The foundations of statistical mechanics from entanglement: Individual states vs. averages

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We consider an alternative approach to the foundations of statistical mechanics, in which subjective randomness, ensemble-averaging or time-averaging are not required. Instead, the universe (i.e. the system together with a sufficiently large environment) is in a quantum pure state subject to a global constraint, and thermalisation results from entanglement between system and environment. We formulate and prove a “General Canonical Principle”, which states that the system will be thermalised for almost all pure states of the universe, and provide rigorous quantitative bounds using Levy’s Lemma.

I. INTRODUCTION

Despite many years of research, the foundations of statistical mechanics remain a controversial subject. Crucial questions regarding the role of probabilities and entropy (which are viewed both as measures of ignorance and objective properties of the state) are not satisfactorily resolved, and the relevance of time averages and ensemble averages to individual physical systems is unclear.

Here we adopt a fundamentally new viewpoint suggested by Yakir Aharonov [1], which is uniquely quantum, and which does not rely on any ignorance probabilities in the description of the state. We consider the global state of a large isolated system, the ‘universe’, to be a quantum pure state. Hence there is no lack of knowledge about the state of the universe, and the entropy of the universe is zero. However, when we consider only part of the universe (that we call the ‘system’), it is possible that its state will not be pure, due to quantum entanglement with the rest of the universe (that we call the ‘environment’). Hence there is an objective ‘lack of knowledge’ about the state of the system, even though we know everything about the state of the universe. In such cases, the entropy of the system is non-zero, even though we have introduced no randomness and the universe itself has zero entropy.

Furthermore, interactions between the system and environment can objectively increase both the entropy of the system and that of the environment by increasing their entanglement. It is conceivable that this is the mechanism behind the second law of thermodynamics. Indeed, as information about the system will tend to leak into (and spread out in) the environment, we might well expect that their entanglement (and hence entropy) will increase over time in accordance with the second law.

The above ideas provide a compelling vision of the foundations of statistical mechanics. Such a viewpoint has been independently proposed recently by Gemmer et al. [2].

In this paper, we address one particular aspect of the above programme. We show that thermalisation is a generic property of pure states of the universe, in the sense that for almost all of them, the reduced state of the system is the canonical mixed state. That is, not only is the state of the system mixed (due to entanglement with the rest of the universe), but it is in precisely the state we would expect from standard statistical arguments.

In fact, we prove a stronger result. In the standard statistical setting, energy constraints are imposed on the state of the universe, which then determine a corresponding temperature and canonical state for the system. Here we consider that states of the universe are subject to arbitrary constraints. We then show that almost every pure state of the universe subject to those constraints is such that the system is in the corresponding generalised canonical state.

Our results are kinematic, rather than dynamical. That is, we do not consider any particular unitary evolution of the global state, and we do not show that thermalisation of the system occurs. However, because almost all states of the universe are such that the system is in a canonical thermal state, we anticipate that most evolutions will quickly carry a state in which the system is not thermalised to one in which it is, and that the system will remain thermalised for most of its evolution.

A key ingredient in our analysis is Levy’s Lemma [3, 4], which plays a similar role to the law of large numbers and governs the properties of typical states in large-dimensional Hilbert spaces. Levy’s Lemma has already been used in quantum information theory to study entanglement and other correlation properties of random states in large bipartite systems [5]. It provides a very powerful tool with which to evaluate functions of randomly chosen quantum states.

The structure of this paper is as follows. In section II we present our main result in the form of a General Canonical Principle. In section III we support this principle with precise mathematical theorems. In section IV we introduce Levy’s Lemma, which is used in sections V and VI to provide proofs of our main theorems. Section VII illustrates these results with the simple example of spins in a magnetic field. Finally, in section VIII we present our conclusions.
II. GENERAL CANONICAL PRINCIPLE

Consider a large quantum mechanical system, ‘the universe’, that we decompose into two parts, the ‘system’ \( S \) and the ‘environment’ \( E \). We will assume that the dimension of the environment is much larger than that of the system. Consider now that the state of the universe obeys some global constraint \( R \). We can represent this quantum mechanically by restricting the allowed states of the system and environment to a subspace \( \mathcal{H}_R \) of the total Hilbert space:

\[
\mathcal{H}_R \subseteq \mathcal{H}_S \otimes \mathcal{H}_E, \tag{1}
\]

where \( \mathcal{H}_S \) and \( \mathcal{H}_E \) are the Hilbert spaces of the system and environment, with dimensions \( d_S \) and \( d_E \) respectively. In standard statistical mechanics \( R \) would typically be a restriction on the total energy of the universe, but here we leave \( R \) completely general.

We define \( \mathcal{E}_R \), the **equiprobable state of the universe corresponding to the restriction** \( R \), by

\[
\mathcal{E}_R = \frac{1}{d_R}, \tag{2}
\]

where \( 1_R \) is the identity (projection) operator on \( \mathcal{H}_R \), and \( d_R \) is the dimension of \( \mathcal{H}_R \). \( \mathcal{E}_R \) is the maximally mixed state in \( \mathcal{H}_R \), in which each pure state has equal probability. This corresponds to the standard intuition of assigning equal a priori probabilities to all states of the universe consistent with the constraints.

We define \( \Omega_S \), the **canonical state of the system corresponding to the restriction** \( R \), as the quantum state of the system when the universe is in the equiprobable state \( \mathcal{E}_R \). The canonical state of the system \( \Omega_S \) is therefore obtained by tracing out the environment in the equiprobable state of the universe:

\[
\Omega_S = \text{Tr}_E(\mathcal{E}_R). \tag{3}
\]

We now come to the main idea behind our paper.

As described in the introduction, we now consider that the universe is in a pure state \( \phi \), and not in the mixed state \( \mathcal{E}_R \) (which represents a subjective lack of knowledge about its state). We prove that despite this, the state of the system is very close to the canonical state \( \Omega_S \) in almost all cases. That is, for almost every pure state of the universe, the system behaves as if the universe were actually in the equiprobable mixed state \( \mathcal{E}_R \).

We now state this basic qualitative result as a general principle, that will subsequently be refined by quantitative theorems:

**General Canonical Principle:** Given a sufficiently small subsystem of the universe, almost every pure state of the universe is such that the subsystem is approximately in the canonical state \( \Omega_S \).

Recalling that the canonical state of the system \( \Omega_S \) is, by definition, the state of the system when the universe is in the equiprobable state \( \mathcal{E}_R \) we can interpret the above principle as follows:

**Principle of Apparently Equal a priori Probability:** For almost every pure state of the universe, the state of a sufficiently small subsystem is approximately the same as if the universe were in the equiprobable state \( \mathcal{E}_R \). In other words, almost every pure state of the universe is locally (i.e. on the system) indistinguishable from \( \mathcal{E}_R \).

For an arbitrary pure state \( |\phi\rangle \) of the universe, the state of the system alone is given by

\[
\rho_S = \text{Tr}_E(|\phi\rangle\langle\phi|). \tag{4}
\]

Our principle states that for almost all states \( |\phi\rangle \in \mathcal{H}_R \),

\[
\rho_S \approx \Omega_S. \tag{5}
\]

Obviously, the above principle is stated qualitatively. To express these results quantitatively, we need to carefully define what we mean by a sufficiently small subsystem, under what distance measure \( \rho_S \approx \Omega_S \), and how good this approximation is. This will be done in the remaining sections of the paper.

We emphasise that the above is a generalised principle, in the sense that the restriction \( R \) imposed on the states of the universe is completely arbitrary (and is not necessarily the usual constraint on energy or other conserved quantities). Similarly, the canonical state \( \Omega_S \) is not necessarily the usual thermal canonical state, but is defined relative to the arbitrary restriction \( R \) by equation (3).

To connect the above principle to standard statistical mechanics, all we have to do is to consider the restriction \( R \) to be that the total energy of the universe is close to \( E \), which then sets the temperature scale \( T \). The total Hamiltonian of the universe \( H_U \) is given by

\[
H_U = H_S + H_E + H_{\text{int}}, \tag{6}
\]

where \( H_S \) and \( H_E \) are the Hamiltonians of the system and environment respectively, and \( H_{\text{int}} \) is the interaction Hamiltonian between the system and environment. In the standard situation, in which \( H_{\text{int}} \) is small and the energy spectrum of the environment is sufficiently dense and uniform, the canonical state \( \Omega_S^{(E)} \) can be computed using standard techniques, and shown to be

\[
\Omega_S^{(E)} \propto \exp \left( -\frac{H_S}{k_B T} \right). \tag{7}
\]

This allows us to state the thermal canonical principle that establishes the validity (at least kinematically) of the viewpoint expressed in the introduction.

**Thermal Canonical Principle:** Given that the total energy of the universe is approximately \( E \), interactions between the system and the rest of the universe are weak, and that the energy spectrum of the universe is sufficiently dense and uniform, almost every pure state of the
universe is such that the state of the system alone is approximately equal to the thermal canonical state $\rho = e^{-\frac{H}{kT}}$, with temperature $T$ (corresponding to the energy $E$).

We emphasise here that our contribution in this paper is to show that $\rho_S \approx \Omega_S$, and has nothing to do with showing that $\Omega_S \propto e^{-\frac{H_S}{kT}}$, which is a standard problem in statistical mechanics.

Finally, we note that the General Canonical Principle applies also in the case where the interaction between the system and environment is not small. In such situations, the canonical state of the system is no longer very strongly on $\Omega_S$. Nevertheless, the general principle remains valid for the corresponding generalised canonical state $\Omega_S$. Furthermore our principle will apply to arbitrary restrictions $R$ that have nothing to do with energy, which may lead to many interesting insights.

III. QUANTITATIVE SETUP AND MAIN THEOREMS

We now formulate and prove precise mathematical theorems corresponding to the General Canonical Principle stated in the previous section.

As a measure of the distance between $\rho_S$ and $\Omega_S$, we use the trace-norm $\|\rho_S - \Omega_S\|_1$, where [6]

$$\|M\|_1 = \text{Tr}|M| = \text{Tr} \sqrt{M^\dagger M},$$

as this distance will be small if and only if it would be hard for any measurement to tell $\rho_S$ and $\Omega_S$ apart. Indeed, $\|M\|_1 = \sup_{\|O\| \leq 1} \text{Tr}(MO)$, where the maximisation is over all operators (observables) $O$ with operator norm bounded by 1.

In our analysis, we also make use of the Hilbert-Schmidt norm $\|M\|_2 = \sqrt{\text{Tr}(M^\dagger M)}$, which is easier to manipulate than $\|M\|_1$. However, we only use this for intermediate calculational purposes, as it does not have the desirable physical properties of the trace-norm. In particular $\|\rho_S - \Omega_S\|_2$ can be small even when the two states are orthogonal for high-dimensional systems.

Throughout this paper we denote by $\langle \cdot \rangle$ the average over states $|\phi\rangle \in \mathcal{H}_R$ according to the uniform distribution. For example, it is easy to see that $\Omega_S = \langle \rho_S \rangle$.

We will prove the following theorems:

**Theorem 1** For a randomly chosen state $|\phi\rangle \in \mathcal{H}_R \subseteq \mathcal{H}_S \otimes \mathcal{H}_E$ and arbitrary $\epsilon > 0$, the distance between the reduced density matrix of the system $\rho_S = \text{Tr}_E(|\phi\rangle\langle \phi|)$ and the canonical state $\Omega_S = \text{Tr}_E \mathcal{E}_R$ is given probabilistically by

$$\text{Prob}[\|\rho_S - \Omega_S\|_1 \geq \eta] \leq \eta',$$

where

$$\eta = \epsilon + \sqrt{\frac{d_S}{d_E}} \eta'.$$

In these expressions, $C$ is a positive constant (given by $C = (18\pi^4)^{-1}$), $d_S$ and $d_E$ are the dimensions of $\mathcal{H}_S$ and $\mathcal{H}_E$ respectively, and $d_E^{\text{eff}}$ is a measure of the effective size of the environment, given by

$$d_E^{\text{eff}} = \frac{1}{\text{Tr} \Omega_E^2} \geq d_S,$$

where $\Omega_E = \text{Tr}_S \mathcal{E}_R$. Both $\eta$ and $\eta'$ will be small quantities, and thus the state will be close to the canonical state with high probability, whenever $d_E^{\text{eff}} \geq d_S$ (i.e. the effective dimension of the environment is much larger than that of the system) and $d_E^{\text{eff}} \eta > 1 \gg \epsilon$. This latter condition can be ensured when $d_E \gg 1$ (i.e. the total accessible space is large), by choosing $\epsilon = d_E^{-1/3}$.

This theorem gives rigorous meaning to our statements in section II about thermalisation being achieved for ‘almost all’ states: we have an exponentially small bound on the relative volume of the exceptional set, i.e. on the probability of finding the system in a state that is far from the canonical state. Interestingly, the exponent scales with the dimension of the space $\mathcal{H}_R$ of the constraint, while the deviation from the canonical state is characterised by the ratio between the system size and the effective size of the environment, which makes intuitive sense.

Theorem 1 provides a bound on the distance between $\rho_S$ and $\Omega_S$, but in many situations we can further improve it. Often the system does not really occupy all of its Hilbert space $\mathcal{H}_S$, and also the estimate of the effective environment dimension $d_E^{\text{eff}}$ may be too small, due to exceptionally large eigenvalues of $\Omega_E = \text{Tr}_S \mathcal{E}_R$. By cutting out these non-typical components (similar to the well-known method of projecting onto the typical subspace), we can optimize the bound obtained, as we will show in Theorem 2. The benefits of this optimization will be apparent in section VII, where we consider a particular example.

**Theorem 2** Assume that there exists some bounded positive operator $X_R$ on $\mathcal{H}_R$ satisfying $0 \leq X_R \leq I$ such that, with $\mathcal{E}_R = \sqrt{X_R \mathcal{E}_R X_R}$,

$$\text{Tr}(\mathcal{E}_R) = \text{Tr}(\mathcal{E}_R X_R) \geq 1 - \delta.$$

(I.e. the probability of obtaining the outcome corresponding to measurement operator $X_R$ in a generalised measurement (POVM) on $\mathcal{E}_R$ is approximately one.)

Then, for a randomly chosen state $|\phi\rangle \in \mathcal{H}_R \subseteq \mathcal{H}_S \otimes \mathcal{H}_E$ and $\epsilon > 0$,

$$\text{Prob}[\|\rho_S - \Omega_S\|_1 \geq \tilde{\eta}] \leq \tilde{\eta}'$$
where
\[
\tilde{\eta} = \epsilon + \sqrt{\frac{d_S}{d_E^\text{eff}}} + 4\sqrt{\delta},
\]  
(15)
\[
\tilde{\eta}' = 2\exp\left(-Cd_R^2\right).
\]  
(16)

Here, \(C\) and \(d_R\) are as in Theorem 1. \(\tilde{d}_S\) is the dimension of the support of \(X_R\) in \(\mathcal{H}_R\), and \(d_E^\text{eff}\) is the effective size of the environment after applying \(X_R\), given by
\[
d_E^\text{eff} = \frac{1}{\text{Tr}\tilde{\Omega}_E^\text{eff}} \geq \frac{d_R}{d_S},
\]  
(17)
where \(\tilde{\Omega}_E = \text{Tr}_S(\tilde{E}_R)\). In many situations \(\delta\) can be made very small, while at the same time improving the relation between system and effective environment dimension. Note that the above is essentially the technique of the smooth (quantum) Rényi entropies \([7, 8]\):

\[
\text{Tr} \tilde{\Omega}_E \text{ is related to } S_0^\text{eff}(\Omega_S) \text{ and } \log d_E^\text{eff} \text{ to } S_2^\text{eff}(\Omega_E). \]

In the process of proving these theorems, we also obtain the following subsidiary results:

1. The average distance between the system’s reduced density matrix for a randomly chosen state and the canonical state will be small whenever the effective environment size is larger than the system. Specifically,
\[
\langle \|\rho_S - \Omega_S\|_1 \rangle \leq \sqrt{\frac{d_S}{d_E^\text{eff}}} \leq \sqrt{\frac{d_S^2}{d_R}},
\]  
(18)
where the effective dimension of the environment \(d_E^\text{eff}\) is given by (12).

2. With high probability, the expectation value of a bounded observable \(O_S\) on the system for a randomly chosen state will be very similar to its expectation value in the canonical state whenever \(d_R \gg 1\). Specifically,
\[
\text{Prob}\left[|\text{Tr}(O_S\rho_S) - \text{Tr}(O_S\Omega_S)| \geq d_R^{1/3}\right] \leq 2\exp\left(-\frac{Cd_R^{1/3}}{\|O_S\|^2}\right),
\]  
(19)
where \(C\) is a constant.

In our analysis we use two alternative methods, with the hope that the different mathematical techniques employed will aid in future exploration of the field.

IV. LEVY’S LEMMA

A major component in the proofs of the following sections is the mathematical theorem known as Levy’s Lemma \([3, 4]\), which states that when a point \(\phi\) is selected at random from a hypersphere of high dimension and \(f(\phi)\) does not vary too rapidly, then \(f(\phi) \approx \langle f \rangle\) with high probability:

\[
\text{Prob}\left[|f(\phi) - \langle f \rangle| \geq \epsilon\right] \leq 2\exp\left(-\frac{2C(d+1)\epsilon^2}{\eta^2}\right)
\]  
(20)

where \(\eta\) is the Lipschitz constant of \(f\), given by \(\eta = \sup |\nabla f|\), and \(C\) is a positive constant (which can be taken to be \(C = (18\pi^3)^{-1}\)).

Due to normalisation, pure states in \(\mathcal{H}_R\) can be represented by points on the surface of a \((2d_R-1)\)-dimensional hypersphere \(S^{2d_R-1}\), and hence we can apply Levy’s Lemma to functions of the randomly selected quantum state \(\phi\) by setting \(d = 2d_R - 1\). For such a randomly chosen state \(|\phi\rangle \in \mathcal{H}_R\), we wish to show that \(\|\rho_S - \Omega_S\|_1 \approx 0\) with high probability.

V. METHOD I: APPLYING LEVY’S LEMMA TO \(\|\rho_S - \Omega_S\|_1\)

In this section, we consider the consequences of applying Levy’s Lemma directly to the distance between \(\rho_S = \text{Tr}_E(|\phi\rangle\langle\phi|)\) and \(\Omega_S\), by choosing
\[
f(\phi) = \|\rho_S - \Omega_S\|_1.
\]  
(21)
in (20). As we prove in appendix A, the function \(f(\phi)\) has Lipschitz constant \(\eta \leq 2\). Applying Levy’s Lemma to \(f(\phi)\) then gives:
\[
\text{Prob}\left[|\|\rho_S - \Omega_S\|_1 - \langle \|\rho_S - \Omega_S\|_1 \rangle| \geq \epsilon\right] \leq 2\exp\left(-Cd_R^2\right).
\]  
(22)

To obtain Theorem 1, we rearrange this equation to get
\[
\text{Prob}\left[\|\rho_S - \Omega_S\|_1 \geq \eta\right] \leq \eta' \leq \eta
\]  
(23)
where
\[
\eta = \epsilon + \langle \|\rho_S - \Omega_S\|_1 \rangle
\]  
(24)
\[
\eta' = 2\exp\left(-Cd_R^2\right).
\]  
(25)

The focus of the following subsections is to obtain a bound on \(\langle \|\rho_S - \Omega_S\|_1 \rangle\). In section VC we show that
\[
\langle \|\rho_S - \Omega_S\|_1 \rangle \leq \sqrt{\frac{d_S}{d_E^\text{eff}}}.
\]  
(26)
where \(d_E^\text{eff}\) is a measure of the effective size of the environment, given by (12). Inserting equation (26) in (24) we obtain Theorem 1.

Typically \(d_R \gg 1\) (the total number of accessible states is large) and hence by choosing \(\epsilon = d_R^{-1/3}\) we can ensure that both \(\epsilon\) and \(\eta'\) are small quantities. When it is also true that \(d_E^\text{eff} \gg d_S\) (the environment is much larger than that of the system) both \(\eta\) and \(\eta'\) will be small quantities, leading to \(\|\rho_S - \Omega_S\|_1 \approx 0\) with high probability.
To obtain Theorem 2, we consider a generalised measurement which has an almost certain outcome for the equiprobable state \( \mathcal{E}_R \in \mathcal{H}_R \), and apply the corresponding measurement operator before proceeding with our analysis. By an appropriate choice of measurement operator, the ratio of the system and environment’s effective dimensions can be significantly improved (as shown by the example in section VII A).

A. Calculating \( \langle \| \rho_S - \Omega_S \|_1 \rangle \)

As mentioned in section III, although \( \| \rho_S - \Omega_S \|_1 \) is a physically meaningful quantity, it is difficult to work with directly, so we first relate it to the Hilbert-Schmidt norm \( \| \rho_S - \Omega_S \|_2 \). The two norms are related by

\[
\| \rho_S - \Omega_S \|_1 \leq \sqrt{d_S} \| \rho_S - \Omega_S \|_2,
\]

as proved in Appendix A.

Expanding \( \| \rho_S - \Omega_S \|_2 \) we obtain

\[
\langle \| \rho_S - \Omega_S \|_2 \rangle \leq \sqrt{\langle \| \rho_S - \Omega_S \|_2^2 \rangle} = \sqrt{\langle \text{Tr}(\rho_S - \Omega_S)^2 \rangle} = \sqrt{\langle \text{Tr} \rho_S^2 \rangle - 2 \text{Tr} \langle \rho_S \Omega_S \rangle + \text{Tr} \Omega_S^2}.
\]

and hence

\[
\langle \| \rho_S - \Omega_S \|_1 \rangle \leq \sqrt{d_S} \langle \text{Tr} \rho_S^2 \rangle - \text{Tr} \Omega_S^2\rangle.
\]

B. Calculating \( \langle \text{Tr}(\rho_S^2) \rangle \)

In this section we show the fundamental inequality

\[
\langle \text{Tr}(\rho_S^2) \rangle \leq \text{Tr} \langle \rho_S \rangle^2 + \text{Tr} \langle \rho_E \rangle^2.
\]

The following calculations and estimates are closely related to the arguments used in random quantum channel coding [9] and random entanglement distillation (see [10]).

To calculate \( \langle \text{Tr}(\rho_S^2) \rangle \), it is helpful to introduce a second copy of the original Hilbert space, extending the problem from \( \mathcal{H}_R \) to \( \mathcal{H}_R \otimes \mathcal{H}_{R'} \) where \( \mathcal{H}_{R'} \subseteq \mathcal{H}_{S'} \otimes \mathcal{H}_{E'} \).

Note that

\[
\text{Tr}_S \rho_S^2 = \sum_k \langle \rho_{kk} \rangle^2 = \sum_{k,l,k',l'} \langle \rho_{kk'} \rho_{l'l'} \rangle \langle kk'|ll' \rangle \langle ll'|kk' \rangle = \text{Tr}_{S'}(\rho_{SS'} \otimes \rho_{S'}),
\]

where \( \text{Tr}_{S'} \) is the flip (or swap) operation \( S \leftrightarrow S' \):

\[
F_{SS'} = \sum_{S,S'} |s'\rangle \langle s'|S \otimes |s\rangle \langle s'|S',
\]

and hence

\[
\text{Tr}_S \rho_S^2 = \text{Tr}_{RR'}(\langle \phi \rangle \langle \phi | \otimes \langle \phi | \otimes \phi \rangle)_{RR'}(F_{SS'} \otimes 1_{EE'}).
\]

So, our problem reduces to the calculation of

\[
V \equiv \langle \langle \phi \rangle \otimes \langle \phi | \otimes \phi \rangle \rangle = \int |\langle \phi | \otimes \langle \phi | \rangle \otimes \phi \rangle | d\phi.
\]

As \( V \) is invariant under operations of the form \( V \rightarrow (U \otimes U)V(U^\dagger \otimes U^\dagger) \) for any unitary \( U \), representation theory implies that

\[
V = \alpha \Pi_{RR'}^{\text{sym}} + \beta \Pi_{RR'}^{\text{anti}},
\]

where \( \Pi_{RR'}^{\text{sym}} \) are projectors onto the symmetric and antisymmetric subspaces of \( \mathcal{H}_R \otimes \mathcal{H}_{R'} \) respectively, and \( \alpha \) and \( \beta \) are constants.

\[
\langle \phi \rangle \otimes \langle \phi | \rangle \frac{1}{\sqrt{2}} (|ab\rangle - |ba\rangle) = 0 \quad \forall a, b, \phi,
\]

it is clear that \( \beta = 0 \), and as \( V \) is a normalised state,

\[
\alpha = \frac{1}{\text{dim}(RR'_\text{sym})} = \frac{2}{d_R(d_R + 1)}
\]

Hence

\[
\langle \phi \rangle \otimes \langle \phi | \rangle = \frac{2}{d_R(d_R + 1)} \Pi_{RR'}^{\text{sym}}
\]

and therefore

\[
\langle \text{Tr}_S \rho_S^2 \rangle = \text{Tr}_{RR'} \left( \left( \frac{2 \Pi_{RR'}^{\text{sym}}}{d_R(d_R + 1)} \right) (F_{SS'} \otimes 1_{EE'}) \right).
\]

To proceed further we perform the substitution

\[
\Pi_{RR'}^{\text{sym}} = \frac{1}{2} (1_{RR'} + F_{RR'}),
\]

where \( F_{RR'} \) is the flip operator taking \( R \leftrightarrow R' \). Noting that \( F_{RR'} = 1_{RR'}(F_{SS'} \otimes 1_{EE'}) \), this gives

\[
\langle \text{Tr}_S \rho_S^2 \rangle = \text{Tr}_{RR'} \left( \frac{1_{RR'}}{d_R(d_R + 1)} \right) (F_{SS'} \otimes 1_{EE'}) + \text{Tr}_{RR'} \left( \frac{1_{RR'}}{d_R(d_R + 1)} \right) (F_{SS'} \otimes 1_{EE'}) \leq \text{Tr}_{RR'} \left( \frac{1}{d_R} \otimes \frac{1}{d_R} \right) (F_{SS'} \otimes 1_{EE'}) \leq \text{Tr}_{SS'} \left( \langle \Omega_S \otimes \Omega_S \rangle F_{SS'} \right) + \text{Tr}_{EE'} \left( \langle \Omega_E \otimes \Omega_E \rangle F_{EE'} \right).
\]

Hence from equation (32),

\[
\langle \text{Tr}_S \rho_S^2 \rangle \leq \text{Tr} \langle \rho_S \rangle^2 + \text{Tr} \langle \rho_E \rangle^2.
\]


C. Bounding $\langle \| \rho_S - \Omega_S \|_1 \rangle$

Inserting the results of the last section in equation (30) we obtain

$$\langle \| \rho_S - \Omega_S \|_1 \rangle \leq \sqrt{d_S \text{Tr}_E \Omega_E^2}$$  \hspace{1cm} (44)

Intuitively, we can understand this equation by defining

$$d_E^{\text{eff}} = \frac{1}{\text{Tr}_E \Omega_E^2},$$  \hspace{1cm} (45)

as the effective dimension of the environment in the canonical state. If all of the non-zero eigenvalues of $\Omega_E$ were of equal weight this would simply correspond to the dimension of $\Omega_E$’s support, but more generally it will measure the dimension of the space in which the environment is most likely to be found. When there is no constraint on the accessible states of the environment, such that $\mathcal{H}_E = \mathcal{H}_S \otimes \mathcal{H}_E$ then $d_E^{\text{eff}} = d_E$.

Denoting the eigenvalues of $\Omega_E$ by $\lambda_E^k$ (with maximum eigenvalue $\lambda_E^{\text{max}}$), it is also interesting to note that

$$\text{Tr}_E \Omega_E^2 = \sum_k (\lambda_E^k)^2 \leq \lambda_E^{\text{max}} \sum_k \lambda_E^k = \max_{\psi_E} \langle \psi_E | \text{Tr}_S \left( \frac{\mathbb{I}_R}{d_R} \right) | \psi_E \rangle = \max_{\psi_E} \sum_s \langle s | \psi_E \rangle \langle \psi_E | s \rangle \leq \frac{d_S}{d_R}.$$  \hspace{1cm} (46)

Hence $d_E^{\text{eff}} \geq d_R/d_S$, and we obtain the final result that

$$\langle \| \rho_S - \Omega_S \|_1 \rangle \leq \sqrt{\frac{d_S}{d_E^{\text{eff}}}} \leq \sqrt{\frac{d_S^2}{d_E}}.$$  \hspace{1cm} (47)

The average distance $\langle \| \rho_S - \Omega_S \|_1 \rangle$ will therefore be small whenever the effective size of the environment is much larger than that of the system ($d_E^{\text{eff}} \gg d_S$).

Inserting the results of equation (47) into equation (24) gives Theorem 1.

D. Improved bounds using restricted subspaces

As mentioned in section III, in many cases it is possible to improve the bounds obtained from Theorem 1 by projecting the states onto a typical subspace before proceeding with the analysis. This can allow one to decrease the effective dimension of the system $d_S$ (by eliminating components with negligible amplitude), and increase the effective dimension of the environment $d_E^{\text{eff}} = (\text{Tr}_E \Omega_E^2)^{-1}$ (by eliminating components of $\Omega_E$ with disproportionately high amplitudes), whilst leaving the equiprobable state $\mathcal{E}_R$ largely unchanged.

To allow for the most general possibility, we consider a generalised measurement operator $X_R$ satisfying $0 \leq X_R \leq \mathbb{1}$ (of which a projector is a special case), which has high probability of being satisfied by $\mathcal{E}_R$, such that

$$\text{Tr}_R(\mathcal{E}_R X_R) \geq 1 - \delta.$$  \hspace{1cm} (48)

We denote the dimension of the support of $X_R$ in $\mathcal{H}_S$ by $d_S^e$, which will play the role of $d_S$ in the revised analysis [11]. The bounds on $\langle \| \rho_S - \Omega_S \|_1 \rangle$ will be optimized by choosing $X_R$ such that $d_S^e$ is as small as possible, and $d_E^{\text{eff}}$ as large as possible.

We also define the sub-normalised states obtained after measurement of $X_R$:

$$|\tilde{\phi}\rangle = \sqrt{X_R} |\phi\rangle,$$  \hspace{1cm} (49)

$$\hat{\mathcal{E}}_R = \sqrt{X_R} \mathcal{E}_R \sqrt{X_R} = \frac{X_R}{d_R},$$  \hspace{1cm} (50)

$$\hat{\Omega}_S = \text{Tr}_E(\hat{\mathcal{E}}_R),$$  \hspace{1cm} (51)

$$\hat{\Omega}_E = \text{Tr}_S(\hat{\mathcal{E}}_R).$$  \hspace{1cm} (52)

Applying the same analysis as in the previous sections to these states, we find

$$\langle \text{Tr}_S \hat{\rho}_S^2 \rangle = \text{Tr}_R \left( \langle \tilde{\phi} \rangle \langle \tilde{\phi} \rangle \otimes \langle \tilde{\phi} \rangle \langle \tilde{\phi} \rangle \right) (F_{SS'} \otimes \mathbb{1}_{EE'})$$

$$= \text{Tr}_R \left( \left( \frac{X_R \otimes X_R}{d_R} \right) \Pi_{SS'}^{\text{sym}} \right) (F_{SS'} \otimes \mathbb{1}_{EE'})$$

$$\leq \text{Tr}_R \left( \left( \frac{X_R}{d_R} \otimes \frac{X_R}{d_R} \right) \Pi_{SS'}^{\text{sym}} \right) (F_{SS'} \otimes \mathbb{1}_{EE'})$$

$$\leq \text{Tr}_R \left( \left( \frac{X_R}{d_R} \otimes \frac{X_R}{d_R} \right) \left( \mathbb{1}_{SS'} \otimes F_{EE'} \right) \right)$$

$$= \text{Tr}_S \hat{\Omega}_S + \text{Tr}_E \hat{\Omega}_E,$$  \hspace{1cm} (53)

where in the second equality we have used the fact that $[\sqrt{X_R} \otimes \sqrt{X_R}, \Pi_{R_R'}^{\text{sym}}] = 0$. From the analogue of equation (30) we can then obtain

$$\langle \| \hat{\rho}_S - \hat{\Omega}_S \|_1 \rangle \leq \sqrt{\frac{d_S}{d_E^{\text{eff}}}},$$  \hspace{1cm} (54)

where (using the analogue of (46))

$$d_E^{\text{eff}} = \frac{1}{\text{Tr}_E \hat{\Omega}_E} \geq \frac{d_R}{d_S}.$$  \hspace{1cm} (55)

To transform this bound on $\langle \| \hat{\rho}_S - \hat{\Omega}_S \|_1 \rangle$ into a bound on $\| \rho_S - \Omega_S \|_1$, we note that

$$\| \rho_S - \Omega_S \|_1 \leq \| \rho_S - \hat{\rho}_S \|_1 + \| \Omega_S - \hat{\Omega}_S \|_1 + \| \hat{\rho}_S - \hat{\Omega}_S \|_1.$$  \hspace{1cm} (56)
We bound $\|\rho_S - \tilde{\rho}_S\|_1$ as follows:

$$
\|\rho_S - \tilde{\rho}_S\|_1 \leq \left\|\langle \phi | \phi \rangle - |\tilde{\phi} \rangle \langle \tilde{\phi} |\right\|_1 \\
\leq \sqrt{2} \left\|\langle \phi | \phi \rangle - \langle \tilde{\phi} | \tilde{\phi} \rangle \right\|_2 \\
= \sqrt{2} \text{Tr}(|\phi \rangle \langle \phi | - |\tilde{\phi} \rangle \langle \tilde{\phi} |)^2 \\
= \sqrt{2(1 - 2\langle \phi | \sqrt{X_R} \phi \rangle^2 + \langle \phi | X_R \phi \rangle^2)} \\
\leq \sqrt{2(1 - \langle \phi | X_R \phi \rangle^2)} \\
\leq \sqrt{4(1 - \text{Tr}(X_R \phi \langle \phi |))},
$$

where in the first inequality we have used the non-increase of the trace-norm under partial tracing, in the second inequality we have used Lemma 6 (Appendix A) and the fact that $|\phi \rangle$ and $|\tilde{\phi} \rangle$ span a two-dimensional subspace, and in the third inequality we have used the fact that $X_R \leq \sqrt{X_R}$ (because $X_R \leq I_R$).

It follows that

$$
\langle \|\rho_S - \tilde{\rho}_S\|_1 \rangle \leq \sqrt{\langle 4(1 - \text{Tr}(X_R |\phi \rangle \langle \phi |)) \rangle} \\
\leq \sqrt{\langle 4(1 - \text{Tr}(X_R |\phi \rangle \langle \phi |)) \rangle} \\
= \sqrt{\langle 4(1 - \text{Tr}(X_R \phi \langle \phi |)) \rangle} \\
\leq 2\sqrt{\delta},
$$

where we have used the concavity of the square root function and equation (48).

In addition, note that from the triangle inequality,

$$
\|\rho_S - \tilde{\rho}_S\|_1 \leq \langle \|\rho_S - \tilde{\rho}_S\|_1 \rangle \\
\leq \langle \|\rho_S - \tilde{\rho}_S\|_1 \rangle \\
\leq 2\sqrt{\delta}.
$$

Inserting these results into the average of equation (56) we get

$$
\langle \|\rho_S - \Omega_S\|_1 \rangle \leq \sqrt{\frac{d_S}{d_E}} + 4\sqrt{\delta}
$$

and inserting this in equation (24) we obtain Theorem 2.

VI. METHOD II: APPLYING LEVY’S LEMMA TO EXPECTATION VALUES

In this section, we describe an alternative method of obtaining bounds on $\|\rho_S - \Omega_S\|_1$ by considering the expectation values of a complete set of observables. The physical intuition is that if the expectations of all observables on two states are close to each other, then the states themselves must be close.

We begin by showing that for an arbitrary (bounded) observable $O_S$ on $S$, the difference in expectation value between a randomly chosen state $\rho_S = \text{Tr}_E(|\phi \rangle \langle \phi |)$ and the canonical state $\Omega_S$ is small with high probability. We then proceed to show that this holds for a full operator basis, and thereby prove that $\rho_S \approx \Omega_S$ with high probability when $d_R \gg d_S^2$.

In this method, Levy’s Lemma plays a far more central role. This approach may be more suitable in some situations, and yields further insights into the underlying structure of the problem.

A. Similarity of expectation values for random and canonical states

Consider Levy’s Lemma applied to the expectation value of an operator $O_S$ on $\mathcal{H}_S$, for which we take

$$
f(\phi) = \text{Tr}(O_S \rho_S).$$

in (20). Let $O_S$ have bounded operator norm $\|O_S\|$ (where $\|O_S\|$ is the modulus of the maximum eigenvalue of the operator). Then the Lipschitz constant of $f(\phi)$ is also bounded, satisfying $\eta \leq 2\|O_S\|$ (as shown in appendix A). We therefore obtain

$$
\text{Prob}[|\text{Tr}(O_S \rho_S) - \langle \text{Tr}(O_S \rho_S) \rangle| \geq \epsilon] \leq 2 \exp \left(-\frac{C \delta \epsilon^2}{\|O_S\|^2}\right).$$

However, note that

$$
\langle \text{Tr}(O_S \rho_S) \rangle = \text{Tr}(O_S \langle \rho_S \rangle) = \text{Tr}(O_S \Omega_S),
$$

and hence that

$$
\text{Prob}[|\text{Tr}(O_S \rho_S) - \text{Tr}(O_S \Omega_S)| \geq \epsilon] \leq 2 \exp \left(-\frac{C \delta \epsilon^2}{\|O_S\|^2}\right).
$$

By choosing $\epsilon = d_R^{-1/3}$ we obtain the result that

$$
\text{Prob}[|\text{Tr}(O_S \rho_S) - \text{Tr}(O_S \Omega_S)| \geq d_R^{-1/3}] \leq 2 \exp \left(-\frac{C \delta d_R^{-1/3}}{\|O_S\|^2}\right).
$$

For $d_R \gg 1$, the expectation value of any given bounded operator for a randomly chosen state will therefore be close to that of the canonical state $\Omega_S$ with high probability [13].

B. Similarity of expectation values for a complete operator basis

Here we consider a complete basis of operators for the system. Rather than Hermitian operators, we find it convenient to consider a basis of unitary operators $U_S^x$. We show that with high probability all of these operators will have (complex) expectation values close to those of the canonical state.

It is always possible to define $d_S^2$ unitary operators $U_S^x$ on the system, labelled by $x \in \{0, 1, \ldots, d_S^2 - 1\}$, such
that these operators form a complete orthogonal operator basis for \( \mathcal{H}_S \) satisfying [14]

\[
\text{Tr}(U_S^x U_S^y) = d_S \delta_{xy},
\]

(66)

where \( \delta_{xy} \) is the Kronecker delta function. One possible choice of \( U_S^x \) is given by

\[
U_S^x = \sum_{s=0}^{d_S-1} e^{2\pi i s (x \mod d_S)/d_S^2} |(s + x) \mod d_S \rangle \langle s|.
\]

(67)

Noting that \( \|U_S^x\| = 1 \forall x \) (due to unitarity), we can then apply equation (64) to \( O_S = U_S^x \) to obtain

\[
\text{Prob}[|\text{Tr}(U_S^x \rho_S) - \text{Tr}(U_S^x \Omega_S)| \geq \epsilon] \leq 2e^{-C d_R \epsilon^2} \forall x.
\]

(68)

Furthermore, as there are only \( d_S \) possible values of \( x \), this implies that

\[
\text{Prob}[\exists x : |\text{Tr}(U_S^x \rho_S) - \text{Tr}(U_S^x \Omega_S)| \geq \epsilon] \leq 2d_S^2 e^{-C d_R \epsilon^2}
\]

(69)

If we take \( \epsilon = d_R^{-1/3} \ll 1 \), note that as the right hand side of (69) will be dominated by the exponential decay \( e^{-C d_R^{1/3}} \), it is very likely that all operators \( U_S^x \) will have expectation values close to their canonical values.

### C. Obtaining a probabilistic bound on \( \|\rho_S - \Omega_S\|_1 \)

As the \( U_S^x \) form a complete basis, we can expand any state \( \rho_S \) as

\[
\rho_S = \frac{1}{d_S} \sum_x C_x(\rho_S) U_S^x
\]

(70)

where

\[
C_x(\rho) = \text{Tr}(U_S^x \rho_S) = \text{Tr}(U_S^x \rho_S)^*.
\]

(71)

Expressing equation (69) in these terms we obtain

\[
\text{Prob}[\exists x : |C_x(\rho_S) - C_x(\Omega)| \geq \epsilon] \leq 2d_S^2 e^{-C d_R \epsilon^2}
\]

(72)

When \( |C_x(\rho_S) - C_x(\Omega_S)| \leq \epsilon \) for all \( x \), an upper bound can be obtained for the squared Hilbert-Schmidt norm [15] as follows:

\[
\|\rho_S - \Omega_S\|_2^2 = \left\| \frac{1}{d_S} \sum_x (C_x(\rho_S) - C_x(\Omega_S)) U_S^x \right\|_2^2
\]

\[
= \frac{1}{d_S^2} \text{Tr} \left( \sum_x (C_x(\rho_S) - C_x(\Omega_S)) U_S^x U_S^x \right)^2
\]

\[
= \frac{1}{d_S} \sum_x |(C_x(\rho_S) - C_x(\Omega_S))^2|
\]

\[
\leq d_S e^2 \epsilon^2
\]

(73)

Hence using the relation between the trace-norm and Hilbert-Schmidt norm (proved in appendix A),

\[
\|\rho_S - \Omega_S\|_1 \leq \sqrt{d_S} \|\rho_S - \Omega_S\|_2 \leq d_S \epsilon.
\]

(74)

Incorporating this result into equation (72) yields

\[
\text{Prob}[\|\rho_S - \Omega_S\|_1 \geq d_S \epsilon] \leq 2d_S^2 e^{-C d_R \epsilon^2}.
\]

(75)

If we choose

\[
\epsilon = \left( \frac{d_S}{d_R} \right)^{1/3}
\]

(76)

we obtain the final result that

\[
\text{Prob}[\|\rho_S - \Omega_S\|_1 \geq \frac{1}{\beta}] \leq 2d_S^2 e^{-C \beta}.
\]

(77)

where

\[
\beta = \left( \frac{d_R}{d_S} \right)^{1/3}
\]

(78)

Note that \( \|\rho_S - \Omega_S\|_1 \approx 0 \) with high probability whenever \( \beta \gg \log_2(d_S) \gg 1 \), and hence when \( d_R \gg d_S^2 \). This result is qualitatively similar to the result obtained using the previous method, although it can be shown that the bound obtained is actually slightly weaker in this case.

### VII. EXAMPLE: SPIN CHAIN WITH \( np \) EXCITATIONS

As a concrete example of the above formalism, consider a chain of \( n \) spin-1/2 systems in an external magnetic field in the +\( z \) direction, where the first \( k \) spins form the system, and the remaining \( n - k \) spins form the environment. We therefore consider a Hamiltonian of the form

\[
H = -\sum_{i=1}^{n} \frac{B}{2} \sigma_z^{(i)}
\]

(79)

where \( B \) is a constant energy (proportional to the external field strength), and \( \sigma_z^{(i)} \) is a Pauli spin operator for the \( i^{th} \) spin.

Under these circumstances, the global energy eigenstates can be divided into orthogonal subspaces dependent on the total number of spins aligned with the field. We consider a restriction to one of these degenerate subspaces \( \mathcal{H}_R \in \mathcal{H}_S \otimes \mathcal{H}_E \) in which \( np \) spins are in the excited state \( |1\rangle \) (opposite to the field) and the remaining \( n(1-p) \) spins are in the ground state \( |0\rangle \) (aligned with the field).

With this setup, \( d_S = 2^k \) and

\[
d_R = \binom{n}{np}.
\]

(80)
Approximating this binomial coefficient by an exponential
(26) (as in Appendix C), gives
\[ d_R \geq \frac{2^n H(p)}{n+1} \tag{81} \]
where \( H(p) = -p \log_2(p) - (1-p) \log_2(1-p) \) (the Shannon
entropy of a single spin).

From Theorem 1,
\[ \text{Prob}[\|\rho_S - \Omega_S\|_1 \geq \eta] \leq \eta', \tag{82} \]
where
\[ \eta = \epsilon + \sqrt{\frac{d_S}{d_R'}}, \tag{83} \]
\[ \eta' = 2 \exp(-C d_R^2). \tag{84} \]

In addition,
\[ \sqrt{\frac{d_S}{d_R'}} \leq \sqrt{\frac{d_S^2}{d_R}} \leq \sqrt{(n+1) 2^{-(nH(p)-2k)/2}}. \tag{85} \]

For an appropriate choice of \( \epsilon \) (e.g. \( \epsilon = d_R^{-1/3} \ll 1 \)), we will obtain \( \|\rho_S - \Omega_S\|_1 \approx 0 \) with high probability whenever
\[ \sqrt{(n+1) 2^{-(nH(p)-2k)/2}} \ll 1 \tag{86} \]
For fixed \( p \), this condition will be satisfied for all suf-

ciently large \( n \gg k \).

We emphasise that our results concern the distance
between \( \rho_S \) and \( \Omega_S \). Computing the precise form of \( \Omega_S \)
is a standard exercise in statistical mechanics, which we
sketch here for completeness.

In the regime where \( n \gg k^2 \), the canonical state \( \Omega_S \)
will take the approximate form
\[ \Omega_S = \sum_s d_S(np - |s|)(n(1-p) - (k-|s|)) |s\rangle \langle s| \]
\[ \approx \sum_s \frac{n!(n-p)^{|s|}(n(1-p))^{k-|s|}}{n^|s| d_S n^k (np)! (n(1-p))!} |s\rangle \langle s| \]
\[ = \sum_s p^{|s|}(1-p)^{k-|s|} |s\rangle \langle s| \tag{87} \]
\[ = (p|1\rangle \langle 1| + (1-p)|0\rangle \langle 0|)^\otimes_k. \tag{88} \]
and hence the canonical state of the system will approxi-
mate that of \( k \) uncorrelated spins, each with a probability
\( p \) of being excited, as expected.

To connect our result to the standard statistical me-
chanical formula,
\[ \Omega_S \propto \exp\left(-\frac{H_S}{k_B T}\right) \tag{89} \]
we use Boltzmann’s formula relating the entropy of the
environment \( S_E(|e|) \) to the number of states \( N_E(|e|) \) of
the environment with a given number of excitations \(|e|\)
to get
\[ S_E(|e|) = k_B \ln N_E(|e|) \]
\[ = k_B \ln \left( \frac{n-k}{|e|} \right) \]
\[ \approx k_B \left( (n-k) \ln(n-k) - |e| \ln |e| \right. \]
\[ \left. - (n-k-|e|) \ln(n-k-|e|) \right), \tag{90} \]
where in the third line we have used Stirling’s approxima-
tion. Defining the temperature in the usual way, and
noting that the energy of the environment is given by
\[ E = |e| B - (n-k) B / 2, \]
we obtain
\[ \frac{1}{T} = \frac{d S_E(E)}{d E \bigg|_{E=(E)}} \]
\[ = \frac{1}{B} \frac{d S_E(|e|)}{d |e| \bigg|_{|e|=(n-k)p}} \]
\[ \approx \frac{k_B}{B} \ln \left( \frac{n-k-|e|}{|e|} \right) \bigg|_{|e|=(n-k)p} \]
\[ = \frac{k_B}{B} \ln \left( \frac{1-p}{p} \right). \tag{91} \]

This formula expresses how the probability \( p \) defines a
temperature \( T \) of the environment. Rearranging equation
(87) to incorporate equation (91) gives the usual
statistical mechanical result
\[ \Omega_S \approx (1-p)^k \sum_s \left( \frac{p}{1-p} \right)^{|s|} |s\rangle \langle s| \]
\[ = (1-p)^k \sum_s \exp\left(-|s| \ln \left( \frac{1-p}{p} \right) \right) |s\rangle \langle s| \]
\[ = (1-p)^k \sum_s \exp\left(-|s| B \right) \frac{1}{k_B T} |s\rangle \langle s| \]
\[ \propto \exp\left(-\frac{H_S}{k_B T}\right). \tag{92} \]

A. Projection on the typical subspace

We can obtain an improved bound on \( \|\rho_S - \Omega_S\|_1 \) by
noting that the system state almost always lies in a typ-
cal subspace with approximately \( kp \) excitations. We
make use of this observation by applying Theorem 2 with
a measurement operator \( X_R \) given by
\[ X_R = \Pi_S \otimes 1_E \tag{93} \]
where \( \Pi_S \) is a projector onto the typical subspace of the
system, in which it contains a number of excitations \(|s|\)
in the range
\[ kp - \xi \leq |s| \leq kp + \xi. \tag{94} \]
It is easy to show, using classical probabilistic arguments (see Appendix B), that
\[
\text{Tr}_R(X_R e_R) = \text{Tr}_S(\Pi_S \Omega_S) \geq 1 - \delta \tag{95}
\]
where
\[
\delta = 2 \exp \left( - \frac{\xi^2}{4 kp(1-p)} \right) \tag{96}
\]
Furthermore, the dimension $d_S$ of the support of $X_R$ on $\mathcal{H}_S$ (which here is simply the dimension of the typical subspace) is shown in Appendix C to be given by
\[
d_S = \sum_{|s|=k p+\xi}^k \left( \begin{array}{c} k \\ s \end{array} \right) \leq (2 \xi + 1) 2^{k H(p) + \xi G(p)} \tag{97}
\]
where
\[
G(p) = \left| \frac{d H(p)}{dp} \right| = \log_2 \left( \frac{p}{1-p} \right) . \tag{98}
\]
From Theorem 2 we obtain:
\[
\text{Prob}\left[ ||\rho_S - \Omega_S||_1 \geq \tilde{\eta} \right] \leq \tilde{\eta}', \tag{99}
\]
where, using $\tilde{d}_S^\text{eff} \geq d_R/d_S$, and inserting the results of equations (81), (96) and (97),
\[
\tilde{\eta} = \epsilon + \sqrt{(n+1)(2\xi + 1)} 2^{(k-n/2) H(p) + \xi G(p)} \tag{100}
\]
\[
\tilde{\eta}' = 2 \exp \left( - C d_R \epsilon^2 \right) . \tag{101}
\]
Choosing $\xi = k^{2/3}$ and $\epsilon = \alpha^{1/3}$ yields
\[
\tilde{\eta} = \sqrt{(n+1)2^{-n H(p)/3}} \tag{102}
\]
\[
\tilde{\eta}' = 2 \exp \left( - C^{2 n H(p)/3} (n+1)^{1/3} \right) . \tag{103}
\]
In the thermodynamic limit in which $p$ is fixed (corresponding to the temperature), the ratio of the system and environment sizes $r = k/(n-k)$ is fixed at some value $r < 1$ (i.e. the system is smaller than the environment), and $n$ tends to infinity, $\eta \to 0$ and $\eta' \to 0$, and hence $\rho_S \to \Omega_S$.

For large (but finite) $n$ the system will be thermalised for almost all states when the system is smaller than the environment (i.e. $r < 1$). Note that as $\eta$ depends exponentially on $(n-2k)$, $\eta \ll 1$ can be achieved with only small differences in the number of spins in the system and environment.

**VIII. CONCLUSIONS**

Let us look back at what we have done. Concerning the problem of thermalisation of a system interacting with an environment in statistical mechanics, there are several standard approaches. One way of looking at it is to say that the only thing we know about the state of the universe is a global constraint such as its total energy. Thus the way to proceed is to take a Bayesian point of view and consider all states consistent with this global constraint to be equally probable. The average over all these states indeed leads to the state of any small subsystem being canonical. But the question then arises what is the meaning of this average, when we deal with just one state. Also, these probabilities are subjective, and this raises the problem of how to argue for an objective meaning of the entropy. A formal way out is that suggested by Gibbs, to consider an ensemble of universes, but of course this doesn’t solve the puzzle, because there is usually only one actual universe. Alternatively, it was suggested that the state of the universe, as it evolves in time, can reach any of the states that are consistent with the global constraint. Thus if we look at time averages, they are the same as the average that results from considering each state of the universe to be equally probable.

To make sense of this image one needs assumptions of ergodicity, to ensure that the universe explores all the available space equally, and of course this doesn’t solve the problem of what the state of the subsystem is at a given time.

What we showed here is that these averages are not necessary. Rather, (almost) any individual state of the universe is such that any sufficiently small subsystem behaves as if the universe were in the equiprobable average state. This is due to massive entanglement between the subsystem and the rest of the universe, which is a generic feature of the vast majority of states. To obtain this result, we have have introduced measures of the effective size of the system, $d_S$, and its environment (i.e. the rest of the universe), $d^\text{eff}_S$, and showed that the average distance between the individual reduced states and the canonical state is directly related to $d_S/d^\text{eff}_S$. Levy’s Lemma is then invoked to conclude that all but an exponentially small fraction of all states are close to the canonical state.

In conclusion, the main message of our paper is that averages are not needed in order to justify the canonical state of a system in contact with the rest of the universe – almost any individual state of the universe is enough to lead to the canonical state. In effect, we propose to replace the Postulate of Equal a priori Probabilities by the Principle of Apparently Equal a priori Probabilities, which states that as far as the system is concerned every single state of the universe seems similar to the average.

We stress once more that we are concerned only with the distance between the state of the system and the canonical state, and not with the precise mathematical form of this canonical state. Indeed, it is an advantage
of our method that these two issues are completely separated. For example, our result is independent of the canonical state having Boltzmannian form, of degeneracies of energy levels, of interaction strength, or of energy (of system, environment or the universe) at all.

In future work [17], we will go beyond the kinematic viewpoint presented here to address the dynamics of thermalisation. In particular, we will investigate under what conditions the state of the universe will evolve into (and spend almost all of its later time in) the large region of its Hilbert space in which its subsystems are thermalised.

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Note added: A very recent independent paper by Goldstein et. al. [18] discusses similar issues to those addressed here.

APPENDIX A: LIPSCHITZ CONSTANTS AND NORM RELATION

Lemma 4 The Lipschitz constant \( \eta \) of the function \( f(\phi) = \|\rho S - \Omega S\|_1 \), satisfies \( \eta \leq 2 \).

Proof: Defining the reduced states \( \rho_1 = \text{Tr}_E(|\phi_1\rangle\langle\phi_1|) \) and \( \rho_2 = \text{Tr}_E(|\phi_2\rangle\langle\phi_2|) \), and using the result that partial tracing cannot increase the trace-norm

\[
|f(\phi_1) - f(\phi_2)|^2 = \|\rho_1 - \Omega S_1\|_1^2 - \|\rho_2 - \Omega S_1\|_1^2 \\
\leq \|\rho_1 - \rho_2\|_1^2 \\
\leq \|\phi_1\|_1^2 - \|\phi_2\|_1^2 \\
= 4 \left( |\langle\phi_1|\phi_2\rangle|^2 \right) \\
\leq 4 \||\phi_1\| - |\phi_2\|^2 \leq 2 \||\phi_1\| - |\phi_2\|^2 \leq 2(\|\phi_1\| - \|\phi_2\|)
\]

Hence \( |f(\phi_1) - f(\phi_2)| \leq 2(\|\phi_1\| - \|\phi_2\|) \), and thus \( \eta \leq 2 \). \( \square \)

Lemma 5 The Lipschitz constant \( \eta \) of the function \( f(\phi) = \text{Tr}(X|\phi\rangle\langle\phi|) \), where \( X \) is any operator on \( \mathcal{H}_R \) with finite operator norm \( \|X\| \), satisfies \( \eta \leq 2 \|X\| \).

Proof:

\[
|f(\phi_1) - f(\phi_2)| = \|\langle\phi_1|X|\phi_1\rangle - \langle\phi_2|X|\phi_2\rangle\| \\
= \frac{1}{2} \left( \|\langle\phi_1| + \langle\phi_2|X|\phi_1\rangle - |\phi_2\rangle\| \\
+ \|\langle\phi_1| + \langle\phi_2|X|\phi_1\rangle + |\phi_2\rangle\| \right) \\
\leq \|X\| \|\phi_1\| + |\phi_2\| \\|\phi_1\| - |\phi_2\| \|X\| \\
\leq 2 \|X\| \|\phi_1\| - |\phi_2\|. \quad (A2)
\]

Lemma 6 For any \( n \times n \) matrix \( M \), \( \|M\|_1 \leq \sqrt{n} \|M\|_2 \).

Proof: If \( M \) has eigenvalues \( \lambda_i \),

\[
\|M\|_1^2 = n^2 \left( \frac{1}{n} \sum_i |\lambda_i|^2 \right) \\
\leq n^2 \frac{1}{n} \sum_i |\lambda_i|^2 = n \|M\|_2 ^2,
\]

by the convexity of the square function. Taking the square-root yields the desired result. \( \square \)

APPENDIX B: PROJECTION ONTO THE TYPICAL SUBSPACE

Lemma 7 Given a system in the canonical state \( \Omega_S \), the probability of it containing a number of excitations \( |s| \) in the range \( kp - \xi \leq |s| \leq kp + \xi \) is given by

\[
\text{Tr}(\Pi_S \Omega_S) \geq 1 - \delta
\]

where

\[
\delta = 2 \exp \left( -\frac{\xi^2}{4kp(1-p)} \right)
\]

Proof: \( \Omega_S \) is essentially a classical probabilistic state, obtained by choosing \( k \) spins at random from a ‘bag’ containing \( np \) excited spins and \( n(1-p) \) un-excited spins without replacement. It is easy to see that this state will lie in the typical subspace with higher probability than if the spins were replaced in the bag after each selection, as the former process is mean reverting, whereas the latter is not. We can bound the probability of lying outside the typical subspace in the case with replacement using Chernoff’s inequality [16] for the sum \( X = \sum_i (s_i - p) \), where \( s_i \in \{0, 1\} \) is the value of the \( i \)th spin. This gives

\[
\text{Prob}\left( |X| > \xi \right) \leq 2e^{-\frac{\xi^2}{4kp\sigma^2}} \quad (B3)
\]

where \( \sigma^2 = kp(1-p) \) is the variance of \( X \). Hence

\[
\text{Prob}\left( |s| - kp > \xi \right) \leq 2e^{-\frac{\xi^2}{4kp\sigma^2}} \quad (B4)
\]

and thus

\[
\text{Tr}(\Pi_S \Omega_S) = 1 - \text{Prob}\left( |s| - kp > \xi \right) \geq 1 - 2e^{-\frac{\xi^2}{4kp\sigma^2}} \quad (B5)
\]

APPENDIX C: EXPONENTIAL BOUNDS ON COMBINATORIAL QUANTITIES

In this appendix we obtain bounds for the combinatorial quantities required to consider the example case of a spin-chain.
From standard probability theory we know that
\[ \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1, \tag{C1} \]
with the maximal term in the sum being obtained when \( k = np \). Hence
\[
\left( \frac{n}{np} \right) p^{np}(1-p)^{n(1-p)} \leq 1 \leq \sum_{k=0}^{n} \left( \frac{n}{np} \right) p^{np}(1-p)^{n(1-p)}. \tag{C2} \]

Noting that
\[ p^{np}(1-p)^{n(1-p)} = 2^{-nH(p)} \tag{C3} \]
where \( H(p) = -p \log_2(p) - (1-p) \log_2(1-p) \), we can rearrange equation (C2) to get
\[ 2^{-nH(p)-\log_2(n+1)} \leq \left( \frac{n}{np} \right) \leq 2^{-nH(p)}. \tag{C4} \]

We also require an upper bound for the dimension of the typical subspace of system \( S \), given by
\[ \tilde{d}_S = \sum_{|s|=kp+\xi} \left( \frac{k}{|s|} \right). \tag{C5} \]

The maximal term in this sum occurs when \( |s| = kp \)

where
\[ \tilde{p} = \begin{cases} \frac{p + \xi/k}{p}, & p < \frac{1}{2} - \frac{\xi}{k} \\ \frac{1}{2}, & \frac{1}{2} - \frac{\xi}{k} \leq \frac{p}{k} \leq \frac{1}{2} + \frac{\xi}{k}, \\ \frac{p}{k}, & p > \frac{1}{2} + \frac{\xi}{k}, \end{cases} \tag{C6} \]

and as the sum consists of \((2\xi + 1)\) terms,
\[ \tilde{d}_S \leq (2\xi + 1) \left( \frac{k}{kp} \right). \tag{C7} \]

Bounding the binomial coefficient by an exponential as above we obtain
\[ \tilde{d}_S \leq (2\xi + 1)2^{kH(\tilde{p})}. \tag{C8} \]

As \( H(p) \) is a concave function of \( p \), we also note that
\[ kH(\tilde{p}) - kH(p) \leq \xi \leq \frac{dH(p)}{dp}. \tag{C9} \]

Defining
\[ G(p) = \frac{dH(p)}{dp} = \log_2 \left( \frac{p}{1-p} \right), \tag{C10} \]
we therefore find that
\[ \tilde{d}_S \leq (2\xi + 1)2^{kH(p) + \xi G(p)}. \tag{C11} \]