STOCHASTIC MACHINE BREAKDOWN AND DISCRETE DELIVERY IN AN IMPERFECT INVENTORY-PRODUCTION SYSTEM

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ABSTRACT. In this paper, we develop an integrated inventory model to determine the optimal lot size and production uptime while considering stochastic machine breakdown and multiple shipments for a single-buyer and single-vendor. Machine breakdown cannot be controlled by the production house. Thus, we assume it as stochastic, not constant. Moreover, we assume that the manufacturing process produces defective items. When a breakdown takes place, the production system follows a no resumption policy. Some defective products cannot be reworked and are discarded from the system. To prevent shortages, we consider safety stock. The model assumes that both batch quantity and the distance between two shipments are identical and that the transportation cost is paid by the buyer. We prove the convexity of the total cost function and derive the closed-form solutions for decision variables analytically. To obtain the optimal production uptime, we determine both the lower and upper bounds for the optimal production uptime using a bisection searching algorithm. To illustrate the applicability of the proposed model, we provided a numerical example and sensitivity analysis.

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1. **Introduction.** The economic order quantity (EOQ) model was first introduced in 1913. Seeking to minimize the total cost, the model generated a balance between holding and ordering costs and determined the optimal order size. Later, the EPQ model considered items produced by machines inside a manufacturing system with a limited production rate, rather than items purchased from outside the factory. Despite their age, both models are still widely used in major industries. Their conditions and assumptions, however, rarely pertain to current real-world environments. To make the models more applicable, different assumptions have been proposed in recent years, including random machine breakdowns, generation of imperfect and scrap items, and discrete shipment orders. The assumption of discrete shipments using multiple batches can make the EPQ model more applicable to real-world problems. For example, Sana et al. [27] formulated an imperfect production process in a volume flexible inventory model, which is just one of numerous studies in this field.

Cárdenas-Barrón [3] derived a basic solution procedure for the EOQ/EPQ model with two backorder costs. Sana [26] developed an inventory model with imperfect-quality products in a three-layer supply chain. Sarkar et al. [33] investigated an economic manufacturing quantity model for an imperfect production process with time-varying demand, inflation, and the time value of money. Sarkar and Moon [34] developed an EPQ model with inflation in an imperfect production system. Cárdenas-Barrón [4] wrote a note on economic order quantity with imperfect quality and quantity discounts. Wee et al. [53] solved a finite horizon EPQ model with backorders. Sarkar et al. [35] developed an EPQ model with a random defective rate, rework process, and backorders for a single-stage production system. Sarkar and Moon [36] considered an imperfect production process for improved quality, setup cost reduction, and variable backorder costs.

Sarker and Khan [37] provided a model in which raw materials were procured from a supplier and processed into finished products in the manufacturing system, and were then dispatched to the buyer in the same batch size. The objective function of that model focused on determining an optimal ordering policy, economic production quantity, and manufacturing batch size. Using an EPQ model in which raw materials were produced and converted into finished products, Diponegoro and Sarker [20] supposed identical intervals to deliver discretely finished batches of items to the customer. Furthermore, that research investigated constant demand and non-constant demand, and the model could accommodate both fixed and variable batch size problems. Siajadi et al. [38] developed a production-inventory model in which several buyers demanded the same product from a single vendor. That study, discussed a model with two or more buyers, and batches were delivered with similar sizes. The authors applied numerical techniques to determine the number of optimal dispatches for a single buyer, and defined the optimal cycle length for buyers/vendors. Zhou and Wang [58] studied an integrated single-vendor single-buyer production-inventory model for a deteriorating item with allowable shortage. Considering multiple products, limited warehouse spaces, and discrete shipment, Pasandideh and Niaki [24] extended the EPQ model to discrete shipments sent from suppliers to retailers. They presented a nonlinear integer program model and used a generic algorithm to solve it.

Manufacturing systems with imperfect and scrapped items have been widely investigated recently and have yielded more applicable models. Chiu et al. [11]...
developed the EPQ model with discrete shipment and generation of imperfect-quality items. They assumed that a portion of imperfect items are scrapped, and that the rest can be repaired or salvaged. At the end of the repair run, repaired items are delivered to customers in discrete batches. In that research, the objective function was to determine the economic production quantity that would minimize the total cost. Pasandideh et al. [25] extended the model of Pasandideh and Niaki [24] with allowable shortages that are completely backlogged. They used a genetic algorithm to solve the problem. Chiu et al. [12] investigated an EPQ model with discrete shipment and generation of imperfect and scrap items. During a production and repair run, the initial delivery of batches was considered to satisfy demand, and the rest were discretely delivered to the customer at the end of the machine repair time. Wee and Widyadana [54] examined a single buyersingle vendor model with discrete shipment and production delay caused by random machine unavailability. They considered unsatisfied demand at the period of machine unavailability as lost sales.

Machine availability throughout a production run is one well-known assumption in the EPQ model. In the real world, this assumption does not hold true; random machine breakdowns and failures occur during production runs. Therefore, Liu and Cao [23] presented an EPQ model with two critical numbers \( (m, M) \) to control setups and shutdowns in a machine. Their assumption is that if the inventory level reaches \( M \), then a machine must be shut down. The machine should start to operate again as the inventory level reaches \( m \). Moreover they considered the breakdown issue in their model such that failure and repair times are random variables. Aboud et al. [1] proposed an inventory model with multiple products and machine unavailability in which unsatisfied demand becomes lost sales. Aboud [2] derived a similar model, with random machine breakdowns and partial backordering, that applies a Markov chain to the problem. Chung and Hou [19] studied random machine breakdowns in which repair times are fixed and machine times-to-failure are a random variable assumed to follow an exponential distribution function. They adopted two no-resumption (NR) and abort/resume (AR) policies to solve the problem. Giri et al. [24] extended an inventory model for unreliable facilities in which the production rate is defined as the decision variable. Also, they assumed that machine breakdown is dependent on the production rate and that production cost was also a function of production rate.

Lin and Gong [22] developed an extended EPQ model with random machine breakdown for a deteriorating product. In their research, repair times were fixed, and shortages were inevitable. Chen and Lo [10] developed an imperfect production system with allowed shortage and product sold out under a warranty period. Chiu et al. [13] determined an optimal production quantity for the EPQ model with scrap, repair, and random machine breakdowns under the NR policy. Chakraborty et al. [8] developed an EPQ model for unreliable manufacturing systems that assume the occurrence of “out-of-control” events and random failures. Chakraborty et al. [9] presented a production inventory model in which the process could shift from an in-control state to an out-of-control state, and machine breakdown could occur during production time. The time that elapsed during a process state shift and breakdowns were both stochastic, random variables. That research considered two separated process inspection policies to decrease both the cost of producing defective items in the out-of-control state and the maintenance cost of machine breakdowns. Chiu [14] presented a robust production-inventory model with an imperfect rework process for
imperfect products. Machine breakdown can occur in that model, and failure time is considered a random variable. Similarly, Chiu et al. [15] investigated the EPQ model under the AR policy. Chiu et al. [16] and Chiu et al. [17] extended the EPQ model with random machine breakdowns and the generation of imperfect quality under the NR policy. Widyadana and Wee [55] presented a production-inventory model for deteriorating items with random machine breakdowns and random repair times. Chiu et al. [18] provided an EPQ model with random machine breakdowns and the generation of imperfect and scrap items. Chakroborty et al. [7] presented an EPQ model in an imperfect production system that used safety stock to prevent shortages in the case of machine breakdowns. Taleizadeh et al. [39] developed a multi-product, multiple-buyer, single-vendor supply chain problem with stochastic demand, variable lead-time, and multi-chance constraint. Moreover some related researches can be found in the works of Taleizadeh et al. [10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. Recently, several related research are found which are in the direction of this paper by Sarkar et al. [29], Sarkar et al. [30], Sarkar and Mahapatra [31], Sarkar and Saren [32], and Sarkar [28]. Several new research directions can be found in the research works by Crdenas-Barrn [5, 6], Widyadana et al. [56], Widyadana and Wee [57], and Taleizadeh et al. [52].

The research just discussed is only part of a larger body of literature. But interestingly, no research in the literature has developed an EPQ model with random machine breakdown, discrete shipment, and imperfect production together. In this paper, which is an extension of Chiu et al. [16, 18], we develop an EPQ model to determine the optimal lot size and production uptime while considering stochastic machine breakdown, multiple shipments, and imperfect production.

2. Problem description and mathematical modeling. Our model considers a manufacturing system that generates imperfect products such that \( x \) percent of manufactured products are defective, and that portion is generated randomly at a production rate of \( d = P_x \). The defective products are not repairable, and at the end of the production uptime, they will be discarded. The constant production rate of items, \( P \), is greater than the annual demand rate \( D \). We assume that stochastic machine breakdown can occur, and the number of breakdowns in a year is a random variable, \( \beta \), that follows the Poisson probability distribution function. When a breakdown occurs, the production system follows the NR policy. In other words, immediately after a breakdown, the repair will be done, and production will not be started until all of the on-hand inventory is depleted. We assume the machine repair time to be constant, and to prevent shortages, we consider safety stock. In this manufacturing system, both batch quantity and the distance between two shipments are identical. Also, the cost of transportation is paid by the buyer. In an EPQ model with a defect rate and no shortage, the production rate of perfect-quality items, \( P - d \), must be greater than the demand rate: \( P - D - d > 0 \).

To determine the optimal production uptime and batch size, the total costs of vendors and buyers should be considered together. In this paper, we use the following notation from Chiu et al. [16].

| Decision variables | Description |
|--------------------|-------------|
| \( t_1 \)          | production uptime when a breakdown does not occur (year). |
| \( q \)            | shipment quantity (units/delivery). |
### Parameters

| Parameter | Description |
|-----------|-------------|
| $A$ | setup cost of vendor ($/setup). |
| $C$ | production cost of vendor ($/unit). |
| $C_s$ | disposal cost of vendor ($/unit). |
| $h$ | holding cost of vendor ($/unit/year). |
| $h_1$ | holding cost for defective units of vendor ($/defective unit/year). |
| $C_t$ | transportation cost of buyer ($/delivery$). |
| $D$ | demand rate of buyer (units/year). |
| $A_1$ | ordering cost of buyer ($/order$). |
| $h_2$ | holding cost of buyer ($/unit/year$). |
| $H_1$ | maximum level of on-hand inventory when machine breakdown does not occur (units). |
| $H_2$ | maximum level of on-hand inventory when machine breakdown occurs (units). |
| $M$ | machine repair time (time unit). |
| $n$ | number of shipments delivered during a cycle when machine breakdown does not occur. |
| $P$ | production rate (units/year). |
| $T$ | cycle length when breakdown does not occur (year). |
| $T'$ | cycle length when breakdown occurs (year). |
| $T_U$ | cycle length for integrated case (year). |
| $t$ | production time before a random breakdown occurs (year). |
| $d$ | time required to deplete all available perfect-quality items when machine breakdown does not occur (year). |
| $t'$ | time required to deplete all available perfect-quality items when machine breakdown occurs (year). |
| $t_r$ | machine repair time (year). |
| $TC(t, q)$ | total inventory costs per cycle when machine breakdown occurs ($/cycle$). |
| $TC(t_1, q)$ | total inventory costs per cycle when machine breakdown does not occur ($/cycle$). |
| $TCU(t_1, q)$ | total inventory costs per unit time for integrated case ($$/year$). |

$t_1$ is the production uptime, and $t$ denotes the time before a breakdown occurs. We investigate two cases, $t < t_1$ and $t \geq t_1$, separately, because machine breakdowns can occur randomly during the production uptime.

2.1. **The first case:** $t > t_1$. In this case, a machine breakdown does not occur during the production uptime. For this case, the on-hand inventories of perfect and defective vendor items are shown in Figures (1) and (2), respectively. Moreover, Figure (3) represents the buyers inventory level when a machine breakdown does not occur.

**Vendor’s cost.** According to figure (1), the setup, variable production, and disposal costs are $SCV(1) = A$, $PC(1) = CPT_1$, and $dCV(1) = C_sPt_1x$, respectively, because $x$ percent of all produced items are scrapped. According to figure (2), the holding cost for defective items is

$$HdCV(1) = \frac{h_1 d(t_1)^2}{2} \tag{1}$$

The holding cost of the safety stock during each cycle is

$$HSCV(1) = hD t_r T \tag{2}$$

To calculate the holding cost of the perfect items, the calculation of number of perfect products per shipment is needed. Thus, the number of perfect items are
calculated and added. To determine the holding cost of perfect items, it first requires determining the average inventory of perfect items in the production uptime which is as follows:

\[
AIPU = \frac{(q + x)T}{2n} + \frac{(q + x + 2x)T}{2n} + \ldots + \frac{(2k - 1)x + q)T}{2n} \\
= \frac{kqT}{2n} + xT \left(1 + 3 + 5 + \ldots + (2k - 1)\right) \\
= \frac{kqT}{2n} + xT \left(\frac{k(1 + 2k - 1)}{2}\right) = \frac{kqT}{2n} + \frac{k^2xT}{2n} \tag{3}
\]

The average inventory in the production downtime is

\[
AIPD = \frac{kxT}{n} + \frac{(kx - q)T}{n} + \frac{(kx - 2q)T}{n} + \ldots + \frac{qT}{n} = \frac{(n - k)}{2n}(q + kx)T \tag{4}
\]

According to (3) and (4), the average inventory of perfect items is

\[
AIPI = \frac{kqT}{2n} + \frac{k^2xT}{2n} + \frac{(n - k)}{2n}\left[qT + \frac{kxT}{2n}\right] \\
= \left[qT + \frac{kxT}{2n}\right](k + n - k) = \frac{qT}{2} + \frac{kxT}{2} \tag{5}
\]

Figure (1), shows that

\[
x = \frac{(P - d)T}{n} - q \tag{6}
\]

\[
T = \frac{nq}{D} \tag{7}
\]
Using (6) and (7) in (5) gives

\[ HCPI = \frac{q}{2} \left[ \frac{nq}{D} + k \left( \frac{nq}{nD} - \frac{(P-d)T}{n} - q \right) \right] \]
\[ = \frac{nq^2}{2D} + nkq \frac{(P-d)nq}{nD} - q \]
\[ = \frac{nq^2}{2D} + nkq \frac{(P-d)}{D} - 1 \]
\[ = \frac{nq^2}{2D} \left[ 1 + k \left( \frac{(P-d)}{D} - 1 \right) \right] \]  
\[ (8) \]

According to Figure (1),

\[ T = \frac{(P-d)t_1}{D} \Rightarrow \frac{T}{t_1} = \frac{(P-d)}{D} = \frac{n}{k} \Rightarrow n = \frac{k((P-d)}{D} \]  
\[ (9) \]

![Figure 2](image-url)

**Figure 2.** The vendors on-hand inventory of defective items when machine breakdown does not occur

Replacing \( n \) in (8) with \( \frac{k(P-d)}{D} \), as shown in (9), gives the average inventory as \( \frac{nq^2}{2D} [1 + n - k] \).

Thus, the holding cost of perfect items is

\[ HCPI = h \left[ \frac{nq^2}{2D} (1 + n - k) \right] \]  
\[ (10) \]

Therefore, the vendor’s total cost is

\[ TCV(t_1, q) = A + C_P t_1 + C_s P t_1 x + \frac{h_1 dt_1^2}{2} + h D t_1 T \]
\[ + \frac{hq^2}{2D} (1 + n - k) \]  
\[ (11) \]
**Figure 3.** The buyers inventory level when machine breakdown does not occur

**Buyer’s cost.** According to Figures (1) and (3), the transportation and fixed ordering costs are $SCB(1) = nC_t$ and $OCB(1) = A_1$, respectively. Moreover the holding cost is

$$HCB(1) = h_2 \left( \frac{qT}{2n} \right) = h_2 \frac{qT}{2} = h_2 \frac{qT}{2} = h_2 \frac{qT}{2}$$  \quad (12)

Therefore, the buyer’s total cost is

$$TCB(t_1, q) = A_1 + nC_t + h_2 \frac{qT}{2}$$  \quad (13)

From (11) and (13), the total cost is

$$TC(t_1, q) = A + A_1 + nC_t + C_P t_1 + C_s P t_1 x + \frac{h_1 dt_1^2}{2} + hD t r T + \frac{h_{nq}^2}{2D} (1 + n - k) + h_2 \frac{q^2}{2D}$$  \quad (14)

According to Figures (1) and (3),

$$T = \frac{(P - d)t_1}{D}$$  \quad (15)

$$k = \frac{D t_1}{q}$$  \quad (16)

$$n = \frac{(P - d)t_1}{q}$$  \quad (17)

As the defective rate $x$, is a random variable with a known probability density function, we can use its expected value. Using all related parameters from (14) to
(17), the expected production-inventory cost per cycle, $E[TC(t_1, q)]$, is

$$E[TC(t_1, q)] = A + A_1 + C Pt_1 + C_s Pt_1 E(x) + \frac{C_t (P - d) t_1}{q} + \frac{h_1 PE(x) t_1^2}{2}$$

$$+ h D g \left( \frac{(P - d) t_1}{D} \right) + h_2 \frac{q^2}{2D} \times \frac{(P - d) t_1}{q}$$

$$+ \frac{h q^2}{2D} \left( \frac{(P - d) t_1}{q} \right) \left( 1 + \frac{(P - d) t_1}{q} - \frac{D t_1}{q} \right)$$  

(18)

This can be simplified to

$$E[TC(t_1, q)] = \left\{ C + C_s E(x) + \frac{C_t (1 - E(x))}{q} + h g (1 - E(x)) + \frac{h q (1 - E(x))}{2D} \right\} P t_1 + \left\{ \frac{h_1 P E(x)}{2} + \frac{h P^2 (1 - E(x))^2}{2D} \right\} t_1^2 + \{ A + A_1 \}$$  

(19)

where $t_r = g$, is the fixed repair time.

2.2. The second case $t \leq t_1$. In this case, a machine failure occurs during the production uptime, and we assume the NR policy. When a machine breakdown occurs, the machine will immediately be repaired, and production will only restart when the inventory level reaches zero. The vendors on-hand inventories of perfect-quality and defective items are shown in Figures (4) and (5), respectively. Moreover, Figure (6) depicts the buyers inventory level in the case of a breakdown.

**Vendor’s cost.** According to figure (4), the setup, variable production, and disposal costs are $SCV(2) = A$, $PC(2) = CP t$, and $dCV(2) = C_s P t x$, respectively. According to figure (5), the holding cost of defective items is

$$H d C V(2) = \frac{h_1 dt^2}{2} + h_1 t d t_r$$  

(20)

And the holding cost of safety stock during each cycle is

$$H S C V(2) = h D t_r T'$$  

(21)

Similar to holding cost of safety stock during each cycle is

$$H C P I = h \left[ \frac{n' q^2}{2D} (1 + n' - k') \right]$$  

(22)

Moreover, the machine repair cost is assumed to be $M$. Therefore, the vendors total cost is

$$T C V(t, q) = A + M + C Pt + C_s P t x + \frac{h_1 dt^2}{2} + h D t_r T'$$

$$+ \frac{h n' q^2}{2D} (1 + n' - k') + h_1 t d t_r$$  

(23)

**Buyer’s cost.** According to Figures (4) and (6), the transportation and fixed ordering costs are $SCB(2) = n C_t$ and $OCB(2) = A_1$, respectively. Moreover the holding cost is

$$H C B(2) = h_2 \left( \frac{q T'}{2n' n'} \right) = h_2 \frac{q T'}{2} = h_2 \frac{n' q^2}{2D}$$  

(24)
Therefore, the buyers total cost is

\[ TCB(t, q) = A_1 + n'C_t + h_2\frac{n'q^2}{2D} \]  

(25)

From (23) and (25), the total cost is

\[ TC(t_1, q) = A + A_1 + M + CP_t + C_x Ptx + \frac{h_1dt^2}{2} + h_1tdt_r + hDt_rT' + \frac{hn'q^2}{2D} \left(1 + n' - k'\right) + h_2\frac{n'q^2}{2D} + n'C_t \]  

(26)

According to Figures (4) and (6),

\[ T' = \frac{(P - d)t}{D} \]  

(27)

\[ k' = \frac{Dt}{q} \]  

(28)

\[ n' = \frac{(P - d)t}{q} \]  

(29)

\[ T' = \frac{n'T}{n} \Rightarrow \frac{T'}{n'} = \frac{T}{n} \]  

(30)

As the defective rate, \( x \), is a random variable with a known probability density function, we can use its expected value. Using all related parameters from (27) to (30), the expected production-inventory cost per cycle, \( E[TC(t, q)] \), is

\[ E[TC(t, q)] = \left( A + A_1 \right) + \left( C + C_x E(x) + h_1gE(x) \right) + \left( \frac{h_1E(x)}{2} + \frac{hP(1 - E(x))^2}{2D} - \frac{h(1 - E(x))}{2} \right) Pt_1^2 \]

\[ + \left( C + C_x E(x) + h_1gE(x) \right) \]

\[ + \left( hq + \frac{C_t}{P} + \frac{hq}{2D} + \frac{h_2g}{2D} \right)(1 - E(x)) \]  

\[ \{ \} Pt_1 \]

(31)

where \( t_r = g \), is the fixed repair time.

Now, (19) and (31) can be rewritten as

\[ E[TC(t_1, q)] = \left( A + A_1 \right) + S_1 t_1 + S_2 t_1^2 \]  

(32)

\[ E[TC(t, q)] = \left( A + A_1 + M \right) + \left( S_1 + h_1gPE(x)t + S_2 t^2 \right) \]  

(33)

where

\[ S_1 = CP + C_x PE(x) + \frac{C_t(1 - E(x))}{q} + hP(1 - E(x)) + \frac{h_2g(1 - E(x))}{2D} \]

\[ + \frac{h_2gP(1 - E(x))}{2D} \]  

(34)

\[ S_2 = \frac{h_1PE(x)}{2} + \frac{hP^2(1 - E(x))^2}{2D} - \frac{hP(1 - E(x))}{2} \]  

(35)

2.3. Integrating the EPQ models with and without breakdowns. Because the defective rate and the number of breakdowns are random variables, the cycle length of the proposed model is not constant. Thus, the expected production-inventory cost per unit time, \( E[TCU(t_1, q)] \), can be obtained as

\[ E[TCU(t_1, q)] = \left\{ \int_0^T E[TC(t, q)]f(t)dt + \int_T^\infty E[TC(t_1, q)]f(t)dt \right\} \]  

\[ E(T_U) \]  

(36)
Figure 4. The vendors on-hand inventory of perfect-quality items in our EPQ model when machine breakdown occurs

Figure 5. The vendors on-hand inventory of defective items when machine breakdown occurs

and

\[ E(T_U) = \int_0^{t_1} E(T')f(t)dt + \int_{t_1}^{\infty} E(T)f(t)dt \] (37)

In this study, we assume that the number of breakdowns per unit time is a random variable that follows a Poisson probability distribution function. Therefore, the time between two breakdowns follows the exponential probability distribution function.
Figure 6. The buyers inventory level when machine breakdown occurs

with parameter $\beta$. According to (37),

$$E(T_U) = \int_0^{t_1} \frac{P(1 - E(x))t}{D} f(t)dt + \frac{P(1 - E(x))t_1}{D} f(t)dt$$

(38)

Given that $\int_0^{t_1} f(t)dt = F(t_1) = 1 - e^{-\beta t_1}$

(39)

then,

$$\int_0^{t_1} t.f(t)dt = -t_1 e^{-\beta t_1} - \frac{1}{\beta} e^{-\beta t_1} + \frac{1}{\beta}$$

(40)

Substituting (39) and (40) into (38) gives

$$E(T_U) = \frac{P(1 - E(x))}{D} \left\{ \int_0^{t_1} tf(t)dt + \int_{t_1}^{\infty} t_1 f(t)dt \right\}$$

$$= \frac{P(1 - E(x))}{D} \left\{ -t_1 e^{-\beta t_1} - \frac{1}{\beta} e^{-\beta t_1} + \frac{1}{\beta} + t_1 e^{-\beta t_1} \right\}$$

$$= \frac{P(1 - E(x))}{\beta D} \left( 1 - e^{-\beta t_1} \right)$$

(41)
Substituting $E[TC(t_1,q)]$, $E[TC(t,q)]$ and $E(T_V)$ (shown in (19), (31), and (41), respectively) into (36), the expected production-inventory cost per unit time becomes the following (see appendix A for detailed calculation)

$$E[TC(t_1,q)] = \frac{(A + A_1)}{P(1 - E(x))(1 - e^{-\beta t_1})} + \frac{S_1D}{P(1 - E(x))} + \frac{h_1gPE(x)D}{P(1 - E(x))}$$

$$+ \frac{MD\beta}{P(1 - E(x))} + \frac{2S_2D}{\beta P(1 - E(x))} - \frac{h_1gPE(x)\beta Dt_1e^{-\beta t_1}}{P(1 - E(x))(1 - e^{-\beta t_1})}$$

$$- \frac{2S_2D_1e^{-\beta t_1}}{P(1 - E(x))(1 - e^{-\beta t_1})} \quad (42)$$

3. **Optimal policy.** To prove the convexity of $E[TC(t_1,q)]$, we use the Hessian matrix to obtain the following derivatives

$$\frac{\partial E(TCU(t_1,q))}{\partial t_1} = \frac{-De^{-\beta t_1}}{(A + A_1)^2} \frac{P(1 - E(x))}{P(1 - E(x))} + \frac{(A + A_1)^2}{P(1 - E(x))} + (1 - e^{-\beta t_1})$$

$$\quad \cdot \left\{ \frac{h_1gPE(x)B + 2S_2}{P(1 - E(x))} \right\} \quad (43)$$

$$\frac{\partial^2 E(TCU(t_1,q))}{\partial t_1^2} = \frac{\beta e^{-\beta t_1}D}{P(1 - E(x))} \left\{ (A + A_1)^2 \frac{(1 - e^{-\beta t_1})}{(1 - e^{-\beta t_1})^3} \right\} + \frac{h_1gPE(x)B}{(1 - e^{-\beta t_1})^3}$$

$$\quad + \frac{2S_2}{(1 - e^{-\beta t_1})^3} \left\{ 2(1 - e^{-\beta t_1}) - \beta t_1(1 + e^{-\beta t_1}) \right\} \quad (44)$$

$$\frac{\partial^2 E(TCU(t_1,q))}{\partial t_1 \partial q} = \frac{\partial^2 E(TCU(t_1,q))}{\partial q \partial t_1} = 0 \quad (45)$$

$$\frac{\partial E(TCU(t_1,q))}{\partial q} = \frac{D}{P(1 - E(x))} \left\{ - \frac{C_1P(1 - E(x))}{q^2} + \frac{hP(1 - E(x))}{2D} \right\}$$

$$\quad + \frac{h_2P(1 - E(x))}{2D}$$

$$\quad = \frac{C_1D}{q^2} + \frac{h}{2} + \frac{h_2}{2} \quad (46)$$

$$\frac{\partial^2 E(TCU(t_1,q))}{\partial q^2} = \frac{2C_1D}{q^3} \quad (47)$$

After substituting (43), (44), (46), and (47) into the following Hessian matrix and making some simplifications, we have

$$\begin{bmatrix} t_1 & q \\ \frac{\partial^2 E(TCU(t_1,q))}{\partial t_1^2} & \frac{\partial^2 E(TCU(t_1,q))}{\partial t_1 \partial q} \\ \frac{\partial^2 E(TCU(t_1,q))}{\partial q \partial t_1} & \frac{\partial^2 E(TCU(t_1,q))}{\partial q^2} \end{bmatrix} \begin{bmatrix} t_1 \\ q \end{bmatrix}$$

$$= \frac{\beta e^{-\beta t_1}D}{P(1 - E(x))(1 - e^{-\beta t_1})^3} \left\{ (A + A_1)^2 \frac{(1 + e^{-\beta t_1})}{(1 + e^{-\beta t_1})^3} \right\}$$

$$\quad + \left\{ (h_1gPE(x)\beta + 2S_2) \left\{ 2(1 - e^{-\beta t_1}) - \beta t_1(1 + e^{-\beta t_1}) \right\} \right\} \quad (48)$$

From (48), because $\frac{\beta e^{-\beta t_1}D}{P(1 - E(x))(1 - e^{-\beta t_1})}$ is greater than zero, $E[TC(t_1,q)]$ will be convex if and only if
\begin{align*}
(A + A_1)\beta^2(1 + e^{-\beta t_1}) + [h_1 g PE(x) \beta + 2S_2] [2(1 - e^{-\beta t_1}) - \beta t_1(1 + e^{-\beta t_1})] & > 0 \tag{49} \\
\text{which can be simplified to} \\
t_1 \leq \frac{(A + A_1)\beta}{h_1 g PE(x) \beta + 2S_2} + \frac{2(1 - e^{-\beta t_1})}{\beta(1 + e^{-\beta t_1})} \tag{50} \\
\text{Hence, } E[TC(t_1, q)] \text{ is convex if and only if} \\
0 \leq t_1 \leq \frac{(A + A_1)\beta}{h_1 g PE(x) \beta + 2S_2} + \frac{2(1 - e^{-\beta t_1})}{\beta(1 + e^{-\beta t_1})} = w(t_1) \tag{51} \\
\text{To determine the optimal values of } t_1 \text{ and } q, \text{ the first derivatives of } E[TC(t_1, q)] \\
\text{with respect to } t_1 \text{ and } q \text{ should be made equal to zero, which gives} \\
\frac{\partial E[TUC(t_1, q)]}{\partial q} = 0 \Rightarrow -\frac{C_i D}{q^2} + \frac{h}{2} + \frac{h_2}{2} = 0 \Rightarrow q^* = \sqrt{\frac{2C_i D}{h + h_2}} \tag{52} \\
\frac{\partial E[TUC(t_1, q)]}{\partial t_1} = \frac{D e^{-\beta t_1}}{P(1 - E(x))(1 - e^{-\beta t_1})^2} \left[ - (A + A_1)\beta^2 \\
+ (1 - e^{-\beta t_1} - \beta t_1) \{h_1 g PE(x)B + 2S_2\} \right] = 0 \tag{53} \\
\text{Because } \frac{D e^{-\beta t_1}}{P(1 - E(x))(1 - e^{-\beta t_1})^2} \text{ is greater than zero, (53) can be rewritten as} \\
(\beta t_1 + e^{-\beta t_1 - 1})(h_1 g PE(x) \beta + 2S_2) - (A + A_1)\beta^2 = 0 \\
\Rightarrow \beta t_1 + e^{-\beta t_1} = \frac{(A + A_1)\beta^2 + h_1 g PE(x) \beta + 2S_2}{h_1 g PE(x) \beta + 2S_2} \\
\Rightarrow \beta t_1 + e^{-\beta t_1} = \frac{(A + A_1)\beta^2}{h_1 g PE(x) \beta + 2S_2} + 1 \tag{54} \\
\text{Assuming } y = \frac{(A + A_1)\beta^2}{h_1 g PE(x) \beta + 2S_2} \\
\beta t_1 + e^{-\beta t_1} = \beta^2 y + 1 \tag{55} \\
\text{To find the optimal run time, we suppose} \\
t^{LL}_1 = \sqrt{2y} = \sqrt{\frac{2(A + A_1)}{h_1 g PE(x) \beta + h_1 PE(x) + \frac{h P(1 - E(x))^2}{h^2} - h P(1 - E(x))}} \tag{56} \\
t^{LU}_1 = \frac{\beta y + \sqrt{\beta^4 y^2 + 8y}}{2} \tag{57} \\
\textbf{Theorem 1.} \text{ The optimal run time must follow the relation } t^{LL}_1 < t^*_1 < t^{LU}_1 \\
\textbf{Proof.} \text{ It is proved in Appendix B.} \qed \\
\textbf{Theorem 2.} \text{ The total cost function } TCU(t_1, q), \text{ is convex.}
Proof. According to (51), \( E[TC(t_1, q)] \) is convex if and only if 0 ≤ \( t_1 \) ≤ \( \frac{(A+A_1)\beta}{h_1gPE(x)\beta+2S_2} + \frac{2(1-e^{-\beta t_1})}{\beta(1+e^{-\beta t_1})} \) = \( w(t_1) \) because both \( \beta \) and \( t_1 \) are positive, and 1 ≤ \( 1 + e^{-\beta t_1} \) ≤ 2. Thus,

\[
w(t_1) > \frac{(A+A_1)\beta}{h_1gPE(x)\beta+2S_2} + \frac{1-e^{-\beta t_1}}{\beta}
\]  

(58)

if \( v = \frac{(A+A_1)}{h_1gPE(x)\beta+2S_2} \), (58) becomes

\[
w(t_1) = v\beta + \frac{1-e^{-\beta t_1}}{\beta}
\]  

(59)

Also, given that \( \frac{\partial E[TCU(t_1, q)]}{\partial t_1} = 0 \),

\[\beta t_1 + e^{-\beta t_1} = \frac{(A+A_1)\beta^2}{h_1gPE(x)\beta+2S_2} + 1 \]  

(60)

or, \( \beta t_1 = v\beta^2 + (1-e^{-\beta t_1}) \) \( t_1 = v\beta + \frac{(1-e^{-\beta t_1})}{\beta} \)  

(61)

Combining (60) and (61), gives

\[
w(t_1) > \left(v\beta + \frac{1-e^{-\beta t_1}}{\beta}\right) = t_1
\]  

(62)

Thus, the total cost function \( TCU(t_1, q) \) is convex. \( \square \)

4. An illustrative numerical example. In this section, we offer a numerical example to illustrate the applicability of our proposed model. Assume that the production and demand rates are \( P = 10000 \) and \( D = 4000 \) units per year, respectively. \( x \) percent of the items produced during the production time could be defective following a uniform probability distribution function over the interval \( [0, 0.2] \). Machine breakdowns might occur during the production uptime. The number of machine failures follows a Poisson distribution function with \( \beta = 0.5 \). The other parameters are \( A = $450 \) per production run, \( A_1 = $150 \) per shipment, \( h = $0.6 \) per item per unit time, \( h_1 = $0.8 \) per defective item per unit time, \( h_2 = $0.9 \) per item per unit time, \( C = $2 \) per item, \( C_s = $0.3 \) per scrapped item, \( C_t = $0.3 \) per item, \( M = $500 \) per each breakdown, and \( t_r = 0.018 \) per year.

From (52), the optimal batch size for each delivery is \( q^* = 730.2967 \). According to (42) and (56), \( E[TCU(t_{1L}^*, q^*)] = $11657.06 \) and \( t_{1L}^* = 0.3985 \). Also, from (42) and (57), \( t_{1U}^* = 0.4188 \) and \( E[TCU(t_{1U}^*, q^*)] = $11656.51 \).

Given that \([t_{1L}^* = 0.3985, t_{1U}^* = 0.4188] \in [0, (t_{1U}^*) = 0.4569] \), obviously the expected total cost function, \( E[TCU(t_1, q)] \), is convex, meeting the required condition. Using Newton’s method and the upper and lower bounds \( (t_{1L}^*, t_{1U}^*) \) as two initial points, the optimal production uptime will be equal to \( t_1^* = 0.4122 \). The optimal expected total cost is shown in Figure (7): \( E[TCU(t_1^*, q^*)] = $11656.35 \).

5. Sensitivity analysis and managerial insights. The optimal values of the decision variables for different values of \( \beta \) are given in Table (1), and the effect of the time between two failures, \( \frac{1}{\beta} \), on the optimal expected total cost per unit time is shown in Figure (8). Table 2 shows a sensitivity analysis of \( t_1^* \), \( q^* \), and \( E[TCU(t_1^*, q^*)] \) for various parameter values. When increases, the value of decreases. Also, the effect of the batch size in each delivery, \( q^* \), on the optimal expected total cost per unit time is shown in Figure (9). Sensitivity analysis of
Table 1. Variations of $\beta$ effects on $t^*_L$, $w(t^*_L)$, $t^*_U$, and $w(t^*_U)$

| $\beta$ | $\beta^{-1}$ | $t^*_L$ | $w(t^*_L)$ | $t^*_U$ | $w(t^*_U)$ |
|---------|--------------|---------|-------------|---------|-------------|
| 0.1     | 10           | 0.398635| 0.406528006| 0.402628| 0.410519    |
| 0.2     | 5            | 0.398597| 0.414274    | 0.40662 | 0.42284     |
| 0.3     | 3.33         | 0.398559| 0.421913    | 0.410651| 0.43396     |
| 0.4     | 2.5          | 0.398521| 0.429443    | 0.41472 | 0.445535    |
| 0.5     | 2            | 0.398483| 0.436868    | 0.418826| 0.456999    |
| 1       | 1            | 0.398294| 0.472429    | 0.439923| 0.512281    |
| 1.1     | 0.909       | 0.398256| 0.479241    | 0.444254| 0.522854    |
| 1.2     | 0.833       | 0.398218| 0.485956    | 0.448623| 0.533239    |
| 1.3     | 0.769       | 0.39818 | 0.492577    | 0.453028| 0.543428    |
| 1.4     | 0.714       | 0.398142| 0.499106    | 0.45747 | 0.553411    |
| 1.5     | 0.667       | 0.398104| 0.505546    | 0.461949| 0.563177    |
| 2       | 0.5         | 0.397915| 0.5365      | 0.484882| 0.608482    |
| 3       | 0.333       | 0.397538| 0.593332    | 0.533358| 0.679759    |
| 4       | 0.25        | 0.397161| 0.645899    | 0.585075| 0.727634    |
| 5       | 0.2         | 0.396786| 0.696877    | 0.639708| 0.762221    |
| 6       | 0.167       | 0.396412| 0.748207    | 0.69691 | 0.794729    |
| 7       | 0.142       | 0.396038| 0.801053    | 0.756338| 0.831822    |
| 8       | 0.125       | 0.395666| 0.855961    | 0.817668| 0.875487    |
| 9       | 0.111       | 0.395295| 0.913066    | 0.880606| 0.925223    |
| 10      | 0.1         | 0.394925| 0.972266    | 0.944891| 0.979797    |

The optimal production time $t^*_1$, and $q^*$ for various parameter values are shown in Figures (10) and (11), respectively.

Figure 7. The behavior of $E[TCU(t_1, q)]$ with respect to $t_1$
Figure 8. The behavior of \[ E[TCU(t_1, q)] \] with respect to \( \frac{1}{\beta} \)

Table 2. Sensitivity analysis of \( t^*_1 \), \( q^* \), and \( E[TCU(t^*_1, q^*)] \) for various parameter values

| Rate of change | \( q^* \) | \( t^*_1 \) | \( E[TCU(t^*_1, q^*)] \) | Rate of change | \( q^* \) | \( t^*_1 \) | \( E[TCU(t^*_1, q^*)] \) |
|---------------|----------|------------|----------------|---------------|----------|------------|----------------|
| \( A \)       | -0.3     | 730.29     | 0.3613         | -0.3          | 730.29   | 0.4121     | 11656.35      |
|               | -0.2     | 730.29     | 0.3789         | -0.2          | 730.29   | 0.4121     | 11656.35      |
|               | 0        | 730.29     | 0.4121         | 0.1           | 730.29   | 0.4121     | 11656.35      |
|               | 0.1      | 730.29     | 0.4278         | 0.2           | 730.29   | 0.4121     | 11656.35      |
|               | 0.2      | 730.29     | 0.4431         | 0.3           | 730.29   | 0.4578     | 11809.83      |
|               | 0.3      | 730.29     | 0.4578         |               |          |            |                |
| \( h_2 \)     | -0.3     | 806.47     | 0.4121         | -0.3          | 730.29   | 0.4121     | 11656.35      |
|               | -0.2     | 778.49     | 0.4121         | -0.2          | 730.29   | 0.4121     | 11656.35      |
|               | -0.1     | 753.24     | 0.4121         | -0.1          | 730.29   | 0.4121     | 11656.35      |
|               | 0        | 730.29     | 0.4121         | 0.1           | 730.29   | 0.4121     | 11656.35      |
|               | 0.1      | 709.32     | 0.4121         | 0.2           | 730.29   | 0.4121     | 11656.35      |
|               | 0.2      | 690.06     | 0.4121         | 0.3           | 730.29   | 0.4121     | 11656.35      |
|               | 0.3      | 672.29     | 0.4121         |               |          |            |                |
| \( C_t \)     | -0.3     | 730.29     | 0.4121         | -0.3          | 730.29   | 0.4121     | 11656.35      |
|               | -0.2     | 730.29     | 0.4121         | -0.2          | 730.29   | 0.4121     | 11656.35      |
|               | -0.1     | 730.29     | 0.4121         | -0.1          | 730.29   | 0.4121     | 11656.35      |
|               | 0        | 730.29     | 0.4121         | 0.1           | 730.29   | 0.4121     | 11656.35      |
|               | 0.1      | 730.29     | 0.4121         | 0.2           | 730.29   | 0.4121     | 11656.35      |
|               | 0.2      | 730.29     | 0.4121         | 0.3           | 730.29   | 0.4121     | 11656.35      |
|               | 0.3      | 730.29     | 0.4121         |               |          |            |                |
| \( P \)       | -0.3     | 730.29     | 0.4121         | -0.3          | 730.29   | 0.4121     | 11656.35      |
|               | -0.2     | 730.29     | 0.4121         | -0.2          | 730.29   | 0.4121     | 11656.35      |
|               | -0.1     | 730.29     | 0.4121         | -0.1          | 730.29   | 0.4121     | 11656.35      |
|               | 0        | 730.29     | 0.4121         | 0.1           | 730.29   | 0.4121     | 11656.35      |
|               | 0.1      | 730.29     | 0.4121         | 0.2           | 730.29   | 0.4121     | 11656.35      |
|               | 0.2      | 730.29     | 0.4121         | 0.3           | 730.29   | 0.4121     | 11656.35      |
|               | 0.3      | 730.29     | 0.4121         |               |          |            |                |
1. The buyers demand rate, vendors setup cost, and vendors holding cost play important roles in determining the optimal production runtime. As shown in Figure (10), the optimal production runtime has a direct, linear relation with the buyers demand rate and vendors setup cost, whereas it has a reverse linear relation with the vendors holding cost.
2. The breakdown rate (number of breakdowns per year) and the buyers transportation cost do not seriously affect the production runtime value. Therefore, machine reliability does not matter to the optimal production time (Figure (10)).

3. When the buyers demand rate and vendors holding cost increase, the optimal shipment quantity increases and decreases, respectively.

4. When the vendors setup production cost changes, the optimal shipment quantity does not change.

5. It is clear from Table (1) that the \([t_{1L}^*, t_{1U}^*]\) interval grows as the machine mean time to failure decreases.

6. **Conclusions.** In this paper, we developed a production-inventory model to determine the optimal production uptime and lot size with stochastic machine breakdown and multiple shipments. We assumed an imperfect production system and adopted the NR policy during machine breakdowns. We first formulated the EPQ model with and without breakdowns. Then, we derived the total production-inventory cost for the integrated system (with/without breakdown). To solve the on-hand problem, we proved the convexity of the total cost function with the help of a classical optimization technique. To obtain the optimal production uptime, we determined its bounds and used a bisection search algorithm. At the end, to illustrate our proposed model, we provided a numerical example and sensitivity analysis. Our model can be used by industry managers in cases of stochastic machine breakdown, which can occur at any time. Industry sectors can use the idea of multiple shipments by which they can incur the optimum cost. Future research could assume an AR policy, which can use stochastic repair times or inspection policies when machine breakdowns occur to reduce the number of breakdowns and reworked items. By considering idea of Sarkar and Mahapatra [51], the model can be extended for the
uncertainty areas. Using the idea of multiple batches and deterioration of products, this model can be extended.

Compliance with ethical standards.

- The manuscript has not been submitted to more than one journal for simultaneous consideration.
- The manuscript has not been published previously (partly or in full), unless the new work concerns an expansion of previous work (please provide transparency on the re-use of material to avoid the hint of text-recycling (self-plagiarism)).
- A single study is not split up into several parts and submitted to various journals or to one journal over time to increase the quantity of submissions (e.g., salami-publishing).
- No data have been fabricated or manipulated (including images) to support our conclusions.
- No data, text, or theories by others are presented as if they were the authors own (plagiarism). Proper acknowledgements to other works has been given (this includes material that is closely copied (near verbatim), summarized, and/or paraphrased), quotation marks are used for verbatim copying of material, and permissions have been secured for material that is copyrighted.
- Authors whose names appear on the submission have contributed sufficiently to the scientific work and therefore share collective responsibility and accountability for the results.

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Appendix A. Deriving the final expected production-inventory cost per unit time

From (47), we know that

$$E[TCU(t_1)] = \frac{1}{E(T_U)} \left[ \int_0^{t_1} E[TC(t)]f(t)dt + \int_{t_1}^{\infty} E[TC(t_1)]f(t)dt \right]$$

(A1)

The first part of the numerator of (A1) can be written as

$$\int_0^{t_1} \left\{ (A + A_1 + M) + [S_1 + h_1gPE(x)]t + S_2t^2 \right\} f(t)dt$$

(A2)

Knowing

$$\int_0^{t_1} f(t)dt = F(t_1) = 1 - e^{-\beta t_1}$$

and

$$\int_0^{t_1} tf(t)dt = -t_1e^{-\beta t_1} + \frac{1}{\beta} (1 - e^{-\beta t_1})$$

and

$$\int_0^{t_1} t^2f(t)dt = -t_1^2e^{-\beta t_1} + \frac{2}{\beta} (t_1e^{-\beta t_1} + \frac{1}{\beta} (1 - e^{-\beta t_1}))$$
and replacing them in (A2), the first part of the numerator of (A1) becomes
\[
(A + A_1 + M)(1 - e^{-\beta t_1})
\]
\[
+ \left( S_1 + h_1 gPE(x) + \frac{2S_2}{\beta} \right) \left( -t_1 e^{-\beta t_1} + \frac{1}{\beta}(1 - e^{-\beta t_1}) \right)
\]
\[
- S_2 t_1^2 e^{-\beta t_1} \tag{A3}
\]
Also, the second part of the numerator of (A1) can be written as
\[
\int_{t_1}^{\infty} \{(A + A_1) + S_1 t_1 + S_2 t_1^2\} f(t) dt = \{(A + A_1) + S_1 t_1 + S_2 t_1^2\} e^{-\beta t_1} \tag{A4}
\]
Substituting (A3) and (A4) into (A1) yields
\[
E[TCU(t_1, q)] = \frac{A + A_1 + (M + \frac{S_1}{\beta} + \frac{h_1 gPE(x)}{\beta}) (1 - e^{-\beta t_1})}{P(1 - E(x)) (1 - e^{-\beta t_1})}
\]
\[
- \frac{h_1 gPE(x) t_1 e^{-\beta t_1} + \frac{2S_2}{\beta^2} (1 - 2e^{-\beta t_1})}{P(1 - E(x)) (1 - e^{-\beta t_1})} \tag{A5}
\]
This can be simplified to
\[
E[TCU(t_1, q)] = \frac{A + A_1 + S_1 D}{P(1 - E(x)) (1 - e^{-\beta t_1})} + \frac{h_1 gPE(x) D}{P(1 - E(x))} + \frac{MD\beta}{P(1 - E(x))} \tag{A6}
\]
\[
+ \frac{2S_2 D}{P(1 - E(x))} - \frac{h_1 gPE(x) \beta D t_1 e^{-\beta t_1}}{P(1 - E(x)) (1 - e^{-\beta t_1})}
\]
\[
- \frac{2S_2 D t_1 e^{-\beta t_1}}{P(1 - E(x)) (1 - e^{-\beta t_1})}
\]

**Appendix B.** Some calculations to prove $t^*_1 < t^*_1 < t^*_1$.

**Proof.** To prove Theorem 2, we use the following proposition presented by Chung (1997).

(a) Let, $g(u) = 1 - u + \frac{u^2}{2} - e^{-u}$, for $u > 0$. Then $g(u) > 0$ for $u > 0$.

(b) $e^{-u} > \left[ \frac{2 - u}{2 + u} \right]$ for $u > 0$

(1) $t^*_1 < t^*_1 < t^*_1$

Let $u = \beta t_1$. Then we have
\[
1 - \beta t_1 + \frac{2(\beta t_1)^2}{2} - e^{-\beta t_1} > 0 \Rightarrow \beta t_1 + e^{-\beta t_1} < 1 + \frac{(\beta t_1)^2}{2} \tag{B1}
\]
or,
\[
\beta^2 y + 1 < 1 + \frac{(\beta t_1)^2}{2} \tag{B2}
\]
Solving (B2) with respect to $t^*_1$ yields
\[
t^*_1 > \sqrt{2} y = t^*_1 \Rightarrow \sqrt{2} y = \frac{2(A + A_1)}{h_1 gPE(x) \beta + 2S_2} \tag{B3}
\]
or,
\[ t_1^* > t_{1L}^* = \sqrt{\frac{2(A + A_1)}{B_1} - h_1P} \]

(2) \[ t_1^* < t_{1U}^* \]

By proposition (b),
\[ e^{-\beta t_1} > \frac{2 - \beta t_1}{2 + \beta t_1} \Rightarrow \beta t_1 + e^{-\beta t_1} > \frac{2 - \beta t_1}{2 + \beta t_1} + \beta t_1 \]

or,
\[ \beta^2 y + 1 > \frac{2 + \beta t_1 + (\beta t_1)^2}{2 + \beta t_1} \Rightarrow \beta^2 y + 1 > 1 + \frac{\beta^2 t_1^2}{2 + \beta t_1} \Rightarrow t_1^2 - \beta t_1 y - 2y < 0 \]

Solving (B6) with respect to \( t_1^* \) yields
\[ t_1^* = \frac{\beta y + \sqrt{\beta^2 y^2 + 8y}}{2} = t_{1U}^* \]

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