CONJUGATE POINTS AND MASLOV INDEX IN LOCALLY
SYMmetric SEMI-RIEMANNIAN MANIFOLDS.

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In memory of Enzo Baldoni, a man of peace.

Abstract. We study the singularities of the exponential map in semi Riemannian locally symmetric manifolds. Conjugate points along geodesics depend only on real negative eigenvalues of the curvature tensor, and their contribution to the Maslov index of the geodesic is computed explicitly. We prove that degeneracy of conjugate points, which is a phenomenon that can only occur in semi-Riemannian geometry, is caused in the locally symmetric case by the lack of diagonalizability of the curvature tensor. The case of Lie groups endowed with a bi-invariant metric is studied in some detail, and conditions are given for the lack of local injectivity of the exponential map around its singularities.

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1. Introduction

The geodesic flow in semi-Riemannian manifolds, i.e., manifolds endowed with a metric tensor which is not positive definite, has features which are quite different from the Riemannian, i.e., positive definite, case. Although the local theory of semi-Riemannian geodesics is totally equivalent to the Riemannian one, when it gets to global properties the situation changes dramatically. Most notably, compact manifolds may fail to be geodesically connected, and the classical Morse theory for geodesics does not apply to the non positive definite case. In this paper we will be concerned with another phenomenon typical of the semi-Riemannian world, which
is the existence of degenerate singularities for the exponential map. Unlike the Riemannian case, degenerate conjugate points may accumulate along a geodesic, and they do not necessarily determine bifurcation. The theoretical occurrence and the relevance of such phenomena has been studied recently in a series of papers; however, no explicit calculation has been carried out so far due to the difficulties in the integration of the geodesic equation. If one wants to study the global geometry of the conjugate locus in a semi-Riemannian manifold, he will find somewhat discouraging the result proven in [11], concerning the distribution of conjugate points along a geodesic. Such set can be arbitrarily complicated: any bounded closed subset of the real line, like Cantor sets or other pathological examples, appears as the set of conjugate instants along spacelike geodesics in conformally flat Lorentzian 3-dimensional manifolds. It is therefore hopeless to be able to develop significant results concerning the geometry of the conjugate locus in the general case of smooth metrics. On the other hand, if one restricts his attention to the case of real-analytic metrics then accumulation does not occur, and higher order methods for analyzing the isolated singularities of the exponential map are available (see [4]). As in the Riemannian case (see [8, 9]), in order to make explicit computation, an important family of examples of analytic semi-Riemannian manifolds to start with is given by the class of Lie groups endowed with an invariant metric. As a first step in this direction, in this paper we will consider the case of (non compact) Lie groups endowed with a bi-invariant semi-Riemannian metric or, more generally, the case of semi-Riemannian locally symmetric spaces. Recall that if $G$ is a semisimple Lie group, then the Killing form of its Lie algebra $\mathfrak{g}$ defines a bi-invariant semi-Riemannian metric on $G$; more generally, given a nondegenerate symmetric bilinear form $B$ on $\mathfrak{g}$ such that $\text{ad}_X$ is $B$-skew symmetric for all $X \in \mathfrak{g}$, then $B$ can be extended to a bi-invariant semi-Riemannian metric on $G$. For instance, if $G$ is semi-simple and non compact, then its Killing form is not definite, and we obtain a non trivial class of examples where the occurrence of several types of nondegeneracies can be detected by explicit computations. The class of Lie groups admitting a bi-invariant semi-Riemannian metric is quite large, and it has been described in [7]. In the present paper we develop an algebraic theory that allows to determine all the singularities of the exponential map of a locally symmetric semi-Riemannian manifold, to characterize which of these singularities are degenerate, and we give a general formula for computing an important integer valued invariant for geodesics called the \textit{Maslov index}. This integer number is given by an algebraic count of the conjugate instants along a geodesic; the notion of Maslov index appears naturally in the infinite dimensional Morse theory for the strongly indefinite functionals, where it plays the role of a generalized Morse index (see [2]).

The Riemannian curvature tensor of a locally symmetric semi-Riemannian manifold $(M, g)$ is parallel, so that the Jacobi equation along a geodesic $\gamma$ is represented, via a parallel trivialization of the tangent bundle $TM$ along $\gamma$, by a second order linear equation with constant coefficients. The singularities of the exponential map of $(M, g)$ are zeroes of solutions of such equations, and they exist when the curvature tensor has real negative eigenvalues (Lemma 3.4). Degeneracies of such singularities correspond to degeneracies of the restriction of the metric tensor $g$ to the generalized eigenspaces of the curvature tensor relative to the real negative eigenvalues (Proposition 2.5 and Corollary 4.9). When $G$ is a Lie group and $h$ is a bi-invariant semi-Riemannian metric on $G$, in which case the geodesics through
the identity are the one-parameter subgroups of $G$, the conjugate points are determined by the purely imaginary eigenvalues of the adjoint map (Proposition 5.11). As in the Riemannian case (see [8]), the multiplicity of each conjugate point in a bi-invariant semi-Riemannian metric is even. In the special case of a bi-invariant Lorentzian metric on a Lie group whose dimension is less than 6, then the Maslov index of a geodesic equals the number of conjugate points (counted with multiplicity) along the geodesic (Proposition 5.12).

The preliminary algebraic results needed to carry out our computations are collected in Section 2. An effort has been made to make the paper self-contained, and, to this aim, in Section 2 we have reproduced the proof of some well known facts (see [5]) about the Jordan form of endomorphisms that are symmetric with respect to non positive definite inner products. New algebraic invariants called Jordan signatures are introduced in Subsection 2.2; these are nonnegative integers associated to each (real) eigenvalue of a $g$-symmetric endomorphism, and they are used in the computation of the contribution to the Maslov index given by the final endpoint. Conjugate points for arbitrary differential systems are defined and discussed in Section 3, where we prove that, for an arbitrary system with constant coefficients, the conjugate instants are determined solely by the real negative eigenvalues of the coefficient matrix.

The Maslov index is computed in Section 4 (Corollary 4.8) using a formula proven in Lemma 4.3 that relates this number with the variation of the extended coindex of a smooth path of symmetric bilinear forms defined on the space of Jacobi fields. Using similar formulas, another symplectic invariant called the Conley–Zehnder index is computed explicitly for systems arising from the Jacobi equation of a locally symmetric semi-Riemannian manifold. Finally, in Section 5 we make some explicit computations in semi-Riemannian Lie groups, and we show how one can extend the results to more general classes of symplectic systems.

2. ALGEBRAIC PRELIMINARIES

We will be concerned with second order linear systems whose matrix of coefficients $A$ is symmetric relatively to a nondegenerate symmetric bilinear form $g$ on $\mathbb{R}^n$, which is not necessarily positive definite. Then, $A$ may not be diagonalizable, and in fact we will show that, when $A$ is the curvature tensor of a semi-Riemannian metric, the occurrence of such circumstance determines the existence of degenerate singularities of the exponential map.

In order to study these singularities and to carry out the necessary computations, it seems natural to use a Jordan basis for the curvature tensor of the semi-Riemannian metric. Following the theory in [5], one proves that in such basis, the matrix representation of the metric has a simple expression (see Proposition 2.5), which allows a direct computation of the Maslov index without employing perturbation arguments. This result will then be used to study the restriction of $g$ to the generalized eigenspaces of $A$ and to define the notion of Jordan signatures.

2.1. Jordan form of $g$-symmetric endomorphisms. Let us introduce our terminology and fix our notations by recalling a few elementary facts concerning the Jordan canonical form for matrices representing linear endomorphisms of $\mathbb{R}^n$. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ a linear endomorphism; when needed, we will consider the $\mathbb{C}$-linear extension of $\Re A$ to an endomorphism of $\mathbb{C}^n$, defined by $\Re A(x + iy) = Ax + iAy$. Given a complex number $z$, we will denote by $\Im(z)$ its imaginary part.
By \( s(A) \) we will mean the spectrum of \( \mathcal{A} \); for \( \lambda \in s(A) \), let \( \mathcal{H}_\lambda(A) \) denote the complex generalized eigenspace of \( A \):

\[
\mathcal{H}_\lambda(A) = \text{Ker}(\mathcal{A} - \lambda)^n.
\]

If \( \lambda \in s(A) \) then obviously \( \exists \in s(A) \); we set:

\[
\mathcal{F}_\lambda(A) = \begin{cases} 
\mathcal{H}_\lambda(A), & \text{if } \lambda \in s(A) \cap \mathbb{R}; \\
\mathcal{H}_\lambda(A) \oplus \mathcal{H}_\lambda(\mathcal{A}), & \text{if } \lambda \in s(A) \setminus \mathbb{R}, 
\end{cases}
\]

so that:

\[
(2.1) \quad \mathbb{C}^n = \bigoplus_{\lambda \in s(A), \Im(\lambda) \geq 0} \mathcal{F}_\lambda(A).
\]

Finally, let \( \mathcal{F}^0_\lambda(A) \) denote the real generalized eigenspace of \( A \):

\[
\mathcal{F}^0_\lambda(A) = \mathcal{F}_\lambda(A) \cap \mathbb{R}^n;
\]

\( \mathcal{F}_\lambda(A) \) is the complexification of \( \mathcal{F}^0_\lambda(A) \), i.e., \( \mathcal{F}_\lambda(A) = \mathcal{F}^0_\lambda(A) + i\mathcal{F}^0_\lambda(A) \), and thus

\[
\mathbb{R}^n = \bigoplus_{\lambda \in s(A), \Im(\lambda) \geq 0} \mathcal{F}^0_\lambda(A).
\]

Clearly, if \( \lambda \in \mathbb{R} \), then \( \dim_{\mathbb{C}}(\text{Ker}(\mathcal{A} - \lambda)) = \dim_{\mathbb{R}}(\text{Ker}(\mathcal{A} - \lambda)) \) and \( \mathcal{F}^0_\lambda(A) = \text{Ker}(\mathcal{A} - \lambda)^n \); we will call the dimension of \( \text{Ker}(\mathcal{A} - \lambda) \) the geometric multiplicity of the eigenvalue \( \lambda \), while the dimension of \( \text{Ker}(\mathcal{A} - \lambda)^n \) will be called the algebraic multiplicity of \( \lambda \).

The spaces \( \mathcal{H}_\lambda(A) \) (and \( \mathcal{F}_\lambda(A) \)) are \( \mathcal{A} \)-invariant, and the restriction \( \mathcal{A}|_{\mathcal{H}_\lambda(A)} \) of \( \mathcal{A} \) to \( \mathcal{H}_\lambda(A) \) is represented in a suitable basis by a matrix which is the direct sum of \( \lambda \)-Jordan blocks, i.e., matrices of the form:

\[
(2.2) \quad \begin{pmatrix}
\lambda & 1 & 0 & 0 & \ldots & 0 \\
0 & \lambda & 1 & 0 & \ldots & 0 \\
\vdots \\
0 & 0 & \ldots & \lambda & 1 \\
0 & 0 & \ldots & 0 & \lambda \\
0 & 0 & \ldots & 0 & 0 & \lambda
\end{pmatrix}.
\]

By direct sum of the \( k_1 \times k_1 \) matrix \( \alpha \) and the \( k_2 \times k_2 \) matrix \( \beta \), we mean the \( (k_1 + k_2) \times (k_1 + k_2) \) matrix given by:

\[
\alpha \oplus \beta = \begin{pmatrix}
\alpha \\
0 \\
\beta
\end{pmatrix}.
\]

We will denote by \( J_k(\lambda) \) a Jordan block of the form (2.2) having size \( k \times k \) when \( k > 1 \); \( J_1(\lambda) \) is defined to be the \( 1 \times 1 \) matrix \( (\lambda) \).

The decomposition of \( \mathcal{A}|_{\mathcal{H}_\lambda(A)} \) into direct sum of \( \lambda \)-Jordan blocks is not unique, but the number of blocks (and their dimension) appearing in this decomposition is fixed, and it is equal to the complex dimension of \( \text{Ker}(\mathcal{A} - \lambda) \). We will now determine the Jordan decomposition of endomorphisms obtained from \( A \) by analytic functional calculus.
In what follows, we will denote by $N_r$ the $r \times r$ nilpotent matrix:

$$N_r = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 & 0 \end{pmatrix}.$$ 

**Lemma 2.1.** Let $\lambda \in \mathbb{C}$, $B = \lambda \cdot \mathbf{I}_r + N_r$ and let $h : U \to \mathbb{C}$ be an analytic function defined on an open $U \subset \mathbb{C}$ containing 0 whose Taylor series $h(x) = \sum_{i=0}^{\infty} a_i x^i$ has radius of convergence $r > |\lambda|$. Then, $h(B) = \sum_{i=0}^{\infty} a_i B^i$ converges, and

$$h(B) = \sum_{i=0}^{r-1} \frac{1}{i!} h^{(i)}(\lambda) N_r^i,$$

where $h^{(i)}$ is the $i$-th derivative of $h$. If $h'(\lambda) \neq 0$, then the canonical Jordan form of $h(B)$ is given by:

$$\begin{pmatrix} h(\lambda) & 1 & 0 & 0 & \ldots & 0 \\ 0 & h(\lambda) & 1 & 0 & \ldots & 0 \\ \vdots \\ 0 & 0 & \ldots & h(\lambda) & 1 \\ 0 & 0 & \ldots & 0 & h(\lambda) \\ 0 & 0 & \ldots & 0 & 0 & h(\lambda) \end{pmatrix}.$$ 

**Proof.** By linearity, it suffices to prove (2.3) for the function $h(x) = x^p$, with $p \in \mathbb{N}$. The proof of the desired equality in this case follows trivially from the binomial formula. The second statement follows now easily, observing that $(h(B) - h(\lambda) I_r)^{r-1}$ is the matrix $h'(\lambda)^{r-1} N_r^{r-1}$. \(\square\)

**Corollary 2.2.** Let $A$ be an endomorphism of $\mathbb{C}^n$ and let $h : U \to \mathbb{C}$ be an analytic function defined on an open $U \subset \mathbb{C}$ containing 0. Assume that the Taylor series of $h$ centered at 0 has radius of convergence $r > |\lambda|$ for all $\lambda \in s(A)$; then, $s(h(A)) = h(s(A))$. \(\square\)

Let us now consider a nondegenerate symmetric bilinear form $g(\cdot, \cdot)$ on $\mathbb{R}^n$; it will be convenient to identify $g$ with the corresponding linear map $^1 \mathbb{R}^n \ni v \mapsto g(\cdot, v) \in \mathbb{R}^{n*}$. Nondegeneracy means that $g$ is an isomorphism, and symmetry means that $g^* = g$. Let $^c g$ denote the unique sesquilinear extension of $g(\cdot, \cdot)$ to $\mathbb{C}^n \times \mathbb{C}^n$; in this case, $^c g$ will be identified with the conjugate linear map $^c g : \mathbb{C}^n \to \mathbb{C}^{n*}$ obtained as the unique conjugate linear extension of $g : \mathbb{R}^n \to \mathbb{R}^{n*}$. Nondegeneracy of $g$ is equivalent to the nondegeneracy of $^c g$, and the symmetry of $g$ is equivalent to $^c g$ being conjugate symmetric, i.e., $^c g(v, w) = \overline{g(w, v)}$ for all $v, w \in \mathbb{C}^n$.

The index $\text{ind}_-(B)$ and the coindex $\text{ind}_+(B)$ of a symmetric bilinear form $B$ defined on a (finite dimensional) real vector space $V$ are defined respectively to be the number of $-1$’s and the number of $1$’s in the canonical matrix representation of $B$ given by Sylvester’s Inertia Theorem; by signature of $B$, denoted by $\sigma(B)$, we

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1In this paper, the superscript " attached to the symbols of spaces or maps will denote duality. When attached to matrices, it will denote the (conjugate) transpose.
will mean the difference $n_+(B) - n_-(B)$. The nullity of $B$ is the dimension of the kernel of $B$, defined by $\text{Ker}(B) = \{ v \in V : B(v, w) = 0 \text{ for all } w \in V \}$.

A subspace $W \subset V$ is said to be $B$-positive (resp., $B$-negative) if $B|_W$ is positive definite (resp., negative definite); a subspace $W \subset V$ will be called $B$-isotropic if $B|_W$ vanishes identically. The index (resp., the coindex) of $B$ is equal to the dimension of a maximal $B$-negative (resp., $B$-positive) subspace of $V$.

**Remark 2.3.** If $B$ is nondegenerate and $W \subset V$ is a $B$-isotropic subspace of $V$, then $n_+(B) \geq \dim(W)$ and $|\sigma(B)| \leq \dim(V) - 2\dim(W)$. Namely, if $W_-$ (resp., $W_+$) is a maximal $B$-negative (resp., $B$-positive) subspace of $V$, then $W_+ \cap W = \{0\}$, hence $\dim(W_+) \leq \dim(V) - \dim(W)$. Moreover, since $B$ is nondegenerate, $\dim(W_+) + \dim(W_-) = \dim(V)$, from which the three inequalities asserted follow easily.

We will assume that $A$ is $g$-symmetric, meaning that $g(Av, w) = g(v, Aw)$ for all $v, w \in \mathbb{R}^n$; in terms of linear maps, this is equivalent to requiring that the following equality holds: $gA = A^g g$. The $g$-symmetry of $A$ is equivalent to the $g$-symmetry of $A$.

**Lemma 2.4.** If $\lambda, \mu \in s(A)$ are such that $\lambda \neq \mu$, then the generalized eigenspaces $\mathcal{H}_\lambda(A)$ and $\mathcal{H}_\mu(A)$ are $g$-orthogonal. If $\lambda \in s(A)$, then the restriction of the bilinear form $g$ to $\mathcal{F}_\lambda(A)$ is nondegenerate, and so is the restriction of $g$ to $\mathcal{F}_\lambda^g(A)$. In particular, if $\lambda \in s(A) \cap \mathbb{R}$, then the restriction of $g$ to $\text{Ker}(A - \lambda)^s$ is nondegenerate.

**Proof.** We show by induction on $k = k_1 + k_2$ that $\text{Ker}(\mathcal{A} - \lambda)^{k_1}$ and $\text{Ker}(\mathcal{A} - \mu)^{k_2}$ are $g$-orthogonal spaces. When $k_1 = k_2 = 1$ it is just a direct computation, namely, for $v \in \text{Ker}(\mathcal{A} - \lambda)$ and $w \in \text{Ker}(\mathcal{A} - \mu)$ one has:

$$\lambda^g(v, w) = \bar{g}(\lambda v, w) = \bar{g}(\mathcal{A} v, w) = \bar{g}(v, \mathcal{A} w) = \bar{g}(v, \mu w) = \bar{g}(v, w)$$

which implies $\bar{g}(v, w) = 0$.

Assume now that $\text{Ker}(\mathcal{A} - \lambda)^{k_1}$ and $\text{Ker}(\mathcal{A} - \mu)^{k_2}$ are $g$-orthogonal spaces for all pairs $k_1$ and $k_2$ such that $k_1 + k_2 < k$; let $s_1, s_2 \geq 1$ be such that $s_1 + s_2 = k$, and let $v \in \text{Ker}(\mathcal{A} - \lambda)^{s_1}$ and $w \in \text{Ker}(\mathcal{A} - \mu)^{s_2}$. Since $(\mathcal{A} - \lambda)v \in \text{Ker}(\mathcal{A} - \lambda)^{s_1-1}$ and $(\mathcal{A} - \mu)w \in \text{Ker}(\mathcal{A} - \mu)^{s_2-1}$, by the induction hypothesis, we have:

$$\bar{g}((\mathcal{A} - \lambda)v, w) = \bar{g}(v, (\mathcal{A} - \mu)w) = 0,$$

and from these two equalities it follows easily $\bar{g}(v, w) = 0$, as in (2.4).

The orthogonality of the generalized eigenspaces shows that (2.1) is in fact a $g$-orthogonal direct decomposition of $C^n$, from which it follows that the restriction of $\bar{g}$ to each $\mathcal{F}_\lambda(A)$ is nondegenerate, since $\bar{g}$ is nondegenerate on $C^n$. Finally, the nondegeneracy of the restriction of $\bar{g}$ on $\mathcal{F}_\lambda(A)$ is equivalent to the nondegeneracy of the restriction of $g$ to $\mathcal{F}_\lambda^g(A)$; in particular, if $\lambda \in \mathbb{R}$, then $g$ is nondegenerate on $\text{Ker}(A - \lambda)^s$. \hfill $\square$

In order to study the restriction of $g$ to the generalized eigenspaces of $A$, we will now determine the form of the matrix representing $g$ in a suitable Jordan basis for $A$. Lemma 2.4 tells us that it is not restrictive to consider the case that $A$ has only

\footnote{With a slight abuse of notations, given a symmetric bilinear form $B$ on a vector space $V$ and given a subspace $W$ of $V$, we will denote by $B|_W$ the restriction of $B$ to $W \times W$.}
two complex conjugate eigenvalues or one real eigenvalue: once the matrix representation $g_\lambda$ of $g|_{F_\lambda^n(A)}$ has been determined for each $\lambda \in s(A)$ with $\Im(\lambda) \geq 0$, then the matrix representation of $g$ will be given by the direct sum of all such $g_\lambda$'s. As a matter of facts, we will only be interested in the case of one real eigenvalue (see Lemma 3.4 below). Using the terminology of [5], we will call a sip matrix an $n \times n$ matrix $\text{Sip}_n$ of the form:

$$\text{Sip}_n = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & 1 \\ 0 & 0 & \ldots & 0 & 1 & 0 \\ \vdots \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}. \tag{2.5}$$

Adapting the proof of [5, Theorem 3.3], we get the following:

**Proposition 2.5.** Let $\lambda$ be a real eigenvalue of $A$, with $r = \dim(\text{Ker}(A - \lambda))$. Then, the real generalized eigenspace can be written as a $g$-orthogonal direct sum:

$$\text{Ker}(A - \lambda)^n = \bigoplus_{i=1}^r V_{\lambda,i},$$

for which the following properties hold:

(a) $g|_{V_{\lambda,i}}$ is nondegenerate for all $i = 1, \ldots, r$;

(b) each $V_{\lambda,i}$ is $A$-invariant;

(c) for all $i$, there exists a basis $v_1^i, \ldots, v_s^i$ of $V_{\lambda,i}$ and a number $\epsilon_i \in \{-1, 1\}$ such that in this basis the matrix representation of $A|_{V_{\lambda,i}}$ is as in (2.2), and the matrix representation of $g|_{V_{\lambda,i}}$ is given by $\epsilon_i \cdot \text{Sip}_n$.

**Proof.** It will suffice to show the existence of a number $\epsilon = \{-1, 1\}$, of a subspace $V \subset \text{Ker}(A - \lambda)^n$ and of a basis $w_1, \ldots, w_s$ of $V$ with the properties:

- $Aw_i = \lambda w_i$ and $Aw_j = w_{j-1} + \lambda w_j$ for $j = 2, \ldots, s$;
- $g(w_j, w_k) = \epsilon \delta_{j+k,s+1}$ for all $j, k = 1, \ldots, s$.

The two properties above imply that $V$ is $A$-invariant and that the restriction $g|_V$ is nondegenerate. The matrix representation of $A|_V$ in the basis $w_1, \ldots, w_s$ is as in (2.2) and the matrix representation of $g|_V$ is $\epsilon \cdot \text{Sip}_n$; the conclusion will follow easily from an induction argument by considering the $g$-orthogonal complement of $V$ in $F_\lambda^n(A)$.

To infer the existence of such a subspace $V$ with the desired basis, let us argue as follows. There exists $s \geq 1$ with the property that $(A - \lambda)^s|_{\text{Ker}(A - \lambda)^n} = 0$ but $(A - \lambda)^{s-1}|_{\text{Ker}(A - \lambda)^n} \neq 0$; since $B = g((A - \lambda)^{s-1}, \cdot)$ is a non zero symmetric bilinear form on $\text{Ker}(A - \lambda)^n$, there must exists a vector $a_1$ such that $B(a_1, a_1) \neq 0$. We can normalize $a_1$ in such a way that $g((A - \lambda)^{s-1}a_1, a_1) = \epsilon$, for some $\epsilon \in \{-1, 1\}$; the case $s = 1$ is concluded by setting $w_1 = a_1$, and we will now assume $s > 1$.

For $j = 1, \ldots, s$, let us define $a_j = (A - \lambda)^{j-1}a_1$ and let $V$ be the space spanned by the $a_j$'s; it is very easy to check that the $a_j$'s are linearly independent, and thus $\dim(V) = s$. For $j + k = s + 1$, we have:

$$g(a_j, a_k) = g((A - \lambda)^{j-1}a_1, (A - \lambda)^{k-1}a_1)$$

$$= g((A - \lambda)^{j+k-2}a_1, a_1) = g((A - \lambda)^{s-1}a_1, a_1) = \epsilon,$$

$$\gamma_{\lambda,i} = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & 1 \\ 0 & 0 & \ldots & 0 & 1 & 0 \\ \vdots \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}. \tag{2.5}$$
while if \( j + k > s + 1 \) we have:

\[
(2.7) \quad g(a_j, a_k) = g((A - \lambda)^{j+k-2}a_1, a_1) = 0.
\]

Now, set \( b_1 = a_1 + \alpha_2a_2 + \ldots + \alpha_s a_s \) and \( b_j = (A-\lambda)^{j-1}b_1 \) for \( j = 1, \ldots, s \). Here the real coefficients \((\alpha_i)_{i=2}^s\) are to be determined in such a way that \( g(b_1, b_j) = 0 \) for all \( j = 1, \ldots, s-1 \), which would imply easily \( g(b_j, b_k) = \varepsilon \delta_{j+k,s+1} \) for all \( j \) and \( k \). Such a choice of the \( \alpha_i \)'s is indeed possible (and unique), namely, the equality \( g(b_1, b_j) = 0 \) is given, in view of (2.6) and (2.7), by:

\[
0 = g(a_1 + \sum_{k=2}^s \alpha_k a_k, a_j + \sum_{k=2}^{s-j} \alpha_k a_{j+k-1}) = g(a_1, a_j) + 2\varepsilon \alpha_{s-j+1} + \text{terms in } \alpha_2, \ldots, \alpha_{s-j},
\]

so that the \( \alpha_i \)'s can be determined recursively by taking \( j = s-1, s-2, \ldots, 1 \) in the above equality. It is easy to check that the \( b_j \)'s form a basis of \( \mathcal{V} \). Finally, set \( w_j = b_{s-j+1} \) for all \( j = 1, \ldots, s \); an immediate computation shows that the \( w_j \)'s have the required properties.

We draw a first immediate conclusion from the above result:

**Corollary 2.6.** If \( \lambda \) is a real eigenvalue of \( A \), then the absolute value of the signature of the restriction of \( g \) to \( \ker((A - \lambda)^n) \) is less than or equal to the dimension of \( \ker(A - \lambda) \). The restriction of \( g \) to the eigenspace \( \ker(A - \lambda) \) is nondegenerate if and only if the algebraic multiplicity and the geometric multiplicity of \( \lambda \) coincide.

**Proof.** Since the signature of \( g \) is additive by \( g \)-orthogonal sums, using the result of Proposition 2.5 it suffices to show that \( \sigma(g|_{\mathcal{V}_{\lambda,i}}) \leq \dim(\ker(A - \lambda)) \). Since \( g|_{\mathcal{V}_{\lambda,i}} \) is represented by the matrix \( \varepsilon_i \cdot \text{Sip}_{n_i} \), then one check immediately that the subspace of \( \mathcal{V}_{\lambda,i} \) generated by the first \( \left[ \frac{n_i}{2} \right] \) vectors of the basis \( v_1^i, \ldots, v_{n_i}^i \) is \( g \)-isotropic. Using Remark 2.3, we get that \( \sigma(g|_{\mathcal{V}_{\lambda,i}}) = 0 \) if \( n_i \) is even, and that \( \left| \sigma(g|_{\mathcal{V}_{\lambda,i}}) \right| = 1 \) if \( n_i \) is odd.

The last statement concerning the nondegeneracy of \( g|_{\ker(A - \lambda)} \) follows immediately from part (c) of Proposition 2.5.

**2.2. Jordan signatures.** We will now introduce the notion of Jordan signatures, which are nonnegative integer invariants associated to a triple \((g, A, \lambda)\), where \( g \) is a nondegenerate symmetric bilinear form on \( \mathbb{R}^n \), \( A \) is a \( g \)-symmetric endomorphism of \( \mathbb{R}^n \) and \( \lambda \) is an eigenvalue of \( A \). For the purposes of this paper, we will consider only the case that \( \lambda \) is real. Given such a triple \((g, A, \lambda)\), write \( \ker(A - \lambda)^n = \bigoplus_{i=1}^r \mathcal{V}_{\lambda,i} \) as in Proposition 2.5, set \( n_i = \dim(\mathcal{V}_{\lambda,i}) \), denote by \( v_1^i, \ldots, v_{n_i}^i \) a basis of \( \mathcal{V}_{\lambda,i} \) as in part (c) of Proposition 2.5 and let \( \varepsilon_i \cdot \text{Sip}_{n_i} \) be the matrix representation of \( g|_{\mathcal{V}_{\lambda,i}} \) relatively to this basis. For \( i = 1, \ldots, r \), define \( \varsigma_i(g, A, \lambda) \) to be the index of the restriction of \( g \) to \( \mathcal{V}_{\lambda,i} \), and define \( g_i(g, A, \lambda) \) as the index of the (degenerate) symmetric bilinear form \( b_{\lambda,i} \) on \( \mathcal{V}_{\lambda,i} \) whose matrix
representation in the given basis is:

\[
\begin{bmatrix}
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & \epsilon_i \\
0 & 0 & 0 & \ldots & 0 & \epsilon_i & 0 \\
0 & 0 & 0 & \ldots & \epsilon_i & 0 & 0 \\
& \vdots & & & & & \\
0 & 0 & \epsilon_i & \ldots & 0 & 0 & 0 \\
0 & \epsilon_i & 0 & \ldots & 0 & 0 & 0 \\
\end{bmatrix}
\]

(2.8)

Finally, set \( \tau_i(g, A, \lambda) = \varrho_i(g, A, \lambda)+1-\varsigma_i(g, A, \lambda) \). With the help of Remark 2.3, such numbers can be computed explicitly as follows:

\[
\varsigma_i(g, A, \lambda) = \begin{cases} 
\frac{n_i}{2}, & \text{if } n_i \text{ is even;} \\
\frac{n_i - 1}{2}, & \text{if } n_i \text{ is odd and } \epsilon_i > 0; \\
\frac{n_i + 1}{2}, & \text{if } n_i \text{ is odd and } \epsilon_i < 0; \\
\end{cases}
\]

(2.9)

\[
\varrho_i(g, A, \lambda) = \begin{cases} 
\frac{n_i - 1}{2}, & \text{if } n_i \text{ is odd;} \\
\frac{n_i}{2} - 1, & \text{if } n_i \text{ is even and } \epsilon_i > 0; \\
\frac{n_i}{2}, & \text{if } n_i \text{ is even and } \epsilon_i < 0. \\
\end{cases}
\]

(2.10)

and

\[
\tau_i(g, A, \lambda) = 1 + \epsilon_i (-1)^{-n_i+1} \in \{0, 1\}.
\]

(2.11)

**Definition 2.7.** The Jordan signatures, \( \varsigma(g, A, \lambda) \), \( \varrho(g, A, \lambda) \) and \( \tau(g, A, \lambda) \) are defined respectively as \( \sum_{i=1}^{r} \varsigma_i(g, A, \lambda) \), \( \sum_{i=1}^{r} \varrho_i(g, A, \lambda) \) and \( \sum_{i=1}^{r} \tau_i(g, A, \lambda) \).

From (2.11), \( 0 \leq \tau(g, A, \lambda) \leq r = \dim(\text{Ker}(A - \lambda)) \); moreover, \( \varsigma(g, A, \lambda) \) coincides with the index of the restriction of \( g \) to \( \text{Ker}(A - \lambda)^n \), and we get:

\[
\tau(g, A, \lambda) = \varrho(g, A, \lambda) + \dim(\text{Ker}(A - \lambda)) - n_- (g|_{\text{Ker}(A - \lambda)^n}).
\]

(2.12)

3. Eigenvalues and Conjugate Points

Let \( I \subset \mathbb{R} \) be an interval and \( t \mapsto a(t), t \mapsto b(t) \) be continuous maps on \( I \) taking values in the space of linear endomorphisms of \( \mathbb{R}^n \). We can give the following general definition:

**Definition 3.1.** Two instants \( t_0, t_1 \in I, t_0 < t_1 \) are said to be conjugate for the second order linear system \( v'' + a(t)v' + b(t)v = 0 \) in \( \mathbb{R}^n \) (we also say that \( t_1 \) is conjugate to \( t_0 \)) if there exists a non identically zero solution \( v \) of the system such that \( v(t_0) = v(t_1) = 0 \). Clearly, the set of such solutions is a vector space whose dimension is less than or equal to \( n \); such dimension is defined to be the multiplicity of the conjugate instant \( t_1 \).
Remark 3.2. Consider the second order linear system \( v'' + a(t)v' + b(t)v = 0 \) in \( \mathbb{R}^n \) with \( a, b : [a, b] \to \text{End}(\mathbb{R}^n) \) continuous maps. There exists \( \varepsilon > 0 \) such that the set \( C = \{ t \in [a, b] : t \text{ is conjugate to } a \} \) does not contain any point of the interval \([a, a + \varepsilon]\). To see this, consider the associated first order system in \( \mathbb{R}^{2n} \):
\[
(v, w)' = X(t) (v, w), \quad \text{with } X = \begin{pmatrix} 0 & I_n \\ -b(t) & -a(t) \end{pmatrix},
\]
and let \( \Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} : [a, b] \to \text{GL}(\mathbb{R}^{2n}) \) be its fundamental solution, i.e., \( \Phi' = X\Phi \) and \( \Phi(a) = I_{2n} \). An instant \( t \) belongs to \( C \) iff \( \Phi_{12}(t) \) is singular; since \( \Phi_{12}(a) = 0_n \) and \( \Phi_{12}'(a) = I_n \), then \( \Phi_{12}(t) \) is positive definite for \( t \in [a, a + \varepsilon] \) when \( \varepsilon > 0 \) is small enough. This proves our assertion.

Remark 3.3. If the coefficients \( a(t) \) and \( b(t) \) are real analytic functions of \( t \), then it is easy to see that the set of conjugate instants is discrete. Namely, if \( v_1, \ldots, v_n \) are linearly independent solutions of the system \( v'' + a(t)v' + b(t)v = 0 \) satisfying \( v_i(t_0) = 0 \) for all \( i \), then by standard regularity arguments each map \( v_i \) is real analytic, and the conjugate instants correspond to the zeroes of the real analytic map \( t \mapsto \det(v_1(t), \ldots, v_n(t)) \). Such map is not identically zero by Remark 3.2.

In case of system with constant coefficients, the existence of conjugate instants is related to the spectrum of the coefficients in a quite straightforward way. For our purposes, we will be interested in the following situation:

**Lemma 3.4.** Let \( A \) be an arbitrarily fixed endomorphism of \( \mathbb{R}^n \). There exists pairs of conjugate instants \( t_0, t_1 \in \mathbb{R} \) for the system \( v'' = Av \) if and only if \( A \) has real negative eigenvalues.

**Proof:** Since the system has constant coefficients, translations preserve its solutions, and therefore it is not restrictive to consider the case \( t_0 = 0 \). We consider the complexified system \( v'' = \bar{A}v \) in \( \mathbb{C}^n \). The first observation is that establishing whether an instant \( t_1 > 0 \) is conjugate to \( 0 \) is equivalent to determining the existence of a complex solution \( v : [0, t_1] \to \mathbb{C}^n \) of this system which is not identically zero and satisfying \( v(0) = v(t_1) = 0 \). Namely, given any such solution, its real part and its imaginary part are solutions of the real system, and they both vanish at \( 0 \) and at \( t_1 \); at least one of the two parts cannot vanish identically.

We can now consider a suitable basis of \( \mathbb{C}^n \) where \( \bar{A} \) is represented by its Jordan form; it is immediate to see that the existence of a non trivial solution of \( v'' = \bar{A}v \) vanishing at two given instants is equivalent to the existence of a non trivial solution in \( \mathbb{C}^k \) of at least one of the systems \( w'' = J_k(\lambda)w \) vanishing at the same two instants. Here \( \lambda \) runs in the spectrum of \( \bar{A} \) and \( J_k(\lambda) \) is any one of the Jordan blocks appearing in the Jordan decomposition of \( \bar{A} \).

It is therefore not restrictive to assume that the spectrum of \( \bar{A} \) consists of a single eigenvalue \( \lambda \in \mathbb{C} \), and that \( \bar{A} \) is represented (in the canonical basis of \( \mathbb{C}^n \)) by the Jordan block \( J_n(\lambda) \) as in (2.2). Let \( v = (v_1, \ldots, v_n) : \mathbb{R} \to \mathbb{C}^n \) be a non trivial solution of \( v'' = J_n(\lambda)v \) vanishing at \( 0 \) and at some other instant \( t_1 > 0 \). Assume that the \( n \)-th component \( v_n : \mathbb{R} \to \mathbb{C} \) of \( v \) is not identically \( 0 \); it is easily computed \( v_n = C_n(e^{\alpha t} - e^{-\alpha t}) \) for some \( C_n \in \mathbb{C} \setminus \{0\} \), where \( \alpha \) is any one of the two complex roots of \( \lambda \). Since \( v_n(t_1) = 0 \), then \( e^{\alpha t_1} = e^{-\alpha t_1} \), i.e., \( 2t_1 \alpha \) is an integer multiple of \( 2\pi i \), i.e., \( \alpha = \frac{2\pi k}{t_1} i \) for some \( k \in \mathbb{Z} \), and therefore \( \lambda = \alpha^2 \) is a negative real number. On the other hand, if \( v_n \) vanishes identically, then one computes easily \( v_{n-1} = C_{n-1}(e^{\alpha t} - e^{-\alpha t}) \), to which the same argument applies, i.e., \( \lambda \in \mathbb{R}^- \) unless \( v_{n-1} \) vanishes identically. An immediate induction argument
completes the proof: if any one of the component $v_k$ of $v$ is not identically zero, then $\lambda \in \mathbb{R}^-$, and we are done. The converse is easy.

By exploiting the argument in the proof of Lemma 3.4 one obtains precise information on the displacement and the number of conjugate instants for the system $v'' = Av$. Let us agree that by the “number of conjugate instants” we mean that each conjugate instant has to be counted with its multiplicity.

**Corollary 3.5.** Let $T > 0$ be fixed and let $A$ be an arbitrary linear endomorphism of $\mathbb{R}^n$. An instant $t_1 \in [0, T]$ is conjugate to 0 for the system $v'' = Av$ if and only if there exists a real negative eigenvalue $\lambda$ of $A$ and a positive integer $k$ such that $t_1 = \frac{k\pi}{\sqrt{|\lambda|}}$. Given such a conjugate instant $t_1$, its multiplicity is given by the sum:

$$\sum_{\lambda} \dim(\ker(A - \lambda)),$$

where the sum is taken over all $\lambda$’s in the real negative spectrum of $A$ of the form $-\frac{k^2 \pi^2}{T^2}$ for some $k \in \mathbb{N} \setminus \{0\}$. The number of conjugate instants to 0 in $[0, T]$ is given by:

$$(3.1) \sum_{\lambda \in s(A) \cap (-\infty, -\frac{\pi^2}{T^2})} \dim(\ker(A - \lambda)) \cdot \left[ \frac{T \sqrt{|\lambda|}}{\pi} \right],$$

where $\lfloor \alpha \rfloor$ denotes the integer part of the real number $\alpha$.

**Proof.** Each $\lambda$-Jordan block of $A$ as in (2.2) gives a contribution of 1 to the multiplicity of the conjugate instant $t_1 = \frac{k\pi}{\sqrt{|\lambda|}}$; namely, the only non trivial solution of $v'' = Av$ vanishing at 0 and at $t_1$ when $A$ is represented by a $\lambda$-Jordan block as in (2.2) with $\lambda < 0$ is given by $v(t) = (C_1 \sin(t \sqrt{|\lambda|}), 0, \ldots, 0)$, for some $C_1 \in \mathbb{R}$ (observe that this fact can be easily obtained from (4.4) and (4.6)).

The conclusion follows easily from the observation that the number of $\lambda$-Jordan blocks appearing in the Jordan form of $A$ equals the dimension of $\ker(A - \lambda)$. □

### 4. Computation of the Maslov Index

Let us fix throughout this section a non degenerate symmetric bilinear form $g$ on $\mathbb{R}^n$, a $g$-symmetric linear endomorphism $A$ of $\mathbb{R}^n$ and a positive instant $T$; the corresponding differential system is:

$$(4.1) \quad v''(t) = Av(t), \quad t \in [0, T].$$

Consider the vector space $\mathbb{R}^n \oplus \mathbb{R}^{n\ast}$ endowed with its canonical symplectic form

$$(4.2) \quad \omega((v, \alpha), (w, \beta)) = \beta(v) - \alpha(w), \quad v, w \in \mathbb{R}^n, \quad \alpha, \beta \in \mathbb{R}^{n\ast}.$$

Equation (4.1) can also be written as a first order system in $\mathbb{R}^n \oplus \mathbb{R}^{n\ast}$, using explicitly the bilinear form $g$, as:

$$(4.3) \quad \begin{pmatrix} v \\ \alpha \end{pmatrix}' = \begin{pmatrix} 0 & g^{-1} \\ gA & 0 \end{pmatrix} \begin{pmatrix} v \\ \alpha \end{pmatrix},$$

from which the symplectic structure of (4.1) appears naturally (see Subsection 5.2).

The endomorphism $X = \begin{pmatrix} 0 & g^{-1} \\ gA & 0 \end{pmatrix}$ of $\mathbb{R}^n \oplus \mathbb{R}^{n\ast}$ belongs to the Lie algebra of
the symplectic group \( \text{Sp}(\mathbb{R}^n \oplus \mathbb{R}^{n*}, \omega) \); the fundamental solution \( \Phi(t) \) of (4.3) is easily computed as the exponential \( \exp(t \cdot X) \):

\[
\Phi(t) = \exp(t \cdot X) = \begin{pmatrix} C(t^2 A) & tS(t^2 A)g^{-1} \\ t g A S(t^2 A) & g C(t^2 A)g^{-1} \end{pmatrix},
\]

where, for \( B \in \text{End}(\mathbb{R}^n) \), we have set:

\[
C(B) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} B^k, \quad S(B) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} B^k.
\]

The conjugate instants of the system (4.1) are precisely the instants \( t \) for which the upper right block of \( \Phi(t) \) is singular, i.e., \( t \in [0, T] \) is a conjugate instant of (4.1) if and only if \( S(t^2 A) \) is singular.

Using Lemma 2.1, we can compute explicitly \( \Phi(t) \) in a Jordan basis for \( A \) using the following:

\[
C(t^2 A) = \begin{pmatrix} \cos \alpha t & \frac{t \sin \alpha t}{2 \alpha} & * & * & \ldots & * \\ 0 & \cos \alpha t & \frac{t \sin \alpha t}{2 \alpha} & * & \ldots & * \\ & \vdots \\ 0 & 0 & \ldots & \cos \alpha t & \frac{t \sin \alpha t}{2 \alpha} & * \\ 0 & 0 & \ldots & 0 & \cos \alpha t & \frac{t \sin \alpha t}{2 \alpha} \end{pmatrix},
\]

and

\[
S(t^2 A) = \begin{pmatrix} \frac{\sin \alpha t}{\alpha t} & \beta(t) & * & * & \ldots & * \\ 0 & \frac{\sin \alpha t}{\alpha t} & \beta(t) & * & \ldots & * \\ & \vdots \\ 0 & 0 & \ldots & \frac{\sin \alpha t}{\alpha t} & \beta(t) & * \\ 0 & 0 & \ldots & 0 & \frac{\sin \alpha t}{\alpha t} & \beta(t) \end{pmatrix},
\]

where \( \beta(t) = \frac{1}{2\alpha^2} \left( \frac{\sin \alpha t}{\alpha t} - \cos t \alpha \right) \).

For all \( t \in \mathbb{R} \), the space:

\[
\ell(t) = \Phi(t) \{0\} \oplus \mathbb{R}^{n*}
\]

is a Lagrangian subspace of \( (\mathbb{R}^n \oplus \mathbb{R}^{n*}, \omega) \), i.e., \( \ell(t) \) is an \( n \)-dimensional subspace on which \( \omega \) vanishes. The map \( t \mapsto \ell(t) \) is a real-analytic map in the Lagrangian Grassmannian \( \Lambda \) of \( (\mathbb{R}^n \oplus \mathbb{R}^{n*}, \omega) \); given a Lagrangian \( L_0 \), we will denote by \( \mu_{L_0} \) the \( L_0 \)-Maslov index. There is a vast literature on the Maslov index, and the most standard reference is [13]; we will use a slightly different definition of Maslov index and we will follow more closely the approach presented in [4]. If we denote by \( \Sigma_{L_0} \) the \( L_0 \)-Maslov cycle, which is the subset of \( \Lambda \) consisting of all Lagrangians \( L \) that are not transversal to \( L_0 \), then roughly speaking the \( L_0 \)-Maslov index of a path \( \ell \) is given by the intersection number of \( \ell \) and \( \Sigma_{L_0} \). When the endpoints of \( \ell \) do not lie on \( \Sigma_{L_0} \), this intersection number can be computed as the class of \( \ell \) in the first relative homology group \( H_1(\Lambda, \Sigma_{L_0}) \cong \mathbb{Z} \). The definition of the Maslov index in the general case is as follows. Assume that \( \ell : [a, b] \to \Lambda \) is a continuous curve for which there exists \( L_1 \in \Lambda \) such that \( L_1 \cap L_0 = L_1 \cap \ell(t) = \{0\} \) for all
Definition 4.1. is a continuous path such that
\( L \in \Lambda \) such that \( L \cap L_1 = \{0\} \), \( \varphi_{L_0,L_1}(L) \) is the symmetric bilinear form on \( L_0 \) given by \( \omega(T_0, \cdot) \), \( T \) being the unique linear map \( T : L_0 \to L_1 \) whose graph
\[
\text{Gr}(T) = \{ x + Tx : x \in L_0 \}
\]
is \( L \). It is not hard to prove that the right hand side of (4.8) does not depend on the choice of \( L_1 \). Moreover, by [4, Corollary 3.5], there exists a unique extension of the \( \mathbb{Z} \)-valued map \( \mu_{L_0} \) above to the set of all continuous curves in \( \Lambda \) which is invariant by fixed endpoints homotopies and additive by concatenation.\(^3\) The Maslov index is also symplectic additive, in the sense that, given symplectic spaces \((V_s, \omega_s)\), \( L_0^s \in \Lambda(V_s, \omega_s) \) and continuous paths \( \ell_s : [0, T] \to \Lambda(V_s, \omega_s) \), with \( s = 1, \ldots, k \), then
\[
\mu_{\bigoplus_{s=1}^k L_0^s}(\ell_s) = \sum_{s=1}^k \mu_{L_0^s}(\ell_s).
\]
Finally, if \( \ell : [a, b] \to \Lambda \) is a continuous path such that \( \dim(\ell(t) \cap L_0) \) is constant on \([a, b]\), then \( \mu_{L_0}(\ell) = 0 \).

Definition 4.1. We will denote by \( \mu(g, A, T) \) the Maslov index of the system \((4.1)\), which is defined as:
\[
\mu(g, A, T) = \mu_{L_0}(\ell), \tag{4.9}
\]
where \( \ell : [0, T] \to \Lambda \) is the smooth curve given in (4.7) and \( L_0 = \{0\} \oplus \mathbb{R}^{n^*} \).

The reader should observe that, when (4.1) comes from the Jacobi equation along a semi-Riemannian geodesic, it is customary in the literature (see [6]) to define the Maslov index of the geodesic as \( \mu_{L_0}(\ell|_{[\varepsilon, T]}\)\), where \( \varepsilon > 0 \) is small enough so that there are no conjugate instants of (4.1) in \([0, \varepsilon]\) (recall Remark 3.3). The contribution to \( \mu_{L_0}(\ell) \) given by the initial instant \( t = 0 \) is easily computed as \(-n_-(g)\) (see also Proposition 4.6), so that \( \mu(g, A, T) \) coincides with \( \mu_{L_0}(\ell|_{[\varepsilon, T]} \)\) \(- n_-(g)\).

The intersections of the curve \( \ell \) in (4.7) with the \( L_0 \)-Maslov cycle occur precisely at the conjugate instants of the system (4.1); when \( g \) is positive definite, then each conjugate instant gives a positive contribution to the computation of the Maslov index, given by its multiplicity. More generally, given a \( C^1\)-curve \( \ell \) in \( \Lambda \) which intersects at \( t = t_0 \) transversally the regular part of the \( L_0 \)-Maslov cycle (in which case such intersection is isolated), the contribution to the Maslov index of \( \ell \) given by \( t_0 \) can be computed as the signature of a certain symmetric bilinear form on \( \ell(t) \cap L_0 \) (see [13]). A conjugate instant \( t_0 \) of the system \( v'' = Av \) will be called nondegenerate if the corresponding intersection with \( \Sigma_{L_0} \) is transverse.

The purpose of this section is to give a formula for computing the Maslov index in the case that \( g \) is arbitrary, in which case the intersection with \( \Sigma_{L_0} \) of the Lagrangian path \( \ell \) given in (4.7) may be degenerate (see Corollary 4.9).

We start with the following:

\(^3\)We briefly observe here that our notion of Maslov index \( \mu_{L_0} \) and the notion of Maslov index \( \mu_{L_0}^{RS} \) discussed in [13], which is a half-integer, differ only in the way of counting the contribution of the endpoints. For a continuous curve \( \gamma : [a, b] \to \Lambda \), the two quantities are related by the following simple identity: \( \mu_{L_0}^{RS}(\gamma) = \mu_{L_0}(\gamma) + \frac{1}{2} \dim(\gamma(a) \cap L_0) - \frac{1}{2} \dim(\gamma(b) \cap L_0) \). In particular, if \( \gamma \) has both endpoints transversal to \( L_0 \), then \( \mu_{L_0}(\gamma) = \mu_{L_0}^{RS}(\gamma) \).
Lemma 4.2. Suppose that \((W_s)^k_{s=1}\) is a family of \(A\)-invariant and \(g\)-orthogonal subspaces of \(\mathbb{R}^n\) such that \(\mathbb{R}^n = \bigoplus_{s=1}^k W_s\); denote by \(A_s : W_s \to W_s\) the restriction of \(A\) to \(W_s\) and by \(g_s\) the restriction of \(g\) to \(W_s \times W_s\).

Then, \(g_s\) is nondegenerate, \(A_s\) is a \(g_s\)-symmetric endomorphism of \(W_s\) for all \(s\), and \(\mu(g, A, T) = \sum_{s=1}^k \mu(g_s, A_s, T)\).

Proof. It follows easily from the symplectic additivity of the Maslov index. Under the assumptions of the Lemma, the symplectic space \((\mathbb{R}^n \oplus \mathbb{R}^{n^*}, \omega)\) is the symplectic direct sum of the spaces \((W_s \oplus W^*_s, \omega)^s\) the Lagrangian space \(L_0\) is the direct sum of the Lagrangians \(\{0\} \oplus W^*_s\), and, by the \(g\)-orthogonality of the \(W_s\), the curve \(\ell\) is the direct sum of curves \(\ell_s\) obtained from the systems \(v'' = A_s v\) in \(W_s\). \(\square\)

Using Lemma 2.4, Proposition 2.5 and Lemma 4.2, it follows that we may restrict our computation of the Maslov index to the case that the spectrum of \(A\) consists of a single eigenvalue \(\lambda\), which is a real negative number, that the Jordan form of \(A\) consists of a single \(\lambda\)-Jordan block, and that the bilinear form \(g\) is represented by a matrix of the form \(\epsilon \cdot S_{\text{ip}_{\lambda}}\) in the canonical basis of \(\mathbb{R}^n\) for some \(\epsilon \in \{-1, 1\}\).

Restriction to this case will simplify some of the computations; the contribution to the Maslov index given by each conjugate instant will be computed using the following:

Lemma 4.3. Let \(t_1 \in [0, T]\) be fixed. If \(C(t_1^2 A)\) is an isomorphism of \(\mathbb{R}^n\), then the Lagrangian \(\ell(t_1)\) is transversal to \(L_1 = \mathbb{R}^n \oplus \{0\}\), and, for \(t\) near \(t_1\), \(\varphi_{L_0, L_1}(\ell(t))\) can be identified with the symmetric bilinear form \(B_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\) given by:

\[
(4.10) \quad B_t = tS(t^2_1 A)C(t^2_1 A)^{-1}g^{-1}.
\]

Proof. Transversality of \(\ell(t_1) = \Phi(t_1)(L_0)\) with \(L_1\) is obviously equivalent to the nonsingularity of the lower right block of \(\Phi(t)\) (see (4.4)). Formula (4.10) is obtained by a straightforward direct calculation. \(\square\)

Lemma 4.3 applies if we assume that the spectrum of \(A\) consists of a single negative real number:

Lemma 4.4. Assume that \(s(A) = \{\lambda\}\), with \(\lambda \in \mathbb{R}^-\), and \(t_1 = \frac{k\pi}{\sqrt{|\lambda|}}\) for some \(k \in \mathbb{N}\). Then, \(C(t_1^2 A)\) is an isomorphism.

Proof. Under the assumption that \(s(A) = \{\lambda\}\), in a Jordan basis for \(A\) the \(n \times n\) matrix \(C(t_1^2 A)\) can be computed explicitly as an upper triangular matrix whose diagonal entries are equal to \(\cos(k\pi/2) = (-1)^k\). Such matrix is nonsingular, and this concludes the proof. \(\square\)

Remark 4.5. Observe that the conclusion of Lemma 4.4 does not hold in general without the assumption that the spectrum of \(A\) consists of a single eigenvalue.

Proposition 4.6. Under the assumptions of Lemma 4.4, the contribution to the Maslov index of each conjugate instant \(t_1 = \frac{k\pi}{\sqrt{|\lambda|}} \in [0, T]\) of (4.1) is given by the signature of \(g\).

\[\text{Here, } W^*_s \text{ is identified with } g(W_s) \subset \mathbb{R}^{n^*}, \text{i.e., with the subspace of } \mathbb{R}^{n^*} \text{ consisting of those linear functionals that vanish on the } g\text{-orthogonal complement of } W_s.\]
Proof. We will assume that the Jordan form of $A$ consists of a unique $\lambda$-Jordan block, and that $\mathbb{R}^n$ has a basis relative to which the matrix representation of $g$ is of the form $\epsilon \cdot \text{Sip}_n$. All the computations that will follow are done using the matrix representations of $A$ and $g$ in such a basis.

By Lemma 4.3 and Lemma 4.4, the contribution to the Maslov index of each conjugate instant is given by the variation of the extended coindex (i.e., coindex plus nullity) of the path of symmetric bilinear forms $B_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given in (4.10). In a Jordan basis for $A$, the symmetric matrix representing $B_t \sim = t S(t^2 A) C(t^2 A)^{-1} g^{-1}$ can be computed easily using (4.5) and (4.6) as:

$$B_t \sim \begin{pmatrix} * & * & \ldots & * & \psi(t) \\ * & * & \ldots & * & \psi(t) \\ \vdots \\ * & \psi(t) & \ldots & 0 & 0 \\ \psi(t) & 0 & \ldots & 0 & 0 \end{pmatrix}$$

where $\psi(t) = \frac{t}{\alpha} \tan(\alpha t)$ and $\alpha = \sqrt{|\lambda|} > 0$. If $\tan(\alpha t) > 0$, then the coindex of $B_t$ equals the coindex of $g$, while if $\tan(\alpha t) < 0$, the coindex of $B_t$ equals the index of $g$; observe that $\tan(\alpha t)$ is negative (resp., positive) in a left (resp., right) neighborhood of $t_1 = \frac{k\pi}{\alpha}$.

If $t_1 \in [0, T]$, then the variation of (extended) coindex of $B_t$ on $[t_1 - \varepsilon, t_1 + \varepsilon]$ is given by:

$$n_+ (B_{t_1+\varepsilon}) - n_+ (B_{t_1-\varepsilon}) = n_+ (g) - n_+ (-g) = n_+ (g) - n_- (g) = \sigma (g). \quad \square$$

The formula for the jump of the extended coindex at the final instant is a little more involved, and it requires an analysis of the matrix representation of the bilinear form at a conjugate instant. Using the notations in Proposition 4.6, if $t_1 = \frac{k\pi}{\alpha}$ for some $k \in \mathbb{N}$, by direct computation involving (4.5) and (4.6) we get:

$$B_{t_1} \sim \begin{pmatrix} * & * & \ldots & * & -\frac{ek\pi}{2\alpha^3} \\ * & * & \ldots & -\frac{ek\pi}{2\alpha^3} & 0 \\ * & * & \ldots & 0 & 0 \\ \vdots \\ * & -\frac{ek\pi}{2\alpha^3} & 0 & \ldots & 0 \\ -\frac{ek\pi}{2\alpha^3} & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$  

We are now ready for the following:

**Proposition 4.7.** Under the assumptions of Lemma 4.4, if $T$ is a conjugate instant of (4.1), i.e., if $T = \frac{k\pi}{\sqrt{|\lambda|}}$ for some $k \in \mathbb{N}$, then its contribution to the Maslov index is given by the Jordan signature $\tau (g, A, \lambda)$.

**Proof.** Using (4.12), the extended coindex of $B_T$ can be computed as follows. In first place, $\dim (\ker (B_T)) = 1$; moreover, we observe that the coindex of $B_T$ is equal to the index of the symmetric bilinear form given in (2.8). Recalling the
definition of the Jordan signatures (2.12), we get:

\[ n_+ (\mathcal{B}_T) + \dim (\ker (\mathcal{B}_T)) - n_+ (\mathcal{B}_{T-\varepsilon}) \]

\[ = \varrho (g, A, \lambda) + \dim (\ker (A - \lambda)) - n_+ (-g) \]

\[ = \varrho (g, A, \lambda) + \dim (\ker (A - \lambda)) - n_- (g) = \tau (g, A, \lambda). \]

This concludes the proof. \( \square \)

Summarizing, we have proved the following:

**Corollary 4.8.** For each conjugate instant \( t \in [0, T] \) of (4.1), denote by \( \mu_t (g, A) \) the contribution to the Maslov index of (4.1) given by \( t \), so that:

\[ \mu (g, A, T) = \sum_{t \in [0, T]} \mu_t (g, A) - n_- (g). \]

Then, denoting by

\[ \mathcal{N}_t = \left\{ \frac{k^2 \pi^2}{t^2} : k \in \mathbb{N} \setminus \{0\} \right\} \subset \mathbb{R}^- , \]

\( \mu_t (g, A) \) is computed as follows:

\[ \mu_t (g, A) = \begin{cases} 
\sum_{\lambda \in \mathfrak{e} (A) \cap \mathcal{N}_t} \sigma \left( g|_{\ker (A - \lambda)^n} \right) , & \text{if } t < T; \\
\sum_{\lambda \in \mathfrak{e} (A) \cap \mathcal{N}_T} \tau (g, A, \lambda) , & \text{if } t = T. \end{cases} \]

Finally, we observe that the contribution to the Maslov index given by each conjugate instant \( t \in [0, T] \) is less than or equal to its multiplicity, due to the inequality on the signature of \( g \) proved in Corollary 2.6, and to the inequality on the Jordan signature \( \tau \) observed at the end of Subsection 2.2.

We conclude with the following observation, which relates the existence of degenerate conjugate instants with the lack of diagonalizability of the coefficients matrix:

**Corollary 4.9.** Let \( t_1 \in [0, T] \) be a conjugate instant of (4.1); then, \( t_1 \) is a non-degenerate conjugate instant if and only if given any real negative eigenvalue \( \lambda \) of \( A \) having the form \( \lambda = \frac{-k^2 \pi^2}{t_1^2} \) for some integer \( k \neq 0 \), the algebraic multiplicity and the geometric multiplicity of \( \lambda \) coincide.

**Proof.** Let us denote by \( P_1 : \mathbb{R}^n \oplus \mathbb{R}^{n^*} \to \mathbb{R}^n \) the projection onto the first summand. The conjugate instant \( t_1 \) is nondegenerate if and only if the restriction of \( g \) to \( P_1 \Phi (t_1) (L_0) \) is nondegenerate (see for instance [4]), i.e., recalling (4.4), if and only if the restriction of \( g \) is nondegenerate on the image of \( S(t_1^2 A) \). A straightforward computations shows that such condition is equivalent to the nondegeneracy of \( g \) to \( \ker (A - \lambda) \) for each eigenvalue \( \lambda \) of \( A \) as in the statement of the Corollary. The conclusion follows at once from the last statement in Corollary 2.6. \( \square \)
5. SOME EXAMPLES AND FINAL REMARKS

5.1. Conley–Zehnder index. The fundamental solution \( t \mapsto \Phi(t) \) of a symplectic system is a smooth curve in the symplectic group; there exists an integer invariant associated to continuous curves in the symplectic group, which is called the Conley–Zehnder index. Given \( \Phi \in \text{Sp}(\mathbb{R}^n \oplus \mathbb{R}^{n*}, \omega) \), then the graph \( \text{Gr}(\Phi) \) of \( \Phi \) is a \( 2n \)-dimensional subspace of \( V^{4n} = (\mathbb{R}^n \oplus \mathbb{R}^{n*}) \oplus (\mathbb{R}^n \oplus \mathbb{R}^{n*}) \). It is easy to see that \( \text{Gr}(\Phi) \) is Lagrangian relatively to the symplectic form \( \bar{\omega} = \omega \oplus (-\omega) \) in \( V^{4n} \), where \( \omega \) is as in (4.2). More precisely:

\[
\bar{\omega} \left[ (v_1, \alpha_1), (v_2, \alpha_2), (w_1, \beta_1), (w_2, \beta_2) \right] = \beta_1(v_1) - \alpha_1(w_1) - \beta_2(v_2) + \alpha_2(w_2).
\]

The diagonal \( \Delta = \left\{ ((v, \alpha), (v, \alpha)) : v \in \mathbb{R}^n, \alpha \in \mathbb{R}^{n*} \right\} \subset V^{4n} \) is also a Lagrangian space relatively to \( \bar{\omega} \), as well as the anti-diagonal \( \Delta^c \):

\[
\Delta^c = \left\{ ((v, \alpha), (-(v, \alpha)) : v \in \mathbb{R}^n, \alpha \in \mathbb{R}^{n*} \right\}.
\]

Definition 5.1. The Conley–Zehnder index \( i_{CZ}(g, A, T) \) of the system (4.1) is defined to be the Maslov index \( \mu_\Delta \) of the curve \( [0, T] \ni t \mapsto \text{Gr}(\Phi(t)) \in \Lambda(V^{4n}, \bar{\omega}) \), where \( \Phi(t) = \exp(tX) \) is the fundamental solution of (4.3):

\[
i_{CZ}(g, A, T) = \mu_\Delta \left( [0, T] \ni t \mapsto \text{Gr}(\Phi(t)) \right).
\]

The Conley–Zehnder index of a symplectic system is a measure of the set of instants \( t \in [0, T] \) at which the graph of the fundamental solution \( \Phi(t) \) is not transversal to \( \Delta \); observe that \( \text{Gr}(\Phi(t)) \) is transversal to \( \Delta \) if and only if \( 1 \not\in s(\Phi(t)) \). The set of instants \( t \in [0, T] \) at which \( \text{Gr}(\Phi(t)) \) is not transversal to \( \Delta \) may fail to be discrete, as we state in the following lemma.

Lemma 5.2. Assume that \( \ker(A) = \{0\} \). Then, the set of instants \( t \in [0, T] \) at which \( \text{Gr}(\Phi(t)) \) is not transversal to \( \Delta \) is finite, and it is given by:

\[
\mathcal{C} = \left\{ t \in [0, T] : -\frac{4k^2\pi^2}{t^2} \in s(A) \text{ for some } k \in \mathbb{N} \setminus \{0\} \right\}.
\]

On the other hand, if \( 0 \in s(A) \) such a set coincides with the whole interval \([0, T] \).

Proof. The proof follows easily from the relations \( s(\exp(tX)) = \exp(s(tX)) \) when \( t \neq 0 \) and \( s(X^2) = s(X)^2 \) obtained from Corollary 2.2 and \( s(X^2) = s(A) \) that comes directly.

\( \square \)

Remark 5.3. Note that, in the very special case of symplectic systems of the form (4.3), the set \( \mathcal{C} \) above is a (proper) subset of the set of conjugate instants of (4.3) (recall Corollary 3.5). There is in general no relation between the two sets.

The Conley–Zehnder index of the fundamental solution of a constant symplectic system is already known in the literature (see for instance [1, Chapter 1], computed using the rotation function in the symplectic group. For systems of the type (4.3), an alternative, direct computation can be made using the Jordan form of \( A \) and the notion of Jordan signatures.

As a consequence of the statements in the Lemma 5.2, it is convenient to reduce the calculation to the case that \( A \) is invertible. To this aim, the following result is needed; its proof is totally analogous to the proof of Lemma 4.2:
Lemma 5.4. Under the assumptions of Lemma 4.2, the Conley–Zehnder index \( \text{ic}_\text{CZ}(g, A, T) \) is given by the sum \( \sum_i \text{ic}_\text{CZ}(g_i, A_i, T) \).

We recall that, if \( \Phi \in \text{Sp}(\mathbb{R}^n \oplus \mathbb{R}^{n*}, \omega) \) has graph which is transversal to \( \Delta \), i.e., if \(-1 \not\in \rho(\Phi)\), then if we identify \( \Delta \) with \( \mathbb{R}^n \oplus \mathbb{R}^{n*} \) via the projection onto the first coordinate, the symmetric bilinear form \( \Phi_{\Delta, \Delta} : \mathbb{R}^n \oplus \mathbb{R}^{n*} \times \mathbb{R}^n \oplus \mathbb{R}^{n*} \to \mathbb{R} \) is given by:

\[
2\omega((I + \Phi)^{-1}(I - \Phi)\cdot, \cdot).
\]

Using the relation \( C(t^2A)^2 = I + t^2AS(t^2A)^2 \), the matrix representation of (5.1) is

\[
\begin{pmatrix}
2tgAS(t^2A)(I + C(t^2A))^{-1}
0
-2tS(t^2A)(I + C(t^2A))^{-1}g^{-1}
\end{pmatrix}.
\]

Lemma 5.5. Let \( W \subset \mathbb{R}^n \) denote the \( g \)-orthogonal space of \( \text{Ker}(A^n) \), let \( \bar{g} \) denote the restriction of \( g \) to \( W \times W \) and let \( \tilde{A} : W \to W \) denote the restriction of \( A \) to \( W \). Then,

\[
\text{ic}_\text{CZ}(g, A, T) = \text{ic}_\text{CZ}(\bar{g}, \tilde{A}, T) + \text{dim}(\text{Ker}A) - \text{dim}(\text{Ker}(A^n)) - \tau(g, A, 0)
\]

where \( A_0 : \text{Ker}(A^n) \to \text{Ker}(A^n) \) is the restriction of \( A \) to \( \text{Ker}(A^n) \) and \( g_0 \) is the restriction of \( g \) to \( \text{Ker}(A^n) \times \text{Ker}(A^n) \). A direct computation involving Lemma 2.1 shows that \( \Phi_0(t) = \exp(tX_0) \), then \( \text{dim}(\text{Gr}(\Phi_0(t)) \cap \Delta) = \text{dim}(\text{Ker}(A)) \) for all \( t \in [0, T] \) while \( \text{dim}(\text{Gr}(\Phi_0(0)) \cap \Delta) = \text{dim}(\text{Ker}(A^n)) \). Hence \( \text{ic}_\text{CZ}(g_0, A_0, T) \) is given by the only contribution of the initial instant \( t = 0 \); in order to compute such contribution, using the symplectic additivity of the Maslov index we will assume that \( \text{dim}(\text{Ker}(A)) = 1 \). Under this assumption, the Jordan form of \( A_0 \) has a single 0-Jordan block of size \( k_0 \times k_0 \), where \( k_0 = \text{dim}(\text{Ker}(A^n)) \), and \( g \) takes the form \( g = e \cdot \text{Sip}_{k_0}, \) with \( e \in \{-1, 1\} \); using (4.5), (4.6) and (5.2) one computes easily the matrix representation of the symmetric bilinear form \( \Phi_{\Delta, \Delta}(\text{Gr}(\Phi_0(t))) \) for \( t > 0 \) near 0, which is the direct sum of two \( k_0 \times k_0 \) symmetric matrices of the form:

\[
\epsilon\begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & 0 & 0
0 & 0 & 0 & \ldots & 0 & 0 & t
0 & 0 & 0 & \ldots & 0 & t & *
0 & 0 & 0 & \ldots & t & * & *
0 & 0 & t & \ldots & * & * & *
0 & t & * & \ldots & * & * & *
\end{pmatrix}
\]

and

\[
\epsilon\begin{pmatrix}
* & * & \ldots & * & * & -t
* & * & \ldots & * & -t & 0
* & * & \ldots & -t & 0 & 0
0 & 0 & \ldots & 0 & 0 & 0
0 & 0 & \ldots & 0 & 0 & 0
0 & 0 & \ldots & 0 & 0 & 0
\end{pmatrix}.
\]

Thus, for \( t > 0 \) near 0, the extended coindex of \( \Phi_{\Delta, \Delta}(\text{Gr}(\Phi_0(t))) \) is easily computed with the help of the Jordan signatures as:

\[
\text{dim}(\text{Gr}(\Phi_0(t)) \cap \Delta) + n_+(\Phi_{\Delta, \Delta}(\text{Gr}(\Phi_0(t))))
\]

\[
= 1 + n_+(\epsilon \cdot \text{Sip}_{k_0-1}) + n_+(-\epsilon \cdot \text{Sip}_{k_0}) = k_0 - \rho(g, A, 0) + \varsigma(g, A, 0).
\]
For \( t = 0 \), the extended coindex of \( \varphi_{\Delta \gamma, \Delta}(\text{Gr}(\Phi_{0}(0))) = \varphi_{\Delta \gamma, \Delta} = \{0\} \) is equal to \( 2k_0 \). Formula (5.3) follows readily using (2.12). □

Lemma 5.5 tells us that, in order to compute the Conley–Zehnder index of (4.1), it suffices to consider the case that \( A \) is invertible.

We are now ready for the following:

**Proposition 5.6.** The contribution to the Conley–Zehnder index of (4.1) given by the initial instant \( t = 0 \) is given by the following formula:

\[
-2n_+(g) + \dim(\text{Ker}A) + \sigma(g|_{\text{Ker}(A^\nu)}) - \tau(g, A, 0) = -2n_+(g) - \rho(g, A, 0) + n_+(g|_{\text{Ker}(A^\nu)}).
\]

If \( t_1 \in ]0, T[ \cap \mathcal{C}, \) then its contribution to the Conley–Zehnder index of (4.1) is given by:

\[
-2 \sum_\lambda \sigma(g|_{\text{Ker}(A-\lambda)^\nu}),
\]

where the sum is taken over all \( \lambda \in \mathfrak{s}(A) \cap \mathbb{R}^- \) of the form \( \lambda = -\frac{4k^2\pi^2}{\tau_1} \) for some \( k \in \mathbb{N} \setminus \{0\} \). The contribution of the final instant \( T \) is given by

\[
2 \sum_\lambda \left( -\rho(g, A, \lambda) + n_-(g|_{\text{Ker}(A-\lambda)^\nu}) \right) = 2 \sum_\lambda \left( -\tau(g, A, \lambda) + \dim(\text{Ker}(A-\lambda)) \right)
\]

where the sum is taken over all \( \lambda \in \mathfrak{s}(A) \cap \mathbb{R}^- \) of the form \( \lambda = -\frac{4k^2\pi^2}{\tau_2} \) for some \( k \in \mathbb{N} \setminus \{0\} \).

**Proof.** The contribution to the Conley–Zehnder index given by the initial instant of the null eigenvalue of \( A \) is computed in Lemma 5.5. We need to compute the contribution to the index given by the initial instant of the reduced symplectic system, i.e., the system in \( W \oplus W^* \) with coefficient matrix:

\[
\tilde{X} = \begin{pmatrix} 0 & \tilde{g}^{-1} \\ \tilde{g}A & 0 \end{pmatrix},
\]

where \( W \) is the \( g \)-orthogonal subspace to \( \text{Ker}(A^\nu) \) and \( \tilde{g} \) is the restriction of \( g \) to \( W \). By Lemma 5.4, we can assume that \( \mathfrak{s}(A) = \{\lambda\} \) and the Jordan form of \( A \) consists of a single block, i.e., \( \dim(\text{Ker}(A - \lambda)) = 1 \). If \( r_0 = \dim(\text{Ker}(A - \lambda)^n) \), the matrix representation of the symmetric bilinear form \( \varphi_{\Delta \gamma, \Delta}(\text{Gr}(\Phi(t))) \) when \( \sin t\alpha \neq 0 \) is the direct sum of two \( r_0 \times r_0 \) symmetric matrices that can be computed using again equations (4.5), (4.6) and (5.2) as:

\[
\epsilon \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 & \mu_1 \\ 0 & 0 & 0 & \ldots & 0 & \mu_1 & * \\ 0 & 0 & 0 & \ldots & \mu_1 & * & * \\ 0 & 0 & 0 & \ldots & * & * & * \\ \vdots \\ 0 & \mu_1 & * & \ldots & * & * & * \\ \mu_1 & * & * & \ldots & * & * & * \end{pmatrix}
\]
and formula (5.4) follows now easily from (5.3).

The matrix representation of two matrices:

\[
\begin{pmatrix}
\ast & \ast & \ast & \ldots & \ast & \ast & \ast \\
\ast & \ast & \ast & \ldots & \ast & \ast & 0 \\
\ast & \ast & \ast & \ldots & \ast & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast \\
0 & 0 & 0 & \ast & \ast & \ast & \ast \\
0 & 0 & 0 & \ast & \ast & \ast & \ast \\
0 & 0 & 0 & \ast & \ast & \ast & \ast \\
0 & 0 & 0 & \ast & \ast & \ast & \ast \\
\end{pmatrix}
\]

(5.7)

\[\epsilon \left( \begin{pmatrix}
\ast & \ast & \ast & \ldots & \ast & \ast & \ast \\
\ast & \ast & \ast & \ldots & \ast & \ast & 0 \\
\ast & \ast & \ast & \ldots & \ast & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast \\
0 & 0 & 0 & \ast & \ast & \ast & \ast \\
0 & 0 & 0 & \ast & \ast & \ast & \ast \\
0 & 0 & 0 & \ast & \ast & \ast & \ast \\
0 & 0 & 0 & \ast & \ast & \ast & \ast \\
\end{pmatrix} \right) \]

where \( \mu_1 = -\frac{2\alpha \sin t_0}{1 + \cos t_0} \) and \( \mu_2 = -\frac{2\sin t_0}{\alpha(1 + \cos t_0)} \). From (5.6) and (5.7) it is easy to see that the contribution of the initial instant is

\[2n_-(\tilde{g}) - 2r_0 = -2n_+(\tilde{g}) = 2(n_+\left(g\mid_{\text{Ker}(A^n)}\right) - n_+(g))\]

and formula (5.4) follows now easily from (5.3).

Assume now \( t_1 \in [0, T] \cap C \) and \( \lambda = -\frac{4k^2\pi^2}{t_1^4} \) for some \( k \in \mathbb{N} \setminus \{0\} \). Then as the matrix representation of \( \varphi_{t_2, \Delta}(\text{Gr}(\Phi(t))) \) is the direct sum of (5.6) and (5.7) it is easy to see that the contribution of \( t_1 \) is \( -2\sigma(\tilde{g}\mid_{\text{Ker}(A-\lambda)^n}) \).

In order to compute the contribution of the final instant first we observe that the matrix representation of \( \varphi_{t_2, \Delta}(\text{Gr}(\Phi(T))) \) when \( T = \frac{2k\pi}{\alpha} \) is the direct sum of the two matrices:

\[
\begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & k\pi & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & k\pi \\
0 & 0 & 0 & \ldots & k\pi & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
\end{pmatrix}
\]

Then the contribution of the final instant is

\[
n_+\left(\varphi_{t_2, \Delta}(\text{Gr}(\Phi(T)))\right) + \dim\left(\text{Ker}(\varphi_{t_2, \Delta}(\text{Gr}(\Phi(T))))\right) = 2(n_+\left(\varphi_{t_2, \Delta}(\text{Gr}(\Phi(T) - \varepsilon))\right) - n_+\left(\varphi_{t_2, \Delta}(\text{Gr}(\Phi(T)))\right))
\]

\[
= 2(\dim(\text{Ker}(A - \lambda)^n) - \varphi(g, A, \lambda) - n_+(g\mid_{\text{Ker}(A-\lambda)^n}))
\]

\[
= 2(-\varphi(g, A, \lambda) + n_+(g\mid_{\text{Ker}(A-\lambda)^n}))
\]

\[
= 2(-\varphi(g, A, \lambda) + \dim(\text{Ker}(A - \lambda)))
\]

This concludes the proof. \( \Box \)

5.2. Maslov index of an arbitrary constant symplectic system. A more general class of differential systems where the notion of Maslov index is naturally defined consists of the so called symplectic systems. Denote by \( \text{Sp}(2n, \mathbb{R}) \) the Lie group consisting of all isomorphisms of \( \mathbb{R}^n \oplus \mathbb{R}^n^* \) preserving the canonical symplectic form of \( \mathbb{R}^n \oplus \mathbb{R}^n^* \), and let \( \text{sp}(2n, \mathbb{R}) \) be its Lie algebra. Recall that a \((2n) \times (2n)\) real matrix \( X \) belongs to \( \text{sp}(2n, \mathbb{R}) \) if and only if \( X \) is written in \( n \times n \) blocks as \( X = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \), where \( B \) and \( C \) are symmetric matrices. We call a symplectic
Proof. Correspond to the conjugate instants of the second order equation in symplectic systems whose coefficient matrix is non singular, the set of conjugate instants is finite.

Namely, given a solution \( v \) of the symplectic system is a smooth curve \( \Phi \) taking values in \( \text{Sp}(2n, \mathbb{R}) \); the Maslov index of a symplectic system is defined to be the \( L_0 \)-Maslov index of the curve \( t \mapsto \Phi(t)(L_0) \in \Lambda \), where \( L_0 \) is the Lagrangian subspace \( \{0\} \oplus \mathbb{R}^{n*} \) of \( \mathbb{R}^n \oplus \mathbb{R}^{n*} \). Similarly, an instant \( t_0 \in [a, b] \) is defined to be conjugate for the system (5.8) if \( \Phi(t_0)(L_0) \) belongs to the \( L_0 \)-Maslov cycle \( \Sigma_{L_0} \); equivalently, \( t_0 \) is conjugate if there exists a non trivial solution \( \begin{pmatrix} v \\ \alpha \end{pmatrix} \) of (5.8) such that \( v(a) = v(t_0) = 0 \). For example, the second order system (4.1) in \( \mathbb{R}^n \) is equivalent to the symplectic system (4.3) in \( \mathbb{R}^n \oplus \mathbb{R}^{n*} \) whose coefficient matrix is the constant curve

\[
X = \begin{pmatrix} 0 & g^{-1} \\ gA & 0 \end{pmatrix};
\]

the notion of Maslov index and conjugate instant for such symplectic system obviously coincide with the corresponding notions for the system (4.1) given in Section 4.

We will now show how to reduce the computation of the Maslov index of a class of constant symplectic systems to the case of a system of the form (4.3), for which the theory developed earlier applies. To this aim, let us fix an element \( X = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} \in \text{sp}(2n, \mathbb{R}) \) and let us consider the corresponding symplectic system as in (5.8). We want to restrict our attention to those symplectic systems for which the set of conjugate instants is discrete, and for this we need the following:

**Lemma 5.7.** Consider a constant symplectic system with matrix coefficients \( X = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} \) on the interval \([0, T]\). If the upper right \( n \times n \) block \( B \) of \( X \) is non singular, the set of conjugate instants is finite.

Conversely, if \( \text{Ker}(A^*) \cap \text{Ker}(B) \neq \{0\} \), then every \( t \in [0, T] \) is conjugate.

**Proof.** If \( B \) is non singular, then the conjugate instants of the symplectic system correspond to the conjugate instants of the second order equation in \( \mathbb{R}^n \):

\[
(5.9) \quad v'' + (BA^*B^{-1} - A)v' - (BC + BA^*B^{-1}A)v = 0.
\]

Namely, given a solution \( v \) of (5.9), the pair \( (v, \alpha) \) with \( \alpha = B^{-1}(v' - Av) \) is a solution of the symplectic system, and this gives a bijection correspondence between the solutions of the first order system and the solutions of (5.9). As observed in Remark 3.3, the conjugate instants of (5.9) form a discrete set. Conversely, if \( \alpha_0 \in \text{Ker}(A^*) \cap \text{Ker}(B) \) is non zero, then the constant \( \begin{pmatrix} v \\ \alpha \end{pmatrix} \equiv \begin{pmatrix} 0 \\ \alpha_0 \end{pmatrix} \) is a non trivial solution of the system for which \( v(a) = v(t_0) = 0 \) for all \( t_0 \in [0, T] \).

In view of the result above, let us now restrict our attention to those constant symplectic systems whose coefficient matrix \( X = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} \) has non singular upper right block \( B \). Let us also assume that the linear map \( B^{-1}A : \mathbb{R}^n \to \mathbb{R}^{n*} \) is self-adjoint, i.e., that \( B^{-1}A = A^*B^{-1} \), and let us denote by \( \phi \) the endomorphism of \( \mathbb{R}^n \oplus \mathbb{R}^{n*} \) which is written in \( n \times n \) blocks as:

\[
\phi = \begin{pmatrix} I & 0 \\ B^{-1}A & I \end{pmatrix}.
\]
An immediate computation shows that, since $B^{-1}A$ is symmetric, $\phi \in \text{Sp}(2n, \mathbb{R})$; moreover, $\phi(L_0) = L_0$. We compute:

$$\widetilde{X} = \phi X \phi^{-1} = \begin{pmatrix} I & 0 \\ B^{-1}A & I \end{pmatrix} \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} \begin{pmatrix} I & 0 \\ -B^{-1}A & I \end{pmatrix} = \begin{pmatrix} 0 & B \\ B^{-1}A^2 + C & 0 \end{pmatrix} \in \text{sp}(2n, \mathbb{R}).$$

Moreover, if $\Phi(t)$ is the fundamental solution of the symplectic system with constant coefficient matrix $X$, the fundamental solution of the symplectic system with coefficient matrix $\widetilde{X}$ is easily computed as:

$$\widetilde{\Phi}(t) = \phi \Phi(t) \phi^{-1}.$$

Since $\phi$ preserves the Lagrangian $L_0$, the conjugate instants of the symplectic systems corresponding to the coefficient matrices $X$ and $\widetilde{X}$ coincide; moreover, the symplectic invariance of the Maslov index implies that also the Maslov indices of the two systems coincide. Let us denote by $A$ the endomorphism of $\mathbb{R}^n$ given by:

$$A = B(\frac{\partial}{\partial t} - B^{-1}A + C) \quad \text{and by} \quad g = B^{-1}.$$

We have proven the following:

**Corollary 5.8.** Let $X = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} \in \text{sp}(2n, \mathbb{R})$ be such that $B$ is non singular and such that $B^{-1}A$ is symmetric. Then, denoting by $A$ the endomorphism of $\mathbb{R}^n$ given in (5.10) and by $g$ the nondegenerate symmetric bilinear form on $\mathbb{R}^n$ given by:

$$g = B^{-1}.$$

A similar reduction is clearly possible for the computation of the Conley–Zehnder index of an arbitrary constant symplectic system.

### 5.3. Bi-invariant metrics on Lie groups.

As a special case of semi-Riemannian locally symmetric manifold, in this section we will consider the case of a Lie group $G$ endowed with a bi-invariant semi-Riemannian metric $h$. We will denote by $\mathfrak{g}$ the Lie algebra of $G$; recall that a (nondegenerate) symmetric bilinear form $h$ on $\mathfrak{g}$ is bi-invariant if and only if $h(\text{ad}_X Y, Z) = -h(Y, \text{ad}_X Z)$ for all $X, Y, Z \in \mathfrak{g}$, where $\text{ad}_X Y = [X, Y]$. A description of Lie algebras admitting semi-Riemannian bi-invariant metrics can be found in [7].

Let us start with the following technical result:

**Lemma 5.9.** Let $\mathfrak{g}$ be a real $n$-dimensional Lie algebra endowed with a bi-invariant nondegenerate symmetric bilinear form $h$. Let $X_1, \ldots, X_n$ be an $h$-orthonormal basis of $\mathfrak{g}$, and set:

$$[X_i, X_j] = \sum_k C_{ij}^k X_k, \quad \epsilon_i = h(X_i, X_i) \in \{\pm 1\}, \quad \epsilon = \epsilon_1 \cdot \ldots \cdot \epsilon_n,$$

and $a_{ij} = \epsilon_i C_{ij}^i$.
for all $i, j = 1, \ldots, n$. Assume that for all choice of (pairwise distinct) indices $i, j, k, l \in \{1, \ldots, n-1\}$ the following identities hold:

\begin{equation}
(5.12) \quad C^n_{ij} C^m_{kl} + C^n_{jk} C^m_{il} + C^n_{ki} C^m_{jl} = 0.
\end{equation}

Then, the characteristic polynomial $P(\lambda)$ of the linear operator $\text{ad}_{X_n} : \mathfrak{g} \to \mathfrak{g}$ is given by:

\begin{equation}
(5.13) \quad P(\lambda) = (-1)^{n-1} \lambda^{n-3} \left( \lambda^2 + \sum_{i<j} \epsilon_i \epsilon_j a_{ij}^2 \right).
\end{equation}

In the above situation, if $\alpha^2 = \sum_{i<j} \epsilon_i \epsilon_j a_{ij}^2 > 0$, then, denoting by $E_{\pm \alpha}$ the (complex) eigenspace of $\text{ad}_{X_n}$ corresponding to the eigenvalue $\pm \alpha$ and by $W_\alpha \subset \mathfrak{g}$ the real part of $E_{i\alpha} \oplus E_{-i\alpha}$ (which is a 2-dimensional subspace of $\mathfrak{g}$), the restriction $h|_{W_\alpha \times W_\alpha}$ is either positive or negative definite.

**Remark 5.10.** We observe that if any two of the indices $i, j, k, l$ are equal, then the identities (5.12) hold automatically; this is easily checked using the anti-symmetry properties satisfied by the coefficients $C^n_{ij}$. From this observation, it follows immediately that the technical assumption (5.12) is satisfied when $n = \dim(\mathfrak{g}) \leq 4$. Moreover, the Jacobi identity satisfied by $[\cdot, \cdot]$ is equivalent to:

\begin{equation}
(5.14) \quad \sum_m \epsilon_m \left( C^n_{ij} C^m_{kl} + C^n_{jk} C^m_{il} + C^n_{ki} C^m_{jl} \right) = 0.
\end{equation}

From this equality it follows that also in the case that $\dim(\mathfrak{g}) = 5$, the assumption (5.12) is satisfied. When $n \geq 6$, the identities (5.12) are not necessarily satisfied; for instance, they are not satisfied in the case of the product $S^3 \times S^3$ endowed with the semi-Riemannian bi-invariant metric $h = h_0 \oplus (-h_0)$, where $h_0$ is the round metric on $S^3$. Finally, we observe that if the metric $h$ is Lorentzian, i.e., if $h$ has index 1, then in the last statement of Lemma 5.9 we can in fact conclude that the restriction $h|_{W_\alpha \times W_\alpha}$ is always positive definite.

**Proof of Lemma 5.9.** In the basis $X_1, \ldots, X_n$, the matrix representing $\text{ad}_{X_n}$ is given by $B = (C^n_{ij})^n_{i,j=1}$; since the last column and the last row of this matrix are zero, we will consider the square matrix of order $n - 1$ obtained removing the last row and the last column. In order to compute the characteristic polynomial of $B$, we observe that if we set $A = (a_{ij})$ then

$$
\det(B - \lambda I) = \epsilon \epsilon_n \det(A - D)
$$

where $D$ is a diagonal matrix of order $n - 1$ with diagonal elements $d_{ii} = \epsilon_i \lambda$. Since $A$ is skew-symmetric, its determinant is zero when $n - 1$ is odd; if $n - 1$ is even then the determinant of $A$ can be computed as the square of the Pfaffian of $A$. Let us recall briefly the notion of Pfaffian. Let $\pi = \{(i_1, j_1), (i_2, j_2), \ldots, (i_r, j_r)\}$ be a partition of $\{1, \ldots, n-1\}$ with $i_k < j_k$ for $k = 1, \ldots, r = \frac{n-1}{2}$, where the order of the pairs is not taken into account. We set $a_\pi = a_{i_1 j_1} a_{i_2 j_2} \ldots a_{i_r j_r}$ and we denote by $\alpha_\pi$ the permutation $(i_1 j_1 i_2 j_2 \ldots i_r j_r)$; the Pfaffian of $A$ is then defined as

$$
\sum_{\pi} \text{sgn}(\alpha_\pi) a_\pi.
$$

In order to get an expression for the characteristic polynomial of $B$, we define $\{k_1 \ldots k_{2t}\}$ with $k_1 < k_2 < \ldots < k_{2t}$, as the Pfaffian of the matrix obtained by
taking the rows and the columns \(k_1, \ldots, k_{2t}\) of the matrix \(A\). Then
\[
P(\lambda) = \text{Det}(B - \lambda I) = \epsilon \epsilon_n \text{Det}(A - D)
\]
\[
= \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} \left( \sum_{k_1 < \ldots < k_{2t}} \{k_1 \ldots k_{2t}\}^2 \epsilon_{k_1} \ldots \epsilon_{k_{2t}} \right) (-\lambda)^{n-2t-1}
\]
where \(k_1, \ldots, k_{2t}\) run in the set \(\{1, \ldots, n-1\}\).

Now, the identity (5.12) is equivalent to \(\{k_1 k_2 k_3 k_4\} = 0\) for all \(k_1, k_2, k_3, k_4 \in \{1, \ldots, n-1\}\), hence we get:
\[
(5.15) \quad P(\lambda) = (-\lambda)^{n-3} \left( \lambda^2 + \sum_{i<j} \epsilon_i \epsilon_j a_{ij}^2 \right).
\]

To show this we observe that the following relation holds:
\[
\{k_1 \ldots k_{2t}\} = \sum_{m<n} \text{sg}(\alpha_{m,n}) \{k_1 k_2 mn\} \{[k_3 k_4 \ldots k_{2t}]_{m,n}\}
\]
\[
= (t-2)\alpha_{k_1 k_2} \{k_3 k_4 \ldots k_{2t}\}
\]
where \(m, n\) take values in \(\{k_3, \ldots, k_{2t}\}\), \([k_3 k_4 \ldots k_{2t}]_{m,n}\) denotes the ordered set obtained by removing \(m\) and \(n\) from the list \((k_3, k_4, \ldots, k_{2t})\), and \(\alpha_{m,n}\) is the permutation
\[
(k_1 k_2 mn[k_3 k_4 \ldots k_{2t}]_{m,n}).
\]
Formula (5.15) is obtained now using induction and the identities (5.12).

When \(\sum_{i<j} \epsilon_i \epsilon_j a_{ij}^2 \neq 0\), at least one of the coefficients \(a_{ij}\) is non null and there exists \(p \in \{1, \ldots, n-1\}\) such that \(\sum_i \epsilon_i a_{ip}^2 \neq 0\); for simplicity we will assume that \(a_{12}\) is not zero. In this case, by using the identities (5.12) is easy to see that the system
\[
T_j = (a_{2j}, -a_{1j}, 0, \ldots, 0, a_{12}, 0, \ldots, 0), \quad j = 3, \ldots, n-1,
\]
where \(a_{12}\) appears in the \(j\)-th position, is a basis of the Eigenspace associated to the zero Eigenvalue of \(A\). Using again the identities (5.12) we get that the vectors
\[
P = (\epsilon_1 a_{1p}, \epsilon_2 a_{2p}, \ldots, \epsilon_{a_{ip}}, \ldots, \epsilon_{n-1} a_{n-1p}),
\]
\[
Q = (\epsilon_1 \sum_i \epsilon_i a_{1i} a_{ip}, \epsilon_2 \sum_i \epsilon_i a_{2i} a_{ip}, \ldots, \epsilon_{a_{ip}} \sum_i \epsilon_i a_{n-1i} a_{ip}),
\]
are orthogonal to every \(T_j\). Furthermore
\[
g(P, P) = \sum_i \epsilon_i a_{ip}^2 \neq 0,
\]
\[
g(P, Q) = 0,
\]
\[
g(Q, Q) = \left( \sum_i \epsilon_i a_{ip}^2 \right) \left( \sum_i \epsilon_i \epsilon_j a_{ij}^2 \right) \neq 0.
\]
Therefore, the system \(\{P, Q\}\) is a basis of \(W_\alpha\). Then, the signature of the restriction of \(h\) to \(W_\alpha\) is computed looking at the sign of the expression:
\[
g(P, P)g(Q, Q) - g(P, Q)^2 = \left( \sum_i \epsilon_i a_{ip}^2 \right)^2 \left( \sum_i \epsilon_i \epsilon_j a_{ij}^2 \right),
\]
which is positive, from which the last statement of the lemma follows. \(\square\)
The geodesics through the identity of a Lie group $G$ endowed with a bi-invariant semi-Riemannian metric $h$ are the one-parameter subgroups of $G$. The covariant derivative of the Levi–Civita connection is given, in the case of left-invariant vector fields $X, Y$ on $G$, by:

\begin{equation}
\nabla_X Y = \frac{1}{2} \text{ad}_X Y = \frac{1}{2} [X, Y],
\end{equation}

and, for $X, Y, Z \in \mathfrak{g} = T_1 G$, the curvature tensor is given by:

\[ R_{XY} Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z = \frac{1}{4} [Z, [X, Y]] = \frac{1}{4} \text{ad}_Z \text{ad}_X Y. \]

**Proposition 5.11.** Let $G$ be an $n$-dimensional real Lie group endowed with a bi-invariant semi-Riemannian metric tensor $h$ and let $\mathfrak{g}$ be its Lie algebra. Let $\gamma : \mathbb{R} \to G$ be a one-parameter subgroup of $G$ with $X = \gamma'(0)$, and let $t_0 \in ]0, +\infty[ \setminus \{0\}$ be fixed. Then:

(a) $\gamma(t_0)$ is conjugate to $\gamma(0) = 1$ along $\gamma$ if and only if the spectrum $s_0 = \text{ad}_X$ of the linear operator $\text{ad}_X : \mathfrak{g} \to \mathfrak{g}$ contains a purely imaginary number of the form $2k\pi t_0^{-1}$ for some $k \in \mathbb{N} \setminus \{0\}$.

If $\gamma(t_0)$ is conjugate to $\gamma(0)$ along $\gamma$, set $\mathcal{K}_{t_0} = \{ k \in \mathbb{N} : 2k\pi t_0^{-1} \in \text{spec}(\text{ad}_X) \}$. Then:

(b) $\gamma(t_0)$ is nondegenerate if and only if $W_{t_0} = \tilde{W}_{t_0}$;

(c) the multiplicity of $\gamma(t_0)$ is given by $\dim(W_{t_0})$, which is an even number;

(d) the contribution of $\gamma(t_0)$ to the Maslov index is $\sigma(h|_{W_{t_0} \times \tilde{W}_{t_0}})$;

(e) if $\sigma(h|_{W_{t_0} \times \tilde{W}_{t_0}}) \neq 0$, then the exponential map $\exp : \mathfrak{g} \to G$ is not locally injective around $t_0 X$.

**Proof.** Using formula (5.16), the Jacobi equation corresponds, via parallel transport along $\gamma$, to the second order equation in $\mathfrak{g}$:

\[ Y''(t) = \frac{1}{4} \text{ad}_X^2 Y(t). \]

By Corollary 2.2, the endomorphism $\frac{1}{4} \text{ad}_X^2$ has a real negative eigenvalue $\lambda$ if and only if $\frac{1}{2} \text{ad}_X$ has the purely imaginary eigenvalues $\pm i\sqrt{-\lambda}$. Hence, part (a) of the thesis follows readily as an application of Corollary 3.5, where $A$ is the $h$-symmetric endomorphism $\frac{1}{2} \text{ad}_X^2$ of $\mathfrak{g} \cong \mathbb{R}^n$. Part (b) follows from Corollary 4.9, part (c) from Corollary 3.5; the observation on the parity of the multiplicity follows from the equalities:

\begin{equation}
\text{dim}_\mathbb{R} \left( \text{Ker}(\text{ad}_X^2 + 4k^2 \pi^2 t_0^{-2}) \right) = \text{dim}_\mathbb{C} \left( \text{Ker}(\text{ad}_X - 2ik\pi t_0^{-1}) \oplus \text{Ker}(\text{ad}_X + 2ik\pi t_0^{-1}) \right) = 2 \text{dim}_\mathbb{C} \left( \text{Ker}(\text{ad}_X - 2ik\pi t_0^{-1}) \right).
\end{equation}
Here the first equality follows from Lemma 2.1 and the second one from the fact that the two involved kernels are conjugate spaces. Part (d) follows from Corollary 4.8. Part (e) is an application of a result on bifurcation of semi-Riemannian geodesics, that can be found in [10].

When \( \dim(G) \leq 5 \) or, more generally, when the structure coefficients of \( g \) satisfy the relations (5.12), the statement of Proposition 5.11 can be improved as follows:

**Proposition 5.12.** Under the hypotheses of Proposition 5.11, let \( X_1, \ldots, X_n \) be an \( h \)-orthonormal basis of \( g \) and set \( \gamma(t) = \exp(tX_n), \) for all \( t \in \mathbb{R} \). If the structure coefficients satisfy the relations (5.12), then there are conjugate points along \( \gamma \) if and only if \( \sum \epsilon_i \epsilon_j a_{ij}^2 > 0 \), where the \( \epsilon_k \)'s and \( a_{rs} \)'s are defined as in the statement of Lemma 5.9. In this case:

(a) every conjugate point along \( \gamma \) is nondegenerate;
(b) every conjugate point has multiplicity equal to 2;
(c) all conjugate points along \( \gamma \) give the same contribution to the Maslov index, which is equal to \( \pm 2 \);
(d) if \( \gamma(t_0) \) is conjugate, then \( \exp \) is not locally injective around \( t_0X_n \).

In particular, by (c), if \( h \) has index 1, i.e., \( (G,h) \) is a Lorentzian group, then the contribution to the Maslov index equals its multiplicity.

**Proof.** It follows readily from Lemma 5.9 and Proposition 5.11, observing that, for \( \lambda \in i\mathbb{R} \setminus \{0\} \), the real generalized eigenspace \( \text{Ker} \left( \text{ad} X_n + \lambda^2 \right)^n \) is the real part of the direct sum of the complex generalized eigenspaces \( \text{Ker} \left( \text{ad} X_n + \lambda \right)^n \) and \( \text{Ker} \left( \text{ad} X_n - \lambda \right)^n \) (observe that this fact follows from Lemma 2.1). Note that, from (5.13), the algebraic multiplicity of each non zero eigenvalue of \( \text{ad} X_n \) is equal to 1; from this observation and from part (b) of Proposition 5.11 we obtain a proof of part (a). Moreover, it follows from Proposition 5.11 that the multiplicity of each negative eigenvalue of \( \text{ad} X_n \) is equal to 2, which proves part (b). Part (c) follows from the last statement in Lemma 5.9 and part (d) of Proposition 5.11. Finally, (d) follows from part (e) in Proposition 5.11. \( \square \)

### 5.4. A final remark.

The results presented in this paper are in striking contrast with several assertions made in a preprint recently appeared [12], where the author attempts a calculation of the Maslov index for an autonomous linear Hamiltonian system\(^5\) using a normal reduction for the coefficient matrix. According to what claimed in [12], the number, the distribution and the contribution to the Maslov index of the conjugate instants would depend also on the non real complex eigenvalues of the curvature tensor (compare with Lemma 3.4, Proposition 3.5 and Corollary 4.8), while the lack of diagonalizability and the role of the generalized eigenspace of the real negative eigenvalues of the curvature tensor has not been recognized. It is also claimed in [12] that the conjugate instants of an arbitrary constant symplectic system do not accumulate at the initial instant (compare with Lemma 5.7).

Arbitrary symplectic changes of coordinates in \( \mathbb{R}^{2n} \) needed for the normal reduction employed in [12] do not preserve the conjugate instants of the system nor the base Lagrangian subspace \( L_0 = \{0\} \oplus \mathbb{R}^n \), so that a suitable correction term

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\(^5\) a “constant symplectic system” in our terminology
to be computed in order to get the correct formulas. Several mistakes in such computation and also in other parts of the preprint have led the author to incorrect conclusions throughout.

The author of [12] has indicated the false address of the University of São Paulo, Brazil, as his own institution; it must be observed that A. Portaluri has no affiliation whatsoever with such institution. The ideas, the methods and the results contained in [12] are entire responsibility of the author’s true institution, which is the Politecnico di Torino, Italy.

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