On the trace-free Einstein equations as a viable alternative to general relativity

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Abstract

The quantum field theoretical prediction for the vacuum energy density leads to a value for the effective cosmological constant that is incorrect by between 60 and 120 orders of magnitude. We review an old proposal of replacing Einstein’s field equations by their trace-free part (the trace-free Einstein equations), together with an independent assumption of energy–momentum conservation by matter fields. While this does not solve the fundamental issue of why the cosmological constant has the value that is observed cosmologically, it is indeed a viable theory that resolves the problem of the discrepancy between the vacuum energy density and the observed value of the cosmological constant. However, one has to check that, as well as preserving the standard cosmological equations, this does not destroy other predictions, such as the junction conditions that underlie the use of standard stellar models. We confirm that no problems arise here: hence, the trace-free Einstein equations are indeed viable for cosmological and astrophysical applications.

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1. Introduction

The interpretation of dark energy is a major puzzle [19]. The gravitational effect of the quantum vacuum is expected to be equivalent to an effective cosmological constant [21], which according to the standard view will cause an accelerated expansion of the universe. However, simple estimates of its expected magnitude are very large, exceeding the observed value by
between 60 and 120 orders of magnitude [20, 4, 11], a blatant contradiction with observations. This indicates a profound discrepancy between general relativity (GR) and quantum field theory (QFT)—a major problem for theoretical physics [17, 20]. This will presumably be resolved when a full-blown theory of quantum gravity is finalized and accepted. However, the field theory view of gravity as a massless spin-2 field, and with the quantum vacuum contributing to the cosmological constant, is a half-way house between GR and a full quantum gravity theory (which will need to have both GR and spin-2 QFT as appropriate limits). We need to resolve that discrepancy, no matter what final quantum gravity theory is adopted.

The key issue is: ‘What are the gravitational effects of the vacuum energy?’ One of the proposals for solving the problem, summarized by Weinberg [20], is use of trace-free Einstein equations (TFE) instead of the standard Einstein’s field equations [7, 8]. If these TFE are adopted, the vacuum energy has no gravitational effect. This does not determine a unique value for the effective cosmological constant, but it does solve the huge discrepancy between theory and observation. However, that resolution is dependent on this form of the field equations giving the same results as the standard equations, not only for the solar system and black holes (which will follow because the vacuum field equations are unchanged), but also on one hand for cosmological models, and on the other for astrophysical objects such as stellar models, where one has standard junction conditions between the interior and exterior solutions. The former must work out as expected (because of Weinberg’s results), but how this happens is not immediately obvious. In the latter case, it is not a priori clear that the usual results should remain true. Indeed, one might even expect a discrepancy between the stellar mass and the mass of the exterior Schwarzschild solution.

In this paper, we confirm that the TFE are compatible with either case, and so are indeed a viable alternative to the GR field equations—with the improvement that, unlike Einstein’s field equations, they do not suffer from the crucial discrepancy with the standard results of QFT, which imply a very large value for the vacuum energy: so large that even solar system results would be affected. Pauli in the 1930s (as quoted by Straumann: [17], pp 6–7) estimated the zero-point energy of the radiation field, assuming that the cut-off of the theory was the electron mass. He then estimated that the radius of the de Sitter universe ($\Lambda \approx \frac{1}{R^2}$) would be smaller than the Earth–Moon distance; actually, one can easily compute that $R \approx 31$ km [17]. So this value drastically affects the solar system, since there would be no solar system. It seems prudent to look for a way out.

2. Basic problem

Classical gravitational dynamics is encoded in Einstein’s field equations (EFE) [7, 8]

$$G_{ab} + \Lambda g_{ab} = R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = \frac{8\pi G}{c^4} T_{ab},$$

(1)

where $\Lambda$ is the cosmological constant. These are subject to the conservation equations

$$\nabla_a G^{ab} = 0 \Rightarrow \nabla_a T^{ab} = 0,$$

(2)

for the total energy–momentum tensor, which guarantee the consistency of the time development of the EFE.

In the standard cosmological application, the metric tensor is assumed to take the spatially homogeneous and isotropic form

$$ds^2 = -c^2 dt^2 + a^2(t) d\sigma^2, \quad u^a = \delta_0^a,$$

(3)

with $a(t)$ a universal time-dependent scale factor, $d\sigma^2$ the metric of a 3-space of constant curvature $k$, and $u^a$ the normalized matter 4-velocity field ($u_au^a = -1$). Because of the
symmetries of the metric, the energy–momentum tensor of the matter necessarily takes a perfect fluid form:

$$T_{ab} = \left( \rho + \frac{p}{c^2} \right) c^2 u_a u_b + p g_{ab},$$

where the matter mass density $\rho$ and the matter isotropic pressure $p$ are related through an equation of state $p = p(\rho)$. Whenever the energy–momentum tensor takes the form (4), the conservation equations (2) give the mass–energy conservation relation

$$\dot{\rho} + 3 \frac{\dot{a}}{a} \left( \rho + \frac{p}{c^2} \right) = 0,$$

where a dot denotes a covariant derivative with respect to proper time, $d/dt$. For the case of the Robertson–Walker metric (3), the EFE (1) reduce to two non-trivial equations: the Raychaudhuri equation

$$3 \frac{\ddot{a}}{a} + 4\pi G \left( \rho + 3 \frac{p}{c^2} \right) - \Lambda c^2 = 0,$$

and the Friedmann equation

$$3 \left( \frac{\dot{a}}{a} \right)^2 = -\frac{3k c^2}{a^2} + 8\pi G \rho + \Lambda c^2.$$

The latter equation is a first integral of the other two; indeed any two of (5)–(7) imply the third. When they are satisfied, all 10 EFE (1) are satisfied.

The key equation as regards gravitational attraction is the Raychaudhuri equation (6), which shows that $(\rho + 3p/c^2)$ is the active gravitational mass density. The equation of state for a vacuum

$$\rho_{\text{vac}} = -\rho_{\text{vac}} c^2$$

shows this is negative:

$$\left( \rho_{\text{vac}} + 3 \frac{p_{\text{vac}}}{c^2} \right) = -2\rho_{\text{vac}},$$

where we can represent the effect of the vacuum in (6) either as a fluid with equation of state (8), or as an effective cosmological constant,

$$\Lambda_{\text{vac}} = \frac{8\pi G}{c^2} \rho_{\text{vac}}.$$

The problem is that the QFT view of the vacuum as an infinite set of oscillators, each with zero-point oscillatory energy $\frac{\hbar}{2} k^2$, gives a diverging value for the vacuum energy $E_{\text{vac}}$. With a suitable high-energy cut-off, the vacuum energy density is estimated by Weinberg [20] to be of the order (in units where $c = 1 = \hbar$)

$$\langle \rho_{\text{vac}} \rangle \simeq 2 \times 10^{71} \text{ GeV}^4,$$

whereas the effective value of the cosmological constant as determined by astronomical observations is of the order

$$\langle \rho_{\text{obs}} \rangle \simeq 10^{-47} \text{ GeV}^4.$$

Because the value (11) is a constant, it has no effect on local dynamics, and so can be subtracted off the total energy density as far as local physics is concerned, but on the standard view, because of (9), it will have a gravitational effect hugely bigger than the observed value (12).

There are a number of ways out of this problem.
(1) Other fields may contribute negative energy densities that will cancel the positive terms, and leave a very small residue as observed; or maybe there is a symmetry implying $\rho_{\text{vac}} + \Lambda_{\text{bare}} = 0$, and one has to have extra fields ('quintessence') to give the observed acceleration. This is indeed possible in principle; for example, each mode of a Dirac field gives a negative contribution to the vacuum energy (hence, this is the option that would be realized via supersymmetry, were supersymmetry not broken in the real universe). But it is very hard to make this work in practice: many fields contribute to the vacuum energy density, and it is highly unlikely they would just happen to cancel the value (11) accurately to 120 decimal places, but not exactly, so as to give (12).

(2) The value in (11) is only an expectation value: maybe we live in a multiverse where $\rho_{\text{vac}}$ takes all sorts of values, and anthropic selection effects determine that we live in a universe domain where its value (12) is very far from the expectation value. This is a possible solution, but many regard it as an act of desperation, abandoning the idea of $\Lambda$ as a fundamental constant in favour of regarding its value as being contingent.

(3) Quantum field theoretical considerations are of no relevance to the quantitative modelling of structure formation in the matter distribution and spatial geometry on cosmological distance scales.

(4) Maybe the EFE (1) are not the true effective equations of gravitational interactions: a variant to the EFE arises from the underlying quantum gravity theory, and negates the gravitational effect of the vacuum.

We explore aspects of the latter option in this paper.

3. Trace-free Einstein gravity

Einstein continually worried about what should go on the right-hand side of the relativistic gravitational field equations. An interesting proposal is to take the trace-free part of the EFE to get the TFE, which is a subset of the EFE that can give back the full EFE with an integration constant. This is an old proposal, essentially initiated by Einstein [9] himself, and developed by many others since, see [20], pp 11–13, and references given there. More recently, it has been developed under the name of ‘unimodular gravity’, see [1, 18, 10, 15] and references therein.

We use a hat to denote the trace-free part of a symmetric tensor: so

\[ \hat{G}_{ab} := R_{ab} - \frac{1}{4} R g_{ab} \]
\[ \hat{T}_{ab} := T_{ab} - \frac{1}{4} T g_{ab} \Rightarrow \hat{G}_a^a = 0, \quad \hat{T}_a^a = 0. \]

On taking its trace-free part, the EFE (1) implies the TFE

\[ \hat{G}_{ab} = \frac{8\pi G}{c^4} \hat{T}_{ab} \Leftrightarrow \hat{R}_{ab} = \frac{8\pi G}{c^4} \left( T_{ab} - \frac{1}{4} T g_{ab} \right). \]

We adopt these as the gravitational field equations, instead of (1). The twice-contracted second Bianchi identity for the Einstein tensor $G_{ab}$ still holds:

\[ \nabla_b G^{ab} = 0. \]

However, now the corresponding divergence relation for the total energy–momentum tensor $T_{ab}$,

\[ \nabla_b T^{ab} = 0, \]

is no longer a consequence of the geometrical identity (14), as in (2): it is a separate assumption.

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5 We thank an unknown referee for this comment.
The gravity theory based on the TFE recovers all the vacuum solutions of the EFE unchanged, so e.g. results from the Schwarzschild and Kerr black hole solutions are still valid. However, it no longer has a cosmological constant problem, as $\Lambda$ does not affect spacetime curvature. Indeed, for a perfect fluid the matter source term is the manifestly trace-free energy–momentum tensor

$$\hat{T}_{ab} = \left( \rho + \frac{p}{c^2} \right) c^2 \left( u_a u_b + \frac{1}{4} g_{ab} \right); \quad (16)$$

hence, matter enters the field equations only in terms of the inertial mass density $(\rho + p/c^2)$, which vanishes for a vacuum. However, as discussed in [20], the theory acquires a new integrability condition: differentiating (13), and using (14) and (15), gives

$$\nabla_b \left( R^{ab} - \frac{1}{4} R g^{ab} \right) = \frac{8 \pi G}{c^4} \nabla_b \left( T^{ab} - \frac{1}{4} T g^{ab} \right) \Rightarrow \nabla_a R = - \frac{8 \pi G}{c^4} \nabla_a T. \quad (17)$$

Integrating, $(R + \frac{8 \pi G}{c^4} T)$ is a constant:

$$R + \frac{8 \pi G}{c^4} T =: 4 \hat{\Lambda}. \quad (18)$$

Substituting into (13) to eliminate $T$ gives back (1), but with a new effective cosmological constant:

$$R_{ab} - \frac{1}{2} R g_{ab} + \hat{\Lambda} g_{ab} = \frac{8 \pi G}{c^4} T_{ab}. \quad (19)$$

So the way it works is as follows: we assume both the TFE (13) and the matter conservation equations (15). The integrability condition (17) follows from these equations. Integrating gives (19). This is not surprising in that we have assumed the validity of the trace-free part of the EFE (1); and (18) is just the trace of those equations. For initial value problems within this trace-free Einstein gravity theory the same consistency result holds as for the full EFE: by (14) the time evolution equations amongst the TFE preserve the constraints along any arbitrary timelike reference congruence threading a given spacetime manifold. Furthermore, as has been pointed out previously, the TFE proposal can be implemented as a dynamical theory where the spacetime volume density $\sqrt{-g}$ is not a dynamical variable [20].

Hence, we have a remarkable result [20]: the TFE (13) together with the differential relations (15) are functionally equivalent to the EFE (19), with the cosmological constant an integration constant $\hat{\Lambda}$ which, in principle, could have any arbitrary positive value (but is observed to be small), and which is unrelated to the vacuum energy $\Lambda_{\text{vac}}$. Note that this is not the same as either

$$\hat{G}_{ab} = \frac{8 \pi G}{c^4} T_{ab} \Rightarrow T = 0, \quad R = \text{constant} \quad (20)$$

(as proposed by Einstein in 1919, see [9]), or

$$G_{ab} = \frac{8 \pi G}{c^4} \hat{\Lambda} g_{ab} \Rightarrow R = 0, \quad T = \text{constant} \quad (21)$$

(which also decouples spacetime curvature from vacuum energy). Each of these proposals has ten field equations, and allows only very restricted forms of matter to occur. The proposal that makes physical sense is (13), where both sides of the equation have been specifically constructed so as to have the same symmetry. Then, we have nine field equations and the energy–momentum tensor of the matter can have a non-zero trace, but only the trace-free part gravitates; this can accommodate generic inhomogeneous matter.

What about experiments? The experimental predictions of the two theories are the same, so no experiment can tell the difference between them, except for one fundamental feature: the EFE (confirmed in the solar system and by binary pulsar measurements to high accuracy)
together with the QFT prediction for the vacuum energy density (confirmed by Casimir force measurements) give the wrong answer by many orders of magnitude; the TFE does not suffer this problem. In this respect, the TFE are strongly preferred by experiment.

This trace-free Einstein gravity theory then should give cosmology without the vacuum energy problem. We explore this in the next section, but we will not deal with the relation of this theory to commonplace variational principle approaches, which is adequately covered elsewhere (see [1, 20, 15] and references there).

4. FLRW cosmologies

This works out in the spatially homogeneous and isotropic case as a special case of the above general theory (as, of course, it has to). That is, starting from (5)–(7), we first determine a trace-free dynamic equation by eliminating $\Lambda$ between (6) and (7), and then derive an integrability condition for this equation. Thereafter, we show how this latter equation can be integrated to recover (6) and (7) with a new effective cosmological constant $\hat{\Lambda}$ as an integration constant, independent of any dynamical values we might assign to the vacuum energy. Thus, this solves the major problem of a vacuum energy many orders of magnitude larger than measured by cosmological observations.

In detail, the trace-free equation is

$$\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 - \frac{kc^2}{a^2} = -4\pi G \left(\rho + \frac{p}{c^2}\right).$$  (22)

The spacetime Ricci curvature scalar $R$ is given by

$$Rc^2 = -\frac{8\pi G}{c^2} T + 4\Lambda c^2 = 8\pi G \left(\rho - 3\frac{p}{c^2}\right) + 4\Lambda c^2 = 6 \left[\frac{\dot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2}\right].$$  (23)

The time derivative of this equation is the needed integrability condition for (22), encoding the fact that $\Lambda$ is a constant (see (17)). It has the form

$$0 = 6 \frac{d}{dt} \left[\frac{\dot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} - \frac{4\pi G}{3} \left(\rho - 3\frac{p}{c^2}\right)\right].$$  (24)

Now (24), being a vanishing total time derivative, can easily be integrated to yield an integration constant $\frac{2}{3} \hat{\Lambda}$. Eliminating $\dot{a}/a$ between (22) and the integral of (24), and including the (now assumed) mass–energy conservation relation (5), one recovers the original dynamic equations (6) and (7), but with a renormalized $\Lambda$. That is, we obtain in this way the Raychaudhuri equation

$$3\frac{\ddot{a}}{a} + 4\pi G \left(\rho + 3\frac{p}{c^2}\right) + \hat{\Lambda} c^2 = 0,$$  (25)

and the Friedmann equation

$$3\left(\frac{\dot{a}}{a}\right)^2 = -\frac{3kc^2}{a^2} + 8\pi G\rho + \hat{\Lambda} c^2.$$  (26)

with an effective cosmological constant $\hat{\Lambda}$—an arbitrary integration constant, unrelated to $\Lambda_{\text{vac}}$. This solves the basic discrepancy between QFT estimates of the energy density of the
quantum vacuum, and the disastrous result if we assume that this is a source term in the Raychaudhuri equation in an obvious way. Thus, we arrive at

**Hypothesis**: the EFE are not the true effective equations of gravity: rather—whatever the underlying quantum theory of gravity—the effective theory is trace-free Einstein gravity, as described above.

In that case, the basic equations of cosmology are (22) and (5), supplemented by an equation of state determining the pressure from the mass density, and (25) and (26) are consequences. The value of the vacuum term $\Lambda_{\text{vac}}$ does not affect the cosmic expansion. Note the following remarkable feature: the active gravitational mass density $\rho_{\text{grav}} = (\rho + 3p/c^2)$ does not occur in either of the equations (22) or (5) that we take as the basis of the dynamics: only the inertial mass density $\rho_{\text{inert}} = (\rho + p/c^2)$ occurs there. Nevertheless, in the end $\rho_{\text{grav}}$ turns out to be the effective gravitational mass density, see (25). What happens is that while (5) determines the time evolution of the energy density, the assumed equation of state determines the time evolution of the pressure, and does so in such a way that (25) results. It is interesting to note that (22) is in the form of the dynamical equations often used for studies of inflationary universe models.\footnote{We thank Roy Maartens for this comment.}

Let us also mention an important point concerning scalar field cosmology. The transformation $T_{ab} \rightarrow T_{ab} + \Lambda g_{ab}$ leaves $\hat{T}_{ab}$ and the conservation equation (15) unchanged only if $\Lambda$ is a constant. In the case of a self-interacting scalar field $\psi$ evolving in a potential $V(\psi)$, it is easily checked for the energy–momentum tensor that

$$T_{ab} = \partial_a \psi \partial_b \psi - \frac{1}{2} \left( \partial_a \partial_b \psi + 2V(\psi) \right) g_{ab} \Rightarrow \hat{T}_{ab} = \partial_a \psi \partial_b \psi - \frac{1}{4} (\partial_a \partial_b \psi) g_{ab}$$

(27)

whatever the potential, the latter of which, therefore, does not appear in the TFE (17), and so has no gravitational effect. Nevertheless, the potential is, of course, of dynamical relevance, since it cannot be eliminated from the conservation equation as long as $\dot{\psi} \neq 0$. For an FLRW scalar field cosmology, (15) yields the equation of motion

$$\dot{\psi} \left( \ddot{\varphi} + 3H \dot{\varphi} + \frac{dV}{d\varphi} \right) = 0.$$  

(28)

The dynamics of a universe containing a scalar field (e.g. during inflation) will then be the same as in standard GR. Note, however, that we now have the freedom to shift the minimum of the potential at will, since $V(\varphi) \rightarrow V(\varphi) + V_0$ leaves the equation of motion unchanged.

5. Junction conditions

To complete the picture of the trace-free Einstein gravity proposal, we need to verify that in the case of a stellar model the interior and exterior solutions match, so that the mass measured outside is the same as the mass of the interior solution; if this was not true, it would be a disaster for these equations, as they would not give standard results for stellar structure models. In the GR context, this issue is settled by the Darmois–Israel junction conditions \cite{6, 12}, which arise from the local embedding relations of Gauß and Codazzi for a timelike or spacelike 3-surface in a four-dimensional spacetime manifold.

5.1. Covariant $3 + 1$ viewpoint

Consider a non-null 3-surface $\Sigma : \{\phi = \text{constant}\}$ (of codimension 1) embedded in a spacetime manifold, with unit normal $n_a$ given by

$$n_a := \nabla_a \phi,$$  

(29)
such that
\[ n_a n^a = \varepsilon = \begin{cases} +1, & \text{when } \Sigma \text{ is timelike} \\ -1, & \text{when } \Sigma \text{ is spacelike} \end{cases} \] (30)
applies. Then
\[ \delta^b_a := \varepsilon n_a n^b + h^b_a \] (31)
defines tensors \( \bot^{a b} := \varepsilon n_a n_b \) and \( h^{a b} \) which respectively project orthogonally and tangentially to \( \Sigma \), with \( 0 = h^a_a n_b \). The covariant derivative of \( n_a \) is given by
\[ \nabla_n n^b = \varepsilon n^b \] (32)
which defines the non-geodesity of \( n_a \) by
\[ \dot{n}_a = \nabla_b n^b \] (33)
and the extrinsic curvature tensor of \( \Sigma \) by
\[ K^{a b} := -h^a_c h^b_d \nabla_c n_d = -D_a n_b, \] (34)
employing \( K := K_a^a \). In the absence of matter surface layers, Darmois’ junction conditions [6] require that across \( \Sigma \) the induced 3-metric and the extrinsic curvature be continuous, i.e.
\[ 0 = [h_a b] \Rightarrow 0 = [D_a], \quad 0 = [\nabla_a], \] (35)
\[ 0 = [K_a b] \Rightarrow 0 = [D_a K_{bc}], \] (36)
employing \( [Q] := Q_+ - Q_- \) to denote the change of any quantity \( Q \) across \( \Sigma \). With (35) and (36), this leads to Israel’s junction conditions [12]
\[ 0 = [n_a n_b G^{a b}], \quad 0 = [n_b h^a_a G^{a b}], \] (37)
As this is a purely geometrical result, it does not make a qualitative difference whether we use (i) the EFE (1), or (ii) the TFE (13) together with (18), to replace the Einstein tensor \( G_{a b} \) in these relations.

5.2. Including a matter surface layer

The previous consideration of the Darmois–Israel junction conditions assumes that there is no matter surface layer on the 3-surface \( \Sigma \) between the two parts of spacetime to be matched. An alternative approach to the derivation of the junction conditions relies on the use of a local coordinate approach. Using the same conventions as in the previous paragraph, in a parametric local representation of \( \Sigma \) given by \( \lambda^a = X^a(\sigma^i) \) (where \( a, b, \ldots = 0, \ldots, 3 \) and \( i, j, \ldots = 1, \ldots, 3 \) the coordinate components of the induced 3-metric are expressed by
\[ \gamma_{i j} := \frac{\partial X^a}{\partial \sigma^i} \frac{\partial X^b}{\partial \sigma^j} g_{a b}. \] (40)
Viewed from the four-dimensional perspective, this is the first fundamental form $h_{ab} = \frac{\partial x^a}{\partial \sigma} \frac{\partial x^b}{\partial \tau} \eta^{ij} = g^{ab} - \epsilon n^a n^b$. If we assume that $n_a$ is smoothly continued into the four-dimensional spacetime along a set geodesics, i.e. such that $\dot{n}_a = 0$, the extrinsic curvature reduces to $K_{ab} = -\nabla_a n_b$ [see (32)]. While $g_{ab}$ is everywhere continuous, its derivative along $n_a$ may not be continuous across $\Sigma$, so that $K_{ab}$ may also be discontinuous.

Now consider a neighbourhood of $\Sigma$ represented in terms of a Gaußian normal coordinate system, where the coordinate $s$ varying in the interval $s_- < s < s_+$ measures the distance to $\Sigma$; we assume $\Sigma$ to be located at $s = 0$. This implies that $n_a = \nabla_a s$. We then denote by $Q' = \partial_s Q = r^a \partial_s Q$ the partial derivative of any quantity $Q$ along the normal direction to $\Sigma$. The change of $Q$ across $\Sigma$ is given by

$$[Q] = Q_+ - Q_- = \overline{Q} := \int_{s_-}^{s_+} Q' \, ds.$$  

(41)

This definition is such that $Q' \to \overline{Q} \delta(s)$ in the limit $s_+ - s_- \to 0$.

In Gaußian normal coordinates, equations (35) and (36) are complemented by the spatial projection [13]

$$h_{ac} h_{bd} G_{cd} \equiv 3 G_{ab} + \epsilon n^a \partial_c (K_{ab} - K h_{ab}) - K K_{ab} + \frac{1}{2} (K^2 + K c_{cd} K^{cd}) h_{ab}. \quad (42)$$

If there is matter localized on $\Sigma$, then its energy–momentum tensor can be computed as

$$S_{ab} = \overline{T}_{ab} = \lim_{s \to 0} \int_{s}^{s'} T_{ab} \, ds,$$  

(43)

which satisfies $0 = S_{ab} n^b$; see [13]. In the infinitesimally thin limit, the four-dimensional energy–momentum tensor on $\Sigma$ is of the form $T_{ab} = S_{ab} \delta(s)$. For the geometry of $\Sigma$ to be well defined, both $S_{ab}$ and $h_{ab}$ should be continuous across $\Sigma$ and thus contain neither a $\delta(s)$ factor nor a discontinuity. The first implies that $\delta R_{abcd}$ and thus $\delta R_{ab}$ have no $\delta(s)$ factor, while the second implies that $K_{ab}$ has no $\delta(s)$ factor. Integration of (42) then implies that

$$h_{ac} h_{bd} \overline{G}_{cd} \equiv \epsilon [K_{ac} b - K h_{ac} b], \quad (44)$$

since only the terms containing a derivative along $s$ contribute to the integral, i.e. $G_{ab} \simeq \epsilon (K_{ab} - K h_{ab})$ to dominant order. Using that $G_{ab} = n_a n_b (G_{cde} n^e) + \epsilon 2 n_a n_b h_{cd} G_{cd} + h_{ac} h_{bd} \overline{G}_{cd}$, and the fact that the integrals along $n^e$ of expressions (33) and (34) vanish even when matter is localized on $\Sigma$, we conclude that

$$\overline{G}_{ab} = \epsilon [K_{ab} - K h_{ab}]. \quad (45)$$

With the standard EFE (1), this would imply that $[K_{ab} - K h_{ab}] = \frac{8 \pi G}{c^4} S_{ab}$, a generalization of (38).

For the TFE (13), we first need to compute the jump in the trace-free Einstein tensor $\hat{G}_{ab}$ by using that $G_{ab} \simeq -2 \epsilon K$, so that

$$\overline{\hat{G}}_{ab} = \epsilon \left[ K_{ab} - K \left( h_{ab} - \frac{1}{2} S_{ab} \right) \right] = \frac{8 \pi G}{c^4} \left( S_{ab} - \frac{1}{4} S g_{ab} \right). \quad (46)$$

While the junction condition for the trace $K$ cannot be extracted from this equation, one has to rely on (17). We decompose the total energy–momentum tensor according to $T_{ab} = T_{ab}^{\text{mat}} + T_{ab}^\Sigma$, with $T_{ab}^\Sigma = S_{ab} \delta(s)$. The hypothesis (15) implies that

$$\nabla_{\tau} T_{ab}^\Sigma = f^a = -\nabla_{\tau} T_{ab}^{\text{mat}}. \quad (47)$$

We assume for simplicity that $T_{ab}^\Sigma$ does not interact with the matter fields, which implies $f^a = 0$ (see e.g. [3] for the treatment of such a case, and [5] for an example in which $\Sigma$ is coupled to a form field allowing a jump in the cosmological constant). Since then $\nabla_{\tau} T_{ab}^{\text{mat}} = 0$, it follows that $n_b (T_{a b}^{\text{mat}}) = 0$, and thus $T_{ab}^{\text{mat}} \perp_{ab} = 0$. Then, (17) implies $n^a \nabla_{a} \left( R + \frac{8 \pi G}{c^4} T \right) = 0,$
so that $R + \frac{8\pi G}{c^2} T$ remains constant across $\Sigma$, and so does the integration constant $\hat{\Lambda}$ (i.e. $\hat{\Lambda}' = 0$). Now, since $R \simeq \varepsilon 2 K'$, we deduce that $\bar{R} = \varepsilon 2 [K] = -\frac{8\pi G}{c^2} S$, where the second equality arises from the constancy of $R + \frac{8\pi G}{c^2} T$, which implies $\bar{R} = -\frac{8\pi G}{c^2} S$. It follows that 

$$\frac{1}{2} \varepsilon [K] g_{ab} = -\frac{8\pi G}{c^2} \frac{1}{2} S g_{ab},$$

so that the junction condition (46) for a matter surface layer reduces to

$$\varepsilon [K_{ab} - K h_{ab}] = \frac{8\pi G}{c^2} S_{ab},$$  \hspace{1cm} (48)

i.e. to the same relation as in classical GR.

In conclusion, the standard GR stellar structure models, with an interior solution matched to an exterior solution across a suitable 3-surface $\Sigma_{1}$, will remain valid in the case of trace-free Einstein gravity. There will not be a problem of the interior and exterior masses not matching. Similar issues arise for the junction conditions in Swiss–Cheese models. Again, the TFE will be acceptable: the usual mass matching condition will be fulfilled, so they do not lead to anomalies here either.

6. Viability of the TFE equations

We have revisited the possibility that the true effective gravitational field equations are given by the TFE, implying that only the trace-free part of the energy–momentum tensor $T_{ab}$ of matter is gravitating. Then, the effective cosmological constant $\hat{\Lambda}$ is a constant of integration that is arbitrarily disposable (as in classical GR), and, hence, is independent of any fundamental value assigned to $\Lambda$ (cf [9], p 196). We do not require a fine tuning $\rho_{\text{vac}} + \Lambda \simeq \Lambda_{\text{obs}}$, because $\rho_{\text{vac}}$ is not gravitating.

Thus, employing the TFE in place of the EFE appears to be a good theoretical assumption to make: any huge $\Lambda_{\text{vac}}$ is powerless to affect cosmology, or indeed the solar system, as the zero point energy will not affect spacetime geometry. The EFE will be as usual, but with $\hat{\Lambda}$ an integration constant that may be small, or may be zero. As observations indicate that this constant corresponds to a particular cosmological length scale ($\hat{\Lambda} \simeq H_{0}^{2}$) that should be determined from initial conditions for our universe; see (17). In that sense, the vacuum energy problem vanishes in the trace-free Einstein gravity proposal, while the almost equality between $\Lambda$ and the Hubble constant, i.e. the coincidence problem, remains.

We have checked here that these equations are compatible with usual cosmological models and also with standard junction conditions, without and with a matter surface layer. In particular, this ensures that stellar structure models will be the same as in GR: there will not be any mass anomaly between the interior and exterior solutions. Hence, the TFE work both for cosmology and for astrophysics.

Overall, this proposal does not solve the issue of why the cosmological constant has the value it has today, but it does resolve the issue of why it does not have the huge value implied by the obvious use of the QFT prediction for the vacuum energy in conjunction with the EFE. The patently incorrect result obtained in this way is a major crisis for theoretical physics, because it suggests a profound contradiction between two of our most successful theories, namely QFT and GR. Use of the TFE instead of the EFE solves that problem.

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