Optimal common resource in majorization-based resource theories

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We address the problem of finding the optimal common resource for an arbitrary family of target states in quantum resource theories based on majorization, that is, theories whose conversion law between resources is determined by a majorization relationship, such as it happens with entanglement, coherence or purity. We provide a conclusive answer to this problem by proving that the majorization lattice is complete. The proof relies heavily on the more geometric construction provided by the Lorenz curves. Our framework includes the case of possibly non-denumerable sets of target states (i.e., target sets described by continuous parameters). In addition, we show that a notion of approximate majorization, which has recently found application in quantum thermodynamics, is in close relation with the completeness of this lattice.

Keywords: quantum resource theories, majorization lattice, optimal common resource

1 Introduction

Quantum resource theories (QRTs) are a very general and powerful framework for studying different phenomena in quantum theory from an operational point of view (see Ref. \cite{1} for a recent review of the topic). Indeed, all QRTs are built from three basic components: free states, free operations and resources. These components are not independent among each other, and they are defined in a way that depends on the physical properties that one wants to describe. In general, for a given QRT, one defines the set of free states $\mathcal{F}$, formed by those states that can be generated without too much effort. Then, an operation $\mathcal{E}$ is said to be free, if it satisfies the condition of mapping free states into free states: $\rho \rightarrow \mathcal{E}(\rho)$ $\equiv$ $\rho \in \mathcal{F} \forall \rho \in \mathcal{F}$. Thus, free operations can be interpreted as the ones that are easy to implement in the lab. Finally, quantum resources are defined as those states that do not belong to the set of free states (i.e., $\rho \notin \mathcal{F}$). These states are the useful ones for doing the corresponding quantum tasks.

Clearly, it is not possible to convert free states into resources by appealing to free operations alone. This is the reason why the term resource theory was coined. In fact, one of the main concerns of the QRTs is the characterization of transformations between resources by means of free operations. Here, we are focused on QRTs for which these transformations are fully characterized by a kind of majorization law between the resources. Precisely, we are interested in QRTs for which $\rho \rightarrow \mathcal{E}(\rho)$ $\equiv$ $x(\rho) \geq x(\sigma)$ or $x(\sigma) \geq x(\rho)$, where $x(\rho)$ and $x(\sigma)$ are probability vectors associated to $\rho$ and $\sigma$, respectively, and $\geq$ means a majorization relation (see e.g. \cite{2} for an introduction to majorization theory). In addition to the characterization of the convertibility of free states by means of free operations \cite{3-8}, majorization theory has been applied to different problems in quantum information such as entanglement criteria \cite{9,10}, majorization uncertainty relations \cite{11,12}, quantum entropies \cite{16,18} and quantum algorithms \cite{19}, among others \cite{20,21,24,25}.

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We restrict to QRTs based on majorization mainly for two reasons. As we have already mentioned, there are several examples of QRTs that satisfy a majorization law (see Tab. 1 and Refs. [3–8]). Thus, the results obtained which are based in the properties of majorization are of great generality, providing a unifying framework for several physical problems. On the other hand, majorization induces a lattice structure [25]. We will show that this allows to introduce the notion of optimal common resource in a very natural way. Before doing that, we stress that the lattice theoretical aspects of majorization theory have not been sufficiently exploited in comparison with other features of it in the area of quantum information. Indeed, the first applications were given in Refs. [26, 27]; only recently, new applications of the majorization lattice have been found [28–34].

Here, we aim to address the following problem. Let us suppose that one wants to have a set of target resources $T$. For obvious practical reasons, it is very useful to find a resource $\rho$, such that it can be converted by means of free operations to any other resource belonging to the target set, that is, $\rho \rightarrow \sigma$ for all $\sigma \in T$. Clearly, the maximal resource (if it exists), by definition, has to perform this task for any QRT and target set. But a more interesting question is whether there exists a state that can carry out the same task, but using the least amount of resources as possible. More precisely, one aims to find a resource $\rho^{ocr}$ such that $\rho^{ocr} \rightarrow \sigma$ for all $\sigma \in T$, and if $\rho$ is another resource satisfying $\rho \rightarrow \sigma$ for all $\sigma \in T$, then either $\rho \rightarrow r^{ocr}$ or $\rho \not\rightarrow r^{ocr}$. If this state exists, we refer to it as the optimal common resource (ocr). In this work, we provide a solution for the problem of finding the optimal common resources for arbitrary target sets of all QRTs based on majorization. This problem was already posed and (partially) solved in Ref. [34], for possibly infinite (but denumerable) target sets of bipartite pure entangled states. Let us stress that our proposal is a twofold extension of that previous work. In the first place, we provide a unifying framework for arbitrary QRTs based on majorization, which includes not only entanglement resource theory, but also the important cases of coherence and purity resource theories. In the second place, we consider the most general case of possibly non-denumerable sets of target resources. This is a powerful extension of previous works, because it allows to apply this technique to target sets which are described by a continuous family of parameters. We provide the answer to this general problem by appealing to the completeness of the majorization lattice. This is an interesting order-theoretic property in itself that, as far as we know, no formal proof was provided in the literature up to now. Thus, we close this gap by giving a proof that relies on the geometrical properties of Lorenz curves associated to target sets.

2 Results

2.1 Majorization lattice

Here, we introduce the majorization lattice and present one of the main results of this work, that is, the completeness property of the majorization lattice.

Let us consider probability vectors whose entries are sorted in non-increasing order, that is,
vectors belonging to the set:

\[
\Delta_d^\downarrow \equiv \left\{ [x_1, \ldots, x_d] : x_i \geq x_{i+1} \geq 0 \text{ and } \sum_{i=1}^d x_i = 1 \right\}.
\] (1)

Geometrically, this set is a convex polytope embedded in the \(d-1\)-probability simplex.

Let us now introduce the notion of majorization between probability vectors (see, e.g. [2]).

**Definition 1.** For given \(x, y \in \Delta_d^\downarrow\), it is said that \(x\) majorizes \(y\), denoted as \(x \succeq y\), if and only if,

\[
\sum_{i=1}^k x_i \geq \sum_{i=1}^k y_i \quad \forall k = 1, \ldots, d-1.
\] (2)

Notice that \(\sum_{i=1}^d x_k = \sum_{i=1}^d y_k\) is trivially satisfied, because \(x\) and \(y\) are probability vectors (so we can discard this condition from the definition of majorization).

The intuitive idea of majorization is that a probability distribution majorizes another one, whenever the former is more concentrated than the latter. In this sense, majorization provides a quantification of the notion of non-uniformity. To fix ideas, let us observe that any probability vector \(x \in \Delta_d^\downarrow\) trivially satisfies the majorization relations: \(e_d \equiv [1, 0, \ldots, 0] \succeq x \succeq \frac{1}{\text{rank} x} \cdot [1, \ldots, 1, 0, \ldots, 0] \succeq [\frac{1}{d}, \ldots, \frac{1}{d}] \equiv u_d\), where \(\text{rank} x\) is the number of positives entries of \(x\), and \(e_d\) and \(u_d\) are the extreme \(d\)-dimensional probability vectors in the sense of maximum non-uniformity (\(e_d\)) and minimum non-uniformity (\(u_d\), i.e. the uniform probability vector), respectively. Let us remark that there are several equivalent definitions of majorization that connect it with the notions of double stochastic matrices, Schur-concave functions and entropies, among others (see e.g. [2]).

Here we are interested in the order-theoretic properties of majorization. Indeed, it can be shown that the set \(\Delta_d^\downarrow\) together with the majorization relation is a **partially ordered set** (POSET, see e.g. [35] for an introduction to order theory). This means that that, for every \(x, y, z \in \Delta_d^\downarrow\) one has

(i) reflexivity: \(x \succeq x\),

(ii) antisymmetry: \(x \succeq y\) and \(y \succeq x\), then \(x = y\), and

(iii) transitivity: \(x \succeq y\) and \(y \succeq z\), then \(x \succeq z\).

Notice that if one leaves the constraint that the entries of the probability vectors are sorted in non-increasing order, then condition (ii) is not valid in general. Instead of this, a weaker version holds, where \(x\) and \(y\) differ only by a permutation of its entries. In such case, majorization gives a preorder because condition (i) and (iii) remain valid.

In general, majorization does not yields a total order for probability vectors belonging to \(\Delta_d^\downarrow\). This is because there exist \(x, y \in \Delta_d^\downarrow\) such that \(x \not\succeq y\) and \(y \not\succeq x\) for any \(d > 2\). In this situation, we say that the probability vectors are incomparable. For instance, it is straightforward to check that \(x = [0.6, 0.16, 0.16, 0.08]\) and \(y = [0.5, 0.3, 0.1, 0.1]\) are incomparable.

There is a visual way to address majorization that consists in appealing to the notion of Lorenz curve [35]. More precisely, for a given \(x \in \Delta_d^\downarrow\) one introduces the set of points \(\left\{(k, \sum_{i=1}^k x_i)\right\}_{k=0}^d\) (with the convention \((0, 0)\) for \(k = 0\)). Then, the Lorenz curve of \(x\), say \(L_x(\omega)\) with \(\omega \in [0, d]\), is obtained by the linear interpolation of these points. At the end, one obtains a non-decreasing and concave polygonal curve from \((0, 0)\) to \((d, 1)\). In this way, given two Lorenz curves of \(x\) and \(y\), if the Lorenz curve of \(x\) is greater than one of \(y\), it implies that \(x\) majorizes \(y\), and viceversa. On the other hand, if the Lorenz curves intersect at least at one point (in addition to the points \((0, 0)\) and
(d, 1), it means that x and y are incomparable. See for example Fig. 1 where the Lorenz curve of e₄, u₄, x = [0.6, 0.16, 0.16, 0.08] and y = [0.5, 0.3, 0.1, 0.1] are plotted. It is clear that e₄ ⪰ x ⪰ u₄ and e₄ ⪰ y ⪰ u₄, but x ⋈ y and y ⋈ x. However, in such case, one can easily realize that there are infinite Lorenz curves below the ones of x and y, and among of all them, there is one which is the greatest one. In the same vein, there are infinitely many Lorenz curves above those of x and y, and there is one which is the lowest one.

These intuitions can be formalized and allow to formulate a notion of infimum and supremum in the general case [25]. Consequently, the definition of majorization lattice is introduced as follows:

**Definition 2.** The quadruple \( L = \langle \Delta_d^\downarrow, \succeq, e_d, u_d \rangle \) defines a bounded lattice order structure, where \( e_d \) is the top element, \( u_d \) is the bottom element and for all \( x, y \in \Delta_d^\downarrow \) the infimum \( x \wedge y \) and the supremum \( x \vee y \) are expressed as in [25] (or see below).

Precisely, the components of the infimum are given by iteration of the formula

\[
(x \wedge y)_k = \min \left\{ \sum_{i=1}^{k} x_i, \sum_{i=1}^{k} y_i \right\} - \min \left\{ \sum_{i=1}^{k-1} x_i, \sum_{i=1}^{k-1} y_i \right\},
\]

for \( k = 1, \ldots, d \) and the convention that summations with the upper index smaller than the lower index are equal to zero. For the supremum, one has to proceed in two steps. First, one has to calculate the probability vector, say \( z \), with components given by

\[
z_k = \max \left\{ \sum_{i=1}^{k} x_i, \sum_{i=1}^{k} y_i \right\} - \max \left\{ \sum_{i=1}^{k-1} x_i, \sum_{i=1}^{k-1} y_i \right\},
\]

In general, this vector does not belong to \( \Delta_d^\uparrow \), because its components are not in a decreasing order. If it is the case that \( z \in \Delta_d^\uparrow \), then \( z = x \vee y \). Otherwise, one has to apply the flatness process (see [25] Lemma 3) in order to get the supremum, as follows. For a probability vector

![Fig. 1: Lorenz curves of e₄ (black), u₄ (gray), x = [0.6, 0.16, 0.16, 0.08] (red) and y = [0.5, 0.3, 0.1, 0.1] (blue). (a) Among all Lorenz curves below the ones of x and y, there exists the greatest Lorenz curve that corresponds to the probability vector x ∧ y = [0.5, 0.26, 0.14, 0.1] (green). (b) Among all Lorenz curves above the ones of x and y, there exists the lowest Lorenz curve that corresponds to the probability vector x ∨ y = [0.6, 0.2, 0.12, 0.08] (cyan).](image-url)
\( w = [w_1, \ldots, w_d]^t \), let \( j \) be the smallest integer in \([2, d]\) such that \( w_j > w_{j-1} \) and let \( k \) be the greatest integer in \([1, j-1]\) such that
\[
 w_{k-1} \geq \frac{\sum_{l=k}^{j} w_l}{j-k+1} = a, \tag{5}
\]
with \( w_0 > 1 \). Then, a flatness probability vector \( w' \) is given by
\[
 w'_l = \begin{cases} 
 a & \text{for } l = k, k+1, \ldots, j \\
 w_l & \text{otherwise} 
\end{cases} \tag{6}
\]
Then, the supremum is obtained in no more than \( d-1 \) iterations, by iteratively applying the above transformations with the input probability vector \( z \) given by (4), until one obtains a probability vector in \( \Delta^d \).

Let us consider a finite set of probability vectors, that is, \( \mathcal{P} = \{x^1, \ldots, x^N\} \) with \( x^i \in \Delta^d \). By appealing to the algebraic properties of the definition of lattice, it is straightforward to show that the infimum and the supremum of \( \mathcal{P} \) always exist, and are given by \( \bigwedge \mathcal{P} = x^1 \land x^2 \land \ldots \land x^N \) and \( \bigvee \mathcal{P} = x^1 \lor x^2 \lor \ldots \lor x^N \). However, if one considers an arbitrary set of probability vectors (which could be infinite), the lattice properties are not strong enough to guarantee the existence of infimum and supremum. If the infimum and supremum exist for arbitrary families, the lattice is said to be complete. Here, we show that the majorization lattice is indeed complete.

**Proposition 1.** Let \( \mathcal{P} \) an arbitrary set of probability vectors such that \( \mathcal{P} \subseteq \Delta^d \). Then, there exist the infimum \( x^{\inf} = \bigwedge \mathcal{P} \) and the supremum \( x^{\sup} = \bigvee \mathcal{P} \) of \( \mathcal{P} \).

In addition, the components of the \( x^{\inf} \) are given by
\[
 x_k^{\inf} = \inf \{S_k\} - \inf \{S_{k-1}\}, \tag{7}
\]
where \( S_k = \{S_k(x) : x \in \mathcal{P}\} \) with \( S_k(x) = \sum_{i=1}^{k} x_i \) for \( k \in \{1, \ldots, d\} \) and \( S_0(x) \equiv 0 \).

On the other hand, to obtain the components of the \( x^{\sup} \), we have first to define the probability vector with components given by
\[
 \bar{x}_k = \sup \{S_k\} - \sup \{S_{k-1}\}. \tag{8}
\]
Then, we compute the upper envelope of the polygonal given by the linear interpolation of the points \( \{(k, S_k(x))\}_{k=0}^{d} \), say \( \bar{L}(\omega) \), by using the Algorithm [1]. Finally, the components of the supremum are given by:
\[
 x_k^{\sup} = \bar{L}(k) - \bar{L}(k-1). \tag{9}
\]

The proof of Proposition 1 is given in [A]. Clearly, when the set is given by two probability vectors in \( \Delta^d \), that is \( \mathcal{P} = \{x, y\} \), the calculus of infimum and supremum of the Proposition 1 reduces to the procedure given in Ref. [25] (see Eqs. (3)–(6)).

### 2.2 Optimal common resource

Now, we are ready to apply the above Proposition to the problem of finding the optimal common resource in QRTs based on majorization.

In the first place, we have to distinguish between two possible cases of QRTs based on majorization. We call direct majorization-based QRTs to those QRTs such that \( \rho \rightarrow \sigma \) iff \( x(\rho) \preceq x(\sigma) \), whereas we call reversed majorization-based QRTs to those that reverse the majorization relation.
(that is, \( \rho \rightarrow \sigma \) iff \( x(\sigma) \geq x(\rho) \)). Notice that entanglement and coherence are of the former type, whereas purity is of the latter one (see Tab. [I]).

Let us consider an arbitrary set of target resources \( \mathcal{T} \). We show now that the problem of finding the optimal common resource of a QRT based on majorization, can be reduced to an application of the completeness of the majorization lattice. Indeed, by directly applying Proposition [I] one finds that the optimal common resource for direct majorization-based QRTs is the supremum of the target set, whereas for reversed majorization-based QRTs it is the infimum. As we have already stressed in the Introduction, this is a twofold extension of the proposal of Ref. [34].

2.3 Infimum and supremum over convex polytopes

Let us illustrate the meaning and relevance of the infimum and supremum discussed above with an interesting example. First, let us note that if \( \mathcal{P} \subseteq \Delta^4_d \) is a convex polytope, then the corresponding infimum and supremum can be computed as the infimum and supremum of the set of vertices, \( \text{vert}(\mathcal{P}) \).

**Lemma 1.** Let \( \mathcal{P} \) be a convex polytope contained in \( \Delta^4_d \), and \( \text{vert}(\mathcal{P}) \) the set of vertices, \( \text{vert}(\mathcal{P}) = \{ v^n \}_{n=1}^N \). Then, the infimum \( x^{\inf} = \bigwedge \mathcal{P} \) and the supremum \( x^{\sup} = \bigvee \mathcal{P} \) of \( \mathcal{P} \) are given by the infimum and supremum elements of \( \text{vert}(\mathcal{P}) \), namely

\[
x^{\inf} = \bigwedge \{ v^n \}_{n=1}^N \quad \text{and} \quad x^{\sup} = \bigvee \{ v^n \}_{n=1}^N.
\]  

(10)

The proof of Lemma 1 is given in [B]. Notice that, although the problem is reduced to the calculation of the infimum and supremum among the extreme points of the convex polytope, \( x^{\inf} \) and \( x^{\sup} \) do not necessarily belong to it (see e.g., Fig. 2(a)). However, we will see an interesting example where the infimum and supremum do belong to the given convex polytope (see e.g., Fig. 2(b)).

![Fig. 2: Infimum and supremum of convex polytopes in \( \Delta^4_d \) (region formed by the convex hull of e3, u3 and \( [\frac{1}{2}, \frac{1}{2}, 0] \)) for (a) \( \mathcal{P} = \{ x \in \Delta^4_d : x = p [0.5, 0.4, 0.1] + (1 - p) [0.55, 0.3, 0.15] \text{ with } p \in [0, 1] \} \) (black line), where \( \bigwedge \mathcal{P} = [0.5, 0.35, 0.15] \) (red hexagon) and \( \bigvee \mathcal{P} = [0.55, 0.35, 0.1] \) (blue square); and (b) \( \mathcal{P} = B_{0.15}([0.525, 0.35, 0.125]) \) (light gray region), where \( \bigwedge \mathcal{P} = [0.45, 0.35, 0.2] \) (red hexagon) and \( \bigvee \mathcal{P} = [0.6, 0.35, 0.05] \) (blue square).](https://example.com/fig2.png)

Let us consider the \( \ell_1 \)-norm \( \epsilon \)-ball centered in \( x^0 \in \Delta^4_d \) intersected with \( \Delta^4_d \), that is, \( B_\epsilon(x^0) = \{ x' \in \Delta^4_d : ||x' - x^0||_1 \leq \epsilon \} \), where \( ||x||_1 = \sum_{i=1}^d |x_i| \) denotes the \( \ell_1 \)-norm of a probability vector. Let us first note that \( \{ x' \in \mathbb{R}^d : ||x' - x^0||_1 \leq \epsilon \} \) is a convex polytope (see Ref. [18]). Then, \( B_\epsilon(x^0) \) is also a convex polytope, because it is the intersection of that convex polytope with \( \Delta^4_d \). Therefore,
by applying Lemma \([\text{I] } \bigwedge B_i(x^0)\) and \([\text{I] } \bigvee B_i(x^0)\) reduces to finding the infimum and supremum of the vertices of \([\text{I] } B_i(x^0)\).

Interestingly enough, our contribution in this paper can be posed in strong connection with the notion of approximate majorization \([\text{I] } \bigwedge \bigvee \text{supremum}\), which has recently found application in quantum thermodynamics \([\text{I] } \bigwedge \bigvee \text{supremum}\). More precisely, the steepest \(\epsilon\)-approximation, \([\text{I] } \bigwedge \bigvee \text{supremum}\), and the flattest \(\epsilon\)-approximation, \([\text{I] } \bigwedge \bigvee \text{supremum}\), of \([\text{I] } \bigwedge \bigvee \text{supremum}\) given in \([\text{I] } \bigwedge \bigvee \text{supremum}\) satisfy that, \([\text{I] } \bigwedge \bigvee \text{supremum}\). Using the definition of infimum and supremum of a given family, it follows that \([\text{I] } \bigwedge \bigvee \text{supremum}\) and \([\text{I] } \bigwedge \bigvee \text{supremum}\) are nothing more than the infimum and the supremum of the corresponding set, respectively, and they can be calculated only from the vertices (Lemma \([\text{I] } \bigwedge \bigvee \text{supremum}\)).

### 3 Concluding remarks

In this paper we gave a solution for the problem of finding an optimal common resource for an arbitrary family of target states of a given a QRT based on majorization like entanglement, coherence or purity (see Tab. \([\text{I] } \bigwedge \bigvee \text{supremum}\)). Our method relies on the completeness properties of the majorization lattice. Indeed, by appealing to the geometrical properties of Lorenz curves, we rigorously proved the completeness of the majorization lattice (Proposition \([\text{I] } \bigwedge \bigvee \text{supremum}\)). Furthermore, we provided concrete algorithms for computing the infimum and supremum of an arbitrary family of states. Our contribution improves previous works (e.g. \([\text{I] } \bigwedge \bigvee \text{supremum}\)), in the sense that our method works for target sets of arbitrary cardinality (i.e., we allow for possibly non-denumerable families of states).

In addition, we showed that the notion of approximate majorization is in strong connection with the property of completeness of the majorization lattice \([\text{I] } \bigwedge \bigvee \text{supremum}\). Indeed, the flattest and steepest approximations are nothing more than the infimum and supremum of the corresponding set, respectively, and they can be calculated only from their vertices (Lemma \([\text{I] } \bigwedge \bigvee \text{supremum}\)).

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### A Proof of Proposition \([\text{I] } \bigwedge \bigvee \text{supremum}\)

In this section we show that for an arbitrary set \(P\) of probability vectors (whose components are arranged in non-increasing order), there exists an infimum \(x^{\text{inf}} \equiv \bigwedge P\) and a supremum \(x^{\text{sup}} \equiv \bigvee P\), with respect to the majorization relation. Furthermore, we provide the algorithms to obtain them.

Let us first introduce some notations and definitions. Let us define the partial sum of the first \(k\) components of a given vector \(x\) as \(S_k(x) \equiv \sum_{i=1}^{k} x_i\) with the convention \(S_0(x) \equiv 0\). Now, let us consider the set formed by all partial sums up to \(k\) that come from probability vectors in \(P\), that is, \(S_k = \{S_k(x) : x \in P\}\) and its infimum \(S_k \equiv \inf S_k\) and supremum \(\bar{S}_k \equiv \sup S_k\). Notice that, for each \(k = 0, \ldots, d\), both \(S_k\) and \(\bar{S}_k\) exist, since each \(S_k\) is a set of real numbers bounded from below by \(\frac{1}{k}\) and above by 1. Finally, let us consider the probability vectors \(x = [S_1, S_2 - S_1, \ldots, S_i - S_{i-1}, \ldots, S_d - S_{d-1}]\)
and $\bar{x} = [\bar{S}_1, \bar{S}_2 - \bar{S}_1, \ldots, \bar{S}_i - \bar{S}_{i-1}, \ldots, \bar{S}_d - \bar{S}_{d-1}]$. Let us prove that from these probability vectors one can obtain the infimum and the supremum, respectively.

Infinum

Let us now prove that $x = x^{\inf}$. To prove that, we appeal to the description of majorization in terms of Lorenz curves. First we show that the curve $L_x(\omega)$ with $\omega \in [0, d]$, formed by the linear interpolation of the points $\{(k, S_k)\}_{k=0}^d$ (notice that $S_0 = 0$ and $\bar{S}_d = 1$) is a Lorenz curve. This is equivalent to prove that $x \in \Delta_d$. We proceed in two steps: (a) $L_x(\omega)$ is non-decreasing i.e., $L_x(k) \leq L_x(k + 1)$ for all $k \in \{0, \ldots, d - 1\}$ (b) $L_x(\omega)$ is concave i.e., $L_x(k) \geq \frac{1}{2} (L_x(k - 1) + L_x(k + 1))$ for all $k \in \{1, \ldots, d - 1\}$. The proofs of both points are given by reductio ad absurdum.

Let us proceed with the proof of (a) $L_x(k) \leq L_x(k + 1)$ for all $k \in \{0, \ldots, d - 1\}$. Let us assume that there exists $k'$ such that $L_x(k') > L_x(k' + 1)$. By construction, there exists a sequence, say $\{L_{x'}(k')\}_{i \in \mathbb{N}}$ with $x' \in \mathcal{P}$, of elements of $\mathcal{S}_{k'}$, that converges to $S_{k'} = L_x(k')$. Let us choose $i'$ big enough such that $L_{x'}(k') - L_x(k' + 1) < \frac{1}{2} (L_x(k') - L_x(k' + 1))$ for all $i \leq i'$. Let us pick one of them, say $i_0$. By definition of $x$, one has $L_{x_{i_0}}(k') \geq L_x(k')$.

Finally, one has $L_{x_{i_0}}(k') > L_x(k') + \frac{1}{2} (L_x(k') - L_x(k' + 1)) \geq L_{x_{i_0}}(k' + 1)$. This is in contradiction with the fact that $L_{x_{i_0}}(k) \leq L_{x_{i_0}}(k + 1)$ for all $k \in \{0, \ldots, d - 1\}$, which is true by definition of Lorenz curve. Then, (a) holds.

Now, we proceed with the proof of (b): $L_x(k) \geq \frac{1}{2} (L_x(k - 1) + L_x(k + 1))$ for all $k \in \{1, \ldots, d - 1\}$. Assume that there exists $k'$ such that $L_x(k') < \frac{1}{2} (L_x(k' - 1) + L_x(k' + 1))$. By construction, there exists a sequence, say $\{L_{x'}(k')\}_{i \in \mathbb{N}}$ with $x' \in \mathcal{P}$, of elements of $\mathcal{S}_{k'}$, that converges to $S_{k'} = L_x(k')$. Let us choose $i'$ big enough such that $L_{x'}(k') - L_x(k' + 1) < \frac{1}{2} (L_x(k') - L_x(k' + 1))$ for all $i \geq i'$. Let us pick one of them, say $i_0$. By definition of $x$, $L_{x_{i_0}}(k' - 1) \geq L_x(k' - 1)$ and $L_{x_{i_0}}(k' + 1) \geq L_x(k' + 1)$. This implies that $L_{x_{i_0}}(k') < \frac{1}{2} (L_x(k' - 1) + L_x(k' + 1)) \leq \frac{1}{2} (L_{x_{i_0}}(k' - 1) + L_{x_{i_0}}(k' + 1))$. But this is in contradiction with the fact that $L_{x_{i_0}}(k) \geq \frac{1}{2} (L_{x_{i_0}}(k - 1) + L_{x_{i_0}}(k + 1))$ for all $k \in \{1, \ldots, d - 1\}$, which is true by definition of Lorenz curve. Then, (b) holds.

Up to now, we have proved that $L_x(\omega)$ is a Lorenz curve that, by construction, satisfies $L_x(\omega) \leq L_x(\omega)$ and $\forall x \in \mathcal{P}$. In other words, we obtain that $x \in \Delta_d$ and $x \succeq x \forall x \in \mathcal{P}$. It remains to be proved that for any $x' \in \Delta_d$ such that $x \succeq x' \forall x \in \mathcal{P}$, one has $x \succeq x'$. In order to do this, we appeal again to the reductio ad absurdum and the notion of Lorenz curve. Let us assume that there exist $x'$ such that $x \succeq x' \forall x \in \mathcal{P}$, but $x \not \succeq x'$. This happens if at least one partial sum of $x'$ is greater than the one of the $x$, say the $k'$ partial sum. In other words, $L_{x'}(k') > L_x(k')$. Choose again a sequence $\{L_{x'}(k')\}_{i \in \mathbb{N}}$ of elements of $\mathcal{S}_{k'}$ that converges to $S_{k'}$. Choose $i'$ big enough such that $L_{x'}(k') - L_x(k') + \frac{1}{2} (L_{x'}(k') - L_x(k'))$ for all $i \geq i'$. Let us pick one of them, say $i_0$, so $L_{x_{i_0}}(k') < L_x(k') + \frac{1}{2} (L_{x'}(k') - L_x(k'))$. But, by hypothesis, one has $L_{x_{i_0}}(k') \leq L_{x_{i_0}}(k)$ for all $k \in \{0, \ldots, d\}$, which is in contradiction with the previous inequality. Thus, there does not exist such $x'$. Therefore, $x = x^{\inf}$.

Supremum

Notice that, according to lattice theory, the arbitrary supremum can be expressed in terms of the arbitrary infimum, and vice versa [40]. This means that our proof of the existence of the infimum for an arbitrary set $\mathcal{P}$ of probability vectors (whose components are arranged in non-increasing order), automatically implies the existence of its supremum $x^{\sup} = \bigwedge\{x' \in \Delta_d : x' \succeq x \forall x \in \mathcal{P}\}$. With this observation we finish our proof that the majorization lattice is complete. Notice that the mere proof of the existence of a supremum, does not guarantee the existence of an algorithm to compute it. Thus, in the sequel, we focus our efforts in providing such an algorithm.

Consider the polygonal curve $L_x(\omega)$, with $\omega \in [0, d]$, formed by the linear interpolation of the points $\{(k, S_k)\}_{k=0}^d$ (notice that $S_0 = 0$ and $\bar{S}_d = 1$). By construction, $L_x(\omega)$ is non-decreasing and
satisfies that $L_{\bar{\omega}}(\omega) \geq L_{\bar{x}}(\omega)$, $\forall \omega \in [0,d]$ and $\forall x \in P$. But, alike $L_{\bar{x}}(\omega)$, $L_{\bar{\omega}}(\omega)$ is not necessarily a Lorenz curve. Thus, it cannot be used to construct the (ordered) probability vector associated to the supremum of the given family. Instead, let us show that the upper envelope of the Lorenz curve. Thus, it cannot be used to construct the (ordered) probability vector associated to the supremum:

Algorithm 1 Upper envelope

```
input: $x \in \mathbb{R}^d$
output: coordinates of the upper envelope of the polygonal curve joining $\{(k, S_k(x))\}_{k=0}^d$
procedure UPPERENV($x$)
    $K \leftarrow \{0\}$  \quad \triangleright \text{Stores the ʻcritical pointsʼ of } x
    $i \leftarrow 0$
    while $i < \text{length}(x)$ do
        $m \leftarrow \{0\}$  \quad \triangleright \text{Stores slope values}
        for $j = i+1 \ldots \text{length}(x)$ do
            $m \leftarrow \text{append} \left\{m, \frac{S_j(x) - S_i(x)}{j-i}\right\}$
        end for
        $k \leftarrow \text{max(position of max}(m))$  \quad \triangleright \text{Finds position of the last maximum slope}
        $K \leftarrow \text{append}\{K, k\}$
        $i \leftarrow k$  \quad \triangleright \text{Updates } i
    end while
    return $\{(k, S_k(x))\}_{k \in K}$  \quad \triangleright \text{Coordinates of the upper envelope}
end procedure
```

Our method to obtain the supremum $x^{sup}$ has three steps: first, we calculate $\bar{x}$; second, we compute the upper envelope of $L_{\bar{x}}(\omega)$, $\bar{L}(\omega)$; third, we compute the elements of $x^{sup}$ as the components of the probability vector associated to the Lorenz curve $\bar{L}(\omega)$. The first and last steps are straightforward. We also provide the Algorithm 1 to find the upper envelope of a polygonal curve with coordinates $\{(k, S_k(x))\}_{k=0}^d$.

Notice that for a given probability vector $\bar{x} \in \mathbb{R}^d$, the output of the Algorithm 1 is a set of points $\{(k, S_k(\bar{x}))\}_{k \in K}$. It is clear that the linear interpolation of these points is a Lorenz curve, say $L_{x^{up}}(\omega)$, which has associated some probability vector $x^{up} \in \Delta^d$. Let us show that $L_{x^{up}}(\omega)$ is equal to the upper envelope of $L_{\bar{x}}(\omega)$. To see that, take two consecutive indices, $k_i, k_{i+1} \in K$. By construction, $L_{\bar{x}}(\omega) = L_{x^{up}}(\omega)$ for $\omega = k_i$ and $\omega = k_{i+1}$. For $\omega \in [k_i, k_{i+1}]$, $L_{x^{up}}(\omega)$ is the linear interpolation and so one has two possibilities: either $k_{i+1} = k_i + 1$ and $L_{x^{up}}(\omega) = L_{x^{up}}(\omega)$ for all $\omega \in [k_i, k_{i+1}]$, or $k_{i+1} > k_i + 1$ and $L_{x^{up}}(\omega) < L_{x^{up}}(\omega)$ for some integer $\omega \in (k_i, k_{i+1})$. In both cases, since the interpolation is linear, there is no concave curve such that $L_{x^{up}}(\omega) \geq L_{x^{up}}(\omega)$ and $L_{x^{up}}(\omega) < L_{x^{up}}(\omega)$ for all $\omega \in (k_i, k_{i+1})$. Since this is case for any $k_i \in K$, $L_{x^{up}}(\omega)$, we necessarily obtain the upper envelope of the polygonal curve joining $\{(k, S_k(x))\}_{k=0}^d$. Then, we have proved that $L_{x^{up}}(\omega) = \bar{L}(\omega)$. This last equality implies in turn that (a) $\bar{L}(\omega)$ it is a Lorenz curve. As a consequence, by construction of $\bar{L}(\omega)$, we also have that $x^{up} = [\bar{L}(1), \bar{L}(2) - \bar{L}(1), \ldots, \bar{L}(i) - \bar{L}(i-1), \ldots, \bar{L}(d) - \bar{L}(d-1)]$ satisfies that $x^{up} \succeq x$ $\forall x \in P$. In addition, we have that $\not\exists x' \in \Delta^d$ such that $L_{x'}(\omega) \geq L_{\bar{x}}(\omega) \forall \omega \in [0,d]$ and
$x^{\text{up}} \succeq x'$. Therefore, (b) holds and $x^{\text{up}} = x^{\text{sup}}$.

**B Proof of Lemma 1**

We prove now that $x^{\text{inf}} \equiv \bigwedge P = \bigwedge \text{vert}(P) \equiv v^{\text{inf}}$ and $x^{\text{sup}} \equiv \bigvee P = \bigvee \text{vert}(P) \equiv v^{\text{sup}}$, that is to say that infimum and supremum can be computed among the set of vertices of the convex polytope.

Let $x$ be an arbitrary probability vector in $P \subseteq \Delta^d$. Since $P$ is a convex polytope, $x$ can be written as a convex combination of the vertices, $x = \sum_{n=1}^{N} p_n v^n$, with $v^n \in \text{vert}(P)$, $p_n \geq 0$ and $\sum_{n=1}^{N} p_n = 1$. For arbitrary $k$, the $k$-partial sum of $x$ gives

$$S_k(x) = \sum_{n} p_n S_k(v^n) \geq S_k(v^{\text{inf}}), \quad \forall k, \forall x \in P,$$

where we have used that, by definition, $v^n \succeq v^{\text{inf}}$, $\forall v^n \in \text{vert}(P)$. On the other hand, since $\text{vert}(P) \subseteq P$ and given that $x^{\text{inf}} \equiv \bigwedge P$, we know by definition of infimum that $v^{\text{inf}} \succeq x^{\text{inf}}$ must hold. Hence, using (11),

$$x \succeq v^{\text{inf}} \succeq x^{\text{inf}}, \quad \forall x \in P.$$

Therefore, by definition of infimum, one has $x^{\text{inf}} = v^{\text{inf}}$.

Analogously, for the supremum one obtains that

$$x^{\text{sup}} \succeq v^{\text{sup}} \succeq x, \quad \forall x \in P,$$

and the desired result follows as before, by definition of supremum, $x^{\text{sup}} = v^{\text{sup}}$.

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