On the limiting procedure by which $SDiff(T^2)$ and $SU(\infty)$ are associated

John Swain

Department of Physics, Northeastern University, Boston, MA 02115, USA

date: April 29, 2004

ABSTRACT

There have been various attempts to identify groups of area-preserving
diffeomorphisms of 2-dimensional manifolds with limits of $SU(N)$ as $N \to \infty$. We discuss the particularly simple case where the manifold concerned is the
two-dimensional torus $T^2$ and argue that the limit, even in the basis commonly
used, is ill-behaved and that the large-N limit of $SU(N)$ is much larger than $SDiff(T^2)$.

I. INTRODUCTION

Groups of area-preserving diffeomorphisms and their Lie algebras have recently been the
focus of much attention in the physics literature. Hoppe [1] has shown that in a suitable
basis, the Lie algebra of the group $SDiff(S^2)$ of area-preserving diffeomorphisms of a sphere
tends to that of $SU(N)$ as $N \to \infty$. Similar arguments have been made associating various
infinite limits of Lie algebras of classical groups with Lie algebras of groups of area-preserving
diffeomorphisms of 2-dimensional surfaces. This has obvious interest in connection with
gauge theories of $SU(N)$ for large $N$. The use of $SU(N)$ for finite $N$ as an approximation to
groups of area-preserving diffeomorphisms has also been used in studies of supermembranes
[2–4] and in particular has been used to argue for their instability. The authors of references
[3] and [4] have especially emphasized the difficulties in relating such infinite limits with Lie
algebras of area-preserving diffeomorphisms. Various authors have considered special limits
and/or large-$N$ limits of other classical Lie algebras [6–10] as relevant for 2-manifolds other
than spheres. The purpose of this Letter is to clarify the nature of the limiting procedure
by which $SU(\infty)$ has been related to $SDiff(T^2)$.

II. THE LIE ALGEBRAS OF $SDIFF(T^2)$

We follow here the treatment of [7], which is particularly clear. The torus $T^2$ is repre-
sented by the plane $\mathbb{R}^2$ with coordinates $x$ and $y$ and the identifications

$$(x, y) = (x + 2\pi, y) \quad (1)$$

and

$$(x, y) = (x, y + 2\pi) \quad (2)$$

A basis for functions on the torus is chosen as

$$Y_{mn}(x, y) = \exp [i(mx + ny)] \quad (3)$$

with $m, n$ running over all integers. The local area-preserving diffeomorphisms are then
generated by the vector fields
\[ L_{mn} = (\epsilon^{ab} \partial_b Y_{mn}) \partial_a = i \exp [i(mx + ny)] (n \partial_x - m \partial_y) \] (4)

with indices \(a, b = 1, 2\). In other words, the divergence-free vector fields are those which are the curl of something else.

The generators clearly close under commutation, with the commutator

\[ [L_{mn}, L_{m',n'}] = (mn' - m'n)L_{m+m',n+n'} \] (5)

### III. THE LIE ALGEBRA OF SU\( (N)\)

To construct the Lie algebra of SU\( (N)\), again following [7], we sketch the basic idea. Fix a positive integer \(N\) and a complex number \(\omega\) such that \(\omega^N = 1\) but \(\omega^r \neq 1\) for \(0 < r < N\). \(\omega\) is called a primitive root of unity. Then we have \(\omega = \exp(2\pi ik/N)\) for some \(k\) relatively prime to \(N\). Now we find unitary, traceless matrices \(g\) and \(h\) such that

\[ hg = \omega gh \] (6)

Then the set of matrices

\[ J_{m,n} = \omega^{mn/2} g^m h^n \] (7)

for \(0 \leq m, n < N\) are linearly independent and are a basis for the \(N \times N\) matrices. \(J_{0,0} = 1\), and all the other \(J_{m,n}\) are traceless and satisfy \(J_{m,n}^\dagger = J_{-m,-n}\). Leaving out \(J_{0,0}\), the scaled matrices \(J'_{m,n} = iN/(2k\pi)J_{m,n}\) generate SU\( (N)\) with the commutation relations

\[ [J'_{m,n}, J'_{m',n'}] = \frac{N}{k\pi} \sin \left( \frac{k\pi}{N} (mn' - m'n) \right) J'_{m+m',n+n'} \] (8)
IV. THE $N \to \infty$ LIMIT

The claim now is that in the limit $N \to \infty$ that the commutation relations in equation III go over to those in equation II. Naively, of course, one would like to argue that as $N \to \infty$,

$$\frac{N}{k\pi} \sin \left(\frac{k\pi}{N}(mn' - m'n)\right) = (mn' - m'n) + O(1/N^2) \quad (9)$$

and drop the terms of order $1/N^2$ and higher. However, let us keep the next term and examine whether or not it can indeed be taken to be small.

$$\frac{N}{k\pi} \sin \left(\frac{k\pi}{N}(mn' - m'n)\right) = (mn' - m'n) - \frac{1}{3!} \left(\frac{k\pi}{N}\right)^2 (mn' - m'n)^3 + \ldots \quad (10)$$

Now consider any choice of $(m, n) = (N/a, 0)$ and $(m', n') = (0, N/b)$ where $a$ and $b$ are arbitrary integers that divide $N$ (including one). Then

$$\frac{(k\pi)^2}{N^2} (mn' - m'n)^3 = \frac{(k\pi)^2}{a^3 b^3} N^4 \quad (11)$$

which is clearly not negligible as $N \to \infty$. It would seem that there are many elements of the Lie algebra of $SU(N)$ which do not belong to $SDiff(T^2)$.

This is in keeping with ideas raised in [11] suggesting that $SU(\infty)$ is much larger than the group of area-preserving diffeomorphisms of a surface, and perhaps describes some sort of theory including topology change. Other work demonstrating that topologically, $SDiff(T^2)$, and indeed all the area-preserving diffeomorphism groups, are inequivalent to $SU(\infty)$ is in [12].

V. ACKNOWLEDGEMENT

The author would like to my colleagues at Northeastern University and the National Science Foundation for their support.
[1] J. Hoppe, Proceedings, Constraints Theory and Relativistic Dynamics (Florence, 1986) eds. G. Longhi and L Lusana (World Scientific, Singapore, 1987) p. 267; Phys. Lett. B215 (1988) 706.

[2] B. deWit, J. Hoppe, and H. Nicolai, Nucl. Phys. B305 (1988) 545.

[3] B. deWit, M. Lüscher, and H. Nicolai, Nucl. Phys. B320 (1989) 133.

[4] B. de Wit and H. Nicolai, Proceedings, 3rd Hellenic School on Elementary Particle Physics (Corfu, 1989) eds. E. N. Argyres et al. (World Scientific, Singapore) p777.

[5] J. Hoppe and P. Schaller, Phys. Lett. B237 (1990) 407.

[6] C. N. Pope and K. S. Stelle, Phys. Lett. B226 (1989) 257.

[7] C. N. Pope and L. J. Romans, Class. Quantum. Grav. 7 (1990) 97.

[8] A. Wolski and J. S. Dowker, J. Math. Phys. 32 (1991) 2304.

[9] B. deWit, U. Marquard, and H. Nicolai, Comm. Math. Phys. 128 (1990) 39.

[10] D. V. Vassilevich, Class. Quantum. Grav. 8 (1991) 2163.

[11] J. Swain, “The Majorana representation of spins and the relation between $SU(\infty)$ and $SDiff(S^2)$”, http://arXiv.org/abs/hep-th/0405004.

[12] J. Swain, “The Topology of $SU(\infty)$ and the Group of Area-Preserving Diffeomorphisms of a Compact 2-manifold”, http://arXiv.org/abs/hep-th/0405003.