Generalised Cumulant Correlators and Hierarchical Clustering

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ABSTRACT

The cumulant correlators, $C_{pq}$, are statistical quantities that generalise the better-known $S_p$ parameters; the former are obtained from the two-point probability distribution function of the density fluctuations while the latter describe only the one-point distribution. If galaxy clustering develops from Gaussian initial fluctuations and a small-angle approximation is adopted, standard perturbative methods suggest a particular hierarchical relationship of the $C_{pq}$ for projected clustering data, such as the APM survey. We establish the usefulness of the two-point cumulants for describing hierarchical clustering by comparing such calculations against available measurements from projected catalogs, finding very good agreement. We extend the idea of cumulant correlators to multi-point generalised cumulant correlators (related to the higher-order correlation functions). We extend previous studies in the highly nonlinear regime to express the generalised cumulant correlators in terms of the underlying “tree amplitudes” of hierarchical scaling models. Such considerations lead to a technique for determining these hierarchical amplitudes, to arbitrary order, from galaxy catalogs and numerical simulations. Knowledge of these amplitudes yields important clues about the phenomenology of gravitational clustering. For instance, we show that three-point cumulant correlator can be used to separate the tree amplitudes up to sixth order. We also combine the particular hierarchical \textit{ansatz} of Bernardeau & Schaeffer (1992) with extended and hyperextended perturbation theory to infer values of the tree amplitudes in the highly nonlinear regime.

Key words: Cosmology: theory – large-scale structure of the Universe – Methods: analytical

1 INTRODUCTION

The Newtonian gravitational force is scale-free. This leads one directly to the suspicion that, at least in the strong clustering limit where memory of any specific length scale set by the initial conditions has been obliterated, the complex pattern of gravity-driven clustering should display relatively simple scaling properties. One way such scaling should manifest itself is in the behaviour of the hierarchy of $N$-point correlation functions (e.g. Peebles 1980). This has led to the formulation of the so-called hierarchical \textit{ansatz} for gravitational clustering which, in its most general form, can be written,

$$\xi_N(\lambda r_1, \ldots, \lambda r_N) = \lambda^{-\gamma(N-1)} \xi_N(r_1, \ldots, r_N)$$

(e.g. Balian & Schaeffer 1989), where $\xi_N$ is the $N$-point correlation function for particles located at positions $r_1\ldots r_N$, and $\gamma$ is the negative slope of two-point correlation function $\xi(r_i, r_j)$. A more restrictive, but still quite general, algebraic form is often used in the literature, in which the correlation function is constructed from linear superposition of all possible topologies of “tree” diagrams connecting $N$ points with $N - 1$ edges. This model permits the association of a number with each distinct tree topology (the tree amplitude) with the amplitudes themselves left arbitrary (Balian & Schaeffer 1989). In other words,

$$\xi_N(r_1, \ldots, r_N) = \sum_{\alpha, N-\text{trees}} Q_{N,\alpha} \sum_{\text{labellings}} (N-1) \prod \xi(r_i, r_j),$$

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where the last term is the two-point correlation function for each pair of particles \((i, j)\). It is important to note that this model is not the only possible way to formulate hierarchical clustering. In general, the amplitudes \(Q_{N,\alpha}\) could depend on the geometry of a specific configuration of \(N\) points, and not just its topology as is assumed here. However, there are some indications, at least at the level of three points, that the amplitudes do become independent of configuration in the highly non-linear regime (Scoccimarro et al. 1998). There are some indications that they become independent of level of non-linearity too at highly non-linear regime (Munshi et al. 1999; see Colombi et al. 1996 for a different view-point).

Many different models of hierarchical scaling, largely consisting of particular choices for the \(Q_{N,\alpha}\), can be found in the literature. In principle, it should be possible to derive these amplitudes from the underlying gravitational physics encoded in the hierarchy of BBGKY equations that describe the dynamical origin of gravity-driven clustering (Davis & Peebles 1977; Fry 1984; Hamilton 1988). However, no general solution to these equations exists, so these various approximate models are employed in order to get some insight into the physics of gravitational clustering in highly nonlinear regime and to aid interpretation of empirical galaxy clustering data.

Despite the increasing availability of large samples of galaxy redshifts, the most precise statistical analyses of galaxy clustering are still performed using projected (angular) samples. The extremely large size of projected galaxy surveys, together with the relatively easy-to-handle effect of projection, makes such samples ideal for the extraction of descriptors such as higher-order cumulants. The APM survey (Maddox et al. 1990; Maddox, Sutherland & Efstathiou 1990; Maddox, Efstathiou & Sutherland 1996), which contains more than 1.3 million objects, is at present the largest angular catalogue available. Such is the amount of useful information in this sample that Gaztanaga (1994) was able to extract estimates of the so-called \(S_p\) parameters (see below), for \(p \leq 9\) from it. Such an analysis is at present beyond the scope of galaxy redshift surveys (Bouchet et al. 1993), at least until the Sloan Digital Sky Survey and Anglo-Australian 2DF survey are completed. Projected catalogs are also free from redshift-space distortions, a fact which can be exploited to yield accurate estimates of real-space clustering phenomena in a relatively unambiguous way. The obvious difficulty associated with projected catalogs is related to the fact that any projected angular scale corresponds to a superposition of physical length scales which have which represent different levels of nonlinearity. Limber (1954) pioneered the study of relating angular correlation functions to their real-space counterparts. These inversion techniques and their small angle approximations are now known to be very efficient for the extraction of information from projected catalogs (e.g. Peebles 1980; Baugh & Efstathiou 1993).

Many statistical studies of large-scale structure focus (in one way or another) on the determination of one-point cumulants for cells of a particular size, which are essentially volume averages of correlation functions over the cell. The second-order cumulant, for example, is the variance of cell-counts, and which is the average of the two-point correlation function over the cell. The standard \(S_p\) parameters (see below) are based on the cumulants of the one-point probability distribution of density fluctuations, \(f_1(\delta)\), and can be expressed in terms of volume averages of the higher-order correlation functions in an analogous fashion. Here and throughout the paper \(\delta(x)\) denotes the dimensionless density contrast, defined by

\[
\delta(x) = \frac{\rho(x) - \rho_0}{\rho_0},
\]

where \(\rho_0\) is the mean density of matter.

Szapudi & Szalay (1997) recently proposed a new class of statistical measures, known as the cumulant correlators, which generalises the set of \(S_p\) parameters mentioned above. The cumulant correlators \(C_{pq}\) are analogous quantities obtained from the two-point (bivariate) distribution \(f_2(\delta_1, \delta_2)\). They form a symmetric matrix, \(C_{pq}\), and estimation of these quantities from data is a very similar task to the extraction of the standard two-point correlation function. Furthermore, using the factorial moments of the discrete count probability density function makes it possible to subtract the effect of Poisson shot noise very efficiently. Szapudi & Szalay (1997) extracted estimates of these quantities for \(p + q \leq 5\) from the APM galaxy survey.

Theoretical work on the cumulant correlators has been performed by Bernardeau (1995), who obtained predictions for the \(C_{pq}\) in three-dimensional real space for clustering developing from Gaussian initial conditions and in the limit of large spatial separations. Interestingly, a relatively simple scaling hierarchy is expected to develop under these conditions, identical to the scaling of their one-point counterparts the \(S_p\) parameters, potentially furnishing a simple and powerful diagnostic of the idea of gravitational instability and, perhaps, of the extent to which galaxy bias affects clustering statistics.

In this paper we study the generalisation of the cumulant correlators to multi-points and for arbitrary order. These statistics are constructed from the quantities \((\delta^p(x_1)\delta^q(x_2)\ldots\delta^s(x_l))\), but while the cumulant correlators have \(l = 2\) with \(p\) and \(q\) arbitrary, the multipoint correlators have no restriction on \(l\) or any of the integers \(p, q, \ldots, s\). We study the dependence of these quantities on the underlying hierarchical tree amplitudes, with a view to testing the different possible hierarchical models using these statistics. We also present the results of a calculation of the quasilinear behaviour of the cumulant correlators in projection, including the effects of observational selection. In Section 2 we outline our main theoretical results in highly nonlinear regime; Section 3 is devoted to the computation of two-point cumulant correlators in projected catalogues in quasi-linear regime. Theoretical results are tested against measurements from the APM survey in Section 4.

## 2 GENERALIZED CUMULANT CORRELATORS IN THE NONLINEAR REGIME

Such are the importance of hierarchical models in the understanding of gravitational clustering that it is important to learn which of choice of tree amplitudes in the model (2) corresponds with reality. This is best tackled by attempting to extract these amplitudes from numerical simulation data. However, the direct determination of higher-order correlations is complicated by
their possible dependence on shape as well as topology. Most numerical studies in this direction are therefore have concentrated on an indirect determination these quantities, using one-point cumulants or two-point cumulant correlators. It is known that cumulant correlators can be used to compute the amplitudes associated with two different types of topologies – the so-called “snake” and “star” topologies – at fourth order (Szapudi & Szalay 1997). However, at higher orders than this the number of different topologies rapidly increases and two-point cumulant correlators are no longer sufficient to effect a unique separation of all tree-amplitudes. The only possible way we can remedy the situation is by moving to the generalised cumulant correlators mentioned above. These are the natural generalisation of their two-point counterparts (Munshi et al. 1998a) and, as we discuss elsewhere, they are also related in a very interesting way to the statistics of collapsed objects, in much the same way as the two-point cumulant correlators are related to the natural bias associated with treating collapsed objects rather than the general distribution of mass (Bernardeau 1992; Munshi et al. 1998a).

The terminology in this field is a little confusing, so it is worth beginning with a brief explanation of notation in order to relate this work with the existing literature. For three points there is only one possible tree, and the unique three-point amplitude is generally denoted \( Q \). As we mentioned in passing, there are two possible topologies for connecting four points (respectively called the “snake” and the “star”). The amplitudes of these tree configurations are usually denoted \( R_a \) and \( R_b \). Following in this vein we introduce, for the distinct topologies at fifth order, the notations \( S_a, S_c \) and \( S_b \) for the “snake”, “star” and “hybrid” cases. At sixth order the number of different topologies is six (Fry 1984) and one can similarly use \( T_a, T_b, T_c, T_d, T_e \) and \( T_f \) to denote the amplitudes associated with them.

The order \( M \) of the multi-point cumulant correlator \( \langle \delta^M(x_1)\delta^M(x_2)\ldots\delta^M(x_i) \rangle \) is defined to be equal to \( p + q + \ldots + s \) and such a quantity of order \( M \) takes contributions from \( M \)-point correlation function. Cumulant correlators of a particular order will depend only on hierarchical amplitudes associated with topologies contributing to the same order, even though the number of spatial points \( l \) on which they depend may vary.

For example, the sixth-order two-point and three-point cumulant correlators will both depend on tree amplitudes associated with six-point correlation functions. For example, there are three different but “degenerate” three-point cumulant correlators of sixth order, a fact which stems from the number of different ways the integer six can be decomposed in three integers, i.e. \( 4 + 1 + 1 = 3 + 2 + 1 = 2 + 2 + 2 = 6 \). All of these three-point cumulant correlators in general can depend on all the six different tree-amplitudes associated with sixth order correlation function. Hence measurement of these three three-point cumulant correlators of sixth order will provide us three equations connecting the six tree amplitudes. The system of three equations with six variables is consequently indeterminate, and can only be solved by including three more equations connecting these amplitudes. But if we complement these equations with information provided by the two-point cumulant correlators of sixth order we get three more equations, corresponding to the fact that \( 1 + 5 = 2 + 4 = 3 + 3 = 6 \) and quantities also depend on exactly the same tree amplitudes. The resulting six equations in six variables will provide us with a fully-determined set of equations which can be solved to yield the amplitudes associated with all topologies of sixth order.

This example can be generalised. Counting the different number of degenerate multi-point cumulant correlators, for a given order and for a specific set of points, one can check that three-point cumulant correlators can in principle be used to determine all tree amplitudes associated with correlation functions up to and including sixth order. We have tabulated the different multi-point cumulants of at a given order to show which can be used to separate out the hierarchical amplitudes associated with correlation functions through equation (2).

### 2.1 Leading-order tree terms

If we express two degenerate two-point cumulant correlators of fourth order in the limit of large separations (or, in other words, the variance evaluated at each point, related to \( \xi^2 \), is much bigger than the covariance between the points, expressed as \( \xi_{ab} \)) we obtain following pair of equations:

\[
\langle \delta^4(x_1)\delta^4(x_2) \rangle = (3R_b + 6R_a)\xi_{ab}\xi^2 \quad \text{(4)}
\]

\[
\langle \delta^2(x_1)\delta^2(x_2) \rangle = 4R_b\xi_{ab}\xi^2 \quad \text{(5)}
\]

As explained earlier these two-point cumulant correlators depend only on the two amplitudes associated with the four-point correlation function, i.e. \( R_a \) and \( R_b \). They can solved simultaneously to compute values of these quantities.

At fifth order there are also two degenerate two-point cumulant correlators which depend on three tree-amplitudes:

\[
\langle \delta^4(x_1)\delta^4(x_2) \rangle = (4S_c + 36S_b + 24S_a)\xi_{ab}\xi^3 \quad \text{(6)}
\]

\[
\langle \delta^3(x_1)\delta^3(x_2) \rangle = (6S_b + 12S_a)\xi_{ab}\xi^3 \quad \text{(7)}
\]

To determine the amplitudes \( S_a, S_b \) and \( S_c \) the two-point cumulant correlators would be insufficient, so we have to move to the three-point quantities instead. These provide us another pair of equations with same tree-amplitudes:

\[
\langle \delta^2(x_1)\delta^2(x_2)\delta^2(x_3) \rangle = (4S_a\xi_{ab}\xi_{bc} + 4S_b\xi_{ab}\xi_{ac} + 4S_b\xi_{ac}\xi_{bc})\xi^2 \quad \text{(8)}
\]

\[
\langle \delta^3(x_1)\delta^3(x_2)\delta^3(x_3) \rangle = ((6S_b + 3S_a)\xi_{ab}\xi_{bc} + \xi_{ac}\xi_{bc}) + (3S_b + 18S_a)\xi_{ab}\xi_{ac}\xi^2 \quad \text{(9)}
\]

Any three of the equations (6) to (9) can be used to determine the tree-amplitudes \( S_a, S_b \) and \( S_c \). The other equation will be consistent but redundant, and can be used as a check. Of course, the one-point cumulant of same order always provide an additional constraint at every order.
There are three degenerate two-point cumulant correlators of sixth order and there are 6 distinct topologies:

\[
\langle \delta^4(x_1) \delta^2(x_2) \rangle = (48T_a + 48T_c + 24T_b + 8T_d) \xi_{ab} \xi_2^4
\]  

(10)

\[
\langle \delta^3(x_1) \delta^3(x_2) \rangle = (36T_b + 36T_a + 9T_c) \xi_{ab} \xi_2^4
\]  

(11)

\[
\langle \delta^5(x_1) \delta^1(x_2) \rangle = (120T_a + 180T_b + 180T_c + 80T_d + 60T_e + 57T_f) \xi_{ab} \xi_2^4
\]  

(12)

Combining these three equations with expressions for the three-point cumulant correlators at sixth order we can have another set of three equations which can be used to used to determine all the relevant tree-amplitudes.

\[
\langle \delta^2(x_1) \delta^2(x_2) \delta^2(x_3) \rangle = (8T_a + 8T_c) (\xi_{ab} \xi_{bc} + \xi_{ab} \xi_{ac} + \xi_{ac} \xi_{bc}) \xi_2^3
\]  

(13)

\[
\langle \delta(x_1) \delta(x_2) \delta^3(x_3) \rangle = ((18T_b + 12T_c + 6T_e) \xi_{ab} \xi_{bc} + (12T_b + 24T_e + 12T_c + 6T_d) \xi_{ab} \xi_{ac} + (6T_b + 12T_a) \xi_{ac} \xi_{bc}) \xi_2^3
\]  

(14)

\[
\langle \delta(x_1) \delta(x_2) \delta^4(x_3) \rangle = (24T_a + 24T_c + 12T_b + 4T_d) (\xi_{ab} \xi_{bc} + \xi_{ab} \xi_{ac} + \xi_{ac} \xi_{bc}) + (4T_f + 24T_e + 36T_d + 8T_d + 24T_a + 120T_b) \xi_{ac} \xi_{bc} \xi_2^3
\]  

(15)

This method can be extended to higher-order correlation functions and, combined with a judicious use of factorial correlators or factorial moments described in Munshi et al. (1998b), it provides the simplest way to determine the hierarchical amplitudes. It is however to be noted that we have neglected the small correction factors pointed out by Boschan et. al. (1994).

### 2.2 Generalised Cumulant Correlators in the Bernardeau-Schaeffer Ansatz

As explained earlier, although all hierarchical models (2) agree on the basic tree structure underpinning the correlation hierarchy describing the matter distribution, specific models differ from each other in the assumptions they make regarding the amplitudes associated with tree-topologies. One such model, which has been studied in great detail, was proposed by Bernardeau & Schaeffer (1992). They assumed that each vertex (or node) in the tree representation of correlation functions (2) carries an weight which depends only on the order of the vertex and on nothing else. Vertices of the same order appearing in different trees will carry exactly same weight. The amplitude of the whole tree is consequently always equal to the product of the weights assigned to each constituent node. Even with this rather simple assumption this model can predict many interesting features associated with collapsed objects, some of which have already been tested successfully against numerical simulation (Munshi et al. 1998b). More sophisticated models have also been constructed by assuming a specific form for the generating functions of these vertex amplitudes. Such models can make further predictions about many body statistics of collapsed objects.

Assuming that the tree amplitudes can be factorized in this way, it is possible to decompose the cumulant correlators of various orders in the following way (Bernardeau 1992; Munshi et. al. 1998a): for the two-point function of arbitrary order

\[
\langle \delta^p(x_a) \delta^q(x_b) \rangle = [C_{p1} \xi_{ab} C_{q1}] \bar{\xi}^{p+q-2};
\]  

(16)

for three points,

\[
\langle \delta^p(x_a) \delta^q(x_b) \delta^r(x_c) \rangle = [C_{q1} \xi_{ab} C_{p11} \xi_{ac} C_{r1} + \xi_{cb} C_{q12} \xi_{ac} C_{r1} + C_{p1} \xi_{ac} C_{r11} \xi_{bc} C_{p1}] \bar{\xi}^{p+q+r-3};
\]  

(17)

for four points

\[
\langle \delta^p(x_a) \delta^q(x_b) \delta^r(x_c) \delta^s(x_d) \rangle = [C_{q1} \xi_{ab} C_{p111} \xi_{ad} C_{r1} + \xi_{cd} C_{q12} \xi_{ad} C_{r1}] + (\text{cyclic permutations}) \bar{\xi}^{p+q+r+s-4};
\]  

(18)

and for five points

\[
\langle \delta^p(x_a) \delta^q(x_b) \delta^r(x_c) \delta^s(x_d) \rangle = [C_{p1111} \xi_{ab} C_{q1} \xi_{ac} C_{r1} + (\text{cyclic permutation}) + C_{p1} \xi_{ab} C_{q11} \xi_{ac} C_{r1} + (\text{cyclic permutation}) + C_{p1} \xi_{ab} C_{q11} \xi_{ac} C_{r1} + (\text{cyclic permutation})] \bar{\xi}^{p+q+r+s+t-5}
\]  

(19)

The vertex functions $C_{p1}$ are simplest to deal with when expressed in terms of their generating functions, e.g.

\[
\mu_1(t) = \sum_{p=1}^{\infty} \frac{C_{p1} t^p}{p!},
\]  

(20)

\[
\mu_2(t) = \sum_{p=1}^{\infty} \frac{C_{p1} t^p}{p!},
\]  

(21)

and so on. With these definitions it is possible to relate the generating functions $\Psi^p(t_1, \ldots, t_p)$ of the cumulant correlators with generating functions of the vertex weights (see Munshi et al. 1998a for more details):

\[
\Psi^{(2)}(t_1, t_2) = \mu_1(-t_1) \xi_{ab} \mu_1(-t_2)
\]  

(22)

\[
\Psi^{(3)}(t_1, t_2, t_3) = \mu_1(-t_1) \xi_{ab} \mu_2(-t_2) \xi_{ac} \mu_1(-t_3) + \ldots (\text{cyclic permutations})
\]  

(23)
Extended perturbation theory and hyper-extended perturbation theory have been suggested by Colombi et al. (1996) and a way that particular predictions using (hyper)extended perturbation techniques. Schaeffer (1992) for the highly non-linear regime with these predictions using (hyper)extended perturbation techniques. An analysis similar to the preceding one has been carried out by Szapudi & Szalay (1993a), using a different ansatz for the amplitudes of all topologies. We can improve upon this situation, by combining predictions of the hierarchical model of Bernard & Schaeffer (1992) for the highly non-linear regime with these predictions using (hyper)extended perturbation techniques. It turns out that this procedure always predicts that “star” topologies carry more weight than “snake” and “hybrid” topologies. For example, in this model, \( R_a \) (4th order) is generated by \( Q = Q_3 \) (third):

\[
R_a = Q^2, \tag{25}
\]

while \( R_b \) is a “new” amplitude at fourth order, which cannot be related to \( Q \). At fifth order,

\[
S_a = Q^3, S_b = QR_b \tag{26}
\]

but \( S_c \) is not constrained by \( Q \) or \( R_a \) or \( R_b \). Likewise,

\[
T_a = Q^4, T_b = Q^2 R_b, T_c = R_s, T_d = Q^2 R_b, \tag{27}
\]

with \( T_f \) free at this level. Tests of these predictions will become possible as larger simulation boxes become available. These will undoubtedly lead us to better understanding of how this model describes the nonlinear regime of gravitational clustering.

### 2.3 Generalised Cumulant Correlators in the Szapudi–Szalay Ansatz

An analysis similar to the preceding one has been carried out by Szapudi & Szalay (1993a), using a different ansatz for the tree-level amplitudes to that of Bernard & Schaeffer (1992). Szapudi & Szalay (1993b) subsequently also looked at the predicted statistics of over-dense cells in this light. In their analysis they assume that amplitudes associated with different topologies but with same number of vertices are always equal. For example this means that, at fourth order, all tree topologies will have an amplitude \( Q_4 \) which is a weighted average of different topologies of fourth order. Since there are 16 distinct configurations of 4 points, of which 4 are “star”, these means 16\( Q_4 \) = \( 4R_a + 4R_b \). Similarly, at fifth order, \( 125Q_5 = 60S_b + 60S_t + 5S_c \) and in general \( Q_{N,α} = Q_N \). The corresponding formulae are:

\[
(\delta^p(x_a)\delta^q(x_b)) = p^{p-1} q^{q-1} Q_{p+q} \xi_{ab} \xi_{p+q}^{-2} \tag{28}
\]

\[
(\delta^p(x_a)\delta^q(x_b)\delta^r(x_c)) = p^{p-1} q^{q-1} r^{r-1} Q_{p+q+r} \left[ p \xi_{ab} \xi_{ac} + q \xi_{ab} \xi_{bc} + r \xi_{ac} \xi_{bc} \right] \xi_{p+q+r}^{-3} \tag{29}
\]

\[
(\delta^p(x_a)\delta^q(x_b)\delta^r(x_c)\delta^s(x_d)) = p^{p-1} q^{q-1} r^{r-1} s^{s-1} Q_{p+q+r+s} \left[ p^2 \xi_{ab} \xi_{ac} \xi_{ad} + (\text{cyclic permutations}) \right] \xi_{p+q+r+s}^{-4} \tag{30}
\]

\[
(\delta^p(x_a)\delta^q(x_b)\delta^r(x_c)\delta^s(x_d)\delta^t(x_e)) = p^{p-1} q^{q-1} r^{r-1} s^{s-1} t^{t-1} Q_{p+q+r+s+t} \left[ p^3 \xi_{ab} \xi_{ac} \xi_{ad} \xi_{ae} + (\text{cyclic permutations}) \right] \xi_{p+q+r+s+t}^{-5} \tag{31}
\]

These results again furnish a simple test of this model. One simply needs to evaluate \( Q_{p+...+t} \) for different values of the particular \( p, q, ..., t \) and check if the parameter \( Q_{p+...+t} \) remains invariant when the individual \( p, q, ..., t \) are changed in such a way that \( p + ... + t \) remains constant.

Using two-point cumulant correlators, Munshi & Melott (1998) have recently tested predictions of these particular versions of the generic hierarchical ansatz against numerical simulations up-to fourth order. More detail studies using three-point factorial correlators up to sixth order will be presented elsewhere.

### 2.4 Hierarchical Amplitudes from Extended and Hyper-extended Perturbation Theory

Extended perturbation theory and hyper-extended perturbation theory have been suggested by Colombi et al. (1996) and Scoccimarro & Frieman (1998) respectively in attempts to obtain more accurate analytic predictions of the \( S_N \) parameters in the highly non-linear regime. Such extensions however are, however, silent concerning the specific contribution to \( S_N \) from distinct topologies. We can improve upon this situation, by combining predictions of the hierarchical model of Bernard & Schaeffer (1992) for the highly non-linear regime with these predictions using (hyper)extended perturbation techniques.
Figure 1. Predictions for hierarchical amplitudes of fourth order $R_a$ and $R_b$ and fifth order $S_a$, $S_b$ and $S_c$ are plotted as a function of lowest order hierarchical amplitude $Q$. Solid lines are predictions from hyper-extended perturbation theory (Scoccimarro & Frieman 1998), dotted lines represent extended perturbation theory (Colombi et al. 1996) and dashed lines represent predictions from specific hierarchical model of Bernardeau & Schaeffer (1992). The bottom most curve in each panel represents snake diagrams $R_a = Q^2$ and $S_a = Q^3$ which are same for all these approximations are same. The triplets of upper curves in left panel represent $R_b$ and in right panel they represent $S_c$ and $S_b$ respectively from top to bottom.

It may not be possible to extend tree-level perturbation techniques straightforwardly by changing spectral index $n$ to effective spectral index $n_{\text{eff}}$ in order to compute multi-point cumulant correlators. In principle, although these quantities can be computed for arbitrary points in the quasi-linear regime they will in general imply that hierarchical amplitudes are depend on shape parameters while, in the highly nonlinear regime hierarchical amplitudes are thought to be independent of shape. At least at the level of three-point this seems to have been already confirmed by some numerical experiments (Scoccimarro et al. 1998). Furthermore, it has been pointed out (Bernardeau 1996) that, in the quasilinear regime, tree-level perturbation theory for one–point cumulant of the smoothed density field $S_3$, implies $S_3 = 3Q$ but for $C_{21}$ is not equal to $2Q$ as one would have expected on the basis of the hierarchical ansatz (which applies to the unsmoothed tree-level perturbation theory). Clearly, one has to be careful about the effect of smoothing on the cumulant correlators. When combining the hierarchical ansatz with extended or hyperextended perturbation theory we have to keep these points in mind. It is nevertheless interesting to combine these two kinds of calculation in this way, as there are otherwise no analytical predictions for these quantities that can be tested against numerical simulations.

The results of this and the previous calculations are shown in Figure 1.

3 QUASI-LINEAR CUMULANT CORRELATORS FROM PROJECTED CATALOGS

In the quasi-linear regime, two-point cumulant correlators have already been studied in great detail using perturbation theory. The extension of such studies to multiple points, as is required for the generalisations we discuss in this paper, are difficult owing to the complicated shape dependence of the hierarchical amplitudes. For this Section, therefore, we shall focus on the (original) two-point version of these statistics.

Estimates of the cumulant correlators were extracted from the APM catalogue by Szapudi & Szalay (1997) and a related theoretical computation of projected cumulant correlators was done by Bernardeau et al. (1997). Let us now concentrate on the APM results in order to illustrate the excellent match between theory and observation.

The usual notation with which the two–point cumulant correlators are expressed is in terms of $C_{pq}$, are defined as (Bernardeau 1995; Szapudi & Szalay 1997)

$$\langle \delta(x_1)^p \delta(x_2)^q \rangle_c = C_{pq} \langle \delta^2(x) \rangle \langle \delta(x_1) \delta(x_2) \rangle^{p+q-2}. \quad (32)$$

An alternative parameterisation, which we shall in fact use, is in terms of the quantities

$$Q_{pq} = \frac{C_{pq}}{p^{p-1}q^{q-1}} \quad (33)$$
We assume the initial density contrast $\delta(x)$ (3) to be Gaussian which means that it can be decomposed into independent Fourier modes $\delta(k)$

$$\delta(x) = \int d^3k \delta(k) \exp(i k \cdot x),$$

(34)

which completely characterized by power spectra $P(k)$

$$\langle \delta(k)\delta(k') \rangle = \delta_0(k + k')P(k).$$

(35)

In such a field, each Fourier mode is statistically independent of the others.

By analogy, for projected catalogs, one define angular cumulant correlators to be

$$\langle \sigma(\gamma_1)\sigma(\gamma_2) \rangle_c = c_{pq} \langle \sigma^2(\gamma_1) \rangle \langle \sigma(\gamma_2) \rangle^{p+q-2},$$

(36)

where $\gamma_1$ and $\gamma_2$ are unit vectors defining positions in the celestial sphere. A quantity $q$ can be defined with respect to $c$ in the same way as analogy with $Q$ is defined with respect to $C$ in equation (33). The quantity $c_{pq}$ is the projected cumulant correlator and $\sigma(\gamma)$ is projected number density of objects:

$$\sigma(\gamma) = n_0 \int_0^\infty r^2 dr F(r) \frac{\rho(r\gamma)}{\rho_0},$$

(37)

where $F(r)$ is the selection function of the catalog and $\rho_0$ the mean density of the universe, $n_0F(r)$ is mean density of observable objects at a radial distance $r$. The projected density field on the sky will usually be smoothed with a smoothing window $W_{\gamma_0}$, the result of which can be written as

$$\sigma_s(\gamma) = n_0 \int W_{\gamma_0}(\gamma_1)d^2\gamma_1 \int r^2 dr F(r) \frac{\rho(r\gamma)}{\rho_0}.$$  

(38)

The function $W_{\gamma_0}$ is unity if the angle between $\gamma$ and a given direction $\gamma_0$ is less than $\theta_0$ and zero otherwise. The projected one–point cumulants

$$s_p = \langle \sigma^p_s(\gamma) \rangle / \langle \sigma^2_s(\gamma) \rangle^{p-1}$$

(39)

are normalized moments of the smoothed one-point projected probability density functions, and coincide with the cumulant correlators in the limit $x_1 \rightarrow x_2$.

The most important assumption that we will be using in our derivation of $c_{pq}$ in projection is the fact that both smoothing angle and the separation angle are small compared to unity. In general, $c_{pq}$ will contain terms which are of higher order in $\langle \sigma(\gamma_1)\sigma(\gamma_2) \rangle$, but we will be neglecting all such terms in our calculations. Detailed perturbative calculations are needed to take higher order loop corrections into account. In this paper we however focus on directly comparing these theoretical predictions against measurements from the APM (Szapudi & Szalay 1997). One of the advantage of using small angle approximation is the separation of effects due to selection function and that of dynamical contributions. Using these approximations, it is possible to show that the projected cumulant correlators can be written:

$$c_{pq} = R_{p+q}\Theta_{pq}; \quad q_{pq} = c_{pq}/q^{p-1}p^{p-1};$$

(40)

where $\Theta_{pq}$ are the cumulant correlators derived from the simplified dynamics of two-dimensional isotropic collapse in a manner we shall shortly describe. We can define the generating function $\beta(y_1, y_2)$ for $\Theta_{pq}$ as

$$\beta(y_1, y_2) = \sum_{p=1, q=1}^\infty \Theta_{pq} \frac{y_1^p y_2^q}{p!q!}.$$  

(41)

Since we are interested only in terms linear in $\langle \delta(\gamma_1)\delta(\gamma_2) \rangle$ we can write $\beta(y_1, y_2) = \beta(y_1)\beta(y_2)$ (Bernardeau 95), where

$$\beta(y_1) = \sum_{p=1}^\infty \Theta_{p1} \frac{y_1^p}{p!},$$

(42)

which means that, in linear order, $\Theta_{pq} = \Theta_{p1}\Theta_{q1}$. Bernardeau (1992) showed that the generating function,

$$G(\tau) = \sum_{p=1}^\infty \nu_p \frac{(-\tau)^p}{p!},$$

(43)

of the tree amplitudes $\nu_p$ that appear in a perturbative expansion of the density distribution satisfy dynamical equations governing collapse of spherical over density, with $\tau$ playing the role of time (Bernardeau 1992; Munshi et al. 1994). It can be shown that generating function for cumulant correlators $\beta(y)$ can be related to $G(\tau)$ by the following relation (Bernardeau 1995):

$$\beta(y) = -y \frac{dG(\beta(y))}{d\tau},$$

(44)
Figure 2. Values of $q_{pq}$ measured from the APM survey by Szapudi & Szalay (1997) as a function of separation angle $\theta$. The solid lines represent tree-level perturbative predictions for small angles.

$$\tau = -y \frac{dG(\tau)}{d\tau}.$$  \hfill (45)

These relations hold only for an unsmoothed initially Gaussian field. In the case of angular smoothing and power law initial condition $P(k) = Ak^n$ one can write

$$\beta(y) = \tau(y) [1 + G(\tau(y))]^{-(n+2)/2}.$$  \hfill (46)

It is easy to see that although the dynamics is determined by two dimensional spherical collapse, the spectral index $n$ is still that of three dimensional power spectrum $P(k)$.

Expanding $\beta(y)$ in Taylor series we obtain the dynamical contribution to the cumulant correlators (Bernardeau et al. 1997; Munshi & Melott 1998)

$$\Theta_{21} = \frac{24}{7} - \frac{1}{2} (n + 2)$$ \hfill (47)

$$\Theta_{31} = \frac{2946}{98} - \frac{195}{14} (n + 2) + \frac{3}{2} (n + 2)^2.$$ \hfill (48)

The factors $R_{p+q}$ in (40) depend on the spectral index and specific form of selection function. They were determined by Gaztanaga (1994)

$$R_p = \frac{m_p^{n-2}(3)M_p(3p - (p-1)(n + 3))}{M_p^{p-1}(3 - n)}$$ \hfill (49)

where

$$M_p(a) = \int_0^\infty dr \, r^{a-1}F^p(r).$$ \hfill (50)

Using the specific form for the selection function $F(r) = Kr^{-0.5} \exp[-(r/D)^2]$ gives $R_3 = 1.19$, $R_4 = 1.52$, $R_5 = 2.00$, as computed by Gaztanaga (1994) and Bernardeau (1995). The slope of the projected two-point correlation function used for comparison is $\gamma = 1.7$.

4 COMPARING ANALYTICAL CALCULATIONS WITH APM MEASUREMENTS

Fully nonlinear calculations of $q_{pq}$ using factorial moments were performed by Szapudi & Szalay (1997), who used 0.23° cells to construct density maps with a magnitude cuts of $b_J = 17$ to 20. Measurements of $q_{pq}$ were presented up-to fifth order. For these magnitude cuts, the angular separation of 1° corresponds approximately to $7h^{-1}\text{Mpc}$, a regime in which where
perturbative calculations should be valid. We have replotted \( q_{0q} \) measured from the APM survey by Szapudi & Szalay (1997) as a function of separation angle \( \theta^\circ \). The smoothing angle remains fixed for all separation angles.

The theoretical predictions discussed above are plotted as straight lines and show reasonable agreement with measured values. Our results are valid in the limit of large separations and in the limit of small angles. These approximations need to be tested with more detailed perturbative calculations and Monte-Carlo simulations. It is clear that dominant contribution will come from the tree level approximation used by us. Simultaneous use of these two approximations may appear self contradictory, but it has been shown by direct Monte-Carlo integration of \( \xi_3(r_1,r_2,r_3) \) in real space that the value of \( C_{21} \) converges to its value in the large separation limit, even when the separation of the two cells is only equal to their diameter (Bernardeau 1995). It has also been shown (Gaztanaga & Bernardeau 1997), that the small-angle approximation is very good for smoothing scales less than 1° and given the errors associated with measurements from galaxy catalogs are large, it provides a very good approximation even for separation as large as 5°. On the other hand, it is expected that contributions coming from terms of higher order in \( <\sigma(\gamma_1)\sigma(\gamma_2)) \) will increase the tree-level values computed here. Finite sample volume corrections tend to reduce the computed value of cumulant correlators and enhance the agreement with tree-level perturbative calculations.

It will be very interesting to include effects of finite volume corrections in measurement of cumulant correlators from galaxy surveys and compare with perturbative calculations beyond tree level. However, the important point we wish to stress here is that projection effects make it difficult to deduce the values of hierarchical amplitudes, such as \( R_a \) and \( R_b \), directly. This means that a comparison with predictions made in real space and in three dimensions with projected cumulant correlators is highly problematic (e.g. Szapudi & Szalay 1997).

It is also interesting to note that, although the smoothing length involved in this measurement is in the quasi-linear regime, the non-linear correction process adopted by Szapudi & Szalay is based on a hierarchical ansatz that applies in the highly nonlinear regime. This nevertheless seems to work very well as the values of \( r_a \) and \( r_b \) (the projected amplitude of two different topologies) derived show a constant variation over all angular separations after correction. However, there may be an error in their paper. They have derived the relations \( r_a = q_{22} \) and \( r_b = 3q_{31} - 2q_{12} \) for large angular separation \( \theta \), as all other terms higher order in \( \xi_3/\xi_2 \) are negligible in this limit. Taking a closer look in their figure we can see that while the measured value of \( q_{22} \) for large angular separations in the lower panel is very close to unity the measured value for \( q_{31} \) in the same limit is close to 2.23 (approximately 0.35 in logarithmic units). Using these values of \( q_{31} \) and \( q_{22} \) we get \( r_a = q_{22} \simeq 1 \) and \( r_b = 3q_{31} - 2q_{12} \simeq 3 \times 2.23 - 2 \times 1 \simeq 4.69 \). However the measured \( r_a \) remains close to 5.5, and the curve which they label \( r_b \) remains close to unity. This indicates that they have probably inadvertently interchanged \( r_a \) and \( r_b \).

5 DISCUSSION

We have shown how to generalise the concept of a cumulant correlator to an arbitrary number of points, and have deconstructed the resulting functions explicitly up to sixth order in large separation limit. These results were obtained without making any specific ansatz for amplitudes associated with different tree-topologies. These results, when combined with results of multi-point factorial correlators derived in Munshi et al. (1998b), allow us to build a direct determination of the hierarchical amplitudes. They can also be used to test the dependence of these parameters on shape parameters in the highly nonlinear regime.

Earlier studies by Szapudi & Szalay (1997) showed that two-point cumulant correlators can separate contributions from tree topologies at fourth order. In this paper we have explicitly shown that if we move from two-point to three-point cumulant correlators it will be possible to separate all tree topologies contributing to fifth and sixth order correlation functions. For higher orders it is necessary to move to four-point cumulant correlators, and so on.

We have also studied the predictions of more particular versions of the hierarchical ansatz which generally assume some specific form for the hierarchical amplitudes. In particular, we have studied an extension of the ansatz given by Bernardeau & Schaeffer (1992) which assumes that, in the highly nonlinear regime, a given tree amplitude can be constructed by multiplying vertex amplitudes constituting the tree. We have also extended the ansatz by Szapudi & Szalay (1992) which is based on replacing each amplitude of the different tree-topologies by an average over different topologies of same order. With our analysis it will be possible to test these results against numerical simulations when simulations with larger dynamic range become available. This will provide an unique way to study gravitational clustering in the highly nonlinear regime.

While using our results of multi-point cumulant correlators derived in highly nonlinear regime it should be realized that although the decomposition is still valid, the amplitudes depend strongly on shape factors. The shape dependence of lowest order tree amplitude \( Q_3 \) has been studied extensively by Scoccimaro et al. (1998). Similar studies of higher order amplitudes will clarify weather these quantities do become independent of shape factors and also more studies are needed to check how they depend on initial power spectra.

We have shown that predictions made for two-point cumulant correlators match very well when compared with measurement from projected catalogs in the quasilinear regime. Our results in the highly nonlinear regime will also be interesting when larger three-dimensional galaxy catalogs are available. It will be possible to separate lower order tree topologies using our method and to test the validity regime of different nonlinear approximations.

Hierarchical amplitudes are also related with the statistics of collapsed objects, i.e. how the over-dense cells are distributed against background matter distribution (Bernardeau & Schaeffer 1992; Munshi et al. 1998a,b) so study of statistics of collapsed objects provide another interesting way to determine the hierarchical amplitudes. The reader is referred to Munshi et al. (1998a,b) for further information about this work.
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REFERENCES

Balian R., Schaefner R., 1989, A& A, 220, 1
Bernardeau F., 1992, ApJ, 392, 1
Bernardeau F., 1994, ApJ, 433, 1
Bernardeau F., 1995, A& A, 301, 309
Bernardeau F., Schaeffer R., 1992, A& A, 255, 1
Bernardeau F., van Waerbeke L., Mellier Y., 1997, A& A, 322, 1
Boschan P., Szapudi I., Szalay A.S., 1994, ApJS, 93, 65
Baugh C.M., Efstathiou G., 1993, MNRAS, 265, 145
Baugh C.M., Gaztananaga E., 1996, MNRAS, 280, L37
Colombi S., Bouchet F.R., Hernquist L., 1996, ApJ, 465, 14
Davis M., Peebles P.J.E., 1977, ApJS, 34, 425
Fry J.N., 1984, ApJ, 279, 499
Gaztananaga E., 1994, MNRAS, 268, 913
Hamilton A.J.S, 1988, ApJ, 332, 67
Juszkiewicz R., Bouchet F.R., Colombi S., 1993, ApJ, 412, L9
Kauffmann G.A.M., Melott A.L. 1992, ApJ 393, 415
Limber D.N., 1954, ApJ, 119, 665
Maddox S.J., Efstathiou G., Sutherland W.J., 1996, MNRAS, 263, 651
Maddox S.J., Sutherland W.J., Efstathiou G., 1990, MNRAS, 246, 433
Maddox S.J., Sutherland W.J., Efstathiou G., Loveday J., 1990, MNRAS, 243, 692
Meiksin A., Szapudi I., Szalay A., 1992, ApJ, 394, 87
Munshi D., Sahni V., Starobinsky A.A., 1994, ApJ, 436, 517
Munshi D., Bernardeau F., Melott A.L., Schaeffer R., 1999, MNRAS, in press
Munshi D., Melott A.L., 1998, preprint/astro-ph/9801011
Munshi D., Coles P., Melott A.L., 1998a, (submitted to MNRAS)
Munshi D., Coles P., Melott A.L., 1998b, (submitted to MNRAS)
Peebles, P.J.E., 1980, The Large Scale Structure of the Universe. Princeton University Press, Princeton
Scoccimarro R., Colombi S., Fry J.N., Frieman J.A., Hivon E., Melott A.L., 1998, ApJ, 496, 586
Scoccimarro R., Frieman J., 1998, preprint, astro-ph/9811184
Szapudi I., Colombi S., 1996, ApJ, 470, 131
Szapudi I., Dalton G., Efstathiou G.P., Szalay A., 1995, ApJ, 444,520
Szapudi I., Szalay A.S., 1993a, ApJ, 408, 43
Szapudi I., Szalay A.S., 1993b, ApJ, 414, 414
Szapudi I., Szalay A.S., 1997, ApJ, 481, L1
Szapudi I., Szalay A.S., Boschan P., 1992, ApJ, 390, 350
Table 1. Multipoint Cumulant Correlators

| Amplitudes | Order | No. of Eq. |
|------------|-------|------------|
| \(\langle \delta^p \rangle\) | \(1\) | \(p+q+\ldots = 1\) | \(1\) |
| \(\langle \delta^p(x_a)\delta^q(x_b)\rangle\) | \(2\) | \(p+q+\ldots = 2\) | \(2\) |
| \(\langle \delta^p(x_a)\ldots\delta^r(x_c)\rangle\) | \(Q\) | \(p+q+\ldots = 3\) | \(3\) |
| \(\langle \delta^p(x_a)\delta^q(x_b)\ldots\delta^r(x_c)\rangle\) | \(R_a, R_b\) | \(p+q+\ldots = 4\) | \(5\) |
| \(\langle \delta^p(x_a)\ldots\delta^q(x_b)\rangle\) | \(S_a, S_b, S_c\) | \(p+q+\ldots = 5\) | \(7\) |
| \(\langle \delta^p(x_a)\ldots\delta^q(x_b)\ldots\delta^r(x_c)\rangle\) | \(T_a, T_b, T_c, T_d, T_e\) | \(p+q+\ldots = 6\) | \(11\) |