Some conformally invariant gap theorems for Bach-flat 4-manifolds

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Abstract
S.-Y. A. Chang, J. Qing, and P. Yang proved an important gap theorem for Bach-flat metrics with round sphere as model case in 2007. In this article, we generalize this result by establishing conformally invariant gap theorems for Bach-flat 4-manifolds with $(\mathbb{CP}^2, g_{FS})$ and $(S^2 \times S^2, g_{prod})$ as model cases. An iteration argument plays an important role in the case of $(\mathbb{CP}^2, g_{FS})$ and the convergence theory of Bach-flat metrics is of particular importance in the case of $(S^2 \times S^2, g_{prod})$. The latter result provides an interesting way to distinguish $(S^2 \times S^2, g_{prod})$ from $(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}, g_{Page})$.

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1 Introduction
Riemannian functionals have played an important role in the study of geometry and topology of smooth manifolds; see Chapter 4 of [2] for a detailed description. In four dimensions, an important example is the Weyl functional defined by

\[ W : g \mapsto \int ||W_g||^2 \, dv_g \]

where $W_g$ denotes the Weyl curvature tensor of $g$, and $\| \cdot \|$ is the norm of $W_g$ as a section of $End(\Lambda^2(M^4))$. Critical points of $W$ are metrics with vanishing Bach tensor $B_{ij}$ defined by

\[ B_{ij} = \nabla^k \nabla^l W_{ijkl} + P^{kl} W_{ijkl} \tag{1.1} \]

where $P$ is the Schouten tensor. Four-manifolds with vanishing Bach tensor are called Bach-flat four-manifolds. They are of importance from the viewpoints of both calculus of variations and differential geometry. Note that Bach-flatness is a conformally invariant condition. See Sect. 2 for more details.
In this article we study the rigidity of oriented, closed Bach-flat four-manifolds with the range of conformal invariants. It turns out that most conformal invariants arise from the Chern–Gauss–Bonnet formula which has a compact form in four dimensions:

$$8\pi^2 \chi(M^4) = \int ||W_g||^2 dv_g + 4 \int \sigma_2(P_g) dv_g$$  \hspace{1cm} (1.2)

where $\chi(M^4)$ is the Euler characteristic of $M^4$ and $\sigma_k(P_g)$ is the $k$-th elementary symmetric polynomial applied to the eigenvalues of $g^{-1}P_g$. It is well-known that $||W_g||^2 dv_g$ is a pointwise conformal invariant in four dimensions and $\chi(M^4)$ is a topological invariant. It follows from (1.2) that $\int \sigma_2(P_g) dv_g$ is a conformal invariant. We shall focus on the following conformal classes on a closed four-manifold $M^4$ defined in [6]:

$$\mathcal{Y}_+^1(M^4) = \{ g : Y(M^4, [g]) > 0 \}$$

and

$$\mathcal{Y}_+^2(M^4) = \{ g \in \mathcal{Y}_+^1(M^4) : \int \sigma_2(P_g) dv_g > 0 \}$$

where $Y(M^4, [g]) > 0$ is the Yamabe invariant of $(M^4, g)$. For $g \in \mathcal{Y}_+^2(M^4)$, the following conformal invariant is defined in [6]

$$\beta(M^4, [g]) = \frac{\int ||W_g|| dv_g}{\int \sigma_2(P_g) dv_g}.$$  

We also recall the definition of the smooth invariant from [6]

$$\beta(M^4) = \inf_{[g]} \beta(M^4, [g]).$$

See Sect. 2 for more details. The conformal invariant $\beta(M^4, [g])$ can be easily calculated for many Einstein four-manifolds. For example, we have

$$\beta(S^4, [g_{S^4}]) = 0, \quad \beta(\mathbb{CP}^2, [g_{FS}]) = 4, \quad \beta(S^2 \times S^2, [g_{prod}]) = 8.$$  

These two invariants $\beta(M^4, [g])$ and $\beta(M^4)$ have played an important role in the study of geometry and topology of four-manifolds; for example, see [4–6]. It has been shown that the range of $\beta$ imposes topological restrictions on the underlying manifolds. The main results of [5] give a (sharp) range for $\beta$ that implies the underlying manifold is the sphere:

**Theorem 1.1** [5] Suppose $M^4$ is oriented. If $g \in \mathcal{Y}_+^2(M^4)$ with

$$\beta(M^4, [g]) < 4,$$  \hspace{1cm} (1.3)

then $M^4$ is diffeomorphic to $S^4$. In particular, if $M^4$ satisfies

$$-\infty < \beta(M^4) < 4,$$

then the same conclusion holds.

Furthermore, if $M^4$ admits a metric with $\beta(M^4, [g]) = 4$, then one of the following must hold:

- $M^4$ is diffeomorphic to $S^4$; or
- $(M^4, g)$ is conformally equivalent to $(\mathbb{CP}^2, g_{FS})$, where $g_{FS}$ is the Fubini-Study metric.

Theorem 1.1 is widely referred to as the conformally invariant sphere theorem in four dimensions. The second part of Theorem 1.1 is a corollary of Theorem C in [5] which provides a classification of Bach-flat metrics with $\beta(M^4, [g]) = 4$:
Theorem 1.2 [5] Suppose \((M^4, g)\) is an oriented Bach-flat four-manifold which is not diffeomorphic to \(S^4\). If \(g \in \mathcal{Y}_2^+ (M^4)\) with \(\beta(M^4, [g]) = 4\), then \((M^4, g)\) is conformally equivalent to \((\mathbb{C}P^2, g_{FS})\), where \(g_{FS}\) is the Fubini-Study metric.

We remark that the condition of Theorem C in [5]

\[
\int ||W_g|| \, dv_g = 4 \int \sigma_2(P_g) \, dv_g
\]

(1.4)

is slightly weaker since \(1.4\) allows \(\int \sigma_2(P_g) \, dv_g\) to be vanishing. In the statement of Theorem 1.2, the condition \(g \in \mathcal{Y}_2^+ (M^4)\) requires \(\int \sigma_2(P_g) \, dv_g\) be positive and thereby rules out the possibility of \((M^4, g)\) being conformally equivalent to a manifold which is isometrically covered by \(S^3 \times S^1\) endowed with the product metric.

The main results of [6] provide a (sharp) characterization of \(\mathbb{C}P^2\) by the range of \(\beta\).

Theorem 1.3 [6] Suppose \(M^4\) is oriented and \(b_2^+ (M^4) > 0\). There is an \(\epsilon > 0\) such that if \(M^4\) admits a metric \(g \in \mathcal{Y}_2^+ (M^4)\) with

\[
4 \leq \beta(M^4, [g]) < 4(1 + \epsilon),
\]

(1.5)

then \(M^4\) is diffeomorphic to \(\mathbb{C}P^2\). In particular, if

\[
\beta(M^4) = 4,
\]

(1.6)

then the same conclusion holds. Moreover, \(\beta(M^4, [g]) = 4\) if and only if \(g \in [g_{FS}]\).

In [7], Chang et al. initiated the study on whether a stronger form of rigidity is valid for Bach-flat metrics with the information of \(\beta\). The main results of [7] distinguish the round sphere from the class of Bach-flat metrics by the information of \(\beta\). See also [14] for a simplified and refined proof.

Theorem 1.4 [7] Let \((M^4, g)\) be an oriented Bach-flat manifold. If \(g \in \mathcal{Y}_2^+ (M^4)\) satisfies

\[
\beta(M^4, [g]) < 4,
\]

(1.7)

then \((M^4, g)\) is conformally equivalent to \((S^4, g_{S^4})\), where \(g_{S^4}\) is the round metric.

From Theorems 1.1, 1.2, 1.3 and 1.4, it is clear that 4 is a “critical” value of \(\beta\) at which the underlying topology can change. The main goal of this paper is to initiate the study of Bach-flat four-manifolds with

\[
\beta(M^4, [g]) \geq 4.
\]

The first theorem of this paper is to distinguish \((\mathbb{C}P^2, g_{FS})\) from the class of Bach-flat four-manifolds by the range of \(\beta\).

Theorem A Let \((M^4, g)\) be an oriented, closed Bach-flat four-manifold with \(b_2^+ (M^4) > 0\). Then there is an \(\epsilon_1 > 0\) such that if \(g \in \mathcal{Y}_2^+ (M^4)\) with

\[
4 \leq \beta(M^4, [g]) < 4(1 + \epsilon_1),
\]

(1.8)

then \((M^4, g)\) is conformally equivalent to \((\mathbb{C}P^2, g_{FS})\), where \(g_{FS}\) is the Fubini-Study metric.

Remark 1.1 The conditions and conclusions in Theorem A are all conformally invariant.

Remark 1.2 The constant \(\epsilon_1\) in Theorem A can be calculated numerically from its proof, though it might not be sharp.
The second theorem of this paper is to distinguish \((S^2 \times S^2, g_{\text{prod}})\) from the class of Bach-flat four-manifolds by the range of \(\beta\).

**Theorem B** Let \((M^4, g)\) be an oriented, closed Bach-flat four-manifold with \(b_2^+(M^4) = b_2^-(M^4) > 0\). Then there is an \(\epsilon_2 > 0\) such that if \(g \in \mathcal{Y}_2^+(M^4)\) with
\[
8 \leq \beta(M^4, [g]) < 8(1 + \epsilon_2),
\]
then \((M^4, g)\) is conformally equivalent to \((S^2 \times S^2, g_{\text{prod}})\), where \(g_{\text{prod}}\) is the standard product metric.

**Remark 1.3** The conditions and conclusions in Theorem B are all conformally invariant.

**Remark 1.4** The constant \(\epsilon_2\) in Theorem B is cannot be computed from the current proof since we argue by contradiction and appeal to a compactness theorem.

**Remark 1.5** The main interest of Theorem B is that it provides a method to distinguish \((S^2 \times S^2, g_{\text{prod}})\) from \((\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, g_{\text{Page}})\). It is clear that both of them are simply connected and satisfy \(b_2^+ = b_2^- = 1\). Theorem B enables us to distinguish them by the conformal invariant \(\beta(M^4, [g])\) which can be calculated from curvature.

The paper is organized as follows. In Sect. 2, we establish the notations and conventions and collect the definitions and properties of \(\beta\) invariants and Bach tensor. In Sects. 3 and 4, Theorems A and B will be proved in detail, respectively.

## 2 Preliminaries

### 2.1 Conformal and smooth invariants

Suppose \((M^4, g)\) is an oriented, closed Riemannian four-manifold. We denote by \(Rm\) the curvature tensor, \(W\) the Weyl tensor, \(Ric\) the Ricci tenor, \(R\) the scalar curvature, respectively. Then we have the well-known decomposition of curvature tensor:
\[
Rm = W + P \otimes g,
\]
where \(\otimes\) is the Kulkarni-Nomizu product and \(P\) is the Schouten tensor defined as
\[
P = \frac{1}{2} \left( Ric - \frac{R}{6} \cdot g \right).
\]
With (2.1), we can write Chern–Gauss–Bonnet formula in four dimensions as
\[
8\pi^2 \chi(M^4) = \int \|W\|^2 \, dv + 4 \int \sigma_2(g^{-1} P) \, dv,
\]
where
- \(\| \cdot \|\) is the norm of Weyl tensor viewed as an endomorphism on the bundle of two-forms.
- There is another norm \(| \cdot |\) of Weyl tensor viewed as a four-tensor. The relation between them is given by
\[
\|W\|^2 = \frac{1}{4} |W|^2.
\]
• $g^{-1} P$ is the $(1, 1)$-tensor from “raising” the subscript of $P$ by contraction with the metric $g$ and $\sigma_2(g^{-1} P)$ is the second elementary symmetric polynomials applied to the eigenvalues of $g^{-1} P$ viewed as a matrix. For the sake of simplicity, we will write $\sigma_2(P)$ in place of $\sigma_2(g^{-1} P)$.

It is well-known that $||W_g||^2 dv_g$ is pointwise conformally invariant and $\chi(M^4)$ is a topological invariant. Therefore, it follows from (2.3) that $\int \sigma_2(P) dv$ is a conformal invariant in four dimensions.

Given a Riemannian manifold $(M^n, g)$ of dimension $n \geq 3$, let $[g]$ denote the equivalence class of metrics pointwise conformal to $g$, and $Y(M^n, [g])$ denote the Yamabe invariant:

$$Y(M^n, [g]) = \inf_{\tilde{g} \in [g]} Vol(\tilde{g})^{-\frac{n-2}{n}} \int R_{\tilde{g}} dv_{\tilde{g}}.$$

We can also express the Yamabe invariant in terms of the first symmetric function of the Schouten tensor: it follows from (2.2) that

$$\sigma_1(P) = \frac{R}{2(n-1)},$$

hence

$$Y(M^n, [g]) = \inf_{\tilde{g} \in [g]} 2(n-1)Vol(\tilde{g})^{-\frac{n-2}{n}} \int \sigma_1(P_{\tilde{g}}) dv_{\tilde{g}}. \quad (2.4)$$

With this interpretation of Yamabe invariant, in dimension four we should view the conformal invariant $\int \sigma_2(P) dv$ as a kind of “second Yamabe invariant”.

Next recall the following definitions from [6]:

$$\mathcal{Y}^+_1(M^4) = \{ g : Y(M^4, [g]) > 0 \}, \quad (2.5)$$

and

$$\mathcal{Y}^+_2(M^4) = \{ g \in \mathcal{Y}^+_1(M^4) : \int \sigma_2(P_g) dv_g > 0 \}. \quad (2.6)$$

For $g \in \mathcal{Y}^+_2(M)$, define the conformal invariant

$$\beta(M^4, [g]) = \frac{\int ||W_g||^2 dv_g}{\int \sigma_2(P_g) dv_g} \geq 0, \quad (2.7)$$

and the smooth invariant

$$\beta(M^4) = \inf_{[g]} \beta(M^4, [g]). \quad (2.8)$$

If $\mathcal{Y}^+_2(M^4) = \emptyset$, we set $\beta(M^4) = -\infty$.

It is noteworthy to point out that the definition of smooth invariant $\beta(M^4)$ is similar to that of $\sigma$-invariant. For a smooth manifold $M^n$, the $\sigma$-invariant is defined as

$$\sigma(M^n) = \sup_{[g]} Y(M^n, [g]).$$

The $\sigma$-invariant was introduced by Schoen in [18] (see also Kobayashi [13]) and has been studied extensively. The $\sigma$-invariant proves to be particularly powerful in dimension three. In three dimensions, we have

$$\sigma(S^3) = \sigma(S^2 \times S^1) = Y(S^3, [g_{S^3}]), \quad \sigma(T^3) = 0.$$
For $m \geq 1$, define

$$\sigma_m = \frac{\sigma(S^3)}{m^{2/3}}.$$  

A remarkable result concerning $\sigma$-invariant which is comparable to Theorem 1.1 is the following theorem established by Bray and Neves in [3]:

**Theorem 2.1** [3] A closed three-manifold with $\sigma > \sigma_2$ is either $S^3$, a connect sum with an $S^2$ bundle over $S^1$, or has more than one nonorientable prime component. In particular, the only closed, simply-connected three-manifold with $\sigma > \sigma_2$ is $S^3$.

Comparing Theorem 2.1 with Theorem 1.1, we remark that $\beta(M^4)$ has played a similar role in four-dimensional geometry and topology as $\sigma$-invariant did in three dimensions.

### 2.2 Bach tensor and Bach-flat metrics

In differential geometry, the study of critical points of curvature functionals is of particular importance. A well-known example is the (normalized) Einstein-Hilbert functional (total scalar curvature):

$$\mathcal{F}: g \rightarrow Vol(g) - \frac{n-2}{n} \int R_g \, dv_g.$$  

The critical points of $\mathcal{F}$ are Einstein metrics. Note that the infimum of $\mathcal{F}$ in a given conformal class is just the Yamabe invariant. In four dimensions, the Riemannian quadratic functionals play an important role in the study of geometry and topology. See Section 4 of [2] for a detailed exposition. From the viewpoint of four-dimensional conformal geometry, it is particularly important to understand the Riemannian functionals which are invariant under conformal change of metrics. A prominent example is the Weyl functional. Let $M$ be a smooth, oriented, closed four-manifold, define the Weyl functional as

$$\mathcal{W}: g \rightarrow \int ||W_g||^2 \, dv_g.$$  

In four dimensions, $||W_g||^2 \, dv_g$ is pointwise conformally invariant and it follows that the Weyl functional is a constant within a conformal class. For a smooth one-parameter family of metrics

$$g_t := g + th + O(t^2),$$  \hfill (2.9)

the first variation of Weyl functional is given by

$$\frac{d}{dt} \bigg|_{t=0} \mathcal{W}(g_t) = \int h^{ij} B_{ij} \, dv_g;$$  \hfill (2.10)

where the Bach tensor $B$ is given by the local coordinate formula in [1,2]

$$B_{ij} = \nabla^k \nabla^l W_{kl} + \frac{1}{2} R^{kl} W_{klj}.$$  \hfill (2.11)

The Bach tensor is symmetric, trace-free and divergence-free. Moreover, the Bach tensor is conformally invariant in the sense that $B_{\tilde{g}} = e^{-2w} B_g$ for $\tilde{g} = e^{2w} g$.

A metric is called Bach-flat if its Bach tensor is vanishing and thereby is a critical point of the Weyl functional. It follows from the conformal invariance of Bach tensor that the Bach-flat condition $B_{ij} = 0$ is also conformally invariant. There are two important classes of Bach-flat metrics:
Self-dual metrics \((W_g = 0)\) are Bach-flat. Indeed, the signature formula implies
\[
\int_M ||W_g||^2 \, dv_g = 2 \int_M ||W_g^-||^2 \, dv_g + 12\pi^2 \tau(M)
\]  
(2.12)
for any Riemannian metric, where \(\tau(M)\) is the signature of \(M\). Hence, self-dual metrics achieve (global) minimum of the Weyl functional and thereby are critical points.

Einstein metrics \((Ric_g = \lambda g)\) are Bach-flat. Indeed, by Bianchi identities, the Bach tensor can be rewritten as
\[
B_{ij} = -\frac{1}{2} \Delta E_{ij} + \frac{1}{6} \nabla_i \nabla_j R - \frac{1}{24} \Delta R g_{ij} - E^{kl} W_{kijl} + E^k_i E^j_k - \frac{1}{4} |E|^2 g_{ij} + \frac{1}{6} R E_{ij},
\]
(2.13)
where \(E_{ij} = R_{ij} - \frac{R}{4} g_{ij}\) is the trace-free Ricci tensor. Then it follows from (2.13) that Einstein metrics are Bach-flat.

In four dimensions, Bach-flat manifolds have been considered as a generalization of Einstein manifolds; see [2]. It is a natural question to understand the relation between Bach-flat condition and Einstein condition. It is an interesting fact that a Bach-flat Kähler four-manifold is locally conformally Einstein near any point with nonzero scalar curvature. The following result is due to Derdziński in [9]:

**Proposition 2.1** [9] If \((M^4, g)\) is Kähler and Bach-flat, then \(\tilde{g} = R_g^{-2} g\) is Einstein near any point with \(R_g \neq 0\). Moreover, if \(R_g > 0\) on \(M^4\), then \((M^4, R_g^{-2} g)\) is an Einstein manifold.

Proposition 2.1 has proved to be fairly useful in constructing Einstein metrics on complex surfaces which don’t admit any Kähler-Einstein metric. In this direction, a remarkable example is the existence of conformally a Kähler, Einstein metric on \(\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}\) by Chen et al. in [8]. It is also known that the Einstein metric on \(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}\) constructed by Page in [16] can be reconstructed similarly. Note that these two Einstein metrics are both conformal to extremal Kähler metrics. Also note that \(S^2 \times S^2\) and \(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}\) are both simply-connected and satisfy \(b_2^+ = b_2^- = 1\). An interesting problem is to distinguish \((S^2 \times S^2, g_{prod})\) and \((\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}, g_{Page})\) with conditions on curvature. Theorem B provides a method by the range of \(\beta\).

Next we explain the interaction between \(\beta(M^4, [g])\) and the Bach-flat condition. As mentioned in the preceding subsection, \(\beta(M^4, [g])\) is of particular importance in studying the geometry and topology of a smooth four-manifold. It is natural to ask whether a stronger form of rigidity holds if we assume some additional conformally invariant conditions on the metric. It turns out that \(\beta(M^4, [g])\) proves to be fairly useful to characterize canonical metrics on Bach-flat four-manifolds. The general philosophy is that the Bach-flat condition can be viewed as an elliptic system of the metric modulo the actions of diffeomorphism group and conformal group and \(\beta(M^4, [g])\) provides information of \(L^2\) norms of the curvature tensor. From there, techniques from elliptic theory can be used to prove rigidity results for Bach-flat four-manifolds. Theorems A and B can be viewed as concrete examples of this idea.

### 3 Proof of Theorem A

In this section, we prove Theorem A. The proof will be divided into several steps. The main strategy can be explained as follows. First, since all conditions are conformally invariant,
we shall choose a proper conformal representative to decode the information. Secondly, we establish the \( L^2 \) estimates which imply that the chosen conformal representative is “close” to \((\mathbb{CP}^2, g_{FS})\) in an \( L^2 \) sense if \( \epsilon_1 \) is chosen to be sufficiently small. Third, we apply Moser-type iteration argument to show that the conformal representative in fact is self-dual and Einstein with positive scalar curvature. Finally, we conclude that \((M^4, g)\) is conformally equivalent to \((\mathbb{CP}^2, g_{FS})\) by a theorem of Poon [17].

### 3.1 Weitzenböck formulas

The following two Weitzenböck identities proved in [4,5] are of importance for our proof of Theorem A.

**Lemma 3.1** [4,5] If \((M^4, g)\) is Bach-flat, then

\[
\int_M \left( 3 \left( |\nabla E|^2 - \frac{1}{12} |\nabla R|^2 \right) + 6tr E^3 + R|E|^2 - 6W_{ijkl}E_{ik}E_{jl} \right) \, dv = 0,
\]

where \( tr E^3 = E_{ij}E_{ik}E_{jk} \), and

\[
\int_M |\nabla W^\pm|^2 \, dv = \int_M \left( 72 \det W^\pm - \frac{1}{2} R|W^\pm|^2 + 2W_{ijkl}^\pm E_{ik}E_{jl} \right) \, dv.
\]

**Remark 3.1** It is clear that Ricci curvature and Weyl curvature are coupled in these identities. To apply both of them simultaneously, we shall choose a conformal representative that involves both Ricci curvature (or scalar curvature) and Weyl curvature.

### 3.2 Conformal representative

In this subsection, we explain the conformal representative which will be used in the proof of Theorem A, the modified Yamabe metric. The modified Yamabe metric was introduced by Gursky in [11].

Let \((M^4, g)\) be a Riemannian four-manifold. Define

\[
F^+_g = R_g - 2\sqrt{6}|W^+_g|,
\]

and

\[
\mathcal{L}_g = -6\Delta_g + R_g - 2\sqrt{6}|W^+_g|.
\]

The operator \( \mathcal{L}_g \) is a variant of conformal Laplacian that satisfies the following conformal transformation law:

\[
\mathcal{L}_{\tilde{g}} \phi = u^{-3} \mathcal{L}_g (\phi u),
\]

where \( \tilde{g} = u^2 g \in \mathfrak{g} \). In analogy to the Yamabe problem, we define the functional

\[
\hat{Y}_g [u] = \langle u, \mathcal{L}_g u \rangle_{L^2} / ||u||^2_{L^4},
\]

and the associated conformal invariant

\[
\hat{Y} (M^4, [g]) = \inf_{u \in W^{1,2}(M,g)} \hat{Y}_g [u].
\]
By the conformal transformation law of $\mathcal{L}_g$, the functional $u \to \widehat{Y}_g[u]$ is equivalent to the Riemannian functional
\[
\widehat{g} = u^2 g \to \text{vol}(\widehat{g})^{-\frac{1}{2}} \int F^+_{\widehat{g}} \, dv_{\widehat{g}}.
\] (3.8)

The motivation for introducing the modified Yamabe metric is explained in the following result (see Theorem 3.3 and Proposition 3.5 of [11]):

**Theorem 3.1** [11] (i) Suppose $M^4$ admits a metric $g$ with $F^+_g \geq 0$ on $M^4$ and $F^+_g > 0$ somewhere. Then $b^+_2(M^4) = 0$. (ii) If $b^+_2(M^4) > 0$, then $M^4$ admits a metric $g$ with $F^+_g \equiv 0$ if and only if $(M^4, g)$ is a Kähler manifold with non-negative scalar curvature. (iii) If $Y(M^4, [g]) > 0$ and $b^+_2(M^4) > 0$, then $\widehat{Y}(M^4, [g]) \leq 0$ and there is a metric $\widehat{g} = u^2 g$ such that
\[
F^+_g = R_{\widehat{g}} - 2\sqrt{6}||W^+_{\widehat{g}}|| \equiv \widehat{Y}(M, [g]) \leq 0
\] (3.9)
and
\[
\int R^2_{\widehat{g}} \, dv_{\widehat{g}} \leq 24 \int ||W^+_{\widehat{g}}||^2 \, dv_{\widehat{g}}.
\] (3.10)

Furthermore, equality is achieved if and only if $F^+_g \equiv 0$ and $R_{\widehat{g}} = 2\sqrt{6}||W^+_{\widehat{g}}|| \equiv \text{const.}$

**Remark 3.2** We shall call the metric realizing $\widehat{Y}(M, [g])$ the modified Yamabe metric and denote it by $g_m$. Note that the conditions in (iii) of Theorem 3.1 are satisfied by the conditions of both Theorems A and B.

It is clear that the construction of the modified Yamabe metric is parallel to that of the Yamabe metric, which implies the existence of metric with constant scalar curvature in a conformal class. The main difference is that the Yamabe metric only involves the trace part of the Riemannian curvature while the modified Yamabe metric involves the Weyl curvature. Moreover, the modified Yamabe metric is related to the Kähler metric as is shown in (ii) of Theorem 3.1. Note that the Fubini-Study metric is a Kähler-Einstein metric. Hence, it is natural to consider the modified Yamabe metric as the conformal representative.

### 3.3 $L^2$ estimates

Recall that the conformal invariant $\beta(M^4, [g])$ is defined in terms of the quotient of total integral of $\sigma_2$-curvature and $L^2$ norm of Weyl tensor. It follows that the $L^2$ estimates for curvature tensor can be readily obtained. Now we recall the following result established in [6].

**Lemma 3.2** [6] Let $(M^4, g)$ be an oriented, closed, Riemannian four-manifold with $b^+_2(M^4) > 0$ and
\[
\beta(M^4, [g]) = 4(1 + \epsilon)
\] (3.11)
for some $0 < \epsilon < 1$, then we have for the modified Yamabe metric $g_m \in [g]$
\[
\int ||W^+_{g_m}||^2 \, dv_{g_m} = \frac{6\epsilon}{2 + \epsilon} \pi^2,
\] (3.12)
\[
\int ||W^-_{g_m}||^2 \, dv_{g_m} = 12\pi^2 + \int ||W^-_{g_m}||^2 \, dv_{g_m},
\] (3.13)
\[ Y(M^4, [g_m]) \geq \frac{24\pi}{\sqrt{2+\epsilon}}, \quad (3.14) \]

\[ \int |E_{g_m}|^2 \, dv_{g_m} \leq 6 \int ||W^-||^2 \, dv_{g_m}, \quad (3.15) \]

\[ \frac{1}{12} \mu_+ Y \leq 3 \int ||W^-||^2 \, dv_{g_m}, \quad (3.16) \]

\[ \frac{1}{24} \int (R_{g_m} - \bar{R}_{g_m})^2 \, dv_{g_m} \leq 3 \int ||W^-||^2 \, dv_{g_m}, \quad (3.17) \]

where

\[ \bar{R}_{g_m} = \int R_{g_m} \, dv_{g_m} / \int dv_{g_m} \]

and \( \mu_+ = -\bar{Y}(M^4, [g]) \geq 0 \) is a constant.

\textbf{Remark 3.3} It is noteworthy that (3.12), (3.13) and (3.14) are all conformally invariant.

Recall that \((\mathbb{C}P^2, g_{FS})\) is self-dual \((W_{g_{FS}}^- \equiv 0)\) and Einstein with \(b_2^+(M) > 0\). Therefore, (3.12) and (3.15) show that \((M^4, g_m)\) is “close” to \((\mathbb{C}P^2, g_{FS})\) in an \(L^2\) sense.

### 3.4 Moser-type iteration argument

In general, for a function \(u\) of \(n\) variables satisfying an elliptic equation (or elliptic inequality), a Moser-type iteration argument will provide \(L^p\) estimates for \(p > n^2/2\) if the \(L^{n/2}\) norm is sufficiently small. In our discussion, we want to apply Moser-type iteration argument to \(||W^-||\). An elliptic inequality is derived from the Bach-flat condition and the \(L^2\) smallness is known from the preceding subsection [see (3.12)]. Our goal is to prove that \(W^- \equiv 0\) if \(\epsilon_1\) is chosen to be sufficiently small.

Before we start the iteration argument, we remark that we may assume \(b_2^+(M) = 1\) and \(b_2^-(M) = 0\) for \(\epsilon_1 < 1\) by using the following lemma established in [6]:

\textbf{Lemma 3.3} [6] \textit{Let} \(M^4\) \textit{be a closed, oriented four-manifold admitting a metric} \(g \in \mathcal{Y}^+_2(M^4)\) \textit{with}

\[ \beta(M^4, [g]) < 8. \]

\textit{If} \(b_2(M^4) > 0\), \textit{then (after possibly changing the orientation) \(b_2^+(M^4) = 1\) and \(b_2^-(M^4) = 0\).}

By abusing notation, from now on, metric in this subsection is the modified Yamabe metric in its conformal class with volume one. From Lemma 3.2, we have

\[ \int |E|^2 \, dv < c(\epsilon_1), \]

\[ \int ||W^-||^2 \, dv < c(\epsilon_1), \]

\[ \int (R - \bar{R})^2 \, dv < c(\epsilon_1), \]

\[ R - 2\sqrt{6}||W^+|| = -\mu_+, \]

where \(\mu_+ \geq 0\) is a constant and \(\mu_+ \rightarrow 0\) and \(c(\epsilon_1) \rightarrow 0\) as \(\epsilon_1 \rightarrow 0\). Note that \(\bar{R}\) is also bounded since the volume is normalized to be one. From now on, we denote by \(c(\epsilon)\) a constant limiting to 0 as \(\epsilon \rightarrow 0\) but \(c(\epsilon)\) may be different from line to line.
We first state an algebraic lemma which is of some independent interest.

**Lemma 3.4**

\[ W_{ijkl} E_{ik} E_{jl} \leq \frac{\sqrt{6}}{3} ||W^\pm|| |E|^2 \]  

(3.19)

**Proof** This lemma is probably known in the work of [5,6]. Here we only sketch the proof of this inequality. Recall the well-known decomposition of Singer-Thorpe:

\[ Riem = \left( \begin{array}{cc} W^+ + \frac{R}{12} Id & B^* \\ B & W^- + \frac{R}{12} Id \end{array} \right) \]  

(3.20)

Note the compositions satisfy

\[ BB^* : \Lambda_+^2 \rightarrow \Lambda_+^2, \quad B^* B : \Lambda_-^2 \rightarrow \Lambda_-^2 \]

Fix a point \( P \in M^4 \), and let \( \lambda_1^\pm \leq \lambda_2^\pm \leq \lambda_3^\pm \) denote the eigenvalues of \( W^\pm \), where \( W^\pm \) are interpreted as endomorphisms of \( \Lambda^2_\pm \). Also denote the eigenvalues of \( BB^* : \Lambda_+^2 \rightarrow \Lambda_+^2 \) by \( b_1^2 \leq b_2^2 \leq b_3^2 \), where \( 0 \leq b_1 \leq b_2 \leq b_3 \). From similar computations of [5], we have

\[ W^\pm_{ijkl} E_{ik} E_{jl} = 4 (W^+, BB^*)_{\Lambda_+^2}, \quad W^-_{ijkl} E_{ik} E_{jl} = 4 (W^-, B^* B)_{\Lambda_-^2} \]  

(3.21)

By Lemma 6 of [15], we have

\[ W^\pm_{ijkl} E_{ik} E_{jl} \leq 4 \sum_{i=1}^3 \lambda_i^\pm b_i^2 \]  

(3.22)

Recall from Lemma 4.2 of [5] that \( |E|^2 = 4 \sum_{i=1}^3 b_i^2 \). For a skew-symmetric \( 3 \times 3 \) matrix \( A \), we have the sharp inequality:

\[ |A(X, X)| \leq \frac{\sqrt{6}}{3} ||A|| |X|^2. \]  

(3.23)

Apply (3.23) to \( A = diag(\lambda_1^\pm, \lambda_2^\pm, \lambda_3^\pm) \) and \( X = (b_1, b_2, b_3) \). We derive

\[ 4 \sum_{i=1}^3 \lambda_i^\pm b_i^2 \leq \frac{\sqrt{6}}{3} ||W^\pm|| |E|^2. \]  

(3.24)

Combining (3.22) and (3.24), we derive the desired inequality. \( \Box \)

Now we finish the proof of Theorem A by establishing the following lemma.

**Lemma 3.5** For sufficiently small \( \epsilon_1 > 0 \), we have \( W^-_{gm} \equiv 0 \) and \( E_{gm} \equiv 0 \).

**Proof** The following basic estimates are from Cauchy-Schwartz inequality:

\[ \int |tr E^3| \, dv \leq \left( \int |E|^2 \, dv \right)^{1/2} \left( \int |E|^4 \, dv \right)^{1/2} \leq c(\epsilon_1) \left( \int |E|^4 \, dv \right)^{1/2}, \]  

(3.25)

\[ \int W^-_{ijkl} E_{ik} E_{jl} \, dv \leq \left( \int ||W^-||^2 \, dv \right)^{1/2} \left( \int |E|^4 \, dv \right)^{1/2} \leq c(\epsilon_1) \left( \int |E|^4 \, dv \right)^{1/2}, \]  

(3.26)
\begin{equation}
\int W^+_{i j k l} E_{i k} E_{j l} \, d v \leq \int \left( \frac{\sqrt{6}}{3} \| W^+ \| - \frac{R}{6} \right) |E|^2 \, d v + \int \frac{R}{6} |E|^2 \, d v
\end{equation}
(3.27)

\begin{equation}
= \frac{\mu_+}{6} \int |E|^2 \, d v + \int \frac{R}{6} |E|^2 \, d v
\end{equation}

\begin{equation}
|\int R |E|^2 \, d v| = \left| \int (R - \bar{R}) |E|^2 \, d v + \bar{R} \int |E|^2 \, d v \right|
\end{equation}

\begin{equation}
\leq \left( \int (R - \bar{R})^2 \, d v \right)^{1/2} \left( \int |E|^4 \, d v \right)^{1/2} + C \int |E|^2 \, d v.
\end{equation}
(3.28)

\begin{equation}
\leq c(\epsilon_1) \left( \int |E|^4 \, d v \right)^{1/2} + C \int |E|^2 \, d v.
\end{equation}

Now apply conformally invariant Sobolev inequality

\begin{equation}
\frac{\sqrt{6}}{6} \left( \int |E|^4 \, d v \right)^{1/2} \leq \int \left( \nabla |E|^2 + \frac{R}{6} |E|^2 \right) \, d v
\end{equation}
(3.29)

to (3.1). We obtain

\begin{equation}
\frac{\sqrt{6}}{6} \left( \int |E|^4 \, d v \right)^{1/2} \leq \frac{1}{12} \int |\nabla R|^2 \, d v - \int 2 \text{tr} E^3 \, d v - \int \frac{R}{6} |E|^2 \, d v
\end{equation}

\begin{equation}
+ \int 2 W^+_{i j k l} E_{i k} E_{j l} \, d v
\end{equation}

\begin{equation}
\leq c(\epsilon_1) \left( \int |E|^4 \, d v \right)^{1/2} + \frac{1}{12} \int |\nabla R|^2 \, d v + C \int |E|^2 \, d v,
\end{equation}
(3.30)

where we have used (3.25), (3.26), (3.27) and (3.28) in the last inequality.

Since \( R - 2 \sqrt{6} \| W^+ \| = -\mu_+ \) is a constant, we differentiate both sides and integrate over the manifold to obtain by Kato inequality

\begin{equation}
\int |\nabla R|^2 \, d v = 24 \int |\nabla W^+|^2 \, d v \leq C \int |\nabla W^+|^2 \, d v.
\end{equation}
(3.31)

Recall the sharp estimate for skew-symmetric matrix:

\begin{equation}
72 \det W^+ \leq 72 \cdot \frac{\sqrt{6}}{18} \cdot \frac{1}{8} \| W^+ \|^3 = \frac{\sqrt{6}}{2} \| W^+ \|^3 = \frac{R + \mu_+}{2} \| W^+ \|^2,
\end{equation}
(3.32)

where we have used \( R - 2 \sqrt{6} \| W^+ \| = -\mu_+ \) in the last equality.

Now apply (3.27), (3.28), (3.32) to (3.2). We easily derive

\begin{equation}
\int |\nabla W^+|^2 \, d v \leq \frac{\mu_+}{2} \int |W^+|^2 \, d v + C \int |E|^2 \, d v + c(\epsilon_1) \left( \int |E|^4 \, d v \right)^{1/2}
\end{equation}
(3.33)

Combining (3.30), (3.31), (3.33), we derive

\begin{equation}
\frac{\sqrt{6}}{6} \left( \int |E|^4 \, d v \right)^{1/2} \leq C\mu_+ \int |W^+|^2 \, d v + c(\epsilon_1) \left( \int |E|^4 \, d v \right)^{1/2} + C \int |E|^2 \, d v
\end{equation}
(3.34)

From Lemma 3.2 and the fact that \( c(\epsilon_1) \to 0 \) as \( \epsilon_1 \to 0 \), we can take sufficiently small \( \epsilon_1 \) to absorb the second term on the right hand side of (3.34) and obtain from (3.13), (3.14),
\( (3.15), (3.16) \)
\[
\left( \int |E|^4 \, dv \right)^{1/2} \leq C \int ||W^-||^2 \, dv. \tag{3.35}
\]

It is now easy to obtain from (3.2):
\[
\frac{Y}{6} \left( \int |W^-|^4 \, dv \right)^{1/2} \leq \int \left( |\nabla W^-|^2 + \frac{R}{6}|W^-|^2 \right) \, dv
- \int \left( \frac{R}{3} - \tilde{R} \right) |W^-|^2 \, dv - \frac{\tilde{R}}{3} \int |W^-|^2 \, dv
\leq c(\epsilon_1) \left( \int |W^-|^4 \, dv \right)^{1/2} + c(\epsilon_1) \left( \int |E|^4 \, dv \right)^{1/2}
- \frac{\tilde{R}}{3} \int |W^-|^2 \, dv
\leq c(\epsilon_1) \left( \int |W^-|^4 \, dv \right)^{1/2} + \left( c(\epsilon_1) - \frac{\tilde{R}}{3} \right) \int |W^-|^2 \, dv \tag{3.36}
\]

The first inequality follows from conformally invariant Sobolev inequality. The second inequality is from Cauchy–Schwartz inequality and (3.18). The third inequality is from (3.35). Recall \( c(\epsilon_1) \to 0 \) as \( \epsilon_1 \to 0 \) and
\[
\tilde{R} \geq Y \geq \frac{24\pi}{\sqrt{2} + \epsilon_1}.
\]

It is clear that we may take sufficiently small \( \epsilon_1 \) to obtain \( W^-_{gm} \equiv 0 \) from (3.36). Moreover, it follows from (3.15) that \( \bar{E}_{gm} \equiv 0 \).

Theorem A can be established now easily. Note that \((M^4, \bar{g}_m)\) is a self-dual Einstein four-manifold with positive scalar curvature. Recall that \( b^+_2(M^4) = 1 \) and \( b^-_2(M^4) = 0 \). Hence, by Theorem A of [17] or Lemma 3.5 of [6], \((M^4, \bar{g})\) is conformally equivalent to \((\mathbb{C}P^2, g_{FS})\).

### 4 Proof of Theorem B

In this section, we prove Theorem B. The proof will be divided into several steps. The main strategy can be explained as follows. First, we establish the \( L^2 \) estimates for the modified Yamabe metric. Second, we recall the compactness and convergence theory of Bach-flat metrics established by Tian and Viaclovsky in [19,20] and explain how it works in our setting. Finally, we argue by contradiction and finish the proof.

#### 4.1 \( L^2 \) estimates

Similar to the proof of Theorem A, we shall first decode the information from \( \beta(M^4, [g]) \) by establishing \( L^2 \) estimates for the modified Yamabe metric. We now prove a lemma which is similar to Lemma 3.2 under the pinching condition of Theorem B.
Lemma 4.1  Let \((M^4, g)\) be a closed, compact oriented Riemannian four-manifold with \(b_2^+(M^4) = b_2^-(M) > 0\) and
\[
\beta(M^4, [g]) = 8(1 + \epsilon) \quad (4.1)
\]
for sufficiently small \(\epsilon > 0\), then we have for the modified Yamabe metric \(g_m \in [g] \)
\[
\int |W^+_{g_m}|^2 dv_{g_m} = \int |W^-_{g_m}|^2 dv_{g_m} = \left( \frac{1 + \epsilon}{3 + 2\epsilon} \right) 32\pi^2, \quad (4.2)
\]
\[
Y(M^4, [g]) \geq 16\pi \sqrt{\frac{3}{3 + 2\epsilon}} \quad (4.3)
\]
\[
\int |E_{g_m}|^2 dv_{g_m} \leq \frac{64\epsilon}{3 + 2\epsilon} 32\pi^2. \quad (4.4)
\]

**Proof**  From Corollary F of [10], \(g \in \mathcal{Y}^+(M)\) implies that \(b_1(M) = 0\). Therefore, \(\chi = 2 + b_2(M) = 2 + 2b_2^+(M)\). Then Chern–Gauss–Bonnet and signature formulas imply
\[
4 \int \sigma_2(P) dv + \int ||W||^2 dv = 8\pi^2(2 + 2b_2^+(M)), \quad (4.5)
\]
\[
\int ||W^+||^2 dv = \int ||W^-||^2 dv. \quad (4.6)
\]
The condition \(\beta(M, [g]) = 8(1 + \epsilon)\) reads
\[
4 \int \sigma_2(P) dv = \frac{1}{2(1 + \epsilon)} \int ||W||^2 dv. \quad (4.7)
\]
From (4.5), (4.6), (4.7), we solve
\[
\int \sigma_2(P) dv = \frac{4\pi^2}{3 + 2\epsilon} (1 + b_2^+(M)), \quad (4.8)
\]
\[
\int ||W^\pm||^2 dv = \frac{(1 + \epsilon) 16\pi^2}{3 + 2\epsilon} (1 + b_2^+(M)).
\]

We first show that \(b_2^+(M) = 1\) if \(\epsilon\) is chosen sufficiently small. We now argue by contradiction. Suppose \(b_2^+(M) \geq 2\). Then from (4.8)
\[
\int \sigma_2(P) dv \geq \frac{12\pi^2}{3 + 2\epsilon}.
\]

Theorem A in [7] (see also [14]) then implies \((M, g)\) is conformally equivalent to \((S^4, g_{S^4})\) if \(\epsilon\) is chosen sufficiently small. This contradicts to the fact \(b_2^+(M) > 0\). Hence, \(b_2^+(M) = 1\) and \(\chi(M) = 4\). Now we have from (4.8)
\[
\int \sigma_2(P) dv = \frac{8\pi^2}{3 + 2\epsilon}, \quad \int ||W^\pm||^2 dv = \left( \frac{1 + \epsilon}{3 + 2\epsilon} \right) 32\pi^2. \quad (4.9)
\]

With same argument from the proof of Lemma 2.7 in [6], we derive
\[
\frac{1}{96} Y^2 \geq \int \sigma_2(P) dv \geq \frac{8\pi^2}{3 + 2\epsilon}. \quad (4.10)
\]
Now we have proved (4.2), (4.3). Note that these two inequalities are conformally invariant.
We now choose the modified Yamabe metric \( g_m \in [g] \) and Theorem 3.1 implies
\[
\frac{1}{24} \int R^2_{g_m} \, dv_{g_m} \leq \int ||W^+_{g_m}||^2 \, dv_{g_m} = \left( \frac{1 + \epsilon}{3 + 2\epsilon} \right) 32\pi^2. \tag{4.11}
\]
Recall by definition and conformal invariance of \( \int \sigma^2(P) \, dv \)
\[
\int \sigma^2(P) \, dv = \frac{1}{96} \int R^2_{g_m} \, dv_{g_m} - \frac{1}{8} \int |E_{g_m}|^2 \, dv_{g_m}.
\]
From (4.9), (4.11) we easily derive (4.4). \( \square \)

### 4.2 Convergence theory of Bach-flat metrics

The convergence theory of Bach-flat metrics has been developed by Tian and Viaclovsky in [19,20]. The most important ingredients are the following \( \epsilon \)-regularity theorem and volume estimate.

**Theorem 4.1** [19,20] Let \((M^4, g)\) be a Bach-flat Riemannian four-manifold with Yamabe constant \( Y > 0 \) and \( g \) is a Yamabe metric in \([g]\). Then there exists positive numbers \( \tau_k \) and \( C_k \) depending on \( Y \) such that, for each geodesic ball \( B_{2r}(p) \) centered at \( p \in M \), if
\[
\int_{B_{2r}(p)} |Rm|^2 \, dv \leq \tau_k, \tag{4.12}
\]
then
\[
\sup_{B_r(p)} |\nabla^k Rm| \leq \frac{C_k}{r^{2+k}} \left( \int_{B_{2r}(p)} |Rm|^2 \, dv \right)^{\frac{1}{2}}. \tag{4.13}
\]

**Theorem 4.2** [19,20] Let \((X, g)\) be a complete, non-compact, 4-manifold with base point \( p \), and let \( r(x) = d(p, x) \), for \( x \in X \). Assume that there exists a constant \( v_0 > 0 \) such that
\[
\text{vol}(B_r(q)) \geq v_0 r^4 \tag{4.14}
\]
holds for all \( q \in X \), assume furthermore that as \( r \to \infty \),
\[
\sup_{S(r)} |Rm_R| = o(r^{-2}), \tag{4.15}
\]
where \( S(r) = \partial B_r(p) \). Assume that the first Betti number \( b_1(X) < \infty \), then \((X, g)\) is an ALE space, and there exists a constant \( v_1 \) (depending on \((X, g))\) so that
\[
\text{vol}(B_r(p)) \leq v_1 r^4. \tag{4.16}
\]

Note that these two theorems apply to Yamabe metrics instead of modified Yamabe metrics. In order to apply them, we need \( L^2 \) estimates for Yamabe metric. This can be done without difficulty. It is easy to see that only (4.4) in Lemma 4.1 is not conformally invariant. However, it is not hard to derive the \( L^2 \) estimates for \( E_{g_Y} \) since the Yamabe metric has “minimizing” property in an \( L^2 \) sense. This argument will be discussed in detail at the beginning of next subsection.
4.3 Completion of the proof

We now finish the proof of Theorem B. Note that (4.2) and (4.3) are both conformally invariant. In addition, recall that
\[
\int \sigma_2(P) \, dv = \frac{1}{96} \int R^2 \, dv - \frac{1}{8} \int |E|^2 \, dv
\]
is conformally invariant and the Yamabe metric realizing the infimum of \( \int R^2 \, dv \) in a conformal class. It then follows from (4.4) that
\[
\int |E_{g_Y}|^2 \, dv_{g_Y} \leq \int |E_{g_m}|^2 \, dv_{g_m} \leq \frac{64\epsilon}{3 + 2\epsilon^2} \pi^2.
\]
(4.17)

We now argue by contradiction. Assume that there exists a sequence of Bach-flat 4-manifolds \((M^4_j, (g_Y)_j)\) which are not conformally equivalent to \((S^2 \times S^2, g_{\text{prod}})\) with \(\epsilon_j \to 0\), where \((g_Y)_j\) denotes the Yamabe metric with \(R_{(g_Y)_j} = 12\). By (4.3), (4.17), we have
\[
Y(M_j, (g_Y)_j) \geq 16\pi \sqrt{\frac{3}{3 + 2\epsilon_j}} \to 16\pi.
\]
(4.18)
\[
\int_{M_j} |E_{(g_Y)_j}|^2 \, dv_{(g_Y)_j} \to 0.
\]
(4.19)

We first prove that there is no point of curvature concentration. To this end, we shall make use of the following lemma to distinguish Euclidean space from other complete non-compact Ricci flat Riemannian manifolds by the comparison of their Yamabe constants. The lemma was proved by Li et al. in [14].

**Lemma 4.2** [14] Suppose \((M^4, g)\) is a complete non-compact Ricci flat Riemannian manifold. Then \((M^4, g)\) is isometric to the Euclidean space \((\mathbb{R}^4, g_E)\) if
\[
Y(M, g) \geq \eta Y(\mathbb{R}^4, g_E)
\]
for some \(\eta > \sqrt{\frac{2}{3}}\).

**Remark 4.1** This lemma shows that there is a gap phenomenon for the Yamabe constants of non-compact Ricci flat Riemannian four-manifolds.

We now argue by contradiction. Note that a uniform lower bound on the Yamabe constants implies a uniform lower bound on the Euclidean volume growth for such sequence of manifolds. Then standing at the point of curvature blow-up, that is, \(p_j \in M_j\) such that
\[
\lambda_j = |Rm_{(g_Y)_j}|(p_j) = \max |Rm_{(g_Y)_j}| \to \infty.
\]
(4.21)

Now consider the sequence of pointed Riemannian manifold \((M_j, g_j, p_j)\) with the re-scaled metric \(g_j = \lambda_j^2 (g_Y)_j\). Therefore, according to the curvature estimates established in [20], there exists a subsequence of \((M_j, g_j, p_j)\) which converges to a complete non-compact manifold \((M_\infty, g_\infty, p_\infty)\) in the Cheeger–Gromov topology. It follows that

- \(Y(M_\infty, g_\infty) \geq 16\pi\) and
- \(\text{Ric}(g_\infty) = 0\).

Here we use a similar argument as in the proof of Lemma 3.4 of [14] to prove \(Y(M_\infty, g_\infty) \geq 16\pi\). Note that
\[
Y(\mathbb{R}^4, g_E) = Y(S^4, g_{S^4}) = 8\sqrt{6}\pi.
\]
(4.22)
Hence,

\[ Y(M_\infty, g_\infty) \geq \frac{\sqrt{6}}{3} Y(\mathbb{R}^4, g_E). \]  

(4.23)

Note \( \frac{\sqrt{6}}{3} > \sqrt{2} \). Hence, Lemma 4.2 implies that \((M_\infty, g_\infty)\) is isometric to Euclidean 4-space, which is a contradiction to the fact that \(|Rm_{g_\infty}|(p_\infty) = 1\). Therefore, there is no point of concentration of curvature.

The convergence theory developed in [19,20] implies that \((M^j, (g_Y)_j)\) converges in \(C^\infty\)-norms to a smooth Einstein manifold \((\widetilde{M}, \widetilde{g})\) satisfying \(\beta(\widetilde{M}, [\widetilde{g}]) = 8\) and \(b_2^+(\widetilde{M}) = 1\). Hence, from Theorem C of [6], \((\widetilde{M}, \widetilde{g})\) is (up to a rescaling) isometric to \((S^2 \times S^2, g_{\text{prod}})\). Recall that \((S^2 \times S^2, g_{\text{prod}})\) is Bach-rigid in the sense that the metric is an isolated Bach-flat metric (see Theorem 7.13 of [12]). It follows that \((M^j, (g_Y)_j)\) must be conformally equivalent to \((S^2 \times S^2, g_{\text{prod}})\) for sufficiently large \(j\), which is a contradiction. Therefore, the proof of Theorem B is complete.

**Remark 4.2** It is not hard to see that we may prove Theorem A with similar argument as we did in the proof of Theorem B. The current proof of Theorem A has two advantages: first, it avoids making use of the general theory of convergence for Bach-flat metrics; second, it is possible to compute a numerical constant of \(\epsilon_1\) if we track the coefficients carefully.

**Remark 4.3** Although it is not possible to get a numerical constant for \(\epsilon_2\) from the current proof, we still know that \(\epsilon_2\) cannot be too large. On \(N = \mathbb{C}P^2#\overline{\mathbb{C}P}^2\), there is an Einstein metric (which is Bach-flat) \(g_{\text{Page}}\) constructed by Page in [16] satisfying

\[ \beta(N, [g_{\text{Page}}]) = 8.4 \ldots. \]

It follows that \(\epsilon_2 < \frac{1}{16}\).

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