On Strong Integrability of the Dressing Cosets

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Abstract. We formulate sufficient conditions for the strong integrability of dressing cosets. We provide several sigma-model backgrounds solving those conditions, some of them are new and some of them were not so far formulated as the dressing cosets. The new models are based on the Drinfeld doubles having the structure of higher-order jet bundles of quadratic Lie groups.

1. Introduction

The dressing cosets are particular nonlinear $\sigma$-models in two dimensions which were originally introduced in [38] in a successful attempt to generalize the so-called Poisson–Lie T-duality [35,36,48]. The first-order Hamiltonian dynamics of the dressing cosets can be described in terms of the dynamical systems referred to as degenerate $\hat{\mathcal{E}}$-models which themselves generalize the so-called non-degenerate $\mathcal{E}$-models. This generalization is substantial, nevertheless it received a relatively little attention in the literature so far although there was some older activity in the field [5,6,22,23,51,52,54,55], just like a more recent one [13,14,20,32,34,47,50].

We believe that the dressing cosets as well as the degenerate $\hat{\mathcal{E}}$-models are going to play an increasing role in future in all those places where their simpler non-degenerate counterparts already play role, like in the theory of integrable $\sigma$-models [1,2,7,10,12,15,18,25–28,45,46,53]. In particular, it is well known that many integrable $\sigma$-models on group manifolds can be in fact interpreted in terms of the non-degenerate $\mathcal{E}$-models [21,29–31,33,40]. However, in the case of the integrable models living on the coset manifold an unified interpretation in terms of the dressing cosets is so far missing. Among other things, we fill this gap, at least partially, in the present paper.

It is important to distinguish between a weak and a strong integrability of a nonlinear $\sigma$-model. The weak integrability means that the equation of
motion of the model can be written in the so-called Lax form with spectral parameter $z$, that is in the form

$$\frac{dL(z)}{d\tau} = [L(z), M(z)],$$

(1.1)

where $L(z)$ and $M(z)$ are $z$-families of matrix-valued\(^1\) functions on the phase space of the $\sigma$-model. The strong integrability then means that every two spectral invariants of the Lax matrix $L(z)$ Poisson commute, for whatever values of the spectral parameters $z$ and $w$. As we shall see later on, this means that the matrix Poisson bracket of the Lax matrix $L(z)$ with $L(w)$ is governed by the so-called $r$-matrix $r(z, w)$, which is an important characteristics of the strongly integrable model.

Sufficient conditions for the weak integrability of the non-degenerate $E$-models were first formulated in [49] and solved in many cases in [40]. They were later extended in [34] to ensure also the strong integrability. In this paper, we formulate the sufficient conditions for the strong integrability of the degenerate $\hat{E}$-models and we solve them for several classes of the dressing cosets. Some of those classes constitute a reinterpretation of the known symmetric spaces $\sigma$-models in terms of the dressing cosets while the others are completely new and are based on high-order jet bundles of quadratic Lie groups interpreted as the Drinfeld doubles. In particular, the jet bundle $J^{2n+1}G$ of a compact semi-simple Lie group $G$ gives rise to the strongly integrable dressing coset describing a suitable interaction of $n$ fields with values in the Lie algebra $\mathcal{G}$ of the group $G$.

The plan of the paper is as follows: In Sect. 2, we review the concept of the non-degenerate $E$-model as well as the sufficient conditions of its integrability (the weak as well as the strong ones). In Sect. 3.1, we review in much technical detail the concepts of the degenerate $\hat{E}$-models as well as that of the dressing cosets while in Sect. 3.2 we start with the presentation of the original results, namely, we formulate the sufficient conditions for the strong integrability of the dressing cosets. In Sect. 4, we interpret a non-deformed, a $\lambda$-deformed and an $\eta$-deformed symmetric space $\sigma$-models as the dressing cosets and we show that the well-known strong integrability of those three theories can be understood in a unified way within our degenerate $\hat{E}$-model formalism. In Sect. 5, we solve our sufficient conditions of the strong integrability for a new family of the dressing cosets which is parametrized by positive integers. Thus, for given $n$, the model describes a suitable integrable interaction of $n$ $\mathcal{G}$-valued fields. Only for $n = 1$ the resulting $\sigma$-model was known previously, it is in fact the pseudo-chiral model of Zakharov and Mikhailov [59, 60]. We determine also the so-called twist function of the generalized pseudo-chiral integrable model for every $n$. In Sect. 6, we provide conclusions and an outlook.

\(^1\)Here, the word “matrix” should be interpreted in a somewhat large sense, in fact, in the present paper $L(z)$ will rather be a differential operator in a loop variable $\sigma$. 
2. Reminder: Non-degenerate $\mathcal{E}$-Models

2.1. The First- and Second-Order Formulations

In this paper, by a Drinfeld double $D$ we shall understand a connected even-dimensional Lie group equipped with a bi-invariant pseudo-Riemannian metric of maximally Lorentzian (split) signature. An $\mathcal{E}$-model, introduced in [29,35–37,48], is a first-order Hamiltonian dynamical system $(\omega, H_\mathcal{E})$ living on the loop group $LD$ of the Drinfeld double. The symplectic form $\omega$ and the Hamiltonian $H_\mathcal{E}$ on $LD$ are, respectively, given by the formulas

$$\omega = -\frac{1}{2} \oint (l^{-1} \delta l \wedge (l^{-1} \delta l))_D,$$

$$H_\mathcal{E} = \frac{1}{2} \oint (l^{-1}, \mathcal{E} l^{-1})_D.$$

Here, $(., .)_D$ is the non-degenerate ad-invariant symmetric bilinear form defined on the Lie algebra $D$ of the Drinfeld double (it is given by the pseudo-Riemannian metric at the group origin). The integration in Eqs. (2.1), (2.2) is over the loop parameter $\sigma$, while the derivative with respect to the loop parameter is denoted by the apostrophe. The symbol $\delta$ stands for the de Rham exterior differential on the loop group $LD$, the closed string configuration $l$ parametrizes $LD$ and, finally, $\mathcal{E} : D \to D$ is an $\mathbb{R}$-linear operator squaring to identity, symmetric with respect to the bilinear form $(., .)_D$ and such that the bilinear form $(., \mathcal{E}., .)_D$ on $D$ is strictly positive definite.

It turns out, that the left action of the group $LD$ on itself is generated by the moment map $j = l'l^{-1}$. The components of the current $j$ then verify the Poisson current algebra

$$\{ (j(\sigma_1), T_1)_D, (j(\sigma_2), T_2)_D \} = (j(\sigma_1), [T_1, T_2])_D \delta(\sigma_1 - \sigma_2) + (T_1, T_2)_D \delta'(\sigma_1 - \sigma_2),$$

where $T_1, T_2$ are arbitrary elements of $D$. The current algebra (2.3) plays a crucial role in the dynamics of the $\mathcal{E}$-models. Indeed, the Poisson brackets (2.3) as well as the explicit form (2.2) of the Hamiltonian give the following first-order equations of motion of the $\mathcal{E}$-model

$$\frac{\partial j}{\partial \tau} = \{ j, H_\mathcal{E} \} = (\mathcal{E}j)' + [\mathcal{E}j, j],$$

(2.4)

Consider the $\mathcal{E}$-model on the Drinfeld double $D$ and let $K \subset D$ be a half-dimensional subgroup such that the Lie subalgebra $\mathcal{K}$ is isotropic with respect to the bilinear form $(., .)_D$. Then, it was shown in [39] that there is a two-dimensional nonlinear $\sigma$-model such that its first-order dynamics can be expressed in terms of the $\mathcal{E}$-model; in particular its first-order Hamiltonian equations of motion are given by (2.4). The action of this $\sigma$-model reads

$$S_\mathcal{E}(l) = \frac{1}{4} \int \delta^{-1} \oint \left( \delta ll^{-1}, [\partial_\sigma ll^{-1}, \delta ll^{-1}]_D \right)_D$$

$$+ \frac{1}{4} \int d\tau \oint \left( W^+_l \partial_+ ll^{-1}, \partial_- ll^{-1} \right)_D - \frac{1}{4} \int d\tau \oint \left( \partial_+ ll^{-1}, W^-_l \partial_- ll^{-1} \right)_D.$$

(2.5)
Here, \( l(\tau, \sigma) \in D \) is a field configuration, \( \partial_{\pm} = \partial_{\tau} \pm \partial_{\sigma} \), \( \delta^{-1} \) is a (symbolic) inverse of the de Rham differential and \( W^\pm \) are the projectors fully characterized by their respective kernels and images

\[
\text{Ker}(W^\pm) = Ad_l(K), \quad \text{Im}(W^\pm) = (1 \pm \mathcal{E})D.
\]  

(2.6)

Note that the quantity \( \delta^{-1} \mathcal{F} \left( \delta l l^{-1}, [\partial_{\sigma} l l^{-1}, \delta ll^{-1}]_D \right) \) is in fact the 1-form \( f(\tau) d\tau \) for some \( f(\tau) \) therefore the first integral in (2.5) is effectively over the time variable.

It may seem, that the \( \sigma \)-model (2.5) lives on the target \( D \) but, actually, it lives on the space of cosets \( D/K \) because it enjoys the gauge symmetry

\[
l(\tau, \sigma) \rightarrow l(\tau, \sigma)k(\tau, \sigma), \quad k(\tau, \sigma) \in K.
\]  

(2.7)

In some cases, the fibration \( D \rightarrow D/K \) admits a global section \( p : D/K \rightarrow D \), in which case we can fix the gauge \( l = p \) in the action (2.5).

Note also that the equations of motion of the \( \sigma \)-model (2.5) can be written as

\[
\partial_{\pm}(W^\pm_\tau \partial_{\pm} ll^{-1}) - \partial_-(W^+_\tau \partial_+ ll^{-1}) - [W^+_\tau \partial_+ ll^{-1}, W^-_\tau \partial_- ll^{-1}]_D = 0.
\]  

(2.8)

The equations of motion (2.4) and (2.8) are related by

\[
j = \frac{1}{2} W^+_\tau \partial_+ ll^{-1} - \frac{1}{2} W^-_\tau \partial_- ll^{-1}.
\]  

(2.9)

### 2.2. Integrability of the Non-degenerate \( \mathcal{E} \)-Models

Suppose that there is the \( \mathcal{E} \)-model based on the Drinfeld double \( D \) and there is also a quadratic Lie algebra \( \mathcal{G} \) together with a one-parametric family of linear operators \( O(z) : D \rightarrow \mathcal{G} \) verifying the following conditions

\[
[O(z)x_+, O(z)x_-]_{\mathcal{G}} = O(z)[x_+, x_-]_{D}, \quad x_\pm \in D, \quad \mathcal{E}x_\pm = \pm x_\pm.
\]  

(2.10)

It was shown in [40,49] that Eq. (2.10) is the sufficient condition for the existence of the Lax pair \( L(z), M(z) \) of this \( \mathcal{E} \)-model. The operators \( L(z), M(z) \) act on the loop Lie algebra \( LG \) and they are given by the formulas

\[
L(z) = \partial_\sigma - \text{ad}^G_{O(z)j},
\]  

(2.11)

\[
M(z) = -\text{ad}^G_{O(z)\mathcal{E}j}.
\]  

(2.12)

The condition (2.10) guarantees that the field equations (2.4) of the \( \mathcal{E} \)-model can be represented in the Lax form with spectral parameter [41]

\[
\{L(z), H_\mathcal{E}\} = \frac{dL(z)}{dt} = [L(z), M(z)].
\]  

(2.13)

We note, that the Poisson bracket \( \{L(z), H_\mathcal{E}\} \) is to be understood as the linear operator with the matrix elements \( \langle \psi, \{L(z), H_\mathcal{E}\} v \rangle \) which are given by the Poisson bracket of the functions \( \langle \psi, L(z)v \rangle \) and \( H_\mathcal{E} \), i.e.,

\[
\langle \psi, \{L(z), H_\mathcal{E}\} v \rangle = \{\langle \psi, L(z)v \rangle, H_\mathcal{E}\}, \quad v \in LG, \quad \psi \in (LG)^*.
\]  

(2.14)

Here, of course, \( v \in LG \) is an arbitrary constant vector and \( \psi \) an arbitrary constant one-form.
The Lax form (2.13) of the field equations guarantees the so-called weak Lax integrability, which means that the spectral invariants (typically the traces of the powers) of the operator $L(z)$ are the integrals of motion. If, however, those integrals of motions Poisson commute among each other, we say that the $\mathcal{E}$-model is strongly Lax integrable [4,43,56].

The sufficient conditions for the strong Lax integrability of $\mathcal{E}$-models were formulated in [34]. It is thus required that it exists a non-degenerate invariant symmetric bilinear form $\langle . , . \rangle_\mathcal{G}$ on $\mathcal{G}$ and there is a two-parametric family of operators $\hat{r}(z, w) : \mathcal{G} \rightarrow \mathcal{G}$ such that it holds

$$
[O^\dagger(z)x, O^\dagger(w)y]_\mathcal{D} + O^\dagger(z)[x, \hat{r}(z, w)y]_\mathcal{G}
+ O^\dagger(w)[\hat{r}(w, z)x, y]_\mathcal{G} = 0, \quad \forall x, y \in \mathcal{G},
$$

$$
(O^\dagger(z)x, O^\dagger(w)y)_\mathcal{D} + (x, \hat{r}(z, w)y)_\mathcal{G} + (\hat{r}(w, z)x, y)_\mathcal{G} = 0, \quad \forall x, y \in \mathcal{G}.
$$

(2.15)

Here, $O(z)^\dagger : \mathcal{G} \rightarrow \mathcal{D}$ is the adjoint of the operator $O(z)$; it is defined by the relation

$$
(O(z)x, y)_\mathcal{G} = (x, O^\dagger(z)y)_\mathcal{D}, \quad \forall x \in \mathcal{D}, y \in \mathcal{G}.
$$

(2.17)

To see, why (2.15) and (2.16) are the sufficient conditions of the strong Lax integrability, we use them, as well as Eq. (2.3), to calculate the Poisson brackets of the matrix elements of the Lax operator $L(z)$

$$(y', L(z)(\sigma_1)x')_\mathcal{G}, (y'', L(w)(\sigma_2)x'')_\mathcal{G} = \{(O(z)j(\sigma_1), x)_\mathcal{G}, (O(w)j(\sigma_2), y)_\mathcal{G}\}
= (j(\sigma_1), [O^\dagger(z)x, O^\dagger(w)y]_\mathcal{D})_\mathcal{G} + (O^\dagger(z)x, O^\dagger(w)y)_\mathcal{D}\partial_{\sigma_1}\delta(\sigma_1 - \sigma_2)
= -\left((O(z)j(\sigma_1), [x, \hat{r}(z, w)y])_\mathcal{G} + (O(w)j(\sigma_2), [\hat{r}(w, z)x, y])_\mathcal{G}\right)\delta(\sigma_1 - \sigma_2)
$$

$$
- \left((x, \hat{r}(z, w)y)_\mathcal{G} + (\hat{r}(w, z)x, y)_\mathcal{G}\right)\partial_{\sigma_1}\delta(\sigma_1 - \sigma_2).
$$

(2.18)

where we have set

$$
x = [y', x']_\mathcal{G}, \quad y = [y'', x'']_\mathcal{G}, \quad x', x'', y', y'' \in \mathcal{G}.
$$

(2.19)

The relation (2.18) can be rewritten in the operator form as

$$
\{L(z) \otimes \text{Id}, \text{Id} \otimes L(w)\} = [r(z, w), L(z) \otimes \text{Id}] - [r^p(w, z), \text{Id} \otimes L(w)],
$$

(2.20)

where the operator $r(z, w)$ acts on the tensor product $L\mathcal{G} \otimes L\mathcal{G}$ and it is defined as

$$
r(z, w) = C_{AB}\text{ad}_{\hat{r}(z, w)T^A} \otimes \text{ad}_{T^B}\delta(\sigma_1 - \sigma_2).
$$

(2.21)

For brevity, we have written in (2.20) $L(z)$ and $L(w)$ rather than $L(z)(\sigma_1)$ and $L(w)(\sigma_2)$. Note that $C_{AB}$ is the inverse matrix of $C^{AB}$ defined by

$$
C^{AB} := (T^A, T^B)_\mathcal{G},
$$

(2.22)

where $T^A$ is some basis of the Lie algebra $\mathcal{G}$ on the choice of which actually the $r$-matrix (2.21) does not depend. The notation $r^p$ means

$$
r^p = \sum_\alpha B_\alpha \otimes A_\alpha.
$$

(2.23)
if $r$ has the form

$$r = \sum_{\alpha} A\alpha \otimes B\alpha \quad (2.24)$$

for some family of linear operators $A\alpha, B\alpha$ acting on $L\mathcal{G}$. The spectral parameter $z$ may even take complex values in which case the crucial map $O(z)$ is considered as the map from $\mathcal{D}$ to $\mathcal{G}^C$.

By the general theory of the Lax integrable system exposed, e.g., in the book [3], the validity of the identity (2.20) in the finite-dimensional context precisely guarantees the strong integrability of the dynamical system possessing the Lax operator $L(z)$. In the infinite-dimensional context, one has to work harder to show this [16,17,44], anyway in the literature on the nonlinear $\sigma$-models the existence of the $r$-matrix satisfying (2.20) is somewhat abusively considered as the condition guaranteeing the strong integrability and we shall also stick to this terminology.

3. Degenerate $\hat{\mathcal{E}}$-Models

3.1. Reminder: Dressing Cosets

Consider again the $\mathbb{R}$-linear involution $\mathcal{E}: \mathcal{D} \to \mathcal{D}$ symmetric with respect to the bilinear form $(\cdot, \cdot)_{\mathcal{D}}$ and such that the bilinear form $(\cdot, \mathcal{E}\cdot)_{\mathcal{D}}$ on $\mathcal{D}$ is strictly positive definite. Furthermore, we suppose

1) There is an isotropic subgroup $F \subset \mathcal{D}$ such that the involution $\mathcal{E}$ commutes with the adjoint action of $F$ on $\mathcal{D}$;
2) The restriction of the bilinear form $(\cdot, \mathcal{E}\cdot)_{\mathcal{D}}$ to the Lie subalgebra $\mathcal{F}$ is strictly positive definite.

In [32], the so-called degenerate $\hat{\mathcal{E}}$-model was associated with the data just described as a slight reformulation of the so-called dressing coset introduced in [37] (in what follows, we use the both terms interchangeably). Thus, the degenerate $\hat{\mathcal{E}}$-model is the first-order Hamiltonian dynamical system obtained by an appropriate symplectic reduction of the dynamical system $(\omega, H_{\hat{\mathcal{E}}})$, where the symplectic form $\omega$ is the standard Kirillov–Drinfeld one

$$\omega = -\frac{1}{2} \oint (l^{-1} \delta l \wedge (l^{-1} \delta l)')_{\mathcal{D}}, \quad (3.1)$$

while the Hamiltonian is given by

$$H_{\hat{\mathcal{E}}}(l) = \frac{1}{2} \oint (j, \hat{\mathcal{E}}j)_{\mathcal{D}}, \quad j := ll'^{-1}. \quad (3.2)$$

Note that the dynamical system $(\omega, H_{\hat{\mathcal{E}}})$ that we are going to reduce symplectically is not the non-degenerate $\mathcal{E}$-model in the sense of the definition given in Sect. 2, in spite of the fact that the symplectic form (3.1) is the same as (2.1). This is because the linear operator $\hat{\mathcal{E}}$ appearing in the Hamiltonian (3.2) is not the same thing as $\mathcal{E}$ although it is obtained from $\mathcal{E}$ in an appropriate way. Actually, $\hat{\mathcal{E}}$ has a nontrivial kernel and, therefore, it cannot be involutive. To define $\hat{\mathcal{E}}$, we first notice that the condition 2) above means that the intersection of the Lie subalgebra $\mathcal{F} \subset \mathcal{D}$ with the linear space $\mathcal{E}\mathcal{F}$ is trivial.
(it contains just $0 \in D$), which means that the Drinfeld double Lie algebra $D$ can be represented as the direct sum of four $\text{Ad}_F$-invariant vector spaces as follows

$$D = V_+ \oplus V_- \oplus \mathcal{F} \oplus \mathcal{E} \mathcal{F}, \quad (3.3)$$

where the subspaces $V_\pm$ are defined by the conditions

$$V_+ \oplus V_- = (\mathcal{F} \oplus \mathcal{E} \mathcal{F})^\perp, \quad \mathcal{E} x_\pm = \pm x_\pm, \quad x_\pm \in V_\pm. \quad (3.4)$$

Note that the notation $\perp$ means the perpendicularity with respect to the bilinear form $(\cdot, \cdot)_D$.

Following the decomposition (3.3), every element $y \in D$ can be unambiguously written as

$$y = y_+ + y_- + y^F + y^\mathcal{E} \mathcal{F}, \quad (3.5)$$

and we can now define the operator $\hat{\mathcal{E}} : D \to D$ by the formula

$$\hat{\mathcal{E}} y := \mathcal{E}(y_+ + y_- + y^\mathcal{E} \mathcal{F}) = y_+ - y_- + \mathcal{E} y^\mathcal{E} \mathcal{F} \in \mathcal{F}^\perp. \quad (3.6)$$

Occasionally, we shall also use the notation

$$y^\perp := y_+ + y_-, \quad (3.7)$$

in particular we write

$$y = y^\perp + y^F + y^\mathcal{E} \mathcal{F} \quad (3.8)$$

and

$$\hat{\mathcal{E}} y = \mathcal{E}(y^\perp + y^\mathcal{E} \mathcal{F}). \quad (3.9)$$

It turns out that the dynamical system $(\omega, H_{\hat{\mathcal{E}}})$ can be indeed symplectically reduced with respect to the left action of the loop group $LF$ on the loop group $LD$. To see it, we have to prove that both the symplectic form (3.1) and the Hamiltonian (3.2) are invariant with respect to the action of $LF$ on the unreduced phase space $LD$.

First we show the $LF$-invariance of the Hamiltonian $H_{\hat{\mathcal{E}}}$. Let $f \in LF$. Then, we have

$$H_{\hat{\mathcal{E}}}(fl) = \frac{1}{2} \oint \left((fl)'(fl)^{-1}, \hat{\mathcal{E}}(fl)'(fl)^{-1}\right)_D \quad (3.10)$$

Note that the term $f' f^{-1}$ disappeared because we see from (3.6) that the image of the operator $\hat{\mathcal{E}}$ belongs to $\mathcal{F}^\perp$. From (3.10), we further find

$$H_{\hat{\mathcal{E}}}(fl) = \frac{1}{2} \oint \left(\text{Ad}_f((l'l^{-1})^\mathcal{F} + (l'l^{-1})^\perp + (l'l^{-1})^\mathcal{E} \mathcal{F}), \hat{\mathcal{E}}(fl)'(fl)^{-1}\right)_D \quad (3.11)$$
Now we show the $LF$-invariance of the symplectic form $\omega$. Let $\phi(\sigma) \in LF$ and consider the vector field $v_\phi$ acting on a function $\Psi$ on $LD$ as

$$(v_\phi \Psi)(l) := \left( \frac{d}{ds} \right)_{s=0} \Psi(e^{s\phi}l).$$

(3.12)

Since $v_\phi$ is the right-invariant vector field on the loop group $LD$, its contraction with the right-invariant Maurer–Cartan form $\delta ll^{-1}$ on $LD$ is

$$\langle \iota_{v_\phi}, \delta ll^{-1} \rangle = \phi.$$

(3.13)

Then, we deduce from Eqs. (3.1) and (3.13)

$$\iota_{v_\phi} \omega = \delta(\delta ll^{-1}, \phi)_{D} = \delta(j, \phi)_{D} = \delta(j^{EF}, \phi)_{D},$$

(3.14)

which means that the quantity $-(j^{EF}, \phi)_{D}$ is the moment map for the left action of the element $\phi$ on $LD$.

Since the form $\omega$ is closed, its Lie derivative with respect to the vector field $v_\phi$ vanishes. Indeed, we have

$$\mathcal{L}_{v_\phi} \omega = (\delta t_{v_\phi} + \iota_{v_\phi} \delta) \omega = \delta(\delta(j^{EF}, \phi)_{D}) = 0.$$  

(3.15)

Now we can perform the symplectic reduction of the dynamical system $(\omega, H_\hat{\mathcal{E}})$ by setting

$$j^{EF} = 0.$$  

(3.16)

The reduced dynamical system is denoted as $(\hat{\omega}, \hat{H}_\hat{\mathcal{E}})$ and it is referred to as the degenerate $\hat{\mathcal{E}}$-model. The reduced phase space $\hat{P}$ can be identified with the space of the left cosets $LF \setminus LD^c$, where $LD^c$ is the space of all elements of the loop group $LD$ verifying the $LF$-invariant constraint (3.16). The reduced Hamiltonian $\hat{H}_\hat{\mathcal{E}}$ is the restriction of the unreduced Hamiltonian $\hat{H}_\mathcal{E}$ from the space $LF \setminus LD$ to the space $LF \setminus LD^c$.

**Remark 3.1.** Note that even if the involution $\mathcal{E}$ satisfies the conditions 1) and 2) above, the Hamiltonian (2.2) of the non-degenerate $\mathcal{E}$-model cannot be reduced because it is not invariant with respect to the left action of the loop group $LF$ on the loop group $LD$. For this reason, the degenerate $\hat{\mathcal{E}}$-model cannot be interpreted as the symplectic reduction of the non-degenerate one.

The equations of motion of the unreduced dynamical system $(\omega, H_\mathcal{E})$ obviously read

$$\frac{\partial j}{\partial \tau} = \{j, H_\mathcal{E}\} = (\mathcal{E}j)' + [\mathcal{E}j, j]_{D},$$

(3.17)

or, in the decomposed form following the decomposition (3.3), as

$$\frac{\partial j^+}{\partial \tau} = j^+_1 + 2[j^+_+, j^-_+] + [\mathcal{E}j^{\mathcal{E}F}, j^+_+] - [j^{\mathcal{E}F}, j^+_+] - [j^{EF}, j^+_+] + j^{EF}, j^- - j^-_+],$$

(3.18a)

$$\frac{\partial j^-}{\partial \tau} = -j^+_1 + 2[j^+_+, j^-_+] + [\mathcal{E}j^{\mathcal{E}F}, j^-_+] + [j^{EF}, j^-_+] - [j^{EF}, j^+_+] + j^{EF}, j^- - j^-_+],$$

(3.18b)

$$\frac{\partial j^{EF}}{\partial \tau} = \mathcal{E}(j^{\mathcal{E}F})' + 2[j^+_+, j^-_+] + [\mathcal{E}j^{\mathcal{E}F}, j^{\mathcal{E}F}] - [j^{\mathcal{E}F}, j^+_+] - j^{\mathcal{E}F}, j^- - j^-_+],$$

(3.18c)

$$\frac{\partial j^{EF}}{\partial \tau} = [\mathcal{E}j^{\mathcal{E}F}, j^{\mathcal{E}F}] - [j^{\mathcal{E}F}, j^+_+] - j^{\mathcal{E}F}, j^- - j^-_+]^{\mathcal{E}F}.$$  

(3.18d)
Imposing the reduction constraint $j^E\mathcal{F} = 0$, we obtain the equations of motions of the degenerate $\hat{E}$-model
\begin{align}
\frac{\partial j_+}{\partial \tau} &= j'_+ + 2[j_+, j_-]^+_+ - [j^\mathcal{F}, j_+]_+, \\
\frac{\partial j_-}{\partial \tau} &= -j'_- + 2[j_+, j_-]_- + [j^\mathcal{F}, j_-]_-, \\
\frac{\partial j^\mathcal{F}}{\partial \tau} &= 2[j_+, j_-]^\mathcal{F}.
\end{align}

(3.19a) (3.19b) (3.19c)

It is perhaps worth pointing out that the quantities $j_{\pm}$ and $j^\mathcal{F}$ appearing in equation (3.19) are not well defined on the reduced phase space $LF\backslash LD^c$ because we find from the identity $j = l'l^{-1}$ that they transform under the action of the loop group $LF$ as
\begin{equation}
\begin{aligned}
j_{\pm} &\to \text{Ad}_f j_{\pm}, \\
j^\mathcal{F} &\to \text{Ad}_f j^\mathcal{F} + \partial_\tau f f^{-1}, \quad f \in LF.
\end{aligned}
\end{equation}

However, equation (3.19) itself is invariant with respect to the transformation (3.20), it can be therefore indeed interpreted as the evolution equation on the reduced phase space.

**Remark 3.2.** Consider the current components $j_{\pm}$ and also two new $\mathcal{F}$-valued fields $A_{\pm}$. Set $\partial_{\pm} := \partial_{\tau} \pm \partial_\sigma$. Then, the following system of equations
\begin{align}
\partial_- j_+ - [A_-, j_+] &= 2[j_+, j_-]^+_+, \\
\partial_+ j_- - [A_+, j_-] &= 2[j_+, j_-]_-^-,
\end{align}

(3.21a) (3.21b)

\begin{equation}
\begin{aligned}
\frac{1}{2} \partial_- A_+ - \frac{1}{2} \partial_+ A_- + \frac{1}{2} [A_+, A_-] &= 2[j_+, j_-]^\mathcal{F}
\end{aligned}
\end{equation}

(3.21c)

is gauge invariant with respect to the gauge transformations
\begin{equation}
\begin{aligned}
j_{\pm} &\to \text{Ad}_f j_{\pm}, \\
A_{\pm} &\to \text{Ad}_f A_{\pm} + \partial_{\pm} f f^{-1}.
\end{aligned}
\end{equation}

(3.22)

Note that here $f \in F$ is the function of both $\tau$ and $\sigma$. Fixing the temporal gauge $A_+ = -A_-$ and setting $A_+ = -A_- = j^\mathcal{F}$, equation (3.21) becomes just the equation of motion (3.19) of the degenerate $\hat{E}$-model. The transformation (3.20) can be then interpreted as the residual time-independent gauge transformation $f(\sigma)$ which preserves the gauge $A_+ = -A_-$. Consider the degenerate $\hat{E}$-model on the Drinfeld double $D$ and let $K \subset D$ be a half-dimensional subgroup such that the Lie subalgebra $K \subset D$ is isotropic with respect to the bilinear form $\langle \cdot, \cdot \rangle_D$. Then, there is a two-dimensional nonlinear $\sigma$-model such that its first-order dynamics can be expressed in terms of this degenerate $\hat{E}$-model; in particular its first-order Hamiltonian equations of motion are given by (3.19). The action of this $\sigma$-model reads [34]
\begin{equation}
\begin{aligned}
S_{\text{WZW}}(l) &= \frac{1}{4} \int \delta^{-1} \phi \left( \delta l l^{-1}, [\partial_\sigma ll^{-1}, \delta ll^{-1}]_D \right)_D \\
&\quad + \frac{1}{4} \int d\tau \phi \left( W_+^+ \partial_+ ll^{-1}, \partial_- ll^{-1} \right)_D
\end{aligned}
\end{equation}
Here, \( W^\pm : D \to D \) are the projectors fully characterized by their respective kernels and images
\[
\text{Ker}(W^\pm) = Ad_l(K), \quad \text{Im}(W^\pm) = V^\pm \oplus \mathcal{F}.
\] (3.24)
It may seem, that the \( \sigma \)-model (3.23) lives on the target \( D \) but, actually, it lives on the space of double cosets \( F \setminus D / K \) because it enjoys the gauge symmetries
\[
l(\tau, \sigma) \to f(\tau, \sigma)l(\tau, \sigma)k(\tau, \sigma), \quad f(\tau, \sigma) \in F, \quad k(\tau, \sigma) \in K.
\] (3.25)

### 3.2. Integrable Dressing Cosets

Suppose that there is the degenerate \( \hat{E} \)-model based on the Drinfeld double \( D \), on the involution \( E \) and on the isotropic subalgebra \( F \subset D \). Moreover, we suppose that there is a quadratic Lie algebra \( G \) which also possesses \( F \) as its subalgebra and that there is a one-parametric family of linear operators \( \hat{O}(z) : V^+ \oplus V^- \to G \) which intertwine the adjoint action of \( F \) on \( V^+ \oplus V^- \) and on \( G \) and verify the following condition
\[
[\hat{O}(z)x^+, \hat{O}(z)x^-]_G = [x^+, x^-]_F \oplus \hat{O}(z)[x^+, x^-]_D, \quad x^\pm \in V^\pm.
\] (3.26)
Here, the element \( [x^+, x^-]_F \) of \( F \) is viewed as the element of \( G \).

It turns out that the condition (3.26) is sufficient for the weak Lax integrability of the degenerate \( \hat{E} \)-model. Indeed, the Lax pair \( L(z), M(z) \) of the operators acting on the loop Lie algebra \( LG \) is given by the formulas
\[
L(z) = \partial_\sigma - \text{ad}^G_{j^F + \hat{O}(z)j^\perp}
\] (3.27)
\[
M(z) = -\text{ad}^G_{\hat{O}(z)j^\perp}
\] (3.28)
as it is straightforward to find out from (3.26) that the dressing coset field equations (3.19) of the degenerate \( \hat{E} \)-model can be indeed represented in the Lax form with the spectral parameter
\[
\frac{dL(z)}{dt} = [L(z), M(z)].
\] (3.29)

To guarantee the strong Lax integrability of the degenerate \( \hat{E} \)-model, we first define
\[
\hat{O}(z)(j^F + j^\perp + j^E_F) := j^F + \hat{O}(z)j^\perp
\] (3.30)
and we supplement the condition (3.26) with the condition that it exists a family of operators \( \hat{r}(z, w) : \mathcal{G} \to \mathcal{G} \) such that it holds
\[
[O^\dagger(z)x, O^\dagger(w)y]_D + O^\dagger(z)[x, \hat{r}(z, w)y]_G + O^\dagger(w)[x, \hat{r}(w, z)y]_G \in \mathcal{F}, \quad \forall x, y \in \mathcal{G},
\] (3.31)
\[
(O^\dagger(z)x, O^\dagger(w)y)_D + (x, \hat{r}(w, z)y)_G + (\hat{r}(w, z)x, y)_G = 0, \quad \forall x, y \in \mathcal{G}.
\] (3.32)
Here, $O^\dagger(z) : \mathcal{G} \rightarrow \mathcal{D}$ is the adjoint of the operator $O(z)$ defined by the relation
\[(O(z)x,y)_\mathcal{G} = (x,O^\dagger(z)y)_\mathcal{D}, \quad \forall x \in \mathcal{D}, y \in \mathcal{G}. \quad (3.33)\]

Let us now see why the conditions (3.31) and (3.32) guarantee the strong integrability. First we remark that the operators $\hat{\sigma}$ intertwine the adjoint action of $\mathcal{F}$ on $V_+ \oplus V_-$ and on $\mathcal{G}$ which implies that the Lax operator $L(z)$ transforms in the adjoint way upon the action of the loop group $LF$. Indeed, using Eq. (3.20), we find
\[
L^f(z) := \partial_\sigma - \text{ad}_{\text{Ad}^O(j^\mathcal{F}\phi + \partial_\sigma j^\mathcal{F}^{-1} + \hat{\sigma})\text{Ad}^O(j^\mathcal{F}^{-1})} = \text{Ad}_f^O \hat{\sigma} = \text{Ad}_f^O \hat{\sigma} = \text{Ad}_f^O L(z) \text{Ad}_f^{-1}. \quad (3.34)
\]

We thus observe that the spectral invariants of the Lax matrix $L(z)$ are invariant with respect to the action of the loop group $LF$ and therefore they restrict to the well-defined functions on the reduced phase space. Moreover, by the general theory of symplectic reduction, the reduced Poisson brackets of the restricted spectral invariants can be evaluated by restricting the unreduced Poisson brackets of the unreduced invariants. In particular, if the unreduced brackets vanish on the constraint surface, the reduced ones vanish. Said in other words, the reduced spectral invariants (which are automatically the integrals of the reduced motion) Poisson commute if the unreduced Poisson brackets of the unreduced spectral invariants vanish on the constraint surface $j^\mathcal{F} = 0$. This occurs precisely if the following condition holds
\[
\{L(z) \otimes \text{Id}, \text{Id} \otimes L(w)\}_{\text{unred}} \approx [r(z,w), L(z) \otimes \text{Id}] - [r^p(w,z), \text{Id} \otimes L(w)]. \quad (3.35)
\]

Here, $r(z,w)$ is some $r$-matrix acting on the space $L\mathcal{G} \otimes L\mathcal{G}$ and the symbol $\approx$ indicates that the left-hand side is equal to the right-hand side on the constraint surface $j^\mathcal{F} = 0$. Said in other words, we first evaluate the unreduced brackets of the Lax operator with itself, then we restrict the result to the constraint surface $j^\mathcal{F} = 0$ and we require that the succession of these two operations gives the right-hand side of (3.35) also restricted to the constraint surface.

If we choose the ansatz (2.21)
\[
r(z,w) = C_{AB} \text{ad}_{\hat{r}(z,w)} T^A \otimes \text{ad}_T B \delta(\sigma_1 - \sigma_2), \quad (3.36)
\]
then, similarly as in Sect. 2.3, we deduce from the conditions (3.31) and (3.32) the validity of the strong integrability condition (3.35). Indeed, we use the current Poisson bracket (2.3) to calculate the unreduced Poisson brackets of the matrix elements of the Lax operator
\[
\{(O(z)j(\sigma_1), x)_{\mathcal{G}}, (O(w)j(\sigma_2), y)_{\mathcal{G}}\}_{\text{unred}}
\]
\[
= \{(j(\sigma_1), O^\dagger(z)z, O^\dagger(w)y)_{\mathcal{D}}\} \partial_\sigma(\sigma_1 - \sigma_2) + (O^\dagger(z)x, O^\dagger(w)y)_{\mathcal{D}} \partial_\sigma(\sigma_1 - \sigma_2)
\]
\[
= -\left(\{(j(\sigma_1), O^\dagger(z)z, \hat{r}(z,w)y)_{\mathcal{G}} + O^\dagger(w)\hat{r}(w,z)x, y)_{\mathcal{G}} + \phi\}_{\mathcal{D}} \partial_\sigma(\sigma_1 - \sigma_2)
\]
\[
- \left(\{(x, \hat{r}(z,w)y)_{\mathcal{G}} + (\hat{r}(w,z)x, y)_{\mathcal{G}}\} \partial_\sigma(\sigma_1 - \sigma_2) \right). \quad (3.37)
\]
Here, $\phi$ is an unspecified element of $F \subset D$ which is eventually irrelevant because
\[
(j(\sigma_1), \phi)_D = (j^{EF}(\sigma_1), \phi)_D \approx 0.
\]
We conclude the argument by remarking that the desired relation (3.35) is nothing but the operator form of equation (3.37).

4. $\sigma$-Models on Symmetric Spaces

In the present section, we test successfully our method by applying it to three $\sigma$-models on symmetric spaces for which the strong integrability has been already proven in the literature. We first represent all those three models as the dressing cosets and then we readily formulate and solve for them the integrability conditions (3.26), (3.31) and (3.32). Our unified treatment does not give exactly the same Lax matrices (nor, for that matter, the same $r$-matrices) that the ones obtained in literature previously, however, our Lax matrices do coincide with those previously known on the primary constraints surface. This means that they are physically equivalent, giving the same set of the integral of motion in involution as the old Lax matrices do.

A question may arise whether our formalism could be adapted to recover exactly the Lax matrices found previously and not just on the constraints surface. In our opinion, this need not be attempted due to both pragmatic and esthetical reasons. The pragmatic reason is based on the fact that working with both old and new Lax matrices is physically equivalent as well as technically similar, notably both of them permit to work off the constraint surface and they just differ in the way how the constraints are included as the parts of them. The esthetical reason is that our new Lax matrices come from the unified treatment based on the degenerate $\hat{\mathcal{E}}$-model formulation of the integrable $\sigma$-models while the old ones were obtained model by model. We believe that this underlying unifying $\hat{\mathcal{E}}$-model formulation is preferable and will also prove to be technically the most efficient one in further investigations.

4.1. The Non-deformed Symmetric Space $\sigma$-Model

Let $\rho : G \to G$ be an involutive automorphism of a quadratic Lie group $G$ and denote $H$ the subgroup of $G$ consisting of fixed points of this automorphism. The tangent involutive automorphism $\rho^* : \mathcal{G} \to \mathcal{G}$ has two eigenspaces: to the eigenvalue $+1$ it corresponds the Lie algebra $\mathcal{H}$ and to the eigenvalue $-1$ it corresponds an $\text{ad}_H$-module $\mathcal{H}^\perp$. As the notation suggests, the eigenspaces $\mathcal{H}$ and $\mathcal{H}^\perp$ are orthogonal to each other with respect to the ad-invariant non-degenerate symmetric bilinear form $(\cdot, \cdot)_G$ on $\mathcal{G}$. We note also that the Lie bracket of two elements from $\mathcal{H}^\perp$ belongs to $\mathcal{H}$.

The space of left cosets $H \backslash G$ is referred to as the symmetric space. We now construct the dressing coset which gives rise to an integrable $\sigma$-model living on the symmetric space target $H \backslash G$. For the Drinfeld double $D$ we take the cotangent bundle $T^*G$ and we view the elements of $D$ as pairs $(g, \gamma), g \in G,
The group multiplication law then reads
\[(g,x)(m,y) = (gm,x + \text{Ad}_g y), \quad g,m \in G, \quad x,y \in \mathcal{G}.\] (4.1)
Furthermore, the unit element $e_D$ of $D = T^*G$ is
\[e_D = (e_G, 0)\] (4.2)
and the inverse element is
\[(g,x)^{-1} = (g^{-1}, -\text{Ad}_{g^{-1}} x).\] (4.3)
The involutive automorphism $\rho : G \to G$ can be naturally lifted to the involutive automorphism $\rho_D : D \to D$ by the formula
\[\rho_D(g,x) := (\rho(g), \rho^* x).\] (4.4)
We denote by $D_H$ the subgroup of the fixed points of the automorphism $\rho_D$. The elements of $D_H$ are obviously the pairs $(h,u)$, $h \in H$, $u \in \mathcal{H}$.

The symmetric non-degenerate ad-invariant bilinear form $(\cdot, \cdot)_D$ on the Lie algebra $D$ is given by
\[
((\xi_0, \xi_1), (\chi_0, \chi_1))_D = (\xi_0, \chi_0)_G + (\xi_1, \chi_0)_G.
\] (4.5)
where $(\cdot, \cdot)_G$ is the symmetric non-degenerate ad-invariant bilinear form on $G$. In this paper, $\mathcal{G}$ will always be semi-simple and $(\cdot, \cdot)_G$ its standard (negatively definite) Killing-Cartan form. Note also that the form $(\cdot, \cdot)_D$ has the split signature $(+, \cdots, +, -, \cdots, -)$.

We shall also need formulas for the left and right Maurer–Cartan forms on $D$
\[l^{-1} dl \equiv (g,x)^{-1} d(g,x) = (g^{-1} dg, \text{Ad}_{g^{-1}} (dx)),\] (4.7)
\[dll^{-1} \equiv d(g,x)(g,x)^{-1} = (dgg^{-1}, dx + [x, dgg^{-1}]),\] (4.8)
and the formula expressing the adjoint action of $D$ on $D$
\[\text{Ad}_{(g,x)}(\xi_0, \xi_1) = (\text{Ad}_g \xi_0, \text{Ad}_g \xi_1 + ax \text{Ad}_g \xi_0).\] (4.9)

Now we define the non-degenerate $\mathcal{E}$-model on $D$ by considering an involution $\mathcal{E} : D \to D$ defined as
\[\mathcal{E}(\xi_0, \xi_1) = -(\xi_1, \xi_0).\] (4.10)
This involution verifies all required properties, it is symmetric with respect to the bilinear form $(\cdot, \cdot)_D$ and the bilinear form $(\cdot, \mathcal{E}. \cdot)_D$ is strictly positive definite. Indeed, we have
\[
((\xi_0, \xi_1), \mathcal{E}(\xi_0, \xi_1))_D = -(\xi_0, \xi_0)_G - (\xi_1, \xi_1)_G.
\] (4.11)
In order, to define the degenerate $\hat{\mathcal{E}}$-model we need the isotropic subgroup $F \subset D$ such that the adjoint action of $F$ on $D$ commutes with the involution $\mathcal{E}$. We choose for $F$ the group $H$, viewed as the subgroup of $D$. We can now
identify the subspaces featuring in the decomposition (3.3) of the Drinfeld double $\mathcal{D}$:

$$(\phi, 0) \in \mathcal{F}, \quad (0, \phi) \in \mathcal{E}\mathcal{F}, \quad (\xi, \mp \xi) \in V_{\pm}, \quad \phi \in \mathcal{H}, \quad \xi \in \mathcal{H}^{\perp}. \quad (4.12)$$

In order to obtain the $\sigma$-model action (3.23) corresponding to our degenerate $\hat{\mathcal{E}}$-model, we have to choose the half-dimensional isotropic subgroup $K \subset \mathcal{D}$. We choose for $K$ the vector space $\mathcal{G}$ viewed as the Abelian Lie group the elements of which have the form $(e^{G}, \gamma) \in \mathcal{D}$. We fix the $K$-part of the gauge symmetry (3.25) by setting $l = g \in G$, we then find

$$W_g^{\pm}(\partial_{\pm}gg^{-1}, 0) = (\partial_{\pm}gg^{-1}, \mp P^{\perp}\partial_{\pm}gg^{-1}) \quad (4.13)$$

and, finally, we conclude from (3.23)

$$S_{\mathcal{E}}(g) = -\frac{1}{2} \int d\tau \oint_{\hat{\mathcal{G}}} \left( P^{\perp}\partial_{+}gg^{-1}, P^{\perp}\partial_{-}gg^{-1} \right) \quad (4.14)$$

Here, $P^{\perp} : \mathcal{G} \to \mathcal{H}^{\perp}$ is the projector with the kernel $\mathcal{H}$. We notice that the action (4.14) has the residual gauge symmetry $g \to hg$, it therefore lives indeed on the symmetric space target $H \backslash G$.

Now we study the integrability of the dressing coset (4.14). Following the analysis of Sect. 3.2, we look for the families of operators $\hat{O}(z)$ and $\hat{r}(z, w)$ verifying the conditions (3.26), (3.31) and (3.32) of Sect. 3.2. We choose the ansatz given by the multiplication by the numerical functions

$$\hat{O}(z)(\xi, \mp \xi) = a_{\pm}(z)\xi, \quad \xi \in \mathcal{H}^{\perp}; \quad (4.15)$$

$$\hat{r}(z, w)\phi = b(z, w)\phi, \quad \hat{r}(z, w)\xi = c(z, w)\xi, \quad \phi \in \mathcal{H}, \quad \xi \in \mathcal{H}^{\perp}. \quad (4.16)$$

We immediately observe that the choice (4.15) respects the condition (3.26) provided it holds

$$a_{\pm}(z)a_{\mp}(z) = 1. \quad (4.17)$$

The dressing coset (4.14) is therefore at least weakly integrable; we may choose without loss of generality

$$a_{\pm}(z) = z^{\pm 1}. \quad (4.18)$$

With the choice (4.18), we find the operators $O^{\dagger}(z)$

$$O^{\dagger}(z)\phi = (0, \phi), \quad \phi \in \mathcal{H}, \quad O^{\dagger}(z)\xi = \frac{1}{2z}(\xi, \xi) - \frac{z}{2}(\xi, -\xi), \quad \xi \in \mathcal{H}^{\perp}. \quad (4.19)$$

The conditions (3.31) and (3.32) then give

$$b(z, w) + b(w, z) = 0 \quad (4.20a)$$

$$c(z, w) + c(w, z) = \frac{1}{2}u(zw) \quad (4.20b)$$

$$b(z, w)z^{\pm 1} + c(w, z)w^{\pm 1} = \frac{1}{2}u(z), \quad (4.20c)$$

where

$$u(x) := x - x^{-1}. \quad (4.21)$$
The system (4.20) does have a (unique) solution given by
\[ b(z, w) = \frac{1}{2} \frac{u(z)u(w)}{u(w/z)}, \quad c(z, w) = \frac{1}{2} \frac{u(w)^2}{u(w/z)}, \tag{4.22} \]
we thus conclude that the symmetric space dressing coset (4.14) is strongly integrable.

Remark 4.1. The strong integrability of the symmetric space \( \sigma \)-model (4.14) was already established in Refs. [9,10]. The Lax matrix of those references is not the same as ours, however, it differs from ours only outside the constraint surface \( j^{\mathcal{E}, \mathcal{F}} = 0 \) (or \( \Pi(0) = 0 \) in the notation of [9,10]). This means that the reduced spectral invariants of our Lax matrix coincide with those of the Lax matrix of Refs. [9,10]. It may be said also that our Lax matrix (2.11) is simpler than that of Refs. [9,10] because it does not contain the variable \( j^{\mathcal{E}, \mathcal{F}} \) but the price to pay for it is that our \( r \)-matrix \( \hat{r}(\lambda, \eta) \) is more complicated than that of Ref. [9,10] because it is not given by a multiple of identity.

4.2. Symmetric Space \( \lambda \)-Deformation
As in the previous section, let \( \rho : G \rightarrow G \) be the involutive automorphism of a semi-simple Lie group \( G \) and denote \( H \) the subgroup of \( G \) consisting of fixed points of this automorphism. However, for the double \( D \) we now take the direct product \( D = G \times G \) and for the automorphism \( \rho_D \) we take
\[ \rho_D(a_L, a_R) := (\rho(a_L), \rho(a_R)), \quad (a_L, a_R) \in G \times G. \tag{4.23} \]
The symmetric non-degenerate ad-invariant bilinear form \((.,.)_D\) on the Lie algebra \( D \) is now given by
\[ \left( (\mu, \mu), (\nu, \nu) \right)_D = (\mu, \nu)_G - (\mu_R, \nu_R)_G, \quad \mu, \nu, \mu_L, \mu_R, \nu_L, \nu_R \in G, \tag{4.24} \]
where \((.,.)_G\) is the (negatively definite) Killing-Cartan form as in Sect. 4.1. Again, the form \((.,.)_D\) has the split signature \((+, \cdots, +, -, \cdots, -)\).

Now we define the non-degenerate \( \mathcal{E} \)-model on \( D \) by considering an involution \( \mathcal{E}_\alpha : \mathcal{D} \rightarrow \mathcal{D} \) defined as
\[ \mathcal{E}_\alpha(\mu_L, \mu_R) = \cosh \alpha (\mu_L, \mu_R) + \sinh \alpha (\mu_D, \mu_D). \tag{4.25} \]
This involution verifies all required properties, it is symmetric with respect to the bilinear form \((.,.)_D\) and the bilinear form \((.,\mathcal{E}_\alpha,.)_D\) is strictly positive definite. Indeed, we have
\[ \left( (\mu, \nu), \mathcal{E}_\alpha(\mu, \nu) \right)_D = -\frac{e^\alpha}{2} (\mu + \nu, \mu + \nu)_G - \frac{e^{-\alpha}}{2} (\mu - \nu, \mu - \nu)_G, \quad \mu, \nu \in G. \tag{4.26} \]

In order to define the degenerate \( \hat{\mathcal{E}} \)-model we need the isotropic subgroup \( F \subset D \) such that the adjoint action of \( F \) on \( D \) commutes with the involution \( \mathcal{E}_\alpha \). We choose for \( F \) the subgroup
\[ F = \{(f, f) \in D, \quad f \in H\}, \tag{4.27} \]
where, as in Sect. 4.1, \( H \) is the subgroup of \( G \) consisting of the fixed points of the involutive automorphism \( \rho : G \to G \). We can now identify the subspaces featuring in the decomposition (3.3) of the Drinfeld double \( D \):

\[
(\phi, \phi) \in \mathcal{F}, \quad (-\phi, \phi) \in \mathcal{E} \mathcal{F}, \quad (\lambda^{\pm 1} \xi, \xi) \in V_{\pm}, \quad \phi \in \mathcal{H}, \quad \xi \in \mathcal{H}^\perp, \quad (4.28)
\]

where

\[
\lambda := \frac{1 - e^{\alpha}}{1 + e^{\alpha}}. \quad (4.29)
\]

In order to obtain the \( \sigma \)-model action (3.23) corresponding to our \( \lambda \)-deformed degenerate \( \hat{\mathcal{E}} \)-model, we have to choose the half-dimensional isotropic subgroup \( K \subset D \). We choose for \( K \) the diagonal subgroup of \( D \), which means

\[
K = \{(x, x) \in D, \ x \in G\}. \quad (4.30)
\]

We fix the \( K \)-part of the gauge symmetry (3.25) by setting \( l_0 = (g, e_G) \in D \), we then find

\[
W_\pm^g \partial_\pm l_0 l_0^{-1} = W_\pm^g (\partial_\pm gg^{-1}, 0)
\]

\[
= -((\lambda^{\pm 1} P^\perp + P) (\text{Ad}_g - \lambda^{\pm 1} P^\perp - P)^{-1} \partial_\pm gg^{-1},
\]

\[
(\text{Ad}_g - \lambda^{\pm 1} P^\perp - P)^{-1} \partial_\pm gg^{-1}) \quad (4.31)
\]

Here, \( P : \mathcal{G} \to \mathcal{H} \) is the projector with the kernel \( \mathcal{H}^\perp \). Finally, we conclude from (3.23)

\[
S_\lambda(g) = \frac{1}{4} \int d\tau \oint \left( \partial_+ gg^{-1}, \partial_- gg^{-1} \right) \Omega + \frac{1}{4} \int \delta^{-1} \oint \left( \delta gg^{-1}, [\partial_\sigma gg^{-1}, \delta gg^{-1}] \right) \Omega
\]

\[
+ \frac{1}{2} \int d\tau \oint \left( \partial_+ gg^{-1}, \frac{1}{\text{Ad}_g(\lambda P^\perp + P) - 1} \partial_- gg^{-1} \right) \Omega. \quad (4.32)
\]

We notice that the action (4.32) has the residual gauge symmetry \( g \to hgh^{-1}, \ h \in H \).

Now we study the integrability of the dressing coset (4.32). Following the analysis of Sect. 3.2, we look for the families of operators \( \hat{O}(z) \) and \( \hat{r}(z, w) \) verifying the conditions (3.26), (3.31) and (3.32) of Sect. 3.2. We choose the ansatz given by the multiplication by the numerical functions

\[
\hat{O}(z)(\lambda^{\pm 1} \xi, \xi) = a_\pm(z) \xi, \quad \xi \in \mathcal{H}^\perp; \quad (4.33)
\]

\[
\hat{r}(z, w) \phi = b(z, w) \phi, \quad \hat{r}(z, w) \xi = c(z, w) \xi, \quad \phi \in \mathcal{H}, \quad \xi \in \mathcal{H}^\perp. \quad (4.34)
\]

We immediately observe that the choice (4.33) respects the condition (3.26) provided it holds

\[
a_+(z)a_-(z) = 1. \quad (4.35)
\]

The dressing coset (4.32) is therefore at least weakly integrable; we may choose without loss of generality

\[
a_\pm(z) = \lambda^{\pm 1/2} z^{\pm 1}. \quad (4.36)
\]
With the choice (4.36), we find the operators $O^\dagger(z)$
\[
O^\dagger(z)(\phi + \xi) = \left( \frac{1}{2} \phi + \frac{u(z\lambda^\pm)}{u(\lambda)} \xi, -\frac{1}{2} \phi + \frac{u(z\lambda^{-\pm})}{u(\lambda)} \xi \right), \quad \phi \in \mathcal{H}, \ \xi \in \mathcal{H}^\perp,
\]
where
\[
u(x) := x - x^{-1}.
\]
The conditions (3.31) and (3.32) then give
\[
b(z, w) + b(w, z) = 0 \quad (4.39a)
\]
\[
c(z, w) + c(w, z) = -\frac{u(zw)}{u(\lambda)} \quad (4.39b)
\]
\[
b(z, w)u(z\lambda^{\pm\frac{1}{2}}) + c(w, z)u(w\lambda^{\pm\frac{1}{2}}) = \mp \frac{1}{2} u(z\lambda^{\pm\frac{1}{2}}). \quad (4.39c)
\]
The system (4.54) does have a (unique) solution given by
\[
b(z, w) = \frac{u(z\lambda^{-\frac{1}{2}})u(w\lambda^{\frac{1}{2}})}{u(\lambda)u(z/w)} - \frac{1}{2}, \quad c(z, w) = \frac{u(w\lambda^{-\frac{1}{2}})u(w\lambda^{\frac{1}{2}})}{u(\lambda)u(z/w)},
\]
we thus conclude that the $\lambda$-deformed symmetric space dressing coset (4.32) is strongly integrable.

**Remark 4.2.** The strong integrability of the symmetric space $\lambda$-deformation (4.32) was already established in Refs. [19,24]. The Lax matrix of those references is not the same as ours, but, again, it differs from ours only outside the constraint surface $j^E_j^F = 0$, which means that the reduced spectral invariants of our Lax matrix coincide with those coming from the Lax matrix of Refs. [19,24].

**4.3. Symmetric Space $\eta$-Deformation**

Let $\rho : G \to G$ be the involutive automorphism of the semi-simple Lie group $G$ and denote $H$ the subgroup of $G$ consisting of fixed points of this automorphism. Suppose moreover, that $\rho$ can be lifted to an involutive automorphism of the complexified group $\rho_C : G^C \to G^C$. For the Drinfeld double, we then take the Lu–Weinstein one [42] which is the complexified group $D = G^C$ viewed as the real group of doubled dimension. The symmetric non-degenerate ad-invariant bilinear form $(.,.)_D$ on the Lie algebra $D$ is now given by an expression which depends on a real positive parameter $\eta$
\[
(\mu_1 + i\nu_1, \mu_2 + i\nu_2)_D = \frac{1}{\eta}(\mu_1, \nu_2)_G + \frac{1}{\eta}(\mu_2, \nu_1)_G, \quad \mu_1, 2, \nu_1, 2 \in \mathcal{G}.
\]
Again, the form $(.,.)_D$ turns out to have the split signature $(+, \cdots, +, -, \cdots, -)$.

Now we define the non-degenerate $E$-model on $D$ by considering an involution $E_\eta : D \to D$ defined as
\[
E_\eta(\mu + i\nu) = -\eta^{-1}\nu - i\eta\mu, \quad \eta > 0, \quad \mu, \nu \in \mathcal{G}.
\]
This involution verifies all required properties, it is symmetric with respect to the bilinear form $(\cdot,\cdot)_D$ and the bilinear form $(\cdot,\mathcal{E}_\eta)_D$ is strictly positive definite. Indeed, we have
\[
(\mu + iv, \mathcal{E}_\eta(\mu + iv))_D = - (\mu, \mu)_G - \eta^{-2}(v, v)_G, \quad \mu, v \in \mathcal{G}.
\] (4.43)

In order to define the degenerate $\mathcal{E}$-model, we choose for the isotropic subgroup $F$ the subgroup $H \subset G \subset G^C$. The adjoint action of $F$ on $\mathcal{D} = \mathcal{G}^C$ then indeed commutes with the involution $\mathcal{E}_\eta$. We can now identify the subspaces featuring in the decomposition (3.3) of the Drinfeld double $\mathcal{D}$:
\[
\phi \in \mathcal{F}, \quad i\phi \in \mathcal{E}\mathcal{F}, \quad \xi \perp i\eta\xi \in \mathcal{V}_\pm, \quad \phi \in \mathcal{H}, \quad \xi \in \mathcal{H}^\perp.
\] (4.44)

In order to obtain the $\sigma$-model action (3.23) corresponding to our $\eta$-deformed degenerate $\mathcal{E}$-model, we have to choose the half-dimensional isotropic subgroup $K \subset D$. We choose for $K$ the subgroup $AN$ of $G^C$ featuring in the Iwasawa decomposition\footnote{Recall, that $A$ is the non-compact part of the complexified Cartan torus of $G$ and $N$ is the nilpotent subgroup generated by the positive roots. In particular, if $G^C$ is $SL(N, \mathbb{C})$, then $AN$ consists of upper-triangular matrices with real positive diagonal.} $G^C = GAN$.

In what follows; it will be convenient to parametrize the elements of the Lie algebra $\mathcal{K}$ in terms of those of the Lie algebra $\mathcal{G}$. This can be achieved via the $\mathbb{R}$-linear Yang-Baxter operator $R : \mathcal{G}^C \to \mathcal{G}^C$ defined as
\[
RE^\alpha = -\text{sign}(\alpha)iE^\alpha, \quad RH^j = 0, \quad [R, i] = 0,
\] (4.45)
where $E^\alpha, H^j$ is the standard Chevalley basis of $\mathcal{G}^C$ and $i$ is viewed as the operator of multiplication by the imaginary unit. Indeed, every element of $\mathcal{K}$ can be then written in a unique way as $(R - i)\chi$, where $\chi \in \mathcal{G}$. We now fix the $K$-part of the gauge symmetry (3.25) by setting $l_0 = g \in G$, we then find
\[
W_g \partial_\pm l_0^{-1} = W_g \partial_\pm gg^{-1} - \eta(R_{g^{-1}} - i)(\eta P^\perp R_{g^{-1}} P^\perp + 1\mathcal{H}^\perp) P^\perp \partial_\pm gg^{-1},
\] (4.46)
where
\[
R_{g^{-1}} = \text{Ad}_g R \text{Ad}_{g^{-1}}.
\] (4.47)

Finally, we conclude from (3.23)
\[
S_\eta(g) = -\frac{1}{2} \int d\tau \, \oint \left( P^\perp \partial_+ gg^{-1} - \eta P^\perp R_{g^{-1}} P^\perp \right) P^\perp \partial_- gg^{-1}
\] (4.48)

We notice that the action (4.48) is the one-parameter deformation of the action (4.14) and it has also the residual gauge symmetry $g \to hg$, $h \in H$.

Now we study the integrability of the $\eta$-deformed dressing coset (4.48). Following the analysis of Sect. 3.2, we look for the families of operators $\hat{O}(z)$ and $\hat{r}(z, w)$ verifying the conditions (3.26), (3.31) and (3.32) of Sect. 3.2. We choose the ansatz given by the multiplication by the numerical functions
\[
\hat{O}(z) \frac{1 \mp i\eta}{\sqrt{1 + \eta^2}} \xi = a_\pm(z)\xi, \quad \xi \in \mathcal{H}^\perp;
\] (4.49)
\[
\hat{r}(z, w)\phi = b(z, w)\phi, \quad \hat{r}(z, w)\xi = c(z, w)\xi, \quad \phi \in \mathcal{H}, \quad \xi \in \mathcal{H}^\perp.
\] (4.50)
We immediately observe that the choice (4.49) respects the condition (3.26) provided it holds
\[ a_+(z)a_-(z) = 1. \] (4.51)
The dressing coset (4.48) is therefore at least weakly integrable; we may choose without loss of generality
\[ a_\pm(z) = z^{\pm 1}. \] (4.52)
With the choice (4.52), we find the operators \( O^\dagger(z) \)
\[ O^\dagger(z)(\phi + \xi) = i\eta\phi + \frac{1}{2}\sqrt{1 + \eta^2}\left((1 + i\eta)z^{-1} - (1 - i\eta)z\right)\xi, \quad \phi \in \mathcal{H}, \ \xi \in \mathcal{H}^\perp. \] (4.53)
The conditions (3.31) and (3.32) then give
\[ b(z, w) + b(w, z) = 0 \] (4.54a)
\[ c(z, w) + c(w, z) = \frac{1 + \eta^2}{2}u(zw) \] (4.54b)
\[ b(z, w)z^{\pm 1} + c(w, z)w^{\pm 1} = \frac{1}{2}u(z) \mp \frac{\eta^2}{2}s(z) \] (4.54c)
where
\[ u(x) := x - x^{-1}, \quad s(x) := x + x^{-1}. \] (4.55)
The system (4.54) does have a unique solution given by
\[ b(z, w) = \frac{1}{2} \frac{u(z)u(w) + \eta^2 s(z)s(w)}{u(w/z)}, \quad c(z, w) = \frac{1}{2} \frac{u(w)^2 + \eta^2 s(w)^2}{u(w/z)}, \] (4.56)
we thus conclude that the \( \eta \)-deformed symmetric space dressing coset (4.48) is strongly integrable.

**Remark 4.3.** The strong integrability of the symmetric space \( \eta \)-deformation (4.48) was already established in Ref. [10]. The Lax matrix of those references is not the same as ours, but, again, it differs from ours only outside the constraint surface \( j_{\mathcal{EF}} = 0 \), which means that the reduced spectral invariants of our Lax matrix coincide with those coming from the Lax matrix of Ref. [10].

### 5. Generalized Pseudo-chiral Models

In all three examples of the integrable dressing cosets treated in Sect. 4, the second term on the right-hand side of the condition (3.26) vanished identically. Now we are going to consider examples where it is no longer the case. The simplest one of those new integrable examples is the pseudo-chiral \( \sigma \)-model of Zakharov and Mikhailov [59,60] characterized by the action
\[ S_\mu(x) = -\frac{1}{2\mu} \int d\tau \oint (\partial_+ x, \partial_- x)_G + \frac{\mu}{3}(x, [\partial_+ x, \partial_- x]_G)_G. \] (5.1)
Here, the field \( x(\tau, \sigma) \) takes values in the Lie algebra \( \mathcal{G} \).
In the present section, we first interpret the pseudo-chiral model (5.1) as the dressing coset based on the Drinfeld double \( D = J^3G \), where \( J^3G \) is the third-order jet bundle of the semi-simple Lie group \( G \). We then obtain more complicated integrable dressing cosets by considering higher-order jet bundles \( J^{2n+1}G \). Those theories are apparently new and they describe an integrable interaction of \( n \mathcal{G} \)-valued fields. They represent the principal examples where our method does not just confirm the old results but provides genuinely new ones.

5.1. Pseudo-chiral Model

Consider the Drinfeld double \( D = J^3G \) introduced in \([40]\), where \( J^3G \) is the third-order jet bundle of the quadratic Lie group \( G \). The group \( J^3G \) can be conveniently parametrized via the right trivialization as the (manifold) direct product \( J^3G = \mathcal{G} \times \mathcal{G} \times \mathcal{G} \times \mathcal{G} \) endowed with the group multiplication law and the inverse element

\[
(g, x_1, x_2, x_3)(h, y_1, y_2, y_3) = (gh, x_1 + g y_1, x_2 + g y_2 + \frac{1}{2} [x_1, g y_1], x_3 + g y_3 + \frac{2}{3} [x_1, g y_2] + \frac{1}{3} [x_2, g y_1] + \frac{1}{6} [x_1, x_1, g y_1]), \tag{5.2}
\]

\[
(g, x_1, x_2, x_3)^{-1} = (g^{-1}, -g^{-1} x_1, -g^{-1} x_2, -g^{-1} x_3 + \frac{1}{3} g^{-1} [x_1, x_2]), \tag{5.3}
\]

where \( g, h \in G \), \( x_i, y_i \in \mathcal{G} \), \( i = 1, 2, 3 \) and

\[
g y_i := \text{Ad}_g y_i. \tag{5.4}
\]

As the vector space, the Lie algebra \( \mathcal{D} \) is the direct sum \( \mathcal{D} = \mathcal{G} \oplus \mathcal{G} \oplus \mathcal{G} \oplus \mathcal{G} \) endowed with the Lie bracket

\[
[(\xi_0, \xi_1, \xi_2, \xi_3), (\chi_0, \chi_1, \chi_2, \chi_3)] \\
= (\{\xi_0, \chi_0\}, [\xi_0, \xi_1] + [\xi_1, \chi_0], [\xi_0, \chi_2] + [\xi_1, \chi_1] + [\xi_2, \chi_0], \\
[\xi_0, \chi_3] + [\xi_1, \chi_2] + [\xi_2, \chi_1] + [\xi_3, \chi_0]). \tag{5.5}
\]

Here, of course, \( \xi_i, \chi_i \in \mathcal{G} \), \( i = 0, 1, 2, 3 \).

For convenience, we add the following useful formulas for the exponential map

\[
\exp(t(0, \xi_1, \xi_2, \xi_3)) = (e_G, t \xi_1, t \xi_2, t \xi_3 + \frac{t^2}{6} [\xi_1, \xi_2]) \in J^3G \tag{5.7}
\]

and for the right-invariant Maurer–Cartan form on \( J^3G \)

\[
dll^{-1} = \left( dgg^{-1}, \text{Ad}_g dX_1, \text{Ad}_g \left(dX_2 + \frac{1}{2} [X_1, dX_1]\right), \right. \tag{5.8}
\]

\[
\left. \text{Ad}_g \left(dX_3 + [X_1, dX_2] + \frac{1}{6} [X_1, [X_1, dX_1]]\right) \right),
\]

\(^3\)The interpretation of the third-order jet bundle of the semi-simple Lie group \( G \) as the Drinfeld double comes from Ref. \([40]\).

\(^4\)The multiplication law (5.2) differs from that given in \([40, 58]\) by suitable normalization conventions, namely \( x_i \in \mathcal{G} \) used in \([58]\) are \( i! \) multiples of our \( x_i \).
where now we have parametrized the element \( l = (g, x_1, x_2, x_3) \) differently, that is in terms of the products of “pure elements” (cf. [58])

\[
l = (g, 0, 0, 0)(e_G, X_1, 0, 0)(e_G, 0, X_2, 0)(e_G, 0, 0, X_3).
\]  
(5.9)

Note that the different parametrizations of the group \( J^3G \) are related by the formulas:

\[
g = g, \quad x_1 = \text{Ad}_g X_1, \quad x_2 = \text{Ad}_g X_2, \quad x_3 = \text{Ad}_g \left( X_3 + \frac{2}{3} [X_1, X_2] \right).
\]  
(5.10)

The split-signature symmetric non-degenerate ad-invariant bilinear form \((\cdot, \cdot)_D\) on the Lie algebra \( D \) is given by

\[
\left( (\xi_0, \xi_1, \xi_2, \xi_3), (\chi_0, \chi_1, \chi_2, \chi_3) \right)_{D} = (\xi_0, \chi_3)g + (\xi_1, \chi_2)g + (\xi_2, \chi_1)g + (\xi_3, \chi_0)g.
\]  
(5.11)

Now we define an appropriate non-degenerate \( \mathcal{E} \)-model on the double \( D \) by choosing the following operator \( \mathcal{E}_\mu : D \to D \)

\[
\mathcal{E}_\mu (\xi_0, \xi_1, \xi_2, \xi_3) = - (\mu^3 \xi_3, \mu \xi_2, \mu^{-1} \xi_1, \mu^{-3} \xi_0).
\]  
(5.12)

To obtain the degenerate \( \hat{\mathcal{E}} \)-model, we use the procedure of Sect. 3.1 and choose the isotropic subalgebra \( F \in D \) as

\[
\mathcal{F} = \{ (\xi_0, 0, 0, 0) \in D, \xi_0 \in G \}.
\]  
(5.13)

Consequently, the spaces \( V_\pm \) are given by

\[
V_\pm = \{ (0, \mu \xi, \mp \xi, 0) \in D, \xi \in G \}.
\]  
(5.14)

We can associate the \( \sigma \)-model action (3.23) to the choices (5.12), (5.13) by choosing the maximally isotropic Lie subalgebra \( K \subset D \) spanned by the elements of \( D \) of the form \((0, 0, \xi_2, \xi_3)\). Fixing the gauge with respect to both \( K \) and \( F \) gauge symmetries as \( l = (e_G, x, 0, 0) \), we find

\[
d(e_G, x, 0, 0)(e_G, x, 0, 0)^{-1} = (0, dx, \frac{1}{2} [x, dx], \frac{1}{6} [x, [x, dx]]),
\]  
(5.15)

therefore the WZ term in the action (3.23) is given by the expression

\[
\frac{1}{4} \int \delta^{-1} \oint_D \left( \delta l^{-1}, [\partial_\sigma l^{-1}, \delta l^{-1}] \right)
= \frac{1}{4} \int \delta^{-1} \oint_D (\delta x, [\partial_\sigma x, \delta x]_G) = \frac{1}{6} \int d\tau \oint_D (x, [\partial_\sigma x, \partial_\tau x]_G).
\]  
(5.16)

Using the formula

\[
\text{Ad}_{(e_G, x, 0, 0)} (0, 0, \xi_2, \xi_3) = (0, 0, \xi_2, \xi_3 + [x, \xi_2]),
\]  
(5.17)

we obtain also

\[
W^\pm_{x} \partial_{\pm} l^{-1} = W^\pm_{x} \partial_{\pm} (e_G, x, 0, 0)(e_G, x, 0, 0)^{-1} = (0, \partial_{\pm} x, \mp \mu^{-1} \partial_{\pm} x, 0),
\]  
(5.18)

therefore the dressing coset action (3.23) finally reads

\[
S_\mu (x) = - \frac{1}{2\mu} \int d\tau \oint_D \left( ([\partial_+ x, \partial_- x]_G + \frac{\mu}{3} (x, [\partial_+ x, \partial_- x]_G) \right).
\]  
(5.19)
We indeed recognize in formula (5.19) the action of the pseudo-chiral model (5.1).

To prove the integrability of the pseudo-chiral model (5.19), we start with the ansatz

\[ \hat{O}(z)(0, \xi_1, \xi_2, 0) = p_1(z)\mu \xi_1 + p_2(z)\mu^2 \xi_2, \quad \xi_1, \xi_2 \in G, \]  

(5.20)

where \( p_1(z) \), \( p_2(z) \) are numerical functions that we look for. The condition (3.26) then gives

\[ p_2^2(z) - p_1^2(z) = p_2(z), \]  

(5.21)

which is solved by

\[ p_1(z) = -z, \quad p_2(z) = -\frac{1}{1 - z^2}. \]  

(5.22)

We find also easily the adjoint operator \( \hat{O}^\dagger(z) : G \to \mathcal{D} \)

\[ \hat{O}^\dagger(z)\xi = (0, p_2(z)\mu^2 \xi, p_1(z)\mu \xi, \xi). \]  

(5.23)

Finally, the operator \( \hat{r}(z, w) : G \to G \) verifying the conditions (3.31) and (3.32) turns out to be also given by the multiplication by the numerical function

\[ \hat{r}(z, w)\xi = \frac{\mu^3}{1 - w^2} \frac{\xi}{z - w}. \]  

(5.24)

In the present context, the polynomial \( 1 - w^2 \) is often referred to as the “twist” function.

### 5.2. Generalized Pseudo-chiral Model with Two Fields

Now we use our method to construct a genuinely new integrable dressing coset \( \sigma \)-model. For that we equip the fifth-order jet bundle \( J^5G \) with the structure of the Drinfeld double. The group \( D = J^5G \) can be conveniently parametrized via the right trivialization as the direct product of manifolds \( J^5G = G \times G \times \cdots \times G \) endowed with the group multiplication law\(^5\) and the inverse element

\[
(g, x_1, x_2, x_3, x_4, x_5)(h, y_1, y_2, y_3, y_4, y_5)
= (gh, x_1 + g y_1, x_2 + g y_2 + \frac{1}{2}[x_1, g y_1], x_3 + g y_3 + \frac{2}{3}[x_1, g y_2] + \frac{1}{3}[x_2, g y_1])
+ \frac{1}{6}[x_1[x_1, g y_1]],
\]

\[
x_4 + g y_4 + \frac{3}{4}[x_1, g y_3] + \frac{1}{2}[x_2, g y_2] + \frac{1}{4}[x_3, g y_1] + \frac{1}{4}[x_1[x_1, g y_2]] + \frac{1}{6}[x_2[x_1, g y_1]]
+ \frac{1}{12}[x_1[x_2, g y_1]]
\]

\(^5\)The multiplication law (5.25) differs from that given in \([40, 58]\) by suitable normalization conventions, namely \( x_i \in G \) used in \([58]\) are \( i! \) multiples of our \( x_i \).
and for the right-invariant Maurer–Cartan form
\[ D \]

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Here, of course, \( \xi J \) elements

\[ g \]

where \( g, h \in G \), \( x_i, y_i \in G, i = 1, \ldots, 5 \)

\[ g y_i := \text{Ad}_g y_i. \] (5.26)

As the vector space, the Lie algebra \( D \) is the direct sum \( D = \underbrace{G \oplus \cdots \oplus G}_{6 \text{ times}} \)

endowed with the Lie bracket

\[ ([\xi_0, \xi_1, \ldots, \xi_5], (\chi_0, \chi_1, \ldots, \chi_5)] = ([\xi_0, \chi_0], [\xi_0, \chi_1] + [\xi_1, \chi_0], [\xi_0, \chi_2] + [\xi_2, \chi_1] + [\xi_2, \chi_0], [\xi_0, \chi_3] + [\xi_1, \chi_2] + [\xi_2, \chi_1] + [\xi_3, \chi_0], [\xi_0, \chi_4] + [\xi_1, \chi_3] + [\xi_2, \chi_2] + [\xi_3, \chi_1] + [\xi_4, \chi_0], [\xi_0, \chi_5] + [\xi_1, \chi_4] + [\xi_2, \chi_3] + [\xi_3, \chi_2] + [\xi_4, \chi_1] + [\xi_5, \chi_0]). \] (5.27)

Here, of course, \( \xi_i, \chi_i \in G, i = 0, 1, \ldots, 5. \)

For convenience, we add the following useful formulas for the exponential map

\[ \exp \left( t(0, \ldots, 0, \xi_j, 0, \ldots, 0) \right) = (e_G, 0, \ldots, 0, t \xi_j, 0, \ldots, 0) \in J^5G, \quad j = 1, \ldots, 5 \] (5.28)

and for the right-invariant Maurer–Cartan form

\[ dll^{-1} = d\tilde{g} \tilde{g}^{-1} + \text{Ad}_{\tilde{g}} \sum_{j=1}^{5} \text{Ad}_{l_{k-1}} \tilde{X}_k d\tilde{X}_j \tilde{X}_j^{-1}, \] (5.29)

where we parametrized the group \( J^5G \) in terms of the products of the “pure elements”

\[ l = \tilde{g} \tilde{X}_1 \tilde{X}_2 \ldots \tilde{X}_5 \equiv (g, 0, 0, 0, 0, 0, 0, 0, 0, 0) \]

\[ \times (e_G, 0, X_2, 0, 0, 0)(e_G, 0, X_3, 0, 0) \]

\[ (e_G, 0, 0, 0, X_4, 0)(e_G, 0, 0, 0, X_5). \] (5.30)

Note also that

\[ d\tilde{g} \tilde{g}^{-1} = (dg^{-1}, 0, 0, 0, 0, 0), \] (5.31)
\[ d\tilde{X}_1 \tilde{X}_1^{-1} = (0,dX_1, \frac{1}{2!}\text{ad}_{X_1}dX_1, \frac{1}{3!}\text{ad}^2_{X_1}dX_1, \frac{1}{4!}\text{ad}^3_{X_1}dX_1, \]
\[ \frac{1}{5!}\text{ad}^4_{X_1}dX_1), \]
\[ (5.32) \]
\[ d\tilde{X}_2 \tilde{X}_2^{-1} = (0,0,dX_2,0, \frac{1}{2!}\text{ad}_{X_2}dX_2, 0), \]
\[ (5.33) \]
\[ d\tilde{X}_3 \tilde{X}_3^{-1} = (0,0,0,dX_3,0,0), \]
\[ d\tilde{X}_4 \tilde{X}_4^{-1} = (0,0,0,0,dX_4,0), \quad d\tilde{X}_5 \tilde{X}_5^{-1} = (0,0,0,0,0,dX_5). \]
\[ (5.34) \]

The split-signature symmetric non-degenerate ad-invariant bilinear form \((.,.)_D\) on the Lie algebra \(D\) is given by
\[ (\xi,\chi)_D = \left( (\xi_0,\xi_1,\ldots,\xi_5), (\chi_0,\chi_1,\ldots,\chi_5) \right)_D = \sum_{j=0}^{5} (\xi_j,\chi_{5-j})_G. \]  
\[ (5.35) \]

Now we define an appropriate non-degenerate \(\mathcal{E}\)-model on the double \(D\) by choosing the following operator \(\mathcal{E}_\mu : D \rightarrow D\)
\[ \mathcal{E}_\mu(\xi_0,\xi_1,\xi_2,\xi_3,\xi_4,\xi_5) = -(\mu^5\xi_5,\mu^3\xi_4,\mu\xi_3,\mu^{-1}\xi_2,\mu^{-3}\xi_1,\mu^{-5}\xi_0). \]  
\[ (5.36) \]

To obtain the degenerate \(\tilde{\mathcal{E}}\)-model, we use the procedure of Sect. 3.1 choose as the isotropic subalgebra \(F \in D\) as
\[ \mathcal{F} = \{(\xi_0,0,0,0,0,0) \in D, \xi_0 \in G\}. \]  
\[ (5.37) \]

Consequently, the spaces \(V_\pm\) are formed by the elements of the form
\[ (0,\mu^3\xi_1,\mu\xi_2,\mp\xi_2,\mp\xi_1,0) \in V_\pm. \]  
\[ (5.38) \]

We can associate the \(\sigma\)-model action (3.23) to the choices (5.36), (5.37) by choosing the maximally isotropic Lie subalgebra \(K \subset D\) formed by the elements of the form
\[ (0,0,0,\xi_3,\xi_4,\xi_5) \in K. \]  
\[ (5.39) \]

Fixing the gauge with respect to the both \(K\) and \(F\) as \(l_0 = \tilde{X}_1 \tilde{X}_2\), we find from (5.29) and (5.34)
\[ d(\tilde{X}_1 \tilde{X}_2)(\tilde{X}_1 \tilde{X}_2)^{-1} = (0,dX_1,dX_2 + \frac{1}{2!}\text{ad}_{X_1}dX_1, \text{ad}_{X_1}dX_2 + \frac{1}{3!}\text{ad}^2_{X_1}dX_1, \]
\[ \frac{1}{2}\text{ad}_{X_2}dX_2 + \frac{1}{2}\text{ad}^2_{X_2}dX_2 + \frac{1}{3!}\text{ad}^3_{X_1}dX_1, \frac{1}{4!}\text{ad}^3_{X_1}dX_1, \frac{1}{2}\text{ad}_{X_1}\text{ad}_{X_2}dX_2 \]
\[ + \frac{1}{3!}\text{ad}^3_{X_1}dX_2 + \frac{1}{5!}\text{ad}^4_{X_1}dX_1), \]
\[ (5.40) \]

therefore the WZ term in the action (3.23) is given by the expression
\[ \frac{1}{4}\int \delta^{-1} \oint \left( \delta l_0 l_0^{-1}, [\partial_\sigma l_0 l_0^{-1}, \delta l_0 l_0^{-1}] \right)_D \]
\[ = -\frac{1}{5!}\int d\tau \oint (X_1, [\text{ad}_{X_1}\partial_\sigma X_1, \text{ad}_{X_1}\partial_\tau X_1])_G \]
\[ + \frac{1}{4}\int d\tau \oint (X_2, [\partial_\sigma X_1, \partial_\tau X_2])_G + [\partial_\sigma X_2, \partial_\tau X_1])_G \]
Using the formula
\[
\text{Ad}_{X_1,X_2}(0, 0, 0, \xi_3, \xi_4, \xi_5) = (0, 0, 0, \xi_3, \xi_4 + [X_1, \xi_3], \xi_5 + [X_1, \xi_4] + [X_2, \xi_3] + \frac{1}{2}[X_1, [X_1, \xi_3]]),
\]
we obtain also
\[
W_{X_1,X_2}^\pm \partial_{\pm} l_0^{-1} = W_{X_1,X_2}^\pm \partial_{\pm} (\tilde{X}_1 \tilde{X}_2)^{-1}
= \left(0, \partial_{\pm} X_1, \partial_{\pm} X_2 + \frac{1}{2} \text{ad}_{X_1} \partial_{\pm} X_1, \mp \mu^{-1} \left(\partial_{\pm} X_2 + \frac{1}{2} \text{ad}_{X_1} \partial_{\pm} X_1\right),
\right)
\]
\[
\mp \mu^{-3} \partial_{\pm} X_1, 0
\]
therefore the dressing coset action (3.23) finally reads
\[
S_\mu(X_1, X_2)
= -\frac{1}{2\mu^3} \int d\tau \oint \left((\partial_+ X_1, \partial_- X_1)_G + \mu^2(U_+, U_-)_G + \mu^3(X_1, [U_+, U_-])_G\right)
+ \frac{1}{15} \int d\tau \oint (\text{ad}_{X_1} \partial_+ X_1, \partial_- X_1)_G
+ \frac{1}{6} \int d\tau \oint (\text{ad}_{X_1} \partial_+ X_1, \partial_- X_1)_G
- (U_+, \text{ad}_{X_1} \partial_+ X_1)_G,
\]
where
\[
U_\pm := \partial_{\pm} X_2 + \frac{1}{2} \text{ad}_{X_1} \partial_{\pm} X_1.
\]
The action (5.44) is the 2-field generalization of the pseudo-chiral model of Zakharov et Mikhailov (5.19).

To prove the integrability of the model (5.44) by using the method developed in Sect. 3.2, we start with the ansatz
\[
\hat{O}(z)(0, \xi_1, \xi_2, \xi_3, \xi_4, 0) = \sum_{j=1}^4 p_j(z)\mu^j \xi_j.
\]
The integrability condition (3.26) then require that the four unknown functions \(p_j(z)\) are solutions of the following system of four equations
\[
p_1^2(z) - p_3^2(z) = p_2(z), \quad p_2^2(z) - p_3^2(z) = p_4(z)
\]
\[
p_1(z)p_2(z) - p_3(z)p_4(z) = p_3(z), \quad p_1(z)p_3(z) - p_2(z)p_4(z) = p_4(z),
\]
or, equivalently,
\[
p_i(z)p_j(z) - p_{5-i}(z)p_{5-j}(z) = p_{i+j}(z), \quad 1 \leq i \leq j \leq i + j \leq 4.
\]
The solution is
\[
p_1(z) = \frac{z^3 - 2z}{z^4 - 3z^2 + 1}, \quad p_2(z) = \frac{z^2 - 1}{z^4 - 3z^2 + 1}.
\]
In the present context, the polynomial
\[ p_3(z) = \frac{z}{z^4 - 3z^2 + 1}, \quad p_4(z) = \frac{1}{z^4 - 3z^2 + 1}, \]  
(5.51)

We find also easily the adjoint operator \( O^\dagger(z) : \mathcal{G} \rightarrow \mathcal{D} \)
\[ O^\dagger(z)\xi = (0, p_4(z)\mu^4\xi, p_3(z)\mu^3\xi, p_2(z)\mu^2\xi, p_1(z)\mu\xi, \xi). \]  
(5.52)

Finally, we need the operator \( \hat{r}(z, w) : \mathcal{G} \rightarrow \mathcal{G} \) verifying the conditions (3.31) and (3.32). We assume that \( \hat{r}(z, w) \) is given by the multiplication by a numerical function \( \rho(z, w) \), that is
\[ \hat{r}(z, w)\xi = \mu^5\rho(z, w)\xi. \]  
(5.53)

This assumption works, because the conditions (3.31) and (3.32) become
\[ \sum_{j=0}^{m} p_{5-j}(z) p_{5+j-m}(w) + p_{5-m}(z)\rho(z, w) + p_{5-m}(w)\rho(w, z) = 0, \]  
(5.54)

where \( m = 1, \ldots, 5 \) and we defined \( p_0(z) = 1 \) and \( p_5(z) = 0 \). The conditions (5.54) are then indeed solved by
\[ \rho(z, w) = \frac{-1}{w^4 - 3w^2 + 1} \frac{1}{z - w}. \]  
(5.55)

In the present context, the polynomial \(-(w^4 - 3w^2 + 1)\) is often referred to as the “twist” function.

5.3. \textbf{n-Field Generalization of the Pseudo-chiral Model}

Taking an integer \( n \), we now equip the \((2n + 1)\)th-order jet bundle \( J^{2n+1}G \) with the structure of the Drinfeld double. The group \( D = J^{2n+1}G \) can be conveniently parametrized via the right trivialization as the direct product of manifolds \( J^{2n+1}G = G \times G \times \cdots \times G \) endowed with the group multiplication law\(^6\) and the inverse element
\[
(g, x_1, \ldots, x_j, \ldots)(h, y_1, \ldots, y_j, \ldots)
= (gh, x_1 + \text{Ad}_g y_1, \ldots, x_j + \sum_{i_1 + \cdots + i_j = j} M_{i_1 \ldots i_j} \text{ad}_{x_{i_1}} \cdots \text{ad}_{x_{i_j}} \text{ad}_g y_{i_j}, \ldots),
\]
\[
(g, x_1, \ldots, x_j, \ldots)^{-1}
= (g^{-1}, -\text{Ad}_{g^{-1}} x_1, \ldots, \sum_{i_1 + \cdots + i_j = j} (-1)^{j} M_{i_1 \ldots i_j} \text{ad}_{g^{-1}} x_{i_1} \cdots \text{ad}_{x_{i_j}} x_{i_1}, \ldots),
\]  
(5.56)

where \( g, h \in G, x_i, y_i \in \mathcal{G}, i = 1, \ldots, 2n + 1 \) and the numerical coefficients \( M_{i_1 \ldots i_l} \) are given by the formula
\[ M_{i_1 \ldots i_l} = \prod_{k=1}^{l} \frac{i_k}{\sum_{m=1}^{k} i_m} \equiv \frac{i_1}{i_1} \times \frac{i_2}{i_1 + i_2} \times \frac{i_3}{i_1 + i_2 + i_3} \times \cdots \times \frac{i_l}{i_1 + \cdots + i_l}. \]  
(5.57)

\(^6\)The multiplication law (5.2) differs from that given in [40,58] by suitable normalization conventions, namely \( x_i \in \mathcal{G} \) used in [58] are \( i! \) multiples of our \( x_i \).
Note also that $i_m$ are strictly positive integers and the sum $\sum_{i_1+\ldots+i_l=j}$ runs over all possible partitions of the integer $j$. The order of the partition also matters, e.g., for the integer $j = 3$ we have the partitions $3$, $2 + 1$, $1 + 1 + 1$ or, in more detail, $i_1 = 3$, for $l = 1$, $i_1 = 2$, $i_2 = 1$ as well as $i_1 = 1$, $i_2 = 2$ for $l = 2$ and $i_1 = i_2 = i_3 = 1$ for $l = 3$. The corresponding coefficients $M_{i_1\ldots i_l}$ read (cf. (5.2))

$$M_3 = 1, \quad M_{12} = \frac{2}{3}, \quad M_{21} = \frac{1}{3}, \quad M_{111} = \frac{1}{6}. \quad (5.58)$$

We remark in particular that $M_{12} \neq M_{21}$ which is another illustration of the fact that the order of the partition does matter.

As the vector space, the Lie algebra $\mathcal{D}$ is the direct sum $\mathcal{D} = \mathcal{G} \oplus \cdots \oplus \mathcal{G}$ (2n+2)-times endowed with the Lie bracket

$$[\xi, \chi]_{\mathcal{D}} := [(\xi_0, \xi_1, \ldots, \xi_{2n+1}), (\chi_0, \chi_1, \ldots, \chi_{2n+1})]_{\mathcal{D}} = ([\xi, \chi]_0, [\xi, \chi]_1, \ldots, [\xi, \chi]_{2n+1}), \quad (5.59)$$

where

$$[\xi, \chi]_m := \sum_{j=0}^{m} [[\xi_j, \chi_{m-j}]_{\mathcal{G}}]_{\mathcal{G}}, \quad m = 0, \ldots, 2n+1. \quad (5.60)$$

Here, of course, $\xi, \chi \in \mathcal{D}$, $\xi_i, \chi_i \in \mathcal{G}$, $i = 0, 1, \ldots, 2n + 1$.

For convenience, we add the following useful formulas for the exponential map

$$\exp (t(0, \ldots, 0, \xi_j, 0, \ldots, 0)) = (e_{\mathcal{G}}, 0, \ldots, 0, t\xi_j, 0, \ldots, 0) \in J^{2n+1} \mathcal{G} \quad (5.61)$$

and for the right-invariant Maurer–Cartan form

$$dll^{-1} = dg^{-1} + \text{Ad}_g \sum_{j=1}^{2n+1} \text{Ad}_{\prod_{k=1}^{j-1} \tilde{X}_k} d\tilde{X}_j \tilde{X}_j^{-1} \quad (5.62)$$

where we parametrized the group $J^{2n+1} \mathcal{G}$ in terms of the products of the “pure elements”

$$l = \tilde{g}\tilde{X}_1\tilde{X}_2\ldots\tilde{X}_{2n+1}$$

$$\equiv (g, 0, \ldots, 0)(e_{\mathcal{G}}, X_1, 0, \ldots, 0)(e_{\mathcal{G}}, 0, X_2, 0, \ldots, 0)\ldots(e_{\mathcal{G}}, 0, \ldots, 0, X_{2n+1}). \quad (5.63)$$

Note also that

$$dgg^{-1} = (dg^{-1}, 0, \ldots, 0), \quad (5.64)$$

$$d\tilde{X}_j\tilde{X}_j^{-1} = (0, \ldots, 0, dX_j, 0, \ldots, 0, \frac{1}{2}[X_j, dX_j], 0, \ldots, 0, \frac{1}{n!}\text{ad}^{n-1}_{X_j}dX_j, 0, \ldots). \quad (5.65)$$
The split-signature symmetric non-degenerate ad-invariant bilinear form $(\cdot, \cdot)_D$ on the Lie algebra $D$ is given by
\[
(\xi, \chi)_D = \left( (\xi_0, \xi_1, \ldots, \xi_{2n+1}), (\chi_0, \chi_1, \ldots, \chi_{2n+1}) \right)_D = \sum_{j=0}^{2n+1} (\xi_j, \chi_{2n+1-j})_G.
\]

(5.66)

Now we define an appropriate non-degenerate $\mathcal{E}$-model on the double $D$ by choosing the following operator
\[
\mathcal{E}_\mu(\xi_0, \xi_1, \ldots, \xi_{2n+1}) = -\left( \mu^{2n+1} \xi_{2n+1}, \mu^{2n-1} \xi_{2n}, \ldots, \mu^3 \xi_{n+2}, \mu^2 \xi_{n+1}, \mu^{-1} \xi_n, \mu^{-3} \xi_{n-1}, \ldots, \mu^{-(2n-1)} \xi_1, \mu^{-(2n+1)} \xi_0 \right).
\]

(5.67)

To obtain the degenerate $\mathcal{F}$-model, we use the procedure of Sect. 3.1 and choose the isotropic subalgebra $\mathcal{F} \subset D$ as
\[
\mathcal{F} = \{ (\xi_0, 0, \ldots, 0) \in D, \xi_0 \in G \}.
\]

(5.68)

Consequently, the spaces $V_{\pm}$ are given by
\[
V_{\pm} = (0, \mu^{2n-1} \xi_1, \ldots, \mu^3 \xi_{n-1}, \mu^2 \xi_n, \mp \xi_n, \mp \xi_{n-1}, \ldots, \mp \xi_1, 0).
\]

(5.69)

We can associate the $\sigma$-model action (3.23) to the choices (5.67), (5.68) by choosing the maximally isotropic Lie subalgebra $\mathcal{K} \subset D$ defined as
\[
\mathcal{K} = \{ \xi \in D, \xi_j = 0, \ j \leq n \}.
\]

(5.70)

that is $\mathcal{K}$ is spanned by the elements of $D$ of the form $(0, \ldots, 0, \xi_{n+1}, \ldots, \xi_{2n+1})$. Fixing the gauge with respect to the both $K$ and $F$ as $l_0 = \bar{X}_1 \bar{X}_2 \ldots \bar{X}_n$ we find
\[
dl_0 l_0^{-1} = \sum_{j=1}^{n} \text{Ad}_{\prod_{k=1}^{j-1} \bar{X}_k} d\bar{X}_j \bar{X}_j^{-1},
\]

(5.71)

therefore the WZ term in the action (3.23) is given by the expression
\[
\frac{1}{4} \int \delta^{-1} f \left( (\delta l_0 l_0^{-1}, [\partial_\sigma l_0 l_0^{-1}, \delta l_0 l_0^{-1}])_D \right),
\]

\[
= \frac{1}{4} \int \delta^{-1} f \sum_{k=1}^{2n} \sum_{j=0}^{2n+1-k} \left( (\delta l_0 l_0^{-1})_k, (\partial_\sigma l_0 l_0^{-1})_j, (\delta l_0 l_0^{-1})_{2n+1-k-j} \right)_G,
\]

(5.72)

where $(\delta l_0 l_0^{-1})_j \in G$ stand for the $j^{th}$ component of the Maurer–Cartan form $\delta l_0 l_0^{-1} \in D$. Furthermore, using the formula
\[
\text{Ad}_{\bar{X}_1 \ldots \bar{X}_n} \mathcal{K} = \mathcal{K},
\]

(5.73)

we obtain also
\[
W_{\bar{X}_j} \partial_\pm l_0 l_0^{-1} = \left( 0, (\partial_\pm l_0 l_0^{-1})_1, \ldots, (\partial_\pm l_0 l_0^{-1})_n, \mp \mu^{-1} (\partial_\pm l_0 l_0^{-1})_n, \ldots, \mp \mu^{-(2n-1)} (\partial_\pm l_0 l_0^{-1})_1, 0 \right).
\]

(5.74)
Inserting formulas (5.74) and (5.72) into the second-order dressing coset action (3.23), we obtain the \( n \)-field generalization of the pseudo-chiral model

\[
S_{\mu}(X_1, \ldots, X_n) = \frac{1}{2} \int d\tau \oint \sum_{j=1}^{n} \mu^{-(2n+1-2j)} \left( (\partial_+ l_0 l_0^{-1})_j, (\partial_- l_0 l_0^{-1})_j \right)_{\mathcal{G}} \\
+ \frac{1}{4} \int d\tau \oint \sum_{j=1}^{n} \left( (\partial_+ l_0 l_0^{-1})_j, (\partial_- l_0 l_0^{-1})_{2n+1-j} \right)_{\mathcal{G}} \\
- \left( (\partial_+ l_0 l_0^{-1})_{2n+1-j}, (\partial_- l_0 l_0^{-1})_j \right)_{\mathcal{G}} \\
+ \frac{1}{4} \int \delta^{-1} \oint \sum_{k=1}^{2n} \sum_{j=0}^{2n+1-k} \left( (\delta l_0 l_0^{-1})_k, [\delta l_0 l_0^{-1}]_j, (\partial_+ l_0 l_0^{-1})_{2n+1-k-j} \right)_{\mathcal{G}} \\
+ \frac{1}{4} \int \delta^{-1} \oint \sum_{k=1}^{2n} \sum_{j=0}^{2n+1-k} \left( (\delta l_0 l_0^{-1})_k, [\delta l_0 l_0^{-1}]_j, (\partial_- l_0 l_0^{-1})_{2n+1-k-j} \right)_{\mathcal{G}}. 
\]

(5.75)

We remark that the action (5.75) of the \( n \)-field pseudo-chiral model employs just the components of the Maurer–Cartan form, which means in fact that it is written in the coordinate-invariant way (the first line correspond to the metric part while the second and third one to the Kalb–Ramond part). Of course, we could express it in the coordinates \( X_j \) given by formulas (5.63) by calculating correspondingly the components of the Maurer–Cartan form from (5.71). However, the result of such calculation is quite cumbersome and, anyway, not very illuminating as it can be observed by looking at the 2-field formula (5.44), where we did work out this procedure in detail.

To prove the integrability of the model (5.75), we start with the ansatz

\[
\hat{O}(z)(0, \xi_1, \xi_2, \ldots, \xi_{2n-1}, \xi_{2n}, 0) = \sum_{j=1}^{2n} p_j(z)\mu^j \xi_j. 
\]

(5.76)

The (weak) integrability condition (3.26) then require that the \( 2n \) unknown functions \( p_j(z) \) are solutions of the following system of \( n^2 \) equations

\[
p_i(z)p_j(z) - p_{2n+1-i}(z)p_{2n+1-j}(z) = p_{i+j}(z), \quad 1 \leq i \leq j \leq i+j \leq 2n. 
\]

(5.77)

The system is overdetermined for \( n > 2 \), nevertheless it does possess the needed one-parameter family of solutions. Actually, we find those solutions quite easily taking inspiration from the quadratic identities holding for the Fibonacci numbers. The result is

\[
p_{2k}(z) = (-1)^k \frac{\cos(n-k+1/2)\theta}{\cos(n+1/2)\theta}, \quad p_{2k+1}(z) = (-1)^k \frac{\sin(n-k+1)\theta}{\cos(n+1/2)\theta}, 
\]

(5.78)

where \( k = 1, \ldots, n \) and

\[
\theta = \frac{\arccos(z/2)}{2}. 
\]

(5.79)

We find also easily the adjoint operator \( O^\dagger(z) : \mathcal{G} \rightarrow \mathcal{D} \)
$O^t(z) \xi = (0, p_{2n}(z) \mu^{2n} \xi, p_{2n-1}(z) \mu^{2n-1} \xi, \ldots, p_2(z) \mu^2 \xi, p_1(z) \mu \xi, \xi)$ \quad (5.80)

Finally, we need the operator $\hat{r}(z, w) : \mathcal{G} \to \mathcal{G}$ verifying the (strong) integrability conditions (3.31) and (3.32). We assume that $\hat{r}(z, w)$ is given by the multiplication by a numerical function $\rho(z, w)$, that is

$$\hat{r}(z, w) \xi = \mu^{2n+1} \rho(z, w) \xi.$$  \quad (5.81)

This assumption works, because the conditions (3.31) and (3.32) become

$$\sum_{j=0}^{m} p_{2n+1-j}(z)p_{2n+1+j-m}(w) + p_{2n+1-m}(z)\rho(z, w) + p_{2n+1-m}(w)\rho(w, z) = 0,$$ \quad (5.82)

where $m = 0, \ldots, 2n + 1$ and we defined $p_0(z) = 1$ and $p_{2n+1}(z) = 0$. The conditions (5.82) are then indeed solved by

$$\rho(z, w) = \frac{(-1)^n \cos \frac{\psi}{2} \times \frac{1}{\sin \frac{\psi}{2} - \sin \frac{\theta}{2}}}{2 \cos (2n + 1) \frac{\psi}{2}}.$$ \quad (5.83)

where

$$z = 2 \sin \frac{\theta}{2}, \quad w = 2 \sin \frac{\psi}{2}.$$ \quad (5.84)

In the present context, the “twist” function is $(-1)^{n+1} \frac{\cos (2n+1) \psi}{\cos \frac{\psi}{2}}$ which is the polynomial function if expressed in the variable $w$.

Remark 5.1. To verify, that the function $\rho(z, w)$ given by the expression (5.83) indeed solves the conditions (5.82), it is useful to use the identity

$$f(\phi) + f(\phi + \alpha) + f(\phi + 2\alpha) + \cdots + f(\phi + n\alpha) = \frac{\sin (n + 1) \frac{\alpha}{2}}{\sin \frac{\alpha}{2}} f \left( \phi + \frac{n\alpha}{2} \right),$$ \quad (5.85)

where $f$ stands either for sin or for cos.

6. Conclusions and Outlook

In this paper, we formulated the sufficient conditions for the strong integrability of the dressing cosets. We have also reformulated the non-deformed, $\lambda$-deformed and $\eta$-deformed symmetric space cosets as the dressing cosets and we have shown that the integrability of those three theories can be established by solving our sufficient conditions. Finally, we have introduced the new class of dressing cosets based on higher-order jet bundles of quadratic Lie groups playing the role of the Drinfeld doubles. Those new theories can be interpreted as the (interacting) $n$-field generalizations of the pseudo-chiral $\sigma$-model of Zakharov and Mikhailov. We have solved our sufficient conditions of the strong integrability also in this case and identified the so-called twist functions of those theories which turn out to have poles only at infinity.

As far as the outlook is concerned, we find appealing to study in future how our results could be put in the perspective of two other frameworks which
are currently used along with the $\mathcal{E}$-models to study integrable $\sigma$-models on group manifolds, namely, the formalism based on the 4d Chern–Simons gauge theory [8] as well as the one based on the structure of the affine Gaudin models [11,57].

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