Towards the Green–Griffiths–Lang conjecture via equivariant localisation

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Abstract

Green, Griffiths and Lang conjectured that for every complex projective algebraic variety \( X \) of general type there exists a proper algebraic subvariety of \( X \) containing all nonconstant entire holomorphic curves \( f : \mathbb{C} \to X \). Using equivariant localisation we develop an iterated residue formula for cohomological pairings on the Demailly–Semple jet bundle. We apply this formula and a strategy of Demailly to give affirmative answer to the Green–Griffiths–Lang conjecture for generic projective hypersurfaces \( X \subset \mathbb{P}^{n+1} \) of degree \( \deg(X) \geqslant n^9 \).

1. Introduction

The Green–Griffiths–Lang (GGL) conjecture [19, 21] states that every projective algebraic variety \( X \) of general type contains a proper algebraic subvariety \( Y \subset X \) such that every nonconstant entire holomorphic curve \( f : \mathbb{C} \to X \) satisfies \( f(\mathbb{C}) \subset Y \). The GGL conjecture is related to the stronger concept of a hyperbolic variety [20]. A projective variety \( X \) is hyperbolic (in the sense of Brody) if there is no nonconstant entire holomorphic curve in \( X \), that is, any holomorphic map \( f : \mathbb{C} \to X \) must be constant. A famous conjecture of Kobayashi from 1970 stipulates that a very general algebraic hypersurface of dimension \( n \) and degree at least \( 2n + 2 \) in the complex projective space \( \mathbb{P}^{n+1} \) is hyperbolic. Hyperbolic algebraic varieties have attracted considerable attention, in part because of their conjectured diophantine properties. For instance, Lang [21] has conjectured that any hyperbolic complex projective variety over a number field \( \mathbb{K} \) can contain only finitely many rational points over \( \mathbb{K} \).

A positive answer to the GGL conjecture has been given for surfaces by McQuillan [22] under the assumption that the second Segre number \( c_2^2 - c_2 \) is positive. Siu in [26–29] developed a strategy to establish algebraic degeneracy of entire holomorphic curves in generic hypersurfaces \( X \subset \mathbb{P}^{n+1} \) of high degree, and also hyperbolicity of such hypersurfaces for even higher degree. Following this strategy combined with techniques of Demailly [11], Diverio, Merker and Rousseau [16] proved that for a generic projective hypersurface \( X \subset \mathbb{P}^{n+1} \) of degree \( \deg(X) > 2n^5 \) the GGL conjecture holds.

A central object in the study of polynomial differential equations on a smooth complex manifold is the bundle \( J_kX \) of \( k \)-jets \( (f', f'', \ldots, f^{(k)}) \) of germs of holomorphic curves \( f : \mathbb{C} \to X \) over \( X \) and the associated Green–Griffiths bundle \( \bigoplus_{m=1}^{\infty} E_{k,m}^{GG} = \mathcal{O}(J_kX) \) of algebraic differential operators [19] whose elements are polynomials \( Q(f', \ldots, f^{(k)}) \) of weighted degree \( m \), where the weight of \( f^{(i)} \) is \( i \). In [11], Demailly introduced the subbundle \( E_{k,m} \subset E_{k,m}^{GG} \) of jets that are invariant under reparameterisation of the source \( \mathbb{C} \). The group \( G_k \) of \( k \)-jets of reparameterisation germs \( (\mathbb{C}, 0) \to (\mathbb{C}, 0) \) at the origin acts fibrewise on \( J_kX \) and \( \bigoplus_{m=1}^{\infty} E_{k,m} = \mathcal{O}(J_kX)^{U_k} \) is the graded algebra of invariant jet differentials under the maximal unipotent subgroup \( U_k \) of \( G_k \). This bundle gives a better reflection of the geometry of entire curves, since

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it only takes care of the image of such curves and not of the way they are parameterised. However, it also comes with a technical difficulty, namely, the reparameterisation group $\mathcal{G}_k$ is nonreductive, and the classical geometric invariant theory of Mumford \cite{Mumford} is not applicable to describe the invariants and the quotient $J_kX/\mathcal{G}_k$; for details see \cite{BC13, Berczi15, Berczi18}.

In \cite{Dem92}, Demailly describes a smooth compactification $X_k$ of $J_kX/\mathcal{G}_k$ — the Demailly–Semple tower — as a tower of projectivised bundles on $X$:

$$X_k \to X_{k-1} \to \cdots \to X_1 \to X_0 = X$$

endowed with projections $\pi_{i,k} : X_k \to X_i$ and canonical line bundles $\pi_{i,k}^*O_{X_i}(1) \to X_k$ whose sections are $\mathcal{G}_k$-invariants. The total space $X_k$ is smooth of dimension $\dim(X_k) = n + k(n - 1)$. According to the work of Demailly \cite{Dem92} and Diverio \cite{Di15, Di18}, the algebraic Morse inequalities developed by Demailly \cite{Dem93} and Trapani \cite{Tr11} then reduce the existence of global invariant jet differentials on $X$ to the positivity of a certain intersection number on the Demailly–Semple tower $X_k \to X_{k-1} \to \cdots \to X_1 \to X_0 = X$.

This paper introduces a new technique to handle the complexity and difficulties of computations with the cohomology ring of the Demailly–Semple tower in \cite{Berczi16}. We integrate along the fibres of $X_k$ using a pull-back argument which allows us to perform fibrewise integration as equivariant push-forward, hence to apply equivariant localisation on the fibre $X_k$ in stages. We transform the localisation formula into an iterated residue formula to express intersection numbers of the canonical bundles $\pi_{i,k}^*O_{X_i}(1)$ on $X_k$ as coefficients of the Laurent expansion of a rational function, and finally, we use the Chern–Weil map to substitute the Chern roots of $TX$ into the weights of the torus action. The idea of introducing iterated residues was motivated by the author’s earlier work \cite{Berczi12}.

**Theorem 1.** Let $X$ be a smooth projective variety of dimension $n$. For $i = 1, \ldots, k$ let $v_i = c_1(\pi_{i,k}^*O_{X_i}(1))$ denote the first Chern class of the $i$th canonical line bundle on the Demailly–Semple tower $X_k$. Let $P = P(v_1, \ldots, v_k) \in H^*(X)[v_1, \ldots, v_k]$ be a polynomial with coefficients in $H^*(X)$, which represents a cohomology element of $X_k$ via pull-back of the coefficients. Then

$$\int_{X_k} P(v_1, \ldots, v_k)$$

$$= \int_X \text{Coeff}_{(z_1 \cdots z_k)^{-1}} \frac{\prod_{2 \leq t_1 \leq t_2 \leq k}(z_{t_1} + z_{t_1+1} + \cdots + z_{t_2})P(z_1, \ldots, z_k)}{\prod_{1 \leq s_1 < s_2 < \cdots < k}(z_{s_1+1} + \cdots + z_{s_2} - z_{s_1})\prod_{j=1}^k(z_1 + \cdots + z_j)^n}$$

$$\times \prod_{j=1}^k s_X\left(\frac{1}{\sum_{i=1}^j z_i}\right),$$

where

$$s_X(t) = \frac{1}{c_X(t)} = 1 + \frac{s_1(X)}{t} + \frac{s_2(X)}{t^2} + \cdots + \frac{s_n(X)}{t^n} + \cdots$$

is the total Segre class (inverse of the total Chern class) of $TX$, and $\text{Coeff}_{(z_1 \cdots z_k)^{-1}}$ stands for the coefficient of $(z_1 \cdots z_k)^{-1}$ in the expansion of the rational expression in the domain $z_1 \ll \cdots \ll z_k$.

If $P(v_1, \ldots, v_k) \in H^{n+k(n-1)}(X_k)$ represents a pure cohomology class of degree equal to $\dim X_k = n + k(n - 1)$, then the coefficient on the right-hand side of this formula is a degree $n$ cohomology class of $X$ expressed as a polynomial in $s_1(X), \ldots, s_n(X)$ and the coefficients of $P$. 
Proving algebraic degeneracy of holomorphic curves on $X$ means finding a nonzero polynomial function $P$ on $X$ such that all entire curves $f : \mathbb{C} \to X$ satisfy $P(f(\mathbb{C})) = 0$. All known methods of proof are based on establishing first the existence of certain algebraic differential equations $P(f, f', \ldots, f^{(k)}) = 0$ of some order $k$, and then the second step is to find enough such equations so that they cut out a proper algebraic locus $Y \subsetneq X$.

This paper focuses on smooth projective hypersurfaces $X \subset \mathbb{P}^{n+1}$. The main technical reason for this is that in this case the Segre classes of $X$ on the right-hand side of Theorem 1.1 are expressible with the degree $d$ of $X$ and the first Chern class $h$ of the hyperplane line bundle over $X$. Then the coefficient on the right-hand side of the formula in Theorem 1.1 becomes a polynomial in $h^n$ with polynomial coefficients in $d,n$, and integration simply means the substitution $h^n = d$.

This paper follows the strategy of [16], but the efficiency of computations with iterated residues allows us prove the GGL conjecture with a sharper exponential bound on the degree of a generic hypersurface. We use the residue formula in Theorem 1.1 to prove the existence of global differential equations of order $k = n$ satisfied by entire holomorphic curves on $X$ with $\deg(X) > 6n^8$. Combined with deformation arguments of [16] (based on earlier works [27, 33]) to get enough independent differential equations, this gives us the following effective degree bound in the GGL conjecture:

**Theorem 1.2.** Let $X \subset \mathbb{P}^{n+1}$ be a generic smooth projective hypersurface of degree $\deg(X) \geq n^{9n}$. Then there exists a proper algebraic subvariety $Y \subsetneq X$ such that every nonconstant entire holomorphic curve $f : \mathbb{C} \to X$ has image contained in $Y$.

A more careful study of the localisation technique of this paper let Darondeau [9] tighten the degree bound to $(5n)^2 n^n$.

In the forthcoming paper [2], we replace the Demailly–Semple bundle with a more sophisticated compactification of $J_kX/G_k$ motivated by the author’s earlier work in global singularity theory [6] on Thom polynomials of singularity classes. We will prove that the GGL conjecture for generic projective hypersurfaces with polynomial degree in the dimension follows from a conjectural positivity property of Thom polynomials, due to Rimányi [25].

More recently, Demailly [13] formulated a generalised version of the GGL conjecture for directed manifolds $(X, V)$, where $V \subset T_X$ is a subbundle, and proved — using holomorphic Morse inequalities and probabilistic methods — that for any projective directed manifold $(X, V)$ with $K_V$ big there is a differential equation $P$ of order $k \gg 1$ such that any entire curve $f$ must satisfy $P(f, f', \ldots, f^{(k)}) = 0$. Merker [23] proved the same for projective hypersurfaces in $\mathbb{P}^{n+1}$ of general type using Riemann–Roch–Hirzebruch. Darondeau [9] adapted techniques of the present paper to study algebraic degeneracy of entire curves in complements of smooth projective hypersurfaces. Most recently Demailly [10] proved the GGL conjecture for directed pairs $(X, V)$ satisfying certain jet stability conditions and announced the proof of the Kobayashi conjecture on the hyperbolicity of very general algebraic hypersurfaces and complete intersections.

### 2. Jet differentials

The central object of this paper is the algebra of invariant jet differentials under reparameterisation of the source space $\mathbb{C}$. For more details see the survey papers [11, 17].
2.1. Invariant jet differentials

Let \( X \) be a complex \( n \)-dimensional manifold and let \( k \) be a positive integer. Green and Griffiths in [19] introduced the bundle \( J_k X \to X \) of \( k \)-jets of germs of parameterised curves in \( X \); its fibre over \( x \in X \) is the set of equivalence classes of germs of holomorphic maps \( f : (\mathbb{C},0) \to (X,x) \), with the equivalence relation \( f \sim g \) if and only if the derivatives \( f^{(j)}(0) = g^{(j)}(0) \) are equal for \( 0 \leq j \leq k \). If we choose local holomorphic coordinates \((z_1, \ldots, z_n)\) on an open neighbourhood \( \Omega \subset X \) around \( x \), the elements of the fibre \( J_k X_x \) are represented by the Taylor expansions

\[
f(t) = x + t f'(0) + \frac{t^2}{2!} f''(0) + \cdots + \frac{t^k}{k!} f^{(k)}(0) + O(t^{k+1})
\]

up to order \( k \) at \( t = 0 \) of \( \mathbb{C}^n \)-valued maps \( f = (f_1, f_2, \ldots, f_n) \) on open neighbourhoods of 0 in \( \mathbb{C} \). Locally in these coordinates the fibre can be written as

\[
J_k X_x = \left\{ (f'(0), \ldots, f^{(k)}(0)/k!) \right\} = (\mathbb{C}^n)^k,
\]

which we identify with \( \mathbb{C}^{nk} \). Note that \( J_1 X = T_X \), but for \( k > 2 \) \( J_k X \) is not a vector bundle over \( X \) since the transition functions are polynomial but not linear, see [11] for details.

Let \( \mathcal{G}_k \) denote the group of \( k \)-jets of local reparameterisations \((\mathbb{C},0) \to (\mathbb{C},0)\)

\[
t \mapsto \varphi(t) = \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_k t^k, \quad \alpha_1 \in \mathbb{C}^*, \alpha_2, \ldots, \alpha_k \in \mathbb{C},
\]

under composition modulo terms \( t^j \) for \( j > k \). This group acts fibrewise on \( J_k X \) by substitution. A short computation shows that this is a linear action on the fibre:

\[
f \circ \varphi(t) = f'(0) \cdot (\alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_k t^k) + \frac{f''(0)}{2!} \cdot (\alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_k t^k)^2 + \cdots + \frac{f^{(k)}(0)}{k!} \cdot (\alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_k t^k)^k \quad \text{modulo } t^{k+1}
\]

so the linear action of \( \varphi \) on the \( k \)-jet \((f'(0), f''(0)/2!, \ldots, f^{(k)}(0)/k!\)) is given by the following matrix multiplication:

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_k \\
0 & \alpha_2 & 2\alpha_1\alpha_2 & \cdots & \alpha_1\alpha_{k-1} + \cdots + \alpha_{k-1}\alpha_1 \\
0 & 0 & \alpha_3 & \cdots & 3\alpha_1^2\alpha_{k-2} + \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \alpha_i^k
\end{pmatrix}
\]

where the \((i,j)\) entry of the matrix is \( \sum_{s_1, \ldots, s_i \in \mathbb{Z}^+} \alpha_{s_1} \cdots \alpha_{s_i} \), for \( 1 \leq i < j \leq k \).

The group \( \mathcal{G}_k \) sits in an exact sequence of groups \( 1 \to U_k \to \mathcal{G}_k \to \mathbb{C}^* \to 1 \), where \( \mathcal{G}_k \to \mathbb{C}^* \) is the morphism \( \varphi \to \varphi'(0) = \alpha_1 \) in the notation used above, and

\[
\mathcal{G}_k = U_k \times \mathbb{C}^*
\]

is a \( \mathbb{C}^* \)-extension of the unipotent group \( U_k \). With the above identification, \( \mathbb{C}^* \) is the subgroup of diagonal matrices satisfying \( \alpha_2 = \cdots = \alpha_k = 0 \) and \( U_k \) is the unipotent radical of \( \mathcal{G}_k \), consisting of matrices of the form above with \( \alpha_1 = 1 \). The action of \( \lambda \in \mathbb{C}^* \) on \( k \)-jets is thus described by

\[
\lambda \cdot (f', f'', \ldots, f^{(k)}) = (\lambda f', \lambda^2 f'', \ldots, \lambda^k f^{(k)}).
\]

We recall from [11] the Green–Griffiths vector bundle \( E_{k,m}^{GG} \) whose fibres are complex-valued polynomials \( Q(f', f'', \ldots, f^{(k)}) \) on the fibres of \( J_k X \) of weighted degree \( m \) with respect to the \( \mathbb{C}^* \) action above, that is, they satisfy the following for any \( \lambda \in \mathbb{C}^* \):

\[
Q(\lambda f', \lambda^2 f'', \ldots, \lambda^k f^{(k)}) = \lambda^m Q(f', f'', \ldots, f^{(k)}).
\]
The fibrewise $G_k$ action on $J_k X$ induces an action on $E_{k,m}^{GG}$. Demailly in [11] defined the bundle of invariant jet differentials of order $k$ and weighted degree $m$ as the subbundle $E_{k,m} \subset E_{k,m}^{GG}$ of polynomial differential operators $Q(f, f', \ldots, f^{(k)})$ which are invariant under $U_k$, that is for any $\varphi \in G_k$

\[ Q((f \circ \varphi)', (f \circ \varphi}'', \ldots, (f \circ \varphi)^{(k)}) = \varphi'(0)^m \cdot Q(f', f''', \ldots, f^{(k)}). \]

We call $E_k = \oplus_m E_{k,m} = (\oplus_m E_{k,m}^{GG})^U_k$ the Demailly–Semple bundle of invariant jet differentials.

### 2.2. Compactification of $J_k X/G_k$

We recall Demailly’s construction from [11] of a smooth relative compactification of the geometric quotient $J_k^{\text{reg}} X/G_k$, where $J_k^{\text{reg}} X \subset J_k X$ is the bundle of regular $k$-jets, that is, $k$-jets such that $f'(0) \neq 0$. This smooth compactification is constructed as an iterated tower of projectivised bundles over $X$. Demailly in [11] uses the term Semi-$k$-jet bundle and in this paper we will call this bundle the Demailly–Semple bundle.

A directed manifold is a pair $(X \times V, \pi : X \to V)$, where $X$ is a manifold of dimension $\dim(X) = n$ and $V \subset T_X$ a subbundle of rank $rk(V) = r$. We associate to $(X, V)$, another directed manifold $(\tilde{X}, \tilde{V})$, where $\tilde{X} = \mathbb{P}(V)$ is the projectivised bundle of lines in $V$ and $\tilde{V}$ is the subbundle of $T_{\tilde{X}}$ defined fibrewise using the natural projection $\pi : \tilde{X} \to X$ as follows:

\[ \tilde{V}_{(x_0, v_0)} = \{ \xi \in T_{\tilde{X}, (x_0, v_0)} \mid \pi_*(\xi) \in \mathbb{C} \cdot v_0 \} \]

for any $x_0 \in X$ and $v_0 \in T_{X, x_0} \setminus \{0\}$. We also have a lifting operator which assigns to a germ of a regular holomorphic curve $f : (\mathbb{C}, 0) \to X$ tangent to $V$ the germ of the holomorphic curve $\tilde{f} : (\mathbb{C}, 0) \to \tilde{X}$ tangent to $\tilde{V}$ defined as $\tilde{f}(t) = (f(t), [f'(t)])$. This construction can be naturally extended to singular curves [11].

Let $X \subset \mathbb{P}^{n+1}$ be a projective hypersurface. Following Demailly [11], we define inductively the $k$-jet bundle $X_k$ and the associated subbundle $V_k \subset T_{X_k}$ by iterating the above construction for $V = T_X$, that is,

\[ (X_0, V_0) = (X, T_X), \quad (X_k, V_k) = (X_{k-1}, \tilde{V}_{k-1}). \]

Therefore,

\[ \dim X_k = n + k(n - 1), \quad \text{rank} V_k = n, \]

and the construction can be described inductively by the following exact sequence

\[ 0 \longrightarrow T_{X_k/X_{k-1}} \longrightarrow V_k \overset{(\pi_k)_*}{\longrightarrow} \mathcal{O}_{X_k}(-1) \longrightarrow 0, \quad (2.2) \]

where $\mathcal{O}_{X_k}(-1)$ is the tautological line bundle on $X_k = \mathbb{P}(V_{k-1})$, $\pi_k : X_k \to X_{k-1}$ is the natural projection and $(\pi_k)_*$ is its differential. Iterating these we get projections $\pi_{j,k} = \pi_{j+1} \circ \cdots \circ \pi_{k-1} \circ \pi_k : X_k \to X_j$ for $j < k$. With this notation $\pi_{0,k} : X_k \to X = X_0$ is a locally trivial holomorphic fibre bundle over $X$ for $k \geq 1$; the fibres $X_{k,x} = \pi_{0,k}^{-1}(x)$ are $k$-stage towers of $\mathbb{P}^{n-1}$ bundles. Indeed, the inductive description of the fibre $X_k = X_{k,x}$ comes from that of the bundle $X_k$ as follows:

\[ (X_0, V_0) = (pt = \{x\}, C^n = T_x X), \quad (X_k, V_k) = (X_{k-1}, \tilde{V}_{k-1}), \]

where the bundle $V_k$ can be again described inductively by the restriction of the exact sequence 2.2 to the fibre:

\[ 0 \longrightarrow T_{X_k/X_{k-1}} \longrightarrow V_k \overset{(\pi_k)_*}{\longrightarrow} \mathcal{O}_{X_k}(-1) \longrightarrow 0. \quad (2.3) \]

We also have an inductively defined $k$-lifiting for germs of holomorphic curves such that $f_{[k]} : \mathbb{C} \to X_k$ is given by $f_{[k]} = \tilde{f}_{[k-1]}$.
THEOREM 2.1 [11]. Suppose that \( n > 2 \). The quotient \( J^\text{reg}_k X/G_k \) has the structure of a locally trivial bundle over \( X \) and there is a holomorphic embedding \( J^\text{reg}_k X/G_k \hookrightarrow X_k \) which identifies \( J^\text{reg}_k X/G_k \) with \( X_k^{\text{reg}} \), that is, the set of points in \( X_k \) of the form \( f_{|k|}(0) \) for some nonsingular \( k \)-jet \( f \). In other words, \( X_k \) is a relative compactification of \( J^\text{reg}_k X/G_k \) over \( X \). Moreover, one has the direct image formula:

\[
(\pi_{0,k})_* \mathcal{O}_{X_k}(m) = \mathcal{O}(E_{k,m}).
\]

3. Equivariant cohomology and localisation

This section is a brief introduction to equivariant cohomology and localisation. For more details, we refer the reader to [6, 7].

Let \( K \cong U(1)^n \) be the maximal compact subgroup of \( T \cong (\mathbb{C}^*)^n \), and denote by \( t \) the Lie algebra of \( K \). Identifying \( T \) with the group \( \mathbb{C}^n \), we obtain a canonical basis of the weights of \( T \): \( \lambda_1, \ldots, \lambda_n \in \mathfrak{t}^* \).

For a manifold \( M \) endowed with the action of \( K \), one can define a differential \( d_K \) on the space \( S^{r^*} \otimes \Omega^*(M)^K \) of polynomial functions on \( t \) with values in \( K \)-invariant differential forms by the formula:

\[
[d_K \alpha](X) = d(\alpha(X)) - \iota(X_M)[\alpha(X)],
\]

where \( X \in \mathfrak{t} \), and \( \iota(X_M) \) is contraction by the corresponding vector field on \( M \). A homogeneous polynomial of degree \( d \) with values in \( r \)-forms is placed in degree \( 2d + r \), and then \( d_K \) is an operator of degree 1. The cohomology of this complex, the so-called equivariant de Rham complex, denoted by \( H^*_T(M) \), is called the \( T \)-equivariant cohomology of \( M \). Elements of \( H^*_T(M) \) are therefore polynomial functions \( \mathfrak{t} \rightarrow \Omega^*(M)^K \) and there is an integration (or push-forward map) \( \int : H^*_T(M) \rightarrow H^*_T(\text{point}) = S^{\mathfrak{t}^*} \) defined as

\[
\left( \int_M \alpha \right)(X) = \int_M \alpha^{[\dim(M)]}(X) \text{ for all } X \in \mathfrak{t},
\]

where \( \alpha^{[\dim(M)]} \) is the differential form top degree part of \( \alpha \). The following theorem is the Atiyah–Bott–Berline–Vergne localisation theorem in the form of [7, Theorem 7.11].

THEOREM 3.1 [1, 8]. Suppose that \( M \) is a compact manifold and \( T \) is a complex torus acting smoothly on \( M \), and the fixed-point set \( M^T \) of the \( T \)-action on \( M \) is finite. Then for any cohomology class \( \alpha \in H^*_T(M) \)

\[
\int_M \alpha = \sum_{f \in M^T} \frac{\alpha^{[0]}(f)}{\text{Euler}^T(T_f M)}.
\]

Here Euler\(^T \)(\( T_f M \)) is the \( T \)-equivariant Euler class of the tangent space \( T_f M \), and \( \alpha^{[0]} \) is the differential form degree 0 part of \( \alpha \).

The right-hand side in the localisation formula considered in the fraction field of the polynomial ring of \( H^*_T(\text{point}) = H^*(BT) = S^{\mathfrak{t}^*} \) (see more on details in [1, 7]). Part of the statement is that the denominators cancel when the sum is simplified.

We start with a toy enumerative example to demonstrate how localisation works.

EXAMPLE 1 (How many lines intersect two given lines and go through a point in \( \mathbb{P}^3 \)?). We think points, lines and planes in \( \mathbb{P}^3 \) as one-, two-, three-dimensional subspaces in \( \mathbb{C}^4 \). For \( R \in \text{Grass}(3, \mathbb{C}^4) \), \( L \in \text{Grass}(1, \mathbb{C}^4) \) define

\[
C_2(R) = \{ V \in \text{Grass}(2,4) : V \subset R \}, \quad C_1(L) = \{ V \in \text{Grass}(2,4) : L \subset V \}.
\]
Standard Schubert calculus says that $C_1(L)$ (respectively, $C_2(R)$) represents the cohomology class $c_1(\tau)$ (respectively, $c_2(\tau)$) where $\tau$ is the tautological rank 2 bundle over $\text{Grass}(2, 4)$, and the answer can be formulated as

$$C_1(L_1) \cap C_1(L_2) \cap C_2(R) = \int_{\text{Grass}(2, 4)} c_1(\tau)^2 c_2(\tau).$$

The fixed-point data for equivariant localisation is the following.

(i) Let the diagonal torus $T^4 \subset \text{GL}(4)$ act on $\mathbb{C}^4$ with weights $\mu_1, \mu_2, \mu_3, \mu_4 \in t^* \subset H^*_T(pt).$

(ii) The induced action on $\text{Grass}(2, 4)$ has $\binom{4}{2}$ fixed points, namely, the coordinate subspaces indexed by pairs in the set $\{1, 2, 3, 4\}$.

(iii) The tangent space of $\text{Grass}(2, 4)$ at the fixed point $(i, j) \in \text{Grass}(2, 4)^T$ is $(\mathbb{C}^2)_{i,j}^* \otimes \mathbb{C}_{s,t}^2$, where $\{s, t\} = \{1, 2, 3, 4\} \setminus \{i, j\}$, and $\mathbb{C}_{i,j}^2 \in \text{Grass}(2, 4)$ is the subspace spanned by the $i, j$ basis. Therefore, the weights on $T_{(i,j)} \text{Grass}$ are $\mu_s - \mu_i, \mu_t - \mu_j$ with $s \neq i, j$.

(iv) The weights of $\tau$ are identified with the Chern roots through the Chern–Weil homomorphism, so $c_i(\tau)$ is represented by the $i$th elementary symmetric polynomial in the weights of $\tau$.

Theorem 3.1 then gives

$$\int_{\text{Grass}(2, 4)} c_1(\tau)^2 c_2(\tau) = \sum_{\sigma \in S_4/S_2} \sigma \cdot \frac{(\mu_1 + \mu_2)^2 \mu_1 \mu_2}{(\mu_3 - \mu_1)(\mu_4 - \mu_1)(\mu_3 - \mu_2)(\mu_4 - \mu_2)} = 1. \quad (3.1)$$

On the right-hand side we sum over all $\binom{4}{2}$ fixed points by taking appropriate permutation of the indices. The sum on the right-hand side turns out to be independent of the factors $\mu_i$.

4. Proof of Theorem 1.1

To evaluate $\int_{X_k} P$ we integrate first along the fibres of the bundle $X_k \to X$, which corresponds to push-forward in cohomology. The fibres of $X_k$ are isomorphic to the $\text{GL}(n)$-module $X_k$, and we would like to perform integration along the fibres using equivariant localisation. However, there is no global $\text{GL}(n)$-action on the bundle $X_k$ and therefore we need first to justify and to explain this step.

4.1. The localisation principle for bundles

Let $\text{EGL}(n) \to \text{BGL}(n)$ be the universal principal $\text{GL}(n)$-bundle. Topologically, $\text{EGL}(n)$ is a contractible space with a free $\text{GL}(n)$-action, and $\text{BGL}(n)$ can be approximated by finite-dimensional algebraic spaces (universal Grassmannians). Let $P \to X$ denote the principal $\text{GL}(n)$-bundle over $X$ and $V = \mathbb{C}^n$ a $\text{GL}(n)$-module such that the maximal torus $T \subset \text{GL}(n)$ acts with weights $\lambda_1, \ldots, \lambda_n$ on $\mathbb{C}^n$, and the tangent bundle $T_X = P \times_{\text{GL}(n)} \mathbb{C}^n$ is the associated bundle. The $\text{GL}(n)$-action on $\mathbb{C}^n$ induces a $\text{GL}(n)$-action on the fibre $X_k$ of the Demailly–Semple bundle, which is the associated bundle:

$$X_k = P \times_{\text{GL}(n)} X_k.$$

Let

$$X_k^{\text{GL}} = \text{EGL}(n) \times_{\text{GL}(n)} X_k$$

denote the universal bundle with fibre $X_k$ over the classifying space $\text{BGL}_n$. For $i = 1, \ldots, k$ we have universal tautological bundle

$$\tau_i = \text{EGL}(n) \times_{\text{GL}(n)} \mathcal{O}_{X_k}(-1)$$
over $\mathcal{X}_k^{GL}$. The classifying map $\rho_k : X \to B\text{GL}(n)$ of the principal bundle $P$ in the diagram

$$
\begin{array}{ccc}
P & \xrightarrow{\hat{\rho}_k} & \mathcal{X}_k^{GL} \\
\downarrow & & \downarrow \\
X & \xrightarrow{\rho_k} & B\text{GL}(n)
\end{array}
$$

induces the following commutative diagram of maps

$$
\begin{array}{ccc}
\mathcal{X}_k & \xrightarrow{\hat{\rho}_k} & \mathcal{X}_k^{GL} \\
\downarrow & & \downarrow \\
X & \xrightarrow{\rho_k} & B\text{GL}(n)
\end{array}
$$

and the induced diagram of cohomology maps is

$$
\begin{array}{ccc}
H^*(X_k) & \xleftarrow{\hat{\rho}_k^*} & H^*(\mathcal{X}_k^{GL}) = H^*_{\text{GL}(n)}(X_k) \\
\downarrow \pi_* & & \downarrow \pi_*^{\text{CW}} \circ \text{Res} \\
H^*(X) & \xrightarrow{\rho_k^* \circ \text{CW}} & H^*(B\text{GL}(n)) = H^*_{\text{GL}(n)}(pt)
\end{array}
$$

where

1. on the tower $X_k \to X_{k-1} \to \cdots \to X$ the tautological bundles satisfy $O_{X_i}(-1) = \hat{\rho}_k^* \tau_i$ for $i = 1, \ldots, k$ and therefore the Chern root $v_i = c_1(\pi^*_{i,k} O_{X_i}(-1)) = \hat{\rho}_k^* c_1^T(\tau_i)$ is the image of the first equivariant Chern classes $u_i$ of $\tau_i$ under $\hat{\rho}_k$;
2. $H^*(\mathcal{X}_k^{GL}) = H^*_{\text{GL}(n)}(X_k)$ and $H^*(B\text{GL}(n)) = H^*_{\text{GL}(n)}(pt)$ are the $\text{GL}(n)$-equivariant cohomology rings of $X_k$ and the point, respectively;
3. $\text{Res} = \pi_*^{\text{GL}(n)}$ is the equivariant push-forward on equivariant cohomology induced by the map $X_k \to pt$. According to Atiyah–Bott and Berline–Vergne the equivariant push-forward map is given by a localisation formula on $X_k$, which we transform into a residue operator in the next section, see Proposition 4.5. Let $P(u_1, \ldots, u_k) \in \mathbb{C}[u_1, \ldots, u_k]$ be a complex polynomial in the Chern roots $u_i = c_1^T(\tau_i)$. The residue operator has the following explicit form:

$$
\text{Res}(P(u)) = \lim_{z \to \infty} \prod_{2 \leq t_1 \leq t_2 \leq k} (z_{t_1} + z_{t_1+1} + \cdots + z_{t_2}) \prod_{1 \leq s_1 < s_2 \leq k} (z_{s_1} - z_{s_1+1} - \cdots - z_{s_2}) \prod_{j=1}^k \prod_{i=1}^n \lambda_i - z_1 - \cdots - z_j
$$

The iterated residue is equal to $(-1)^k$ times the coefficient of $(z_1 \cdots z_k)^{-1}$ after expansion of the rational expression on the contour $|z_1| \ll |z_2| \ll \cdots \ll |z_k|$;
4. CW is the Chern–Weil map which substitutes the Chern roots of $TX$ into the weights $\lambda_1, \ldots, \lambda_n$;
5. $\pi_*$ is integration along the fibre.

When $P(v_1, \ldots, v_k) \in H^*(X)[v_1, \ldots, v_k]$ is not just a complex polynomial, but a polynomial in the Chern roots $-v_i = c_1(\pi^*_{i,k} O_{X_i}(-1))$ (note the negative sign!) with coefficients in $H^*(X)$, we use the fact that, since the fibre $X_k$ of $X_k$ is a smooth projective variety, the push-forward map $\pi_*$ is a $H^*(X)$-module homomorphism, that is,

$$
\pi_*(\pi^* \beta \cdot \alpha) = \beta \cdot \pi_* \alpha \text{ holds for any } \alpha \in H^*(X_k), \beta \in H^*(X).
$$

This, together with the commutativity of the diagram tells us that integration along the fibre of $X_k$ is given by first applying the residue operation to $P(u_1, \ldots, u_k)$ followed by the substitution
of the Chern roots of $X$ into the weights $\lambda_i$ of the torus action, leaving the coefficients of $P$ as they are. That is, since $\tilde{\rho}_k^* P(u_1, \ldots, u_k) = P(-v_1, \ldots, -v_k)$, we have

$$\pi_* P(-v_1, \ldots, -v_k) = \tilde{\rho}_k^* \pi_*^{GL} P(u_1, \ldots, u_k) = \text{Res}(P(u)) |_{\sigma_i(\lambda_1, \ldots, \lambda_n) \to e_i(T_X)}, \quad (4.1)$$

where $\sigma_i$ is the $i$th elementary symmetric polynomial.

**Remark 1.** The described localisation principle for integration on $X_k$ works in a more general context. Let $E = P \times_G F \to X$ be a locally trivial fibre bundle with fibre $F$ and structure group $G$, associated to the principal $G$-bundle $P \to X$. Let $F^G = EG \times_G F$ denote the universal bundle with fibre $F$ over the classifying space $BG$. We have similar pull-back and cohomology map diagrams in this more general setup

$$
\begin{align*}
E & \xrightarrow{\tilde{\rho}} F^G \quad & H^*(E) & \xrightarrow{\tilde{\rho}^*} H^*(F^G) = H_G^*(F) \\
X & \xrightarrow{\rho} BG \quad & H^*(X) & \xrightarrow{\rho^*} H^*(BG) = H_G^*(pt)
\end{align*}
$$

where, again, $\pi_*^G$ is the equivariant push-forward to the point calculated using localisation. If $\alpha = \tilde{\rho}^* (\alpha^G)$ is in the image of $\tilde{\rho}^*$ (we call such classes equivariant fibre classes) then by commutativity of the diagram we get

$$\pi_* \alpha = \rho^* \pi_*^G \alpha^G.$$  

Assume that the fixed-point components of the action of the maximal torus $T$ of $G$ has only isolated fixed points and let $F^T$ denote the fixed-point set. Substituting the localisation formula we get

$$\pi_* \alpha = \rho^* \sum_{f \in F^T} \frac{\alpha^G|_f}{\text{Euler}^T(T_f F)}. \quad (4.2)$$

Formula 4.2 holds without further geometric assumption on the fibre $F$. However, in applications we need to know if a class $\alpha \in H^*(E)$ is an equivariant fibre class or not, and moreover, find a corresponding equivariant class $\alpha^G \in H^*_G(F)$. Working with towers of Grassmannian bundles, such as the Demailly–Semple bundle, provides an immediate solution, since as we have seen, Chern classes of the tautological bundles over $E$ are images of the corresponding equivariant Chern classes of tautological bundles over $F^G$. Moreover, in this special case the cohomology map $\rho^*$ can be identified with the Chern–Weil homomorphism.

In general, a convenient geometric assumption is that $F$ is equivariantly formal. This means that every $\alpha \in H^*(F)$ has an equivariant extension, that is, the natural map $H^*_G(F) \to H^*(F)$ is surjective. Equivalently,

$$H^*(E) = H^*(P \times_G F) = H^*_G(P \times F) = H^*_G(P) \otimes H^*_G(pt) \otimes H^*_G(F) = H^*(X) \otimes H^*_G(pt) \otimes H^*_G(F)$$

holds, that is, $H^*(E)$ is a free $H^*(X)$ module with generators $\tilde{\rho}^*(b_1), \ldots, \tilde{\rho}^*(b_m)$ for some $b_1, \ldots, b_m \in H^*_G(F)$ whose restrictions form a basis of $H^*(F)$. In this case any cohomology class $\alpha \in H^*(E)$ can be written as

$$\alpha = \tilde{\rho}^*(\alpha^G) \cdot \pi^* \beta \quad \text{for some } \alpha^G \in H^*_G(F), \beta \in H^*(X)$$

and the integration formula has the simple form

$$\pi_* (\alpha) = \beta \cdot \rho^* \sum_{f \in F^T} \frac{\alpha^G|_f}{\text{Euler}^T(T_f F)}.$$  

Our referee kindly drew our attention to the recent paper of Tu [32] where the same localisation argument is described when $F$ is equivariantly formal. However, as we already
pointed out, we only use a special set-up where $F$ is smooth complex variety formed as a tower of projective bundles. Smooth complex varieties are equivariantly formal, but we use more than that, namely the explicit descriptions of $\rho^*, \rho^*$ given above only apply for tower of Grassmannian and flag bundles such as $X_k$. Note that the present paper was submitted before the first appearance of [32] and our approach is independent of [32].

4.2. Equivariant localisation on the fibre of the Demailly–Semple tower

This section is the main technical part of the paper. We develop an iterated residue formula for equivariant integration on $X_k$, the fibre of the Demailly–Semple bundle. We use equivariant localisation to calculate the equivariant push-forward map $\pi_k^{GL} : H^*_GL(n)(X_k) \to H^*_GL(n)(pt)$ for Chern monomials of the rank $k$ tautological bundle over $X_k$.

We start by recalling the construction of $X_k$. For $k = 1$ we have $X_1 = \mathbb{P}(T_{X,x})$ and (2.2) specialises as

$$0 \longrightarrow T_{X_1} \longrightarrow V_1 \longrightarrow O_{X_1}(-1) \longrightarrow 0. \quad (4.3)$$

Let $\{e_1, \ldots, e_n\}$ be an eigenbasis for the $T$-action on $V_0 = T_{X,x}$ with weights $\lambda_1, \ldots, \lambda_n$. As (4.3) is $T$-equivariant, the weights on $V_1|_{[e_j]}$ at the fixed point $[e_j] = [0 : \ldots : 0 : 1 : 0 : \ldots : 0] \in X_1$ are $\lambda_j$ and $\lambda_i - \lambda_j$ for $i \neq j$.

Now (2.2) restricted to the fibre $X_k$ gives us

$$0 \longrightarrow T_{X_k}/X_{k-1} \longrightarrow V_k \longrightarrow O_{X_k}(-1) \longrightarrow 0. \quad (4.3)$$

Locally $V_k$ is the direct sum of the two bundles on the ends. Fix a point $y \in X_k$, and let $V_{k-1, \pi(y)}$ denote the fibre of $V_{k-1}$ at the point $\pi(y) \in X_{k-1}$, where $\pi = \pi_{k,k-1}$. If $y$ is a fixed point of the $T$-action on $X_k$, then $\pi(y)$ is a fixed point on $X_{k-1}$, and therefore $V_{k-1, \pi(y)}$ is $T$-invariant, acted on by $T$ with weights $w_1, \ldots, w_n \in \text{Lin}(\lambda_1, \ldots, \lambda_n)$ in the eigenbasis $e_1, \ldots, e_n$. Here

$$\text{Lin}(\lambda_1, \ldots, \lambda_n) = \{a_1 \lambda_1 + \cdots + a_n \lambda_n : a_1, \ldots, a_n \in \mathbb{C}\}$$

denotes the vector space of complex linear forms of $\lambda_1, \ldots, \lambda_n$. By definition $X_k = \mathbb{P}(V_{k-1})$; let $z$ be the fixed line corresponding to the weight $w_j$. The weights on $T_{X_k}/X_{k-1,z} = T_{\mathbb{P}(V_{k-1, \pi(z)})}$ at $z$ are $w_i - w_j$ for $i \neq j$, and the weight on the tautological bundle $O_{X_k}(-1)$ at $z \in X_k$ is $w_j$, so the weights on $V_{k,z}$ are

$$w_i - w_j \text{ for } i = 1, \ldots, n, i \neq j, \text{ and } w_j.$$ 

Therefore, a fixed point $z = F_{w_1, \ldots, w_k}$ is characterised by a sequence $(w_1, \ldots, w_k)$ of weights $w_i \in \text{Lin}(\lambda_1, \ldots, \lambda_n), i = 1, \ldots, k$, where

1. $w_1 \in S = \{\lambda_1, \ldots, \lambda_n\}$;
2. for $i \geq 2$ $w_i \in S(w_1, \ldots, w_{i-1}) = \{w_{i-1}, w - w_{i-1} : w \in S(w_1, \ldots, w_{i-2})\} \neq \emptyset$

and $A^\neq = A \setminus \{0\}$ denotes the set of nonzero elements of $A$.

Here $S(w_1, \ldots, w_{i-1})$ collects the weights of the $T$-action on the fibre $V_{i-1, F_{w_1, \ldots, w_{i-1}}}$. For $n = k = 3$ we have collected the fixed-point data in Table 1.

In general, we get by induction the following.

**Lemma 4.1.** Let $1 \leq i \leq k$ and $w_j \in S(w_1, \ldots, w_{j-1})$ for $1 \leq j \leq i$. Then

$$S(w_1, \ldots, w_i) = \{\lambda_j - w_1 - \cdots - w_1, w_1 - w_2 - \cdots - w_i, \ldots, w_{i-1} - w_1, w_i : 1 \leq j \leq n\} \setminus \{(w_t + w_{t+1} + \cdots + w_i) : 2 \leq t \leq i\},$$

where for $i = 1$ we define the subtracted set to be the empty set.
Proof. For $i = 1$ and $w_1 = \lambda_r$ for some $1 \leq r \leq n$, the weights are

$$S(\lambda_r) = \{\lambda_j - \lambda_r, \lambda_r : j \neq r\} = \{\lambda_j - w_1, w_1 : 1 \leq j \leq n\} \neq 0$$

as stated. Assume the lemma holds for $i - 1$, and use (2) above at the description of the weights:

$$S(w_1, \ldots, w_i) = \{\lambda_j - w_1 - \cdots - w_i, w_1 - w_2 - \cdots - w_i, \ldots, w_{i-1} - w_i, w_i : 1 \leq j \leq n\} \neq 0$$

\[ \{-(w_i + w_{i+1} + \cdots + w_r) : 2 \leq t \leq i \} \]

as stated. □

Remark 2. Note that there is exactly one element of the set

$$\{\lambda_j - w_1 - \cdots - w_i, w_1 - w_2 - \cdots - w_i, \ldots, w_i - w_i, w_i : 1 \leq j \leq n\}$$

which is equal to zero for every choice of $w_1, \ldots, w_i$. We exclude this element by taking the nonzero part of this in set in Lemma 4.1.

The fixed-point set on the fibre $\mathcal{X}_k$ is then $\mathcal{F}_k = \{F_{w_1, \ldots, w_k} : w_i \in S(w_1, \ldots, w_{i-1})\}$. Proposition 3.1 applied to the fibre of the Demailly–Semple bundle gives

**Proposition 4.2.** Let $\mathcal{X}_k$ be the fibre of the Demailly–Semple $k$-jet bundle over $X$ at $x \in X$ endowed with the induced $T = (\mathbb{C}^*)^n$ action from $T_{\mathcal{X}, x}$. Then for any $\alpha \in H^*_T(\mathcal{X}_k)$

$$\int_{\mathcal{X}_k} \alpha = \sum_{F_{w_1, \ldots, w_k} \in \mathcal{F}_k} \frac{\alpha^0(F_{w_1, \ldots, w_k})}{\prod_{j=1}^k \prod_{w \in S(w_1, \ldots, w_{j-1})} (w - w_j)}.$$

Proof. The equivariant Euler class of the tangent bundle of $\mathcal{X}_k$ at the fixed point $F_{w_1, \ldots, w_k}$ is the product of the weights in the tangent directions:

$$\text{Euler}_T(T_{F_{w_1, \ldots, w_k}, \mathcal{X}_k}) = \prod_{j=1}^k \text{Euler}_T(T_{F_{w_1, \ldots, w_j}, \mathcal{P}(V_{j-1, F_{w_1, \ldots, w_{j-1}}}))$$

and the weights on $V_{j-1, F_{w_1, \ldots, w_{j-1}}}$ are collected in $S(w_1, \ldots, w_{j-1})$. □

In particular, we have the following.

**Table 1.** Weights on the Demailly–Semple bundle for $n = k = 3$.

| $\nu_0$ | $\nu_1$ | $\nu_2$ |
|---------|---------|---------|
| $S(\lambda_1, \lambda_1)$ | $\{\lambda_1, \lambda_2 - 2\lambda_1, \lambda_3 - 2\lambda_1\}$ | $S(\lambda_1, \lambda_1)$ |
| $S(\lambda_1)$ | $\{\lambda_1, \lambda_2 - \lambda_1, \lambda_3 - \lambda_1\}$ | $S(\lambda_1, \lambda_2 - \lambda_1)$ |
| $S(\lambda_1, \lambda_3 - \lambda_1)$ | $\{2\lambda_1 - \lambda_2, \lambda_2 - \lambda_1, \lambda_3 - \lambda_2\}$ | $S(\lambda_1, \lambda_3 - \lambda_1)$ |
| $S(\lambda_1, \lambda_2 - \lambda_2)$ | $\{2\lambda_1 - \lambda_3, \lambda_2 - \lambda_3, \lambda_3 - \lambda_1\}$ | $S(\lambda_1, \lambda_2 - \lambda_2)$ |
| $S(\lambda_2, \lambda_3 - \lambda_2)$ | $\{\lambda_1 - \lambda_2, \lambda_2 - \lambda_1, \lambda_3 - \lambda_2\}$ | $S(\lambda_2, \lambda_3 - \lambda_2)$ |
| $S(\lambda_3, \lambda_1 - \lambda_3)$ | $\{\lambda_1 - \lambda_3, \lambda_2 - \lambda_3, \lambda_3 - \lambda_2\}$ | $S(\lambda_3, \lambda_1 - \lambda_3)$ |
| $S(\lambda_3, \lambda_2 - \lambda_3)$ | $\{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \lambda_3 - \lambda_2\}$ | $S(\lambda_3, \lambda_2 - \lambda_3)$ |
| $S(\lambda_3, \lambda_3)$ | $\{\lambda_1 - 2\lambda_3, \lambda_2 - 2\lambda_3, \lambda_3\}$ | $S(\lambda_3, \lambda_3)$ |
Corollary 4.3. Let \( u_i = c_1(\pi^*_k \mathcal{O}_{X_k}(-1)), \) \( i = 1, \ldots, k \) denote the first Chern classes of the tautological line bundles on \( X_k \), and let \( \alpha(u_1, \ldots, u_k) \) be a homogeneous polynomial of degree \( \dim(X_k) = k(n - 1) \). Then \( \alpha^{[0]}(F_{w_1, \ldots, w_k}) = \alpha(w_1, \ldots, w_k) \) and therefore

\[
\int_{X_k} \alpha(u_1, \ldots, u_k) = \sum_{F_{w_1, \ldots, w_k} \in \mathcal{B}_k} \frac{\alpha(w_1, \ldots, w_k)}{\prod_{j=1}^{k} \prod_{\omega \in S(w_1, \ldots, w_{j-1})}(w - w_j)^{\gamma_j}}.
\]

4.3. Transforming the localisation formula into iterated residue

In this section we prove Theorem 1.1 by transforming the right-hand side of the formula in Corollary 4.3 into an iterated residue, motivated by [6]. This will enable us to make effective calculations with the cohomology ring of the Demailly–Semple bundle and to prove positivity of the intersection numbers coming up in the Morse inequalities mentioned in the introduction (see (5.2)).

The final formula will involve the notion of an iterated residue at infinity (cf., for example, [30]). Let \( \omega_1, \ldots, \omega_N \) be affine linear forms on \( \mathbb{C}^k \); denoting the coordinates by \( z_1, \ldots, z_k \), this means that we can write \( \omega_i = a_i^0 z_1 + \cdots + a_i^k z_k \) for \( i = 1, \ldots, N \). We will use the shorthand \( h(z) \) for a function \( h(z_1, \ldots, z_k) \), and \( dz \) for the holomorphic \( k \)-form \( dz_1 \wedge \cdots \wedge dz_k \).

Now, let \( h(z) \) be an entire function, and define the iterated residue at infinity as follows:

\[
\text{Res}_{z_1 = \infty} \ldots \text{Res}_{z_k = \infty} \frac{h(z)}{\prod_{i=1}^{N} \omega_i} = \left( \frac{1}{2\pi i} \right)^k \int_{|z_i| = R_i} \ldots \int_{|z_k| = R_k} \frac{h(z)}{\prod_{i=1}^{N} \omega_i} dz_1 \ldots dz_k,
\]

where \( 1 < R_1 \ll \cdots \ll R_k \). The torus \( \{ |z_m| = R_m; \ m = 1 \cdots k \} \) is oriented in such a way that \( \text{Res}_{z_1 = \infty} \ldots \text{Res}_{z_k = \infty} \frac{dz}{(z_1 \cdots z_k)} = (-1)^k \). We will also use the following simplified notation:

\[
\text{Res}_{z = \infty} \equiv \text{Res}_{z_1 = \infty} \ldots \text{Res}_{z_k = \infty}.
\]

In practice, one way to compute the iterated residue (4.4) is the following algorithm (cf., for example, [30]): for each \( i \), use the expansion

\[
\frac{1}{\omega_i} = \sum_{j=0}^\infty (-1)^j \left( \frac{a_i^0 + a_i^1 z_1 + \cdots + a_i^{q(i)-1} z_{q(i)-1}}{a_i^{q(i)} z_{q(i)}} \right)^j,
\]

where \( q(i) \) is the largest value of \( m \) for which \( a_i^m \neq 0 \), then multiply the product of these expressions with \((-1)^k h(z_1 \cdots z_k)\), and then take the coefficient of \( z_1^{-1} \cdots z_k^{-1} \) in the resulting Laurent series.

The second option to compute iterated residues is working step by step. First, take the residue with respect to \( z_k \) by applying the Residue Theorem on \( \mathbb{C} \cup \{0\} \); the residue at \( z_k = \infty \) is minus the sum of the residues at the finite poles \( w_j = -1/a_j^k(a_j^0 + \cdots + a_j^{k-1} z_{k-1}) \) for those factors where \( a_j^k \neq 0 \). In particular, if these poles are pairwise different then

\[
\text{Res}_{z_k = \infty} \frac{h(z)}{\prod_{i=1}^{N} \omega_i} = \sum_{\text{sum}} \frac{\prod_{j \neq \gamma} \omega_j^{-1}}{\prod_{i=1}^{N} \omega_i} \frac{-h(z_1, \ldots, z_k, w_j)}{w_i(z_1, \ldots, z_{k-1}, w_j)}.
\]

Example 2. The rational expression \( 1/(z_1(z_1 - z_2)) \) has two different Laurent expansions, but on \( |z_1| \ll |z_2| \) we use \( 1/(z_1 z_1 - z_2)) = \sum_{i=0}^\infty (-1)^i (z_1^{-1} z_2^{i+1}) \) to get \( \text{Res}_{z_1 = \infty} 1/(z_1 - z_2) = 1 \). Another example is \( \text{Res}_{z_1 = \infty} 1/(z_1 z_2)(2z_1 - z_2) \) has two different Laurent expansions, but on \( |z_1| \ll |z_2| \) we use \( 1/(z_1 z_2)(2z_1 - z_2) = \text{coeff}_{(z_1 z_2) - 1} (1/z_2^2)(1 + (z_1/z_2) + (z_1^2/z_2^2) + \cdots) \).
Example 3. Let us revisit our toy example, Example 1. Define the differential form

$$
\omega = \frac{-(z_2 - z_1)^2(z_1 + z_2)^2z_1z_2}{\prod_{i=1}^4(\mu_i - z_1)\prod_{i=1}^4(\mu_i - z_2)}
$$

This is a meromorphic form in $z_2$ on $\mathbb{P}^1$ with poles at $z_2 = \mu_i$, $1 \leq i \leq 4$ and $z_2 = \infty$. The poles at $\mu_i$ are nondegenerate and therefore applying the Residue Theorem we get

$$
\text{Res}_{z_2=\infty} \omega = -\frac{4}{\prod_{j=1}^4(\mu_j - z_1)\prod_{j=1}^4(\mu_j - \mu_i)} + \sum_{i=1}^4 \frac{(\mu_i - z_1)(\mu_i + z_1)^2\mu_i\mu_j}{\prod_{j\neq i}(\mu_j - \mu_i)}
$$

Repeating the same with $z_1$ we get

$$
\text{Res}_{z_1=\infty} \text{Res}_{z_2=\infty} \omega = \sum_{i=1}^4 \sum_{j\neq i} \frac{(\mu_i - \mu_j)(\mu_i + \mu_j)^2\mu_i\mu_j}{\prod_{k\neq i,j}(\mu_k - \mu_j)\prod_{j\neq i}(\mu_j - \mu_i)}
$$

On the other hand, using the above algorithm by expanding the rational form $\omega$ we get

$$
\text{Res}_{z_1=\infty} \text{Res}_{z_2=\infty} \omega = 2
$$

giving us the desired result 1 for the integral.

We start with the following iterated Residue Theorem on projective spaces.

**Proposition 4.4.** For a polynomial $P(u)$ on $\mathbb{C}$ we have

$$
\sum_{i=1}^n \frac{P(\lambda_i)}{\prod_{j\neq i}(\lambda_j - \lambda_i)} = \text{Res}_{z=\infty} \frac{P(z)}{\prod_{j=1}^n(\lambda_j - z)} dz. \quad (4.7)
$$

**Proof.** We compute the residue on the right-hand side of (4.7) using the Residue Theorem on the projective line $\mathbb{C} \cup \{\infty\}$. This residue is a contour integral, whose value is minus the sum of the $z$-residues of the form in (4.7). These poles are at $z = \lambda_j$, $j = 1 \ldots n$, and after cancelling the signs that arise, we obtain the left-hand side of (4.7). \qed

Now we prove the following iterated residue formula for the cohomology pairings of $X_k$.

**Proposition 4.5.** Let $k \geq 2$ and let $X_k$ be the fibre of the Demailly–Semple k-jet bundle. Let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ denote the weights of diagonal action of $T = (\mathbb{C}^*)^n$ on $\mathbb{C}^n$ which induces a $T$-action on $X_k$. Let $P(u_1, \ldots, u_k) \in H^k_{T}(\mathfrak{g}(n-1))$ be a homogeneous equivariant class of degree $\text{dim}(X_k) = k(n-1)$ in $u_i = c_1(\pi_{i,n}^*O_{X_k}(1))$. Then

$$
\int_{X_k} P(u) = \text{Res}_{z=\infty} \frac{\prod_{1 \leq i \leq 1 \leq k} (z_1 + z_{1+1} + \cdots + z_t)}{\prod_{1 \leq a \leq s_1 < s_2 \leq k} (z_{a_1} - z_{a_1+1} - \cdots - z_{a_2})} \frac{P(z_1, \ldots, z_k)}{\prod_{j=1}^k \prod_{i=1}^n(\lambda_i - z_1 - \cdots - z_j)} dz.
$$
Proof. We use that $X_k \to X_{k-1}$ is a $\mathbb{P}^{n-1}$ bundle over $X_{k-1}$ and therefore integration on $X_k$ can be achieved first integrating over the fibre followed by integration over $X_{k-1}$. That is, the fixed points on $X_k$ fibre over the fixed-point set on $X_{k-1}$ and using Corollary 4.3 we get

$$
\int_{X_k} P(u) = \sum_{F_{w_1,\ldots,w_k} \in \tilde{\mathcal{S}}_k} \frac{P(w_1,\ldots,w_k)}{\prod_{j=1}^k \prod_{w \in S(w_1,\ldots,w_{j-1}), w \neq w_j} (w - w_j)}
= \sum_{F_{w_1,\ldots,w_{k-1}} \in \tilde{\mathcal{S}}_{k-1}} \frac{1}{\prod_{j=1}^{k-1} \prod_{w \in S(w_1,\ldots,w_{j-1})} (w - w_j)} \cdot \sum_{w_k \in S(w_1,\ldots,w_{k-1})} \frac{P(w_1,\ldots,w_k)}{\prod_{w \in S(w_1,\ldots,w_{k-1}), w \neq w_k} (w - w_k)}.
$$

Recall that the weights of the $T$ action on $\pi^{-1}(F_{w_1,\ldots,w_{k-1}}) = \mathbb{P}(V_{k-1},F_{w_1,\ldots,w_{k-1}})$ are collected in the set $S(w_1,\ldots,w_{k-1}) \subset \text{Lin}(\lambda_1,\ldots,\lambda_n)$, so by Proposition 4.4 the second sum, which is the integral on the fibre $\pi^{-1}(F_{w_1,\ldots,w_{k-1}}) \cong \mathbb{P}^{n-1}$ can be written as a residue

$$
\sum_{w_k \in S(w_1,\ldots,w_{k-1})} \frac{P(w_1,\ldots,w_k)}{\prod_{w \in S(w_1,\ldots,w_{k-1}), w \neq w_k} (w - w_k)} = \text{Res}_{z_k = \infty} \frac{P(w_1,\ldots,w_{k-1},z_k)}{\prod_{w \in S(w_1,\ldots,w_{k-1})} (w - z_k)}.
$$

Iterating this on the Demaini–Seample tower $X_k \to X_{k-1} \to \cdots \to X_1 \to \{x\}$ using Proposition 4.4 in every step we get

$$
\int_{X_k} P(u) = \text{Res}_{z = \infty} \frac{P(z_1,\ldots,z_k)}{\prod_{j=1}^k \prod_{w \in S(z_1,\ldots,z_{j-1})} (w - z_j)}.
$$

Using the description of $S(z_1,\ldots,z_i)$ from Lemma 4.1 we get the desired iterated residue formula

$$
\int_{X_k} P(u) = \text{Res}_{z = \infty} \frac{\prod_2 \leq t_1 \leq t_2 \leq k - z_{[t_1, t_2]} P(z_1,\ldots,z_k)}{\prod_{1 \leq s_1 < s_2 \leq k} (z_{s_1} - z_{[s_1+1, s_2]}) \prod_{j=1}^k \prod_{i=1}^n (\lambda_i - z_{[1, \ldots, j]})}.
$$

Note that the term corresponding to $t_1 = t_2 = j$ in the numerator cancels out with the term $-z_j$ in the denominator coming from the unique zero element in the set $\{\lambda_1 - z_1 - \cdots - z_{j-1}, z_1 - z_2 - \cdots - z_i, \ldots, z_{j-2} - z_{j-1}, z_{j-1} : 1 \leq s \leq n\}$ in Lemma 4.1 (see Remark 2). □

4.4 Proof of Theorem 1.1

Proposition 4.5, together with the localisation principle described in §4.1, gives us Theorem 1.1. Indeed, applying the residue operator of Proposition 4.5 and the Chern–Weil homomorphism in (4.1) we get

$$
\int_{X_k} P(-v) = \int_{X_k} \text{Res}_{z = \infty} \frac{\prod_2 \leq t_1 \leq t_2 \leq k - z_{[t_1, t_2]} P(z_1,\ldots,z_k)}{\prod_{1 \leq s_1 < s_2 \leq k} (z_{s_1} - z_{s_1+1} - \cdots - z_{s_2})} \times \frac{P(z_1,\ldots,z_k)}{\prod_{j=1}^k \prod_{i=1}^n (\lambda_i - z_1 - \cdots - z_j)^{\sigma_i(\lambda_1,\ldots,\lambda_n) \cdot c_i(T_X)}} dz,
$$

where $\sigma_i$ is the $i$th elementary symmetric polynomial. Now we replace the variables $z_i$ by $-z_i$ for $i = 1,\ldots,k$. This changes the sign of the iterated residue by $(-1)^k$ as it corresponds to changing the orientation of the contour circles and moreover, since $z_i$ stands for the first Chern class of the tautological bundle $u_i = c_1(\pi_{i,k}^*O_{X_i}(-1))$, $-z_i$ corresponds to its dual $c_1(\pi_{i,k}^*O_{X_i}(1))$. This changes the sign of the iterated residue by $(-1)^k$ and therefore

$$
\int_{X_k} P(-v) = \text{Res}_{z = \infty} \frac{\prod_2 \leq t_1 \leq t_2 \leq k - z_{[t_1, t_2]} P(z_1,\ldots,z_k)}{\prod_{1 \leq s_1 < s_2 \leq k} (z_{s_1} - z_{s_1+1} - \cdots - z_{s_2})} \times \frac{P(z_1,\ldots,z_k)}{\prod_{j=1}^k \prod_{i=1}^n (\lambda_i - z_1 - \cdots - z_j)^{\sigma_i(\lambda_1,\ldots,\lambda_n) \cdot c_i(T_X)}} dz.
$$
whose image under $\tilde{\rho}^*$ is $v_i$. So we get

$$
\int_{X_k} P(v) = \text{Res}_{z=\infty} \frac{(-1)^k \prod_{2 \leq t_1 \leq t_2 \leq k} z_{[t_1 \ldots t_2]} P(z_1, \ldots, z_k) dz}{\prod_{1 \leq s_1 < s_2 \leq k} (-z_{s_1} + z_{[s_1+\ldots+s_2]}) \prod_{j=1}^{k} \prod_{i=1}^{n} (\lambda_i + z_{[1\ldots j]})} |_{\sigma_i(\lambda_1, \ldots, \lambda_n) \to c_i(T_X)}.
$$

(4.8)

We use the shorthand notation

$$
x_{[i \ldots j]} = x_i + x_{i+1} + \cdots + x_j
$$

(4.9)

for the sum of entries of a constant vector $(x_i, \ldots, x_j)$. To get the final formula of Theorem 1.1, we note that the terms involving the factors $\lambda_i$ in (4.8) can be rewritten as

$$
\frac{1}{\prod_{i=1}^{n} (\lambda_i + z_{[1\ldots j]})} = \frac{1}{z_{[1\ldots j]} c(1/z_{[1\ldots j]})} = \frac{s_X(1/z_{[1\ldots j]})}{z_{[1\ldots j]}},
$$

where

$$
s_X \left( \frac{1}{z_{[1\ldots j]}} \right) = 1 + \frac{s_1(X)}{z_{[1\ldots j]}} + \frac{s_2(X)}{z_{[1\ldots j]}^2} + \cdots + \frac{s_n(X)}{z_{[1\ldots j]}^n}
$$

is the total Segre class of $X$ evaluated at $1/z_{[1\ldots j]}$. Finally, $(-1)^k$ cancels out after taking the iterated residue and we get Theorem 1.1.

4.5. An example: Euler characteristic of the jet differentials bundle

In the rest of the present paper focus on projective hypersurfaces $X \subset \mathbb{P}^{n+1}$ and we use our iterated residue formula to prove the existence of global sections of some twisted jet differentials bundle. In order to prove this we follow [16] and use Morse inequalities to reduce the question to the positivity of some appropriately defined tautological integral over the Demailly–Semple bundle.

However, Theorem 1.1 gives iterated residue formula for other topological invariants of the jet differentials bundle as well, here we give the formula for the Euler characteristic

$$
\chi(X, E_k) = \sum_{i=0}^{n} (-1)^i \dim H^i(X, E_k)
$$

of the invariant jet differentials bundle $E_k = \oplus_{m=0}^{\infty} E_{k,m}$. This can be expressed with the Chern character of $E_k$ and the Todd class of $X$ as the integral

$$
\chi(X, E_k) = \int_X [\text{ch}(E_k) \cdot \text{Td}(T_X)]_n,
$$

where $[\cdot]_n$ is the degree $n$ part of the expression. Note that according to Theorem 2.1 $(\pi_{0,k})_* \mathcal{O}_{X_k}(m) = \mathcal{O}(E_{k,m})$ holds and localisation on the Demailly–Semple tower then gives the following iterated residue formula.

**Corollary 4.6.** For $(a_1, \ldots, a_k) \in \mathbb{Z}^k$ define the following line bundle on $X_k$:

$$
\mathcal{O}_{X_k}(a) = \pi_{1,k}^* \mathcal{O}_{X_1}(a_1) \otimes \pi_{2,k}^* \mathcal{O}_{X_2}(a_2) \otimes \cdots \otimes \mathcal{O}_{X_k}(a_k)
$$

which is a subbundle of $\mathcal{O}_{X_k}(a_1 + \cdots + a_k)$ and therefore $(\pi_{0,k})_* \mathcal{O}_{X_k}(a) \subset \mathcal{O}(E_{a_1 + \cdots + a_k, m})$. 


Then
\[
\chi(X, O_{X_k}(a)) = \int_X \frac{\text{Res}_{z \to \infty} (-1)^k \prod_{2 \leq t_1 \leq t_2 \leq k} (z_{t_1} + z_{t_1 + 1} + \cdots + z_{t_2}) \text{ch}(O_{X_k}(a)) \cdot Td(T_X)}{\prod_{1 \leq s_1 < s_2 \leq k} (-z_{s_1} + z_{s_1 + 1} + \cdots + z_{s_2}) \prod_{j=1}^k (z_1 + \cdots + z_j)^n} \cdot \prod_{j=1}^k s_X \left( \frac{1}{z_1 + \cdots + z_j} \right),
\]
where
\[
\text{ch}(O_{X_k}(a)) = e^{a_1 z_1 + \cdots + a_k z_k} \text{ and } Td(T_X) = 1 + \frac{1}{2} c_1 + \frac{1}{12} (c_1^2 + c_2) + \cdots
\]

5. Proof of Theorem 1.2

Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $\deg X = d$ and let $X_k$ denote the $k$-level Demailly–Semple bundle on $X$. We start with recalling the following classical major result which connects jet differentials to the GGL conjecture.

**Theorem 5.1** (Fundamental vanishing theorem [11, 19, 26]). Assume that there exist integers $k, m > 0$ and an ample line bundle $A \to X$ such that
\[
H^0(X_k, O_{X_k}(m) \otimes \pi^* A^{-1}) \simeq H^0(X, E_{k,m} \otimes A^{-1}) \neq 0
\]
has nonzero elements and let $Z \subset X_k$ be the base locus of these sections. Then every nonconstant entire holomorphic curve $f : \mathbb{C} \to X$ necessarily satisfies $f[k](\mathbb{C}) \subset Z$. In other words, for every global $\mathbb{G}_k$-invariant differential equation $P$ vanishing on an ample divisor, every entire holomorphic curve $f$ must satisfy the algebraic differential equation $P(f(t), \ldots, f^{(k)}(t)) \equiv 0$.

Note, that by [15, Theorem 1],
\[
H^0(X, E_{k,m} \otimes A^{-1}) = 0
\]
holds for all $m \geq 1$ if $k < n$, so we must restrict our attention to the range $k \geq n$. In fact, we will assume that $k = n$ in this section.

To control the order of vanishing of these differential forms along the ample divisor we choose $A$ to be, as in [16], a proper twist of the canonical bundle of $X$. Recall that the canonical bundle of the smooth, degree $d$ hypersurface $X$ is
\[
K_X = O_X(d - n - 2)
\]
which is ample as soon as $d \geq n + 3$. The following theorem summarises the results of [16, §3].

**Theorem 5.2** (Algebraic degeneracy of entire curves [16]). Assume that $n = k$, and there exist a $\delta = \delta(n) > 0$ and $D = D(n, \delta)$ such that
\[
H^0(X_n, O_{X_n}(m) \otimes \pi^* K_X^{-\delta m}) \simeq H^0(X, E_{n,m} \otimes K_X^{-\delta m}) \neq 0
\]
whenever $\deg(X) > D(n, \delta)$ for some $m \gg 0$. Then the GGL conjecture holds whenever
\[
\deg(X) > \max \left( D(n, \delta), \frac{n^2 + 2n}{\delta} + n + 2 \right).
\]

For $(a_1, \ldots, a_k) \in \mathbb{Z}^k$ define the following line bundle on $X_k$:
\[
O_{X_k}(a) = \pi_{1,k}^* O_{X_1}(a_1) \otimes \pi_{2,k}^* O_{X_2}(a_2) \otimes \cdots \otimes O_{X_k}(a_k).
\]
We use notation \( |a| = a_1 + \cdots + a_k \). Then \( \mathcal{O}_{X_k}(a) \) is a subsheaf of \( \mathcal{O}_{X_k}(|a|) \). Theorem 5.2 accompanied with the following theorem gives us Theorem 1.2.

**Theorem 5.3.** Let \( X \subset \mathbb{P}^{n+1} \) be a smooth complex hypersurface of degree \( d = \deg(X) > 6n^{8n} \). If \( a_i = n^{8(n+1-i)} \) and \( \delta = 1/2n^{8n} \) then
\[
H^0 \left( X; \mathcal{O}_X(m|a|) \otimes \pi_{0,k}^{-\delta[a]} \right) \simeq H^0 \left( X; E_{n,m|a|} \otimes K_X^{-\delta[a]} \right) \neq 0
\]
is nonzero for \( m \gg 1 \).

To prove Theorem 5.3 we use the algebraic Morse inequalities of Demailly and Trapani and replace the cohomological computations of [16] with the study of the iterated residue. Let \( L \to X \) be a holomorphic line bundle over a compact Kähler manifold of dimension \( n \) and \( E \to X \) a holomorphic vector bundle of rank \( r \). Demailly in [12] proved the following theorem.

**Theorem 5.4 (Algebraic Morse inequalities [12, 31]).** Suppose that \( L = F \otimes G^{-1} \) is the difference of the nef line bundles \( F,G \). Then for any nonnegative integer \( q \in \mathbb{Z}_{\geq 0} \)
\[
\sum_{j=0}^{q} (-1)^{q-j} h^j(X, L^\otimes m \otimes E) \leq \frac{r}{n!} \sum_{j=0}^{q} (-1)^{q-j} \binom{n}{j} F^{n-j} \cdot G^j + o(m^n).
\]
In particular, \( q = 1 \) asserts that \( L^\otimes m \otimes E \) has a global section for \( m \) large provided
\[
F^n - nF^{n-1}G > 0.
\]

**Proposition 5.5** [11, Proposition 6.16; 15, Proposition 3.2].

(i) If \( a_1 \geq 3a_2, \ldots, a_{k-2} \geq 3a_{k-1}, \) and \( a_{k-1} \geq 2a_k \geq 0 \), then line bundle \( \mathcal{O}_{X_k}(a) \) is relatively nef over \( X \). If, moreover,
\[
a_1 \geq 3a_2, \ldots, a_{k-2} \geq 3a_{k-1} \text{ and } a_{k-1} > 2a_k > 0
\]
holds, then \( \mathcal{O}_{X_k}(a) \) is relatively ample over \( X \).

(ii) Let \( \mathcal{O}_X(1) \) denote the hyperplane divisor on \( X \). If (5.1) holds, then \( \mathcal{O}_{X_k}(a) \otimes \pi_{0,k}^* \mathcal{O}_X(l) \) is nef, provided that \( l \geq 2|a| \), where \( |a| = a_1 + \cdots + a_k \).

For \( d \geq n+3 \) the canonical bundle \( K_X \simeq \mathcal{O}_X(d-n-2) \) is ample, and therefore we have the following expressions for \( \mathcal{O}_{X_k}(a) \) and \( \mathcal{O}_{X_k}(a) \otimes \pi_{0,k}^{-\delta[a]} \) as a difference of two nef line bundles over \( X \):
\[
\mathcal{O}_{X_k}(a) = (\mathcal{O}_{X_k}(a) \otimes \pi_{0,k}^{-\delta[a]} \mathcal{O}_X(2|a|)) \otimes (\pi_{0,k}^* \mathcal{O}_X(2|a|))^{-1}
\]
\[
\mathcal{O}_{X_k}(a) \otimes \pi_{0,k}^{-\delta[a]} = (\mathcal{O}_{X_k}(a) \otimes \pi_{0,k}^{-\delta[a]} \mathcal{O}_X(2|a|)) \otimes (\pi_{0,k}^* \mathcal{O}_X(2|a|) \otimes \pi_{0,k}^* \mathcal{O}_X^{-\delta[a]})^{-1}.
\]

According to the Morse inequalities the positivity of the following intersection product:
\[
I(n,k,a,\delta) = (\mathcal{O}_{X_k}(a) \otimes \pi_{0,k}^{-\delta[a]} \mathcal{O}_X(2|a|))^{(n+k)(n-1)}
\]
\[
- (n+k(n-1))(\mathcal{O}_{X_k}(a) \otimes \pi_{0,k}^{-\delta[a]} \mathcal{O}_X(2|a|))^{(k+1)(n-1)} \cdot (\pi_{0,k}^{-\delta[a]} \mathcal{O}_X(2|a|) \otimes \pi_{0,k}^{-\delta[a]})
\]
implies that
\[
\mathcal{O}_{X_k}(m \cdot a) \otimes \pi_{0,k}^{-\delta[a]} \mathcal{O}_X(2|a|) \subset \mathcal{O}_{X_k}(m|a|) \otimes \pi_{0,k}^{-\delta[a]} \mathcal{O}_X(2|a|)
\]
has a global section for \( m \) large enough (where \( m \cdot a = (ma_1, \ldots, ma_k) \)) and this proves Theorem 5.3.
Let \( h = c_1(O_X(1)) \) denote the first Chern class of the tautological line bundle \( O_X(1) \), \( c_l = c_l(T_X) \) for \( l = 1, \ldots, n \), and \( u_s = c_1(O_X(1)) \) for \( s = 1, \ldots, k \). Then \( c_1(K_X) = -c_1 = (d - n - 2)h \), and the intersection product (5.2) becomes the polynomial
\[
I_{n,k,a,\delta}(u_1, \ldots, u_k, h) = (a_1 u_1 + \cdots + a_k u_k + 2|a|h)^{(k+1)(n-1)}
\] \times (a_1 u_1 + \cdots + a_k u_k + S_{n,k,\delta,d}|a|h),
\]
(5.3)
where \( S_{n,k,\delta,d} = 2 - (n + k(n - 1))(2 + \delta(d - n - 2)) \).

The Chern classes of \( X \) are expressible with \( d, h \) using the following identity:
\[
(1 + h)^{n+2} = (1 + dh)c(X),
\]
where \( c(X) = c(T_X) \) is the total Chern class of \( X \). Applying Theorem 1.1 (and using the shorthand notations in (4.9)) we arrive at

**Proposition 5.6.** Let \( I(n, k, a, \delta) \) be the intersection number on the Demailly–Semple bundle defined in (5.2). Then with the notation (5.3):
\[
I(n, k, a, \delta) = \int_X \text{Res} \frac{(-1)^k \prod_{1 \leq i_1 \leq t_2 \leq k} (z_{i_1}^{j_2}) \prod_{j=1}^{k} (z_{1, j} + dh)I_{n,k,a,\delta}(z, h) \, dz}{\prod_{1 \leq s_1 < s_2 \leq k} (-z_{s_1} + z_{s_1+1} + \cdots + z_{s_2}) \prod_{j=1}^{k} (z_{1, j} + h)^{n+2}}.
\]

This formula has the pleasant feature that it expresses the aimed intersection number directly in terms of \( n, k, a, d, \delta \). Indeed, according to \( \S \, 4.1 \), the result of the iterated residue is a polynomial in \( n, k, a, d, \delta \), and integrating over \( X \) simply means a substitution \( d = h^n \).

**5.1. Computations with the iterated residue for \( n = k \)**

From now on we assume that \( n = k \), focusing on Theorem 5.3. The iterated residue is formally a contour integral, but as we have explained in \( \S \, 4.3 \), it simply means an expansion of the rational expression respecting the order \( 1 \ll |z_1| \ll \cdots \ll |z_k| \). Using the notation introduced in (4.9) we have the following expansions in Proposition 5.6.

1. \( \frac{1}{z_{1, j1}} = \frac{1}{z_{j1}} (1 - \frac{z_{1, j1}}{z_{j1}})^{-1} = \frac{1}{z_{j1}} (1 + \frac{z_{1, j1}}{z_{j1}} + \frac{(z_{1, j1})^2}{z_{j1}} + \cdots) \) for \( j = 1 \), where for \( j = 1 \) we define \( z_{1, j1} = 0 \).
2. \( \frac{z_{1, t2}}{z_{1, t2} + z_{t1} + \cdots + z_{t2}} = 1 + \frac{z_{t1}}{z_{t2}} (1 + \frac{z_{1, t2}}{z_{t1}}) - \frac{(z_{1, t2})^2}{z_{t1}} + \cdots \) for \( 1 \leq t_1 < t_2 \leq n \).

For \( n = k \) we use the notation \( I_{n,a,\delta,d}(z, h) = (a_1 z_1 + \cdots + a_n z_n + 2|a|h)^{n-1} \) for the form and by (5.3)
\[
I_{n,a,\delta,d}(z, h) = (a_1 z_1 + \cdots + a_n z_n + 2|a|h)^{n-1} (a_1 z_1 + \cdots + a_n z_n + S_{n,\delta}|a|h - n^2 \delta|a|h),
\]
where
\[
S_{n,\delta} = 2 - 2n^2 + n^2(n + 2)\delta.
\]

Substituting these into Proposition 5.6 we get the following:
\[
I(n, a, \delta, d) = (-1)^n \int_X \text{Res} \prod_{j=1}^{n} \left( 1 + \frac{z_{[1, \cdots, j-1]} + dh}{z_j} \right)_{A^0(z)}
\]
\[
\prod_{1 \leq t_1 < t_2 \leq n} \left( 1 + \frac{2z_{t1}}{z_{t2}} (1 + \frac{z_{t1}}{z_{t2}} - \frac{z_{t1} z_{t1+1} \cdots z_{t2-1}}{z_{t2}} + \cdots) \right)_{A^1(z)} \prod_{j=1}^{n} \left( 1 - \frac{z_{[1, \cdots, j-1]} + h}{z_j} + \cdots \right)_{A^2(z)} \right)_{A^n(z)}
\]
\[
(a_1 z_1 + \cdots + a_n z_n + 2 |a| h) n^2 - 1 \left( (a_1 z_1 + \cdots + a_n z_n + S_{n, \delta} |a| h - n^2 \delta |a| d h) \right) \left( z_1 \cdots z_n \right)^n \frac{dz}{B(z)} \tag{5.4}
\]

Let

\[ A(z) = A^0(z) A^1(z) A^2(z) \]

denote the product of the first three rational expressions for short.

**Definition 1.** Fix a basis \( \{ e_1, \ldots, e_n \} \) of \( \mathbb{Z}^n \). For a lattice point \( i = (i_1, \ldots, i_n) \in \mathbb{Z}^n \) we call

\[ D(i) = ni_1 + (n - 1)i_2 + \cdots + i_n \]

the defect of \( i \). The positive lattice semigroup is defined as

\[ \Lambda^+ = \bigoplus_{i < j} \mathbb{Z}^{\geq 0} (e_i - e_j) \oplus \bigoplus_{i=1}^n \mathbb{Z}^{\leq 0} e_i. \]

The negative lattice points are elements of \( \Lambda^- = -\Lambda^+ \). Finally, for \( a, b \in \mathbb{Z}^n \) we say that \( a \geq b \) if there is a \( c \in \Lambda^+ \) with \( b + c = a \).

We now prove the following theorem which together with the Morse inequalities gives us Theorem 5.3.

**Theorem 5.7.** Let \( a_i = n^{8(n+1-i)} \) and \( \delta = \frac{1}{2n^8} \). Then \( I(n, a, \delta, d) > 0 \) if \( d > 6n^{8n} \).

For \( i \in \mathbb{Z}^n, j, k \in \mathbb{Z} \) let \( A_{z^i d^j h^k} \) denote the coefficient of \( z^i d^j h^k \) in \( A(z) \), and use similar notations for coefficients in \( B(z) \).

**Lemma 5.8.** \( A_{z^i (dh)^m} = \text{coeff}_{z^i (dh)^m} A(z) = 0 \) unless \( i \in \Lambda^+ \), for any \( m \geq 0 \).

**Proof.** From (5.4) we see that all monomials appearing after multiplying out is the product of terms of the form \( \frac{z_i}{z_j}, \frac{h}{z_j} \) and \( \frac{dh}{z_j} \) with \( 1 \leq i < j \leq n \), this implies the result. \( \square \)

Let us step back a bit looking at formula (5.4). The residue is by definition the coefficient of \( 1/z_1 \ldots z_n \) in the appropriate Laurent expansion of the big rational expression in \( z_1, \ldots, z_n, n, d, h \) and \( \delta \), multiplied by \( (-1)^n \). We can therefore omit the \( (-1)^n \) factor and simply compute the corresponding coefficient. The result is a polynomial in \( n, d, h, \delta \), and in fact, a relatively easy argument shows that it is a polynomial in \( n, d, \delta \) multiplied by \( h^n \).

Indeed, giving degree 1 to \( z_1, \ldots, z_n, h \) and 0 to \( n, d, \delta \), the rational expression in the residue has total degree 0. Therefore the coefficient of \( 1/z_1 \ldots z_n \) has degree \( n \), so it has the form \( h^n p(n, d, \delta) \) with a polynomial \( p \). Since \( d \) appears only as a linear factor next to \( h \), the degree of \( p \) in \( d \) is \( n \).

Moreover, \( \int_X h^n = d \), so the integration over \( X \) is simply a substitution \( h^n = d \), resulting the equation \( I(n, a, \delta, d) = dp(n, a, \delta, d) \), where

\[ p(n, a, \delta, d) = p_n(n, a, \delta) d^n + \cdots + p_1(n, a, \delta) d + p_0(n, a, \delta) \]

is a polynomial in \( d \) of degree \( n \). The goal is to show that \( p_n \) dominates the rest of the polynomials, that is, to prove the following.
The next goal is to compute the leading coefficient \( p_n > 0 \) and \( |p_{n-l}| < 3n^{8l/n}p_n \).

Theorem 5.7 is a straightforward consequence of this Proposition by applying the following elementary statement.

Lemma 5.10 (Fujiwara bound). If \( p(d) = p_n d^n + p_{n-1} d^{n-1} + \cdots + p_1 d + p_0 \in \mathbb{R}[d] \) satisfies the inequalities

\[
p_n > 0; \quad |p_{n-l}| < D^l |p_n| \quad \text{for} \quad l = 1, \ldots, n
\]

then \( p(d) > 0 \) for \( d > 2D \).

5.2. Estimation of the leading coefficient

The next goal is to compute the leading coefficient \( p_n(n, \delta) \). For \( i = (i_1, \ldots, i_n) \in \mathbb{Z}^n \) let \( \Sigma i = i_1 + \cdots + i_n \) denote the sum of its coordinates. From (5.3)

\[
p_n = \sum_{\Sigma i=0} B_{\Sigma} A_{\Sigma-i-1(dh)^n} - n^2 |\delta| \sum_{\Sigma i=-1} B_{\Sigma} A_{\Sigma-i-1(dh)^n-1}, \tag{5.5}
\]

where \( (1, \ldots, 1) \). Note that, according to Lemma 5.8, some terms on the right-hand side are 0, since we have not made any restrictions on the relation of \( i \) to \( \Lambda^+ \).

There is a dominant term on the right-hand side, corresponding to \( i = (0, \ldots, 0) \) in the first sum:

\[
B_0 = \frac{B_{\Sigma} A_{\Sigma}}{n^2} = (a_1 \cdots a_n)^n \binom{n^2}{n, \ldots, n}. \tag{5.6}
\]

Here \( \binom{m_1}{m_1 \cdots m_s} = \frac{m!}{m_1! \cdots m_s!} \) denotes the multinomial coefficient equal to the coefficient of \( x_1^{m_1} \cdots x_s^{m_s} \) in \( (x_1 + \cdots + x_s)^m \). We show that the absolute sum of the remaining terms is less than this dominant term, implying a lower bound for \( p_n \) when \( \delta, \mathbf{a} \) as in Theorem 5.7. According to the choice \( a_i = n^{8(n+1-i)} \) we have for \( \Sigma i = 0 \)

\[
B_{\Sigma} = \binom{n^2}{i_1 + n, \ldots, i_n + n} a_{i_1+n}^{i_1+n} \cdots a_{i_n+n}^{i_n+n} < a_1^1 \cdots a_n^1 \tag{5.7}
\]

On the other hand \( A_{\Sigma-i-1(dh)^n} = 0 \) unless \( D(i) \leq 0 \) in which case

\[
A_{\Sigma-i-1(dh)^n} = \sum_{i_1+i_2=-1} A_{i_1} A_{i_2}^2 < 2^{-D(i)} n^{-3D(i)} \tag{5.8}
\]

holds according to the following two lemmas which will be repeatedly used in the forthcoming parts of the proof.

Lemma 5.11. We have the following estimations.

(i) \( \sharp \{ i \in \Lambda^+: \Sigma i = 0, D(i) = i \} \leq (n-1)^i \).

(ii) Let \( i \in \Lambda^+, \Sigma i = 0 \) be fixed and let \( s \) be a positive integer. Then

\[
\sharp \{ (i_1, \ldots, i_s) \in (\Lambda^+)^s : i_1 + \cdots + i_s = i \} \leq s^{D(i)}.
\]

Proof. Let \( i = \sum_{j=1}^{s} (e_{i_j} - e_{i_j+1}) \) be the unique decomposition of \( i \) into the sum of positive simple roots. We have \( n - 1 \) positive simple roots which gives the first inequality. For the second part note that each summand can be put into any of the \( s \) multi-indices \( i_1, \ldots, i_s \) which gives us the second inequality.
Note that for $\Sigma i = 0$ $i \in \Lambda^-$ \ {0} implies that $D(i) < 0$, which means that $A_i^1 = A_i^2 = A_i^3 = 0$ for $D(i) < 0$, as expected, since all these coefficients are 0 unless $i \in \Lambda^-.$

**Lemma 5.12.** Let $\Sigma i = 0$. Then $A_i^1, A_i^2 < n^{3D(i)}$.

**Proof.** Denoting by $z^{i(t_1, t_2)}$ the monomial we pick from the term corresponding to $t_1, t_2$ we get by definition

$$|A_i^1| = \sum_{\sum_{t_1, t_2} i(t_1, t_2) = 1} \prod_{1 \leq t_1 < t_2 \leq n} \text{coeff}_{z^{i(t_1, t_2)}} \left( 1 + \frac{z_{t_1}}{z_{t_2}} \left( 1 + \frac{2z_{t_1} - z_{[t_1+1\ldots t_2-1]}}{z_{t_2}} \right) - \ldots \right)$$

$$\leq \sum_{\sum_{t_1, t_2} i(t_1, t_2) = 1} \prod_{t_1, t_2} 2 \cdot \text{comb}(i^+(t_1, t_2)),$$

where $i(t_1, t_2) = i^+(t_1, t_2) - m_{t_1, t_2} e_{t_2}$ for unique $i^+ \in (\mathbb{Z}^0)^n$, $m_{t_1, t_2} \in \mathbb{Z}^0$, and for $j = (j_1, \ldots, j_n) \in (\mathbb{Z}^0)^n$ we define $\text{comb}(j) = (j_1^{+1} \cdots j_n^{+1})$. Note that $\text{comb}(j)$ is a summand in $n^{\Sigma j} = (1 + \cdots + 1)^{\Sigma j}$ and therefore $\text{comb}(j) < n^{\Sigma j}/2$. Hence for $i^+(t_1, t_2) \neq 0$ we have

$$\text{comb}(i^+(t_1, t_2)) < \frac{1}{2} n^{\Sigma i^+(t_1, t_2)} \leq \frac{1}{2} n^{D(i)}(t_1, t_2) \text{ so } \prod_{t_1, t_2} 2 \cdot \text{comb}(i^+(t_1, t_2)) < n^{D(i)}.$$ 

On the other hand, Lemma 5.11 with $s = \binom{n}{2}$ gives

$$\sum_{i = \sum_{t_1, t_2} i(t_1, t_2)} 1 < \binom{n}{2}^{D(i)}$$

and therefore

$$|A_i^1| < n^{D(i)} \binom{n}{2}^{D(i)} < n^{3D(i)}.$$ 

Similarly, if we denote by $z^{i(j)} h^{-\Sigma i(j)}$ the term which we pick from the $j$th term of $A^2$ then

$$|A_i^2| \leq \sum_{j=1}^n \prod_{i(j)=i} \text{coeff}_{z^{i(j)}} \left( 1 - \frac{z_{j_1} + \cdots + z_{j_{m+1}}}{z_{j}} + \left( \frac{z_{j_1} + \cdots + z_{j_{m+1}}}{z_{j}} \right)^2 - \ldots \right)^{n+2}$$

$$\leq \sum_{j=1}^n \prod_{i(j)=i} \sum_{s_1(j) + \cdots + s_{m+2}(j) = i(j)} \prod_{1 \leq j \leq n, 1 \leq m \leq n+2} \text{comb}(s_m^+(j)).$$

Since $\Sigma i = 0$, we do not have $h$ in the numerator and therefore $s_1(j) = 0$ for any $j$. Lemma 5.11 gives again

$$\sum_{j=1}^n \sum_{i(j)=i} 1 < ((n-1)(n+1))^{D(i)},$$

whereas for $s_m^+(j) \neq 0$ we have $\text{comb}(s_m^+(j)) < \frac{1}{2} n^{D(s_m(j))}$ as before, giving us

$$|A_i^2| < ((n-1)n(n+1))^{D(i)} < n^{3D(i)}$$

which proves Lemma 5.12. □
Substituting inequalities (5.7) and (5.8) into (5.5) and using Lemma 5.11 we get
\[
\sum_{\Sigma i = 0}^{n^2} B_{\Sigma i} A_{\Sigma i - 1} (d h)^n < \sum_{i=1}^{n^2} \sum_{i \neq 0, \Sigma i = 0, i \in \Lambda^+ \cap \mathbb{N}} \left( \frac{2}{n^5} \right)^i B_0 = \sum_{i=1}^{n^2} \left( \frac{2}{n^5} \right)^i B_0 \sum_{i \neq 0, \Sigma i = 0, i \in \Lambda^+ \cap \mathbb{N}} 1
\]
\[
< \sum_{i=1}^{n^2} \left( \frac{2}{n^5} \right)^i n^i B_0 < \frac{1}{4} B_0.
\]

(5.9)

We can handle the second sum of the right-hand side in (5.5) in a similar fashion. For \( \Sigma i = -1\), and \( e_j = (0, \ldots, 1^j, \ldots, 0) \) the \( j \)-th coordinate vector we have

\[
A_{\Sigma i - 1} (d h)^n = \sum_{j_2=1}^{n} \sum_{j_1 \leq j_2} \sum_{e_{j_1} = -1 - e_{j_1}}^{e_{j_1} = 0} \sum_{i_1, i_2 \in \Lambda^+} A_{i_1}^1 A_{i_2}^2
\]

holds because we have to sum over all terms coming from \( A^0 \) in (5.4). So applying Lemmas 5.11 and 5.12 again, we get

\[
|A_{\Sigma i - 1} (d h)^n| < \sum_{j=1}^{n} \sum_{j_1 \leq j_2} 2^{-D(i+e_{j_1})} n^{-3D(i+e_{j_1})} < \sum_{i \leq j \leq n} 2^{-D(i)+n+1-j} n^{-1-3D(i)+n+1-j}.
\]

Then, similar to (5.7), for \( \Sigma i = -1\)

\[
B_{\Sigma i} (d h) = \left( n^2 - 1 \right)^{i_1 + n, \ldots, i_n + n} a_1^{i_1 + n} \cdots a_n^{i_n + n} < n^{8D(i)} B_0
\]

(5.10)

and therefore by the first part of Lemma 5.11 we get

\[
\left| \sum_{\Sigma i = -1}^{\infty} B_{\Sigma i} (d h) A_{\Sigma i - 1} (d h)^n \right| < \sum_{\Sigma i = -1}^{\infty} \sum_{i \leq j \leq n} 2^{-D(i)+n+1-j} n^{5D(i)-3n+3+j-2} B_0
\]

\[
< \frac{1}{n} \sum_{i=1}^{\infty} \sum_{D(i) = -1, \Sigma i = 0} \left( \frac{n^3}{2} \right)^{-i} B_0 < \frac{1}{n} \sum_{i=1}^{\infty} \left( \frac{2}{n^5} \right)^i B_0 < \frac{1}{4n^2} B_0.
\]

(5.11)

Since \( \delta = \frac{1}{2\pi n^e} \) and \( a_i = n^{6(n+1-i)} \), we have \( \delta|a| < 1 \) so substituting (5.9) and (5.11) into (5.5) we get

\[
p_n > \frac{1}{2} B_0 > 0
\]

(5.12)

proving the first statement of Proposition 5.9.

5.3. Estimation of the coefficients \( p_{n-l}(n, a, \delta) \)

In this subsection we study the coefficients \( p_{n-l}(n, a, \delta) = \text{coeff}_{d_n+1-l} I(n, a, \delta, d) \) for \( 1 \leq l \leq n \) to prove the second part of Proposition 5.9. From (5.4)
\[ p_{n-l}(n, a, \delta) = \sum_{s=0}^{l} \sum_{\Sigma i = -s} B_{z^i h^s} A_{z^{-i-1} h^{l-s}(dh)^{n-l}} - n^2 \delta [a] \sum_{s=1}^{l+1} \sum_{\Sigma i = -s-1} B_{z^i h^s (dh)} A_{z^{-i-1} h^{l-s} (dh)^{n-l-1}}. \]  

(5.13)

**Lemma 5.13.** Let \( a_i = n^{8(n+1-i)}, \delta = 1/2n^8 \). The dominant term in (5.13) is \( B_{z^i h^s} A_{z^{-i-1} (dh)^{n-l}} \) where

\[ i(l) = (0, \ldots, 0, -1, \ldots, -1) = -e_{n-l+1} - \cdots - e_n, \]

(5.14)

that is, the sum of the other terms in (5.13) is smaller than half of this dominant term and hence

\[ |p_{n-l}(n, a, \delta)| < \frac{3}{2} |B_{z^i h^s} A_{z^{-i-1} (dh)^{n-l}}|. \]

We devote the rest of this section to the proof of this lemma. We start with studying terms of the first sum in (5.13). For \( \Sigma i = -s \)

\[ B_{z^i h^s} = \left( \binom{n^2}{s, i_1 + n_1, \ldots, i_n + n} + \binom{n^2 - 1}{s - 1, i_1 + n_1, \ldots, i_n + n} \right) (2|a|)^s a_i^{i_1+n} \]

and therefore

\[ |B_{z^i h^s}| < 2n^2 \left( \binom{n^2}{s, i_1 + n_1, \ldots, i_n + n} (2|a|)^s a_i^{i_1+n} \ldots a_n^{i_n+n}. \]

(5.16)

Note that many of the terms in (5.13) vanish, because \( A_{z^{-i-1} h^{l-s} (dh)^{n-l}} = 0 \) unless \(-i - 1 \in \Lambda^+\).

Using (5.16) and the closed form for \( B_0 \) in (5.6) we can estimate from above this dominant term as

\[ |B_{z^i h^s} A_{z^{-i-1} (dh)^{n-l}}| < 2n^2 \left( \binom{n^2}{l, n-1, \ldots, n-1, n_1, \ldots, n_{n-l}} (2|a|)^l a_1^{i_1} \ldots a_n^{i_n} a_{n-l+1}^{n-l} \ldots a_n^{n-1} \right) \]

\[ < 2n^2 (2|a|)^l n^{-d(l+1)} B_0 < n^{8ln} B_0. \]

(5.17)

When \( i \neq i(l) \) the right-hand side of (5.16) can be estimated using the trivial inequality between multinomial coefficients:

\[ |B_{z^i h^s}| < n^{8D(i-i(l))} (2|a|)^{s-l} B_{z^i h^l}. \]

(5.18)

Since \( A_{z^{-i-1} h^{l-s} (dh)^{n-l}} = 0 \) if \(-i - 1 \not\in \Lambda^+\), for the nonvanishing terms \(-i - 1 \in \Lambda^+\) must hold and therefore \( D(i(l) - i) = D(e_1 + \cdots + e_{n-l} - i - 1) \geq 0 \). On the other hand, by (5.4) for \( \Sigma i = -s \)

\[ A_{z^{-i-1} h^{l-s} (dh)^{n-l}} = \sum_{1 \leq j_1 < \cdots < j_l \leq n} \sum_{m_1 \leq j_1, \ldots, m_l \leq j_l} \sum_{1 \leq i_1 + i_2 = -1 \leq m_1 \ldots \leq m_l} A_{z^{i_1} A_{z^{i_2 h^{l-s}}}, \}

(5.19)

where in this summation we pick \( z_{m_i}/z_{j_i} \) from the \( j_i \)th term of \( A_0 \), and \( dh/z_s \) from the \( s \)th term if \( s \not\in \{j_1, \ldots, j_l\} \). Note that \( \Sigma i_1 = 0 \) and \( \Sigma i_2 = s - l \), otherwise the corresponding coefficients are zero.
Lemma 5.14. Let $\Sigma i = -s$. Then
\begin{equation}
|A_{x^{-1}h^{-s}(dh)^{n-l}}|^2 < n^{3D(i) + s}.
\end{equation}

Proof. The proof is analogous to the proof of Lemma 5.12: we first allocate $s$ factors in the denominator of $z^1$ and pair all of them with $h$ in the numerator; we can choose these $s$ factors less than $n^s$ different ways. Then repeat the argument in the proof of Lemma 5.12. \hfill \Box

Applying Lemmas 5.12 and 5.14 we get the following upper bound:
\begin{equation}
|A_{x^{-1}h^{-s}(dh)^{n-l}}| < \sum_{1 \leq j_1 < \cdots < j_l \leq n} \sum_{m_1 \leq j_1, \ldots, m_l \leq j_l} \sum_{i_1, i_2 \in \Lambda^+} n^{-3(D(i) + (n+1-m_1) + \cdots + (n+1-m_l) + l - s)}
\end{equation}
\begin{equation}
< \sum_{1 \leq m_1 < \cdots < m_l \leq n} \sum_{i_1, i_2 \in \Lambda^+} (n+1-m_1)(n-m_{l-1}) \cdots (n-l+2-m_1)n^{-3(D(i) + (n+1-m_1) + \cdots + (n+1-m_l) + l - s)}
\end{equation}
\begin{equation}
< \sum_{1 \leq m_1 < \cdots < m_l \leq n} \sum_{i_1, i_2 \in \Lambda^+} n^{-3D(i-i(l)) + l - s},
\end{equation}
where we used the following inequality for $1 \leq m_1 < \cdots < m_l \leq n$:
\begin{equation}
(n+1-m_1)(n-m_{l-1}) \cdots (n-l+2-m_1)n^{-(n+1-m_1) - \cdots - (n+1-m_l)} \leq n^{l-l + (l-1) - \cdots - 1} = n^{D(i(l))}.
\end{equation}

Applying Lemma 5.11 again we get
\begin{equation}
|A_{x^{-1}h^{-s}(dh)^{n-l}}| < \sum_{1 \leq m_1 < \cdots < m_l \leq n} \sum_{1+e_{m_1} + \cdots + e_{m_l} \in \Lambda^-} 2^{-D(i) - (n+1-m_1) - \cdots - (n+1-m_l)} n^{3(D(i(i(l))-1) + l - s)}
\end{equation}
\begin{equation}
< \sum_{1 \leq m_1 < \cdots < m_l \leq n} \sum_{1+e_{m_1} + \cdots + e_{m_l} \in \Lambda^-} 2^{D(i(i(l))-1)} n^{3(D(i(i(l))-1) + l - s)},
\end{equation}
where, again, $(n+1-m_1) + \cdots + n+1-m_l > 1+2+\cdots+l = -D(i(l))$. Putting (5.18) and (5.20) together we can estimate the first sum in (5.13) as follows:

\begin{equation}
\sum_{s=0}^{l} \sum_{\sum i = -s \atop i \neq i(l)} B_{x^h} \sum_{s=0}^{l} B_{x^h} A_{x^{-1}h^{-s}(dh)^{n-l}}< \sum_{s=0}^{l} \sum_{1 \leq m_1 < \cdots < m_l \leq n} \sum_{1+e_{m_1} + \cdots + e_{m_l} \in \Lambda^-} 2^{D(i(i(l))-1)} n^{D(i(i(l))-1) + l - s}(2|a|)^{s-l} B_{x^h} A_{x^{-1}h^{-s}(dh)^{n-l}}.
\end{equation}
To summarise our results, since 

\[
\sum_{i=0}^{l} \sum_{\substack{1 \leq m_1 < \cdots < m_l \leq n \\
i + e_{m_1} + \cdots + e_{m_l} \in \Lambda^-}} n^{-4D(i(l)-i)} n^{(8n-1)(s-l)} B_{z^{i(l)}h^{l}}.
\]

(5.21)

Observe that

for \( \Sigma = -l \) we have

\[
i + e_{m_1} + \cdots + e_{m_l} \in \Lambda^- \Rightarrow D(i + e_{m_1} + \cdots + e_{m_l}) = D(i) + (n + 1 - m_1)
\]

\[
\cdots + (n + 1 - m_l) \leq 0 \Rightarrow D(i - i(l)) \leq (m_1 + l - n - 1) + (m_2 + l - n - 2)
\]

+ \cdots + (m_l - n).

Therefore using the temporary notation \( r_i = m_i + l - n - i \leq 0 \), we get

\[
\{1 \leq m_1 < \cdots < m_l \leq n : i + e_{m_1} + \cdots + e_{m_l} \in \Lambda^- \} = \{r_1, \ldots, r_l \leq 0 : r_1 + \cdots + r_l > D(i - i(l)) \} = lD(i(l)-i)\]

(5.22)

for \( \Sigma = -s > -l \), clearly

\[
\{1 \leq m_1 < \cdots < m_l \leq n : i + e_{m_1} + \cdots + e_{m_l} \in \Lambda^- \} \leq \nu \{1 \leq m_1 < \cdots < m_l \leq n : \underbrace{i - e_n - \cdots - e_{n-l+s+1} + e_{m_1} + \cdots + e_{m_l}}_{\Sigma = -l} \leq m_1 + l - n - 1 \}
\]

Substituting these into (5.21) we get

\[
\left| \sum_{s=0}^{l} \sum_{\substack{1 \leq m_1 < \cdots < m_l \leq n \\
i + e_{m_1} + \cdots + e_{m_l} \in \Lambda^-}} B_{z^{i(l)}h^{l}} A_{z^{i(l)}h^{l}} (dh)^{n-l} \right|
\]

\[
< \sum_{s=0}^{l} \sum_{\substack{1 \leq m_1 < \cdots < m_l \leq n \\
i + e_{m_1} + \cdots + e_{m_l} \in \Lambda^-}} lD(i(l)-i) + 1 + \cdots + (l-s) n^{-4D(i(l)-i)} n^{(8n-1)(s-l)} B_{z^{i(l)}h^{l}}.
\]

\[
< \sum_{s=0}^{l} \sum_{m=1}^{\infty} \sum_{\substack{1 \leq m_1 < \cdots < m_l \leq n \\
i + e_{m_1} + \cdots + e_{m_l} \in \Lambda^-}} n^{-3m + (7n-1)(s-l)} B_{z^{i(l)}h^{l}} < \sum_{s=0}^{l} \sum_{m=1}^{\infty} n^{-2m + (7n-1)(s-l)} B_{z^{i(l)}h^{l}}
\]

\[
< \sum_{s=0}^{l} \frac{1}{8} n^{(7n-1)(s-l)} B_{z^{i(l)}h^{l}} < \frac{1}{4} B_{z^{i(l)}h^{l}}.
\]

To summarise our results, since \( A_{z^{i(l)}h^{l}} (dh)^{n-l} = 1 \), we get

\[
\left| \sum_{s=0}^{l} \sum_{\substack{1 \leq m_1 < \cdots < m_l \leq n \\
i + e_{m_1} + \cdots + e_{m_l} \in \Lambda^-}} B_{z^{i(l)}h^{l}} A_{z^{i(l)}h^{l}} (dh)^{n-l} \right| < \frac{1}{4} B_{z^{i(l)}h^{l}} A_{z^{i(l)}h^{l}} (dh)^{n-l}.
\]

(5.23)
The analogous computation for the second sum in (5.13) shows that for $\delta = 1/2n^8n$, $a_i = n^{8(n+1−i)}$ ($i = 1, \ldots, n$) we have

$$n^2\delta^{|\alpha|} \sum_{s=1}^{l+1} \sum_{\Sigma = s-1} B_{z_i} h^*(dh_j) A_{z_i-r} h^*(dh)_{n-1} < 3 B_{z_i} h^*(dh)_0 < \frac{3}{2} n^{8n} B_0 < 3n^{8n} |p_n|.$$  

(5.24)

Then (5.23) and (5.24) gives Lemma 5.13. Combined with (5.17) and (5.12) we get

$$|p_{n−l}| < \frac{3}{2} B_{z_i} h^*(dh_j) A_{z_i-r} h^*(dh)_{n-1} < \frac{3}{2} n^{8n} B_0 < 3n^{8n} |p_n|$$

which proves Proposition 5.9. This together with the Fujiwara bound in Lemma 5.10 proves Theorem 5.7. The Morse inequalities in Theorem 5.4 then give Theorem 5.3 and finally Theorems 5.2 and 5.3 together give Theorem 1.2.

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