On \( p \)-adic cascade equations of hydrodynamic type in modeling fully developed turbulence

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Abstract

A \( p \)-adic hydrodynamic type equation with two integrals of motion is proposed. It can be considered as a model cascade equation for energy dissipation in fully developed turbulence. Some of special cases of the proposed equation are detailed and it is shown that for a specific choice of parameters they have stationary solutions that correspond to the 2/3 Kolmogorov-Obukhov law. Possible further studies of the proposed model are discussed.

Keywords: turbulence, energy cascade processes, shell models, \( p \)-adic analysis, \( p \)-adic models

1 Introduction

\( p \)-Adic analysis is an adequate tool for describing systems that have an explicit or hidden hierarchical structure. Over the past 3 decades, the \( p \)-adic analysis apparatus has been actively used to model a different class of systems, such as spin glasses, proteins, models of the genetic code, dynamic systems, socio-economic systems, etc., (for a review see [1]). One of the applications of \( p \)-adic analysis is ultrametric modeling of cascade processes of energy dissipation in fully developed turbulence, outlined in [2] [3]. The main idea of work was [2] to consider the equations of the hierarchical cascade model in ultrametric space, implemented by the field of \( p \)-adic numbers. In this case, such a property of fully developed turbulence as multifractality is naturally ensured, and it is possible to
obtain the 2/3 Kolmogorov-Obukhov law \[4, 5, 6\]. A further modification of the nonlinear ultrametric equation introduced in \[2\] was proposed in work \[3\]. It was found that, using the ultrametric wavelet analysis, one can get a family of exact solutions for this modified equation.

The equations proposed in \[2, 3\] can be considered as some approximations of the Navier-Stokes equations written in some basis, after truncing a number of modes and after a possible \(p\)-adic parameterization. Note that the question of the existence of integrals of motion of the proposed equations was not discussed in works \[2, 3\]. Nevertheless, the Navier-Stokes equations in the absence of dissipation and external forces have two integrals of motion – energy and hydrodynamic helicity (or enstrophy in the 2d case). For this reason, \(p\)-adic equations, claiming a possible description of cascade processes of energy dissipation in fully developed turbulence, must have the structure of equations of hydrodynamic type with two integrals of motion. Recall that the term "systems of hydrodynamic type" was introduced by Obukhov \[7\] for quadratically non-linear systems that, while keeping only homogeneous terms, satisfy the following conditions: regularity (conservation of phase volume) and the existence of at least one quadratic integral of motion. The first hierarchical models of the hydrodynamic type with one integral of motion, which describes the cascade process of energy dissipation and gives the 2/3 law, was proposed by Obukhov \[7, 8, 9\] and Desnyanskii and Novikov \[10\]. A hierarchical model of a hydrodynamic type with two integrals of motion was first proposed by Gledzer \[11\] (see also \[12, 13\]), then developed by Ohkitani and Yamada \[14, 15\] and is now called the GOY model (see, for example, \[16, 17, 18, 19\]). All these models belong to the class of the so-called shell models. The basic idea behind shell models is to divide the spectral space of the velocity field into concentric spheres of growing radii \(k_i = k_0 \lambda^i\), where \(\lambda\) is the parameter characterizing the ratio of the characteristic scales of turbulent eddies. The set of modes contained in a single sphere of fixed radius is called the shell. In shell models only a few modes are retained in each shell. The equations for the mode components of the velocity field are a system of hydrodynamic type with two integrals of motion containing only terms that link only \(s\) of the nearest modes. For Obukhov model and Desnyanskii and Novikov model the value \(s = 1\), while for GOY model \(s = 2\).

In this article, we propose a general \(p\)-adic model that has the structure of a hydrodynamic type system with two integrals of motion. In Section 2, we present the general form of the equations of this model. In Section 3, we consider some special cases of the general \(p\)-adic equation proposed in Section 2 with different values of \(s\) and various forms of the second integral of motion. In this case the \(p\)-adic equation is reduced to equations of shell models type. We show that for certain relations between the parameters the equation of the model has a stationary solution corresponding to the 2/3 law.
2 p-Adic hydrodynamic type equation with two integrals of motion

Let us consider the Navier-Stokes equation for an incompressible viscous fluid, formally written in the form in which the term with pressure is excluded:

\[
\partial_t v_i = -v_j \partial_j v_i + (\partial_i \partial_j)^{-1} \partial_i ((\partial_j v_k)(\partial_k v_j)) + \nu \partial_i \partial_i v_i + (\delta_{ij} - (\partial_i \partial_j)^{-1} \partial_i) f_j, \quad \partial_j v_j = 0. \tag{1}
\]

In Eq. (1), \(i, j, k, l = 1, 2, 3\), \(v_i = (x, t)\) is the velocity field, \(\nu\) is the kinematic viscosity, \(f_i\) is the density of external forces and summation over repeated indices is assumed. Let us assume that there exists a certain countable orthonormal basis of vector functions \(\left\{ e_a^{(i)} \right\} \), \(\int d^3 \vec{r} e_a^{(i)} e_b^{(i)} = \delta_{ab}\) (\(a, b\) are multi-indices) such that \(\partial_i e_a^{(i)} = 0\). To simplify the notation, we consider the functions \(\left\{ e_a^{(i)} \right\}\) to be real. Then, by decomposing the functions \(v_i\) and \(f_i\) into a sum over a set of basis functions \(\left\{ e_a^{(i)} \right\}\)

\[
v_i = \sum_a v_a e_a^{(i)}, \quad f_i = \sum_a f_a e_a^{(i)}.
\]

one can write the Eq. (1) in the form of a system of linked equations

\[
\dot{v}_a = \sum_{b,c} K_{abc} v_b v_c - \sum_j \nu_{ab} v_b + \sum_j \phi_{ab} f_b. \tag{2}
\]

In the absence of viscosity and external forces, the energy per unit mass \(E = \frac{1}{2} \int d^3 x v^2\) and the hydrodynamic helicity \(H = \int d^3 x \varepsilon_{ijk} v_i \partial_j v_k\) are the integrals of the motion of the Eq. (1). Therefore, the Eqs. (2) for \(\nu = 0\) and \(f_a = 0\) must also have two integrals of motion, which are

\[
E = \frac{1}{2} \sum_a v_a^2, \tag{3}
\]

and

\[
H = \frac{1}{2} \sum_{a,b} h_{ab} v_a v_b, \quad h_{ab} = \int d^3 x \varepsilon_{ijk} e_a^{(i)} \partial_j e_{b}^{(k)}. \tag{4}
\]

In this case Eqs. (2) are a system of hydrodynamic type with two integrals of motion and its most general form is

\[
\dot{v}_a = \sum_{b,c,d} \varepsilon_{abcd} h_{dc} v_b v_c - \sum_b \nu_{ab} v_b + \sum_b \phi_{ab} f_b, \tag{5}
\]

where \(h_{ab} = h_{ba}\) and \(\varepsilon_{abc}\) is totally anti-symmetric under the exchange of indices.
Next, we represent the $p$-adic parameterization of Eqs. (5). First, recall the definition of $p$-adic number. Let $\mathbb{Q}$ be a field of rational numbers and let $p$ be a fixed prime number. Any rational number $x \neq 0$ is uniquely represented as

$$x = \pm p^n \frac{a}{b},$$

where $n$ is an integer, and $a, b$ are natural numbers that are not divisible by $p$ and have no common multipliers. The $p$-adic norm $|x|_p$ of number $x \in \mathbb{Q}$ is defined by the equalities $|x|_p = p^{-n}$, $|0|_p = 0$. The field of $p$-adic numbers $\mathbb{Q}_p$ is defined as a completion of the field of rational numbers $\mathbb{Q}$ by $p$-adic norm $|x|_p$. Any $p$-adic number can be represented as a series converging by $p$-adic norm:

$$x = p^{-n} \left( a_0 + a_1 p + a_2 p^2 + \ldots \right),$$

The norm on $\mathbb{Q}_p$ induces the metric $d(x, y) = |x - y|_p$ which is ultrametric, i.e. satisfies the strong triangle inequality $\forall x, y, z \ d(x, y) \leq \max \{d(y, z), d(x, z)\}$. We will denote: $B_i(a) = \{ x \in \mathbb{Q}_p : |x - a|_p \leq p^i \}$ – a ball of radius $p^i$ centered at point $a$, $S_i(a) = \{ x \in \mathbb{Q}_p : |x - a|_p = p^i \}$ – a sphere of radius $p^i$ centered at point $a$, $B_i \equiv B_i(0)$, $S_i \equiv S_i(0)$. On $\mathbb{Q}_p$ there exists a unique (up to a factor) Haar measure $d_p x$ which is invariant with respect to translations $d_p (x + a) = d_p x$. We assume that $d_p x$ is a full measure; that is,

$$\int_{\mathbb{Z}_p} d_p x = 1. \quad (7)$$

Under this hypothesis the measure $d_p x$ is unique. For more information about $p$-adic numbers, $p$-adic analysis and its applications, see [20, 21].

For what follows, we also define the class $W^{\alpha}_1$ ($\alpha \geq 0$) of complex functions $f(x)$ on $\mathbb{Q}_p$ satisfying the following conditions:

1) $|\varphi(x)| \leq C \left( 1 + |x|_p^\alpha \right)$, where $C$ is a real positive number;

2) there exists a natural number $l$ such that $\varphi(x + x') = \varphi(x)$ for any $x \in \mathbb{Q}_p$ and any $x' \in \mathbb{Q}_p$, $|x'|_p \leq p^l$. A function $\varphi(x)$ satisfying such condition is called locally constant, and the number $l$ is called the exponent of local constancy of a function.

For our purposes, we can assume that the number $p$ is a natural number $p = m > 2$. In this case $\mathbb{Q}_p$ is a ring of $m$-adic numbers $\mathbb{Q}_m$ with the pseudonorm $|x|_m$, which also induces on $\mathbb{Q}_m$ the ultrametrics $d(x, y) = |x - y|_m$. [22]

Let $v(x, t)$ be a function on $\mathbb{Q}_p \times \mathbb{R}$. We consider the equation of the following structure

$$\frac{\partial v(x, t)}{\partial t} = \int_{\mathbb{Q}_p} m(x) d_p y \int_{\mathbb{Q}_p} m(z) d_p z \int_{\mathbb{Q}_p} m(z') d_p z' \epsilon(x, y, z') h(z', z) v(y, t) v(z, t)$$

$$- \int_{\mathbb{Q}_p} m(y) d_p y v(x, y) v(y, t) + f(x), \quad (8)$$
where $m(x)$ is some function $\mathbb{Q}_p \to \mathbb{R}_+$ which is locally integrable with respect to the Haar measure $d_p x$, $\varepsilon(x, y, z)$ is a completely antisymmetric function $\mathbb{Q}_p \times \mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{R}$ and $h(x, y)$ is a symmetric function $\mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{R}$. The Eq. (8) with $\nu(x, y) = 0$, $f(x) = 0$ has the following two integrals of motion:

$$E = \int_{\mathbb{Q}_p} m(x) d_p x v^2(x, t), \quad H = \int_{\mathbb{Q}_p} m(x) d_p x \int_{\mathbb{Q}_p} m(y) d_p y h(x, y) v(x, t) v(y, t).$$  \tag{9}$$

Eq. (8) is a general $p$-adic equation of hydrodynamic type with two integrals of motion. We consider special cases of this equation in the next section.

3 \quad \textit{$p$-Adic cascade equations on $B_r$}

The study of the properties of the model based on the Eq. (8) is related to the choice of functions $m(x), \varepsilon(x, y, z), h(x, y), \lambda(x, y), f(x)$. In this section, we choose the function $m(x)$, which determines the integration measure in Eq. (8) as follows

$$m(x) = \left( \Omega \left( |x|_p \right) \left( 1 - p^{-1} \right) + \left( 1 - \Omega \left( |x|_p \right) \right) \right),$$

where we use the notation well known in analysis:

$$\Omega \left( |x|_p p^{-i} \right) = \begin{cases} 1, & |x|_p \leq p^i \\ 0, & |x|_p > p^i \end{cases}.$$ 

So $v(x, t)$ is a function on $B_r \times \mathbb{R}$ rather than on $\mathbb{Q}_p \times \mathbb{R}$. The points $x \in B_r \subset \mathbb{Q}_p$ parameterize the velocity field modes. We assume that for $y \in \mathbb{Z}_p$ the points $x$ and $x + y$ parameterize the same mode of the velocity field, thus modes are parameterized by $p$-adic balls of unit radius in $B_r$ or, equivalently, by points in $B_r/\mathbb{Z}_p$. Also we assume that each mode corresponds to a turbulent eddies and the scale $l_i$ of the eddies corresponding to any unit ball in $S_i$ is determined by the radius of $S_i$: $l_i = l_0 p^{-i}$, where $l_0$ is the largest scale of the cascade. In this case, all modes corresponding to eddies of the same scale $l_i$ give the same contribution to energy and hydrodynamic helicity.

Further, we assume that any eddy of $l_i$ can interact only with eddies of scales $l_{i-s}, \ldots, l_{i-1}, l_{i+1}, \ldots, l_{i+s}$. Moreover, the function $\varepsilon(x, y, z)$ in the Eq. (8), which determines this interaction, has the form

$$\varepsilon(x, y, z) = \sigma(x, y) \sigma(y, z) \sigma(z, x),$$

$$\sigma(x, y) = \sigma_s \left( |x|_p, |y|_p \right).$$
\[ = |x - y|^{-\alpha} \left( \Omega \left( \frac{|x|}{y} p^{-s} \right) \Omega \left( \frac{|y|}{x} p^s \right) - \Omega \left( \frac{|y|}{y} p^{-s} \right) \right), \]

and \( \alpha \) is some real parameter. We also assume that all other functions in the Eq. (8) are depend only on the \( p \)-adic norm: \( h(x, y) = h \left( |x|_p, |y|_p \right), \nu(x, y) = \nu \left( |x|_p, |y|_p \right), f(x) = f \left( |x|_p \right). \) Moreover, functions \( f \left( |x|_p \right) \in W_0^0 (B_r) \) and \( h \left( |x|_p, |y|_p \right), \nu \left( |x|_p, |y|_p \right) \in W_0^0 (B_r) \times W_0^0 (B_r). \)

Within the framework of the assumptions made, the Eq. (8) has the form

\[
\frac{\partial v(x, t)}{\partial t} = \int_{Q_p} d\mu(y) \int_{Q_p} d\mu(z) \int_{Q_p} d\mu(z') \times S \left( |x|_p, |y|_p \right) S \left( |y|_p, |z|_p \right) S \left( |z|_p, |x|_p \right) v \left( |y|_p, t \right) h \left( |z'|_p, |z|_p \right) v \left( |z|_p, t \right) \]

\[ - \int_{Q_p} d\mu(y) \int_{Q_p} d\mu(x) \nu \left( |x|_p, |y|_p \right) v(y, t) + f \left( |x|_p \right). \quad (10) \]

where we denote \( d\mu(x) \equiv m(x) d_p x. \) The Cauchy problem for the Eq. (10) can be defined in the class of functions \( v(x, t) = v \left( |x|_p, t \right) \) from \( W_0^0 \text{norm} (B_r) \times \mathbb{R} \), where \( W_0^0 \text{norm} (B_r) \) is the intersection of the class \( W_0^0 (B_r) \) with the class of functions that depend only on the norm \( |x|_p \).

Let \( e_i(x) = \Omega \left( |x|_p p^{-i} \right) - \Omega \left( |x|_p p^{-i+1} \right) \) be the characteristic function of the sphere \( S_i, \)

\( i > 0 \) and \( e_0(x) = \Omega \left( |x|_p \right) \) be the characteristic function of \( \mathbb{Z}_p. \) Let us expand the functions \( v(x), h \left( |x|_p \right), S \left( |x|_p, |y|_p \right), m \left( |x|_p \right), f \left( |x|_p \right) \) in basis \( \{e_0(x), e_1(x), \ldots, e_n(x)\} \)

\[ v(x) = \sum_{i=0}^{r} V_i e_i(x), \quad h \left( |x|_p, |y|_p \right) = \sum_{i,j=0}^{r} \nu_{i,j} e_i(x) e_j(y), \]

\[ \nu \left( |x|_p, |y|_p \right) = \sum_{i,j=0}^{r} h_{i,j} e_i(x) e_j(y), \quad f \left( |x|_p \right) = \sum_{i=0}^{r} f_i e_i(x), \]

\[ S \left( |x|_p, |y|_p \right) = \sum_{i=0}^{r} \sum_{j=i-s}^{i-1} p^{-\alpha i} e_i(x) e_j(y) - \sum_{i=0}^{r} \sum_{j=i+1}^{i+s} p^{-\alpha j} e_i(x) e_j(y), \]

and we accept that \( e_i \equiv 0 \) for \( i < 0 \) and \( i > r. \) Then it can be shown that the Eq. (10) is equivalent to the following system of equations for \( V_i: \)

\[ \tilde{V}_i = h_0 \left( 1 - p^{-1} \right)^3 p^{(\gamma + 2)s} \]
\[
	\times \sum_{j,k,l=0}^{2s} \theta_{s,j} \theta_{s,k} \theta_{s,l} \delta_{j+k+l,3s} p^{-\alpha (\max\{s,j\} + \max\{s,k\} + \max\{s,l\}) - j - k} V_{i+j+s} \sum_{m=0}^{r} h_{i-k+s,m} p^m V_m
\]

\[-\sum_{j=0}^{r} \nu_{i,j} V_j + f_i,
\]

(11)

where

\[
\theta_{i,j} \equiv \begin{cases} 
1, & i > j \\
-1, & i < j \\
0, & i = j 
\end{cases}
\]

and here and below it is accepted that \(V_i \equiv 0\) for \(i < 0\) and \(i > r\). The Eqs. (11) for \(\nu_{i,j} = 0, f_i = 0\) have two integrals of motion:

\[
E = \sum_{i=0}^{r} E_i, \quad E_i = (1 - p^{-1}) p^i V_i^2,
\]

(12)

\[
H = (1 - p^{-1})^2 \sum_{i,j=0}^{r} p^i p^j h_{i,j} V_i V_j
\]

(13)

Note that the factor \((1 - p^{-1}) p^i\) in (12) appears because the \(p\)-adic sphere of radius \(p^i\) \((i > 0)\) is a union of \((1 - p^{-1}) p^i\) \(p\)-adic balls of unit radius. Physically, this means that at the hierarchical level \(i\), there is \((1 - p^{-1}) p^i\) self-similar eddies, each of which corresponds to some \(p\)-adic unit ball. The velocity component for each such eddy is equal to \(V_i\) and therefore the total contribution of all such eddies to integrals of motion is proportional to the factor \((1 - p^{-1}) p^i V_i^2\).

Next, we consider a specific choice of function \(h \left(|x|_p, |y|_p\right)\) of the form

\[
h \left(|x|_p, |y|_p\right) = h_0 \left|\frac{x}{y}\right|_p \Omega \left(\frac{x}{|y|_p}\right) \Omega \left(\frac{y}{|x|_p}\right)
\]

(14)

corresponding to the diagonal form of the second integral of motion \(H\). It is easy to verify that when choosing \(s = 2, \alpha = 0\) the Eqs. (11) go over to the equations

\[
\dot{V}_i = h_0 (1 - p^{-1})^3 p^{(\gamma + 2)i}
\]

\[
\times \left(p^3 \left(p^{\gamma - p^{2\gamma}}\right) V_{i+1} V_{i+2} + \left(p^{\gamma - p^{-\gamma}}\right) V_{i-1} V_{i+1} + p^{-3} \left(p^{-2\gamma} - p^{-\gamma}\right) V_{i-2} V_{i-1}\right)
\]

\[-\sum_{j=0}^{r} \nu_{i,j} V_j + f_i.
\]

(15)

Note that the quadratic term in Eqs. (15) although similar, do not coincide with corresponding term in the equations of the GOY model, which have the following general form (see, for example, [16, 17, 18, 19]):
\[ \dot{v}_i = a \lambda^i \left( v_{i+1}v_{i+2} - \varepsilon v_{i-1}v_{i+1} + \varepsilon - \frac{1}{\lambda^2} v_{i-2}v_{i-1} \right) - v_i v_i + f_i. \]  
\hspace{1cm} (16)

(here \( \lambda > 1, \varepsilon, a \) are parameters of model), since the integrals of motion (12) and (13) of Eqs. (15) are different in form from the integrals of motion of Eqs. (16):

\[ E = \sum_i V_i^2, \quad H = \sum_i (\varepsilon - 1)^{-1} V_i^2. \]

Note that the correspondence between (15) and (16) takes place for \( \gamma = -\frac{1}{2}, p = \lambda \) and after a scaling \( V_i = p^{-\frac{1}{2}i}v_i \). One can show that with a suitable choice of the coefficients \( \nu_{i,j} \) and \( f_i \) the system of Eqs. (15) has a stationary solution

\[ V_i = cp^{-\frac{5}{6}i} \]  
\hspace{1cm} (17)

if one of the conditions (18)

\[ \gamma = -\frac{1}{2} \text{ or } \gamma = -\frac{1}{4} \]  
\hspace{1cm} (18)

is satisfied. The stationary solution (17) leads to the following formula for the total energy of the modes of the \( i \)-th level

\[ E_i \sim p^j V_i^2 \sim p^{-\frac{2}{3}i}, \]

which corresponds to the 2/3 law.

We now consider the Eqs. (11) when choosing (14), \( s = 3 \) and \( \alpha = 0 \). In this case, the Eqs. (11) have the form

\[ \dot{V}_i = h_0 \left( 1 - p^{-1} \right)^3 p^{(\gamma+2)i} \]

\[ \times \left( p^5 \left( p^{3\gamma} - p^{2\gamma} \right) V_{i+3}V_{i+2} + p^4 \left( p^{3\gamma} - p^\gamma \right) V_{i+3}V_{i+1} + p^3 \left( p^{2\gamma} - p^\gamma \right) V_{i+2}V_{i+1} + p \left( p^{-\gamma} - p^\gamma \right) V_{i-1}V_{i+2} + \right. \]

\[ + \left. \left( p^{-\gamma} - p^\gamma \right) V_{i-1}V_{i+1} + p^{-1} \left( p^{-2\gamma} - p^\gamma \right) V_{i-2}V_{i+1} + p^{-3} \left( p^{-\gamma} - p^{-2\gamma} \right) V_{i-1}V_{i-2} + p^{-4} \left( p^{-\gamma} - p^{-3\gamma} \right) V_{i-1}V_{i-3} \right) \]

\[ + p^{-5} \left( p^{-2\gamma} - p^{-3\gamma} \right) V_{i-2}V_{i-3} \],  
\hspace{1cm} (19)

where the terms corresponding to dissipation and external forces are omitted. It can be shown that a model with a quadratic term of the form (19) for \( \gamma = \frac{5}{2} \) and \( \gamma = \frac{5}{4} \) also has a stationary solution (17), which corresponds to the 2/3 law.
In conclusion, we present the case of a model with an off-diagonal second integral of motion $H$. The following choice

$$h \left( |x|_p, |y|_p \right) = \frac{1}{2} h_0 |x|^\gamma \left( \Omega \left( \frac{|x|}{|y|_p}, p^{-1} \right) \Omega \left( \frac{|y|}{|x|_p}, p \right) + \Omega \left( \frac{|y|}{|x|_p}, p^{-1} \right) \Omega \left( \frac{|x|}{|y|_p}, p \right) \right),$$

leads to the following expression for $H$:

$$H = h_0 p^{-1} \sum_{i=0}^{r} p^{(\gamma+2)i} V_i V_{i-1}. \tag{10}$$

In this case, from (10) we get the equations with quadratic term of the form

$$\dot{V}_i = h_0 \left( 1 - p^{-1} \right)^3 p^{(\gamma+3)i} \times \sum_{j,k,l=0}^{2s} \theta_{s,j} \theta_{s,k} \theta_{s,l} \delta_{j+k+l,3s} p^{-\alpha (\text{max}\{s,j\} + \text{max}\{s,k\} + \text{max}\{s,l\}) + j - i} \times V_{i+j-s} p^{(\gamma+1)(-k+s)} \left( p^{-1} V_{i-k+s-1} + \theta_{i-k+s} p^{\gamma+1} V_{i-k+s+1} \right), \tag{20}$$

where

$$\theta_i \equiv \begin{cases} 1, & i \geq 0 \\ 0, & i < j \end{cases}. \tag{21}$$

In the case $s = 2$ Eqs. (20) have the form

$$\dot{V}_i = h_0 \left( 1 - p^{-1} \right)^3 p^{(\gamma+3)i} \times \left[ p^{-3} \left( p^{-\gamma-2} V_{i-2} V_{i-2} + V_{i-2} V_{i-2} - p^{-2\gamma-3} V_{i-1} V_{i-3} - (1 - \delta_{i,1}) p^{-\gamma-1} V_{i-1} V_{i-1} \right) + p^{-\gamma-2} V_{i+1} V_{i+2} + (1 - \delta_{i,0}) V_{i+1} V_{i+1} - (1 - \delta_{i,n}) p^\gamma V_{i-1} V_{i-2} - p^{2\gamma+2} V_{i-1} V_{i+2} + p^3 \left( (1 - \delta_{i,n-1}) p^{2\gamma+1} V_{i+1} V_{i+1} + p^{3\gamma+3} V_{i+3} V_{i+3} - p^7 V_{i+2} V_{i+2} - p^{2\gamma+2} V_{i+2} V_{i+2} \right) \right]. \tag{21}$$

It also can be shown that model (21) has a stationary solution corresponding to the 2/3 law for $\gamma = -1$, $\gamma = -\frac{3}{2}$ and $\gamma = -\frac{5}{4}$.

### 4 Concluding remarks

The main goal of this short article is to give another important direction in the application of $p$-adic analysis, namely, the parameterization of hydrodynamic-type systems with two integrals of motion. Such an equation was written in the form (8) and in the future it can be used to study new types of models of cascade processes of energy dissipation in fully developed turbulence. It was shown that with a special choice of functions included...
in this equation, it leads to a class of shell-type equations of the form (11). It was also shown that such a class of equations for a specific choice of parameters has a stationary solutions that is consistent with Kolmogorov-Obukhov law. At the same time, we are not discussing problems related to the stability of stationary solutions of the models under consideration, and this is one of the direction of future studies. In this connection, it is also of interest to study different models with off-diagonal forms of the second integral of motion.

The class of equations considered in Section 3 has discrete scale invariance, since the norm of the $p$-adic number trivially has the scaling property: $|px|_p = p^{-1}|x|_p$. However, these equations (10) are not translationally invariant on $\mathbb{Q}_p$. Note that the general equation (11) allows constructing translationally invariant hierarchical models that are determined by the choice of the functions $\nu(x, y, z)$ and $\nu(x, y)$, incoming in this equation. For example, the translation-invariant model is obviously obtained by choosing

$$\nu(x, y, z) = \sigma(x - y) \sigma(y - z) \sigma(z - x), \quad h(x, y) = h(|x - y|_p)$$

where $\sigma(x)$ is the odd function $\mathbb{Q}_p \rightarrow \mathbb{R}$: $\sigma(x) = \sigma(-x)$. Such a class of models may also be of potential interest for future studies.

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