ON ADDITIVE REPRESENTATION FUNCTIONS

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ABSTRACT. Let $\mathcal{A} = \{a_1 < a_2 < a_3 \ldots < a_n < \ldots\}$ be an infinite sequence of integers and let $R_2(n) = |\{(i,j) : a_i + a_j = n, a_i, a_j \in \mathcal{A} \mid i \leq j\}|$. We define $S_k = \sum_{i=1}^{k} (R_2(2i) - R_2(2i - 1))$. We prove that, if $L^1$ norm of $S_k$ is small then $L^1$ norm of $\frac{S_k}{n}$ is large.

1. INTRODUCTION

Let $\mathcal{A} = \{a_1, a_2, \ldots\}$ be an infinite sequence of non-negative integers. Let $n \in \mathbb{N}_0$, denote the number of solutions of $a_i + a_j = n$, and $a_i + a_j = n \ (i \leq j)$ by $R_1(n)$ and $R_2(n)$, respectively. More precisely:

$$R_1(n) = \sum_{a_i + a_j = n} 1;$$
$$R_2(n) = \sum_{a_i + a_j = n \ i \leq j} 1;$$

Also define, $R_3(n) = \sum_{a_i + a_j = n \ i < j} 1$.

It is easy to check that if $\mathcal{A}$ is a full set or a complement of a finite set inside the set of natural numbers then all $R_1, R_2, R_3$ are monotonically increasing. Here we are interested in inverse problems. In other words how the monotonicity of one of the representation function affects the cardinality of the set $\mathcal{A}$. Erdős, Sárközy and Sós [8, 9] and the first author [2] studied the monotonicity properties of the functions $R_1$, $R_2$, $R_3$. It turns out that monotonicity of these three functions differs significantly.

Erdős, Sárközy and Sós [8] proved that $R_1(n)$ can be monotonically increasing from a certain point only in a trivial way:

**Theorem A.** If $R_1(n + 1) \geq R_1(n)$ for all large $n$, then $\mathbb{N} \setminus \mathcal{A}$ is a finite set.

The analogous conclusion is not true in case of $R_2$. If we define $\mathcal{A}(N) = |\mathcal{A} \cap [1, N]|$, then first author [2] proved that,

**Theorem B.** If $R_2(n + 1) \geq R_2(n)$ for all large $n$, then $\mathcal{A}(N) = N + O(\log N)$

That is to say, the complement set of $\mathcal{A}$ is of order $O(\log N)$ at most. The following result was also proved in [2]:

**Theorem C.** If $\mathcal{A}(N) = o\left(\frac{N}{\log N}\right)$, then the function $R_3(n)$ can not be eventually increasing.

Again Erdos, Sarkozy and Sos [8] proved another result related to $R_2$:

**Theorem D.** If $\lim_{n \to \infty} \frac{n - \mathcal{A}(n)}{n} = +\infty$, then we have,

$$\limsup_{n \to \infty} \left(\sum_{k=1}^{n} (R_2(2k) - R_2(2k + 1))\right) = \infty$$

The result is tight, in fact they gave an example of a sequence $\mathcal{A}$ where $n - \mathcal{A}(n) > c \log n$ (for large $n$ and fixed constant $c$) and $\limsup_{N \to \infty} \frac{\sum_{k=1}^{N} (R_2(2k) - R_2(2k + 1))}{N} = +\infty$.

In [2], Tang and Chen gave a quantative version of Theorem D. To state the theorem, let us define a few notations:

$$S(n) = \sum_{k \leq n} (R_2(2k) - R_2(2k + 1))$$
$$m(N) = N(\log N + \log \log N).$$



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Then $L^\infty$ norm of $S(n)$ is defined by

$$T(N) = \max_{n \leq N} S(n) = \max_{n \leq N} \sum_{k \leq n} (R_2(2k) - R_2(2k + 1)).$$

They proved that, when the ratio $\frac{T(N)}{\sigma(N)}$ is bounded above by a small enough fixed constant, then $T(N)$ and $\frac{N - A(N)}{\log N}$ satisfies a simple inequality. More precisely:

**Theorem E.** If

1. $T(N) < \frac{1}{36} \sigma(N)$, then,

2. $T(N) > \frac{1}{80e} \frac{N - \sigma(N)}{\log N} - \frac{11}{4} - \frac{N_1}{8}$

Where $N_1$ is a fixed positive integer, depending only on $\mathcal{A}$

It is easy to see that under the condition (1), Theorem E implies Theorem D

Now Set

$$S^+(n) = \max \{S(n), 0\}$$

and

$$T^+(N) = \max_{n \leq N} \{S^+(n)\}$$

**Note:** $T(N)$ and $T^+(N)$ are same unless all $\{S(n)\}_{n \leq N}$ are negative.

In this paper we again assume that $\frac{T(N)}{\sigma(N)}$ is bounded above and prove an improved inequality where we replace $L^\infty$ norm of $S(n)$ by $L^1$ norm of $\sum_{n \leq N} \frac{S(n)}{n}$. More precisely:

**Theorem 1.** Let $\mathcal{A}$ be an infinite sequence of positive integers and there exists $N_0$ such that $T(N) < \frac{1}{36} \sigma(N)$ for all $N \geq N_0$. Then there exists a constant $N_1 > 0$ such that

3. $\sum_{n=1}^{m(N)} \frac{S^+(n)}{n} > \frac{1}{10e} (N - \sigma(N)) - \frac{1}{7} \log N - c_1.$

for some constant $c_1$ depending on the first few elements of $\mathcal{A}$.

**Corollary 1.** If (3) holds then $T^+(N) > \frac{1}{11e} \frac{N - \sigma(N)}{\log N} - \frac{1}{7}$

So if at least one of $S(n)$ is non-negative, then $T^+(N)$ indeed equals to $T(N)$. In that case Corollary 1 gives Theorem E with a better constant. Corollary 1 also implies the following corollaries:

**Corollary 2.** If $\limsup_{N \to \infty} \frac{N - \sigma(N)}{\log N} = +\infty$, then

$$T(N) \geq \min \{\frac{1}{36} \sigma(N), \frac{1}{11e} \frac{N - \sigma(N)}{\log N}\}$$

for all large $N$.

**Corollary 3.** If $\limsup_{N \to \infty} \frac{N - \sigma(N)}{\log N} = +\infty$, then $\limsup_{n \to \infty} S_n = +\infty$.

**Remark 1.** In the proof, we assume that $N_1$, mentioned in Theorem 1 is sufficiently large. But as the reader can notice, one can take any $N_1$, which satisfies the following:

Define $N$ by: $\sigma(N) > 40$,
then $N_1$ can be the smallest integer $N$ satisfying

$$(0.98)^N < \left(\frac{40}{eN}\right)^N.$$
2. Notations and preliminary lemmas:

Set
\[ f(z) = \sum_{a \in \mathcal{A}} z^a, \text{ for } |z| < 1 \]

Then
\[ f(z)^2 = \sum_{n=1}^{\infty} R(n)z^n. \]

Let \( \chi_\mathcal{A} \) be the characteristic function of \( \mathcal{A} \), i.e.
\[ \chi(n) = \begin{cases} 1, & \text{if } n \in \mathcal{A} \\ 0, & \text{else} \end{cases} \]

For a positive real number \( Y \), define
\[ \psi(Y) = f(e^{-Y}) = \sum_{a \in \mathcal{A}} e^{-aY} \]
and
\[ g(N) = 1 + 4(1 - e^{-2Y}) \sum_{k=1}^{\infty} S_k e^{-2kY}. \]

Before proving the theorem we shall prove a few lemmas which we shall be using later.

**Lemma 1.** Let \( 0 < x \leq 1 \) be a real number. Then
\[ \sum_{n=0}^{\infty} 2^n x^{2^n} \leq \frac{x(1+x)}{1-x} \]

**Proof.** Note that
\[ 2^n x^{2^n} \leq 2 \sum_{n=1}^{\infty} \sum_{j=2^n}^{2^{n+1}-1} x^j. \]
Summing over \( n = 1 \) to \( \infty \),
\[ \sum_{n=1}^{\infty} 2^n x^{2^n} \leq 2 \sum_{j=2}^{\infty} x^j = \frac{2x^2}{1-x} \]
Adding \( x \) (corresponding to \( n = 0 \)) on both sides, we get the result. \( \square \)

Note that \( \psi : (0, \infty) \to \mathbb{R} \) be a continuous function, which is positive and increasing. It can be shown that if the complement of \( \mathcal{A} \) is finite then
\[ \psi(Y)^2 = (2Y + O(\frac{1}{Y})) \psi(Y) \]
and conversely.

We aim to prove that, even if \( \Psi(Y) \) satisfies a hypothesis, slightly weaker than (4), then some conclusion can be arrived about \( \Psi(Y) \).

Let \( f : \mathbb{R} \to [0, \infty) \) be an function. For any real number \( y \), and integer \( \alpha \geq 0 \), define \( F(y, \alpha) \) by recurrence, as follows:
\[ F(y; 0) = 0; \]
and
\[ F(y, \alpha + 1) = \frac{f(y 2^{\alpha+1}) + F(y, \alpha)}{2}. \]

In other words,
\[ F(y; \alpha) = \frac{f(2^\alpha y)}{2} + \frac{f(2^{\alpha-1} y)}{4} + \frac{f(2^{\alpha-2} y)}{8} + \ldots + \frac{f(2 y)}{2^\alpha}. \]

**Lemma 2.** If \( (\psi(Y))^2 \geq 2Y \exp(-f(Y)) \psi(Y^2) \) for all \( Y \geq N_0 \), then
\[ \psi(y 2^\alpha) \geq y 2^\alpha \exp(-F(y; \alpha))(\frac{\psi(y)}{y})^{\frac{1}{2^\alpha}} \]
for all integers \( \alpha \geq 0 \), and a fixed real number \( y \geq N_0 \)
Proof. For $\alpha = 0$, both sides are equal; For the general case by induction,

\[
(\psi(y^{2^{\alpha+1}}))^2 \geq 2y^{2^{\alpha+1}} \exp(-f(y^{2^{\alpha+1}}))\psi(y^{2^\alpha})
\]

\[
= y^{2^{\alpha+2}} \exp(-f(y^{2^{\alpha+1}}) - F(y; \alpha))(\psi(y))^{\frac{1}{y^{2^{\alpha+1}}}}
\]

\[
= (r^{2^\alpha+1} \exp(-F(y, \alpha + 1))(\psi(y))^{\frac{1}{y^{2^{\alpha+1}}}})^2
\]

and hence the result. \hfill \Box

We will see the implication of Lemma 1 and Lemma 2 in the proof of Theorem 1. Before doing so we need one more fact.

Intuitively it makes sense that if $T(Y)(= T([Y]))$ is 'small' then $g(Y)$ should also be 'small'. We will show that is the case in the next lemma:

**Lemma 3.** Let $g(Y)$ and $T(Y)$ are defined as above. Then

\[
g(Y) < T(Y) + 10
\]

for all $Y \geq 40$. Further if $T(N) \leq \frac{1}{36}\phi(N)$ for all integers $N \geq N_0$ then there exists $N_2 \geq N_0$ such that

\[
g(Y) \leq \psi\left(\frac{Y}{2}\right)
\]

for all $Y \geq N_2$.

**Proof.** First take $N = [Y]$. Then we have:

\[
(1 - e^{-\frac{\alpha}{Y}}) \sum_{k=1}^{\infty} S_k e^{-\frac{2k}{Y}} = \sum_{k=1}^{m(N)} (R_2(2k) - R_2(2k + 1))e^{-\frac{2k}{Y}}
\]

\[
= \sum_{k=1}^{m(N)} (R_2(2k) - R_2(2k + 1))e^{-\frac{2k}{Y}}
\]

\[
+ \sum_{k=m(N)}^{\infty} (R_2(2k) - R_2(2k + 1))e^{-\frac{2k}{Y}};
\]

\[
= \Sigma_1 + \Sigma_2; \text{ Say;}
\]

Now

\[
\Sigma_1 = \sum_{k=1}^{m(N)} (R_2(2k) - R_2(2k + 1))e^{-\frac{2k}{Y}}
\]

\[
= (1 - e^{-\frac{\alpha}{Y}}) \sum_{k=1}^{m(N)-1} S_k e^{-\frac{2k}{Y}} + S_{m(N)}e^{-\frac{m(N)}{Y}};
\]

\[
\leq \max_{k \leq m(N)} S_k + 2;
\]

\[
= T(N) + 2.
\]

Also

\[
\Sigma_2 = \sum_{k=m(N)+1}^{\infty} (R_2(2k) - R_2(2k + 1)) e^{-\frac{2k}{Y}}
\]

Using $|R_2(2k) - R_2(2k + 1)| < k$ and $m(N) = N(\log N + \log \log N)$, and also considering the fact that $h_1(x) = xe^{-\frac{2x}{Y}}$ is a decreasing function of $x \geq N$ we get,

\[
\Sigma_2 \leq \int_{m(N)}^{\infty} xe^{-\frac{2x}{Y}} dx;
\]

\[
< \frac{3}{4 \log N} \quad \text{if } N > 80.
\]
So
\[ g(N) = 1 + 4(\Sigma_1 + \Sigma_2); \]
\[ < 1 + 4(T(N) + \frac{3}{4\log N} + 2) \quad \text{for } N \geq N_0; \]
\[ = 4T(N) + 9 + \frac{3}{\log N}. \]
Which proves the first part of the lemma since \( g \) is an increasing function.
To prove the second part note that
\[ g(Y) \leq \frac{1}{9} \mathcal{A}(Y) + 10 \]
for \( N \geq 100. \) Then the fact that
\[ \Psi(Y) > \sum_{a \in \mathcal{A}} e^{-\frac{a}{Y}} > e^{-2} \sum_{a \in \mathcal{A}} 1 = \frac{1}{e^2} \mathcal{A}(Y), \]
proves the result. \( \square \)

3. **Proof of Theorem**

We observe, by comparing the coefficients of \( \alpha^n \) from both sides, that
\[ f(\alpha^2) = \frac{1 - \alpha}{2\alpha} (f(\alpha))^2 + 2 \sum_{k=1}^{\infty} (R_2(2k) - R_2(2k + 1)) \alpha^{2k} - \frac{1 + \alpha}{2\alpha} f(-\alpha)^2. \]
Since \( \alpha > 0 \) this gives,
\[ f(\alpha^2) \leq \frac{1 - \alpha}{2\alpha} f(\alpha)^2 + 2 \sum_{k=1}^{\infty} (R_2(2k) - R_2(2k + 1)) \alpha^{2k} \quad (5) \]
Now considering the right hand side of the summation, we get:
\[ \sum_{k=1}^{\infty} (R_2(2k) - R_2(2k + 1)) \alpha^{2k} = \sum_{k=1}^{\infty} (S_k - S_{k-1}) \alpha^{2k} \]
\[ = \sum_{k=1}^{\infty} S_k (\alpha^{2k} - \alpha^{2k+1}) \]
\[ = (1 - \alpha^2) \sum_{k=1}^{\infty} S_k \alpha^{2k}. \]
Thus we get, from (5),
\[ f(\alpha^2) \leq \frac{1 - \alpha}{2\alpha} f(\alpha)^2 + 2(1 - \alpha^2) \sum_{k=1}^{\infty} S_k \alpha^{2k} \quad (6) \]
Now putting \( \alpha = e^{\frac{-1}{Y}} \), we get
\[ \psi(Y) \leq \frac{1}{2} (\psi(\frac{Y}{2}) + \frac{1}{Y}) (\psi(Y))^2 + 2(1 - e^{-2}) \sum_{k=1}^{\infty} S_k e^{-\frac{k}{Y}}. \]
Since \( \psi(Y) \leq Y \), this gives
\[ 2Y \psi(Y) \leq (\psi(Y))^2 + Y g(Y) \quad (7) \]
Thus
\[ (\psi(Y))^2 \geq 2Y \psi(Y) - Y g(Y). \quad (8) \]
Now using the second part of lemma 3, we get
\[ \psi(Y)^2 \geq Y \psi(Y) \quad \forall Y \geq N_0 \]
Thus, \( \psi \) satisfies lemma 2, with \( f(Y) = \log 2. \) Thus
\[ F(y; \alpha) = (\log 2)(1 - \frac{1}{2\alpha + 1}) \]
Thus,  
\[ \psi(y_0 2^\alpha) \geq \frac{y_0}{2} \frac{1}{2^{\frac{1}{2^\alpha}}} \left( \frac{\psi(y_0)}{y_0} \right)^{\frac{1}{2^\alpha}} \]  
for \( N_0 \leq y_0 < 2N_0 \)

This gives, since \( \psi(N_0) \geq 1 \), and if \( \alpha \) is sufficiently large, say \( \alpha \geq \alpha_0 \),

\[ \psi(y_0 2^\alpha) \geq (0.49) y_0 2^\alpha \]

for \( y_0 \geq N_0 \). Thus choosing \( \alpha \) suitably and defining \( y_0 = \frac{y}{N_0} \) so that \( N_0 \leq y_0 < 2N_0 \), we get

\[ \psi(Y) \geq (0.49) Y \]  
for all \( Y \geq N_1 (= N_0 2^\alpha) \).

Thus, from (8)

\[ \psi(Y)^2 \geq 2Y \psi(Y) \left( 1 - \frac{g(Y)}{0.49Y} \right). \]

for all \( Y \geq N_1 \). Since \( \frac{g(Y)}{Y} < \frac{1}{\alpha} \), lemma[2] is satisfied with \( f(Y) = 2.3 \frac{g(Y)}{Y} \)

Then

\[ F(y; \alpha) = \frac{2.3}{2^{\alpha+1}Y} (g(2^\alpha y) + g(2^{\alpha-1} y) + \ldots + g(2y)) \]

This gives

\[ \psi(y_1 2^\alpha) \geq y_1 2^\alpha \exp(-F(y_1; \alpha)) \left( \frac{\psi(y_1)}{y_1} \right)^{\frac{1}{2^\alpha}}; \]

for \( N_1 \leq y_1 < 2N_1 \).

Note that in this case

\[
g(2^\alpha y_1) + g(2^{\alpha-1} y_1) + \ldots + g(2y) = \alpha + \frac{8}{2^\alpha y_1} \sum_{k=1}^{\infty} S^+_k \sum_{n=0}^{\alpha} \frac{1}{\alpha} \frac{2^n e^{-\frac{n+1}{n+2}}}{1 - e^{-\frac{2}{2^\alpha y_1}}} \]

\[
\leq \alpha + \frac{8}{2^\alpha y_1} \sum_{k=1}^{\infty} S^+_k \frac{2^\alpha y_1}{k}; \quad \text{by Lemma[1]}
\]

That implies for large enough \( \alpha \) and fixed \( y_1 \) with \( N_1 \leq y_1 < 2N_1 \),

\[
\frac{\psi(y_1 2^\alpha)}{y_1 2^\alpha} \geq \exp(-F(y_1; \alpha) + \frac{c}{y_1 2^\alpha})
\]

for some constant \( c \) depending on \( \mathcal{A} \). For example, you can choose \( c = \inf_{N_1 \leq y < 2N_1} y \log \left( \frac{\psi(y)}{y} \right) \).

Taking logarithm on both sides

\[
F(y_1; \alpha) = \frac{c}{y_1 2^\alpha} > (1 - \frac{\psi(y_1 2^\alpha)}{y_1 2^\alpha}),
\]

Or,

\[
\frac{2.3}{2^{\alpha+1}y_1} (\alpha + \frac{8}{2^\alpha y_1} \sum_{k=1}^{\infty} S^+_k) - \frac{c}{y_1 2^\alpha} \geq \frac{1}{\alpha} \left( y_1 2^\alpha - \mathcal{A}(y_1 2^\alpha) \right).
\]

As before choosing \( \alpha \) suitably, so that \( N_1 \leq y_1 (= \frac{N}{2^\alpha}) < 2N_1 \), it implies that

\[
\sum_{k=1}^{\infty} \frac{S^+_k}{k} > \frac{1}{10e} (N - \mathcal{A}(N)) - \frac{1}{7} \log N - c_1
\]

for large enough \( N \) and fixed constant \( c_1 \) depending on \( \mathcal{A} \). Which proves Theorem[1]
Remark 2. Also we solved a question raised by Sárközy (see [10], Problem 5, Page 337): Does there exist a set \( \mathcal{A} \subseteq \mathbb{N} \) such that \( \mathcal{A} \) is an infinite set and \( R_1(n + 1) \geq R_1(n) \) holds on a sequence of integers \( n \) whose density is 1? Here we show that the answer to this question is positive by giving a simple example.

Let \( \mathbb{B} \) be an infinite Sidon set of even integers and \( \mathcal{A} = \mathbb{N} \setminus \mathbb{B} \);

Put

\[
Y = (\mathbb{B} + \mathbb{B}) \cup \mathbb{B} \quad \text{and} \quad X = \mathbb{N} \setminus Y;
\]

Then,

\[
R_1(n + 1) \geq R_1(n) \quad \text{for all} \ n \in X.
\]

To see this, let

\[
f(z) = \sum_{a \in \mathcal{A}} z^a \quad \text{and} \quad g(z) = \sum_{b \in \mathbb{B}} z^b,
\]

Then

\[
\sum_{n=1}^{\infty} (R_1(n) - R_1(n-1)) z^n = (1 - z) f(z)^2 = (1 - z) (z - g(z))^2
\]

\[
= \frac{z^2}{(1 - z)} + (1 - z) (g(z))^2 - 2zg(z)
\]

let

\[
r_1(n) = \sum_{\substack{b_1 = b_2 = \ldots = 1, \ \ b_i \in \mathbb{A}, b_j \in \mathbb{B}}} 1
\]

So \( R_1(n + 1) \geq R_1(n) \) iff coefficient of \( z^{n+1} \) in \( (1 - z) (f(z))^2 \) is non negative.

Now coefficient of \( z^{2k} \) is

\[
1 + r_1(2k) - r_1(2k - 1) - 2X_{\mathbb{B}}(2k - 1)
\]

and coefficient of \( z^{2k+1} \) is

\[
1 + r_1(2k + 1) - r_1(2k) - 2X_{\mathbb{B}}(2k)
\]

Then it is clear from the above choice of \( X \) and \( \mathcal{A} \) that, \( R_1(n + 1) \geq R_1(n) \) for all \( n \) in \( X \).

For example we can take \( \mathbb{B} = \{2, 4, 8, 16, 32, \ldots, 2^m, \ldots\} \). Then \( \mathbb{B} \) is infinite and \( X \) is of density 1.

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ABSTRACT. Let \( \mathcal{A} = \{a_1 < a_2 < a_3 \ldots < a_r < \cdots \} \) be an infinite sequence of non-negative integers and let \( R_2(n) = |\{(i,j): a_i + a_j = n, a_i, a_j \in \mathcal{A}\}| \). We define \( S_k = \sum_{i=1}^{k} (R_2(2i) - R_2(2i+1)) \). We prove that if \( L^\infty \) norm of \( S_k' (= \max\{S_k,0\}) \) is small, then \( L^1 \) norm of \( S_k' \) is large.

1. INTRODUCTION

Let \( \mathcal{A} = \{a_1, a_2, \cdots \} (0 \leq a_1 < a_2 < \cdots) \) be an infinite sequence of non-negative integers. For \( n \in \mathbb{N}_0 \), Define

\[
R_1(n) = R_1(\mathcal{A}, n) = \sum_{a_i + a_j = n} 1
\]

\[
R_2(n) = R_2(\mathcal{A}, n) = \sum_{a_i + a_j = n} 1.
\]

Now, it is easy to check that if \( \mathcal{A} \) is a full set or a complement of a finite set inside the set of natural numbers, then \( R_1 \) and \( R_2 \) are monotonically increasing. Here we are interested in inverse problems, i.e., how the monotonicity of the representation functions affects the cardinality of the set \( \mathcal{A} \).

The question of characterisation of the set \( \mathcal{A} \), under the condition that either \( R_1(n) \) or \( R_2(n) \) is monotonic, was raised by Erdős, Sárközy and Sós [8]. Also see [9] and [2].

In [3], Erdős, Sárközy and Sós proved that \( R_1(n) \) can be monotonically increasing from a certain point, only in a trivial way. See [8] and [2] for the following theorem.

Theorem A. If \( R_1(n+1) \geq R_1(n) \) for all large \( n \), then \( \mathbb{N} \setminus \mathcal{A} \) is a finite set.

While analogous conclusion is not true in case of \( R_2 \), if, we define

\[
\mathcal{A}(N) = |\mathcal{A} \cap [1,N]|,
\]

then the first author [2] proved the following theorem:

Theorem B. If \( R_2(n+1) \geq R_2(n) \) for all large \( n \), then \( \mathcal{A}(N) = N + O(\log N) \).

In other words, If \( R_2(n) \) is monotonic, then the complement set of \( \mathcal{A} \) is almost of order \( O(\log N) \).

In the first part of this paper we shall focus on the function \( R_2 \) and quantities related to monotonicity of it. Also in Section 6, we shall make a remark concerning a question raised by Sárközy [10], related to monotonicity of \( R_1 \).

In [9] Erdős, Sárközy and Sós proved

Theorem C. If

\[
\lim_{n \to +\infty} \frac{n - \mathcal{A}(n)}{\log n} = +\infty,
\]

then we have,

\[
\limsup_{N \to +\infty} \sum_{k=1}^{N} (R_2(2k) - R_2(2k+1)) = +\infty.
\]

The assumption \( 4 \) in the above theorem can not be relaxed. In fact Erdős, Sárközy and Sós [9] constructed a sequence \( \mathcal{A} \) where \( (n - \mathcal{A}(n)) > c \log n \) (for large \( n \) and fixed constant \( c \)) and

\[
\limsup_{N \to +\infty} \sum_{k=1}^{N} (R_2(2k) - R_2(2k+1)) < +\infty.
\]
In [4], Tang and Chen gave a quantitative version of Theorem C. Before we state their theorem, let us define a few notations.

\[ S_n = \sum_{k \leq n} (R_2(2k) - R_2(2k + 1)), \]
\[ m(N) = N(\log N + \log \log N). \]

Also \( L^\infty \) norm of \( S_n \), denoted by \( T(N) \), is defined as follows:

\[ T(N) = \max_{n \leq m(N)} S_n = \max_{n \leq m(N)} \sum_{k \leq n} (R_2(2k) - R_2(2k + 1)). \]

In [4] the authors proved that, when the ratio \( \frac{T(N)}{A(N)} \) is bounded above by a small enough fixed constant, then \( T(N) \) and \( \frac{N - A(N)}{\log N} \) satisfies a simple inequality. More precisely,

**Theorem D.** If \( T(N) \) be defined as in (6) and \( \frac{T(N)}{A(N)} < \frac{1}{36} \) for all large enough \( N \), then there exists a constant \( c > 0 \), depending only on \( A \), such that

\[ T(N) > \frac{1}{80e} \frac{N - A(N)}{\log N} - \frac{11}{4} - \frac{C}{8}. \]

It is easy to see that, under the condition (7), Theorem D implies Theorem C.

Now, set

\[ S_+^n = \max \{ S_n, 0 \}, \]
and

\[ T^+(N) = \max_{n \leq m(N)} \{ S_+^n \}. \]

**Note:** \( T(N) \) and \( T^+(N) \) are same unless all the elements of the set \( \{ S_n : n \leq m(N) \} \) are negative.

In this paper, we again assume that \( \frac{T(N)}{A(N)} \) is bounded above and prove an improved version of (8) where we replace \( L^\infty \) norm of \( S(n) \) by \( L^1 \) norm of \( S_+^n \). More precisely, we prove the following theorem:

**Theorem 1.** Let \( \mathcal{A} \) be an infinite sequence of positive integers and there exists \( N_0 \) such that \( T(N) < \frac{1}{36} \mathcal{A}(N) \) for all \( N \geq N_0 \). Then there exists a constant \( c_1 > 0 \), depending on \( \mathcal{A} \), such that

\[ \sum_{n=1}^{m(N)} \frac{S_+^n}{n} > \frac{1}{10e} (N - \mathcal{A}(N)) - \frac{1}{4} \log N - c_1. \]

for all large enough \( N \).

**Corollary 1.** If (7) in Theorem D holds, then for any \( \varepsilon > 0 \),

\[ T^+(N) > \frac{1}{10e + \varepsilon} \frac{N - \mathcal{A}(N)}{\log N} - \frac{1}{4}, \]

for any large enough \( N \).

So, if at least one of \( S(n) \) is non-negative, then \( T^+(N) \) indeed equals \( T(N) \). In that case, Corollary 1 gives Theorem D with a better constant. Corollary 1 also implies the following:

**Corollary 2.** If \( \limsup_{N \to +\infty} \frac{N - \mathcal{A}(N)}{\log N} = +\infty \), then

\[ \limsup_{n \to +\infty} \{ S_n \} = +\infty. \]
2. Generating Functions:

It is more natural to consider the problem in terms of generating function. Set
\[ f(z) = \sum_{a \in A} a z^a, \quad |z| < 1. \]

Then,
\[ f(z)^2 = \sum_{n=1}^{+\infty} R(n) z^n. \]

For any positive real number \( Y \), define
\[ \psi(Y) = f(e^{-\frac{Y}{2}}) = \sum_{a \in A} e^{-\frac{aY}{2}}, \tag{12} \]
and
\[ g(Y) = 1 + 4(1 - e^{-\frac{Y}{2}}) \sum_{k=1}^{+\infty} S_k e^{-\frac{2kY}{2}}. \tag{13} \]

**Theorem 2.** Let \( g(Y) \) and \( \psi(Y) \) be defined as above. Also assume
\[ g(Y) \leq \min\{\psi(Y), 1 - e^{-\frac{Y}{2}}\} \tag{14} \]
for all sufficiently large positive real number \( Y \). Then
\[ \psi(Y) \geq Y \exp \left( -\frac{3}{2Y} \left( \log_2 Y + \frac{16}{Y} \sum_{k=1}^{+\infty} S_k e^{-\frac{2kY}{2}} \right) - \frac{c}{Y} \right) \tag{15} \]
for some positive constant \( c \) depending only on first few elements of \( A \).

In Section 3, we will give a proof of Theorem 2. In Section 4, we will show how Theorem 2 follows from Theorem 2.

3. Notations and preliminary lemmas:

Consider a function \( h : \mathbb{R} \mapsto [0, +\infty) \). For any real number \( Y \) and integer \( \alpha \geq 0 \) define \( H(Y; \alpha) \) by recurrence, as follows:
\[ H(Y; 0) = 0 \]
\[ H(Y; \alpha) = \frac{h(Y)}{2} + \frac{h(Y/2)}{4} + \frac{h(Y/4)}{8} + \cdots + \frac{h(Y/2^{\alpha-1})}{2^\alpha} \]
\[ = \sum_{j=0}^{\alpha-1} \frac{1}{2^{j+1}} h\left( \frac{Y}{2^j} \right) \quad \text{for integer } \alpha \geq 1. \tag{15} \]

Also
\[ H(Y) = \sum_{j=0}^{+\infty} \frac{1}{2^{j+1}} h\left( \frac{Y}{2^j} \right). \tag{16} \]

**Lemma 1.** If \( h(Y) \) and \( H(Y; \alpha) \) is defined as above and
\[ (\psi(Y))^2 \geq 2Y \exp(-h(Y)) \psi(Y) \tag{17} \]
for all real number \( Y \geq \tilde{N}_0 \), then for every integer \( \alpha \geq 0 \),
\[ \psi(Y) \geq Y \exp(-H(Y; \alpha)) \left( \psi\left( \frac{Y}{2} \right) \right)^{2^\alpha} \tag{18} \]
for any real number \( Y \geq 2^\alpha \tilde{N}_0 \).
Proof. For \( \alpha = 0 \), both sides are equal.
For the general case, suppose it is true for \( \alpha = \alpha_0 \). Then
\[
(\psi(Y))^2 \geq 2Y \exp(-h(Y))\psi\left(\frac{Y}{2}\right)
\]
\[
= Y^2 \exp\left(-h(Y) - H\left(\frac{Y}{2}, \alpha_0\right)\right) \left(\frac{\psi\left(\frac{Y}{2^{\alpha_0+1}}\right)}{Y}\right)^{\frac{1}{\alpha_0}}
\]
for \( Y \geq 2N_1 \)
\[
= \left(Y \exp(-H(Y, \alpha + 1)) \left(\frac{\psi\left(\frac{Y}{2^{\alpha_0+1}}\right)}{Y}\right)^{\frac{1}{\alpha_0}}\right)^2
\]
and hence the result. \( \square \)

Lemma 2. There exist a \( c > 0 \) such that, if \( Y \) is large enough, then we have
\[
\left(\frac{\psi\left(\frac{Y}{2^\alpha}\right)^{2\alpha}}{Y}\right)^{\frac{1}{2\alpha}} \geq \exp\left(-\frac{c}{Y}\right)
\]
for some \( \alpha \leq \log_2 Y \).

Proof. Now fix an interval \([a, 2a]\) so that \( \psi(a) \geq 1 \).
Then choose \( \alpha \) suitably so that \( \frac{Y}{2^\alpha} \in [a, 2a] \). In that case, we have
\[
\left(\frac{\psi\left(\frac{Y}{2^\alpha}\right)^{2\alpha}}{Y}\right)^{\frac{1}{2\alpha}} \geq \left(\frac{1}{2a}\right)^{2a}.
\]
This proves the lemma. \( \square \)

Lemma 3. Let \( 0 < x < 1 \) be a real number. Then
\[
\sum_{n=0}^{+\infty} 2^n x^{2^n} \leq \frac{2x}{1-x}.
\]

Proof. Note that
\[
2^n x^{2^n} \leq 2 \sum_{2^n-1 < j \leq 2^n} x^j.
\]
Summing over \( n = 1 \) to \( +\infty \),
\[
\sum_{n=1}^{+\infty} 2^n x^{2^n} \leq 2 \sum_{j=2}^{+\infty} x^j = \frac{2x^2}{1-x}
\]
Adding \( x \) (corresponding to \( n = 0 \)) on both sides, we get the result. \( \square \)

Lemma 4. In the notation of Lemma 1, let \( h(Y) = d\left(\frac{\psi(Y)}{Y}\right) \) for some fixed positive constant \( d \), to be chosen later. Then
\[
H(Y, \alpha) \leq \frac{d}{2Y} \left(\alpha + \frac{8}{Y} \sum_{k=1}^{+\infty} \frac{S_k^+ e^{-Y^{2k+1}}}{1 - e^{-Y^{2k+1}}}\right)
\]

Proof. set \( x = e^{-Y^{2k+1}} \).
\[
g(Y) = 1 + 4(1 - e^{-Y^{2k+1}}) \sum_{k=1}^{+\infty} S_k^+ x^k.
\]
\[
\leq 1 + \frac{8}{Y} \sum_{k=1}^{+\infty} S_k^+ x^k.
\]
Then
\[ H(Y; \alpha) = \sum_{j=0}^{\alpha-1} \frac{1}{2j+1} h \left( \frac{Y}{2j} \right) \]
\[ \leq \sum_{j=0}^{\alpha-1} \frac{d}{2Y} \left( 1 + \frac{8}{Y} \sum_{k=1}^{\infty} S_k^+ 2^j x^{2j} \right) \]
\[ \leq \frac{d}{2Y} \left( \alpha + \frac{8}{Y} \sum_{k=1}^{\infty} S_k^+ \sum_{j=0}^{\infty} 2^j x^{2j} \right) \]
\[ = \frac{d}{2Y} \left( \alpha + \frac{8}{Y} \sum_{k=1}^{\infty} S_k^+ \frac{2^k}{1-x^2} \right). \]

\[ \square \]

4. PROOF OF THEOREM 2:

It is easy to verify the following inequality by comparing the coefficients of \( z^n \) from both sides.

\[ f(z^2) = \frac{1-z}{2z}(f(z))^2 + 2 \sum_{k=1}^{\infty} (R_2(2k) - R_2(2k+1))z^{2k} - \frac{(1+z)}{2z} f(-z)^2. \]

If \( z > 0 \), this gives,

\[ f(z^2) \leq \frac{1-z}{2z} f(z)^2 + 2 \sum_{k=1}^{\infty} (R_2(2k) - R_2(2k+1))z^{2k}. \]

Now, considering the right hand side of the summation, we get

\[ \sum_{k=1}^{\infty} (R_2(2k) - R_2(2k+1))z^{2k} = \sum_{k=1}^{\infty} (S_k - S_{k-1})z^{2k} \]
\[ = \sum_{k=1}^{\infty} S_k (z^{2k} - z^{2k+2}) - S_0 z^2 \]
\[ \leq (1-z^2) \sum_{k=1}^{\infty} S_k z^{2k}. \]

Thus, from (20) we get

\[ f(z^2) \leq \frac{1-z}{2z} f(z)^2 + 2(1-z^2) \sum_{k=1}^{\infty} S_k z^{2k}. \]

Now putting \( z = e^{-1} \), we get

\[ \psi \left( \frac{Y}{2} \right) \leq \frac{1}{2} \left( \frac{1}{Y} + \frac{1}{Y^2} \right) (\psi(Y))^2 + 2(1-e^{-2}) \sum_{k=1}^{\infty} S_k e^{-2k}. \]

Since \( \psi(Y) \leq Y \), this gives

\[ 2Y \psi \left( \frac{Y}{2} \right) \leq (\psi(Y))^2 + Yg(Y). \]

Thus,

\[ (\psi(Y))^2 \geq 2Y \psi \left( \frac{Y}{2} \right) - Yg(Y). \]

Lemma 5. If \( g(Y) \leq \psi \left( \frac{Y}{2} \right) \), then for all large enough real number \( Y \)
\[ \psi(Y) \geq 0.49Y. \]

Proof. Since \( g(Y) \leq \psi \left( \frac{Y}{2} \right) \), using (21) we get

\[ (\psi(Y))^2 \geq Y \psi \left( \frac{Y}{2} \right). \]
Then (17) in Lemma 7 holds with \( h(Y) = \log 2 \).
In that case
\[
H(Y) = \sum_{0 \leq j \leq \infty} \frac{1}{2^j} h\left( \frac{Y}{2^j} \right) = \log 2.
\]

This gives, by Lemma 1 and Lemma 2
\[
\psi(Y) \geq (0.49)Y
\]
if \( Y \) is large enough.

Thus, combining (21) and Lemma 5 we get
\[
\psi(Y) \geq Y \left( \frac{Y}{2} \right) \left( 1 - \frac{g(Y)}{0.49Y} \right)
\]
for sufficiently large \( Y \).
Since \( \frac{g(Y)}{Y} < \frac{1}{16} \), equation (17) in Lemma 7 is satisfied with \( h(Y) = 2.3 \frac{g(Y)}{Y} \).
Hence Lemma 7 and Lemma 7 together give the following:
\[
\psi(Y) \geq Y \exp \left( -\frac{2.3}{2Y} \left( \alpha + \frac{16}{Y} \sum_{k=1}^{\infty} S_k \left( \frac{e^{-2kN}}{1 - e^{-2N}} \right) \right) \right) \left( \frac{\psi \left( \frac{Y}{2} \right) 2^{\alpha}}{Y} \right)^{\frac{1}{2}}.
\]
Hence Theorem 2 follows from (23) and Lemma 2.

5. PROOF OF THEOREM 1

Lemma 6. Let \( g(N) \) and \( T(N) \) are defined as in (13) and (6). Then
(a) \( g(N) < 4T(N) + 40 \)
for all real number \( N \geq 40 \).
(b) Further, if \( T(N) \leq \frac{1}{16} \varphi(N) \) for all integers \( N \geq N_0 \), then there exists \( N_2 \geq N_0 \) such that
\[
g(N) \leq \psi \left( \frac{N}{2} \right)
\]
for all real number \( N \geq N_2 \).

Proof. We know \( S_k \leq T(N) \) for \( k \leq m(N) \) and \( S_k \leq \frac{k^2}{2} \) for \( k \geq m(N) \).
Now
\[
g(N) = 1 + 4(1 - e^{-2N}) \left\{ \sum_{k \leq m(N)} S_k e^{-\frac{2k}{N}} + \sum_{k > m(N)} S_k e^{-\frac{2k}{N}} \right\}
= 1 + 4(1 - e^{-2N}) \left\{ \Sigma_3 + \Sigma_4 \right\}, \quad \text{say.}
\]
\[
\Sigma_3 \leq \sum_{k=0}^{\infty} T(N) e^{-\frac{2k}{N}} = T(N) \frac{1}{1 - e^{-2N}}.
\]
\[
\Sigma_4 \leq \sum_{k>m(N)} \frac{k^2}{2} e^{-\frac{2k}{N}} \leq \int_{m(N)}^{+\infty} \frac{x^2}{2} e^{-\frac{2x}{N}} dx \leq 10
\]
and hence (a). To prove (b) note that
\[
g(N) \leq \frac{1}{9} \varphi(N) + 10
\]
for \( N \geq 100 \). Then, the fact that
\[
\Psi \left( \frac{N}{2} \right) > \sum_{a \in \varphi} e^{-\frac{a}{N}} > e^{-2} \sum_{a \in \varphi} 1 = \frac{1}{e^2} \varphi(N)
\]
proves the result.
Lemma 4 shows condition (14) of Theorem 2 are satisfied. Hence

\[ \frac{\psi(N)}{N} \geq \exp \left( \frac{2.3}{2N} \left( \log_2 N + \frac{16}{N} \sum_{k=1}^{\infty} S_k^+ \frac{e^{-2k}}{1 - e^{-2k}} \right) - \frac{c}{N} \right) \]

for some constant \( c \) depending on \( \mathcal{A} \).

Taking logarithm on both sides,

\[ \frac{2.3}{2N} \left( \log_2 N + \frac{16}{N} \sum_{k=1}^{\infty} S_k^+ \frac{e^{-2k}}{1 - e^{-2k}} \right) + \frac{c}{N} > \log \left( 1 - \frac{\psi(N)}{N} \right) > \left( 1 - \frac{\psi(N)}{N} \right)^{\frac{1}{N}}. \]

Or,

\[ \frac{2.3}{2N} \left( \log_2 N + \frac{16}{N} \sum_{k=1}^{m(N)} S_k^+ \frac{e^{-2k}}{1 - e^{-2k}} \right) + 10 + \frac{c}{N} \geq \frac{1}{e} \left( \frac{N - \mathcal{A}(N)}{N} \right). \]

Now, \( \frac{e^{-t}}{1 - e^{-t}} \leq \frac{1}{t} \) and hence we can replace \( \frac{e^{-2k}}{1 - e^{-2k}} \) by \( \frac{2k}{N} \).

\[ \frac{2.3}{2N} \left( \log_2 N + \frac{16}{N} \sum_{k=1}^{\infty} S_k^+ \frac{2k}{k} + 10 \right) + \frac{c}{N} \geq \frac{1}{e} \left( \frac{N - \mathcal{A}(N)}{N} \right). \]

It implies that

\[ \frac{1}{10e} (N - \mathcal{A}(N)) > \frac{1}{8} \log_2 N - e. \]

for large enough \( N \) and fixed constant \( e \) depending on \( \mathcal{A} \). This proves Theorem 7.

6. Monotonicity of \( R_1(n) \) on dense set of integers

Remark 1. Also, we solved a question raised by Sárközy (see [10]). [Problem 5, Page 337]. His question was the following:

Does there exist an infinite set \( \mathcal{A} \subset \mathbb{N} \) such that \( \mathbb{N} \setminus \mathcal{A} \) is also an infinite and \( R_1(n + 1) \geq R_1(n) \) holds on a sequence of integers \( n \) whose density is 1?

Here we show that the answer to this question is positive by giving a simple example.

A Sidon set is a set of positive integers such that the sums of any two terms are all different, i.e., \( R_2(n) \leq 1 \) for the corresponding \( R_2 \) function. By [11], it is possible to construct sidon sequence of order \( (n \log n)^{\frac{1}{2}} \).

Now, let \( \mathcal{B} \) be an infinite Sidon set of even integers and \( \mathcal{A} = \mathbb{N} \setminus \mathcal{B} \);

Put \( Y = (\mathcal{B} \cup \mathcal{B}) \cup \mathcal{B} \) and \( X = \mathbb{N} \setminus Y \);

Then,

\[ R_1(n + 1) \geq R_1(n) \quad \text{for all } n \in X. \]

To see this, let

\[ f(z) = \sum_{a \in \mathcal{A}} z^a \quad \text{and } g(z) = \sum_{b \in \mathcal{B}} z^b. \]

Then,

\[ \sum_{n=1}^{\infty} (R_1(n) - R_1(n-1)) z^n = (1 - z)f(z)^2 \]

\[ = (1 - z) \left( \frac{z}{1 - z} - g(z) \right)^2 \]

\[ = \frac{z^2}{(1 - z)} + (1 - z)(g(z))^2 - 2zg(z). \]

Again, let

\[ r_1(n) = \sum_{b_i = b_j = n} 1, \quad b_i \in \mathcal{B}, b_j \in \mathcal{B}. \]
So, $R_1(n + 1) \geq R_1(n)$ iff coefficient of $z^{n+1}$ in $(1 - z)(f(z))^2$ is non negative.

Now coefficient of $z^{2k}$ is $1 + r_1(2k) - r_1(2k - 1) - 2\chi_{A}(2k - 1)$

and coefficient of $z^{2k+1}$ is $1 + r_1(2k + 1) - r_1(2k) - 2\chi_{A}(2k)$

Then, it is clear from the above choice of $X$ and $A$ that $R_1(n + 1) \geq R_1(n)$ for all $n$ in $X$.

For example, we can take $\mathcal B = \{2, 4, 8, 16, 32, \ldots, 2^m, \ldots\}$. Then $\mathcal B$ is infinite and $X$ is of density 1.

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