Morse theory and Euler characteristic of sections of spherical varieties

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Chapter 1

Introduction

A classical result by D. Bernstein (cf. Bernstien) asserts that if $p(x_1, \ldots, x_n) = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha x^\alpha$, (where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $x^\alpha = (x_1^{\alpha_1}, \ldots, x_n^{\alpha_n})$) is a Laurent polynomial with generic coefficients then, Euler characteristic (in homotopy sense) of the hypersurface $\{x \in \mathbb{C}^n | p(x) = 0\}$ is equal to $n!\text{Vol}(\Delta)$ where $\Delta$ is the Newton polyhedron of $p$ i.e. the convex hull of points $\alpha \in \mathbb{Z}^n$ with $c_\alpha \neq 0$. One can put this result in a more fancy way as follows:

Let $p(x) = \sum_{\alpha \in S} c_\alpha x^\alpha$ with $S = \{\alpha_1, \ldots, \alpha_N\} \subset \mathbb{Z}^N$ and let $\pi : \mathbb{C}^* \to \mathbb{C}^* \subset GL(N, \mathbb{C})$ be the finite dimensional representation of algebraic torus $\mathbb{C}^*$ defined by $\pi(x) = \text{diag}(x^{\alpha_1}, \ldots, x^{\alpha_N})$ and consider the hyperplane $L = \{x \in \mathbb{C}^n | f(x) = 0\}$ where $f(x_1, \ldots, x_N) = \sum_{i=1}^N \alpha_i x_i$, then if $\alpha_i$ are generic enough, the Euler characteristic of hyperplane section $L \cap \pi(\mathbb{C}^*)$ is equal to $n!\text{Vol}(\Delta)$, where $\Delta$ is convex hull of $\{\alpha_1, \ldots, \alpha_N\}$.

In this thesis, we try to generalize the above result to actions of other algebraic groups.

On the other hand, there is the famous theorem of Kuchnierenko which states that the degree of $\pi(\mathbb{C}^*)$ sitting inside $\mathbb{C}^*$ is equal to $n!\text{Vol}(\Delta)$. Hence Bernstein’s result indeed claims that upto a sign, Euler characteristic of a generic hyperplane section of $\pi(\mathbb{C}^*)$ is equal to $\text{deg}(\pi(\mathbb{C}^*))$ as a subvariety of $\mathbb{C}^N$.

We will use some variant of Morse theory to prove our results. For the manifold we take a submanifold of $\mathbb{C}^N$ and for the Morse function we take a linear functional on $\mathbb{C}^N$. This is different from the classical Morse theory since neither the manifold is assumed to be compact nor the function is proper.

Morse theory relates the Euler characteristic of sections and the number
of critical points. Let us explain the idea:

First consider the real case. Let \( f \) be a Morse function from a compact manifold \( M \) to \( \mathbb{R} \). From Morse theory we know how the topology of the sets \( M_{\leq a} = f^{-1}((-\infty, a]) \) changes as \( a \) passes a critical value. More precisely, suppose \( c \) is the only critical value between \( a \) and \( b \), \( a \leq b \) and \( f^{-1}(c) \) contains only one critical point \( p \). Then \( M_{\leq b} \) has the homotopy type of \( M_{\leq a} \) with a cell of some dimension \( \lambda \) attached. \( \lambda \), called the index of the critical point \( p \), is in fact equal to the number of negative eigen values of the matrix of second derivative of \( f \).

As for the complex case, let \( f : M \to \mathbb{C} \) be a Morse function from a \( d \)-dimensional complex manifold \( M \) to \( \mathbb{C} \). We do not assume that \( M \) is compact (as there are no non-constant holomorphic functions on compact manifolds). But suppose that we can apply Morse theory to the real part of \( f \). If \( f \) does not have any critical points, Morse theory tells us that \( f \) defines a fibration of \( M \) over \( \mathbb{C} \), that is for any value of \( c \) in the range of \( f \) we have

\[
M \cong f^{-1}(c) \times \mathbb{C},
\]

and hence for the Euler characteristic \( \chi \) (in the homotopy sense) we have:

\[
\chi(M) = \chi(f^{-1}(c)) \cdot \chi(\mathbb{C}) = \chi(f^{-1}(c))
\]

, as \( \mathbb{C} \) has the same homotopy type as a point and \( \chi \) of a point is 1. But usually functions do have critical points. In this case, away from the set of critical points \( f \) is a fibration. Moreover Mores theory tells us about the topology of fibres at critical values: roughly speaking, as one moves from a regular value to a critical value, the topology of the fibre changes in such a way that a sphere of real dimension \( d \) vanishes to a point. So in terms of Euler characteristic we have:

\[
\chi(M) = \chi(f^{-1}(c)) + \text{correction terms coming from critical points},
\]

where \( c \) is a regular value. As the Euler characteristic of a punctured sphere of real dimension \( d \) is \((-1)^d\), we get

\[
\chi(M) = \chi(f^{-1}(c)) + (-1)^d \cdot \text{the number of critical points}.
\]

Let us denote the number of critical points of \( f \) on \( M \) by \( \mu(M, f) \). We can formulate the above result as
Theorem 1.0.1 (Formula for Euler characteristic). Let $M$ be a (closed) complex algebraic submanifold of $\mathbb{C}^n$ of complex dimension $d$, and $f$ a generic complex linear functional on $\mathbb{C}^n$. Let $c$ be a regular value for $f|_M$, we have

$$\chi(f^{-1}(c)) = \chi(M) + (-1)^{d+1} \cdot \mu(f, M).$$

The thesis is made up of two parts. The first part deals with Morse theory. It develops a variant of Morse theory for $f : M \to \mathbb{R}$ where $f$ is not necessarily proper and $M$ is an algebraic submanifold of $\mathbb{R}^N$ such that $f$ satisfies certain transversality condition with respect to $M$. This can be roughly interpreted as $f$ does not have any critical point at infinity of $M$.

From this Morse theoretic results we can then derive a formula for the Euler characteristic of hyperplane sections of algebraic submanifolds of $\mathbb{C}^N$, in terms of number of critical points of the linear functional defining the hyperplane.

We will approach this variant of Morse theory in several ways:

1. Stratified Morse theory: We will derive our required result from the Thom-Mather-Whitney stratification theory. They basically generalized the first theorem of classical Morse theory to the stratified spaces.

2. Generalized Morse theory of Palais and Smale: One can also derive our result from the Generalized Morse theory of Palais and Smale. Their theory is a generalization of classical Morse theory to the Hilbert manifolds. Their application of their theory is for infinite dimensional manifolds of geodesics (cf. Generalized Morse theory or Morse theory on Hilbert manifolds). Interestingly this was also the original motivation of Morse himself.

3. Finally, in case our manifold is an orbit of a Lie group action we will give direct proofs for the results. In fact as will be explained in its place, our results obtained are stronger than the results for a general stratification: for orbits of Lie group actions we can prove that the sections are diffeomorphic (rather than only homeomorphic).

The main objective of the second part is to apply the formula for the Euler characteristic obtained in the previous part to prove a generalization of Bernstein’s theorem to the so-called actions with spherical orbits.

We start by discussing the classical results for torus actions, Kouchnirenko’s theorem and its generalization to spherical varieties.

---

1 The generalization of the second theorem of Morse theory to stratified space was done by Goresky and McPherson, cf. Stratified Morse Theory.
Next we attempt to generalize Bernstein’s Euler characteristic theorem to representations of reductive algebraic groups (Section 1). Let \( G \) be a complex connected reductive algebraic group and \( \pi : G \to GL(n, \mathbb{C}) \) a faithful representation. To apply Morse theory we need \( \pi(G) \) to be a closed subset of \( M(n, \mathbb{C}) \), the vector space of \( n \times n \) matrices. We prove a criterion for \( \pi(G) \) to be closed (Proposition 1) and from the formula for Euler characteristic we obtain

**Theorem 1.0.2.** Let \( \pi : G \to GL(N, \mathbb{C}) \) be a faithful representation of a \( d \)-dimensional complex connected reductive group \( G \). Suppose \( \pi(G) \) is closed in \( M(N, \mathbb{C}) \) that is origin belongs to the convex hull of weights of \( \pi \). Then for \( f \) a generic linear functional on \( M(N, \mathbb{C}) \) and \( c \) a generic complex number we have

\[
\chi(\{ x \in \pi(G) | f(x) = c \}) = (-1)^{d+1} \cdot \mu(f, \pi(G)).
\]

We can then prove Bernstein’s Euler characteristic theorem by showing that the degree (as a subvariety) of a generic orbit of \( \mathbb{C}^n \) acting on \( \mathbb{C}^N \) is equal to the number of critical points of a generic functional restricted to this orbit. Next we investigate in what other group actions one has that the number of critical points on an orbit is equal to the degree. We will carefully consider the case of representations of \( SL_2(\mathbb{C}) \) and see that the number of critical points on an orbit is NOT equal to degree in this case. In fact, it turns out that equality of number of critical points and degree is a special situation and is usually not true. Our main result will be that it is indeed true for linear actions with so called *generic spherical orbits*. These kind of actions already contain all linear torus actions.

**Theorem 1.0.3.** Let \( G \) acts linearly on \( \mathbb{C}^N \) such that generic orbits are spherical and closed. Let \( X \) be a generic orbit and let \( f \) be a generic linear functional on \( \mathbb{C}^N \). Then

\[
\deg(X) = \mu(X, f).
\]

**Corollary 1.0.4 (the main theorem).** Let \( G \) acts linearly on \( \mathbb{C}^N \) such that generic orbits are spherical and closed. Let \( X \) be a generic orbit and let \( f \) be a generic linear functional on \( \mathbb{C}^N \) and \( c \) a generic complex number. Then

\[
\chi(f^{-1}(c) \cap X) = \chi(X) + (-1)^{\dim(X)+1} \cdot \deg(X).
\]

\(^2\)Bernstein himself, proved his result using similar methods, cf. Bernstein
I should mention that, being a homogeneous space of $G$, $\chi(X)$ is usually zero and in case it is not zero there is a simple formula for $\chi(X)$ in terms of stabilizer subgroup of $X$ and the Weyl group of $G$ (see Proposition ).

In section we give the proof of the above result. Interesting enough, all representations with generic spherical orbits had already been classified by I. Arzhantsev. His list of indecomposable actions with spherical orbits, includes all torus actions as well as about 30 more examples. We will examine all the examples and verify again directly that the degree is equal to the number of critical points. Main concrete example will be:

**Theorem 1.0.5.** The Euler characteristic of a generic hyperplane section of $SL(n, \mathbb{C}) \subset M(n, \mathbb{C})$ is equal to $-1^n \cdot n$.

Finally, in the last section, we give a formula, in general, for the number of critical points in terms of degree and the intersection numbers of Chern classes. The number of critical points would be then equal to degree if all the terms corresponding to intersection numbers Chern classes cancel each other out.
Chapter 2

Morse theory

2.1 Basic definitions and the classical Morse theory

In this section we briefly go over the basic definitions and state the two classical theorems of Morse theory.

Throughout this section, $M$ is a $d$-dimensional smooth manifold and $f : M \rightarrow \mathbb{R}$ is a smooth function.

**Definition 2.1.1.** A $p \in M$ is called a critical point of $f$ if $df(p) = 0$. More generally if $f : M \rightarrow N$ is a differentiable map between smooth manifolds, a point $p \in M$ is called a critical point of $f$ if the derivative $df(p) : T_pM \rightarrow T_{f(p)}N$ is not surjective. Points which are not critical are called regular.

**Definition 2.1.2.** A $c \in \mathbb{R}$ is called a critical value if $f^{-1}(c)$ contains a critical point, otherwise $c$ is called a regular value. A critical value which has only one inverse image is called a simple critical value.

The celebrated theorem of Sard Milnor, says that almost every value is regular, i.e. the set of critical values is of measure zero.

The second derivative $d^2f$ at each point $p \in M$ is a bilinear map on $T_pM \times T_pM$. If one fixes a local coordinate system at $p$ on $M$, this bilinear map is represented by the $d \times d$ matrix $[\partial^2 f / \partial x_i \partial x_j(p)]$ of second order partial derivatives of $f$. This matrix is called the Hessian matrix of $f$ at $p$ with respect to the local coordinate system chosen.
Definition 2.1.3. A critical point \( p \in M \) is called non-degenerate, if \( d^2f \) is a non-degenerate bilinear form, or equivalently if the Hessian matrix, with respect to some coordinate system, is invertible. A function, all whose critical points are non-degenerate is called a Morse function. If all the critical values of \( f \) are simple \( f \) is called a simple Morse function.

Definition 2.1.4. Let \( p \in M \) be a non-degenerate critical point. The number of negative eigenvalues of the Hessian matrix of \( f \) at \( p \), in some local coordinate, is called the index of the critical point \( p \). One can easily see that this number is independent of the local coordinate chosen.

It can be proved that the set of simple Morse functions is a dense open set in the space of all \( C^\infty \) functions. Any function after an arbitrarily small perturbation becomes a Morse function with simple critical values.

The key lemma in determination of local behaviour of functions at non-degenerate critical points is the so-called Morse lemma (cf. Milnor). It is an essential step in the proof of Morse theorems.

Theorem 2.1.1 (Morse Lemma). Let \( f \) be smooth function on \( \mathbb{R}^n \) such that origin \( O \) is a non-degenerate critical point of \( f \) and \( f(O) = 0 \). Then with a smooth change of coordinates in a neighbourhood of the origin \( f \) will have the form

\[
f(x_1, \ldots, x_n) = -x_1^2 - x_2^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots x_n^2.
\]

where \( k \) is the index of the critical point \( O \).

Now we are ready to state the Morse theorems. They are concerned with the topology of the sets \( M_{\leq a} = f^{-1}((-\infty, a]) \), that is all the points whose \( f \) is below \( a \). The proof can be found for example in Milnor or Hirsch.

Theorem 1 (Classical Morse theory part A). Let \( M \) be a \( C^\infty \) manifold and let \( f : M \to \mathbb{R} \) be a Morse function. Suppose there are no critical values in the interval \( [a, b] \). Then the subsets \( M_{\leq a} \) and \( M_{\leq b} \) have the same homotopy type. In other words, as \( c \) varies between the open interval between two adjacent critical values, the homotopy type of \( M_\leq \) remains constant.

Theorem 2 (Classical Morse theory part B). Let \( M \) be a \( C^\infty \) manifold and let \( f : M \to \mathbb{R} \) be a simple Morse function. Let \( c \) be the only critical value in the interval \( [a, b] \). Then \( M_b \) has the homotopy type of \( M_a \) with a cell of dimension \( \lambda \) attached, where \( \lambda \) is the index of the critical point \( f^{-1}(c) \).
We should remark that it is not crucial in the above theorem to assume $c$ is a simple critical value. In general, we get one cell attached for each critical point in the inverse image of $c$.

### 2.2 Main theorems of the chapter

#### 2.2.1 A variant of Morse theory

Suppose $M \subset \mathbb{R}^n$ is a (closed) algebraic submanifold and $f : \mathbb{R}^n \to \mathbb{R}$ is a linear functional, such that $f|_M$ is a Morse function. In this section we formulate a Morse theory which says that for good enough $f$ (with respect to $M$) we have analogues of theorems A and B of the classical Morse theory for $f|_M$. \[1\]

The first part of Morse theory (theorem A) says that the homotopy type of the sets $M_{\leq c}$ remains constant as long as $c$ is varying between two adjacent critical points. Let us see an example where this is not true if $M$ is not compact.

**Example 2.2.1.** Let $M \subset \mathbb{R}^2$ be the right part of the graph of the function $y = 1/x$. and let $f(x, y) = y$. As you see $M_{\leq c} = \emptyset$ for $c \leq 0$ and $M_{\leq c}$ consists of a single point for $c > 0$, i.e. the homotopy type of $M_{\leq c}$ changes although $c$ is not a critical value for $f$.

In the above example there is no critical point $p \in M$ with $f(p) = 0$ but the line $y = 0$ is tangent to $M$ at infinity suggesting that we can think of $c = 0$ be a critical value corresponding to a point at infinity of $M$.

The philosophy is that one can repeat the Morse theory for non-compact manifolds (or non-proper functions) if there are no critical points at infinity. But of course, one should make it more precise what one means by $f|_M$ has no critical point at infinity.

We now state the theorems, later on we discuss the condition of having no critical point at infinity. As we will see almost all the linear functionals in $\mathbb{R}^n$ behave well and have no critical point at infinity for given algebraic submanifold $M$.

---

\[1\] As we will see, $M$ being algebraic is not crucial. We only need that closure of $M$ in projective space admits a Whitney stratification with finite number of strata.
Theorem 2.2.1 \( (A') \). Let \( M \) be a (closed) algebraic submanifold of \( \mathbb{R}^n \) and \( f \) a generic linear functional on \( \mathbb{R}^n \). Suppose \( f|_M \) does not have any critical value in the interval \([a,b]\). Then the set \( M_{\leq a} \) and \( M_{\geq b} \) have the same homotopy type.

From Theorem \( A' \), one can prove a more general form of it, i.e. when we have a function from \( M \) to \( \mathbb{R}^k \).

Theorem 2.2.2 \( (A', \text{the general form}) \). Let \( M \) a (closed) algebraic submanifold of \( \mathbb{R}^n \) and \( f \) a generic linear function from \( \mathbb{R}^n \) to \( \mathbb{R}^k \). Suppose \( U \subset \mathbb{R}^k \) is an open set that contains no critical values. Then \( f : f^{-1}(U) \cap M \to U \) is a fibration.

As soon as theorem \( A' \) is established, repeating the proof of second part of the classical Morse theory (theorem B) we obtain

Theorem 2.2.3 \( (B') \). Let \( M \) be a (closed) algebraic submanifold of \( \mathbb{R}^n \) and \( f \) a generic linear functional on \( \mathbb{R}^n \) let \( c \) be the only critical value in the interval \([a,b]\). Then \( M_b \) has the homotopy type of \( M_a \) with a cell of dimension \( \lambda \) attached, where \( \lambda \) is the index of the critical point \( f^{-1}(c) \).

Proof. Let \( p \) be the critical point \( f^{-1}(c) \). Following the proof of Morse theorem (cf. Morse Theory, J. Milnor, Theorem 3.2, Chap.1, [Milnor]) one can find \( \epsilon > 0 \) and arbitrarily small neighbourhood \( U \subset \mathbb{R}^n \) and \( \tilde{f} : \mathbb{R}^n \to \mathbb{R} \), such that: \( \tilde{f} = f \) outside \( U \) and inside \( U \), \( \tilde{f} \) is defined in the following way:

In \( U \) consider the coordinate system \( u^1, \ldots, u^n \) such that:

1. \( u^1(p) = u^n(p) = 0 \)
2. \( Y \cap U \subset \{u^{m+1} = \cdots = u^n = 0\} \)
3. \( f|_Y(x) = c - (u^1)^2 - \cdots - (u^\lambda)^2 + (u^{\lambda+1})^2 + \cdots + (u^m)^2, \) where \( m = \dim(M) \).

Now define \( \tilde{f} \) in \( U \) by:

\[
\tilde{f} = f - \mu((u^1)^2 + \cdots + (u^\lambda)^2 + 2(u^{\lambda+1})^2 + \cdots + 2(u^m)^2 + (u^{m+1})^2 + \cdots + (u^n)^2)
\]

where \( \mu : \mathbb{R} \to \mathbb{R} \) is smooth and \( \mu(0) > \epsilon, \mu(r) = 0, \forall r \geq 2\epsilon \) and \( -1 < \mu'(r) \leq 0, \forall r \). It is clear from defenition of \( \tilde{f} \) that it is smooth on whole \( \mathbb{R}^{2n} \).

As a function on \( M \), \( \tilde{f} \) has the following properties:
1. \( \tilde{f}^{-1}(\infty, c - \epsilon] = f^{-1}(\infty, c - \epsilon] \)

2. critical points of \( \tilde{f} \) and \( f_1 \) on \( M \) are the same.

3. \( \tilde{f} \) has no critical value in \([c - \epsilon, c + \epsilon]\).

4. (Main Property) \( \tilde{f}^{-1}(\infty, c + \epsilon] \) has homotopy type of \( f^{-1}(\infty, c + \epsilon] \) with a cell of dimension \( \lambda \) attached.

Now \( \tilde{f} \) satisfies the conditions of theorem A' on \([c - \epsilon, c + \epsilon]\) as \( f = \tilde{f} \) outside \( U \). Hence we conclude that

\[
\tilde{f}^{-1}(\infty, c - \epsilon] \sim \tilde{f}^{-1}(\infty, c + \epsilon].
\]

And from the property number 4 of \( \tilde{f} \)

\[
f^{-1}(\infty, c + \epsilon] \sim \tilde{f}^{-1}(\infty, c + \epsilon] \text{ with a cell of dimension \( \lambda \) attached.}
\sim \tilde{f}^{-1}(\infty, c - \epsilon] \text{ with a cell of dimension \( \lambda \) attached.}
= f^{-1}(\infty, c - \epsilon] \text{ with a cell of dimension \( \lambda \) attached.}
\]

\[\square\]

2.2.2 Formula for the Euler characteristic of sections

In this subsection, we apply the above theorems to obtain a formula for the Euler characteristic of hyperplane sections of algebraic submanifolds.

\( M \) will denote a (closed) complex algebraic submanifold of \( \mathbb{C}^n \cong \mathbb{R}^{2n} \) of complex dimension \( m \). To make \( M \) compact, we can either consider its closure in \( \mathbb{R}P^{2n} \) or \( \mathbb{C}P^n \). Since our Morse theoretic theorems A' and B' deal with real projective space, we prefer to take the closure of \( M \) in \( \mathbb{R}P^{2n} \) which we denote by \( \overline{M} \). By theorem , there is a stratification \( \mathcal{A} \) of \( \overline{M} \) with finite number of algebraic strata such that \( M \) itself is a union of strata.

The definitions of critical point and non-degenerate critical point for the complex functions is verbatim to the real functions.

**Notation 2.2.1.** We denote the number of critical points of a function \( f \) from \( M \) to \( \mathbb{R} \) (or \( \mathbb{C} \)) by \( \mu(M, f) \) (whenever this is finite).

We prove
Theorem 2.2.4 (Formula for Euler characteristic). Let $M$ be a (closed) complex algebraic submanifold of $\mathbb{C}^n$ of complex dimension $m$, and $f$ a generic complex linear functional on $\mathbb{C}^n$. Let $c$ be a regular value for $f|_M$, we have
\[
\chi(f^{-1}(c)) = \chi(M) + (-1)^{m+1} \cdot \mu(f,M).
\]

Proof. Write $f$ as $f_1 + i \cdot f_2$. Obviously, $f_1$ and $f_2$ are $\mathbb{R}$-linear functionals on $\mathbb{C}^n \cong \mathbb{R}^{2n}$. To prove the theorem we apply the theorems $A'$ and $B'$ to $f_1$.

We need few observations about the critical points of $f$ and $f_1$. Note that by Cauchy-Riemann relations, $p$ is a critical point of $f|_M$ if and only if, it is a critical point of $f_1|_M$.

Theorem 2.2.5 (Complex Morse Lemma). Let $f$ be a holomorphic function on $\mathbb{C}^m$ such that origin $O$ is a non-degenerate critical point of $f$ and $f(O) = 0$. Then with a smooth change of coordinates in a neighbourhood of the origin $f$ will have the form
\[
f(x_1, \ldots , x_m) = x_1^2 + x_2^2 + \cdots + x_m^2.
\]

Proof. The proof is verbatim to the real Morse lemma. Only notice that any non-degenerate complex quadratic form, after a linear change of coordinates, can be put in the form $x_1^2 + \cdots + x_m^2$.

Corollary 2.2.6. Let $f$ be a holomorphic function on $U \subset \mathbb{C}^m$ and $f_1 = \text{Re}(f)$. Then any non-degenerate critical point of $f_1$ (which is automatically a critical point of $f$) has index equal to $m$.

Proof. Simple calculation shows that the Hessian matrix for the function $x^2 - y^2$ on $\mathbb{R}^2$ (which is the real part of the function $z^2$) at origin is $\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$. i.e. there is one positive and one negative eigen value. From this we can easily see that the index of zero as a non-degenerate critical point of $\text{Re}(x_1^2 + \cdots + x_m^2)$ is $m$.

Now since $f$ is an algebraic function on $M$ it has only finitely many critical points on $M$, also as we mentioned before, for generic $f$, all $\{x \in \mathbb{C}^n \mid f(x) = c\}$, $(\forall c \in \mathbb{C})$, are transverse to all strata at infinity in $M$.

Take the regular values $c,d \in \mathbb{R}$ such that $f_1^{-1}(\infty, c]$ does not contain any critical point and $f_1^{-1}(\infty, d]$ contains all critical points of $f_1$ on $M$. It follows easily from the above corollary of complex Morse lemma that the
index of a critical point $p$ of the function $f_1$ on $M$ is equal to $m$, the complex dimension of the manifold.

By repeated application of theorems $A'$ and $B'$ (for all the critical values) we get

$$M = f_1^{-1}(-\infty, +\infty)$$
$$\sim f_1^{-1}(-\infty, d]$$
$$\sim f_1^{-1}(-\infty, c]$$

with cells of real dimension $d$ attached

where $\sim$ means homotopy equivalent.

In terms of the Euler characteristic

$$\chi(M) = \chi(f_1^{-1}(-\infty, c]) + (-1)^m \cdot \mu(M, f_1).$$

But by Theorem A' (general form), applied to $f$, $f_1^{-1}(-\infty, c]$ has the same homotopy type as $f^{-1}(c)$ (because $f : M \to \mathbb{C}$ is a fibration restricted to $f_1^{-1}(-\infty, c]$), so we get

$$\chi(M) = \chi(f^{-1}(c)) + (-1)^m \cdot \mu(f, M).$$

or

$$\chi(f^{-1}(c)) = \chi(M) + (-1)^{m+1} \cdot \mu(f, M).$$

\[\square\]

2.3 Critical points at infinity, Whitney stratified sets and the proof of theorem A'

The genericity condition for $f$ in the above theorems is that $f$ has no critical points at infinity. We are going to make this precise. A point being regular point can be stated as a transversality condition: a point $p \in M$ with $f(p) = c$ is regular iff $H_c = \{x \in \mathbb{R}^n | f(x) = c\}$ is transverse to $M$ at $p$. In a similar way one can say that $f|_M$ does not have any critical point at infinity iff level sets of $f$ are transverse to $M \setminus M \subset \mathbb{R}P^n$. But $M$ is not necessarily a manifold. in order to resolve this problem we introduce notion of a stratified set and a Whitney stratified set.
2.3.1 Whitney stratification

In many places in math, we deal with objects such as algebraic varieties which are not manifolds but are union of manifolds glued together, stratification theory provides a general framework for doing analysis on this objects.

**Definition 2.3.1 (Stratified set).** Let \( Y \) be a closed subset of a smooth manifold \( X \). A collection \( \mathcal{A} \) of disjoint, locally closed submanifolds \( S_i \subset Y \) \((i \in I)\) is called a *stratification of \( Y \)* iff

1. \( Y = \bigcup_{i \in I} S_i \).
2. \( S_i \cap S_j \neq \emptyset \) only if \( i = j \) or \( S_i \subset \overline{S_j} \) or \( S_j \subset \overline{S_i} \).

The submanifolds \( S_i \) are then called *strata* and \( Y \) is called a *stratified set*.

Whitney realized that the notion of a stratification, in general, is too wild to expect a good theory for. One needs some conditions to ensure that the strata are glued together in a regular way. To this end, he imposed his conditions (A) and (B) on a stratification to guarantee the so called *topological triviality along the strata*, i.e. if we slice the stratified set and move the slice along some strata, the topological picture does not change.

**Definition 2.3.2 (Whitney stratified set).** Let \( Y \) be a closed subset of \( \mathbb{R}^n \). A stratification \( \mathcal{A} \) of \( Y \) is called a *Whitney stratification* iff for any pair of strata \( A \) and \( B \in \mathcal{A} \) with \( A \subset \overline{B} \) we have the following conditionds (A) and (B) satisfied

(A) For any point \( a \in A \) and a sequence \( \{b_i\}(i = 1, 2, \ldots) \) of points in \( B \), assume that \( \lim_{i \to \infty} b_i = a \) and there is a limit plane \( L = \lim_{i \to \infty} T_{b_i} B \subset T_a \mathbb{R}^n \). Then we have \( T_a A \subset L \).

(B) For any point \( a \in A \) and sequences \( \{a_i \in A\} \) and \( \{b_i \in B\} \) assume that \( \lim_{i \to \infty} a_i = \lim_{i \to \infty} b_i = a \). Write \( \overline{b_i a_i} \) for the secant line through \( b_i \) and \( a_i \). Think of \( \overline{b_i a_i} \) as a subspace of \( T_{b_i} \mathbb{R}^n \). Assume that there are limit plane \( L = \lim_{i \to \infty} T_{b_i} B \subset T_a \mathbb{R}^n \) and limit line \( K = \lim_{i \to \infty} \overline{b_i a_i} \subset T_a \mathbb{R}^n \). Then we have: \( K \subset L \).

One can prove that Whitney conditions (A) and (B) are invariant under diffeomorphisms of \( \mathbb{R}^n \). Hence we can speak of a Whitney stratified subset of a smooth manifold.
It is easy to see that condition (B) implies (A) and hence, logically speaking, it is redundant. But there are still reasons that authors prefer to mention both of them together.

Whitney stratification is important because one can prove (cf. Goresky-McPherson p.)

1. Whitney stratifications are locally topologically trivial along the strata.
2. Any closed analytic subset of an analytic manifold admits a Whitney stratification.
3. Whitney stratified spaces can be triangulated.
4. The transversal intersection of two Whitney stratified spaces is again a Whitney stratified space, whose strata are the intersections of the strata of the two spaces. Also the Cartesian product of two Whitney stratified spaces is a Whitney stratified space, the strata being the product of strata of the stratifications.

and the most important example of Whitney stratification for us is the case of algebraic varieties:

**Theorem 2.3.1 (cf. ).** Any (closed) algebraic subset $M$ of an algebraic manifold $X$ (either over $\mathbb{C}$ or $\mathbb{R}$) admits a Whitney stratification with finite number of algebraic strata. Moreover if we fix an algebraic subvariety $V$ of $M$, we can choose the stratification such that $V$ becomes a union of strata.

Let $M \subset \mathbb{R}^n$ be a (closed) algebraic submanifold and let $\mathcal{A} = \{X_0, X_1, \ldots, X_r\} (X_0 = M)$ be a finite Whitney stratification of $\overline{M} \subset \mathbb{P}^n$. As usual let $f(x) = \sum_{i=1}^n f_i x_i$ be a linear functional on $\mathbb{R}^n$. Consider $\mathbb{P}^n$ as $\mathbb{R}^n \cup \mathbb{P}^{n-1}$. $f$ then defines a projective (n-2)-plane $H_{\infty} = \{(x_1 : \ldots : x_n) \in \mathbb{P}^{n-1} | f(x_1, \ldots, x_n) = 0\} \subset \mathbb{P}^n$. This is in fact, the intersection of the projective hyperplanes $H_c = \{x \in \mathbb{R}^n | f(x) = c\}$.

Now we are ready to state the genericity condition $f$ has to satisfy with respect to $M$. In next subsection we prove that

**Theorem 2.3.2.** If the linear functional $f$ is such that $H_{\infty}$ is transverse to all the strata $X_1, \ldots, X_n$ at infinity, then the implications of the theorems $A'$ and $B'$ hold for $f|_M$.

**Remark 2.3.1.** Given a submanifold $X \subset \mathbb{P}^n$, almost any plane is transverse to $X$, same is true if we have a finite number of submanifolds. Hence for a generic functional $f$ the above transversality condition is satisfied.
2.3.2 Thom’s isotopy lemma and the proof of the main theorem

There is a generalization of the theorem (A) of classical Morse theory to Whitney stratified space. It is the so-called Thom’s first isotopy lemma.

**Theorem 2.3.3 (Thom’s First Isotopy Lemma).** Let $Y$ be a subset of a smooth manifold $X$ with Whitney stratification $A$ and let $f : X \to P$ be a smooth map into another smooth manifold $P$, such that for each stratum $S \in A$, $f|_S$ is a submersion and $f|_{S \cap Y}$ is a proper map. Then $f : Y \to P$ is locally trivial over $P$.

The situation in the isotopy lemma is not exactly as in the theorem (A’). Because even though $M \subset \mathbb{R}P^n$ is a compact Whitney stratified set, $f : M \to \mathbb{R}$ can not be extended in a good way to the whole $M$ to obtain a proper map on strata. Although with a simple trick we can convert the situation so that we can apply the isotopy lemma.

Using the isotopy lemma, we prove the following theorem which immediately implies theorem (A’).

**Theorem 2.3.4 (Main theorem).** Let $X$ be a compact smooth manifold and $D$ a domain in $\mathbb{R}^k$ and $F : X \times D \to X$ a smooth map such that $F_z : X \to X$ is diffeomorphism for any $z \in \mathbb{R}^k$. Suppose $M \subset X$ is a compact submanifold of $X$ of codimension $k$. $F_z(M)$ can be thought of as a family of submanifolds of $X$ parametrized by $\mathbb{R}^k$.

Also suppose $Y \subset X$ is a Whitney stratified subset of $X$ such that the following transversality condition holds:

$\forall z \in D$ we have $F_z(M)$ is transversal to all strata $S$ in $A$.

Then for all $z_1, z_2 \in D$, $F_{z_1}(M) \cap Y$ is homeomorphic to $F_{z_2}(M) \cap Y$ under an stratum preserving homeomorphism.

**Remark 2.3.2.** To obtain theorem (A’) from the above theorem, we take $X$ to be $\mathbb{R}P^n$, $k = 1$, $M = \{x \in \mathbb{R}^n | f(x) = 0\} \cong \mathbb{R}P^{n-1}$ and $D = [a, b]$. For $x = (x_0, \ldots, x_n)$ define

$$F(x, r) = (x_0, \ldots, x_{n-1}, x_n/r).$$

Then $F_z(M)$ is simply the closure of a level set of $f$, i.e.

$$F_z(M) = \overline{f^{-1}(z)}.$$
Hence the theorem implies that \( f^{-1}(z) \cap M \) (for \( a \leq z \leq b \)) are homeomorphic under a stratum preserving map. In particular, \( f^{-1}(z) \cap M \) are homeomorphic.

**Proof.** Consider \( \hat{F} : M \times D \to X \times \mathbb{R}^k \) given by \( \hat{F}(m, z) = (F(m, z), z) \). Obviously \( \hat{F} \) is an embedding. Now consider \( Y \times D \subset X \times \mathbb{R}^k \) as a Whitney stratified space with product stratification \( A \times \{\mathbb{R}\} \) (cf. intersection-start). Since by assumption \( F_z(M) \) is transversal to all strata in \( Y \), so \( \hat{F} \) is transversal to all strata in \( Y \times D \). Thus inverse image of of starta in \( Y \times D \) under \( \hat{F} \) gives a Whitney stratification for \( \hat{F}^{-1}(Y \times D) \subset M \times D \) (cf. intersection-of-strat). Now consider projection \( t : M \times \mathbb{R}^k \to \mathbb{R}^k \) as a function on \( \hat{F}^{-1}(Y \times \mathbb{R}^k) \). Then since for any \( z \), \( F_z : M \to X \) is an embedding and \( F_z(M) \) is transversal to all strata in \( Y \) so \( t \) is submersion restricted to any stratum of \( \hat{F}^{-1}(Y \times D) \), and thus by Thom’s First Isotopy Lemma, \( t : \hat{F}^{-1}(Y \times D) \to D \) is a locally trivial fibration. Notice \( t^{-1}(z) = \{(m, z) | F_z(m) \in Y\} = F_z(M) \cap Y \). So for \( z_1, z_2 \in D, F_{z_1}(M) \cap Y \) is homeomorphic to \( F_{z_2}(M) \cap Y \) under a stratum preserving homeomorphism. \( \square \)

### 2.4 Generalized Morse theory of Palais and Smale and an alternative proof of the theorems \( A' \) and \( B' \)

To be able to apply Morse theory to infinite dimensional spaces of loops and geodesics, Palais and Smale developed a generalized Morse theory (cf. Palais-Smale and Palais). Since infinite dimensional manifolds are not compact, their theory can be also applied to the non-compact finite dimensional manifolds.

Following Palais-Smale (Generalized Morse theory), we give a brief account of their theory:

Let \( M \) be a \( C^2 \)-Riemannian manifold modeled on a separable Hilbert space (hence it can be infinite dimensional). Let \( f : M \to \mathbb{R} \) be a \( C^2 \) function. Assume that \( f \) satisfies the following extra condition

\( (C) \) If \( \{x_i\} \) is a sequence in \( M \) on which \( |f| \) is bounded and \( \|\nabla f(x_i)\| \) converges to zero, then there is a critical point of \( f \) in the closure of the set \( \{x_1, x_2, \ldots \} \).
Theorem 2.4.1. Let $M$ and $f$ satisfy the condition (C) above and assume that all the critical points of $f$ are non-degenerate. Then

1. For any real numbers $a < b$ there are only finitely many critical points of $f$ in $M_{a,b} = \{ x \in M | a < f(x) < b \}$, hence the critical values of $f$ are isolated.

2. Let $a$ and $b$ be regular values of $f$ and suppose that among the critical points of $f$ in $M_{a,b}$ there are $r$ having finite index. Let the indices of these critical points be $\lambda_1, \ldots, \lambda_r$. Then $M_{\leq b}$ has the homotopy type of $M_{\leq a}$ with $r$ cells of dimensions $\lambda_1, \ldots, \lambda_r$ attached.

We now prove that in the situation of theorem trans-condition, the condition (C) is satisfied and hence Palais-Smale theory gives an alternative proof of out theorems (A′) and (B′).

Proposition 2.4.2. Let $M$ be a (closed) algebraic submanifold of $\mathbb{R}^n$ of dimension $d$ and $A = \{ X_0, X_1, \ldots, X_r \}$ ($X_0 = M$) a finite Whitney stratification for $\overline{A} \subset \mathbb{R}P^n$. Suppose $f$ is a linear functional such that $H_{\infty} = \{ (x_1: \ldots : x_n) \in \mathbb{R}P^{n-1} | f(x_1, \ldots, x_n) = 0 \}$ is transverse to all the strata $X_1, \ldots, X_n$ at infinity. Then the condition (C) holds for $M$ and $f$.

Proof. Let $\{ x_i \}$ be the sequence in the condition (C). If $\{ x_i \}$ is bounded there is nothing to prove. So assume it is not bounded. Without loss of generality assume $x_i$ converges to $x \in \overline{M} \setminus M$ and $\lim_{i \to \infty} \| \nabla f_{|M}(x_i) \| = 0$. Now the sequence of tangent planes $T_{y_i}M$ should have a convergent subsequence in the Grassmanian $Gr(n,d)$ of $d$-planes in $\mathbb{R}^n$ as Grassmanian is compact. Again without loss of generality assume that $T_{x_i}M$ converges to $L \in Gr(n,d)$. Since $\| \nabla f_{|M}(x_i) \|$ goes to zero as $i$ goes to infinity, $\text{Angle}(\nabla f, T_{x_i}M)$, the angle between $\nabla f$ and $T_{x_i}M$, should also tend to zero. But since $\text{Angle}(\nabla f, \cdot)$ is continuous we must have $\text{Angle}(\nabla f, L) = 0$, that is $L \subset H_{\infty}$. By Whitney condition (A) if $\lim_{i \to \infty} T_{x_i}M = L$ and $\lim_{i \to \infty} x_i = x$ then $T_x X \subset L$, where $X$ is the strata containing $x$. Hence $T_x X \subset H_{\infty}$. But this is a contradiction as $X$ and $H_{\infty}$ are transverse. □

2.5 Morse theory for the orbits of a Lie group action

In this section we reconsider the Morse theory of the section ref for the case of orbits of Lie group actions. We will give stronger versions (in the sense
to be explained) of the isotopy lemma and our main theorem for this case as well as direct self-contained proofs.

Orbits of actions of Lie groups provide interesting examples of Whitney stratifications.

**Theorem 2.5.1 (cf.).** Let a Lie group \( G \) acts smoothly on a manifold \( X \) and let \( Y \subset X \) be a closed invariant subset consisting of a finite number of orbits. Then the decomposition of \( Y \) into orbits is a Whitney stratification of \( Y \).

Morse theory for stratified sets (Thom’s first isotopy lemma) guarantees that the level sets of a Morse function on a stratified set are homeomorphic under a stratum preserving homeomorphism, provided that certain transversality condition holds. Our generalization of isotopy lemma also asserts that intersections of a stratified set with a moving family of submanifolds are homeomorphic as long as certain transversality condition holds.

But in general, we can not talk about this sections be diffeomorphic under a stratum preserving diffeomorphism as the Whitney stratification does not carry enough smooth structure. The following famous example illustrates this.

**Example 2.5.1.** Let \( \gamma : \mathbb{R} \to (0, \infty) \) be a smooth non-constant function and let \( Z \subset \mathbb{R}^3 \) be defined by the equation

\[
xy(x + y)(x - \gamma(z) \cdot y) = 0.
\]

\( Z \) is stratified by the \( z \)-axis which we call it \( X \) and its complement (denoted by \( Y \)). This is in fact a Whitney stratification. Adding \( \mathbb{R}^4 \setminus A \) as a third stratum we obtain a Whitney stratification of \( \mathbb{R}^3 \). Now consider the function \( \pi_z \), projection on \( z \) coordinate. The level sets of \( \pi_z \) are planes parallel to the \( xy \)-plane. The intersection of a level set with the strata consists of four lines \( x \)-axis, \( y \)-axis, the line \( x + y = 0 \) and the line \( x - \gamma(z) \cdot y = 0 \).

It is obvious that the topological picture of the sections of the stratification remains the same as \( z \) is changing, but now let us see if the picture is the same in the smooth category, i.e. if one can find a stratum preserving diffeomorphism between two different level sets \( \pi_z^{-1}(c_1) \) and \( \pi_z^{-1}(c_2) \). This is in fact impossible because if there is such a diffeomorphism, its derivative at origin will be a linear transformation in \( \mathbb{R}^2 \) which leaves the three lines \( x \)-axis, \( y \)-axis and \( x + y = 0 \) invariant and maps \( x - \gamma(c_1) \cdot y = 0 \) to...
\[ x - \gamma(c_2) \cdot y = 0. \] But this is impossible since a linear transformation preserve the cross ratio of a collection of 4 lines through origin in \( \mathbb{R}^2 \) while the cross ration of our four lines is changing according to \( \gamma(z) \) (unless \( \gamma \) is constant).

In the proof of the first theorem of classical Morse theory (Theorem A) one uses the gradient vector field of the Morse function \( f \) and constructs a vector field such that \( f \) increases with constant speed along trajectories of the vector field. The flow of this vector field then gives us the required diffeomorphism. To prove the isotopy lemma, one basically follow the same idea but one needs the vector fields be tangent to the strata as well so that the flow of the vector fields preserve the strata. But the strata of a Whitney stratification are not attached together in a smooth way hence we can only construct \emph{continuous} vector fields (rather than smooth) tangent to the strata and satisfying our required properties. Thus the flows of the vector fields give us only homeomorphisms.

But in the case the strata are the orbits of a Lie group action, there is a priori lots of smooth vector fields tangent to all the orbits, i.e. generating vector fields of the action. This enables us to prove stronger version of our isotopy lemma (i.e. diffeomorphism instead of homeomorphism).

The following can be considered as \emph{Thoms’ first isotopy lemma for Lie group actions}.

**Proposition 2.5.2.** Let a Lie group \( G \) act smoothly on a manifold \( X \) and let \( Y \subset X \) be an invariant subset with compact closure. Also assume \( f : X \to \mathbb{R} \) is a smooth function, suppose \( [a, b] \subset \mathbb{R} \) such that \( \forall c \in [a, b], f^{-1}(c) \) is transversal to all orbits in \( Y \), then there exists vector field \( v \) with compact support on \( X \) satisfying:

1. For any \( y \in Y, v(y) \) is tangent to orbit of \( y \).

2. Lie derivative of \( f \) along \( v \) is equal to 1 at any point \( y \in Y \).

Existence of \( v \) then implies that for \( c_1 \text{ and } c_2 \in [a, b], f^{-1}(c_1) \cap Y \) is diffeomorphic to \( f^{-1}(c_2) \cap Y \) under an orbit preserving diffeomorphism of whole space \( X \).

**Proof.** Let \( y \in \overline{Y} \) then,

Claim: There exist an open neighbourhood \( U_y \) of \( y \) and a smooth non zero vector field \( \xi_y \) on \( U \) such that \( \xi_y \) at any \( x \) in \( U \) is tangent to orbits of \( x \) and Lie derivative of \( f \) along \( \xi_y \) is non zero on \( U \).
Proof of Claim: For any $\xi \in \mathfrak{g} = T_eG$, let $v_\xi$ be the left invariant vector field on $X$ generated by $\xi$ (i.e. $v_\xi(x) = d\lambda(e)(\xi)$ where $\lambda : G \rightarrow X, \lambda(g) = g.x$). $v_\xi$ at $x$ is obviously tangent to orbit of $x$. Since $\xi \mapsto v_\xi(y)$ is surjective, as a map from $\mathfrak{g}$ to $T_yO_y$, there exist $\xi \in \mathfrak{g}$ such that Lie derivative of $f$ along $v_\xi$ at $y = L_{v_\xi}f(y) = df(d\lambda(e)(\xi))$ is non zero . Let’s fix $y$ and denote $v_\xi$ by $\xi_y$. By continuity we can find a neighbourhood $U_y$ of $y$ such that $L_{\xi_y}f$ is non zero on $U_y$.

Let $v_y = \xi_y/(L_{\xi_y}f)$, then obviously Lie derivative of $v_y$ on $U_y$ is 1. Now we use partition of unity to patch all $v_y$s together to get a compactly supported vector field $v$ on a neighbourhood of $Y$ such that Lie derivative of $f$ along $v$ is 1, flow of $v$ then gives us the desired diffeomorphism between $f^{-1}(c_1) \cap Y$ and $f^{-1}(c_2) \cap Y$ (for $c_1$ and $c_2$ in $[a, b]$).

$Y \subset X$ is compact . Take neighbourhood $W_y$ of $y$ such that $\overline{W_y} \subset U_y$, and take $W_{y_1}, \ldots, W_{y_n}$ among $W_y$s such that they cover $\overline{Y}$. Let $\eta_1, \ldots, \eta_n$ be partition of unity corresponding to open sets $\{U_{y_1}, \ldots, U_{y_n}\}$ such that $\text{supp}(\eta_i) \subset W_{y_i}$, each $\eta_i$ can be smoothly extended to $X$ by letting $\eta_{y_i}$ to be zero outside $U_{y_i}$. Let $v = \sum \eta_{y_i}v_{y_i}$ then $v$ is defined everywhere and $\text{supp}(v) \subset \bigcup W_{y_i}$ which is compact. so $v$ is compactly supported.

\[
L_vf(x) = df(x)(\sum \eta_{y_i}(x).v_{y_i}(x)) \\
= df(x)(\sum_{\{i|x \in W_{y_i}\}} \eta_{y_i}(x).v_{y_i}(x)) \\
= \sum_{\{i|x \in W_{y_i}\}} \eta_{y_i}(x).L_{v_{y_i}}(x) \\
= \sum_{\{i|x \in W_{y_i}\}} \eta_{y_i}(x) \\
= 1
\]

Using the same idea that used was above before we can prove a stronger version of our main theorem in subsection

The following theorem is the main theorem we need and is basically a generalized version of the above proposition where instead of level set of function $f$ i.e. $f^{-1}(z)$ we consider a smooth family of submanifolds $M_z$ parametrized by $z \in \mathbb{R}$. It would be useful to generalize the situation and
consider the family to be parametrized by \( \mathbb{R}^k \), this is given by a smooth map \( F : X \times \mathbb{R}^k \to X \) where \( f_z : M \to X \) is a diffeomorphism for each \( z \in \mathbb{R}^k \) and we consider the smooth family \( M_z \) to be \( f_z(M) \).

More precisely:

**Theorem 2.5.3 (Main Theorem).** Let \( X \) be a compact real manifold and \( F : X \times \mathbb{R}^k \to X \) smooth such that \( f_z : X \to X \) is diffeomorphism for any \( z \in \mathbb{R}^k \). Suppose \( M \subset X \) is a compact submanifold of \( X \) of codimension \( k \). Also assume that the Lie group \( G \) acts on \( X \) smoothly and let \( Y \subset X \) be a closed invariant subset, and finally assume that the following transversality condition holds:

\[
\forall z \in D = I_1 \times I_2 \times \cdots \times I_k \subset \mathbb{R}^k \text{ and } \forall y \in Y, \text{ we have } f_z(M) \text{ is transversal to } O_y, \text{ (where } O_y = \text{ orbit of } y \text{ and } I_i \text{'s are closed intervals).}
\]

then there exist vector fields \( v_1, \ldots, v_k \) on \( X \times \mathbb{R}^k \) satisfying:

1. \( v_i \) has compact support \( (i = 1, \ldots, k) \).
2. If \( \hat{F}(x,z) = (F(x,z),z) \) and \( \hat{M} = \hat{F}(M \times \mathbb{R}^k) \) then \( \forall (x,z) \in \hat{M}, \ v_i(x,z) \in T_{(x,z)}\hat{M}, \) i.e. \( v_i \) are tangent to \( \hat{M} \), \( (i = 1, \ldots, k) \).
3. For any \( (x,z) \in X \times D, v_i(x,z) \in T_{x}O_z \times \mathbb{R}^k = T_{(x,z)}(O_x \times \mathbb{R}^k), \) i.e. \( v_i \) are tangent to orbits of \( G \) acting on \( X \times D \) (where \( G \) acts trivially on \( \mathbb{R}^k \).)
4. (Main Property) \( L_{v_1} t_j(x,z) = \delta_{ij}, \forall (x,z) \in \hat{M} \cap (Y \times D), \) where \( t_j : X \times \mathbb{R}^k \to \mathbb{R}, \) \( t_j(x,z) = z_j = j\text{-th component of } z, \) and \( L \) is for Lie derivative.

In addition the existence of \( v_i \) implies that all \( f_z(M) \cap Y \) are diffeomorphic for \( z \in D \).

**Proof.** Suppose \( v_i \) are constructed satisfying (1) to (4). Let \( \phi_i \) be flow of \( v_i \), since \( v_i \) are compactly supported \( \phi_i^t \) is defined for all \( t \). Note that \( v_i \) is tangent to \( \hat{M} \), so defines a vector field on \( \hat{M} \) and so \( \phi_i^t \) maps \( \hat{M} \) to \( \hat{M} \). Now consider \( \psi : \hat{M} \times \mathbb{R}^k \to \hat{M} \), given by \( \psi(\hat{x}, t_1, \ldots, t_k) = \phi_1^{t_1}(\phi_2^{t_2}(\ldots \phi_k^{t_k}(\hat{x}) \ldots)) \), for \( \hat{x} \in \hat{M} \) and \( (t_1, \ldots, t_k) \in D \).

We show that \( \psi_z : \hat{M} \to \hat{M} \) is a diffeomorphism, where \( z = (t_1, \ldots, t_k) \) which maps \( M_{z_0} \cap (Y \times D) \) onto \( M_{z_0+z} \cap (Y \times D) \) for \( z_0 \) and \( z_0 + z \in D \). This is because of (1) to (4). Since \( v_i \) are tangent to orbits of \( G \) acting on
$X \times \mathbb{R}^k$ and $Y \times \mathbb{R}^k$ is a union of orbits so $\phi^t_1, \ldots, \phi^t_k$ map $Y \times \mathbb{R}^k$ onto itself and so does $\psi_z$. Same reasoning shows that $\psi_z$ maps $\hat{M}$ onto itself. Now Lie derivative of $t_i$ along $v_j$ is equal to $\delta_{ij}$, this means that $t_i(\psi_z(x)) = t_i(x) + z_j$, where $z = (z_1, \ldots, z_n)$. So $\psi_z$ maps $M_{z_0} \cap \hat{M} = \{ x \in \hat{M} : t_i(x) = z_i \}$ onto $M_{z_0 + z} \cap \hat{M} = \{ x \in \hat{M} : t_i(x) = z_0 + z_i \}$

Now we show how to construct $v_i$s satisfying (1) to (4). Recall $\hat{F}(x, z) = (F(x, z), z)$. Let $\chi_i(x, z) = d\hat{F}(x, z)(0, e_i)$ be the velocity vector fields, where $(0, e_i) \in T_x X \times \mathbb{R}^k$ and $e_i$ is the $i$-th standard basis element in $\mathbb{R}^k$. Then since $\hat{F}(M \times \mathbb{R}^k) = \hat{M}$, $\chi_i$ are tangent to $\hat{M}$ and so $\chi_i$ restricted to $\hat{M}$ defines a vector field on it.

Now take $(x, z) \in \hat{M} \cap (Y \times D), let (x, z) = \hat{F}(p, z)$ where $p \in M$ and let $\gamma_1, \ldots, \gamma_m \in \mathfrak{g}$ be vector fields on a neighbourhood $U$ of $p$ in $M$ such that $\gamma_1, \ldots, \gamma_m$ generate $T_y M$, $\forall y \in U, (m = \dim(M))$, and let $\nu_i(F(y, z), z) = d\hat{F}(y, z)(\gamma_i(y), 0)$ be the image of $\gamma_i$ under $\hat{F}$. Since $\hat{F}_z$ is embedding $\nu_1, \ldots, \nu_m$ give a basis for $T_{\hat{F}(y, z)}(\hat{F}(M)), \forall y \in U$.

Now since $O_x$ is transversal to $F_2(M), \forall x \in Y, \forall z \in D$, one can find $\xi_1, \ldots, \xi_k \in \mathfrak{g} = \text{Lie algebra of } G$ such that the invariant vector fields on $X \times \mathbb{R}^k$ generated by $\xi_1, \ldots, \xi_k$ (which we again denote them by $\xi_1, \ldots, \xi_k$) together with $\nu_1, \ldots, \nu_m$ and $\partial/\partial t_1, \ldots, \partial/\partial t_k$ form a basis for $T_{(x, z)}(X \times \mathbb{R}^k)$, where $(\partial/\partial t_1, \ldots, \partial/\partial t_k)$ is the standard basis for $\mathbb{R}^k$. Thus there exists a neighbourhood $U$ of $(x, z)$ in $X \times \mathbb{R}^k$ such that $\xi_1(y, \zeta), \ldots, \xi_k(y, \zeta)$ together with $\nu_i(y, \zeta), \ldots, \nu_m(y, \zeta)$ and $\partial/\partial t_1, \ldots, \partial/\partial t_k$ form a basis for $T_{(y, \zeta)}(X \times \mathbb{R}^k)$, for all $(y, \zeta) \in U$. Now let vector fields $V_i^{(x, z)}$ be the projection of $\chi_i(y, \zeta)$ on the space generated by $\xi_1, \ldots, \xi_k$ and $\partial/\partial t_1, \ldots, \partial/\partial t_k$, having definition of $\chi_i$ in mind, one can write then $V_i^{(x, z)}(y, \zeta) = \alpha^1(y, \zeta)\xi^1(y) + \cdots + \alpha^k(y, \zeta)\xi^k(y) + \partial/\partial t_i$.

$V_i^{(x, z)}(y, \zeta)$ are vector fields on $U$ such that:

1. $V_i^{(x, z)}(y, \zeta) \in T_y O_{y, \zeta} \times \mathbb{R}^k$
2. $L_{t_j}V_i^{(x, z)}(y, \zeta) = \delta_{ij}$ because $L_{t_j}\chi_i = \delta_{ij}$ and $L_{t_j}\nu_i = \delta_{ij}$
3. $V_i^{(x, z)}(y, \zeta) \in T_{(y, \zeta)}\hat{M}$ for $(y, \zeta) \in \hat{M}$ because $\chi_i$ and $\nu_i$ are tangent to $\hat{M}$.

So for any $(x, z) \in \hat{M} \cap (Y \times D)$ we get a neighborhood $U(x, z) \in X \times \mathbb{R}^k$ and vector fields $V^{(x, z)}_i (i = 1, \ldots, k)$ defined on $U(x, z)$. Notice that $\hat{F}(M \times D)$
is compact, so \( \widehat{M} \cap (Y \times D) \subset \widehat{F}(M \times D) \) is compact, \( Y \) being closed.

Take \( W_{(x,z)} \subset W'_{(x,z)} \subset U_{(x,z)} \) such that \( \overline{W'} \subset W' \) and \( \overline{W'} \subset U \), with \( \overline{W'} \) and \( \overline{W'} \) compact. Since \( \widehat{M} \cap (Y \times D) \) is compact finitely many of \( W \)s cover \( \widehat{M} \cap (Y \times D) \), denote them by \( W_i \), \( i = 1, \ldots, r \). Let \( \eta_1, \ldots, \eta_r \) and \( \eta \) be partition of unity for \( X \times \mathbb{R}^k \) corresponding to open cover \( W'_1, \ldots, W'_r \) and \( (\bigcup_{i=1}^r \overline{W}_i)^c \), (since \( \bigcup \overline{W}_i \subset \bigcup W'_i, W'_1, \ldots, W'_r \) and \( (\bigcup \overline{W}_i)^c \) cover the whole space \( X \times \mathbb{R}^k \)). Denote vector fields \( V_j(x,z) \) corresponding to open set \( U_i \) by \( V_j^i(x,z) \), \( j = 1, \ldots, k \). Since \( \text{supp}(V_j^i) \subset U_i \) and \( \text{supp}(\eta_i) \subset W'_i \) and \( \overline{W}'_i \subset U_i \) then \( \eta_i(x,z)V_j^i(x,z) \) are globally defined and smooth everywhere on \( X \times \mathbb{R}^k \) with \( \text{supp}(\eta_i V_j^i) \subset \bigcup W'_i \). Define \( v_j(y,z) = \sum_{i=1}^r \eta_i(y,z)V_j^i(y,z) \), then \( v_j \) are smooth everywhere and \( \text{supp}(v_j) \subset \bigcup W'_i \). For \( (y,z) \in \bigcup W_i \), \( L_I v_j(y,z) = \sum_{i:(y,z) \in W'_i} \eta_i(y,z)L_I V_j^i(y,z) = \sum_{i:(y,z) \in W'_i} \eta_i(y,z) \delta_{kj} = \delta_{kj} \), because \( \eta_i \)s are partition of unity with respect to \( W'_i \) and \( (y,z) \notin (\bigcup \overline{W}_i)^c \).

So for \( (y,z) \in \bigcup W_i = \) an open set in \( \widehat{M}(Y \times D) \) we have \( L_I v_j(y,z) = \delta_{ij} \) and since \( \text{supp}(v_j) \subset \bigcup W'_i \subset \overline{W}_i \), which is compact, \( v_j \)s are compactly supported. finally because of (1) to (3) for \( V_j^i \)s we have:

1. \( v_j(y,z) \in \mathcal{T}_y O_Y \times \mathbb{R}^k \).

2. \( v_j(y,z) \in T_{(y,z)} \widehat{M} \).

and we just proved:

3. \( L_I v_j(y,z) = \delta_{ij} \) for \( (y,z) \in \) a neighbourhood of \( \widehat{M} \cap (Y \times D) \).
Chapter 3

Euler Characteristic of Sections of Orbits of Algebraic Groups

3.1 Algebraic torus actions

Notation: We denote the direct product of $n$ copies of multiplicative group of complex numbers by $\mathbb{C}^*$. Let $x = (x_1, \ldots, x_n)$ be an element of this group and let $k = (k_1, \ldots, k_n)$ be an $n$-tuple of integers. We denote the monomial $x_1^{k_1} \ldots x_n^{k_n}$ simply by $x^k$.

As we mentioned in the beginning, there is a beautiful theorem due to D.N. Bernstien (cf. Bernstein and Askold) regarding the topology of hypersurfaces defined in $\mathbb{C}^*$. Let us recall the theorem.

Definition 3.1.1. Let $f(x_1, \ldots, x_n)$ be a Laurent polynomial (i.e. negative powers of $x_i$’s are allowed) with complex coefficients. To each monomial $c_k x^k$ we can assign a point $k = (k_1, \ldots, k_n)$ in $\mathbb{Z}^n$. The Newton polyhedron $\Delta$ of $f$ is defined to be the convex hull of the points $k$ in $\mathbb{Z}^n$ corresponding to monomials of $f$.

Newton polyhedron is a very nice combinatorial invariant of $f$ which contains a lot of information about geometry and topology of geometric objects defined by polynomials. It can be thought as the generalization of degree of a one variable polynomial. The philosophy is that in the generic cases, one can describe all the discrete topological and geometric invariants of $f$ in terms of its Newton polyhedra.

For each polyhedron $\Delta \subset \mathbb{R}^n$ and a covector $\xi$ we define the polyhedron $\Delta^\xi$ to be the face of the polyhedron $\Delta$ on which $\xi$ attains a minimum (in
particular, \( \Delta^0 = \Delta \). If \( \Delta \) is Newton polyhedron of a polynomial \( f \), then the sum of monomials corresponding to points in \( \Delta^\xi \) is denoted by \( f^\xi \).

**Definition 3.1.2.** We say that a system of Laurent polynomials \( f_1, \ldots, f_k \) is *non-singular* for their Newton polyhedra, if for any covector \( \xi \in \mathbb{R}^n \) the following condition holds: for any solution \( z \) of the system \( f_1^\xi(x) = \ldots = f_k^\xi(x) = 0 \) in \( \mathbb{C}^n \), differentials \( df_i^\xi \) \( (i = 1, \ldots, k) \) are linearly independent.

One can show that non-singularity condition is a generic condition, i.e. in the space of all systems of functions \( f_1, \ldots, f_k \) with fixed Newton polyhedra \( \Delta_1, \ldots, \Delta_k \), non-singularity condition holds for every system except for a set of measure zero.

**Theorem 3.1.1 (Bernstein Bernstein).** Let \( f(x_1, \ldots, x_n) \) be a Laurent polynomial which is non-singular for its Newton polyhedron \( \Delta \). Let \( X = \{ f(x_1, \ldots, x_n) = 0 \} \). Then \( \chi(X) = n! \cdot \text{Vol}(\Delta) \).

One can generalize the above theorem to complete intersections of hypersurfaces. We need a bit of notation. Let \( \Delta_1, \ldots, \Delta_n \) be \( n \)-polyhedra in \( \mathbb{R}^n \). Let \( V(\Delta_1, \ldots, \Delta_n) \) denotes the mixed volume \( \text{Vol} \) of these polyhedra. Now let \( F(x_1, \ldots, x_k) \) be the Taylor series of an analytic function in \( k \) variables at the point \( 0 \). We wish to define the number \( F(\Delta_1, \ldots, \Delta_k) \). If \( F \) is a monomial of degree \( n \), \( F(x_1, \ldots, x_n) = x_1^{n_1} \cdots x_k^{n_k} \), we put

\[
F(\Delta_1, \ldots, \Delta_k) = n!V(\Delta_1, \ldots, \Delta_1, \ldots, \Delta_k, \ldots, \Delta_k).
\]

where each \( \Delta_i \) is repeated \( n_i \) times in the mixed volume. One extends the definition to a homogeneous polynomial \( F \) of degree \( n \) by linearity and for a power series \( F \) one defines

\[
F(\Delta_1, \ldots, \Delta_k) = n!F_n(\Delta_1, \ldots, \Delta_k).
\]

where \( F_n \) is the homogeneous part of \( F \) of degree \( n \). We then have

**Theorem 3.1.2 (Bernstein Bernstein).** Let \( X \) be the variety defined on \( \mathbb{C}^n \) by a non-degenerate system of equations \( f_1 = \cdots = f_k = 0 \) with Newton polyhedra \( \Delta_1, \ldots, \Delta_k \). Then

\[
\chi(X) = \prod_{i=1}^k \Delta_i (1 + \Delta_i)^{-1}.
\]
Bernstein’s theorem can be put in a more fancy way as follow: let \( p(x) = \sum_{\alpha \in S} c_{\alpha} x^{\alpha} \) with \( S = \{\alpha_1, \ldots, \alpha_N\} \subset \mathbb{Z}^N \) be a Laurant polynomial. As before we denote its Newton polyhedron by \( \Delta \). Let \( \pi : \mathbb{C}^* \to \mathbb{C}^N \subset GL(n, \mathbb{C}) \) be a finite dimensional representation of algebraic torus \( \mathbb{C}^* \) given by \( \pi(x_1, \ldots, x_n) = \text{diag}(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}) \) and consider the hyperplane \( L = \{x \in \mathbb{C}^N | f(x) = 0\} \) where \( f(x_1, \ldots, x_n) = \sum_{i=1}^n \alpha_i x_i \), then if \( \alpha_i \) are generic enough, the Euler characteristic of hyperplane section \( L \cap \pi(\mathbb{C}^*) \) is equal to \( n!Vol(\Delta) \).

On the other hand there is a remarkable formula due to Kushnierenk o (Kushnierenko) giving the number of solutions of a system of Laurant polynomials.

**Theorem 3.1.3 (Kouchnierenko).** Let \( f_1, \ldots, f_n \) be \( n \) Laurant polynomials in \( n \) variables with the same Newton polyhedron \( \Delta \). If the coefficients of the polynomials are generic enough, then the number of solutions of the system \( f_1(x) = \ldots = f_n(x) = 0 \) is equal to \( n!Vol(\Delta) \).

There is a generalization of Kouchnierenko’s theorem to any system of generic polynomials (not necessarily with the same Newton polyhedra) due to D. Bernstein

**Theorem 3.1.4 (Bernstein).** Let \( f_1, \ldots, f_n \) be \( n \) Laurant polynomials in \( n \) variables with Newton polyhedra \( \Delta_1, \ldots, \Delta_n \). If the coefficients of the polynomials are generic enough, then the number of solutions of the system \( f_1(x) = \ldots = f_n(x) = 0 \) is equal to \( n!V(\Delta_1, \ldots, \Delta_n) \).

Kouchnierenko’s theorem can also be formulated using representations of \( \mathbb{C}^* \). Let us recall notion of degree of a subvariety \( V \) of \( \mathbb{C}^n \) (or \( \mathbb{C}P^n \)): if \( \text{dim}(V) = d \) then the number of intersections of \( V \) and a generic plane of codimension \( d \) is constant called degree of the subvariety \( V \). Now let \( \pi : \mathbb{C}^* \to \mathbb{C}^N \subset GL(n, \mathbb{C}) \) be the representation of \( \mathbb{C}^* \) defined by monomials corresponding to the the points in Newton polyhedron of \( f_1, \ldots, f_n \). \( \mathbb{C}^* \) gets embedded in \( \mathbb{C}^N \) via \( \pi \). Hence

**Theorem 3.1.5.** Let us \( \mathbb{C}^* \) acts on \( \mathbb{C}^N \) via a linear representation \( \pi \). Let \( \Delta \) denotes the convex hull of weights of this representation. Then degree of a generic orbit of this action is equal to \( n!Vol(\Delta) \).
3.2 Generalization of the Kouchnierenko’s theorem and the spherical varieties

3.2.1 Generalization of Kouchnierenko’s theorem to the representations of reductive groups

Let $G$ be a connected $n$-dimensional complex reductive group and let $\pi$ be an $N$-dimensional holomorphic representation of this group. Consider the systems $f_1(x) = f_2(x) = \cdots = f_n(x) = 0$, $x \in G$, where each $f_i$ is a linear combination of matrix entries of the representation $\pi$. All the systems that lie outside of a certain algebraic hypersurface in the space of such systems, have the same number of roots which we denote by $N(\pi)$.

Let $t$ and $t^*$ be the Lie algebra and dual Lie algebra of a maximal torus $T$ in $G$.

**Definition 3.2.1.** The Newton polyhedron $\Delta$ of the representation $\pi$ is defined to be the convex hull of its weights.

One can construct a homogeneous polynomial $V$ of degree $n$ on the cone of all convex bodies in $t^*$, i.e. $V$ is the restriction to the diagonal of an $n$-linear function on the cone of convex bodies, such that

**Theorem 3.2.1 (Kazarnovski(1986)).** $N(\pi)$ is equal to $n! \cdot V(\Delta)$. Or in other words, the degree of the subvariety $\pi(G)$ of $M(N, \mathbb{C})$, vector space of all $N \times N$ matrices, is equal to $n! \cdot V(\Delta)$.

For the construction of $V$ and the proof of this theorem look at Kazarnovskii.

3.2.2 Further generalization of the Kouchnierenko’s theorem: spherical varieties

When we have a $N$-dimensional representation $\pi$ of $G$, we can think of $G$ acting on the space of matrices $M(N, \mathbb{C})$ via left multiplication, or we can let $G \times G$ acts on $M(N, \mathbb{C})$ by left-right multiplication $((g, h) \cdot m = g \cdot m \cdot h^{-1})$. In both cases $\pi(G)$ would be the orbit of identity. Hence Kazarnovskii’s theorem gives a combinatorial formula for the degree of this orbit.
It is interesting to see if there is similar combinatorial formula for a bigger class of orbits of reductive groups. Brion (cf. Brion) has discovered such a formula for the so-called Spherical varieties. To give statement of his theorem we need to introduce some notations and definitions:

**Definition 3.2.2.** Let $G$ be connected reductive group. A homogeneous space $G/H$ is called spherical if a Borel subgroup of $G$ has a dense orbit in $G/H$. Similarly a $G$-variety $X$ (i.e. a variety together with an algebraic action of $G$) is called spherical if a Borel subgroup of $G$ has a dense orbit.

Let us see how Spherical varieties generalize some of the previously considered, interesting examples of group actions. In all the examples $G$ is a complex reductive group.

**Example 3.2.1.** Let $G = \mathbb{C}^*$. Then the Borel subgroup of $G$ is $G$ itself. Hence $G$ is a spherical homogeneous space with natural left action of $G$ on itself, and spherical $G$-variety are just toric varieties by definition.

**Example 3.2.2.** Suppose $\sigma$ is an involution of $G$ (that is an automorphism of order two of $G$). Let $H$ be the subgroup of all elements of $G$ fixed by $\sigma$. The homogeneous space $G/H$ is called a symmetric variety. Origin of this notion, perhaps, is from differential geometry. It is a manifold with high number of symmetries and hence interesting object to be studied. It is well-known that every symmetric variety is a spherical homogeneous space.

**Example 3.2.3.** Let $G \times G$ act on $G$ by left-right multiplication. From the so-called Bruhat decomposition, one knows that $B \times B$ (which is a Borel subgroup of $G \times G$) has an open dense orbit in $G$. Hence $G$ is a $G \times G$-spherical homogeneous space. This example is, in fact, what Kazarnovskii considered.

Let $V$ be the representation space of a connected reductive complex group $G$. Let $K$ be a maximal compact subgroup of $G$. Action of $G$ on $V$ naturally induces an action on the projective space $\mathbb{P}(V)$. Choose $K$-invariant Kähler structure on the $\mathbb{P}(V)$, which is in particular, a symplectic manifold (the symplectic form $\omega$ being the being the imaginary part of the Kähler form). Define a map $\mu$ from $X$ to $\mathfrak{k}^*$ (the dual of the Lie algebra $\mathfrak{k}$ of $K$) by $\mu(x)(A) = (\tilde{x} \cdot A \tilde{x})(\tilde{x} \cdot x)^{-1}$ where $x \in \mathbb{P}(V)$; $\tilde{x}$ is a representation of $x$ in $V$; $A \in \mathfrak{k}$. It is easy to see that $\mu$ is $K$-invariant and its differential $d\mu$ satisfies $d\mu_x(\xi)(A) =$
\( \omega_\epsilon(\xi, A_x) \) for every \( x \in X, \xi \in T_x \mathbb{P}(V), A \in \mathfrak{k} \). This means that \( \mu \) is a moment map for the symplectic action of \( K \) on \( \mathbb{P}(V) \) (cf. GS).

If \( X \) is any closed smooth algebraic subvariety of \( \mathbb{P}(V) \) which is \( G \)-stable, then \( X \) inherits a \( K \)-invariant Kähler structure, and the restriction of \( \mu \) to \( X \) is still a moment map for the induced symplectic structure on \( X \).

The image \( \mu(X) \) has a nice convexity property. As \( \mu(X) \) is a \( K \)-stable subset of \( \mathfrak{k}^* \) (\( K \) acting on \( \mathfrak{k}^* \) via coadjoint representation), it is described by its intersection with a fundamental domain \( C \) of \( K \) acting on \( \mathfrak{k}^* \). We can choose \( C \) to be a Weyl chamber in the dual \( \mathfrak{t}^* \) of Lie algebra of a maximal torus of \( K \).

**Theorem 3.2.2 (Kirwan, GS).** The intersection \( \Delta = \mu(X) \cap C \) is a convex polyhedron with rational vertices with respect to the weight lattice.

\( \Delta \) is usually called the moment polyhedron. For the case of action of algebraic torus \( \mathbb{C}^* \), the moment polyhedron is same as the Newton polyhedron. One can think of moment polyhedron as Newton polyhedron for spherical varieties.

Let \( R \) be a root system of \( G \) (with respect to the maximal torus associated to \( \mathfrak{t} \)), \( R^+ \) the set of positive roots defined by the choice of \( C \), and \( E \) the set of all positive roots which are orthogonal to \( \Delta \). We denote by \( \rho \) half the sum of all the positive roots.

The following theorem due to Brion (cf. Brion) give the degree of \( X \) as the integral of a certain function on the moment polyhedron.

**Theorem 3.2.3 (Brion (19??)).** The degree of an \( n \)-dimensional spherical subvariety \( X \) of \( \mathbb{P}(V) \) is equal to

\[
 n! \int_{\Delta} \prod_{\alpha \in R^+ \setminus E} (\gamma, \alpha)/(\rho, \alpha) d\gamma
\]

Following a suggestion by A.G. Khovanskii, A. Okounkov has define a bigger polyhedron \( \tilde{\Delta} \) over the moment polyhedron \( \Delta \) (i.e. it projects onto \( \Delta \)) such that the above formula becomes \( \text{deg}(X) = n! \text{Vol}(\tilde{\Delta}) \) (cf. Okounkov).

Let \( G \) be the torus \( \mathbb{C}^* \). Since \( G \) is abelian it is equal to its Borel subgroup and hence all homogeneous spaces of \( G \) are spherical. A spherical \( G \)-variety then is a variety which has a dense \( G \) orbit. So If \( G \) is the torus, spherical \( G \)-varieties are exactly toric varieties. Consider a diagonal representations of
torus $G$ on $V = \mathbb{C}^N$ with weights $\omega_1, \cdots, \omega_N$. As before $G$ acts on $\mathbb{P}(V)$, let $X$ be the closure of the orbit of $(1 : \cdots : 1) \in \mathbb{P}(V)$. One can easily see that the moment polyhedron $\Delta$ in this case is exactly the Newton polyhedron, i.e. the convex hull of $\omega_1, \cdots, \omega_N$ and also the above formula for degree of $X$ reduces to $n! \text{Vol} (\Delta)$ yielding the Kouchnierenko’s theorem.

One can also obtain Kazarnovskii’s result as a special case of the formula for degree of spherical varieties. For this, one considers action of $G \times G$ on $G$ via multiplication from left and right. Let $B$ denote a Borel subgroup of $G$. From a well-known result in algebraic groups (i.e. Bruhat decomposition), $B \times B$ which is a Borel subgroup of $G \times G$ has a dense orbit in $G$ (the big Bruhat cell) and hence $G$ is a $G \times G$-spherical orbit. Let $\pi$ be an $N$-dimensional faithful representation of $G$. Then $G \times G$ can act on $M(N, \mathbb{C})$ by multiplication from left and right and $\pi(G)$ is the orbit of identity. This action obviously induces an action of $G \times G$ on the projective space $\mathbb{P}(M(N, \mathbb{C}))$. Let $X$ be the closure in $\mathbb{P}(M(N, \mathbb{C}))$ of the orbit of identity (that is image of $\pi(G)$ in the projective space). Then $X$ is a spherical variety since $\pi(G)$ is a spherical $G \times G$ homogeneous space. If $\omega_1, \cdots, \omega_t$ are weights of the representation $\pi$, one can show that the moment polyhedron $\Delta$ is the convex hull of the points $(\omega_i, -\omega_i), i = 1, \cdots, t$ living in $t^* \oplus t^*$, that is the dual of the Lie algebra of a maximal torus of $G \times G$ (here $t$ is the Lie algebra of a maximal torus of $G$). If we project $\Delta$ on $t^*$ we obtain $\Delta'$ the convex hull of weights $\omega_1, \cdots, \omega_t$. This $\Delta'$ is the polyhedron appeared in the Kazarnovskii’s formula. In fact, if one rewrites the Brion’s formula as an integral over $\Delta'$ rather than $\Delta$, one will recover Kazarnovskii’s formula.

### 3.3 Euler characteristic of hyperplane sections, the number of critical points and degree of a subvariety.

#### 3.3.1 Euler characteristic and the number of critical points

Let us recall the theorem Eulerchar-number-of-critical-points, relating Euler characteristic of a generic hyperplane section and the number of critical points of a linear functional. Let $f$ be a linear functional on $\mathbb{C}^N$ and $Y \subset \mathbb{C}^N$ some submanifold, We denote by $\mu(f, Y)$, the number of critical points of
Theorem 3.3.1. Let $X$ be a smooth closed algebraic subset of $\mathbb{C}^N$ of dimension $d$ such that $\bar{X} \subset \mathbb{C}P^N$ has a Whitney stratification with finite number of algebraic strata. Let $f$ be a generic linear functional $f$ on $\mathbb{C}^N$ and $c$ a generic complex number. We then have

$$\chi(\{x \in X | f(x) = c\}) = \chi(X) + (-1)^{(d+1)} \cdot \mu(f, X).$$

Now suppose a complex Lie group $G$ acts linearly on $\mathbb{C}^N$, the action obviously extends to $\mathbb{C}P^N$ by letting $G$ act trivially on the last homogeneous coordinate. Let $X$ be a smooth closed orbit of $G$ in $\mathbb{C}^N$ such that $\bar{X} \subset \mathbb{C}P^N$ consists of a finite number of $G$-orbits. Then it is well-known that $\bar{X}$ admits a Whitney stratification with $G$-invariant strata (i.e. each stratum is union of orbits). So we can apply the above theorem to get

Theorem 3.3.2. Let a complex Lie group $G$ act linearly on $\mathbb{C}^N$ (and hence on $\mathbb{C}P^N$). Suppose $X \subset \mathbb{C}^N$ is a closed $G$-orbit of dimension $d$ whose closure in $\mathbb{C}P^N$ consists of a finite number of orbits. Then for a generic linear functional $f$ on $\mathbb{C}^N$ and a generic complex number $c$ we have

$$\chi(\{x \in X | f(x) = c\}) = \chi(X) + (-1)^{(d+1)} \cdot \mu(f, X).$$

Let us consider an interesting special case of this theorem. It turns out that in this case the conditions of the above theorem are easy to verify. Let $\pi : G \to GL(N, \mathbb{C})$ be a faithful representation of a complex connected reductive group $G$. One can define a linear action of $G \times G$ on $M(N, \mathbb{C})$, the vector space of matrices, by $(g, h) \cdot m = \pi(g) \cdot m \cdot \pi(h)^{-1}$. Then $G \times G$-orbit of identity will be simply $\pi(G)$.

The following theorem tells us what chi of $\pi(G)$ is.

Proposition 3.3.3. If $G$ is a complex connected reductive group then Euler characteristic of $G$ is equal to zero.

Proof. By Iwasawa decomposition $G$ has the same homotopy type as its maximal compact subgroup $K$. In particular $\chi(G) = \chi(K)$. Any non-zero left invariant vector field on the compact Lie group $K$ gives a everywhere non-zero vector field on $K$ and hence by Poincare-Hopf theorem $\chi(K) = 0$. 

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It is an easy lemma to prove that if one has a homomorphism from an algebraic groups \( G_1 \) to another algebraic group \( G_2 \), image of \( G_1 \) in \( G_2 \) is always a closed subset of \( G_2 \), hence in our case \( \pi(G) \) is always closed subset of \( GL(N, \mathbb{C}) \), but it may not be a closed subset of \( M(N, \mathbb{C}) \). Next propositions, provide a nice criterion to check when \( \pi(G) \) is a closed subset of \( M(N, \mathbb{C}) \), namely \( \pi(G) \) is closed in \( M(N, \mathbb{C}) \) iff origin belongs to the interior of the convex hull of weights of the representation.

To prove the general case, we need the special case of torus

**Proposition 3.3.4.** Let \( \pi : \mathbb{C}^n \to GL(N, \mathbb{C}) \) be a representation of an algebraic torus with weights \( \omega_1, \cdots, \omega_N \in \mathbb{Z}^n \). Then \( \pi(\mathbb{C}^n) \) is a closed subset of \( M(N, \mathbb{C}) \) iff origin belongs to the interior of the convex hull of the weights \( \omega_1, \cdots, \omega_N \).

**Proof.** See Vinberg-Popov. \( \square \)

**Proposition 3.3.5.** Let \( \pi : G \to GL(N, \mathbb{C}) \) be a representation of a reductive group \( G \). Then \( \pi(G) \) is a closed subset of \( M(N, \mathbb{C}) \) iff \( \pi(T) \) is closed in \( M(N, \mathbb{C}) \) that is iff origin belongs to the interior of the convex hull of the weights of the representation \( \pi \).

**Proof.** Let \( T \) be a maximal torus of \( G \) and \( B \) a Borel subgroup containing \( T \), let \( B=TU \) be a Levi decomposition and let \( K \) be a maximal compact subgroup of \( G \) such that \( G = TUK \). We may assume that \( \pi(B) \) is contained in the usual upper triangular Borel subgroup of \( GL(N, \mathbb{C}) \) and \( \pi(T) \) is contained in the diagonal matrices, and \( \pi(U) \) is contained in the upper triangular matrices with 1’s down the diagonal. Suppose \( g_n = \pi(t_n)\pi(u_n)\pi(k_n) \) is a sequence in \( \pi(G) \) that converges to \( g \) in \( M(N, \mathbb{C}) \). We want to show that \( g \) is in \( \pi(G) \). Since \( \pi(G) \) is closed in \( GL(N, \mathbb{C}) \), as was explained before, it suffices to show that \( g \) is in \( GL(N, \mathbb{C}) \). Suppose not. Since \( \pi(K) \) is compact, we may assume that \( \pi(k_n) \) converges to an element \( k \) in \( GL(N, \mathbb{C}) \). We may thus assume that \( g_n = \pi(t_n)\pi(u_n) \). This is a sequence in the standard Borel subgroup of \( GL(N, \mathbb{C}) \) converging to an element \( g \) not in \( GL(N, \mathbb{C}) \). This implies that \( \pi(t_n) \) converges to an element not in \( \pi(T) \), contradicting the assumption that \( \pi(T) \) is closed in \( M(N, \mathbb{C}) \). \( \square \)

As was mentioned in previous section, \( G \) under left-right action of \( G \times G \) is an example of so-called spherical homogeneous space, that is \( B \times B \), which is a Borel subgroup of \( G \times G \), has a dense orbit in \( G \) (Bruhat decomposition). Of
course this equally applies to $\pi(G)$ i.e. $\pi(G)$ is a spherical $G \times G$ homogeneous space. It is a well-known result in the theory of spherical varieties that any compactification of a spherical homogeneous space, consists of a finite number of orbits. In our case it means that closure of $\pi(G)$ in $\mathbb{C}P^N$ consists of a finite number of $G \times G$ orbits.

Now we restate the theorem about the Euler characteristic of generic hyperplane sections in the special case of a reductive group embedded in vector space of matrices.

**Theorem 3.3.6.** Let $\pi : G \to GL(N, \mathbb{C})$ be a faithful representation of a $d$-dimensional complex connected reductive group $G$. Suppose $\pi(G)$ is closed in $M(N, \mathbb{C})$ that is origin belongs to the convex hull of weights of $\pi$. Then for $f$ a generic linear functional on $M(N, \mathbb{C})$ and $c$ a generic complex number we have

$$
\chi(\{x \in \pi(G) | f(x) = c\}) = (-1)^{(d+1)} \cdot \mu(f, \pi(G)).
$$

### 3.3.2 Proof of the Bernstein’s theorem

Suppose the torus $\mathbb{C}^*^n$ is acting on $\mathbb{C}^N$ via a diagonal faithful representation $\pi$. We can view $\pi$ as a homomorphism $\pi : \mathbb{C}^*^n \to \mathbb{C}^*^N \subset GL(N, \mathbb{C})$, as the image of $\pi$ lies in the subspace of diagonal matrices. so we can think of the torus $\mathbb{C}^*^n$ as embedded in a bigger torus $\mathbb{C}^*^N$ via the homomorphism $\pi$.

As was mentioned before in section, one can reformulate the Bernstein’s theorem regarding Euler characteristic of a generic hypersurface in $\mathbb{C}^*^n$ as Euler characteristic of a generic hyperplane section of the torus $\mathbb{C}^*^n$ embedded in $\mathbb{C}^N$ is equal to $(-1)^{(n+1)}$ times its degree as a subvariety of $\mathbb{C}^N$.

First of all we claim that we can assume that the image of $\mathbb{C}^*^n$ in $\mathbb{C}^N$ is closed, or equivalently the origin is in the interior of the Newton polyhedron of $\pi$. If not, we shift the Newton polyhedron so that origin lies inside the polyhedron. Shifting the polyhedron corresponds to multiplying all the monomials by a fixed monomial. Since we are considering a hypersurface in $\mathbb{C}^*^n$, multiplying everything by a fixed monomial does not change the hypersurface in $\mathbb{C}^*^n$, i.e. without loss of generality we can assume origin is inside the Newton polyhedron. As we mentioned in the previous section this means that the image of $\mathbb{C}^*^n$ in $\mathbb{C}^N$ is closed. We are now in the position to use theorem , to get that Euler characteristic of a hyperplane section is equal to $-1^{(n+1)} \cdot \mu(f, \pi(\mathbb{C}^*^n))$, where $f$ is a generic linear function defining the hyperplane in $\mathbb{C}^N$. So the only thing remains to complete the proof of the
Bernstein’s theorem is to show that number of critical points of a generic linear functional restricted to the torus is equal to its degree.

**Proposition 3.3.7.** Let \( \pi : \mathbb{C}^* \to \mathbb{C}^N \subset GL(N, \mathbb{C}) \) be a diagonal faithful representation of \( \mathbb{C}^* \). Let \( f \) be a generic linear functional on \( \mathbb{C}^N \). Then the number \( \mu(f, \pi(\mathbb{C}^*)) \) of critical points of \( f|_{\pi(\mathbb{C}^*)} \) is equal to the degree of \( \pi(\mathbb{C}^*) \) as a subvariety of \( \mathbb{C}^N \).

**Proof.** The linear functional \( f \) correspond to a Laurant polynomial on the torus \( \mathbb{C}^n \) which we denote by \( F(x_1, \ldots, x_n) \). A point \( \pi(x), x \in \mathbb{C}^* \) is a critical point for \( f|_{\pi(\mathbb{C}^*)} \) iff \( x \) is a critical point of \( F \), i.e. if \( x \) is a solution of the system of equations:

\[
\begin{align*}
\frac{\partial F}{\partial x_1}(x) &= 0 \\
&\vdots \\
\frac{\partial F}{\partial x_n}(x) &= 0
\end{align*}
\]

Let \( \Delta \) be the Newton polyhedron of \( F \) and denote by \( \Delta_i \) the Newton polyhedron of \( \frac{\partial F}{\partial x_i} \). By Bernstein-Kuchnirenko theorem we know that the number of solutions of the above system is equal to \( n! \cdot V(\Delta_1, \ldots, \Delta_n) \), where \( V \) denotes the mixed volume of convex bodies in \( \mathbb{R}^n \).

Notice that since the origin is inside the \( \Delta \), \( \Delta_i \) is just \( \Delta \) shifted in the direction \(-e_i\), where \( e_1, \ldots, e_n \) is the standard basis for the Lattice \( \mathbb{Z}^N \). But mixed volume is invariant under shifting of the convex bodies so the number of solutions is equal to \( n! \cdot V(\Delta, \ldots, \Delta) = n! \cdot Vol(\Delta) \). By Kouchnirenko’s theorem the last number is equal to the degree which proves that the number of critical points is equal to the degree. \( \square \)

### 3.3.3 How one can relate the number of critical points and the degree of an orbit

As usual suppose a complex Lie group \( G \) acts linearly on \( \mathbb{C}^N \). In this section we try to compare the number of critical points of a generic linear functional \( f \) restricted to an orbit \( X \) and the degree of the variety \( X \).

Let us recall some basic definitions from the theory of Lie group actions. For any \( x \in \mathbb{C}^N \) there is a linear map \( T : g \to T_xO_x \), where \( O_x \) denotes the orbit of \( x \), defined by

\[
T(\xi) = \frac{\partial}{\partial t} \bigg|_{t=0} exp(t \cdot \xi) \cdot x.
\]
Let $Vec(C^N)$ denote the vector space of all vector fields on $C^N$. We can get a linear map $v : g \rightarrow Vec(C^N), \xi \mapsto v_\xi$, by defining $v_\xi(x) = T(\xi)(x)$. This is the so-called generating vector field of $\xi$. Each generating vector field is tangent to the orbits.

As usual let $f$ be a linear functional on $C^N$. $f$ can be differentiated along vector fields. Derivative of $f$ along a vector field $v \in Vec(C^N)$ at a point $x$ is $df(x)(v)$. Let $M \subset C^N$ be a submanifold. If derivative of $f$ at a point $x \in M$ along any vector field tangent to $M$ is zero then $df_M(x) = 0$, i.e. $x$ is a critical point for $f$ restricted to $M$.

Now for $\xi \in g$ consider the hyperplane $H_{\xi,f} = \{ x \in C^N | df(v_\xi(x)) = 0 \}$ and let $H_f = \bigcap_{\xi \in g} H_{\xi,f}$. Let $X$ be an orbit and let $x \in X$. Since generating vector fields span the tangent spaces to the orbits we have $df_{|X}(v_\xi(x)) = 0, \forall \xi \in g$ iff $x$ is a critical point of $f_{|X}$. In other words, the critical points of $f_{|X}$ are exactly the points in $H_f \cap X$. Next proposition tells us what is dimension of $H_f$ as a linear subspace of $C^N$.

**Proposition 3.3.8.** If $d$ is the maximum dimension of the orbits then $\dim(H_f) = N - d$, i.e. $H_f$ has complementary dimension to the orbits of maximum dimension.

**Proof.** Let $x \in C^N$ be a point such that its orbit $O_x$ has maximal dimension equal to $d$. Let $v_1, \ldots, v_d$ be generating vector fields of the Lie algebra vectors $\xi_1, \ldots, \xi_d$ respectively such that $v_1(x), \ldots, v_d(x)$ give a basis for $T_xO_x$. Since $v_1, \ldots, v_d$ are independent at $x$, they are independent in a neighbourhood $U$ of $x$ and since $d$ is the maximum dimension of orbits, we get that for $y \in U$, $v_1(y), \ldots, v_d(y)$ give a basis for $T_yO_y$. From definition of $H_f$ it follows then that $H_f \cap U = H_{\xi_1,f} \cap \cdots H_{\xi_d,f} \cap U$. That is $H_f \cap U$ has codimension $d$, which implies that $H_f$ itself has codimension $d$. \qed

Let us say a few words about the orbits with maximum dimension for actions of reductive groups. It turns out that for a linear action of a complex reductive group, almost all the orbits are the same and one can talk about the notion of generic orbit, more precisely

**Theorem 3.3.9 (Vinberg-Popov p.).** For a linear action of a complex reductive group $G$ on $C^N$, there exists a subgroup $S$ of $G$ and an open dense subset $U$ of $C^N$, such that stabilizer of any $x \in U$ is conjugate to $S$.

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Such an $S$ is called *stabilizer in general position* and the orbits in $U$ are called *generic orbits*. It is not difficult to show that generic orbits are exactly the orbits with maximum dimension. Hence for a linear action of a reductive group $\dim(H_f) = \dim(S)$.

Since the degree of a subvariety in $\mathbb{C}^N$ is its number of intersection points with a generic plane of complementary dimension, the above discussion then suggests that for an orbit $X$ with maximum dimension and a generic linear functional $f$, $\deg(X) = \mu(f, X)$. But unfortunately this is not true. The reason is that while $f$ is generic the plane $H_f$ is not generic and no matter what $f$ is, $H_f$ and $X$ can always have intersection points at infinity. What we intend to prove is that $\deg(X) = \mu(f, X)$ in some good cases. The following is a simple example that $X$ and $H_f$ always have intersection at infinity and hence $\deg(X)$ and $\mu(f, X)$ are different. In this example the group is not reductive.

**Example 3.3.1.** Consider the highest weight representation $V_n$ of $SL_2(\mathbb{C})$. As usual, this is realized as the vector space generated by the symbols $x^n$, $x^{n-1}y$, $\ldots$, $y^n$. Let $B$ be the Borel subgroup of upper triangular matrices, and $U$ the unipotent subgroup of triangular matrices with 1’s on the diagonal. Since $x^n$ is an eigen vector for the action of $B$, it is the highest weight vector. Weyl group of $SL_2(\mathbb{C})$ has 2 elements and $w = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is the non-trivial element in it and $y^n = w \cdot x^n$.

Let us consider the $U$-orbit of $y^n$ which we denote by $X$

$$X = U \cdot y^n = \{(x + uy)^n | \forall u \in \mathbb{C}\}.$$ 

We identify $V_n$ and $\mathbb{C}^{n+1}$ via the basis $\{x^n, \ldots, y^n\}$ and represent vectors in $V_n$ by $n + 1$-tuples of numbers. Then

$$X = U \cdot y^n = \{(1, u, \ldots, u^n) | \forall u \in \mathbb{C}\}.$$ 

Obviously $X$ is a subvariety of degree $n$ of $V_n$. Now let us see what is closure of $B$-orbit of $y^n$ in $\mathbb{C}P^n$. We represent points in the projective space by homogeneous their coordinates. Letting $u$ go infinity, it can be easily verified that the closure of $X$ in $\mathbb{C}P^n$ is

$$\bar{X} = X \cup \{(0 : \ldots : 0 : 1 : 0)\}.$$ 

Let $f \in V_n^*$ be a linear functional. Take the vector $\xi = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathfrak{b}$, the Lie algebra of $B$. We want to compute the generating vector field of this
vector and the hyperplane $H_{\xi, f}$ corresponding to $f$ and this vector. From definition we have

$$v_\xi(x) = \frac{\partial}{\partial t} |_{t=0} \exp(t \cdot \xi) \cdot x.$$ 

One has $\exp(t \cdot \xi) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \in B$ and

$$\exp(t \cdot \xi) \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & t & t^2 & \cdots & t^n \\ 0 & 1 & 2t & \cdots & 2t^{n-1} \\ \vdots \\ 0 & \cdots & 0 & 1 & nt \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + t \cdot x_2 + \cdots \\ x_2 + 2t \cdot x_3 + \cdots \\ \vdots \\ x_n + nt \cdot x_{n+1} + \cdots \\ x_{n+1} \end{bmatrix}.$$ 

Hence for $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n+1} \end{bmatrix}$,

$$v_\xi(x) = \frac{\partial}{\partial t} |_{t=0} \exp(t \cdot \xi) \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} x_2 \\ 2x_3 \\ \vdots \\ nx_{n+1} \\ 0 \end{bmatrix}.$$ 

Represent $f$ by $(f_1, \ldots, f_{n+1})$ in the dual basis for $V_n^*$. Then

$$H_{f, \xi} = \{ x \in V_n | f(v_\xi(x)) = 0 \}$$

$$= \{ x \in V_n | f_1 \cdot x_2 + 2f_2 \cdot x_3 + \cdots + nf_n \cdot x_{n+1} = 0 \}$$

every element of $X$ is of the form $(1, u, \ldots, u^n)$. Remembering the formula for $v_\xi$, the critical points correspond to the solutions of

$$\sum_{i=1}^n if_iu^i = 0$$

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Notice that the point at infinity \((0 : \ldots : 0 : 1 : 0) \in \bar{X} \setminus X\) satisfies the equation \(f_1 \cdot x_2 + 2f_2 \cdot x_3 + \cdots + nf_n \cdot x_{n+1} = 0\) and hence lies on the closure of \(H_f\). Since \(\text{deg}(X) = n\), \(H_f\) intersects \(X\) in at most \(n - 1\) finite points, which are in fact the critical points of \(f|_X\).

But for torus actions, the number of critical points is equal to the degree of orbit. As was discussed before, this in fact follows from the theorems of Bernstein and Kouchnierenko. In case of a diagonal action of a torus on \(\mathbb{C}^N\), any point all whose components are non-zero has generic orbit.

**Proposition 3.3.10.** Let a torus \(\mathbb{C}^*\) acts on \(\mathbb{C}^N\) via a diagonal representation. Let \(X\) be a generic orbit and \(f\) a generic linear functional. Then we have \(\text{deg}(X) = \mu(f, X)\).

**Proof.** This was proved during the proof of Bernstein theorem ( ). □

In the next section, we intend to show that this is true in the more general case of actions with generic spherical orbits, more precisely

**Theorem 3.3.11.** Suppose a connected reductive group \(G\) acts linearly on \(\mathbb{C}^N\) such that generic orbits are closed and spherical. Let \(X\) be a generic orbit. Then for a generic linear functional \(f\) on \(\mathbb{C}^N\) we have

\[
\text{deg}(X) = \mu(X, f).
\]

### 3.3.4 A remark on the number of critical points and incidence correspondence

Let \(\overline{X}\) be a smooth subvariety in \(\mathbb{C}P^n\) and let \(X = \overline{X} \cap \mathbb{C}^n\) be its affine part. Let \(f\) be a generic linear functional on \(\mathbb{C}^n\). \(\text{deg}(\overline{X})\) is, in fact, degree of the map \(f|_X : X \to \mathbb{C}\). On the other hand, \(\mu(X, f)\), the number of critical points of \(f|_X\), can also be realized as the degree of a map but on the so-called incidence variety. Let \(\mathbb{C}P^{n*}\) be the Grassmanian of \((n - 1)\)-planes in \(\mathbb{C}P^n\).

**Definition 3.3.1.** The incidence variety \(\Phi\) of \(\overline{X}\) is

\[
\Phi = \{(p, H)|p \in \overline{X}, H \in \mathbb{C}P^{n*} \text{ and } T_p\overline{X} \subset H\}.
\]

We have the natural maps \(\pi_1 : \Phi \to X, \pi_2(p, H) = p\) and \(\pi_2 : \Phi \to \mathbb{C}P^{n*}, \pi_2(p, H) = H\). \(\pi_2(\Phi)\) is called the dual variety of \(\overline{X}\) and can be thought of as the variety of all tangent hyperplanes to \(\overline{X}\).
Proposition 3.3.12. \( \mu(f, X) = deg(\pi_2) \).

Proof. \( x \in \mathbb{C}^n \) is a critical point of \( f|_X \iff df|_X(x) = 0 \iff T_xX \subset ker(f) \). Let \( ker(f) = H \in \mathbb{C}P^n \), then \( T_xX \subset H \iff (x, H) \in \Phi \), i.e. \( (x, H) \in \pi_2^{-1}(H) \). \( \square \)

3.3.5 The Example of \( SL_2(\mathbb{C}) \)

In this example we look at image of \( SL_2(\mathbb{C}) \) embedded in some space of matrices and compute its degree, Euler characteristic of hyperplane sections and the number of critical points of a generic linear functional on image of \( SL_2(\mathbb{C}) \). We will see that in this case, the degree is not equal to the number of critical points.

As is known all irreducible representation of \( SL_2(\mathbb{C}) \) are \( V_n = \text{Sym}(\mathbb{C}^2)^n \) for any natural number \( n \). Obviously \( \dim(V_n) = n + 1 \) and as \( SL_2(\mathbb{C}) \) is a simple group all its non-trivial representations are faithful. These irreducible representations call also be realized as \( V_n = \text{vector space of all homogeneous polynomials in } x \text{ and } y \text{ of degree } n \), and \( SL_2(\mathbb{C}) \) acts by \( g \cdot f(x, y) = f(g^{-1}(x, y)) \). \( V_n \) is in fact the highest weight representation of \( SL_2(\mathbb{C}) \) corresponding to the weight \( n \).

We will use the first realization of \( V_n \) namely \( Sym(\mathbb{C}^2)^n \). Denote by \( \pi \) the homomorphism from \( SL_2(\mathbb{C}) \) to \( GL(Sym(\mathbb{C}^2)^n) \). Let \( \{x, y\} \) be a basis for \( \mathbb{C}^2 \). Then all the monomials \( \{x^i y^j | i, j \geq 0; i + j = n\} \) give a basis for \( V_n \) and \( SL_2(\mathbb{C}) \) acts according to \( g \cdot (x^i y^j) = (ax + by)^i(cx + dy)^j \), where \( g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \).

One can compute the matrix of \( \pi(g) \) and see that each entry of this matrix is a homogeneous polynomial of degree \( n \) in \( a, b, c \) and \( d \) and all possible monomials of degree \( n \) in \( a, b, c \) and \( d \) appear in the entries. Hence the degree of \( \pi(SL_2(\mathbb{C})) \) will be the number of solutions of the system of equations

\[
\begin{align*}
P_1(a, b, c, d) &= C_1 \\
P_2(a, b, c, d) &= C_2 \\
P_3(a, b, c, d) &= C_3 \\
ad - bc &= 1
\end{align*}
\]

where each \( P_i \) is a generic homogeneous polynomial in \( a, b, c \) and \( d \).

We will use Kouchnierenko theorem to find the number of solutions of such a system. Since \( ad - bc \) is a non-degenerate quadratic form, after a suitable
linear change of coordinates (namely substitute $a$ and $d$ with $a + id$ and $a - id$ respectively and substitute $b$ and $c$ with $ib + c$ and $ib - c$ respectively), it can be put in the form $a^2 + b^2 + c^2 + d^2$. It is evident that after this change of coordinates the polynomials $P_i$ will remain a generic polynomial of degree $n$. With abuse of notation we denote the polynomials $P_i$ after this change of coordinates again by $P_i$.

Let $\Delta$ denote the polytope in $\mathbb{R}^4$ whose vertices are the origin and the standard basis vectors. The 4 dimensional volume of $\Delta$ is $1/4!$. Then the Newton polyhedron of $a^2 + b^2 + c^2 + d^2$ is $2\Delta$ and the Newton polyhedron of $P_i(a, b, c, d) - C_i$ is equal to $n\Delta$. From the Bernstein-Kouchninenko theorem (Theorem ) it follows that the number of the solutions of the above system is equal to

$$4! V(2\Delta, n\Delta, n\Delta, n\Delta) = 4! \cdot 2n^3 \cdot Vol(\Delta)$$

$$= 4! \cdot 2n^3 \cdot 1/4!$$

$$= 2n^3$$

That is, the degree of $\pi(SL_2(\mathbb{C}))$ in $M(n + 1, \mathbb{C})$, the vector space of $(n + 1) \times (n + 1)$ matrices, is equal to $2n^3$.

On the other hand, Euler characteristic, denoted by $\chi$, of a hyperplane section of $\pi(SL_2(\mathbb{C}))$ is equal to Euler characteristic of $\{(a, b, c, d)| ad - bc = 1; P(a, b, c, d) = C\}$, where $P$ is a generic homogeneous polynomial of degree $n$ and $C$ is some generic constant number. As before with a linear change of coordinates we can work with $a^2 + b^2 + c^2 + d^2 = 1$ instead of $ad - bc = 1$. With abuse of notation, we denote the polynomial $P$ after the change of coordinate by $P$ again. So it sufices to find $\chi(Z)$ where

$$Z = \{(a, b, c, d)| a^2 + b^2 + c^2 + d^2 = 1; P(a, b, c, d) = C\}.$$ 

Recall that Euler characteristic is an additive function i.e. if $X = A \cup B$ and $A \cap B = \emptyset$ then $\chi(X) = \chi(A) = \chi(B)$. We partition $Z = \{(a, b, c, d)| ad - bc = 1; P(a, b, c, d) = C\}$ into pieces whose Euler characteristic can be computed using Bernstein’s theorem.

Let $S \subset \{a, b, c, d\}$ and define $Z_S \subset Z$ to be the set of points in $Z$ that exactly their coordinates belonging to $S$ are zero, e.g. $Z_\emptyset = \{(a, b, c, d)| a^2 + b^2 + c^2 + d^2 = 1; P(a, b, c, d); a, b, c, d \neq 0\}$ and $Z_a = \{(a, b, c, d)| a^2 + b^2 + c^2 + d^2 = 1; P(a, b, c, d); a, b, c, d \neq 0\}$. Then $Z$ is the disjoint union of $Z_S$’s ($S \subset \{a, b, c, d\}$). From additivity of the Euler characteristic we obtain

$$\chi(Z) = \sum_{S \subset \{a, b, c, d\}} \chi(Z_S).$$
It is evident that
\[ \chi(Z_a) = \chi(Z_b) = \chi(Z_c) = \chi(Z_d). \]
and
\[ \chi(Z_{ab}) = \chi(Z_{bc}) = \chi(Z_{cd}) = \chi(Z_{ad}) = \chi(Z_{ac}) = \chi(Z_{bd}). \]
and
\[ \chi(Z_{abc}) = \chi(Z_{bcd}) = \chi(Z_{acd}) = \chi(Z_{abd}). \]
Hence
\[ \chi(Z) = \chi(Z_\emptyset) + 4\chi(Z_a) + 6\chi(Z_{ab}) + 4\chi(Z_{abc}). \]
Next we compute each of the terms separately using generalized version of Bernstein’s theorem. In the following multiplicative notation refers to mixed volume of polyhedra.

1. \( \chi(Z_\emptyset) \) is the 4th degree term in the expansion of
   \[ 4!(2\Delta - (2\Delta^2) + (2\Delta)^3)(n\Delta - (n\Delta)^2 + (n\Delta)^3) \]
   That is
   \[ 4!(2n^3\Delta^4 + 4n^2\Delta^4 + 8n\Delta^4) = 4!(2n^3 + 4n^2 + 8n)Vol_4(\Delta) = 2n^3 + 4n^2 + 8n \]

2. Let \( \Delta' \) be the polyhedron in \( \mathbb{R}^3 \) whose vertices are the origin and the standard basis vectors. 3 dimensional volume of \( \Delta' \) is 1/3!. \( \chi(Z_a) \) is then the 3rd degree term in the expansion of
   \[ 3!(2\Delta' - (2\Delta'^2))(n\Delta' - (n\Delta')^2) \]
   That is
   \[ 3!(-2n^2\Delta'^3 - 4n\Delta'^3) = 3!(-2n^2 - 4n)Vol_3(\Delta') = -2n^2 - 4n \]

3. Let \( \Delta'' \) be the polyhedron in \( \mathbb{R}^2 \) whose vertices are the origin and the standard basis vectors. 2 dimensional volume of \( \Delta' \) is 1/2!. \( \chi(Z_{ab}) \) is then the 2nd degree term in the expansion of
   \[ 2! \cdot 2\Delta'' \cdot n\Delta'' \]
   That is
   \[ 2! \cdot 2n\Delta''^2) = 2! \cdot 2nVol_2(\Delta'') = 2n \]
4. Finally note that for generic polynomial $P$, $Z_{abc}$ is empty since $a^2 + b^2 + c^2 + d^2 = 1$ and $a = b = c = 0$ implies $d = \pm 1$ and $\pm 1$ is not a root of a generic polynomial $P(0, 0, 0, d) - C$.

Putting every thing together, we obtain

$$
\chi(Z) = \chi(Z_0) + 4\chi(Z_a) + 6\chi(Z_{ab}) + 4\chi(Z_{abc})
$$

$$
= (2n^3 + 4n + 8n) + 4(-2n^2 - 4n) + 6(2n) + 4 \cdot 0
$$

$$
= 2n^3 - 4n^2 + 4n
$$

Hence Euler characteristic of a generic hyperplane section of $\pi(SL_2(\mathbb{C}))$ in $M(n + 1, \mathbb{C})$, the vector space of $(n + 1) \times (n + 1)$ matrices, is equal to $2n^3 - 4n^2 + 4n$. Of course this is not in general equal to $2n^3$, although for $n = 1$, i.e. the natural representation of $SL_2(\mathbb{C})$ the two numbers are coincide

$$
2 \cdot 1^3 - 4 \cdot 1^2 + 4 \cdot 1 = 2 \cdot 1^3 = 2.
$$

More generally, we will show that, this is the case for the natural representation of $SL(n, \mathbb{C})$. In general, as is the case in this example, the main terms of the formulae for degree and the number of critical points are the same. We will see this in the last section (section ).

### 3.4 Actions with spherical orbits and the main theorem

In this section we intend to prove that for a group action with generic spherical orbits, the number of critical points of a generic functional on a generic orbit is equal to the degree of the orbit. Before we discuss the proof we need some preliminaries regarding spherical varieties.

#### 3.4.1 Some preliminaries form the theory of spherical varieties

Recall that a $G$-variety is a spherical variety if a Borel subgroup of $G$ has a dense open orbit.

An important property of the spherical varieties is that one can approach any point in the closure of an orbit with a one parameter subgroup. Next theorem is even a stronger result. The proof can be found in Brion-Luna-Vust. First we need the definition of an equivariant embedding.
Definition 3.4.1. Let \(G/H\) be a homogeneous space. An equivariant embedding of the homogeneous space \(G/H\) is a pair \((X, i)\), where \(X\) is a variety with a \(G\) action and \(i\) is a \(G\)-equivariant embedding \(i : G/H \to X\) with \(i(G/H)\) an open dense subset of \(X\).

Theorem 3.4.1 (Brion-Luna-Vust). Let \(G/H\) be a spherical homogeneous space. Let \(x \in G/H\) then there exists a torus \(T_x \subset G\), not necessarily unique, such that in any equivariant embedding \(Y\) of \(G/H\), closure of the \(T_x\)-orbit of \(x\) intersects all the \(G\)-orbits.

Let us see how the above theorem implies that one can approach any point in the closure of any orbit with a one parameter subgroup: suppose \(y \in G/H\). By the above theorem, let \(g \cdot y\) belong to the intersection of closure of the \(T_x\)-orbit of \(x\) and \(G\)-orbit of \(y\). Hence there is a one dimensional subtorus \(S\) (one parameter subgroup) of \(T_x\) such that \(S \cdot x\) approaches \(g \cdot y\). Now if \(S' = g \cdot S \cdot g^{-1}\) and \(x' = g \cdot x\) then \(S'\) is a one parameter subgroup of \(G\) and \(S' \cdot x'\) approaches \(y\).

Corollary 3.4.2. Any spherical variety consists of a finite number of \(G\)-orbits.

Proof. Closure of \(T_x\)-orbit of \(x\), is a toric variety. It is well-known that a toric variety consists of a finite number of torus orbits. But closure of this \(T_x\)-orbit intersects all the \(G\)-orbits and hence there should exist a finite number of \(G\)-orbits.

In fact even the number of Borel orbits is finite

Theorem 3.4.3. In a spherical variety, for a Borel subgroup \(B\), there are only a finite number of \(B\) orbits.

For the proof, look at Brion-Luna-Vust.

3.4.2 The main theorem

Now we are in the position to state and prove the theorem regarding the degree of a generic spherical orbit and the number of critical points.
Theorem 3.4.4. Let $G$ acts linearly on $\mathbb{C}^N$ such that generic orbits are spherical and closed. Let $X$ be a generic orbit and let $f$ be a generic linear functional on $\mathbb{C}^N$. Then

$$\deg(X) = \mu(X, f).$$

Proof. As we saw in section , $p \in X$ is a critical point of $f|_X$ iff for any generating vector field $v_\xi (\xi \in g)$ on $X$, $f(v_\xi(x)) = 0$, i.e. $p$ belongs to the hyperplane $H_f = \{ x \in \mathbb{C}^n | f(v_\xi(x)) = 0 \}$. By Proposition , $H_f$ has codimension equal to the dimension of $X$. Hence intersection number of $\overline{X}$ and $\overline{H_f}$ (that is closures of $X$ and $H$ in $\mathbb{C}P^n$) is equal to $\deg(X)$. If $f$ is generic, the critical points of $f|_X$ are non-degenerate and hence $\mu(f, X) = \sharp(H_f \cap X)$. So we only need to show that $X$ and $\overline{H_f}$ do not have any intersection at infinity., i.e. $X \cap H_f = \overline{X} \cap \overline{H_f}$.

We proceed the proof by contradiction. Suppose there exists $z \in \overline{X} \cap \overline{H_f} \setminus X \cap H_f$. Let $B$ be the Borel subgroup of $G$ with a dense orbit. Choose coordinates in $\mathbb{C}^n$ such that $B$ acts by upper triangular matrices. Define

$$V_i = \{ (x_1 : x_2 : \ldots : x_i : 1 : 0 : \ldots : 0) \in \mathbb{C}P^n \}.$$ 

where we have used homogeneous coordinates to represent points in the projective space. Then $V_i \cong \mathbb{C}^i$ and we have the cell decomposition $\mathbb{C}P^n = V_n \cup \cdots \cup V_1$.

Let the linear functional $f$ be given by $f(x) = \sum_{i=1}^n f_i x_i$, where $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$. Define a linear functional on $V_i$ by

$$\tilde{f}_i(x) = \sum_{j=1}^i f_j x_j,$$

where $x = (x_1 : x_2 : \ldots : x_i : 1 : 0 : \ldots : 0) \in V_i$. Suppose $z = (z_1 : z_2 : \ldots : z_k : 1 : 0 : \ldots : 0) \in V_k$. We intend to show:

$$\sum_{j=1}^{k+1} f_j z_j = 0 \quad (\text{that is } \tilde{f}_k(z) = -f_{k+1}) \quad \text{and } z \text{ is a critical point } \tilde{f}_k \text{ restricted to } B \cdot z, \text{ the Borel orbit of } z \text{ (that is } -f_{k+1} \text{ is a critical value).}$$

But by Sard’s theorem, for generic $f$, $-f_{k+1}$ is not a critical value. Since $\overline{X}$ consists of a finite number of Borel orbits $-f_{k+1}$ is not a critical value for $\tilde{f}_i$ restricted to a Borel orbit. Hence there is no point $z \in \overline{X} \cap \overline{H_f} \setminus X \cap H_f$.

It only remains to prove the above claim. Next lemma proves the first claim, i.e. if $z \in \overline{X} \cap \overline{H_f}$ then $\sum_{j=1}^n f_j z_j = \tilde{f}_k(z) + f_{k+1} = 0$. 47
Notice that from the theorem, we can approach $z$ with a one-parameter subgroup $\alpha$ of $G$.

Let $GL(n, \mathbb{C})$ acts in the usual way on $\mathbb{C}^n$. Let $z \in \mathbb{C}P^n \setminus \mathbb{C}^n$ be a point at infinity. Suppose there is a point $x \in \mathbb{C}^n$ and a one-parameter subgroup $exp(t \cdot \alpha), \alpha \in gl(n, \mathbb{C})$, with $\alpha$ diagonalizable, such that $exp(t \cdot \alpha) \cdot x$ converges to $z$ as $t$ goes to infinity. Denote the generating vector field of $\alpha$ on $\mathbb{C}^n$ by $v_\alpha$. Let $f$ be a linear functional on $\mathbb{C}^n$ and as before $H_{f,\alpha}$ be the hyperplane $\{y \mid df(v_\alpha(y) = 0\}$, we have

**Lemma 3.4.5.** $z \in H_{f,\alpha}$ implies that $z \in \{y \mid f(y) = 0\}$. Roughly speaking: if we can approach a point $z$ at infinity with a one-parameter subgroup $\alpha$ and if as we approach $z$ the derivative of $f$ along $\alpha$ is zero, then $f$ lies on the hyperplane at infinity defined by $f$.

**Proof.** The proof is based on direct calculation of $v_\alpha$. Choose coordinates in $\mathbb{C}^n$ such that $\alpha$ becomes diagonal. So let us assume that $\alpha = (\alpha_1, \ldots, \alpha_n)$. For $y = (Y_1, \ldots, y_n)$ we have

\[
v_\alpha(y) = \frac{\partial}{\partial s} \bigg|_{s=0} exp(s \alpha) \cdot y = \frac{\partial}{\partial s} \bigg|_{s=0} (e^{\alpha_1 \cdot s} \cdot \ldots \cdot e^{\alpha_n \cdot s}) \cdot (y_1, \ldots, y_n) = (\alpha_1 y_1, \ldots, \alpha_n y_n)
\]

Now let $x = (x_1, \ldots, x_n)$ then

\[
exp(t \alpha) \cdot x = (e^{t \alpha_1} \cdot x_1, \ldots, e^{t \alpha_n} \cdot x_n) = (e^{t \alpha_1} \cdot x_1 : \ldots : e^{t \alpha_n} \cdot x_n : 1) = (e^{t(\alpha_1 - m)} \cdot x_1 : \ldots : e^{t(\alpha_n - m)} \cdot x_n : e^{-t \cdot m})
\]

where $m = \min(\alpha_1, \ldots, \alpha_n)$. Note that all $\alpha_i - m \geq 0$ and hence $e^{t(\alpha_i - m)}$ is either 1 or approaches 0 as $t \to \infty$. Also since $\lim_{t \to \infty} exp(t \alpha) \cdot x = z \notin \mathbb{C}^n$ then $\lim_{t \to \infty} e^{-t \cdot m} = 0$ so $m > 0$ and we have

\[
z_i = \begin{cases} 
0 & \text{if } \alpha_i \neq m \\
x_i & \text{if } \alpha_i = m
\end{cases}
\]

Let $f(y) = \sum_{i=1}^n f_i y_i$. Suppose $z \in \overline{H_{f,\alpha}}$, then $\sum_{i=1}^n f_i \alpha_i z_i = 0$ where $z = (z_1 : \ldots : z_n : 0)$. But $z_i = 0$ if $\alpha_i \neq m$, hence

\[
0 = \sum_{i=1}^n f_i \alpha_i z_i
\]
\[ \sum_{\{i|\alpha_i=m\}} f_i \alpha_i z_i \]
\[ = m \cdot \sum_{\{i|\alpha_i=m\}} f_i z_i \]
\[ = m \cdot \sum_{i=1}^{n} f_i z_i \]

Since \( m > 0 \), this implies that \( \sum_{i=1}^{n} f_i z_i = 0 \) as was required. \( \square \)

Next we will show that \( z \in X \cap H \setminus X \cap H_f \) implies that \( z \) is a critical point of \( \tilde{f}_k B \cdot z \). For this, we need to calculate the generating vector fields of Borel elements on \( V_i \)'s.

Let \( B \subset GL(n,\mathbb{C}) \) be the subgroup of all upper triangular matrices. Natural action of \( GL(n,\mathbb{C}) \) on \( \mathbb{C}^n \) extends to \( \mathbb{C}P^n \) by letting it act trivially on the last homogenous component. One can easily see that

**Proposition 3.4.6.** The action of \( B \) on \( \mathbb{C}P^n \) respects the cell decomposition \( \mathbb{C}P^n = V_n \cup \cdots \cup V_1 \).

Let \( \xi \in \mathfrak{b} \), the Lie algebra of \( B \) and \( \text{exp}(t \cdot \xi) = [b_{ij}(t)] \) the corresponding one-parameter subgroup. The diagonal elements of \( \text{exp}(t \cdot \xi) \) are homomorphisms \( b_{ii}(t) = e^{c_i t} \). Next simple lemma gives a formula for the generating vector field of \( \xi \) on the cells \( V_i \).

**Lemma 3.4.7.** Let \( z \in V_k = \{(x_1 : x_2 : \cdots : x_k : 1 : 0 : \cdots : 0) \in \mathbb{C}P^n \} \) and let \( v_\xi \) denote the generating vector field of \( \xi \) on \( V_k \) then

\[ v_\xi(z) = \left( \sum_{j=1}^{k} b'_{1j}(0)z_j - c_{k+1}z_1 : \sum_{j=2}^{k} b'_{2j}(0)z_j - c_{k+1}z_2 : \cdots , b'_{kk}(0)z_k - c_{k+1}z_k : 1 : 0 : \cdots : 0 \right) \]

In particular if \( k = n \), i.e. generating vector field on \( \mathbb{C}^n \)

\[ v_\xi(z) = \left( \sum_{j=1}^{n} b'_{1j}(0)z_j , \sum_{j=2}^{n} b'_{2j}(0)z_j , \cdots , b'_{nm}(0)z_k \right) \]

**Proof.** Just from the definition we have

\[ v_\xi(z) = \frac{\partial}{\partial t}_{|t=0} \text{exp}(t\xi) \cdot z. \]

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Also from the definition of the action we have

\[ \exp(t\xi) \cdot z = \left( \sum_{j=1}^{k} b_{1j}(t)z_j : \sum_{j=2}^{k} b_{2j}(t)z_j : \ldots, b_{k+1k+1}(t) : 0 : \ldots : 0 \right). \]

\[ = \left( \sum_{j=2}^{k} \frac{b_{2j}(t)z_j}{b_{k+1k+1}(t)} : \ldots, 1 : 0 : \ldots : 0 \right). \]

Differentiating with respect to \( t \) we get

\[ \frac{\partial}{\partial t} \bigg|_{t=0} \exp(t\xi) \cdot z = \left( \sum_{j=1}^{k} \partial(b'_{1j}(0) \cdot b_{k+1k+1}(0) \cdot b_{k+1k+1}(0))b_{k+1k+1}(0)^2 : \ldots, 1 : 0 : \ldots : 0 \right). \]

but \( b_{k+1k+1}(0) = 1, b_{ij}(0) = \delta_{ij} \) and \( b'_{k+1k+1}(0) = c_{k+1} \) so

\[ v_{\xi}(z) = \left( \sum_{j=1}^{k} b'_{1j}(0)z_j - c_{k+1}z_1 : \sum_{j=2}^{k} b'_{2j}(0)z_j - c_{k+1}z_2 : \ldots, b'_{kk}(0)z_k - c_{k+1}z_k : 1 : 0 : \ldots : 0 \right). \]

Having every thing in hand, we know see that \( z \) is a critical point of \( \tilde{f}_{k|B,z} \).

Let \( v_{\xi} \) be the generating vector field \( \xi \in b \) on \( \mathbb{C}^n \). From the above lemma (lemma )

\[ v_{\xi}(y) = \left( \sum_{j=1}^{n} b'_{1j}(0)y_j, \sum_{j=2}^{n} b'_{2j}(0)y_j, \ldots, b'_{nn}(0)y_k \right) \]

where \( y = (y_1, \ldots, y_n) \). Since \( \forall \xi, z \in \overline{H_{\xi,f}} = \{ y | f(v_{\xi}(y)) = 0 \} \) we have

\[ \sum_{i=1}^{n} \sum_{j=i}^{n} f_i \cdot b'_{ij}(0) \cdot z_j = 0. \]

On the other hand, since we can approach \( z \) with a one parameter subgroup from lemma it follows that \( z \in \overline{\{ y | f(y) = 0 \} } \), i.e. \( \sum_{i=1}^{n} f_i z_i = 0. \) Subtract \( c_{k+1} \cdot \sum_{i=1}^{n} f_i z_i \) from the left side of ghaf we get

\[ 0 = \sum_{i=1}^{n} \sum_{j=i}^{n} f_i(b'_{ij}(0) \cdot z_j - c_{k+1}z_i). \]
\[ \sum_{i=1}^{k+1} \sum_{j=i}^{k+1} (b'_{ij}(0) \cdot z_j - c_{k+1}z_i) \cdot z_j - c_{k+1}z_i) \cdot (b'_{k+1k+1}(0)z_{k+1} - c_{k+1}x_{k+1}) \cdot z_i \]

Since \( v_\xi(z) = 0 \) for all \( \xi \in \mathfrak{b} \) we conclude that \( z \) is a critical point of \( \tilde{f}_k|B \cdot z \).

This finishes the proof of the main theorem. \( \square \)

Finally, from theorem we then obtain

**Corollary 3.4.8 (the main theorem).** Let \( G \) acts linearly on \( \mathbb{C}^N \) such that generic orbits are spherical and closed. Let \( X \) be a generic orbit and let \( f \) be a generic linear functional on \( \mathbb{C}^N \) and \( c \) a generic complex number. Then

\[ \chi(f^{-1}(c) \cap X) = \chi(X) + (-1)^{\dim(X)+1} \cdot \deg(X). \]

**Example 3.4.1 (the main example).** Let \( G = SL_2(\mathbb{C}) \) consider its irreducible representations \( V_n = Sym^2(\mathbb{C}^n) \). This gives an embedding \( \pi : SL_2(\mathbb{C}) \hookrightarrow M(n+1, \mathbb{C}) \). As we mentioned before, \( G \times G \) acts on \( M(n+1, \mathbb{C}) \) by left-right multiplication and \( \pi(G) \) is a spherical orbit. Unfortunately, one can show that it is a generic orbit of this action only for \( n = 2 \) which correspond to natural representation of \( SL_2(\mathbb{C}) \). According to our previous calculation of the number of critical points and the degree for \( SL_2(\mathbb{C}) \) embedded in \( M(n+1, \mathbb{C}) \) in section SL2, degree is equal to \( 2n^3 \) while the number of critical points \( \mu(f, SL_2(\mathbb{C})) = 4n^3 - 6n^2 + 4n \). In case of \( n = 1 \) these two numbers are equal: \( 4 - 6 + 4 = 2 \). Which agrees with the main theorem main-theorem, since \( SL_2(\mathbb{C}) \) is a closed generic spherical orbit of the action of \( SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \) on \( M(2, \mathbb{C}) \) (it is closed simply because it is given by \( det = 1 \)).

More generally one can see that for \( G = SL(n, \mathbb{C}) \), action of \( G \times G \) on \( M(n, \mathbb{C}) \) is an action with closed generic spherical orbits and \( SL(n, \mathbb{C}) \) itself is a generic spherical orbit. We can verify this as follows: let \( a \in M(n, \mathbb{C}) \) be an invertible element. It is obvious that the \( G \times G \) orbit of \( a \) is \( \{x|det(x) = det(a)\} \) which is a hypersurface in \( M(n, \mathbb{C}) \). The \( G \times G \)-Stabilizer of \( a \) is
also \{ (g, m \cdot g \cdot m^{-1} | g \in G \}. Now let \( b \) be another invertible matrix. One can easily see that \( x \cdot \text{Stab}(a) \cdot x^{-1} = \text{Stab}(b) \) where \( x = (1, ba^{-1}) \in G \times G \). So all the orbits of invertible elements are generic orbits and of course \( SL(n, \mathbb{C}) \) is the orbit of identity.

Noting that \( \dim(SL(n, \mathbb{C})) = n^2 - 1 \) and \( \chi(SL(n, \mathbb{C})) = 0 \) (cf. theorem), the main theorem in this case then implies that

**Theorem 3.4.9.** The Euler characteristic of a generic hyperplane section of \( SL(n, \mathbb{C}) \subset M(n, \mathbb{C}) \) is equal to \(-1^{n^2} \cdot n\).

### 3.4.3 Classification of modules with spherical orbits

In Arzhantsev has classified all the representations with generic spherical orbits. We quote the main results of his paper here in this subsection.

Let \( G \) be a connected reductive algebraic group over an algebraically closed field \( K \) of zero characteristic, and \( G^s \) denote the maximal connected semisimple subgroup of \( G \).

**Definition 3.4.2.** Let \( X \) be an irreducible algebraic variety. We shall say that an action \( G : X \) is an action with generic spherical orbits if there exists an open subset \( X_0 \subset X \) such that for any \( x \in X_0 \) the orbit \( Gx \) is spherical.

In [Corollary 1]Arzhantsev1 it is shown that if a \( G \)-module \( V \), is a module with generic spherical orbits, then in fact all \( G \)-orbits are spherical.

Below we list some basic facts, without proof, about actions with spherical orbits.

1. Any trivial \( G \)-action is an action with spherical orbits.
2. Suppose that for an action \( G : X \) a stabilizer in general position exists, see [sec. 7.3]Vinberg-Popov. (This is always the case for linear actions.) Denote this subgroup by \( H \). The action \( G : X \) is an action with spherical orbits iff \( H \) is a spherical subgroup of \( G \).
3. Rosenlicht’s theorem Rosenlicht, implies that an action \( G : X \) is an action with spherical orbits iff \( K(X)^G = K(X)^B \), where \( K(X)^G \) and \( K(X)^B \) denote the subfields of rational function on \( X \) invariant under \( G \) and \( B \) respectively.
4. A \( G \)-module \( V \) is called a spherical module if it is spherical as a \( G \)-variety, i.e. if a Borel subgroup has a dense open orbit in \( V \). It is shown that any module with spherical orbits can be realized as a spherical module after an extension of the group \( G \) by a central torus.
(5) Let $G_1 : X_1$ and $G_2 : X_2$ be actions with spherical orbits. Then the action $(G_1 \times G_2) : (X_1 \times X_2)$ is an action with spherical orbits.

Because of the above fact (5), to classify the modules with spherical orbits, it suffices to consider only indecomposable modules.

**Definition 3.4.3.** A $G$-module $V$ is *indecomposable* if there exists no proper decompositions $G^s = G_1^s \times G_2^s$ and $V = V_1 \oplus V_2$ such that $(g_1, g_2)(v_1, v_2) = (g_1 v_1, g_2 v_2)$ for any $g = (g_1, g_2) \in G^s$ and any $v = (v_1, v_2) \in V$.

Yet, there is another way one can obtain modules with spherical orbits without any cost, i.e. extension of the group $G$ by a torus.

**Definition 3.4.4.** We say that a $G'$-module $V$ is obtained from a $G$-module $V$ by a torus extension if there exists a torus $T$ acting on $V$ such that $T$- and $G$-actions commute and $G' = TG$.

It is clear that any $G$-module $V$ is obtained by a torus extension from the $G^s$-module $V$.

**Lemma 3.4.10.** Suppose that $V$ is a $G$-module with spherical orbits and a $G'$-module $V$ is obtained from this module by a torus extension. Then $V$ is a $G'$-module with spherical orbits.

*Proof.* Let $H$ be the generic isotropy subgroup for the action $G : V$. By assumption, $H$ is spherical in $G$. Then any subgroup of $G'$ containing $H$ is spherical in $G'$. Hence a generic isotropy subgroup for the $G'$-module $V$ is spherical. \qed

Now we are ready to state the classification theorem for $G$-modules with spherical orbits.

**Theorem 3.4.11.** All indecomposable $G$-modules with spherical orbits are either indicated in Tables 1-3 or are obtained from the indicated modules by a torus extension.
| $G$          | weights | dim $V$ | $\mathcal{H}$       | codim |
|-------------|---------|---------|----------------------|-------|
| 0           | $\{e\}$ | 0       | 1                    | 0     |
| 1           | $SL(n)$ | $\phi_1$ | $n$                 | $A_{n-2} + R_{n-1}$ | 0     |
| 2           | $\Lambda^2 SL(2n)$ | $\phi_2$ | $2n^2 - n$ | $C_n$       | 1     |
| 3           | $\Lambda^2 SL(2n + 1)$ | $\phi_2$ | $2n^2 + n$ | $C_n + R_{2n}$ | 0     |
| 4           | $S^2 SL(2n)$ | $2\phi_1$ | $2n^2 + n$ | $D_n$       | 1     |
| 5           | $S^2 SL(2n + 1)$ | $2\phi_1$ | $2n^2 + 3n + 1$ | $B_n$       | 1     |
| 6           | $SO(2n)$ | $\phi_1$ | $2n$               | $B_{n-1}$ | 1     |
| 7           | $SO(2n + 1)$ | $\phi_1$ | $2n + 1$           | $D_n$       | 1     |
| 8           | $Spin(7)$ | $\phi_3$ | 8                   | $G_2$       | 1     |
| 9           | $Spin(9)$ | $\phi_4$ | 16                  | $B_3$       | 1     |
| 10          | $Spin(10)$ | $\phi_4$ | 16                  | $B_3 + R_8$ | 0     |
| 11          | $Sp(2n)$ | $\phi_1$ | $2n$               | $C_{n-1} + R_{2n-1}$ | 0     |
| 12          | $G_2$ | $\phi_1$ | 7                   | $A_2$       | 1     |
| 13          | $E_6$ | $\phi_1$ | 27                  | $F_4$       | 1     |
|   | G                  | weights                       | dim $V$ | $\mathcal{H}$                             | codim |
|---|-------------------|-------------------------------|---------|-------------------------------------------|-------|
| 14| $SL(2) \times K^*$ | $\phi_1 \otimes \epsilon + \phi_1 \otimes \epsilon^{-1}$ | 4       | $t_1$                                     | 1     |
| 15| $SL(n) \times K^*$, $n > 2$ | $\phi_1 \otimes \epsilon^a + \phi_1 \otimes \epsilon^b$, $a \neq b$ | $2n$    | $A_{n-3} + t_1 + R_{2(n-2)}$ | 0     |
| 16| $SL(n)$, $n > 2$ | $\phi_1 + \phi_{n-1}$ | $2n$    | $A_{n-2}$ | 1     |
| 17| $SL(2n+1)$ | $\phi_1 + \phi_2$ | $(2n+1)(n+1)$ | $C_n$ | 1     |
| 18| $SL(2n+1) \times K^*$ | $\phi_1 \otimes \epsilon + \phi_{2n-1} \otimes \epsilon^b$, $a \neq nb$ | $(2n+1)(n+1)$ | $C_{n-1} + t_1 + R_{2(n-2)}$ | 0     |
| 19| $SL(2n)$ | $\phi_1 + \phi_2$ | $n(2n+1)$ | $C_{n-1} + R_{2n-1}$ | 1     |
| 20| $SO(8)$ | $\phi_1 + \phi_3$ | 16 | $G_2$ | 2     |
| 21| $Sp(2n) \times K^*$ | $\phi_1 \otimes \epsilon + \phi_1 \otimes \epsilon^{-1}$ | $4n$ | $C_{n-1} + t_1$ | 1     |
| 22| $SL(n) \times SL(m)$, $n > m$ | $\phi_1 \otimes \phi_1$ | $nm$ | $A_{n-m-1} + A_{m-1} + R_{nm-m^2}$ | 0     |
| 23| $SL(n) \times SL(n)$ | $\phi_1 \otimes \phi_1$ | $n^2$ | $A_{n-1}$ | 1     |
| 24| $SL(2) \times Sp(2n)$ | $\phi_1 \otimes \phi_1$ | $4n$ | $C_{n-1} + A_1$ | 1     |
| 25| $SL(3) \times Sp(2n) \times K^*$, $n > 1$ | $\phi_1 \otimes \phi_1 \otimes \epsilon$ | $6n$ | $C_{n-2} + A_1 + t_1 + R_{2n-1}$ | 0     |
| 26| $SL(4) \times Sp(4)$ | $\phi_1 \otimes \phi_1$ | 16 | $C_2$ | 1     |
| 27| $SL(n) \times Sp(4)$, $n > 4$ | $\phi_1 \otimes \phi_1$ | $4n$ | $A_{n-5} + C_2 + R_{4(n-4)}$ | 0     |
### Table 3

| $G$                                                                 | weights                                                                 | dim $V$ | codim |
|---------------------------------------------------------------------|-------------------------------------------------------------------------|---------|-------|
| 28 $SL(n) \times SL(n) \times K^*$                                  | $\phi_1 \otimes \epsilon + \phi_1 \otimes \psi_1$                      | $n(n+1)$| 1     |
|                                                                     | $\phi_1 \otimes \epsilon + \phi_{n-1} \otimes \psi_{n-1}$             |         |       |
| 29 $SL(n+1) \times SL(n) \times K^*$                               | $\phi_1 \otimes \epsilon^n + \phi_1 \otimes \psi_1 \otimes \epsilon^{-1}$ | $(n+1)^2$| 1     |
| 30 $SL(n+1) \times SL(n) \times K^* \times K^*$, $n > 1$             | $\phi_1 \otimes \epsilon_1 + \phi_n \otimes \psi_{n-1} \otimes \epsilon_2$ | $(n+1)^2$| 0     |
| 31 $SL(n) \times SL(m) \times K^*$, $n > m + 1$                     | $\phi_1 \otimes \epsilon^a + \phi_1 \otimes \psi_1 \otimes \epsilon^b$, $a \neq b$ | $n(m+1)$| 0     |
| 32 $SL(n) \times SL(m) \times K^*$, $n > m + 1 > 2$                 | $\phi_1 \otimes \epsilon^a + \phi_{n-1} \otimes \psi_{m-1} \otimes \epsilon^b$, $a \neq -b$ | $n(m+1)$| 0     |
| 33 $SL(n) \times SL(m) \times K^*$, $n < m$                        | $\phi_1 \otimes \epsilon^a + \phi_1 \otimes \psi_1 \otimes \epsilon^b$, $a \neq 0$ | $n(m+1)$| 0     |
|                                                                     | $\phi_1 \otimes \epsilon^a + \phi_{n-1} \otimes \psi_{m-1} \otimes \epsilon^b$, $a \neq 0$ |         |       |
| 34 $SL(n) \times SL(2) \times SL(m)$, $n > 2$, $m > 2$             | $\phi_1 \otimes \psi_1 + \psi_1 \otimes \tau_1$                       | $2(n+m)$| 0     |
| 35 $SL(n) \times SL(2) \times Sp(2m)$, $n > 2$, $m \geq 1$         | $\phi_1 \otimes \psi_1 + \psi_1 \otimes \tau_1$                       | $2(n+2m)$| 1     |
| 36 $Sp(2n) \times SL(2) \times Sp(2m)$, $n, m \geq 1$              | $\phi_1 \otimes \psi_1 + \psi_1 \otimes \tau_1$                       | $4(m+n)$| 2     |
| 37 $SL(2) \times Sp(2n) \times K^*$                                | $\phi_1 \otimes \epsilon + \phi_1 \otimes \psi_1$                    | $2(2n+1)$| 1     |

**Comments to the Tables.** The column ”$G$” contains a reductive group $G$. In Table 1 the linear group $\Lambda^2 SL(n)$ is the image of $SL(n)$ under the action in the second exterior power of the tautological representation, and $S^2 SL(n)$ is the same thing with respect to the second symmetric power.

In the column ”weights” the highest weights of the $G$-module are indicated.

For the group $G_1 \times G_2$ the weight $\phi \otimes \psi$ corresponds to the tensor product of simple $G_1$- and $G_2$-modules with highest weights $\phi$ and $\psi$ respectively. The symbol + denotes a direct sum of modules. If $G^s$ is the product of several simple groups, then their fundamental weights are denoted successively by
letters $\phi_i$, $\psi_i$ and $\tau_i$. The fundamental weight of the central torus is denoted by $\epsilon$ (for a two-dimensional torus – by $\epsilon_1$ and $\epsilon_2$).

In the column ”dim $V$” the dimension of the module is shown.

In Tables 1 and 2 the column ”$H$” contains the type of the tangent algebra $H$ of the generic isotropy subgroup $H$ for our module. Here $t_1$ is the tangent algebra of the one-dimensional central torus in $H$, and $R_k$ is the tangent algebra of the $k$-dimensional unipotent radical of $H$. The information of this column is taken from Elashvili’s tables el1, el2.

In the last column the codimension of a generic $G$-orbit in $V$ is shown.

### 3.4.4 Closer look at the actions in the list

There is a nice criterion by V. Popov, which determines when the generic orbits of a linear action are closed, namely

**Theorem 3.4.12 (cf. Popov).** Let $H$ be a stabilizer in general position for a $G$-module $V$. Then generic orbits are closed in $V$ if $H$ is a reductive subgroup of $G$.

Hence we can see that in the list of examples in the previous section, the actions numbered 0, 2, 4, 5, 6, 7, 12, 14, 16, 17, 19, 20, 21, 23, 24, 26, 28, 29, 35, 36, 37 have closed generic orbits. These are in fact the actions such that codimension of a generic orbit is $\geq 1$.

Now, we examine these ones to see how the generic orbits look like topologically, in each case.

**Notation 3.4.1.** We denote by $\langle \cdot, \cdot \rangle$ the standard inner product on $\mathbb{C}^n$, i.e. for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$. We also denote by $\omega$ the standard symplectic form on on $\mathbb{C}^{2n}$, that is for $x = (x_1, \ldots, x_{2n})$ and $y = (y_1, \ldots, y_{2n})$, $\omega(x, y) = \sum_{i=1}^{n} x_i y_i - \sum_{i=n+1}^{2n} x_i y_i$.

**Definition 3.4.5.** Let $V$ be a $G$-variety. A *polynomial invariant* for the action of $G$ is a polynomial $P$ on $V$ such that

$$P(x) = P(g \cdot x), \forall x \in V, \forall g \in G.$$
invariant $P$ in each case and the generic orbits are given by $\{P(x) = c\}$ for some constant $c \in \mathbb{C}$.

0. $G = \{e\}$ and $V = \mathbb{C}$. Every point in $V$ is an orbit and $P(x) = x$.

2. $G = SL(2n, \mathbb{C})$ and $V = \Lambda^2(\mathbb{C}^{2n}) = \text{vector space of skew-symmetric matrices over } \mathbb{C}$. $SL(2n, \mathbb{C})$ acts on $V$ by left-right multiplication.

$$g \cdot m = g^t \cdot m \cdot g.$$ Polynomial invariant is $det$.

4. and 5. $G = SL(n, \mathbb{C})$ and $V = Sym^2(\mathbb{C}^n) = \text{vector space of symmetric matrices over } \mathbb{C}$. $SL(n, \mathbb{C})$ acts on $V$ by left-right multiplication. Again the polynomial invariant is $det$.

6. and 7. $G = SO(n, \mathbb{C})$ and $V = \mathbb{C}^n$. $SO(n, \mathbb{C})$ acts on $V$ via natural representation. $P(x) = \langle x, x \rangle$ is the polynomial invariant.

12. $G = G_2$ and $V = \mathbb{C}^7$. This representation is the highest weight representation of $G_2$ corresponding to the weight $\omega_1$ (first fundamental weight). It is the smallest representation of $G_2$ and is called the standard representation. One can show that the action of $G_2$ on $V$ preserves a non-degenerate quadratic form (cf. Fulton-Harris p. 355).

13. $G = E_6$ and $G$ acts on $V = \mathbb{C}^2 \otimes \mathbb{C}^2$. This representation is the highest weight representation of $E_6$ corresponding to the weight $\omega_1$ (first fundamental weight). It can be shown that the invariant polynomial in this case is of degree 3 (cf. ). This is the only example in the list that its invariant has degree 3.

14. $G = SL(2, \mathbb{C}) \times \mathbb{C}^*$ and $V = \mathbb{C}^2 \otimes \mathbb{C} \oplus \mathbb{C}^2 \otimes \mathbb{C} = \mathbb{C}^2 \oplus \mathbb{C}^2$. $G$ acts by

$$(g, k) \cdot (v, w) = (k \cdot g \cdot v, k^{-1} \cdot g \cdot w).$$

where $v, w = in \mathbb{C}^2$. Invariant polynomial is $P(v, w) = det(v, w)$ ( $v$ and $w$ thought of as column vectors).

16. $G = SL(n, \mathbb{C}), n > 2$ and $G$ acts on $V = \mathbb{C}^n \oplus (\mathbb{C}^n)^*$ by

$$g \cdot (v, w) = (g \cdot v, (g^{-1})^t \cdot w).$$

Invariant polynomial is $P(v, w) = \langle v, w \rangle$.

17. and 19. $G = SL(n, \mathbb{C})$ and $V = \mathbb{C}^n \oplus \Lambda^2(\mathbb{C}^n)$. $G$ acts by

$$g \cdot (v, m) = (g \cdot v, g \cdot m).$$

Invariant polynomial is $P(v, m) = det(m)$.

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20. \( G = SO(8, \mathbb{C}) \) and \( G \) acts on \( V = \mathbb{C}^8 \oplus (\mathbb{C}^8)^\oplus \) by
\[
g \cdot (v, w) = (g \cdot v, (g^{-1})^t \cdot w).
\]

There are two invariant polynomials \( P(v, w) = \langle v, v \rangle \) and \( Q(v, w) = \langle w, w \rangle \).

21. \( G = Sp(2n, \mathbb{C}) \times \mathbb{C}^* \) and \( G \) acts on \( V = \mathbb{C}^{2n} \oplus \mathbb{C}^{2n} \) by
\[
(g, k) \cdot (v, w) = (k \cdot g \cdot v, k^{-1} \cdot g \cdot w)
\].

Invariant polynomial is \( P(v, w) = \omega(v, w) \).

23. \( G = SL(n, \mathbb{C}) \times SL(n, \mathbb{C}) \) and \( G \) acts on \( V = \) vector space of all \( n \times n \) matrices by
\[
(g_1, g_2) \cdot m = g_1 \cdot m \cdot g_2^t.
\]

Invariant polynomial is \( P(m) = \det(m) \). This is the example considered in section , Example .

24. \( G = SL(2, \mathbb{C}) \times Sp(2n, \mathbb{C}) \) and \( G \) acts on \( V = \mathbb{C}^2 \otimes \mathbb{C}^{2n} = \) vector space of all \( 2 \times 2n \) matrices by
\[
(g_1, g_2) \cdot m = g_1 \cdot m \cdot g_2^t.
\]

If \( w_1 \) and \( w_2 \) are the rows of \( m \) then the invariant polynomial is \( P(m) = \omega(w_1, w_2) \).

26. \( G = SL(4, \mathbb{C}) \times Sp(4, \mathbb{C}) \) and \( G \) acts on \( V = \mathbb{C}^4 \otimes \mathbb{C}^4 = \) vector space of all \( 4 \times 4 \) matrices, by
\[
g \cdot (v, m) = (g \cdot v, g \cdot m).
\]

Invariant polynomial is \( P(m) = \det(m) \).

28. \( G = SL(n, \mathbb{C}) \times SL(n, \mathbb{C}) \times \mathbb{C}^* \) and \( G \) acts on \( V = \mathbb{C}^n \oplus (\mathbb{C}^n)^\otimes 2 \) by
\[
(g_1, g_2, k) \cdot (v, m) = (k \cdot g_1 \cdot v, g_1 \cdot m \cdot g_2^t).
\]

Invariant polynomial is \( P(v, m) = \det(m) \).

29. \( G = SL(n+1, \mathbb{C}) \times SL(n, \mathbb{C}) \times \mathbb{C}^* \) and \( G \) acts on \( V = \mathbb{C}^{n+1} \oplus \mathbb{C}^{n+1} \otimes \mathbb{C}^n \) by
\[
(g_1, g_2, k) \cdot (v, m) = (k^n \cdot g_1 \cdot v, g_1 \cdot m \cdot g_2^t).
\]

Invariant polynomial is \( P(v, m) = ?? \).

35. \( G = SL(n, \mathbb{C}) \times SL(2, \mathbb{C}) \times Sp(2m, \mathbb{C}), n > 2, m \geq 1 \) and \( G \) acts on \( V = \mathbb{C}^n \otimes \mathbb{C}^2 \oplus \mathbb{C}^2 \otimes \mathbb{C}^{2m} \) by
\[
(g_1, g_2, g_3) \cdot (m_1, m_2) = (g_1 \cdot m_1 \cdot g_2^t, g_2 \cdot m_2 \cdot g_3^t).
\]
Invariant polynomial is $P(m_1, m_2) = \omega(w_1, w_2)$, where $w_1$ and $w_2$ are the rows of $m_2$.

36. $G = Sp(2n, \mathbb{C}) \times SL(2, \mathbb{C}) \times Sp(2m, \mathbb{C}), n, m \geq 1$ and $G$ acts on $V = \mathbb{C}^{2n} \otimes \mathbb{C}^2 \oplus \mathbb{C}^2 \otimes \mathbb{C}^{2m}$ by

$$(g_1, g_2, g_3) \cdot (m_1, m_2) = (g_1 \cdot m_1 \cdot g_2^t, g_2 \cdot m_2 \cdot g_3^t).$$

There are two invariant polynomials $P(m_1, m_2) = \omega(v_1, v_2)$ and $Q(m_1, m_2) = \omega(w_1, w_2)$, where $v_1, v_2, w_1$ and $w_2$ are the rows of $m_1$ and $m_2$ respectively.

37. $G = SL(2, \mathbb{C}) \times Sp(2n, \mathbb{C}) \times \mathbb{C}^*$ and $G$ acts on $V = \mathbb{C}^2 \oplus \mathbb{C}^2 \otimes \mathbb{C}^{2n}$ by

$$(g_1, g_2, k) \cdot (v, m) = (k \cdot g_1 \cdot v, g_1 \cdot m \cdot g_2^t).$$

Invariant polynomial is $P(v, m) = \omega(w_1, w_2)$, where $w_1$ and $w_2$ are the rows of $m$.

3.4.5 Euler characteristic of sections and the number of critical points of generic functionals for a hypersurface defined by a non-degenerate quadratic form.

Let $Q(x_1, \ldots, x_n)$ be a non-degenerate quadratic form in $n$ variables over complex numbers. From linear algebra we know that with a linear change of basis we can assume $Q(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2$. Let $X = \{x \in \mathbb{C}^n | Q(x) = c\}$ for some generic constant number $c$. Obviously $\text{deg}(X) = 2$ since a plane of complementary dimension to $X$ is a line, and generically a line intersects $X$ in two points.

As usual let $f$ be a generic linear functional on $\mathbb{C}^n$ and $(f_1, \ldots, f_n)$ the coordinates of $f$ in the dual basis. By Lagrange’s multipliers, $x = (x_1, \ldots, x_n) \in X$ is a critical point for $f|_X$ iff $Q(x) = c$ and there exists scalar $\lambda$ such that $\nabla f(x) = \lambda \cdot \nabla Q(x)$. Notice that $\nabla f = (f_1, \ldots, f_n)$ and $\nabla Q(x) = (2x_1, \ldots, 2x_n)$. Hence $x$ is a critical point iff

$$(x_1, \ldots, x_n) = (f_1/2\lambda, \ldots, f_n/2\lambda)$$

where $\lambda$ is chosen such that $Q(f_1/2\lambda, \ldots, f_n/2\lambda) = c$. From the latter equation one obtains $\lambda = \pm (Q(f_1, \ldots, f_n)/c)^{1/2}$. If $f$ is generic there are two solutions for $\lambda$ and accordingly two solutions for $x$, that is there are two critical points for $f|_X$.
Next we verify the formula for the Euler characteristic of sections of \( X \), it is well-known that

**Proposition 3.4.13.** Let \( Q(x) \) be a non-degenerate quadratic form in \( n \) variables over \( \mathbb{C} \). For generic \( c \in \mathbb{C} \) the hypersurface \( Q^{-1}(c) \) has the homotopy type of a sphere of real dimension \( n - 1 \).

Let \( f^{-1}(d) \) be a generic level set of \( f \), that is a hyperplane in \( \mathbb{C}^n \). We are interested in the topology of the intersection of \( X \) and this hyperplane, i.e.

\[
X \cap f^{-1}(d) = \{(x_1, \ldots, x_n)|Q(x_1, \ldots, x_n) = c, f_1x_1 + \cdots + f_nx_n = d\}.
\]

Without loss of generality suppose \( f_n \neq 0 \), then \((d - f_1x_1 - \cdots - f_{n-1}x_{n-1})/f_n = x_n \). The set \( X \cap f^{-1}(d) \) is then homeomorphic to

\[
S = \{(x_1, \ldots, x_{n-1})|Q(x_1, \ldots, x_{n-1}, (d - f_1x_1 - \cdots - f_{n-1}x_{n-1})/f_n) = c\}.
\]

The equation defining \( S \) is a non-degenerate quadratic form in \( x_1, \ldots, x_{n-1} \) and hence \( S \) has the homotopy type of sphere of real dimension \( n - 2 \). From this, one can easily verify the formula

\[
\chi(X \cap f^{-1}(d)) = \chi(X) + (-1)^{\dim(X)+1} \cdot \deg(X).
\]

### 3.4.6 Euler characteristic of sections and the number of critical points of generic functionals for the subvariety \( \{det = \text{constant}\} \) in the space of matrices.

There is a standard non-degenerate bilinear form on the vector space \( M(n, \mathbb{C}) \) of square matrices namely, \((A, B) = tr(A \cdot B)\). Let \( f \) be a functional on \( M(n, \mathbb{C}) \) then there is a matrix \( F \) such that \( f(A) = (F, A), \forall A \in M(n, \mathbb{C}) \).

Take a number \( c \) and consider the variety \( X = \{A \in M(n, \mathbb{C})|det(A) = c\} \). We want to see how many critical points \( f|_X \) have. As usual one uses Lagrange’s multipliers: \( M \) is a critical point for iff \( det(M) = c \) and there exists scalar \( \lambda \) such that \( \nabla f(M) = \lambda \cdot \nabla det(M) \), where gradient is taken with respect to the bilinear form \((\cdot, \cdot)\). Let us compute these gradients: since \( f \) is linear and \( f(A) = (F, A) \), we have \( \nabla f \) is constantly equal to \( F \). As
for $\nabla det$ we know that derivative of determinant at identity is trace, i.e. $d(det)_I(V) = tr(V)$ where $I$ is the identity matrix. So

$$d(det)_M(V) = \frac{\partial}{\partial t}_{|t=0} det(M + tV)$$
$$= det(M) \cdot \frac{\partial}{\partial t}_{|t=0} det(I + tM^{-1}V)$$
$$= det(M) \cdot d(det)_I(M^{-1}V)$$
$$= det(M) \cdot tr(M^{-1}V)$$
$$= tr(det(M)M^{-1} \cdot V).$$

Hence $\nabla det(M) = det(M)M^{-1}$. So $M$ is a critical point iff $det(M) = c$ and there exists $\lambda$ such that

$$F = \lambda \cdot det(M) \cdot M^{-1}.$$ 

Taking determinant of both sides and solving for $\lambda$ we obtain

$$det(F)/c^{n-1} = \lambda^n.$$ 

and hence if $F$ is generic, there are $n$ solutions for $\lambda$ and accordingly $n$ solutions for $M$, that is $f|_X$ has $n$ critical points.

As determinant is a degree $n$ polynomial, degree of $X$ as a subvariety is $n$ so we see that in this case the number of critical points and degree are the same.

### 3.4.7 Example of $E_6$ acting on $\mathbb{C}^{27}$

As one sees in the list of actions with spherical orbits, the 27 dimensional standard representation of $E_6$ is a module with spherical orbits. This is the only example in the list that the invariant polynomial is of degree 3 (cubic).

Let us briefly explain how one can construct this representation and the invariant cubic polynomial.

Let $\mathbb{O}$ be the 8 dimensional algebra of Cayley numbers over $\mathbb{R}$. One constructs a Jordan algebra $J$ of $3 \times 3$ Hermitian matrices over $\mathbb{O}$. This is defined as the set of matrices of the form

$$x = \begin{bmatrix} \alpha & a & b \\ \bar{a} & \beta & c \\ \bar{b} & \bar{c} & \gamma \end{bmatrix}.$$ 

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with $\alpha, \beta$ and $\gamma \in \mathbb{R}$ and $a, b$ and $c$ in $\mathbb{O}$. The product $o$ in $\mathbb{J}$ is given by

$$xoy = \frac{1}{2}(xy + yx).$$

where the product in the right hand side are usual matrix multiplication. This algebra is commutative but not associative and satisfies the identity $(xox)oy = (xox)o(yox)$. One can define $tr$ and $det$ functions on $\mathbb{J}$ analogous to $tr$ and $det$ of matrices over fields. They are defined by

$$tr(x) = \alpha + \beta + \gamma.$$

$$det(x) = \alpha\beta\gamma + tr(a(\bar{c}b)) - \alpha N(c) - \beta N(b) - \gamma N(a).$$

where $N$ denotes norm of a octonion which is sum of squares of its coordinates.

Note that $tr(x)$ and $det(x)$in $\mathbb{R}$. It is not true in general that $det(xoy) = det(x) \cdot det(y)$, but one can prove that this is the case if $x$ and $y$ generate an associative subalgebra of $\mathbb{J}$. As one can readily verify, $\mathbb{J}$ is a $27 = 3 \cdot 8 + 3$ dimensional vector space over $\mathbb{R}$ and det is a homogenous cubic polynomial in 27 variables.

Let us consider $\mathbb{J}_C$, the complexification of $\mathbb{J}$. It is a 27 dimensional vector space over $\mathbb{C}$. It is well-known that

**Theorem 3.4.14.** 1. The exceptional complex algebraic group $F_4$ is the group of automorphisms of the vector space $\mathbb{J}_C$ which preserve the scalar product $(x, y) = tr(xoy)$ and the scalar triple product $(x, y, z) = tr((xoy)oz)$.

2. The exceptional complex algebraic group $E_6$ is the group of automorphisms of the vector space $\mathbb{J}_C$ which preserve $det$.

This action of $E_6$ on $\mathbb{J}_C$ is called the standard representation of $E_6$. As one can see in the list, it is an action with spherical orbits. One can show that in this representation the stabilizer of a generic point is isomorphic to $F_4$.

Our theorem on Euler characteristic of sections in the case of orbits of this representation then becomes

**Theorem 3.4.15.** Let $X$ be a generic orbit in the standard representation of $E_6$, then $\chi(X) = 0$ and

$$\chi(\text{a generic hyperplane section of } X) = -3.$$
Proof. Since the stabilizer of a generic orbit is $F_4$, $X$ as a homogeneous space is $E_6/F_4$. \( \text{rank}(E_6) = 6 \) and \( \text{rank}(F_4) = 4 \) so \( \chi(E_6/F_4) = 0 \) (cf. Theorem ). Since \( \text{det} \) is constant along the orbits and \( \text{det}^{-1}(c) \) is connected, we see that $X$ is equal to \( \text{det}^{-1}(c) \) for some \( c \in \mathbb{C} \) and, as \( \text{det} \) is a cubic homogeneous polynomial, \( \deg(X) = 3 \). Now from our main theorem on Euler characteristic of section we get

\[
\chi(\text{a generic hyperplane section of } X) = 0 + (-1)^{26+1} \cdot 3 = -3.
\]

\( \square \)
Chapter 4

Chern classes, Euler characteristic and the number of critical points

In this section we will mention some basic facts about Chern classes and then explain formulae for Euler characteristic of hypersurfaces and the number of critical points of functions in terms of intersection numbers of Chern classes.

4.1 Basic Facts

Let $M$ be a real differentiable manifold and $V$ a complex vector bundle of rank $n$ over $M$. There is a natural way of associating a sequence of characteristic cohomology classes to $V$, namely Chern classes of $V$, such that $c_i(V) \in H^i(M, \mathbb{Z}), i = 0, \ldots, n$ ($c_0(V)$ is defined to be 1) and the following are satisfied

1. If $V$ is a trivial bundle then $c_i(V) = 0$ for $i = 1, \ldots, n$.
2. If $f : M \to N$ is a smooth map between manifolds and $E$ a complex vector bundle over $N$ then

$$c_i(f^*E) = f^*(c_i(E)).$$

where $f^*E$ is the pull-back bundle over $M$ and $f^*$ on the right hand side is the induced map on the cohomology.

Total Chern class of $V$ denoted $c(V)$ is the sum $1 + c_1(V) + \cdots + c_k(V)$ of all the Chern classes.
3. If $E$ and $F$ are complex vector bundles over $M$ then

$$c(E \oplus F) = c(E) \cdot c(F).$$

This is known as the *Whitney product formula*.

There are many different ways to construct Chern classes. For construction of Chern classes one can refer to Griffith-Harris p., Bott-Tu p. or Hirzebruch p.

Now, let $M$ be a complex manifold. Chern classes of the tangent bundle $TM$ are simply called Chern classes of $M$. The last Chern class of $M$ is called the *Euler class*. It has the property that its integral is equal to $2\pi$ times that Euler characteristic of $M$.

Now, let $M$ be a compact complex manifold. One knows that there is a 1-1 correspondence between divisor classes and the line bundles on $M$. If $L$ is a line bundle and $\sigma$ a section of $L$, then the divisor class corresponding to $L$ is given by:

$$D = \sigma \cap M.$$

**Proposition 4.1.1 (cf Griffith-Harris).** Let $M$ be a compact complex manifold and $L$ a line bundle. The divisor class $D$ corresponding to $M$ is the Poincare dual to the first Chern class $c_1(L)$.

### 4.2 Chern classes of complete intersections of hypersurfaces

Let $D$ be a divisor on $M$ and $L$ its corresponding line bundle. We want to represent the Chern classes of $D$ in terms of intersection numbers of Chern classes of $L$ and $TM$. In particular we will get a formula for the Euler characteristic of $D$. The following proposition tells us how one can relate tangent bundle of $D$, tangent bundle of $M$ and line bundle $M$.

**Proposition 4.2.1 (cf Hirzebruch).** Let $N$ denote the normal bundle to $D$. Then we have $N_D = L_D$ and hence $TM_D = TD \oplus L|_D$. From the Whitney product formula we then obtain

$$c(TM_D) = c(TD) \cdot c(L|_D).$$
Let us denote the Poincare duality map by $P : H^*(M, \mathbb{Z}) \to H_*(M, \mathbb{Z})$. Note that $c(TM_D)$ is simply $c(TM) \cdot P(D)$, where the dot denotes the cup product of cohomology classes. We also know that $c(L) = 1 + c_1(L) = 1 + P(D)$. Hence

$$c(TM) \cdot P(D) = c(TD) \cdot (1 + P(D)).$$

Solving the above equation for $c(TD)$ we obtain

**Proposition 4.2.2.** Let $M$ be a compact complex manifold of dimension $n$ and $D$ divisor. Then the Chern classes of $TD$, the tangent bundle of $D$, can be computed in terms of cup products of Chern classes of $M$ and $P(D)$ by

$$c(TD) = c(TM) \cdot P(D) \cdot (1 + P(D))^{-1},$$

where we interpret $(1 + P(D))^{-1}$ as the Taylor series $1 - P(D) + P(D)^2 - \cdots$ and terms of degree higher than $n$, the dimension of the manifold are zero. Equating degree $n$ terms in the above we obtain the formula for the Euler class

$$e(TD) = c_{n-1}(TD) = \sum_{i=0}^{n-1} (-1)^{n-i+1} \cdot c_i(M) \cdot P(D)^{n-i}.$$

One can generalize the above formula to intersections of several hypersurfaces. We state the proposition without proof. The proof follows the same lines as the previous Proposition.

**Proposition 4.2.3.** Let $D_1, \ldots, D_k$ be $k$ transversally intersecting divisors on $M$ and $L_1, \ldots, L_k$ their corresponding line bundles. Let $Y = D_1 \cap \ldots \cap D_k$. One has

$$TM_Y = L_1 \oplus \cdots \oplus L_k \oplus TY.$$

and hence

$$c(Y) = c(M) \cdot \prod_{i=1}^{k} P(D_i)P((1 + D_i)^{-1}).$$

For example, from the above Proposition we can get a formula for Euler characteristic of transversal intersection of two divisors, namely

$$e(D_1 \cap D_2) = \text{sum of the degree } n \text{ terms in } (P(D_1) \cdot P(D_1) \cdot (1 + P(D_1)) \cdot (1 + P(D_2)) \cdot c(M)

= \sum_{2 \leq i+j \leq n} (-1)^{i+j} D_1^i \cdot D_2^j \cdot c_{n-i-j}(M)$$
4.3 Euler characteristic of affine sections and the number of critical points

Now let \( L \) be an ample line bundle on \( M \).

Let \( M \subset \mathbb{C}P^N \) be a smooth projective subvariety of dimension \( n \). Let \( S \) be the God-given line bundle \( S \) on \( \mathbb{C}P^N \), namely universal subbundle. The fibre above each \( x \in \mathbb{C}P^N \) is simply the line through origin representing that point in the projective space. Restriction of \( S \) to \( M \) gives a line bundle \( L \) on \( M \). Each projective hyperplane in \( \mathbb{C}P^N \) defines a homology class and one can show that it is in fact the dual of the first Chern class of \( S \) and hence hyperplanes represent the divisor class corresponding to \( S \). Consequently intersection of \( M \) and a hyperplane in \( \mathbb{C}P^N \) represents the divisor class corresponding to \( L \). Let us denote the divisor class of hyperplane sections of \( M \) by \( D \). Just from the definition of degree of a subvariety we have

\[
\text{deg}(M) = D^n.
\]

where the power notation in the left hand side refers to the intersection product of homology classes.

Considering the dual of the formula for the Euler class we get

\[
\chi(D) = (-1)^{n+1}\text{deg}(M) + \sum_{i=1}^{n-1} (-1)^{n-i+1} \cdot c_i(M) \cdot P(D)^{n-i}.
\]

Now let \( D' = D \cap \mathbb{C}^N \) be the affine part of \( D \). We can also write down a formula for Euler characteristic of \( D' \). Denote by \( H \) the projective hyperplane at infinity. We then have \( D = D' \cup (D \cap H) \). By additivity of Euler characteristic we then obtain

\[
\chi(D') = \chi(D) - \chi(D \cap H)
\]

Substituting the formula for the Euler characteristic of the intersection of two divisors we get

\[
\chi(D') = \chi(D) - \sum_{2 \leq i+j \leq n} (-1)^{i+j} D^i \cdot H^j \cdot c_{n-i-j}(M).
\]

Note that \( H \) is in the same cohomology class as \( D \) thus

\[
\chi(D') = \chi(D) - \sum_{2 \leq k \leq n} (-1)^k D^k \cdot c_{n-k}(M).
\]
Finally substituting the formula for $\chi(D)$ we obtain

**Proposition 4.3.1.** Let $D$ be a divisor on a projective subvariety $M \subset \mathbb{C}P^N$. Let $D' = D \cap \mathbb{C}^N$ be the affine part of $D$. Then

$$
\chi(D') = (-1)^{n+1} \deg(M) + \sum_{i=1}^{n-1} (-1)^{n-i+1} \cdot c_i(M) \cdot D^{n-i} - \sum_{2 \leq k \leq n} (-1)^k D^k \cdot c_{n-k}(M).
$$

As A.G. Khovanskii has shown in Askold, using the above formula for Euler characteristic of affine hyperplane sections and the knowledge of Chern classes of projective toric varieties, one can give another proof of Bernstein’s theorem. In fact, one shows that in Euler-Chern and when $M$ is a projective toric variety, all the terms except $\deg(M)$ cancels out and one is left with

$$
\chi(D') = (-1)^{n+1} \deg(M).
$$

On the other hand, from Theorem Euler-char-formula we have

$$
\chi(D') = \chi(M \cap \mathbb{C}^N) + (-1)^{n+1} \cdot \mu(f, M \cap \mathbb{C}^N),
$$

where $\mu(f, M \cap \mathbb{C}^N)$ is the number of critical points of a generic linear functional $f$ on affine part of $M$.

Note that $\chi(M \cap \mathbb{C}^N) = \chi(M) - \chi(M \cap H)$ and $\chi(M \cap H) = \chi(D)$. Hence

$$
\chi(D') = \chi(M) - \chi(D) + (-1)^{n+1} \cdot \mu(f, M \cap \mathbb{C}^N).
$$

Now comparing Euler-critical and Euler-Chern we can get a formula for the number of critical points in terms of degree of $M$ and intersection numbers of $D$ and the Chern classes of $M$.

**Proposition 4.3.2.** Let $M \subset \mathbb{C}P^N$ be a smooth projective subvariety, and $f$ a generic linear functional on $\mathbb{C}^N$. Then $\mu(f, M \cap \mathbb{C}^N)$, the number of critical points of $f$ restricted to $M \cap \mathbb{C}^N$, can be obtained from

$$
\mu(f, M \cap \mathbb{C}^N) = (-1)^{n+1}(\chi(M) - 2\chi(D) + \chi(D^2))
$$

$$
= (-1)^{n+1}(c_n(X) - 2 \cdot \sum_{i=0}^{n-1} (-1)^{n-i+1} \cdot c_i(M) \cdot D^{n-i}) + \sum_{2 \leq k \leq n} (-1)^k D^k \cdot c_{n-k}(M)
$$