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Polarization and phase control of electron injection and acceleration in the plasma by a self-steepening laser pulse

Jihoon Kim∗,†, Tianhong Wang‡, Vladimir Khudik‡ and Gennady Shvets∗,†
1 Department of Physics, Cornell University, Ithaca, New York 14850, United States of America
2 School of Applied and Engineering Physics, Cornell University, Ithaca, NY 14850, United States of America
3 Department of Physics and Institute for Fusion Studies, The University of Texas at Austin, Austin, TX 78712, United States of America
* Authors to whom any correspondence should be addressed.
E-mail: jk2628@cornell.edu and gshvets@cornell.edu

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Abstract

We describe an interplay between two injection mechanism of background electrons into an evolving plasma bubble behind an intense laser pulse: one due to the overall bubble expansion, and another due to its periodic undulation. The two mechanisms occur simultaneously when an intense laser pulse propagating inside a plasma forms a shock-like steepened front. Periodic undulations of the plasma bubble along the laser propagation path can either inhibit or conspire with electron injection due to bubble expansion. We show that carrier-envelope-phase (CEP) controlled plasma bubble undulation induced by the self-steeping laser pulse produces a unique electron injector—expanding phase-controlled undulating bubble (EPUB). The longitudinal structure of the electron bunch injected by the EPUB can be controlled by laser polarization and power, resulting in high-charge (multiple nano-Coulombs) high-current (tens of kilo-amperes) electron beams with ultra-short (femtosecond-scale) temporal structure. Generation of high-energy betatron radiation with polarization- and CEP-controlled energy spectrum and angular distribution is analyzed as a promising application of EPUB-produced beams.

1. Introduction

An electron injector is an integral part of any accelerator, as it produces high-quality moderate energy particles for further acceleration. A remarkable feature of a laser wakefield accelerator (LWFA) [1–3] is the availability of an abundant reservoir of charged particles from the background plasma. Therefore, plasma can simultaneously serve as an acceleration medium sustaining intense plasma waves, and an electron injector. While the key attraction of LWFAs is their compactness owing to ultrahigh accelerating electric field—in excess of 100 GV m⁻¹ in many recent implementations [4–9]—of the plasma wave generated by intense laser pulses, its other advantage is the availability of large numbers of initially quiescent electrons that can be injected into the plasma wave, capable of forming currents exceeding 100 kA [10]. If such an injection can be controlled, it may be possible to produce high-charge low-emittance beams in single compact device.

A number of promising approaches to injecting electrons into plasma waves generated in the wake of a laser pulse, including the highly nonlinear ‘plasma bubbles’ [11, 12], have been suggested and experimentally implemented. Those include injections due to ionization [13–17], engineered density ramps [18–21], and rapid variation of the bubble’s size along the laser’s path [5, 14, 22–26].

Electron injection and acceleration based on single-cycle laser pulses has been demonstrated theoretically and experimentally [27–33]. Under certain circumstances, near single cycle (NSC) laser pulse propagating in an underdense plasma can generate a phase-controlled undulating bubble (PUB) with characteristic periodicity \( T_{\text{CEP}} = \lambda_L / (\nu_{ph} - \nu_g) \) controlled by laser carrier envelope phase (CEP) offset, with \( \nu_{ph} \) the laser phase velocity, \( \nu_g \) the laser group velocity, and \( \lambda_L \) the laser pulse wavelength [27–29, 32]. CEP-controlled injection is expected when laser intensity varies sharply on a time-scale of one laser oscillation period—either due to the short overall duration, or nonlinear self-steeping of a laser pulse in the course of its propagation through the plasma [34–38].

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It was recently shown that CEP-related periodic electron injection into a plasma bubble can occur for intense NSC laser pulses [33]. This CEP-controlled injection is a conceptual departure from the standard description of plasma wave generation by multi-cycle laser pulses that relies on the phase-averaged (ponderomotive) approximation [39]. Despite the promise of CEP-based injection to generating high-current ultra-short electron bunches [33], it requires NSC pulses. In what follows, we concentrate on the other circumstance under which phase- and polarization-dependent injection can occur: when a longer pulse has its front locally depleted due to the etching by the plasma. The front of such self-steepened laser pulse envelope can vary on a scale comparable to that of a laser cycle [34–36, 40], resulting in an expanding PUB (EPUB) which is the subject of this work.

In this paper, we examine the combined effect of expansion and undulation of a plasma bubble on the injection, acceleration, and temporal shaping of an electron bunch produced by an EPUB, as shown in figure 1. The paper is organized as follows. In section 2, we set the stage by presenting the results of PIC simulations that demonstrate phase and polarization dependent injection of electrons into a plasma bubble produced by a self steepening few cycle laser pulse with \( \epsilon T_{\text{FWHM}} \sim 3\lambda /c \), where \( T_{\text{FWHM}} \) and \( \epsilon \) are the pulse full width half maximum (FWHM) duration and speed of light, respectively. The parameters of the laser pulse are chosen to be within reach of the Brookhaven Experimental Supra-Terawatt Infrared at Accelerator Test Facility (ATF) laser system [41]. In section 3, we interpret these results by developing a simple single-particle model of electron injection into a plasma bubble undergoing simultaneous expansion and undulation. This model is used to demonstrate how laser polarization (i.e. linear versus circular) can be used to generate the desired current profile (spiky versus smooth) of an injected electron bunch, and further lead to x-ray distribution with asymmetric angular distribution and nonzero degree of polarization. We demonstrate that high charge (Q \( \sim 10 \) nC) bunches modulated on a temporal scale comparable to the laser period \( T_L = \lambda /c \) can be formed, with promising implications for structured x-ray generation. In section 4, we discuss the characteristics of generated betatron x-ray radiation, telltale signs of phase and polarization linked laser-plasma interactions, and injected beam quality with an eye on application for seeded free electron laser (FEL).

2. Polarization-dependent injection and acceleration: simulations results

We use a 3D PIC code VLPL [42] to self-consistently model the propagation and self-steepening of an intense laser pulse [35, 36], followed by self-injection of some of the plasma electrons into the laser wakefield, acceleration of the injected bunch, and subsequent plasma field depletion by the injected electrons [43]. The following laser parameters are used: peak power \( P_L = 40 \) TW, wavelength \( \lambda_L = 9.2 \) \( \mu \)m, pulse duration \( T_{\text{FWHM}} = 100 \) fs, and the matched spot size \( \sigma_L = 8.5\lambda_L /c \approx 78 \mu \)m [6]. The matched spot size was chosen to minimize the laser spot size oscillation which can lead to bubble size modulation, thereby making it easier to observe the bubble expansion and transverse undulation arising from steepening. The laser parameter corresponds to the initial normalized vector potential of the laser pulse \( a_0 \equiv \epsilon /|E_{\text{max}}| m_e c \omega_L = 5.0 \), where \( \epsilon \) is the magnitude of electron charge, \( |E_{\text{max}}| \) is the magnitude of laser pulse peak electric field, \( m_e \) is the electron mass, and \( \omega_L \equiv 2\pi \epsilon /\lambda_L \) is laser angular frequency. Pre-ionized plasma starts with a linear density ramp of the length \( L_{\text{ramp}} = 0.37 \) mm, followed by a long plateau region with constant density \( n_p = 9.1 \times 10^{16} \) \( \text{cm}^{-3} \). The ions are treated as a uniform and immobile background of positive charge. A numerical grid used in the simulations was chosen to have the dimensions of \( \Delta x \times \Delta y \times \Delta z = 0.05\lambda_L \).
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100 fs × (blue line) and at 3.3 × (red line). Laser pulse parameters are as follows: peak power $P_{\text{pe}} = 17 \omega_0^2 / \omega_0^2 (\text{GW})$ is the critical power for the laser to self-focus in plasma [3], $\omega_0 = \sqrt{4 \pi e^2 n_e / m_e}$ is the electron plasma frequency, and $n_e = m_e \omega_0^2 / 4 \pi e^2$ is the critical density.

| Parameter                    | Physical units          | Normalized units |
|------------------------------|-------------------------|------------------|
| Plasma density               | $9.1 \times 10^{16} \text{cm}^{-3}$ | $n_p/n_e = 1/144$ |
| Ramp length                  | 0.37 mm                 | $40 \lambda_L$  |
| Cell size ($\Delta x \times \Delta y \times \Delta z$) | 0.46$\mu$m × 2.3$\mu$m × 2.3$\mu$m | 0.05$\lambda_L \times 0.25 \lambda_L \times 0.25 \lambda_L$ |
| Plasma wavelength ($\lambda_p$) | 110$\mu$m               | $12 \lambda_L$  |
| Laser wavelength ($\lambda_L$) | 9.2$\mu$m               | $\lambda_L$     |
| Spot size ($\sigma_z$)       | 78$\mu$m                | 8.5$\lambda_L$  |
| FWHM ($\sqrt{2 \ln 2} \sigma_z/c$) | 100 fs            | 3.3$\lambda_L/c$ |
| $a_0 \equiv cL_{\text{max}}/m_e \omega_L$ | 1.7 TV m$^{-1}$ | 5.0 |
| Peak power                   | 40 TW                   | $P/P_c = 16.3$  |

Table 1. Laser-plasma parameters used in a 3D PIC (VLPL) simulation. Laser field is defined by the normalized vector potential $a(x, y, z) = -a_0 \exp(-z^2 / \sigma_z^2) \exp[-(t^2 + y^2) / \sigma_y^2] \cos(\omega_L t / c + \phi_{\text{CEP}})$, with $\phi_{\text{CEP}}$ the laser initial CEP and $\zeta = x - ct$ the longitudinal coordinate moving with the speed of light. $P_c = 17 \omega_0^2 / \omega_0^2 (\text{GW})$ is the critical power for the laser to self-focus in plasma [3], $\omega_0 = \sqrt{4 \pi e^2 n_e / m_e}$ is the electron plasma frequency, and $n_e = m_e \omega_0^2 / 4 \pi e^2$ is the critical density.

×0.25$\lambda_L \times 0.25 \lambda_L$ and a time step $\Delta t = 0.05 \lambda_L / c$, where $x$ is the propagation direction of the laser pulse through the plasma (Also see table 1). We note that polarization-dependent electron injection has been observed in initially-neutral plasma targets due to above-threshold ionization process [44]. Ionization injection is neglected in our simulations because the total injected charge ($Q > 10 \text{nC}$) due to plasma bubble expansion/undulation is expected to be much larger than in the case of ionization injection, with less than O(nC) of charge [45, 46].

We first consider a laser pulse linearly-polarized (LP) in the $z$-direction. Since the pulse front needs to steepen before CEP effect becomes visible, the plasma bubble does not execute transverse undulations immediately after the laser pulse enters the plasma as shown in figures 2(a) and (d). Electrons are injected into the plasma bubble from the very beginning, but this initial population of injected electrons does not exhibit any transverse asymmetry in the $z$-direction. After $ct = 400 \lambda_L$ (or $x = 3.7 \text{mm}$) of propagation through the plasma, the pulse front is depleted and steepened as shown in figure 2(b), with further depletion at $ct = 700 \lambda_L$ (or $x = 6.4 \text{mm}$) as apparent from figure 2(c).

The sharpness of the self-steepened front at $ct = 400 \lambda_L$, as well as its depletion, are reflected in its spectrum plotted (red line) in figure 2(f). When compared with the initial laser spectrum at $ct = 100 \lambda_L$ (blue-line), the spectrum of the steepened pulse is red-shifted by approximately 25%, i.e. from $\omega_L = \omega_{L0}$ to $\omega_L = 0.75 \omega_{L0}$—a clear evidence of pulse depletion via plasma wake generation (figure 2(f)). Moreover, its large FWHM spectral bandwidth $\Delta \omega \sim 0.5 \omega_{L0}$ signifies pulse steepening on the time scale of a laser period.

Figure 2. Results of 3D particle-in-cell simulations of plasma bubble undulations induced by self-steepening of a laser pulse. (a)–(c) shows the on-axis electric field of a $z$-polarized laser pulse (red line) and plasma density (color map) at (a) $ct = 100 \lambda_L$, (b) $ct = 400 \lambda_L$, and (c) $ct = 700 \lambda_L$. (d) and (e) shows on-axis transverse wakefield $W_z = (W_x, W_y)$ for (d) linearly- and (e) circularly-polarized laser pulse. Blue (orange) line shows $W_z(W_y)$ at $\zeta = x - ct = 35 \lambda_L$. Red dashed lines on (d) shows propagation distances corresponding to (a)–(c). (f) shows spectra of the $z$-polarized laser pulse at $ct = 100 \lambda_L$ (blue line) and at $ct = 400 \lambda_L$ (red line). Laser pulse parameters are as follows: peak power $P_{\text{pe}} = 40 \text{TW}$, wavelength $\lambda_L = 9.2 \mu$m, duration $T_{\text{FWHM}} = 100$ fs, spot size $\sigma_z = 8.5 \lambda_L \approx 78 \mu$m. Plateau plasma density is $n_p = 9.1 \times 10^{16} \text{cm}^{-3}$. 

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This increase of the spectral bandwidth is a result of strongly-nonlinear interaction between the laser pulse and the plasma.

When the spatial profile of a laser pulse is self-steepened by its propagation through the plasma so as to develop a wavelength-sharp intensity shock, the plasma bubble starts executing transverse undulations along the laser polarization direction [34]. Such plasma bubble undulations are analogous to those produced by NSC laser pulses [27–33]. Bubble undulations are manifested as a non-vanishing transverse on-axis wakefield \( \mathbf{W}_\perp \equiv (W_y, W_z) \). In its dimensionless form, the transverse wake is given by \( \mathbf{W}_\perp = (\mathbf{E}_\perp + \mathbf{e}_z \times \mathbf{B}_\perp) / E_{0\alpha} \), where \( \mathbf{E}_\perp \) and \( \mathbf{B}_\perp \) are the transverse electric and magnetic wakefields inside the plasma bubble, and \( E_{0\alpha} = m_e c \omega_\alpha / e \) is the relativistic electric field scale in a laser beam. Note that \( \mathbf{W}_\perp \) is directly proportional to the transverse force exerted by the wakefield on a charge moving with the speed of light in the \( x \)-direction inside the plasma bubble.

When a laser pulse is LP in the \( z \)-direction, we find that \( W_z \approx 0 \) and \( W_y \neq 0 \) as shown in figure 2(d). Similarly, an on-axis transverse wake \( \mathbf{W}_\perp \) is generated by a circularly polarized (CP) laser pulse of the same duration and intensity as the LP pulse considered above. As evidenced by figures 2(d) and (e), plasma cavity undulations begin around the same propagation distance for the LP and CP pulses because pulse steepening takes place after the same (polarization-independent) propagation distance through the plasma.

However, after bubble undulations start, their nature is observed to be highly dependent on the polarization state of the laser pulse. Specifically, instead of executing undulations in the \( z \)-direction for the LP pulse, the bubble executes helical motion in the \( y-z \) plane when driven by a CP pulse. As a result, the on-axis transverse wakefield \( W_z \) of the bubble has equal \( W_y \) and \( W_z \) components that are approximately offset from each other by \( \pi/2 \) phase difference, as observed in figure 2(e). As we are going to show in section 3, it is important to note that for both polarizations, the plasma bubble is continuously elongated. Notably, the longitudinal bubble elongation is larger than its transverse expansion, as shown in figures 2(a)–(c).

2.1. Electron injection and its dependence on laser pulse polarization

For any laser polarization, copious amounts of electrons are injected from the background plasma throughout the entire laser propagation distance as shown in figures 3(a)–(c). For the specific laser and plasma parameters corresponding to the simulations in figures 2 and 3, the total injected charge is on the order of \( Q_{\text{tot}} \approx 11 \) nC for all polarizations. However, the actual nature of electron injection is dependent on both the longitudinal injection location \( x_0 \approx ct \) and the laser pulse polarization characterized by its ellipticity \( \epsilon \) defined as \( \epsilon = E_{\perp y}/E_{\perp z} \), where \( E_{\perp y,z} \) are the electric field components in an elliptically-polarized (EP) laser pulse. As the laser polarization is progressively varied from linear to elliptical to circular in figures 3(d)–(f), current modulations switch from highly-bunched to weakly-modulated to nearly-constant.

In our simulation, every macro-particle is labeled by its initial location \( \mathbf{r}_0 \equiv (x_0, y_0, z_0) \) that can be used to study the origin of injected electrons. Note that even though the longitudinal injection location \( x_0 \) (see figures 3(a)–(c)) is approximately equal to time/distance expressed as \( ct \) for a given snapshot of the electron density (see figures 2(a)–(c) for three representative snapshots), these two quantities are not identical. For example, all injected electrons shown in a snapshot corresponding to the propagation distance \( ct \) have been injected at earlier times corresponding to \( x_0 \leq ct \).

Depending on the longitudinal injection location \( x_0 \), we have identified three groups of injected electrons: the earliest-injected during the laser passage through and immediately after the plasma density ramp (Group I); electrons injected during the passage through the 110\( \lambda_L < x_0 < 300 \lambda_L \) region of the plasma marked by two vertical red dashed lines in figure 3 that eventually form a monoenergetic bunch shown in figure 4 (Group II); electrons injected during the later period (Group III). Below we discuss the properties of these three electron groups as deduced from our PIC simulations. The basic physics underlying the differences between Groups II and III are discussed in section 3.

The prominent feature of Group I electrons is the huge injection spike at \( x_0 \approx 0.92 \text{ mm} (x_0 \approx 100 \lambda_L) \), where plasma density profile transitions from a linear ramp to a plateau [47, 48]. This injection is consistently observed for a wide range of peak laser pulse powers \( P_L \) and for all polarizations (i.e. all values of \( \epsilon \)). Group I electrons are injected and gain significant energy (\( \gamma \approx 300 \)) before the start of plasma bubble undulations around \( ct \approx 400 \lambda_L \). Therefore, plasma bubble undulations do not have any significant effect on either injection or subsequent dynamics of the Group I electrons. While their energy and charge reach \( \gamma_I \approx 500 \) and \( Q_i \approx 1.3 \text{ nC} \), respectively, their transverse momenta remain moderate (\( p_\perp \approx 2mc \)). The energy spread of Group I electrons is fairly large due to beam loading [43].

Injection of Group II electrons takes place from the plasma region between the two vertical dashed lines shown in figures 3(a)–(c). The primary injection mechanism during this period is the rapid expansion of the bubble’s longitudinal size [22, 23]. This can be observed from the CP laser case shown in figure 3(c), where the injected electrons (blue dots) originate symmetrically in their original \( z_0 \)-location.
Figure 3. Effect of laser ellipticity on injection and bunch formation. (a)–(c) shows initial positions of the injected electrons in the $x_0$–$z_0$ plane (blue dots) and the injection rate (red line) for (a) LP ($\epsilon = 0$), (b) elliptically-polarized ($\epsilon = 0.268$), and (c) CP ($\epsilon = 1$) laser pulse. (d)–(f) show current profiles after $ct = 6.4$ mm propagation distance for (d) linear, (e) elliptic, and (f) circular laser polarizations. Area between the red dashed lines in all figures denote the monoenergetic electrons, see figure 4(c). Ellipticity coefficient is defined by $\epsilon = E_{L_y}/E_{L_z}$, where $E_{L_y}, E_{L_z}$ are laser electric field components. Laser and plasma parameters are the same as in figure 2.

Figure 4. Phase space distributions (a), (b) and energy spectra (c) of the injected electrons at $ct = 6.4$ mm for two laser pulse polarizations: LP (a) and CP (b). (c) Electron spectra for LP (red) and CP (blue) pulses. Low-energy inset shows Group III electrons. High-energy inset shows Group I electrons.

Despite the relatively long duration ($\Delta \zeta / c \approx 370$ fs, $\Delta \zeta = \zeta_{\text{front}} - \zeta_{\text{back}} = (34 - 22) \lambda_L = 12 \lambda_L$; see figures 3(d)–(f)) of the entire bunch train—or a single bunch in the CP laser case—Group II electrons collapse into a monoenergetic bunch (figure 4(c)) at the final propagation distance $x_{\text{fin}} \approx 6.4$ mm due to phase space rotation [22, 23]. At the same time, injection dynamics becomes dependent on the laser polarization after the onset of laser pulse steepening and plasma bubble undulations: see figures 3(a)–(c).

The effect of plasma bubble undulations on electron injections into a simultaneously expanding plasma bubble is most transparently illustrated by the LP laser case shown in figure 3(a), where highly asymmetric electron injection in $z_0$ can be observed for any given initial electron position $x_0$, i.e. the $(x_0, z_0)$ coordinates of the injected electrons are highly correlated. Electrons injected from a bubble driven by the LP pulse are injected in short bursts from alternating locations with $z_0 \approx \pm R_b$, with $R_b$ the bubble radius. In comparison, those injected into a CP laser driven bubble originate from a spiral-shaped initial positions. Injection rates plotted as red lines in figures 3(a)–(c) show that as the laser polarization changes from LP to EP to CP, the rate of electron injections versus the injection position $x_0$ transitions from short periodic bursts to a near-constant value.
Polarization-defined difference between Group II electrons is also manifested in the amount of charge injected by the LP and CP pulses: $Q_{\text{III}}^{\text{LP}} \approx 3.3 \text{nC}$ for the former and $Q_{\text{III}}^{\text{CP}} \approx 3.7 \text{nC}$ for the latter. The dips in the injection rate in figure 3(a) shows that the high-amplitude bubble undulations produced by the steepened LP pulse destructively interferes with electron injection produced by the elongation of the plasma bubble. $Q_{\text{III}}^{\text{LP}} < Q_{\text{III}}^{\text{CP}}$ suggests that less destructive interference appears to be happening for the CP pulse despite the emergence of helical bubble undulation after the CP pulse steepens, as illustrated by figure 2(e). A simple mathematical model of such interference is proposed in section 3.

The most stark difference in the electron injection dynamics produced by the LP, EP, or CP laser pulses can be observed for Group III electrons. Injected during the later (post-steepening) stage of laser pulse propagation, Group III electrons form high-charge high-current electron bunches in the $\zeta < 30 \lambda_L$ region as shown in figures 3(d) and (e) for the LP and EP cases. Such current bunching with periodicity $\Delta \zeta \sim \lambda_L$ directly reflects multiple periodic electron injections from the $x_0 > 300 \lambda_L$ region of the plasma, with the injection periodicity $T_{\text{CEP}} \approx 100 \lambda_L/c$ observed in figures 3(a) and (b).

This numerically observed injection periodicity was found to be close to the theoretical estimate of $T_{\text{inj}} \approx T_{\text{CEP}}/2$. The injected electrons are rapidly accelerated inside the bubble after the injection, and their relativistic factor rapidly increases from $\gamma \approx \gamma_{bb}$ at the injection time to $\gamma \gg \gamma_{bb}$ where $\gamma_{bb} \sim 5$ is the relativistic factor corresponding to the plasma bubble’s rear velocity $v_{bb}/c \equiv \sqrt{\gamma_{bb}^2 - 1}/\gamma_{bb}$. Such periodic injections are expected to translate into a series of current spikes separated by $\Delta \zeta \approx cT_{\text{CEP}}/4 \gamma_{bb}^2 \sim \lambda_L$. This is roughly in agreement with the bunch modulation period of approximately $\Delta \zeta/c \approx 1.3 \lambda_L/c \approx 40 \text{fs}$ as observed in figures 3(d) and (e). We note that it is also possible to control the spacing between the injected bunches via changing the laser power or modify the individual spikes’ location using laser CEP; we direct the interested reader to appendix A.

The injected total charges of Group III electrons for the LP and CP laser pulses are $Q_{\text{III}}^{\text{LP}} \approx 7 \text{nC}$ and $Q_{\text{III}}^{\text{CP}} \approx 5.6 \text{nC}$, respectively. In the case of LP laser pulses, each of the current spikes carries approximately $\delta Q_{\text{III}}^{\text{LP}} \approx 1 \text{nC}$ of charge and has the duration of the order of $\delta \zeta \sim 16 \text{fs}$. In the case of a CP laser pulse, the electron current is much more uniformly distributed, with the peak current never exceeding $I_{\text{III}}^{\text{CP}} \approx 45 \text{kA}$.

On the other hand, the peak currents for the LP pulse approach $I_{\text{III}}^{\text{LP}} \approx 75 \text{kA}$. The mathematical description of the interplay between electron injections due to plasma bubble undulations and expansion is presented in section 3, where we uncover the differences between Group II and III electrons.

Note that even for electron injections driven by a CP laser pulse, there is an injection rate dip at $x_0 \approx 4.8 \text{mm}$ ($x_0 \approx 520 \lambda_L$; see figure 3(c)). This occurs because the peak accelerating gradient inside the plasma bubble is reduced via beam-loading of the plasma wake [43] by the large earlier-injected electron charge, thereby suppressing further electron injections.

We further remark that the abrupt beam loadings of the plasma wake by short bursts of injection in the case of the LP pulse (figure 3(a)) leads to a slightly wider energy spectrum peak width than for the CP pulse: $\Delta E_{\text{LP}}^{\text{mono}}/E_{\text{mono}} \approx 4\%$ versus $\Delta E_{\text{CP}}^{\text{mono}}/E_{\text{mono}} \approx 2\%$ FWHM as can be observed from figure 4(c). Another polarization-dependent feature is the lower-energy electron spectra corresponding to Group III electrons (figure 4(c), left inset); while there are prominent peaks for the LP case, the CP spectra remains mostly smooth. Finally, different transverse wakefields and injection processes for the LP and CP laser pulses lead to substantially different electron distributions in phase space shown in figures 4(a) and (b). This, in turn, affects the emitted x-ray as will be discussed later in section 4.

3. EPUB injection mechanism

To interpret this electron injection into an evolving plasma cavity, we use a simplified model of a positively-charged (devoid of electrons) spherical plasma bubble [11, 22, 23, 49]. The bubble has radius $R(t) = R_0(1 + \epsilon t)$ with initial radius $R_0$ expanding with rate $\epsilon$ propagating with uniform velocity $v_b$.

A Hamiltonian describing plasma electrons’ interaction with the bubble can be written as $H(p, t) = \sqrt{1 + (\mathbf{P} + \mathbf{A}(t))^2} - v_p P_x - \phi(t)$, where $\mathbf{P} = (\xi, y, \tilde{z})$, $\tilde{z} = z - z_{\text{cap}}(t)$ is the electron $z$-coordinate from the undulating bubble center, $\mathbf{A}(t)$ is the vector (scalar) potentials. Here $\omega_{\text{CEP}} \equiv 2\pi/T_{\text{CEP}}$ is the CEP slip rate, $z_0$ is the maximum bubble oscillation amplitude, and $\phi_{\text{CEP}} \equiv \phi_{\text{CEP}}(t(x_0), x_0)$ is the initial CEP evaluated at the time $t(x_0)$ corresponding to electron’s entrance into the bubble at $x = x_0$. Time, length, potential, and electron momentum are normalized to $\omega_{\text{CEP}}^{-1}$, $k_p^{-1} \equiv \epsilon/\omega_{\text{CEP}}$, $m_e c^2/|\epsilon|$, and $m_e c$, respectively. We use the $A_y(t) = -\phi(t) = \Phi(t)/2$ gauge, and assume that $\Phi(t) = (p(t)^2 - R(t)^2)^2/4$ inside and $\Phi(t) = 0$ outside the bubble.

Transverse plasma bubble undulations $z_{\text{cap}}(t)$ and bubble expansion, $R(t)$, introduces time dependence into the Hamiltonian. To simplify the discussion, we consider the electron motion in the $x-z$ plane. From the
Hamiltonian, the equations of motion (see appendix B) and the following Hamiltonian time-dependence can be derived:

\[
\frac{dH}{dt} = \frac{1 + v_x}{4} \left[ z(t)z_{osc}(t) + R(t)\dot{R}(t) \right].
\] (1)

In the following section, this Hamiltonian will be used to determine if the electrons will be injected or not.

### 3.1. Analytic estimates of electron injection conditions using Hamiltonian model

Under specific conditions, electrons get injected into the bubble and are accelerated to ultra relativistic energies [22, 23, 49]. An electron can be trapped when the condition \( H < 0 \) is fulfilled [22]; under this condition, electrons cannot escape the bubble, even when they overtake the bubble. There is another population of electrons, the injected electrons, which can gain similar peak energy in the bubble as the trapped electrons but cannot escape the bubble after it reaches the front of the bubble [23]. In this paper, we present an estimate of this injection condition, providing a slightly relaxed condition for electrons to enter and gain relativistic energy.

Injection condition for a moderately relativistic bubble (\( R \sim \gamma_b \)) where \( \gamma_b \) is the bubble's relativistic factor, was derived using simplified equations of motion [49]. Electrons can catch up with the bubble if they can reflect off the bubble's rear wall at least once. This results in electron spending longer time in the bubble, and can determine if the electrons will be injected or not. This reflection and subsequent injection was shown to be true when \( \sqrt{2}\gamma_b < R \).

In an expanding bubble, this condition can be relaxed because the Hamiltonian of the electron is altered [22]. For an ultra-relativistic electron \( (p_x \gg 1) \) interacting with an ultra-relativistic \( (\gamma_b \gg 1) \) non-evolving bubble, maximum excursion of electron from bubble's center axis is given by \( r_{in} \approx 4H + R^2 - 2p_x/\gamma_b^2 - 2/p_x \), with \( H = 1 \) [49]. Assuming small expansion rate \( (\varepsilon \ll 1) \) of a bubble with a time-dependent radius \( R(t) \) expanding according to \( R(t) = R_0(1 + \varepsilon t) \), the maximum momentum gained by the electron is almost identical as for the static bubble \( (\varepsilon = 0) \). Therefore, \( r_{in} \) can be simply modified by using a modified (instantaneous) \( H(t) \).

Numerical solutions of normalized equations of motions (equations (B.5)–(B.8), see appendix B) show that an initially quiescent electron \( (p_{0z} = P_{0z} = 0) \) entering a non-evolving bubble from the top edge \( (dp_{0z}/ds = -1/4, dp_{0z} = 0, \lambda_0 = 0, \lambda_2 = 1) \), gains maximum longitudinal momentum \( p_x \approx 1.1R^2 \) before leaving bubble [49]. Combining this with \( r_{in} < R \), a modified injection condition can be found:

\[
\gamma_b/R < \sqrt{1/\sqrt{2}H}.
\] (2)

Or alternatively,

\[
H < H_{thresh} \approx 0.6R^2/\gamma_b^2.
\] (3)

For an ultra-relativistic bubble where \( \gamma_b \gg R \), this will only hold when \( H \approx 0 \), in accordance with the previous result [23], and electron trapping [22] becomes necessary. For an initially quiescent electron \( H_0 = 1 \) to get injected, the change of Hamiltonian, \( \Delta H = H(t) - H_{thresh} = H_{thresh} - 1 \) needs to hold.

When an electron interacts with an undulating bubble, its Hamiltonian can further increase or decrease, since the term \( z_{osc} \) in \( dH/dt \) will change sign according to period \( T_{CEP} \) [33]. This change of electron Hamiltonian can suppress or trigger electron injection at unmatched (matched) undulation phases, \( \phi_{CEP} = -\pi/2 \) \( (\pi/2) \) for \( \tilde{z} = -R \) and \( \phi_{CEP} = \pi/2 \) \( (-\pi/2) \) for \( \tilde{z} = R \), by increasing (decreasing) the Hamiltonian above (below) the injection threshold. The combined effect of expansion and undulation can result in periodic electron injections from the background plasma, which in turn modifies the injected bunch current profile.

To illustrate the modified injection condition and the effect of bubble expansion and undulation, we solve equations of motion (equations (B.1)–(B.4), see appendix B) for four initially quiescent electrons entering the bubble at \( t = 0, \xi = x - v_xt = 0, z = \pm R \) (figure 5). The bubble oscillates with a period \( T_{CEP} = 50 \) and also expands at different rate until \( t_{CEP} = 20 \). For a slow-expanding bubble(EB) that does not undergo undulations (dashed lines in figure 5(a): \( \zeta = 0 \)), the expansion rate \( \varepsilon = 0.001 \) is insufficient to cause injection. For a finite undulation amplitude (solid lines in figure 5(a): \( \zeta = 1.5/k_b \)), injection can be enabled, but only for certain undulation phases \( \phi_{CEP} \) or impact parameters \( \zeta_0 \). For example, for a given undulation phase, injection into an EPUB is enabled for the electron with \( \zeta_0 = R_0 \) (red solid line), but not for the electron with \( \zeta_0 = -R_0 \) (orange solid line). This example of a slowly-expanding undulating plasma bubble emulates the injection mechanism of Group III electrons described earlier in section 2.

For a fast-EB exemplified by figure 5(b), where the expansion rate is chosen as \( \varepsilon = 0.003 \), bubble undulations can play the opposite role of suppressing electron injections. For such expansion rate, electrons
Figure 5. Electron interaction with an evolving bubble: single-particle simulations. (a) and (b) show trajectories of background electrons interacting with slowly (a) and rapidly (b) expanding plasma bubbles with (solid lines) and without (dashed lines) simultaneous bubble undulations. Bubble expansion/undulation is marked by black/blue arrows. All electrons start at initial positions \((\xi_0 = 8/k_p, z_0 = \pm 6/k_p)\) (stars), and the bubble initial boundary without undulation is shown in thick gray lines. Bubble parameters are as follows: \(R_0 = 6/k_p, \gamma_b = 6, \varepsilon = 0.001\) (a), \(\varepsilon = 0.003\) (b), \(z_u = 1.5/k_p, T_{CEP} = 50/\omega_p, \phi_{CEP} = -\pi/2\). The lengths in the figure are in units of \(k_p^{-1}\).

Figure 6. Hamiltonian analysis of injection process. (a) and (b) shows time-evolution of the electron Hamiltonian \(H(t)\) for slow-EB (a) and fast-EB (b). Initial conditions are the same as in figure 5. Black dashed lines show injection threshold Hamiltonian \(H_{\text{thresh}}\). (c), (d) Time-evolution of the electron Hamiltonian increment \(\Delta H^{(1)}(t)\) for undulating \((z_u = 1.5/k_p)\) slow-EB (c) and fast-EB (d) for impact parameter \(\tilde{z}_0 = +R\). The lines denote \(\Delta H^{(1)}\) (equation (5), black solid), \(\Delta H_{\text{exp}}\) (black dashed), and \(\Delta H_{\text{thresh}} = H_{\text{thresh}} - 1\) (red). Injection is enabled for those \(0 < t < T_{CEP}\) inside blue-shaded regions. Bubble parameters are the same as in figure 5. The differences can be understood more quantitatively by tracking the electron Hamiltonian. For slow-EB, Electron Hamiltonian does not decrease below \(H_{\text{thresh}}\) without undulation, regardless of initial impact parameters (figure 6(a), green dashed, purple dashed). Only the electron experiencing matched phase

are injected when the bubble is not undulating (dashed lines). Similar to the slow-EB case, the electron with ‘matched’ impact parameter \(z_0 = R_0\) (red solid line) is injected, but the one with the ‘unmatched’ impact parameter \(z_0 = -R_0\) (orange solid line) is prevented from injection by a finite-amplitude plasma bubble undulation. This example of a rapidly-expanding undulating plasma bubble emulates the injection suppression mechanism of Group II electrons described in section 2.

This difference can be understood more quantitatively by tracking the electron Hamiltonian. For slow-EB, Electron Hamiltonian does not decrease below \(H_{\text{thresh}}\) without undulation, regardless of initial impact parameters (figure 6(a), green dashed, purple dashed). Only the electron experiencing matched phase
undulation and expansion (figure 6(a), red) can have \( H < H_{\text{thresh}} \) and get injected, in accordance with particle trajectory shown in figure 5(a).

For fast-EB, expansion is fast enough so the Hamiltonian decreases below \( H_{\text{thresh}} \) even without undulation (figure 6(b), green dashed, purple dashed). For this scenario, electron experiencing the unmatched phase undulation and expansion is the only one that is not injected (figure 6(b), orange) since its Hamiltonian increases above \( H_{\text{thresh}} \).

We have discussed electron injection from an evolving bubble for matched and unmatched phases. To better understand injection of background particles for all different phases, we give an estimate for final Hamiltonian for electrons that are most likely to get injected (\( \pm = \pm R \)) as the bubble propagates using estimates from equations (B.5)–(B.8). We compute Hamiltonian of electrons located on the sinusoidal trajectory defined by \( x_0 = v_0 t, y_0 = 0, z_0 = \pm R + z_{\text{osc}}(t) \). Electrons located on this trajectory graze the bubble boundary, entering the bubble at its edge. Change in Hamiltonian of electrons entering bubble at different time can be estimated by integrating:

\[
\Delta H = \int \left[ p_z(t)z_{\text{osc}}(t) - \frac{1}{4} \frac{p_x^2}{R(t)} \right] dt,
\]

where the integral is calculated along the electron trajectory.

To lowest order in bubble oscillation amplitude and expansion rate, we can use the quantities from a non-evolving bubble (\( v_0 = 0, \epsilon = 0 \)) to estimate change in Hamiltonian. Assuming passage time of electron through the bubble, \( T_{\text{pass}} \), is much smaller than oscillation period, \( T_{\text{CEP}} \), \( z_{\text{osc}} \approx -z_0 \omega_{\text{CEP}} \sin(\omega_{\text{CEP}} t_{\text{enter}}) \), with the time enters the bubble denoted by \( t_{\text{enter}} \). Furthermore, if the bubble radius varies slowly (\( \epsilon \ll 1 \)), \( R(t)R(t) \approx R^2 \epsilon \). This simplifies the integral to \( \Delta H \approx \Delta z_{\text{osc}} \omega_{\text{CEP}} \sin(\omega_{\text{CEP}} t_{\text{enter}}) \Delta p_z + (\Delta t_{\text{exit}} + \Delta x/R^2 \epsilon)^2/4 \) where \( \Delta p_z \) is the exit transverse momentum of the electron, \( \Delta t_{\text{exit}} \) is the time it takes the electron to reach the back of the bubble, and \( \Delta x \) is the longitudinal distance electron travels during its interaction with the bubble.

To estimate \( \Delta x + \Delta t_{\text{exit}} \) and \( \Delta p_z \), we solve equations (B.5)–(B.8) for initial conditions (\( dP_0/ds = \mp 1/4 \), \( dP_0/ds = 0, x_0 = 0, z_0 = \pm 1 \)). Re-scaling the variables (\( \Delta x, \Delta z \)) to (\( \Delta \xi, \Delta \zeta \)) and noting \( \Delta \zeta = \Delta x - \Delta t_{\text{exit}} \), one can obtain \( \Delta \xi + \Delta t_{\text{exit}} = 5.4R \) when the electron reaches maximum excursion from axis near the back at \( r = r_{\text{cm}} \). Similar procedure converting \( \Delta p_z \) to \( \Delta p_x \) can be carried out to calculate \( \Delta p_x = \mp 0.16R^2 \). Using these results, the Hamiltonian increment for an electron entering the bubble from the edge (\( z_0 = z_0 - z_{\text{osc}}(t) = \pm R \)) is given by:

\[
\Delta H^{(1)} = -1.35\epsilon R^2_0 \pm 0.16z_0 \omega_{\text{CEP}} R_0^2 \times \sin(\omega_{\text{CEP}} t_{\text{enter}}),
\]

where the two signs correspond to the two impact parameters \( z_0 = \pm R \). Equation (5) contains two terms with distinct behaviors: the bubble expansion contribution \( \Delta H_{\text{exp}} = -1.35\epsilon R_0^2 \) and the bubble undulation contribution \( \Delta H_{\text{osc}} = \pm 0.16z_0 \omega_{\text{CEP}} \sin(\omega_{\text{CEP}} t_{\text{enter}}) R_0^2 \). The former depends on the expansion rate of the bubble and does not depend on the time \( t_{\text{enter}} \) of the electron encounter with the bubble. On the contrary, the latter term depends on the bubble undulation amplitude, and is periodic in \( t_{\text{enter}} \) with a period \( T_{\text{CEP}} \). Therefore, bubble undulations can affect electron injection in two ways. Constructive contribution of bubble oscillations to electrons injection occurs when its expansion rate \( \epsilon \) is not sufficiently large to cause injection: \( 1 + \Delta H_{\text{exp}} > H_{\text{thresh}} \). In that case (see black dashed and red solid lines in figure 6(c)), finite \( \Delta H_{\text{osc}} \) can further reduce \( H \) and cause electron injection. If the oscillation amplitude of \( \Delta H_{\text{osc}} \) is large enough, bubble oscillations can reduce the Hamiltonian below \( H_{\text{thresh}} \) for some injection times \( t_{\text{enter}} \). As indicated by blue shading in figure 6(c), this would result in periodic electron injections during the time periods when \( \Delta H^{(1)} \) (solid black line) drops below the injection threshold \( H_{\text{thresh}} \) (solid red line). In principle, the intervals of electron injection can be as short as possible because the time intervals during which \( H < H_{\text{thresh}} \) can be arbitrarily short. Therefore, electrons injected owing to constructive contribution of bubble oscillations correspond to Group III electrons previously described in section 2.

Destructive contribution of bubble oscillations to electrons injection occurs when \( \epsilon \) is large enough to cause injection on its own: \( 1 + \Delta H_{\text{exp}} < H_{\text{thresh}} \) as indicated by black dashed and red solid lines in figure 6(d). However, finite \( \Delta H_{\text{osc}} \) will periodically increase \( \Delta H \) thereby suppressing injection for at least some time periods shaded white in figure 6(d). We note that the injection periods (blue-shaded region in figure 6(d)) cannot become arbitrarily short because the time periods during which injection is suppressed cannot be longer than \( T_{\text{CEP}}/2 \). Electrons injected owing to destructive contribution of bubble oscillations correspond to Group II electrons.

The two injection scenarios describing the emergence of Group II and III electrons predict that the injection process is periodic along the propagation direction, with a spatial period \( L_{\text{CEP}} \approx \epsilon T_{\text{CEP}} \). However, only the first scenario enables electron injection over a propagation distance \( L_{\text{inj}} < L_{\text{CEP}}/2 \) through the plasma. This finding illuminates the reason for the short durations \( \Delta \xi \) of the beamlets comprising Group
III electrons: because the injected electrons (moving with relativistic speed \( \approx c \)) are slowly slipping with respect to the back of the plasma bubble (moving with the speed \( \approx v_{bb} \)), plasma electron injected over a distance \( L_{inj} \) are compressed into ultra-short bunches with \( \Delta \zeta \approx L_{inj}/2\gamma_{bb}^2 \), where \( \gamma_{bb} = 1/\sqrt{1 - v_{bb}^2/c^2} \). Not surprisingly, ultra-short bunches with \( \Delta \zeta \approx L_{inj}/2\gamma_{bb}^2 \) were observed only for Group III electrons in our PIC simulations, as shown in figure 3(d).

Up to now, we have only considered \( \Delta H \) in the case of a plasma bubble undulating in one direction. Plasma bubble whose center moves along a helix can be also generated by a circularly polarized laser pulse. Under this circumstance, the bubble centroid would always have a finite undulation speed, although the direction of its transverse motion would be continuously changing. The helical nature of bubble undulations has a direct impact on electron injection into the bubble because at any point in time, there are some directions from which injection is either suppressed or enhanced. The effect of helical undulations of the bubble centroid on the temporal structure of the accelerated beam is investigated below in section 3.2. We note that it is also possible to use elliptically polarized laser to drive the bubble, which can generate a bubble whose center follows a helix elongated in a specific transverse direction.

3.2. Polarization-dependent current profiles: particle swarm simulations

The analytic calculations presented above demonstrate the feasibility of controlling electron injection via the combination of plasma bubble expansion and undulation. For analytic tractability, the calculations in section 3.1 are limited to those electrons that are most likely to be injected, i.e. having an impact parameter \( r_0 \equiv \sqrt{x_0^2 + y_0^2} \approx R \) matched to the unperturbed bubble radius \( R \). In fact, background electrons with different impact parameters interact with an evolving bubble; some of these electrons get injected into it. Therefore, in order to interpret the results of our PIC simulations presented in section 2, it is necessary to simulate a large-volume ‘swarm’ of background plasma electrons interacting with a plasma bubble. This was done by seeding test particles into a three dimensional volume spanning a wide range of initial conditions \((R < x_0 < 80, -10.5 < y_0 < 10.5)\) and launching three different evolving spherical potentials that can capture and accelerate particles: (1) an EB, (2) an expanding and helically undulating bubble (EHUB), and (3) an expanding and linearly undulating sphere (ELUB).

In the EHUB case, the center of the bubble moves transversely according to \( z_{osc}(t) \equiv z_0 \cos(\omega_{CEP} t + \phi_{CEP}), y_{osc}(t) \equiv z_0 \sin(\omega_{CEP} t + \phi_{CEP}), \) and the 3D particle equations of motion are similar to equations (B.1)–(B.4) derived for a particle traveling in the \( x-z \) plane, but now contains two more differential equations. These can be derived by taking the partial derivatives of the canonical momentum and coordinate in the \( y \) direction. Finally, the EB case can be retrieved by assuming no transverse bubble undulation \((z_0 = 0)\).

As can be seen from the injection rate (figures 7(a) and (b)), only the ELUB injection rate is periodically modulated. While the injection rate of an EHUB does not exhibit modulation, the transverse location from which they originate from does show periodic modulation (figure 7(a)). Injected electron distribution in figures 7(a) and (b) and injection rate in figure 7(b) are periodic. Their initial location projected onto \( x-z \) plane has an approximate periodicity of \( \Delta x_0 = v_{bb} T_{CEP} \approx 50 \), in agreement with the time dependence of \( \Delta H(t) \). We note that injection rate for ELUB has periodicity \( v_{bb} T_{CEP}/2 \), since the injection process happens twice, at \( \tilde{z} \approx \pm R \), for each undulation period.

After the electrons are injected into the bubble, they quickly gain relativistic energy from the accelerating field and move at ultra relativistic velocity. Because the bubble phase velocity is slower than that of the injected electrons, electrons will advance through the bubble after acceleration to ultra relativistic energy. This slippage of the back of the bubble from the injected bunches determines the longitudinal structure of the injected bunch in the case of a linearly undulating bubble. One can estimate the periodicity of the bunch modulation via converting injection periodicity to that in the frame moving with the speed of light, \( (\zeta = x - ct) \). The rear of the bubble moves at \( v_{bb} = v_{bb} - R \tilde{z} \) slower than bubble velocity because of bubble expansion. While the CEP phase slips one cycle, the back of the bubble slips away from the ultra-relativistic particles by distance \( \Delta \zeta = (c - v_{bb}) T_{CEP} \). Because there are two injection per one oscillation, injected bunch forms a structure with longitudinal modulation \( \Delta \zeta = (c - v_{bb}) T_{CEP}/2 \approx T_{CEP}/4\gamma_{bb}^2 \) (figure 7(d)). We note that this is the formula that was cited for bunch modulation periodicity in section 2.

3.3. Effect of bubble undulations on the betatron radiation emitted by injected/accelerated electrons

Laser-wakefield generated electrons can emit collimated, high-brightness \( x \)-rays via betatron radiation [50]. Because the injected electrons form beams with femtosecond-scale durations, the resulting pulses of betatron radiation have a similarly ultrashort temporal format and, potentially, tunable polarization imparted by that of the laser pulse [51–53]. Such \( x \)-rays have been used to image various targets with fine details. Some of the recent examples include irregular eutectics in the aluminum–silicon (Al–Si) system [54], as well as various
biological samples [55]. Below we demonstrate that the EPUB injection mechanism provides a new approach to controlling x-ray polarization, intensity, and angular distribution.

Our particle swarm simulation shows that electrons injected by an expanding and undulating bubble gain more transverse momentum than those injected into a merely expanding bubble. We plot the electron distribution in the $(\gamma, p^\perp_{\text{max}})$ space at propagation distance of $t = 400$ for expanding (figure 8(a)) and expanding and linearly undulating (figure 8(b)) bubbles. Here, $p^\perp_{\text{max}} = \sqrt{2 \gamma \epsilon^\perp}$ denotes the maximum possible transverse momentum derived from the transverse energy $\epsilon^\perp = p^2_\perp / 2 \gamma + r^2 / 4$ [56].
We observe from figure 8(a) that the electrons injected into a merely expanding bubble do not spread out in the \((\gamma, p_{\perp}^{\max})\) phase space, forming a line-like feature. On the contrary, those injected by an ELUB significantly spread out in phase space, as can be observed from figure 8(b). This difference has direct consequences for the radiated x-ray spectra because the frequency range of the x-ray photon energy emitted by the electrons is determined by the critical frequency \(\omega_{\text{crit}}\) that can be estimated as [56]:

\[
\omega_{\text{crit}} \approx \frac{3\omega_{p}}{\sqrt{8}} p_{\perp}^{3/2}.
\] (6)

Therefore, an undulating bubble could potentially yield higher-energy x-rays because of the larger values of \(p_{\perp}\). Also, a wider angular spread is predicted for x-ray generated from undulating bubble because synchrontron-like radiation has an opening angle of \(\theta_{\text{emission}} \approx 0\) [57].

To compare x-ray emission from the non-undulating and undulating bubble, we have calculated the betatron radiation from a swarm of relativistic test particle trajectories according to the standard expression [57, 58]:

\[
\frac{d^2 I}{d\omega d\Omega} = \frac{e^2 \omega^2}{4\pi^2 c} \int_{-\infty}^{\infty} \mathbf{n} \times \mathbf{\beta} e^{\pm i(n \cdot r(t)/c)} dt^2.
\] (7)

The integration was carried out using an in-house code Simple Incoherent Radiation Calculation (SIRC) [59]. We compute the Stokes parameters according to the formula \(I = S_0 = E_\gamma \cdot E_\gamma^* + E_z \cdot E_z^*\), \(S_1 = E_y \cdot E_\gamma^* - E_\gamma \cdot E_y^*\), \(S_2 = E_\gamma \cdot E_z^* - E_z \cdot E_\gamma^*\), \(S_3 = -i(E_y \cdot E_z^* - E_z \cdot E_y^*)\). The polarization degree is expressed as \(P = \sqrt{S_1^2 + S_2^2 + S_3^2}/S_0\) [60], and the fraction of vertical and horizontal polarization may be characterized by \(P_1 = S_1/S_0\).

Because electrons are symmetrically injected into an EB, they undergo betatron oscillations in the \(y-z\) plane without any directional preference, and without significant transverse momentum spread, as indicated in figure 8(a) by their vertical clustering. Therefore, their betatron (x-ray) emission is symmetric and confined in small angular region with FWHM \(\Delta \theta_y \approx \Delta \theta_z \approx 10\) mrad, as observed in figure 8(c). The normalized Stokes parameter \(S_1/S_0\) characterizing the degree of linear polarization of the resulting x-rays is plotted in figure 8(d). We observe that the x-rays are essentially un-polarized (\(S_1/S_0 \approx 0\)) near the axis (\(\theta_y = \theta_z \approx 0\): inside the white circle), where the radiation intensity is the highest. While direction-dependent linear polarization is observed at larger emission angles, the number of such x-rays is small.

The situation is qualitatively different for the electrons injected into an ELUB undulating along the \(z\)-direction. These electrons, initially trapped near the axis, subsequently experience transverse kicks in the undulation direction and start executing large betatron oscillations that are predominantly along the \(z\)-direction. Consequently, the x-rays are emitted with anisotropic angular distribution as shown in figure 8(e): \(\Delta \theta_y \ll \Delta \theta_z \approx 70\) mrad. Moreover, the on-axis x-rays are strongly polarized in the undulation direction, as can be observed in figure 8(f): \(S_1/S_0 \approx -0.5\). We observe that the angular intensity distribution of x-rays plotted in figure 8 strongly correlates with their polarization properties. In the case of an ELUB, the x-rays are primarily polarized in the \(z\)-direction according to figure 8(f), and their angular distribution is anisotropic, i.e., also elongated in the same direction according to figure 8(e). Likewise, the lack of x-ray polarization in the EB case correlates with their isotropic angular distribution.

As discussed earlier, the spectra of the emitted x-rays extend to higher energies for the linearly-undulating (ELUB) bubble vs the elongating bubble without undulations (EB), as indicated by the red (for ELUB) vs black (for EB) lines in figure 8(g). Also, the on-axis x-rays from the ELUB are more than 50% polarized in the undulation direction, in comparison to almost unpolarized on-axis x-ray from the EB non-undulating bubble: see the inset of figure 8(g). Notably, the resulting high degree of linear polarization is achieved without introducing a tilt in the laser pulse front, or using asymmetric laser intensity distribution [61]. Instead, the x-rays are linearly polarized because they are emitted via betatron radiation by trapped/accelerated plasma electrons subjected to the transverse wake \(W_{\perp} \approx e_z W_z\) that originates from laser polarization controlled undulations of a plasma bubble.

4. Potential applications and telltale signs of phase-dependent laser-plasma acceleration

In this section, we discuss possible applications of electron injection and acceleration using the EPUK approach. Those include highly efficient generation of high-charge high-energy electron beams and the control of the electron beam pointing by the CEP offset. We will also describe x-ray generation by the electrons produced using differently-polarized laser pulses, discussing the effect of the laser polarization and its CEP offset on the x-ray properties, such as their angular distribution. Unlike section 3.3, where
single-particle simulations of electron injection and acceleration in prescribed fields were used, here we use first-principles PIC simulations to model x-ray generation. Therefore, collective effects such as wake depletion and subsequent transition from LWFA to PWFA regimes, are properly accounted for. Many of the observables described below, such as finite electron pointing angle from the laser axis and angular asymmetry of the emitted betatron radiation, can be used as telltale signs of phase- and polarization-dependent effects in ultra-intense laser plasma interactions. In the rest of this section, we assume the same laser and plasma parameters as used in section 2, listed in the caption of figure 2 and table 1. Energy spectra of the accelerated electrons are shown in figure 4(c). For conceptual simplicity, only linear and circular polarizations of the driver laser are considered in this section.

4.1. Efficient conversion of laser pulse energy into electron kinetic energy

One of the key desirable metrics of any laser plasma wakefield accelerator (LPWA) scheme is the high energy conversion efficiency of the laser energy into the energy of accelerated electrons. It has been shown that few-cycle driven lasers can efficiently transfer their energy into the injected electrons. In our simulations presented in figure 2, electrons are injected almost as soon as the pulse enters plasma, injecting extremely high charge ($Q \sim 10 \text{nC}$) into the wakefield. The large injected charge can efficiently convert the wakefield energy excited by the pulse into electron kinetic energy. Furthermore, the pulse is almost depleted by the time the monoenergetic peak reaches $\sim 200 \text{MeV}$ as shown in figure 4(c). Therefore, the depletion and dephasing lengths are well-matched, and almost no pulse energy is wasted because of the injected electrons entering the decelerating portion of the bubble. This results in high efficiency ($E_{\text{beam}}/E_{\text{laser}} \sim 50\%$, $E_{\text{laser}}$ is the laser pulse initial energy and $E_{\text{beam}}$ is the injected electron beam’s kinetic energy) of laser energy conversion into electron kinetic energy. Below we discuss how the resulting high-charge ultra-relativistic near-monoenergetic electron bunch can be utilized for producing large amounts of polarization-dependent and CEP-dependent x-rays via betatron radiation.

4.2. Radiation generation and beam asymmetry

After the formation of a quasi-monoenergetic high-charge bunch, the laser pulse energy is depleted and can no longer excite a strong wake at the propagation distance $c t_1 = 1000 \lambda_e = 9.2 \text{ mm}$, as shown in figures 9(b) and (d). However, the large accelerated charge can now excite its own plasma wake afterwards, resulting in a transition from the LWFA into the plasma wakefield accelerator (PWFA) regime as illustrated in figures 9(c) and (e) for $c t_2 = 1500 \lambda_e = 13.8 \text{ mm}$. A further acceleration of the trapped electrons injected during the later LWFA stages ensues, leading to larger spread but higher peak of electrons energy. According to equation (6), higher-energy electrons can produce more energetic x-ray photons via betatron radiation. The transition to PWFA regime results in much larger numbers and energies of the x-rays produced inside the $0 < x < c t_2$ (red line) region than in the $0 < x < c t_1$ (blue line) region, as shown in figure 9(a). After laser
depletion, bubble undulations essentially stop, as can be observed from a more symmetric plasma bubble shape at \( t = t_2 \) than at \( t = t_1 \); compare figures 9(b) and (c). Nevertheless, several laser polarization- and phase-dependent observables persist even after the laser pulse depletion, i.e. after the LWFA-to-PWFA transition.

First, the electron beam acquires an elongated shape along the direction of the laser polarization, as well as a CEP-dependent average transverse tilt, as shown in figures 9(f) and (g). Specifically, the FWHM angular electron spread in the \( y \)- and \( z \)-directions is given by \( \delta \theta_y = (4 \text{ mrad}, 8 \text{ mrad}) \) after the propagation distance of \( c t_1 \) through the plasma. Therefore, the angular spreading of the electron beam is anisotropic, i.e. more extended along the laser polarization direction \( z \). This anisotropic angular beam spread is responsible for the similarly anisotropic spread of the emitted x-rays shown in the insets of figure 9(a).

Moreover, we note that the average pointing direction of the beam, characterized by \( \langle \Theta_y, \Theta_z \rangle \), depends on the CEP offset. As shown in figures 9(f) and (g), the change from \( \phi_{CEP} = 0 \) to \( \phi_{CEP} = \pi/2 \) reverses the overall deflection angle of ultra-relativistic (\( \gamma > 380 \)) electrons from \( \langle \Theta_z \rangle(0) \approx -1 \text{ mrad} \) to \( \langle \Theta_z \rangle(\pi/2) \approx 1 \text{ mrad} \). Even though \( \langle \Theta_y \rangle \) is smaller than \( \Delta \Theta_z \), by nearly an order of magnitude, it still appears to be an experimentally measurable quantity [28, 40, 65].

Second, the emitted x-rays retain their elongated angular distribution (see the two insets in figure 9(a)) even after propagating the distance of \( c t_2 \) through the plasma. For example, the ratio of the angular spreads along \( \Delta \theta_y^{(2)} \approx 13 \text{ mrad} \) and perpendicular to \( \Delta \theta_y^{(1)} \approx 6 \text{ mrad} \) the laser polarization direction is \( \Delta \theta_y^{(2)}/\Delta \theta_y^{(1)} \approx 2 \) at \( t = t_2 \). While this ratio is somewhat smaller than the one observed at \( t = t_1 \) \( (\Delta \theta_y^{(2)}/\Delta \theta_y^{(1)} \approx 3) \), it can be measured and used for estimating the magnitude of the electrons betatron trajectories [66, 67].

Finally, we analyze the dependence of the angular distribution \( I(\Theta_y, \Theta_z) \) of the beam-generated x-ray flux on the CEP offset \( \phi_{CEP} \). If \( I(\Theta_y, \Theta_z) \) is indeed CEP-dependent, then it can be used as a telltale sign of phase-dependent beam dynamics in the plasma bubble. Because multiple electrons undergo several betatron oscillations inside the plasma bubble after their injection, and those electrons may not be fully correlated in their motion, it is not \textit{a priori} obvious that the absolute laser phase (characterized by \( \phi_{CEP} \)) will have a measurable impact on the angular distribution of the emitted x-rays. Broadly speaking, we are investigating if there is an imprint of the absolute phase of a collection of low-energy laser photons onto a collection of high-energy x-ray photons generated via a fairly complex and indirect up-conversion process of the former. Here \( h \omega_{x, \text{ray}} \approx 1 \text{ keV} \) while \( h \omega_{\text{LWFA}} \approx 0.11 \text{ eV} \), i.e. such up-conversion is inherently an extremely high-order process.

Surprisingly, we find that there is a small but non-negligible CEP dependence on the angular x-ray asymmetry, i.e. \( I \equiv I(\Theta_y, \Theta_z; \phi_{CEP}) \) can be expressed as \( I = I_0(\Theta_y, \Theta_z) + \delta I(\Theta_y, \Theta_z; \phi_{CEP}) \), where \( I_0 \) is CEP-independent even function of \( (\Theta_y, \Theta_z) \) and \( \delta I \) the small CEP-dependent quantity satisfying \( |\delta I| \ll I_0 \). Not only does the angular x-ray distribution have a significantly larger spread in the \( z \)-direction than in the \( y \)-direction (i.e. \( I_0(\Theta_y, \Theta_z) \) is anisotropic and polarization-dependent, as presented in figure 9(a) earlier), but also its small asymmetric deviation from \( I_0 \) is a function of the laser CEP offset.

To better visualize the dependence of the asymmetry function \( \delta I \) on \( \phi_{CEP} \), we introduce and plot two angle-integrated x-ray fluxes in figures 10(a) and (b): \( \hat{I}_y(\phi_{CEP}) \equiv \int I(\Theta_y, \Theta_z; \phi_{CEP}) \, d\Theta_z \) and \( \hat{I}_z(\phi_{CEP}) \equiv \int I(\Theta_y, \Theta_z; \phi_{CEP}) \, d\Theta_y \). These quantities are plotted in figures 10(a) and (b) for three values of \( \phi_{CEP} = 0, \pi/2, \pi \). While the dependence of \( \hat{I}_y \) on the CEP offset is negligible, there is a small but visible dependence of \( \hat{I}_z \) on \( \phi_{CEP} \) (figure 10(b)). The corresponding differences between the integrated x-ray fluxes calculated for \( \phi_{CEP} = 0 \) and \( \phi_{CEP} = \pi \), defined as the CEP contrasts \( \delta I_y(\Theta_y; \phi_{CEP}) \equiv I_y(\Theta_y; 0) - I_y(\Theta_y; \pi) \) and \( \delta I_z(\Theta_z; \phi_{CEP}) \equiv I_z(\Theta_z; 0) - I_z(\Theta_z; \pi) \), are plotted in figure 10(c). The two plots clearly show that \( |\delta I_z| \gg |\delta I_y| \), confirming that there is a much larger CEP contrast for the x-rays emitted in the LP direction.

4.3. LP dependence of the spectral brightness and angular distribution of the x-ray flux

As noted earlier, different polarizations of the driving laser pulse result in distinct phase space distributions of the accelerated electrons: see figures 4(a) and (b) for the comparison between the CP and LP cases. Consequently, the radiated photon spectra are also expected to vary accordingly. For example, a LWFA driven by a CP pulse generates more photons at the lower energy range, but an LP-driven LWFA generates more photons at higher energies as shown in figure 11(a). Likewise, the angular distribution of the x-ray flux produced by the betatron radiation of the accelerated electrons is also polarization-dependent. Confirming our findings from the swarm simulations (see figures 8(c)–(g)), an LP driver generates x-ray with angular distribution that is strongly elongated in the laser polarization direction, as shown in figure 11(b). Notably, the elongation direction does not depend on the CEP offset, as was shown earlier in figures 10(a) and (b).

While a CP laser driver also generates angularly-anisotropic x-ray flux elongated in one direction, the elongation direction itself can, in principle, be dependent on the CEP offset of the laser pulse. The CEP effect
on the angular x-ray distribution \( I(\theta_y, \theta_z; \phi_{\text{CEP}}) \) in the CP case is illustrated in figures 11(c) and (d), where the intensity ellipse is rotated from the \(-45^\circ\) position for \( \phi_{\text{CEP}} = 0 \) to \(+45^\circ\) position for \( \phi_{\text{CEP}} = \pi/2 \). This is because the angle at which highest energy electrons are injected rotates according to the CEP offset of the CP pulse, leading to most intense betatron radiation generated at a different angle.

Such asymmetric CEP-controlled distribution of x-rays occurs despite the fact that the electron injection is mostly continuous in the case of CP driver. Nevertheless, injected electrons ‘remember’ the direction of the laser pulse at the time of laser pulse steepening. This direction, which is determined by the CEP offset of a CP-polarized laser, is reflected by the direction of a helical transverse wake \( W_\perp \) at the back of the bubble, where electron injection takes place. Characterization of the phase of CP laser pulses has been proposed in the case of ultra-intense lasers [68], but it remains an open area of active research for multi terawatt class lasers. We envision that betatron radiation may enable the characterization of CEP offsets of CP laser pulses.

4.4. Summary of experimental signatures for different LP

We summarize some experimental verification that could be accessed in a laboratory in order to verify our findings, with an eye on confirming the polarization-dependent injection and acceleration process (table 2).

As mentioned in section 2, the lower-energy electron spectra profile shows stark difference depending on the LP. The presence of Group III electrons lead to prominent peaks in the lower-energy wings of the electron spectra in the LP case in contrast to the relatively flat low-energy energy spectra for the CP case (figure 4(c), left inset). Another key observable is the betatron radiation intensity distribution. For LP, the injected electrons preferentially execute betatron oscillation in the LP direction regardless of CEP, and will have an elongated x-ray intensity distribution in the polarization direction. In contrast, the intensity in the CP case will rotate according to CEP (figures 11(c) and (d)).

Figure 10. CEP-dependent angular x-ray flux distributions for an LP laser. (a) and (b) shows angle-averaged x-ray flux \( \bar{I}_{(\theta)} \) integrated over \( \theta_y \) and \( \theta_z \), respectively. Solid lines denote (a) \( \bar{I}_{(\theta_y; \phi_{\text{CEP}})} \) and (b) \( \bar{I}_{(\theta_z; \phi_{\text{CEP}})} \). Line colors denote the following: \( \phi_{\text{CEP}} = 0 \) (blue), \( \pi/2 \) (red), and \( \pi \) (yellow). (c) shows CEP contrasts between \( \phi_{\text{CEP}} = 0 \) and \( \phi_{\text{CEP}} = \pi \), with different colors denoting \( \delta \bar{I}_{(\theta_y)} \) (red line) and \( \delta \bar{I}_{(\theta_z)} \) (blue line). Propagation distance at which these x-rays were simulated was \( c t = 1000\lambda_L \), counting photons with energy \( h\omega_{x-ray} < 3 \text{ keV} \).

Figure 11. CEP dependence of the angular distribution of betatron x-rays for CP and LP laser drivers at propagation distance of \( c t = 1000\lambda_L \) for photon energy \( h\omega_{x-ray} < 3 \text{ keV} \). (a) shows x-ray spectra for LP (red) and CP (blue) laser pulses, (b)–(d) shows x-ray intensity distribution for photons below 3 keV generated by LP driver \( \phi_{\text{CEP}} = 0 \) (b), CP driver \( \phi_{\text{CEP}} = 0 \) (c) and CP driver, \( \phi_{\text{CEP}} = \pi/2 \) (d). White lines: \( 135^\circ \) (c), \( 45^\circ \) (d). Parameters: same as in figure 9. Propagation distance: \( c t = 1000\lambda_L \). Photons energy: \( h\omega_{x-ray} < 3 \text{ keV} \).
4.5. Discussion of beam quality and potential use as seeded FEL

The temporal beam modulation makes the EPUB-generated beam an interesting candidate for seeded FEL using LPWA [16, 69]. However, stringent requirements on beam quality such as energy spread of the whole beam at a sub-percent level and O(nm) emittance as well as bunching length of O(nm) are required for such uses [69].

After the end of LWFA stage at 6.4 mm propagation (figure 2(c)), the injected electron beam forms a monoenergetic peak with FWHM of 4% for LP and 2% for CP. Furthermore, the emittance for the monoenergetic electrons $|\gamma - 400| < 20$ are 2.2 mm-mrad-$\pi$ for LP and 3.3 mm-mrad-$\pi$ for CP. For the whole relativistic part of the beam ($\gamma > 200$), the emittance increases to 2.4 mm-mrad-$\pi$ for LP and 6.7 mm-mrad-$\pi$ for CP.

As of now, the EPUB-generated beam quality is not suitable for FEL [70]. However, unlike the sophisticated injection schemes requiring a separate electron beam driver and a density modulated downramp formed by multiple laser pulses [16, 69], EPUB is simple, requiring only a plasma jet and a short-pulse laser. Consequently, it can be easily accessed by smaller university-scale laboratories and has less source of experimental error compared to the above mentioned schemes. If the beam quality can be improved, the EPUB generated beam could be a promising alternative source for seeded FEL. Future work to optimize emittance and energy spectra of the entire bunch will be carried out, and the extent to which the bunching wavelength can be modified will also be explored. Given the large charge ($Q > 10 \text{nC}$) of the beam, interesting opportunities may arise by further manipulation of the beam, such as choosing a small fraction of beam by collimating (and thus reducing the emittance) or selecting a small energy slice of the beam (and thus reducing the energy spread).

5. Conclusions

In this paper, we proposed the concept of an EPUB, in which a self-steepened few-cycle multi-TW laser pulse produces an expanding and undulating plasma bubble in its wake. EPUB can trap electron bunch with spatiotemporal structure. The degree of bunching is controlled via changing laser polarization, alternating between highly modulated high-current beam or a flat-current beam with fs scale modulation.

PIC simulation shows that indeed a structured beam with larger charge (O(nC)) is trapped. Appreciable fraction of the injected beam forms a highly mono energetic energy peak, and a large fraction (50%) of laser pulse energy is efficiently transferred to the injected electrons. The modulation period can be altered via changing laser power, and the precise location of some of the injected bunch can be controlled via changing laser CEP.

Experimental observables to determine the laser polarization and CEP are proposed. By measuring electron beam pointing or x-ray intensity distribution, the absolute CEP of the laser can be retrieved. Furthermore, x-ray intensity distribution and lower-energy electron spectra shows stark contrast depending on the laser polarization, and may be accessed by experiments to confirm the polarization-dependence of EPUB.

EPUB injection and acceleration scheme is not limited to mid-infrared (e.g. $\text{CO}_2$) lasers, and can be applied to shorter wavelength few-cycle TW class lasers. This will provide an efficient way to generate bright betatron x-ray source in the soft x-ray regime switching between pulsed and continuous configuration with controllable duration and time-delay. It can also serve as a way to calibrate or retrieve the CEP of any laser system that does not have phase stabilization.

High-flux polarized x-ray radiation (including soft x-rays with energies below $h\omega_{\text{x-ray}} < 1 \text{keV}$) is of great importance for element-specific studies in a variety of scientific fields, including wet cell biology [71], condensed matter physics, extreme ultraviolet optics technology, and warm dense matter [71–74]. Of particular interest for time-resolved x-ray absorption spectroscopy are femtosecond broadband features [72] of betatron radiation. With improvement in beam quality and change in modulation length which is a subject of future work, our scheme is also a promising candidate for seeded FELs. It is likely that our scheme will be of interest in generating linearly-polarized femtosecond broadband x-rays, seeded FEL, and other applications.

### Table 2. Summary of potential experimental signatures.

| Spikes in lower-energy spectra | Linearly polarized (LP) | Circularly polarized (CP) |
|-------------------------------|------------------------|--------------------------|
| Betatron intensity elongation  | Yes                    | In polarization direction | No                       |
|                               |                        | Rotates according to CEP  |

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*Note: The table content is not transcribed as it contains specific numerical data that is not relevant to the overall context of the discussion.*
**Data availability statement**

The data cannot be made publicly available upon publication because the cost of preparing, depositing and hosting the data would be prohibitive within the terms of this research project. The data that support the findings of this study are available upon reasonable request from the authors.

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**Appendix A. Control of the injected/accelerated bunch profile using laser intensity and CEP offset**

In this appendix, we describe additional PIC simulations demonstrating that the current profile of the injected and accelerated electron bunch can be further controlled by the intensity and the CEP of the laser pulse. The same spatial/temporal laser parameters as in section 2 (see the caption of figure 2 and table 1) and two laser powers (low: \( P_L^{(1)} = 40 \)TW and high: \( P_L^{(2)} = 50 \)TW) are used.

While bunches are injected into the bubble throughout multiple undulation cycles, the absolute value of the CEP offset \( \phi_{\text{CEP}} \) of a few-cycle pulse described in section 2 \((cT_{\text{FWHM}}/\lambda_L \sim 3)\) can have direct effect on the temporal profile of the injected electron bunch. To analyze the dependence of the current profile on \( \phi_{\text{CEP}} \), we carried out PIC simulations for the low-power case \((P_L = P_L^{(1)})\) laser pulse with \( \phi_{\text{CEP}} = 0 \) and \( \phi_{\text{CEP}} = \pi/2 \). Because \( \phi_{\text{CEP}} \) controls the phase of the bubble undulation \( z_{\text{osc}}(t) \) in the laser polarization direction \( z \), and that of the transverse wake \( W_z(t, \zeta) \), electron injection times are also \( \phi_{\text{CEP}} \)-dependent. This can be observed in figures A1(a) and (b), where the injection rates and the current profiles are plotted as a black \((\phi_{\text{CEP}} = 0)\) and blue \((\phi_{\text{CEP}} = \pi/2)\) lines in figures A1(a) and (b), respectively.

Not surprisingly, only Group II and III electrons (see section 2.1 for the definitions of the three electron groups) are affected by the CEP offset because bubble undulations can either suppress or enhance electron injections in a phase-dependent manner. Namely, the locations of the troughs in the \( \phi_{\text{CEP}} = 0 \) case become locations of peaks for \( \phi_{\text{CEP}} = \pi/2 \) case. On the other hand, Group I (i.e. earlier injected electrons) are not significantly controlled by the absolute value of \( \phi_{\text{CEP}} = \pi/2 \) because (i) their undulation amplitude is small, and (ii) bubble undulations neither suppress nor enhance electron injections. Other CEP-independent effects, such as beam loading, play a more dominant role in determining injection dynamics. For completeness, the amplitude of the transverse wake \( W_z(t, \zeta) \) at the fixed position \( \zeta_0 = 35\lambda_L \) near the rear of the bubble is plotted in figure A1(c) for the two CEP phases as a black line \((\phi_{\text{CEP}} = 0)\) and a blue line \((\phi_{\text{CEP}} = \pi/2)\). The two curves are shifted in time by \( \pi/2 \) according to figure A1(c).

In addition, the transverse wake is plotted in figure A1(c) for a more intense laser pulse with the peak power \( P_L = P_L^{(2)} \). As expected, the transverse wake (orange line) exceed in magnitude that produced by the lower-intensity laser with \( P_L = P_L^{(2)} \) (black and blue lines). However, the oscillation period \( T_{\text{CEP}} \) of the transverse wake is the same for both laser powers. This is expected because the expression for \( T_{\text{CEP}} \) is intensity-independent: \( T_{\text{CEP}} \approx \lambda_L/(\nu_{ph} - \nu_b) \sim (\lambda_L/c)(\omega_b^2/\omega_L^2) \). Therefore, the injection period is also intensity-independent as can be seen in figure A1(d), where the injection rates along the laser propagation direction \( x \) are plotted for \( P_L = P_L^{(1)} \) (black line) and \( P_L = P_L^{(2)} \) (orange line); the same \( \phi_{\text{CEP}} = 0 \) was used in both simulations.

The modulation period \( \Delta \zeta \) of the injected electrons is, however, highly sensitive to the velocity \( \nu_{bb} \) of the back of the expanding plasma bubble, which in turn depends on the rate of plasma bubble expansion according to \( \nu_{bb}/c = \nu_b/c - k_p R_b \varepsilon \). As shown in section 2.1, the resulting compression effect of the injected electron bunches results in their periodicity given by \( \Delta \zeta \approx cT_{\text{CEP}}/4\nu_{bb}^2 \), where \( \gamma_{bb} = 1/\sqrt{1 - \nu_{bb}^2/c^2} \). These dependencies suggest that, by keeping plasma density and laser frequency same but increasing the pulse power, one can increase the bubble radius \( R_b \) and the expansion rate \( \varepsilon \). This would effectively slow down the back of the plasma bubble and result in a smaller \( \gamma_{bb} \), leading to reduced bunch compression and increased current modulation period \( \Delta \zeta \).

By plotting the location \( \zeta_{bb} \) of the back of the plasma bubble for the two laser powers in figure A1(f), we indeed confirm that \( \nu_{bb} \) becomes slower for \( P_L = P_L^{(2)} \) (orange line) that for \( P_L = P_L^{(1)} \) (black line).
Figure A1. Injection control using LP laser CEP and power. (a) and (b) shows injection rate (a) and current (b) for different CEP with same laser power (40 TW). (d) and (e) shows injection rate (d) and current (e) for different power with same CEP. (c) shows transverse wake at $\zeta = 35\lambda_L$ and (f) shows bubble rear position. Black line shows quantities for 40 TW $\phi_{\text{CEP}} = 0$ laser pulse, blue line shows quantities for 40 TW, $\phi_{\text{CEP}} = \pi/2$ laser pulse, and orange line shows quantities for 50 TW, $\phi_{\text{CEP}} = 0$ laser pulse.

Quantitatively, $\gamma^{(1)}_{bb} \approx 5$ and $\gamma^{(2)}_{bb} \approx 4.5$. This results in a longer bunch modulation period for the higher-power laser pulse: we observe from figure A1(e) that the $20\lambda_L < \zeta < 33\lambda_L$ window contains 8 current peaks for the $P_L = P^{(1)}_L$ case (corresponding to $\Delta \zeta^{(1)}/c \approx 40$ fs), and only six current peaks for the $P_L = P^{(2)}_L$ case (corresponding to $\Delta \zeta^{(2)}/c \approx 55$ fs). Therefore, both the period and the absolute timing of laser-accelerated femtosecond electron bunches can be controlled using the CEP offset and the peak power of a few-cycle laser pulse.

Appendix B. Equation of motions from the moving frame Hamiltonian

In this appendix, we introduce the equations of motion for the electron interacting with a moving bubble which we use throughout the text.

From the moving frame Hamiltonian in section 3, the following equations of motion can be derived using $dP/dt = -\partial H/\partial P$, $dP/dt = \partial H/\partial P$:

\[ \frac{d\xi}{dt} = \frac{p_x}{\gamma} - v_b, \]  
\( \text{(B.1)} \)

\[ \frac{dp_x}{dt} = -\frac{1}{4} \frac{X^2}{\gamma} + \frac{1}{2} \frac{p_x}{\gamma^2} - \frac{1}{4} \frac{P_z}{\gamma}, \]  
\( \text{(B.2)} \)

\[ \frac{dz}{dt} = p_z / \gamma, \]  
\( \text{(B.3)} \)

\[ \frac{dp_z}{dt} = -\frac{(v_x + 1)\tilde{z}}{4}, \]  
\( \text{(B.4)} \)

which are used to numerically compute the electron trajectories in the test-particle simulations.

A convenient way to describe electron interaction with a non-evolving plasma bubble has been derived [49] under which time, length, and momentum are normalized again from that of equations (B.1)–(B.4): $s = t/R$, $X = \xi/R$, $Z = z/R$, $P_x = p_x/R^2$, $P_z = p_z/R^2$. We note that the quantity $k_0$ is already the unit-less bubble radius normalized by $k_0^{-1}$. Dropping terms of $O(R^{-4})$, $O(\gamma^{-2})$ from equations (B.1)–(B.4) for a non-evolving bubble, differential equations independent of bubble radius can be found, which we reproduce here for convenience:

\[ \frac{dp_x}{ds} = -\frac{X}{2} + \frac{P_x}{\sqrt{p_x^2 + p_z^2}} Z \]  
\( \text{(B.5)} \)
These equations can be numerically solved using initial conditions appropriate for electrons that are most likely to get injected (i.e. at the edge of the bubble), namely $X_0 = 0$, $s_0 = 0$, $Z = \pm 1$. The numerical solutions of these equations when the electron reaches the back of the bubble ($X^2 + Z^2 = 1$) can be re-scaled to find physical values. For a slowly evolving bubble, these equations are convenient ways to estimate the zeroth order quantities, as was done in section 3.

**ORCID iDs**

Jihoon Kim  &  https://orcid.org/0000-0002-4826-238X

Tianhong Wang  &  https://orcid.org/0000-0002-5934-8031
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