DISCRETE GROUPS WITHOUT FINITE QUOTIENTS

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Abstract. We construct an infinite discrete subgroup of the isometry group of $\mathbb{H}^3$ with no finite quotients other than the trivial group.

It is well-known that every finitely generated linear group is residually finite [9]. Finite generation is definitively necessary, as it is already made apparent by the group $\mathbb{Q}$. However, people working with Kleinian groups, that is with discrete groups of isometries of hyperbolic spaces $\mathbb{H}^n$, might find examples as $\mathbb{Q}$ to be kind of pathological. In fact, it is well-known that discreteness of a group of isometries of hyperbolic space imposes non-trivial algebraic conditions. For example, centralisers of infinite order elements in discrete Kleinian groups are virtually abelian. Or, more to the point, while $\text{PSL}_2 \mathbb{Q} \subset \text{PSL}_2 \mathbb{R} \subset \text{Isom}(\mathbb{H}^2)$ is simple [5, 7], it is easy to see, using small cancellation arguments, that there are no infinite, simple, and discrete subgroups of $\text{Isom}(\mathbb{H}^n)$ (compare with 3 [4]). Also, Kleinian groups are mostly studied in low dimensions, and in that setting further algebraic restrictions do arise. For instance, the classification of 2-dimensional orbifolds implies that all discrete subgroups of $\text{Isom} \mathbb{H}^2$ are residually finite.

The goal of this note is to present examples of discrete subgroups of $\text{Isom}(\mathbb{H}^3)$ which fail to be residually finite. In fact, they don’t have any non-trivial finite quotients whatsoever.

Example 1. There is an infinite discrete subgroup $G \subset \text{Isom}(\mathbb{H}^3)$ without finite non-trivial quotients.

As we just said, having no finite quotients, the group $G$ in Example 1 clearly fails to be residually finite. Examples of discrete, non-residually finite subgroups of $\text{Isom}(\mathbb{H}^3)$ have been previously constructed by Agol [1]. Both Agol’s examples and the group in Example 1 have torsion. We present next an example, a variation of Agol’s example, showing that there are also torsion free discrete non-residually finite subgroups of $\text{Isom}(\mathbb{H}^3)$:

Example 2. There is a torsion free discrete subgroup $G \subset \text{Isom}(\mathbb{H}^3)$ which is not residually finite.

The remaining of this note is devoted to discuss these two examples.
In the course of our discussion we feel free to use standard facts of hyperbolic geometry as one might find in classical texts such as [6, 8]. It will also be convenient to see our groups as fundamental groups of infinite, locally finite, graph of groups. We refer to standard texts like [10] for basic facts about graphs of groups.

Example 1. We give an algebraic description of a group $G$, then we prove that it has no finite quotients, and we finally show that it is isomorphic to a discrete subgroup of $\text{PSL}_2\mathbb{C}$.

Let $T$ be the maximal rooted binary tree. Denote by $V$ and $E$ the sets of vertices and edges respectively, let $\ast$ be the root of $T$ and, for $v \in V$, let $|v| \in \mathbb{N}$ be the distance from $v$ to $\ast$. We orient the edges of $T$ so that they point to the root and for $e \in E$ we let $e^+$ be its terminal vertex. Given a vertex $v \in V$ with $|v| \geq 1$ let $e_0(v)$ the edge leaving $v$ and pointing out of $v$ and label the two edges pointing into $v$ by $e_1(v)$ and $e_2(v)$.

Consider from now the group

$$G = \left\{ g_e | e \in E \right\} \left\{ g_{e_0(v)}^{3+|v|}, g_{e_0(v)}g_{e_1(v)}^{-1}g_{e_2(v)}^{-1} | v \in V \text{ with } |v| \geq 1 \right\}. $$

The group $G$ also admits a description as the fundamental group

$$G = \pi_1(T)$$

of a graph of groups $T$ with underlaying graph $T$ with vertex groups

$$G_v = \left\{ g_{e_0(v)}, g_{e_1(v)}, g_{e_2(v)} \right\} \left\{ g_{e_0(v)}^{3+|v|}, g_{e_1(v)}^{4+|v|}, g_{e_2(v)}^{4+|v|}, g_{e_0(v)}g_{e_1(v)}^{-1}g_{e_2(v)}^{-1} \right\}$$

if $v \neq \ast$, with

$$G_\ast = \left\{ g_{e_1(\ast)}, g_{e_2(\ast)} \right\} \left\{ g_{e_1(\ast)}^{4+|e^+|}, g_{e_2(\ast)}^{4+|e^+|} \right\},$$

and with edge groups

$$G_e = \left\{ g_e \right\} \left\{ g_e^{4+|e^+|} \right\}.$$ 

We are going to think of the group $G$ as the nested union of a sequence of subgroups. The easiest way to describe these subgroups is as the fundamental groups

$$G^n = \pi_1(T^n)$$

of the subgraph of groups $T^n \subset T$ corresponding to the ball of radius $n - 1$ around the root $\ast$. Alternatively, $G^n$ is the subgroup of $G$ generated by all those elements $g_{e_0(v)}$ with $|v| \leq n$. We have

$$G^1 \subset G^2 \subset G^3 \subset \ldots, \quad G = \bigcup_{n \in \mathbb{N}} G^n.$$

Claim. The group $G^n$ is generated by the set $S^n = \{ g_{e_0(v)} \text{ with } |v| = n \}$.
Proof. Since \( G^n \) is generated by \( S_1 \cup S_2 \cup \cdots \cup S_n \) and hence by \( G^{n-1} \cup S^n \), we can argue by induction on \( n \). Therefore, it suffices to prove that \( S^{n-1} \) is contained in the group generated by \( S^n \). Well, given \( v \in V \) with \( |v| = n - 1 \) let \( v_1 \) and \( v_2 \in V \) be the initial vertices of the edges \( e_1(v) \) and \( e_2(v) \). Given that \( e_1(v) = \rho_0(v_1) \) and \( e_2(v) = \rho_0(v_2) \) the presentation of the group gives us:

\[
ge_{e_0(v)} = \rho_{e_0(v_2)} \rho_{e_0(v_1)}
\]

The claim then immediately follows. □

We are now ready to prove that \( G \) has no finite quotients:

Claim. \( G \) has no finite non-trivial quotients.

Proof. Let \( H \) be a finite group and \( \pi : G \to H \) be a homomorphism. We need to prove that \( \pi \) is trivial. Now, denote by \( |H| \) the order of \( H \), let \( k \in \mathbb{N} \) be arbitrary, and let \( n > k \) with

\[
3 + n \equiv 1 \mod (|H|).
\]

From the presentation of the group \( G \) we get that the elements in the set \( S^n = \{ \rho_{e_0(v)} \text{ with } |v| = n \} \) have order \( 3 + n \) in \( G \). Since \( |H| \) and \( 3 + n \) are prime to each other, we get that the elements in \( S^n \) are killed by \( \pi \):

\[
S^n \subset \ker(\pi).
\]

Now, from the previous claim we also get that the subgroup \( G^n \) is contained in the kernel of \( \pi \), meaning that also \( G^k \subset \ker(\pi) \). Since \( G \) is the union of the subgroups \( G^k \) we get that \( G \subset \ker(\pi) \). We have proved the claim. □

All is left now is to show that \( G \) admits discrete and faithful representations into \( \text{PSL}_2 \mathbb{C} \). We are going to construct the desired representation as a limit of representations of the groups \( G^n \).

Claim. For each \( n \geq 0 \) there is a discrete and faithful representation \( \rho_n : G^n \to \text{PSL}_2 \mathbb{C} \) such that the restriction of \( \rho_n \) to \( G^{n-1} \) agrees with \( \rho_{n-1} \) for every \( n \geq 1 \).

Moreover, the limit set of the group \( \rho_n(G_v) \) bounds a round disk in the discontinuity domain of \( \rho_n(G^n) \) for every vertex group \( v \) with \( |v| = n \).

The final assertion of the Claim serves to be able to argue by induction – the real point is the first assertion because it allows us to define

\[
\rho : G \to \text{PSL}_2 \mathbb{C}
\]

satisfying \( \rho(g) = \rho_n(g) \) if \( g \in G^n \). This representation is faithful because each one of the representations \( \rho_n \) is. The same argument holds true for discreteness because already the first group \( \rho(G_1) \) is non-elementary [2].

All that it is left is to prove the claim.
Proof of the claim. We will argue by induction. First note that
\[ G^0 = G_* = \langle g_{e_1(s)}, g_{e_2(s)} | g_{e_1(s)}^4, g_{e_2(s)}^4 \rangle \]
is isomorphic to the \((4, 4, \infty)\)-triangle group:
\[ H_0 = \langle a, b | a^4, b^4 \rangle. \]
We can thus take \( \rho_0 : G_0 \to \text{PSL}_2 \mathbb{C} \) to be the standard fuchsian representation and take the said disk to be any one of the two connected components of the discontinuity domain of \( \rho_0(G_0) \).

Suppose that the claim holds true for \( n - 1 \). For each vertex \( v \in V \) with \( |v| = n - 1 \) let \( \Delta_v \subset \mathbb{S}^2 = \partial_{\infty} \mathbb{H}^3 \) be the disk in the discontinuity domain bounded by the limit set of \( \rho_{n-1}(G_v) \). Note that \( \Delta_v \) is \( \rho_{n-1}(G_v) \)-invariant and that no translate of \( \Delta_v \) under \( \rho_{n-1}(G^{n-1}) \) meets \( \Delta_v \) for another vertex \( w \neq v \) with \( |w| = n - 1 \). The disk \( \Delta_v \) is the boundary at infinity of a hyperbolic half-space \( H_v \) - let \( \Delta'_v \) be the hyperbolic plane bounding \( H_v \).

Suppose now that we have a vertex \( z \in V \) with \( |z| = n \) and denote by \( z^+ \) the terminal vertex of the edge \( e = e_0(z) \). We identify the edge group \( G_{e_0(z)} \) with the corresponding subgroup of the vertex group \( G_{z^+} \). The group \( \rho_{n-1}(G_{e_0(z)}) \) is cyclic and has a unique fixed points \( \alpha \in \Delta_v \) and \( \alpha' \in \Delta'_v \).

For \( L > 0 \) let \( x \in [\alpha', \alpha) \) be the point at distance \( L \) from \( \alpha' \) and let \( D \) be the hyperbolic plane containing \( x \) and perpendicular to the ray \( [x, \alpha) \). The plane \( D \) is \( \rho_{n-1}(G_{e_0(z)}) \)-invariant.

Consider now the edge group \( G_{e_0(z)} \) as a subgroup of \( G_z \). The action of \( G_{e_0(z)} \) on \( D \) via \( \rho_{n-1} \) extends to a discrete action of the triangle group \( G_z \) and thus to an action of \( G^{(n-1)} \ast G_{e_0(z)} G_z \).

Proceeding in this way with all vertices \( z \in V \) with \( |z| = n \) we then get a representation
\[ \rho_L : G^n \to \text{PSL}_2 \mathbb{C} \]
depending on the parameter \( L \). By construction all of these constructions extend the representation \( \rho_{n-1} \) and it follows from the Klein-Maskit combination theorem that for \( L \) large enough the representation \( \rho_L \) is discrete and satisfies the additional desired claim. \( \square \)

This concludes the discussion of Example 1.

Example 2. Let \( T \) be a once holed torus and let \( \alpha \) and \( \beta \) be simple curves in \( T \) which intersect (transversely) in a single point. Let also \( U \) be a regular neighborhood of \( \beta \times \{0\} \) in the 3-manifold \( T \times [-1, 1] \), \( \mu \) the meridian of \( U \), and \( \beta' \subset \partial U \) the longitude of \( U \) isotopic to \( \beta \times \{1\} \) in \( T \times [-1, 1] \setminus U \). Finally, for \( n \in \mathbb{N} \) let \( M_n \) be manifold obtained from \( T \times [-1, 1] \setminus U \) by Dehn-filling the curve \( n\mu + \beta' \). The curve \( \beta' \) intersects the new meridian \( n \) times, which means that it represents the \( n \)-th power of the soul of the new solid torus. We get thus the presentation
\[ \pi_1(M_n) = \pi_1(T) \ast_{(\beta)} \pi_1(U) \cong \langle a, b, c | b = c^n \rangle \]
with respect to which the curve $\partial T \times \{0\}$ corresponds to the conjugacy class of $[a,b] = [a,c^n]$.

Now, the pair $(M_n, \partial T \times [-1,1])$ is a pared manifold, which means that the group $\pi_1(M_n)$ admits a geometrically finite representation

$$\rho_n : \pi_1(M_n) \to \text{PSL}_2 \mathbb{C} = \text{Isom}_+(\mathbb{H}^3)$$

with $\rho_n([a,b]) = g \in \text{PSL}_2 \mathbb{C}$ where $g(z) = z + 1$.

For $t \in \mathbb{R}$ let $h_t \in \text{PSL}_2 \mathbb{C}$ be the parabolic element $h_t(z) = z + ti$. Now, a standard combination argument implies that for a sufficiently fast growing sequence $t_n \to \infty$, the group

$$G = \langle \cup_n h_{t_n} \rho_n(\pi(M_n)) h_{t_n}^{-1} \mid n \in \mathbb{N} \rangle$$

generated by the union of the groups $h_{t_n} \rho_n(\pi(M_n)) h_{t_n}^{-1}$ is discrete.

Notice now that for each $n$ the element $g \in G$ can be written as

$$g = h_t g h_t^{-1} = h_t \rho_n([a,b]) h_t^{-1} = h_t \rho_n([a,c^n]) h_t^{-1}.$$ 

It follows that if $H$ is an arbitrary finite group and if $\pi : G \to H$ is a homomorphism then

$$\pi(g) = \pi(h_t) [\pi(\rho_n(a), \rho|H| c)|H|] \pi(h_t)^{-1}$$

$$= \pi(h_t) [\pi(\rho_n(a), \text{Id}_H)] \pi(h_t)^{-1}$$

$$= \text{Id}_H$$

where $|H|$ is the order of $H$. Having proved that $g \in G$ belongs to the kernel of every homomorphism to a finite group, we have showed that $G$ is not residually finite. This concludes the discussion of Example 2.

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