Abstract

We give a unified RTT presentation of (super)-Yangians \( Y(\mathfrak{g}) \) for \( \mathfrak{g} = \mathfrak{so}(n), \mathfrak{sp}(2n) \) and \( \mathfrak{osp}(m|2n) \).

MSC number: 81R50, 17B37
1 Introduction

The Yangian $Y(\mathfrak{a})$ based on a simple Lie algebra $\mathfrak{a}$ is defined \cite{1,2} as the homogeneous quantisation of the algebra $\mathfrak{a}[u] = \mathfrak{a} \otimes \mathbb{C}[u]$ endowed with its standard bialgebra structure, where $\mathbb{C}[u]$ is the ring of polynomials in the indeterminate $u$. There exists for the Yangian $Y(\mathfrak{a})$ three different realisations, due to Drinfel’d \cite{1,2,3}. In the first realisation the Yangian is generated by the elements $J_0^\mathfrak{a}$ of the Lie algebra and a second set of generators $J_1^\mathfrak{a}$ in one-to-one correspondence with $J_0^\mathfrak{a}$ realising a representation space thereof. The second realisation is given in terms of generators and relations similar to the description of a loop algebra as a space of maps. However in this realisation no explicit formula for the comultiplication is known in general, except in the $sl(2)$ case \cite{4}. The third realisation uses the Faddeev–Reshetikhin–Takhtajan (FRT) formalism \cite{5}, but it is only established in the cases of classical Lie algebras.

The FRT formalism is also used as the original definition of the super Yangian $Y(gl(M|N))$ \cite{6,7}. The purpose of this paper is to define the Yangian for the orthosymplectic Lie superalgebras via the FRT formalism. As a by-product, we exhibit a unified construction which encompasses the three cases $\mathfrak{g} = so(M)$, $\mathfrak{g} = sp(N)$ and $\mathfrak{g} = osp(M|N)$. A key feature in this procedure is the explicit expression of a “quantum determinant”-like central element which coincides with that given by Drinfel’d in the $\mathfrak{g} = so(M)$ case \cite{1}.

Note that a first attempt for an FRT formulation of Yangians based on $so(M)$ and $sp(N)$ was done by Olshanski et al. \cite{8,9}. However, it led to the notion of twisted Yangians, which indeed are deformations of loop algebras on $so(M)$ and $sp(N)$, but appear as Hopf coideals rather than Hopf algebras. The same feature holds for twisted super Yangians, corresponding to $osp(M|N)$ superalgebras \cite{10}.

Known rational solutions of the Yang–Baxter equation involve $R$-matrices of the form (i) $R(u) = \mathbb{I} + \frac{P}{u}$ and (ii) $R(u) = \mathbb{I} + \frac{P}{u} - \frac{K}{u + K}$ \cite{11,12,13}. The first case, where $P$ is defined as the (super)-permutation map, is known to define the Yangians $Y(sl(N))$ and $Y(sl(M|N))$ via the FRT formalism \cite{5,14}. In the case (ii), $K$ is a partial (super)-transposition of $P$. Some $R$-matrices of this type occur as factorised $S$-matrices of quantum field models in two dimensions exhibiting the $so(M)$ symmetry \cite{11}.

We will show that the $R$-matrix $R(u) = \mathbb{I} + \frac{P}{u} - \frac{K}{u + K}$ can be used to define the Yangian $Y(\mathfrak{g})$ within the RTT formalism, for $\mathfrak{g} = osp(M|N)$ ($N$ even) as well as for the cases of $\mathfrak{g} = so(M)$ or $\mathfrak{g} = sp(N)$ ($N$ even). We prove that the algebra defined this way is indeed a quantisation of $\mathfrak{g}[u]$ endowed with its canonical bialgebra structure.

The letter is organised as follows. In Section \ref{sec:2} after some definitions, we introduce for each $\mathfrak{g}$ a rational $R$-matrix expressed in terms of the (super)-permutation and of its partial transposition. We check that it satisfies the (super) Yang–Baxter equation in all cases. In Section \ref{sec:3} we define a (super)-algebra through the RTT formalism. We establish that the quotient of this algebra by the quantum determinant-like central element is the Yangian $Y(\mathfrak{g})$, as defined in \cite{1,2}.
2 General setting

Let $gl(M\mid N)$ be the $\mathbb{Z}_2$-graded algebra of $(M+N)\times(M+N)$ matrices $X_{ij}$. Let $\theta_0 = \pm 1$. The $\mathbb{Z}_2$-gradation is defined by $(-1)^{[i]} = \theta_0$ if $1 \leq i \leq M$ and $(-1)^{[i]} = -\theta_0$ if $M+1 \leq i \leq M+N$. We will always assume that $N$ is even. The following construction yields the $osp(M\mid N)$ Yangian, and it will lead to the non-super Yangians by taking $N = 0$, $\theta_0 = 1$ (orthogonal case) or $M = 0$, $\theta_0 = -1$ (symplectic case).

**Definition 2.1** For each index $i$, we introduce a sign $\theta_i$

\[
\theta_i = \begin{cases} 
+1 & \text{for } 1 \leq i \leq M + \frac{N}{2} \\
-1 & \text{for } M + \frac{N}{2} + 1 \leq i \leq M + N
\end{cases}
\]

(2.1)

and a conjugate index $\bar{i}$

\[
\bar{i} = \begin{cases} 
M + 1 - i & \text{for } 1 \leq i \leq M \\
2M + N + 1 - i & \text{for } M + 1 \leq i \leq M + N
\end{cases}
\]

(2.2)

In particular $\theta_i \theta_{\bar{i}} = \theta_0 (-1)^{[i]}$.

As usual $E_{ij}$ denotes the elementary matrix with entry 1 in row $i$ and column $j$ and zero elsewhere.

**Definition 2.2** For $A = \sum_{ij} A^{ij} E_{ij}$, we define the transposition $t$ by

\[
A^t = \sum_{ij} (-1)^{[i][j]} \theta_i \theta_j A^{ij} E_{\bar{j}i} = \sum_{ij} (A^{ij})^t E_{ij}
\]

(2.3)

It satisfies $(A^t)^t = A$ and, for $\mathbb{C}$-valued matrices, $(AB)^t = B^t A^t$.

We shall use a graded tensor product, i.e. such that, for $a$, $b$, $c$ and $d$ with definite gradings, $(a \otimes b)(c \otimes d) = (-1)^{[b][c]} ac \otimes bd$.

**Definition 2.3** Let $P$ be the (super)permutation operator (i.e. $X_{21} \equiv PX_{12}P$)

\[
P = \sum_{i,j=1}^{M+N} (-1)^{[j]} E_{ij} \otimes E_{ji}
\]

(2.4)

and

\[
K \equiv P^{t_1} = \sum_{i,j=1}^{M+N} (-1)^{[i]} \theta_i \theta_j E_{\bar{j}i} \otimes E_{ji}
\]

(2.5)

where $t_1$ is the transposition in the first space of the tensor product. In particular $P_{21} = P_{12}$ and $K_{21} = K_{12}$.

We define the $R$-matrix

\[
R(u) = I + \frac{P}{u} - \frac{K}{u + \kappa}
\]

(2.6)
Proposition 2.4 The matrix $R(u)$ satisfies

\begin{align*}
R_{12}^1(-u - \kappa) &= R_{12}(u), \quad \text{(crossing symmetry)} \tag{2.7} \\
R_{12}(u) R_{12}(-u) &= (1 - 1/u^2)I, \quad \text{(unitarity)} \tag{2.8}
\end{align*}

provided that $2\kappa = (M - N - 2)\theta_0 = (\alpha_0 + 2\rho, \alpha_0)/2$, where $\rho$ is the super Weyl vector and $\alpha_0$ the longest root.

Proof: we use the fact that the operators $P$ and $K$ satisfy

\begin{align*}
P^2 &= I, \quad PK = KP = \theta_0 K, \quad \text{and} \quad K^2 = \theta_0 (M - N)K \tag{2.9}
\end{align*}

\[\blacksquare\]

Theorem 2.5 The $R$-matrix $[2.6]$ satisfies the super Yang–Baxter equation

\begin{align*}
R_{12}(u) R_{13}(u + v) R_{23}(v) = R_{23}(v) R_{13}(u + v) R_{12}(u) \tag{2.10}
\end{align*}

for $2\kappa = (M - N - 2)\theta_0$, where the graded tensor product is understood.

Proof: we use the following relations obeyed by the matrices $P$ and $K$

\begin{align*}
P_{13} K_{23} &= K_{12} P_{13} \quad &K_{13} K_{12} &= P_{23} K_{12} \\
P_{12} P_{23} K_{12} &= \theta_0 P_{13} K_{12} \quad &P_{12} K_{23} K_{12} &= \theta_0 K_{13} K_{12} \\
K_{12} K_{13} K_{23} &= \theta_0 P_{13} K_{23} \quad &K_{12} P_{23} K_{12} &= K_{12} K_{12} \tag{2.11}
\end{align*}

These relations are obtained by direct computation using the definition of the matrices $P$ and $K$.

\[\blacksquare\]

In the case related to $so(N)$, this solution of the Yang–Baxter equation with spectral parameter was found in [11]. It is also one of the cases explored in [12].

3 Yangians

We consider the Hopf (super)algebra $\mathcal{U}(R)$ generated by the operators $T_{(n)}^{ij}$, for $1 \leq i, j \leq M + N$, $n \in \mathbb{Z}_{\geq 0}$, encapsulated into a $(M + N) \times (M + N)$ matrix

\begin{align*}
T(u) &= \sum_{n \in \mathbb{Z}_{\geq 0}} T_{(n)} u^{-n} = \sum_{i,j=1}^{M+N} \sum_{n \in \mathbb{Z}_{\geq 0}} T_{(n)}^{ij} u^{-n} E_{ij} = \sum_{i,j=1}^{M+N} T^{ij}(u) E_{ij} \tag{3.1}
\end{align*}

and $T_{(0)}^{ij} = \delta_{ij}$. One defines $\mathcal{U}(R)$ by imposing the following constraints on $T(u)$

\begin{align*}
R_{12}(u - v) T_1(u) T_2(v) = T_2(v) T_1(u) R_{12}(u - v) \tag{3.2}
\end{align*}
with the matrix $R(u)$ defined in (2.6).

The explicit commutation relations between the generating operators $T^{ij}(u)$ read

$$
[T^{ij}(u), T^{kl}(v)] = \frac{(-1)^{|k||i|+|k||j|}}{u - v} \left( T^{kj}(v)T^{il}(u) - T^{kj}(u)T^{il}(v) \right) + \frac{1}{u - v + \kappa} \sum_p \left( \delta_{ik} (-1)^{|p|+|j|+|k|} \theta_{i\theta_p} T^{pj}(u) T^{pl}(v) 

- \delta_{lj} (-1)^{|k||j|+|k||i|+|i|} \theta_{p\theta_j} T^{kp}(v) T^{ip}(u) \right)
$$

(3.3)

that is, in terms of the generators $T^{ij}_{(n)}$

$$
[T^{ij}_{(r+2)}, T^{kl}_{(s)}] + [T^{ij}_{(r)}, T^{kl}_{(s+2)}] = 2[T^{ij}_{(r+1)}, T^{kl}_{(s+1)}] - \kappa [T^{ij}_{(r+1)}, T^{kl}_{(s)}] + \kappa [T^{ij}_{(r)}, T^{kl}_{(s+1)}] + (-1)^{|k||i|+|k||j|} T^{kj}_{(s+1)} T^{il}_{(r+1)} - T^{kj}_{(s+1)} T^{il}_{(r+1)} 

+ \kappa T^{kj}_{(s)} T^{il}_{(r)} - \kappa T^{kj}_{(r)} T^{il}_{(s)} + \sum_p \left( \delta_{ik} (-1)^{|p|+|j|+|k|+|i|} \theta_{i\theta_p} (T^{pj}_{(r+1)} T^{pl}_{(s)} - T^{pj}_{(r+1)} T^{pl}_{(s+1)}) 

- \delta_{lj} (-1)^{|k||j|+|k||i|+|i|} \theta_{p\theta_j} (T^{kp}_{(s)} T^{ip}_{(r+1)} - T^{kp}_{(s+1)} T^{ip}_{(r+1)}) \right)
$$

(3.4)

where $r, s \geq -2$ with by convention $T^{ij}_{(n)} = 0$ for $n < 0$.

The Hopf algebra structure of $U(R)$ is given by [5]

$$
\Delta(T(u)) = T(u) \otimes T(u) \quad \text{i.e.} \quad \Delta(T^{ij}(u)) = \sum_{k=1}^{M+N} T^{ik}(u) \otimes T^{kj}(u)
$$

(3.5)

$$
S(T(u)) = T(u)^{-1} \quad ; \quad \epsilon(T(u)) = \mathbb{I}_{M+N}
$$

(3.6)

**Theorem 3.1** The operators generated by $C(u) = T^i(u - \kappa)T(u)$ lie in the centre of the algebra $U(R)$ and $C(u) = c(u) \mathbb{I}$. Furthermore, $\Delta(c(u)) = c(u) \otimes c(u)$ and the two-sided ideal $\mathcal{I}$ generated by $C(u) - \mathbb{I}$ is also a coideal. The quotient $U/\mathcal{I}$ is then a Hopf algebra.

**Proof:** We first prove that $C(u)$ is diagonal. Indeed, the relation (3.2) implies

$$
K_{12} T_1(u - \kappa) T_2(u) = T_2(u) T_1(u - \kappa) K_{12}
$$

(3.7)

from which it follows, after having transposed in space $p$

$$
\sum_{ijkl} (-1)^{|k|} T^i(u - \kappa)^{ij} T(u)^{jl} E_{ik} \otimes E_{kl} = \sum_{pqrs} (-1)^{|p|+|q|+|r|+|s|} T(u)^{pq} T^i(u - \kappa)^{qr} E_{sr} \otimes E_{ps}
$$

(3.8)

Therefore, one has

$$
\sum_j T^i(u - \kappa)^{ij} T(u)^{jl} = \delta_{il} c(u) \quad \text{or} \quad C(u) = c(u) \mathbb{I}
$$

(3.9)
We define

\[ C(u) T_2(v) = T_1(u - \kappa) T_1(u) T_2(v) = T_1(u - \kappa) R_{12}^{-1}(u - v) T_2(v) T_1(u) R_{12}(u - v) \]  

(3.10)

where we have used the unitarity and crossing properties (2.8) and (2.7) of \( R(u) \). Now using the transposition of the relation (3.2) in space 1 and the crossing property of \( R(u) \), one can derive the following exchange relation:

\[ T_1(u - \kappa) R_{12}^{-1}(u - v) T_2(v) = T_2(v) R_{12}^{-1}(u - v) T_1(u - \kappa) \]  

(3.11)

Hence

\[ C(u) T_2(v) = T_2(v) R_{12}^{-1}(u - v) T_1(u - \kappa) T_1(u) R_{12}(u - v) \]

\[ = T_2(v) R_{12}^{-1}(u - v) C(u) R_{12}(u - v) \]  

(3.12)

Since \( C(u) = c(u) \mathbb{1} \), one obtains easily \( C(u) T_2(v) = T_2(v) C(u) \).

From the defining relations of \( C(u) \) the coproduct of \( c(u) \) is straightforwardly obtained as \( \Delta(c(u)) = c(u) \otimes c(u) \) which shows that \( I \) is a coideal. It is interesting to note that this is precisely the structure of the coproduct of the quantum determinant whenever such an object has been constructed. 

At order \( u^{-1} \) the equation \( C(u) = \mathbb{1} \) yields the relation \( T_{(1)}^u + T_{(1)} = 0 \). Note that those linear relations \( T_{(1)}^{ij} + T_{(1)}^{ij} = 0 \) for which \( i \neq j \) were already implied by the commutation relations (3.4). At higher orders, \( C(u) = \mathbb{1} \) induces relations with the generic form \( T_{(n)}^u + T_{(n)} = \mathcal{F}(T_{(m)}, m < n) \) where \( \mathcal{F} \) is a quadratic function.

In particular, once the exchange relations (3.3) (for \( r = s = 0 \)) are taken into account, the generators \( T_{(1)}^{ij} \) exhibit the structure of the Lie (super) algebra \( \mathfrak{g} \).

**Definition 3.2** Let \( \mathfrak{g} \) be a finite dimensional complex simple Lie (super) algebra. We define the bialgebra \( \mathfrak{g}[u] \) as \( \mathfrak{g} \otimes \mathbb{C}[u] \) endowed with the Poisson cobracket \( \delta \) defined by

\[ \delta f(u, v) = 2 \left[ \mathbb{1} \otimes f(v) + f(u) \otimes \mathbb{1}, \frac{\mathfrak{C}}{u - v} \right] \]  

(3.13)

where \( \mathfrak{C} \) is the tensorial Casimir element of \( \mathfrak{g} \) associated with a given non-degenerate invariant bilinear form \( \mathfrak{B} \), and \( f : \mathbb{C} \to \mathfrak{g} \) is a polynomial map, i.e. an element of \( \mathfrak{g}[u] \).

**Theorem 3.3** Let \( \mathfrak{g} \) be a finite dimensional complex simple Lie (super) algebra of type \( \text{so}(M) \), \( \text{sp}(N) \), \( \text{osp}(M|N) \). Let \( \mathcal{U}(R) \) be the Hopf algebra with generators \( T(u) \) subject to the relations (3.2) and Hopf structure (3.3)-(3.0). The quotient of the algebra \( \mathcal{U}(R) \) by the two-sided ideal \( I \) generated by \( C(u) = T^u(u - \kappa) T(u) = \mathbb{1} \) (i.e. \( c(u) = 1 \)) is a homogeneous quantisation of \( (\mathfrak{g}[u], \delta) \).

**Proof:** We define \( \mathcal{U}_h \) as the algebra generated by the generating functional \( \tilde{\mathcal{I}}(u) \)

\[ \tilde{\mathcal{I}}(u) = \frac{1}{\hbar} \left( T(u/h) - 1 \right) \]  

(3.14)
and the identity, the relations being derived from those of $\mathcal{U}(R)$, i.e.

$$
[\tilde{t}_1(u), \tilde{t}_2(v)] = \left[ \tilde{t}_1(u) + \tilde{t}_2(v), \frac{P}{u-v} \right] - \frac{\hbar}{u-v} (P\tilde{t}_1(u)\tilde{t}_2(v) - \tilde{t}_1(u)\tilde{t}_2(v)P) \\
- \left[ \tilde{t}_1(u) + \tilde{t}_2(v), \frac{K}{u-v+h\kappa} \right] + \frac{\hbar}{u-v+h\kappa} (K\tilde{t}_1(u)\tilde{t}_2(v) - \tilde{t}_1(u)\tilde{t}_2(v)K).
$$

(3.15)

Thus the relations in $\mathcal{U}_h/(h\mathcal{U}_h)$ are

$$
[\tilde{t}_1(u), \tilde{t}_2(v)] = \left[ \tilde{t}_1(u) + \tilde{t}_2(v), \frac{P-K}{u-v} \right].
$$

(3.16)

The equation $C(u) = \mathbb{I}$ expressed in $\mathcal{U}_h$ generate a two-sided ideal $\mathcal{I}_h$, which now induces relations with the generic form $\tilde{t}_i^{(n)} + \tilde{t}_j^{(n)} = hF(\tilde{t}_i^{(m)}, m < n)$ where $F$ is a quadratic function. In the quotient algebra $\mathcal{U}_h/(h\mathcal{U}_h)$ this becomes equivalent to the standard linear symmetrisation relation $J^{(n)}_i + J^{(n)}_j = 0$ for the generators of the loop algebra $\mathfrak{g}[u]$, so that $\mathcal{U}_h/(h\mathcal{U}_h) \simeq \mathcal{U}(\mathfrak{g}[u])$ as algebras, for $\mathcal{U}_h \equiv \mathcal{U}/\mathcal{I}_h$. This characterises $\mathcal{U}_h$ as a quantisation of the algebra $\mathcal{U}(\mathfrak{g}[u])$.

We now examine the coproduct structure in order to recognise it as a quantisation of the cocommutator $\delta$, namely

$$
\frac{\Delta - \Delta^{op}}{\hbar} \left( \tilde{t}(u) \right) \mod h = \delta \left( \tilde{t}(u) \right) \mod h
$$

(3.17)

From (3.5), the order $u^{-n}$ of the $(i, j)$ entry of the left hand side of this formula reads

$$
\frac{\Delta - \Delta^{op}}{\hbar} \tilde{t}^{(m)} \mod h = \sum_{r=0}^{m} \left( \tilde{t}_r \otimes \tilde{t}_{(m-r)} - \tilde{t}_{(r)} \otimes \tilde{t}_{(m-r)} \right) \mod h.
$$

(3.18)

Now, denoting generically $\tilde{t}^a = \tilde{t}^j - (\tilde{t}^j)^t$ and $E_a = E_{ij} - (E_{ij})^t$, and using $\tilde{t} = \tilde{t}^t \mod h$, we can symmetrise and get

$$
\sum_c \frac{\Delta - \Delta^{op}}{\hbar} \tilde{t}^{(m)} E_c \mod h = \sum_{a,b} \sum_{r=0}^{m} \tilde{t}_r^a \otimes \tilde{t}_{(m-r)}^b [E_a, E_b] \mod h
$$

$$
= \sum_{a,b} \sum_{r=0}^{m} \tilde{t}_r^a \otimes \tilde{t}_{(m-r)}^b f_{ab}^c E_c \mod h.
$$

(3.19)

The right hand side of the formula (3.17) can be computed once one recalls that $\mathcal{C} = \sum_{ab} \mathfrak{B}_{ab} t^a \otimes t^b$.

One obtains

$$
\delta(t^{a}_{(m)}) = \sum_{a,b} \sum_{r=0}^{m} t^{a}_{(r)} \otimes t^{b}_{(m-r)} f^{c}_{ab}
$$

(3.20)

where the $t^{a}_{(m)}$ denote the generators of the loop algebra $\mathfrak{g}[u]$. Since the modes of $(\tilde{t}(u) \mod h)$ coincide with the $t^{a}_{(m)}$ and the structure constants $f^{c}_{ab}$ and $f^{c}_{ab}$ are identified through the bilinear form $\mathfrak{B}$, one gets the desired result (3.17).

Therefore the Hopf algebra $\mathcal{U}(R)/\mathcal{I} \equiv \mathcal{U}_{h=1}$ is a quantisation of $\mathcal{U}(\mathfrak{g}[u])$ and $\Delta$ is a quantisation of $\delta$.  

\[ \boxed{} \]
From the above theorem, we are naturally led to the following definition:

**Definition 3.4** We define the Yangian of $osp(M|N)$ as $Y(\mathfrak{g}) \equiv \mathcal{U}(R)/\mathcal{I}$. Explicitly, its defining relations are given by

$$ R_{12}(u - v) T_1(u) T_2(v) = T_2(v) T_1(u) R_{12}(u - v) , $$

$$ C(u) = T^t(u - \kappa) T(u) = \mathbb{I}, $$

where $R_{12}(u) = \mathbb{I} + \frac{P}{u} - \frac{K}{u + \kappa}. $

For $N = 0$ or $M = 0$, this definition is consistent with the one of Drinfel’d [1, 2] for the $so(M)$ and $sp(N)$ cases respectively.

**Remark:** The explicit $R$-matrices for the Yangians $Y(so(N))$ and $Y(sp(N))$ can be obtained by taking the scaling limit $q \to 1, \ z = q^z \to 1$ keeping $u$ fixed, of the evaluated trigonometric $R$-matrices of $U_q(so(N))$ and $U_q(sp(N))$ computed in [13]. Similarly, one can show that the $R$-matrix of $Y(osp(1|2))$ is the scaling limit of the evaluated trigonometric $R$-matrix of $U_q(osp(1|2))$ [16].

### 4 Twisted Yangians and reflection algebras

We would finally like to comment upon a possible connection between the notions of twisted Yangians and reflection algebras within the framework of this Yangian construction.

Following the lines of [8, 9] (see also [10] for the supersymmetric case), we define on $\mathcal{U}(R)$:

$$ \tau[T(u)] = T^t(-u - \kappa) $$

which reads for the super-Yangian generators:

$$ \tau(T^{ab}(u)) = (-1)^{|a||b| + 1} \theta_a \theta_b T^{\bar{a}\bar{b}}(-u - \kappa) $$

$\tau$ is an algebra automorphism, as a direct consequence of unitarity, crossing symmetry and the property $R_{t_1 t_2}(u) = R(u)$ which itself comes from $P^{t_1 t_2} = P$.

The twisted super-Yangian $\mathcal{U}(R)^{tw}$ is the subalgebra generated by $S(u) = \tau[T(u)]T(u)$, with $\tau$ given in (4.2). $S(u)$ obeys the following relation:

$$ R_{12}(u - v) S_1(u) R_{12}(u + v) S_2(v) = S_2(v) R_{12}(u + v) S_1(u) R_{12}(u - v) $$

It is easy to show that $\mathcal{U}(R)^{tw}$ is a coideal in $\mathcal{U}(R)$.

Similarly, one introduces the notion of reflection algebras $\mathcal{S}(R)$, generated by

$$ B(u) = T^{-1}(-u)T(u) $$

which obeys the same relation (4.3), interpreted here as a reflection equation. $\mathcal{S}(R)$ is also a coideal of $\mathcal{U}(R)$. This type of algebras have been originally introduced in [17] for the Yangian $Y(N)$, based
on \( \mathfrak{gl}(N) \), and play an important role in integrable systems with boundaries (see e.g. [18]). However in the coset \( \mathcal{U}(R)/\mathcal{Z} \), one has \( B(u) = S(u) \), so that \( S(R) \) and \( \mathcal{U}(R)^{tw} \) are two versions of the same Hopf coideal in \( \mathcal{U}(R) \). The situation is here different from the case of the Yangian \( Y(N) \). Indeed the twisted Yangians \( Y^\pm(N) \) and the boundary algebras \( B(N, \ell) \) are known to be different for \( N > 2 \), whilst for \( N = 2 \) one has \( B(2, 0) = Y^-(2) \) and \( B(2, 1) = Y^+(2) \) [19].

Acknowledgements: We would like to thank A. Molev and V. Tolstoy for discussions and comments.

References

[1] V.G. Drinfel’d, Hopf algebras and the quantum Yang–Baxter equation, Soviet. Math. Dokl. 32 (1985) 254–258.
[2] V.G. Drinfel’d, Quantum Groups, Proceedings Int. Cong. Math. Berkeley, California, USA (1986) 798–820.
[3] V.G. Drinfel’d, A new realization of Yangians and quantized affine algebras, Soviet. Math. Dokl. 36 (1988) 212–216.
[4] A.I. Molev, Yangians and their applications, Handbook of Algebra, vol. 3, Elsevier, to appear.
[5] L.D. Faddeev, N.Yu. Reshetikhin and L.A. Takhtajan, Quantization of Lie groups and Lie algebras, Leningrad Math. J. 1 (1990) 193–225.
[6] M.L. Nazarov, Quantum Berezinian and the classical Capelli identity, Lett. Math. Phys. 21 (1991) 123–131.
[7] R.B. Zhang, The \( \mathfrak{gl}(M|N) \) super Yangian and its finite dimensional representations, Lett. Math. Phys. 37 (1996) 419–434.
[8] G.I. Olshanski, Twisted Yangians and infinite dimensional Lie algebras, in “Quantum groups”, Lecture Notes in Math. 1510 (P. Kulish ed.), pp. 104–120, NY 1992.
[9] A. Molev, M. Nazarov and G. Olshanski, Yangians and classical Lie algebras, Russ. Math. Surveys 51 (1996) 205–282, hep-th/9409025.
[10] C. Briot and E. Ragoucy, Twisted superYangians and their representations, preprint LAPTH-875/01, math.QA/0111308.
[11] Al. B. Zamolodchikov, Al. B. Zamolodchikov, Relativistic factorized S-matrix in two dimensions having \( O(N) \) isotropic symmetry, Nucl. Phys. B133 (1978) 525–535 and Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field models, Ann. Phys. 120 (1979) 253–291.
[12] P.P. Kulish, E.K. Sklyanin, *Solutions of the Yang–Baxter equation*, Zap. Nauchn. Sem. LOMI, 95 (1980) 129–160 and J. Sov. Math. 19 (1982) 1596–1620.

[13] A.P. Isaev, *Quantum groups and Yang–Baxter equations*, Phys. Part. Nucl. 26 (1995) 501–526.

[14] A.N. Kirilov and N.Yu. Reshetikhin, *The Yangians, Bethe Ansatz and combinatorics*, Lett. Math. Phys. 12 (1986) 199-208.

[15] M. Jimbo, *Quantum R-matrix for the generalized Toda system*, Commun. Math. Phys. 102 (1986), 537–547.

[16] S. Khoroshkin and V. Tolstoy, *Twisting of quantum (super)algebras. Connection of Drinfeld’s and Cartan–Weyl realizations for quantum affine algebras*, hep-th/9404036.

[17] E.K. Sklyanin, *Boundary conditions for integrable quantum systems*, J. Phys. A21 (1988) 2375–2389.

[18] M. Mintchev, E. Ragoucy and P. Sorba, *Spontaneous symmetry breaking in the gl(N)-NLS algebra*, J. Phys. A34 (2001) 8345–8364, hep-th/0104079.

[19] A. Molev and E. Ragoucy, *Representations of boundary algebras*, Rev. Math. Phys. 14 (2002) 317–342, math.QA/0107213