A New Mechanism for Massive Binary Black-Hole Evolution

Kimitake Hayasaki

Yukawa Institute for Theoretical Physics, Kyoto University, Sakyo-ku, Kyoto 606-8502

kimitake@yukawa.kyoto-u.ac.jp

(Received 2008 May 17; accepted 2008 September 10)

Abstract

It is still unknown how a binary black hole (BBH) evolves after its semi-major axis has reached the sub-parsec scale, where dynamical friction with neighboring stars is no longer effective (the so-called final-parsec problem). In this paper, we propose a new mechanism by which a massive BBH can naturally coalesce within a Hubble time. We consider the evolution of a BBH with a triple disk composed of an accretion disk around each black hole and one circumbinary disk surrounding them. While the circumbinary disk removes the orbital angular momentum of the BBH via a binary-disk resonant interaction, the mass transfer from the circumbinary disk to each black hole adds some fraction of its angular momentum to the orbital angular momentum of the BBH. We find that there is a critical value of the mass-transfer rate where extraction of the orbital angular momentum from the BBH is balanced with the addition of orbital angular momentum to the BBH. The semi-major axis of the BBH decays with time, whereas the orbital eccentricity of the BBH grows with time, if the mass-transfer rate is smaller than the critical one, and vice versa. Its evolutionary timescale is characterized by the product of the viscous timescale of the circumbinary disk and the ratio of the total black-hole mass to the mass of the circumbinary disk. Most massive BBHs are able to merge within a Hubble time by the proposed mechanism, which helps to solve the final parsec problem.

Key words: accretion, accretion disks — binaries:general — black hole physics — galaxies:nuclei

1. Introduction

Massive black holes in galactic nuclei are considered to have co-evolved with their host galaxies (Ferrarese & Merritt 2000; Gebhardt et al. 2000; Magorrian et al. 1998). Since galaxies are well-known to evolve through frequent mergers, this strongly suggests that black-hole growth is mainly caused by black-hole mergers and a subsequent accretion of gas (Yu & Tremaine 2002; Di Matteo et al. 2005). If so, a massive binary black hole (BBH) is inevitably formed before the black holes merge by emitting gravitational radiation. Even if there are transiently triple-massive black holes in a galactic nucleus, the system finally settles down to the formation of a massive BBH by merging of two black holes or by ejecting one black hole from the system via a gravitational slingshot (Iwasawa et al. 2006). Recent hydrodynamic simulations showed the rapid BBH formation within several Gys by interactions between the black holes and the surrounding stars and gas in a gas-rich galaxy merger (Mayer et al. 2007).

It is widely accepted that a massive BBH mainly evolves via three stages (Bagelman et al. 1980; Yu 2002). Firstly, each black hole sinks independently towards the center of the common gravitational potential due to dynamical friction with the neighboring stars. When the separation between two black holes becomes less than 1 pc or so, the angular-momentum loss by dynamical friction slows down due to depletion of the stars on orbits intersecting the BBH (Saslaw et al. 1974). This is the second evolutionary stage. Finally, the BBH coalesces rapidly if the semi-major axis decreases to the point where the emission of the gravitational radiation becomes an efficient mechanism to remove the orbital angular momentum of the BBH. The transition from the second stage to the final stage is, however, considered to be a bottleneck of the evolutionary path for the BBH to coalesce because of cutting off the supply of the stars on intersecting orbits. This is called the final parsec problem (see Merritt & Milosavljević 2005 for a review).

Many authors have tackled this problem in the context of the interaction between the black holes and the stars, but there has still been extensive discussions (Roos 1981; Makino 1997; Quinlan & Hernquist 1997; Milosavljević & Merritt 2001; Makino & Funato 2004; Merritt et al. 2007; Sesana et al. 2007). There is another possible way to extract the energy and the angular momentum from a BBH by the interaction between black holes and the gas surrounding them (e.g., Armitage & Natarajan 2005). In some cases of circular binaries with an extreme mass ratio, the secondary black hole could be embedded in the gas disk around the primary black hole, and then migrate to the primary black hole. This kind of the binary-disk interaction could be a candidate to resolve the final parsec problem (Ivanov et al. 1999; Gould & Rix 2000; Armitage & Natarajan 2002).

Hayasaki et al. (2007) found that if a BBH is surrounded by a gaseous disk with the nearly Keplerian rotation (i.e., the circumbinary disk: CBD), the gas can be transferred from the CBD to each black hole. The mass transfer leads to the formation of an accretion disk around each black hole (Hayasaki et al. 2008), and then the BBH system finally has a triple disk composed of an accretion disk around each black hole and a CBD as a mass reservoir around the binary (see figure 1 for an artistic impression of the triple-disk system around the BBH). There is, however, little known about how the BBH evolves in such a triple-disk system. We therefore consider the evolution of a massive BBH interacting with a triple disk. Although we mainly discuss the case of a massive BBH system,
Artistic impression of a massive binary black hole (BBH) with a triple disk on a subparsec scale of a merged galactic nucleus, illustrated by K. Morimoto. The BBH is surrounded by a circumbinary disk (CBD), from which the gas transfers to the central binary, and then an accretion disk is formed around each black hole.

our results can be applied to other possible contexts, such as compact binaries, young binary star formation, and extrasolar planet formation because of the scaling nature.

The plan of this paper is organized as follows. In section 2, we describe the derivation of the basic equations governing the orbital evolution of a BBH and the formulation of the gravitational torque of the BBH, which acts on the CBD. The solutions for their evolutionary equations are then reported in section 3. Section 4 is devoted to discussions, and finally we summarize the results in section 5.

2. Basic Equations

We assume that two black holes are gravitationally bounded and their motions follow Kepler’s third law, by which the BBH is defined. The total energy of the BBH, $E_b$, is written as

$$E_b = -\frac{GM_{bh}\mu}{2a},$$

where $a$ is the semi-major axis of the BBH and $\mu = M_1 M_2 / M_{bh}$ is its reduced mass. Here, $M_{bh} = M_1 + M_2$ is the total black-hole mass, $M_1$ is the primary black-hole mass, and $M_2$ is the secondary black-hole mass. By differentiating both sides of the above equation with time, the energy dissipation rate of the orbital motion can be obtained as

$$\frac{\dot{E}_b}{E_b} = -\frac{\dot{a}}{a} \frac{M_1}{M_1} + \frac{M_2}{M_2},$$

where $\dot{M}_1$ is the mass-accretion rate of the primary black hole and $\dot{M}_2$ is the mass-accretion rate of the secondary black hole. The dot over each physical quantity shows the time differentiation, unless otherwise noted.

The orbital angular momentum of the BBH $J_b$ is written as

$$J_b = \mu a^2 \Omega_b \sqrt{1 - e^2},$$

where $e$ is the orbital eccentricity and $\Omega_b$ is the angular frequency of the BBH. The relationship between the orbital energy, $E_b$, and the orbital angular momentum, $J_b$, can be written from equations (1) and (3) as

$$E_b = -\frac{\Omega_b J_b}{2\sqrt{1 - e^2}}.$$  

The change rate of the orbital angular momentum, $\dot{J}_b$, can be written as

$$\frac{\dot{J}_b}{J_b} = -\frac{\dot{a}}{2a} - \frac{e \dot{e}}{(1 - e^2)} \frac{M_{bh} \dot{M}_1}{2M_{bh}} + \frac{M_1 \dot{M}_2}{M_1} + \frac{M_2 \dot{M}_2}{M_2},$$

where $\dot{M}_{bh} = \dot{M}_1 + \dot{M}_2$ is the total mass-accretion rate. The orbital evolution of the BBH is generally governed by
equations (1)–(5). Assuming that the timescale of the total mass-accretion rate is much longer than the timescale of the energy dissipation and angular-momentum loss, the second and third terms on the right-hand side of equations (2) and (5) can be neglected. Therefore, a set of basic equations of the orbital evolution is finally given by

$$\frac{\dot{a}}{a} = -\frac{\dot{E}_b}{E_b},$$

and

$$\frac{e\dot{e}}{1-e^2} = -\frac{\dot{J}_b}{2E_b},$$

respectively.

2.1. Interaction between Disk and Binary Black Hole

We consider the resonant interaction between BBH and CBD. The CBD is assumed to be geometrically thin, to be aligned with the orbital plane of the BBH, and have nearly Keplerian rotation with no self-gravitation. The binary potential can be expanded by a Fourier double series,

$$\Phi(r, \theta, t) = \sum_{m,j} \phi_{ml}(r) \exp[i(m\theta - l\Omega_b t)],$$

where \(m\) is the azimuthal number and \(l\) is the time-harmonic number. The potential component, \(\phi_{ml}\), can be written as

$$\phi_{ml}(r) = \frac{1}{2\pi^2} \int_0^{2\pi} d(\Omega_b t) \int_0^{2\pi} \Phi \cos(m\theta - l\Omega_b t)d\theta,$$

which gives rise to a corotational resonance at the radius where \(\Omega_p = \Omega\) and the Lindblad resonances at radii \(\Omega_p = \Omega \pm (\kappa/m)\). Here, \(\Omega_p\) is the pattern frequency of the binary potential, which is defined by

$$\Omega_p = \frac{1}{m} \Omega_b,$$

(10)

\(\kappa\) is the epicyclic frequency, and the upper and lower sign correspond to the outer Lindblad resonance (OLR) and the inner Lindblad resonance (ILR), respectively. The radii of these resonances for the CBD are given by

$$r_{\text{OLR}} = \left(\frac{m+1}{l}\right)^{2/3} a,$$

(11)

and

$$r_{\text{ILR}} = \left(\frac{m}{l}\right)^{2/3} a.$$

(12)

Here, the epicyclic frequency is \(\kappa \sim \Omega\) for nearly Keplerian disks. The standard formula for the torques at the LR is given by Goldreich and Tremaine (1979),

$$T_{ml}^{\text{LRs}} = \frac{m(m+1)\pi^2 \Sigma (\lambda - 2m)^2 \phi_{ml}^2}{3l^2 \Omega_b^2},$$

(13)

where \(\lambda = d \ln \phi_{ml}/d \ln r\) for the OLR in the CBD (cf. Artymowicz & Lubow (1994)). Similarly, the torque at the CR is written as

$$T_{ml}^{\text{CR}} = \frac{2m^3 \pi^2 \Sigma \phi_{ml}^2}{3l^2 \Omega_b^2},$$

(14)

Note that the angular momentum is added from the binary to the CBD via the LRs and the CR. On the other hand, the viscous torque formula derived by Lin and Papaloizou (1986) is written as

$$T_{\text{vis}} = 3\pi \alpha_{SS} \Sigma \Omega^2 \frac{r^4}{r^2},$$

(15)

where \(\alpha_{SS}\) is the Shakura-Sunyaev viscosity parameter (Shakura & Sunyaev 1973). Since the viscous torque removes angular momentum from the CBD, it is truncated and forms a gap between the CBD and the binary if the viscous torque is less than the resonant torque. The inner edge of the CBD is formed at the given resonance radius where the viscous torque is balanced with the resonant torque:

$$T_{\text{vis}} = \sum_{m,l} T_{ml}^{\text{OLR}} + \sum_{m,l} T_{ml}^{\text{ILR}} + \sum_{m,l} T_{ml}^{\text{CR}} \approx \sum_{m,l} T_{ml}^{\text{OLR}},$$

(16)

where the summation is taken over all combinations of \((m,l)\) that give the same resonance radius. Actually, criterion (16) is determined only by the viscous torque and the torques from the OLR of the lowest-order potential component, because the torques from the OLR dominate those from the CR in the CBD and high-order potential components contribute little to the total torque, even if the eccentricity of the binary is not small (Goldreich & Tremaine 1979; Artymowicz et al. 1991; Artymowicz & Lubow 1994). Following Artymowicz and Lubow (1994), the inner edge of the CBD, \(r_{in}\), is approximately determined by the \((m,l) = (2,1)\) resonant torque in a binary with a low eccentricity: \(r_{in} \approx (m + 1/l)^{2/3} a \approx 2.08a\). It has been confirmed by Hayasaki et al. (2007) that this value is also applicable for a BBH with a moderate eccentricity of \(e = 0.5\). Therefore, the BBH transfers most of the angular momentum to the CBD via the 1:3 resonant radius, \(r_{in} \approx 2.08a\). Below, the set of \((m,l)\) is regarded as \((2,1)\), unless otherwise noted.

3. Orbital Evolution of Massive Binary Black Holes

The long-term evolution of the semi-major axis and of the orbital eccentricity are mainly driven by the resonant interaction between the BBH and the CBD. In this section, we firstly describe the evolutionary relation between the semi-major axis and the orbital eccentricity, the evolution of the semi-major axis, the evolution of the orbital eccentricity, and finally the effect of mass transfer from the CBD on the orbital evolution of the BBH.

3.1. Relation between Semi-Major Axis and Orbital Eccentricity

As the motion of the BBH slowly varies with time by resonantly interacting with the CBD, there is an adiabatic invariant that can be regarded as the orbital angular momentum of the BBH (Landau & Lifshitz 1960). In such a system, the energy dissipation rate of the BBH is proportional to the product of the change rate of the adiabatic invariant and the characteristic frequency of the system, e.g., \(\Omega_p\) (cf. Lubow & Artymowicz 1996):
The above equation can be integrated, and then

$$
e^{-68 K. Hayasaki [Vol. 61,}
$$

From equations (4), (10), and (17), equation (6) is rewritten as

$$a_{0} = \frac{a}{a_{0}}.$$

which determines the evolution of the semi-major axis of the BBH. Using the above equation, equation (7) is rewritten as

$$e \hat{e} = \left(1 - \frac{1}{m} \sqrt{1 - e^2}\right) \frac{J_{b}}{J_{b}}.$$

From equations (18) and (19), we can obtain a differential equation relating $a$ to $e$,

$$\frac{\dot{a}}{a} = -2 \frac{l}{m} \frac{\hat{e} e}{\sqrt{1 - e^2}} \left(1 - \frac{1}{m} \sqrt{1 - e^2}\right).$$

The above equation can be integrated, and then $a$ can be expressed as a function of $e$,

$$a = a_{0} \left(1 - \frac{1}{m} \sqrt{1 - e^2}\right) \left(1 - \frac{1}{m} \sqrt{1 - e^2}\right)^2,$$

where $a_{0}$ and $e_{0}$ are the initial value of the semi-major axis and the orbital eccentricity, respectively. Figure 2 exhibits the evolution of the semi-major axis, $a$, normalized by $a_{0}$ as a function of the orbital eccentricity, $e$. It is noted from the figure that the semi-major axis decays with time as the orbital eccentricity grows with time, and vice versa. When the BBH evolves towards a black-hole merger, the eccentricity growth is faster than the orbital decay, because the orbital eccentricity becomes more rapidly close to 1.0 than the semi-major axis is close to 1.0.

3.2. Evolution of Semi-Major Axis

Assuming that the CBD completely absorbs the orbital angular momentum of the BBH and is a quasi-steady state, the change rate of the orbital angular momentum, $\dot{J}_{b}$, can be written as $\dot{J}_{b} = -J_{CBD}$, where $J_{CBD}$ is the change rate of the angular momentum of the CBD,

$$\frac{\dot{a}}{a} = -2 \frac{l}{m} \frac{J_{CBD}}{J_{b}} \sqrt{1 - e^2}.$$

Since the orbital angular momentum is always transferred from the BBH to the CBD via the inner edge of the CBD $r_{in}$, $J_{CBD}$ can be approximately estimated as

$$J_{CBD} \simeq T_{vis} r_{in} = 3 \left(\frac{m + 1}{l}\right)^{1/3} \left(\frac{1 + q^2}{a_{SS}}\right)^{1/2} \frac{M_{CBD}}{M_{bh}} \frac{1}{\sqrt{1 - e^2}} \frac{J_{b}}{\tau_{CBD}},$$

where $q$ is the mass ratio of the secondary black hole to the primary black hole, the scope of which is $0.1 \leq q \leq 1.0$ because the CBD is not truncated, and thus the thick-disc model is broken down if $q$ is less than 0.1. Also, $M_{CBD} \sim \pi r_{in}^2 \Sigma$, and $\tau_{CBD}$ is the viscous timescale of the CBD when $r = r_{in}$:

$$\tau_{vis} \sim 4.8 \times 10^3 \left(\frac{\pi}{c}\right)^{1/3} \frac{\left(m + 1\right)^{1/3} \left(\frac{1 + q^2}{a_{SS}}\right)^{1/2} \frac{M_{CBD}}{M_{bh}}}{\left(1 - \frac{1}{m} \sqrt{1 - e^2}\right)^{1/2}} \left(\frac{M_{bh}}{M_{\odot}}\right)^{1/3} \left(\frac{a_{SS}}{a_{0}}\right)^{1/2} \left[\text{yr}\right].$$

Figure 3 represents the characteristic timescale, $\tau_{c}$, of the massive BBH evolution as function of the black-hole mass in units of the solar mass. Here, we set $T_{0} = 10^4 K$ and $a_{SS} = 0.1$, which are typical values for a disk in active galactic nuclei (AGNs). The ratio of the CBD mass to the black-hole mass is set as $M_{CBD}/M_{bh} = 10^{-2}$, which ensures that the CBD is stable for self-gravitation, because Toomre’s $Q$ value is $Q = (M_{CBD}/M_{bh})^{-1} (H/r) > 1$ (Toomre 1964). The solid line denotes $t_{c}$ in the range $M_{bh} = 10^6 - 10^8 M_{\odot}$. As can be seen in figure 3, $t_{c}$ is the longer as the black hole is more massive. Integrating equation (25),

$$\frac{a}{a_{0}} = \left(1 - \frac{l}{t_{c}}\right)^{2}.$$

Fig. 2. Evolution of the semi-major axis, $a$, as a function of the orbital eccentricity, $e$. The semi-major axis, $a$, is normalized by the initial value of the semi-major axis, $a_{0}$. The initial value of the orbital eccentricity is indicated by $e_{0}$. The solid line, the dashed line, and the dotted line are the $a$-$e$ relations of $e_{0} = 0.0$, $e_{0} = 0.5$, and $e_{0} = 0.9$, respectively.
which determines the evolution of the semi-major axis of BBH.

Figure 4a shows the evolution of the semi-major axis $a$ normalized by the initial value of the semi-major axis $a_0$. It is noted from the figure that the semi-major axis rapidly decays with time regardless of the orbital eccentricity, and the BBH finally merges at $t = t_c$.

3.3. Evolution of Orbital Eccentricity

Substituting equation (25) into equation (20) can be written as

$$
\frac{e \dot{e}}{\sqrt{1-e^2}} \left( 1 - \frac{l}{m} \sqrt{1-e^2} \right) = \frac{m}{T} \frac{1}{t_c-t}.
$$

which determines the evolution of the orbital eccentricity. Integrating both sides, we can obtain the evolutionary timescale of the orbital eccentricity,

$$
t/t_c = 1 - \left[ \left( 1 - \frac{l}{m} \sqrt{1-e_0^2} \right) / \left( 1 - \frac{l}{m} \sqrt{1-e^2} \right) \right].
$$

The evolution of the orbital eccentricity, $e$, is

$$
e = \sqrt{1 + \eta(t)} \left[ 1 - \eta(t) \right],
$$

where $\eta$ is

$$
\eta(t) = \frac{m}{T} \left[ 1 - \left( 1 - \frac{l}{m} \sqrt{1-e_0^2} \right) \left( \frac{t_c-t}{t_c} \right) \right].
$$

Figure 4b represents the evolution of the orbital eccentricity of the BBH in the case of $e_0 = 0.0$, $e_0 = 0.5$, and $e_0 = 0.9$, respectively. This figure clearly shows that the orbital eccentricity grows with time and is close to 1.0 within $t = t_c$, regardless of the initial value of the orbital eccentricity. The higher value of the initial eccentricity becomes more rapidly close to 1.0.

3.4. Effect of Mass Transfer from Circumbinary Disk

Hayasaki et al. (2007) found that the mass transfer from the CBD to each black hole occurs every binary orbit. Since the transferred mass has a relative angular momentum to each black hole, an accretion disk is formed around each black hole (Hayasaki & Mineshige 2008). In this section, we investigate the effect of mass transfer on the evolution of a massive BBH.
The averaged torque, which is added to the accretion disks by mass transfer during one orbital period is given by

\[ J_T \simeq (\dot{M}_T)^2 \frac{\Omega_{\text{in}}}{l} = \left( \frac{m + 1}{l} \right)^{1/3} \frac{(1 + q)^2 (\dot{M}_T)}{q} \sqrt{\frac{M_{\text{bh}}}{M_l}} \frac{J_b}{\sqrt{1 - e^2}}, \tag{33} \]

where \( \dot{M}_T \) denotes the averaged mass-transfer rate during one orbital period, which can be expressed as the sum of the averaged mass-transfer rate to the primary black hole (the primary transfer rate, \( \dot{M}_{T,1} \)) and the averaged mass-transfer rate to the secondary black hole (the secondary transfer rate, \( \dot{M}_{T,2} \)). Unless otherwise noted, we call \( \dot{M}_T \) the mass-transfer rate. The accumulated mass around each black hole accretes onto each black hole, by which the angular momentum is transferred outward, and is finally added to the binary due to the tidal torque (cf. Kato et al. 2008). The torque added to the binary by this process, \( J_{\text{add}} \), can be estimated as

\[ J_{\text{add}} = J_{\text{pbhd}} + J_{\text{shbd}}, \tag{34} \]

where \( J_{\text{pbhd}} \) and \( J_{\text{shbd}} \) are the torque added from the accretion disk around the primary black hole to the BBH (the primary torque) and the torque added from the accretion disk around the secondary black hole to the BBH (the secondary torque), respectively. By defining the ratio of the primary transfer rate to the secondary transfer rate, \( q_T \equiv \dot{M}_{T,2}/\dot{M}_{T,1} \), the primary torque can be written as

\[ J_{\text{pbhd}} = (\dot{M}_{T,1})^2 \frac{P_b}{r_{\text{vis},1}} \sqrt{GM_1 r_{c,1}} \left( 1 - \sqrt{\frac{r_{\text{ms},1}}{r_c}} \right) \simeq \frac{(\dot{M}_{T,1})^2 P_b}{1 + q_T} \sqrt{GM_1 r_{c,1}}, \tag{35} \]

where \( P_b \) is the orbital period of the BBH, \( r_{\text{vis},1} \) is the viscous timescale measured at the circularization radius \( r_{c,1} \) of the accretion disk around the primary black hole, and \( r_{\text{ms},1} \) is the radius of the marginally stable circular orbit of the primary black hole. Here, we approximate \( J_{\text{pbhd}} \) by assuming that \( r_{\text{ms},1} \) is much smaller than \( r_{c,1} \). Similarly, the secondary torque can be written as

\[ J_{\text{shbd}} = q_T (\dot{M}_{T,2})^2 \frac{P_b}{r_{\text{vis},2}} \sqrt{GM_2 r_{c,2}}, \tag{36} \]

where \( r_{\text{vis},2} \) is the viscous timescale measured at the circularization radius, \( r_{c,2} \), of the accretion disk around the secondary black hole. Substituting equations (33) and (35)–(36) into equation (34), we can obtain

\[ J_{\text{add}} = f J_T. \tag{37} \]

Here, \( f \) is the parameter, which is defined as

\[ f = \frac{2 \pi \alpha_{\text{SS}}}{1 + q_T} \left[ 1 + q_T \frac{c_{s,2}}{c_{s,1}} \right] \left( \frac{m + 1}{l} \right)^{-1/3} \left( \frac{c_{s,1}}{v_b} \right)^2, \tag{38} \]

where \( v_b \) is the orbital velocity of the BBH; \( c_{s,1} \) and \( c_{s,2} \) are the sound velocity of the primary and secondary disk, respectively. For simplicity, we assume that \( c_{s,1} \equiv c_{s,2} \) and the ratio of \( c_{s,1} \) to \( v_b \) is constant during the evolution of the BBH. The parameter \( f \) shows how much the torque of the mass transfer is converted to the torque of the BBH. When \( f \) equals to 1.0, the torque of the mass transfer is completely converted to the torque of the BBH. The value of \( f \) is, however, substantially much smaller than 1.0 because the orbital velocity is generally much faster than the sound velocity at the circularization radius.

The angular-momentum balance in the BBH system can be expressed as

\[ J_b = - J_{\text{CBD}} + J_{\text{add}} = - m \frac{1}{l} \frac{1}{t_c} \left( 1 - \frac{\dot{M}_{T}}{M_{\text{crit}}} \left( \frac{a}{a_0} \right)^{1/2} \right) \left( \frac{a}{a_0} \right)^{-1/2} \times \frac{J_b}{\sqrt{1 - e^2}}, \tag{39} \]

and thus, from equation (25), the differential equation of the semi-major axis can be written as

\[ \frac{da}{dt} = - \frac{2}{t_c} \left[ 1 - \frac{\dot{M}_{T}}{M_{\text{crit}}} \left( \frac{a}{a_0} \right)^{1/2} \right] \left( \frac{a}{a_0} \right)^{-1/2}, \tag{40} \]

where \( \dot{M}_{\text{crit}} \) is the critical mass-transfer rate, which can be defined as

\[ \dot{M}_{\text{crit}} = \frac{3 M_{\text{CBD}}}{f \alpha_{\text{SS}} \dot{M}_{T}} \frac{M_{\text{bh}}}{t_c}. \tag{41} \]

Here, we use equations (26) and (27). Integrating both sides of equation (40), we obtain

\[ \frac{a}{a_0} = \left( \frac{\dot{M}_{T}}{M_{\text{crit}}} \right)^{-2} \times \left[ 1 - \left( 1 - \frac{\dot{M}_{T}}{M_{\text{crit}}} \right) \exp \left( \frac{t}{t_c} \frac{\dot{M}_{T}}{M_{\text{crit}}} \right) \right]^2. \tag{42} \]

Figure 5a displays the evolution of the semi-major axis of a massive BBH with a triple disk. It is noted from the figure that the semi-major axis, \( a \), decays with time when \( M_{\text{crit}} > \dot{M}_T \), while the semi-major axis increases with time when \( M_{\text{crit}} < \dot{M}_T \). In the limit of \( \dot{M}_T/M_{\text{crit}} \ll 1.0 \), the dotted line corresponds to the solid line of figure 4a. The growth rate of the semi-major axis is more steep when \( \dot{M}_T/M_{\text{crit}} = 2.0 \) than when \( \dot{M}_T/M_{\text{crit}} = 1.25 \). This means that the semi-major axis more rapidly grows with time, since accretion disks are more massive.

Similarly, the differential equation of the orbital eccentricity, \( e \), is modified by the mass transfer as follows:
where $h_P$ is smaller than 1.0 when $h_P$ is larger than 1.0 when $M$. Which conditions are more promising, $h_P$, the dotted line corresponds to the solid line of figure 4b. By integrating both sides of the above equation, we can obtain the evolutionary equation of the orbital eccentricity,

$$e = \sqrt{(1 + \eta(t))(1 - \eta(t))},$$

where $\eta$ is

$$\eta(t) = \frac{m}{l} - \frac{m}{l} \left(1 - \frac{l}{m} \sqrt{1 - e_0^2}\right) \frac{\langle M_T \rangle}{M_{\text{crit}}},$$

$$\times \left[1 - \left(1 - \frac{\langle M_T \rangle}{M_{\text{crit}}} \exp \left(\frac{t\langle M_T \rangle}{t_c M_{\text{crit}}}\right)\right)^{-1}\right].$$

Figure 5b represents the evolution of the orbital eccentricity of a massive BBH with a triple disk. The figure shows that the orbital eccentricity decays with time when $\langle M_T \rangle/M_{\text{crit}} > 1.0$, whereas it grows with time when $\langle M_T \rangle/M_{\text{crit}} < 1.0$. This is independent of the initial value of the orbital eccentricity, $e_0$. The rate of evolution is more rapid because $\langle M_T \rangle/M_{\text{crit}}$ is larger than 1.0 when $\langle M_T \rangle/M_{\text{crit}} > 1.0$ and because $\langle M_T \rangle/M_{\text{crit}}$ is smaller than 1.0 when $\langle M_T \rangle/M_{\text{crit}} < 1.0$. In the limit of $\langle M_T \rangle/M_{\text{crit}} \ll 1.0$, the dotted line corresponds to the solid line of figure 4b.

Which conditions are more promising, $\langle M_T \rangle/M_{\text{crit}} < 1$ or $\langle M_T \rangle/M_{\text{crit}} > 1$? The answer is $\langle M_T \rangle/M_{\text{crit}} < 1$. In order to confirm this answer, we consider the minimum value of the critical mass-transfer rate in what follows. If $t_c$ is longer than a Hubble time, the binaries never merge within a Hubble time. We therefore confine our argument to the case in which $t_c$ is shorter than a Hubble time. Since the possible value of $t_c$ is then a Hubble time at the maximum, the minimum value of the critical mass transfer rate can be estimated as

$$M_{\text{crit},\text{min}} = \frac{q}{(1 + q)^2} \frac{m}{l} \left(\frac{m + 1}{l}\right)^{-1/3} \frac{1}{f_{\text{max}}} \frac{M_{\text{bh}}}{t_{\text{H}}},$$

where $t_{\text{H}} = 1/H_0 \sim 1.37 \times 10^{10}$ yr is a Hubble time $f_{\text{max}}$ is the maximum value of $f$, which can be expressed as

$$f_{\text{max}} = 2\pi a_{\text{SS}} \left(\frac{m + 1}{l}\right)^{-1/3} \left(\frac{c_{\text{b},\text{min}}}{v_{\text{b},\text{min}}}\right)^2.$$

Here, $v_{\text{b},\text{min}}$ is

$$v_{\text{b},\text{min}} = \sqrt{\frac{GM_{\text{bh}}}{a_{h}}} = \frac{2(1 + q)}{q^{1/2}} \frac{M_{\text{bh}}}{a_{h}} \sigma,$$

where $a_{h} = G\mu/4\sigma^2$ is the hardening radius where the binding energy of the BBH exceeds the kinetic energy of the stars, and $\sigma$ is the 1D velocity dispersion of the stars in the galactic nucleus (Quinlan 1996). Assuming that $\sigma$ approximately equals the observed stellar velocity dispersion, we obtain from the best current estimate of the $M-\sigma$ relation (Merritt & Milosavljević 2005)

$$\sigma = \frac{200 \text{km s}^{-1}}{(1.66 \pm 0.24)\beta} \left(\frac{M_{\text{bh}}}{10^8 M_\odot}\right)^{\beta} \sim 180 \text{km s}^{-1} \left(\frac{M_{\text{bh}}}{10^8 M_\odot}\right)^{\beta}$$

(49)
with $\beta = 1/(4.86 \pm 0.43)$. Substituting equations (47)–(49) into (46), we can obtain

$$M_{\text{crit, min}} \sim 0.1 \left( \frac{0.1}{\alpha_{\text{SS}}} \right) \left( \frac{180 \text{ km s}^{-1}}{c_{s,1}} \right)^2 \times \left( \frac{M_{\text{bh}}}{10^8 M_{\odot}} \right)^{2\beta+1} \left( \frac{M_{\odot}}{\text{yr}} \right).$$

(50)

for $m = 2$ and $l = 1$. On the other hand, the Eddington accretion rate, $M_{\text{Edd}}$, is

$$M_{\text{Edd}} = \frac{4\pi GM_{\text{bh}}}{c^2} \sim 0.2 \left( \frac{M_{\text{bh}}}{10^8 M_{\odot}} \right) \left( \frac{M_{\odot}}{\text{yr}} \right).$$

(51)

where $c$ is the velocity of light and $\kappa_T$ is the Thompson opacity coefficient. The ratio of $M_{\text{crit, min}}$ to $M_{\text{Edd}}$ can be written as

$$\frac{M_{\text{crit, min}}}{M_{\text{Edd}}} \sim 0.5 \left( \frac{200 \text{ km s}^{-1}}{c_{s,1}} \right)^2 \left( \frac{M_{\text{bh}}}{10^8 M_{\odot}} \right)^{2\beta} \left( \frac{0.1}{\alpha_{\text{SS}}} \right).$$

(52)

for $m = 2$ and $l = 1$.

Since the sound velocity, $c_{s,1} \sim 10 \text{ km s}^{-1}$, in the outer region of a typical AGN disk, $M_{\text{crit, min}}$ is always larger than $M_{\text{Edd}}$ for a supermassive black hole with mass in the $10^8 M_{\odot}$ to $10^9 M_{\odot}$ range. Hayasaki et al. (2007) actually confirmed that the mass-transfer rate is sub-Eddington rate if the gas is supplied to the CBD at the Eddington rate. It is, therefore, promising that most massive BBHs evolve towards orbital decay with the growth of the orbital eccentricity.

4. Discussion

We now consider how a massive BBH with a triple disk evolves through the binary-disk interaction. In the long-term, the BBH resonantly interacts with the CBD at the OLR radius, via which the orbital angular momentum is mostly extracted by the CBD. In the short-term, the gas is transferred from the CBD to each component of the BBH. This process is repeated every binary orbit during the evolution of the BBH, by which the accretion disk is formed around each black hole, and then each accretion disk viscously evolves. Some fraction of the angular momentum of each accretion disk is finally converted to the orbital angular momentum of the BBH due to the tidal torque of each black hole. When the mass-transfer rate corresponds to a critical value, no BBH evolves, because the angular-momentum exchange caused by these binary-disk interactions is balanced.

The direction of BBH evolution is determined by the critical mass-transfer rate. If the mass-transfer rate is smaller than the critical mass-transfer rate, the separation between two black holes decays with time, whereas the orbital eccentricity grows with time, and is finally close to 1.0. The grown-up eccentricity will then cause the black hole to plunge into the CBD, resulting in an enlargement of the gap between the central binary and the CBD (cf. Artymowicz & Lubow 1994). This direct interaction would make the semi-major axis shorten because of the friction between the black hole and the gas in the CBD.

What happens if the mass-transfer rate is large enough to be over the Eddington accretion rate? Since the mass-transfer rate is then larger than the critical mass-transfer rate, the semi-major axis rapidly increases with time, whereas the binary orbit become circularized with time. When the binary separation reaches the hardening radius, the growth of the semi-major axis will be again stalled by the dynamical friction with the neighboring stars. Thus, the BBH is long-lived in this case.

Where is the orbital angular momentum of the BBH finally going? The angular momentum absorbed by the CBD is
gradually transferred outward with the gas in the CBD. The gas around the BBH will, therefore, be dispersed if no gas is supplied to the CBD. However, if gas is supplied to the CBD during the BBH evolution, the outwardly transferred angular momentum will be removed by a supply source. This is essentially the same problem as how the gas is supplied to AGNs. Again, if the gas is supplied to the CBD at any rate, the CBD is likely to be a quasi-steady state. Such a state is, however, considered to be different from the steady state of the accretion disk around a single black hole. The orbital evolution of the BBH should therefore be solved with the evolution of the CBD. We will tackle this problem by a numerical simulation in a forthcoming paper, where both the full torque of the BBH and the evolution of the CBD will be taken into account.

Figure 6 shows the coalescent timescale of a massive BBH with $M_{bh} = 10^8 M_\odot$ and $M_{CBD}/M_{bh} = 10^{-2}$. Two black holes first evolve towards the shrinkage of the binary separation due to dynamical friction with the neighboring stars (Begelman et al. 1980). Next, the orbital evolution is stalled at $a_h$, where the evolutionary timescale by the dynamical friction is over a Hubble time, even if the emission of the gravitational wave dissipates the binding energy of the BBH. However, the BBH can track a new evolutionary path due to the binary-disk interaction. The new evolutionary path is denoted by the solid thick line of figure 6. In addition, the rapid growth of the orbital eccentricity allows connection to the final evolutionary path where the BBH evolves towards the coalescence by emitting gravitational radiation, since the timescale of the final evolutionary path in an eccentric binary is a factor of $(1-e^2)^{7/2}$ times shorter than that in a circular binary (Peters 1964). The combination of the orbital decay and the eccentricity growth makes it possible for the massive BBH to coalesce with a Hubble time.

One of the most interesting problems is how many massive BBHs are in the Universe. It is noted from figures 3 and 6 that the BBH more rapidly coalesces as the black-hole mass is less massive and the CBD is more massive. This suggests that a supermassive BBH with masses of more than $\sim 10^8-9 M_\odot$ is long-lived as a binary. Also, a relatively less-massive BBH with masses of $10^5-7 M_\odot$ would sequentially merge in the early Universe, and grow to a more massive, single black hole. The host galaxies with such grown black holes would suffer a major merger, resulting in a supermassive BBH formation. If so, we can infer, by using the $M$–$a$ relation, that such supermassive BBHs are observed in well-developed galaxies with gas-rich nuclei.

5. Conclusions

For the purpose of resolving the final parsec problem, we considered the evolution of a massive BBH with a triple disk on the subparsec scale. Our main conclusions are summarized as follows:

1. The orbital evolution of the BBH is characterized by $t_c \sim t_{vis,0} (M_{CBD}/M_{bh})^{-1}$.
2. There is a critical mass-transfer rate, which determines the evolutionary direction of the BBH with a triple disk. When the mass-transfer rate is less than the critical one, the semi-major axis decays with time, whereas the orbital eccentricity grows with time. The orbital eccentricity is rapidly close to 1.0 during $t_c/2$, even if the BBH is initially on a circular orbit, while the semi-major axis is 0.0 during $t_c$, regardless of the orbital eccentricity. The high value of the orbital eccentricity results in a rapid merger within the Hubble time through the emission of gravitational radiation.
3. When the mass-transfer rate is larger than the critical one, the semi-major axis grows with time, whereas the orbital eccentricity decays with time. In such a system, the semi-major axis reaches to the hardening radius of the BBH, typically $a_h \sim 1$ pc when $M_{bh} = 10^8 M_\odot$, and its orbit is completely circularized.
4. Since the minimum value of the critical mass-transfer rate is larger than the Eddington accretion rate of a massive black hole with mass in the $10^6 M_\odot$ to $10^9 M_\odot$ range, as long as the evolutionary timescale is shorter than the Hubble time, it is plausible that the mass-transfer rate is smaller than the critical one. Most BBH, therefore, prefer to evolve towards a rapid coalescence within the Hubble time.

K.H. thanks an anonymous referee for useful comments and suggestions. K.H. is grateful to Shin Mineshige, Atsuo T. Okazaki, Shigehiro Nagataki, Keitaro Takahashi, Yuji Senndouda, Norita Kawanaka, Ryouji Kawabata, Kohta Murase, Kiki Viderdayanty, and Junichi Aoi for helpful discussions, and Kanako Morimoto for the beautiful illustration. K.H. also thanks YITP in Kyoto University, where this work was extensively discussed during the YITP-W-05-11 on 2005 September 20–21, the YITP-W-06-20 on 2007 February 13–15, and the YITP-W-07-20 on January 8–11, 2008. The calculations reported here were performed using the facility at the Centre for Astrophysics & Supercomputing at Swinburne University of Technology, Australia and at YITP of Kyoto University. This work has been supported in part by the Grants-in-Aid of the Ministry of Education, Science, Culture, and Sport and Technology (MEXT; 30374218 K.H), and by the Grant-in-Aid for the 21st Century COE Scientific Research Programs on “Topological Science and Technology” and “Center for Diversity and Universality in Physics” from MEXT.

References

Armitage, P. J., & Natarajan, P. 2002, ApJ, 567, L9
Armitage, P. J., & Natarajan, P. 2005, ApJ, 634, 921
Artyomowicz, P., & Lubow, S. H. 1994, ApJ, 421, 651
Artyomowicz, P., Clarke, C. J., Lubow, S. H., Pringle, J. E. 1991, ApJ, 370, L35
Begelman, M. C., Blandford, R. D., & Rees, M. J. 1980, Nature, 287, 307
Di Matteo, T., Springel, V., & Hernquist, L. 2005, Nature, 433, 604
Ferrarese, L., & Merritt, D. 2000, ApJ, 539, L9
Gebhardt, K., et al. 2000, ApJ, 539, L13
Gebhardt, K., et al. 2002, ApJ, 567, L9
Gebhardt, K., et al. 2000, ApJ, 539, L13
Gebhardt, K., et al. 2000, ApJ, 539, L13
Goldreich, P., & Tremaine, S. 1979, ApJ, 233, 857
Gould, A., & Rix, H.-W. 2000, ApJ, 532, L29
Hayasaki, K., & Mineshige, S. 2008, in AIP Conf. Proc., 1016, Origin of matter and evolution of galaxies: The 10th International Symposium on Origin of Matter and Evolution of Galaxies: From the Dawn of Universe to the Formation of Solar System, ed. T. Suda, T. Nozawa, A. Ohnishi, K. Kato, M. Y. Fujimoto, T. Kajino, & S. Kubono (New York: AIP), 406
Hayasaki, K., Mineshige, S., & Ho, L. C. 2008, ApJ, 682, 1134
Hayasaki, K., Mineshige, S., & Sudou, H. 2007, PASJ, 59, 427
Ivanov, P. B., Papaloizou, J. C. B., & Polnarev, A. G. 1999, MNRAS, 307, 79
Iwasawa, M., Funato, Y., & Makino, J. 2006, ApJ, 651, 1059
Kato, S., Fukue, J., & Mineshige, S. 2008, Black-Hole Accretion Disks: Towards a New Paradigm, (Kyoto: Kyoto Univ. Press)
Landau, L. D., & Lifshitz, E. M. 1960, Mechanics (Oxford: Pergamon).
Lin, D. N. C., & Papaloizou, J. 1986, ApJ, 307, 395
Lubow, S. H., & Artymowicz, P. 1996, In Evolutionary Processes in Binary Stars, ed. R. A. M. Wijers, M. B. Davies & C. A. Tout (Dordrecht: Kluwer Academic Publishers), 53
Magorrian, J., et al. 1998, AJ, 115, 2285
Makino, J. 1997, ApJ, 478, 58
Makino, J., & Funato, Y. 2004, ApJ, 602, 93
Matsubayashi, T., Makino, J., & Ebisuzaki, T. 2007, ApJ, 656, 879
Mayer, L., Kazantzidis, S., Madau, P., Colpi, M., Quinn, T., & Wadsley, J. 2007, Science, 316, 1874
Merritt, D & Ekers, R. D. 2002, Science, 297, 1310
Merritt, D., Mikkola, S., & Szell, A. 2007, ApJ, 671, 53
Merritt, D., & Milosavljević, M. 2005, Living Rev. Relativity, 8, 8
Milosavljević, M., & Merritt, D. 2001, ApJ, 563, 34
Peters, P. C. 1964, Phys. Rev., 136, 1224
Quinlan, G. D. 1996, New Astron., 1, 35
Quinlan, G. D., & Hernquist, L. 1997, New Astron., 2, 533
Roos, N. 1981, A&A, 104, 218
Saslaw, W. C., Valtonen, M. J., & Aarseth, S. J. 1974, ApJ, 190, 253
Sesana, A., Haardt, F., & Madau, P. 2007, ApJ, 660, 546
Shakura, N. I., & Sunyaev, R. A. 1973, A&A, 24, 337
Toomre, A. 1964, ApJ, 139, 1217
Yu, Q. 2002, MNRAS, 331, 935
Yu, Q., & Tremaine, S. 2002, MNRAS, 335, 965