KAELER STRUCTURES ON $K_C/(P, P)$

MENG-KIAT CHUAH

Abstract

Let $K$ be a compact connected semi-simple Lie group, let $G = K_C$, and let $G = KAN$ be an Iwasawa decomposition. Given a $K$-invariant Kähler structure $\omega$ on $G/N$, there corresponds a pre-quantum line bundle $L$ on $G/N$. Following a suggestion of A.S. Schwarz, in a joint work with V. Guillemin, we study its holomorphic sections $\mathcal{O}(L)$ as a $K$-representation space. We define a $K$-invariant $L^2$-structure on $\mathcal{O}(L)$, and let $H_\omega \subset \mathcal{O}(L)$ denote the space of square-integrable holomorphic sections. Then $H_\omega$ is a unitary $K$-representation space, but we find that not all unitary irreducible $K$-representations occur as subrepresentations of $H_\omega$. This paper serves as a continuation of that work, by generalizing the space considered. Instead of working with $G/N = G/(B, B)$, where $B$ is a Borel subgroup containing $N$, we consider $G/(P, P)$, for all parabolic subgroups $P$ containing $B$. We carry out similar construction, and recover in $H_\omega$ the unitary irreducible $K$-representations previously missing. As a result, we use these holomorphic sections to construct a model for $K$: a unitary $K$-representation in which every irreducible $K$-representation occurs with multiplicity one.

1991 Mathematics Subject Classification. Primary 53C55.

Keywords: Lie group, Kähler, line bundle.

1 INTRODUCTION

Let $K$ be a compact connected semi-simple Lie group, let $G = K_C$ be its com-
plexification, and let $G = KAN$ be an Iwasawa decomposition. Since $G$ and $N$ are complex Lie groups, $G/N$ is a complex manifold, and $G$ acts on $G/N$ by left action. Let $T$ be the centralizer of $A$ in $K$, so that $H = TA$ is a Cartan subgroup of $G$. Since $H$ normalizes $N$, there is a right action of $H$ on $G/N$. We shall often be interested in the maximal compact group action of $K \times T$. We let $\mathfrak{g}, \mathfrak{k}, \mathfrak{h}, \mathfrak{t}, \mathfrak{a}, \mathfrak{n}$ denote the Lie algebras of $G, K, H, T, A, N$ respectively.

The following scheme of geometric quantization was suggested by A.S. Schwarz [12]: Equip $G/N$ with a suitable $K$-invariant Kaehler structure $\omega$, and consider the pre-quantum line bundle $L$ associated to $\omega$ ([5], [11]). The Chern class of $L$ is $[\omega]$, and $L$ comes with a connection $\nabla$ whose curvature is $\omega$, as well as an invariant Hermitian structure $<, >$. We denote by $O(L)$ the space of holomorphic sections on $L$. The $K$-action on $G/N$ lifts to a $K$-representation on $O(L)$. Let $\mu$ be the $K \times A$-invariant measure on $G/N$, which is unique up to non-zero constant. Given a holomorphic section $s$ of $L$, we consider the integral

$$\int_{G/N} < s, s > \mu .$$

Let $H_{\omega} \subset O(L)$ denote the holomorphic sections in which this integral converges. Since $\mu$ is $K$-invariant, $H_{\omega}$ becomes a unitary $K$-representation space. It was hoped in [12] that every irreducible $K$-representation occurs with multiplicity one in $H_{\omega}$ (called a model by I.M. Gelfand [7]).

By the method of highest weight, the irreducible $K$-representations can be labeled by the dominant integral weights in $\mathfrak{t}^*$, up to isomorphism. In a joint work with V. Guillemin [4], we carry out this construction, but find that no matter how $\omega$ is chosen, the irreducibles whose highest weights lie on the wall of the Weyl chamber do not occur in the Hilbert space $H_{\omega}$. Therefore, not all unitary $K$-irreducibles occur in $H_{\omega}$. The present paper follows a suggestion of V. Guillemin ([4] p.192), by modifying the space $G/N$ to more general classes of homogeneous spaces. As a result, we manage to recover the unitary $K$-irreducibles previously missing.

Let $B = HN$ be the Borel subgroup of $G$. Observe that $(B, B) = N$, hence
$G/N = G/(B,B)$. With this in mind, we can generalize the class of homogeneous spaces considered to $G/(P,P)$, for $P$ a parabolic subgroup of $G$ containing $B$. Since $P$ is a complex Lie group, so is $(P,P)$; hence $G/(P,P)$ is a complex manifold. Clearly $G$ acts on $G/(P,P)$ on the left, and we shall see that a complex subgroup of $H$ normalizes $(P,P)$, and hence acts on $G/(P,P)$ on the right.

Let $W \subset t^*$ denote the open Weyl chamber, and $\overline{W}$ its closure. We say that $\sigma \subset \overline{W}$ is a cell if there exists a subset $S$ of the positive simple roots $\Delta$ such that
\begin{equation}
\sigma = \{ x \in \overline{W} ; (x, S) = 0, (x, \Delta \backslash S) > 0 \},
\end{equation}
where the pairing used is the Killing form. This way, $\overline{W}$ is a disjoint union of the cells of various dimensions. Using the Killing form and the almost complex structure, it is convenient to regard the cell $\sigma$ as contained in any of the spaces $\mathfrak{h}, \mathfrak{t}, \mathfrak{a}, \mathfrak{h}^*, \mathfrak{t}^*, \mathfrak{a}^*$, depending on the context. The cell $\sigma$ defines a subalgebra $\mathfrak{h}_\sigma$ of $\mathfrak{h}$, by taking complex linear span of $\sigma$. Similarly, the subalgebras $\mathfrak{t}_\sigma, \mathfrak{a}_\sigma$ are defined by intersecting $\mathfrak{h}_\sigma$ with $\mathfrak{t}, \mathfrak{a}$ respectively. These subalgebras define the subgroups $H_\sigma, T_\sigma, A_\sigma$ of $H, T, A$ respectively. A bijective correspondence between the cells $\{\sigma\}$ and the parabolic subgroups $\{P\}$ containing $B$ is given by Langlands decomposition ([10] p.132)
\begin{equation}
P = MA_\sigma N_\sigma.
\end{equation}

Fix a parabolic subgroup $P$ containing $B$, with $\sigma$ its corresponding cell. Since $H_\sigma$ is the normalizer of $(P,P)$ in $H$, it acts on $G/(P,P)$ on the right. Out of the action of the complex group $G \times H_\sigma$, we shall consider the action of the maximal compact group $K \times T_\sigma$ on $G/(P,P)$. We shall show that

**Theorem I**  Let $\omega$ be a $K$-invariant Kaehler structure on $G/(P,P)$. Then $\omega$ is $K \times T_\sigma$-invariant if and only if it has a potential function.

Though we shall be interested mostly in Kaehler structures, Theorem I holds also for degenerate $(1,1)$-form $\omega$. In the next theorem, we shall derive a necessary and
sufficient condition for a $(1,1)$-form $\omega$ to be Kaehler. Let $\omega$ be a $K \times T_\sigma$-invariant $(1,1)$-form, so that

$$\omega = \sqrt{-1} \partial \bar{\partial} F,$$

for some function $F$ on $G/(P,P)$. Averaging by the compact group $K$ if necessary, we may assume that $F$ is $K$-invariant. Let $K^\sigma$ be the centralizer of $T_\sigma$ in $K$. It defines a compact semi-simple subgroup $K^\sigma_{ss}$ of $K$, given by $K^\sigma_{ss} = (K^\sigma, K^\sigma)$. We shall show that, as real manifolds and $K \times H_\sigma$-spaces,

$$G/(P,P) = (K/K^\sigma_{ss}) \times A_\sigma. \tag{1.3}$$

Therefore, the potential function $F$, being $K$-invariant, can be regarded as a function on $A_\sigma$. Since the exponential map identifies the vector space $a_\sigma$ with $A_\sigma$, $F$ becomes a function on $a_\sigma$. The almost complex structure identifies the dual spaces $a^\sigma_\sigma \cong t^*_\sigma$, hence the Legendre transform of $F$ can be written as

$$L_F : a_\sigma \longrightarrow t^*_\sigma.$$

The significance of this map will become apparent shortly, when we study the moment map. We write $\log : A_\sigma \longrightarrow a_\sigma$ for the inverse of the exponential map.

The $K$-action on $G/(P,P)$ preserving $\omega$ is Hamiltonian: there exists a unique moment map

$$\Phi : G/(P,P) \longrightarrow \mathfrak{k}^*$$

corresponding to this action. Since $\Phi$ is $K$-equivariant, $(1.3)$ implies that it is determined by its value on $A_\sigma \subset (K/K^\sigma_{ss}) \times A_\sigma$, where $A_\sigma$ is imbedded as its product with the identity coset $eK^\sigma_{ss} \in K/K^\sigma_{ss}$. Meanwhile, since $\mathfrak{k}$ is semi-simple, the Killing form on $\mathfrak{k}$ is non-degenerate; which induces the inclusion $t^* \subset \mathfrak{k}^*$ from $t \subset \mathfrak{k}$.

**Theorem II** Let $\omega$ be a $K \times T_\sigma$-invariant $(1,1)$-form on $G/(P,P)$. Then its moment map $\Phi$ and its potential function $F$ satisfy $\Phi(a) = \frac{1}{2} L_F(\log a) \in t^*_\sigma$ for all $a \in A_\sigma$. Further, $\omega = \sqrt{-1} \partial \bar{\partial} F$ is Kaehler if and only if:
(i) $F \in C^\infty(a_{\sigma})$ is strictly convex; and

(ii) The image of $\frac{1}{2}L_F$ is contained in the cell $\sigma \subset t^*_{\sigma}$; i.e. $\Phi(A_{\sigma}) \subset \sigma$.

Since a $K \times T_{\sigma}$-invariant Kaehler structure $\omega$ has a potential function $F$, it is exact. Therefore, it is in particular integral. Let $L$ be the line bundle on $G/(P, P)$ whose Chern class is $[\omega] = 0$, equipped with a connection $\nabla$ whose curvature is $\omega$ ([3],[1]). The topology of $L$ is trivial, but the connection $\nabla$ gives rise to interesting geometry on the holomorphic sections of $L$. We recall that $L$ is equipped with an invariant Hermitian structure $\langle, \rangle$. Let $\mu$ be a $K \times A_{\sigma}$-invariant measure on $G/(P, P)$. We consider the integral
\[
\int_{G/(P, P)} \langle s, s \rangle \mu,
\]
for holomorphic sections $s$ of $L$. As we shall see in Theorem III, convergence of this integral is determined by the image of the moment map. The $K \times T_{\sigma}$-action on $G/(P, P)$ lifts to a $K \times T_{\sigma}$-representation on $\mathcal{O}(L)$, the space of holomorphic sections of $L$. We similarly define $H_{\omega} \subset \mathcal{O}(L)$ to be the holomorphic sections in which (1.4) converges. Since $\mu$ is $K$-invariant, $H_{\omega}$ becomes a unitary $K$-representation space.

For a dominant integral weight $\lambda$, let $\mathcal{O}(L)_{\lambda}$ be the holomorphic sections in $L$ that transform by $\lambda$ under the right $T_{\sigma}$-action. Since the left $K$-action commutes with the right $T_{\sigma}$-action, $\mathcal{O}(L)_{\lambda}$ is a $K$-representation space. Let $\sigma$ be the cell corresponding to the parabolic subgroup $P$, and let $\overline{\sigma}$ be its closure. Then

**Theorem III** The irreducible $K$-representation with highest weight $\lambda$ occurs in $\mathcal{O}(L)$ if and only if $\lambda \in \overline{\sigma}$. For $\lambda \in \overline{\sigma}$, it occurs with multiplicity one, and is given by $\mathcal{O}(L)_{\lambda}$. Further, $\mathcal{O}(L)_{\lambda}$ is contained in $H_{\omega}$ if and only if $\lambda$ lies in the image of the moment map.

With this result, it is now clear that in [4], the singular representations are never contained in $H_{\omega}$:
When $P = B$, $\sigma$ becomes the open Weyl chamber $W$. Then Theorem II says that $\Phi(A_\sigma) \subset W$; and by $K$-equivariance, $\Phi(G/(P, P)) = Ad^*_K(\Phi(A_\sigma))$ does not intersect the wall $\overline{W} \setminus W$. Consequently, by Theorem III, the irreducible representations $O(L)_\lambda$ with highest weight $\lambda \in \overline{W} \setminus W$ cannot be contained in $H_\omega$.

Similarly, for general parabolic subgroup $P$, not all $O(L)_\lambda$ are contained in $H_\omega$: For $\lambda \in \overline{\sigma} \setminus \sigma$, Theorems II and III say that $O(L)_\lambda$ exists non-trivially but is not contained in $H_\omega$.

We shall see that, however, for a suitable Kaehler structure $\omega$ on $G/(P, P)$, the image of the moment map intersects $\overline{\sigma}$ in all of $\sigma$. This way, by Theorem III, all the $K$-irreducibles $O(L)_\lambda$ with highest weights $\lambda \in \sigma$ are contained in $H_\omega$. As an application, we provide a geometric construction of a unitary $K$-representation, containing all the irreducibles with multiplicity one.

**Acknowledgement** The author would like to thank V. Guillemin, R. Sjamaar and D. Vogan for many helpful suggestions. The referee has helped to clarify some definitions and notations used in this paper.
2 KAELER STRUCTURES ON $G/(P, P)$

The main purpose of this section is to prove Theorem I. Since $K$ is connected and semi-simple, so is $G = K_C$. Let $P$ be a parabolic subgroup of $G$ containing $B$, and $\sigma$ the cell corresponding to $P$. They are related by Langlands decomposition (1.2)

$$P = MA_\sigma N_\sigma,$$

where $A_\sigma$ is the subgroup described in §1. Then $A_\sigma \subset A, N_\sigma \subset N$, where $A, N$ come from Iwasaw decomposition of $G$. Further, $A_\sigma$ normalizes $N_\sigma$, and is the centralizer of $MA_\sigma$ in $A$. Therefore, $H_\sigma = T_\sigma A_\sigma$ is the normalizer of $(P, P) = (M, M)N_\sigma$ in $H$, which induces a natural right $H_\sigma$-action on $G/(P, P)$. We shall give another description of $G/(P, P)$, which reflects this right action better.

Since $G$ is semi-simple, the Killing form is non-degenerate. Let $a_\sigma^\perp$ be the orthocomplement of $a_\sigma$ with respect to the Killing form in $a$, and $A_\sigma^\perp \subset A$ the corresponding subgroup induced by $a_\sigma^\perp$. We construct $t_\sigma^\perp, T_\sigma^\perp, h_\sigma^\perp, H_\sigma^\perp$ similarly. Let $K^\sigma$ be the subgroup of $K$ given by

$$K^\sigma = \{ k \in K ; \; kt = tk \text{ for all } t \in T_\sigma \}.$$

Let $K^\sigma_{ss} = (K^\sigma, K^\sigma)$ be the corresponding compact semi-simple Lie group. Then

$$\quad (K^\sigma_{ss})_{C} = K^\sigma_{ss} A_\sigma^\perp (M \cap N)$$

is an Iwasawa decomposition of the complexified group $(K^\sigma_{ss})_{C}$. Since $N = (M \cap N) N_\sigma$, it follows from (2.1) that

$$\quad K^\sigma_{ss} A_\sigma^\perp N = (K^\sigma_{ss})_{C} N_\sigma$$

$$\quad = (K^\sigma_{C})_{ss} N_\sigma$$

$$\quad = (MA_\sigma, MA_\sigma) N_\sigma$$

$$\quad = (M, M) N_\sigma$$

$$\quad = (P, P).$$
Then, the Iwasawa decomposition $G = KAN$ and (2.2) imply that

(2.3) \[ G/(P, P) = (K/K_{ss}^\sigma) \times A_\sigma, \]

as real manifolds and $K \times H_\sigma$-spaces. With this description, the right action of $H_\sigma = T_\sigma A_\sigma$ is clear: $T_\sigma$ acts on $(K/K_{ss}^\sigma) \times A_\sigma$ simply because it commutes with $K_{ss}^\sigma$ and $A_\sigma$, while $A_\sigma$ acts on $(K/K_{ss}^\sigma) \times A_\sigma$ by group multiplication on itself. We shall be concerned with the $K \times T_\sigma$-action on $G/(P, P)$.

Since $N = (B, B) \subset (P, P)$, there is a fibration

(2.4) \[ \pi : G/N \longrightarrow G/(P, P). \]

It follows from $G = KAN$ and (2.3) that the fiber of $\pi$ is $K_{ss}^\sigma \times A_\sigma^\bot$. Further, $\pi$ sends every right $H$-orbit in $G/N$ to a right $H_\sigma$-orbit in $G/(P, P)$, by contracting each $H_\sigma^\bot$-coset to a point.

Given a $K$-invariant Kaehler structure $\omega$ on $G/(P, P)$, we want to show that it is invariant under the right $T_\sigma$-action if and only if it has a potential function. Our strategy is to work on the $(1,1)$-form $\pi^* \omega$ on $G/N$ using results in [4], then transfer this result back to $\omega$. Let $V$ be the orthocomplement of $t$ in $\mathfrak{k}$ with respect to the Killing form, so that $\mathfrak{k} = \mathfrak{t} \oplus V$. The Killing form also induces $t^* \subset \mathfrak{t}^*$ from $\mathfrak{t} \subset \mathfrak{k}$. If $F$ is a function on $A$, then by the exponential map, it becomes a function on $\mathfrak{a}$. Using the almost complex structure, $\mathfrak{a}^* \cong \mathfrak{t}^*$. Therefore, the Legendre transform of $F$ can be written as

(2.5) \[ L_F : \mathfrak{a} \longrightarrow \mathfrak{t}^*. \]

Given $\xi \in \mathfrak{k}$, we let $\xi^\sharp$ denote its infinitesimal vector field on $G/N$ induced by the $K$-action. Let $J$ be the almost complex structure on $G/N$. For $\eta = J\xi \in \mathfrak{a}$, where $\xi \in \mathfrak{t}$, we define $\eta^\sharp$ to be the vector field $J\xi^\sharp$. Let $a \in A \subset KA = G/N$. Then its tangent space is $T_a(G/N) = \mathfrak{h}_a^\sharp \oplus \mathfrak{v}_a^\sharp$. We recall the following result from [4]:

**Proposition 2.1** [4] Let $\omega$ be a $K \times T$-invariant $(1,1)$-form on $G/N$. Then
\[ \omega = \sqrt{-1} \partial \bar{\partial} F, \text{ where } F \in C^\infty(A) \text{ by } K\text{-invariance. It satisfies } \omega(\eta^i, V^j)_a = 0. \] The \( K \)-action is Hamiltonian, with moment map \( \Phi : G/N \rightarrow \mathfrak{k}^* \) satisfying

(i) \( \Phi(a) \in \mathfrak{t}^* \) for all \( a \in A \subset KA = G/N; \)

(ii) \( \Phi : A \rightarrow \mathfrak{t}^* \) is given by \( \Phi(a) = \frac{1}{2} L_F(\log a). \)

Let \( m = \dim \sigma, n = \dim \mathfrak{t} \). Let \( \{\lambda_1, ..., \lambda_r\} \) be the positive roots of \( \mathfrak{g} \), where \( \{\lambda_1, ..., \lambda_n\} \) are simple. Here \( m \leq n \leq r \). Then \( \dim V = 2r \), and \( \dim \mathfrak{t} = n + 2r \). In the following proposition, we give a useful decomposition of \( V \). Recall that we define the cell \( \sigma \) in (1.1) using a subset \( S \) of the positive simple roots \( \Delta \). By switching the roles of \( S \) and \( \Delta \setminus S \), we can define another cell \( \sigma' \), with dimension \( n - m \). We call \( \sigma' \) the complementary cell to \( \sigma \). Let \( J \) be the almost complex structure on \( \mathfrak{k} \oplus \mathfrak{a} = \mathfrak{g}/\mathfrak{n} \).

**Proposition 2.2** Let \( \sigma, \sigma' \) be complementary cells of dimensions \( m, n - m \) respectively, where \( m \leq n \leq r = \frac{1}{2} \dim V \). There exists a decomposition \( V = \bigoplus_i V_i \) into two dimensional subspaces \( V_i \). Each \( V_i \) is preserved by \( J \) and satisfies \( [V_i, V_i] \subset \mathfrak{t} \).

Further,

(i) \( \mathfrak{t}^\perp_{\sigma'} = \bigoplus_m [V_i, V_i] \),

(ii) \( \mathfrak{t}^\perp_{\sigma} = \bigoplus_{m+1}^n [V_i, V_i] \).

If \( \omega \) is a \( K \times T \)-invariant \((1,1)\)-form on \( G/N \), then \( \omega(V_i^a, V_j^a) = 0 \) for all \( i \neq j, a \in A \subset KA = G/N \).

**Proof:** Let \( \{\lambda_1, ..., \lambda_r\} \) be the positive roots of \( \mathfrak{g} \), indexed such that the first \( n \) of them are simple. Further, we can require that

\[ (\lambda_i, \sigma) > 0, \ (\lambda_i, \sigma') = 0; \ i = 1, ..., m, \]

and

\[ (\lambda_i, \sigma) = 0, \ (\lambda_i, \sigma') > 0; \ i = m + 1, ..., n, \]

where the pairing taken is the Killing form.
Let $g_{\pm i}$ be the root spaces corresponding to $\pm \lambda_i$. Then there exist $\xi_{\pm i} \in g_{\pm i}$ such that
\begin{equation}
\{ \zeta_i = \xi_i - \xi_{-i} \, , \, \gamma_i = \sqrt{-1}(\xi_i + \xi_{-i}) \}_i=1,...,r
\end{equation}
form a basis of $V$ ([8] p.421). Here $\{ \zeta_i, \gamma_i \}$ are orthogonal to $t$ because the root spaces $g_i$ are orthogonal to $h$. Further, $\{ \xi_{\pm i} \}$ can be chosen such that $[\zeta_i, \gamma_i] \in t$, and is dual to $\lambda_i \in t^*$ with respect to the Killing form. We define
\[ V_i = R(\zeta_i, \gamma_i). \]
Then $[V_i, V_i] \subset t$. Let $J$ be the almost complex structure on $k \oplus a = g/N$. From (2.6), it follows that $J$ sends $\zeta_i$ to $\gamma_i$, and sends $\gamma_i$ to $-\zeta_i$. Therefore, each $V_i$ is preserved by $J$.

For $i = 1, ..., m$, $(\lambda_i, \sigma') = 0$. Since $[\zeta_i, \gamma_i]$ is dual to $\lambda_i$, it follows that $[\zeta_i, \gamma_i] \in t^\perp_{\sigma'}$. Hence $[V_i, V_i] \subset t^\perp_{\sigma'}$. But the dual vectors of $\lambda_1, ..., \lambda_m$ form a basis of $t^\perp_{\sigma'}$, hence $t^\perp_{\sigma'} = \oplus [V_i, V_i]$.

For $i = m + 1, ..., n$, $(\lambda_i, \sigma) = 0$. By similar argument, $t^\perp_{\sigma} = \oplus [V_i, V_i]$.

Let $\omega$ be a $K \times T$-invariant $(1,1)$-form on $G/N$. Suppose that $i \neq j$; we want to show that $\omega(V_i^\sharp, V_j^\sharp)_a = 0$ for $a \in A \subset KA = G/N$. Let $p : k \rightarrow t$ be the orthogonal projection, annihilating $V$. Let $\xi \in V_i, \eta \in V_j$. From (2.6), it follows that $[\xi, \eta]$ is either 0 or in $V_k$, depending on whether $\lambda_i + \lambda_j$ is some positive root $\lambda_k$. In any case,
\begin{equation}
p[\xi, \eta] = 0 \, ; \, \xi \in V_i, \eta \in V_j.
\end{equation}

Let $\Phi : G/N \rightarrow t^*$ be the moment map corresponding to the $K$-action preserving $\omega$. Then $\Phi(a) \in t^*$, by Proposition 2.1. Consequently,
\[ \omega(\xi^\sharp, \eta^\sharp)_a = (\Phi(a), [\xi, \eta]) \]
\[ = (\Phi(a), p[\xi, \eta]) \text{ since } \Phi(a) \in t^* \]
\[ = 0. \]

Therefore, $\omega(V_i^\sharp, V_j^\sharp)_a = 0$ for $i \neq j$. This proves the proposition. \qed
Let $\omega$ be a $K \times T_\sigma$-invariant Kaehler structure on $G/(P, P)$. Let $\pi$ be the fibration in (2.4). Then $\pi^*\omega$ is a $K \times TA^\perp_\sigma$-invariant $(1,1)$-form on $G/N$. By Proposition 2.1, it has the form

$$\pi^*\omega = \sqrt{-1}\partial \bar{\partial} f,$$

where $f$ is a $K$-invariant function on $G/N$. Since $G/N = KA$, $f \in C^\infty(A)$. We shall show that $f$ can be replaced with another function $F$ which is in the image of

$$\pi^* : C^\infty(G/(P, P)) \longrightarrow C^\infty(G/N),$$

so that we get a potential function for $\omega$.

Let $\sigma$ be the cell which corresponds to $P$ by (1.2), and $\sigma'$ its complementary cell. Then $\sigma'$ defines subgroups $H_{\sigma'}, T_{\sigma'}, A_{\sigma'}$ of $H, T, A$ respectively. By taking the ortho-complements of the Lie algebras $\mathfrak{h}_{\sigma'}, \mathfrak{t}_{\sigma'}, \mathfrak{a}_{\sigma'}$, we construct the subgroups $H^\perp_{\sigma'}, T^\perp_{\sigma'}, A^\perp_{\sigma'}$ as before. Note in particular that $A = A^\perp_{\sigma'} A^\perp_{\sigma}$. Define $F \in C^\infty(A)$ by

$$(2.8) \quad F = \rho^* f, \quad \rho : A \longrightarrow A^\perp_{\sigma'} \longrightarrow A;$$

where $\rho$ is the composite function of the submersion $A \longrightarrow A^\perp_{\sigma'}$ annihilating $A^\perp_{\sigma}$, followed by the inclusion $A^\perp_{\sigma'} \longrightarrow A$. By $G/N = KA$, $F$ extends uniquely to be a $K \times TA^\perp_\sigma$-invariant function on $G/N$. Note that $F$ is in the image of $\pi^*$. We define the $K \times TA^\perp_\sigma$-invariant $(1,1)$-form

$$\Omega = \sqrt{-1}\partial \bar{\partial} F.$$

We shall show that

$$(2.9) \quad \Omega = \pi^*\omega.$$  

Here both $\Omega$ and $\pi^*\omega$ are $K \times TA^\perp_\sigma$-invariant. Since $G/N = KA_{\sigma'} A_{\sigma}^\perp$, we only have to compare them at $a \in A_{\sigma'}^\perp$. Also, Proposition 2.1 says that $\mathfrak{h}_a^\perp$ and $V_a^\perp$ are complementary with respect to both $\Omega_a$ and $\pi^*\omega_a$. Therefore, (2.9) will follow if we can show that

$$(2.10) \quad \Omega(\xi^\perp, \eta^\perp)_a = \pi^*\omega(\xi^\perp, \eta^\perp)_a; \quad \xi, \eta \in \mathfrak{h} \text{ or } \xi, \eta \in V, a \in A_{\sigma'}^\perp.$$
This will be checked by the following two lemmas. Recall that $L_F, L_f : a \to \mathfrak{t}^*$ are the Legendre transforms of $F$ and $f$, described in (2.3).

**Lemma 2.3** $\Omega(\xi^\sharp, \eta^\sharp)_{a} = \pi^* \omega(\xi^\sharp, \eta^\sharp)_{a}$ for all $\xi, \eta \in V, a \in A^\perp_{\sigma'}$.

*Proof:* By Proposition 2.2, the spaces $(V_1)^\sharp_{a}, \ldots, (V_r)^\sharp_{a}$ are pairwise complementary with respect to $\Omega_a$ and $\pi^* \omega_a$, $a \in A^\perp_{\sigma'}$. Therefore, to prove the statement in this lemma, we may consider $\xi, \eta \in V_i$ for each component $V_i$ separately. Since each $V_i$ is two dimensional, it suffices to consider $\xi = \zeta_i, \eta = \gamma_i$. Let $\Phi_F, \Phi_f : G/N \to \mathfrak{t}^*$ be the moment maps of the $K$-actions preserving $\Omega, \pi^* \omega$ respectively. We recall from Proposition 2.1 that $\Phi_F(a) = \frac{1}{2} L_F(\log a), \Phi_f(a) = \frac{1}{2} L_f(\log a)$. We follow the indices $i = 1, \ldots, r$ used in Proposition 2.2, as well as the cells $\sigma, \sigma'$ of dimensions $m, n - m$ respectively. In what follows, we break up our arguments into three cases, according to the different values of the index $i$.

**Case 1:** $i = 1, \ldots, m$.

\[
\Omega(\zeta^\sharp_i, \gamma^\sharp_i)_{a} = (\Phi_F(a), [\zeta_i, \gamma_i]) = (\frac{1}{2} L_F(\log a), [\zeta_i, \gamma_i]).
\]

By Proposition 2.2, $[\zeta_i, \gamma_i] \in \mathfrak{t}^\perp_{\sigma'}$, for $i = 1, \ldots, m$. By (2.8), $L_F(\log a)$ and $L_f(\log a)$ agree on $\mathfrak{t}^\perp_{\sigma'}$, for $a \in A^\perp_{\sigma'}$. Therefore, the last expression is

\[
(\frac{1}{2} L_f(\log a), [\zeta_i, \gamma_i]) = (\Phi_f(a), [\zeta_i, \gamma_i]) = \pi^* \omega(\zeta^\sharp_i, \gamma^\sharp_i)_{a}.
\]

**Case 2:** $i = m + 1, \ldots, n$.

We recall (2.4), which implies that

\[
[v, \zeta_i] = \sqrt{-1}(\lambda_i, v)\gamma_i, \quad [v, \gamma_i] = -\sqrt{-1}(\lambda_i, v)\zeta_i
\]

for all $v \in \mathfrak{t}$. Therefore, the Lie algebra $\mathfrak{k}^\sigma$ of $K^\sigma$ is given by

\[
\mathfrak{k}^\sigma = \{\xi \in \mathfrak{k} ; [\xi, \sigma] = 0\} = \mathfrak{t} \oplus_{(\lambda_i, \sigma) = 0} V_i.
\]
The center of this Lie algebra is \( t_\sigma \), hence the semi-simple Lie algebra \( k_\sigma^{ss} \) is given by

\[
\begin{align*}
k_\sigma^{ss} &= t_\sigma^\perp \oplus_{(\lambda_i, \sigma) = 0} V_i.
\end{align*}
\]

For \( i = m + 1, \ldots, n \), \( (\lambda_i, \sigma) = 0 \); hence \( \zeta_i, \gamma_i \in k_\sigma^{ss} \). But \( K_\sigma^{ss} \) is in the fiber of \( \pi \), so \( \iota(\xi^\sharp)\pi^* \omega_a = 0 \) for all \( \xi \in V_i \).

We shall show that

\[
\iota(\xi^\sharp) \Omega_a = 0
\]

for all \( \xi \in V_i \). Since each \( V_i \) is two dimensional, this will follow if we can show that \( \Omega(\zeta^\sharp_i, \gamma^\sharp_i)_a = 0 \), for \( i = m + 1, \ldots, n \). But

\[
\Omega(\zeta^\sharp_i, \gamma^\sharp_i)_a = \left( \frac{1}{2} L_F(\log a), [\zeta_i, \gamma_i] \right) = 0,
\]

since \( [\zeta_i, \gamma_i] \in t_\sigma^\perp \) and by (2.8), \( L_F(\log a) \) vanishes there.

**Case 3:** \( i = n + 1, \ldots, r \).

From Cases 1, 2, we see that \( L_F(\log a), L_f(\log a) \in t^* \) agree on the spaces \( t_\sigma^\perp, t_{\sigma'}^\perp \).

Since \( t = t_\sigma^\perp \oplus t_{\sigma'}^\perp \), it follows that \( L_F(\log a) = L_f(\log a) \in t^* \). Therefore,

\[
\begin{align*}
\Omega(\zeta^\sharp_i, \gamma^\sharp_i)_a &= (\Phi_F(a), [\zeta_i, \gamma_i]) \\
&= (\frac{1}{2} L_F(\log a), [\zeta_i, \gamma_i]) \\
&= (\frac{1}{2} L_f(\log a), [\zeta_i, \gamma_i]) \\
&= (\Phi_f(a), [\zeta_i, \gamma_i]) \\
&= \pi^* \omega(\zeta^\sharp_i, \gamma^\sharp_i)_a.
\end{align*}
\]

This proves Lemma 2.3. \( \square \)

**Lemma 2.4** \( \Omega(\xi^\sharp, \eta^\sharp)_a = \pi^* \omega(\xi^\sharp, \eta^\sharp)_a \) for all \( \xi, \eta \in h, a \in A_{\sigma'}^\perp \).

**Proof:** Let \( h_\sigma, h_{\sigma'} \) denote the subalgebras of \( h \), by taking the complex linear spans of \( \sigma, \sigma' \) respectively. Let \( h_\sigma^\perp, h_{\sigma'}^\perp \) denote their orthocomplements with respect to the Killing form. Then \( h = h_\sigma^\perp \oplus h_{\sigma'}^\perp \).

**Case 1:** \( \xi, \eta \in h_\sigma^\perp \).
Let $\iota : H^\perp_{\sigma'} \hookrightarrow H$ denote the inclusion. From (2.8), we get

$$\sqrt{-1} \partial \bar{\partial} (\iota^* F) = \sqrt{-1} \partial \bar{\partial} (\iota^* f),$$

where $\partial, \bar{\partial}$ are Dolbeault operators on $H^\perp_{\sigma'}$ here. Therefore, given $a \in A_{\sigma'}^\perp \subset H^\perp_{\sigma'}$,

$$\Omega(\xi^\sharp, \eta^\sharp)_a = (\iota^* \Omega)(\xi^\sharp, \eta^\sharp)_a = (\sqrt{-1} \partial \bar{\partial} (\iota^* F))(\xi^\sharp, \eta^\sharp)_a = (\sqrt{-1} \partial \bar{\partial} (\iota^* f))(\xi^\sharp, \eta^\sharp)_a = (\iota^* \pi^* \omega)(\xi^\sharp, \eta^\sharp)_a = \pi^* \omega(\xi^\sharp, \eta^\sharp)_a.$$

**Case 2:** $\xi \in h^\perp_{\sigma}$.

We shall show that

$$(2.13) \quad \iota(\xi^\sharp) \pi^* \omega_a = \iota(\xi^\sharp) \Omega_a = 0,$$

which completes the proof of this lemma. Since $\pi^* \omega$ and $\Omega$ are $(1,1)$-forms, it suffices to check (2.13) for $\xi \in t^\perp_{\sigma}$.

The fiber of $\pi$ is $K^\sigma_{ss} \times A^\perp_{\sigma}$, which contains $H^\perp_{\sigma}$. Therefore,

$$\iota(\xi^\sharp) \pi^* \omega_a = 0.$$

We observe that, as complex manifolds,

$$H = C^n / Z^n, \quad H^\perp_{\sigma} = C^{n-m} / Z^{n-m}, \quad H^\perp_{\sigma'} = C^m / Z^m,$$

and $H = H^\perp_{\sigma} H^\perp_{\sigma'}$. We introduce complex coordinates $\{z_1, ..., z_m\}$ on $H^\perp_{\sigma'}$ as well as $\{z_{m+1}, ..., z_n\}$ on $H^\perp_{\sigma}$; so that $H$ adopts the product coordinates. Let $z = x + \sqrt{-1} y$, and we let $x, y$ be the coordinates on $T, A$ respectively. From $H = TA, G/N = KA$ and $T \subset K$, we get a natural holomorphic imbedding $\iota : H \hookrightarrow G/N$. Then $\iota^* F$, being $T$-invariant, is a function on $y$ only. For simplicity we still denote it as $F$. It follows from (2.8) that

$$\frac{\partial F}{\partial y_i} = 0 \text{ for } i = m + 1, ..., n.$$
Therefore, for $a \in A^\perp_\sigma$,

$$\iota(\xi^2)(\iota^*\Omega)_a = \iota(\xi^2)(\sqrt{-1}\partial\bar{\partial}F)_a$$

(2.14)

$$= \iota(\xi^2)(\frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 F}{\partial y_j \partial y_k} dx_j \wedge dy_k)$$

$$= \iota(\xi^2)(\frac{1}{2} \sum_{j,k=1}^{m} \frac{\partial^2 F}{\partial y_j \partial y_k} dx_j \wedge dy_k).$$

On the other hand, since $\xi \in t^\perp_\sigma$, the vector field $\xi^2$ on $H$ is of the form

$$\xi^2 = \sum_{m+1}^{n} c_i \frac{\partial}{\partial x_i}.$$ 

This, together with (2.14), imply that

$$\iota(\xi^2)\Omega_a = 0.$$

This proves (2.13). Combining the results in Cases 1, 2, we have proved Lemma 2.4. 

\[\square\]

Lemmas 2.3 and 2.4 imply (2.10), and hence (2.9). Namely, we have shown that given a $K \times T_\sigma$-invariant Kaehler structure $\omega$ on $G/(P, P)$, there exists a function $F$, which is in the image of $\pi^*$ by virtue of (2.8), such that

$$\pi^*\omega = \sqrt{-1}\partial\bar{\partial}F.$$

Since $F$ is in the image of $\pi^*$, and since $\pi^*$ is injective, it follows that $\omega$ itself has a potential function.

Conversely, suppose that a $K$-invariant Kaehler structure $\omega$ on $G/(P, P)$ has a potential function $F$. Averaging by the compact group $K$ if necessary, we may assume that $F$ is $K$-invariant. But by (2.3), this means that $F$ is just a function on $A_\sigma$, and is automatically $K \times T_\sigma$-invariant. Then $\omega$ is also $K \times T_\sigma$-invariant. This proves Theorem I.

We note that our arguments do not require $\omega$ to be positive definite. Namely, Theorem I holds even if $\omega$ is merely a $K$-invariant $(1,1)$-form. In the next section, we use the moment map to derive a necessary and sufficient condition for a $K \times T_\sigma$-invariant $(1,1)$-form to be Kaehler.
3  MOMENT MAP

Let $\omega$ be a $K \times T_\sigma$-invariant $(1,1)$-form on $G/(P, P)$, with moment map

$$\Phi : G/(P, P) \longrightarrow k^*$$

corresponding to the Hamiltonian action of $K$ on $G/(P, P)$ preserving $\omega$. It is easy to see that this action is Hamiltonian; either from the semi-simplicity of $K$ ([8], §26), or from the fact that $\omega = \sqrt{-1} \partial \bar{\partial} F$ implies $\omega = d\beta$ for some $K$-invariant real 1-form $\beta$ ([8], Theorem 4.2.10). We shall study the moment map $\Phi$, and derive a necessary and sufficient condition for $\omega$ to be Kaehler.

Suppose now that $\omega$ is a $K \times T_\sigma$-invariant Kaehler structure. We want to derive the two conditions stated in Theorem II. By Theorem I, $\omega$ has a potential function $F$. Averaging by $K$ if necessary, we may assume that $F$ is $K$-invariant. By (2.3), $G/(P, P) = (K/K^\sigma_{ss}) \times A_\sigma$; so the $K$-invariant function $F$ is just a function on $A_\sigma$. Let $\pi$ be the fibration in (2.4). Then

$$\Phi \circ \pi : G/N \longrightarrow k^*$$

is the moment map corresponding to the $K$-action on $(G/N, \pi^* \omega)$. Recall that $P$ corresponds to a cell $\sigma$ via (1.2). Also, $G/N = KA$ and $G/(P, P) = (K/K^\sigma_{ss}) \times A_\sigma$ induce the inclusions

$$A \hookrightarrow \{e\} \times KA = G/N , \quad A_\sigma \hookrightarrow \{eK^\sigma_{ss}\} \times A_\sigma \subset (K/K^\sigma_{ss}) \times A_\sigma = G/(P, P) .$$

Therefore, we can regard $A$ and $A_\sigma$ as contained in $G/N$ and $G/(P, P)$ respectively. Note that $\pi(A) = A_\sigma$. From Proposition 2.1, we see that

$$(\Phi \circ \pi)(A) \subset t^* .$$

Since the fibration $\pi$ sends $A$ to $A_\sigma$, it follows that $\Phi(A_\sigma) \subset t^*$. By $K$-equivariance of $\Phi$, $\Phi|_{A_\sigma}$ determines $\Phi$ entirely. The exponential map from $a_\sigma$ to $A_\sigma$ is a diffeomorphism, and we let log be its inverse. This way, the potential function $F$ becomes a
function on $a_\sigma$. Then, by the almost complex structure, $a_\sigma^* \cong t_\sigma^*$. Consequently, the Legendre transform of $F$ is

$$L_F : a_\sigma \to t_\sigma^*.$$  

We shall show that

$$\Phi : A_\sigma \to t^*$$

is given by $\Phi(a) = \frac{1}{2}L_F(\log a)$ for all $a \in A_\sigma$. Let

$$\iota : H_\sigma \to G/(P, P)$$

be the natural holomorphic imbedding of $H_\sigma = T_\sigma A_\sigma$. Then $\iota^*\omega$ is a $T_\sigma$-invariant Kaehler structure on $T_\sigma A_\sigma$, with potential function $\iota^*F$. For simplicity, we still write $\iota^*F$ as $F$. Let $m$ be the dimension of the cell $\sigma$. Then, as a complex manifold, $H_\sigma = \mathbb{C}^m/\mathbb{Z}^m$. Therefore, we can introduce complex coordinates $\{z_1, \ldots, z_m\}$ on $H_\sigma$, where

$$H_\sigma = \mathbb{C}^m/\mathbb{Z}^m = \{z_1, \ldots, z_m\}, \quad T_\sigma = \mathbb{R}^m/\mathbb{Z}^m = \{x_1, \ldots, x_m\}, \quad A_\sigma = \mathbb{R}^m = \{y_1, \ldots, y_m\}, \quad z_i = x_i + \sqrt{-1}y_i.$$  

Since $F$ is $T_\sigma$-invariant, it is a function on $y$ only. Then $\iota^*\omega$ becomes (here $\partial, \bar{\partial}$ are Dolbeault operators on $H_\sigma$)

$$\iota^*\omega = \sqrt{-1}\partial\bar{\partial}F = \frac{1}{2} \sum_{j,k=1}^{m} \frac{\partial^2 F}{\partial y_j \partial y_k} dx_j \wedge dy_k,$$  

where $F \in C^\infty(\mathbb{R}^m)$. Since $\omega$ is Kaehler, so is $\iota^*\omega$; and (3.2) says that $\iota^*\omega$ is Kaehler if and only if the Hessian matrix of $F$ is positive definite, i.e. $F$ is strictly convex.

The moment map $\Phi$ of the $K$-action on $(G/(P, P), \omega)$ restricts to be the moment map $\Phi'$ of the $T_\sigma$-action on $(T_\sigma A_\sigma, \iota^*\omega)$. Let

$$\beta = -\frac{1}{2} \sum_{j=1}^{m} \frac{\partial F}{\partial y_j} dx_j$$

be a $T_\sigma$-invariant 1-form on $T_\sigma A_\sigma$. From (3.2), it follows that $d\beta = \iota^*\omega$, so the moment
map $\Phi'$ of the $T_\sigma$-action is

$$
(\Phi'(ta), \xi) = -(\beta, \xi^\sharp)(ta)
= \left(\frac{1}{2} \sum_{j=1}^m \frac{\partial F}{\partial y_j} dx_j, \sum_{k=1}^m \xi_k \frac{\partial}{\partial x_k}\right)(ta)
= \frac{1}{2} \sum_{j=1}^m \frac{\partial F}{\partial y_j}(a) \xi_j
= \frac{1}{2} (L_F(a), \xi),
$$

where $ta \in T_\sigma A_\sigma, \xi \in \mathfrak{t} = \mathbb{R}^m$. Our computation identifies $a$ with $A$ by the exponential map, so in fact $\Phi'(ta) = \frac{1}{2} L_F(\log a)$ for all $ta \in T_\sigma A_\sigma$. But $\Phi$ and $\Phi'$ agree on $A_\sigma$, so $\Phi(a) = \frac{1}{2} L_F(\log a)$. Hence $\Phi(A_\sigma) \subset t^*_\sigma$. We claim further that $\Phi(A_\sigma) \subset \sigma$.

Let $V_i \subset V \subset \mathfrak{t}$ be the subspaces constructed in Proposition 2.2, and let $\{\zeta_i, \gamma_i\} \in V_i$ be the vectors in (2.6). Recall that these indices are made with respect to the positive roots $\{\lambda_i\}$. Since $G/(P, P) = (K/K_{ss}) \times A_\sigma$, the infinitesimal vector fields $\zeta_i^\sharp, \gamma_i^\sharp$ on $G/(P, P)$ are non-zero if and only if $\zeta_i, \gamma_i \notin \mathfrak{t}_{ss}^\sigma$. By (2.12), this is equivalent to $(\lambda_i, \sigma) > 0$. Let $J$ be the almost complex structure in $G/(P, P), a \in A_\sigma$, and $(\lambda_i, \sigma) > 0$ so that $\zeta_i^\sharp, \gamma_i^\sharp \neq 0$. By (2.13), $J\zeta_i = \gamma_i$. Since $\omega$ is Kaehler,

$$
0 < \omega(\zeta_i^\sharp, J\zeta_i^\sharp)_a
= \omega(\zeta_i^\sharp, \gamma_i^\sharp)_a
= (\Phi(a), [\zeta_i, \gamma_i])
= (\Phi(a), \lambda_i).
$$

(3.3)

We have shown that, for all $a \in A_\sigma$, $(\Phi(a), \lambda_i) > 0$ whenever $\lambda_i$ is a positive root satisfying $(\lambda_i, \sigma) > 0$. This, together with $\Phi(A_\sigma) \subset t^*_\sigma$, imply that $\Phi(A_\sigma) \subset \sigma$, as claimed.

We have shown that if $\omega$ is Kaehler, then the two conditions stated in Theorem II have to be satisfied. We next show that, conversely, these two conditions are sufficient for $\omega$ to be Kaehler.

Recall that the infinitesimal vector field $\xi^\sharp$ on $G/(P, P)$ vanishes if $\xi \in \mathfrak{t}_{ss}^\sigma$. Hence the tangent space at $a \in A_\sigma \subset G/(P, P)$ is spanned by $(\mathfrak{t}_{ss}^\sigma)_a^\bot, (a_\sigma)^\sharp_a$. Here we define $\eta^\sharp$ for $\eta = J\xi \in a_\sigma$ by $\eta^\sharp = J\xi^\sharp$, where $\xi \in \mathfrak{t}_\sigma$. However, it follows from (2.12) that

$$
\mathfrak{t}_{ss}^\sigma = \mathfrak{t}_\sigma \oplus (\lambda_i, \sigma) > 0 V_i,
$$
where $V_i$ is the space described in Proposition 2.2. Here the distinct $V_i$ are orthogonal to one another, due to the orthogonality of the root spaces $g_i$ ([8], p.166).

Consequently, the tangent space at $a \in A_\sigma \subset G/(P,P)$ is

\[(3.4)\]

$$T_a(G/(P,P)) = (h_{\sigma})_a^\sharp \oplus_{(\lambda_i,\sigma) > 0} (V_i)_a^\sharp.$$  

We claim that $\omega(h_{\sigma}^\sharp, V_i^\sharp)_a = \omega(V_i^\sharp, V_j^\sharp)_a = 0$, for $i \neq j$:

Since $J$ preserves $h_{\sigma}$ and $V_i$, and $\omega$ is a $(1,1)$-form, the first part follows if we can show that $\omega(t_{\sigma}^\sharp, V_i^\sharp)_a = 0$. Let $p : t \rightarrow t$ be the orthogonal projection, annihilating $V$. Let $\xi \in t_{\sigma}, \eta \in V_i$. Then $p[\xi, \eta] = 0$, by (2.14). Since $\Phi(a) \in t^*$ for $a \in A$,

$$\omega(\xi^\sharp, \eta^\sharp)_a = (\Phi(a), [\xi, \eta]) = (\Phi(a), p[\xi, \eta]) = 0.$$  

Hence $\omega(h_{\sigma}^\sharp, V_i^\sharp)_a = 0$. For $i \neq j$, it follows from (2.7) that $p[V_i, V_j] = 0$. So, by similar argument, $\omega(V_i^\sharp, V_j^\sharp)_a = 0$ as claimed.

Therefore, by $K$-invariance of $\omega$ and (3.4), the positive definite condition of $\omega$ follows if we can check that

\[(3.5)\]

$$\omega(\xi^\sharp, J\xi^\sharp)_a > 0 \ ; \ \xi \in h_{\sigma} \text{ or } \xi \in V_i, (\lambda_i, \sigma) > 0, a \in A_\sigma.$$  

But they follow from the two conditions of Theorem II: Condition (i) of Theorem II implies that the expression in (3.2) is positive definite and hence (3.3) holds for $\xi \in h_{\sigma}$. Condition (ii) of Theorem II implies that $\omega(\Phi(a), \lambda_i) > 0$ whenever $(\lambda_i, \sigma) > 0$, so it follows from (3.3) that (3.3) holds for $\xi \in V_i$. This proves Theorem II.
Fix a $K \times T_\sigma$-invariant Kaehler structure $\omega$ on $G/(P, P)$. By Theorem I, $\omega$ has a potential function $F$. Recall that $P$ determines the subgroup $A_\sigma$ by (1.2). By $K$-invariance and (2.3), we can regard $F$ as a function on $A_\sigma$. In particular, the expression $\omega = \sqrt{-1} \partial \bar{\partial} F$ also implies that $\omega$ is exact. Hence $\omega$ is integral, and there exists a complex line bundle $L$ on $G/(P, P)$ whose Chern class is $[\omega] = 0$, equipped with a connection $\nabla$ whose curvature is $\omega$, as well as an invariant Hermitian structure $<, >$ (5, 11). The line bundle $L$ is trivial since $[\omega] = 0$, but the connection $\nabla$ gives rise to interesting geometry. We say that a section $s$ is holomorphic if $\nabla s$ annihilates anti-holomorphic vector fields on $G/(P, P)$. We shall show that the $K \times T_\sigma$-action on $G/(P, P)$ lifts to a $K \times T_\sigma$-representation on the space of holomorphic sections of $L$. To do this, we shall construct a global trivialization of $L$. The following topological property of $G/(P, P)$ is useful in this construction:

\textbf{Lemma 4.1} \quad H^1(G/(P, P), C) = 0.

\textit{Proof:} \quad By (2.3), $G/(P, P) = (K/K_{ss}^\sigma) \times A_\sigma$. Since $A_\sigma$ is Euclidean, it suffices to show that $H^1(K/K_{ss}^\sigma, C) = 0$.

The fibration $K \rightarrow K/K_{ss}^\sigma$ induces a long exact sequence of homotopy groups,

(4.1) \quad \ldots \rightarrow \pi_1(K) \rightarrow \pi_1(K/K_{ss}^\sigma) \rightarrow \pi_0(K_{ss}^\sigma) \rightarrow \ldots

However, by (\cite{2} p.223),

$$\pi_1(K) \cong \ker(\exp : t \rightarrow T)/\mathbb{Z}(\text{roots of } \mathfrak{k}).$$

Therefore, since $K$ is semi-simple, $\pi_1(K)$ is finite. By compactness of $K_{ss}^\sigma$, $\pi_0(K_{ss}^\sigma)$ is finite. Hence $\pi_1(K/K_{ss}^\sigma)$, being caught in the middle in (1.1), is also finite. It follows that

$$H^1(K/K_{ss}^\sigma, C) \cong \text{Hom}(\pi_1(K/K_{ss}^\sigma), C) = 0,$$

which proves the lemma. \hfill \Box
We return to our pre-quantum line bundle $L$ on $G/(P, P)$, corresponding to the $K \times T_{\sigma}$-invariant Kaehler structure $\omega$. Let $\beta$ be the 1-form $-\sqrt{-1}\partial F$, so $d\beta = \omega$. We claim that

**Proposition 4.2** There exists a non-vanishing section $s_o$ on $L$, with the property

(4.2) \[ \beta = \frac{1}{\sqrt{-1}} \nabla_{s_o} s_o. \]

This section is unique up to a non-zero constant multiple, and is holomorphic. Up to a non-zero constant,

\[ <s_o, s_o> = e^{-F}. \]

**Proof:** Since $[\omega] = 0$, $L$ is a trivial bundle; so there exists a nowhere zero section $s_1$ of $L$. Let

\[ \alpha = \frac{1}{\sqrt{-1}} \nabla_{s_1} s_1. \]

By the definition of the curvature form on $L$, $d\alpha = \omega$; so $d(\beta - \alpha) = 0$. Since $H^1(G/(P, P), \mathbb{C}) = 0$, there exists a complex-valued function $f$ such that $\beta = \alpha + df$. Let $s_o = (\exp \sqrt{-1}f)s_1$. Then

\[ \frac{1}{\sqrt{-1}} \nabla_{s_o} s_o = \frac{1}{\sqrt{-1}} \nabla_{s_1} s_1 + df = \beta. \]

This proves the existence of a holomorphic section $s_o$ satisfying (4.2).

Suppose that $s_1$ and $s_2$ are two sections satisfying this formula. Let $h = \frac{s_2}{s_1}$. Then

\[ \frac{1}{\sqrt{-1}} \nabla_{s_2} s_2 = \frac{1}{\sqrt{-1}} \nabla_{s_1} s_1 + \frac{1}{\sqrt{-1}} d\log h, \]

which implies that $h$ is a constant. Hence, up to a constant, the solution of (4.2) is unique.

If $v$ is an anti-holomorphic vector field, then

\[ \frac{1}{\sqrt{-1}} \nabla_{v s_o} s_o = \iota(v)\beta = 0, \]
as $\beta$ is a form of type $(1,0)$. Hence $s_o$ is holomorphic. Since $\beta$ is $K \times T_\sigma$-invariant, $s_o$ induces a $K \times T_\sigma$-representation on the space of holomorphic sections on $L$, where $s_o$ is $K \times T_\sigma$-invariant. Namely, given a holomorphic section $f s_o$ of $L$ (note $s_o$ is non-vanishing), $K \times T_\sigma$ acts by

$$\tag{4.3}\ L_k^* R_t^*(f s_o) = (L_k^* R_t^* f) s_o \ ; \ k \in K, t \in T_\sigma,$$

where $L_k^* R_t^* f$ denotes the standard action on the holomorphic functions lifted from the $K \times T_\sigma$-action on $G/(P, P)$. Hence $s_o$ defines a $K \times T_\sigma$-equivariant trivialization.

For this section $s_o$, we now show that $< s_o, s_o > = e^{-F}$. By $K$-invariance, it suffices to show that this is the case when restricted to $A_\sigma$. Let $\sigma$ be the cell corresponding to the parabolic subgroup $P$, and let $m$ be the dimension of $\sigma$. We write $T_\sigma A_\sigma = \mathbb{C}^m/\mathbb{Z}^m = \{z_1, ..., z_m\}$ as in (3.1), so that $F$, being $T_\sigma$-invariant, is a function on $y$ only. Let $i : T_\sigma A_\sigma \to G/(P, P)$ be the natural inclusion. Then

$$\tag{4.4} i^* \beta = -\sqrt{-1} \partial F = \frac{1}{2} \sum_1^m \frac{\partial F}{\partial y_i} dz_i.$$ 

Let $\nabla_i = \nabla_{\beta_i}$. Then

$$\frac{\partial}{\partial y_i} < s_o, s_o >= < \nabla_i s_o, s_o > + < s_o, \nabla_i s_o > .$$

However, by (1.2) and (1.4),

$$\nabla_i s_o \over s_o = \sqrt{-1} (\beta, \frac{\partial}{\partial y_i}) = -\frac{1}{2} \frac{\partial F}{\partial y_i}$$

so

$$\frac{\partial}{\partial y_i} \log < s_o, s_o > = -\frac{\partial F}{\partial y_i}.$$

Therefore, up to a non-zero constant multiple,

$$< s_o, s_o > = e^{-F}.$$ 

This proves the proposition. \qed
Let $\mathcal{O}(L)$ denote the space of holomorphic sections of the line bundle $L$ on $G/(P, P)$. By Proposition 4.2, $s_o$ induces a $K \times T_\sigma$-representation on $\mathcal{O}(L)$, given by (13), where $s_o$ is $K \times T_\sigma$-invariant. Let $\lambda \in t_\sigma^*$ be a dominant integral weight, and let $\mathcal{O}(L)_\lambda$ denote the holomorphic sections that transform by $\lambda$ under the right $T_\sigma$-action. Since this action commutes with the left $K$ action, $\mathcal{O}(L)_\lambda$ is a $K$-subrepresentation of $\mathcal{O}(L)$. We now show that the $K$-finite vectors in $\mathcal{O}(L)$ decompose into $\{\mathcal{O}(L)_\lambda ; \lambda \in \sigma\}$ as irreducible $K$-representations with highest weights $\lambda$. Using the holomorphic section $s_o$ of Proposition 4.2, it suffices to consider the holomorphic functions $\mathcal{O}(G/(P, P))$; since

$$\mathcal{O}(G/(P, P)) \otimes s_o = \mathcal{O}(L)$$

is a $K \times T_\sigma$-equivariant trivialization.

Recall that $\overline{W}$ is the closure of the Weyl chamber $W$, and $\sigma \subset \overline{W}$ is the cell corresponding to $P$. Let $\overline{\sigma}$ denote its closure in $\overline{W}$. For a dominant integral weight $\lambda \in t^*$, let $\mathcal{O}_\lambda \subset \mathcal{O}(G/(P, P))$ denote the holomorphic functions that transform by $\lambda$ under the right $T_\sigma$-action. Since the right $T_\sigma$-action commutes with the left $K$-action, each $\mathcal{O}_\lambda$ is a $K$-representation space.

**Proposition 4.3** The irreducible $K$-representation with highest weight $\lambda$ occurs in $\mathcal{O}(G/(P, P))$ if and only if $\lambda \in \overline{\sigma}$. For $\lambda \in \overline{\sigma}$, it occurs with multiplicity one, and is given by $\mathcal{O}_\lambda$.

**Proof:** The fibration $\pi$ of (24) induces an injection of holomorphic functions,

$$\pi^* : \mathcal{O}(G/(P, P)) \rightarrow \mathcal{O}(G/N).$$

This map intertwines with the $K \times T_\sigma$-action.

Let $\lambda$ be a dominant integral weight, but suppose that $\lambda \notin \overline{\sigma}$. We shall show that the $K$-irreducible with highest weight $\lambda$ does not occur in $\mathcal{O}(G/(P, P))$. By the Borel-Weil theorem, the $K$-irreducible with highest weight $\lambda$ occurs in $\mathcal{O}(G/N)$ with multiplicity one, and can be taken as the holomorphic functions in $G/N$ that...
transform by $\lambda$ under the right $T$-action. We denote this space by $V_\lambda \subset \mathcal{O}(G/N)$. Since $\pi^*$ is injective, it suffices to show that

\begin{equation}
\pi^* \mathcal{O}(G/(P, P)) \cap V_\lambda = 0.
\end{equation}

Since $\lambda \not\in \sigma$, $(\lambda, \xi) \neq 0$ for some $\xi \in t_\sigma^\perp$. Let $0 \neq f \in V_\lambda$. Then the right action $R_\xi^*$ on $V_\lambda$ satisfies

\begin{equation}
R_\xi^* f = (\lambda, \xi) f \neq 0.
\end{equation}

Since $T_\sigma^\perp$ is in the fiber of $\pi$, the image of $\pi^*$ is $T_\sigma^\perp$-invariant. Therefore, (4.3) says that $f$ cannot be in the image of $\pi^*$. This proves (4.5).

Conversely, suppose that $\lambda \in \sigma$ is a dominant integral weight. We again let $V_\lambda \subset \mathcal{O}(G/N)$ be the holomorphic functions that transform by $\lambda$. By the Borel-Weil theorem, $V_\lambda$ is an irreducible representation with highest weight $\lambda$, and such irreducible occurs with multiplicity one. Therefore, to complete the proof of Proposition 4.3, we need to show

\begin{equation}
V_\lambda \subset \pi^* \mathcal{O}(G/(P, P)), \lambda \in \sigma.
\end{equation}

Recall from (2.2) that the fiber of $\pi$ is $(K_{ss}^\sigma)_C/(M \cap N) = K_{ss}^\sigma \times A_\sigma^\perp$. Choose a fiber of $\pi$, and let $i : K_{ss}^\sigma A_\sigma^\perp \hookrightarrow G/N$ be a holomorphic $K_{ss}^\sigma \times A_\sigma^\perp$-equivariant imbedding as this fiber of $\pi$. Let $f \in V_\lambda$. We claim that $f$ is constant on this fiber:

By applying the Borel-Weil theorem on $(K_{ss}^\sigma)_C/(M \cap N) = K_{ss}^\sigma \times A_\sigma^\perp$, we see that $i^* f$, which is right $T_\sigma^\perp$-invariant since $(\lambda, t_\sigma^\perp) = 0$, has to be a constant function. Hence $f$ is constant on that fiber, as claimed.

Since our argument is independent of the choice of fiber and element of $V_\lambda$, we conclude that every element of $V_\lambda$ is constant on every fiber of $\pi$. This implies (4.7), and Proposition 4.3 is now proved. □

24
We have shown that the irreducible $K$-representation with highest weight $\lambda$ occurs in $O(L)$ if and only if $\lambda \in \mathfrak{g}$. For $\lambda \in \mathfrak{g}$, it occurs with multiplicity one, and is given by $O(L)_\lambda$. We shall decide which of these irreducible $K$-representations are square-integrable, in the following sense.

From the description $G/(P, P) = (K/K_{ss}) \times A_{\sigma}$, we see that there is a $K \times A_{\sigma}$-invariant measure $\mu$ on $G/(P, P)$, which is unique up to non-zero constant. Given a holomorphic section $s$ of $L$, we consider the integral

$$\int_{G/(P, P)} \langle s, s \rangle \, \mu.$$ 

Let $H_\omega \subset O(L)$ be the holomorphic sections in which this integral converges. Since the Hermitian structure $\langle , \rangle$ and $\mu$ are $K$-invariant, $H_\omega$ becomes a unitary $K$-representation space. The next proposition shows which irreducible $K$-representations occur in $H_\omega$.

Let $\lambda \in \mathfrak{g}$ be a dominant integral weight. Let

$$\Phi : G/(P, P) \rightarrow \mathfrak{k}^*$$

be the moment map of the $K$-action on $(G/(P, P), \omega)$. Recall that $O_\lambda$ and $O(L)_\lambda$ are respectively the holomorphic functions and sections that transform by $\lambda \in t_{\sigma}^*$ under the right $T_{\sigma}$-action.

**Proposition 4.4** Let $s \in O(L)_\lambda$. Then $s \in H_\omega$ if and only if $\lambda$ is in the image of the moment map.

**Proof:** Let $s_o$ be the unique holomorphic section of Proposition 4.2. Therefore, $\langle s_o, s_o \rangle = e^{-F}$, where $F$ is the potential function of $\omega$. Since $s_o$ is non-vanishing and $K \times T_{\sigma}$-invariant,

$$O(L)_\lambda = O_\lambda \otimes s_o.$$ 

Therefore, we are reduced to showing that $f \in O_\lambda$ satisfies

$$\int_{G/(P, P)} |f|^2 e^{-F} \, \mu < \infty$$

(4.8)
if and only if $\lambda$ is in the image of $\Phi$.

Here $\mu$ is the product of a $K$-invariant measure $dk$ on $K/K_{ss}$ and a Haar measure $da$ on $A_\sigma$. By the exponential map, the measure $da$ on $A_\sigma$ can be identified with the Lebesgue measure $dy$ on $R^m$, where $m = \dim \sigma$. Given $k \in K$, the left $K$-action on $\mathcal{O}_\lambda$, $L_k^*: \mathcal{O}_\lambda \to \mathcal{O}_\lambda$, is

$$(L_k^* f)(p) = f(kp).$$

Let $f_1, ..., f_N$ be a basis of $\mathcal{O}_\lambda$ which is orthonormal with respect to the (unique) $K$-invariant inner product on $\mathcal{O}_\lambda$. Given an element $f = \sum c_i f_i$ of $\mathcal{O}_\lambda$,

$f(ky) = (L_k^* f)(y) = \sum c_i a_{ir}(k) f_r(y),$

where $a_{ir}(k)$ is the $ir$th matrix coefficient of the $K$-representation on $\mathcal{O}_\lambda$ with respect to the basis above. Thus

$$\int |f(ky)|^2 dk = \sum c_i \overline{c_j} (\int a_{ir}(k) \overline{a_{js}(k)} dk) f_r(y) \overline{f_s(y)},$$

where the integrals are taken over $K/K_{ss}$. However, by Peter-Weyl the inner integral is equal to

$$\frac{1}{N} \delta_{ij} \delta_{rs},$$

(E p.186) so the integral (4.8) reduces to

$$\frac{1}{N} \|f\|^2 \int_{R^m} \sum |f_r(y)|^2 e^{-F(y)} dy,$$

where $\|f\|$ is the norm of $f$ with respect to the given $K$-invariant inner product structure on $\mathcal{O}_\lambda$. However, each of the functions $f_r(y)$ transforms under the infinitesimal $t_\sigma$-action according to the character $\lambda \in t_\sigma^*$, and therefore, being holomorphic, transforms under the action of $h_\sigma = (t_\sigma)_C$ according to the complexified character $\lambda_C \in h_\sigma^*$. In particular, $|f_r(y)|^2$ is a constant multiple of $e^{2\lambda(y)}$. Hence if $f \neq 0$, (4.9) is a constant multiple of the integral

$$\int_{R^m} e^{-F(y)+2\lambda(y)} dy.$$
However, this integral converges if and only if $2\lambda$ is in the image of the Legendre transform of $F$ ([4], Appendix); or equivalently if and only if $\lambda$ is in the image of the moment map. This proves the proposition.

\[ \square \]

With this result, Theorem III follows. We see from Theorems II, III that not all irreducibles are contained in $H_\omega$: The irreducible representation $O(L)_\lambda$ with highest weight $\lambda$ satisfies $O(L)_\lambda \subset H_\omega$ if and only if $\lambda \in \frac{1}{2}L_F(a_\sigma) \subset \sigma$. This necessarily excludes $\lambda \in \sigma \setminus \sigma$. However, in the next section, we shall see that the potential function $F$ can be constructed such that $\frac{1}{2}L_F(a_\sigma) = \sigma$, and hence $O(L)_\lambda \subset H_\omega$ for all $\lambda \in \sigma$. 

27
5 CONSTRUCTION OF A MODEL

Let $P$ be a parabolic subgroup of $G$, and let $\sigma$ its corresponding cell of dimension $m$, given in (1.2). There exist dominant fundamental weights $\alpha_1, \ldots, \alpha_m \in a_\sigma^*$ (p.498) such that

$$\sigma = \{ \sum_{i=1}^m y_i \alpha_i ; y_i > 0 \}.$$  

Let $F_P : a_\sigma \rightarrow \mathbb{R}$ be defined by

$$(5.1) \quad F_P(v) = \sum_{i=1}^m e^{\alpha_i(v)}.$$  

Then $F_P \in C^\infty(a_\sigma)$ is strictly convex, and the image of its Legendre transform is exactly $\sigma$. Therefore, the moment map $\Phi$ satisfies $\Phi(A_\sigma) = \sigma$. Extend $F_P$ to $G/(P,P)$ by $K$-invariance, and it follows from Theorem II that

$$\omega_P = \sqrt{-1} \partial \bar{\partial} F_P$$

is a Kaehler structure on $G/(P,P)$. Let $L_P$ be the corresponding line bundle, described before. For a dominant integral weight $\lambda$, we let $O(L_P)_{\lambda}$ denote the holomorphic sections of $L_P$ that transform by $\lambda$ under the right $T_\sigma$-action. Let $H_{\omega_P}$ be the holomorphic sections that are square-integrable under (1.4), so that it is a unitary $K$-representation space. By Theorem III, $O(L_P)_{\lambda}$ is an irreducible $K$-representation with highest weight $\lambda$, whenever $\lambda \in \sigma$. Further, since $\Phi(A_{\sigma}) = \sigma$, $O(L_P)_{\lambda} \subset H_{\omega_P}$ whenever $\lambda \in \sigma$.

We repeat this geometric construction among all the parabolic subgroups $P$ containing the fixed Borel subgroup $B = HN$. In each case, we use $F_P$ in (5.1) as the potential function for the Kaehler structure $\omega_P$ on $G/(P,P)$. Then the direct sum

$$\bigoplus_{B \subset P} H_{\omega_P}$$

is a model in the sense of I.M. Gelfand [7]: a unitary $K$-representation where all irreducibles occur with multiplicity one.
References

[1] R. Abraham and J. Marsden, *Foundations of Mechanics, 2nd. ed.*, Addison-Wesley, 1985.

[2] T. Brocker and T. Dieck, *Representations of compact Lie groups*, Springer-Verlag, N.Y. 1985.

[3] C. Chevalley, *Theory of Lie Groups*, Princeton U. Press, Princeton 1946.

[4] M.K. Chuah and V. Guillemin, *Kaehler structures on $K_C/N$*, Contemporary Math. 154 : The Penrose transform and analytic cohomology in representation theory (1993), 181-195.

[5] V. Guillemin and S. Sternberg, *Geometric quantization and multiplicities of group representations*, Invent. Math. 67 (1982), 515-538.

[6] V. Guillemin and S. Sternberg, *Symplectic techniques in physics*, Cambridge U. Press, Cambridge 1991.

[7] I.M. Gelfand and A. Zelevinski, *Models of representations of classical groups and their hidden symmetries*, Funct. Anal. Appl. 18 (1984), 183-198.

[8] S. Helgason, *Differential Geometry, Lie groups, and symmetric spaces*, Academic Press, 1978.

[9] S. Helgason, *Groups and Geometric Analysis*, Academic Press, 1984.

[10] A. Knapp, *Representation Theory of Semisimple Groups*, Princeton U. Press, Princeton 1986.

[11] B. Kostant, *Quantization and unitary representations*, Lecture Notes in Math. 170, Springer 1970, 87-208.

[12] H.S. La, P. Nelson, A.S. Schwarz, *Virasoro Model Space*, Comm. Math. Phys. 134 (1990), 539-554.
DEPARTMENT OF APPLIED MATHEMATICS, NATIONAL CHIAO TUNG UNIVERSITY, HSINCHU, TAIWAN.

E-mail address: chuah@math.nctu.edu.tw