ | $x$ | $\beta = 0.5$ | $\beta = 0.75$ |
|-----|-----|---------------|---------------|
|     |     | RPS           | ADM           | VIM           | RPS           | ADM           | VIM           |
| 0.2 | 0.25| 0.0820103     | 0.0820103     | 0.07320989   | 0.0738078    | 0.0738078    | 0.07162351   |
|     | 0.5 | 0.328041      | 0.328041      | 0.29283956   | 0.2952311    | 0.2952311    | 0.28649408   |
|     | 0.75| 0.738093      | 0.738093      | 0.65888901   | 0.664270     | 0.664270     | 0.64461167   |
| 0.4 | 0.25| 1.312165      | 1.312165      | 1.17135823   | 1.1809243    | 1.1809243    | 1.14597630   |
|     | 0.5 | 0.371691      | 0.371691      | 0.33121168   | 0.3320456    | 0.3320456    | 0.31990430   |
|     | 0.75| 0.836304      | 0.836304      | 0.74522628   | 0.747103     | 0.747103     | 0.71978468   |
| 0.6 | 0.25| 1.486763      | 1.486763      | 1.32484671   | 1.3281823    | 1.3281823    | 1.27961720   |
|     | 0.5 | 0.411323      | 0.411323      | 0.36754806   | 0.369096     | 0.369096     | 0.35299034   |
|     | 0.75| 0.925478      | 0.925478      | 0.82698314   | 0.830466     | 0.830466     | 0.79422825   |
| 0.8 | 0.25| 1.645294      | 1.645294      | 1.47019225   | 1.476385     | 1.476385     | 1.41196134   |
|     | 0.5 | 0.449754      | 0.449754      | 0.40271647   | 0.407679     | 0.407679     | 0.38652789   |
|     | 0.75| 1.011947      | 1.011947      | 0.90611205   | 0.917277     | 0.917277     | 0.86968775   |
|     | 1   | 1.799017      | 1.799017      | 1.61086586   | 1.630715     | 1.630715     | 1.54611160   |

RPS: residual power series; ADM: Adomain decomposition method; VIM: variational iteration method.
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