THE ISOMORPHISM PROBLEM FOR SCHUBERT VARIETIES

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Abstract. Schubert varieties in the full flag variety of Kac-Moody type are indexed by elements of the corresponding Weyl group. We give a practical criterion for when two such Schubert varieties (from potentially different flag varieties) are isomorphic, in terms of the Cartan matrix and reduced words for the indexing Weyl group elements. As a corollary, we show that two such Schubert varieties are isomorphic if and only if there is an isomorphism between their integral cohomology rings that preserves the Schubert basis.

1. Introduction

Kac-Moody flag varieties are central objects of study in geometry, topology, and representation theory. In particular, the finite-type Kac-Moody flag varieties are the usual generalized flag varieties associated to semisimple Lie groups. While generalized flag varieties are finite-dimensional, Kac-Moody flag varieties of non-finite type are infinite-dimensional. However, all Kac-Moody flag varieties can be realized as ind-varieties stratified by finite-dimensional Schubert varieties [Kum87]. Schubert varieties are themselves important examples of algebraic varieties, and their singularities are closely connected with the representation theory of the corresponding Kac-Moody groups and algebras. As a result, their geometry has been closely studied (see, for instance, the surveys [BL00 AB16]).

We are interested in the isomorphism problem for Schubert varieties: when are two Schubert varieties isomorphic as algebraic varieties? This natural geometric question was first raised for Schubert varieties by Develin, Martin, and Reiner [DMR07]. They show that two partition varieties (a subclass of type A Schubert varieties introduced by Ding [Din97]) are isomorphic if and only if their integral cohomology rings are isomorphic. Aside from this, and the case of toric Schubert varieties which we cover below, the question does not seem to have been pursued further, even in type A or other finite types. In this paper, we give a complete solution to the isomorphism problem for Schubert varieties in full flag varieties of any Kac-Moody type over \( \mathbb{C} \).

To describe this solution, it is helpful to recall some basic facts about Schubert varieties. We follow the conventions from [Kum02]. Recall that the starting data for constructing a Kac-Moody Lie algebra (and subsequently it’s Kac-Moody group, full flag variety, and Schubert varieties) is a (generalized) Cartan matrix, which is an integer matrix \( A := [A_{st}]_{(s,t) \in S^2} \) indexed by some finite set \( S \), such that for any \( s,t \in S \),

1. \( A_{st} = 2 \) if \( s = t \),
2. \( A_{st} \leq 0 \) if \( s \neq t \), and
3. \( A_{st} = 0 \) if and only if \( A_{ts} = 0 \).

Let \( G := G(A) \) and \( B := B(A) \) denote the Kac-Moody group and Borel subgroup of a Cartan matrix \( A \). The full flag variety corresponding to \( A \) is the quotient \( X = X(A) := G/B \). The Weyl group \( W := W(A) \) of \( A \) is the crystallographic Coxeter group generated by \( S \), and
satisfying relations \((st)^{m_{st}} = e\), where \(m_{ss} = 1\), and

\[
m_{st} = \begin{cases} 
2 & \text{if } A_{st}A_{ts} = 0 \\
3 & \text{if } A_{st}A_{ts} = 1 \\
4 & \text{if } A_{st}A_{ts} = 2 \\
6 & \text{if } A_{st}A_{ts} = 3 \\
\infty & \text{if } A_{st}A_{ts} \geq 4 
\end{cases}
\]

for all \(s, t \in S\). The pair \((W, S)\) forms a Coxeter system, and the elements of \(S\) are referred to as the **simple reflections** (or **simple transpositions**) in \(W\). For every \(w \in W\), the **Schubert variety** \(X(w, A)\) is defined to be the closure of \(BwB/B\) in \(X\). It is well known that \(X(w, A)\) is an irreducible finite dimensional complex variety of dimension \(\ell(w)\), where \(\ell : W \to \mathbb{Z}_{\geq 0}\) is the length function of the Coxeter system \((W, S)\). A product \(w = s_1 \cdots s_k\) of simple reflections \(s_1, \ldots, s_k\) in \(W\) is a **reduced word** if \(k = \ell(w)\). Every element of \(W\) can be written as a reduced word.

If \(w = s\) is a simple reflection, then \(X(w, A) \cong \mathbb{P}^1\), and hence all one-dimensional Schubert varieties are isomorphic, independent of \(A\). The case of two-dimensional Schubert varieties is more interesting:

**Example 1.1.** Suppose \(X(w, A)\) is a two-dimensional Schubert variety, so \(w = st \in S\) for some \(s \neq t\). Then \(X(w, A)\) is a Zariski-locally-trivial \(\mathbb{P}^1\)-bundle over \(\mathbb{P}^1\). The cohomology ring \(H^*(X(w, A); \mathbb{Z})\) is the free \(\mathbb{Z}\)-module generated by the Schubert classes \(\xi_u\) for \(u \in \{e, s, t, w\}\), where \(\xi_u\) has degree \(2\ell(u)\) and \(e\) is the identity. The ring structure is determined by the relations \(\xi_s^2 = 0, \xi_s : \xi_t = \xi_w,\) and \(\xi_t^2 = -A_{st}\xi_w\). From this, it follows that \(X(w, A)\) is the Hirzebruch surface \(\Sigma_n\), where \(n = -A_{st}\).

It is well-known that \(\Sigma_n \cong \Sigma_m\) if and only if \(m = n\). Hence \(X(w, A) \cong X(w', A')\) with \(w' = s't'\) if and only if \(A_{st} = A'_{s't'}\). In particular, the isomorphism type of \(X(w, A)\) depends only on the value of \(A_{st}\), not on \(A_{ts}\).

Let \(\leq\) denote the Bruhat order for the Coxeter system \((W, S)\), and define the **support** of an element \(w \in W\) to be the set

\[S(w) := \{ s \in S : s \leq w \} .\]

A simple reflection \(s \in S\) belongs to \(S(w)\) if and only if \(s\) appears in some (or equivalently, every) reduced word of \(w\). Inspired by the example of Hirzebruch surfaces, we define:

**Definition 1.2.** Let \(A\) and \(A'\) be Cartan matrices over \(S\) and \(S'\) respectively. Let \(w \in W(A)\) and \(w' \in W(A')\). We say the pair \((w, A)\) and \((w', A')\) are **Cartan equivalent** if there is a bijection \(\sigma : S(w) \to S(w')\) such that the following are satisfied:

(a) There are reduced words \(w = s_1 \cdots s_k\) and \(w' = t_1 \cdots t_k\) such that \(\sigma(s_i) = t_i\) for all \(1 \leq i \leq k\).

(b) If \(ss' \leq w\) for \(s \neq s' \in S(w)\), then \(A_{ss'} = A'_{\sigma(s)\sigma(s')}\).

Although it’s not immediately obvious from the definition, we show in the next section (see Corollary 2.3) that Cartan equivalence is an equivalence relation (and in particular is symmetric). We also refer to the bijection \(\sigma : S(w) \to S(w')\) in Definition 1.2 as a Cartan equivalence.

Our main result is that, somewhat surprisingly, two Schubert varieties \(X(w, A)\) and \(X(w', A')\) are isomorphic if and only if \((w, A)\) and \((w', A')\) are Cartan equivalent. As
part of the proof, we give a cohomological characterization of isomorphism as well. Recall that Schubert varieties are stratified by their Schubert subvarieties, and this stratification implies that the Schubert classes form a basis for the integral cohomology ring \( H^*(X(w, A)) := H^*(X(w, A); \mathbb{Z}) \).

**Theorem 1.3.** Let \( A \) and \( A' \) be Cartan matrices over \( S \) and \( S' \) respectively. Let \( w \in W(A) \) and \( w' \in W(A') \). Then the following are equivalent.

1. The pairs \((w, A)\) and \((w', A')\) are Cartan equivalent.
2. The Schubert varieties \( X(w, A) \) and \( X(w', A') \) are algebraically isomorphic.
3. There is a graded ring isomorphism \( \phi : H^*(X(w, A)) \to H^*(X(w', A')) \) which sends the Schubert basis for \( H^*(X(w, A)) \) to the Schubert basis for \( H^*(X(w', A')) \).

Theorem 1.3 can be readily applied in many situations. To illustrate this, we give several examples and applications in Section 1.1. We remark that the theorem does not hold if we drop the requirement in part (3) that the isomorphism preserve Schubert bases. Indeed, returning to Example 1.1, two Hirzebruch surfaces \( \Sigma_n \) and \( \Sigma_m \) have isomorphic integral cohomology rings if and only if they are diffeomorphic, which happens if and only if \( m = n \mod 2 \).

Hirzebruch surfaces are also examples of toric varieties. In general, a Schubert variety \( X(w, A) \) is a toric variety if and only if \( w = s_1 \cdots s_k \) for distinct simple reflections \( s_1, \ldots, s_k \in S \) [Kar13]. It follows from this that toric Schubert varieties are toric manifolds (smooth compact toric varieties). It is well known that isomorphism classes of toric varieties are determined by the combinatorial data in their associated fans [Ful93]. In addition, a result of Masuda states that toric manifolds are isomorphic if and only if their equivariant cohomology rings are weakly isomorphic [Mas08]. Both of these criteria apply to toric Schubert varieties in particular. It is an open question as to whether toric manifolds are cohomologically rigid, in the sense that any two toric manifolds with isomorphic cohomology rings are diffeomorphic or homeomorphic. We can ask the same question for Schubert varieties:

**Question 1.4.** Suppose \( X(w, A) \) and \( X(w', A') \) are both smooth, and \( H^*(X(w, A)) \) and \( H^*(X(w', A')) \) are isomorphic as graded rings. Are \( X(w, A) \) and \( X(w', A') \) diffeomorphic or homeomorphic?

If integral cohomology is replaced with rational cohomology \( H^*(X(w, A); \mathbb{Q}) \), then Question 1.4 has a negative answer, as all Hirzebruch surfaces have isomorphic cohomology rings over \( \mathbb{Q} \). Another counterexample is provided by the flag varieties of finite types \( B_n \) and \( C_n \), since the cohomology rings are isomorphic over \( \mathbb{Q} \), but not over \( \mathbb{Z} \) [BS02, Bor53, EC95].

Finally, recall that if \( A \) is a Cartan matrix over \( S \), then for any subset \( J \subseteq S \) there is a partial flag variety \( X^J := \mathcal{G}(A)/\mathcal{P}(A)_J \), where \( \mathcal{P}(A)_J \) is the parabolic subgroup generated by the elements of \( J \). Partial flag varieties are also stratified by Schubert varieties \( X^J(w, A) \), where \( X^J(w, A) \) is defined as the closure of \( Bw\mathcal{P}_J/\mathcal{P}_J \) in \( X^J \). Partial flag varieties include familiar examples such as the Grassmannians. Next we show the notion of Cartan equivalence is neither necessary nor sufficient for distinguishing Schubert varieties in partial flag varieties.

**Example 1.5.** Consider the Cartan matrices of types \( A_3 \) and \( C_2 \) given by

\[
A_3 = \begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 2 \\
\end{pmatrix}
\quad \text{and} \quad
C_2 = \begin{pmatrix}
2 & -2 \\
-1 & 2 \\
\end{pmatrix}
\]
over index sets \( S = \{s_1, s_2, s_3\} \) and \( S' = \{s'_1, s'_2\} \) respectively. First consider the case
where \( J = \{s_2, s_3\} \) and \( J' = \{s'_2\} \) with \( w = s_3 s_2 s_1 \) and \( w' = s'_1 s'_2 s'_1 \). Then \( X^J(w, A_3) \) and \( X^{J'}(w', C_3) \) are both isomorphic to the projective space \( \mathbb{P}^3 \). However \( |S(w)| = 3 \) and \( |S(w')| = 2 \) and hence \( (w, A_3) \) and \( (w', C_3) \) cannot be Cartan equivalent.

Conversely, consider \( w = s_2 s_1 s_3 s_2 \) and \( J = \{s_1, s_3\} \). Then \( X^0(w, A_3) = X(w, A_3) \) is singular, whereas \( X^J(w, A_3) \) is the Grassmannian \( \text{Gr}(2, 4) \), and hence is smooth. Clearly \((w, A_3)\) is Cartan equivalent to itself; however \( X^0(w, A_3) \not\cong X^J(w, A_3) \).

The class of Schubert varieties of partial flag varieties is much broader than the class of Schubert varieties in full flag varieties. It is not clear whether the isomorphism problem for this broader class should have a simple combinatorial solution like Cartan equivalence. We leave this as an open problem.

In proving Theorem 1.3 we need to show that a Cartan equivalence can be constructed from the isomorphism between cohomology rings. To do this, we prove that the Cartan matrix entries \( A_{st} \) for \( st \leq w \) and the reduced words for \( w \) can be recovered from \( H^*(X(w), A) \) along with its Schubert basis. This gives a procedure to solve a related problem of independent interest: constructing a presentation of a Schubert variety (as a Schubert variety) solely from geometric data. We outline this procedure in Section 4.1.

1.1. **Examples and applications of Theorem 1.3.** In this section, we illustrate the potential applications of Theorem 1.3 with several examples. We start with a basic example of how the theory works:

**Example 1.6.** Let \( A \) be the following Cartan matrix over \( S = \{s_1, s_2, s_3, s_4\} \):

\[
\begin{pmatrix}
2 & -1 & -1 & 0 \\
-1 & 2 & -3 & 0 \\
-2 & -1 & 2 & -5 \\
0 & 0 & -3 & 2
\end{pmatrix}
\]

Let \( w \) be the reduced word \( s_2 s_3 s_2 s_1 s_4 \) in \( W(A) \). To illustrate Definition 1.2 part (2), we list all elements \( s_i s_j \leq w \) in the corresponding positions within the matrix \( A \) and then highlight the relevant data in \( A \) needed to determine the Cartan equivalence class of \((w, A)\).

\[
\begin{pmatrix}
\vdots \\
s_{2s_1} & s_{2s_3} & s_{2s_4} \\
s_{3s_1} & s_{3s_2} & s_{3s_4} \\
s_{4s_1} & s_{4s_2} & s_{4s_4}
\end{pmatrix} \Rightarrow 
\begin{pmatrix}
2 & * & * & 0 \\
-1 & 2 & -3 & 0 \\
-2 & -1 & 2 & -5 \\
0 & 0 & * & 2
\end{pmatrix}
\]

The “*” entries correspond to pairs of indices \((s_i, s_j), i \neq j\), such that \( s_i s_j \not\leq w \). Suppose \( A' \) is another Cartan matrix over \( S \) which agrees with \( A \) on all the non-starred entries. By Lemma 2.2, the word \( s_{2s_3} s_{2s_1} s_4 \) will be reduced in \( W(A') \) as well, so \((w, A)\) and \((w, A')\) will be Cartan equivalent. By Theorem 1.3, we would then have \( X(w, A) \cong X(w', A') \).

More generally, if a Cartan matrix \( A' \) contains a submatrix of the form on the right in Equation (1) (where the “*” entries can be any number), then there will be an element \( w' \in W(A') \) such that \((w', A')\) is Cartan equivalent to \((w, A)\), and hence \( X(w, A) \cong X(w', A') \).

One of the advantages of Theorem 1.3 is that it makes it easy to determine whether Schubert varieties in different types are isomorphic. For instance:
Example 1.7. Consider the Cartan matrices
\[
A_3 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \quad \text{and} \quad B_3 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}
\]
of types $A_3$ and $B_3$ over $S = \{s_1, s_2, s_3\}$. Theorem 1.3 implies that
\[
X(s_1s_2s_3, A_3) \cong X(s_1s_2s_3, B_3)
\]
and
\[
X(s_3s_2s_1, A_3) \not\cong X(s_3s_2s_1, B_3).
\]
Also note that
\[
X(s_2s_1s_3, A_3) \not\cong X(s_1s_3s_2, A_3).
\]

Note that every Schubert variety in this example is smooth and has the same Poincaré polynomial. Hence the varieties in this example cannot be distinguished by these properties. In the last example, we have a case where $X(w, A) \not\cong X(w^{-1}, A)$.

For any $A$ is a Cartan matrix indexed by $S$, and $J \subseteq S$, let $A_J := [A_{st}]_{(s,t) \in J^2}$ denote the induced Cartan matrix over $J$. The group $W(A_J)$ can be thought of as the subgroup of $W(A)$ generated by $J$, and is typically denoted by $W_J$. If $w \in W(A)$, then $w \in W(A_{S(w)})$. Something that shows up in the previous example is that the isomorphism type of $X(w, A)$ depends only on $A_{S(w)}$. In fact, $X(w, A) \cong X(w, A_{S(w)})$. While this follows from Theorem 1.3, it can also be easily proved without it (see for instance [RS16, Lemma 4.8]). We say $w \in W(A)$ is fully supported if $S(w) = S$.

Finite and affine type Cartan matrices are classified by Dynkin diagrams, which are graphs with simple edges and decorated multiedges (See Figure 1). The vertex set of the Dynkin diagram is the index set $S$ of the Cartan matrix $A$, and the edge or multiedge between vertices $s$ and $t$ determines the matrix values $A_{st}$ and $A_{ts}$. For any Cartan matrix $A$, we can also consider the Coxeter graph of $A$, which is the graph with vertex set $S$, and $m_{st} - 2$ edges between vertices $s$ and $t$. In finite and affine types, the Coxeter graph is the Dynkin diagram with decorations removed from the multiedges.

A particularly interesting case to look at is finite versus infinite type. Recall that a Cartan matrix $A$ is simply-laced if $A_{st} \in \{0, -1\}$ for all $s \neq t \in S$. Equivalently, a Cartan matrix is simply-laced if the exponents $m_{st}$ of the Coxeter relations are either 2 or 3, i.e. the Coxeter graph is simple. Given a Cartan matrix $A$, it is convenient to define the simple Coxeter graph $\Gamma(A) := (S, E)$ to be the graph with vertex set $S$ and edges $(s, t) \in E$ if and only if $A_{st} \neq 0$. In other words, $\Gamma(A)$ is the underlying simple graph of the Coxeter graph of $A$.

Lemma 1.8. If $X(w, A) \cong X(w', A')$, then $\Gamma(A_{S(w)}) \cong \Gamma(A_{S(w')})$.

Proof. Let $\sigma : S(w) \to S(w')$ be a Cartan equivalence. If $s, t \in S(w)$, then either $st \leq w$ or $ts \leq w$. If $st \leq w$, then $A_{st} = A'_{\sigma(s)\sigma(t)}$, while if $ts \leq w$ then $A_{ts} = A'_{\sigma(t)\sigma(s)}$. Using the fact that $A_{st} = 0$ if and only if $A_{ts} = 0$, we conclude that $A_{st} = 0$ if and only if $A'_{\sigma(s)\sigma(t)} = 0$. Therefore $\sigma$ is a graph isomorphism.

Corollary 1.9. If $X(w, A)$ is isomorphic to a Schubert variety in a finite type flag variety, then $\Gamma(A_{S(w)})$ is a finite type Coxeter graph. In particular, if $A_{S(w)}$ is simply-laced, then $X(w, A)$ is isomorphic to a Schubert variety of finite type if and only if $A_{S(w)}$ is of finite type.
| Finite type | Dynkin diagram | $\text{Aut}(A)$ | $\text{Aut}(\Gamma(A))$ |
|-------------|----------------|-----------------|------------------------|
| $A_n$       | ![Dynkin diagram](A_n) | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| $B_n$ ($n \geq 3$) | ![Dynkin diagram](B_n) | $\mathbb{Z}_1$ | $\mathbb{Z}_2$ |
| $C_n$ ($n \geq 2$) | ![Dynkin diagram](C_n) | $\mathbb{Z}_1$ | $\mathbb{Z}_2$ |
| $D_4$       | ![Dynkin diagram](D_4) | $S_3$ | $S_3$ |
| $D_n$ ($n \geq 5$) | ![Dynkin diagram](D_n) | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| $E_6$       | ![Dynkin diagram](E_6) | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| $E_7$       | ![Dynkin diagram](E_7) | $\mathbb{Z}_1$ | $\mathbb{Z}_1$ |
| $E_8$       | ![Dynkin diagram](E_8) | $\mathbb{Z}_1$ | $\mathbb{Z}_1$ |
| $F_4$       | ![Dynkin diagram](F_4) | $\mathbb{Z}_1$ | $\mathbb{Z}_2$ |
| $G_2$       | ![Dynkin diagram](G_2) | $\mathbb{Z}_1$ | $\mathbb{Z}_2$ |

| Affine type | Dynkin diagram | $\text{Aut}(A)$ | $\text{Aut}(\Gamma(A))$ |
|-------------|----------------|-----------------|------------------------|
| $\tilde{A}_n$ | ![Dynkin diagram](tilde_A_n) | $I_2(n)$ | $I_2(n)$ |
| $\tilde{B}_n$ ($n \geq 3$) | ![Dynkin diagram](tilde_B_n) | $\mathbb{Z}_1$ | $\mathbb{Z}_2$ |
| $\tilde{C}_n$ ($n \geq 2$) | ![Dynkin diagram](tilde_C_n) | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| $\tilde{D}_4$ | ![Dynkin diagram](tilde_D_4) | $S_4$ | $S_4$ |
| $\tilde{D}_n$ ($n \geq 5$) | ![Dynkin diagram](tilde_D_n) | $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ | $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ |
| $\tilde{E}_6$ | ![Dynkin diagram](tilde_E_6) | $S_3$ | $S_3$ |
| $\tilde{E}_7$ | ![Dynkin diagram](tilde_E_7) | $\mathbb{Z}_1$ | $\mathbb{Z}_2$ |
| $\tilde{E}_8$ | ![Dynkin diagram](tilde_E_8) | $\mathbb{Z}_1$ | $\mathbb{Z}_2$ |
| $\tilde{F}_4$ | ![Dynkin diagram](tilde_F_4) | $\mathbb{Z}_1$ | $\mathbb{Z}_2$ |
| $\tilde{G}_2$ | ![Dynkin diagram](tilde_G_2) | $\mathbb{Z}_1$ | $\mathbb{Z}_2$ |
| $\tilde{A}_2$ | ![Dynkin diagram](tilde_A_2) | $\mathbb{Z}_1$ | $\mathbb{Z}_2$ |
| $\tilde{A}_{2n}$ | ![Dynkin diagram](tilde_A_2n) | $\mathbb{Z}_1$ | $\mathbb{Z}_2$ |
| $\tilde{A}_{2n+1}$ | ![Dynkin diagram](tilde_A_2n+1) | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| $\tilde{D}_2$ | ![Dynkin diagram](tilde_D_2) | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| $\tilde{D}_3$ | ![Dynkin diagram](tilde_D_3) | $\mathbb{Z}_1$ | $\mathbb{Z}_2$ |
| $\tilde{E}_6$ | ![Dynkin diagram](tilde_E_6) | $\mathbb{Z}_1$ | $\mathbb{Z}_2$ |

Figure 1. Dynkin diagrams and automorphism groups of all finite and affine Lie types. $S_n$ denotes the symmetric group, and $I_2(n)$ denotes the dihedral group.
Proof. It is easy to verify from Figure 1 that the underlying simple graph of a finite type Dynkin diagram is also a finite type Dynkin diagram. If $X(w, A) \cong X(w', A')$ where $A'$ has finite type, then by Lemma 1.8, the graph $\Gamma(A_{S(w)})$ is isomorphic to $\Gamma(A'_{S(w')})$ and hence has finite type. If $A_{S(w)}$ is simply-laced, then $\Gamma(A_{S(w)})$ is the Coxeter graph of $A_{S(w)}$, and hence $A_{S(w)}$ is of finite type.

Example 1.10. The Coxeter graph of affine type $\tilde{A}_n$ is a cycle for $n \geq 2$. Hence if $w \in W(\tilde{A}_n)$ is fully supported, then $X(w, A_n)$ is not isomorphic to any Schubert variety of finite type. The same is true for affine types $\tilde{D}_n$ and $\tilde{E}_n$.

Example 1.11. If $A_{S(w)}$ is not simply-laced, then it's possible for $X(w, A_{S(w)})$ to be isomorphic to a finite-type Schubert variety, even if $A_{S(w)}$ is not of finite type. For instance, consider the Cartan matrices

$$\tilde{A}_1 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \text{ and } C_2 = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$$

of affine type $\tilde{A}_1$ and finite type $C_2$ over index set $\{s_1, s_2\}$. Then $X(s_1s_2, \tilde{A}_1) \cong X(s_1s_2, C_2)$.

Example 1.12. The criterion in Definition 1.2 simplifies if $A$ is simply-laced, since $A_{st} = A_{ts}$ for all $s, t$. More generally, suppose that $A$ and $A'$ are symmetric Cartan matrices, in the sense that $A_{st} = A_{ts}$ for all $s, t \in S$ and let $w \in W(A)$, $w' \in W(A')$. If $X(w, A) \cong X(w', A')$, then the Cartan matrices $A_{S(w)}$ and $A'_{S(w')}$ are the same up to permutation of rows and columns. In other words, if $X(w, A)$ and $X(w', A')$ are isomorphic, then the flag varieties $\mathcal{X}(A_{S(w)})$ and $\mathcal{X}(A'_{S(w')})$ are isomorphic, with an isomorphism that identifies $X(w, A)$ and $X(w', A')$.

We can use Lemma 1.8 for comparison to other classes of Schubert varieties as well.

Example 1.13. The finite type $A$ Schubert varieties are the best studied class of Schubert varieties. By Lemma 1.8, a Schubert variety $X(w, A)$ is isomorphic to a Schubert variety of finite type $A$ if and only if

1. the simple Coxeter graph $\Gamma(A_{S(w)})$ is a disjoint union of paths, and
2. for every $s \leq w$, we have $A_{st} \in \{0, -1\}$.

We can also use Theorem 1.3 to calculate the isomorphism classes of Schubert varieties in a fixed Kac-Moody flag variety. For any Cartan matrix $A$ and $w \in W(A)$, let

$$\text{Isom}(w, A) := \{w' \in W(A) : X(w, A) \cong X(w', A)\}$$

denote the isomorphism class of $X(w, A)$ within the Kac-Moody flag variety $\mathcal{X}(A)$.

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1 Symmetric is not the same as symmetrizable, a common condition imposed on Cartan matrices when studying the representation theory of Kac-Moody Lie algebras.
Example 1.14. The flag variety $X(A_3)$ (where $A_3$ is the Cartan matrix over $S = \{s_1, s_2, s_3\}$ as in Example [1.7]) has 14 isomorphism classes of Schubert varieties:

| $\ell(w)$ | $\text{Isom}(w, A_3)$ |
|-----------|---------------------|
| 0         | \{1\}               |
| 1         | $\{s_1, s_2, s_3\}$ |
| 2         | $\{s_1s_3\}, \{s_1s_2, s_2s_3, s_3s_2\}$ |
| 3         | $\{s_1s_2s_1, s_2s_3s_2\}, \{s_1s_3s_2\}, \{s_2s_1s_3\}, \{s_1s_2s_3, s_3s_2s_1\}$ |
| 4         | $\{s_1s_2s_3s_2, s_3s_2s_3s_2\}, \{s_2s_1s_2s_3, s_2s_3s_2s_1\}, \{s_2s_1s_3s_2\}$ |
| 5         | $\{s_2s_1s_2s_3s_2, s_2s_3s_2s_1s_2\}, \{s_3s_2s_1s_2s_3\}$ |
| 6         | $\{s_3s_2s_1s_3s_2s_3\}$ |

In types $A_4$ and $A_5$ there are 54 and 316 Schubert isomorphism classes respectively.

When $w \in W(A)$ is not fully supported, it’s possible for $X(w, A)$ to be isomorphic to $X(w', A)$ where $S(w) \neq S(w')$. For instance, $X(s, A) \cong X(t, A)$ for all $s, t \in S$. So in Example [1.14] $|\text{Isom}(s_1, A_3)| = 3$, and in general $|\text{Isom}(s, A)| = |S|$. However, in Example [1.14] all the fully supported elements $w$ have isomorphism classes of size 1 or 2. In fact, the sets $\text{Isom}(w, A)$ when $A$ is of finite and affine type and $w$ is fully supported are surprisingly small. To explain this, we can bound the size of the sets $\text{Isom}(w, A)$ in terms of the automorphism groups of $\Gamma(A)$. Recall that a diagram automorphism of a Cartan matrix $A$ with index set $S$ is a bijection $\sigma : S \to S$ such that $A_{st} = A_{\sigma(s)\sigma(t)}$. We let $\text{Aut}(A)$ denote the group of diagram automorphisms of $A$. Diagram automorphisms play an important role in the classification of automorphisms of Kac-Moody groups [CM05, CC93, KW92]. We let $\text{Aut}(\Gamma(A))$ denote the graph automorphism group of $\Gamma(A)$, i.e. the set of bijections $\sigma : S \to S$ such that $A_{st} = 0$ if and only if $A_{\sigma(s)\sigma(t)} = 0$. Since $\Gamma(A)$ is the Dynkin diagram of the Cartan matrix where every off-diagonal non-zero entry of $A$ is changed to $-1$, $\text{Aut}(\Gamma(A))$ is also the automorphism group of some Cartan matrix.

Corollary 1.15. If $A$ is a Cartan matrix over $S$ and $w \in W(A)$ is fully supported, then

$$|\text{Isom}(w, A)| \leq |\text{Aut}(\Gamma(A))|.$$  

Furthermore, if $A$ is symmetric, then

$$|\text{Isom}(w, A)| \leq |\text{Aut}(A)|.$$  

Proof. If $w \in W(A)$ is fully-supported, then a Cartan equivalence $\sigma : S \to S$ between $(w, A)$ and $(w', A)$ is an element of $\text{Aut}(\Gamma(A))$.

If $A$ is symmetric, and $s \neq t \in S$, then either $st \leq w$, in which case $A_{st} = A_{\sigma(s)\sigma(t)}$, or $ts \leq w$, in which case $A_{st} = A_{ts} = A_{\sigma(t)\sigma(s)} = A_{\sigma(s)\sigma(t)}$. So $\sigma$ will be in $\text{Aut}(A)$. \[\square\]

The automorphisms groups of $A$ for $A$ finite and (untwisted) affine types are well-known, and are shown in Figure [11]. As can be verified from the table, if $A$ is finite or affine, then $\Gamma(A)$ is also finite or affine, so $\text{Aut}(\Gamma(A))$ can be determined from the table as well. It follows that for all simple finite and affine types except affine type $A_6$, $|\text{Isom}(w, A)| \leq 24$ for all fully supported $w$, and for many simple finite and affine types, the bound is $|\text{Isom}(w, A)| \leq 2$. 

1.2. Outline of paper. The remainder of this paper focuses on proving Theorem 1.3. In Section 2, we review some basic facts on Coxeter groups and apply them to Cartan equivalences. In Section 3, we recall the definition of the algebraic structure on Schubert varieties using the Kac-Moody Lie algebra, and prove (1) implies (2). In Section 4, we study the cohomology ring of Schubert varieties, and prove (2) implies (3) and (3) implies (1). An explanation of how to construct a presentation of a Schubert variety from geometric data is given in Subsection 4.1. We use [Kum02] as the primary reference for background material throughout the paper.

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2. Combinatorics of Cartan equivalence

In this section, we establish some basic combinatorial properties of Cartan equivalence. As usual, a word over alphabet $S$ is a sequence $(s_1, \ldots, s_k)$ with $s_i \in S$ for $1 \leq i \leq k$. As mentioned in the introduction, if $A$ is a Cartan matrix over index set $S$, then $(s_1, \ldots, s_k)$ is a reduced word if there is no way to write $s_1 \cdots s_k \in W(A)$ as a product of fewer than $k$ elements of $S$. For any $w \in W(A)$, let $RW(w)$ denote the set of reduced words of $w$. Normally when working with $W(A)$, a word $(s_1, \ldots, s_k)$ is written as $s_1 \cdots s_k$, so the expression $s_1 \cdots s_k$ can refer to a word or to an element of $W(A)$ depending on the context. We use the same convention in this paper, but we’ll also use the sequence notation for words when there is potential for confusion.

We need the following special case of the subword property for Bruhat order.

**Lemma 2.1.** Let $A$ be a Cartan matrix over a finite set $S$. Let $w \in W(A)$ and suppose $s, t \in S(w)$. If $A_{st} < 0$ then the following are equivalent:
- $st \leq w$.
- The element $s$ appears before $t$ in some reduced word for $w$.
- The element $s$ appears before $t$ in any reduced word for $w$.

Otherwise, if $A_{st} = 0$, then $st = ts \leq w$.

**Proof.** Special case of [BB05, Theorem 2.2.2].

The definition of Cartan equivalence ostensibly requires us to find a reduced expression for $w$ which corresponds to a reduced expression for $w'$. However, any reduced expression will do:

**Lemma 2.2.** Let $A$ and $A'$ be Cartan matrices on $S$ and $S'$ respectively and let $w \in W(A)$. Let $(s_1, \ldots, s_k)$ be a reduced word for $w \in W(A)$, and suppose $\sigma : S(w) \to S'$ is an injection satisfying the condition that $A_{st} = A'_{\sigma(s)\sigma(t)}$ whenever $st \leq w$. Then $(\sigma(s_1), \ldots, \sigma(s_k))$ is a reduced word for $w' := \sigma(s_1) \cdots \sigma(s_k) \in W(A')$, and $(w, A)$ is Cartan equivalent to $(w', A')$. Furthermore, $(t_1, \ldots, t_k) \in RW(w)$ if and only if $(\sigma(t_1), \ldots, \sigma(t_k)) \in RW(w')$.

**Proof.** Let $m_{st}$ and $m'_{st}$ be the Coxeter exponents for $A$ and $A'$ respectively. A word $(t_1, \ldots, t_k)$ with $t_1, \ldots, t_k \in S'$ is non-reduced if and only if it is possible to apply Coxeter relations

\[
\begin{align*}
(s, t, \ldots) & = (t, s, \ldots) \\
(\overbrace{m_{st}}^m t, \ldots) & = (\overbrace{m'_{st}}^m t, \ldots)
\end{align*}
\]
for simple reflections $s, t \in S'$, to get a word $(t'_1, \ldots, t'_k)$ with $t'_i = t'_{i+1}$ for some $1 \leq i < k$. Suppose $(\sigma(s_1), \ldots, \sigma(s_k))$ contains an alternating subword $(\sigma(s), \sigma(t), \ldots)$ of length $m_{\sigma(s)\sigma(t)}$. If $m_{\sigma(s)\sigma(t)} \geq 3$, then $st \leq w$ and $ts \leq w$, and so $A_{st} = A'_{\sigma(s)\sigma(t)}$ and $A_{ts} = A'_{\sigma(t)\sigma(s)}$. If $m_{\sigma(s)\sigma(t)} = 2$, then $st \leq w$, and hence $A_{st} = A'_{\sigma(s)\sigma(t)} = 0$, and so $A_{ts} = 0 = A'_{\sigma(t)\sigma(s)}$ as well. Thus in either case, $m_{st} = m'_{\sigma(s)\sigma(t)}$ and so any Coxeter relation that can be applied to $(\sigma(s_1), \ldots, \sigma(s_k))$ can also be applied to $(s_1, \ldots, s_k)$, giving another reduced word for $w$. Since $(s_1, \ldots, s_k)$ is reduced, we’ll never get a word of the form $(s'_1, \ldots, s'_k)$ with $s'_i = s'_{i+1}$ by applying Coxeter relations. So the same is true of $(\sigma(s_1), \ldots, \sigma(s_k))$, and thus $(\sigma(s_1), \ldots, \sigma(s_k))$ is reduced as well. The fact that $(w, A)$ and $(w', A')$ are Cartan equivalence follows immediately.

Similarly, if $(s_1, \ldots, s_k)$ contains an alternating subword $(s, t, \ldots)$ of length $m_{st}$, then the same argument shows that $m_{st} = m'_{\sigma(s)\sigma(t)}$ and hence any Coxeter relation that can be applied to $(s_1, \ldots, s_k)$ can also be applied to $(\sigma(s_1), \ldots, \sigma(s_k))$. Since $\text{RW}(w)$ and $\text{RW}(w')$ are exactly the words we get by applying Coxeter relations to $(s_1, \ldots, s_k)$ and $(\sigma(s_1), \ldots, \sigma(s_k))$, it follows that $(t_1, \ldots, t_k) \in \text{RW}(w)$ if and only if $(\sigma(t_1), \ldots, \sigma(t_k)) \in \text{RW}(w')$. 

\section{Corollary 2.3.} Cartan equivalence is an equivalence relation.

\textbf{Proof.} Clearly Cartan equivalence is reflexive. Suppose that $\sigma : S(w) \to S(w')$ is a Cartan equivalence as in Definition 2.2. By definition, there is a reduced word $s_1 \cdots s_n$ for $w$ such that $\sigma(s_1) \cdots \sigma(s_n)$ is a reduced word for $w'$. Suppose $s't' \leq w'$, where $s' = \sigma(s)$ and $t' = \sigma(t)$. By Lemma 2.1, if $A'_{t't'} \neq 0$, then $s'$ occurs before $t'$ in $\sigma(s_1) \cdots \sigma(s_n)$. So $st \leq w$, and hence $A_{st} = A_{\sigma^{-1}(w')} = A'_{s't'}$. If $A'_{t't'} = 0$, then at least one of $st$ or $ts \leq w$. In both cases $A_{\sigma^{-1}(w')} = A_{st} = A'_{s't'}$, since if $ts \leq w$, then $A_{ts} = A'_{s't'} = 0$, implying $A_{st} = 0$. So $\sigma^{-1}$ is also a Cartan equivalence, and Cartan equivalence is symmetric.

For transitivity, suppose $\sigma : S(w) \to S(w')$ and $\tau : S(w') \to S(w'')$ are Cartan equivalences from $(w, A)$ to $(w', A')$ and $(w', A')$ to $(w'', A'')$ respectively. Let $s_1, \ldots, s_n$ be a reduced word for $w$. By Lemma 2.2, $\sigma(s_1) \cdots \sigma(s_n)$ is a reduced word for $w'$, and therefore $\tau(\sigma(s_1)) \cdots \tau(\sigma(s_n))$ is a reduced word for $w''$. If $st \leq w$, then $\sigma(s)\sigma(t) \leq w'$. So $A_{st} = A'_{\sigma(s)\sigma(t)} = A''_{\tau(\sigma(s))\tau(\sigma(t))}$ for all $st \leq w$, and hence $\tau \circ \sigma : S(w) \to S(w'')$ is a Cartan equivalence between $(w, A)$ and $(w'', A'')$. 

Another corollary of Lemma 2.2 is that Cartan equivalences preserve Bruhat intervals:

\section{Corollary 2.4.} Suppose $(w, A)$ and $(w', A')$ are Cartan equivalent under the bijection $\sigma : S(w) \to S(w')$. For any $v \leq w$, there is $v' \leq w'$ such that $(v, A)$ and $(v', A')$ are Cartan equivalent under an induced bijection $\sigma|_{S(v)} : S(v) \to S(\sigma(v))$. Furthermore, this correspondence gives a poset isomorphism $\sigma : [e, w] \to [e, w']$.$^*$

\textbf{Proof.} Fix a reduced word $w = s_1 \cdots s_k$ and let $v \leq w$. Then there is a subsequence $(i_1, \ldots, i_m)$ for which $v = s_{i_1} \cdots s_{i_m}$ is a reduced word for $v$. Let $v' := \sigma(s_{i_1}) \cdots \sigma(s_{i_m})$. It follows from Lemma 2.2 that $(v, A)$ is Cartan equivalent to $(v', A')$ and $v' \leq w'$. Also by Lemma 2.2, the element $v'$ is independent of the choice of reduced word for $v$. So $\sigma$ induces a function $\sigma : [e, w] \to [e, w']$. If $u \leq v$, then there is another subsequence $(j_1, \ldots, j_k)$ such that $u = s_{j_1} \cdots s_{j_k}$ is a reduced word for $u$, and hence $\sigma(u) = \sigma(s_{j_1}) \cdots \sigma(s_{j_k}) \leq \sigma(v)$, so $\sigma$ is order-preserving. The function $\sigma^{-1} : [e, w'] \to [e, w]$ induced by $\sigma^{-1}$ is an inverse to $\sigma$, so $\sigma$ is a bijection.
The isomorphism problem for Schubert varieties 11

3. Cartan equivalence and Kac-Moody Lie algebras

In this section we prove that condition (1) implies condition (2) from Theorem 1.3 which we state as its own proposition:

**Proposition 3.1.** If \((w, A)\) and \((w', A')\) are Cartan equivalent, then \(X(w, A) \cong X(w', A')\) as algebraic varieties.

The general idea of the proof of Proposition 3.1 is to use the Cartan equivalence between \((w, A)\) and \((w', A')\) to construct a linear map \(\pi : V \to V'\) between certain vector spaces \(V\) and \(V'\) for which the algebraic structures of \(X(w, A)\) and \(X(w', A')\) are realized as closed embeddings into \(\mathbb{P}(V)\) and \(\mathbb{P}(V')\) respectively. Restricting \(\pi\) to \(X(w, A)\) will then give an algebraic bijection between \(X(w, A)\) and \(X(w', A')\). Since these varieties are normal, they will be isomorphic. We begin by recalling the construction of a Kac-Moody Lie algebra \(g(A)\) from a Cartan matrix \(A\) over a finite set \(S\) of size \(n\) as given in [Kum02] or [Kac90]. Let \(\mathfrak{h}(A)\) be a complex vector space of dimension \(2n - \text{rank}(A)\). Let \(\{h_s\}_{s \in S} \subset \mathfrak{h}(A)\) denote a set of linearly independent vectors with corresponding simple root vectors \(\{\alpha_s\}_{s \in S} \subset \mathfrak{h}^*(A)\) satisfying

\[
\alpha_t(h_s) = A_{st}.
\]

The **Kac-Moody Lie algebra** \(g(A)\) is the Lie algebra generated by \(\mathfrak{h}\), along with elements \(\{e_s\}_{s \in S}, \{f_s\}_{s \in S}\), satisfying Lie bracket relations:

1. \([h, e_s] = 0\),
2. \([h, f_s] = \alpha_s(h) e_s\) and \([h, f_s] = -\alpha_s(h) f_s\) for all \(h \in \mathfrak{h}(A)\) and \(s \in S\),
3. \([e_s, f_t] = \delta_{st} h_s\),
4. \(\text{ad}(e_s)^{1-A_{st}}(e_t) = 0\) for all \(s \neq t \in S\), and
5. \(\text{ad}(f_s)^{1-A_{st}}(f_t) = 0\) for all \(s \neq t \in S\).

The algebra \(g(A)\) has a triangular decomposition

\[
g(A) = \mathfrak{n}^- (A) \oplus \mathfrak{h}(A) \oplus \mathfrak{n}^+ (A),
\]

where \(\mathfrak{n}^\pm (A)\) are the subalgebras generated by \(\{e_s\}_{s \in S}\) and \(\{f_s\}_{s \in S}\) respectively. More specifically, the algebra \(\mathfrak{n}^+(A)\) is the free Lie algebra generated by \(\{e_s\}_{s \in S}\) satisfying the relations in (4) above. Similarly, \(\mathfrak{n}^-(A)\) is freely generated by the \(\{f_s\}_{s \in S}\) subject to the relations in (5).

In the proof of Proposition 3.1, we want to be able to work with Kac-Moody Lie algebras without specifying a Cartan matrix. To do this, consider the set the variables \(\{a_{st}\}_{(s,t) \in S^2}\) and let \(R_0 = \mathbb{C}[a_{st} : (s, t) \in S^2]\) denote the polynomial ring generated by these variables. Let \(\mathcal{R}\) be the free (associative non-commutative) \(R_0\)-algebra generated by symbols \(\tilde{f}_s, \tilde{h}_s,\) and \(\tilde{e}_s\), for \(s \in S\). We use \(\tilde{f}^I\) to denote an arbitrary non-commutative monomial in the variables \(\{\tilde{f}_s\}_{s \in S}, y^I\) to denote a non-commutative monomial in the variables \(\{\tilde{h}_s\}_{s \in S}\) and \(\{\tilde{e}_s\}_{s \in S}\), and \(x^I\) to denote a non-commutative monomial in all three families of variables. A general element of \(\mathcal{R}\) can then be written as

\[
\sum_I g_I(a) x^I,
\]

where each \(g_I(a)\) is an element of \(R_0\), and \(g_I(a) = 0\) for all but finitely many \(I\). We say an element of \(\mathcal{R}\) is **independent of** \(a_{st}\) if each monomial in the coefficients \(g_I(a)\) do not contain \(a_{st}\) as a factor. We say that an element of \(\mathcal{R}\) is a **normal form** if it can be written as

\[
\sum_{I,J} g_{IJ}(a) \tilde{f}^I y^J.
\]
In other words, every monomial is ordered so that all $\tilde{f}_s$’s precede all $\tilde{h}_s$’s and $\varepsilon_s$’s. Given a Cartan matrix $A = [A_{st}]_{(s,t) \in S^2}$ over $S$, there is a morphism $\phi(A) : R \to U(\mathfrak{g}(A))$ which sends
\[
(f_s, h_s, \varepsilon_s) \mapsto (f_s, h_s, \varepsilon_s) \quad \text{and} \quad a_{st} \mapsto A_{st} \text{ for all } (s, t) \in S^2.
\]
Note that $a_{ss}$ is a variable in $R_0$. However, $\phi(A)(a_{ss}) = 2$ for all Cartan matrices $A$. Hence the reader can replace $a_{ss}$ with 2 in all the proceeding calculations.

Let $\mathfrak{g}'(A)$ denote the commutator subalgebra of $\mathfrak{g}(A)$. The morphism $\phi(A)$ maps $R$ surjectively onto $U(\mathfrak{g}'(A))$, the universal enveloping algebra of $\mathfrak{g}'(A)$, since it is generated by $\{f_s, h_s, e_s \mid s \in S\}$. Given an element $\tau \in R$, we say that $\nu \in R$ is a normal form of $\tau$ if:

(a) $\nu$ is a normal form, and
(b) $\phi(A)(\nu) = \phi(A)(\tau)$ for all Cartan matrices $A$ over $S^2$.

Using the relations (2) and (3) from the definition of $\mathfrak{g}(A)$, it is clear that every element of $R$ has a normal form\footnote{This is analogous to the isomorphism $U(\mathfrak{g}(A)) \cong U(\mathfrak{n}^- (A)) \oplus U(\mathfrak{n}^+ (A))$. Working with $R$ allows us to use this isomorphism without specifying the Cartan matrix $A$.}.

Given $\tau \in R$, we construct a specific normal form $\eta(\tau)$ as follows:

- if $\varepsilon_s \tilde{f}_t$ or $\tilde{h}_s \tilde{f}_t$ does not occur in any monomial of $\tau$, then set $\eta(\tau) = \tau$.
- Otherwise find the rightmost occurrence of $\varepsilon_s \tilde{f}_t$ or $\tilde{h}_s \tilde{f}_t$ in each monomial of $\tau$, and either
  - replace $\varepsilon_s \tilde{f}_t$ with $\tilde{f}_t \varepsilon_s + \delta_{st} \tilde{h}_s$, or
  - replace $\tilde{h}_s \tilde{f}_t$ with $\tilde{f}_t \tilde{h}_s - a_{st} \tilde{f}_t$.
- Repeat for each monomial in the resulting sum.

We call $\eta(\tau)$ the normal form of $\tau$.

**Example 3.2.** Let $S = \{s_1, s_2, s_3\}$ and $\tau = \tilde{h}_1 \varepsilon_2 \varepsilon_3 \tilde{f}_2$. Applying the algorithm for $\eta(\tau)$ yields:
\[
\tau = \tilde{h}_1 \varepsilon_2 (\varepsilon_3 \tilde{f}_2) \rightarrow \tilde{h}_1 (\tilde{f}_2 \varepsilon_3) = \tilde{h}_1 (\tilde{f}_2) \varepsilon_3
\]
\[
= \tilde{h}_1 (\tilde{f}_2 \varepsilon_2 + \tilde{h}_2) \varepsilon_3 = (\tilde{h}_1 \tilde{f}_2) \varepsilon_2 \varepsilon_3 + \tilde{h}_1 \tilde{h}_2 \varepsilon_3
\]
\[
= (\tilde{f}_2 \tilde{h}_1) \varepsilon_2 \varepsilon_3 - a_{12} \tilde{f}_2 \varepsilon_2 \varepsilon_3 + \tilde{h}_1 \tilde{h}_2 \varepsilon_3 = \eta(\tau).
\]

**Lemma 3.3.** Let $\tau$ be an element of $R$ which is independent of $a_{st}$ with $s \neq t$. Suppose that, in every monomial of $\tau$, every $\tilde{f}_t$ occurs to the left of every $\tilde{h}_s$ and $\tilde{f}_s$. Then $\eta(\tau)$ is also independent of $a_{st}$.

**Proof.** An $a_{st}$ is created in the above procedure when we switch
\[
\tilde{h}_s \tilde{f}_t \rightarrow \tilde{f}_t \tilde{h}_s - a_{st} \tilde{f}_t
\]
and $\tilde{h}_s$ is created whenever we switch
\[
\varepsilon_s \tilde{f}_s \rightarrow \tilde{f}_s \varepsilon_s + \tilde{h}_s.
\]
When the hypothesis of the lemma holds, all $\tilde{h}_s$’s occur to the right of all $\tilde{f}_t$’s in every monomial of $\tau$. If we always apply the above steps to the rightmost occurrence of $\varepsilon_r \tilde{f}_u$ or $\tilde{h}_r \tilde{f}_u$, then we never create a $\tilde{h}_s$ to the left of any $\tilde{f}_t$. Thus we will never create any $a_{st}$’s. Since $\tau$ is independent of $a_{st}$, we conclude that $\eta(\tau)$ is also independent of $a_{st}$.\qed
Example 3.2 illustrates that if $f_t$ occurs to the right of $h_s$ in $\tau$, then $\eta(\tau)$ may dependent on $a_{st}$ even if $\tau$ does not.

For the rest of this section, we work with two Cartan matrices $A$ and $A'$ over finite sets $S$ and $S'$ with $|S| = |S'|$. We will also fix a bijection $\sigma : S \to S'$ and for the sake of notational simplicity, we let $A'_{st} := A'_{\sigma(s)\sigma(t)}$ for all $s, t \in S$ (note that we are not assuming that $A_{st} = A'_{st}$). Let $\mathfrak{g}(A)$ and $\mathfrak{g}(A')$ denote the corresponding Kac-Moody algebras to $A$ and $A'$. We will use the notation above for the generators of $\mathfrak{g}(A)$, and refer to the generators of $\mathfrak{g}(A')$ by

$$h'_s := h_{\sigma(s)}, \quad e'_s := e_{\sigma(s)}, \quad \text{and} \quad f'_s := f_{\sigma(s)}.$$ 

Note that we always use $\mathfrak{g}(A')$ instead of $\mathfrak{g}'$ to avoid confusion with the commutator subgroup of $\mathfrak{g} = \mathfrak{g}(A)$.

Recall that a representation for $\mathfrak{g}(A)$ is said to be integrable if $e_s$ and $f_s$ are locally nilpotent for every $s \in S$. Given a dominant integral weight $\lambda$, we let $L^\max(\lambda) = L^\max(\lambda, A)$ denote the maximal integrable $\mathfrak{g}(A)$ module of highest weight $\lambda$.

Given two Cartan matrices $A$ and $A'$ over $S$ and $S'$ as above, we write $A \leq A'$ to mean that $A_{st} \leq A'_{st}$ for all $s, t \in S$ (in other words, $|A_{st}| \geq |A'_{st}|$). For the next few lemmas, we assume the following hypotheses:

**Hypothesis 3.4.**  
(i) $A$ and $A'$ are Cartan matrices over $S$ and $S'$ with $A \leq A'$.
(ii) $\lambda \in \mathfrak{h}(A)^+$ and $\lambda' \in \mathfrak{h}(A')^+$ are dominant integrable weights such that $\lambda(h_s) \geq \lambda'(h'_s)$ for all $s \in S$.
(iii) $V = L^\max(\lambda, A)$ and $V' = L^\max(\lambda', A')$ are the maximal integrable modules with highest weights $\lambda$ and $\lambda'$.
(iv) $\omega$ and $\omega'$ are highest weight vectors for $V$ and $V'$ respectively.

**Lemma 3.5.** Suppose hypotheses 3.4 hold. Then:

(a) There are surjective Lie algebra morphisms

$$\psi^+ : \mathfrak{n}^+(A) \to \mathfrak{n}^+(A') \quad \text{and} \quad \psi^- : \mathfrak{n}^-(A) \to \mathfrak{n}^-(A')$$

mapping $e_s \mapsto e'_s$ and $f_s \mapsto f'_s$ respectively.

(b) There is a surjective $\mathfrak{n}^-(A)$-module morphism $\pi : V \to V'$ sending $\omega$ to $\omega'$, where $V'$ is regarded as a $\mathfrak{n}^-(A)$-module using $\psi^-$.

**Proof.** Part (a), the fact that $\psi^+ : \mathfrak{n}^+(A) \to \mathfrak{n}^+(A')$ is well-defined and surjective follows from the fact that $1 - A_{st} \geq 1 - A'_{st}$ for all $s, t \in S$. In particular, the relation $\text{ad}(e'_s)^{1-A_{st}}(e'_t) = 0$ holds in $\mathfrak{n}^+(A')$. Similar argument holds for the map $\psi^- : \mathfrak{n}^-(A) \to \mathfrak{n}^-(A')$.

For part (b), recall that $L^\max(\lambda, A)$ is the quotient of the Verma module (see [Kum02 Definition 2.1.1 and Definition 2.1.5])

$$M(\lambda, A) := \mathcal{U}(\mathfrak{n}^-(A)) \otimes_{\mathbb{C}} \mathbb{C}_{\lambda}$$

by the $\mathfrak{g}(A)$-module $M^1(\lambda, A)$ generated by $f_s^{\lambda(h_s)+1}v_{\lambda}$, where $s \in S$ and $v_{\lambda}$ is the cyclic weight vector $1 \otimes 1$ in $M(\lambda, A)$. By [Kum02 Lemma 2.1.6], the action of $\mathfrak{n}^+$ sends the generators $f_s^{\lambda(h_s)+1}v_{\lambda}$ to zero, so that $M^1(\lambda, A)$ is also the $\mathcal{U}(\mathfrak{n}^-(A))$-module generated by $f_s^{\lambda(h_s)+1}v_{\lambda}$. By part (a), for any cyclic weight vector $v_{\lambda'}$ of $M(\lambda', A')$, there is a surjective $\mathcal{U}(\mathfrak{n}^-(A))$-module map $\pi : M(\lambda, A) \to M(\lambda', A')$ sending $v_{\lambda} \mapsto v_{\lambda'}$. Now

$$\pi(f_s^{\lambda(h_s)+1}v_{\lambda}) = (f'_s)^{\lambda(h_s)+1}v_{\lambda'}$$
belongs to $M^1(\lambda', A')$ for all $s \in S$ since $\lambda(h_s) + 1 \geq \lambda'(h'_s) + 1$. Thus we get an induced morphism $\pi : L^{\text{max}}(\lambda, A) \to L^{\text{max}}(\lambda', A')$ on the quotients. \hfill \square

We now recall the root system of a Kac-Moody algebra $g(A)$ and the inversion set of an element $w \in W(A)$. Let $Q = Q(A) := \bigoplus_{s \in S} \mathbb{Z} \alpha_s \subset h^*$ denote the root lattice and let $R(A) \subseteq Q$ denote the root system of $g(A)$. We can decompose the set $R(A) = R^+(A) \sqcup R^-(A)$ where $R^+(A)$ and $R^-(A)$ denote the subsets of positive and negative roots respectively. We have that $n^+(A) = \bigoplus_{\alpha \in R^+(A)} g(A)_\alpha$ where $g(A)_\alpha$ is the root space corresponding to $\alpha$. The Weyl group $W(A)$ (which is generated by $S$) acts on $Q$ by

$$s(\alpha_t) := \alpha_t - \alpha_t(h_s) \alpha_s$$

for $s, t \in S$. Given $w \in W(A)$, define the inversion set

$$I(w) := \{ \alpha \in R^+(A) : w^{-1}(\alpha) \in R^-(A) \}.$$  

Note that the inversion set is finite of size $\ell(w)$ and lies inside the sublattice $\bigoplus_{s \in S(w)} \mathbb{Z} \alpha_s$.

**Lemma 3.6.** Let $A$ and $A'$ be Cartan matrices over $S$ and $S'$ with $w \in W(A)$ and $w' \in W(A')$. Suppose that $w$ is Cartan equivalent to $w'$ under the bijection $\sigma : S(w) \to S(w')$. Then the induced isomorphism

$$\sigma : \bigoplus_{s \in S(w)} \mathbb{Z} \alpha_s \to \bigoplus_{s' \in S(w')} \mathbb{Z} \alpha'_{s'}$$

given by $\alpha_s \mapsto \alpha'_{\sigma(s)}$ identifies $I(w)$ with $I(w')$.

**Proof.** Let $w = s_1 \cdots s_k$ be a reduced expression. Then $I(w) = \{ \beta_1, \ldots, \beta_k \}$, where

$$\beta_\ell = s_1 \cdots s_{\ell-1}(\alpha_{s_\ell}).$$

Given $1 \leq \ell \leq k$, we want to show that $\sigma(\beta_\ell)$ belongs to $I(w')$. First note that if $st \leq w$, then $A_{st} = A'_{\sigma(s)\sigma(t)}$ and hence

$$\sigma(s(\alpha_t)) = \sigma(\alpha_t - A_s \alpha_s) = \alpha'_{\sigma(t)} - A'_{\sigma(s)\sigma(t)} \alpha'_{\sigma(s)} = \sigma(s)(\alpha'_{\sigma(t)}).$$

In particular, $\sigma(s_{\ell-1}(\alpha_{s_\ell})) = \sigma(s_{\ell-1})(\alpha'_{\sigma(s_\ell)})$. For the purpose of induction, suppose that

$$\sigma(s_m \cdots s_{\ell-1}(\alpha_{s_\ell})) = \sigma(s_m) \cdots \sigma(s_{\ell-1})(\alpha'_{\sigma(s_\ell)}).$$

Let $s_{m-1} = s$ and write $s_m \cdots s_{\ell-1}(\alpha_{s_\ell}) = \sum_{t \in S} c_t \alpha_t$ where $c_t \in \mathbb{Z}$. If $st \not\leq w$, then $t$ does not appear to the right of any $s$ in the reduced expression for $w$. This implies the coefficient $c_t = 0$ for all $t \in S$ such that $st \not\leq w$. Hence

$$\sigma(s_m \cdots s_{\ell-1}(\alpha_{s_\ell})) = \sum_{t \in S} c_t \sigma(s_{m-1}(\alpha_t)) = \sum_{t \in S} c_t \sigma(s_{m-1})(\alpha'_{\sigma(t)})$$

$$= \sigma(s_{m-1}) \left( \sum_{t \in S} c_t \alpha'_{\sigma(t)} \right) = \sigma(s_{m-1}) \sigma(s_m \cdots s_{\ell-1}(\alpha_{s_\ell})).$$

By induction, we have

$$\sigma(\beta_\ell) = \sigma(s_1 \cdots s_{\ell-1}(\alpha_{s_\ell})) = \sigma(s_1) \cdots \sigma(s_{\ell-1})(\alpha'_{\sigma(s_\ell)}).$$

Since $\sigma(s_1) \cdots \sigma(s_k)$ is a reduced expression for $w'$, we have $\sigma(\beta_\ell) \in I(w')$. \hfill \square
For \( w \in W(A) \), define
\[
\mathfrak{n}^+(A)_w := \bigoplus_{\alpha \in I(w)} \mathfrak{g}(A)_\alpha.
\]

Since \( I(w) \) is closed under bracket, \( \mathfrak{n}^+(A)_w \) is a finite-dimensional nilpotent Lie algebra. We now add two additional hypotheses:

**Hypothesis 3.7.**

(i) Suppose that \( w \in W(A) \) such that \( A_{st} = A'_{st} \) for all \( st \leq w \).

(ii) Let \( w' \in W(A') \) denote the element which is Cartan equivalent to \( w \) under the bijection \( \sigma : S \to S' \).

**Lemma 3.8.** Suppose that hypotheses 3.4 and 3.7 hold. Then \( \psi^+ \) induces an isomorphism \( \mathfrak{n}^+(A)_w \to \mathfrak{n}^+(A')_{w'} \).

**Proof.** Because \( \mathfrak{n}^+(A) \) is generated by \( \{e_s\}_{s \in S} \), it is spanned by Lie monomials in these same variables. Since each monomial is contained in a root space, we have that \( \psi^+(\mathfrak{g}(A)_\alpha) \subseteq \mathfrak{g}(A')_{\sigma(\alpha)} \) where \( \sigma : S(w) \to S(w') \) is given in Lemma 3.6. The lemma now follows from Lemma 3.6. \( \square \)

**Lemma 3.9.** Let \( \exp : \mathfrak{g}(A) \to \mathcal{G}(A) \) denote the exponential map and suppose that hypotheses 3.4 and 3.7 hold. Further suppose that \( \lambda(h_s) = \lambda'(h'_s) \) for all \( s \in S \). Then
\[
\pi(\exp(z) v \cdot \omega) = \exp(\psi^+(z)) \sigma(v) \cdot \omega'
\]
for all \( z \in \mathfrak{n}^+(A) \) and \( v \leq w \).

**Proof.** Let \( v = s_1 \ldots s_k \) be a reduced expression. If \( st \not\subseteq w \) then \( st \not\subseteq v \), so \( \sigma(s_1) \ldots \sigma(s_k) \) is also a reduced expression for \( \sigma(v) \). By [Kum02, Definition 1.3.2 (5)], the action of \( s \) (resp. \( \sigma(s) \)) on \( V \) (resp. \( V' \)) is given by
\[
\exp(f_s) \exp(-e_s) \exp(f_s) \quad (\text{resp. } \exp(f'_s) \exp(-e'_s) \exp(f'_s)).
\]
Since \( V \) is integrable, for any \( s \in S \) and \( u \in V \), there is an integer \( N_0 \) such that
\[
\exp(-e_s) \cdot u = \left( \sum_{k=0}^{N} \frac{(-e_s)^k}{k!} \right) \cdot u \quad \text{and} \quad \exp(f_s) \cdot u = \left( \sum_{k=0}^{N} \frac{f_s^k}{k!} \right) \cdot u
\]
for all \( N \geq N_0 \). The same applies to \( \exp(-e'_s) \) and \( \exp(f'_s) \) when acting on \( V' \). Since \( V \) and \( V' \) are highest-weight modules, we also have similar expressions for \( \exp(z) \) and \( \exp(\psi^+(z)) \). Thus we can find an integer \( N >> 0 \) such that
\[
\exp(z) v \cdot \omega = \exp(z)(s_1 \ldots s_k) \cdot \omega = \left( \sum_{j=0}^{N} \frac{z^j}{j!} \right) \cdot \left( \sum_{(l_1,m_1,n_1) \in [N]^3} \frac{f_{s_1}^{l_1}(-e_{s_1})^{m_1} f_{s_1}^{n_1}}{(l_1!)(m_1!(n_1!))} \right) \ldots
\]
and
\[
\exp(\psi^+(z)) \sigma(v) \cdot \omega' = \left( \sum_{j=0}^{N} \frac{\psi^+(z)^j}{j!} \right) \cdot \left( \sum_{(l_1,m_1,n_1) \in [N]^3} \frac{(f'_{s_1})^{l_1}(-e'_{s_1})^{m_1}(f'_{s_1})^{n_1}}{(l_1!)(m_1!(n_1!))} \right) \ldots \]
and
\[
\ldots \left( \sum_{(l_k,m_k,n_k) \in [N]^3} \frac{(f'_{s_k})^{l_k}(-e'_{s_k})^{m_k}(f'_{s_k})^{n_k}}{(l_k!)(m_k!(n_k!))} \right) \cdot \omega',
\]
Proof of Proposition 3.1. Let \( \sigma \). In particular, hypotheses 3.7 will hold. Assume (now with loss of generality) that every point of \( S \) that the variety structure on \( S \) does not occur in \( \Phi \), Lemma 3.3 implies that \( \phi(A')(\eta)' = \phi(A')(\eta_0)' \). Letting 
\[
\eta_0 = \sum_{I,J} c_{IJ} g_{IJ}(a) \tilde{f}_I^I.
\]
Note that \( h_s \omega = \lambda(h_s) \omega \) and \( e_s \omega = 0 \) for all \( s \in S \). Hence, for any \( I \) in the above sum, we have that \( \phi(A)(y^I) \omega = c_I \omega \) for some \( c_I \in \mathbb{Z} \). Since \( \lambda(h_s) = \lambda'(h_s) \), we also have \( \phi(A')(y^I) \omega' = c_I \omega' \). Letting 
\[
\eta_0 = \sum_{I,J} c_{IJ} g_{IJ}(a) \tilde{f}_I^I
\]
yields 
\[
\phi(A)(\Phi) \cdot \omega = \phi(A)(\eta_0) \cdot \omega \quad \text{and} \quad \phi(A')(\Phi) \cdot \omega' = \phi(A')(\eta_0) \cdot \omega'.
\]
If \( st \leq w \) then \( st \leq v \), and this implies that \( f_0 \) occurs to the left of every \( f_s \) in \( \Phi \) by Lemma 2.1. Since \( h_s \) does not occur in \( \Phi \), Lemma 3.3 implies that \( \eta(\Phi) \) is independent of \( a_{st} \). This also implies that \( \eta_0 \) is independent of \( a_{st} \). On the other hand, if \( st \leq w \), then \( A_{st} = A'_{st} \). So 
\[
\eta_0 = \sum_{I,J} c_{IJ} g_{IJ}(A) \tilde{f}_I^I = \sum_{I,J} c_{IJ} g_{IJ}(A') \tilde{f}_I^I.
\]
It follows that \( \phi(A')(\eta_0) = \psi^{-1}(\phi(\eta_0)) \). Finally, the map \( \pi \) is \( \psi^{-1} \)-equivariant, so 
\[
\pi(\exp(z) v \cdot \omega) = \pi(\phi(A)(\Phi) \cdot \omega) = \pi(\phi(A)(\eta_0) \cdot \omega)
\]
\[
= \psi^{-1}(\phi(A)(\eta_0)) \pi(\omega) = \phi(A')(\eta_0) \omega'
\]
\[
= \phi(A')(\Phi) \omega' = \exp(\psi^+(z)) \sigma(v) \cdot \omega'.
\]

\[ \square \]

**Proof of Proposition 3.1.** Let \( A \) and \( A' \) be Cartan matrices over \( S \) and \( S' \) respectively and that \( \sigma : S(w) \to S(w') \) gives a Cartan equivalence between \( (w, A) \) and \( (w', A') \). Without loss of generality, we can assume that \( S = S(w) \) and \( S' = S(w') \) (for instance, see [RS16, Lemma 4.8]). In particular, hypotheses 3.7 will hold. Assume (now with loss of generality) that \( A \leq A' \). By [Kum02, Definition 7.1.19], the stable variety structure on \( X(w, A) \) is induced (meaning that there is a closed embedding) by taking the map 
\[
G(A)/B(A) \to \mathbb{P}(V) : g \mapsto g \cdot \omega,
\]
where \( B(A) \) is the Borel subgroup of \( G(A) \), and \( \omega \) is the highest weight vector of \( V = L^{\max}(\lambda, A) \) for a large enough dominant weight \( \lambda \). By possibly increasing \( \lambda \), we can assume that the variety structure on \( X(w', A') \) is induced by taking the map 
\[
G(A')/B(A') \to \mathbb{P}(V') : g \mapsto g \cdot \omega',
\]
where \( \omega' \) is the highest weight of \( V' = L^{\max}(\lambda', A') \) for a dominant weight \( \lambda' \) with \( \lambda'(h_{\sigma(s)}) = \lambda(h_s) \). Hence all parts of Hypothesis 3.4 are satisfied.

Consider the map \( \pi : X(w, A) \to \mathbb{P}(V') \) induced by \( \pi : V \to V' \). By [Kum02, 6.2.E.1], every point of \( X(w, A) \) can be written uniquely as \( \exp(x) v \cdot \omega \) for some \( v \leq w \) and
Let \( x \in n^+(A)_v \). By Corollary 2.1 and Lemmas 3.8 and 3.9, the map \( \pi \) induces a bijection \( X(w, A) \to X(w', A') \). Since \( X(w', A') \) is a normal variety (see [Kum02, Theorem 8.2.2 (b)]), the map \( \pi \) restricted to \( X(w, A) \) must be an algebraic isomorphism.

Now suppose that \( A \not\preceq A' \). Define \( \bar{A} \) by \( \bar{A}_{st} = \min(A_{st}, A'_{st}) \). By Lemma 2.2, there is an element \( \bar{w} \) in \( W(\bar{A}) \) such that \( (\bar{w}, \bar{A}) \) is Cartan equivalent to both \( (w, A) \) and \( (w', A') \). We also have \( \bar{A} \leq A \) and \( \bar{A} \leq A' \) and hence the above argument implies \( X(w, A) \cong X(\bar{w}, \bar{A}) \cong X(w', A') \).

\[ \square \]

4. The cohomology ring of Schubert varieties

In this section, we finish the proof of Theorem 1.3 by showing that condition (2) implies condition (3), and that condition (3) implies condition (1). Let \( A \) be a Cartan matrix over \( S \), and let \( w \in W(A) \). Recall that \( B = B(A) \) is the Borel subgroup of the Kac-Moody group \( \mathcal{G}(A) \). The Schubert variety \( X(w, A) \) has a stratification given by its decomposition into Schubert cells

\[ X(w, A) = \overline{B w \mathcal{B}} / \mathcal{B} = \bigcup_{w \leq w} B w \mathcal{B} / \mathcal{B}. \]

Let \( x_v \) denote the fundamental class of the Schubert subvariety \( X(v, A) = \overline{B v \mathcal{B}} / \mathcal{B} \subseteq X(w, A) \) in the integral homology group \( H_{2\ell(v)}(X(w, A)) := H_{2\ell(v)}(X(w, A), \mathbb{Z}) \). Equivalently, if we consider \( X(v, A) \) as a cycle in the Chow group \( \text{Ch}_*(X(w, A)) \), then

\[ x_v = \text{cl}(X(v, A)) \]

where \( \text{cl} : \text{Ch}_*(X(w, A)) \to H_*(X(w, A)) \) denotes the cycle map between Chow groups and homology. It is well known that the Schubert homology classes \( \{x_v\}_{v \leq w} \) form a \( \mathbb{Z} \)-basis of \( H_*(X(w, A)) \). The corresponding Schubert basis \( \{\xi_v\}_{v \leq w} \) in integral cohomology \( H^*(X(w, A)) := H^*(X(w, A), \mathbb{Z}) \) is defined (using the identification \( H^*(X(w, A)) \cong \text{Hom}_{\mathbb{Z}}(H_*(X(w, A)), \mathbb{Z}) \)) by

\[ \xi_v(x_u) := \delta_{vu}. \]

We now prove that condition (2) implies condition (3) in Theorem 1.3.

**Proposition 4.1.** Let \( A \) and \( A' \) be Cartan matrices with \( w \in W(A) \) and \( w' \in W(A') \) and suppose that \( \phi : X(w, A) \to X(w', A') \) is an algebraic isomorphism. Then the induced map

\[ \phi^* : H^*(X(w', A')) \to H^*(X(w, A)) \]

is a graded ring isomorphism that identifies Schubert bases.

**Proof of Proposition 4.1.** Let \( \text{Ch}_*(X(w, A)) \) and \( \text{Ch}_*(X(w', A')) \) denote the Chow groups of \( X(w, A) \) and \( X(w', A') \). Since \( \phi \) is an algebraic isomorphism, the induced isomorphism of Chow groups \( \phi_* : \text{Ch}_*(X(w, A)) \to \text{Ch}_*(X(w', A')) \) preserves the cone of effective classes. By [FMS93, Corollary to Theorem 1], any effective class is a nonnegative \( \mathbb{Z} \)-linear combination of Schubert cycles. Hence the Schubert cycles form the minimal extremal rays of the effective cone, so \( \phi_* \) maps Schubert cycles to Schubert cycles. Since the Schubert classes in cohomology are dual to the Schubert classes in homology, \( \phi^* \) maps Schubert classes to Schubert classes. \( \square \)

To finish the proof of Theorem 1.3 we just need to show that condition (3) implies condition (1):

\[ \text{This idea effectively goes back to [Hir84]; see also [KN98].} \]
Proposition 4.2. Let $A$ and $A'$ be Cartan matrices with $w \in W(A)$ and $w' \in W(A')$. Suppose there is graded ring isomorphism

$$\phi : H^*(X(w, A)) \to H^*(X(w', A'))$$

which identifies Schubert bases. Then $(w, A)$ is Cartan equivalent to $(w', A')$.

For the proof of Proposition 4.2, suppose we have two Schubert varieties $X(w, A)$ and $X(w', A')$ with an isomorphism $\phi$ between cohomology rings as in the proposition. Let $E := \{\xi_v\}_{v \leq w}$ denote the Schubert basis of $H^*(X(w, A))$. As sets, there is a bijection $E \to [e, w]$ by mapping $\xi_u \mapsto u$. Define $\bar{S}(w) := E \cap H^2(X(w, A))$. The bijection $E \to [e, w]$ sends $\xi_u$ to $s$, and hence identifies $\bar{S}(w)$ with $S(w)$. We similarly define $E'$ and $\bar{S}(w')$ for $H^*(X(w', A'))$. By assumption, the isomorphism $\phi$ restricted to $E$ gives a degree-preserving bijection $\phi : E \to E'$. Let $\sigma : S(w) \to S(w')$ be the bijection corresponding to $\phi|_{\bar{S}(w)}$, so in particular $\sigma(\xi_s) = \xi_{\sigma(s)}$ for any $s \in S(w)$. To prove Proposition 4.2, we will show that $\sigma$ is a Cartan equivalence. For this, we need to show that all of the relevant entries of the Cartan matrix and the reduced word structure of $w \in W(A)$ can be reconstructed from the ring structure of $H^*(X(w, A))$.

Recall the set of roots $R(A) \subseteq Q(A) = \bigoplus_{s \in S} \mathbb{Z} \alpha_s \subseteq \mathfrak{h}^*(A)$. For notational simplicity, we denote $R := R(A)$ and $R^+ := R^+(A)$. A root $\beta \in R$ is said to be a real root if $\beta = v(\alpha_s)$ for some $v \in W(A)$ and $s \in S$. The Weyl group $W(A)$ acts on the dual space $\mathfrak{h}(A)$ by

$$s(h) := h - \alpha_s(h) h_s$$

for $s \in S$ and $h \in \mathfrak{h}(A)$, where $h_s \in \mathfrak{h}(A)$, $s \in S$ are vectors as in Section 3 (so $\alpha_t(h_s) = A_{st}$).

The restriction of this action to $\bigoplus_{s \in S} \mathbb{Z} h_s$ is the dual action to the action of $W(A)$ on $Q(A)$. For a real root $\beta = v(\alpha_s)$, the corresponding coroot is $\beta' := v(h_s) \in \mathfrak{h}(A)$, and the corresponding reflection is $s_\beta := vsv^{-1} \in W(A)$. Note that $\alpha'_t = h_t$ and $s_{\alpha_t} = t$ for all $t \in S$. Finally, we pick fundamental coweights $\{\omega_s\}_{s \in S} \subseteq \mathfrak{h}^*(A)$ satisfying the formula

$$\omega_t(h_t) := \delta_{st}.$$ 

The main computational tool used to prove Proposition 4.2 is Chevalley’s (non-equivariant) formula for multiplying Schubert classes by simple Schubert classes. We use here the version from [Kum02, Theorem 11.1.7 (i)].

Proposition 4.3. (Chevalley’s formula) For any $\xi_s \in \bar{S}(w)$ and $\xi_u \in E$,

$$\xi_s : \xi_u = \sum \omega_s(u^{-1}(\beta')) \xi_{s \beta u}$$

where the sum is over all real roots $\beta \in R^+$ such that $\ell(s_{\beta u}) = \ell(u) + 1$ and $s_{\beta u} \leq w$.

Note that if $\ell(s_{\beta u}) = \ell(u) + 1$, then $u^{-1}(\beta') \in R^+$ and hence $\omega_s(u^{-1}(\beta'))$ is a nonnegative integer. For any $F \in H^*(X(w, A))$, define the support of $F$ as

$$\text{Supp}(F) := \{\xi_v \in E : F(x_v) \neq 0\}.$$ 

In other words, if we write $F = \sum_{v \leq w} c_v \xi_v$, then $\xi_v \in \text{Supp}(F)$ if and only if $c_v \neq 0$. Let $\prec$ be the partial order on $E$ generated by the covering relations $\xi_u \prec \xi_v$ for $u$ and $v$ such that $\xi_v \in \text{Supp}(\xi_s \xi_u)$ for some $s \in S(w)$. In the next lemma, we show that $\prec$ corresponds to Bruhat order $\leq$ on $[e, w]$.

Lemma 4.4. $\xi_u \prec \xi_v$ if and only if $u \leq v$. 

Proof. It suffices to consider covering relations. Recall that \( u \leq v \) if and only if \( v = s_\beta u \) with \( \ell(v) = \ell(u) + 1 \) for some real root \( \beta \in R^+ \). Hence \( \xi_u \prec \xi_v \) immediately implies \( u \leq v \). Conversely, if \( u \leq v \) with \( \ell(v) = \ell(u) + 1 \), then \( v = s_\beta u \) where \( u^{-1}(\beta^\vee) \in R^+ \). So there exists \( s \in S(w) \) for which \( \omega_s(u^{-1}(\beta^\vee)) > 0 \), implying that \( \xi_v \in \text{Supp}(\xi_s \cdot \xi_u) \). Hence \( \xi_u \prec \xi_v \).

Next we show that Cartan matrix entries \( A_{st} \) for \( st \leq w \) can be recovered from products of simple Schubert classes.

**Lemma 4.5.** Let \( s, t \in S(w) \) such that \( s \neq t \). Then:

1. \( \text{Supp}(\xi_s \xi_t) \cap \text{Supp}(\xi_t^2) \neq \emptyset \) if and only if \( st \leq w \) and \( st \neq ts \). In this case,
   \[
   \text{Supp}(\xi_s \xi_t) \cap \text{Supp}(\xi_t^2) = \{ \xi_{st} \} \quad \text{and} \quad A_{st} = -\xi_t^2(x_{st}).
   \]

2. \( \text{Supp}(\xi_s \xi_t) \cap (\text{Supp}(\xi_t^2) \cup \text{Supp}(\xi_s^2)) = \emptyset \) if and only if \( ts = st \). In this case, \( A_{st} = 0 \).

**Proof.** By Chevalley’s formula, we have

\[
\xi_t^2 = \sum_{s': st \leq w \atop s' \neq t} \omega_t(t(\alpha_{s'}^\vee)) \xi_{s't} + \sum_{s'': ts'' \leq w \atop ts'' \neq st''} \omega_t(\alpha_{s''}^\vee) \xi_{st''},
\]

In the second sum, since \( t \neq s'' \), \( \omega_t(\alpha_{s''}^\vee) = 0 \). So

\[
\xi_t^2 = \sum_{s': st \leq w \atop s' \neq t} \omega_t(\alpha_{s'}^\vee - A_{s't} \alpha_t^\vee) \xi_{s't} = \sum_{s': st \leq w \atop s' \neq t} -A_{s't} \xi_{s't}.
\]

Since \( A_{s't} = 0 \) if and only if \( s't = ts' \), we conclude that

\[
\text{Supp}(\xi_t^2) = \{ \xi_{s't} \mid s' \in S(w) \text{ such that } s't \neq ts' \text{ and } s't \leq w \}.
\]

If \( s \neq t \), then

\[
\xi_{st} = \sum_{s': st \leq w \atop s' \neq t} \omega_s(t(\alpha_{s'}^\vee)) \xi_{s't} + \sum_{s'': ts'' \leq w \atop ts'' \neq st''} \omega_s(\alpha_{s''}^\vee) \xi_{st''}
\]

\[
= \sum_{s': st \leq w \atop s' \neq t} \omega_s(\alpha_{s'}^\vee - A_{s't} \alpha_t^\vee) \xi_{s't} + \sum_{s'': ts'' \leq w \atop ts'' \neq st''} \omega_s(\alpha_{s''}^\vee) \xi_{st''}.
\]

If \( st \leq w \) then the first sum is \( \xi_{st} \), and otherwise the first sum is 0. If \( st \neq ts \), and \( ts \leq w \), then the second sum is \( \xi_{ts} \), while otherwise it’s 0. So we conclude that \( \text{Supp}(\xi_s \xi_t) \subseteq \{ \xi_{st}, \xi_{ts} \} \) with \( \xi_{st} \in \text{Supp}(\xi_s \xi_t) \) if and only if \( st \leq w \). This proves part (1). For part (2), note that either \( st \leq w \) or \( ts \leq w \). So if \( \text{Supp}(\xi_t^2) \cup \text{Supp}(\xi_s^2) \) contains neither \( \xi_{st} \) or \( \xi_{ts} \), we must have \( st = ts \). Conversely, if \( st = ts \), then \( \xi_{st} = \xi_{ts} \) cannot belong to either \( \text{Supp}(\xi_t^2) \) or \( \text{Supp}(\xi_s^2) \). Since \( \text{Supp}(\xi_s \xi_t) \) will always contain one of \( \xi_{st} \) or \( \xi_{ts} \), this proves part (2).

**Corollary 4.6.** Let \( s, t \in S(w) \) such that \( s \neq t \). If \( st \leq w \), then \( A_{st} = A'_{\sigma(s)\sigma(t)} \).

**Proof.** First suppose that \( s, t \in S(w) \) commute. By Lemma 4.5 part (2), we have

\[
\text{Supp}(\xi_s \xi_t) \cap (\text{Supp}(\xi_t^2) \cup \text{Supp}(\xi_s^2)) = \emptyset.
\]

Since \( \phi \) is a graded ring isomorphism identifying Schubert bases, we also have that

\[
\text{Supp}(\phi(\xi_s \phi(\xi_t))) \cap (\text{Supp}(\phi(\xi_t)^2) \cup \text{Supp}(\phi(\xi_s)^2)) = \emptyset.
\]
This implies that \( \sigma(s), \sigma(t) \) also commute and \( A_{st} = A'_{\sigma(s)\sigma(t)} = 0 \). If \( s, t \) do not commute, then Lemma 4.5 part (1) implies

\[
\text{Supp}(\xi_s \xi_t) \cap \text{Supp}(\xi_t^2) = \{\xi_{st}\}.
\]

Applying the map \( \phi \) to this equation gives

\[
\text{Supp}(\phi(\xi_s)\phi(\xi_t)) \cap \text{Supp}(\phi(\xi_t)^2) = \{\phi(\xi_{st})\}.
\]

Since this set is nonempty, Lemma 4.5 part (1) also implies that \( \sigma(s)\sigma(t) \leq w' \) and \( \sigma(s), \sigma(t) \) do not commute. Moreover, we know that

\[
\text{Supp}(\phi(\xi_s)\phi(\xi_t)) \cap \text{Supp}(\phi(\xi_t)^2) = \{\xi_{\sigma(s)\sigma(t)}\},
\]

so \( \phi(\xi_{st}) = \xi_{\sigma(s)\sigma(t)} \). Since \( A_{st} \) is the coefficient of \( \xi_{st} \) in \( -\xi_t^2 \), and \( A'_{\sigma(s)\sigma(t)} \) is the coefficient of \( \xi_{\sigma(s)\sigma(t)} \) in \( -\xi_t^2 = \phi(-\xi_t^2) \), we conclude that \( A_{st} = A'_{\sigma(s)\sigma(t)} \).

One consequence of Lemma 2.2 and Corollary 4.6 is that if \( w = s_1 \cdots s_k \) is a reduced expression, then \( \sigma(s_1) \cdots \sigma(s_k) \in W(A') \) is a reduced expression Cartan equivalent to \( w \). What remains to be shown is that \( w' = \sigma(s_1) \cdots \sigma(s_k) \). In order to show this, we next show how to reconstruct the descent sets of elements in \( W(A) \) from the ring structure of \( H^*(X(w,A)) \). For any \( J \subseteq S \), let \( W_J \) denote Coxeter subgroup generated by \( J \) and let \( W_J \) denote the set of minimal length coset representatives for the cosets \( W(A)/W_J \). The right descent set of \( u \in W(A) \) is

\[
D_R(u) := \{ s \in S : \ell(us) = \ell(u) - 1 \}.
\]

If \( u \leq w \), then \( D_R(u) \subseteq S(w) \). It is well known that \( s \in D_R(u) \) if and only if \( u \notin W^s \), and that \( D_R(u) \) is the set of simple reflections \( s \) for which there are reduced expressions \( u = s_1 \cdots s_k \) with \( s_k = s \). For any subset \( J \subseteq S(w) \), let \( H^J \) be the subring of \( H^*(X(w,A)) \) generated by \( \{ \xi_s : s \in S(w) \setminus J \} \), and let

\[
E^J := \bigcup_{F \in H^J} \text{Supp}(F)
\]

be the support set of \( H^J \) in \( E \).

**Lemma 4.7.** The map \( E \to [e,w] \) given by \( \xi_u \mapsto v \) restricts to a bijection \( E^J \to W^J \cap [e,w] \).

**Proof.** We first show that \( \{ v \in [e,w] : \xi_v \in E^J \} \) is a subset of \( W^J \). Suppose \( u \in W^J \cap [e,w] \), \( s \in S(w) \setminus J \). By Chevalley’s formula, if \( \xi_v \in S(\xi_s \xi_u) \) then \( v = s_\beta u \) for some real root \( \beta \in R^+ \) with \( \ell(v) = \ell(u) + 1 \) and \( \omega_s(u^{-1}(\beta^\vee)) \neq 0 \). Suppose \( v \) is of this form. If \( v \notin W^J \), then there is some simple reflection \( s' \in D_R(v) \cap J \), and consequently there is a reduced expression \( v = s_1 \cdots s_k \) for \( v \) with \( s_k = s' \in J \). But \( u = s_1 \cdots s_k \) where \( s_\ell \) means that \( s_\ell \) is omitted from the product, and \( u \in W^J \), so we must have that \( \ell = k \), and ultimately \( v = us' \). Hence \( s_{u^{-1}\beta} = u^{-1}s_\beta u = u^{-1}v = s' \), and since the correspondence between roots and reflections is a bijection, \( u^{-1}\beta = \alpha_{s'} \) and \( u^{-1}\beta^\vee = \alpha_{s'}^\vee \). Since \( s \notin J \), \( \omega_s(u^{-1}(\beta^\vee)) = \omega_s(\alpha_{s'}^\vee) = 0 \), so if \( v \notin W^J \), then \( \xi_v \notin \text{Supp}(\xi_s \xi_u) \).

Because \( H^*(X(w,A)) \) is graded, \( \xi_u \in E^J \) for \( s \in S(w) \) if and only if \( s \in W^J \). We’ve shown that if \( u \in W^J \cap [e,w] \), \( s \in S(W) \setminus J \), and \( \xi_v \in \text{Supp}(\xi_s \xi_u) \), then \( v \in W^J \), so we conclude by induction on degree that \( \{ v \in [e,w] : \xi_v \in E^J \} \subseteq W^J \).

For the converse, we show that \( \{ \xi_v : v \in W^J \cap [e,w] \} \) is a subset of \( E^J \) by induction on length. Indeed, suppose that \( v \in W^J \cap [e,w] \), and write \( v = s_1 \cdots s_k \). Let \( u := s_1v = s_2 \cdots s_k \). Clearly \( u \in W^J \), and by induction we can assume that \( \xi_u \in E^J \). Since \( \alpha_{s_1} \) is an inversion of \( v \), we have that \( \alpha_{s_1} \in R^+ \cap vR^- \). Hence \( v^{-1}(\alpha_{s_1}) \in v^{-1}R^+ \cap R^- \) which
implies \( v^{-1}(-\alpha_{s_1}) \in v^{-1}R^- \cap R^+ \). Now [Kum02 Exercise 1.3.E] says that, since \( v \in W^J \), the set \( v^{-1}R^- \cap R^+ \subseteq R^+ \setminus R^+_J \), where \( R^+_J \) is the subset of the positive roots in the span of \( \{\alpha' : s' \in J\} \). Thus

\[
u^{-1}(\alpha_{s_1}) = u^{-1}s_1(-\alpha_{s_1}) = v^{-1}(-\alpha_{s_1}) \in R^+ \setminus R^+_J.
\]

We conclude that there exists \( s \notin J \) such that \( \omega_s(u^{-1}(\alpha'_{s_1})) \neq 0 \). Consequently \( \xi_v \in \text{Supp}(\xi_s\xi_u) \), implying \( \xi_v \in E^J \).

We now define

\[
\tilde{D}_R(\xi_u) := \{\xi_s \in \tilde{S}(w) : \xi_u \notin E^s\}.
\]

**Lemma 4.8.** For any \( u \leq w \), the bijection \( \tilde{S}(w) \rightarrow S(w) \) given by \( \xi_s \mapsto s \) restricts to a bijection \( \tilde{D}_R(\xi_u) \rightarrow D_R(\xi_u) \).

**Proof.** Since \( s \in D_R(u) \) if and only if \( u \notin W^s \), the lemma follows immediately from Lemma 4.7.

**Lemma 4.9.** For every \( \xi_s \in \tilde{D}_R(\xi_v) \), there exists a unique \( \xi_u \in E^s \) for which \( \xi_v \in \text{Supp}(\xi_s\xi_u) \). Moreover, \( us = v \).

**Proof.** Fix \( \xi_s \in \tilde{D}_R(\xi_v) \). By Lemma 4.8, \( s \in D_R(v) \) and hence \( u = vs \leq v \). The element \( u \) is the unique element less than \( v \) such that \( \ell(u) = \ell(v) + 1 \) and \( u \in W_v \). Hence Lemma 4.4 and 4.7 imply \( \xi_v \in \text{Supp}(\xi_s\xi_u) \).

Recall that if \( w \in W(A) \), then \( RW(w) \) is the set of reduced words of \( w \). Lemma 4.9 implies we can inductively define \( \tilde{RW}(\xi_v) \) by setting \( \tilde{RW}(\xi_e) := \{\epsilon\} \) where \( \epsilon \) denotes the empty sequence, and

\[
\tilde{RW}(\xi_v) := \left\{ (\xi_{s_1}, \ldots, \xi_{s_m}, \xi_v) : \begin{array}{l}
\xi_s \in \tilde{D}_R(\xi_u), (\xi_{s_1}, \ldots, \xi_{s_m}) \in \tilde{RW}(\xi_u) \text{ for } u \\
\xi_v \in \text{Supp}(\xi_s\xi_u) & \text{the unique element such that } \xi_u \in E^s \text{ and } \n\end{array} \right\}.
\]

**Lemma 4.10.** For any \( v \in [e, w] \), the bijection \( \tilde{S}(w) \rightarrow S(w) \) given by \( \xi_s \mapsto s \) induces a bijection between \( \tilde{RW}(\xi_e) \) and \( RW(v) \).

**Proof.** \( RW(v) \) is the set of sequences of the form \( (s_1, \ldots, s_k) \) where \( s_k \in D_R(v) \) and \( (s_1, \ldots, s_{k-1}) \subseteq RW(v \cdot s_k) \). The lemma follows by induction on length using Lemmas 4.8 and 4.9.

**Proof of Proposition 4.2.** Suppose that \( \phi : H^*(X(w, A)) \rightarrow H^*(X(w', A')) \) is a graded ring isomorphism which identifies Schubert classes. In particular, the restricted map \( \phi|_{H^2(X(w, A))} \) induces a bijection \( \sigma : S(w) \rightarrow S(w') \). Let \( w = s_1 \cdots s_k \) be a reduced expression. Lemma 4.10 implies that \( (\xi_{s_1}, \ldots, \xi_{s_k}) \in \tilde{RW}(\xi_w) \). Since \( \phi \) is an isomorphism that identifies Schubert classes, \( \phi(\text{Supp}(F)) = \text{Supp}(\phi(F)) \) for any \( F \in H^*(X(w, A)) \). As a result, \( \phi(E^J) = (E^J)^{\sigma(J)} \) for any \( J \subseteq S(w) \) and hence \( \phi(\tilde{D}_R(\xi_v)) = \tilde{D}_R(\phi(\xi_v)) \). Also we have that \( \xi_w \in \text{Supp}(\xi_s\xi_u) \) if and only if \( \phi(\xi_v) \in \text{Supp}(\phi(\xi_s)\phi(\xi_u)) = \text{Supp}(\xi_{\sigma(s)}\phi(\xi_u)) \).

So Lemma 4.9 implies \( (\xi_{\sigma(s_1)}, \ldots, \xi_{\sigma(s_k)}) \in \tilde{RW}(\phi(\xi_w)) \). Since \( \phi \) is graded, \( \phi(\xi_w) = \xi_w' \), and hence \( \sigma(s_1) \cdots \sigma(s_k) \in RW(w') \) by Lemma 4.10. Finally, Corollary 4.6 says that \( A_{st} = A_{\sigma(s)\sigma(t)} \) for all \( st \leq w \). Thus \( w \) and \( w' \) are Cartan equivalent.

Together, Propositions 3.1, 4.1, and 4.2 complete the proof of Theorem 1.3.
4.1. Constructing a presentation of a Schubert variety. In section, we describe how to construct a presentation of a Schubert variety from geometric data. Suppose we have a variety $X$ which is isomorphic to a Schubert variety $X(w, A)$ for some $w \in W(A)$, but the element $w$ and matrix $A$ are unknown. We want to find a Cartan matrix $A'$ and $w' \in W(A')$ such that $X \cong X(w', A')$. This can be done if we know the integral cohomology ring $H^*(X; \mathbb{Z})$, and the effective cone $X$ in $H^*(X) = H^*(X; \mathbb{Z})$. Indeed, let $E$ be the generators of the extremal rays of the effective cone. Since $X$ is isomorphic to a Schubert variety, we know that $E$ is the Schubert basis for $H^*(X)$. The proof of Proposition 4.2 then gives a procedure to construct $w'$ and $A'$ from $H^*(X)$ and $E$. Specifically, for any $F \in H^*(X)$, we can write $F = \sum_{\xi \in E} c_\xi \xi$ for some integers $c_\xi$ and define

$$\text{Supp}(F) := \{\xi \in E \mid c_\xi \neq 0\}.$$ 

Let $\tilde{S} := E \cap H^2(X)$. We first construct a Cartan matrix $A'$ over the set $\tilde{S}$ using Lemma 4.5. Let $\zeta_1, \zeta_2 \in \tilde{S}$ and consider the following cases (where products denote the product in $H^*(X)$):

1. If $\zeta_1 = \zeta_2$, then set $A'_{\zeta_1, \zeta_2} = 2$.
2. If $\zeta_1 \neq \zeta_2$ and $\text{Supp}(C_{\zeta_1 \zeta_2}) \cap (\text{Supp}(\zeta_1^2) \cup \text{Supp}(\zeta_2^2)) = \emptyset$, then set $A'_{\zeta_1, \zeta_2} = A'_{\zeta_2, \zeta_1} = 0$.
3. If $\zeta_1 \neq \zeta_2$ and $\text{Supp}(C_{\zeta_1 \zeta_2}) \cap \text{Supp}(\zeta_2^2) = \{\nu\}$, then write $\zeta_2^2 = \sum_{\xi \in E} c_\xi \xi$, and set $A'_{\zeta_1, \zeta_2} = -c_s$.
4. If $\zeta_1 \neq \zeta_2$, $\text{Supp}(C_{\zeta_1 \zeta_2}) \cap \text{Supp}(\zeta_1^2) = \emptyset$, and $\text{Supp}(\zeta_2^2) \cap \text{Supp}(\zeta_1^2) \neq \emptyset$, then set $A'_{\zeta_1, \zeta_2}$ to any negative integer.

By Lemma 4.5, exactly one of these cases must hold for each pair $(\zeta_1, \zeta_2) \in \tilde{S}^2$. In addition, if $\zeta : S \to \tilde{S} : s \mapsto \zeta_s$, then $A_{st} = A'_{\zeta_s, \zeta_t}$ for all $st \leq w$.

Next, observe that for any $\xi \in E$, we can construct $\tilde{D}_R(\xi)$ and $\tilde{RW}(\xi)$ strictly in terms of $H^*(X)$ and $E$, without referring to $w$ or $A$. Indeed, for any $J \subseteq \tilde{S}$ we can define $H^J$ to be the subalgebra of $H^*(X)$ generated by $J$, and $E^J$ to be the set $\bigcup_{F \in H^J} \text{Supp}(F) \subseteq E$. We can then define $\tilde{D}_R(\xi)$ and $\tilde{RW}(\xi)$ as in the previous section, except that instead of making a distinction between $\xi_s$ and $s$, we write $\xi \in \tilde{S}$ in place of both of them (so for instance, we’d define $\tilde{D}_R(\xi) = \{\xi \in \tilde{S} : \xi \notin E(\zeta)\}$).

Let $\xi$ be the unique element of $E$ of highest degree, let $(\zeta_1, \ldots, \zeta_k) \in \tilde{RW}(\xi)$, and let $w' := \zeta_1 \cdots \zeta_k \in W(A')$. Then $\xi$ must be $\xi_w$, and if we write $\zeta_i = \xi_s_i$, then $s_1 \cdots s_k$ is a reduced word for $w$ by Lemma 4.10. Thus $\xi$ is a Cartan equivalence between $(w, A)$ and $(w', A')$, and so $X = X(w, A) \cong X(w', A')$.

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