SIMULTANEOUS DETERMINATION OF TWO COEFFICIENTS IN THE RIEMANNIAN HYPERBOLIC EQUATION FROM BOUNDARY MEASUREMENTS

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ABSTRACT. In this paper we consider the inverse problem of determining on a compact Riemannian manifold the electric potential and the absorption coefficient in the wave equation with Dirichlet data from measured Neumann boundary observations. This information is enclosed in the dynamical Dirichlet-to-Neumann map associated to the wave equation. We prove in dimension $n \geq 2$ that the knowledge of the Dirichlet-to-Neumann map for the wave equation uniquely determines the absorption coefficient and the electric potential and we establish Hölder-type stability.

1. INTRODUCTION AND MAIN RESULTS

Let $(M, g)$ be an $n$-dimensional ($n \geq 2$) compact Riemannian manifold with smooth boundary $\partial M$ where $g$ denotes a Riemannian metric of class $C^\infty$. We let $\Delta$ denote the Laplace-Beltrami operator on $M$. A summary of the main Riemannian geometric notions needed in this paper is provided in Section 2. In this paper we study an inverse problem for the wave equation in the presence of an absorption coefficient and an electric potential. Given $T > 0$, we denote $Q = M \times (0, T)$ and $\Sigma = \partial M \times (0, T)$. We consider the following initial boundary value problem for the wave equation with a potential $q$ and an absorption coefficient $a$,

\begin{equation}
\begin{aligned}
&\left( \partial_t^2 - \Delta + a(x)\partial_t + q(x) \right) u = 0 \quad \text{in } Q, \\
u(\cdot, 0) = \partial_t u(\cdot, 0) = 0 \quad \text{in } M, \\
u = f \quad \text{on } \Sigma.
\end{aligned}
\end{equation}

Here $a, q : M \to \mathbb{R}$ are real valued functions in $L^\infty(M)$ and $f \in H^1(\Sigma)$.

1.1. Well-posedness and direct problem. For this paper, we use many of the notational conventions in [8]. Let $(M, g)$ be a (smooth) compact Riemannian manifold with boundary of dimension $n \geq 2$. We refer to [18] for the differential calculus of tensor fields on Riemannian manifolds. If we fix local coordinates $x = (x^1, \ldots, x^n)$ and let $(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n})$ denote the corresponding tangent vector fields, the inner product and the norm on the tangent space $T_pM$ are given by

\[ g(X, Y) = \langle X, Y \rangle = \sum_{j,k=1}^n g_{jk} X^j Y^k, \]

\[ |X| = \langle X, X \rangle^{1/2}, \quad X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}, \quad Y = \sum_{i=1}^n Y^i \frac{\partial}{\partial x^i}. \]
If $f$ is a $C^1$ function on $M$, we define the gradient of $f$ as the vector field $\nabla f$ such that

$$X(f) = \langle \nabla f, X \rangle$$

for all vector fields $X$ on $M$. In local coordinates, we have

$$\nabla f = \sum_{i,j=1}^{n} g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j},$$

where $(g^{ij})$ is the inverse of the tensor $g$. The metric tensor $g$ induces the Riemannian volume $dv^n = (\det g)^{1/2} dx^1 \wedge \cdots \wedge dx^n$. We denote by $L^2(M)$ the completion of $C^\infty(M)$ with respect to the usual inner product

$$\langle u, v \rangle = \int_M u(x) \overline{v(x)} \, dv^n, \quad u, v \in C^\infty(M).$$

The Sobolev space $H^1(M)$ is the completion of $C^\infty(M)$ with respect to the norm $\| \cdot \|_{H^1(M)}$.

$$\|u\|^2_{H^1(M)} = \|u\|^2_{L^2(M)} + \|\nabla u\|^2_{L^2(M)}.$$

The normal derivative is given by

$$\partial_n u := \langle \nabla u, \nu \rangle = \sum_{j,k=1}^{n} g^{jk} \nu_j \frac{\partial u}{\partial x^k},$$

where $\nu$ is the unit outward vector field to $\partial M$. Moreover, using covariant derivatives (see [14]), it is possible to define coordinate invariant norms in $H^k(M), k \geq 0$.

With these definitions in mind, we consider the following initial boundary value problem for the wave equation

$$\begin{cases}
(\partial_t^2 - \Delta + a(x)\partial_t + q(x)) v(x,t) = F(x,t) & \text{in } Q, \\
v(x,0) = 0, \quad \partial_t v(x,0) = 0 & \text{in } M, \\
v(x,t) = 0 & \text{on } \Sigma.
\end{cases} \tag{1.4}$$

We know this problem is well-posed, since we have the following existence and uniqueness result, see [19].

**Lemma 1.1.** Let $T > 0$, $a \in L^\infty(M)$, and $q \in L^\infty(M)$. Assuming $F \in H^1(0,T;L^2(M))$ such that $F(\cdot,0) = 0$ in $M$, then there exists a unique solution $v$ to (1.4) such that

$$v \in C^2(0,T;L^2(M)) \cap C^1(0,T;H^1_0(M)) \cap C(0,T;H^2(M)).$$

Furthermore, there is a constant $C > 0$ such that

$$\|\partial_t v(\cdot,t)\|_{L^2(M)} + \|\nabla v(\cdot,t)\|_{L^2(M)} \leq C \|F\|_{L^2(Q)}, \tag{1.5}$$

and

$$\|\partial_t^2 v(\cdot,t)\|_{L^2(M)} + \|\nabla \partial_t v(\cdot,t)\|_{L^2(M)} + \|\Delta v(\cdot,t)\|_{L^2(M)} \leq C \|F\|_{H^1(0,T;L^2(M))}. \tag{1.6}$$

A proof of the following lemma may be found for instance in [15].

**Lemma 1.2.** Let $f \in H^1(\Sigma)$ be a function such that $f(x,0) = 0$ for all $x \in \partial M$. There exists an unique solution

$$u \in C^1(0,T;L^2(M)) \cap C(0,T;H^1(M)) \tag{1.7}$$

to the problem (1.7). Furthermore, there is a constant $C > 0$ such that

$$\|\partial_n u\|_{L^2(\Sigma)} \leq C \|f\|_{H^1(\Sigma)}. \tag{1.8}$$
1.2. Inverse problem and Main result. From the physical viewpoint, our inverse problem consists in determining the properties (e.g. an absorption coefficient) of an inhomogeneous medium by probing it with disturbances generated on the boundary. The measurements are responses of the medium to these disturbances which are measured on the boundary, and the goal is to recover the potential $q(x)$ and the absorption coefficient $a(x)$ which describes the property of the medium. Here we assume that the medium is quiet initially, and $f$ is a disturbance which is used to probe the medium. Roughly speaking, the data is $\partial_{\nu} u$ measured on the boundary for different choices of $f$.

We may define the Dirichlet to Neumann (D-N) map associated with hyperbolic problem (1.1) by
\begin{equation}
\Lambda_{a,q}(f) = \partial_{\nu} u, \quad f \in \mathcal{H}_0^1(\Sigma) = \{f \in H^1(\Sigma), \ f(\cdot,0) = 0 \text{ on } \partial M\}.
\end{equation}
Therefore the Dirichlet-to-Neumann map $\Lambda_{a,q}$ defined by (1.9) is continuous. We denote by $\|\Lambda_{a,q}\|$ its norm in $\mathcal{L}(\mathcal{H}_0^1(\Sigma); L^2(\Sigma))$.

For a Riemannian manifold $(M, g)$ with boundary $\partial M$, we denote by $\nabla$ the Levi-Civita connection on $(M, g)$. For a point $x \in \partial M$, the second quadratic form of the boundary
\[ \Pi(\xi, \xi) = \langle \nabla_{\xi^\nu} \xi, \xi \rangle, \quad \xi \in T_x(\partial M), \]
is defined on the space $T_x(\partial M)$. We say that the boundary is strictly convex if the form is positive-definite for all $x \in \partial M$ (see [32]).

**Definition 1.3.** We say that the Riemannian manifold $(M, g)$ (or that the metric $g$) is simple in $M$, if $\partial M$ is strictly convex with respect to $g$, and for any $x \in M$, the exponential map $\exp_x : \exp^{-1}(M) \to M$ is a diffeomorphism. The latter means that every two points $x, y \in M$ are joined by a unique geodesic smoothly depending on $x$ and $y$.

Note that if $(M, g)$ is simple, one can extend it to a simple manifold $M_1$ such that $M_1^{\text{int}} \supset M$.

Let us now introduce the admissible sets of absorption coefficients $a$ and electric potentials $q$. Let $m_1, m_2 > 0$ and $\eta > n/2$ be given, set
\begin{equation}
\mathcal{A}(m_1, \eta) = \{a \in W^{2,\infty}(M), \|a\|_{H^\eta(M)} \leq m_1\},
\end{equation}
and
\begin{equation}
\mathcal{Q}(m_2) = \{q \in W^{2,\infty}(M), \|q\|_{H^\eta(M)} \leq m_2\}.
\end{equation}
Introduce one more notation. Given $x \in M$ and a 2-plane $\pi \subset T_x M$, denote by $K(x, \pi)$ the sectional curvature of $\pi$ at $x$. For $\xi \in T_x M$ with $|\xi| = 1$, put
\[ K(x, \xi) = \sup_{\pi, \xi \in \pi} K(x, \pi), \quad K^+(x, \xi) = \max\{0, K(x, \xi)\}.
\]
Define the following characteristic:
\[ k^+(M, g) = \sup_{\gamma} \int_{\ell_1}^{\ell_2} tK^+(\gamma(t), \gamma(t)) dt, \]
where $\gamma : [\ell_1, \ell_2] \to M$, ranges in the set of all unit speed geodesic in $M$.

The main results of this paper are as follows.

**Theorem 1.4.** Let $(M, g)$ be a simple compact Riemannian manifold with boundary of dimension $n \geq 2$ such that $k^+(M, g) < 1$, and let $T > \text{Diam}_g(M)$. There exist $C > 0$ and $\gamma \in (0, 1)$ such that for any $a_1, a_2 \in \mathcal{A}(m_1, \eta)$ and $q_1, q_2 \in \mathcal{Q}(m_2)$ coincide near the boundary $\partial M$, the following estimate holds true
\begin{equation}
\|a_1 - a_2\|_{L^2(M)} + \|q_1 - q_2\|_{L^2(M)} \leq C \|\Lambda_{a_1, q_1} - \Lambda_{a_2, q_2}\| \gamma.
\end{equation}
where $C$ depends on $M$, $m_1, m_2$, $\eta$ and $n$.

By Theorem 1.4 we can readily derive the following uniqueness result

**Corollary 1.5.** Let $(M, g)$ be a simple compact Riemannian manifold with boundary of dimension $n \geq 2$, such that $k^+(M, g) < 1$ and let $T > \text{Diam}_g(M)$, we have that $\Lambda_{a_1, q_1} = \Lambda_{a_2, q_2}$ implies $a_1 = a_2$ and $q_1 = q_2$ almost everywhere in $M$.

### 1.3. Relation to the literature

In recent years significant progress has been made for the problem of identifying one coefficient in the euclidean hyperbolic equation $(g_{ij} = \delta_{ij})$. In [29], Rakesh and Symes prove that the D-to-N map determines uniquely the time-independent potential in a wave equation. Ramm and Sjöstrand [30] has extended the result in [29] to the case of time-dependent potentials. Isakov [16] has considered the simultaneous uniqueness determination of a zeroth order coefficient and an absorption coefficient. A key ingredient in the existing results is the construction of complex geometric optics solutions of the wave equation, concentrated along a line, and the relationship between the hyperbolic D-to-N map and the X-ray transform play a crucial role. In [25] Pestov propose a linear procedure based on the boundary control method for determining both coefficients, absorption and speed, for the wave equation.

For the stability estimates, Sun [36] established in the Euclidean case stability estimates for potentials from the Dirichlet-to-Neumann map. In [6] the authors consider the stability in an inverse problem of determining the potential $q$ entering the wave equation in a bounded smooth domain of $\mathbb{R}^d$ from boundary observations. The observation is given by a hyperbolic (dynamic) Dirichlet to Neumann map associated to a wave equation and prove a log-type stability estimate in determining $q$ from a partial Dirichlet to Neumann map. For the wave equation with a lower order term $q(t, x)$, Waters [38] proves that we can recover the X-ray transform of time dependent potentials $q(t, x)$ from the dynamical Dirichlet-to-Neumann map in a stable way. He derive conditional Hölder stability estimates for the X-ray transform of $q(t, x)$.

In the case of Riemannian wave equation, Bellassoued and Dos Santos Ferreira [8] seek stability estimates in the inverse problem of determining the potential or the velocity in a wave equation posed in a simple riemannian wave equation $(M, g)$ from measured Neumann boundary observations. The authors prove in dimension $n \geq 2$ that the knowledge of the Dirichlet-to-Neumann map for the wave equation uniquely determines the electric potential and they show a Hölder-type stability in determining the potential. Similar results for the determination of velocities close to 1 is also given.

In [33] and [34] Stefanov and Uhlmann considered the inverse problem of determining a Riemannian metric $g$ on a Riemannian manifold $(M, g)$ with boundary from the hyperbolic Dirichlet-to-Neumann map associated to solutions of the wave equation $(c^2_t - \Delta_g)u = 0$. A Hölder type of conditional stability estimate was proven in [33] for metrics close enough to the Euclidean metric in $C^k$, $k \geq 1$ or for generic simple metrics in [34]. It is clear that one cannot hope to uniquely determine the metric $g = (g_{jk})$ from the knowledge of the Dirichlet-to-Neumann map $\Lambda_{g, a, q}$. As was noted in [33], the Dirichlet-to-Neumann map is invariant under a gauge transformation of the metric $g$. Namely, given a diffeomorphism $\psi : M \rightarrow M$ such that $\psi|_{\partial M} = \text{Id}$ one has $\Lambda_{\psi^* g, a, q} = \Lambda_{g, a, q}$ where $\psi^* g$ denotes the pullback of the metric $g$ under $\psi$.

In [24], Montalto studies the stability of simultaneously recovering the Riemannian metric $g$, a covector field $b$ and a potential $q$ in a Riemannian manifold $M$ from the boundary measurements modeled by the Dirichlet-to-Neumann map. He shows that, assuming the metric is close to a generic simple metric, and the two covector close, a conditional Hölder-type stability for the recovery holds up to the natural gauge transformations that fix the boundary. This result generalizes the results in [8] and [33].
In [3] Belishev and Kurylev gave an affirmative answer to the general problem of finding a smooth metric from the Dirichlet-to-Neumann map. Their approach is based on the boundary control method introduced by Belishev [2] and uses in an essential way an unique continuation property. Unfortunately it seems unlikely that this method would provide stability estimates even under geometric and topological restrictions. Their method also solves the problem of recovering g through boundary spectral data. The boundary control method gave rise to several refinements of the results of [3]: one can cite for instance [21], [19] and [1].

The importance of control theory for inverse problems was first understood by Belishev [2]. He used control theory to develop the first variant of the boundary control (BC) method. Later, the idea based on control theory was combined with the geometrical ones. The importance of the geometry for inverse problems follows the fact that any elliptic second-order differential operator gives rise to a Riemannian metric in the corresponding domain. The role of this metric becomes clearer if we consider the solutions of the corresponding wave equation. Indeed, these waves propagate with the unit speed along geodesics of this Riemannian metric. These geometric ideas were introduced to the boundary control method in [21], [19].

In this paper, the inverse problem under consideration is whether, for a fixed metric g, the knowledge of the Dirichlet-to-Neumann map \( \Lambda_{g,a,q} \) on the boundary uniquely determines the electric potential \( q \) and the absorption coefficient \( a \).

Uniqueness properties for local Dirichlet-to-Neumann maps associated with the wave equation are rather well understood (e.g., Belishev [2], Katchlov, Kurylev and Lassas [19], Kurylev and Lassas [21]) but stability for such operators is far from being apprehended. For instance, one may refer to Isakov and Sun [17] where a local Dirichlet-to-Neumann map yields a stability result in determining a coefficient in a subdomain. There are quite a few works on Dirichlet-to-Neumann maps, so our references are far from being complete: see also Eskin [12]-[13], Uhlmann [37] as related papers.

The main goal of this paper is to study the stability of the inverse problem for the dynamical anisotropic wave equation. The approach that we develop is a dynamical approach. It is based on the consideration of the wave equation and involves various techniques to study an initial-boundary value problem for the hyperbolic equation. In this paper we prove a Hölder-type estimate which shows that a dispersion term \( q \) and the absorption coefficient \( a \) depends stably on the Dirichlet-to-Neumann map. Our approach here is different from [24] in order to prove a stability estimate without a smallness assumption for the coefficients. The main idea is to probe the medium by real geometric optics solutions of the wave equation, concentrated along a geodesic line, starting on one side of the boundary, and measure responses of the medium on other side of the boundary and using directly a stability estimate for the geodesic X-ray transform without passing by the normal operator as in [24].

The outline of the paper is as follows. In section 2 we give an important stability estimate for the geodiscal ray transform. In section 3 we construct special geometrical optics solutions to wave equations with potential and absorption coefficients. In section 4 and 5, we establish stability estimates for the absorption coefficient and the electric potential. The appendix A is devoted to the study of the Poisson kernel in the tangent sphere bundle.

2. Stability estimate for the geodesical X-ray transform

2.1. Geodesical ray transform on a simple manifold. The geodesic X-ray transform of a function is defined by integrating over geodesics. It is naturally arises in linearization of the problem of determining a coefficient in partial differential equation. The X-ray transform also arises in Computer Tomography,
Positron Emission Tomography, geophysical imaging in determining the inner structure of the Earth, ultrasound imaging. Uniqueness result and stability estimates of the geodesic X-ray transform were obtained by Mukhometov [25] for simple surface. For simple manifolds of any dimension this result was proven in [33], see also V. A. Sharafutdinov’s book [32]. In his paper Dairbekov generalized this result for nontrapping manifolds without conjugate points [10]. Fredholm type inversion formulas were given in [27] by Pestov and Uhlmann.

In this section we first collect some formulas needed in the rest of this paper and introduce the geodesical X-ray transform on the manifolds we will be using. Let \((M, g)\) be a Riemannian manifold, for \(x \in M\) and \(\xi \in T_x M\) we let \(\gamma_{x,\xi}\) denote the unique geodesic starting at the point \(x\) in the direction \(\xi\). By

\[
SM = \{(x, \xi) \in TM; |\xi| = 1\}, \quad S^* M = \{(x, p) \in T^* M; |p| = 1\},
\]

we denote the sphere bundle and co-sphere bundle of \(M\). The exponential map \(\exp_x : T_x M \to M\) is given by

\[
(2.1) \quad \exp_x(v) = \gamma_{x,\xi}(|v|), \quad \xi = \frac{v}{|v|}.
\]

A compact Riemannian manifold \((M, g)\) with boundary is called a convex non-trapping manifold, if it satisfies two conditions:

(i) the boundary \(\partial M\) is strictly convex, i.e., the second fundamental form of the boundary is positive definite at every boundary point,

(ii) all geodesics having finite length in \(M\), i.e., for each \((x, \xi) \in SM\), the maximal geodesic \(\gamma_{x,\xi}(t)\) satisfying the initial conditions \(\gamma_{x,\xi}(0) = x\) and \(\dot{\gamma}_{x,\xi}(0) = \xi\) is defined on a finite segment with extremities \(\ell_-(x, \xi)\) and \(\ell_+(x, \xi)\). We recall that a geodesic \(\gamma : [a, b] \to M\) is maximal if it cannot be extended to a segment \([a - \varepsilon_1, b + \varepsilon_2]\), where \(\varepsilon_1 \geq 0\) and \(\varepsilon_1 + \varepsilon_2 > 0\).

An important subclass of convex non-trapping manifolds are simple manifolds. We say that a compact Riemannian manifold \((M, g)\) is simple if it satisfies the following properties

(a) \((M, g)\) is convex and non-trapping,

(b) there are no conjugate points on any geodesic.

A simple \(n\)-dimensional Riemannian manifold is diffeomorphic to a closed ball in \(\mathbb{R}^n\), and any pair of points in the manifold are joined by an unique geodesic.

Let \((x, \xi) \in SM\), there exist a unique geodesic \(\gamma_{x,\xi}\) associated to \((x, \xi)\) which is maximally defined on a finite interval \([\ell_-(x, \xi), \ell_+(x, \xi)]\), with \(\gamma_{x,\xi}(\ell_{\pm}(x, \xi)) \in \partial M\). We define the geodesic flow \(\phi_t\) as following

\[
(2.2) \quad \phi_t : SM \to SM, \quad \phi_t(x, \xi) = (\gamma_{x,\xi}(t), \dot{\gamma}_{x,\xi}(t)), \quad t \in [\ell_-(x, \xi), \ell_+(x, \xi)],
\]

and \(\phi_t\) is a flow, that is, \(\phi_t \circ \phi_s = \phi_{t+s}\).

Now, we introduce the submanifolds of inner and outer vectors of \(SM\)

\[
(2.3) \quad \partial_\pm SM = \{(x, \xi) \in SM; x \in \partial M, \pm \langle \xi, \nu(x) \rangle < 0\},
\]

where \(\nu\) is the unit outer normal to the boundary. Note that the manifolds \(\partial_+ SM\) and \(\partial_- SM\) have the same boundary \(S(\partial M)\), and \(\partial SM = \partial_+ SM \cup \partial_- SM\). We denote by \(C^\infty(\partial_+ SM)\) be space of smooth functions on the manifold \(\partial_+ SM\). Thus we can define two functions \(\ell_\pm : SM \to \mathbb{R}\) which satisfy

\[
\ell_-(x, \xi) \leq 0, \quad \ell_+(x, \xi) \geq 0, \quad \ell_+(x, \xi) = -\ell_-(x, -\xi), \quad \ell_-(x, \xi) = 0, \quad (x, \xi) \in \partial_+ SM, \quad \ell_+(x, \xi) = 0, \quad (x, \xi) \in \partial_- SM,
\]
\[ \ell_-(\phi_t(x, \xi)) = \ell_-(x, \xi) - t, \quad \ell_+(\phi_t(x, \xi)) = \ell_+(x, \xi) - t. \]

For \((x, \xi) \in \partial_+ SM\), we denote by \(\gamma_{x,\xi} : [0, \ell_+(x, \xi)] \rightarrow M\) the maximal geodesic satisfying the initial conditions \(\gamma_{x,\xi}(0) = x\) and \(\dot{\gamma}_{x,\xi}(0) = \xi\).

Concerning smoothness properties of \(\ell_{\pm}(x, \xi)\), we can see that these functions are smooth near a point \((x, \xi)\) such that the geodesic \(\gamma_{x,\xi}(t)\) intersects \(\partial M\) transversely for \(t = \ell_{\pm}(x, \xi)\). By strict convexity of \(\partial M\), the functions \(\ell_{\pm}(x, \xi)\) are smooth on \(TM \setminus T(\partial M)\). In fact, all points of \(TM \cap T(\partial M)\) are singular for \(\ell_{\pm}\); since some derivatives of these functions are unbounded in a neighbourhood of such points. In particular, \(\ell_+\) is smooth on \(\partial_+ SM\), see Lemma 4.1.1 of [32].

The Riemannian scalar product on \(T_x M\) induces the volume form on \(S_x M\), denoted by \(d\omega_x(\xi)\) and given by

\[ d\omega_x(\xi) = \sqrt{|g|} \sum_{k=1}^n (-1)^k \xi^k d\xi^1 \wedge \cdots \wedge d\xi^k \wedge \cdots \wedge d\xi^n. \]

As usual, the notation \(\wedge\) means that the corresponding factor has been dropped. We introduce the volume form \(d\nu^{2n-1}\) on the manifold \(SM\) by

\[ d\nu^{2n-1}(x, \xi) = d\omega_x(\xi) \wedge d\nu^n, \]

where \(d\nu^n\) is the Riemannian volume form on \(M\). By Liouville’s theorem, the form \(d\nu^{2n-1}\) is preserved by the geodesic flow. The corresponding volume form on the boundary \(\partial SM = \{(x, \xi) \in SM, \ x \in \partial M\}\) is given by

\[ d\sigma^{2n-2} = d\omega_x(\xi) \wedge d\sigma^{n-1}, \]

where \(d\sigma^{n-1}\) is the volume form of \(\partial M\).

Let \(L^2_\mu(\partial_+ SM)\) be the space of square integrable functions with respect to the measure \(\mu(x, \xi) \, d\sigma^{2n-2}\) with \(\mu(x, \xi) = |\langle \xi, \nu(x) \rangle|\). This Hilbert space is endowed with the scalar product

\[ (u, v)_\mu = \int_{\partial_+ SM} u(x, \xi) \overline{v(x, \xi)} \mu(x, \xi) \, d\sigma^{2n-2}. \]

The ray transform (also called geodesic \(X\)-ray transform) on a convex non trapping manifold \(M\) is the linear operator

\[ \mathcal{I} : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(\partial_+ SM), \]

defined by the equality

\[ \mathcal{I} f(x, \xi) = \int_0^{\ell_+(x, \xi)} f(\gamma_{x,\xi}(t)) \, dt, \quad (x, \xi) \in \partial_+ SM. \]

The right-hand side of (2.6) is a smooth function on \(\partial_+ SM\) because the integration limit \(\ell_+(x, \xi)\) is a smooth function on \(\partial_+ SM\). The ray transform on a convex non trapping manifold \(M\) can be extended as a bounded operator

\[ \mathcal{I} : H^k(M) \longrightarrow H^k(\partial_+ SM), \]

for every integer \(k \geq 1\), see Theorem 4.2.1 of [32].
2.2. Inverse inequality for the geodesical ray-transform. This subsection concerns the problem of inverting the ray transform.

Let $R$ be the curvature tensor of the Levi-Civita connection $\nabla_X$ defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z, \quad X, Y, Z \in \mathcal{T}M.$$  

For a point $x \in M$ and a two-dimensional subspace $\pi \subset T_x M$, the number

$$K(x, \pi) = \frac{\langle R(\xi, \eta)\eta, \xi \rangle}{|\xi|^2|\eta|^2 - \langle \xi, \eta \rangle^2},$$

is independent of the choice of the basis $\xi, \eta$ for $\pi$. It is called the sectional curvature of the manifold $M$ at the point $x$ and in the two-dimensional direction $\pi$.

For $(x, \xi) \in \mathcal{T}M$, we set $K(x, \pi) = \sup_{\pi \ni \xi} K(x, \pi)$ and

$$K^+(x, \xi) = \max\{0, K(x, \xi)\}.$$  

For a simple compact Riemannian manifold $(M, g)$, we set

$$k^+(M, g) = \sup \{ \int_0^{\ell} \frac{k(t)}{\gamma(t)} dt, (x, \xi) \in \partial_+ SM \}.$$  

The aim of this section is to prove the following theorem.

**Theorem 2.1.** For every simple compact Riemannian manifold $(M, g)$ with $k^+(M, g) < 1$, there exist $C > 0$ such that the following stability estimate

$$\|f\|_{L^2(M)} \leq C\|If\|_{H^1(\partial_+ SM)},$$

holds for any $f \in H^1(M)$.

By density arguments, it suffices to prove the theorem for $f \in C^\infty(M)$. Indeed, if $f \in H^1(M)$, then we can find a sequence $(f_k)_k$ in $C^\infty(M)$ converging towards $f$ in $H^1(M)$. The ray transform $\mathcal{I}$ is a bounded operator from $H^1(M)$ into $H^1(\partial_+ SM)$, so the sequence $(\mathcal{I}f_k)_k$ converges towards $\mathcal{I}f$ in $H^1(\partial_+ SM)$. Applying the theorem for $f_k$ and passing to the limit as $k \to +\infty$, we deduce that $\|f\|_{L^2(M)} \leq C\|\mathcal{I}f\|_{H^1(\partial_+ SM)}$.

Before starting the proof of the theorem, we need to introduce some notions and notations. We will use the Einstein summation convention to abbreviate the notations. When an index variable appears twice in a simple term and is not otherwise defined, it implies summation of that term over all the values of the index. For example, for $i \in 1, 2, 3$, $c_i x^i$ means $c_1 x^1 + c_2 x^2 + c_3 x^3$.

Let $\tau^r_s M$ be the bundle of tensors of degree $(r, s)$ on $M$. Its sections denoted by $T^r_s$ are called tensor fields of degree $(r, s)$. Let $U$ be a domain of $M$. We denote by $C^\infty(\tau^r_s M, U)$ the $C^\infty(U)$-module of smooth sections of the bundle $\tau^r_s M$ over $U$. The notation $C^\infty(\tau^r_s M, M)$ will usually be abbreviated to $C^\infty(\tau^r_s M)$. If $(x^1, \ldots, x^n)$ is a local coordinate system defined in a domain $U$, then any tensor field $u \in C^\infty(\tau^r_s M, U)$ can be uniquely represented as

$$u = u^{i_1, \ldots, i_r}_{j_1, \ldots, j_s} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s},$$

where $u^{i_1, \ldots, i_r} \in C^\infty(U)$ are called the coordinates of the field $u$ in the given coordinate system. We will usually abbreviate (2.11) as follows

$$u = (u^{i_1, \ldots, i_r}_{j_1, \ldots, j_s}).$$
The bundles $\tau^s_M$ and $\tau^p_M$ are dual to each other and, consequently, $C^\infty(\tau^s_M)$ and $C^\infty(\tau^p_M)$ are mutually
dual $C^\infty(M)$-modules. This implies in particular that a field $u \in C^\infty(\tau^s_0 M)$ can be considered as a $C^\infty(M)$-
multilinear mapping $u : C^\infty(\tau^s_0 M) \times \cdots \times C^\infty(\tau^1_0 M) \rightarrow C^\infty(\tau^1_0 M)$. Consequently, a given connection $\nabla$
on $M$ defines the $C$-linear mapping (denoted by the same letter)
\[(2.13) \quad \nabla : C^\infty(\tau^r_0 M) \rightarrow C^\infty(\tau^1_0 M)\]
by the formula $(\nabla v)(u) = \nabla_u v$. The tensor field $\nabla v$ is called the covariant derivative of the vector field
$v$ (with respect to the given connection). The covariant differenciation defined on vector fields can be transferred to tensor fields (see [32] Theorem 3.2.1 pp. 85)
\[(2.14) \quad \nabla : C^\infty(\tau^s_0 M) \rightarrow C^\infty(\tau^r_{s+1} M)\]
such that (2.14) coincides with the mapping (2.13), for $r = 1$ and $s = 0$ and that for a field
\[u = u^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s},\]
the field $\nabla u$ is defined by
\[\nabla u = \nabla_k u^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} \frac{\partial}{\partial x^k} \otimes \cdots \otimes \frac{\partial}{\partial x^r} \otimes dx^{i_1} \otimes \cdots \otimes dx^{j_s} \otimes dx^k,\]
where
\[\nabla_k u^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} = \frac{\partial}{\partial x^k} u^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} + \sum_{m=1}^{r} \Gamma^p_{kj_m} u^{i_1,\ldots,i_{m-1}p,i_{m+1},\ldots,i_r}_{j_1,\ldots,j_s} - \sum_{m=1}^{s} \Gamma^p_{jm} u^{i_1,\ldots,i_r}_{j_1,\ldots,j_{m-1}p,j_{m+1},\ldots,j_s},\]
where $\Gamma^p_{kj}$ is the Christoffel symbol.

Now, we will extend this covariant differenciation for tensors on $TM$. If $(x^1,\ldots,x^n)$ is a local coordinate
system defined in a domain $U \subset M$, then we denote by $\delta_\xi$ the coordinate vector fields and by $dx^i$ the coordinate covector fields. We recall that the coordinates of a vector $\xi \in T_x M$ are the coefficients of the expansion $\xi = \xi^i \frac{\partial}{\partial x^i}$. Let $p$ be the projection on $M$. On the domain $p^{-1}(U) \subset TM$, the family
of the functions $(x^1,\ldots,x^n,\xi^1,\ldots,\xi^n)$ is a local coordinate system which is called associated with
the system $(x^1,\ldots,x^n)$. A local coordinate system on $TM$ will be called a natural coordinate system if it is
associated with some local coordinate system on $M$. In the sequel, we will use only such coordinate systems on
$TM$. The algebra of tensor fields of the manifold $TM$ is generated locally by the coordinate fields
$\delta_\xi, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial \xi^i}, dx^i, d\xi^i$. A tensor $u \in T_{s,r}(x,\xi) (TM)$ of degree $(r, s)$ at a point $(x, \xi) \in TM$ is called semibasic if in
some (and so, in any) natural coordinate system, it can be represented as:
\[u = u^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} \frac{\partial}{\partial \xi^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial \xi^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}.\]
We will abbreviate this equality to
\[u = (u^{i_1,\ldots,i_r}_{j_1,\ldots,j_s}).\]
We denote by $\beta^r_s M$ the subbundle in $\tau^r_s (TM)$ made of all semibasic tensors of degree $(r, s)$. Note that
$C^\infty(\beta^r_s M) = C^\infty(TM)$. The elements of $C^\infty(\beta^0_s M)$ are called the semibasic vector fields, and the elements
of $C^\infty(\beta^r_0 M)$ are called semibasic covector fields. Tensor fields on $M$ can be identified with the semibasic
tensor fields on $TM$ whose components are independent of the second argument $\xi$. Thus we obtain the canonical imbedding
\[(2.17) \quad t : C^\infty(\tau^r_s M) \subseteq C^\infty(\beta^r_s M).\]
Note that $t(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i}$ and $t(dx^i) = dx^i$. 

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For \( u \in C^\infty(\beta^r_s M) \), we define two semibasic tensor fields \( \nabla^v u \) and \( \nabla^h u \) by the formulas

\[
\nabla^v u = \nabla^v_k u_{j_1, \ldots, j_r} \frac{\partial}{\partial \xi^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial \xi^{j_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_r} \otimes dx^k,
\]

where

\[
\nabla^v_k u_{j_1, \ldots, j_s} = \frac{\partial}{\partial \xi^k} u_{j_1, \ldots, j_s},
\]

and

\[
\nabla^h u = \nabla^h_k u_{j_1, \ldots, j_s} \frac{\partial}{\partial \xi^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial \xi^{j_s}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s} \otimes dx^k,
\]

where

\[
\nabla^h_k u_{j_1, \ldots, j_s} = \frac{\partial}{\partial \xi^k} u_{j_1, \ldots, j_s} + \sum_{m=1}^s \Gamma^p_{kj_m} u_{j_1, \ldots, j_{m-1}, j_m+1, \ldots, j_s} - \sum_{m=1}^s \Gamma^p_{kj_m} u_{j_1, \ldots, j_m-1, j_{m+1}, \ldots, j_s}.
\]

We thus obtain two well-defined differential operators \( \nabla^v, \nabla^h : C^\infty(\beta^r_s M) \rightarrow C^\infty(\beta^r_{s+1} M) \) that are respectively called the vertical and horizontal covariant derivatives.

For \( u \in C^\infty(TM) \) the covariant field \( \nabla^v u \in C^\infty(T^*M) \) given in a coordinates system by

\[
\nabla^v u = (\nabla^v_k u) dx^k,
\]

\[
\nabla^h_k u = \frac{\partial u}{\partial \xi^k} - \Gamma^p_{kq} \xi^q \frac{\partial u}{\partial \xi^p}.
\]

The vertical covariant derivative of \( u \) is given by

\[
\nabla^v u = (\nabla^v_k u) dx^k, \quad \nabla^v_k u = \frac{\partial u}{\partial \xi^k}.
\]

We can show that these derivatives satisfy the following commutation formulas (for more details, see [32], pp. 95).

\[
\nabla^v_k \nabla^v_l = \nabla^v_l \nabla^v_k.
\]

We can also prove the following relations

\[
\nabla^h_k \xi^i = 0, \quad \nabla^v_k \xi^i = \delta^i_k.
\]

As can be easily shown, \( \nabla^v \) and \( \nabla^h \) are well-defined first-order differential operators. In particular, they extend naturally to the Sobolev space \( H^1(\beta^r_s M) \).

The vertical divergence, \( \nabla^v \text{div} \), and the horizontal divergence, \( \nabla^h \text{div} \), of a semibasic vector field \( V \) are defined by

\[
\nabla^v \text{div}(V) = \nabla^v_k v^k, \quad \nabla^h \text{div}(V) = \nabla^h_k v^k.
\]
To prove Theorem 2.1, we also need the two following divergence formulas (see [32], p 101).

\textit{Gauss-Ostrogradskii formula} of the vertical divergence

\begin{equation}
\int_{SM} \mathbf{v} \cdot \mathbf{W} \; dv^{2n-1} = (n-2) \int_{SM} \langle W, \xi \rangle \; dv^{2n-1}, \quad W \in C^\infty(TM),
\end{equation}

\textit{Gauss-Ostrogradskii formula} of the horizontal divergence

\begin{equation}
\int_{SM} \frac{h}{n} \mathbf{v} \cdot \mathbf{V} \; dv^{2n-1} = \int_{\partial SM} \langle V, \nu \rangle \; d\sigma^{2n-2}, \quad V \in C^\infty(TM),
\end{equation}

Let $H$ denote the vector field associated with the geodesic flow $\phi_t$. For $u \in C^\infty(SM)$ and $(x, \xi) \in SM$, we have

\begin{equation}
Hu(x, \xi) = \frac{d}{dt} u(\phi_t(x, \xi))|_{t=0}.
\end{equation}

and we call it the differentiation along the geodesics. In coordinate form, we have

\begin{equation}
H = \xi^i \frac{\partial}{\partial x^i} - \Gamma^i_{jk} \xi^j \frac{\partial}{\partial \xi^k} = \xi^i \left( \frac{\partial}{\partial x^i} - \Gamma^p_{iq} \xi^q \frac{\partial}{\partial \xi^p} \right) = \xi^i \nabla_i.
\end{equation}

Now we consider the \textit{Pestov identity}, which is the basic energy identity that has been used since the work of Mukhometov [25] in most injectivity proofs of ray transforms in absence of real-analyticity or special symmetries. For a function $u \in C^\infty(TM)$, we have

\begin{equation}
2 \frac{h}{n} \langle \nabla u, \nabla Hu \rangle = \frac{h}{n} \langle \nabla u \rangle^2 + \frac{h}{n} \langle \mathbf{v} \rangle \cdot \mathbf{W} - \langle R(\xi, \nabla u)\xi, \nabla u \rangle.
\end{equation}

Here $H$ is the geodesic vector field associated with the geodesic flow given by (2.29), $R$ is the curvature tensor, and the semibasic vector $V$ and $W$ are given by

\begin{equation}
V = \langle \nabla u, \nabla u \rangle \xi - \langle \xi, \nabla u \rangle \nabla u,
\end{equation}

\begin{equation}
W = \langle \xi, \nabla u \rangle \nabla u.
\end{equation}

We introduce the function $u : SM \rightarrow \mathbb{R}$ defined by

\begin{equation}
u(x, \xi) = \int_0^{\ell_+(x, \xi)} f(\gamma_{x, \xi}(t))dt.
\end{equation}

It satisfies the boundary conditions

\begin{equation}
u = \mathcal{I} f, \quad \text{on } \partial_+ SM,
\end{equation}

and, since $\ell_+(x, \xi) = 0$ for $(x, \xi) \in \partial_- SM$, we have

\begin{equation}
u = 0, \quad \text{on } \partial_- SM.
\end{equation}

\textbf{Lemma 2.2.} \textit{The function $u$ given by (2.34) is smooth on $TM \setminus T(\partial M)$ and has the following properties:}

1. $u$ is homogeneous function of degree $-1$ in $\xi$.
2. $u$ satisfies the following kinetic equation $Hu = -f$.
3. $u$ satisfies the following equation $H \nabla u = -\frac{h}{\nabla u}$. 
Proof. Item (1) is immediate from the relations $\ell_+(x, \lambda \xi) = \frac{1}{\lambda} \ell_+(x, \xi)$ and $\gamma_{x, \lambda t}(t) = \gamma_{x, \xi}(\lambda t)$ for any $\lambda > 0$. Then

$$u(x, \lambda \xi) = \int_0^{\lambda \xi + (x, \xi)} f(\gamma_{x, \xi}(\lambda t)) dt = \frac{1}{\lambda} u(x, \xi).$$

To prove item (2). For $s \in \mathbb{R}$ sufficiently small, we set $x_s = \gamma_{x, \xi}(s)$ and $\xi_s = \gamma_{x, \xi}(s)$. Then, $\gamma_{x, \xi}(t) = \gamma_{x, \xi}(t + s)$ and $\ell_+(x_s, \xi_s) = \tau_+(x, \xi) - s$. So,

$$u(\gamma_{x, \xi}(s), \gamma_{x, \xi}(s)) = u(x_s, \xi_s) = \int_0^{\ell_+(x_s, \xi_s)} f(\gamma_{x, \xi}(t + s)) dt = \int_s^{\ell_+(x, \xi)} f(\gamma_{x, \xi}(t)) dt.$$

Differentiating with respect to $s$ and taking $s = 0$, we obtain that

$$\frac{\partial u}{\partial x_i} \gamma_{x, \xi}^i(0) + \frac{\partial u}{\partial \xi_i} \gamma_{x, \xi}^i(0) = -f(x).$$

Since we have $\gamma_{x, \xi}(0) = x$, $\gamma_{x, \xi}^i(0) = \xi$ and $\gamma_{x, \xi}^i(0) = -\Gamma_{jk}^i(x) \xi^j \xi^k$, then

$$\xi^i \frac{\partial u}{\partial x_i} - \Gamma_{jk}^i(x) \xi^j \xi^k \frac{\partial u}{\partial \xi_i} = -f(x).$$

Thus we have $Hu = -f$.

To prove item (3), applying the operator $\nabla$ to the kinetic equation, we obtain $\nabla(Hu) = -\nabla f = 0$. It follows that

$$0 = \nabla(Hu) = \nabla_j(\xi^k \nabla_i u) dx^j = (\nabla_j \xi^i) \nabla_i u dx^j + \xi^i (\nabla_j \nabla_i u) dx^j.$$

From (2.24) and (2.25), we get

$$0 = \nabla u + H(\nabla u).$$

The proof is complete. \qed

To prove the Theorem 2.1 we will also need the following lemma (see [32], pp. 124, for the proof).

Lemma 2.3. Let $(M, g)$ be a simple Riemannian manifold and $K^+$ given by (2.3). Let a semibasic tensor field $u \in C^c(\beta^0, M)$ satisfies boundary condition (2.36), then the following inequality

$$(2.37) \quad \int_M K^+(x, \xi) |u(x, \xi)|^2 dv^{2n-1} \leq k^+ \int_M |Hu(x, \xi)|^2 dv^{2n-1}.$$

holds true. Here $k^+ = k^+(M, g)$ is given by (2.9).

Proof of Theorem 2.1. We suppose, for a moment, that we have proved the equality

$$(2.38) \quad \int_M \left[ \frac{h}{|\nabla u|^2} - \langle R(\xi, \nabla u) \xi, \nabla u \rangle + (n - 2) |Hu|^2 \right] dv^{2n-1} = -\int_{\partial_+ M} \langle V, \nu \rangle d\sigma^{2n-2},$$

and the estimates

$$(2.39) \quad |\int_{\partial_+ M} \langle V, \nu \rangle d\sigma^{2n-2}| \leq C \|f\|^2_{H^1(\partial_+ SM)},$$

and

$$(2.40) \quad |\int_M \langle R(\xi, \nabla u) \xi, \nabla u \rangle dv^{2n-1}| \leq k^+ \int_M |Hu|^2 dv^{2n-1}.$$
Combining (2.40), (2.39) and (2.38) with item (3) of Lemma 2.2, we find

\begin{equation}
(1 - k^+) \int_{SM} |\nabla u|^2 \, dv^{2n-1} + (n - 2) \int_{SM} |Hu|^2 \, dv^{2n-1} \leq C\|f\|_{H^1(\tilde{\partial}, SM)}^2.
\end{equation}

For \( k^+ < 1 \) and \( n \geq 2 \), we deduce the estimate

\begin{equation}
\int_{SM} |\nabla u|^2 \, dv^{2n-1} \leq C\|f\|_{H^1(\tilde{\partial}, SM)}^2.
\end{equation}

In view of the definition of \( H \), in (2.30), there exists a constant \( C \) such that

\[ |Hu|^2 \leq C|\nabla u|^2. \]

Using the item (2) of Lemma 2.2 and (2.30), we conclude that

\[ \|f\|_{L^2(SM)}^2 = \int_{SM} |Hu|^2 \, dv^{2n-1} \leq C \int_{SM} |\nabla u|^2 \, dv^{2n-1} \leq C\|f\|_{H^1(\tilde{\partial}, SM)}^2, \]

and the Theorem 2.1 is done.

Now, we come back to prove (2.38), (2.39) and (2.40). First, we start with (2.40). By (2.8) we find

\begin{equation}
|R(\xi, \nabla u)\xi, \nabla u)| \leq k^+ (x, \xi)|\nabla u(x, \xi)|^2, \quad \forall (x, \xi) \in SM.
\end{equation}

Furthermore, the lemma 2.3 combined with (2.43) gives the following estimate

\[ \int_{SM} |R(\xi, \nabla u)\xi, \nabla u)| \, dv^{2n-1} \leq k^+ \int_{SM} |H\nabla u|^2 \, dv^{2n-1} \]

\[ \leq k^+ \int_{SM} |\nabla u|^2 \, dv^{2n-1}, \]

This completes the proof of (2.40).

We prove now (2.38). Since we have \( Hu = -f \) and \( \nabla f = 0 \), then the Pestov’s identity (2.31) gives

\begin{equation}
|\nabla u|^2 + \text{div}(V) + \text{div}(W) - \langle R(\xi, \nabla u)\xi, \nabla u) \rangle = 0.
\end{equation}

Avoiding eventual singularities of \( u \) on \( T\tilde{\partial}M \), we will consider the variety \( M_\rho \) defined by

\[ M_\rho = \{ x \in M, \quad d_g(x, \partial M) \geq \rho \}, \]

where \( \rho > 0 \). In some neighbourhood of \( \partial M \), the function \( x \mapsto d_g(x, \partial M) \) is smooth and \( \partial M_\rho \) is strictly convex for sufficiently small \( \rho > 0 \). The function \( u \) is smooth on \( SM_\rho \) since \( SM_\rho \subset SM \setminus S(\partial M) \). Integrating (2.45) over \( SM_\rho \) and using the formula divergence (2.27) and (2.28), we find

\[ \int_{SM_\rho} \left[ \frac{h}{|\nabla u|^2} - \langle R(\xi, \nabla u)\xi, \nabla u \rangle \right] \, dv^{2n-1} = -\int_{SM_\rho} \left[ \frac{h}{\text{div}(V) + \text{div}(W)} \right] \, dv^{2n-1} \]

\[ = -\int_{\partial SM_\rho} \langle V, \nu \rangle \, d\sigma^{2n-2} - (n - 2) \int_{SM_\rho} \langle W, \xi \rangle \, dv^{2n-1}, \]

where \( \nu = \nu_\rho(x) \) is the unit vector of the outer normal to the boundary of \( M_\rho \). In view of (2.33), we have

\[ \langle W, \xi \rangle = \langle \xi, \nabla u \rangle^2 = |Hu|^2. \]

Hence, we obtain the equality

\begin{equation}
\int_{SM_\rho} \left[ \frac{h}{|\nabla u|^2} - \langle R(\xi, \nabla u)\xi, \nabla u \rangle + (n - 2)|Hu|^2 \right] \, dv^{2n-1} = -\int_{\partial SM_\rho} \langle V, \nu \rangle \, d\sigma^{2n-2}.
\end{equation}
Note that all the integrands of (2.47) are smooth on $S$. Taking finally, it remains to prove the estimate (2.39). In view of the boundary condition

\[
\text{Now, we wish to pass to the limit as } \rho \to 0. \text{ We will apply the Lebesgue dominated convergence theorem. Denote by } \chi_\rho \text{ the characteristic function of the set } SM_\rho \text{ and by } p \text{ the projection } p : \partial SM \to \partial SM_\rho, \quad p(x, \xi) = (x', \xi'), \text{ where } x' \text{ is such that the geodesic } \gamma_{xx'} \text{ has length } \rho \text{ and intersects } \partial M \text{ orthogonally at } x \text{ and } x', \text{ and } \xi' \text{ is obtained by the parallel translation of the vector } \xi \text{ along } \gamma_{xx'.} \text{ So the equality } (2.46) \text{ becomes}
\]

\[
(2.47) \quad \int_{SM} \left[ \frac{h}{|\nabla u|^2} - \langle R(\xi, \nabla u)\xi, \nabla u \rangle + (n - 2)|Hu|^2 \right] \chi_\rho \, dv^2n-1 = - \int_{\partial SM} \langle V, \nu \rangle p_*(d\sigma^{2n-2}).
\]

Note that all the integrands of (2.47) are smooth on $SM \setminus \partial SM$ and so, they converge towards their values almost everywhere, when $\rho \to 0$. Since the functions $|\nabla u|^2$ and $|Hu|^2$ are positive and the second function satisfies (2.43), then the left side of (2.47) converges as $\rho \to 0$. To apply the Lebesgue dominated convergence theorem in (2.47), it remains to prove that $|\langle V, \nu \rangle p_*$ is bounded by a summable function on $\partial SM$ which does not depend on $\rho$.

For $(x, \xi) \in \partial SM$, we put

\[
\begin{aligned}
\frac{h}{\nabla \tan u} &= \frac{h}{\nabla u} - \frac{h}{\langle \nabla u, \nu \rangle} \nu, \\
\frac{v}{\nabla \tan u} &= \frac{v}{\nabla u} - \frac{v}{\langle \nabla u, \xi \rangle} \xi.
\end{aligned}
\]

We see that

\[
\frac{h}{\langle \nabla \tan u, \nu \rangle} = \frac{v}{\langle \nabla \tan u, \xi \rangle} = 0.
\]

Then $\frac{h}{\nabla \tan}$ and $\frac{v}{\nabla \tan}$ are in fact differential operators on $\partial SM$ and $\frac{h}{\nabla \tan} u$, $\frac{v}{\nabla \tan} u$ are completely determined by the restriction $u|_{\partial SM}$ of $u$ on $\partial SM$.

For $(x, \xi) \in \partial SM$, by a simple computation we obtain

\[
(2.48) \quad \langle V, \nu \rangle = \langle \frac{h}{\nabla \tan} u, \frac{v}{\nabla \tan} u \rangle \langle \xi, \nu \rangle - \langle \frac{h}{\nabla \tan} u, \langle \xi \rangle \nabla \tan u, \nu \rangle.
\]

From (2.34), we can see that the derivatives $\frac{h}{\nabla \tan} u$ and $\frac{v}{\nabla \tan} u$ are locally bounded on $\partial SM$. It is important that the right-hand side of (2.48) does not contain $\langle \frac{h}{\nabla u}, \nu \rangle$ and $\langle \frac{v}{\nabla u}, \xi \rangle$.

Taking $\rho \to 0$ in the equality (2.46), we have

\[
\int_{SM} \left[ \frac{h}{|\nabla u|^2} - \langle R(\xi, \nabla u)\xi, \nabla u \rangle + (n - 2)|Hu|^2 \right] \, dv^2n-1 = - \int_{\partial SM} \langle V, \nu \rangle \, d\sigma^{2n-2}.
\]

Finally, it remains to prove the estimate (2.39). In view of the boundary condition $u = I f$ on $\partial_+ SM$ and $u = 0$ on $\partial_- SM$ we obtain

\[
\int_{\partial SM} \langle V, \nu \rangle \, d\sigma^{2n-2} = \int_{\partial_+ SM} \left( \langle \frac{h}{\nabla \tan} (I f), \frac{v}{\nabla \tan} (I f) \rangle \langle \xi, \nu \rangle - \langle \frac{h}{\nabla \tan} (I f), \langle \xi \rangle \nabla \tan (I f), \nu \rangle \right) \, d\sigma^{2n-2}
\]

\[
= \int_{\partial_+ SM} Q(I f) \, d\sigma^{2n-2}.
\]

where $Q u$ is a quadratic form in variables $\frac{h}{\nabla \tan} u$ and $\frac{v}{\nabla \tan} u$ and hence, $Q$ is a quadratic first-order differential operator on the manifold $\partial_+ SM$. Consequently, there exists a constant $C$ such that we have

\[
\left| \int_{\partial_+ SM} Q(I f) \, d\sigma^{2n-2} \right| \leq C \| I f \|^2_{H^1(\partial_+ SM)}.
\]

This completes the proof of the Theorem 2.1.
3. GEOMETRIC OPTICS SOLUTIONS FOR THE DAMPED WAVE EQUATION

The main result in this section is Lemma 3.2, which ensures the existence of a family of solutions of the wave equation.

The WKB expansion method is a classical way to construct a special solution with a large parameter of wave systems. It is based on the assumption that the solution of the wave equation, can be sought as an expansion in powers of the frequency. This expansion arises here as a power series depending on the small parameter $h$, which represents the relative wavelength of the initial conditions. Introducing it in a scalar wave equation leads to a system of coupled equations governing the behavior of the phase (eikonal equation) and of the amplitudes of the different expansion coefficients (transport equations).

Denote by $\text{div} X$ the divergence of a vector field $X \in H^1(M, TM)$ on $M$, i.e. in local coordinates (see pp. 42, [19]),

$$\text{(3.1) } \text{div} X = \frac{1}{\sqrt{|g|}} \sum_{i=1}^n \partial_i \left( \sqrt{|g|} X^i \right), \quad X = \sum_{i=1}^n X^i \partial_i, \quad |g| = \det g.$$  

If $X \in H^1(M, TM)$ the divergence formula reads

$$\text{(3.2) } \int_M \text{div} X \, dv^n = \int_{\partial M} \langle X, \nu \rangle \, d\sigma^{n-1},$$

and for a function $f \in H^1(M)$ Green’s formula reads

$$\text{(3.3) } \int_M \text{div} X f \, dv^n = -\int_M \langle X, \nabla f \rangle \, dv^n + \int_{\partial M} \langle X, \nu \rangle f \, d\sigma^{n-1}.$$  

Then if $f \in H^1(M)$ and $w \in H^2(M)$, the following identity holds

$$\text{(3.4) } \int_M \Delta w f \, dv^n = -\int_M \langle \nabla w, \nabla f \rangle \, dv^n + \int_{\partial M} \partial_w w \, d\sigma^{n-1}.$$  

In this section we give a construction of geometric optics solutions to the wave equation which concentrated along geodesic curves in space-time. We are, however, dealing with damped equations. For $T > \text{Diam}_g(M)$, let $M_1$ a simple Riemannian manifold and, $\varepsilon > 0$ such that

$$\text{(3.5) } M_1 \ni M, \quad T > \text{Diam}_g(M_1) + 2\varepsilon.$$  

The absorption coefficients $a_1, a_2$ and the potentials $q_1$ and $q_2$ may be extended to $M_1$.

Let $y \in \partial M_1$. Denote points in $M_1$ by $(r, \xi)$ where $(r, \xi)$ are polar normal coordinates in $M_1$ with center $y$. That is

$$\text{(3.6) } x = \exp_y(r\xi), \quad r = d_g(y, x) > 0, \quad \xi \in S_yM_1 = \{ \xi \in T_yM_1, \; |\xi| = 1 \}.$$  

In these coordinates (which depend on the choice of $y$) the metric takes the form

$$\tilde{g}(r, \xi) = dr^2 + g_0(r, \xi),$$

where $g_0(r, \xi)$ is a smooth positive definite metric. For any function $w$ compactly supported in $M$, we set for $r > 0$ and $\xi \in S_yM_1$

$$\tilde{w}(r, \xi) = w(\exp_y(r\xi)),$$

where we have extended $w$ by 0 outside $M$.

Finally we denote

$$S^+_yM_1 = \{ \xi \in S_yM_1, \; \langle \nu(y), \xi \rangle < 0 \}.$$
3.1. Eikonal and transport equations. We start with the following lemma which give a solution of an eikonal equation

**Lemma 3.1.** For \( y \in \partial M_1 \), we denote the function \( \varrho(x) = d_g(y, x) \). Then \( \varrho \in C^2(M) \) and satisfies the eikonal equation

\[
|\nabla \varrho|^2 = g^{ij} \frac{\partial \varrho}{\partial x_i} \frac{\partial \varrho}{\partial x_j} = 1, \quad \forall x \in M.
\]

**Proof.** By the simplicity assumption, since \( y \notin M \), we have \( \varrho \in C^\infty(M) \), and in polar normal coordinates, we get

\[
\varrho(r, \xi) = r = d_g(y, x).
\]

The proof is complete. \( \square \)

Let us introduce the following spaces:

\[
V(Q) = \left\{ \theta \in H^3(0, T; L^2(M)) \cap H^1(0, T; H^2(M)) \mid \partial_x^j \theta(\cdot, 0) = \partial_x^j \theta(\cdot, T) = 0, \ j = 0, 1, 2. \right\},
\]

and

\[
W(Q) = \left\{ \psi \in W^{2,\infty}(Q) \mid \partial_t \psi \in W^{2,\infty}(Q) \right\},
\]

equipped with the norms:

\[
\|\theta\|_{V(Q)} := \|\theta\|_{H^1(0,T;H^2(M))} + \|\theta\|_{H^3(0,T;L^2(M))}, \quad \theta \in V(Q),
\]

\[
\|\psi\|_{W(Q)} := \|\psi\|_{W^{2,\infty}(Q)} + \|\partial_t \psi\|_{W^{2,\infty}(Q)}, \quad \psi \in W(Q).
\]

We want to find a function \( \theta \in V(Q) \) which solves the first transport equation

\[
\partial_t \theta + \langle d \varrho, d\theta \rangle + \frac{1}{2} (\Delta \varrho) \theta = 0, \quad \forall t \in \mathbb{R}, \ x \in M,
\]

where \( \varrho \) is given by Lemma 3.1.

Moreover for \( a \in W^{2,\infty}(M) \), we need to find a function \( \psi_a \in W(Q) \) which solves the second transport equation

\[
\partial_t \psi_a + \langle d \varrho, d\psi_a \rangle + \frac{a}{2} \psi_a = 0, \quad \forall t \in \mathbb{R}, \ x \in M.
\]

The first step is to solve the transport equation (3.11). Recall that if \( f(r) \) is any function of the geodesic distance \( r \), then

\[
\Delta \varrho f(r) = f''(r) + \frac{\alpha^{-1}}{2} \frac{\partial \alpha}{\partial r} f'(r).
\]

Here \( \alpha = \alpha(r, \xi) \) denotes the square of the volume element in geodesic polar coordinates. The transport equation (3.11) becomes

\[
\frac{\partial \varrho}{\partial t} + \frac{\partial \varrho}{\partial r} \frac{\partial \varrho}{\partial r} + \frac{1}{4} \alpha^{-1} \frac{\partial \alpha}{\partial r} \frac{\partial \varrho}{\partial r} = 0.
\]

Thus \( \varrho \) satisfies

\[
\frac{\partial \varrho}{\partial t} + \frac{\partial \varrho}{\partial r} \frac{\partial \varrho}{\partial r} + \frac{1}{4} \alpha^{-1} \frac{\partial \alpha}{\partial r} = 0.
\]
Let \( \phi \in C^\infty_c(\mathbb{R}) \) and \( \Psi \in H^2(S_y^+M) \). Let us write \( \tilde{\theta} \) in the form

\[
(3.16) \quad \tilde{\theta}(t, r, \xi) = \alpha^{-1/4}\phi(t - r)\Psi(\xi).
\]

Direct computations yield

\[
(3.17) \quad \frac{\partial\tilde{\theta}}{\partial t}(t, r, \xi) = \alpha^{-1/4}\phi'(t - r)\Psi(\xi),
\]

and, we find

\[
(3.18) \quad \frac{\partial\tilde{\theta}}{\partial r}(t, r, \xi) = -\frac{1}{4}\alpha^{-5/4}\frac{\partial\phi}{\partial r}(t - r)\Psi(\xi) - \alpha^{-1/4}\phi'(t - r)\Psi(\xi).
\]

Finally, (3.18) and (3.17) yield

\[
(3.19) \quad \frac{\partial^2\tilde{\theta}}{\partial t\partial r}(t, r, \xi) + \frac{\partial\tilde{\theta}}{\partial r}(t, r, \xi) = -\frac{1}{4}\alpha^{-1}\tilde{\theta}(t, r, \xi)\frac{\partial\phi}{\partial r}.
\]

Now if we assume that \( \text{supp}(\phi) \subset (0, \epsilon) \), then for any \( x = \exp_y(r\xi) \in M \), it is easy to see that

\[
\tilde{\theta}(0, r, \xi) = \tilde{\theta}(T, r, \xi) = 0, \quad j = 0, 1, 2, \quad T - r > \epsilon.
\]

For the second transport equation (3.12), in polar coordinates, takes the form

\[
(3.20) \quad \frac{\partial\tilde{\psi}_a}{\partial t} + \frac{\partial\tilde{\psi}_a}{\partial r} + \frac{1}{2}\tilde{a}(r, y, \xi)\tilde{\psi}_a = 0,
\]

where \( \tilde{a}(r, y, \xi) := a(\theta_{r}(y, \xi)) \). Thus \( \tilde{\psi}_a \) satisfies

\[
(3.21) \quad \frac{\partial\tilde{\psi}_a}{\partial t} + \frac{\partial\tilde{\psi}_a}{\partial r} + \frac{1}{2}\tilde{a}(r, y, \xi)\tilde{\psi}_a = 0.
\]

Thus, we can choose \( \tilde{\psi}_a \) as following

\[
(3.22) \quad \tilde{\psi}_a(t, y, r, \xi) = \exp\left(-\frac{1}{2} \int_{0}^{t} \tilde{a}(r - s, y, \xi) ds\right).
\]

Since \( \tilde{a} \in W^{2,\infty}(M) \), we get \( \psi_a \in \mathcal{W}(Q) \). Hence (3.12) is solved.

### 3.2. WKB-solutions of the wave equation

We introduce the function

\[
(3.23) \quad \varphi(x, t) := \varrho(x) - t, \quad x \in M, \quad t \in (0, T),
\]

where \( \varrho \) is given by Lemma 3.1

**Lemma 3.2.** Let \( a \in W^{2,\infty}(M) \), \( q \in W^{2,\infty}(M) \), and \( \theta \in \mathcal{V}(Q) \), \( \psi_a \in \mathcal{W}(Q) \) solve respectively (3.11) and (3.12). Then for all \( h > 0 \) small enough, there exists a solution

\[
(3.24) \quad u(x, t) = \theta(x, t)\tilde{\psi}_a(x, t)e^{i\varphi(x, t)/h} + r_h(x, t),
\]

the remainder \( r_h(x, t) \) is such that

\[
r_h(x, t) = 0, \quad (x, t) \in \Sigma,
\]

where \( \Sigma \) is the singular sheet of the wave equation.
By a simple computation, we have
\[ r_h(x,0) = \partial_t r_h(x,0) = 0, \quad x \in M. \]
Furthermore, there exist \( C > 0, h_0 > 0 \) such that, for all \( h \leq h_0 \) the following estimates hold true.

\[
\sum_{k=0}^{2} \sum_{j=0}^{k} \| \partial^j_t r_h(\cdot, t) \|_{H^{k-j}(M)} \leq C \| \theta \|_{V(Q)}.
\]

The constant \( C \) depends only on \( T \) and \( M \) (that is \( C \) does not depend on \( a \) and \( h \)).

**Proof.** Let \( r(x, t; h) \) solves the following homogenous boundary value problem
\[
\begin{aligned}
&\left\{ \begin{array}{l}
(\partial_t^2 - \Delta + a(x) \partial_t + q(x)) r(x,t) = V_h(x,t) \quad \text{in } Q, \\
r(x,0) = \partial_t r(x,0) = 0, \\
r(x,t) = 0
\end{array} \right. \\
\end{aligned}
\]
where the source term \( V_h \) is given by
\[
V_h(x,t) = - \left( \partial_t^2 - \Delta + a(x) \partial_t + q(x) \right) \left( (\theta \psi_a)(x,t)e^{i\varphi/h} \right).
\]
To prove our Lemma it would be enough to show that \( r \) satisfies the estimates (3.25).

By a simple computation, we have
\[
-V_h(x,t) = e^{i\varphi(x,t)/h} \left( \partial_t^2 - \Delta + a(x) \partial_t + q(x) \right) ((\theta \psi_a)(x,t))
- 2i \frac{h}{\varphi} e^{i\varphi(x,t)/h} \theta_a(x,t) \left( \partial_t \theta + \langle d\theta, d\varphi \rangle \right) (x,t)
- \frac{1}{h^2} \theta \psi_a(x,t) e^{i\varphi(x,t)/h} (1 - |d\varphi|^2).
\]
Taking into account (3.7)-(3.11) and (3.12), the right-hand side of (3.28) becomes
\[
V_h(x,t) = -e^{i\varphi(x,t)/h} \left( \partial_t^2 - \Delta + a(x) \partial_t + q(x) \right) ((\theta \psi_a)(x,t))
- e^{i\varphi(x,t)/h} V_0(x,t).
\]
Since \( \theta \in V(Q) \) and \( \psi_a \in W(Q) \) we deduce that \( V_0 \in H^1_0(0,T; L^2(M)) \). Furthermore, there is a constant \( C > 0 \), such that
\[
\|V_0\|_{L^2(Q)} + \|\partial_t V_0\|_{L^2(Q)} \leq C \|\theta\|_{V(Q)}.
\]
By Lemma 1.1 we find
\[
\sum_{k=0}^{2} \sum_{j=0}^{k} \| \partial^j_t r_h(\cdot, t) \|_{H^{k-j}(M)} \leq C \| \theta \|_{V(Q)}.
\]
Since the coefficients \( a \) and \( q \) do not depend on \( t \), the function
\[
r_h^*(x,t) = \int_0^t r_h(x,s) ds,
\]
solves the mixed hyperbolic problem (3.26) with the right side
\[
V_h^*(x,t) = \int_0^t V_h(x,s) ds = -ih \int_0^t V_0(x,s) \partial_s \left( e^{i\varphi(x,s)/h} \right) ds.
\]
Integrating by part with respect to \( s \), we conclude that
\[
\|V_h^*\|_{L^2(Q)} \leq C h \|\theta\|_{V(Q)}.
\]
and by (1.5), we get
\[ \| r_h(\cdot,t) \|_{L^2(M)} = \| \hat{r}_h \|_{L^2(M)} \leq C h \| \theta \|_{\mathcal{V}(Q)}. \]
Since \( \| V_h \|_{L^2(Q)} + h \| \hat{V}_h \|_{L^2(Q)} \leq C \| \theta \|_{\mathcal{V}(Q)}, \) by using again the energy estimates for the problem (3.26), obtain
\[ \| \hat{\partial}_t r_h(\cdot,t) \|_{L^2(M)} + \| \nabla r_h(\cdot,t) \|_{L^2(M)} \leq C \| \theta \|_{\mathcal{V}(Q)}. \]
and by (1.6), we have
\[ \| \hat{\partial}_t r_h(\cdot,t) \|_{L^2(M)} + \| \nabla r_h(\cdot,t) \|_{L^2(M)} \leq C h^{-1} \| \theta \|_{\mathcal{V}(Q)}. \]
Collecting (3.32)–(3.33) we get (3.25). The proof is complete. □

By similar way, we can prove the following Lemma:

**Lemma 3.3.** Let \( a \in W^{2,\infty}(M), \) \( q \in W^{2,\infty}(M), \) and \( \theta \in \mathcal{V}(Q), \) \( \psi_{-a} \mathcal{W}(Q) \) solve respectively (3.11) and (3.12) (with \( a \) replaced by \( -a \)). Then for all \( h > 0 \) small enough, there exists a solution
\[ u(x,t;h) \in C^2(0,T;L^2(M)) \cap C^1(0,T;H^1(M)) \cap C(0,T;H^2(M)) \]
of the wave equation
\[ (\hat{\partial}_t^2 - \Delta - a(x) \hat{\partial}_t + q(x))u = 0, \quad \text{in} \quad Q, \]
with the final condition
\[ u(T,x) = \hat{\partial}_t u(T,x) = 0, \quad \text{in} \quad M, \]
of the form
\[ u(x,t;h) = \theta(x,t)\psi_{-a}(x,t)e^{i\varphi(x,t)/h} + r_h(x,t), \]
the remainder \( r_h(t,x) \) is such that
\[ r_h(x,t) = 0, \quad (x,t) \in \Sigma, \]
\[ r_h(x,T) = \hat{\partial}_t r_h(x,T) = 0, \quad x \in M. \]
Furthermore, there exist \( C > 0, h_0 > 0 \) such that, for all \( h \leq h_0 \) the following estimates hold true.
\[ \sum_{k=0}^{2} \sum_{j=0}^{k} h^{k-1} \| \hat{\partial}_t^j r_h(\cdot,t) \|_{H^{k-j}(M)} \leq C \| \theta \|_{\mathcal{V}(Q)}. \]
The constant \( C \) depends only on \( T \) and \( M \) (that is \( C \) does not depend on \( a \) and \( h \)).

4. **Stable determination of the absorption coefficient**

In this section, we prove the stability estimate of the absorption coefficient \( a \). We are going to use the geometrical optics solutions constructed in the previous section; this will provide information on the geodesic ray transform of the difference of two absorption coefficients.
4.1. Preliminary estimates. The main purpose of this section is to present a preliminary estimate, which relates the difference of two absorption coefficients to the Dirichlet-to-Neumann map. As before, we let $a_1$, $a_2 \in \mathcal{A}(m_1, \eta)$ and $q_1, q_2 \in \mathcal{Q}(m_2)$ such that $a_1 = a_2$, $q_1 = q_2$ near the boundary $\partial M$. We set

$$a(x) = (a_1 - a_2)(x), \quad q(x) = (q_1 - q_2)(x).$$

Recall that we have extended $a_1, a_2$ as $W^{2,\infty}(M_1)$ in such a way that $a = 0$ and $q = 0$ on $M_1 \setminus M$. We denote $\psi_{a_2} \in \mathcal{W}(Q)$ and $\psi_{-a_1} \in \mathcal{W}(Q)$ the solutions of (3.12) respectively with $a = a_2$ and $a = -a_1$ given by (3.22), and set

$$\psi_a(x, t) = \psi_{a_2}(x, t)\psi_{-a_1}(x, t).$$

**Lemma 4.1.** Let $T > 0$. There exist $C > 0$ such that for any $\theta_j \in \mathcal{W}(Q)$, $j = 1, 2$, satisfying the transport equation (3.7), the following estimate holds true:

$$| \int_0^T \int_M a(x)(\theta_2 \bar{\theta}_1)(x, t)\psi_a(x, t) \, dx \, dt | \leq C (h + h^{-2}\|\Lambda_{a_1, q_1} - \Lambda_{a_2, q_2}\|) \|\theta_1\|_{\mathcal{W}(Q)}\|\theta_2\|_{\mathcal{W}(Q)}$$

for all $h \in (0, h_0)$.

**Proof.** First, if $\theta_2$ satisfies (3.11), $\psi_{a_2}$ satisfies (3.12), and $h < h_0$, Lemma 3.2 guarantees the existence of a geometrical optics solution $u_2$

$$u_2(x, t) = (\theta_2 \psi_{a_2})(x, t)e^{i\varphi(x, t)/h} + r_{2,h}(x, t),$$

to the wave equation corresponding to the coefficients $a_2$ and $q_2$,

$$\left(\partial_t^2 - \Delta + a_2(x)\partial_t + q_2(x)\right) u(x, t) = 0 \quad \text{in } Q, \quad u(\cdot, 0) = \partial_t u(\cdot, 0) = 0 \quad \text{in } M,$$

where $r_{2,h}$ satisfies

$$h^{-1}\|r_{2,h}(. , t)\|_{L^2(M)} + \|\partial_t r_{2,h}(. , t)\|_{L^2(M)} + \|\nabla r_{2,h}(. , t)\|_{L^2(M)} \leq C\|\theta_2\|_{\mathcal{W}(Q)},$$

$$r_{2,h}(x, t) = 0, \quad \forall (x, t) \in \Sigma.$$

Moreover

$$u_2 \in C^2(0, T; L^2(M)) \cap C^1(0, T; H^1(M)) \cap C(0, T; H^2(M)).$$

Let us denote by $f_h$ the function

$$f_h(x, t) = (\theta_2 \psi_{a_2})(x, t)e^{i\varphi(x, t)/h}, \quad (x, t) \in \Sigma,$$

and we consider $v$ the solution of the following non-homogenous boundary value problem

$$\left\{ \begin{array}{l}
(\partial_t^2 - \Delta + a_1(x)\partial_t + q_1(x)) v = 0, \quad \text{in } Q, \\
v(x, 0) = \partial_t v(x, 0) = 0, \quad \text{in } M, \\
v(x, t) = u_2(x, t) := f_h(x, t), \quad \text{on } \Sigma.
\end{array} \right.$$

We let $w = v - u_2$. Therefore, $w$ solves the following homogenous boundary value problem

$$\left\{ \begin{array}{l}
(\partial_t^2 - \Delta + a_1(x)\partial_t + q_1(x)) w(x, t) = a(x)\partial_t u_2(x, t) + q(x)u_2(x, t) \quad \text{in } Q, \\
w(x, 0) = \partial_t w(x, 0) = 0, \quad \text{in } M, \\
w(x, t) = 0, \quad \text{on } \Sigma.
\end{array} \right.$$

Using the fact that $a(x)\partial_t u_2 + q(x)u_2 \in W^{1,1}(0, T; L^2(M))$ with $u_2(\cdot, 0) = \partial_t u_2(\cdot, 0) \equiv 0$, by Lemma 1.1 we deduce that

$$w \in C^1(0, T; L^2(M)) \cap C(0, T; H^2(M) \cap H^1_0(M)).$$
Integrating by parts and using Green’s formula (3.4), we find
\[
\int_0^T \left( \frac{\partial^2 \varphi}{\partial t^2} - \Delta - a_1(x) \frac{\partial \varphi}{\partial t} + q_1(x) \right) u_1(x, t) \, dt = 0, \quad (x, t) \in Q,
\]
having the special form
\[
(4.6) \quad u_1(x, t) = (\theta_1 \psi_{-a_1})(x, t) e^{i \varphi(x, t) / h} + r_{1,h}(x, t),
\]
which corresponds to the coefficients \(-a_1\) and \(q_1\), where \(r_{1,h}\) satisfies for \(h < h_0\)
\[
(4.7) \quad h^{-1} \| r_{1,h} \|_{L^2(M)} + \| \frac{\partial}{\partial t} r_{1,h} \|_{L^2(M)} + \| \nabla r_{1,h} \|_{L^2(M)} \leq C \| \theta_1 \|_{V(Q)}.
\]
Integrating by parts and using Green’s formula (3.4), we find
\[
(4.8) \quad \int_0^T \int_M \left( \frac{\partial^2 \varphi}{\partial t^2} - \Delta + a_1(x) \frac{\partial \varphi}{\partial t} + q_1(x) \right) w \, dv \, dt = \int_0^T \int_M a(x) \frac{\partial \varphi}{\partial t} w_2 \, dv \, dt
\]
\[
+ \int_0^T \int_M q(x) w_2 \, dv \, dt = - \int_0^T \int_M \frac{\partial}{\partial t} \varphi w_1 \, ds \, dt.
\]
Taking (4.8), (4.6) into account, we deduce
\[
(4.9) \quad - \int_0^T \int_M a(x) \frac{\partial \varphi}{\partial t} w_2 \, dv \, dt = \int_0^T \int_M \left( \Lambda_{a_1,q_1} - \Lambda_{q_2,q_2} \right) f_h(x, t) \varphi(x, t) \, ds \, dt
\]
\[
+ \int_0^T \int_M q(x) w_2 \, dv \, dt
\]
where \(g_h \) is given by
\[
g_h(x, t) = (\theta_1 \psi_{-a_1})(x, t) e^{i \varphi(x, t) / h}, \quad (x, t) \in \Sigma.
\]
It follows from (4.9), (4.6) and (4.3) that
\[
(4.10) \quad i h^{-1} \int_0^T \int_M a(x) \left( \frac{\partial \varphi}{\partial t} \right) (\theta_2 \varphi_{-a_2})(x, t) \psi_{-a_1}(x, t) \, dv \, dt =
\]
\[
\int_0^T \int_M \varphi_h \left( \Lambda_{a_1,q_1} - \Lambda_{q_2,q_2} \right) f_h \, ds \, dt - i h^{-1} \int_0^T \int_M a(x) \left( \frac{\partial \varphi}{\partial t} \right) (\theta_2 \varphi_{a_2})(x, t) \varphi_h(x, t) \, dv \, dt
\]
\[
+ \int_0^T \int_M a(x) \frac{\partial \varphi}{\partial t} (\theta_2 \varphi_{a_2})(x, t) \, dv \, dt + \int_0^T \int_M a(x) \frac{\partial \varphi}{\partial t} (\theta_2 \varphi_{a_2})(x, t) \varphi_h(x, t) \, dv \, dt
\]
\[
+ \int_0^T \int_M a(x) \frac{\partial \varphi}{\partial t} r_{2,h}(x, t) \psi_{a_1}(x, t) \, dv \, dt + \int_0^T \int_M a(x) \frac{\partial \varphi}{\partial t} r_{2,h}(x, t) \varphi_h(x, t) \, dv \, dt
\]
\[
+ \int_0^T \int_M q(x) w_2 \, dv \, dt = \int_0^T \int_M \varphi_h \left( \Lambda_{a_1,q_1} - \Lambda_{q_2,q_2} \right) f_h \, ds \, dt + \mathcal{R}_h.
\]
In view of (4.7) and (4.4), we have
\[
(4.11) \quad | \mathcal{R}_h | \leq C \| \theta_1 \|_{V(Q)} \| \theta_2 \|_{V(Q)}.
\]
On the other hand, by the trace theorem, we find

\[
\left| \int_0^T \int_{\partial M} \left( \lambda_{a_1, q_1} - \lambda_{a_2, q_2} \right) (x) \gamma_h \, d\sigma \, dt \right| \leq \| \lambda_{a_1, q_1} - \lambda_{a_2, q_2} \| H^1(\Sigma) \| \gamma_h \| L^2(\Sigma)
\]

(4.12)

The estimate (4.2) follows easily from (4.10), (4.11) and (4.12). This completes the proof of the Lemma.

**Lemma 4.2.** There exists \( C > 0 \) such that for any \( \Psi \in H^2(S_y M_1) \), the following estimate

\[
\left| \int_{S^+_y M_1} \left( \exp \left( -\frac{1}{2} \mathcal{L}(a)(y, \xi) \right) - 1 \right) \Psi(\xi) \, d\omega_y(\xi) \right| 
\leq C \left( h + h^{-2} \| \lambda_{a_1, q_1} - \lambda_{a_2, q_2} \| \right) \| \Psi \|_{H^2(S_y M_1)}
\]

holds for any \( y \in \partial M_1 \).

We use the notation

\[ S^+_y M_1 = \{ \xi \in S_y M_1 : \langle \nu, \xi \rangle < 0 \} \]

**Proof.** We take two solutions to (3.11) of the form

\[
\tilde{\theta}_1(t, r, \xi) = \alpha^{-1/4} \phi(t - r) \Psi(\xi),
\]

\[
\tilde{\theta}_2(t, r, \xi) = \alpha^{-1/4} \phi(t - r).
\]

Now we change variable in the left term of (4.1), \( x = \exp_y(r \xi) \), \( r > 0 \) and \( \xi \in S_y M_1 \), we have

\[
\int_0^T \int_M a(x)(\tilde{\theta}_1 \theta_2)(x, t) \psi_a(x, t) \, dv^n \, dt
\]

(4.14)

\[
= \int_0^T \int_{S^+_y M_1} \tilde{\alpha}(r, y, \xi) (\tilde{\theta}_1 \tilde{\theta}_2)(r, t, \xi) \tilde{\psi}_a(t, r, \xi) \alpha^{1/2} \, dr \, d\omega_y(\xi) \, dt
\]

\[
= \int_0^T \int_{S^+_y M_1} \tilde{\alpha}(r, y, \xi) \phi^2(t - r) \tilde{\psi}_a(t, r, \xi) \Psi(\xi) \, dr \, d\omega_y(\xi) \, dt
\]

\[
= \int_0^T \int_{S^+_y M_1} \tilde{\alpha}(t - \tau, r, \xi) \phi^2(\tau) \tilde{\psi}_a(t, \tau, \xi) \Psi(\xi) \, d\tau \, d\omega_y(\xi) \, dt
\]

By the support properties of the function \( \phi \), we get that the left-hand side term in (4.14) reads

\[
\int_{S^+_y M_1} \phi^2(\tau) \left[ \exp \left( -\frac{1}{2} \int_0^T \tilde{a}(s - \tau, y, \xi) \, ds \right) - 1 \right] \Psi(\xi) \, d\tau \, d\omega_y(\xi) =
\]
\[ \int_{S^+_y M_1} \left[ \exp \left( -\frac{1}{2} \int_0^{\ell_+ (y, \xi)} \tilde{a}(s, y, \xi) \, ds \right) - 1 \right] \Psi(\xi) \mu(y, \xi) \, d\omega_y(\xi). \]

Then, by (4.14) and (4.2) we get

\[ (4.15) \quad \int_{S^+_y M_1} \left( \exp \left( -\frac{1}{2} \mathcal{I}(a)(y, \xi) \right) - 1 \right) \Psi(\xi) \, d\omega_y(\xi) \leq C \left( h + h^{-2} \| \Lambda_{\alpha_1 q_1} - \Lambda_{\alpha_2 q_2} \| \right) \| \Psi \|_{H^2(S^+_y M_1)}. \]

This completes the proof of the Lemma.

4.2. **End of the proof of the stability estimate of the absorption coefficient.** Let us now complete the proof of the stability estimate of the absorption coefficient.

We define the Poisson kernel of \( B(0, 1) \subset T_y M_1 \), i.e.,

\[ P(\theta, \xi) = \frac{1 - |\theta|^2}{\alpha_n |\theta - \xi|^n}, \quad \theta \in B(0, 1); \quad \xi \in S^+_y M_1. \]

For \( 0 < \kappa < 1 \), we define \( \Psi_\kappa : S^+_y M_1 \times S^+_y M_1 \rightarrow \mathbb{R} \) as

\[ (4.16) \quad \Psi_\kappa(\theta, \xi) = P(\kappa \theta, \xi). \]

We have the following Lemma (see Appendix A for the proof).

**Lemma 4.3.** Let \( \Psi_\kappa \) given by (4.16), \( \kappa \in (0, 1) \). Then we have the following properties:

\[ (4.17) \quad 0 \leq \Psi_\kappa(\theta, \xi) \leq 2 \frac{1 - \kappa n}{\alpha_n (1 - \kappa)^{n-1}}, \quad \forall \kappa \in (0, 1), \quad \forall \xi, \theta \in S^+_y M_1. \]

\[ (4.18) \quad \int_{S^+_y M_1} \Psi_\kappa(\theta, \xi) \, d\omega_y(\xi) = 1, \quad \forall \kappa \in (0, 1), \quad \forall \theta \in S^+_y M_1. \]

\[ (4.19) \quad \int_{S^+_y M_1} \Psi_\kappa(\theta, \xi) |\theta - \xi| \, d\omega_y(\xi) \leq C (1 - \kappa)^{1/2 \nu}, \quad \forall \kappa \in (0, 1), \quad \forall \theta \in S^+_y M_1. \]

\[ (4.20) \quad \| \Psi_\kappa(\theta, \cdot) \|^2_{H^2(S^+_y M_1)} \leq \frac{C}{(1 - \kappa)^{n+3}}, \quad \forall \kappa \in (0, 1), \quad \forall \theta \in S^+_y M_1. \]

**Lemma 4.4.** Let \( \alpha_i \in \mathcal{A}(m_1, \alpha), \; q_i \in \mathcal{P}(m_2), \; i = 1, 2 \). There exist \( C > 0, \; \delta > 0, \; \beta > 0 \) and \( h_0 > 0 \) such that

\[ (4.21) \quad |\mathcal{I}(a)(y, \theta)| \leq C \left( h^{-\delta} \| \Lambda_{\alpha_2 q_2} - \Lambda_{\alpha_1 q_1} \| + h^3 \right), \quad \forall (y, \theta) \in \partial^+_S M_1, \]

for any \( h \leq h_0 \). Here \( C \) depends only on \( M, T, m_1 \) and \( m_2 \).

**Proof.** Let \((y, \theta) \in \partial^+_S M_1 \) be a fixed and let \( \Psi_\kappa \) be the positive function given by (4.16). We extend \( \mathcal{I}(a) \) by zero in \( \partial_- S M \), then, we have

\[ (4.22) \quad \left| \exp \left( -\frac{1}{2} \mathcal{I}(a)(y, \theta) \right) - 1 \right| = \left| \int_{S^+_y M_1} \Psi_\kappa(\theta, \xi) \left[ \exp \left( -\frac{1}{2} \mathcal{I}(a)(y, \theta) \right) - 1 \right] \, d\omega_y(\xi) \right| \]

\[ \leq \int_{S^+_y M_1} \Psi_\kappa(\theta, \xi) \left[ \exp \left( -\frac{1}{2} \mathcal{I}(a)(y, \theta) \right) - \exp \left( -\frac{1}{2} \mathcal{I}(a)(y, \xi) \right) \right] \, d\omega_y(\xi) \]
The proof of Lemma 4.4 is complete. □

Therefore, since we have
\[
\left| \exp\left(-\frac{1}{2} I(a)(y,\theta)\right) - \exp\left(-\frac{1}{2} I(a)(y,\xi)\right) \right| \leq C \left| I(a)(y,\theta) - I(a)(y,\xi) \right|,
\]
and using the fact that
\[
\left| I(a)(y,\theta) - I(a)(y,\xi) \right| \leq C |\theta - \xi|,
\]
we deduce upon applying Lemma 4.2 with \( \Psi = \Psi_{\kappa}(\theta,\cdot) \) the following estimation
\[
\left| \exp\left(-\frac{1}{2} I(a)(y,\theta)\right) - 1 \right| \leq C \int_{S_{y}M_{1}} \Psi_{\kappa}(\theta,\xi) |\theta - \xi| \ d\omega_{y}(\xi) + C \left( h^{-2} \| \Lambda_{a_{2},q_{2}} - \Lambda_{a_{1},q_{1}} \| + h \right) \| \Psi_{\kappa} \|_{H^{2}(S_{y}M_{1})}^{2}.
\]

On the other hand, by (4.20) and (4.19), we have the following inequality
\[
\left| \exp\left(-\frac{1}{2} I(a)(y,\theta)\right) - 1 \right| \leq C \left( 1 - \kappa \right)^{1/2n} + C \left( h^{-2} \| \Lambda_{a_{2},q_{2}} - \Lambda_{a_{1},q_{1}} \| + h \right) \left( 1 - \kappa \right)^{-(3+n)}.
\]

Selecting \( (1 - \kappa) \) small such that 
\[
(1 - \kappa)^{1/2n} = h(1 - \kappa)^{-(n+3)},
\]
that is 
\[
(1 - \kappa) = h^{2n/1+2n^{2}+6n},
\]
we find two constants \( \delta > 0 \) and \( \beta > 0 \) such that
\[
\left| \exp\left(-\frac{1}{2} I(a)(y,\theta)\right) - 1 \right| \leq C \left[ h^{-\delta} \| \Lambda_{a_{2},q_{2}} - \Lambda_{a_{1},q_{1}} \| + h^{\beta} \right].
\]

Now, using the fact that 
\[
|X| \leq e^{M} |e^{X} - 1| \text{ for any } |X| \leq M,
\]
we deduce that
\[
\left| -\frac{1}{2} I(a)(y,\theta) \right| \leq e^{M_{1}T} \left| \exp\left(-\frac{1}{2} I(a)(y,\theta)\right) - 1 \right|.
\]

Hence, we conclude that for all \( \theta \in S_{y}M_{1} \) and \( y \in e \) we have
\[
\left| I(a)(y,\theta) \right| \leq C \left( h^{-\delta} \| \Lambda_{a_{2},q_{2}} - \Lambda_{a_{1},q_{1}} \| + h^{\beta} \right).
\]

The proof of Lemma 4.4 is complete. □

Integrating the estimate (4.21) over \( \partial_{+}SM_{1} \), with respect to \( \mu(y,\theta) \ d\sigma_{2n-2}(y,\theta) \), then minimizing on \( h \), we get
\[
\|I(a)\|_{L^{2}(\partial_{+}SM_{1})} \leq C \| \Lambda_{a_{2},q_{2}} - \Lambda_{a_{1},q_{1}} \|^{\frac{\delta}{n+2}}.
\]

With respect to the Theorem 2.1 we have
\[
\|a\|_{L^{2}(M)} \leq C \|I(a)\|_{H^{1}(\partial_{+}SM_{1})}.
\]

By interpolation inequality and (2.7), we have
\[
\|I(a)\|_{H^{1}(\partial_{+}SM_{1})} \leq C \|I(a)\|_{L^{2}(\partial_{+}SM_{1})} \|I(a)\|_{H^{2}(\partial_{+}SM_{1})}
\]
\[
\leq C \|I(a)\|_{L^{2}(\partial_{+}SM_{1})}.
\]

Combining with (4.24) and (4.25), we get
\[
\|a\|_{L^{2}(M)} \leq C \| \Lambda_{a_{2},q_{2}} - \Lambda_{a_{1},q_{1}} \|^{\frac{\delta}{n}},
\]
We denote $s_0 = \frac{\beta}{2(\beta + \delta)}$.

Moreover, let $\eta_0 \in (n/2, \eta)$, by Sobolev embedding and interpolation inequality, there exists $\delta \in (0, 1)$ such that

$$
\|a\|_{C^0(M)} \leq \|a\|_{H^{\eta_0}(M)} \leq C {\|a\|_{L^2(M)}^{\delta} \|a\|^{1-\delta}_{H^{\eta}(M)}} \leq C \|\Lambda_{a_2,q_2} - \Lambda_{a_1,q_1}\|^{\delta s_0}.
$$

This completes the proof of the Hölder stability estimate of the absorption coefficient.

## 5. Stable Determination of the Electric Potential

In this section, we prove a stability estimate for the electric potential $q$. We use the stability result obtained for the absorption coefficient $a$. Like in the previous section, we let $a_1, a_2 \in \mathcal{A}(m_1, \eta)$ and $q_1, q_2 \in \mathcal{P}(m_2)$ such that $a_1 = a_2, q_1 = q_2$ near the boundary $\partial M$. We set

$$
a(x) = (a_1 - a_2)(x), \quad q(x) = (q_1 - q_2)(x).
$$

Recall that we have extended $a_1, a_2$ as $W^{2,\infty}(M_1)$ in such a way that $a = 0$ and $q = 0$ on $M_1 \setminus \partial M$.

We denote $\psi_{a_2} \in \mathcal{W}(Q)$ and $\psi_{-a_1} \in \mathcal{W}(Q)$ the solutions of (3.12) respectively with $a = a_2$ and $a = -a_1$ given by (3.22), and set

$$
\psi_{a}(x, t) = \psi_{a_2}(x, t)\psi_{-a_1}(x, t).
$$

We have a prelemmary estimate.

**Lemma 5.1.** Let $T > 0$. There exist $C > 0$ such that for any $\theta_j \in \mathcal{V}(Q)$, $j = 1, 2$, satisfying the transport equation (3.17), the following estimate holds true:

$$
\int_0^T \int_M q(x)(\theta_2 \overline{\theta}_1)(x,t) \, dv^n \, dt \leq C \Big( h + h^{-1} \|a\|_{C^0(M)} + h^{-\frac{\delta}{2}} \|\Lambda_{a_1,q_1} - \Lambda_{a_2,q_2}\| \Big) \|\theta_1\|_{\mathcal{V}(Q)} \|\theta_2\|_{\mathcal{V}(Q)}
$$

for all $h \in (0, h_0)$.

**Proof:** From the equality (4.10) and the expressions (4.6) and (4.3), it follows that

$$
\int_0^T \int_M q(x)(\theta_2 \overline{\theta}_1)(x,t) \psi_a(x,t) \, dv^n \, dt = - \int_0^T \int_{\partial M} \overline{\theta}_h (\Lambda_{a_1,q_1} - \Lambda_{a_2,q_2}) f_h \, d\sigma^{n-1} \, dt
$$

$$
+ i h^{-1} \int_0^T \int_M a(x)(\theta_2 \overline{\theta}_1)(x,t) (\psi_{a_2} \psi_{-a_1})(x,t) \, dv^n \, dt + i h^{-1} \int_0^T \int_M a(x)(\theta_2 \psi_{a_2})(x,t) \nu_{1,h} e^{i\varphi/h} \, dv^n \, dt
$$

$$
- \int_0^T \int_M a(x) \overline{\psi}_t(x,t) (\overline{\theta}_1 \psi_{a_1})(x,t) \, dv^n \, dt - \int_0^T \int_M a(x) \overline{\psi}_t(x,t) \overline{\theta}_{1,h} e^{i\varphi/h} \, dv^n \, dt
$$

$$
- \int_0^T \int_M a(x) \overline{\psi}_t(x,t) r_{2,h}(x,t) e^{-i\varphi/h} \, dv^n \, dt - \int_0^T \int_M a(x) \overline{\psi}_t(x,t) \nu_{1,h}(x,t) e^{i\varphi/h} \, dv^n \, dt
$$

$$
- \int_0^T \int_M q(x)(\overline{\theta}_1 \psi_{-a_1})(x,t) r_{2,h}(x,t) e^{-i\varphi/h} \, dv^n \, dt - \int_0^T \int_M q(x)(\theta_2 \psi_{a_2})(x,t) \nu_{1,h}(x,t) e^{i\varphi/h} \, dv^n \, dt
$$

$$
- \int_0^T \int_M q(x)(\overline{\theta}_1 \psi_{-a_1})(x,t) \nu_{1,h}(x,t) e^{i\varphi/h} \, dv^n \, dt
$$

We set

$$
\int_0^T \int_M q(x)(\theta_2 \overline{\theta}_1)(x,t) \psi_a(x,t) \, dv^n \, dt = - \int_0^T \int_{\partial M} \overline{\theta}_h (\Lambda_{a_1,q_1} - \Lambda_{a_2,q_2}) f_h \, d\sigma^{n-1} \, dt + \mathcal{R}_h' \text{.}
$$
In view of (4.7) and (4.4), we have
\begin{equation}
|\mathcal{R}_h| \leq C \left( h + h^{-1} \|a\|_{C^0(M)} \right) \|\theta_1\|_{\mathcal{V}(Q)} \|\theta_2\|_{\mathcal{V}(Q)}.
\end{equation}

On the other hand, by the trace theorem, we find
\begin{equation}
\left| \int_0^T \int_{\partial M} (\Lambda_{a_1,q_1} - \Lambda_{a_2,q_2}) (f_h) \frac{\partial \varphi}{\partial n} \, \text{d}t \right| \leq \|\Lambda_{a_1,q_1} - \Lambda_{a_2,q_2}\| \|f_h\|_{H^1(S)} \|\varphi\|_{L^2(S)}
\end{equation}

Furthermore
\begin{equation}
\int_0^T \int_M q(x)(\theta_2 \varphi_1)(x,t) \, \text{d}v \, dt = \int_0^T \int_M q(x)(\theta_2 \varphi_1)(x,t)(1 - \psi_a(x,t)) \, \text{d}v \, dt
\end{equation}
\begin{equation}
+ \int_0^T \int_M q(x)(\theta_2 \varphi_1)(x,t)\psi_a(x,t) \, \text{d}v \, dt
\end{equation}
and since
\begin{equation}
|1 - \psi_a(x,t)| \leq C \|a\|_{C(M)}
\end{equation}
we deduce that
\begin{equation}
\left| \int_0^T \int_M q(x)(\theta_2 \varphi_1)(x,t)(1 - \psi_a(x,t)) \, \text{d}v \, dt \right| \leq C \|a\|_{C^0(M)} \|\theta_1\|_{\mathcal{V}(Q)} \|\theta_2\|_{\mathcal{V}(Q)}.
\end{equation}

Combining with (5.4), (5.5) and (5.6), the estimate (5.2) follows.

This completes the proof of the Lemma.

**Lemma 5.2.** There exists \( C > 0 \) such that for any \( \Psi \in H^2(S_y M_1) \), the following estimate
\begin{equation}
\int_{S_y M_1} Tq(y,\xi) \Psi(\xi) \, \text{d}\omega_y(\xi) \leq C \left( h^2 + h^{-1} \|a\|_{C^0(M)} + h^{-2} \|\Lambda_{a_1,q_1} - \Lambda_{a_2,q_2}\| \right) \|\Psi\|_{H^2(S_y M_1)},
\end{equation}
holds for any \( y \in \partial M_1 \).

**Proof.** As in Lemma 4.2, we take two solutions to (3.11) of the form
\begin{equation}
\tilde{\theta}_1(t,r,\xi) = \alpha^{-1/4} \phi(t-r) \Psi(\xi),
\end{equation}
\begin{equation}
\tilde{\theta}_2(t,r,\xi) = \alpha^{-1/4} \phi(t-r).
\end{equation}

We change variable in the left term of (5.2), \( x = \exp_y(r\xi), \, r > 0 \) and \( \xi \in S_y M_1 \). We have
\begin{equation}
\int_0^T \int_M q(x)(\tilde{\theta}_1 \tilde{\theta}_2)(x,t) \, \text{d}v \, dt
\end{equation}
\begin{equation}
= \int_0^T \int_{S_y M_1} \int_0^{\ell_+(y,\xi)} q(r,y,\xi)(\tilde{\theta}_1 \tilde{\theta}_2)(t,r,\xi) \alpha^{1/2} \, \text{d}r \, \text{d}\omega_y(\xi) \, dt
\end{equation}
\begin{equation}
= \int_0^T \int_{S_y M_1} \int_0^{\ell_+(y,\xi)} q(r,y,\xi) \phi^2(t-r) \Psi(\xi) \, \text{d}r \, \text{d}\omega_y(\xi) \, dt.
\end{equation}

By support properties of the function \( \phi \), we get
\begin{equation}
\int_0^T \int_M q(x)(\tilde{\theta}_1 \tilde{\theta}_2)(x,t) \, \text{d}v \, dt = \left( \int_\mathbb{R} \phi^2(t) \, dt \right) \int_{S_y M_1} Tq(y,\xi) \Psi(\xi) \, \text{d}\omega_y(\xi).
\end{equation}
Then, by (5.2), we get
\[
\| \int_{S_y M_1} I(q(y, \xi)) \Psi(\xi) \, d\omega_y(\xi) \| \leq C \left( h + h^{-1} \| a \|_{C^0(M)} + h^{-3} \| \Lambda_{\alpha, q_1} - \Lambda_{\alpha, q_2} \| \right) \| \Psi \|_{H^2(S^+_y M_1)}.
\]
This complete the proof. \qed

Let us now complete the proof of the stability estimate of the electric potential. We recall the definition of the Poisson kernel of \( B(0, 1) \subset T_y M_1 \), i.e.,
\[
P(\theta, \xi) = \frac{1 - |\theta|^2}{\alpha_3 |\theta - \xi|^n}, \quad \theta \in B(0, 1); \quad \xi \in S_y M_1.
\]
For \( 0 < \kappa < 1 \), we define \( \Psi_\kappa : S_y M_1 \times S_y M_1 \to \mathbb{R} \) as
\[
(5.10) \quad \Psi_\kappa(\theta, \xi) = P(\kappa \theta, \xi).
\]

Lemma 5.3. Let \( a_i \in \mathcal{A}(m_1, \alpha), \, q_i \in \mathcal{Q}(m_2), \, i = 1, \, 2 \). There exist \( C > 0, \, \delta > 0, \, \beta > 0 \) and \( h_0 > 0 \) such that
\[
(5.11) \quad |I(q)(y, \theta)| \leq C \left( h^{-\delta} \| \Lambda_{\alpha_2, q_2} - \Lambda_{\alpha_1, q_1} \| + h^{-\delta} \| a \|_{C^0(M)} + h^\beta \right), \quad \forall (y, \theta) \in \partial^+ SM_1,
\]
for any \( h \leq h_0 \). Here \( C \) depends only on \( M, \, T, \, m_1 \) and \( m_2 \).

Proof. We fix \((y, \theta) \in \partial^+ SM_1\) and let the positive function \( \Psi_\kappa \) given by (5.10), we have
\[
(5.12) \quad |I(q)(y, \theta)| = \left| \int_{S_y M_1} \Psi_\kappa(\theta, \xi)I(q)(y, \theta) \, d\omega_y(\xi) \right| \
\leq \left| \int_{S_y M_1} \Psi_\kappa(\theta, \xi) \left( I(q)(y, \theta) - I(q)(y, \xi) \right) \, d\omega_y(\xi) \right| \
+ \left| \int_{S_y M_1} \Psi_\kappa(\theta, \xi) I(q)(y, \xi) \, d\omega_y(\xi) \right|.
\]
Since we have
\[
\left| I(q)(y, \theta) - I(q)(y, \xi) \right| \leq C |\theta - \xi|,
\]
then we deduce upon applying Lemma 5.2 with \( \Psi = \Psi_\kappa(\theta, \cdot) \) that we have the following estimation
\[
|I(q)(y, \theta)| \leq C \int_{S_y M_1} \Psi_\kappa(\theta, \xi) |\theta - \xi| \, d\omega_y(\xi) \
+ C \left( h + h^{-1} \| a \|_{C^0(M)} + h^{-3} \| \Lambda_{\alpha_2, q_2} - \Lambda_{\alpha_1, q_1} \| \right) \| \Psi_\kappa \|_{H^2(S_y M_1)}^2.
\]
On the other hand, by (4.20) and (4.19), we obtain
\[
|I(q)(y, \theta)| \leq C (1 - \kappa)^{1/2n} + C \left( h^{-3} \| \Lambda_{\alpha_2, q_2} - \Lambda_{\alpha_1, q_1} \| + h^{-1} \| a \|_{C^0(M)} + h \right) (1 - \kappa)^{-3(n+1)}.
\]
Take \( (1 - \kappa) \) small such that \((1 - \kappa)^{1/2n} = h(1 - \kappa)^{-(n+3)}\), that is \((1 - \kappa) = h^{2n/1+2n^2+6n}\), we find two constants \( \delta > 0 \) and \( \beta > 0 \) such that
\[
|I(q)(y, \theta)| \leq C \left[ h^{-\delta} \| \Lambda_{\alpha_2, q_2} - \Lambda_{\alpha_1, q_1} \| + h^{-\delta} \| a \|_{C^0(M)} + h^\beta \right].
\]
The proof of Lemma 5.3 is complete. \qed
Integrating the estimate (5.11) over \( \partial_+ SM_1 \), with respect to \( \mu(y, \theta) \) \( d\sigma^{2n-2}(y, \theta) \), then minimizing on \( h \), we get
\[
\|I(q)\|_{L^2(\partial_+ SM_1)} \leq C(\|A_{a_2, q_2} - A_{a_1, q_1}\| + \|a\|_{c^0(M)})^{\frac{a}{s_1}}.
\]

By interpolation inequality, we get
\[
\|I(q)\|^2_{H^1(\partial_+ SM_1)} \leq C\|I(q)\|_{L^2(\partial_+ SM_1)}\|I(q)\|_{H^2(\partial_+ SM_1)}
\leq C\|I(q)\|_{L^2(\partial_+ SM_1)}.
\]

From the Theorem 2.1, it follows that
\[
\|q\|_{L^2(M)} \leq C(\|A_{a_2, q_2} - A_{a_1, q_1}\| + \|d\|_{c^0(M)})^{\frac{a}{s_1}}.
\]

Using the estimate (4.26), we conclude that
\[
\|q\|_{L^2(M)} \leq C\|A_{a_2, q_2} - A_{a_1, q_1}\|^{s_1},
\]
where \( s_1 \in (0, 1) \). This completes the proof of the Theorem 1.4.

**APPENDIX A. PROOF OF LEMMA 4.3**

We define the Poisson kernel of \( B(0, 1) \subset T_yM_1 \), i.e.,
\[
P(\theta, \xi) = \frac{1 - |\theta|^2}{\alpha_n|\theta - \xi|^n}, \quad \theta \in B(0, 1); \quad \xi \in S_yM_1.
\]

For \( 0 < \kappa < 1 \), we define \( \Psi_\kappa : S_yM_1 \times S_yM_1 \to \mathbb{R} \) as
\[
\Psi_\kappa(\theta, \xi) = P(\kappa \theta, \xi).
\]

Let \( P_0 \) the Poisson kernel for the Euclidian unit ball \( B_0(0, 1) \subset \mathbb{R}^n \) i.e.,
\[
P_0(\theta, \xi) = \frac{1 - |\theta||\xi|}{\alpha_n|\theta - \xi||^{n+1}}, \quad \theta \in B_0(0, 1); \quad \xi \in \mathbb{S}^{n-1}.
\]

where \( |\cdot|_0 \) is the Euclidian norm of \( \mathbb{R}^n \). From the well known properties of \( P_0 \), we have
\[
\int_{\mathbb{S}^{n-1}} P_0(\kappa \hat{\theta}, \hat{\xi})d\omega_0(\hat{\xi}) = 1, \quad \text{for all } \kappa \in (0, 1), \quad \hat{\theta} \in \mathbb{S}^{n-1}.
\]

Let \( \gamma = \sqrt{2} := (g_{ij}) \) be definite symmetric positive matrix such that \( \gamma^2 = (g_{ij}) \), then we get
\[
P(\theta, \xi) = \frac{1 - |\gamma^{-1}\theta|_0^2}{\alpha_n|\gamma^{-1}\theta - \gamma^{-1}\xi|_0^n}, \quad \theta, \xi \in S_yM.
\]

we deduce from the change of variable \( \hat{\xi} = \gamma^{-1}\xi \)
\[
\int_{S_yM_1} P(\kappa \theta, \xi)d\omega_y(\xi) = \int_{S_yM_1} P_0(\kappa \gamma^{-1}\theta, \gamma^{-1}\xi)d\omega_y(\xi)
= \frac{1}{\det \gamma} \int_{\mathbb{S}^{n-1}} P_0(\kappa \gamma^{-1}\theta, \hat{\xi})(\det \gamma)d\omega_0(\hat{\xi}) = 1.
\]

This complete the proof of (4.18). Let now
\[
V_\theta = \left\{ \xi \in S_yM_1, \quad |\kappa \theta - \xi| \leq (1 - \kappa)^{1/2n} \right\}.
\]
By a simple computation, we have

and then we get (4.19).

(A.2) \[ \int_{S_{\mu M_1}} \Psi_\kappa(\theta, \xi)|\theta - \xi|d\omega_\xi(\xi) \leq \int_{S_{\mu M_1}} \Psi_\kappa(\theta, \xi)|\kappa \theta - \xi|d\omega_\xi(\xi) + (1 - \kappa) \]

\[ + \int_{V_\delta} \Psi_\kappa(\theta, \xi)|\kappa \theta - \xi|d\omega_\xi(\xi) + \int_{S_{\mu M_1} \setminus V_\delta} \Psi_\kappa(\theta, \xi)|\kappa \theta - \xi|d\omega_\xi(\xi) + (1 - \kappa) \]

\[ (1 - \kappa)^{1/2n} + 4(1 - \kappa)^{1/2} + (1 - \kappa) \leq C(1 - \kappa)^{1/2n}, \]

and then we get (4.19).

By a simple computation, we have

\[ |\nabla^k_\xi P(\kappa \theta, \xi)| \leq C \frac{1 - \kappa^2}{|\kappa \theta - \xi|^{n+k}}, \quad k = 1, 2, \]

we deduce that

(A.3) \[ \|\Psi_\kappa(\theta, \cdot)\|_{H^2(S_{\mu M_1})} \leq C \frac{1 - \kappa^2}{(1 - \kappa)^{n+4}} \int_{S_{\mu M_1}} \frac{1 - \kappa^2}{|\kappa \theta - \xi|^n}d\omega_\xi(\xi) \leq \frac{C}{(1 - \kappa)^{n+3}}. \]

This complete the proof of (4.20).

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