Form factors of the spin-1 analogue of the eight-vertex model

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The twenty-one-vertex model, the spin-1 analogue of the eight-vertex model, is considered on the basis of free-field representations of vertex operators in the $2 \times 2$-fold fusion solid on solid (SOS) model and vertex–face transformation. The tail operators, which translate corner transfer matrices of the twenty-one-vertex model into those of the fusion SOS model, are constructed by using free bosons and fermions for both diagonal and off-diagonal matrix elements with respect to the ground-state sectors. Form factors of any local operators are therefore obtained in terms of multiple integral formulae, in principle. As the simplest example, the two-particle form factor of the spin operator is calculated explicitly.

Subject Index A10

1. Introduction

In this paper, we consider the spin-1 analogue of Baxter’s eight-vertex model [1], on the basis of the vertex-operator approach [2]. The model is often called the twenty-one-vertex model since the $R$ matrix has twenty-one nonzero elements. The eight-vertex model is related to the spin-$\frac{1}{2}$ anisotropic Heisenberg spin chain. Lashkevich and Pugai [3] found that the correlation functions of the eight-vertex model can be obtained by using the free-field realization of the vertex operators in the eight-vertex solid on solid (SOS) model [4], with insertion of the nonlocal operator $\Lambda$, called the “tail operator”. In Ref. [5], Lashkevich obtained integral formulae for form factors of the eight-vertex model.

There has been some research that generalizes the work of Refs. [3,5]. The vertex-operator approach for higher-spin generalization of the eight-vertex model was presented in Ref. [6]. For higher-rank generalization, the integral formulae for correlation functions of Belavin’s $(\mathbb{Z}/n\mathbb{Z})$-symmetric model [7] were presented in Ref. [8], and the form-factor formulae were presented in Ref. [9].

We are interested in the form factors, originally defined as matrix elements of some local operators, in the twenty-one-vertex model. In this paper, we will construct tail operators for both diagonal and off-diagonal matrix elements with respect to the ground-state sectors.

Let us mention the trigonometric-limit cases of the elliptic vertex model. In Ref. [10], spontaneous polarization formulae of the higher-spin analogue of the six-vertex model, the trigonometric limit of the eight-vertex model, were obtained by using Bethe ansatz method. Idzumi [11,12] reproduced these formulae for the spin-1 case in terms of the vertex-operator formalism. In the critical limit,
the spin-$^k_2$ (isotropic) Heisenberg spin chain is described by the level-$k$ Wess–Zumino–Witten model [13], whose central charge is given by $c = \frac{3k}{k+2}$. Since $c = 1$ for the spin-$^1_2$ case, the eight-vertex model can be described in terms of one boson. The spin-$1$ analogue of the eight-vertex model (twenty-one-vertex model) can be described in terms of one boson and one fermion, because $c = \frac{3}{2} = 1 + \frac{1}{2}$ for $k = 2$. Actually, Idzumi [11,12] and Bougourzi and Weston [14] constructed level-$2$ irreducible highest-weight representations of the quantum affine Lie algebra $\hat{U}_q(\mathfrak{sl}_2)$ in terms of one boson and one fermion.

Let us turn to the elliptic case. Baxter [15–17] found the vertex–face transformation that relates the eight-vertex model and the SOS model. Boos et al. [18] proposed a conjectural formula for multipoint correlation functions of the $Z$-invariant (inhomogeneous) eight-vertex model. The restricted SOS (RSOS) model was constructed in Ref. [19]. The higher-spin generalization of the RSOS model was introduced in Refs. [20,21] on the basis of the fusion procedure. Kojima et al. [6] constructed a vertex-operator formalism for the higher-spin analogue of the eight-vertex model, by using vertex–face transformation onto a $k \times k$ fusion SOS model.

The present paper was written in a self-contained manner so that Sects. 2 and 3, except Sect. 3.5, are reviews of previous related works. In Sect. 2, we review the basic objects of the twenty-one-vertex model, the corresponding fusion face model [20,21], the vertex–face correspondence of these two models, and the tail operators that translate correlation functions and form factors of the fusion SOS model into those of the twenty-one-vertex model. Some detailed definitions of the models concerned are listed in Appendix A. In Sect. 3, we introduce free-field representations for the $2 \times 2$ fusion SOS model. The type-I and type-II vertex operators, the tail operators, and the corner transfer matrix (CTM) Hamiltonian can be realized in terms of bosons and fermions. Form factors of the twenty-one-vertex model can be obtained by these objects, in principle. Section 4 is devoted to derivation of the integral formulae for the form factors of the twenty-one-vertex model. In order to show the relevance of our present method, we calculate the simplest form factor of the local operator $S^1_z$ in Sect. 4.2. In Sect. 5, we give some concluding remarks. Useful operator product expansion (OPE) formulae and commutation relations for basic operators are given in Appendix B. In Appendix C, the details of derivation are given for the free-field representations of the tail operators off-diagonal with respect to the ground-state sectors.

2. Basic objects

The present section aims to formulate the problem, thereby fixing the notation.

2.1. Theta functions

The Jacobi theta functions with two pseudoperiods $1$ and $\tau$ ($\text{Im} \, \tau > 0$) are defined as follows:

$$\vartheta\left[\begin{array}{c} a \\ b \end{array}\right](u; \tau) := \sum_{m \in \mathbb{Z}} \exp\left\{\pi \sqrt{-1} \left( m + a \right) \left[\left( m + a \right) \tau + 2 (u + b) \right]\right\}, \quad (2.1)$$

for $a, b \in \mathbb{R}$. In what follows, we use the symbols $\theta_1(u; \tau), \ldots, \theta_4(u; \tau)$ when $(a, b) = \left( \frac{1}{2}, -\frac{1}{2} \right), \left( \frac{1}{2}, 0 \right), (0, 0), (0, \frac{1}{2})$ in (2.1), respectively. Let $r > 2$ and $\epsilon > 0$ be fixed, and let

$$h_j^{(r)}(u) := \theta_j \left( \frac{u}{t}; \frac{\pi \sqrt{-1}}{\epsilon t} \right), \quad (j = 1, 2, 3, 4)$$
for \( t > 0 \). We put \( h^{(r)}_j(u) = h_j(u) \) for simplicity. We will use the abbreviations

\[
[u] := x^{2u^2 - u} \Theta_{x^{2r}}(x^{2u}), \quad [u] := x^{2u^2 - u} \Theta_{x^{2r}}(-x^{2u}),
\]

\[
\|u\| := x^{2u^2} \Theta_{x^{2r}}(x^{2u^2 + r}), \quad \|u\| := x^{2u^2} \Theta_{x^{2r}}(-x^{2u^2 + r}),
\]

(2.2)

where

\[
\Theta_q(z) = (z; q)_\infty (q z^{-1}; q)_\infty (q; q)_\infty = \sum_{m \in \mathbb{Z}} q^{m(m-1)/2} (-z)^m,
\]

\[
(z; q_1, \ldots, q_m)_\infty = \prod_{i_1, \ldots, i_m \geq 0} (1 - zq_1^{i_1} \cdots q_m^{i_m}).
\]

Note that

\[
h_1(u) = \sqrt{\frac{4}{\pi}} \exp \left( -\frac{4}{x} \right) [u], \quad h_2(u) = \sqrt{\frac{4}{\pi}} \|u\|,
\]

\[
h_3(u) = \sqrt{\frac{4}{\pi}} \|u\|,
\]

where \( x = e^{-t} \).

In the present paper, we often use the following abbreviations:

\[
[r'] = r - 1, \quad [r''] = r - 2, \quad [u]' = [u] |_{r' \rightarrow r - 1}, \quad [u]'' = [u] |_{r'' \rightarrow r - 2},
\]

and so on.

### 2.2. Spin-1 analogue of the eight-vertex model

The twenty-one-vertex model is constructed from the original spin-1/2 eight-vertex model by the fusion procedure. Let

\[
R(u)v_{\varepsilon_1} \otimes v_{\varepsilon_2} = \sum_{\varepsilon_1', \varepsilon_2'} v_{\varepsilon_1'} \otimes v_{\varepsilon_2'} R(u)^{\varepsilon_1 \varepsilon_2}_{\varepsilon_1' \varepsilon_2'}
\]

be the \( R \) matrix of the eight-vertex model. The nonzero elements of the \( R \) matrix are given as follows:

\[
R(u)^{e e}_{e e} = \frac{1}{\bar{\kappa}(u)} \frac{h_2^{(2r)}(1) h_2^{(2r)}(u)}{h_2^{(2r)}(0) h_2^{(2r)}(1 - u)},
\]

\[
R(u)^{e - e}_{e - e} = -\frac{1}{\bar{\kappa}(u)} \frac{h_2^{(2r)}(1) h_1^{(2r)}(u)}{h_2^{(2r)}(0) h_1^{(2r)}(1 - u)},
\]

\[
R(u)^{-e e}_{e - e} = \frac{1}{\bar{\kappa}(u)} \frac{h_1^{(2r)}(1) h_2^{(2r)}(u)}{h_2^{(2r)}(0) h_1^{(2r)}(1 - u)},
\]

\[
R(u)^{-e - e}_{e e} = -\frac{1}{\bar{\kappa}(u)} \frac{h_1^{(2r)}(1) h_1^{(2r)}(u)}{h_2^{(2r)}(0) h_2^{(2r)}(1 - u)},
\]

(2.4)

where

\[
\bar{\kappa}(u) = \zeta \frac{\rho(z)}{\rho(z^{-1})}, \quad \left( z = \zeta^2 = x^{2u}, x = e^{-t} \right)
\]

\[
\rho(z) = (x^2 z; x^4, x^{2r})_\infty (x^{2r+2}; x^4, x^{2r})_\infty (x^{2r}; x^4, x^{2r})_\infty.
\]

(2.5)

Let

\[
R^{(1, 1)}(u)v_{j_1} \otimes v_{j_2} = \sum_{j_1', j_2' = -1}^{1} v_{j_1'} \otimes v_{j_2'} R^{(1, 1)}(u)^{j_1' j_2'}_{j_1 j_2}
\]

(2.6)
be the $R$ matrix of the twenty-one-vertex model. This $R^{(1,1)}(u)$ can be obtained from $R(u)$ in terms of the fusion procedure. The following property,

$$PR^{(1,1)}(1) = -R^{(1,1)}(1), \quad P(x \otimes y) = y \otimes x,$$

(2.7)

is important in the fusion procedure. The explicit expressions of the matrix elements of the $R$ matrix of the twenty-one-vertex model are given in Appendix A.

We assume that the parameters $u$, $e$, and $r$ in (2.4) and (A2) lie in the so-called principal regime:

$$\epsilon > 0, \quad r > 2, \quad 0 < u < 1.$$

(2.8)

This is the antiferroelectric region of the parameters. The twenty-one-vertex model has three kinds of ground states labeled by $i$ for $i = 0, 1, 2$. Accordingly, there are three spaces of physical states $\mathcal{H}^{(i)} (i = 0, 1, 2)$. Here, the space $\mathcal{H}^{(i)}$ is the $\mathbb{C}$-vector space spanned by the half-infinite pure tensor vectors of the forms

$$v_{s_1} \otimes v_{s_2} \otimes v_{s_3} \otimes \ldots \quad \text{with} \quad s_j \in \{-1, 0, 1\}, \quad \text{for} \quad j = 1, 2, 3, \ldots$$

(2.9)

and

$$s_j = \begin{cases} 1 - i & (j \equiv 0 \mod 2) \quad \text{for} \quad j \gg 0. \\ i - 1 & (j \equiv 1 \mod 2) \end{cases}$$

(2.10)

Note that $\mathcal{H}^{(i)}$ is isomorphic to the level-2 highest-weight module of the affine Lie algebra $A_1^{(1)}$, with the highest weight

$$\lambda_i := (2 - i) \Lambda_0 + i \Lambda_1 \quad (i = 0, 1, 2),$$

respectively. Here, the $\Lambda_i$ ($i = 0, 1$) denote the fundamental weights of $A_1^{(1)}$.

Let $\mathcal{H}^{*(i)}$ be the dual of $\mathcal{H}^{(i)}$ spanned by the half-infinite pure tensor vectors of the forms

$$\ldots \otimes v_{s_{-2}} \otimes v_{s_{-1}} \otimes v_{s_0} \quad \text{with} \quad s_j \in \{-1, 0, 1\}, \quad \text{for} \quad j = 0, -1, -2, \ldots$$

(2.11)

and

$$s_j = \begin{cases} 1 - i & (j \equiv 0 \mod 2) \quad \text{for} \quad j \ll 0. \\ i - 1 & (j \equiv 1 \mod 2) \end{cases}$$

(2.12)

Let us consider the so-called low-temperature limit $x \to 0$ of (A2) with fixed $\zeta = x^u$. Then the $R^{(1,1)}(u)$ behaves as

$$R^{(1,1)}(u) = x^{s_1 s_2} \sim \zeta^{H(s_1, s_2)} \delta_{s_1} \delta_{s_2} (x \to 0)$$

(2.13)

where

$$H(s, s') = |s + s'| = \begin{cases} 0 & (s, s') = (\pm 1, \mp 1) \quad (0, 0)) \\
1 & (s, s') = (\pm 1, 0) \quad (0, \pm 1) \\
2 & (s, s') = (\pm 1, \pm 1) \end{cases}$$

(2.14)

Thus, the southeast corner transfer matrix behaves as

$$A_{SEC}^{(i)}(u) = x^{s_1 s_2 \ldots} \sim \zeta^{H_{CTM}^{(i)}} \delta_{s_1} \delta_{s_2} \ldots : \mathcal{H}^{(i)} \to \mathcal{H}^{(i)}$$

(2.15)

in the low-temperature limit $x \to 0$, where

$$H_{CTM}^{(i)} (s_1, s_2, \ldots) = \sum_{j=1}^{\infty} j H(s_j, s_{j+1}).$$

(2.16)
We assume that (2.15) is valid not only for the low-temperature limit \( x \to 0 \) but also for finite \( 0 < x < 1 \).\(^1\) Likewise, the other three types of corner transfer matrices are introduced as follows:

\[
A_{\text{NE}}^{(i)}(u) : \quad \mathcal{H}^{(i)} \to \mathcal{H}^{(*)_{i}},
\]

\[
A_{\text{NW}}^{(i)}(u) : \quad \mathcal{H}^{(*)_{i}} \to \mathcal{H}^{(*)_{i}},
\]

\[
A_{\text{SW}}^{(i)}(u) : \quad \mathcal{H}^{(*)_{i}} \to \mathcal{H}^{(i)},
\]

where NE, NW, and SW stand for the northeast, northwest, and southwest corners. It seems to be rather general \([1]\) that the product of four CTMs in the infinite lattice limit is independent of \( u \):

\[
\rho^{(i)} = A_{\text{SE}}^{(i)}(u)A_{\text{SW}}^{(i)}(u)A_{\text{NW}}^{(i)}(u)A_{\text{NE}}^{(i)}(u) = x^{2H_{\text{CTM}}}.
\]

The trace of \( \rho^{(i)} \) coincides with the principally specialized character of \( \lambda \), up to some factors \([22]\):

\[
\chi^{(i)} := \text{tr}_{\mathcal{H}^{(i)}} \left( x^{2H_{\text{CTM}}} \right) = x^{j} \chi_{\lambda_{j}}(x) = \begin{cases} (-x^{2}; x^{2})_{\infty} (-x^{4}; x^{4})_{\infty} & (i = 0, 2) \\ (-x^{2}; x^{2})_{\infty} (-x^{4}; x^{4})_{\infty} & (i = 1) \end{cases}.
\]

We introduce the type-I vertex operator by the following half-infinite transfer matrix:

\[
\Phi^{j}(u_{1} - u_{2}) = \begin{array}{|c|c|c|c|c|}
\hline
u_{1} & j & \vdots \\
\hline
u_{2} & u_{2} & u_{2} & u_{2} \\
\hline
\end{array}
\]

Then the operator (2.20) is an intertwiner from \( \mathcal{H}^{(i)} \) to \( \mathcal{H}^{(2-i)} \). The type-I vertex operators satisfy the following commutation relation:

\[
\Phi^{j_{1}}(u_{1}) \Phi^{j_{2}}(u_{2}) = \sum_{j_{1}', j_{2}'} R^{(1,1)}(u_{1} - u_{2})^{j_{1}j_{2}}_{j_{1}'j_{2}'} \Phi^{j_{2}'}(u_{2}) \Phi^{j_{1}'}(u_{1}).
\]

When we consider an operator related to the “creation–annihilation” process, we need another type of vertex operators, the type-II vertex operators that satisfy the following commutation relations:

\[
\Psi^{*_j_{2}}(u_{2}) \Psi^{*_j_{1}}(u_{1}) = \sum_{j_{1}', j_{2}'} \Psi^{*_j_{2}}_{j_{1}'}(u_{1}) \Psi^{*_j_{1}}_{j_{2}'}(u_{2}) S^{(1,1)}(u_{1} - u_{2})^{j_{1}j_{2}}_{j_{1}'j_{2}'},
\]

\[
\Phi^{j_{1}}(u_{1}) \Psi^{*_j_{2}}(u_{2}) = -\Psi^{*_j_{2}}(u_{2}) \Phi^{j_{1}}(u_{1}).
\]

where

\[
S^{(1,1)}(u) = R^{(1,1)}(u)|_{r \to -r - 2}.
\]

Furthermore, the type-I vertex operator \( \Phi^{j}(u) \), and the type-II vertex operators \( \Psi^{*_j}(v) \) and \( \rho^{(i)} \) introduced in (2.18) satisfy the homogeneity relations

\[
\Phi^{j(2-i,i)}(u)\rho^{(i)} = \rho^{(2-i)}\Phi^{j(2-i,i)}(u - 2), \quad \Psi^{*_j(2-i,i)}(u)\rho^{(i)} = \rho^{(2-i)}\Psi^{*_j(2-i,i)}(u - 2).
\]

\(^1\) Note that the \( u \)-dependence of \( R^{(1,1)}(u) \) is actually \( \xi \)-dependence, where \( \xi = x^{u} \). Since the eigenvalues \( \lambda_{\xi} \) of \( A^{(i)}_{\text{SE}}(u) \) should be invariant under the shift \( u \mapsto u + 2\pi \sqrt{-1} \log x \), we have \( \lambda_{\xi} = \xi^{n_{r}}(n_{r} \in \mathbb{Z}) \). Owing to the discreteness property of eigenvalues, (2.15) should be valid even for finite \( 0 < x < 1 \), in the sense of a similarity transformation.
and the normalization conditions
\[
\sum_{j=-1}^{1} \Phi_j^{(2,i-1)}(u) \Phi_j^{(2,i)}(u) = 1, \quad \Psi_j^{(2,i-1)}(u') \Psi_j^{(2,i)}(u) = \frac{\delta_j^i}{1 - x^{-2}z/z'} + O(1).
\]

Here, \( z = x^{2u} \), \( z' = x^{2u'} \) and
\[
\lambda \Phi_j^{(2,i)}(u) = \Phi_j^{(2,i)}(u - 1), \quad \lambda \Psi_j^{(2,i)}(u) = \Psi_j^{(2,i)}(u - 1),
\]
and \( \lambda \) and \( \lambda^* \) are appropriate constants defined later.

### 2.3. 2 \times 2 fusion SOS model

The SOS model was introduced by Baxter [15–17] in order to solve the eight-vertex model. The state variables of the SOS model take integer values. A pair \((a, b)\) is admissible if \( b = a \pm 1 \). Let \((a, b)\) be the state variables at adjacent sites. Then the pair \((a, b)\) is admissible. For \((a, b, c, d)\), let \( W[\begin{array}{cc}a & d \\ b & c \end{array}] \) be the Boltzmann weight of the SOS model for the state configuration \([a, b, c, d]\) round a face. Here, the four states \(a, b, c, d\) are ordered clockwise from the SE corner. In this model, \( W[\begin{array}{cc}a & d \\ b & c \end{array}] = 0\), unless the four pairs \((a, b), (a, d), (b, c), (d, c)\) are admissible. Nonzero Boltzmann weights are given as follows:

\[
W \begin{bmatrix} k+2 & k+1 \\ k+1 & k \end{bmatrix} = \frac{1}{\kappa(u)},
W \begin{bmatrix} k & k+1 \\ k+1 & k \end{bmatrix} = \frac{1}{\kappa(u)} [1][k+u],
W \begin{bmatrix} k & k+1 \\ k+1 & k \end{bmatrix} = -\frac{1}{\kappa(u)} [u][k+1],
\]

The twenty-one-vertex model can be transformed into a \(2 \times 2\) fusion SOS model in terms of the vertex–face correspondence. Let \((a, b)\) be the state variables of the \(2 \times 2\) fusion SOS model at adjacent sites. Then \(b = a \pm 2\), or \( b = a \). In what follows, we denote \( b \sim a \) when \( b - a \in \{-2, 0, 2\}\). Nonzero Boltzmann weights \( W_{22}(u) \) are given in Appendix A.

Here we again assume that the parameters \(u, \epsilon,\) and \(r\) in (2.28) and (A3) lie in (2.8). This region of the parameters is called regime III in the SOS-type model. For \( k, l \in \mathbb{Z} \) and \( i = 0, 1, 2 \), let \( \mathcal{H}_{i,k}^{(l)} \) be the space of admissible paths \((k_0, k_1, k_2, \ldots)\) such that

\[
k_0 = k, \quad k_{j+1} \sim k_j \quad \text{for} \quad j = 0, 1, 2, \ldots ,
\]

and

\[
k_j = \begin{cases} l + i & (j \equiv 0 \mod 2) \\ l + 2 - i & (j \equiv 1 \mod 2) \end{cases} \quad \text{for} \quad j \gg 0.
\]

Also, let \( \mathcal{H}_{i,k}^{(l)} \) be the space of admissible paths \((\ldots, k_{-2}, k_{-1}, k_0)\) such that

\[
k_0 = k, \quad k_{j-1} \sim k_j \quad \text{for} \quad j = 0, -1, -2, \ldots ,
\]

and

\[
k_j = \begin{cases} l + i & (j \equiv 0 \mod 2) \\ l + 2 - i & (j \equiv 1 \mod 2) \end{cases} \quad \text{for} \quad j \ll 0.
\]
After gauge transformation \[20,21\], the Boltzmann weights \( W_{22}(u) \) in the so-called low-temperature limit \( x \to 0 \) behave as

\[
W_{22} \begin{bmatrix} a & b \\ c & d \end{bmatrix} u \sim \delta_{bd} \zeta^{1/2} |c-a|.
\]

(2.33)

Here we take the limit \( x \to 0 \) with \( \zeta = x^{i} \) fixed. Let \( A_{l,k}^{(i)}, B_{l,k}^{(i)}, C_{l,k}^{(i)}, \) and \( D_{l,k}^{(i)} \) be the SE, SW, NW, and NE corner transfer matrices. Then the SE corner transfer matrix behaves as follows:

\[
A_{l,k}^{(i)}(u) \gets \zeta_{l,k}^{(i)} \rightarrow \mathcal{H}_{l,k}^{(i)},
\]

in the low-temperature limit \( x \to 0 \), where

\[
\mathcal{H}_{l,k}^{(i)}(k_0, k_1, k_2, \ldots) = \frac{1}{2} \sum_{j=1}^{\infty} \left| k_{j+1} - k_{j-1} \right|.
\]

Likewise, the other three types of corner transfer matrices are introduced as follows:

\[
B_{l,k}^{(i)}(u) : \mathcal{H}^{(i)} \rightarrow \mathcal{H}^{(i)},
\]

\[
C_{l,k}^{(i)}(u) : \mathcal{H}^{(i)} \rightarrow \mathcal{H}^{(i)},
\]

\[
D_{l,k}^{(i)}(u) : \mathcal{H}^{(i)} \rightarrow \mathcal{H}^{(i)}.
\]

(2.36)

It seems to be rather general \[1\] that the product of four CTMs in the infinite lattice limit is independent of \( u \):

\[
\rho_{l,k}^{(i)} = A_{l,k}^{(i)}(u)B_{l,k}^{(i)}(u)C_{l,k}^{(i)}(u)D_{l,k}^{(i)}(u) = [k] x^{2H_{l,k}^{(i)}}.
\]

(2.37)

Let \( k = l + i + 2j \) (mod 4), where \( i = 0, 1, 2 \) and \( j = 0, 1 \). Then the trace of \( \rho_{l,k}^{(i)} \) can be given as follows \[23\]:

\[
\lambda_{l,k}^{(i)} := \text{tr}_{\mathcal{H}_{l,k}^{(i)}} \left( \rho_{l,k}^{(i)} \right) = [k] x^{2 H_{l,k}^{(i)}} x^{\frac{r}{2} 2^{i} + \frac{r''}{2} k^2}.
\]

(2.38)

Here \( c_{\lambda,j}^{(i)}(x^4) \) is the string function \[24\], up to some factors. We change the definitions for the present purpose as follows:

\[
c_{\lambda,0}^{(i)} (x^4) = c_{\lambda,2}^{(i)} (x^4) = \frac{(-x^2; x^4)_{\infty}}{(x^4; x^4)_{\infty}},
\]

\[
c_{\lambda,1}^{(i)} (x^4) = c_{\lambda,3}^{(i)} (x^4) = x^{\frac{1}{2} (-x^4; x^4)_{\infty}},
\]

\[
c_{\lambda,j}^{(i)} (x^4) = 0 \quad (\text{for } j \neq i \text{ mod } 2).
\]

Besides (2.39), we have the following symmetry:

\[
c_{\lambda,j}^{(i)} (x^4) = c_{\lambda,j+4}^{(i)} (x^4) = c_{\lambda,2-j}^{(i-1)} (x^4).
\]

(2.40)
From (2.38), (2.39), (2.40), and (2.19), we obtain the following relation [6]:

$$
\sum_{k \in l+i+2\mathbb{Z}} \chi_{l,k}^{(i)} = [l]'' \chi^{(i)}.
$$

(2.41)

We introduce the type-I vertex operator by the following half-infinite transfer matrix:

$$
\Phi(u_1 - u_2)^{c'}_k = u_1^{c'}\Phi_1^k u_2^{c} u_2^{c} u_1^{c}
$$

(2.42)

Then the operator (2.42) is an intertwiner from $\mathcal{H}_{l,k}^{(i)}$ to $\mathcal{H}_{l,k'}^{(2-i)}$. The type-I vertex operators satisfy the following commutation relation:

$$
\Phi(u_1)^c \Psi(u_2) = \sum_d W_{22} \left[ \begin{array}{cc} c & d \\ b & a \end{array} \right] u_1 - u_2 \Phi(u_2)^c \Phi(u_1)^d.
$$

(2.43)

The free-field realization of $\Phi(u_2)^b$ was constructed in Ref. [6]; see Sect. 3.2.

The type-II vertex operators should satisfy the following commutation relations:

$$
\Psi^* (u_2)^d \Phi(u_1)^a = \sum_b \Psi^* (u_1)^c \Psi^* (u_2)^b W''_{22} \left[ \begin{array}{cc} c & d \\ b & a \end{array} \right] u_1 - u_2, 
$$

(2.44)

$$
\Phi(u_1)^c \Psi^* (u_2)^d = -\Psi^* (u_2)^d \Phi(u_1)^c,
$$

(2.45)

where

$$
W''_{22} \left[ \begin{array}{cc} c & d \\ b & a \end{array} \right] v = \left. W_{22} \left[ \begin{array}{cc} c & d \\ b & a \end{array} \right] v \right|_{r \to r-2}.
$$

(2.46)

Furthermore, the type-I vertex operator $\Phi(u_2)^{c'}$, and the type-II vertex operators $\Psi^*(u)^d$ and $\rho^{(i)}_{l,k}$ introduced in (2.37) satisfy the homogeneity relations

$$
\Phi^{(2-i,i)} (u_2)^{c'} \rho^{(i)}_{l,k} [k] = \frac{\rho^{(2-i)}_{l,k'}}{[k']} \Phi^{(2-i,i)} (u - 2)^{c'}_k, \quad \Psi^* (2-i,i)(u)^d \rho^{(i)}_{l,k} = \frac{\rho^{(2-i)}_{l,k'}}{[k']} \Psi^* (2-i,i) (u - 2)^d,
$$

(2.47)

and the normalization conditions

$$
\sum_{k' \sim k} \Phi^*(i,2-i)(u)^{c'}_k \Phi(2-i,i)(u)^k = 1, \quad \Psi^*(i,2-i)(u)^d \Psi^* (2-i,i)(u)^d = \frac{\delta^d_i}{1 - x^{-2z/z'}} + O(1).
$$

(2.48)

Here, $z = x^{2u}$, $z' = x^{2u'}$, and the dual vertex operators $\Phi^*(u)^c$ and $\Psi^*(u)^d$ will be defined in Sect. 3.
Fig. 1. Pictorial representation of vertex–face correspondence.

\[ \sum_{\varepsilon = \pm} \frac{k}{k'} = \delta_{k'}^{k}, \quad \sum_{k' = k + 1} \frac{c'}{u} k' = \delta_{c'}^{c} \]

Fig. 2. Pictorial representation of the dual intertwining vectors.

2.4. Vertex–face correspondence

Baxter [15–17] introduced the intertwining vectors that relate the eight-vertex model and the SOS model. Let

\[ \tau(u)_{k + 1}^{k} = \sum_{\varepsilon = \pm} v_{\varepsilon} \tau^{\varepsilon}(u)_{k + 1}^{k} = \frac{f(u)}{\sqrt{2}} \left[ h_{3}^{(2r)}(k \mp u) \right], \]

where the scalar function \( f(u) \) satisfies

\[ h_{1}(u) f(u) f(u - 1) = 1. \]

Then the following relation holds (cf. Fig. 1):

\[ R(u_{1} - u_{2}) \tau(u_{1})_{a}^{d} \otimes \tau(u_{2})_{a}^{d} = \sum_{b} \tau(u_{1})_{b}^{c} \otimes \tau(u_{2})_{b}^{c} W \left[ \begin{array}{c|c} c & d \\ b & a \end{array} \right] \left[ u_{1} - u_{2} \right]. \] (2.50)

Note that the present intertwining vectors are different from the ones used in Refs. [15–17], which relate the \( R \) matrix of the eight-vertex model in the disordered phase and the Boltzmann weights \( W \) of the SOS model in regime III.

Let us introduce the dual intertwining vectors (see Fig. 2) satisfying

\[ \sum_{\varepsilon = \pm} \frac{k}{k'} = \delta_{k'}^{k}, \quad \sum_{k' = k + 1} \frac{c'}{u} k' = \delta_{c'}^{c}. \] (2.51)

From (2.50) and (2.51), we have (cf. Fig. 3)

\[ \tau^{*}(u_{1})_{c}^{b} \otimes \tau^{*}(u_{2})_{b}^{d} R(u_{1} - u_{2}) = \sum_{d} W \left[ \begin{array}{c|c} c & d \\ b & a \end{array} \right] \left[ u_{1} - u_{2} \right] \tau^{*}(u_{1})_{b}^{c} \otimes \tau^{*}(u_{2})_{c}^{d}. \] (2.52)

Intertwining vectors that relate the twenty-one-vertex model and the \( 2 \times 2 \) fusion SOS model can be constructed by the fusion procedure. In what follows, let us denote these vectors for the fusion models by \( t(u)_{k}^{k'} \) and \( t^{*}(u)_{k}^{k'} \). The explicit expressions of these fused intertwining vectors are given in Appendix A.
Fig. 3. Vertex–face correspondence by dual intertwining vectors.

Let

$$t''(u)_i^j := t^*(u; \epsilon, r - 2)_i^j.$$  \hfill (2.53)

Then we have

$$R^{(1,1)} (u_1 - u_2) t (u_1)_{a}^{d} \otimes t (u_2)_{b}^{c} = \sum_b t (u_1)_b^{c} \otimes t (u_2)_a^{b} W_{22} \left[ \begin{array}{cc} c & d \\ a & b \end{array} \right] u_1 - u_2$$  \hfill (2.54)

and

$$t''(u)_c^{b} \otimes t''(u)_a^{d} S^{(1,1)} (u_1 - u_2) = \sum_d W''_{22} \left[ \begin{array}{cc} c & d \\ a & b \end{array} \right] t''(u)_d^{a} \otimes t''(u)_c^{b}. \hfill (2.55)$$

Let us introduce the following symbol:

$$L \left[ \begin{array}{cc} a'_0 & a'_1 \\ a_0 & a_1 \end{array} | u_0 \right] := \sum_{j=-1}^{\infty} t^*_j (-u_0)^{a_0}_a t^j (-u_0)^{a'_0}_{a'_1}. \hfill (2.56)$$

Then, from (A7),

$$L \left[ \begin{array}{cc} a_0 & a'_1 \\ a_0 & a_1 \end{array} | u_0 \right] = \delta_0^{a'_1}. \hfill (2.57)$$

The explicit expressions of $L$ are given in Appendix A.

Assume that $0 < \Re(u_0) < 2$. Then it follows from (A4) and (A6) that, for $i = 0, 1, 2$,

$$\left| t^*_i (-u_0)^{l+2-i}_{l+i} \right| \sim 1 \quad (x \to 0) \hfill (2.58)$$

is much greater than other products $t^*_j (-u_0)^{l+2-i}_{j+i} (-u_0)^{l+2-i}_{j+i} (j \neq i)$ in the low-temperature limit. Thus, the boundary condition $\mathcal{H}^{(i)}$ of the twenty-one-vertex model (2.10) corresponds to that of $\mathcal{H}_{l,k}^{(i)}$ of the $2 \times 2$ fusion SOS model (2.30).

2.5. Tail operators and commutation relations

Tail operators were originally introduced in Refs. [3,5], in order to translate the correlation functions of the eight-vertex model into those of the SOS model. Tail operators for the higher-spin case were constructed in Ref. [6], and those for the higher-rank case were constructed in Refs. [8,9].

Let us introduce the intertwining operators between $\mathcal{H}^{(i)}$ and $\mathcal{H}_{l,k}^{(i)}$:

$$T (u_0)^{jk} = \prod_{j=1}^{\infty} t^{x_j} (-u_0)^{k_{j-1}}_{k_j} : \mathcal{H}^{(i)} \rightarrow \mathcal{H}_{l,k}^{(i)}; \hfill (2.59)$$

$$T (u_0)^{lk} = \prod_{j=1}^{\infty} t^{x_j} (-u_0)^{k_{j-1}}_{k_j} : \mathcal{H}_{l,k}^{(i)} \rightarrow \mathcal{H}^{(i)}. \hfill (2.59)$$
From (A5) and (A8), the following intertwining relations hold:

\[
T(u_0)^{j^k} \Phi^j(u) = \sum_k t^j (u - u_0)^{k'}_k \Phi(u)^{k'}_k T(u)^{lk},
\]

(2.60)

\[
T(u_0)^{l^k} \Phi(u)^{k'}_k = \sum_{j=1}^1 t^*_j (u - u_0)^{k'}_k \Phi(j) T(u_0)^{lk}.
\]

(2.61)

The tail operator is defined by the product of these two objects (see Fig. 4):

\[
\Lambda(u_0)^{l^k'}_l = T(u_0)^{l^k} T(u_0)^{lk} : \mathcal{H}^{(i)}_l \rightarrow \mathcal{H}^{(i)}_{l^k}'.
\]

(2.62)

From (2.60), (2.61), and (2.62), we have

\[
\Lambda(u_0)^{l^k'}_l \Phi(u)^{k'}_k = \sum_{d=c} L \left[ \begin{array}{cc} c & d \\ b & a \end{array} \right] u_0 - u \Phi(u)^{k'}_k \Lambda(u_0)^{l^d}_l.
\]

(2.63)

Furthermore, consider the algebra

\[
\Psi^*_j(u) T(u_0)^{l^k} = \sum_{l' \sim l} T(u_0)^{l'k} \Psi^*(u)^{l'j} (u - u_0 - \Delta u_0)^{l'}_l,
\]

(2.64)

\[
\Psi^*(u)^{l'j} T(u_0)^{lk} = \sum_k T(u)^{lk} \Psi^*(u)^{l'j} (u - u_0 - \Delta u_0)^{l'}_l.
\]

(2.65)

This algebra is consistent with (2.60)–(2.61) for any value of \(\Delta u_0\). The value of \(\Delta u_0\) will be fixed in the next section. From (2.64), (2.65), and (2.62), we have

\[
\Psi^*(u)^{l''l'} \Lambda(u_0)^{l^k'}_l = \sum_{l_1} L'' \left[ \begin{array}{cc} l'' & l' \\ l_1 & l \end{array} \right] u_0 + \Delta u - u \Lambda(u_0)^{l''k'}_l \Psi^*(u)^{l'_l}.
\]

(2.66)

In what follows, we suppress \(l\)-dependence to denote \(\Lambda(u_0)^{l^k'}_l\) by \(\Lambda(u_0)^{k'}_k\). From (2.59), (2.62), and (2.66), we have

\[
\Lambda(u_0)^{l^k'}_l = \prod_{j=0}^\infty L \left[ \begin{array}{cc} k'_{j+1} & k'_{j+1} \\ k_j & k_j \end{array} \right] u_0.
\]

(2.67)

It is obvious from (2.57) that we have

\[
\Lambda(u_0)^{l^k}_l = \delta^l_l.
\]

(2.68)

The relation (2.41) implies that

\[
\text{tr}_{\mathcal{H}^{(i)}} \left( \rho \right)^{(i)} = \frac{1}{[l]''} \sum_{k\ell+i+2\mathbb{Z}} \text{tr}_{\mathcal{H}_l^{(i)}} \left( \rho_{l,k}^{(i)} \right).
\]

(2.69)
Insert unity (2.68) into the RHS of (2.69). Then we have

\[
\text{tr}_{\omega(i)} \left( \rho^{(i)} \right) = \sum_{k \in \mathbb{L}^+ + l + 2} \text{tr}_{\omega(i)} \left( \frac{\rho_{i,k}^{(i)}}{[l]''} T(u_0)^l T(u)_l \right)
\]

\[
= \sum_{k \in \mathbb{L}^+ + l + 2} \text{tr}_{\omega(i)} \left( T(u)_l \frac{\rho_{i,k}^{(i)}}{[l]''} T(u_0)^l \right). \tag{2.70}
\]

Thus, in what follows, we assume that

\[
\rho^{(i)} = \sum_{k \in \mathbb{L}^+ + l + 2} T(u)_l \frac{\rho_{i,k}^{(i)}}{[l]''} T(u_0)^l. \tag{2.71}
\]

3. Free-field realization

One of the most standard ways to calculate correlation functions and form factors is the vertex-operator approach [2] on the basis of free-field representation. The face-type elliptic quantum group \( B_{q,\lambda}(\widehat{sl}_2) \) was introduced in Ref. [25]. The elliptic algebra \( U_{q,\rho}(\widehat{sl}_2) \) associated with fusion SOS models was defined in Ref. [26], and its free-field representations were constructed in Refs. [26,27]. Using these representations, we derive the free-field representation of the tail operator in this section.

3.1. Bosons and fermions

Let us consider the bosons \( \beta_m (m \in \mathbb{Z} \setminus \{0\}) \) with the commutation relations

\[
[\beta_m, \beta_{m'}] = m \frac{[r^m_m]}{[r_m]} \delta_{m + m', 0}. \tag{3.1}
\]

Here the symbol \([a]_x \) stands for \((x^a - x^{-a})/(x - x^{-1})\). The relation between the present \( \beta_m \) and the bosons \( a_m \) in Ref. [6] is as follows:

\[
\beta_m = \begin{cases} 
\frac{m[r^m_m]}{[2m]_x} a_m & (m > 0) \\
\frac{m_l - 2m}{[2m]_x} a_m & (m < 0)
\end{cases} \tag{3.2}
\]

We will deal with the bosonic Fock spaces \( \mathcal{F}^{(i)}_{l,k} (l, k \in \mathbb{Z}) \) generated by \( \beta_{-m} (m > 0) \) and \( e^\alpha, e^\beta \) over the vacuum vectors \( |l, k\rangle \):

\[
\mathcal{F}^{(i)}_{l,k} = \mathbb{C} [\beta_{-1}, \beta_{-2}, \ldots] \otimes \left( \oplus_{n \in \mathbb{Z}} \mathbb{C} e^{\lambda_i + n \alpha + m \beta} \right) |l, k\rangle,
\]

where

\[
\beta_m |l, k\rangle = 0 (m > 0),
\]

\[
e^{\pm \alpha} |l, k\rangle = |l, k \pm 2\rangle,
\]

\[
e^{\pm \beta} |l, k\rangle = |l \pm 2, k\rangle.
\]

Let \( K \) and \( L \) be the operators that act diagonally on \( \mathcal{F}^{(i)}_{l,k} \):

\[
K |l, k\rangle = k |l, k\rangle, \quad L |l, k\rangle = l |l, k\rangle.
\]

Furthermore, let us consider the fermions

\[
\phi (w) = \sum_m \phi_m w^{-m} \tag{3.3}
\]
with the anticommutation relations
\[
\{\phi_m, \phi_{m'}\} = \delta_{m,-m'}x^{2m} + x^{-2m}x + 1.
\] (3.4)

We refer to \(\phi_m\) for \(m \in \mathbb{Z} + \frac{1}{2}\) as Neveu–Schwarz fermions, and \(\phi_m\) for \(m \in \mathbb{Z}\) as Ramond fermions. Let
\[
\mathcal{F} = \begin{cases} \mathbb{C}[\phi_{-\frac{1}{2}}, \phi_{\frac{1}{2}}, \ldots] & (\text{for } i = 0, 2) \\ \mathbb{C}[\phi_{-1}, \phi_{-2}, \ldots] & (\text{for } i = 1) \end{cases}
\]
be the fermionic Fock space.

Note that the following anticommutation relation holds:
\[
\{\phi(w_1), \phi(w_2)\} = \frac{1}{x + x^{-1}} \left( \delta \left( \frac{x^2 w_2}{w_1} \right) + \delta \left( \frac{x^2 w_1}{w_2} \right) \right).
\] (3.5)

Here we use \(\phi_0^2 = 1/(x + x^{-1})\) for the Ramond fermion sector.

The total space of states \(\mathcal{H}_{i,k}^{(i)}\) is isomorphic to
\[
\mathcal{H}_{i,k}^{(i)} = \mathcal{F}_{i,k}^{(i)} \otimes \mathcal{F}.
\] (3.6)

### 3.2. Free-field realization of type-I vertex operators

Let us introduce the following basic operators:
\[
\Phi_1(u) = z^\nu : \exp \left( - \sum_{m \neq 0} \frac{\beta_m}{m} z^{-m} \right) : e^{\alpha_L}L + \frac{\nu}{2}K,
\]
\[
A(v) = w^\nu : \exp \left( \sum_{m \neq 0} \frac{\beta_m}{m} w^{-m} \right) : e^{-\alpha_L}w^{-\frac{1}{2}}L - \frac{\nu}{2}K \phi(w),
\] (3.7)

where \(z = x^{2u}\), \(w = x^{2v}\). For some useful OPE formulae and commutation relations, see Appendix B.

Then the type-I vertex operators (half-transfer matrices) on \(\mathcal{H}_{i,k}^{(i)}\) can be realized in terms of bosons and fermions:
\[
\Phi(u)^{k+1} = \frac{[1]}{[k][k+1]} \Phi_1(u),
\]
\[
\Phi(u)^{k} = \frac{[2]}{[k-1][k+1]} \Phi_1(u) X(u),
\]
\[
\Phi(u)^{k-2} = \frac{[1]}{[k][k-1]} \Phi_1(u) X(u)^2,
\] (3.8)

where
\[
X(u) = \oint_C \frac{dw}{2\pi \sqrt{-1}w} A(v) \frac{[v-u-K]}{[v-u-1]}.
\] (3.9)

Considering the denominators \([v-u-1]\) together with the OPE formulae (B2), the expressions (3.8) have poles at \(w = x^{\pm(2+2nr)}z (n \in \mathbb{Z}_{\geq 0})\). The integral contour \(C\) for \(w\)-integration is an anticlockwise circle such that all integral variables lie in the common convergence domain; i.e., the contour \(C\) encircles the poles at \(w = x^{2+2nr}z (n \in \mathbb{Z}_{\geq 0})\), but not the poles at \(w = x^{-2-2nr}z (n \in \mathbb{Z}_{\geq 0})\).
Let
\[ Y(u) = -\oint_C \frac{dw}{2\pi\sqrt{-1}w} A(v) \frac{[v - u + 2 - K]}{[v - u + 1]} . \] (3.10)

Then we can rewrite (3.8) as follows:
\[ \Phi(u)^{k+2} = \frac{[1]}{[k][k+1]} \Phi_1(u), \]
\[ \Phi(u)^k = \frac{[2]}{[k-1][k+1]} Y(u) \Phi_1(u), \] (3.11)
\[ \Phi(u)^{k-2} = \frac{[1]}{[k][k-1]} Y(u)^2 \Phi_1(u). \]

Note that
\[ \Phi(u)^{k'} : \mathcal{H}^{(i)}_{l,k} \longrightarrow \mathcal{H}^{(2-i)}_{l,k'} . \] (3.12)

These type-I vertex operators satisfy the following commutation relations on \( \mathcal{H}^{(i)}_{l,k} \):
\[ \Phi(\Phi_1)^{a} \Phi(\Phi_1)^{b} = \sum_d W_{22} \begin{bmatrix} c & d \\ b & a \end{bmatrix} \Phi(\Phi_1)^{c} \Phi(\Phi_1)^{d} . \] (3.13)

Dual vertex operators are likewise defined as follows:
\[ \Phi^*(u)^{k} = \frac{i}{\lambda} \Phi_1 (u - 1), \]
\[ \Phi^*(u)^{k+2} = \frac{i}{\lambda} \Phi_1 (u - 1) X (u - 1), \]
\[ \Phi^*(u)^{k-2} = \frac{i}{\lambda} \Phi_1 (u - 1) X (u - 1)^2. \] (3.14)

Here the normalization factor can be determined as
\[ \lambda = \left( \frac{x^{2r}; x^{2r}}{x + x^{-1}} \right)^2 \left( \frac{x^{2r}; x^{2r}}{x^{-1}} \right)^3 \left( \frac{x^{2r}; x^{2r}}{x} \right)^{-3}, \]
such that \( \Phi(u)^{k} \) and \( \Phi^*(u)^{k'} \) satisfy the inversion relation:
\[ \sum_{k'} \Phi^*(u)^{k'} \Phi(u)^{k} = 1. \] (3.15)

As explained below (3.9), the integral contour \( C = C_u \) actually depends on \( u \). In Eqs. (3.14), the \( w \)-integration contour \( C_{u-1} \) of \( X(u-1) \) encircles the poles at \( w = x^{2n}z (n \in \mathbb{Z}_{\geq 0}) \), but not the poles at \( w = x^{-4-2n}z (n \in \mathbb{Z}_{\geq 0}) \). Note that
\[ \Phi^*(u)^{k'} : \mathcal{H}^{(i)}_{l,k'} \longrightarrow \mathcal{H}^{(2-i)}_{l,k}. \] (3.16)

A level-2 representation of the elliptic algebra \( U_{x,p}(sl_2) \) was obtained in terms of one free boson and one free fermion in Refs. [28,29].
3.3. Free-field realization of type-II vertex operators

Let us introduce the following basic operators:

\[ \Psi_1^*(u) = z^{\frac{r}{2m}} : \exp \left( \sum_{m \neq 0} \frac{[r m]_x}{m} \beta_m z^{-m} \right) : e^{\beta z^{\frac{r}{2m}} L - \frac{1}{2} K}, \tag{3.17} \]

\[ B(v) = w^{\frac{r}{2m}} : \exp \left( - \sum_{m \neq 0} \frac{[r m]_x}{m} \beta_m w^{-m} \right) : e^{-\beta w^{\frac{r}{2m}} L + \frac{1}{2} K} \phi(w), \]

\[ X^*(u) = \oint_{C'} \frac{dw}{2\pi i w} B(v) \frac{[v - u + L]''}{[v - u + 1]''}, \]

\[ Y^*(u) = - \oint_{C'} \frac{dw}{2\pi i w} B(v) \frac{[v - u + L - 2]'''}{[v - u - 1]'''}. \tag{3.18} \]

The integral contour \( C' \) for \( X^*(u) \) encircles the poles at \( w = x^{-2+2mr} z \ (n \in \mathbb{Z}_{\geq 0}) \), but not the poles at \( w = x^{-2} z \ (n \in \mathbb{Z}_{\geq 0}) \).

Then the type-II vertex operators on \( \mathcal{H}_{l,k}^{(i)} \) can be realized in terms of bosons and fermions:

\[ \Psi^*(u)_{l}^{i+2} = \Psi_1^*(u), \]

\[ \Psi^*(u)_{l}^{i} = \Psi_1^*(u) X^*(u) = Y^*(u) \Psi_1^*(u), \tag{3.19} \]

\[ \Psi^*(u)_{l}^{i-2} = \Psi_1^*(u) X^*(u)^2 = Y^*(u)^2 \Psi_1^*(u), \]

where \( z = x^2, \ w = x^2 \). For some useful OPE formulae and commutation relations, see Appendix B.

Note that

\[ \Psi^*(u)_{l}^{i'} : \mathcal{H}_{l,k}^{(i)} \rightarrow \mathcal{H}_{l,k}^{(2-i)}. \tag{3.20} \]

These type-II vertex operators satisfy the following commutation relations on \( \mathcal{H}_{l,k}^{(i)} \):

\[ \Psi^*(u_2)_{c}^{d} \Psi^*(u_1)_{a}^{b} = \sum_b W_{22}^{ab} \left[ c \atop b \right] \left[ d \atop a \right] u_1 - u_2 \Psi^*(u_1)_{b}^{c} \Psi^*(u_2)_{a}^{d}. \tag{3.21} \]

Dual vertex operators are likewise defined as follows:

\[ \Psi(u)_{l}^{i+2} = \frac{1}{\lambda^*} \frac{[1]''}{[l]''[l + 1]''} \Psi_1^*(u - 1), \]

\[ \Psi(u)_{l}^{i} = \frac{1}{\lambda^*} \frac{[2]''}{[l - 1]''[l + 1]''} \Psi_1^*(u - 1) X^*(u - 1), \tag{3.22} \]

\[ \Psi(u)_{l}^{i-2} = \frac{1}{\lambda^*} \frac{[1]''}{[l - 1]''[l + 1]''} \Psi_1^*(u - 1) X^*(u - 1)^2. \]

Here the normalization factor can be determined as

\[ \lambda^* = - \frac{(x^{2r'; x^{2r''}})_{\infty} (x^{2r'; x^{2r''}})_{\infty}}{(x + x^{-1}) (x^{-2}; x^{2r''})_{\infty} (x^{2r''}; x^{2r''})_{\infty}^4}, \]

such that \( \Psi(u)_{l}^{i'} \) and \( \Psi^*(u)_{l}^{i''} \) satisfy the inversion relation:

\[ \Psi \left( u' \right)_{l}^{i'} \Psi^* \left( u'' \right)_{l}^{i'} = \frac{\delta_{l}^{i''}}{1 - x^{-2} z/z'} + O(1). \tag{3.23} \]
For later convenience, we also introduce another type of basic operator:

\[
W(v) = w^{\frac{2}{2}} \exp\left( -\sum_{m\neq 0} \left[ \frac{2m}{r} \right]_{x} \beta_{m} w^{-m} \right) : e^{-\alpha \beta_{m} \frac{k - \frac{1}{m}}{2}} :. \tag{3.24}
\]

For useful OPE formulae and commutation relations, see Appendix B.

### 3.4 Free-field realization of tail operators: diagonal sectors

Another ingredient of the present scheme is the tail operators \( \Lambda(u_{0})_{l_{l}}^{k'k} \). In this paper we use a different normalization from the one used in Ref. [6]. Thus we briefly explain how to derive free-field representations of \( \Lambda(u_{0})_{l_{l}}^{k'k} \).

First let \( l' = l \), i.e., diagonal with respect to the ground-state sectors. When \( k' \leq k - 2 \), let us consider \( (2.63) \) for \( (a, b, c) = (k, k + 2, k') \):

\[
\Lambda (u_{0})_{k+2}^{k'} \Phi(u)_{k}^{k+2} = \sum_{k'' \sim k'} L \left[ \begin{array}{ccc}
{k'} & {k''} \\
{k} & {k - 2} \\
{u_{0} - u} & {u_{0} - u}
\end{array} \right] \Phi(u)_{k}^{k'} \Lambda (u_{0})_{k}^{k''}. \tag{3.25}
\]

Here, we briefly denote \( \Lambda(u_{0})_{k}^{h'k} \) by \( \Lambda(u_{0})_{k}^{k'} \). It follows from (A9) that \( L(u_{0} - u) \) has simple poles at \( u_{0} - u = \pm \frac{1}{2} \). Note that

\[
\left[ u_{0} - u + \frac{1}{2} \right] L \left[ \begin{array}{ccc}
{k'} & {k''} \\
{k} & {k - 2} \\
{u_{0} - u} & {u_{0} - u}
\end{array} \right] \bigg|_{u_{0} = u - \frac{1}{2}}
\]

for \( k'' = k' \) and \( k' = 2 \) are all equal. Thus, if we assume that the LHS of (3.25) has no pole at \( u_{0} = u - \frac{1}{2} \), we have the following necessary conditions:

\[
\sum_{k'' \sim k'} \Phi(u)_{k}^{k'} \Lambda (u - \frac{1}{2})_{k}^{k''} = 0, \tag{3.26}
\]

i.e.,

\[
\frac{[1] \Phi_{1}(u) \Lambda \left( u - \frac{1}{2} \right)^{k-2}_{k} }{[k' - 2][k' - 1]} + \frac{[2] \Phi_{1}(u) X_{k} \Lambda \left( u - \frac{1}{2} \right)^{k'}_{k} }{[k' + 1][k' - 1]} + \frac{[1] \Phi_{1}(u) X_{k} (u) \Lambda \left( u - \frac{1}{2} \right)^{k''}_{k} }{[k' + 1][k' + 2]} = 0.
\]

(3.27)

Let \( k' = k - 2 \). Then the LHS of (3.27) contains \( \Lambda (u - \frac{1}{2})^{k}_{k} = 1 \). By changing \( k' = k - 2, k - 4, k - 6, \ldots \), we can solve (3.27) iteratively as follows:

\[
\Lambda (u_{0})_{k}^{2-2s} = (-X (u_{0} + \frac{1}{2}))^{s} \frac{[s + 1][k - 2s][k - s + 1]}{[1][k][k + 1]}. \tag{3.28}
\]

Here we use the identity:

\[
\frac{[1][s + 1][k - s + 1]}{[k - 2s - 1]} - \frac{[2][s + 2][k - 2s - 2][k - s]}{[k - 2s - 1][k - 2s - 3]} + \frac{[1][s + 3][k - s - 1]}{[k - 2s - 3]} = 0.
\]

Furthermore, we can check that (3.28) for generic \( u_{0} \) satisfies (3.25).

Equation (3.28) gives expressions of \( \Lambda (u_{0})_{k}^{k'} \) for \( k' \leq k \). When \( k' > k \), we should realize another free-field representation of \( \mathcal{H}^{(i)}_{k} \) on the Fock space \( \mathcal{F} \). Then \( \Lambda (u_{0})_{k}^{k + 2s} \) can be identified with \( \Lambda (u_{0})_{k}^{k - 2s} \), in addition to the identification \( \Phi(u)_{k}^{k'} \) and \( \Phi (u)_{k}^{k'} \) with \( \Phi(u)_{k}^{k'} \) and \( \Phi^{*} (u)_{k}^{k'} \), respectively. Note that the expression (3.28) was obtained in Ref. [6] for the general spin-\( K/2 \) (\( K \times K \)-fused) SOS model.
Correlation functions in the twenty-one-vertex model can be constructed in terms of type-I vertex operators of the fusion SOS model and tail operators as follows:

\[
\frac{1}{\chi^{(i)}} \text{tr}_{\tau^{(i)}}^{(i)} \left( \Phi_{j_1}^* (u_1) \cdots \Phi_{j_n}^* (u_n) \Phi^j_n (u_n) \cdots \Phi^j_l (u_1) \rho^{(i)} \right) = \frac{1}{\chi^{(l)}} \sum_{k \in \mathbb{Z}} \text{tr}_{\tau^{(l)}} \left( T (u_0)^{l/k} \Phi_{j_1}^* (u_1) \cdots \Phi_{j_n}^* (u_n) \Phi^j_n (u_n) \cdots \Phi^j_l (u_1) T (u_0)^{l/k} \left[ \rho \rho^{(l/k)} \right]^{l/k} \right) = \frac{1}{\chi^{(l')}} \sum_{k, k_1, \ldots, k_{2n}} t_{j_1}^* (u_1 - u_0)^k \cdots t_{j_n}^* (u_n - u_0)^k \cdots t_{j_1} (u_1 - u_0)^k \times \text{tr}_{\tau^{(l')}} \left( \Phi^* (u_1)^{k_{2n}} \cdots \Phi^* (u_n)^{k_{2n}+1} \Phi (u_n)^{k_{2n}} \cdots \Phi (u_1)^{k_{2n}} \Lambda (u_0)^{k_{2n}} \rho^{(l/k)} \right). (3.29)
\]

Here, the sum on the third line is taken over \( \{k, k_{2n}, \ldots, k_1 | k_1 \sim k_2, \ldots, k_{2n} \sim k; k \in l + i + 2\mathbb{Z} \} \), and we use (2.60), (2.61), (2.71), and (2.62).

3.5. Free-field realization of tail operators: off-diagonal sectors

In this subsection, let us consider the tail operators for \( \Lambda (u_0)^{l/k} \) with \( l' \neq l \), i.e., off-diagonal with respect to the ground-state sectors.

Let \( k' \equiv k \). Then we have \( l' = l \) from (2.68). Let \( k' < k \) (resp. \( k' > k \)) and \( k' \equiv k \) (mod 2). Then we have

\[
\Lambda (u_0)^{l/k'} = 0, (3.30)
\]

unless \( l' \leq l \) (resp. \( l' \geq l \)). Actually, if \( \Lambda (u_0)^{l/k'} \neq 0 \) for, e.g., \( k' < k \) and \( l' > l \), there must exist a number \( j \) such that \( k'_j = k_j \) and therefore \( k'_m = k_m \) for \( \forall m, j \), which implies \( l' = l \) from (2.68).

Let \( k' < k \) and \( l' = l \) on (2.66). Firstly let \( l'' = l + 2 \). Then Eq. (2.66) reduces to

\[
\Psi^* (u_1)^{l+2} \Lambda (u_0)^{l/k} \Lambda (u_0)^{l+2} \Psi^* (u_1)^{l+2}. (3.31)
\]

This relation holds from (3.28).

Secondly let \( l'' = l \). Then Eq. (2.66) reduces to

\[
\left[ \Psi^* (u_1), \Lambda (u_0)^{l/k} \right] = \Lambda (u_0)^{l/k} \Psi^* (u_1)^{l+2} \left[ \begin{array}{l} l + 2 \\ l \\ u_0 + \Delta u - u \end{array} \right]. (3.32)
\]

Since \( \Psi^* (u_1)^{l} = Y (u) \Psi^* (u_1)^{l+2} \) and (3.31), Eq. (3.32) implies

\[
[Y (u), \Lambda (u_0)^{l/k}] = \Lambda (u_0)^{l/k} \left[ \begin{array}{l} l'' \\ l + 2 \\ u_0 + \Delta u - u - \frac{1}{2} \end{array} \right] \Psi^* (u_1)^{l+2} \Psi^* (u_1)^{l+2}. (3.33)
\]

Thus we find \( \Delta u = 0 \) and

\[
\Lambda (u_0)^{l-2k-2s} = \left[ \begin{array}{l} l'' \\ l + 2 \\ u_0 + \Delta u - u - \frac{1}{2} \end{array} \right] \Psi^* (u_1)^{l+2} \Psi^* (u_1)^{l+2} \left[ \begin{array}{l} l \\ [s] \\ [s + 1][k - s + 1][k - s][k - 2s] \\ \frac{\partial[0][1]^2[k][k + 1]}{[1]^2} \end{array} \right]. (3.34)
\]
Here $\partial[0] = (x^{2r} ; x^{2r})_{\infty}$, and

$$W_-(u_0) = W \left( u_0 - \frac{r - 3}{2} \right). \quad (3.35)$$

Thirdly let $l'' = l - 2$. Then Eq. (2.66) reduces to

$$\begin{align*}
\left[ \Psi^*(u)_{l''}^{l'-2}, \Lambda(u_0)_{l_k}^{k'} \right] &= \Lambda(u_0)_{l_k+2}^{l'-2k'} \Psi^*(u)_{l''}^{l'+2L''} \left[ \begin{array}{c} l-2 \\ l' \\ u_0 - u \end{array} \right] \\
&+ \Lambda(u_0)_{l_k}^{l'-2k'} \Psi^*(u)_{l''}^{l'+2L''} \left[ \begin{array}{c} l-2 \\ l' \\ u_0 - u \end{array} \right]. \quad (3.36)
\end{align*}$$

By solving (3.36) we find

$$\Lambda(u_0)_{l_k}^{l'-4k-2s} = \frac{[l]' [s][s+1][k-s+1][k-s][k-2s]}{[1]''} \frac{\partial[0][1]^2[k][k+1]}{\partial[0][1]^2[k][k+1]} \times X^* \left( u_0 + \frac{1}{2} \right) W_-(u_0) \left( -X \left( u_0 - \frac{1}{2} \right) \right)^{s-1}. \quad (3.37)$$

In general, we obtain

$$\Lambda(u_0)_{l_k}^{l'-2k-2s} = \frac{[l]' [s][s+1][k-s+1][k-s][k-2s]}{[1]''} \frac{\partial[0][1]^2[k][k+1]}{\partial[0][1]^2[k][k+1]} \times X^* \left( u_0 + \frac{1}{2} \right)^{l'-1} W_-(u_0) \left( -X \left( u_0 - \frac{1}{2} \right) \right)^{s-1}. \quad (3.38)$$

For details of the derivation, see Appendix C.

Equation (3.38) is valid for $k < k$ and $l' < l$. When $k' > k$ and $l' > l$, we should construct the free-field representation of $\Lambda(u_0)^{l'-k'}_{l-k}$ on another realization of $H_{l,k}^{(i)}$ on the Fock space $\mathcal{F}_{-l,-k}^\phi \otimes \mathcal{F}_\Phi$. Then $\Lambda(u_0)^{l'+2k+2s}_{l_k}$ can be identified with $\Lambda(u_0)^{l-2k-2s}_{l'-k}$, in addition to the identification $\Phi(u)^{k'}_k$ and $\Psi^*(u)^{l'}_l$ with $\Phi(u)^{k'}_k$ and $\Psi^*(u)^{l'}_l$, respectively.

### 3.6 Free-field realization of the CTM Hamiltonian

We can realize the CTM Hamiltonian of the $2 \times 2$ fusion SOS model in terms of free fields as follows:

$$H_{l,k}^{(i)} = H_{a}^{(i,k)} + H_{\phi}^{(i)}, \quad (3.39)$$

where

$$\begin{align*}
\frac{1}{2} H_{a}^{(i,k)} &= \sum_{n=1}^{\infty} \frac{[r m]_x}{[r^n m]_x} \beta_m \beta_m + \frac{1}{4} \left( \frac{r}{2r''} L^2 - KL + \frac{r''}{2r} K^2 \right), \\
\frac{1}{2} H_{\phi}^{(i)} &= \sum_{n>0} n \frac{x + x^{-1}}{x^{2n} + x^{-2n}} \phi_n \phi_n + \frac{i (2 - i)}{8}. \quad (3.40)
\end{align*}$$

Let us examine the validity of these expressions. First of all, (3.39) satisfies the homogeneity relation

$$\Phi^{(2-i,i)}(u)^{k'}_k \times H_{l,k}^{(i)} \times \Phi^{(2-i,i)}(u - 2)^{k'}_k = \Phi^{(2-i,i)}(u)^{k'}_k \times H_{l,k}^{(2-i)} \times \Phi^{(2-i,i)}(u - 2)^{k'}_k, $$

$$\Psi^{(2-i,i)}(u)^{l'}_l \times H_{l,k}^{(i)} \times \Psi^{(2-i,i)}(u - 2)^{l'}_l = \Psi^{(2-i,i)}(u)^{l'}_l \times H_{l,k}^{(2-i)} \times \Psi^{(2-i,i)}(u - 2)^{l'}_l. \quad (3.41)$$
Secondly, the traces on the bosonic/fermionic Fock space are given as follows:

\[
\begin{align*}
\text{tr}_{\mathcal{F}_\phi^{(i)}}(x^2H^{(i)}_\phi) &= x^{r}\frac{r^{-2}l^2-k l^*+\nu^\prime k^2}{2\nu^\prime l^2} \times \begin{cases} 
\hat{\lambda}_1 + \hat{\lambda}_2 & (i = 0, 2) \\
\hat{\lambda}_1 - \hat{\lambda}_2 & (i = 1) 
\end{cases}, \\
(3.42)
\end{align*}
\]

which implies (2.69). From these checks, we conclude that \( H^{(i)}_{t, k} = \mathcal{F}^{(i)}_{t, k} \otimes \mathcal{F}_\phi^{(i)} \) and \( \rho^{(i)}_{t, k} = [k]^{2H^{(i)}_{t, k}} \).

The fermionic trace formulae are given as follows [11,12]:

\[
F^{(i)}(w_1, w_2) := \text{tr}_{\mathcal{F}_\phi^{(i)}}(\phi(w_1) \phi(w_2) : x^2H^{(i)}_\phi)
\]

\[
= \begin{cases} 
\sum_{n=0}^{\infty} \left( \frac{x^2 w_1}{w_2} \right)^m - \left( \frac{x^2 w_2}{w_1} \right)^m & (i = 0, 2) \\
\frac{x^{1/2}(-x^2 ; x^4)^{\infty}}{x + x^{-1}} \sum_{m=0}^{\infty} \left( \frac{x^2 w_1}{w_2} \right)^m - \left( \frac{x^2 w_2}{w_1} \right)^m & (i = 1) 
\end{cases}.
\]

(3.43)

4. Form factors

4.1. Integral formulae

We are now in a position to write down the integral formulae for form factors, matrix elements of some local operators. For simplicity, let us choose \( S^c_i \) at the center site of the lattice as a local operator:

\[
S^c_i = \sum_{j=-1}^{j} j E_{jj}^{(1)}.
\]

(4.1)

The free-field representation of \( S^c_i \) is given by

\[
\hat{S}^c_i = \sum_{j=-1}^{j} j \Phi^*_j(u) \Phi^j(u).
\]

(4.2)

The corresponding form factors with \( m \) "charged" particles are:

\[
F_m^{(i)} \left( S^c_i ; u_1, \ldots, u_m \right)_{j_1 \ldots j_m} = \frac{1}{\chi^{(i)}} \text{Tr}_{\mathcal{F}^{(i)}} \left( \Psi^*_{j_1} (u_1) \cdots \Psi^*_{j_m} (u_m) S^c_i \rho^{(i)} \right).
\]

(4.3)

Note that the local operator (4.1) commutes with the type-II vertex operators because of (4.2) and (B11)–(B15).

From the construction in Sect. 3, we can rewrite (4.3) as follows:

\[
F_m^{(i)} \left( S^c_i ; u_1, \ldots, u_m \right)_{j_1 \ldots j_m}
\]

\[
= \sum_{l_1, \ldots, l_m} l^{* n}_{j_1} (u_1 - u_0)^{l_1}_{j_1} \cdots l^{* n}_{j_m} (u_m - u_0)^{l_m}_{l_{m-1}} F_m^{(i)} \left( S^c_i ; u_1, \ldots, u_m \right)_{l_1 \cdots l_m},
\]

(4.4)

where

\[
F_m^{(i)} \left( S^c_i ; u_1, \ldots, u_m \right)_{l_1 \cdots l_m}
\]

\[
= \frac{1}{\chi^{(i)}} \sum_{k_{1} l_{1} + i \in (2)} \sum_{k_{1} l_{2} j = -1} \frac{1}{j} \text{Tr}_{\mathcal{F}_{t, k}^{(i)}} \left( \Phi^* (u)^{l_{m-1}}_{l_1} \Phi (u)^{l_m}_{l_{m-1}} \Delta (u_0)^{l_{m-2}}_{l_{m-1}} \frac{[k] x^{2H^{(i)}_{t, k}}}{[l]^n} \right).
\]

(4.5)
Note that Eq. (4.4) can be inverted as follows:

\[ F_m^{(i)} (S^e_1; u_1, \ldots, u_m)_{j_1, \ldots, j_m} = \sum_{j_1, \ldots, j_m} t^{j_1, \ldots, j_m} (u_1 - u_0)^{j_1} \cdots t^{j_m, j_m} (u_m - u_0)^{j_m-1} F_m^{(i)} (S^e_1; u_1, \ldots, u_m)_{j_1, \ldots, j_m}. \] (4.6)

Free-field representations of the tail operators \( \Lambda(u_{1 \, k}) \) have been constructed in Sect. 3, and those of all other operators \( \Phi(u_{1 \, k}), \Phi^*(u_{1 \, k}), \Psi(u_{1 \, k})^*, \) and \( H_{1 \, k}^{(i)} \) on (4.5) were also given in Sect. 3. Integral formulae for form factors of any local operators can therefore be obtained for form factors of the spin-1 analogue of the eight-vertex model, in principle.

4.2. Calculation of two-point form factors

It is very difficult to obtain general integral formulae (4.4), as Lashkevich said in Ref. [5]. In order to show the relevance of the present scheme, we calculate the simplest form factor of the local operator \( S^e_1 \) in this subsection.

Let us consider (4.5) for \( i = 2, m = 2, l_1 = l - 2, \) and \( l_2 = l - 4. \) Since \( l_2 < l, \) the tail operator \( \Lambda(u_{0 \, k}) \) vanishes unless \( k_2 < k. \) Thus, the sum with respect to \( k_1 \) and \( k_2 \) should be taken over \( (k_1, k_2) = (k - 2, k - 4), \) \( (k + 2, k - 2), \) \( (k, k - 2). \) We notice that the form factors (4.4) in the twenty-one-vertex model should be \( u_0- \) independent. For simplicity of calculation, let \( u_0 \rightarrow u - \frac{3}{2}. \) By taking the sum with respect to \( j = \pm 1 \) and \( (k_1, k_2), \) we have

\[ F_2^{(2)} (S^e_1; u_1, u_2)_{l_1 - 2l_2 - 4} = \frac{1}{2x^{(2)} \lambda} \{ [0][2][u - u_0 - \frac{5}{2}] \sum_{k \in \mathbb{Z} \setminus \{2\}} \frac{[k + u - u_0 - \frac{3}{2}]}{[k + 1]} \}
\]

\[ \times \int_{C_{\infty}} \frac{du_2}{2\pi i} \frac{v_2 - u_0 - \frac{5}{2} + l}{v_2 - u_0 + \frac{5}{2} + l} \int_{C_{\infty}} \frac{du_1}{2\pi i} \frac{v_1 - u - k}{v_1 - u_0 + \frac{5}{2}} \times T (u_1, u_2, u, u_0, v_1, v_2), \] (4.7)

where \( T (u_1, u_2, u, u_0, v_1, v_2) \) is a trace function

\[ T (u_1, u_2, u, u_0, v_1, v_2) = \text{Tr}_{H_{1 \, k}^{(2)}} (\Psi^e_1 (u_1) \Psi^e_1 (u_2) \Phi (u - 1) \Phi (u) A (v_1) B (v_2) W_-(u_0) x^{2H_{1 \, k}^{(2)}}). \] (4.8)

Here, the integral contour \( C \) encircles the poles at \( x^{2r} z \) and \( x^{1+2r} z_0 (n \geq 0) \) but not \( x^{-2} - 2r z \) nor \( x^{2} - z^{(n+1)} z_0 (n \geq 0); \) the integral contour \( C' \) encircles the poles at \( x^{-1} + 2r^* z_0 (n \geq 0). \)

From the expression of the fermionic trace (3.43), the integral with respect to \( w_1 \) can be performed as follows:

\[ F_2^{(2)} (S^e_1; u_1, u_2)_{l_1 - 2l_2 - 4} = \frac{1}{2x^{(2)} \lambda} \{ [0][2][u - u_0 - \frac{5}{2}] \sum_{k \in \mathbb{Z} \setminus \{2\}} \frac{[k + u - u_0 - \frac{3}{2}]}{[k + 1]} \}
\]

\[ \times \left( \int_{C_{\infty}} \frac{dw_2}{2\pi i} - \int_{C_{\infty}} \frac{dw_2}{2\pi i} \right) G (v_2) \times \text{Tr}_{H_{1 \, k}^{(2)}} (\Psi^e_1 (u_1) \Psi^e_1 (u_2) \Phi (u - 1) \Phi (u) W (v_2) W_-(u_0) x^{2H_{1 \, k}^{(2)}}), \] (4.9)
where
\[
G(v_2) = \frac{(-x^2; x^4\infty) [v_2 - u_0 + \frac{r + 3}{2} + l]}{[v_2 - u_0 + \frac{r + 1}{2}]^\prime} [v_2 - u + \frac{r}{2} - k] [v_2 - u_0 + \frac{r + 5}{2}] [v_2 - u + \frac{r}{2}] [v_2 - u_0 + \frac{r - 3}{2}].
\]

Thus, the difference of the two integrals with respect to \( w_2 \) on (4.9) can be evaluated by the residue at \( w_2 = x^{-r}z \) and \( w_2 = x^{1-r}z_0 \). The former residue vanishes because of \( \Phi_1(u - 1)W(v_2) = 0 \) at \( v_2 = u - \frac{r}{2} \). Hence we have
\[
F_2^{(2)}(S^\tau_1; u_1, u_2)_{l-2l-4} = \frac{1}{2x^2} \lambda \sum_{k \equiv l(2)} \{ (u_0 - 0 - \frac{r}{2}) [l - 3][u - u_0 + \frac{1}{2}] \} \frac{(-x^2; x^4\infty)}{[v_2 - u_0 + \frac{r + 3}{2} + l]} [v_2 - u + \frac{r}{2} - k] [v_2 - u_0 + \frac{r + 5}{2}] [v_2 - u + \frac{r}{2}] [v_2 - u_0 + \frac{r - 3}{2}].
\]
\[
\times \frac{[0] [u - u_0 - \frac{r}{2}] [l - 3]}{[u - u_0 + \frac{1}{2}] [u - u_0 + \frac{1}{2}] [l - 3]} \sum_{k \equiv l(2)} \{ k + u - u_0 - \frac{3}{2} \} \frac{(-x^2; x^4\infty)}{[v_2 - u_0 + \frac{r + 3}{2} + l]} [v_2 - u + \frac{r}{2} - k] [v_2 - u_0 + \frac{r + 5}{2}] [v_2 - u + \frac{r}{2}] [v_2 - u_0 + \frac{r - 3}{2}].
\]
\[
\times Tr_{\Psi^*_1} \left( \Phi_1(u_1) \Phi_1(u_2) \Phi_1(u - 1) \Phi_1(u) W_-(u_0 - 1) W_-(u_0) x^{2H_0(r, k)} \right), \quad (4.10)
\]

where \( \partial[0] = (x^{2r''}; x^{2r''}) \).

By using OPE formulæ in Appendix B and the method of trace calculation explained in Ref. [2], we obtain
\[
F_2^{(2)}(S^\tau_1; u_1, u_2)_{l-2l-4} = cx^{-7 - \frac{11}{2}r - \frac{r}{2}r''} \frac{3r}{2} \frac{2r}{2r''}
\]
\[
\times [l - 3] \sum_{k \equiv l(2)} \frac{[k]}{[u_1 + u_2 - 2u_0]} \left( \frac{r + l - k}{2r'' - kl + \frac{r''k^2}{2r'}} \right)
\]
\[
\times \left( x^{2r'}/z_1; x^{2r''} \right) \left( x^{2r'}/z_1; x^{2r''} \right) \left( x^{2r'}/z_1; x^{2r''} \right) \left( x^{2r'}/z_1; x^{2r''} \right)
\]
\[
\times \prod_{j=1}^{2} f^*(u_j - u_0) \left( x^{2r'}/z_1; x^{2r''} \right), \quad (4.11)
\]

where
\[
c = \frac{\left( x^2; x^4 \right)^2 \left( x^2; x^4 \right)^2 \left( x^2; x^4 \right)^2 \left( x^2; x^4 \right)^2}{\left( x^2; x^4 \right)^2 \left( x^2; x^4 \right)^2 \left( x^2; x^4 \right)^2 \left( x^2; x^4 \right)^2}
\]
\[
\times \left( x^{2r'}/x^{2r''} \right)^3 \left( x^{2r'}/x^{2r''} \right)^3 \left( x^{2r'}/x^{2r''} \right)^3 \left( x^{2r'}/x^{2r''} \right)^3.
\]

and
\[
f^*(u) = \frac{1}{(x^{-1}; x^{2r''}) \left( x^{2r'}/x^{2r''} \right)^3 \left( x^{2r'}/x^{2r''} \right)^3 \left( x^{2r'}/x^{2r''} \right)^3 \left( x^{2r'}/x^{2r''} \right)^3}.
\]

By substituting
\[
\sum_{k \equiv l(2)} \frac{[k]}{[u_1 + u_2 - 2u_0]} \left( \frac{r + l - k}{2r'' - kl + \frac{r''k^2}{2r'}} \right)
\]
\[
= x \frac{1}{r''(u_1 + u_2 - 2u_0)^2 + u_1 + u_2 - 2u_0} \left( l + u_1 + u_2 - 2u_0 \right)^{2} (2u_0 - u_1 - u_2) \right) \left( l + u_1 + u_2 - 2u_0 \right)^{2} (2u_0 - u_1 - u_2) \right) \left( l + u_1 + u_2 - 2u_0 \right)^{2} (2u_0 - u_1 - u_2) \right)
\]
\[
2 \text{ Note that the contour } x^{-r'} C' \text{ does not encircle the point } w_2 = x^{5-r}z_0.
\]
into (4.11), we get
\[
F_2^{(2)}(S_1^z; u_1, u_2)_{l=2l-4} = \frac{\pi x^{-r''/2}}{2\epsilon r''}c(u_1, u_2, u) [u_1 + u_2 - 2u_0]_2 \\
\times \left\{ h_1^{(2r'')} (2l + u_1 + u_2 - 2u_0 - 3) h_2^{(2r'')} (u_1 + u_2 - 2u_0 + 3) \\
- h_2^{(2r'')} (2l + u_1 + u_2 - 2u_0 - 3) h_1^{(2r'')} (u_1 + u_2 - 2u_0 + 3) \right\},
\]
where
\[
c(u_1, u_2, u) = cx^{-\frac{11}{15} - \frac{r}{r''}} z_{\frac{r}{r''}} z_{\frac{2r}{r''}} x^{2r''}
\times \left( x^{-2z_2/z_1}; x^{2r''} \right)_\infty \left( x^{2z_2/z_1}; x^4 \right)_\infty \left( x^{2z_1/z_2}; x^4 \right)_\infty \\
\times x^{-\frac{1}{20}(u_1+u_2-2u_0)^2-u_1+u_2-2u_0} \prod_{j=1}^2 \frac{x^* (u_j - u_0)}{(x^{-2z_2/z_1}; x^2)_\infty (x^{4z_j/z_1}; x^2)_\infty}.
\]

Note that
\[
t'''' (u_1 - u_0) \begin{bmatrix} 1 \end{bmatrix} (u_2 - u_0) \begin{bmatrix} 2 \end{bmatrix} - t'''' (u_1 - u_0) \begin{bmatrix} 1 \end{bmatrix} (u_2 - u_0) \begin{bmatrix} 2 \end{bmatrix}
\]
\[
= h_1^{(2r'')} (2l + u_1 + u_2 - 2u_0 - 3) h_1^{(2r'')} (u_1 - u_2 - 2u_0 + 3) h_2^{(2r'')} (0) h_2^{(2r'')} (0)
\]
\[
= \frac{4h_1^{(2r'')} (u_1 - u_0 + \frac{1}{2}) h_1^{(2r'')} (u_2 - u_0 + \frac{1}{2}) h_1^{(2r'')} (u_1 - u_0 + \frac{3}{2}) h_1^{(2r'')} (u_2 - u_0 + \frac{3}{2})}{4h_1^{(2r'')} (u_1 - u_0 + \frac{1}{2}) h_1^{(2r'')} (u_2 - u_0 + \frac{1}{2}) h_1^{(2r'')} (u_1 - u_0 + \frac{3}{2}) h_1^{(2r'')} (u_2 - u_0 + \frac{3}{2})},
\]
\[
t'''' (u_1 - u_0) \begin{bmatrix} 2 \end{bmatrix} (u_2 - u_0) \begin{bmatrix} 2 \end{bmatrix} - t'''' (u_1 - u_0) \begin{bmatrix} 3 \end{bmatrix} (u_2 - u_0) \begin{bmatrix} 3 \end{bmatrix}
\]
\[
= \frac{h_2^{(2r'')} (2l + u_1 + u_2 - 2u_0 - 3) h_2^{(2r'')} (u_1 - u_2 - 2u_0 + 3) h_2^{(2r'')} (0) h_2^{(2r'')} (0)}{4h_1^{(2r'')} (u_1 - u_0 + \frac{1}{2}) h_1^{(2r'')} (u_2 - u_0 + \frac{1}{2}) h_1^{(2r'')} (u_1 - u_0 + \frac{3}{2}) h_1^{(2r'')} (u_2 - u_0 + \frac{3}{2})},
\]
where \( h'''' (u) := h_j(u) \mid_{r'' \rightarrow r'' - 2} (j = 1, 2, 3, 4) \). Thus, Eq. (4.13) can be reduced as follows:
\[
F_2^{(2)}(S_1^z; u_1, u_2)_{l=2l-4}
\]
\[
= \left[ u_1 - u_0 + \frac{1}{2}\right]^{\prime\prime} \left[ u_2 - u_0 + \frac{1}{2}\right]^{\prime\prime} \left[ u_1 - u_0 + \frac{3}{2}\right]^{\prime\prime} \left[ u_2 - u_0 + \frac{3}{2}\right]^{\prime\prime} c(u_1, u_2, u) [u_1 + u_2 - 2u_0]_2
\times \left\{ t'''' (u_1 - u_0) \begin{bmatrix} 1 \end{bmatrix} (u_2 - u_0) \begin{bmatrix} 2 \end{bmatrix} - t'''' (u_1 - u_0) \begin{bmatrix} 1 \end{bmatrix} (u_2 - u_0) \begin{bmatrix} 2 \end{bmatrix}
\right.
\times \frac{h_2^{(2r'')} (u_1 + u_2 - 2u_0 + 3)}{h_1^{(2r'')} (u_2 - u_1 - 2)} - \frac{h_1^{(2r'')} (u_1 + u_2 - 2u_0 + 3)}{h_2^{(2r'')} (u_2 - u_1 - 2)}
\times \left\{ t'''' (u_1 - u_0) \begin{bmatrix} 2 \end{bmatrix} (u_2 - u_0) \begin{bmatrix} 2 \end{bmatrix} - t'''' (u_1 - u_0) \begin{bmatrix} 3 \end{bmatrix} (u_2 - u_0) \begin{bmatrix} 3 \end{bmatrix} \right\}.
\]

By comparing (4.15) and (4.6), we obtain
\[
F_2^{(2)}(S_1^z; u_1, u_2)_{\pm 1, \pm 1} = \pm d(u_1, u_2, u) [u_1 + u_2 - 2u + 3]_2 \frac{h_2^{(2r'')} (u_1 + u_2 - 2u + 3)}{h_1^{(2r'')} (u_2 - u_1 - 2)}
\]
\[
F_2^{(2)}(S_1^z; u_1, u_2)_{\pm 1, \pm 1} = \mp d(u_1, u_2, u) [u_1 + u_2 - 2u + 3]_2 \frac{h_1^{(2r'')} (u_1 + u_2 - 2u + 3)}{h_2^{(2r'')} (u_2 - u_1 - 2)}.
\]
where

\[
\begin{align*}
    d (u_1, u_2, u) &= c \left( \frac{x^{2r''}; x^{2r''}}{x^{2r''+2}; x^{4r''}} \right) _{\infty} \frac{(u_1 - u_2)^2}{r''^2 + 6 + \frac{4}{r'z_2^2} + \frac{1}{r''z_2^2} + \frac{r'}{r''z_2} + z_2 - 1} \times \left( x^{-2z_2/z_1}; x^{2r''} \right) _{\infty} \left( x^{2r'} z_1/z_2; x^{2r''} \right) _{\infty} \left( x^{2z_2/z_1}; x^4 \right) _{\infty} \left( x^{2z_1/z_2}; x^4 \right) _{\infty} \\
    &\times \prod_{j=1}^{2} \left( x^{-2z_2/z_j}; x^2 \right) _{\infty} \left( x^{4z_j/z_2}; x^2 \right) _{\infty}.
\end{align*}
\]

Note that the nonzero components of \( F_2^{(2)} (S_1^z; u_1, u_2) \) on (4.16) have poles at \( z_2 = x^4 z_1 \), which is consistent with the relation (2.26)–(2.27).

### 4.3. Trigonometric limit

Let us examine the trigonometric limit \( r \to \infty \). The trigonometric limit of the twenty-one-vertex model, the spin-1 analogue of the eight-vertex model, is called the nineteen-vertex model. The operator algebra of the nineteen-vertex model can be constructed in terms of level-2 irreducible highest-weight representations of \( U_q (\widehat{sl}_2) \). In what follows, we use the same letters for both the elliptic model and its trigonometric-limit model, e.g., \( S(u) \) denotes the \( S \) matrix for both the twenty-one-vertex and the nineteen-vertex models.

Unfortunately, we have no results for form factors of the nineteen-vertex model. Thus, let us examine Smirnov’s axioms [30], which form factors of integrable models should satisfy. Using the \( S \)-matrix symmetry relation (2.22), the following relations should hold:

\[
F_2^{(2)} (S_1^z; u_2, u_1)_{\pm 1, \mp 1} = \sum_{j=-1}^{1} F_2^{(2)} (S_1^z; u_1, u_2)_j \cdot S (u_1 - u_2)^j \cdot \frac{1}{S (u_1 + u_2)}.
\]

From (4.17), we have

\[
\mathcal{F}_2^{(2)} (S_1^z; u_2, u_1) = S (u_1 - u_2) \mathcal{F}_2^{(2)} (S_1^z; u_1, u_2),
\]

where

\[
\mathcal{F}_2^{(2)} (S_1^z; u_1, u_2) := F_2^{(2)} (S_1^z; u_1, u_2)_{1, -1} - F_2^{(2)} (S_1^z; u_1, u_2)_{-1, 1},
\]

and

\[
S(u) := S(u)_{-1, 1} - S(u)_{1, -1} \rightarrow - \frac{(x \xi^{-1} - x^{-1} \xi)}{(x \xi - x^{-1} \xi)} \cdot \frac{(x^2 \xi - x^{-2} \xi^{-1})}{(x^2 \xi^{-1} - x^{-2} \xi)} \quad (r \to \infty).
\]

Here, \( \xi = x^u \).

From (4.16), \( F_2^{(2)} (S_1^z; u_1, u_2)_{\pm 1, \mp 1} \to 0 \) in the limit \( r \to \infty \), which is consistent with the charge conservation under \( U_q (\widehat{sl}_2) \)-symmetry. On the other hand, after appropriate redefinition, we have

\[
F_2^{(2)} (S_1^z; u_1, u_2)_{\pm 1, \mp 1} \sim \frac{\pm A \left( \frac{1}{z_1/z} \right)}{(\zeta_1/\zeta_2 - \zeta_2/\zeta_1)} \cdot \frac{(x^{2z_1/z_2}; x^4)_{\infty} (x^{2z_1/z_2}; x^4)_{\infty} (u_1 + u_2 - 2u + 3)_2}{(\zeta_1/\zeta_2 - \zeta_2/\zeta_1)} (x^{2z_1/z_2}; x^4)_{\infty} (x^{2z_1/z_2}; x^4)_{\infty} \cdot \left( x^{2z_2/z_1}; x^4 \right)_{\infty} \left( x^{2z_1/z_2}; x^4 \right)_{\infty}^{1/2} \prod_{j=1}^{2} \left( x^{-2z_2/z_j}; x^2 \right) _{\infty} \left( x^{4z_j/z_2}; x^2 \right) _{\infty} \quad (r \to \infty)
\]

where \( A \) is some constant and \( \zeta_j = x^u (j = 1, 2) \). Thus, our formula (4.19) satisfies the \( S \)-matrix symmetry relation (4.18).
Furthermore, from the homogeneity relation (2.25), the following cyclicity relation should hold:

\[ F_2^{(2)}(S^z_1; u_1 - 2, u_2)_{j_1,j_2} = F_2^{(0)}(S^z_1; u_2, u_1)_{j_2,j_1}. \]  (4.20)

From (4.20), we have

\[ F_2^{(2)}(S^z_1; u_1 - 2, u_2) = F_2^{(2)}(S^z_1; u_2, u_1). \]  (4.21)

Here we use the relation \( F_2^{(0)}(S^z_1; u_1, u_2)_{j_1,j_2} = - F_2^{(2)}(S^z_1; u_1, u_2)_{j_1,j_2} \). Note that our formula (4.19) satisfies the cyclicity relation (4.21) when \( u = (u_1 + u_2 + 3)/2 \).

Hence, we conclude that the trigonometric limit of the two-point form factors of the local operator \( S^z_1 \) satisfy Smirnov’s axioms and therefore give appropriate \( q \)-difference equations of level 0.

## 5. Concluding remarks

In this paper, we have derived integral formulae for form factors of the twenty-one-vertex model. For that purpose, we constructed free-field representations of the type-I vertex operators \( \Phi(u)^{k \rightarrow k'} \) and the type-II vertex operators \( \Psi^* u \rho^{l \rightarrow l'} \) in the 2 \( \times \) 2 fusion SOS model, the tail operators \( \Lambda(u_0)^{l \rightarrow l'} \), and the corner transfer Hamiltonian \( H_{l,k}^{(i)} \). Our integral formulae for the form factors of \( S^z_1 \) are given by (4.4)–(4.5), which are given in terms of the \( m \)-fold multiple integrals.

Our approach is based on some assumptions. We assumed that the vertex operator algebra (2.60)–(2.61), (2.64)–(2.65), and (2.71) correctly describes the intertwining relations between the twenty-one-vertex model and the 2 \( \times \) 2 fusion SOS model. We also assumed that the free-field representations (3.28), (3.38), and (3.39)–(3.40) provide relevant representations of the vertex-operator algebra. Using the present formalism, we can obtain the integral formulae for any form factors of any local operators in the twenty-one-vertex model, in principle. However, as Lashkevich said in Ref. [5], it is very difficult to obtain general formulae for form factors. In order to show the relevance of the present scheme, we calculated the simplest form factor of the local operator \( S^z_1 \) in Sect. 4.2. We also show in Sect. 4.3 that our form-factor formulae satisfy appropriate \( q \)-difference equations of level 0 in the trigonometric limit.

Here we wish to refer to correlation functions in the twenty-one-vertex model. A correlation function is a special example of a form factor; however, it is not the simplest one. Let us recall (4.5). In order to calculate the form factor/correlation function of the spin operator \( S^z_1 \), we have to perform the sum with respect to \( k_1 \) and \( k_2 \), the state variables of the SOS model. There are only three nonzero terms \( (k_1, k_2) = (k - 2, k - 4), (k - 2, k - 2), (k, k - 2) \) for calculation of two-point form factors, whereas there are 9 (= 3 \( \times \) 3) nonzero terms for that of correlation functions. This is why we have calculated not the correlation functions but the two-point form factors of the spin operator \( S^z_1 \) as examples.

We expect to find appropriate Smirnov axiomatic structures [30], S-matrix symmetry, cyclicity, and annihilation pole condition, besides some analytic properties, in the form factors (4.4)–(4.5). For that purpose, we should construct multipoint form factors. We wish to address this issue in a separate paper.

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3 A similar specialization of the value of \( u \) was also needed for Lashkevich’s formula [5], the case of the spin-1/2 model.

4 Even after using the symmetry of \( k \mapsto -k \), we should take the sum with respect to five terms. Thus, the correlation function is not the simplest example.
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Appendix A. Definitions of the models concerned

A.1. R matrix of the spin-1 analogue of the eight-vertex model

Let $R^{(s, s')}(u)$ ($s, s' = \frac{1}{2}, 1, \frac{3}{2}, \ldots$) be the $R$ matrix of vertically $2s$-fold and horizontally $2s'$-fold fusion of $R^{(1, \frac{1}{2})}(u)$, the $R$ matrix of the eight-vertex model. Then the nonzero elements of $R^{(1, \frac{1}{2})}(u)$ are given as follows:

$$
R^{(1, \frac{1}{2})}(u)_{\pm \pm 1}^{\pm \pm 1} = \frac{1}{\bar{k}_{1,2}(u)} \frac{\theta_2^2 \left( \frac{1}{2} \right) \theta_2 \left( \frac{u}{2} \right)}{\theta_2^2 (0) \theta_2 \left( \frac{2-u}{2} \right)} ,
$$

$$
R^{(1, \frac{1}{2})}(u)_{\pm \pm 1}^{\pm \pm 1} = \frac{1}{\bar{k}_{1,2}(u)} \frac{\theta_2^2 \left( \frac{1}{2} \right) \theta_1 \left( \frac{u}{2} \right)}{\theta_2^2 (0) \theta_1 \left( \frac{2-u}{2} \right)} ,
$$

$$
R^{(1, \frac{1}{2})}(u)_{0\pm 0}^{\pm 0} = \frac{1}{\bar{k}_{1,2}(u)} \frac{\theta_2 \left( \frac{1}{2} \right) \theta_2 \left( \frac{1-u}{2} \right)}{\theta_2 (0) \theta_1 (2-u)},
$$

$$
R^{(1, \frac{1}{2})}(u)_{0\pm 1}^{0\pm 1} = \frac{1}{\bar{k}_{1,2}(u)} \frac{\theta_1 \left( \frac{1}{2} \right) \theta_2 \left( \frac{1-u}{2} \right)}{\theta_2 (0) \theta_1 (2-u)},
$$

where $\theta_i \left( \frac{u}{2} \right) = \theta_i \left( \frac{u}{2} ; \frac{2}{2} \right)$, and

$$
\bar{k}_{1,2}(u) = \left( x^{-1}z \right)^{-\frac{1}{2} - \frac{u'}{2}} \left( z \cdot x^{2r} \right)_{\infty} \left( x^{2r} z^{-1} \cdot x^{2r} \right)_{\infty}.
$$

The case $(s, s') = (1, 1)$ is of interest in the present study. There are twenty-one nonzero elements of $R^{(1,1)}(u)$, so that the spin-1 analogue of the eight-vertex model is also called the twenty-one-vertex model. The explicit expressions of the nonzero elements of $R^{(1,1)}(u)$ are given as follows:

$$
R^{(1,1)}(u)_{\pm \pm 1}^{\pm \pm 1} = \frac{1}{\bar{k}_{2,2}(u)} \left( \theta_2^4 \left( \frac{1}{2} \right) \theta_2 \left( \frac{u}{2} \right) \theta_2 \left( \frac{1+u}{2} \right) \theta_1 \left( \frac{1+u}{2} \right) \theta_2 \left( \frac{2-u}{2} \right) \theta_2 \left( \frac{1-u}{2} \right) \theta_1 \left( \frac{1-u}{2} \right) \theta_2 \left( \frac{2-u}{2} \right) \theta_1 \left( \frac{2-u}{2} \right) \theta_2 \left( \frac{u}{2} \right) \right),
$$

$$
R^{(1,1)}(u)_{0\pm 1}^{0\pm 1} = \frac{1}{\bar{k}_{2,2}(u)} \theta_1 \theta_2 \left( \frac{1}{2} \right) \theta_2 \left( \frac{u}{2} \right) \theta_2 \left( \frac{2-u}{2} \right) \theta_1 \theta_2 \left( \frac{2-u}{2} \right) = R^{(1,1)}(u)_{0\pm 1}^{0\pm 1}.
$$
In what follows, we use the following symbols:

\[ R^{(1,1)}(u)^{\pm 10}_{\pm 10} = \frac{1}{\kappa_{2,2}(u)} \frac{\theta^2}{\theta^2_2} \left( \frac{1}{2} \right) \theta_1 \theta_2 \left( \frac{u}{2r} \right) \theta_1 \theta_2 \left( \frac{u^*}{2r} \right) = R^{(1,1)}(u)^{00}_{00} \]

\[ R^{(1,1)}(u)^{\pm 11}_{\pm 11} = \frac{1}{\kappa_{2,2}(u)} \left( \frac{\theta^2}{\theta^2_2} \left( \frac{1}{2} \right) \theta_1 \theta_2 \left( \frac{u}{2r} \right) \theta_1 \theta_2 \left( \frac{u^*}{2r} \right) \right) \]

\[ R^{(1,1)}(u)^{00}_{00} = -\frac{1}{\kappa_{2,2}(u)} \left( \frac{\theta^2}{\theta^2_2} \left( \frac{1}{2} \right) \theta_1 \theta_2 \left( \frac{u}{2r} \right) \theta_1 \theta_2 \left( \frac{u^*}{2r} \right) \right) \]

(A2)

Here,

\[ \kappa_{2,2}(u) = z^{-u} \left( \frac{x^{2r}; x^{2r}}{x^{2r}; x^{2r}} \right) \left( \frac{x^{2r}; x^{2r}}{x^{2r}; x^{2r}} \right) \left( \frac{x^{2r}; x^{2r}}{x^{2r}; x^{2r}} \right) \]

Note that some of components are modified by symmetrization of the \( R \) matrix.

In this article, we assume that the parameters \( v, \epsilon, \) and \( r \) lie in the so-called principal regime (2.8).

### A.2. Boltzmann weights of the \( 2 \times 2 \) fusion SOS model

In what follows, we use the following symbols:

\[
\begin{pmatrix}
  u \\
  m \\
\end{pmatrix} = [u]_m \begin{pmatrix}
  [u]_m \\
  [m]_m \\
\end{pmatrix}, \\
[u]_m = [u][u-1] \cdots [u-m+1].
\]

Let \( W_{22} \) be the Boltzmann weights of the \( 2 \times 2 \) fusion SOS model, and let

\[
W_{22} \begin{pmatrix}
  c & d \\
  b & a \\
\end{pmatrix} = \kappa^{(2,2)}(u) \begin{pmatrix}
  2 - u \\
  2 \\
\end{pmatrix} W_{22} \begin{pmatrix}
  c & d \\
  b & a \\
\end{pmatrix}
\]

be unnormalized weights. Then the nonzero \( \overline{W}_{22} \) are given as follows:

\[
\overline{W}_{22} \begin{pmatrix}
  k + 4 & k + 2 \\
  k + 2 & k \\
\end{pmatrix} = \begin{pmatrix}
  2 - u \\
  2 \\
\end{pmatrix},
\]

\[
\overline{W}_{22} \begin{pmatrix}
  k + 2 & k + 2 \\
  k + 2 & k \\
\end{pmatrix} = \frac{[1 - u][k + 1 + u]}{[1][k + 1]},
\]

\[
\overline{W}_{22} \begin{pmatrix}
  k + 2 & k \\
  k & k \\
\end{pmatrix} = \frac{[1 - u][k + 1 + u]}{[1][k + 1]},
\]

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\[
W_{22} \left[ \begin{array}{cc}
  k \pm 2 & k \pm 2 \\
  k & k
\end{array} \right] u = \frac{[k \pm 3]}{[k \pm 1]} \left[ 1 - u \right] / 2,
\]
\[
W_{22} \left[ \begin{array}{cc}
  k \pm 2 & k \\
  k \pm 2 & k
\end{array} \right] u = \frac{[k \mp 1]}{[k \pm 1]} \left[ 1 - u \right] / 2,
\]
\[
W_{22} \left[ \begin{array}{cc}
  k & k \pm 2 \\
  k \pm 2 & k
\end{array} \right] u = \frac{\pm k + u + 1}{2} \left[ \begin{array}{c}
  2 \\
  \pm k + 1 \\
  2
\end{array} \right],
\]
\[
W_{22} \left[ \begin{array}{cc}
  k & k \pm 2 \\
  k \mp 2 & k
\end{array} \right] u = \frac{\pm k + 2}{2} \left[ \begin{array}{c}
  u + 1 \\
  \pm k \\
  2
\end{array} \right],
\]
\[
W_{22} \left[ \begin{array}{cc}
  k & k \\
  k \mp 2 & k
\end{array} \right] u = \frac{-[k \mp 1][u][k \mp u]}{[2][k][k \mp 1]},
\]
\[
W_{22} \left[ \begin{array}{cc}
  k & k \\
  k & k \pm 2
\end{array} \right] u = \frac{-[2][k \mp 2][u][k \mp u]}{[1][k-1][k+1]},
\]
\[
W_{22} \left[ \begin{array}{cc}
  k & k \\
  k & k
\end{array} \right] u = \frac{[k - 1 + u][k - u]}{[k][k - 1]} + \frac{[k - 1][k + 2]}{[k][k + 1]} \left[ 1 - u \right] / 2. \quad \text{(A3)}
\]

Note that some of the weights are modified by symmetrization of the Boltzmann weights. In this paper, we consider the so-called regime III in the model, i.e., \( 0 < u < 1 \).

### A.3. Fused intertwining vectors

For \( k' = k, k \pm 2 \), let
\[
t(u)_k = \sum_{j=-1}^{1} v_j t^j(u)_k,
\]
\[
t(u)_{k \pm 2} = \frac{1}{2h_1(u + 1/2)} \left[ \begin{array}{c}
  h_3^{(2r)}(k \mp u + \frac{3}{2}) h_3^{(2r)}(k \mp u + \frac{1}{2}) \\
  2 h_4(1) h_4(k \mp u + \frac{1}{2}) \\
  h_4^{(2r)}(k \mp u + \frac{3}{2}) h_4^{(2r)}(k \mp u + \frac{1}{2})
\end{array} \right], \quad \text{(A4)}
\]
\[
t(u)_k = \frac{1}{2h_1(u + 1/2)} \left[ \begin{array}{c}
  h_3^{(2r)}(k - u + \frac{1}{2}) h_3^{(2r)}(k + u + \frac{1}{2}) \\
  2 h_4(k) h_4(u + \frac{1}{2}) \\
  h_4^{(2r)}(k - u + \frac{1}{2}) h_4^{(2r)}(k + u + \frac{1}{2})
\end{array} \right].
\]

Then the following relation holds:
\[
R^{(1,1)} (u_1 - u_2) t(u_1)_a^d \otimes t(u_2)_b^c = \sum_b t(u_1)_b^c \otimes t(u_2)_a^b W_{22} \left[ \begin{array}{c}
  c \\
  d \\
  a
\end{array} \right] (u_1 - u_2). \quad \text{(A5)}
\]
The dual intertwining vectors are given as follows:

\[ t^*(u)_k^{k'} = \sum_{j=-1}^{1} v^{*j} t^*_j(u)_k^{k'} \]
\[ t^*(u)_{k \pm 2} = \frac{h_4^{(2r)}(k \pm u \pm \frac{1}{2}) - h_3^{(2r)}(k \pm u \pm \frac{1}{2})}{2h_1(u - \frac{1}{2})h_1(k)h_1(k+1)} \]
\[ t^*_1(u)_k^{k'} = -\frac{h_4^{(2r)}(k + u + \frac{1}{2})h_4^{(2r)}(k - u + \frac{1}{2})}{2h_1(u - \frac{1}{2})h_1(k)h_1(k+1)} - \frac{h_4^{(2r)}(k - u - \frac{1}{2})h_4^{(2r)}(k + u - \frac{3}{2})}{2h_1(u - \frac{1}{2})h_1(k)h_1(k-1)}. \] (A6)
\[ t^*_0(u)_k^{k'} = \frac{h_4(u - \frac{1}{2})h_4(k+1) + h_4(k-1)}{2h_1(u - \frac{1}{2})h_1(k)} \left( \frac{h_4(k+1)}{h_1(k+1)} + \frac{h_4(k-1)}{h_1(k-1)} \right), \]
\[ t^*_1(u)_k^{k'} = -\frac{h_3^{(2r)}(k + u + \frac{1}{2})h_3^{(2r)}(k - u + \frac{3}{2})}{2h_1(u - \frac{1}{2})h_1(k)h_1(k+1)} - \frac{h_3^{(2r)}(k - u - \frac{1}{2})h_3^{(2r)}(k + u - \frac{3}{2})}{2h_1(u - \frac{1}{2})h_1(k)h_1(k-1)}. \]

The intertwining vectors and their dual vectors satisfy the following inversion relations:

\[ \sum_{j=-1}^{1} t_j^*(u)_k^{k'} t^j(u)_k^{k''} = \delta_{k''}^{k'}, \quad \sum_{k' \neq k} t_j^*(u)_k^{k'} t^j(u)_k^{k''} = \delta_{j}^{j'}. \] (A7)

Then the following relation holds:

\[ t^*(u_1)_c^b \otimes t^*(u_2)_b^a R^{(1,1)}(u_1 - u_2) = \sum_d W_{222} \begin{bmatrix} c & d \\ b & a \end{bmatrix} u_1 - u_2 \] \[ t^*(u_1)_c^a \otimes t^*(u_2)_c^d. \] (A8)

The explicit expressions of the \( L \)-operators defined by (2.56) are given as follows:

\[ L \begin{bmatrix} k' & k' \mp 2 \\ k & k \mp 2 \end{bmatrix} u_0 = \begin{bmatrix} \pm \frac{k+k'}{2} \\ \pm k \end{bmatrix} \begin{bmatrix} u_0 \pm \frac{k-k'+1}{2} \\ 2 \end{bmatrix}, \]
\[ \begin{bmatrix} \pm \frac{k-k'}{2} \\ \pm k \end{bmatrix} \begin{bmatrix} u_0 \pm \frac{k+k'-1}{2} \\ 2 \end{bmatrix}. \] (A9)
\[ L \begin{bmatrix} k' & k' \mp 2 \\ k & k \mp 2 \end{bmatrix} u_0 = \begin{bmatrix} \pm k \end{bmatrix} \begin{bmatrix} u_0 \pm \frac{k-k'+1}{2} \\ 2 \end{bmatrix}, \]
\[ \begin{bmatrix} \pm k \end{bmatrix} \begin{bmatrix} u_0 \pm \frac{k+k'-1}{2} \\ 2 \end{bmatrix} \]
\[ = \begin{bmatrix} \frac{k+k'}{2} \end{bmatrix} \begin{bmatrix} u_0 \pm \frac{k-k'+1}{2} \end{bmatrix} \begin{bmatrix} u_0 \pm \frac{k+k'+1}{2} \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} k \end{bmatrix} \begin{bmatrix} u_0 \pm \frac{k-k'+1}{2} \end{bmatrix}. \] (2)
\[ L \begin{bmatrix} k' & k' \\ k & k \end{bmatrix} u_0 = \begin{bmatrix} \frac{k\pm k'}{2} \end{bmatrix} \begin{bmatrix} u_0 \pm \frac{k-k'+1}{2} \end{bmatrix} \begin{bmatrix} u_0 \pm \frac{k-k'+1}{2} \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} k \end{bmatrix} \begin{bmatrix} u_0 \pm \frac{k-k'+1}{2} \end{bmatrix}. \] (11)
Appendix B. OPE formulae and commutation relations

In this appendix, we list some useful formulae for the basic operators. In what follows, we denote $z = x^{2u}$, $w = x^{2v}$.

First, useful OPE formulae are:

\[
\Phi_1(u)\Phi_1(v) = z^{r^2} \frac{x^2 w/z; x^{2r}}{(x^2 w/z; x^{2r})_\infty} : \Phi_1(u)\Phi_1(v) :, \quad (B1)
\]

\[
\Phi_1(u)A(v) = z^{-r^2} \frac{x^{2r} w/z; x^{2r}}{(x^2 w/z; x^{2r})_\infty} : \Phi_1(u)A(v) :, \quad (B2)
\]

\[
A(v)\Phi_1(u) = w^{-r^2} \frac{x^{2r} z/w; x^{2r}}{(x^2 z/w; x^{2r})_\infty} : A(v)\Phi_1(u) :, \quad (B3)
\]

\[
\hat{A}(u)\hat{A}(v) = z^{r^2} \frac{x^2 w/z; x^{2r}}{(x^2 w/z; x^{2r})_\infty} : \hat{A}(u)\hat{A}(v) :, \quad (B4)
\]

\[
A(u)A(v) = z^{-r^2} \frac{x^2 w/z; x^{2r}}{(x^2 w/z; x^{2r})_\infty} : A(u)A(v) : + f (z, w) : \hat{A}(u)\hat{A}(v) :, \quad (B5)
\]

\[
\Psi_1^*(u)\Psi_1^*(v) = z^{r^2} \frac{x^{-2} w/z; x^{2r}}{(x^{-2} w/z; x^{2r})_\infty} : \Psi_1^*(u)\Psi_1^*(v) :, \quad (B6)
\]

\[
\Psi_1^*(u)B(v) = z^{-r^2} \frac{x^{2r} w/z; x^{2r}}{(x^{-2} w/z; x^{2r})_\infty} : \Psi_1^*(u)B(v) :, \quad (B7)
\]

\[
B(v)\Psi_1^*(u) = w^{-r^2} \frac{x^{2r} z/w; x^{2r}}{(x^{-2} z/w; x^{2r})_\infty} : \Psi_1^*(u)B(v) :, \quad (B8)
\]

\[
\hat{B}(u)\hat{B}(v) = z^{r^2} \frac{x^{-2} w/z; x^{2r}}{(x^{-2} w/z; x^{2r})_\infty} : \hat{B}(u)\hat{B}(v) :, \quad (B9)
\]

\[
B(u)B(v) = z^{-r^2} \frac{x^{-2} w/z; x^{2r}}{(x^{-2} w/z; x^{2r})_\infty} : B(u)B(v) : + f (z, w) : \hat{B}(u)\hat{B}(v) :, \quad (B10)
\]

\[
\Phi_1(u)\Psi_1^*(v) = \frac{1}{z} \frac{1}{1 - w/z} : \Phi_1(u)\Psi_1^*(v) : = -\Psi_1^*(v)\Phi_1(u), \quad (B11)
\]

\[
\Phi_1(u)B(v) = z \left(1 - \frac{w}{z}\right) : \Phi_1(u)B(v) : = -B(v)\Phi_1(u), \quad (B12)
\]

\[
\Psi_1^*(u)A(v) = \frac{1}{z} \frac{1}{1 - w/z} : \Psi_1^*(u)A(v) : = -A(v)\Psi_1(u), \quad (B13)
\]

\[
A(u)B(v) = \frac{1}{z} \frac{1}{1 - w/z} : A(u)B(v) : + f (z, w) : \hat{A}(u)\hat{B}(v) :, \quad (B14)
\]

\[
B(v)A(u) = \frac{1}{w} \frac{1}{1 - z/w} : B(v)A(u) : + f (w, z) : \hat{B}(v)\hat{A}(u) :, \quad (B15)
\]

\[
W(v)\Phi_1(u) = w^{2r} \frac{x^{r-2} z/w; x^{2r}}{(x^{r+2} z/w; x^{2r})_\infty} : W(v)\Phi_1(u) :, \quad (B16)
\]
Here \( \hat{A}(v) \) and \( \hat{B}(v) \) denote the fermion contraction

\[
\hat{A}(v) = w^{r''} : \exp \left( \sum_{m \neq 0} \beta_m \frac{w^{-m}}{m} \right) : e^{-\alpha \frac{1}{2} L - r'' K}.
\]

\[
\hat{B}(v) = w^{r''} : \exp \left( - \sum_{m \neq 0} \left[ \frac{rm}{m} \beta_m \frac{w^{-m}}{m} \right] \right) : e^{-\beta \frac{1}{2} L + \frac{1}{2} K},
\]

and

\[
f(z, w) = \frac{1}{x + x^{-1}} \sum_{m > 0} \left( \left( \frac{x^2 w}{z} \right)^m + \left( \frac{x^{-2} w}{z} \right)^m \right).
\]

From these, we obtain the following commutation relations:

\[
\Phi_1(u) \Phi_1(v) = \frac{[v - u + 1]}{[u - v + 1]} \Phi_1(v) \Phi_1(u),
\]

\[
A(v) \Phi_1(u) = \frac{[v - u + 1]}{[u - v + 1]} \Phi_1(u) A(v),
\]

\[
[u - v + 1] A(u) A(v) = [u - v - 1] A(v) A(u),
\]

\[
\Psi_1^*(u) \Psi_1^*(v) = \frac{[v - u - 1]''}{[u - v - 1]''} \Psi_1^*(v) \Psi_1^*(u),
\]

\[
B(v) \Psi_1^*(u) = \frac{[v - u - 1]''}{[u - v - 1]''} \Psi_1^*(u) B(v),
\]

\[
[u - v - 1]'' B(u) B(v) = [u - v + 1]' B(v) B(u),
\]

\[
\Phi_1(u) \Psi_1^*(v) = -\Psi_1^*(v) \Phi_1(u),
\]

\[
\Phi_1(u) B(v) = B(v) \Phi_1(u),
\]

\[
\Psi_1^*(u) A(v) = A(v) \Psi_1^*(u),
\]

\[
[A(u), B(v)] = \frac{1}{(x + x^{-1})(z - w)} : \hat{A}(u) \hat{B}(v) : \left( \delta \left( \frac{x^2 w}{z} \right) + \delta \left( \frac{x^{-2} w}{z} \right) \right),
\]

\[
W(v) \Phi_1(u) = \frac{[u - v + 2'']}{[v - u + 2''] \Phi_1(u) W(v)},
\]
Appendix C. Free-field representations of $\Lambda (u_0)_{lk}$

Consider the LHS of (3.32) with $k' = k-2$.

\[
\left[ \Phi^*(u)^{l,k-2}, \Lambda (u_0)_{lk}^{l,k-2} \right] = c_1 \int_C \frac{dw}{2\pi \sqrt{-1}w} \int_C \frac{dw'}{2\pi \sqrt{-1}w'} \left[ B(v), A(v') \right] \Phi^*_l(u) \\
\times \frac{[v - u + l]^\nu}{[v - u - \frac{1}{2} - k]^\nu} \frac{[v - u - \frac{1}{2} - k]^\nu}{[v - u - \frac{3}{2} - k]^\nu},
\]

(C1)

where

\[ c_s = (-1)^s \frac{[s + 1][k - 2s][k - s - 1]}{[1][k][k + 1]}. \]

Using (B30), the integral with respect to $w'$ on (C1) can be evaluated by the substitution $w' = x^{\pm 2} w$. The result is as follows:

\[
\left[ \Phi^*(u), \Lambda (u_0)_{lk}^{l,k-2} \right] = \frac{c_1}{x^{-2} - x^2} \int_{C'} \frac{dw}{2\pi \sqrt{-1}w} \left( F \left( v - \frac{r}{2} \right) W \left( v - \frac{r}{2} \right) - F \left( v + \frac{r}{2} \right) W \left( v + \frac{r}{2} \right) \right) \Phi^*_l(u) \\
= \frac{c_1}{x^{-2} - x^2} \left( \int_{x^{-r}C'} \frac{dw}{2\pi \sqrt{-1}w} - \int_{x^{-r}C'} \frac{dw}{2\pi \sqrt{-1}w} \right) F(v) W(v) \Phi^*_l(u),
\]

(C2)

where

\[ F(v) = \frac{[v - u - \frac{r}{2} + l]^\nu}{[v - u - \frac{1}{2} - k]^\nu} \frac{[v - u - \frac{r}{2} + l]^\nu}{[v - u - \frac{3}{2} - k]^\nu}. \]

The integral with respect to $w$ on (C2) can be evaluated by the residue at $w = x^r z$ and $w = x^{3-r} z_0$. The former residue vanishes because of (B18). Thus, from (3.33), we have

\[
\frac{c_1}{x^{-2} - x^2} \left[ u_0 - u + \frac{1}{2} + l \right]^\nu \left[ u_0 - u - \frac{1}{2} \right]^\nu \frac{\partial}[0] W_-(u_0) = \Lambda (u_0)_{lk, l+2k}^{l+2k} \left[ u_0 + \Delta u - u + l + \frac{1}{2} \right]^\nu.
\]

Hence, we conclude that $\Delta u = 0$ and

\[
\Lambda (u_0)_{lk, l+2k}^{l+2k} = \frac{[l + 2]^\nu [2][k - 1][k - 2]}{[1]^\nu (x^{-2} - x^2) \delta[0][1][k + 1]} W_-(u_0).
\]

(C3)

Let us summarize the result as follows:

\[
\left[ \Phi^*(u), X \left( u_0 + \frac{1}{2} \right) \right] \Phi^*_l(u) = \frac{1}{x^{-2} - x^2} W_-(u_0) \Phi^*_l(u) \left[ u_0 - u + \frac{1}{2} + l \right]^\nu \left[ u_0 - u - \frac{1}{2} \right]^\nu \frac{\partial}[0].
\]

(C4)
Consider the LHS of (3.32) with $k' = k - 2s$.

\[
\left[ \Psi^*(u)_i^l, \Lambda (u_0)^{(k')}_{lk} \right]
\]

\[
= c_s \left( \int_{C'} \frac{dw}{2\pi \sqrt{-1}w} \prod_{j=1}^{s-1} \int_{C} \frac{dw'_j}{2\pi \sqrt{-1}w'_j} \right) \left[ B(v), A (v'_1) \cdots A (v'_{s-1}) \right] \Psi^*_1(u) \times \left[ \frac{[v' - u + l]^s}{[v' - u - 1]^s} \prod_{i<j} [v'_i - v'_j + 1] \prod_{i=1}^s [v'_i - u - \frac{1}{2} - (k - s + 1)] \right]
\]

\[
= c_s \left( \frac{s!}{s!} \int_{C'} \frac{dw}{2\pi \sqrt{-1}w} \prod_{j=1}^{s-1} \int_{C} \frac{dw'_j}{2\pi \sqrt{-1}w'_j} \right) W_-(u_0) \times \left( \frac{[u_0 - u + \frac{1}{2} + l]^s}{[u_0 - u - \frac{1}{2}]^s} \prod_{i<j} [v'_i - v'_j + 1] \prod_{i=1}^s [v'_i - u - \frac{1}{2} - (k - s + 1)] \right)
\]  

Using (C4) and the commutation relation (B32), we have

\[
\left[ \Psi^*(u)_i^l, \Lambda (u_0)^{(k')}_{lk} \right]
\]

\[
= c_s \left( \frac{s!}{s!} \int_{C'} \frac{dw}{2\pi \sqrt{-1}w} \prod_{j=1}^{s-1} \int_{C} \frac{dw'_j}{2\pi \sqrt{-1}w'_j} \right) W_-(u_0) \times \left( \frac{[u_0 - u + \frac{1}{2} + l]^s}{[u_0 - u - \frac{1}{2}]^s} \prod_{i<j} [v'_i - v'_j + 1] \prod_{i=1}^s [v'_i - u - \frac{1}{2} - (k - s + 1)] \right)
\]

From (3.33) with $\Delta u = 0$ and (C6), we obtain (3.34).

Consider the LHS of (3.36) with $k' = k - 2s$.

\[
\left[ \Psi^*(u)_i^l, \Lambda (u_0)^{(k')}_{lk} \right]
\]

\[
= c_s \left( \int_{C_a} \frac{dw_a}{2\pi \sqrt{-1}w_a} \prod_{j=1}^s \int_{C} \frac{dw'_j}{2\pi \sqrt{-1}w'_j} \right) \left[ B(v_1) B(v_2), A (v'_1) \cdots A (v'_{s}) \right] \Psi^*_1(u) \times \left[ \frac{[v_1 - u + l - 2]^s}{[v_1 - u - 1]^s} \prod_{i<j} [v'_i - v'_j + 1] \prod_{i=1}^s [v'_i - u - \frac{1}{2} - k] \right]
\]

\[
= c_s \left( \frac{s!}{s!} \int_{C_a} \frac{dw_a}{2\pi \sqrt{-1}w_a} \prod_{j=1}^s \int_{C} \frac{dw'_j}{2\pi \sqrt{-1}w'_j} \right) W_-(u_0) \times \left( \frac{[v_1 - u + l - 2]^s}{[v_1 - u - 1]^s} \prod_{i<j} [v'_i - v'_j + 1] \prod_{i=1}^s [v'_i - u - \frac{1}{2} - (k - s + 1)] \right)
\]  

(C7)
Using (C4) and the commutation relations (B32), (B34), we have

\[
\left[ \Psi^* (u)^{l-2}_{l-2}, \Lambda (u_0)^{k}_{l+k} \right] = \frac{c_s}{x^2 - x^2} \sum_{s=0}^{s-1} \frac{[2]^{l-2}_{l-2}}{[s]^{l-2}_{l-2}} \left[ u_0 - u - \frac{l}{2} + \frac{1}{2} \right]^{l-2}_{l-2} \frac{[s-k]}{\partial [0]}
\]

\[
\times \int_\mathcal{C} \frac{dw_1}{2\pi \sqrt{-1}w_1} \prod_{j=1}^{s-1} \frac{dw_j}{2\pi \sqrt{-1}w_j} B(v_1) W_-(u_0) A(v_{s-1}) \cdots A(v_1) \Psi^* (u)
\]

\[
\times \left[ \frac{v_1 - u_0 - \frac{1}{2}}{s} \right]^{l-2}_{l-2} \left[ v_1 - u + l - \frac{1}{2} \right]^{l-2}_{l-2} \left[ v_1 - u_0 - \frac{1}{2} - (k - s + 1) \right]^{l-2}_{l-2}
\]

(C8)

Using (C8) and

\[
L'' \left[ \begin{array}{ccc} l-2 & l & l \\ l & l & u_0 - u \end{array} \right] = \frac{[2]^{l-2}_{l-2}}{[l+1]^{l-2}_{l-2}} \left[ u_0 - u + \frac{1}{2} \right]^{l-2}_{l-2},
\]

\[
L'' \left[ \begin{array}{ccc} l-2 & l & l \\ l+2 & l & u_0 - u \end{array} \right] = \frac{[1]^{l-2}_{l-2}}{[l+1]^{l-2}_{l-2}} \left[ u_0 - u + \frac{1}{2} \right]^{l-2}_{l-2},
\]

the relation (3.36) reduces to

\[
\frac{[1]^{l-2}_{l-2}}{[l+2]^{l-2}_{l-2}} \Lambda (u_0)^{l-2}_{l+2k} = \frac{c_s}{x^2 - x^2} \sum_{s=0}^{s-1} \frac{[s]^{l-2}_{l-2}}{[s-1]^{l-2}_{l-2}} \left[ u_0 - u + \frac{1}{2} \right]^{l-2}_{l-2} \left[ v_1 - u + l - \frac{1}{2} \right]^{l-2}_{l-2} \left[ v_1 - u_0 + \frac{1}{2} \right]^{l-2}_{l-2} \left[ v_1 - u_1 - \frac{1}{2} \right]^{l-2}_{l-2}
\]

\[
\times B(v_1) W_-(u_0) X (u_0 - \frac{1}{2})^{s-1} = \frac{[1]^{l-2}_{l-2}}{[u_0 - u + \frac{1}{2}]^{l-2}_{l-2}} \Lambda (u_0)^{l-2}_{l+k} \Psi^* (u).
\]

(C9)

Using the commutation relation (B34) and the addition theorem

\[
\frac{[l+1]^{l-2}_{l-2}}{[1]^{l-2}_{l-2}} \left[ v_1 - u_0 - \frac{1}{2} \right]^{l-2}_{l-2} \left[ v_1 - u + l - \frac{1}{2} \right]^{l-2}_{l-2} = \frac{[1]^{l-2}_{l-2}}{[1]^{l-2}_{l-2}} \left[ u_0 - u - \frac{1}{2} \right]^{l-2}_{l-2} \left[ v_1 - u_0 + \frac{1}{2} \right]^{l-2}_{l-2} \left[ v_1 - u - \frac{1}{2} \right]^{l-2}_{l-2}
\]

\[
= \left[ u_0 - u + l + \frac{1}{2} \right]^{l-2}_{l-2} \left[ v_1 - u_0 - \frac{1}{2} + \frac{1}{2} \right]^{l-2}_{l-2} \left[ v_1 - u_0 + \frac{1}{2} + \frac{1}{2} \right]^{l-2}_{l-2},
\]

the relation (C9) reduces to (3.37).

Repeating similar procedures, we can derive the general expression (3.38).

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