Stochastic logarithmic Schrödinger equations: energy regularized approach

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Abstract

In this paper, we prove the global existence and uniqueness of the solution of the stochastic logarithmic Schrödinger (SlogS) equation driven by additive noise or multiplicative noise. The key ingredient lies on the regularized stochastic logarithmic Schrödinger (RSlogS) equation with regularized energy and the strong convergence analysis of the solutions of (RSlogS) equations. In addition, temporal Hölder regularity estimates and uniform estimates in energy space $H^1(\mathcal{O})$ and weighted Sobolev space $L^2_\alpha(\mathcal{O})$ of the solutions for both SlogS equation and RSlogS equation are also obtained.

\textbf{Keywords:} stochastic Schrödinger equation, logarithmic nonlinearity, energy regularized approximation, strong convergence

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1. Introduction

The deterministic logarithmic Schrödinger equation has wide applications in quantum mechanics, quantum optics, nuclear physics, transport and diffusion phenomena, open quantum system, Bose–Einstein condensations and so on (see e.g. \cite{1, 6, 14, 16, 19, 20}). It takes the form of

$$\partial_t u(t, x) = i\Delta u(t, x) + i\lambda u(t, x) \log(|u(t, x)|^2) + iV(t, x, |u|^2)u(t, x), \quad x \in \mathbb{R}^d, \quad t > 0,$$

$$u(0, x) = u_0(x), \quad x \in \mathbb{R}^d,$$

where $\Delta$ is the Laplacian operator on $\mathcal{O} \subset \mathbb{R}^d$ with $\mathcal{O}$ being either $\mathbb{R}^d$ or a bounded domain with homogeneous Dirichlet or periodic boundary condition, $t$ is time, $x$ is spatial coordinate, $\lambda \in \mathbb{R}/\{0\}$ characterizes the force of nonlinear interaction, and $V$ is a real-valued function. While retaining many of the known features of the linear Schrödinger equation, Bialynicki-Birula and Mycielski show that only such a logarithmic nonlinearity satisfies the condition of separability of noninteracting systems (see \cite{8}). The logarithmic nonlinearity makes the logarithmic Schrödinger equation unique among nonlinear wave equations. For instance, the longtime dynamics of the logarithmic Schrödinger equation is essentially different from the Schrödinger equation. There is

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a faster dispersive phenomenon when \( \lambda < 0 \) and the convergence of the modulus of the solution to a universal Gaussian profile (see [2]), and no dispersive phenomenon when \( \lambda > 0 \) (see [8]).

In this paper, we are mainly focus on the well-posedness of the following stochastic logarithmic Schrödinger (SlogS) equation,

\[
du(t) = i\Delta u(t)dt + i\lambda u(t) \log(|u(t)|^2)dt + \tilde{g}(u) \ast dW(t), \ t > 0
\]

\[u(0) = u_0,
\]

where \( W(t) = \sum_{k \in \mathbb{N}^+} Q^k \epsilon_k \beta_k(t) \), \( \{\epsilon_k\}_{k \in \mathbb{N}^+} \) is an orthonormal basis of \( L^2(\mathcal{O}; \mathbb{C}) \) with \( \{\beta_k\}_{k \in \mathbb{N}^+} \) being a sequence of independent Brownian motions on a probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \).

Here \( \tilde{g} \) is a continuous function and \( \tilde{g}(u) \ast dW(t) \) is defined by

\[
\tilde{g}(u) \ast dW(t) = -\frac{1}{2} \sum_{k \in \mathbb{N}^+} |Q^k \epsilon_k|^2 \left( |g(|u|^2)|^2u \right) dt
\]

\[-i \sum_{k \in \mathbb{N}^+} g(|u|^2)g'(|u|^2)|u|^2uIm(Q^k \epsilon_k)Q^k \epsilon_k dt + ig(|u|^2)udW(t)\]

if \( \tilde{g}(x) = ig(|x|^2)x \), and by

\[
\tilde{g}(u) \ast dW(t) = dW(t)
\]

if \( \tilde{g} = 1 \). We would like to remark that when \( W \) is \( L^2(\mathcal{O}; \mathbb{R}) \)-valued and \( \tilde{g}(x) = ig(|x|^2)x \), \( \tilde{g}(u) \ast dW(t) \) is just the classical Stratonovich integral.

The SlogS equation (1) could be derived from the deterministic model by using Nelson’s mechanics [18]. Applying the Madlung transformation \( u(t, x) = \sqrt{\rho(t, x)}e^{iS(t, x)} \), [17] obtains a fluid expression of the solution as follows,

\[
\partial_t S(t, x) = -|\nabla S(t, x)|^2 - \frac{1}{4} \frac{\delta I}{\delta \rho}(\rho(t, x)) + \lambda \log(\rho) + V(t, x, \rho(t, x)),
\]

\[
\partial_t \rho(t, x) = -2\text{div}(\rho(t, x)\nabla S(t, x)), \ S(0, x_0) = S_0(x_0), \ \rho(0) = \rho_0,
\]

where \( I(\rho) = \int_{\mathcal{O}} |\nabla \log(\rho)|^2 \rho dx \) is the Fisher information. If \( V \) is random and fluctuates rapidly, the term \( iWu \) can be approximated by some multiplicative Gaussian noise \( \tilde{g}(u)W \), which plays an important role in the theory of measurements continuous in time in open quantum systems (see e.g. [3]). Then we could use the inverse of Madlung transformation and formally obtain the stochastic logarithmic Schrödinger equation

\[
\partial_t u(t, x) = i\Delta u(t, x) + i\lambda u(t, x) \log(|u(t, x)|^2) + \tilde{g}(u(t, x))W(t, x), \ x \in \mathcal{O}, \ t > 0
\]

\[u(0, x) = u_0(x), \ x \in \overline{\mathcal{O}}.
\]

The main assumption on \( W \) and \( \tilde{g} \) is stated as follows.

**Assumption 1.** The diffusion operator is the Nemystkii operator of \( \tilde{g} \). Wiener process \( W \) and \( \tilde{g} \) satisfies one of the following condition,

Case 1. \( \{W(t)\}_{t \geq 0} \) is \( L^2(\mathcal{O}; \mathbb{C}) \)-valued and \( \tilde{g} = 1 \);

Case 2. \( \{W(t)\}_{t \geq 0} \) is \( L^2(\mathcal{O}; \mathbb{C}) \)-valued, \( \tilde{g}(x) = ig(|x|^2)x \), \( g \in C_b^2(\mathbb{R}) \) and satisfies the growth condition

\[
\sup_{x \in [0, \infty)} |g(x)| + \sup_{x \in (0, \infty)} |g'(x)| + \sup_{x \in [0, \infty)} |g''(x)x^2| \leq C_g,
\]
Case 3. \( \{W(t)\}_{t \geq 0} \) is \( L^2(\mathbb{O}; \mathbb{R}) \)-valued, \( \tilde{g}(x) = ig(|x|^2)x, \ g \in \mathcal{C}_b^1(\mathbb{R}) \) and satisfies the growth condition

\[
\sup_{x \in [0, \infty)} |g(x)| + \sup_{x \in [0, \infty)} |g'(x)x| \leq C_g.
\]

Assumption 2. Assume that \( g \) satisfies

\[
(x + y)(g(|x|^2) - g(|y|^2)) \leq C_g |x - y|, \quad x, y \in [0, \infty).
\]

When \( W(t) \) is \( L^2(\mathbb{O}; \mathbb{C}) \)-valued, we in addition assume that \( g \) satisfies following one-side Lipschitz continuity

\[
|\tilde{g}(x) - \tilde{g}(y)|g(|x|^2)|x|^2x - g'(|y|^2)g(|y|^2)y| \leq C_g |x - y|^2, \quad x, y \in \mathbb{C}.
\]

A typical example is \( \tilde{g}(u) = u \), and then Eq. (11) becomes the SlogS equation driven by linear multiplicative noise in [4].

There are two main difficulties in proving the well-posedness of the SlogS equation. On one hand, the random perturbation in SlogS equation destroys a lot of physical conservation laws, like the mass and energy conservation laws in Case 1 and Case 2, and the energy conservation law in Case 3. Similar phenomena has been observed in stochastic nonlinear Schrödinger equation with polynomial nonlinearity (see [13]). On the other hand, the logarithmic nonlinearity in SlogS equation is not locally Lipschitz continuous. The contraction mapping arguments via Strichartz estimates (see e.g. [2, 13, 15]) for stochastic nonlinear Schrödinger equation with smooth nonlinearity are not applicable here. We only realize that in Case 2, when the driving noise is a linear multiplicative noise (\( \tilde{g}(u) = u \)), [3] uses a rescaling technique, together with maximal monotone operator theory to obtain a unique global mild solution in some Orlicz space. As far as we know, there has no results concerning the well-posedness of the SlogS equation driven by additive noise or general multiplicative noise.

To show the well-posedness of the considered model, we introduce an energy regularized problem inspired by [2] where the authors use the regularized problem to study error estimates of numerical methods for deterministic logarithmic Schrödinger equation. The main idea is firstly constructing a proper approximation of \( \log(|x|^2) \) denoted by \( f_\epsilon(|x|^2) \). Then it induces the regularized entropy \( F_\epsilon \) which is an approximation of the entropy \( F(\rho) = \int_\mathbb{O}(\rho \log(\rho) - \rho)dx \), where \( \rho = |u|^2 \). The RSlogS equation is defined by

\[
du = i\Delta u' dt + i u' f_\epsilon(|u'|^2) dt + \tilde{g}(u') * dW(t),
\]

whose regularized energy is \( \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}F_\epsilon(u) \). Denoting \( \mathbb{H} := L^2 = L^2(\mathbb{O}; \mathbb{C}) \) with the product \( (u, v) := \int_\mathbb{O} Re(\bar{u}v)dx \), for \( u, v \in \mathbb{H} \), we obtain the existence and uniqueness of the solution of regularized SlogS equation by proving \( \epsilon \)-independent estimate in \( \mathbb{H} := W^{1,2} \) and the weighted \( L^2 \)-space \( L^2_\alpha := \{ v \in L^2 \mid x \mapsto (1 + |x|^2)^\alpha v(x) \in L^2 \} \) with the norm \( \|v\|_{L^2_\alpha(\mathbb{R}^d)} := \|(1 + |x|^2)^\alpha v(x)\|_{L^2(\mathbb{R}^d)} \). Then we are able to prove that the limit of \( \{u'\}_{\epsilon > 0} \) is convergent to a unique stochastic process \( u \) which is shown to be the unique mild solution of (11). Meanwhile, the sharp convergence rate of \( \{u'\}_{\epsilon > 0} \) is given when \( \mathbb{O} = \mathbb{R}^d \), or \( \mathbb{O} \) is a bounded domain in \( \mathbb{R}^d \) equipped with homogenous Dirichlet or periodic boundary condition. Our main result is formulated as follows.

**Theorem 1.** Let \( T > 0 \), Assumptions [1] and [2] hold, \( u_0 \in \mathbb{H}^1 \cap L^2_\alpha, \ alpha \in (0, 1) \), be \( F_0 \) measurable and has any finite \( p \)-th moment. Assume that \( \sum_{i \in \mathbb{N}^+} \|Q^i_\epsilon e_i\|_{L^2_\alpha}^2 + \|Q^i_\epsilon e_i\|_{\mathbb{H}^1}^2 < \infty \) when \( \tilde{g} = 1 \) and that \( \sum_{i \in \mathbb{N}^+} \|Q^i_\epsilon e_i\|_{\mathbb{H}^1}^2 + \|Q^i_\epsilon e_i\|_{W^{1,\infty}}^2 < \infty \) when \( \tilde{g}(x) = ig(|x|^2)x \). Then there exists a unique
mild solution $u \in C([0,T];\mathbb{H})$ of Eq. (1). Moreover, for $p \geq 2$, there exists $C(Q,T,\lambda,p,u_0) > 0$ such that

$$\mathbb{E}\left[\sup_{t \in [0,T]} \|u(t)\|_{\mathbb{H}^1}^p\right] + \mathbb{E}\left[\sup_{t \in [0,T]} \|u(t)\|_{L^2}^p\right] \leq C(Q,T,\lambda,p,u_0).$$

When $W(t)$ is $L^2(\Omega;\mathbb{R})$-valued, the well-posedness of SlogS equation with a super-linearly growing diffusion coefficient is also proven (see Theorem 3).

The reminder of this article is organized as follows. In section 2, we introduce the RSlogS equation and show the local well-posedness of RSlogS equation driven by both additive and multiplicative noise. Section 3 is devoted to $\epsilon$-independent estimate of the mild solution in $\mathbb{H}^1$ and $L^2$ of the RSlogS equation. In section 4, we prove the main result by passing the limit of the sequence of the regularized mild solutions and providing the sharp strong convergence rate. Several technique details are postponed to the Appendix. Throughout this article, $C$ denotes various constants which may change from line to line.

2. Regularized SLogS equation

In this section, we show the well-posedness of the solution for Eq. (4) (see Appendix for the definition of the solution). We would like to remark that there are several choices of the regularization function $f_\epsilon(|x|^2)$. For instance, one may take $f_\epsilon(|x|^2) = \log(|x|^2 + \epsilon)$ (see Lemma 8 in Appendix for the necessary properties) or $f_\epsilon(|x|^2) = \log(|x|^2 + \epsilon)$ (see e.g., [2] and references therein for more choices of regularization functions). If the regularization function $f_\epsilon$ enjoys the same properties of $\log(|x|^2 + \epsilon)$, then one can follow our approach to obtain the well-posedness of Eq. (4). In the following, we first present the local well-posedness of Eq. (4), and then derive global existence and uniform estimate of its solution. For simplicity, we always assume that $0 < \epsilon \ll 1$.

2.1. Local well-posedness of Regularized SLogS equation

In this part, we give the detailed estimates to get the local well-posedness in $\mathbb{H}^2$ of Eq. (4) if $d \leq 3$ via the regularization function $f_\epsilon(|x|^2) = \log(|x|^2 + \epsilon)$. In the case of $d \geq 3$, one could use the regularization function like $f_\epsilon(|x|^2) = \log(\frac{|x|^2 + \epsilon}{|x|^2})$ to get the local well-posedness in $\mathbb{H}^1$. Assume that $u_0 \in \mathbb{H}^2$ when using the regularization $\log(|x|^2 + \epsilon)$ and that $u_0 \in \mathbb{H}^1$ when applying the regularization $\log(\frac{|x|^2 + \epsilon}{|x|^2})$.

Let $\tau \leq T$ be an $\mathcal{F}_T$-stopping time. And we call $v \in M^p_{\mathcal{F}_T}(\Omega; C([0,\tau];\mathbb{H}^2))$, if there exists $\{\tau_n\}_{n \in \mathbb{N}^+}$ with $\tau_n \nearrow \tau$ as $n \to \infty$ a.s., such that $v \in M^p_{\mathcal{F}_T}(\Omega; C([0,\tau_n];\mathbb{H}^2))$ for $n \in \mathbb{N}^+$. Next we show the existence and uniqueness of the local mild solution (see Definition 1 in Appendix).

For the sake of simplicity, let us ignore the dependence on $\epsilon$ and write $u_R := u_{R\epsilon}$, where $u_R$ is the solution of the truncated equation

$$du_R = i\Delta u_R dt + i\lambda \Theta_R(u_R, t) u_R f_\epsilon(|u_R|^2) dt - \frac{1}{2} \Theta_R(u_R, t) \sum_{k \in \mathbb{N}^+} |Q^2 e_k|^2 \left(g(|u_R|^2)^2 u_R\right) dt$$

(5)
\[-i\Theta_R(u_R, t) \sum_{k \in \mathbb{N}^+} g(|u_R|^2)g'(|u_R|^2)|u_R|^2 u_R Im(Q^k e_k) Q^k e_k dt \]
\[+ \Theta_R(u_R, t)ig(|u_R|^2)u_R dW(t). \]

Here, \( \Theta_R(u, t) := \theta_R(\|u\|_{C([0, t]; \mathbb{H}^2)}) \), \( R > 0 \), with a cut-off function \( \theta_R \), that is, a positive \( C^\infty \) function on \( \mathbb{R}^+ \) which has a compact support, and
\[
\theta_R(x) = \begin{cases} 
0, & \text{for } x \geq 2R, \\
1, & \text{for } x \in [0, 1]. 
\end{cases}
\]

**Lemma 1.** Let Assumption [\[\] hold, \( d \leq 3 \), and \( f_\lambda(|x|^2) = \log(|x|^2 + \epsilon) \). Assume in addition that \( g \in C^\infty([-\infty, \infty]) \) when \( W(t) \) is \( L^2(\Omega; C([-\infty, \infty]) \)-valued and that \( g \in C^\infty([-\infty, \infty]) \) when \( W(t) \) is \( L^2(\Omega; \mathbb{R}) \)-valued.

Suppose that the \( Q \)-Wiener process \( W(t) \) satisfies \( \sum_{k \in \mathbb{N}^+} \|Q^k c_k\|_{\mathbb{H}^2}^2 < \infty \), and \( u_0 \in \mathbb{H}^2 \) is \( \mathcal{F}_0 \)-measurable and has any finite \( p \)th moment. Then there exists a unique global solution to [\[\] with continuous \( \mathbb{H}^2 \)-valued path.

**Proof** Let \( S(t) = \exp(it\Delta) \) be the \( \mathcal{C}_0 \)-group generated by \( i\Delta \). For fixed \( R > 0 \), we use the following notations, for \( t \in [0, T] \),
\[
\Gamma^R_{det} u(t) := i \int_0^t S(t - s) \left( \Theta_R(u, s) \lambda f_\lambda(|u(s)|^2) u(s) \right) ds,
\]
\[
\Gamma^R_{mod} u(t) := -\frac{1}{2} \int_0^t S(t - s) \left( \Theta_R(u, s) \sum_{k \in \mathbb{N}^+} |Q^k c_k|^2 \left(|g(|u(s)|^2)|^2 u(s)\right) \right) ds,
\]
\[-i \int_0^t S(t - s) \left( \Theta_R(u, s) \sum_{k \in \mathbb{N}^+} g(|u(s)|^2) g'(|u(s)|^2) |u(s)|^2 u(s) Im(Q^k c_k) Q^k c_k \right) ds,
\]
\[
\Gamma^R_{Sto} u(t) := i \int_0^t S(t - s) \left( \Theta_R(u, s) g(|u(s)|^2) u(s) \right) dW(s).
\]

We look for a fixed point of the following operator given by
\[
\Gamma^R u(t) := S(t)u_0 + \Gamma^R_{det} u(t) + \Gamma^R_{mod} u(t) + \Gamma^R_{Sto} u(t), \quad u \in \mathcal{M}^R(\Omega; C([0, r]; \mathbb{H}^2)),
\]
where \( r \) will be chosen later. The unitary property of \( S(\cdot) \) yields that
\[
\|S(\cdot)u_0\|_{\mathcal{M}^R(\Omega; C([0, r]; \mathbb{H}^2))} \leq \|u_0\|_{\mathbb{H}^2}.
\]

Now, we define a stopping time \( \tau = \inf\{t \in [0, T]: \|u\|_{C([0, t]; \mathbb{H}^2)} \geq 2R\} \land r \). By using the properties of \( \log(\cdot) \) and the Sobolev embedding \( \mathbb{H}^2 \hookrightarrow L^\infty(\Omega) \), we have
\[
\|\Gamma^R_{det} u\|_{C([0, r]; \mathbb{H}^2)} \leq CR|\lambda| \max(|\log(\epsilon)|, |\log(\epsilon + 4R^2)|) \left( \sup_{t \in [0, r]} \|u(t)\|_{\mathbb{H}^2} \right)
\]
\[+ CR|\lambda|(1 + \epsilon^{-1}) \left( \sup_{t \in [0, r]} \|u(t)\|_{\mathbb{H}^2} + \sup_{t \in [0, r]} \|u(t)\|_{\mathbb{H}^2}^2 \right)
\]
\[\leq C(\epsilon, \lambda) r(1 + 2R + R^2),
\]
and
\[
\|\Gamma^R_{mod} u\|_{C([0, r]; \mathbb{H}^2)} \leq C(\lambda, g) r \sum_{k \in \mathbb{N}^+} \|Q^k c_k\|_{\mathbb{H}^2}^2 \left(R + R^3 + R^5\right).
\]
The Burkholder inequality yields that for $p \geq 2,$
\[
\|\Gamma_{\det}^R u\|_{M^p_{\text{F}}(\Omega; C([0, r]; \mathbb{H}^2))} + \|\Gamma_{\text{mod}}^R u\|_{M^p_{\text{F}}(\Omega; C([0, r]; \mathbb{H}^2))} \leq C(\lambda, g) r \sum_{k \in \mathbb{N}^+} \|Q^\frac{3}{2} e_k\|_{L^2}^2 \left( R + R^3 + R^5 \right).
\]

The Burkholder inequality yields that for $p \geq 2,$
\[
\|\Gamma_{\text{sto}}^R u\|_{M^p_{\text{F}}(\Omega; C([0, r]; \mathbb{H}^2))} \leq C(g) \left( \mathbb{E} \left[ \int_0^r \sum_{k \in \mathbb{N}^+} \|g(|u(s)|^2)u(s)Q^\frac{3}{2} e_k\|_{L^2}^2 ds \right] \right)^{\frac{1}{2}}
\leq C(g) r^{\frac{3}{2}} \left( \sum_{k \in \mathbb{N}^+} \|Q^\frac{3}{2} e_k\|_{L^2}^2 \right)^{\frac{1}{2}} \left( R + R^3 \right).
\]
Therefore, $\Gamma^R$ is well-defined on $M^p_{\text{F}}(\Omega; C([0, r]; \mathbb{H}^2)).$

Now we turn to show the contractivity of $\Gamma^R$. Let $u_1, u_2 \in M^p_{\text{F}}(\Omega; C([0, r]; \mathbb{H}^2)),$ and define the stopping times $\tau_j = \inf \{ t \in [0, T] : \|u_j\|_{C([0, r]; \mathbb{H}^2)} \geq 2R \} \wedge r,$ $j = 1, 2.$ For a fixed $\omega,$ let us assume that $\tau_1 \leq \tau_2$ without the loss of generality. Then direct calculation leads to
\[
\|\Gamma_{\det}^R u_1 - \Gamma_{\det}^R u_2\|_{C([0, r]; \mathbb{H}^2)} \leq C(\lambda, r) \left( \left\| \Theta_R(u_2, \cdot) - \Theta_R(u_1, \cdot) \right\|_{L^p(\Omega; C([0, r]; \mathbb{H}^2))} \right) + C(\lambda) r \left( \left\| u_1 - u_2 \right\|_{C([0, r]; \mathbb{H}^2)} \right) \notag
\]
\[
\leq C(\lambda, r) \left( \left\| \Theta_R(u_2, \cdot) - \Theta_R(u_1, \cdot) \right\|_{C([0, r]; \mathbb{H}^2)} \right) + C(\lambda) r \left( \left\| u_1 - u_2 \right\|_{C([0, r]; \mathbb{H}^2)} \right) \leq C(\lambda, \epsilon) r \left( \left\| u_1 - u_2 \right\|_{C([0, r]; \mathbb{H}^2)} \right) \notag
\]
and
\[
\|\Gamma_{\text{mod}}^R u_1 - \Gamma_{\text{mod}}^R u_2\|_{C([0, r]; \mathbb{H}^2)} \leq C(\lambda, \epsilon) \sum_{k \in \mathbb{N}^+} \|Q^\frac{3}{2} e_k\|_{L^2} \left\| u_1 - u_2 \right\|_{C([0, r]; \mathbb{H}^2)} (1 + R^3).
\]
By applying the Burkholder inequality, we obtain
\[
\|\Gamma_{\text{sto}}^R u_1 - \Gamma_{\text{sto}}^R u_2\|_{M^p_{\text{F}}(\Omega; C([0, r]; \mathbb{H}^2))} \leq C(r) \left( \left\| \Theta_R(u_1, \cdot) - \Theta_R(u_2, \cdot) \right\|_{L^p(\Omega; C([0, r]; \mathbb{H}^2))} \right) + C(r) \left( \left\| u_1 - u_2 \right\|_{C([0, r]; \mathbb{H}^2)} \right) \notag
\]
\[
\leq C(\lambda, \epsilon) \sum_{k \in \mathbb{N}^+} \|Q^\frac{3}{2} e_k\|_{L^2}^2 \left\| u_1 - u_2 \right\|_{M^p_{\text{F}}(\Omega; C([0, r]; \mathbb{H}^2))} (1 + R^3).
\]
where $L^2_2$ is the space of Hilbert–Schmidt operators form $U_0 = Q^\frac{3}{2}(L^2(\Omega))$ to $\mathbb{H}^2.$ Combining all the above estimates, we have
\[
\|\Gamma^R u_1 - \Gamma^R u_2\|_{M^p_{\text{F}}(\Omega; C([0, r]; \mathbb{H}^2))} \leq C(\lambda, \epsilon) \left( \sum_{k \in \mathbb{N}^+} \|Q^\frac{3}{2} e_k\|_{L^2}^2 \right) \left( 1 + \|u_1 - u_2\|_{M^p_{\text{F}}(\Omega; C([0, r]; \mathbb{H}^2))} \right) (1 + R^3),
\]
which implies that there exists a small $r > 0$ depending on $Q, R, \lambda, \epsilon$ such that $\Gamma^R$ is a strict contraction in $M_{(r)}^R(\Omega, C((0, r]; \mathbb{H}^2))$ and has a fixed point $u^{R, 1}$ satisfying $\Gamma^R(u^{R, 1}) = u^{R, 1}$.

Assume that we have found the fixed point on each interval $[(l - 1)r, kr]$, $l \leq k$ for some $k \geq 1$. Define

$$u^{R, k} = S(t)u_0 + \Gamma^R_{det}u^{R, k} + \Gamma^R_{mod}u^{R, k} + \Gamma^R_{Sto}u^{R, k},$$
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on $[0, kr]$. In order to extend $u^{R, k}$ to $[kr, (k + 1)r]$, we repeat the previous arguments to show that on the interval $[kr, (k + 1)r]$, there exists a fixed point of the map $\Gamma^R$ defined by

$$\Gamma^R u(t) := S(t)u^{R, k}(kr) + \Gamma^R_{det}u(t) + \Gamma^R_{mod}u(t) + \Gamma^R_{Sto}u(t), u \in M_{(r)}^R(\Omega, C([0, r]; \mathbb{H}^2)).$$

Here we use the following notations,

$$\Gamma^R_{det}u(t) := \int_{0}^{t} S(t - s)\left(\Theta_R(u, k, s)\lambda f_l((u(s)^2)u(s))\right)ds,$$

$$\Gamma^R_{mod}u(t) := -\frac{1}{2}\int_{0}^{t} S(t - s)\left(\Theta_R(u, k, s)\sum_{j \in \mathbb{N}^+} |Q^k_j|e_j^2 \left(\left|g((u(s)^2)|g'(|u(s)^2)|u(s)^2u(s))Im(Q^k_j)e_j^2\right|\right)\right)ds,$$

$$\Gamma^R_{Sto}u(t) := \int_{0}^{t} S(t - s)\left(\Theta_R(u, k, s)g((u(s)^2))u(s)\right)dW^W(s),$$

where $t \in [0, r], W^W(s) = W(s) + W^k(s) - W(kr), u \in M_{(r)}^R(\Omega, C([0, r]; \mathbb{H}^2))$ and

$$\Theta_R(u, k, s) = \theta_R(\|u^{R, k}\|_{C([0, kr], \mathbb{H}^2)} + \|u\|_{C([0, r], \mathbb{H}^2)}).$$

For any different $v_1, v_2 \in M_{(r)}^R(\Omega, C([0, r]; \mathbb{H}^2))$, we define the stopping times $\tau_i = \inf\{t \in [0, T - kr] : \|u^{R, k}\|_{C([0, kr], \mathbb{H}^2)} + \|v_i\|_{C([0, kr], \mathbb{H}^2)} \geq 2R\} \wedge r, i = 1, 2$ and assume that $\tau_1 \leq \tau_2$ for the convenience. Then the same procedures yield that this map is a strict contraction and has a fixed point $u^{R, k+1}$ for a small $r > 0$. Now, we define a process $u^R$ as $u^R(t) := u^{R, k}(t)$ for $t \in [0, kr]$ and $u^R(t) := u^{R, k+1}(t)$ for $t \in [kr, kr + 1]$. It can be checked that $u^R$ satisfies the uniqueness of the mild solution can be obtained by repeating the previous arguments.

**Proposition 1.** Let the condition of Lemma 1 hold. There exists a unique local mild solution to (1) with continuous $\mathbb{H}^2$-valued path. And the solution is defined on a random interval $[0, \tau^*_R(u_0, \epsilon)]$, where $\tau^*_R(u_0, \epsilon, \omega)$ is a stopping time such that $\tau^*_R(u_0, \epsilon, \omega) = +\infty$ or $\lim_{t \to +\infty} \|u^R(t)\|_{\mathbb{H}^2} = +\infty$.

**Proof** When $\tilde{g} = 1$, one can follow the same steps in the proof of [13, Theorem 3.1] to complete the proof. It suffices to consider the multiplicative noise case. Let $\{u^R\}_{R > 0} \in M_{(r)}(\Omega, C([0, T]; \mathbb{H}^2))$ be a sequence of solution constructed in Lemma 1. And define a stopping time sequence $\tau_R := \inf\{t \in [0, T] : \|u^R\|_{C([0, t], \mathbb{H}^2)} \geq R\} \wedge T$. Then $\tau_R > 0$ is well-defined since $\|u^R\|_{C([0, t], \mathbb{H}^2)}$ is an increasing, continuous and $\mathcal{F}_t$-adapted process. We claim that if $R_1 \leq R_2$, then $\tau_{R_2} \leq \tau_{R_2}$ and $u^{R_1} = u^{R_2}, a.s. \text{ on } [0, \tau_{R_2}]$. Let $\tau_{R_2, R_1} := \inf\{t \in [0, T] : \|u^{R_2}\|_{C([0, t], \mathbb{H}^2)} \geq R_1\} \wedge T$. Then it holds that $\tau_{R_2, R_1} \leq \tau_{R_2}$ and $\Theta_{R_1}(u^{R_2}, t) = \Theta_{R_2}(u^{R_2}, t)$ on $t \in [0, \tau_{R_2, R_1}]$. This implies that $\{\tau_{R_2, R_1}\}$ is a solution of (5) and that $u^{R_1} = u^{R_2}, a.s. \text{ on } [t \leq \tau_{R_2, R_1}]$. Thus we conclude that $\tau_{R_1} \leq \tau_{R_2, R_1}, \text{ a.s.}$ and that $u^{R_1} = u^{R_2}$ for $t \leq \tau_{R_1}$.
Now consider the triple \( \{ u, (\tau R)_{\mathbb{R}^N^+}, \tau \infty \} \) defined by \( u(t) := u^R(t) \) for \( t \in [0, \tau R] \) and \( \tau \infty = \sup_{t \in \mathbb{R}^N^+} \tau R \). From Lemma 1, we know that \( u \in M^R_{\mathbb{R}^N^+}(\Omega, C([0, \tau]; \mathbb{H}^2)) \) satisfies 4 for \( t \leq \tau R \). The uniqueness of the local solution also holds. If we assume that \((u, \tau, (v, \sigma)) \) are local mild solutions of (4), then \( u(t) = v(t) \), a.s. on \( \{ t < \sigma \land \tau \} \). Let \( R_1, R_2 \in \mathbb{N}^+ \). Set \( \tau_{R_1, R_2} := \inf \{ t \in [0, T] : \max \{ \| u \|_{C([0,t]; \mathbb{H}^2)}, \| v \|_{C([0,t]; \mathbb{H}^2)} \} \geq R_1 \} \land \sigma_{R_2} \land \sigma_{R_2} \). Then we have on \( \{ t \leq \tau_{R_1, R_2} \} \), \((u, \tau_{R_1, R_2}), (v, \tau_{R_1, R_2}) \) are local mild solutions of (4). The uniqueness in Lemma 1 leads to \( u = v \) on \( \{ t \leq \tau_{R_1, R_2} \} \). Letting \( R_1, R_2 \to \infty \), we complete the proof. 

If we assume that \( g \in \mathcal{C}_S^m(\mathbb{R}) \), \( u_0 \in \mathbb{H}^s \), \( s \geq 2 \), following the same procedures, we can also obtain the local existence of the solution \( u_0 \), in \( \mathcal{C}([0, \tau^*]; \mathbb{H}^s) \) when \( d \leq 3 \). When \( d \geq 3 \), one needs to use another regularization function \( \log(\frac{|\hat{x}|^2 + \epsilon^2}{m}) \) and additional assumption on \( g \). In this case, we can get the local well-posedness in \( M^R_{\mathbb{R}^N^+}(\Omega; C([0, \tau^*]; \mathbb{H}^1)) \) based on Lemma 8 in Appendix and previous arguments. Since its proof is similar to that in Proposition 4, we omit these details here and leave them to readers.

**Proposition 2.** Let Assumption 1 hold, and \( f_\epsilon(|x|^2) = \log(\frac{|\hat{x}|^2 + \epsilon^2}{m}) \). Suppose that \( d \in \mathbb{N}^+ \), \( u_0 \in \mathbb{H}^2 \) is \( \mathcal{F}_0 \)-measurable and has any finite \( p \)-th moment for \( p \geq 1 \). Assume in addition that Q-Wiener process \( W \) satisfies \( \sum_{k \in \mathbb{N}^+} \| \hat{Q}^2 \epsilon_k \|_L^2 + \| \hat{Q}^2 \epsilon_k \|_H^2 < \infty \) when \( \hat{g} = 1 \), and \( \sum_{k \in \mathbb{N}^+} \| \hat{Q}^2 \epsilon_k \|_{W^{1, \infty}} + \| \hat{Q}^2 \epsilon_k \|_{H^1} < \infty \) when \( \hat{g}(x) = \log(|x|^2)x \). Then there exists a unique local mild solution to (4) with continuous \( \mathbb{H}^s \)-valued path. And the solution is defined on a random interval \([0, \tau^*(u_0, \epsilon, \omega)) \), where \( \tau^*(u_0, \epsilon, \omega) \) is a stopping time such that \( \tau^*(u_0, \epsilon, \omega) = +\infty \) or \( \lim_{t \to \tau^*} \| u(t) \|_{\mathbb{H}^1} = +\infty \).

**2.2. Global existence and uniform estimate of regularized SlogS equation**

Due to the blow-up alternative results in section 2, it suffices to prove that \( \sup_{t \in [0, \tau^*]} \| u(t) \|_{\mathbb{H}^p} < \infty \), or \( \sup_{t \in [0, \tau^*]} \| u(t) \|_{\mathbb{H}^p} = \infty \), a.s. under corresponding assumptions. In the following, we present several a priori estimates in strong sense to achieve our goal. To simplify the presentation, we omit some procedures like mollifying the unbounded operator \( \Delta \) and taking the limit on the regularization parameter. More precisely, the mollifier \( \Theta_m \), \( m \in \mathbb{N}^+ \) may be defined by the Fourier transformation (see e.g. 13)

\[
\hat{\Theta}_m v(\xi) = \hat{\Theta}_m(\frac{|\xi|}{m}) \hat{v}(\xi), \quad \xi \in \mathbb{R}^d,
\]

where \( \hat{\Theta} \) is a positive \( \mathcal{C}^\infty \) function on \( \mathbb{R}^+ \), has a compact support satisfying \( \theta(x) = 0 \), for \( x \geq 2 \) and \( \theta(x) = 1 \), for \( 0 \leq x \leq 1 \). Another choice of mollifier is via Yosida approximation \( \Theta_m := m(m - \Delta)^{-1} \) for \( m \in \mathbb{N}^+ \) (see e.g. 15). This kind of procedure is introduced to make that the Itô formula can be applied rigorously to deducing several a priori estimates. If \( \mathcal{O} \) becomes a bounded domain equipped with periodic or homogeneous Dirichlet boundary condition, the mollifier can be chosen as the Galerkin projection, and the approximated equation becomes the Galerkin approximation (see e.g. 2, 11-14).

In this section, we assume that \( u_0 \) has finite \( p \)-moment for all \( p \geq 1 \) for simplicity. We will also use Assumption 1 to obtain the global existence of the mild solution of the regularized SlogS equation. When \( d \leq 3 \) and \( f_\epsilon(x) = \log(x + \epsilon) \), we assume that \( \sum_{k} \| \hat{Q}^2 \epsilon_k \|_{L^2}^2 < \infty \). When \( d \in \mathbb{N}^+ \) and \( f_\epsilon(x) = \log(\frac{|x|^2}{x_m}) \), we assume that \( \sum_{k} \| \hat{Q}^2 \epsilon_k \|_{W^{1, \infty}}^2 < \infty \) for the multiplicative noise.

**2.3. A priori estimates in \( L^2 \) and \( \mathbb{H}^1 \)**

**Lemma 2.** Let \( T > 0 \). Under the condition of Proposition 4 or Proposition 2, assume that \((u^*, \tau^*) \) be a local mild solution in \( \mathbb{H}^1 \). Then for any \( p \geq 2 \), there exists a positive constant
$$C(Q, T, \lambda, p, u_0) > 0$$ such that

$$\mathbb{E} \left[ \sup_{t \in [0, \tau^* \land T]} \|u^e(t)\|^p \right] \leq C(Q, T, \lambda, p, u_0).$$

**Proof** Take any stopping time $$\tau < \tau^* \land T$$, a.s. Using the Itô formula to $$M^k(u^e(t))$$, where $$M(u^e(t)) := \|u^e(t)\|^2$$ and $$k \in \mathbb{N}^+$$ or $$k \geq 2$$, we obtain that for $$t \in [0, \tau]$$ and the case $$\tilde{g} = 1$$,

$$M^k(u^e(t)) = M^k(u_0^e) + 2k(k-1) \int_0^t M^{k-2}(u^e(s)) \sum_{i \in \mathbb{N}^+} \langle u^e(s), Q^{\tilde{g}}_i e_i \rangle^2 ds$$

$$+ k \int_0^t M^{k-1}(u^e(s)) \sum_{i \in \mathbb{N}^+} \|Q^{\tilde{g}}_i e_i\|^2 ds + 2k \int_0^t M^{k-1}(u^e(s)) \langle u^e(s), dW(s) \rangle,$$

and for $$t \in [0, \tau]$$ and the case $$\tilde{g}(x) = ig(|x|^2)x$$,

$$M^k(u^e(t)) = M^k(u_0^e) + 2k(k-1) \int_0^t M^{k-2}(u^e(s)) \sum_{i \in \mathbb{N}^+} \langle u^e(s), ig(|u^e(s)|^2)u^e(s)Q^{\tilde{g}}_i e_i \rangle^2 ds$$

$$+ 2k \int_0^t M^{k-1}(u^e(s)) \langle u^e(s), ig(|u^e(s)|^2)u^e(s) dW(s) \rangle + 2k \int_0^t M^{k-1}(u^e(s))$$

$$\times \langle u^e(s), -i \sum_{k \in \mathbb{N}^+} \text{Im}(Q^{\tilde{g}}_k e_k)Q^{\tilde{g}}_k g(|u^e(s)|^2)g(|u^e(s)|^2)u^e(s)^2 u^e(s) \rangle ds.$$

In particular, if $$W(t, x)$$ is real-valued and $$\tilde{g}(x) = ig(|x|^2)x$$, we have $$M^k(u^e(t)) = M^k(u_0^e)$$, for $$t \in [0, T]$$ a.s. By Assumption 1 and conditions in Propositions 1 and 2 using the martingale inequality, the Hölder inequality and the Young inequality and Gronwall’s inequality, we achieve that for all $$k \geq 1$$,

$$\sup_{t \in [0, \tau]} \mathbb{E} \left[ M^k(u^e(t)) \right] \leq C(T, k, u_0, Q).$$

Next, taking the supreme over $$t$$ and repeating the above procedures, we have that

$$\mathbb{E} \left[ \sup_{t \in [0, \tau]} M^k(u^e(t)) \right] \leq C(k)$$

$$\leq C(k) E \left[ \int_0^T M^{k-2}(u^e(s)) \sum_{i \in \mathbb{N}^+} (1 + \|u^e(s)\|^2) \|Q^{\tilde{g}}_i e_i\|^2 ds \right]$$

$$+ C(k) E \left[ \left( \int_0^T M^{2k-2}(u^e(s))(1 + \|u^e(s)\|^4) \sum_{i \in \mathbb{N}^+} \|Q^{\tilde{g}}_i e_i\|^2 ds \right)^{\frac{1}{2}} \right],$$

where $$U = L^2$$ for additive noise case and $$U = L^\infty$$ for multiplicative noise case. Applying the estimate of $$\mathbb{E} \left[ M^k(u^e(t)) \right]$$, we complete the proof by taking $$p = 2k$$. \[ \square \]

**Lemma 3.** Let $$T > 0$$. Under the condition of Proposition 1 or Proposition 4, assume that $$(u^e, \tau^*_e)$$ is a local mild solution in $$\mathbb{H}^1$$. Then for any $$p \geq 2$$, there exists $$C(Q, T, \lambda, p, u_0) > 0$$ such that

$$\mathbb{E} \left[ \sup_{t \in [0, \tau^*_e \land T]} \|u^e(t)\|_{L^2}^p \right] \leq C(Q, T, \lambda, p, u_0).$$
Proof Take any stopping time $\tau < \tau^* \land T$, a.s. Applying the Itô formula to the kinetic energy $K(u^*(t)) := \frac{1}{2}\|\nabla u^*(t)\|^2$, for $k \in \mathbb{N}^+$ and using integration by parts, we obtain that for $t \in [0, \tau]$, 

\[
K^k(u^*(t)) = K^k(u_0^*) + k \int_0^t K^{k-1}(u^*(s))\langle \nabla u^*(s), \i\mathbb{R} f(u^*(s)) \rangle Re(\bar{u}^*(s) \nabla u^*(s)) u^*(s) \, ds \\
+ \frac{1}{2}(k-1) \int_0^t K^{k-2}(u^*(s)) \sum_{i \in \mathbb{N}^+} \langle \nabla u^*(s), \nabla Q^\perp \nabla \nabla \epsilon_i \rangle^2 \, ds \\
+ \frac{k}{2} \int_0^t K^{k-1}(u^*(s)) \sum_{i \in \mathbb{N}^+} \langle \nabla Q^\perp \nabla \nabla \epsilon_i, \nabla Q^\perp \nabla \nabla \epsilon_i \rangle \, ds \\
+ k \int_0^t K^{k-1}(u^*(s)) \langle \nabla u^*(s), \nabla dW(s) \rangle 
\]

for the additive noise case, and

\[
K^k(u^*(t)) = K^k(u_0^*) + k \int_0^t K^{k-1}(u^*(s))\langle \nabla u^*(s), \i\mathbb{R} f(u^*(s)) \rangle Re(\bar{u}^*(s) \nabla u^*(s)) u^*(s) \, ds \\
+ \frac{1}{2}(k-1) \int_0^t K^{k-2}(u^*(s)) \sum_{i \in \mathbb{N}^+} \langle \nabla u^*(s), \i\nabla g(|u^*(s)|^2) u^*(s) \nabla Q^\perp \nabla \nabla \epsilon_i \rangle^2 \, ds \\
+ \frac{k}{2} \int_0^t K^{k-1}(u^*(s)) \sum_{i \in \mathbb{N}^+} \langle \nabla g(|u^*(s)|^2) u^*(s) \nabla Q^\perp \nabla \nabla \epsilon_i, \nabla g(|u^*(s)|^2) u^*(s) \nabla Q^\perp \nabla \nabla \epsilon_i \rangle \, ds \\
+ k \int_0^t K^{k-1}(u^*(s)) \langle \nabla u^*(s), \i\nabla g(|u^*(s)|^2) u^*(s) dW(s) \rangle \\
+ k \int_0^t K^{k-1}(u^*(s)) \langle \Delta u^*(s), \sum_{i \in \mathbb{N}^+} |Q^\perp \nabla \nabla \epsilon_i|^2 g(|u^*(s)|^2)^2 u^*(s) \rangle \, ds \\
- k \int_0^t K^{k-1}(u^*(s)) \langle \Delta u^*(s), \i \sum_{i \in \mathbb{N}^+} g(|u^*(s)|^2) g'(|u^*(s)|^2) |u^*(s)|^2 u^*(s) dW(s) \rangle \\
+ k \int_0^t K^{k-1}(u^*(s)) \langle \Delta u^*(s), \i \sum_{i \in \mathbb{N}^+} g(|u^*(s)|^2) g'(|u^*(s)|^2) |u^*(s)|^2 u^*(s) Im(Q^\perp \nabla \nabla \epsilon_i) \rangle \, ds 
\]

for the multiplicative noise case. Applying integration by parts, in the multiplicative noise case, we further obtain

\[
K^k(u^*(t)) \leq K^k(u_0^*) + k \int_0^t K^{k-1}(u^*(s))\langle \nabla u^*, \i\mathbb{R} f(u^*(s)) \rangle Re(\bar{u}^* \nabla u^*) u^*(s) \, ds \\
+ C_k \int_0^t K^{k-2}(u^*(s)) \sum_{i \in \mathbb{N}^+} \left( \langle \nabla u^*, g(|u^*(s)|^2) u^* \nabla Q^\perp \nabla \nabla \epsilon_i \rangle^2 + \langle \nabla u^*, g'(|u^*(s)|^2) Re(\bar{u}^* \nabla u^*) u^* Q^\perp \nabla \nabla \epsilon_i \rangle^2 \right) \, ds \\
+ C_k \int_0^t K^{k-1}(u^*(s)) \sum_{i \in \mathbb{N}^+} \left( \|g(|u^*(s)|^2) u^* \nabla Q^\perp \nabla \nabla \epsilon_i \|^2 + \|g(|u^*(s)|^2) \nabla u^* Q^\perp \nabla \nabla \epsilon_i \|^2 + \|g'(|u^*(s)|^2) Re(\bar{u}^* \nabla u^*) u^* Q^\perp \nabla \nabla \epsilon_i \|^2 \right) \, ds \\
+ k \int_0^t K^{k-1}(u^*(s)) \langle \nabla u^*, \i\nabla g(|u^*(s)|^2) u^* dW(s) \rangle 
\]

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+ C_k \int_0^t K^{k-1}(u^*(s)) \sum_{i \in \mathbb{N}^+} \left( |\langle \nabla u^*, (\nabla Im(Q^\frac{1}{2} e_1)Q^\frac{1}{2} e_1 + \nabla Q^\frac{1}{2} e_1 I_m(Q^\frac{1}{2} e_1))g'(|u^*|^2)g(|u^*|^2)|u^*|^2 u^*| \right) \\
+ |\langle \nabla u^*, Im(Q^\frac{1}{2} e_1)Q^\frac{1}{2} e_1 g''(|u^*|^2)g(|u^*|^2)|u^*|^2 u^* Re(\bar{u}^* \nabla u^*)| \\
+ |\langle \nabla u^*, Im(Q^\frac{1}{2} e_1)Q^\frac{1}{2} e_1 |g'(|u^*|^2)|^2|u^*|^2 u^* Re(\bar{u}^* \nabla u^*)| \\
+ |\langle \nabla u^*, Im(Q^\frac{1}{2} e_1)Q^\frac{1}{2} e_1 g'(|u^*|^2)g(|u^*|^2)(2Re(\bar{u}^* \nabla u^*)u^* + |u^*|^2 \nabla u^*) \rangle | ds.

By using the property of $g$ in Assumption [I] and conditions on $Q$, and applying Hölder’s, Young’s, and Burkholder’s inequalities, we achieve that for some small $\epsilon_1 > 0$,

$$
\mathbb{E} \left[ \sup_{r \in [0, t]} K^k(u^*(r)) \right] \\
\leq \mathbb{E}[K^k(u_0^*)] + C_k |\lambda| \mathbb{E} \left[ \int_0^t \sup_{r \in [0, s]} K^{k-1}(u^*(r)) \| \nabla u^*(r) \|^2 \| ds \right] \\
+ C_k T \mathbb{E} \left[ \sup_{r \in [0, t]} K^{k-2}(u^*(r)) \sum_{i \in \mathbb{N}^+} \| \nabla u^*(r) \|^2 (1 + \| \nabla u^*(r) \|^2 + \| u^*(r) \|^2) \| Q^\frac{1}{2} e_1 \|^2 \|ds \right] \\
+ C_k \mathbb{E} \left[ \left( \int_0^t K^{2k-2}(u^*(s)) \sum_{i \in \mathbb{N}^+} \| \nabla u^*(r) \|^2 (1 + \| \nabla u^*(r) \|^2 + \| u^*(r) \|^2) \| \nabla Q^\frac{1}{2} e_1 \|^2 \|ds \right)^2 \right] \\
\leq \mathbb{E}[K^k(u_0^*)] + C(T, \lambda, k, \epsilon_1, Q) + \epsilon_1 \mathbb{E} \left[ \sup_{r \in [0, t]} K^k(u^*(r)) \right] \\
+ C_k \mathbb{E} \left[ \int_0^t \sup_{r \in [0, s]} K^k(u^*(r)) dr \right],
$$

where $\| Q^\frac{1}{2} e_1 \|^2 = \| Q^\frac{1}{2} e_1 \|_{L^2}^2$ for the additive noise case and $\| Q^\frac{1}{2} e_1 \|^2 = \| Q^\frac{1}{2} e_1 \|_{H^1}^2 + \| Q^\frac{1}{2} e_1 \|_{W^{1, \infty}}^2$ for the multiplicative noise case. Applying Gronwall’s inequality, we complete the proof by taking $p = 2k$.

From the above proofs of Lemmas 2 and 3, it is not hard to see that to obtain $\epsilon$-independent estimates, the boundedness restriction $\sup_{x \geq 0} |g(x)| < \infty$ may be not necessary in the case that $W(t, x)$ is real-valued. We present such result in the following which is the key of the global well-posedness of an SlogS equation with super-linear growth diffusion in next section.

**Lemma 4.** Let $T > 0$ and $(u^*, \tau^*_T)$ be a local mild solution in $H^s$, $s \geq 1$ for any $p \geq 1$. Assume that $u_0^* \in H^1 \cap L^2_{\alpha}$, for some $\alpha \in (0, 1]$, is $F_0$-measurable and has any finite $p$th moment, and $W(t, x)$ is real-valued with $\sum_{i \in \mathbb{N}^+} \| Q^\frac{1}{2} e_1 \|_{H^1}^2 + \| Q^\frac{1}{2} e_1 \|_{W^{1, \infty}}^2 < \infty$. Let $\bar{g}(x) = \frac{1}{2}g(|x|^2)x$, $g \in C_0^1(\mathbb{R}) \cap C(\mathbb{R})$ satisfy the growth condition and the embedding condition,

$$
\sup_{x \in [0, \infty)} |g'(x)x| \leq C_g, \\
\| v g(|v|^2) \|_{L^q} \leq C_d (1 + \| v \|_{H^1} + \| v \|_{L^2_{\alpha}}),
$$

for some $q \geq 2$, where $C_g > 0$ depends on $g$, and $C_d > 0$ depends on $\mathcal{O}$, $d$, $\| v \|$. Then it holds that $M(u^*_T) = M(u_0)$ for $t \in (0, \tau^*_T)$. Furthermore, there exists a positive constant $C(Q, T, \lambda, p, u_0)$ such that

$$
\mathbb{E} \left[ \sup_{t \in [0, \tau^*_T \wedge T]} \| u^*(t) \|_{H^1}^p \right] \leq C(Q, T, \lambda, p, u_0).
$$

**Proof** The proof is similar to those of Lemmas 2 and 3. We only need to modify the estimation involved with $g$. The mass conservation is not hard to be obtained since the calculations in
Lemmas 2 only use the assumptions that \( g \in C(\mathbb{R}) \) and \( W(t, x) \) is real-valued. Therefore, we focus on the estimate in \( \mathbb{H}^1 \). We only show estimation about \( \mathbb{E}\left[K^k(u'(t))\right] \) since the proof on \( \mathbb{E}\left[\sup_{t \in [0, T')} K^k(u'(t))\right] \) is similar. Then following the same steps in Lemma 3 we get that for \( \frac{1}{q} + \frac{1}{p} = \frac{1}{2} \),

\[
\mathbb{E}\left[K^k(u'(t))\right] \\
\leq \mathbb{E}\left[K^k(u_0')\right] + k \mathbb{E}\left[\int_0^t K^{k-1}(u'(s))\frac{1}{2}\|\nabla u'(s)\|^2 \frac{1}{2}Re(\bar{u}'\nabla u')u'(s)ds\right] \\
+ C_k \mathbb{E}\left[\int_0^t K^{k-1}(u'(s)) \sum_{i \in \mathbb{N}^+} \left( \left(\nabla u', g(|u'|^2)Re(u'(\nabla u')u')e_i\right)^2 + \left(\nabla u', g'(|u'|^2)Re(u'(\nabla u')u')Q^\frac{1}{2}e_i\right)^2 \right)ds\right] \\
+ C_k \mathbb{E}\left[\int_0^t K^{k-1}(u'(s)) \sum_{i \in \mathbb{N}^+} \left( \|g(|u'|^2)u'\|_{L^q}^2 \|Q^\frac{1}{2}e_i\|^2_{L^{q'}} + \|\nabla u'\|^2 \|Q^\frac{1}{2}e_i\|^2 \right)ds\right].
\]

Applying the embedding condition on \( g \), mass conservation law and the procedures in the proof of Proposition 3, we complete the proof by taking \( p = 2k \) and Gronwall’s inequality. □

The embedding condition is depending on the assumption on \( Q \) and \( d \). One example which satisfies the embedding condition on \( g \) and is not bounded is \( g(x) = \log(c + x) \), for \( x \geq 0 \) and \( c > 0 \). Let us verify this example on \( \mathcal{O} = \mathbb{R}^d \). If the domain \( \mathcal{O} \) is bounded, one can obtain the similar estimate. We apply the Gagliardo–Nirenberg interpolation inequality and get that for \( q > 2 \) and small enough \( \eta > 0 \),

\[
\|g(|v|^2)v\|_{L^q}^q \leq \int_{c+|v|^2 \geq 1} \left| \log(c + |v|^2) \right| |v|^q dx + \int_{c+|v|^2 \leq 1} \left| \log(c + |v|^2) \right| |v|^q dx \\
\leq C \left( \|v\|_{L^q}^q + \|v\|_{L^{2+\eta}}^{2+\eta} + \|v\|_{L^{2+\eta}}^{q-\eta} \right) \\
\leq C \left( \|v\|^{1-\alpha_0}_q \|\nabla v\|_{L^\alpha} + \|v\|^{1-\alpha_0}_q \|\nabla v\|_q^{1+\eta} + \|v\|^{1-\alpha_0}_q \|\nabla v\|_q^{1-\alpha_0} \right) \\
\leq C \left( \|\nabla v\|^{q+\alpha_0}_q + \|\nabla v\|^{q+\alpha_0}_q + \|\nabla v\|^{q+\alpha_0}_q \right),
\]

where \( \alpha_0 = \frac{d(1+\eta)-2}{2q(1+\eta)} \), \( \alpha_1 = \frac{d(1+\eta)-2}{2q(1+\eta)} \) and \( \alpha_2 = \frac{d(1+\eta)-2}{2q(1+\eta)} \) satisfies \( \alpha_i \in (0, 1), \ i = 0, 1, 2 \). When \( q = 2 \), similar calculations, together with the interpolation inequality in Lemma 5, yield that

\[
\|g(|v|^2)v\|_2^2 \leq C \left( \|v\|^2 + \|v\|_{L^{2+2\eta}}^{2+2\eta} + \|v\|_{L^{2-2\eta}}^{2-2\eta} \right) \\
\leq C \left( \|\nabla v\|^2 + \|\nabla v\|^{(2+2\eta)\alpha_1} + \|v\|_{L^{2+2\eta}}^{\frac{4q}{d}} \right),
\]

where \( \alpha_1 = \frac{d(2(1+\eta)-2)}{4(1+\eta)} \) and \( \alpha_2 \in \left( \frac{d-2}{2-2\eta}, 1 \right) \).

2.4. \( L^2_q \)-estimate and modified energy

Beyond the \( L^2 \) and \( \mathbb{H}^1 \) estimates, we also need the uniform boundedness in \( L^2_q \), \( \alpha > 0 \) to show the strong convergence of \( \{u^\prime\}_c>0 \) when \( \mathcal{O} = \mathbb{R}^d \). We would like to mention that when \( \mathcal{O} \)
is a bounded domain, such estimate in $L^2_\alpha$, $\alpha > 0$ is not necessary. To this end, the following useful weighted interpolation inequality is introduced.

**Lemma 5.** Let $d \in \mathbb{N}^+$ and $\eta \in (0, 1)$. Then for $\alpha > \frac{dn}{2 - 2\eta}$, it holds that for some $C = C(d) > 0$,

$$
\|v\|_{L^{2-\eta}} \leq C\|v\|^{1-\frac{dn}{d-2\eta}}\|v\|^{\frac{dn}{d-2\eta}}_\alpha, \quad v \in L^2 \cap L^\alpha_\alpha.
$$

**Proof** Using the Cauchy-Schwarz inequality and $\eta \in (0, 1)$, let

$$
\|v\|_{L^{2-\eta}} \leq \int_{|x| \leq r} |v(x)|^{2-2\eta}dx + \int_{|x| \geq r} \frac{|x|^{\alpha(2-2\eta)}|v(x)|^{2-2\eta}}{|x|^{\alpha(2-2\eta)}}dx
\leq Cr^{\eta}\|v\|^{2-2\eta} + C\|v\|^{2-2\eta}_{L^\alpha_\alpha} \left( \int_{|x| \geq r} \frac{1}{|x|^{\alpha(2-2\eta)}}dx \right)^{\eta}
\leq Cr^{\eta}\|v\|^{2-2\eta} + Cr^{-\alpha(2-2\eta)/2}\|v\|^{2-2\eta}_{L^\alpha_\alpha}.
$$

Let $r = \left(\frac{\|v\|_{L^2_\alpha}}{\|v\|_{L^{2-\eta}}}\right)^{\frac{1}{\eta}}$, we complete the proof. \qed

**Proposition 3.** Let $T > 0$, $\mathcal{O} = \mathbb{R}^d$, $d \in \mathbb{N}^+$, Under the condition of Proposition 7 or Proposition 8, let $(u^\tau, \tau^\tau)$ be a local mild solution in $\mathbb{H}^s$, $s \geq 1$ for any $p \geq 1$. Let $u_0 \in L^p_\alpha \cap \mathbb{H}^1$, for some $\alpha \in (0, 1]$, have any finite $p$th moment. Then the solution $u^\tau$ of regularized problem satisfies for $\alpha \in (0, 1]$,

$$
\mathbb{E}\left[ \sup_{t \in [0,\tau^\tau \wedge T]} \|u^\tau(t)\|_{L^\alpha_\alpha}^{2p} \right] \leq C(Q, T, \lambda, p, u_0).
$$

**Proof** We first introduce the stopping time

$$
\tau_R = \inf \left\{ t \in [0, T] : \sup_{s \in [0, t]} \|u^\tau(s)\|_{L^\alpha_\alpha} \geq R \right\} \wedge \tau^\tau,
$$

then show that $\mathbb{E}\left[ \sup_{t \in [0,\tau^\tau]} \|u^\tau(t)\|_{L^\alpha_\alpha}^{2p} \right] \leq C(T, u_0, Q, p)$ independent of $R$. After taking $R \to \infty$, we get $\tau_R = \tau^\tau$, a.s. For simplicity, we only prove uniform upper bound when $p = 1$.

Taking $0 < t \leq t_1 \leq \tau_R$, and applying the Itô formula to $\|u^\tau\|_{L^\alpha_\alpha}^2 = \int_{\mathcal{O}} (1 + |x|^2)^\alpha |u^\tau|^2 dx$, we get

$$
\|u^\tau(t)\|_{L^\alpha_\alpha}^2 = \|u_0^\tau\|_{L^\alpha_\alpha}^2 + \int_0^t 2((1 + |x|^2)^\alpha u^\tau(s), i\Delta u^\tau(s))ds + \int_0^t 2\sum_{i \in \mathbb{N}^+} ((1 + |x|^2)^\alpha Q^\tau e_i, Q^\tau e_i)ds
$$

$$
+ \int_0^t 2((1 + |x|^2)^\alpha u^\tau(s), if_\varepsilon(|u^\tau(s)|^2)u^\tau(s))ds + \int_0^t 2((1 + |x|^2)^\alpha u^\tau(s), dW(s))
$$

for additive noise case, and

$$
\|u^\tau(t)\|_{L^\alpha_\alpha}^2 = \|u_0^\tau\|_{L^\alpha_\alpha}^2 + 2\int_0^t ((1 + |x|^2)^\alpha u^\tau(s), i\Delta u^\tau(s) + if_\varepsilon(|u^\tau(s)|^2)u^\tau(s))ds
$$

$$
- \int_0^t ((1 + |x|^2)^\alpha u^\tau(s), (g(|u^\tau(s)|^2))^2 u^\tau(s)) \sum_i |Q^\tau e_i|^2 ds
$$

for multiplicative noise case.
Lemma 6. Lemma 6 and Proposition 4 in Appendix.

is the local mild solution in $H^1$ in time and loss of uniform estimate in $H^2$ deterministic case. The main reason is that the rough driving noise leads to low $H^\alpha$ regularity regularization $f_\alpha$ for $\epsilon$ solution of (4) enjoys Corollary 1.

for additive noise case, and estimate of $u$ for multiplicative noise case. Then Young’s and Gronwall’s inequalities, together with a priori estimate for multiplicative noise case. Using integration by parts, then taking supreme over $t \in [0, t_1]$ and applying the Burkholder inequality, we deduce

$$
E \left[ \sup_{t \in [0, t_1]} \|u'(t)\|_{L^2}^2 \right] \leq E \left[ \|u_0'\|_{L^2}^2 \right] + C_\alpha E \left[ \int_0^{t_1} \langle (1 + |x|^2)^{\alpha-1}x u'(s), \nabla u'(s) \rangle ds \right] + C \left( E \left[ \int_0^{t_1} \sum_{i \in \mathbb{N}^+} \langle (1 + |x|^2)^{\alpha} u'(s) \rangle \|x u'(s)\|^2 \| (1 + |x|^2)^{\alpha} \frac{Q^2}{Q} x_i \|^2 ds \right] \right)^{\frac{1}{2}}
$$

for additive noise case, and

$$
E \left[ \sup_{t \in [0, t_1]} \|u'(t)\|_{L^2}^2 \right] \leq E \left[ \|u_0'\|_{L^2}^2 \right] + C_\alpha E \left[ \int_0^{t_1} \langle (1 + |x|^2)^{\alpha-1}x u'(s), \nabla u'(s) \rangle ds \right] + C \left( E \left[ \int_0^{t_1} \sum_{i \in \mathbb{N}^+} \langle (1 + |x|^2)^{\alpha} u'(s) \rangle \|Q^2 u'(s)\|^2 \|Q^2 x_i \|^2 ds \right] \right)^{\frac{1}{2}}
$$

for multiplicative noise case. Then Young’s and Gronwall’s inequalities, together with a priori estimate of $u'$ in $\mathbb{H}^1$, lead to the desired result.

Corollary 1. Under the condition of Lemma 6, the solution $u'$ of regularized problem satisfies for $\alpha \in (0, 1]$,

$$
E \left[ \sup_{t \in [0, \tau^*_\epsilon \wedge T]} \|u'(t)\|^2_{L^2} \right] \leq C(Q, T, \lambda, p, u_0).
$$

It is not possible to obtain the uniform bound of the exact solution in $L^2$ for $\alpha \in (1, 2]$ like the deterministic case. The main reason is that the rough driving noise leads to low Hölder regularity in time and loss of uniform estimate in $\mathbb{H}^2$ for the mild solution. We can not expect that the mild solution of 1 enjoys $\epsilon$-independent estimate in $\mathbb{H}^2$. More precisely, we prove that applying the regularization $f_\epsilon(|x|^2) = \log(|x|^2 + \epsilon)$ in Proposition 1 one can only expect $\epsilon$-dependent estimate in $\mathbb{H}^2$. We omit the tedious calculation and procedures, and present a sketch of the proof for Lemma 6 and Proposition 7 in Appendix.

Lemma 6. Let $T > 0$ and $d = 1$. Under the condition of Proposition 7, assume that $(u', \tau^*_\epsilon)$ is the local mild solution in $\mathbb{H}^1$. In addition assume that $\sum_{i \in \mathbb{N}^+} \|Q^2 x_i \|_{L^2}^{1/2} < \infty$ for the multiplicative noise. Then for any $p \geq 2$, there exists a positive $C(Q, T, \lambda, p, u_0)$ such that

$$
E \left[ \sup_{t \in [0, \tau^*_\epsilon \wedge T]} \|u'(t)\|^2_{L^2} \right] \leq C(Q, T, \lambda, p, u_0)(1 + \epsilon^{-2p}).
$$
Proposition 4. Assume that $\mathcal{O} = \mathbb{R}^d$. Let $T > 0$ and $d = 1$. Under the condition of Proposition 4, assume that $(u^\epsilon, \tau^\epsilon_\ast)$ is the local mild solution in $H^1$. Let $u_0 \in L^2_\mathcal{O}$, for some $\alpha \in (1, 2]$. In addition assume that $\sum_i \|Q^\ast e_i\|^2_{L^2_\mathcal{O}} < \infty$ in the additive noise case and that $\sum_{i \in \mathbb{N}^+} \|Q^\ast e_i\|^2_{W^{2, \infty} \mathcal{O}} < \infty$ for the multiplicative noise. Then the solution $u^\epsilon(t)$ of regularized problem satisfies for $\alpha \in (1, 2]$,

$$
\mathbb{E} \left[ \sup_{t \in [0, \tau^\epsilon_\ast \wedge T]} \|u^\epsilon(t)\|^2_{L^2_\mathcal{O}} \right] \leq C(Q, T, \lambda, p, u_0)(1 + \epsilon^{-2p}).
$$

The above results indicate that both spatial and temporal regularity for SLogS equation are rougher than deterministic LogS equation.

In the following, we present the behavior of the regularized energy for the RSLogS equation. When applying $f_\epsilon(|x|^2) = \log(|x|^2 + \epsilon)$, the modified energy of Eq. (4) becomes

$$H_\epsilon(u^\epsilon(t)) := K(u^\epsilon) - \frac{\lambda}{2} F_\epsilon(|u^\epsilon|^2)$$

with $F_\epsilon(|u^\epsilon|^2) = \int_{\mathcal{O}} \left((\epsilon + |u^\epsilon|^2) \log(\epsilon + |u^\epsilon|^2) - |u^\epsilon|^2 - \epsilon \log \epsilon \right) dx$. When using another regularization $f_\epsilon(|u|^2) = \log(\frac{|u|^2}{1 + |u|^2\epsilon})$, its regularized entropy in the modified energy $H_\epsilon$ becomes

$$F_\epsilon(|u|^2) = \int_{\mathcal{O}} \left(|u|^2 \log(\frac{|u|^2 + \epsilon}{1 + |u|^2\epsilon}) + \epsilon \log(|u|^2 + \epsilon) - \frac{1}{\epsilon} \log(\epsilon|u|^2 + 1) - \epsilon \log(\epsilon) \right) dx.$$

In general, the modified energy is defined by the regularized entropy $\tilde{F}(\rho) = \int_{\mathcal{O}} \int_0^\rho f_\epsilon(s) ds dx$, where $f_\epsilon(\cdot)$ is a suitable approximation of $\log(\cdot)$. We remark that the regularized energy is well-defined when $\mathcal{O}$ is a bounded domain. The additional constant term $\epsilon \log(\epsilon)$ ensures that the regularized energy is still well-defined when $\mathcal{O} = \mathbb{R}^d$. We leave the proof of Proposition 5 in Appendix.

Proposition 5. Let $T > 0$. Under the condition of Proposition 4 or Proposition 4, assume that $(u^\epsilon, \tau^\epsilon_\ast)$ is the local mild solution in $H^1$. Then for any $p \geq 2$, there exists a positive constant $C(Q, T, \lambda, p, u_0)$ such that

$$\mathbb{E} \left[ \sup_{t \in [0, \tau^\epsilon_\ast \wedge T]} |H_\epsilon(u^\epsilon(t))|^p \right] \leq C(Q, T, \lambda, p, u_0).$$

Below, we present the global existence of the unique mild solution for Eq. (4) in $H^2$ based on Proposition 4, Lemma 6, Proposition 2 and Lemma 2 as well as a standard argument in Appendix.

Proposition 6. Let Assumption 2 hold and $(u^\epsilon, \tau^\epsilon_\ast)$ be the local mild solution in Proposition 4 or Proposition 4. Then the mild solution $u^\epsilon$ in $H^1$ is global, i.e., $\tau^\epsilon_\ast = +\infty$, a.s. In addition assume that $d = 1$ in Proposition 4 the mild solution $u^\epsilon$ in $H^2$ is global.

3. Well-posedness for SLogS equation

Based on the a priori estimates of the regularized problem, we are going to prove the strong convergence of any sequence of the solutions of the regularized problem. This immediately implies that the existence and uniqueness of the mild solution in $L^2$ for SLogS equation.
3.1. Well-posedness for SLogS equation via strong convergence approximation

In this part, we not only show the strong convergence of a sequence of solutions to the regularized SLogS equation but also give the explicit strong convergence rate. The strong convergence rate of the regularized SLogS equation will make a great contribution to the numerical analysis of numerical schemes for the SLogS equation. And this topic will be studied in a companion paper. For the strong convergence result, we only present the mean square convergence rate since the schemes for the SLogS equation. And this topic will be studied in a companion paper. For the

In the multiplicative noise case, Assumption 2 is needed to obtain the strong convergence rate of the solution of the SLogS equation via strong convergence approximation. We remark that the assumption can be weakened if one only wants to obtain the strong convergence instead of deriving a convergence rate. Some sufficient condition for \( \epsilon_n \) in Assumption 2 is

\[
\left| g'(|x|^2)g(|x|^2)|x|^2 - g'(|y|^2)g(|y|^2)|y|^2 \right| \leq C_g |x - y|^2, \quad x, y \in \mathbb{C}
\]

or

\[
\left| g'(|x|^2)g(|x|^2)|x|^2 - g'(|y|^2)g(|y|^2)|y|^2 \right| \leq C_g |x - y|^2, \quad x, y \in \mathbb{C}.
\]

Functions like \( \frac{1}{|x|^2 + \epsilon} \) with \( \epsilon > 0 \), etc., will satisfy Assumption 2.

The main idea of the proof lies on showing that for a decreasing sequence \( \{\epsilon_n\}_{n \in \mathbb{N}^+} \) satisfying \( \lim_{n \to \infty} \epsilon_n = 0 \), the sequence \( \{u_n\}_{n \in \mathbb{N}^+} \) must be a Cauchy sequence in \( C([0,T]; L^p(\Omega; \mathbb{H})) \), \( p \geq 2 \). As a consequence, we obtain that there exists a limit process \( u \) in \( C([0,T]; L^p(\Omega; \mathbb{H})) \) which is shown to be independent of the sequence \( \{u_n\}_{n \in \mathbb{N}^+} \) and is the unique mild solution of the mild form of \( \text{(4)} \).

[Proof of Theorem 1] Based on Proposition 1 we can construct a sequence of mild solutions \( \{u_n\}_{n \in \mathbb{N}^+} \) of Eq. \( \text{(4)} \) with \( f_{\epsilon_n}(|x|^2) = \log(\frac{\epsilon_n^2 + |x|^2}{2\epsilon_n}) \). Here the decreasing sequence \( \{\epsilon_n\}_{n \in \mathbb{N}^+} \) satisfies \( \lim_{n \to \infty} \epsilon_n = 0 \). We use the following steps to complete the proof. For simplicity, we only present the details for \( p = 2 \) since the procedures for \( p > 2 \) are similar.

Step 1: \( \{u_n\}_{n \in \mathbb{N}^+} \) is a Cauchy sequence in \( L^2(\Omega; C([0,T]; \mathbb{H})) \).

Fix different \( n, m \in \mathbb{N}^+ \) such that \( n < m \). Subtracting the equation of \( u_n \) from the equation of \( u_m \), we have that

\[
d(u_m - u_n) = i\Delta(u_m - u_n)dt + i\lambda(f_{\epsilon_n}(|u_m|^2)u_m - f_{\epsilon_n}(|u_n|^2)u_n)dt
\]

for additive noise case, and

\[
d(u_m - u_n) = i\Delta(u_m - u_n)dt + i\lambda(f_{\epsilon_n}(|u_m|^2)u_m - f_{\epsilon_n}(|u_n|^2)u_n)dt
\]

\[
- \frac{1}{2} \sum_{k \in \mathbb{N}^+} |Qx_k|^2 \left( g(|u_m|^2)u_m - g(|u_n|^2)u_n \right) dt
\]

\[
- i \sum_{k \in \mathbb{N}^+} \Im(Qx_k)Qx_k \left( g'(|u_m|^2)g(|u_m|^2)u_m - g'(|u_n|^2)g(|u_n|^2)u_n \right) dt
\]

\[
+ i \left( g(|u_m|^2)u_m - g(|u_n|^2)u_n \right) dW(t)
\]

for multiplicative noise case. Then using the Itô formula to \( \|u_m(t) - u_n(t)\|^2 \), the properties of \( f_{\epsilon_n} \) in Lemma 8, the mean value theorem and the Gagliardo–Nirenberg interpolation inequality, we obtain that for \( \eta(2 - d) \leq 2 \),

\[
\|u_m(t) - u_n(t)\|^2 \leq C_m|u_m(t) - u_n(t)|^2 + C_m|u_m(t) - u_n(t)|^2
\]

for additive noise case, and
\[
\begin{align*}
&= \int_0^t 2(u'^m - u^m, i\lambda f_{\epsilon_m}(|u'|^2)u^m - f_{\epsilon_m}(|u'|^2)u^m)ds \\
&\leq \int_0^t 4|\lambda||u'^m(s) - u^m(s)||^2 ds + 4|\lambda|\int_0^t |Im(u'^m(s) - u^m(s), (f_{\epsilon_m}(|u'|^2) - f_{\epsilon_m}(|u'|^2))u^m)|ds \\
&\leq \int_0^t 6|\lambda||u'^m(s) - u^m(s)||^2 ds + 4|\lambda|\int_0^t ||u'(s) - u^m(s)||_{L^2} \frac{(|\epsilon_m - \epsilon_m)|u'|^2}{\epsilon_m + |u'|^2} ds \\
&\quad + 2|\lambda|\int_0^t ||\log(1 + \frac{(|\epsilon_m - \epsilon_m)|u'|^2}{1 + \epsilon_m|u'|^2})||u'|^2 ds \\
&\leq \int_0^t 6|\lambda||u'^m(s) - u^m(s)||^2 ds + 4|\lambda|\epsilon_m^2 \int_0^t ||u'(s) - u^m(s)||_{L^1} ds + 2|\lambda|C\epsilon_m^2 \int_0^t ||u'^{2+2\eta'}|_{L^{2+2\eta'}} ds
\end{align*}
\]
for additive noise case, and
\[
\|u'^m(t) - u^m(t)\|^2
\]
\[
= \int_0^t 2(u'^m - u^m, i\lambda f_{\epsilon_m}(|u'|^2)u^m - f_{\epsilon_m}(|u'|^2)u^m)ds \\
- \int_0^t \langle u'^m - u^m, \sum_{k \in \mathbb{N}^+} |Q\hat{\varphi}_e|^2 (g(|u'|^2)^2 u'^m - g(|u'|^2)^2 u^m) \rangle ds \\
- 2\int_0^t \langle u'^m - u^m, i \sum_{k \in \mathbb{N}^+} Im(Q\hat{\varphi}_e)Q\hat{\varphi}_e (g'(|u'|^2)g(|u'|^2)|u'^m|^2 u^m - g'(|u'|^2)g(|u'|^2)|u^m|^2 u'^m) \rangle ds \\
+ 2\int_0^t \langle u'^m - u^m, i (g(|u'|^2)u'^m - g(|u'|^2)u^m) dW(s) \rangle \\
+ \int_0^t (g(|u'|^2)u'^m - g(|u'|^2)u^m, \sum_{k \in \mathbb{N}^+} |Q\hat{\varphi}_e|^2 (g(|u'|^2)u'^m - g(|u'|^2)u^m) ds \\
\leq \int_0^t 4|\lambda||u'^m(s) - u^m(s)||^2 ds + 4|\lambda|\epsilon_m^2 \int_0^t ||u'(s) - u^m(s)||_{L^1} ds + 2|\lambda|C\epsilon_m^2 \int_0^t ||u'^{2+2\eta'}|_{L^{2+2\eta'}} ds
\]
for multiplicative noise case. By using (2) and (3) in Assumption 2 and the assumptions on \(Q\), we have that
\[
\|u'^m(t) - u^m(t)\|^2
\]
\[
\leq \int_0^t 4|\lambda| + C(g, Q)||u'^m(s) - u^m(s)||^2 ds + 4|\lambda|\epsilon_m^2 \int_0^t ||u'(s) - u^m(s)||_{L^1} ds \\
+ 2|\lambda|C\epsilon_m^2 \int_0^t ||u'^{2+2\eta'}|_{L^{2+2\eta'}} ds + \int_0^t \langle u'^m - u^m, i (g(|u'|^2)u'^m - g(|u'|^2)u^m) dW(s) \rangle.
\]
Next we show the strong convergence of the sequence \(\{u'^n\}_{n \in \mathbb{N}^+}\) in the following different cases.
Case 1: $\mathcal{O}$ is a bounded domain. By using the Hölder inequality $\|u^m(t) - u^n(t)\|_{L^1} \leq |\mathcal{O}|^{\frac{1}{2}} \|u^m - u^n\|_{L^2}$ on $\mathcal{O}$ and $\mathcal{W}$, and using the Gronwall’s inequality, we get

\[
\sup_{t \in [0,T]} \|u^m(t) - u^n(t)\|^2 \leq C(\lambda, T, |\mathcal{O}|)(\epsilon_n + \epsilon^\gamma_n)(1 + \sup_{t \in [0,T]} \|u^n\|_{L^{2+2\eta}}^{2+2\eta})
\]

for additive noise case. In the multiplicative noise case, taking supreme over $u$ and then taking expectation on $\mathcal{W}$, together with the Burkholder and Young inequalities, we get that for a small $\kappa > 0$,

\[
\mathbb{E}\left[ \sup_{t \in [0,T]} \|u^m(t) - u^n(t)\|^2 \right] \leq C(\lambda, T, |\mathcal{O}|, Q)(\epsilon_n + \epsilon^\gamma_n) + C\mathbb{E}\left[ \sup_{t \in [0,T]} \left| \int_0^t (\epsilon_n + \epsilon^\gamma_n) \left( g(|u^m|^2)u^m - g(|u^n|^2)u^n \right) dW(s) \right| \right]
\]

\[
\leq C(\lambda, T, |\mathcal{O}|, Q)(\epsilon_n + \epsilon^\gamma_n) + C\mathbb{E}\left[ \int_0^T \sum_i \|\dot{Q}^\gamma_i e_i\|_{L^\infty}^2 \|u^m - u^n\|^2 ds \right]^\frac{1}{2}
\]

\[
\leq C(\lambda, T, |\mathcal{O}|, Q)(\epsilon_n + \epsilon^\gamma_n) + C\mathbb{E}\left[ \sup_{t \in [0,T]} \|u^m(t) - u^n(t)\|^2 \right]
\]

\[
+ C(\kappa)\mathbb{E}\left[ \int_0^T \sum_i \|\dot{Q}^\gamma_i e_i\|_{L^\infty}^2 \|u^m - u^n\|^2 ds \right].
\]

Taking $\kappa < \frac{1}{2}$, we have that

\[
\mathbb{E}\left[ \sup_{t \in [0,T]} \|u^m(t) - u^n(t)\|^2 \right] \leq C(Q, T, \lambda, p, u_0, |\mathcal{O}|)(\epsilon_n + \epsilon^\gamma_n).
\]

Case 2: $\mathcal{O} = \mathbb{R}^d$. Since $u_0 \in L^2_p$, $\alpha \in (0, 1]$, using the interpolation inequality in Lemma $\mathbb{S}$ implies that for any $\eta \in (0, 1)$ and $\alpha > \frac{\eta}{(1-\eta)}$ (i.e., $\eta \in (0, \frac{1}{2+2\gamma})$),

\[
\|u^m(t) - u^n(t)\|^2 \leq \int_0^t 4\|u^m(s) - u^n(s)\|^2 ds + 4|\lambda| \int_0^t \epsilon_n \|u^m(s) - u^n(s)\|_{L^{2+2\eta}}^{1-\eta} ds
\]

\[
+ 2|\lambda| C\epsilon_n \int_0^t \|u^n\|_{L^{2+2\eta}}^{2+2\eta} ds
\]

\[
\leq \int_0^t 4|\lambda| \|u^m(s) - u^n(s)\|^2 ds + 2|\lambda| \int_0^t \|u^m(s) - u^n(s)\|^2 + 2|\lambda| C\epsilon_n \int_0^t \|u^n\|_{L^{2+2\eta}}^{2+2\eta} ds
\]

\[
+ 2|\lambda| C\epsilon_n \int_0^t \|u^n\|_{L^{2+2\eta}}^{2+2\eta} ds
\]

\[
\leq \int_0^t 6|\lambda| \|u^m(s) - u^n(s)\|^2 ds + 2|\lambda| C\epsilon_n \int_0^t \|u^n\|_{L^{2+2\eta}}^{1-\eta} ds
\]

\[
+ 2|\lambda| C\epsilon_n \int_0^t \|u^n\|_{L^{2+2\eta}}^{2+2\eta} ds
\]

for additive noise case, and

\[
\|u^m(t) - u^n(t)\|^2
\]

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\[
\leq \int_0^t C\|u^{en}(s) - u^{en}(s)\|^2 ds + C\varepsilon_m^2 \int_0^t \|u^{en}\|_{L^2}^{2-2\eta - \frac{2\alpha}{d+2}} ds
\]
\[
+ 2|\lambda| C\varepsilon_n^2 \int_0^t \|u^{en}\|_{L^{2+2\eta}}^{2+2\eta} ds + \left| \int_0^t \langle u^{en} - u^{en}, i\left( g(\|u^{en}\|^2)u^{en} - g(\|u^{en}\|^2)u^{en}\right) dW(s) \right|
\]
for multiplicative noise case. Then taking supreme over $t$, taking expectation, using (2), Lemma 2 and Proposition 3 and applying Gronwall’s inequality, we have that for $\alpha \in (0, 1], \eta \in (0, \frac{2\alpha}{2\alpha+1})$ and $\eta'(d-2) \leq 2$,
\[
\mathbb{E}\left[ \sup_{t \in [0,T]} \|u^{en}(t) - u^{en}(t)\|^2 \right] \leq C(T, Q, u_0, g)\left( \sup_{t \in [0,T]} \|u^{en}\|_{L^2}^{2-2\eta - \frac{2\alpha}{d+2}} + \|u^{en}\|_{L^{2+2\eta}}^{2+2\eta} \right) (\varepsilon_n^2 + \varepsilon_n')
\]
\[
\leq C(T, Q, u_0, g, \alpha, \eta)\varepsilon_n' \min(\eta, \eta').
\]

Step 2: The limit process $u$ of $\{u^{en}\}_{n \in \mathbb{N}^+}$ in $M_T^1(C([0,T]; H))$ satisfies (1) in mild form. We use the multiplicative noise case to present all the detailed procedures. It suffices to prove that each term in the mild form of RSLogS equation (1) converges to the corresponding part in
\[
S(t)u_0 + i\lambda \int_0^t S(t-s)\log(\|u^2\|) ds - \frac{1}{2} \int_0^t S(t-s)(g(\|u^2\|^2))^2 u \sum_k |Q_{\frac{3}{2}}e_k|^2 ds
\]
\[
- i \int_0^t S(t-s)g(\|u^2\|^2)u |\sum_k Im(Q_{\frac{3}{2}}e_k)Q_{\frac{3}{2}}e_k ds + i \int_0^t S(t-s)g(\|u^2\|^2) dW(s)
\]
\[
:= S(t)u_0 + V_1 + V_2 + V_3 + V_4.
\]
We first claim that all the terms $V_1 - V_4$ make sense. By Lemma 3 and Proposition 3, we have that for $p \geq 2$,
\[
\sup_n \mathbb{E}\left[ \sup_{t \in [0,T]} \|u^{en}(t)\|^p_{H^1} \right] + \sup_n \mathbb{E}\left[ \sup_{t \in [0,T]} \|u^{en}(t)\|^p_{L^2} \right] \leq C(u_0, T, Q).
\]
By applying the Fourier transform and Parseval’s theorem, using the Fatou theorem and strong convergence of $(u^{en})_{n \in \mathbb{N}^+}$ in $M_T^1(C([0,T]; H))$, we obtain
\[
\mathbb{E}\left[ \sup_{t \in [0,T]} \|u(t)\|^p_{H^1} \right] + \mathbb{E}\left[ \sup_{t \in [0,T]} \|u(t)\|^p_{L^2} \right]
\]
\[
\leq \sup_n \mathbb{E}\left[ \sup_{t \in [0,T]} \|u^{en}(t)\|^2_{H^1} \right] + \sup_n \mathbb{E}\left[ \sup_{t \in [0,T]} \|u^{en}(t)\|^2_{L^2} \right] \leq C(u_0, T, Q).
\]
Then the Gagliardo–Nirenberg interpolation inequality yields that for small $\eta', \eta > 0$,
\[
\|\log(|u|^2)u\|^2 = \int_{|u|^2 \geq 1} (\log(|u|^2))^2|u|^2 dx + \int_{|u|^2 \leq 1} (\log(|u|^2))^2|u|^2 dx
\]
\[
\leq \int_{|u|^2 \geq 1} |u|^{2+2\eta'} dx + \int_{|u|^2 \leq 1} |u|^{2-2\eta} dx
\]
\[
\leq C(\|u\|_{L^{2+2\eta}}^{2+2\eta'} + \|u\|_{L^{2-2\eta}}^{2-2\eta})
\]
\[
\leq C(\|u\|_{L^{2-2\eta}}^{2-2\eta} + \|\nabla u\|^{d\eta'} \|u\|^{2d\eta + 2d\eta'}).
\]
When $\mathcal{O} = \mathbb{R}^d$, we use the weighted version of the interpolation inequality in Lemma 3 to deal with the term $\|u\|_{L^{2-2\eta}}^{2-2\eta}$, and have that for small $\eta < \frac{2\alpha}{2\alpha + d}$,

$$\| \log(\|u\|^2) u \|^2 \leq C(\|\nabla u\|^{d^*}\|u\|^{2n^*+2-d^*} + \| u \|_{L_{x,t}^{\infty}}^{\frac{2}{\alpha}} \| u \|^{2-2\eta - \frac{2\eta}{\alpha}}).$$

This implies that $V_1$ makes sense in $M^2_t(C([0, T]; \mathbb{H}))$ by Proposition 3, Lemmas 2 and 4. Meanwhile, we can show that $V_2 - V_4 \in M^2_t(C([0, T]; H))$ by using the Minkowski and Burkerholder inequalities due to our assumption on $g$ and $Q$.

Next, we show that the mild form of $u^n$ converges to $S(t)u_0 + V_1 + V_2 + V_3 + V_4$. To prove that

$$\lim_{n \to \infty} E \left[ \sup_{t \in [0, T]} \left\| \int_0^t S(t-s)f_n(\|u^n\|^2)u^n ds - V_1 \right\|^2 \right] = 0,$$

we use the following decomposition of $f_n(\|u^n\|^2)u^n - \log(\|u\|^2) u$. When $|u| > |u^n|$,

$$f_n(\|u^n\|^2)u^n - \log(\|u\|^2) u = (f_n(\|u^n\|^2) - f_n(\|u\|^2))u^n + f_n(\|u\|^2)(u^n - u) + (f_n(\|u\|^2) - \log(\|u\|^2)) u,$$

and when $|u| < |u^n|$,

$$f_n(\|u^n\|^2)u^n - \log(\|u\|^2) u = (\log(\|u^n\|^2) - \log(\|u\|^2))u + \log(\|u^n\|^2)(u^n - u) + (f_n(\|u^n\|^2) - \log(\|u^n\|^2))u^n.$$

For convenience, let us show the estimate for $|u| > |u^n|$, the other case will be estimated in a similar way. By using the Hölder inequality and the mean-valued theorem, we have that for small $\gamma > 0$,

$$\left| (f_n(\|u^n\|^2) - f_n(\|u\|^2)) u^n \right| \leq \left( \frac{|u|^2 - |u^n|^2}{\epsilon_n + |u^n|^2} \right)^{\frac{1}{2} + \frac{1}{2} \gamma} (f_n(\|u^n\|^2) - f_n(\|u\|^2))^{\frac{1}{2} + \frac{1}{2} \gamma} u^n \leq \left( |u| - |u^n| \right)^{\frac{1}{2} + \frac{1}{2} \gamma} (|u| + |u^n|)^{\frac{1}{2} + \frac{1}{2} \gamma} \frac{|u^n|}{(\epsilon_n + |u^n|^2)^{\frac{1}{2} + \frac{1}{2} \gamma}} (f_n(\|u^n\|^2) + f_n(\|u\|^2))^{\frac{1}{2} + \gamma}.$$

Then it implies that for small enough $\eta > 0$,

$$\int_{|u| > |u^n|} \left| (f_n(\|u^n\|^2) - f_n(\|u\|^2)) u^n \right|^2 dx \leq \int_{|u| > |u^n|} \frac{|u|^2}{(\epsilon_n + |u^n|^2)^{1-\gamma}} |u - u^n|^{1-\gamma} (|u| + |u^n|)^{1-\gamma} (f_n(\|u^n\|^2) + f_n(\|u\|^2))^{1+\gamma} dx$$

$$\leq \int_{|u| > |u^n|, \epsilon_n + |u^n|^2 \leq 1} \frac{|u|^2}{(\epsilon_n + |u^n|^2)^{1-\gamma}} |u - u^n|^{1-\gamma} (|u| + |u^n|)^{1-\gamma} (f_n(\|u^n\|^2) + f_n(\|u\|^2))^{1+\gamma} dx$$

$$+ \int_{|u| > |u^n|, \epsilon_n + |u^n|^2 \geq 1} \frac{|u|^2}{(\epsilon_n + |u^n|^2)^{1-\gamma}} |u - u^n|^{1-\gamma} (|u| + |u^n|)^{1-\gamma} (f_n(\|u^n\|^2) + f_n(\|u\|^2))^{1+\gamma} dx$$

$$\leq C \int_{|u| > |u^n|, \epsilon_n + |u^n|^2 \leq 1} \frac{|u|^2}{(\epsilon_n + |u^n|^2)^{1-\gamma}} |u - u^n|^{1-\gamma} (|u| + |u^n|)^{1-\gamma} (\epsilon_n + |u^n|^2) - \eta + (\epsilon_n + |u|^2) \eta) dx$$

$$+ C \int_{|u| > |u^n|, \epsilon_n + |u^n|^2 \geq 1} \frac{|u|^2}{(\epsilon_n + |u^n|^2)^{1-\gamma}} (|u| + |u^n|)^{1-\gamma} u + u^n |u|^{1-\gamma} (\epsilon_n + |u|^2)^\eta dx.$$
Now choosing $2 - 2\gamma + 2\eta \leq 2$, using the H"{o}lder inequality and the weighted interpolation inequality in Lemma \[5\] we have that for $\alpha > \frac{\eta d}{1 + \gamma - 2\eta}$,

\[
\int_{|u| > |u^\alpha|} \left| (f_{\alpha_n}(|u^\alpha|^2) - f_{\alpha_n}(|u|^2))u^\alpha \right|^2 \, dx \\
\leq C \int_{\mathbb{R}^d} |u^\alpha|^2 |u^\alpha - u|^2 |u - u^\alpha|^{1 - \gamma} \, dx \\
+ C \int_{\mathbb{R}^d} |u^\alpha|^2 |u - u^\alpha|^{1 - \gamma} (|u| + |u'|)^{1 - \gamma} \, dx \\
\leq C \|u - u^\alpha\|_{L^{1+\gamma}} \left( |u|^{2 - 2\eta} + |u|^2 \gamma \right) \|u + |u'|^{1 - \gamma} \right\|_{L^\frac{2}{1+\gamma}} \\
\leq C \|u - u^\alpha\|_{L^{1+\gamma}} \left( |u|^{1+\gamma} + \|u\|_{L^{\frac{4}{1+\gamma}}} \right) \\
\leq C \|u - u^\alpha\|_{L^{1+\gamma}} \left( |u|^{1+\gamma} + \|u\|_{L^{\frac{4}{1+\gamma}}} \right) \\
\leq C \|u - u^\alpha\|_{L^{1+\gamma}} \left( |u|^{1+\gamma} + \|u\|_{L^{\frac{4}{1+\gamma}}} \right).
\]

For the term $f_{\alpha_n}(|u|^2)(u^\alpha - u)$, we similarly have that for $\eta' < \frac{2}{\min(d-2,0)}$ and $\eta < \frac{2\alpha}{2\alpha - d}$,

\[
\int_{|u| > |u^\alpha|} \left| f_{\alpha_n}(|u|^2)(u^\alpha - u) \right|^2 \, dx \\
\leq \int_{|u| > |u^\alpha|, |u| + |u'| \leq 1} (\epsilon_n + |u|^2) \frac{\gamma}{2} u^\alpha - u|^2 \, dx \\
\leq C \|u^\alpha - u\| \left( |u^\alpha - u| + \|u\|_{L^{2+\gamma}} + \|u^\alpha\|_{L^{2-\gamma}} \right) \\
\leq C \|u^\alpha - u\| \left( |u^\alpha - u| + \|\nabla u\|^\alpha_{L^2} + \|u\|^{1+\gamma - \frac{d\alpha}{2}} + \|u\|^{\frac{2\alpha}{2\alpha - d}} \right).
\]

For the term $(f_{\alpha_n}(|u|^2) - \log(|u|^2))u$, the mean-valued theorem, the property that $\log(1 + |x|) \leq |x|$ and the Gagliardo--Nirenberg interpolation inequality

\[
\|u\|_{L^1(\mathbb{R}^d)} \leq C_1 \|u\|_{L^\alpha} \|u\|_{L^\alpha}^{\alpha}, \quad q = \frac{2d}{d - 2\alpha}, \quad \alpha \in (0,1]
\]

with $q = 2\eta' + 2$, yield that for $\eta'(d-2) \leq 2$ and $\eta \leq \frac{\alpha}{2\alpha - d}$,

\[
\int_{\Omega} \left| (f_{\alpha_n}(|u|^2) - \log(|u|^2))u \right|^2 \, dx \\
\leq C \|u\|_{L^1} \left( \|u\|_{L^\alpha} + \|u\|_{L^\alpha}^{\alpha} \right) \|u\|_{L^\alpha}^{2-\gamma} \\
\leq C \|u\|_{L^1} \left( \|u\|_{L^\alpha} + \|u\|_{L^\alpha}^{\alpha} \right) \|u\|_{L^\alpha}^{2-\gamma} \\
\leq C \|u\|_{L^1} \left( \|u\|_{L^\alpha} + \|u\|_{L^\alpha}^{\alpha} \right) \|u\|_{L^\alpha}^{2-\gamma}.
\]

Combining the above estimates, using the a priori estimate of $u^\alpha$ and $u$ in Lemmas \[2\] and \[3\] and Proposition \[3\] and applying the strong convergence of $u^\alpha$ \[5\], we obtain that

\[
\lim_{n \to \infty} \mathbb{E} \left[ \sup_{t \in [0,T]} \left\| \int_0^t S(t-s) f_{\alpha_n}(|u^\alpha|^2)u^\alpha - \log(|u|^2)uds \right\|^2 \right] = 0.
\]

The Minkowski inequality, \[2\] and \[3\] yield that

\[
\left\| - \int_0^t S(t-s) \frac{1}{2} (g(|u^\alpha|^2))^2 u^\alpha \sum_k |Q^k_{\alpha k}|^2 ds - V_2 \right\|
\]
\[
\begin{align*}
\leq \sum_k \|Q_{\geq}^\ast e_k\|_{\mathcal{L}_\infty}^2 \int_0^T \|g(|u^{\varepsilon_n}|^2)\|^2 u^{\varepsilon_n} - g(|u|^2)^2 u\| ds \\
\leq CgT \sum_k \|Q_{\geq}^\ast e_k\|_{\mathcal{L}_\infty}^2 \sup_{t \in [0,T]} \|u^{\varepsilon_n}(t) - u(t)\|,
\end{align*}
\]

and
\[
\begin{align*}
\left\| -i \int_0^t S(t - s)(g'(|u^{\varepsilon_n}|^2)g(|u^{\varepsilon_n}|^2)|u^{\varepsilon_n}|^2 u^{\varepsilon_n} + \sum_k Im(Q_{\geq}^\ast e_k)Q_{\geq}^\ast e_k ds - V_3 \right\|
\leq CgT \sum_k \|Q_{\geq}^\ast e_k\|_{\mathcal{L}_\infty}^2 \sup_{t \in [0,T]} \|u^{\varepsilon_n}(t) - u(t)\|.
\end{align*}
\]

The Burkholder inequality and the unitary property of \(S(\cdot)\) yield that
\[
\begin{align*}
\mathbb{E}\left[ \sup_{t \in [0,T]} \left\| \int_0^t S(t - s)g(|u^{\varepsilon_n}|^2)u^{\varepsilon_n} dW(s) - V_4 \right\|^2 \right]
\leq C \mathbb{E}\left[ \int_0^T \sum_k \|Q_{\geq}^\ast e_k\|_{\mathcal{L}_\infty}^2 \|g(|u^{\varepsilon_n}|^2)u^{\varepsilon_n} - g(|u|^2)^2 u\| ds \right] \\
\leq C \sup_{t \in [0,T]} \|u^{\varepsilon_n}(t) - u(t)\|^2.
\end{align*}
\]

Combining the above estimates and the strong convergence of \(u^{\varepsilon_n}\), we complete the proof of step 2.

Step 3: \(u\) is independent of the choice of the sequence of \(\{u^{\varepsilon_n}\}_{n \in \mathbb{N}^+}\). Assume that \(\tilde{u}\) and \(u\) are two different limit processes of two different sequences of \(\{u^{\varepsilon_n}\}_{n \in \mathbb{N}^+}\) and \(\{u^{\varepsilon_m}\}_{m \in \mathbb{N}^+}\), respectively. Then by step 2, they both satisfy Eq. (11). By repeating the procedures in step 1, it is not hard to obtain that \(\tilde{u} = u\).

The procedures in the above proof immediately yield the following convergence rate result for \(u^\ast\) in the regularized problem (1) and the Hölder regularity estimate of \(u^\ast\) and \(u^0\).

**Corollary 2.** Let the condition of Theorem 1 hold. Assume that \(u^\ast\) is the mild solution in Proposition 2 \(\varepsilon \in (0, 1)\). For \(p \geq 2\), there exists \(C(Q, T, \lambda, p, u_0) > 0\) such that for any \(\eta^2/d(2) \leq \eta \leq 2\),
\[
\mathbb{E}\left[ \sup_{t \in [0,T]} \|u(t) - u^\ast(t)\|^p \right] \leq C(Q, T, \lambda, p, u_0)(\varepsilon^\frac{d}{2\eta} + \varepsilon^\frac{d}{p})
\]
when \(\mathcal{O}\) is bounded domain, and
\[
\mathbb{E}\left[ \sup_{t \in [0,T]} \|u(t) - u^\ast(t)\|^p \right] \leq C(Q, T, \lambda, p, u_0, \alpha)(\varepsilon^\frac{d}{2\eta + \alpha} + \varepsilon^\frac{d}{p})
\]
when \(\mathcal{O} = \mathbb{R}^d\).

**Corollary 3.** Let the condition of Theorem 1 hold. Assume that \(u^\ast\) is the mild solution in Proposition 2 \(\varepsilon \in (0, 1)\) and \(u^0\) is the mild solution of Eq. (1). For \(p \geq 2\), there exists \(C(Q, T, \lambda, p, u_0) > 0\) such that for \(\varepsilon \in (0, 1]\),
\[
\mathbb{E}\left[ \|u^\ast(t) - u^\ast(s)\|^p \right] \leq C(Q, T, \lambda, p, u_0) |t - s|^\frac{p}{\eta}.
\]

**Proof** By means of the mild form of \(u^\ast\), \(\varepsilon \in [0, 1]\), the priori estimates of \(u^\ast\) in \(H^1 \cap L^p_0\) in Lemmas 3 and 4 and in Step 2 of the proof of Theorem 1 and the Burkholder inequality, we obtain the desirable result. 

\[\square\]
3.2. Well-posedness of SlogS equation with super-linearly growing diffusion coefficients

In this part, we extend the scope of \( \tilde{g} \), which allows the diffusion with super-linear growth, for the well-posedness of SlogS equation driven by conservative multiplicative noise. For instance, it includes the example \( \tilde{g}(x) = \log(1 + |x|^2) \), for \( c > 0 \).

**Theorem 2.** Let \( W(t) \) be \( L^2(\Omega; \mathbb{R}) \)-valued and \( g \in C^1_b(\mathbb{R}) \cap C(\mathbb{R}) \) satisfy the growth condition and the embedding condition,

\[
\sup_{x \in [0, \infty)} |g'(x)x| \leq C_g,
\]

\[
\|v g(v^2)\|_{L^1} \leq C_d(1 + \|v\|_{L^2} + \|v\|_{L^2})
\]

for some \( q \geq 2, \alpha \in [0, 1] \), where \( C_g > 0 \) depends on \( g \), \( C_d > 0 \) depends on \( \alpha \), \( d \), \( v \in \mathbb{E}^1 \cap L^2 \). Assume that \( d = 1, u_0 \in \mathbb{E}^1 \cap L^2, \alpha \in (0, 1] \), and \( \sum_{i \in \mathbb{N}^+} \|Q^i e_i\|_{L^2}^2 + \|Q^i e_i\|_{W^{1, \infty}}^2 < \infty \).

Then there exists a unique mild solution \( u \) in \( C([0, T]; \mathbb{E}) \) for Eq. (1) satisfying

\[
\mathbb{E}\left[ \sup_{t \in [0, T]} \|u(t)\|_{L^2}^p \right] + \mathbb{E}\left[ \sup_{t \in [0, T]} \|u(t)\|_{L^2}^p \right] \leq C(Q, T, \alpha, p, u_0).
\]

**Proof** By Proposition 3 and Lemma 4, we can introduce the truncated sample space

\[
\Omega_R(t) := \left\{ \omega: \sup_{s \in [0, t]} \|u^m(s)\|_{L^\infty} \leq R, \sup_{s \in [0, t]} \|u^m(s)\|_{L^\infty} \leq R \right\},
\]

where \( n \leq m \). The Gagliardo–Nirenberg interpolation inequality in \( d = 1 \), the priori estimate in \( \mathbb{E}^1 \) and the continuity in \( L^2 \) of \( u^m \) imply that \( u^m \) are continuous in \( L^\infty \) a.s. Define a stopping time

\[
\tau_R := \inf\{ t > 0 : \min_{s \in [0, t]} \|u^m(s)\|_{L^\infty}, \sup_{s \in [0, t]} \|u^m(s)\|_{L^\infty} \geq R \} \land T.
\]

Then on \( \Omega_R(T) \), we have \( \tau_R = T \). Let us take \( f(x) = \log(x + \epsilon), x > 0 \) for convenience. It is obvious that \( \Omega_R(t) \to \Omega \) as \( R \to \infty \) and that for any \( p \geq 1 \),

\[
\mathbb{P}\left( \sup_{t \in [0, T]} \min_{s \in [0, t]} \|u^m(t)\|_{L^\infty}, \|u^m(t)\|_{L^\infty} \geq R \right) \leq C \frac{1}{R^p} \mathbb{E}\left[ \sup_{t \in [0, T]} \|u^m(t)\|_{L^\infty}^p \right] + \mathbb{E}\left[ \|u^m(t)\|_{L^\infty}^p \right].
\]

Step 1: \( \{u^m\}_{m \in \mathbb{N}^+} \) forms a Cauchy sequence in \( \mathcal{M}_f(\Omega; C([0, T]; \mathbb{E}^m)) \). Following the same steps like the proof of Theorem 1, applying the Itô formula on \( \Omega_R(t) \) for \( t \in (0, \tau_R) \) yields that

\[
\|u^m(t) - u^m(t)\|_{L^2}^2 \leq \int_0^t 4|\lambda| \|u^m(s) - u^m(s)\|^2 ds + \int_0^t \|u^m(s) - u^m(s)\|_{L^2} ds
\]

\[
+ \int_0^t \sum_{k \in \mathbb{N}^+} \langle Q^i e_k\rangle (g(|u^m|^2) - g(|u^m|^2)) u^m, (g(|u^m|^2) - g(|u^m|^2)) u^m \rangle ds
\]

\[
+ \int_0^t \langle u^m - u^m, i \left( g(|u^m|^2) u^m - g(|u^m|^2) u^m \right) dW(s) \rangle.
\]
Taking expectation on $\Omega_R(t)$ yields that
\[ E\left[ I_{\Omega_R(t)}\|u^{\epsilon_m}(t) - u^{\epsilon_n}(t)\|^2 \right] \]
\[ \leq \int_0^t 4|\lambda| E\left[ I_{\Omega_R(t)}\|u^{\epsilon_m}(s) - u^{\epsilon_n}(s)\|^2 \right] ds + 4|\lambda| \epsilon_n^\frac{1}{2} \int_0^t E\left[ I_{\Omega_R(t)}\|u^{\epsilon_m}(s) - u^{\epsilon_n}(s)\|_{L^1} \right] ds \]
\[ + \int_0^t \sum_{k \in \mathbb{N}^+} |Q^{\frac{1}{2}} e_k|_{L^\infty}^2 E\left[ I_{\Omega_R(t)}(g(\|u^{\epsilon_m}\|^2) - g(\|u^{\epsilon_n}\|^2))u^{\epsilon_m}, (g(\|u^{\epsilon_m}\|^2) - g(\|u^{\epsilon_n}\|^2))u^{\epsilon_n} \right] ds. \]

Making use of the assumptions on $g$, we get
\[ E\left[ I_{\Omega_R(t)}\|u^{\epsilon_m}(t) - u^{\epsilon_n}(t)\|^2 \right] \leq \int_0^t 4|\lambda| E\left[ I_{\Omega_R(t)}\|u^{\epsilon_m}(s) - u^{\epsilon_n}(s)\|^2 \right] ds + 4|\lambda| \epsilon_n^\frac{1}{2} \int_0^t E\left[ I_{\Omega_R(t)}\|u^{\epsilon_m}(s) - u^{\epsilon_n}(s)\|_{L^1} \right] ds \]
\[ + C(Q) \int_0^t E\left[ I_{\Omega_R(t)} \left( |u^{\epsilon_m}|^2 + |u^{\epsilon_n}|^2 \right) \right] ds \]
\[ \leq \int_0^t 4|\lambda| E\left[ I_{\Omega_R(t)}\|u^{\epsilon_m}(s) - u^{\epsilon_n}(s)\|^2 \right] ds + 4|\lambda| \epsilon_n^\frac{1}{2} \int_0^t E\left[ I_{\Omega_R(t)}\|u^{\epsilon_m}(s) - u^{\epsilon_n}(s)\|_{L^1} \right] ds \]
\[ + C(Q) \int_0^t E\left[ (1 + R^4)I_{\Omega_R(t)}\|u^{\epsilon_m}(t) - u^{\epsilon_n}(t)\|^2 \right] ds. \]

If $\mathcal{O}$ is bounded, then Hölder inequality and Gronwall’s inequality yield that
\[ E\left[ I_{\Omega_R(t)}\|u^{\epsilon_m}(t) - u^{\epsilon_n}(t)\|^2 \right] \leq C(u_0, Q, T, \lambda) e^{(1 + R^4) T} \epsilon_n. \]

On the other hand, the Chebyshev inequality and the a priori estimate lead to
\[ E\left[ I_{\Omega_R(t)}\|u^{\epsilon_m}(t) - u^{\epsilon_n}(t)\|^2 \right] \leq (\mathbb{P}(\Omega_R(t)))^\frac{1}{\gamma} \left( E\left[ \|u^{\epsilon_m}(t) - u^{\epsilon_n}(t)\|^{2\gamma} \right] \right)^\frac{1}{\gamma}, \]
where $\frac{1}{\gamma} + \frac{1}{2\gamma} = 1$. From the above estimate, choosing $p \gg p_1$ and denote by $\kappa = \frac{p}{p_1}$, we conclude that
\[ E\left[ \|u^{\epsilon_m}(t) - u^{\epsilon_n}(t)\|^2 \right] \leq C(u_0, Q, T, \lambda, p, p_1) \left( \epsilon_n^{(1 + R^4) T} + R^{-\kappa} \right). \]

Then one may take $R = (\frac{c}{T} |\log(\epsilon)|)^{\frac{1}{\kappa}}$ for $c \in (0, 1)$ and get
\[ E\left[ \|u^{\epsilon_m}(t) - u^{\epsilon_n}(t)\|^2 \right] \leq C(u_0, Q, T, \lambda, p, p_1) \left( \epsilon_n^{1 - \kappa} + (\frac{c}{T} |\log(\epsilon_n)|)^{-\frac{1}{\kappa}} \right). \]

By further applying the Burkholder inequality to the stochastic integral, we achieve that for any $\kappa > 0$,
\[ E\left[ \sup_{t \in [0, T]} \|u^{\epsilon_m}(t) - u^{\epsilon_n}(t)\|^2 \right] \leq C(u_0, Q, T, \lambda, p, p_1) |\log(\epsilon_n)|^{-\frac{1}{\kappa}}. \]

When $\mathcal{O} = \mathbb{R}^d$, we just repeat the procedures in the proof of the case that $g$ is bounded and obtain that for $\eta \in (0, \frac{2\alpha}{2\alpha + 2\gamma}]$ and $\alpha \in (0, 1]$,
\[ E\left[ I_{\Omega_R(t)}\|u^{\epsilon_m}(t) - u^{\epsilon_n}(t)\|^2 \right] \leq C(u_0, Q, T, \lambda, p, p_1) |\log(\epsilon_n)|^{-\frac{1}{\kappa}}. \]
\[ \frac{1}{2} \int_0^t S(t-s)(g(|u^\varepsilon|^2))u^\varepsilon \sum_k |Q_k^\varepsilon e_k|^2 ds - V_2 \geq 0. \]

The Minkowski inequality and the properties of \( g \) yield that on \( \Omega_{R_1}(t) \), for a small enough \( \eta_1 > 0 \),

\[ -\frac{1}{2} \int_0^t S(t-s)(g(|u^\varepsilon|^2))u^\varepsilon \sum_k |Q_k^\varepsilon e_k|^2 ds - V_2 \geq 0. \]

On the other hand, for any \( p_2 > 0 \),

\[ \mathbb{E} \left[ \| u_{R_1}^\varepsilon(t) \|_2^p \right] - \frac{1}{2} \int_0^t S(t-s)(g(|u^\varepsilon|^2))u^\varepsilon \sum_k |Q_k^\varepsilon e_k|^2 ds - V_2 \leq C(u_0, Q, T, p_2) R_1^{-p_2}. \]

Taking \( R_1 = O(\| \epsilon_n \|^{-2/(2+\kappa_1)}) \), \( \kappa_1 < \kappa \), we have that

\[ \lim_{n \to \infty} \sup_{t \in [0, T]} \mathbb{E} \left[ \left\| \frac{1}{2} \int_0^t S(t-s)(g(|u^\varepsilon|^2))u^\varepsilon \sum_k |Q_k^\varepsilon e_k|^2 ds - V_2 \right\|^2 \right] = 0, \]
which immediately implies that
\[
\lim_{n \to \infty} \mathbb{E} \left[ \sup_{t \in [0,T]} \left\| - \frac{1}{2} \int_0^t S(t-s)(g(|u^n|)u^n)\,dW(s) - V_2 \right\|^2 \right] = 0.
\]
The Burkholder inequality and the unitary property of \(S(\cdot)\) yield that
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \left\| \int_0^t i S(t-s)g(|u^n|)u^n\,dW(s) - V_4 \right\|^2 \right] \\
\leq C \mathbb{E} \left[ \sum_k \left\| Q_k u^n \right\|^2_{L^\infty} \left\| g(|u^n|)u^n - g(|u|^2)u \right\|^2 ds \right] \\
\leq C(1 + R_1^2) \mathbb{E} \left[ \sup_{t \in [0,T]} \| u^n(t) - u(t) \|^2 \right].
\]
On the other hand, the Chebyshev inequality, together with the a priori estimate of \(u^n\), implies that
\[
\mathbb{E} \left[ \sup_{t \geq \tau_{R_1}} \left\| \int_0^t i S(t-s)g(|u^n|)u^n\,dW(s) - V_4 \right\|^2 \right] \\
= \mathbb{E} \left[ \sup_{t \in [0,T]} \mathbb{I}_{\tau_{R_1}} \left\| \int_0^t S(t-s)g(|u^n|)u^n\,dW(s) - V_4 \right\|^2 \right] \\
\leq C(u_0, Q, T, p) R_1^{-p}. \\
\]
Taking \(R_1 = \mathcal{O}(\log(\epsilon_n)^{\frac{1}{1+2(p-1)}})\), \(\kappa_1 < \kappa\), we have that
\[
\lim_{n \to \infty} \mathbb{E} \left[ \sup_{t \in [0,T]} \left\| \int_0^t i S(t-s)g(|u^n|)u^n\,dW(s) - V_4 \right\|^2 \right] = 0.
\]
Combining the above estimates and the strong convergence of \(u^n\), we complete the proof. \(\square\)

**Remark 1.** One may extend the scope of \(\bar{g}\) to an abstract framework by similar arguments. Here the assumption \(d = 1\) lies on the fact that in \(H^1\) is an algebra by Sololev embedding theorem. When considering the case \(d \geq 2\), one may use \(H^s, s > \frac{d}{2}\) as the underlying space for the local well-posedness. However, as stated in Lemma 6, it seems impossible to get the uniform bound of \(u^n\) in \(H^s\) for \(s \geq 2\).

### 4. Appendix

The original problem and regularized problem can be rewritten into the equivalent evolution forms
\[
\begin{align*}
\frac{du}{dt} &= A\,dt + F(u)\,dt + G(u)\,dW(t), \\
\quad u(0) &= u_0, \tag{9}
\end{align*}
\]
where \(A = i\Delta\), \(F\) is the Nemystkii operator of drift coefficient function, and \(G\) are the Nemystkii operator of diffusion coefficient function. Then the mild solution of the above evolution is defined as follows.
Definition 1. A continuous \( H \)-valued \( F_t \) adapted process \( u \) is a solution to (9) if it satisfies \( \mathbb{P} \)-a.s for all \( t \in [0, T] \),

\[
u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds + \int_0^t G(u(s))dW(s),
\]

where \( S(t) \) is the \( C_0 \)-group generated by \( A \).

Definition 2. A local mild solution of (10) is \((u, \tau) := (u, \tau_n, \tau)\) satisfying \( \tau_n \uparrow \tau \), a.s., as \( n \to \infty \), \( u \in \mathcal{M}_F^p(\Omega; C([0, \tau); H^p]), s > 0, p \geq 1 \) and that

\[
u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds
+ \int_0^t S(t-s)G(u(s))dW(s), \text{ a.s.,}
\]

for \( t \leq \tau_n \) in \( H^2 \) for \( n \in \mathbb{N}^+ \). Solutions of (9) are called unique, if

\[
\mathbb{P}\left(u_1(t) = u_2(t), \forall t \in [0, \sigma_1 \wedge \sigma_2]\right) = 0.
\]

for all local mild solution \((u_1, \sigma_1)\) and \((u_2, \sigma_2)\). The local solution \((u, \tau)\) is called a global mild solution if \( \tau = T, \text{a.s. and } u \in \mathcal{M}_F^p(\Omega; C([0, T]; H^p)) \).

Lemma 7. Let \( \epsilon \in (0, 1) \). Then \( f_\epsilon(x) = \log(|x|^2 + \epsilon), x \in \mathbb{C} \), satisfies

\[
|\text{Im}[(f_\epsilon(x_1) - f_\epsilon(x_2))(\bar{x}_1 - \bar{x}_2)]| \leq 4|x_1 - x_2|^2,
\]

Proof Without loss of generality, we assume that \( 0 < |x_2| \leq |x_1| \). Notice that

\[
\text{Im}[(f_\epsilon(x_1) - f_\epsilon(x_2))(\bar{x}_1 - \bar{x}_2)] = \frac{1}{2}(\log(\epsilon + |x_1|^2) - \log(\epsilon + |x_2|^2))\text{Im}(\bar{x}_1x_2 - \bar{x}_2x_1).
\]

Direct calculation yields that

\[
|\text{Im}(\bar{x}_1x_2 - \bar{x}_2x_1)| \leq 2|x_2||x_1 - x_2|.
\]

Using the fact that

\[
|\log(\epsilon + |x_1|^2) - \log(\epsilon + |x_2|^2)| = 2|\log((\epsilon + |x_1|^2)^{\frac{1}{2}}) - \log((\epsilon + |x_2|^2)^{\frac{1}{2}})|,
\]

we obtain

\[
|\text{Im}[(f_\epsilon(x_1) - f_\epsilon(x_2))(\bar{x}_1 - \bar{x}_2)]|
\leq 2|\log((\epsilon + |x_1|^2)^{\frac{1}{2}}) - \log((\epsilon + |x_2|^2)^{\frac{1}{2}})||x_2||x_1 - x_2|.
\]

The mean value theorem leads to the desired result.

\[
\text{Lemma 8. Let } \epsilon \in (0, 1). \text{ Then } f_\epsilon(|x|^2) = \log\left(\frac{|x|^2 + \epsilon}{1 + \epsilon |x|^2}\right) \text{ satisfies the following properties,}
\]

\[
|f_\epsilon(|x|^2)| \leq |\log(\epsilon)|,
|d_{|x^2}f_\epsilon(|x|^2)| \leq \frac{2(1 - \epsilon^2)|x|}{(\epsilon + |x|^2)(1 + \epsilon |x|^2)},
|\text{Im}[f_\epsilon(|x|^2)x_1 - f_\epsilon(|x|^2)x_2](\bar{x}_1 - \bar{x}_2)| \leq 4(1 - \epsilon^2)|x_1 - x_2|^2.
\]

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where we use the following estimation, for any small enough $\eta > \alpha > 0$, inequality, we have that for small $q > 0$,

$$|F_\epsilon(|u'(t)|^2)| = \int_\Omega \left[ (\epsilon + |u'(t)|^2) \log(\epsilon + |u'(t)|^2) - |u'(t)|^2 - \epsilon \log(\epsilon) \right] dx$$

$$\leq |u'(t)|^2 + \int_\Omega |u'(t)|^2 \log(\epsilon + |u'(t)|^2) dx + \left| \int_\Omega \epsilon (\log(\epsilon + |u'(t)|^2) - \log(\epsilon)) dx \right|$$

$$\leq 2 |u'(t)|^2 + \|u'(t)\|^2_{L^{2q}} + \|u'(t)\|^2_{L^{2q}},$$

where we use the following estimation, for any small enough $\eta > 0$,

$$\int_\Omega |u'(t)|^2 \log(\epsilon + |u'(t)|^2) dx$$

$$= \int_{\epsilon + |u'|^2 \geq 1} f_\epsilon(|u'|^2) |u'|^2 dx + \int_{\epsilon + |u'|^2 \leq 1} f_\epsilon(|u'|^2) |u'|^2 dx$$

$$\leq \int_{\epsilon + |u'|^2 \geq 1} \left( \epsilon + |u'|^2 \right)^{2q} |u'|^2 dx + \int_{\epsilon + |u'|^2 \leq 1} \left( \epsilon + |u'|^2 \right)^{-2q} |u'|^2 dx$$

$$\leq \|u\|^2_{L^{2q}} + \|u\|^2_{L^{2q}},$$

Then by the Gagliardo–Nirenberg interpolation inequality in a bounded domain $\Omega$, i.e.,

$$\|u\|_{L^q(\Omega)} \leq C_1 \|u\|^{1-\alpha} \|\nabla u\|^{\alpha} + C_2 \|u\|, \quad q = \frac{2d}{d - 2\alpha}, \quad \alpha \in (0, 1],$$

we have that

$$\int_\Omega F_\epsilon(|u'(s)|^2) dx \leq \int_\Omega \left( (\epsilon + |u'|^2) \log(\epsilon + |u'|^2) - |u'|^2 - \epsilon \log(\epsilon) \right) dx$$

$$\leq C(1 + \|u\|^2_{L^{2q}}) \leq C(\|u\|^2 + \|\nabla u\|^\frac{d}{d - 2\alpha} \|u\|^\frac{2d}{d - 2\alpha}).$$

Taking $p$th moment and applying Lemma 3, we complete the proof for the case that $\Omega$ is a bounded domain.

When $\Omega = \mathbb{R}^d$, we need to control $\|u\|_2$. By using the weighted interpolation inequality in Lemma 3 with $\alpha > \frac{d\eta}{2 - 2q}, \alpha \in (0, 1]$, and applying the Gagliardo–Nirenberg interpolation inequality,

$$\|u\|_{L^q(\mathbb{R}^d)} \leq C_1 \|u\|^{1-\alpha} \|\nabla u\|^{\alpha}, \quad q = \frac{2d}{d - 2\alpha}, \quad \alpha \in (0, 1],$$

where $q = 2\eta + 2$. Based on Lemma 3 and Lemma 8 we complete the proof by using the Young inequality and taking $p$th moment.

[Sketch Proof of Lemma 6] Due to the loss of the regularity of the solution in time, we can not establish the bound in $\mathbb{H}^2$ through $\frac{\partial}{\partial t}$ like in the deterministic case. According to
Lemma 3, it suffices to bound $\|\Delta u^r\|^2$. We present the procedures of the estimation of $E[\|\Delta u\|^2]$ for the conservative multiplicative noise case. One can easily follow the procedures to obtain the estimate of $E[\sup_{t \in [0, T]} \|\Delta u(t)\|^2]$ for both additive and multiplicative noises.

By using the Itô formula to $\|\Delta u^r\|^2$ we obtain that

$$
\|\Delta u^r(t)\|^2 = \|\Delta u^r(0)\|^2 + 2 \int_0^t \langle \Delta u^r(s), II_{det} \rangle ds + 2 \int_0^t \langle \Delta u^r(s), II_{mod} \rangle ds + 2II_{Str},
$$

where

$$
II_{Sto} := \int_0^t \langle \Delta u^r, 1g(\|u^r\|^2)\nabla u^r \nabla dW(s) \rangle + \int_0^t \langle \Delta u^r, i4g(\|u^r\|^2)u^r \Delta dW(s) \rangle
$$
$$
+ \int_0^t \langle \Delta u^r, i2g'(\|u^r\|^2)Re(\bar{u}^r \nabla u^r) \nabla u^r dW(s) \rangle + \int_0^t \langle \Delta u^r, i2g'(\|u^r\|^2)Re(\bar{u}^r \Delta u^r) u^r dW(s) \rangle
$$
$$
+ \int_0^t \langle \Delta u^r, 2g'(\|u^r\|^2)\nabla |u^r|^2 u^r dW(s) \rangle.
$$

$$
II_{det} := i\Delta^2 u^r + i\lambda f(\|u^r\|^2)\Delta u^r dt + i4\lambda f'(\|u^r\|^2)Re(\bar{u}^r \nabla u^r) \nabla u^r
$$
$$
+ i4\lambda f''(\|u^r\|^2)(Re(\bar{u}^r \nabla u^r))^2 u^r
$$
$$
+ i2\lambda f'(\|u^r\|^2)Re(\bar{u}^r \Delta u^r) u^r,
$$

and $II_{mod}$ is the summation of all terms involving the second derivative of the Itô modified term produced by the Stratonovich integral. Here for simplicity, we omit the presentation of the explicit form for $II_{Str}$. Taking expectation and using the Gagliardo–Nirenberg interpolation inequality $\|\nabla v\|_{L^q} \leq C\|\Delta v\|^{\frac{q}{2}}\|\nabla v\|^{\frac{q}{2}}$ in $d = 1$, we obtain that

$$
E[\|\Delta u^r(t)\|^2]
$$
$$
\leq E[\|\Delta u^r(0)\|^2] + C(\lambda, p)e^{-\frac{2\lambda}{\|u^r\|}} E\left[ \int_0^t \|\Delta u^r(r)\| (1 + \|\Delta u^r\|^2) dr \right]
$$
$$
+ C(\lambda, p) E\left[ \int_0^t \|u^r(0)^2\| \left( \|\sum_i \Delta \bar{Q} e_i \| \|\sum_i \|Q e_i\|^2(\|u^r\|^2)^2 u^r\| \|\sum_i |Q e_i|^2(\|u^r\|^2)^2 \nabla u^r\| \|\sum_i |Q e_i|^2(\|u^r\|^2)^2 \nabla u^r\|^2 \| \right) dr \right]
$$
$$
+ C(\lambda, p) E\left[ \int_0^t \|\Delta u^r\| \left( \|\sum_i |Q e_i|^2(\|u^r\|^2)^2 \nabla u^r\| \|\sum_i |Q e_i|^2(\|u^r\|^2)^2 \nabla u^r\|^2 \| \right) dr \right]
$$
$$
+ C(\lambda, p) E\left[ \int_0^t \|\sum_i (\|g(\|u^r\|^2)^2 u^r\|) \|\sum_i |Q e_i|^2(\|u^r\|^2)^2 \nabla u^r\|^2 \| \right) dr \right]
$$
$$
+ C(\lambda, p) E\left[ \int_0^t \|\sum_i (\|g(\|u^r\|^2)^2 u^r\|) \|\sum_i |Q e_i|^2(\|u^r\|^2)^2 \nabla u^r\| \|\sum_i |Q e_i|^2(\|u^r\|^2)^2 \nabla u^r\|^2 \| \right) dr \right]
$$

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\[ := \mathbb{E}\left[ \| \Delta u'(0) \|^2 \right] + C(\lambda, p) \varepsilon^{-\frac{1}{2}} \mathbb{E}\left[ \int_0^t \| \Delta u'(r) \| (1 + \| \nabla u' \|^2) dr \right] + C(\lambda, p) \mathbb{E}\left[ \int_0^t |A(r)| dr \right] + C(\lambda, p) \mathbb{E}\left[ \int_0^t |B(r)| dr \right]. \]

Now applying the Hölder inequality, using the properties of \( g \), using the Gagliardo–Nirenberg interpolation inequality, we obtain that for a small \( \eta > 0 \),

\[ A(r) \leq \sum_i \| \Delta Q_i^2 e_i \|_{L^\infty} \| Q_i^2 e_i \|_{L^\infty} \|(g(|u'|^2))^2 u'\|_{L^\infty} + \sum_i \| \nabla Q_i^2 e_i \|_{L^\infty} \|(g(|u'|^2))^2 u'\|_{L^\infty} \]

\[ + \sum_i \| \nabla Q_i^2 e_i \|_{L^\infty} \| Q_i^2 e_i \|_{L^\infty} \|(g(|u'|^2)) \nabla u'\|_{L^4} + \sum_i \| Q_i^2 e_i \|_{L^\infty} \| Q_i^2 e_i \|_{L^\infty} \|(g(|u'|^2)) \nabla u'\|_{L^4} \]

\[ + \sum_i \| \nabla Q_i^2 e_i \|_{L^\infty} \| Q_i^2 e_i \|_{L^\infty} \| g(|u'|^2)^2 u'\|_{L^\infty} \| \nabla u'\|_{L^4}. \]

Similarly, we have that for a small \( \eta > 0 \),

\[ B(r) \leq \sum_i \| g(|u'|^2) \|_{L^\infty} \| \nabla u'\|_{L^4} \| Q_i^2 e_i \|_{L^\infty} \| \Delta Q_i^2 e_i \|^2 \]

\[ + \| g'(|u'|^2) u' \|_{L^\infty} \| Q_i^2 e_i \|_{L^\infty} \| \Delta Q_i^2 e_i \|^2 + \| g'(|u'|^2) u' \|_{L^\infty} \| \nabla u'\|_{L^4} \| Q_i^2 e_i \|_{L^\infty} \]

\[ + \| g''(|u'|^2) u' \|_{L^\infty} \| \Delta Q_i^2 e_i \|^2 + \| g'(|u'|^2) u' \|_{L^\infty} \| \nabla u'\|_{L^4} \| Q_i^2 e_i \|_{L^\infty}. \]

Combining the above estimates, and using the Young inequality and Gronwall inequality imply that

\[ \mathbb{E}\left[ \| \Delta u'(t) \|^2 \right] \leq C(u_0, T, Q, p, \eta)(1 + \varepsilon^{-2}). \]

Now, taking supreme over \( t \), then taking expectation, and applying the Burkholder inequality to the \( III_{st,0} dW(t) \), we achieve that for sufficient small \( \eta > 0 \),

\[ \mathbb{E}\left[ \sup_{\tau \in [0, T]} \| \Delta u'(t) \|^2 \right] \leq C(u_0, T, Q, \eta)(1 + \varepsilon^{-2}). \]

**[Proof of Proposition 4]** We follow the steps in the proof of Proposition 3 to present the proof in the case of \( p = 2 \). For convenience, we present the proof for the multiplicative noise
case. Applying the Itô formula to $\|u_t\|_{L^2}^2 = \int_{\mathbb{R}^d} (1 + |x|^2)^{\alpha} |u_t|^2 \, dx$, using integration by parts, then taking supreme over $t$, and applying Burkholder inequality, we deduce that

$$
E\left[ \sup_{t \in [0,T]} \|u_t(t)\|_{L^2}^2 \right] \leq E\left[ \|u_0\|_{L^2}^2 \right] + 2\alpha E\left[ \int_0^T \left| \left( (1 + |x|^2)^{\alpha-1} xu_t(s), \nabla u_t^\prime \right) \right| \, ds \right]
$$

$$
+ E\left[ \sup_{t \in [0,T]} \left| \left( (1 + |x|^2)^{\alpha} u_t(s), i g(|u_t(s)|^2) u_t(s) dW(s) \right) \right| \right]
$$

$$
\leq E\left[ \|u_0\|_{L^2}^2 \right] + C\alpha E\left[ \int_0^T \left| \left( (1 + |x|^2)^{\alpha-1} xu_t(s), \nabla u_t^\prime \right) \right| \, ds \right]
$$

$$
+ CE\left[ \int_0^T \sum_{i \in \mathbb{N}^+} \left| \left( (1 + |x|^2)^{\alpha} u_t(s), i g(|u_t(s)|^2) u_t(s) Q^\prime \xi_i \right) \right|^2 \, ds \right].
$$

By Hölder’s inequality, for $\alpha \in (1, 2]$, we have that

$$
\left| \left( (1 + |x|^2)^{\alpha-1} xu_t^\prime, \nabla u_t^\prime \right) \right| \leq C\|u_t^\prime\|_{L^2}^2 \left( (1 + |x|^2)^{\frac{\alpha}{2} - \frac{1}{2}} \nabla u_t^\prime \right).
$$

Integration by parts and Hölder’s inequality yield that for some small $\eta > 0$,

$$
\left| \left( (1 + |x|^2)^{\frac{\alpha}{2} - \frac{1}{2}} \nabla u_t^\prime \right)^2 = \left( (1 + |x|^2)^{\alpha-1} \nabla u_t^\prime, \nabla u_t^\prime \right)
$$

$$
= -\left( (1 + |x|^2)^{\alpha-1} u_t^\prime, \Delta u_t \right) - 2(\alpha - 1)\left( (1 + |x|^2)^{\alpha-2} x \nabla u_t^\prime, u_t^\prime \right)
$$

$$
\leq \|u_t^\prime\|_{L^2}^{2} \Delta u_t + C(\eta)(\alpha - 1)\|u_t^\prime\|_{L^2}^{2} + \eta|\alpha - 1|\|u_t^\prime\|_{L^2}^{2} + \eta|\alpha - 1|\|u_t^\prime\|_{L^2}^{2} - \frac{3}{2} (\alpha - 3, 0).
$$

Combining the above estimates in Proposition 3 and using Young’s inequality, we achieve that

$$
E\left[ \sup_{t \in [0,T]} \|u_t(t)\|_{L^2}^2 \right] \leq e^{CT} (1 + \epsilon^{-1}), \quad \text{if } \alpha \in (1, \frac{3}{2}],
$$

$$
E\left[ \sup_{t \in [0,T]} \|u_t(t)\|_{L^2}^2 \right] \leq e^{CT} (1 + \epsilon^{-\frac{2}{3}}), \quad \text{if } \alpha \in \left(\frac{3}{2}, 2\right],
$$

$$
E\left[ \sup_{t \in [0,T]} \|u_t(t)\|_{L^2}^2 \right] \leq e^{CT} (1 + \epsilon^{-2}), \quad \text{if } \alpha = 2.
$$

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