On Generalized Edge Corona Product of Graphs

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December 14, 2017

Abstract

Let $G$ be a simple graph with $m$ edges and $H_i$, $1 \leq i \leq m$, be simple graphs too. The generalized edge corona product of graphs $G$ and $H_1, \ldots, H_m$, denoted by $G \circ (H_1, \ldots, H_m)$, is obtained by taking one copy of graphs $G, H_1, \ldots, H_m$ and joining two end vertices of $i$–th edge of $G$ to every vertex of $H_i$, $1 \leq i \leq m$. In this paper, some results regarding the $k$–distance chromatic number of generalized edge corona product of graphs are presented. Also, as a consequence of our results, we compute this invariant for the graphs $K_n \circ (H_1, \ldots, H_m)$, $T \circ (H_1, \ldots, H_m)$ and $K_{m,n} \circ (H_1, \ldots, H_m)$. Moreover, the dominating set, the domination number and the independence number of any connected graph $G$ and arbitrary graphs $H_i$, $1 \leq i \leq |E(G)|$, are evaluated under generalized edge corona operation.

1 Introduction

Let $G$ be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. We denote the shortest distance between two vertices $v_i$ and $v_j$ in $G$ by $d_G(v_i, v_j)$. The eccentricity $\varepsilon(v)$ of a vertex $v$ in $G$ is defined as the maximum distance between $v$ and any other vertex of $G$. The maximum eccentricity over all vertices of $G$ is called the diameter of $G$ and denoted by $D(G)$ and the minimum eccentricity among all vertices of $G$ is called the radius of $G$ and denoted by $r(G)$. 


Let $G$ be a simple graph with $m$ edges and $H_1, H_2, \ldots, H_m$ be $m$ simple graphs. The \textit{generalized edge corona product}, denoted by $G \diamond (H_1, \ldots, H_m)$, is the graph obtained by taking one copy of graphs $G, H_1, H_2, \ldots, H_m$ and then joining two end-vertices of the $i$-th edge $e_i$ of of $G$ to every vertex of $H_i$ for $1 \leq i \leq m$. In particular, if $H_1, \ldots, H_m$ are isomorphic graphs, then the generalized edge corona becomes to the well-known \textit{edge corona product} of two graphs $G$ and $H$ denoted by $G \diamond H$ [1, 6, 10]. As an example of edge corona of two graphs see Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{example.png}
\caption{The left graph is $G$ and the right graph is $G \diamond K_2$}
\end{figure}

It follows from definition of the edge corona product that for two graphs $G$ and $H$ with $|V(G)| = n_1$, $|E(G)| = m_1$, $|V(H)| = n_2$ and $|E(H)| = m_2$, the graph $G \diamond H$ has $n_1 + m_1 n_2$ vertices and $m_1 (1 + m_2 + 2n_2)$ edges. This fact allows us to recognize immediately that the associative law never holds, that is $G_1 \diamond (G_2 \diamond G_3) \not\cong (G_1 \diamond G_2) \diamond G_3$. Indeed the first one has $n_1 + m_1 n_2 + m_1 m_2 n_3 + 2m_1 n_2 n_3 + m_1 n_3$ vertices while the second graph has $n_1 + m_1 n_2 + m_1 m_2 n_3$ vertices and these numbers are never equal. So, the associative law never holds for generalized edge corona as well.

A $k$-\textit{distance coloring} of a graph $G$ is a vertex coloring of $G$ such that no two vertices lying at distance less than or equal to $k$ in $G$ are assigned the same color [2, 3, 5]. The $k$-\textit{distance chromatic number} of $G$ is the minimum number of colors necessary to $k$-distance color $G$, and is denoted by $\chi_{\leq k}(G)$. We note that proper coloring is a particular case of $k$-distance coloring, where $k = 1$ and the chromatic number in this case denoted by $\chi(G)$.

A set $D$ of vertices in a graph $G$ is a \textit{dominating set} if every vertex in $V(G) - D$ is adjacent to at least one vertex in $D$. The \textit{domination number} $\gamma(G)$ is the number of vertices in a smallest dominating set for $G$. Dominating sets in graphs are natural models for facility location problems in operational research. These problems are concerned with the location of one or more facilities in a way that optimizes a certain objective function such as minimizing transportation cost, providing equitable service to customers.
and capturing the largest market share [4, 5, 9].

In this paper, the chromatic number of the generalized edge corona product of graphs are computed. Furthermore, the upper and lower bounds for 2−distance chromatic number and some results regarding the case $k = 3$ have been obtained. As a consequence, these quantities for the edge corona product of some graphs are calculated. In addition, the dominating set, the domination number and the independence number of generalized edge corona product of any connected graph $G$ and arbitrary graphs $H_i$, $1 \leq i \leq |E(G)|$, are obtained.

2 $k$−Distance Chromatic Number of $G \diamond (H_1, \ldots, H_m)$

Let $G$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. We start this section by giving the result relative to 1-distance chromatic number (chromatic number) of $G \diamond (H_1, \ldots, H_m)$.

Theorem 1. The chromatic number of $G \diamond (H_1, \ldots, H_m)$ is given by

$$
\chi(G \diamond (H_1, \ldots, H_m)) = \max_{1 \leq i \leq m} \{\chi(G), \chi(H_i) + 2\}.
$$

Proof. Clearly $\chi(G \diamond (H_1, \ldots, H_m)) \geq \chi(G)$. Also, since every vertex $u \in V(H_i)$, $1 \leq i \leq m$, is adjacent to both ends of $e_i = v_i v_{i+1}$, we have $\chi(G \diamond (H_1, ..., H_m)) \geq \max \{\chi(H_i) + 2, 1 \leq i \leq m\}$ and so, $\chi(G \diamond (H_1, ..., H_m)) \geq \max \{\chi(G), \chi(H_i) + 2\}$.

We now consider the reverse inequality, let $t = \max_{1 \leq i \leq m} \{\chi(G), \chi(H_i) + 2\}$ and let us color the vertices $v_i$, $1 \leq i \leq n$, of $G$ with $t$ different colors $c_1, c_2, \ldots, c_t$. If $v_i$ and $v_{i+1}$ have colors $c_i$ and $c_{i+1}$, respectively, then the graph $H_i$ can be colored by using the set of colors $\{c_1, c_2, \ldots, c_{i-1}, c_{i+2}, \ldots, c_t\}$. Hence, we arrived at

$$
\chi(G \diamond (H_1, ..., H_m)) \leq t = \max_{1 \leq i \leq m} \{\chi(G), \chi(H_i) + 2\}
$$

and the proof is completed. \qed

Our next step is to study the particular case $k = 2$. In this case, we will give the exact value of $\chi_{\leq 2}(T \diamond (H_1, ..., H_{|E(T)|}))$ and an upper bound for $\chi_{\leq 2}(K_n \diamond (H_1, ..., H_{|E(K_n)|}))$. As a direct consequence, $\chi_{\leq 2}(T \diamond H)$ and
\( \chi_{\leq 2}(K_{n_1} \diamond H) \) will be determined. Following Kishore and Sunitha \[7\], a tree \( T \) on \( n \) vertices can be considered as composed of \( r \) branches or \( r \) star graphs connected together to form a single component either by edges or clinging together.

**Theorem 2.** Let \( T \) be a tree with a vertex \( v \) of maximum degree \( \Delta \), \( m \) edges and \( n \) vertices. If \( H_{i_k} \) are corresponding graph to the edge \( e_{i_k} = vv_i, 1 \leq k \leq \Delta, \) in \( T \diamond (H_1, ..., H_m) \) and \( n_{i_k} \) be the order of \( H_{i_k}, 1 \leq k \leq \Delta, \) then 
\[
\chi_{\leq 2}(T \diamond (H_1, ..., H_m)) = \Delta + 1 + \sum_{k=1}^{\Delta} n_{i_k}.
\]

**Proof.** Consider \( T \) as a subgraph of \( T \diamond (H_1, ..., H_m) \). We can easily observe that 2−distance chromatic number of \( T \) is \( \Delta + 1 \). Let there be \( r \) connected star graphs for \( T \). We start by the vertex \( v \) of the maximum degree \( \Delta \) in \( T \). Obviously, all the vertices adjacent to \( v \) should be colored different from \( v \) and also from each others. Hence we need \( \Delta + 1 \) colors. Since other vertices have degrees less than or equal to \( \Delta \), there is no more colors needed. We are repeating this process till all the vertices in the \( r \) branches of \( T \) are colored. Next we consider all graphs \( H_{i_k}, 1 \leq k \leq \Delta \). Since all of these graphs have the same neighbor \( v \), they have to be colored differently. Also it is clear that no pairs of vertices of each \( H_{i_k} \) could be colored the same. Moreover, by definition of generalized edge corona and 2−distance coloring, the vertices of each \( H_{i_k} \) should be colored differently from all the \( \Delta + 1 \) colors used before. Similarly, since other vertices of \( T \) have degrees less than or equal to \( \Delta \), there is no more colors needed to color all the vertices of \( H_{i_k}'s \) in the \( r \) branches of \( T \). So in general, we arrived at 
\[
\chi_{\leq 2}(T \diamond (H_1, ..., H_m)) = \Delta + 1 + \sum_{k=1}^{\Delta} n_{i_k}.
\]
and the theorem is proved. \( \square \)

**Corollary 1.** Let \( T \) be a tree with maximum degree \( \Delta \) and \( H \) be a graph of order \( n \). Then 
\[
\chi_{\leq 2}(T \diamond H) = (n + 1)\Delta + 1.
\]

A set of edges in a graph \( G \) is called independent or a matching if no two edges have a vertex in common. The size of any largest matching in \( G \) is called the matching number of \( G \) and is denoted by \( \nu(G) \).

Before proving Theorem 3, we record the following simple lemma:

**Lemma 1.** The number of independent edges in \( K_n \) is 
\[
\nu(K_n) = \begin{cases} 
\frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \\
\frac{n-1}{2} & \text{if } n \equiv 1 \pmod{2}
\end{cases}
\]

4
Theorem 3. Let $K_n$ be a complete graph of order $n$ and $H_i$ be graphs with $n_i$ vertices, $1 \leq i \leq |E(K_n)|$. Then,

$$\chi_{\leq 2}(K_n \diamond (H_1, \ldots, H_{|E(K_n)|})) \leq \begin{cases} n(t+1) & 2 \nmid n \\ n(t+1) - t & 2 \mid n \end{cases}$$

where $t = \max_{1 \leq i \leq |E(K_n)|}\{n_i\}$.

Proof. It is clear that we need $n$ colors for coloring the vertices of $K_n$. Since all pairs $u, v$ of vertices of $H_i$, $1 \leq i \leq |E(K_n)|$, can be connected with at least a path of length two, using a vertex of $K_n$, every pair of distinct vertices of $H_i$ should be colored differently. Let $t = \max_{1 \leq i \leq |E(K_n)|}\{n_i\}$. By Lemma 1, if $n$ is even, then the set of nonempty graphs $H_i$ can be divided into $n - 1$ classes of size $n/2$ which their vertices could be colored using the same set of colors, and if $n$ is odd, then we have $n$ classes of size $n - 1$ of $H_i$ which could be assigned the same colors. So, in general, if $n$ is even we need at most $t(n-1)$ new colors and if $n$ is odd, we need at most $tn$ new colors and the result follows.

Applying Theorem 3, we obtain the following corollary.

Corollary 2. Let $K_{n_1}$ be a complete graph of order $n_1$ and $H$ be a graph with $n_2$ vertices. Then we have

$$\chi_{\leq 2}(K_{n_1} \diamond H) = \begin{cases} n_1(n_2 + 1) & 2 \nmid n_1 \\ n_1(n_2 + 1) - n_2 & 2 \mid n_1 \end{cases}$$

Theorem 4. Let $G$ be a graph with a vertex $v$ of maximum degree $\Delta$ and $H_{i_k}$ be corresponding graph to the edge $e_{i_k} = vv_i$, $1 \leq k \leq \Delta$, in $G \diamond (H_1, \ldots, H_m)$ and $n_{i_k}$ be order of $H_{i_k}$, $1 \leq k \leq \Delta$. Then,

$$\Delta + 1 + \sum_{k=1}^{\Delta} n_{i_k} \leq \chi_{\leq 2}(G \diamond (H_1, \ldots, H_{|E(G)|})) \leq n(t+1),$$

where $|V(G)| = n$ and $t = \max_{1 \leq i \leq |E(G)|}\{|V(H_i)|\}$.

Proof. By taking into account that the maximum degree of $G \diamond (H_1, \ldots, H_{|E(G)|})$ is $\Delta + \sum_{k=1}^{\Delta} n_{i_k}$ and the fact that $\Delta(G \diamond (H_1, \ldots, H_{|E(G)|})) + 1 \leq n(t+1)$, we obtain the lower bound.

On the other hand, suppose $H_i$ be a graph corresponding to the edge $e_i$ of $G$. Clearly, every pair of vertices of $H_i$ should be colored differently.
In the worst case, if \( G \) is a complete graph, then there is a path of length two between every vertex of \( H_i \) and every vertex of \( G \). Therefore, the upper bound is obtained by applying the Theorem \( \text{3} \). 

By considering Theorems \( \text{2} \) and \( \text{3} \) we can conclude that the above inequalities, are sharp.

**Corollary 3.** Let \( G \) be a graph with maximum degree \( \triangle \). Then,

\[
(n_2 + 1)\triangle + 1 \leq \chi_{\leq 2}(G \odot H) \leq n_1(n_2 + 1)
\]

where \( n_1 \) and \( n_2 \) are the number of vertices of \( G \) and \( H \), respectively.

By considering Corollaries \( \text{1} \) and \( \text{2} \) we can conclude that the above inequalities, are sharp. We now present some results on 3−distance chromatic number of some families of graphs.

**Lemma 2.** \( D(G \odot (H_1, \ldots, H_{|E(G)|})) \leq D(G) + 2 \).

**Theorem 5.** The 3−distance chromatic number of \( K_n \odot (H_1, \ldots, H_{|E(K_n)|}) \) is

\[
\chi_{\leq 3}(K_n \odot (H_1, \ldots, H_{|E(K_n)|})) = n + \frac{n(n - 1)}{2} \sum_{i=1}^{n} n_i.
\]

*Proof.* By Lemma \( \text{2} \), the diameter of \( K_n \odot (H_1, \ldots, H_{|E(K_n)|}) \) is at most three. So all vertices of \( K_n \odot (H_1, \ldots, H_{|E(K_n)|}) \) have to be assigned different colors. \( \square \)

The next corollary now follows directly from Theorem \( \text{5} \).

**Corollary 4.** The 3−distance chromatic number of \( K_n \odot H \) is

\[
\chi_{\leq 3}(K_n \odot H) = n + \frac{n(n - 1)}{2}|V(H)|.
\]

**Theorem 6.** \( \chi_{\leq 3}(K_{m,n} \odot (H_1, \ldots, H_{mn})) = m + n + \sum_{i=1}^{mn} n_i \), where \( n_i \) is the number of vertices of \( H_i \). In particular, \( \chi_{\leq 3}(K_{m,n} \odot H) = m + n + mn_2 \), where \( n_2 \) is the number of vertices of \( H \).

*Proof.* The result follows from the fact that \( D(K_{m,n} \odot (H_1, \ldots, H_{mn})) \leq 3 \). \( \square \)
3 Domination and Independence Numbers of 
$G \circ (H_1, \ldots, H_m)$

Let $G$ be a graph with $|E(G)| = m$. The aim of this section is to compute the domination and independence numbers of $G \circ (H_1, \ldots, H_m)$. To present this result, we need some definitions. Recall that a vertex cover of a graph $G$ is a set of vertices such that each edge of $G$ is incident to at least one vertex of the set. The minimum cardinality of this set is the vertex covering number which is denoted by $\beta(G)$. For every $e_i = u_i v_i \in E(G)$, $1 \leq i \leq m$, by $e_i + H_i$ we denote the subgraph of $G \circ (H_1, \ldots, H_m)$ which obtained by joining two ends of the edge $e_i$ with all vertices of $H_i$, where $H_i$ is a graph corresponding to the edge $e_i$ of $G$.

**Theorem 7.** Suppose $G$ is a connected graph with $m$ edges and $H_i, 1 \leq i \leq m$ are graphs. Then $D \subseteq V(G \circ (H_1, \ldots, H_m))$ is a dominating set of $G \circ (H_1, \ldots, H_m)$ if and only if $V(e_i + H_i) \cap D$ is a dominating set of $e_i + H_i$ for every $e_i \in E(G)$.

*Proof.* Let $D$ be a dominating set in $G \circ (H_1, \ldots, H_m)$ and $e_i = u_i v_i \in E(G)$. If $u_i$ (or $v_i$) belongs to $D$, then $\{u_i\}$ (or $\{v_i\}$) is a dominating set of $e_i + H_i$, i.e. $V(e_i + H_i) \cap D$ is a dominating set of $e_i + H_i$. Suppose that $D$ consists of no vertices of $u_i$ and $v_i$. Let $x \in V(e_i + H_i) \setminus \{D \cup \{u_i, v_i\}\}$. Since $D$ is a dominating set of $G \circ (H_1, \ldots, H_m)$, there exists $y \in D$ such that $xy \in E(G \circ (H_1, \ldots, H_m))$. Clearly, $y \in V(H_i + e_i) \cap D$ and $xy \in E(H_i + e_i)$. It follows that $V(H_i + e_i) \cap D$ is a dominating set of $e_i + H_i$.

Conversely, suppose that $V(H_i + e_i) \cap D$ is a dominating set of $e_i + H_i$ for every $e_i \in E(G)$, $1 \leq i \leq m$. Since $G$ is connected, it immediately concludes that $D$ is a dominating set of $G \circ (H_1, \ldots, H_m)$.

**Theorem 8.** Let $G$ be a connected graph and $H_i, 1 \leq i \leq |E(G)|$ be arbitrary graphs. Then $\gamma(G \circ (H_1, \ldots, H_m)) = \beta(G)$.

*Proof.* Let $C$ be a vertex cover of $G$. By definition, we can observe that the intersection of two of the sets $V(H_i + e_i)$ and $C$ is $\{u_i\}$ or $\{v_i\}$ or both of them. All the previous cases are dominating set of $H_i + e_i$. By Theorem 7, $C$ is a dominating set of $G \circ H$. So $\gamma(G \circ (H_1, \ldots, H_m)) \leq |C| = \beta(G)$.

To complete the proof, let $D^*$ be a minimum dominating set of $G \circ H$. By Theorem 7, $V(H_i + e_i) \cap D^*$ is a dominating set of $H_i + e_i$ for every $e_i \in E(G)$. Therefore, $D^*$ contains either at least one of the vertices $u_i$ and $v_i$ or exactly
one vertex $x$ of $H_i$. If $x \in D^*$, then without loss of generality, we replace $x$ by $u_i$ in $D^*$. By continuing this process, we reach a new minimum dominating set of $D^*_{\text{new}}$, whose all members belong to $V(G)$. It is clear that $D^*_{\text{new}}$ is also a vertex cover of $G$. Thus, $\gamma(G \diamond (H_1, \ldots, H_m)) = |D^*_{\text{new}}| \geq \beta(G)$. Therefore, $\gamma(G \diamond (H_1, \ldots, H_m)) = \beta(G)$.

Corollary 5. Let $G$ be a connected graph and $H$ be any graph. Then the domination number of $G \diamond H$ is $\beta(G)$.

A set $S$ of vertices in a graph $G$ is an independent set if and only if there is no edge in $E(G)$ between any two vertices in $S$. A maximum independent set is an independent set of largest possible size for $G$. This size is called the independence number of $G$, and denoted $\alpha(G)$.

Theorem 9. For a connected graph $G$ and arbitrary graphs $H_i, 1 \leq i \leq m$, $\alpha(G \diamond (H_1, \ldots, H_m)) = \sum_{i=1}^{m} \alpha(H_i)$.

Proof. Let $H_i$ be a graph corresponding to the edge $e_i = uv$. Clearly all vertices of $H_i$ are independent from all vertices of $H_j, i \neq j$. Let $S_{H_i}$ be a maximum independent set of $H_i$. Since the generalized edge corona operation on $G$ and all $H_i$ does not make any edge between two vertices of $S_{H_i}$, $S = \bigcup_{i=1}^{m} S_{H_i}$ is an independent set of $G \diamond (H_1, \ldots, H_m)$. So $\alpha(G \diamond (H_1, \ldots, H_m)) \geq \sum_{i=1}^{m} \alpha(H_i)$.

On the other hand, let $S$ be a maximum independent set of $G \diamond (H_1, \ldots, H_m)$. If $S$ contains no vertices of $G$, then the proof is complete. Otherwise, consider $e_i = uv \in E(G)$ such that $v \in S$. Clearly, in this case $\alpha(H_i) = 1$ and so we can remove $v$ from $S$ and add an arbitrary vertex of $H_i$ to $S$. This procedure is continued until all elements of $S$ belong to $V(H_i)$. So, $\alpha(G \diamond (H_1, \ldots, H_m)) \leq \sum_{i=1}^{m} \alpha(H_i)$, which will complete the proof.

Corollary 6. For a connected graph $G$ and an arbitrary graph $H$, the independence number of $G \diamond H$ is $\alpha(G \diamond H) = m\alpha(H)$, where $m$ is the number of edges of $G$.

4 Conclusion

We determined some invariants of the generalized edge corona product of some graphs such as:
the chromatic number as a particular case of $k$-distance chromatic number,
dominating set, domination number and the independence number of generalized edge corona product of graphs.

Furthermore, the bounds for $2-$distance chromatic number and some results regarding the case $k = 3$ were obtained. As a consequence, these invariants for the edge corona product of two graphs are calculated. These experiences can help us to reduce the problem of computing properties of big graphs to the problem of computing some parameters of the factor graphs.

References

[1] Chithra, K. P., Germina, K. A., and Sudev, N. K., On the Sparing Number of the Edge-Corona of Graphs, *Int. J. Comput. Appl.*, (2015), 118, (1), 1–5.

[2] Fertin, G., Godard, E., and Raspaud, A., Acyclic and $k$-distance coloring of the grid, *Inform. Process. Lett.*, (2003), 87, (1), 51–58.

[3] Georges, J. P., Mauro, D. M., and Stein, M. I., Labeling products of complete graphs with a condition at distance two, *SIAM J. Discrete Math.*, (2000), 14, (1), 28–35.

[4] Go, C. E., and Canoy, S. R., Domination in the corona and join of graphs, *Int. Math. Forum* (2011), 6, (16), 763–771.

[5] Gonzalez Yero, I., Kuziak, D., and Rond, A., on Aguilar, Coloring, location and domination of corona graphs, *Aequat. Math.*, (2013), 86, 1–21.

[6] Hou, Y., and Shiu, W-C., The spectrum of the edge corona of two graphs, *Electron. J. Linear Algebra*, (2010), 20, 586–594.

[7] Kishore, A. and Sunitha, M. S., Injective chromatic sum and injective chromatic polynomials of graphs, *Gen. Math. Notes*, 2013, 18, (2), 55–66.

[8] Y. Luo and W. Yan, Spectra of the generalized edge corona of graphs, *Discrete Math. Algorithm. Appl.*, 1 (2017) 1–10.

[9] Quadras, J., Albert, S. M. M., Domination parameters in coronene torus network, *Math. Comput. Sci.*, (2015), 9, 169–175.
[10] I. Rezaee Abdolhosseinizadeh, F. Rahbarnia, M. Tavakoli, A. R. Ashrafi,
Some vertex-degree-based topological indices under edge corona product, *Italian J. of Pure and Applied Mathematics* 38 (2017), 81–91.