Constants in Markov’s and Bernstein inequality on a finite interval in $\mathbb{R}$

Grzegorz Sroka

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Abstract
In this paper we demonstrate the constants in the pointwise Bernstein inequality

$$|P^{(\alpha)}(x)| \leq \left( \frac{2n}{\sqrt{(x-a)(b-x)}} \right)^{\alpha} ||P||_{[a,b]},$$

for the $\alpha-$th derivative of an algebraic polynomial in $L^\infty-$norms on an interval in $\mathbb{R}$, where $\alpha \geq 3$. This result was obtained using the tools of theory of pluripotential and we apply it to get the main result which is a new generalization of V. A. Markov’s type inequalities

$$||P^{(\alpha)}||_p \leq C^{1/p} \left( \frac{2}{b-a} \right)^{\alpha} ||T_n^{(\alpha)}||_{[-1,1]} n^{2/p} ||P||_p,$$

for the $\alpha-$th derivative of an algebraic polynomial in $L^p$ norms, where $p \geq 1$. In particular, we show that for any $\alpha \geq 3$ the constant $C$ in the V. A. Markov inequality satisfies the condition $C \leq 8 \left( \frac{32 \cdot 3,94741 \cdot \pi M a^2}{3 \sqrt{3}} \right)^{1/p}$.

Keywords Bernstein inequality · V. A. Markov’s inequality · Siciak’s extremal function · Constant in polynomial inequality

1 Introduction

Let $\mathbb{R}[x]$ be the ring of algebraic polynomials in one real variable and $\mathbb{R}_n[x] = \{ P \in \mathbb{R}[x] : \deg P \leq n \}$. We set

\[ \mathbb{R}_n[x] = \{ P \in \mathbb{R}[x] : \deg P \leq n \} \]
\[ \|P\|_{[a,b]} := \max_{x \in [a,b]} |P(x)|, \quad P \in \mathbb{R}[x], \quad [a, b] \subset \mathbb{R}. \]

\[ \|P\|_p = \|P\|_{L^p([a,b])} := \left( \int_a^b |P(x)|^p dx \right)^{1/p}, \quad 1 \leq p < \infty. \]

The Bernstein inequality (cf. [25]) holds for the interior points of the interval \([a, b]\) and is of the following form

\[ |P'(x)| \leq \frac{n}{\sqrt{(x-a)(b-x)}} \cdot \|P\|_{[a,b]} \quad (1) \]

The inequality is exact in the sense that for any fixed point \(x \in (a, b)\) as \(n \to \infty\) the formula (cf. [25])

\[ \sup_{P \in \mathbb{R}[x], \ p \neq 0} \frac{|P'(x)|}{\|P\|_{[a,b]} \cdot \sqrt{(x-a)(b-x)}} = n(1 + o(1)) \]

is satisfied.

For sets on the real line the general form of the Bernstein’s inequality is given in (cf. [7]) and (cf. [36–38]). Let \(E \subset \mathbb{R}\) be a compact set, then we get

\[ |P'(x)| \leq n\pi \omega_E(x) \cdot \|P\|_E, \quad x \in \text{Int}(E), \quad (2) \]

for algebraic polynomials \(P\) of degree at most \(n = 1, 2, \ldots\) Let \(E \subset \mathbb{R}\) and \(\omega_E(x)\) be the density of \(\mu_E\) with respect to Lebesgue measure wherever it exists. The measure \(\mu_E\) is called the equilibrium measure of \(E\) (cf. [18, 38]). In a special case when \(E = [-1, 1]\) the inequality (1) becomes the original Bernstein inequality:

\[ |P'(x)| \leq \frac{n}{\sqrt{1-x^2}} \cdot \|P\|_{[-1,1]} \quad (3) \]

because \(\omega_{[-1,1]}(x) = \frac{1}{\pi \sqrt{1-x^2}}\).

Let us recall some well known results on Markov’s inequality.

By the Markov theorem (cf. [22]) we obtain that for any polynomial \(P\)

\[ \|P'\|_{[a,b]} \leq \frac{2}{b-a} n^2 \cdot \|P\|_{[a,b]}. \quad (4) \]

For the Chebyshev polynomial of the form

\[ T_n(x) = \cos \left[ n \arccos \left( \frac{2x - a - b}{b - a} \right) \right] \]

we get an equality.
V. A. Markov (cf. [24], p.141.) generalized A. A. Markov’s inequality to derivatives of an arbitrary order $P^{(\alpha)}(x)$. He proved that for any $P$ and any $\alpha = 1, \ldots, n$

$$\|P^{(\alpha)}\|_{[a,b]} \leq \left(\frac{2}{b-a}\right)^{\alpha} \|T_n^{(\alpha)}\|_{[-1,1]} \cdot \|P\|_{[a,b]} = \left(\frac{2}{b-a}\right)^{\alpha} T_n^{(\alpha)}(1) \cdot \|P\|_{[a,b]}$$

$$= \left(\frac{2}{b-a}\right)^{\alpha} \cdot \left(\frac{(n^2-1^2) \cdots (n^2-(\alpha-1)^2)}{(2\alpha-1)!!}\right) \|P\|_{[a,b]}.$$ (5)

Furthermore, in this case, the extremal polynomial is the Chebyshev polynomial $T_n(x)$.

Inequality (4) was generalized by Schwab (cf. [23, 29]) who proved that for a polynomial $P$ of total degree $n$, on a finite interval, we have

$$\|P'\|_{L^2} \leq 2\sqrt{3} \frac{n^2}{|b-a|} \|P\|_{L^2}.$$ 

The exact constant in the univariate Markov inequality in $L^2$–norm was investigated in (cf. [27]). The authors of this article proved that for any polynomial $P$ of total degree $n = 1, 2, 3, 4$, :

$$\|P'\|_{L^2} \leq \frac{2\sqrt{C_n}}{b-a} \|P\|_{L^2},$$

where $C_1 = 3$, $C_2 = 15$, $C_3 = \frac{45+\sqrt{1605}}{2}$ and $C_4 = \frac{105+3\sqrt{805}}{2}$.

G. Sroka (cf. [35]) showed the following important result:

$$\|P^{(\alpha)}\|_{L^p(-1,1)} \leq (C_\alpha (p+1)\alpha^2)^{1/p} \|T_n^{(\alpha)}\|_{[-1,1]} \|P\|_{L^p(-1,1)}, \quad P \in \mathbb{R}_n[x],$$

where constants $C_\alpha$ are bounded and $T_n(x) = \cos(n \arccos x)$ ($x \in [-1, 1]$) are the Chebyshev polynomials of the first kind. Furthermore, he proved that $C_\alpha \leq \frac{12}{\sqrt{2}} e^2$ for $\alpha \geq 3$.

Recently, M. Baran and P. Ozorka (see P. Ozorka’s PhD thesis) or (cf. [1]) obtained, applying different methods, the inequality:

$$\|P^{(\alpha)}\|_{L^p(-1,1)} \leq B_p \|T_n^{(\alpha)}\|_{L^p(-1,1)} \frac{n^2}{\alpha} \|P\|_{L^p(-1,1)},$$

which is sharp for $\alpha \geq 3$ with $B_p$ independent of $n$, and $P \in \mathbb{R}_n[x]$, for $1 \leq p \leq 2$.

It is worth mention that the problem of the V. Markov inequality in $L^p$ norms is a special case of the general question of calculating the constant $C(p, q, \alpha, n)$ in

$$\|P^{(\alpha)}\|_{L^p([-1,1])} \leq C(p, q, \alpha, n) \|P\|_{L^q([-1,1])}, \quad P \in \mathbb{R}_n[x].$$

This research area was considered by many mathematicians, e.g., Glazyrina (cf. [17]), Simonov (cf. [33]), L. Białas-Cież and G. Sroka (cf. [11]), G. Sroka (cf. [35]) and M. Baran, P. Ozorka (cf. [1]).
In (cf. [14]) P. Borvein and V. Totik extended Markov and Bernstein inequalities (for \( \alpha = 1 \)) to arbitrary subsets of \([-1, 1]\) and \([-\pi, \pi]\), respectively. In (cf. [15]) P. Borwein generalized the inequality (1) and (3) on disjoint intervals. In addition, in papers (cf. [26, 34]), an estimation of the constant in the Markov’s inequality for a simplex using minimal polynomials were introduced as a novel benchmark problem. Markov’s-type polynomial inequalities (or inverse inequalities), as well as Bernstein inequalities, are often found in many areas of applied mathematics, including popular numerical solutions of differential equations. Proper estimates of optimal constants in both types of inequalities can help to improve the bounds of numerical errors. More information on this topic can be found in the papers by M. Oszust, G. Sroka (cf. [26]), M. Baran and L. Biaś-Cież (cf. [3]) and references.

2 The crucial tools

First let us recall some well known theorems and examples.

*Siciak’s ekstremal function* on a compact subset \( E \) of \( \mathbb{C} \) is defined by:

\[
\Phi(E, z) := \sup \left\{ \frac{|P(z)|^{1/n}}{\|P\|_E^{1/n}} : P \in \mathbb{R}_n[x], \deg P = n \geq 1, P|_E \not\equiv 0 \right\}, \ z \in \mathbb{C},
\]

\( \| \cdot \|_E \) is the maximum norm on \( E \).

We refer to (cf. [4–6, 8, 9, 12, 19, 30–32]) for definitions and basic properties connected with this important tool in pluripotential theory and its applications to approximation theory.

**Theorem A** (cf. [4, 19, Lemma 5.4.2]) For any \( z \in \mathbb{C} \), we have such a formula for Siciak’s extremal function in this case on the interval \([-1, 1]\):

\[
\Phi([-1, 1], z) = h \left( \frac{1}{2} |z - 1| + \frac{1}{2} |z + 1| \right),
\]

(6)

where \( h(t) = t + \sqrt{t^2 - 1} \) for \( t \geq 1 \).

Before we continue, we remind Cauchy’s inequality.

Let \( \tilde{D}(x_0, r) \) be a closed circle centered at the point \( x_0 \) and radius \( r > 0 \).

**Theorem B** Let \( P \in \mathbb{R}_n[x] \). Then for any \( x_0 \in \mathbb{C} \) and for \( 1 \leq \alpha \leq n, r > 0 \)

\[
\left| P^{(\alpha)}(x_0) \right| \leq \alpha! \| P \|_{\tilde{D}(x_0, r)} \frac{1}{r^\alpha}.
\]

(7)

We will use the following well known Bernstein–Walsh’s inequality.

**Proposition 1** For any \( x_0 \in \mathbb{C} \), \( P \in \mathbb{R}_n[x] \), and \( r > 0 \)

\[
\| P \|_{\tilde{D}(x_0, r)} \leq \left( \sup_{\|z - x_0\|_\infty \leq r} \Phi\left([a, b], z\right) \right)^n
\]
\[ \| P \|_{[a,b]} = \left( \sup_{\| z-x_0 \|_\infty \leq r} \Phi \left( \left[ -1, 1 \right], \frac{2}{b-a} z - \frac{b+a}{b-a} \right) \right)^n \| P \|_{[-1,1]} . \] (8)

**Observation 1** Considering the inequality: \( \Phi([−1, 1], z) \leq \Phi(\bar{B}, z) \) (cf. [8], Proof of theorem 1.3, the fact (2.7)), we can notice that in case of a closed Euclidean unit ball \( \bar{B} \subset \mathbb{R}^2 \) the result concerning Bernstein inequality will be the same as in case of interval \([−1, 1]\).

**Theorem 1** For any \( x_0 \in (a, b) \) and \( r \in \mathbb{R}_+ := (0, +\infty) \), we have:

\[ \sup_{\| z \|_\infty \leq r} \Phi ([a, b], z + x_0) \leq h \left( 1 + \frac{\sqrt{r^2}}{(x_0-a)(b-x_0)} \right) \]

\[ = \frac{r}{\sqrt{(x_0-a)(b-x_0)}} + \sqrt{1 + \frac{r^2}{(x_0-a)(b-x_0)}} . \] (9)

**Proof** From (6) it follows that:

\[ \sup_{\| z \|_\infty \leq r} \Phi ([a, b], z + x_0) = \sup_{\| z \|_\infty \leq r} h \left( \frac{1}{b-a} |z + x_0 - b| + \frac{1}{b-a} |z + x_0 - a| \right) . \]

We denote

\[ L = \frac{1}{b-a} |z + x_0 - b| + \frac{1}{b-a} |z + x_0 - a| \]

\[ = \frac{1}{b-a} \sqrt{(x+x_0-b)^2 + y^2} + \frac{1}{b-a} \sqrt{(x+x_0-a)^2 + y^2} \]

for \( z = x + iy \). If \( |z| \leq r \) then \( x^2 + y^2 \leq r^2 \).

Hence we get \( L \leq \frac{1}{b-a} \sqrt{r^2 + 2x(x_0-b) + (x_0-b)^2} + \frac{1}{b-a} \sqrt{r^2 + 2x(x_0-a) + (x_0-a)^2} \). Let

\[ f(x) = \frac{1}{b-a} \sqrt{r^2 + 2x(x_0-b) + (x_0-b)^2} + \frac{1}{b-a} \sqrt{r^2 + 2x(x_0-a) + (x_0-a)^2} \]

Then \( f'(x) = \frac{1}{b-a} \left( \frac{x_0-a}{\sqrt{(x_0-a)^2 + 2(x_0-a)x + r^2}} + \frac{x_0-b}{\sqrt{(x_0-b)^2 + 2(x_0-b)x + r^2}} \right) \).

If we solve the equation \( f'(u) = 0 \), we obtain \( u = \frac{(a+b-2x_0)r^2}{2(a-x_0)(b-x_0)} \).

We observe that the domain of the function \( f \) (for \( x_0 \in (a, b) \) and \( r > 0 \)) is the interval \( \left[ \frac{-r^2-(x_0-a)^2}{2(x_0-a)}, \frac{-r^2-(x_0-b)^2}{2(x_0-b)} \right] \). Thus if \( u \) belongs to the domain of \( f \), then

\[ f \left( \frac{(a+b-2x_0)r^2}{2(a-x_0)(b-x_0)} \right) = 1 + \frac{r^2}{(x_0-a)(b-x_0)} \]

for \( x_0 \in (a, b) \).
It is easy to check that the values of the function \( f \) at the endpoints of the interval are
\[
\inf_{x_0 \in (a, b)} \frac{x_0^2 + r^2 - ab + x_0 (a+b)}{(a-b)(x_0-a)}
\]
respectively and for \( x_0 \in (a, b) \) we have
\[
\sqrt{1 + \frac{x_0^2 + r^2 - ab + x_0 (a+b)}{(a-b)(x_0-a)}} \geq 0
\]
Combining the last inequality with (11) for \( r = t \sqrt{(x_0-a)(b-x_0)} \) in (9), we have
\[
\sup_{\|z\| \leq t \sqrt{(x_0-a)(b-x_0)}} \Phi([a, b], z) \leq t + \sqrt{1 + t^2}
\]
for every \( t > 0 \).

Combining the last inequality with (11) for \( r = t \sqrt{(x_0-a)(b-x_0)} \) we obtain the assertion of the theorem.

\[\Box\]

**Theorem 2** Let \( P \in \mathbb{R}_n[x] \). Then for every \( x_0 \in (a, b) \) and for every \( 1 \leq \alpha \leq n \),
\[
\left| P^{(\alpha)}(x_0) \right| \leq \alpha! \left( \frac{1}{\sqrt{(x_0-a)(b-x_0)}} \right)^{\alpha} \inf_{t > 0} \left\{ \left( t + \sqrt{1 + t^2} \right)^n t^{-\alpha} \right\} \| P \|_{[a,b]}.
\]
(10)

**Proof** Applying Cauchy’s (7) and Bernstein–Walsh’s (8) inequalities, we get
\[
\left| P^{(\alpha)}(x_0) \right| \leq \frac{\alpha!}{\sqrt{\pi}} \left( \sup_{\|z-x_0\| \leq r} \Phi([a, b], z) \right)^n \| P \|_{[a,b]}.
\]
(11)

Notice that putting \( r = t \sqrt{(x_0-a)(b-x_0)} \) in (9), we have
\[
\sup_{\|z\| \leq t \sqrt{(x_0-a)(b-x_0)}} \Phi([a, b], z + x_0) \leq t + \sqrt{1 + t^2}
\]
for every \( t > 0 \).

Combining the last inequality with (11) for \( r = t \sqrt{(x_0-a)(b-x_0)} \) we obtain the assertion of the theorem.

\[\Box\]

**Lemma 1** For any \( t > 0 \),
\[
\inf_{t > 0} \left\{ \left( t + \sqrt{1 + t^2} \right)^n t^{-\alpha} \right\} = \begin{cases} 
2^\alpha & \text{for } \alpha = n, \\
\left( \frac{\alpha}{\sqrt{\pi}} \right)^\alpha \left( 1 + \frac{\alpha}{n} \right) \left( \frac{1}{\sqrt{1 - \frac{\alpha^2}{n^2}}} \right)^n \left( \sqrt{1 - \frac{\alpha^2}{n^2}} \right)^\alpha & \text{for } 1 \leq \alpha \leq n-1.
\end{cases}
\]
**Proof** Let \( f(t) = \left( t + \sqrt{t^2 + 1} \right)^n t^{-\alpha} \), where \( 1 \leq \alpha \leq n - 1 \). Then

\[
f'(t) = t^{-\alpha-1} \left( t + \sqrt{t^2 + 1} \right)^{n-1} \left[ nt \left( 1 + \frac{t}{\sqrt{t^2 + 1}} \right) - \alpha \left( t + \sqrt{t^2 + 1} \right) \right].
\]

After solving the equation \( f'(x) = 0 \) we get \( x = \frac{\alpha}{\sqrt{n^2 - \alpha^2}} = \sqrt{\frac{n}{1 - (\frac{n}{\alpha})^2}} \). Since at \( x = \frac{\alpha}{\sqrt{n^2 - \alpha^2}} \) \( f' \) changes sign from negative to positive, \( f(t) \) has a local minimum, at this point, equal to

\[
\inf_{t>0} \left\{ (t + \sqrt{1 + t^2})^n t^{-\alpha} \right\} = f \left( \frac{\sqrt{\alpha}}{\sqrt{1 - (\frac{n}{\alpha})^2}} \right) = \left( \frac{n}{\alpha} \right)^\alpha \left[ (1 + \frac{\alpha}{n}) \cdot \frac{1}{\sqrt{1 - (\frac{n}{\alpha})^2}} \right]^n
\]

\[
\left( \frac{\sqrt{1 - (\frac{n}{\alpha})^2}}{\frac{n}{\alpha}} \right)^\alpha = e^{\frac{\alpha}{\alpha^2}}.
\]

**Corollary 1** If \( \alpha \geq 1 \) is fixed, then \( \lim_{n \to \infty} n^{-\alpha} \left( \frac{n}{\alpha} \right)^\alpha \left[ (1 + \frac{\alpha}{n}) \cdot \frac{1}{\sqrt{1 - (\frac{n}{\alpha})^2}} \right]^n \]

\[
\left( \frac{\sqrt{1 - (\frac{n}{\alpha})^2}}{\frac{n}{\alpha}} \right)^\alpha = e^{\frac{\alpha}{\alpha^2}}.
\]

**Lemma 2** Let \( m(t) = t + \sqrt{1 + t^2} \). Then for any \( t \in (0, 1] \), we obtain

\[
\left( t + \sqrt{1 + t^2} \right)^{1/t} \leq e
\]

which is equivalent to the inequality \( t + \sqrt{1 + t^2} \leq e^t \), for \( t \in (0, 1] \).

**Proof** Let consider the following function \( f(t) = \left( t + \sqrt{1 + t^2} \right)^{1/t} - e \). Observe that the interval \((0, 1]\) is contained in the domain of the function \( f \). We prove that the function \( f \) is decreasing on the interval \((0, 1]\). To this end, \( f(t) = \left( \frac{\sqrt{1 + t^2}}{1 + t^2} \right)^{1/t} \left( t - \sqrt{1 + t^2} \ln \left( \frac{1 + t^2}{1 + t^2} \right) \right) / t^2 \sqrt{1 + t^2} \). Let define \( g(t) = t - \sqrt{1 + t^2} \ln \left( \frac{1 + t^2}{1 + t^2} \right) \). We have \( g(0) = 0 \), \( g'(t) = -\frac{t \ln \left( \frac{1 + t^2}{1 + t^2} \right)}{\sqrt{1 + t^2}} < 0 \) for \( t \in (0, \infty) \). This implies that \( f'(t) < 0 \) for \( t \in (0, 1] \), so the function \( f \) is decreasing on the interval \((0, 1]\). On the other hand, using de l’Hospital’s rule we easily see that \( \lim_{t \to 0^+} \left( t + \sqrt{1 + t^2} \right)^{1/t} = e \). Therefore \( f(t) < 0 \) for \( t \in (0, 1] \), this concludes the proof of the inequality.

**Lemma 3** For any \( \alpha \geq 3 \), we have

\[
e^{\alpha + 1} < 4^\alpha.
\]

**Proof** As \( e < 2.8 \), therefore \( e^{\alpha + 1} < (2.8)^{\alpha + 1} \). We will prove that for \( \alpha \geq 3 \), \( (2.8)^{\alpha + 1} < 4^\alpha \) which is equivalent to inequality 2. \( \left( \frac{4}{2.8} \right)^\alpha \) for \( \alpha \geq 3 \). As the
function \( f(\alpha) = \left(\frac{4}{\pi^8}\right)^\alpha \) is increasing, it suffices to show that \( 2.8 < \left(\frac{4}{\pi^8}\right)^3 \) which is clearly true, because \( \left(\frac{4}{\pi^8}\right)^3 \approx 2.9. \)

\[ \square \]

### 3 On Bernstein inequality for the line segment \([a, b] \subseteq \mathbb{R}\)

By (10) we are able to deduce the following result of this paper:

**Theorem 3** Let \( P \in \mathbb{R}_n[x] \). Then for any \( x_0 \in (a, b) \) and for any \( \alpha \geq 3 \) we obtain

\[
\left| P^{(\alpha)}(x_0) \right| \leq \frac{\alpha!}{\alpha^\alpha} \left( \frac{\alpha}{n} + \sqrt{1 + \left( \frac{\alpha}{n} \right)^2} \right)^n \left( \frac{n}{\sqrt{(x_0 - a)(b - x_0)}} \right)^\alpha \| P \|_{[a,b]}
\]

\[
\| P \|_{[a,b]} \leq \left( \frac{2n}{\sqrt{(x_0 - a)(b - x_0)}} \right)^\alpha \| P \|_{[a,b]}
\]

**Proof** From Lemma 2 for \( \alpha \leq n \) we get

\[
\left( \frac{\alpha}{n} + \sqrt{1 + \left( \frac{\alpha}{n} \right)^2} \right)^n = \left( \frac{\alpha}{n} + \sqrt{1 + \left( \frac{\alpha}{n} \right)^2} \right)^{n/\alpha} \leq e^{\alpha/\alpha}.
\]

Hence from Theorem 2 for \( t = \alpha/n \) we get the estimates

\[
\left| P^{(\alpha)}(x_0) \right| \leq \frac{\alpha!}{\alpha^\alpha} \left( \frac{\alpha}{n} + \sqrt{1 + \left( \frac{\alpha}{n} \right)^2} \right)^n \left( \frac{n}{\sqrt{(x_0 - a)(b - x_0)}} \right)^\alpha \| P \|_{[a,b]}
\]

\[
\leq \frac{\alpha!}{\alpha^\alpha} e^{\alpha/\alpha} \left( \frac{n}{\sqrt{(x_0 - a)(b - x_0)}} \right)^\alpha \| P \|_{[a,b]}
\]

Finally, using inequality \( \alpha! < e \left( \frac{\alpha}{2} \right)^\alpha \) for any \( 3 \geq \alpha \in \mathbb{N} \) and Lemma 3, we obtain

\[
\left| P^{(\alpha)}(x_0) \right| \leq \frac{\alpha!}{\alpha^\alpha} e^{\alpha/\alpha} \left( \frac{n}{\sqrt{(x_0 - a)(b - x_0)}} \right)^\alpha \| P \|_{[a,b]}
\]

\[
< \left( \frac{2n}{\sqrt{(x_0 - a)(b - x_0)}} \right)^\alpha \| P \|_{[a,b]},
\]

and the proof is complete. \( \square \)

**Remark 1** In P. Borwein’s and T. Erdélyi book (cf. [13]) (p. 258-260), we can find a sketch of the proof that is a slightly weaker version of P. Bernstein’s inequality on \([-1, 1]\) for \( \alpha \)–th derivatives based on the M. A. Lachance’a argument from 1984 (cf.
His method is based on Bernstein–Szegö’s inequality and Bernstein’s inequality for trigonometric polynomials.

**Remark 2** Bernstein’s pointwise inequality (cf. [13]) for \( \alpha \)-th derivatives was used in article (cf. [11]) to show V.A. Markov’s inequality in \( L^p \) norms with Jacobi’s weighs. Such an inequality was also applied in the manuscript (cf. [35]) to show the main result concerning transferring the classic V.A. Markov’s inequality into the case of integral norms. In the manuscript (cf. [28]) the authors use Bernstein’s inequality (3) to prove that Gauss-Jacobi (-Lobatto) nodes of suitable order are \( L^\infty \)-norming meshes for algebraic polynomials, in a wide range of Jacobi parameters. In (cf. [39]) it is shown that finite-dimensional univariate function spaces satisfying a Bernstein-like inequality admit norming meshes.

**Remark 3** Theorem 3 is a special case of the Bernstein’s inequality for \( \alpha \)-th derivatives for convex bodies (compact sets, convex with nonempty interior) in \( \mathbb{R} \). For example, if we set \( \alpha = -b \) in the Theorem 3 then we have a central-symmetric body convex (i.e. its interior contains 0). If we assume \( 0 < a < b \), in the Theorem 3 we get a non central-symmetric convex body. It is worth mention that A. Kroó and S. Révész (cf. [20]) etc. investigated Bernstein and Markov inequalities in uniform norms for convex bodies.

### 4 Auxiliary lemma

In this section we show a result which will be needed in the proof of the main result. A crucial idea is a factorization of operator of \( \alpha \)-th derivative from \( (\mathbb{R}_n[x], || \cdot ||) \) to itself by the space \( (\mathbb{R}_n[x], || \cdot ||_{[a,b]}) \).

We begin with the following inequality.

**Example 1** The numerical computations carried out using Mathematica programs imply that the constant \( J \) in the inequality:

\[
\int_0^1 \left[ t(1-t) \right]^{\frac{p}{2}} + n^{p\alpha} [(2\alpha - 1)!]^p \cdot \left[ n^2(n^2 - 1) \cdots (n^2 - (\alpha - 1)^2) \right]^{-p} \cdot \left[ n^2(n^2 - 1) \cdots (n^2 - (\alpha - 1)^2) \right]^{-p} \cdot \left[ n^2(n^2 - 1) \cdots (n^2 - (\alpha - 1)^2) \right]^{-p} \cdot \left[ n^2(n^2 - 1) \cdots (n^2 - (\alpha - 1)^2) \right]^{-p} \cdot \left[ n^2(n^2 - 1) \cdots (n^2 - (\alpha - 1)^2) \right]^{-p} dt \leq J
\]

should be between 1 and 3, 94741.

**Lemma 4** For each \( p \geq 1 \) such that \( p\alpha > 1 \) and for an arbitrary \( P \in \mathbb{R}_n[x] \), we have the inequality

\[
\left\| P^{(\alpha)} \right\|_p \leq B(\alpha, n, p) \left\| P \right\|_{[a,b]},
\]

where

\[
B(\alpha, n, p) := 2^{\alpha+1/p} (b - a)^{1/p-\alpha} J^{1/p}
\]
\[
\frac{n^2(n^2 - 1) \ldots (n^2 - (\alpha - 1)^2)}{[n^2(n^2 - 1) \ldots (n^2 - (\alpha - 1)^2)]^{\frac{2}{\alpha p}} \sin \left(\frac{2\pi}{\alpha p}\right)} \leq \frac{2^p}{\pi^p} \frac{(2\alpha - 1)!!}{[(2\alpha - 1)!!]^p}.
\]

**Proof** Let us recall that
\[
\min(A, B) \leq 2 \left[\frac{1}{A} + \frac{1}{B}\right]^{-1}, A, B > 0.
\] (12)

Applying a version of Bernstein’s inequality from Theorem 3 and V. A. Markov’s inequality (5), we get the following inequality (for \(x \in (a, b)\))
\[
\left|P^{(\alpha)}(x)\right|^p \leq \min \left[\left(\frac{2n}{\sqrt{(x-a)(b-x)}}\right)^{\alpha p}, \left(\frac{2}{b-a}\right)^{\alpha p}, \left(T_n^{(\alpha)}(1)\right)^p\right] \cdot \|P\|_{[a,b]}^p.
\]

This reduces to
\[
\left|P^{(\alpha)}(x)\right|^p \leq \min \left[\left(\frac{2n}{\sqrt{(x-a)(b-x)}}\right)^{\alpha p}, \left(\frac{2}{b-a}\right)^{\alpha p}, \left(n^2(n^2 + 1) \ldots (n^2 + (\alpha - 1)^2)\right)^{\alpha p} \frac{(2\alpha - 1)!!}{[(2\alpha - 1)!!]^p}\right] \cdot \|P\|_{[a,b]}^p.
\]

Hence, by (12), we obtain
\[
\left|P^{(\alpha)}(x)\right|^p \leq 2^{p\alpha + 1} n^{p\alpha} \left[\left((x-a)(b-x)\right)^{\alpha p} + n^{p\alpha} (b-a)^{p\alpha} \frac{(2\alpha - 1)!!}{[(2\alpha - 1)!!]^p}\right]^p \cdot \left(\frac{n^2(n^2 - 1) \ldots (n^2 - (\alpha - 1)^2)}{(2\alpha - 1)!!}\right)^{\alpha p} \cdot \|P\|_{[a,b]}^p.
\]

Integrating both sides of the last inequality gives one
\[
\|P^{(\alpha)}(x)\|^p_{[a,b]} \leq 2^{p\alpha + 1} n^{p\alpha} \int_{a}^{b} \left[\left((x-a)(b-x)\right)^{\alpha p} + n^{p\alpha} (b-a)^{p\alpha} \frac{(2\alpha - 1)!!}{[(2\alpha - 1)!!]^p}\right]^p \cdot \left(\frac{n^2(n^2 - 1) \ldots (n^2 - (\alpha - 1)^2)}{(2\alpha - 1)!!}\right)^{\alpha p} \cdot \|P\|_{[a,b]} d x.
\]

Now, by using the substitution \(x = a + (b-a)t, t \in [0, 1]\) we obtain
\[
\|P^{(\alpha)}(t)\|^p_{[a,b]} \leq 2^{p\alpha + 1} n^{p\alpha} (b-a)^{1-p\alpha} \int_{0}^{1} \left[\left((1-t)(b)\right)^{\alpha p} + n^{p\alpha} [(2\alpha - 1)!!]^p \cdot \left(\frac{n^2(n^2 - 1) \ldots (n^2 - (\alpha - 1)^2)}{(2\alpha - 1)!!}\right)^{\alpha p}\right]^{-1} \cdot \|P\|_{[a,b]} d t.
\]

hence we have
\[
\|P^{(\alpha)}(t)\|^p_{[a,b]} \leq 2^{p\alpha + 1} n^{p\alpha} (b-a)^{1-p\alpha} \int_{0}^{1} \]
\[
\left[ t^{p\alpha/2} + n^{p\alpha} \left(2\alpha - 1\right)!! \right]^p \cdot \left[ n^2(n^2 - 1) \cdots (n^2 - (\alpha - 1)^2) \right]^{-p} \right]^{-1} dt \left\| P \right\|_{[a,b]}^p.
\]

Now, by using the substitution \( y = t^{\alpha/2}n^2(n^2 - 1) \cdots (n^2 - (\alpha - 1)^2) \) we calculate
\[
\int_0^1 \left[ t^{p\alpha/2} + n^{p\alpha} \left(2\alpha - 1\right)!! \right]^p \cdot \left[ n^2(n^2 - 1) \cdots (n^2 - (\alpha - 1)^2) \right]^{-p} \right]^{-1} dt
= \frac{n^{2p} (n^2 - 1)^p \cdots (n^2 - (\alpha - 1)^2)^p}{[n^2(n^2 - 1) \cdots (n^2 - (\alpha - 1))^2]^{\frac{2}{\alpha}}} \cdot \frac{\alpha}{\alpha} \cdot \frac{1}{n^{2p} (2\alpha - 1)!! \alpha^{\frac{2}{\alpha}}} \leq \frac{n^{2p} (n^2 - 1)^p \cdots (n^2 - (\alpha - 1)^2)^p}{[n^2(n^2 - 1) \cdots (n^2 - (\alpha - 1))^2]^{\frac{2}{\alpha}}} \sin \left(\frac{2\pi}{p\alpha}\right) \left\| P \right\|_{[a,b]}.
\]
as claimed
\[
\left\| P^{(\alpha)} \right\|_p \leq 2^{\alpha+1/p} \left( b - a \right)^{1/p - \alpha} \frac{n^2(n^2 - 1) \cdots (n^2 - (\alpha - 1)^2)}{[n^2(n^2 - 1) \cdots (n^2 - (\alpha - 1))^2]^{\frac{2}{p\alpha}}} \left(\frac{\frac{2\pi}{p\alpha}}{\sin \left(\frac{2\pi}{p\alpha}\right)}\right)^\frac{1}{\alpha} \n^2 \left[ (2\alpha - 1)!! \right]^{\frac{\alpha}{p\alpha}} \left\| P \right\|_{[a,b]}.
\]

**Observation 2** For fixed \( n \in \mathbb{N} \) and \( \alpha \geq 1 \) we have
\[
\lim_{p \to \infty} B(n, \alpha, p) = \left( \frac{2}{b - a} \right)^\alpha \cdot \frac{n^2(n^2 - 1) \cdots (n^2 - (\alpha - 1)^2)}{2\alpha - 1}.
\]

**Remark 4** The Bojanov conjecture (cf. [10]) asserts that for \( \alpha \) th derivative it should be true that
\[
\left\| P^{(\alpha)} \right\|_{L^p([-1,1])} \leq \left\| T_n^{(\alpha)} \right\|_{L^p([-1,1])} \left\| P \right\|_{[-1,1]}, \quad P \in \mathbb{R}_n[x], \quad p \geq 1.
\]

Applying the method of proof of Lemma 4 we can derive (for \( \alpha = 1 \)) inequalities equivalent to the Bojanov’s inequalities (13), which are true in this case.
Theorem 4 (cf. [16]) Suppose that $1 \leq p \leq q \leq \infty$ and that $E = [a, b]^N$. Then there is a universal constant $M$ such that:

$$||D^\alpha P||_{L^q(E)} \leq 8^N \left( \frac{Mn^2}{b-a} \right)^{|\alpha|+N \left( \frac{1}{p} - \frac{1}{q} \right)} ||P||_{L^p(E)}$$

for all polynomials $P \in \mathbb{R}_n[x]$, with $n \geq 1$. In case $p = q$, the factor $8^N$ may be replaced by 1.

5 On V. A. Markov inequality for the line segment $[a, b] \subset \mathbb{R}$

Applying Lemma 4 and Nikol’skii inequality for $q = \infty$, $N = 1$, $\alpha = 0$, we get the following result.

Theorem 5 Let $P \in \mathbb{R}_n[x]$. Then for arbitrary $p \geq 1$ and for all $\alpha \geq 3$ we have

$$\|P^{(\alpha)}\|_p \leq C(\alpha, n, p) \|P\|_p,$$

where

$$C(\alpha, n, p) = B(\alpha, n, p)D(\alpha, n, p)$$

$$= 2^{\alpha+1/p} (b-a)^{1/p-\alpha} \frac{n^2(n^2-1) \ldots (n^2-(\alpha-1)^2)}{[n^2(n^2-1) \ldots (n^2-(\alpha-1)^2)]^{1/p}} \frac{2\pi}{\sin \left( \frac{2\alpha \pi}{\alpha p} \right)} n^{2/p} \left[ \frac{(2\alpha-1)!!}{(2\alpha-1)!!} \right]^{2/p} 8J^{1/p} \left( \frac{Mn^2}{b-a} \right)^{1/p},$$

for $D(\alpha, n, p) = 8J^{1/p} \left( \frac{Mn^2}{b-a} \right)^{1/p}$ and

$$\lim_{p \to \infty} C(\alpha, n, p) = \lim_{p \to \infty} B(\alpha, n, p)D(\alpha, n, p) = 8 \cdot \left( \frac{2}{b-a} \right)^{\alpha} \frac{n^2(n^2-1) \ldots (n^2-(\alpha-1)^2)}{(2\alpha-1)!!}. $$

Corollary 2 For an arbitrary $\alpha \geq 3$ and for all $p \geq 1$ we have bound

$$C(\alpha, n, p) \leq B_p||T^{(\alpha)}_n||_{[-1,1]} n^{2/p},$$

where we can take

$$B_p([a, b]) = \left( \frac{2}{b-a} \right)^{\alpha} \left( \frac{16\alpha^2 MJ \sqrt{8p\pi}}{3\sqrt{3}} \right)^{1/p}.$$
**Proof** We have

\[
C(\alpha, n, p) \leq (2MJ 8^p)^{\frac{1}{p}} \left(\frac{2}{b-a}\right)^\alpha n^{-\frac{2}{2p} \cdot \alpha} n^{\frac{4}{p}} \left(\frac{2\pi}{\alpha p}\right)^{\frac{1}{p}} \left[(2\alpha - 1)!!\right]^{\frac{2}{\alpha p}} \left|T_n^{(\alpha)}\right|_{[-1,1]}
\]

Since \(\frac{2}{\alpha} \leq \frac{2\pi}{\alpha p} \leq \frac{2\pi}{3}\) for \(\alpha \geq 3, p \geq 1\), we get

\[
\frac{2\pi}{\alpha p} \leq \sup_{t \in (0, \frac{2\pi}{3})} \frac{t}{\sin(t)} = \frac{4\pi}{3\sqrt{3}}
\]

Hence, we obtain

\[
C(\alpha, n, p) \leq \left(\frac{2}{b-a}\right)^\alpha \left(\frac{8n^2MJ 8^p\pi}{3\sqrt{3}}\right)^{1/p} \cdot \left(2^\alpha \alpha^!\right)^{\frac{2}{\alpha p}} \left|T_n^{(\alpha)}\right|_{[-1,1]}
\]

\[
\leq \left(\frac{2}{b-a}\right)^\alpha \left(\frac{32n^2MJ 8^p\pi}{3\sqrt{3}}\right)^{1/p} \cdot \alpha^{\frac{2}{\alpha p}} \left|T_n^{(\alpha)}\right|_{[-1,1]}
\]

\[
\leq \left(\frac{2}{b-a}\right)^\alpha \left(\frac{32\alpha^2MJ 8^p\pi}{3\sqrt{3}}\right)^{1/p} \cdot \left|T_n^{(\alpha)}\right|_{[-1,1]}
\]

\[
\leq \left(\frac{2}{b-a}\right)^\alpha \cdot \left(\frac{32\alpha^2MJ 8^p\pi}{3\sqrt{3}}\right)^{1/p} n^{2/p} \left|T_n^{(\alpha)}\right|_{[-1,1]}
\]

\[
= B_p([a, b]) n^{2/p} \left|T_n^{(\alpha)}\right|_{[-1,1]}
\]

\[
\square
\]

**Corollary 3** For an arbitrary \(\alpha \geq 3\) and for all \(p \geq 1\) we have the bound

\[
C(\alpha, n, p) \leq B_p n^{2/p} \left(\frac{n^{2\alpha}}{(2\alpha - 1)!!}\right) \leq B_p n^{2/p} \left(\frac{n^{2\alpha}}{\alpha!}\right),
\]

where we can take

\[
B_p([a, b]) = \left(\frac{2}{b-a}\right)^\alpha \cdot \left(\frac{32\alpha^2MJ 8^p\pi}{3\sqrt{3}}\right)^{1/p}.
\]
The next, a non-obvious, corollary follows from Corollary 1 and earlier results by M. Baran, P. Ozorka (cf. [1]), and M. Baran, A. Kowalska, P. Ozorka (cf. [3] and was discussed mainly in the case $p = 2$). This indicates that in the case of $L^p$ norm the line segment $[a, b] \subset \mathbb{R}$ possesses the V. A. Markov property in the sense considered nad M. Baran, L. Białas-Cież (cf. [2, 5]).

**Corollary 4** For a fixed $p \geq 1$ there exists a constant $C_p$ such that for all $\alpha \geq 3$ we obtain a Vladimir Markov’s type inequality:

$$||P^{(\alpha)}||_p \leq C_p \frac{\alpha 2^{\alpha}}{\alpha !} ||P||_p, \quad P \in \mathbb{C}[x].$$

**Declarations**

**Conflicts of interest** The authors declare that they have no competing interests

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