GENERALIZED HERMITE-HADAMARD-FEJER TYPE INEQUALITIES FOR GA-CONVEX FUNCTIONS VIA FRACTIONAL INTEGRAL

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Abstract. In this paper, it is a function that is a GA-convex differentiable for a new identity. As a result of this identity some new and general fractional integral inequalities for differentiable GA-convex functions are obtained.

1. Introduction

The classical or the usual convexity is defined as follows:
A function \( f : \emptyset \neq I \subseteq \mathbb{R} \rightarrow \mathbb{R} \), is said to be convex on \( I \) if inequality

\[
 f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)
\]

holds for all \( x, y \in I \) and \( t \in [0,1] \).

A number of papers have been written on inequalities using the classical convexity and one of the most fascinating inequalities in mathematical analysis is stated as follows

\[
 f\left(\frac{a + b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2},
\]

where \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a convex mapping and \( a, b \in I \) with \( a \leq b \). Both the inequalities hold in reversed direction if \( f \) is concave. The inequalities stated in (1.1) are known as Hermite-Hadamard inequalities.

For more results on (1.1) which provide new proof, significantly extensions, generalizations, refinements, counterparts, new Hermite-Hadamard-type inequalities and numerous applications, we refer the interested reader to [4]-[6],[8, 10, 11, 13, 14] and the references therein.

The usual notion of convex function have been generalized in diverse manners. One of them is the so called GA-convex functions and is stated in the definition below.

Definition 1. [10] [11] A function \( f : I \subseteq \mathbb{R} = (0, \infty) \rightarrow \mathbb{R} \) is said to be GA-convex function on \( I \) if

\[
 f\left(x^\lambda y^{1-\lambda}\right) \leq \lambda f\left(x\right) + (1-\lambda) f\left(y\right)
\]

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holds for all \( x, y \in I \) and \( \lambda \in [0, 1] \), where \( x^\lambda y^{1-\lambda} \) and \( \lambda f(x) + (1-\lambda)f(y) \) are respectively the weighted geometric mean of two positive numbers \( x \) and \( y \) and the weighted arithmetic mean of \( f(x) \) and \( f(y) \).

The definition of GA-convexity is further generalized as s-GA-convexity in the second sense as follows.

Definition 2.\cite{17} A function \( f : I \subseteq \mathbb{R} = (0, \infty) \rightarrow \mathbb{R} \) is said to be s-GA-convex function on \( I \) if
\[
\left( x^\lambda y^{1-\lambda} \right) \leq \lambda^s f(x) + (1-\lambda)^s f(y)
\]
holds for all \( x, y \in I \) and \( \lambda \in [0, 1] \) and for some \( s \in (0, 1] \).

For the properties of GA-convex functions and GA-\( s \)-convex functions, we refer the reader to \cite{9 12 14 15 16 17 21 22} and the reference there in.

Most recently, a number of findings have been seen on Hermite-Hadamard type integral inequalities for GA-convex and for GA-\( s \)-convex functions.

Zang at al. in \cite{20} established the following Hermite-Hadamard type integral inequalities for GA-convex function.

Theorem 1.\cite{20} Let \( f : I \subseteq \mathbb{R}^+ \rightarrow \mathbb{R} \) be differentiable on \( I^o \), and \( a, b \in I \) with \( a < b \) and \( f' \in L[a, b] \). If \( |f'|^q \) is GA-convex on \([a, b]\) for \( q \geq 1 \), then
\[
\int_a^b \left| bf(b) - af(a) - \int_a^b f(x)\,dx \right| \leq \frac{|(b-a)A(a,b)|^{1-1/q}}{2^{1/q}}
\]
\[
\times \left\{ \left[ L(a, b^2, a^2) - a^2 \right] |f'(a)|^q + \left[ b^2 - L(a^2, b^2) \right] |f'(b)|^q \right\}^{1/q}.
\]

Theorem 2.\cite{20} Let \( f : I \subseteq \mathbb{R}^+ \rightarrow \mathbb{R} \) be differentiable on \( I^o \), and \( a, b \in I \) with \( a < b \) and \( f' \in L[a, b] \). If \( |f'|^q \) is GA-convex on \([a, b]\) for \( q > 1 \), then
\[
\int_a^b \left| bf(b) - af(a) - \int_a^b f(x)\,dx \right| \leq (\ln b - \ln a)
\]
\[
\times \left[ L(2a^q, b^q, a^q) - a^2(2q-1) \right]^{1-1/q} \left[ A(|f'(a)|^q, |f'(b)|^q) \right]^{1/q}.
\]

Theorem 3. Let \( f : I \subseteq \mathbb{R}^+ \rightarrow \mathbb{R} \) be differentiable on \( I^o \), and \( a, b \in I \) with \( a < b \) and \( f' \in L[a, b] \). If \( |f'|^q \) is GA-convex on \([a, b]\) for \( q > 1 \) and \( 2q > p > 0 \), then
\[
\int_a^b \left| bf(b) - af(a) - \int_a^b f(x)\,dx \right| \leq \frac{(\ln b - \ln a)^{1-1/q}}{p^{1/q}}
\]
\[
\times \left[ L(2q-p, a^q(p-1), b^q(p-1)) \right]^{1-1/q}
\]
\[
\times \left\{ \left[ L(a^p, b^p, a^p) - a^p \right] |f'(a)|^q + [b^p - L(a^p, b^p)] |f'(b)|^q \right\}^{1/q}.
\]

Theorem 4. Suppose that \( f : I \subseteq \mathbb{R} = (0, \infty) \rightarrow \mathbb{R} \) is s-GA-convex function in the second sense, where \( s \in (0, 1] \) and let \( a, b \in [0, \infty), a < b \). If \( f \in L[a, b] \), then the following inequalities hold
\[
2^{s-1} f \left( \sqrt{ab} \right) \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x}\,dx \leq \frac{f(a) + f(b)}{s + 1}.
\]
the constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in $(1.1)$. 

If $f$ in Theorem 5 is GA-convex function, then we get the following inequalities.

(1.6) \[ f\left(\sqrt{ab}\right) \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \leq \frac{f(a) + f(b)}{2}. \]

For more results on GA-convex function and s-GA-convex function see e.g [9, 17, 21] and [22].

**Definition 3.** A function $f : I \subseteq \mathbb{R} = (0, \infty) \rightarrow \mathbb{R}$ is said to be geometrically symmetric with respect to $\sqrt{ab}$ if the inequality $g\left(\frac{ab}{x}\right) = g(x)$ holds for all $x \in [a, b]$

**Definition 4.** Let $f \in L[a, b]$. The right-hand side and left-hand side Hadamard fractional integrals $J_{a^+}^\alpha f$ and $J_{b^-}^\alpha f$ of order $\alpha > 0$ with $b > a \leq 0$ are defined by

\[
J_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} f(t) \frac{dt}{t}, \quad x > a
\]
\[
J_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{\alpha-1} f(t) \frac{dt}{t}, \quad x < b
\]
respectively where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1} \, dt$.

**Lemma 1.** For $0 < \theta \leq 1$ and $0 < a \leq b$ we have

\[ |a^\theta - b^\theta| \leq (b - a)^\theta. \]

In [1] D. Y. Hwang found out a new identity and by using this identity, established a new inequalities. In this paper, we established a new identity similar to that’s identity in [1] and then we obtained some new and general integral inequalities for differentiable GA-convex functions using this lemma.

2. **Main result**

Throughout this section, let $\|g\|_{\infty} = \sup_{x \in [a, b]} |g(x)|$, for the continuous function $g : [a, b] \rightarrow \mathbb{R} = (0, \infty) \rightarrow \mathbb{R}$ be differentiable mapping $I^\alpha$, where $a, b \in I$ with $a \leq b$, and $h : [a, b] \rightarrow (0, \infty)$ be differentiable mapping. In the following sections, for convenience, let the notion $L(t) = a^tG^{1-t}$, $U(t) = b^tG^{1-t}$ and $G = G(a, b) = \sqrt{ab}$.

**Lemma 2.** If $f' \in L(a, b)$ then the following inequality holds:

(2.1) \[ |h(b) - 2h(a)| \frac{f(a)}{2} + h(b)\frac{f(b)}{2} - \int_a^b f(x)h'(x) \, dx \]
\[
\frac{\ln b - \ln a}{4} \left\{ \int_0^1 \left[ 2h (a^t G^{1-t}) - h(b) \right] f' (a^t G^{1-t}) a^t G^{1-t} \, dt 
+ \int_0^1 \left[ 2h (b^t G^{1-t}) - h(b) \right] f' (b^t G^{1-t}) b^t G^{1-t} \, dt \right\}.
\]

**Proof.** By the integration by parts, we have

\[
I_1 = \int_0^1 \left[ 2h (a^t G^{1-t}) - h(b) \right] d (f (a^t G^{1-t}))
= \left[ 2h (a^t G^{1-t}) - h(b) \right] f (a^t G^{1-t}) \bigg|_0^1
- 2 \ln \left( \frac{a}{G} \right) \int_0^1 f (a^t G^{1-t}) h' (a^t G^{1-t}) a^t G^{1-t} \, dt
\]

and

\[
I_2 = \int_0^1 \left[ 2h (b^t G^{1-t}) - h(b) \right] d (f (b^t G^{1-t}))
= \left[ 2h (b^t G^{1-t}) - h(b) \right] f (b^t G^{1-t}) \bigg|_0^1
- 2 \ln \left( \frac{b}{G} \right) \int_0^1 f (b^t G^{1-t}) h' (b^t G^{1-t}) b^t G^{1-t} \, dt
\]

Therefore

\[
\frac{I_1 + I_2}{2} = \left[ h(b) - 2h(a) \right] \frac{f(a) + f(b)}{2} + \frac{\ln b - \ln a}{2} \left\{ \int_0^1 f (a^t G^{1-t}) h' (a^t G^{1-t}) a^t G^{1-t} \, dt 
+ \int_0^1 f (b^t G^{1-t}) h' (b^t G^{1-t}) b^t G^{1-t} \, dt \right\}
\]

This complete the proof. \(\square\)

**Lemma 3.** For \(a, G, b > 0\), we have

\[
\zeta_1 (a, b) = \int_0^1 t a^t G^{1-t} \left| 2h (a^t G^{1-t}) - h(b) \right| \, dt
\]

\[
\zeta_2 (a, b) = \int_0^1 (1-t) a^t G^{1-t} \left| 2h (a^t G^{1-t}) - h(b) \right| \, dt + \int_0^1 (1-t) b^t G^{1-t} \left| 2h (b^t G^{1-t}) - h(b) \right| \, dt
\]
Proof. Continuing inequality (2.1) in Lemma 1 we obtain:

\[ \zeta_3(a, b) = \int_0^1 t b^t G^{1-t} \left| 2h (b^t G^{1-t}) - h(b) \right| dt \]

**Theorem 5.** Let \( f : I \subseteq \mathbb{R} = (0, \infty) \rightarrow \mathbb{R} \) be differentiable mapping \( I^o \), where \( a, b \in I^o \) with \( a < b \). If the mapping \( |f'| \) is GA-convex on \( [a, b] \), then the following inequality holds:

\[ \left| \left[ h(b) - 2h(a) \right] \frac{f(a)}{2} + h(b) \frac{f(b)}{2} - \int_a^b f(x)h'(x)dx \right| \]

\[ \leq \frac{\ln b - \ln a}{4} \left\{ \int_0^1 2h (a^t G^{1-t}) - h(b) \left| f' (a^t G^{1-t}) a^t G^{1-t} \right| dt \right. \]

\[ + \left. \int_0^1 2h (b^t G^{1-t}) - h(b) \left| f' (b^t G^{1-t}) b^t G^{1-t} \right| dt \right\} \]

where \( \zeta_1(a, b), \zeta_2(a, b), \zeta_3(a, b) \) are defined in Lemma 2.

Proof. Continuing inequality (2.1) in Lemma 1

\[ \left| \left[ h(b) - 2h(a) \right] \frac{f(a)}{2} + h(b) \frac{f(b)}{2} - \int_a^b f(x)h'(x)dx \right| \]

\[ \leq \frac{\ln b - \ln a}{4} \left\{ \int_0^1 2h (a^t G^{1-t}) - h(b) \left| f' (a^t G^{1-t}) a^t G^{1-t} \right| dt \right. \]

\[ + \left. \int_0^1 2h (b^t G^{1-t}) - h(b) \left| f' (b^t G^{1-t}) b^t G^{1-t} \right| dt \right\} \]

Using \( |f'| \) is GA-convex in (2.7)

\[ \left| \left[ h(b) - 2h(a) \right] \frac{f(a)}{2} + h(b) \frac{f(b)}{2} - \int_a^b f(x)h'(x)dx \right| \]

\[ \leq \frac{\ln b - \ln a}{4} \left\{ \int_0^1 2h (a^t G^{1-t}) - h(b) \left| t \cdot f'(a) + (1 - t) |f'(G)| \right| a^t G^{1-t} dt \right. \]

\[ + \left. \int_0^1 2h (b^t G^{1-t}) - h(b) \left| t \cdot f'(b) + (1 - t) |f'(G)| \right| b^t G^{1-t} dt \right\} \],

by (2.8) and Lemma 2, this proof is complete. \( \square \)

**Corollary 1.** \( g : [a, b] \rightarrow [0, \infty) \) be continuous positive mapping and symmetric to \( \sqrt{ab} \). Let \( h(x) = \int_a^x \left[ (\ln t)^{\alpha-1} + (\ln t)^{\beta-1} \right] \frac{g(t)}{t} dt \) for all \( t \in [a, b] \) and \( \alpha > 0 \) in Teorem 5, we obtain:

\[ \left( \frac{f(a) + f(b)}{2} \right) \left[ J_\alpha^a g(b) + J_\alpha^b g(a) \right] - \left[ J_\alpha^a (fg)(b) + J_\alpha^b (fg)(a) \right] \]
By left side of inequality (2.8) in Theorem 5, when we write

\[ \|f(a)\|_\infty \leq \frac{\ln b - \ln a}{2\alpha + 1} \|g\|_\infty \left[ C_1(\alpha) |f'(a)| + C_2(\alpha) |f'(G)| + C_3(\alpha) |f'(b)| \right] \]

where

\[ C_1(\alpha) = \int_0^1 [(1 + t)^\alpha - (1 - t)^\alpha] t a^t G^{1-t} dt \]

\[ C_2(\alpha) = \int_0^1 (1 - t) [(1 + t)^\alpha - (1 - t)^\alpha] [a^t G^{1-t} + b^t G^{1-t}] dt \]

\[ C_3(\alpha) = \int_0^1 [(1 + t)^\alpha - (1 - t)^\alpha] t b^t G^{1-t} dt \]

Specially in (2.9) and using Lemma 2, for \(0 < \alpha \leq 1\) we have:

\[ \left| \left( \frac{f(a) + f(b)}{2} \right) [J_+ a^t g(b) + J_- b^t g(a)] - [J_+ a^t (fg)(b) + J_- b^t (fg)(a)] \right| \]

\[ \leq \frac{\ln b - \ln a}{2} \|g\|_\infty \left[ C_1(\alpha) |f'(a)| + C_2(\alpha) |f'(G)| + C_3(\alpha) |f'(b)| \right] \]

where

\[ C_1(\alpha) = \int_0^1 t^{\alpha + 1} a^t G^{1-t} \]

\[ C_2(\alpha) = \int_0^1 [(1 + t) t^\alpha a^t G^{1-t} + (1 - t) t^\alpha b^t G^{1-t}] dt \]

\[ C_3(\alpha) = \int_0^1 t^{\alpha + 1} b^t G^{1-t} \]

**Proof.** By left side of inequality (2.8) in Theorem 5, when we write \(h(x) = \int_a^x \left[ (\ln \frac{b}{x})^{\alpha - 1} + (\ln \frac{x}{a})^{\alpha - 1} \right] \frac{g(t)}{t} dt \) for all \(t \in [a, b]\), we have

\[ \left| \Gamma(\alpha) \left( \frac{f(a) + f(b)}{2} \right) [J_+ a^t g(b) + J_- b^t g(a)] - \Gamma(\alpha) [J_+ a^t (fg)(b) + J_- b^t (fg)(a)] \right| \]

On the other hand, right side of inequality (2.8)

\[ \leq \frac{\ln b - \ln a}{4} \left\{ \left| \int_0^1 \frac{1}{2} \left[ \ln \frac{b}{x}\right]^{\alpha - 1} \left[ (\ln \frac{b}{x})^{\alpha - 1} + (\ln \frac{x}{a})^{\alpha - 1} \right] \frac{g(x)}{x} dx \right| \left| t |f'(a)| + (1 - t) |f'(G)| \right| a^t G^{1-t} dt \]
In the last inequality, we have

\[ \frac{\ln b - \ln a}{4\Gamma(\alpha)} \left\{ \int_0^1 \left[ \left( \ln \frac{b}{x} \right)^{\alpha-1} + \left( \ln \frac{x}{a} \right)^{\alpha-1} \right] \frac{g(x)}{x} dx \right\} \left( t |f'(b)| + (1 - t) |f'(G)| b^{\gamma - 1} dt \right) \]  

and

\[ \frac{\ln b - \ln a}{4\Gamma(\alpha)} \|g\|_{\infty} \left\{ \int_0^1 \left[ \left( \ln \frac{b}{x} \right)^{\alpha-1} + \left( \ln \frac{x}{a} \right)^{\alpha-1} \right] \frac{1}{x} dx \right\} \left( t |f'(b)| + (1 - t) |f'(G)| b^{\gamma - 1} dt \right) \]

for all \( t \in [0, 1] \). By (2.10), (2.11) and (2.12), we have

\[ \left( \frac{f(a) + f(b)}{2} \right) \left[ J_{a^+}^\alpha g(b) + J_{b^-}^\alpha g(a) \right] - [J_{a^+}^\alpha (fg)(b) + J_{b^-}^\alpha (fg)(a)] \]

\[ \leq \frac{\ln b - \ln a}{4\Gamma(\alpha)} \left\{ \int_0^1 \left[ \left( \ln \frac{b}{x} \right)^{\alpha-1} + \left( \ln \frac{x}{a} \right)^{\alpha-1} \right] \frac{g(x)}{x} dx \right\} \left( t |f'(b)| + (1 - t) |f'(G)| b^{\gamma - 1} dt \right) \]

\[ + \frac{\ln b - \ln a}{4\Gamma(\alpha)} \|g\|_{\infty} \left\{ \int_0^1 \left[ \left( \ln \frac{b}{x} \right)^{\alpha-1} + \left( \ln \frac{x}{a} \right)^{\alpha-1} \right] \frac{1}{x} dx \right\} \left( t |f'(b)| + (1 - t) |f'(G)| b^{\gamma - 1} dt \right) \]

In the last inequality,
\begin{align*}
&\int_{\alpha t^G-t}^{bG^{-1}} \left[ \left( \ln \frac{b}{x} \right)^{\alpha-1} + \left( \ln \frac{x}{a} \right)^{\alpha-1} \right] \frac{1}{x} \, dx = \int_{\alpha t^G-t}^{bG^{-1}} \left( \ln \frac{b}{x} \right)^{\alpha-1} \frac{1}{x} \, dx + \int_{\alpha t^G-t}^{bG^{-1}} \left( \ln \frac{x}{a} \right)^{\alpha-1} \frac{1}{x} \, dx \\
&\quad = \frac{2}{\alpha} \left( \ln b - \ln a \right)^{\alpha} \left[ (1 + t)^{\alpha} - (1 - t)^{\alpha} \right].
\end{align*}

By Lemma 1, we have
\begin{align*}
&\int_{\alpha t^G-t}^{bG^{-1}} \left[ \left( \ln \frac{b}{x} \right)^{\alpha-1} + \left( \ln \frac{x}{a} \right)^{\alpha-1} \right] \frac{1}{x} \, dx = \int_{\alpha t^G-t}^{bG^{-1}} \left( \ln \frac{b}{x} \right)^{\alpha-1} \frac{1}{x} \, dx + \int_{\alpha t^G-t}^{bG^{-1}} \left( \ln \frac{x}{a} \right)^{\alpha-1} \frac{1}{x} \, dx \\
&\quad \leq \frac{2}{\alpha} \left( \ln b - \ln a \right)^{\alpha} \mu^\alpha.
\end{align*}

A combination of (2.13) and (2.14), we have (2.9). This complete is proof. □

**Corollary 2.** In Corollary 1,

(1) If \( \alpha = 1 \) is in corollary, we obtain following Hermite-Hadamard-Fejer Type inequality for GA-convex function which is related in (2.10):

\begin{align*}
&\int_{\alpha t^G-t}^{bG^{-1}} \left[ \left( \ln \frac{b}{x} \right)^{\alpha-1} + \left( \ln \frac{x}{a} \right)^{\alpha-1} \right] \frac{1}{x} \, dx = \int_{\alpha t^G-t}^{bG^{-1}} \left( \ln \frac{b}{x} \right)^{\alpha-1} \frac{1}{x} \, dx + \int_{\alpha t^G-t}^{bG^{-1}} \left( \ln \frac{x}{a} \right)^{\alpha-1} \frac{1}{x} \, dx \\
&\quad \leq \frac{2}{\alpha} \left( \ln b - \ln a \right)^{\alpha} \mu^\alpha
\end{align*}

where for \( a, b, G > 0 \), we have

\begin{align*}
C_1(1) &= \frac{1}{t} \int_0^t a^t G^{-1} \, dt = \frac{2}{\ln b - \ln a} \left\{ -a + \frac{4a}{\ln b - \ln a} - \frac{8a - 8G}{(\ln b - \ln a)^2} \right\} , \\
C_2(1) &= \frac{1}{t} \int_0^t (1 - t) a^t G^{-1} \, dt + \frac{1}{t} \int_0^t (1 - t) b^t G^{-1} \, dt = \frac{2}{\ln b - \ln a} \left\{ \frac{2(a + b + 2G)}{\ln b - \ln a} + \frac{8(a - b)}{(\ln b - \ln a)^2} \right\} , \\
\text{and} \\
C_3(1) &= \frac{1}{t} \int_0^t t^2 a^t G^{-1} \, dt = \frac{2}{\ln b - \ln a} \left\{ b - \frac{4b}{\ln b - \ln a} + \frac{8b - 8G}{(\ln b - \ln a)^2} \right\} .
\end{align*}

(2) If \( g(x) = 1 \) is in corollary, we obtain following Hermite-Hadamard-Fejer Type inequality for GA-convex function which is related in (2.9):

\begin{align*}
&\left| \left( \frac{f(a) + f(b)}{2} \right) - \frac{\Gamma(\alpha + 1)}{2(\ln b - \ln a)^\alpha} \left[ J^\alpha_a f(b) + J^\alpha_b f(a) \right] \right| \\
&\quad \leq \frac{\ln b - \ln a}{2^{\alpha + 2}} \left[ C_1(\alpha) |f'(a)| + C_2(\alpha) |f'(G)| + C_3(\alpha) |f'(b)| \right].
\end{align*}

(3) If \( g(x) = 1 \) and \( \alpha = 1 \) is in corollary, we obtain following Hermite-Hadamard-Fejer Type inequality for GA-convex function which is related in (2.10):

\begin{align*}
&\left| \left( \frac{f(a) + f(b)}{2} \right) - \frac{\Gamma(\alpha + 1)}{2(\ln b - \ln a)^\alpha} \left[ J^\alpha_a f(b) + J^\alpha_b f(a) \right] \right| \\
&\quad \leq \frac{\ln b - \ln a}{2^{\alpha + 2}} \left[ C_1(\alpha) |f'(a)| + C_2(\alpha) |f'(G)| + C_3(\alpha) |f'(b)| \right].
\end{align*}
6, we obtain:

where for 

(\ln) \leq \mid (ln α + 1) = 1. If the mapping \mid f'\mid is GA-convex on [a, b], then the following inequality holds:

\mid \mid (h(b) - 2h(a)) \frac{f(a)}{2} + h(b)\frac{f(b)}{2} - \int_{a}^{b} f(x)h'(x)dx \mid \leq \frac{\ln b - \ln a}{4} \left[ C_1(1) |f'(a)| + C_2(1) |f'(G)| + C_3(1) |f'(b)| \right].

**Theorem 6.** Let \( f: I \subseteq \mathbb{R}^+ = (0, \infty) \rightarrow \mathbb{R} \) be differentiable mapping \( I^0 \), where \( a, b \in I \) with \( a < b \), and \( g: [a, b] \rightarrow [0, \infty) \) be continuous positive mapping and symmetric to \( \sqrt{ab} \) and \( \frac{1}{2} + \frac{1}{q} = 1 \). If the mapping \( |f'|^q \) is GA-convex on \( [a, b] \), then the following inequality holds:

(2.19) \quad \left| \left( \frac{f(a) + f(b)}{2} \right) \left( \frac{g(a)}{a} \mid f'(a) \mid ^q + \frac{g(b)}{b} \mid f'(b) \mid ^q \right) \right| \leq \frac{\ln b - \ln a}{4} \left[ C_1(1) \| f' \| + C_2(1) \| f'(G) \| + C_3(1) \| f'(b) \| \right].

**Proof.** Continuing from (2.7) in Theorem 5, we use Holder Inequality and we use that \( |f'|^q \) is GA-convex. Thus this proof is complete. \( \square \)

**Corollary 3.** Let \( h(x) = \int_{a}^{x} \left( \frac{\ln \frac{a}{t}}{t} \right)^{\alpha - 1} + \left( \frac{\ln \frac{b}{t}}{t} \right)^{\alpha - 1} \right) \frac{g(t)}{t} dt \) for all \( t \in [a, b] \) in Theorem 6, we obtain:

(2.20) \quad \left| \left( \frac{f(a) + f(b)}{2} \right) \left( J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a) \right) - \left( J_{a+}^\alpha (fg)(b) + J_{b-}^\alpha (fg)(a) \right) \right| \leq \frac{\ln b - \ln a}{2^{\alpha+1} \Gamma (\alpha + 1)} \left( \frac{2^{\alpha+2} - 2^2}{\alpha + 1} \right) \left[ C_1(\alpha, q) \mid f'(a) \mid ^q + C_2(\alpha, q) \mid f'(G) \mid ^q + C_3(\alpha, q) \mid f'(b) \mid ^q \right].

where for \( q > 1 \)

\begin{align*}
C_1(\alpha, q) &= \int_{0}^{1} \left[ (1 + t)^\alpha - (1 - t)^\alpha \right] t a^q G^q(1-t) dt \\
C_2(\alpha, q) &= \int_{0}^{1} \left[ (1 + t)^\alpha - (1 - t)^\alpha \right] \left( 1 - t \right) \left( a^q G^q(1-t) + b^q G^q(1-t) \right) dt
\end{align*}
Corollary 4. When \( \alpha = 1 \) and \( g(x) = \frac{1}{\ln b - \ln a} \) is taken in Corollary 3, we obtain:

\[
\left| \left( \frac{f(a) + f(b)}{2} \right) \right| \leq \frac{(\ln b - \ln a)^{1+\frac{1}{q}}}{2^{\frac{1}{q}+\frac{1}{r}}} \left[ C_1 (1, q) |f'(a)|^q + C_2 (1, q) |f'(G)|^q + C_3 (1, q) |f'(b)|^q \right]^{\frac{1}{q}}.
\]

This proof is complete.
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