Geometry of Four-vector Fields on Quaternionic Flag Manifolds.

Philip Foth & Frederick Leitner

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Abstract

The purpose of this paper is to describe certain natural 4-vector fields on quaternionic flag manifolds, which geometrically determine the Bruhat cell decomposition. These structures naturally descend from the symplectic group $\text{Sp}(n)$, and are related to the dressing action given by the Iwasawa decomposition of the general linear group over the quaternions, $\text{GL}_n(\mathbb{H})$.

1 Introduction

In this paper we wish to describe certain natural 4-vector fields on quaternionic flag manifolds. In the context of the Poisson geometry, a bi-vector field is penultimate in the study of the geometry of the underlying manifold. Analogously, we make use of a 4-vector field, closed under the Schouten bracket with itself, which we call a $\text{quatrisson}$ structure, to reveal the internal structures of certain natural spaces arising in geometry, namely quaternionic flag manifolds. A more general definition involving a multi-vector field was first given in [1]. Another generalization, the Nambu-Poisson structure, was studied in [16].

Quaternionic flag manifolds possess natural group invariant quatrisson structures, and the study of the geometry of the flag manifolds can be pursued in the natural setup of quatrisson 4-vector fields and tetraplectic structures [6]. In particular, we describe the so-called Bruhat quatrisson 4-vector fields on quaternionic flag manifolds where the leaf decompositions coincide with the Bruhat decompositions of $\text{GL}_n(\mathbb{H})$ defined purely combinatorially. We also show that the existence of the Bruhat decomposition leads to a description of the tetraplectic leaves in the group $\text{Sp}(n)$ in terms of the dressing action on the group.

Drinfeld [4], Lu and Weinstein [11], Semenov-Tian-Shansky [14], and Soibelman [15] first described this setup in the context of standard Poisson geometry, and this viewpoint has been elaborated by many others. Several important features of the Poisson geometry of flag manifolds readily translate to our situation, including Schubert calculus and a version of generalized hamiltonian
dynamics. We suggest that further studies of these structures might lead to interesting results related to the geometry and (equivariant) differential calculus on quaternionic flag manifolds as well as quantum groups.

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2 Quaternionic matrices and flags

We begin with some generalities on quaternionic matrices for which we define the following subgroups of $\text{GL}_n(\mathbb{H})$:

- $\mathcal{R} := \{\text{diag}(r_1, \ldots, r_n) \mid r_i \in \mathbb{R}_+\}$
- $\mathcal{U} := \{\text{upper triangular matrices with 1's along the diagonal}\}$
- $\mathcal{V}_w := \{U \in \mathcal{U} \mid (P_wUP_w^{-1})^t \in \mathcal{U}\}$ here $P_w$ denotes the permutation matrix $(P_w)_{i,j} = \delta_{i,w(j)}$ for $w \in S_n$
- $\mathcal{D} := \{\text{diag}(d_1, \ldots, d_n) \mid d_i \in \mathbb{H}^*\}$
- $\mathcal{B} := \mathcal{UD}$

Now for $G \in \text{GL}_n(\mathbb{H})$ we recall the strict Bruhat normal form of $G$ as:

$$G = UDP_wV$$

Here all the matrices are uniquely determined: the matrix $D = \text{diag}(d_1, \ldots, d_n)$ belongs to $\mathcal{D}$, $P_w$ is, as usual, the permutation matrix corresponding to $w \in S_n$; both $U$ and $V$ belong to $\mathcal{U}$; and we further require that $P_wVw^{-1}$ is lower triangular with 1's along the diagonal, i.e. $V \in \mathcal{V}_w$. This decomposition allows us to define the Dieudonné determinant as the residue of $\text{sgn}(w) \cdot \prod d_i$ in $\mathbb{H}/[\mathbb{H}, \mathbb{H}] = \mathbb{R}$. Moreover, by means of the strict Bruhat normal form, we obtain the Bruhat decomposition:

$$\text{GL}_n(\mathbb{H}) = \bigsqcup_{w \in S_n} BP_w \mathcal{V}_w.$$  

Denoting $Z_w := \{BP_w \mathcal{V}_w\}$, we obtain from the Bruhat decomposition a parameterization of $\text{GL}_n(\mathbb{H})$ by $S_n$. The condition that $V^w := wVw^{-1}$ is lower triangular implies in the case of $w = e$ that $V^e = V$ must be both upper and lower triangular hence equals the identity matrix and hence $\text{dim}_\mathbb{R}(Z_e) = \text{dim}_\mathbb{R}(\mathcal{B})$. Taking $w$ to be the longest permutation, $w_l = (n \ (n-1) \cdots \ 2 \ 1)$, rotates the matrix $V$ by $180^\circ$ so that it is lower triangular. As no further conditions on the entries of $V$ are imposed, we have that $\text{dim}_\mathbb{R}(Z_{w_l}) = 4n^2$. In general, we
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define the length of a permutation $w$, $\text{len}(w)$, to be the minimal number of adjacent transpositions required in a factorization of the permutation. One readily sees that the maximal number of non-zero entries allowed in $V_w$, so that $V_w$ is lower triangular, is exactly $\text{len}(w)$ so that $\dim_R(Z_w) = 4 \cdot \text{len}(w) + \dim_R(B)$. We will see later that the entries of $V$ give coordinates on the Bruhat cells of quaternionic flags.

Denoting conjugate transpose by $(\cdot)^*$, we have the Lie group:

$$\text{Sp}(n) := \{ g \in \text{GL}_n(\mathbb{H}) \mid g^*g = e \}$$

We also identify the corresponding Lie algebra:

$$\text{sp}_n := \{ X \in \text{gl}_n(\mathbb{H}) \mid X + X^* = 0 \}$$

with $T_x\text{Sp}(n)$ for $x \in \text{GL}_n(\mathbb{H})$ by left translation. One knows that matrices in $\text{Sp}(n)$ have Dieudonné determinant 1, and thus lie inside the semi-simple group:

$$\text{SL}(n) := \{ g \in \text{GL}_n(\mathbb{H}) \mid \det(g) = 1 \}$$

We define the *spheroid* to be $\Sigma := \text{Sp}(1)^n \simeq (S^3)^n$, whose elements are of the form $\text{diag}(\exp(s_1), \ldots, \exp(s_n))$ where the $s_i$ are purely quaternionic (no real component). This is a subgroup of $\mathcal{D}$, and we have, in fact, that:

$$\mathcal{D} = \Sigma \times \mathcal{R}$$

We denote the corresponding Lie algebra by $\mathfrak{s}$. The full quaternionic flag of $\mathbb{H}^n$, which we denote $F_n$, can now be identified as $F_n \simeq \Sigma \backslash \text{Sp}(n) \simeq \mathcal{B} \backslash \text{GL}_n(\mathbb{H})$. Using the second identification of $F_n$, we denote $C_w := \mathcal{B} \backslash Z_w$, which we call the *Bruhat cells* of the flag. By our discussion above, we see that $\dim_R(C_w) = 4 \cdot \text{len}(w)$.

**Example.** Consider the space $\mathbb{H}P^1$ identified as $\mathbb{H}P^1 \simeq \Sigma \backslash \text{Sp}(2) \simeq S^4$ from which we obtain the fibration:

$$\Sigma \simeq S^3 \times S^3 \simeq \text{Sp}(1) \times \text{Sp}(1) \rightarrow \text{Sp}(2) \downarrow \mathbb{H}P^1 \simeq S^4.$$

The Bruhat decomposition yields a decomposition of $\Sigma \backslash \text{Sp}(2) \simeq \mathbb{H}P^4$ into the cells $C_{(12)}$ and $C_e$ which have real dimensions 4 and 0 respectively. We view the cells under the identifications that $C_{(12)} \simeq \mathcal{N}^2 = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \simeq \mathbb{H}$ and $C_e$ is the North pole.

Recall that $S^4$ has neither a symplectic nor a complex structure (nor even an almost complex [13]). This is one of the reasons for introducing the tetraplectic structure in $\mathbb{H}$.
3 Quatrisson and tetraplectic structures

Recall that a symplectic manifold is a manifold equipped with a closed non-degenerate 2-form. We also recall that a Poisson manifold is a manifold equipped with a bi-vector field that induces a Lie algebra structure on the space of smooth functions, compatible with the commutative product of functions via the Leibniz rule. In the case of quaternionic flags, we make use of the following structures which reflect the underlying geometry:

**Definition 3.1** Let $X$ be a real orientable manifold of dimension $4m$. A tetraplectic structure on $X$ is a four-form, $\psi$ satisfying:
1) $\psi$ is closed ($d\psi = 0$)
2) $\psi^m$ is a volume form

We call the pair $(X, \psi)$ a tetraplectic manifold. A map $\phi: (X, \psi) \to (X', \psi')$ is called tetraplectic if $\phi^* \psi' = \psi$. If, in addition, $\phi$ is a diffeomorphism, then we call $\phi$ a tetraplectomorphism.

**Example.** Let $\psi$ be an $\text{Sp}(2)$-invariant volume form on $S^4$. Then $(\mathbb{HP}^1, \psi)$ is a teraplectic manifold. In fact, in [6] the construction of invariant tetraplectic structures on all quaternionic flag manifolds was given.

One can define a standard Poisson structure on a manifold by giving a bi-vector field whose the Schouten bracket with itself is zero. However, in order to reflect the geometry of our situation we will make use of 4-vector fields, for which we recall the following [7]:

**Proposition 3.2** Denoting $\bigwedge^i \chi(M)$ the space of $i$-vector fields on $M$, there exists a unique bracket, called the Schouten bracket:

$$[\cdot, \cdot]: \bigwedge^p \chi(M) \times \bigwedge^q \chi(M) \to \bigwedge^{p+q-1} \chi(M)$$

which extends the usual Lie bracket of vector fields and is an $\mathbb{R}$-linear operation satisfying the following identities:
1) $[P, Q] = (-1)^{pq}[Q, P]$ (Anti-Symmetry)
2) $[P, Q \wedge R] = [P, Q] \wedge R + (-1)^{pq+q} Q \wedge [P, R]$ (Leibniz)
3) $(-1)^{p(q-1)}[P, [Q, R]] + (-1)^{q(p-1)}[Q, [R, P]] + (-1)^{r(q-1)}[R, [P, Q]] = 0$ (Jacobi)

We recall that in [6] the authors use the vanishing of the Schouten bracket of a $p$-vector field $\xi$ with itself, $[\xi, \xi] = 0$, to define the Generalized Poisson Structures (GPS). For etymologic-semantic reasons, we give the following definition:

**Definition 3.3** Let $M$ be a manifold, and let $\xi$ be a 4-vector field on $M$ satisfying $[\xi, \xi] = 0$. We call $\xi$ a quatrisson structure on $M$ and the pair $(M, \xi)$ aquatrisson manifold.
Definition 3.4 For two quatrisson manifolds \((X, \xi)\) and \((X', \xi')\) a map \(\phi : X \to X'\) is called a quatrisson map if for any quadruple of functions \(f_i \in C^\infty(X')\), \(1 \leq i \leq 4\) the following identity holds:
\[
\xi(d\phi^* f_1 \wedge d\phi^* f_2 \wedge d\phi^* f_3 \wedge d\phi^* f_4) = \phi^* \xi'(df_1 \wedge df_2 \wedge df_3 \wedge df_4).
\]
i.e. \(\phi^*(\xi) = \xi'\).

Definition 3.5 Let \(\xi\) be a 4-vector field on a 4m-dimensional manifold, \(M\). Then we call \(\xi\) non-degenerate if \(\xi^m\) is a nowhere vanishing 4m-vector field.

If \(\xi\) is a non-degenerate vector field on \(M\), then \(\xi\) induces a surjection \(\bigwedge^4 T_x^*M \to T_xM\) for all \(x \in M\), obtained by contraction with \(\xi\). If \((M, \xi)\) is quatrisson, we define the rank of \(\xi\) at \(x \in M\) as the dimension of the image of this map. One can see that a quatrisson structure \(\xi\) on \(M\) is non-degenerate if the rank of \(\xi\) at any point of \(M\) is equal to the dimension of \(M\).

Lemma 3.6 Letting \((M, \xi)\) be as above, the rank of \(\xi\) at any \(x \in M\) is divisible by 4.

Proof. This is an easy exercise in multi-linear algebra for the reader. \(\square\)

Definition 3.7 Let \(M\) be a manifold equipped with a 4-vector field \(\xi\). We say that a smooth 4l-dimensional submanifold, \(L\), is a tetraplectic leaf in \(M\) if:
1) \(\xi\) comes from \(\bigwedge^4 \chi(L)\) at all points of \(L\)
2) \(\xi\) is non-degenerate on \(L\)
3) \(L\) is not properly included in any other such submanifold of \(M\)
4) the four-form, \(\psi\), given by \(i_\psi \xi^m = \psi^{m-1}\), defines a tetraplectic structure on \(M\).

To each triple of functions \(f=(f_1, f_2, f_3)\), we can associate a “hamiltonian” vector field \(X_f\) given by \(i(df_1 \wedge df_2 \wedge df_3)\xi\). Then we get the characteristic distribution of \(M\). Unlike in the Poisson case, we cannot expect in general that \((M, \xi)\) is stratified as a union of smooth tetraplectic leaves, even if \(\xi\) is quatrisson, see Example 8 in \([7]\). However, the particular case of this result for quaternionic flag manifolds will follow later.

Example. Let \(F_n\) be a quaternionic flag manifold considered as a tetraplectic manifold with an \(\text{Sp}(n)\)-invariant 4-form \(\psi\). The corresponding 4-vector field, \(\chi\), defined by \(i_\chi(\psi^m) = \psi^{m-1}\), is quatrisson and \(\text{Sp}(n)\)-invariant. We refer to \(\chi\) as the invariant quatrisson structure of \(F_n\). \(\square\)

4 Quatrisson structures on \(\mathbb{H}P^1\).

For our flag manifolds, we construct a Bruhat quatrisson structure explicitly by analogy to the Poisson case as in \([11]\) or \([12]\). We begin by defining a quatrisson 4-vector field, \(\kappa\), on \(\text{Sp}(2)\), which we show descends to the quotient \(\Sigma / \text{Sp}(2)\).
We begin by defining an element $\Lambda$ of $\wedge^4 sp_2$ in terms of the following basis for $sp_2$:

$$
E = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}, \quad S_x = \begin{pmatrix}
0 & x \\
x & 0
\end{pmatrix}, \\
H_x = \begin{pmatrix}
x & 0 \\
0 & -x
\end{pmatrix}, \quad M_x = \begin{pmatrix}
x & 0 \\
0 & x
\end{pmatrix}
$$

where $x$ is one of $\{1, j, k\}$. For convenience we denote $S_{-x} := -S_x$, $H_{-x} := -H_x$, and $M_{-x} := -M_x$. We now can note the following commutator relations:

$$
[M_x, E] = 0 \quad [H_x, E] = 2 \cdot S_x \quad [S_x, E] = 2 \cdot H_x
$$

where $x$ is one of $\{1, j, k\}$. For convenience we denote $S_{-x} := -S_x$, $H_{-x} := -H_x$, and $M_{-x} := -M_x$. We now can note the following commutator relations:

$$
[M_x, E] = 0 \quad [H_x, E] = 2 \cdot S_x \quad [S_x, E] = 2 \cdot H_x
$$

We may now define:

$$
\Lambda := E \wedge S_i \wedge S_j \wedge S_k.
$$

and denoting by $\Lambda^L$ and $\Lambda^R$ the left and right invariant 4-vector fields on $Sp(2)$ with value $\Lambda$ at the identity element, we let:

$$
\kappa = \Lambda^L - \Lambda^R.
$$

**Proposition 4.1** The 4-vector field $\kappa$ is a quatrisson structure on $Sp(2)$. Moreover, $\kappa$ descends to a vector field, $\aleph$, on $\mathbb{HP}^1 \simeq \Sigma \backslash Sp(2)$ inducing a $\Sigma$-invariant quatrisson structure on $\mathbb{HP}^1$ called the Bruhat quatrisson structure.

**Proof.** The fact that $\kappa$ is a quatrisson structure on $Sp(2)$ is nothing more than the fact that $[\kappa, \kappa] \in \wedge^7 \chi(\text{Sp}(2)) = 0$. To show that $\aleph$ is $\Sigma$-invariant, and hence descends, we may apply the same formalism of the Poisson case and show that for any $X \in \mathfrak{s}$ we have $\text{ad}_X(\Lambda) = [X, \Lambda] = 0$. This follows readily from the above commutator relations and the Leibniz rule of the Schouten bracket. It is clear that $[\aleph, \aleph] = 0$. $\square$

We can make use of the Bruhat decomposition to describe the vector field explicitly. As above, we denote by $C_\alpha$ and $C_{(12)}$ the cells of $F_n$ corresponding to
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the North pole and the \( \mathbb{H} \) components. It is clear, that at the North pole \( \kappa \) is the zero vector. For \( x \in \mathcal{C}_{(12)} \) we choose a convenient coset representative in \( \text{Sp}(2) \), namely:

\[
k_x = \frac{1}{\sqrt{1 + \rho^2}} \begin{pmatrix} -\bar{v} & 1 \\ 1 & v \end{pmatrix},
\]

where \( \rho = |v| = \sqrt{v\bar{v}} \). In fact, the identification of \( S^4 \) as a natural \( \text{SO}(5) \simeq \text{Sp}(2)/\mathbb{Z}/2 \)-invariant submanifold of \( \mathbb{R}^5 \) with \( \mathbb{H} \) plus the point at infinity using stereographic projection sends \( v \in \mathbb{H} \) to a point in \( S^4 \) at the height \( 1 - \frac{1}{1 + |v|^2} \)

and the same \( \text{Sp}(1) \)-angular coordinate.

To compute \( \mathcal{N} \) at \( x \), we identify \( T_x \text{Sp}(2) \) with \( T_e \text{Sp}(2) \) by right translations so that we have \( \Lambda^R = \Lambda \) and \( \Lambda^L \) is simply conjugation of \( \Lambda \) by \( k_x \). We have thus expressed \( \mathcal{N} = f(v) \partial_{v_1} \wedge \partial_{v_2} \wedge \partial_{v_3} \wedge \partial_{v_4} \) in terms of the coordinates \( v = v_1 + v_2 \cdot i + v_3 \cdot j + v_4 \cdot k \). We would like more natural coordinates for \( \mathbb{H} \), namely

if \( v = \rho \cdot \exp(\theta_1 \cdot i)\exp(\theta_2 \cdot j)\exp(\theta_3 \cdot k) \in \mathbb{H} \), then we have \( \mathcal{N} = \frac{g(\rho)}{\rho^3} \partial_\rho \wedge \Theta \),

for \( \Theta = \partial_{\theta_1} \wedge \partial_{\theta_2} \wedge \partial_{\theta_3} \) - the \( \text{Sp}(1) \)-invariant 3-vector field on \( \text{Sp}(1) \simeq S^3 \). To find \( g(\rho) \), we divide \( \mathcal{N} \) by its value at \( v = 0 \), the South pole. After a computer assisted computation\(^1\) we see that \( g(\rho) = \frac{(1 + 3\rho^4)}{(1 + \rho^2)^3} \). Thus we have proved that:

**Proposition 4.2** The invariant quatrisson structure on \( S^4 \simeq \mathbb{H}P^1 \) is given by:

\[
\chi := \frac{(1 + \rho^2)^4}{\rho^3} \partial_\rho \wedge \Theta
\]

and the Bruhat quatrisson structure is given by:

\[
\mathcal{N} = \frac{(1 + \rho^2)(1 + 3\rho^4)}{\rho^3} \partial_\rho \wedge \Theta.
\]

In particular we have that:

\[
\mathcal{N} = \frac{1 + 3\rho^4}{(1 + \rho^2)^3} \chi.
\]

One can easily see that \( \mathcal{N} \) has rank four everywhere except for at the North pole, where it vanishes. Thus the two cells are characterized by the rank of \( \mathcal{N} \).

5 Quatrisson structures on flag manifolds

Following \[12\] we will produce the quatrisson structure on the full flag of \( \mathbb{H}^n \) by way of the so-called multiplication formula. By analogy to the \( \text{Sp}(2) \) case, for \( 1 \leq p < q \leq n \) we denote by \( E_{p,q} \) the quaternionic matrix whose entries are 0’s everywhere except in the \( (p,q) \)-th position which is 1, and the \( (q,p) \)-th

\[^1\]We thank Klaus Lux and Stephane Lafortune for help with this.
position which is $-1$. We also let $S_{x}^{p,q}$ denote the matrix with 0’s everywhere except in the $(p,q)$-th and $(q,p)$-th positions where the entries are $x$ where $x$ is again chosen from $\{i, j, k\}$. Similarly, these matrices are clearly in the Lie algebra, $\mathfrak{sp}_n$, of $\text{Sp}(n)$, and correspond to “positive roots” (i.e. pairs of integers $1 \leq p < q \leq n$) as in [12]. We define $\Lambda \in \bigwedge^4 T_e \text{Sp}(n)$ by:

$$
\Lambda = \sum_{p<q} E^{p,q} \wedge S_{i}^{p,q} \wedge S_{j}^{p,q} \wedge S_{k}^{p,q}.
$$

Then if $\Lambda^L$ and $\Lambda^R$ are the right and left invariant 4-vector fields on $\text{Sp}(n)$ with the values $\Lambda$ at the identity element on $\text{Sp}(n)$, we let:

$$\kappa = \Lambda^L - \Lambda^R.$$

Unlike the $\text{Sp}(2)$ case, when $n > 2$, one can readily check that $\kappa$ will not be a quartrisson structure on $\text{Sp}(n)$ by making use of the Leibniz rule and commutator relations similar to those as above and noting that there will be some terms that will not cancel. However, we still have:

**Proposition 5.1** The 4-vector field $\kappa$ descends to $\Sigma \backslash \text{Sp}(n)$, inducing a $\Sigma$-invariant quartrisson structure, $\aleph$, called the Bruhat quartrisson structure.

**Proof.** For $\kappa$ to descend and be invariant we need to show that both the left and right translations by elements of the Spheroid leave $\kappa$ invariant, meaning that the adjoint action by the Spheroid on $\Lambda$ is trivial. This can be checked similarly to the $n = 2$ case of Proposition 4.1. One can also directly check that $[\aleph, \aleph] = 0$ on $\Sigma \backslash \text{Sp}(n)$, which will also follow from Proposition 5.6. \(\square\)

We recall:

**Definition 5.2** Let $H$ be a Lie group equipped with a multiplicative 4-vector field $\mu$, which acts on a quartrisson manifold $(P, \xi)$:

$$\beta : H \times P \rightarrow P.$$

We say that $H$ acts multiplicatively if, denoting the corresponding translation maps:

$$\beta_h : P \rightarrow P \hspace{1cm} \beta_y : H \rightarrow P$$

$$y \mapsto h \cdot y \hspace{1cm} h \mapsto h \cdot y$$

we have:

$$\xi(h \cdot x) = \beta_h \cdot \xi(x) + \beta_x \cdot \mu(h).$$

We sometimes say that the actions is multiplicative with respect to the direct sum 4-vector field $\mu \oplus \xi$ on $H \times P$.

and notice the following fact (cf. [11], [8]):
Lemma 5.3 The 4-vector field $\kappa$ on $\text{Sp}(n)$ is multiplicative.

Proposition 5.4 Let $\mathcal{H}$ be the Bruhat quatrisson structure on $\Sigma\setminus\text{Sp}(n)$. The action map:

$$\text{Sp}(n) \times \Sigma\setminus\text{Sp}(n) \to \Sigma\setminus\text{Sp}(n) : (g, h) \mapsto g \cdot h$$

is multiplicative with respect to the four-vector field $\kappa \oplus \mathcal{H}$ on $\text{Sp}(n) \times (\Sigma\setminus\text{Sp}(n))$.

Proof. Straightforward. $\square$

We will also make use of the following embeddings:

$$f_{r,r+1} : \text{Sp}(2) \to \text{Sp}(n)$$

where $1 \leq r < n$ and for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the matrix $A_{r,r+1}$ is given by:

$$\begin{pmatrix}
Id & 0 \\
 a & b \\
 c & d \\
 0 & Id
\end{pmatrix} \leftarrow r^{th} \text{ row}$$

Lemma 5.5 The embeddings $f_{r,r+1} : \text{Sp}(2) \hookrightarrow \text{Sp}(n)$ respect the multiplicative 4-vector fields $\kappa$.

Proof. Straightforward. $\square$

Proposition 5.6 Every tetraplectic leaf $L$ of $\text{Sp}(n)$ lies entirely in some $Z_w$.

If $L_w$ is a tetraplectic leaf containing the permutation matrix $P_w$ corresponding to some $w \in S_n$, and we write $w = \prod_{i=1}^{m} \tau_i$ as a minimal product of adjacent transpositions, we have a tetraplectomorphism:

$$F_w : L_{\tau_1} \times \cdots \times L_{\tau_m} \to L_w$$

$$(l_1, \ldots, l_m) \mapsto l_1 l_2 \cdots l_m$$

Moreover, for $\sigma \in \Sigma$, the tetraplectic leaf through $\sigma P_w$ equals $\sigma L_w$.

Proof. (cf. [12], [13].) Immediately follows from the discussion above. $\square$

More explicitly, one can follow [12] to identify $L_w$ with the $V_w$ - orbit of $P_w$, and in the next section, we will define and exploit the analogues of the dressing action [14] to get clearer picture of the tetraplectic leaves. In any case, we have the following:

Theorem 5.7 The tetraplectic leaf decomposition of the quaternionic flag manifold $F_n \simeq B\setminus\text{GL}_n(\mathbb{H})$ arising from the Bruhat quatrisson structure coincides with the Bruhat cell decomposition.

Proof. The important point is that any tetraplectic leaf in $\text{Sp}(n)$ under the quotient map $\text{Sp}(n) \to \Sigma\setminus\text{Sp}(n)$ maps tetraplectomorphically onto a Bruhat cell as follows from the results in this section. $\square$
6 Quatrisson action and intrinsic derivative

We elaborate on some general notions related to group actions in the quatrisson context where we recall the notation set forth in Definition 5.2 and assume that we have a multiplicative action. Denoting the Lie algebra of $H$ by $\mathfrak{h}$, we let:

$$\gamma : \mathfrak{h} \to \chi(P)$$

be the usual Lie algebra anti-homomorphism, and recall the intrinsic derivative of $\xi$ at $e$:

$$d_e \xi : \mathfrak{h} \to \bigwedge^4 \mathfrak{h}.$$ 

We also define the 4-bracket $[\cdot,\cdot,\cdot,\cdot]$ on $\mathfrak{h}^*$ to be the dual of $d\xi_e$. The next statement and its proof are analogous to Theorem 2.6 of [11].

**Theorem 6.1** In the above situation for each $X \in \mathfrak{h}$ we have:

$$\mathcal{L}_\gamma(X)\xi = \wedge^4 (d_e\mu)(X).$$

Moreover, for any 1-forms $\omega_i$ for $1 \leq i \leq 4$ on $P$ we have:

$$\mathcal{L}_\gamma(X)\xi(\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4) = \langle [\zeta_1,\zeta_2,\zeta_3,\zeta_4],X \rangle$$

where $\zeta_i$ is the $\mathfrak{h}^*$-valued function on $P$ defined by:

$$\langle \zeta_i,X \rangle = \langle \omega_i,\gamma(X) \rangle \quad \text{for} \ X \in \mathfrak{h}$$

and $[\zeta_1,\zeta_2,\zeta_3,\zeta_4]$ denotes the point-wise 4-bracket in $\mathfrak{h}^*$.

7 Dressing action

The Iwasawa decomposition of $GL_n(\mathbb{H}) = \mathcal{RU}Sp(n) = Sp(n)\mathcal{R}\mathcal{U}$ allows us to define:

**Definition 7.1** The dressing action of $\mathcal{R}\mathcal{U}$ on $Sp(n)$ is the map $\mathcal{R}\mathcal{U} \times Sp(n) \to Sp(n)$ given by $(G,K) \mapsto K'$ where $G \cdot K = K' \cdot R \cdot U$ for the unique $R \in \mathcal{R}$ and $U \in \mathcal{U}$.

Our goal of this section is to relate the orbits of the dressing action with the tetraplectic leaves of the group $Sp(n)$. Notice that we have restricted the usual dressing action to $\mathcal{R}\mathcal{U}$ since we will be only concerned with $\mathcal{R}\mathcal{U}$ orbits of the dressing action in the remainder. Finally, we can state the main result of this section.

**Theorem 7.2** The tetraplectic leaves of $\kappa$ on $Sp(n)$ are the orbits of the dressing action of $\mathcal{R}\mathcal{U}$ on $Sp(n)$. 
Proof. We already know that the leaves are parametrized by $S_n$ and $\Sigma$. More precisely, we define the center of any leaf as the element $P_w \sigma$, where $w$ as usual is the permutation matrix corresponding to $w \in S_n$, and $\sigma \in \Sigma$. The dressing action can be rewritten as $(G, K) \mapsto GKG' \in \text{Sp}(n)$, for $G, G' \in \mathcal{R}U$, which leaves us in the same (open) submanifold of the Bruhat decomposition.

Taking $K = \sigma P_w$, we see that the orbit of a dressing action on a cell remains in that cell as there are no permutations appearing in $G$ or $G'$. Further, the fact that the orbit is contained within a single leaf follows from $G$ and $G'$ being upper triangular with real diagonal, and thus the dressing action does not introduce any non-trivial elements of $\Sigma$.

For the opposite inclusion, suppose we are given two points, $K_1, K_2$, of a tetraplectic leaf. As the $K_i$ are in the same leaf, this implies that the $K_i$'s have the same permutation type, $w$, in the Bruhat decomposition, so we write $K_1 = B_1 P_w V_1$ for some $B_1 \in \mathcal{B}$ and some $V_1 \in V_w$. Then we have $B_2 B_1^{-1} K_1 = K_2 V_2 V_1^{-1}$ with $V_2 V_1^{-1} \in V_w$. Now, as $B_i \in \mathcal{B}$, we may write:

$$B_i = \text{diag}(d_{i1}, \ldots, d_{in}) \text{diag}(r_{i1}, \ldots, r_{in}), \quad r_{ij} \in \mathbb{R}^+, \quad d_{ij} \in \text{Sp}(1).$$

But as the orbits are parametrized by $\Sigma$, we know that $d_{ij}$ corresponds to $d_{ij}'$, which implies that $B_2 B_1^{-1}$ must be in $\mathcal{R}$ from which it follows that the $K_i$'s lie in the same orbit.

Another possible proof of the above result can be obtained using the infinitesimal computations near the centers of each leaf [11], [14]. Once we know that the tetraplectic leaves go along the orbits of the dressing action infinitesimally, the analyticity of the manifolds in question will provide a global coincidence.

We have established that the orbits of the dressing action of $\mathcal{R}U$ on $\text{Sp}(n)$ coincide with the tetraplectic leaves induced by the 4-vector field $\kappa$, and these are permuted by the action of $\Sigma$. Therefore we have obtained a geometric orbit picture for any tetraplectic leaf or a Bruhat cell, in $F_n$.

8 Further remarks.

First of all, the approach that we pursued in the present paper can be easily extended to all partial quaternionic flag manifolds, in particular the Grassmannians and projective spaces.

It would be interesting to express the dressing action as a quartrisson action, with respect to a multiplicative 4-vector field on $\mathcal{R}U$. While it is clear that such a structure exists, it is not easy to write down a local expression. It seems plausible that a suitable generalization of Lu-Ratiu construction [10] would help.

Evens and Lu [5] showed that the Kostant harmonic forms [9] on complex flag manifolds have a Poisson harmonic nature with respect to the Bruhat Poisson structure. It would be interesting to see how their ideas can be applied to our situation. One can use the operator $\partial_\kappa = -d \circ \iota_\kappa + \iota_\kappa \circ d + \iota_\sigma$ to define $\text{Sp}(n)$-harmonic forms on the quaternionic flag manifolds. Here $\sigma$ is the modular tri-vector field given by $d(\iota_\kappa \psi^m) = \iota_\sigma \psi^m$, and $\psi^m$ is a $\text{Sp}(n)$-invariant
volume form on $F_n$. Analogously to the $T$-equivariant cohomology of complex flag manifolds, one can consider the $\Sigma$-equivariant cohomology. Another possibility is to consider quaternionic flag manifolds as fixed point sets of certain natural involutions on complex partial flag manifolds, where the dimensions of the subspaces are even, and restrict certain subalgebra of forms.

Another possible venue to pursue is to study the hamiltonian type dynamics associated with the quaternionic structures. In particular, it seems that to determine a system subject to a $\Sigma$-action which preserves a hamiltonian, we may need fewer integrals than in the standard Poisson case. We suspect that certain symmetric spaces such as quaternionic Grassmannians will have the property that an invariant quaternionic structure is compatible with the Bruhat quaternion structure, i.e. $[\chi, R] = 0$. This would lead to a generalized bi-hamiltonian type systems, which are worth investigating.

The 4-bracket on $\mathfrak{h}^*$ that we briefly mentioned in Section 6, gives rise to a certain deformed algebra of functions on $H$ (by way of the Kontsevich formality theorem) where the deformation parameter $\hbar$ now has degree 2. This implies that the $m_2$ term in the operadic expansion is just the standard multiplication, $m_3$ is trivial, and $m_4$ is determined by the bracket. This is the first natural occurrence of the generalized quantum group setup that we are aware of, and thus it seems plausible that it would lead to new interesting algebraic structures.

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