PRINCIPAL POLARIZATIONS ON PRODUCTS OF ELLIPTIC CURVES

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Dedicated to Sevin Recillas

Abstract. An abelian variety admits only a finite number of isomorphism classes of principal polarizations. The paper gives an interpretation of this number in terms of class numbers of definite Hermitian forms in the case of a product of elliptic curves without complex multiplication. In the case of a self-product of an elliptic curve, as well as in the two-dimensional case, classical class number computations can be applied to determine this number.

1. Introduction

Let $X$ be an abelian variety of dimension $g$ over the field $\mathbb{C}$ of complex numbers. The Néron-Severi group of $X$ is by definition the quotient of the group of line bundles on $X$ modulo the subgroup of line bundle which are algebraically equivalent to zero,

$$NS(X) = \text{Pic}(X)/\text{Pic}^0(X).$$

A polarization of $X$ is by definition an element $l \in NS(X)$ which is represented by an ample line bundle $L$ on $X$. By a slight abuse of notation we denote the polarization defined by $L$ often by $L$ itself. The pair $(X, L)$ is called a polarized abelian variety. Every polarization $L$ of $X$ induces an isogeny $\varphi_L : X \to \hat{X}$ of $X$ onto its dual abelian variety $\hat{X} = \text{Pic}^0(X)$, defined by

$$\varphi_L(x) = t_x^*L \otimes L^{-1},$$

where $t_x : X \to X$ denotes the translation by $x$. A polarization on $X$ is called principal, if $\varphi_L$ is an isomorphism.

Two polarizations $L_1$ and $L_2$ on $X$ are said to be equivalent if there is an automorphism $\tau \in \text{Aut}(X)$ such that

$$\tau^*L_2 \sim L_1,$$

where $\sim$ denotes algebraic equivalence. Consider the set

$$P(X) = \{\text{equivalence classes of principal polarizations on } X\}.$$
According to a theorem of Narasimhan-Nori \[3\], \( P(X) \) is a finite set. It is the aim of this note to compute the number \( \#P(X) \) in the case of a product of elliptic curves without complex multiplication,

\[
X = E_1 \times \cdots \times E_n.
\]

According to Lemma \[2.2\] we may assume that the \( E_i \) are pairwise isogenous. The main result is Theorem \[3.5\] which says that in this case there is a bijection between \( P(X) \) and the set of equivalence classes of Hermitian forms of a certain type.

We can then apply results on class numbers of Hermitian or quadratic forms in order to compute the cardinality of the set \( P(X) \) in certain cases. If for example \( E \) is an elliptic curve without complex multiplication and

\[
X = E \times \cdots \times E,
\]

then

\[
\#P(X) = h(n)
\]

(see Theorem \[4.1\], where \( h(n) \) denotes the number of equivalence classes of positive definite integral quadratic forms of rank \( n = \dim X \). For \( n \leq 25 \), the class number \( h(n) \) is known. We deduce that for \( n \leq 7 \) the abelian variety \( X \) admits no principal polarization apart from the canonical one. For \( n = 8 \), there is another principal polarization and we have for example \( \#P(X) = 8 \) for \( n = 16 \) and \( \#P(X) = 297 \) for \( n = 24 \). Moreover, it is a consequence of the mass formula of Minkowski-Siegel that the number \( \#P(X) \) is unbounded for \( n \to \infty \).

For another application of Theorem \[3.5\] consider

\[
X = E_1 \times E_2
\]

with isogenous elliptic curves \( E_1 \) and \( E_2 \) without complex multiplication admitting an isogeny of minimal positive degree \( d \). Here there is a bijection between \( P(X) \) and the set of equivalence classes of primitive positive definite integral quadratic forms of rank 2 and determinant \( d \). The corresponding class number \( \tilde{h}(d) \) has been computed by Hayashida in \[4\], also in order to compute the number of classes of principal polarizations of \( E_1 \times E_2 \) as above, but with a different approach. It is a consequence of this result that the number \( \#P(X) \) is unbounded for \( d \to \infty \).

The main idea for the proof of Theorem \[3.5\] is as follows: The canonical principal polarization of \( X \) induces a bijection between \( P(X) \) and the set of equivalence classes of symmetric automorphisms of \( X \). Via the analytic representation of \( X \) and a suitable choice of bases this set can be considered as a set of equivalence classes of Hermitian forms.
2. Generalities

Let $X$ denote a complex abelian variety of dimension $n$. In order to compute the number $\#P(X)$ we may assume that $X$ admits at least one principal polarization, say $L_0$, which is fixed in the sequel. This polarization induces an anti-involution on the ring $\text{End}(X)$ of endomorphisms of $X$, the Rosati involution defined by

$$\alpha \mapsto \alpha' = \varphi_{L_0}^{-1} \hat{\alpha} \varphi_{L_0},$$

for every $\alpha \in \text{End}(X)$, where $\hat{\alpha} : \hat{X} \to \hat{X}$ denotes the dual endomorphism. Denote by

$$\text{End}^s(X) = \{ \alpha \in \text{End}(X) \mid \alpha' = \alpha \}$$

the abelian subgroup of symmetric endomorphisms. Similarly $\text{Aut}^s(X)$ is defined. Finally, set

$$\text{Aut}^s(X)^+ = \{ \alpha \in \text{Aut}^s(X) \mid \alpha \text{ totally positive} \},$$

where totally positive means that all roots of the minimal polynomial are positive. The group $\text{Aut}(X)$ acts on the set $\text{Aut}^s(X)^+$ by

$$(\varphi, \alpha) \mapsto \varphi' \alpha \varphi$$

for all $\varphi \in \text{Aut}(X)$ and $\alpha \in \text{Aut}^s(X)^+$. For the proof of the following proposition we refer to [1], Proposition 5.2.1 and Theorem 5.2.4.

**Proposition 2.1.** With the notation above we have

1. The map

$$\epsilon : \text{NS}(X) \to \text{End}^s(X), \quad L \mapsto \varphi_{L_0}^{-1} \varphi_L$$

is an isomorphism of groups.

2. The map $\epsilon$ induces a bijection

$$P(X) \sim \text{Aut}^s(X)^+ / \sim,$$

where $\text{Aut}^s(X)^+ / \sim$ denotes the set of equivalence classes with respect to the above action.

A polarization $L$ on $X$ is called reducible if $L = L_1 \otimes L_2$ with ample line bundles $L_1$ and $L_2$, or equivalently, if there are abelian subvarieties $X_1$ and $X_2$ of $X$ such that there is an isomorphism of polarized abelian varieties

$$(X, L) \cong (X_1, L_1) \times (X_2, L_2),$$

where here $L_i$ denotes the restriction of $L$ to $X_i$ for $i = 1, 2$.

**Lemma 2.2.** Let $(X_i, L_i)$ be principally polarized abelian varieties for $i = 1, 2$. If $\text{Hom}(X_1, X_2) = 0$, then every principal polarization of $X_1 \times X_2$ is reducible.
Proof. The assumption implies \( \text{End}(X) = \text{End}(X_1) \oplus \text{End}(X_2) \). The principal polarization \( p_1^* L_1 \otimes p_2^* L_2 \) induces an isomorphism \( \text{NS}(X) \cong \text{End}^*(X) = \text{End}^*(X_1) \oplus \text{End}^*(X_2) \). Together with Proposition 2.1 (2) this gives the assertion. \( \square \)

Now let \( E_i \) be complex elliptic curves and consider the abelian variety \( X = E_1 \times \cdots \times E_n \).

It admits the canonical principal polarization \( L_0 = p_1^* \mathcal{O}_{E_1}(0) \otimes \cdots \otimes p_n^* \mathcal{O}_{E_n}(0) \), where \( p_i : X \to E_i \) denotes the \( i \)-th projection. Lemma \ref{lemma} implies that, in order to determine all principal polarizations of \( X \), we may assume that the elliptic curves \( E_i \) are pairwise isogenous.

For the computation of \( \#P(X) \) we introduce suitable period matrices of \( X \). For \( i = 1, \ldots, n \) there is an element \( z_i \) in the upper half plane \( \mathcal{H} \) such that \( E_i = \mathbb{C}/\Lambda_i \) with \( \Lambda_i = \mathbb{Z} + z_i \mathbb{Z} \).

Then the following matrix is a period matrix for the abelian variety \( X \) with respect to the canonical basis \( e_1, \ldots, e_n \) of \( \mathbb{C}^n \) and a suitable basis \( \lambda_1, \ldots, \lambda_{2n} \) of the lattice \( \Lambda \):

\[
\Pi = (I_n \, Z),
\]

where \( I_n \) denotes the unit matrix of degree \( n \) and \( Z = \text{diag}(z_1, \ldots, z_n) \) satisfies

\[
Z^t = Z \quad \text{and} \quad \text{Im} Z > 0.
\]

The first Chern class of the polarization \( L_0 \) can be considered as an alternating form \( E_{L_0} \) on the lattice \( \Lambda \). With respect to the basis \( \lambda_1, \cdots, \lambda_{2n} \) of \( \Lambda \) the alternating form \( E_{L_0} \) is given by the following matrix, also denoted by \( E_{L_0} \):

\[
E_{L_0} = \begin{pmatrix}
0 & I_n \\
-I_n & 0
\end{pmatrix}.
\]

For any \( \varphi \in \text{End}(X) \) the analytic and rational representations

\[
A_{\varphi} := \rho_a(\varphi) \in M_n(\mathbb{C}) \quad \text{and} \quad R_{\varphi} := \rho_r(\varphi) \in M_{2n}(\mathbb{Z})
\]

satisfy the following equation

\[
A_{\varphi} \Pi = \Pi R_{\varphi}.
\]

Conversely, any pair of matrices \((A, R) \in M_n(\mathbb{C}) \times M_{2n}(\mathbb{Z})\) satisfying \( (2) \) defines an endomorphism of \( X \). The following lemma computes the analytic representation of the Rosati involution of \( \varphi \) in terms of the chosen bases.

**Lemma 2.3.**

\[
A_{\varphi'} = \text{Im} \, Z \cdot \overline{A_{\varphi}} \cdot (\text{Im} \, Z)^{-1}.
\]
Proof. It is well-known (see [1], Proposition 5.1.1) that the rational representation \( R_{\varphi'} \) of \( \varphi' \) is the adjoint matrix of \( R_\varphi \) with respect to the alternating form \( E_{L_0} \), which implies

\[
R_{\varphi'} = E_{L_0}^{-1} R_\varphi^t E_{L_0}
\]

The matrix \( \begin{pmatrix} \Pi & \Pi \end{pmatrix} \) is invertible, since \( \Pi \) is a period matrix. Hence (2) applied to \( \varphi \) and \( \varphi' \) implies

\[
R_\varphi = \begin{pmatrix} \Pi & \Pi \end{pmatrix}^{-1} \begin{pmatrix} A_\varphi & 0 \\ 0 & A_\varphi' \end{pmatrix} \begin{pmatrix} \Pi & \Pi \end{pmatrix}
\]

and

\[
\begin{pmatrix} A_{\varphi'} & 0 \\ 0 & A_{\varphi'} \end{pmatrix} = \begin{pmatrix} \Pi & \Pi \end{pmatrix} R_{\varphi'} \left( \begin{pmatrix} \Pi & \Pi \end{pmatrix} \right)^{-1}
\]

Inserting (3) and the transpose of (4) into (5), we obtain

\[
\begin{pmatrix} A_{\varphi'} & 0 \\ 0 & A_{\varphi'} \end{pmatrix} = S \begin{pmatrix} A_\varphi' & 0 \\ 0 & A_\varphi \end{pmatrix} S^{-1}
\]

with

\[
S = \begin{pmatrix} \Pi & \Pi \\ \Pi & \Pi \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \left( \begin{pmatrix} \Pi^t & \Pi^t \end{pmatrix} \right) = 2i \begin{pmatrix} 0 & \text{Im} Z \\ -\text{Im} Z & 0 \end{pmatrix}.
\]

For the last equation we used (1). This implies the assertion. \( \square \)

3. \( E_i \) without complex multiplication

Let the notation be as at the end of the last section. In particular \( E_i = \mathbb{C}/\Lambda \), \( i = 1, \ldots, n \) are pairwise isogenous complex elliptic curves. In this section we assume in addition that the \( E_i \) are without complex multiplication which means that \( \text{End}(E) \simeq \mathbb{Z} \). Then

\[
\text{Hom}(E_i, E_j) \simeq \mathbb{Z}
\]

for all \( i, j = 1, \cdots, n \). Denote by \( \tau_{ij} : E_i \to E_j \) an isogeny of minimal positive degree, say \( d_{ij} \). We identify \( \tau_{ij} \) with its analytic representation, i.e., we consider it as a complex number.

Lemma 3.1. The complex conjugate \( \overline{\tau_{ij}} \) represents a homomorphism \( E_j \to E_i \), denoted by the same symbol, such that

\[
\overline{\tau_{ij}} \tau_{ij} = \deg(\tau_{ij}) 1_{E_i} \quad \text{and} \quad \tau_{ij} \overline{\tau_{ij}} = \deg(\tau_{ij}) 1_{E_j}.
\]

\( \overline{\tau_{ij}} \) is a homomorphism of minimal positive degree.

Proof. There exists a homomorphism \( \overline{\tau_{ji}} : E_j \to E_i \) such that \( \overline{\tau_{ji}} \tau_{ij} = \deg(\tau_{ij}) 1_{E_i} \) and \( \tau_{ij} \overline{\tau_{ji}} = \deg(\tau_{ij}) 1_{E_j} \). In particular

\[
\overline{\tau_{ji}} \tau_{ij} = \deg(\tau_{ij}).
\]
On the other hand, since the rational representation is the direct sum of the analytic representation and its complex conjugate,

\[ \tau_{ij} \tau_{ij} = \det(R_{\tau_{ij}}) = \deg(\tau_{ij}). \]

This implies that the analytic representation of the homomorphism \( \tilde{\tau}_{ij} \) is given by the complex number \( \tau_{ij} \) and thus completes the proof of the lemma.

Any homomorphism \( \varphi_{ij} : E_i \to E_j \) can be written as \( \varphi_{ij} = d_{ij} \tau_{ij} \) with a uniquely determined integer \( d_{ij} \). This remark together with Lemma 3.1 imply the following proposition.

**Proposition 3.2.** The analytic representation induces an isomorphism

\[ \text{End}(X) \xrightarrow{\cong} \mathcal{M}_n(X), \quad \varphi \mapsto A_\varphi \]

where \( \mathcal{M}_n(X) \) denotes the ring of all matrices of the form

\begin{equation}
A_\varphi = \begin{pmatrix}
d_{11} & d_{12} \tau_{12} & \cdots & d_{1n} \tau_{1n} \\
d_{21} \tau_{12} & d_{22} & \cdots & d_{2n} \tau_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
d_{n1} \tau_{1n} & d_{n2} \tau_{2n} & \cdots & d_{nn}
\end{pmatrix}
\end{equation}

with \( \tau_{ij} \) as above and \( d_{ij} \in \mathbb{Z} \) for all \( i, j \).

**Lemma 3.3.** For any \( \varphi \in \text{End}(X) \) we have, with respect to the chosen bases,

\[ \det A_\varphi \in \mathbb{Z}. \]

**Proof.** The matrix is of the form \( A_\varphi = (\varphi_{ij}) \) with homomorphisms \( \varphi_{ij} : E_i \to E_j \), which are identified with their analytic representations. By definition of the determinant,

\[ \det A_\varphi = \sum_{\sigma \in S_n} \text{sign}(\sigma) \varphi_{1\sigma(1)} \varphi_{2\sigma(2)} \cdots \varphi_{n\sigma(n)}. \]

Now any \( \sigma \in S_n \) is a product of cycles \( \sigma = \sigma_\nu \cdots \sigma_1 \). But any cycle \( \sigma_\nu \) of length \( s_\nu \), say, is of the form

\[ \sigma_\nu = (k, \sigma(k), \sigma^2(k), \cdots, \sigma^{s_\nu-1}(k)). \]

Hence the corresponding homomorphism \( \varphi_{k\sigma(k)} \varphi_{\sigma(k)\sigma^2(k)} \cdots \varphi_{\sigma^{s_\nu-1}(k),k} \) is an endomorphism of \( E_k \). So by assumption any cycle is an element of \( \text{End}(E_i) = \mathbb{Z} \) for some \( i \). This implies that \( \det A_\varphi \) is an integer as a sum of products of integers. \( \square \)

As in the last section, \( \Pi = (I_n, Z) \) with \( Z = \text{diag}(z_1, \cdots, z_n) \) is a period matrix for \( X = E_1 \times \cdots \times E_n \). If \( \{e_1, \ldots, e_n\} \) denotes the
products of elliptic curves

canonical basis of \( \mathbb{C}^n \), we introduce a new basis \( \{ f_1, \ldots, f_n \} \) of \( \mathbb{C}^n \) by setting
\[
f_i = \sqrt{\text{Im}(z_i)} e_i
\]
for \( i = 1, \ldots, n \). Since \( \text{Im}(z_i) > 0 \) for all \( i \),
\[
T = \text{diag}(\sqrt{\text{Im}(z_1)}, \ldots, \sqrt{\text{Im}(z_n)})
\]
is a well-defined matrix of \( GL_n(\mathbb{R}) \). The period matrix of \( X \) with respect to the new basis \( \{ f_i \} \) of \( \mathbb{C}^n \) and the old basis \( \{ \lambda_j \} \) of \( \Lambda \) is
\[
\tilde{\Pi} = T^{-1} \Pi.
\]
Using this it is an easy consequence of Lemma 2.3 that the analytic representation \( A_{\varphi'} \) of the Rosati transform \( \varphi' \) of \( \varphi \) with respect to these bases is given by
\[
A_{\varphi'} = A_{\varphi}^t.
\]
Thus we can conclude

**Lemma 3.4.** For \( \varphi \in \text{End}(X) \) the following conditions are equivalent

1. \( \varphi \) is symmetric with respect to the Rosati involution,
2. The analytic representation \( A_{\varphi} \) with respect to the bases \( \{ f_i \} \) of \( \mathbb{C}^n \) and \( \{ \lambda_j \} \) of \( \Lambda \) is a Hermitian matrix.

Note that Lemma 3.4 does not assume that the elliptic curves are without complex multiplication. For the proof of the lemma a normalization, different from the one used above, turns out to be more convenient (see Remark 3.6).

Consider now the set
\[
M_n^+(X) = \{ A \in M_n(X) \mid A^t = A, \ A > 0, \ \text{det} \ A = 1 \}.
\]
Two matrices \( A_1, A_2 \in M_n^+(X) \) are called equivalent, if there is an invertible matrix \( T \in M_n(X) \) such that
\[
A_2 = T^t A_1 T.
\]
Note that this equivalence is just the matrix version of the usual equivalence of Hermitian forms. Recall that \( P(X) \) denotes the set of equivalence classes of principal polarizations of \( X \).

**Theorem 3.5.** There is a bijection
\[
P(X) \to M_n^+(X) / \sim,
\]
where \( M_n^+(X) / \sim \) denotes the set of equivalence classes of Hermitian matrices in \( M_n^+(X) \).

**Proof.** We claim first that the analytic representation with respect to the basis \( \{ f_i \} \) induces a bijection \( \text{Aut}^+(X) / \sim \to M_n^+(X) \). For the proof note that by Lemma 3.4 an endomorphism \( \alpha \) of \( X \) is symmetric if and only if \( A_{\alpha} \) is Hermitian. It is totally positive if
and only if all zeros of the minimal polynomial are positive, i.e., \( A_\alpha \) is positive definite. On the other hand, \( \alpha \) is an automorphism if and only if \( \deg(\alpha) = 1 \). But \( \deg(\alpha) \) equals the determinant of the rational representation \( R_\alpha \). So (4) implies

\[
\det A_\alpha \cdot \det A_\alpha = 1.
\]

By Lemma 3.3 \( \det A_\alpha \in \mathbb{Z} \), which implies \( \det A_\alpha = \pm 1 \) and thus \( = 1 \), if \( A_\alpha \) is positive definite. This implies the assertion.

It is clear that the bijection \( Aut^*(X)^+ \to M_n^+(X) \) is compatible with the equivalence relations. Hence Proposition 2.1 completes the proof of the theorem. \( \square \)

**Remark 3.6.** One can use the same method in order to prove an analogous result for a product of pairwise isogenous elliptic curves with complex multiplication. This has been done in the thesis of P. Schuster, written under the supervision of the author (see [7]). Again the idea is to use suitable bases in order to interpret the set \( P(X) \) in terms of classes of Hermitian forms. In the case of dimension 2 and of self-products of dimension 3 Schuster applies class number formulas of Hashimoto-Koseki in order to compute the number \( \#P(X) \).

**4. Self-products of an elliptic curve \( E \)**

Let \( E \) be an elliptic curve over \( \mathbb{C} \) without complex multiplication and consider the abelian variety

\[
X = E \times \cdots \times E
\]

of dimension \( n \). In this case we may choose \( \tau_{ij} = 1_E \) for all \( i, j \). Then the following theorem is a special case of Theorem 3.3.

**Theorem 4.1.** There is a bijection of \( P(X) \) with the set of equivalence classes of positive definite unimodular integral quadratic forms.

**Corollary 4.2.** For \( n \leq 7 \), there is no principal polarization on \( X \) apart from \( L_0 \).

**Proof.** For \( n \leq 7 \), there is only one positive definite unimodular integral quadratic form (see [2]). \( \square \)

There is an extensive literature about the number \( h(n) \) of classes of positive definite unimodular integral quadratic forms of rank \( n \) (see [2] and the literature quoted there). According to Theorem 4.1 this number can be interpreted as the number \( \#P(X) \) of equivalence classes of principal polarizations of \( X \). In particular \( h(n) \) has been computed for \( n \leq 25 \) by Kneser, Niemeier, Conway-Sloane and Borchards (see [2], table 2.2). Together with Theorem 4.1 this gives
Corollary 4.3.

| n   | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|-----|----|----|----|----|----|----|----|----|----|
| #P(X)| 2  | 2  | 2  | 2  | 4  | 5  | 8  |    |    |

| n   | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
|-----|----|----|----|----|----|----|----|----|----|
| #P(X)| 9  | 13 | 16 | 28 | 40 | 68 | 117| 297| 665|

The mass formula of Siegel-Minkowski gives an estimate for the number $h(n)$ (see [2] or [3]). As a consequence one gets for example that $h(32) \geq 80.000.000$ and moreover that $h(n)$ tends to infinity if $n \to \infty$.

Remark 4.4. The theory of unimodular definite integral quadratic forms distinguishes between even and odd ones (or of type I and type II). It would be interesting to see whether this distinction has a geometric meaning for the corresponding principal polarizations.

5. Abelian surfaces $E_1 \times E_2$

In this section consider

$$X = E_1 \times E_2$$

with isogenous elliptic curves $E_1$ and $E_2$ without complex multiplication and assume that

$$\tau : E_1 \to E_2$$

is an isogeny of minimal positive degree $d \geq 2$. According to Theorem 3.5 there is a bijection between $P(X)$ and the set of classes of Hermitian forms $M_2^e(X)^+/\sim$, where

$$M_2^e(X)^+ = \{ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} \tau & a_{22} \end{pmatrix} | a_{ij} \in \mathbb{Z}, a_{11} > 0, \det A = 1 \}.$$

In order to compute the corresponding class number, recall from Lemma 3.1 that the complex conjugate $\overline{\tau}$ represents an isogeny $E_2 \to E_1$ of minimal positive degree $d$, and consider the map

$$\Phi : \text{End}(E_1 \times E_2) \to \text{End}(E_1 \times E_1), \quad \varphi \mapsto (1_{E_1} \times \overline{\tau}) \varphi (1_{E_1} \times \tau)$$

In terms of the analytic representation with respect to the basis $\{ f_j \}$ of section 3, the map $\Phi$ is given by

$$A_\varphi \mapsto \Phi(A_\varphi) = \begin{pmatrix} 1 & 0 \\ 0 & \overline{\tau} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \overline{\tau} \\ a_{12} \tau & a_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12}d \\ a_{12}d & a_{22}d \end{pmatrix}.$$ 

So $\Phi(A_\varphi)$ is an integral quadratic form of determinant $d$. Recall that a integral $(2 \times 2)$-matrix $(m_{ij})$ is called primitive, if $\gcd(m_{ij} | i, j = 1, 2) = 1$. Consider the set of integral quadratic forms given as matrices by

$$\widetilde{M}_2^e(X)^+ = \{ B = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix} | B \text{ primitive, } b_{11} > 0, \det B = d \}.$$

As usual, two integral quadratic forms $B_1$ and $B_2$ are called equivalent if there is a $T \in GL_2(\mathbb{Z})$ such that $B_2 = T^t B_1 T$. Clearly this defines an
equivalence relation on the set $\tilde{M}_2^s(X)^+$. It is well-known that there are only finitely many equivalence classes of such forms (in fact, this is also a consequence of the following theorem). Let $\tilde{h}(d)$ denote the corresponding class number:

$$\tilde{h}(d) = #(\tilde{M}_2^s(X)^+ / \sim).$$

**Theorem 5.1.** Let $X = E_1 \times E_2$ with elliptic curves $E_1$ and $E_2$ admitting an isogeny of minimal positive degree $d$. Then

$$\#P(X) = \tilde{h}(d).$$

**Proof.** According to Theorem 3.5 it is sufficient to show that the map $\Phi : M_2^s(X)^+ \rightarrow \tilde{M}_2^s(X)^+$ is compatible with the equivalence relations and induces a bijection of the equivalence classes.

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \in M_2^s(X)^+$. Then $\Phi(A) = \begin{pmatrix} a_{11} & a_{12}d \\ a_{12}d & a_{22}d \end{pmatrix}$ is a primitive matrix, since $\gcd(a_{11}, a_{12}d, a_{22}d) = \gcd(a_{11}, \gcd(a_{11}, a_{22})d) = \gcd(a_{11}, d) = 1$,

where we used the fact that $\det(A) = 1$.

Next we claim that for $A_1, A_2 \in M_2^s(X)^+$,

$$A_1 \sim A_2 \iff \Phi(A_1) \sim \Phi(A_2).$$

Let $T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$ be an invertible matrix of the ring $M_2(X)$ with $A_2 = T^t A_1 T$. Then $\Phi(A_2) = \tilde{T}^t \Phi(A_1) \tilde{T}$ with $\tilde{T} = \begin{pmatrix} t_{11} & t_{12}d \\ t_{21} & t_{22} \end{pmatrix}$. This implies $\Phi(A_1) \sim \Phi(A_2)$, since $\det(\tilde{T}) = \det(T)$.

Conversely, given $B_1 = \begin{pmatrix} b_{11} & b_{12}d \\ b_{12}d & b_{22}d \end{pmatrix}$ and $B_2 = \begin{pmatrix} c_{11} & c_{12}d \\ c_{12}d & c_{22}d \end{pmatrix} = \Phi(A_2)$ with $B_2 = T^t B_1 T$, $T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$. This means in particular

$$c_{11} = t_{11}^2 b_{11} + d(2t_{11}t_{21}b_{12} + t_{21}^2 b_{22})$$

$$c_{12}d = t_{11}t_{12}b_{11} + d(t_{12}t_{21}b_{12} + t_{11}t_{22}b_{12} + t_{21}t_{22}b_{22})$$

Since $B_1$ and $B_2$ are primitive, this implies

$$t_{12} \equiv 0 \mod d,$$

from which we get $A_1 \sim A_2$ by reading the above computation upside down.

Hence $\Phi$ induces a map $\Phi : M_2^s(X)^+ / \sim \rightarrow \tilde{M}_2^s(X)^+ / \sim$. It remains to show that $\Phi$ is bijective.

Clearly $\Phi$ is injective. To see that it is surjective, let $B = (b_{ij}) \in \tilde{M}_2^s(X)^+$. According to an elementary result for binary quadratic forms (see e.g., [7], p. 132) there are integers $x_0, y_0$, prime to each other, such that $\gcd(b_{11}x_0^2 + 2b_{12}x_0y_0 + b_{22}y_0^2, d) = 1$. Choose $x_1, y_1 \in \mathbb{Z}$ with
\[ x_0y_1 - y_0x_1 = 1. \]

Using this we see that, replacing \( B \) by \( U^tBU \) with \( U = \begin{pmatrix} x_0 & x_1 \\ y_0 & y_1 \end{pmatrix} \), we may assume \( \gcd(b_{11}, d) = 1 \). So \( b_{11}r + ds = 1 \) with integers \( r, s \). Let \( T = \begin{pmatrix} 1 & -b_{12}r \\ 0 & 1 \end{pmatrix} \). Then

\[
B \sim T^tBT = \begin{pmatrix} b_{11} & b_{12}sd \\ b_{12}sd & b_{11}b_{12}r^2 - 2b_{12}r + b_{22} \end{pmatrix} \in \text{Im}(\Phi),
\]
since \( d \) divides \( b_{12} \) and \( b_{22} \). This means that \( \Phi \) is surjective.

The class number \( \tilde{h}(d) \) has been computed by T. Hayashida in [4], actually for the same purpose as here, namely the determination of the number of classes of principal polarization on \( E_1 \times E_2 \) as above. However the way to associate a quadratic form of \( \tilde{M}_2(X) \) to a principal polarization and thus the proof of Theorem 5.1 was completely different. I will not repeat the actual class numbers here, but refer to [4] for details. Note only that the formulas imply that the number \( \#P(X) \) is unbounded as \( d \to \infty \).

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