On \(n\)-th class preserving automorphisms of \(n\)-isoclinism family

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MS received 8 May 2017; revised 27 November 2017; accepted 11 December 2017; published online 18 December 2018

Abstract. Let \(G\) be a finite group and let \(M\) and \(N\) be two normal subgroups of \(G\). Let \(\text{Aut}_M(N)(G)\) denote the group of all automorphisms of \(G\) which fix \(N\) element-wise and act trivially on \(G/M\). Let \(n\) be a positive integer. In this article, we have shown that if \(G\) and \(H\) are two \(n\)-isoclinic groups, then there exists an isomorphism from \(\text{Aut}_{\gamma_n+1}(G)\) to \(\text{Aut}_{\gamma_n+1}(H)\), which maps the group of \(n\)-th class preserving automorphisms of \(G\) to the group of \(n\)-th class preserving automorphisms of \(H\). Also, for a nilpotent group \(G\) of class \((n + 1)\), if \(\gamma_{n+1}(G)\) is cyclic, then we prove that \(\text{Aut}_{\gamma_{n+1}}(G)\) is isomorphic to the group of inner automorphisms of a quotient group of \(G\).

Keywords. Finite group; inner automorphism; \(n\)-isoclinism; \(n\)-th class preserving automorphism.

2010 Mathematics Subject Classification. 20D15, 20D45.

1. Introduction

In 1940, Hall [4] introduced the notion of isoclinism to have a satisfactory classification of finite \(p\)-groups. Isoclinism is an equivalence relation in the class of all groups and the notion holds for finite as well as infinite groups. The notion of isoclinism is weaker than isomorphism or in other words, isoclinism is a more general equivalence relation in the class of all groups and all the abelian groups form one equivalence class. Roughly speaking, two groups are isoclinic if their central quotients are isomorphic and their commutator subgroups are also isomorphic. More precisely, if \(G\) and \(H\) are two groups, we say \(G\) is isoclinic to \(H\) if there exists an isomorphism \(\alpha : G/Z(G) \rightarrow H/Z(H)\) and an isomorphism \(\beta : \gamma_2(G) \rightarrow \gamma_2(H)\) such that the following diagram commutes:

\[
\begin{array}{ccc}
G/Z(G) \times G/Z(G) & \xrightarrow{\alpha \times \alpha} & H/Z(H) \times H/Z(H) \\
\downarrow \gamma_G & & \downarrow \gamma_H \\
\gamma_2(G) & \xrightarrow{\beta} & \gamma_2(H),
\end{array}
\]

where \(Z(G)\) and \(\gamma_2(G)\) denote the center and the commutator subgroup of the group \(G\). The maps \(\gamma_G\) and \(\gamma_H\) are defined as

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\[ \gamma_G(g_1 Z(G), g_2 Z(G)) = [g_1, g_2] \]
for all \( g_1, g_2 \in G \) and
\[ \gamma_H(h_1 Z(H), h_2 Z(H)) = [h_1, h_2] \]
for all \( h_1, h_2 \in H \).

The notion of isoclinism has played a very important role in the classification of finite \( p \)-groups (see [4, 5]). Many authors have investigated the various group theoretical properties which are invariant under isoclinism (see [5, 6, 8]). Later, Hall [5] generalized the notion of isoclinism to that of isologism, which is in fact an isoclinism with respect to certain varieties of groups.

Let \( n \) denote a positive integer. In 1986, Hekster [6] conceptualized and studied the notion of \( n \)-isoclinism over the variety of all nilpotent groups of class at most \( n \). He has also done an extensive study of the various group theoretical properties which are invariant under \( n \)-isoclinism. He has shown that most of the known results for isoclinism carry over to \( n \)-isoclinism. In recent years, some mathematicians also studied various subgroups of the group of all automorphisms of \( n \)-isoclinic groups, then there exists an isomorphism
\[ \phi : \text{Aut}_{Z_1(G)}(G) \to \text{Aut}_{Z_1(H)}(H) \] such that \( \phi(\text{Aut}_{Z_1}(G)) = \text{Aut}_{Z_1}(H) \). In this article, we study the subgroups of the automorphism group which are invariant for an \( n \)-isoclinism family. More precisely, in Theorem 3.3, we prove that if \( G \) and \( H \) are two finite non abelian \( n \)-isoclinic groups, then there exists an isomorphism
\[ \Psi : \text{Aut}_{Z_{n+1}(G)}(G) \to \text{Aut}_{Z_{n+1}(H)}(H) \] such that \( \Psi(\text{Aut}_{Z_n}(G)) = \text{Aut}_{Z_n}(H) \), where \( \gamma_{n+1}(G) \) and \( Z_{n}(G) \) denote the \((n + 1)\)-th terms of the lower and the upper central series of group \( G \), respectively. Rai’s Theorem A of [8] and the extension of Yadav’s Theorem 4.1 of [9] are obtained as Corollary 3.4.

Rai [8] also proved that if \( G \) is a finite \( p \)-group of nilpotency class two, then
\[ \text{Aut}_{Z_1(G)}(G) = \text{Inn}(G) \] if and only if \( \gamma_2(G) \) is cyclic. In section 3, we generalize this result in the following two ways:

(1) We consider a finite nilpotent group (not just a finite \( p \)-group of nilpotency class 2).
(2) We study \( \text{Aut}_{Z_n(G)}^M(G) \) for a central subgroup \( M \) of \( G \) which contains \( \gamma_{n+1}(G) \).

In Theorem 3.5, we prove that if \( G \) is a finite non abelian group of nilpotency class \((n + 1)\) with a central subgroup \( M \) such that \( \gamma_{n+1}(G) \subseteq M \), then
\[ \text{Aut}_{Z_{n+1}(G)}^M(G) \cong \text{Inn}(G/Z_{n-1}(G)) \]
if and only if \( M_{p_i} \) is cyclic for each \( p_i \in \pi(G/Z_{n-1}(G)) \), where \( M_{p_i} \) denotes the \( p_i \)-primary component of the group \( M \) and \( \pi(G) \) denotes the set of all prime divisors of the order of a group \( G \). We obtain Rai’s Theorem B(2) of [8] as Corollary 3.6.
2. Notations and preliminaries

In this section, we recall a few definitions and some known results. Let $G$ be a finite group. We denote the identity of a group by $1$. Throughout the article, $n$ denotes an integer and $p$ denotes a prime number. The lower central series of a group $G$ is the series $\gamma_1(G) \geq \gamma_2(G) \geq \cdots$ defined as $\gamma_1(G) = G$ and $\gamma_{n+1}(G) = [\gamma_n(G), G]$ for all $n \geq 1$. Note that each $\gamma_i(G)$ is a characteristic subgroup of $G$ and $\gamma_2(G)$ is called the commutator subgroup of $G$. A group $G$ is said to be nilpotent of class $n$ if $\gamma_n(G) \neq 1$ and $\gamma_{n+1}(G) = \{1\}$.

The upper central series of $G$ is a sequence of normal subgroups $\{1\} = Z_0(G) \leq Z_1(G) \leq Z_2(G) \leq \cdots$ such that $Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$. From the definition of upper central series, it follows that $g \in Z_n(G)$ if and only if $[g, g_1, \ldots, g_n] = 1$, where $g_i \in G$ for $1 \leq i \leq n$. Hence, $[Z_n(G), \gamma_n(G)] = \{1\}$. With this observation, it is easy to see that there exists a well-defined map

$$\gamma(n, G): \frac{G}{Z_n(G)} \times \cdots \times \frac{G}{Z_n(G)} \to \gamma_{n+1}(G)$$

given by

$$\gamma(n, G)(g_1Z_n(G), \ldots, g_{n+1}Z_n(G)) = [g_1, g_2, \ldots, g_{n+1}].$$

We recall the notions of $n$-homoclinism and $n$-isoclinism which are introduced by Hall [5] and extensively investigated by Hekster [6].

Let $G$ and $H$ be two finite groups. A pair $(\alpha, \beta)$ is called $n$-homoclinism between $G$ and $H$ if $\alpha: G/Z_n(G) \to H/Z_n(H)$ is a group homomorphism and $\beta: \gamma_{n+1}(G) \to \gamma_{n+1}(H)$ is a group homomorphism such that the following diagram commutes:

$$\begin{array}{ccc}
G/Z_n(G) \times \cdots \times G/Z_n(G) & \xrightarrow{\alpha^{n+1}} & H/Z_n(H) \times \cdots \times H/Z_n(H) \\
\downarrow \gamma(n, G) & & \downarrow \gamma(n, H) \\
\gamma_{n+1}(G) & \xrightarrow{\beta} & \gamma_{n+1}(H).
\end{array}$$

Observe that the notion of $n$-homoclinism generalizes the notion of homomorphism. In fact, $0$-homoclinism is nothing but homomorphism. If $\alpha$ is surjective, then $\beta$ is surjective and if $\beta$ is injective, then $\alpha$ is injective. We say $G$ and $H$ are $n$-isoclinic if $\alpha$ and $\beta$ are isomorphisms. In this case, the pair $(\alpha, \beta)$ is called $n$-isoclinism between $G$ and $H$.

An $n$-isoclinism is an equivalence relation on the class of all groups and an equivalence class is called an $n$-isoclinism family.

Now we recall some well-known results on $n$-isoclinism ($n$-homoclinism).

Lemma 2.1 [6, Lemma 3.8]. Let $(\alpha, \beta)$ be an $n$-homoclinism from $G$ to $H$ and let $x \in \gamma_{n+1}(G)$. Then the following statements hold:

1. $\alpha(xZ_n(G)) = \beta(x)Z_n(H)$.
2. For $g \in G$ and $h \in \alpha(gZ_n(G))$, $\beta(g^{-1}xg) = h^{-1}\beta(x)h$.

We also recall the following theorem of Hekster.
Theorem 2.2 [6, Theorem 3.12]. Let \((\alpha, \beta)\) be an \(n\)-isoclinism from \(G\) to \(H\). Then for all \(i \geq 0\),
\[
\beta(\gamma_{n+1}(G) \cap Z_i(G)) = \gamma_{n+1}(H) \cap Z_i(H).
\]

The exponent of a group \(G\) is the smallest positive integer \(n\) such that \(g^n = 1\) for all \(g \in G\). We denote the exponent of a group \(G\) by \(\exp(G)\).

The following result is due to \[2\].

Lemma 2.6 [2, Proposition 1.3]. Let \(G\) and \(H\) be two finite abelian \(p\)-groups and let \(\exp(H) | \exp(G)\). Then \(\hom(G, H) \simeq G\) if and only if \(H\) is cyclic.

3. \(n\)-th Class preserving automorphisms of \(n\)-isoclinism family

Throughout the section, \(n\) denotes a positive integer. In this section, we study the subgroups of the automorphism group of \(n\)-isoclinic groups. Rai’s Theorem A of [8] is a special case of Theorem 3.3.

Let \(G\) be a nilpotent group of class \((n + 1)\). Then in Theorem 3.5, we obtain a necessary and sufficient condition on \(\gamma_{n+1}(G) \cap Z_i(G)\) such that \(\aut_{\gamma_{n+1}(G) \cap Z_i(G)}(G)\) is isomorphic to the group of inner automorphisms of a quotient group of \(G\). Rai’s Theorem B(2) of [8] is obtained as a corollary to Theorem 3.5.

Note that the subgroups of the automorphism group, which we study in this section, are trivial for an abelian group. Thus, we may further assume that groups under consideration are non-abelian.

Lemma 3.1. Let \(G\) and \(H\) be two finite groups and \((\alpha, \beta)\) be an \(n\)-isoclinism from \(G\) to \(H\). Then the following statements are true:
(1) For each \( f \in \text{Aut}_{\gamma_{n+1}(G)}(G) \), the map \( \theta_f : H \to H \) is given by \( \theta_f(h) = h\beta(g^{-1}f(g)) \), where \( g \in G \) such that \( \alpha(gZ_n(G)) = hZ_n(H) \), is well defined.

(2) For all \( h \in H \), \( h^{-1}\theta_f(h) \in \gamma_{n+1}(H) \).

(3) If \( f_1, f_2 \in \text{Aut}_{\gamma_{n+1}(G)}(G) \) such that \( f_1 \neq f_2 \), then \( \theta_{f_1} \neq \theta_{f_2} \).

**Proof.**

(1) Suppose \( g_1Z_n(G) = g_2Z_n(G) \) for some \( g_1, g_2 \in G \). Then \( g_1^{-1}g_2 \in Z_n(G) \). As \( f \) fixes \( Z_n(G) \) element-wise, \( f(g_1^{-1}g_2) = g_1^{-1}g_2 \). This implies that \( g_1^{-1}f(g_1) = g_2^{-1}f(g_2) \).

Hence, \( h\beta(g_1^{-1}f(g_1)) = h\beta(g_2^{-1}f(g_2)) \).

(2) Trivial.

(3) Let \( f_1, f_2 \in \text{Aut}_{\gamma_{n+1}(G)}(G) \) and let \( f_1 \neq f_2 \). Then there exists \( 1 \neq g \in G \) such that \( f_1(g) \neq f_2(g) \). Therefore, \( g^{-1}f_1(g) \neq g^{-1}f_2(g) \). Since \( \beta \) is injective, \( \beta(g^{-1}f_1(g)) \neq \beta(g^{-1}f_2(g)) \). Hence, \( \theta_{f_1}(h) \neq \theta_{f_2}(h) \), where \( h \in H \) such that \( \alpha(gZ_n(G)) = hZ_n(H) \).

\( \square \)

**Theorem 3.2.** Let \( G \) and \( H \) be two finite groups and \( (\alpha, \beta) \) be an \( n \)-isoclinism from \( G \) to \( H \). Then for each \( f \in \text{Aut}_{\gamma_{n+1}(G)}(G) \), the map \( \theta_f \in \text{Aut}_{\gamma_{n+1}(H)}(H) \), where \( \theta_f \) is defined as in Lemma 3.1(1).

**Proof.** By Lemma 3.1, \( \theta_f \) is well-defined and \( h^{-1}\theta_f(h) \in \gamma_{n+1}(H) \). Let \( h_1, h_2 \in H \) and \( g_1, g_2 \in G \) such that \( \alpha(g_1Z_n(G)) = h_1Z_n(H) \) and \( \alpha(g_2Z_n(G)) = h_2Z_n(H) \). Then \( \alpha(g_1g_2Z_n(G)) = h_1h_2Z_n(H) \) and

\[
\theta_f(h_1h_2) = h_1h_2\beta(g_2^{-1}g_1^{-1}f(g_1g_2)) = h_1(h_2\beta(g_2^{-1}g_1^{-1}f(g_1)g_2^{-1}g_1g_2^{-1}f(g_2))) = h_1(h_2\beta(g_2^{-1}g_1^{-1}f(g_1)g_2^{-1}g_1g_2^{-1}f(g_2))h_2\beta(g_2^{-1}g_1^{-1}f(g_2))).
\]

Note that \( g_1^{-1}f(g_1) \in \gamma_{n+1}(G) \). Therefore, by using Lemma 2.1(2),

\[
\beta(g_2^{-1}g_1^{-1}f(g_1)g_2^{-1}g_1g_2^{-1}f(g_2)) = h_2^{-1}\beta(g_1^{-1}f(g_1))h_2.
\]

Hence, \( \theta_f(h_1h_2) = h_1\beta(g_1^{-1}f(g_1))h_2\beta(g_2^{-1}f(g_2)) = \theta_f(h_1)\theta_f(h_2) \). Therefore, \( \theta_f \) is an endomorphism.

In order to prove that \( \theta_f \) is an automorphism, it is enough to prove that \( \theta_f \) is injective. Let \( \theta_f(h) = 1 \) for some \( h \in H \). Then \( h\beta(g^{-1}f(g)) = 1 \), where \( g \in G \) such that \( \alpha(gZ_n(G)) = hZ_n(H) \). Hence, \( h = (\beta(g^{-1}f(g)))^{-1} \in \gamma_{n+1}(H) \). Without loss of generality, we may assume that \( g \in \gamma_{n+1}(G) \). If not, then there exists \( g' \in \gamma_{n+1}(G) \) such that \( \beta(g') = h \) and \( g'Z_n(G) = gZ_n(G) \). Hence, we can replace \( g \) by \( g' \).

Now observe that \( 1 = h\beta(g^{-1}f(g)) = h\beta(g^{-1})\beta(f(g)), g \in \gamma_{n+1}(H) \). Hence, \( \beta(f(g)) = h^{-1}\beta(g) \in \gamma_{n+1}(H) \). Furthermore, \( hZ_n(H) = \alpha(gZ_n(G)) = \beta(g)Z_n(H) \) implies that \( h^{-1}\beta(g) \in Z_n(H) \). Therefore, \( \beta(f(g)) \in Z_n(H) \cap \gamma_{n+1}(H) \). By using Theorem 2.2, we have \( f(g) \in Z_n(G) \cap \gamma_{n+1}(G) \). As \( f \) fixes \( Z_n(G) \) element-wise, \( f(g) = g \) and \( h = 1 \). Hence, \( \theta_f \) is an isomorphism.

Also, if \( h \in Z_n(H) \), then \( g \in Z_n(G) \), where \( g \in G \) such that \( \alpha(gZ_n(G)) = hZ_n(H) \). Therefore, \( \theta_f(h) = h \). Hence, \( \theta_f \in \text{Aut}_{\gamma_{n+1}(H)}(H) \). \( \square \)
Theorem 3.3. Let $G$ and $H$ be two finite groups and let $(\alpha, \beta)$ be an $n$-isoclinism from $G$ to $H$. Then there exists an isomorphism $\Psi : \text{Aut}_{Z_n(G)}^{\gamma_{n+1}(G)} \rightarrow \text{Aut}_{Z_n(H)}^{\gamma_{n+1}(H)}$ such that $\Psi(\text{Aut}^n(G)) = \text{Aut}^n(H)$.

Proof. Define $\Psi : \text{Aut}_{Z_n(G)}^{\gamma_{n+1}(G)} \rightarrow \text{Aut}_{Z_n(H)}^{\gamma_{n+1}(H)}$ such that $\Psi(f) = \theta_f$, where $\theta_f$ is the map defined in Lemma 3.1(1). By Theorem 3.2, $\theta_f \in \text{Aut}_{Z_n(H)}^{\gamma_{n+1}(H)}(H)$.

First, we show that $\Psi$ is a group homomorphism. Let $f_1, f_2 \in \text{Aut}_{Z_n(G)}^{\gamma_{n+1}(G)}$. Then we need to show that $\theta_{f_1 \circ f_2} = \theta_{f_1} \circ \theta_{f_2}$. Consider $h \in H$. Then

$$(\theta_{f_1} \circ \theta_{f_2})(h) = \theta_{f_1}(h \beta(g^{-1} f_2(g))),$$

where $g \in G$ such that $\alpha(gZ_n(G)) = hZ_n(H)$. Note that

$$\alpha(f_2(g)Z_n(G)) = \alpha(gZ_n(G))\alpha(g^{-1} f_2(g)Z_n(G)) = hZ_n(H)\alpha(g^{-1} f_2(g)Z_n(G)).$$

Since $g^{-1} f_2(g) \in \gamma_{n+1}(G)$, by Lemma 2.1, we have

$$\alpha(g^{-1} f_2(g)Z_n(G)) = \beta(g^{-1} f_2(g))Z_n(H).$$

Therefore, $\alpha(f_2(g)Z_n(G)) = h \beta(g^{-1} f_2(g))Z_n(H)$. Thus

$$(\theta_{f_1} \circ \theta_{f_2})(h) = \theta_{f_1}(h \beta(g^{-1} f_2(g))) = h \beta(g^{-1} f_2(g))\beta(f_2(g)^{-1}f_1(f_2(g))) = \theta_{f_1 \circ f_2}(h).$$

Hence, $\Psi$ is a group homomorphism.

To prove that $\Psi$ is an isomorphism, define a map $\Phi$ from $\text{Aut}_{Z_n(H)}^{\gamma_{n+1}(H)}$ to $\text{Aut}_{Z_n(G)}^{\gamma_{n+1}(G)}$ such that $\Phi(\theta) = f_\theta$, where $f_\theta : G \rightarrow G$ is given by $f_\theta(g) = g\beta^{-1}(h^{-1} \theta(h))$, where $h \in H$ such that $\alpha^{-1}(hZ_n(H)) = gZ_n(G)$. Using the fact that $\alpha$ and $\beta$ are isomorphisms, one can prove that $\Phi$ is a group homomorphism and $\Phi(\theta_f) = f$ for $f \in \text{Aut}_{Z_n(G)}^{\gamma_{n+1}(G)}$. Therefore, $(\Phi \circ \Psi)(f) = \Phi(\theta_f) = f$. Hence, $\Phi \circ \Psi = I$. Similarly, one can show that $\Psi \circ \Phi = I$. Thus $\Psi$ is an isomorphism.

Next we show that $\Psi(\text{Aut}^n(G)) = \text{Aut}^n(H)$. Consider $f \in \text{Aut}^n(G) \subseteq \text{Aut}_{Z_n(G)}^{\gamma_{n+1}(G)}$. Clearly, $\theta_f \in \text{Aut}_{Z_n(H)}^{\gamma_{n+1}(H)}(H)$ and $\theta_f(h) = h \beta(g^{-1} f(g))$, where $g \in G$ such that $\alpha(gZ_n(G)) = hZ_n(H)$. Also, for $g \in G$, there exists $x \in \gamma_n(G)$ such that $f(g) = x^{-1} gx$. Therefore, $\theta_f(h) = h \beta(g^{-1} x^{-1} gx)$. We know that if $x \in \gamma_n(G)$, then $x$ is the product of some commutators of weight $n$. Let us first assume that $x = [g_1, \ldots, g_n]$ for $1 \leq i \leq n$. $g_i \in G$ and $\alpha(g_i Z_n(G)) = h_i Z_n(H)$. Then by the commutativity of the diagram in $n$-isoclinism, one can observe that $\beta(g, x) = [h, y]$, where $y \in \gamma_n(H)$ is $y = [h_1, \ldots, h_n]$. Hence $\theta_f(h) = y^{-1}hy$. For proof in the general case, let us first assume that $x$ is the product of two commutators of weight $n$. Let $x_1 = [g_1, \ldots, g_n]$ and $x_2 = [g'_1, \ldots, g'_n]$, where
Theorem 3.5. Let $G$ be a finite non abelian group of nilpotency class $n$ and let $M = (M_{p_1},...,M_{p_s})$ be a central subgroup of $G$ such that $M_{p_i}$ is cyclic for each $p_i = \pi_i$. Then by using induction on the number of the product of the commutators of weight $n$. Hence, we have $\Psi(\text{Aut}_{\mathbb{Z}}^n(G)) \subseteq \text{Aut}_{\mathbb{Z}}^n(H)$. Similarly, we can show that $\Phi(\text{Aut}_{\mathbb{Z}}^n(H)) \subseteq \text{Aut}_{\mathbb{Z}}^n(G)$.

COROLLARY 3.4 [8, Theorem A]

Let $G$ and $H$ be two finite isoclinic groups. Then there exists an isomorphism $\Psi : \text{Aut}_{\mathbb{Z}}^n(G) \to \text{Aut}_{\mathbb{Z}}^n(H)$ such that $\Psi(\text{Aut}_{\mathbb{Z}}^n(G)) = \text{Aut}_{\mathbb{Z}}^n(H)$.

Proof. Put $n = 1$ in Theorem 3.3.

Let $G$ be a finite non abelian group and let $M$ be a central subgroup of $G$. Let $M_p$ denote the $p$-primary component of $M$, where $p$ is a prime divisor of the order of $M$. In [8], Rai proved that if $G$ is a finite $p$-group of nilpotency class two, then $\text{Aut}_{\mathbb{Z}}^2(G) = \text{Inn}(G)$ if and only if $\gamma_2(G)$ is cyclic. We generalize this result to a nilpotent group of class $(n+1)$. Also, we have shown that the result is true for any central subgroup $M$ of $G$ which contains $\gamma_{n+1}(G)$.

Theorem 3.5. Let $G$ be a finite non abelian group of nilpotency class $(n+1)$ and let $M$ be a central subgroup of $G$ such that $\gamma_{n+1}(G) \subseteq M$. Then

$$\text{Aut}_{\mathbb{Z}}^n(G) \simeq \text{Inn}(G/Z_{n-1}(G))$$

if and only if $M_{p_i}$ is cyclic for each $p_i = \pi_i$. Let $\pi(G/Z_n(G)) = \{p_1,\ldots,p_r\}$ and let

$$G/Z_n(G) \simeq H_{p_1} \times H_{p_2} \times \cdots \times H_{p_r},$$

where $H_{p_i}$ denotes the $p_i$-primary component of $G/Z_n(G)$. Similarly, let $\pi(M) = \{p_1,\ldots,p_r,q_1,\ldots,q_s\}$ and

$$M \simeq M_{p_1} \times \cdots \times M_{p_r} \times M_{q_1} \times \cdots \times M_{q_s},$$

where $M_{p_i}$ and $M_{q_j}$ denote the $p_i$-primary and $q_j$-primary components of $M$, respectively. Clearly, $\exp(H_{p_i}) | \exp(M_{p_i})$ for all $1 \leq i \leq r$. Now by Theorem 2.5, we have

$$\text{Aut}_{\mathbb{Z}}^n(G) \simeq \text{Hom}(G/Z_n(G), M)$$

$$= \text{Hom} \left( \prod_{i=1}^r H_{p_i}, \prod_{i=1}^r M_{p_i} \times \prod_{j=1}^s M_{q_j} \right)$$
\[
\simeq \text{Hom} \left( \prod_{i=1}^{r} H_{p_i}, \prod_{i=1}^{r} M_{p_i} \right)
\]
\[
\simeq \prod_{i=1}^{r} \text{Hom}(H_{p_i}, M_{p_i}).
\]

By Lemma 2.6, \(\text{Hom}(H_{p_i}, M_{p_i}) \simeq H_{p_i}\) if and only if \(M_{p_i}\) is cyclic. Hence,

\[
\text{Aut}_{\gamma_n(G)}^M(G) \simeq \prod_{i=1}^{r} H_{p_i} \simeq G/Z_n(G) \simeq \text{Inn}(G/Z_{n-1}(G))
\]
if and only if \(M_{p_i}\) is cyclic for all \(1 \leq i \leq r\). \(\square\)

**COROLLARY 3.6**

Let \(G\) be a finite non abelian \(p\)-group of class \((n+1)\). Then \(\text{Aut}_{\gamma_n(G)}^{\gamma_{n+1}(G)}(G) \simeq \text{Inn}(G/Z_{n-1}(G))\) if and only if \(\gamma_{n+1}(G)\) is cyclic.

Rai’s Theorem B(2) of [8] follows from Corollary 3.6 by taking \(n = 1\).

**Acknowledgements**

The author would like to thank the referee/referees for their valuable suggestions. This research is partially supported by SERB-DST Grant YSS/2015/001567.

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**Communicating Editor:** B Sury