AUTOMORPHISM GROUPS OF AFFINE VARIETIES WITHOUT NON-ALGEBRAIC ELEMENTS

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ABSTRACT. Given an affine algebraic variety $X$, we prove that if the neutral component $\text{Aut}^0(X)$ of the automorphism group consists of algebraic elements, then it is nested, i.e., is a direct limit of algebraic subgroups. This improves our earlier result \cite{5}. To prove it, we obtain the following fact. If a connected ind-group $G$ contains a closed connected ind-subgroup $H \subset G$ with a geometrically smooth point, and for any $g \in G$ some power of $g$ belongs to $H$, then $G = H$.

1. Introduction

In this note we work over an algebraically closed field of characteristic zero $\mathbb{K}$. We study the automorphism groups of affine varieties. It is well known that these groups can be larger than any algebraic group. For example, the automorphism group $\text{Aut}(\mathbb{A}^n)$ of the affine $n$-space $\mathbb{A}^n$ contains a copy of a polynomial ring in $n - 1$ variables, hence it is infinite-dimensional for $n \geq 2$.

In \cite{9} Shafarevich introduced the notion of the infinite-dimensional algebraic group, which is currently called the ind-group and showed that $\text{Aut}(\mathbb{A}^n)$ has the structure of the ind-group. Later it was shown that $\text{Aut}(X)$ has a natural structure of an ind-group for any affine variety $X$, see \cite[Section 5]{4} and also \cite[Section 2]{6}.

We call an element $g$ of the automorphism group $\text{Aut}(X)$ algebraic if there is an algebraic subgroup $G$ of the ind-group $\text{Aut}(X)$ that contains $g$. We also denote by $\mathbb{G}_a$ the additive group of the field and by $\mathcal{U}(X) \subset \text{Aut}(X)$ the (possibly trivial) subgroup generated by all the $\mathbb{G}_a$-actions. It is usually called the special automorphism group and is also denoted by $\text{SAut}(X)$.

In \cite{5} we proved that for the subgroup $\text{Aut}_{\text{alg}}(X) \subset \text{Aut}(X)$ generated by all connected algebraic subgroups the following conditions are equivalent:

- $\mathcal{U}(X)$ is abelian;
- all elements of $\text{Aut}_{\text{alg}}(X)$ are algebraic;
- the subgroup $\text{Aut}_{\text{alg}}(X) \subset \text{Aut}(X)$ is a closed nested ind-subgroup, i.e., is a direct limit of algebraic subgroups;
- $\text{Aut}_{\text{alg}}(X) = T \ltimes \mathcal{U}(X)$, where $T$ is a maximal subtorus of $\text{Aut}(X)$, and $\mathcal{U}(X)$ is closed in $\text{Aut}(X)$.

In this paper we prove that this result can be partially extended from $\text{Aut}_{\text{alg}}(X)$ to the connected component $\text{Aut}^0(X)$. More precisely, we have the following result which is proved in Section 4.

Theorem 1.1. Let $X$ be an affine variety. The following conditions are equivalent:

1. all elements of $\text{Aut}^0(X)$ are algebraic;
2. the subgroup $\text{Aut}^0(X) \subset \text{Aut}(X)$ is a closed nested ind-subgroup;

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In [6] this theorem is proved for algebraic surfaces with a nontrivial group \( \mathcal{U}(X) \).

The key observation in our proof is as follows. Under condition (1) for any element of \( \text{Aut}^0(X) \) some power of it belongs to \( T \ltimes \mathcal{U}(X) \), see Lemma 4.1. In Section 3 we prove that an ind-group \( G \) coincides with its ind-subgroup \( H \) if for any element of \( G \) some power of it lies in \( H \), see Theorem 3.1. We also need a certain smoothness condition on \( H \), which is fulfilled if \( H \) is nested.

In Section 5 we also state some observations about the group of automorphisms of a rigid affine variety, i.e., an affine variety that admits no \( \mathbb{G}_a \)-actions.

2. Preliminaries

2.1. Ind-groups. The notion of an ind-group goes back to Shafarevich who called these objects infinite dimensional groups (see [9]). We refer to [4] for basic notions in this context.

Definition 2.1. By an affine ind-variety we mean an injective limit \( V = \varinjlim V_i \) of an ascending sequence \( V_0 \hookrightarrow V_1 \hookrightarrow V_2 \hookrightarrow \ldots \) such that the following holds:

1. \( V = \bigcup_{k \in \mathbb{N}} V_k \);
2. each \( V_k \) is an affine algebraic variety;
3. for all \( k \in \mathbb{N} \) the embedding \( V_k \hookrightarrow V_{k+1} \) is closed in the Zariski topology.

For simplicity we will call an affine ind-variety simply an ind-variety. An ind-variety \( V \) has a natural topology: a subset \( S \subset V \) is called open (resp. closed) if \( S_k := S \cap V_k \subset V_k \) is open (resp. closed) for all \( k \in \mathbb{N} \). A closed subset \( S \subset V \) has a natural structure of an ind-variety and is called an ind-subvariety.

The product of ind-varieties \( X = \varinjlim X_i \) and \( Y = \varinjlim Y_i \) is defined as \( \varinjlim X_i \times Y_i \). A morphism between ind-varieties \( V = \bigcup_k V_k \) and \( W = \bigcup_m W_m \) is a map \( \phi : \hat{V} \to W \) such that for every \( k \in \mathbb{N} \) there is an \( m \in \mathbb{N} \) such that \( \phi(V_k) \subset W_m \) and that the induced map \( V_k \to W_m \) is a morphism of algebraic varieties. This allows us to give the following definition.

Definition 2.2. An ind-variety \( G \) is said to be an ind-group if the underlying set \( G \) is a group such that the map \( G \times G \to G, (g,h) \mapsto gh^{-1} \), is a morphism.

A closed subgroup \( H \) of \( G \) is a subgroup that is also a closed subset. Then \( H \) is again an ind-group with respect to the induced ind-variety structure. A closed subgroup \( H \) of an ind-group \( G = \varinjlim G_i \) is called an algebraic subgroup if \( H \) is contained in some \( G_i \).

The next result can be found in [1, Section 5].

Proposition 2.3. Let \( X \) be an affine variety. Then \( \text{Aut}(X) \) has the structure of an ind-group such that a regular action of an algebraic group \( G \) on \( X \) induces a homomorphism of ind-groups \( G \to \text{Aut}(X) \).

Two ind-structures \( V = \varinjlim V_i \) and \( V' = \varinjlim V'_i \) are called equivalent, if the identity map \( \varinjlim V_i \to \varinjlim V'_i \) is an isomorphism of ind-varieties. One also calls \( \varinjlim V'_i \) an admissible filtration of the ind-variety \( V = \varinjlim V_i \).

Definition 2.4 ([3, Definition 1.9.4]). A point \( p \) in an ind-variety \( V \) is called geometrically smooth, if there exists an admissible filtration \( V = \varinjlim V_i \) such that \( p \) is a smooth point of \( V_i \) for each \( i \).
An element \( g \in \text{Aut}(X) \) is called \emph{algebraic} if there is an algebraic subgroup \( G \subset \text{Aut}(X) \) such that \( g \in G \). An ind-group \( G = \varprojlim G_i \) is called \emph{nested} if \( G_i \) is an algebraic group for \( i = 1, 2, \ldots \).

2.2. Lie algebras of ind-groups. For an ind-variety \( V = \bigcup_{k \in \mathbb{N}} V_k \) we can define the tangent space in \( x \in V \) in the obvious way: we have \( x \in V_k \) for \( k \geq k_0 \), and \( T_x V_k \subset T_x V_{k+1} \) for \( k \geq k_0 \), and then we define

\[
T_x V := \bigcup_{k \geq k_0} T_x V_k,
\]

which is a vector space of at most countable dimension.

For an ind-group \( G \), the tangent space \( T_e G \) has a natural structure of a Lie algebra which is denoted by \( \text{Lie} G \), see [7, Section 4] and [4, Section 2] for details.

2.3. \( \mathbb{G}_a \)-actions. Given an affine variety \( X \), we denote by \( \text{Aut}_{\text{alg}}(X) \subset \text{Aut}(X) \) the subgroup generated by all connected algebraic subgroups of the automorphism group \( \text{Aut}(X) \).

An element \( u \in \text{Aut}(X) \) is called \emph{unipotent} if \( u \) belongs to an algebraic subgroup of \( \text{Aut}(X) \) isomorphic to \( \mathbb{G}_a \). We denote the automorphism subgroup of \( \text{Aut}(X) \) generated by all the unipotent elements by \( U(X) \).

3. IND-SUBGROUP WITH POWERS OF ELEMENTS

In this section we explore the situation when an ind-subgroup contains some power of any element of the ind-group and prove Theorem 3.1.

**Theorem 3.1.** Let \( G \) be a connected ind-group and \( H \subset G \) be a closed connected ind-subgroup with a geometrically smooth point. Assume that for any \( g \in G \) there exists \( d \in \mathbb{N} \) such that \( g^d \in H \). Then \( G = H \).

By [4, Theorem 0.1.1] and [4, Remark 2.2.3] there exist ind-structures \( G = \varprojlim G_i \) and \( H = \varprojlim H_i \) such that each \( G_i \) and \( H_i \) is an irreducible subset containing the identity. Moreover, since there exists a geometrically smooth point \( p \in H \), then every point in \( H \) is geometrically smooth, and we may assume that each \( H_i \) is smooth at the identity.

**Remark 3.2.** Any nested ind-group is geometrically smooth at each point. However, to our knowledge, this property is not proven for arbitrary ind-groups. For example, a stronger property of being \emph{strongly smooth} does not hold for the ind-group \( \text{Aut} (\mathbb{A}^2) \), see [4, Corollary 14.1.2]. More generally, this group does not admit a filtration by normal varieties.

Consider the multiplication map

\[
\mu_d: G^d = G \times \cdots \times G \to G, \ (g_1, \ldots, g_d) \mapsto g_1 \cdots g_d.
\]

Its differential is the linear map

\[
d\mu_d: (\text{Lie} G)^d = \text{Lie} G \times \cdots \times \text{Lie} G \to \text{Lie} G.
\]

We have the following statement.

**Lemma 3.3.** Given \((x_1, \ldots, x_d) \in (\text{Lie} G)^d\), the following holds:

\[
d\mu_d((x_1, \ldots, x_d)) = x_1 + \cdots + x_d.
\]
Proof. By linearity,
\[ d\mu_d((x_1, \ldots, x_d)) = \sum_i d\mu_d((0, \ldots, 0, x_i, 0, \ldots, 0)). \]
We claim that \( d\mu_d((0, \ldots, 0, x_i, 0, \ldots, 0)) = x_i \). Indeed, let us denote
\[ s_i: G \to \underbrace{G \times \cdots \times G}_{d \text{ times}}, \quad g \mapsto (id, \ldots, g_{\text{i-th position}}, \ldots, id). \]
The composition \( \mu \circ s_i \) is the trivial automorphism of \( G \). Hence,
\[ d(\mu \circ s_i): \text{Lie } G \to \underbrace{\text{Lie } G \oplus \cdots \oplus \text{Lie } G}_{d \text{ times}} \]
is the identity map, where the first map in (2) is given by the embedding into the i-th coordinate. Therefore, we conclude that \( d\mu_d((0, \ldots, 0, x_i, 0, \ldots, 0)) = x_i \). Now, from (1) it follows that
\[ d\mu_d((x_1, \ldots, x_d)) = \sum_i x_i. \]
□

Definition 3.4. We denote \( \phi_d: G \to G, \ g \mapsto g^d \). It is an endomorphism of an ind-variety.

Corollary 3.5. The differential \( d\phi_d: \text{Lie } G \to \text{Lie } G \) satisfies
\[ d\phi_d(x) = d \cdot x \]
for any \( x \in \text{Lie } G \).

Proof. Consider an embedding
\[ s: G \to \underbrace{G \times \cdots \times G}_{d \text{ times}}; \quad g \mapsto (g, \ldots, g). \]
Its differential is the embedding
\[ ds: \text{Lie } G \to \underbrace{\text{Lie } G \oplus \cdots \oplus \text{Lie } G}_{d \text{ times}}; \quad x \mapsto (x, \ldots, x). \]
Since \( \phi_d = \mu_d \circ s \), by Lemma 3.3
\[ d\phi_d(x) = d\mu_d((x, \ldots, x)) = d \cdot x. \]
□

Definition 3.6. For each \( d, k \in \mathbb{N} \) we denote
\[ X_{d,k} = \phi_d^{-1}(H_k) = \{ g \in G \mid g^d \in H_k \} \subset G. \]

Lemma 3.7. (1) The subset \( X_{d,k} \) is closed in \( G \) for any \( d, k \in \mathbb{N} \).
(2) For any closed algebraic subset \( A \subset G \) there exist \( d, k \in \mathbb{N} \) such that \( A \subset X_{d,k} \).

Proof. The map \( \phi_d \) is a morphism of ind-varieties, so the first statement follows from \( X_{d,k} = \phi_d^{-1}(H_k) \).

The increasing sequence of closed subsets
\[ X_{d,1} \subset X_{d,2} \subset \ldots \subset X_{d,i} \subset \ldots \]
exhausts \( G \), hence \( A \subset X_{d,i} \) for some \( i \in \mathbb{N} \). We may take \( d = i! \) and \( k = i \) to get the second assertion. See also [4 Theorem 1.3.3].
□
Proof of Theorem 1.1. Denote the restriction of $\phi_d: G \to G$, $g \mapsto g^d$, to $X_{d,k}$ by $\phi_{d,k}$. Then

$$\phi_{d,k}: X_{d,k} \to H_k, \ g \mapsto g^d.$$  

Its differential map at the identity,

$$d(\phi_{d,k})_{\text{id}}: T_{\text{id}}X_{d,k} \to T_{\text{id}}H_k,$$

is given by $x \mapsto d \cdot x$ due to Corollary 3.3. This map has trivial kernel and is surjective due to $H_k \subset X_{d,k}$. So, $\dim T_{\text{id}}X_{d,k} = \dim T_{\text{id}}H_k$.

Since $H_k$ is smooth at the identity, $\dim T_{\text{id}}H_k = \dim H_k$. Let $Y$ be the union of irreducible components of $X_{d,k}$ containing the identity. From $H_k \subset Y$ and $\dim T_{\text{id}}Y = \dim H_k$ we infer that $Y = H_k$. Thus, the set $X_{d,k}$ contains $H_k$ as an irreducible component, and other components do not contain the identity.

By Lemma 3.7 for any $i \in \mathbb{N}$ there exist $d, k \in \mathbb{N}$ such that $G_i \subset X_{d,k}$. Since $G_i$ is irreducible and contains the identity, $G_i$ is a subset of the only irreducible component of $X_{d,k}$ which contains the identity, namely, $H_k$. We conclude that $G \subseteq H$. \qed

4. Neutral component without non-algebraic elements

In this section we assume that $\text{Aut}^\circ(X)$ consists of algebraic elements. By [5, Main Theorem], $\mathcal{U}(X)$ is an abelian unipotent ind-group (which is trivial, one-dimensional, or infinite-dimensional), and the subgroup $\text{Aut}_{\text{alg}}(X)$ generated by connected algebraic subgroups equals $T \times \mathcal{U}(X)$, where $T$ is a maximal algebraic torus.

Lemma 4.1. For any algebraic element $g \in \text{Aut}^\circ(X)$ there exists $d \in \mathbb{N}$ such that $g^d \in T \times \mathcal{U}(X)$.

Proof. The Zariski closure of $\{g^n \mid n \in \mathbb{Z}\}$ is an abelian algebraic group, which we denote by $G$. The subgroup $G^\circ$ is of finite index in $G$, so we may denote $d = |G/G^\circ|$ and we have $g^d \in G^\circ$. Since $G^\circ$ is a connected algebraic group, $G^\circ \subset T \times \mathcal{U}(X)$. The claim follows. \qed

Remark 4.2. By [2, Theorem 1.1], for any algebraic group $G$ there is a finite subgroup $H \subset G$ such that $G = H \cdot G^\circ$. Thus, any algebraic element of $\text{Aut}(X)$ is a product of an element of $\text{Aut}_{\text{alg}}(X)$ and a finite order one.

As we have mentioned above, $\mathcal{U}(X)$ is a direct limit of its unipotent algebraic subgroups, i.e., $\mathcal{U}(X) = \lim \mathcal{U}(X)_k$, where each $\mathcal{U}(X)_k$ is a closed unipotent algebraic subgroup of $\mathcal{U}(X)$. We set $\overline{\mathcal{U}(X)}_k = \mathcal{U}(X)$ for each $k$ if $\mathcal{U}(X)$ is itself an algebraic group.

Proof of Theorem 1.1. Assume that all elements of $\text{Aut}^\circ(X)$ are algebraic. By [5, Theorem 1.3], $\text{Aut}_{\text{alg}}(X)$ equals $T \times \mathcal{U}(X)$. By Lemma 4.1 we may apply Theorem 3.1 to $G = \text{Aut}^\circ(X)$ and $H = T \times \mathcal{U}(X)$ and conclude that $\text{Aut}^\circ(X) = T \times \mathcal{U}(X)$. This proves the implication $(1) \Rightarrow (3)$. The implications $(3) \Rightarrow (2) \Rightarrow (1)$ are obvious. \qed

Corollary 4.3. Let $X$ be an affine algebraic variety without $\mathbb{G}_a$-actions such that $\text{Aut}^\circ(X)$ consists of algebraic elements. Then $\text{Aut}^\circ(X)$ is an algebraic torus of dimension at most $\dim X$.

Proof. In this case $\text{Lie Aut}^\circ(X) = t$, so $\text{Aut}^\circ(X)$ is finite-dimensional. Then $\text{Aut}^\circ(X)$ is a connected algebraic group and is defined by $\text{Lie T}$. \qed

Remark 4.4. If $X$ does not admit $\mathbb{G}_a$- and $\mathbb{G}_m$-actions, then Theorem 1.1 can be obtained from [1, Proposition 3.6]. Indeed, in this case all elements of $\text{Aut}^\circ(X)$ are of finite order. Hence, for some $n \in \mathbb{N}$ the subset of elements of order at most $n$, which is a closed, contains the identity as a limit point.
5. THE AUTOMORPHISM GROUP OF A RIGID VARIETY

In this section we do not assume that $\text{Aut}^o(X)$ consists of algebraic elements. Assume that an affine variety $X$ is rigid, i.e., admits no $\mathbb{G}_a$-actions. By [5, Main Theorem], all $\mathbb{G}_m$-actions on $X$ commute, hence $\text{Aut}_{\text{alg}}(X) = T$ is an algebraic torus.

**Proposition 5.1.** Each element of $\text{Aut}^o(X)$ commutes with $T$.

*Proof.* The torus $T$ is a normal closed subgroup in $\text{Aut}^o(X)$. Consider the action of $\text{Aut}^o(X)$ on $T$ by conjugations. Since the group of automorphisms of the algebraic torus $T$ of dimension $n$ seen as an algebraic group is isomorphic to $\text{GL}(n, \mathbb{Z})$, we obtain the homomorphism $\text{Aut}^o(X) \to \text{GL}(n, \mathbb{Z})$. Since $\text{GL}(n, \mathbb{Z})$ is discrete, the image of $\text{Aut}^o(X)$ is trivial. The assertion follows. □

**Corollary 5.2.** Each element of $\text{Aut}^o(X)$ is contained in an abelian group $A \times T$, where $A$ is a cyclic group.

**Remark 5.3.** Any maximal abstract abelian subgroup $G$ of $\text{Aut}^o(X)$ is an at most countable extension of $T$. Indeed, $G$ coincides with its centralizer, hence is a closed ind-subgroup ([8, Lemma 2.4]). Further, $G$ contains $T$, and by [3, Theorem B] the connected component $G^o$ is algebraic. So, $G^o = T$.

In particular, the only maximal connected abelian ind-subgroup of $\text{Aut}^o(X)$ is $T$.

**Question 5.4.** Given a rigid affine variety $X$, what can we say about the subset of algebraic elements of $\text{Aut}^o(X)$?

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