Relative index theorem in $K$-homology

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Abstract. We prove an analog of Gromov–Lawson type relative index theorems for $K$-homology classes.

Introduction

Let $M$ and $M'$ be two manifolds coinciding outside some subsets $Q \subset M$ and $Q' \subset M'$ (i.e., $M \setminus Q$ and $M' \setminus Q'$ are identified with each other), and let $D$ and $D'$ be elliptic operators on $M$ and $M'$, respectively, coinciding on $M \setminus Q \simeq M' \setminus Q'$. The difference $\text{ind } D - \text{ind } D'$ of their indices is called the relative index of $D$ and $D'$. A relative index theorem is a statement of the following type: the relative index is independent of the structure of $M$ and $M'$, as well as of $D$ and $D'$, on the set where they coincide, i.e., on $M \setminus Q$; in other words, to compute the relative index, it suffices to know the structure of $D$ and $D'$ on $Q$ and $Q'$, respectively. Such theorems are trivial for smooth closed manifolds (owing to the existence of a local index formula; e.g., see [1]), but they are informative in more general cases. For example, a relative index theorem for Dirac operators on complete noncompact Riemannian manifolds was proved in the famous paper [2] by Gromov and Lawson. For further examples, we refer the reader to the paper [3], where the relative index theorem was proved in a rather general abstract framework that not only included many of the earlier known theorems as special cases but also permitted one to obtain a number of index formulas for elliptic differential operators and Fourier integral operators on manifolds with singularities (see [4]). Note, however, that index is not the only homotopy invariant of elliptic operators, and hence it is of interest to obtain locality theorems for broader sets of invariants. There are various directions in which to generalize the relative index theorem. For example, Bunke [5] considered Dirac operators acting on sections of projective bundles of Hilbert $B$-modules over a complete noncompact Riemannian manifold, where $B$ is a $C^*$-algebra, and obtained a relative index theorem for such operators, the index being an element of the $K$-group of $B$. Here we solve a different problem. If the elliptic operators in question are local with respect to some $C^*$-algebra $A$, then it is natural to ask how the corresponding classes in the $K$-homology of $A$ vary under a “local” variation of the operator. Here the algebra $A$ is not assumed to be commutative, and accordingly, localization is based on ideals in $A$. It turns out—which is the main result of the paper—that this variation obeys the same laws as the relative index
in the “classical” theorems does. That is why we still refer to our theorem as a “relative index theorem,” even though it deals with $K$-homology classes rather than the index. All results are stated in terms of Fredholm modules; for the standard construction that assigns a Fredholm module to an elliptic operator, we refer the reader to the literature (e.g., see [6]).

1. $K$-homology

Recall the definition of $K$-homology groups of a $C^*$-algebra $A$ (see [6] Chap. 8)). A Fredholm module over $A$ is a triple $x = (\rho, H, F)$, where $H$ is a Hilbert space, $\rho: A \to \mathfrak{B}(H)$ is a representation of $A$ on $H$, and $F \in \mathfrak{B}(H)$ is an operator such that $[F, \rho(\varphi)] \sim 0$ $\forall \varphi \in A$ (locality), $F \approx F^*$, $F^2 \approx 1$, (1)

where $\sim$ stands for equality modulo compact operators and $\approx$ for equality modulo locally compact operators, i.e., operators $C$ such that the operators $\rho(\varphi)C$ and $C\rho(\varphi)$ are compact for every $\varphi \in A$. Two Fredholm modules $(\rho, H, F_0)$ and $(\rho, H, F_1)$ corresponding to one and the same representation $\rho$ are said to be homotopic if they can be embedded in a family $(\rho, H, F_t)$, $t \in [0, 1]$, of Fredholm modules such that the function $t \mapsto F_t$ is operator norm continuous. A Fredholm module is said to be degenerate if all relations in (1) are satisfied exactly rather than modulo (locally) compact operators. We say that two Fredholm modules $x$ and $x'$ are equivalent if there exists a degenerate module $x''$ such that the modules $x \oplus x''$ and $x' \oplus x''$ are unitarily equivalent to homotopic Fredholm modules. The set of equivalence classes of Fredholm modules is denoted by $K^1(A)$; the direct sum of modules induces a structure of an abelian group on $K^1(A)$, which is called the (odd) $K$-homology group of $A$. The definition of the even $K$-homology group $K^0(A)$ is completely similar; here one considers graded Fredholm modules, i.e., ones equipped with the following additional structure: the space $H$ is $\mathbb{Z}_2$-graded, $H = H_+ \oplus H_-$, the representation $\rho$ is even (i.e., preserves the grading, $\rho(A)H_\pm \subset H_\pm$), and the operator $F$ is odd (i.e., $FH_+ \subset H_-$ and $FH_- \subset H_+$).

The results stated below hold for $K^0(A)$ as well as $K^1(A)$, and it is tacitly assumed throughout that all Fredholm modules involved are graded in the first case. For brevity, we often write $\varphi$ rather than $\rho(\varphi)$; which representation is meant is always clear from the context.

2. Fredholm modules agreeing on an ideal

Let $x = (\rho, H, F)$ and $\bar{x} = (\bar{\rho}, \bar{H}, \bar{F})$ be Fredholm modules over $A$, and let $J \subset A$ be an ideal. The orthogonal projections $P: H \to H_0$, where $H_0 = JH \subset H$, and $\bar{P}: \bar{H} \to \bar{H}_0$, where $\bar{H}_0 = J\bar{H}$, commute with the action of $A$.

**Definition 1.** Given an operator $T: H_0 \to \bar{H}_0$ intertwining the representations $\rho$ and $\bar{\rho}$, preserving the grading in the graded case, and satisfying $TPFP^* \approx \bar{P}\bar{F}P$, we say that $x$ and $\bar{x}$ agree on the ideal $J$.

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1The subspace $JH$, as well as the subspaces $J_1H$ and $J_2H$ considered below, is closed. This is a special case of the general assertion that the subspace $BH$ of a Hilbert space $H$ equipped with a representation of a $C^*$-algebra $B$ is closed (see [6] pp. 25–26, Sec. 1.9.17).
3. Cutting and pasting

Let $J_1, J_2 \subset A$ be ideals such that $J_1 + J_2 = A$. Let $x$ and $\bar{x}$ be Fredholm modules over $A$ agreeing on the ideal $J = J_1 \cap J_2$, and assume that the representations $\rho$ and $\bar{\rho}$ are nondegenerate (i.e., $AH = H$ and $A\bar{H} = \bar{H}$). Then one can define a Fredholm module $x \circ \bar{x}$ obtained, informally speaking, by “pasting together over $J$ the part of $x$ corresponding to $J_1$ with the part of $\bar{x}$ corresponding to $J_2$.” To this end, we represent $F$ (and, in a similar way, $\bar{F}$) by a $3 \times 3$ matrix associated with the decomposition of $H$ into the direct orthogonal sum of the $A$-invariant subspaces $H_0 = JH$, $H_1 = J_1 H \oplus H_0$ (the orthogonal complement), and $H_2 = J_2 H \oplus H_0$, $H = H_1 \oplus H_0 \oplus H_2$ (in this particular order).

We denote the orthogonal projection onto $H_j$ by $P_j$, $j = 0, 1, 2$. Note that $\varphi P_1 FP_2 = \varphi_1 P_1 FP_2 = P_1 \varphi_1 FP_2 \sim P_1 \varphi_1 P_2 = 0$ for any $\varphi \in A$; i.e., $P_1 FP_2 \approx 0$, and likewise $P_2 FP_1 \approx 0$, so that the desired representation can be written out as

$$F \approx \begin{pmatrix} a & b & 0 \\ b^* & c & d \\ 0 & d^* & e \end{pmatrix}, \quad \bar{F} \approx \begin{pmatrix} \bar{a} & \bar{b} & 0 \\ \bar{b}^* & \bar{c} & \bar{d} \\ 0 & \bar{d}^* & \bar{e} \end{pmatrix}, \quad a = a^*, \ c = c^*, \ e = e^*, \ \bar{a} = \bar{a^*}, \ \bar{c} = \bar{c^*}, \ \bar{e} = \bar{e^*}, \quad (2)$$

where all entries are local. The condition that $x$ and $\bar{x}$ agree on $J$ acquires the form $TcT^* \approx \bar{c}$. To simplify the notation, we identify $H_0$ with $\bar{H}_0$ via $T$; then we no longer write out $T$ explicitly, and the agreement condition on $J$ becomes $c \approx \bar{c}$.

Set

$$H \circ \bar{H} = H_1 \oplus H_0 \oplus \bar{H}_2, \ \rho \circ \bar{\rho} = \rho|_{H_1 \oplus H_0} \oplus \bar{\rho}|_{\bar{H}_2}, \ F \circ \bar{F} = \begin{pmatrix} a & b & 0 \\ b^* & c & d \\ 0 & d^* & e \end{pmatrix}. \quad (3)$$

PROPOSITION 1. The Fredholm module $x \circ \bar{x} = (\rho \circ \bar{\rho}, H \circ \bar{H}, F \circ \bar{F})$ over $A$ is well defined by formulas (2).

PROOF. In terms of the matrix in (2), the condition $F^2 \approx 1$ becomes

$$a^2 + bb^* \approx 1, \ ab + bc \approx 0, \ cd + de \approx 0, \ d^*d + e^2 \approx 1, \ bd \approx 0, \quad (4)$$

$$\varphi b^*b \sim \varphi_1(1 - c^2), \ \varphi dd^* \sim \varphi_2(1 - c^2) \quad \forall \varphi \in A, \quad (5)$$

and the last condition in (4) is satisfied automatically ($\varphi bd = (\varphi b)d \sim (b\varphi_1)d = b(\varphi_1 d) = 0$), while condition (5) follows from the fact that $b^*b + dd^* + c^2 \approx 1$. Similar relations hold for $\bar{F}$. To prove the proposition, it suffices to verify that $(F \circ \bar{F})^2 \sim 1$. (The other conditions in (4) obviously hold for $x \circ \bar{x}$.) The verification, after squaring the matrix, is reduced to routine calculations using the relation $c \approx \bar{c}$ and also relations (2)–(5) for $F$ and $\bar{F}$. For example, for the entry in the second line and second row, we obtain $\varphi ((F \circ \bar{F})^2)_{22} = \varphi (b^*b + c^2 + dd^*) \sim \varphi_1(1 - c^2) + \varphi c^2 + \varphi_2(1 - c^2) = \varphi_1, \ \forall \varphi \in A.$

The Fredholm module $\bar{x} \circ x$ is defined in a similar way.

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2In specific examples, some of the subspaces $H_0$, $H_1$, and $H_2$ may prove to be trivial (zero). Our argument remains valid in this case, but the result is not of much interest.

3From now on, for an arbitrary $\varphi \in A$, by $\varphi_1 \in J_1$ and $\varphi_2 \in J_2$ we denote arbitrary elements such that $\varphi = \varphi_1 + \varphi_2$.

4Here 1 stands for the identity operators on relevant subspaces.
4. Relative index theorem

Now we are in a position to state our main result.

**Theorem 1.** Under the assumptions of Sec. 3, one has

\[ [x ⨿ \tilde{x}] - [x] = [\tilde{x}] - [\tilde{x} ⨿ x], \]

where \([y] \in K^*(A)\) is the element defined by a Fredholm module \(y\).

Identity (6) means that the difference of \(K\)-homology classes resulting from the nonagreement of Fredholm modules over the ideal \(J_2\) is independent of the structure of these modules over the ideal \(J_1\) (where they agree).

**Remark 1.** As far as the author is aware, the result is new even for a commutative algebra \(A\) in that relation (6) is established for elements of the \(K\)-homology group rather than for the indices of the operators in question. (Note, however, that this was essentially done “behind the scenes” in [5] for the case in which \(A\) is a function algebra on a complete noncompact Riemannian manifold and the Fredholm modules correspond to some Dirac type operators.) The classical relative index theorems can be obtained from our result if one assumes that \(A\) is a unital function algebra: it suffices to use the homomorphism \(\text{ind} : K_0(A) \to K_0(C) \simeq \mathbb{Z}\) corresponding to the natural embedding of \(C\) in \(A\). Thus, the theorem stated above can be viewed as a natural generalization of relative index theorems in the framework of noncommutative geometry.

Note also that a similar theorem holds in Kasparov’s \(KK\)-theory. It is considered in a separate paper [7]. The reason for separate analysis is that although these two theorems do overlap, they are not corollaries of one another. Namely, both theorems deal with elements of \(KK(A, B)\), but

- (This paper) \(B = C\), and \(A\) is arbitrary.
- (7) \(B\) is arbitrary, and \(A\) is unital.

Moreover, even the construction of the module \(F ⨿ \tilde{F}\) in [7] is different: in the present paper, we use projections, but in [7] we are forced to use a partition of unity in \(A\), because appropriate projections do not necessarily exist in Kasparov \((A, B)\)-modules.

**Outline of the proof.** It suffices to deform the Fredholm module \(x \oplus \tilde{x}\) to a module that is unitarily equivalent to the module \((x ⨿ \tilde{x}) \oplus (\tilde{x} ⨿ x)\). The homotopy is given by the family of Fredholm modules \((\rho ⨿ \tilde{p}, H \oplus \tilde{H}, F_t)\), \(t \in [0, \pi/2]\), where the operator \(F_t\) is specified in the direct sum decomposition \(H \oplus \tilde{H} = H_1 \oplus H_0 \oplus H_2 \oplus \tilde{H}_1 \oplus \tilde{H}_0 \oplus \tilde{H}_2\) by the 6 \(\times\) 6 block matrix

\[
F_t = \begin{pmatrix}
a & b & 0 & 0 & 0 & 0 \\
b^* & c & d \cos t & 0 & 0 & -\tilde{d} \sin t \\
0 & d^* \cos t & e & 0 & d^* \sin t & 0 \\
0 & 0 & 0 & a & \tilde{b} & 0 \\
0 & 0 & d \sin t & \tilde{b}^* & c & \tilde{d} \cos t \\
0 & -\tilde{d}^* \sin t & 0 & 0 & \tilde{d}^* \cos t & \tilde{c}
\end{pmatrix}. \tag{7}
\]
The first and second conditions in (1) are obvious for $F_t$, and the third condition ($F_t^2 = 1$) can be verified by routine computations. Next, $F_0 = F \oplus \tilde{F}$ and

$$F_{\pi/2} = \begin{pmatrix} a & b & 0 & 0 & 0 & 0 \\ b^* & c & 0 & 0 & 0 & -d \\ 0 & 0 & e & 0 & d^* & 0 \\ 0 & 0 & 0 & \tilde{a} & \tilde{b} & 0 \\ 0 & 0 & d & \tilde{b}^* & c & 0 \\ 0 & -\tilde{d}^* & 0 & 0 & 0 & \tilde{c} \end{pmatrix} = U^* \begin{pmatrix} a & b & 0 & 0 & 0 & 0 \\ b^* & c & \tilde{d} & 0 & 0 & 0 \\ 0 & \tilde{d}^* & \tilde{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{a} & \tilde{b} & 0 \\ 0 & 0 & 0 & b^* & c & d \\ 0 & 0 & 0 & 0 & d^* & e \end{pmatrix} U, \quad (8)$$

where the unitary operator

$$U : H \oplus \tilde{H} \equiv H_1 \oplus H_0 \oplus H_2 \oplus \tilde{H}_1 \oplus H_0 \oplus \tilde{H}_2 \longrightarrow H_1 \oplus H_0 \oplus \tilde{H}_2 \oplus \tilde{H}_1 \oplus H_0 \oplus H_2 \equiv (H \oplus \tilde{H}) \oplus (\tilde{H} \oplus H)$$

interchanges the third and sixth components and then multiplies the third component by $-1$. Thus, $F_{\pi/2} = U^*((F \oplus \tilde{F}) \oplus (\tilde{F} \oplus F))U$, as desired. \qed

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