A strengthening of the spectral chromatic critical edge theorem: Books and theta graphs

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Abstract

A graph is color-critical if it contains an edge whose removal reduces its chromatic number. Let $T_{n,k}$ be the Turán graph with $n$ vertices and $k$ parts. Given a graph $H$, let $ex(n,H)$ be the Turán number of $H$. Simonovits’ chromatic critical edge theorem states that if $H$ is color-critical with $\chi(H) = k + 1$, then there exists an $n_0(H)$ such that $ex(n,H) = |E(T_{n,k})|$ and the Turán graph $T_{n,k}$ is the only extremal graph provided $n \geq n_0(H)$. Nikiforov proved a spectral chromatic critical edge theorem. It asserts that if $H$ is color-critical and $\chi(H) = k + 1$, then there exists an $n_0(H)$ (which is exponential with $|V(H)|$) such that $ex_sp(n,H) = \rho(T_{n,k})$ and $T_{n,k}$ is the only extremal graph provided $n \geq n_0(H)$, where $\rho(G)$ is the spectral radius of $G$ and $ex_sp(n,H) = \max\{\rho(G) : |V(G)| = n \text{ and } HG \nsubseteq G\}$. In addition, if $H$ is either a complete graph or an odd cycle, then $n_0(H)$ is linear with $|V(H)|$. A book graph $B_r$ is a set of $r$ triangles sharing a common edge and a theta graph $\theta_r$ is a graph which consists of two vertices connected by three internally disjoint paths with length one, two, and $r$. Notice that both $B_r$ and $\theta_r$ are color-critical. In this article, we prove that if $\rho(G) \geq \rho(T_{n,2})$, then $G$ contains a book $B_r$ with $r > \frac{4}{13}n$ unless $G = T_{n,2}$. Similarly, we prove that if $\rho(G) \geq \rho(T_{n,2})$, then $G$ contains a theta graph $\theta_r$ with $r > \frac{n}{10}$ for odd $r$ and $r > \frac{n}{7}$ for even $r$ unless $G = T_{n,2}$. Our results imply that $n_0(H)$ in the spectral chromatic
critical edge theorem is linear with $|V(H)|$ for book graphs and theta graphs. Our result for book graphs can be viewed as a spectral version of an Erdős conjecture (1962) stating that every $n$-vertex graph with $|E(G)| > \frac{1}{6} |E(T_{n,2})|$ contains a book graph $B_r$ with $r > \frac{n}{6}$. Moreover, our result for theta graphs yields that every graph with $\rho(G) > \rho(T_{n,2})$ contains a cycle of length $t$ for each $t \leq \frac{n}{7}$. This is related to an open question by Nikiforov (2008) which asks for the maximum $c$ such that every graph of large enough order $n$ with $\rho(G) > \rho(T_{n,2})$ contains a cycle of length $t$ for every $t \leq cn$.

**KEYWORDS**
book, chromatic critical edge theorem, consecutive cycles, spectral extrema, theta graph

**MATHEMATICS SUBJECT CLASSIFICATION**
05C50, 05C35

1 | INTRODUCTION

For a graph $H$, the *Turán number* $ex(n, H)$ is maximum possible number of edges in an $n$-vertex $H$-free graph. A graph on $n$ vertices is an *extremal graph* for $H$ if it is $H$-free and contains exactly $ex(n, H)$ edges. The *Turán graph* $T_{n,k}$ is the balanced complete $k$-partite graph with $n$ vertices. An important motivation of investigating Turán numbers is that they are very useful for Ramsey theory. The original statements can be found in [12]. So far, the exact values of $ex(n, H)$ for most $H$ are unknown. Turán’s Theorem [38] gives that $ex(n, K_{k+1}) = |E(T_{n,k})|$ for $n \geq k + 1$ and $T_{n,k}$ is the unique extremal graph. The Turán number $ex(n, H)$ is known for several families of acyclic $H$, see $ex(n, P_k)$ [14, 18, 22], $ex(n, k \cdot P_2)$ [14], $ex(n, k \cdot P_3)$ [6, 42] and $ex(n, k \cdot K_{1,k})$ [23]. More extremal results on forests can be found in [24, 40]. Füredi and Gunderson [20] determined $ex(n, C_{2k+1})$ for all $k$ and $n$ and characterized all extremal graphs. To be precise, they proved that $ex(n, C_{2k+1}) = |E(T_{n,2})| = \left\lfloor \frac{n^2}{4} \right\rfloor$ for $n \geq 4k - 2$.

Let $A(G)$ be the *adjacency matrix* of a graph $G$. The *spectral radius* $\rho(G)$ is the largest eigenvalue of $A(G)$. Nikiforov [33] initiated the study of *Brualdi-Solheid-Turán type problem* which asks for the maximum spectral radius of an $H$-free graph with $n$ vertices. Define

$$ex_{sp}(n, H) = \max \{\rho(G) : |V(G)| = n \text{ and } H \not\subseteq G\}.$$ 

A graph $G$ is said to be *spectral extremal* for $H$ if $G$ is an $H$-free graph of order $n$ and $\rho(G) = ex_{sp}(n, H)$. We use $Ex_{sp}(n, H)$ to denote the set of the spectral extremal graphs for $H$. 
In the past decade, much attention has been paid to the above spectral extremal problem, see $\text{ex}_{sp}(n, K_{k+1})$ [28, 41], $\text{ex}_{sp}(n, K_{sp})$ [28, 32], $\text{ex}_{sp}(n, P_k)$ [33], $\text{ex}_{sp}(n, \bigcup_{i=1}^{k} P_i)$ [7], $\text{ex}_{sp}(n, \bigcup_{i=1}^{k} S_d)$ [8], $\text{ex}_{sp}(n, C_4)$ [28, 44], $\text{ex}_{sp}(n, C_6)$ [45], $\text{ex}_{sp}(n, C_{2k+1})$ [33], $\text{ex}_{sp}(n, F_k)$ [9] (where $F_k$ is a friendship graph), $\text{ex}_{sp}(n, W_{2k+1})$ [10W_{2k+1}] (where is an odd wheel), (outer) planar graphs [25, 37]. For more results on spectral extremal graph theory, see [1, 26, 27, 34].

A graph is color-critical if it contains an edge whose deletion reduces its chromatic number. Füredi and Gunderson [20] named the following result as chromatic critical edge theorem.

**Theorem 1.1** (Simonovits, [36]). Let $H$ be a color-critical graph with $\chi(H) = k + 1$. Then there exists a number $n_0(H)$ such that $T_{n,k}$ is the only extremal graph with respect to $\text{ex}(n, H)$ provided $n \geq n_0(H)$.

For plenty of Turán-type results, researchers managed to prove their spectral analogues, for example, spectral Erdős-Stone-Bollobás theorem [30] and spectral version of saturation problem [31]. By Nikiforov’s result (see Theorem 2, [31]), one can get the following spectral chromatic critical edge theorem.

**Theorem 1.2.** Let $H$ be a color-critical graph with $\chi(H) = k + 1$. Then there exists an $n_0(H)$ such that $n \geq n_0(H)$, then $T_{n,k}$ is the only extremal graph with respect to $\text{ex}_{sp}(n, H)$.

We claim that if $n \geq k$ and $T_{n,k}$ is an extremal graph with respect to $\text{ex}_{sp}(n, H)$, then it is also an extremal graph with respect to $\text{ex}(n, H)$. To see this, let $G^*$ be an extremal graph with respect to $\text{ex}(n, H)$ and set $s = \left\lfloor \frac{n}{k} \right\rfloor$. Note that $\rho(T_{n,k}) = \frac{1}{2}(n - 2s - 1 + \sqrt{(n - 2s - 1)^2 + 4s(s + 1)(k - 1)})$. By a well-known inequality $\rho(G) \geq \frac{2|E(G)|}{n}$ due to Collatz and Sinogowitz [11], we have

$$|E(G^*)| \leq \frac{n}{2}\rho(G^*) \leq \frac{n}{2}\rho(T_{n,k}) = \frac{1}{4}(n(n - 2s - 1)$$

$$+ n\sqrt{(n - 2s - 1)^2 + 4s(s + 1)(k - 1)}).$$

On the other hand,

$$|E(T_{n,k})| = \frac{1}{2}\sum d(v) = \frac{1}{2}(n(n - 2s - 1) + s(s + 1)k).$$

By a direct calculation, one can show that $|E(G^*)| \leq \frac{n}{2}\rho(T_{n,k}) < |E(T_{n,k})| + 1$ for $n \geq k$. It follows that $|E(G^*)| \leq |E(T_{n,k})|$ and $T_{n,k}$ is also an extremal graph with respect to $\text{ex}(n, H)$. Thus we have the following fact.

**Fact 1.1.** Theorem 1.2 implies Theorem 1.1 except for the uniqueness.

We are interested in the following problem which is very useful for investigating the existence of desired subgraphs.
Problem 1.1. Characterize color-critical graphs $H$ such that $n_0(H)$ is linear with $|V(H)|$, where $n_0(H)$ is defined in Theorem 1.2?

Up to now, it is known that $n_0(H)$ is linear with $|V(H)|$ for $H$ being a complete graph or an odd cycle, see [28] and [29]. More precisely, $n_0(K_{r+1}) = r$ and $n_0(C_{2r-1}) \leq 640r$. The goal of this article is to show that $n_0(H)$ is linear with $|V(H)|$ for $H$ being a book graph or a theta graph. To introduce our results, we need following definitions.

A book graph $B_r$ is a graph which consists of $r$ triangles sharing a common edge. Obviously, $B_r$ is color-critical with $\chi(B_r) = 3$. The following is our first result.

Theorem 1.3. $Ex_p(n, B_{r+1}) = \{T_n, 2\}$ for $n \geq \frac{13}{2}r$.

The size of a book is the number of triangles in it, i.e., the size of $B_r$ equals $r$. The size of the largest book in a graph $G$ is called the booksize of $G$. In 1962, Erdős [15] initiated the study of books in graphs. Since then, books have attracted considerable attention in extremal graph theory (see, e.g., [4, 16, 17, 21]). Erdős [15] proposed the following conjecture: the booksize of a graph $G$ on $n$ vertices with $\rho(G) > n/2$ is greater than $n/6$. This conjecture was proved by Edwards in an unpublished manuscript [13] and independently by Khadžiivanov and Nikiforov in [21]. Moreover, they constructed a graph with $n = 6r$ vertices and more than $|E(T_n, 2)|$ edges such that its booksize is $\frac{n}{6} + 1$. This implies that the conjectured booksize $\frac{n}{6}$ is best possible.

Since we focus on spectral extremal problems, it is natural to ask what is the minimum booksize of a graph $G$ on $n$ vertices with $\rho(G) > \rho(T_n, 2)$? We have the following result.

Corollary 1.1. For arbitrary positive integer $n$, the booksize of a graph $G$ on $n$ vertices with $\rho(G) \geq \rho(T_n, 2)$ is greater than $\frac{n}{6.5}$, unless $G \cong T_n, 2$.

Recall that $\frac{n}{2} - \frac{1}{2} \rho(T_n, k) < |E(T_n, k)| + 1$. If $|E(G)| > |E(T_n, 2)|$, then

$$\rho(G) \geq \frac{2|E(G)|}{n} \geq \frac{2(|E(T_n, 2)| + 1)}{n} > \rho(T_n, 2).$$

Thus, we propose a stronger problem than Erdős’ conjecture for further research.

Problem 1.2. For arbitrary positive integer $n$, if $G$ is a graph of order $n$ with $\rho(G) > \rho(T_n, 2)$, is it true that the booksize of $G$ is greater than $\frac{n}{6}$?

A generalized theta graph $\theta(l_1, \ldots, l_t)$ is the graph obtained by connecting two vertices with $t$ internally disjoint paths of lengths $l_1, \ldots, l_t$, where $l_t \leq \cdots \leq l_1$ and $l_2 \geq 2$. In particular, $\theta(l, l) \cong C_{2l}$. Therefore, the problem of determining $ex(n, \theta(l_1, \ldots, l_t))$ generalizes the problem of determining $ex(n, C_{2l})$. If $l_1 = \cdots = l_t$, then Faudree and Simonovits [19] showed that a $ex(n, \theta(l, \ldots, l)) = O_{t, 1}(n^{1+1/l})$. Recently, Bukh and Tait [5] improved the above result. With the restriction $t = 3$, we get a theta graph $\theta(p, q, r)$. For the lower bound of $ex(n, \theta(4, 4, 4))$, Verstraëte and Williford [39] proved a lower bound whose order of magnitude is $n^{5/4}$. In this article, we focus on a class of theta graphs $\theta(1, 2, r + 1)$. For convenience, we use $\theta_{r+1}$ to denote
\(\theta(1, 2, r + 1)\). Obviously, \(\chi(\theta_{r+1}) = 3\) and \(\theta_{r+1}\) is color-critical. Our second main result is the following theorem.

**Theorem 1.4.** \(Ex_{sp}(n, \theta_{r+1}) = \{T_{n,2}\}\) for \(n \geq 10r\) if \(r\) is odd and \(n \geq 7r\) if \(r\) is even.

The study of consecutive cycles is an important topic in extremal graph theory. From [3], we know that if \(G\) is a graph of order \(n\) with \(|E(G)| > |E(T_{n,2})|\), then \(G\) contains a cycle of length \(t\) for every \(t \leq \left\lfloor \frac{n + 1}{2} \right\rfloor\). In 2008, Nikiforov [29] studied the spectral condition for the existence of consecutive cycles. He proved that if \(G\) is a graph of order \(n\) with \(\rho(G) > \rho(T_{n,2})\), then \(G\) contains a cycle of length \(t\) for every \(t \leq \frac{n}{320}\). Moreover, Nikiforov proposed the following problem.

**Problem 1.3** (Nikiforov, [29]). Determine the maximum constant \(c\) such that for all positive \(\varepsilon < c\) and large enough \(n\), every graph \(G\) of order \(n\) with \(\rho(G) > \rho(T_{n,2})\) contains a cycle of length \(t\) for every \(t \leq (c - \varepsilon)n\).

Nikiforov’s result implies that \(c > \frac{1}{320}\). Moreover, Nikiforov [29] constructed a graph \(S_{n,k}\), which is the join of a complete graph of order \(k = \left\lfloor \frac{(3 - \sqrt{5})n}{4} \right\rfloor\) with an empty graph of order \(n - k\). One can see that \(\rho(S_{n,k}) > \frac{n}{2} \geq \rho(T_{n,2})\). However, \(S_{n,k}\) contains no cycle of length greater than \(2k\) (if \(S_{n,k}\) contains a cycle \(C\) of length at least \(2k + 1\), then \(C\) contains at least \(k + 1\) vertices out of the \(k\)-clique. Since any two of these \(k + 1\) vertices cannot be consecutive in \(C\), \(C\) contains at least \(k + 1\) vertices of the \(k\)-clique, a contradiction). This implies that \(c \leq \frac{3 - \sqrt{5}}{2} < \frac{1}{2.5}\). Recently, Ning and Peng [35] improved Nikiforov’s result by proving \(c > \frac{1}{160}\). By Theorem 1.4, for every even \(r\) and every graph \(G\) of order \(n \geq 7r\), if \(\rho(G) > \rho(T_{n,2})\), then \(G\) contains a \(\theta_{r+1}\). It follows that \(G\) contains \(\theta_3, \theta_5, ..., \theta_{r+1}\). Note that \(\theta_{r+1}\) contains \(C_3, C_{t+2}\) and \(C_{t+3}\) for each \(t \in \{2, 4, ..., r\}\). Thus, we get the following result.

**Corollary 1.2.** Let \(n\) be an arbitrary positive integer and \(G\) be a graph of order \(n\) with \(\rho(G) > \rho(T_{n,2})\). Then \(G\) contains a cycle of length \(t\) for every \(t \leq \frac{n}{7}\) unless \(G \cong T_{n,2}\).

Corollary 1.2 indicates that \(c > \frac{1}{7}\) without the assumption \(n\) being sufficiently large. Inspired by Nikiforov’s problem, one may ask whether we can remove the condition \(n\) being large enough in Problem 1.3, which is the following problem.

**Problem 1.4.** Determine the maximum constant \(c’\) such that for any positive integer \(n\), every graph \(G\) of order \(n\) with \(\rho(G) > \rho(T_{n,2})\) contains a copy of \(\theta_{r+1}\) for every \(r \leq c’n\).

Theorem 1.4 implies that \(c’ \geq \frac{1}{10}\). Moreover, combining Theorem 1.3, Theorem 1.4 and Fact 1.1, we essentially determine the Turán numbers of \(B_{r+1}\) and \(\theta_{r+1}\).

**Corollary 1.3.** Let \(r\) be a positive integer. Then we have the following statements.
(i) \( \text{ex}(n, B_{r+1}) = \left\lfloor \frac{n^2}{4} \right\rfloor \) for \( n \geq \frac{13}{2} r; \)

(ii) \( \text{ex}(n, \emptyset_{r+1}) = \left\lfloor \frac{n^2}{4} \right\rfloor \) for \( n \geq 10r \) if \( r \) is odd and \( n \geq 7r \) if \( r \) is even.

The rest of the article is organized as follows. In Section 2, we introduce some lemmas. In Sections 3 and 4, we give the proofs of Theorems 1.3 and 1.4 respectively.

2 | PRELIMINARIES

Let \( G \) be a simple graph. For a vertex \( u \in V(G) \) and a subgraph \( H \subseteq G \) (possibly \( u \notin V(H) \)), we use \( N_H(u) \) to denote the set of neighbors of \( u \) in \( V(H) \) and \( d_H(u) \) to denote \( |N_H(u)| \). If \( uv \in E(G) \), then we write \( u \sim v \). The subgraph of \( G \) induced by a vertex subset \( S \) is denoted by \( G[S] \).

Let \( G \) be a \( P_{r+2} \)-free graph of order \( n \). Then \( |E(G)| \leq \frac{r}{2} n \), and equality holds if and only if \( G \) is a union of disjoint copies of \( K_{r+1} \).

Lemma 2.2 (Zhai et al. [43]). Let \( G = \langle X, Y \rangle \) be a bipartite graph, where \( |X| \geq r \) and \( |Y| \geq r - 1 \geq 1 \). If \( G \) does not contain a path of length \( 2r \) with both endpoints in \( X \), then

\[
|E(G)| \leq (r - 1)|X| + r|Y| - r(r - 1).
\]

Equality holds if and only if \( G \cong K_{|X|,|Y|} \), where \( |X| = r \) or \( |Y| = r - 1 \).

Assume that \( H \in \{B_{r+1}, \emptyset_{r+1}\} \) and \( G \) is an extremal graph with respect to \( \text{ex}_p(n, H) \). Let \( X = (x_1, ..., x_n)^T \) be the Perron vector of \( G \) and \( u^* \in V(G) \) with \( x_{u^*} = \max \{x_i : i = 1, ..., n\} \). Let \( A = N_G(u^*), B = V(G) \setminus (A \cup \{u^*\}) \) and

\[
\gamma(u^*) = |A| + 2e(A) + e(A, B).
\]

Then we have the following lemma.

Lemma 2.3. \( |A| \geq \left\lfloor \frac{n}{2} \right\rfloor \) and \( \gamma(u^*) \geq \left\lfloor \frac{n^2}{4} \right\rfloor \). Moreover, if \( e(A) = 0 \) then \( G \cong T_{n,2} \).

Proof. Since \( T_{n,2} \) is \( H \)-free and \( G \) attains the maximum spectral radius, we have

\[
\rho(G) \geq \rho(T_{n,2}) = \sqrt{\frac{n}{2} \left\lfloor \frac{n}{2} \right\rfloor} = \sqrt{\frac{n^2}{4}} > \frac{n - 1}{2}.
\]
Note that $\rho(G)x_{u^*} = \sum_{u \in A} x_u \leq |A|x_{u^*}$. Then $|A| \geq \rho(G) > \frac{n-1}{2}$, which implies that

$$|A| \geq \left\lfloor \frac{n}{2} \right\rfloor. \tag{1}$$

By using eigen-equation again, we have

$$\rho^2(G)x_{u^*} = \rho(G) \sum_{u \in A} x_u + \sum_{u \in A} \sum_{v \in B} x_v$$

$$= |A|x_{u^*} + \sum_{u \in A} d_A(u)x_u + \sum_{v \in B} d_A(v)x_v$$

$$\leq |A|x_{u^*} + 2e(A)x_{u^*} + e(A,B)x_{u^*}$$

$$= \gamma(u^*)x_{u^*}.$$

It follows that

$$\gamma(u^*) \geq \rho^2(G) \geq \left\lfloor \frac{n^2}{4} \right\rfloor. \tag{2}$$

Now if $e(A) = 0$, then

$$\gamma(u^*) = |A| + e(A,B) \leq |A| + |A||B| = |A|(|B| + 1) \leq \left\lfloor \frac{n^2}{4} \right\rfloor,$$

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$$\gamma(u^*) = |A| + e(A,B) \leq |A| + |A||B| = |A|(|B| + 1) \leq \left\lfloor \frac{n^2}{4} \right\rfloor.$$

since $|A| + (|B| + 1) = n$. Together with (2), we have $\gamma(u^*) = \left\lfloor \frac{n^2}{4} \right\rfloor$. Consequently, all the inequalities in (3) must be equalities. Therefore, $e(A,B) = |A||B|$. Recall inequality (1). We have $|A| = \left\lfloor \frac{n}{2} \right\rfloor$ and $|B| + 1 = \left\lfloor \frac{n}{2} \right\rfloor$. This implies that $G$ contains $T_{n,2}$ as a spanning subgraph. Furthermore, $e(B) = 0$ as we assume $G$ is $B_{r+1}$-free or $\partial_{r+1}$-free. Therefore, $G \cong T_{n,2}$ as desired. □

### 3 SPECTRAL EXTREMA FOR BOOK GRAPHS

In this section, we give the proof of Theorem 1.3. Suppose that $G$ is an extremal graph with respect to $\text{ex}_{\text{sp}}(n, B_{r+1})$. Let $X = (x_1, \ldots, x_n)^t$ be the Perron vector of $G$ and $u^*$ be a vertex with $x_{u^*} = \max\{x_i : i = 1, \ldots, n\}$. Since $T_{n,2}$ contains no $B_{r+1}$, for $n \geq \frac{13}{2}r$ we have

$$\rho(G) \geq \rho(T_{n,2}) = \sqrt{\frac{n^2}{4}} > 3r. \tag{4}$$

**Proof of Theorem 1.3.** Let $A = N_G(u^*)$ and $B = V(G) \setminus (A \cup \{u^*\})$. If $e(A) = 0$, then $G \cong T_{n,2}$ by Lemma 2.3, and the theorem follows. In the following, we assume that $e(A) \neq 0$. □
Claim 3.1. \( d_A(u) \leq r \) for each \( u \in A \), and hence \( e(A) \leq \frac{r}{2}|A| \).

Proof. Note that \( G[A] \) is \( K_{1,r+1} \)-free. Otherwise, a \( K_{1,r+1} \) together with the vertex \( u^* \) form a book \( B_{r+1} \). Therefore, the maximum degree of \( G[A] \) is at most \( r \) and the claim follows.

Claim 3.2. \( d_B(u_1) + d_B(u_2) \leq |B| + r - 1 \) for any \( u_1u_2 \in E(A) \).

Proof. If \( d_B(u_1) + d_B(u_2) \geq |B| + r \) for some \( u_1u_2 \in E(A) \), then we get that

\[
|B| + r \leq d_B(u_1) + d_B(u_2) = |N_B(u_1) \cup N_B(u_2)| + |N_B(u_1) \cap N_B(u_2)| \\
\leq |B| + |N_B(u_1) \cap N_B(u_2)|.
\]

Therefore, \( |N_B(u_1) \cap N_B(u_2)| \geq r \). The edge \( u_1u_2 \) together with \( u^* \) and an \( r \)-subset of \( N_B(u_1) \cap N_B(u_2) \) form a book \( B_{r+1} \), a contradiction.

Claim 3.3. \( x_{u_1} + x_{u_2} \leq \frac{|B| + r + 1}{\rho(G) - r} x_{u^*} \) for any \( u_1u_2 \in E(A) \).

Proof. Suppose that \( x_u + x_v = \max_{uv \in E(A)}(x_u + x_v) \). Now we only need to show that \( x_u + x_v \leq \frac{|B| + r + 1}{\rho(G) - r} x_{u^*} \). By Claim 3.1, we have

\[
\rho(G)x_u = x_{u^*} + \sum_{w \in N_A(u)} x_w + \sum_{w \in N_B(u)} x_w \leq x_{u^*} + rx_v + d_B(u)x_{u^*}
\]

and

\[
\rho(G)x_v = x_{u^*} + \sum_{w \in N_A(v)} x_w + \sum_{w \in N_B(v)} x_w \leq x_{u^*} + rx_u + d_B(v)x_{u^*}.
\]

Combining the above two inequalities and using Claim 3.2, we obtain

\[
(\rho(G) - r)(x_u + x_v) \leq 2x_{u^*} + (d_B(u) + d_B(v))x_{u^*} \leq (|B| + r + 1)x_{u^*}.
\]

Therefore, the claim follows.

We use \( \varepsilon(A, B) \) to denote the number of nonadjacent vertex-pairs between \( A \) and \( B \), that is, \( \varepsilon(A, B) = |A||B| - e(A, B) \). For the sake of simplicity, we use \( d_B(u) \) instead of \( \varepsilon([u], B) \), where \( u \in A \). Now we have the following claim.

Claim 3.4. \( \varepsilon(A, B) \geq \frac{1}{r} e(A)(|B| - r + 1) \).

Proof. By Claim 3.2, \( d_B(u_1) + d_B(u_2) \leq |B| + r - 1 \) for \( u_1u_2 \in E(A) \). Then
\[ \overline{d_B}(u_1) + \overline{d_B}(u_2) \geq 2|B| - (|B| + r - 1) = |B| - r + 1. \]

For each vertex \( u \in A \), since \( d_A(u) \leq r \), we can see that \( \overline{d_B}(u) \) appears at most \( r \) times in \( \sum_{u_1, u_2 \in E(A)} (\overline{d_B}(u_1) + \overline{d_B}(u_2)) \). Thus

\[
\bar{e}(A, B) = \sum_{u \in A} \overline{d_B}(u) \geq \frac{1}{r} \sum_{u_1, u_2 \in E(A)} (\overline{d_B}(u_1) + \overline{d_B}(u_2)) \geq \frac{1}{r} e(A)(|B| - r + 1),
\]

as required. \( \square \)

Now we give the proof of Theorem 1.3. We know that

\[
\rho^2(G)x^* = |A|x^* + \sum_{u \in A} d_A(u)x_u + \sum_{v \in B} d_A(v)x_v
= |A|x^* + \sum_{u_1, u_2 \in E(A)} (x_u + x_{u_2}) + \sum_{v \in B} d_A(v)x_v
\leq |A|x^* + e(A)\frac{|B| + r + 1}{\rho(G) - r}x_u + e(A, B)x^*_u,
\]

where the last inequality follows from Claim 3.3. Note that \( e(A, B) = |A|||B| - \bar{e}(A, B) \). Together with Claim 3.4, it follows that

\[
\rho^2(G) \leq |A| + e(A)\frac{|B| + r + 1}{\rho(G) - r} + |A|||B| - \frac{1}{r} e(A)(|B| - r + 1)
= |A|(|B| + 1) + \left(\frac{|B| + r + 1}{\rho(G) - r} - \frac{|B| - r + 1}{r}\right)e(A). \tag{5}
\]

If \( \rho(G) > r\left(1 + \frac{|B| + r + 1}{|B| - r + 1}\right) \), then

\[
\frac{|B| + r + 1}{\rho(G) - r} < \frac{|B| - r + 1}{r}
\]

and so \( \rho^2(G) < |A|(|B| + 1) \leq \left\lceil \frac{n^2}{4} \right\rceil \), which contradicts the inequality (4).

If \( \rho(G) = r\left(1 + \frac{|B| + r + 1}{|B| - r + 1}\right) \), then \( \rho^2(G) \leq |A|(|B| + 1) \leq \left\lceil \frac{n^2}{4} \right\rceil \). Recall inequality (4). We have

\[
\rho^2(G) = \left\lceil \frac{n^2}{4} \right\rceil.
\]

Lemma 2.3 implies that \( |A| = \lceil \frac{n}{2} \rceil \) and \( |B| + 1 = \lceil \frac{n}{2} \rceil \). Furthermore, by \( \rho(G) = r\left(1 + \frac{|B| + r + 1}{|B| - r + 1}\right) \), we obtain

\[
\rho^2(G) = \left(\frac{2r}{n/2 - r}\right)^2 \left(\frac{n}{2}\right)^2 \leq \left(\frac{2r}{n/2 - r}\right)^2 \left(\frac{n}{2}\right)^2 \leq \left(\frac{2r}{n/2 - r}\right)^2 \rho^2(G) \leq \rho^2(G), \tag{6}
\]

as \( n \geq \frac{13}{2}r \). Hence, \( \left(\frac{n}{2}\right)^2 = \left\lceil \frac{n^2}{4} \right\rceil \), which implies that \( n \) is even. Consequently,

\[
\frac{2r}{n/2 - r} = \frac{2r}{n/2 - r} \leq \frac{8}{9}.
\]
Therefore, inequality (6) is strict, a contradiction.

If \( \rho(G) < r \left( 1 + \frac{|B| + r + 1}{|B| - r + 1} \right) \), then \( \frac{|B| + r + 1}{\rho(G) - r} > \frac{|B| - r + 1}{r} \). Recall that \( \rho(G) > 3r \) by (4) and \( e(A) \leq \frac{1}{2} |A| \) by Claim 3.1. Then, by (5), we have

\[
\rho^2(G) \leq |A|(|B| + 1) + \left( \frac{|B| + r + 1}{\rho(G) - r} - \frac{|B| - r + 1}{r} \right) \frac{r}{2} |A|
\]

\[
< |A|(|B| + 1) + \left( \frac{|B| + r + 1}{2r} - \frac{|B| - r + 1}{r} \right) \frac{r}{2} |A|
\]

\[
= \frac{3}{4} |A|(|B| + 1 + r)
\]

\[
\leq \frac{3}{4} \left( \frac{(n + r)^2}{4} \right),
\]

as \( |A| + |B| + 1 = n \). Since \( n \geq \frac{13}{2} r \), we have \( \rho^2(G) < \frac{3}{4} \left( \frac{(n + r)^2}{4} \right) \leq \left( \frac{n^2}{4} \right) \), which contradicts (4).

This completes the proof of Theorem 1.3.

4 | SPECTRAL EXTREMA FOR THETA GRAPHS

In this section, we give the proof of Theorem 1.4. Suppose that \( G \) is an extremal graph with respect to \( \text{ex}_{\theta}(n, \theta_{r+1}) \). Let \( X = (x_1, ..., x_n)^t \) be the Perron vector of \( G \) and \( u^* \in V(G) \) with \( x_{u^*} = \max\{x_i : i = 1, ..., n\} \). Let \( A = N_G(u^*), B = V(G) \setminus (A \cup \{u^*\}) \) and

\[
\gamma(u^*) = |A| + 2e(A) + e(A, B).
\]

**Proof of Theorem 1.4.** There are two cases. \( \square \)

**Case 1. \( r \) is odd.**

Notice that \( n \geq 10r \). If \( r = 1 \), then \( \theta_{r+1} \) is \( B_{r+1} \). By Theorem 1.3, \( \text{ex}_{\theta}(n, \theta_2) = \{T_{n,2}\} \) for \( n \geq 7 \). In the following, we may assume that \( r \geq 3 \) and prove a number of claims.

**Claim 4.1.** \( |B| \geq 2r \). Moreover, \( |E(H)| \leq \frac{r}{2} |V(H)| \) for each component \( H \) of \( G[A] \).

**Proof.** Since \( G \) is \( \theta_{r+1} \)-free, \( G[A] \) is \( P_{r+2} \)-free. By Lemma 2.1, \( |E(H)| \leq \frac{r}{2} |V(H)| \) for each component \( H \) of \( G[A] \), as claimed. It follows that \( e(A) \leq \frac{r}{2} |A| \) and then

\[
\gamma(u^*) = |A| + 2e(A) + e(A, B) \leq (1 + r + |B|)|A| = (1 + r + |B|)(n - 1 - |B|).
\]

Suppose that \( |B| \leq 2r - 1 \). Note that \( n \geq 10r \). Write \( b = |B| \) and \( f(b) = (1 + r + b)(n - 1 - b) \). Then \( f(b) \) is concave down and \( f(2r - 1) \) achieves
the maximum value as \( |B| \leq 2r - 1 \). Therefore, \( \gamma(u^*) \leq 3r \cdot (n - 2r) < \left| \frac{n^2}{4} \right| \) which contradicts (2). Therefore, \( |B| \geq 2r \).

An \textit{A-path} of \( G \) is a path consisting of edges in \( E(A, B) \) with both endpoints in \( A \). The length of a path \( P \) is denoted by \( \ell(P) \).

\textbf{Claim 4.2.} There exists an \textit{A-path} of length \( 3r - 1 \) in \( G \).

\textbf{Proof.} By Lemma 2.3, we know that \( |A| \geq \left| \frac{n^2}{2} \right| \). Suppose that there exists no \textit{A-path} of length \( 3r - 1 \). Then by Lemma 2.2,

\[
e(A, B) \leq \frac{3r - 3}{2} |A| + \frac{3r - 1}{2} |B| - \frac{(3r - 1)(3r - 3)}{4}.
\]

Recall that \( |A| + 2e(A) \leq (1 + r) |A| \) and \( |B| \geq 2r \). We obtain

\[
\gamma(u^*) \leq \frac{5r - 1}{2} |A| + \frac{3r - 1}{2} |B| - \frac{(3r - 1)(3r - 3)}{4} = \frac{5r - 1}{2} (|A| + |B| + 1) - r|B| - \frac{9r^2 - 2r + 1}{4} \leq \frac{(5r - 1)n}{2} - 2r^2 - \frac{9}{4} r^2 + \frac{r}{2}.
\]

Since \( n \geq 10r \), we have \( \gamma(u^*) \leq \frac{5n^2}{2} - \frac{17}{4} r^2 < \left| \frac{n^2}{4} \right| \), a contradiction. The claim follows.

The set of internal vertices of a path \( P \) is denoted by \( V_{\text{in}}(P) \). For two vertices \( u, v \in V(P) \), the distance between \( u \) and \( v \) in \( P \) is denoted by \( d_P(u, v) \). A vertex of a \( \theta_{r+1} \) is said to be a \textit{head}, if it is of degree two and it belongs to a triangle of \( \theta_{r+1} \). The following two claims give some structural properties of \( G \).

\textbf{Claim 4.3.} Let \( P \) be an \textit{A-path} in \( G \) and \( u \in A \setminus V(P) \). The following statements hold.

\begin{itemize}
  \item [(i)] If \( \ell(P) \geq r + 3 \) and \( u \sim u' \) for some \( u' \in A \setminus V_{\text{in}}(P) \), then \( \varepsilon([u, u'], B) \geq 3 \).
  \item [(ii)] If \( \ell(P) \geq 2r + 2 \) and \( u \sim u' \sim u'' \) for some \( u', u'' \in A \setminus V(P) \), then \( \varepsilon([u], B) \geq r + 1 \).
\end{itemize}

\textbf{Proof.} (i) Let \( P = u_i v_i u_2 v_2 \cdots u_{i+1} v_{i+1} u_{i+1} \) be an \textit{A-path} of length \( r + 3 \), where \( u_i \in A \) and \( v_i \in B \). Pick two pairs of vertices \( \{v_1, v_{i+1}\} \) and \( \{v_2, v_{i+2}\} \). Clearly, \( d_P(v_i, v_{i+1}) = r - 1 \) for \( i \in \{1, 2\} \). If \( e([u, u'], \{v_i, v_{i+1}\}) \geq 3 \), then there are two independent edges, say \( uv_i, u'v_{i+1}, u''v_{i+2} \), in \( E([u, u'], \{v_i, v_{i+1}\}) \), thus \( u_i u_{i+1} v_{i+1} \cdots u_{i+1} v_{i+2} u' \) is an \textit{A-path} of length \( r + 1 \). Combining the triangle \( u'u'' \), we get a \( \theta_{r+1} \) with its head \( u^* \), a contradiction. Therefore, \( e([u, u'], \{v_i, v_{i+1}\}) \leq 2 \) and so \( \varepsilon([u, u'], \{v_i, v_{i+1}\}) \geq 2 \) for \( i \in \{1, 2\} \).

If \( r \geq 5 \), then \( \{v_1, v_{i+1}\} \cap \{v_2, v_{i+2}\} = \emptyset \), and thus \( \varepsilon([u, u'], B) \geq 4 \), as required. Now assume \( r = 3 \). It suffices to show \( \varepsilon([u, u'], \{v_1, v_2, v_3\}) \geq 3 \). Note that
\[ \bar{\varepsilon}([u, u'], [v_1, v_2]) \geq 2 \] and \[ \bar{\varepsilon}([u, u'], [v_2, v_3]) \geq 2. \] If \( \bar{\varepsilon}([u, u'], [v_1, v_2]) \leq 2 \), then \( \bar{\varepsilon}([u, u'], [v_2]) = 2 \) and \( \bar{\varepsilon}([u, u'], [v_1, v_3]) = 0 \). This implies that \( e([u, u'], [v_1, v_3]) = 4 \). Since \( u' \in A \setminus V_{in}(P) \), either \( u' \notin \{u_1, u_2, u_3\} \) or \( u' \notin \{u_2, u_3, u_4\} \). Without loss of generality, assume that \( u' \notin \{u_1, u_2, u_3\} \), then \( u'u^*v_1u \) is a path of length 4. Together with the triangle \( u'v_3, u \), we get a \( \vartriangle_4 \) with its head \( v_3 \), a contradiction. Therefore, \( \bar{\varepsilon}([u, u'], B) \geq 3 \).

(ii) Let \( P = u_1v_1u_2v_2 \cdots u_{r+1}v_{r+1}u_{r+2} \) be an \( A \)-path of length \( 2r + 2 \), where \( u_i \in A \) and \( v_j \in B \). If \( u \) is adjacent to some \( v_j \), then select a vertex \( v_j \in V(P) \) such that \( d_P(v_i, v_j) = r - 3 \). Let \( P_{v_i \rightarrow v_j} \) be the subpath of \( P \) with endpoints \( v_i \) and \( v_j \). Then \( u'u^*v_{j+1}u^* \) is a path of length \( r + 1 \). Combining the triangle \( u'u^*u^* \), we get a \( \vartriangle_{r+1} \) with its head \( u^* \), a contradiction. Therefore, \( N_G(u) \cap \{v_1, ..., v_{r+1}\} = \emptyset \) and the result follows.

\[ \text{Claim 4.4.} \] Let \( P \) be an \( A \)-path in \( G \) starting from some vertex \( u \in A \). If \( \ell(P) \geq r - 1 \) and \( u \sim u' \sim u'' \) for some \( u', u'' \in A \), then either \( u' \in V(P) \) or \( u'' \in V(P) \).

\[ \text{Proof.} \] Let \( P_{u \rightarrow u_m} \) be an \( A \)-path of length \( r - 1 \) with endpoints \( u \) and \( u'' \). If \( u', u'' \notin V(P_{u \rightarrow u_m}) \), then \( u'u''v_{j+1}u^* \) is a path of length \( r - 1 \). Combining the triangle \( u'u''u^* \), we get a \( \vartriangle_{r+1} \) with its head \( u^* \), a contradiction. Hence, the claim holds.

A component \( H \) of \( G[A] \) is said to be \textit{good}, if \( 2|E(H)| + e(V(H), B) < |V(H)||B| \). The following claim gives a characterization of good components in \( G[A] \).

\[ \text{Claim 4.5.} \] Let \( H \) be a component of \( G[A] \) and \( P \) be an \( A \)-path of length \( 3r - 1 \) in \( G \). If one of the following conditions holds, then \( H \) is a good component.

(i) \( |V(H)| = 2 \) and \( V(H) \not\subseteq V(P) \);
(ii) \( |V(H)| \geq 3 \) and \( V^*(H) \not\subseteq V(P) \), where \( V^*(H) = \{v \in V(H) : d_H(v) \geq 2\} \).

\[ \text{Proof.} \] (i) Let \( V(H) = \{u, u'\} \). Since \( V(H) \not\subseteq V(P) \), without loss of generality, we may assume that \( u \notin V(P) \). If \( r \geq 5 \), then \( \ell(P) = 3r - 1 \geq 2r + 4 \). Note that \( r \) is odd and each \( A \)-path is of length even. No matter \( u' \in V(P) \) or not, there exists an \( A \)-path \( P' \) of length \( r + 3 \) with \( P' \subseteq P \) and \( u' \notin V_{in}(P') \). By Claim 4.3, we have \( \bar{\varepsilon}([u, u'], B) \geq 3 \). It follows that

\[ 2|E(H)| + e(V(H), B) \leq 2 + |V(H)||B| - 3 < |V(H)||B|, \]

and hence \( H \) is a good component.

If \( r = 3 \), then \( \ell(P) = 8 \). Let \( P = u_1v_1 \cdots u_4v_4u_5 \), where \( u_i \in A \) and \( v_i \in B \). If \( u' \neq u_3 \) (where \( u_3 \) is the central vertex of \( P \)), then there exists an \( A \)-path \( P'' \) of length \( r + 3 \) with \( P'' \subseteq P \) and \( u' \notin V_{in}(P'') \). Similar to above, \( H \) is good. If \( u' = u_3 \), then \( u \sim u_3 \), and \( \bar{\varepsilon}([u, u_3], [v_1, v_3]) \geq 2 \) (otherwise, there are two independent edges, say \( uv_1, u_3v_2 \), in \( E([u, u_3], [v_1, v_2]) \); thus, \( uv_1u_3v_2u_3 \) is an \( A \)-path of length 4 and then there exists a \( \vartriangle_4 \) with its head \( u^* \)). By symmetry, we also have \( \bar{\varepsilon}([u, u_3], [v_3, u_4]) \geq 2 \). It follows that

\[ 2|E(H)| + e(V(H), B) \leq 2 + |V(H)||B| - 4 < |V(H)||B|, \]
as desired.
(ii) We first show \( V(H) \cap V(P) = \emptyset \). Suppose to the contrary, then \( H \) contains a path with one endpoint in \( V(H) \cap V(P) \) and the other in \( V^*(H) \setminus V(P) \), as \( V*(H) \notin V(P) \); furthermore, we can find an edge \( uu' \) with \( u \in V(H) \cap V(P) \) and \( u' \in V^*(H) \setminus V(P) \). By the definition of \( V^*(H) \), we have \( d_H(u') \geq 2 \). Let \( u'' \in N_{H(u')} \setminus \{u\} \). Note that \( \ell(P) = 3r - 1 \). Then, there exists an \( A \)-path \( P' \) of length \( r - 1 \) such that \( P' \subseteq P \) and \( u \) is an endpoint of \( P' \). Since \( u' \notin V(P) \), we have \( u'' \notin V(P') \), and so \( u'' \in V(P') \subseteq V(P) \) by Claim 4.4. Recall that \( \ell(P) = 3r - 1 \) and \( u, u'' \in V(H) \cap V(P) \). Then, \( u \) and \( u'' \) separate \( P \) into at most three subpaths. Since all of them are \( A \)-paths and have even lengths, one of them has length at least \( r + 1 \). Consequently, we can find an \( A \)-path \( P'' \) of length \( r - 1 \) with one endpoint in \( \{u, u''\} \) and the other in \( V(P) \setminus \{u, u''\} \). By Claim 4.4 we have \( u'' \in V(P) \), a contradiction. Therefore, \( V(H) \cap V(P) = \emptyset \).

Note that \( Pr_r = 3 - \frac{1}{2} \geq 2r + 2 \) and \( |V(H)| \geq 3 \). If \( H \cong K_{1,s} \) with \( s \geq 2 \), then for each pendant vertex \( u \in V(H) \) there exist \( u', u'' \in V(H) \) with \( u \sim u' \sim u'' \). Since \( u, u', u'' \notin V(P) \), by Claim 4.3 we have \( \sigma(\{u\}, B) \geq r + 1 \). Hence,

\[
2|E(H)| + e(V(H), B) \leq 2s + |V(H)||B| - (r + 1)s < |V(H)||B|,
\]

as required. If \( H \) is not a star, then for each vertex \( u \in V(H) \) there exist \( u', u'' \in V(H) \) with \( u \sim u' \sim u'' \). Again by Claim 4.3, we have \( \sigma(\{u\}, B) \geq r + 1 \). Recall Claim 4.1. We have

\[
2|E(H)| + e(V(H), B) \leq r|V(H)| + |V(H)||B| - (r + 1)|V(H)| < |V(H)||B|,
\]

as required. \( \square \)

For a subset \( A' \subseteq A \), an \( A' \)-path of \( G \) is a path consisting of edges in \( E(A', B) \) with both endpoints in \( A' \). Claim 4.2 states that there exists an \( A \)-path of length \( 3r - 1 \) in \( G \). The following claim gives a stronger characterization.

**Claim 4.6.** Let \( P \) be an \( A \)-path of length \( 3r - 1 \) in \( G \) and \( A' = A \setminus V(P) \). Then there exists an \( A' \)-path \( P' \) of length \( 3r - 1 \).

**Proof.** We know that \( |B| \geq 2r, |A| \geq \left\lceil \frac{n}{2} \right\rceil \geq 5r \) and \( |A\setminus A'| = |V(P) \cap A| = \frac{3r + 1}{2} \). Then \( |B| > \frac{3r - 3}{2} \) and \( |A'| = |A| - \frac{3r + 1}{2} > \frac{3r - 1}{2} \). If \( G \) does not contain an \( A' \)-path of length \( 3r - 1 \), then by Lemma 2.2,

\[
e(A', B) \leq \frac{3r - 3}{2} |A'| + \frac{3r - 1}{2} |B| - \frac{(3r - 1)(3r - 3)}{4}.
\]

Moreover,

\[
e(A \setminus A', B) \leq |A \setminus A'||B| = \frac{3r + 1}{2} |B|.
\]

By Claim 4.1, we have \( 2e(A) \leq r|A| \). Combining these inequalities, we can see that
\[
\gamma(u^*) = |A| + 2e(A) + e(A, B) \\
\leq (1 + r)|A| + \frac{3r - 3}{2}|A'| + 3r|B| - \frac{(3r - 1)(3r - 3)}{4} \\
= (1 + r)|A| + \frac{3r - 3}{2}(|A| - \frac{3r + 1}{2}) + 3r|B| - \frac{(3r - 1)(3r - 3)}{4} \\
= \frac{5r - 1}{2}|A| + 3r|B| - \frac{3r(3r - 3)}{2} \\
= 3r(|A| + |B| + 1) - \frac{r + 1}{2}|A| - \frac{3r(3r - 1)}{2} \\
< 3rn - \frac{rn}{4} - 3r^2,
\]

where the last inequality follows from \(|A| \geq \frac{n}{2}\) and \(3r - 1 > 2r\). Therefore, \(\gamma(u^*) < \frac{11}{4}rn - 3r^2 < \left[\frac{n^2}{4}\right]\) for \(n \geq 10r\), a contradiction. The claim holds.

To complete the proof, we only need to show \(e(A) = 0\) by Lemma 2.3. Suppose to the contrary that \(e(A) \neq 0\). Then \(G[A]\) contains nontrivial components. Let \(A_0 = \{u \in A : d_{A}(u) = 0\}\) and \(A_1 = A \setminus A_0\). If all nontrivial components of \(G[A]\) are good components, then \(2e(A_1) + e(A_1, B) < |A_1||B|\). It follows that

\[
\gamma(u^*) = |A| + 2e(A_1) + e(A, B) < |A| + (|A_1| + |A_0||B|) = |A|(|B| + 1) \leq \left[\frac{n^2}{4}\right],
\]

which contradicts Lemma 2.3. Thus, \(G[A]\) contains a nontrivial component \(H\) which is not good. By Claim 4.6, there exist two \(A\)-paths, \(P\) and \(P'\), such that \(\ell(P) = \ell(P') = 3r - 1\) and \(V(P) \cap V(P') \cap A = \emptyset\). Furthermore, by Claim 4.5, if \(|V(H)| = 2\) then \(V(H) \subseteq V(P)\) and \(V(H) \subseteq V(P')\); and if \(|V(H)| \geq 3\) then \(V^*(H) \subseteq V(P)\) and \(V^*(H) \subseteq V(P')\). In each case, we get a contradiction to the fact that \(V(P) \cap V(P') \cap A = \emptyset\). This completes the proof of Case 1.

**Case 2.** \(r\) is even.

Now, \(r \geq 2\) and \(n \geq 7r\). We first prove four claims.

**Claim 4.7.** If \(P\) is an \(A\)-path of length \(r\) starting from \(u_i\), then \(N_A(u_i) \subseteq V(P)\).

**Proof.** Let \(P = u_1v_1 \cdots u_kv_ku_{k+1}\), where \(u_i \in A\) and \(v_i \in B\). Then \(P^* = P + u_{k+1}u^*\) is a path of length \(r + 1\). If there exists a vertex \(u \in N_A(u_i) \setminus V(P)\), then \(P^*\) and the triangle \(u_iuu^*\) consist a \(\theta_{r+1}\) with its head \(u\), a contradiction. Therefore, \(N_A(u_i) \subseteq V(P)\).

**Claim 4.8.** \(|B| \geq r\). Moreover, \(G\) contains an \(A\)-path of length \(2r\).

**Proof.** We first show \(|B| \geq r\). Suppose to the contrary that \(|B| \leq r - 1\). Since \(G\) is \(\theta_{r+1}\)-free, \(G[A]\) is \(P_{r+2}\)-free. Then by Lemma 2.1, \(2e(A) \leq r|A|\). It follows that

\[
\gamma(u^*) = |A| + 2e(A) + e(A, B) \leq (1 + r)|A| + |A||B| \\
= (1 + r + |B|(n - 1 - |B|)).
\]
Since \( n \geq 7r \) and \( |B| \leq r - 1 \), we can see that \( f(|B|) = (1 + r + |B|)(n - 1 - |B|) \) is concave down and \( f(r - 1) \) achieves the maximum value as \( |B| \leq r - 1 \). Hence, \( \gamma(u^*) \leq 2r(n - r) < \left\lfloor \frac{n^2}{4} \right\rfloor \), a contradiction. Therefore, we get \( e(A, B) \leq (r - 1)|A| + r|B| - r(r - 1) \).

Furthermore,

\[
\gamma(u^*) \leq (1 + r)|A| + e(A, B) \leq 2r|A| + r|B| - r(r - 1) = 2r(|A| + |B| + 1) - r(|B| + r + 1).
\]

Since \(|A| + |B| + 1 = n\) and \(|B| \geq r\), we have \( \gamma(u^*) < 2rn - 2r^2 \leq \left\lfloor \frac{n^2}{4} \right\rfloor \), a contradiction. Therefore, \( G \) contains an \( A \)-path of length \( 2r \). 

Claim 4.9. Let \( P \) be an \( A \)-path of length \( 2r \) in \( G \) and \( A' = A \setminus V(P) \). Then there also exists an \( A' \)-path \( P' \) of length \( 2r \).

Proof. Let \( P = u_1v_1 \cdots u_rv_ru_{r+1} \), where \( u_i \in A \) and \( v_i \in B \). Then \(|A \setminus A'| = |V(P) \cap A| = r + 1\). As \(|A| \geq \left\lfloor \frac{n}{2} \right\rfloor \) and \( n \geq 7r \), we get \(|A'| = |A| - (r + 1) > r\). If \( G \) does not contain an \( A' \)-path of length \( 2r \), then by Lemma 2.2,

\[
e(A', B) \leq (r - 1)|A'| + r|B| - r(r - 1).
\]

We first show \( e(A', A \setminus A') = 0 \). Note that \( A \setminus A' = \{u_1, \ldots, u_{r+1}\} \). Suppose that \( uu_i \in E(A) \) for some \( u \in A' \) and \( u_i \in A \setminus A' \). Without loss of generality, we may assume that \( i \leq \frac{r}{2} + 1 \). Then \( u_1v_1 \cdots u_{i+1}v_{i+1}u_2^*u_2 \) is a path of length \( r + 1 \). Together with the triangle \( u_1u_2u_2^* \), we get a \( A' \) with its head \( u_2 \), a contradiction. Hence, \( e(A', A \setminus A') = 0 \).

Next, we show \( e(A \setminus A') \leq \frac{r^2}{8} \). Let

\[
V_{11} = \left\{ u_i : 1 \leq i \leq \frac{r}{2} + 1 \right\}, \quad V_{12} = \left\{ u_i : \frac{r}{2} + 1 \leq i \leq r + 1 \right\}.
\]

If \( u_iu_j \in E(G) \) for some \( i, j \) with \( 1 \leq i < j \leq \frac{r}{2} + 1 \), then \( P' = u_1v_1 \cdots u_{j-1}v_{j-1}u_j^*u_j \) is an \( A \)-path of length \( r \) starting from \( u_j \). By Claim 4.7, \( N_A(u_j) \subseteq V(P') \) and then \( u_j \in V(P') \), a contradiction. Therefore, we get \( e(V_{11}) = 0 \). Using the same argument, we can also show \( e(V_{12}) = 0 \). Hence, \( G[A \setminus A'] \) is a bipartite graph with an isolated vertex \( u_{\frac{r}{2} + 1} \). Furthermore, for each \( i \leq \frac{r}{2} \), \( P'' = u_1v_1 \cdots u_{i+1}v_{i+1}u_i^*u_i \) is an \( A \)-path of length \( r \) starting from \( u_i \). Again by Claim 4.7, if \( u_iu_j \in E(G) \) for some \( j \geq i + \frac{r}{2} + 1 \), then \( N_A(u_i) \subseteq V(P'') \). This implies that \( u_j \in V(P'') \), a contradiction. Now one can see that \( N_{A \setminus A'}(u_i) \subseteq \{ u_j : \frac{r}{2} + 2 \leq j \leq \frac{r}{2} + i \} \) for each \( i \leq \frac{r}{2} \) (where \( N_{A \setminus A'}(u_i) = \emptyset \)). Thus, \( d_{A \setminus A'}(u_i) \leq i - 1 \) for each \( i \leq \frac{r}{2} \). It follows that
\[ e(A \setminus A') = \sum_{i=1}^{r} d_{A \setminus A'}(u_i) \leq \sum_{i=1}^{r} (i - 1) = \frac{r(r - 2)}{8} < \frac{r^2}{8}. \]

Note that \( G[A'] \) is \( P_{r+2} \)-free. Then by Lemma 2.1, \( 2e(A') \leq r|A'|. \) Thus

\[ 2e(A) = 2e(A') + 2e(A \setminus A') < r|A'| + \frac{r^2}{4}. \] (8)

Note that

\[ e(A \setminus A', B) \leq |A \setminus A'||B| = (r + 1)|B|. \] (9)

Combining (7), (8) with (9), we have

\[ \gamma(u^*) = |A| + 2e(A) + e(A', B) + e(A \setminus A', B) \]
\[ \leq |A| + (2r - 1)|A'| + (2r + 1)|B| - \left( \frac{3}{4}r^2 - r \right) \]
\[ = |A| + (2r - 1)(|A| - (r + 1)) + (2r + 1)|B| - \left( \frac{3}{4}r^2 - r \right) \]
\[ = 2r(|A| + |B| + 1) + (|B| + 1) - \left( \frac{11}{4}r^2 + 2r \right) \]
\[ \leq \left( 2r + \frac{1}{2} \right)n - \left( \frac{11}{4}r^2 + 2r \right), \]

where the last inequality follows from \( |B| + 1 = n - |A| \leq \frac{n}{2}. \) Since \( n \geq 7r \) and \( r \geq 2, \) we have \( \gamma(u^*) \leq \left( 2r + \frac{1}{2} \right)n - \left( \frac{11}{4}r^2 + 2r \right) < \left[ \frac{n^2}{4} \right], \) a contradiction. The claim holds. \( \square \)

**Claim 4.10.** Let \( H \) be a nontrivial component of \( G[A]. \) Then \( 2|E(H)| + e(V(H), B) \leq |V(H)||B|. \) Moreover, if the equality holds, then \( H \cong K_{r+1}. \)

**Proof.** By Claim 4.8, there exists an \( A \)-path \( P \) with \( \ell(P) = 2r. \) We first show that either \( V(H) \subseteq V(P) \) or \( V(H) \cap V(P) = \emptyset. \) Suppose to the contrary that we can find an edge \( uu' \) with \( u \in V(H) \cap V(P) \) and \( u' \in V(H) \setminus V(P). \) Since \( \ell(P) = 2r, \) there exists an \( A \)-path \( P' \) of length \( r \) such that \( V(P') \subseteq V(P) \) and \( u \) is an endpoint of \( P'. \) By Claim 4.7, \( N_A(u) \subseteq V(P') \) and so \( u' \in V(P') \subseteq V(P), \) a contradiction. Hence, \( V(H) \subseteq V(P) \) or \( V(H) \cap V(P) = \emptyset. \)

Now by Claim 4.9, there exists another \( A \)-path \( P'' \) of length \( 2r \) with \( V(P) \cap V(P'') \cap A = \emptyset. \) Similar to the above, either \( V(H) \subseteq V(P'') \) or \( V(H) \cap V(P'') = \emptyset. \) It follows that \( V(H) \cap V(P) = \emptyset \) or \( V(H) \cap V(P'') = \emptyset. \) Without loss of generality, assume that \( V(H) \cap V(P) = \emptyset. \)

Let \( P = u_1v_1\cdots u_rv_ru_{r+1}, \) where \( u_i \in A \) and \( v_i \in B. \) We need to show that \( N_G(u) \cap \{v_1, \ldots, v_r\} = \emptyset \) for each \( u \in V(H). \) Suppose to the contrary that \( u \sim v_i \) for some \( u \in V(H) \) and some \( i \in \{1, \ldots, r\}. \) Combining the edge \( uv_i \) with a subpath of \( P \) of length \( r - 1 \) starting from \( v_i, \) we can get an \( A \)-path of length \( r \) starting from \( u. \) By Claim 4.7, we have \( N_A(u) \subseteq V(P), \) which contradicts the fact that \( V(H) \cap V(P) = \emptyset. \)
Now, we have \( d_B(u) \leq |B| - r \) for each \( u \in V(H) \). Thus \( e(V(H), B) \leq |V(H)||(|B| - r) \). Since \( H \) is \( P_{r+2} \)-free, by Lemma 2.1 we have \( 2|E(H)| \leq r|V(H)| \), with equality if and only if \( H \cong K_{r+1} \). It follows that \( 2|E(H)| + e(V(H), B) \leq |V(H)||B| \), and if equality holds then \( H \cong K_{r+1} \). This completes the proof. \( \square \)

To complete the proof of Case 2 of Theorem 1.4, it remains to show \( e(A) = 0 \) by Lemma 2.3. By Claim 4.10, \( 2|E(H)| + e(V(H), B) \leq |V(H)||B| \) for any component \( H \) of \( G[A] \). Furthermore, if \( G[A] \) contains a good component, then we have \( 2e(A) + e(A, B) < |A||B| \). It follows that

\[
\gamma(u^*) = |A| + 2e(A) + e(A, B) < |A||(|B| + 1) \leq \left\lfloor \frac{n^2}{4} \right\rfloor,
\]
a contradiction. Hence, all the components of \( G[A] \) are not good, that is,

\[
2|E(H)| + e(V(H), B) = |V(H)||B| \tag{10}
\]
for every component \( H \) of \( G[A] \). This implies that

\[
\gamma(u^*) = |A| + 2e(A) + e(A, B) = |A| + |A||B| = |A||(|B| + 1) \leq \left\lfloor \frac{n^2}{4} \right\rfloor,
\]

Combining Lemma 2.3, we have \( \gamma(u^*) = \left\lfloor \frac{n^2}{4} \right\rfloor \), \( |A| = \left\lfloor \frac{n}{2} \right\rfloor \), and\n
\[
|B| = n - 1 - |A| > r + 1.
\]

Now suppose that \( e(A) \neq 0 \) and let \( H \) be a nontrivial component of \( G[A] \). Then by (10) and Claim 4.10, we have \( H \cong K_{r+1} \). Note that \( G \) is \( \partial_{r+1} \)-free. One can see that \( d_B(w) \leq 1 \) for each \( w \in B \). Since \( r + 1 < |B| \), we obtain

\[
2|E(H)| + e(V(H), B) \leq 2|E(K_{r+1})| + |B| = r(r + 1) + |B| < r|B| + |B| = |V(H)||B|,
\]
which contradicts (10). Therefore, \( e(A) = 0 \). The proof of Case 2 is complete.

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