Abstract

Chiral skyrmions are stable particle-like solutions of the Landau-Lifshitz equation for ferromagnets with the Dzyaloshinskii-Moriya interaction (DMI), characterized by a topological number. We study the profile of an axially symmetric skyrmion and give exact formulas for the solution of the corresponding far field and near field equations, in the asymptotic limit of small DMI constant (alternatively large anisotropy). The matching of these two fields leads to a formula for the skyrmion radius as a function of the DMI constant. The derived solutions show the different length scales which are present in the skyrmion profiles. The picture is thus created of a chiral skyrmion that is born out of a Belavin-Polyakov solution with infinitesimally small radius, as the DMI constant is increased from zero. The skyrmion retains the Belavin-Polyakov profile over and well-beyond the core, before it assumes an exponential decay; the profile of an axially-symmetric Belavin-Polyakov solution of unit degree plays the role of the universal core profile of chiral skyrmions.

Keywords:
Magnetic skyrmion, Micromagnetics, Dzyaloshinskii-Moriya interaction

1. Introduction

Magnetic skyrmions were predicted to be stabilized in ferromagnets with the Dzyaloshinskii-Moriya interaction (DMI) [4][2]. They have been observed in Dzyaloshinskii-Moriya (DM) materials and techniques have been developed for individual skyrmions to be created and annihilated in a controlled manner [23]. Skyrmions are examples of topological magnetic solitons in ferromagnetic films that exhibit particle-like behavior in the sense that they are localized robust entities both regarding their statics and their dynamical behavior. This makes them attractive for theoretical studies in order to understand details of their behavior while it also gives them a strong potential for applications [12].

Magnetic solitons [17], such as magnetic bubbles and vortices [20,14], have been investigated theoretically and experimentally, and their global features (such as topology and qualitative morphology) have been observed to an extent. It is though only in recent years that experimental techniques have been developed that offer sufficient resolution for the observation of detailed features of the skyrmion profile [24,5,18,26]. The details of the skyrmion profile determine to a large extent and sometimes crucially the properties of the skyrmion [8] and is thus essential for the manipulation of individual skyrmions.
Skyrmions can be found as solutions of the Landau-Lifshitz equation in the presence of DMI by numerical methods. The existence of such solutions has been rigorously proved \cite{21, 19}, but so far no analytic formula for the skyrmion profile has grown out of rigorous mathematical reasoning. Instead, an ad-hoc ansatz based on explicit domain wall profiles \cite{7} has been suggested and is widely used to examine structural and dynamic properties, see, e.g., \cite{24, 29, 8}. Further trial profiles have been tested to this end \cite{27, 28}.

In this paper we derive formulas for the skyrmion profile by employing asymptotic methods that give analytic approximations for the solutions of the Landau-Lifshitz equation. Our methods are valid for the case of small DMI constant or large anisotropy and they can readily be extended to the case of a large external field. The derived solutions show the detailed features and the different length scales which are present in the skyrmion profile. The role of DMI for the existence of skyrmion solutions and the role of the Belavin-Polyakov solution as a universal limit of skyrmion profiles are revealed.

The availability of mathematically derived formulas will facilitate the comparison of experimentally observed profiles, particularly focusing on some of their special features, and may be useful for a variety of other purposes. Specifically, the skyrmion profile enters in an essential way in formulas for dynamical phenomena \cite{22, 15}, for example, skyrmion translation and rotation modes \cite{25}, and it is crucial for quantitative calculations. Finally, the methods developed in this paper are potentially useful in the search for solutions of the Landau-Lifshitz equation in cases of skyrmion dynamics.

The paper is arranged as follows. The basic equations are presented in section 2. The far field is analyzed in section 3, where it is calculated as an integral and also as a series, each serving a different purpose. Section 4 is devoted to the calculation of the near field. The matching of the near and far fields is performed in section 5 and the skyrmion radius is calculated as a function of the small DMI constant in section 6. A summary and discussion of results are presented in section 7. Finally, an outline for an existence proof of the skyrmion is presented in Appendix A.

2. The basic equations

2.1. Magnetization vector

We assume a thin film of a ferromagnetic material lying on the \(xy\)-plane. The micromagnetic structure is described via the magnetization vector \(m = m(x, y)\) with a fixed magnitude normalized to unity, \(m^2 = 1\). We will assume a ferromagnet with exchange interaction, a Dzyaloshinskii-Moriya interaction (DMI), and an anisotropy of the easy-axis type perpendicular to the film, governed by the normalized energy

\[
E_\epsilon(m) = \int \left( \frac{\partial_\mu m \cdot \partial_\mu m}{2} + \frac{1 - m^2}{2} + \epsilon \hat{e}_\mu \cdot (\partial_\mu m \times m) \right) \, dx
\]

with summation over repeated indices \(\mu = 1, 2\) and \(\hat{e}_1, \hat{e}_2, \hat{e}_3\) are the unit vectors for the magnetization in the respective directions. Static magnetization fields are local minimizers of \(E_\epsilon\) satisfying the normalized Landau-Lifshitz equation

\[
m \times h_\epsilon = 0
\]

where the effective field

\[
h_\epsilon = \partial_\mu \partial_\mu m + m_3 \hat{e}_3 - \epsilon \hat{e}_\mu \times \partial_\mu m
\]

is minus the variational gradient of \(E_\epsilon = E_\epsilon(m)\). We measure lengths in units of the domain wall width \(\ell_w = \sqrt{A/K}\), where \(A\) is the exchange and \(K\) the anisotropy constant. The equation contains a single parameter

\[
\epsilon = 2 \frac{\ell_w}{\ell_S} = \frac{D}{\sqrt{AK}}
\]
defined via an additional length scale of this model $\ell_S = D/(2K)$, where $D$ is the DMI constant. The lowest energy (ground) state is the spiral for $\epsilon > 4/\pi$ and the ferromagnetic state for $\epsilon < 4/\pi$ [2, 9]. Isolated chiral skyrmions occur in the ferromagnetic regime as local energy minimizers in a nontrivial homotopy class [4, 2, 3, 21, 19].

Let us consider the angles $(\Theta, \Phi)$ for the spherical parametrization of the magnetization vector, and the polar coordinates $(r, \phi)$ for the film plane. We assume an axially symmetric skyrmion with $\Phi = \phi + \pi/2$ and $\Theta = \Theta(r)$, called a Bloch skyrmion. All subsequent calculations remain valid (actually identical) if, instead of the bulk DMI term in Eq. (2) and a Bloch skyrmion, we consider the interfacial DMI term and a Néel skyrmion, $\Phi = \phi$. The equation for the profile $\Theta = \Theta(r)$ is the same for both skyrmions, therefore the following calculations apply equally to Bloch and Néel skyrmions.

2.2. Stereographic projection

We define the field

$$u(r) = \tan \frac{\Theta(r)}{2}$$

which is the modulus of the stereographic projection of the magnetization vector. The equation for $u$ reads

$$u'' + \frac{u'}{r} + u \left( \frac{u}{r^2} u'^2 + 1 \right) - \frac{2uu'^2}{u^2 + 1} + \frac{u^2 - 1}{u^2 + 1} u + 2 \epsilon \frac{u^2}{r^2(u^2 + 1)} = 0.$$  (5)

where the prime denotes differentiation with respect to $r$. A well-known solution of this equation has been obtained for the case that only the exchange interaction is present, that is, when the last two terms in Eq. (7) are absent. This is the axially-symmetric Belavin-Polyakov solution of unit degree

$$u(r) = \frac{a}{r},$$  (6)

where $a$ is an arbitrary complex constant and $|a|$ gives the radius of the skyrmion core [1].

In order to remove the singularity at $r = 0$, we define the field

$$v(s) = su, \quad s = \frac{r}{2},$$  (9)

which transforms Eq. (7) to the quasilinear equation (linear in the highest derivative)

$$\ddot{v} + \dot{v} \frac{3v^2 - s^2}{s^2} - \frac{2v^2}{v^2 + s^2} + 4 \frac{v^2}{v^2 + s^2} v = -4 \epsilon \frac{v^2}{v^2 + s^2}.$$  (10)

Passing to the variable $s$ from $r$ is a matter of convenience.

In order to display the findings of our analysis, we find numerically the skyrmion profiles for various values of the parameter $\epsilon$ by employing a relaxation algorithm for solving the Landau-Lifshitz equation [2, 16] as well as by solving numerically Eq. (10) using a shooting method. In the latter method we start integration at $s = 0$ choosing $v(s = 0) = v_0$, where $v_0$ is chosen arbitrarily, $\dot{v}(s = 0) = 0$ and we integrate up to large values of $s$. We seek and actually find a $v_0$ such that $v(s)$ decreases monotonically tending to zero for large $s$. The results of both methods agree, but we only obtain a result by the shooting method for small
The function $v(s)$ is monotonically decreasing and the value at the skyrmion center $v_0 = v(0)$ decreases with decreasing $\epsilon$. The coordinate $s$ is measured in units of twice the domain wall width ($2\ell_w$).

The field $v$ takes a finite non-zero value at $s = 0$ indicating that the stereographic field $u$ diverges as $1/r$ at $r \to 0$, similar to the skyrmion in Eq. (8). The value $v(0)$ decreases for larger $\epsilon$, i.e., for large anisotropy or small DM parameter. For small $s$, the skyrmion profile is well approximated by the profile (8). Smaller $v(0)$ indicates a smaller skyrmion radius, as will be made precise in Sec. 5. Fig. 2 depicts the magnetization vector components for the skyrmion profiles of Fig. 1. The radius of the skyrmion is clearly seen to decrease with decreasing $\epsilon$ in agreement with our calculations below.

We are seeking a skyrmion profile as a solution of Eq. (10) for which $v(s)$ is positive and it tends to zero at spatial infinity. Aiming at neglecting the term that contains $\dot{v}^2$, we require

$$v \ll 1, \quad \dot{v} \ll v, \quad \ddot{v} \ll \sqrt{\epsilon}, \quad \text{for } \epsilon \ll 1. \quad (11)$$

Then Eq. (10) simplifies to the semilinear equation (linear in all derivatives),

$$\ddot{v} + \frac{\dot{v}}{s} \frac{3v^2 - s^2}{v^2 + s^2} + 4\frac{v^2 - s^2}{v^2 + s^2} v = -4\epsilon \frac{v^2}{v^2 + s^2}. \quad (12)$$
We solve Eq. (12) asymptotically, by obtaining two separate linear equations, one for the far field and one for the near field, and matching their solutions through an overlap subdomain of the independent variable \( s \).

The matching condition yields the relation between the boundary value \( v(0) = v_0 \) and the small parameter \( \epsilon \). The significance of this relationship is that the value \( v_0 \) equals the radius of the skyrmion in the asymptotic limit. Indeed, \( \Theta = \frac{\pi}{2} \) at the skyrmion radius, thus \( v = s \). In the limit \( \epsilon \to 0 \), the value of \( v \) at which \( v(s) = s \), converges to \( v_0 \) (see Eq. (51)).

We verify a posteriori, that the profile obtained satisfies the conditions (11).

3. The far-field

Under the additional assumption \( v \ll s \), which defines the range of the far-field, Eq. (12) simplifies to the far-field equation

\[
\ddot{v} - \frac{s}{s} - 4v = 0. \tag{13}
\]

3.1. A growing solution

The indicial equation of Eq. (13) has roots 0 and 2. According to the Frobenius theory [6], Eq. (13) has a power series solution that begins with the power of the larger root. In our case, this series solution is

\[
v_3(s) = \sum_{n=1}^{\infty} \frac{s^{2n}}{(n-1)!n!}. \tag{14}
\]

The series has infinite radius of convergence and exhibits exponential growth. This solution is the gateway to the calculation of the decaying solution in the form of an integral formula in Sec. 3.2 and a series in Sec. 3.3.

For large \( s \), we use the Stirling approximation for the factorial \( n! \sim n^n e^{-n} \sqrt{2\pi n} \) and have

\[
(n-1)!n! = \frac{(n!)^2}{n} \sim 2\pi n^{2n} e^{-2n} = 2\pi \exp(2n \ln n - 2n). \tag{15}
\]

Inserting this into the expression for \( v_3 \), we obtain

\[
v_3(s) \sim \frac{1}{2\pi} \sum_{n=1}^{\infty} \exp[2n(ln s - \ln n + 1)]. \tag{16}
\]

We pass to the integral expression of the series by introducing the scaled Dirac comb

\[
\Delta_{1/s}(x) = \sum_{n\in\mathbb{N}} \delta(x - n/s),
\]

i.e., an atomic measure that has unit mass at each point where \( sx \) is a positive integer. Then

\[
v_3 \sim \frac{1}{2\pi} \int_0^{\infty} \exp[2xs(ln s - \ln xs + 1)] \Delta_{1/s}(dx) = \frac{1}{2\pi} \int_0^{\infty} \exp[s(-2x \ln x + 2x)] \Delta_{1/s}(dx). \tag{17}
\]

By the sampling property of the scaled Dirac comb, we may replace \( \Delta_{1/s}(dx) \) by \( s \, dx \) in the asymptotic limit of large \( s \). Thus

\[
v_3 \sim \frac{s}{2\pi} \int_0^{\infty} \exp[2xs(1 - \ln x)] \, dx. \tag{18}
\]

The maximizer of the exponent is \( x = 1 \) and the second derivative of the exponent at the maximizer equals -2. According to the steepest descent formula for the asymptotic evaluation of integrals,

\[
v_3 \sim \frac{1}{2} \sqrt{\frac{s}{\pi}} e^{2s}, \quad s \gg 1. \tag{19}
\]
3.2. The decaying solution: integral formula

We adopt the forms
\[ v_3 = A(s) e^{2s}, \quad v_d(s) = B(s) e^{-2s} \]  
(20)

where
\[ A(s) \sim \frac{1}{2} \sqrt{\frac{s}{\pi}}, \]  
(21)
and \( B(s) \) is to be determined. The Wronskian of the two solutions \( v_3 \) and \( v_d \) is
\[ W = \begin{vmatrix} v_3 & v_d \\ \dot{v}_3 & \dot{v}_d \end{vmatrix}. \]  
(22)

Differentiating Eq. (22) and using Eq. (13) we find
\[ \dot{W} = \int_0^\infty \frac{t}{s} \frac{e^{-4t}}{A^2(t)} \, dt = \frac{s}{A^2}, \]  
(23)
in which we normalized the constant factor to equal unity. Inserting the two solution forms (20) in Eq. (23), we obtain the equation
\[ AB - B\dot{A} - 4AB = -s \Rightarrow \frac{d}{ds} \left( \frac{B}{A} \right) - 4 \frac{B}{A} = -\frac{s}{A^2} \]  
(24)
which is integrated to obtain
\[ B(s) e^{-4s} = A(s) \int_0^\infty \frac{t e^{-4t}}{A^2(t)} \, dt. \]  
(25)

The asymptotic expression of \( B(s) \) is
\[ B(s) \sim \frac{1}{2} \sqrt{\frac{s}{\pi}}, \quad s \gg 1. \]  
(28)

The asymptotic expression of \( B(s) \) is obtained by using the forms (20) and (21) in Eq. (27). The general solution of Eq. (13) is a linear combination of the growing solution \( v_3 \) and the decaying solution \( v_d \). Clearly, the decaying solution is unique up to a constant factor.
3.3. The decaying solution: series formula

In accordance with Frobenius theory, a second solution \( \tilde{v}_4 \) of Eq. (13) is given by the expression

\[
\tilde{v}_4 = av_3 \ln s + v_5
\]  

(29)

where \( v_5 \) is a power series starting with the power having as exponent the smallest root of the indicial equation. This is zero in our case, thus \( v_5 \) starts with nonzero constant term,

\[
v_5(s) = b_0 + \sum_{n=1}^{\infty} b_n s^{2n}.
\]  

(30)

The constant \( a \) and the coefficients \( b_n \) are obtained by substituting this expression in Eq. (13). We obtain the equation

\[
\ddot{v}_5 - \frac{\dot{v}_5}{s} - 4v_5 = -2a \left( \frac{v_3}{s} - \frac{\dot{v}_3}{s^2} \right).
\]  

(31)

We substitute \( v_3 \) from Eq. (14) and \( v_5 \) from Eq. (30) and we balance coefficients of the same order. For the lowest order terms we make the choice

\[
b_0 = 1 \quad \text{and we find} \quad a = 2.
\]

The coefficient balance for all other terms gives

\[
(n - 1)b_n = b_{n-1} - \frac{2n - 1}{(n - 1)!n!}, \quad n = 2, 3, 4, \ldots.
\]  

(32)

Defining

\[
p_k = \frac{1}{(k - 1)k}, \quad q_k = \frac{2k - 1}{(k - 1)!k!}
\]  

(33)

the balance of coefficients is written succinctly as

\[
b_k = p_k b_{k-1} - q_k p_k, \quad k = 2, 3, 4, \ldots
\]  

(34)

We determine the coefficients \( b_n \) by multiplying Eq. (34) by \( p_{k+1} \cdots p_n \) and summing all resulting equations for \( k = 2, \ldots, n, \)

\[
b_n = p_2 \cdots p_n b_1 - \sum_{k=2}^{n} q_k p_k \cdots p_n, \quad n = 2, 3, 4, \ldots
\]

We note that

\[
p_2 \cdots p_n = \frac{1}{n!(n-1)!}, \quad q_k p_k \cdots p_n = \frac{2k - 1}{(k-1)!k(n-1)!n!},
\]  

thus

\[
b_n = \frac{1}{(n - 1)!n!} \left[ b_1 - \sum_{k=2}^{n} \frac{2k - 1}{k(k-1)} \right].
\]  

(35)

We define

\[
\tilde{\xi}_n = \frac{1}{2} \sum_{k=2}^{n} \left( \frac{1}{k} + \frac{1}{k-1} \right), \quad n = 2, 3, 4, \ldots
\]  

(36)

and we write the result as

\[
b_n = \frac{b_1 - 2\tilde{\xi}_n}{n!(n-1)!}, \quad n = 2, 3, 4, \ldots
\]  

(37)
Substituting Eq. (37) in Eq. (30) we finally have
\[ v_5 = 1 + b_1 s^2 + \sum_{n=2}^{\infty} \frac{b_1 - 2\xi_n}{n!(n-1)!} s^{2n}. \]  

(38)

Putting our results together we have that the function
\[ \tilde{v}_4 = 1 + b_1 \sum_{n=1}^{\infty} \frac{s^{2n}}{(n-1)!n!} + 2(s) \sum_{n=1}^{\infty} \frac{s^{2n}}{(n-1)!n!} - 2 \sum_{n=2}^{\infty} \xi_n s^{2n}, \]  

(39)

where \( b_1 \) is arbitrary, satisfies the far field equation (13). Notice that the first series in this expression of \( \tilde{v}_4 \), is the function \( v_3 \), our first solution of the far-field equation. The decaying solution \( v_d \) of Eq. (26) is a linear combination of \( v_3 \) and \( \tilde{v}_4 \).

**Theorem 2.** The decaying solution of the far-field equation, calculated up to an arbitrary constant factor, is given by the function
\[ v_4 = 1 + (2\gamma - 1)s^2 + 2 \sum_{n=1}^{\infty} \frac{(\ln s - \ln n)s^{2n}}{(n-1)!n!} - 2 \sum_{n=2}^{\infty} \xi_n s^{2n}, \]  

(40)

where \( \gamma \approx 0.5772 \) is the Euler-Mascheroni constant and
\[ \xi_n = \frac{1}{2} \sum_{n=2}^{\infty} \left( \frac{1}{k} + \frac{1}{k - 1} \right) - \left( \ln n + \gamma - \frac{1}{2} \right) \sim O \left( \frac{1}{n} \right), \quad n = 2, 3, 4, \ldots. \]  

(41)

The scaling constant in the relation \( v_d = Cv_4 \) is given by
\[ C = v_d(0) = \lim_{s \to 0^+} v_3(s) \int_s^{\infty} \frac{t}{v_3^2(t)} \mathrm{d}t = \frac{1}{2}. \]  

(42)

**Proof.** The function \( v_4 \) is obtained from the solution \( \tilde{v}_4 \) of the far-field equation by choosing \( b_1 = 2\gamma - 1 \), therefore \( v_4 \) satisfies the far-field equation.

In order to show the decay of \( v_4 \) it suffices to show that it cannot attain the growth of \( v_3 \). Indeed, any solution of the far-field equation is a linear combination \( c_1 v_3 + c_2 v_d \) and any solution that does not have the growth of \( v_3 \) must satisfy \( c_1 = 0 \) and thus be a multiple of the decaying solution \( v_d \). We mirror our procedure for determining the growth of \( v_3 \). The terms of the first of the two series in the expression of \( v_4 \) (see Eq. (40)) are the terms of \( v_3 \), multiplied by the factor \( \ln s - \ln n \). Using the Dirac comb to pass from the series to an integral as in our asymptotic calculation of the growing solution \( v_3 \), the series becomes
\[ \sim \frac{s}{2\pi} \int_0^{\infty} (\ln x) \exp[2sx(1 - \ln x)] \mathrm{d}x. \]  

(43)

The integrand equals zero at the maximizer of the exponent, \( x = 1 \). It follows from the steepest descent method for the asymptotic evaluation of integrals, that the series does not attain the growth of \( v_3 \), in other words, the ratio of the series over \( v_3 \) converges to zero, as \( s \to \infty \). The same is true for the ratio of the second series in Eq. (40) over \( v_3 \) where,
\[ \xi_n = \tilde{\xi}_n - \left( \ln n + \gamma - \frac{1}{2} \right) \sim O \left( \frac{1}{n} \right), \quad n \gg 1. \]  

(44)
In order to calculate the numerical value of the constant $C$, given by the above integral expression in the limit $s \to 0$, we use the series expansion of $v^3$ to write the integrand as
\[
\frac{t}{v^3(t)} = \frac{1}{t^3} + \phi(t), \quad \phi(t) = O(t^{-1}) \quad \text{as} \quad t \to 0,
\] (45)
where $\phi(t)$ decays exponentially as $t$ grows. The contribution of $\phi$ to the value of $C$ is clearly zero. The contribution of $\frac{1}{t^3}$ is obtained by direct calculation. This proves the theorem.

4. The near-field

In the near field, we replace $v$ in Eq. (12) with its initial value $v_0$. The approximation is valid for the range of $s$ over which $v_0 - v \ll v_0$. The equation obtained in this way,
\[
\ddot{v} + \frac{3v_0^2 - s^2}{v_0^2 + s^2} \frac{\dot{v}}{s} = -4\frac{v_0^2 - s^2}{v_0^2 + s^2}v_0 - 4\epsilon \frac{v_0^2}{v_0^2 + s^2},
\] (46)
is integrable by the integrating factor
\[
\mu(s) = \frac{s^3}{(v_0^2 + s^2)^2}.
\]
We obtain
\[
\frac{d}{ds} \left[ \frac{s^3}{(v_0^2 + s^2)^2} \dot{v} \right] = -4\frac{v_0^2 - s^2}{(v_0^2 + s^2)^3}v_0 s^3 - 4\epsilon \frac{v_0^2 s^3}{(v_0^2 + s^2)^3},
\]
which integrates to
\[
\frac{s^4}{(v_0^2 + s^2)^2} \dot{v} = -\epsilon \frac{s^4}{(v_0^2 + s^2)^2} + 2v_0 \left[ \ln \left( 1 + \frac{s^2}{v_0^2} \right) - \frac{v_0^2 + 2s^2}{(v_0^2 + s^2)^2} \right].
\]
We finally have
\[
\frac{\dot{v}}{s} = -\epsilon + 2v_0 \left[ \frac{(v_0^2 + s^2)^2}{s^4} \ln \left( 1 + \frac{s^2}{v_0^2} \right) - \frac{v_0^2 + 2s^2}{s^2} \right].
\] (47)
The constant of integration has been judiciously chosen to eliminate the fourth-order singularity $s^{-4}$ on the right.

We proceed to the integration of Eq. (47). We make the change of variables $w(\tau) = v/v_0$ and $\tau = s^2/v_0^2$. Then Eq. (47) becomes
\[
\frac{dw}{d\tau} = -\frac{\epsilon}{2v_0} + v_0^2 \left[ \ln(1 + \tau) + 2 \frac{\ln(1 + \tau)}{\tau} + \frac{\ln(1 + \tau)}{\tau^2} - \frac{1}{\tau} - 2 \right]
\] (48)
We integrate Eq. (48) and obtain
\[
w(\tau) = 1 - \frac{\epsilon}{2v_0} + v_0^2 \left[ \tau \ln(1 + \tau) - \frac{1}{\tau} \ln(1 + \tau) + 1 - 2\text{Li}_2(\tau) - 3\tau \right]
\] (49)
where
\[
\text{Li}_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}
\]
Figure 3: The numerically calculated profiles of skyrmions for three values of the parameter $\epsilon$, as indicated in each figure, are shown by black dots. The blue solid line shows the far field approximation of the skyrmion profile given by Eq. (40), which is a solution of Eq. (13). The arbitrary factor of the far field has been chosen such that it fits the numerical data at large $s$. The red solid line shows the near field approximation of Eq. (47), or Eq. (49), which is a solution of Eq. (46). The free parameter $v_0$ of the near field has been chosen so that it agrees with the numerical data at $s = 0$. The green dotted line is $v = s$ and it cuts the skyrmion profile at the skyrmion radius. The skyrmion radius is decreasing for decreasing $\epsilon$ and it is seen graphically that it is approximately equal to $v_0$ for small $\epsilon$.

is the polylogarithm function and the constant of integration has been chosen so that $w(0) = 1$. Keeping the dominant terms for large $\tau$ we have

$$w(\tau) = 1 - (\epsilon + 6v_0) \frac{\tau}{2} + v_0^2 \tau \ln(\tau)$$

(50)

which, in the original variables, gives the near field

$$v_N(s) = v_0 - \left(\epsilon + 6v_0 + 4v_0 \ln v_0\right) \frac{s^2}{2} + 2v_0 s^2 \ln s.$$  

(51)

Fig. 3 shows the numerically calculated skyrmion profiles for three values of $\epsilon$ together with the corresponding approximations for the far field in Eq. (40) and the near field in Eq. (49). The arbitrary constant of the near and far fields are chosen to agree with the numerical solutions. The near field approximation turns to an increasing function of $s$ past its domain of validity and this part is not plotted in the figure. When $\epsilon$ is small enough a matching between the near field and the far field can be done through the near field Eq. (51) and the far field Eq. (53), shown below, without the use of numerical data.

5. Asymptotic matching

The far field equation has been derived under the assumptions $v_0^2 \ll s^2$ and $\dot{v} \ll s$. A condition for the validity of the near-field approximation, consistent with inequality (11) is $\epsilon s^2 \ll v_0$. We also require $v_0 < \epsilon$, which is automatically satisfied in our calculation below. Thus, both the far-field and the near-field asymptotic solutions are valid in the internal layer

$$v_0^2 \ll s^2 \ll \frac{v_0}{\epsilon} < 1, \quad \epsilon \to 0.$$  

(52)

We observe that both the far field equation obtained from Eq. (13), in which terms of order $O(s^4)$ or $O(s^4 \ln s)$ are neglected,

$$v_F(s) = a_0 \left[1 + (2\gamma - 1)s^2 + 2s^2 \ln s\right]$$  

(53)
and the near field Eq. (51) combine a constant term with $s^2 \ln s$ and $s^2$. Matching the coefficients of these terms in the two equations yields

$$\epsilon = -4v_0 (\gamma + 1 + \ln v_0),$$

or, more compactly

$$\epsilon = -4v_0 \ln \left( \frac{v_0}{\alpha} \right), \quad \alpha = e^{-(\gamma+1)} \approx 0.2065. \quad (55)$$

The last equation determines implicitly the value of $v_0$ as a function of the parameter $\epsilon$.

We have solved numerically by a shooting method Eq. (10) for various values of $\epsilon$ and found the values $v_0 = v(0)$ for the skyrmion profiles. In Fig. 4 we plot the value $v_0$ as a function of $\epsilon$ and compare the numerical data with formula (55). The agreement is excellent for small values of $\epsilon \lesssim 0.2$. For these values of $\epsilon$ the skyrmion profile is that of the Belavin-Polyakov soliton (8) for a range of $s$ far exceeding the skyrmion radius, while the profile tail is exponentially decaying for large $s$ as shown in Eq. (20).

The equations for both the near and far field in which terms of order $O(s^4)$ or $O(s^4 \ln s)$ are neglected and their derivatives are

$$v(s) = v_0 \left[ 1 + 2 s^2 \ln s + (2\gamma - 1)s^2 + \cdots \right], \quad \dot{v}(s) = 4v_0 [s \ln s + \gamma + \cdots] \quad (56)$$

each holding in its respective domain, $v_0 \ll s$ for the far-field and $s^2 \ln s \ll 1$ for the near-field.

6. Skyrmion radius

6.1. Skyrmion radius from asymptotic matching

A formula for the skyrmion radius for small values of $\epsilon$ can be determined from the above results. In order to see this note that the skyrmion radius is the solution of the equation $v(s) = s$. From Eq. (56) we find that $v(s) \approx v_0$, for small values of $s$, that is, the skyrmion radius $R$ is approximately at $s = v_0$. Since lengths for $s$ are measured in units of $2\ell_w$ we have

$$R = 2v_0 \quad (57)$$
in units $\ell_w$. We can now write Eq. (55) for the skyrmion radius as

$$\epsilon = -2R \ln \left( \frac{R}{2a} \right), \quad \epsilon \ll 1.$$  \hspace{1cm} (58)

For $R \ll 1$ the logarithm in Eq. (58) may be approximated by the Lambert $W$ function [13], for which the equation can be inverted in terms of elementary functions, i.e.,

$$R \approx -\frac{\epsilon}{2 \ln \left( \frac{\epsilon}{4\alpha} \right)} \approx -\frac{\epsilon}{2 \ln \epsilon}$$  \hspace{1cm} (59)

for $\epsilon \ll 1$. For values of $\epsilon \gtrsim 0.25$ Eq. (58) overestimates the skyrmion radius while it gives no result for $\epsilon \gtrsim 0.3$. Numerical results show that the skyrmion radius increases with increasing $\epsilon$ and it diverges to infinity for $\epsilon \to 4/\pi$, in agreement with theoretical results. The latter behavior is not captured by formula (58). The study of the regime for $\epsilon \gtrsim 0.3$ using the methods introduced in this paper is left to another study.

Restoring physical constants in Eq. (59) we have

$$R \approx -\frac{D}{K} \ln \left( \frac{D}{\sqrt{AK}} \right), \quad D \ll \sqrt{AK}.$$  \hspace{1cm} (60)

This shows that the skyrmion radius decreases for decreasing DMI constant or increasing anisotropy. In the limit $D \to 0$ or $K \to \infty$, the skyrmion radius $R$ goes to zero. A form of Eq. (58) which is more instructive as it implicates the length scales of this model, is

$$R \approx -\frac{2\ell_S}{\ln \left( \frac{2\ell_S}{\ell_w} \right)}.$$  \hspace{1cm} (61)

We will finally apply the above formulas and give specific examples. The regime of small $\epsilon$ can be obtained for small enough values of the DMI parameter $D$. Furthermore, for a regime where the skyrmion has larger radius we assume a small anisotropy $K$, which gives a large $\ell_w$ (domain wall width scale). As a first example let us choose $\epsilon = 0.6$ so that the formulas for the near and the far field give good approximations as seen in Fig. 3. From Fig. 4 we have a skyrmion radius $R \approx 2v_0 \approx 0.36\ell_w$. As a second example let us choose $\epsilon = 0.2$ which falls within the range of validity of the asymptotic matching and Eq. (55). We then have a skyrmion radius $R \approx 2v_0 \approx 0.04\ell_w$.

6.2. Universal core profile and energy asymptotics

Our approach indicates that the profile of an axially-symmetric Belavin-Polyakov solution of unit degree plays the role of the universal core profile of chiral skyrmions $m_\epsilon$ in the asymptotic regime $\epsilon \ll 1$. The formula for the skyrmion radius in Eq. (58) identifies a scaling law for the size of the skyrmion core. It is therefore natural to rescale space by the approximate skyrmion radius $R_\epsilon = -\frac{\epsilon}{2 \ln \epsilon}$ from Eq. (59) to obtain a non-collapsing family of magnetization fields

$$\hat{m}_\epsilon(y) = m_\epsilon(x) \quad \text{where} \quad y = x/R_\epsilon.$$  \hspace{1cm} (62)

Taking into account Eq. (9), rescaling $t = s/R_\epsilon$ entails the following rescaling of $v = v_\epsilon$

$$\hat{v}_\epsilon(t) = \frac{v_\epsilon(R_\epsilon t)}{R_\epsilon} \quad \text{where} \quad t = \frac{|y|}{2}.$$  \hspace{1cm} (63)
According to Eq. (56) and Eq. (57) we have $\hat{v}(t) \to \frac{1}{2}$ uniformly for bounded $t$. Consequently the fields $\hat{m}_\epsilon$ are uniformly approximated on compact subsets of $\mathbb{R}^2$ by the normalized Belavin-Polyakov solution $\phi$, i.e., the rotated stereographic map used in [21][11]. Upgrading this approximation property in terms of integral norms supports the conjectured asymptotics of minimal skyrmion energies $E_\epsilon = E_\epsilon(m_\epsilon)$

$$E_\epsilon - 4\pi \sim \frac{\epsilon^2}{\ln \epsilon} \tag{64}$$

in the regime $\epsilon \ll 1$. The upper bound has been established in [11] by means of appropriate trial functions. A matching lower bound requires an ansatz-free argument that is not at hand, but we shall outline a heuristics based on scaling and convergence. Eq. (62) yields the rescaled energy

$$\hat{E}_\epsilon(\hat{m}_\epsilon) = \int_{\mathbb{R}^2} \left[ \frac{\partial_\mu \hat{m} \cdot \partial_\mu \hat{m}}{2} + \frac{\epsilon^2}{2 \ln \epsilon} \left( \frac{1 - \hat{m}_3^2}{4 \ln \epsilon} - \hat{e}_3 \cdot (\partial_\mu \hat{m} \times \hat{m}) \right) \right] \, dy \tag{65}$$

with $\hat{E}_\epsilon(\hat{m}_\epsilon) = E_\epsilon(m_\epsilon)$. The key property of the integrand of $\hat{E}_\epsilon(\hat{m}_\epsilon)$ is that the prefactor to anisotropy and DMI is proportional to the expected energy gain in Eq. (64). The Derrick-Pohozaev identity [10] implies a balance of anisotropy and DMI

$$\int_{\mathbb{R}^2} \frac{1 - \hat{m}_3^2}{2 \ln \epsilon} \, dy = \int_{\mathbb{R}^2} \hat{e}_3 \cdot (\partial_\mu \hat{m}_\epsilon \times \hat{m}_\epsilon) \, dy. \tag{66}$$

Claiming strong convergence $\hat{m}_\epsilon - \phi \to 0$ in $H^1 \cap L^4(\mathbb{R}^2)$, which is consistent with the limited decay properties of the Belavin-Polyakov solution $\phi$, the arguments in [11] ensure convergence of the renormalized DMI term, i.e.,

$$\lim_{\epsilon \searrow 0} \int_{\mathbb{R}^2} \hat{e}_3 \cdot (\partial_\mu \hat{m}_\epsilon \times \hat{m}_\epsilon) \, dy = \int_{\mathbb{R}^2} \hat{e}_3 \cdot (\partial_\mu \phi \times (\phi - \hat{e}_3)) \, dy = -8\pi. \tag{67}$$

The resulting bounds on the anisotropy term in Eq. (66) feature in particular the well-known logarithmic divergence of mass for degree one solitons in the pure exchange model. But more importantly, going back to Eq. (65), we obtain the precise energy asymptotics

$$E_\epsilon = 4\pi \left( 1 - \frac{\epsilon^2}{2 \ln \epsilon} \right) + o \left( \frac{\epsilon^2}{\ln \epsilon} \right) \tag{68}$$

for $\epsilon \ll 1$, provided of course that minimal energies are attained by axially-symmetric skyrmions. A fully rigorous argument quantifying all approximation steps and providing suitable compactness properties of rescaled skyrmion configurations is beyond the scope of this discussion and deferred to future studies.

7. Summary of results and discussion

We have derived exact formulas by means of asymptotic methods for the profile of a chiral skyrmion in a model with exchange, easy-axis anisotropy and Dzyaloshinskii-Moriya interaction (bulk or interfacial). Our results create the picture that the chiral skyrmion is born out of a Belavin-Polyakov (BP) solution with infinitesimally small radius as the DMI constant is increased from zero (or the anisotropy constant is decreased from infinity). The basis of all calculations is Eq. (10) for a scalar field $v$ which is related to the polar angle $\Theta$ of the axially-symmetric magnetization field $m$ by

$$\Theta(r) = 2 \arctan \left( \frac{v(s)}{s} \right) \quad \text{where} \quad s = r^2. \tag{69}$$
The field \( v \) is derived from the modulus of the stereographic projection of the magnetization vector, transformed in a way that the BP solution of unit degree given by \( 2 \arctan(a/r) \) with a free scaling factor \( a \) is expressed as a constant function \( v \equiv a/2 \).

The profile \( v(s) \) at large distances (skyrmion tail), which we call the far field, has a form defined solely by exchange and anisotropy and is therefore exponentially approaching the perpendicular magnetization direction (ferromagnetic state). The length scale of this part of the profile is the domain wall width \( \ell_w \). We find an expression for the profile in this region in the form of a series by the Frobenius method, given in Eq. (40). We also find the asymptotic form of this series as a decaying exponential, given in Eq. (27).

The profile in the central region (skyrmion core) has the form of a BP solution. When the parameter \( \epsilon \), given in Eq. (4), is small (i.e., for small DMI parameter or for large anisotropy) we find a modification to the BP profile as a solution of Eq. (46). We thus obtain a precise form of the profile in the region of the skyrmion core and beyond that as \( \epsilon \to 0 \). We call this profile, given in Eq. (49) and approximated in Eq. (51), the near field.

By matching the formulas for the skyrmion profile obtained at the central region (near field) and at large distances (far field) we obtain the value of a parameter directly connected to the skyrmion radius as a function of \( \epsilon \). The result, given in Eq. (58), shows that the skyrmion radius goes to zero as \( \epsilon \to 0 \). The dependence of the radius on \( \epsilon \) supports an asymptotic relation for the minimal skyrmion energy, given in Eq. (64).

The matching of the near and far fields, in the asymptotic limit of \( \epsilon \to 0 \), leads to the same equation for both fields, shown in Eqs. (51) and (53), not only on the overlap region, but also beyond it. The asymptotic coincidence of the two fields reaches back to \( s = 0 \), in agreement to the profile in the rightmost entry of Fig. 3, in which the far field alone is a good approximation both far and near. The contribution of the near field equation is yet still crucial; it determines the arbitrary factor of the far field equation, a factor that depends entirely on the strength of the Dzyaloshinskii-Moriya term. The role of DMI for the existence of skyrmion solutions is, thus, revealed. It perturbs the Belavin-Polyakov skyrmion near the position \( r = 0 \) in such a way that the matching with the tail of the skyrmion profile is possible.

A subtle technical point pertains to the term that contains the factor \( \dot{v}^2 \) in Eq. (10). This makes the equation nonlinear in the derivatives of \( v \) setting the stage for the dominance of this term and the unlimited growth of \( v \). We have used the “smallness” conditions (11) to control this growth; and since the smallness of \( \dot{v} \) is enhanced by squaring it, the term becomes negligible. In the Appendix, we outline a proof of existence of the skyrmion, under the condition of small \( v_0 \), in spite of the presence of the potentially growing term in question.

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Appendix A. Outline of existence proof of the skyrmion

We will outline a proof that the solution of Eq. (10) with initial values \( v(0) = v_0 \) and \( \dot{v}(0) = 0 \) exists. We consider a closed interval of positive initial values \( A = [v_0^-, v_0^+] \). Eq. (10) gives that \( \ddot{v}(0) = -(v_0 + \epsilon) \), thus, \( v(s) \) is initially decreasing. We further consider the following three subsets of \( A \).
A1: Before reaching the value \( v = 0 \), the solution \( v(s) \) becomes increasing at some finite value of \( s \in (0, \infty) \).

A2: The solution reaches the value \( v(s) = 0 \) at some finite value of \( s \) (it is shown below that \( \dot{v}(s) \neq 0 \)).

A3: \( v(s) \to 0 \) as \( s \to \infty \).

With the additional conditions (1) \( v^+_0 \in A_1 \) and \( v^-_0 \in A_2 \), that is realizable as we have tested numerically, and (2) \( v(s) \) cannot converge monotonically to a nonzero value as \( s \to \infty \), that is easily deduced from Eq. (10), the three sets are mutually disjoint and they clearly constitute a partitioning of the interval \( A \).

Proving that the sets \( A_1 \) and \( A_2 \) are open in the closed set \( A \) implies that \( A_3 \) is nonempty, since the closed interval \( A \) cannot be the union of two nonempty, disjoint open sets. Hence the skyrmion exists.

The openness of \( A_2 \) is guaranteed by the fact that the graph of \( v(s) \) cuts the \( s \)-axis transversely; this is because \( v(s) = 0 \) and \( \dot{v}(s) = 0 \) hold simultaneously only for the zero solution of Eq. (10). The openness of \( A_1 \) is a more subtle issue. It is threatened by the simultaneous holding of \( \dot{v} = 0 \) and \( \ddot{v} = 0 \) at some point \( s = s^* \). However, at any such point, Eq. (10) produces \( \ddot{v}(s^*) = 8s^*v/(v^2 + s^*2) > 0 \). A positive third derivative is incompatible with the function \( v(s) \) being decreasing to the left of \( s = s^* \). This completes the proof.

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