A Sharp Liouville Theorem for Elliptic Operators *

Enrico Priola †
Department of Mathematics, University of Torino, Torino, 10123, Italy
Feng-Yu Wang ‡
School of Mathematical Sci. and Lab. Math. Com. Sys., Beijing Normal University, Beijing 100875, China
and
Department of Mathematics, Swansea University, Singleton Park, SA2 8PP, UK

February 16, 2010

Abstract

We introduce a new condition on elliptic operators $L = \frac{1}{2} \triangle + b \cdot \nabla$ which ensures the validity of the Liouville property for bounded solutions to $Lu = 0$ on $\mathbb{R}^d$. Such condition is sharp when $d = 1$. We extend our Liouville theorem to more general second order operators in non-divergence form assuming a Cordes type condition.

AMS subject Classification: 35J15, 47D07.
Keywords: Liouville theorem, space-time harmonic functions.

1 Introduction

Let

$$L = \frac{1}{2} \sum_{i,j=1}^{d} q_{ij}(x) D_{ij} + \sum_{i=1}^{d} b_i(x) D_i$$

be a uniformly elliptic second order differential operator on $\mathbb{R}^d$ with continuous coefficients $q_{ij}$ and $b_i$ (here $D_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$ and $D_i = \frac{\partial}{\partial x_i}$, $1 \leq i, j \leq d$). Recall that a smooth real

*Supported in part by NNSFC(10721091) and the 973-Project and by the Italian M.I.U.R. research project “Equazioni di Kolmogorov”.
†enrico.priola@unito.it
‡wangfy@bnu.edu.cn; F.Y.Wang@swansea.ac.uk
function $u$ on $\mathbb{R}^d$ is called $L$-harmonic if $Lu = 0$ holds on $\mathbb{R}^d$. An operator $L$ is said to possess the Liouville property when all bounded $L$-harmonic functions are constant (or, equivalently, when a two-sided Liouville theorem holds for $L$). Such property is also of interest for the study of non-linear PDEs of the form $\Delta u + F(u) = 0$ (see e.g. [1, 2]).

There are a plenty of results on the Liouville property. Let $\lambda_0 > 0$ be the ellipticity constant of $L$. A typical condition implying the Liouville property is the following (see e.g. [3, 6, 7]):

\[
\frac{1}{2\lambda_0}\|q(x) - q(x + h)\|^2 + 2\langle b(x + h) - b(x), h \rangle \leq 0, \quad x, h \in \mathbb{R}^d
\]

(given a $d \times d$ real matrix $A$, we denote by $\|A\|$ its Hilbert-Schmidt norm; moreover $\langle \cdot, \cdot \rangle$ is the Euclidean inner product in $\mathbb{R}^d$). However this is not completely satisfactory for two reasons. The first one is that when $b(x)$ is constant the matrix $q(x)$ must be constant as well. This is a restriction since it is known that the Liouville property holds when $b$ is constant and $q(x)$ is variable (see [5, Corollary 4.1] which is a consequence of invariant Harnack inequalities).

The second weak point of (1.1) is that when $q(x)$ is the identity, i.e., we are considering $L_0 = \frac{1}{2}\Delta + b \cdot \nabla$, such hypothesis is not optimal even when $d = 1$. The aim of this note is to find out a sharp and easy to check criterion ensuring the Liouville property for $L_0$. Our condition is sharp when $d = 1$; indeed if this does not hold one can construct counterexamples of operators $L_0$ without the Liouville property.

We prove our Liouville type theorem in the more general setting of elliptic operators $L$, with $q(x)$ variable, imposing an additional Cordes type condition (see [4]).

To explain the motivation of our desired condition for the Liouville property, let us start with a one-dimensional example

\[
L_0 = \frac{1}{2} \frac{d^2}{dx^2} + \frac{x}{2 + x^2} \left( \delta + \frac{2}{\log(2 + x^2)} \right) \frac{d}{dx},
\]

where $\delta$ is a constant. It is easy to see that a harmonic function of $L_0$ has the form

\[
u(x) = c_1 + c_2 \int_0^x \frac{dr}{(2 + r^2)^{\delta} \log^2(2 + r^2)}, \quad x \in \mathbb{R},
\]

where $c_1, c_2$ are constants. Thus, all bounded harmonic functions are constant if any only if $\delta < 1/2$. In order to reduce this condition to a usual monotonicity condition on the drift $b(x) = \frac{x}{2 + x^2} \left( \delta + \frac{2}{\log(2 + x^2)} \right)$, we note that (using also that $b$ is odd)

\[
\lim_{s \to \infty} \sup_{|x - y| = s} (x - y) (b(x) - b(y)) = \lim_{s \to \infty} s (b(s/2) - b(-s/2)) = 4\delta.
\]

Then the statement can be reformulated as all bounded $L_0$-harmonic functions are constant if and only if

\[
\lim_{s \to \infty} \sup_{|x - y| = s} (x - y) (b(x) - b(y)) < 2.
\]
In general, let e.g. \( L_0 = \frac{1}{2} \Delta + b \cdot \nabla \) on \( \mathbb{R}^d \), we may wish to prove the Liouville property of \( L_0 \) under the following hypothesis

\[
\limsup_{s \to \infty} \sup_{|x-y|=s} \langle x - y, b(x) - b(y) \rangle < 2.
\]

This follows immediately from our main result.

## 2 Main theorem

We prove a Liouville type theorem for bounded space-time harmonic functions. Recall that a smooth function \( u \) on \([0, \infty) \times \mathbb{R}^d\) is called space-time harmonic for \( L \), if

\[
\partial_t u + Lu = 0
\]

holds. To state our main result, we make the following assumptions.

**(H)**

(i) The coefficients \( b(x) \) and \( q(x) \) are continuous, and, for any \( \lambda > 0 \), \( \omega(s) := \sup_{|x-y|\leq s}\{\lambda \|q(x) - q(y)\|^2 + 2\langle x-y, b(x) - b(y) \rangle\} \) satisfies

\[
\int_0^1 \frac{\omega(s)}{s} ds < \infty;
\]

(ii) there exist two constants \( 0 < \lambda_0 < \Lambda_0 \) such that

\[
\lambda_0 |h|^2 \leq \sum_{i,j=1}^n q_{ij}(x)h_i h_j \leq \Lambda_0 |h|^2, \quad x, h \in \mathbb{R}^d.
\]

**Theorem 2.1.** Assume (H). If

\[
\limsup_{s \to \infty} \sup_{|x-y|=s} \langle x - y, b(x) - b(y) \rangle < 2\lambda_0 - \frac{d}{2}(\Lambda_0 - \lambda_0),
\]

then any bounded space-time harmonic function for \( L \) is constant.

**Proof.** We will suitably apply [6, Theorem 3.6]. To this purpose, we have to consider a coupling for \( L \). By (2.1) we may take constants \( \mu, s_0 > 0 \) and \( s_1 \in \mathbb{R} \) such that \( \mu < \lambda_0 \) and

\[
\sup_{|x-y|=s} \langle x - y, b(x) - b(y) \rangle \leq s_1 < 2\mu - \frac{1}{2}d(\Lambda_0 - \mu), \quad s \geq s_0.
\]

Define a symmetric positive definite matrix \( \sigma(x) \), such that \( \sigma(x)^2 + \mu I = q(x) \), \( x \in \mathbb{R}^d \). Clearly we have \( \sigma^2(x) \geq (\lambda_0 - \mu)I \). We construct a coupling as in Section 3.1 of...
replacing the ellipticity constant \( \lambda_0 \) with \( \mu \) (note that under our assumptions the associated diffusion process does not explode). Applying [6, Lemma 3.3] we deduce that

\[
\| \sigma(x) - \sigma(y) \|^2 \leq \frac{1}{4(\lambda_0 - \mu)} \| q(x) - q(y) \|^2, \quad x, y \in \mathbb{R}^d.
\]

Combining this with (H)(i) for \( \lambda = \frac{1}{4(\lambda_0 - \mu)} \), we obtain

\[
(2.3) \quad \| \sigma(x) - \sigma(y) \|^2 + 2 \langle x - y, b(x) - b(y) \rangle \leq \omega(|x - y|) \text{ for } x, y \in \mathbb{R}^d, \text{ and } \int_{0}^{s_0} \frac{\omega(s)}{s} ds < \infty.
\]

On the other hand, since \( \sigma(x)^2 \leq (\Lambda_0 - \mu) I \), we have \( 0 \leq \sigma(x) \leq (\Lambda_0 - \mu)^{1/2} I \), for any \( x \in \mathbb{R}^d \). Thus

\[
-(\Lambda_0 - \mu)^{1/2} I \leq \sigma(x) - \sigma(y) \leq (\Lambda_0 - \mu)^{1/2} I, \quad x, y \in \mathbb{R}^d.
\]

We deduce that \( 0 \leq (\sigma(x) - \sigma(y))^2 \leq (\Lambda_0 - \mu) I \) and so

\[
\| \sigma(x) - \sigma(y) \|^2 = \text{Tr} \left[ (\sigma(x) - \sigma(y))^2 \right] \leq d(\Lambda_0 - \mu), \quad x, y \in \mathbb{R}^d.
\]

Combining this with (2.2) we obtain

\[
\| \sigma(x) - \sigma(y) \|^2 + 2 \langle x - y, b(x) - b(y) \rangle \leq 2s_1 + d(\Lambda_0 - \mu) =: s_2 < 4\mu, \quad |x - y| \geq s_0.
\]

From this and (2.3) we conclude that

\[
\| \sigma(x) - \sigma(y) \|^2 + 2 \langle x - y, b(x) - b(y) \rangle \leq |x - y| g(|x - y|), \quad x, y \in \mathbb{R}^d
\]

holds for

\[
g(s) := \frac{\omega(s)}{s} 1_{[0,s_0]}(s) + \frac{s_2}{s} 1_{(s_0,\infty)}, \quad s > 0.
\]

Since by (H)

\[
c := \int_{0}^{s_0} g(s) ds < \infty,
\]

we have

\[
\int_{0}^{\infty} \exp \left( - \frac{1}{4\mu} \int_{0}^{s} g(s) ds \right) ds \geq \int_{1}^{\infty} \exp \left( - \frac{1}{4\mu} \int_{0}^{s_0} g(s) ds \right) \exp \left( - \frac{1}{4\mu} \int_{s_0}^{s} g(s) ds \right) ds
\]

\[
\geq e^{-c_1} \int_{1}^{\infty} s^{-s_2/4\mu} ds = \infty
\]

since \( s_2 < 4\mu \). Applying [6, Theorem 3.6], we get the assertion.
References

[1] Barlow, M. T., On the Liouville property for divergence form operators. Canad. J. Math. 50 (1998), no. 3, 487-496.

[2] Berestycki, H., Caffarelli, L., Nirenberg, L., Further qualitative properties for elliptic equations in unbounded domains. Dedicated to Ennio De Giorgi. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 25 (1997), no. 1-2, (1998) 69-94.

[3] Bertoldi M., Fornaro S. Gradient estimates in parabolic problems with unbounded coefficients. Studia Math. 165 (2004), no. 3, 221-254.

[4] Cordes, H. O. Uber die erste Randwertaufgabe bei quasilinearen Differentialgleichungen zweiter Ordnung in mehr als zwei Variablen. (German) Math. Ann. 131 (1956), 278-312.

[5] Krylov, N. V., Safonov, M. V., A property of the solutions of parabolic equations with measurable coefficients. Izv. Akad. Nauk SSSR Ser. Mat. 44 (1980), no. 1, 161-175.

[6] Priola, E., Wang, Feng-Yu, Gradient estimates for diffusion semigroups with singular coefficients. J. Funct. Anal. 236 (2006), 244-264.

[7] Priola, E., Zabczyk, J., Liouville theorems for nonlocal operators. J. Funct. Anal., 216 (2004), 455-490.