Worst possible sub-directions in high-dimensional models

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Abstract: We examine the rate of convergence of the Lasso estimator of lower dimensional components of the high-dimensional parameter. Under bounds on the \( \ell_1 \)-norm on the worst possible sub-direction these rates are of order \( \sqrt{|J| \log p/n} \) where \( p \) is the total number of parameters, \( J \subset \{1, \ldots, p\} \) represents a subset of the parameters and \( n \) is the number of observations. We also derive rates in sup-norm in terms of the rate of convergence in \( \ell_1 \)-norm. The irrepresentable condition on a set \( J \) requires that the \( \ell_1 \)-norm of the worst possible sub-direction is sufficiently smaller than one. In that case sharp oracle results can be obtained. Moreover, if the coefficients in \( J \) are small enough the Lasso will put these coefficients to zero. This extends known results which say that the irrepresentable condition on the active set (the set where coefficients are exactly zero) implies no false positives. We further show that by de-sparsifying one obtains fast rates in supremum norm without conditions on the worst possible sub-direction. The main assumption here is that approximate sparsity is of order \( \sqrt{n}/\log p \). The results are extended to M-estimation with \( \ell_1 \)-penalty for generalized linear models and exponential families for example. For the graphical Lasso this leads to an extension of known results to the case where the precision matrix is only approximately sparse. The bounds we provide are non-asymptotic but we also present asymptotic formulations for ease of interpretation.

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1. Introduction

We consider estimation bounds for parameters of interest in high-dimensional models. We apply the M-estimation procedure with \( \ell_1 \)-penalty and show that under certain conditions one-dimensional parameters can be estimated with rate \( \sqrt{\log p/n} \), where \( p \) is the total number of parameters and \( n \) is the number of observations. More generally, for a subset \( J \subset \{1, \ldots, p\} \) the group of parameters with index in \( J \) can be estimated with rate \( \sqrt{|J| \log p/n} \) in \( \ell_2 \)-norm. For this to happen, the “worst possible sub-direction” is required to have a bounded \( \ell_1 \)-norm. If this \( \ell_1 \)-norm is less than one, we obtain oracle rates \( \sqrt{|J \cap S| \log p/n} \) where \( S \) is the set of active parameters or a sparse approximation thereof. Taking \( J \) to be the set \( S^c \) gives variable selection results under an irrepresentable condition.

By de-sparsifying one obtains fast rates (of order \( 1/\sqrt{n} \) under certain conditions) for one-dimensional parameters without conditions on the \( \ell_1 \)-norm of the worst possible sub-direction. The de-sparsified estimator can moreover be used for the construction of asymptotic confidence intervals for parameters of interest. This study is an
intermediate step towards this end. We investigate the rates and conditions for remainder terms to be negligible. Global convergence (e.g. in $\ell_1$-norm) is generally sufficient for the latter. However, in high-dimensional models which are not very sparse, global convergence does not happen. For example, when estimating a $p \times p$ precision matrix (where there are actually $p(p-1)/2$ parameters), the global rate in $\ell_1$-norm will not be faster than $p\sqrt{\log p/n}$. To handle such cases, we show that the irrepresentable condition on the set of small coefficients yield rates in the sup-norm.

1.1. Related work

The motivation of this study is founded in [8] where asymptotic confidence intervals for the elements of a precision matrix are studied, based on the graphical Lasso. This work uses results from [15], which in turn relies on irrepresentable conditions implying that with high probability there are no false positives. In this paper we extend such a result to the case where the model is only approximately sparse which is possibly more appropriate in the context of confidence intervals and testing.

The literature on a semi-parametric approach to confidence intervals and testing in high dimensions is expanding quickly. An important reference is [23] and further work can be found in [9], [10], [20] and the papers [2], [3], and [4]. Our work presents rates in sup-norm for Lasso estimators and is in that aspect related to [12] although our conditions are based on worst possible sub-directions instead of incoherence. Also related is [22] but our work does not rely on irrepresentable conditions, i.e., the $\ell_1$-norm of the worst possible sub-direction is allowed to be larger than one (but if it is smaller we reproduce variable selection results). Irrepresentable conditions for variable selection were introduced in [13] and [24]. Our formulation shows these are conditions on worst possible sub-directions. We moreover extend the situation to models which are only approximately sparse.

1.2. Organization of the paper

The paper is organized as follows. In Section 2 we consider the linear model with fixed design and the Lasso estimator. We derive in Subsection 2.1 rates for a single coefficient and in Subsection 2.2 rates in (weighted) sup-norm. We consider desparsifying the Lasso in Subsection 2.3, leading to improved rates. We also discuss thresholding yielding a re-sparsified estimator. The results are based on approximate worst possible sub-directions using a Lasso but one can also apply a Dantzig selector. This is discussed in Subsection 2.4. Subsection 2.5 gives rates for groups of variables. Sharp oracle inequalities as well as variable selection results are derived. This leads to a further refinement in Subsection 2.6 where we prove that under certain irrepresentable conditions the Lasso will estimate small coefficients as being zero. In the final part of this section, Subsection 2.7, we present results for a de-sparsified estimator of a group of variables.

In the remainder of the paper worst possible sub-directions are in terms of theoretical (unknown) quantities, which means they do not immediately lead to a desparsifying procedure. We remark that de-sparsifying is nevertheless possible (see also [20] and [8]) but a full discussion goes beyond the scope of this paper. Section 3 gives a result analogous to the one of Subsection 2.5 for groups of variables for the case of random design. Here, worst possible sub-directions are taken in terms of the population inner-product matrix. Section 4 studies general loss functions. In Section
we discuss the remainder term for the linear model with random design (Subsection 5.1), the generalized linear model (Subsection 5.2) and exponential families (Subsection 5.3). Then we move to Brouwer’s fixed point theorem for deriving rates for estimators defined as solution of a system of equations. This theorem provides a way to handle the situation where the global rate is not fast enough to deal with the remainder term. We apply this in Section 7 to derive rates in sup-norm from the KKT conditions. Finally, we examine in Section 8 the remainder term of the graphical Lasso as an example. The approach there is as in [15] but with the extension to models which are only approximately sparse. Section 9 contains all proofs.

The results in this paper are presented in a non-asymptotic form. To simplify their interpretation, we present a separate asymptotic formulation at various stages, where we assume “standard” asymptotic scenarios.

2. The linear model with fixed design

Let $Y$ be an $n$-vector of response variables and $X$ a fixed $n \times p$ design matrix and consider the model

$$Y = X\beta^0 + \epsilon,$$

where $\epsilon$ is unobservable noise and $\beta^0$ is a $p$-vector of unknown coefficients. For a vector $v \in \mathbb{R}^n$ we write (with some abuse of notation) $\|v\|_n^2 := v^T v/n$. The Lasso estimator ([16]) is

$$\hat{\beta} := \hat{\beta}(\lambda) := \min_{\beta \in \mathbb{R}^p} \left\{ \|Y - X\beta\|_n^2 + 2\lambda \|\beta\|_1 \right\}.$$  (2.2)

Here, $\lambda$ is a tuning parameter which may be chosen data-dependent (e.g. when using the square root Lasso introduced in [1]). Typically, $\lambda$ is chosen of order $\sqrt{\log p/n}$ and proportional to some estimate of the noise level $\sigma := (\mathbb{E}\|\epsilon\|_n^2)^{1/2}$.

The estimator $\hat{\beta}$ satisfies the Karush-Kuhn-Tucker or KKT conditions

$$-X^T(Y - X\hat{\beta})/n + \lambda \hat{z} = 0$$

(2.3)

where $\hat{z}_j = \text{sign}(\hat{\beta}_j)$ if $\hat{\beta}_j \neq 0$ and $\|\hat{z}\|_\infty \leq 1$. Thus, $\hat{\beta}^T \hat{z} = \|\hat{\beta}\|_1$ and

$$Y^T(Y - X\hat{\beta})/n = \|Y - X\hat{\beta}\|_n^2 + \lambda \|\hat{\beta}\|_1.$$  (2.4)

These equalities will play a key role in our proofs.

2.1. Bounds for a single parameter

Let $j \in \{1, \ldots, p\}$ be some index. We define (approximate) worst possible sub-directions with help of the Lasso, where we regress $X_j$ on the set of all other variables $X_{-j} := \{X_k : k \neq j\}$ with $\ell_1$-penalty on the coefficients:

$$\hat{\gamma}_j := \hat{\gamma}_j(\lambda_j) := \arg \min_{\gamma_j \in \mathbb{R}^{p-1}} \left\{ \|X_j - X_{-j}\gamma_j\|_n^2 + 2\lambda_j \|\gamma_j\|_1 \right\}.$$  

We leave the choice of the tuning parameter $\lambda_j$ free at this stage, but will indicate in Corollary 2.1 that the square root Lasso gives well-scaled bounds.

Define $\hat{\tau}_j^2 := \|X_j - X_{-j}\hat{\gamma}_j\|_n^2$ and $\hat{\tau}_j^2 := \hat{\tau}_j^2 + \lambda_j \|\hat{\gamma}_j\|_1$. Let $\hat{c}_{k,j} := -\gamma_{k,j}$, $k \neq j$ and $\hat{c}_{j,j} := 1$. Note that $\hat{r}_j = \|X\hat{c}_j\|_n$. Inspired by semi-parametric theory (see e.g.
below we introduce sets $T_{j,e}$ and $T_{j,\text{rem}}$ which we discuss in Remark 2.1 following the lemma. The sup-script “e” stands for the noise term $\epsilon$. The subscript “rem” stands for “remainder”: under certain conditions terms with this subscript are of smaller order than the other terms.

**Lemma 2.1.** Let $\gamma_j$ be obtained using the Lasso as described above. Let

$$
T_{j,e} := \{ \hat{c}_j^T X^T \epsilon \}/n \leq \lambda_j \gamma_j \}, \quad T_{j,\text{rem}} := \{ \lambda_j \| \hat{\beta}_{-j} - \beta^0_{-j} \|_1 \leq \lambda_{j,\text{rem}} \gamma_j \}.
$$

On $T_{j,e} \cap T_{j,\text{rem}}$ it holds that

$$
|\hat{\beta}_j - \beta^0_j| \leq (\lambda_e + \lambda_{j,\text{rem}}) \gamma_j / \hat{\tau}_j^2 + \lambda \| \hat{\theta}_j \|_1.
$$

The following result is moreover useful when $|\beta^0_j|$ is small. If $|\gamma_j|_1 < 1$ and $\lambda(1 - |\gamma_j|_1) > (\lambda_e + \lambda_{j,\text{rem}}) \gamma_j$, then on $T_{j,e} \cap T_{j,\text{rem}}$

$$
\left( \lambda(1 - |\gamma_j|_1) - (\lambda_e + \lambda_{j,\text{rem}}) \gamma_j \right) |\hat{\beta}_j| \leq (\lambda_e + \lambda_{j,\text{rem}}) \gamma_j + \lambda \| \hat{\epsilon}_j \|_1 |\beta^0_j|.
$$

**Remark 2.1.** Note that if $\mathbb{E} \epsilon = 0$ and $\mathbb{E} \epsilon \epsilon^T = \sigma^2 I$, then $\text{var}(\hat{c}_j^T X^T \epsilon \epsilon^T \epsilon \epsilon^T \epsilon) = \sigma^2 \gamma_j^2 / n$. Hence, with $\lambda_{j,e} = \mathcal{O}(\sigma_{e,2}/\sqrt{n})$ large enough the set $T_{j,e}$ will have large probability. The set $T_{j,\text{rem}}$ will e.g. have large probability under compatibility conditions with $\lambda_{j,\text{rem}} = \mathcal{O}(\lambda_{j,\text{rem}})$ where $s$ is the number of non-zero $\beta^0_j$ or some sparse approximation thereof, see e.g. [5] and [7], and see also Corollary 2.2.

**Corollary 2.1.** Let us apply the square root Lasso

$$
\hat{\gamma}_j := \arg \min_{\gamma_j \in \mathbb{R}^p} \left\{ \| X_j - X_{-j} \gamma_j \|_2 + \lambda_0 \| \gamma_j \|_1 \right\}
$$

where $\lambda_0$ is a “universal” tuning parameter ([1]). Then $\lambda_j = \lambda_0 \gamma_j$ and so $\lambda_j \| \hat{\beta}_{-j} - \beta^0_{-j} \|_1 / \gamma_j = \lambda_0 \| \hat{\beta}_{-j} - \beta^0_{-j} \|_1$ which can be bounded by $\lambda_0 \| \hat{\beta} - \beta^0 \|_1$. Define now $T_{j,e}$ as in Lemma 2.1 and $T_{j,\text{rem}} := \{ \lambda_0 \| \hat{\beta} - \beta^0 \|_1 \leq \lambda_{j,\text{rem}} \}$. We then have on $T_{j,e} \cap T_{j,\text{rem}}$

$$
\gamma_j^2 \| \hat{\beta}_j - \beta^0_j \|_1 / \gamma_j \leq \lambda_{j,e} + \lambda_{j,\text{rem}} + \lambda \| \hat{\epsilon}_j \|_1 / \gamma_j.
$$

As noted in Remark 2.1 we typically can take $\lambda_{j,\text{rem}} \asymp \lambda_0 s$ and moreover typically $\lambda_0 = \mathcal{O}(\sqrt{\log p/n})$ and $\lambda = \mathcal{O}(\sigma_{e,2}/\sqrt{n})$. Hence, under suitable conditions on the sparseness $s$ the remainder term $\lambda_{j,\text{rem}}$ is negligible. Note that the first term $\lambda_{j,e}$ needs to be chosen be proportional on the standard deviation $\sigma_{e}$ of the noise $\epsilon$ to ensure that $T_{j,e}$ has large probability. The tuning $\lambda$ has to be chosen proportional to (an estimator of) $\sigma_{e}$ as well. Thus, all three terms scale with $\sigma_{e}$.

**Standard asymptotic scenario I** For a better reading of the bounds we present them in an asymptotic formulation. In all asymptotic formulations in this section we assume $\mathbb{E} \epsilon = 0$ and $\mathbb{E} \epsilon \epsilon^T = \sigma^2 I$ where $\sigma_{e} = \mathcal{O}(1)$. Suppose that $\hat{\tau}_j^2 = \mathcal{O}(1)$ (which is true of the columns of $X$ are normalized so that $\|X_j\|_2 = 1$), $1/\hat{\tau}_j^2 = \mathcal{O}(1)$ (a restricted eigenvalue or compatibility condition), $\lambda = \mathcal{O}(\sqrt{\log p/n})$ (the standard choice), that with $\lambda_{j,e} = \mathcal{O}(n^{-1/2})$ and $\lambda_{j,\text{rem}} = \mathcal{O}(n^{-1/2})$ the probability of $T_{j,e} \cap T_{j,\text{rem}}$ goes to one (this follows from the moment conditions on the noise and when $\lambda_0 = \mathcal{O}(\sqrt{\log p/n})$ from $s = \mathcal{O}(\sqrt{n}/\log p)$ and a compatibility condition). Then Lemma 2.1 gives that

$$
|\hat{\beta}_j - \beta^0_j| = \mathcal{O}(n^{-1/2}) + \lambda \| \hat{\theta}_j \|_1
$$

and if $|\gamma_j|_1 < 1$

$$
|\hat{\beta}_j| \leq \frac{\| \hat{\epsilon}_j \|_1 + \mathcal{O}(1)}{1 - |\gamma_j|_1 + \mathcal{O}(1)} |\beta^0_j|.
$$
2.2. A bound for the (weighted) $\ell_\infty$-norm

The above can be applied for each $j$. Let $\tilde{C} := (\tilde{c}_1, \ldots, \tilde{c}_p)$, $\tilde{\Theta} := (\tilde{\theta}_1, \ldots, \tilde{\theta}_p)$, and $\tilde{T} := \text{diag}(\tilde{\tau}_1, \ldots, \tilde{\tau}_p)$, $\hat{T} := \text{diag}(\hat{\tau}_1, \ldots, \hat{\tau}_p)$. Define for a matrix $A$ the $\ell_1$-operator norm

$$\|A\|_1 := \max_j \sum_k |A_{k,j}|.$$ 

We only present the result when using the square root Lasso because of its elegant scaling. We obtain a bound for the $\ell_\infty$-estimation error in terms of the $\ell_1$-estimation error. The weighted $\ell_\infty$-norm will be of interest when the residual variances $\tilde{\tau}_j^2$ are not balanced for the various values of $j$. Note also that we may use the bound $\tilde{\tau}_j/\hat{\tau}_j \leq 1$ for all $j$.

**Lemma 2.2.** Define

$$\mathcal{T}_{\text{all}} := \{\|c^T X \hat{C} \hat{T}^{-1}\|_\infty \leq \lambda_\epsilon\}.$$ 

When using the square root Lasso as in Corollary 2.1, we have on $\mathcal{T}_{\text{all}}$

$$\|\tilde{\beta} - \beta^0\|_\infty \leq (\lambda_\epsilon + \lambda_0 \|\tilde{\beta} - \beta^0\|_1) \|\hat{T}^{-2}\|_\infty + \lambda \|\hat{\Theta}\|_1$$

and

$$\|\hat{T}^{-1} \tilde{T}^2 (\tilde{\beta} - \beta^0)\|_\infty \leq \lambda_\epsilon + \lambda_0 \|\tilde{\beta} - \beta^0\|_1 + \lambda \|\hat{C} \hat{T}^{-1}\|_1.$$ 

**Standard asymptotic scenario II** Assume that $\|\hat{T}^{-1}\|_\infty = O(1)$ and that for $\lambda_\epsilon = O(\sqrt{\log p/n})$ the probability of $\mathcal{T}_{\text{all}}$ goes to one and for $\lambda = O_P(\sqrt{\log p/n})$ it holds that $\lambda_0 \|\tilde{\beta} - \beta^0\|_1 = O_P(\sqrt{\log p/n})$. Then we get

$$\|\tilde{\beta} - \beta^0\|_\infty = O_P(\sqrt{\log p/n}) + \lambda \|\hat{\Theta}\|_1.$$ 

2.3. De-sparsifying the Lasso and re-sparsifying

The de-sparsified Lasso is defined in [20] as

$$\hat{b}_j = \hat{\beta}_j + \hat{\theta}_j^T X^T (Y - X \hat{\beta})/n, \quad j = 1, \ldots, p$$

or in matrix notation

$$\hat{b} = \hat{\beta} + \hat{\Theta}^T X^T (Y - X \hat{\beta})/n.$$ 

The de-sparsified Lasso removes the term involving the $\ell_1$-norm of the worst possible sub-direction. In that sense, it removes the bias due to the $\ell_1$-penalty on this sub-direction. We prove this for completeness, but the result is as in [20] where also the details concerning the resulting (asymptotic) normality of the de-sparsified Lasso are presented.

**Lemma 2.3.** Let $\hat{\Theta}$ be obtained using the square root Lasso. Then

$$\hat{T}^{-1} \tilde{T}^2 (\hat{b} - \beta^0) = \hat{T}^{-1} \tilde{T}^2 \hat{\Theta}^T X^T c/n + \text{rem},$$

where $\|\text{rem}\|_\infty \leq \lambda_0 \|\tilde{\beta} - \beta^0\|_1$. Hence for a fixed $j$, on

$$\mathcal{T}_{j,\epsilon} \cap \mathcal{T}_{\text{rem}} := \{c_j^T X^T c/n \leq \lambda_{j,\epsilon} \tilde{\tau}_j, \quad \lambda_0 \|\tilde{\beta} - \beta^0\|_1 \leq \lambda_{\text{rem}}\},$$

we have

$$\tilde{\tau}_j^2 |\hat{b}_j - \beta^0_j|/\tilde{\tau}_j \leq \lambda_{j,\epsilon} + \lambda_{\text{rem}}$$

5
and on

\[ T_{\text{all}} \cap T_{\text{rem}} := \{ \| \hat{T}^{-1} \hat{C}^T X^T \epsilon \|_\infty / n \leq \lambda_c, \ \lambda_0 \| \hat{\beta} - \beta^0 \|_1 \leq \lambda_{\text{rem}} \} \]

it holds that

\[ \| \hat{T}^{-1} \hat{T}_2 (\hat{b} - \beta^0) \|_\infty \leq \lambda_c + \lambda_{\text{rem}}. \]

**Re-sparsifying** One may want to re-sparsify the de-sparsified Lasso \( \hat{b} \) using some threshold \( \lambda_{\text{sparse}} \) giving the estimator

\[ \hat{b}_{j, \text{sparse}} := \hat{b}_j 1\{ |\hat{b}_j| > \lambda_{\text{sparse}} \hat{\tau}_j / \hat{\tau}_j^2 \}, \ j = 1, \ldots, p. \]

As we see from Lemma 2.3 this new estimator can improve the \( \ell_\infty \)-bounds of the Lasso and has under sparsity conditions \( \ell_q \)-bounds similar to the Lasso (1 \( \leq q < \infty \)).

**Standard asymptotic scenario** Under Scenario I we have

\[ |\hat{b}_j - \beta^0_j| = O_P(n^{-1/2}), \]

and under Scenario II

\[ \| \hat{b} - \beta^0 \|_\infty = O_P(\sqrt{\log p / n}). \]

This can be used for exact recovery of the relevant active set where the coefficients are sufficiently larger than \( \sqrt{\log p / n} \) in absolute value. In other words, it leads to exact recovery without assuming an irrepresentable condition. For the re-sparsified estimator we also take \( \lambda_{\text{sparse}} = \sqrt{\log p / n} \). Then under Scenario II

\[ \| \hat{b}_{\text{sparse}} - \beta^0 \|_\infty = O_P(\sqrt{\log p / n}). \]

Assuming as an example that for some fixed \( r \in (0, 1) \) it holds that \( \sum_{j=1}^p |\beta^0_j|^r = \mathcal{O}(1) \), then we obtain under Scenario II

\[ \| \hat{b}_{\text{sparse}} - \beta^0 \|_q = O_P(s^{\frac{1}{2}} \sqrt{\log p / n}), \ 1 \leq q \leq \infty \]

where \( s = (\sqrt{n / \log p})^r \).

2.4. **Using the Dantzig selector**

Instead of defining approximate projections with help of the Lasso, we may also use the Dantzig selector, which gives a new definition of \( \hat{\gamma}_j \):

\[ \hat{\gamma}_j := \arg \min_{\gamma_j \in \mathbb{R}^{p-1}} \| \gamma_j \|_1 \text{ s.t. } \| X_j^T (X_j - X_{-j} \gamma_j) \|_\infty / n \leq \lambda_j. \]

The result of Lemma 2.4 below is quite similar to the one of Corollary 2.1.

**Lemma 2.4.** Let \( \hat{\gamma}_j \) be obtained using the Dantzig selector. Assume \( \lambda_j \| \hat{\gamma}_j \|_1 < \hat{\tau}_j^2 \).

On

\[ T_{j, \epsilon} \cap T_{j, \text{rem}} := \{ |\hat{c}_j^T X^T \epsilon | / n \leq \lambda_{j, \epsilon} \hat{\gamma}_j, \ \lambda_j \| \hat{\beta}_{-j} - \beta_{-j}^0 \|_1 / \hat{\tau}_j \leq \lambda_{j, \text{rem}} \} \]

we have

\[ \left[ \frac{\hat{\tau}_j^2 - \lambda_j \| \hat{\gamma}_j \|_1}{\hat{\tau}_j} \right] |\hat{\beta}_j - \beta_j^0| \leq \lambda_{j, \epsilon} + \lambda_{j, \text{rem}} + \lambda \| \hat{\gamma}_j \|_1 / \hat{\tau}_j. \]
2.5. A bound for a group of variables

Recall the \( \ell_1 \) operator norm \( \|A\| := \max_j \sum_k |a_{k,j}| \) of a matrix \( A \). We define its \( \ell_1 \) norm as \( \|A\|_1 := \sum_j \sum_k |a_{k,j}| \) for a given subset of the variables. Let \( J \subset \{1, \ldots, p\} \) be a given subset of the \( \beta \) variables. We let \( X_J := \{X_j\}_{j \in J} \) and \( X_{-J} := \{X_j\}_{j \not\in J} \). Moreover, we write \( \lambda_J := \{\lambda_j\}_{j \in J} \in \mathbb{R}^{|J|} \) and use the same notation for \( \{\beta_j(\{j \in J\})\}_{j=1}^p \) and likewise for \( \beta_{-J} = \beta_{J^c} \). We let

\[
\hat{\Gamma}_J := \arg\min_{\Gamma_J} \left\{ \text{trace}(X_J - X_{-J} \Gamma_J)^T(X_J - X_{-J} \Gamma_J) + \lambda_{J,0}\|\Gamma_J\|_1 \right\}.
\]

In other words, each column \( \hat{\gamma}_{j,J} \) of \( \hat{\Gamma}_J \) is obtained by performing a Lasso of \( X_J \) on \( X_{-J} \) with tuning parameter \( \lambda_{J,0} \). We define \( \hat{C}_J := X \hat{C}_J = X_J - X_{-J} \hat{\Gamma}_J \) so that \( \|\hat{C}_J\|_1 = 1 + \|\hat{\Gamma}_J\|_1 \). We introduce the smallest eigenvalue

\[
\hat{\phi}_J^2 := \min\{\|X \hat{C}_J \beta_J\|^2_n : \|\beta_J\|_2 = 1\}.
\]

The compatibility constant is

\[
\hat{\phi}_J^2 (L,S) := \min\{|S|\|X \beta\|^2_n : \|\beta_S\|_1 \leq L, \|\beta_S\|_1 = 1\},
\]

see [17] or [7].

Let us make three remarks. Firstly, we note that \( \|\hat{\Gamma}_J\|_1 \) is generally not directly comparable to \( \max_{j \in J} \|\hat{\gamma}_j\|_1 \) (\( \hat{\gamma}_j, j = 1, \ldots, p \), defined in Subsection 2.1). Secondly, in view of the scaling in Part I of Theorem 2.1 below, a matrix version of the square root Lasso would be to take \( \lambda_{J,0} = \lambda_0 \hat{\phi}_J \). And thirdly, it is easy to see that

\[
\hat{\phi}_J^2 \leq \hat{\phi}_J^2 (L,J \cap S), \forall \ L, S.
\]

Theorem 2.1 below may be applied to general and hence also relatively small sets \( J \). In that case one may want to replace \( \phi(L,J \cap S) \) by \( \hat{\phi}_J \) so that the constant \( L \) defined in the theorem no longer plays any role. (Moreover, we let \( \phi(\infty, S) = \hat{\phi}_J \).)

In Part I of the theorem we establish bounds under relatively weak (see (2.5)) or no (see (2.6)) conditions on the (approximate) worst possible sub direction \( \hat{\Gamma}_J \). Part II of the theorem assumes \( \|\hat{\Gamma}_J\|_1 \) is sufficiently small and presents oracle bounds.

**Theorem 2.1.** Let \( \hat{\Gamma}_J \) be obtained using the Lasso.

**Part I** Define

\[
\tilde{T}_{J,e} := \{|\beta_J^T \hat{C}_J \epsilon|/n \leq \hat{\lambda}_{J,e} \sqrt{|J|} \|X \hat{C}_J \beta_J\|_n, \ \forall \ \beta_J \in \mathbb{R}^{|J|} \}.
\]

If \( \lambda_{J,0} |J| \|\hat{\Gamma}_J\|_1 < \hat{\phi}_J^2 \), then we have on \( \tilde{T}_{J,e} \)

\[
\left( 1 - \frac{\lambda_{J,0} |J| \|\hat{\Gamma}_J\|_1}{\hat{\phi}_J^2} \right) \|X \hat{C}_J (\hat{\beta}_J - \beta_J^0)\|_n \leq \sqrt{|J|} \left( \hat{\lambda}_{J,e} \hat{\phi}_J + \lambda_{J,0} \|\hat{\beta}_{-J} - \beta_{-J}^0\|_1 + \lambda \|\hat{C}_J\|_1 \right).
\]

Furthermore on \( \tilde{T}_{J,e} \)

\[
\|X \hat{C}_J (\hat{\beta}_J - \beta_J^0)\|_n \leq \sqrt{|J|} \left( \hat{\lambda}_{J,e} \hat{\phi}_J + (\lambda + \lambda_{J,0} \|\hat{\beta} - \beta_J^0\|_1) \|\hat{C}_J\|_1 \right)
\]

which holds without assuming some bound for \( \|\hat{\Gamma}_J\|_1 \).

**Part II** Define

\[
\mathcal{T}_{J,e} := \{\|\hat{C}_J X^T \epsilon\|_\infty/n \leq \hat{\lambda}_{J,e} \}
\]
and 
\[ T_{J,\text{rem}} := \{ \lambda_{J,0} \| \hat{\beta} - \beta^0 \|_1 \leq \lambda_{J,\text{rem}} \}. \]
Assume \( \| \hat{\Gamma}_J \|_1 < 1 \) and that in fact for some \( \lambda_{1} \geq 0 \)
\[ \lambda(1 - \| \hat{\Gamma}_J \|_1) \geq \lambda_{J,e} + \lambda_{J,\text{rem}} \| \hat{C}_J \|_1 + \lambda_1. \]
Let
\[ L := \frac{\lambda_{J,e} + (\lambda + \lambda_{J,\text{rem}}) \| \hat{C}_J \|_1 + \lambda \| \hat{\Gamma}_J \|_1 + \lambda_1}{\lambda(1 - \| \hat{\Gamma}_J \|_1) - (\lambda_{J,e} + \lambda_{J,\text{rem}} \| \hat{C}_J \|_1) - \lambda_1}. \]
Let \( \beta \in \mathbb{R}^p \) be arbitrary and let \( S := \{ j : \beta_j \neq 0 \} \) be its active set. Then on
\( T_{J,e} \cap T_{J,\text{rem}} \),
\[ \| X \hat{C}_J (\hat{\beta}_J - \beta_j^0) \|_n^2 + 2 \left( \lambda(1 - \| \hat{\Gamma}_J \|_1) - (\lambda_{J,e} + \lambda_{J,\text{rem}} \| \hat{C}_J \|_1) - \lambda_1 \right) \| \hat{\beta}_J - \beta_j \|_1 \leq \frac{4\lambda^2 |J \cap S|}{\phi^2(L, J \cap S)} + \| X \hat{C}_J (\beta_j - \beta_j^0) \|_n^2. \]

**Remark 2.2.** Although (2.6) is a rougher bound than (2.5) (assuming the condition on \( \| \hat{\Gamma}_J \|_1 \)) it is more preferable. We have stated (2.5) because for desparsifying we actually need this more refined result (see Lemma 2.5).

**Corollary 2.2.**

**Sharp oracle inequality** If we take \( J = \{1, \ldots, p\} \) (and \( \lambda_{1,0} = 0 \)) we recover from Part II of Theorem 2.1 the sharp oracle inequality: on \( T_e := \{ \| c^T X \|_\infty / n \leq \lambda_e \} \) with \( \lambda_c < \lambda \):
\[ \| X (\hat{\beta} - \beta_j^0) \|_n^2 + 2 \left( \lambda - \lambda_c - (\lambda_c - \lambda_1) \right) \| \hat{\beta}_J - \beta_{J,0} \|_1 \leq \frac{4\lambda^2 |S|}{\phi^2(L, S)} + \| X (\beta - \beta_j^0) \|_n^2 \]
with \( L = (\lambda + \lambda_c + \lambda_1) / (\lambda - \lambda_c - \lambda_1) \). This corresponds to results in [11] or [19].

**Small coefficients** If we take \( J = S^c \) where \( S \) is such that \( \text{rank}(X_S) = |S| \), and we choose \( \lambda_{1,0} = 0 \) we find from Part II of Theorem 2.1: on \( T_e \): under the irrepresentable condition on the set \( S^c \)
\[ \| \hat{\Gamma}_{S^c} \|_1 \leq (\lambda - \lambda_c) / (\lambda + \lambda_c) \]
we have
\[ \left( \lambda - \lambda_c - \| \hat{\Gamma}_J \|_1 (\lambda + \lambda_c) \right) \| \hat{\beta}_{S^c} \|_1 \leq \| X \hat{C}_{S^c} \beta_j^0 \|_n^2. \] (2.7)
(We took \( 2\lambda_1 = \lambda - \lambda_c - \| \hat{\Gamma}_J \|_1 (\lambda + \lambda_c) \).) This bound generalizes the bound for small values of \( |\beta_j^0| \) presented in Lemma 2.1. To bound \( \| X \hat{C}_{S^c} \beta_j^0 \|_n^2 \) one may want to use
\[ \| X \hat{C}_{S^c} \beta_j^0 \|_n^2 \leq \| \beta_j^0 \|_2^2 \| \hat{C}_{S^c} X^T X \hat{C}_{S^c} \|_2^2 / n \]
where \( \| A \|_2^2 \) is the largest eigenvalue of a positive semi-definite matrix \( A \). Moreover
\[ \| \hat{C}_{S^c} X^T X \hat{C}_{S^c} \|_2^2 / n \leq \| X \hat{C}_{S^c} \|_2^2 / n + 2\lambda_{J,0} |J| . \] (2.8)
Alternatively (and for a comparison with Theorem 2.2 below) one may use the bound
\[ \| X \hat{C}_{S^c} \beta_j^0 \|_n^2 \leq \| \beta_j^0 \|_1 \| | \hat{C}_{S^c} \beta_j^0 \|_\infty / n \leq \| \beta_j^0 \|_1 \| \hat{C}_{S^c} \beta_j^0 \|_\infty \| \hat{C}_{S^c} X^T X \hat{C}_{S^c} \|_1 / n. \]

**Single coefficients** If we take \( J = \{ j \} \), \( \lambda_{1,0} := \lambda_j \) and use the first result (2.5) of Part I of Theorem 2.1 we find on \( T_{J,e} \cap T_{J,\text{rem}} := \{ \| c_j^T X \| / n \leq \lambda_c \hat{\tau}_j, \lambda_j \| \hat{\beta}_{-j} - \beta_{-j}^0 \|_1 / \hat{\tau}_j \leq \lambda_{J,\text{rem}} \} \)
\[ (\hat{\tau}_j^2 - \lambda_j \| \hat{\tau}_j \|_1) \| \hat{\beta}_j - \beta_j^0 \|_1 / \hat{\tau}_j \leq \lambda_c + \lambda_{J,\text{rem}} + \lambda \| \hat{c}_j \|_1 / \hat{\tau}_j \]
which is similar to Lemma 2.4. With a more refined handling of the cross terms in the matrix computations one can of course also recover the result of Lemma 2.1.
Standard asymptotic scenario III

The constant $\hat{\lambda}_{J,\epsilon}$ in Part I of Theorem 2.1 can be taken of order $1/\sqrt{n}$. If the design is normalized (diag($X^TX)/n = I$) the constant $\lambda_{J,\epsilon}$ in Part II can generally be taken of order $1/\sqrt{n}$ when $J$ is finite. Else we take $\lambda_{J,\epsilon} = \lambda_\epsilon \|\hat{C}_J\|_1$ with $\lambda_\epsilon = O(\sqrt{\log p/n})$. Note that $\hat{C}_J^T X T \epsilon/n$ has covariance matrix $\hat{C}_J^T X T X \hat{C}_J/n$, whose maximal eigenvalue can be bounded as in (2.8). We suppose that the probability of $T_\epsilon := \{\|\epsilon^T X\|_\infty/n \leq \lambda_\epsilon\}$ goes to one for $\lambda_\epsilon = O_p(\sqrt{\log p/n})$ suitably chosen. We assume $\lambda = O_p(\sqrt{\log p/n})$ and $\lambda_{J,\epsilon} = O_p(\sqrt{\log p/n})$, that $1/\hat{\phi}_J = O(1)$ and that the largest eigenvalue of $X_T X/n$ is $O(1)$. Then we find from Part II of Theorem 2.1, under the assumption $\|\hat{\Gamma}_J\|_1 \leq (\lambda - \lambda_\epsilon)/(\lambda + \lambda_\epsilon) + o(1)$, that

$$\|\hat{\beta}_J - \beta_J^0\|_2^2 = O_p\left(\log p |J \cap S|/n + \|\beta_J - \beta_J^0\|_2^2\right),$$

where for suitable $\lambda_{\text{thres}} = O(\sqrt{\log p/n})$ the vector $\beta$ is the sparse approximation

$$\beta_j := \beta_j^0 [\{\beta_j^0 > \lambda_{\text{thres}}\}, \forall j,$$

and $S := \{j : |\beta_j^0| > \lambda_{\text{thres}}\}$ is its active set.

2.6. Variable selection in the approximately sparse case

The irrepresentable condition on the set of inactive variables is commonly used to show that the $\ell_1$-penalized estimator $\hat{\beta}$ has no false positives. Inequality (2.7) in Corollary 2.2 shows that under irrepresentable conditions on a set $S^c$ where the coefficients of $\beta_j^0$, $j \in S^c$, are small, the estimated coefficients $|\hat{\beta}_j|$, $j \in S^c$ will be small as well. We now show that under stronger bounds for $\|\hat{\Gamma}_{S^c}\|_1$ actually $\hat{\beta}_j$ will be zero for $j \in S^c$. This result thus extends the situation to the approximately sparse case where there may be many non-zero but small coefficients. It can be a step towards local uniformity and away from super-efficiency and may be useful for building confidence intervals. Note that $X \hat{C}_{S^c} \beta_{S^c}^0$ is the part of $X \beta_{S^c}$ left over after projecting it on $X_S$.

**Theorem 2.2.** Let $\hat{\beta}$ be the unique solution of the KKT conditions (2.3). Let $T_\epsilon := \{\|X_T \epsilon\|_\infty/n \leq \lambda_\epsilon\}$. Consider some set $S \subset \{1, \ldots, p\}$ with $\text{rank}(X_S) = |S|$ and define $\hat{\Gamma}_{S^c} := (X_S^T X_S)^{-1}X_S^T X_{S^c}$ and $X \hat{C}_{S^c} = X_{S^c} - X_S \hat{\Gamma}_{S^c}$. Suppose

$$(\lambda + \lambda_\epsilon) \|\hat{\Gamma}_{S^c}\|_1 + \|\hat{C}_{S^c}^T X \hat{C}_{S^c} \beta_{S^c}^0\|_\infty/n \leq \lambda - \lambda_\epsilon.$$

Then on $T_\epsilon$ we have $\hat{\beta}_j = 0$ for all $j \notin S$.

2.7. De-sparsifying a group of variables

We define the group de-sparsified estimator

$$\hat{b}_j := \hat{\beta}_j + (\hat{C}_J^T X_T X_J)^{-1} \hat{C}_J^T X_T (Y - X \hat{\beta})$$

assuming the above used matrix inverse exists.

Again we shall need the smallest eigenvalue

$$\hat{\phi}_2^2 := \min\{\|X \hat{C}_J \hat{\beta}_J\|_2^2 : \|\beta_J\|_2 = 1\}$$

which is bounded from below by the compatibility constant $\hat{\phi}_2^2(\|\hat{\Gamma}\|_1, J)$. 


Lemma 2.5. Let
\[ \mathcal{T}_{J, e} := \{ \| \beta J^T \hat{C}_j X^T \epsilon \|_n \leq \tilde{\lambda}_{J, e} \sqrt{|J|} \| X \hat{C}_j \beta_j \|_n, \forall \beta_j \in \mathbb{R}^{|J|} \}. \]

If \( \lambda_{J, 0} |J| \| \hat{\phi}_J \|_1 / \hat{\phi}_J < 1 \), then on \( \mathcal{T}_{J, e} \)
\[ (1 - \lambda_{J, 0} |J| \| \| \hat{\phi}_J \|_1) \| X \hat{C}_j (\hat{b}_J - \beta_0^j) \|_n \leq \sqrt{|J|} \hat{\lambda}_{J, e} \hat{\phi}_J + \lambda_{J, 0} \| \hat{\beta}_J - \beta_0^J \|_1. \]

Let furthermore
\[ \mathcal{T}_{J, e}^* := \{ \| \hat{C}_j^T X^T \epsilon \|_2 / \sqrt{n} \leq \sqrt{|J|} \lambda_{J, e} \}. \]

Then on \( \mathcal{T}_{J, e}^* \),
\[ \| \hat{C}_j^T X^T X_j (\hat{b}_J - \beta_0^j) \|_2 / \sqrt{n} \leq \sqrt{|J|} \lambda_{J, e} \hat{\phi}_J + \lambda_{J, 0} \| \hat{\beta}_J - \beta_0^J \|_1. \]

Standard asymptotic scenario IV. Suppose that \( 1 / \hat{\phi}_J = \mathcal{O}(1) \) and that for a suitable \( \lambda = \mathcal{O}(\sqrt{\log p / n}) \) it holds that \( \lambda_{J, 0} \| \hat{\beta} - \beta_0 \|_1 = \mathcal{O}(n^{-1/2}) \) (recall this typically holds if approximate sparsity is of order \( \sqrt{n} / \log p \)) and that for suitable \( \lambda_{J, e} = \mathcal{O}(n^{-1/2}) \) the probability of \( \mathcal{T}_{J, e}^* \) tends to one. Then
\[ \| \hat{C}_j^T X^T X_j (\hat{b}_J - \beta_0^j) \|_2 / \sqrt{n} = \mathcal{O}(\sqrt{|J| / n}). \]

If we assume moreover for suitable \( \tilde{\lambda}_{J, e} = \mathcal{O}(n^{-1/2}) \) the probability of \( \mathcal{T}_{J, e} \) tends to one and that \( \| \hat{\Gamma}_j \|_1 \sqrt{|J| \log p / n} / \hat{\phi}_J = \mathcal{O}(1) \) is suitably small, then
\[ \| \hat{b}_J - \beta_0^j \|_2 = \mathcal{O}(\sqrt{|J| / n}). \]

3. Random design

We study the linear model (2.1) but now with \( X \) a random matrix with i.i.d. rows with distribution \( P \). Again \( \hat{\beta} \) is the Lasso estimator defined in (2.2). We let \( X_{-J} \Gamma_j \) be the projection of \( X_j \) on \( X_{-J} \) in \( L_2(P) \) and \( X C_J := X_j - X_{-J} \Gamma_j \). Then by the same arguments as in Theorem 2.1 we find Theorem 3.1 below. To avoid digressions we omit the counterpart of the first inequality (2.5) in Theorem 2.1 as it has no direct counterpart for de-sparsifying since we now use theoretical projections which are usually unknown (as \( P \) is usually unknown).

With some abuse of notation, we let \( \hat{\phi}_J \) now be the smallest eigenvalue of the matrix \( C_j^T X^T X C_j / n \). We note that the formulation in Theorem 3.1 is again in terms of the (now random) norm \( \| \cdot \|_n \) and (now random) compatibility constants \( \hat{\phi}(L, S) \).

In our results of the next section for general loss functions we use an alternative approach, leading in the case of random design to a formulation where the difference between the \( L_2(P) \) norm and the empirical norm \( \| \cdot \|_n \) ends up in the remainder term (see Subsection 5.1).

Theorem 3.1. Define
\[ \mathcal{T}_{J, e} := \{ \| \beta J^T \hat{C}_j X^T \epsilon \|_n \leq \tilde{\lambda}_{J, e} \sqrt{|J|} \| X \hat{C}_j \beta_j \|_n, \forall \beta_j \in \mathbb{R}^{|J|} \}, \]
\[ \mathcal{T}_{J, e} := \{ \| \hat{C}_j^T X^T \epsilon \|_n \leq \lambda_{J, e} \}, \]
\[ \mathcal{T}_{J, e} := \{ \| X^T X C_J \|_n \| \hat{\beta} - \beta_0 \|_1 / n \leq \lambda_{J, e} \}. \]
We have on $T_{J,e} \cap T_{J,\text{rem}}$
\[ \|XC_J(\hat{\beta}_J - \beta^0_J)\|_n \leq \frac{\sqrt{J}}{\phi_J} \left[ \lambda_{J,e} \hat{\phi}_J + (\lambda + \lambda_{J,\text{rem}})\|C_J\|_1 \right]. \]

Assume $\|\Gamma_J\|_1 < 1$ and that in fact that for some $\lambda_1 \geq 0$
\[ \lambda(1 - \|\Gamma\|_1 - \lambda_1) > \lambda_{J,e} + \lambda_{J,\text{rem}}\|C_J\|_1 + \lambda_1. \]

Let
\[ L := \frac{\lambda_{J,e} + (\lambda + \lambda_{J,\text{rem}})\|C_J\|_1 + \lambda\|\Gamma_J\|_1 + \lambda_1}{\lambda(1 - \|\Gamma_J\|_1) - (\lambda_{J,e} + \lambda_{J,\text{rem}}\|C_J\|_1) - \lambda_1}. \]

Then on $T_{J,e} \cap T_{J,\text{rem}}$,
\[ \|XC_J(\hat{\beta}_J - \beta^0_J)\|_n^2 + 2\left( \lambda(1 - \|\Gamma_J\|_1) - (\lambda_{J,e} + \lambda_{J,\text{rem}}\|C_J\|_1) - \lambda_1 \right)\|\hat{\beta}_J - \beta_J\|_1 \leq \frac{4\lambda^2|J \cap S|}{\phi^2(L, J \cap S)} + \|XC_J(\beta_J - \beta^0_J)\|_n^2. \]

4. General loss functions

Let $X_1, \ldots, X_n$ be independent observations in some observation space $\mathcal{X}$. For $d \in \mathbb{N}$ and a function $f : \mathcal{X} \to \mathbb{R}^d$ we use the notation
\[ P_n f := \frac{1}{n} \sum_{i=1}^n f(X_i), \quad P f := \mathbb{E} P_n f. \]

Consider a convex subset $\Theta$ of $\mathbb{R}^p$ and a loss function $\rho_\theta : \mathcal{X} \to \mathbb{R}$ with derivative $\dot{\rho}_\theta := \partial \rho_\theta / \partial \theta, \theta \in \Theta$. We examine the $\ell_1$-penalized M-estimator
\[ \hat{\theta} := \arg\min_{\theta \in \Theta} \left\{ P_n \rho_\theta + \lambda \|\theta\|_{1,\text{off}} \right\}, \]

where $\|\theta\|_{1,\text{off}} = \sum_{j \notin J} |\theta_j|$ is the $\ell_1$-penalty on the parameters $\{\theta_j\}_{j \notin J}$ and $J \subset \{1, \ldots, p\}$ is a fixed set. The subscript "off" refers to the set of parameters $\{\theta_j\}_{j \notin J}$ which are susceptible to being turned off (i.e. set to zero). The set $J$ contains the indexes of parameters that are not penalized. We assume that $\hat{\theta}$ is the unique solution of the KKT conditions
\[ P_n \dot{\rho}_\hat{\theta} + \lambda \hat{\varepsilon} = 0, \quad (4.1) \]

where $\hat{\varepsilon}_j = \text{sign}(\hat{\theta}_j)$ if $\hat{\theta}_j \neq 0$, $j \notin J$, $\hat{\varepsilon}_j = 0$ if $j \in J$ and $\|\hat{\varepsilon}\|_{\infty} \leq 1$.

We define $\theta^0$ as the solution of $P \dot{\rho}_{\theta^0} = 0$ (assumed to exist and be unique) and assume that
\[ \mathcal{I} := I(\theta^0) := \partial P \dot{\rho}_{\theta^0} / \partial \theta^T|_{\theta = \theta^0} \]
exists and is invertible. Its smallest eigenvalue is denoted by $\phi_{\theta^0}^2$. We define
\[ \text{rem}(\theta - \theta^0) := \mathcal{I}(\theta - \theta^0) - P_n(\dot{\rho}_\theta - \dot{\rho}_{\theta^0}). \]

The behaviour of this remainder term is studied in the next sections.

We let
\[ \Gamma_J := \mathcal{I}^{-1}_{J,J} \mathcal{I}_{-J,J}. \]
and for all $j \in J$ and all $k \in \{1, \ldots, p\}$
\[
C_{j,k,j} = \begin{cases} 
1 & k = j, \\
0 & k \neq j, k \in J . \\
-\gamma_{j,k,j} & k \notin J . 
\end{cases}
\]

We let \(\| \Gamma_j \|_{1, \text{off}} := \max_{j \in J} \sum_{k \in J \setminus J} |\Gamma_{j,k,j}| \) and \(\| C_j \|_{1, \text{off}} := 1 + \| \Gamma_j \|_{1, \text{off}} = \max_{j \in J} \sum_{k \in J} |c_{j,k,j}|\). We write
\[
w := -P_n \hat{\theta}_0 .
\]

Note that if \(nP_n \hat{\theta}_0\) is a well-specified log-likelihood then under regularity \(nP[w w^T] = \mathcal{I}\) is the Fisher information and \(nP[C_j^T w w^T C_j] = \mathcal{I}_{J,j} - \Gamma_j^T \mathcal{I}_{-J,j} \Gamma_j\).

In following theorem we derive rates for groups (Part I) and sharp oracle results under conditions on the worst possible sub-direction \(\Gamma_j\) (Part II). The latter generalizes sharp oracle results for M-estimators presented in [18].

**Theorem 4.1.** We define
\[
\mathcal{T}_{\text{rem}} := \{||\text{rem}(\hat{\theta} - \theta^0)||_\infty \leq \lambda_{\text{rem}}\}.
\]

**Part I** Let
\[
\mathcal{T}_{J,w} := \{||\hat{\theta}_0^j C_j^T w|| \leq \lambda_{J,w} \sqrt{|J|} \sqrt{\hat{\theta}_0^j (\mathcal{I}_{J,j} - \Gamma_j^T \mathcal{I}_{-J,j} \Gamma_j) \theta_j}, \ \forall \theta_j \in \mathbb{R}^{|J|}\}.
\]

Then on \(\mathcal{T}_{J,w} \cap \mathcal{T}_{\text{rem}}\)
\[
(\hat{\theta}_j - \theta^0)^T (\mathcal{I}_{J,j} - \Gamma_j^T \mathcal{I}_{-J,j} \Gamma_j)(\hat{\theta}_j - \theta^0) \leq \left(\lambda_{J,w} + \lambda \|C_j\|_{1, \text{off}} + \lambda_{\text{rem}} \|C_j\|_1 \right)^2 |J|/\phi_0^2.
\]

**Part II** Let
\[
\mathcal{T}_{J,w} := \{||C_j^T w||_\infty \leq \lambda_{J,w}\},
\]

Assume that \(\| \Gamma_j \|_{1, \text{off}} < 1\) and in fact that for some \(\lambda_1 \geq 0\)
\[
\lambda(1 - || \Gamma_j ||_{1, \text{off}}) > \lambda_{J,w} + \lambda_{\text{rem}} ||C_j||_1 + \lambda_1 .
\]

Let \(\theta \in \mathbb{R}^p\) be a vector with \(S := \{\theta_j \neq 0\} \supset J\). Then on \(\mathcal{T}_{J,w} \cap \mathcal{T}_{\text{rem}}\)
\[
(\hat{\theta}_j - \theta^0)^T (\mathcal{I}_{J,j} - \Gamma_j^T \mathcal{I}_{-J,j} \Gamma_j)(\hat{\theta}_j - \theta^0) + 2 \left(\lambda(1 - || \Gamma_j ||_{1, \text{off}}) - (\lambda_{J,w} + \lambda_{\text{rem}} ||C_j||_1) - \lambda_1 \right) ||\hat{\theta}_j - \beta_j||_1 \leq 4\lambda^2 |J \cap S|/\phi_0^2 + (\theta_j - \theta^0)^T (\mathcal{I}_{J,j} - \Gamma_j^T \mathcal{I}_{-J,j} \Gamma_j)(\theta_j - \theta^0).
\]

Theorem 4.1 follows using the same arguments as those used for the proof of Theorem 2.1. It can be applied to obtain a global oracle inequality (\(J = \{1, \ldots, p\}\)), a result for small coefficients (\(J = S^c\)) and rates for single parameters (\(J = \{j\}\)). We note however that unlike Corollary 2.2 all these results involve he remainder term \(\lambda_{\text{rem}}\) which then needs to be handled using separate arguments (e.g. applying results from [21], [7] or [14], see also the next section).

We also formulate an extension of Theorem 2.2.
Theorem 4.2. Let \( \hat{\beta} \) be the unique solution of the KKT conditions (4.1). Consider some set \( S \supset J \) and suppose \( \hat{\theta}_S \in \Theta \) is the solution of the KKT conditions under the restriction that the coefficients are zero outside the set \( S \):

\[
P_n(\hat{\beta}_S) = 0,
\]

(4.2)

where \( \hat{z}_{j,S} = \text{sign}(\hat{\theta}_{j,S}) \) if \( \hat{\theta}_{j,S} \neq 0 \), \( j \in S \setminus J \), \( \hat{z}_{j,S} = 0 \) if \( j \in J \) and \( \| \hat{z}_S \|_\infty \leq 1 \). Let \( T_w := \{ \| w \|_\infty \leq \lambda_w \} \) and \( T_{\text{rem}} := \{ \| \text{rem}(\hat{\theta}_S - \theta^0) \|_\infty \leq \lambda_{\text{rem}} \} \). Suppose
\[
\lambda \| \hat{\Gamma}_S \|_{1,\text{off}} + (\lambda_w + \lambda_{\text{rem}}) \| \hat{\Gamma}_S \|_1 + \| (I_{S^c,S^c} - \Sigma_{S^c,S^c}) \theta^0_{S^c} \|_\infty \leq \lambda - \lambda_w - \lambda_{\text{rem}}.
\]

Then on \( T_w \cap T_{\text{rem}} \) we have \( \hat{\theta}_j = 0 \) for all \( j \notin S \).

5. The remainder in terms of global norms

5.1. The linear model with random design

In the linear model, we write \( \theta := \beta \) and we have
\[
\rho_\beta(X_i, Y_i) = (Y_i - X_i \beta)^2, \ i = 1, \ldots, n,
\]

where with some abuse of notation \( X_i \) is now the \( i \)-th row of \( X \) (i.e. we use the notation \( X_i \) for a row and \( X_j \) for a column, the distinction only in the notation for observations and variables: \( i \) for observations and \( j \) for variables). Moreover,
\[
P_n(\hat{\beta} - \beta^0) = -X^T(Y - X \hat{\beta})/n = -X^T \epsilon/n + \Sigma(\hat{\beta} - \beta^0)
\]

where \( \Sigma := X^T X/n \). It follows that for the case of random design
\[
P_n(\hat{\beta} - \beta^0) = \hat{\Sigma}(\hat{\beta} - \beta^0)
\]

and \( I = \Sigma_0 := \mathbf{E} \Sigma \). Hence
\[
\text{rem}(\beta - \beta^0) = - (\hat{\Sigma} - \Sigma_0)(\beta - \beta^0)
\]

and
\[
\| \text{rem}(\beta - \beta^0) \|_\infty \leq \| \hat{\Sigma} - \Sigma_0 \|_\infty \| \hat{\beta} - \beta^0 \|_1.
\]

5.2. Generalized linear models with random design

We let
\[
\rho_\beta(X_i, Y_i) := \rho(Y_i, X_i \beta), \ i = 1, \ldots, n
\]

where \( \{X_i\}_{i=1}^n \) are i.i.d. \( p \)-dimensional row-vectors and \( \{Y_i\}_{i=1}^n \) are i.i.d. response vectors with values in some set \( \mathcal{Y} \subseteq \mathbb{R} \). Assume that for all \( z, \tilde{z} \)
\[
|\hat{\rho}(y, z) - \rho(y, \tilde{z})| \leq L |z - \tilde{z}|, \ \forall \ y \in \mathcal{Y}
\]

(this can be made into a local condition) and that \( \| X \|_\infty \leq K_X \) Then
\[
\| \text{rem}(\beta - \beta^0) \|_\infty \leq K_X L \| X(\hat{\beta} - \beta^0) \|_n^2 + \| \Sigma_{\beta^0} - \Sigma_{\hat{\beta}^0} \|_\infty \| \hat{\beta} - \beta^0 \|_1.
\]

Here
\[
\Sigma_{\beta^0} := X^T W_{\beta^0} X/n, \ \Sigma_{\hat{\beta}^0} := \mathbf{E} \Sigma_{\beta^0}
\]

where \( W_{\beta^0} := \text{diag}((\hat{\rho}(Y_i, X_i \beta^0))_{i=1}^n) \).

Standard asymptotic scenario V Suppose \( \hat{\rho}(Y_i, X_i \beta^0) \geq \eta > 0 \) almost surely. Assume \( K_X = O(1) \), \( L = O(1) \) as well as \( 1/\eta = O(1) \). Let \( \lambda = O_P(\sqrt{\log p/n}) \) and suppose \( ||X(\hat{\beta} - \beta^0)||_n = O_P(\lambda \sqrt{n}) \) and \( ||\hat{\beta} - \beta^0||_1 = o_P(\lambda s) \) (see e.g. [7]). Then \(
\| \text{rem}(\hat{\beta} - \beta^0) \|_\infty = o_P(n^{-1/2}) \) for \( s = o(\sqrt{n} / \log p) \).
5.3. Exponential families

Let $X_1, \ldots, X_n$ be i.i.d. with distribution $P$. We consider the loss function

$$\rho_\theta(x) := -\sum_{j=1}^{p} \psi_j(x)\theta_j + d(\theta)$$

where $\theta \in \Theta$ with $\Theta$ a convex subset of $\mathbb{R}^p$. Moreover, $d(\theta)$ is a twice differentiable convex function satisfying $\dot{d}(\theta_0) = P\psi$. We assume existence of

$$\mathcal{I} := I(\theta^0) := \ddot{d}(\theta^0)$$

and we assume that $\mathcal{I}^{-1}$ exists and let $\phi^2_0$ be the smallest eigenvalue of $\mathcal{I}$.

In this case

$$w := -P_n\dot{\rho}_{\theta^0} = (P_n - P)\psi$$

and

$$\text{rem}(\theta - \theta^0) = \mathcal{I}(\theta)(\theta - \theta^0) - [\dot{d}(\theta) - \dot{d}(\theta_0)].$$

If we assume

$$\|\ddot{d}(\theta) - \ddot{d}(\theta^0)\|_{\infty} = \mathcal{O}(\|\theta - \theta^0\|_1) \tag{5.1}$$

then

$$\|\text{rem}(\theta - \theta^0)\|_{\infty} \leq \|\theta - \theta_0\|_1\|\ddot{d}(\theta) - \ddot{d}(\theta^0)\|_{\infty} = \mathcal{O}(\|\theta - \theta^0\|_1^2).$$

Furthermore, for exponential families

$$d(\theta) = \log \left( \int \exp \left[\sum_{j=1}^{p} \psi_j \theta_j \right] d\mu \right), \quad \theta \in \Theta,$$

where $\mu$ is some dominating measure for $P$. When the model is well-specified it holds that the density $p_0 := dP/d\mu$ is equal to $p_0 = \exp[-\rho_{\theta^0}]$. Then (5.1) holds if $\|\log p_0\|_{\infty} = \mathcal{O}(1)$, $\max_j \|\psi_j\|_{\infty} = \mathcal{O}(1)$ and $\|\theta - \theta^0\|_1 = \mathcal{O}(1)$.

**Standard asymptotic scenario VI** Suppose that $\lambda = \mathcal{O}_P(\sqrt{\log p/n})$ and that $\|\hat{\theta} - \theta^0\|_1 = \mathcal{O}_P(\lambda s)$ and $\|\hat{\theta} - \theta^0\|_2 = \mathcal{O}_P(\lambda \sqrt{s})$ (see e.g. [18] for such results for general high-dimensional models). Then $\|\text{rem}(\theta - \theta^0)\|_{\infty} = o_P(n^{-1/2})$ for $s = o(n^{1/4}/\log p)$.

6. Brouwer’s fixed point theorem

The remainder term $\text{rem}(\hat{\theta} - \theta^0)$ will generally only be small if $\hat{\theta}$ is close enough to $\theta^0$. If the global rate of convergence is too slow we need a technique different from the one of the previous section to deal with the remainder term. Here, Brouwer’s fixed point theorem can be useful. The idea is from [15].

**Lemma 6.1.** Let $\hat{\theta}$ be the unique solution in $\mathbb{R}^p$ of the estimating equations

$$\hat{\theta} - \theta^0 = G(\text{rem}(\hat{\theta} - \theta^0)) + v(\hat{\theta} - \theta^0) + u_0$$

where $G : \mathbb{R}^p \to \mathbb{R}^p$ and $v : \mathbb{R}^p \to \mathbb{R}^p$ are functions and $u_0 \in \mathbb{R}^p$ is a constant vector. Let $q \geq 1$. Suppose that for some constants $K$ and $\varepsilon$

$$G(B_q(1)) \in B_q(K),$$
\[ \|u_0\|_q \leq K\varepsilon, \quad \sup_\theta \|v(\theta - \theta^0)\|_q \leq K\varepsilon \]

and that
\[ \|\text{rem}(B_\theta(3K\varepsilon))\|_q \leq \varepsilon. \]

Then \[ \|\hat{\theta} - \theta^0\|_q \leq 3K\varepsilon. \]

7. The irrepresentable condition and rates in sup-norm

We showed in Theorems 2.2 and 4.2 that an irrepresentable condition on the set of small variables can be used to show that these parameters are estimated as being zero. However, in some cases there are no ready-to-use results to handle the remainder term. One may then need rather strong conditions to deal with this. We now show that irrepresentable conditions can lead to convergence in sup-norm. The remainder term then needs to be small for sup-norm neighbourhoods which are smaller than the $\ell_1$- or $\ell_2$- neighbourhoods these imply. The idea is as in [15] but with the extension that we only need approximate sparsity (i.e. there may be many non-zero but small coefficients).

In the next theorem we use the notation of Section 4, and assume that \( \hat{\theta} \) is the unique solution of the KKT conditions (4.1). The proof is based on Lemma 6.1.

**Theorem 7.1.** Let \( S \supset \mathcal{J} \) and suppose the solution \( \tilde{\theta}_S \in \Theta \) of the restricted KKT conditions (4.2) exists. Assume for some \( \kappa_{S^c} \geq 1 \)
\[ \|I_{S^c,S^c} - \Gamma_{S^c}^T I_{S,S} \Gamma_{S^c}\|_1 \leq \kappa_{S^c}, \]
for some \( K_S \geq 1 \)
\[ \|I_{S,S}\|_1 \leq K_S \]
and that for some \( \lambda_{\text{thres}} \)
\[ \|\theta^0_{S^c}\|_\infty \leq \lambda_{\text{thres}}. \]

Let
\[ T_w := \{\|w\|_\infty \leq \lambda_w\}, \]
\[ T_{\text{rem}} := \{\|\text{rem}(\theta_S - \theta^0)\|_\infty \leq \lambda_{\text{rem}} \forall \theta_S \in \Theta : \|\theta_S - \theta^0\|_\infty \leq 3K(\lambda_w + \lambda)\}. \]

Suppose
\[ \lambda\|\Gamma_S\|_{1,\text{off}} + (\lambda_w + \lambda_{\text{rem}})\|\Gamma_{S^c}\|_1 + \kappa_{S^c}\lambda_{\text{thres}} \leq \lambda - \lambda_w - \lambda_{\text{rem}}. \]

Then on \( T_w \cap T_{\text{rem}} \), we have \[ \|\hat{\theta}_S - \theta^0_S\|_\infty \leq 3K_S(\lambda_w + \lambda) \] and \[ \|\hat{\theta}_{S^c}\|_\infty = 0. \]

8. Estimating a precision matrix

We investigate the remainder term for the graphical Lasso. The approach is again similar to [15]. Our main extension is that we no longer assume that the truth is exactly sparse (in the sense of having many parameters exactly equal to zero) but only approximately sparse (i.e. having a sparse approximation). This extension may be important when applying the results for obtaining confidence intervals (see [8]) because approximate sparsity appears more in line with the concept of honesty for confidence intervals.
Let $X$ be an $n \times p$ matrix with i.i.d. rows with distribution $P$. We let $\hat{\Sigma} := X^TX/n$ and $\Sigma_0 := \mathbf{E}\hat{\Sigma}$. We assume that $\Theta_0 := \Sigma_0^{-1}$ exists. The matrix $\Theta_0$ is called the precision matrix. We consider the estimator

$$\hat{\Theta} = \arg \min_{\Theta \text{ p.s.d.}} \text{trace}(\hat{\Sigma}\Theta) - \log \det(\Theta) + 2\lambda\|\Theta\|_{1,\text{off}},$$

where $\|\Theta\|_{1,\text{off}} := \sum_j \sum_{k\neq j} |\Theta_{k,j}|$. (We note that $\Theta$ is now the parameter (not the parameter space) and the parameter space is the set of all positive semi-definite (p.s.d.) matrices.) Observe this corresponds to a loss function from the exponential family, namely

$$\rho_\Theta(x) = -\sum_{j,k} \psi_{k,j}(x)\Theta_{k,j} + d(\Theta), \quad \psi_{k,j}(x) = -x_kx_j, \quad d(\Theta) = -\log \det(\Theta).$$

Consider the KKT-conditions

$$\hat{\Sigma} - \hat{\Theta}^{-1} + \lambda \hat{Z} = 0$$

where $\hat{Z}_{j,j} = 0$ for all $j$ and for $j \neq k$, $\hat{Z}_{j,k} := \text{sign}(\hat{\Theta}_{j,k})$ when $\hat{\Theta}_{j,k} \neq 0$. Moreover $\|\hat{Z}\|_\infty \leq 1$. We define $W = \hat{\Sigma} - \Sigma_0$ and write this as

$$W - (\hat{\Theta}^{-1} - \Theta_0^{-1}) + \lambda \hat{Z} = 0.$$

Note that for $\hat{\Delta} := \hat{\Theta} - \Theta_0$

$$-(\hat{\Theta}^{-1} - \Theta_0^{-1}) = \hat{\Theta}^{-1}\hat{\Delta}\Theta_0^{-1}$$

$$:= \Theta_0^{-1}\hat{\Delta}\Theta_0^{-1} - \text{rem}(\hat{\Delta}),$$

where

$$\text{rem}(\Delta) := \left((\Theta_0 + \Delta)^{-1} - \Theta_0^{-1} + \Theta_0^{-1}\Delta\Theta_0^{-1}\right)$$

is the remainder term.

We let $\Theta_S$ be a symmetric positive definite matrix which has zeroes outside the set $S$. Moreover, $\Theta_S^0$ denotes the matrix $\Theta_0$ with all entries in $S^c$ set to zero (this matrix may not be positive semi-definite) and $\Theta_S^0 := \Theta_0 - \Theta_S^0$. We moreover write

$$\|\Theta_S^0\|_0 := \max_j \{|k: (k,j) \in S, \Theta_0,k,j \neq 0\}|.$$

**Lemma 8.1.** Suppose that for some $\epsilon_0$ and $\eta_0$

$$\|\Theta_0^{-1}\|_1 (\epsilon_0 \|\Theta_S^0\|_0 + \|\Theta_S^0\|_1) \leq \eta_0 < 1.$$

Then for $\|\Theta_S - \Theta_0\|_\infty \leq \epsilon_0$ we have

$$\text{rem}(\Theta_S - \Theta_0) \leq \eta_0 \epsilon_0 \|\Theta_0^{-1}\|_1^2 / (1 - \eta_0).$$

**Standard asymptotic scenario VII** We combine Lemma 8.1 with Theorem 7.1, where in the latter we use the vectorized versions of the matrix parameter $\Theta$ to define the matrices involved. Suppose that $\|\Theta_0^{-1}\|_1 = O(1)$ (recall $\Theta_0^{-1} = \Sigma_0$), $\|I_{S^c}^T - \Gamma_{S^c}^T\|_1 = O(1)$ and $\|I_{S^c}^T\|_1 = O(1)$. Assume that the probability of $T_w$ (with $w$ the vectorized version of $W$) tends to one for $\lambda_w = O(\sqrt{\log p/n})$ and let $\lambda = O_p(\sqrt{\log p/n})$ be suitably large. Let $S := \{j,k: |\Theta_{0,j,k}| > \lambda_{\text{thres}}\}$ where
\[ \lambda_{\text{thr}} = O(\sqrt{\log p/n}) \] is suitably small. Assume moreover \( \| \Theta^0 \|_1 = d \) (this is up to small coefficients the maximal edge degree of the matrix \( \Theta_0 \)) with \( d = O(\sqrt{n} / \log p) \) and that \( \| \Theta^0 \|_1 = O(\lambda d) \). Finally assume that \( \| \Gamma^S \|_1 < 1 \) is sufficiently small (but not necessarily tending to zero). Then \( \| \hat{\Theta} - \Theta_0 \|_1 = O_p(\sqrt{\log p/n}) \) and moreover \( \hat{\Theta}_{j,k} = 0 \) for all \( (j,k) \in S \). An example of an approximately sparse case where this result can be applied is where the columns of \( \Theta_0 \) have a uniformly finite \( \ell_r \)-“norm” for some \( 0 < r < 1 \), i.e. where

\[ \max_j \sum_k |\Theta_{0,k,j}|^r = O(1). \]

9. Proofs

9.1. Proofs for Section 2: The linear model with fixed design

We first prove the results for a single parameter.

**Proof of Lemma 2.1.** Define for \( t \in \mathbb{R} \), \( \hat{\beta}(t) := \hat{\beta} + t \hat{c}_j \). By the KKT conditions (see (2.3)) the sub-gradient of \( \| Y - X \hat{\beta}(t) \|_2^2 / 2 + \lambda \| \beta(t) \|_1 \) at \( t = 0 \) is equal to zero, i.e.

\[ -(\hat{c}_j X^T (Y - X \hat{\beta})/n + \lambda \hat{c}_j^T \hat{z}) = 0 \]

where \( \hat{z}_j = \text{sign}(\hat{\beta}_j) \) if \( \hat{\beta}_j \neq 0 \) and moreover \( \| \hat{z} \|_\infty \leq 1 \). This can be rewritten to

\[ -(X_j - X_{-j} \hat{\gamma}_j)^T \epsilon/n + (X_j - X_{-j} \hat{\gamma}_j)^T X_j (\hat{\beta}_j - \beta^0) / n \]

\[ + (X_j - X_{-j} \hat{\gamma}_j)^T X_{-j} (\hat{\beta}_{-j} - \beta^0_{-j}) / n = \lambda \hat{c}_j^T \hat{z}. \]

But from the KKT conditions, using the counterpart of (2.4) for \( \hat{\gamma}_j \),

\[ (X_j - X_{-j} \hat{\gamma}_j)^T X_j / n = \hat{\gamma}^2_j. \]

Moreover, again by the KKT conditions, using the counterpart of (2.3) for \( \hat{\gamma}_j \),

\[ \|(X_j - X_{-j} \hat{\gamma}_j)^T X_{-j} \|_\infty / n \leq \lambda_j. \]

It is also clear that

\[ |\hat{c}_j^T \hat{z}| \leq \| \hat{c}_j \|_1. \]

We thus find on \( T_{j,c} \) that

\[ \hat{\gamma}^2_j |\hat{\beta}_j - \beta^0_j| \leq \lambda_j \epsilon \hat{\gamma}_j + \lambda_j \| \hat{\beta}_{-j} - \beta^0_{-j} \|_1 + \lambda \| \hat{c}_j \|_1. \]

Whence the first result. The second result follows from

\[ (\hat{\beta}_j - \beta^0_j) \hat{c}_j^T \hat{z} \geq |\hat{\beta}_j| - |\beta^0_j| - |\hat{\gamma}_j| \| \hat{\beta}_{-j} - \beta^0_{-j} \|. \]

\[ \square \]

**Proof of Lemma 2.2.** This follows immediately from Corollary 2.1. \[ \square \]

**Proof of Lemma 2.3.** Let \( \hat{\Sigma} := X^T X/n \) be the Gram matrix. By the KKT conditions for \( \{ \hat{\gamma}_j \} \), we have

\[ \hat{\Sigma} \hat{\Theta} - I = \lambda_0 \hat{Z} \hat{T} \hat{T}^{-2}, \]
where \( \hat{Z}_{j,j} = 0 \) for all \( j \) and for \( k \neq j \), \( \hat{z}_{k,j} = \text{sign}(\hat{\theta}_{k,j}) \) when \( \hat{\theta}_{k,j} \neq 0 \). Moreover \( \|\hat{Z}\|_{\infty} \leq 1 \). Therefore

\[
\hat{T}^{-1}\hat{T}^2(\hat{\beta} - \beta^0) = \hat{T}^{-1}\hat{T}^2(\beta - \beta^0) + \hat{T}^{-1}\hat{T}^2\hat{\Theta}^T X^T(Y - X\hat{\beta})/n
\]

\[
= \hat{T}^{-1}\hat{T}^2\hat{\Theta}^T X^T\epsilon/n - \hat{T}^{-1}\hat{T}^2(\hat{\Theta}^T\hat{\Sigma} - I)(\hat{\beta} - \beta^0)
\]

\[
= \hat{T}^{-1}\hat{T}^2\hat{\Theta}^T X^T\epsilon/n - \lambda_0\hat{Z}^T(\hat{\beta} - \beta^0).
\]

Clearly, \( \|\hat{Z}^T(\hat{\beta} - \beta^0)\|_{\infty} \leq \|\hat{Z}\|_{\infty}\|\hat{\beta} - \beta^0\|_1 \leq \|\hat{\beta} - \beta^0\|_1 \). \(\Box\)

**Proof of lemma 2.4.** As in the proof of Lemma 2.1

\[
(X_j - X_{-j}\hat{\gamma}_j)^T \epsilon/n + (X_j - X_{-j}\hat{\gamma}_j)^T X_j(\hat{\beta}_j - \beta^0_j)/n
\]

\[
+(X_j - X_{-j}\hat{\gamma}_j)^T X_{-j}(\hat{\beta}_{-j} - \beta^0_{-j})/n = \lambda\epsilon^T\hat{c}_j,
\]

Now by definition

\[
\|(X_j - X_{-j}\hat{\gamma}_j)^T X_{-j}(\hat{\beta}_{-j} - \beta^0_{-j})/n \leq \lambda_j\|\hat{\beta}_{-j} - \beta^0_{-j}\|_1.
\]

Furthermore

\[
(X_j - X_{-j}\hat{\gamma}_j)^T X_j/n = \hat{\tau}_j^2 + (X_j - X_{-j}\hat{\gamma}_j)^T X_{-j}\hat{\gamma}_j/n \geq \hat{\tau}_j^2 - \lambda_j\|\hat{\gamma}_j\|_1.
\]

Continuing as in the proof of Lemma 2.1, we get on \( T_j,\epsilon \)

\[
\hat{\tau}_j^2|\hat{\beta}_j - \beta^0_j| \leq \lambda_j\hat{\tau}_j + \lambda_j\|\hat{\beta}_{-j} - \beta^0_{-j}\|_1 + \lambda\|\hat{c}_j\|_1 + \lambda\|\hat{\gamma}_j\|_1|\hat{\beta}_j - \beta^0_j|.
\]

\(\Box\)

The main result for a group of parameters has a somewhat more involved proof.

**Proof of Theorem 2.1.** Recall the KKT conditions

\[-X^T(Y - X\hat{\beta})/n + \lambda\hat{\epsilon} = 0 \]

where \( \hat{\epsilon} = \text{sign}(\hat{\beta}_j) \) if \( \hat{\beta}_j \neq 0 \) and \( \|\hat{\epsilon}\|_{\infty} \leq 1 \). Multiply by \((\hat{\beta}_J - \beta^0_J)^T\hat{C}_J^T\) to find that

\[-(\hat{\beta}_J - \beta^0_J)^T\hat{C}_J^T X^T(Y - X\hat{\beta})/n + \lambda(\hat{\beta}_J - \beta^0_J)^T\hat{C}_J^T\hat{\epsilon} = 0.\]

Rewrite this to

\[-(\hat{\beta}_J - \beta^0_J)^T\hat{C}_J^T X^T\epsilon/n + (\hat{\beta}_J - \beta^0_J)^T\hat{C}_J^T X^T\hat{C}_J(\hat{\beta}_J - \beta^0_J)/n
\]

\[+(\hat{\beta}_J - \beta^0_J)^T\hat{C}_J^T X_{-j}\left((\hat{\beta}_{-j} - \beta^0_{-j}) + \hat{\Gamma}_j(\hat{\beta}_J - \beta^0_J)\right)/n
\]

\[+\lambda\|\hat{\beta}_j\|_1 - \lambda(\beta_j)^T\hat{\epsilon}_J - \lambda(\hat{\beta}_J - \beta^0_J)^T\hat{\Gamma}_J^T\hat{\epsilon}_{-j} = 0.\]

By the KKT conditions for \( \hat{\Gamma}_j \)

\[
\|(X\hat{C}_J)^T X_{-j}\|_{\infty}/n \leq \lambda_0.
\]

We therefore get

\[
(\hat{\beta}_J - \beta^0_J)^T\hat{C}_J^T X^T\hat{C}_J(\hat{\beta}_J - \beta^0_J)/n \leq ||(\hat{\beta}_J - \beta^0_J)^T\hat{C}_J^T X\epsilon||/n
\]

\[+(\lambda||\hat{\beta}_{-j} - \beta^0_{-j}\|_1 + \lambda\|\hat{\Gamma}_J\|_1)||\hat{\beta}_J - \beta^0_J\|_1.
\]

18
This gives by (9.1) with $\hat{\beta}_J = \beta_J^0$ and we use that
\[
\|\hat{\beta}_J - \beta_J^0\|_1^2 \leq |J|^2 \|\hat{C}_J (\hat{\beta}_J - \beta_J^0)\|_n^2 / \hat{\sigma}_J^2.
\]

**Part I** To obtain the first equation, we apply (9.1) with $\hat{\beta}_J = \beta_J^0$ and we use that
\[
\|\hat{\beta}_J - \beta_J^0\|_1^2 \leq |J|^2 \|\hat{C}_J (\hat{\beta}_J - \beta_J^0)\|_n^2 / \hat{\sigma}_J^2.
\]

Then we get on $T_{J, \epsilon}$
\[
(1 - \frac{\lambda_{J,0} |J| \hat{\sigma}_J}{\hat{\sigma}_J^2}) \|\hat{\beta}_J - \beta_J^0\|_1^2 \leq \lambda_{J, \epsilon} \sqrt{|J|^2} \|\hat{C}_J (\hat{\beta}_J - \beta_J^0)\|_n^2.
\]

Then we obtain
\[
\lambda_{J, \epsilon} \sqrt{|J|^2} \|\hat{C}_J (\hat{\beta}_J - \beta_J^0)\|_n^2 \leq \lambda_{J, \epsilon} \sqrt{|J|^2} \|\hat{C}_J (\hat{\beta}_J - \beta_J^0)\|_n^2.
\]

This gives the first result (2.5).

The second result (2.6) follows from (9.1) by similar arguments and the bounds $\|\hat{\beta}_J - \beta_J^0\|_1 \leq \|\hat{\beta} - \beta^0\|_1$ and $\|\hat{\beta}_J - \beta_J^0\|_1 \leq \|\hat{\beta} - \beta^0\|_1.$

**Part II** We may also use (9.1) to obtain that on $T_{J, \epsilon}$
\[
(\hat{\beta}_J - \beta_J)^T \hat{C}_J^T X^T \hat{C}_J (\hat{\beta}_J - \beta_J^0) / n \tag{9.2}
\]
\[
\leq \left[ \lambda_{J, \epsilon} + \lambda_{J, 0} \lambda_{J, \epsilon} + \lambda \|\hat{C}_J\|_1 \right] \|\hat{\beta}_J - \beta_J\|_1 - \|\hat{\beta}_J - \beta_J\|_1.
\]

If
\[
\|\hat{C}_J (\hat{\beta}_J - \beta_J^0)\|_n^2 + 2 \lambda_{J, 0} \|\hat{\beta}_J - \beta_J^0\|_1 \leq \|\hat{C}_J (\hat{\beta}_J - \beta_J)\|_n^2,
\]
we are done. So let us assume in the rest of the proof that
\[
\|\hat{C}_J (\hat{\beta}_J - \beta_J^0)\|_n^2 + 2 \lambda_{J, 0} \|\hat{\beta}_J - \beta_J^0\|_1 \geq \|\hat{C}_J (\hat{\beta}_J - \beta_J)\|_n^2.
\]

Then we have
\[
(\hat{\beta}_J - \beta_J)^T \hat{C}_J^T X^T \hat{C}_J (\hat{\beta}_J - \beta_J^0) / n = \frac{1}{2} \|\hat{C}_J (\hat{\beta}_J - \beta_J^0)\|_n^2
\]
\[
+ \frac{1}{2} \|\hat{C}_J (\hat{\beta}_J - \beta_J^0)\|_n^2 - \frac{1}{2} \|\hat{C}_J (\hat{\beta}_J - \beta_J)\|_n^2
\]
\[
\geq \frac{1}{2} \|\hat{C}_J (\hat{\beta}_J - \beta_J)\|_n^2 - \lambda_{J, 0} \|\hat{\beta}_J - \beta_J^0\|_1 \|\hat{\beta}_J - \beta_J^0\|_1.
\]

This gives by (9.2)
\[
\left[ \lambda - \lambda_{J, \epsilon} - \lambda_{J, 0} \|\hat{\beta} - \beta^0\|_1 \|\hat{C}_J\|_1 - \lambda \|\hat{C}_J\|_1 - \lambda_{J, \epsilon} \right] \|\hat{\beta}_J - \beta_J^0\|_1
\]
\[
\leq \left[ \lambda_{J, \epsilon} + \lambda_{J, 0} \|\hat{\beta} - \beta^0\|_1 \|\hat{C}_J\|_1 + \lambda \|\hat{C}_J\|_1 \right] \|\hat{\beta}_J - \beta_J^0\|_1.
\]
and hence on $T_{J,e} \cap T_{J,\text{rem}}$

\[
\left( \lambda(1 - \|\hat{\Gamma}_J\|_1) - (\lambda_{J,e} + \lambda_{J,\text{rem}}\|\hat{C}_J\|_1) - \lambda_1 \right) \|\hat{\beta}_{J,S}\|_1 \\
\leq (\lambda_{J,e} + (\lambda + \lambda_{J,\text{rem}})\|\hat{C}_J\|_1 + \lambda_1)\|\hat{\beta}_{J,S} - \beta_{J,S}\|_1.
\]

Moreover

\[
\|\hat{\Gamma}_J(\hat{\beta}_J - \beta_J)\|_1 \leq \|\hat{\Gamma}_J\|_1\|\hat{\beta}_J - \beta_J\|_1 = \|\hat{\Gamma}_J\|_1\|\hat{\beta}_{J,S}\|_1 + \|\hat{\Gamma}_J\|_1\|\hat{\beta}_{J,S} - \beta_{J,S}\|_1.
\]

Thus

\[
\|\hat{\beta}_{J,S}\|_1 + \|\hat{\Gamma}_J(\hat{\beta}_J - \beta_J)\|_1 \leq (1 + \|\hat{\Gamma}_J\|_1)\|\hat{\beta}_{J,S}\|_1 + \|\hat{\Gamma}_J\|_1\|\hat{\beta}_{J,S} - \beta_{J,S}\|_1 \\
\leq L\|\hat{\beta}_{J,S} - \beta_{J,S}\|_1.
\]

It follows that

\[
\|\hat{\beta}_{J,S} - \beta_{J,S}\|_1 \leq \frac{\sqrt{|J \cap S|}}{\phi(L, J \cap S)}\|X\hat{C}_J(\hat{\beta}_J - \beta_J)\|_n.
\]

But then

\[
\frac{1}{2}\|X\hat{C}_J(\hat{\beta}_J - \beta_J^0)\|_n^2 + \frac{1}{2}\|X\hat{C}_J(\hat{\beta}_J - \beta_J)\|_n^2 - \frac{1}{2}\|X\hat{C}_J(\beta_J - \beta_J^0)\|_n^2 \\
+ \left[ \lambda(1 - \|\hat{\Gamma}_J\|_1) - (\lambda_{J,e} + \lambda_{J,\text{rem}}\|\hat{C}_J\|_1) - \lambda_1 \right] \|\hat{\beta}_J - \beta_J\|_1 \\
\leq 2\lambda \frac{\sqrt{|J \cap S|}}{\phi(L, J \cap S)}\|X\hat{C}_J(\hat{\beta}_J - \beta_J)\|_n \\
\leq 2\lambda^2 \frac{|J \cap S|}{\phi^2(L, J \cap S)} + \frac{1}{2}\|X\hat{C}_J(\hat{\beta}_J - \beta_J)\|_n^2.
\]

This gives

\[
\|X\hat{C}_J(\hat{\beta}_J - \beta_J^0)\|_n^2 + 2\left[ \lambda(1 - \|\hat{\Gamma}_J\|_1) - (\lambda_{J,e} + \lambda_{J,\text{rem}}\|\hat{C}_J\|_1) - \lambda_1 \right] \|\hat{\beta}_J - \beta_J\|_1 \\
\leq \frac{4\lambda^2|J \cap S|}{\phi^2(L, J \cap S)} + \|X\hat{C}_J(\beta_J - \beta_J^0)\|_n^2.
\]

\[
\square
\]

We now show that under certain conditions the estimator puts values in $S^c$ to zero.

**Proof of Theorem 2.2.** Let $\hat{\beta}_S$ be the solution of the KKT conditions under the restriction that the coefficients are zero outside the set $S$:

\[
-X_S^T(Y - X_S\hat{\beta}_S)/n + \lambda\hat{z}_S = 0,
\]

where for $j \in S$, $\hat{z}_{j,S} = \text{sign}(\hat{\beta}_{j,S})$ if $\hat{\beta}_{j,S} \neq 0$ and where $\|\hat{z}_S\|_{\infty} \leq 1$. Define

\[
\lambda\hat{z}_{S^c} := X_{S^c}^T(Y - X_S\hat{\beta}_S)/n.
\]

Then

\[
\lambda\hat{z}_{S^c} = \lambda\hat{\Gamma}_{S^c}^T\hat{z}_S + \hat{C}_{S^c}^T X^T X\hat{C}_{S^c}\beta^0_{S^c}/n + \hat{C}_{S^c}^T X^T \eta/n.
\]
It follows that on \( T \)
\[ \| \tilde{z}_S \|_\infty \leq 1. \]

Hence \( \tilde{\beta}_S \) is a solution of the unrestricted KKT conditions (2.3) and hence \( \tilde{\beta}_S = \hat{\beta} \).

The final proof of this section concerns the de-sparsification of a group of variables.

**Proof of Lemma 2.5.** We have
\[
\hat{C}_J^T X^T J X J (\hat{b}_J - \beta_0^J) / n = \hat{C}_J^T X^T J (\hat{\beta}_J - \beta_0^J) / n \\
+ \hat{C}_J X^T \epsilon / n - \hat{C}_J^T X (\hat{\beta}_J - \beta_0^J) / n - \hat{C}_J^T X (\hat{\beta}_{-J} - \beta_0^J) / n \\
= \hat{C}_J X^T \epsilon / n - \hat{C}_J^T X (\hat{\beta}_J - \beta_0^J) / n - \hat{C}_J^T X (\hat{\beta}_{-J} - \beta_0^J) / n.
\]

The result now follows using the same arguments as for deriving (2.5) and (2.6) of Theorem 2.1.

\( \square \)

**9.2. Proof for Section 3: Random design.**

**Proof of Theorem 3.1.** This follows using the same arguments as in the proof of Theorem 2.1.

\( \square \)

**9.3. Proof for Section 4: General loss functions**

**Proof of Theorem 4.1.** By the KKT conditions (4.1)
\[
-w + \mathcal{I}(\hat{\theta} - \theta^0) - \text{rem}(\hat{\theta} - \theta^0) + \lambda \hat{\epsilon} = 0.
\]

Multiplying by \((\hat{\theta}_J - \theta_J)^T C_J^T\) and rewriting gives
\[
-(\hat{\theta}_J - \theta_J)^T C_J^T w + (\hat{\theta}_J - \theta_J)(\mathcal{I}_{J,J} - \Gamma_{J,J}^T \mathcal{J}_{J,J} \Gamma_{J,J})(\hat{\theta}_J - \theta_J^0) \\
+ \lambda(\hat{\theta}_J - \theta_J)^T \hat{\epsilon}_J + \lambda(\hat{\theta}_J - \theta_J)^T \Gamma_{J,J}^T \hat{\epsilon}_{-J} \\
-(\hat{\theta}_J - \theta_J)^T \text{rem}_{-J}(\hat{\theta} - \theta^0) - (\hat{\theta}_J - \theta_J^0)^T \Gamma_{J,J}^T \text{rem}_{-J}(\hat{\theta} - \theta^0) = 0.
\]

The results now follow in the same manner as for Theorem 2.1.

\( \square \)

**Proof of Theorem 4.2.** This follows from the same arguments as those used for proving Theorem 2.2.

\( \square \)

**9.4. Proof for Section 6: Brouwer’s fixed point theorem**

**Proof of Lemma 6.1.** Let \( F(\delta) := G(\text{rem}(\delta)) + v(\delta) + u_0 \). Then for \( \| \delta \|_q \leq 3K\epsilon \)
\[
\| F(\delta) \|_q \leq \| G(\text{rem}(\delta)) \|_q + \| v(\delta) \|_q + \| u_0 \|_q \\
\leq K \| \text{rem}(\delta) \|_q + 2K\epsilon \\
\leq 3K\epsilon.
\]

By Brouwer’s fixed point theorem there exists a \( \hat{\delta} \) with \( \| \hat{\delta} \|_q \leq 3K\epsilon \) such that
\[ F(\hat{\delta}) = \hat{\delta}. \]

But since \( \hat{\theta} \) is unique we must have \( \hat{\theta} - \theta^0 = \hat{\delta} \).

\( \square \)
9.5. Proof for Section 7: The irrepresentable condition and rates in sup-norm

Proof of Theorem 7.1. Throughout the proof we assume we are on $T_{\nu} \cap T_{\text{rem}}$. Recall that for a vector $\theta \in \mathbb{R}^p$, the vector $\tilde{\theta}_S$ is either the $|S|$-dimensional vector $\{\theta_j\}_{j \in S}$ or the $p$-dimensional vector $\{\theta_j|j \in S\}$, whichever is appropriate. Recall also that $\tilde{\theta}_S$ is a solution of

$$P_n(\tilde{\theta}_S) + \lambda \tilde{z}_S = 0,$$

where $\tilde{z}_{j,S} = \text{sign}(\tilde{\theta}_{j,S})$ if $\tilde{\theta}_{j,S} \neq 0$, $j \in S \setminus J$, $\tilde{z}_{j,S} = 0$ if $j \in J$ and $\|\tilde{z}_S\|_\infty \leq 1$.

We have

$$-w_S + \mathcal{I}_{S,S}(\tilde{\theta}_S - \theta_0^S) + \mathcal{I}_{S,S'} \theta_{S'}^0 - \text{rem}_S(\tilde{\theta}_S - \theta_0^S) + \lambda \tilde{z}_S = 0.$$

In other words

$$\tilde{\theta}_S - \theta_0^S = \mathcal{I}_{S,S}^{-1}(\text{rem}(\tilde{\theta}_S - \theta_0^S)) + \nu(\tilde{\theta}_S - \theta_0^S) + u_0,$$

where

$$\nu(\tilde{\theta}_S - \theta_0^S) = \mathcal{I}_{S,S}^{-1}(w_S - \lambda z_S),$$

with $z_{j,S} = \text{sign}(\tilde{\theta}_{j,S})$ if $\tilde{\theta}_{j,S} \neq 0$, $j \in S \setminus J$, $z_{j,S} = 0$, $j \in J$, $\|z_S\|_\infty \leq 1$, and

$$u_0 = -\Gamma_{S'} \theta_{S'}^0.$$

Hence

$$\sup_{\theta_S} \|\nu(\tilde{\theta}_S - \theta_0^S)\|_\infty \leq K_S(\lambda_S + \lambda).$$

and

$$\|u_0\|_\infty \leq \|\Gamma_{S'}\|_1 \lambda_{S'} \leq K_S \lambda_{S'} \leq K_S \lambda \leq K_S(\lambda + \lambda_w).$$

Let $\delta_S$ be a vector satisfying $\|\delta_S\|_\infty \leq 3K_S(\lambda_w + \lambda)$. Since $\lambda_{S'} \leq K_S(\lambda_w + \lambda)$, we also have $\|\delta_S - \theta_0^S\|_\infty \leq 3K_S(\lambda_w + \lambda)$ and so

$$\|\text{rem}(\delta_S + \theta_0^S)\|_\infty \leq \lambda_{\text{rem}} \leq \lambda_w + \lambda.$$

By Lemma 6.1 (with $G = \mathcal{I}_{S,S}^{-1}$ and with $\text{rem}(\cdot) := \text{rem}(\cdot)$ in Lemma 6.1 now taken as $\text{rem}(\delta_S) = \text{rem}(\delta_S + \theta_0^S)$) we may now conclude that

$$\|\tilde{\theta}_S - \theta_0^S\|_\infty \leq 3K_S(\lambda_w + \lambda).$$

We now have

$$\tilde{\theta}_S - \theta_0^S = \mathcal{I}_{S,S}^{-1}\left(w_S - \lambda \tilde{z}_S - (\text{rem}(\tilde{\theta}_S - \theta_0^S))_S\right) - \Gamma_{S'} \theta_{S'}^0,$$

where

$$\|\text{rem}(\tilde{\theta}_S - \theta_0^S)\|_\infty \leq \lambda_{\text{rem}}.$$

Thus

$$P_n(\tilde{\theta}_S)_{S'} = -w_{S'} + \mathcal{I}_{S',S}(\tilde{\theta}_S - \theta_0^S) + \mathcal{I}_{S,S'} \theta_{S'}^0 - \text{rem}_{S'}(\tilde{\theta}_S - \theta_0^S) = -C_{S'}^T w - \Gamma_{S'} \left(\lambda \tilde{z}_S + (\text{rem}(\tilde{\theta}_S - \theta_0^S))_{S'}\right) - \mathcal{I}_{S',S} \Gamma_{S'} \theta_{S'} + \mathcal{I}_{S,S'} \theta_{S'}^0 - (\text{rem}(\tilde{\theta}_S - \theta_0^S))_{S'}.$$
It follows that
\[ \| P_n(\hat{\theta}^0_s) \|_{\infty} \leq \lambda w(1 + \| \Gamma \|_1) + \lambda_{\text{rem}} \| C \|_1 + \| \Gamma \|_{1,\text{off}} \lambda + \kappa \lambda_{\text{thres}} \leq \lambda. \]
Therefore, if we define
\[ \lambda \hat{z} := -P_n(\hat{\theta}^0_s) \]
we get that \( \hat{\theta}_S \) is the solution of
\[ P_n\hat{\theta}_S + \lambda \hat{z} = 0, \]
where \( \hat{z}_j = \text{sign}(\hat{\theta}_{j,S}) \) if \( \hat{\theta}_{j,S} \neq 0, j \notin J \), \( \hat{z}_j = 0, j \in J \), and \( \| \hat{z} \|_{\infty} \leq 1 \), i.e. \( \hat{\theta}_S \) is a solution of (4.1). Since the solution is unique we must have \( \hat{\theta} = \theta^0_S \).

\[ \square \]

9.6. Proof for Section 8: Estimating a precision matrix

**Proof of Lemma 8.1.** Let \( \Delta := \Theta - \Theta_0 \). It holds that
\[ \text{rem}(\Delta) = (\Theta^{-1}_0 \Delta)^2 (I + \Theta^{-1}_0 \Delta)^{-1} \Theta^{-1}_0. \]
But
\[ \|(I + \Theta^{-1}_0 \Delta)^{-1}\|_1 \leq \sum_{m=0}^{\infty} \|(\Theta^{-1}_0 \Delta)^m\|_1 \]
and
\[ \|(\Theta^{-1}_0 \Delta)^m\|_1 \leq \|\Theta^{-1}_0 \Delta\|_1 \leq (\|\Theta^{-1}_0 \|_1 \|\Delta\|_1)^m. \]
We have
\[ \|\Theta - \Theta_0\|_1 \leq \varepsilon_0 \|\Theta^0\|_0 + \|\Theta^0_{S^c}\|_1. \]
It follows that
\[ \|\Theta^{-1}_0 \|_1 \|\Theta - \Theta_0\|_1 \leq \|\Theta^{-1}_0 \|_1 \left(\varepsilon_0 \|\Theta^0\|_0 + \|\Theta^0_{S^c}\|_1\right) \leq \eta_0 \]
and so
\[ \|(I + \Theta^{-1}_0 \Delta)\|_1 \leq 1/(1 - \eta_0). \]
We moreover have
\[ \|e_j^T(\Theta^{-1}_0 (\Theta - \Theta_0))^2\|_1 \leq \|e_j^T \Theta^{-1}_0 (\Theta - \Theta_0)\Theta^{-1}_0 \varepsilon_0 \leq \|\Theta^{-1}_0 (\Theta - \Theta_0)\Theta^{-1}_0 \varepsilon_0 \]
\[ \leq (\varepsilon_0 \|\Theta^0\|_0 + \|\Theta^0_{S^c}\|_1) \|\Theta^{-1}_0\|_1 \|\varepsilon_0 \leq \eta_0 \varepsilon_0 \|\Theta^{-1}_0\|_1. \]
Also
\[ \|(I + \Theta^{-1}_0 (\Theta - \Theta_0))^{-1} \Theta^{-1}_0 e_k\|_\infty \leq \|(I + \Theta^{-1}_0 (\Theta - \Theta_0))^{-1}\|_1 \|\Theta^{-1}_0\|_1 \]
\[ \leq \|\Theta^{-1}_0\|_1/(1 - \eta_0) \]
So we find
\[ \|\text{rem}(\Delta)\|_\infty \leq \eta_0 \varepsilon_0 \|\Theta^{-1}_0\|_1^2/(1 - \eta_0). \]
\[ \square \]

23
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