DRAWING NON-PLANAR GRAPHS WITH CROSSING-FREE SUBGRAPHS*

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THE PROBLEM

Instance: a pair $<G,S>$ such that

- $G$ is a **non-planar** graph
- $S$ is a **planar spanning** subgraph of $G$

Question: does there exist a **straight-line drawing** of $G$ in which the edges of $S$ not involved in any crossing?

$<G,S> = <K_{3,3}, \text{hamiltonian path}>$
BAD CROSSINGS
GOOD CROSSINGS
MOTIVATIONS

- different groups of edges may have different semantics/importance

- a visual interface might attempt to display more important edges in a planar way

- try to maintain the overview of the whole graph

Routing Tree for node Q
TWO SETTINGS

Straight-line setting
1. edges of $S$ and of $G \setminus S$ are straight-line segments

Polyline setting
1. edges of $S$ are straight-line segments
2. edges of $G \setminus S$ have at most $k$-bends

**straight-line compatible drawings**

**k-bend compatible drawings**
RESULTS: STRAIGHT-LINE SETTING

Always positive instances

1. spiders

2. caterpillars

3. BFS-trees
   - every graph admits such planar spanning subgraphs
RESULTS: STRAIGHT-LINE SETTING

Negative instances

- even if S is a binary tree

Efficient testing and drawing algorithm

- when S is triconnected
STRAIGHT-LINE: SPIDERS

\[ u \in S \]

\[ u \in G \setminus S \]
Augmentation of $G$ and $S$ with dummy vertices and edges

Order $L$
- suitable traversal of the augmented version of $S$ started at $u_1$

$$L = \{u_1, s_1, v_{11}, v_{12}, v_{13}, t_1, u_2, s_2, v_{21}, t_2, u_3, s_3, v_{31}, t_3, u_4, s_4\}$$
CONSTRUCTION:

- consider a quarter of circumference $C$
- split $C$ in $|L|-1$ equally spaced sectors
- each vertex is suitably drawn along the rays delimiting each sector
STRAIGHT-LINE: CATERPILLARS

CONSTRUCTION:

• vertices of the spine
CONSTRUCTION:

- leaf vertices are drawn on a convex curve
**STRAIGHT-LINE: CATERPILLARS**

*S is crossing free:*

1. S is drawn planar
2. edges between leaves inside $P$
3. edges between spine and leaf vertices do not cross $S$

**Nice property:**

$$\frac{d_{\min}}{d_{\max}} = \Omega(n^{-1})$$
RESULTS: NEGATIVE INSTANCES

1. $G = K_{13}$ and $S$ is the complete rooted ternary tree

2. $G = K_{22}$ and $S$ is the complete unrooted binary tree
RESULTS: NEGATIVE INSTANCES

Case 1

Case 2

\[ S \subseteq \mathcal{G} \]

\[ r \in S \]  \[ u \in G \setminus S \]
RESULTS: NEGATIVE INSTANCES

Case 1

Case 1.1

Case 1.2
RESULTS: NEGATIVE INSTANCES

Case 1.1
RESULTS: BFS-TREES

**KEY PROPERTY:**
edges of $G \setminus S$ connect vertices belonging to the **same level** or to **consecutive levels**
- construction similar to the caterpillar case

BFS-trees can be constructed in linear time

the drawing may require $\Omega(2^n)$-area
Necessary and sufficient conditions:

1. each edge $e$ of $G \setminus S$ connects vertices belonging to the same face of $S$
2. there exists a face $f$ of $S$ containing 3 vertices that do not separate the endpoints of any edge of $G \setminus S$
Testing Condition 2:

Auxiliary labelled biconnected outerplane graph $G_f$ for each face $f$ of $S$

- **EMPTY** faces of $G_f$ contain vertices satisfying Cond 2
- **FULL** faces of $G_f$ do not
CASE 1: edge $e$ splits a single EMPTY face $F$ of $G_f$

UPDATE: replace $F$ with 2 EMPTY faces $F'$ and $F''$
STRAIGHT-LINE: TRICONNECTED GRAPHS

Testing Condition 2:

\[ G_f \]

CASE 2: edge e crosses a set of inner edges of \( G_f \)

UPDATE: form a new **FULL** face \( F_x \)
(and remove the previous EMPTY and FULL faces traversed)

\[ G = (V,E) \quad S=(V,W) \]

**COND2: \( O(|E\setminus W| \times |V|) \)-time**
RESULTS: POLYLINE SETTING

- grid K-bend compatible drawings of Trees
  - 1-bend
  - 3-bend
  - RAC 4-bend

right angle crossings
POLYLINE: 3-BEND TREES

- augment $S$ to an embedded biconnected planar graph $S^*$

- dummy vertices belong to the same face $F^*$
POLYLINE: 3-BEND TREES

- **straight-line grid drawing** of $S^*$ [Kant’96]
  - the leaf vertices of $S^*$ have the same $y$-coordinate $Y$
- bend points for the edges of $G\setminus S$ have either $y$-coordinate $Y$ or $Y-1$
OPEN PROBLEMS FOR FUTURE WORK

- STRAIGHT-LINE SETTING
  - What is the complexity of the problem when $S$ is not a spanning triconnected graph?
  - Give a characterization of which spanning trees of a complete graph can be always realized.

- POLYLINE SETTING
  - What is the optimal area requirement for grid $k$-bend compatible drawings when $S$ is a spanning trees?
THANK YOU!