Some exercises with the Lasso and its compatibility constant

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Abstract We consider the Lasso for a noiseless experiment where one has observations $X\beta^0$ and uses the penalized version of basis pursuit. We compute for some special designs the compatibility constant, a quantity closely related to the restricted eigenvalue. We moreover show the dependence of the (penalized) prediction error on this compatibility constant. This exercise illustrates that compatibility is necessarily entering into the bounds for the (penalized) prediction error and that the bounds in the literature therefore are - up to constants - tight. We also give conditions that show that in the noisy case the dominating term for the prediction error is given by the prediction error of the noiseless case.

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1 Introduction

Let $X \in \mathbb{R}^{n \times p}$ be an $n \times p$ matrix and $\beta^0 \in \mathbb{R}^p$ be a fixed vector. We consider the Lasso for the noiseless case

$$\beta^* := \arg \min_{\beta \in \mathbb{R}^p} \mathcal{L}(\beta),$$

with

$$\mathcal{L}(\beta) := \|X(\beta - \beta^0)\|_2^2 + 2\lambda\|\beta\|_1.$$

Aim in this note is to show that the upper bounds for $\|X(\beta^* - \beta^0)\|_2^2$ given in the literature (see Section 3 for some references) are also lower bounds, in the sense that there are designs where an upper bound is tight, possibly up to constants. The upper bounds that we consider depend on the so-called compatibility constant $\hat{\phi}_2^S(S)$ which we define in Definition 1.1 below. In [Zhang et al., 2014] it is shown that for a given sparsity level, there is a design and a lower bound for the mean prediction error in the noisy case, that holds for any polynomial time algorithm. This lower bound is close to the known upper bounds and in particular shows that compatibility conditions or restricted eigenvalue conditions cannot be avoided. Our aim is to make this visible for the Lasso by presenting some explicit expressions. This helps to understand why compatibility is playing a crucial role and also to understand the concept itself. Our results follow from straightforward computation for some special cases of design.

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1In the noiseless case the results apply when $\|X\beta\|_2^2$ ($\beta \in \mathbb{R}^p$) is replaced by any other quadratic form $\beta^T \Sigma \beta$ ($\beta \in \mathbb{R}^p$) with $\Sigma$ a given $p \times p$ matrix. The “sample size” $n$ is playing the role of the rank of $\Sigma$. 
We will show that the upper bounds involving compatibility constants (here given in Section 3) match the lower bounds “up to constants” or even “asymptotically exactly” for certain designs. The designs we consider are in our view not atypical. Therefore, our conclusion is that there is not much space for improvement of the existing upper bounds.

Note that we consider a noiseless version of the Lasso. When examining lower bounds this is reasonable, as one may expect that adding noise will not improve the performance of the Lasso. We will moreover show in Section 2 that for certain designs, the “bias” \( \|X(\beta^* - \beta^0)\|_2^2 \) of the noisy Lasso is the dominating term, so that bounds for the noiseless case immediately carry over to the noisy case.

In order to be able to define the compatibility constant \( \hat{\phi}^2(S) \) we introduce here some notation. For \( S \subset \{1, \ldots, p\} \) and a vector \( \beta \in \mathbb{R}^p \) let \( \beta_{j,S} := \beta_j \{ j \in S \} \in \mathbb{R}^p \). We apply the same notation for the \(|S|\)-dimensional vector \( \{ \beta_j \}_{j \in S} \). We moreover write \( \beta_{-S} := \beta_{S^c} \) where \( S^c \) is the complement of the set \( S \). If \( S \) consists of a single variable, say \( S = \{j\} \) we write \( \beta_{-S} := \beta_{-j} \).

Definition 1.1 The compatibility constant (see van de Geer [2007] or van de Geer [2016] and its references) is

\[
\hat{\phi}^2(L, S) := \min \left\{ |S| \|X\beta\|_2^2 : \|\beta_S\|_1 = 1, \|\beta_{-S}\|_1 \leq L \right\}.
\]

The constant \( L \geq 1 \) is called a stretching factor. For \( L = 1 \) we write \( \hat{\phi}^2(S) := \hat{\phi}^2(1, S) \). When \( S = \{1, \ldots, p\} \) we let \( \hat{\phi}^2(S) := \min\{|S|\|X\beta_S\|_2^2 : \|\beta_S\|_1 = 1\} \). For \( S = \emptyset \) we set \( |S| / \hat{\phi}^2(S) = 0 \).

The compatibility constant \( \hat{\phi}^2(L, S) \) with stretching constant \( L > 1 \) can play a role when considering the noisy situation. In this paper however, we mainly study the noiseless case and take \( L = 1 \). A noisy case where \( L \) can be taken equal to 1 is considered in Section 2.

It is sometimes helpful to consider \( \hat{\Gamma}^2(S) := |S| / \hat{\phi}^2(S) \) as the effective sparsity\footnote{A better terminology is perhaps to call \( \hat{\Gamma}^2(S) \) the effective non-sparsity} at the set \( S \) (van de Geer [2016]). Two sets should be compared in terms of their effective sparsity rather than in terms of their compatibility constants, in the sense that that we prefer sets \( S \) with \( \hat{\Gamma}^2(S) \) small.

The compatibility constant \( \hat{\phi}^2(S) \) depends on the set \( S \) and clearly also on the design \( X \) through the Gram matrix \( \hat{\Sigma} := X^TX \). We express the latter dependence in our notation by the “hat”. This is a habit coming from the case of random design, where \( \hat{\Sigma} \) is an estimator of \( \mathbb{E} \Sigma \) (in statistics, estimators are commonly denoted with a “hat”). However, to avoid a cumbersome notation, not all quantities depending on \( X \) with be furnished with a “hat”.
1.1 Notation

Let $X_j$ denote the $j$-th column of $X$ ($j = 1, \ldots, p$). The Gram matrix is $\Sigma := X^T X$.

The active set (or support set) of $\beta_0$ is $S_0 := \{j : \beta_0^j \neq 0\}$. If $j \in S_0$ we call $j$- or $X_j$ - an active variable. When $\hat{\phi}^2(S_0) > 0$ one says that the null space property holds [Donoho and Tanner [2005]). The cardinality of $S_0$ is denoted by $s_0 := |S_0|$. We moreover write the cardinality of the set $S_0^c$ of inactive variables as $m_0 := p - s_0$.

1.2 Organization of the paper

Section 2 shows how the results for the noiseless case carry over to the noisy case when the Gram matrix (or an approximation thereof) has bounded maximal eigenvalue and $\sqrt{n}\lambda$ is large ($\sqrt{n}\lambda \to \infty$). Such a choice for the tuning parameter $\lambda$ corresponds to $p$ large, as can be the case in most of the following sections (Sections 7, 8, 10, 11, 13, and the last result of Section 14). Section 3 states some upper bounds for the (penalized) prediction error of the noiseless Lasso. These bounds are not novel, but as constants may now come into play, we have re-derived them with an eye on the constants for the special situation with no noise. Section 4 has some considerations about the design: we assume it to be “fair” as defined there. Then, in the rest of the paper, we take the first two variables as being among the active ones. In Section 5 we present the structure (design and coefficients) for these first two variables. Section 6 considers the case $p = s_0 = 2$: it has no inactive variables. This is extended in Section 7 where $p = s_0 = 2N$ (for some $N \in \mathbb{N}$) is even. The next step is to start adding inactive variables. Section 8 contains a trivial case, where the inactive variables are orthogonal to the active ones. Section 9 has $s_0 = 2$ and $m_0 = 1$ and the single inactive variable is a linear combination of the two active ones plus an orthogonal term: the active variables are so to speak the “parents” of the inactive one. Section 10 extends this to $s_0 = 2N$ even and $m_0 = 1$. Section 11 returns to the case $s_0 = 2$, but now $m_0$ is arbitrary. The active variables are again “parents” of all the inactive ones. In Section 12 we take $s_0$ as well as $m_0$ equal to 2, but now part of the correlation between the two inactive variables is unique to those two, i.e., their correlation is not solely due to having the active ones as common “parents”. Section 13 extends this to $s_0 = m_0 = 2N$. In Section 14 the active variables are a linear combination of the inactive ones plus orthogonal term: the inactive ones are now presented as the “parents” of the active ones instead of the other way around. Section 15 contains the proofs.

For a symmetric matrix $A$ we let $\Lambda_{\min}(A)$ be its smallest and $\Lambda_{\max}(A)$ be its largest eigenvalue.

For two constants $u$ and $v$ we let $u \lor v := \max\{u, v\}$ (and $u \land v := \min\{u, v\}$). For $N \in \mathbb{N}$ and a vector $w \in \mathbb{R}^N$ and a real-valued function $f$ we define the
vector $f(w)$ as $f(w) := (f(w_1), \ldots, f(w_N))^T$.

2 The noisy case

This section studies the noisy model

$$Y = X\beta^0 + \epsilon,$$

where $Y$ is an $n$-vector of observations and with $\epsilon = (\epsilon_1, \ldots, \epsilon_n)^T$ containing i.i.d. $\mathcal{N}(0, 1/n)$-distributed noise variables. We will compare the noisy Lasso

$$\hat{\beta} := \arg\min_{\beta \in \mathbb{R}^p} \left\{ \|Y - X\beta\|_2^2 + 2\lambda \|\beta\|_1 \right\}$$

with the noiseless Lasso

$$\beta^* := \arg\min_{\beta \in \mathbb{R}^p} \left\{ \|X(\beta - \beta^0)\|_2^2 + 2\lambda \|\beta\|_1 \right\}.$$

We show in the next two theorems that under certain conditions on the design the “bias” $\|X(\hat{\beta} - \beta^0)\|_2$ is of larger order (in probability) than the “estimation error” $\|X(\hat{\beta} - \beta^*)\|_2$ (where “bias” and “estimation error” are here to be understood in generic terms). By the triangle inequality

$$\|X(\hat{\beta} - \beta^0)\|_2 \geq \|X(\hat{\beta} - \beta^*)\|_2 - \|X(\hat{\beta} - \beta^*)\|_2$$

this implies a high probability lower bound for the prediction error $\|X(\hat{\beta} - \beta^0)\|_2$ of the noisy Lasso in terms of the prediction error $\|X(\beta^* - \beta^0)\|_2$ of the noiseless Lasso.

**Theorem 2.1** Let $\|X_j\|_2 \leq 1$ for all $j$, and let $0 < \alpha < 1$ and $0 < \alpha_1 < 1$ be fixed and $\lambda_0 := \sqrt{2\log(2p/\alpha)/n}$. Let $\eta \lambda > \lambda_0$ for some $0 \leq \eta < 1$. Then with probability at least $1 - \alpha - \alpha_1$

$$\|X(\hat{\beta} - \beta^*)\|_2 \leq \sqrt{\frac{\Lambda_{\max}(\hat{\Sigma})}{n\lambda^2(1-\eta)^2}} \|X(\beta^* - \beta^0)\|_2 + \sqrt{\frac{2\log(1/\alpha_1)}{n}}.$$

**Asymptotics** We see we may choose $\lambda \asymp \sqrt{\log p/n}$. Then, for $p \to \infty$ and $\Lambda_{\max}(\hat{\Sigma}) = \mathcal{O}(1)$ we get

$$\|X(\hat{\beta} - \beta^*)\|_2 = o_p(1)\|X(\beta^* - \beta^0)\|_2 + \mathcal{O}_p(1/\sqrt{n}).$$

In general the largest eigenvalue $\Lambda_{\max}(\hat{\Sigma})$ may be large, and may be hard to control, for example when the Gram matrix $\hat{\Sigma}$ comes from random design. We now let $\Sigma_0$ be some approximation of $\hat{\Sigma}$, for example a population version $\mathbb{E}\hat{\Sigma}$ of $\Sigma_0$ in the case of random design.

We use the notation $\|\hat{\Sigma} - \Sigma_0\|_\infty := \max_{j,k} |\hat{\Sigma}_{j,k} - \Sigma_{0,j,k}|$. 

4
Theorem 2.2 Let \( \|X_j\|_2 \leq 1 \) for all \( j \), and let \( 0 < \alpha < 1 \) and \( 0 < \alpha_1 < 1 \) be fixed and \( \lambda_0 := \sqrt{2 \log(2p/\alpha)/n} \). Let \( \eta \lambda > \lambda_0 \) for some \( 0 \leq \eta < 1 \). Suppose that

\[
\xi := \|\hat{\Sigma} - \Sigma_0\|_\infty \|\beta^* - \beta^0\|_1 < \lambda(1 - \eta). \tag{1}
\]

Then with probability at least \( 1 - \alpha - \alpha_1 \)

\[
\|X(\hat{\beta} - \beta^*)\|_2 \leq \frac{S(\beta^* - \beta^0)}{\lambda_{\max}(\Sigma_0)} \left( \frac{\|X(\beta^* - \beta^0)\|^2_2 + \xi \|\beta^* - \beta^0\|_1}{\lambda(1 - \eta) - \xi} \right)^{1/2} + \frac{2 \log(1/\alpha_1)}{n}.
\]

Condition (1) is a condition requiring the \( \ell_1 \)-error of \( \beta^* \) to be small. In an asymptotic setup, it typically needs sparsity \( s_0 \) of small order \( \sqrt{n/\log p} \). However, in the case of Gaussian random design for example and \( \Sigma_0 = E\Sigma \) one may apply more careful bounds to prove a result that does not require such sparsity conditions.

3 Upper bounds

There are several upper bounds in the literature. The one we will mainly apply is along the lines of Theorem 6.1 in B"uhlmann and van de Geer [2011], with some refinements. The result is given in Lemma 3.1. There are however more general bounds in literature, in particular sharp oracle bounds as in Koltchinskii et al. [2011] (see also Giraud [2014], Theorem 4.1 or van de Geer [2016], Theorem 2.2). We present these in Lemma 3.2.

The upper bounds follow from the KKT (Karush-Kuhn-Tucker) conditions

\[
\hat{\Sigma}(\beta^* - \beta^0) + \lambda z^* = 0.
\]

Here \( z^* \in \partial \|\beta^*\|_1 \) with \( \partial \|\beta\|_1 \) the sub-differential of the mapping \( \beta \mapsto \|\beta\|_1, \beta \in \mathbb{R}^p \). In other words \( \beta^*^T z^* = \|\beta^*\|_1 \) and \( \|z^*\|_\infty \leq 1 \).

Here are the upper bounds for the prediction error we will use. They include upper bounds for \( \|\beta^*\|_1 \) and \( \|\beta^*_{-S_0}\|_1 \).

Lemma 3.1 It holds that

\[
\|X(\beta^* - \beta^0)\|^2_2 + \lambda \|\beta^*\|_1 \leq \lambda \|\beta^0\|_1,
\]

and

\[
\|X(\beta^* - \beta^0)\|^2_2 + 2\lambda \|\beta^*_{-S_0}\|_1 \leq \frac{\lambda^2 s_0}{\phi^2(S_0)}.
\]

The next lemma contains the more general sharp oracle inequalities for the prediction error.
Lemma 3.2 The prediction error \( \|X(\beta^* - \beta^0)\|_2^2 \) satisfies the bound
\[
\|X(\beta^* - \beta^0)\|_2^2 \leq \hat{U}(\beta^0),
\]
where
\[
\hat{U}(\beta^0) = \min\{\hat{U}_I(\beta^0), \hat{U}_{II}(\beta^0), \hat{U}_{III}(\beta^0)\}
\]
with
\[
\hat{U}_I(\beta^0) := \frac{\lambda^2 s_0}{\hat{\phi}^2(S_0)} \wedge \lambda \|eta^0\|_1
\]
\[
\hat{U}_{II}(\beta^0)
:= \min_{S} \left\{ \left( \frac{\lambda^2 |S|}{4\hat{\phi}^2(S)} + \frac{\lambda^2 |S|}{4\hat{\phi}^2(S)} + \lambda \|eta^0\|_1 \right)^2 \vee 2\lambda \|eta^0\|_1 \right\}
\]
\[
\hat{U}_{III}(\beta^0)
:= \min_{S} \min_{\beta} \left\{ \|X(\beta - \beta^0)\|_2^2 + \frac{\lambda^2 |S|}{\hat{\phi}^2(S)} + 2\lambda \|eta - \beta^0\|_1 \vee 4\lambda \|eta - \beta^0\|_1 \right\}
\].

Clearly, if
\( \circ \) the minimum over \( S \) in the definition of \( \hat{U}_{II}(\beta^0) \) is attained in \( S_0 \),
\( \circ \) the minimum over \( (S, \beta) \) in the definition of \( \hat{U}_{III}(\beta^0) \) is attained in \( (S_0, \beta^0) \), and
\( \circ \lambda^2 s_0/\hat{\phi}^2(S_0) \leq \lambda \|eta^0\|_1 \), then
\[
\hat{U}_I(\beta^0) = \hat{U}_{II}(\beta^0) = \hat{U}_{III}(\beta^0) = \frac{\lambda^2 s_0}{\hat{\phi}^2(S_0)}.
\]

This will be the case in most of the examples we consider in this paper, that is, we do not explore the power of the sharp oracle inequalities of Lemma 3.2. Instead, we mainly compare exact results for the (penalized) prediction error with the bounds of Lemma 3.1.

Remark 3.1 Clearly, Lemma 3.2 implies the bound \( \hat{U}_I(\beta^0) \). Further, by restricting \( S \) in the minimization giving \( \hat{U}_{II}(\beta^0) \) to \( S \in \{S_0, \emptyset\} \) one sees
\[
\hat{U}_{II}(\beta^0) \leq \frac{\lambda^2 s_0}{\hat{\phi}^2(S_0)} \vee 2\lambda \|eta^0\|_1.
\]

In other words, up to a factor “2”, the bound \( \hat{U}_{II}(\beta^0) \) improves upon \( \hat{U}_I(\beta^0) \). Similarly, taking \( \beta = \beta^0 \) in the minimization giving \( \hat{U}_{III}(\beta^0) \) one finds
\[
\hat{U}_{III}(\beta^0) \leq \left( \frac{\lambda^2 |S|}{\hat{\phi}^2(S)} + 2\lambda \|eta^0 - \beta^0\|_1 \right) \vee 4\lambda \|eta^0 - \beta^0\|_1,
\]
that is, up to a factor “2”, \( \hat{U}_{III}(\beta^0) \) improves upon \( \hat{U}_I(\beta^0) \). Note also that
\[
\hat{U}_{III}(\beta^0) \leq \min_{S} \left\{ \|X(b_S - \beta^0)\|_2^2 + \frac{\lambda^2 |S|}{\hat{\phi}^2(S)} \right\}
\]
where (for every set \( S \)) \( Xb_S \) is the projection of \( X\beta^0 \) on the space spanned by \( \{X_j\}_{j \in S} \).
4 Some considerations about the design

**Definition 4.1** We say that $X$ has normalized columns if for any $j$ it holds that $\|X_j\|_2 = 1$. We then call the design normalized.

**Definition 4.2** We say that $X$ has no aligned columns if for any $j \neq k$, and any constant $b$ it holds that $X_j \neq bX_k$.

**Definition 4.3** We say that $X$ is a fair design if it is normalized and has no aligned columns.

The reason for requiring normalized design is that when the columns in $X$ have different lengths, say the length of the first column $X_1$ is much smaller than that of the others, then in effect the first variable gets a heavy penalty as compared to the others. By taking $\|X_1\|_2$ extremely small, one can force the Lasso to choose $\beta_1^*$ extremely small, thus creating an unfair situation.

With normalized design, no aligned columns means that $X_j \neq \pm X_k$ for all $j \neq k$.

As we will see, one of the reasons why in the rest of the paper we assume that there are at least two active variables is the following:

**Lemma 4.1** There is no fair design such that $\hat{\phi}(\{1\}) = 0$.

5 Assumption about the first two variables

In what follows we consider throughout the case where $\beta_1^0 \geq \beta_2^0 > 0$ so that the first two variables are among the active ones. Moreover, we assume

$$(X_1, X_2)^T(X_1, X_2) = \begin{pmatrix} 1 & -\hat{\rho} \\ -\hat{\rho} & 1 \end{pmatrix},$$

where $0 < \hat{\rho} = -X^T X_2 < 1$ is minus the inner product between $X_1$ and $X_2$. Although we do not insist that $X_1$ and/or $X_2$ are centered, we sometimes refer to $-\hat{\rho}$ as the correlation between $X_1$ and $X_2$. The negative correlation is to be seen in relation with both $\beta_1^0$ and $\beta_2^0$ positive. It is so to speak the more difficult case for the Lasso.

Throughout the paper, we set

$$\hat{\rho}^2 := 1 - \hat{\rho}.$$

Fair design as defined in the previous section is related to using the penalty $\lambda \|\beta\|_1$ with equal weights for all coefficients. But linear combinations of the columns in $X$ are of course generally not normalized. We obviously have for example $\|X_1 + X_2\|_2^2 = 2\hat{\rho}^2$ which is less than 1 when $\hat{\rho}^2 < 1/2$. As we will see this is roughly the main ingredient when constructing exact results depending on compatibility constants.
6 Results for $p = s_0 = 2$

In this section $p$ equals 2 so that $X = (X_1, X_2)$. One may argue that this is not exactly a high-dimensional situation (for which the Lasso is designed) and therefore of limited interest. However, lower bounds for the low-dimensional situation can easily be extended to higher dimensions (trivially for example, by adding inactive variables orthogonal to the active ones, see Section 8). If the irrepresentable condition holds, the Lasso will not select inactive variables (see Zhao and Yu [2006]) which brings us back to a lower-dimensional situation. Lemmas 14.2 and 14.3 are examples where the Lasso ignores inactive variables that are correlated with the active ones.

Lemma 6.1 We have 
\[ \hat{\phi}^2(\{1\}) = 1 - \hat{\rho}^2 = \hat{\varphi}^2(2 - \hat{\varphi}^2). \]

Moreover 
\[ \hat{\rho}^2(S_0) = \hat{\varphi}^2, \quad \hat{\Gamma}^2(S_0) = \frac{2}{\hat{\varphi}^2}. \]

In the case considered here ($p = 2$) the minimal eigenvalue $\Lambda_{\min}(\hat{\Sigma})$ of the Gram matrix $\hat{\Sigma}$ is 
\[ \Lambda_{\min}(\hat{\Sigma}) = 1 - \hat{\rho} = \hat{\varphi}^2. \]

Thus, the compatibility constant $\hat{\phi}^2(S_0)$ is just another expression for this minimal eigenvalue. Lemma 6.1 gives an example in a higher-dimensional case, where the compatibility constant can be (much) larger than $\Lambda_{\min}(\hat{\Sigma})$, and in fact also (much) larger than the restricted eigenvalue as defined in Bickel et al. [2009].

Lemma 6.2 Consider the following three cases:

Case 1: $\lambda/\hat{\varphi}^2 \leq \beta_2^0$

Case 2: $\beta_2^0 \leq \lambda/\hat{\varphi}^2 \leq \beta_2^0 + (\beta_1^0 - \beta_2^0)/\hat{\varphi}^2$

Case 3: $\lambda/\hat{\varphi}^2 \geq \beta_2^0 + (\beta_1^0 - \beta_2^0)/\hat{\varphi}^2$.

Then we have 
\[ \|X(\beta^* - \beta^0)\|_2^2 = \begin{cases} 
2\lambda^2/\hat{\varphi}^2 & \text{in Case 1} \\
\hat{\varphi}^2(2 - \hat{\varphi}^2)(\beta_2^0)^2 + \lambda^2 & \text{in Case 2} \\
\|X\beta^0\|_2^2 & \text{in Case 3} 
\end{cases} \]

and 
\[ \beta^* = \begin{cases} 
\left( \begin{array}{c} \beta_1^0 - \lambda/\hat{\varphi}^2 \\ \beta_2^0 - \lambda/\hat{\varphi}^2 \\
\beta_1^0 - (1 - \hat{\varphi}^2)\beta_2^0 - \lambda \\
0 \\
0 
\end{array} \right) & \text{in Case 1} \\
\left( \begin{array}{c} \beta_1^0 - (1 - \hat{\varphi}^2)\beta_2^0 - \lambda \\
0 \\
0 
\end{array} \right) & \text{in Case 2} \\
\left( \begin{array}{c} 0 \\
0 \\
0 
\end{array} \right) & \text{in Case 3} 
\end{cases} \]
Corollary 6.1 Lemma 6.2 reveals that in Case 1
\[ \|X(\beta^* - \beta^0)\|_2^2 + \lambda \|\beta^*\|_1 = \lambda \|\beta^0\|_1, \]
and, invoking Lemma 6.1,
\[ \|X(\beta^* - \beta^0)\|_2^2 = \frac{\lambda^2 s_0}{\hat{\phi}^2(S_0)}. \]
This corresponds exactly to the bounds in Lemma 3.1.

Corollary 6.2 It may be of interest to consider the intersection of the cases in Lemma 6.2. We see that
\[ \|X(\beta^* - \beta^0)\|_2^2 = \begin{cases} 2\lambda \beta_2^0 & \text{in Case } 1 \cap 2 : \beta_2^0 = \lambda/\hat{\phi}^2 \\ 2\lambda \beta_1^0 - ((\beta_1^0)^2 - (\beta_2^0)^2) & \text{in Case } 2 \cap 3 : \lambda/\hat{\phi}^2 = \beta_2^0 + (\beta_1^0 - \beta_2^0)/\hat{\phi}^2. \end{cases} \]
Thus, the bound \( \hat{U}_{III}^0 \) in Lemma 3.2 is tight in Case 1 \( \cap \) 2.

Corollary 6.3 When \( \beta_1^0 = \beta_2^0 \), the union of cases gives
\[ \|X(\beta^* - \beta^0)\|_2^2 = \begin{cases} 2\lambda^2/\hat{\phi}^2 & \text{in Case } 1 \cup 2 : \lambda/\hat{\phi}^2 \leq \beta_2^0 \\ 2\hat{\phi}^2(\beta_2^0)^2 & \text{in Case } 2 \cup 3 : \lambda/\hat{\phi}^2 \geq \beta_2^0. \end{cases} \]

Remark 6.1 On may verify that Case 2 has
\[ \|X(\beta^* - \beta^0)\|_2^2 = \|X(b_{11}) - \beta^0\|_2^2 + \lambda^2, \]
where \( Xb_{11} \) is the projection of \( X\beta^0 \) on \( X_1 \). This can be compared with (2) (following from \( U_{III}^0 \) defined in Lemma 3.2) in Remark 3.1.

Remark 6.2 In Case 3, we have \( (\lambda - (\beta_1^0 - \beta_2^0))/\hat{\phi}^2 = \beta_2^0 > 0 \). This implies \( \beta_1^0 - \beta_2^0 < \lambda \). Note moreover that this case illustrates that the bound (2) in Remark 3.1 (and hence \( U_{III}^0 \) defined in Lemma 3.2) can be tight.

Remark 6.3 The case \( \hat{\rho} = 1 \) is not treated in Lemma 6.2. It corresponds to Case 2 with \( \hat{\phi}^2 \downarrow 0 \).

7 Results for \( p = s_0 = 2N \)

The results of the previous section are easily extended to a larger active set \( S_0 \). We assume \( S_0 = \{1, 2, \ldots, s_0\} \) with \( s_0 \) even, say \( s_0 = 2N \) (with \( N \in \mathbb{N} \) and \( 2N \leq n \)). Moreover we again assume \( p = s_0 \). Then
\[ X = (X_1, X_2, \ldots, X_{2N-1}, X_{2N}). \]
We split the design into \( N \) matrices of dimension \( n \times 2 \).
Lemma 7.1 Consider fair design with (for $k \in \{1, \ldots, N\}$) $(X_{2k-1}, X_{2k})$ orthogonal to the space spanned by the remaining columns. Assume that $\hat{\rho}_k := -X_{2k-1}^T X_{2k} > 0$ and write $\hat{\phi}_k^2 := 1 - \hat{\rho}_k$ for all $k \in \{1, \ldots, N\}$. Then

$$\Lambda_{\min}(\hat{\Sigma}) = \min_k \hat{\phi}_k^2$$

and

$$\hat{\phi}^2(S_0) = \frac{N}{\|1/\hat{\phi}^2\|_1} \geq \Lambda_{\min}(\hat{\Sigma}), \quad \hat{T}^2(S_0) = 2\|1/\hat{\phi}^2\|_1.$$ 

Moreover, for $S = \{2, 4, \ldots, 2N\}$

$$\hat{\phi}^2(S) = \frac{N}{\|(1 - \hat{\rho}^2)^{-1}\|_1}.$$ 

Remark 7.1 The restricted eigenvalue (Bickel et al. [2009]) is defined as

$$\hat{\kappa}^2(S_0) = \min \left\{ \frac{\|X_\beta S - X_\beta S_0\|_2^2}{\|\beta S\|_2^2} : \|\beta S\|_1 \leq \|\beta S_0\|_1 \right\}.$$ 

In the case we are considering in this section, where $S_0 = \{1, \ldots, p\}$, one obviously has $\hat{\kappa}^2(S_0) = \Lambda_{\min}(\hat{\Sigma})$. Therefore, in the situation of Lemma 7.1 $\hat{\kappa}^2(S_0) \leq \hat{\phi}^2(S_0)$ and the difference can be substantial.

The next lemma is again an illustration of the tightness of the upper bounds in Lemma 3.1.

Lemma 7.2 Consider design as in Lemma 7.1. Suppose that for all $k$, $\beta^0_{2k-1} \geq \beta^0_{2k} \geq \lambda/\hat{\phi}_k^2$. Then

$$\|X(\hat{\beta} - \beta^0)\|_2^2 = \frac{\lambda^2 s_0}{\hat{\phi}^2(S_0)} = 2\lambda^2\|1/\hat{\phi}^2\|_1$$

and

$$\|X(\hat{\beta} - \beta^0)\|_2^2 + \lambda\|\hat{\beta}\|_1 = \lambda\|\beta^0\|_1.$$ 

Remark 7.2 For the special case of Lemma 7.2 with equality $\beta^0_{2k} = \lambda/\hat{\phi}_k^2$ for all $k$, have for $S := \{2, 4, \ldots, 2N\}$

$$\|X(\hat{\beta} - \beta^0)\|_2^2 = 2\lambda\|\beta S\|_1,$$

showing tightness of $U_{II}(\beta^0)$.

8 A trivial extension to $m_0 > 0$

Recall that $X_{-S_0}$ contains the $m_0 := |S_0'|$ inactive variables. If these are orthogonal to the active ones the results are trivially as for the case $m_0 = 0$. As an example, let us take $s_0 = 2$. 

10
Lemma 8.1 Let again $S_0 = \{1, 2\}$ and suppose that $X_{S_0}^T X_{-S_0} = 0$. Then
\[
\hat{\phi}^2(S_0) = \hat{\varphi}^2
\]
and for $\beta_1^0 \geq \beta_2^0 \geq \lambda/\hat{\varphi}^2$,
\[
\|X(\beta^* - \beta^0)\|_2^2 + \lambda \|\beta^*\|_1 = \lambda \|\beta^0\|_1
\]
and
\[
\|X(\beta^* - \beta^0)\|_2^2 = \frac{\lambda^2 s_0}{\hat{\phi}^2(S_0)} = \frac{2\lambda^2}{\hat{\varphi}^2}.
\]
By the same argument, one may always extend in what follows the non-active set with variables that are orthogonal the ones considered.

9 A result for $s_0 = 2, m_0 = 1$

We now add one inactive variable, that is we take $S_0 = \{1, 2\}$ and $S_0^c = \{3\}$.

Lemma 9.1 Suppose that
\[
X_3 = C(X_1 + X_2)/2 + U
\]
where $C$ is a constant satisfying $C > 1$ and $C^2 \hat{\varphi}^2/2 < 1$, and $U$ is a vector with $U^T (X_1, X_2) = 0$. Define
\[
\hat{\tau}^2 := 1 - C^2 \hat{\varphi}^2/2.
\]
Then
\[
\hat{\phi}^2(S_0) = \hat{\varphi}^2 \hat{\tau}^2, \quad \hat{\Gamma}^2(S_0) := \frac{s_0}{\hat{\phi}^2(S_0)} = \frac{2}{\hat{\varphi}^2} + \frac{C^2}{\hat{\tau}^2}.
\]
For $\beta_1^0 \geq \beta_2^0 \geq \lambda/\hat{\varphi}^2 + \lambda C(C - 1)/(2\hat{\tau}^2)$ we have
\[
\|X(\beta^* - \beta^0)\|_2^2 + 2\lambda \|\beta^*_{-S_0}\|_1 = \frac{\lambda^2 s_0}{\hat{\phi}^2(S_0)} - \frac{\lambda^2}{\hat{\tau}^2}.
\]
The above lemma shows that the second upper bound of Lemma 3.1 is a term $\lambda^2/\hat{\tau}^2$ too large. However, this term can be small. An example is given in the next corollary.

Corollary 9.1 Take in Lemma 9.1 the constant $C = 2$. Then for $\hat{\varphi}^2 < 1/2$
\[
\hat{\phi}^2(S_0) = \hat{\varphi}^2(1 - 2\hat{\varphi}^2),
\]
and so
\[
\hat{\Gamma}^2(S_0) = \frac{2}{\hat{\varphi}^2} + \frac{4}{1 - 2\hat{\varphi}^2},
\]
and for $\beta_1^0 > \beta_2^0 \geq \lambda(1 - \hat{\varphi}^2)/(\hat{\varphi}^2(1 - 2\hat{\varphi}^2)$ we have
\[
\|X(\beta^* - \beta^0)\|_2^2 + 2\lambda \|\beta^*_{-S_0}\|_1 = \frac{2\lambda^2}{\hat{\varphi}^2} + \frac{3\lambda^2}{1 - 2\hat{\varphi}^2}.
\]
In other words, the bound in lemma 3.1 has a factor “4” whereas the exact result has a factor “3”. For \( \varphi^2 \downarrow 0 \) we see that the upper bound is asymptotically tight, as then \( 2\lambda^2/\varphi^2 \) is the leading term. Conversely, for \( \varphi^2 \uparrow 1/2 \) the upper bound is asymptotically a factor 4/3 too large.

10 A result for \( s_0 = 2N, m_0 = 1 \)

We have seen in the previous section that the upper bound of Lemma 3.1 can be off, for example by a factor 4/3 asymptotically. The question arises whether in a generalized setting this factor increases when \( s_0 \) increases. If this is not the case, the non-tightness of the bound is really only a matter of constants. In this section we show in an example that the gap between the upper bounds of Lemma 3.1 and the exact bound does not depend on \( s_0 \).

**Lemma 10.1** Let \( S_0 = \{1, \ldots, 2N\} \), \( S'_0 = \{2N + 1\} \) and 

\[
(X_{2k-1}, X_{2k})^T(X_{2k-1}, X_{2k}) := \begin{pmatrix} 1 & -\hat{\rho}_k \\ -\hat{\rho}_k & 1 \end{pmatrix}, \quad k = 1, \ldots, N,
\]

where each \( \hat{\rho}_k \) is between 0 and 1. Then we define \( \hat{\varphi}_k^2 := 1 - \hat{\rho}_k \), \( k = 1, \ldots, N \). Further, assume that \( (X_{2k-1}, X_{2k}) \) is orthogonal to \( \{X_j\}_{j \in S_0 \setminus \{2k-1, 2k\}} \) for all \( k \). Let 

\[
X_{2N+1} = C \sum_{j=1}^{2N} X_j / s_0 + U
\]

where \( C > 1 \), \( C^2 \sum_{k=1}^{N} 2\hat{\varphi}_k^2 / s_0^2 < 1 \) and \( U \) is orthogonal to \( \{X_j\}_{j \in S_0} \). Write \( \hat{\tau}^2 := 1 - C^2 \sum_{k=1}^{N} 2\hat{\varphi}_k^2 / s_0^2 \). Then for \( \hat{\varphi}^2 = (\hat{\varphi}_1^2, \ldots, \hat{\varphi}_N^2) \)

\[
\hat{\Gamma}^2(S_0) = 2\|1/\hat{\varphi}^2\|_1 + \frac{C^2}{\hat{\tau}^2}.
\]

Moreover, for \( \beta_{2k-1}^0 \geq \beta_{2k}^0 \geq \lambda / \hat{\varphi}_k^2 + \lambda C (C - 1)/(s_0 \hat{\tau}^2) \), \( k = 1, \ldots, N \), we have

\[
\|X(\beta^* - \beta^0)\|_2^2 + 2\lambda \|\beta^*_S0\|_1 = \lambda^2 \hat{\Gamma}^2(S_0) - \lambda^2 / \hat{\tau}^2.
\]

**Corollary 10.1** When \( \hat{\varphi}_1^2 = \cdots = \hat{\varphi}_N^2 := \hat{\varphi}_0^2 \) (say) in Lemma 10.1 and \( C = 2 \) one gets

\[
\hat{\Gamma}^2(S_0) = \frac{s_0}{\varphi_0^2} + \frac{4}{\hat{\tau}^2},
\]

with \( \hat{\tau}^2 = 1 - 4\hat{\varphi}_0^2 / s_0 \). For \( \beta_{2k-1}^0 \geq \beta_{2k}^0 \geq \lambda / \hat{\varphi}_k^2 + 2\lambda / (\hat{\tau}^2 s_0) \) for all \( k \), we get

\[
\|X(\beta^* - \beta^0)\|_2^2 + 2\lambda \|\beta^*_S0\|_1 = \frac{\lambda^2 s_0}{\varphi_0^2} - \frac{\lambda^2}{\hat{\tau}^2}.
\]

So with \( \hat{\varphi}_0^2 \) kept fixed the gap of Lemma 3.1 decreases with \( s_0 \).
11 A result for $s_0 = 2$ and $m_0$ possibly large

We now set $S_0 = \{1, 2\}$ and $S_0^c := \{3, \ldots, 2 + m_0\}$ where $m_0$ is possibly large (in an asymptotic sense it may be of order $1/\lambda$ say).

**Lemma 11.1** Suppose
\[ X_{2+k} = C_k (X_1 + X_2)/2 + U_k, \quad k = 1, \ldots, m_0, \]
where, for $k = 1, \ldots, m_0$, the constant $C_k$ has $C_k > 1$ but $C_k^2 \hat{\varphi}^2/2 < 1$, and where the vector $U_k$ is orthogonal to $\{X_1, X_2, \{U_j\}_{j \neq k}\}$.

Let for each $k \in \{1, \ldots, m_0\}$, the constant $\hat{\tau}_k^2$ be given by
\[ \hat{\tau}_k^2 = 1 - C_k^2 \hat{\varphi}^2. \]

Then
\[ \hat{\Gamma}^2(S_0) = \frac{2}{\hat{\varphi}^2} + \frac{m_0}{\hat{\tau}_k^2}. \]

Moreover, if $\beta^0 \geq \beta_2^0 \geq \lambda/\hat{\varphi}^2 + \lambda \sum_{k=1}^{m_0} C_k (C_k - 1)/(2 \hat{\tau}_k^2)$, it holds that
\[ \|X(\beta^* - \beta^0)\|_2^2 + 2\lambda \|\beta^*_{S_0}\|_1 = \lambda^2 \hat{\Gamma}(S_0) - \lambda^2 \|\hat{\tau}\|_1 \]

**Corollary 11.1** If we take $C_k = 2$ for all $k \in \{1, \ldots, m_0\}$ we obtain
\[ \hat{\Gamma}^2(S_0) = \frac{2}{\hat{\varphi}^2} + \frac{4m_0}{1 - 2\hat{\varphi}^2}, \]
and
\[ \|X(\beta^* - \beta^0)\|_2^2 + 2\lambda \|\beta^*_{S_0}\|_1 = \frac{2\lambda^2}{\hat{\varphi}^2} + \frac{3\lambda^2 m_0}{1 - 2\hat{\varphi}^2}. \]

The upper bound of Lemma 3.1 is off no more than a factor $4/3$.

12 Some results for $s_0 = m_0 = 2$

In this section, the active set is again $S_0 = \{1, 2\}$ and the non-active one is $S_0^c = \{3, 4\}$. Thus, both $s_0$ and $m_0 := p - s_0$ are equal to 2.

In Section 9 we have seen that the upper bound of Lemma 3.1 can be too large, but that the gap is small when the main term is due to highly negatively correlated active variables. In this section, we consider first a setup similar to the one in Section 9. Again, the upper bounds are not tight but the gap can be small. Unlike the previous section, the main terms in the bound in this section are now not necessarily determined by the negative correlations in the active set.

**Lemma 12.1** Let
\[ X_3 = C(X_1 + X_2)/2 + U + V, \quad X_4 = C(X_1 + X_2)/2 + U - V, \]
where $C > 1$, $C^2 \varphi^2/2 < 1$, $U^T X_{S_0} = V^T X_{S_0} = 0$ and $U^T V = 0$. Set
\[ \hat{\tau}^2 := U^T U \]
where $0 < \hat{\tau}^2 < 1 - C^2\hat{\phi}^2/2$. Then

$$\hat{\phi}^2(S_0) = \frac{\hat{\phi}^2\hat{\tau}^2}{C^2\hat{\phi}^2/2 + \hat{\tau}^2}, \quad \hat{\Gamma}^2(S_0) = \frac{2}{\hat{\phi}^2} + \frac{C^2}{\hat{\tau}^2}.$$  

Let $\beta_0^1 \geq \beta_0^2 \geq \lambda/\hat{\phi}^2 + \lambda(C - 1)/(2\hat{\tau}^2)$. Then

$$\|X(\beta^* - \beta^0)\|^2_2 + 2\lambda\|\beta^*_S - S_0\|_1 = \lambda^2\hat{\Gamma}^2(S_0) - \frac{\lambda^2}{\hat{\tau}^2}.$$  

We can also have a look what happens if in the above lemma, we let $\hat{\tau}^2 = 0$ instead of $> 0$. Then the compatibility constant $\hat{\phi}^2(S_0)$ is zero. In this case, the prediction error $\|X(\beta^* - \beta^0)\|^2_2$ is in a sense still under control, but the penalized prediction error $\|X(\beta^* - \beta^0)\|^2_2 + 2\lambda\|\beta^*_S - S_0\|_1$ can show the “slow rate”.

**Lemma 12.2** Let

$$X_3 = C(X_1 + X_2)/2 + V, \quad X_4 = C(X_1 + X_2)/2 - V,$$

where $C > 1$, $C^2\hat{\phi}^2/2 < 1$ and $V^T X_{S_0} = 0$. Then

$$\hat{\phi}^2(S_0) = 0.$$  

Moreover when $\beta_0^1 \geq \beta_0^2 \geq \lambda/\hat{\phi}^2$ we find

$$\|X(\beta^* - \beta^0)\|^2_2 = \frac{2\lambda^2}{\hat{\phi}^2},$$

$$\|X(\beta^* - \beta^0)\|^2_2 + 2\lambda\|\beta^*_S - S_0\|_1 = \frac{4\lambda\beta_0^0}{C} - 2\lambda^2\left(\frac{2}{C} - 1\right).$$

Note that if in the above lemma $C = 2$ we arrive at the bound

$$\|X(\beta^* - \beta^0)\|^2_2 + 2\lambda\|\beta^*_S - S_0\|_1 = 2\lambda\beta_0^0$$

and with $C = 4$ we get

$$\|X(\beta^* - \beta^0)\|^2_2 + 2\lambda\|\beta^*_S - S_0\|_1 = \lambda\beta_0^0 + \frac{\lambda^2}{\hat{\phi}^2}.$$  

The next lemma has the situation of Lemma 12.2 but now with $C = 1$ instead of $C > 1$. This is an example where the minimizer of $\mathcal{L}(\cdot)$ is not unique.

**Lemma 12.3** Let

$$X_3 = (X_1 + X_2)/2 + V, \quad X_4 = (X_1 + X_2)/2 - V,$$

where $V^T X_{S_0} = 0$. Then

$$\hat{\phi}^2(S_0) = 0.$$
Moreover when $\beta_1^0 \geq \beta_2^0 \geq \lambda/\hat{\varphi}^2$, we find that the vector

$$
\beta^* = \begin{pmatrix}
\beta_1^0 - \lambda/\hat{\varphi}^2 - \beta_3^*\\
\beta_2^0 - \lambda/\hat{\varphi}^2 - \beta_3^*\\
\beta_3^*\\
\beta_3^*
\end{pmatrix}
$$

is for all $0 \leq \beta_3^* \leq \beta_2^0 - \lambda/\hat{\varphi}^2$ a minimizer of $L(\cdot)$ and we have

$$
\|X(\beta^* - \beta^0)\|_2^2 = \frac{2\lambda^2}{\hat{\varphi}^2}.
$$

$$
\|X(\beta^* - \beta^0)\|_2^2 + 2\lambda\|\beta^*_{-S_0}\|_1 \leq 4\lambda\beta_2^0 - \frac{2\lambda^2}{\hat{\varphi}^2}.
$$

13 The case $s_0 = m_0 = 2N$

Suppose $S_0 = \{1, \ldots, 2N\}$ and $S_0^c = \{2N + 1, \ldots, 4N\}$. We can easily extend the situation of Section 12, where $N = 1$, to $N > 1$ by assuming $N$ mutually orthogonal blocks of variables. This extension is trivial but nevertheless useful as it moves us away from a very low-dimensional situation.

**Lemma 13.1** Set for $k = 1, \ldots, N$

$$(X_{2k-1}, X_{2k})^T(X_{2k-1}, X_{2k}) = \begin{pmatrix} 1 & -\hat{\rho}_k \\ -\hat{\rho}_k & 1 \end{pmatrix}, \ \hat{\varphi}_k := 1 - \hat{\rho}_k,$$

and $(X_{2k-1}, X_{2k})$ orthogonal to $\{X_j\}_{j \in S_0 \setminus \{2k-1, 2k\}}$. Let for $k = 1, \ldots, N$

$$X_{2N+2k-1} = C_k(X_{2k-1} + X_{2k}) + U_k + V_k, X_{2N+2k} = C_k(X_{2k-1} + X_{2k}) + U_k - V_k,$$

where $C_k > 1$ and $C_k^2\hat{\varphi}^2/2 < 1$, $(U_k, V_k)$ orthogonal to $\{X_j\}_{j \in S_0 \setminus \{2k-1, 2k\}}$ as well as to $\{(U_j, V_j)\}_{j \neq k}$, and $U_k^TV_k = 0$. Let $\hat{\tau}_k^2 := U_k^TU_k$ with $0 < \hat{\tau}_k^2 < 1 - C_k^2\hat{\varphi}_k^2$. Then

$$\hat{\Gamma}^2(S_0) = 2\sum_{k=1}^N 1/\hat{\varphi}_k^2 + \sum_{k=1}^N C_k^2/\hat{\tau}_k^2.$$

If, for $k = 1, \ldots, N$, $\beta_{2k-1}^0 \geq \beta_{2k}^0 \geq \lambda/\hat{\varphi}^2 + \lambda C_k(C_k - 1)/(2\hat{\tau}_2)$ we obtain

$$
\|X(\beta^* - \beta^0)\|_2^2 + 2\lambda\|\beta^*_{-S_0}\|_1 = \lambda^2\hat{\Gamma}^2(S_0) - \lambda^2\|1/\hat{\tau}^2\|_1.
$$

14 Further results with $s_0 = 2$

In the previous sections with $S_0 = \{1, 2\}$ we assume that each inactive variable is a given a linear combination of the active ones plus an orthogonal term. In this section, we assume the situation is the other way around: each active
variable is a given linear combination of the inactive ones plus an orthogonal term.

We first examine a case where the compatibility constant is zero, and the presence of non-active variables has big impact on the prediction error, even when the negative correlation \( \hat{\rho} \) between active variables is small. Afterwards, this situation is slightly adjusted to one with positive compatibility constant, but the upper bounds are then a factor too large.

The next lemma has compatibility constant \( \hat{\phi}^2(S_0) \) equal to zero.

**Lemma 14.1** Let \( S_0^c = \{3, 4\} \) \((m_0 = 2)\) and

\[
(X_3, X_4)^T(X_3, X_4) = \begin{pmatrix} 1 & -\hat{\theta} \\ -\hat{\theta} & 1 \end{pmatrix},
\]

Assume that for some vector \((\gamma_3, \gamma_4)^T = \gamma_{-S_0} \in \mathbb{R}^2\) with \( 1/2 < \gamma_3 < 1 \) and \( \gamma_4 = 1 - \gamma_3 \).

\[
X_1 = X_{-S_0} \gamma + V, \quad X_2 = X_{-S_0} \gamma - V,
\]

where \( X_{-S_0} := \{X_j\}_{j \notin S_0} \) and where \( V^T X_{-S_0} = 0 \). Then

\[
\hat{\phi}^2(S_0) = 0
\]

and

\[
\hat{\varphi}^2 = 2(1 - 4\gamma_3(1 - \gamma_3)) + 4\gamma_3(1 - \gamma_3)\hat{\psi}^2,
\]

where \( \hat{\psi}^2 := 1 - \hat{\theta} \).

Furthermore, if \( 2\gamma_4 \beta_2^0 \geq \lambda/\hat{\psi}^2 \) we have

\[
\|X(\beta^* - \beta^0)\|_2^2 = \frac{2\lambda^2}{\hat{\psi}^2}
\]

and

\[
\|X(\beta^* - \beta^0)\|_2^2 + 2\lambda\|\beta^*_{-S_0}\|_1 = 4\lambda\beta_2^0 - \frac{2\lambda^2}{\hat{\psi}^2} \geq 4\lambda\gamma_3\beta_2^0.
\]

The above lemma illustrates that when the compatibility condition fails, the prediction error \( \|X(\beta^* - \beta^0)\|_2^2 \) can be as large as \( 4\lambda\gamma_3\beta_2^0 \) where \( \gamma_4 < 1/2 \), even when the correlation \( -\hat{\rho} \) between \( X_1 \) and \( X_2 \) is not close to \(-1\), i.e., even when \( \hat{\varphi}^2 \) is not close to zero (as \( \hat{\varphi}^2 > 2(1 - 4\gamma_3(1 - \gamma_3)) \)).

We now consider two situations where the compatibility constant is positive. Moreover, there are no false positives, i.e. \( \|\beta^*_{-S_0}\|_1 = 0 \). Indeed, in the two Lemmas 14.2 and 14.3 the irrepresentable condition (Zhao and Yu [2006]) holds.

**Lemma 14.2** Let \( S_0^c : \{3, 4\} \) \((m_0 = 2)\) and

\[
(X_3, X_4)^T(X_3, X_4) = \begin{pmatrix} 1 & -\hat{\theta} \\ -\hat{\theta} & 1 \end{pmatrix},
\]

and write \( \hat{\psi}^2 := 1 - \hat{\theta} \). Assume that

\[
X_1 = C(X_3 + X_4)/2 + V, \quad X_2 = C(X_3 + X_4)/2 - V,
\]
where $V^T X_{-S_0} = 0$ and $C > 1$, $C^2 \hat{\psi}^2 / 2 < 1$. Then
\[
\hat{\phi}^2(S_0) = (C - 1)^2 \hat{\psi}^2, \quad \hat{\Gamma}^2(S_0) = \frac{2}{(C - 1)^2 \hat{\psi}^2}.
\]
Moreover, $\hat{\phi}^2 = C^2 \hat{\psi}^2$, and for $\beta_0^2 \geq \lambda / \hat{\phi}^2$,
\[
\|X(\beta^* - \beta^0)\|_2^2 = \frac{2\lambda^2}{\hat{\phi}^2} = \frac{\lambda^2 \hat{\Gamma}^2(S_0) (C - 1)^2}{C^2}
\]
and $\|\beta^*_{-S_0}\|_1 = 0$.

In other words, the upper bound $\lambda^2 |S_0| / \hat{\phi}^2(S_0)$ is a factor $C^2 / (C - 1)^2$ too large in this case.

In the last result of this paper, we again let $s_0 = 2$ but now $m_0$ is arbitrary. Moreover, we assume that the inactive variables are orthogonal to each other.

**Lemma 14.3** Let $S_0 = \{1, 2\}$, $\hat{\Sigma}_{-S_0,-S_0} = I$ and
\[
X_1 = CX_{-S_0} \gamma_{-S_0} + V, \quad X_2 = X_{-S_0} \gamma_{-S_0} - V,
\]
where $X_{-S_0} := \{X_j\}_{j \notin S_0}$ and where $V^T X_{-S_0} = 0$. Assume moreover $\|\gamma_{-S_0}\|_1 = 1$, $2C^2 \|\gamma_{-S_0}\|_2 < 1$ and $\|\gamma_{-S_0}\|_\infty \leq C \|\gamma_{-S_0}\|_2$. Then
\[
\hat{\phi}^2(S_0) = 2 \min_{\|\beta_{-S_0}\|_1 \leq 1} \|C \gamma_{-S_0} - \beta_{-S_0}\|_2^2, \quad \hat{\phi}^2 = 2C^2 \|\gamma_{-S_0}\|_2^2
\]
and moreover for $\beta_1^0 \geq \beta_2^0 \geq \lambda / \hat{\phi}^2$
\[
\|X(\beta^* - \beta^0)\|_2^2 = \frac{2\lambda^2}{\hat{\phi}^2}.
\]

**Corollary 14.1** An example of a vector $\gamma_{-S_0}$ and constant $C$ in Lemma 14.3 is
\[
\gamma_{-S_0} = \left(\frac{1}{m_0}, \cdots, \frac{1}{m_0} \right)^T, \quad m_0 \times 1
\]
and $1 < C^2 < m_0 / 2$. Then
\[
\hat{\phi}^2(S_0) = 2(C - 1)^2 / m_0, \quad \hat{\Gamma}^2(S_0) = \frac{m_0}{(C - 1)^2}
\]
and
\[
\|X(\beta^* - \beta^0)\|_2^2 = \frac{\lambda^2 m_0}{C^2} = \frac{\lambda^2 \hat{\Gamma}^2(S_0) (C - 1)^2}{C^2}.
\]
So again there is a gap with Lemma 3.1, but it is small for $C$ large.
15 Proofs

In the proofs, we sometimes use the following notation. The matrix with columns in $S \subset \{1, \ldots, p\}$ is written as $X_S := \{X_j\}_{j \in S}$ and $X_{-S} := \{X_j\}_{j \notin S}$ has its columns in $S^c$. The order in the columns is taken increasing in the index (i.e., we remove some columns and otherwise keep the original ordering). We write
\[
\hat{\Sigma}_{S,S} := X_S^T X_S, \quad \hat{\Sigma}_{-S,S} := X_{-S}^T X_S,
\]
\[
\hat{\Sigma}_{S,-S} := X_S^T X_{-S}, \quad \hat{\Sigma}_{-S,-S} := X_{-S}^T X_{-S}.
\]

In the proofs of results from Section 6 and onwards we present explicit expressions for the minimizer $\beta^*$ showing it is the solution of the KKT conditions. One may check that the solution is unique in each case except for Lemma 12.3.

15.1 Proof of the results in Section 2

Theorem 2.1 and its proof are stated as Problem 2.4 in van de Geer [2016]. Here, we present a complete proof. For this we need some auxiliary lemmas.

Lemma 15.1 It holds that
\[
\|X(\hat{\beta} - \beta^*)\|^2_2 + \lambda\|\hat{\beta}\|_1 - \lambda\hat{\beta}^T z^* \leq (\hat{\beta} - \beta^*)^T X^T \epsilon.
\]

Proof of Lemma 15.1. By the KKT conditions for $\hat{\beta}$
\[-X^T (Y - X\hat{\beta}) + \lambda\hat{\epsilon} = 0,
\]
where $\hat{\epsilon} \in \partial\|\hat{\beta}\|_1$. In other words
\[
\hat{\Sigma}(\hat{\beta} - \beta^0) + \lambda\hat{\epsilon} = X^T \epsilon.
\]
By the KKT conditions for $\beta^*$
\[
\hat{\Sigma}(\beta^* - \beta^0) + \lambda z^* = 0.
\]
Hence, taking the difference
\[
\hat{\Sigma}(\hat{\beta} - \beta^*) + \lambda(\hat{\epsilon} - z^*) = X^T \epsilon.
\]
Multiply by $(\hat{\beta} - \beta^*)^T$ to find
\[
\|X(\hat{\beta} - \beta^*)\|^2_2 + \lambda(\hat{\beta} - \beta^*)^T (\hat{\epsilon} - z^*) = (\hat{\beta} - \beta^*)^T X^T \epsilon.
\]
But
\[
(\hat{\beta} - \beta^*)^T (\hat{\epsilon} - z^*) = \hat{\beta}^T (\hat{\epsilon} - z^*) + \beta^* (z^* - \hat{\epsilon}).
\]
Both terms are non-negative: since $\hat{\beta}^T z^* \leq \|\hat{\beta}\|_1 \|z^*\|_\infty \leq \|\hat{\beta}\|_1$ we have
\[
\hat{\beta}^T (\hat{\epsilon} - z^*) = \|\hat{\beta}\|_1 - \hat{\beta}^T z^* \geq 0.
\]
and by the same argument
\[ \beta^* (z^* - \hat{z}) = \| \beta^* \|_1 - \beta^* \hat{z} \geq 0. \]

Dropping the term \( \| \beta^* \|_1 - \beta^* \hat{z} \) therefore yields
\[ \| X(\hat{\beta} - \beta^*) \|_2^2 + \lambda \| \hat{\beta} \|_1 - \lambda \hat{\beta}^T z^* \leq (\hat{\beta} - \beta^*)^T X^T e. \]

Recall that the vector \( \beta^* \) satisfies the KKT conditions
\[ \hat{\Sigma}(\beta^* - \beta^0) + \lambda z^* = 0, \]
where \( z^* \in \partial \| \beta^* \|_1 \).

Define
\[ \bar{S}^* := \{ j : |z^*_j| \geq 1 - \eta \}. \]

Note that \( \bar{S} \supset \bar{S}^* \) where \( \bar{S}^* \) is the active set of \( \beta^* \). We write \( \bar{s}^* := |\bar{S}^*| \).

**Lemma 15.2** It holds that
\[ \bar{s}^* \leq \frac{\Lambda_{\text{max}}(\hat{\Sigma})}{\lambda^2 (1 - \eta)^2} \| X(\beta^* - \beta^0) \|_2^2. \]

**Proof of Lemma 15.2.** By the KKT conditions for \( \beta^* \) it is true that
\[ \lambda^2 \| z^* \|_2^2 = (\beta^* - \beta^0)^T \hat{\Sigma}^2 (\beta^* - \beta^0) \leq \Lambda_{\text{max}}(\hat{\Sigma}) \| X(\beta^* - \beta^0) \|_2^2. \]

On the other hand
\[ \| z^* \|_2^2 \geq \| z^*_{\bar{S}^*} \|_2^2 \geq (1 - \eta)^2 \bar{s}^*. \]

Hence
\[ \bar{s}^* \leq \frac{\Lambda_{\text{max}}(\hat{\Sigma})}{\lambda^2 (1 - \eta)^2} \| X(\beta^* - \beta^0) \|_2^2. \]

Define the random variable
\[ V^2(\bar{S}^*) := \max_{\| X\beta_{\bar{S}^*} \|_2 = 1} |\beta_{\bar{S}^*}^T X^T e|. \]

Define moreover the vector \( X_{\bar{S}^*} \hat{\gamma}_{\bar{S}^*} \) as the projection of \( X\hat{\beta} \) on the space spanned by the columns of \( X_{\bar{S}^*} \) and let \( w \) be the random variable
\[ w := \left\| \left[ X\hat{\beta} - X_{\bar{S}^*} \hat{\gamma}_{\bar{S}^*} \right]^T e \right\|_{\infty} / \| \hat{\beta}_{\bar{S}^*} \|_1. \]

**Lemma 15.3** We have
\[ \| X(\hat{\beta} - \beta^*) \|_2^2 + 2(\eta \lambda - w) \| \hat{\beta}_{\bar{S}^*} \|_1 \leq V^2(\bar{S}^*). \]
Proof of Lemma 15.3. By Pythagoras’ theorem, and using that $S_* \subset \tilde{S}_*$

$$
\|X(\hat{\beta} - \beta^*)\|_2^2 = \|X_{\hat{S}_*} \hat{\gamma}_{\hat{S}_*} - X\beta^*\|_2^2 + \|X\hat{\beta} - X_{\hat{S}_*} \hat{\gamma}_{\hat{S}_*}\|_2^2.
$$

Therefore, in view of Lemma 15.3

$$
\|X_{\hat{S}_*} \hat{\gamma}_{\hat{S}_*} - X\beta^*\|_2^2 + \|X\hat{\beta} - X_{\hat{S}_*} \hat{\gamma}_{\hat{S}_*}\|_2^2 + \lambda\|\hat{\beta}\|_1 - \lambda \beta^T z^*
$$

$$
\leq \left[ X_{\hat{S}_*} \hat{\gamma}_{\hat{S}_*} - X\beta^* \right]^T \epsilon + \left[ X\hat{\beta} - X_{\hat{S}_*} \hat{\gamma}_{\hat{S}_*} \right]^T \epsilon.
$$

By the Cauchy-Schwarz inequality

$$
\left[ X_{\hat{S}_*} \hat{\gamma}_{\hat{S}_*} - X\beta^* \right]^T \epsilon \leq \mathbf{V}(\tilde{S}_*)\|X_{\hat{S}_*} \hat{\gamma}_{\hat{S}_*} - X\beta^*\|_2.
$$

Moreover, by the definition of $w$

$$
\left[ X\hat{\beta} - X_{\hat{S}_*} \hat{\gamma}_{\hat{S}_*} \right]^T \epsilon \leq w\|\hat{\beta}_{-\tilde{S}_*}\|_1.
$$

On the other hand, $|z_j^*| \leq 1 - \eta$ for all $j \notin \tilde{S}_*$ and hence

$$
\|\hat{\beta}_{-\tilde{S}_*}\|_1 - z_j^T \hat{\beta}_{-\tilde{S}_*} \geq \eta\|\hat{\beta}_{-\tilde{S}_*}\|_1.
$$

We thus arrive at

$$
\|X_{\hat{S}_*} \hat{\gamma}_{\hat{S}_*} - X\beta^*\|_2^2 + \|X\hat{\beta} - X_{\hat{S}_*} \hat{\gamma}_{\hat{S}_*}\|_2^2 + \eta\lambda\|\hat{\beta}_{-\tilde{S}_*}\|_1
$$

$$
\leq \mathbf{V}(\tilde{S}_*)\|X_{\hat{S}_*} \hat{\gamma}_{\hat{S}_*} - X\beta^*\|_2 + w\|\hat{\beta}_{-\tilde{S}_*}\|_1
$$

$$
\leq \mathbf{V}^2(\tilde{S}_*)/2 + \|X_{\hat{S}_*} \hat{\gamma}_{\hat{S}_*} - X\beta^*\|_2^2/2 + w\|\hat{\beta}_{-\tilde{S}_*}\|_1
$$

or

$$
\|X_{\hat{S}_*} \hat{\gamma}_{\hat{S}_*} - X\beta^*\|_2^2 + 2\|X\hat{\beta} - X_{\hat{S}_*} \hat{\gamma}_{\hat{S}_*}\|_2^2 + 2(\eta\lambda - w)\|\hat{\beta}_{-\tilde{S}_*}\|_1 \leq \mathbf{V}^2(\tilde{S}_*).
$$

But then also

$$
\|X(\hat{\beta} - \beta^*)\|_2^2 + 2(\eta\lambda - w)\|\hat{\beta}_{-\tilde{S}_*}\|_1 \leq \mathbf{V}^2(\tilde{S}_*).
$$

\[\square\]

Lemma 15.4 Let $\lambda_0 := \sqrt{2\log(2p/\alpha)/n}$. It holds that with probability at least $1 - \alpha$ that

$$
w \leq \lambda_0.
$$

Proof of Lemma 15.4. Write the singular value decomposition of $X_{\tilde{S}_*}$ as

$$
X_{\tilde{S}_*} = \tilde{P}_s \tilde{\Lambda}_s^{1/2} \tilde{Q}_s
$$
where $P^T_\sigma P_\sigma = I$, $Q^T_\sigma Q_\sigma = I$ and $\Lambda_\sigma$ the diagonal matrix of eigenvalues of $X^T_\sigma X_\sigma$. Since $X\hat{\beta} = X_\sigma \hat{\beta}_\sigma + X_{-\sigma} \hat{\beta}_{-\sigma}$, we see that

$$X\hat{\beta} - X_\sigma \hat{\gamma}_\sigma = (I - P_\sigma P^T_\sigma)X_\sigma \hat{\beta}_{-\sigma}.$$ 

Hence

$$\left[ X\hat{\beta} - X_\sigma \hat{\gamma}_\sigma \right]^T \epsilon = (X_{-\sigma} \hat{\beta}_{-\sigma})^T (I - P_\sigma P^T_\sigma) \epsilon.$$ 

Thus

$$w \leq \|X^T_\sigma (I - P_\sigma P^T_\sigma) \epsilon\|_\infty.$$ 

The diagonal elements of the matrix

$$X^T_\sigma (I - P_\sigma P^T_\sigma) X_{-\sigma}$$

are projected versions of the columns of $X_{-\sigma}$ and hence at most $\max_{j \in S_\sigma} \|X_j\|_2^2$, which is by assumption at most 1. It follows that each element of the vector $\sqrt{n}X^T_\sigma (I - P_\sigma P^T_\sigma) \epsilon$ is normally distributed with mean zero and variance at most 1. The dimension of this vector is at most $p$. Now use that for standard normal random variables $W_1, \ldots, W_p$, and for any $t > 0$,

$$\mathbb{P}(\max_{1 \leq j \leq p} |W_j| > \sqrt{2(\log(2p) + t)}) \leq p \mathbb{P}(|W_1| > \sqrt{2(\log(2p) + t)}) \leq 2p \exp[-(\log(2p + t)] = \exp[-t].$$

Apply this with $t = \log(1/\alpha)$.

**Lemma 15.5** We have

$$\mathbb{P}(\mathbf{V}(\hat{S}_\sigma) \geq \sqrt{s_\sigma/n + 2\log(1/\alpha_1)/n}) \leq \alpha_1.$$ 

**Proof of Lemma 15.5.** Let $\chi^2_T$ be chi-squared random variable with $T$ degrees of freedom. Lemma 1 in [Laurent and Massart, 2000] says that for all $t > 0$

$$\mathbb{P}(\chi^2_T \geq T + 2\sqrt{Tt} + 2t) \leq \exp[-t].$$

Since $T + 2\sqrt{Tt} + 2t \leq (\sqrt{T} + \sqrt{2t})^2$ we find

$$\mathbb{P}(\chi_T \geq \sqrt{T} + \sqrt{2t}) \leq \exp[-t].$$

Apply this with $t = \log(1/\alpha_1)$.

**Proof of Theorem 2.1.** We know by Lemma 15.4 that with probability at least $1 - \alpha$

$$w \leq \lambda_0$$

and from Lemma 15.5 with probability at least $1 - \alpha_1$

$$\mathbf{V}(\hat{S}_\sigma) \leq \sqrt{s_\sigma/n} + \sqrt{2\log(1/\alpha_1)/n}.$$
By Lemma [15.2]
\[ \bar{s}_* \leq \frac{\Lambda_{\text{max}}(\hat{\Sigma})}{\lambda^2(1-\eta)^2} \|X(\beta^* - \beta^0)\|_2^2. \]

Hence with probability at least \(1 - \alpha_1\)
\[ V(\bar{S}_*) \leq \sqrt{\frac{\Lambda_{\text{max}}(\hat{\Sigma})}{n\lambda^2(1-\eta)}} \|X(\beta^* - \beta^0)\|_2 + \sqrt{\frac{2\log(1/\alpha_1)}{n}}. \]

Combine this with Lemmas [15.1] and [15.3] and invoke the condition \(\eta \lambda > \lambda_0\) to complete the proof. \(\square\)

**Lemma 15.6** Suppose that
\[ \|\hat{\Sigma} - \Sigma_0\|_\infty \|\beta^* - \beta^0\|_1 < \lambda(1-\eta). \]

Then
\[ \bar{s}_* \leq \frac{\Lambda_{\text{max}}(\Sigma_0) \left( \|X(\beta^* - \beta^0)\|_2^2 + \|\hat{\Sigma} - \Sigma_0\|_\infty \|\beta^* - \beta^0\|_1^2 \right)}{\left( \lambda(1-\eta) - \|\hat{\Sigma} - \Sigma_0\|_\infty \|\beta^* - \beta^0\|_1 \right)^2}. \]

**Proof of Lemma 15.6.** We start again with the KKT conditions for \(\beta^*\)
\[ \hat{\Sigma}(\beta^* - \beta^0) + \lambda z^* = 0. \]

Then
\[ \Sigma_0(\beta^* - \beta^0) + (\hat{\Sigma} - \Sigma_0)(\beta^* - \beta^0) = -\lambda z^*. \]

But for all \(j\)
\[ |((\hat{\Sigma} - \Sigma_0)(\beta^* - \beta^0))_j| \leq \|\hat{\Sigma} - \Sigma_0\|_\infty \|\beta^* - \beta^0\|_1 \]

so
\[ |(\Sigma_0(\beta^* - \beta^0))_j| \geq \lambda|z^*_j| - \|\hat{\Sigma} - \Sigma_0\|_\infty \|\beta^* - \beta^0\|_1. \]

If \(|z^*_j| > (1 - \eta)\) we get
\[ \lambda|z^*_j| - \|\hat{\Sigma} - \Sigma_0\|_\infty \|\beta^* - \beta^0\|_1 > \lambda(1-\eta) - \|\hat{\Sigma} - \Sigma_0\|_\infty \|\beta^* - \beta^0\|_1 > 0. \]

Thus
\[ \sum_{j \in \bar{S}_*} \left( \lambda z_j - \|\hat{\Sigma} - \Sigma_0\|_\infty \|\beta^* - \beta^0\|_1 \right)^2 \geq \bar{s}_* \left( \lambda(1-\eta) - \|\hat{\Sigma} - \Sigma_0\|_\infty \|\beta^* - \beta^0\|_1 \right)^2. \]

On the other hand
\[ \sum_{j \in \bar{S}_*} |(\Sigma_0(\beta^* - \beta^0))_j|^2 \leq \Lambda_{\text{max}}(\Sigma_0)(\beta^* - \beta^0)^T \Sigma_0(\beta^* - \beta^0). \]

\[ \leq \Lambda_{\text{max}}(\Sigma_0) \left( \|X(\beta^* - \beta^0)\|_2^2 + \|\hat{\Sigma} - \Sigma_0\|_\infty \|\beta^* - \beta^0\|_1^2 \right). \]
Hence
\[
\bar{s}_* \leq \Lambda_{\max}(\Sigma_0) \left( \left\| X(\beta^* - \beta^0) \right\|_2^2 + \left\| \hat{\Sigma} - \Sigma_0 \right\|_{\infty} \left\| \beta^* - \beta^0 \right\|_1^2 \right) \cdot \left( \lambda (1 - \eta) - \left\| \hat{\Sigma} - \Sigma_0 \right\|_{\infty} \left\| \beta^* - \beta^0 \right\|_1 \right)^{-2}.
\]

**Proof of Theorem 2.2.** We have by Lemma 15.6
\[
\bar{s}_* \leq \Lambda_{\max}(\Sigma_0) \left( \left\| X(\beta^* - \beta^0) \right\|_2^2 + \left\| \hat{\Sigma} - \Sigma_0 \right\|_{\infty} \left\| \beta^* - \beta^0 \right\|_1^2 \right) \cdot \left( \lambda (1 - \eta) - \left\| \hat{\Sigma} - \Sigma_0 \right\|_{\infty} \left\| \beta^* - \beta^0 \right\|_1 \right)^{-2}.
\]

So with probability at least 1 - \(\alpha_1\),
\[
\mathbb{V}(\bar{S}_*) \leq \frac{\Lambda_{\max}(\Sigma_0) \left( \left\| X(\beta^* - \beta^0) \right\|_2^2 + \left\| \hat{\Sigma} - \Sigma_0 \right\|_{\infty} \left\| \beta^* - \beta^0 \right\|_1^2 \right)^{1/2}}{\left( \lambda (1 - \eta) - \left\| \hat{\Sigma} - \Sigma_0 \right\|_{\infty} \left\| \beta^* - \beta^0 \right\|_1 \right)^{1/2}} + \sqrt{\frac{2 \log(1/\alpha_1)}{n}}.
\]

The proof can be completed along the same lines as the proof of Theorem 2.1. \(\square\)

### 15.2 Proofs for Section 3

**Proof of Lemma 3.1.** By the KKT conditions
\[
\hat{\Sigma}(\beta^* - \beta^0) + \lambda z^* = 0, \; z^* \in \partial \| \beta^* \|_1.
\]

Hence
\[
0 \leq \left\| X(\beta^* - \beta^0) \right\|_2^2 = (\beta^* - \beta^0)^T \hat{\Sigma}(\beta^* - \beta^0) = \lambda(\beta^0 - \beta^*)^T z^* \leq \lambda \| \beta^0 \|_1 - \lambda \| \beta^* \|_1.
\]

Therefore the first bound of the lemma holds. Continuing with (3) and applying the definition of the compatibility constant \(\hat{\phi}^2(S_0)\) one finds
\[
0 \leq \left\| X(\beta^* - \beta^0) \right\|_2^2 \leq \lambda \| \beta^0 \|_1 - \lambda \| \beta^* \|_1 \leq \lambda \| \beta^0 - \beta_{S_0}^* \|_1 - \lambda \| \beta_{+S_0}^* \|_1 \leq \lambda \sqrt{s_0} \| X(\beta^* - \beta^0) \|_2 \hat{\phi}(S_0) - \lambda \| \beta_{+S_0}^* \|_1 \leq \lambda^2 s_0 / (2\hat{\phi}^2(S_0)) + \left\| X(\beta^* - \beta^0) \right\|_2^2 / 2 - \lambda \| \beta_{+S_0}^* \|_1.
\]
This yields the second bound of the lemma. \qed

**Proof of Lemma 3.2.** The first minimum $\hat{U}_1(\beta^0)$ for the prediction error follows from Lemma 3.1.

We recall the KKT conditions
\[
\hat{\Sigma}(\beta^* - \beta^0) + \lambda z^* = 0, \ z^* \in \partial \|\beta^*\|_1.
\]

For the second minimum $\hat{U}_2(\beta^0)$, let $S \subseteq \{1, \ldots, p\}$ be arbitrary. We note that when $\|X(\beta^* - \beta^0)\|^2_2 - 2\lambda \|\beta^0_S\|_1 \leq 0$ there is nothing to prove here. So let us assume $\|X(\beta^* - \beta^0)\|^2_2 - 2\lambda \|\beta^0_S\|_1 \geq 0$. Then we have by the KKT conditions
\[
0 \leq \|X(\beta^* - \beta^0)\|^2_2 - 2\lambda \|\beta^0_S\|_1
\leq \lambda \|\beta^0\|_1 - \lambda \|\beta^*_1\| - 2\lambda \|\beta^0_S\|_1
\leq \lambda \|\beta^*_S - \beta^0_S\|_1 - \lambda \|\beta^*_S - \beta^0_S\|_1
\leq \lambda \|\beta^*_S - \beta^0_S\|_1 - \lambda \|\beta^*_S - \beta^0_S\|_1.
\]

By the definition of the compatibility constant we now find
\[
\|X(\beta^* - \beta^0)\|^2_2 - 2\lambda \|\beta^0_S\|_1
\leq \lambda \sqrt{|S|} \|X(\beta^* - \beta^0)\|^2_2/\hat{\phi}(S) - \lambda \|\beta^*_S\|_1 - \lambda \|\beta^0_S\|_1
\leq \lambda \sqrt{|S|} \|X(\beta^* - \beta^0)\|^2_2/\hat{\phi}(S) - \lambda \|\beta^0_S\|_1.
\]

It follows that
\[
\left(\|X(\beta^* - \beta^0)\|^2_2 - \lambda \sqrt{|S|}/(2\hat{\phi}(S))\right)^2
\leq \lambda^2 |S|/(4\hat{\phi}^2(S)) + \lambda \|\beta_S\|_1.
\]

We now turn to the third minimum $\hat{U}_3(\beta^0)$. For any $\beta$
\[
(\beta^* - \beta)^T \hat{\Sigma}(\beta^* - \beta) + (\beta^* - \beta)^T z^* = 0.
\]

We have
\[
(\beta^* - \beta)^T \hat{\Sigma}(\beta^* - \beta) = \|X(\beta^* - \beta^0)\|^2_2/2 - \|X(\beta - \beta^0)\|^2_2/2
\]
\[
+ \|X(\beta^* - \beta)\|^2_2/2.
\]

Let $S \subseteq \{1, \ldots, p\}$. If $(\beta^* - \beta)^T \hat{\Sigma}(\beta^* - \beta^0) - 2\lambda \|\beta_S\|_1 \leq 0$ we are done. On the other hand, if $(\beta^* - \beta)^T \hat{\Sigma}(\beta^* - \beta^0) - 2\lambda \|\beta_S\|_1 \geq 0$ we get
\[
0 \leq (\beta^* - \beta)^T \hat{\Sigma}(\beta^* - \beta^0) - 2\lambda \|\beta_S\|_1
\leq \lambda \|\beta^*_S - \beta^0_S\|_1 - \lambda \|\beta^*_S - \beta^0_S\|_1
\leq \lambda \|\beta^*_S - \beta^0_S\|_1 - \lambda \|\beta^*_S - \beta^0_S\|_1
\leq \lambda \|\beta^*_S - \beta^0_S\|_1 - \lambda \|\beta^*_S - \beta^0_S\|_1.
\]

We can apply the definition of the compatibility constant to find
\[
(\beta^* - \beta)^T \hat{\Sigma}(\beta^* - \beta^0) \leq \lambda \sqrt{|S|} \|X(\beta^* - \beta)\|^2_2/\hat{\phi}(S) + \lambda \|\beta_S\|_1
\leq \lambda^2 |S|/(4\hat{\phi}^2(S)) + \|X(\beta^* - \beta)\|^2_2/2 + \lambda \|\beta_S\|_1,
\]

24
which gives
\[
\|X(\beta^* - \beta^0)\|_2^2/2 - \|X(\beta - \beta^0)\|_2^2/2 + \|X(\beta^* - \beta)\|_2^2/2 \\
\leq \lambda^2|\mathcal{S}|/(2\hat{\phi}^2(\mathcal{S})) + \|X(\beta^* - \beta)\|_2^2/2 + \lambda\|\beta - \beta^\dagger\|_1.
\]
\[
\]
\[
\]
\[
\]
\[
\]
\[
\]

15.3 Proof of the lemma in Section 4

**Proof of Lemma 4.1**. Suppose on the contrary that \( \hat{\phi} \{ \{1\} \} = 0 \). Then there exists a \( \gamma_{-1} \) with \( \|\gamma_{-1}\|_1 = 1 \) such that \( X_1 = X_{-1}\gamma_{-1} \). This gives
\[
1 = \|X_1\|_2^2 = \|X_{-1}\gamma_{-1}\|_2^2.
\]
We show that this is not possible. We let \( X_{-1} \) be an \( n \times m_0 \)-matrix and prove the result by induction in \( m_0 \).
\- \( m_0 = 1 \): Trivial.
\- \( m_0 = 2 \): Let \( \hat{\theta} := X_2^TX_3 \). Assume without loss of generality that \( \gamma_{-1}^T = (\gamma_2, \gamma_3) \) has both its components non-negative. Then \( \gamma_3 = 1 - \gamma_2 \) and
\[
\|X_{-1}\gamma_{-1}\|_2^2 = \gamma_2^2 + (1 - \gamma_2)^2 + 2\gamma_2(1 - \gamma_2)\hat{\theta} \\
= 1 + 2\gamma_2(1 - \gamma_2)\hat{\theta}
\]
This can only be equal to 1 if \( \gamma_2 = 0 \) or \( \gamma_2 = 1 \) or \( \hat{\theta} = -1 \), all cases which we excluded.
\- Induction step: suppose it is true for the value \( m_0 - 1 \): for all \( \gamma_{-1} \) with \( \gamma_{j_0} = 0 \) for some \( j_0 \in \{2, \ldots, m_0 + 1\} \) and with \( \|\gamma\|_1 = 1 \) it holds that \( \|X_{-1}\gamma_{-1}\|_2^2 < 1 \). Let \( \gamma_{-1}^T = (\gamma_2, \ldots, \gamma_{m_0+1}) \) be a vector with \( \|\gamma_{-1}\|_1 = 1 \) and with \( |\gamma_{m_0+1}| < 1 \). Then we know by induction that either \( \|X_{-1}\gamma_{-1} - X_{m_0+1}\gamma_{m_0+1}\|_2 < (1 - |\gamma_{m_0+1}|) \) or there is a \( j_0 \in \{2, \ldots, m_0\} \) such that \( |\gamma_{j_0}| = 1 - |\gamma_{m_0+1}| \). In the last case all values \( j \in \{2, \ldots, m_0\} \) other than \( j_0 \) must be zero so it brings us back to the case \( m_0 = 2 \). In the first case we have by the triangle inequality
\[
\|X_{-1}\gamma_{-1}\|_2 \leq \|X_{-1}\gamma_{-1} - X_{m_0+1}\gamma_{m_0+1}\|_2 + |\gamma_{m_0+1}| \\
< (1 - |\gamma_{m_0+1}|) + |\gamma_{m_0+1}| = 1.
\]

\[
\]
\[
\]

15.4 Proofs for Section 6

**Proof of Lemma 6.1**. The coefficient of the projection of \( X_1 \) on \( X_2 \) is
\[
\arg \min_{\beta_2 \in \mathbb{R}} \|X_1 - X_2\beta_2\|_2^2 = -\hat{\rho}.
\]
Since \( 0 < \hat{\rho} < 1 \) we thus find
\[
\hat{\phi}^2(\{1\}) := \min_{|\beta_2| \leq 1} \|X_1 - X_2\beta_2\|_2^2 = \|X_1 + \hat{\rho}X_2\|_2^2 = 1 - \hat{\rho}^2.
\]

\[
\]
As \( \varphi^2 = (1 - \hat{\rho}) \) we have

\[
1 - \hat{\rho}^2 = (1 - \hat{\rho})(1 + \hat{\rho}) = (1 - \hat{\rho})(2 - (1 - \hat{\rho})) = \varphi^2(2 - \varphi^2).
\]

The second result follows from symmetry arguments: the minimum of \( \|X_1\beta_1 + X_2\beta_2 \) over \( |\beta_1| + |\beta_2| = 1 \) is reached at equal values for \( \beta_1 \) and \( \beta_2 \). \( \square \)

**Proof of Lemma 6.2.** One readily verifies that \( 0 \leq \beta_1^* \leq \beta_1^0 \) and \( 0 \leq \beta_2^* \leq \beta_2^0 \). Let \( \Delta_1 := \beta_1^0 - \beta_1^* \) and \( \Delta_2 := \beta_2^0 - \beta_2^* \). Recall the KKT conditions

\[
\hat{\Sigma}\Delta = \lambda z^*, \ z^* \in \partial\|\beta^*\|_1.
\]

- **Case 1:** \( \lambda/\varphi^2 \leq \beta_2^0 \). For \( \Delta_1 = \Delta = \lambda/\varphi^2 \)

\[
\hat{\Sigma}\Delta = \lambda \left( \frac{1}{1} \right).
\]

Since

\[
\left( \frac{\beta_1^*}{\beta_2^*} \right) = \left( \frac{\beta_1^0 - \Delta_1}{\beta_2^0 - \Delta_2} \right)
\]

has both its components non-negative, it is a solution of the KKT conditions, in fact it is the unique solution.

- **Case 2:** \( \beta_2^0 < \lambda/\varphi^2 \leq \beta_2^0 + (\beta_1^0 - \beta_2^0)/\varphi^2 \) With \( \Delta_1 = \lambda + (1 - \varphi^2)\beta_2^0 = \lambda + \hat{\rho}\beta_2^0 \) and \( \Delta_2 = \beta_2^0 \) we obtain

\[
\hat{\Sigma}\Delta = \left( \frac{1}{1 - \hat{\rho}} \right) \left( \frac{\lambda + \hat{\rho}\beta_2^0}{1} \right) = \lambda \left( \frac{1}{z_2^*} \right)
\]

with \( z_2^* = -\hat{\rho} + (1 - \hat{\rho}^2)\beta_2^0/\lambda \). As \( |z_2^*| \leq 1 \) and \( \beta_1^* = \beta_1^0 - \Delta_1 \geq 0, \beta_2^* = 0 \), we see that indeed \( \beta^* \) is the solution of the KKT conditions.

- **Case 3:** \( \lambda/\varphi^2 > \beta_2^0 + (\beta_1^0 - \beta_2^0)/\varphi^2 \). With \( \Delta_1 = \beta_1^0 \) and \( \Delta_2 = \beta_2^0 \) we obtain

\[
\hat{\Sigma}\Delta = \left( \frac{\beta_1^0 - \hat{\rho}\beta_2^0}{-\hat{\rho}\beta_1^0 + \beta_2^0} \right) = \lambda \left( \frac{z_1^*}{z_2^*} \right)
\]

where \( 0 < z_1^* = (\beta_1^0 - \beta_2^0 + \varphi^2 \beta_2^0)/\lambda \leq 1 \) and \( \lambda z_2^* = \leq (\hat{\rho}\beta_2^0 + \beta_2^0)/\lambda = \varphi^2 \beta_2^0/\lambda \leq \lambda \) and \( \lambda z_2^* = \leq (\varphi^2 \beta_1^0 - (\beta_1^0 - \beta_2^0))/\lambda \geq -1 + \varphi^2 \beta_2^0/\lambda \geq -\lambda \).

Hence the KKT conditions hold for \( \beta_1^* = \beta_2^* = 0 \). \( \square \)

**15.5 Proofs for Section 7**

**Proof of Lemma 7.1.** The expression for the minimal eigenvalue \( \Lambda_{\min}(\hat{\Sigma}) \) is trivial. Then, by orthogonality

\[
\|X\beta\|_2^2 = \sum_{k=1}^{N} \|X\beta_{2k-1} + X\beta_{2k}\|_2^2
\]
and by the arguments of Lemma 6.1 for all $k$

$$\min_{|\beta_{2k-1}| + |\beta_{2k}| = 1} \|X\beta_{2k-1} + X\beta_{2k}\|_2^2 = \hat{\varphi}_k^2 / 2.$$ 

For any vector $v \in \mathbb{R}^N$

$$\|v\|_2^2 = \left(\sum_{k=1}^{N} \frac{|v_k|}{\hat{\varphi}_k}\right) \leq \left(\sum_{k=1}^{N} \frac{v_k^2 \hat{\varphi}_k^2}{\hat{\varphi}_k^2}\right) = \left(\sum_{k=1}^{N} \frac{1}{\hat{\varphi}_k^2}\right)^{-1},$$

and this gives

$$\min_{\|v\|_1 = 1} \|v\|_2^2 = \left(\sum_{k=1}^{N} \frac{1}{\hat{\varphi}_k^2}\right)^{-1} = \|1/\hat{\varphi^2}\|_1^{-1}.$$ 

So

$$\min_{\|\beta\|_1 = 1} \|X\beta\|_2^2 = \min_{\|v\|_1 = 1} \sum_{k=1}^{N} \frac{|v_k|}{\hat{\varphi}_k} \leq \left(\sum_{k=1}^{N} \frac{v_k^2 \hat{\varphi}_k^2}{\hat{\varphi}_k^2}\right) = \left(\sum_{k=1}^{N} \frac{1}{\hat{\varphi}_k^2}\right)^{-1} \|1/\hat{\varphi^2}\|_1^{-1}.$$

The expression for $\hat{\varphi}^2(S)$ follows by similar arguments. 

**Proof of Lemma 7.2.** By Lemma 7.1 the compatibility constant is

$$\hat{\varphi}^2(S_0) = N \|1/\hat{\varphi^2}\|_1^{-1}.$$ 

This gives by Lemma 3.1 (recall $|S_0| = 2N$)

$$\|X(\beta^* - \beta^0)\|_2^2 \leq 2\lambda^2 \|1/\varphi^2\|_1.$$ 

On the other hand, by the orthogonality and the decomposability of the $\ell_1$-norm, the Lasso problem can also be decomposed, giving in view of Lemma 6.2 for each $k$,

$$\|X_{2k-1}(\beta_{2k-1}^* - \beta_{2k-1}^0) + X_{2k}(\beta_{2k}^* - \beta_{2k}^0)\|_2^2 = \frac{2\lambda^2}{\hat{\varphi}_k^2}$$

and

$$\|X_{2k-1}(\beta_{2k-1}^* - \beta_{2k-1}^0) + X_{2k}(\beta_{2k}^* - \beta_{2k}^0)\|_2^2 + \lambda(\|\beta_{2k-1}^*\| + |\beta_{2k}^*|) = \lambda(\|\beta_{2k-1}^0\| + |\beta_{2k}^0|)$$

where we used the assumption $\lambda/\hat{\varphi}_k^2 \leq \beta_{2k}^0 \leq \beta_{2k-1}^0$. Thus

$$\|X(\beta^* - \beta^0)\|_2^2 = \frac{2\lambda^2}{\hat{\varphi}_k^2} \sum_{k=1}^{N} 1/\varphi_k^2 = 2\lambda^2 \|1/\hat{\varphi^2}\|_1$$

and

$$\|X(\beta^* - \beta^0)\|_2^2 + \lambda\|\beta^*\|_1 = \lambda\|\beta^0\|_1.$$ 

\[\square\]
15.6 Proof of the lemma in Section 8

Proof of Lemma 8.1. Obviously for all $\beta_{S_0}$

$$\arg\min \left\{ \|X\beta_{S_0} - X\beta_{-S_0}\|_2^2 : \|\beta_{-S_0}\|_1 \leq 1 \right\} = 0.$$ 

So the result of the lemma follows immediately from Lemmas 6.1 and 6.2. \(\square\)

15.7 Proof of the lemma in Section 9

Proof of Lemma 9.1. It holds by symmetry arguments that for all $\beta_{3} \in \mathbb{R}$

$$\min_{|\beta_1| + |\beta_2| = 1} \|X_1\beta_1 + X_2\beta_2 - X_3\beta_3\|_2^2 = \|(X_1 + X_2)/2 - X_3\beta_3\|_2^2.$$ 

Moreover

$$\gamma_3 := \arg\min_{\beta_3 \in \mathbb{R}} \|X_1\beta_1 + X_2\beta_2 - X_3\beta_3\|_2^2$$

$$= \arg\min_{\beta_3 \in \mathbb{R}} \|X_1 + X_2 - 2X_3\beta_3\|_2^2$$

$$= \arg\min_{\beta_3 \in \mathbb{R}} \|(X_1 + X_2)(1 - C\beta_3) - 2U\beta_3\|_2^2$$

$$= \arg\min_{\beta_3 \in \mathbb{R}} \left\{ 2\hat{\phi}^2(1 - C\beta_3)^2 + 4\tau^2\beta_3^2 \right\}$$

$$= \frac{C(2\hat{\phi}^2)}{4\hat{\tau}^2 + C^2(2\hat{\phi}^2)}.$$ 

Since $|\gamma_3| < 1$, we conclude that

$$\hat{\phi}^2(S_0)/s_0 = \frac{\|(X_1 + X_2)/2 - X_3\gamma_3\|_2^2/n}{\frac{1}{4} \times \frac{(2\hat{\phi}^2)(4\hat{\tau}^2)}{4\hat{\tau}^2 + C^2(2\hat{\phi}^2)}}$$

$$= \frac{1}{4} \times \frac{(2\hat{\phi}^2)\hat{\tau}^2}{\hat{\tau}^2 + C^2\hat{\phi}^2/2}$$

$$= \hat{\phi}^2\hat{\tau}^2/2$$

where in the last step we used that $\hat{\tau}^2 + C^2\hat{\phi}^2/2 = 1$. Since $s_0 = 2$ we conclude that $\hat{\phi}^2(S_0) = \hat{\phi}^2\hat{\tau}^2$.

To arrive at the second result, we write $\beta^*_1 = \beta^*_1 - \Delta_1$ and $\beta^*_2 = \beta^*_2 - \Delta_2$. We have

$$\hat{\Sigma} = \begin{pmatrix} 1 & -\hat{\rho} & C\hat{\phi}^2/2 \\ -\hat{\rho} & 1 & C\hat{\phi}^2/2 \\ C\hat{\phi}^2/2 & C\hat{\phi}^2/2 & 1 \end{pmatrix}.$$ 

For $\beta^*_3 = \lambda(2C - 1)/\hat{\tau}^2$, $\Delta_1 = \Delta_2 = C\beta^*_3 + \lambda/\hat{\phi}^2$ we find

$$\hat{\Sigma} \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ -\beta^*_3 \end{pmatrix} = \begin{pmatrix} C\hat{\phi}^2\Delta_1 - C\hat{\phi}^2\beta^*_3/2 \\ \hat{\phi}^2\Delta_1 - C\hat{\phi}^2\beta^*_3/2 \\ C\hat{\phi}^2\Delta_1 - \beta^*_3 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$
Since \( 0 \leq \beta_1^* \leq \beta_1^0 \) and \( 0 \leq \beta_2^* \leq \beta_2^0 \) and \( \beta_3^* > 0 \), the vector \( \beta^* \) is indeed the solution of the KKT conditions. \( \square \)

### 15.8 Proof of the lemma in Section 10

**Proof of Lemma 10.1.** Let 
\[
\gamma := \arg \min \left\{ \|X\beta_S_0 - X_{2N+1}\beta_{2N+1}\|_2^2 : \|\beta_S_0\|_1 = 1, \ |\beta_{2N+1}| \leq 1 \right\}.
\]

By straightforward computations in a spirit similar to the one used in the proof of Lemma 9.1, one finds for \( k = 1, \ldots, N \)
\[
\gamma_{2k-1} = \gamma_{2k} = \frac{C^2/(N\|1/\hat{\varphi}^2\|_1) + 2\hat{\tau}^2\hat{\varphi}_k^2/\|1/\hat{\varphi}^2\|_1}{2C^2/\|1/\hat{\varphi}^2\|_1 + 4\hat{\tau}^2}
\]
and moreover
\[
\gamma_{2N+1} = \frac{2C\|1/\hat{\varphi}^2\|_1}{2C^2/\|1/\hat{\varphi}^2\|_1 + 4\hat{\tau}^2}.
\]

Inserting these values one sees
\[
\hat{\varphi}^2(S_0)/s_0 = \frac{\|X\gamma_{S_0} - X_{2N+1}\gamma_{2N+1}\|_2^2}{\hat{\tau}^2/\|1/\hat{\varphi}^2\|_1} = \frac{\hat{\tau}^2/\|1/\hat{\varphi}^2\|_1}{2\hat{\tau}^2 + C^2/\|1/\hat{\varphi}^2\|_1}.
\]

The second result of the lemma also follows from similar arguments as used in the proof of Lemma 9.1. The minimizing values are
\[
\beta_{2N+1}^* = \lambda(C - 1)/\hat{\tau}^2
\]
and for \( k = 1, \ldots, N \)
\[
\beta_{2k-1}^0 - \beta_{2k-1}^* - C\beta_{2N+1}^*/2 = \beta_{2k}^0 - \beta_{2k}^* - C\beta_{2N+1}^*/2 = \lambda/\hat{\varphi}_k^2.
\]
\( \square \)

### 15.9 Proof of the lemma in Section 11

**Proof of Lemma 11.1.** This follows by similar arguments as used in the proof of Lemma 9.1. \( \square \)

### 15.10 Proofs for Section 12

**Proof of Lemma 12.1.** We minimize
\[
\|X_1\beta_1 + X_2\beta_2 - X_3\beta_3 - X_4\beta_4\|_2^2
\]
By symmetry arguments, we know $eta_3 = eta_4$. Hence
\[ (\beta_3 - \beta_4)^2 (1 - C \beta_3 + \beta_4/2) + (\beta_3 + \beta_4)^2 \tilde{\tau}^2 \]

This implies $eta_3 = \beta_4$. So we minimize
\[ |X_1 (\beta_1 - C \beta_3) + X_2 (\beta_2 - C \beta_3)|^2 + (2 \beta_3)^2 \tilde{\tau}^2. \]

By symmetry arguments, we know $\beta_1 = \beta_2$ say both $+1/2$. Then we need to minimize
\[ |(X_1 + X_2) (1/2 - C \beta_3)|^2 + (2 \beta_3)^2 \tilde{\tau}^2 = (1/2 - C \beta_3)^2 2 \phi^2 + (2 \beta_3)^2 \tilde{\tau}^2. \]

The minimizing value for $2 \beta_3$ is
\[ 2 \gamma_3 = \frac{1}{2} \frac{2 C (2 \phi^2)}{C^2 (2 \phi^2) + 4 \tilde{\tau}^2}. \]

In other words
\[ \min \left\{ \|X \beta\|_2^2 : \|\beta_{S_0}\|_1 = 1, \|\beta_{\bar{S}_0}\|_1 \leq 1 \right\} = \frac{1}{4} \frac{2 \phi^2 \tilde{\tau}^2}{C^2 (2 \phi^2) + 4 \tilde{\tau}^2}. \]

Hence
\[ \hat{\phi}^2 (S_0) = \frac{1}{2} \frac{2 \phi^2 \tilde{\tau}^2}{C^2 (2 \phi^2) + 4 \tilde{\tau}^2}. \]

For the second result, we check the KKT conditions with $\beta_3^* = \beta_4^*$ and $\Delta_1 - C \beta_3^* = \Delta_2 - C \beta_3^* = \lambda/\phi^2$ where $\Delta_1 = \beta_1^0 - \beta_4^*$ and $\Delta_2 = \beta_2^0 - \beta_4^*$. It holds that
\[ \Sigma = \begin{pmatrix}
1 & -\hat{\beta} & C \phi^2/2 & C \phi^2/2 \\
-\hat{\beta} & 1 & C \phi^2/2 & C \phi^2/2 \\
C \phi^2/2 & C \phi^2/2 & 1 & C^2 \phi^2 + 2 \tilde{\tau}^2 - 1 \\
C \phi^2/2 & C \phi^2/2 & C^2 \phi^2 + 2 \tilde{\tau}^2 - 1 & 1
\end{pmatrix}. \]

Hence
\[ \Sigma \begin{pmatrix}
\Delta_1 \\
\Delta_2 \\
-\beta_3 \\
-\beta_4
\end{pmatrix} = \begin{pmatrix}
C \phi^2 \Delta_1/2 - C \phi^2 \beta_3 \\
C \phi^2 \Delta_1/2 - C \phi^2 \beta_3 \\
C \phi^2 \Delta_1 - (C^2 \phi^2 + 2 \tilde{\tau}^2) \beta_3 \\
C \phi^2 \Delta_1 - (C^2 \phi^2 + 2 \tilde{\tau}^2) \beta_3
\end{pmatrix} = \begin{pmatrix}
\lambda/\phi^2 + C \beta_3 \\
\lambda/\phi^2 + C \beta_3 \\
(\lambda/\phi^2 + C \beta_3)/2 - C \phi^2 \beta_3 \\
(\lambda/\phi^2 + C \beta_3)/2 - C \phi^2 \beta_3
\end{pmatrix} = \begin{pmatrix}
\lambda \\
\lambda \\
C \lambda - 2 \tilde{\tau}^2 \beta_3 \\
C \lambda - 2 \tilde{\tau}^2 \beta_3
\end{pmatrix}. \]

\[ \lambda \begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix}. \]
Thus $\beta^*$ is the solution of the KKT conditions. \qed

**Proof of Lemma 12.2.** Along similar lines as in the proof of Lemma 12.1 one finds $\beta_3^* = \beta_4^*$ and

$$\beta_3^* - \beta_4^* - C\beta_3^* = \beta_1^* - \beta_1^* - C\beta_3^* = \lambda/\hat{\varphi}^2$$

but now $\beta_3^*$ is the largest possible value such that $\beta_3^* - \beta_3^* \leq \beta_3^*$. It follows that $\beta_1^* = \beta_1^* - \beta_2^*$, $\beta_2^* = 0$, $\beta_3^* = (\beta_3^* - \lambda/\hat{\varphi}^2)/C$. \qed

**Proof of Lemma 12.3.** The Gram matrix is now

$$\hat{\Sigma} = \begin{pmatrix} 1 & -\hat{\rho} & \hat{\varphi}^2/2 & \hat{\varphi}^2/2 \\ -\hat{\rho} & 1 & \hat{\varphi}^2/2 & \hat{\varphi}^2/2 \\ \hat{\varphi}^2/2 & \hat{\varphi}^2/2 & 1 & \hat{\varphi}^2 - 1 \\ \hat{\varphi}^2/2 & \hat{\varphi}^2/2 & \hat{\varphi}^2 - 1 & 1 \end{pmatrix}. $$

Hence, with $\Delta_1 = \Delta_2 = \beta_3^* + \lambda/\hat{\varphi}^2$ we find

$$\hat{\Sigma} \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ -\beta_3^*_\ |
\end{pmatrix} = \begin{pmatrix} \hat{\varphi}^2(\beta_3^* + \lambda/\hat{\varphi}^2) - \hat{\varphi}^2\beta_3^* \\ \hat{\varphi}^2(\beta_3^* + \lambda/\hat{\varphi}^2) - \hat{\varphi}^2\beta_3^* \\ \hat{\varphi}^2(\beta_3^* + \lambda/\hat{\varphi}^2) - \hat{\varphi}^2\beta_3^* \\ \hat{\varphi}^2(\beta_3^* + \lambda/\hat{\varphi}^2) - \hat{\varphi}^2\beta_3^* \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$ 

Since for $0 \leq \beta_3^* \leq -\lambda/\hat{\varphi}^2$ it holds that $\beta_1^* = \beta_1^* - \beta_3^* - \lambda/\hat{\varphi}^2 \geq 0$, $\beta_2^* = \beta_2^* - \beta_3^* - \lambda/\hat{\varphi}^2 \geq 0$ and $\beta_3^* \geq 0$, the vector $\beta^*$ is indeed the solution of the KKT conditions. With this value one finds the result for the prediction error and the bound for $\|\beta^* - S_0\|_1$. \qed

**15.11 Proof of the lemma in Section 13**

**Proof of Lemma 13.1.** This follows from the same arguments as used in the proofs of Lemmas 12.1 and 12.1. \qed

**15.12 Proofs for Section 14**

**Proof of Lemma 14.1.** Observe first that

$$\hat{\rho} = 1 - 2\gamma_{-S_0}^T \hat{\Sigma}_{-S_0, -S_0} \gamma_{-S_0}$$

$$= 1 - 2\left(\gamma_3^2(1 - \gamma_3)^2 - 2\gamma_3(1 - \gamma_3)\hat{\theta}\right)$$

$$= 1 - 2\left(1 - 2\gamma_3(1 - \gamma_3)(1 + \hat{\theta})\right)$$

$$= -1 + 4\gamma_3(1 - \gamma_3)(1 + \hat{\theta}).$$
Therefore
\[
\varphi^2 = 1 - \rho \\
= 2 - 4\gamma_3(1 - \gamma_3)(1 + \theta) \\
= 2(1 - 4\gamma_3(1 - \gamma_3)) + 4\gamma_3(1 - \gamma_3)\psi^2.
\]
So \(\varphi^2 > \psi^2\).

The Gram matrix is now
\[
\hat{\Sigma} = \begin{pmatrix}
1 & -\rho & \hat{\theta}_3 & \hat{\theta}_4 \\
-\rho & 1 & \hat{\theta}_3 & \hat{\theta}_4 \\
\hat{\theta}_3 & \hat{\theta}_3 & 1 & -\theta \\
\hat{\theta}_4 & \hat{\theta}_4 & -\theta & 1
\end{pmatrix}
\]
where \(\hat{\theta}_3 := \gamma_3 - (1 - \gamma_3)\hat{\theta}\) and \(\hat{\theta}_4 = (1 - \gamma_3) - \gamma_3\hat{\theta}\). Then for \(\beta^*_1 = \beta^0_1 - \Delta_1 := \beta^0_1 - \beta^0_2, \beta^*_2 = \beta_2 - \Delta_2 := 0, \beta^*_3 = 2\gamma_3\beta^0_2 - \lambda/\psi^2\) and \(\beta^*_4 = 2(1 - \gamma_3)\beta^0_2 - \lambda/\psi^2\) we get
\[
\hat{\Sigma} \begin{pmatrix}
\Delta_1 \\
\Delta_2 \\
-\beta^*_3 \\
-\beta^*_4
\end{pmatrix}
= \begin{pmatrix}
\varphi^2\beta^0_2 - \hat{\theta}_3\beta^*_3 + \hat{\theta}_4\beta^*_4 \\
\varphi^2\beta^0_2 - \hat{\theta}_3\beta^*_3 + \hat{\theta}_4\beta^*_4 \\
2\beta^0_2\hat{\theta}_3 + \beta^*_3 - \theta\beta^*_4 \\
2\beta^0_2\hat{\theta}_4 - \theta\beta^*_3 + \beta^*_4
\end{pmatrix}
= \lambda \begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix}.
\]
So \(\beta^*\) is the solution of the KKT conditions. \(\square\)

**Proof of Lemma 14.2.** It is straightforward to calculate
\[
\varphi^2 = C^2\hat{\psi}^2.
\]
To find \(\hat{\phi}^2(S_0)\) we minimize
\[
\|X_1\beta_1 + X_2\beta_2 - X_{-S_0}\beta_{-S_0}\|_2^2
\]
on \(0 < \beta_1 = 1/2 < 1\) and \(\|\beta_{-S_0}\|_1 \leq 1\). Symmetry arguments yield \(\beta_1 = 1/2\). We then minimize
\[
\|X_3 + X_4 - X_{S_0}\beta_{-S_0}\|_2^2
\]
on \(\|\beta_{-S_0}\|_1 \leq 1\). This gives that the entries in \(\beta_{-S_0}\) are equal to 1/2 and hence \(\hat{\phi}^2(S_0) = (C - 1)^2\hat{\psi}^2\).

In view of Lemmas 6.2 and 14.2 it suffices to show that \(\beta^*_S = 0\) corresponds to the unique solution of the KKT conditions. We have with \(\Delta_1 = \Delta_2 = \lambda/\varphi^2\)
\[
\hat{\Sigma}_{-S_0,S_0} \begin{pmatrix}
\Delta_1 \\
\Delta_2
\end{pmatrix}
= \begin{pmatrix}
C\hat{\psi}^2/2 & C\hat{\psi}^2/2 \\
C\hat{\psi}^2/2 & C\hat{\psi}^2/2
\end{pmatrix}
\begin{pmatrix}
\lambda/\varphi^2 \\
\lambda/\varphi^2
\end{pmatrix}
= \begin{pmatrix}
C\lambda\hat{\psi}^2/\varphi^2 \\
C\lambda\hat{\psi}^2/\varphi^2
\end{pmatrix}
= \lambda \begin{pmatrix}
1/C \\
1/C
\end{pmatrix}.
\]
since $\hat{\varphi}^2 = C^2 \hat{\psi}^2$. So the KKT conditions are satisfied, with $z^*_S = 1/C$. The solution is unique because for $\gamma - S_0 = (C, C)^T/2$ it holds that $\|\gamma - S_0\|_1 > 1$. $\square$

**Proof of Lemma 14.3.** Note first that indeed $\hat{\rho} = 1 - 2C^2 \|\gamma - S_0\|_2^2 > 0$ since $2C^2 \|\gamma - S_0\|_2^2 < 1$. It follows that $\hat{\varphi}^2 = 2C^2 \|\gamma - S_0\|_2^2$. We have

$$||\beta_1 X_1 + \beta_2 X_2 - X_{-S_0} \beta_{-S_0}||^2_2$$

is minimized over $|\beta_1| + |\beta_2| = 1$ at $\beta_1 = \beta_2 = 1/2$ and

$$||X_{-S_0}(C\gamma - S_0 - \beta_{-S_0})||^2_2 = ||C\gamma - S_0 - \beta_{-S_0}||^2_2.$$ 

To obtain the prediction error, in view of Lemmas 6.2 and 14.2, it suffices to show that $\beta^*_S = 0$ and $\Delta_1 = \beta^0_1 - \beta^*_1 = \lambda/\hat{\varphi}^2$, $\Delta_2 = \beta^0_1 - \beta^*_1 = \lambda/\hat{\varphi}^2$ is the unique solution of the KKT conditions. We have

$$\tilde{\Sigma}_{-S_0, S_0} \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} = C\gamma - S_0 (\Delta_1 + \Delta_2) = 2C\lambda \gamma - S_0 / \hat{\varphi}^2 = \lambda \gamma - S_0 / C \|\gamma - S_0\|_2^2 = \lambda z^*_S,$$

where $\|z^*_S\|\infty = ||\gamma - S_0\|\infty / (C \|\gamma - S_0\|_2^2) \leq 1$. The solution is unique because $\|\gamma - S_0\|_1 > 1$. Another way to see it is by noting that for any $\|z_S\|\infty \leq 1$

$$||\tilde{\Sigma}_{-S_0, S_0} \tilde{\Sigma}_{-S_0, S_0} z_S||\infty = ||\gamma - S_0 (z_1 + z_2)||\infty / \hat{\varphi}^2 = ||\gamma - S_0||\infty / (C \|\gamma - S_0\|_2^2) < 1$$

i.e., the irrepresentable condition holds.

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