On Domatic and Total Domatic Numbers of Product Graphs

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Abstract

A domatic (total domatic) k-coloring of a graph G is an assignment of k colors to the vertices of G such that each vertex contains vertices of all k colors in its closed neighborhood (neighborhood). The domatic (total domatic) number of G, denoted d(G) (d_t(G)), is the maximum k for which G has a domatic (total domatic) k-coloring. In this paper, we show that for two non-trivial graphs G and H, the domatic and total domatic numbers of their Cartesian product G □ H is bounded above by \( \max\{|V(G)|, |V(H)|\} \) and below by \( \max\{d(G), d(H)\} \). Both these bounds are tight for an infinite family of graphs. Further, we show that if H is bipartite, then \( d_t(G □ H) \) is bounded below by \( 2 \min\{d_t(G), d_t(H)\} \) and \( d(G □ H) \) is bounded below by \( 2 \min\{d(G), d_t(H)\} \). These bounds give easy proofs for many of the known bounds on the domatic and total domatic numbers of hypercubes [8, 31] and the domination and total domination numbers of hypercubes [16, 22] and also give new bounds for Hamming graphs. We also obtain the domatic (total domatic) number and domination (total domination) number of n-dimensional torus \( \prod_{i=1}^{n} C_{k_i} \) with some suitable conditions to each \( k_i \), which turns out to be a generalization of a result due to Gravier [13] and give easy proof of a result due to Klavžar and Seifter [23].

Key Words: Cartesian product, Domatic number, Hamming graph, Injective coloring, Torus, Total domatic number.

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1 Introduction

All graphs considered in this paper are finite, simple, undirected and do not contain an isolated vertex. Let \( P_n, C_n \) and \( K_n \) respectively denote the path, the cycle and the complete graph on \( n \) vertices. Let \( \delta(G) \) and \( \Delta(G) \) denote the minimum and maximum degree of a graph G respectively. The neighborhood \( \mathcal{N}(x) \) of a vertex \( x \) is \( \{u: ux \in E(G)\} \) and its closed neighborhood \( \mathcal{N}[x] \) is \( \mathcal{N}(x) \cup \{x\} \). For \( S \subseteq V(G) \), let \( \langle S \rangle \) denote the subgraph induced by \( S \) in \( G \). For any graph \( G \), let \( \overline{G} \) denote the complement of \( G \). Let \( |n| \) be the
set of consecutive integers \( \{1, 2, \ldots, n\} \) and \( \mathbb{Z}_n \) be the set of congruence classes of integers modulo \( n \).

The Cartesian product of two graphs \( G \) and \( H \), denoted \( G \square H \), is a graph whose vertex set is \( V(G) \times V(H) = \{(x, y) : x \in V(G) \text{ and } y \in V(H)\} \) and two vertices \((x_1, y_1)\) and \((x_2, y_2)\) of \( G \square H \) are adjacent if and only if either \( x_1 = x_2 \) and \( y_1y_2 \in E(H) \) or \( y_1 = y_2 \) and \( x_1x_2 \in E(G) \). For any vertex \( u \in V(G) \), \( \{\{u\} \times V(H)\} \) is isomorphic to \( H \). It is called the \( H \)-layer of \( u \) and is denoted by \( H_u \). For any vertex \( v \in V(H) \), \( \{V(G) \times \{v\}\} \) is isomorphic to \( G \) called the \( G \)-layer of \( v \) and is denoted by \( G_v \). For \( d \geq 2 \), let \( G_1 \square G_2 \square \cdots \square G_d \). For \( r_i \geq 3 \), we call \( \square C_{r_i} \) a \( d \)-dimensional torus. For \( n \geq 1, q \geq 2 \), the Hamming graph \( H_{n,q} \) is \( \square_{i=1}^n K_q \). The special case \( H_{n,2} \) is a hypercube of dimension \( n \), denoted as \( Q_n \).

A domatic (total domatic) \( k \)-coloring of a graph \( G \) is an assignment of \( k \) colors to the vertices of \( G \) such that each vertex contains vertices of all \( k \) colors in its closed neighborhood (neighborhood). The domatic (total domatic) number of \( G \), denoted \( d(G) \) \((d_t(G))\), is the maximum \( k \) for which \( G \) has a domatic (total domatic) \( k \)-coloring.

Let \( D \subset V(G) \), if \( N(D) \supseteq V(G) \setminus D \) then \( D \) is a dominating set of \( G \) and if \( N(D) = V(G) \) then \( D \) is a total dominating set of \( G \). The domination (total domination) number of a graph \( G \) is the cardinality of a smallest dominating (total dominating) set of \( G \) and is denoted \( \gamma(G) \) \((\gamma_t(G))\). In any domatic (total domatic) coloring of a graph \( G \), each color class is a dominating (total dominating) set of \( G \). Thus the domatic and total domatic numbers can be also seen in the following way. The domatic (total domatic) number of \( G \) is the maximum number of classes of a partition of \( V(G) \) such that each class is a dominating (total dominating) set of \( G \). There is considerable literature on domination and total domination in graphs. See for instance, \([6,13,20,21,23]\) and a survey of selected topics by Henning \([19]\).

The concept of domatic number and total domatic number was introduced by Cockayne et al., in \([10]\) and \([9]\) respectively, and investigated further in \([1,2,5,8,12,18,24,26,32,33]\). In \([33]\), Zelinka have shown the existence of graphs with very large minimum degree have a total domatic number 1. Chen et al., \([8]\) and Goddard and Henning \([12]\) have studied total domatic coloring under the names coupon coloring and thoroughly dispersed coloring respectively. The motivation for study of total domatic coloring and its applications were mentioned by Chen et al., in \([8]\). Further, they showed that every \( d \)-regular graph \( G \) has \( d_t(G) \geq (1 - o(1))\sqrt{d} / \log d \) as \( d \rightarrow \infty \), and the proportion of \( d \)-regular graphs \( G \) for which \( d_t(G) \leq (1 + o(1))\sqrt{d} / \log d \) tends to 1 as \( |V(G)| \rightarrow \infty \). In \([12]\), Goddard and Henning have shown that the total domatic number of a planar graph cannot exceed 4 and conjectured that every planar triangulation \( G \) on four or more vertices has \( d_t(G) \) at least 2. There are some partial answers to this conjecture by Akbari et al., \([1]\) and Nagy \([26]\). For a bipartite graph \( G \), Heggenes and Telle \([18]\) shown that deciding whether \( d_t(G) \geq 2 \) is NP-complete. In \([24]\), Koivisto et al., shown that it is NP-complete to decide whether \( d_t(G) \geq 3 \) where \( G \) is a bipartite planar graph of bounded maximum degree. Also, they have shown that if \( G \) is split
or \( k \)-regular graph for \( k \geq 3 \), then it is NP-complete to decide whether \( d_t(G) \geq k \).

In \cite{8}, Chen et al., mentioned that for any graph \( G \), it would be interesting to determine any relations between \( d_t(G) \) and \( d_t(G \square G) \). More generally, for any graphs \( G \) and \( H \), we start to determine the relationship between \( d_t(G) \), \( d_t(H) \) and \( d_t(G \square H) \). In this direction, we prove that if at least one among \( G \) or \( H \) is bipartite, then \( G \square H \) has a total domatic coloring with \( 2 \min\{d_t(G), d_t(H)\} \) colors. As consequences, we show that when \( n \) is a power of 2, the total domatic number of the hypercubes \( Q_n \) and \( Q_{n+1} \) is \( n \) and the torus \( \bigsquare_{i=1}^{n} C_{4r_i} \) is \( 2n \). Also, for any positive integer \( d \) and with some suitable conditions to each \( k_i \), we show that the total domatic number and total domination number of the torus \( \bigsquare d_{i=1}^{d} C_{k_i} \) is \( 2d \) and \( (\prod_{i=1}^{d} k_i)/2d \) respectively. In addition, we obtain similar bounds and results for the domatic number and domination number of \( G \square H \). Also, we prove that the domatic and total domatic numbers of \( G \square H \) is upper bounded by \( \max\{|V(G)|, |V(H)|\} \), and lower bounded by \( \max\{d(G), d(H)\} \).

The concept of injective coloring was introduced by Hahn et al., in \cite{15} and further studied in \cite{7} \cite{11} \cite{25}. An injective \( k \)-coloring of a graph \( G \) is an assignment of \( k \) colors to the vertices of \( G \) such that any two vertices in the neighborhood of each vertex have distinct colors. The minimum \( k \) for which such a coloring exists is the injective chromatic number of \( G \), denoted \( \chi_i(G) \). Also, the injective chromatic number of a graph \( G \) can be seen in the following way. The common neighbor graph \( G^{(2)} \) of \( G \) has the same vertex set \( V(G) \) and any two vertices \( u, v \) are adjacent in \( G^{(2)} \) if there is a path of length 2 joining \( u \) and \( v \) in \( G \). It is noted that \( \chi_i(G) = \chi(G^{(2)}) \). The square of a graph \( G \), denoted \( G^2 \), has the same vertex set \( V(G) \) and edge set \( E(G) \cup E(G^{(2)}) \).

We obtain a lower bound for the domatic and total domatic number of \( H_{n-1,q} \) and \( H_{n,q} \) respectively, when \( n \) is a power of 2 and \( q \) at least 2. In \cite{8}, Chen et al., determined the injective chromatic number of \( H_{n,q} \), where \( q \) is a prime power and \( n = \frac{q^k-1}{q-1} \), for some positive integer \( k \). As a consequence, we obtain a lower bound for the domatic and total domatic numbers of \( H_{n-1,q} \) and \( H_{n,q} \) respectively for some more values of \( n \) when \( q \) is a prime power.

## 2 Preliminaries

It is easy to see that \( d(G) \leq \delta(G) + 1 \) and \( d_t(G) \leq \delta(G) \) for every graph \( G \). We will call the graphs which attain these bounds as domatically full and totally domatically full respectively. Regular total domatically full graphs are also called rainbow graphs (see, \cite{27} \cite{29}). We first make some easy observations on rainbow graphs. Examples of rainbow graphs include cycles \( C_n \) where \( n \equiv 0 \pmod{4} \), \( K_{n,n} \), \( K_n \square K_2 \), etc.

**Proposition 2.1.** Let \( G \) be an \( r \)-regular total domatically full graph. Every \( r \)-total domatic coloring of \( G \), say \( f : V(G) \to [r] \) satisfies the following.
(i) Each color class of \( f \) has the same size \( \frac{|V(G)|}{r} \). (\cite{29,32})

(ii) Each color class of \( f \) has an even number of vertices and \( G \) contains a perfect matching.

(iii) \( r \) divides \( |V(G)| \) and \( r^2 \) divides \( |E(G)| \). (\cite{29})

(iv) \( \gamma_t(G) = \frac{|V(G)|}{r} \).

(v) \( d_t(G) = \chi_t(G) = r \).

\textbf{Proof.} (i) Let \( V_1 \) and \( V_2 \) be two color classes of \( c_1 \) and \( c_2 \), respectively, where \( c_1, c_2 \in [r] \).
Each vertex of \( V_1 \) is adjacent to exactly one vertex of \( V_2 \) and vice versa and thus \( |V_1| = |V_2| \).
Hence \( f \) partitioned \( V(G) \) into \( r \) classes having the same size \( \frac{|V(G)|}{r} \). (see, \cite{29,32}).

(ii) For any vertex \( x \in V(G) \), there exists exactly one neighbor of \( x \) having the color \( f(x) \).
Thus each color class \( V_i \) of \( f \) induces a perfect matching in \( \langle V_i \rangle \) and hence \( |V_i| \) is even. Also, \( G \) contains a perfect matching which is the union of perfect matchings of all color classes.

(iii) Clearly, \( r \) divides \( |V(G)| \) and \( r^2 \) divides \( |E(G)| \) which follows from the fact that \( |E(G)| = \frac{|V(G)|}{2} \) and \( \frac{|V(G)|}{r} \) is even (see, \cite{29}).

(iv) Each color class \( V_i \) of \( f \) is a total dominating set, thus \( \gamma_t(G) \leq \frac{|V(G)|}{r} \). Also, \( \gamma_t(G) \geq \frac{|V(G)|}{\Delta(G)} = \frac{|V(G)|}{r} \), since any set \( S \) of size smaller than \( \frac{|V(G)|}{\Delta(G)} \) can dominate at most \( |S| \Delta(G) < |V(G)| \) vertices.

(v) Clearly, \( f \) is also an injective coloring and the proof follows from \( \chi_t(G) \geq \Delta(G) = r \). \( \square \)

A complete characterization of regular domatically full graphs was done by Zelinka \cite{30}.
Examples of regular domatically full graphs include cycles \( C_n \) where \( n \equiv 0 \pmod{3} \), \( K_n \), etc.

\textbf{Theorem 2.2.} \cite{30} An \( r \)-regular graph \( G \) is domatically full if and only if \( r + 1 \) divides \( |V(G)| \) and \( G \) has an \((r + 1)\)-coloring \( f \) such that each color class of \( f \) is an independent set of size \( \frac{|V(G)|}{r+1} \) and the subgraph induced by the vertices of any two color classes of \( f \) is a perfect matching.

The domatic coloring of an \( r \)-regular graph is closely associated with the proper injective coloring which follows in Corollary \cite{23,33} We say that an injective coloring is proper if no two adjacent vertices get the same color.

\textbf{Corollary 2.3.} Let \( G \) be an \( r \)-regular graph. \( G \) is domatically full if and only if \( G \) has a proper \((r + 1)\)-injective coloring. Also, \( \gamma(G) = \frac{|V(G)|}{r+1} \).

\section{Domatic and total domatic numbers of Cartesian products}

The union of two disjoint dominating sets of \( G \) should be a total dominating set for \( G \) and thus \( \left\lfloor \frac{d(G)}{2} \right\rfloor \leq d_t(G) \leq d(G) \leq 2d_t(G) + 1 \). In any total domatic coloring of \( G \), each
vertex should receive its own color from a neighbor, thus \( d_t(G) \leq \left\lfloor \frac{|V(G)|}{2} \right\rfloor \) and also, \( d(G) \leq |V(G)| \). Hence it follows that, for any two graphs \( G \) and \( H \), \( \left\lfloor \frac{1}{2} \max\{d(G), d(H)\} \right\rfloor \leq \left\lfloor \frac{1}{2}d(G \square H) \right\rfloor \leq d_t(G \square H) \leq \left\lfloor \frac{1}{2}d(G \square H) \right\rfloor \leq \left\lfloor \frac{1}{2}|V(G)||V(H)| \right\rfloor \) and \( d(G \square H) \leq |V(G \square H)| = |V(G)||V(H)| \). We improve these bounds given above for any two graphs in Theorem \ref{thm:main}.

**Theorem 3.1.** For any two graphs \( G \) and \( H \) without an isolated vertex, we have

\[
\max\{d(G), d(H)\} \leq d_t(G \square H) \leq d(G \square H) \leq \max\{|V(G)|, |V(H)|\}.
\]

**Proof.** Let \( G \) and \( H \) be graphs of order \( m \) and \( n \) respectively. Now, let us prove the upper bound for \( d(G \square H) \). Without loss of generality, let \( n \geq m \). Let us consider the coloring of \( G \square H \) as filling the cells of \( m \times n \) grid with colors. For a cell \((i, j)\), \( 1 \leq i \leq m, 1 \leq j \leq n \), call the set of cells in the \( i^{th} \) row and \( j^{th} \) column as a cross-hair at \((i, j)\). There are \( mn \) cross-hairs, one corresponding to each cell of the grid. Each cross-hair has \( m + n - 1 \) cells. If there is a \( k \)-domatic coloring, then each cross-hair contains all \( k \) colors occurs at least once.

**Claim.** In any domatic coloring of \( G \square H \), each color should appears in at least \( m \) cells.

Suppose a color \( c_1 \) appears in less than \( m \) cells, then there exists a row \( i \) as well as a column \( j \) in the grid which do not contain \( c_1 \). In this case, the cross-hair at \((i, j)\) does not contain \( c_1 \) and hence the coloring is not a domatic coloring. Thus the claim holds. Since each color should appears in at least \( m \) cells and there are \( mn \) cells in the grid, the maximum possible value of \( k \) in any domatic \( k \)-coloring is \( n \). Thus \( d(G \square H) \leq n = \max\{|V(G)|, |V(H)|\} \).

Now, let us prove the lower bound for \( d_t(G \square H) \). Let \( r \) and \( s \) be the domatic numbers of \( G \) and \( H \) respectively. Suppose \( r \geq s \). Let \( D_1, D_2, \ldots, D_r \) be a domatic partition of \( V(G) \). For \( 1 \leq i \leq r \), any vertex \( u \in D_i \) and \( v \in V(H) \), let us define a coloring \( f \) for the vertices of \( G \square H \) by \( f((u, v)) = i \). Since \( H \) does not have an isolated vertex, \( f \) is a total domatic coloring of \( G \square H \) with \( r \) colors and thus \( d_t(G \square H) \geq r = \max\{d(G), d(H)\} \).

If at least one of these graphs \( G \) and \( H \) is disconnected, then the upper bound can be improved by considering the smallest size of the components of \( G \) and \( H \).

The bounds given in Theorem \ref{thm:main} are tight for the graphs mentioned in Corollary \ref{cor:main}.

**Corollary 3.2.** Let \( m, n \) be two integers greater than \( 1 \) and \( G \) be a graph of order \( m \) without an isolated vertex. If \( m \leq n \), then \( d_t(G \square K_n) = d(G \square K_n) = n \). In particular, \( d_t(K_m \square K_n) = d(K_m \square K_n) = \max\{m, n\} \).

**Proof.** By Theorem \ref{thm:main} we have \( n = \max\{d(G), d(K_n)\} \leq d_t(G \square K_n) \leq d(G \square K_n) \leq \max\{m, n\} = n \).

The tightness of the lower bound for \( d_t(G \square H) \) in Theorem \ref{thm:main} can be also seen by considering \( G \cong K_n \) and \( H \cong K_2 \). More generally, in all cases when \( G \) is domatically full and \( \delta(H) = 1 \), the lower bound is attained for \( d_t(G \square H) \).
Corollary 3.3. If $G$ is a domatically full graph and $H$ is a graph with minimum degree 1, then $d_t(G \Box H) = d(G)$.

Proof. We have $d_t(G \Box H) \geq d(G)$ by Theorem 3.1. The upper bound follows since $d_t(G \Box H) \leq \delta(G \Box H) = \delta(G) + 1 = d(G)$.

The tightness of the upper bound in Theorem 3.1 can also be seen by considering $G \cong K_n$ and $H \cong K_2$. Theorem 3.4 demonstrates the same in a more general case.

Theorem 3.4. Let $r, s_0, s_1, \ldots, s_{r-1}$ be positive integers such that $r \geq 2$, $s_0 \leq s_1 \leq \cdots \leq s_{r-1}$. If $G$ is a graph with total domatic number at least $s_0$ and $H$ is a graph which contains $K_{s_0,s_1,\ldots,s_{r-1}}$ as a spanning subgraph, then $d_t(G \Box H) \geq rs_0$. If each $s_i$ is equal to $s_0$ and $|V(G)| \leq rs_0$, then $d_t(G \Box H) = |V(H)| = rs_0$ and $d_t(H \Box H) = |V(H)|$. If $G$ is a graph with domatic number at least $s_0$, then the same results hold for $d(G \Box H)$.

Proof. Let $G$ be a graph with total domatic number at least $s_0$ and $H$ be a graph which contains the spanning subgraph $H'$, namely $K_{s_0,s_1,\ldots,s_{r-1}}$. Let $U_i, i \in \mathbb{Z}_{s_0}$ be the color classes corresponding to $s_0$-total domatic coloring of $G$. We label the vertices in each color class $U_i$ of $G$ as $\{u_{ij} : j \in \mathbb{Z}_{U_i}\}$ and label the vertices of $k$th part of $H'$, $k \in \mathbb{Z}_r$ as $\{v_{kl} : l \in \mathbb{Z}_{s_k}\}$. Let us define a coloring $f$ for the vertices of $G \Box H'$ in the following way:

$$f((u_{ij}, v_{kl})) = (r(i + l) + k) \mod rs_0.$$  

The vertex $(u_{ij}, v_{kl})$ should be adjacent to the vertices $\{(u_{ij'}, v_{kl}) : i' \in \mathbb{Z}_{s_0},$ for some $j' \in \mathbb{Z}_{U_i}\}$ in the layer $G_{v_{kl}}$ and $\{(u_{ij}, v_{k'l'}) : k' \in \mathbb{Z}_r, l' \in \mathbb{Z}_{s_k}\}$ in the layer $H_{u_{ij}}$. Note that $\{(r(i + l') + k') \mod rs_0 : k' \in \mathbb{Z}_r, l' \in \mathbb{Z}_{s_k}\} = \mathbb{Z}_{rs_0}$. The set of colors in the neighbors of $(u_{ij}, v_{kl})$ are $\{(r(i' + l) + k) \mod rs_0 : i' \in \mathbb{Z}_r, l' \in \mathbb{Z}_{s_k}\} = \mathbb{Z}_{rs_0}$ on $s_k \geq s_0$. Thus, each vertex of $G \Box H'$ sees all the colors $\mathbb{Z}_{rs_0}$ of its open neighborhood. Since $G \Box H'$ is a spanning subgraph of $G \Box H$, we have $d_t(G \Box H) \geq d_t(G \Box H') \geq rs_0$.

For $r \geq 2$, if each $s_i$ is equal to $s_0$, then the coloring $f$ mentioned above yields that $d_t(G \Box H) \geq d_t(G \Box H') \geq |V(H)|$ and by Theorem 3.1, we have $d_t(G \Box H) \leq \max\{|V(G)|,|V(H)|\} = |V(H)|$. Since $d_t(G) \geq s_0$, by above arguments we get $d_t(H \Box H) = |V(H)|$.

If $G$ is a graph with domatic number at least $s_0$, then the same proof mentioned above will work for $d(G \Box H)$ in such a way that each vertex of $G \Box H'$ should contain the vertices of all the colors $\mathbb{Z}_{rs_0}$ in its closed neighborhood. Thus $d(G \Box H) \geq d(G \Box H') \geq rs_0$. Also, $d(G \Box H) = |V(H)| = rs_0$ when each $s_i$ is equal to $s_0$ and $|V(G)| \leq rs_0$. 

Remark 3.1. A graph $H$ contains $K_{s_0,s_1,\ldots,s_{r-1}}$, $s_0 \leq s_1 \leq \cdots \leq s_{r-1}$, as a spanning graph if and only if the components of $\overline{H}$ can be grouped into $r$ parts such that each part has at least $s_0$ vertices.

Next, let us consider the Cartesian product of complete graphs and cycles. The upper bound given in Theorem 3.1 is not tight for cycles of length larger than the size of the complete graphs.
Figure 1: 3-total domatic coloring of $G \Box K_2$, where $G$ is a Peterson graph

**Proposition 3.5.** Let $m, n$ be two integers such that $m > n \geq 3$, we have $d_t(C_m \Box K_n) = n$.

**Proof.** Let $U = \{u_i : i \in \mathbb{Z}_m\}$ and $V = \{v_j : j \in \mathbb{Z}_n\}$ be the vertices of $C_m$ and $K_n$ respectively. Suppose there exists an $(n + 1)$-total domatic coloring for $C_m \Box K_n$. Since $C_m \Box K_n$ is $(n + 1)$-regular, the neighbors of each vertex should be colored distinctly and any edge of the same color class cannot be in the layer $\{u_i\} \Box K_n$ for all $i, 0 \leq i \leq m - 1$. Let us consider an edge $(u_i, v_0)(u_{i+1}, v_0)$ in $C_m \Box \{v_0\}$ such that both the vertices are colored $c_1$. For all $x \in \{u_i-1, u_i, u_{i+1}, u_{i+2}\}$, the vertices of $\{x\} \Box K_n, (u_i-2, v_0)$ and $(u_i+3, v_0)$ cannot be colored with color $c_1$, otherwise there exists a vertex which sees the same color $c_1$ in its two neighbors. If we choose the other edge for the color class $c_1$ either $(u_{i+3}, y)(u_{i+4}, y)$ or $(u_{i-2}, y)(u_{i-3}, y)$ for some $y \in \{v_1, \ldots, v_{n-1}\}$, then it covers the vertices of exactly 3 new layers $\{z\} \Box K_n$ for all $z \in \{u_{i+3}, u_{i+4}, u_{i+5}\}$ or $z \in \{u_{i-2}, u_{i-3}, u_{i-4}\}$ respectively. Otherwise, an edge will covers 4 new layers of $K_n$ in $C_m \Box K_n$. Thus an edge which chosen first has covered 4 layers of $K_n$ and the subsequent edges covers either 3 or 4 layers of $K_n$, likewise we can choose at most $\left\lfloor \frac{m-1}{3} \right\rfloor$ edges for a color class. By (i) of Proposition 2.1 size of an each color class equals $\frac{mn}{n+1} \leq 2 \left\lfloor \frac{m-1}{3} \right\rfloor \leq \frac{2m}{3}$ which in turns $n \leq \frac{2(n+1)}{3}$ and yields $n \leq 2$, a contradiction. Hence $d_t(C_m \Box K_n) \leq n$ and lower bound follows from $d_t(C_m \Box K_n) \geq \max\{d(C_m), d(K_n)\} = n$. \hfill $\Box$

Now, let us consider the bounds for the domatic and total domatic numbers of $G \Box K_2$.

**Proposition 3.6.** Let $G$ be a graph without an isolated vertex, we have $d(G) \leq d_t(G \Box K_2) \leq 2d_t(G)+1 \leq 2d(G)+1$ and $d(G \Box K_2) \leq 2d(G)+1$.

**Proof.** Let us consider the graph $G \Box K_2$. By Theorem 3.1 $d_t(G \Box K_2) \geq \max\{d(G), 2\} = d(G)$. Let $k = d_t(G \Box K_2)$ and let $v_1, v_2$ be the vertices of $K_2$. Let $D_1, D_2, \ldots, D_k$ be a total domatic partition of $V(G \Box K_2)$. It is easy to observe that for any two $i, j \in [k]$, the set $\{u : (u, v) \in D_i \cup D_j\}$ is a total dominating set of $G$. Thus $d(G) \geq d_t(G) \geq \left\lfloor \frac{k}{2} \right\rfloor$ which in turns that $k = d_t(G \Box K_2) \leq 2d_t(G) + 1 \leq 2d(G) + 1$. Similar arguments hold for the domatic partition of $V(G \Box K_2)$ and we get $d(G \Box K_2) \leq 2d(G) + 1$. \hfill $\Box$

There is an example of graph $G$ such that $d_t(G \Box K_2) > d(G)$. Let $G$ be a Peterson graph, we have $d_t(G) = d(G) = 2$ and $d_t(G \Box K_2) = 3$. The 3-total domatic coloring of $G \Box K_2$ has
been shown in Figure 1. By (iii) of Proposition 2.1, 4-total domatic coloring is not possible as $4^2 \nmid 40$, where $|E(G \boxtimes K_2)| = 40$. The upper bound $2d_t(G) + 1$ for $d_t(G \boxtimes K_2)$ is tight for graphs $K_n$, $n$ odd and $C_{3k}$, $4 \nmid k$. We could not find a graph $G$ such that $d_t(G \boxtimes K_2) = 2d(G)$. This leads us to ask the following question.

**Problem 3.7.** Let $G$ be a graph without an isolated vertex, find the smallest constant $c$ such that $d_t(G \boxtimes K_2) \leq cd(G) + O(1)$.

Let $G$ be the complement of a perfect matching. It is easy to observe that $d(G) = \frac{|V(G)|}{2}$ and $d(G \boxtimes K_2) = |V(G)| = 2d(G)$. Also, we could not find a graph $G$ such that $d(G \boxtimes K_2) = 2d(G) + 1$. This leads us to ask a question: Is there any graph $G$ such that $d(G \boxtimes K_2) > 2d(G)$?

Now, let us consider a lower bound for the domatic and total domatic numbers of Cartesian product of a graph and a bipartite graph in terms of its domatic and total domatic numbers. The bound given in Theorem 3.8 has been applied multiple times in this paper.

**Theorem 3.8.** Let $G$ be a graph and $H$ be a bipartite graph, we have

(i) $d_t(G \square H) \geq 2 \min \{d_t(G), d_t(H)\}$ and

(ii) $d(G \square H) \geq 2 \min \{d(G), d_t(H)\}$.

**Proof.** (i) Let $G$ be a graph of order $m$ and $H$ be a bipartite graph of order $n$ with bipartition $[X, Y]$. Let $\{u_i : i \in \mathbb{Z}_m\}$ and $\{v_j : j \in \mathbb{Z}_n\}$ be the vertices of $G$ and $H$ respectively. Let $k = \min \{d_t(G), d_t(H)\}$, there exists a total domatic coloring for $G$ and $H$ with $k$ colors. Let $g$ and $h$ be a $k$-total domatic coloring of $G$ and $H$ respectively, and let $\mathbb{Z}_k$ be the $k$ colors. Now, let us define a coloring $f$ for the vertices of $G \square H$. For $i \in \mathbb{Z}_m$, and $j \in \mathbb{Z}_n$,

$$f((u_i, v_j)) = \begin{cases} (2g(u_i) + 2h(v_j)) \pmod{2k} & \text{if } v_j \in X \\ (2g(u_i) + 2h(v_j) + 1) \pmod{2k} & \text{if } v_j \in Y. \end{cases}$$

For any vertex $u_i \in V(G)$ and $v_j \in X$, by the coloring $f$, the colors seen by the vertex $(u_i, v_j)$ from the neighbors of $u_i$ in $G_{v_j}$ are $\{(2s + 2h(v_j)) \pmod{2k} : s \in \mathbb{Z}_k\} = \{2l : l \in \mathbb{Z}_k\}$ and from the neighbors of $v_j$ in $H_{u_i}$ are $\{(2g(u_i) + 2t + 1) \pmod{2k} : t \in \mathbb{Z}_k\} = \{2l + 1 : l \in \mathbb{Z}_k\}$. Similarly, for any vertex $u_i \in V(G)$ and $v_j \in Y$, the colors seen by the vertex $(u_i, v_j)$ from the neighbors of $u_i$ in $G_{v_j}$ are $\{2l + 1 : l \in \mathbb{Z}_k\}$ and from the neighbors of $v_j$ in $H_{u_i}$ are $\{2l : l \in \mathbb{Z}_k\}$. Each vertex $(u_i, v_j)$ in $G \square H$ sees all the colors $\mathbb{Z}_{2k}$ in its open neighborhood and thus $f$ is a total domatic coloring using $2k$ colors. Hence $d_t(G \square H) \geq 2k = 2 \min \{d_t(G), d_t(H)\}$.

(ii) Let $k = \min \{d(G), d_t(H)\}$. Let $g$ be a $k$-domatic coloring of $G$ and $h$ be a $k$-total domatic coloring of $H$. For any vertex $u_i \in V(G)$ and $v_j \in X$, by the coloring $f$ defined in Equation (3.1), the colors seen by the vertex $(u_i, v_j)$ from the closed neighborhood of $u_i$ in $G_{v_j}$ are $\{(2s + 2h(v_j)) \pmod{2k} : s \in \mathbb{Z}_k\} = \{2l : l \in \mathbb{Z}_k\}$ and from the open neighborhood of $v_j$ in $H_{u_i}$ are $\{(2g(u_i) + 2t + 1) \pmod{2k} : t \in \mathbb{Z}_k\} = \{2l + 1 : l \in \mathbb{Z}_k\}$. Similarly, for any vertex $u_i \in V(G)$ and $v_j \in Y$, the colors seen by the vertex $(u_i, v_j)$ from
the closed neighborhood of $u_i$ in $G_{v_j}$ are $\{2l + 1 : l \in \mathbb{Z}_k\}$ and from the open neighborhood of $v_j$ in $H_{u_i}$ are $\{2l : l \in \mathbb{Z}_k\}$. Each vertex $(u_i, v_j)$ in $G \square H$ contains vertices of all the colors $\mathbb{Z}_{2k}$ in its closed neighborhood and thus $f$ is a domatic coloring using $2k$ colors. Hence $d(G \square H) \geq 2\min\{d(G), d(H)\}$.

If at least one of these graphs $G$, $H$ is disconnected, then apply the same technique to each component of $G \square H$ separately. □

The bound given in Theorem 3.8 for $d(G \square H)$ is tight, which follows by taking $G \cong K_n$ and $H \cong K_{n,n}$ and also, the bound for $d_t(G \square H)$ is tight by taking $G \cong K_{2n}$ and $H \cong K_{n,n}$. One of the simplest examples such that a strict inequality holds in Theorem 3.8 is $G$ when the closed neighborhood of $u$ includes $H$. Hence there exists at least $k$ rows and $n - \sqrt{2n}$ columns of $G$. For the graph $G$ Theorem 3.8 is not true for all graphs in general. For $G \cong K_2 \square K_3$, we have $d_t(G) = 3$.

Since $G \square G$ is 6-regular, the neighbors of each vertex should be colored distinctly when $d_t(G \square G) = 6$. There exists a color which occurs at least twice in the subgraph $K_3 \square K_3$ of $G \square G$ and hence there exists a vertex which contains the same color in its two neighbors, a contradiction. Thus $d_t(G \square G) < 6 = 2d_t(G)$. We can extend the above idea to show that there exists graph $G$ such that $d_t(G \square G) \leq d(G \square G) \leq d_t(G) + \sqrt{2d_t(G)}$.

**Proposition 3.9.** For the graph $G = K_n \square K_2$, $d_t(G \square G) \leq d(G \square G) \leq d_t(G) + \sqrt{2d_t(G)}$.

**Proof.** From Corollary 3.2 we know that $d_t(G) = d(G) = n$ and we will show that $d(G \square G) \leq n + \sqrt{2n}$. Suppose that there exists a domatic coloring $f$ of $G \square G$ using more than $n + \sqrt{2n}$ colors, then there exists a color (call it red) which appears in at most $4(n - \sqrt{2n})$ vertices of $G \square G$. Otherwise, we get a contradiction, since $4(n - \sqrt{2n} + 1)(n + \sqrt{2n} + 1) = 4(n^2 + 1) > |V(G \square G)|$.

Observe that $G \square G$ is isomorphic to $H \square C_4$, where $H = K_n \square K_n$. We label the vertices of $C_4$ with $\mathbb{Z}_4$ and the copy of $H$ corresponding to vertex $i$ of this $C_4$ will be called $H_i$. Let $R_i$ denote the set of vertices in $H_i$ which are colored red by $f$. Since $|R_0 \cup \cdots \cup R_3| \leq 4(n - \sqrt{2n})$, the smallest of them, without loss of generality say $R_0$, has at most $n - \sqrt{2n}$ vertices. Let $k = n - |R_0|$, and note that $k \geq \sqrt{2n}$.

The coloring $f$ on $H_0$ can be represented by an $n \times n$ matrix $M_0$. Since $|R_0| = n - k$, there exists at least $k$ rows and $k$ columns of $M_0$ which do not contain any red vertex. Hence the $k^2$ vertices corresponding to the $k \times k$ submatrix determined by the above rows and columns are neither red, nor they see a red neighbor in $H_0$. Hence each of them have a red neighbor in $H_1$ or $H_3$. Since each vertex in $V(H_1) \cup V(H_3)$ is adjacent to exactly one vertex in $V(H_0)$, we can conclude that $|R_1 \cup R_3| \geq k^2$. But now, since $|R_2| \geq |R_0|$, we get $|R_0 \cup \cdots \cup R_3| \geq 2(n - k) + k^2$. Since $k^2 - 2k$ is an increasing function of $k$ for $k \geq 1$, the above lower bound is at least $4n - 2\sqrt{2n}$ for any $k \geq \sqrt{2n}$. This contradicts our earlier upper bound on $|R_0 \cup \cdots \cup R_3|$. □

One of the consequences of Theorem 3.8 is Corollary 3.10.
Corollary 3.10. Let $d$ be a positive integer, $n$ and $k$ be powers of $2$, $k \leq n$. If $G$ is a graph with $d(G) \geq kd$ and for $1 \leq i \leq n$, $H_i$ is a bipartite graph such that $d_t(H_i) \geq d$, then $d_t(\bigoplus_{i=1}^{n} H_i) \geq nd$ and $d(G \bigoplus (\bigoplus_{i=1}^{n-k} H_i)) \geq nd$.

Proof. For each $j$ taken in the increasing order $1, 2, \ldots, \frac{n}{k}$, repeated application of Theorem 3.8 to pairs of bipartite graphs that have a total domatic coloring with $jd$ colors yields the bipartite graphs having a total domatic coloring with $2jd$ colors. Finally, we get $d_t(\bigoplus_{i=1}^{n} H_i) \geq nd$. Now, let $G$ be a graph such that $d(G) \geq kd$. First we group $H_1, H_2, \ldots, H_{n-k}$ into groups of size $k$ each and apply the above argument to get the bipartite graphs $H'_1, H'_2, \ldots, H'_{\frac{n}{k}-1}$ with $d_t(H'_j) \geq kd$, $1 \leq j \leq \frac{n}{k} - 1$. Now, consider the graphs in this order $G, H'_1, H'_2, \ldots, H'_{\frac{n}{k}-1}$. Apply Theorem 3.8(ii) to the first pair and Theorem 3.8(i) to the remaining pairs of graphs repeatedly as mentioned above to obtain the final result. Thus $d(G \bigoplus (\bigoplus_{i=1}^{n-k} H_i)) \geq \left(\frac{n}{k}\right)kd = nd$. \hfill \Box

4 Hamming graphs

Let us start this section by re-derive the results for the domatic number of the hypercubes $Q_{n-1}$ and $Q_n$ [31] and the total domatic number of the hypercube $Q_n$ [8], when $n$ is a power of 2.

Corollary 4.1. Let $n$ be a powers of 2, we have $d_t(Q_n) = d_t(Q_{n+1}) = n$ and $d(Q_{n-1}) = d(Q_n) = n$.

Proof. It is easy to see that $d_t(Q_n) \leq n$ and $d(Q_{n-1}) \leq n$. Since $d_t(K_2) = 1$, $d(K_2) = 2$, $Q_n \cong K_2 \bigoplus K_2$ and $Q_{n-1} \cong K_2 \bigoplus (K_2 \bigoplus K_2)$, by Corollary 3.10 we get $d_t(Q_n) \geq n$ and $d(Q_{n-1}) \geq n$. Suppose $d_t(Q_{n-1}) = n + 1$, by (iii) of Proposition 2.1 $n + 1$ should divides $2^{n+1}$, a contradiction. Thus $d_t(Q_{n-1}) \leq n$. Also, $d_t(Q_{n+1}) \geq d_t(Q_n) = n$. Similarly, by Theorem 2.2 we get $d(Q_n) \leq n$ and $d(Q_n) \geq d(Q_{n-1}) = n$. \hfill \Box

In [17], Havel obtained that $d(Q_6) = 5$. Also, $5 = d(Q_6) \leq d_t(Q_7) \leq \left\lfloor \frac{2^7}{24} \right\rfloor = 5$. We obtain a new lower bound for $d(Q_n)$ and $d_t(Q_n)$ for some values of $n$ in Corollary 4.2.

Corollary 4.2. Let $k, n$ be positive integers such that $n \geq 2^k7 - 1$, we have $d(Q_n) \geq 2^k5$ and $d_t(Q_{n+1}) \geq 2^k5$.

Proof. Let $G \cong Q_6$ and $H_i \cong Q_7$, for $1 \leq i \leq 2^k$. Clearly, $d(G) = 5$ and $d_t(H_i) = 5$. By Corollary 3.10, we get $d(G \bigoplus (\bigoplus_{i=1}^{2^k} H_i)) \geq 2^k5$ and $d_t(\bigoplus_{i=1}^{2^k} H_i) \geq 2^k5$. Since $n \geq 2^k7 - 1$, we have $d(Q_n) \geq 2^k5$ and $d_t(Q_{n+1}) \geq 2^k5$. \hfill \Box

When $n$ is a power of 2, we can also re-derive the following results on the domination and total domination numbers of the hypercubes $Q_{n-1}$ [16] and $Q_n$ [22] respectively.
Table 1: Bounds on $\gamma(n), \gamma_t(n), d(n)$ and $d_t(n), 1 \leq n \leq 17$. Lower bound for $d(n)$ follows from Corollary 3.10, $d_t(n)$ follows from Corollary 4.1 and Corollary 4.2.

| $n$ | $\gamma(Q_n)$ | $\gamma_t(Q_n)$ | $d(Q_n)$ | $d_t(Q_n)$ |
|-----|----------------|-----------------|-----------|------------|
| 1   | 1              | 2               | 2         | 1          |
| 2   | 2              | 2               | 2         | 2          |
| 3   | 2              | 4               | 4         | 2          |
| 4   | 4              | 4               | 4         | 4          |
| 5   | 7              | 8               | 4         | 4          |
| 6   | 12             | 14              | 5         | 4          |
| 7   | 16             | 24              | 8         | 5          |
| 8   | 32             | 32              | 8         | 8          |
| 9   | 62             | 64              | 8         | 8          |
| 10  | 107-120        | 124             | 8-9       | 8          |
| 11  | 180-192        | 214-240         | 8-11      | 8-9        |
| 12  | 342-380        | 360-384         | 8-11      | 8-11       |
| 13  | 598-704        | 684-760         | 10-13     | 8-11       |
| 14  | 1171-1408      | 1196-1408       | 10-13     | 10-13      |
| 15  | $2^{11}$       | 2342-2816       | 16        | 10-13      |
| 16  | $2^{12}$       | $2^{12}$        | 16        | 16         |
| 17  | 7377-2$^{13}$  | $2^{13}$        | 16-17     | 16         |

**Corollary 4.3.** [16, 22] For a positive integer $k$, $\gamma(Q_{2^k-1}) = 2^{2^k-k-1}$, $\gamma_t(Q_{2^k}) = 2^{2^k-k}$, $\gamma(Q_{2^k}) \leq 2^{2^k-k}$ and $\gamma_t(Q_{2^k+1}) \leq 2^{2^k-k+1}$.

**Proof.** By Corollary 2.3, 4.1 and (iv) of Proposition 2.1, we get $\gamma(Q_{2^k-1}) = 2^{2^k-k-1}$ and $\gamma_t(Q_{2^k}) = 2^{2^k-k}$ respectively. Also, by Corollary 4.1, we have $\gamma(Q_{2^k}) \leq 2^{2^k-k} = 2^{2^k-k+1}$ and $\gamma_t(Q_{2^k+1}) \leq 2^{2^k-k+1}$.

Note that $\gamma(Q_{2^k}) = 2^{2^k-k}$ follows from the sphere bound mentioned in [28] and $\gamma_t(Q_{2^k+1}) = 2^{2^k-k+1}$ follows from the result proved by Azarija et al., [3] namely, $\gamma_t(Q_{n+1}) = 2\gamma(Q_n)$.

In Table 1, we have mentioned the present best bounds for $\gamma(n)$ (see, Table 1 in [4,28]) and $\gamma_t(n)$ follows from the result $\gamma_t(n) = 2\gamma(Q_{n-1})$ [3]. The bounds for $d(n)$ and $d_t(n)$ follows from Corollary 4.1 (also see, [8,31]) and Corollary 4.2.

Now, let us start obtain a lower bound for the domatic and total domatic numbers of Hamming graphs $H_{n-1,q}$ and $H_{n,q}$ respectively when $n$ is a power of 2 and $q \geq 2$.

**Theorem 4.4.** Let $q$ be an integer greater than 1 and $n$ a power of 2, we have $d(H_{n-1,q}) \geq n \left\lceil \frac{q}{2} \right\rceil$ and $d_t(H_{n,q}) \geq n \left\lceil \frac{q}{2} \right\rceil$.

**Proof.** Let $K_{\left\lceil \frac{q}{2} \right\rceil, \left\lfloor \frac{q}{2} \right\rfloor}$ be a complete bipartite subgraph of $K_q$. Any coloring which assigns same set of $\left\lceil \frac{q}{2} \right\rceil$ different colors to each part of $K_{\left\lceil \frac{q}{2} \right\rceil, \left\lfloor \frac{q}{2} \right\rfloor}$ is a total domatic coloring with $\left\lceil \frac{q}{2} \right\rceil$ colors. Since $n$ is a power of 2, by Corollary 3.10, we have $d_t(H_{n,q}) = d_t(\bigcup_{i=1}^{n} K_q) \geq d_t(\bigcup_{i=1}^{n} K_{\left\lceil \frac{q}{2} \right\rceil, \left\lfloor \frac{q}{2} \right\rfloor}) = n \left\lceil \frac{q}{2} \right\rceil$. Also, $d(K_q) = q \geq 2 \left\lceil \frac{q}{2} \right\rceil$, by Corollary 3.10, we have $d(H_{n-1,q}) = d(\bigcup_{i=1}^{n-1} K_q) \geq d(K_q \bigcup (\bigcup_{i=1}^{n-2} K_{\left\lceil \frac{q}{2} \right\rceil, \left\lfloor \frac{q}{2} \right\rfloor})) \geq n \left\lceil \frac{q}{2} \right\rceil$. 

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Note that an equality holds for the graphs $H_{1,q}$, $H_{2,2q}$, and the problem remains open for $H_{n,q}$ $n \geq 4$ except $H_{n,2}$.

In [8] Chen et al., obtained the injective chromatic number of $H_{n,q}$ as follows.

**Theorem 4.5.** [8] Let $k, n$ be positive integers. If $q$ is a prime power and $n = \frac{(q^k - 1)}{(q-1)}$, then $\chi_i(H_{n,q}) = \chi(H_{n,q}^2) = q^k$.

**Corollary 4.6** is an immediate consequence of Theorem 3.1 and Theorem 4.5.

**Corollary 4.6.** Let $k, n$ be positive integers. If $q$ is a prime power and $n = \frac{(q^k - 1)}{(q-1)}$, then $d(H_{n,q}) = q^k$ and $d_t(H_{n+1,q}) \geq q^k$.

**Proof.** By Theorem 4.5, we have $\chi_i(H_{n,q}) = \chi(H_{n,q}^2) = q^k = n(q - 1) + 1$. Since $H_{n,q}$ is $n(q - 1)$-regular and there is a proper injective coloring with $n(q - 1) + 1$ colors, by Corollary 2.3 we get $d(H_{n,q}) = q^k$. By Theorem 3.1 we have $d_t(H_{n+1,q}) = d_t(H_{n,q} \square K_q) \geq d(H_{n,q}) = q^k$. □

As a consequence of Theorem 3.1 4.4 and Corollary 3.10 4.6 we get Corollary 4.7

**Corollary 4.7.** Let $n$ be a positive integer and $q$ be a prime power. If $k$ is the largest positive integer such that $\frac{q^k}{q-1} \leq n$, $j$ is the smallest positive integer such that $q^k \leq 2^j \left\lfloor \frac{q}{2} \right\rfloor$ and $i$ is the largest positive integer such that $\frac{q^i - 1}{q-1} + (2^i - 1)2^j \leq n$, then $d(H_{n,q}) \geq 2^i q^k$ and $d_t(H_{n+1,q}) \geq 2^i q^k$.

**Proof.** Let $n' = \frac{q^k}{q-1}$ and $G \cong H_{n',q}$. For $0 \leq l < 2^i$, let $G_l \cong H_{2^i,q}$. It is clear from the proof of Theorem 4.4 there exists a spanning bipartite subgraph of $G_l$ that have a total domatic coloring with $2^i \left\lfloor \frac{q}{2} \right\rfloor$ colors, let it be $H_l$. Clearly, $d_t(H_l) \geq 2^i \left\lfloor \frac{q}{2} \right\rfloor \geq q^k$ and by Corollary 4.6, $d(G) = q^k$. By Corollary 3.10 we get $d(G \square (\bigcup_{l=1}^{2^i} H_l)) \geq 2^i q^k$. Since $n \geq n' + (2^i - 1)2^j$, we have $d(H_{n,q}) \geq d(H_{n',q} \square (\bigcup_{l=1}^{2^i-1} G_l)) \geq d(G \square (\bigcup_{l=1}^{2^i-1} H_l)) \geq 2^i q^k$. By Theorem 3.1 we have $d_t(H_{n+1,q}) \geq d(H_{n,q}) \geq 2^i q^k$. □

## 5 Tori

In this section, we examine the $d$-dimensional tori which are domatically and total dominically full. First we obtain some sufficient conditions for the tori which are total dominically full.

**Corollary 5.1.** Let $d, k_1, k_2, \ldots, k_d$ be positive integers such that $d$ is a power of 2 and $k_i \equiv 0 \pmod{4}$, $1 \leq i \leq d$, we have $d_t(\bigcup_{i=1}^{d} C_{k_i}) = 2d$ and $\gamma_t(\bigcup_{i=1}^{d} C_{k_i}) = \frac{d}{2}$.
In the remaining part of this section, we try to generalize this result to larger collections of tori.

In [13], Gravier independently obtained the total domination number of some tori by the concept of periodic tiling which mentioned in [14].

**Theorem 5.2.** [13] Let \( d, k_1, k_2, \ldots, k_d \) be positive integers such that \( d \geq 2 \) and \( k_i \equiv 0 \pmod{4} \), for \( 1 \leq i \leq d \), we have \( \gamma_t(\Box \bigcup_{i=1}^{d} C_{k_i}) = (\prod_{i=1}^{d} k_i)/2d \). Moreover, if \( d \) is even and for any positive integer \( k_i \), \( 1 \leq i \leq d \) such that \( k_i \equiv 0 \pmod{2d} \), then this equality still holds.

By vertex transitivity of the tori, one can obtain a total domatic coloring of the tori mentioned in Theorem 5.2 with \( 2d \) colors. Hence this tori are total domatically full. We extend this to larger classes of tori. Moreover, our proof is much shorter.

**Theorem 5.3.** Let \( d, k_1, k_2, \ldots, k_d \) be positive integers. If \( k_d \) is congruent to 0 \pmod{4} and the remaining \( k_i \)'s are congruent to 0 \pmod{2d} \), then \( d_t(\Box \bigcup_{i=1}^{d} C_{k_i}) = 2d \).

**Proof.** Let \( G \cong \Box \bigcup_{i=1}^{d} C_{k_i} \), where \( d, k_i \)'s are positive integers. Since \( \delta(G) = 2d \), it is enough to give a \( 2d \)-total domatic coloring for \( G \). Let the vertices of \( G \) be \( \{x = (x_1, x_2, \ldots, x_d) : x_i \in \mathbb{Z}_{k_i}, 1 \leq i \leq d\} \). Now, let us define a coloring \( f \) by

\[
  f(x) = \begin{cases} 
  \sum_{i=1}^{d-1} i x_i \pmod{2d} & \text{if } x_d \equiv 0 \text{ or } 1 \pmod{4} \\
  (\sum_{i=1}^{d-1} i x_i) + d \pmod{2d} & \text{if } x_d \equiv 2 \text{ or } 3 \pmod{4}.
  \end{cases}
\]

Let \( k \) be the color of the vertex \( x = (x_1, x_2, \ldots, x_d) \) defined by \( f \). The set of neighbors of \( x \) are \( \{(x_1, \ldots, x_{i-1}, x_i-1, x_{i+1}, \ldots, x_d), (x_1, \ldots, x_{i-1}, x_i+1, x_{i+1}, \ldots, x_d) \} : 1 \leq i \leq d \}. \) The set of colors in the neighbors of \( x \) are the union of \( \{k-i, k+i\} \) along the dimension \( i, 1 \leq i \leq d-1 \) and \( \{k, k+d\} \) along the dimension \( d \) which equals \( \mathbb{Z}_{2d} \). Thus each vertex contains vertices of all \( 2d \) colors \( \mathbb{Z}_{2d} \) in its open neighborhood and \( f \) is a \( 2d \)-total domatic coloring for \( G \). Hence, \( d_t(\Box \bigcup_{i=1}^{d} C_{k_i}) = 2d \).

In Theorem 5.3, \( d \)-dimensional tori are total domatically full when a cycle of length is a multiple of \( 4 \) and other cycles length are a multiple of \( 2d \) but this condition can be further generalized. We obtain a sufficient condition for the total domatically full tori in Corollary 5.4 which generalize the results mentioned in Corollary 5.1 and Theorem 5.3.

**Corollary 5.4.** Let \( d = 2^p q \), \( q \) be an odd integer and \( p \geq 0 \). If \( 2^p \) number of \( k_i \)'s are congruent to 0 \pmod{4} and the remaining \( k_i \)'s are congruent to 0 \pmod{2q} \), then \( d_t(\Box \bigcup_{i=1}^{d} C_{k_i}) = 2d \).

**Proof.** Let \( G \cong \Box \bigcup_{i=1}^{d} C_{k_i} \), where \( d, k_i \)'s are positive integers. For \( p = 0 \), the proof follows from Theorem 5.3. Let us consider \( p \geq 1 \). Since \( \Box \bigcup_{i=1}^{d} C_{k_i} \) is transitive with respect to the product, among \( 2^p \) number of \( k_i \)'s are congruent to 0 \pmod{4} \), without loss of generality, choose those \( k_i \)'s are \( k_q, k_{2q}, \ldots, k_d \). Now, split the graph \( \Box \bigcup_{i=1}^{d} C_{k_i} \) into product of \( 2^p \) smaller
product of $q$ graphs each as $(\square C_{k_i}) \square \ldots \square (\square C_{k_i})$, where $d = 2^p q$. By Theorem 5.3, each smaller product graph $G_j$ has the total domatic number $2q$ for $1 \leq j \leq 2^p$. By Corollary 5.10 we have $d_t(\square C_{k_i}) = d_t(\square G_j) = 2^p 2q = 2d$.

The Corollary 5.5 is an immediate consequence of Corollary 5.4 and (iv) of Proposition 2.1 which turns to be a generalization of the result given in Corollary 5.1 and a result due to S. Gravier [13] for the $d$-dimensional tori given in Theorem 5.2.

**Corollary 5.5.** Let $d = 2^p q$, $q$ be an odd integer and $p \geq 0$. If $2^p$ number of $k_i$'s are congruent to 0 (mod 4) and the remaining $k_i$'s are congruent to 0 (mod 2q), then $\gamma_t(\square C_{k_i}) = (\prod_{i=1}^{d} k_i)/2d$.

Note that, the family of tori given in Corollary 5.5 contains the family of tori as mentioned in Theorem 5.2. Suppose the dimension of the torus is 12; Corollary 5.5 finds $\gamma_t$ for the tori $\square C_{p_i}$, where each $p_i$ is congruent to 0 (mod 4) and the remaining $p_i$'s are congruent to 0 (mod 6) but Theorem 5.2 finds $\gamma_t$ for the torus $\square C_{k_i}$, where each $k_i \equiv 0$ (mod 24).

Corollary 5.6 is an immediate consequence of Corollary 5.4, 5.5.

**Corollary 5.6.** Let $d = 2^p q$, $q$ be an odd integer and $p \geq 0$ and let $G_i$ be a Hamiltonian graph, $1 \leq i \leq d$. If $2^p$ number of $G_i$'s have $|V(G_i)| \equiv 0$ (mod 4) and the remaining $G_i$'s have $|V(G_i)| \equiv 0$ (mod 2q), then $d_t(\square G_i) \geq 2d$ and $\gamma_t(\square C_{k_i}) \leq (\prod_{i=1}^{d} |V(G_i)|)/2d$.

Next, we obtain a sufficient condition for the domatically full tori.

**Theorem 5.7.** Let $d, k_1, k_2, \ldots, k_d$ be positive integers. If each $k_i$ is congruent to 0 (mod 2d + 1), then $d(\square C_{k_i}) = 2d + 1$.

**Proof.** Let $G \cong \square C_{k_i}$, where $d, k_i$'s are positive integers and each $k_i \equiv 0$ (mod 2d + 1). Since $\delta(G) + 1 = 2d + 1$, it is enough to give a (2d + 1)-domatic coloring for $G$. Let the vertices of $G$ be $\{x = (x_1, x_2, \ldots, x_d) : x_i \in \mathbb{Z}_{k_i}, 1 \leq i \leq d\}$. Now, let us define a coloring $f$ by

$$f(x) = \sum_{i=1}^{d} i x_i \mod (2d + 1).$$

Let $k$ be the color of the vertex $x = (x_1, x_2, \ldots, x_d)$ defined by $f$. The set of neighbors of $x$ are $\{(x_1, \ldots, x_{i-1}, x_i - 1, x_{i+1}, \ldots, x_d), (x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_d) : 1 \leq i \leq d\}$. The set of colors in the neighbors of $x$ are $\{k - i, k + i\}$ along the dimension $i, 1 \leq i \leq d$ which equals $\mathbb{Z}_{2d+1} \setminus \{k\}$. Thus each vertex contains vertices of all 2d + 1 colors $\mathbb{Z}_{2d+1}$ in its closed neighborhood. Hence $f$ is a (2d + 1)-domatic coloring for $G$ and $d(\square C_{k_i}) = 2d + 1$. 

\[\square\]
As a consequence of Theorem 5.7 and Corollary 2.3, we get Corollary 5.8 which re-
derive the result proved by Klavžar and Seifter [23].

**Corollary 5.8.** [23] Let \(d, k_1, k_2, \ldots, k_d\) be positive integers. If each \(k_i\) is congruent to 0 (mod \(2d + 1\)), then \(\gamma(\square_{i=1}^{d} C_{k_i}) = (\prod_{i=1}^{d} k_i)/(2d + 1)\).

Corollary 5.9 is an immediate consequence of Theorem 5.7 and Corollary 5.8.

**Corollary 5.9.** Let \(d\) be a positive integer and let \(G_i\) be a Hamiltonian graph, \(1 \leq i \leq d\). If each \(G_i\) have \(|V(G_i)| \equiv 0 \pmod{2d + 1}\), then \(d(\square_{i=1}^{d} G_i) \geq 2d + 1\) and \(\gamma(\square_{i=1}^{d} G_i) \leq (\prod_{i=1}^{d} |V(G_i)|)/(2d + 1)\).

### 6 Conclusion and open problems

Accepting the invitation by Chen, Kim, Tait and Verstraete [8] to determine any relationships between \(d_t(G \Box G)\) and \(d_t(G)\), we started this investigation aiming to find good lower and upper bounds to \(d_t(G \Box G)\) in terms of \(d_t(G)\). In this paper, we have made improvements to the easy lower bound \(d_t(G \Box G) \geq d_t(G)\). Firstly we showed that if \(\delta(G) \geq 1\), then \(d_t(G \Box G) \geq d(G)\). We also showed that \(d_t(G \Box G) \geq 2d_t(G)\) if \(G\) is bipartite. Bipartiteness is necessary for the above lower bound. We can show the existence of infinite families of non-bipartite graphs where \(d_t(G \Box G) = d_t(G) + \sqrt{2d_t(G)}\). Nevertheless, we have indirectly used this result for non-bipartite graphs in the form \(d_t(G \Box G) \geq 2d_t(G')\), where \(G'\) is a spanning bipartite subgraph of \(G\).

In contrast, we haven’t been able to prove any upper bound for \(d_t(G \Box G)\) in terms of \(d_t(G)\). We know of graphs \(G\) (\(G = K_{2n+1}\) for example) where \(d_t(G \Box G) = 2d_t(G) + 1\). We conjecture that it is the maximum possible.

**Conjecture 6.1.** For any two graphs \(G\) and \(H\) without an isolated vertex,

\[d_t(G \Box H) \leq 2 \max\{d_t(G), d_t(H)\} + 1.\]

A weaker form of the above conjecture, \(d_t(G \Box H) \leq 2 \max\{d(G), d(H)\} + 1\), is also an interesting open problem. As far as we know, the version of Conjecture 6.1 for domatic number is also open.

**Conjecture 6.2.** For any two graphs \(G\) and \(H\) without an isolated vertex,

\[d(G \Box H) \leq 2 \max\{d(G), d(H)\} + 1.\]

If this upper bound is true, then it is also tight. To see that, consider the cycle \(C_n\), where \(n\) is a multiple of 5 but not 3. By Theorem 5.7, we have \(d(C_n \Box C_n) = 5 = 2d(C_n) + 1\).
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References

[1] S. Akbari, M. Motiei, S. Mozaffari, and S. Yazdanbod, *Cubic graphs with total domatic number at least two*, Discuss. Math. Graph Theory 38 (2018), 75–82.

[2] H. Aram, S. M. Sheikholeslami, and L. Volkmann, *On the total domatic number of regular graphs*, Trans. Comb. 1 (2012), 45–51.

[3] J. Azarija, M. A. Henning, and S. Klavžar, *(Total) domination in prisms*, Electron. J. Combin. 24 (2017), 1–19.

[4] R. Bertolo, P. R. J. Östergård, and W. D. Weakley, *An updated table of binary/ternary mixed covering codes*, J. Combin. Des. 12 (2004), 157–176.

[5] I. Bouchemakh and S. Ouatiki, *On the domatic and the total domatic numbers of the 2-section graph of the order-interval hypergraph of a finite poset*, Discrete Math. 309 (2009), 3674–3679.

[6] B. Brešar, T. R. Hartinger, T. Kos, and M. Milanič, *On total domination in the Cartesian product of graphs*, Discuss. Math. Graph Theory 38 (2018), 963–976.

[7] Y. Bu, D. Chen, A. Raspaud, and W. Wang, *Injective coloring of planar graphs*, Discrete Appl. Math. 157 (2009), 663–672.

[8] B. Chen, J. H. Kim, M. Tait, and J. Verstraete, *On coupon colorings of graphs*, Discrete Appl. Math. 193 (2015), 94–101.

[9] E. J. Cockayne, R. M. Dawes, and S. T. Hedetniemi, *Total domination in graphs*, Networks 10 (1980), 211–219.

[10] E. J. Cockayne and S. T. Hedetniemi, *Towards a theory of domination in graphs*, Networks 7 (1977), 247–261.

[11] D. W. Cranston, S.-J. Kim, and G. Yu, *Injective colorings of sparse graphs*, Discrete Math. 310 (2010), 2965–2973.

[12] W. Goddard and M. A. Henning, *Thoroughly dispersed colorings*, J. Graph Theory 88 (2018), 174–191.

[13] S. Gravier, *Total domination number of grid graphs*, Discrete Appl. Math. 121 (2002), 119–128.

[14] S. Gravier, M. Mollard, and C. Payan, *Variations on tilings in the Manhattan metric*, Geom. Dedicata 76 (1999), 265–273.

[15] G. Hahn, J. Kratochvíl, J. Širáň, and D. Sotteau, *On the injective chromatic number of graphs*, Discrete Math. 256 (2002), 179–192.
[16] F. Harary and M. Livingston, *Independent domination in hypercubes*, Appl. Math. Lett. 6 (1993), 27–28.

[17] I. Havel, *Domination in n-cubes with diagonals*, Math. Slovaca 48 (1998), 105–115.

[18] P. Heggernes and J. A. Telle, *Partitioning graphs into generalized dominating sets*, Nordic J. Comput. 5 (1998), 128–142.

[19] M. A. Henning, *A survey of selected recent results on total domination in graphs*, Discrete Math. 309 (2009), 32–63.

[20] M. A. Henning and D. F. Rall, *On the total domination number of Cartesian products of graphs*, Graphs Combin. 21 (2005), 63–69.

[21] P. T. Ho, *A note on the total domination number*, Util. Math. 77 (2008), 97–100.

[22] S. M. Johnson, *A new lower bound for coverings by rook domains*, Util. Math. 1 (1972), 121–140.

[23] S. Klavžar and N. Seifter, *Dominating Cartesian products of cycles*, Discrete Appl. Math. 59 (1995), 129–136.

[24] M. Koivisto, P. Laakkonen, and J. Lauri, *NP-completeness results for partitioning a graph into total dominating sets*, Theoret. Comput. Sci. 818 (2020), 22–31.

[25] B. Lužar, R. Škrekovski, and M. Tancer, *Injective colorings of planar graphs with few colors*, Discrete Math. 309 (2009), 5636–5649.

[26] Z. L. Nagy, *Coupon-coloring and total domination in Hamiltonian planar triangulations*, Graphs Combin. 34 (2018), 1385–1394.

[27] S. Oh, H. Yoo, and T. Yun, *Rainbow graphs and switching classes*, SIAM J. Discrete Math. 27 (2013), 1106–1111.

[28] G. J. M. Wee, *Improved sphere bounds on the covering radius of codes*, IEEE Trans. Inform. Theory 34 (1988), 237–245.

[29] A. J. Woldar, *Rainbow graphs*, Codes and designs (Columbus, OH, 2000), Ohio State Univ. Math. Res. Inst. Publ., vol. 10, de Gruyter, Berlin, 2002, pp. 313–322.

[30] B. Zelinka, *Domatically critical graphs*, Czechoslovak Math. J. 30 (1980), 486–489.

[31] B. Zelinka, *Domatic numbers of cube graphs*, Math. Slovaca 32 (1982), 117–119.

[32] B. Zelinka, *Regular totally domatically full graphs*, Discrete Math. 86 (1990), 71–79.

[33] B. Zelinka, *Total domatic number and degrees of vertices of a graph*, Math. Slovaca 39 (1989), 7–11.