About an efficiency functional implementing the principle of least effort

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Abstract

A probabilistic functional of efficiency has been proposed recently in order to implement the principle of least effort and to derive Zipf-Pareto’s laws with a calculus of variation. This work is a further investigation of this efficiency measure from mathematical point of view. We address some key mathematical properties of this functional such as its unicity, its robustness against small variation of probability distribution and its relationship with inequality as well as probabilistic uncertainty. In passing, a method for calculating non-negative continuous (differential) entropy is proposed based upon a generalized definition of informational entropy called varentropy.

Keywords: Least effort, Maximum efficiency, Zipf’s law, Pareto’s law, Entropy
1) Introduction

Achieving more by doing less. Everybody would agree with this proverb describing a common truth many living systems if not all. Guillaume Ferrero, a French philosopher, interpreted the mental inertia of human being using this truth he referred to as principle of least effort (PLE) [1]. Later on Zipf applied it to a quantitative study of language property, writing: “The power laws in linguistics and in other human systems reflect an economical rule: everything carried out by human being and other biological entities must be done with least effort (at least statistically)” [2][3]. This principle of least effort has inspired huge effort to derive power laws from the idea of minimization of effort, or to check the applicability of PLE to different living systems. The reader is referred to the brief review in [4] and to the references there in. Nevertheless, to our knowledge, there is no, to date, derivation of power law directly from PLE through calculus of variation minimizing a measure of effort. This is quite understandable because ‘effort’ is a very vague concept whose meaning and nature differ in different domains. It can be an expenditure of (mechanical or biological) energy, of time, of information, of an amount of money and so on. Sometimes effort can be very abstract things such as spiritual, mental and mindful attempt. The absence of a precise definition of ‘effort’ is, to our opinion, the main obstacle to the implementation of PLE with variational calculus.

Regarding derivation of power laws through variational approach, it is worth mentioning a series of constructive efforts in the past several decades in the framework of generalized statistics including the nonextensive statistics based on Thallis and Rényi entropies [5]-[8], κ-statistics [9], superstatistics and others [8][10]. The main idea here is to extend the Boltzmann-Gibbs statistical mechanics with the conventional logarithmic entropy and exponential probability, to more general formalisms allowing the derivation of power law probability distributions by using variational calculus of maximum entropy [5]-[9] or by considering the effect of thermal fluctuation [10]. It is worth noticing that this development of statistics theory is mainly guided by the principle of maximum informational entropy [11], and not directly associated with the idea of least effort.

We recently proposed a derivation of Zipf and Pareto laws using a calculus of variation implementing PLE [12]. The aim is to yield power law directly from PLE using an efficiency functional (referred to as Zipf-Pareto efficiency). In this approach, ‘least effort’ is implemented by ‘maximal efficiency’ following two considerations. First, efficiency is easier to be defined than effort. For the performance of living systems, we can in fact borrow the concept of efficiency of thermal engine in thermodynamics. Second, maximum efficiency is in practice
equivalent to least effort for the following reason: efficiency is proportional to the ratio of outcome (achievement) to effort (expenditure), maximum efficiency implies either maximization of outcome for given effort or minimization of effort for given outcome. We explain below briefly that these two extremum principles can be formulated in a single calculus of variation.

The idea of our previous work is the following: a living agent, or a system composed of living agents, can be regarded as a working engine spending heat $Q$ (expenditure) and providing mechanical work $W$ (achievement) with an efficiency defined by $\eta = \frac{W}{Q}$, or, inversely, spending a work $W$ and creating heat $Q$, with an efficiency defined by $\eta' = \frac{Q}{W}$. As $\eta$ and $\eta'$ have similar physical and mathematical properties [12], $\eta = \frac{W}{Q}$ will be considered throughout this work.

For living systems, the achievement $W$ can be anything they desire to obtain or fulfill, such as food, income, wealth, city population, firm size, frequency of events and words, information and so forth. Similarly, the expenditure $Q$ can also be anything a living agent consume in order to achieve what he desires, such as physical or even mental effort, energy, time, materials, money, information, etc. PLE applied to the efficiency $\eta = \frac{W}{Q}$ then means the maximization of $W$ for given $Q$ or the minimization of $Q$ for given $W$. This discussion gives a glimpse of the systems to which the idea of the present work can be considered applicable.

In the previous work [12], the efficiency functional was derived from a general nonadditive property of the thermodynamic efficiency of heat engine. A brief review of that approach is given below. The aim of this work is to investigate some important properties of the ZP efficiency, including its unicity given the nonadditivity of the efficiency, its robustness against small variation of probability distribution and its relationship with inequality as well as probabilistic uncertainty and entropy. In passing, we address a century old problem of negative continuous (differential) entropy using a generalized definition of informational entropy called varentropy [22][23].

2) The probabilistic functional of efficiency

It is well known that the efficiency in thermodynamics is not an additive quantity. Suppose two Carnot engines, the first engine absorbs an energy $Q_1$ from the hot heat bath, delivers a work $W_1$, and rejects an energy $Q_2$ to a cold heat bath, with an efficiency $\eta_1 = \frac{W_1}{Q_1} = 1 - \frac{Q_2}{Q_1}$;
and the second engine absorbs the energy $Q_2$, delivers a work $W_2$, and rejects an energy $Q_3$, getting the efficiency $\eta_2 = \frac{W_2}{Q_2} = 1 - \frac{Q_3}{Q_2}$. The overall efficiency $\eta$ of the ensemble of two engines is defined by

$$\eta = \frac{W_1 + W_2}{Q_1} = 1 - \frac{Q_3}{Q_1}$$

It is straightforward to calculate

$$\eta = \eta_1 + \eta_2 - \eta_1 \eta_2.$$  

(1)

Lord Kelvin rewrite Eq.(1) as $(1 - \eta) = (1 - \eta_1)(1 - \eta_2)$. This form of the efficiency nonadditivity helped Kelvin to discover the absolute temperature, which in turn leads to the discovery of the second law of thermodynamics about 170 years ago.

Notice that Carnot engine is an ideal machine functioning in reversible process without loss of energy. Real engines always lose a part of energy by friction, vibration or thermal radiation. A real engine cannot transform all the heat cost into work with the ideal relationship $Q_1 - Q_2 = W_1$. We introduce in this case a loss coefficient $a$ in such a way to write

$$\eta = \eta_1 + \eta_2 + a \eta_1 \eta_2$$  

(2)

or $(1 + a \eta) = (1 + a \eta_1)(1 + a \eta_2)$. The reversible case of Carnot engine is recovered for $a = -1$.

Now if a large number of real engines (living agents) functioning randomly altogether form a single big system, and all agents in the ensemble are making effort to get as much as possible a measurable quantity represented by a random variable $X$ having $w$ discrete values $x_i$ with $i = 1,2,...,w$. More they get that quantity, larger is $x_i$. This quantity can be any achievement like food, income, wealth, city population, frequency of events etc. as mentioned above. At equilibrium (or stationary) states of the whole systems, all agents are distributed over the whole range of $X$ with $n_i$ agents at the value $x_i$. We have $\sum_{i=1}^{w} n_i = N$.

The probability $p_i$ of finding an agents at the value $x_i$ is $p_i = \frac{n_i}{N}$ obeying the normalization $\sum_{i=1}^{w} p_i = 1$.

Due to the statistical nature of the model with a large number of agents distributed over all the values of $X$, it is reasonable to suppose that the total efficiency $\eta_i$ of the agents on the value

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1 https://en.wikipedia.org/wiki/Second_law_of_thermodynamics
depends on the number \( n_i \) with \( \eta_i = f(n_i) \) or on the probability distribution \( \eta_i = f(p_i) \).

The average efficiency \( \eta \) of the whole system reads \( \eta = \sum_{i=1}^{w} p_i \eta_i \).

Now let us separate the whole ensemble of agents into two independent subsystems \( A \) and \( B \), with efficiency \( \eta_k(A) \) and \( \eta_j(B) \), respectively. The probability distribution of the agents in \( A \) is \( p_k(A) \) and that in \( B \) is \( p_j(B) \). The probability distribution of the whole ensemble can be written as

\[
p_i = p_k(A)p_j(B) \quad \text{with} \quad i = kj
\]

We choose Eq.(2) as the efficiency nonadditivity. This implies a total efficiency given by

\[
\eta_i = \eta_k(A) + \eta_j(B) + a \eta_k(A) \eta_j(B)
\]

or \((1 + a \eta_i) = [1 + a \eta_k(A)][1 + a \eta_j(B)]\).

It can be proved (proof given below) that Eq.(3) and Eq.(4) uniquely leads to \((1 + a \eta_i) = p_i^b \) or \( \eta_i = \frac{p_i^{b-1}}{a} \). From some mathematical consideration \([12]\), we choose \( b = -a \). The average efficiency of the whole ensemble of \( N \) agents reads

\[
\eta = \sum_{i=1}^{w} p_i \eta_i = \frac{\sum p_i^{1-a} - 1}{a}
\]

in which the normalization \( \sum p_i = 1 \) is considered. The concave property of this formula is shown in Figure 1 for \( w = 2 \) and several values of \( a \) in the interval \( 0 \leq a \leq 1 \). \( \eta \) is a monotonically increasing function of \( a \) and concave in this domain with its extremum at \( p = \frac{1}{2} \). It becomes a constant independent of probability distribution when \( a \to 1 \). But this case does not happen because \( a \) will be limited to \( 0 < a < 0.5 \) for mathematical consideration.

In our previous work \([12]\), the efficiency of Eq.(5) was used to implement PLE following the idea that if the maximum efficiency is achieved, then the least effort implies the achievement of the system, represented by the average \( X = \sum p_i x_i \), should be maximal as well. In other words, the maximum of \( \eta \) and the maximum of \( X \) are mutually conditioned by PLE, which is thus implemented in the calculus of variation \( \delta(\eta + cX) = 0 \) where \( c \) is a positive multiplier.

This calculus straightforwardly leads to Pareto law \( p(X > x) = \left(\frac{x_{\text{min}}}{x}\right)^{\frac{1}{a}-1} \) and Zipf law \( x_r = \frac{x_1}{r^\alpha} \) where \( r \) is the rank of the achievement \( x_r \) (frequency of words for instance), \( \alpha = a \gamma \) and \( \gamma \) a constant characterizing the relationship between the derivative of Pareto law \( p'(X > x) \) and the rank \( r \) of \( x : p'(X > x) \propto r^\gamma \). The reader is referred to \([12]\) for the details of the variational calculus.
Figure 1. Variation of the efficiency as a function of the probability distribution over two states $p_1 = p, p_2 = 1 - p$ for different values of $a$ in the interval $0 \leq a \leq 1$. $\eta$ is a monotonically increasing function of $a$ and concave in this domain with its extremum at $p = \frac{1}{2}$.

Following the universal character of PLE for all living systems and the starting hypothesis of this work that living agents can be considered as thermal engine in term of their efficiency, we hope that this approach is meaningful to most, if not all, living systems and systems driven by living agents that obey PLE and show power law behaviors. We can cite, among others, economic systems, linguistic systems, informational and social networks and so on, which are systems driven by the effort of human beings to achieve something. For example, in economic system in term of income, income is the outcome $X$ people are looking for with effort. People must try in general to achieve more income in doing less effort, leading to Pareto law in the distribution of income [13]. In the linguistic systems, similar thing happens since people try not to be talkative (statistically, of course) and to express as much information as possible by using as less words as possible. This behavior tends to increase the frequency of individual words in a text of a given number of words, as if the frequency of words was the thing people are looking for with effort, leading to Zipf law of frequency-rank distribution [2][3]. In term of city population in a an economic and political society, people tend to move to large cities for many reasons, as if the size of city was something they are trying to obtain with of course as less
effort as possible, leading to Zipf-Pareto laws of size distribution [4][14]. The last similar example we think worth mentioning is the informational networks where the Zipf-Pareto laws is ubiquitous under the reign of preferential attachment [15]. This rule is a typical behavior of least effort or of maximum efficiency in the quest for information. Its derivation from PLE will be discussed in a coming work.

We would like to remark the Zipf-Pareto laws are often relevant to Pareto’s 80/20 rule\(^2\), or same rule with different proportion, 90/10 for instance). This rule is very helpful for having a glimpse of the ubiquity of Zipf-Pareto type distributions and the underlying PLE and maximum efficiency you can hardly ignore even in daily life. For example, if you have a large number of phone numbers in your calling list, you can easily check that 80% (even more) of your call are made to only 20% (even less) of the phone numbers of the list. Same behavior with your mailing list. And if you have a large number of friends, you can easily check a small percentage (perhaps 10%) of your friends occupy most (perhaps 90%) of your time spent with friends. Similar phenomena happen with the books you keep in your library, shoes and cloths if you have many. All this in order to say that PLE should be meaningful for so many living systems.

Our question about the applicability of our approach is that to what extent the efficiency functional \(\eta\) can offer a statistical metric of the effectiveness or performance of living systems. The reader will see in what follows that the value of \(\eta\) can be easily calculated for a system if its distribution laws is known. The following analysis of \(\eta\), in relation with entropy as a measure of disorder, is carried out in order to better understand its attribute and potentials as a measure of performance.

3) Uniqueness of the efficiency functional

Now we provide a proof of uniqueness of the functional Eq.(5) if Eq.(3) and Eq.(4) are given. Eq.(4) can be written as

\[
1 + a \eta_i(X) = 1 + a \eta_k(X) + a \eta_j(Y) + a^2 \eta_k(X) \eta_j(Y).
\]

The relationship suggests naturally that the sought-for function is of class \(C^2\) at least, and its form correspond to Taylor expansion at the order 2 in two variables with restrictions to get the right form. For consequently, we propose the generating function \(G_X = E(s^X)\) updated at two discrete random variables.

We remind the properties of this function of a single variable:

\(^2\)https://en.wikipedia.org/wiki/Pareto_principle
**Proposition**  \( G_X(s) = E(s^X) \) is at least defined on \([-1,1]\), \( G_X(1) = 1 \), with the following power expansion series:

\[
G_X(s) = \sum_{k=1}^{\infty} p_k s^k.
\]

The generating function \( G_X = E(s^X) \) of \( X \) is continuous and indefinitely differentiable on \([-1,1]\). We propose the following function:

\[
G_{X,Y}(x,y) = G_{X,Y}(1,1) + (x - 1) \frac{\partial G_{X,Y}}{\partial x}(1,1) + (y - 1) \frac{\partial G_{X,Y}}{\partial y}(1,1) + (x - 1)(y - 1) \frac{\partial^2 G_{X,Y}}{\partial x \partial y}(1,1) + o\|(x - 1), (y - 1)\|^2
\]

since \( G \) has a Taylor expansion at the point \((1, 1)\). Moreover, the factorial moments of order \( k \) are related to the values of \( G \) at the point \((1,1)\). Hence the first partial derivative with respect to \( x \) at the point \((1,1)\) appearing in the formula above can be identified with \( E(X) \), respectively \( E(Y) \) and the second partial derivative in \( x \) and \( y \) with \( E(XY) = E(X)E(Y) \) when \( X \) and \( Y \) are independent.

Thus to get Eq. (4), we have to assert that \( G \) has no moment of order 2 in each single variable and \( x - 1 = y - 1 = a \). The relationship Eq.(3) is clearly satisfied in the case where \( X \) and \( Y \) are independent for \( G_{X+Y} = G_XG_Y \) since in the power series expansion, it appears the term

\[
p_k = \frac{G^k(0)}{k!} \quad p_j = \frac{G^j(0)}{j!}, \quad \text{with} \quad p_i = p_k p_j, i = j + k
\]

We summarize: Assume that \( X \) and \( Y \) are independent discrete random variables without moments of order 2, then the Taylor expansion of the generating function \( G_{(X,Y)} \) at the order 2 at the point \((1,1)\) in choosing \( a = x - 1 = y - 1 \) satisfies the relationship Eq.(3) et Eq.(4).

Finally, we can recover Eq.(5), \( (1 + a \eta_i) = p_i^b \) in using the following lemma applied to \( p_i = f(\eta_i) \).

**Lemma:**

Let \( f \) be a positive continuous function on the \( \sigma \)-algebra \( B(R^+) \) satisfying the condition

\[
\forall x \in R^+, \forall y \in R^+, f(xy) = f(x)f(y),
\]

Then there is \( a > 0 \) such that \( \forall x \in R^+, f(x) = x^a \).

**Proof**
Set $\forall x \in \mathbb{R}^+, g(x) = \ln(f(e^x))$, $g$ is continuous on $\mathbb{R}^+$, we have for every $x \in \mathbb{R}^+$ and $y \in \mathbb{R}^+$, the following relationship:

$$g(x + y) = \ln(f(e^{x+y})) = \ln(f(e^x e^y)) = \ln(f(e^x)f(e^y)),$$

after the hypothesis in lemma.

Therefore, we deduce $g(x + y) = g(x) + g(y)$.

Consequently, $\forall n \in \mathbb{N}^*, g(n) = ng(1)$, $g(1) = g\left(\frac{1}{n} + \cdots + \frac{1}{n}\right) = ng\left(\frac{1}{n}\right)$, et $g\left(\frac{1}{n}\right) = \frac{g(1)}{n}$.

Thus, $\forall p \in \mathbb{N}^*, \forall q \in \mathbb{N}^*, g\left(\frac{p}{q}\right) = g\left(\frac{1}{q} + \cdots + \frac{1}{q}\right) = pg\left(\frac{1}{q}\right) = p \frac{g(1)}{q}$.

As $\forall x \in \mathbb{R}^+$, there is a sequence $(p_n, q_n)$ such that $\lim_{n \to \infty} \frac{p_n}{q_n} = x$, like $g$ is continuous at $x$,

$$\lim_{n \to \infty} g\left(\frac{p_n}{q_n}\right) = g(x) = \lim_{n \to \infty} p_n \left(\frac{g(1)}{q_n}\right) = xg(1).$$

Thus $\forall X \in \mathbb{R}^+$, we set $x = \ln X$, then we have

$$f(X) = f(e^x) = e^{g(x)} = e^{xg(1)} = (e^x)^{g(1)} = X^a,$$

In setting $a = g(1)$, the uniqueness of Eq.(5) is proven.

This function naturally generalizes in the framework of continuous random variables in replacing the generating function by the function of moments $M_X(s) = E(e^{sX})$ (we recover $G_X(s) = M_X(\ln s)$) in applying for transfer’s theorem on a compact support.

4) Lesche stability of the efficiency functional

If a physical quantity is a continuous function of some variables such as time, position, energy, probability distribution etc., this quantity should undergo smooth variation if the variables smoothly change following small evolution or perturbation of the system in consideration. This condition has been used as a criterion to verify if a mathematical definition of physical quantity is robust against small modification of the variables [16]. In the past several years, this criterion allowed the investigation of several generalizations of functionals such as entropy, information and statistical means, and an importance conclusion telling that not all analytic and concave functionals are Lesche stable or robust against small variation of probability [17]. In what follows, we provide a proof of the robustness (stability) of the functional Eq.(5) against small variation of the probability distribution.

Definition. A Function $C$ defined on $A = \cup_{n \in \mathbb{N}^*}(p \in [0,1]^n, \sum_{i=1}^n p_i = 1)$ is said to be stable (robust) if $C$ has the following property
∀ε > 0, ∃ δ > 0, ∀ N ∈ N*, ∀ p, p' ∈ A ∩ RN, ||p - p'||1 < δ ⇒ \left| \frac{c(p) - c(p')}{c_{N, max}} \right| < ε.

C_{N, max} is defined by C_{N, max} = \max\{|c(p)|, p ∈ A ∩ RN\}.

**Proposition.**
For all α ∈ ]0,1[, the Efficiency \( p \mapsto E(p) \) given by Eq.(5) is stable.

For the proof of this proposition, we have need of the following lemmas.

**Lemma 1.**
Let α ∈ ]0,1[, N ∈ N*, \( p = (p_i), p' = (p_i') \) ∈ [0,1]N. We have
\[
\sum_{i=1}^{N} |p_i^{1-α} - (p_i')^{1-α}| \leq N^α(||p - p'||1)^{1-α}
\]

**Proof.**
We can suppose that \( p \neq p' \) and we put
\( I_1 = \{ i \in \{1, \cdots, N\}, p_i' < p_i \} \) and \( I_2 = \{ i \in \{1, \cdots, N\}, p_i < p_i' \} \).

We have \( I_1 \neq \emptyset, I_2 \neq \emptyset \), and for all \( i \in I_1 \).
\[
1 = \left( \frac{p_i'}{p_i} \right) + \left( 1 - \frac{p_i'}{p_i} \right) \leq \left( \frac{p_i'}{p_i} \right)^{1-α} + \left( 1 - \frac{p_i'}{p_i} \right)^{1-α} = \left( \frac{p_i'}{p_i} \right)^{1-α} + |1 - \frac{p_i'}{p_i}|^{1-α},
\]

We deduce that \( (p_i)^{1-α} - (p_i')^{1-α} \leq |p_i - p_i'|^{α-1} \) and by symmetry, that all \( i \in I_2 \), deduce that \( (p_i)^{1-α} - (p_i')^{1-α} \leq |p_i - p_i'|^{1-α} \).

We conclude that for \( i \in \{1, \cdots, N\}, |(p_i)^{1-α} - (p_i')^{1-α}| \leq |p_i - p_i'|^{1-α} \).

New, we have, \( \sum_{i=1}^{N} |(p_i)^{1-α} - (p_i')^{1-α}| \leq \sum_{i=1}^{N} |p_i - p_i'|^{1-α} = \sum_{i=1}^{N} 1 \times |p_i - p_i'|^{1-α} \), and by Holder’s Inequality, (because, \( 1 = a + (1 - a) = \frac{1}{1/a} + \frac{1}{1/(1-a)} \)), we obtain
\[
\sum_{i=1}^{N} 1 \times |p_i - p_i'|^{1-α} \leq \left( \sum_{i=1}^{N} \frac{1}{1/a} \right)^a \left( \sum_{i=1}^{N} |p_i - p_i'|^{1-α} \right)^{1-α} = N^α \left( \sum_{i=1}^{N} |p_i - p_i'| \right)^{1-α} = N^α (||p - p'||1)^{1-α}
\]

**Lemma 2.**
For all α ∈ ]0,1[, the function \( x \mapsto \frac{x^α}{1-x^α} \) is bounded on [2, +∞[ by M > 0.

Indeed if α ∈ ]0,1[, the function \( x \mapsto \frac{x^α}{1-x^α} \) is continuous on [2, +∞[ and \( \lim_{x \to +\infty} \frac{x^α}{1-x^α} = 1 \).

**Proof of the proposition.**
Using the Lagrange multiplier method, we prove that:
\[
E_{N, max} = E((1/N), \cdots, (1/N)) = \frac{1-N^α}{α}.
\]
Let $\varepsilon > 0$, there exists $\delta = \left(\frac{\varepsilon}{M}\right)^{1/(1-a)} > 0$, such that for all $N \in \mathbb{N}^\ast \setminus \{1\}$, $p, p' \in A \cap \mathbb{R}^N$, $\|p - p'\|_1 < \delta$ implies

$$\frac{E(p) - E(p')}{E_{N,\text{max}}} = \frac{\sum_{i=1}^{N} (p_i^{1-a} - (p_i')^{1-a})}{|1 - N^a|} \leq \sum_{i=1}^{N} \left| p_i^{1-a} - (p_i')^{1-a} \right| |1 - N^a| \leq N^a \left( \sum_{i=1}^{N} |p_i - p_i'| \right)^{1-a} |1 - N^a| \leq N^a (\|p - p'\|_1)^{1-a} \frac{|1 - N^a|}{M \delta^{1-a}} = \varepsilon$$

The above proposition is proved.

5) **Negative ZP efficiency?**

In the work [12], the efficiency functional Eq.(5) is supposed to be positive in the case of discrete distribution by physical consideration since negative efficiency in thermodynamics would mean an engine absorbing heat but doing negative work or consuming work. This is not meaningful from the viewpoint of physics. This positivity is guaranteed by Eq.(5) in the case of discrete values of $X$ and may be lost when $X$ becomes continuous, meaning that Eq.(5) is calculated with continuous probability distribution $\rho(x)$

$$\eta = \int_{x_{\text{min}}}^{x_{\text{max}}} \rho(x) \frac{\rho(x)^{1-a} - 1}{a} \, dx$$

The negative efficiency turns out to be possible depending on the distribution (see below). In many domains out of physics, negative efficiency may make sense if the output is negative. In economy for example, if a person or a company is indebted during a given period, negative efficiency is useful to reflect the reality. The following analysis keeping negative efficiency is motivated by this consideration.
Figure 2. Evolution of the efficiency of Eq.(6) calculated for Pareto’s distribution in the interval $0 < a < 0.5$ or $\infty < \beta < 1$. $\eta$ diverges for $a=0.5$ ($\beta = 1$) and $a=0$ ($\beta = \infty$). $\eta = 0$ for $a = 0.1945$ or $\beta = 4.14$, and $\eta < 0$ in the range $0 < a < 0.1945$ or $\infty < \beta < 4.14$.

As shown in [12], the PLE implies the maximization of the efficiency Eq.(5), which leads to Pareto’s distribution

$$ P(X > x) = \left( \frac{x_m}{x} \right)^{\beta} \quad (7) $$

where $P(X > x)$ is the probability of finding a person with income larger than a value $x$, $x_m$ the smallest income and $\beta = \frac{1}{a} - 1$ a positive constant characterizing the distribution. We have $0 < a < 1$ and $0 < \beta < \infty$. With this distribution, the efficiency Eq.(6) can be calculated with the Pareto PDF $\rho(x) = \frac{\beta}{x^{\beta+1}}$ where we suppose $x_m = 1$ and $x_{max} = \infty$ for simplicity. The result reads $\eta = \frac{1}{a} \left[ \left( \frac{a}{1-a} \right)^{a} \frac{1-a}{1-2a} - 1 \right] = (\beta + 1) \left[ \beta^{\frac{-1}{\beta+1}} \cdot \frac{\beta}{\beta-1} - 1 \right]$ which is referred to as Zipf-Pareto (ZP) efficiency and plotted in Figure 2 in the range $0 < a < 0.5$. The range $0.5 < a < 1$ is excluded due to negative means [12].

The ZP efficiency increases with increasing $a$ or decreasing $\beta$, and diverges for $a=0.5$ ($\beta = 1$) and $a=0$ ($\beta = \infty$). $\eta = 0$ for $a = 0.1945$ or $\beta = 4.14$, and $\eta < 0$ in the range $0 <
\[ a < 0.1945 \text{ or } 4.14 < \beta < \infty. \] We can get further insight into these behaviors of the efficiency in the following discussion in relation with the concept of inequality.

6) Efficiency and inequality

The relationship between efficiency and inequality is a hot topic in economy [18][19][20]. According to a classical point of view in economics, economic efficiency or growth is positively correlated with inequality. This belief comes from the ideas of incentives. Inequality implies opportunity and prospect of higher incomes to save, invest, work hard, and innovate. Without these opportunity and prospect, firms and individuals will reduce effort resulting in lower economic growth. The economic disaster in some former communist countries making egalitarian experiments is an example against redistributive policies. The same reasoning makes sense in educational system as well. Inequality in educational levels implies, at least statistically, opportunity to receive higher education, to attain more knowledge, higher social classes, higher incomes, honor and appreciation and so on. It is imaginable that if everybody in a population has the same educational level, the effort to achieve more education, more knowledge and skill will be reduced, leading to low efficiency marked by low production of knowledge and innovation. This incentive mechanism has its place in every system of living agents where effort is needed to achieve higher desirable output.

However, a question naturally arises about to what extent this incentive mechanism works. In practice, it will be hardly convincing to claim that an economic system is most efficient when only one person has all the wealth and the rest of the population has nothing, or that an educational system has maximal efficiency with only one person very educated and the rest of the population without any education \((G=1)\). The reality is much messier than the pure incentive reasoning. We can have an idea about this messiness from a figure of the evolution of Gini coefficient after the World War II on the page of [21]. In this figure, we see many countries (USA, China, India, UK etc.) experienced increasing inequality accompanied by long-term growth. On the other hand, we see obvious counterexamples with the countries such as France, Germany, Japan etc. showing inequality coefficient decreasing during the period of long-term growth. The existence of different theories on economic growth is a reflection of the diversity of viewpoints in this domain [20][21].

Actually, the economic growth depends on so many non-economic or extrinsic factors like political situation, social stability, technological level and progress, human capital and so on, that sometimes the intrinsic factors such as incentive may be hidden, making it difficult to figure out from empirical data to what degree the economic performance depends on the intrinsic
factors. For this reason, it is of our interest to study the behavior of ZP efficiency in relation with inequality in term of output measured by Gini coefficient $G$ which can be related to the Pareto law by $G = \frac{1}{2^{\beta - 1}}$. The result is plotted in Figure 3 where we see a positive correlation between ZP efficiency $\eta$ and Gini coefficient $G$. $\eta$ is very large when $G \to 1$, the case of maximal inequality. $\eta$ decreases with decreasing $G$ and becomes negative when $\eta$ is smaller than a given value determined by the Pareto index $\beta = 4.14$.

![Figure 3. Variation of ZP efficiency $\eta$ as a function of the Gini coefficient $G$ indicating inequality ($G=0$ absolute equality and $G=1$ maximum inequality). $\eta$ becomes negative in the interval $0 < G < 0.14$ corresponding to $0 < a < 0.1945$ or $4.14 < \beta < \infty$, and very large when $\eta \to 1$ (maximum inequality).](image.png)

The main conclusion of Figure 3 is that the ZP efficiency, being positively correlated with inequality, should be regarded as an intrinsic quantity free from the complex extrinsic factors influencing the functioning of a living system, at least for the systems showing Zipf-Pareto laws. This is understandable because this efficiency has been derived from a purely mathematical property, a nonadditivity that is ubiquitous to the ratio between output and input [24]. This observation is a little disadvantageous for ZP efficiency, implying that one must be careful when using it to measure the efficiency in a real system. Is ZP efficiency only a theoretical measure with fundamental interest? This is an issue worthy to be investigate further.
7) Relationship with some entropies

In order to get more insight into the efficiency Eq(5) or (6), it is of interest to compare it to entropy which is another important quantity to characterize the dynamics of the systems undergoing probabilistic processes. In what follows, we only show the results with two entropies: Boltzman-Shannon entropy\(^3\) and varentropy [22][23], the former being a standard measure of information or probabilistic uncertainty, and the latter a maximizable measure of uncertainty for any probability distribution, which is not the case of Boltzman-Shannon entropy which is only maximized for either uniform or exponential distribution [22].

![Graph showing ZP efficiency and Shannon entropy](https://en.wikipedia.org/wiki/Entropy_(information_theory)#Characterization)

Figure 4. Evolution of ZP efficiency \(\eta = (\beta + 1)(\beta^{-\frac{1}{\beta+1}} \cdot \frac{\beta}{\beta-1} - 1)\) compared to the evolution of the Shannon entropy Eq.(8) as a function of \(a\) and \(\beta\) in the interval \(1 < \beta < \infty\). We see a positive correlation between \(\eta\) and \(S_{BS}\), both of them decreasing with decreasing \(\beta\). When \(\beta = 1\), \(\eta\) diverges, but the \(S_{BS}\) is finite. \(S_{BS}\) becomes negative in the interval \(\ln\beta > 1 + \frac{1}{\beta}\) or \(\beta > 3.59\).

- **Boltzman-Shannon entropy**

  For Zipf-Pareto laws with the PDF \((x) = \frac{\beta}{x^{\beta+1}}\), following the usual continuous Boltzmann-Shannon entropy \(S_{BS}\), we have

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\(^3\) https://en.wikipedia.org/wiki/Entropy_(information_theory)#Characterization
\[ S_{BS} = - \int_1^\infty \rho(x) \ln(\rho(x)) \, dx = -\ln\beta + 1 + \frac{1}{\beta} \]  

(9)

This result is plotted in Figure 4 for comparison with ZP efficiency \( \eta \).

We observe a positive correlation between \( \eta \) and \( S_{BS} \), both of them decreasing with decreasing \( \beta \). When \( \beta = 1 \), \( \eta \) diverges but the \( S_{BS} \) is finite. The trouble takes place for the interval \( \ln\beta > 1 + \frac{1}{\beta} \) or \( \beta > 3.59 \) where \( S_{BS} \) becomes negative. This is a case where Boltzmann-Shannon entropy becomes questionable as a measure of probabilistic uncertainty since the uncertainty is widely considered positive. Jaynes [25] has investigated this problem of \( S_{BS} \) encountered with continuous distribution and tried to save the situation by defining a continuous version of Boltzmann-Shannon entropy: 

\[ S_{BS}^C = - \int_1^\infty \rho(x) \ln\left(\frac{\rho(x)}{m(x)}\right) \, dx + \ln N \]  

where \( m(x) \) is called invariant measure of the density of discrete values of \( x \) when their total number \( N \to \infty \) [25][26]. The divergent term \( \ln N \) cannot be kept in a proper definition of entropy measure. If it is wiped out, we can write 

\[ S_{BS}^C = S_{BS} + \int_1^\infty \rho(x) \ln m(x) \, dx \]  

Now it is possible to choose the invariant measure \( m(x) \) in an appropriate (a little bit ad hoc) way for \( S_{BS}^C \) to be positive (see below). But one should not forget that \( S_{BS}^C \) contains an infinite term \( \ln N \). This awkward term arises in the derivation of the continuous entropy by using the Shannon formula for discrete entropy \( S_{BS} = - \sum_i p_i \ln p_i \). If one substitutes \( p_i = \rho dx \) in the discrete \( S_{BS} \), a divergent term \(-\ln dx \propto \ln N\) is inevitable. As a matter of fact, this discrete entropy formula has been derived from some postulates including the positivity of \( S_{BS} \) and \( p_i \leq 1 \). These postulates being no more valid for continuous distribution, the Shannon formula cannot be used for continuous distribution. One simply cannot replace a discrete probability with \( p_i = \rho dx \) in the Shannon formula, as has been done by Boltzman and Shannon themselves as well as Jaynes [25][26].

In what follows, we will show that the correct formula of \( S_{BS} \) or others for continuous distribution can be derived from a general definition of informational entropy called varentropy [22][23]. Varentropy is defined from scratch in a variational form inspired by the second law of thermodynamics. In this way, we can easily obtain the continuous version of \( S_{BS} \) without using the rather obscure measure \( m(x) \) proposed by Jaynes [25].

- **Varentropy \( S_V \)**

The varentropy is defined by mimicking the variational form of the entropy of the second law of thermodynamics [22]. This is a definition from scratch without any prerequisite or postulate on the property of entropy. The motivations was to look for an uncertainty measure
that is optimal or maximal for any probability distribution. Although $S_{BS}$ has been widely used as a universal uncertainty measure for any probability distribution, it is only maximal for exponential and uniform distribution. The question is whether it is possible to define a more general measure of uncertainty that is just $S_{BS}$ for exponential distribution but better or more optimal than $S_{BS}$ for other distributions.

The answer is such a measure does exist and is given by $\delta S_V = A(\delta \bar{x} - \delta \bar{x}) = A \int x \delta \rho(x) dx$ for a continuous distribution $\rho(x)$, where $S_V$ is the uncertainty or entropy measure named varentropy and $A$ is a constant determining the dimension of the uncertainty measure. If the variable $x$ is energy $E$, we have $\delta S_V = A(\delta \bar{E} - \delta \bar{E}) = \frac{1}{T} \delta U - \delta W$, where $A = \frac{1}{T}$ the inverse temperature, $\bar{E}$ the internal energy, $\delta W = \delta \bar{E}$ is the external work and $S_V$ is just the entropy of the second law of thermodynamics. The maximization of $S_V$ is then a natural consequence of the second law of thermodynamics. This entropy turns out to be equivalent to a generalized entropy defined in [27] from mathematical consideration based upon the principle of maximum entropy.

Now for exponential distribution $\rho(x) = \frac{1}{Z} e^{-x}$ where $x$ is a dimensionless variable and $Z$ the normalization constant, it is straightforward to get $\delta S_V = -A \int \ln(Z\rho) \delta \rho dx = -A \int (\ln Z + \ln \rho) \delta \rho dx = -A\delta \int \rho \ln \rho dx = \delta [-A \int \rho (\ln \rho - \ln m) dx]$ where $m(x)$ is some function satisfying $\int \rho \ln m(x) dx = C$ a constant for $\delta C = 0$. Let $A=1$, we obtain the continuous Boltzmann–Shannon entropy we denote by $S^B_S$ in keeping the index $V$ to indicate its origin from varentropy:

$$S^B_S = -\int \rho \ln \frac{\rho}{m} dx$$

(10)

where $m$ is the invariant measure introduced by Jaynes. But this entropy formula does not suffer from the divergent term $\ln N$ in the formula of Jaynes [25][26].

The formula Eq.(10) can take another equivalent form if we keep the term $\ln Z$ in the above calculation, i.e. $\delta S^B_S = -A \int (\ln Z + \ln \rho) \delta \rho dx = -A\delta \int \rho (\ln Z + \ln \rho) dx = \delta [-A \int \rho (\ln (Z\rho) - \ln m) dx]$ which implies $S^B_S = -\int \rho \frac{Z\rho}{m} dx$ (A=1). Using Pareto distribution $\rho(x) = \frac{\beta}{x^{\beta+1}}$, it is easy to calculate $S^B_S = 1 + \frac{1}{\beta} + C$. Let $C = -1$, we get

$$S^B_S = \frac{1}{\beta} = \frac{a}{1 - a} \geq 0$$

(11)

which will be plotted in Figure 5.
It is noteworthy that, as well known, $S_{BS}^V$ can be maximized to obtain uniform and exponential distributions. Hence, it is not at maximum for power law distributions. For a discrete power law $p_i = \frac{1}{Z} x_i^{-1/b}$, the maximized entropy is $S_V = \frac{\sum_i p_i^{1-b} - 1}{1-b}$ which is positive and zero for non-probabilistic case [22]. This entropy looks like Tsallis entropy $S_T = \frac{\sum_i p_i^{q-1}}{1-q}$ but is essentially different. $S_T$ is a generalized $q$-logarithmic entropy (while $S_V$ is not) and tends to Shannon entropy when $q \to 1$, but $S_V$ tends to zero for $b \to 0$ and diverges for $b \to 1$ [22].

![Figure 5. Evolution of ZP efficiency $\eta = (\beta + 1)[\beta^{-\frac{1}{\beta} + 1} \cdot \beta_{\beta-1} - 1]$ compared to the continuous $S_{BS}^V = \frac{1}{\beta} = \frac{a}{1-a}$ and $S_V^P = \frac{\beta^{a} + 1}{\beta(\beta-1)}$. Both entropies increase from zero with increasing $a$ or decreasing $\beta$ in the intervals $0 < a < 0.5$ or $1 < \beta < \infty$. So they have positive correlation with ZP efficiency. However, at $a \to 0.5$ or $\beta \to 1$, $S_{BS}^V$ is finite and $S_V^P$ tends to infinity. So $S_V^P$ is closer to $\eta$ than $S_{BS}^V$.](image)

Now for continuous power law distribution $\rho(x) = \frac{1}{Z} x^{-1/b}$, the continuous varentropy $S_V^P$ should be calculated as follows: $\delta S_V^P = A \int (Z\rho)^{-b} \delta \rho dx = A \int \delta \left(\frac{Z^{-b}}{1-b} \rho^{1-b}\right) dx = \delta\{\frac{A}{1-b} \int (Z^{-b} \rho^{1-b} - m) dx\}$. Let $A = 1$, this continuous varentropy reads
where the function \( m \) is such that \( \int \rho m(x) \, dx = C \) is a constant of the variation, i.e., \( \delta C = 0 \). For Pareto PDF \( \rho(x) = \frac{\beta}{x^{\beta+1}} \), let \( Z = \frac{1}{\beta} \), \( b = \frac{1}{\beta+1} \) in Eq.(12), we obtain

\[
S^p_V = \int_1^{+\infty} \beta \frac{1}{\beta+1} \cdot \frac{\beta}{\beta+1} \cdot \rho(x) \, dx - C = \frac{\beta}{\beta} \left( \frac{\beta}{\beta+1} \right) = \frac{\beta+1}{\beta} \left( \frac{\beta}{\beta-1} - C \right).
\]

Let \( C = 1 \), we obtain a positive varentropy

\[
S^p_V = \frac{\beta+1}{\beta (\beta-1)} \geq 0
\]

This continuous varentropy of Pareto distribution is plotted in Figure 5 in comparison with ZP efficiency \( \eta \) and the continuous \( S_{BS} = \frac{1}{\beta} = \frac{a}{1-a} \) for Pareto distribution. We observe that the efficiency is positively correlated to entropy as a measure of dynamical uncertainty or disorder.

By the way, this result also reveals a positive correlation between inequality and dynamic disorder. In other words, larger inequality implies larger disorder of the dynamic process.

8) Conclusion

An efficiency functional implementing the principle of least effort in order to derive Zipf-Pareto’s laws is further investigated from the mathematical point of view. Specifically, we studied its unicity and stability (robustness) against small variation of probability distribution. It is shown that this efficiency functional is the unique one required by a typical nonadditivity relationship of efficiency in thermodynamics and the condition of independent distribution of probability between subsystems. The behavior of this efficiency calculated for Pareto distribution was compared to the Gini coefficient (inequality) and probabilistic uncertainty (disorder) measured by two entropies. For this purpose, a method of calculation of continuous entropy is proposed to avoid negative entropy of continuous probability distribution, which was a long open question in information theory. It is observed that the Zipf-Pareto efficiency is positively correlated to Gini coefficient and entropy. Put it differently, the Zipf-Pareto efficiency increases with increasing inequality and uncertainty of a large number of living agents. This result reveals that inequality has also a positive correlation with dynamic uncertainty (disorder).

This work is part of the tentative to get deeper understanding of the approach of maximum efficiency as an implementation of PLE. The objective of this work is, on the one hand, to
provide a variational calculus to derive power laws from PLE, and, on the other hand, to find a statistical metric of effectiveness or of performance of the systems composed of or driven by a large number of living agents all obeying, at least statistically, the rule of least effort. The first goal seems to have been reached, but the approach needs extension in order to account for more complicated distribution laws than single power laws of Zipf-Pareto distributions [12]. While a great deal of theoretical as well as empirical work are necessary for the second goal.

**Data availability**

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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Appendix

Here we show how to find out the value of $a$ for the ZP efficiency calculated for Pareto distribution to vanish:

$$\eta(a) = \frac{1}{a \left( \frac{a}{1-a} \right)^a \frac{1-a}{1-2a} - 1} = 0$$

This equation is only defined on the open interval $J = \left( 0, \frac{1}{2} \right) U \left( \frac{1}{2}, 1 \right)$. Hence for $a \neq \left\{ 0, \frac{1}{2}, 1 \right\}$, this equation boils down:

$$F(a) = \frac{1}{a} \left[ \frac{a^a(1-a)^{-a+1}}{1-2a} - \frac{1-2a}{1-2a} \right] = 0;$$

Therefore,

$$a^a(1-a)^{-a+1} - (1-2a) = 0.$$ 

Thus, it is natural to study the following function $g$ defined by:

$$g(x) = x^x(1-x)^{1-x} - (1-2x) \text{ sur } J.$$ 

Corollary:

Let be $g: \left[ 0, \frac{1}{2} \right] \to R$, defined by $g(0) = 0$ and $g(x) = x^x(1-x)^{1-x} - (1-2x)$, then on the one hand, $g$ is continuous and indefinitely differentiable on $\left( 0, \frac{1}{2} \right]$, and on the other hand there is a unique $a$ in $\left( 0, \frac{1}{2} \right)$ such that $g(a) = 0$. We just provide the arguments of proof of Corollary:

Proof is based on the Intermediate values’ theorem applied at the differential $g'$ on the one hand over the open interval $\left( \lim_{x \to 0} g'(x) = -\infty, g'\left( \frac{1}{2} \right) = 2 \right]$ that contains 0 and on the other hand applied at $g$ on the open interval $[g(b) < 0, g(\frac{1}{2})=2]$ which also contains 0. We give an approximate value of $a = 0.194513$ for $\eta = 0$. 

