Generic hyperplane section of curves and an application to regularity bounds in positive characteristic

Edoardo Ballico
Department of Mathematics
University of Trento
38050 Povo (TN)
Italy
ballico@science.unitn.it

Chikashi Miyazaki
Nagano National College of Technology
716 Tokuma, Nagano 381-8550
Japan
miyazaki@cc.nagano-nct.ac.jp

Abstract

This paper investigates the Castelnuovo-Mumford regularity of the generic hyperplane section of projective curves in positive characteristic case, and yields an application to a sharp bound on the regularity for nondegenerate projective varieties.

1991 Mathematics Subject Classification. Primary 14B15; Secondary 13D40, 14H45, 14N05
1 Introduction

The purpose of this paper is to study an upper bound of the index of regularity of a generic hyperplane section of projective curves and its application to sharp regularity bounds for projective varieties.

For a projective scheme $X \subset \mathbb{P}^N_K$, we define the Castelnuovo-Mumford regularity $\text{reg}(X)$ as the smallest integer $m$ such that $H^i(\mathbb{P}^N_K, \mathcal{I}_X(m-i)) = 0$ for all $i \geq 1$, see, e.g., [6]. The interest in this concept stems partly from the well-known fact: The regularity $\text{reg}(X)$ is the smallest integer $m$ such that the minimal generators of the $n$-th syzygy module of the defining ideal $I$ of $X$ occur in degree $\leq m + n$ for all $n \geq 0$.

In particular, for a zero-dimensional scheme $S \subset \mathbb{P}^N_K$, we define the index of regularity $i(S)$ of $S$ as the smallest integer $t$ such that $H^1(\mathbb{P}^N_K, \mathcal{I}_S(t)) = 0$. We remark that $\text{reg}(S) = i(S) + 1$.

Throughout this paper, for a rational number $\ell \in \mathbb{Q}$, we write $\lceil \ell \rceil$ for the minimal integer which is larger than or equal to $\ell$, and $\lfloor \ell \rfloor$ for the maximal integer which is smaller than or equal to $\ell$.

Let $S \subset \mathbb{P}^N_K$ be a generic hyperplane section of a nondegenerate projective curve $C \subset \mathbb{P}^N_K$ over an algebraically closed field $K$. Then $S$ has the uniform position property in case $\text{char}(K) = 0$, see [8], while the property does not necessarily hold in case $\text{char}(K) > 0$, see [19]. Instead, even for the positive characteristic case, $S$ has the linear semi-uniform position property introduced in [1], see §2 for the definition. The linear semi-uniform position has an important role in studying the positive characteristic case.

For example, by studying the $h$-vectors of a zero-dimensional scheme $S$ in linear semi-uniform position, we have an upper bound on the index of regularity, that is, $i(S) \leq \lceil (\deg(S) - 1)/N \rceil$, see, e.g., [1], [8]. Also, there are some known facts on the sharpness of the above bound. If a zero-dimensional scheme $S \subset \mathbb{P}^N_K$ lies on a rational normal curve, then we have an equality, $i(S) = \lceil (\deg(S) - 1)/N \rceil$. On the other hand, we assume that a zero-dimensional scheme $S \subset \mathbb{P}^N_K$ is in uniform position and $\deg(S)$ is large enough. If the equality $i(S) = \lceil (\deg(S) - 1)/N \rceil$ holds, then $S$ lies on a rational normal curve, see, e.g., [14, 22].
In Section 2, we consider a generic hyperplane section $S \subset \mathbb{P}^N_K$ of a non-degenerate projective curve over an algebraically closed field $K$ such that $S$ does not have the uniform position property. So we always focus on the case $\text{char}(K) > 0$. First, we will show that, under the condition that $N \geq 3$ and $\deg(S)$ is large enough, if $S$ does not have the uniform position property, then $i(S) \leq \lceil (\deg(S) - 1)/N \rceil - 1$ in (2.1) and (2.2). The lemmas are technically key results of this paper. As in classical Castelnuovo’s method, we will show the assertion of the lemmas, and in fact, the linear semi-uniform position property will be useful for this proof. Then we apply the lemmas to the main result of this section, see Theorem 2.3. Let $S \subset \mathbb{P}^N_K$ be a generic hyperplane section of a nondegenerate projective curve with $\deg(S)$ large enough. Without assuming $S$ is in uniform position, if the equality $i(S) = \lceil (\deg(S) - 1)/N \rceil$ holds, then $S$ lies on a rational normal curve. Finally we describe a results on the index of regularity for a generic hyperplane section of very strange curves, see Proposition 2.6.

In Section 3, we study the Castelnuovo-Mumford regularity of projective varieties as an application of §2. In recent years upper bounds on the Castelnuovo-Mumford regularity of a projective variety $X \subset \mathbb{P}^N_K$ have been given by several authors in terms of $\dim(X)$, $\deg(X)$, $\text{codim}(X)$, and $k(X)$, see, e.g., [10, 15, 18], where $k(X)$ is the Ellia-Migliore-Miró Roig number measuring the deficiency module, or sometimes called as the Rao module, see §3 for the definition. A regularity bound $\text{reg}(X) \leq \lceil (\deg(X) - 1)/\text{codim}(X) \rceil + \max\{k(X) \dim(X), 1\}$ is known for a nondegenerate projective variety $X$, see [10, 18]. Conversely, under the assumption that a nondegenerate projective variety $X$ is ACM, that is, the coordinate ring of $X$ is Cohen-Macaulay, if $\text{reg}(X) \leq \lceil (\deg(X) - 1)/\text{codim}(X) \rceil + 1$ and $\deg(X)$ is large enough, then $X$ is a variety of minimal degree, see [16, 20]. Moreover, there gives a classification of nondegenerate projective non-ACM varieties $X$ attaining a regularity bound $\text{reg}(X) = \lceil (\deg(X) - 1)/\text{codim}(X) \rceil + k(X) \dim(X)$. In [14], under the assumption that $\deg(X)$ is large enough and $\text{char}(K) = 0$, it is shown that a projective non-ACM variety having the equality must be a curve on a rational ruled surface, that is, on a Hirzeburch surface. In §3, we show the corresponding result in the positive characteristic case as an application of (2.3), see Theorem 3.2.

The first named author was partially supported by MURST and GN-
SAGA of CNR (Italy). The second named author was partially supported by Grant-in-Aid for Scientific Research (no. 09740042), Ministry of Education, Science, Sports and Culture (Japan).

2 Regularity of a Generic Hyperplane Section of Projective Curves in Positive Characteristic

Let $K$ be an algebraically closed field with $\text{char}(K) = p > 0$.

In this section we will show that if $S \subset \mathbb{P}_K^N$ is a generic hyperplane section of an integral curve with $\deg(S)$ large enough, then either $S$ is in uniform position or $i(S) \leq \lceil (\deg(S) - 1)/N \rceil - 1$. Here the index of regularity $i(S)$ of $S$ is defined as the smallest integer $t$ such that $H^1(\mathbb{P}_K^N, I_S(t)) = 0$. (Notice that $\text{reg}(S) = i(S) + 1 = a(R) + 2$, where $R$ is the coordinate ring of $S$ and $a(R)$ is an $a$-invariant of $R$, that is, $a(R) = \max \{ \ell \mid [H^1_{\mathfrak{m}_R}(R)]_\ell \neq 0 \}$.)

A zero-dimensional scheme $S \subset \mathbb{P}_K^N$ is called in uniform position if $H_Z(t) = \max \{ \deg(Z), H_S(t) \}$ for all $t$, for any subscheme $Z$ of $S$, where $H_Z$ and $H_S$ denote the Hilbert function of $Z$ and $S$ respectively.

A zero-dimensional scheme $S$, spanning $\mathbb{P}_K^N$, is called in linear semi-uniform position if there are integers $v(i, S)$, simply written as $v(i)$, $0 \leq i \leq N$ such that every $i$-plane $L$ in $\mathbb{P}_K^N$ spanned by linearly independent $i + 1$ points of $S$ contains exactly $v(i)$ points of $S$. A generic hyperplane section of a nondegenerate projective integral curve is in linear semi-uniform position, see [1]. We say $S$ is in linear general position if $v(i) = i + 1$ for all $i \geq 1$.

Let $S$ be a zero-dimensional scheme of $\mathbb{P}_K^N$ in linear semi-uniform position. Then $v(i + 1) \geq (v(1) - 1)v(i) + 1$ for $0 \leq i \leq N - 1$, see [4]. Also, we have, by [1] (or see [18]), $i(S) \leq \lceil (\deg(S) - 1)/N \rceil$.

Further, we note that “uniform position” implies “linear general position” and that “linear general position” implies “linear semi-uniform position”.
Lemma 2.1  Let $S \subset \mathbb{P}^N_K$ be a generic hyperplane section of a nondegenerate projective integral curve $C \subset \mathbb{P}^{N+1}_K$ with $d = \deg(C)$. Assume that $N \geq 3$ and $d \geq 25$. If $v(1) \geq 3$, then $i(S) \leq \lceil (d - 1)/N \rceil - 1$.

Proof. The assumption $v(1) \geq 3$ yields $v(i) \geq 2^{i+1} - 1$ for $0 \leq i \leq N$. Put $v = v(N - 1)$ and $w = v(N - 2)$. Note that $w \geq 2^{N-1} - 1$, $v \geq (v(1) - 1)v(N - 2) + 1 \geq 2w + 1$ and $d \geq 2v + 1 \geq 2^{N+1} - 1$.

We have only to show that $H^0(\mathcal{O}_{\mathbb{P}^N_K}(\ell)) \rightarrow H^0(\mathcal{O}_S(\ell))$ is surjective, where $\ell = \lceil (d - 1)/N \rceil - 1$. For any fixed point $P \in S$, we will show that there is a union of $\ell$ hyperplanes $F = H(1) \cup \cdots \cup H(\ell)$ in $\mathbb{P}^N_K$ such that $S \cap F = S \{P\}$, as in the classical Castelnuovo’s method for finite sets in linear general position.

First, let us take a hyperplane $H(1)$ which contains exactly $v$ points of $S \{P\}$ from the linear semi-uniform position property. Then $H(1)$ does not contain $P$.

Next, let us fix an $(N - 2)$-plane $L$ in $H(1)$ such that $L$ contains exactly $w$ points of $S \cap H(1)$. Put $\ell_1 = \lceil (d - v - 1)/(v - w) \rceil + 1$. Now we will inductively construct hyperplanes $H(2), \cdots, H(\ell_1)$ such that the number of points of $(S \{P\}) \cap (H(1) \cup \cdots \cup H(i))$ is $v + (i - 1)(v - w)$ for $i = 1, \cdots, \ell_1$. In fact, since $d - 1 - v - (i - 1)(v - w) \geq v - w$ for $i \leq \ell_1 - 1$, there exists a point $Q$ in $S \{P\} \cup H(1) \cup \cdots \cup H(i))$ such that a hyperplane $M$ spanned by $L$ and $Q$ does not contain $P$. Then $M$ contains exactly $v - w$ points of $S \{P\} \cup H(1) \cup \cdots \cup H(i))$ from the linear semi-uniform position property. So we take $H(i + 1) = M$. Thus the union of $\ell_1$ hyperplanes $H(1) \cup \cdots \cup H(\ell_1)$ contains $v + (\ell_1 - 1)(v - w)$ points of $S$ and does not contain $P$. Also, we note that $S \{P\} \cup H(1) \cup \cdots \cup H(\ell_1))$ consists of at most $v - w - 1$ points.

However, we see that $S \{P\} \cup H(1) \cup \cdots \cup H(\ell_1))$ consists of exactly $v - w - 1$ points. In fact, if the number of the remaining points were less than $v - w - 1$, then the hyperplane spanned by $M$ and a point from $S \{P\} \cup H(1) \cup \cdots \cup H(\ell_1))$ would contain at most $v - 1$ points of $S$, which contradicts with $v(N - 1) = v$. Thus we also have that there exist a hyperplane $G$ containing the $(N - 2)$-plane $L$, all the remaining points of $S \{P\} \cup H(1) \cup \cdots \cup H(\ell_1))$ and the point $P$. Of course $S \cap G$ consists of exactly $v$ points including $P$. 

5
Since \( S \cap G \) is in linear semi-uniform position in \( G \cong \mathbb{P}^{N-1}_K \), there are \( \ell_2 \) hyperplanes \( M(\ell_1 + 1), \cdots, M(\ell_2) \) of \( \mathbb{P}^{N-1}_K \) such that the union of them contains the remaining points and does not contain \( P \), where \( \ell_2 = \lceil (v - 1)/(N - 1) \rceil = \lceil (v - 2)/(N - 1) \rceil + 1 \). Thus we can take \( \ell_2 \) hyperplanes \( H(\ell_1 + 1), \cdots, H(\ell_2) \) of \( \mathbb{P}^N_K \) as desired. Note that we used a fact from [1] that \( \mathcal{H}^0(\mathcal{O}_{\mathbb{P}^{N-1}_K}(t)) \rightarrow \mathcal{H}^0(\mathcal{O}_{\mathbb{P}^{N-1}_K}(t)) \) is surjective for all \( t \geq \lceil (v - 1)/(N - 1) \rceil \), not necessarily for \( t = \lceil (v - 1)/(N - 1) \rceil - 1 \), without using the hypothesis of the induction on \( N \). So, if necessary, we may need to take a (possibly reducible) hypersurface \( F(1) \) of degree \( \ell_2 \) in place of the union of \( \ell_2 \) hyperplanes, and then go on the similar proof.

Therefore we have \( S \cap (H(1) \cup \cdots \cup H(\ell_1) \cup \cdots H(\ell_1 + \ell_2)) = S \setminus \{P\} \) (or \( S \cap (H(1) \cup \cdots \cup H(\ell_1) \cup F(1)) = S \setminus \{P\} \)).

Thus the proof is reduced to an arithmetic question. In other words, we need to prove \( \ell_1 + \ell_2 \leq \ell \), namely,

\[
\left\lceil \frac{d - 1}{N} \right\rceil - \left\lfloor \frac{d - v - 1}{v - w} \right\rfloor - \left\lfloor \frac{v - 2}{N - 1} \right\rfloor \geq 3.
\]

Moreover, from the above argument, we remark that \( d = v + \ell_1(v - w) \).

First, assume that \( N \geq 5 \). Since \( v - w \geq w + 1 \geq 4(N - 1) \), it suffices to show that \( (d - 1)/N - (d - v - 1)/4(N - 1) - (v - 2)/(N - 1) \geq 3 \). In fact, we easily have this inequality by reducing it to the case \( d = 2v + 1 \). Hence we proved the case \( N \geq 5 \).

Second, assume that \( N = 4 \). The inequality \( \lceil (d - 1)/4 \rceil - \lceil (d - v - 1)/(v - w) \rceil - \lceil (v - 2)/3 \rceil \geq 3 \) holds except for the case \((d, v, w) = (32, 15, 7)\) or \((33, 15, 7)\). But both cases contradict with \( d = v + \ell_1(v - w) \). Hence we proved the case \( N = 4 \).

Finally, assume that \( N = 3 \). Then we have \( \lceil (d - 1)/3 \rceil - \lceil (d - v - 1)/(v - w) \rceil - \lceil (v - 2)/2 \rceil \geq 3 \) except for the case \( w = 3 \) and \((d, v) = (25, 7), (25, 8), (25, 10), (25, 12), (28, 7)\) under the condition \( d \geq 25 \). But all the exceptional cases contradict with \( d = v + \ell_1(v - w) \). Hence we proved the case \( N = 3 \).
Lemma 2.2 Let $S \subset \mathbb{P}_K^N$ be a generic hyperplane section of a nondegenerate projective integral curve $C \subset \mathbb{P}_K^{N+1}$ with $d = \deg(C)$. Assume that $N \geq 3$ and $d \geq 23$. If $v(1) = 2$ and $v(2) \geq 4$, then $i(S) \leq \lceil (d-1)/N \rceil - 1$.

Proof. In fact, by (2.2), the assumption in (2.2) yields that $\deg(C) = 2^k$ for some $k \geq N$ and $v(i, S) = 2^i$ for all $i \leq N - 1$ since $d \geq 23$. In particular, $v(N - 1) = 2^{N-1}$ and $v(N - 2) = 2^{N-2}$.

First assume that $N \geq 5$. Just by copying the proof of (2.1) as in the Castelnuovo’s method, we see that the proof is reduced to show an inequality $\lceil (2^k - 1)/N \rceil - \lfloor (2^k - 2^{N-1} - 1)/(2^N - 2^{N-2}) \rfloor - \lfloor (2^{N-1} - 2)/(N - 1) \rfloor \geq 3$, namely,

\[
\frac{2^k - 1}{N} \geq 2^{k-N+2} - 1 + \frac{2^{N-1} - 1}{N - 1},
\]

which is easily shown. Hence we proved the case $N \geq 5$.

Next assume that $N = 3$. As in the classical Castelnuovo’s method, we will take a union of hyperplanes with containing $S$ and without containing $P$.

First let us take a hyperplane $H(1)$ with containing exactly 4 points of $S \setminus \{P\}$.

Now we will inductively construct hyperplanes $H(2), \ldots, H(\ell_1)$ such that the number of points of $(S \setminus \{P\}) \cap G(i)$ is $4i$ for $i = 1, \ldots, \ell_1$, where $\ell_1 = 2^{k-3}$ and $G(i) = H(1) \cup \cdots \cup H(i)$. For any $i = 1, \ldots, \ell_1 - 1$, we will show that there exists a hyperplane $H(i+1)$ with containing exactly 4 points of $S \setminus \{P \cup G(i)\}$. In fact, take 2 points $Q_1$ and $Q_2$ in $S \setminus \{P \cup G(i)\}$. Then there exists a point $Q_3$ from $S \setminus \{P, Q_1, Q_2 \cup G(i)\}$ such that the hyperplane spanned by $Q_1, Q_2$ and $Q_3$ does not contain any points of $S \cap \{P \cup G(i)\}$, since the number of points of $S \setminus \{P, Q_1, Q_2 \cup G(i)\}$ is larger than that of $S \cap \{P \cup G(i)\}$.

So the number of the remaining point of $S \setminus \{P \cup H(1) \cup \cdots \cup H(\ell_1)\}$ is $2^{k-1} - 1$. Next we will inductively construct hyperplanes $H(\ell_1+1), \ldots, H(\ell_1+\ell_2)$ for some $\ell_2 \leq \lceil (2^{k-1})/3 \rceil$, satisfying that $S \setminus \{P\} = S \cap (H(1) \cup \cdots \cup H(\ell_1+\ell_2))$. In fact, assume that we already take hyperplanes $H(1), \ldots, H(i)$ for $i \geq \ell_1$ satisfying some suitable condition. If the number of the remaining points of $S \setminus \{P \cup G(i)\}$, where $G(i) = H(1) \cup \cdots \cup H(i)$, is larger than 3, we can take the hyperplane $H(i+1)$ spanned by appropriate 3 points from
we can go on this process if $S$ contains at least two points of $S$. If the number of points of $L$ such that the hyperplane $M$ contains exactly 4 points of $S$, then we take such $L$ such that $H(i + 1) = H(i + 2)$ contains the remaining 3 points of $S \setminus (\{P\} \cup G(i))$ and does not contain $P$. If the number of the remaining points of $S \setminus (\{P\} \cup G(i))$ is either 1 or 2, then we take a hyperplane $H(i + 1)$ such that $H(i + 1)$ contains the remaining 1 or 2 points of $S \setminus (\{P\} \cup G(i))$ and does not contain $P$.

Thus the proof is reduced to an arithmetic question as in (2.1). Namely, $\ell_1 + \ell_2 \leq \lceil (2^k - 1)/3 \rceil - 1$, in other words,

$$\left\lfloor \frac{2^k - 1}{3} \right\rfloor - 2^{k-3} - \left\lfloor \frac{2^{k-1}}{3} \right\rfloor \geq 1.$$ 

Then we easily see the inequality except for the case $k = 3, 4$.

Hence we proved the case $N = 3$.

Finally assume that $N = 4$. Again we will prove as in the classical Castelnuovo’s method.

First let us take hyperplane $H(1)$ with containing exactly 8 points of $S \setminus \{P\}$.

Now we will inductively construct hyperplanes $H(2), \ldots, H(\ell_1)$ for some integer $\ell_1 \leq \lceil (2^k + 1)/7 \rceil$ such that $S \cap (H(i + 1) \setminus G(i))$ contains at least 7 points and does not contain $P$, where $G(i) = H(1) \cup \cdots \cup H(i)$. In fact, take 2 points $Q_1$ and $Q_2$ from $S \setminus (\{P\} \cup G(i))$. Then there exists a point $Q_3$ in $S \setminus (\{P, Q_1, Q_2\} \cup G(i))$ such that the 2-plane $L$ spanned by $Q_1, Q_2$ and $Q_3$ does not contain any points of $S \cap (\{P\} \cup G(i))$ if the number of points of $S \setminus (\{P, Q_1, Q_2\} \cup G(i))$ is larger than that of $S \cap (\{P\} \cup G(i))$. In other words, we can take such $L$ if $S \setminus (\{P\} \cup G(i))$ contains at least $2^{k-1} + 2$ points. Thus the 2-plane $L$ contains exactly 4 points of $S \setminus (\{P\} \cup G(i))$, and we put $S \cap L = \{Q_1, \ldots, Q_4\}$. Then there exists a point $Q_5$ from $S \setminus (\{P, Q_1, \ldots, Q_4\} \cup G(i))$ such that the hyperplane $M$ spanned by the point $Q_5$ and the 2-plane $L$ contains at least two points of $S \setminus (\{P, Q_1, \ldots, Q_4\} \cup G(i))$ without containing $P$, if the number of points of $S \setminus (\{P, Q_1, \ldots, Q_4\} \cup G(i))$ minus 2 is larger than that of $S \cap (\{P\} \cup G(i))$. In this case we put $H(i + 1) = M$. In other words, we can go on this process if $S \setminus (\{P\} \cup G(i))$ contains at least $2^{k-1} + 4$ points.
Thus we constructed a union of hyperplanes \( G(\ell_1) = H(1) \cup \cdots \cup H(\ell_1) \) such that \( G(\ell_1) \) contains at least \( 2^{k-1} - 4 \) points of \( S \) and does not contain \( P \) for some \( \ell_1 \leq \lfloor (2^{k-1} + 1)/7 \rfloor \).

So the number of the remaining point of \( S \setminus \{P\} \cup H(1) \cup \cdots \cup H(\ell_1) \) is at most \( 2^{k-1} + 3 \). Next we will inductively construct hyperplanes \( H(\ell_1 + 1), \ldots, H(\ell_1 + \ell_2) \) for some integer \( \ell_2 \leq \lfloor (2^{k-1} + 1)/4 \rfloor - 1 \), satisfying that \( S \setminus \{P\} = S \cap (H(1) \cup \cdots \cup H(\ell_1 + \ell_2)) \). Assume that we already take hyperplanes \( H(1), \cdots, H(i) \) for \( i \geq \ell_1 \) satisfying some suitable condition.

If the number of the remaining points of \( S \setminus \{P\} \cup G(i) \) is larger than 6, we can take a hyperplane \( H(i + 1) \) containing at least 4 points of \( S \setminus G(i) \) and without containing \( P \). So the number of \( S \cap (H(i+1) \setminus G(i)) \) is at least 4, and possibly more. If the number of the remaining points of \( S \setminus \{P\} \cup G(i) \) is 6, then we take hyperplanes \( H(i + 1), H(i + 2) \) and \( H(i + 3) \) with \( H(i + 1) \cup H(i + 2) \cup H(i + 3) \) containing the remaining 6 points of \( S \setminus \{P\} \cup G(i) \) and without containing \( P \). If the number of the remaining points of \( S \setminus \{P\} \cup G(i) \) is either 3, 4 or 5, then we take hyperplanes \( H(i + 1) \) and \( H(i + 2) \) with \( H(i + 1) \cup H(i + 2) \) containing the remaining 3, 4 or 5 points of \( S \setminus \{P\} \cup G(i) \) and without containing \( P \). If the number of the remaining points of \( S \setminus \{P\} \cup G(i) \) is either 1 or 2, then we take a hyperplane \( H(i + 1) \) with containing the remaining 1 or 2 points of \( S \setminus \{P\} \cup G(i) \) and without containing \( P \). Thus we see that there exist hyperplanes \( H(\ell_1 + 1), \cdots, H(\ell_1 + \ell_2) \) as desired.

Thus the proof is reduced to an arithmetic question as in (2.1). Namely, \( \ell_1 + \ell_2 \leq \lfloor (2^{k-1} - 1)/4 \rfloor - 1 \), in other words,

\[
\left\lfloor \frac{2^k - 1}{4} \right\rfloor - \left\lfloor \frac{2^{k-1} + 1}{7} \right\rfloor - 2^{k-3} \geq 1.
\]

Then we easily see the inequality.

Hence we proved the case \( N = 4 \). \(\square\)

**Theorem 2.3** Let \( S \subset \mathbb{P}_K^N \) be a generic hyperplane section of a nondegenerate projective integral curve \( C \subset \mathbb{P}_K^{N+1} \) with \( d = \deg(C) \). If \( d \geq \max\{N^2 + 2N + 2, 25\} \) and \( i(S) = \lceil (d - 1)/N \rceil \), then \( S \) lies on a rational normal curve.
Proof. For the case \( N = 2 \), the corresponding result as in [21, (3.2)] on the \( h \)-vector for the positive characteristic case is true, see [3, (1.1)] or [7, 9]. So the assertion follows from the proof of [14, (2.5)].

We may assume that \( N \geq 3 \) and that the Uniform Position Lemma fails for the curve \( C \). Note that \( d \geq 25 \). Then, by [19, (2.5)], \( C \) satisfies either (i) every secant of \( C \) is a multisecant, that is, \( v(1) \geq 3 \), or (ii) every plane spanned by three points contains one more point of \( C \), that is, \( v(1) = 2 \) and \( v(2) \geq 4 \). Therefore, by (2.1) and (2.2), we obtain that \( i(S) \leq \left\lceil \frac{(d-1)}{N} \right\rceil - 1 \).

So we exclude the case.

Hence the assertion is proved. ✷

Lemma 2.4 Let \( S \subset \mathbb{P}^2_K \) be a generic hyperplane section of a nondegenerate integral space curve \( C \) with \( d = \deg(C) \). If \( v(1) \geq 4 \), then \( i(S) \leq \left\lceil \frac{(d-1)}{N} \right\rceil - 1 \).

Proof. Put \( v = v(1) \). Following the Castelnuovo’s method, we will have the corresponding proof as in (2.1). For any fixed point \( P \in S \), we have only to show that there is a union of \( \ell \) lines \( F = L(1) \cup \cdots \cup L(\ell) \) in \( \mathbb{P}^N_K \) such that \( S \cap F = S \setminus \{P\} \), where \( \ell = \left\lceil \frac{(d-1)}{2} \right\rceil - 1 \).

First, let us take a line \( L(1) \) which contains exactly \( v \) points of \( S \setminus \{P\} \) from the linear semi-uniform position property. Then \( L(1) \) does not contain \( P \).

Next, let us fix a point \( Q \) of \( L(1) \) and put \( \ell_1 = \left\lceil \frac{(d-v-1)}{(v-1)} \right\rceil \). Then we can construct lines \( L(2), \ldots, L(\ell_1) \), by taking inductively a line \( L(i+1) \) with containing \( Q \) and without containing any points of \( \{P\} \cup L(1) \cup \cdots \cup L(i) \setminus \{Q\} \) for \( 1 \leq i \leq \ell_1 - 1 \).

Moreover, since \( S \setminus (\{P\} \cup L(1) \cup \cdots \cup L(\ell_1)) \) consists of at most \( v-2 \) points (and in fact exactly \( v-2 \) points), we can take appropriate \( v-2 \) lines \( L(\ell_1 + 1), \ldots, L(\ell_1 + v - 2) \) with containing the remaining points of \( S \setminus \{P\} \) and without containing \( P \).

Thus the proof is reduced to an arithmetic question. In other words, \( \ell_1 + v - 2 \leq \ell \), namely, \( \left\lceil \frac{(d-1)}{2} \right\rceil - \left\lceil \frac{(d-v-1)}{(v-1)} \right\rceil - v + 1 \geq 0 \), which is easily shown.

Hence the assertion is proved. ✷
Lemma 2.5 Let $S \subset \mathbb{P}^2_K$ be a generic hyperplane section of a nondegenerate integral space curve $C$ with $d = \deg(C)$. If $v(1) = 3$ and $d \geq 24$, then $i(S) \leq \lceil (d-1)/2 \rceil - 1$.

Proof. Following the Castelnuovo’s method, we will have the corresponding proof as in (2.2), the case $N = 3$. For any fixed point $P \in S$, we have only to show that there is a union of $\ell$ lines $F = L(1) \cup \cdots \cup L(\ell)$ in $\mathbb{P}^N_K$ such that $S \cap F = S \setminus \{P\}$, where $\ell = \lceil (d-1)/2 \rceil - 1$.

First, let us take a line $L(1)$ which contains exactly 3 points of $S \setminus \{P\}$ from the linear semi-uniform position property. Then $L(1)$ does not contain $P$.

Put $\ell_1 = \lceil (d-4)/6 \rceil + 1$. Then we can construct lines $L(2), \ldots, L(\ell_1)$, by taking inductively a line $L(i+1)$ without containing any points of $\{P\} \cup L(1) \cup \cdots \cup L(i)$ for $1 \leq i \leq \ell_1 - 1$.

Moreover, since $S \setminus (\{P\} \cup L(1) \cup \cdots \cup L(\ell_1))$ consists of at most $\lceil (d+1)/2 \rceil$ points, we can take appropriate $\ell_2$ lines $L(\ell_1 + 1), \ldots, L(\ell_2)$ with containing the remaining points of $S \setminus \{P\}$ and without containing $P$, where $\ell_2 = \lceil (d+3)/4 \rceil$.

Thus the proof is reduced to an arithmetic question. In other words, $\ell_1 + \ell_2 \leq \ell$, namely, $\lceil (d-1)/2 \rceil - \lceil (d-4)/6 \rceil - \lceil (d+3)/4 \rceil \geq 2$, which is easily shown for $d \geq 24$.

Hence the assertion is proved.

We say that a nondegenerate projective integral curve $C$ is very strange if a generic hyperplane section $S$ of $C$ is not in linear general position.

Proposition 2.6 Let $S \subset \mathbb{P}^N_K$ be a generic hyperplane section of a nondegenerate projective integral curve $C \subset \mathbb{P}^{N+1}_K$. Assume that $C$ is very strange. If $d = \deg(C) \geq 25$, then $i(S) \leq \lceil (d-1)/N \rceil - 1$

Proof. It immediately follows from (2.1), (2.2), (2.4), (2.5) and the proof of (2.3). □
3 An Application to a Sharp Bound on the Castelnuovo-Mumford Regularity

Let $K$ be an algebraically closed field. Let $S = K[x_0, \cdots, x_N]$ be the polynomial ring and $m = (x_0, \cdots, x_N)$ be the irrelevant ideal. Let $X$ be a projective scheme of $\mathbb{P}^N_K = \text{Proj}(S)$. For an integer $m$, $X$ is said to be $m$-regular if $H^i(\mathbb{P}^N_K, I_X(m - i)) = 0$ for all $i \geq 1$. The Castelnuovo-Mumford regularity of $X \subset \mathbb{P}^N_K$ is the least such $m$ and is denoted by $\text{reg}(X)$.

Let $k$ be a nonnegative integer. Then $X$ is called $k$-Buchsbaum if the graded $S$-module $M^i(X) = \bigoplus_{\ell \in \mathbb{Z}} H^i(\mathbb{P}^N_K, I_X(\ell))$, called the deficiency module of $X$, is annihilated by $m^k$ for $1 \leq i \leq \dim(X)$, see, e.g., [12, 13]. On the other hand, $X$ is called strongly $k$-Buchsbaum if $X \cap V$ has the $k$-Buchsbaum property for any complete intersection $V$ of $\mathbb{P}^N_K$ with $\text{codim}(X \cap V) = \text{codim}(X) + \text{codim}(V)$, possibly $V = \mathbb{P}^N_K$. So “strongly $k$-Buchsbaum” implies “$k$-Buchsbaum”. Further we call the minimal nonnegative integer $n$, if there exists, such that $X$ is $n$-Buchsbaum (resp. strongly $n$-Buchsbaum), as the Ellia-Migliore-Miró Roig number (resp. the strongly Ellia-Migliore-Miró Roig number) of $X$ and denote by $k(X)$ (resp. $\bar{k}(X)$), see [14]. In case $X$ is not $k$-Buchsbaum for all $k \geq 0$, then we put $k(X) = \bar{k}(X) = \infty$. Note that $k(X) < \infty$ if and only if $\bar{k}(X) < \infty$. Moreover it is equivalent to saying that $X$ is locally Cohen-Macaulay and equi-dimensional.

Upper bounds on the Castelnuovo-Mumford regularity of a projective variety $X$ are given in terms of $\dim(X)$, $\deg(X)$, $\text{codim}(X)$, $k(X)$ and $\bar{k}(X)$. Moreover, in case $\text{char}(K) = 0$, the extremal cases for the bounds are classified under a certain assumption.

**Proposition 3.1** Let $X$ be a nondegenerate projective variety in $\mathbb{P}^N_K$. Assume that $X$ is not ACM, that is, $k(X) \geq 1$. Then

(a) $\text{reg}(X) \leq \lceil (\deg(X) - 1)/\text{codim}(X) \rceil + k(X) \dim(X)$.

(b) $\text{reg}(X) \leq \lceil (\deg(X) - 1)/\text{codim}(X) \rceil + \bar{k}(X) \dim(X) - \dim(X) + 1$.

Furthermore, assume that $\text{char}(K) = 0$ and $\deg(X) \geq 2 \text{codim}(X)^2 + \text{codim}(X) + 2$. If the equality $\text{reg}(X) = \lceil (\deg(X) - 1)/\text{codim}(X) \rceil + k(X) \dim(X)$ holds, then $X$ is a curve on a rational ruled surface.
Proof. See [14, 15, 18].

Now we will study the extremal case for the inequality in (3.1) in positive characteristic case. We assume that a variety is not ACM, see [16] for the ACM case.

**Theorem 3.2** Let $X$ be a nondegenerate projective variety in $\mathbb{P}^N_K$ with $k(X) \geq 1$. Assume that either $\text{char}(K) = 0$ and $\deg(X) \geq \text{codim}(X)^2 + 2\text{codim}(X) + 2$, or $\text{char}(K) = p > 0$ and $\deg(X) \geq \max\{2\text{codim}(X)^2 + \text{codim}(X) + 2, 25\}$.

(a) If the equality $\text{reg}(X) = \lceil (\deg(X) - 1)/\text{codim}(X) \rceil + k(X) \dim(X)$ holds, then $X$ is a curve on a rational ruled surface.

(b) If the equality $\text{reg}(X) = \lceil (\deg(X) - 1)/\text{codim}(X) \rceil + \bar{k}(X) \dim(X) - \dim(X) + 1$ holds, then $X$ is a curve on a rational ruled surface.

Proof. We will prove (a). The proof of (b) is similar as in (a), which is left to the readers.

First we assume that $\text{char}(K) = p > 0$ and $\deg(X) \geq \max\{2\text{codim}(X)^2 + \text{codim}(X) + 2, 25\}$. The lemmas (2.5), (2.6), (2.7) and (2.8) in [14] work for the case $\text{char}(K) = p > 0$, although an assumption $\text{char}(K) = 0$ is mentioned in [14]. However, for the positive characteristic case, we cannot apply [14, (2.5)] as an inductive step, because a generic hyperplane section of an integral curve is not necessarily in uniform position. In other words, the corresponding proof as in [14] works for the positive characteristic case, except for the Uniform Position Lemma.

Thus, by applying Theorem 2.3 in place of [14, (2.5)], we have the assertion.

On the other hand, for the case $\text{char}(K) = 0$ and $\deg(X) \geq \text{codim}(X)^2 + 2\text{codim}(X) + 2$, we use [17, (3.3)] in place of [14, (2.6),(2.8)]. (Notice that [17, (3.3)] is a consequence of the “Socle Lemma”, see [11], and cannot be applied for the positive characteristic case.) Hence we have the assertion.
References

[1] E. Ballico, On singular curves in positive characteristic, Math. Nachr. 141 (1989), 267 – 273.

[2] E. Ballico, On the general hyperplane section of a curve in char. p., Rendiconti Istito Mat. Univ. Trieste 22 (1990), 117 – 125.

[3] E. Ballico, On the general hyperplane section of a projective curve, Beiträge zur Algebra und Geometrie, 39 (1998) 85 – 96.

[4] E. Ballico and A. Cossidente, On the generic hyperplane section of curves in positive characteristic, J. Pure Appl. Algebra, 102 (1995), 243 – 250.

[5] E. Ballico and K. Yanagawa, On the $h$-vector of a Cohen-Macaulay domain in positive characteristic, Comm. Algebra 26 (1998), 1745 – 1756.

[6] D. Eisenbud, Commutative algebra with a view toward algebraic geometry, GTM 150, Springer, 1995.

[7] A. Geramita and J. Migliore, Hyperplane sections of a smooth curve in $\mathbb{P}^3$, Comm. Algebra 17 (1989), 3129 – 3164.

[8] J. Harris (with D. Eisenbud), Curves in projective space, Les Presses de l’Université de Montréal, 1982.

[9] J. Herzog, N. V. Trung and G. Valla, On hyperplane sections of reduced and irreducible variety of low codimension, J. Math. Kyoto 34 (1994), 47 – 71.

[10] L. T. Hoa and C. Miyazaki, Bounds on Castelnuovo-Mumford regularity for generalized Cohen-Macaulay graded rings, Math. Ann. 301 (1995), 587 – 598.

[11] C. Huneke and B. Ulrich, General hyperplane sections of algebraic varieties, J. Algebraic Geometry, 2 (1993), 487 – 505.

[12] J. Migliore, Introduction to liaison theory and deficiency modules, Progress in Math. 165, Birkhäuser, 1998.
[13] J. Migliore and R. Miró Roig, On $k$-Buchsbaum curves in $\mathbb{P}^3$, Comm. Algebra. 18 (1990), 2403 – 2422.

[14] C. Miyazaki, Sharp bounds on Castelnuovo-Mumford regularity, to appear in Trans. Amer. Math. Soc.

[15] C. Miyazaki and W. Vogel, Bounds on cohomology and Castelnuovo-Mumford regularity, J. Algebra, 185 (1996), 626 – 642.

[16] U. Nagel, On the defining equations and syzygies of arithmetically Cohen-Macaulay varieties in arbitrary characteristic, J. Algebra 175 (1995), 359 – 372.

[17] U. Nagel, Arithmetically Buchsbaum divisors on varieties of minimal degree, to appear in Trans. Amer. Math. Soc.

[18] U. Nagel and P. Schenzel, Degree bounds for generators of cohomology modules and Castelnuovo-Mumford regularity, to appear in Nagoya Math. J.

[19] J. Rathmann, The uniform position principle for curves in characteristic $p$, Math. Ann. 276 (1987), 565 – 579.

[20] N. V. Trung and G. Valla, Degree bounds for the defining equations of arithmetically Cohen-Macaulay varieties, Math. Ann. 281 (1988), 209 – 218.

[21] K. Yanagawa, Castelnuovo’s Lemma and $h$-vectors of Cohen-Macaulay homogeneous domains, J. Pure Appl. Algebra 105 (1995), 107 – 116.

[22] K. Yanagawa, On the regularities of arithmetically Buchsbaum curves, Math. Z. 226 (1997), 155 – 163.