In this letter we study the effects of a noncommutative in the phase space of an empty (4+1) Kaluza-Klein universe with cosmological constant. We analyze the effects of the noncommutative deformations on the cosmological constant. Finally we comment on the possibility that the origin of the cosmological constant in this model, is related to the noncommutativity between the 4 dimensional scale factor and the compact extra dimension.

PACS numbers: 04.20.Fy, 04.50.h, 98.80.Qc

I. INTRODUCTION

The cosmological constant problem has been studied for a long time in different scenarios and has remain as one of the major open problems in modern physics [1]. It is associated with the cosmic acceleration and in a more general context with the dark energy problem [2]. There is a belief that the solution will come from an unconventional approach (i.e. arguments are given that UV/IR mixing mechanism is needed [1]).

An unconventional approach is the old idea of noncommutative spacetime [3], a natural feature of noncommutative quantum field theory is UV/IR mixing [4]. The central idea behind noncommutativity is to expresses uncertainty in the simultaneous measurement of any pair of conjugate variables, such as position and momentum. This idea was revived at the beginning of this century [5, 6]. This renewed interest in noncommutativity slowly but steadily permeated in the realm of gravity, from which several approaches to noncommutative gravity [7, 8] were proposed. The end result of these formulations of noncommutative gravity is a highly nonlinear theory and finding solutions to the corresponding noncommutative field equations is complicated.

It is argued that noncommutativity can affect the evolution of the universe making the study of noncommutative cosmological models an interesting testing ground for noncommutativity [9]. An approach to find noncommutative cosmological models, is to derive them from the full noncommutative theory of gravity, this might seem as a fruitless endeavor, but in order to avoid difficulties in the study of the possible influence of noncommutativity in relation with the cosmological constant, we will use the ideas in [10]. The main ingredients of this proposal are: minisuperspace approach to cosmology whose variables are the 3-metric components in a finite configuration space. This formalism has the advantage that the inclusion of matter is straightforward. By considering these models one freezes out degrees of freedom and the canonical quantization of these minisuperspace models gives the Wheeler-DeWitt equation (WDW). A general analysis suggest that conditions can be found to justify the minisuperspace approach and presume the behavior of the wave function as fundamental [11]. If one considers string theory, general relativithen the WDW equation corresponds to an $S$-wave approximation [12]. Secondly, noncommutative space-time has the consequence that the fields do not commute. In a more specific manner this is due to the Moyal product [5, 6]. To introduce these elements we take into account the procedure to generalize usual quantum mechanics to the noncommutative version [13]. Having the WDW equation to describe the quantum evolution of the universe and being the “coordinates” of these models the fields, it was assumed that the variables do not commute. Then, an effective noncommutativity was defined in the minisuperspace from which the quantum evolution of the cosmological model was studied [10]. In particular in the last few years there have been several attempts to study the possible effects of noncommutativity in the cosmological scenario. In [14] it is argued that there is a possible relation between the 4D cosmological constant and the noncommutative parameter of the compactified space in string theory, also in [15] noncommutativity is introduced in a 5 dimensional Kaluza-Klein universe in order to study the hierarchy problem. In [16], evidence is presented of the relationship between late time acceleration in dilaton cosmology and the noncommutative parameters. Furthermore, a more direct relation in connection with the cosmological constant problem has been addressed in [17], where it is shown that by means of minisuperspace noncommutativity a small cosmological constant arises, and seems to alleviate the discrepancy between the calculated and observed vacuum energy density.

The aim of this paper is to analyze the effects of the
noncommutative deformation on the cosmological constant $\Lambda$. We start with an empty (4+1) dimensional Kaluza-Klein universe with cosmological constant and an FRW metric. We will implement the noncommutative effects, by introducing a deformation in the phase space constructed from the minisuperspace variables and their conjugate momenta. From the resulting noncommutative model we find a relationship between the cosmological constant $\Lambda$ and the noncommutative parameters $\theta$ and $\beta$.

II. DEFORMED PHASE SPACE COSMOLOGY.

As already mentioned in the introduction, we will work with an empty (4+1) theory of gravity with cosmological constant $\Lambda$

$$I = \int \sqrt{-g} (R - \Lambda) dt d^3x d\rho,$$  

(1)

where $\{t, x^i\}$ are the coordinates of the 4-dimensional space time and $\rho$ represents the coordinate of the fifth dimension. We are interested in Kaluza-Klein cosmology, so an FRW type metric is assumed, which is of the form

$$ds^2 = -dt^2 + \frac{a^2(t)dr^2+\phi^2(t)d\rho^2}{(1+\frac{\kappa^2 r^2}{4})} + \frac{d^2x}{4},$$  

(2)

where $\kappa = 0, \pm 1$ and $a(t), \phi(t)$ are the scale factors of the universe and the compact dimension. Substituting this metric into the action (1) and integrating over the spatial dimensions, we obtain an 4-dimensional effective lagrangian

$$L = \frac{1}{2} \left[a\phi \dot{a}^2 + a^2 \ddot{a} - \kappa a \phi + \frac{1}{3} \Lambda a^3 \phi\right].$$  

(3)

For the purposes of simplicity and calculations, we can rewrite this lagrangian in a more convenient way

$$L = \frac{1}{2} \left[(\dot{x}^2 - \omega^2 x^2) - (\dot{y}^2 - \omega^2 y^2)\right],$$  

(4)

where the new variables where defined as

$$x = \frac{1}{\sqrt{8}} \left(a^2 + a\phi - \frac{3\kappa}{\Lambda}\right), \quad y = \frac{1}{\sqrt{8}} \left(a^2 - a\phi - \frac{3\kappa}{\Lambda}\right),$$  

(5)

and $\omega = -\frac{2\Lambda}{3}$. The equations of motion are those derived from varying the action action (4). The Hamiltonian constraint for the model is calculated from the action and is given by

$$H = \frac{1}{2} \left[(\dot{x}^2 + \omega^2 x^2) - (\dot{y}^2 + \omega^2 y^2)\right] = 0,$$  

(6)

which describes an isotropic oscillator-ghost-oscillator system. A full analysis of the quantum behavior of this model is presented in [13].

As is well known, there are different approaches to introduce noncommutativity to physical theories. In particular, to study noncommutative cosmology, there exist a well explored path to introduce noncommutativity into a cosmological setting [10]. In this set up the noncommutativity is realized in the minisuperspace variables.

A different approach to study quantum effects is based on the deformation of the phase space. The deformation of the phase space structure is achieved through the the Moyal brackets, which are based on the Moyal product. However, a more appropriate way to introduce the deformation is by means of the Poisson brackets rather than the Moyal ones.

The most conventional way to understand the noncommutativity between the phase space variables (minisuperspace variables) is by replacing the usual product of two arbitrary functions with the Moyal product (or star product) as

$$(f \ast g)(x) = \exp \left[\frac{1}{2} \alpha^{ab}(1) \partial_a \partial_b\right] f(x_1)g(x_2)|_{x_1 = x_2 = x},$$  

(7)

such that

$$\alpha_{ab} = \begin{pmatrix} \theta_{ij} & \delta_{ij} + \sigma_{ij} \\ -\delta_{ij} - \sigma_{ij} & \beta_{ij} \end{pmatrix},$$  

(8)

where the $\theta$ and $\beta$ are 2 x 2 antisymmetric matrices and represent the noncommutativity in the coordinates and momenta respectively, and $\sigma$ can be written as a combination of $\theta$ and $\beta$. With this product law a straightforward calculations gives

$$\{x_i, x_j\} = \theta_{ij}, \quad \{x_i, p_j\} = \delta_{ij} + \sigma_{ij}, \quad \{p_i, p_j\} = \beta_{ij}.$$  

(9)

The noncommutative deformation has been applied to the minisuperspace variables as well as to the corresponding canonical momenta, this type of noncommutativity can be motivated by string theory correction to gravity [10, 13]. In the rest of this paper we will use for the noncommutative parameters $\theta_{ij} = -\theta_{ij}$ and $\beta_{ij} = \beta_{ij}$.

If we consider the following change of variables in the classical phase space $\{u, v, p_u, p_v\}$

$$\dot{v} = v - \frac{\theta}{2} p_u, \quad \dot{u} = u - \frac{\theta}{2} p_v, \quad \dot{p}_v = p_v + \frac{\beta}{2} u, \quad \dot{p}_u = p_u - \frac{\beta}{2} v,$$  

(10)

it can be verify that if $\{u, v, p_u, p_v\}$ obey the usual Poisson algebra then

$$\{v, \dot{u}\} = \theta, \quad \{u, \dot{p}_v\} = \{\dot{v}, \dot{u}\} = 1 + \sigma, \quad \{p_v, p_u\} = \beta.$$  

(11)

Now that we have defined the deformed phase space, we can see the effects on the proposed cosmological model. From the action (4), we can obtain the hamiltonian constraint (9), inserting relations (11), we can construct the WDW equation.
by a closer inspection of the equation, it is convenient to make the following definitions
\[
\omega^2 = \frac{4(\beta - \theta \omega^2)^2 + 4(\omega^2 - \beta^2/4)}{(4 - \omega^2 \theta^2)^2}
\]
(13)

\[A_\hat{u} = -\frac{2(\beta - \theta \omega^2)}{4 - \omega^2 \theta^2} \hat{u}, \quad A_\hat{v} = \frac{2(\beta - \theta \omega^2)}{4 - \omega^2 \theta^2} \hat{v},\]

with these definitions, we can rewrite (12) in a much simpler and suggestive form
\[
H = \left\{ \left(\hat{p}_u - A_\hat{u} \right)^2 + \omega^2 \hat{u}^2 \right\} - \left\{ \left(\hat{p}_v - A_\hat{v} \right)^2 + \omega^2 \hat{v}^2 \right\},
\]
(14)

which is a two dimensional anisotropic ghost-oscillator. From (14) we can see that the terms \((p_u - A_u)\) can be associated to a minimal coupling term as is done in electromagnetic theory. If we perform the Poisson brackets \(\{p_u - A_u, p_v - A_v\}\) the result is that a component of a \(B\)-field is obtained. From this vector potential we find that \(B = \frac{4(\beta - \theta \omega^2)}{4 - \omega^2 \theta^2} \) and the vector potential \(A\) can be rewritten as \(A_\hat{u} = -\frac{\beta}{2} \hat{v}\) and \(A_\hat{v} = \frac{\beta}{2} \hat{u}\). On the other hand, we already know from (11) that \(\{p_u, \hat{p}_u\} = \beta\) and if we set \(\theta = 0\) in the above equation for \(B\) we can conclude that the deformation of the momentum plays a role analogous to a magnetic field.

III. DISCUSSION

We found that \(\omega\) is defined in terms of the cosmological constant, then modifications to the oscillator frequency will imply modifications to the effective cosmological constant. In our case we have done a deformation of the phase space of the theory, by introducing a modification to the momenta and to the minisuperspace coordinates, this gives two new fundamental constants \(\theta\) and \(\beta\). As expected we obtain a different functional dependence for frequency \(\omega\) and the magnetic \(B\) field as functions of \(\beta\) and \(\theta\). With this in mind, we can construct a new frequency \(\tilde{\omega}\) in terms of \(\omega^2\) and the cyclotron term \(B^2/4\),
\[
\tilde{\omega}^2 = \omega^2 - B^2/4 = \frac{4(\omega^2 - \beta^2/4)}{4 - \omega^2 \theta^2}.
\]
(15)

This \(\tilde{\omega}\) was obtained by a definition of the effective cosmological constant \(\tilde{\Lambda}_{eff}\) as is was done in Section II, to finally get a redefinition of the effective cosmological constant due to noncommutative parameters
\[
\tilde{\Lambda}_{eff} = \frac{4(\Lambda_{eff} + \frac{3}{8} \beta^2)}{4 - \frac{3}{8} \theta^2 | \Lambda_{eff} |}.
\]
(16)

Let us start with the case \(\beta = 0\), this should be equivalent to the noncommutative minisuperspace model, we get an effective cosmological constant given by
\[
\tilde{\Lambda}_{eff} = \frac{4 \Lambda_{eff}}{(4 - \frac{3}{8} \theta^2 | \Lambda_{eff} |)},
\]
(17)

we can see from (17), that the noncommutative parameter \(\theta\) can not take the place of the cosmological constant, but depending on the value of \(\theta\) the effective cosmological constant \(\tilde{\Lambda}_{eff}\) is modified. This result was obtained in [15] and used to present a solution to the Hierarchy problem. The author assumed \(M_{EW}\) to be the natural cutoff in the commutative model, then \(\Lambda_{eff} \sim M_{EW}^4\) and Planck’s scale as the cutoff in the noncommutative model \(\Lambda_{eff} \sim M_P^4\). With this assumptions and Eq. (17), the two scales are related by the parameter \(\theta\), this gives an explanation to the Hierarchy problem in the context of the Wheeler-DeWitt equation by assuming that the only fundamental scale is the \(M_{EW}\) scale, and the Planck scale is a consequence of deforming the minisuperspace coordinates of the theory.

Now we turn our attention to the case where there is no deformation on the coordinates. Taking the noncommutative parameter \(\theta = 0\) we have that the frequency and the effective cosmological constant are given by
\[
\tilde{\omega}^2 = \omega^2 - \beta^2/4, \quad \text{and} \quad \tilde{\Lambda}_{eff} = \Lambda_{eff} + \frac{3}{8} \beta^2.
\]
(18)

From the last equation we get the most interesting result of this paper. We can see that noncommutative parameter \(\beta\) and \(\Lambda_{eff}\) compete to give the effective cosmological constant \(\tilde{\Lambda}_{eff}\). If we consider the case of a flat universe with a vanishing \(\Lambda_{eff}\) we see that \(\tilde{\Lambda}_{eff} = \frac{3}{8} \beta^2\). This gives the origin of the cosmological constant in connection with the deformation parameter. As already argued, the presence of the deformation parameters \(\theta\) and \(\beta\) gives an alternative approach to the origin of the cosmological constant, in this approach the cosmological constant is given by the parameter \(\beta\) and is considered a new fundamental constant. In this case, one trace can the origin of \(\Lambda\) to the noncommutative deformation between the canonical momenta associated to the 4 dimensional scale factor and the canonical momenta associated to the compact extra dimension. Recently, some evidence on the possibility that the effects of the phase space deformation could be related to the late time acceleration of the universe as well as to the cosmological constant were presented [15].

Interestingly, in the particular case when \(\beta = \omega^2 \theta\) we find that frequency reduces to \(\tilde{\omega}^2 = \omega^2\) and the magnetic
potential energy vanishes as the effective magnetic field $B = 0$. Then we have that $\Lambda_{eff} = \Lambda_{eff}$ and in this case even as we have done a deformation on the minisuperspace of the theory, the effects cancel out and the resulting theory behaves as in the commutative theory.

ACKNOWLEDGMENTS

M.S. supported by DAIP grant 18/10 and CONACYT grants 62253, 135023. S. P. P. is supported by CONACyT graduate grant. E.M. is partially supported by PROMEP grants 103.5/10/6209.

[1] J. Polchinski, “The cosmological constant and the string theory landscape”, arXiv hep-th:0603249.
[2] J. -Q. Xia, M. Viel, JCAP 0904 (2009) 002. [arXiv:0901.0605 [astro-ph.CO]].
[3] H. Snyder, Phys. Rev. 71, 38 (1947).
[4] M. R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. 73, 977 (2001).
[5] N. Seiberg and E. Witten, JHEP 9909:032 (1999).
[6] J. Wess, Commun. Math. Phys. 219, 247-257 (2001).
[7] H. García-Compeán, O. Obregón, C. Ramírez and M. Sabido, Phys. Rev. D 68 (2003) 044015; H. García-Compeán, O. Obregón, C. Ramírez and M. Sabido, Phys. Rev. D 68 (2003) 045010.
[8] V. P. Nair, Nucl. Phys. B 651, 313 (2003); M.A. Cardella and Daniela Zanon, Class. Quant. Grav. 20, L95 (2003); A.H. Chamseddine, J. Math. Phys. 44, 2534 (2003); J.W. Moffat, Phys. Lett. B 491 (2000) 345; Phys. Lett. B 493 (2000) 142; E. Harikumar and V. O. Rivelles, Class. Quant. Grav. 23, 7551 (2006); P. Aschieri, C. Blohmann, M. Dimitrijevic, F. Meyer, P. Schupp and J. Wess, Class. Quant. Grav. 22, 3511 (2005); P. Aschieri, M. Dimitrijevic, F. Meyer and J. Wess, Class. Quant. Grav. 23, 1883 (2006); L. Alvarez-Gaume, F. Meyer and M. A. Vazquez-Mozo, Nucl. Phys. B 753, 92 (2006).
[9] W. Guzman, M. Sabido, J. Socorro, Phys. Rev. D76, 087302 (2007); W. Guzman, M. Sabido, J. Socorro, Phys. Lett. B697, 271-274 (2011); O. Obregón, I. Quiros, Phys. Rev. D84, 044005 (2011).
[10] H. García-Compeán, O. Obregón and C. Ramírez, Phys. Rev. Lett. 88, 161301 (2002).
[11] J. J. Halliwell, in Proceedings of the 13th International Conference on General Relativity, edited by R. J. Gleisser, C. N. Kozameh, and O. M. Moreschi (IOP Publishing, Bristol, 1993).
[12] I. Susskind and J. Uglum, arXiv:hep-th/9410074.
[13] L. Mezincescu, “Star Operation in Quantum Mechanics,” arXiv:hep-th/0007046.
[14] R. Kallosh, arXiv:hep-th/0405246.
[15] A. Rezaei-Aghdam, F. Darabi, A. R. Rastkar, Phys. Lett. B615, 141-145 (2005). [gr-qc/0412089].
[16] B. Vakili, P. Pedram, S. Jalalzadeh, Phys. Lett. B687, 119-123 (2010).
[17] O. Obregon, M. Sabido, E. Mena, Mod. Phys. Lett. A24, 1907-1914 (2009).
[18] B. Vakili, P. Pedram, S. Jalalzadeh, Phys. Lett. B687, 119-123 (2010); S. Pérez-Payán, M. Sabido and C. Yee, “Effects of deformed phase space on scalar field cosmology”, submitted to Phys. Lett. B 2011.