Characteristic Parameters in Integrated Photoelasticity: An Application of Poincaré’s Equivalence Theorem

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Abstract

The Poincaré Equivalence Theorem states that any optical element which contains no absorbing components can be replaced by an equivalent optical model which consists of one linear retarder and one rotator only, both of which are uniquely determined. This has many useful applications in the field of Optics of Polarized Light. In particular, it arises naturally in attempts to reconstruct spatially varying refractive tensors or dielectric tensors from measurements of the change of state of polarization of light beams passing through the medium, a field which is known as Tensor Tomography. A special case is Photoelasticity, where the internal stress of a transparent material may be reconstructed from knowledge of the local optical tensors by using the stress-optical laws. – We present a rigorous approach to the Poincaré Equivalence Theorem by explicitly proving a matrix decomposition theorem, from which the Poincaré Equivalence Theorem follows as a corollary. To make the paper self-contained we supplement a brief account of the Jones matrix formalism, at least as far as linear retarders and rotators are concerned. We point out the connection between the parameters of the Poincaré-equivalent model to previously introduced notions of the Characteristic Parameters of an optical model in the engineering literature. Finally, we briefly illustrate how characteristic parameters and Poincaré-equivalent models naturally arise in Photoelasticity.

Keywords: Poincaré Equivalence Theorem, Matrix Decomposition, Characteristic Parameters, Equivalent Optical Models, Jones matrix formalism, Photoelasticity

1 Introduction

The propagation of an electromagnetic wave through a material medium is the result of the interaction between the fundamental fields \( E \) and \( B \) with a macroscopic number of microscopic sources constituting the bulk matter. In principle, these sources must be incorporated into the dynamics of the total system by appropriate interaction terms. However, if the frequency bandwidth of the light under consideration is sufficiently far away from the resonance frequencies of the macroscopic medium, any photon-atom encounter is only transient; in this case we may refer to the medium as a passive optical element. Its macroscopic effect on impinging radiation may be summarized by introducing a (space- and time-dependent) refractive index.

There are essentially two classes of such optical elements: Firstly, polarizers, which absorb, or at least attenuate, one of two given orthogonal polarization forms of the light passing through the medium; and secondly, retarders, which introduce a phase retardation between the components of these polarization forms, but otherwise preserve the total intensity of the light. In principle, a polarized light beam can be analyzed with respect to any two orthogonal polarization forms. However, in practice, two cases are of particular importance: The linear retarder, which introduces a phase lag between the components of linearly analyzed light, and the rotator, which does the same with respect to the circularly polarized components of a polarized light beam.

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The description of the passage of light through optical elements depends on whether the light is polarized or not. For unpolarized or partially polarized light, a description in terms of the Stokes parameters [1] (for a definition, see e.g. [2]) is appropriate; if these are arranged into four-dimensional real vectors such that optical elements act on it by means of real (4, 4) matrices we speak of the Mueller formalism [3, 4]. If, on the other hand, the light beam is completely polarized, the polarization state of the transverse degrees of freedom of the electric field can be conveniently arranged into a two-dimensional complex vector, while optical elements now act on it by means of complex matrices. This formalism is called the Jones calculus [5, 6, 7, 8, 9, 10, 11, 12, 13]. The above-mentioned concepts prove to be useful not only within a classical context, but appropriate generalizations are used in Quantum Optics as well: The unitary representation of beam splitters in the Jones matrix formalism acting on quantum-mechanical mode operators was discussed in [14]. A broad introduction to optical elements in linear optical networks within a quantum-optical context was recently given in [16].

Within the framework of the Jones calculus, polarizers are described by Hermitean matrices, while retarders are represented by unitary ones. The latter reflects the fact that retarders preserve the total intensity of the light. The fact that unitary matrices have the group property then implies that an arbitrary sequence of retarders is again described by a unitary matrix. Such a sequence could be a pile of birefringent plates, each with constant rotation of principal axes and constant phase retardation. But we can also consider the limit of infinitely many, infinitely thin, birefringent plates: This situation describes a medium which exhibits a spatially varying dielectric tensor \( \epsilon_{ij} \), or more generally, refractive tensor \( n_{ij} \). In this case, the phase velocity depends not only on the location within the medium, but also on the polarization state of the (locally-plane) wave at this point. This corresponds to the most general case of an inhomogeneous anisotropic medium.

Such a scenario arises naturally in the field of Photoelasticity [17, 18, 19, 20, 21, 22, 23, 24]: Experience shows that certain materials, such as glasses and polymers, are optically isotropic and homogeneous when unloaded, but exhibit local anisotropy when strained by an external load. The relation between the resulting dielectric tensor \( \epsilon_{ij} \) and the stress tensor \( \sigma_{ij} \) in the interior of the medium is called the stress-optical law; its basic form has been discovered long ago by Maxwell [25]. Integrated Photoelasticity is concerned with the reconstruction of optical tensors, and via the stress-optical law, of stress tensors, from data sets which are acquired by sending polarized light through the loaded specimen at many different angles, thereby measuring the change in the state of polarization. For this method to work, the material must be sufficiently transparent to ignore absorption within the medium. Since tensorial quantities are reconstructed, these methods belong to the field of Tensor Tomography.

In the examples above, the medium is given by a pile of retarders and/or rotators, possibly in the infinitesimal limit. The previous discussion implies that, within the Jones formalism, such a medium may be described by a unitary (unimodular) matrix \( U \). It turns out that such a matrix always determines – and, in turn, is fully determined by – a set of characteristic directions [26] and a characteristic phase retardation between them. These characteristic parameters [26] have been introduced within the context of Photoelasticity from an engineering point of view, and are not necessarily standard in the mathematical literature [a rigorous mathematical definition will be given in section 3]. If the matrix \( U \) is interpreted as belonging to an optical element of the kind as discussed above, the characteristic parameters can be given an operational meaning: The primary characteristic directions determine those planes of linear polarization at the entry into the medium for which the state of polarization at the emergence from the medium is again linear. The secondary characteristic directions determine the planes of linear polarization of the emerging light, if the incident light was linearly polarized in the primary characteristic directions; in general, they differ from the primary ones. It turns out [sections 3, 7] that there are always two orthogonal primary and two orthogonal secondary characteristic directions. Light which is linearly polarized along the two primary directions travels with different phase velocities, though, so that both waves emerge with a phase difference – the characteristic phase retardation.

It is an important task to reconstruct the optical properties of the medium, i.e. the matrix \( U \), from measurements of the characteristic parameters. To this end, relations between the parameters of a normal form of unitary matrices and the characteristic parameters may be derived [22, 23, 26]. However, these relations only determine the squares of sines or cosines of angles and therefore the actual computation of these quantities is rather involved. It is at this point where the Poincaré Equivalence Theorem [27] comes into play: This theorem states that any optical medium which is described by a unitary unimodular matrix can be replaced by an optically equivalent model which consists of one retarder followed by
one rotator, or the other way round; in each case, both elements are uniquely determined. Then it can be shown easily that the characteristic parameters of the optical medium coincide with the optical parameters of the equivalent model, i.e. with the principal directions and phase retardation for the equivalent retarder, and the rotation angle for the equivalent rotator; but from the latter, the matrix parameters of the equivalent model, i.e. with the principal directions and phase retardation for the optical medium, can be shown easily that the characteristic parameters of the optical medium coincide with the optical parameters of the equivalent retarder and rotator; or the other way round; in each case, both elements are uniquely determined. Then it follows as a corollary [section 5]. To put this result into a context, and to make the paper reasonably self-contained, we provide a brief overview of the description of polarized light, and optical elements acting on it, in terms of the Jones calculus; this is done in section 2.

The second objective of this paper is to clarify the relation between the (operationally defined) characteristic parameters of a transparent medium, and the optical parameters of the Poincaré-equivalent model. To this end we recapitulate in section 3 the notions of characteristic directions and characteristic phase retardation for non-absorbing optical elements. We also prove the existence of characteristic directions for any unitary unimodular (2, 2) matrix in a standard parametrisation. The relation between characteristic parameters and the parameters of the Poincaré-equivalent retarder and rotator then are analyzed in section 7. Finally, in section 8 we outline the basic ideas of three-dimensional photoelasticity in order to illustrate the context in which a specimen with continuous variation of optical tensors presents itself, and how the Poincaré-equivalent model may be important to the objective of tensor tomography. In section 9 we summarize our results.

## 2 Polarized Light in the Jones Matrix Formalism

In this paper we only consider completely polarized classical light. Strictly speaking, this is an idealization, which can be approximately realized only if 1) the light is strictly monochromatic, and 2) the complex degree of coherence [2] of two orthogonal components of the electric field always has unit modulus. Physically, this means that the two orthogonal components have a well-defined and constant phase relation for all times.

The transverse electric field of a general elliptically polarized light beam propagating in the 3-direction takes the form

$$
E(x, t) = a_1 \cos(kz - \omega t + \delta_1) \mathbf{e}_1 + a_2 \cos(kz - \omega t + \delta_2) \mathbf{e}_2,
$$

(1)

where $a_1, a_2$ are real amplitudes, $u = \omega/k$ is the phase velocity in the (isotropic and homogeneous) medium, $\delta_1$ and $\delta_2$ are constant phases, and $\mathbf{e}_1$ and $\mathbf{e}_2$ are real unit vectors in the direction of the $x$- and $y$-axes. Since the action of optical elements on light beams is accomplished by linear transformations, it is admissible to assume that the beam has unit relative intensity [21] $a_1^2 + a_2^2 = 1$. We now choose a fixed location within the beam, say, $z = 0$, and express (1) as the real part of the complex vector

$$
E = e^{-i\omega t} \left\{ a_1 e^{i\delta_1} \mathbf{e}_1 + a_2 e^{i\delta_2} \mathbf{e}_2 \right\}.
$$

(2)

The quantity in curly brackets defines the polarization form of the light; the associated complex 2-vector

$$
\nu_l = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a_1 e^{i\delta_1} \\ a_2 e^{i\delta_2} \end{pmatrix},
$$

(3)

is called the Jones vector of the light beam with respect to linear polarization along the directions $\mathbf{e}_1$ and $\mathbf{e}_2$. We use a subscript $l$ to denote the fact that the Jones vector (3) denotes linearly polarized components.

As can be seen, the polarization form and its associated Jones vector are defined only up to a global phase, i.e. Jones vectors $v$ and $e^{i\Phi}v$ are equivalent, since the global phase $\Phi$ can always be reabsorbed into the total phase $e^{-i\omega t}$ governing the time dynamics.
Eq. (2) represents the decomposition of the (complex form of the) electric field vector into two linearly polarized components whose amplitudes are \( a_1 \), \( a_2 \), and whose relative phase is \( \delta \equiv \delta_2 - \delta_1 \). This decomposition is convenient to derive the Jones matrix of a linear retarder. To study the rotator it is necessary to analyze the light with respect to complex basis vectors \( e_+ \) and \( e_- \) which represent right-handed and left-handed circular polarization, respectively, which are defined as

\[
\mathbf{e}_\pm = \frac{1}{\sqrt{2}} (\mathbf{e}_1 \pm i\mathbf{e}_2),
\]

\[
(e_+, e_-) = (\mathbf{e}_1, \mathbf{e}_2) \mathbf{M}, \quad \mathbf{M} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.
\]  

(4a)  

(4b)

We recall that right-handed = left-circular, and vice versa. The electric field vector (2) can be expanded in terms of the basis \((e_+, e_-)\),

\[
\mathbf{E} \sim (\mathbf{e}_1, \mathbf{e}_2) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (e_+, e_-) \begin{pmatrix} v_+ \\ v_- \end{pmatrix},
\]

such that the components \( v_+ \) and \( v_- \) make up the Jones vector \( v_c \) of the light in the basis (4),

\[
v_c = \begin{pmatrix} v_+ \\ v_- \end{pmatrix} = \mathbf{M}^\dagger v_l = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.
\]

(6)

We use a subscript \( c \) to point out that the Jones vector (6) denotes circularly polarized components.

Evidently, the matrix \( \mathbf{M} \) mapping the basis vectors \((\mathbf{e}_1, \mathbf{e}_2)\) for linear polarization to those \((e_+, e_-)\) for circular polarization is unitary. Indeed, this is true for any transformation between basis vectors of different polarization forms.

We are now in a position to derive the Jones matrices for linear retarders and rotators.

### 2.1 The linear retarder in the Jones calculus

A **linear retarder** is an optical element made of a material which exhibits homogeneous optical anisotropy along a given direction of light transmission [2, 20, 21]. Perpendicular to the direction of passage of light, the retarder has two distinct orthogonal directions, called the *fast* and the *slow* axis, respectively. The phase velocity of a plane wave which is linearly polarized along the fast axis is greater than for waves polarized along the slow axis. This means that, at the point of emergence from the retarder, the component along the slow axis ("slow component") has acquired a phase lag \( \delta \) with respect to the component along the fast axis ("fast component"). Let us assume that the fast/slow axis is oriented along the \( x/y \)-axis. If the Jones vector of light prior to entry into the retarder is

\[
v = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},
\]

then after emergence from the retarder it has changed to

\[
v' = e^{i\Phi_g} \begin{pmatrix} c_1 \\ e^{i\delta/2} c_2 \end{pmatrix}.
\]

(8)

The exponential \( e^{i\delta} \) indeed represents a phase lag of the local harmonic oscillator at \( z = 0 \) vibrating in the \( y \)-direction, because of our convention to count the phase of the time dynamics as \( e^{-i\omega t} \). Furthermore, \( \Phi_g \) represents a global phase which is picked up by any wave passing through the retarder, irrespective of its polarization form. Since global phases are immaterial in our present discussion, it is common to adjust them for maximum convenience, or otherwise discard them altogether, which is what we shall do throughout. Physically, a global change of phase could be accomplished by passage of the light through an *isotropic* medium whose optical path length corresponds to the desired phase difference.

Since the global phase in (8) can be chosen freely, we may extract and discard a global phase \( \Phi_g = \frac{\delta}{2} \), in which case the transformed Jones vector reads

\[
v' = \begin{pmatrix} e^{-i\delta/2} c_1 \\ e^{i\delta/2} c_2 \end{pmatrix}.
\]

(9)
It is now clear that the matrix which accomplishes the transformation \( v \to v' \) must take the form

\[
\mathbf{J}_{\text{lin}}(0, \delta) = \begin{pmatrix} e^{-i\delta/2} & 0 \\ 0 & e^{i\delta/2} \end{pmatrix}.
\] (10)

Our notation \( \mathbf{J}_{\text{lin}}(0, \delta) \) reflects the fact that the phase difference between the components is \( \delta \), while the angle between the fast axis and the \( x \)-direction is \( \theta = 0 \).

It is easy to derive the Jones matrix \( \mathbf{J}_{\text{lin}}(\theta, \delta) \) of a linear retarder whose fast axis makes a nonvanishing angle with the \( x \)-axis. To this end we introduce the rotation matrix

\[
\mathbf{R}(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.
\] (11)

This is an \( SO(2) \)-matrix representing a passive rotation by \( \theta \) about the \( z \)-axis, that is to say, the basis vectors and vector components transform according to

\[
\mathbf{E} \sim (\mathbf{e}_1, \mathbf{e}_2) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (\mathbf{e}'_1, \mathbf{e}'_2) \begin{pmatrix} v'_1 \\ v'_2 \end{pmatrix}, \quad (\mathbf{e}'_1, \mathbf{e}'_2) = (\mathbf{e}_1, \mathbf{e}_2) \mathbf{R}(\theta)^T,
\] (12a)

\[
\begin{pmatrix} v'_1 \\ v'_2 \end{pmatrix} = \mathbf{R}(\theta) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \mathbf{R}(\theta)^T = \mathbf{R}(\theta)^{-1} = \mathbf{R}(-\theta).
\] (12b)

The group property implies that a succession of two rotations is again a rotation; furthermore, two rotations around the same axis commute. Thus we have

\[
\mathbf{R}(\theta_1 + \theta_2) = \mathbf{R}(\theta_1) \mathbf{R}(\theta_2) = \mathbf{R}(\theta_2) \mathbf{R}(\theta_1).
\] (13)

Using the rotation matrices it is easy to derive the Jones matrix \( \mathbf{J}_{\text{lin}}(\theta, \delta) \) for a linear retarder with nonvanishing angle \( \theta \): We only need to keep in mind that in the reference frame of the fast \( (\mathbf{e}'_1) \) and slow \( (\mathbf{e}'_2) \) axes, the retarder must act with a Jones matrix (10) on the components \( v'_1 \) and \( v'_2 \). Thus,

\[
\begin{aligned}
\mathbf{E} \sim (\mathbf{e}_1, \mathbf{e}_2) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &\to (\mathbf{e}'_1, \mathbf{e}'_2) \mathbf{J}_{\text{lin}}(0, \delta) \begin{pmatrix} v'_1 \\ v'_2 \end{pmatrix} \\
&= (\mathbf{e}_1, \mathbf{e}_2) \mathbf{R}(\theta)^T \mathbf{J}_{\text{lin}}(0, \delta) \mathbf{R}(\theta) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (\mathbf{e}_1, \mathbf{e}_2) \mathbf{J}_{\text{lin}}(\theta, \delta) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},
\end{aligned}
\] (14)

where we have used eqs. (12). It follows that

\[
\mathbf{J}_{\text{lin}}(\theta, \delta) = \mathbf{R}(-\theta) \mathbf{J}_{\text{lin}}(0, \delta) \mathbf{R}(\theta),
\] (15)

or explicitly,

\[
\mathbf{J}_{\text{lin}}(\theta, \delta) = \begin{pmatrix} \cos \frac{\delta}{2} - i \sin \frac{\delta}{2} \cos 2\theta & -i \sin \frac{\delta}{2} \sin 2\theta \\ -i \sin \frac{\delta}{2} \sin 2\theta & \cos \frac{\delta}{2} + i \sin \frac{\delta}{2} \cos 2\theta \end{pmatrix}.
\] (16)

Equation (15) is indeed valid for Jones matrices \( \mathbf{J} \) of arbitrary optical devices whose preferred axes are related by an orthogonal transformation, since all we have used in the derivation of (15) was the law of passive coordinate transformations onto a coordinate system whose axes coincided with the preferred axes of the optical device.

We note that the Jones matrices of the linear retarder, (10) and (15), are unitary unimodular matrices, since we have chosen the global phase \( \Phi_g \) in (10) in such a way that \( \det \mathbf{J}_{\text{lin}}(0, \delta) = 1 \).

### 2.2 The rotator in the Jones calculus

The rotator is an optical element which introduces a phase lag \( \delta \) of the left-handed relative to the right-handed component of the complex electric field vector \( \mathbf{E} \). If the electric field is expanded in terms of the basis (4), the left-handed / right-handed components are just the elements \( v_- \) and \( v_+ \) in the Jones vector \( v_c \) given in eq. (6). Thus, after passage through the rotator we have

\[
v_c = \begin{pmatrix} v_- \\ v_+ \end{pmatrix} \to v'_c = \begin{pmatrix} e^{-i\delta/2} v_+ \\ e^{i\delta/2} v_- \end{pmatrix}.
\] (17)
Evidently, the linear map which accomplishes the transformation (17) is precisely the matrix $J_{\text{lin}}(0, \delta)$ of the linear retarder given in (10); thus, in the basis (4) we simply have
\[ v'_c = J_{\text{lin}}(0, \delta) v_c . \tag{18} \]

Using eq. (6) which effects the transformation between the components for linear and circular polarization, we can easily transform eq. (18) back into the Cartesian frame $(e_1, e_2)$, in which the Jones vector is given by (3),
\[ v'_l = M J_{\text{lin}}(0, \delta) M^\dagger v_l = J_{\text{rot}}(-\frac{\delta}{2}) v_l . \tag{19} \]

This determines the Jones matrix $J_{\text{rot}}(-\frac{\delta}{2})$ of the rotator in the Cartesian basis,
\[ J_{\text{rot}}(-\frac{\delta}{2}) = \begin{pmatrix} \cos \frac{\delta}{2} & -\sin \frac{\delta}{2} \\ \sin \frac{\delta}{2} & \cos \frac{\delta}{2} \end{pmatrix} = R(-\frac{\delta}{2}) , \tag{20} \]

where the rotation matrix $R$ was defined in (11).

The reason for our seemingly strange notation convention becomes clear if we recall that a positive phase lag of the left-handed component with respect to the right-handed component should rotate the plane of polarization of a linearly polarized light beam in a counter-clockwise sense; this is a positive angle $+\delta/2$ in the $xy$-plane, if we look into the oncoming wave, i.e. from positive towards negative $z$-values. It is easy to check this:
\[ J_{\text{rot}}(-\frac{\delta}{2}) \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} = \begin{pmatrix} \cos \left( \theta + \frac{\delta}{2} \right) \\ \sin \left( \theta + \frac{\delta}{2} \right) \end{pmatrix} . \tag{21} \]

Here, $[\cos \theta, \sin \theta]^T$ is the Jones vector of a normalized, linearly polarized beam, and $\theta$ is the initial angle of inclination of the plane of polarization. After passage through the rotator, the plane of polarization has rotated by $+\delta/2$.

This finishes our discussion of linear retarders and rotators in the Jones calculus.

3 Characteristic parameters of $SU(2)$ matrices

As mentioned in the Introduction, the terms 'Characteristic Directions', 'Characteristic Phase Difference' and 'Characteristic Parameters' were first introduced by Aben within the framework of Photoelasticity [26]. We now wish to explain their meaning.

Consider a non-absorbing optical medium whose Jones matrix $U$ is unitary. If $U$ happens to have a determinant different from 1, we can always extract and discard a global phase, so that $U$ may be assumed to be unimodular, $\det U = 1$, and hence an element of $SU(2)$. A parametrisation of $SU(2)$ matrices is given by [30]
\[ U = \begin{pmatrix} \cos \theta e^{i\phi} & -\sin \theta e^{i\chi} \\ \sin \theta e^{-i\chi} & \cos \theta e^{-i\phi} \end{pmatrix} , \quad \theta \in [0, \frac{\pi}{2}] , \quad \phi, \chi \in [0, 2\pi) . \tag{22} \]

This parametrisation is unique in the interior of the interval $[0, \frac{\pi}{2}]$, i.e. for $0 < \theta < \frac{\pi}{2}$. If $\theta$ takes one of the boundary values, parts of the parameter space must undergo identification in order to make the map between parameters and matrices unique: For $\theta = 0$ we must identify points $(\theta, \phi, \chi) = (0, 0, \chi)$, $\chi \in [0, 2\pi)$. For $\theta = \frac{\pi}{2}$, the points $(\theta, \phi, \chi) = (\frac{\pi}{2}, \phi, 0)$, $\phi \in [0, 2\pi)$, must be identified. This agreement makes the mapping between parameter space and matrices bijective.

Using this parametrisation we prove the

**Theorem 3.1 (Characteristic directions).** For every unimodular unitary matrix $U$ of the form (22) there exist two real vectors
\[ w_m = \begin{pmatrix} \cos \gamma_m \\ \sin \gamma_m \end{pmatrix} , \quad m = 1, 2 \tag{23} \]
such that the vectors $Uw_m$ are real vectors times an overall phase,

$$Uw_m = e^{i\Phi_m} \begin{pmatrix} \cos \gamma'_m \\ \sin \gamma'_m \end{pmatrix} = e^{i\Phi_m} w'_m , \quad m = 1, 2.$$  \hspace{1cm} (24)

The angles $\gamma_m$ are solutions of the equation

$$\tan 2\gamma_m = -\frac{\sin(\phi + \chi) \sin 2\theta}{\cos^2 \theta \sin 2\phi - \sin^2 \theta \sin 2\chi} ,$$  \hspace{1cm} (25)

such that

$$\gamma_1 \in [-\frac{\pi}{4}, \frac{\pi}{4}] \quad \text{and} \quad \gamma_2 = \gamma_1 + \frac{\pi}{2}.$$  \hspace{1cm} (26)

The vectors $w_m, m = 1, 2$ as given in (23) span the primary characteristic directions of the matrix $U$.

The angles $\gamma'_m$ are solutions of the equation

$$\tan 2\gamma' = \frac{\sin(\phi - \chi) \sin 2\theta}{\cos^2 \theta \sin 2\phi + \sin^2 \theta \sin 2\chi} ,$$  \hspace{1cm} (27)

such that

$$\gamma'_1 \in [-\frac{\pi}{4}, \frac{\pi}{4}] \quad \text{and} \quad \gamma'_2 = \gamma'_1 + \frac{\pi}{2}.$$  \hspace{1cm} (28)

The vectors $w'_m, m = 1, 2$ as given in (24) span the secondary characteristic directions of the matrix $U$.

The angles $\gamma_m$ and $\gamma'_m$ are well-defined if and only if

$$(\theta, \phi) \not\in \left\{ (0, 0), (0, \frac{\pi}{2}), (0, \pi), (0, \frac{3\pi}{2}) \right\} ,$$  \hspace{1cm} (29a)

$$(\theta, \chi) \not\in \left\{ \left( \frac{\pi}{2}, 0 \right), \left( \frac{\pi}{2}, \frac{\pi}{2} \right), \left( \frac{\pi}{2}, \pi \right), \left( \frac{\pi}{2}, \frac{3\pi}{2} \right) \right\} .$$  \hspace{1cm} (29b)

If one of the parameter values (29) is taken, the right-hand side of eqs. (25, 27) becomes the ill-defined expression $\frac{0}{0}$, and no angles $\gamma_m, \gamma'_m$ can be determined.

Proof:

If the statement (24) holds, then the quotient $w'_m / w'_{m2}$, or equivalently, the product $w'_{m1}(w'_{m2})^*$, must be real, for $m = 1, 2$. This requires that

$$\Im w'_{m1}(w'_{m2})^* = 0 .$$  \hspace{1cm} (30)

We now express $(w'_m)_{1,2}$ as $(Uw_m)_{1,2}$, respectively, and impose the condition (30). Under the conditions (29) this leads to eq. (25), which determines an angle $2\gamma_1$ lying between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$. Then, the angle $2\gamma_1 + \pi$ also satisfies (25). It follows that $\gamma_1$ lies between $-\frac{\pi}{4}$ and $+\frac{\pi}{4}$, and the second solution is $\gamma_2 = \gamma_1 + \frac{\pi}{2}$. Thus, there are two primary characteristic directions, and they are orthogonal.

To determine the angles $\gamma'_m$, we invert (24),

$$U^\dagger w'_m = e^{-i\Phi_m} w_m .$$  \hspace{1cm} (31)

This has the form of eq. (24), and we can therefore use the same line of arguments as before, if we put the matrix $U^\dagger$ into the same form as (22),

$$U \rightarrow U^\dagger \quad \Rightarrow \quad \theta \rightarrow -\theta, \quad \phi \rightarrow -\phi, \quad \chi \rightarrow \chi .$$  \hspace{1cm} (32)

If the replacements (32) are used in (25) we obtain (27), again under the conditions (29). — It is easy to check that, whenever one of the values in (29) is taken, the right-hand sides in both (25) and (27) are ill-defined. — This finishes the proof.  

The physical meaning of this is as follows: The existence of two solutions for (24) implies that there exist two perpendicular directions for the plane of linearly polarized light, given by the solutions (23), such that if light polarized in this direction enters the device it will emerge linearly polarized again; this is a consequence of the fact that the components of the transformed Jones vector have the same overall
phase. This statement is true only for the light as it emerges behind the apparatus; in the interior of the device, the light is elliptically polarized in general. Furthermore, the phase velocity of a linearly polarized wave entering the device at the angle $\gamma_1$ differs from the one entering at $\gamma_2$, so that both waves will emerge with a phase difference $\Delta$, see eq. (36) below.

We now turn to the determination of the phases $\Phi_m$ in (24). We find

**Proposition 3.2 (Determination of phases).** The phases $\Phi_m$, $m = 1, 2$ satisfy the equations

\[
\begin{align*}
\cos^2 \Phi_m &= \cos^2 \theta \cos^2 \phi + \sin^2 \theta \cos^2 \chi, \\
\sin^2 \Phi_m &= \cos^2 \theta \sin^2 \phi + \sin^2 \theta \sin^2 \chi, \\
\tan^2 \Phi_m &= \frac{\sin^2 \phi + \sin^2 \chi \tan^2 \theta}{\cos^2 \phi + \cos^2 \chi \tan^2 \theta}
\end{align*}
\]

and

\[
\cos 2\Phi_m = \cos^2 \theta \cos 2\phi + \sin^2 \theta \cos 2\chi .
\]

**Proof:**

We perform the matrix multiplication $Uw_m$ with the form (22) for $U$, but the angles $\gamma_m$ need not be known explicitly. On the right-hand side of this equation we have $e^{i\Phi_m} w'_m$. We now expand this equation into real and imaginary parts, obtaining two equations which on their right-hand sides contain $\cos \Phi_m w'$ and $\sin \Phi_m w'$, respectively. If the component equations are squared and added, we obtain the relations (33a, 33b). To arrive at (33c), we divide (33a) by (33b) and then divide the whole equation by a common factor $\cos^2 \theta$. Subtracting (33b) from (33a) then yields (34).

It is important to point out that the phase changes $\Phi_m$ in general do not coincide with the real phases of the light as it passes through the real device, for in the latter case there is also an additional global phase which is accumulated by any light beam that travels through the apparatus. This global phase has been discarded in the construction of the Jones matrices of the retarder and the rotator, in order to make these matrices unimodular. However, the global phase is the same for light of any polarization form, and therefore cancels out if we construct the phase difference between light beams which enter the device along the primary characteristic directions. This phase difference does not depend on unobservable parameters and can therefore be measured. In principle, it can be evaluated from (33c): This equation has four solutions, given by

\[
\Phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \quad \phi + \pi, \quad -\Phi, \quad -\Phi + \pi .
\]

One of the solutions $\Phi_1$ and $\Phi_2$ must be chosen from the set $\{\Phi, \Phi + \pi\}$, while the second one must take values in $\{-\Phi, -\Phi + \pi\}$. In both cases we arrive at the conclusion that

\[
\Delta = 2\Phi_2 \mod \pi .
\]

At this point a remark concerning the results (28) and (36) are in order:

Eq. (27) determines the directions of polarization of the emerging light. However, it does not determine whether $Uw_1$ is equal to $w'_1$ or $w'_2$, and similarly for $Uw_2$. This is reflected in the indeterminacy of the phase difference in (36): The modulus of $\Delta$ is determined only up to $\pi$, while the sign is undetermined. It is at this point where the decomposition theorem 4.1 and the ensuing Poincaré Equivalence Theorem prove very useful, since they allow to remove this indeterminacy completely, provided the measurements on the sample are capable of distinguishing between the fast and the slow axis of the equivalent retarder [see sections 5, 7].

For every optical device described by a unitary unimodular matrix $U$, the set of quantities $(\gamma_1, \gamma'_1, \Delta)$, are called the characteristic parameters of the model.

4 Matrix Equivalence and the Poincaré Equivalence Theorem

In theorem 4.1 we show that every two-dimensional unitary unimodular matrix $U$ has a unique decomposition in terms of a minimal number of 'basic building blocks'. There are two of these building blocks:
The first is given by the rotation matrices $\mathbf{R}(\xi)$ as defined in (11); the second is given by diagonal unitary unimodular matrices
\[
\mathbf{D}(\delta) = \begin{pmatrix}
e^{i\delta} & 0 \\ 0 & e^{-i\delta}
\end{pmatrix}, \quad \delta \in [0, 2\pi).
\] (37)

After that, in theorem 5.1 we deduce the Poincaré Equivalence Theorem from theorem 4.1.

**Theorem 4.1 (Decomposition).** Let $U$ be a unitary unimodular matrix in the parametrisation (22). Then the following statements are true:

1.) If the angle $\theta$ in (22) lies in the interior of the interval $[0, \frac{\pi}{2})$, $0 < \theta < \frac{\pi}{2}$, then there exist angles $\alpha, \beta \in [0, \pi)$, $\delta \in [0, 2\pi)$, (38)

\[
U = \mathbf{R}(-\frac{\alpha + \beta}{2}) \mathbf{D}(\delta) \mathbf{R}(\frac{\alpha - \beta}{2}) 
\] (39a)

\[
= \begin{pmatrix}
\cos \frac{\alpha + \beta}{2} & -\sin \frac{\alpha + \beta}{2} \\
\sin \frac{\alpha + \beta}{2} & \cos \frac{\alpha + \beta}{2}
\end{pmatrix}
\begin{pmatrix}
e^{i\delta} & 0 \\ 0 & e^{-i\delta}
\end{pmatrix}
\begin{pmatrix}
\cos \frac{\alpha - \beta}{2} & \sin \frac{\alpha - \beta}{2} \\
-\sin \frac{\alpha - \beta}{2} & \cos \frac{\alpha - \beta}{2}
\end{pmatrix}
\] (39b)

\[
= \begin{pmatrix}
\cos \delta \cos \beta + i \sin \delta \cos \alpha & -\cos \delta \sin \beta + i \sin \delta \sin \alpha \\
\cos \delta \sin \beta + i \sin \delta \sin \alpha & \cos \delta \cos \beta - i \sin \delta \cos \alpha
\end{pmatrix}.
\] (39c)

The angles (38) are uniquely determined, and there is no way to represent $U$ by less than three matrices of the kind (11) and (37).

II.) For $\theta = 0$ we have
\[
U = \mathbf{D}(\phi).
\] (40a)

This is a unique minimal decomposition in terms of one diagonal matrix (37).

A representation in terms of three matrices, form (39a), may be given for parameter values
\[
\alpha = \beta = 0, \quad \delta = \phi,
\] (40b)

which is unique provided that
\[
\phi \notin \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}.
\] (40c)

If $\phi \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$, a decomposition in terms of three matrices is not unique, since one of the angles $\alpha$ or $\beta$ is left unspecified. However, the choice of parameters (40b) is unique in that it always corresponds to the minimal decomposition (40a).

III.) For $\theta = \frac{\pi}{2}$ we have
\[
U = \mathbf{R}(-\frac{\pi}{2}) \mathbf{D}(-\chi).
\] (41a)

This is a unique minimal decomposition in terms of one diagonal matrix (37) and one rotation matrix (11).

A representation in terms of three matrices, form (39a), may be given for parameter values
\[
\alpha = \beta = \frac{\pi}{2}, \quad \delta = -\chi,
\] (41b)

which is unique provided that
\[
\chi \notin \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}.
\] (41c)

If $\chi \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$, a decomposition in terms of three matrices is not unique, since one of the angles $\alpha$ or $\beta$ is left unspecified. However, the choice of parameters (41b) is unique in that it always corresponds to the minimal decomposition (41a).

\textbf{Proof:}

We first must confirm that the right-hand sides of (39a, 40a, 41a) indeed lie in $SU(2)$. But this is clear, since the adjoints of these matrices are equal to their inverses, and the determinant is equal to
one in each case. So, we know that these matrices comprise a subset of SU(2); but we do not yet know whether this subset comprises the whole of SU(2). This is what we are going to show next.

To this end we first focus on decompositions in terms of three matrices, (39a). Let \( (\theta, \phi, \chi) \) be a parameter triple pertaining to a given form (22). If a decomposition (39a) of \( U \) exists, there must exist a triple \( (\alpha, \beta, \delta) \) such that the real and imaginary parts of each component of the matrices (22) and (39c) coincide. This yields four equations

\[
\begin{align*}
\cos \delta \cos \beta &= \cos \theta \cos \phi, \\
\cos \delta \sin \beta &= \sin \theta \cos \chi, \\
\sin \delta \cos \alpha &= \cos \theta \sin \phi, \\
\sin \delta \sin \alpha &= -\sin \theta \sin \chi.
\end{align*}
\]

We now show that the system (42) is indeed solvable for the triple \( (\alpha, \beta, \delta) \), and that the solution is unique under the circumstances described in case I.) – III.) above.

We first examine case I.): On dividing (42c) by (42a), and (42d) by (42b), we find

\[
\begin{align*}
tan \phi &= \frac{\tan \delta \cos \alpha \cos \beta}{\cos \theta \cos \phi}, \\
tan \chi &= -\tan \delta \frac{\sin \alpha \sin \beta}{\sin \theta \sin \chi}.
\end{align*}
\]

For this step to be admissible we had to use the fact that \( \cos \theta, \sin \theta \neq 0 \), which is true by assumption. Furthermore, we square and add (42a) and (42b), and (42c) and (42d), and divide the resulting equations. This yields the formula

\[
\tan^2 \delta = \frac{\sin^2 \phi + \sin^2 \chi \tan^2 \theta}{\cos^2 \phi + \cos^2 \chi \tan^2 \theta},
\]

which determines \( \tan \delta \) up to a sign from known quantities \( \phi, \chi \) and \( \theta \). However, we require [see (38)] that \( \sin \alpha, \sin \beta \) always be nonnegative. Then the second equation in (43) implies that the sign of \( \tan \delta \) is the negative of the sign of \( \tan \chi \). Thus, \( \delta \) is determined up to multiples of \( \pi \). Now, since \( \sin \beta \) must be nonnegative, eq. (42b) determines the sign of \( \cos \delta \), and therefore \( \delta \) is uniquely determined in \([0, 2\pi)\).

Now that \( \delta \) is determined, \( \alpha \) and \( \beta \) are uniquely determined by the system (42). In fact they are determined to lie in the interval \([0, \pi)\), for, we have used the condition \( \sin \beta \geq 0 \) as well as the positivity of \( \frac{\sin \alpha \sin \beta}{\sin \theta \sin \chi} \) in (43) for the determination of \( \delta \).

Now let us examine case II.): Let \( \theta = 0 \). We first treat the case \( \phi \not\in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\} \). Then the system (42) becomes

\[
\begin{align*}
\cos \delta \cos \beta &= \cos \phi, \\
\cos \delta \sin \beta &= 0, \\
\sin \delta \cos \alpha &= \sin \phi, \\
\sin \delta \sin \alpha &= 0.
\end{align*}
\]

Since none of \( \sin \phi, \cos \phi \) is zero, eqs. (45a) and (45c) imply that \( \sin \delta, \cos \delta \neq 0 \). But then eqs. (45b), (45d) imply that \( \sin \alpha = \sin \beta = 0 \), and due to (38) it follows that \( \alpha = \beta = 0 \). In this case, (45a) and (45c) imply that \( \delta = \phi \). This gives the minimal decomposition (40a), which coincides with form (39a) for the choice of parameters (40b).

Now we briefly discuss the cases where \( \phi \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\} \). We proceed along the same lines as above, that is to say, using the system (45) as our starting point. For \( \cos \phi = 0 \) we arrive at the decomposition

\[
U = R(-\frac{\pi}{2}) D(\phi) R(-\frac{\pi}{2}),
\]

where \( \alpha = 0 \) and \( \beta \) is left unspecified. This indeterminacy corresponds to the non-uniqueness of the parametrisation \( (\theta, \phi, \chi) \), eq. (22), when the parameter \( \theta \) takes values \( 0, \frac{\pi}{2} \) in the boundary of the parameter space. Eq. (46a) contains three matrices for \( \beta \neq 0 \), but only one for \( \beta = 0 \), since in this
case $\mathbf{R}(0) = \mathbb{I}_2$. The choice $\beta = 0$ therefore defines the minimal decomposition which is unique if $\beta$ is restricted [eq. (38)] to lie in the admissible range $[0, \pi)$. This choice of parameters coincides with (40b), and the resulting form for $U$ coincides with (40a).

Analogously, for $\sin \phi = 0$ we have the decomposition

$$U = \mathbf{R}(-\frac{\alpha}{2}) \mathbf{D}(\phi) \mathbf{R}(\frac{\alpha}{2})$$

with $\beta = 0$ and $\alpha$ left unspecified. For general $\alpha$, (46b) provides a decomposition in terms of three matrices. A minimal decomposition, in terms of one matrix only, is obtained for the choice $\alpha = 0$, which is unique again, since $\alpha$ is restricted to lie in $[0, \pi)$. Again, the parameter values $\alpha = \beta = 0$ corresponding to the minimal decomposition coincide with (40b), and the resulting minimal form is (40a).

Now we examine case III.): Let $\theta = \frac{\pi}{2}$ but $\chi \notin \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ at first. Then the system (42) becomes

$$\cos \delta \cos \beta = 0$$

$$\cos \delta \sin \beta = \cos \chi$$

$$\sin \delta \cos \alpha = 0$$

$$\sin \delta \sin \alpha = -\sin \chi$$

Since $\sin \chi, \cos \chi \neq 0$ by assumption, we must have $\sin \delta, \cos \delta \neq 0$ from (47b, 47d). But then (47a, 47c) imply that $\cos \alpha = \cos \beta = 0$, or $\alpha = \beta = \frac{\pi}{2}$, due to (38). Then (47b, 47d) imply that $\delta = -\chi$, so all angles are uniquely determined, and the decomposition (41a) follows.

Now we discuss the remaining cases, $\chi \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$. This proceeds exactly as for eqs. (46a, 46b). For $\cos \chi = 0$ we obtain the decomposition

$$U = \mathbf{R}(-\frac{\pi/2 + \beta}{2}) \mathbf{D}(-\chi) \mathbf{R}(\frac{\pi/2 - \beta}{2})$$

with $\alpha = \frac{\pi}{2}$ and $\beta$ left unspecified. This becomes minimal for $\beta = \frac{\pi}{2}$, corresponding to (41b), and leads to eq. (41a). For $\sin \chi = 0$ we have

$$U = \mathbf{R}(\frac{\pi/2 - \beta}{2}) \mathbf{D}(\chi) \mathbf{R}(\frac{\pi/2 + \beta}{2})$$

with $\beta = \frac{\pi}{2}$ and unspecified $\alpha$. The minimal decomposition is obtained for $\alpha = \frac{\pi}{2}$, corresponding to (41b), which leads to

$$U = \mathbf{R}(-\frac{\pi}{2}) \mathbf{D}(\chi)$$

However, in the case at hand we have $\mathbf{D}(\chi) = \mathbf{D}(-\chi)$, so that the minimal form is again equal to (41a).

This finishes the proof of theorem 4.1.

5 The Poincaré Equivalence Theorem

We are now in a position to deduce the Poincaré Equivalence Theorem from our decomposition theorem 4.1. Let us assume that $U$ is the Jones matrix of a non-absorbing optical medium. As discussed before, $U$ is unitary and may be assumed to be unimodular, possibly after discarding a global phase. Then $U$ has a unique decomposition (39a) or (40a) or (41a). For the purposes of the Poincaré Equivalence Theorem, we use the decomposition (39a) in terms of three matrices, so that

$$U = \mathbf{R}(-\xi) \mathbf{D}(\delta) \mathbf{R}(\zeta)$$

Here we have introduced two angles

$$\xi = \frac{\alpha + \beta}{2}$$

$$\zeta = \frac{\alpha - \beta}{2}$$

and it is understood that $\xi$ and $\zeta$ are always chosen according to the minimal decompositions (39a, 40b, 41b). In the next step we take into account that the diagonal matrix $\mathbf{D}(\delta)$ is formally identical to the
Jones matrix of a linear retarder with fast axis along the $x$-direction and a phase lag of $-2\delta$ of the slow with respect to the fast axis,

$$D(\delta) = j_{\text{lin}}(0, -2\delta) \ ,$$

according to (10). We can now rewrite (49) as

$$U = R(-\xi) j_{\text{lin}}(0, -2\delta) R(\xi) R(-\xi + \xi) = j_{\text{lin}}(\xi, -2\delta) R(\xi - \xi)$$

$$= R(-\xi + \xi) R(-\xi) j_{\text{lin}}(0, -2\delta) R(\xi) R(\xi - \xi) j_{\text{lin}}(\xi, -2\delta) \ .$$

Here we have used (15) and the group property (13) of the rotation matrices.

In eqs. (52), the rotation matrices have been used to transform the Jones matrices of linear retarders at different angles of their fast axes into each other. However, in eq. (20) we pointed out that a rotation matrix $R(-\pi/2)$ is formally identical to the Jones matrix of a rotator $j_{\text{rot}}(-\pi/2)$. Then the right-hand sides of eqs. (52a, 52b) may be rewritten as

$$U = j_{\text{lin}}(\pi/2 + \beta, -2\delta) j_{\text{rot}}(-\beta) \ ,$$

$$= j_{\text{rot}}(-\beta) j_{\text{lin}}(\pi/2 + \beta, -2\delta) \ .$$

From the proof of theorem 4.1 we know that the decompositions (52, 53) are unique provided that the parameters ($\theta, \phi, \chi$) in the parametrisation (22) of $U$ are restricted according to (29).

If one of the parameter sets in (29) is taken, the matrices in (49) are no longer uniquely determined. From the proof of theorem 4.1 we know that in these cases either $\alpha$ or $\beta$ remains unspecified. However, the phase retardation $\Delta = 2\delta$ is always well-defined. It is easy to translate the condition that ($\theta, \phi$) or ($\theta, \chi$) take one of the values in (29a, 29b), respectively, into a condition for the optical parameters of the linear retarder and rotator in eqs. (53). This can be done with the help of (46a, 46b, 48a, 48b), or also (40a) and (41a). It is easy to check that the decomposition is non-unique if and only if the phase retardation $\Delta$ of the linear retarder in the equivalent optical model is an even or odd integer multiple of $\pi$. Physically this means that the retarder in these cases can be realized by a stack of half-wave plates.

In eqs. (52), (53) we have proven the

**Theorem 5.1 (Poincare Equivalence Theorem).** For every non-absorbing passive optical medium (represented by a unimodular unitary matrix) there exists an optically equivalent model which is built from one linear retarder and one rotator only. These elements may be arranged in any order, and in both cases the phase lag $\Delta = 2\delta$ is the same.

If $\Delta$ is not a multiple of $\pi$, $\alpha$ and $\beta$ are uniquely determined.

If $\Delta$ is a multiple of $\pi$ (corresponding to a stack of half-wave plates), one of the angles $\alpha$ or $\beta$ is left unspecified. However, $\alpha$ and $\beta$ may still be chosen according to (40b) and (41b), corresponding to the minimal decompositions (40a) and (41a).

### 6 Jones vectors, Stokes parameters, and the Poincaré sphere

In eqs. (2, 3) we have defined the polarization form and the associated Jones vector of a completely polarized beam of light. We mentioned that these quantities are defined only up to a global phase, so that Jones vectors $v$ and $e^{i\phi}v$ are equivalent.

An alternative description of the light beam in terms of the so-called *Stokes parameters* removes this ambiguity: The reason is that, in contrast to the Jones vector, the Stokes parameters are quadratic in the (complex) field strength and can, in principle, be determined by four intensity measurements, in conjunction with a linear polarizer and a quarter-wave plate, or equivalents [2, 31]. The Stokes parameters can be motivated by observing that, for a wave propagating in the $z$-direction, the scalar products $e_1 \cdot E = a_1$ and $e_2 \cdot E = a_2$ are the complex amplitudes of electric field strength linearly polarized in the $x$- and $y$-direction, respectively. The phase difference between these components accounts for the (in general) elliptical polarization form of the light beam. Then the Stokes parameters are defined in terms
of these quantities as

\[
\begin{align*}
s_0 &= |\mathbf{e}_1 \cdot \mathbf{E}|^2 + |\mathbf{e}_2 \cdot \mathbf{E}|^2 = a_1^2 + a_2^2, \\
s_1 &= |\mathbf{e}_1 \cdot \mathbf{E}|^2 - |\mathbf{e}_2 \cdot \mathbf{E}|^2 = a_1^2 - a_2^2, \\
s_2 &= 2 \Re \left( [\mathbf{e}_1 \cdot \mathbf{E}^* (\mathbf{e}_1 \cdot \mathbf{E})] = 2 a_1 a_2 \cos(\delta_2 - \delta_1), \\
s_3 &= 2 \Im \left( [\mathbf{e}_1 \cdot \mathbf{E}^* (\mathbf{e}_1 \cdot \mathbf{E})] = 2 a_1 a_2 \sin(\delta_2 - \delta_1). 
\end{align*}
\]

The electric field vector of the polarized light beam, expressed by (3) or (54), describes an ellipse whose principal axes are not in the \(x\)- and \(y\)-direction in general. The first principal axis, with length \(2a\), makes an angle \(\psi\) with the \(x\)-axis, while the second principal axis, being perpendicular to the first, has a length \(2b\). The shape and orientation of the ellipse can be described by the angle \(\chi\) such that \(\tan \chi = \pm b/a\). It is then a matter of elementary computation to show that the Stokes parameters (54) can be expressed in terms of these angles as [2]

\[
\begin{align*}
s_1 &= s_0 \cos 2\chi \cos 2\psi, \\
s_2 &= s_0 \cos 2\chi \sin 2\psi, \\
s_3 &= s_0 \sin 2\chi.
\end{align*}
\]

Evidently, the system (55) can be interpreted as the Cartesian coordinates of a point on a sphere of radius \(s_0\), the so-called Poincaré sphere [27]. The significance of the Poincaré sphere lies in the fact that its points are in 1-1 correspondence with the distinct states of polarization of a completely polarized light beam with constant intensity; the ambiguity in the Jones matrix formalism, in which a continuous infinity of Jones vectors corresponded to physically equivalent polarization forms, has been removed. One can show that right- (left-) handed circular polarization is represented by the north (south) pole on the Poincaré sphere; whilst the states of linear polarization correspond to the points in the equatorial plane. All other points describe a general elliptical polarization. Each of the equatorial points represents a different angle of the plane of linear polarization; as a consequence, an optical rotator, changing just this angle, but preserving the linear polarization form, may be regarded as a rotation \(R_3\) about the \(z\)-axis on the Poincaré sphere. Similarly, an optical retarder, capable of transforming a linear polarization state into a general elliptical one, will correspond to a rotation about an angle \(2\psi\) whose axis lies in the \(xy\)-plane of the coordinate system. However, any such rotation can be obtained as a sequence \(R_3 R_2 R_3^{-1}\) of rotations about the \(z\)-axis, the momentary \(y\)-axis, and the new \(z\)-axis again.

More generally, it can be shown that the action of any loss-less optical element corresponds to a proper rotation on the Poincaré sphere. From this fact, the Poincaré Equivalence Theorem can be inferred immediately: For, let \(U\) denote the unitary matrix of the optical element in the Jones matrix formalism, and let \(R(U)\) denote the corresponding rotation on the Poincaré sphere. We know that every proper rotation can be decomposed into three Euler rotations, which may be chosen as a sequence \(R_1 R_2 R_3\) of three rotations about the momentary \(z\)- and \(y\)-axes. From the previous paragraph we know that the factors in this decomposition represent the actions of rotators and linear retarders, so that the sequence \(R_3 R_2 R_3\) corresponds, in the Jones matrix formalism, to a sequence of optical elements \(\mathbf{R}' \mathbf{D} \mathbf{R}\), which is of course just eq. (39a). In this way, the decomposition of \(SU(2)\)-matrices into 'basic building blocks' as in the Poincaré Equivalence Theorem can be traced back to the Euler decomposition of proper rotations on the Poincaré sphere.

The parameters in (54) are classical quantities, of course. It is possible to define associated quantum Stokes parameters together with a quantum Poincaré sphere; these concepts are reviewed in [16], for example. In [15], an application of the quantum Stokes parameters to squeezed light has been presented.

### 7 Relation between Characteristic Parameters and the Poincaré-Equivalent Model

In section 3 we have shown that every optical device which (in the framework of Jones calculus) can be represented by a unitary unimodular \((2, 2)\) matrix \(U\) possesses two orthogonal directions, \(w_1\) and \(w_2\), such that linearly polarized light whose plane of polarization on entry into the apparatus is \(w_1\) or \(w_2\) will emerge linearly polarized again, with polarization directions \(w'_1\) and \(w'_2\). Furthermore, we saw that
linearly polarized light entering at the two primary characteristic directions emerges with different phases; an expression for the relative phase difference was given in (36), but we were not able to determine it uniquely.

On the other hand, in theorem 5.1 we showed that every such device was optically equivalent to a succession of linear retarders and rotators. Since a linear retarder has two preferred spatial directions (the fast and slow axis) and a well-defined phase retardation between them, this suggests that the characteristic directions might be closely related to the fast and slow axes, while the phase difference (36) might be identified with the retardation of the retarder.

We now show that this is indeed so: We first treat the unique cases, i.e. parameters \((\theta, \phi, \chi)\) satisfy (29), or equivalently, the phase lag \(\Delta = -2\delta\) of the equivalent linear retarder is not a multiple of \(\pi\). In this case the linear retarder has precisely two preferred directions, namely the fast and slow optical axis. Only at these directions preserves the linear retarder the state of linear polarization; any other direction of linear polarization on entry will emerge as elliptical polarization form. On using the decomposition (49) it is then clear that the primary characteristic directions are determined by the condition

\[
\begin{align*}
R(\zeta) w_m &= e_m, & m &= 1, 2 , \\
e_1 &= (1, 0)^T, & e_2 &= (0, 1)^T. 
\end{align*}
\] (56)

Therefore we must have

\[
w_1 = R(-\zeta) e_1 = \begin{bmatrix} \cos \left(\frac{\alpha - \beta}{2}\right) \\ \sin \left(\frac{\alpha - \beta}{2}\right) \end{bmatrix}, \quad w_2 = R(-\zeta) e_2 = \begin{bmatrix} -\sin \left(\frac{\alpha - \beta}{2}\right) \\ \cos \left(\frac{\alpha - \beta}{2}\right) \end{bmatrix},
\] (57)

where we have defined that \(w_1\) be the direction corresponding to the faster phase velocity. Similarly, the secondary characteristic directions must be

\[
w' = R(-\zeta) e_1 = \begin{bmatrix} \cos \left(\frac{\alpha + \beta}{2}\right) \\ \sin \left(\frac{\alpha + \beta}{2}\right) \end{bmatrix}, \quad w' = R(-\zeta) e_2 = \begin{bmatrix} -\sin \left(\frac{\alpha + \beta}{2}\right) \\ \cos \left(\frac{\alpha + \beta}{2}\right) \end{bmatrix}.
\] (58)

Again, we have defined that \(w'\) be the secondary characteristic direction corresponding to the faster phase velocity. A comparison between (57) and (23), and (58) and (24) now shows that we should set

\[
\gamma_1 = \frac{\alpha - \beta}{2} = \zeta , \quad \gamma_2 = \gamma_1 + \frac{\pi}{2} ,
\] (59a)

and

\[
\gamma'_1 = \frac{\alpha + \beta}{2} = \zeta , \quad \gamma'_2 = \gamma'_1 + \frac{\pi}{2} ,
\] (59b)

where we have used (50).

Light which travels through the medium with linear polarization along \(w_1, w_2\) acquires a phase (relative to a global phase \(\Phi_y\)) of

\[
\Phi_1 = \delta , \quad \Phi_2 = -\delta .
\] (60)

Then the total phase difference between directions \(w_1\) and \(w_2\) is

\[
\Delta = \Phi_2 - \Phi_1 = -2\delta .
\] (61)

The arguments leading to eqs. (57–61) clearly are based on intuition rather than rigorous derivation. In order to be rigorous we have to show that these equations are consistent with the results of theorems 3.1, 3.2 and eq. (36) in section 3. More precisely, we must show that the angles \(\gamma_m\) as defined in (59a) satisfy the equations (25), while \(\gamma'_m\) as defined in (59b) must satisfy (27). This can be done easily, on using the assumption that parameters \((\theta, \phi, \chi)\) satisfy (29): To show that the first statement is true it is sufficient to expand \(\tan(\alpha - \beta)\) in terms of sines, cosines of \(\alpha + \beta\), and multiply both nominator and denominator of the resulting fraction with \(\cos \delta \sin \delta\); it is at this point where we need the condition that \(\cos \delta, \sin \delta \neq 0\). The result can be easily recast in the form of eq. (25). The second statement linking (59b) to (27) is confirmed in the same way. Finally, eqs. (33a, 33b) can be derived immediately from the system (42). We note that we have removed the indeterminacy concerning the assignment between \(\Phi_m\)
and \( \pm \delta \) by decreeing that the subscript 1 should refer to the fast axis of the equivalent retarder. This assignment is now well-justified since it is based on the distinction between different values of a physically measurable quantity, namely the phase velocity along the axes. Previously, in section 3, we did not yet possess the physical justification for such a deliberate choice.

We now turn to discuss the non-unique decompositions: From section 5 we know that in these cases the phase retardation of the equivalent linear retarder is always a multiple of \( \pi \); in particular, it is always well-defined. What is left unspecified is one of the angles \( \alpha, \beta \) the phase retardation of the equivalent linear retarder is always a multiple of \( \pi \). This is physically reasonable, for, whenever \( \Delta = k\pi, k \in \mathbb{Z} \), the retarder transforms any linearly polarized light into linearly polarized light again, and hence any direction (before entry into the medium) is a primary characteristic direction (and any direction behind the medium is a secondary one). This is clearly consistent with the eqs. (25, 27): In the comment following (29) we remarked that (25, 27) remained ill-defined if parameter values (29) were taken. This simply means that there are no preferred directions in this case, and any two orthogonal directions before and behind the medium may serve as primary and secondary characteristic directions.

We finish this section by writing down the decompositions (53) of the equivalent optical model in terms of the characteristic parameters: In the case of unique decompositions, eqs. (49) and (53) read

\[
U = J_{\text{rot}}(-\gamma'_{1}) J_{\text{lin}}(0, \Delta) J_{\text{rot}}(\gamma_{1})
\]

\[
= J_{\text{lin}}(\gamma'_{1}, \Delta) J_{\text{rot}}(\gamma_{1} - \gamma'_{1})
\]

\[
= J_{\text{rot}}(\gamma_{1} - \gamma'_{1}) J_{\text{lin}}(\gamma_{1}, \Delta)
\]

where we have used (59) and (61). For \( \theta = 0 \), the minimal decomposition (40a) can be expressed as

\[
U = J_{\text{lin}}(0, \Delta), \quad \Delta = k\pi, \quad k \in \mathbb{Z}
\]

For \( \theta = \frac{\pi}{2} \), the minimal decomposition (41a) can be expressed as

\[
U = J_{\text{rot}}(-\frac{\pi}{2}) J_{\text{lin}}(\Delta), \quad \Delta = k\pi, \quad k \in \mathbb{Z}
\]

### 8 Basic Ideas of Integrated Photoelasticity

In this section we want to outline, in a single paragraph, how optical media with spatially dependent dielectric tensors naturally emerge in Photoelasticity.

Consider a medium (such as glass or certain polymers) which is optically homogeneous and isotropic, with a dielectric tensor \( \epsilon_{ij} \) when strain-free, but which becomes inhomogeneous and anisotropic when subject to an external load. The dielectric tensor \( \epsilon_{ij} \) then is related to the stress tensor \( \sigma_{ij} \) of the material according to the stress-optical law

\[
\epsilon_{ij} = \epsilon \delta_{ij} + C_1 \sigma_{ij} + C_2 \text{tr}(\sigma) \delta_{ij}, \quad i,j = 1,2,3
\]

where \( C_1, C_2 \) are stress-optical constants. The medium is assumed to occupy a bounded region in space. Polarized light is sent through the specimen at many different angles, and from the change of the state of polarization the characteristic parameters of the model, for a given direction of the incident light beam, are retrieved. Light is assumed to propagate through the medium along straight lines, an approximation which is acceptable as long as the stress-induced birefringence is weak. Under these circumstances, the Maxwell equations describing the propagation of a plane wave with locally varying state of polarization along the \( z \)-direction of the medium can be cast into the form

\[
\frac{d}{dz} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{i \omega}{2 \epsilon \sqrt{\epsilon}} \begin{pmatrix} \epsilon_{11} - \epsilon & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} - \epsilon \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}
\]

where \( v_1(z), v_2(z) \) are the components of the spatially dependent Jones vector for fixed \( x \) and \( y \), and \( \epsilon_{ij}(z), i,j = 1,2 \) are the components of the projection of the local dielectric tensor onto the \( xy \)-plane, at position \((x,y,z)\). The point \((x,y)\) in the \( xy \)-plane characterizes the location of entry of the light beam (which is assumed to have very small diameter) into the medium. Eqs. (64) may be expressed in an equivalent way by introducing a unitary propagation matrix \( U(z) \) such that

\[
v(z) = U(z, z_i) v(z_i)
\]
where \( z_i \) is any point before the medium. This matrix must satisfy a differential equation analogous to (64),
\[
\frac{d}{dz} U(z, z_i) = i H(z) U(z, z_i)
\]
where the matrix \( H \) is given on the right-hand side of (64). If \( z_f \) is a point behind the medium, the matrix \( U(z_f, z_i) \) is obtained by integrating (66); it determines the change of polarization form of the light beam passing through the medium between points \((x, y, z_i)\) and \((x, y, z_f)\) along a straight line. Measurements of this change in the polarization determine the characteristic parameters, and in turn, the equivalent model in either of the forms (62), for the given ray of light. Repeating the procedure for many rays at different angles yields data sets which contain the characteristic parameters for any possible direction of the light beam. From these data we attempt to reconstruct the local dielectric tensor in the interior of the medium. This problem, called three-dimensional Photoelasticity, is still unsolved in its most general form, and the Poincaré Equivalence Theorem as given in one of the forms above may play an important role in this goal.

9 Summary

In this paper we have proven a decomposition theorem for two-dimensional unitary unimodular matrices which represent a non-absorbing passive optical medium in the framework of the Jones calculus. From this decomposition theorem, the Poincaré Equivalence Theorem can be derived as a corollary. The latter states that every non-absorbing optical medium is optically equivalent to a succession of linear retarders and rotators, in any order. Except for few cases, the equivalent linear retarder and rotator are uniquely determined from data characterizing the original medium. We have shown that the parameters of these equivalent elements have a simple interpretation in terms of measurable quantities, the so-called Characteristic Parameters of an optical model, which have been known in the engineering literature for some time. The relation between these Characteristic Parameters and the Poincaré-equivalent optical model has been fully clarified. Finally, we have illustrated the context in which the Poincaré-equivalent model is expected to become a useful tool: Within the framework of three-dimensional Photoelasticity, the decomposition theorems proven in this paper may turn out to be important to the goal of reconstructing local dielectric tensors from measurements of the global change of the polarization state of light transmitted through an inhomogeneous and anisotropic medium with spatially varying refractive tensor.

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