A Hike in the Phases of the 1-in-3 satisfiability

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We summarise our results for the random $\epsilon$-1-in-3 satisfiability problem, where $\epsilon$ is a probability of negation of the variable. We employ both rigorous and heuristic methods to describe the SAT/UNSAT and Hard/Easy transitions.

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\section{I. INTRODUCTION}

At the Les Houches “Complex Systems” school we learned about spin glasses and methods by which to study their statistical properties (G. Parisi), also the applications of statistical physics to combinatorial optimization problems were introduced (R. Monasson). Following the acquired knowledges we studied the average case of 1-in-3 SAT problem. We applied rigorous methods to obtain algorithmic and probabilistic bounds on the SAT/UNSAT and Hard/Easy transitions. We also employed the cavity method to obtain a more complete picture of the problem. Our study went beyond a simple exercise, and led to several interesting results and a separate publication of three of the authors \cite{1}. Here we summarise shortly the methods and give the main results.

1-in-3 SAT is a boolean satisfaction problem. Each formula consists of a set of variables and clauses, and each clause contains 3 literals. A clause is satisfied if exactly one of the literals is True, and the formula is satisfiable (SAT) if there is any assignment of variables to True or False, such that every clause is satisfied. 1-in-3 SAT has important similarities to other constraint-satisfaction and graph-theoretical problems. It is a canonical NP-complete problem \cite{2} and thus relates to a host of practically relevant problems. In particular we consider here an ensemble of formulas parameterised by $\gamma$ and $\epsilon$, where $\gamma$ is the mean connectivity of variables in the ensemble, and $\epsilon \in [0, 1/2]$ is the probability that variables appear as a negative literal in the interactions. We then study the SAT/UNSAT and Easy/Hard transition curves in this space, as the number of variables $N \rightarrow \infty$.

This parameterisation is motivated by the knowledge that in many related problems there exists a sharp transition in the typical-case behaviour, from a SAT to an UNSAT regime as the parameters are varied; and a more heuristic Easy-Hard transition, where in a Easy phase many “local” algorithms work in a polynomial time, whereas in the Hard phase they need an exponential time. Previous work in (symmetric) 1-in-3 SAT ($\epsilon = 1/2$) demonstrated that SAT/UNSAT transition is sharp at the threshold $\gamma_{\text{sym}}^* = 1$, and not accompanied by any Hard region. However, for Exact Cover (i.e. positive 1-in-3-SAT, $\epsilon = 0$) this threshold is difficult to determine, with only upper \cite{3} and lower bounds \cite{4} to the transition being known, and the presence of a Hard region is suspected. Studying the threshold behaviour in $\gamma$, for a continuum of problems parameterised by $\epsilon$, allows us to better understand the origin of these differences and the nature of the two transitions.

\section{II. RESULTS}

Figure\textsuperscript{1} outlines our present knowledge on the phase diagram of the $\epsilon$-1-in-3-SAT problem. Results are consistent with previous statements restricted to the special cases $\epsilon = 0$ \cite{4,5}, and $\epsilon = 1/2$ \cite{3}.

A rigorous analysis of the Unit Clause Propagation (UC) and Short Clause Heuristics (SCH) algorithms \cite{3,5} led to upper (dashed line) and lower (shifted to $x$-axis) bounds on the SAT/UNSAT threshold. This result identifies two regions in which the problem is known to be Easy-UNSAT and Easy-SAT, and surprisingly for $\epsilon > 0.273$ the upper and lower bounds coincide, proving the existence of a range of $\epsilon$ in which, at all values of $\gamma$, formulas are almost surely Easy. A second rigorous upper bound for the SAT/UNSAT transition is obtained by considering the First Moment Method on the 2-core \cite{3}, and a third one for the Easy-UNSAT region by an algorithmic method of embedding formulas in the less constrained 3-XOR-SAT problem \cite{6} (out of scale in the figure). Both these bounds are an improvement w.r.t. Unit Clause for small values of $\epsilon$, as the UC line diverges for $\epsilon \rightarrow 0$. 

\textsuperscript{1} arXiv:cond-mat/0702421v1 [cond-mat.stat-mech] 18 Feb 2007
The other results on the phase diagram are obtained by the cavity method \[7\]. We worked both in the assumptions of replica symmetry (RS: existence of a single pure state) and one-step replica symmetry breaking \[8\] (1RSB: existence of exponentially-many pure states). Furthermore we checked the stability of the solutions thereby obtained \[9\].

The RS solution is able to identify the phase transition when \(\epsilon\) is large, but is proven to be unstable at smaller \(\epsilon\) (solid line with triangle marks in fig. 1). The 1RSB solution we explored describes clusters which contain solutions (have zero energy) and disregards their size (entropy) \[7\]. A 1RSB solution of this kind exists in a region with \(\epsilon \lesssim 0.21\), this gives an indication for the Easy/Hard-SAT transition. The common interpretation is that the existence of many states (clusters) is an intrinsic (i.e. algorithm-independent) reason for the average computational hardness \[10, 11\].

There exists a region in which neither RS nor 1RSB assumptions appear to hold sway, in this region we can however guess that 1RSB, as a mean-field assumption, provides an upper bound to the SAT/UNSAT transition curve \[12\].

III. CONCLUDING REMARKS

Several questions arise out of this study, also concerning the relation with previous works on 1-in-3 SAT, Exact Cover and \(K\)-SAT, both from Statistical-Physics and algorithmic perspectives.

The nature of the interactions, being highly constrained (only 3 out of 8 configurations satisfy a clause, and fixing 2 variables could violate a clause), leads to remarkable properties. The first of them is the success of clause-decimation methods (UC and SCH). The second one, quite unexpected, is the arising of “hard contradictions”: above the light gray region in figure, the solution to 1RSB cavity equations at zero energy becomes singular. The relative success of the SCH algorithm by comparison with the RS cavity method is also a novel result. We found a region where SCH is able to find a satisfying assignment almost surely in polynomial time, and yet from the Statistical-Physics point of view the replica symmetry is broken.

Other algorithmic issues should be investigated in the 1-in-3 SAT problem. Particularly, the performance of Belief Propagation \[13\] and Survey Propagation \[11\] (related resp. to RS and 1RSB cavity interpretation). Furthermore, it is interesting to compare our results with the behaviour of the structurally-affine \((2 + p)\)-SAT problem \[14\].

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