REMARKS ON DEFORMED AND UNDEFORMED
KNIZHNIK-ZAMOLODCHIKOV EQUATIONS.

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Abstract. Deformed and undeformed KZ equations are considered for \( k = 0 \). It is shown that they allow the same number of solutions, one being the asymptotics of others. Essential difference in analytical properties of the solutions is explained.

In this paper certain beautiful mathematical structures will be explained which arise when studying the deformations of famous Knizhnik-Zamolodchikov equations [1]. These deformations were proposed in [2]. We shall be considering the case of zero central extension for which the deformed equations coincide with Form Factors Equation [3] as it has been explained in [4]. We shall show that there exists exactly the same number of solutions to deformed equations as to usual ones although the properties of the solutions are quite different.

The stress will be done on the braiding properties of the solutions. We shall show that the deformation leads to trivialization of braiding. That is opposite to the usual belief that the deformation should lead to more complicated braiding. The trivialization of braiding is responsible for the application of the deformed equations to the theory of particles, i.e. to the description of form factors. Also, it is closely connected with certain “quantization” of Riemann surfaces which will be discussed below.

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1. Knizhnik-Zamolodchikov equations for \( k = 0 \).

Knizhnik-Zamolodchikov equations for the affine algebra \( \hat{sl}(N) \) can be written as follows:

\[
\left( \frac{k + N}{2} \frac{d}{d\lambda_i} + \sum_{j \neq i} \hat{r}_{i,j}(\lambda_i - \lambda_j) \right) f(\lambda_1, \cdots, \lambda_m) = 0
\]

where \( k \) is the central charge, a finite-dimensional representation of \( sl(N) \) is associated to every \( \lambda_i \), \( \hat{r}_{i,j} \) is classical r-matrix acting in the tensor product of i-th and j-th representations:

\[
\hat{r}_{i,j}(\lambda_i - \lambda_j) = \frac{K^{ab}t^a_i \otimes t^b_j}{\lambda_i - \lambda_j},
\]
$K$ is Killing form, $t^a_i, t^b_j$ are generators of $sl(N)$ acting in corresponding representations. The function $f$ belongs to the tensor product of the spaces of the representations.

We shall be considering the following particular case of the equations: the algebra under consideration will be $sl(2)$, we shall deal only with fundamental two-dimensional representations, we take rather unusual value of the central charge: $k = 0$. Also we change the definition of $r_{i,j}$ to

$$r_{i,j} (\lambda_i - \lambda_j) = \frac{1}{2} \frac{P_{i,j}}{\lambda_i - \lambda_j}$$

($P_{i,j}$ is permutation) which differs in our case from (1) by $\frac{1}{4} I$. With that change of r-matrix some irrelevant multiplier ($\prod_{i<j}(\lambda_i - \lambda_j)^{1/4}$) is removed from the solutions. We restrict ourselves with those solutions which are singlets (belong to one-dimensional representation) under the global $sl(2)$. So, we deal with the equations

$$\frac{d}{d\lambda_i} + \sum_{i \neq j} r_{i,j} (\lambda_i - \lambda_j) f(\lambda_1, \cdots, \lambda_{2n}) = 0$$

with $r_{i,j}$ given by (2).

As it has been mentioned the consideration of $k = 0$ is rather unusual, so we have to comment on it. Usually $k \in \mathbb{Z}_+$ are considered which are responsible for the applications to the Conformal Field Theory, $\lambda_i$ then are points in the coordinate space. One can try to apply the deformed KZ equations which are to appear soon to certain lattice deformation of WZNW model (may be to the one described in [5]). In that case, certainly, $k \in \mathbb{Z}_+$ should be considered for the deformed equations either. However, the applications of the deformed equations we are interested in are completely different: $\lambda_i$ for us will belong rather to the momentum space describing the rapidities of physical particles. There are good reasons for $k$ to equal zero in that case. One can ask about the meaning of undeformed KZ equations in this situation. It is a complicated question the answer to which is not quite clear now. The following can be said, however. The classical limit of the deformed equations should describe not the ultraviolet limit in a sense of perturbations of CFT [6], but rather certain quasiclassical limit for asymptotically free field theories which is connected with periodical problem for classical integrable equations. This understanding follows from the logic of the paper [4]. It should be emphasized that the periodical problem is used not in usual fashion as regularization of space interval, but rather as that associated with monodromy around one point in the space-time. So, in certain sense the situation is close to that considered in [7] where the finite-dimensional quantum group responsible for the quantization of WZNW model arises from the monodromy of classical equation. In our case the infinite-dimensional quantum group (double of Yangian [8,11,4]) responsible for the quantization of asymptotically free massive field theory [9,10,11,4] should arise from the monodromy of classical integrable equation of KDV type. In the $sl(2)$ case the periodical problem in question is closely connected with hyper-elliptic surfaces (HES) which parametrize the solutions. So, no wonder that the solutions of (3) are closely related to HES. Continuing these reasonings we would like to identify the particles in quantum theory with the moduli of classical periodical problem. But still much work remains to be done in this direction.
Anyway, we consider $k = 0$. So, let us write down the solutions to the equations (3). Using the usual basis in two-dimensional space:

$$e^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

we shall denote the components of $f(\lambda_1, \cdots, \lambda_{2n})$ in the natural basis of the tensor product

$$e^{\varepsilon_1, \cdots, \varepsilon_{2n}} = e^{\varepsilon_1} \otimes \cdots \otimes e^{\varepsilon_{2n}}$$

by $f(\lambda_1, \cdots, \lambda_{2n})_{\varepsilon_1, \cdots, \varepsilon_{2n}}$. We are looking for the singlet solutions of the equations which means in particular that $\sum \varepsilon_i = 0$. For each particular component $f(\lambda_1, \cdots, \lambda_{2n})_{\varepsilon_1, \cdots, \varepsilon_{2n}}$ the choice of $\varepsilon_1, \cdots, \varepsilon_{2n}$ induces a partition of $\Lambda = \{\lambda_1, \cdots, \lambda_{2n}\}$ into $\Lambda^+ = \{\lambda_j : \varepsilon_j = +\}$ and $\Lambda^- = \{\lambda_j : \varepsilon_j = -\}$. Different solutions will be parametrized by the sets $\gamma_1, \cdots, \gamma_{n-1}$ which will be specified later. The solutions look as follows

$$f^{\gamma_1, \cdots, \gamma_{n-1}}(\lambda_1, \cdots, \lambda_{2n})_{\varepsilon_1, \cdots, \varepsilon_{2n}} = \prod_{\lambda^+ \in \Lambda^+, \lambda^- \in \Lambda^-} (\lambda^+ - \lambda^-)^{-1} \det|| \int_{\gamma_i} \omega_j(\tau|\Lambda^-|\Lambda^+) d\tau ||_{(n-1)\times(n-1)}$$

where $\omega_j$ are the following differentials on the HES $w^2 = P(\tau) \equiv \prod(\tau - \lambda_i)$:

$$\omega_j(\tau|\Lambda^-|\Lambda^+) = \frac{1}{\sqrt{P(\tau)}}$$

$$\times \left\{ \prod_{\lambda^-} (\tau - \lambda^-) \left[ \frac{d}{d\tau} \frac{\prod(\tau - \lambda^-)}{\tau - \lambda_j} \right]_0 + \prod_{\lambda^+} (\tau - \lambda^+) \left[ \frac{d}{d\tau} \frac{\prod(\tau - \lambda^+)}{\tau - \lambda_j} \right]_0 \right\}$$

where $[ ]_0$ means that only the polynomial part of the expression in brackets is taken. The differentials $\omega_j$ are of the second kind: they have singularities at $\infty^\pm$, but their residues at the infinities are equal to zero. It should be mentioned that the singular part of the differential $\omega_j$ is independent of the partition of $\Lambda$:

$$\omega_j(\tau|\Lambda^-|\Lambda^+) - \omega_j(\tau|\Lambda^+|\Lambda^-) = \text{of the first kind}$$

First kind differentials in our case are of the type: $\tau^j / \sqrt{P(\tau)}$, $0 \leq j \leq n - 2$. The contours $\gamma_1, \cdots, \gamma_{n-1}$ are arbitrary cycles on the HES (notice that its genus $g$ equals $n - 1$). There are $2n - 1$ evident cycles $c_i : \ c_i$ contains two branching points $(\lambda_i, \lambda_{i+1})$ $i = 1, \cdots, 2n - 1$, but only $2n - 2$ of them are independent due to the relation

$$c_1 + c_3 + \cdots + c_{2n-1} \sim 0$$

which implies that

$$\sum_{i=1}^{n} \int_{c_{2i-1}} \omega_j = 0, \ \forall j$$

where $\sum_{i=1}^{n} \int_{c_{2i-1}} \omega_j = 0, \ \forall j$.
The canonical $a$ and $b$ cycles can be introduced as follows: $a_i = c_{2i-1}, b_i = c_{2i}, i = 1, \cdots, n-1$. So, there are $C_g^2$ independent solutions to (3). It can be said that the solutions are parametrized by $\wedge^g \mathcal{J}$ where $\mathcal{J}$ is Jacobi variety of the HES. On the other hand $\mathcal{J}$ is exactly the place where $g$ independent local integrals of classical periodical problem associated with $sl(2)$-type integrable equations live. That is why the above formulae are very suggestive from the point of view of the connection between local and nonlocal ($sl(2)$) symmetries in classical theory (these problems are discussed in [12]). It should be said also that the formulae (5) present a variant of those given in [13]. They are more explicit due to the peculiarity of $k = 0$ case.

Now, let us turn to the problem of braiding. The equations (3) are invariant under the permutation $\lambda_i \leftrightarrow \lambda_j$ and simultaneous permutation of the associated spaces. Let us denote the operation of the analytical continuation $\lambda_i \leftrightarrow \lambda_{i+1}$ and permutation of corresponding spaces by $B_{i,i+1}$. Then we are supposed to have a formula of the type:

$$B_{i,i+1} = \sum_{\gamma_1, \cdots, \gamma_{n-1}} C_{\gamma_1, \cdots, \gamma_{n-1}} f^{\gamma_1, \cdots, \gamma_{n-1}}(\lambda_1, \cdots, \lambda_2n)$$

where $C$ are some constants. It is well known nowadays that the braiding is described by the finite-dimensional quantum group $SL(2)_q$ [14], in our case $q = -1$. On the other hand due to the explicit formulae (5) braiding allows different interpretation being a consequence of recalculation of the canonical bases of homology of HES after the automorphism $\lambda_i \leftrightarrow \lambda_{i+1}$ is applied. So, we have an amusing connection between $SL(2)_{-1}$ and very classical mathematical problem.

2. Deformed case (Form factors for $su(2)$-invariant Thirring model).

Let us consider the deformed case now. First, we introduce the quantum $R$-matrix acting in $C^2 \otimes C^2$:

$$R_{1,2} = \frac{\beta - \pi i P_{1,2}}{\beta - \pi i}$$

where $P_{1,2}$ is permutation. This definition is different from the definition of crossing-symmetrical $S$-matrix used in [3,4] by certain factor. For the goals of the present paper we can ignore this factor which is responsible only for the multiplication of solutions of the deformed equations by $\prod_{i,j} \zeta(\beta_i - \beta_j)$ with some rather complicated function $\zeta$ [3]. Classical limit corresponds to $\beta = \frac{2\pi i}{h} \lambda$, $h \to +0$. Evidently,

$$R_{1,2}(\beta) = I - hr_{1,2}(\lambda) + O(h^2)$$

in this limit.

For $k = 0$ the deformed KZ equations [2] coincide essentially with the Form Factors Equations [3]. But for our goals it is more convenient to write them in the form used in [2]:

$$F(\beta_1, \cdots, \beta_j + 2\pi i, \cdots, \beta_{2n}) = R_{2n,j}(\beta_{2n} - \beta_j - 2\pi i) \cdots R_{j+1,j}(\beta_{j+1} - \beta_j - 2\pi i) \times R_{1,j}(\beta_j - \beta_j) \cdots R_{j-1,j}(\beta_{j-1} - \beta_j) \times F(\beta_1, \cdots, \beta_j, \cdots, \beta_{2n})$$

(7)
where $F \in (C^2)^{\otimes 2n}$. Formally, the equations (7) turn into (3) for the function

\[ f(\lambda_1, \cdots, \lambda_{2n}) \simeq_{h \to +i0} C(h)F\left(\frac{2\pi i \lambda_1}{h}, \cdots, \frac{2\pi i \lambda_{2n}}{h}\right) \]

where $C(h)$ is certain normalization constant which might be needed. One point should be emphasized here. We did very formal and potentially misleading computation. The real connection between the solutions of deformed and undeformed equations can be very complicated: the formula (8) can hardly present a real limit, more probably it is asymptotical formula (which is exactly the case in the situation under consideration), but there may be even no mathematically clear correspondence at all as it happens to be in $q$-deformed case for $|q| = 1$ (the solutions of classical KZ equations with trigonometrical $r$-matrix [15] do not describe even the asymptotics of the form factors in Sine-Gordon model).

The equations (7) allow explicit analytical solutions [3] which were found even before then those of KZ equations. But before proceeding further we have to introduce the notion of quasiconstant. Evidently, the solutions of (7) can be multiplied not only by constants, but also by arbitrary $2\pi i$-periodic functions of $\beta_j$. However the multiplication by arbitrary functions of the kind can disturb the braiding properties, that is why we restrict ourselves with $2\pi i$-periodic and symmetric functions.

The ring of these functions is generated by

\[ z_i \equiv \sigma_i(\exp(\beta_1), \cdots, \exp(\beta_{2n})) \]

where $\sigma_i$ is elementary symmetrical polynomial of degree $i$.

Now we shall describe the solutions. It should be mentioned that only special ones were used for applications to form factors which have to satisfy additional requirements. However in implicit form the general solutions are contained in [3].

The usual base of the tensor product (4) is not useful for our goals, so we start with introduction of another basis, $w^{\varepsilon_1, \cdots, \varepsilon_{2n}}(\beta_1, \cdots, \beta_{2n})$. The elements of this basis satisfy the relations

\[
R_{i,i+1}(\beta_i - \beta_{i+1})w^{\varepsilon_1, \cdots, \varepsilon_{i-1}, 1, \varepsilon_i, \varepsilon_{i+1}, \varepsilon_{i+2}, \cdots, \varepsilon_{2n}}(\beta_1, \cdots, \beta_{i-1}, \beta_i, \beta_{i+1}, \cdots, \beta_{2n}) = P_{i,i+1}w^{\varepsilon_1, \cdots, \varepsilon_{i-1}, 1, \varepsilon_i, \varepsilon_{i+1}, \varepsilon_{i+2}, \cdots, \varepsilon_{2n}}(\beta_1, \cdots, \beta_{i-1}, \beta_i+1, \beta_{i+1}, \cdots, \beta_{2n})
\]

The vectors $w$ are explicitly described in [3]. For us it is essential that

\[ w^{\varepsilon_1, \cdots, \varepsilon_{2n}}(\beta_1, \cdots, \beta_{2n}) \to e^{\varepsilon_1, \cdots, \varepsilon_{2n}} \]

as $\beta_k = \frac{2\pi i}{h}\lambda_k$, $h \to 0$. So, in the limit $w$ reproduce usual basis of $(C^2)^{\otimes 2n}$. Let us denote the components of $F(\beta_1, \cdots, \beta_{2n})$ with respect to the basis $w$ by $F(\beta_1, \cdots, \beta_{2n})_{\varepsilon_1, \cdots, \varepsilon_{2n}}$. There are many solutions of (7) which are parametrized by sets of integers $\{k_1, \cdots, k_{n-1}\} : 1 \leq k_1 < k_2 < \cdots < k_{n-1} \leq 2n - 1$. We again consider only one-dimensional with respect to global $sl(2)$ solutions. The formulae for the solutions look as follows:

\[
F^{k_1, \cdots, k_{n-1}}(\beta_1, \cdots, \beta_{2n})_{\varepsilon_1, \cdots, \varepsilon_{2n}} = \prod (\beta^+ - \beta^-)^{-1} \times \det || (z_{2n})^{-1/2} z_k \int_{-\infty}^{\infty} \Omega_j(\alpha|B^-|B^+)e^{(n-k_i)\alpha} d\alpha ||_{(n-1) \times (n-1)}
\]
the operation of analytical continuation 

\[ \beta \]

by \( R \)

Consider that more formally. The equations (7) are invariant under the permutation

\[ \text{equations being a meromorphic functions to have nontrivial braiding}. \]

Let us consider that more formally. The equations (7) are invariant under the permutation \( \lambda_i = \lambda_j \). So, it is very hard for the solutions of deformed equations being a meromorphic functions to have nontrivial braiding. Let us consider that more formally. The equations (7) are invariant under the permutation \( \beta_i \leftrightarrow \beta_j \) with simultaneous multiplication of \( F \) by \( R_{i+1|i}(\beta_{i+1} - \beta_i) \). Let us denote the operation of analytical continuation \( \beta_i \leftrightarrow \beta_j \) together with multiplication of \( F \) by \( R_{i+1|i}(\beta_{i+1} - \beta_i) \) by \( B_{i,j} \).

\[
B_{i,j} = \sum_{k_1', \ldots, k_{n-1}'} C_{k_1', \ldots, k_{n-1}'}^{k_1, \ldots, k_{n-1}} F^{k_1', \ldots, k_{n-1}'}(\beta_1, \ldots, \beta_{2n})
\]

where \( \sigma_m \) are elementary symmetrical polynomials of degree \( m \). The limitations on \( k_1, \ldots, k_{n-1} \) are due to the requirement of the convergency of the integrals, notice that

\[
\Gamma\left(\frac{1}{4} + \frac{\alpha}{2\pi i}\right)\Gamma\left(\frac{1}{4} - \frac{\alpha}{2\pi i}\right) \sim_{\alpha \to \pm \infty} |\alpha|^{-1/2} \exp\left(-\frac{1}{2} |\alpha|\right)
\]

The quasiconstant terms in the formula (9) are introduced for the sake of better classical limit.

Just like in undeformed case the real number of independent solutions is less than the number of partitions \( k_1, \ldots, k_{n-1} \) due to the following remarkable identity [3]:

\[
\sum_{i=1}^{n} z_{2i-1} \int_{-\infty}^{\infty} \Omega_j(\alpha|B^-|B^+) e^{(n-(2i-1))\sigma_i} d\alpha = 0
\]

\[
\forall j, B^-, B^+
\]

This identity can be considered as exact deformation of (6). After the relation (11) is imposed we find \( C_{2n-2}^{n-1} \) independent solutions of (7) which is exactly the same as in undeformed case. The physically important solutions of (7) which describe the energy-momentum tensor for \( su(2) \)-invariant Thirring model correspond to the following special choice of \( k_1, \ldots, k_{n-1}; k_i = 2i-1, \ i = 1, \ldots, n-1 \).

Let us now turn to the problem of braiding. Notice first that the functions (9) are meromorphic functions of all their arguments with essential singularities at infinity. This property differs them from the solutions of undeformed equations which have branching points at \( \lambda_i = \lambda_j \). So, it is very hard for the solutions of deformed equations being a meromorphic functions to have nontrivial braiding. Let us consider that more formally. The equations (7) are invariant under the permutation \( \beta_i \leftrightarrow \beta_j \) with simultaneous multiplication of \( F \) by \( R_{i+1|i}(\beta_{i+1} - \beta_i) \). Let us denote the operation of analytical continuation \( \beta_i \leftrightarrow \beta_j \) together with multiplication of \( F \) by \( R_{i+1|i}(\beta_{i+1} - \beta_i) \) by \( B_{i,j} \). We are supposed to have a relation of the kind:

\[
B_{i,j} = \sum_{k_1', \ldots, k_{n-1}'} C_{k_1', \ldots, k_{n-1}'}^{k_1, \ldots, k_{n-1}} F^{k_1', \ldots, k_{n-1}'}(\beta_1, \ldots, \beta_{2n})
\]
But it is quite clear from (9) that

\[ C_{k_1', \ldots, k_{n-1}'}^{k_1, \ldots, k_{n-1}} = \delta_{k_1', \ldots, k_{n-1}'}^{k_1, \ldots, k_{n-1}} \]

so, there is no mixing different solutions braiding for the equations (7), and we could impose the symmetry condition from the very beginning as it is done in [3]. This remarkable difference between the solutions of deformed and undeformed equations makes us to consider more carefully the classical limit \( \beta = \frac{2\pi i}{h} \), \( h \to +i0 \).

3. Classical limit.

To start this section we shall first formulate the results of it. There is one-to-one correspondence between the solutions of deformed and undeformed equations. But the correspondence is asymptotical: every particular solution of (3) describes the asymptotics of one solution of (7) when \( h \to +i0 \). The asymptotical character of the correspondence explains the differences of the properties of the solutions.

Before proceeding further let us illustrate the last statement with simple example. Consider the function:

\[ f(x) = \text{ch}(x/2)\Gamma(\alpha + \frac{x}{2\pi i})\Gamma(\alpha - \frac{x}{2\pi i}) \]

which satisfies the functional equation:

\[ f(x + 2\pi i) = \frac{2\pi i\alpha + x}{2\pi i(1 - \alpha) + x}f(x) \]

(12)

Evidently, \( f(x) \) is meromorphic function of \( x \) with essential singularity at the infinity. Suppose now \( x \in \mathbb{R} \) and consider the asymptotics \( x = \Lambda y, \Lambda \to \infty \):

\[ f(\Lambda y) \sim \Lambda^\kappa (|y|)^\kappa, \quad \kappa = 2\alpha - 1 \]

as a consequence of Stirling formula. So, the asymptotics is not analytical in \( y \) (contains \( |y| \)). Consider now \( y > 0 \), calculate the asymptotics, and then continue it analytically the result being:

\[ f_0(y) = \Lambda^\kappa (y)^\kappa \]

which has branching point at \( y = 0 \) and is not even but satisfies the formal limit of the equation (12):

\[ \frac{d}{dy}f_0(y) = \kappa \frac{y}{y}f_0(y) \]

This is the type of phenomena we deal with. The mechanism responsible for them is the concentration of infinite sequences of poles into cuts.

Let us calculate now the asymptotics of (9) for \( \beta_j = \frac{2\pi i}{h} \lambda_j, \quad h \to +i0 \). We have first to order \( \lambda_j \) in certain way, let it be \( \lambda_1 < \lambda_2 < \cdots < \lambda_{2n} \). The main problem is in estimating of the integrals in the determinant. The way of doing that is explained in [4,16]. Let us take one of the integrals:

\[ (z_{2n})^{-1/2}z_{2k_i} \int_{-\infty}^{\infty} \Omega_j(\alpha|B^-|B^+)e^{(n-k_i)\alpha}d\alpha \]
We have rescaled \( \beta_j \) as \( \frac{2\pi i}{h} \lambda_j \), let us also rescale \( \alpha \) as \( \frac{2\pi i}{h} \tau \). The function \( \Gamma(\frac{1}{4} + \frac{\alpha - \beta_j}{2\pi i}) \Gamma(\frac{1}{4} - \frac{\alpha - \beta_j}{2\pi i}) \) behaves as \( h \to +0 \) as follows:

\[
\Gamma\left(\frac{1}{4} + \frac{\alpha - \beta_j}{2\pi i}\right) \Gamma\left(\frac{1}{4} - \frac{\alpha - \beta_j}{2\pi i}\right) \sim |h|^{1/2} (|\tau - \lambda_j|)^{-1/2} \exp\left(-\frac{1}{2|h|} |\tau - \lambda_j|\right)
\]

Hence, we have very rapid exponential behaviour, so, let us concentrate on it for a moment neglecting the powers. Inside the segment \( \lambda_m < \tau < \lambda_{m+1} \) the exponential behaviour of the integrand is the following

\[
\exp\left(\frac{1}{|h|} ((2n - m - k_i)\tau + 1/2 \sum_{k=1}^{m} \lambda_k - 1/2 \sum_{k=m+1}^{2n} \lambda_k)\right)
\]

which means that for \( 2n - m < k_i \) the integral is estimated due to the integration by parts by the value of the integrand at the left, while for \( 2n - m > k_i \) – at the right end of the segment. The segment \( \lambda_{2n-k_i} < \tau < \lambda_{2n-k_i+1} \) is of special importance: here is the only place where the integration survives. Certainly these reasonings should be supported by study of the power contributions. Due to (10) there are singularities at \( \tau = \lambda_i \), but fortunately they are integrable, so, they only change \( h \) in the integration by parts estimation to \( h^{1/2} \). Combining all that, and performing exact calculation we draw the following conclusion: the main contribution to the asymptotics is provided by the integral over \( \lambda_{2n-k_i} < \tau < \lambda_{2n-k_i+1} \), the contributions from \( \lambda_m < \tau < \lambda_{m+1} \), \( m > 2n - k_i + 1, m < 2n - k_i - 1 \) are exponentially small, while the integrals over \( \lambda_{2n-k_i+1} < \tau < \lambda_{2n-k_i+1+1} \) are of the order \( h^{1/2} \) with respect to the leading one. So, the final result is the following:

\[
(z_{2n})^{-1/2} z_{k_i} \int_{-\infty}^{\infty} \Omega_j(\alpha|B^-|B^+) e^{(n-k_i)\alpha} d\alpha
\]

\[
= h^{-(j-1)} \left[ \int_{\lambda_{2n-k_i+1}}^{\lambda_{2n-k_i}} \omega_j(\tau|\Lambda^-|\Lambda^+) d\tau + \mathcal{O}(h^{1/2}) \right]
\]

We used the fact that the polynomial part of \( \Omega_j \) evidently tends to that of \( \omega_j \) in the limit. Evidently,

\[
\int_{\lambda_m}^{\lambda_{m+1}} \omega_j(\tau|\Lambda^-|\Lambda^+) d\tau = 1/2 \int_{c_i} \omega_j(\tau|\Lambda^-|\Lambda^+) d\tau
\]

where \( c_i \) are the cycles on HES introduced in Section 1.

Thus,

\[
F^{k_1, \ldots, k_{n-1}}(\beta_1, \ldots, \beta_{2n}) = (1/2)^{n-1} h^{(n-1)(n-2)/2} \left[ f^{\gamma_1, \ldots, \gamma_{n-1}}(\lambda_1, \ldots, \lambda_{2n}) + \mathcal{O}(h^{1/2}) \right]
\]

where \( \gamma_i = c_{2n-k_i} \). The asymptotical formula (13) proves the statements formulated at the beginning of this section. Actually, the asymptotical correspondence is so explicit that it makes us to suspect certain structure of “quantized” HES underlying it. The differentials

\[
\alpha^j \prod_{j=1}^{2n} \Gamma\left(\frac{1}{4} + \frac{\alpha - \beta_j}{2\pi i}\right) \Gamma\left(\frac{1}{4} - \frac{\alpha - \beta_j}{2\pi i}\right) d\alpha, \quad 0 \leq j \leq n - 2
\]
play the role of first kind (regular holomorphic) differentials. The differentials

$$\Omega_j(\alpha|B^-|B^+)d\alpha, \quad 1 \leq j \leq n - 1$$

for any particular partition into $B^-$ and $B^+$ can be taken for the base of holomorphic differentials singular at infinity. The singular part of the differentials is independent on the partition:

$$\Omega_j(\alpha|B^-|B^+)d\alpha - \Omega_j(\alpha|B'^-|B'^+)d\alpha = \text{"first kind"}$$

There is the following correspondence between the integrals:

$$\int_{c_{2n-1}} \leftrightarrow z_{2n}^{-1/2} \int_{-\infty}^{+\infty} \exp((n-l)\alpha)$$

The relation (11) is very important from that point of view.

**Conclusion.**

Let us formulate several challenging problems arising in the connection with the matters considered in the paper.

1. As it has been already mentioned there should be a connection between $k = 0$ KZ equations and finite-gap solutions of classical integrable equations. This connection should imply beautiful relation between local integrals and nonlocal ones. The latter are associated to monodromy being described in our case by $\hat{sl}(2)$ while the first ones are associated with flows on the Jacobi variety of the HES. This is exactly the kind of things we deal with for $k = 0$ KZ equations.

2. The solutions of $k = 0$ KZ equations provide enough amount of data to describe the HES: all the periods of holomorphic differentials can be extracted from them. It would be interesting to understand more about this connection of algebraic and analytic objects.

3. When the connection between the loop algebra and HES is understood it should be possible to understand the “quantization” of HES mentioned above, because algebraic mechanism responsible for that is the deformation of the loop algebra into the double of Yangian.

4. The problem of generalization of our results is also important. HES arise in connection with $\hat{sl}(2)$. In the paper [17] (see also[16]) the form factors of $su(N)$-invariant Thirring model were calculated which satisfy from modern point of view the deformed $sl(N), k = 0$ KZ equations. The solutions are given in terms of integrals which can be regarded as deformations of periods of differentials on the surfaces with the branching points of order $N$. Is it possible to deal with more general type of surfaces and what kind of algebraic structure corresponds to them?
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