AN EXPLICIT FORMULA FOR HECKE L-FUNCTIONS

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Abstract. In this paper an explicit formula is given for a sequence of numbers. The positivity of this sequence of numbers implies that zeros in the critical strip of the Euler product of Hecke polynomials, which are associated with the space of cusp forms of weight $k$ for Hecke congruence subgroups, lie on the critical line.

1. Introduction

Let $k$ and $N$ be positive integers with $k > 2$, and let $\chi$ be a Dirichlet character of modulus $N$ with $\chi(-1) = (-1)^k$ and with conductor $f$. We denote by $S_k(N, \chi)$ the space of all cusp forms of weight $k$ and character $\chi$ for the Hecke congruence subgroup $\Gamma_0(N)$ of level $N$. That is, $f$ belongs to $S_k(N, \chi)$ if and only if $f$ is holomorphic in the upper half-plane, satisfies
\[
  f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)
\]
for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, satisfies the usual regularity conditions at the cusps of $\Gamma_0(N)$, and vanishes at each cusp of $\Gamma_0(N)$.

The Hecke operators $T_n$ are defined by
\[
  (T_n f)(z) = \frac{1}{n} \sum_{ad=n} \chi(a) a^k \sum_{0 \leq b < d} f\left(\frac{az+b}{d}\right). \tag{1.1}
\]
A function $f$ in $S_k(N, \chi)$ is called a Hecke eigenform if
\[
  T_n f = \lambda(n) f
\]
for all positive integers $n$ with $(n, N) = 1$. The Fricke involution $W$ is defined by
\[
  (W f)(z) = N^{-k/2} z^{-k} F(-1/Nz),
\]
and the complex conjugation operator $K$ is defined by
\[
  (K f)(z) = \bar{f}(\bar{z}).
\]

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Set $\tilde{W} = KW$. An element $f$ in $S_k(N, \chi)$ is called a newform if it is an eigenfunction of $\tilde{W}$ and of all the Hecke operators $T_n$. An element $f$ in $S_k(N, \chi)$ is called an oldform if there is an element $f'$ in $S_k(N', \chi_{N'})$ such that $f(z) = f'(dz)$, where $N', d$ are positive integers satisfying $d|N'$ and $dN'|N$ and where $\chi_{N'}$ is the Dirichlet character of modulus $N'$ induced by $\chi$.

Let $f$ be a newform in $S_k(N, \chi)$ normalized so that its first Fourier coefficient is 1. Then it has the Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} \lambda(n)e^{2\pi inz}$$

with the Fourier coefficients equal to the eigenvalues of Hecke operators. Since $f$ is an eigenfunction of the involution $\tilde{W}$, we can assume that $\tilde{W}f = \eta f$ for constant $\eta$. Let

$$L_f(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}$$

for $\Re s > \frac{k+1}{2}$. It has the Euler product

$$L_f(s) = \prod_p (1 - \lambda(p)p^{-s} + \chi(p)p^{k-1-2s})^{-1}. \tag{1.2}$$

If we denote

$$\xi_f(s) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma\left(\frac{k-1}{2} + s\right)L_f\left(\frac{k-1}{2} + s\right), \tag{1.3}$$

then $\xi_f(s)$ is an entire function and satisfies the functional identity

$$\xi_f(s) = i^k \bar{\eta} \xi_f(1-\bar{s}).\tag{1.4}$$

When $\chi$ is primitive, we have $\eta = \tau(\bar{\chi})\lambda(N)N^{-k/2}$ with $\tau(\chi)$ being the Gauss sum for $\chi$; see Iwaniec [6].

Let

$$L_N(s) = \prod_{p|N} \det|1 - T(p)p^{-s} + \chi(p)p^{k-1-2s}I|^{-1} \tag{1.5}$$

where $I$ is the identity map acting on the space $S_k(N, \chi)$, and let

$$\xi_N(s) = N^{gs/2}(2\pi)^{-gs}\Gamma^g\left(\frac{k-1}{2} + s\right)L_N\left(\frac{k-1}{2} + s\right), \tag{1.6}$$

where $g$ is the dimension of the space $S_k(N, \chi)$. We will show that $\xi_N(s)$ is an entire function, and that zeros of $\xi_N(s)$ in the strip $0 < \Re s < 1$ appear in pairs $\rho$ and $1 - \bar{\rho}$.

Let

$$\tau_N(n) = \sum_{\rho} [1 - (1 - \frac{1}{\rho})^{-n}]$$

for $n = 1, 2, \cdots$, where the sum on $\rho$ runs over all zeros of $\xi_N(s)$ taken in the order given by $|3\rho| < T$ for $T \to \infty$ with a zero of multiplicity $\ell$ appearing $\ell$ times in the list. If $\rho = 0$ is a zero of $\xi_N(s)$, then $(1 - 1/\rho)^{-n}$ in (1.6) is interpreted to be 0.

In [2], Bombieri and Lagarias generalized a criterion of the author for the Riemann hypothesis [8] and obtained the following useful theorem.
Theorem 1.1. (Bombieri-Lagarias [2]) Let \( \mathcal{R} \) be a set of complex numbers \( \rho \), whose elements have positive integral multiplicities assigned to them, such that \( 1 \notin \mathcal{R} \) and

\[
\sum_{\rho} \frac{1 + |\Re \rho|}{(1 + |\rho|)^2} < \infty.
\]

Then the following conditions are equivalent:

1. \( \Re \rho \leq \frac{1}{2} \) for every \( \rho \) in \( \mathcal{R} \);
2. \( \sum_{\rho} \Re [1 - (1 - \frac{1}{\rho})^{-n}] \geq 0 \) for \( n = 1, 2, \ldots \).

As a corollary of Theorem 1.1 we obtain a criterion for the location of all non-trivial zeros of Hecke \( L \)-functions associated with all Hecke eigenforms which form an orthonormal basis in \( S_k(N, \chi) \).

Corollary 1.2. All zeros of \( \xi_N(s) \) in the strip \( 0 < \Re s < 1 \) lie on the critical line \( \Re s = 1/2 \) if, and only if, \( \tau_N(n) \geq 0 \) for all positive integers \( n \).

In order to state the main theorem, we need an explicit formula for the trace \( \text{tr} T(p^\ell) \) of Hecke operators \( T(p^\ell) \) acting on the space \( S_k(N, \chi) \) for all primes \( p \nmid N \) and for \( \ell = 1, 2, \ldots \). This formula is given by the Eichler-Selberg trace formula obtained in Oesterlé [12] (cf. Serre [13]). We denote by \( \varphi \) the Euler \( \varphi \)-function. Let

\[
\psi(N) = N \prod_{p \mid N} (1 + 1/p),
\]

and let \( \chi(\sqrt{n}) = 0 \) if \( n \) is not the square of an integer.

Theorem 1.3. (Théorème 3', [12]) For every positive integer \( n \), the trace \( \text{tr}(T(n)) \) of the Hecke operator \( T(n) \) acting on the space \( S_k(N, \chi) \) is given by

\[
\text{tr}(T(n)) = n^{\frac{k-1}{12}} \chi(\sqrt{n}) \sum_{\rho \neq \overline{\rho}, \rho^k - 1 - \rho^{k-1}} \sum_{m \in \mathbb{Z}^+} \frac{h((t^2 - 4n)/m^2)}{w((t^2 - 4n)/m^2)} \mu(t, n, m)
\]

\[
- \frac{1}{2} \sum_{0 < d | n} \min(d^{k-1}, (n/d)^{k-1}) \sum_{c \mid N, (c, N/c) = 1} \varphi((c, N/c)) \chi(y)
\]

where \( \rho, \overline{\rho} \) are the roots of \( x^2 - tx + n = 0 \), where the integer \( y \equiv d \pmod{c} \), \( y \equiv n/d \pmod{N/c} \), where

\[
\mu(t, n, m) = \frac{\psi(N)}{\psi(N/\gcd(N, m))} \sum_{x \equiv 0 \pmod{N}} \chi(x),
\]

and where \( h(f) \) and \( w(f) \) are respectively the class number and the number of units in the ring of integers of the imaginary quadratic field of discriminant \( f < 0 \).
Let $\gamma = 0.5772 \cdots$ be Euler's constant, let
\[
\Lambda(m) = \begin{cases} 
\ln p, & \text{if } m \text{ is a positive power of a prime } p; \\
0, & \text{otherwise},
\end{cases}
\]
and let $d(m)$ be the number of positive divisors of $m$. For $m|N$ and $f|m$, we denote by $\nu_m$ the dimension of the subspace generated by all newforms in $S_k(m, \chi_m)$, where $\chi_m$ is the Dirichlet character of modulus $m$ induced by the Dirichlet character $\chi$ of modulus $N$. In this paper we obtain the following explicit formula for the $\tau_N(n)'s$. The weight 2 case of this formula was obtained in Li [9].

**Theorem 1.4.** Let $\tau_N(n)$ be given in (1.6). Then we have
\[
\tau_N(n) = \frac{n}{2} \ln \left( N\nu_N \prod_{f|m, m|N} m^{\nu_m d(N/m)} \right) - ng(\ln 2 + \gamma + \frac{2}{k+1})

- \sum_{l=1}^{n} \binom{n}{l} \frac{(-1)^{l+1}}{(l-1)!} \sum_{m=1}^{\infty} \frac{\Lambda(m)}{m^{\frac{2}{k+1}}} B(m)(\ln m)^{l-1}

+ ng \sum_{l=1}^{\infty} \frac{k+1}{l(2l+k+1)} + g \sum_{m=2}^{n} \binom{n}{m} \sum_{l=1}^{\infty} \frac{(-1)^{m}}{(l+k-1/2)^m}.
\]
for all positive integers $n$, where $B(p^\ell) = tr(T(p^\ell)) - \chi(p)p^k tr(T(p^{\ell-2}))$ for $p \nmid N$.

This paper is organized as follows: An arithmetic formula is obtained in section 2 for a sequence of numbers, whose positivity implies that nontrivial zeros of Hecke $L$-functions associated with newforms lie on the critical line. This formula will be used in the proof of Theorem 1.4. In section 3 we give some preliminary results for the proof of Theorem 1.4. Finally, Theorem 1.4 is proved in section 4.

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### 2. Explicit formulas for Hecke $L$-functions

Let $f$ be a normalized newform in $S_k(N, \chi)$, and let $\xi_f(s)$ be given in (1.3). Put
\[
\tau_f(n) = \sum_{\rho} [1 - \left( 1 - \frac{1}{\rho} \right)^{-n}]
\]
for $n = 1, 2, \cdots$, where the sum is over all the zeros of $\xi_f(s)$ in the order given by $|3\rho| < T$ for $T \to \infty$ with a zero of multiplicity $\ell$ appearing $\ell$ times in the list.

Assume that $f$ is a normalized newform in $S_k(N, \chi)$. For each prime number $p$, let $\alpha_p$ and $\beta_p$ be the two roots of $T^2 - \lambda(p)T + \chi(p)p^{k-1}$ where $\lambda(p)$ is given in (1.2). Put
\[
b_f(p^m) = \begin{cases} 
\lambda(p)^m, & \text{if } p|N; \\
\alpha_p^m + \beta_p^m, & \text{if } (p, N) = 1.
\end{cases}
\]

The following arithmetic formula for $\tau_f(n)$ generalizes an arithmetic formula of Bombieri and Lagarias [2] for the Riemann zeta function.
Theorem 2.1. Assume that \( f \) is a normalized newform in \( S_k(N, \chi) \). If \( \tau_f(n) \) is given in (2.1), then we have
\[
\tau_f(n) = n \left( \ln \frac{\sqrt{N}}{2\pi} - \gamma \right) - \sum_{j=1}^{n} \binom{n}{j} \frac{(-1)^{j-1}}{(j-1)!} \sum_{l=1}^{\infty} \frac{\Lambda(l)}{l} b_f(k)(\ln l)^{j-1} \\
+ n \left( \frac{-2}{k+1} + \sum_{l=1}^{\infty} \frac{k+1}{l(2l+k+1)} \right) + \sum_{j=2}^{n} \binom{n}{j} (-1)^j \sum_{l=1}^{\infty} \frac{1}{(l + \frac{k-1}{2})^j},
\]
for \( n = 1, 2, \cdots \).

Lemma 2.2. (see [10]) Let \( F(x) \) be a function defined on the real line \( \mathbb{R} \) such that
\[
2F(x) = F(x + 0) + F(x - 0)
\]
for all \( x \in \mathbb{R} \), such that \( F(x) \exp((\epsilon + 1/2)|x|) \) is integrable and of bounded variation on \( \mathbb{R} \) for a constant \( \epsilon > 0 \), and such that \( (F(x) - F(0))/x \) is of bounded variation on \( \mathbb{R} \). Then
\[
\sum_{\rho} \Phi(\rho) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{k/2}} [b_f(n)F(\ln n) + \overline{b}_f(n)F(-\ln n)] \\
- \int_{0}^{\infty} (F(x) + F(-x)) \frac{e^{-kx/2}}{1 - e^{-x}} - 2F(0) \frac{e^{-x}}{x} \, dx,
\]
where the sum on \( \rho \) runs over all zeros of \( \xi_f(s) \) in the order given by \( |\Im\rho| < T \) for \( T \to \infty \), and
\[
\Phi(s) = \int_{-\infty}^{\infty} F(x)e^{(s-1/2)x} \, dx.
\]

Lemma 2.3. ([4] [5] [11]) Let \( f \) be a newform in \( S_k(N, \chi) \). Then there an absolute effective constant \( c > 0 \) such that \( L_f(s) \) has no zeros in the region
\[
\{ s = \sigma + it : \sigma \geq 1 - \frac{c}{\ln(N + 1 + |t|)} \},
\]
where \( L_f \) is given in (1.2).

Lemma 2.4. (Lemma 2 of [2]) For \( n = 1, 2, \cdots \), let
\[
F_n(x) = \begin{cases} 
\frac{e^{x/2}}{\sum_{j=1}^{n} \binom{n}{j} \frac{x^{j-1}}{(j-1)!}}, & \text{if } -\infty < x < 0; \\
n/2, & \text{if } x = 0; \\
0, & \text{if } 0 < x.
\end{cases}
\]
Then
\[
\Phi_n(s) = 1 - \left( 1 - \frac{1}{s} \right)^n
\]
where \( \Phi_n \) is related to \( F_n \) by the relation
\[
\Phi_n(s) = \int_{-\infty}^{\infty} F_n(x)e^{(s-1/2)x} \, dx.
\]
Proof of Theorem 2.1. Since \( \xi_f(s) \) is an entire function of order one and satisfies the functional identity \( \xi_f(s) = i^k \tilde{\xi}_f(1 - \bar{s}) \), we have

\[
\xi_f(s) = i^k \tilde{\xi}_f(1) \prod_{\rho} (1 - s/\rho)
\]

where the product is over all the zeros of \( \xi_f(s) \) in the order given by \( |3\rho| < T \) for \( T \to \infty \). If \( \varphi_f(z) = \xi_f(1/(1 - z)) \), then

\[
\frac{\varphi_f'(z)}{\varphi_f(z)} = \sum_{n=0}^{\infty} \tau_f(n + 1) z^n
\]

where the coefficients \( \tau_f(n) \) are given in (2.1).

For a sufficiently large positive number \( X \) that is not an integer, let

\[
F_{n,X}(x) = \begin{cases} 
F_n(x), & \text{if } -\ln X < x < \infty; \\
\frac{1}{2} F_n(-\ln X), & \text{if } x = -\ln X; \\
0, & \text{if } -\infty < x < -\ln X
\end{cases}
\]

where \( F_n(x) \) is given in Lemma 2.4. Then \( F_{n,X}(x) \) satisfies all conditions of Lemma 2.2. Let

\[
\Phi_{n,X}(s) = \int_{-\infty}^{\infty} F_{n,X}(x) e^{(s-1/2)x} dx.
\]

By Lemma 2.2, we obtain that

\[
\sum_{\rho} \Phi_{n,X}(\rho) = 2 F_{n,X}(0) \ln \frac{\sqrt{N}}{2\pi} - \sum_{l=1}^{\infty} \frac{\Lambda(l)}{l^{s/2}} [b_f(l) F_{n,X}(\ln l) + \bar{b}_f(l) F_{n,X}(-\ln l)]
\]

\[
- \int_{0}^{\infty} \left( \frac{F_{n,X}(x) + F_{n,X}(-x)}{1 - e^{-x}} e^{-kx/2} - 2 F_{n,X}(0) \frac{e^{-x}}{x} \right) dx,
\]

where the sum on \( \rho \) runs over all zeros of \( \xi_f(s) \) in the order given by \( |3\rho| < T \) for \( T \to \infty \). It follows that

\[
\lim_{X \to \infty} \sum_{\rho} \Phi_{n,X}(\rho)
\]

\[
= n \left( \ln \frac{\sqrt{N}}{2\pi} - \gamma \right) - \sum_{j=1}^{\infty} \binom{n}{j} (-1)^{j-1} \sum_{l=1}^{\infty} \frac{\Lambda(l)}{l^{j/2}} \bar{b}_f(k)(\ln l)^{j-1}
\]

\[
+ n \left( -\frac{2}{k+1} + \sum_{l=1}^{\infty} \frac{k+1}{l(2l+k+1)} \right) + \sum_{j=2}^{\infty} \binom{n}{j} (-1)^{j-1} \sum_{l=1}^{\infty} \frac{1}{(l + \frac{2j-1}{2})^2}.
\]

We have

\[
\Phi_n(s) - \Phi_{n,X}(s) = X^{-s} \sum_{j=1}^{n} \binom{n}{j} (-1)^{j-1} \sum_{m=0}^{j-1} \frac{(\ln X)^{j-m-1}}{(j-m-1)!} s^{-m-1}
\]

\[
= X^{-s} \sum_{j=1}^{n} \binom{n}{j} (-\ln X)^{j-1} \frac{1}{(j-1)!} + O \left( \frac{(\ln X)^{n-2}}{|s|^2} X^{-R_s} \right)
\]

(2.4)
for \( \Re s > 0 \).

Let \( \rho \) be any zero of \( \xi_f(s) \). By Lemma 2.3, we have

\[
\frac{c}{\ln(N + 1 + |\rho|)} \leq \Re \rho \leq 1 - \frac{c}{\ln(N + 1 + |\rho|)}
\]

for a positive constant \( c \). An argument similar to that made in the proof of (3.9) of [2] shows that

\[
(2.5) \sum_{\rho} X^{-\Re \rho} \frac{1}{|\rho|^2} \ll e^{-c' \sqrt{\ln X}}
\]

for a positive constant \( c' \).

Since

\[
\sum_{\rho} X^{-\rho} \frac{1}{\rho} = \sum_{\rho} X^{-(1-\bar{\rho})} - \frac{1}{X} \sum_{\rho} \frac{X^{\bar{\rho}}}{\bar{\rho}} + O \left( \sum_{\rho} \frac{X^{-(1-\Re \rho)}}{|\rho|^2} \right)
\]

\[
= -\frac{1}{X} \sum_{\rho} \frac{X^{\bar{\rho}}}{\bar{\rho}} + O \left( e^{-c' \sqrt{\ln X}} \right),
\]

and since

\[
\lim_{X \to \infty} \frac{\ln(X)^{j-1}}{X} \sum_{\rho} \frac{X^{\bar{\rho}}}{\bar{\rho}} = 0
\]

for \( j = 1, 2, \cdots, n \) by Theorem 4.2 and Theorem 5.2 of [11], we have

\[
(2.6) \lim_{X \to \infty} \frac{\ln(X)^{j-1}}{X} \sum_{\rho} \frac{X^{-\rho}}{\rho} = 0
\]

for \( j = 1, 2, \cdots, n \). It follows from (2.4), (2.5) and (2.6) that

\[
\lim_{X \to \infty} \sum_{\rho} \Phi_{n,X}(\rho) = \sum_{\rho} \Phi_{n}(\rho).
\]

Hence, we have

\[
\sum_{\rho} \Phi_{n}(\bar{\rho}) = n \left( \ln \frac{\sqrt{N}}{2\pi} - \gamma \right) - \sum_{j=1}^{n} \binom{n}{j} \frac{(-1)^{j-1}}{(j-1)!} \sum_{l=1}^{\infty} \frac{\Lambda(l)}{l^{j+1}} b_f(k) \ln(l)^{j-1}
\]

\[
+ n \left( \frac{-2}{k+1} + \sum_{l=1}^{\infty} \frac{k+1}{l(2l+k+1)} \right) + \sum_{j=2}^{n} \binom{n}{j} (-1)^{j} \sum_{l=1}^{\infty} \frac{1}{(l+\frac{k-1}{2})^j}
\]

Since zeros of \( \xi_f(s) \) appear in pairs \( \rho \) and \( 1 - \bar{\rho} \), we have

\[
\sum_{\rho} \Phi_{n}(\bar{\rho}) = \sum_{\rho} \left[ 1 - \frac{1}{\bar{\rho}} \right]^n
\]

\[
= \sum_{\rho} \left[ 1 - \frac{1}{1 - \rho} \right]^n
\]

\[
= \sum_{\rho} \left[ 1 - (1 - 1/\rho)^{-n} \right] = \tau_f(n)
\]

for \( n = 1, 2, \cdots \), where the sum is over all the zeros of \( \xi_f(s) \) in the order given by \( |\Im \rho| < T \) for \( T \to \infty \) with a zero of multiplicity \( \ell \) appearing \( \ell \) times in the list.

This completes the proof of the theorem. □
3. Preliminary results

A fundamental result of Hecke asserts that a basis \( \{ f_1, f_2, \ldots, f_g \} \) in \( S_k(N, \chi) \) exists which consists of eigenfunctions of all the Hecke operators \( T(n) \) with \((n,N) = 1\); see Theorem 6.21 in Iwaniec [6]. We can assume that each \( f_j \) is either a normalized newform in \( S_k(N, \chi) \) or coming from a normalized newform in a lower level. For \( j = 1, \ldots, g \), we choose \( g_j = f_j \) if \( f_j \) is a normalized newform in \( S_k(N, \chi) \), and \( g_j = f'_j \) if \( f_j \) is an oldform in \( S_k(N, \chi) \) and if \( f'_j \) is a normalized newform in \( S_k(N', \chi_{N'}) \) for some divisor \( N' \) of \( N \) such that \( f_j(z) = f'_j(dz) \) for some positive integer \( d|N/N' \), where \( \chi_{N'} \) is the Dirichlet character of modulus \( N' \).

Let

\[
\xi_H(s) = \prod_{j=1}^g \xi_{g_j}(s)
\]

where \( \xi_{g_j}(s) \) is defined as in (1.3). Since \( \xi_{g_j}(s) \) is an entire function and satisfies the functional identity \( \xi_{g_j}(s) = w_j \bar{\xi}_{g_j}(1-\bar{s}) \) for a constant \( w_j \), the function \( \xi_H(s) \) is entire and satisfies the functional identity

\[
\xi_H(s) = w \bar{\xi}_H(1-\bar{s})
\]

for a constant \( w \). Put

\[
\tau_H(n) = \sum_{j=1}^g \tau_{g_j}(n),
\]

where \( \tau_{g_j}(n) \) is defined similarly as in (2.1). If \( \varphi_H(z) = \xi_H(1/(1-z)) \), then we have

\[
\frac{\varphi'_H(z)}{\varphi_H(z)} = \sum_{n=0}^{\infty} \tau_H(n+1)z^n
\]

by (2.3).

**Lemma 3.1.** For all positive integers \( n \) with \((n,N) = 1\), we have

\[
(T(n)f_j)(z) = \lambda_{g_j}(n)f_j(z)
\]

for \( j = 1, 2, \ldots, g \), where \( \lambda_{g_j}(n) \) is the eigenvalue of \( T(n) \) acting on \( g_j(z) \).

**Proof.** If \( f_j \) is a newform, the stated identity is trivially true.

Next, we assume that \( f_j \) is an oldform. Let \( g_j \) be a normalized newform in \( S_k(N', \chi_{N'}) \) for some divisor \( N' \) of \( N \) with \( f|N' \) such that \( f_j(z) = g_j(dz) \) for some positive integer \( d|N/N' \). Since \((n,N) = 1\), by (1.1) we have

\[
(T(n)f)(z) = \frac{1}{n} \sum_{ad=n} \chi(a) \sum_{0 \leq b < d} f \left( \frac{az+b}{d} \right)
\]
for any function \( f \) in \( S_k(N, \chi) \). Thus, we have

\[
(T(n)f_j)(z) = \frac{1}{n} \sum_{\alpha \delta = n} \chi(\alpha) \alpha^k \sum_{0 \leq \beta < \delta} g_j \left( \frac{\alpha dz + d\beta}{\delta} \right).
\]

Since \((n, N) = 1\), \(d|N\) and \(\alpha \delta = n\), we have \((\delta, d) = 1\). Let \(r_\beta\) be remainder of \(d\beta\) modulo \(\delta\). Then \(\{r_\beta : 0 \leq \beta < \delta\} = \{0, 1, \cdots, \delta - 1\}\). Since \(g_j \in S_k(N', \chi_{N'})\), we have

\[
g_j \left( \frac{\alpha dz + d\beta}{\delta} \right) = g_j \left( \frac{\alpha dz + r_\beta}{\delta} \right).
\]

Note that \(\chi(t) = \chi_{N'}(t)\) for any positive integer \(t\) with \((t, N) = 1\). It follows that

\[
(T(n)f_j)(z) = \frac{1}{n} \sum_{\alpha \delta = n} \chi_{N'}(\alpha) \alpha^k \sum_{0 \leq \beta < \delta} g_j \left( \frac{\alpha dz + \beta}{\delta} \right)
= (T(n)g_j)(w) = \lambda_{g_j}(n)g_j(w) = \lambda_{g_j}(n)\bar{f}_j(z)
\]

where \(w = dz\).

This completes the proof of the lemma. \(\square\)

**Lemma 3.2.** Let \(\xi_N(s)\) be given in (1.5). Then \(\xi_N(s)\) is an entire function, and its zeros in the strip \(0 < \Re s < 1\) appear in pairs \(\rho\) and \(1 - \rho\).

**Proof.** For \(j = 1, 2, \cdots, g\), let \(g_j\) be a normalized newform in \(S_k(N_j, \chi_j)\) where \(\chi_j\) is the Dirichlet character of modulus \(N_j\) induced by the Dirichlet character \(\chi\) of modulus \(N\). Then we can write

\[
L_{g_j}(s) = \prod_p \left( 1 - \lambda_{g_j}(p)p^{-s} + \chi_j(p)p^{k-1-2s} \right)^{-1}.
\]

Since \(f_1, f_2, \cdots, f_g\) are eigenfunctions of all the Hecke operators \(T(n)\) with \((n, N) = 1\), by Lemma 3.1 we have

\[
\det \left| 1 - T(p)p^{-s} + \chi(p)p^{k-1-2s} I \right| = \prod_{j=1}^{g} \left( 1 - \lambda_{g_j}(p)p^{-s} + \chi(p)p^{k-1-2s} \right)
\]

for any prime \(p \nmid N\). Let \(L_N(s)\) be given in (1.4). Then we have

\[
L_N(s) = \prod_{j=1}^{g} \prod_{p \nmid N} \left( 1 - \lambda_{g_j}(p)p^{-s} + \chi(p)p^{k-1-2s} \right)^{-1}.
\]

Since \(\chi(p) = \chi_j(p)\) for \(j = 1, \cdots, g\) when \(p \nmid N\), we have

\[
L_N(s) = \prod_{j=1}^{g} (L_{g_j}(s) \prod_{p \nmid N_j} \left( 1 - \lambda_{g_j}(p)p^{-s} \right) \prod_{p \nmid N_j, p \nmid N} \left( 1 - \lambda_{g_j}(p)p^{-s} + \chi_j(p)p^{k-1-2s} \right)).
\]

Since

\[
\xi_{g_j}(s) = N_j^{s/2}(2\pi)^{-s} \Gamma \left( \frac{k-1}{2} + s \right) L_{g_j} \left( \frac{k-1}{2} + s \right),
\]

AN EXPLICIT FORMULA FOR HECKE L-FUNCTIONS
we have
\[ \xi_H(s) = A^{s/2} N^{gs/2} (2\pi)^{-g} \Gamma^g \left( \frac{k-1}{2} + s \right) \prod_{j=1}^{g} L_{g_j} \left( \frac{k-1}{2} + s \right) \]

where \( A = N^{-g} \prod_{j=1}^{g} N_j \). It follows that
\[ (3.6) \quad \xi_N(s) = \xi_H(s) A^{-s/2} \times \prod_{j=1}^{g} \left( \prod_{p|N_j} (1 - \lambda_{g_j}(p) p^{\frac{k-1}{2} - s}) \prod_{p|N, p|N_j} (1 - \lambda_{g_j}(p) p^{\frac{k-1}{2} - s} + \chi_j(p) p^{-2s}) \right). \]

This implies that \( \xi_N(s) \) is an entire function.

Assume \( p|N_j \). By Theorem 3 of Li [7], if \( \chi_j \) is not a character mod \( N_j/p \) then \( |\lambda_{g_j}(p)| = p^{\frac{k-2}{2}} \), and if \( \chi_j \) is a character mod \( N_j/p \) then \( \lambda_{g_j}(p) = 0 \) when \( p^2|N_j \) and \( \lambda_{g_j}(p) = p^{\frac{k-2}{2}} \) when \( p^2 \nmid N_j \). It follows that all zeros of the function
\[ \prod_{j=1}^{g} \prod_{p|N_j} (1 - \lambda_{g_j}(p) p^{\frac{k-1}{2} - s}) \]
lie on the lines \( \Re s = 0, -1/2 \).

Assume that \( p \nmid N_j \) and \( p|N \). Since the two roots of the polynomial \( 1 - \lambda_{g_j}(p) p^{\frac{k-1}{2}} z + \chi_j(p) z^2 \) for \( p \nmid N_j \) are of absolute value one by the Ramanujan conjecture which was proved in Théorème 8.2 of Deligne [3], all zeros of the function
\[ \prod_{j=1}^{g} \prod_{p|N_j, p|N} \left( 1 - \lambda_{g_j}(p) p^{\frac{k-1}{2} - s} + \chi_j(p) p^{-2s} \right) \]
lie on the line \( \Re s = 0 \). Therefore, it follows from (3.2) and (3.6) that zeros of \( \xi_N(s) \) in the critical strip \( 0 < \Re s < 1 \) appear in pairs \( \rho, 1 - \overline{\rho} \).

This completes the proof of the lemma. □

**Proof of Corollary 1.2.** Let \( \tau_N(n) \) be defined by (1.6) for all positive integers \( n \). Since \( \xi_H(s) \) is an entire function of order one, by (3.6) \( \xi_N(s) \) is an entire function of order one. This implies that
\[ \sum_{\rho} \frac{1 + |\Re \rho|}{(1 + |\rho|)^2} < \infty, \]
where the sum is over all zeros \( \rho \) of \( \xi_N(s) \). Thus, conditions of Theorem 1.1 are satisfied. Since by (3.6) all zeros of \( \xi_N(s) \) outside the strip \( 0 < \Re s < 1 \) lie on the lines \( \Re s = 0, -1/2 \), Theorem 1.1 implies that all zeros of \( \xi_N(s) \) in the critical strip \( 0 < \Re s < 1 \) satisfy \( \Re s \leq 1/2 \) if, and only if, \( \tau_N(n) \geq 0 \) for all positive integers \( n \). By Lemma 3.2, all zeros of \( \xi_N(s) \) in the critical strip \( 0 < \Re s < 1 \) appear in pairs \( \rho \) and \( 1 - \overline{\rho} \). Thus, \( \tau_N(n) \geq 0 \) for all positive integers \( n \) if, and only if, \( \Re \rho \leq 1/2 \) and \( \Re (1 - \overline{\rho}) \leq 1/2 \) for all zeros \( \rho \) of \( \xi_N(s) \) in the critical strip \( 0 < \Re s < 1 \). That is, all zeros of \( \xi_N(s) \) in the strip \( 0 < \Re s < 1 \) lie on the critical line \( \Re s = 1/2 \) if, and only if, \( \tau_N(n) \geq 0 \) for all positive integers \( n \).

This completes the proof of the corollary. □
**Lemma 3.3.** Let $p$ be a prime, and let $\alpha$ be a complex number. Then we have

$$1 - \alpha p^{-s} = c_p s^\epsilon \prod_{\rho} (1 - s/\rho)$$

where the product on $\rho$ is over all nonzero zeros of $1 - \alpha p^{-s}$ taken in the order given by $|\rho| < T$ for $T \to \infty$ and where $c_p = 1 - \alpha, \epsilon_p = 0$ if $\alpha \neq 1$ and $c_p = \ln p, \epsilon_p = 1$ if $\alpha = 1$.

**Proof.** Since $1 - \alpha p^{-s}$ is an entire function of order one, by Hadamard’s factorization theorem there is a constant $a$ such that

$$(3.7) \quad 1 - \alpha p^{-s} = c_p e^{as} s^\epsilon \prod_{\rho} (1 - s/\rho) e^{s/\rho}$$

where the product is over all nonzero zeros of $1 - \alpha p^{-s}$. Let $\alpha = e^{it}$ with $0 \leq \Re t < 2\pi$. Then the zeros of $1 - \alpha p^{-s}$ are $i(t + 2k\pi)/\ln p, k = 0, \pm 1, \pm 2, \cdots$. Since

$$\sum_{k=1}^{\infty} \left( -i \ln p \frac{-i \ln p}{t + 2k\pi} + \frac{1}{t - 2k\pi} \right) = 2it \ln p \sum_{k=1}^{\infty} \frac{1}{(2k\pi)^2 - t^2}$$

is absolutely convergent, by using (3.7) we can write

$$(3.8) \quad 1 - \alpha p^{-s} = c_p e^{hs} s^\epsilon \prod_{\rho} (1 - s/\rho)$$

for a constant $h$, where the product runs over all nonzero zeros $\rho$ of $1 - \alpha p^{-s}$ taken in the order given by $|\rho| < T$ for $T \to \infty$. By taking logarithmic derivative of both sides of (3.8) with respect to $s$ we get

$$(3.9) \quad \frac{\alpha \ln p}{p^s - \alpha} = h + \frac{\epsilon_p}{s} + \sum_{\rho} \frac{1}{s - \rho}.$$ 

By letting $s \to \infty$ in (3.9) we find that $h = 0$. Then the stated identity follows from (3.8).

This completes the proof of the lemma. □

**Lemma 3.4.** Let $\alpha, p$ be given in Lemma 3.3, and let $s = (1 - z)^{-1}$. Then we have

$$\frac{d}{dz} \ln(1 - \alpha p^{-s}) = \sum_{n=0}^{\infty} \left( \sum_{\rho} [1 - (1 - 1/\rho)^{-n+1}] \right) z^n$$

for $z$ in a small neighborhood of the origin, where the sum on $\rho$ is over all zeros of $1 - \alpha p^{-s}$ taken in the order given by $|\rho| < T$ for $T \to \infty$.

**Proof.** By Lemma 3.3 we have

$$(3.10) \quad \frac{d}{dz} \ln(1 - \alpha p^{-s}) = \frac{\epsilon_p}{1 - z} + \sum_{\rho} \frac{1}{(1 - \rho) + \rho z} \frac{1}{1 - z}$$
where the sum on $\rho$ is over all nonzero zeros of $1 - \alpha p^{-s}$. Since
\[
\frac{1}{(1 - \rho) + \rho z} \frac{1}{1 - z} = \frac{1}{1 - \rho} \left\{ \sum_{k=0}^{\infty} \left( \frac{-\rho}{1 - \rho} \right)^k z^k \right\} \left\{ \sum_{l=0}^{\infty} z^l \right\}
\]
\[
= \sum_{n=0}^{\infty} \left[ 1 - (1 - \rho)^{-n-1} \right] z^n
\]
for $z$ in a small neighborhood of the origin, the stated identity follows from (3.10).

Note that, if $s = \rho = 0$ is a zero of $1 - \alpha p^{-s}$, then $(1 - 1/\rho)^{-n-1}$ is interpreted to be 0 for all positive integers $n$.

This completes the proof of the lemma. □

Remark. By (3.6), Lemma 3.4, (2.3) and (3.1) we have
\[
\frac{d}{dz} \log \left( A^{s/2} \xi_N(s) \right) = \sum_{n=0}^{\infty} \tau_N(n+1) z^n
\]
with $s = (1 - z)^{-1}$, where the $\tau_N(n)$'s are given in (1.6). Then, by Lemma 3.2, to prove that all zeros of $\xi_N(s)$ in the strip $0 < \Re s < 1$ lie on the critical line $\Re s = 1/2$ it is enough to find an upper bound for each $\tau_N(n)$ which implies that the above series is analytic for $|z| < 1$.

Lemma 3.5. Let $\alpha, p$ be given in Lemma 3.3, and let $s = (1 - z)^{-1}$. Then we have
\[
\frac{d}{dz} \ln(1 - \alpha p^{-s}) = \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \binom{n+1}{j+1} (-1)^j \frac{\ln p}{j!} \sum_{m=1}^{\infty} \ln p^m \alpha^m (\ln p^m)^j \right) z^n
\]
for $z$ in a small neighborhood of the origin.

Proof. Let $m$ be any positive integer. By using mathematical induction on $n$, we can show that
\[
(3.11) \quad \frac{d^n}{dz^n} [(1 - z)^{-m} p^{-ks}]_{z=0} = p^{-k} \sum_{j=0}^{n} \binom{n}{j} (n + m - 1) \cdots (j + m)(-\ln p^k)^j
\]
for $n = 1, 2, \cdots$. By using the formula (3.11) with $m = 2$ we find that
\[
\frac{d}{dz} \ln(1 - \alpha p^{-s}) = (1 - z)^{-2} \ln p \sum_{k=1}^{\infty} \alpha^k p^{-ks}
\]
\[
= \sum_{n=0}^{\infty} \frac{z^n}{n!} \left( \sum_{k=1}^{\infty} \alpha^k \frac{\ln p}{p^k} \sum_{j=0}^{n} \binom{n}{j} (n+1) \cdots (j+2)(-\ln p^k)^j \right)
\]
\[
= \sum_{n=0}^{\infty} \frac{z^n}{n!} \left( \sum_{j=0}^{n} \binom{n+1}{j+1} (-1)^j \frac{\ln p}{j!} \sum_{k=1}^{\infty} \alpha^k (\ln p^k)^j \right)
\]
This completes the proof of the lemma. □
Lemma 3.6. Let \( p \) be a prime, and let \( \alpha \) be a complex number. Then we have
\[
\sum_{\rho} [1 - (1 - 1/\rho)^{-n}] = \sum_{j=1}^{n} \binom{n}{j} \frac{(-1)^{j-1}}{(j-1)!} \sum_{m=1}^{\infty} \frac{\ln p}{p^m} \alpha^m (\ln p^m)^{j-1}
\]
for \( n = 1, 2, \cdots \), where the sum on \( \rho \) is over all zeros of \( 1 - \alpha p^{-s} \) taken in the order given by \( |3| \rho < T \) for \( T \to \infty \) with a zero of multiplicity \( \ell \) appearing \( \ell \) times in the list.

Proof. The stated identity follows from Lemma 3.4 and Lemma 3.5. \( \square \)

Lemma 3.7. For \( n = 1, 2, \cdots \), we have
\[
\tau_N(n) = \sum_{j=1}^{g} \tau_{g_j}(n) + \sum_{l=1}^{n} \binom{n}{l} \frac{(-1)^{l-1}}{(l-1)!} \sum_{m=1}^{\infty} \frac{\Lambda(m)}{m} (\sum_{j=1}^{g} b_{g_j}(m))(\ln m)^{l-1}
\]
where \( b_{g_j}(m) \) is given as in (2.2).

Proof. Let \( \tau_H(n) \) be given in (3.3). By (2.1), (3.1) and (3.6) we have
\[
(3.12) \quad \tau_N(n) = \tau_H(n) + \sum_{j=1}^{g} \sum_{p|N_j} \sum_{\rho_j} [1 - (1 - 1/\rho_j)^{-n}] + \sum_{p|N_j, p|N} \sum_{\alpha_j} [1 - (1 - 1/\alpha_j)^{-n}],
\]
where the sum on \( \rho_j \) is over all zeros of \( 1 - \lambda_{g_j}(p)p^{1/s} \) with \( p|N_j \) and where the sum on \( \alpha_j \) is over all zeros of \( 1 - \lambda_{g_j}(p)p^{1/s} + \chi_j(p)p^{-2s} \) with \( p \nmid N_j, p|N \). By Lemma 3.6 we have
\[
(3.13) \quad \sum_{\rho_j} [1 - (1 - 1/\rho_j)^{-n}] = \sum_{l=1}^{n} \binom{n}{l} \frac{(-1)^{l-1}}{(l-1)!} \sum_{m=1}^{\infty} \frac{\ln p}{p^{m+1}} b_{g_j}(p^m)(\ln p^m)^{l-1}
\]
and
\[
(3.14) \quad \sum_{\alpha_j} [1 - (1 - 1/\alpha_j)^{-n}] = \sum_{l=1}^{n} \binom{n}{l} \frac{(-1)^{l-1}}{(l-1)!} \sum_{m=1}^{\infty} \frac{\ln p}{p^{m+1}} b_{g_j}(p^m)(\ln p^m)^{l-1}.
\]
It follows from (3.13) and (3.14) that
\[
(3.15) \quad \sum_{p|N_j} \sum_{\rho_j} [1 - (1 - 1/\rho_j)^{-n}] + \sum_{p|N_j, p|N} \sum_{\alpha_j} [1 - (1 - 1/\alpha_j)^{-n}]
\]
\[
= \sum_{p|N} \sum_{l=1}^{n} \binom{n}{l} \frac{(-1)^{l-1}}{(l-1)!} \sum_{m=1}^{\infty} \frac{\ln p}{p^{m+1}} b_{g_j}(p^m)(\ln p^m)^{l-1}.
\]
By (3.15) we have
\[
(3.16) \quad \sum_{j=1}^{g} \left( \sum_{p|N_j} \sum_{\rho_j} [1 - (1 - 1/\rho_j)^{-n}] + \sum_{p|N_j, p|N} \sum_{\alpha_j} [1 - (1 - 1/\alpha_j)^{-n}] \right)
\]
\[
= \sum_{l=1}^{n} \binom{n}{l} \frac{(-1)^{l-1}}{(l-1)!} \sum_{m=1}^{\infty} \frac{\Lambda(m)}{m+1} (\sum_{j=1}^{g} b_{g_j}(m))(\ln m)^{l-1}.
\]
The stated identity then follows from (3.12).

This completes the proof of the lemma. \( \square \)
4. Proof of Theorem 1.4

We define an operator $S$ acting on the space $S_k(N, \chi)$ by

\begin{align}
S(1) &= 2I \\
S(p) &= T(p) \\
S(p^m) &= T(p^m) - \chi(p)p^{k-1}T(p^{m-2})
\end{align}

for $m = 2, 3, \cdots$.

**Lemma 4.1.** For each prime $p \mid N$, the trace $\text{tr}(S(p^m))$ of $S(p^m)$ acting on the space $S_k(N, \chi)$ is given by

\begin{align}
\text{tr}(S(p^m)) = \sum_{j=1}^{g} b_{g_j}(p^m)
\end{align}

for $m = 0, 1, 2, \cdots$.

**Proof.** Let $p$ be any prime. It follows from the recursion formula (see (6.25) in Iwaniec [6])

\[ T(p^{m+1}) = T(p)T(p^m) - \chi(p)p^{k-1}T(p^{m-1}) \]

that

\begin{align}
S(p^{m+1}) &= S(p)S(p^m) - \chi(p)p^{k-1}S(p^{m-1})
\end{align}

for $m = 1, 2, \cdots$.

Let $p$ be any prime not dividing $N$. When $m = 0$, we have

\[ S(p^m)f_j = 2f_j = b_{g_j}(p^m)f_j. \]

When $m = 1$, we have

\[ S(p^m)f_j = T(p)f_j = \lambda_{g_j}(p)f_j = b_{g_j}(p^m)f_j \]

by Lemma 3.1. Assume that

\[ S(p^m)f_j = b_{g_j}(p^m)f_j \]

for all integers $m \leq l$. When $m = l + 1$, we have

\[ S(p^m)f_j = (S(p)S(p^l) - \chi(p)p^{k-1}S(p^{l-1}))(f_j) \]

\[ = (b_{g_{j}}(p)b_{g_j}(p^l) - \chi(p)p^{k-1}b_{g_j}(p^{l-1}))(f_j) \]

\[ = b_{g_j}(p^m)f_j. \]

By mathematical induction the identity

\[ S(p^m)f_j = b_{g_j}(p^m)f_j \]

holds for all nonnegative integers $m$. Since $\{f_1, \cdots, f_g\}$ is a basis for $S_k(N, \chi)$, we have

\[ \text{tr}(S(p^m)) = \sum_{j=1}^{g} b_{g_j}(p^m) \]

for $m = 0, 1, 2, \cdots$.

This completes the proof of the lemma. $\square$

From Lemma 4.1 and the definition (4.1) we obtain the following corollary.
Corollary 4.2. For each prime $p \nmid N$, we have
\[
\sum_{j=1}^{g} b_{g_j}(p^m) = \text{tr}(T(p^m)) - \chi(p)p^{k-1} \text{tr}(T(p^{m-2})
\]
for $m = 0, 1, 2, \cdots$.

Proof of Theorem 1.4. By Lemma 3.7 we have
\[
(4.4)
\]
\[
\tau_N(n) = \sum_{j=1}^{g} \tau_{g_j}(n) + n \sum_{l=1}^{n} \left( \frac{n}{l} \right) \frac{(-1)^{l-1}}{(l-1)!} \sum_{m=1}^{\infty} \frac{\Lambda(m)}{m^{k+1}} \left( \sum_{j=1}^{g} b_{g_j}(m) \right) (\ln m)^{l-1}
\]
for $n = 1, 2, \cdots$. Since $g_j$ is a normalized newform in $S_k(N_j, \chi)$, by Theorem 2.1 we have
\[
(4.5)
\]
\[
\tau_{g_j}(n) = n \left( \ln \frac{\sqrt{N_j}}{2\pi} - \gamma \right) - \sum_{l=1}^{\infty} \left( \frac{n}{l} \right) \frac{(-1)^{l-1}}{(l-1)!} \sum_{m=1}^{\infty} \frac{\Lambda(m)}{m^{k+1}} \left( \sum_{j=1}^{g} b_{g_j}(m) \right) (\ln m)^{l-1}
\]
+ $n \left( -\frac{2}{k+1} + \sum_{l=1}^{\infty} \frac{k+1}{l(l+2k+1)} \right) + \sum_{m=2}^{n} \left( \frac{n}{m} \right) (-1)^{m} \sum_{l=1}^{\infty} \frac{1}{(l+\frac{k-1}{2})^m}$

By using (4.4) and (4.5) we obtain that
\[
(4.6)
\]
\[
\tau_N(n) = \frac{n}{2} \ln(N_1 \cdots N_g) - \sum_{l=1}^{\infty} \left( \frac{n}{l} \right) \frac{(-1)^{l-1}}{(l-1)!} \sum_{m=1}^{\infty} \frac{\Lambda(m)}{m^{k+1}} \left( \sum_{j=1}^{g} b_{g_j}(m) \right) (\ln m)^{l-1}
\]
\[
- ng[\ln 2\pi + \gamma + \frac{2}{k+1} - \sum_{l=1}^{\infty} \frac{k+1}{l(l+2k+1)}] + g \sum_{m=2}^{n} \left( \frac{n}{m} \right) \sum_{l=1}^{\infty} \frac{(-1)^{m}}{(l+\frac{k-1}{2})^m}.
\]

By the argument in the proof of Theorem 5 in Atkin and Lehner [1], we have
\[
(4.7)
\]
\[
\prod_{N_j \neq N, 1 \leq j \leq g} N_j = \prod_{m|N} m^{v_m d(N/m)}.
\]

We also have
\[
(4.8)
\]
\[
\prod_{N_j = N, 1 \leq j \leq g} N_j = N^{v_N}.
\]

By (4.6), (4.7) and (4.8) we have
\[
\tau_N(n) = \frac{n}{2} \ln \left( N^{v_N} \prod_{l|m, m|N} m^{v_m d(N/m)} \right) - ng[\ln 2\pi + \gamma + \frac{2}{k+1}]
\]
\[
- \sum_{l=1}^{n} \left( \frac{n}{l} \right) (-1)^{l-1} \sum_{m=1}^{\infty} \frac{\Lambda(m)}{m^{k+1}} \left( \sum_{j=1}^{g} b_{g_j}(m) \right) (\ln m)^{l-1}
\]
\[
+ ng \sum_{l=1}^{\infty} \frac{k+1}{l(l+2k+1)} + g \sum_{m=2}^{n} \left( \frac{n}{m} \right) \sum_{l=1}^{\infty} \frac{(-1)^{m}}{(l+\frac{k-1}{2})^m}.
\]
The stated identity then follows from Corollary 4.2. This completes the proof of the theorem. □

Remark. If $\chi$ is a primitive Dirichlet character of modulus $N$, we define

$$L_N(s) = \prod_p \det |1 - T(p)p^{-s} + \chi(p)p^{k-1-2s}I|^{-1}$$

where $I$ is the identity map acting on the space $S_k(N, \chi)$ and where the product on $p$ runs over all prime numbers. Let

$$\xi_N(s) = N^{gs/2}(2\pi)^{-gs/2}\Gamma_g\left(\frac{k-1}{2} + s\right)L_N\left(\frac{k-1}{2} + s\right),$$

where $g$ denotes the dimension of the space $S_k(N, \chi)$. Since the space $S_k(N, \chi)$ contains only newforms (see §6.7 of Iwaniec [6]), we have $\xi_N(s) = \xi_H(s)$ by (3.6). Hence, $\xi_N(s)$ is an entire function and satisfies the functional identity

$$\xi_N(s) = w\xi_N(1 - \overline{s}),$$

where $w$ is a constant.

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