SUBGROUPS, HYPERBOLICITY AND COHOMOLOGICAL DIMENSION FOR TOTALLY DISCONNECTED LOCALLY COMPACT GROUPS

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Abstract. This article is part of the program of studying large-scale geometric properties of totally disconnected locally compact groups, TDLC-groups, by analogy with the theory for discrete groups. We provide a characterisation of hyperbolic TDLC-groups, in terms of homological isoperimetric inequalities. This characterisation is used to prove that, for hyperbolic TDLC-groups with rational discrete cohomological dimension $\leq 2$, hyperbolicity is inherited by compactly presented closed subgroups. As a consequence, every compactly presented closed subgroup of the automorphism group $\text{Aut}(X)$ of a negatively curved locally finite 2-dimensional building $X$ is a hyperbolic TDLC-group, whenever $\text{Aut}(X)$ acts with finitely many orbits on $X$. Examples where this result applies include hyperbolic Bourdon’s buildings.

1. Introduction

A locally compact group $G$ is totally disconnected if the identity is its own connected component. Hereafter, we use TDLC-group as a shorthand for totally disconnected locally compact group.

Large-scale properties of a TDLC-group $G$ can be addressed by investigating a family of quasi-isometric locally finite connected graphs which are known as Cayley-Abels graphs of $G$; see §3.1 for the definition and further details. Therefore, the theory of TDLC-groups becomes amenable to many tools from geometric group theory (see [3, 5, 22] for example) and the notion of hyperbolic group carries over to the realm of TDLC-groups.

The motivation for this work is to gain a better understanding of the interaction between the geometric properties of the TDLC-group $G$ and its cohomological properties by analogy with the discrete case. An investigation of this type was initiated in [12, 10] where the rational discrete cohomology for TDLC-groups has been introduced and the authors have shown that many well-known properties that hold for discrete groups can be transferred to the context of TDLC-groups (in some cases after substantial work).

For a TDLC-group $G$, the representation theory used in [12] leans on the notion of discrete $\mathbb{Q}[G]$-module, that is a $\mathbb{Q}[G]$-module $M$ such that the action $G \times M \to M$ is continuous when $M$ carries the discrete topology. In the case that $G$ is discrete, any $\mathbb{Q}[G]$-module is discrete. Because of the divisibility of $\mathbb{Q}$, the abelian category $\mathbb{Q}[G]_{\text{dis}}$ of discrete $\mathbb{Q}[G]$-modules has enough projectives. As a consequence, the notions of rational discrete cohomological dimension, denoted by $\text{cd}_{\mathbb{Q}}(G)$, and type $\text{FP}_n$ can be introduced for every TDLC-group $G$ in the category $\mathbb{Q}[G]_{\text{dis}}$ (see §2.3 for the necessary background). This opens up the possibility of investigating TDLC-groups by imposing some cohomological finiteness conditions.

The main result of this article is a subgroup theorem for hyperbolic TDLC-groups of rational discrete cohomological dimension at most 2.

Key words and phrases. hyperbolic groups, totally disconnected locally compact groups, homological finiteness, cohomological dimension 2, compactly presented.
Theorem 1.1. Let $G$ be a hyperbolic TDLC-group with $\text{cd}_Q(G) \leq 2$. Every compactly presented closed subgroup $H$ of $G$ is hyperbolic.

This theorem generalises the following two results for discrete groups:

- Finitely presented subgroups of hyperbolic groups of integral cohomological dimension less than or equal to two are hyperbolic. This is a result of Gersten [16, Theorem 5.4] which can be recovered as a consequence of the inequality $\text{cd}_Q(\_ \_ \_) \leq \text{cd}_Z(\_ \_ \_)$.

- Finitely presented subgroups of hyperbolic groups of rational cohomological dimension less than or equal to two are hyperbolic. This is a recent result in [2] which is the analogue of Theorem 1.1 in the discrete case.

We remark that Brady constructed an example of a discrete hyperbolic group of integral cohomological dimension three that contains a finitely presented subgroup that is not hyperbolic [7]. Hence the the dimensional bound on the results stated above is sharp.

In the framework of discrete groups, it is a result of Gersten that $\text{FP}_2$ subgroups of hyperbolic groups of integral cohomological dimension at most two are hyperbolic [16, Theorem 5.4]. We raise the following question:

**Question 1.** Does Theorem 1.1 remain true if $H$ is of type $\text{FP}_2$ in $Q[H]_{\text{dis}}$ but not compactly presented?

It is well known that if $X$ is a locally finite simplicial complex then the group of simplicial automorphisms $\text{Aut}(X)$ endowed with the compact open topology is a TDLC-group [9, Theorem 2.1]. If, in addition, $X$ admits a $\text{CAT}(-1)$ metric then $\text{Aut}(X)$ is a hyperbolic TDLC-group with $\text{cd}_Q(\text{Aut}(X)) \leq \dim(X)$.

Corollary 1.2. Let $X$ be a locally finite 2-dimensional simplicial $\text{CAT}(-1)$-complex. If $\text{Aut}(X)$ acts with finitely many orbits on $X$, then every compactly presented closed subgroup of $\text{Aut}(X)$ is a hyperbolic TDLC-group.

A discrete version of Corollary 1.2 was proved in [18, Corollary 1.5] using combinatorial techniques. There are different sources of complexes $X$ satisfying the hypothesis of Corollary 1.2 and such that $\text{Aut}(X)$ is a non-discrete TDLC group. For example:

- Bourdon’s building $I_{p,q}$, $p \geq 5$ and $q \geq 3$, is the unique simply connected polyhedral 2-complex such that all 2-cells are right-angled hyperbolic $p$-gons and the link of each vertex is the complete bipartite graph $K_{q,q}$. These complexes were introduced by Bourdon [6]. The natural metric on $I_{p,q}$ is $\text{CAT}(-1)$ and $\text{Aut}(I_{p,q})$ is non-discrete.

- For an integer $k$ and a finite graph $L$, a $(k, L)$-complex is a simply connected 2-dimensional polyhedral complex such that all 2-dimensional faces are $k$-gons and the link of every vertex is isomorphic to the graph $L$. A result of Świątkowski [23, Main Theorem (1)] provides sufficient conditions on the graph $L$ guaranteeing that if $k \geq 4$ then $\text{Aut}(X)$ is a non-discrete group for any $(k, L)$-complex $X$. It is a consequence of Gromov’s link condition, that a $(k, L)$-complex admits a $\text{CAT}(-1)$-structure for any $k$ sufficiently large.

In order to prove Theorem 1.1, we follow ideas from Gersten [16]. We introduce the concept of weak $n$-dimensional linear isoperimetric inequality for TDLC-groups, which is a homological analogue in higher dimensions of linear isoperimetric inequalities. Profinite groups are characterised as TDLC-groups satisfying the weak 0-dimensional linear isoperimetric inequality: see Section 4. The weak 1-dimensional linear isoperimetric inequality is called from here on the weak linear isoperimetric inequality. The following result generalises for TDLC-groups a well-known characterisation of hyperbolicity in the discrete case [16, Theorem 3.1].
Theorem 1.3. A compactly generated TDLC-group $G$ is hyperbolic if and only if $G$ is compactly presented and satisfies the weak linear isoperimetric inequality.

The property of satisfying the weak $n$-dimensional linear isoperimetric inequality is inherited by closed subgroups under some cohomological finiteness conditions.

Theorem 1.4. Let $G$ be a TDLC-group of type $\text{FP}_\infty$ with $\text{cd}_Q(G) = n + 1$ that satisfies the weak $n$-dimensional linear isoperimetric inequality. Then every closed subgroup $H$ of $G$ of type $\text{FP}_{n+1}$ satisfies the weak $n$-dimensional linear isoperimetric inequality.

The major part of the paper is devoted to the proof of Theorem 1.4. The proof relies on the strategy developed in [2] where the authors replace some topological arguments from [16, 19] with algebraic arguments. These arguments are accessible in the category of rational discrete modules over TDLC-groups.

It is a simple verification that Theorem 1.1 follows by Theorems 1.4 and 1.3.

**Proof of the Theorem** [1.1] Since $G$ is hyperbolic, Theorem 1.3 implies that $G$ satisfies the weak linear isoperimetric inequality. By Theorem 1.4, $H$ also satisfies the weak linear isoperimetric inequality. We can then apply Theorem 1.3 again to conclude the proof. □

**Organisation.** Preliminary definitions regarding TDLC-groups and rational discrete modules are given in Section 2. Then Section 3 consists of definitions and some preliminary results on Cayley-Abels graphs, compact presentability and hyperbolicity for TDLC-groups. Section 4 introduces the weak $n$-dimensional linear isoperimetric inequality. Section 5 is devoted to the proof of Theorem 1.4. Finally, Section 6 relates hyperbolicity and the weak linear isoperimetric inequality and contains the proof of Theorem 1.3.

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2. TDLC-GROUPS AND RATIONAL DISCRETE $G$-MODULES

Throughout this section $G$ always denotes a TDLC-group. Note that a TDLC-group is Hausdorff. Discrete groups are TDLC-groups. Profinite groups are precisely compact TDLC-groups [23, Proposition 0]. A fundamental result about the structure of TDLC-groups is known as van Dantzig’s Theorem:

**Theorem 2.1** (van Dantzig’s Theorem, [25]). The family of all compact open subgroups of a TDLC-group $G$ forms a neighbourhood system of the identity element.

Note that every Hausdorff topological group admitting such a local basis is necessarily TDLC. Hence the conclusion of van Dantzig’s Theorem characterises TDLC-groups in the class of Hausdorff topological groups.

For example, the non-Archimedean local fields $\mathbb{Q}_p$ and $\mathbb{F}_q((t))$ admit, respectively, the following local basis at the identity element:

1. $\{ p^n \mathbb{Z}_p \mid n \in \mathbb{N} \}$, where $\mathbb{Z}_p = \{ x \in \mathbb{Q}_p \mid |x| \leq 1 \} = \{ x \in \mathbb{Q}_p \mid |x| < p \}$ is compact and open;
2. $\{ t^n \mathbb{F}_q[[t]] \mid n \in \mathbb{N} \}$, where the norm is defined by $q^{-\text{ord}(f)}$. 

3.
2.1. **Rational discrete $G$-modules.** Let $\mathbb{Q}$ denote the field of rational numbers, and let $\mathbb{Q}[G] \text{-mod}$ be the category of abstract left $\mathbb{Q}[G]$-modules and their homomorphisms. A left $\mathbb{Q}[G]$-module $M$ is said to be **discrete** if the stabiliser

$$G_m = \{ g \in G \mid g \cdot m = m \},$$

of each element $m \in M$ is an open subgroup of $G$. Equivalently, the action $G \times M \to M$ is continuous when $M$ carries the discrete topology. The full subcategory of $\mathbb{Q}[G] \text{-mod}$ whose objects are discrete $\mathbb{Q}[G]$-modules is denoted by $\mathbb{Q}[G] \text{dis}$. It was shown in [12] that $\mathbb{Q}[G] \text{dis}$ is an abelian category with enough injectives and projectives.

2.2. **Permutation $\mathbb{Q}[G]$-modules in $\mathbb{Q}[G] \text{dis}$.** Let $\Omega$ be a non-empty left $G$-set. For $\omega \in \Omega$ let $G_\omega$ denote the pointwise stabiliser. The $G$-set $\Omega$ is called **discrete** if all pointwise stabilisers are open subgroups of $G$, and $\Omega$ is called **proper** if all pointwise stabilisers are open and compact.

The $\mathbb{Q}$-vector space $\mathbb{Q}[\Omega]$ - freely spanned by a discrete $G$-set $\Omega$ - carries a canonical structure of discrete left $\mathbb{Q}[G]$-module called the **discrete permutation $\mathbb{Q}[G]$-module induced by $\Omega$**.

Note that a discrete permutation $\mathbb{Q}[G]$-module in $\mathbb{Q}[G] \text{dis}$ is a coproduct

$$\mathbb{Q}[\Omega] \cong \coprod_{\omega \in \mathcal{R}} \mathbb{Q}[G/G_\omega],$$

in $\mathbb{Q}[G] \text{dis}$, where $\mathcal{R}$ is a set of representatives of the $G$-orbits in $\Omega$, and $\Omega$ is a discrete $G$-set.

A proper permutation $\mathbb{Q}[G]$-module is a discrete $\mathbb{Q}[G]$-module of the form $\mathbb{Q}[\Omega]$ where $\Omega$ is a proper $G$-set.

A proper permutation $\mathbb{Q}[G]$-module is a projective object in $\mathbb{Q}[G] \text{dis}$; see [12]. The arguments of this article rely on the following characterisation of projective objects in $\mathbb{Q}[G] \text{dis}$, a non-trivial result that in particular relies on Maschke’s theorem on irreducible representations of finite groups, and Serre’s results on Galois cohomology.

**Proposition 2.2** ([12 Corollary 3.3]). Let $G$ be a TDLC-group. A discrete $\mathbb{Q}[G]$-module $M$ is projective in $\mathbb{Q}[G] \text{dis}$ if, and only if, $M$ is a direct summand of a proper permutation $\mathbb{Q}[G]$-module in $\mathbb{Q}[G] \text{dis}$.

Throughout the article, we only consider resolutions consisting of discrete permutation $\mathbb{Q}[G]$-modules, and we refer to this type of resolutions as **permutation resolutions in $\mathbb{Q}[G] \text{dis}$**. Analogously, a resolution that consists only of proper permutation modules is called a proper permutation resolution in $\mathbb{Q}[G] \text{dis}$. When the category is clear from the context, we will omit the term “in $\mathbb{Q}[G] \text{dis}$”.

2.3. **Rational discrete homological finiteness.** Following [12], we say that a TDLC-group $G$ is of **type $\text{FP}_n$** ($n \in \mathbb{N}$) if there exists a partial proper permutation resolution in $\mathbb{Q}[G] \text{dis}$

$$\begin{array}{cccccc}
\mathbb{Q}[\Omega_n] & \longrightarrow & \mathbb{Q}[\Omega_{n-1}] & \longrightarrow & \cdots & \longrightarrow & \mathbb{Q}[\Omega_0] & \longrightarrow & \mathbb{Q} & \longrightarrow & 0
\end{array}$$

of the trivial discrete $\mathbb{Q}[G]$-module $\mathbb{Q}$ of finite type, i.e., every discrete left $G$-set $\Omega_i$ is finite modulo $G$ or equivalently $\mathbb{Q}[\Omega_i]$ is finitely generated. The group $G$ is of type $\text{FP}_\infty$ if it is $\text{FP}_n$ for every $n \in \mathbb{N}$. Notice that having type $\text{FP}_0$ is an empty condition for a TDLC-group $G$. On the other hand, having type $\text{FP}_1$ is equivalent to be compactly generated (see [12 Proposition 5.3]) and compact presentation implies type $\text{FP}_2$. 

The rational discrete cohomological dimension of $G$, $\text{cd}_\mathbb{Q}(G) \in \mathbb{N} \cup \{\infty\}$, is defined to be the minimum $n$ such that the trivial discrete $\mathbb{Q}[G]$-module $\mathbb{Q}$ admits a projective resolution

\begin{equation}
0 \rightarrow P_n \xrightarrow{\partial_n} P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow \mathbb{Q} \rightarrow 0
\end{equation}

in $\mathbb{Q}[G]_{\text{dis}}$ of length $n$. The rational discrete cohomological dimension reflects structural information on a TDLC-group $G$. For example, $G$ is profinite if and only if $\text{cd}_\mathbb{Q}(G) = 0$.

By composing the notions above, one says that $G$ is of type FP if

(i) $G$ is of type FP$\infty$, and

(ii) $\text{cd}_\mathbb{Q}(G) = d < \infty$.

For a TDLC-group $G$ of type FP, the trivial left $\mathbb{Q}[G]$-module $\mathbb{Q}$ possesses a projective resolution $(P_\bullet, \partial_\bullet)$ which is finitely generated and concentrated in degrees 0 to $d$. It is not known whether $(P_\bullet, \partial_\bullet)$ can be assumed to be a proper permutation resolution of finite length.

2.4. Restriction of scalars. Let $H$ be a closed subgroup of the TDLC-group $G$. It follows that $H$ is a TDLC-group and in particular the category $\mathbb{Q}[H]_{\text{dis}}$ is well defined. The restriction of scalars from $\mathbb{Q}[G]_{\text{dis}}$ to $\mathbb{Q}[H]_{\text{dis}}$ preserves discretness. In other words there is a well defined restriction functor

\begin{equation}
\text{res}_G^H(\cdot): \mathbb{Q}[G]_{\text{dis}} \rightarrow \mathbb{Q}[H]_{\text{dis}},
\end{equation}

obtained by restriction of scalars via the natural map $\mathbb{Q}[H] \hookrightarrow \mathbb{Q}[G]$. The restriction is an exact functor which maps projectives to projectives. Indeed, for every proper permutation $\mathbb{Q}[G]$-module $\mathbb{Q}[\Omega]$, the discrete $\mathbb{Q}[H]$-module $\text{res}_G^H(\mathbb{Q}[\Omega])$ is still a proper permutation module in $\mathbb{Q}[H]_{\text{dis}}$. To simplify notation, for a discrete $\mathbb{Q}[G]$-module $M$, we may write $M$ for $\text{res}_G^H(M)$ when the meaning is clear.

3. Cayley-Abels graphs, Compact presentability and Hyperbolicity

3.1. Compactly generated TDLC-groups and Cayley-Abels graphs. In this article a graph is a 1-dimensional simplicial complex, hence graphs are undirected, without loops, and without multiple edges between the same pair of vertices.

A locally compact group is said to be compactly generated if there exists a compact subset that algebraically generates the whole group.

Proposition 3.1. \cite[Theorem 2.2]{20} A TDLC-group $G$ is compactly generated if and only if it acts vertex transitively with compact open vertex stabilisers on a locally finite connected graph $\Gamma$.

A graph with a $G$-action as in the proposition above is called a Cayley-Abels graph for $G$. In \cite{20} these graphs are referred to as rough Cayley graphs but the notion of Cayley-Abels graph traces back to Abels \cite{1}.

As soon as the compactly generated TDLC-group $G$ is non-discrete, the $G$-action on a Cayley-Abels graph is never free. That is to say, the action always has non-trivial vertex stabilisers. Nevertheless, these large but compact stabilisers play an important role in the study of the cohomology of $G$: they give rise to proper permutation $\mathbb{Q}[G]$-modules.

A consequence of van Dantzig’s Theorem is the following.

Proposition 3.2. For a TDLC-group $G$ the following statements are equivalent:

1. $G$ is compactly generated.

2. There exists a compact open subgroup $K$ of $G$ and a finite subset $S$ of $G$ such that $K \cup S$ generates $G$ algebraically.
There exists a finite graph of profinite groups \((A, \Lambda)\) with a single vertex, together with a continuous open surjective homomorphism \(\phi: \pi_1(A, \Lambda, \Xi) \to G\) such that \(\phi|_{A_v}\) is injective for all \(v \in V(\Lambda)\).

**Proof.** Note that if \(C\) is a compact set generating \(G\) and \(K\) is a compact open subgroup of \(G\) then there is a finite subset \(S \subset G\) such that the collection of left cosets \(\{sK | s \in S\}\) covers \(C\). Hence, by van Dantzig’s Theorem, (1) implies (2). To show that (2) implies (3), consider the graph of groups with a single vertex and an edge for each element of \(S\). The vertex group is \(K\), and each edge group is \(K \cap K^s\) with morphisms the inclusion and conjugation by \(s\): see [12, Proposition 5.10, proof of (a)]. That (3) implies (1) is immediate since \(G\) is a quotient of the compactly generated TDLC-group \(\pi_1(A, \Lambda, \Xi)\).  

\[\square\]

Note that in the terminology of the third statement of the above proposition, a Cayley-Abels graph for \(G\) can be obtained by considering the quotient of the (topological realisation as a 1-dimensional simplicial complex of the) universal tree of \((A, \Lambda)\) by the kernel of \(\phi\).

### 3.2. Quasi-isometry for TDLC-groups and Hyperbolicity

The edge-path metric on a Cayley-Abels graph \(\Gamma\) of a TDLC-group \(G\) induces a left-invariant pseudo-metric on \(G\), by pulling back the metric of the \(G\)-orbit of a vertex of \(\Gamma\). In the following proposition, we denote this pseudo-metric by \(\text{dist}_\Gamma\).

Following [13], an action of a topological group \(G\) on a (pseudo-) metric space \(X\) is **geometric** if it satisfies:

- (Isometric) The action is by isometries;
- (Cobounded) There is \(F \subset X\) of finite diameter such that \(\bigcup_{g \in G} gF = X\);
- (Locally bounded) For every \(g \in G\) and bounded subset \(B \subset X\) there is a neighborhood \(V\) of \(g\) in \(G\) such that \(VB\) is bounded in \(X\); and
- (Metrically proper) The subset \(\{g \in G: \text{dist}_X(x, gx) \leq R\}\) is relatively compact in \(X\) for all \(x \in X\) and \(R > 0\).

The following version of the Švarc-Milnor Lemma is a consequence of work by Cornulier and de la Harpe on locally compact groups; see [13, Corollary 4.B.11 and Theorem 4.C.5].

**Proposition 3.3.** Let \(G\) be a TDLC-group, let \(X\) be a geodesic (pseudo-) metric space, and let \(x \in X\). Suppose there exists a geometric action of \(G\) on \(X\). Then there is a Cayley-Abels graph \(\Gamma\) for \(G\) such that the map between the pseudo-metric spaces

\[(G, \text{dist}_\Gamma) \to (X, \text{dist}_X), \quad x \mapsto gx\]

is a quasi-isometry.

This proposition implies the following result from [20, Theorem 2.7].

**Corollary 3.4.** The Cayley-Abels graphs associated to a compactly generated TDLC-group are all quasi-isometric each other.

This quasi-isometric invariance of Cayley-Abels graphs allows us to define geometric notions for compactly generated TDLC-groups such as ends, number of ends or growth, by considering quasi-isometric invariants of a Cayley-Abels graph associated to \(G\).

**Definition 3.1.** A TDLC-group \(G\) is defined to be hyperbolic if \(G\) is compactly generated and some (hence any) Cayley-Abels graph of \(G\) is hyperbolic.

For an equivalent definition of hyperbolic TDLC-group using (standard) Cayley graphs over compact generating sets see [4] for details.
3.3. **Compactly presented TDLC-groups.** A locally compact group is said to be *compactly presented* if it admits a presentation \( \langle K \mid R \rangle \) where \( K \) is a compact subset of \( G \) and there is a uniform bound on the length of the relations in \( R \). Observe that being compactly presented implies being compactly generated. There are also an equivalent definition of compact presentation [12, § 5.8] based on van Dantzig’s Theorem in the context of Proposition 3.2.

**Corollary 3.5.** [13] A TDLC-group \( G \) is compactly presented if and only if

1. there exists a finite graph of profinite groups \((A, \Lambda)\) with a single vertex together with a continuous open surjective homomorphism \( \phi: \pi_1(A, \Lambda, \Xi) \to G \) such that \( \phi|_{A_v} \) is injective for all \( v \in V(A) \), and
2. the kernel of \( \phi \) is finitely generated as a normal subgroup.

**Proof.** Note that the if direction is immediate since \( \pi_1(A, \Lambda, \Xi) \) is compactly presented. Indeed, a group presentation of \( \pi_1(A, \Lambda, \Xi) \) has as generators the formal union of the vertex group and a finite number of elements corresponding to the edges of the graph. The set of relations consists of the multiplication table of the vertex group and the HNN-relations; note that all these relations have length at most four. Since the kernel of \( \phi \) is finitely generated as a normal subgroup, it follows that \( G \) is compactly presented.

For the only if direction, since \( G \) is compactly presented, in particular it is compactly generated and hence there is a finite graph of profinite groups \((A, \Lambda)\) with the required properties for (1). It remains to show that the kernel of \( \phi \) is finitely generated as a normal subgroup. By [12, Proposition 5.10(b)], \( \text{ker}(\phi) \) is a discrete subgroup of \( \pi_1(A, \Lambda, \Xi) \). Since \( \pi_1(A, \Lambda, \Xi) \) is compactly generated and \( G \) is compactly presented, [13, Proposition 8.A.10(2)] implies that \( \text{ker}(\phi) \) is compactly generated as a normal subgroup; by discreteness it follows that \( \text{ker}(\phi) \) is finitely generated as a normal subgroup. \( \square \)

**Proposition 3.6.** A TDLC-group \( G \) is compactly presented if and only if there exists a simply connected cellular \( G \)-complex \( X \) with compact open cell stabilisers, finitely many \( G \)-orbits of cells of dimension at most 2, and such that elements of \( G \) fixing a cell setwise fixes it pointwise (no inversions).

A \( G \)-complex with the properties stated in the above proposition is called a topological model of \( G \) of type \( \mathbb{F}_2 \).

**Proof of Proposition 3.6.** The equivalence of compact presentability and the existence of a topological model for \( \mathbb{F}_2 \) follows from standard arguments. That compact presentability is a consequence of the existence of the topological model follows directly from [8, I.8, Theorem 8.10]; for compact presentability implying the existence of such a complex see for example [11, Proposition 3.4]. \( \square \)

The following result is well known for discrete hyperbolic groups. The proof in [8, III, Proposition 3.21] carries over for hyperbolic TDLC-groups by considering the Rips complex on a Cayley-Abels graph instead of the standard Cayley graph.

**Proposition 3.7.** Let \( G \) be a hyperbolic TDLC-group. Then \( G \) acts on a simplicial complex \( X \) such that:

1. \( X \) is finite dimensional, contractible and locally finite;
2. \( G \) acts simplicially, cell stabilisers are compact open subgroups, and there are finitely many \( G \)-orbits of cells.
3. \( G \) acts transitively on the vertex set of \( X \).

In particular, the topological realisation of the barycentric subdivision of \( X \) is a topological model for \( \mathbb{F}_2 \), and hence \( G \) is compactly presented.
4. Weak $n$-dimensional isoperimetric inequality

4.1. (Pseudo-)Norms on vector spaces. Given a vector space $V$ over a subfield $F$ of the complex numbers, a pseudo-norm on $V$ is a nonnegative-valued scalar function $\|\cdot\| : V \to \mathbb{R}_+$ with the following properties:

(N1) (Subadditivity) $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in V$;

(N2) (Absolute Homogeneity) $\|\lambda \cdot v\| = |\lambda| \|v\|$ for all $\lambda \in F$ and $v \in V$.

A pseudo-norm $\|\cdot\|$ on a vector space $V$ is said to be a norm if it satisfies the following additional property:

(N3) (Point-separation) $\|v\| = 0, v \in V \Rightarrow v = 0$.

Let $f : (V, \|\cdot\|_V) \to (W, \|\cdot\|_W)$ be a linear function between pseudo-normed vector spaces. We say that $f$ is bounded if there exists a constant $C > 0$ such that $\|f(v)\|_W \leq C \|v\|_V$ for all $v \in V$. In such a case, we write $\|\cdot\|_V \preceq f^* \|\cdot\|_W$ when the constant $C$ is irrelevant. Two different norms $\|\cdot\|$ and $\|\cdot\|'$ on $V$ are said to be equivalent, $\|\cdot\| \sim \|\cdot\|'$, if $\|\cdot\| \preceq \|\cdot\|' \preceq \|\cdot\|$. From here on the relation $\preceq$ will be denoted as $\leq$.

4.2. $\ell_1$-norm on permutation $\mathbb{Q}[G]$-modules. Let $\mathbb{Q}[\Omega]$ be a permutation $\mathbb{Q}[G]$-module. In particular, $\mathbb{Q}[\Omega]$ is a $\mathbb{Q}$-vector space with linear basis $\Omega$. Therefore, the nonnegative-valued function

$$\|\cdot\|^{\Omega}_1 : \mathbb{Q}[\Omega] \to \mathbb{Q}_+, \quad \text{s.t.} \quad \sum_{\omega \in \Omega} \alpha_\omega \omega \mapsto \sum_{\omega \in \Omega} |\alpha_\omega|,$$

defines a norm on $\mathbb{Q}[\Omega]$. As usual, we shall refer to $\|\cdot\|_1^{\Omega}$ as the $\ell_1$-norm on $\mathbb{Q}[\Omega]$. Notice that $\|\cdot\|_1^{\Omega}$ is $G$-equivariant.

**Proposition 4.1.** Let $\phi : \mathbb{Q}[\Omega] \to \mathbb{Q}[\Omega']$ be a morphism of finitely generated permutation $\mathbb{Q}[G]$-modules. Then $\|\cdot\|_1^{\Omega'} \preceq \phi^* \|\cdot\|_1^{\Omega}$.

**Proof.** This is a consequence of the $G$-invariance of the $\ell_1$-norm and the fact that the modules are finitely generated. Indeed, the morphism $\phi$ is described by a finite matrix $A = (a_{ij})$ with entries in $\mathbb{Q}[G]$. Consider the $\ell_1$-norm $\|\cdot\|_1$ on $\mathbb{Q}[G]$ and let $C = \max |a_{ij}|$. Then $\|\phi(x)\|_1^{\Omega'} \leq C \|x\|_1^{\Omega}$ for every $x \in \mathbb{Q}[\Omega]$. \qed

The above proposition will be used for discrete permutation modules over $\mathbb{Q}[G]$.

4.3. Filling pseudo-norms on discrete $\mathbb{Q}[G]$-modules. Let $M$ be a finitely generated discrete $\mathbb{Q}[G]$-module. Since $\mathbb{Q}[G]\text{dis}$ has enough projectives, there exists a finitely generated proper permutation $\mathbb{Q}[G]$-module $\mathbb{Q}[\Omega]$ mapping onto $M$, that is, $\mathbb{Q}[\Omega] \xrightarrow{\partial} M$ and $G$ acts on $\Omega$ with compact open stabilisers and finitely many orbits. The filling pseudo-norm $\|\cdot\|_\partial$ on $M$ induced by $\partial$ is defined as

$$\|m\|_\partial = \inf\{\|x\|_1^{\Omega} \mid x \in \mathbb{Q}[\Omega], \partial(x) = m\}.$$
One easily verifies that $\|\cdot\|_\beta$ is subadditive and absolutely homogeneous. Note that
\begin{equation}
\|\cdot\|_\beta \preceq \|\cdot\|_\Omega^1.
\end{equation}
It is an observation that an $\ell_1$-norm on a finitely generated discrete permutation $G$-module $Q[\Omega]$ is equivalent to a filling norm.

**Proposition 4.2.** Morphisms between finitely generated discrete $Q[G]$-modules are bounded with respect to filling pseudo-norms.

**Proof.** Let $f: M \to N$ be a morphism of finitely generated discrete $Q[G]$-modules. Since $M$ and $N$ are both finitely generated in $Q[G]$ dis, there exist morphisms $Q[\Omega_1] \xrightarrow{\partial_1} M$ and $Q[\Omega_2] \xrightarrow{\partial_2} N$ such that each $Q[\Omega_i]$ is a finitely generated proper permutation module. By the universal property of $Q[\Omega_1]$ as a projective object, there is $\phi: Q[\Omega_1] \to Q[\Omega_2]$ such that the following diagram commutes:

\[
\begin{array}{ccc}
Q[\Omega_1] & \xrightarrow{\phi} & Q[\Omega_2] \\
\partial_1 & & \partial_2 \\
M & \xrightarrow{f} & N \\
\end{array}
\]

For any $m \in M$ and any $\varepsilon > 0$, let $x_m \in Q[\Omega_1]$ such that $\partial_1(x_m) = m$ and $\|x_m\|_{\Omega_1} \leq \partial_1 \|m\|_{\Omega_1} + \varepsilon$. Since $f(m) = \partial_2(\phi(x))$, one has

\[
\|f(m)\|_{\partial_2} \leq \partial_2 \|\phi(x)\|_{\Omega_2} \leq \|x\|_{\Omega_1} \leq \partial_1 \|m\|_{\Omega_1} + \varepsilon.
\]

By Proposition 4.1, $\|x\|_{\Omega_1} \leq \partial_1 \|m\|_{\Omega_1} + \varepsilon$. Since $\varepsilon$ is arbitrary, we deduce $\|f\|_{\partial_2} \preceq f \|\cdot\|_{\partial_1}$.

By considering the identity function on a finitely generated discrete $Q[G]$-module $M$, the previous proposition implies:

**Corollary 4.3.** Let $G$ be a TDLC-group. Any two filling pseudo-norms on a finitely generated discrete $Q[G]$-module $M$ are equivalent. In particular, all the filling pseudo-norms on a finitely generated proper permutation $Q[G]$-module $Q[\Omega]$ are equivalent to $\|\cdot\|_{\Omega}^1$, and therefore they are all norms.

The former implies that each finitely generated discrete $Q[G]$-module $M$ admits a unique filling pseudo-norm up to equivalence. Therefore, by abuse of notation, we denote by $\|\cdot\|_M$ any filling pseudo-norm of $M$ and we refer to $\|\cdot\|_M$ as the filling pseudo-norm of $M$.

**4.4. Undistorted submodules.** Let $M$ be a discrete $Q[G]$-module with a norm $\|\cdot\|$ and let $N$ be a finitely generated discrete $Q[G]$-submodule of $M$. Then $N$ is said to be undistorted with respect to $\|\cdot\|$ if the restriction of $\|\cdot\|$ to $N$ is equivalent to a filling norm on $N$. In the case that $M$ is finitely generated and $N$ is undistorted with respect to the filling norm $\|\cdot\|_M$ we shall simply say that $N$ is undistorted in $M$.

We note that in general it is not the case that finitely generated submodules of $M$ are undistorted; we refer the reader to Section 6 for counter-examples.

**Proposition 4.4.** Let $G$ be a TDLC-group. The filling pseudo-norm $\|\cdot\|_P$ of a finitely generated projective discrete $Q[G]$-module $P$ is a norm. Moreover, if $P$ is a direct summand of a finitely generated proper permutation module $Q[\Omega]$, then $P$ is undistorted in $Q[\Omega]$. 
Proof. Let $Q[\Omega]$ be a finitely generated proper permutation module such that $P$ is a direct summand of $Q[\Omega]$; see Proposition 2.2. Let $\iota: P \to Q[\Omega]$ be the inclusion and let $\pi: Q[\Omega] \to P$ be the projection such that $\pi \circ \iota = id_P$. Proposition 4.2 implies $\|\cdot\|_P^0 \leq \|\cdot\|_P$ and $\|\cdot\|_P \leq \|\cdot\|_1^0$ on $P$. The former inequality implies that $\|\cdot\|_P$ is a norm, and both of them imply that $\|\cdot\|_P \sim \|\cdot\|_1^0$ on $P$. 

More generally, this argument shows that a direct summand of any finitely generated discrete $Q[G]$-module, with the filling norm, is undistorted.

We conclude the section with a technical result about bounded morphisms that will be used later and relies on the proof of the previous proposition.

**Proposition 4.5.** Let $G$ be a TDLC-group and $H$ a closed subgroup of $G$. Let $M$ be a finitely generated and projective $Q[G]$-module in $Q[G]\dis$ with filling norm $\|\cdot\|_M$. Regard $M$ as a $Q[H]$-module via restriction, and suppose that $N$ is a finitely generated direct summand of $M$ in $Q[H]\dis$. Then $N$ is an undistorted $Q[H]$-module of $M$ with respect to the norm $\|\cdot\|_M$.

Proof. The $Q[H]$-module $N$ is projective since the restriction of $M$ is projective and hence $N$ is a direct summand of a projective $Q[H]$-module.

By Proposition 4.2, $M$ can be assumed to be a finitely generated proper permutation $Q[G]$-module $Q[\Omega]$. Note that the restriction of $Q[\Omega]$ is a proper permutation $Q[H]$-module.

Since $N$ is finitely generated, there exists an $H$-subset $\Sigma$ of $\Omega$ such that $\Sigma/H$ is finite and $N$ is a $Q[H]$-submodule of $Q[\Sigma]$. Since $N$ and $Q[\Sigma]$ are direct summands of $Q[\Omega]$ as $Q[H]$-modules, it follows that $N$ is a direct summand of the finitely generated proper permutation $Q[H]$-module $Q[\Sigma]$.

Proposition 4.3 implies that the pseudo-norm $\|\cdot\|_N$ is a norm and $\|\cdot\|_N \sim \|\cdot\|_1^\Sigma$ on $N$. Since $\|\cdot\|_1^\Sigma = \|\cdot\|_1^0$ on $Q[\Sigma]$, it follows that $\|\cdot\|_N \sim \|\cdot\|_1^0$ on the elements of $N$. 

4.5. **Weak n-dimensional linear isoperimetric inequality.** Let $G$ be a TDLC-group of type $FP_{n+1}$. Then there exists a partial proper permutation resolution

$$Q[\Omega_{n+1}] \xrightarrow{\delta_{n+1}} Q[\Omega_n] \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_2} Q[\Omega_1] \xrightarrow{\delta_1} Q[\Omega_0] \xrightarrow{\delta_0} Q \xrightarrow{\partial} 0$$

of finite type, i.e. it consists of finitely generated discrete $Q[G]$-modules. We say that $G$ satisfies the weak $n$-dimensional linear isoperimetric inequality if $ker(\delta_n)$ is an undistorted submodule of $Q[\Omega_n]$. The special case for $n = 1$ is referred as the weak linear isoperimetric inequality.

Note that, by Proposition 4.2, $\|\cdot\|_1^\Omega \geq \|\cdot\|_{ker(\delta_n)}$ where $\iota: ker(\delta_n) \to Q[\Omega_n]$ is the inclusion. Hence, the weak $n$-dimensional linear isoperimetric inequality is equivalent to the existence of a constant $C > 0$ such that $\|\cdot\|_{ker(\delta_n)} \leq C \|\cdot\|_1^{\Omega_n}$ on $ker(\delta_n)$.

The proof of the following proposition is an adaption of the proof of [19] Theorem 3.5 that we have included for the reader’s convenience.

**Proposition 4.6.** For a TDLC-group $G$ of type $FP_{n+1}$, the property of satisfying the weak linear $n$-dimensional isoperimetric inequality is independent of the choice of the proper permutation resolution of finite type in $Q[G]\dis$.

Proof. Let $(Q[\Omega_i], \delta_i), (Q[\Lambda_i], \delta_i)$ be a pair of proper permutation resolutions of $Q$ which contain finitely generated modules for degrees $i = 0, \ldots, n + 1$. Suppose $G$ satisfies the weak $n$-dimensional linear isoperimetric inequality with respect to the resolution $(Q[\Lambda_i], \delta_i)$. Hence there is $C > 0$ such that

$$\|x\|_{ker(\delta_n)} \leq C \|x\|_1^{\Lambda_n}.$$
for all \( x \in \ker(\delta_n) \).

Since any two projective resolutions of \( Q \) are chain homotopy equivalent, there exist chain maps \( f: (Q[\Omega], \partial_1) \to (Q[\Lambda], \delta_1) \) and \( g: (Q[\Lambda], \delta_1) \to (Q[\Omega], \partial_1) \), and a 1-differential \( h: (Q[\Omega], \partial_1) \to (Q[\Omega], \partial_1) \) such that

\[
\delta_{n+1} \circ h_{n+1} + h_n \circ \partial_n = g_n \circ f_n - \text{Id}.
\]

Diagrammatically, one has

\[
\begin{array}{c}
Q[\Omega]_{n+1} \xrightarrow{h_n} Q[\Omega]_n \xrightarrow{\partial_n} Q[\Omega]_{n-1} \\
\downarrow g_{n+1} \quad \quad \downarrow f_n \quad \downarrow g_{n-1} \quad \downarrow f_{n-1} \\
Q[\Lambda]_{n+1} \xrightarrow{\delta_{n+1}} Q[\Lambda]_n \xrightarrow{\delta_n} Q[\Lambda]_{n-1}
\end{array}
\]

Since \( g_{n+1}, f_n \) and \( h_n \) are morphisms between finitely generated discrete \( Q[G] \)-modules, Proposition 4.2 applies and, therefore, the constant \( C \) defined above can be assumed to satisfy:

(D1) \( \|g_{n+1}(\lambda)\|_{\Omega_{n+1}}^1 \le C \|\lambda\|_{\Lambda_{n+1}}^1 \), for all \( \lambda \in Q[\Lambda_{n+1}] \);

(D2) \( \|f_n(\omega)\|_{\Lambda_n}^1 \le C \|\omega\|_{\Omega_n}^1 \), for all \( \omega \in Q[\Omega_n] \); and

(D3) \( \|h_n(\omega)\|_{\Omega_{n+1}}^1 \le C \|\omega\|_{\Omega_n}^1 \), for all \( \omega \in Q[\Omega_n] \).

We prove below that there is a constant \( D > 0 \) such that for any \( \alpha \in \ker(\partial_n) \) and \( \epsilon > 0 \)

\[
\|\alpha\|_{\ker(\partial_n)} \le D \|\alpha\|_{\Omega_n}^1 + D\epsilon.
\]

Then it follows that \( G \) satisfies the weak \( n \)-dimensional linear isoperimetric inequality with respect to the resolution \( (Q[\Omega], \partial_1) \) by letting \( \epsilon \to 0 \).

Let \( \alpha \in \ker(\partial_n) \) and \( \epsilon > 0 \). By the diagram (4.7), it follows that \( f_n(\alpha) \in \ker(\delta_n) = \delta_{n+1}(Q[\Lambda_{n+1}]) \). Since \( Q[\Lambda_{n+1}] \) is finitely generated, we can consider the filling-norm \( \|\cdot\|_{\ker(\delta_n)} \) to be induced by \( \delta_{n+1} \). Therefore, by the definition of the filling norm \( \|\cdot\|_{\ker(\delta_n)} \) there is \( \beta \in Q[\Lambda_{n+1}] \) such that \( \delta_{n+1}(\beta) = f_n(\alpha) \) and

\[
\|\beta\|_{\Lambda_{n+1}}^1 \le \|f_n(\alpha)\|_{\ker(\delta_n)} + \epsilon.
\]

By evaluating \( \alpha \) in Equation 4.6 we can write

\[
\begin{align*}
\alpha &= g_n(f_n(\alpha)) - \delta_{n+1}(h_n(\alpha)) \\
&= g_n(\delta_{n+1}(\beta)) - \delta_{n+1}(h_n(\alpha)) \\
&= \delta_{n+1}(g_n(\beta) - h_n(\alpha)).
\end{align*}
\]

Hence

\[
\begin{align*}
\|\alpha\|_{\ker(\partial_n)} &\le \|g_n(\beta) - h_n(\alpha)\|_{\Omega_{n+1}}^1 \\
&\le \|g_n(\beta)\|_{\Omega_{n+1}}^1 + \|h_n(\alpha)\|_{\Omega_{n+1}}^1 \\
&\le C\|\beta\|_{\Lambda_{n+1}}^1 + C\|\alpha\|_{\Omega_n}^1 \\
&\le C\|f_n(\alpha)\|_{\ker(\delta_n)} + C\epsilon + C\|\alpha\|_{\Omega_n}^1 \\
&\le C\|f_n(\alpha)\|_{\Lambda_n}^1 + C\|\alpha\|_{\Omega_n}^1 + C\epsilon \\
&\le C\|\alpha\|_{\Omega_n}^1 + C\|\alpha\|_{\Omega_n}^1 + C\epsilon
\end{align*}
\]

by inequality 4.8.

4.6. Weak 0-Dimensional Isoperimetric Inequality and Profinite Groups.

As previously mentioned, a group is profinite if and only if it is a compact TDLC-group [23 Proposition 0]. The following statement is a simple application of the definitions of this section.
Proposition 4.7. Let $G$ be a TDLC-group. Then $G$ is compact if and only if it is compactly generated and satisfies a weak 0-dimensional isoperimetric inequality.

The only if direction of the proposition is immediate. Indeed, if $G$ is a compact TDLC-group, then the trivial $G$-module $\mathbb{Q}$ is projective in $\mathcal{Q}[G]\text{-}\text{dis}$. In this case, one can read the weak 0-dimensional isoperimetric inequality from the resolution $0 \to \mathbb{Q} \to \mathbb{Q} \to 0$.

For the rest of the section, suppose that $G$ is a TDLC-group satisfying a weak 0-dimensional isoperimetric inequality. Let $\Gamma$ be a Cayley-Abels graph of $G$, let $\text{dist}$ be the combinatorial path metric on the set of vertices $V$ of $\Gamma$, and let $E$ denote the set of edges of $\Gamma$. In order to prove that $G$ is profinite, it is enough to show that $V$ is finite.

Choose an orientation for each edge of $\Gamma$ and consider the augmented rational cellular chain complex of $\Gamma$,

$$Q[E] \xrightarrow{s} Q[V] \xrightarrow{t} \mathbb{Q} \to 0.$$ 

Since $\Gamma$ is Cayley-Abels graph, this is a proper normal permutation resolution.

Following ideas in [44], define a partial order $\preceq$ on $Q[E]$ as follows. For $\nu, \mu \in Q[E]$, $\nu = \sum_{e \in E} t_e \epsilon$ and $\mu = \sum_{e \in E} s_e \epsilon$, then $\nu \preceq \mu$ if and only if $t_e \leq s_e$ for every $e \in E$. Observe that if $\nu \preceq \mu$ then $\|\nu\|_E^F = \|\mu - \nu\|_E^F + \|\nu\|_E^F$; in particular $\|\nu\|_E^F \leq \|\mu\|_E^F$. An element $\nu \in Q[E]$ is called integral if $t_e \in \mathbb{Z}$ for each $e$. Define analogously $\preceq$ on $Q[V]$.

Lemma 4.8. Suppose that $\mu \in Q[E]$ is integral and $\delta(\mu) = m(v-u)$ where $u, v \in V$ and $m$ is a positive integer. Then there is an integral element $\nu \in Q[E]$ such that $\delta(\nu) = v-u$ and $\nu \preceq \mu$ and $\|\nu\|_E^F \geq \text{dist}(u,v)$.

Sketch of the proof. Suppose $\mu = \sum_{e \in E} s_e \epsilon$. Consider a directed multigraph $\Xi$ (multiple edges between distinct vertices are allowed) with vertex set $V$ and such that for each $e \in E$ if $s_e \geq 0$ then there are $|s_e|$ edges from $a$ to $b$ where $\delta(e) = b-a$; and if $s_e < 0$ then there are $|s_e|$ from $b$ to $a$. The degree sum formula for directed multigraphs implies that $u$ and $v$ are in the same connected component of $\Xi$. It is an exercise to show that there is a directed path $\gamma$ from $u$ to $v$ in $\Xi$ that can be assumed to be injective on vertices. The path $\gamma$ induces an element $\nu \in Q[E]$ such that if $\nu = \sum_{e \in E} t_e \epsilon$ then $t_e = \pm 1$ and $\nu \preceq \mu$. Moreover $\gamma$ induces a path in $\Gamma$ from $u$ to $v$ and hence $\|\nu\|_E^F \geq \text{dist}(u,v)$.

Suppose, for a contradiction, that $V$ is an infinite set. Fix $v_0 \in V$. For every $n \in \mathbb{N}$, let $v_n \in V$ such that $\text{dist}(v_0, v_n) \geq n$. Note that such a vertex $v_n$ always exists since $\Gamma$ is locally finite and connected. Let $\alpha_n = v_n - v_0$ and observe that $\alpha_n \in \ker(\delta)$ and $\|\alpha_n\|_V^2 = 2$. We will show that $\|\alpha_n\|_{\ker(\delta)} \geq n$ for every $n$, and hence $G$ cannot satisfy a weak 0-dimensional isoperimetric inequality. Fix $n \in \mathbb{N}$, and let $\mu = \sum_{e \in E} s_e \epsilon \in Q[E]$ such that $\delta(\mu) = \alpha_n = v_n - v_0$. Then there is $m \in \mathbb{N}$ such that $m\mu$ is integral. Since $\delta(m\mu) = m(v_n - v_0)$, Lemma 4.8 implies that there is $\nu_1 \in Q[E]$ such that $\delta(\nu_1) = v_n - v_0$ and $\nu_1 \preceq m\mu$ and $\|\nu\|_1 \geq \text{dist}(v_0, v_n)$. Let $\mu_1 = m\mu - \nu_1$ and note that $\mu_1$ is integral, $\delta(\mu_1) = (m-1)(v_n - v_0)$, and $\|m\mu_1\|_E^F = \|\mu_1\|_E^F + \|\nu_1\|_E^F \geq \|\mu_1\|_E^F + \text{dist}(v_0, v_n)$. An induction argument on $m$ then proves that $\|m\mu_1\|_E^F \geq m\text{dist}(v_0, v_n)$ and hence $\|\mu\|_E^F \geq \text{dist}(v_0, v_n)$.

5. Proof of Subgroup Theorem

The proof of the theorem relies on the following lemma. Let $G$ be a TDLC-group of type $\text{FP}_n$ and $H$ a closed subgroup of $G$ of type $\text{FP}_n$. 


Lemma 5.1. There are partial proper permutation resolutions

\[ Q[\Omega_n] \xrightarrow{\delta_n} Q[\Omega_{n-1}] \rightarrow \cdots \rightarrow Q[\Omega_0] \rightarrow Q \rightarrow 0, \]

\[ Q[\Sigma_n] \xrightarrow{\partial_n} Q[\Sigma_{n-1}] \rightarrow \cdots \rightarrow Q[\Sigma_0] \rightarrow Q \rightarrow 0 \]

of \( Q \) in \( Q[H] \text{dis} \) and \( Q[G] \text{dis} \) respectively, satisfying the following properties.

1. \( \Omega_0, \ldots, \Omega_n \) are finitely generated \( H \)-sets;
2. \( \Sigma_0, \ldots, \Sigma_n \) are finitely generated \( G \)-sets;
3. restricting the \( G \)-action on each \( \Sigma_i \) to \( H \), \( \Omega_i \) is an \( H \)-subset of \( \Sigma_i \) via \( \iota_i : \Omega_i \rightarrow \Sigma_i \);
4. the diagram

\[
\begin{array}{c}
\ker(\delta_n) \xrightarrow{Q[\iota_n]} Q[\Omega_n] \xrightarrow{\delta_n} \cdots \xrightarrow{Q[\iota_0]} Q[\Omega_0] \xrightarrow{\delta_0} Q \rightarrow 0 \\
\ker(\partial_n) \xrightarrow{Q[\iota_n]} Q[\Sigma_n] \xrightarrow{\partial_n} \cdots \xrightarrow{Q[\iota_0]} Q[\Sigma_0] \xrightarrow{\partial_0} Q \rightarrow 0
\end{array}
\]

of \( Q[H] \)-modules commutes;
5. \( \text{coker}(\ker(\delta_n) \to \ker(\partial_n)) \) is a projective \( Q[H] \)-module.

Proof. Take a partial proper permutation resolution

\[ Q[\Sigma_n] \xrightarrow{\partial_n} Q[\Sigma_{n-1}] \rightarrow \cdots \rightarrow Q[\Sigma_0] \rightarrow Q \rightarrow 0 \]

of \( Q \) in \( Q[G] \text{dis} \). We construct the required resolution

\[ Q[\Omega_n] \xrightarrow{\delta_n} Q[\Omega_{n-1}] \rightarrow \cdots \rightarrow Q[\Omega_0] \rightarrow Q \rightarrow 0 \]

in \( Q[H] \text{dis} \) by induction on \( n \). So suppose we have already constructed a diagram

\[
\begin{array}{c}
\ker(\delta_{n-1}) \xrightarrow{Q[\iota_{n-1}]} Q[\Omega_{n-1}] \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{Q[\iota_0]} Q[\Omega_0] \xrightarrow{\delta_0} Q \rightarrow 0 \\
\ker(\partial_{n-1}) \xrightarrow{Q[\iota_{n-1}]} Q[\Sigma_{n-1}] \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{Q[\iota_0]} Q[\Sigma_0] \xrightarrow{\partial_0} Q \rightarrow 0
\end{array}
\]

satisfying the conditions for \( n - 1 \) (this is trivial for the base case \( n = 0 \)).

Write \( \iota \) for the induced map \( \ker(\delta_{n-1}) \to \ker(\partial_{n-1}) \); by hypothesis, there is a map \( \pi : \ker(\partial_{n-1}) \to \ker(\delta_{n-1}) \) such that \( \pi \iota \) is the identity on \( \ker(\delta_{n-1}) \). Since \( H \) has type \( \text{FP}_n \), \( \ker(\delta_{n-1}) \) is finitely generated; pick a finite generating set \( x_1, \ldots, x_k \) and pick a preimage \( y_i \) of each element \( x_i \) in \( Q[\Sigma_n] \); via the map \( Q[\Sigma_n] \xrightarrow{\partial_n} \ker(\delta_{n-1}) \xrightarrow{\pi} \ker(\delta_{n-1}) \). Each \( y_i \) is a finite sum \( \sum_{j=1}^k a_{ij} \alpha_{ij} \) with \( \alpha_{ij} \in \Sigma_n \) and \( a_{ij} \in Q \). Now let \( \Omega_n \) be the (finitely generated) \( H \)-subset of \( \Sigma_n \) generated by the \( \alpha_{ij} \). We get an induced map \( \pi \partial_n Q[\iota_n] : Q[\Omega_n] \rightarrow \ker(\delta_{n-1}) \) extending the commutative diagram as required; it only remains to check condition 5.
To see this, consider the following commutative diagram in $\mathbb{Q}[H] \text{dis}$

\[
\begin{array}{cccccc}
0 & \rightarrow & \ker(\delta_n) & \rightarrow & \mathbb{Q}[\Omega_n] & \rightarrow & \ker(\delta_{n-1}) \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \ker(\partial_n) & \rightarrow & \mathbb{Q}[\Sigma_n] & \rightarrow & \ker(\partial_{n-1}) \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \text{coker}(\iota') & \rightarrow & \text{coker}(\mathbb{Q}[\iota_n]) & \rightarrow & \text{coker}(\iota) \rightarrow 0
\end{array}
\]

Note that the diagram consists of exact rows and exact columns. Since $\mathbb{Q}[\Omega_n^H]$ is a direct summand of $\mathbb{Q}[\Sigma_n^G]$ in $\mathbb{Q}[H] \text{dis}$, it follows that each $\text{coker}(\mathbb{Q}[\iota_n])$ is projective; $\text{coker}(\iota)$ is projective by hypothesis. Then exactness of the bottom row implies that $\text{coker}(\iota')$ is projective.

Remark 5.2. In $\mathbb{Q}[G] \text{dis}$, it is possible to develop a homological mapping cylinder argument analogous to [2] Proposition 4.1 that yields a similar conclusion to Lemma 5.1 but only for open subgroups of $G$. This argument was developed in a preliminary version of this article.

Proof of Theorem 1.4. Since $G$ and $H$ have type $FP_n$, we may use the partial proper permutation resolutions described in Lemma 5.1, we keep the notation from there. Because $G$ has type $FP_{n+1}$ and $\cd_0(G) = n + 1$, $\ker(\partial_n)$ is finitely generated (in $\mathbb{Q}[G] \text{dis}$) and projective; because $H$ has type $FP_{n+1}$ and $\text{coker}(\iota')$ is projective, $\ker(\partial_n)$ is a finitely generated (in $\mathbb{Q}[H] \text{dis}$) summand of $\text{coker}(\partial_n)$. So:

1. $\|\_\|_{\ker(\delta_n^H)} \sim \|\_\|_{\ker(\partial_n^G)}$ on the elements of $\ker(\delta_n^H)$, by Proposition 4.5.
2. $\|\_\|_{\Omega_n^H} \sim \|\_\|_{\Sigma_n^G}$ on the elements of $\mathbb{Q}[\Omega_n^H]$, because $\Omega_n$ is a subset of $\Sigma_n$;
3. $\|\_\|_{\ker(\partial_n^G)} \sim \|\_\|_{\Sigma_n^G}$ on the elements of $\ker(\partial_n^G)$, because $G$ satisfies the weak $n$-dimensional linear isoperimetric inequality.

Therefore $\|\_\|_{\ker(\delta_n^H)} \sim \|\_\|_{\Omega_n^H}$ on the elements of $\ker(\delta_n^H)$, i.e. $H$ satisfies the weak $n$-dimensional isoperimetric inequality. 

6. Weak linear isoperimetric inequality and hyperbolicity

The notion of linear isoperimetric inequality was used to characterise discrete hyperbolic groups by Gersten [15]. Different generalisations of Gersten’s result have been presented by various authors; see for example [17], [21] and [19]. In particular, Manning and Groves [17] reformulated Gersten’s argument to provide a homological characterisation of simply connected hyperbolic 2-complexes by means of a homological isoperimetric inequality. Here we use results from [17] to provide an analogue characterisation of hyperbolic TDLC-groups.

Let $X$ be a complex with $i$-skeleton denoted by $X^{(i)}$. Consider the cellular chain complex $(C_\bullet(X, \mathbb{Q}), \partial_\bullet)$ of $X$ with rational coefficients. Each vector space $C_i(X, \mathbb{Q})$ is $\mathbb{Q}$-spanned by the collection of $i$-cells $\sigma$ of $X$. An $i$-chain $\alpha$ is a formal linear combination $\sum_{\sigma \in X^{(i)}} r_\sigma \sigma$ where $r_\sigma \in \mathbb{Q}$. The $\ell_1$-norm on $C_i(X, \mathbb{Q})$ is defined as $\|\alpha\|_{\ell_1} = \sum |r_\sigma|$, where
where $|\cdot|$ denotes the absolute value function on $\mathbb{Q}$.

**Definition 6.1** ([17] Definition 2.18 Combinatorial path). Let $X$ be a complex. Suppose $I$ is an interval with a cellular structure. A combinatorial path $I \rightarrow X^{(1)}$ is a cellular path sending 1-cells to either 1-cells or 0-cells. A combinatorial path is a combinatorial path with equal endpoints.

From here on, to simplify notation, the 1-chain induced by a combinatorial loop $c$ in $X$ is denoted by $c$ as well.

**Definition 6.2** ([17] Definition 2.28 Linear Homological isoperimetric inequality). Let $X$ be a simply connected complex. We say that $X$ satisfies the linear homological isoperimetric inequality if there is a constant $C > 0$ such that for any combinatorial loop $c$ in $X$ there is some $\sigma \in C_2(X, \mathbb{Q})$ with $\partial(\sigma) = c$ satisfying

$$\|\sigma\|_{X,2} \leq K\|c\|_{1,1}.$$  

**Definition 6.3.** Let $G$ be a compactly presented TDLC-group. There exists a simply connected $G$-complex $X$ with compact open cell stabilisers, the 2-skeleton $X^{(2)}$ is compact modulo $G$, the $G$-action is cellular and an element in $G$ fixing a cell setwise fixes it already pointwise. The group $G$ satisfies the linear homological isoperimetric inequality if $X$ does.

The above definition is independent of the choice of $X$ as a consequence of Proposition 4.5, the fact that a compactly presented TDLC-group has type FP, and the following statement.

**Proposition 6.1.** Suppose $G$ is a compactly presented TDLC-group and $X$ is a topological model of $G$ of type $F_2$. Then $G$ satisfies the weak linear isoperimetric inequality if and only if $X$ satisfies the linear homological isoperimetric inequality.

**Proof.** The augmented cellular chain complex $(C_*(X, \mathbb{Q}), \partial_*)$ of $X$ is a proper partial permutation resolution of $\mathbb{Q}$ of type $F_2$. The module $C_1(X, \mathbb{Q})$ is a proper permutation module and we can take as its filling norm $\|\cdot\|_{C_1}$ the $\ell_1$-norm induced by $G$-set of $i$-cells.

The weak linear isoperimetric inequality means that the filling norm $\|\cdot\|_{Z_1}$ of $Z_1(X, \mathbb{Q})$ is equivalent to the restriction of $\|\cdot\|_{C_1}$ to $Z_1(X, \mathbb{Q})$. Hence there is a constant $C > 0$ such that $\|\cdot\|_{Z_1} \leq C\|\cdot\|_{C_1}$ on $Z_1(X, \mathbb{Q})$. To prove the linear homological isoperimetric inequality is enough consider non-trivial combinatorial loops, the inequality is trivial otherwise. Let $c$ be a non-trivial combinatorial loop and let $\mu \in C_2(X)$ such that $\partial(\mu) = c$ and $\|\mu\|_{C_2} \leq \|c\|_{Z_1} + 1$. In particular, $\|\mu\|_{C_2} \leq \|c\|_{Z_1} + \|c\|_{C_1}$ since $\|c\|_{C_1}$ is a positive integer. It follows that $\|\mu\|_{C_2} \leq (C+1)\|c\|_{C_1}$ for any non-trivial combinatorial loop.

Conversely, suppose that $X$ satisfies the linear homological isoperimetric inequality for a constant $C$. Then the filling norm on $\gamma \in Z_1(X, \mathbb{Q})$ is given by $\|\gamma\|_{Z_1} = \inf\{\|\mu\|_{C_1} : \mu \in C_2(X, \mathbb{Q}), \partial(\mu) = \gamma\}$. Hence $\|\cdot\|_{Z_1} \leq C\|\cdot\|_{C_1}$ on $Z_1(X, \mathbb{Q})$. On the other hand, since the inclusion $Z_1(X, \mathbb{Q}) \hookrightarrow C_1(X, \mathbb{Q})$ bounded, there is another constant $C'$ such that $\|\cdot\|_{C_1} \leq C'\|\cdot\|_{Z_1}$ on $Z_1(X, \mathbb{Q})$. Therefore the norms $\|\cdot\|_{Z_1}$ and $\|\cdot\|_{C_1}$ are equivalent on $Z_1(X, \mathbb{Q})$.

Below we recall a characterisation of hyperbolic simply connected 2-complexes from [17].

**Proposition 6.2.** ([17] Proposition 2.23, Lemma 2.29, Theorem 2.30) Let $X$ be a simply connected 2-complex.

1. If $X^{(1)}$ is hyperbolic, then $X$ satisfies the linear homological isoperimetric inequality.
(2) If there is a constant $M$ such that the attaching map for each 2-cell in $X$ has length at most $M$, and $X$ satisfies a linear homological isoperimetric inequality; then $X^{(1)}$ is hyperbolic.

Proof of Theorem 1.3. Let $G$ be a compactly generated TDLC-group. Suppose that $G$ is hyperbolic. By Proposition 6.1 $G$ is compactly presented and there is a topological model $X$ of $G$ of type $F_2$. By Proposition 3.3, the 1-dimensional complex $X^{(1)}$ is quasi-isometric to a Cayley-Abels graph of $G$. It follows that $X^{(1)}$ is hyperbolic. Hence, Propositions 6.1 and 6.2 imply that $G$ satisfies the weak linear isoperimetric inequality.

Conversely, suppose that $G$ is compactly presented and satisfies the weak linear isoperimetric inequality. Proposition 3.6 implies that there is a topological model $X$ of $G$ of type $F_2$. By Proposition 6.1, $X$ satisfies the linear homological isoperimetric inequality. Since the $G$-action on the 2-skeleton $X^{(2)}$ has finitely many $G$-orbits of 2-cells, there is a constant $M$ such that the attaching map for each 2-cell in $X$ has length at most $M$. Then Proposition 6.2 implies that $X^{(1)}$ is hyperbolic. By Proposition 3.3, the Cayley-Abels graphs of $G$ are hyperbolic. □

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