Models for MADM with Single-Valued Neutrosophic 2-Tuple Linguistic Muirhead Mean Operators

Jie Wang 1, Jianping Lu 1, Guiwu Wei 1,*, Rui Lin 2,* and Cun Wei 1,3

1 School of Business, Sichuan Normal University, Chengdu 610101, China; JW970326@163.com (J.W.); luip2002@163.com (J.L.); weicun1990@163.com (C.W.)
2 School of Economics and Management, Chongqing University of Arts and Sciences, Chongqing 402160, China
3 School of Statistics, Southwestern University of Finance and Economics, Chengdu 611130, China
* Correspondence: weiguiwu1973@sicnu.edu.cn (G.W.); linrui@cqw.edu.cn (R.L.)

Received: 24 April 2019; Accepted: 11 May 2019; Published: 17 May 2019

Abstract: In this article, we expand the Muirhead mean (MM) operator and dual Muirhead mean (DMM) operator with single-valued neutrosophic 2-tuple linguistic numbers (SVN2TLNs) to propose the single-valued neutrosophic 2-tuple linguistic Muirhead mean (SVN2TLMM) operator, the single-valued neutrosophic 2-tuple linguistic weighted Muirhead mean (SVN2TLWMM) operator, the single-valued neutrosophic 2-tuple linguistic dual Muirhead mean (SVN2TLDMM) operator, and the single-valued neutrosophic 2-tuple linguistic weighted dual Muirhead mean (SVN2TLWDMM) operator. Multiple attribute decision making (MADM) methods are then proposed using these operators. Finally, we utilize an applicable example for green supplier selection in green supply chain management to prove the proposed methods.

Keywords: multiple attribute decision making (MADM); single-valued neutrosophic 2-tuple linguistic set (SVN2TLSs); SVN2TLMM operator; SVN2TLDMM operator; green supplier selection; green supply chain management

1. Introduction

In order to effectively depict the fuzziness and uncertainty information in real multiple attribute decision making (MADM) problems, Smarandache [1,2] proposed the use of neutrosophic sets (NSs), which have attracted the attention of many scholars. The main advantage of NSs is their capacity to denote inconsistent and indeterminate information. An NS has more potential power than any other fuzzy mathematical tool, such as the fuzzy set [3], the intuitionistic fuzzy set (IFS) [4], and the interval-valued neutrosophic fuzzy set (IVIFS) [5]. However, it is hard to use NSs to solve practical MADM problems. Therefore, Wang et al. [6,7] proposed the use of a single-valued neutrosophic set (SVNS) and an interval neutrosophic set (INS), which can include much more information than fuzzy sets, IFSs, and IVIFSs. Ye [8] proposed the use of MADM with the correlation coefficients of SVNSs. Broumi and Smarandache [9] investigated the correlation coefficients of interval neutrosophic numbers (INNs). Biswas et al. [10] proposed the use of single-valued neutrosophic number TOPSIS (Technique for Order Preference by Similarity to an Ideal Solution) models. Liu et al. [11] developed the generalized Hamacher operations for SVNSs. Sahin and Liu [12] presented the maximizing deviation method using neutrosophic settings. Ye [13] defined some similarity measures of INSs. Zhang et al. [14] defined some interval neutrosophic information aggregating operators. Ye [15] proposed the use of a simplified neutrosophic set (SNS), which included SVNSs and INSs. Many researchers have given their attention to SNSs. For example, Peng et al. [16] presented some basic operational laws of simplified neutrosophic number (SNNs) and
proposed the use of simplified neutrosophic aggregation operators. Additionally, Peng et al. [17] studied an outranking method to handle simplified neutrosophic information, and then Zhang et al. [18] presented an extended version of Peng’s method using an interval neutrosophic environment. Liu and Liu [19] developed a generalized weighted power operator with SVNNs. Deli and Subas [20] discussed a method to rank SVNNs. Peng et al. [21] proposed the use of multi-valued neutrosophic sets and defined some power operators for multiple attribute group decision making (MAGDM). Zhang et al. [22] defined the weighted correlation coefficient for INNs. Chen and Ye [23] proposed the use of Dombi operations with SVNNs. Liu and Wang [24] proposed the use of the SVN normalized weighted Bonferroni mean (WBM). Wu et al. [25] proposed the use of prioritized operator and cross-entropy with SNSs in MADM problems. Li et al. [26] developed some SVNN Heronian mean operators in MADM problems. Xu et al. [27] proposed the use of the TORDIM (an acronym in Portuguese for Interactive Multi-Criteria Decision Making) method for SVN MADM.

Even though SVNSs have been widely used in some areas, all the existing methods are unsuitable for expressing the truth-membership, indeterminacy-membership, and falsity-membership of an element to a 2-tuple linguistic term set, which can affect a decisionmaker’s confidence level when they are making evaluations. In order to overcome this limit, Wu et al. [28] defined the basic concept of single-valued neutrosophic 2-tuple linguistic sets (SVN2TLSs) to cope with this problem on the basis of the SVNSs [6] and 2-tuple linguistic term set [29,30]. Therefore, how to aggregate these single-valued neutrosophic 2-tuple linguistic numbers (SVN2TLNs) is an interesting issue. To solve it, we propose the use of some Muirhead mean (MM) operators with SVN2TLNs. In order to do this, the remainder of this paper is presented as follows: In Section 2, we introduce the concept of SVN2TLSs. In Section 3, we develop some MM operators with SVN2TLNs. In Section 4, we develop some MADM models with SVN2TLNs based on these operators. In Section 5, we present a numerical example to select green suppliers with SVN2TLNs in order to illustrate the method proposed. Section 6 finishes this paper with some concluding remarks.

2. Preliminaries

Wu et al. [28] proposed the use of the concept of SVN2TLSs based on the SVNSs [6] and 2-tuple linguistic term sets [29,30].

2.1. Single-Valued Neutrosophic 2-Tuple Linguistic Sets

**Definition 1 (28).** A SVN2TLS \( A \) in \( X \) is given as follows:

\[
A = \left\{(s_{\theta(x)}, \rho), (T_\theta(x), I_\theta(x), F_\theta(x), x \in X)\right\}
\]

where \( s_{\theta(x)} \in S \), \( T_\theta(x) \in [0,1] \), \( I_\theta(x) \in [0,1] \), and \( F_\theta(x) \in [0,1] \), with corresponding condition \( 0 \leq T_\theta(x) + I_\theta(x) + F_\theta(x) \leq 3 \), \( \forall x \in X \). The values \( T_\theta(x) \), \( I_\theta(x) \), and \( F_\theta(x) \) represent, respectively, the truth-membership, the indeterminacy-membership, and the falsity-membership of the element \( x \) to the linguistic variable \( (s_{\theta(x)}, \rho) \).

For convenience, Wu et al. [28] called \( \bar{a} = \left\{(s_a, \rho_a), (T_a, I_a, F_a)\right\} \) a single-valued neutrosophic 2-tuple linguistic number, where \( T_a \in (0,1) \), \( I_a \in (0,1) \), \( F_a \in (0,1) \), \( 0 \leq T_a + I_a + F_a \leq 3 \) \( s_{\theta(x)} \in S \), and \( \rho \in [-0.5,0.5] \).
Definition 2 ([31]). Let \( \tilde{a}_1 = \left( (s_{a_1}, \rho_1), (T_{a_1}, I_{a_1}, F_{a_1}) \right) \) and \( \tilde{a}_2 = \left( (s_{a_2}, \rho_2), (T_{a_2}, I_{a_2}, F_{a_2}) \right) \) be two SVN2TLNs, \( S(\tilde{a}_1) = \Delta \left( \Delta^{-1} \left( s_{\theta_{a_1}}, \rho_1 \right) \left( \frac{2 + T_{a_1} - I_{a_1} - F_{a_1}}{3} \right) \right) \), \( S(\tilde{a}_2) \in [0, 1] \) and \( S(\tilde{a}_2) = \Delta \left( \Delta^{-1} \left( s_{\theta_{a_2}}, \rho_2 \right) \left( \frac{2 + T_{a_2} - I_{a_2} - F_{a_2}}{3} \right) \right) \), \( S(\tilde{a}_2) \in [0, 1] \) be the scores values of \( \tilde{a}_1 \) and \( \tilde{a}_2 \), respectively, and let \( H(\tilde{a}_1) = \Delta \left( \Delta^{-1} \left( s_{\theta_{a_1}}, \rho_1 \right) \left( T_{a_1} - F_{a_1} \right) \right) \), \( H(\tilde{a}_2) \in [-t, t] \) be the accuracy degrees of \( \tilde{a}_1 \) and \( \tilde{a}_2 \), respectively. Then, if \( S(\tilde{a}_1) < S(\tilde{a}_2) \), \( \tilde{a}_1 < \tilde{a}_2 \); if \( S(\tilde{a}_1) = S(\tilde{a}_2) \), then (1) if \( H(\tilde{a}_1) = H(\tilde{a}_2) \), \( \tilde{a}_1 = \tilde{a}_2 \), and (2) if \( H(\tilde{a}_1) < H(\tilde{a}_2) \), \( \tilde{a}_1 < \tilde{a}_2 \).

Definition 3 ([32]). Let \( \tilde{a}_1 = \left( (s_{a_1}, \rho_1), (T_{a_1}, I_{a_1}, F_{a_1}) \right) \) and \( \tilde{a}_2 = \left( (s_{a_2}, \rho_2), (T_{a_2}, I_{a_2}, F_{a_2}) \right) \) be two SVN2TLNs; then, the following is true:

1. \( \tilde{a}_1 \oplus \tilde{a}_2 = \left( \Delta \left( \Delta^{-1} \left( s_{\theta_{a_1}}, \rho_1 \right) + \Delta^{-1} \left( s_{\theta_{a_2}}, \rho_2 \right) \right) \left( T_{a_1} + T_{a_2} - T_{a_1} - T_{a_2} + I_{a_1} - I_{a_2} + F_{a_1} - F_{a_2} \right) \right) \);
2. \( \tilde{a}_1 \odot \tilde{a}_2 = \left( \Delta \left( \Delta^{-1} \left( s_{\theta_{a_1}}, \rho_1 \right) \Delta^{-1} \left( s_{\theta_{a_2}}, \rho_2 \right) \right) \left( T_{a_1} T_{a_2} - T_{a_1} + I_{a_1} I_{a_2} - I_{a_1} - I_{a_2} + F_{a_1} F_{a_2} + F_{a_1} F_{a_2} \right) \right) \);
3. \( \lambda \tilde{a}_1 = \left( \Delta \left( \Delta^{-1} \left( s_{\theta_{a_1}}, \rho_1 \right) \right) \left( 1 - (1 - T_{a_1})^\lambda, (I_{a_1})^\lambda, (F_{a_1})^\lambda \right) \right), \lambda > 0; \)
4. \( (\tilde{a}_1)^\lambda = \left( \Delta \left( \Delta^{-1} \left( s_{\theta_{a_1}}, \rho_1 \right) \right) \left( (T_{a_1})^\lambda, 1 - (1 - I_{a_1})^\lambda, 1 - (1 - F_{a_1})^\lambda \right) \right), \lambda > 0, \)

where \( \Delta^{-1} \) is the function of converting the 2-tuple linguistic variables to the exact numbers and \( \Delta \) is the function of converting the computing results to the 2-tuple linguistic variables.

2.2. MM Operators

Muirhead [33] proposed the use of the MM operator. Tang et al. [34] developed some interval-valued Pythagorean fuzzy Muirhead mean operators. Wang et al. [35] proposed the use of some picture fuzzy Muirhead mean operators in MADM problems.

Definition 4 ([33]). Let \( a_j (j = 1, 2, \ldots, n) \) be a set of nonnegative real numbers, and \( P = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n \) be a vector of parameters if:

\[
\text{MM}^p (a_1, a_2, \ldots, a_n) = \frac{1}{n!} \sum_{S_n} \left( \prod_{j=1}^{n} a_{\sigma(j)}^p \right) \left( \sum_{p} \prod_{j=1}^{n} a_{\sigma(j)}^p \right) \left( \sum_{p} \prod_{j=1}^{n} a_{\sigma(j)}^p \right).
\]

\( \text{MM}^p \) is the MM operator, where \( \sigma(j) (j = 1, 2, \ldots, n) \) is any permutation of \( \{1, 2, \ldots, n\} \) and \( S_n \) is the set of all permutations of \( \{1, 2, \ldots, n\} \).

3. Some Muirhead Mean Operators with SVN2TLNs
3.1. The Single-Valued Neutrosophic 2-Tuple Linguistic Muirhead Mean (SVN2TLMM) Operator

This section covers MM and its fusing with SVN2TLNs and proposes the SVN2TLMM operator.

**Definition 5.** Let $\tilde{a}_j = \left( (s_j, \rho_j), (T_j, I_j, F_j) \right) (j = 1, 2, \ldots, n)$ be a set of SVN2TLNs. The SVN2TLMM operator is

$$SVN2TLMM^p (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left( \frac{1}{n!} \left( \bigoplus_{\sigma \in \Sigma} \left( \bigotimes_{j=1}^{n} \tilde{a}^{p_j}_{\sigma(j)} \right) \right) \right)^{1 \over \sum_{j=1}^{n} p_j}.$$  \hspace{1cm} (3)

**Theorem 1.** Let $\tilde{a}_j = \left( (s_j, \rho_j), (T_j, I_j, F_j) \right) (j = 1, 2, \ldots, n)$ be a set of SVN2TLNs. The aggregated value by using SVN2TLMM operators is also a SVN2TLN where

$$SVN2TLMM^p (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left( \frac{1}{n!} \left( \bigoplus_{\sigma \in \Sigma} \left( \bigotimes_{j=1}^{n} \tilde{a}^{p_j}_{\sigma(j)} \right) \right) \right)^{1 \over \sum_{j=1}^{n} p_j},$$  \hspace{1cm} (4)

**Proof.** Based on the exponential operation laws of SVN2TLNs, we can derive

$$\tilde{a}^{p_j}_{\sigma(j)} = \left( \Delta \left( \left( \Delta^{-1} (s_j, \rho_j) \right)^{p_j} (T_{\sigma(j)})^{p_j}, 1 - (1 - I_{\sigma(j)})^{p_j}, 1 - (1 - F_{\sigma(j)})^{p_j} \right) \right).$$  \hspace{1cm} (5)

Therefore, by utilizing the multiplication operation laws of SVN2TLNs, the $\bigotimes_{j=1}^{n} \tilde{a}^{p_j}_{\sigma(j)}$ can be derived as follows:

$$\bigotimes_{j=1}^{n} \tilde{a}^{p_j}_{\sigma(j)} = \left( \prod_{j=1}^{n} (T_{\sigma(j)})^{p_j}, 1 - \prod_{j=1}^{n} (1 - I_{\sigma(j)})^{p_j}, 1 - \prod_{j=1}^{n} (1 - F_{\sigma(j)})^{p_j} \right).$$  \hspace{1cm} (6)
Therefore, according to the addition operation laws of SVN2TLNs, we can obtain

\[
    \bigoplus_{\sigma \in \Sigma_2} \left( \bigotimes_{j=1}^{n} \tilde{a}_{\sigma(j)}^p \right) = \left[ \Delta \left( \sum_{\sigma \in \Sigma_2} \left( \prod_{j=1}^{n} (\Delta^{-1}(s_j, \rho_j))^p \right) \right) \right],
\]

(7)

Furthermore, based on the scalar-multiplication operation of SVN2TLNs, we can derive

\[
    \frac{1}{n!} \left( \bigoplus_{\sigma \in \Sigma_2} \left( \bigotimes_{j=1}^{n} \tilde{a}_{\sigma(j)}^p \right) \right) = \left[ \Delta \left( \frac{1}{n!} \left( \sum_{\sigma \in \Sigma_2} \left( \prod_{j=1}^{n} (\Delta^{-1}(s_j, \rho_j))^p \right) \right) \right) \right],
\]

(8)

Therefore, the aggregated value by using SVN2TLMM operators can be listed as follows:

\[
    \text{SVN2TLMM}^p(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left( \frac{1}{n!} \left( \bigoplus_{\sigma \in \Sigma_2} \left( \bigotimes_{j=1}^{n} \tilde{a}_{\sigma(j)}^p \right) \right) \right) \sum_{\mu=0}^{\hat{p}_j} \sum_{\nu=0}^{\hat{p}_j} \left[ \Delta \left( \frac{1}{\mu!} \left( \sum_{\sigma \in \Sigma_2} \left( \prod_{j=1}^{n} (\Delta^{-1}(s_j, \rho_j))^p \right) \right) \right) \right],
\]

(9)

Therefore, Equation (4) is kept. In Equations (4)–(9), the $\Delta^{-1}$ is the function of converting the 2-tuple linguistic variables to the exact numbers and $\Delta$ is the function of converting the computing results to the 2-tuple linguistic variables.

Then, we need to prove that Equation (4) is a SVN2TLN. We need to prove two conditions as follows:
\begin{align*}
\text{① } & 0 \leq T \leq 1, 0 \leq I \leq 1, 0 \leq F \leq 1 \\
\text{② } & 0 \leq T + I + F \leq 3
\end{align*}

Let
\begin{align*}
T &= 1 - \prod_{\sigma \in S_n} 1 - \prod_{j=1}^{n} (T_{\sigma(j)})^{p_j} \\
I &= 1 - \prod_{\sigma \in S_n} 1 - \prod_{j=1}^{n} (1 - I_{\sigma(j)})^{p_j} \\
F &= 1 - \prod_{\sigma \in S_n} 1 - \prod_{j=1}^{n} (1 - F_{\sigma(j)})^{p_j}
\end{align*}

Proof. ① Because $0 \leq T_{\sigma(j)} \leq 1$, we get

\begin{equation}
0 \leq (T_{\sigma(j)})^{p_j} \leq 1 \quad \text{and} \quad 0 \leq 1 - \prod_{j=1}^{n} (T_{\sigma(j)})^{p_j} \leq 1. \quad (10)
\end{equation}

Then,

\begin{equation}
0 \leq \prod_{\sigma \in S_n} 1 - \prod_{j=1}^{n} (T_{\sigma(j)})^{p_j} \leq 1 \quad (11)
\end{equation}

\begin{equation}
0 \leq 1 - \prod_{\sigma \in S_n} 1 - \prod_{j=1}^{n} (T_{\sigma(j)})^{p_j} \leq 1. \quad (12)
\end{equation}

That means $0 \leq T \leq 1$, so ① is maintained. Similarly, we can find that $0 \leq I \leq 1, 0 \leq F \leq 1$.

② Because $0 \leq T \leq 1, 0 \leq I \leq 1, 0 \leq F \leq 1$, $0 \leq T + I + F \leq 3$.

□

Example 1. Let $\{ (s_3, 0), (0.7, 0.5, 0.3) \}, \{ (s_4, 0), (0.8, 0.6, 0.2) \}, \{ (s_5, 0), (0.6, 0.7, 0.1) \}$ be three SVN2TLNs, and $P = (0.3, 0.4, 0.2)$; then, according to Equation (4), we have
$$SVN2TLMM^{0.3,0.4,0.2} \left( \left\{ (s_3,0),(0.7,0.5,0.3) \right\}, \left\{ (s_4,0),(0.8,0.6,0.2) \right\}, \left\{ (s_2,0),(0.6,0.7,0.1) \right\} \right)$$

$$= \left\{ \Delta \left( \frac{1}{n!} \left( \sum_{\sigma \in S_n} \prod_{j=1}^{n} \left( \frac{\Delta^{-1}(s_j, \rho_j)}{p_j} \right)^{p_j} \right)^{p_j} \left( \frac{1}{\sum_{j=1}^{n} p_j} \right) \right) \right\},$$

$$= \left\{ 1 - \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - \frac{1}{\sum_{j=1}^{n} p_j} \right) \left( 1 - \frac{1}{\sum_{j=1}^{n} p_j} \right) \right)^{p_j} \right\}$$
Then, we can identify some properties of the SVN2TLMM operator.

**Property 1. (Idempotency)** If \( \tilde{\alpha}_{\sigma(j)} = \left( (s_j, \rho_j), (T_{\sigma(j)}, I_{\sigma(j)}, F_{\sigma(j)}) \right) \) \((j = 1, 2, \ldots, n)\) are equal, then
\[ \mathbf{SVN2TLMM}^p \left( \tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n \right) = \tilde{a}. \]  \hfill (13)

**Proof.** Because \( \tilde{a}_{\sigma(j)} = \tilde{a} = \left( (s, \rho), (T, I, F) \right) \), then

\[
\begin{align*}
\mathbf{SVN2TLMM}^p \left( \tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n \right) &= \left\{ \frac{1}{n!} \left( \sum_{\sigma \in \mathcal{S}_n} \left( \prod_{j=1}^{n} \left( \Delta^{-1} (s, \rho) \right)^{p_j} \right) \right) \right\} ^{1/n} \\
&= \left\{ \frac{1}{n!} \left( \sum_{\sigma \in \mathcal{S}_n} \left( \prod_{j=1}^{n} \left( 1 - T^{p_j} \right) \right) \right) ^{1/n} \right\} ^{1/n} \\
&= \left( \Delta \left( \Delta^{-1} (s, \rho) \right), (T, I, F) \right) = \tilde{a}.
\end{align*}
\]

\( \square \)

**Property 2.** *(Monotonicity)* Let \( \tilde{a}_j = \left( (s_j, \rho_j), (T_{a_j}, I_{a_j}, F_{a_j}) \right) \) \((j = 1, 2, \ldots, n)\) and \( \tilde{b}_j = \left( (s_j, \rho_j), (T_{b_j}, I_{b_j}, F_{b_j}) \right) \) \((j = 1, 2, \ldots, n)\) be two sets of SVN2TLNs. If \( \Delta^{-1} \left( S_j, \rho_j \right) \leq \Delta^{-1} \left( S_{b_j}, \rho_{b_j} \right) \) and \( T_{a_j} \leq T_{b_j} \) and \( I_{a_j} \geq I_{b_j} \) and \( F_{a_j} \geq F_{b_j} \) hold for all \( j \), then

\[ \mathbf{SVN2TLMM}^p \left( \tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n \right) \leq \mathbf{SVN2TLMM}^p \left( \tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n \right). \]  \hfill (14)

**Proof.** Let \( \mathbf{SVN2TLMM}^p \left( \tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n \right) = \left( (s_a, \rho_a), (T_a, I_a, F_a) \right) \) and \( \mathbf{SVN2TLMM}^p \left( \tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n \right) = \left( (s_b, \rho_b), (T_b, I_b, F_b) \right) \). Given that \( \Delta^{-1} \left( S_a, \rho_a \right) \leq \Delta^{-1} \left( S_b, \rho_b \right) \), we can obtain
\[ \prod_{j=1}^{n} (\Delta^{-1}(S_a, \rho_a))^{p_j} \leq \prod_{j=1}^{n} (\Delta^{-1}(S_h, \rho_h))^{p_j} \]  
(15)  
\[ \sum_{\sigma \in S_h} \left( \prod_{j=1}^{n} (\Delta^{-1}(S_a, \rho_a))^{p_j} \right) \leq \sum_{\sigma \in S_h} \left( \prod_{j=1}^{n} (\Delta^{-1}(S_h, \rho_h))^{p_j} \right). \]  
(16)  
Therefore,  
\[ \Delta \left( \frac{1}{n!} \sum_{\sigma \in S_h} \left( \prod_{j=1}^{n} (\Delta^{-1}(S_a, \rho_a))^{p_j} \right) \right)^{1/n} \leq \Delta \left( \frac{1}{n!} \sum_{\sigma \in S_h} \left( \prod_{j=1}^{n} (\Delta^{-1}(S_h, \rho_h))^{p_j} \right) \right)^{1/n} \]  
(17)  
That means \((S_a, \rho_a) \leq (S_h, \rho_h)\). Given that \(T_a \leq T_h\), we can also obtain  
\[ \prod_{j=1}^{n} T_{a_j}^{p_j} \leq \prod_{j=1}^{n} T_{h_j}^{p_j} \]  
(18)  
\[ \prod_{j=1}^{n} \left( 1 - \prod_{j=1}^{n} T_{a_j}^{p_j} \right)^{1/n} \geq \prod_{j=1}^{n} \left( 1 - \prod_{j=1}^{n} T_{h_j}^{p_j} \right)^{1/n} \]  
(19)  
\[ \left( 1 - \prod_{j=1}^{n} T_{a_j}^{p_j} \right)^{1/n} \left( \sum_{\sigma \in S_h} \prod_{j=1}^{n} (\Delta^{-1}(S_a, \rho_a))^{p_j} \right)^{1/n} \leq \left( 1 - \prod_{j=1}^{n} T_{a_j}^{p_j} \right)^{1/n} \left( \sum_{\sigma \in S_h} \prod_{j=1}^{n} (\Delta^{-1}(S_h, \rho_h))^{p_j} \right)^{1/n}. \]  
(20)  
That is \(T_a \leq T_h\). Similarly, we can obtain \(I_a \geq I_h\) and \(F_a \geq F_h\).  
If \(\Delta^{-1}(S_a, \rho_a) < \Delta^{-1}(S_h, \rho_h)\) and \(T_a \leq T_h\), \(I_a \geq I_h\) and \(F_a \geq F_h\) ,  
SVN2TLMM\(^p\) \((\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) < SVN2TLMM\(^p\) \((\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n)\)  
If \(\Delta^{-1}(S_a, \rho_a) = \Delta^{-1}(S_h, \rho_h)\) and \(T_a < T_h\), \(I_a > I_h\) and \(F_a > F_h\),  
SVN2TLMM\(^p\) \((\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = SVN2TLMM\(^p\) \((\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n)\)  
If \(\Delta^{-1}(S_a, \rho_a) = \Delta^{-1}(S_h, \rho_h)\) and \(T_a = T_h\), \(I_a = I_h\) and \(F_a = F_h\),  
SVN2TLMM\(^p\) \((\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = SVN2TLMM\(^p\) \((\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n)\).  
So, Property 2 is correct. \(\square\)  

**Property 3. (Boundedness)** Let \(\tilde{a}_j = \{ (s_j, \rho), (T_j, I_j, F_j) \} (j = 1, 2, \ldots, n)\) be a set of SVN2TLNs. If  
\[ \tilde{a}_i^+ = \left( \max_j (s_j, \rho), (\max_j (T_j), \min_j (I_j), \min_j (F_j)) \right) \]  
and  
\[ \tilde{a}_i^- = \left( \min_j (s_j, \rho), (\min_j (T_j), \max_j (I_j), \max_j (F_j)) \right), \]  
then  
\[ \tilde{a}^- \leq SVN2TLMM\(^p\) (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq \tilde{a}^+. \]  
(21)  
From Property 1:
\[
\text{SVN2TLMM}^p (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \tilde{a}^-
\]
\[
\text{SVN2TLMM}^p (\tilde{a}_1^+, \tilde{a}_2^+, \ldots, \tilde{a}_n^+) = \tilde{a}^+.
\]

From Property 2:
\[
\tilde{a}^- \leq \text{SVN2TLMM}^p (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq \tilde{a}^+.
\]

3.2. The Single-Valued Neutrosophic 2-Tuple Linguistic Weighted Muirhead Mean (SVN2TLWMM) Operator

In actual MADM, it is important to consider attribute weights. This section proposes the use of a SVN2TLWMM operator as follows:

**Definition 6.** Let \( \tilde{a}_j = \left\{ (s_j, \rho_j), (T_j, I_j, F_j) \right\} (j = 1, 2, \ldots, n) \) be a set of SVN2TLNs with a weight vector of \( w = (w_1, w_2, \ldots, w_n)^T \) , thereby satisfying \( w_i \in [0,1] \) and \( \sum_{i=1}^{n} w_i = 1 \) , and let \( P = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n \) be a vector of parameters if

\[
\text{SVN2TLWMM}^p_{\text{me}} (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left( \frac{1}{n!} \left( \bigotimes_{\sigma \in S_n} \left( \bigotimes_{j=1}^{n} w_{\sigma(j)}^{p_{\sigma(j)}} \tilde{a}_{\sigma(j)}^{p_{\sigma(j)}} \right) \right) \right)^{1/\sum_{i=1}^{n} p_i}. \quad (22)
\]

SVN2TLWMM\(^p_{\text{me}}\) is the single-valued neutrosophic 2-tuple linguistic MM, where \( \sigma(j)(j = 1, 2, \ldots, n) \) is any permutation of \( \{1, 2, \ldots, n\} \) and \( S_n \) is the set of all permutations of \( \{1, 2, \ldots, n\} \).

**Theorem 2.** Let \( \tilde{a}_j = \left\{ (s_j, \rho_j), (T_j, I_j, F_j) \right\} (j = 1, 2, \ldots, n) \) be a set of SVN2TLNs. The aggregated value by using SVN2TLWMM operators is also a SVN2TLN where
Proof. From the exponential operation of the SVN2TLNs, we can derive
\[
\tilde{a}_{\sigma(j)} = \left\{ \Delta\left( \left( \Delta^{-1} \left( s_j, \rho_j \right) \right)^p \right), \left\{ T^p_{\sigma(j)}, 1 - (1 - I_{\sigma(j)})^p, 1 - (1 - F_{\sigma(j)})^p \right\} \right\}. \tag{24}
\]
Therefore, by utilizing the scalar-multiplication operation laws of the SVN2TLNs, the \( n w_{\sigma(j)} \tilde{a}_{\sigma(j)}^p \) can be derived as
\[
\begin{align*}
\Delta \left( n w_{\sigma(j)} \left( \Delta^{-1} \left( s_j, \rho_j \right) \right)^p \right), \\
1 - \left( 1 - T^p_{\sigma(j)} \right)^{n w_{\sigma(j)}}, \\
1 - \left( 1 - I_{\sigma(j)} \right)^{n w_{\sigma(j)}}, \\
1 - \left( 1 - F_{\sigma(j)} \right)^{n w_{\sigma(j)}} \end{align*}
\]
Therefore, by utilizing the addition operation of the SVN2TLNs, we can get
\[
\bigotimes_{j=1}^{n} \text{nw}_{\alpha(j)} \tilde{\alpha}_{\sigma(j)} = \Delta \left( \prod_{j=1}^{n} \left( \text{nw}_{\alpha(j)} \left( \Delta^{-1} \left( s_{j}, \rho_{j} \right) \right)^{p_{j}} \right) \right),
\]

\[
= \left\{ \prod_{j=1}^{n} \left( 1 - \left( 1 - T_{\alpha(j)}^{p_{j}} \right)^{\text{nw}_{\alpha(j)}} \right) \right\} \cdot \left\{ \prod_{j=1}^{n} \left( 1 - \left( 1 - I_{\alpha(j)}^{p_{j}} \right)^{\text{nw}_{\alpha(j)}} \right) \right\}.
\] (26)

Therefore, by utilizing the addition operation of the SVN2TLNs, we can get
\[
\bigoplus_{\sigma \in S_{n}} \left( \bigotimes_{j=1}^{n} \text{nw}_{\alpha(j)} \tilde{\alpha}_{\sigma(j)} \right) = \Delta \left( \sum_{\sigma \in S_{n}} \left( \prod_{j=1}^{n} \left( \text{nw}_{\alpha(j)} \left( \Delta^{-1} \left( s_{j}, \rho_{j} \right) \right)^{p_{j}} \right) \right) \right),
\]

\[
= \left\{ \prod_{\sigma \in S_{n}} \left( 1 - \left( 1 - T_{\sigma(j)}^{p_{j}} \right)^{\text{nw}_{\sigma(j)}} \right) \right\} \cdot \left\{ \prod_{\sigma \in S_{n}} \left( 1 - \left( 1 - I_{\sigma(j)}^{p_{j}} \right)^{\text{nw}_{\sigma(j)}} \right) \right\}.
\] (27)

Furthermore, based on the scalar-multiplication operation of the SVN2TLNs, we can derive
\[
\frac{1}{n!} \left( \bigoplus_{\sigma \in S_{n}} \left( \bigotimes_{j=1}^{n} \text{nw}_{\alpha(j)} \tilde{\alpha}_{\sigma(j)} \right) \right) = \Delta \left( \frac{1}{n!} \sum_{\sigma \in S_{n}} \left( \prod_{j=1}^{n} \left( \text{nw}_{\alpha(j)} \left( \Delta^{-1} \left( s_{j}, \rho_{j} \right) \right)^{p_{j}} \right) \right) \right),
\]

\[
= \left\{ \prod_{\sigma \in S_{n}} \left( 1 - \left( 1 - T_{\sigma(j)}^{p_{j}} \right)^{\text{nw}_{\sigma(j)}} \right) \right\} \cdot \left\{ \prod_{\sigma \in S_{n}} \left( 1 - \left( 1 - I_{\sigma(j)}^{p_{j}} \right)^{\text{nw}_{\sigma(j)}} \right) \right\} \cdot \left\{ \prod_{\sigma \in S_{n}} \left( 1 - \left( 1 - F_{\sigma(j)}^{p_{j}} \right)^{\text{nw}_{\sigma(j)}} \right) \right\}.
\] (28)

Therefore, the aggregated value by using SVN2TLWMM operators can be listed as follows:
Therefore, Equation (23) is kept. In Equations (23)–(29), $\Delta^{-1}$ is the function of converting the 2-tuple linguistic variables to the exact numbers and $\Delta$ is the function of converting the computing results to the 2-tuple linguistic variables.

Then we need to prove that Equation (23) is a SVN2TLN.

1. $0 \leq T \leq 1, 0 \leq I \leq 1, 0 \leq F \leq 1$
2. $0 \leq T + I + F \leq 3$.

**Proof.** Let

$$T = \left( 1 - \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - T_{\sigma(j)}^{p_j} \right)^{w_{\sigma(j)}} \right) \right)^{\frac{1}{n!}} \sum_{j=1}^{n} p_j$$

$$I = \left( 1 - \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - I_{\sigma(j)}^{p_j} \right)^{w_{\sigma(j)}} \right) \right)^{\frac{1}{n!}} \sum_{j=1}^{n} p_j$$

$$F = \left( 1 - \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - F_{\sigma(j)}^{p_j} \right)^{w_{\sigma(j)}} \right) \right)^{\frac{1}{n!}} \sum_{j=1}^{n} p_j$$

Because $0 \leq T_{\sigma(j)} \leq 1$, we get
\[
0 \leq T_{\sigma(j)}^p \leq 1 \quad \text{and} \quad 0 \leq 1 - \left(1 - T_{\sigma(j)}^p\right)^{nw_{\sigma(j)}} \leq 1.
\] (30)

Then,

\[
0 \leq 1 - \prod_{j=1}^{n} \left(1 - \left(1 - T_{\sigma(j)}^p\right)^{nw_{\sigma(j)}}\right) \leq 1
\] (31)

\[
0 \leq \left(\prod_{\sigma \in S_{n}} \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - T_{\sigma(j)}^p\right)^{nw_{\sigma(j)}}\right)\right)\right)^{\frac{1}{n!}} \leq 1
\] (32)

\[
0 \leq \left(1 - \left(\prod_{\sigma \in S_{n}} \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - T_{\sigma(j)}^p\right)^{nw_{\sigma(j)}}\right)\right)\right)\right)^{\frac{1}{n!}} \left[\sum_{j=1}^{n} p_{j}\right] \leq 1.
\] (33)

That means \(0 \leq T \leq 1\), so ① is maintained.

Similarly, we can get \(0 \leq I \leq 1, 0 \leq F \leq 1\).

② Because \(0 \leq T \leq 1, 0 \leq I \leq 1, 0 \leq F \leq 1, 0 \leq T + I + F \leq 3\).

□

Example 2. Let \(\{(s_3, 0), (0.7, 0.5, 0.3)\}, \{(s_4, 0), (0.8, 0.6, 0.2)\}\) and \(\{(s_2, 0), (0.6, 0.7, 0.1)\}\) be three SVN2TLNs, \(P=(0.3,0.4,0.2)\) and \(w=(0.4,0.3,0.3)\), then according to (23) we have:

\[
\text{SVN2TLWMM}_{(0.3,0.4,0.2)}^{(0.4,0.3,0.3)}\left(\{(s_3, 0), (0.7, 0.5, 0.3)\}, \{(s_4, 0), (0.8, 0.6, 0.2)\}\right)
\]

\[
\begin{aligned}
&= \left(\frac{\sum_{j=1}^{n} \left(\sum_{\sigma \in S_{n}} \left(\prod_{j=1}^{n} \left(1 - \left(1 - T_{\sigma(j)}^p\right)^{nw_{\sigma(j)}}\right)\right)\right)^{\frac{1}{n!}}}{\sum_{j=1}^{n} p_{j}}\right)^{\frac{1}{n!}} \\
&= \left(\frac{\sum_{j=1}^{n} \left(\prod_{\sigma \in S_{n}} \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - T_{\sigma(j)}^p\right)^{nw_{\sigma(j)}}\right)\right)\right)^{\frac{1}{n!}}}{\sum_{j=1}^{n} p_{j}}\right)^{\frac{1}{n!}} \\
&= \left(\frac{\sum_{j=1}^{n} \left(\prod_{\sigma \in S_{n}} \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - I_{\sigma(j)}^p\right)^{nw_{\sigma(j)}}\right)\right)\right)^{\frac{1}{n!}}}{\sum_{j=1}^{n} p_{j}}\right)^{\frac{1}{n!}} \\
&= \left(\frac{\sum_{j=1}^{n} \left(\prod_{\sigma \in S_{n}} \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - F_{\sigma(j)}^p\right)^{nw_{\sigma(j)}}\right)\right)\right)^{\frac{1}{n!}}}{\sum_{j=1}^{n} p_{j}}\right)^{\frac{1}{n!}}
\end{aligned}
\]
We will now discuss some properties of the SVN2TLWMM operator.

**Property 4. (Monotonicity)** Let \( \tilde{a}_j = \left\{ (s_{a_j}, \rho_{a_j}), (T_{a_j}, I_{a_j}, F_{a_j}) \right\} (j = 1, 2, \ldots, n) \) and \( \tilde{b}_j = \left\{ (s_{b_j}, \rho_{b_j}), (T_{b_j}, I_{b_j}, F_{b_j}) \right\} (j = 1, 2, \ldots, n) \) be two sets of SVN2TLNs. If \( \Delta^{-1}(s_{a_j}, \rho_{a_j}) \leq \Delta^{-1}(s_{b_j}, \rho_{b_j}) \) and \( T_{a_j} \leq T_{b_j} \) and \( I_{a_j} \geq I_{b_j} \) and \( F_{a_j} \geq F_{b_j} \) hold for all \( j \), then:

\[
\text{SVN2TLWMM}_{nv}^p (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq \text{SVN2TLWMM}_{nv}^p (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n)
\]  

(34)

The proof is similar to SVN2TLMM. It is omitted here.

**Property 5. (Boundedness)** Let \( \tilde{a}_i = \left\{ (s_i, \rho_i), (T_i, I_i, F_i) \right\} (i = 1, 2, \ldots, n) \) be a set of SVN2TLNs. If

\[
\tilde{a}^- = \left( \max_i (S_i, \rho_i), \left( \max_i (T_i), \min_i (I_i), \min_i (F_i) \right) \right)
\]

and

\[
\tilde{a}^+ = \left( \min_i (S_i, \rho_i), \left( \min_i (T_i), \max_i (I_i), \max_i (F_i) \right) \right)
\]

then:

\[
\tilde{a}^- \leq \text{SVN2TLWMM}_{nv}^p (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq \tilde{a}^+
\]

(35)

From Theorem 2, we get:

\[
\text{SVN2TLWMM}_{nv}^p (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)
\]

\[
= \left\{ \frac{1}{n!} \sum_{\sigma \in S_n} \left( \prod_{j=1}^{n} \min_{\sigma(j)} \left( \min \Delta^{-1} (s_j, \rho_j) \right)^{p_j} \right) \right\} \frac{1}{\sum_{j=1}^{n} p_j}
\]

\[
= \left\{ \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - \min \Delta^{-1} (s_j, \rho_j) \right) \right) \right\} \frac{1}{\sum_{j=1}^{n} p_j}
\]

\[
= \left\{ \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - (1 - \min \Delta^{-1} (s_j, \rho_j) \right) \right) \right\} \frac{1}{\sum_{j=1}^{n} p_j}
\]

\[
\left\{ \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - (1 - \max \Delta^{-1} (s_j, \rho_j) \right) \right) \right\} \frac{1}{\sum_{j=1}^{n} p_j}
\]

\[
\left\{ \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - (1 - \max \Delta^{-1} (s_j, \rho_j) \right) \right) \right\} \frac{1}{\sum_{j=1}^{n} p_j}
\]  

(36)
SVN2TLWMM$_{m}^{\rho}\left(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n\right)$

$$\Delta\left(\frac{1}{n!}\sum_{\sigma \in S_n}\left(\prod_{j=1}^{n}nw_{\sigma(j)}\left(\max\Delta^{-1}\left(s_j,\rho_j\right)^{\rho}\right)\right)\right)^{\frac{1}{n!}}$$

$$\left(1-\prod_{\sigma \in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\max T_{\sigma(j)}^{\rho}\right)_{\mathsf{w}_{\sigma(j)}}\right)\right)\right)^{\frac{1}{n!}}$$

$$\left(1-\prod_{\sigma \in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\min I_{\sigma(j)}^{\rho}\right)_{\mathsf{w}_{\sigma(j)}}\right)\right)\right)^{\frac{1}{n!}}$$

$$\left(1-\prod_{\sigma \in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\min F_{\sigma(j)}^{\rho}\right)_{\mathsf{w}_{\sigma(j)}}\right)\right)\right)^{\frac{1}{n!}}$$

From Property 4, we get:

$$\tilde{a}^{-}\leq\text{SVN2TLWMM}_{m}^{\rho}\left(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n\right)\leq\tilde{a}^{+}$$

(38)

It is obvious that SVN2TLMM operators lacks the property of idempotency.

### 3.3. The Single-Valued Neutrosophic 2-Tuple Linguistic Dual Muirhead Mean (SVN2TLDMM) Operator

Qin and Liu [36] proposed the use of the dual Muirhead mean (DMM) operator.

**Definition 7.** Let $a_j (j = 1, 2, \ldots, n)$ be a set of non-negative real numbers and $P = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$ be a vector of parameters if:

$$\text{DMM}^{\rho}(a_1, a_2, \ldots, a_n) = \frac{1}{\sum_{j=1}^{n}p_j} \left(\prod_{\sigma \in S_n} \sum_{j=1}^{n}p_ja_{\sigma(j)}\right)^{\frac{1}{n!}}$$

(39)

DMM$^{\rho}$ is the dual Muirhead mean (DMM) operator, where $\sigma(j) (j = 1, 2, \ldots, n)$ is any permutation of $\{1, 2, \ldots, n\}$, and $S_n$ is the set of all permutations of $\{1, 2, \ldots, n\}$.

In this section we will propose the SVN 2-tuple linguistic DMM (SVN2TLDMM) operator.

**Definition 8.** Let $\tilde{a}_j = \left\langle \left(s_j, \rho_j\right), \left(T_j, I_j, F_j\right)\right\rangle (j = 1, 2, \ldots, n)$ be a set of SVN2TLNs and let $P = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$ be a vector of parameters if:

$$\text{SVN2TLDMM}^{\rho}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \frac{1}{\sum_{j=1}^{n}p_j} \left(\bigotimes_{\sigma \in S_n} \left(\prod_{j=1}^{n}p_j\tilde{a}_{\sigma(j)}\right)\right)^{\frac{1}{n!}}$$

(40)
where $\sigma(j)(j=1,2,\ldots,n)$ is any permutation of $\{1,2,\ldots,n\}$ and $S_n$ is the set of all permutations of $\{1,2,\ldots,n\}$.

**Theorem 3.** Let $\bar{a}_j = \left\{ (s_j, \rho_j), (T_j, I_j, F_j) \right\} (j=1,2,\ldots,n)$ be a set of SVN2TLNs. The aggregated value by using the SVN2TLDMM operators is also a SVN2TLN where:

\[
\text{SVN2TLDMM}^p(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n)
\]

\[
= \frac{1}{\sum_{j=1}^n p_j} \left( \bigotimes_{\sigma \in S_n} \left( p_j \tilde{\sigma}(j) \right) \right) \frac{1}{n!} \left( \sum_{j=1}^n p_j \sigma(j) \right) \frac{1}{n!} \left( \prod_{j=1}^n (1 - P_j \sigma(j)) \right) \frac{1}{n!} \left( \prod_{j=1}^n (1 - I_j \sigma(j)) \right) \frac{1}{n!} \left( \prod_{j=1}^n (1 - F_j \sigma(j)) \right)
\]

(41)

**Proof.** From the multiplication operation laws of SVN2TLNs depicted in Definition 3, we can obtain:

\[
p_j a_{\sigma(j)} = \left\{ p_j \sigma^{-1}(s_j, \rho_j), \left( 1 - T_{\sigma(j)} \right)^{p_j}, \left( 1 - I_{\sigma(j)} \right)^{p_j}, \left( 1 - F_{\sigma(j)} \right)^{p_j} \right\}
\]

(42)

Therefore, according to the addition operation of SVN2TLNs, we can derive:

\[
\bigotimes_{j=1}^n \left( p_j a_{\sigma(j)} \right) = \left\{ \sum_{j=1}^n \left( p_j \sigma^{-1}(s_j, \rho_j) \right), \left( 1 - \prod_{j=1}^n \left( 1 - T_{\sigma(j)} \right)^{p_j} \right), \left( 1 - \prod_{j=1}^n \left( 1 - I_{\sigma(j)} \right)^{p_j} \right), \left( 1 - \prod_{j=1}^n \left( 1 - F_{\sigma(j)} \right)^{p_j} \right) \right\}
\]

(43)

Therefore, based on the multiplication operation of SVN2TLNs, we can get:

\[
\bigotimes_{\sigma \in S_n} \left( \bigotimes_{j=1}^n \left( p_j a_{\sigma(j)} \right) \right) = \left\{ \prod_{\sigma \in S_n} \left( \sum_{j=1}^n \left( p_j \sigma^{-1}(s_j, \rho_j) \right) \right), \left( 1 - \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^n \left( 1 - T_{\sigma(j)} \right)^{p_j} \right) \right), \left( 1 - \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^n \left( 1 - I_{\sigma(j)} \right)^{p_j} \right) \right), \left( 1 - \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^n \left( 1 - F_{\sigma(j)} \right)^{p_j} \right) \right) \right\}
\]

(44)

Furthermore, by utilizing the exponential operation of SVN2TLNs we can derive:
Therefore, the aggregated results using the SVN2TLDM operator can be shown as:

\[
\text{SVN2TLDM}^p(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \frac{1}{\sum_{j=1}^n P_j} \left( \bigotimes_{\sigma \in \mathcal{S}_x} \left( \bigoplus_{j=1}^n \left( p_j \Delta^{-1} \left( s_j, \rho_j \right) \right) \right) \right) \cdot \Delta \left( \prod_{\sigma \in \mathcal{S}_x} \left( \prod_{j=1}^n \left( p_j \Delta^{-1} \left( s_j, \rho_j \right) \right) \right) \right)^{\frac{1}{n!}},
\]

(45)

Therefore, (41) is kept. In Equations (41)–(46), the symbol “\( \Delta^{-1} \)” is the function of converting the 2-tuple linguistic variables to the exact numbers and “\( \Delta \)” is the function of converting the computing results to the 2-tuple linguistic variables.

We now need to prove that (41) is a SVN2TLN. We need to prove the two conditions:

1. \( 0 \leq T \leq 1, 0 \leq I \leq 1, 0 \leq F \leq 1 \)
2. \( 0 \leq T + I + F \leq 3 \)

Let

\[
T = 1 - \left( \prod_{\sigma \in \mathcal{S}_x} \left( \prod_{j=1}^n \left( 1 - T_{\sigma(j)} \right)^{p_j} \right) \right) \cdot \prod_{\sigma \in \mathcal{S}_x} \left( \prod_{j=1}^n \left( 1 - I_{\sigma(j)} \right)^{p_j} \right) \cdot \prod_{\sigma \in \mathcal{S}_x} \left( \prod_{j=1}^n \left( 1 - F_{\sigma(j)} \right)^{p_j} \right)
\]

\[
I = \left( \prod_{\sigma \in \mathcal{S}_x} \left( \prod_{j=1}^n \left( 1 - I_{\sigma(j)} \right) \right) \right) \cdot \prod_{\sigma \in \mathcal{S}_x} \left( \prod_{j=1}^n \left( 1 - F_{\sigma(j)} \right)^{p_j} \right)
\]

\[
F = \left( \prod_{\sigma \in \mathcal{S}_x} \left( \prod_{j=1}^n \left( 1 - F_{\sigma(j)} \right) \right) \right) \cdot \prod_{\sigma \in \mathcal{S}_x} \left( \prod_{j=1}^n \left( 1 - I_{\sigma(j)} \right)^{p_j} \right)
\]
Since \(0 \leq T_{\sigma(j)} \leq 1\), we get:

\[
0 \leq \left(1 - T_{\sigma(j)}\right)^{p_j} \leq 1 \quad \text{and} \quad 0 \leq 1 - \prod_{j=1}^{n} \left(1 - T_{\sigma(j)}\right)^{p_j} \leq 1
\]

(47)

\[
0 \leq 1 - \prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - T_{\sigma(j)}\right)^{p_j}\right) \leq 1
\]

(48)

Then:

\[
0 \leq 1 - \left(1 - \prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - T_{\sigma(j)}\right)^{p_j}\right)\right) \frac{1}{\sum_{j=1}^{n} p_j} \leq 1
\]

(49)

That means \(0 \leq T \leq 1\), so \(1\) is maintained.

Similarly, we can get \(0 \leq I \leq 1, 0 \leq F \leq 1\)

(2) Because \(0 \leq T \leq 1, 0 \leq I \leq 1, 0 \leq F \leq 1\), \(0 \leq T + I + F \leq 3\).

**Example 3.** Let \(\{s_1, 0\}, \{0.7, 0.5, 0.3\}, \{s_2, 0\}, \{0.6, 0.7, 0.1\}\) be three SVN2TLNs, and \(P = (0.3, 0.4, 0.2)\); then, according to Equation (41), we have

\[
\text{SVN2TLDMM}^{(0.3, 0.4, 0.2)} \left(\{s_1, 0\}, \{0.7, 0.5, 0.3\}\right) = \left(\{s_2, 0\}, \{0.6, 0.7, 0.1\}\right)
\]

\[
= \beta \left[ \frac{1}{\sum_{j=1}^{n} p_j} \prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - T_{\sigma(j)}\right)^{p_j}\right) \frac{1}{\sum_{j=1}^{n} p_j} \right],
\]

\[
= \left[ 1 - \prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - T_{\sigma(j)}\right)^{p_j}\right) \right] \frac{1}{\sum_{j=1}^{n} p_j} \left[ 1 - \prod_{\sigma \in S_n} \left(1 - I_{\sigma(j)}\right)^{p_j}\right] \frac{1}{\sum_{j=1}^{n} p_j}.
\]

Then:

\[
0 \leq 1 - \prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - T_{\sigma(j)}\right)^{p_j}\right) \frac{1}{\sum_{j=1}^{n} p_j} \leq 1
\]

(48)
Similar to the SVN2TLMM operator, we can get the properties as follows:

**Property 6. (Idempotency)** If \( \tilde{a}_{\sigma(j)} = \left( (s_j, \rho_j), (T_{\sigma(j)}, I_{\sigma(j)}, F_{\sigma(j)}) \right) \) \((j = 1, 2, \ldots, n)\) are equal, then

\[ \text{SVN2TLDM}^p(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \tilde{a}. \quad (50) \]

**Property 7. (Monotonicity)** Let \( \tilde{a}_j = \left( (s_j, \rho_j), (T_{a_j}, I_{a_j}, F_{a_j}) \right) \) \((j = 1, 2, \ldots, n)\) and \( \tilde{b}_j = \left( (s_j, \rho_j), (T_{b_j}, I_{b_j}, F_{b_j}) \right) \) \((j = 1, 2, \ldots, n)\) be two sets of SVN2TLNs. If

\[ \Delta^{-1}(S_{a_j}, \rho_{a_j}) \leq \Delta^{-1}(S_{b_j}, \rho_{b_j}) \text{ and } T_{a_j} \leq T_{b_j} \text{ and } I_{a_j} \geq I_{b_j} \text{ and } F_{a_j} \geq F_{b_j} \]

hold for all \( j \), then

\[ \text{SVN2TLDM}^p(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq \text{SVN2TLDM}^p(\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n). \quad (51) \]

**Property 8. (Boundedness)** Let \( \tilde{a}_i = \left( (s_i, \rho_i), (T_i, I_i, F_i) \right) \) \((i = 1, 2, \ldots, n)\) be a set of SVN2TLNs. If

\[ \tilde{a}_i^+ = \left( \max_i (S_i, \rho_i), \left( \max_i (T_i), \min_i (I_i), \min_i (F_i) \right) \right) \]

and

\[ \tilde{a}_i^- = \left( \min_i (S_i, \rho_i), \left( \min_i (T_i), \max_i (I_i), \max_i (F_i) \right) \right), \]

then

\[ \tilde{a}^- \leq \text{SVN2TLDM}^p(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq \tilde{a}^+. \quad (52) \]
3.4. The Single-Valued Neutrosophic 2-Tuple Linguistic Weighted Dual Muirhead Mean (SVN2TLWDMM) Operator

In actual MADM, it is important to consider attribute weights. This section proposes the use of a SVN2TLWDMM operator.

**Definition 9.** Let \( \tilde{a}_j = \left( \left( s_j, \rho_j \right), \left( T_j, I_j, F_j \right) \right) \) \( (j = 1, 2, \ldots, n) \) be a set of SVN2TLNs with a weight vector of \( w = (w_1, w_2, \ldots, w_n)^T \), thereby satisfying \( w_i \in [0,1] \) and \( \sum_{i=1}^{n} w_i = 1 \), and let \( P = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n \) be a vector of parameters if

\[
\text{SVN2TLWDMM}^p_{\text{nw}} (\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \frac{1}{\sum_{j=1}^{n} p_j} \left( \bigotimes_{\sigma \in S_n} \left( \bigoplus_{j=1}^{n} \left( p_j \tilde{a}_{\sigma(j)}^{m_{w(j)}} \right) \right) \right)^{\frac{1}{n}} \tag{53}
\]

where \( \sigma(j)(j = 1, 2, \ldots, n) \) is any permutation of \( \{1, 2, \ldots, n\} \) and \( S_n \) is the set of all permutations of \( \{1, 2, \ldots, n\} \).

**Theorem 4.** Let \( \tilde{a}_j = \left( \left( s_j, \rho_j \right), \left( T_j, I_j, F_j \right) \right) \) \( (j = 1, 2, \ldots, n) \) be a set of SVN2TLNs. The aggregated value by using SVN2TLWDMM operators is also a SVN2TLN where

\[
\text{SVN2TLWDMM}^p_{\text{nw}} (\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)
\]

\[
= \frac{1}{\sum_{j=1}^{n} p_j} \left( \bigotimes_{\sigma \in S_n} \left( \bigoplus_{j=1}^{n} \left( p_j \tilde{a}_{\sigma(j)}^{m_{w(j)}} \right) \right) \right)^{\frac{1}{n}}
\]

\[
\Delta \left[ \sum_{j=1}^{n} p_j \left( \prod_{\sigma \in S_n} \left( \sum_{j=1}^{n} \left( p_j \left( \Delta^{-1} \left( s_j, \rho_j \right) \right)^{m_{w(j)}} \right) \right) \right)^{\frac{1}{n}} \right]
\]

\[
1 - \prod_{\sigma \in S_n} \left( \sum_{j=1}^{n} \left( p_j \left( 1 - T_{\sigma(j)}^{m_{w(j)}} \right)^{p_j} \right) \right)^{\frac{1}{n}} \sum_{j=1}^{n} p_j
\]

\[
1 - \prod_{\sigma \in S_n} \left( \sum_{j=1}^{n} \left( p_j \left( 1 - I_{\sigma(j)}^{m_{w(j)}} \right)^{p_j} \right) \right)^{\frac{1}{n}} \sum_{j=1}^{n} p_j
\]

\[
1 - \prod_{\sigma \in S_n} \left( \sum_{j=1}^{n} \left( p_j \left( 1 - F_{\sigma(j)}^{m_{w(j)}} \right)^{p_j} \right) \right)^{\frac{1}{n}} \sum_{j=1}^{n} p_j
\]

**Proof.** From the exponential operation laws of SVN2TLNs depicted in Definition 3, we can ascertain that

\[
a_{\sigma(j)}^{m_{w(j)}} = \left( \Delta \left( \left( \Delta^{-1} \left( s_j, \rho_j \right) \right)^{m_{w(j)}} \right), T_{\sigma(j)}^{m_{w(j)}}, 1 - \left( 1 - I_{\sigma(j)}^{m_{w(j)}} \right)^{p_j}, 1 - \left( 1 - F_{\sigma(j)}^{m_{w(j)}} \right)^{p_j} \right) \tag{55}
\]
Then, based on the scalar-multiplication operation laws of SVN2TLNs, we can derive

\[
p_{j}^{nw_{\sigma(j)}} = \left\{ \Delta \left( p_{j} \left( \Delta^{-1} \left( s_{j}, \rho_{j} \right) \right)^{nw_{\sigma(j)}} \right), \right.
\]

\[
\left[ 1 - \left( 1 - T_{\sigma(j)}^{nw_{\sigma(j)}} \right)^{p_{j}}, \left( 1 - I_{\sigma(j)}^{nw_{\sigma(j)}} \right)^{p_{j}} \right].
\]

(56)

Therefore, according to the addition operation laws of SVN2TLNs, we can get

\[
\bigoplus_{j=1}^{n} \left( p_{j}^{nw_{\sigma(j)}} \right) = \left\{ \Delta \left( \sum_{j=1}^{n} \left( p_{j} \left( \Delta^{-1} \left( s_{j}, \rho_{j} \right) \right)^{nw_{\sigma(j)}} \right) \right), \right.
\]

\[
\left[ 1 - \prod_{j=1}^{n} \left( 1 - T_{\sigma(j)}^{nw_{\sigma(j)}} \right)^{p_{j}}, \prod_{j=1}^{n} \left( 1 - I_{\sigma(j)}^{nw_{\sigma(j)}} \right)^{p_{j}} \right].
\]

(57)

Therefore, by utilizing the multiplication operation laws of SVN2TLNs, we can derive

\[
\bigotimes_{\sigma \in S_{n}} \left( \bigoplus_{j=1}^{n} \left( p_{j}^{nw_{\sigma(j)}} \right) \right) = \left\{ \Delta \left( \prod_{\sigma \in S_{n}} \left( \sum_{j=1}^{n} \left( p_{j} \left( \Delta^{-1} \left( s_{j}, \rho_{j} \right) \right)^{nw_{\sigma(j)}} \right) \right) \right), \right.
\]

\[
\left[ \prod_{\sigma \in S_{n}} \left( 1 - \prod_{j=1}^{n} \left( 1 - T_{\sigma(j)}^{nw_{\sigma(j)}} \right)^{p_{j}} \right), \right.
\]

\[
1 - \prod_{\sigma \in S_{n}} \left( 1 - \prod_{j=1}^{n} \left( 1 - I_{\sigma(j)}^{nw_{\sigma(j)}} \right)^{p_{j}} \right),
\]

\[
1 - \prod_{\sigma \in S_{n}} \left( 1 - \prod_{j=1}^{n} \left( 1 - F_{\sigma(j)}^{nw_{\sigma(j)}} \right)^{p_{j}} \right) \right].
\]

(58)

Furthermore, by using the exponential operation laws of SVN2TLNs, we can get
Therefore, the fused results using the SVN2TLWDMM operator can be shown as follows:

\[
\begin{align*}
&\left(\bigotimes_{\sigma \in \Sigma_{n}} \left( \bigoplus_{j=1}^{n} \left( P_{j} a_{\sigma(j)}^{\alpha_{\sigma(j)}} \right) \right) \right)_{\frac{1}{a!}}
\end{align*}
\]

\[
\Delta \left( \prod_{\sigma \in \Sigma_{n}} \left( \sum_{j=1}^{n} \left( P_{j} \left( \Delta^{-1} \left( S_{j}, P_{j} \right) \right)^{\alpha_{\sigma(j)}} \right) \right) \right)_{\frac{1}{a!}},
\]

\[
= \left\{ \begin{array}{l}
\prod_{\sigma \in \Sigma_{n}} \left( 1 - \prod_{j=1}^{n} \left( 1 - T_{\sigma(j)}^{\alpha_{\sigma(j)}} \right) P_{j} \right)_{\frac{1}{a!}}, \\
1 - \left( \prod_{\sigma \in \Sigma_{n}} \left( 1 - \prod_{j=1}^{n} \left( 1 - I_{\sigma(j)}^{\alpha_{\sigma(j)}} \right) P_{j} \right) \right)_{\frac{1}{a!}}, \\
1 - \left( \prod_{\sigma \in \Sigma_{n}} \left( 1 - \prod_{j=1}^{n} \left( 1 - F_{\sigma(j)}^{\alpha_{\sigma(j)}} \right) P_{j} \right) \right)_{\frac{1}{a!}}. \\
\end{array} \right. 
\]

(59)

\[
\frac{1}{\sum_{j=1}^{n} P_{j} \left( \prod_{\sigma \in \Sigma_{n}} \left( \sum_{j=1}^{n} \left( P_{j} \left( \Delta^{-1} \left( S_{j}, P_{j} \right) \right)^{\alpha_{\sigma(j)}} \right) \right) \right)_{\frac{1}{a!}}}
\]

\[
\Delta \left( \prod_{\sigma \in \Sigma_{n}} \left( \sum_{j=1}^{n} \left( P_{j} \left( \Delta^{-1} \left( S_{j}, P_{j} \right) \right)^{\alpha_{\sigma(j)}} \right) \right) \right)_{\frac{1}{a!}},
\]

\[
= \left\{ \begin{array}{l}
\prod_{\sigma \in \Sigma_{n}} \left( 1 - \prod_{j=1}^{n} \left( 1 - T_{\sigma(j)}^{\alpha_{\sigma(j)}} \right) P_{j} \right)_{\frac{1}{a!}}, \\
1 - \left( \prod_{\sigma \in \Sigma_{n}} \left( 1 - \prod_{j=1}^{n} \left( 1 - I_{\sigma(j)}^{\alpha_{\sigma(j)}} \right) P_{j} \right) \right)_{\frac{1}{a!}}, \\
1 - \left( \prod_{\sigma \in \Sigma_{n}} \left( 1 - \prod_{j=1}^{n} \left( 1 - F_{\sigma(j)}^{\alpha_{\sigma(j)}} \right) P_{j} \right) \right)_{\frac{1}{a!}}. \\
\end{array} \right. 
\]

(60)

Therefore, the fused results using the SVN2TLWDMM operator can be shown as follows:

\[
\text{SVN2TLWDMM}_{\alpha}^{*} \left( \tilde{a}_{1}, \tilde{a}_{2}, \cdots, \tilde{a}_{n} \right)
\]

\[
= \frac{1}{\sum_{j=1}^{n} P_{j} \left( \prod_{\sigma \in \Sigma_{n}} \left( \sum_{j=1}^{n} \left( P_{j} \left( \Delta^{-1} \left( S_{j}, P_{j} \right) \right)^{\alpha_{\sigma(j)}} \right) \right) \right)_{\frac{1}{a!}}}
\]

\[
\Delta \left( \prod_{\sigma \in \Sigma_{n}} \left( \sum_{j=1}^{n} \left( P_{j} \left( \Delta^{-1} \left( S_{j}, P_{j} \right) \right)^{\alpha_{\sigma(j)}} \right) \right) \right)_{\frac{1}{a!}},
\]

\[
= \left\{ \begin{array}{l}
\prod_{\sigma \in \Sigma_{n}} \left( 1 - \prod_{j=1}^{n} \left( 1 - T_{\sigma(j)}^{\alpha_{\sigma(j)}} \right) P_{j} \right)_{\frac{1}{a!}}, \\
1 - \left( \prod_{\sigma \in \Sigma_{n}} \left( 1 - \prod_{j=1}^{n} \left( 1 - I_{\sigma(j)}^{\alpha_{\sigma(j)}} \right) P_{j} \right) \right)_{\frac{1}{a!}}, \\
1 - \left( \prod_{\sigma \in \Sigma_{n}} \left( 1 - \prod_{j=1}^{n} \left( 1 - F_{\sigma(j)}^{\alpha_{\sigma(j)}} \right) P_{j} \right) \right)_{\frac{1}{a!}}. \\
\end{array} \right. 
\]

(60)

Therefore, Equation (54) is kept. In Equations (54)–(60), \( \Delta^{-1} \) is the function of converting the 2-tuple linguistic variables to the exact numbers and \( \Delta \) is the function of converting the computing results to the 2-tuple linguistic variables.

Then, we need to prove that Equation (54) is a SVN2TLN. We need to prove the two conditions:

\( 0 \leq T \leq 1, 0 \leq I \leq 1, 0 \leq F \leq 1 \) \hspace{1cm} (2) \( 0 \leq T + I + F \leq 3 \)

\( \square \)

**Proof.** Let
\[ T = 1 - \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - T_{\sigma(j)}^{m_{\sigma(j)}} \right)^{p_j} \right) \left( \frac{1}{n!} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}} \]

\[ I = \left( 1 - \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - I_{\sigma(j)}^{m_{\sigma(j)}} \right)^{p_j} \right) \right)^{\frac{1}{\sum_{j=1}^{n} p_j}} \]

\[ F = \left( 1 - \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - F_{\sigma(j)}^{m_{\sigma(j)}} \right)^{p_j} \right) \right)^{\frac{1}{\sum_{j=1}^{n} p_j}} \]

(1) Because \( 0 \leq T_{\sigma(j)} \leq 1 \), we can get

\[ 0 \leq \left( 1 - T_{\sigma(j)}^{m_{\sigma(j)}} \right)^{p_j} \leq 1 \text{ and } 0 \leq 1 - \prod_{j=1}^{n} \left( 1 - T_{\sigma(j)}^{m_{\sigma(j)}} \right)^{p_j} \leq 1. \] (61)

Then,

\[ 0 \leq \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - T_{\sigma(j)}^{m_{\sigma(j)}} \right)^{p_j} \right) \left( \frac{1}{n!} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}} \leq 1 \] (62)

\[ 0 \leq 1 - \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - T_{\sigma(j)}^{m_{\sigma(j)}} \right)^{p_j} \right) \left( \frac{1}{n!} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}} \leq 1. \] (63)

That means \( 0 \leq T \leq 1 \), so (1) is maintained.

Similarly, we can get \( 0 \leq I \leq 1, 0 \leq F \leq 1 \).

(2) Because \( 0 \leq T \leq 1, 0 \leq I \leq 1, 0 \leq F \leq 1, 0 \leq T + I + F \leq 3. \) □

**Example 4.** Let \( \{(s_2, 0), (0.7, 0.5, 0.3)\}, \{(s_4, 0), (0.8, 0.6, 0.2)\}, \{(s_2, 0), (0.6, 0.7, 0.1)\} \) be three SVN2TLNs, and \( P = (0.3, 0.4, 0.2) \) and \( w = (0.4, 0.3, 0.3) \); then, according to Equation (54), we have...
SVN2TLWDMM \((0,3,0,0,4,0.2)\) \(\{(s_3, 0), (0.7, 0.5, 0.3)\}, \{(s_4, 0), (0.8, 0.6, 0.2)\}\)

\[
\left\{ \sum_{j=1}^{n} p_j \left( \Delta^{-1} \left( s_j, \rho_j \right) \right)^{\omega_{\sigma(j)}} \right\}^{\frac{1}{n!}}
\]
We will now discuss some properties of the SVN2TLWMM operator.
Property 9. (Monotonicity) Let \( \tilde{a}_j = \left( s_j, \rho_j, T_j, I_j, F_j \right) \) (j = 1, 2, \ldots, n) and \( \tilde{b}_j = \left( s_j, \rho_j, T_j, I_j, F_j \right) \) (j = 1, 2, \ldots, n) be two sets of SVN2TLNs. If \( \Delta^{-1} \left( s_j, \rho_j \right) \leq \Delta^{-1} \left( s_j, \rho_j \right) \) and \( T_j \leq T_j \) and \( I_j \geq I_j \) and \( F_j \geq F_j \), hold for all \( j \), then

\[
\text{SVN2TLWDMM}^p_{nw} \left( \tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n \right) \leq \text{SVN2TLWDMM}^p_{nw} \left( \tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n \right).
\]

(64)

The proof is similar to SVN2TLWMM. It is omitted here.

Property 10. (Boundedness) Let \( \tilde{a}_j = \left( s_j, \rho_j, (T_j, I_j, F_j) \right) \) (j = 1, 2, \ldots, n) be a set of SVN2TLNs. If \( \tilde{a}^+ = \left( \max \left( S_j, \rho_j \right), \left( \max \left( T_j \right), \min \left( I_j \right), \min \left( F_j \right) \right) \right) \) and \( \tilde{a}^- = \left( \min \left( S_j, \rho_j \right), \left( \min \left( T_j \right), \max \left( I_j \right), \max \left( F_j \right) \right) \right) \), then

\[
\tilde{a}^- \leq \text{SVN2TLWDMM}^p_{nw} \left( \tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n \right) \leq \tilde{a}^+.
\]

(65)

From Theorem 4:

\[
\text{SVN2TLWDMM}^p_{nw} \left( \tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n \right)
\]

\[
= \left( \Delta \frac{1}{\sum_{j=1}^{n} P_j} \left( \prod_{\sigma \in S_n} \left( \sum_{j=1}^{n} p_j \left( \min \Delta^{-1} \left( s_j, \rho_j \right) \right)^{\min \left( I_j, \rho_j \right)} \right) \right)^{1/\alpha} \right) \left( 1 - \left( 1 - \prod_{\sigma \in S_n} \left( 1 - \min \left( T^{\min \left( I_j, \rho_j \right)} \right) \right)^{1/P_j} \right) \right)^{1/\alpha} \left( 1 - \left( 1 - \prod_{\sigma \in S_n} \left( 1 - \max \left( I^{\min \left( I_j, \rho_j \right)} \right) \right)^{1/P_j} \right) \right)^{1/\alpha} \left( 1 - \left( 1 - \prod_{\sigma \in S_n} \left( 1 - \max \left( F^{\min \left( I_j, \rho_j \right)} \right) \right)^{1/P_j} \right) \right)^{1/\alpha}
\]

(66)
SVN2TLWDMM$^P_{nw} \left( \tilde{a}_1^+, \tilde{a}_2^+, \cdots, \tilde{a}_n^+ \right)$

\[
\Delta \left( \frac{1}{\sum_{j=1}^n p_j} \left( \prod_{\sigma \in S^e} \left( \sum_{j=1}^n \left( \max \Delta^{-1} \left( s_j, p_j \right) \right)^{w_{\sigma(j)}^n} \right) \right) \right)^{\frac{1}{M}} 

= \frac{1}{\sum_{j=1}^n p_j} \left( 1 - \prod_{\sigma \in S^e} \left( 1 - \prod_{j=1}^n \left( 1 - \min I_{\sigma(j)}^n \right)^{w_{\sigma(j)}^n} \right) \right) \right)^{\frac{1}{M}} \left( 1 - \prod_{j=1}^n \left( 1 - \min F_{\sigma(j)}^n \right)^{w_{\sigma(j)}^n} \right) \right)^{\frac{1}{M}} \right) \right) \right)
\]

(67)

From Property 9:

\[
\tilde{a}^- \leq SVN2TLWDMM^P_{nw} \left( \tilde{a}_1, \tilde{a}_2^+, \cdots, \tilde{a}_n^+ \right) \leq \tilde{a}^+.
\]

(68)

It is obvious that the SVN2TLWDMM operator lacks the property of idempotency.

4. Numerical Example and Comparative Analysis

4.1. Numerical Example

The green supplier selection is a classic MADM problem [37–39]. Therefore, in this section we use a numerical example to select green suppliers in green supply chain management with SVN2TLNs in order to show the proposed method. There are five possible green suppliers $A_i (i = 1, 2, 3, 4, 5)$ to be selected. We selected four attributes to assess these possible green suppliers: $G_1$ is the product quality factor, $G_2$ is the environmental factor, $G_3$ is the delivery factor, and $G_4$ is the price factor. These five possible green suppliers $A_i (i = 1, 2, 3, 4, 5)$ are to be assessed with SVN2TLNs by the decisionmaker using the above four attributes, whose weighting vectors $\omega = (0.2, 0.3, 0.4, 0.1)$ are listed in Table 1.

Table 1. Single-valued neutrosophic 2-tuple linguistic number (SVN2TLN) decision matrix.

|     | $G_1$       | $G_2$       | $G_3$       | $G_4$       |
|-----|-------------|-------------|-------------|-------------|
| $A_1$ | $<\text{s0,0},(0.7,0.4,0.6)>$ | $<\text{s0,0},(0.4,0.5,0.6)>$ | $<\text{s0,0},(0.4,0.5,0.6)>$ | $<\text{s0,0},(0.6,0.5,0.3)>$ |
| $A_2$ | $<\text{s0,0},(0.5,0.4,0.2)>$ | $<\text{s0,0},(0.6,0.5,0.4)>$ | $<\text{s0,0},(0.6,0.5,0.4)>$ | $<\text{s0,0},(0.8,0.3,0.5)>$ |
| $A_3$ | $<\text{s0,0},(0.5,0.4,0.2)>$ | $<\text{s0,0},(0.7,0.5,0.6)>$ | $<\text{s0,0},(0.5,0.6,0.3)>$ | $<\text{s0,0},(0.4,0.5,0.2)>$ |
| $A_4$ | $<\text{s0,0},(0.8,0.4,0.6)>$ | $<\text{s0,0},(0.7,0.4,0.3)>$ | $<\text{s0,0},(0.6,0.5,0.3)>$ | $<\text{s0,0},(0.6,0.4,0.6)>$ |
| $A_5$ | $<\text{s0,0},(0.5,0.6,0.4)>$ | $<\text{s0,0},(0.7,0.4,0.7)>$ | $<\text{s0,0},(0.6,0.5,0.2)>$ | $<\text{s0,0},(0.8,0.6,0.8)>$ |

We can now use the approach developed for selecting green suppliers in green supply chain management.
Step 1. According to SVN2TLNs $r_{ij} (i=1,2,3,4,5, j=1,2,3,4)$, we can aggregate all SVN2TLNs $r_{ij}$ by using the SVN2TLWMM (SVN2TLWDMM) operator to get the SVN2TLNs $A_i (i=1,2,3,4,5)$ of the green suppliers $A_i$. Supposing that $P=(1,1,1,0)$, the aggregating results are shown in Table 2.

Table 2. The aggregating results of the green suppliers by the single-valued neutrosophic 2-tuple linguistic dual Muirhead mean (SVN2TLWMM) (single-valued neutrosophic 2-tuple linguistic weighted dual Muirhead mean (SVN2TLWDMM)) operator.

| SVN2TLWMM | SVN2TLWDMM |
|-----------|-------------|
| $A_1$     | $(s_2, -0.3762), (s_1, -0.3943)$ |
| $A_2$     | $(s_2, -0.4702), (s_2, -0.0122)$ |
| $A_3$     | $(s_1, 0.4358), (s_2, 0.2592)$ |
| $A_4$     | $(s_2, 0.1847), (s_2, 0.4730)$ |
| $A_5$     | $(s_1, -0.1410), (s_1, 0.0225)$ |

Step 2. In accordance with the aggregating results in Table 2, the score values of the green suppliers are shown in Table 3.

Table 3. The score values of the green suppliers.

| SVN2TLWMM | SVN2TLWDMM |
|-----------|-------------|
| $A_1$     | $(s_2, -0.3762), (s_1, -0.3943)$ |
| $A_2$     | $(s_2, -0.4702), (s_2, -0.0122)$ |
| $A_3$     | $(s_1, 0.4358), (s_2, 0.2592)$ |
| $A_4$     | $(s_2, 0.1847), (s_2, 0.4730)$ |
| $A_5$     | $(s_1, -0.1410), (s_1, 0.0225)$ |

Step 3. According to the score values listed in Table 3, the order of the green suppliers are listed in Table 4. The best green supplier is $A_4$.

Table 4. Ordering of the green suppliers.

| Ordering |
|----------|
| SVN2TLWMM     | $A_4 > A_1 > A_2 > A_3 > A_5$ |
| SVN2TLWDMM    | $A_4 > A_1 > A_3 > A_2 > A_5$ |

4.2. Influence of the Parameter on the Final Result

In order to show the effects on the ranking results by altering the parameters of $P$ in the SVN2TLWMM (SVN2TLWDMM) operators, the results are listed in Tables 5 and 6.

Table 5. Ranking results for different parameters of the SVN2TLWMM operator.

| P       | $s(A_1)$ | $s(A_2)$ | $s(A_3)$ | $s(A_4)$ | $s(A_5)$ |
|---------|----------|----------|----------|----------|----------|
| $(1,0,0)$ | $(s_2, 0.3096)$ | $(s_1, 0.2931)$ | $(s_0, 0.2036)$ | $(s_0, -0.1905)$ | $(s_0, -0.1063)$ |
| $(1,1,0)$ | $(s_2, -0.4419)$ | $(s_0, -0.4987)$ | $(s_0, 0.1192)$ | $(s_0, -0.0706)$ | $(s_0, -0.1410)$ |
| $(1,1,1)$ | $(s_2, -0.3762)$ | $(s_0, -0.4702)$ | $(s_0, 0.4358)$ | $(s_0, 0.1847)$ | $(s_0, -0.2159)$ |
| $(2,2,2)$ | $(s_2, -0.3620)$ | $(s_0, -0.4811)$ | $(s_0, 0.4095)$ | $(s_0, 0.1700)$ | $(s_0, -0.2159)$ |
| $(2,0,0)$ | $(s_2, -0.2505)$ | $(s_0, -0.2515)$ | $(s_0, -0.3744)$ | $(s_0, 0.3621)$ | $(s_0, 0.3912)$ |
| $(3,0,0)$ | $(s_2, -0.0101)$ | $(s_0, 0.0035)$ | $(s_0, -0.1482)$ | $(s_0, -0.3787)$ | $(s_0, -0.2759)$ |

Ordering: $A_4 > A_1 > A_2 > A_3 > A_5$
Table 6. Ranking results for different parameters of the SVN2TLWDMM operator.

| P   | s(A1)   | s(A2)   | s(A3)   | s(A4)   | s(A5)   | Ordering          |
|-----|---------|---------|---------|---------|---------|------------------|
| (1,0,0) | (s1, 0.2756) | (s1, 0.0001) | (s1, −0.0497) | (s4, −0.4654) | (s1, 0.0155) | A4 > A1 > A2 > A3 > A5 |
| (1,1,0) | (s1, 0.3874) | (s1, −0.0806) | (s2, 0.0341) | (s1, −0.0287) | (s1, −0.0515) | A4 > A1 > A3 > A2 > A5 |
| (1,1,1) | (s1, −0.3943) | (s1, −0.0122) | (s2, 0.2592) | (s1, 0.4730) | (s1, 0.0225) | A4 > A1 > A3 > A2 > A5 |
| (2,2,2) | (s1, −0.2159) | (s1, 0.2825) | (s1, 0.2287) | (s1, −0.4771) | (s1, −0.0165) | A1 > A4 > A2 > A3 > A5 |
| (2,0,0) | (s1, 0.2100) | (s1, −0.0916) | (s1, −0.1583) | (s2, 0.2577) | (s1, −0.1059) | A4 > A1 > A2 > A3 > A5 |

4.3. Comparative Analysis

We can now compare our proposed method with TOPSIS methods using the single-valued neutrosophic linguistic numbers (SVNLNs) proposed by Ye [32]. The comparative results are listed in Table 7.

Table 7. Ordering of the green suppliers.

| Ordering | TOPSIS with SVNLNs | A4 > A1 > A2 > A3 > A5 |
|----------|--------------------|------------------------|

In Table 7, we can see that we get the same best green supplier, and only two of the methods’ ranking results are slightly different. This shows that the method we proposed is reasonable and effective. However, the existing TOPSIS methods with SVNLNs [32] do not consider the relationship information among the arguments being aggregated and therefore cannot eliminate the influence of unfair arguments on decision results. Our proposed SVN2TLWMM and SVN2TLWDMM operators consider the relationship information among the arguments being aggregated.

5. Conclusions

In this paper, we investigated MADM problems with SVN2TLNs. Then, we utilized the Muirhead mean operator and dual Muirhead mean operator to develop some Muirhead mean operators with SVN2TLNs: the SVN2TLMM operator, the SVN2TLWMM operator, the SVN2TLDMM operator, and the SVN2TLWDMM operator. The main properties of these proposed operators were investigated. We then used these operators to propose some models for MADM problems with SVN2TLNs.

The case study for green supplier selection shows that the proposed MADM method is practical and effective. The advantages of the proposed approach are as follows:

(1) The proposed approach is based on SVN2TLNs, which are suitable to be used in real life situations. SVN2TLNs have the capacity to deal with imprecise and vague information. They are suitable for expressing the truth-membership, indeterminacy-membership, and falsity-membership of an element to a 2-tuple linguistic term, which can affect the decisionmaker’s confidence level when they are making the evaluation. Therefore, the decisionmakers may find it more flexible and convenient to express their opinions as SVN2TLNs. The existing operation rules and comparison rules were contrasted and discussed.

(2) Some Muirhead mean operators with SVN2TLNs were developed: the SVN2TLMM operator, the SVN2TLWMM operator, the SVN2TLDMM operator, and the SVN2TLWDMM operator. The main characteristics of these proposed operators were investigated.

(3) Some methods were established to solve MADM problems with SVN2TLNs, and the evaluation results turned out to be reasonable.

(4) MADM methods based on such operators with SVN2TLNs are novel decision making methods, and these methods were applied to green supplier selection in this study. Furthermore, MADM methods based on such operators with SVN2TLNs are not only easy to calculate, but can also realize the reasonable and stable ranking of alternatives.
In future studies, the application of the proposed aggregating operators of SVN2TLNs need to be studied in many other uncertain and fuzzy environments [40–43] and extended to other application domains [44,45].

**Author Contributions:** Jie Wang, Jianping Lu, Guiwu Wei, Rui Lin and Cun Wei conceived and worked together to achieve this work, Jie Wang compiled the computing program by Matlab and analyzed the data, Jie Wang and Guiwu Wei wrote the paper. Finally, all the authors have read and approved the final manuscript.

**Funding:** The work was supported by the National Natural Science Foundation of China under Grant No. 71571128 and the Humanities and Social Sciences Foundation of Ministry of Education of the People’s Republic of China (14YJCZH091). The APC was funded by Humanities and Social Sciences Foundation of Ministry of Education of the People’s Republic of China (14YJCZH091).

**Conflicts of Interest:** The authors declare no conflicts of interest.

**References**

1. Smarandache, F. *A Unifying Field in Logics: Neutrosophy: Neutrosophic Probability, Set and Logic*; American Research Press: Rehoboth, DE, USA, 1999.
2. Smarandache, F. *A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability and Statistics*, 3rd ed.; American Research Press: Rehoboth, DE, USA, 2003.
3. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, 8, 338–353.
4. Atanassov, K. Intuitionistic fuzzy sets. *Fuzzy Syst. Set.* **1986**, 20, 87–96.
5. Atanassov, K.; Gargov, G. Interval valued intuitionistic fuzzy sets. *Fuzzy Syst. Set.* **1989**, 31, 343–349.
6. Wang, H.; Smarandache, F.; Zhang, Y.Q.; Sunderraman, R. Single-valued neutrosophic sets. *Multi-space Multistuct.* **2010**, 4, 410–413.
7. Wang, H.; Smarandache, F.; Zhang, Y.Q.; Sunderraman, R. Interval neutrosophic sets and logic: Theory and applications in computing. *Hexis: Phoenix, AZ, USA*, 2005.
8. Ye, J. Multicriteria decision-making method using the correlation coefficient under single-valued neutrosophic environment. *Int. J. Syst. Sci.* **2013**, 42, 386–394.
9. Broumi, S.; Smarandache, F. Correlation Coefficient of Interval Neutrosophic Set. *Appl. Mech. Mater.* **2013**, 436, 511–517.
10. Biswas, P.; Pramanik, S.; Giri, B.C. TOPSIS method for multi-attribute group decision-making under single-valued neutrosophic environment. *Neural Comput. Appl.* **2016**, 27, 727–737.
11. Liu, P.D.; Chu, Y.C.; Li, Y.W.; Chen, Y.B. Some generalized neutrosophic number Hamacher aggregation operators and their application to Group Decision Making. *Int. J. Fuzzy Syst.* **2014**, 16, 242–255.
12. Ahin, R.S.; Liu, P.D. Maximizing deviation method for neutrosophic multiple attribute decision making with incomplete weight information. *Neural Comput. Appl.* **2016**, 27, 2017–2029.
13. Ye, J. Similarity measures between interval neutrosophic sets and their applications in multicriteria decision-making. *J. Intell. Fuzzy Syst.* **2014**, 26, 165–172.
14. Zhang, H.Y.; Wang, J.Q.; Chen, X.H. Interval neutrosophic sets and their application in multicriteria decision making problems. *Sci. Word J.* **2014**, 1–15.
15. Ye, J. A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets. *J. Intell. Fuzzy Syst.* **2014**, 26, 2459–2466.
16. Peng, J.J.; Wang, J.Q.; Wang, J.; Zhang, H.Y.; Chen, X.H. Simplified neutrosophic sets and their applications in multicriteria group decision-making problems. *Int. J. Syst. Sci.* **2016**, 47, 2342–2358.
17. Peng, J.J.; Wang, J.Q.; Zhang, H.Y.; Chen, X.H. An outranking approach for multi-criteria decision-making problems with simplified neutrosophic sets. *Appl. Soft Comput.* **2014**, 25, 336–346.
18. Zhang, H.; Wang, J.Q.; Chen, X.H. An outranking approach for multi-criteria decision-making problems with interval-valued neutrosophic sets. *Neural Comput. Appl.* **2016**, 27, 615–627.
19. Liu, P.D.; Xi, L. The neutrosophic number generalized weighted power averaging operator and its application in multiple attribute group decision making. *Int. J. Mach. Learn. Cybernet.* **2018**, 9, 347–358. doi:10.1007/s13042-016-0508-0.
20. Deli, I.; Subas, Y. A ranking method of single-valued neutrosophic numbers and its applications to multi-attribute decision making problem. *Int. J. Mach. Learn. Cybern.* **2017**, 8, 1309–1322.
21. Peng, J.J.; Wang, J.Q.; Wu, X.H.; Wang, J.; Chen, X.H. Multi-valued neutrosophic sets and power aggregation operators with their applications in multi-criteria group decision-making problems. *Int. J.*
22. Zhang, H.Y.; Ji, P.; Wang, J.Q.; Chen, X.H. An improved weighted correlation coefficient based on
integrated weight for interval neutrosophic sets and its application in multi-criteria decision-making
problem. *Int. J. Comput. Intell. Syst.* 2015, 8, 1027–1043.

23. Chen, J.Q.; Ye, J. Some Single-Valued Neutrosophic Dombi Weighted Aggregation Operators for Multiple
Attribute Decision-Making. *Symmetry* 2017, 96, 82. doi:10.3390/sym9060082.

24. Liu, P.D.; Wang, Y.M. Multiple attribute decision making method based on single-valued neutrosophic
normalized weighted Bonferroni mean. *Neural Comput. Appl.* 2014, 25, 2001–2010.

25. Wu, X.H.; Wang, J.Q.; Peng, J.J.; Chen, X.H. Cross-entropy and prioritized aggregation operator with
simplified neutrosophic sets and their application in multi-criteria decision-making problems. *J. Intell.
Fuzzy Syst.* 2016, 18, 1104–1116.

26. Li, Y.; Liu, P.; Chen, Y. Some Single-Valued Neutrosophic Number Heronian Mean Operators and Their
Application in Multiple Attribute Group Decision Making. *Informatica* 2016, 27, 85–110.

27. Xu, D.S.; Wei, C.; Wei, G.W. TODIM method for single-valued Neutrosophic multiple attribute decision
making. *Information* 2017, 8, 125.

28. Wu, Q.; Zhou, L.G.; Chen, H.Y.; Guan, X.J. Some new Hamacher aggregation operators under
single-valued neutrosophic 2-tuple linguistic environment and their applications to multi-attribute group
decision making. *Comput. Ind. Eng.* 2018, 116, 144–162.

29. Herrera, F.; Martínez, L. A 2-tuple fuzzy linguistic representation model for computing with words. *IEEE
Trans. Fuzzy Syst.* 2000, 8, 746–752.

30. Herrera, F.; Martínez, L. An approach for combining linguistic and numerical information based on the
2-tuple fuzzy linguistic representation model in decision-making. *Int. J. Uncertain. Fuzziness Knowl. Based
Syst.* 2000, 8, 539–562.

31. Ye, J. Some aggregation operators of interval neutrosophic linguistic numbers for multiple attribute
decision making. *J. Intell. Fuzzy Syst.* 2014, 27, 2231–2241.

32. Ye, J. An extended TOPSIS method for multiple attribute group decision making based on single-valued
neutrosophic linguistic numbers. *J. Intell. Fuzzy Syst.* 2015, 28, 247–255.

33. Muirhead, R.F. Some methods applicable to identities and inequalities of symmetric algebraic functions
of n letters. *Proc. Edinb. Math. Soc.* 1902, 21, 144–162.

34. Tang, X.Y.; Wei, G.W.; Gao, H. Models for Multiple Attribute Decision Making with Interval-Valued
Pythagorean Fuzzy Muirhead Mean Operators and Their Application to Green Suppliers Selection.
*Informatica* 2019, 30, 153–186.

35. Wang, R.; Wang, J.; Gao, H.; Wei, G. Methods for MADM with Picture Fuzzy Muirhead Mean Operators
and Their Application for Evaluating the Financial Investment Risk. *Symmetry* 2019, 11, 6.

36. Qin, J.Q.; Liu, X.W. 2-tuple linguistic Muirhead mean operators for multiple attribute group decision
making and its application to supplier selection. *Kybernetes* 2017, 45, 2–29.

37. Wang, J.; Wei, G.W.; Wei, Y. Models for Green Supplier Selection with Some 2-Tuple Linguistic
Neutrosophic Number Bonferroni Mean Operators. *Symmetry* 2018, 10, 131.

38. Gao, H. Pythagorean Fuzzy Hamacher Prioritized Aggregation Operators in Multiple Attribute Decision
Making. *J. Intell. Fuzzy Syst.* 2018, 35, 2229–2245.

39. Li, Z.; Gao, H.; Wei, G. Methods for Multiple Attribute Group Decision Making Based on Intuitionistic
Fuzzy Dombi Hamy Mean Operators. *Symmetry* 2018, 10, 574.

40. Li, Z.X.; Wei, G.W.; Gao, H., Methods for Multiple Attribute Decision Making with Interval-Valued
Pythagorean Fuzzy Information. *Mathematics* 2018, 6, 228.

41. Kuo, M.S.; Liang, G.S. A soft computing method of performance evaluation with MCDM based on
interval-valued fuzzy numbers. *Appl. Soft Comput.* 2012, 12, 476–485.

42. Wu, L.; Wei, G.; Gao, H.; Wei, Y. Some Interval-Valued Intuitionistic Fuzzy Dombi Hamy Mean Operators
and Their Application for Evaluating the Elderly Tourism Service Quality in Tourism Destination.
*Mathematics* 2018, 6, 294.

43. Li, Z.X.; Wei, G.W.; Lu, M., Pythagorean Fuzzy Hamy Mean Operators in Multiple Attribute Group
Decision Making and Their Application to Supplier Selection. *Symmetry-Basel* 2018, 10, (10), 205.
44. Wei, Y.; Qin, S.; Li, X.; Zhu, S.; Wei, G. Oil price fluctuation, stock market and macroeconomic fundamentals: Evidence from China before and after the financial crisis. *Financ. Res. Lett.* **2019**, *30*, 23–29.

45. Wei, Y.; Yu, Q.; Liu, J.; Cao, Y. Hot money and China’s stock market volatility: Further evidence using the GARCH-MIDAS model. *Phys. A Stat. Mech. Its Appl.* **2018**, *492*, 923–930.

© 2019 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).