On the Higgs potential in the MS$_3$IESM

A Mondragón, M Mondragón and U J Saldaña Salazar
Instituto de Física, Universidad Nacional Autónoma de México, Apdo. Postal 20-364, 01000, México D.F., México.
E-mail: ulisesjesus@fisica.unam.mx

Abstract. We will briefly present a new derivation of the most general form of the Higgs potential with the highest degree of $S_3$ flavour symmetry. The discussion will be made in the framework of the Minimal $S_3$-Invariant Extension of the Standard Model (MS$_3$IESM). In this form of the theory, the Higgs sector has three flavoured Higgs fields that are $SU(2)_L$ doublets and belong to a reducible $1 \oplus 2$ representation of the $S_3$ flavour symmetry. Requiring the highest degree of $S_3$ flavour symmetry, the Higgs potential, in its most general form, has only six self-couplings.

1. Introduction
The Standard Model (SM), which successfully describes the fundamental interactions between elementary particles, is being experimentally tested in its still unknown scalar sector. It has become of major importance to experimentally proof the existence of the SM single massive scalar field. The explanation, to whether the massive nature of fundamental particles is being originated from their Yukawa interactions with a scalar field, depends upon the latter experimental observation. Apart from the unknown scalar sector, there are yet a few other open questions that emerge from the fact that there are three generations of matter. It is desirable, that any flavour extension of the SM, should have a unified treatment for both the scalar sector and the flavoured Yukawa interactions.

This paper is organized as follows. In section 2, we briefly comment on what the flavour puzzle is and its relation to the Higgs mechanism. In section 3, we discuss and sketch the construction of the Minimal $S_3$-Invariant Extension of the Standard Model (MS$_3$IESM). In section 4, the derivation of the most general $S_3$-invariant Higgs potential is presented. In section 5, we summarize and conclude with some remarks.

2. The flavour puzzle and its relation to the Higgs mechanism
In 1973, Kobayashi and Maskawa first predicted that the SM needed to have at least three generations of matter, in order to theoretically accommodate Charge-Parity (CP) violation without having any further complications [1]. Nevertheless, a sign of having only three families was not found until later on, when the experiments operating at the electron-positron colliders Stanford Linear Collider (SLC) and Large Electron-Positron collider (LEP) by looking at $Z$ decays showed that there are only three flavours of active neutrinos [2]. The latter fact seems to be pointing out that matter appears in three generations. This came to be known as the flavour puzzle of high energy physics. Why are there three generations of matter? Do we have
any other information regarding this problem? In order to start moving efficiently along the direction of the real explanation, the right questions have to be asked.

In the following subsections the features of the flavour puzzle and how they are related with the Higgs mechanism are briefly commented.

2.1. Some features of the flavour puzzle

There are some phenomenological aspects, that clearly show interrelations between the different families. It is through the study of this interrelations, that we can come closer to find, a solution to the flavour puzzle.

- **Flavour mixing:**
  The experimentally observed flavour oscillations, for both the quark and lepton sector, imply, that weak eigenstates are not mass eigenstates. Matter generations may then be considered as independent entities that interrelate only through their dynamics. The proportion of this interrelationship or mixing is expressed by an angle. The reported mixing angles in the Particle Data Group (PDG) parametrization, for both the quark and lepton sector, are:

- Quark weak mixing angles at 2σ [3]:
  - $\theta_{12}^q = 13.021^\circ \pm 0.039^\circ$,
  - $\theta_{23}^q = 2.350^\circ \pm 0.052^\circ$,
  - $\theta_{13}^q = 0.199^\circ \pm 0.008^\circ$;

- Lepton weak mixing angles with normal neutrino mass hierarchy at 1σ [4]:
  - $\theta_{12}^l = 33.95^\circ ^{+1.05^\circ}_{-0.93^\circ}$,
  - $\theta_{23}^l = 46.14^\circ ^{+3.46^\circ}_{-3.44^\circ}$,
  - $\theta_{13}^l = 6.54^\circ ^{+1.59^\circ}_{-1.41^\circ}$.

In 1967, Sakharov proposed a set of physical conditions on the early universe evolution that could give an explanation to the nowadays observed baryon asymmetry [5]. One of these physical conditions is the presence of CP violation. The SM with massive neutrinos has two possible sources of CP violation: the quark and lepton mixing matrices. As Kobayashi and Maskawa first noticed [1], a 3×3 complex unitary matrix has one non-removable phase besides the three mixing angles. It is through this phase that one is able to introduce CP violation into the SM weak processes. So this means, that apart from the mixing angles, flavour oscillations between three mass states also bring about a desired source for CP violation.

- **Mass spectrum:**
  Fundamental fermions in the SM without the Higgs mechanism are massless. Once the Yukawa interactions are introduced, through the spontaneous electroweak symmetry breaking, which implies a non zero vacuum expectation value for the Higgs boson, fermions become massive.

At present time, all known fundamental fermions are of massive nature. A semi-log plot of the most recent charged fermions mass values [3] divided over the top quark mass, see Figure 1, clearly shows a strong hierarchical structure. Neutrino masses can not be seen in Figure 1 due to their smallness; their presence in the semi-log plot would only strengthen the strong hierarchical structure. In which way the mass distribution could help us to discern the underlying flavour structure? This feature of the flavour puzzle requires to be approached in a very careful manner. It is here where flavour theories depart from each other.

Therefore, what may be seen from these aspects of the flavour puzzle is that all the known reciprocal relations between matter generations are observed in the weak sector. This last statement is what motivates the next subsection.
2.2. The relation with the Higgs mechanism

The SM gauge group is given by $G_{\text{SM}} = SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$. The transition, by spontaneous symmetry breaking, from the SM gauge group, $G_{\text{SM}}$, to $SU(3)_C \otimes U(1)_{\text{EM}}$ is produced by the Higgs mechanism. Through this mechanism, the massive vector bosons and the fundamental fermions acquire their mass. Besides this, a massive scalar boson is left. The experimentally elusive remnant field is called the Higgs boson.

In the Higgs mechanism the local symmetry group that gets spontaneously broken is $SU(2)_L$. It is from this symmetry group from which the characteristics of the flavour puzzle emerge. In fact, the strong relation of these features with the Higgs mechanism, as already seen by F. Wilczek in 1977 [6], can be clearly seen in the limit of massless fundamental fermions, because the weak mixing angles lose all meaning, and thereby, without the Higgs mechanism the aspects of the flavour puzzle vanish. Thus, a complete study of the flavour puzzle should require a careful examination of its features along with the Higgs mechanism.

3. The Minimal $S_3$-Invariant Extension of the Standard Model (MS$_3$IESM)

A vast literature exists on trying to solve the flavour puzzle by using a flavour symmetry, whether continuous or discrete, but given its size, we only refer to some recent reviews on non-Abelian discrete flavour symmetries [7–9]. It has been shown in [10] that by demanding simplicity to a flavour extension of the SM the resulting flavour framework is that of the MS$_3$IESM. The same result can be obtained by a different approach; somewhat similar to the one shown in [11]. Through the following subsections the derivation of the essential ingredients of the MS$_3$IESM will be presented, after that, the construction of the flavour framework is briefly sketched.

3.1. Derivation of $S_3$ as the flavour symmetry

In section 2, it was highlighted the importance of asking the right questions in order to efficiently work towards the solution of the flavour puzzle. If a flavour symmetry really exists, then, the aspects there mentioned could help to point out what the flavour symmetry group is. It is a well known fact, that when particles are described by a symmetry group, their mass distributes according to the irreducible representation (irrep) they belong. Any strong evidence could then come from the mass spectrum.

Matter generations are found to be replicated over each individual sector of fundamental particles: $u$ type quarks, $d$ type quarks, charged leptons and neutrinos. Therefore, the flavour
symmetry could be acting independently on each fermion sector.

How is mass being distributed along each independent fermion sector? According to this question, the mass spectrum, see Figure 1, transforms into what we call the flavour spectrum, see Figure 2, which was already shown in [10]. Neutrino masses were not considered in the plot due to the uncertainty in their individual mass values, although we believe their masses also distribute as the charged fermions masses.

From this picture, a clear pattern of group symmetry representations of doublets and singlets, $2 \oplus 1$, emerges; which suggests to accommodate the particles in these representations. For the massive neutrinos, depending on whether they have normal or inverted hierarchy, we believe that two kinds of pattern should exist: $2 \oplus 1$ or $1 \oplus 2$, respectively.

In section 2, it was stated that a complete study of the flavour puzzle should require a careful examination of its aspects along with the Higgs mechanism. This means, that it is necessary to survey the mass spectrum in relation to the Higgs mechanism: how things are before and after its introduction. Before the introduction of the Yukawa interactions families are undistinguishable. A flavour symmetry naturally arises from the SM matter Lagrangian as an invariance under permutations of the three families. Seen in this way, this fact points naturally to a symmetry group which represents the permutations of three objects. $S_3$, the smallest non-Abelian discrete symmetry group, naturally describes this scenario. This particular characteristic of $S_3$ of expressing the similarity of three different flavoured fermions, with regard to their interactions with gauge bosons, was already stressed out in [12]. After the Higgs mechanism breaks the gauge symmetry, families become distinguishable because their experimental observed masses are different. The invariance under permutations of the three families is now hidden. Could these two seemingly unrelated facts be connected? Let us explore this question.

A group theory analysis reveals that $S_3$ has three distinct irreps: two singlets, $1_S$ and $1_A$, symmetric and antisymmetric, respectively, and one doublet, $2$. A space of three families implies the use of the three dimensional real representation of $S_3$, which has as invariant subspaces: a one dimensional space corresponding to the $1_S$ and a two dimensional space, orthogonal to the singlet’s space, corresponding to the $2$. Hence, a three dimensional representation of $S_3$, $3$, reduces to a direct sum of $S_3$ irreps: $2 \oplus 1_S$. This means, that, after the introduction of the $S_3$-invariant Yukawa interactions, the mass spectra would then have the flavour structure $2 \oplus 1$. The latter fact is in very good agreement with the experimentally determined flavour spectrum, see Figure 2. As a consequence, $S_3$ then becomes the symmetry which naturally reconciles, without any further assumptions, two seemingly unrelated facts: three undistinguishable families before the introduction of the Yukawa interactions with three discernible massive families after the Higgs mechanism.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{flavour_structure.png}
\caption{The flavour structure unveiled from the charged fermions mass spectrum. The indices I, II and III refer to the family index. Plot and legend taken from Figure 27.1 in [10].}
\end{figure}
3.2. Construction of the flavour framework

An addition of a flavour symmetry to the SM is done through two theoretical elements: the Yukawa Lagrangian and the Higgs potential. When adding a flavour symmetry to the SM the first step corresponds to the assignment of the particles to the irreps of the flavour symmetry group. From the flavour spectrum, see Figure 2, it is evident how the assignment should be done: the first two families go into the doublet representation and the third family to the symmetric singlet representation. The second step is to assign the Higgs weak doublet field to a flavour irrep. The SM has a single weak doublet Higgs field, therefore, in order to construct $S_3$ invariant terms, in both the Yukawa Lagrangian and the Higgs potential, the weak doublet Higgs field must be assigned to the symmetric singlet irrep of $S_3$. Subsequent to the breaking of the gauge symmetry by the Higgs mechanism, the generic mass matrix for Dirac fermions that results:

$$M_f = \begin{pmatrix}
\mu_1 & 0 & 0 \\
0 & \mu_1 & 0 \\
0 & 0 & \mu_3
\end{pmatrix},$$

shows a degenerate spectrum in the first two families, which is in clear disagreement with the experimentally observed mass spectrum. The latter result was already observed and proof as a theorem in [12]. The mass differences between the first two families can be explained by breaking the flavour symmetry [13]. But, if $S_3$ is to be preserved, how to incorporate the mass differences without breaking the flavour symmetry? If two more Higgs weak doublets are added, in a $S_3$ doublet irrep, in addition to the one from the SM, it can be shown that the generic mass matrix for Dirac fermions transforms into [14]:

$$M_f = \begin{pmatrix}
\mu_1 + \mu_2 & \mu_4 & \mu_5 \\
\mu_4 & \mu_1 - \mu_2 & \mu_6 \\
\mu_7 & \mu_8 & \mu_3
\end{pmatrix},$$

which implies not only a non-degenerate mass spectrum but also a mixing between the families. Furthermore, as an important consequence, the concepts of flavour and generations are now extended to a more fundamental level, because now, the Higgs sector also has flavour as well as all fermions.

When finding a correct parametrization of the mass matrices in terms of their mass eigenvalues, the mixing angles may now be expressed in terms of mass ratios. This new possibility is usually considered the primary objective of constructing flavour frameworks [15–24]. Accidentally, this intention of preserving the flavour symmetry, which meant the introduction of two more SM Higgs fields, satisfies the condition of needing at least three Higgs weak doublets, in order to be able to express the fermion mixing angles in terms of mass ratios [21].

4. Derivation of the most general $S_3$-invariant Higgs potential

Within the MS$_3$IESM, which is the most general $S_3$-invariant Higgs potential?, and, in which sense is it being referred as the most general? Work on this topic, the most general $S_3$-invariant Higgs potential, has been done before [12,14,21,25–33]. A direct comparison between the expressions of the distinct authors of the aforecited literature shows a confusing scenario: different numbers of invariant terms and self-couplings. Any comparisons between theory and experiment depend upon the form of the Higgs potential, therefore, in order to avoid misleading conclusions a search for an answer to both questions is needed.

What are the prominent attributes of the most general expression? In which sense the term general is to be understood? We propose the following convention:

- that it has the highest degree of flavour symmetry, and, within this
- the highest possible degree of arbitrariness.
We give now a systematic procedure, following the previous convention, to construct the Higgs potential:

1. In order to guarantee the highest degree of flavour symmetry, all the linearly independent \( S_3 \)-invariant terms with two and four Higgs weak doublet fields ought to be found, \( n = 2 \):
   - \( 1_S \otimes 1_S \)
   - \( 2 \otimes 2 \)\(_S \)

\( n = 4 \):
   - \( 1_S \otimes 1_S \otimes 1_S \otimes 1_S \)
   - \( [(1_S \otimes 2) \otimes (1_S \otimes 2)]_S \)
   - \( [(1_S \otimes 2) \otimes (2 \otimes 2)]_S \)
   - \( (2 \otimes 2) \)\( _S \) \( \otimes (2 \otimes 2) \)\( _S \)
   - \( (2 \otimes 2)_A \otimes (2 \otimes 2)_A \)
   - \( [(2 \otimes 2)_A \otimes (2 \otimes 2)_A]_S \),

where the braces [ ] refer to the action of just taking the \( x \) part of the tensor product.

2. Make an explicit convention of where to place the symmetric and antisymmetric \( S_3 \) doublet components.

\[
H_D = \begin{pmatrix} H_{DA} \\ H_{DS} \end{pmatrix},
\]

this matters, because, the mass matrix for Dirac fermions depends on this convention.

3. For each \( S_3 \)-invariant term, make all the different contractions of independent weak indexes, \( X_w, X_{w'} \), for example:

   - \( (2 \otimes 2)_S \otimes (2 \otimes 2)_S = 1_S^3 \)
   - \( (2_w \otimes 2_w)_S \otimes (2_{w'} \otimes 2_{w'})_S = 1_S^3 \otimes G_{SM} \)
   - \( (2_w \otimes 2_{w'})_S \otimes (2_w \otimes 2_{w'})_S = 1_S^3 \otimes G_{SM} \)
   - \( (2_w \otimes 2_{w'})_S \otimes (2_{w'} \otimes 2_w)_S = 1_S^3 \otimes G_{SM} \),

the next expressions clearly show how different structures may emerge from the same \( S_3 \)-invariant term in terms of the weak doublet Higgs fields:

   - \( \frac{1}{2}(H^1_{1w}H_{1w} + H^1_{2w}H_{2w})^2 \)
   - \( \frac{1}{2}[(H^1_{1w}H_{1w})^2 + (H^1_{2w}H_{2w})^2 + (H^1_{1w}H_{2w})^2 + (H^1_{2w}H_{1w})^2] \)
   - \( \frac{1}{2}[(H^1_{1w}H_{1w})^2 + (H^1_{2w}H_{2w})^2 + (H^1_{1w}H_{2w})^2 + (H^1_{2w}H_{1w})^2] \)

where the convention \( H_{1w} = H_{DAw}, H_{2w} = H_{DSw} \) was taken in order to simplify the notation.

4. Assign the same self-coupling to each different weak indexes contraction of the same \( S_3 \)-invariant term.

The necessary elements to carry out all the aforementioned steps are: compute the direct products of \( S_3 \) irreps and carefully make explicit the weak, \( SU(2)_L \), indexes in all terms.

The expression for the most general \( S_3 \)-invariant Higgs potential, already shown in [10], has only six independent self-couplings and it is given by:

\[
V_H = \mu^2 S_3 H_S + \mu^2 D(H^1_S H_1 + H^1_H H_2) + a(H^1_S H_S)^2 + b f_{ijk} [(H^1_S H_1)(H^1_H H_k) + h.c.] + c[(H^1_S H_1)(H^1_H H_S) + (H^1_H H_2)(H^1_H H_S) + (H^1_H H_1)^2 + (H^1_H H_2)^2 + (H^1_S H_S)^2 + (H^1_S H_2)(H^1_H H_1 + H^1_H H_2)] + d[2(H^1_H H_1 + H^1_H H_2)^2 + (H^1_H H_1 - H^1_H H_2)^2 + (H^1_H H_2)^2 + (H^1_H H_2 - H^1_H H_1)^2] + e(H^1_H H_2 - H^1_H H_1)^2 + f[(H^1_H H_1 - H^1_H H_2)^2 + 2(H^1_H H_1 + H^1_H H_2)^2 - 2(H^1_H H_2 - H^1_H H_1)^2],
\]
where \(a, b, c, \ldots, f\) are constants; \(\mu^2_S, \mu^2_D\) are negative real numbers; 1, 2, and \(S\) are the flavour indexes for the components of the doublet and the symmetric singlet irrep of \(S_3\), respectively; \(f_{112} = f_{121} = f_{211} = -f_{222} = 1\), all other tensor components vanish.

5. Conclusions
The flavour puzzle still remains an open problem of high energy physics. However, the addition of a flavour symmetry to the SM is the simplest approach known in order to try to give an explanation to the flavour puzzle. We have emphasized the importance of looking for the right questions in order to start moving efficiently along the direction of the real explanation.

Some of the aspects of the flavour puzzle, from our point of view, were discussed in Section 2. It was highlighted the strong relation with the Higgs mechanism that exists, such that any complete study of the flavour puzzle should require a careful examination of its aspects along with it.

A different approach to derive the essential ingredients of the MS\(_3\)IESM was presented. The latter was made by taking into consideration some of the features of the flavour puzzle along with the Higgs mechanism, in particular, the mass spectrum, see Figure 1, which is the strongest suggestion of flavour structure we have nowadays. The mass spectrum was related with a flavour structure through the question: how is mass being distributed along each independent fermion sector?, see Figure 2. A \(2 \oplus 1\) structure can be readily seen from it. When studying the latter along with the Higgs mechanism, it was necessary to reconcile two seemingly unrelated facts: three undistinguishable families before the introduction of the Yukawa interactions with three discernible massive families after the breaking of the gauge symmetry by the Higgs mechanism. From this quest, we found that: \(S_3\) is the only symmetry that naturally reconciles two seemingly unrelated facts without any further assumptions. The scalar sector was extended, by introducing two more Higgs weak doublets into the SM in a flavour doublet irrep of \(S_3\), and this led to a non-degenerate mass spectra with flavour mixing; furthermore, the concepts of flavour and generations were extended to a more fundamental level, because the Higgs sector was flavoured as well as the fermion sector.

Finally, the steps in the computation of the most general \(S_3\)-invariant Higgs potential were presented and each one of those steps was illustrated with an example. This calculation is important for a meaningful and significative comparison of theory and experiment, in this respect, any conclusions on the Higgs phenomenology depend strongly on having the form of the Higgs potential which is in agreement with the flavour symmetry proposed.

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