CONVERGENCE ERROR ESTIMATES AT LOW REGULARITY
FOR TIME DISCRETIZATIONS OF KDV

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Abstract. We consider various filtered time discretizations of the periodic Korteweg–de Vries equation: a filtered exponential integrator, a filtered Lie splitting scheme as well as a filtered resonance-based discretization and establish convergence error estimates at low regularity. Our analysis is based on discrete Bourgain spaces and allows to prove convergence in $L^2$ for rough data $u_0 \in H^s$, $s > 0$ with an explicit convergence rate.

1. Introduction

We consider the Korteweg–de Vries (KdV) equation

$$\partial_t u(t, x) + \partial_x^3 u(t, x) = -\frac{1}{2} \partial_x u^2(t, x), \quad (t, x) \in \mathbb{R} \times T$$

with initial data $u(0, x) = u_0(x)$. In the last decades a large variety of numerical schemes was proposed to approximate the time dynamics of KdV solutions; see, e.g., [4, 6, 7, 8, 5, 12, 16, 18, 19, 20]. Their error analysis is so far restricted to smooth Sobolev spaces and requires smooth solutions $u$ at least in $H^s$ with $s > 3/2$. For a long time it was therefore an open question whether convergence, even with arbitrarily small rate, can be achieved for rough data

$$u_0 \in H^s, \quad 0 < s \leq 3/2.$$ (2)

The aim of this paper is to address this question.

In this paper, we consider the filtered exponential integrator

$$u^{n+1} = e^{-\tau \partial_x^3} \left[ u^n - \frac{\tau}{2} \phi_1(\tau \partial_x^3) \Pi_\tau \partial_x (\Pi_\tau u^n)^2 \right], \quad \phi_1(\delta) = \frac{e^\delta - 1}{\delta},$$

the filtered Lie splitting (or Lawson method)

$$u^{n+1} = e^{-\tau \partial_x^3} \left[ u^n - \frac{\tau}{2} \Pi_\tau \partial_x (\Pi_\tau u^n)^2 \right],$$

as well as the filtered version of the resonance-based scheme introduced in [5]

$$u^{n+1} = e^{-\tau \partial_x^3} u^n - \frac{1}{6} \Pi_\tau \left( e^{-\tau \partial_x^3} \partial_x^{-1} \Pi_\tau u^n \right)^2 + \frac{1}{6} \Pi_\tau e^{-\tau \partial_x^3} \left( \partial_x^{-1} \Pi_\tau u^n \right)^2,$$

where the projection operator $\Pi_\tau$ is defined by the Fourier multiplier

$$\Pi_\tau = \chi(-i \partial_x \tau^{-1}),$$

with $\chi = 1_{[-1, 1]}$.

The unfiltered Strang splitting scheme for KdV was analysed in [6, 7]; under the assumption that the nonlinear part, i.e., Burger’s equation $\partial_t u = -\frac{1}{2} \partial_x u^2$, is solved exactly, second-order convergence rate for $H^{r+5}$ solutions could be established in $H^r$ for any $r \geq 1$ (the same analysis would give first order convergence for $H^{r+3}$ solutions for the Lie splitting). With the aid of a Rusanov scheme, which allows to handle the derivative in Burger’s nonlinearity, error estimates for $H^3$ solutions could be furthermore obtained in [16]. In [4], where a finite difference scheme is studied for the equation on the real line $\mathbb{R}$, a convergence result is obtained for data in $H^s$, $s \geq 3/4$. Thereby,
convergence of order $1/42$ holds under the CFL condition $\Delta t \leq \Delta x^3$ in case of $s = 3/4$. The latter convergence analysis is, however, restricted to the real line as it heavily relies on a smoothing effect on $\mathbb{R}$ which does not hold on the torus $\mathbb{T}$. The unfiltered resonance based discretisation, that is (5) with $\Pi_r = 1$, was originally introduced in [5] to allow better convergence rates for rougher data than classical schemes. More precisely, first-order convergence in $H^1$ for solutions in $H^3$ can be established ([5]). Another unfiltered resonance based discretisation of embedded type was recently introduced in [20] which allows first-order convergence in $H^{1/2+\epsilon}$ for solutions in $H^{3/2+\epsilon}$ for any $\epsilon > 0$. The convergence analysis in [5, 20], based on energy type estimates and standard product rules in Sobolev spaces, would not allow to handle data verifying (2) on the torus (even at the price of a reduced convergence rate) since at least Lipschitz solutions are needed for the argument. The situation is even worse for the unfiltered exponential integrator (3) or the Lie splitting (4) without Friedrichs or Rusanov corrections for Burgers (as used in [4, 16]) since the energy method is unconclusive and the schemes seem unstable.

The aim of this paper is to handle in a unified way the three filtered schemes (3), (4), (5) and to provide convergence estimates which allow to deal with rough data (2). In context of nonlinear Schrödinger equations low regularity estimates could be recently established with the aid of discrete Strichartz type estimates (on $\mathbb{R}^d$) and Bourgain type estimates (on $\mathbb{T}$); see [10, 9, 14, 13, 15]. In context of the KdV equation (1) our analysis will still rely on the discrete Bourgain spaces introduced in [14]. Nevertheless, as in the analysis of the continuous PDE, in order to recover the loss of derivative in Burger’s nonlinearity some new substantial developments are needed. The presence of the filter $\Pi_\tau$ will be crucial to avoid a loss of derivative and to reproduce at the discrete level the favorable frequency interactions of the KdV equation.

To deal with all three schemes (3)-(5) at the same time we introduce

$$\Psi_\tau(v) = -\frac{1}{2} \int_0^\tau \psi_1(s, \partial_x) \partial_x (\psi_2(s, \partial_x)v)^2 ds,$$

(7)

where $\psi_1(s, \partial_x)$ and $\psi_2(s, \partial_x)$ are Fourier multipliers with bounded symbols $\psi_{1,2}(s, \partial_x) \in \{1, e^{\pm s\partial_x^3}\}$. This notation allows us to express the schemes (3), (4) and (5) in the compact way

$$u^{n+1} = e^{-\tau \partial_x^3} [u^n + \Pi_\tau \Psi_\tau(\Pi_\tau u^n)],$$

(8)

where the choice

$$\psi_1(s, \partial_x) = e^{s\partial_x^3}, \quad \psi_2(s, \partial_x) = 1$$

corresponds to the exponential integrator (3), while setting

$$\psi_1(s, \partial_x) = 1, \quad \psi_2(s, \partial_x) = 1$$

yields the Lie splitting (4) and

$$\psi_1(s, \partial_x) = e^{s\partial_x^3}, \quad \psi_2(s, \partial_x) = e^{-s\partial_x^3}$$

(9)

leads to the resonance based scheme (5).

The filtered scheme (8) with the corresponding choice of filter function can be seen as a classical exponential integrator/Lie splitting/resonance based discretisation applied to the projected KdV equation

$$\partial_t u_\tau + \partial_x^3 u_\tau = -\frac{1}{2} \Pi_\tau \partial_x (\Pi_\tau u_\tau)^2, \quad u_\tau(0) = \Pi_\tau u_0.$$  

(10)

Our main convergence result is the following:

**Theorem 1.1.** For every $T > 0$ and $u_0 \in H^{s_0}$, $s_0 \geq 0$, $\int T u_0 = 0$, let $u \in C([0,T],H^{s_0}) \cap X^0(T)$ (we shall define this space in Section 2) be the exact solution of (1) with initial datum $u_0$ and
denote by \( u^n \) the sequence defined by the scheme (8). Then, we have the following error estimate: there exist \( \tau_0 > 0 \) and \( C_T > 0 \) such that for every step size \( \tau \in (0, \tau_0] \)
\[
\| u^n - u(t_n) \|_{L^2} \leq C_T \max(\tau^{\frac{3}{4}}, \tau), \quad 0 \leq n\tau \leq T. \tag{11}
\]

Here, we are able to establish a convergence result with explicit convergence rate for any initial data in \( H^{s_0}, \ s_0 > 0 \). For \( s_0 \geq 3 \), we recover a classical first order convergence result. Note that even in case of smooth solutions \( s_0 > 3 \), the convergence analysis of the schemes (3) and (4) will require the use of discrete Bourgain spaces. This is due to the fact that the bilinear estimates in these spaces are crucial to overcome the derivative in the right hand side in the stability analysis. For the resonance based scheme (5) (and its unfiltered counterpart, i.e., \( \Pi_n = 1 \)), we recover a classical first order convergence result with explicit convergence rate for any initial data in \( H^{s_0}, \ s_0 > 3 \) since they are needed to estimate \( \| u \|_{H^{s_0}} \), uniformly in \( \tau \). Further we further write a ~ b if \( a \lesssim b \). When we want to emphasize that \( C \) depends on an additional parameter \( \gamma \), we write \( a \lesssim_{\gamma} b \). Further, we denote \( \langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}} \).

Outline of the paper. The idea is to first analyse the difference between the original KdV equation (1) and its projected counterpart (10) on the continuous level; see Section 2. This will then allow us to analyse the time discretisation error introduced by the discretisation (8) applied to the projected equation (10); see Section 4 and 5. In Section 3, we introduce the appropriate discrete Bourgain spaces for the KdV equation and establish their main properties. The proof of the crucial bilinear estimate stated in Lemma 3.3 is postponed to Section 6.

Notations. For two expressions \( a \) and \( b \), we write \( a \lesssim b \) whenever \( a \leq C b \) holds with some constant \( C > 0 \), uniformly in \( \tau \in (0,1] \) and \( K \geq 1 \). We further write \( a \sim b \) if \( a \lesssim b \lesssim a \). When we want to emphasize that \( C \) depends on an additional parameter \( \gamma \), we write \( a \lesssim_{\gamma} b \). Further, we denote \( \langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}} \).

2. Error between the solutions of the exact and projected equation

In this section we establish an estimate on the difference between the solutions of the original KdV equation (1) and its projected counterpart (10). This will yield a bound on
\[
\| u(t) - u_{\tau}(t) \|_{L^2}.
\]

We shall first recall the main tools that are used to prove local well-posedness at low regularity for KdV on the torus [2, 11, 3] since they are needed to estimate \( \| u(t) - u_{\tau}(t) \|_{L^2} \).

Let us recall the definition of Bourgain spaces in the setting of the KdV equation. A tempered distribution \( u(t,x) \) on \( \mathbb{R} \times \mathbb{T} \) belongs to the Bourgain space \( X^{s,b} \) if its following norm is finite
\[
\| u \|_{X^{s,b}} = \left( \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} (1 + |k|)^{2s} (1 + |\sigma - k|^3)^{2b} |\tilde{u} (\sigma,k)|^2 \, d\sigma \right)^{\frac{1}{2}},
\]
where \( \tilde{u} \) is the space-time Fourier transform of \( u \):
\[
\tilde{u}(\sigma,k) = \int_{\mathbb{R} \times \mathbb{T}} e^{-i\sigma t - ikx} u(t,x) \, dt \, dx.
\]
We shall also use a localized version of this space. For \( I \subset \mathbb{R} \) being an open interval, we say that \( u \in X^{s,b}(I) \) if \( \| u \|_{X^{s,b}(I)} < \infty \), where
\[
\| u \|_{X^{s,b}(I)} = \inf \{ \| \tilde{u} \|_{X^{s,b}}, \tilde{u}|_I = u \}.
\]
When \( I = (0,T) \) we will often simply use the notation \( X^{s,b}(T) \). We refer for example to [14] Lemma 2.1 for some useful properties of these spaces in this setting (and to [2] and [17] for more details). A particularly useful property is the embedding \( X^{s,b} \subset C(\mathbb{R}, H^{s}) \) for \( b > 1/2 \). In the case of the KdV equation on the torus, in order to resolve the derivative in the nonlinearity, we are forced to work with the borderline space which is at the level of \( b = 1/2 \) (cf. [11, 3]). To get a space
with good properties, we work with the smaller space $X^s$, which has the same scaling properties in time as $X^{s,\frac{1}{2}}$, defined by the following norm:

$$
\|u\|_{X^s} = \|u\|_{X^{s,\frac{1}{2}}} + \|\langle k \rangle^s \tilde{u}\|_{L^1(\sigma)}.
$$

(12)

We define more precisely $X^s$ as the space of space-time tempered distributions such that $\tilde{u}(\sigma, 0) = 0$ and the above norm is finite. In a similar way, we get a localized version $X^s(I)$ or $X^s(T)$ if $I = (0, T)$ by setting

$$
\|u\|_{X^s(I)} = \inf\{\|\Pi\|_{X^s}, \Pi|_I = u\}.
$$

The main well-posedness result for (1) reads:

**Theorem 2.1.** For every $T > 0$ and $u_0 \in L^2$, $\int_T u_0 = 0$, there exists a unique solution $u$ of (1) such that $u \in X^0(T)$. Moreover, if $u_0 \in H^{s_0}$, $s_0 > 0$, then $u \in X^{s_0}(T)$.

Note that we have $X^0(T) \subset C([0, T], L^2)$ and $X^{s_0}(T) \subset C([0, T], H^{s_0})$. The result also holds true for initial data of some negative regularity, nevertheless, since we have chosen to measure the convergence of our numerical schemes in the natural $L^2$ norm, we shall not use these more general results.

We refer to [11, 2, 3] for the detailed proof, nevertheless, we shall recall the main ingredients since we will later use related arguments at the discrete level.

**Proof.** The existence in short time is first proven by a fixed point argument on the following truncated problem:

$$
v \mapsto \Phi(v)
$$

such that

$$
\Phi(v)(t) = \chi(t) e^{-t\partial_x^3} u_0 - \frac{1}{2\chi(t)} \int_0^t e^{-(t-s)\partial_x^3} \partial_x \left( \chi \left( \frac{s}{\sigma} \right) u(s) \right) ds,
$$

(13)

where $\chi \in [0, 1]$ is a smooth compactly supported function which is equal to 1 on $[-1, 1]$ and supported in $[-2, 2]$. For $|t| \leq \delta \leq 1/2$, a fixed point of the above equation gives a solution of the original Cauchy problem, denoted by $u$. The parameter $\delta > 0$ is chosen in the proof to get a contraction.

The basic properties of the spaces $X^{s,b}$ and $X^s$ that are needed are the following:

**Lemma 2.2.** For $\eta \in C^\infty_c(\mathbb{R})$, we have that

$$
\|\eta(t) e^{-t\partial_x^3} f\|_{X^s} \lesssim \|f\|_{H^s}, \quad s \in \mathbb{R}, f \in H^s(\mathbb{R}),
$$

$$
\|\eta(t) e^{-t\partial_x^3} f\|_{X^{s,b'}} \lesssim_{\eta, b, b'} T^{b-b'} \|f\|_{X^{s,b}}, \quad s \in \mathbb{R}, -\frac{1}{2} < b' \leq b < \frac{1}{2}, 0 < T \leq 1,
$$

$$
\|\chi(t) \int_0^t e^{-(t-s)\partial_x^3} F(s) ds\|_{X^s} \lesssim_{\eta, b} \|F\|_{Y^s}, \quad s \in \mathbb{R}
$$

(14, 15, 16)

where $Y^s$ is the space defined by the norm

$$
\|F\|_{Y^s} = \|F\|_{X^{s,-\frac{1}{2}}} + \|\langle k \rangle^s (\sigma - k^3)^{-1}\tilde{F}\|_{L^2(\sigma L^3)}.
$$

The other ingredient is the following crucial bilinear estimate.

**Lemma 2.3.** For $s \geq 0$, and $u \in X^s$, we have the estimate:

$$
\|\partial_x (u^2)\|_{Y^s} \lesssim \|u\|_{X^{s,\frac{1}{2}}} \|u\|_{X^{0,\frac{1}{2}}} + \|u\|_{X^{0,\frac{1}{2}}} \|u\|_{X^{s,\frac{1}{4}}}.
$$

The difficult part is to prove the estimate for $s = 0$ (it actually holds true for $s > -1/2$), it is afterwards easy to get the estimate for $s > 0$. Note that by definition of our space $X^s$, we have that $\int_T u(t, \cdot) = 0$. By polarization, we easily deduce that we also have for example

$$
\|\partial_x (uv)\|_{Y^0} \lesssim \|u\|_{X^{0,\frac{1}{2}}} \|v\|_{X^{0,\frac{1}{2}}} + \|v\|_{X^{0,\frac{1}{2}}} \|u\|_{X^{0,\frac{1}{4}}}.
$$

(17)
By using (14) and (16), we get that for $v \in X^0$

$$\|\Phi(v)\|_{X^0} \leq C \left( \|u_0\|_{L^2} + \left\| \partial_x \left( \frac{t}{\delta} \right) v \right\|_{Y^0}^2 \right),$$

where $C > 0$ is independent of $u_0$ and $\delta$. Then, we can use Lemma 2.3, to get

$$\|\Phi(v)\|_{X^0} \leq C \left( \|u_0\|_{L^2} + \chi \left( \frac{t}{\delta} \right) \left\| \chi \left( \frac{t}{\delta} \right) v \right\|_{X^0, \frac{1}{2}} \right),$$

and we finally deduce from (15) that

$$\|\Phi(v)\|_{X^0} \leq C \left( \|u_0\|_{L^2} + \delta^\epsilon \|v\|_{X^0}^2 \right),$$

where $\epsilon$ is any number in $(0, 1/6)$. By using the same ingredients, we also get that for every $v, w \in X^0$, we have that

$$\|\Phi(v) - \Phi(w)\|_{X^0} \leq C \delta^\epsilon \left( \|v\|_{X^0} + \|w\|_{X^0} \right) \|v - w\|_{X^0}$$

for some $C > 0$ independent of $\delta$ and $u_0$.

Consequently, by taking $R = 2C\|u_0\|_{L^2}$, we get that there exists $\delta > 0$ sufficiently small that depends only on $\|u_0\|_{L^2}$, such that $\Phi$ is a contraction on the closed ball $B(0, R)$ of $X^0$. This proves the existence of a fixed point for $\Phi$ and hence the existence of a solution $u$ of (1) on $[0, \delta]$. By using again Lemma 2.2 and Lemma 2.3, we also have for $s > 0$, that

$$\|\Phi(v)\|_{X^s} \leq C\|u_0\|_{H^s} + C\delta^\epsilon \|v\|_{X^0} \|v\|_{X^s},$$

such that if $u_0$ is in $H^s$ then we also have that $u \in X^{s,b}([0, \delta])$. Since the $L^2$ norm is conserved for (1), we can reiterate the construction on $[\delta, 2\delta]$ and so on to get a global solution. We thus obtain a solution $u$ with $u \in X^{s,b}(T)$ for every $T$.

\[\square\]

Let us now consider the projected equation (10). A straightforward adaptation of the previous proof yields the following global well-posedness result.

**Proposition 2.4.** For $u_0 \in H^{s_0}$, $s_0 \geq 0$ and $\tau \in [0, 1]$, there exists a unique solution $u_\tau$ of (10) such that $u_\tau \in X^{s_0}(T)$ for every $T > 0$. Moreover, for every $T > 0$, there exists $M_T > 0$ such that for every $\tau \in (0, 1]$, we have the estimate

$$\|u_\tau\|_{X^{s_0}(T)} \leq M_T.$$

**Remark 2.5.** Note that, since $\Pi_\tau^2 = \Pi_\tau$, we have that $\Pi_\tau u_\tau$ solves the same equation (10) with the same initial data as $u_\tau$ and hence we have by uniqueness that

$$\Pi_\tau u_\tau(t) = u_\tau(t) \quad \text{for all} \quad t \geq 0.$$

We shall also need an estimate with more $b$ regularity:

**Corollary 2.6.** For every $T \geq 1$ and $u_0 \in H^{s_0}$, $s_0 \geq 0$, $\int_T u_0 = 0$, there exists $M_T > 0$ such that for every $\tau \in (0, 1]$, we have the estimate

$$\|u_\tau\|_{X^{s_0,1}(T)} \leq \frac{M_T}{\tau^3}.$$

**Proof.** For $\delta > 0$, small enough (depending only on $T$ and $\|u_0\|_{H^{s_0}}$) we have that $u_\tau$ coincides with the following fixed point $U_\tau \in X^{s_0}$:

$$U_\tau(t) = \chi(t)e^{-t\partial_x^2} \Pi_\tau u_0 - \frac{1}{2} \chi(t) \Pi_\tau \int_0^t e^{-(t-s)\partial_x^2} \partial_x \left( \Pi_\tau \chi \left( \frac{s}{\delta} \right) U_\tau(s) \right)^2 ds,$$
where $\chi \in [0, 1]$ is a smooth compactly supported function which is equal to 1 on $[-1, 1]$ and supported in $[-2, 2]$. We thus get that

$$\|U_\tau\|_{X^{s_0,1}} \lesssim \|u_0\|_{L^2} + \left\| \partial_x \Pi_\tau \left( \Pi_\tau \chi \left( \frac{t}{\delta} \right) U_\tau \right) \right\|_{X^{s_0,0}}.$$ 

Here we have used again (14) and the following general estimate for Bourgain spaces:

$$\left\| \chi(t) \int_0^t e^{-(t-s)\partial_x^2} F(s) \, ds \right\|_{X^{s,b}} \lesssim \eta, b \left\| F \right\|_{X^{s,b-1}}$$

for $b \in (1/2, 1]$. We refer again to [11, 2, 3] and the book [17]. To estimate the right hand side, we use that $\Pi_\tau$ projects on frequencies $|k| \leq \tau^{-\frac{1}{4}}$. Together with the generalized Leibniz rule this yields that

$$\|U_\tau\|_{X^{s_0,1}} \lesssim \|u_0\|_{L^2} + \frac{1}{\tau^\frac{3}{4}} \left\| (\partial_x)_{s_0} \Pi_\tau \chi \left( \frac{t}{\delta} \right) U_\tau \right\|_{L^4(\mathbb{R} \times T)}^2.$$ 

To conclude, we use the Strichartz estimate for KdV on the torus (which is actually used for the proof of Lemma 2.3, we again refer to [11, 2, 3]) which reads

$$\|u\|_{L^4(\mathbb{R} \times T)} \lesssim \|u\|_{X^{0, \frac{1}{2}}}$$

(we will prove a discrete version of this estimate in Section 6). This yields

$$\|U_\tau\|_{X^{s_0,1}} \lesssim \|u_0\|_{L^2} + \frac{1}{\tau^\frac{3}{4}} \|U_\tau\|_{X^{s_0,0}}^2.$$ 

By iterating the argument, we thus deduce that

$$\|u_\tau\|_{X^{s_0,1}(T)} \leq \frac{M_T}{\tau^\frac{3}{4}}$$

thanks to Proposition 2.4.

We can also easily get the following estimate on the difference $\|u(t) - u_\tau(t)\|_{L^2}$ which was the aim of this section.

**Proposition 2.7.** For $u_0 \in H^{s_0}$, $s_0 \geq 0$, $\int_T u_0 = 0$, and every $T > 0$, there exists $C_T > 0$ such that for every $\tau \in (0, 1)$, we have the estimate

$$\|u - u_\tau\|_{X^{0}(T)} \leq C_T T^{-\frac{s_0}{2}}.$$ 

Since $X^0(T) \subset C([0, T], L^2)$, we have in particular that

$$\sup_{t \in [0, T]} \|u(t) - u_\tau(t)\|_{L^2} \leq C_T T^{-\frac{s_0}{2}}.$$ 

**Proof.** For some $\delta > 0$ sufficiently small, we first observe that $u \in X^{s_0}(T)$ the solution of (1) coincides on $[0, \delta]$ with the fixed point of $\Phi$ defined in (13) which belongs to $X^{s_0}$. We shall (by abuse of notation) still denote by $u$ this fixed point. In a similar way, $u_\tau \in X^{s_0}(T)$ coincides on $[0, \delta]$ with the fixed point of $\Phi_\tau$ in $X^0$ that we shall still denote by $u_\tau$, where

$$\Phi_\tau(v)(t) = \chi(t)e^{-t\partial_x^2} \Pi_\tau u_0 - \frac{1}{2} \chi(t)\Pi_\tau \int_0^t e^{-(t-s)\partial_x^2} \partial_x \left( \Pi_\tau \chi \left( \frac{s}{\delta} \right) v(s) \right)^2 ds.$$
With these notations, we thus get that

\[ u(t) - u_\tau(t) = \chi(t)e^{-t\Delta}(1 - \Pi_\tau)u_0 - \frac{1}{2}\chi(t)(1 - \Pi_\tau)\int_0^t e^{-(t-s)\Delta} \frac{\partial^3}{\partial x^3} \left( \chi \left( \frac{s}{\delta} \right) u(s) \right)^2 \, ds \]

\[ - \frac{1}{2}\chi(t)\Pi_\tau \int_0^t e^{-(t-s)\Delta} \frac{\partial^3}{\partial x^3} \left( \chi \left( \frac{s}{\delta} \right) (1 - \Pi_\tau)u(s) \right)^2 \, ds \]

\[ - \chi(t)\Pi_\tau \int_0^t e^{-(t-s)\Delta} \frac{\partial^3}{\partial x^3} \left( \chi \left( \frac{s}{\delta} \right) (1 - \Pi_\tau)u(s) \chi \left( \frac{s}{\delta} \right) \Pi_\tau u(s) \right) \, ds \]

\[ - \frac{1}{2}\chi(t)\Pi_\tau \int_0^t e^{-(t-s)\Delta} \frac{\partial^3}{\partial x^3} \left( \chi \left( \frac{s}{\delta} \right) \Pi_\tau (u(s) + u_\tau(s)) \chi \left( \frac{s}{\delta} \right) \Pi_\tau (u(s) - u_\tau(s)) \right) \, ds. \quad (18) \]

Thanks to the definition of \( \Pi_\tau \), we have that

\[ \|(1 - \Pi_\tau)f\|_{L^2} \leq \frac{2}{\pi} \|f\|_{H^0}, \quad \forall f \in H^0 \]

and thus

\[ \|(1 - \Pi_\tau)f\|_{X^0} \leq \frac{2}{\pi} \|f\|_{X^0}, \quad \forall f \in X^0. \]

Consequently, by using this observation and again (14), (15) as well as Lemma 2.3, we obtain that

\[ \|u - u_\tau\|_{X^0} \leq \frac{2}{\pi} \|u_0\|_{H^0} + \frac{2}{\pi} \left\| \chi(t) \int_0^t e^{-(t-s)\Delta} \frac{\partial^3}{\partial x^3} \left( \chi \left( \frac{s}{\delta} \right) u(s) \right)^2 \right\|_{X^0} \]

\[ + \|u\|_{X^0} \|u\|_{X^0} + \|u_\tau\|_{X^0} + \|u_\tau\|_{X^0} \|u - u_\tau\|_{X^0} \]

\[ \leq \frac{2}{\pi} \left( \|u_0\|_{H^0} + \|u\|_{X^0} + \|u_\tau\|_{X^0} \right) \|u - u_\tau\|_{X^0}. \]

Let us fix \( M_T \) independent of \( \tau \in (0, 1) \) and \( \delta \in (0, 1] \) such that

\[ \|u\|_{X^0} + \|u_\tau\|_{X^0} \leq M_T, \]

we then obtain that

\[ \|u - u_\tau\|_{X^0} \leq \frac{2}{\pi} \left( \|u_0\|_{H^0} + M_T^2 \right) + 2\delta^\pi M_T \|u - u_\tau\|_{X^0}. \]

By taking \( \delta \) sufficiently small so that \( 2\delta^\pi M_T < 1/4 \), we then obtain that

\[ \|u - u_\tau\|_{X^0} \leq C_T \frac{2}{\pi} \]

which gives the desired estimate on \([0, \delta]\]. We can then iterate the argument to get the estimate on the full interval \([0, T]\).

\[ \square \]

3. DISCRETE BOURGAIN-KdV SPACES

In order to perform error estimates at low regularity, we shall develop at the discrete level the harmonic analysis tools used in Section 2. Definitions and properties of discrete Bourgain spaces were introduced (in the context of the nonlinear Schrödinger equation) in [14]. Nevertheless, as in the continuous case, we need additional results in order to handle the KdV equation, namely we shall introduce the discrete counterpart of the space \( X^s \), study its properties and prove a bilinear estimate analogous to the one of Lemma 2.3.

For sequences of functions \((u^n(x))_{n \in \mathbb{Z}}\), we define the Fourier transform \( \widehat{u^n}(\sigma,k) \) by

\[ \mathcal{F}_{n,x}(u^n)(\sigma,k) = \widehat{u^n}(\sigma,k) = \tau \sum_{m \in \mathbb{Z}} \hat{u}^m(k) e^{im\pi \sigma}, \quad \hat{u}^m(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u^n(x) e^{-ikx} \, dx. \]

Parseval’s identity then reads

\[ \|\hat{u^n}\|_{L^2}^2 = \|u^n\|_{H^0}^2, \quad (19) \]
where
\[ \|\tilde{u}^n\|_{L^2}^2 = \int_{-\pi}^{\pi} \sum_{k \in \mathbb{Z}} |\tilde{u}^n(\sigma,k)|^2 \, d\sigma, \quad \|u^n\|_{L^2}^2 = \tau \sum_{m \in \mathbb{Z}} \int_{-\pi}^{\pi} |u^n(x)|^2 \, dx. \]

We define the discrete Bourgain spaces \(X^s_{\tau} \) for \( s \geq 0, b \in \mathbb{R}, \tau > 0\) by
\[ \|u^n\|_{X^s_{\tau}} = \left\| (k)^s (d_{\tau}(\sigma + k^3))^b \tilde{u}^n(\sigma,k) \right\|_{L^2_{\tau}}, \tag{20} \]
where \( d_{\tau}(\sigma) = \frac{e^{i\sigma \tau} - 1}{\tau} \). Note that \( d_{\tau} \) is 2\( \pi/\tau \) periodic and that uniformly in \( \tau \), we have \( |d_{\tau}(\sigma)| \sim |\sigma| \) for \( |\tau\sigma| \leq \pi \). Since \( |d_{\tau}(\sigma)| \lesssim \tau^{-1} \), we also have that the discrete spaces satisfy the embeddings
\[ \|u^n\|_{X^s_{\tau}} \lesssim \frac{1}{\tau^{b-b'}} \|u^n\|_{X^s_{\tau}}, \quad b \geq b'. \tag{21} \]

Some useful more technical properties are gathered in the following lemma:

**Lemma 3.1.** For \( \eta \in C_c^\infty(\mathbb{R}) \) and \( \tau \in (0,1) \), we have that
\[ \|\eta(n\tau)e^{-n\tau \partial_x^3}f\|_{X^s_{\tau}} \lesssim_{\eta,b} \|f\|_{H^s}, \quad s \in \mathbb{R}, b \in \mathbb{R}, f \in H^s, \tag{22} \]
\[ \|\eta(n\tau)u^n\|_{X^s_{\tau}} \lesssim_{\eta,b} \|u^n\|_{X^s_{\tau}}, \quad s \in \mathbb{R}, b \in \mathbb{R}, u^n \in X^s_{\tau}, \tag{23} \]
\[ \|\eta(n\tau)\|_{X^s_{\tau}} \lesssim_{\eta,b,b'} \tau^{1-b'} \|u^n\|_{X^{s,b'}}, \quad s \in \mathbb{R}, -\frac{1}{2} < b' \leq b < \frac{1}{2}, 0 < T = N\tau \leq 1, N \geq 1. \tag{24} \]

In addition, for
\[ U^n(x) = \eta(n\tau)\tau \sum_{m=0}^n e^{-i(m-n)\tau \partial_x^3} u^m(x), \]
we have
\[ \|U^n\|_{X^s_{\tau}} \lesssim_{\eta,b} \|u^n\|_{X^{s,b-1}_{\tau}}, \quad s \in \mathbb{R}, b > 1/2. \tag{25} \]

We stress that all given estimates are uniform in \( \tau \).

The proof directly follows from the ones of [14, Lemma 3.4]. Indeed, it suffices to observe that
\[ \|u^n\|_{X^s_{\tau}} = \|e^{i\tau \partial_x^3} u^n\|_{H^s_{\tau}}, \]
where
\[ \|u^n\|_{H^s_{\tau}} := \|\langle d_{\tau}(\sigma)\rangle^b \langle k \rangle^s \tilde{u}^n(\sigma,k)\|_{L^2_{\tau}} \]
and the proofs only use the properties of the space \( H^s_{\tau} \).

The next step that we shall need in order to handle the KdV equation is to adapt (16) in the case \( b = 1/2 \). We first define the discrete counterparts \( X^s_{\tau} \) of the \( X^s \) space. We say that a sequence of function \( (u^n(x))_n \in l^2_\tau L^2 \) such that \( \int_{-\pi}^{\pi} u^n = 0, \forall n \) is in \( X^s_{\tau} \) for \( s \geq 0 \) if the following norm is finite
\[ \|u^n\|_{X^s_{\tau}} = \|u^n\|_{X^{s,\frac{1}{2}}_{\tau}} + \|\langle k \rangle^s \tilde{u}(\sigma,k)\|_{L^2_{\tau}(\sigma)} \]
and in the same way, we also define \( Y^s_{\tau} \) by
\[ \|F^n\|_{Y^s_{\tau}} = \|F^n\|_{X^{s,-\frac{1}{2}}_{\tau}} + \left\| \frac{\langle k \rangle^s}{\langle d_{\tau}(\sigma + k^3)\rangle} F^n(\sigma,k) \right\|_{L^2_{\tau}(\sigma)} \].

**Lemma 3.2.** We have the following properties:

1. We have the embedding \( X^s_{\tau} \subset l^\infty(\mathbb{Z}, H^s(\mathbb{T})) \);
\[ \sup_n \|u^n\|_{H^s(\mathbb{T})} \lesssim \|u^n\|_{X^s_{\tau}}, \quad s \in \mathbb{R}, (u^n)_n \in X^s_{\tau}; \tag{26} \]
(2) Let us define for \((u^n)_n \in Y^s_r\), and \(\eta \in \O_c^\infty(\R)\)

\[
U^n(x) := \eta(n\tau)\tau \sum_{m=0}^n e^{-(n-m)\tau \partial^3} u^m(x),
\]

then, we have

\[
\|U^n\|_{X^s_t} \lesssim \eta \|u^n\|_{Y^s}, \quad s \in \R.
\]  

The above estimates are uniform for \(\tau \in (0, 1]\).

Proof. We first prove (26). By definition of our Fourier transforms, we have that for every \(k \in \Z\), and every \(m \in \Z\), we have

\[
\hat{u}^m(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{u}^n(\sigma, k) e^{-im\tau\sigma} \, d\sigma
\]

and hence

\[
|\hat{u}^m(k)| \leq \frac{1}{2\pi} \|\hat{u}^n(\cdot, k)\|_{L^1(\sigma)}.
\]

Consequently, by taking the \(l^2\) norm in \(k\) and by using the Bessel identity, we obtain

\[
\|u^n\|_{L^2(\T)} \lesssim \|\hat{u}^n(\cdot, k)\|_{l^2(k) L^1(\sigma)} \leq \|u^n\|_{X^0_t}.
\]

This gives (26) for \(s = 0\), and the general case follows by replacing \(u^n\) by \(\langle \partial_x \rangle^s u^n\).

Let us now prove (28). Again, we give the proof for \(s = 0\), the general case just follows by applying \(\langle \partial_x \rangle^s\) to the two sides of (27). Let us set

\[
F^n(x) = e^{+n\tau\partial^3} U^n(x), \quad f^n(x) = e^{+n\tau\partial^3} u^n(x)
\]

so that

\[
F^n(x) = \eta(n\tau)\tau \sum_{m=0}^n f^m.
\]

We shall first prove that

\[
\|F^n\|_{H^\frac{1}{2}} + \|\hat{F^n}\|_{l^2(k) L^1(\sigma)} \lesssim \|f^n\|_{H^\frac{1}{2}} + \left\| \frac{1}{\langle d\tau(\sigma) \rangle} \hat{f^n} \right\|_{l^2(k) L^1(\sigma)}
\]

which is equivalent to

\[
\|U^n\|_{X^0_t} + \|\hat{U^n}\|_{l^2(k) L^1(\sigma)} \lesssim \|u^n\|_{Y^0}.
\]

By direct computation, we find that

\[
\tilde{F}^n(\sigma, k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\tau\sigma_0}}{\tau} d\tau(\sigma_0) \hat{F^n}(\sigma_0, k) (g(\sigma) - e^{-i\tau\sigma_0} g(\sigma - \sigma_0)) \, d\sigma_0,
\]

where \(g(\sigma) = F_r(\eta(n\tau))(\sigma)\). Note that \(g\) is fastly decreasing in the sense that

\[
\left| \langle d\tau(\sigma) \rangle^K g(\sigma) \right| \lesssim 1,
\]

where the estimate is uniform in \(\tau \in (0, 1]\) and \(\sigma\) for every \(K\). We then split,

\[
\tilde{F}^n(\sigma, k) = \int_{|\sigma_0| \leq 1} + \int_{|\sigma_0| \geq 1} := \tilde{F}^n_1 + \tilde{F}^n_2.
\]

For \(|\sigma_0| \leq 1\), we can use the Taylor formula and the fast decay of \(g\) to get that

\[
\left| \langle d\tau(\sigma) \rangle^K \left( g(\sigma) - e^{-i\tau\sigma_0} g(\sigma - \sigma_0) \right) \right| \lesssim \frac{1}{\langle d\tau(\sigma - \sigma_0) \rangle^K}.
\]
Therefore, we obtain that
\[
\langle d_\tau(\sigma) \rangle \frac{1}{2} |\mathcal{F}_2^n| \lesssim \int_{|\sigma_0| \leq 1} \frac{1}{\langle d_\tau(\sigma_0) \rangle^K} |\tilde{f}_n(\sigma_0, k)| d\sigma_0.
\]
By choosing $K$ large enough we thus find that
\[
\|F_1^n\|_{H^ \frac{1}{2} L^2} + \|\tilde{F}_n\|_{l^2(k)L^1(\sigma)} \lesssim \int_{|\sigma_0| \leq 1} \|\tilde{f}_n(\sigma_0, \cdot)\|_{L^2} + \int_{|\sigma_0| \leq 1} \|\tilde{f}_n(\sigma_0, k) d\sigma_0\| \lesssim \|f_n\|_{H^ \frac{1}{2} L^2} + \left\| \frac{1}{\langle d_\tau(\sigma_0) \rangle} \tilde{f}_n \right\|_{l^2(k)L^1(\sigma)},
\]
where we have used that $|\sigma_0| \leq 1$ and Cauchy-Schwarz (for the first term).

It remains to bound the second term $F_2^n$. We write
\[
\langle d_\tau(\sigma) \rangle \frac{1}{2} |\mathcal{F}_2^n| \lesssim \int_{|\sigma_0| \geq 1} \frac{1}{\langle d_\tau(\sigma) \rangle^K} |\tilde{f}_n(\sigma_0, k)| d\sigma_0 + \int_{|\sigma_0| \geq 1} \frac{1}{\langle d_\tau(\sigma_0) \rangle} \tilde{f}_n(\sigma_0, k) \frac{1}{\langle d_\tau(\sigma_0 - \sigma_0) \rangle^K} d\sigma_0.
\]
where we have used that $\langle d_\tau(\sigma) \rangle \frac{1}{2} |g(\sigma)| \lesssim \langle d_\tau(\sigma) \rangle^{-K}$ for the first term and $\langle d_\tau(\sigma) \rangle \frac{1}{2} |g(\sigma - \sigma_0)| \lesssim \langle d_\tau(\sigma_0) \rangle \frac{1}{2} \langle d_\tau(\sigma - \sigma_0) \rangle^{-K}$ for the second one with $K$ large enough. By taking the $L^2$ norm in $\sigma$ and by using the Young inequality for convolutions for the second term, we get that
\[
\|\langle d_\tau(\sigma) \rangle \frac{1}{2} \tilde{F}_2^n (\cdot, k)\|_{L^2(\sigma)} \lesssim \left\| \frac{1}{\langle d_\tau(\sigma) \rangle} \tilde{f}_n(\cdot, k) \right\|_{L^1(\sigma)} + \left\| \frac{1}{\langle d_\tau(\sigma) \rangle} \tilde{f}_n(\cdot, k) \right\|_{L^2(\sigma)}.
\]
Finally, by taking the $l^2$ norm in $k$, we obtain that
\[
\left\| \tilde{F}_2^n \right\|_{H^ \frac{1}{2} l^2(k)} \lesssim \left\| \frac{1}{\langle d_\tau(\sigma) \rangle} \tilde{f}_n \right\|_{l^2(k)L^1(\sigma)} + \|f_n\|_{H^ \frac{1}{2} l^2(k)}.
\]
From the fast decay of $g$, we also have by similar arguments that
\[
|\tilde{F}_2^n(\sigma, k)| \lesssim \int_{|\sigma_0| \geq 1} \frac{1}{\langle d_\tau(\sigma) \rangle^K} |\tilde{f}_n(\sigma_0, k)| d\sigma_0 + \int_{|\sigma_0| \geq 1} \frac{1}{\langle d_\tau(\sigma_0) \rangle} \tilde{f}_n(\sigma_0, k) \frac{1}{\langle d_\tau(\sigma_0 - \sigma_0) \rangle^K} d\sigma_0.
\]
By taking the $L^1$ norm in $\sigma$ and then the $l^2$ norm in $k$, we thus find that
\[
\|\tilde{F}_2^n(\sigma, k)\|_{l^2(k)L^1(\sigma)} \lesssim \left\| \frac{1}{\langle d_\tau(\sigma) \rangle} \tilde{f}_n \right\|_{l^2(k)L^1(\sigma)}.
\]
Gathering (31), (32) and (33), we finally get (29), this ends the proof of (28).

The next result, we will need is the discrete counterpart of Lemma 2.3:

**Lemma 3.3.** For every $s \geq 0$, there exists $C > 0$ such that for every $(u^n)_n$, $(v^n)_n \in X^s_T$, we have the estimate
\[
\|\partial_x \Pi_x (\Pi_x u^n \Pi_x v^n)\|_{Y^s_{\tau}} \leq C \left( \|u^n\|_{X^{s, \frac{1}{2}}_{\tau}} \|v^n\|_{X^{s, \frac{1}{2}}_{\tau}} + \|v^n\|_{X^{s, \frac{1}{2}}_{\tau}} \|u^n\|_{X^{s, \frac{1}{2}}_{\tau}} \right).
\]

Note that as in the continuous case, the above estimate does not involve space derivatives in the right hand-side. The use of the projections $\Pi_x$ is crucial to get this property. Since the understanding of the proof of this lemma is not essential to understand the error estimates, we postpone it to Section 6.

The last property we shall need is to relate the discrete and the continuous Bourgain norms for the sequence defined by $u^n = u_\tau(t_n)$ where $u_\tau$ is the solution of (10) given by Proposition 2.4. We
shall still denote by $u_\tau$ an extension of $u_\tau \in X^{s_0}$ which coincides with $u_\tau$ on $[-4T,4T]$ and such that thanks to Proposition 2.4 and Corollary 2.6

$$\|u_\tau\|_{X^{s_0}} + \tau^{\frac{1}{2}}\|u_\tau\|_{X^{s_0,1}} \leq M_T$$

(34)

for some $M_T$ independent of $\tau \in (0,1]$.

**Lemma 3.4.** Let $T \geq 1$ and let $u_\tau$ be an extension as above of the solution of (10) given by Proposition 2.4. Then, there exists $C_T > 0$ such that for every $\tau \in (0,1]$, we have the estimate

$$\sup_{s \in [-4\tau,4\tau]} \|u_\tau(t_n + s)\|_{X^{s_0,\frac{1}{2}}} \leq C_T.$$  

**Proof.** Let us set $f(\cdot) = \langle \partial_x \rangle^{s_0} e^{-it\partial_x^2} u_\tau(\cdot + s)$ and $f^n(x) = f(n\tau, x)$, it suffices to prove that

$$\|f^n\|_{H^\frac{1}{2}L^2} \lesssim \|f\|_{H^\frac{1}{2}L^2} + \tau^{\frac{1}{2}}\|f\|_{H^1L^2}.$$  

Then we can conclude from (34).

The discrete Fourier transform of the sequence $(f_m)_m$ is by definition given by

$$\tilde{f}^m(\sigma, k) = \tau \sum_{n \in \mathbb{Z}} f(n\tau, k) e^{i\tau\sigma}.$$  

We thus have by Poisson’s summation formula that

$$\tilde{f}^n(\sigma, k) = \sum_{m \in \mathbb{Z}} f\left(\sigma + \frac{2\pi}{\tau} m, k\right), \quad \sigma \in [-\pi/\tau, \pi/\tau].$$  

Therefore,

$$\langle d_\tau(\sigma) \rangle^{\frac{1}{2}} \tilde{f}^n(\sigma, k) = d_\tau(\sigma) \tilde{f}(\sigma, k) + \sum_{m \in \mathbb{Z}, m \neq 0} \langle d_\tau(\sigma) \rangle^{\frac{1}{2}} \tilde{f}\left(\sigma + \frac{2\pi}{\tau} m, k\right).$$  

Since, we have $|d_\tau(\sigma)| \lesssim |\sigma|$, we get from Cauchy–Schwarz that

$$|\langle d_\tau(\sigma) \rangle^{\frac{1}{2}} \tilde{f}^n(\sigma, k)| \lesssim |\langle \sigma \rangle^{\frac{1}{2}} \tilde{f}(\sigma, k)|^2 + \sum_{\mu \neq 0} \frac{1}{|\sigma + \frac{2\pi}{\tau} \mu|^2} \sum_{m \in \mathbb{Z}, m \neq 0} \left|\langle \sigma + \frac{2\pi}{\tau} m \rangle^2 \langle d_\tau(\sigma) \rangle \tilde{f}\left(\sigma + \frac{2\pi}{\tau} m, k\right)\right|^2.$$  

We then observe that for $\sigma \in [-\pi/\tau, \pi/\tau]$, $\mu \neq 0$ we have that

$$\left|\frac{\sigma + \frac{2\pi}{\tau} \mu}{\tau} \right| \geq \frac{\pi |\mu|}{\tau}$$

so that

$$\sum_{\mu \neq 0} \frac{1}{|\sigma + \frac{2\pi}{\tau} \mu|^2} \lesssim \tau^2.$$  

By using that $|d_\tau(\sigma)| \leq 2/\tau$, we thus find that

$$|\langle d_\tau(\sigma) \rangle^{\frac{1}{2}} \tilde{f}^n(\sigma, k)| \lesssim |\langle \sigma \rangle^{\frac{1}{2}} \tilde{f}(\sigma, k)|^2 + \tau \sum_{m \in \mathbb{Z}, m \neq 0} \left|\langle \sigma + \frac{2\pi}{\tau} m \rangle^2 \tilde{f}\left(\sigma + \frac{2\pi}{\tau} m, k\right)\right|^2.$$  

By integrating with respect to $\sigma \in [-\pi/\tau, \pi/\tau]$ and summing over $k$, we thus obtain that

$$\|f^n\|_{H^\frac{1}{2}L^2} \lesssim \|f\|_{H^\frac{1}{2}L^2} + \tau^{\frac{1}{2}}\|f\|_{H^1L^2}.$$  

This ends the proof. 

□
4. Error estimate of the time discretisation of the modified projected equation

In this section we derive an estimate on the time discretisation error introduced by the discretisation (8) applied to the projected equation (10). This will give an estimate on

\[ \|u_\tau(t_n) - u^n\|_{L^2}. \]

Let us denote by \( \Phi^\tau \) the numerical flow of (8) and by \( \varphi^\tau_\tau \) the exact flow of the projected KdV equation (1). Then we have

\[ \varphi^\tau_\tau(u_\tau(t_n)) = u_\tau(t_n + t) \quad \text{and} \quad u^{n+1} = \Phi^\tau(u^n). \]

The mild solution of the projected KdV equation (10) is given by Duhamel’s formula

\[ u_\tau(t_n + \tau) = \varphi^\tau_\tau(u_\tau(t_n)) = e^{-\tau \partial_3^3} u_\tau(t_n) - \frac{1}{2} e^{-\tau \partial_3^3} \int_0^\tau e^{\tau \partial_3^3} \Pi_\tau \partial_\tau (\Pi_\tau u_\tau(s) + \tilde{v}(s, \partial_\tau) u_\tau(t_n + s))^2 \, ds. \] (35)

With the aid of the notation (7) we can furthermore express the numerical flow \( \Phi^\tau \) applied to some function \( v \) as follows

\[ \Phi^\tau(v) = e^{-\tau \partial_3^3} v - \frac{1}{2} e^{-\tau \partial_3^3} \int_0^\tau \psi_1(s, \partial_{\tau}) \Pi_\tau \partial_{\tau} (\Pi_\tau \psi_2(s, \partial_{\tau}) v)^2 \, ds. \] (36)

4.1. Local error analysis. Taking the difference between (35) and (36) we see (by iterating Duhamel’s formula replacing \( \tau \) by \( s \) in (35)) that the local error

\[ \mathcal{E}(\tau, t_n) = e^{\tau \partial_3^3} (\varphi^\tau_\tau(u_\tau(t_n)) - \Phi^\tau(u_\tau(t_n))) \]

takes the form

\[ \mathcal{E}(\tau, t_n) = -\frac{1}{2} \int_0^\tau \left[ e^{\tau \partial_3^3} - \psi_1(s, \partial_{\tau}) \right] \Pi_\tau \partial_{\tau} (\Pi_\tau \psi_2(s, \partial_{\tau}) u_\tau(t_n))^2 \, ds \]

\[ - \frac{1}{2} \int_0^\tau e^{\tau \partial_3^3} \Pi_\tau \partial_{\tau} \left[ \Pi_\tau (u_\tau(t_n + s) - \psi_2(s, \partial_{\tau}) u_\tau(t_n)) \right] \Pi_\tau \left[ u_\tau(t_n + s) + \psi_2(s, \partial_{\tau}) u_\tau(t_n) \right] \, ds \]

\[ = -\frac{1}{2} \int_0^\tau \left[ e^{\tau \partial_3^3} - \psi_1(s, \partial_{\tau}) \right] \Pi_\tau \partial_{\tau} (\Pi_\tau \psi_2(s, \partial_{\tau}) u_\tau(t_n))^2 \, ds \]

\[ - \frac{1}{2} \int_0^\tau e^{\tau \partial_3^3} \Pi_\tau \partial_{\tau} \left[ \Pi_\tau (e^{\tau \partial_3^3} - \psi_2(s, \partial_{\tau}) u_\tau(t_n)) \right] \Pi_\tau \left[ u_\tau(t_n + s) + \psi_2(s, \partial_{\tau}) u_\tau(t_n) \right] \, ds \]

\[ + \frac{1}{4} \int_0^\tau e^{\tau \partial_3^3} \Pi_\tau \partial_{\tau} \left[ \right. \int_0^\tau e^{-(s-\xi) \partial_3^3} \Pi_\tau \partial_\tau (\Pi_\tau u_\tau(t_n + \xi))^2 \, d\xi \left. \right] \Pi_\tau \left[ u_\tau(t_n + s) + \psi_2(s, \partial_{\tau}) u_\tau(t_n) \right] \, ds. \] (37)

4.2. Global error analysis. Let \( e^{n+1} = u_\tau(t_{n+1}) - u^{n+1} \) denote the time discretisation error. By inserting zero in terms of \( \pm \Phi^\tau(u_\tau(t_n)) \) we obtain

\[ e^{n+1} = \varphi^\tau_\tau(u_\tau(t_n)) - \Phi^\tau(u^n) \]

\[ = \varphi^\tau_\tau(u_\tau(t_n)) - \Phi^\tau(u_\tau(t_n)) + \Phi^\tau(u_\tau(t_n)) - \Phi^\tau(u^n) \]

\[ = e^{\tau \partial_3^3} e^n + \mathcal{J}^\tau(e^n, u_\tau(t_n)) + e^{-\tau \partial_3^3} \mathcal{E}(\tau, t_n) \]

\[ = \sum_{\ell=0}^n e^{-\tau \partial_3^3} \mathcal{J}^\tau(e^\ell, u_\tau(t_\ell)) + \sum_{\ell=0}^n e^{-\tau \partial_3^3} \mathcal{E}(\tau, t_\ell), \]
where

$$J^\tau(e^n, u_\tau(t_n)) := \frac{1}{2} e^{-\tau \partial_x^3} \int_0^\tau \psi_1(s, \partial_x) \Pi_\tau \partial_x \left[ (\Pi_\tau \psi_2(s, \partial_x) e^n) (\Pi_\tau \psi_2(s, \partial_x)(-e^n + 2u_\tau(t_n))) \right] ds$$

(38)

and the local error $E(\tau, t_n) = e^{\tau \partial_x^3} (\varphi^\tau_\tau(u_\tau(t_n)) - \Phi^\tau(u_\tau(t_n)))$ is given by

$$E(\tau, t_n) = \sum_{j=1}^3 E_j^\tau(t_n)$$

(39)

with (see (37))

$$E_1^\tau(t_n) = \frac{1}{2} \int_0^\tau \left[ e^{\partial_x^3} - \psi_1(s, \partial_x) \right] \Pi_\tau \partial_x (\Pi_\tau \psi_2(s, \partial_x) u_\tau(t_n))^2 ds$$

(40)

$$E_2^\tau(t_n) = -\frac{1}{2} \int_0^\tau e^{\partial_x^3} \Pi_\tau \partial_x \left[ \Pi_\tau (e^{\partial_x^3} - \psi_2(s, \partial_x)) u_\tau(t_n) \right] \Pi_\tau [u_\tau(t_n + s) + \psi_2(s, \partial_x) u_\tau(t_n)] ds$$

(41)

$$E_3^\tau(t_n) = \frac{1}{4} \int_0^\tau e^{\partial_x^3} \Pi_\tau \partial_x \left( \int_0^\tau e^{-(\tau-\epsilon)\partial_x^3} \Pi_\tau \partial_x (\Pi_\tau u_\tau(t_n + \xi))^2 d\xi \right) \Pi_\tau [u_\tau(t_n + s) + \psi_2(s, \partial_x) u_\tau(t_n)] ds$$

(42)

In order to use global Bourgain spaces, we use again $\eta$ a smooth and compactly supported function, which is one on $[-1, 1]$ and supported in $[-2, 2]$ and we consider $e^n$ that will solve for $n \in \mathbb{Z}$ the following fixed point:

$$e^{n+1} = e(n-\ell) \tau \partial_x^3 J^\tau(\eta \left[ e^{-\tau \partial_x^3} - \psi_1(s, \partial_x) \right] \Pi_\tau \partial_x (\Pi_\tau \psi_2(s, \partial_x) u_\tau(t_n))^2 + e(n-\ell+1) \tau \partial_x^3 \eta(t_\ell) E(\tau, t_\ell),$$

(43)

where $J^\tau$ and $E$ are now defined by (38), (39) with $u_\tau$ replaced by a global extension satisfying the estimate (34) and $T_1 > 0, T_1 \leq 1 \leq T$ will be chosen sufficiently small. We observe that for $0 \leq n \leq N_1$, where $N_1 = \lceil \frac{T_1}{T} \rceil$, a solution of the above fixed point coincides with $u_\tau(t_n) - u^n$.

With these new definitions, we have the following estimate on the global error:

**Proposition 4.1.** There exists $C_T > 0$ such that for every $\tau \in (0, 1]$, we have the estimate

$$\tau^{-1} \|E(\tau, t_n)\|_{Y^0_{\tau}} \leq C_T \tau^\alpha, \quad \alpha = \min \left( 1, \frac{s_0}{3} \right).$$

**Proof.** Thanks to (39), we estimate each of the $E_j^\tau(t_n)$.

For $E_1^\tau(t_n)$, since $e^{\partial_x^3} - \psi_i, i = 1, 2$ and $\Pi_\tau$ are Fourier multipliers in the space variable, and since $\Pi_\tau$ projects on frequencies less than $\tau^{-\frac{1}{2}}$, we observe that for any function $(F(t_n))_n$, we have by Taylor expansion that

$$\sup_{s \in [-\tau, \tau]} \left\| \left[ e^{\partial_x^3} - \psi_1(s, \partial_x) \right] \Pi_\tau F(t_n) \right\|_{Y^0_{\tau}} \lesssim \tau^\alpha \|F(t_n)\|_{Y^0_{\tau}}.$$

(44)

Therefore, we get that

$$\tau^{-1} \|E_1^\tau(t_n)\|_{Y^0_{\tau}} \leq \tau^\alpha \sup_{s \in [0, \tau]} \left\| \Pi_\tau \partial_x (\Pi_\tau \psi_2(s, \partial_x) u_\tau(t_n))^2 \right\|_{Y^0_{\tau}}.$$

Then by using Lemma 3.3 and the fact that $\Pi_\tau, \psi_2$ are bounded Fourier multiplier (in space), we get that

$$\tau^{-1} \|E_1^\tau(t_n)\|_{Y^0_{\tau}} \leq \tau^\alpha \left\| u_\tau \right\|^2_{X^0_{\tau}}.$$
By using Lemma 3.4, we finally get that
\[ \tau^{-1}\|\mathcal{E}_1^\tau(t_n)\|_{Y_n^0} \leq C_T\tau^\alpha. \]

For \( \mathcal{E}_2^\tau \) defined in (41), by using again Lemma 3.3 we get that
\[ \tau^{-1}\|\mathcal{E}_2^\tau(t_n)\|_{Y_n^0} \leq \sup_{s\in[0,\tau]}\left\| \Pi_\tau (e^{s\partial_x^2} - \psi_2(s, \partial_x)) u_\tau(t_n) \right\|_{X_n^{0,\frac{1}{2}}} \left\| \Pi_\tau [u_\tau(t_n + s) + \psi_2(s, \partial_x)u_\tau(t_n)] \right\|_{X_{\tau}^{0,\frac{1}{2}}} . \]

Consequently, by using again (44) and Lemma 3.4, we find again that
\[ \tau^{-1}\|\mathcal{E}_2^\tau(t_n)\|_{Y_n^0} \lesssim C_T\tau^\alpha. \]

In a similar way, for \( \mathcal{E}_3^\tau \) defined in (42), we first use Lemma 3.3 to get that
\[ \tau^{-1}\|\mathcal{E}_3^\tau(t_n)\|_{Y_n^0} \lesssim \sup_{s\in[0,\tau]} \left( \left\| \int_0^\tau e^{-(s-\xi)\partial_x^2} \Pi_\tau \partial_x (\Pi_\tau u_\tau(t_n + \xi))^2 d\xi \right\|_{X_n^{0,\frac{1}{2}}} \left\| \Pi_\tau [u_\tau(t_n + s) + \psi_2(s, \partial_x)u_\tau(t_n)] \right\|_{X_{\tau}^{0,\frac{1}{2}}} \right). \] (45)

By using again the property of \( \Pi_\tau \), we first write that in the case \( s_0 \leq 1 \), we have
\[ \tau^{-1}\|\mathcal{E}_3^\tau(t_n)\|_{Y_n^0} \leq \tau\tau^{-\frac{1}{4}} \sup_{s, \xi\in[0,\tau]} \left\| (\Pi_\tau u_\tau(t_n + \xi))^2 \right\|_{X_n^{0,\frac{1}{2}}} \left\| \Pi_\tau [u_\tau(t_n + s) + \psi_2(s, \partial_x)u_\tau(t_n)] \right\|_{X_{\tau}^{0,\frac{1}{2}}} . \]

Next, thanks thanks to the property (21) of discrete Bourgain spaces, we get that
\[ \tau^{-1}\|\mathcal{E}_3^\tau(t_n)\|_{Y_n^0} \leq \tau^\frac{1}{2}\tau^{-\frac{1}{4}} \sup_{s, \xi\in[0,\tau]} \left\| (\Pi_\tau u_\tau(t_n + \xi))^2 \right\|_{X_n^{0,\frac{1}{2}}} \left\| \Pi_\tau [u_\tau(t_n + s) + \psi_2(s, \partial_x)u_\tau(t_n)] \right\|_{X_{\tau}^{0,\frac{1}{2}}} . \]

Next, by using the discrete Strichartz estimate (50), we get that
\[ \left\| (\Pi_\tau u_\tau(t_n + \xi))^2 \right\|_{X_n^{0,\frac{1}{2}}} \lesssim \|u_\tau(t_n + \xi)\|_{X_n^{0,\frac{1}{2}}}^2 . \]

Consequently, by using again Lemma 3.4, we get that
\[ \tau^{-1}\|\mathcal{E}_3^\tau(t_n)\|_{Y_n^0} \leq C_T\tau^{-\frac{1}{4}} + \frac{1}{\tau}. \]

If \( s_0 \geq 1 \), we can directly use (45), (21) and (50) to get that
\[ \tau^{-1}\|\mathcal{E}_3^\tau(t_n)\|_{Y_n^0} \leq \tau^\frac{1}{2} \sup_{s, \xi\in[0,\tau]} \left\| \Pi_\tau \partial_x u_\tau(t_n + \xi) \right\|_{X_n^{0,\frac{1}{2}}} \left\| u_\tau(t_n + \xi) \right\|_{X_n^{0,\frac{1}{2}}} \left\| (\Pi_\tau u_\tau(t_n + s) + \psi_2(s, \partial_x)u_\tau(t_n)) \right\|_{X_{\tau}^{0,\frac{1}{2}}} \left\| u_\tau(t_n) \right\|_{X_{\tau}^{0,\frac{1}{2}}} \right) \]
and therefore
\[ \tau^{-1}\|\mathcal{E}_3^\tau(t_n)\|_{Y_n^0} \leq C_T\tau^\frac{3}{4} . \]

We thus have obtained that
\[ \tau^{-1}\|\mathcal{E}_3^\tau(t_n)\|_{Y_n^0} \leq C_T\tau^{-\frac{a}{3}} \]
if \( s_0 \leq 3/2 \).

If \( s_0 > 3/2 \), since \( H^{s_0} \subset W^{1,\infty} \), we easily get directly from (42) that
\[ \tau^{-1}\|\mathcal{E}_3^\tau(t_n)\|_{Y_n^0} \leq \tau^{-1}\|\mathcal{E}_3^\tau(t_n)\|_{X_n^{0,0}} \lesssim \tau \sup_{\xi, s\in[0,\tau]} \left\| \Pi_\tau \partial_x u_\tau(t_n + \xi) \right\|_{L^2} \left\| \partial_x u_\tau(t_n + \xi) \right\|_{L^\infty} \left\| u_\tau(t_n + s) \right\|_{L^\infty} + \left\| u_\tau(t_n) \right\|_{L^\infty} \right). \]

This yields by using the property of \( \Pi_\tau \) that for \( s_0 \leq 2 \), we have
\[ \tau^{-1}\|\mathcal{E}_3^\tau(t_n)\|_{Y_n^0} \leq \tau^{-\frac{2-a}{3}} \left( C_T \lesssim \tau^{\frac{3}{2}} + \frac{a}{3} C_T \right) \]
while
\[ \tau^{-1}\|\mathcal{E}_3^\tau(t_n)\|_{Y_n^0} \leq \tau C_T \]

if $s_0 \geq 2$.

Summarizing all the cases, we obtain that

$$\tau^{-1} \| E^T_3(t_n) \|_{Y^n} \leq C_T \tau^\alpha.$$ 

This ends the proof. $\square$

5. PROOF OF THEOREM 1.1

We are now in a position to give the proof of Theorem 1.1. We first observe that thanks to Proposition 2.7, we have from the triangle inequality that

$$\| u(t_n) - u^n \|_{L^2} \leq \| u(t_n) - u(t_n) \|_{L^2} + \| u(t_n) - u^n \|_{L^2} \leq C_T \tau^{\frac{s_0}{2}} + \| u(t_n) - u^n \|_{L^2}. \quad (46)$$

To get the error estimates of Theorem 1.1 for $t_n \leq T_1$, it thus suffices to estimate $\| e^n \|_{X^n}$ thanks to (26) where $e^n$ solves the fixed point (43). By using (28), we get that

$$\| e^n \|_{X^n} \leq \tau^{-1} \| J^T \left( \eta \left( \frac{t_n}{T_1} \right) e^n, \eta \left( \frac{t_n}{T_1} \right) u(t_n) \right) \|_{Y^n} + \tau^{-1} \| E(\tau, t_n) \|_{Y^n},$$

and hence, thanks to Proposition 4.1 we have that

$$\| e^n \|_{X^n} \leq \tau^{-1} \| J^T \left( \eta \left( \frac{t_n}{T_1} \right) e^n, \eta \left( \frac{t_n}{T_1} \right) u(t_n) \right) \|_{Y^n} + C_T \tau^\alpha,$$

where $\alpha = \min(1, s_0/3)$. Next we estimate $\tau^{-1} \| J^T \left( \eta \left( \frac{t_n}{T_1} \right) e^n, \eta \left( \frac{t_n}{T_1} \right) u(t_n) \right) \|_{Y^n}$. From the expression (38), we get by using Lemma 3.3 (and the fact that $\psi_1, \psi_2$ are bounded Fourier multipliers) that

$$\tau^{-1} \| J^T \left( \eta \left( \frac{t_n}{T_1} \right) e^n, \eta \left( \frac{t_n}{T_1} \right) u(t_n) \right) \|_{Y^n} \leq \| \eta \left( \frac{t_n}{T_1} \right) e^n \|_{X^n_{\tau^\frac{1}{2}}} + \| \eta \left( \frac{t_n}{T_1} \right) u(t_n) \|_{X^n_{\tau^\frac{1}{2}}}.$$ 

Consequently, by using (15) and Lemma 3.4, we get that

$$\tau^{-1} \| J^T \left( \eta \left( \frac{t_n}{T_1} \right) e^n, \eta \left( \frac{t_n}{T_1} \right) u(t_n) \right) \|_{Y^n} \leq C_T T_1^\alpha \| e^n \|_{X^n} + C_T T_1^\alpha \| e^n \|_{X^n}^2.$$ 

for some $C_T > 0$ independent of $\tau \in (0, 1]$ and $T_1 \in (0, 1]$. This yields

$$\| e^n \|_{X^n} \leq C_T \left( \tau^\alpha + T_1^\alpha \| e^n \|_{X^n} + T_1^\alpha \| e^n \|_{X^n}^2 \right).$$

We thus get for $T_1$ sufficiently small that

$$\| e^n \|_{X^n} \leq C_T \tau^\alpha.$$ 

This proves the desired estimate (11) for $0 \leq n \leq N_1 = T_1/\tau$. We can then iterate in a classical way the argument on $T_1/\tau \leq n \leq 2T_1/\tau$ and so on to get the final estimate for $0 \leq n \leq T/\tau$. 

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6. Proof of Lemma 3.3

In this section, we shall prove Lemma 3.3. We adapt the proof in [2], [11], the main difficulty is to check that because of the frequency localization induced by the filter \( \Pi_r \), the favorable frequency interaction of the KdV equation is kept at the discrete level.

The first step is to prove the following Strichartz estimate which has its own interest.

**Lemma 6.1.** There exists \( C > 0 \) such that for every \( u^n \in X^0_r \), and \( \tau \in (0,1] \), we have the estimate

\[
\| \Pi_r u^n \|_{L^4 L^1} \leq C \| u^n \|_{X^0_r}.
\]

**Proof.** We first use a Littlewood-Paley type decomposition, we write

\[
1_{(-\pi,\pi)}(\sigma) = \sum_{m \geq 0} 1_{m}(\sigma),
\]

where \( 1_{m} \) is supported in \( 2^m \leq |\sigma| < 2^{m+1} \cap [-\pi, \pi) \) (the sum is actually finite). Next, we extend \( 1_{m} \) on \( \mathbb{R} \) by \( 2\pi/\tau \) periodicity so that

\[
1 = \sum_{m \geq 0} 1_{m}(\sigma).
\]

By using this decomposition, we expand

\[
\Pi_r u^n = \sum_{m \geq 0} u^n_m,
\]

where

\[
\tilde{u}^n_m(\sigma, k) = \tilde{u}^n(\sigma, k) 1_m(\sigma, k)
\]

and we have set

\[
1_m(\sigma, k) = 1_m(\sigma + k^3) 1_r 1|k| \leq 1 \, (k).
\]

Note that by our definition \( 1_m(\cdot, k) \) is \( 2\pi/\tau \) periodic.

We then write

\[
\| \Pi_r u^n \|_{L^4 L^1}^2 = \| (\Pi_r u^n)^2 \|_{L^2}^2 \leq 2 \sum_{p, q \geq 0} \| u^n_p u^n_{p+q} \|_{L^2 L^2}
\]

and hence from the Bessel identity we have that

\[
\| \Pi_r u^n \|_{L^4 L^1}^2 \leq 2 \sum_{p, q \geq 0} \| \tilde{u}^n_p * \tilde{u}^n_{p+q} \|_{L^2 L^2},
\]

where

\[
\tilde{u}^n_p * \tilde{u}^n_{p+q}(\sigma, k) = \sum_{k'} \int_{\sigma'} \tilde{u}^n_p(\sigma', k') \tilde{u}^n_{p+q}(\sigma - \sigma', k - k') d\sigma'.
\]

We thus need to estimate \( \| \tilde{u}^n_p * \tilde{u}^n_{p+q}(\sigma, k) \|_{L^2 L^2} \). We shall handle differently the \( l^2 \) norm for \( |k| \leq 2^\beta \) and \( |k| \geq 2^\beta \) for \( \beta \) to be chosen.

For \( |k| \leq 2^\beta \), we write

\[
\| \tilde{u}^n_p * \tilde{u}^n_{p+q}(\sigma, k) \|_{L^2} \leq \sum_{k'} \left\| \int_{\sigma'} \tilde{u}^n_p(\sigma', k') u^n_{p+q}(\sigma - \sigma', k - k') d\sigma' \right\|_{L^2}.
\]

and we use the Young inequality for convolution to obtain

\[
\| \tilde{u}^n_p * \tilde{u}^n_{p+q}(\sigma, k) \|_{L^2} \leq \sum_{k'} \left\| \tilde{u}^n_p(\cdot, k') \right\|_{L^1} \left\| u^n_{p+q}(\cdot, k - k') \right\|_{L^2}.
\]
By the frequency localization of $\tilde{u}_p^\alpha$, we have by Cauchy-Schwarz that
\[
\left\| \tilde{u}_p^\alpha(\cdot, k') \right\|_{L^1(\sigma)} \leq 2^\frac{\beta}{2} \left\| \tilde{u}_p^\alpha(\sigma, k') \right\|_{L^2(\sigma)}.
\]
Therefore we obtain by using also Cauchy-Schwarz for the sum in $k_1$ that
\[
\left\| \tilde{u}_p^\alpha \ast \tilde{u}_{p+q}^\alpha(\sigma, k) \right\|_{L^2(\sigma)} \lesssim 2^\frac{\beta}{2} \left\| \tilde{u}_p^\alpha \right\|_{L^2(\sigma)} \left\| \tilde{u}_{p+q}^\alpha \right\|_{L^2(\sigma)}.
\]
This yields
\[
\left\| \tilde{u}_p^\alpha \ast \tilde{u}_{p+q}^\alpha \right\|_{L^2(\sigma)^2(|k| \geq 2^\beta)} \lesssim 2^\frac{\beta}{2 + \frac{\beta}{2}} \left\| \tilde{u}_p^\alpha \right\|_{L^2(\sigma)^2} \left\| \tilde{u}_{p+q}^\alpha \right\|_{L^2(\sigma)^2}.
\tag{49}
\]
For $|k| \geq 2^\beta$, we write that
\[
\left\| \tilde{u}_p^\alpha \ast \tilde{u}_{p+q}^\alpha \right\|_{L^2(\sigma)^2(|k| \geq 2^\beta)} = \left\| \sum_{k'} \int_{\sigma'} \tilde{u}_p^\alpha(\sigma', k') \tilde{u}_{p+q}^\alpha(\sigma - \sigma', k - k') \mathbb{1}_p(\sigma', k') \mathbb{1}_{p+q}(\sigma - \sigma', k - k') d\sigma' \right\|_{L^2(\sigma)^2(|k| \geq 2^\beta)}
\]
and we get from Cauchy-Schwarz that
\[
\left\| \tilde{u}_p^\alpha \ast \tilde{u}_{p+q}^\alpha \right\|_{L^2(\sigma)^2(|k| \geq 2^\beta)} \lesssim \left( \sum_{k'} \int_{\sigma'} |\tilde{u}_p^\alpha(\sigma', k')|^2 |\tilde{u}_{p+q}^\alpha(\sigma - \sigma', k - k')|^2 d\sigma' \right)^{\frac{1}{2}} \left( \mathbb{1}_p \ast \mathbb{1}_{p+q}(\sigma, k) \right)^{\frac{1}{2}}_{L^2(\sigma)^2(|k| \geq 2^\beta)}.
\]
This yields
\[
\left\| \tilde{u}_p^\alpha \ast \tilde{u}_{p+q}^\alpha \right\|_{L^2(\sigma)^2(|k| \geq 2^\beta)} \lesssim \left\| \mathbb{1}_p \ast \mathbb{1}_{p+q} \right\|_{L^2(\sigma)^2} \left\| \tilde{u}_p^\alpha \right\|_{L^2(\sigma)^2} \left\| \tilde{u}_{p+q}^\alpha \right\|_{L^2(\sigma)^2}.
\]
We thus need to estimate
\[
\left\| \mathbb{1}_p \ast \mathbb{1}_{p+q} \right\|_{L^\infty(\sigma, |k| \geq 2^\beta)} \text{ where }
\]
\[
\mathbb{1}_p \ast \mathbb{1}_{p+q}(\sigma, k) = \sum_{k'} \int_{\sigma'} \mathbb{1}_p(\sigma', k') \mathbb{1}_{p+q}(\sigma - \sigma', k - k') d\sigma'.
\]
From the definitions of $\mathbb{1}_m$, we have a non-zero integral if $|\sigma' + k'\beta - \frac{m_1 \pi}{2}| \leq 2^{p+1}$ and $|\sigma - \sigma' + (k-k')\beta - \frac{m_2 \pi}{2}| \leq 2^{p+1+1}$ for some $m_1, m_2 \in \mathbb{Z}$ which means that $\sigma' + k'\beta \in E_p$ and $\sigma - \sigma' + (k-k')\beta \in E_{p+q}$ where we have set $E _k = \bigcup_{m \in \mathbb{Z}} \left[ -\frac{2^{p+1} + 2^{p+1}}{\beta}, \frac{2^{p+1} + 2^{p+1}}{\beta} \right]$. Note that since $|k|, |k'| \leq \frac{\pi}{\beta}$ the number of intervals in $E_p$ and $E_{p+q}$ yielding a nontrivial contribution is $O(1)$, we can thus take $N = O(1)$ independent of $\tau$. For a given $k'$, we observe that if the integral is not zero then it is bounded by $O(2^p)$. Moreover, to evaluate the number of non zero terms in the sum, we see that, we must have
\[
\sigma + k^3 + (k-k')^3 \in E_{p+q} + E_p
\]
which is equivalent to
\[
k' - k^2 \in \frac{1}{3} \left( -\frac{\sigma}{k} - k^2 + 2^\beta (E_{p+q} + E_p) \right)
\]
since $|k| \geq 2^\beta$. This yields
\[
\left( k' - k \right)^2 \leq -\frac{k^2}{4} + \frac{1}{3} \left( -\frac{\sigma}{k} - k^2 + 2^\beta (E_{p+q} + E_p) \right)
\]
which means that $k'$ must be restricted to a finite number $N$ of intervals of length smaller than $O(2^{\frac{2p+2-\beta}{2}})$. We thus find that
\[
\left\| \mathbb{1}_p \ast \mathbb{1}_{p+q} \right\|_{L^\infty(\sigma, |k| \geq 2^\beta)} \lesssim 2^\frac{p+q-\beta}{4}.
and hence
\[ \|u^n_p * \tilde{u}^n_{p+q}\|_{L^2(\sigma)t^2} \lesssim 2^{\frac{3p+q-\beta}{4}} \left( \|u^n_p\|_{L^2(\sigma)t^2} + \|\tilde{u}^n_{p+q}\|_{L^2(\sigma)t^2} \right). \]

Thanks to the last estimate and (49), we can then optimize the choice of \( \beta \). We take \( \beta = \frac{p+q}{3} \) and we deduce that
\[ \|u^n_p * \tilde{u}^n_{p+q}\|_{L^2(\sigma)t^2} \lesssim 2^{\frac{q+q}{3}} \left( \|u^n_p\|_{L^2(\sigma)t^2} + \|\tilde{u}^n_{p+q}\|_{L^2(\sigma)t^2} \right). \]

We then get from (48) that
\[ \|\Pi_r u^n\|_{L^2_t L^4_x}^2 \lesssim \sum_{q \geq 0} 2^{-\frac{q}{2}} \sum_{p \geq 0} 2^{\frac{p}{2}} \left( \|u^n_p\|_{L^2(\sigma)t^2}^2 + 2^{\frac{(p+q)}{3}} \|\tilde{u}^n_{p+q}\|_{L^2(\sigma)t^2} \right). \]

We finally conclude by using Cauchy-Schwarz for the sum in \( q \) and the fact that
\[ \sum_{l} \left( 2^{\frac{l}{2}} \|l^n_l\|_{L^2(\sigma)t^2} \right)^2 \lesssim \|u^n\|_{X^{1,\frac{4}{3}}}. \]

We can then deduce from Lemma 6.1 that

**Corollary 6.2.** For every \( s \geq 0 \), there exists \( C > 0 \) such that for every \( u^n, v^n \in X^{1,\frac{4}{3}}_r \), and \( \tau \in (0,1] \), we have the estimate
\[ \|\langle \partial_x \rangle^s (\Pi_r u^n \Pi_r v^n)\|_{L^2_t L^2_x} \leq C \|u^n\|_{X^{1,\frac{4}{3}}_r}. \]

**Proof.** We observe that:
\[ \|\langle \partial_x \rangle^s (\Pi_r u^n \Pi_r v^n)\|_{L^2_t L^2_x} = \left\| (\langle k \rangle^s (\Pi_r u^n * \Pi_r v^n)(\sigma, k)) \right\|_{L^2_t L^2_x} \]
\[ \lesssim \left( \|\langle k \rangle^s |\Pi_r u^n|\|_{L^2_t L^2_x} + \|\Pi_r \tilde{v}^n| \right\|_{L^2_t L^2_x} \] \[ \lesssim \left\| \langle k \rangle^s |\Pi_r u^n|\|_{L^2_t L^2_x} \right\|_{L^2_t L^2_x} \] \[ \lesssim \|a^n b^n\|_{L^2_t L^2_x}, \]
where \( a^n(x), b^n(x) \) are such that
\[ \tilde{a}^n(\sigma, k) = \langle k \rangle^s |\Pi_r u^n|, \quad \tilde{b}^n(\sigma, k) = |\Pi_r \tilde{v}^n|. \]

By using Cauchy-Schwarz and Lemma 6.1, we get that
\[ \|\langle k \rangle^s |\Pi_r u^n|\|_{L^2_t L^2_x} \leq \|\Pi_r a^n\|_{L^2_t L^4_x} \|\Pi_r n^n\|_{L^2_t L^4_x} \lesssim \|a^n\|_{X^{0,\frac{4}{3}}_r} \|b^n\|_{X^{0,\frac{4}{3}}_r} \lesssim \|u^n\|_{X^{0,\frac{4}{3}}_r} \|v^n\|_{X^{0,\frac{4}{3}}_r}. \]

The symmetric term can be handled with a similar argument. This ends the proof. 

We are now in position to give the proof of Lemma 3.3.
Proof of Lemma 3.3. We give the proof for $s = 0$. The general case $s > 0$ can be easily deduced with the same type of arguments as above. We shall thus estimate $\|\partial_x \Pi_\tau (\Pi_\tau u^n \Pi_\tau v^n)\|_{\mathcal{X}_\tau^{0,-\frac{1}{2}}}$. We use a duality argument:

$$
\|\partial_x \Pi_\tau (\Pi_\tau u^n \Pi_\tau v^n)\|_{\mathcal{X}_\tau^{0,-\frac{1}{2}}} = \sup_{\|u^n\|_{\mathcal{X}_\tau^{0,-\frac{1}{2}}} \leq 1} \left| \tau \sum_n \int_\mathbb{R} \Pi_\tau u^n \Pi_\tau v^n \partial_x u^n \, dx \right|
$$

We then set

$$
\tilde{a}^n(\sigma, k) = (d_\tau(\sigma + k^3))^\frac{1}{2} \|\Pi_\tau u^n(\sigma, k)\|, \\
\tilde{b}^n(\sigma, k) = (d_\tau(\sigma + k^3))^\frac{1}{2} \|\Pi_\tau v^n(\sigma, k)\|, \\
\tilde{c}^n(\sigma, k) = (d_\tau(\sigma + k^3))^\frac{1}{2} \|\Pi_\tau w^n(-\sigma, -k)\|
$$

and we shall estimate

$$
I = \sum_{k, k'} \int \mathbb{R} m_\tau(\sigma, \sigma', k, k') \tilde{a}^n(\sigma', k') \tilde{b}^n(\sigma - \sigma', k - k') \tilde{c}^n(\sigma, k) \, d\sigma' \, d\sigma,
$$

where

$$
m_\tau(\sigma, \sigma', k, k') = \frac{|k|}{(d_\tau(\sigma' + k^3))^{\frac{1}{2}} (d_\tau(\sigma - \sigma' + (k - k')^3))^{\frac{1}{2}} (d_\tau(\sigma + k^3))^{\frac{1}{2}}}.
$$

Note that we have $\sigma, \sigma' \in [-\pi/\tau, \pi/\tau]$ and by the choice of $\Pi_\tau$, $|k|^3, |k'|^3 \leq \tau^{-1}$.

We first assume that $|\sigma + k^3|$ or $|\sigma' + k'^3|$ is bigger than $\epsilon/\tau$ for some $\epsilon > 0$ independent of $\tau$ to be chosen. Let us assume that it is the first one (the other case being symmetric), then $|d_\tau(\sigma + k^3)|^{\frac{1}{2}} \gtrsim \tau^{-\frac{1}{2}}$ and therefore,

$$
m_\tau(\sigma, \sigma', k, k') \lesssim \frac{\tau^{\frac{1}{2}}}{(d_\tau(\sigma' + k^3))^{\frac{1}{2}} (d_\tau(\sigma - \sigma' + (k - k')^3))^{\frac{1}{2}} (d_\tau(\sigma + k^3))^{\frac{1}{2}}} \lesssim \frac{1}{(d_\tau(\sigma' + k^3))^{\frac{1}{2}} (d_\tau(\sigma - \sigma' + (k - k')^3))^{\frac{1}{2}}}.
$$

When both $|\sigma + k^3|$ and $|\sigma' + k'^3|$ are smaller than $\epsilon/\tau$, since

$$
\sigma - \sigma' + (k - k')^3 = \sigma + k^3 - (\sigma' + k^3) - 3kk'(k - k'),
$$

we have that

$$
|\sigma - \sigma' + (k - k')^3| \leq \frac{2\epsilon}{\tau} + \frac{6}{\tau} < \frac{2\pi}{\tau}
$$

by choosing $\epsilon$ sufficiently small. Therefore in this situation we have that

$$
|d_\tau(\sigma + k^3)| \gtrsim |\sigma + k^3|, |d_\tau(\sigma' + k'^3)| \gtrsim |\sigma' + k'^3|, |d_\tau(\sigma - \sigma' + (k - k')^3)| \gtrsim |\sigma - \sigma' + (k - k')^3|.
$$

We are thus in a situation very close to the continuous case, we have

$$
m_\tau(\sigma, \sigma', k, k') \lesssim \frac{|k|}{(\sigma' + k'^3)^{\frac{1}{2}} (\sigma - \sigma' + (k - k')^3)^{\frac{1}{2}} (\sigma + k^3)^{\frac{1}{2}}}.
$$
and since
\[ \sigma' + k^3 - (\sigma + k^3) + \sigma - \sigma' + (k - k')^3 = -3kk'(k - k') \]
we deduce that
\[ \max(|\sigma' + k^3|, |\sigma + k^3|, |\sigma - \sigma' + (k - k')^3|) \geq 3|k| |k'| |k - k'|. \] (52)

Let us assume that the largest one above is $|\sigma + k^3|$, the other cases being similar. Then we get
\[
m_{r}(\sigma, \sigma', k, k') \lesssim \frac{|k|^\frac{1}{2}}{(k')\frac{1}{2}(k - k')\frac{1}{2} (d_r(\sigma' + k^3))\frac{1}{2} (d_r(\sigma - \sigma' + (k - k')^3))}\lesssim\frac{1}{(d_r(\sigma' + k^3))\frac{1}{2} (d_r(\sigma - \sigma' + (k - k')^3))}\]
by using that $|k| \leq |k'| + |k - k'|$ to get the last line. The above estimate is similar to (51). We can thus estimate $I$ by $II +$ symmetric terms where
\[ II = \sum_{k, k'} \int_{\sigma, \sigma'} \tilde{a}_n(\sigma', k') \tilde{b}_n(\sigma - \sigma', k - k') \tilde{c}_n(\sigma, k) \, d\sigma d\sigma', \]
where we have set
\[ \tilde{a}_n(\sigma, k) = |\Pi_r w^n(\sigma, k)|, \quad \tilde{b}_n(\sigma, k) = |\Pi_r v^n(\sigma, k)|. \]

Going back to the physical space, we get that
\[ II = \tau \sum_{n} \int_{\Gamma} \tilde{a}_n \tilde{b}_n \tilde{c}_n \, dx \]
and hence from the H"older inequality, we find
\[ II \leq \|\tilde{a}_n\|_{L^1} \|\tilde{b}_n\|_{L^1} \|\tilde{c}_n\|_{L^1}. \]

This yields thanks to Lemma 6.1,
\[ II \lesssim \|\alpha^n\|_{X_{\tau}^{0, \frac{1}{2}}} \|\beta^n\|_{X_{\tau}^{0, \frac{1}{2}}} \|\tilde{c}_n\|_{L^2} \lesssim \|u^n\|_{X_{\tau}^{0, \frac{1}{2}}} \|v^n\|_{X_{\tau}^{0, \frac{1}{2}}} \|w^n\|_{X_{\tau}^{0, \frac{1}{2}}}. \]

We thus finally get that
\[ I \leq \left( \|u^n\|_{X_{\tau}^{0, \frac{1}{2}}} \|v^n\|_{X_{\tau}^{0, \frac{1}{2}}} + \|u^n\|_{X_{\tau}^{0, \frac{1}{2}}} \|v^n\|_{X_{\tau}^{0, \frac{1}{2}}} \|w^n\|_{X_{\tau}^{0, \frac{1}{2}}} \right) \|	ilde{w}_n\|_{X_{\tau}^{0, \frac{1}{2}}} \]
from which we deduce the estimate of $\|\partial_x \Pi_r (\Pi_r u^n \Pi_r v^n)\|_{X_{\tau}^{0, -\frac{1}{2}}}$. We use again a duality argument, we take $(w^n)_n$ such that $\|\tilde{w}_n\|_{H^{2}(k)L^\infty(\sigma)} \leq 1$ and hence, we have to estimate with $\alpha^n$ and $\beta^n$ as above
\[ III = \sum_{k, k'} \int_{\sigma, \sigma'} m_{r}^1(\sigma, \sigma', k, k') \tilde{a}_n(\sigma', k') \tilde{b}_n(\sigma - \sigma', k - k') \tilde{c}_n(\sigma, k) \, d\sigma d\sigma' \]
with
\[ m_{r}^1(\sigma, \sigma', k, k') = \frac{|k|}{(d_r(\sigma' + k^3))^{\frac{1}{2}} (d_r(\sigma - \sigma' + (k - k')^3))^{\frac{1}{2}} (d_r(\sigma + k^3))}. \]

Again, if $|\sigma' + k^3| \geq \epsilon/\tau$, we have since $|k| \leq \tau^{-\frac{1}{4}}$ that
\[ m_{r}^1(\sigma, \sigma', k, k') \lesssim \frac{1}{(d_r(\sigma - \sigma' + (k - k')^3))^{\frac{1}{2}} (d_r(\sigma + k^3))}. \] (53)
Going back to the physical space and using the Hölder inequality, we estimate this part of $\text{III}$ by

$$
\|a^n\|_{l_t^2 L^2_x} \|\beta^n\|_{l_t^2 L^4_x} \left\| \mathcal{F}_{\alpha, k \rightarrow n, x} \left( \frac{\Pi_{\tau} w^n}{\langle d_{\tau}(\sigma + k^3) \rangle} \right) \right\|_{l_t^2 L^4}
$$

which is bounded thanks to Lemma 6.1 by

$$
\|a^n\|_{l_t^2 L^2_x} \|\beta^n\|_{l_t^2 L^2_x} \left\| \frac{\tilde{w}^n}{\langle d_{\tau}(\sigma + k^3) \rangle} \right\|_{l_t^2 L^2(\sigma)} \lesssim \|u^n\|_{X^0_{\tau} \frac{3}{2}} \|v^n\|_{X^0_{\tau} \frac{3}{2}} \left\| \frac{\tilde{w}^n}{\langle d_{\tau}(\sigma + k^3) \rangle} \right\|_{l_t^2 L^2(\sigma)} \lesssim \|u^n\|_{X^0_{\tau} \frac{3}{2}} \|v^n\|_{X^0_{\tau} \frac{3}{2}} \left\| \tilde{w}^n \right\|_{l_t^2 L^\infty(\sigma)}
$$

since $4/3 > 1$ which is the desired estimate.

If $|\sigma + k^3| \geq \epsilon/\tau$, we have

$$
m^4_1(\sigma, \sigma', k, k') \lesssim \tau^{-\frac{1}{3}} \frac{1}{\langle d_{\tau}(\sigma' + k^3) \rangle^{\frac{1}{3}} \langle d_{\tau}(\sigma - \sigma' + (k - k')^3) \rangle^{\frac{1}{3}} \langle |d_{\tau}(\sigma + k^3)| + \frac{1}{\tau} \rangle}.
$$

From the same arguments as above using Lemma 6.1, we then get that this contribution in $\text{III}$ can be estimated by

$$
\tau^{-\frac{1}{3}} \|u^n\|_{X^0_{\tau} \frac{3}{2}} \|v^n\|_{X^0_{\tau} \frac{3}{2}} \left\| \frac{\tilde{w}^n}{\langle d_{\tau}(\sigma + k^3) \rangle + \frac{1}{\tau}} \right\|_{l_t^2 L^2(\sigma)} \lesssim \|u^n\|_{X^0_{\tau} \frac{3}{2}} \|v^n\|_{X^0_{\tau} \frac{3}{2}} \left\| \tilde{w}^n \right\|_{l_t^2 L^\infty(\sigma)},
$$

where for the last estimate, we have used that

$$
\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \frac{1}{\langle |d_{\tau}(\sigma + k^3)| + \frac{1}{\tau} \rangle^2} \lesssim \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \frac{1}{1 + |\sigma|^2 + \frac{1}{\tau^2}} d\sigma \lesssim \tau.
$$

It remains to handle the case $|\sigma + k^3| \leq \epsilon/\tau$, $|\sigma' + k^3| \leq \epsilon/\tau$. By choosing $\epsilon$ sufficiently small as before, we have in this case that

$$
m^4_1(\sigma, \sigma', k, k') \lesssim \frac{|k|}{\langle |\sigma' + k^3| \rangle^{\frac{1}{3}} \langle |\sigma - \sigma' + (k - k')^3| \rangle^{\frac{1}{3}} \langle |\sigma + k^3| \rangle^{\frac{1}{3}}}.
$$

We shall thus use again the property (52). We shall consider the two cases:

- if $\max(|\sigma' + k^3|, |\sigma + k^3|, |\sigma - \sigma' + (k - k')^3|) = |\sigma' + k^3|$ (the case that the max is $|\sigma - \sigma' + (k - k')^3|$ is symmetric). Then we get from (52) that

$$
m^4_1(\sigma, \sigma', k, k') \lesssim \frac{1}{\langle d_{\tau}(\sigma - \sigma' + (k - k')^3) \rangle^{\frac{1}{3}} \langle d_{\tau}(\sigma + k^3) \rangle}.
$$

which is similar to (53). We can thus estimate this contribution to $\text{III}$ in the same way as previously.

- if $\max(|\sigma' + k^3|, |\sigma + k^3|, |\sigma - \sigma' + (k - k')^3|) = |\sigma + k^3|$. We observe that (52) gives

$$
|\sigma + k^3| \gtrsim |k||k'||k - k'| \gtrsim |k|^2
$$

and we therefore get that

$$
m^4_1(\sigma, \sigma', k, k') \lesssim \frac{|k|}{\langle d_{\tau}(\sigma' + k^3) \rangle^{\frac{1}{3}} \langle d_{\tau}(\sigma - \sigma' + (k - k')^3) \rangle^{\frac{1}{3}} \langle |d_{\tau}(\sigma + k^3)| + |k|^2 \rangle}.
$$
We thus get by using again Lemma 6.1, get that this contribution in III can be estimated by
\[ \left\| u^n \right\|_{X^F_{\alpha,1}} \left\| v^n \right\|_{X^F_{\alpha,1}} \left\| \frac{|k| \tilde{w}^n}{\langle |d_\tau (\sigma + k^3) | + |k|^2 \rangle} \right\|_{L^2(k)L^2(\sigma)} \left\| |k| \tilde{w}^n \right\|_{L^2(k)L^2(\sigma)} \left\| u^n \right\|_{L^2(k)L^\infty(\sigma)} \]
To conclude, we use that
\[ \left\| \frac{|k| \tilde{w}^n}{\langle |d_\tau (\sigma + k^3) | + |k|^2 \rangle} \right\|_{L^2(k)L^2(\sigma)} \lesssim \left\| \tilde{w}^n \right\|_{L^2(k)L^2(\sigma)} \]
since
\[ \int_{\pi}^{2\pi} \frac{1}{\langle |d_\tau (\sigma + k^3) | + |k|^2 \rangle^2} d\sigma \lesssim \int_{\pi}^{2\pi} \frac{1}{1 + |\sigma|^2 + |k|^4} d\sigma \lesssim \frac{1}{|k|^2} \]
Gathering all the above estimates, we arrive at
\[ \left\| \frac{1}{|d_\tau (\sigma + k^3)|} F_{n,x,\sigma,k} (\partial_x \Pi_\tau (\Pi_\tau u^n \Pi_\tau v^n)) \right\|_{L^2(k)L^1(\sigma)} \lesssim \left\| u^n \right\|_{X^F_{\alpha,1}} \left\| v^n \right\|_{X^F_{\alpha,1}} + \left\| u^n \right\|_{X^F_{\alpha,1}} \left\| v^n \right\|_{X^F_{\alpha,1}} \]
This ends the proof of Lemma 3.3.

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