On adjoint homological Selmer modules for SL$_2$-representations of knot groups

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Abstract. We introduce the adjoint homological Selmer module for an SL$_2$-representation of a knot group, which may be seen as a knot theoretic analogue of the dual adjoint Selmer module for a Galois representation. We then show finitely generated torsion-ness of our adjoint Selmer module, which are widely known as conjectures in number theory, and give some concrete examples.

Introduction

In this paper, we continue our study on the interplay between knot theory and number theory, and introduce an adjoint Selmer module for the adjoint representation of a knot group representation. To pursue the analogy more strictly, we especially consider the Fitting ideal of the adjoint Selmer module for the universal deformation of a knot group representation, which may be seen as an analogue of the algebraic $p$-adic $L$-function associated with the adjoint Selmer module for the universal deformation of a Galois representation. We then show finitely generated torsion-ness of our adjoint Selmer module, which are widely known as conjectures in number theory. Main ingredients of our proof are Euler characteristics and $\gamma$-regular representations.

Let us recall the notion of a $\gamma$-regular representation in 3-dimensional hyperbolic geometry ([Por97, p. 3.3]). For simplicity, we consider the case for a hyperbolic knot. Let $K$ be a hyperbolic knot in the 3-sphere $S^3$ and let $E_K := S^3 \setminus \text{int}(V_K)$ be the knot exterior, where $V_K$ is a tubular neighborhood of $K$. Let $G_K := \pi_1(E_K)$ be the knot group and let $D_K := \pi_1(\partial E_K)$. Let $\rho : G_K \to \text{SL}_2(\mathbb{C})$ be an irreducible representation such that $\rho |_{D_K}$ is hyperbolic or parabolic. Let $\text{Ad}(\rho)$ be the adjoint representation of $\rho$, namely, the representation space is $\text{sl}_2(\mathbb{C})$ on which $G_K$ acts by $\text{Ad}(\rho)(X) := \rho(g) X \rho(g)^{-1}$ for $g \in G_K$ and $X \in \text{sl}_2(\mathbb{C})$. For a simple closed curve $\gamma$ on $\partial E_K$, let $I_\gamma$ be the subgroup of $D_K$ generated by $\gamma$, and we assume that $\rho(\gamma) \neq \pm I$. The representation $\rho$ is said to be $\gamma$-regular if the natural homomorphism

$$\varphi_\gamma : H_1(I_\gamma; \text{Ad}(\rho)) \to H_1(G_K; \text{Ad}(\rho))$$

is surjective.

Let $X(G_K)$ be the character variety of SL$_2(\mathbb{C})$-representations of $G_K$ and $X_0(G_K)$ be the irreducible component of $X(G_K)$ containing $\chi_0 := \text{tr}(\rho_0)$, where $\rho_0$ is a lift of the holonomy representation of the complete hyperbolic structure on int($E_K$). Suppose $\chi_\rho := \text{tr}(\rho) \in X_0(G_K)$. Joan Porti [Por97] proved that when $\rho$ is $\gamma$-regular, the evaluation map

$$X_0(G_K) \to \mathbb{C}; \chi \mapsto \chi(\gamma)$$

gives an analytic isomorphism on a neighborhood of $\chi_\rho$. In particular, $\rho_0$ is $\gamma$-regular for any $\gamma(\neq 1) \in D_K$ and the above parametrization around $\chi_0$ is due to Thurston ([Thu79]).

Let $\mathbb{Q}$ be the field of all algebraic numbers, $S$ be a finite set of primes, and $R$ be the ring of $S$-integers in some finite algebraic number field. Since $X_0(G_K)$ is defined over a finite algebraic
number field ([LR10]), \( \mathbb{Q} \)-rational points are dense in \( X_0(G_K) \) and such a point corresponds to a representation \( \rho: G_K \to \text{SL}_2(R) \). For example, hyperbolic Dehn surgery points are such algebraic points. So it may be natural to ask the integral \( \gamma \)-regularity for \( \rho: G_K \to \text{SL}_2(R) \), namely, whether the natural \( R \)-homomorphism

\[
\varphi_{\gamma,R}: H_1(I_\gamma; \text{Ad}(\rho)) \to H_1(G_K; \text{Ad}(\rho))
\]

is surjective or not. It amounts to study the \( R \)-module \( \text{Coker}(\varphi_{\gamma,R}) \). It turns out, from the viewpoint of arithmetic topology ([Mor12]), that the \( R \)-module \( \text{Coker}(\varphi_{\gamma,R}) \) may be regarded as an analogue of the dual adjoint Selmer module in Iwasawa theory for \( p \)-adic Galois representations ([Gre89], [Hid00a], [Hid00b, Chapter 5], [Hid06]). See [Kit+18] for the analogy between twisted knot modules and dual Selmer modules.

We also give some concrete examples and guess a relation between the well-known topological invariants (Porti’s torsions). Further relations between the Iwasawa class formula and Iwasawa invariants are expected, and generalizations such as symmetric power representations and quantum representations are also expected.

This paper is organized as follows. In Section 1, we introduce the notion of an adjoint homological \( \gamma \)-Selmer module for any \( \text{SL}_2 \)-representation of \( G_K \) over any integral domain \( A \). In Section 2, we study its \( A \)-module structure for a Noetherian unique factorization domain \( A \) and give a non-trivial example for the figure-eight knot. In Section 3, we give an example for a lift \( \rho_0 \) of the holonomy representation of the figure eight knot where \( \rho_0 \) is not \( \mu \)-regular over the number ring \( \mathbb{Z}[1 + \sqrt{-3}] / 2 \) for a meridian \( \mu \). In Section 4, we consider the case of universal deformations, which are studied in [Kit+18], [Mor+17], and [TTU21]. Lastly, in Section 5, by using the parameters of the universal deformation and the character variety of the one-dimensional representation of knot groups, namely, the parameter \( t \) of the Alexander polynomial, we generalize the adjoint homological \( \gamma \)-Selmer module to the case of two-variable.

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1. Adjoint homological \( \gamma \)-Selmer modules

Let \( K \) be a knot in the 3-sphere \( S^3 \). Let \( V_K \) be a tubular neighborhood of \( K \) and let \( E_K := S^3 \setminus \text{int}(V_K) \) be the knot exterior. Let \( G_K := \pi_1(E_K) \) be the knot group of \( K \) and let \( D_K := \pi_1(\partial V_K) = \pi_1(\partial E_K) \). For \( \gamma \in D_K \), let \( I_\gamma \) be the subgroup of \( D_K \) generated by \( \gamma \).

Let \( A \) be an integral domain with quotient field \( Q(A) \). Let \( \rho: G_K \to \text{SL}_2(A) \) be a representation of \( G_K \). Let \( \text{Ad}(\rho) \) denote the adjoint representation of \( \rho \), namely, the representation space is the \( A \)-module \( V_\rho := \text{sl}_2(A) \) on which \( G_K \) acts by \( \text{Ad}(\rho(g))(X) := \rho(g)X\rho(g)^{-1} \) for \( g \in G_K \) and \( X \in \text{sl}_2(A) \). For \( \gamma \in D_K \), the natural homomorphism \( I_\gamma \subset D_K \to G_K \) induces the \( A \)-homomorphism of group homology groups with coefficients in \( \text{Ad}(\rho) \):

\[
\varphi_\gamma : H_1(I_\gamma, \text{Ad}(\rho)) \to H_1(G_K, \text{Ad}(\rho)).
\]

We then define the adjoint homological \( \gamma \)-Selmer module\(^1\) attached to the representation \( \rho \) by the

\(^1\)Selmer modules in number theory are defined by using cohomology groups and so our homological Selmer modules are analogues of the dual of adjoint Selmer modules in number theory (cf. [Gre89], [Hid00b], [Hid06]).
A-module

\[ \text{Sel}_h^\gamma(\Ad(\rho)) := \text{Coker}(\varphi_\gamma). \]

We say that \( \rho \) is \( \gamma \)-regular over \( A \) if \( \text{Sel}_\gamma^h(\Ad(\rho)) = \{0\} \). It is easy to see that \( \text{Sel}_\gamma^h(\Ad(\rho)) \) and \( \text{Sel}_{\gamma_2}^h(\Ad(\rho)) \) are \( A \)-isomorphic if \( \gamma_1 \) and \( \gamma_2 \) are conjugate in \( G_K \). In particular, when \( \gamma \) is a meridian of a knot \( K \), we simply write \( \text{Sel}_h^h(\Ad(\rho)) \) for \( \text{Sel}_\gamma^h(\Ad(\rho)) \) and call it the adjoint homological Selmer module attached to \( \rho \).

Let \( \rho_{Q(A)} : G_K \to \SL_2(Q(A)) \) be the representation obtained from \( \rho \) by the extension of scalars.

**Lemma 1.1.** Suppose that \( \text{Sel}_\gamma^h(\Ad(\rho_{Q(A)})) = 0 \). Then \( \text{Sel}_\gamma^h(\Ad(\rho)) \) is a finitely generated torsion \( A \)-module.

**Proof.** Since \( G_K \) is a finitely presented group and \( \Ad(\rho) \) is a free \( A \)-module of finite rank, \( H_1(G_K, \Ad(\rho)) \) is a finitely generated \( A \)-module and hence \( \text{Sel}_\gamma^h(\Ad(\rho)) \) is so. So it suffices to show that \( \text{Sel}_\gamma^h(\Ad(\rho)) \) is a torsion \( A \)-module. Let \( \text{tor}(\text{Sel}_\gamma^h(\Ad(\rho))) \) denote the torsion \( A \)-submodule of \( \text{Sel}_\gamma^h(\Ad(\rho)) \). We note

\[ \text{tor}(\text{Sel}_\gamma^h(\Ad(\rho))) = \text{Ker}(\text{Sel}_\gamma^h(\Ad(\rho)) \to \text{Sel}_\gamma^h(\Ad(\rho)) \otimes_A Q(A)). \]

By the definition of \( \text{Sel}_\gamma^h(\Ad(\rho)) \), we have the exact sequence of \( A \)-modules

\[ H_1(I_\gamma, \Ad(\rho)) \xrightarrow{\iota_\gamma} H_1(G_K, \Ad(\rho)) \to \text{Sel}_\gamma^h(\Ad(\rho)) \to 0. \]

Since \( Q(A) \) is a flat \( A \)-module and \( H_1(X, \Ad(\rho_{Q(A)})) = H_1(X, \Ad(\rho)) \otimes_A Q(A) \) for \( X = I_\gamma, G_K \), tensoring \( Q(A) \) with (1.1.2) over \( A \) and the assumption yield

\[ \text{Sel}_\gamma^h(\Ad(\rho)) \otimes_A Q(A) = \text{Sel}_\gamma^h(\Ad(\rho_{Q(A)})) = 0. \]

By (1.1.1) and (1.1.3), we have \( \text{tor}(\text{Sel}_\gamma^h(\Ad(\rho))) = \text{Sel}_\gamma^h(\Ad(\rho)) \), and hence the assertion follows. \( \square \)

## 2. Presentations of adjoint homological Selmer modules

We keep the same notations as in Section 1. In this Section, we assume that \( A \) is a Noetherian unique factorization domain or a formal power series over a field. We study the \( A \)-module structure of the adjoint homological Selmer module \( \text{Sel}_\gamma^h(\Ad(\rho)) \).

We take a Wirtinger presentation of \( G_K \):

\[ G_K = \langle g_1, \ldots, g_n \mid r_1 = \cdots = r_{n-1} = 1 \rangle, \]

where \( g_1, \ldots, g_n \) \((n \geq 2)\) represent meridians of a knot \( K \), and let \( \rho : G_K \to \SL_2(A) \) be a representation. Let \( F \) be the free group on the words \( g_1, \ldots, g_n \) and let \( \pi : A[F] \to A[G_K] \) be the natural homomorphism of group rings. We write the same \( g_i \) for the image of \( g_i \) in \( G_K \). Let \( V_\rho \) be the representation space

\[ V_\rho := \sl_2(A) = Av_1 \oplus Av_2 \oplus Av_3, \]

where

\[ v_1 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ v_2 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ v_3 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \]
In the following, taking $\gamma = g_1$, we shall compute a presentation matrix of $\text{Sel}^h(\text{Ad}(\rho)) = \text{Sel}^h(\text{Ad}(\rho))$ over $A$, under some (mild) assumptions.

First, we compute $H_1(I_\gamma, \text{Ad}(\rho))$. Let $W_1$ be the CW complex attached to the presentation $\langle g_1 | - \rangle$, which is homotopically equivalent to the circle $S^1$. We consider the chain complex $C_*(W_1, V_\rho)$:

$$0 \rightarrow C_1(W_1, V_\rho) \xrightarrow{d_1} C_0(W_1, V_\rho) \rightarrow 0,$$

defined by $C_1(W_1, V_\rho) := V_\rho$, $C_0(W_1, V_\rho) := V_\rho$ and $d_1 := \text{Ad}(\rho)(g_1) - I$. So we have

$$H_1(I_\gamma, \text{Ad}(\rho)) = H_1(W_1, V_\rho) = \text{Ker}(d_1) = \{ v \in V_\rho \mid \rho(g_1)v = v\rho(g_1) \}.$$  

Next, we compute $H_1(G_K, \text{Ad}(\rho))$. Let $W_2$ be the CW complex attached to the presentation (2.1). We note that the knot exterior $E_K$ and the CW complex $W_2$ are homotopically equivalent by Whitehead’s theorem because they are both the Eilenberg-MacLane space $K(G_K, 1)$. We consider the chain complex $C_*(W_2, V_\rho)$:

$$0 \rightarrow C_2(W_2, V_\rho) \xrightarrow{\partial_2} C_1(W_2, V_\rho) \xrightarrow{\partial_1} C_0(W_2, V_\rho) \rightarrow 0,$$

defined by

$$\left\{ \begin{array}{l}
C_2(W_2, V_\rho) := (V_\rho)^{(n-1)}, \\
C_1(W_2, V_\rho) := (V_\rho)^{\oplus n}, \\
C_0(W_2, V_\rho) := V_\rho,
\end{array} \right.$$

where $\partial_2 : A[F] \rightarrow A[F]$ denotes the Fox derivative over $A$, extended from $\mathbb{Z}$ ([Fox53]), and $\partial_2$ is regarded as a (big) $(n-1) \times n$ matrix whose $(i, j)$-entry is the $3 \times 3$ matrix $\text{Ad}(\rho) \circ \pi \left( \frac{\partial_{r_j}}{\partial g_i} \right)$. So we have

$$H_1(G_K, \text{Ad}(\rho)) = H_1(W_2, V_\rho) = \text{Ker}(\partial_1)/\text{Im}(\partial_2).$$

By (2.2), (2.3) and the definition of $\varphi_\gamma : H_1(I_\gamma, \text{Ad}(\rho)) \rightarrow H_1(G_K, \text{Ad}(\rho))$, we can regard $\varphi_\gamma$ as the composition of an inclusion map

$$\text{Ker}(d_1) \hookrightarrow \text{Ker}(\partial_1); \quad v \mapsto \left( \begin{array}{c} v \\ 0 \\ \vdots \\ 0 \end{array} \right)$$

and the natural $A$-homomorphism $\text{Ker}(\partial_1) \rightarrow \text{Ker}(\partial_1)/\text{Im}(\partial_2)$. So we have the isomorphism of $A$-modules

$$\text{Sel}^h(\text{Ad}(\rho)) = \text{Ker}(\partial_1)/(\text{Im}(\partial_2) + \text{Ker}(d_1)),$$

and the exact sequence of $A$-modules:

$$0 \rightarrow \text{Sel}^h(\text{Ad}(\rho)) \rightarrow V_\rho^{\oplus n}/(\text{Im}(\partial_2) + \text{Ker}(d_1)) \rightarrow V_\rho^{\oplus n}/\text{Ker}(\partial_1) \rightarrow 0.$$  

In the following, we make the assumptions

$$\left\{ \begin{array}{l}
(\text{A1}) \text{Ker}(d_1) = Av_0 \text{ for some } v_0 \in \text{Ker}(d_1), \\
(\text{A2}) \partial_1 \text{ induces the } A\text{-isomorphism } V_\rho^{\oplus n}/\text{Ker}(\partial_1) \cong V_\rho,
\end{array} \right.$$
Let \( f : V_\rho \rightarrow V_\rho \) be the \( A \)-homomorphism defined by
\[
f(v_1) = v_0, \ f(v_2) = O, \ f(v_3) = O \ (O: \text{zero matrix}).
\]
Then we have
\[
\Ker(d_1) = \Im(f).
\]
Let \( \delta : V_\rho^\oplus n = V_\rho^\oplus (n-1) \oplus V_\rho \rightarrow V_\rho^\oplus n \) be the \( A \)-homomorphism defined by
\[
\delta \begin{pmatrix} v \\ w \end{pmatrix} = \partial_2(v) + \begin{pmatrix} f(w) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
\]
Then we have
\[
\Im(\delta) = \Im(\partial_2) + \Ker(d_1).
\]
By (2.4) and (A2), we have the exact sequence of \( A \)-modules
\[
0 \rightarrow \Sel^b(\Ad(\rho)) \rightarrow V_\rho^\oplus n / \Im(\delta) \rightarrow V_\rho \rightarrow 0.
\]
We consider the \( 3n \)-basis of \( V_\rho^\oplus n \)
\[
\mathbf{v}^1 := \begin{pmatrix} v_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \mathbf{v}^1 := \begin{pmatrix} v_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \mathbf{v}^1 := \begin{pmatrix} v_3 \\ 0 \\ \vdots \\ 0 \end{pmatrix},
\]
\[
\mathbf{v}^2 := \begin{pmatrix} 0 \\ v_1 \\ \vdots \\ 0 \end{pmatrix}, \ \mathbf{v}^2 := \begin{pmatrix} 0 \\ v_2 \\ \vdots \\ 0 \end{pmatrix}, \ \mathbf{v}^2 := \begin{pmatrix} 0 \\ v_3 \\ \vdots \\ 0 \end{pmatrix},
\]
\[
\vdots
\]
\[
\mathbf{v}^n := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v_1 \end{pmatrix}, \ \mathbf{v}^n := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v_2 \end{pmatrix}, \ \mathbf{v}^n := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v_3 \end{pmatrix}.
\]
Then the presentation matrix \( D \) of \( \delta \) with respect to the basis \( \{\mathbf{v}^j_h\} \) \((1 \leq h \leq 3, 1 \leq j \leq n)\) is given by the following form:
\[
D = \begin{pmatrix}
A_{1,1} & A_{2,1} & \cdots & A_{n,1} \\
A_{1,2} & A_{2,2} & \cdots & A_{n,2} \\
\vdots & \vdots & \ddots & \vdots \\
A_{1,n-1} & A_{2,n-1} & \cdots & A_{n,n-1} \\
B & O & \cdots & O
\end{pmatrix}.
\]
Here, $A_{ji}$ is the presentation matrix of $\text{Ad}(\rho(\partial r_i/\partial g_j))$: $V_\rho \to V_\rho$, with respect to the basis $\{v_1, v_2, v_3\}$ and $B$ is the presentation matrix of $f : V_\rho \to V_\rho$, with respect to the basis $\{v_1, v_2, v_3\}$.

Summing up the above discussion, we have the following theorem.

**Theorem 2.7.** Let the notations and assumptions be as above. We have the exact sequence of $A$-modules
\[ 0 \to \text{Sel}^b(\text{Ad}(\rho)) \to V_\rho^\oplus \to \text{Im}(\delta) \to \text{ker}(\partial_1) \to V_\rho \to 0, \]
where a presentation matrix of $\delta : V_\rho^\oplus \to V_\rho^\oplus$ is given by the matrix $D$ in (2.6), and for an integer $d \geq 0$, the $d$-th Fitting ideal of $\text{Sel}^b(\text{Ad}(\rho))$ over $A$ is generated by the greatest common divisor of all $(d + 3)$-minors of $D$.

**Proof.** The latter assertion follows from that $V_\rho$ is a free $A$-module of rank 3 (cf. [Kaw12, p. 7.2]).

**Example 2.8.** Let $K$ be the figure-eight knot, whose knot group is given by
\[ G_K = \langle g_1, g_2 \mid g_1g_2g_1^{-1}g_2g_1 = g_2g_1g_2^{-1}g_1^{-1}g_2 \rangle. \]

We consider the case of $\gamma$ being the meridian $\mu = g_1$. Let $\rho_A : G_K \to \text{SL}_2(A)$ be the representation given by the following (cf. [Kit+18, Section 4], [Ril84]):
\[ \rho_A(g_1) = \begin{pmatrix} x + \sqrt{x^2 - 1} & 1 \\ 0 & x - \sqrt{x^2 - 1} \end{pmatrix}, \]
\[ \rho_A(g_2) = \begin{pmatrix} x + \sqrt{x^2 - 1} & 0 \\ -\{(x^2 - y(x) - 2) & x - \sqrt{x^2 - 1} \end{pmatrix}, \]
where $A = \mathbb{C}[\sqrt{x - 1}]$, and $y(x) = \frac{x^2 + 1 + \sqrt{(x^2 - 1)(x^2 - 5)}}{2}$. We can rewrite these matrices as
\[ \rho_A(g_1) = \begin{pmatrix} (s^2 + 1) + \sqrt{(s^2 + 1)^2 - 4} & 1 \\ 0 & (s^2 + 1) - \sqrt{(s^2 + 1)^2 - 4} \end{pmatrix}, \]
\[ \rho_A(g_2) = \begin{pmatrix} (s^2 + 1) + \sqrt{(s^2 + 1)^2 - 4} & 0 \\ -\{(s^2 + 1)^2 - y(s^2 + 1) - 2) & (s^2 + 1) - \sqrt{(s^2 + 1)^2 - 4} \end{pmatrix}, \]
where we use $s := \sqrt{x - 1}$. Since
\[ \text{ker}(\partial_1) = A \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus A \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \simeq A^\oplus 3, \]
we have $V_{\rho_A}^\oplus / \text{ker}(\partial_1) \simeq V_{\rho_A}$, and so the assumption (A2) is satisfied. Now, let
\[ P_\rho := \frac{(s^2 + 1)^2 - 4}{2} \begin{pmatrix} \sqrt{(s^2 + 1)^2 - 4} & 1 \\ 0 & \sqrt{(s^2 + 1)^2 - 4} \end{pmatrix} \in \mathfrak{sl}_2(A). \]
Since
\[ H_1(I_\mu, \text{Ad}(\rho_A)) = A \cdot P_\rho, \]
we can take \( v_0 \) as \( P_\rho \) and so the assumption (A1) is satisfied.

Next, let us see the presentation matrix \( D \) in (2.6):
\[
D = \begin{pmatrix}
\text{Ad}(\rho_A) \left( \frac{\partial r}{\partial g_1} \right) & \text{Ad}(\rho_A) \left( \frac{\partial r}{\partial g_2} \right) \\
1 & \frac{\sqrt{(s^2+1)^2-4}}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Then the greatest common divisor \( L_\mu(x) \in \mathbb{C}[s] \) of all 3-minors of \( D \) is given by
\[ L_\mu(x) = \sqrt{\{(s^2+1)^2 - 1\}\{(s^2+1)^2 - 5\}} \equiv s \in \mathbb{C}[s], \]
and so \( V_{\rho_A}^{\oplus 2}/\text{Im}(\partial_1) \simeq A^{\oplus 3} \oplus A/sA \). Hence by Theorem 2.7, we have
\[ \text{Sel}_\mu^h(\text{Ad}(\rho_A)) \simeq A/sA \simeq \mathbb{C}. \]

**Example 2.9.** Let \( K \) be the figure-eight knot. Next, we consider the case of \( \gamma \) being the preferred longitude \( \lambda \) corresponding to \( \mu = g_1 \). Let \( \rho_A : G_K \to \text{SL}_2(A) \) be the representation given by the following:
\[
\rho_A(g_1) = \begin{pmatrix}
x + \sqrt{x^2-1} \\
x - \sqrt{x^2-1}
\end{pmatrix},
\]
\[
\rho_A(g_2) = \begin{pmatrix}
x + \sqrt{x^2-1} \\
-(x^2 - y(x) - 2) & x - \sqrt{x^2-1}
\end{pmatrix},
\]
where \( A = \mathbb{C} \left[\left[ x - \sqrt{\frac{5}{2}} \right] \right] \), and \( y(x) = \frac{x^2+1+\sqrt{(x^2-1)(x^2-5)}}{2} \). We can rewrite these matrices as
\[
\rho_A(g_1) = \begin{pmatrix}
x + \sqrt{x^2-1} \\
x - \sqrt{x^2-1}
\end{pmatrix},
\]
\[
\rho_A(g_2) = \begin{pmatrix}
x + \sqrt{x^2-1} \\
-(x^2 - y(x) - 2) & x - \sqrt{x^2-1}
\end{pmatrix},
\]
where we use \( s := x - \sqrt{\frac{5}{2}} \). Similarly as Example 2.8, since \( \text{Ker}(\partial_1) \simeq A^{\oplus 3} \), we have \( V_{\rho_A}^{\oplus 2}/\text{Ker}(\partial_1) \simeq V_{\rho_A} \) and so the assumption (A2) is satisfied.

Now, let
\[
P_\rho := \begin{pmatrix}
\sqrt{(s+\sqrt{\frac{5}{2}})^2-4} \\
0
\end{pmatrix} \in \text{sl}_2(A),
\]
where
and let $\lambda$ be the longitude with respect to $\mu = g_1$, and $h_{\rho}^{(1)}(\tilde{\lambda})$ the reference generator of $H^1(G_K, \text{Ad}(\rho_A))$ with respect to a lift $\tilde{\lambda}$ in the universal covering of $\lambda$. Then we have

$$T_\lambda \cdot h_{\rho}^{(1)}(\tilde{\lambda})(\tilde{\mu}) = T_\mu \cdot h_{\rho}^{(1)}(\tilde{\mu})(\tilde{\mu}),$$

(cf. [Por97, Proposition 3.18] for cohomology), where $T_\mu := \frac{1}{2} \left\lfloor \left( s + \sqrt{\frac{5}{2}} \right)^2 - 1 \right\rfloor \left\lfloor \left( s + \sqrt{\frac{5}{2}} \right)^2 - 5 \right\rfloor$, $T_\lambda := 5 - 2 \left( s + \sqrt{\frac{5}{2}} \right)^2 \in A$, and by [Por97, p.69]

$$h_{\rho}^{(1)}(\tilde{\mu})(\tilde{\mu}) = \left( \begin{array}{cc}
\frac{\partial M(s)}{\partial s} & \frac{1}{M(s)} \\
0 & -\frac{1}{M(s)}
\end{array} \right) = \left( \begin{array}{cc}
\sqrt{\left(s+\sqrt{\frac{5}{2}}\right)^2-4} & 0 \\
0 & \frac{1}{\sqrt{\left(s+\sqrt{\frac{5}{2}}\right)^2-4}}
\end{array} \right) \in \text{sl}_2(A),$$

where $M(s) = \frac{(s+\sqrt{\frac{5}{2}})+\sqrt{(s+\sqrt{\frac{5}{2}})^2-4}}{2} \in A$ is the positive eigenvalue of $\rho_A(\mu)$. So we have

$$\langle h_{\rho}^{(1)}(\tilde{\lambda}), P_{\rho} \otimes \tilde{\mu} \rangle = \frac{T_\mu}{T_\lambda} \cdot \langle h_{\rho}^{(1)}(\tilde{\mu}), P_{\rho} \otimes \tilde{\mu} \rangle = \frac{T_\mu}{T_\lambda}.$$ 

Hence, it is natural to take $v_0$ as $\frac{T_\lambda}{T_\mu} \cdot P_{\rho}$.

Next, let us see the presentation matrix

$$\left( \begin{array}{cc}
\text{Ad}(\rho_A) \left( \frac{\partial r}{\partial g_1} \right) & \text{Ad}(\rho_A) \left( \frac{\partial r}{\partial g_2} \right) \\
\frac{T_\lambda}{T_\mu} \cdot \frac{T_\lambda}{T_\mu} \cdot \sqrt{\left(s+\sqrt{\frac{5}{2}}\right)^2-4} & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{array} \right).$$

Then the greatest common divisor $L_\lambda(x) \in \mathbb{C}[[s]]$ of all 3-minors of this matrix is given by

$$L_\lambda(x) = T_\lambda = s \in \mathbb{C}[[s]],$$

and so $V_{\rho_A}^h / \text{Im}(\delta) \simeq A^3 \oplus A/sA$. Hence by Theorem 2.7, we have

$$\text{Sel}_h^h(\text{Ad}(\rho_A)) \simeq A/sA \simeq \mathbb{C}.$$ 

**Example 2.10.** Let $K$ be the knot $5_2$, whose knot group is given by

$$G_K = \langle g_1, g_2 \mid g_1 g_2 g_1^{-1} g_2^{-1} g_1 g_2 g_1 = g_2 g_1 g_2 g_1^{-1} g_2^{-1} g_1 g_2 \rangle.$$ 

Let $\rho_A : G_K \rightarrow \text{SL}_2(A)$ be the representation given by the following:

$$\rho_A(g_1) = \left( \begin{array}{cc}
\frac{x+\sqrt{x^2-4}}{2} & \frac{1}{2} \\
0 & \frac{x-\sqrt{x^2-4}}{2}
\end{array} \right),$$

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where \( A = \mathbb{C}[[x - \beta]] \), \( \beta = 2.546 \cdots \in \mathbb{C} \) is the simple root satisfying the equation
\[
\beta^4 + 2\beta^3 - 5\beta^2 - 14\beta - 7 = 0,
\]
and \( y = y(x) \in A \) is the unique solution satisfying the equation
\[
y^3 - (x^2 + 1)y^2 + (3x^2 - 2)y - 2x^2 + 1 = 0
\]
and
\[
y(\beta) = \xi,
\]
where \( \xi = 3.132 \cdots \in \mathbb{C} \) is the simple root satisfying the equation
\[
\xi^4 - 6\xi^3 + 11\xi^2 - 6\xi - 1 = 0.
\]

We can rewrite these matrices as
\[
\rho_A(g_1) = \begin{pmatrix}
\frac{(s^2 + \beta) + \sqrt{(s^2 + \beta)^2 - 1}}{2} & \frac{1}{2} \\
0 & \frac{(s^2 + \beta) - \sqrt{(s^2 + \beta)^2 - 1}}{2}
\end{pmatrix},
\]
\[
\rho_A(g_2) = \begin{pmatrix}
\frac{(s^2 + \beta) + \sqrt{(s^2 + \beta)^2 - 1}}{2} & 0 \\
-\{(s^2 + \beta)^2 - y(s^2 + \beta) - 2\} & \frac{(s^2 + \beta) - \sqrt{(s^2 + \beta)^2 - 1}}{2}
\end{pmatrix},
\]

where we use \( s := \sqrt{x - \beta} \). Similarly as Example 2.8, since \( \text{Ker}(\partial_1) \simeq A^{\oplus 3} \), we have \( V_{\rho_A}^{\oplus 2}/\text{Ker}(\partial_1) \simeq V_{\rho_A} \) and so the assumption (A2) is satisfied. Now, let
\[
P_\rho := \begin{pmatrix}
\frac{\sqrt{(s^2 + \beta)^2 - 1}}{2} \\
0 & \frac{1}{2} - \frac{\sqrt{(s^2 + \beta)^2 - 1}}{2}
\end{pmatrix} \in \text{sl}_2(A),
\]
Since
\[
H_1(I_\mu, \text{Ad}(\rho_A)) = A \cdot P_\rho,
\]
we can take \( v_0 \) as \( P_\rho \) and so the assumption (A1) is satisfied. Then the greatest common divisor \( L_\mu(s) \in \mathbb{C}[[s]] \) of all 3-minors of \( D \) associated with the meridian is given by
\[
L_\mu(x) = \sqrt{(s^2 + \beta)^4 - \beta^4} \equiv s \in \mathbb{C}[[s]].
\]
and by similar argument as Example 2.9, the greatest common divisor \( L_\lambda(s) \in \mathbb{C}[[s]] \) of all 3-minors of \( D \) associated with the longitude is given by
\[
L_\lambda(s) = 5(s^2 + \beta)^4 y - 10(s^2 + \beta)^4 - 5(s^2 + \beta)^2 y^2 - 7(s^2 + \beta)^2 y + 31(s^2 + \beta)^2 + 7y^2 - 7y - 21 \equiv 1 \in \mathbb{C}[[s]].
\]
Hence by Theorem 2.7, we have
\[
\text{Sel}_\mu^h(\text{Ad}(\rho_A)) \simeq A/sA \simeq \mathbb{C}, \quad \text{Sel}_\lambda^h(\text{Ad}(\rho_A)) = 0.
\]
3. Adjoint homological Selmer modules for holonomy representations

Let $K$ be a hyperbolic knot in $S^3$ and let $\text{hol}: G_K \to \text{PSL}_2(\mathbb{C})$ be the holonomy representation attached to the complete hyperbolic structure on $\text{int}(E_K)$. It is known that $\text{hol}$ can be lifted to a representation $\rho_0: G_K \to \text{SL}_2(\mathcal{O}_{F_0,S_0})$, where $\mathcal{O}_{F_0,S_0}$ denotes the ring of $S_0$-integers of a finite algebraic number field $F_0$ and a finite set $S_0$ of finite primes of $F$, and that $\rho_0$ is irreducible over $\mathbb{C}$ ([CS83, p. 3], [MR03, p. 3.2]).

Let $X(G_K)$ be the character variety of $\text{SL}_2(\mathbb{C})$-representations of $G_K$ and let $X_0(G_K)$ be the irreducible component of $X(G_K)$ containing $\chi_0 := \text{tr}(\rho_0)$. Since $X_0(G_K)$ is defined over a finite algebraic number field ([LR10]), $\mathbb{Q}$-rational points are dense in $X(G_K)$, where $\mathbb{Q}$ is the field of all algebraic numbers. For example, Dehn surgery points, which yield closed hyperbolic 3-manifolds, are such algebraic points. Let $\chi_{\rho}$ be a $\mathbb{Q}$-rational point in $X_0(G_K)$ which corresponds to, up to conjugation, a representation $\rho_{O,F,S}: G_K \to \text{SL}_2(\mathcal{O}_{F,S})$, $\chi_{\rho_{O,F,S}} = \text{tr}(\rho_{O,F,S})$, where $\mathcal{O}_{F,S}$ is the ring of $S$-integers of a finite algebraic number field $F$ and $S$ is a finite set of primes of $F$. We write $\rho_{\mathcal{C}}$ for the scalar extension of $\rho_{O,F,S}$ to $\mathbb{C}$. We assume that $\rho_{\mathcal{C}}$ is irreducible.

The following theorem is shown by Porti:

**Theorem 3.1** ([Por97, p. 3.3]). Let the notations and assumptions be as above. The following assertions hold.

1. Suppose that $\rho_{O,F,S}|_{D_K}$ is hyperbolic. Then $\rho_{\mathcal{C}}$ is $\gamma$-regular, namely, $\text{Sel}_\gamma^h(\text{Ad}(\rho_{\mathcal{C}})) = \{0\}$ for some $\gamma \in D_K$ if and only if $\chi_{\rho_{O,F,S}}$ is a smooth and reduced point of $X_0(G_K)$.

2. Suppose that $\rho_{O,F,S}|_{D_K}$ is parabolic. Let $\gamma \in D_K$ with $\rho_{O,F,S}(\gamma) \neq \pm 1$. Then $\rho_{\mathcal{C}}$ is $\gamma$-regular if and only if $\chi_{\rho_{O,F,S}}$ is a reduced point of $X_0(G_K)$. In particular, $(\rho_0)_{\mathcal{C}}$ is $\gamma$-regular for any $\gamma \neq 1$.

In these cases, the evaluation map

$$X_0(G_K) \to \mathbb{C}; \chi \mapsto \chi(\gamma)$$

is an analytic isomorphism on a neighborhood of $\chi_{\rho_{O,F,S}}$.

**Theorem 3.2.** Let $\rho_{O,F,S}: G_K \to \text{SL}_2(\mathcal{O}_{F,S})$ be a representation, and assume that the scalar extension $\rho_{\mathcal{C}}: G_K \to \text{SL}_2(\mathbb{C})$ of $\rho_{O,F,S}$ is a $\gamma$-regular irreducible representation. Then $\text{Sel}_\gamma^h(\text{Ad}(\rho_{O,F,S}))$ is a finitely generated torsion $\mathcal{O}_{F,S}$-module. In particular, $\text{Sel}_\gamma^h(\text{Ad}(\rho_{\mathcal{C}}))$ is a finitely generated torsion $\mathcal{O}_{F_0,S_0}$-module for any $\gamma \neq 1$.

**Proof.** Let $\rho_F: G_K \to \text{SL}_2(F)$ be the representation obtained from $\rho_{O,F,S}$ by the extension of scalar to $F$. Since $\text{Sel}_\gamma^h(\text{Ad}(\rho_F))$ is an $F$-vector space and $\text{Sel}_\gamma^h(\text{Ad}(\rho_F)) \otimes_F \mathbb{C} = \text{Sel}_\gamma^h(\text{Ad}(\rho_{\mathcal{C}})) = 0$, $\text{Sel}_\gamma^h(\text{Ad}(\rho_F)) = 0$. By Lemma 1.1, $\text{Sel}_\gamma^h(\text{Ad}(\rho_{O,F,S}))$ is a finitely generated torsion $\mathcal{O}_{F,S}$-module. $\Box$

**Example 3.3.** Let $K$ be the figure-eight knot, whose knot group is given by

$$G_K = \langle g_1, g_2 \mid g_1 g_2^{-1} g_1^{-1} g_2 g_1 = g_2 g_1 g_2^{-1} g_1^{-1} g_2 \rangle.$$  

It is known that a lifting of the holonomy representation attached to the complete hyperbolic structure on $S^3 \setminus K$ is given, up to conjugation, by the following ([Ril84]):

$$\rho_0 : G_K \to \text{SL}_2(\mathcal{O}_{F_0}); \quad \rho_0(g_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho_0(g_2) = \begin{pmatrix} 1 & \sqrt{10} \\ 2 & 1 \end{pmatrix}.$$
where \( O_{F_0} = \mathbb{Z}[\frac{1+\sqrt{-3}}{2}] \) is the ring of integers of \( F_0 = \mathbb{Q}(\sqrt{-3}) \).

Since \( g_1 \neq 1 \) in \( D_K \), by Theorem 3.2, \( \text{Sel}^h(\text{Ad}(\rho_0)) \) is a finitely generated torsion \( O_{F_0} \)-module. It is easy to see (by putting \( x = 2 \) in Example 2.8) that assumptions (A1) and (A2) are satisfied, and by the straightforward computation and Theorem 2.7, we have

\[
\text{Sel}^h(\text{Ad}(\rho_0)) \simeq O_{F_0}/\sqrt{-3} O_{F_0}.
\]

**Remark 3.4.** In view of the analogy with number theory (Iwasawa theory), our homological Selmer modules are expected to have some relations with Reidemeister torsions associated with adjoint representations. In [Por97, Chapter 4], Porti introduced the torsion function \( T_{(E_K, \mu)} \) on \( X_0(G_K) \). According to his concrete computation worked out for the figure eight knot, we have

\[
T_{(E_K, \mu)}(\rho_0) = (\pm 1/2 \sqrt{(x^2-1)(x^2-5)})|_{x=2} = \pm \sqrt{-3}/2.
\]

See also [Por18, p. 5.2.1]. Similar result holds for knot 5_2. It would be interesting to pursue connections between our homological Selmer modules and Porti’s torsions.

### 4. Adjoint homological Selmer modules for universal deformations

We keep the same notations and assumptions as before. In this Section, we focus on the case of universal deformations, which is introduced in [Mor+17] and [Kit+18]. For this reason, we first consider adjoint homological Selmer modules for residual representations.

**Proposition 4.1.** There are infinitely many maximal ideal \( p \) of \( O_{F,S} \) such that \( \overline{\rho} := \rho_{O_{F,S}} \mod p : G_K \to \text{SL}_2(\mathbb{F}_p) \) is an absolutely irreducible representation, where \( \mathbb{F}_p := O_F/p \), and \( \text{Sel}_{\gamma}(\text{Ad}(\overline{\rho})) = 0 \).

**Proof.** Let \( p \) be any maximal ideal of \( O_{F,S} \). Assume that \( \overline{\rho} := \rho_{O_{F,S}} \mod p : G_K \to \text{SL}_2(\mathbb{F}_p) \) is not absolutely irreducible. Then there is a finite extension \( F' \) of \( F \) and a maximal ideal \( p \) of the ring of integers \( O_{F'} \) of \( F' \) lying over \( p \) such that the representation \( \overline{\rho}' \) obtained from \( \overline{\rho} \) by the scalar extension to \( \mathbb{F}_p := O_{F'/p} \) is not an irreducible representation of \( G_K \). It suffices to show that the representation \( \rho_{F'} \) obtained from \( \rho_F \) by the scalar extension to \( F' \) is not irreducible, for it contradicts to the irreducibility of \( \rho \). Since \( \overline{\rho}' \) is assumed to be reducible, there is a non-trivial \( G_K \)-invariant subspace \( V \) of \( \mathbb{F}_p^{\oplus 2} \), namely, \( \dim_{\mathbb{F}_p} V = 1 \). Let \( \Lambda \) be the inverse image of \( V \) under the natural homomorphism \( O_{F'}^{\oplus 2} \to \mathbb{F}_p^{\oplus 2} \) and let \( W := \Lambda \otimes_{O_{F'}} F' \). It is easy to see that \( W \) is a subrepresentation of \( \rho_{F'} \) over \( F' \). Let \( \Lambda_p := \Lambda \otimes_{O_{F'}} O_p \) and \( V_p := \Lambda_p \otimes_{O_p} F_p' \), where \( O_p \) and \( F_p' \) are \( p \)-adic completions of \( O_{F'} \) and \( F' \) at \( p \), respectively. Then \( \Lambda_p \) is a free \( O_p \)-submodule of \( O_p^{\oplus 2} \) and we have

\[
\dim_{\mathbb{F}_p} W = \dim_{\mathbb{F}_p} V_p = \text{rank}_{O_p} \Lambda_p = \dim_{\mathbb{F}_p} V = 1.
\]

Therefore \( W \) is a non-trivial subrepresentation of \( \rho_{F'} \). Hence we showed that \( \overline{\rho} := \rho_{O_{F,S}} \mod p \) is absolutely irreducible for any maximal ideal \( p \) of \( O_{F,S} \).

On the other hand, by Theorem 3.2, there are only finitely many maximal ideal \( p \) of \( O_{F,S} \) such that \( \text{Sel}_{\gamma}(\text{Ad}(\rho_{O_{F,S}})) \otimes_{O_{F,S}} \mathbb{F}_p \neq 0 \). Take a sufficiently large finite set \( T \) of finite primes of \( F \) containing \( S \) such that \( O_{F,T} \) is a principal ideal domain and \( \text{Sel}_{\gamma}(\text{Ad}(\rho_{O_{F,T}})) \otimes_{O_{F,T}} \mathbb{F}_p = 0 \) for any maximal ideal \( p \) outside \( T \). We note that maximal ideals of \( O_{F,T} \) correspond bijectively to maximal ideals of \( O_{F,S} \) outside \( T \). Let \( \rho_{O_{F,T}} \) be the representation obtained from \( \rho_{O_{F,S}} \) by the scalar extension to \( O_{F,T} \). Let \( G \) denote \( I_K \) or \( G_K \). Let \( p \) be any maximal ideal of \( O_{F,T} \) and \( \mathbb{F}_p := O_{F,T}/p \). Consider the chain complex of \( O_{F,T} \)-modules

\[
(4.1.1) \quad C_2(G, \text{Ad}(\rho_{O_{F,T}})) \xrightarrow{\partial_2} C_1(G, \text{Ad}(\rho_{O_{F,T}})) \xrightarrow{\partial_1} C_0(G, \text{Ad}(\rho_{O_{F,T}}))
\]
and the chain complex of $\mathbb{F}_p$-modules

\[
(4.1.2) \quad C_2(G, \text{Ad}(\overline{\rho})) \xrightarrow{\partial_2} C_1(G, \text{Ad}(\overline{\rho})) \xrightarrow{\partial_1} C_0(G, \text{Ad}(\overline{\rho})) ,
\]

where $\overline{\rho} := \rho_{\mathcal{O}_{F,T}} \mod \mathfrak{p}$ and $\overline{\partial}_i := \partial_i \mod \mathfrak{p}$. From (4.1.1) we have the exact sequence

\[0 \to \text{Im}(\partial_2) \to \text{Ker}(\partial_1) \to H_1(G, \text{Ad}(\rho_{\mathcal{O}_{F,T}})) \to 0.\]

By taking the tensor product over $\mathcal{O}_{F,T}$ with $\mathbb{F}_p$, we have the exact sequence

\[\text{Im}(\partial_2) \otimes_{\mathcal{O}_{F,T}} \mathbb{F}_p \to \text{Ker}(\partial_1) \otimes_{\mathcal{O}_{F,T}} \mathbb{F}_p \to H_1(G, \text{Ad}(\rho_{\mathcal{O}_{F,T}})) \otimes_{\mathcal{O}_{F,T}} \mathbb{F}_p \to 0.\]

Since $\mathcal{O}_{F,T}$ is a principal ideal domain, $\text{Im}(\partial_1)$ and $\text{Ker}(\partial_1)$ are free and so we have

\[\text{Im}(\partial_2) \otimes_{\mathcal{O}_{F,T}} \mathbb{F}_p = \text{Im}(\overline{\partial}_2), \text{Ker}(\partial_1) \otimes_{\mathcal{O}_{F,T}} \mathbb{F}_p = \text{Ker}(\overline{\partial}_1).\]

So, comparing with (4.1.2), we have

\[
(4.1.3) \quad H_1(G, \text{Ad}(\rho_{\mathcal{O}_{F,T}})) \otimes_{\mathcal{O}_{F,T}} \mathbb{F}_p = H_1(G, \text{Ad}(\overline{\rho})).
\]

Consider the exact sequence of $\mathcal{O}_{F,T}$-modules

\[H_1(I_K, \text{Ad}(\rho_{\mathcal{O}_{F,T}})) \to H_1(G_K, \text{Ad}(\rho_{\mathcal{O}_{F,T}})) \to \text{Sel}^h_i(\text{Ad}(\rho_{\mathcal{O}_{F,T}})) \to 0.\]

Taking the tensor product over $\mathcal{O}_{F,T}$ with $\mathbb{F}_p$ and by (4.1.3), we have the exact sequence

\[H_1(I_K, \text{Ad}(\overline{\rho})) \to H_1(G_K, \text{Ad}(\overline{\rho})) \to \text{Sel}^h_i(\text{Ad}(\rho_{\mathcal{O}_{F,T}})) \otimes_{\mathcal{O}_{F,T}} \mathbb{F}_p \to 0\]

and hence

\[\text{Sel}^h_i(\text{Ad}(\rho_{\mathcal{O}_{F,T}})) \otimes_{\mathcal{O}_{F,T}} \mathbb{F}_p = \text{Sel}^h_i(\text{Ad}(\overline{\rho})).\]

By the choice of $T$, we have $\text{Sel}^h_i(\text{Ad}(\overline{\rho})) = 0$.

Here, let us recall the universal deformation for knot group representations. For a local ring $A$, we denote by $\mathfrak{m}_A$ the maximal ideal of $A$. Let $k$ be a field with characteristic $\text{char}(k) \neq 2$, and $\mathcal{O}$ a complete discrete valuation ring with $\mathcal{O}/\mathfrak{m}_\mathcal{O} = k$. Let $\overline{\mathfrak{p}} : G_K \to \text{SL}_2(k)$ be a representation. The pair $(R, \rho)$ is called a deformation of $\overline{\rho}$ when $R$ is a complete local $\mathcal{O}$-algebra with $R/\mathfrak{m}_R = k$, and $\rho : G_K \to \text{SL}_2(R)$ is a representation such that $\rho \mod \mathfrak{m}_R = \overline{\rho}$. Moreover, the pair $(R_{\overline{\rho}}, \rho)$ is called a universal deformation of $\overline{\rho}$ if $(R_{\overline{\rho}}, \rho)$ is a deformation of $\overline{\rho}$, and for all deformation $(R, \rho)$ of $\overline{\rho}$, there exists $\psi : R_{\overline{\rho}} \to R$ such that $\psi \circ \rho \approx \rho$. Here, $\rho_1 \approx \rho_2$ means there exists $U \in I_2 + M_2(\mathfrak{m}_R)$ such that $\rho_2(g) = U\rho_1(g)U^{-1}$ for all $g \in G_K$.

![Diagram](diagram.png)

By the universal property, a universal deformation $(R_{\overline{\rho}}, \rho)$ of $\overline{\rho}$ is unique (if it exists) up to strict equivalence, and we call $R_{\overline{\rho}}$ the universal deformation ring of $\overline{\rho}$. 

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Now, let \( p \) be a maximal ideal of \( \mathcal{O}_F \) satisfying the properties in Proposition 4.1. Let \( \mathcal{O}_p \) be the \( p \)-adic completion of \( \mathcal{O}_F \). By [Mor+17, Theorem 2.2.2], we have the universal deformation \( \rho: G_K \rightarrow \text{SL}_2(\mathbb{R}_\rho) \) of \( \rho \). We assume that the universal deformation ring \( \mathcal{R}_\rho \) is a unique factorization domain.

**Theorem 4.2.** For \( \gamma \in D_K \) with \( \gamma \neq 1 \), the adjoint homological \( \gamma \)-Selmer module \( \text{Sel}^h_{\gamma}(\text{Ad}(\rho)) \) is a finitely generated torsion \( \mathcal{R}_\rho \)-module.

**Proof.** By assumptions and direct calculations, we have \( \dim H_0(E_K) = 0 \) and \( \dim H_0(\gamma) = 1 \) over \( \mathbb{F}_p \) and the quotient field \( Q(\mathcal{R}_\rho) \) of \( \mathcal{R}_\rho \). By the exact sequence of relative homology, we have

\[
0 \rightarrow \text{Sel}^h_{\gamma} \rightarrow H_1(E_K, \gamma) \rightarrow H_0(\gamma) \rightarrow H_0(E_K) \rightarrow 0
\]

over \( \mathbb{F}_p \) and \( Q(\mathcal{R}_\rho) \). By Proposition 4.1, since \( \text{Sel}^h_{\gamma}(\text{Ad}(\overline{\rho})) = 0 \), we have

\[
\dim H_1(E_K, \gamma, \text{Ad}(\overline{\rho})) = \dim H_0(\gamma, \text{Ad}(\overline{\rho})) - \dim H_0(E_K, \text{Ad}(\overline{\rho})) = 1,
\]

and by the Euler characteristic argument, we have

\[
\dim H_2(E_K, \gamma, \text{Ad}(\overline{\rho})) = \dim H_1(E_K, \gamma, \text{Ad}(\overline{\rho})) = 1.
\]

Next, let \((W, \gamma')\) be the 2-dimensional CW complex \( W \) (also defined in Section 2) with one vertex \( \gamma' \), which is homotopically equivalent with \((E_K, \gamma)\). Consider the boundary map \( \partial'_2: C_2(W, \gamma') \rightarrow C_1(W, \gamma') \) of the chain complex over \( \mathbb{F}_p \) and \( Q(\mathcal{R}_\rho) \). By (4.2.1), we have

\[
\text{corank}_{\mathbb{F}_p}(\partial'_2) = \text{corank}_{Q(\mathcal{R}_\rho)}(\partial'_2) = 0 \text{ or } 1.
\]

Hence, we have

\[
\dim H_2(E_K, \gamma, \text{Ad}(\rho_{Q(\mathcal{R}_\rho)})) = 0 \text{ or } 1,
\]

and by the Euler characteristic argument, we have

\[
\dim H_1(E_K, \gamma, \text{Ad}(\rho_{Q(\mathcal{R}_\rho)})) = \dim H_2(E_K, \gamma, \text{Ad}(\rho_{Q(\mathcal{R}_\rho)})) = 0 \text{ or } 1.
\]

By (4.2.1), we have

\[
\dim H_1(E_K, \gamma, \text{Ad}(\rho_{Q(\mathcal{R}_\rho)})) = \dim H_0(\gamma, \text{Ad}(\rho_{Q(\mathcal{R}_\rho)})) - \dim H_0(E_K, \text{Ad}(\rho_{Q(\mathcal{R}_\rho)})) = 1,
\]

and so \( \text{Sel}^h_{\gamma}(\text{Ad}(\rho_{Q(\mathcal{R}_\rho)})) = 0 \). Hence, \( \text{Sel}^h_{\gamma}(\text{Ad}(\rho)) \) is a finitely generated torsion \( \mathcal{R}_\rho \)-module. \( \square \)

**Remark 4.3.** The representation \( \rho_A: G_K \rightarrow \text{SL}_2(\mathbb{C}[\sqrt{x-1}]) \) in Example 2.8 is a candidate of the universal representation by regarding the representation over the power series \( \mathbb{Z}_p[[\sqrt{x-1}]] \) of the \( p \)-adic integers for some prime number \( p \). However, there still remains a rigorous discussion for the case of power series having square root in the variable.

One way to avoid this problem is to replace the parameter \( x = \text{tr}(\rho_A(g_1)) \) of the character variety by another parameter, such as \( y = \text{tr}(\rho_A(g_1g_2)) \), so that we may regard \( \rho_A \) as the representation over the power series \( \mathbb{Z}_p[[y-1]] \), and we obtain

\[
\text{Sel}^h_{\rho}(\text{Ad}(\rho_A)) \simeq \mathbb{Z}_p[[y-1]]/(y-1)\mathbb{Z}_p[[y-1]].
\]

See Example 5.3, where such kind of problem does not occur.
5. Two-variable adjoint homological Selmer modules for universal deformations

Next, let us consider two-variable adjoint homological Selmer modules for universal deformations. Note that in our situation, “two-variable” means the parameters of the universal deformation and the character variety of one-dimensional representations of knot groups, namely the parameter \( t \) of the Alexander polynomial. Let \( A \) be an integral domain. Let \( \rho: G_K \to \text{SL}_2(A) \) be a representation of \( G_K \). We define the two-variable chain complex associated with \( \rho \) for the knot complement \( X_K \) as follows. Let \( \tilde{X} \to X_K \) be the universal cover of \( X_K \). Let \( \alpha : G_K \to G_K^{ab} \simeq \mathbb{Z} = \langle t \rangle \) be the abelianization homomorphism. Then \( V_\rho[t^{\pm 1}] = A[t^{\pm 1}] \otimes_A V_\rho \) is a right \( A[G_K] \)-module via

\[(a(t) \otimes v)_g := a(t) \cdot \alpha(g) \otimes v, \rho(g),\]

where \( a(t) \in A[t^{\pm 1}], v \in V_\rho \) and \( g \in G_K \).

We define the two-variable \( \rho \)-twisted chain complex \( C_\bullet(X_K; V_\rho[t^{\pm 1}]) \) of \( X_K \) by

\[C_\bullet(X_K; V_\rho[t^{\pm 1}]) := V_\rho[t^{\pm 1}] \otimes_{A[G_K]} C_\bullet(\tilde{X}),\]

and the two-variable \( \rho \)-twisted homology \( H_i(X_K; V_\rho[t^{\pm 1}]) \) by

\[H_i(X_K; V_\rho[t^{\pm 1}]) := H_i(C_\bullet(X_K; V_\rho[t^{\pm 1}])).\]

Since the knot complement \( X_K \) is the Eilenberg-MacLane space \( K(G_K, 1) \), we denote \( H_i(X_K; V_\rho[t^{\pm 1}]) \) by \( H_i(G_K; V_\rho[t^{\pm 1}]) \).

For the case of the group \( I_n \), comparing with (2.2), the set \( \{ v(t) \in V_\rho[t^{\pm 1}] \mid \rho(g_1)v(t) = t \cdot v(t)\rho(g_1) \} \) vanishes. Hence, we define the two-variable adjoint homological \( \gamma \)-Selmer module attached to the representation \( \rho \) by the \( A[t^{\pm 1}] \)-module

\[\text{Sel}^h_\gamma(\text{Ad}(\rho)[t^{\pm 1}]) := H_1(G_K; V_\rho[t^{\pm 1}]).\]

For the finitely generated torsion-ness of \( \text{Sel}^h_\gamma(\text{Ad}(\rho)[t^{\pm 1}]) \), similar Euler characteristic arguments hold as [Kit+18, Theorem 3.2.4]. Note that the condition \( \text{det}(t \cdot \rho(g) - I) \neq 0 \) for some \( g \in G_K \) and \( \Delta_K(\rho; t) \neq 0 \) always holds for the universal deformation \( \rho \).

**Theorem 5.1.** Let \( (R_\gamma, \rho) \) be the universal deformation and assume that the universal deformation ring \( R_\gamma \) is a Noetherian unique factorization domain. For \( \gamma \in D_K \) with \( \gamma \neq 1 \), the adjoint homological \( \gamma \)-Selmer module \( \text{Sel}^h_\gamma(\text{Ad}(\rho)[t^{\pm 1}]) \) is a finitely generated torsion \( R_\gamma[t^{\pm 1}] \)-module.

From now on, we keep the same notations as in Section 2. We study the \( A[t^{\pm 1}] \)-module structure of the adjoint homological Selmer module \( \text{Sel}^h_\gamma(\text{Ad}(\rho)[t^{\pm 1}]) \).

We take again a Wirtinger presentation of \( G_K \):

\[G_K = \langle g_1, \ldots, g_n \mid r_1 = \cdots = r_{n-1} = 1 \rangle,\]

where \( g_1, \ldots, g_n \) \((n \geq 2)\) represent meridians of a knot \( K \), and let \( \rho: G_K \to \text{SL}_2(A) \) be a representation. Let \( V_\rho[t^{\pm 1}] \) be the representation space

\[V_\rho[t^{\pm 1}] := \text{sl}_2(A[t^{\pm 1}]) = A[t^{\pm 1}]v_1 \oplus A[t^{\pm 1}]v_2 \oplus A[t^{\pm 1}]v_3,\]

where

\[v_1 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ v_2 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ v_3 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.\]
In the following, taking $\gamma = g_1$, we shall compute a presentation matrix of \( \text{Sel}^h(\text{Ad}(\rho)[t^{\pm 1}]) = \text{Sel}^h(\text{Ad}(\rho)[t^{\pm 1}]) \) over \( A[t^{\pm 1}] \), under some (mild) assumptions.

Similar discussion as [Kit+18, Section 3] holds by replacing the coefficient \( V_{\rho} \) to \( V_{\rho}[t^{\pm 1}] \), and the boundary maps \( \partial_2 \) to \( \partial_2[t^{\pm 1}] := (\text{Ad}(\rho) \circ \Psi \left( \delta \right)) \), where \( \Psi := (\rho \otimes \alpha) \circ \pi : A[F] \rightarrow M_2(\text{Ad}(\rho)[t^{\pm 1}]) \) is an \( A \)-algebra homomorphism, and \( \alpha : G_K \rightarrow \mathbb{Z} \) is an abelianization.

Similarly as Theorem 2.7, we have the following theorem.

**Theorem 5.2.** Let the notations and assumptions be as above. Then a presentation matrix of \( \text{Sel}^h(\text{Ad}(\rho)[t^{\pm 1}]) \) is given by the matrix \( \partial_2[t^{\pm 1}] \), and for an integer \( d \geq 0 \), the \( d \)-th Fitting ideal of \( \text{Sel}^h(\text{Ad}(\rho)[t^{\pm 1}]) \) over \( A[t^{\pm 1}] \) is generated by the greatest common divisor of all \( (d + 3) \)-minors of \( \partial_2[t^{\pm 1}] \).

We give herewith a concrete example, which provides a non-trivial result.

**Example 5.3.** Let \( K \) be the figure-eight knot, whose knot group is given by

\[
G_K = \langle g_1, g_2 \mid g_1g_2^{-1}g_1^{-1}g_2g_1 = g_2g_1g_2^{-1}g_1^{-1}g_2 \rangle.
\]

Let \( \rho : G_K \rightarrow \text{SL}_2(\mathbb{C}) \) be the representation

\[
\rho(g_1) = \begin{pmatrix}
\sqrt{\frac{2}{\sqrt{2}}} & 1 \\
0 & \sqrt{\frac{2}{\sqrt{2}}} - 1
\end{pmatrix},
\]

\[
\rho(g_2) = \begin{pmatrix}
\sqrt{\frac{2}{\sqrt{2}}} & \sqrt{\frac{2}{\sqrt{2}}} - 1 \\
\frac{5}{4} + \sqrt{-15} & 0
\end{pmatrix}.
\]

Consider a residual representation \( \overline{\rho} : G_K \rightarrow \text{SL}_2(\mathbb{F}_{53}) \) induced by \( \rho \) such that

\[
\overline{\rho}(g_1) = \begin{pmatrix} 19 & 1 \\ 0 & 14 \end{pmatrix},
\]

\[
\overline{\rho}(g_2) = \begin{pmatrix} 19 & 0 \\ 44 & 14 \end{pmatrix}.
\]

Then we have the universal deformation \( \rho_R : G_K \rightarrow \text{SL}_2(R) \) of \( \overline{\rho} \) given by the following:

\[
\rho_R(g_1) = \begin{pmatrix} x + \sqrt{x^2 - 1} & 1 \\ 0 & x - \sqrt{x^2 - 1} \end{pmatrix},
\]

\[
\rho_R(g_2) = \begin{pmatrix} x + \sqrt{x^2 - 1} & 0 \\ -(x^2 - y(x) - 2) & x - \sqrt{x^2 - 1} \end{pmatrix},
\]

where \( R = \mathbb{Z}_{53}[\left[ x - \sqrt{\frac{5}{4}} \right] \], and \( y(x) = \frac{x^2 + 1 + \sqrt{(x^2 - 1)(x^2 - 5)}}{2} \). Similarly as Example 2.9, we can rewrite these matrices as

\[
\rho_A(g_1) = \begin{pmatrix} \frac{(s + \sqrt{\frac{5}{4}}) + \sqrt{(s + \sqrt{\frac{5}{4}})^2 - 4}}{2} & 1 \\ 0 & \frac{(s + \sqrt{\frac{5}{4}}) - \sqrt{(s + \sqrt{\frac{5}{4}})^2 - 4}}{2} \end{pmatrix},
\]

...
\[ \rho_A(g_2) = \begin{pmatrix} \frac{s + \sqrt{\frac{5}{2}} + \sqrt{(s + \sqrt{\frac{5}{2}})^2 - 4}}{2} & 0 \\ - \left( \left( s + \sqrt{\frac{5}{2}} \right)^2 - y \left( s + \sqrt{\frac{5}{2}} \right) - 2 \right) & \frac{(s + \sqrt{\frac{5}{2}})^2 - 4}{2} \end{pmatrix}, \]

where we use \( s := x - \sqrt{\frac{5}{2}} \).

Next, let us see the presentation matrix \( \partial_2[t^\pm1] \) in Theorem 5.2. Then the greatest common divisor \( \Phi(s,t) \in \mathbb{Z}_{53}[[s]][t^\pm1] \) of all 3-minors of \( \partial_2[t^\pm1] \) is given by

\[ \Phi(s,t) = (t - 1) \left[ t^2 - \left\{ 2 \left( s + \sqrt{\frac{5}{2}} \right)^2 - 3 \right\} t + 1 \right] \in \mathbb{Z}_{53}[[s]][t^\pm1]. \]

Hence, we have

\[ \text{Sel}_h^\lambda(\text{Ad}(\rho_R)[t^\pm1]) \cong R[t^\pm1]/(t - 1) \left[ t^2 - \left\{ 2 \left( s + \sqrt{\frac{5}{2}} \right)^2 - 3 \right\} t + 1 \right] R[t^\pm1]. \]

Moreover, by Example 2.9, we have

\[ \frac{\Phi(s,t)}{t - 1} \bigg|_{t=1} = L_\lambda(s). \]

**Example 5.4.** Let \( K \) be the knot 5_2, whose knot group is given by

\[ G_K = \langle g_1, g_2 \mid g_1 g_2 g_1^{-1} g_2^{-1} g_1 g_2 g_1 = g_2 g_1 g_2 g_1^{-1} g_2^{-1} g_1 g_2 \rangle. \]

Consider a residual representation \( \overline{\rho} \colon G_K \to \text{SL}_2(\mathbb{F}_{17}) \) such that

\[ \overline{\rho}(g_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \]
\[ \overline{\rho}(g_2) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}. \]

Then we have the universal deformation \( \rho_R \colon G_K \to \text{SL}_2(R) \) of \( \overline{\rho} \) given by the following:

\[ \rho_R(g_1) = \begin{pmatrix} \frac{x + \sqrt{x^2 - 4}}{2} & 1 \\ 0 & \frac{x - \sqrt{x^2 - 4}}{2} \end{pmatrix}, \]
\[ \rho_R(g_2) = \begin{pmatrix} \frac{x + \sqrt{x^2 - 4}}{2} & 0 \\ -(x^2 - y(x) - 2) & \frac{x - \sqrt{x^2 - 4}}{2} \end{pmatrix}, \]

where \( R = \mathbb{Z}_{17}[[x - \beta]], \beta = \sqrt{-1} \cdot 1.00098 \cdots \in \mathbb{Z}_{17} \) is the simple root satisfying the equation

\[ 20\beta^6 - 126\beta^4 + 196\beta^2 + 343 = 0, \]

and \( y = y(x) \in R \) is, by Hensel’s lemma, the unique solution satisfying the equation

\[ y^3 - (x^2 + 1)y^2 + (3x^2 - 2)y - 2x^2 + 1 = 0 \]
and
\[ y(\beta) = \xi, \]
where \( \xi = -2.493 \cdots \in \mathbb{Z}_{17} \) is the simple root satisfying the equation
\[ 2\xi^3 - 2\xi^2 - 11\xi + 16 = 0. \]

We can rewrite these matrices as
\[
\rho_A(g_1) = \begin{pmatrix}
\frac{(s + \beta) + \sqrt{(s + \beta)^2 - 4}}{2} & 1 \\
0 & \frac{(s + \beta) - \sqrt{(s + \beta)^2 - 4}}{2}
\end{pmatrix},
\]
\[
\rho_A(g_2) = \begin{pmatrix}
\frac{(s + \beta) + \sqrt{(s + \beta)^2 - 4}}{2} & 0 \\
-(s + \beta)^2 - y(s + \beta) - 2 & \frac{(s + \beta) - \sqrt{(s + \beta)^2 - 4}}{2}
\end{pmatrix},
\]
where we use \( s := x - \beta \).

Next, let us see the presentation matrix \( \partial_2[t^{\pm 1}] \) in Theorem 5.2. Then the greatest common divisor \( \Phi(x,t) \in \mathbb{Z}_{17}[[s]][t^{\pm 1}] \) of all 3-minors of \( \partial_2[t^{\pm 1}] \) is given by
\[
\Phi(s,t) = (t - 1)[2(s + \beta)^4y - 4(s + \beta)^4 - 2(s + \beta)^2y^2 - 2(s + \beta)^2y + 10(s + \beta)^2 + 2y^2 - 2y - 6]t^2
+ \{(s + \beta)^4y - 2(s + \beta)^4 - (s + \beta)^2y^2 - 3(s + \beta)^2y + 11(s + \beta)^2 + 3y^2 - 3y - 9\}t
+ \{2(s + \beta)^4y - 4(s + \beta)^4 - 2(s + \beta)^2y^2 - 2(s + \beta)^2y + 10(s + \beta)^2 + 2y^2 - 2y - 6\}
\in \mathbb{Z}_{17}[[s]][t^{\pm 1}].
\]

Hence, we have
\[
\text{Sel}_\lambda^\theta(\text{Ad}(\rho_R)[t^{\pm 1}]) \simeq R[t^{\pm 1}]/\Phi(s,t)R[t^{\pm 1}].
\]

Moreover, by Example 2.10, we have
\[
\Phi(s,t) \bigg|_{t=1} = L_\lambda(s).
\]

Based on these examples, we have the following conjecture:

**Conjecture 5.5.** Let \((R_\tau, \rho)\) be the universal deformation and assume that the universal deformation ring \( R_\tau \) is a Noetherian unique factorization domain. Let \( \Phi(s,t) \in R_\tau[t^{\pm 1}] \) be the greatest common divisor of all 3-minors of \( \partial_2[t^{\pm 1}] \) and \( L_\lambda(s) \in R_\tau \) be the greatest common divisor of all 3-minors of \( D \) in (2.6) associated with the longitude. Then we have
\[
\Phi(s,t) \bigg|_{t=1} = L_\lambda(s).
\]

This conjecture is considered as a knot-theoretic analogue of Theorem 1 for two-variable adjoint Selmer groups in Hida-Tilouine-Urban [HTU97] (see also [Hid00a, Theorem 6.3], [Hid06, Chapter 5]). It would be interesting to pursue this analogy in detail.
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