COEXISTENCE OF BOUNDED AND UNBOUNDED GEOMETRY FOR AREA-PRESERVING MAPS

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Abstract. The geometry of the period doubling Cantor sets of strongly dissipative infinitely renormalizable Hénon-like maps has been shown to be unbounded by M. Lyubich, M. Martens and A. de Carvalho, although the measure of unbounded “spots” in the Cantor set has been demonstrated to be zero.

We show that an even more extreme situation takes place for infinitely renormalizable area-preserving Hénon-like maps: bounded and unbounded geometries coexist with both phenomena occurring on subsets of positive measure in the Cantor sets.

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1. Introduction

The period doubling cascade is one of the fundamental scenarios of transition from periodic to chaotic dynamics in one-dimensional systems. This cascade accumulates on a dynamical system that admits an attracting invariant Cantor set. The properties of these period doubling Cantor sets are very well understood. In the late 1970’s Feigenbaum [F1, F2] and, independently, Coulet and Tresser [CT, TC], discovered numerically universal geometric properties of these Cantor sets.

M. Feigenbaum, P. Coulet and C. Tresser introduced renormalization in dynamics to explain the observed geometrical universality.

Date: July 31, 2018.
Renormalization, viewed as an operator on a class of dynamical systems, maps one such system into another one which corresponds to a rescaled version of a higher iteration of the original system acting on a subset of its phase space. This renormalization operator typically has a hyperbolic horseshoe; the dynamics of the systems on the stable leaves converges to an orbit (which can be periodic), and the behavior of the renormalization operator around this orbit determines the asymptotic small scale properties. This explains the observed universality.

The renormalization technique has been generalized to many types of dynamics. However, a rigorous study of universality has been surprisingly difficult and technically sophisticated. It has only been thoroughly carried out in the case of one-dimensional maps, on the interval or the circle, see [AL, FMP, He, L, Ma, McM, MS, S, VSK, Y].

One of the strong properties of infinitely renormalizable maps is rigidity: asymptotically, on smaller and smaller scales, there is a universal geometry around the invariant Cantor set. All period doubling Cantor sets are topologically equivalent, but a priori there is no reason to believe that these conjugating homeomorphisms can have some smoothness. However, it is well-known that the period doubling Cantor sets in one-dimensional dynamics are rigid: there are smooth conjugations.

Many numerical and physical experiments show that exactly the same universal geometry from one-dimensional dynamics occurs also in some dissipative higher dimensional systems. Surprisingly, the rigidity phenomenon is more delicate in higher dimensions.

Recently, one-dimensional techniques have been extended to strongly dissipative perturbations of one-dimensional systems, such as Hénon maps, , see [CEK1, CLM, LM1, LM2, Haz, HLM]. Strongly dissipative two-dimensional Hénon-like maps can be thought of as two-dimensional perturbations of one-dimensional systems. In [CEK1] and [CLM] two renormalization schemes were developed for strongly dissipative Hénon-like maps at the accumulation of period doubling which explain the universal geometry present in these Hénon-like maps. Surprisingly, the period doubling Hénon-like Cantor sets are not smoothly conjugated to their one-dimensional counterpart, see [CLM, LM1].

Nevertheless, the geometry of the one-dimensional period doubling Cantor set is still present. The conjugations between the Cantor sets of strongly dissipative Hénon-like maps is almost everywhere, with respect to the natural measure on the Cantor set, smooth. This phenomenon is called Probabilistic Rigidity, see [LM1, LM3]. Small scale geometry has a probabilistic nature in higher dimensions.
We will say that a Cantor set has bounded geometry, if the distance between neighboring sets in the \( n \)-th generation cover is commensurable to their diameters. \[ \text{[LM1]} \] and \[ \text{[LM3]} \] demonstrate that the unbounded geometry is present in the Cantor sets of strongly dissipative Hénon-like maps, but the measure of pieces in the \( n \)-th cover with unbounded geometry disappears as \( n \) increases.

The other extreme case is that of the area-preserving maps. Area-preserving maps at accumulation of period doubling are observed by several authors in the early 80’s, see \[ \text{[DP, HI, BCGG, Bo, CEK2, EKW1]} \]. In \[ \text{[EKW2]} \] Eckmann, Koch and Wittwer introduced a period doubling renormalization scheme for area preserving maps and described the hyperbolic behavior of the renormalization operator in a neighborhood of a renormalization fixed point. In particular, they observed universality for maps at the accumulation of period doubling.

It was shown in \[ \text{[GJ1]} \] that the maps in this Eckmann-Koch-Wittwer universality class do have a period doubling Cantor set and the Lyapunov exponents of dynamics restricted to this Cantor set are zero. \[ \text{[GJM]} \] later demonstrated that for maps in a certain strong stable manifold of the renormalization fixed point, a manifold with finite codimension, their period doubling Cantor sets are rigidly conjugate by a \( C^{1+\alpha} \) coordinate change with \( \alpha \geq 0.237 \):

\[ \textbf{Rigidity for Area-preserving Maps.} \] The period doubling Cantor sets of area-preserving maps in the universality class of the Eckman-Koch-Wittwer renormalization fixed point are smoothly conjugate.

In this paper we consider the geometry of the period doubling Cantor sets for area-preserving infinitely renormalizable maps and demonstrate that the situation is more complicated than for the dissipative ones. All \( n \)-th level covers contain subsets of pieces with both bounded and unbounded geometry, the measure of these subsets stays positive as \( n \) increases:

\[ \textbf{Coexistence of Bounded and Unbounded Geometry.} \] The period doubling Cantor sets of area-preserving maps in the universality class of the Eckman-Koch-Wittwer have bounded and unbounded geometry both on subsets of positive measure.

The exact statements about the measure of bounded and unbounded pieces are contained in Theorems A, B and C.
2. Preliminaries

Given a domain $D \subset \mathbb{C}^2$, let $D \subset \text{int}(D \cap \mathbb{R}^2)$ be compactly contained in the real slice. Assume $(0,0) \in D$. An area-preserving map $F : D \to F(D) \subset \mathbb{R}^2$ will mean a real analytic map which has a holomorphic extension to $D$, continuous on the boundary of $D$, and is an exact symplectic diffeomorphism onto its image with the following properties

1) $T \circ F \circ T = F^{-1}$, where $T(x,u) = (x,-u)$ (reversibility),
2) $\partial u y \neq 0$ with $(y,v) = F(x,u)$ (twist condition).

The collection of such maps is denoted by $\text{Cons}(D)$. It has been shown in [EKW2] that the set $\text{Cons}(D)$ can be identified with a Banach space $A(D_s)$ of real symmetric functions $s : D_s \subset \mathbb{C}^2 \mapsto \mathbb{C}$ holomorphic on some domain $D_s$, continuous on the boundary of $D_s$. Specifically, $F \in \text{Cons}(D)$ is generated by $s$:

$$
(2.1) \quad \left( \begin{array}{c} x \\ -s(y,x) \end{array} \right) \mapsto \left( \begin{array}{c} y \\ s(x,y) \end{array} \right).
$$

In [GJ2] Gaidashev and Johnson construct simply connected domain $D_s \subset \mathbb{C}^2$ and $D \subset \mathbb{C}$, and adapt the renormalization scheme from [EKW2]. This renormalization scheme is defined on a neighborhood $B$ of $s^* \in A(D_s)$, where $s^*$ corresponds to the Eckman-Koch-Wittwer fixed point. There are analytic functions

$$
F \mapsto \lambda_F \in (-\infty,0)
$$

and

$$
F \mapsto \mu_F \in (0,\infty),
$$
called the rescaling, which are used to renormalize $F$ (or $s$). The renormalization operator $R$ is defined by

$$
RF = \Lambda_F^{-1} \circ F \circ F \circ \Lambda_F, \quad \text{where} \quad \Lambda_F : (x,u) \mapsto (\lambda_F x, \mu_F u).
$$

At the level of the generating functions, the renormalization operator $\mathcal{R} : \mathcal{B} \subset A(D_s) \mapsto A(D_s)$ is defined as follows (see [GJ2]):

$$
(2.2) \quad \mathcal{R}[s](x,y) = \mu^{-1} s(z(x,y), \lambda y),
$$

where $z$ is the unique symmetric ($z(x,y) = z(y,x)$) solution of

$$
(2.3) \quad s(\lambda x, z(x,y)) + s(\lambda y, z(x,y)) = 0,
$$

and $\lambda$ and $\mu$ are fixed by the normalization conditions $\mathcal{R}[s](1,0) = 0$, which implies $z(1,0) = z(0,1) = 1$, and $\partial_1 \mathcal{R}[s](1,0) = 1$ which implies $\mu = \partial_1 z(1,0)$. 

The results from [GJ2] which will be used in the sequel are collected in the following Theorem, Lemma, and Proposition. All proofs in [GJ2] are done for the operator $R$ in the neighborhood $B$ in the space $A(D_s)$, however, it is proved in [GJ2] that the map

\[ I : s \mapsto (y_s, s \circ h_s), \]

where $y(s)$ is the unique solution of $0 = u + s(y, x)$, and $(h_s(x, u) = (x, y_s(x, u))$, is an analytic embedding, and the set $\mathcal{F} := I(B) \subset \operatorname{Cons}(D)$ is a Banach submanifold of the space $O(D)$ of functions $F : D \mapsto \mathbb{C}^2$ holomorphic on $D$, continuous on $\partial D$. The following is a Theorem from [GJ2], reformulated for the submanifold $\mathcal{F}$.

**Theorem 2.1.** There exists $F_* \in \mathcal{F}$ such that

1) $F_*$ is a hyperbolic fixed point of the renormalization operator $R : \mathcal{F} \mapsto \operatorname{Cons}(D)$.
2) $F_*$ has a two-dimensional unstable manifold in $\mathcal{F}$.
3) $F_*$ has a codimension two stable manifold $W^s(F_*)$ in $\mathcal{F}$.
4) $F_*$ has a codimension three strong stable manifold $W^{ss}(F_*) \subset W^s(F_*)$.
6) There exist a distance function $d$ on $W^{ss}(F_*)$, and $\nu < 0.126$ such that for every $F, \tilde{F} \in W^{ss}(F_*)$

\[ d(RF, R\tilde{F}) \leq \nu \cdot d(F, \tilde{F}). \]

In particular,

\[ d(R^n F, F_*) \leq \nu^n \cdot d(F, F_*). \]

7) The one dimensional family defined by $F_t = \phi_t^{-1} \circ F_* \circ \phi_t$, where $\phi_t : D \mapsto \phi_t(D) \subset \mathbb{R}^2$ is the diffeomorphism defined by

\[ \phi_t(x, u) = (x + tx^2, \frac{u}{1 + 2tx}), \]

for $|t|$ sufficiently small, is contained in the stable manifold $W^s(F_*)$ and is transversal to the strong stable manifold $W^{ss}(F_*) \subset W^s(F_*)$ and intersects only in $F_*$.

8) The map $\Lambda_F$ depends analytically on $F$.

**2.1. The ratchet phenomenon.** Twist maps have a property called the *ratchet phenomenon*. It means that for any twist map $F(x, u) = (y, v)$ satisfying

\[ \frac{\partial y}{\partial u} \geq a > 0 \]

there are horizontal cones $\Theta_h$ and vertical cones $\Theta_v$ such that if $p' \in p + \Theta_v$ then $F(p') \in F(p) + \Theta_h$ and that the angle of the cones depend
only on $a$, see e.g. Lemma 12.1 of \cite{Go}. The same is true for negative twist maps with
\[
\frac{\partial y}{\partial u} \leq a < 0.
\]
More precisely a positive twist map maps any point $p'$ in the half cone $p + \Theta^+_v$ into the half cone $F(p) + \Theta^+_v$ and a negative twist map maps any point $p'$ in $p + \Theta^-_v$ into $F(p) + \Theta^-_v$.

![Diagram](Figure 2.1. The ratchet phenomenon for a positive twist map. A negative twist map reverses the signs.)

3. The Cantor set

In this section we recall the construction of the invariant Cantor set for infinitely renormalizable maps. As in dimension one, it is a Cantor set on which the map acts like the dyadic adding machine. The construction is done via an iterated function system.

We will use the notation
\[
\psi_0^n = \Lambda_{R^{n-1}F} : D \to \mathbb{R}^2
\]
and
\[
\psi_1^n = R^{n-1}F \circ \Lambda_{R^{n-1}F} : D \to \mathbb{R}^2,
\]
here, $D$ denotes the real slice of $\mathcal{D}$. Observe,
\[
R^nF = (\psi_0^n)^{-1} \circ R^{n-1}F \circ R^{n-1}F \circ \psi_0^n.
\]
The convergence of $R^nF \to F_*$ and Theorem \ref{Thm2.1}(6) immediately implies
Lemma 3.1. For every $F \in W^s(F_*)$ and any $k \geq 0$,

$$\lim_{n \to \infty} ||\psi_n^{1}(F) - \psi_{0,1}(F_*)||_{C^k} = 0.$$

The above Lemma shows a crucial difference between conservative and dissipative infinitely renormalizable maps: in the conservative case, the rescaling maps converge to a diffeomorphism. In the dissipative case, the corresponding rescaling $\psi_1^n$ converge to a degenerate map, a map with one-dimensional image.

The construction of the Cantor set in the conservative case is exactly the same as in the dissipative case. The difference lies in the asymptotic behavior of the rescalings.

Let

$$\Psi^{20} = \psi_0^1 \circ \psi_0^2, \quad \Psi^{21} = \psi_0^1 \circ \psi_1^2, \quad \Psi^{21} = \psi_1^1 \circ \psi_0^2, \quad \ldots \quad \Psi^{n} = \psi_{w^n}^1 \circ \cdots \circ \psi_{w^1}^1 : \text{Dom}(R^nF) \to \text{Dom}(F).$$

and, proceeding this way, construct, for any $w = (w_1, \ldots, w_n) \in \{0,1\}^n$, $n \geq 1$, the maps $\Psi^n_w(F)$ will also be used to emphasize the map under consideration.

The transformations $\Psi^n_w$ will be referred to as the renormalization microscope.

Lemma 3.2. For $F \in \mathcal{F}$ there are analytically defined simply connected domains $B_0(F) \subset D$ and $B_1(F) \subset D$ such that

\begin{equation}
B_1(F) \cap B_0(F) = \emptyset,
\end{equation}

and

\begin{equation}
F^2(B_0(F)) \cap B_0(F) \neq \emptyset.
\end{equation}

Moreover,

$$\psi_i(B_0(RF) \cup B_1(RF)) \subset B_i(F),$$

with $i \in \{0,1\}$.

Additionally, the sets $B_0(F)$ and $B_1(F)$ contain a period 2 hyperbolic orbit of $F$, $p_0^F \in B_0(F), p_1^F = F(p_0^F) \in B_1(F)$, which is the unique such orbit in $B_0(F) \cup B_1(F)$.

Proposition 3.3. There exists a neighborhood $U$ of $F_*$ and $0 < \theta_1 < \theta_2 < 1$ such that for $F \in W^s_{\text{loc}}(F_*)$,

\begin{equation}
W^s_{\text{loc}}(F_*) = W^s(F_*) \cap U
\end{equation}

we have

$$\theta_1^4 \cdot |v| \leq |D\Psi^4_w(x,u)v| \leq \theta_2^4 \cdot |v|$$
for every $w \in \{0, 1\}^4$ and $(x, u) \in B_0(F) \cup B_1(F)$. Moreover,

\begin{equation}
\frac{\theta_2 \nu}{\theta_1} < 1.
\end{equation}

\textbf{Remark 3.1.} The following estimates are obtained in \cite{GJ2}.

\begin{equation}
\theta_1 \geq 0.061
\end{equation}

\begin{equation}
\theta_2 \leq 0.249
\end{equation}

\begin{equation}
\nu < 0.126
\end{equation}

\begin{equation}
\frac{\theta_2 \nu}{\theta_1} < 0.515
\end{equation}

\textbf{Remark 3.2.} The estimate (3.4) plays a crucial role in the proof of the Rigidity Theorem \ref{rigidity}. The numbers $\theta_2$, $\theta_1$ and $\nu$ are bounds on the maximal expansion and contraction in the renormalization microscope, and a bound on the spectral radius of renormalization. As it turns out, the Rigidity Theorem \ref{rigidity} can be proved under the condition

\[
\frac{\text{maximal expansion} \cdot \text{spectral radius}}{\text{maximal contraction}} < 1.
\]

The derivative of renormalization at the fixed point is a compact operator. In particular, rigidity can be proved on a finite codimension subspace where the contraction is strong enough. The numerical estimates from \cite{GJ2} show that only the weakest stable direction is not strong enough. Luckily, this weakest stable direction corresponds to a one-dimensional family of analytically conjugated maps.

The following Lemma follows directly from Proposition \ref{Prop:MaxExpContraction}.

\textbf{Lemma 3.4.} For every $F \in W_{\text{loc}}^s(F_*)$ there exists $C > 0$ such that for any word $w \in \{0, 1\}^n$, $n \geq 1$,

\[
\|D\Psi_w^n\| \leq C\theta_2^n
\]

where $\theta_2 < 1$ is given in Proposition \ref{Prop:MaxExpContraction} and (3.6).

Define the pieces of the $n$th-level or $n$th-scale as follows. For each $w \in \{0, 1\}^n$ let

\[
B_{w0}^n \equiv B_{w0}^n(F) = \Psi_w^n(B_0(R^n F))
\]

and

\[
B_{w1}^n \equiv B_{w1}^n(F) = \Psi_w^n(B_1(R^n F)).
\]
The set of words \( \{0,1\}^n \) can be viewed as the additive group of residues mod \( 2^n \) by letting
\[
w \mapsto \sum_{k=0}^{n-1} w_{k+1} 2^k.
\]
Let \( p: \{0,1\}^n \to \{0,1\}^n \) be the operation of adding 1 in this group. The following has been proved in \([GJM]\).

**Lemma 3.5.** For every \( F \in W^s(F_*) \) and \( n \geq 1 \)
1) The above families of pieces are nested:
\[
B^n_w \subset B^{n-1}_w, \quad w \in \{0,1\}^n, \quad \nu \in \{0,1\}.
\]
2) The pieces \( B^n_w, \ w \in \{0,1\}^{n+1} \setminus \{1^{n+1}\} \), are pairwise disjoint.
3) Under \( F \), the pieces are permuted as follows.
\[
F(B^n_w) = B^n_{p(w)},
\]
unless \( p(w) = 0^{n+1} \). If \( p(w) = 0^{n+1} \), then \( F(B^n_w) \cap B^n_{0^{n+1}} \neq \emptyset \).

The union of all pieces of level \( n \) will be denoted by \( \mathcal{B}^n \):
\[
\mathcal{B}^n \equiv \mathcal{B}_F^n = \bigcup_{w \in \{0,1\}^{n+1}} B^n_w.
\]

**Lemma 3.4** implies:

**Lemma 3.6.** For every \( F \in W^s_{loc}(F_*) \) there exists \( C > 0 \) such that for all \( w \in \{0,1\}^{n+1} \), diam \( B^n_w \leq C \theta_2^n \).

Let
\[
C_F = \bigcap_{n=1}^{\infty} \bigcup_{w \in \{0,1\}^n} B^n_w.
\]
Let us also consider the dyadic group \( \{0,1\}^\infty = \lim_{n \to \infty} \{0,1\}^n \). The elements of \( \{0,1\}^\infty \) are infinite sequences \((w_1w_2\ldots)\) of symbols 0 and 1.
that can be also represented as formal power series

\[ w \mapsto \sum_{k=0}^{\infty} w_{k+1} 2^k. \]

The integers \( \mathbb{Z} \) are embedded into \( \{0,1\}^\infty \) as finite series. The adding machine \( p : \{0,1\}^\infty \to \{0,1\}^\infty \) is the operation of adding 1 in this group. The discussion above yields:

**Theorem 3.7.** For every \( F \in W_{\text{loc}}^s(F_*) \) the set \( \mathcal{C}_F \) is an invariant Cantor set. The map \( F \) acts on \( \mathcal{C}_F \) as the dyadic adding machine \( p : \{0,1\}^\infty \to \{0,1\}^\infty \). The conjugacy between \( p \) and \( F|_{\mathcal{C}_F} \) is given by the following homeomorphism \( h_F : \{0,1\}^\infty \to \mathcal{C}_F : \)

\[ h_F : w = (w_1 w_2 \ldots) \mapsto \bigcap_{n=1}^{\infty} B_n^w w_{n+1}. \]

Furthermore, \( \mathcal{C}_F \) has Lebesgue measure zero with

\[ \text{HD}(\mathcal{C}_F) \leq -\frac{\log 2}{\log \theta_2} \leq 0.795. \]

The invariant Cantor sets \( \mathcal{C}_F \) are the counterpart of the period doubling Cantor sets in one-dimensional dynamics and strongly dissipative Hénon-like maps, see [CLM, GST, Mi]. The dynamics of \( F \) restricted to \( \mathcal{C}_F \) is conjugated to the adding machine. The adding machine is uniquely ergodic. Let \( \mu \) be the unique invariant measure of \( F \) restricted to \( \mathcal{C}_F \): \( \mu(B_n^w) = \frac{1}{2^{n+1}} \).

The proof of the following theorem appears in [GJM].

**Theorem 3.8.** The measure \( \mu_F \) of every map \( F \in W^s(F_*) \) has a single characteristic exponent, \( \chi = 0 \).

The most important result concerning the period doubling Cantor sets for area-preserving maps is their rigidity in the strong universality class \( W^{ss}(F_*) \):

**Theorem 3.9.** (Rigidity). Let \( F, G \) be any two maps in \( W^{ss}(F_*) \), then there exists \( \alpha > 0.237 \), such that the topological conjugacy between \( F|_{\mathcal{C}_F} \) and \( G|_{\mathcal{C}_G} \) extends to a \( C^{1+\alpha} \) diffeomorphism of neighborhoods of the Cantor sets.
4. EXPANSION AND CONTRACTION IN THE PIECES

4.1. **Vanishing hyperbolicity of periodic orbits.** Let \( D \subset \mathbb{R}^2 \) be the domain of analyticity of \( F \), and suppose that \( p \) is a point whose orbit \( O(p) \subset D \). Recall, the definition of the upper Lyapunov exponent of \((p,v) \in D \times \mathbb{R}^2\) with respect to \( F \):

\[
\chi(p,v;F) \equiv \lim_{i \to -\infty} \frac{1}{i} \log \|DF^i(p)v\|,
\]

where \( \| \cdot \| \) is some norm in \( \mathbb{R}^2 \). The maximal Lyapunov exponent of \( p \in D \) with respect to \( F \) is defined as

\[
\chi(p;F) \equiv \sup_{\|v\|=1} \chi(p,v;F).
\]

The following Lemma about the existence of hyperbolic fixed points for maps in a small neighborhood of the renormalization fixed point map \( F^* \) is a restatement of a result from [GJ1]. The proof of the Lemma is computer-assisted (see [GJ]).

**Lemma 4.1.** Let \( W^s_{loc}(F_*) \) be as in (3.3). Every map \( F \in W^s_{loc}(F_*) \) possesses a hyperbolic fixed point \( p^F \in D \), such that

1) \( \pi_x p^F \in (0.577606201171875, 0.577629923820496) \), and \( \pi_u p^F = 0 \),

2) \( DF(p^F) \) has two negative eigenvalues:

\[
e^F_- \in (-0.486053466796875, -0.48602294921875),
e^F_+ \in (-2.0576171875, -2.057373046875),
\]

corresponding to the following two eigenvectors:

\[
s^F \in [1.0, (0.77978515625, 0.779815673828125)], \quad u^F = T(s^F).
\]

An immediate consequence of this Lemma is existence of hyperbolic \( 2^n \)-th periodic orbits for maps in \( W^s_{loc}(F_*) \). Let \( O_n(F) \) denote such \( 2^n \)-th periodic orbit for \( F \in W^s_{loc}(F_*) \), specifically:

\[
O_n(F) = \bigcup_{i=0}^{2^n-1} F^i(\Psi^{F}_{p^F}(p^{F^n})),
\]

where \( p^{F^n} \) is the fixed point of \( F_n = R^n[F] \in W^s_{loc}(F_*) \). We will also denote

\[
p^F_{0^n} = \Psi^{F}_{p^{F^n}}(p^{F^n}), \quad p^F_w = F^{\sum_{i=1}^{n} w_i 2^{i-1}}(p^F_{0^n}).
\]

Consider the stable and unstable invariant direction fields on \( O_n(F) \). At every point \( p^F_w, w \in \{0,1\}^n \) of \( O_n(F) \), these directions are given by

\[
u^F_w = D \left( F^{\sum_{i=1}^{n} w_i 2^{i-1} \Lambda_{n,F}}(p^F_{0^n}) \right) u^F_{p^F_w} = D\Psi^F_w(p^{F^n}) u^F_{p^F_w},
\]

\[
s^F_w = D \left( F^{\sum_{i=1}^{n} w_i 2^{i-1} \Lambda_{n,F}}(p^F_{0^n}) \right) s^F_{p^F_w} = D\Psi^F_w(p^{F^n}) s^F_{p^F_w}.
\]
Lemma 4.2. The set $\bigcup_{n=0}^{\infty} O_n(F) \cup C_F$ is in the set of regular points for $F$, specifically,

1) The decomposition

$$\mathbb{R}^2 = E_-(p_w^F) \bigoplus E_+(p_w^F) \equiv \text{span}\{s_w^F\} \bigoplus \text{span}\{u_w^F\}, \ w \in \{0, 1\}^n$$

is invariant under

$$DF: D \times \mathbb{R}^2 \mapsto \mathbb{R}^2.$$ 

The Lyapunov exponents

$$\chi^F(n; F) \equiv -\chi^F_+(n; F) = \frac{\log |e^F_n|}{2^n}, \ x \in O_n(F),$$

where $e^F_n$ is as in Lemma 4.1, exist, are $F$-invariant, and

$$\lim_{i \to \infty} \frac{1}{i} \log \left\{ \frac{\|DF^i(x)v\|}{\|v\|} \right\} = \chi^F_{\pm}(n; F),$$

uniformly for all $v \in E_+(p_w^F) \setminus \{0\}$;

2) For all $1 \leq k < 2^n$ and $v \in \mathbb{R}^2$,

$$|e^F_n||v| < \|DF^k(p_0^F)v\| < |e^F_n||v|.$$

Proof. 1) Let $i = q2^n + k$, $k = 2^j_1 + 2^j_2 + \ldots + 2^j_m < 2^n$, then

$$DF^i(p_0^F)s_0^n = DF^{k+q2^n}(p_0^F)s_0^n = DF^k(F^{q2^n}(p_0^F)) \cdot DF^{q2^n}(p_0^F)s_0^n$$

$$= DF^k(p_0^F) \cdot D\left(\Psi^F_{0^n} \circ F^q \circ (\Psi^F_{0^n})^{-1}\right) \cdot (p_0^F)s_0^n$$

$$= DF^k(p_0^F) \cdot D\Psi^F_{0^n} \cdot F^q \circ (\Psi^F_{0^n})^{-1} \cdot (p_0^F)$$

$$\cdot DF_n^q \left(\Psi^F_{0^n} \circ F^q \circ (\Psi^F_{0^n})^{-1}\right) \cdot (p_0^F) = DF^{k+q2^n}(p_0^F)s_0^n$$

Denote $C_n$ and $c_n$ - upper and lower bounds on the derivative norm of $F$ on $O_n(F)$. Then

$$c_n^k|e^-_n|^q\|s_0^n\| \leq \|DF^i(p_0^F)s_0^n\| \leq C_n^k|e^-_n|^q\|s_0^n\|,$$

(4.3)
and,

\[
\lim_{i \to \infty} \frac{1}{i} \log \left[ \frac{\|DF^i(p_{0^n}F) s_{0^n}^F\|}{\|s_{0^n}^F\|} \right] \leq \lim_{i \to \infty} \frac{k}{i} \log C_n + q \log \{|e_{F_n}^-|\} = \frac{\log \{|e_{F_n}^-|\}}{2^n},
\]

\[
\lim_{i \to \infty} \frac{1}{i} \log \left[ \frac{\|DF^i(p_{0^n}F) s_{0^n}^F\|}{\|s_{0^n}^F\|} \right] \geq \lim_{i \to \infty} \frac{k}{i} \log C_n + q \log \{|e_{F_n}^-|\} = \frac{\log \{|e_{F_n}^-|\}}{2^n},
\]

therefore, the limit

\[
\lim_{i \to \infty} \frac{1}{i} \log \left[ \frac{\|DF^i(p_{0^n}F) s_{0^n}^F\|}{\|s_{0^n}^F\|} \right]
\]

exists, and is equal to

\[
\chi_-(n;F) \equiv \frac{\log \{|e_{F_n}^-|\}}{2^n}.
\]

A similar computation demonstrates that

\[
\lim_{i \to \infty} \frac{1}{i} \log \left[ \frac{\|DF^i(p_{0^n}F) u_{0^n}^F\|}{\|u_{0^n}^F\|} \right] = \frac{\log \{|e_{F_n}^+|\}}{2^n} = -\chi_-(n;F).
\]

2) The result is immediate by uniform hyperbolicity of the orbit $O_n(F)$.

\[\square\]

4.2. Bounds on expansion and contraction in the pieces. In this subsection we will obtain bounds on expansion and contraction in pieces of a fixed level, Lemmas 4.3, 4.4 and 4.5. These bounds will be used to prove positive measure of both bounded and unbounded geometry.

Lemma 4.3. There exist sequences of constants $C_n$ and $\gamma_m$ such that for all $F \in W_{\text{loc}}^s(F_*)$, all $0 \leq k \leq 2^n$, all $m \geq 0$ and all $w \in \{0, 1\}^{m+n+1}$ such that $F^{2^n}(B_{m+n}^w) = B_{m+n}^w F^{2^n}(w)$,

\[
\|DF^k|_{B_{m+n}^w} - DF^k(p_w)\| \leq C_n \theta_{2}^m \gamma_{m+1} \leq 2 \gamma_{m+1} C_n \theta_{2}^m \gamma_{m} = \gamma_{m+1} \leq 2 \gamma_{m} C_n \theta_{2}^m \gamma_{m} = 1.
\]

where $C_{n+1} \leq 2 \gamma_{m} C_n$, and $\lim_{m \to \infty} \gamma_{m} = 1$.

Proof. By first order approximation of the eigenvalues of $DF^k$ and using Lemma 3.6, we can see that for all $x \in B_{m+n}^w$

\[
\|DF^k|_{B_{m+n}^w} - DF^k(x)\| \leq C_n \theta_{2}^m \gamma_{m+1} \leq 2 \gamma_{m+1} C_n \theta_{2}^m \gamma_{m} = \gamma_{m+1} \leq 2 \gamma_{m} C_n \theta_{2}^m \gamma_{m} = 1.
\]
for some $C_n$, all $1 \leq k \leq 2^n$ and all $m \geq 0$. We have, for any $2^n \leq j = 2^n + k \leq 2^{n+1}$ and any set $B_{m+n+1}^{m+n+1}$ such that $F_{m+n+1}^{2^{n+1}}(B_{m+n+1}^{m+n+1}) = B_{m+n+1}^{m+n+1}$ and any $y \in B_{m+n+1}^{m+n+1}$,

$$
\|DF^j|_{B_{m+n+1}^{m+n+1}} - DF^j(y)\|
\leq \|DF^k|_{B_{m+n+1}^{m+n+1},p_{2^n}(w)} \cdot DF^{2^n}|_{B_{m+n+1}^{m+n+1}} - DF^k(F^{2^n}(y)) \cdot DF^{2^n}(y)\|
\leq \|DF^k|_{B_{m+n+1}^{m+n+1},p_{2^n}(w)} \cdot DF^{2^n}(y)\| + \|DF^{2^n}|_{B_{m+n+1}^{m+n+1}} - DF^{2^n}(y)\|.
$$

Now since $B_{m+n+1}^{m+n+1} \subset B_{m+n}^{m+n}$ we can use (4.5) twice to bound $\|DF^k|_{B_{m+n+1}^{m+n+1},p_{2^n}(w)}$ and $\|DF^{2^n}|_{B_{m+n+1}^{m+n+1}} - DF^{2^n}(y)\|$ by $C_n \theta^{m+n+1}$ and get

$$
\|DF^j|_{B_{m+n+1}^{m+n+1}} - DF^j(y)\| \leq C_n \theta^{m+n+1} \left( \|DF^{2^n}(y)\| + \|DF^k|_{B_{m+n+1}^{m+n+1},p_{2^n}(w)}\| \right).
$$

Next,

$$
\|DF^j|_{B_{m+n+1}^{m+n+1}} - DF^j(y)\| \leq
\leq C_n \theta^{m+n+1} \left( \|DF^{2^n}(y)\| + \|DF^k|_{B_{m+n+1}^{m+n+1},p_{2^n}(w)}\| \right)
\leq C_n \theta^{m+n+1} \left( \|DF^{2^n}(y) - DF^{2^n}(p_{2^n}(w))\| + \|DF^{2^n}(p_{2^n}(w))\| \right)
\leq C_n \theta^{m+n+1} \left( C_n \theta^{m+n+1} \|DF^{2^n}(p_{2^n}(w))\| \right)
\leq C_n \theta^{m+n+1} \left( C_n \theta^{m+n+1} \|DF^{2^n}(p_{2^n}(w))\| \right).
$$

Pick $m$ large enough so that $C_n \theta^{m+n} \leq \|DF^{2^n}|_{B_{m+n}^{m+n}}\|$ and let $v \in \{0,1\}^{2m+n+1}$ be an extension of $w$, i.e. $v^{m+n+1} = w$ where $v^{m+n+1}$ denotes the restriction of $v$ to the first $m + n + 1$ symbols. Notice that for all non-negative integers $l$ the points $p_{2^{m+n+1}(v)}$ are contained
inside $B_{w}^{n+m}$. Then, using hyperbolicity of $O(p_\nu)$,

$$|e_+^{F_{2m+n}}| = \|DF^{2m+n+1}(p_\nu)\|$$

$$= \prod_{l=0}^{2^n-1} \|DF^{2m+n+1}(p_{D^l2m+n+1(w)})\|$$

$$\geq \prod_{l=0}^{2^n-1} \|DF^{2n}(p_{D^l2m+n+1(w)})\|$$

$$\geq \prod_{l=0}^{2^n-1} \left(\|DF^{2n}|_{B_{w}^{m+n}}\| - C_n\theta_2^{m+n}\right),$$

Therefore,

$$\|DF^{2n}(p_w)\| \leq \|DF^{2n}|_{B_{w}^{m+n}}\| \leq |e_+^{F_{2m+n}}|^{2^n} + C_n\theta_2^{m+n}$$

and similarly for $\|DF^{2n}(p_{\nu^{2n}(w)})\|$. Finally,

$$\|DF^j|_{B_{w}^{m+n+1}} - DF^j(y)\| \leq 2 \left(e_+^{F_{2m+n}}\right)^{2^n} + 2C_n\theta_2^{m+n}) C_2\theta_2^{m+n+1}.$$}

We can now set

\begin{equation}
C_{n+1} = 2 \left(e_+^{F_{2m+n}}\right)^{2^n} + 2C_n\theta_2^{m+n}) C_n.
\end{equation}

It follows that there exists $\gamma_m$, satisfying $\lim_{m\to\infty} \gamma_m = 1$, such that

$$C_{n+1} = 2\gamma_m C_n.$$

The recursive formula (4.6) bounding the growth of $C_n$ allows us to prove the following:

**Corollary 4.4.** There exists a constant $C$ such that for $m \geq \tilde{m}$, where $\tilde{m} \in \mathbb{N}$ is the smallest solution of

\begin{equation}
\sup_{F \in W_{loc}^s(F_\nu)} 2 \left(e_+^{F}\right)^{2^n} + 2C\theta_2^{\tilde{m}} < 3,
\end{equation}

we have $C_n \leq C3^n$ in $B^{m+n}$ for all $n \in \mathcal{N}$.

**Proof.** First, the bound is clearly true for $n = 0$ by choosing $C = C_0$. 


Next, assume that the bound is true for some $C_n$. Then using (4.6) we have

$$C_{n+1} \leq 2 \left( |e_+^{F_{2n-n}}|^2 + 2C_3^n \theta_2^{n+1} \right) C_3^n$$

$$\leq \sup_{F \in W^{s}_{\text{loc}}(F_*)} 2 \left( |e_+^F|^2 + 2C_3^n \theta_2 \right) C_3^n$$

$$\leq C3^{n+1},$$

so the bound is also true for $C_{n+1}$. By induction the bound is true for all $n$. \hfill \Box

Remark 4.1. The quantity $C$ can be bounded by the second derivatives of $F$ by a first order approximation of $DF$ around the point $p^m_w$, however we do not have an estimate of this.

Corollary 4.5.

(4.8) \[ \lim_{m \to \infty} \|DF^k|_{B^{m+n}} - DF^k(p_w)\| = 0 \]

uniformly for all $k \leq 2^n$ and $n \in \mathbb{N}$. In particular, there is a constant $A$, such that

(4.9) \[ \|DF^k|_{B^{m+n}}\| < A \]

for all $m > \bar{m}$, $1 \leq k < 2^n$ and $n \in \mathbb{N}$.

Proof. Using Lemma 4.3 and Corollary 4.4 we have for all $m > \bar{m}$

$$\|DF^k|_{B^{m+n}} - DF^k(p_w)\| \leq C_3^n \theta_2^{n+n}$$

$$= C_3^n \theta_2^{m+n}$$

$$= C(3\theta_2)^n \theta_2^m$$

Since $3\theta_2 \leq 1$ this vanishes as $m \to \infty$.

The second claim follows from the uniform hyperbolicity of the orbits $O_{m+n+1}(F)$:

$$\|DF^k(p_w)\| \leq \|DF^{2n}(p_w)\| \leq |e_+^{F_{n-1}}|.$$

\hfill \Box

5. Unbounded geometry

5.1. Unbounded geometry near the tip. We will demonstrate existence of unbounded geometry for the fixed point map $F_*$. Since by the Rigidity Theorem 3.9 dynamics of all $F \in W^{s}(F_*)$ on their Cantor sets is conjugate to that of $F_*$ by a $C^{1+\alpha}$ transformation, identical results hold for all maps in the strong universality class of $F_*$. 
**Definition 5.1.** We will say that an infinitely renormalizable Hénon-like map has \( D \)-bounded geometry if \( \text{diam}(B_{wi}) \lesssim_D d(B_{w1}, B_{w0}) \), \( i \in \{0, 1\}, \ w \in \{0, 1\}^n, \ n \in \mathbb{N} \), where \( \lesssim_D \) stands for commensurability with a constant \( D \). If no such \( D \) exists, then we say that the geometry is unbounded.

We begin with the following simple observation. The point \( \tau = (0, 0) \) is the fixed point of \( \psi_0 \). We will refer to this point as the *tip*. One can also find the fixed point of the second map, \( \psi_1 \), in the iterated function system.

**Lemma 5.1.** \( T(F_*(\tau)) \in B_1 \) is the fixed point of \( \psi_1 \)

**Proof.** We have according to (2.1)

\[
F_*(\tau) = \left( \begin{array}{c}
1 \\
1
\end{array} \right) \in B_1.
\]

and \( T(F_*(\tau)) = (1, -s_*(0, 1)) \). According to the midpoint equation (2.3), together with the normalization \( z(1, 0) = z(0, 1) = 1 \),

\[
-s_*(0, 1) = s_*(\lambda_*, 1),
\]

and, therefore, according to (2.1), the preimage of \( T(F_*(\tau)) = (1, s_*(\lambda_*, 1)) \) under \( F_*(\tau) = (\lambda_*, -s_*(1, \lambda_*)) \). By the fixed point equation (2.2) we have \( -s_*(1, \lambda_*) = -\mu_0 s_*(0, 1) \), and the preimage of \( T(F_*(\tau)) \) is the point \( (\lambda_*, -\mu_0 s_*(0, 1)) \).

We conclude that \( (1, -s_*(0, 1)) = T(F_*(\tau)) \) is the fixed point of the contraction \( \psi_1 \):

\[
T(F_*(\tau)) = F_*(\lambda_*, -\mu_0 s_*(0, 1))
\]

\[
= F_*(1, -s_*(0, 1))
\]

\[
= \psi_1(1, -s_*(0, 1)) \in B_1.
\]

\( \square \)

**Proposition 5.2.** The geometry of \( C_F \) is unbounded near the tip.

**Proof.** Consider the two pieces \( B_{10} \) and \( B_{11} \). We have, using \( \psi_0^2(\tau) = \tau \),

\[
B_{10} \ni \psi_1(\psi_0(\tau)) = F_*(\lambda_*(\tau)) = F_*(\tau) = \left( \begin{array}{c}
1 \\
1
\end{array} \right).
\]

On the other hand, the fixed point of \( \psi_1 \), \( T(F_*(\tau)) = \psi_1(T(F_*(\tau))) \in B_{11} \). Now, consider the pieces \( B_{0^{11}} \) and \( B_{0^{10}} \). We have

\[
(\lambda_*, -\mu_0 s_*(0, 1)) = (\lambda_*, s_*(\lambda_*, 1)) \in B_{0^{11}} \text{ and } (\lambda_*, \mu_0^n s_*(0, 1)) \in B_{0^{10}}.
\]

Therefore,

\[
\text{dist}(B_{0^{11}}, B_{0^{10}}) < 2 \mu_0^n |s_*(0, 1)| \sim |\mu_0|^n,
\]
while, for sufficiently large $n$, 
$$\text{diam}(B_{0^{n+1}}) \succ_D \text{diam}(B_{0^{n+0}}) > \min \{ \text{diam}(\pi_x B_{11}), \text{diam}(\pi_x B_{10}) \} |\lambda^n|,$$
where 
$$D = \max \left\{ \frac{\text{diam}(\pi_x B_{11})}{\text{diam}(\pi_x B_{10})}, \frac{\text{diam}(\pi_x B_{10})}{\text{diam}(\pi_x B_{11})} \right\}.$$ 

The conclusion follows. \hfill \Box

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig51.png}
\caption{Unbounded geometry near the tip.}
\end{figure}

Figure 5.1 illustrates the idea behind the proof of unbounded geometry near the tip. Relative to their sizes the pieces $B_{011}$ and $B_{010}$ are closer together, thinner and wider than their preimages $B_{11}$ and $B_{10}$ due to the different scalings of $\psi_0$ in the vertical and horizontal directions. This geometry will only become more extreme at $B_{0^{n+1}}$ and $B_{0^{n+0}}$ as more and more iterates of $\psi_0$ are applied.

5.2. Positive measure of unbounded geometry. Consider the pieces $B_{0^{m+n+1}}$ and $B_{0^{m+n+1}}$. Both pieces are subsets of the piece $B_{0^{m+n+1}}$. By the proof of Proposition 5.2 there is a constant $K$ such that 
$$d(B_{0^{m+n+1}}, B_{0^{m+n+1}}) \leq K \mu_{\star}^{m+n+1},$$ 
$$\text{diam}(B_{0^{m+n+1}}) \geq K^{-1}|\lambda_{\star}|^{m+n+1}$$
for $i = 0, 1$. By Corollary 4.5 for any $m \geq \tilde{m}$ we have $\|DF^k|_{B_{0^{m+n+1}}} \| \leq A$ for all $k$ satisfying $0 \leq k < 2^n$ where $A$ is some constant. With this
and the above bounds we get
\[ d(F^k(B^{m+n+1}_{0^m+n+10}), F^k(B^{m+n+1}_{0^m+n+11})) \leq \|DF^k|_{B^{m+n}_{0^m+n+1}}\|d(B^{m+n+1}_{0^m+n+10}, B^{m+n+1}_{0^m+n+11}) \leq AK\mu^m_{m+n+1}, \]
\[ \text{diam}(F^k(B^{m+n+1}_{0^m+n+11})) \geq \|DF^k|_{B^{m+n}_{0^m+n+1}}\|^{-1} \text{diam}(B^{m+n+1}_{0^m+n+1}) \geq A^{-1}K^{-1}|\lambda_s|^{m+n+1}\]

Putting these two together we get
\[ \frac{d(B^{m+n+1}_{m+n+10}, B^{m+n+1}_{m+n+11})}{\text{diam}(B^{m+n+1}_{m+n+11})} \leq \frac{AK\mu^m_{m+n+1}}{A^{-1}K^{-1}|\lambda_s|^{m+n+1}} = A^2K^2 \frac{\mu^m_{m+n+1}}{|\lambda_s|^{m+n+1}}\]

and since \( \frac{\mu}{|\lambda_s|} < 1 \) this approaches 0 as \( n \to \infty \). Therefore all these pieces have unbounded geometry. The total measure of these pieces is
\[ 2 \cdot \frac{2^n}{2^{m+n+2}} = 2^{-m-1} > 0. \]
We have therefore proven

**Theorem A.** Let \( F \in W^s_{loc}(F_s) \) and let \( \gamma = \mu_s|\lambda_s|^{-1} < 1 \). Then for any \( m \geq m \) and any \( n \geq 0 \) the measure of all pieces in \( B^{m+n+1} \) with
\[ \frac{d(B^{m+n+1}_{u0}, B^{m+n+1}_{u1})}{\text{diam}(B^{m+n+1}_{u1})} \leq D\gamma^{m+n+1}\]
is at least \( 2^{-m-1} \).

6. Bounded geometry

6.1. Positive measure of bounded geometry. The bounds obtained in Lemma 4.3 can also be used straightforwardly to demonstrate that pieces in the orbit of certain length of the two central pieces \( B^{n+m}_{0^m+n} \) and \( B^{n+m}_{0^m+n+1} \) have bounded geometry. The total measure of these iterates is nonzero.

**Theorem B.** For all \( m \geq m \), where \( m \) is as in (4.7), and all \( F \in W^s_{loc}(F_s) \), the measure of pieces \( B^{n+m+1}_{w} \in B^{n+m+1} \) with bounded geometry is at least \( 2^{-m-1} \).

**Proof.** To simplify notation, let
\[ K_{n,m} = \frac{d(B^{n+m+1}_{0^m+n+10}, B^{n+m+1}_{0^m+n+11})}{\text{diam}(B^{n+m+1}_{0^m+n+11})}. \]

Since pieces \( B^{n+m+1}_{0^m+n+10} \) and \( B^{n+m+1}_{0^m+n+11} \) are linear rescalings of pieces \( B^0 \) and \( B^1 \), \( K_{n,m} \) is clearly bounded: there exists a constant \( K \), such that
\[ K^{-1} \leq K_{n,m} \leq K. \]
Let \( m \geq \bar{m} \) where \( \bar{m} \) is as in (4.7). Then for any \( 0 \leq k < 2^n \) we have, by Lemma 4.3

\[
\frac{d(B_{p^k}^{n+m+1}(0^{n+m+1}), B_{p^k}^{n+m+1}(0^{n+m+1}))}{\text{diam}(B_{p^k}^{n+m+1}(0^{n+m+1}))} \leq \|F_k|_{B_{p^k}^{n+m+1}}\|^2 K
\]

\[
\leq (|e_{+}^{F_{n+m}}| + C_3^n \theta_2^n)^2 K
\]

\[
\leq (|e_{+}^{F_{n+m}}| + C_3^n \theta_2^n)^2 K
\]

since \( 3\theta_2 < 1 \). Furthermore, since \( F \in W_{sloc}^s(F_*) \) it follows that \( |e_{+}^{F_{n+m}}| \) is also bounded. Thus

\[
\frac{d(B_{p^k}^{n+m+1}(0^{n+m+1}), B_{p^k}^{n+m+1}(0^{n+m+1}))}{\text{diam}(B_{p^k}^{n+m+1}(0^{n+m+1}))}
\]

is bounded from above by some constant \( D \). The lower bound is attained analogously using the fact that

\[
\frac{d(B_{p^k}^{n+m+1}(0^{n+m+1}), B_{p^k}^{n+m+1}(0^{n+m+1}))}{\text{diam}(B_{p^k}^{n+m+1}(0^{n+m+1}))} \geq \|F_k|_{B_{p^k}^{n+m+1}}\|^{-2} K^{-1}
\]

Thus all the pieces \( B_{p^k}^{n+m+1}(0^{n+m+1}) \) have bounded geometry for \( 0 \leq k < 2^n \). The total measure of such pieces in \( B^{n+m+1} \) is \( 2 \cdot \frac{2^n}{2^{n+m+2}} = 2^{-m-1} \) and is independent of \( n \).

\[\square\]

### 6.2. Further bounds on geometry.

We will now give bounds on geometry of some of the pieces not in the immediate orbit of length \( 2^n \) of the central pieces.

The next Lemma describes bounds from above on pieces in the orbits of “centrally located” pieces \( B_{0^m w}^{m+n} \) where \( w \in \{0,1\}^m \).

**Lemma 6.1.** For all \( F \in W_{sloc}^s(F_*) \), all \( m \geq \bar{m} \), where \( \bar{m} \) is as in (4.7), any \( n \geq 0 \), any \( w \in \{0,1\}^m \) and any \( k \) satisfying \( 0 \leq k < 2^n \) we have

\[
\frac{d(F_k(B_{0^m w}^{m+n}), F_k(B_{0^m w}^{m+n}))}{\text{diam}(F_k(B_{0^m w}^{m+n}))} \leq K_m
\]

where \( K_m \) is some constant independent of \( n \).

**Proof.** According to Corollary 4.5 \( \|DF_k|_{B_{0^m w}^{m+n}}\| \) is bounded for all \( k < 2^n \) and all \( m \geq \bar{m} \).
First, since $||\psi_0|| = |\lambda|$ and $B_{0^n w1}^{m+n} \subset B_{0^n w1}^{m+n-1}$ for $i = 0, 1$ we have
\[
d(F_k(B_{0^n w0}^{m+n}), F_k(B_{0^n w1}^{m+n})) \leq ||DF_k|_{B_{0^n w}^{m+n-1}}||d(B_{0^n w0}, B_{0^n w1}) \leq ||DF_k|_{B_{0^n w}^{m+n-1}}|||\lambda|^n d(B_{0^n w0}, B_{0^n w1}).
\]

Next we consider a lower bound on the diameter of $B_{0^n w1}^{m+n}$. For this let $D = \min_{wi \in \{0,1\}^{m+1}} \text{diam}(\pi_x(B_{wi}^m))$, i.e. $D$ is the width of the piece that is smallest after projecting onto the $x$-axis. Then $\text{diam}(B_{w1}^m) \geq D$ for all $wi \in \{0,1\}^{m+1}$. It follows that $\text{diam}(B_{0^n w1}^{m+n}) \geq |\lambda|^n D$ and hence we get
\[
\text{diam}(F_k(B_{0^n w1}^{m+n})) \geq ||DF_k|_{B_{0^n w}^{m+n-1}}||^{-1} \text{diam}(B_{0^n w1}^{m+n}) \geq ||DF_k|_{B_{0^n w}^{m+n-1}}|||\lambda|^n D
\]
where we have used that $\min_{|v|=1} ||DF_k|_{B_{0^n w}^{m+n-1}}(v)|| = ||DF_k|_{B_{0^n w}^{m+n-1}}||^{-1}$ by the fact that $F$ is a symplectomorphism and since $\text{det} DF_k = 1$ everywhere for all $k$. Putting it all together we have
\[
d(F_k(B_{0^n w0}^{m+n}), F_k(B_{0^n w1}^{m+n})) \leq \frac{|\lambda|^n ||DF_k|_{B_{0^n w}^{m+n-1}}||d(B_{0^n w0}, B_{0^n w1})}{|\lambda|^n ||DF_k|_{B_{0^n w}^{m+n-1}}||^{-1} \text{diam}(B_{w1}^m)} \leq ||DF_k|_{B_{0^n w}^{m+n-1}}||^2 \frac{d(B_{w0}^{m}, B_{w1}^{m})}{D} \leq K_m.
\]

The next Lemma describes bounds from below.

**Lemma 6.2.** For all $F \in W^s_{loc}(F_*)$, given an $I \in \mathbb{N}$ and $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that for any fixed $m > M$ and for all $n \in \mathbb{N}$, for at least $I$ values of $k$ the following holds
\[
\frac{\cos(\tau)}{||DF_k|_{B_{0^n w}^{m-1}}||^2 \text{diam}(B_{0^n w1}^{m+n})} \leq \frac{d(\psi_0^m(F_k(B_{0^n w0}^m)), \psi_0^m(F_k(B_{0^n w1}^m)))}{\text{diam}(\psi_0^m(F_k(B_{0^n w1}^m)))},
\]
where $\tau = \max\{\pi/2 - \alpha, \beta + \varepsilon\}$, and $\alpha$ and $\beta$ are the angles of the vertical and horizontal cones, respectively, given by the ratchet phenomenon.

**Proof.** First, we have
\[
\text{diam}(\psi_0^m(F_k(B_{0^n w1}^m))) \leq |\lambda|^n \text{diam}(F_k(B_{0^n w1}^m)) \leq |\lambda|^n ||DF_k|_{B_{0^n w}^{m-1}}|| \text{diam}(B_{0^n w1}^{m}).
\]
Next, to find a lower bound on $d(\psi^n(F^k(B_{0^m0}^m)), \psi^n(F^k(B_{0^m1}^m)))$ let $q_0^k \in F^k(B_{0^m0}^m)$ and $q_1^k \in F^k(B_{0^m1}^m)$ be such that

$$d(\psi^n(F^k(B_{0^m0}^m)), \psi^n(B_{0^m1}^m)) = d(\psi^n(q_0^k), \psi^n(q_1^k))$$

$$= \sqrt{\lambda^{2n}w_x^2 + \mu^{2n}w_y^2}$$

$$\geq |\lambda|^n|w_x|$$

where $q_1^k - q_0^k = (w_x, w_y) = w$. Let points $p_i^k$ be defined by

$$F^k(p_i^k) = q_i^k.$$ 

Notice, that since the ratio of the width of the sets $B_{0^m0}^m$ to their length is $O(\mu^n/|\lambda|^m)$, for every $\delta > 0$ there exists an $M \in \mathbb{N}$ such that for all $m > M$ the vectors $p_0^k p_1^k$ and $p_0^{k+1} p_1^{k+1}$ lie in the horizontal cone of opening $\delta$. Furthermore, for every $\varepsilon > 0$ and $I \in \mathbb{N}$ there exists $\delta > 0$ such that the angle between the vectors $F(q_0^k) F(q_1^k)$ and $q_0^{k+1} q_1^{k+1}$ does not exceed $\varepsilon$ for all $0 < k < 2I$.

By the ratchet phenomenon it follows that either $q_1^k$ is outside the vertical cone of $q_0^k$, or, otherwise, $F(q_1^k)$ is contained in the horizontal cone of $F(q_0^k)$. Thus, either $q_1^k$ is outside the vertical cone of $q_0^k$, or, otherwise, $q_1^{k+1}$ is contained in a horizontal cone of $q_0^{k+1}$ with an opening $\beta + \varepsilon$.

With this we get for at least $I$ of the values of $k$

$$|\lambda|^n|w_x| \geq |\lambda|^n||w|| \cos(\tau)$$

$$\geq |\lambda|^n \cos(\tau) d(F^k(B_{0^m0}^m), F^k(B_{0^m1}^m))$$

$$\geq |\lambda|^n \cos(\tau) \|D F^k|_{B_{0^m1}^m}^{-1}\|d(B_{0^m0}^m, B_{0^m1}^m).$$

Putting these together we obtain the required bound. \(\square\)

Lemmas 6.1 and 6.2 immediately result in the following

**Corollary 6.3.** For all $F \in W^s_{loc}(F_*)$, given an $I \in \mathbb{N}$ and $\varepsilon > 0$ there exists $M \in \mathbb{N}$ and a constant $K_m > 0$, independent of $n$, such that for any fixed $m > M$ and for all $n \in \mathbb{N}$, for at least $I$ values of $k$ the following holds

$$K_m^{-1} \leq \frac{d(B_{0^n p^k(0^{m})0}^{m+n}, B_{0^n p^k(0^{m})1}^{m+n})}{\text{diam}(B_{0^n p^k(0^{m})1}^{m+n})} \leq K_m.$$ 

The following is a straightforward consequence of Corollary 6.3.

**Theorem C.** For any $I \in \mathbb{N}$ there exists a positive integer $M$ such that for all $F \in W^s_{loc}(F_*)$, all $m > M$ and all $n \in \mathbb{N}$, the measure of the pieces $B_{w}^{m+n} \in B^{n+m}$ with bounded geometry is at least $I2^{-m-1}$. 
Proof. Repeat the proof of Theorem B with $B_{0^p k (0^m) 0}$ substituted for the trivially bounded pieces $B_{0^p (0^m)}$. Conclude that the measure of the pieces with the bounded geometry is $2 \cdot \frac{I^2_m}{m+2} = I^{2-m-1}$. □

Remark 6.1. Theorem C can be used as an alternative to proving positive measure of bounded geometry, possibly yielding a higher measure. However $M \geq \bar{m}$ where $\bar{m}$ is as in (4.7) and therefore would require estimates on these to make the comparison.

Remark 6.2. If for some $m \geq \bar{m}$ we could find precise enough bounds on the positions of all $B^m_w$ the techniques of Theorems A, B and C can be combined to find the exact measures of bounded and unbounded geometry.

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