The moduli in a 4D N=1 heterotic compactification on an elliptic CY, as well as in the dual F-theoretic compactification, break into ”base” parameters which are even (under the natural involution of the elliptic curves), and ”fiber” or twisting parameters; the latter include a continuous part which is odd, as well as a discrete part. We interpret all the heterotic moduli in terms of cohomology groups of the spectral covers, and identify them with the corresponding F-theoretic moduli in a certain stable degeneration. The argument is based on the comparison of three geometric objects: the spectral and cameral covers and the ADE del Pezzo fibrations. For the continuous part of the twisting moduli, this amounts to an isomorphism between certain abelian varieties: the connected component of the heterotic Prym variety (a modified Jacobian) and the F-theoretic intermediate Jacobian. The comparison of the discrete part generalizes the matching of heterotic 5brane / F-theoretic 3brane impurities.
Introduction

The classical string model for $N = 1$ supersymmetry in 4D is the compactification of the heterotic string on a Calabi-Yau $Z$ with a vector bundle $V = V_1 + V_2$ which breaks part of the $E_8 \times E_8$ symmetry. In this paper we describe the moduli in a heterotic compactification on an elliptic CY, as well as in the dual F-theoretic compactification. These moduli include ”base” parameters which are even (under the natural involution of the elliptic curves), and ”fiber” or twisting parameters; the latter include a continuous part which is odd, as well as a discrete part. We will interpret all the heterotic moduli in terms of cohomology groups of spectral covers, and identify them with the corresponding F-theoretic moduli in a certain stable degeneration of the $K3$ fibers. The argument will actually be based on the comparison of three geometric objects: the spectral and cameral covers and the ADE del Pezzo fibrations. The full twisting moduli are given, on the heterotic side, by a Prym variety constructed from the cameral cover. This Prym has a discrete group of components, each of which is an abelian variety. On the F-theory side the continuous part of the moduli is identified with the intermediate Jacobian of the del Pezzo fibration. The full moduli are given by the Deligne cohomology, an extension of a discrete group by the intermediate Jacobian. We show that the two abelian varieties making up the continuous parts are isomorphic, in any dimension and for any group. As to the discrete parts, we are able to identify the F-theory side as a subgroup of finite index in the heterotic side. Assuming that a couple of group cohomologies vanish, this becomes an isomorphism between the discrete parts as well. The comparison of the discrete parts generalizes the matching of heterotic 5brane / F-theoretic 3brane impurities.

Let us describe first some background for the problem. Two aspects of recent progress involve the consideration of the associated moduli space [1], [2], [3], [4], [5], and the occurrence of non-perturbative five-branes in the vacuum as correction term in the anomaly cancellation

$$c_2(Z) = c_2(V_1) + c_2(V_2) + n_5 f.$$

This progress was made possible by considering the case that $Z$ is elliptically fibered $\pi : Z \rightarrow B$ (of fibre $f$ which the five-branes wrap) over a two-fold base $B$. The motivation for requiring an elliptic fibration was twofold: first there exists then a dual $F$-theory description by considering a four-fold $X^4$ which is correspondingly fibered by $K3$ over $B$, thus extending adiabatically over the base $B$ the duality in 8D between the heterotic string on $T^2$ and $F$-theory over $K3$ [6]. Here the $K3$ is assumed to be elliptically fibered over $\mathbb{P}^1$; so this is actually a type IIB compactification on this $\mathbb{P}^1$ with varying dilaton+RR-scalar $\lambda + ie^{-\phi}$ reflecting the complex structure modulus of the elliptic ($F$-theory) fibre. The 18 deformations of this type of $K3$ correspond then to the 16 Wilson lines on the heterotic side + the complex structure parameter of the (heterotic) elliptic $T^2 +1$ (the Kahler parameter of the $T^2$) sharing the moduli space $SO(2, 18; \mathbb{Z}) \backslash SO(2, 18; \mathbb{R})/SO(2; \mathbb{R}) \times SO(18; \mathbb{R})$; the size of the $\mathbb{P}^1$ corresponds with the heterotic dilaton. For consistency of such a four-fold compactification, tadpole cancellation requires [6] that a number $n_3$ of space-time filling three-branes has to be turned on (we don’t turn on gauge bundles inside the seven-brane [10])

$$n_3 = \frac{\chi(X^4)}{24}$$
which precise number could then be matched with \( n_5 \), in the case of \( V \) a \( G = E_8 \) bundle leaving no unbroken gauge group. To make the identification one has to consider that the base \( B^3 \), the compactification space of the type IIB string theory with varying dilaton, is in turn fibered by \( \mathbb{P}^1 \) over the base \( B^2 \) common with the heterotic side, concretely \( n_{1/2} = 6c_1 \pm t \) with \( t = c_1(T) \) the cohomology class characterizing the \( \mathbb{P}^1 \) fibration of \( B^3 \) over \( B^2 \), the \( \mathbb{P}^1 \) bundle being given as projectivization of the vector bundle \( \mathcal{O} \oplus T \) (the analogue of the well known relation [3] in the 6D case, corresponding to \( B \) now a \( \mathbb{P}^1 \), between bundles of instanton number \((12+n,12-n)\) and \( F \)-theory for a Calabi-Yau three-fold over the Hirzebruch surface \( F_n \)).

The other motivation was that by working with elliptically fibered \( Z \) one can adiabatically extend the known results about moduli spaces of \( G \)-bundles over an elliptic curve \( E = T^2 \), of course taking into account that such a fiberwise description of the isomorphism class of a bundle leaves definitely room for twisting along the base \( B \). The latter possibility actually involves a two-fold complication: there is a continuous as well as a discrete part of these data. It is quite easy to see this for \( G = SU(n) \): in this case \( V \) can be constructed via push-forward of the Poincare bundle on the spectral cover \( C \times_B Z \), possibly twisted by a line bundle \( N \) over the spectral surface \( C \) (an \( n \)-fold cover of \( B \) (via \( \pi \) lying in \( Z \)), whose first Chern class (projected to \( B \)) is known from the condition \( c_1(V) = 0 \). So \( N \) itself is known up to the following two remaining degrees of freedom: first a class in \( H^{1,1}(C) \) which projects to zero in \( B \) (the discrete part), and second an element of \( Jac(C) := Pic_0(C) \) (the continuous part; the moduli odd under the elliptic involution). We will see below how to generalize this to other groups.

The continuous part is expected [1] to correspond on the \( F \)-theory side to the odd moduli, related there to the intermediate Jacobian \( J^3(X^4) \) of dimension \( h^{2,1} \), so that the following picture emerges. The moduli space \( \mathcal{M} \) of the bundles is fibered \( \mathcal{M} \to \mathcal{Y} \), with fibre \( Jac(C) \). There is a corresponding picture on the \( F \)-theory side: ignoring the Kähler classes (on both sides), the moduli space there is again fibered. The base is the moduli space of those complex deformations which fix a certain complex structure of \( Z \); the fibre is the intermediate Jacobian \( J^3(X^4) = H^3(X,\mathbb{R})/H^3(X,\mathbb{Z}) \) In total, \( h^{2,1}(Z) + h^1(Z,adV) + 1 = h^{2,1} + h^{2,1} \). (Unspecified Hodge numbers refer to \( X^4 \)). The fibre moduli are ‘odd’, the deformations belonging to the base ‘even’.

The discrete part should correspond to the possibility of turning on four-flux in an \( M \)-respectively \( F \)-theory compactification, as will be described in more detail below. This must, of course, be included in a description of the fibre data of \( \mathcal{M} \to \mathcal{Y} \). (The continuous and discrete part together describe for the four-fold what is known as its Deligne cohomology, which in the case considered is an extension of \( H^{2,2}(X^4,\mathbb{Z}) \) by \( J^3(X^4) \); here Hodge cohomology with integer coefficients will refer to the obvious intersection).

We start in section 1 by establishing some notation while briefly reviewing the match of the number of the base moduli, coming from \( \mathcal{Y} \). This is done for \( SU(n) \) following [1],[3], and then for \( E_8 \) following [3]. We then discuss the match of brane impurities, without odd moduli/flux, for \( E_8 \) [3] and for \( SU(n) \) [4]. Later in the paper we will go on and consider the match of brane impurities including the full twisting degrees of freedom for \( E_8 \). We will also give an interpretation (section 1.2) of all the bundle moduli \( H^1(Z,adV) \), even or odd under the involution, in terms of even respectively odd cohomology of the spectral surface, including an interpretation of the \( \mathbb{Z}_2 \) equivariant index of [4] as giving essentially the holomorphic Euler characteristic of the spectral surface.
In section 2 a dictionary is established between the geometry of the spectral surface and the full $F$-theory moduli, including the discrete data. This is again done in two steps: first we show how one can consider the device of spectral cover (considered in \[1\] for $G$ in the A-series) respectively the del Pezzo construction (considered in \[11\] for $G$ in the E-series) when suitably generalized as alternative and effectively related descriptions of one and the same thing for any bundle.

Our main results concerning the identification of the Prym and the Deligne cohomology are obtained in section 3. Our point of view is that while the bundle of del Pezzos exists only for some groups, and the spectral cover depends on some non-canonical choices, there exists in complete generality one naturally defined object which we use to relate the various strands: this is the cameral cover. The reason this is the right object is because the distinguished Prym, a certain extension of a discrete group by an abelian variety which is attached to the cameral cover, is isomorphic to the group of twisting data. Having recalled this isomorphism \[2\], we then go on and relate the distinguished Prym to the analogous groups attached to the spectral cover and the del Pezzo fibration, whose connected components are, respectively, the Prym-Tyurin variety and the relative intermediate Jacobian.

In section 4 we go on to the connection with the $F$-theory side \[3\]. We consider the stable degeneration $X^4 \to X^4_{\text{deg}} = W_1 \cup Z W_2$ where the $W_i$ are fibered by del Pezzo surfaces over $B$. The 8D picture involves a $K3$ degenerating into the union of two rational elliptic surfaces (here still called del Pezzo and denoted $dP_i$). The base of the fibration is the union of two projective lines intersecting in a point $Q$ over which a common elliptic curve $E$ is fibered; roughly speaking the two $E_8$ contributions in the $K3$ are separated; note that the transcendental lattice of the $K3$ is $E_8 \oplus E_8 \oplus H$, with $H$ the 2-dimensional hyperbolic plane piece, which leads to the 18-dimensional space $S := SO(2,18)/SO(2) \times SO(18)$ divided by the appropriate discrete group; one specializes then to two $E_8$ singularities at positions $z = 0, \infty$ in the $P^1$ base, which after the 'separation' in two surfaces are again resmoothed; imagine to take (for the $dP_9$’s to come alive) the two $f_4, g_6$ parts at $z = 0, \infty$ of the original Weierstrass data $f_8, g_12$ of the $K3$. This corresponds on the heterotic side to the large area degeneration of a $T^2$ of the same complex structure parameter as $E$ \[4\]. Intuitively speaking again, imagine that the $H$ and its counterpart in $S$ above correspond to the degrees of freedom represented by the complex structure modulus $\tau$ and the area ($+B$-field) modulus $\rho$ of $E$; then in the $\rho \to i\infty$ limit one finds in the corresponding boundary component of the quotient (discrete\$S$) the two spaces $(W \setminus (E_i \otimes \Lambda_c))$, 'glued' together by $\tau(E_1) = \tau(E_2)$, describing the moduli of the two $dP_9$’s ($\Lambda_c$ the coroot lattice of $E_8$, $W$ the Weyl group). The heterotic invariant $n_5 = c_2Z - c_2V_1 - c_2V_2$ is then mirrored on the F-theory side by $n_3 = -\frac{x(2)}{24} + \frac{x(W_1)}{24} + \frac{x(W_2)}{24}$. Note that, as necessary for the relation to the abelian variety $\text{Jac}(\mathcal{C})$, the relevant intermediate Jacobian of the $W_i$ in the stable degeneration is abelian (note that the intermediate Jacobian of the fourfold $X$ itself is also abelian, as the $h^{3,0}(X) = 0$). Let us remark further that we will consider large area for the base $B$ to remain within the realm of classical geometry. Furthermore we will assume that the unbroken gauge group is ADE.

We close by pointing out that not only the occurrence of the four-flux modifies the required number of three-branes but that also the three-branes refine in some sense the quantization condition for the four-flux, excluding for example the simplest choice of
fulfilling the integrality congruence \([12]\) by setting the flux equal to \(\frac{1}{2}c_2(X^4)\): just as this would violate supersymmetry \([13]\) in case of non-primitiveness of \(c_2\) it violates it also, one sees here, due to the negative number of three-branes which would be needed.

1 Bundle moduli and brane impurities

We review first some of the setup of \([1]\) concerning the spectral cover construction for \(V\) an \(SU(n)\) vector bundle. Then we show how the even respectively odd bundle moduli are reflected in the even respectively odd cohomology of the spectral surface. Finally we collect some results on the comparison of bundle moduli and F-theory moduli respectively of the brane impurities for \(G = SU(n), E_8\).

1.1 Spectral cover for \(G = SU(n)\)

Let \(V\) be an \(SU(n)\) vector bundle over the elliptically fibered heterotic Calabi-Yau threefold \(\pi : Z \to B\) with section \(\sigma\) of normal bundle \(L^{-1} = K_B\). We also think of \(\sigma\) as element of \(H^2(Z)\), and then \(\sigma|_\sigma = -c_1|_\sigma\) (unspecified Chern classes will always refer to \(B\)). Let \(\mathcal{M}\) be a line bundle over \(B\) of \(c_1(\mathcal{M}) = \eta\), \(C\) the locus \(s = 0\) for the section \(s = a_0z^n + a_2z^{n-2}x + a_3z^{n-3}y + \ldots + a_nx^n/2\) of \(\mathcal{O}(\sigma)^n \otimes \mathcal{M}\) and \(\mathcal{N}\) a (‘twisting’) line bundle over \(C\) so that

\[
V = \pi_{2*}(\mathcal{N} \otimes \mathcal{P}_B) \tag{1.1}
\]

(where the (relative) Poincare line bundle is restricted to \(C \times_B Z\), \(\mathcal{N}\) understood as being pulled back by \(\pi_1\));

\[
c_2(V) = \sigma \eta + \omega \tag{1.2}
\]

where \(\omega \in H^4(B)\), i.e. \(\omega = c_2(V|_B)\) and \(\eta = \pi_2 c_2 V\).

Because of the triviality of \(\mathcal{P}\) when restricted to \(\sigma\) one gets using Grothendieck-Riemann-Roch for \(\pi : C \times_B \sigma(B) = C \to B\) that

\[
\pi_*(e^{c_1(\mathcal{N})}Td(C)) = ch(V)Td(B) \tag{1.3}
\]

With the condition \(c_1(V) = 0\) one finds

\[
c_1(\mathcal{N}) = -\frac{1}{2}(c_1(C) - \pi^*c_1(B)) + \gamma \tag{1.4}
\]

where \(\gamma \in H^{1,1}(C, \mathbb{Z})\) with \(\pi_\gamma = 0 \in H^{1,1}(B, \mathbb{Z})\) (actually \(\gamma\) can be half-integral). One has then

\[
\omega = -\frac{n^3 - n c_1^2}{6} + \frac{n}{8}(\eta - nc_1) + \frac{1}{2}\pi_\gamma(\gamma^2) \tag{1.5}
\]

Actually the last two terms combine: the only general elements of \(H^{1,1}(C, \mathbb{Z})\) are \(\sigma|_C\)

\footnote{the last term is \(x^{(n-3)/2}y\) for \(n\) odd; actually \(n = 0(2)\) and \(\eta \equiv c_1(2)\) was assumed in \([1]\): \(a_r \in \Gamma(B, \mathcal{M} \otimes \mathcal{L}^{-r})\).} and \(\pi^*\beta\) (for \(\beta \in H^{1,1}(B, \mathbb{Z})\)), which have because of \(C = n\sigma + \pi^*\eta\) the relation \(\pi_\gamma(\sigma|_C) =...\)
\[ \pi_*\sigma(n\sigma + \pi^*\eta) = \pi_*\sigma(-nc_1 + \pi^*\eta) = \eta - nc_1; \] 
so \( \gamma = \lambda(n\sigma - \pi^*(\eta - nc_1)) \) (with \( \lambda \) half-integral if \( (K_C^{-1} \otimes K_B)^{1/2} \) does not exist as a line bundle) and \( \pi_*(\gamma^2) = -\lambda^2 n\eta(\eta - nc_1) \). 
So for the generator \( \gamma_0 \), say, corresponding to \( \lambda = 1/2 \), the term completely disappears leaving the \( \eta \)-independent piece 
\[ \omega(V_{\gamma_0}) = -\frac{n^3 - nc_1^2}{6} \frac{1}{4} \] 
(1.6)

1.2 Bundle moduli and F-theory moduli for \( G = SU(n), E_8 \)

**SU(n)**

In \([1]\) a natural map (for \( G \) general) was given between the (even) bundle parameters and the F-theory parameters (as far as complex structure is concerned). Their number on the heterotic side was also checked by an index computation. Of more immediate concern for our purposes is the expression of this number in terms of the spectral surface.

Let us first recall that the moduli space \( \mathcal{M} \) of the bundle is fibered \( \mathcal{M} \rightarrow \mathcal{Y} \) with fibre \( \text{Jac}(C) \). This picture emerges if you consider the Leray spectral sequence for \( \pi : Z \rightarrow B \), which gives for the first-order deformations \([1] H^1(Z, adV) \)
\[ 0 \rightarrow H^1(B, R^0\pi_*adV) \rightarrow H^1(Z, adV) \rightarrow H^0(B, R^1\pi_*adV) \rightarrow 0 \]
(1.7)
where the first term shows the tangent space to the fibre whereas the last term exhibits the space whose projectivization gives the global sections of the global moduli object \( \mathcal{W} \rightarrow B \) (cf. \([1]\) and section 2).

Now we will consider \( n_e - n_o \), the difference of bundle moduli even respectively odd under the involution \( \tau \). We will establish the relation 
\[ n_e - n_o = h^{2,0}(C) - h^{1,0}(C) \]
(1.8)

Before we come to it a number of remarks are in order. First, just as in 8D the spectral points in the elliptic curve represent the degrees of freedom of the bundle, one should expect by the principle of adiabatic extension that the deformations (whose number is \( h^{d,0}(C) \) for a \( d \) dimensional spectral object, as its normal bundle equals its canonical bundle) of the spectral object in its Calabi-Yau will represent the even deformations. So it came actually out in the 6D case where the quaternionic dimension of the vector bundle moduli space was identified \([1]\) with the genus of the spectral curve as in 6D the relation \( C = n\sigma + \pi^*\eta \) means \( C = n\sigma + kE \) with \([1]\) \( k = c_2(V) \in \mathbb{Z} \) and \( E \) the elliptic fibre of \( Z = K3 \); so \( K_C = C^2 = 2nc_2(V) - 2n^2 \) giving the result for \( \gamma_C \). The authors of \([1]\) went on to a 4D compactification. In a case by case analysis, for \( B \) a Hirzebruch surface, they matched the number of holomorphic two-forms on the spectral surface \( C \) with the corresponding relevant part of the complex deformations of the corresponding F-theory four-fold \( X^4 \). They did this by counting monomials preserving the type of singularity corresponding to the unbroken gauge group. (This, like the computations in \([1]\), concerns a case without

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3As our base \( B \) will always be rational and \( C \) will be generically smooth the first-order deformations will actually be unobstructed \([1]\).

4note that \( k - 2n = \sigma \cdot C = h^0(B, \pi_*V) = h^0(B, R^1\pi_*V) = h^1(K3, V) = -\chi(K3, V) = c_2(V) - 2n \)
odd moduli corresponding to $H^{1,0}(C)$ respectively $H^{2,1}(X^4)$. Now one computes with $C = n\sigma + \pi^*\eta, c_2(Z) = (c_2 + c_1^2) + 10c_1^2 + 12\sigma c_1, \sigma^2 = -\sigma c_1, \pi^*\pi^* = id, \pi^*\sigma = 1$ and Noethers theorem that

$$1 + h^{2,0}(C) - h^{1,0}(C) = \frac{c_2(C) + c_1^2(C)}{12}|_C = \frac{c_2(Z)C + 2C^3}{12}$$

$$= n + \frac{n^3 - n}{6}c_1^2 + \frac{n}{2}\eta(\eta - nc_1) + nc_1$$

(1.9)

Now as the ordinary index $\chi(Z, adV)$ vanishes by Serre duality one has to consider the $\tau$-equivariant index $\chi_\tau(Z, adV) = \sum_i (-1)^i Tr_{H^i(Z, adV)} \tau$. Using the projector $\frac{1 + \tau}{2}$ one computes for

$$I = -\frac{1}{2} \chi_\tau(Z, adV) = -\sum_i (-1)^i Tr_{H^i(Z, adV)} \frac{1 + \tau}{2} = n_e - n_o$$

(1.10)

(note that Serre duality interchanges now the even and odd subspaces of the respective cohomologies) that [7]

$$I = rk - \sum_j \int_{U_j} c_2(V) = n - 1 - 4\omega_{\gamma=0} + \eta c_1$$

(1.11)

(the $U_j$ denote the two fixed point sets).

$E_8$

One can match [9] the number of moduli with the $F$-theory side, including a number $n_o$ of odd moduli, provided we use an identification $n_o = h^{2,1}(X^4)$ for the odd moduli.

One has for the heterotic base deformations (assuming that $Z$ has a smooth Weierstrass model which is general [13], i.e has only one section, so that $h^{1,1}(Z) = h^{1,1}(B) + 1 = c_2 - 1 = 11 - c_1^2$)

$$h^{2,1}(Z) = h^{1,1}(Z) - \frac{\chi(Z)}{2} = 11 + 29c_1^2$$

(1.12)

as the smooth $Z$ has $\chi(Z) = -60c_1^2$ [16]. Furthermore one counts the moduli $h^1(Z, adV) = I + 2n_o$ of the bundle $V$ by applying [17] an index computation first used in this specific form in [4] with

$$I = 16 + 332c_1^2 + 120t^2$$

(1.13)

So that in total

$$1 + h^{2,1}(Z) + h^1(Z, adV) = 28 + 361c_1^2 + 120t^2 + 2n_o$$

(1.14)

For the $F$-theory four-fold $X^4$ one finds with $\frac{\chi}{6} - 8 = h^{1,1} - h^{2,1} + h^{3,1}$ [8] and $h^{3,1} = 2 + h^{1,1}(B) = 12 - c_1^2$ (reflecting that there is no unbroken gauge group) the final matching

$$h^{3,1} + h^{2,1} = \frac{\chi}{6} - 20 + 2c_1^2 + 2h^{2,1} = 28 + 361c_1^2 + 120t^2 + 2h^{2,1}$$

(1.15)

using $c_1^3(B^3) = 6c_1^2 + 2t^2$ [8] and $\frac{\chi}{24} = 12 + 15c_1^3(B^3)$ [8].
1.3 Brane impurities

For \( G = E_8 \) the number \( n_5 \) of non-perturbative heterotic fivebranes occurring because of anomaly cancellation is

\[
n_5 = c_2(Z) - (c_2(V_1) + c_2(V_2)) = 12 + 10c_1^2 - (\omega_1 + \omega_2)
\]

(1.16)

using \( c_2(Z) = \eta_Z \sigma + \omega_Z \) with \( \omega_Z = 12 + 10c_1^2 \) and \( \eta_Z = 12c_1 = \eta_1 + \eta_2 \) in analogy to the decomposition of \( c_2(V) \) (remember \( \eta_1/2 = 6c_1 \pm t \)). Note that the \( \sigma \)-carrying parts like \( \sigma \eta \) are adjusted to cancel, as in the 6D story, so the remaining pure \( H^4(B) \) parts carry the relevant information. Now \( n_5 \) was computed \([1]\) using the relation

\[
\omega_{1/2} = -40c_1^2 - 15t^2 \mp 45tc_1
\]

(1.17)

to be

\[
n_5 = 12 + 90c_1^2 + 30t^2
\]

(1.18)

This was matched, in case \( \gamma = 0 \) and no moduli are odd under the fibre involution, with the number of space-time filling three-branes \( n_3 = \chi(X_4^{\text{24f}}) \) of a dual F-theory compactification on the elliptically fibered four-fold \( X^4 \rightarrow B^3 \).

\( SU(n) \)

The foregoing has the following generalization to \( G = SU(n) \) (cf. \([10]\)). Now one is working for the sake of computation of cohomological data (not for doing \( F \)-theory on it) on the fiberwise resolved four-fold.

Using this device one computes now \( h^{1,1} = 2 + h^{1,1}(B) + 16 - rk \) which leads with the index formula \([1]\)

\[
I - rk = -4(c_2V - \eta \sigma) + \eta c_1 = -4\omega_Z + 4n_5 + 12c_1^2
\]

(1.19)

and \( h^{2,1} = n_o \) and

\[
h^{3,1} = h^{2,1}(Z) + I + n_o + 1 = 12 + 29c_1^2 + I + h^{2,1}
\]

(1.20)

to (using \( \frac{X}{24} = 2 + \frac{1}{4}(40 + 28c_1^2 + I - rk) = 12 + 10c_1^2 - \omega_Z + n_5 = n_5 \))

\[
\frac{X}{24} = 2 + \frac{1}{4}(40 + 28c_1^2 + I - rk) = 12 + 10c_1^2 - \omega_Z + n_5 = n_5
\]

(1.21)

2 Heuristic considerations: continuous and discrete data

A convenient point to start with is a preliminary comparison of the discrete data. The most notable features are:

i) shifted integrality (to half-integrality),
ii) restriction to the subspace $\text{ker} \, \pi$ respectively primitiveness and
iii) correction contribution in $n_5$ respectively $n_3$.

ad i) let $G_i$, $i = 1, 2$, be the projections associated with the stable degeneration $X^4 \to X^4_{\text{deg}} = W_1 \cup_Z W_2$. The analogy in the data concerned with the discrete part of the twisting degrees of freedom (cf. below) is represented in the following juxtaposition: on the heterotic side one has

$$
\gamma = \frac{c_1(C) - \pi^*c_1(B)}{2} + c_1(N), \quad (2.1)
$$

where the last term is an element of integral cohomology whereas the square root $(K_C^{-1} \otimes K_B)^{1/2}$ does not necessarily exist as a line bundle. Similarly one has on the F-theory side

$$
G = \frac{c_2}{2} + \alpha \quad (2.2)
$$

where $\alpha \in H^4(X, \mathbb{Z})$, but $c_2$ is not necessarily even. Strictly speaking one should consider here the projected $G_i$ ($i = 1, 2$).

ad ii) the $G$ admissible in an $N = 1$ supersymmetric compactification are in $\text{ker}(J \wedge \cdot)$ [13]. The last condition comes down for the relevant projected classes in $H^{2,2}(W)$ to the following: on the heterotic side the actual spectral cover construction will in the $E_8$ case involve the corresponding fibration of $dP_8$ surfaces over $B$ (the section of $dP_9$ blown down); now, the embeddings of the 8D heterotic elliptic curves in the 8D del Pezzos patch together to an embedding of $Z$ in the $W_i$. This gives rise to maps $H^*(W) \to H^*(Z) \to H^{*-2}(B)$ by restriction respectively integration over the fibre; but for the $dP_8$ the anticanonical class given by the elliptic curve $E$ is ample, so actually the $\text{ker}(J \wedge \cdot)$ condition is reflected in a $\text{ker}(H^{2,2}(W) \to H^{2,2}(Z))$ condition, respectively, if one combines with the integration over the fibre, in a $\text{ker}(H^*(W) \to H^*(Z) \to H^{*-2}(B))$ condition; one has then to divide out the class dual to $S_b$, the del Pezzo fibre of $pr : W \to B$, corresponding to a differential form supported on the base, which is mapped to zero in the integration over the $\pi : Z \to B$ fibre. So finally the space we are concerned with is the $(\text{ker} : W \to B)/S_b\mathbb{Z}$ part in $H^{2,2}$ (cf. the second diagram below). So the primitiveness condition is the analogue of the condition $\text{ker}(\pi : H^{1,1}(C, \mathbb{Z}) \to H^{1,1}(B, \mathbb{Z})$ on $\gamma$.

ad iii) note that a typical value for $G$ such as $\frac{1}{2}c_2(X^4)$ gives non negative $G^2$ whereas $\gamma^2$ is negative by the Hodge index theorem. So, comparing the heterotic contribution of $\gamma^2$ in eq. (1.5)

$$
n_5(\gamma) = n_5(\gamma = 0) + \frac{1}{2}\pi_*(\gamma^2) \quad (2.3)
$$

with the formula [13]

$$
n_3 = \frac{\chi(X^4)}{24} - \frac{1}{2}G^2, \quad (2.4)
$$
we are led to expect an association letting \( \gamma_i \) correspond with \( G_i \) giving

\[
\pi^i_*(\gamma_i^2) = -G_i^2.
\]

This would fit in and actually complete the general scheme of a duality dictionary beyond the previously considered cases of relating \( h^{2,0}(C) \) and \( h^{3,1}(X^4) \) respectively elements of \( H^{1,0}(C) \) and \( H^{2,1}(X^4) \) in a satisfying way (cf. [4],[1], section 1 and the introduction).

Together with the proposed identification of the discrete moduli one gets a dictionary of elements related by a \((1,1)\) Hodge shift

| \( C \) | \( X^4 \) |
|------|------|
| \( H^{2,0} \) | \( H^{3,1} \) |
| \( H^{1,0} \) | \( H^{2,1} \) |
| \( H^{1,1} \) | \( H^{2,2} \) |

where in the first line the deformations of \( X^4 \) preserving the given type of singularity (corresponding with the unbroken gauge group; actually we will consider the parts in the \( W_i \)) are understood, in the second line a part of the relative jacobian (see below) is understood, and in the last line the subspaces \( \ker \pi_* \) respectively \( \ker (J \wedge \cdot) \).

A naive way to obtain the association of \( \gamma_i \) with \( G_i \) is via the cylinder map [31], which we describe below (3.16). This replaces each point in \( C \) by a (complex projective) line \( L \) lying above it in the del Pezzo. Indeed, \( L^2 = -1 \), suggesting the desired relation \((2.5)\). Unfortunately, this fails, as the right hand side of \((2.3)\) gets contributions also from distinct lines which intersect in the del Pezzo. The full story is a bit more subtle. We will see that \( H^1(C) \) breaks into several isotypic pieces (five of them, for \( E_8 \)). The values of \( \gamma \) coming from bundles all live in one of these isotypic pieces, the distinguished piece. On this piece the cylinder map changes the intersection numbers not by \(-1\) but by a factor of \(-60\) (for \( E_8 \)). Furthermore, we will see (in section (3.2)) that the cylinder map itself is divisible (by 60) on this distinguished piece, so the correct association sends \( \gamma \) to \( \frac{1}{60} \) times its cylinder.

We will actually have to insert an intermediate step in relating the data from the spectral surface \( C_i \) \((i = 1,2)\), corresponding to the bundle \( V_i \) to \( W_i \). For this note first that the (even) deformations of \( V_i \) correspond to those deformations of \( W_i \) which preserve fiberwise the elliptic curve \( E \) common with the heterotic side, so preserving in total the Calabi-Yau \( Z \) common to the \( W_i \): their number is given by the dimension of \( H^1(W_i, T_{W_i} \otimes O(-Z)) \cong H^{3,1}(W_i) \). These are the deformations in \( H^{3,1}(X^4) \) which are relevant to the respective bundle. Second, under the stable degeneration \( J^3(X) \) splits off the abelian varieties \( J^3(W_i) \), which contain the pieces relevant for the comparison. Third this construction interprets those elements of \( H^{2,2}_{prim}(X^4, Z) \) that are captured by the corresponding parts in the \( W_i \) cohomology (for the relation of the primitiveness condition to the \( W, Z \) geometry cf. above).

To include also the continuous part in the picture we have to discuss more fully the issue of the twisting data which can occur in piecing together bundles over \( Z \) if we have a description for bundles over an elliptic curve \( E \). In this paragraph we will review the description of the twisting data based on spectral covers, as given in [3]. In the remainder of the paper we will work instead with the description given in [2], based on the cameral
cover. So let $\mathcal{M}_E$ be the moduli space of semistable $G$-bundles over $E$, respectively $\mathcal{M}_{Z|B}$ the relative object (for $G = E_8$ we will not allow cuspidal fibers) and finally $\Xi$ the (locally existing) universal bundle over $E \times \mathcal{M}_E^0$ respectively over $Z \times B \mathcal{M}_{Z|B}^0$ (the superscript 0 denoting the smooth locus of the moduli spaces, where then such a universal object exists locally. Now a $G$-bundle over $Z$ which is fiberwise (over the open subset of $B$ over which lie smooth fibers) semistable gives a section (over that subset) of $\mathcal{M}_{Z|B}$. Our aim is to describe conversely the (possibly obstructed) existence and (non)-uniqueness of a bundle given such a section. The idea is of course, as far as possible, to pull-back a universal bundle. Now consider first the situation in a fibre: to $\Xi$ the is an associated abelian group scheme of automorphism groups $\text{Aut}(\Xi)$ over $\mathcal{M}_E^0$ (of associated sheaf of sections, say, $A$). The set of possible universal bundles over $E \times \mathcal{M}_E^0$ (if the obstruction in $H^2(\mathcal{M}_E^0, A)$ vanishes) is then rotated through under the elements of $H^1(\mathcal{M}_E^0, A)$. In the relative version where a section $s$ of $\mathcal{M}_{Z|B}^0 \to B$ is given the relevant space is correspondingly $H^1(B, A_B(s))$.

To make this explicit, let us start with $G = SU(n)$ and display the decomposition of the twisting data $H^1(A_B)$ into the continuous part (the relative jacobian $J(C/B)$) and the discrete part (the multiples of the $\gamma$ class). Both groups are taken up to possible finite group discrepancies. Additionally, in the last line we must allow half-integral cohomology.

\[
\begin{array}{ccccccc}
0 & \downarrow & 0 & \to & J(C/B) & \to & Pic_0 C & \to & Pic_0 B & \to & 0 \\
0 & \downarrow & 0 & \to & H^1(A_B) & \to & Pic C & \to & Pic B & \to & 0 \\
0 & \downarrow & 0 & \to & \gamma \mathbb{Z} & \to & H^{1,1}(C, \mathbb{Z}) & \to & H^{1,1}(B, \mathbb{Z}) & \to & 0 \\
0 & \downarrow & 0
\end{array}
\]

Now, when one tries to transfer these results to the (D- and especially to the) E-series, one faces at first the following problem. For the E-series one does not describe the bundles via the spectral cover construction but instead via the associated del Pezzo fibration, giving not a covering of $B$ but a fibration over it by surfaces. This is related to the following crucial fact (cf. [19]): consider the type IIA string on an elliptic K3 with ADE singularity times $\mathbb{T}^2$; the $N = 1$ content of this 4D $N = 4$ theory includes three adjoint chiral fields $X, Y, Z$ (with superpotential $\text{Tr}[[X,Y],Z]$), whose Cartan vevs (Higgs branch) correspond to blowing up respectively deforming the singularity respectively giving Wilson lines to the ADE gauge group on $T^2$; the R-symmetry induces an equivalence of the corresponding moduli spaces. This gives the main theorem on the structure of the moduli space $\mathcal{M}_G$ of flat $G$-bundles on an elliptic curve (cf. [1], and also [2] for partial identifications of the relevant mirror map in this connection).

Concretely let us take as elliptic curve $E = P_{1,2,3}[6]$ of equation $e := z^6 + x^3 + y^2 + \mu xyz = 0$ leading (with $s$ of sect. 1.1) to the deformation $e + vs$ of the $SU(n)$ singularity showing at the same time Looijenga’s moduli space $\{a_0, a_2, a_3, \ldots, a_n\} \in \mathbb{P}^{n-1}$ of flat $SU(n)$ bundles over $E$ as well as the 0D spectral geometry consisting of $n$ points $(e = 0) \cap (s = 0)$ on $E$. Note that in this case of the $A_n$ group it is possible to effectively replace a 2D geometry of
P$^1$'s by the zero dimensional representatives as $v$ occurs only linear and so in the process of period integral evaluation to describe the variation of Hodge structure relevant here can be integrated out. For the general phenomenon relating even (0D to 2D, of symmetric intersection form) or odd (1D to 3D, of antisymmetric intersection form) cohomology cf. [23]; the same relation underlies the extraction [24] of the 1D Seiberg-Witten curve from the 3D periods of a Calabi-Yau and the relation between $K3$ singularities and gauge groups $ADE$.

By contrast the same decoupling phenomenon does not take place for the the $E_k$ case: there one finds instead for the deformation $e + \sum_{i=1}^{6} a_i v_i z^{6-i} + b_2 v^2 x^2 + b_3 v^3 y + b_4 v^4 x$ of zero locus $dP_8 = P_{1,1,2,3}[6]$ showing the 2D spectral geometry of the del Pezzo surface with $H^{1,1}(dP, Z)^{1_E} = E_8$ and moduli space $P_{1,2,3,4,5,6,2,3,4}$.(Correspondingly there occurs a situation involving the $E$ groups, where the Coulomb branch of an $N = 2$ system does not reduce to a Riemann surface but a description in terms of 3-form periods has to be given [19].)

It is interesting to note that this phenomenon admits also a representation-theoretic explanation. The Weyl group of $A_n$ admits a small permutation representation, the $(n+1)$-dimensional one, which decomposes into the sum of only two irreducible representations: the trivial one and the weights, $Z[W/W_0] \cong 1 \oplus \Lambda$. By contrast every permutation representation of $W_{E_n}$ contains at least three irreducible constituents. This means that the cohomology of every associated spectral cover will contain some additional pieces. To get an object with the right cohomology, i.e. the distinguished isotypic piece mentioned above, one must either go up in dimension, or restrict attention to classes which transform correctly under some correspondences.

Note that the the effective replacing of the $P^1$ classes by points accounts for the missing dimensions causing the mentioned $(1,1)$ shift in cohomology when comparing the dual results. Namely the description in the $E_k$ case is already well adapted to the F-theory picture of having a fibration $W \to B$ (for each bundle) of del Pezzo surfaces over $B$. As mentioned the 8D heterotic elliptic curve is contained in the both 8D del Pezzo, so is the $Z$ in the $W$, giving rise to maps $H^s(W) \to H^s(Z) \to H^{s-2}(B)$ and to the diagram involving now the intermediate jacobian $J^3$ (cf [3] for the analogous situation in 6D) and interpreting $H^1(A_B)$ as (relative) Deligne cohomology$^5$ (which relates to $J^3$ as Pic to Pic$^0$)

\[
\begin{array}{cccccc}
0 & \rightarrow & J^3(W/B) & \rightarrow & J^3(W) & \rightarrow & J(B) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & H^1(A_B) & \rightarrow & H^2_0(W, Z) & \rightarrow & H^2(W, Z)/SZ & \rightarrow & H^{1,1}(B, Z) & \rightarrow & 0 \\
\end{array}
\]

\[
H^3(\cdot, Z) \rightarrow H^3(\cdot, C)/F^2 H^3(\cdot, C) \rightarrow H^4_0(\cdot) \rightarrow H^4(\cdot, Z) \rightarrow H^4(\cdot, C)/F^2 H^4(\cdot, C)
\] (2.6)

\[\text{compare the long exact sequence in Deligne cohomology}\]
To actually establish this picture we will proceed in three steps:

I) first we describe a generalized spectral cover construction for arbitrary group $G$ (or in particular, an ADE group.) The main point one has to take into account is, that for $G \neq SU(n)$ the fibre of the spectral covering is of a more ’entangled’ nature as the covering group is no longer the symmetric group.

II) Then we make use of the connection between ADE root systems and del Pezzo surfaces (cf. for example [25], [26], [27], [28]), the root system describing a certain part of the $H^{1,1}(dP, \mathbb{Z})$ (denoting the obvious intersection), so that the variation of the fibre of the spectral cover over $B$ describes the variation of certain (-1) curves $l$ in their variation in a family of surfaces over $B$ (expressing the effective replacement of these lines by points, causing the (1,1)-shift). This leads also to the necessary relation between $G_i^2$ and $l^2 \pi^* \gamma^2 = -\pi^* \gamma^2$.

III) Thirdly we make the transition to the F-theory side, which for example in the case $G = E_8$ consists just in relating the $dP_8$ fibre from the heterotic side to the $dP_9$ fibre from the F-theory side (blowing down a section brings one back to $dP_8$). In general this comes down [1] to see that the occurrence of a section $\theta : B \to X$ of $H$-singularities ($H$ the commutant of $G$ in $E_8$) causes the split off from $J^3(X)$ of a certain factor involving cycles of special behavior along $\theta$. Remember that under the stable degeneration $J^3(X)$ splits off the abelian varieties $J^3(W_i)$, which contain the pieces relevant for the comparison.

Before entering the general description let us give a short account of step I) in the framework described already above. Actually we will see the spectral cover as parametrization of exceptional lines in a surface fibration over $B$. This occurs by taking into account the description of the ’enlarged’ root system in surface cohomology (cf. the appendix). Note that as the same moduli space $W_G$ parametrizes $G$ bundles over an elliptic curve $E$ and del Pezzo surfaces $dP_G$ (with $E = -K$ fixed) one gets by adiabatic extension over the base $B$ that to the bundle $V$ over $Z$ corresponds a fibration $W_G^{het} \to B$ of $dP_G$ surfaces via pulling back the universal object (now the universal surface not the universal bundle) along the section $s : B \to W_G = \mathcal{M}_{Z/B}$.

Let us conclude this section with a description of the spectral cover construction for $G$ an ADE group. The covering $C \to B$ will be (cf. [29], [3]) locally modelled on (pulled back from) the covering of degree $d = \frac{|W|}{|W_0|}$

$$
\begin{align*}
\mathcal{T} &= (E \otimes \Lambda_c)/W_0 \\
\downarrow \\
\mathcal{M} &= (E \otimes \Lambda_c)/W
\end{align*}
$$

(2.7)

where $W_0 \subset W$ is the stabilizer of an element $\lambda \in \mathfrak{h}$ whose Weyl orbit spans $\mathfrak{h}$ over $\mathbb{C}$; for example $\lambda$ could be (the dual of) the maximal root $\tilde{\alpha}$ with $W\lambda = R$.

---

6 $\mathfrak{h}$ the Cartan Lie algebra, $R$ the set of roots, $d = |R|$, $Q(R)$ the lattice generated by them, $\Lambda_c$ the coroot lattice in the Cartan (denoted $\Lambda$ in [3]; $\tilde{\alpha}$ the quasi-miniscule, non-miniscule weight
The evaluation on \( \lambda \) gives the epimorphisms
\[
(C[W/W_0] \otimes \mathbf{C})^* \to (\mathcal{O}_{E \otimes \Lambda_c} \otimes \mathbf{C} h)^W
\]
where the sheaf of \( W \)-invariant sections of the trivial vector bundle \( \mathcal{O}_{E \otimes \Lambda_c} \otimes \mathbf{C} h \) is coherent on \( M = (\mathbf{E} \otimes \Lambda)/W \) and locally free over \( M^0 \), being equal there to the cotangent bundle \( \Omega_{M^0} = \text{Lie} \mathbf{A} \). The ensuing map \( \text{Lie} J(C) = H^1(C, \mathcal{O}_T) \to H^1(\text{Lie} \mathbf{A}) = \text{Lie} H^1(\mathbf{A}) \) presents the continuous part of \( H^1(\mathbf{A}) \) as quotient of the Jacobian \( J(C) \), the Prym-Tyurin variety of the spectral cover. For \( G = SU(n) \) respectively \( SO(2n) \) it reduces to the ordinary Jacobian respectively Prym variety (for the general construction cf. Appendix and [30]).

3 Bundle moduli via cover constructions

There are three geometric objects, each of which can be used to encode the even moduli of the heterotic theory. These are the spectral cover, the cameral cover, and the del Pezzo fibration. We will first recall the description of these objects and show that they represent equivalent data. We will then proceed to the identification of the relevant parts of their cohomologies. We fix throughout the heterotic space \( Z \), which is fibered over a base manifold \( B \), with elliptic fibers which we assume are in Weierstrass form. Although the bases \( B \) which arise in actual heterotic compactifications are severely restricted, we will not need to make any such assumptions about \( B \).

3.1 The three “covers”

Spectral covers

The first and most familiar object is the spectral cover \( \overline{\mathcal{C}} : \overline{B} \to B \) which was already mentioned in the previous section. For any subgroup \( W_0 \subset W \), we can consider a spectral cover which is locally modelled on the covering \( (\text{Hom}(\Lambda, E))/W_0 \to (\text{Hom}(\Lambda, E))/W \), of degree \( d = |W|/|W_0| \). Here \( \Lambda \) is the character lattice of \( G \) (dual to the \( \Lambda_c \) used in the previous section) and \( W_0 \subset W \) is an arbitrary subgroup, for example \( W_0 \) could be the stabilizer of an element \( \lambda \in \mathbf{h} \). One important case is the cover of smallest degree, corresponding to a minuscule weight \( \lambda \). For groups \( G = A_n, D_n, E_6, E_7, E_8 \), this is a branched cover of degree \( n, 2n, 27, 56, 240 \) respectively. The opposite extreme is when \( W = 0 \). There are also intermediate possibilities, such as the one corresponding to the maximal root \( \lambda = \tilde{\alpha} \) with \( W \lambda = \tilde{R} \). In the context of \( G = E_8 \) we will refer to the smallest cover simply as “the” spectral cover, although other covers also occur naturally (cf. equation (3.9).)

7 occurring in the exponential sequence \( 0 \to \Lambda \to \text{Lie} \mathbf{A} \to \mathcal{A} \to 0 \), where \( \mathcal{A} = \mathcal{A}^0 \) for example on the (Zariski) open subset of split bundles, where \( \mathcal{A} = \mathbf{C}^{*r} \); also \( \mathcal{A}_B(s) = \mathcal{A}_B^0(s) \) for a generic section and \( G \neq A_1, A_2 \) (and no cuspidal fibres for \( G = E_8 \)

8 as the identification \( W_{A_n} \cong S_{r+1} \) shows a certain ‘disentangledness’ of the fibre structure in that case, in contrast to the cases where \( W \) is only a subgroup of a symmetric group, showing that the fibre elements are not on equal footing

9 As we will from now on have to distinguish two 'covers' associated to \( B \) we switch to the notation \( \overline{B} \) for the spectral cover (previously denoted \( C \)), and \( B \) for the cameral cover.
Our third geometric object is not a cover, but a fibration $\tilde{B} \to B$. It is modelled on a $W$-cover $h : h \to h/W$. Equivalently, over the open set of nice points of $B$, it is modelled on $G/T \to G/N$. Here $T$ is a maximal torus and $N$ is the normalizer in $G$ of $T$, so points of $G/N$ parametrize the possible maximal tori, while a point of $G/T$ includes the additional choice of a Borel subgroup containing $T$, which amounts to the same as the choice of a chamber in the corresponding Cartan. The extension to all of $B$ is modelled on an appropriate partial compactification $\overline{G/T} \to \overline{G/N}$ constructed in \cite{[2]}. We have seen that the even part of the heterotic data amounts to the data of a spectral cover; this is the same as specifying the cameral cover $\tilde{B} \to B$ together with a $W$-equivariant map $v : \Lambda \times \tilde{B} \to \tilde{Z}$ (or equivalently, a family of maps $v_\lambda : \tilde{B} \to \tilde{Z}$ depending linearly on $\lambda \in \Lambda$) commuting with the projection to $B$. Each such $v_\lambda$ induces in particular a map $\pi_\lambda : B \to \tilde{B}$ where $\tilde{B}$ is the spectral cover corresponding to the subgroup $W_0 \subset W$ which fixes $\lambda$. For $\lambda$ in the interior of a chamber, $\pi_\lambda$ can be expected to be a birational isomorphism between $B$ and the largest spectral cover, given by $W_0 = 0$. The main difference between the $W_0 = 0$ spectral cover and $\tilde{B}$ is that the former sits, fiber by fiber, inside $\text{Hom}(\Lambda, E)$, while the latter sees only information along the base $B$ and is correspondingly an abstract (unembedded) cover.

**Del Pezzo fibrations** Our third geometric object is not a cover, but a fibration $\pi : U \to B$ whose fibers are complex surfaces. Let us describe how this is constructed for $G = E_8$, where the fibers are $E_8$ del Pezzo surfaces, obtained from $\mathbb{P}^2$ by blowing up successively 8 (distinct or possibly infinitesimally near) points. The Picard group of such a surface, or its second cohomology $H^2(dP_8, \mathbb{Z})$, is a rank-9 lattice, generated by the class $L$ (pullback of a line in $\mathbb{P}^2$) and the 8 exceptional curves $E_i$. It contains the anticanonical class $F := 3L - \sum_i E_i$, with $F^2 = 1$. Correspondingly, sections of the anticanonical system form a pencil of elliptic curves passing through a base point $p$. We will normalize the embedding of an elliptic curve into its del Pezzo by requiring that the zero point $\sigma$ of the elliptic curve map to the base point $p$ of the del Pezzo. The primitive cohomology, or the orthogonal complement $H^2_0(dP_8, \mathbb{Z})$ of $F$, is isomorphic to the $E_8$ weight lattice $\Lambda$, generated by the classes $\lambda_i := E_i - E_{i+1}$, $i = 1, \ldots, 7$ and $\lambda_8 := L - E_1 - E_2 - E_3$. In fact the Picard group is the direct sum of $\Lambda$ and $\mathbb{Z}F$. In particular, $L$ can be expressed as a linear combination of $F$ and the $\lambda_i$. Explicitly, we find:

$$L - 3F = -(5\lambda_1 + 10\lambda_2 + 15\lambda_3 + 12\lambda_4 + 9\lambda_5 + 6\lambda_6 + 3\lambda_7 + 8\lambda_8)$$

(3.9)

Now given the Weierstrass elliptic fibration $\pi_Z : Z \to B$ with section $\sigma : B \to Z$ and an $E_8$ bundle on $Z$ whose restriction to each fiber is semistable and regularizable (in the sense of \cite{[2]}), we get a cameral cover $\tilde{B} \to B$, hence a pullback family $\pi_{\tilde{Z}} : \tilde{Z} := Z \times_B \tilde{B} \to \tilde{B}$ together with a map $v : \Lambda \times \tilde{B} \to \tilde{Z}$. We assume also that the image of this map does not contain any singular points of singular elliptic fibers. Starting with this data, we will construct a del Pezzo bundle $U \to B$ as well as an embedding of $Z$ into $U$.

---

10points of $G/N$ parametrize regular centralizers in $G$, i.e. abelian subgroups whose dimension equals the rank of $G$ and which are the commutant of some element of $G$. The maximal tori are regular centralizers. An example of a regular centralizer which is not a maximal torus is the commutant in $G = SL(n)$ of a nilpotent element made up of a single Jordan block.
We start by constructing a $\mathbb{P}^2$-bundle $\tilde{U}_0$ over $\tilde{B}$ containing $\tilde{Z}$ as a family of Weierstrass cubics. This is obtained as projectivization of the rank-3 vector bundle $(\pi_\tilde{Z})_*\mathcal{L}$ for some line bundle $\mathcal{L}$ on $\tilde{Z}$, of degree 3 on each elliptic fiber. The simplest choice, $\mathcal{L}_0 := \tilde{\pi}^* (O_Z(3\sigma))$, does not work. Instead, we need to take
\[
\mathcal{L} := \tilde{\pi}^* (O_Z(2\sigma)) \otimes O_Z(v((L-3F) \times \tilde{B})), \tag{3.10}
\]
where $L-3F$ is given by (3.9). Now over $\tilde{B}$ we have 8 sections $\nu_{\lambda_i}$ of $\tilde{Z}$, and hence of $\tilde{U}_0$, labelled by the basis $\lambda_i \in \Lambda$. We will blow them up sequentially in $\tilde{U}_0$. The fibers of the resulting family $\tilde{U}' \to \tilde{B}$ are almost del Pezzo surfaces: they become del Pezzos when line configurations of type ADE are blown down to produce ADE singularities. This can be done simultaneously for the whole family: replace $\tilde{\pi}$ when line configurations of type ADE are blown down to produce ADE singularities.

We have just seen that the even heterotic data, consisting of an $E_8$ spectral cover (or equivalently of a cameral $\tilde{B}$ plus maps $\nu : \Lambda \times \tilde{B} \to \tilde{Z}$), determine a del Pezzo fibration $\pi : U \to B$. Conversely, given $Z$ and $U$ we recover the spectral cover $\bar{B} \to B$: it parametrizes the lines in the moving del pezzo fibers of $U$. Similarly, a point of the cameral $\tilde{B}$ corresponds to an 8-tuple of disjoint lines $E_i$ in the del Pezzo, or equivalently to an isomorphism (preserving the intersection forms) from $\Lambda$ to $H^2_0(dP_8, \mathbb{Z})$.

The upshot then is that our three types of data: a spectral cover, a del Pezzo fibration, and a cameral cover with map $\nu$, are equivalent to each other when $G = E_8$. The obvious analogue works for type $E_n$: a del Pezzo of type $E_n$ is the successive blowup of $\mathbb{P}^2$ at $n \leq 8$ (distinct or possibly infinitesimally close) points. The character lattice $\Lambda$ of $E_n$ is still isomorphic, as an abelian group with intersection form, to the primitive cohomology group $H^2_0(dP_n, \mathbb{Z})$. For type $A_n$ or $D_n$ we use the fact that the corresponding character lattices can be embedded into the $E_n$ lattice as the orthogonal complement of an appropriate fundamental weight (corresponding to one of the ends of the Dynkin diagram). We will therefore define a del Pezzo fibration of type $A_n$ or $D_n$ to be a del Pezzo fibration $\pi : U \to B$ of type $E_n$ together with a section of the family of $E_n$ lattices $R^2_0, \mathbb{Z}$ which, in each fiber, is in the $W$ orbit of that fundamental weight. (For $A_n$, for example, this additional data consists, in each fiber, of specifying the pullback of a line of the original $P^2$.) This extends the correspondence between cameral covers (plus $\nu$), spectral covers, and del Pezzo fibrations from the $E_n$ case to the ADE case. We can therefore refer to either of these three types of data as spectral data. In the sequel we will, however, concentrate on the case $G = E_8$.

### 3.2 The matching of cohomologies
Since the heterotic moduli are mapped to the family of spectral data, it is crucial to understand the fiber of this map, or the space of twisting data for a given cover. Two descriptions of this fiber are in the literature. In [2] it was seen that this fiber can be identified with the space of principal $G$-Higgs bundles $(P_B, C)$ on $B$. The space of principal $G$-Higgs bundles has nothing to do with our elliptic fibration $Z$: the definition involves only the base $B$. So the space of principal $G$-Higgs bundles with given cameral cover $\tilde{B}$ should be describable in terms of $B, \tilde{B}$ only. Such a description was indeed found in [3], Theorem 12: if this space is non-empty, it is a principal homogenous space over a certain subgroup of $\text{Hom}_W(\Lambda, \text{Pic}(\tilde{B}))$. More precisely, for compact, connected $B$, this subgroup is the distinguished Prym of the cameral cover, or the kernel, $\text{Prym}_A(\tilde{B}) := \ker(cl)$, of the natural homomorphism

$$cl : \text{Hom}_W(\Lambda, \text{Pic}(\tilde{B})) \to H^2(W, T).$$

Here as before, $\Lambda$ is the character lattice of $G$, and $\text{Hom}_W$ is the group of $W$-equivariant homomorphisms. This Prym has both a continuous and a discrete part. (The name "distinguished Prym" was applied in [3] and [2], somewhat ambiguously, both to this Prym and to its connected component.) A similar description of the fiber (modulo some equivalences for irregular bundles) is given in [5] in terms of the object $H^1(A_B)$ described in the previous subsection. For regular bundles, the sheaf $A_B$ of automorphisms along the fiber, described in [5], is the same as the sheaf $C$ of regular centralizers of [3], [2].

Two further versions of the Prym can be constructed from our other geometric objects: the Prym-Tyurin variety $\text{Prym}(\overline{B}/B)$, an abelian subvariety of $\text{Pic}^0(\overline{B})$ whose construction we recall below, and the relative intermediate Jacobian $J^3(U/B)$. Our purpose here is to compare these three. We will find that $\text{Prym}(\overline{B}/B)$ and $J^3(U/B)$ can be identified (up to a finite group) with the continuous part of $\text{Prym}_A(\tilde{B})$, and that the discrete part too (again, up to a possible discrepancy of a finite group) can be identified in terms of cohomology groups of $\overline{B}$ and $U$. (We remind the reader that our del Pezzo fibration $U$, and hence also its intermediate Jacobian $J^3(U/B)$, live in the heterotic theory. It is therefore possible to obtain their exact relationship to the cameral or spectral covers by purely geometric means. It is the intermediate Jacobian $J^3(X)$, which arises in the F-theory, which is related to all three of our objects only in the stable degeneration limit.)

We are going to compare the relevant parts of the cohomologies of our three geometric objects. It is convenient to discuss the rational cohomology first, then to pass to integer coefficients, and finally to consider the cohomology with $\mathbb{R}/\mathbb{Z}$ coefficients, which gives the abelian varieties.

**Rational cohomology of the cameral cover**

Let us start with the rational cohomology $H^i(\tilde{B}, Q)$ of the cameral cover, which is a representation of $W$. As such, it decomposes into *isotypic pieces*:

$$H^i(\tilde{B}, Q) = \bigoplus_{\rho} M^\rho \otimes H^i_{\rho},$$

---

$^{11}$A principal $G$-Higgs bundle on $B$ is a principal $G$-bundle $P_B$ together with a family $C \subset \text{Ad}P_B$ of regular centralizers in the adjoint bundle $\text{Ad}P_B$. Regular centralizers were defined in the previous footnote. Actually, what is identified with the space of principal $G$-Higgs bundles is the space of *regularized* principal bundles with given spectral data. In well-behaved situations, e.g. when the restriction of our bundle to each elliptic fiber is already regular, the regularization is unique and so can be omitted.
where $\rho$ runs through the irreducible representations of $W$, the multiplicity space $M^\rho$ is just the $Q$ vector space in which $\rho$ acts, and

$$H^i_\rho := \text{Hom}_W(\rho, H^i(\tilde{B}, Q)).$$

We will be concerned mostly with two of these pieces, where $\rho$ is either the representation $\Lambda$ of $W$ on the weights of $G$, or the trivial representation $1$.

**Rational cohomology of the spectral cover**

The Weyl group does not act on the spectral cover $\mathcal{B} = \tilde{B}/W_0$, nor therefore on its cohomology. Nevertheless, we can decompose $H^i(\mathcal{B}, Q)$ into its isotypic pieces (cf. [29]) simply by using one of the projections $\pi_\lambda : \tilde{B} \rightarrow \mathcal{B}$ to embed $H^i(\mathcal{B}, Q)$ into $H^i(\tilde{B}, Q)$. We write this as

$$H^i(\mathcal{B}, Q) = \bigoplus_\rho M^\rho_B \otimes H^i_\rho,$$

with multiplicity space $M^\rho_B \subset M^\rho$ which is the stabilizer of $W_0$ in $M^\rho$. Up to conjugation, this is independent of the map $\pi_\lambda$ used. The two representations $\rho = \Lambda, 1$ have the property that they are present in the cohomology of every spectral cover $\mathcal{B}$: the multiplicity space $M^\rho_B$ is always non-zero (cf. [3]).

For every irreducible representation $\rho$ of $W$ there is a correspondence $D_\rho$ on $\mathcal{B}$ which induces on $H^i(\mathcal{B})$ (a certain multiple $-q$ of) the projection to a $\rho$-isotypic piece. A general formula for such correspondences is given in section 12 of [29] in terms of an integral vector $v \in M^\rho$ which is fixed by $W_0$, and a $W$-invariant pairing on $M^\rho$. It was seen there that the multiple is given by:

$$- q = \left[ \frac{|W : W_0|}{\dim \rho} \right] |v|^2. \quad (3.12)$$

Essentially the same result in the case $\rho = \Lambda$ was obtained earlier by Kanev. (We would like to point out that the analogous correspondence on $\tilde{B}$ exists, but is much simpler, so it is not needed explicitly: it is given in the obvious way as a combination of the actions of elements $w \in W$ with weights given by the ($\mathbb{Z}$-valued!) characters $\text{Tr}_w(\rho)$. Our point here is that even though there is no $W$-action on $\mathcal{B}$, the $D_\rho$ survive on the quotient covers $\mathcal{B}$ as a sort of replacement for the $W$-action.)

We work this out explicitly in the case of $G = E_8$ and the standard, degree 240, spectral cover $\mathcal{B}$ whose points parametrize lines on the del Pezzo fibers of $U$. We will describe explicitly some of the correspondences on the isotypic pieces. Let $D_k$ be the family of line pairs in $\mathcal{B} \times \mathcal{B}$ with intersection number $k$; $D_k$ is non empty only for $k = -1, \ldots, 3$. Under the action of the ring generated by the $D_k$, we find that the $W$-module $Q[W/W_0]$, the local system $\pi_*Q$, as well as the cohomology $H^* (\mathcal{B}, Q)$ each decomposes into five pieces corresponding to representations of dimensions 1, 35, 84, 8, 112; the first three are even, the last two odd under the action of the Bertini involution $D_3$. The most useful correspondences will be the combinations

$$D := \sum kD_k, J := \sum D_k, D' := D - J. \quad (3.13)$$

The eigenvalues of $D$ are 240 on 1, -60 on 8 = $\Lambda$, and 0 on the other pieces; on the other hand, the only nonzero eigenvalue of $J$ is 240 on 1. It follows that $D'$ acting on
$H^*(\overline{B})$ is $-q = \frac{240}{8} |\alpha|^2 = -60$ times the projection to the $\Lambda$ piece, in accordance with the general formula (3.12). In particular, the image of $D'$ acting on $H^i(\overline{B},Q)$ is just the distinguished piece $H^i_\Lambda$, while the image of $D$ acting on $H^i(\overline{B},Q)$ is $H^i_\Lambda \oplus H^i_1$. (The correspondence $i$ used in the description given by Kanev \[31\], \[30\] and recalled in the Appendix below is $i = D + D_{-1}$, where $D_{-1} = \Delta$ acts as the identity, so the eigenvalues of $i$ are shifted by 1 from the eigenvalues of $D$. Note that $D$ and $D'$ differ only on the trivial piece $H^*(B)$; Kanev can thus work with the correspondence $i$ (which is equivalent to working with $D$) and get away with it, because he takes the base $B$ to be $P^1$ which has no $H^1$. We are in a situation where we do care about the possible contribution of the cohomology of $B$, so we are forced to work with the ”correct” correspondence $D'$.)

This description has obvious analogues for the ADE groups. We refer the reader to \[30\] for some of the details.

**Cohomology of the del Pezzo fibration**

Elementray topological considerations show that for each $i$ the cohomology group $H^{i+2}(U,Z)$ decomposes as the sum of subgroups:

$$H^{i+2}(U,Z) = H^{i+2}(B,Z) \oplus H^{i-2}(B,Z) \oplus H^i_0 \oplus H^i_1.$$  

(3.14)

We will see below that the ”pure” term $H^i_0$ decomposes further, as:

$$H^i_0 = H^i(B,Z) \oplus H^i_0(U).$$  

(3.15)

We will refer to the summand $H^i_0(U)$ as the reduced cohomology of $U$. We will see in the next paragraph that, over $Q$, it can be identified with the distinguished piece $H^i_\Lambda$ which occurred in the cohomologies of $\overline{B}$ and $B$.

To describe the decomposition (3.14), we use the projection $\pi : U \to B$, the section $\sigma : B \to Z$, the inclusion $j : Z \to \overline{U}$, and the projection $\pi_Z : Z \to B$. In terms of these maps, we can describe the first two summands more accurately as:

$$\pi^*(H^{i+2}(B,Z)) \oplus j_\ast \sigma_\ast(H^{i-2}(B,Z)).$$

Let $\alpha := \pi^\ast \sigma^\ast j^\ast$, $\beta := j_\ast \sigma_\ast \pi_\ast$ be the projections onto these summands. They satisfy $\alpha^2 = \alpha$, $\beta^2 = \beta$, $\beta \alpha = 0$ but, unfortunately, $\alpha \beta \neq 0$. Nevertheless, their images do fit into a direct sum decomposition (over $\mathbb{Z}$!). In fact, we can find a set of three orthogonal projections, namely $\alpha(1-\beta), \beta, (1-\alpha)(1-\beta)$. Their images are our summands $H^{i+2}(B,Z)$, $H^{i-2}(B,Z)$ and $H^i_0 \oplus H^i_1$. (Note that $\alpha$, $\alpha(1-\beta)$ have the same image, since $\alpha(x) = \alpha(1-\beta)\alpha(x)$.)

For the comparison with $H^i_\Lambda$ we will use the cylinder map:

$$c := i_\ast p^\ast : H^i(\overline{B},Z) \to H^{i+2}(U,Z),$$

(3.16)

where $p : P \to \overline{B}$ is the natural $P^1$-bundle, and $i : P \to U$ embeds each $P^1$ as a line in the del Pezzo. In the opposite direction we have

$$c^\ast := p_\ast i^\ast : H^{i+2}(U,Z) \to H^i(\overline{B},Z).$$

In \[31\] Kanev shows the surjectivity of $c$ (over $Q$) when the base $B$ is $P^1$. The generalization we will prove is that, after composition with $(1-\alpha)/(1-\beta)$, the projected
cylinder map \((1 - \alpha)(1 - \beta)c = (1 - \alpha)c\) induces a surjection of \(H^i(\overline{B}, \mathbb{Q})\) onto \(H_p^{i+2} \otimes \mathbb{Q}\). Further, this is compatible with the isotypic decomposition of \(H^i(\overline{B}, \mathbb{Q})\) into five pieces: the cylinder map takes the distinguished piece \(H^i_\Lambda \subset H^i(\overline{B}, \mathbb{Q})\) isomorphically to \(H_0^{i+2}\); while the trivial piece \(H^i_1\) goes isomorphically to \(H^i(B, \mathbb{Q})\). In fact, the composition \(cc^*\) acts as multiplication by 240 on \(H^i(B, \mathbb{Q})\) and by \(-60\) on \(H_0^{i+2}\), while \(c^*c\) acts by 240 on \(\mathbf{1}\), by \(-60\) on \(\Lambda\), and by 0 on the other pieces. (Concerning the trivial piece \(H^i_1\), we point out that there are really two distinct ways we could have used to map it into \(H^{i+2}(U, \mathbb{Z})\), namely via \(j_* \pi^*_Z\) or via \(i_* p^* \pi^*\). The difference \(i_* p^* \pi^* - 240j_* \pi^*_Z\) is in the image of \(\alpha\).

**Local systems**

All in all, we have three ways to realize the lattice \(\Lambda\):

1. As the isotypic piece \(\text{Hom}_W(\Lambda, \mathbb{Z}[W])\)
2. As the image \(D'[\mathbb{Z}[W/W_0]]\) of the correspondence \(D'\) of (3.13)
3. As the reduced part \(H^2_B(S, \mathbb{Z})\) of the cohomology of a del Pezzo-8 surface \(S\).

These extend to three isomorphic local systems over our base \(B\):

1. \(\Lambda_B := \text{Hom}_W(\Lambda, \bar{\pi}_Z \mathbb{Z}[\bar{B}]\)
2. \(\Lambda_B := D'\bar{\pi}_Z \mathbb{Z}[\bar{B}]\)
3. \(\Lambda_U := (R^2 \pi_* Z_U)_0\)

(Here by "local system" we mean a locally constant sheaf on the open subset of \(B\) where the covers are unramified, extended by (inclusion), to all of \(B\).) Indeed, the identification of (1) and (3) is immediate using the self-duality of \(\Lambda\). The isomorphism from (3) to (2) is induced by (the restriction to the reduced piece of) \(c^*\). The inverse map is (the restriction to the \(\Lambda\) piece of) \(-c/60\), which is defined over the integers. (This is equivalent to checking that the image of \(D'\) acting on \(\mathbb{Z}[W/W_0]\) is the same as the image of \(\Lambda\) in \(\mathbb{Z}[W/W_0]\).)

**Spectral sequences**

The cohomology of the covers can be computed in terms of local systems on \(B\):

\[ H^i(\overline{B}, \mathbb{Z}) = H^i(B, \bar{\pi}_Z \mathbb{Z}) \]
\[ H^i(\overline{B}, \mathbb{Z}) = H^i(B, \pi_* Z) \]

The cohomology of \(U\), on the other hand, requires the complex of sheaves \(R\pi_* Z\); in other words, it comes out of the Leray spectral sequence for \(\pi\). But the projections \(\alpha\) and \(\beta\) of (3.14) act on \(R\pi_* Z\), so each summand in (3.14) is again computed as the cohomology of a local system on \(B\). In particular, \(H_p^{i+2} = (1 - \alpha)(1 - \beta)H^{i+2}(U, \mathbb{Z}) = H^i(B, R^2 \pi_* Z)\). But this local system is, globally, a direct sum of a trivial piece (coming from the anticanonical divisors \(F\) in the del Pezzos) and the \(\Lambda\) piece: \(R\pi_* Z = ZF \oplus \Lambda_U\).

Therefore, its cohomology decomposes as claimed in (3.13). The reduced cohomology \(H_0^{i+2}(U)\) can therefore be described as \(H^i\) of \(B\) with coefficients in either of the local systems \(\Lambda_B, \Lambda_B\), or \(\Lambda_U\).
Next we want to describe the distinguished pieces of the cohomologies of $\tilde{B}$ and $\overline{B}$:

$$H^i_\Lambda(\tilde{B}, \mathbb{Z}) := \text{Hom}_W(\Lambda, H^i(\tilde{B}, \mathbb{Z})), \quad H^i(\overline{B}, \mathbb{Z}) := D'H^i(\overline{B}, \mathbb{Z}).$$

It is convenient to introduce the two spectral sequences computing the $W$-equivariant cohomology $H^{i+j}_W := H^{i+j}(\text{Hom}(\Lambda, \tilde{\pi}_*\mathbb{Z}_{\mathbb{B}})).$ Their $E_2$ terms are, respectively,

$$1E^{ij} = H^i(W, \text{Hom}(\Lambda, H^j(\tilde{B}, \mathbb{Z})))$$

and

$$2E^{ij} = H^j(B, H^i(W, \text{Hom}(\Lambda, \tilde{\pi}_*\mathbb{Z}_{\mathbb{B}}))).$$

From $2E$ we get edge homomorphisms

$$H^{i+2}_0(U, \mathbb{Z}) = H^i(B, \Lambda_{\mathbb{B}}) = H^i(B, \text{Hom}_W(\Lambda, \tilde{\pi}_*\mathbb{Z}_{\mathbb{B}})) \to H^i_W,$$

while $1E$ gives

$$H^i_W \to \text{Hom}_W(\Lambda, H^i(\tilde{B}, \mathbb{Z})).$$

Similarly for $\overline{B}$ we find maps

$$H^i_\Lambda(\overline{B}, \mathbb{Z}) = D'H^i(B, \tilde{\pi}_*\mathbb{Z}) \to H^i(B, D'\tilde{\pi}_*\mathbb{Z}) = H^{i+2}_0(U, \mathbb{Z})$$

which can also be analyzed via spectral sequences. At any rate, we see that the natural map between the distinguished pieces for $\overline{B}$ and $\tilde{B}$ factors:

$$H^i_\Lambda(\overline{B}, \mathbb{Z}) \to H^{i+2}_0(U, \mathbb{Z}) \to H^i_W \to H^i_\Lambda(\tilde{B}, \mathbb{Z}). \quad (3.17)$$

In order to go further, we need to compute some group cohomologies. In general, all the higher cohomologies are finite abelian groups. A useful observation is that $H^1(W, \text{Hom}(\Lambda, \mathbb{Z}[W]))$ vanishes. (This is the same as

$$H^0(W, \text{Hom}(\Lambda, \mathbb{Z}[W]) \otimes (R/\mathbb{Z})) / H^0(W, \text{Hom}(\Lambda, \mathbb{Z}[W]) \otimes (R)).$$

But since $W$ permutes a $\mathbb{Z}$-basis of $\mathbb{Z}[W]$, all $W$-invariants in the torus come from $W$-invariants in the vector space.) This implies that the $2E^{1j}$ terms vanish, while $2E^{0j} = H^j(B, \Lambda_{\mathbb{B}}) = H^{i+2}_0(U, \mathbb{Z})$. In particular we find:

$$H^0(U, \mathbb{Z}) = H^1_W, \quad 0 \to H^1(U, \mathbb{Z}) \to H^2_W \to H^0(B, H^2(W, \text{Hom}(\Lambda, \tilde{\pi}_*\mathbb{Z}_{\mathbb{B}}))). \quad (3.18)$$

In order to simplify $1E$, we will assume that our group $G$ is of adjoint type, e.g. this holds for our main interest, $G = E_8$. This means that $\Lambda$ is the root lattice, and $\Lambda^* := \text{Hom}(\Lambda, \mathbb{Z})$ is the full weight lattice (of the Langlands dual group, but this is immaterial for groups of types ADE). In this case one sees easily that $1E^{i0} = H^i(W, \Lambda^*) = 0$ for $i = 0, 1$. By an explicit calculation, we checked this also when $i = 2$ for adjoint $G$ of type $A_n, D_n$ or $E_8$. The next term, $1E^{30} = H^3(W, \Lambda^*)$ is precisely the right hand side $H^2(W, T)$ of (3.11). It would be nice to know that this vanished, but we have not computed it yet. Our conclusions for $1E$ are:

$$H^1_W = \text{Hom}_W(\Lambda, H^1(\tilde{B}, \mathbb{Z})), \quad (3.19)$$
and, under the assumption that $H^2(W, T) = 0$,

$$0 \rightarrow H^1(W, \text{Hom}(\Lambda, H^1(\tilde{B}, \mathbb{Z}))) \rightarrow H^2_W \rightarrow \text{Hom}_W(\Lambda, H^2(\tilde{B}, \mathbb{Z})) \rightarrow H^2(W, \text{Hom}(\Lambda, H^1(\tilde{B}, \mathbb{Z})))$$

(3.20)

**The twisting data**

We are finally ready to prove our main result, the identification of the distinguished Prym variety $\text{Prym}_\Lambda(\tilde{B})$, which precisely parametrizes the full twisting data in the heterotic theory, with the Deligne cohomology of the del Pezzo fibration. As we have already emphasized several times before, this is an identification between two objects in the heterotic theory. When we go, in the next section, to an appropriate boundary component, the Deligne cohomology of the del Pezzo fibration will be reinterpreted as the fiber, or twisting data, in the fibration of the F-theory moduli. The appropriate boundary component is the locus where the T-dualities are killed, so that the relation between the heterotic and F-theoretic fibers can indeed be expected to be described there by classical geometry.

Our main point here is that the continuous data is actually determined by the discrete data which we have already analyzed. The maps between the lattices in (3.17) are, of course, morphisms of Hodge structures. Therefore, they induce maps between the Abelian varieties made out of those lattices. We recall that the family of twisting data for the heterotic theory is given by $\text{Prym}_\Lambda(\tilde{B})$, defined in (3.11). Its connected part is the Abelian variety made out of the lattice

$$^1E_{01} = \text{Hom}_W(\Lambda, H^1(\tilde{B}, \mathbb{Z})).$$

(3.21)

We want to compare this with the intermediate Jacobian $J^3(U/B) := J^3_0(U)$. But this is the Abelian variety made out of the lattice $H^3_0(U, \mathbb{Z})$, so the desired isomorphism follows immediately from (3.18) and (3.19). The analogous result when $B = \mathbb{P}^1$ and $G = E_6, E_7$ was proved by Kanev [31]. The result in case $G = E_8$ and $B$ is 1-dimensional is announced in [3].

Next, the discrete part, i.e. the group of connected components of $\text{Prym}_\Lambda(\tilde{B})$. From (3.11) we have:

$$0 \rightarrow \text{Comp}(\text{Prym}_\Lambda(\tilde{B})) \rightarrow \text{Comp}(\text{Hom}_W(\Lambda, \text{Pic}(\tilde{B}))) \rightarrow H^2(W, T).$$

(3.22)

Now from the long-exact sequences of $W$-cohomology of the two short exact sequences:

$$0 \rightarrow \text{Pic}^0(\tilde{B}) \rightarrow \text{Pic}(\tilde{B}) \rightarrow H^{11}(\tilde{B}, \mathbb{Z}) \rightarrow 0$$

and

$$0 \rightarrow H^1(\tilde{B}, \mathbb{Z}) \rightarrow H^1(\tilde{B}, \mathbb{R}) \rightarrow \text{Pic}^0(\tilde{B}) \rightarrow 0$$

we deduce:

$$0 \rightarrow H^1(W, \text{Hom}(\Lambda, H^1(\tilde{B}, \mathbb{Z}))) \rightarrow \text{Comp}(\text{Hom}_W(\Lambda, \text{Pic}(\tilde{B}))) \rightarrow$$

$$\rightarrow \text{Hom}_W(\Lambda, H^{11}(\tilde{B}, \mathbb{Z})) \rightarrow H^1(W, \text{Hom}(\Lambda, \text{Pic}^0(\tilde{B}))).$$

(3.23)

Noting the isomorphism of $H^1(W, \text{Hom}(\Lambda, \text{Pic}^0(\tilde{B})))$ with $H^2(W, \text{Hom}(\Lambda, H^1(\tilde{B}, \mathbb{Z})))$, we can map the entire sequence (3.23) into (3.20). The first and last terms match exactly,
while between the third terms we get an inclusion. We have therefore identified the discrete data $Comp(Prym_\Lambda(\tilde{B})) = Comp(Hom_W(\Lambda, Pic(\tilde{B})))$ with the subspace of $H^2_W$ which is of Hodge type $(1,1)$. From (3.18), we see that the group of components of the Deligne cohomology, $H^{22}_0(U, \mathbb{Z})$, is thus identified with a subgroup of finite index in $Comp(Prym_\Lambda(\tilde{B}))$. We would get an actual isomorphism if we had an isomorphism in (3.18); for instance, this would follow if we knew that $H^2(W, Hom(\Lambda, \mathbb{Z}[W])) = 0$. We have also made the assumption that $H^2(W, T) = 0$. If this turns out to be non-zero, there could be a finite discrepancy, though this seems rather unlikely: heuristically, the group of components would be reduced according to (3.22), while $H^2_W$ would be similarly reduced due to the non-zero $1E^{30}$ term, and the effects are likely to cancel so that the isomorphism could be preserved. In fact, in case $G = E_6, E_7, E_8$ and $B = \mathbb{P}^1$, one indeed gets an isomorphism, according to (3).

To summarize, we have proved that the continuous part of the twisting data is given by the relative intermediate Jacobian $J^3(U/B)$, and that the discrete part of the twisting data contains $H^{22}_0(U, \mathbb{Z})$ as a subgroup of finite index. This finite index is zero if, as we expect, the two group cohomologies $H^2(W, T)$ and $H^2(W, Hom(\Lambda, \mathbb{Z}[W]))$ both vanish.

4 F-theory considerations

4.1 Transition to F-theory

In the representation of the bundle via the del Pezzo construction respectively in the stable degeneration on the $F$-theory side the data are already in a form appropriate to comparison. In the case of $E_8$ bundles one has just to blow down the section of the $dP_9$ fibre on the $F$-theory side to get the $dP_8$ fibre of the heterotic side showing the relation of the cohomologies and hence of the intermediate jacobians.

Now for a bundle of group $H \neq E_8$ the section $\theta : B \to X^4$ of $G$-singularities in the $F$-theory setup correspond to having a bundle of unbroken gauge group $G$, i.e. an $H$ bundle where $H$ is the commutant of $G$ in $E_8$, over the heterotic Calabi-Yau $Z$ respectively having a section $s : B \to \mathcal{W}_H = \mathcal{M}_{Z/B}$ (at least locally over the dense open subset of $B$ over which the fibres correspond to semistable bundles) or, as the fibre of $\mathcal{W}_H$ over $b \in B$ parametrizes the corresponding del Pezzo surfaces, a bundle $W^{het}_H \to B$ of del Pezzo surfaces $dP_H$ fibered over $B$. We would like to see that the factor whose split off from $H^{2,1}(X^4)$ is caused by the local data along $\theta(B)$ is captured by $H^{2,1}(W^{het}_H)$. The question is local in the $dP$ fibre and global along $B$. So locally at $\theta$, i.e. at the singularity along $B$, the picture in the $K3$ fibre of $X^4 \to B$ respectively the $dP$ fibre on the heterotic side is the same if one considers heterotically actually a $dP_8$ fibration with $G$ singularity instead of the $dP_H$ fibration: this can be done as we have an $ADE$ system of rational (-2) curves lying in $H^{1,1}(K3, \mathbb{Z})$ as well as in $H^{1,1}(dP_8, \mathbb{Z})^+ \med{P}$ (in the case of the E-series, say; $F$ the elliptic curve representing the ample anticanonical divisor); note that the complex structures for $dP_H$ are given by homomorphisms $H^{1,1}(dP_H, \mathbb{Z})^+ \med{P} \to F$ and the complex structures for $dP_8$ keeping the $G$ singularity are similarly given by the

12We assume $G$ to be simply-laced; otherwise one would have to consider the monodromy operation on the vanishing cycles of a corresponding $F$-theory singularity, which are realized as outer automorphisms of the Dynkin diagram of the $F$-theory singularity leaving the non-simply-laced quotient as unbroken gauge group [22],[23].
corresponding homomorphisms for $dP_8$ mapping the $G$ system of rational (-2) curves to zero (i.e. they essentially describe a mapping for the $H$-part). The matching of the cohomologies gives the matching of the intermediate jacobians.

Note that as far as complex structure deformations are concerned the distribution into deformations of $Z$ respectively those deformations $H^1(W_i, T_{W_i} \otimes \mathcal{O}(-Z)) \cong H^{3,1}(W_i)$ of $W_i$, which preserve $Z$, reflects the well known distribution of the deforming monomials of the defining $F$-theory equation for $X^4$ between those which are ”middle-polynomials” and the rest.

4.2 A remark on the four-flux

We close with a remark on the four-flux. We saw above how the identification of the necessary number of three-branes (from tadpole cancellation) is modified in presence of the four-flux [18] and how this is reflected on the heterotic side. As a further example of the mutual interference of consistency conditions we will see now how the fact that we take the space-time filling branes into consideration refines the usual four-flux quantization condition.

Let us consider a $N = 2$ compactification to 3D of $M$-theory on a Calabi-Yau fourfold $X$; this corresponds, in a certain limit for elliptically fibered $X$, to a $N = 1$ F-theory compactification to 4D. In [12] it was shown that one has as quantization law for the four-flux $G = \frac{1}{2\pi} dC$ not the naive integrality of $G$ but $G = \frac{c_2}{2} + \alpha$ where $\alpha \in H^4(X, \mathbb{Z})$, i.e. in case $c_2$ is not even one is not free in the decision to turn on four-flux or not or, to formulate it differently, the possible ’0-value’ has changed from 0 to $\frac{c_2}{2}$. We will see below that to achieve the wanted amount of supersymmetry in a consistent compactification, $\alpha$ is even further restricted than to be merely an integral class. For example the easiest way to solve the congruence, namely to put $G = \frac{c_2}{2}$, is thereby (besides by possible non-primitiveness) ruled out if one wants to achieve $N = 1$ supersymmetry, i.e. there is actually no such thing in general as some simplest (and may be even shifted) ’0-value’.

First recall that one has from the necessity of tadpole-cancellation, that a number $n_3$ of spacetime-filling branes has to be turned on $\chi_24 = n_3$ (here and in the following we have to assume $\chi \geq 0$). If one actually includes both degrees of freedom one finds [18]

$$\frac{\chi}{24} = n_3 + \frac{1}{2} \int G \wedge G \quad (4.24)$$

Furthermore we have $\int c_2^2 = 480 + \frac{\chi}{3}$ so the Euler number completely cancels out and one has

$$n_3 = -60 - \frac{1}{2} \int \alpha^2 + \alpha c_2 \quad (4.25)$$

and, because of $n_3 \geq 0$ to keep the supersymmetry [7],

$$\int \alpha^2 + \alpha c_2 \leq -120 \quad (4.26)$$

On the other hand one has from the self-duality of $G$ that $G^2 = \int G \wedge G \geq 0$ so that we get finally the following bounds

$$-120 - \frac{\chi}{12} \leq \alpha^2 + \alpha c_2 \leq -120 \quad (4.27)$$
For example one has for \( \alpha \in c_{1/2} \) that \( \alpha^2 \leq -120 \), which as remarked especially rules out \( G = \frac{Q_2}{2} \); or, to give an example where a condition \( \alpha c_2 \neq 0 \) occurs, for \( \alpha = D^2 \) with a divisor which contributes to the superpotential (i.e. \(-\frac{1}{24}\alpha c_2 = \chi(D, \mathcal{O}_D) = 1\), cf. [34] and [35]) one finds \( D^4 \leq -96 \), i.e. again a contradiction as the left hand side vanishes for the vertical \( D \) (more generally you find for \( \alpha = D^2 \) the condition \( \chi(D, \mathcal{O}_D) \geq 5 \) or equivalently \( \chi(\mathcal{O}(D)) \leq -5 \)). Note conversely that for a divisor \( D \) contributing to the superpotential \( D^2 \cdot \alpha = 12 \) from \( G|_D = 0 \) [36].

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## Appendix

### A Root systems and del Pezzo surfaces

Useful references are [25], [26], [27], [28] in general and especially [31], [30].

First one makes the transition (cf. below) from the lattice \( L = Q(R) \subset h^* \) to its extension \( N(L, \lambda) = L \oplus [l]Z \) related to the embedding of the root system into \( H^{1,1}(dP, Z) \), where in addition to the blowing up classes also the line class of the original \( P^2 \) lives (or equivalently, as a certain linear combination with the other classes, the further class \( f = -K \)).

The fact that \( C \rightarrow B \) is modelled (\( W \)-equivariantly with fibre the Weyl orbit \( W\lambda \)) on (2.8) translates to the representation of the Prym-Tyurin variety as the image \( P(C, i) \) of the endomorphism \( i - 1 \) of \( J(C) \) coming from the correspondence \( G = D - \Delta \) (\( \Delta \) the diagonal, so \( i \) coming from \( D \)) given (outside the ramification locus) by

\[
G(x) = \sum_{j=1}^{d} (x, l_j) l_j \tag{A.28}
\]

(using the identification of the fibre \( W\lambda \cong Wl = \{(l_j)_{j=1,..,d=|R|}\} \); for the \( D \) correspondence the sum goes over \( l_j \neq x \), it is of degree \( n = \text{deg}_C D(x) = (l, \sum_j l_j) + 1 = d + 1 \); the occurring scalar products are for example \( 0, \pm 1 \) in the E-series up to \( E_6 \) and include further 2 and even 3 for \( E_7 \) respectively. \( E_8 \)).

Generalizing the involution \( i^2 = 1 \), i.e. \((i - 1)(i + 1) = 0\), of the D-series (cf. below), which leads to the ordinary Prym \( J^-(C) \), one has \((i - 1)(i + q - 1) = 0\) with \( q = -(\lambda|\lambda)\frac{4}{r} \in \mathbb{N} \); so \( q = 1 \leftrightarrow D = 0 \leftrightarrow W\lambda = W\omega_1 \) in the A-series, \( q = 2 \) for \( D \), and \( W\omega_1 \) orbit and generally \( q = 2\frac{|R|}{r} = 2h \) for \( \lambda = \bar{\alpha} \), the maximal root \( (W\lambda = R, h \text{ the Coxeter number}) \).

As pointed out above, one makes first the transition from the lattice \( L = Q(R) \subset h^* \) to its extension \( N(L, \lambda) = L \oplus [l]Z \) related to the embedding into the extended root system respectively to the one into \( H^{1,1}(dP, Z) \).

The \( W \)-operation is linearly extended by \( w(kl) = k(w\lambda - \lambda) + kl \). With \( \kappa = \sum_w w(l) \) one has \( N(L, \lambda)_Q = L_Q \oplus \kappa Q, l = \lambda + c\kappa \) and the relation of orbits \( W\lambda = (\lambda_i)_i \) and
\( Wl = (l_1 = \lambda + \alpha \kappa)_i \), so that the projection on the first summand \( p : N(L, \lambda)_Q \to L_Q \)
gives a \( W \)-equivariant bijection \( Wl \to W\lambda \). Furthermore the symmetric, bilinear, \( W \)-invariant, negative definite form \((\cdot, \cdot)\) on \( L_Q \) extends to a unique symmetric, bilinear, \( W \)-invariant form \((\cdot, \cdot)\) on \( N(L, \lambda)_Q \) with \((\alpha, l) = (\alpha, \lambda)\) for \( \alpha \in L_Q \) and \((l, l) = -1\). One has \((l_\alpha, l_\beta) = (\alpha|\beta) + 1\) and of course \((\alpha|\alpha) = 0, \pm 1\) apart from \((\alpha|\pm \alpha) = \mp 2\).

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