A NEW METHOD FOR PROVING SOME INEQUALITIES RELATED TO SEVERAL SPECIAL FUNCTIONS

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Abstract

In this paper we present a new approach to proving some exponential inequalities involving the sinc function. Power series expansions are used to generate new polynomial inequalities that are sufficient to prove the given exponential inequalities.

Keywords: Exponential inequalities, sinc function, polynomial bounds, power series expansions

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1. Introduction and preliminaries

Inequalities involving trigonometric and inverse trigonometric functions play an important role and have many applications in science and engineering, see [1], [8]−[10], [18], [19]. Of special interest are inequalities with sinc function, i.e. sinc \( x = \frac{\sin x}{x} \) \( (0 < x \leq \frac{\pi}{2}) \). It is well-known that the sinc function is often used in signal processing, optics, radio transmission, sound recording, etc.

Starting from JORDAN’s inequality [1],

\[
\frac{2}{\pi} \leq \frac{\sin x}{x} \leq 1, \quad 0 < x \leq \frac{\pi}{2},
\]

and continuing with the polynomial bounds [2], [3], [4], [5], [12], some exponential bounds have recently been considered [23].

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In [13] and [16] the inequalities of the following form were studied:

\[
\left( 1 - \frac{4(\pi - 2)}{\pi^3} x^2 \right) ^{\alpha_j x^3 + \beta_j x^2 + \gamma_j x + \delta_j} \geq \frac{\sin x}{x},
\]
(1)

and

\[
\left( 1 - \frac{4(\pi - 2)}{\pi^3} x^2 \right) ^{\alpha_j x^3 + \beta_j x^2 + \gamma_j x + \delta_j} \leq \frac{\sin x}{x},
\]
(2)

for \(x \in \left(0, \frac{\pi}{2}\right)\) where \(\alpha_j, \beta_j, \gamma_j, \delta_j (j = 1, 2)\) are specified real coefficients.

In this paper we present a new approach to proving exponential inequalities of the above form. Using the power series expansions of the corresponding functions and some newly developed approximation techniques, we reduce exponential inequalities to the corresponding polynomial inequalities that are more easily analysed and proved.

The application of our method is illustrated in the proofs of the inequalities (1) and (2), where the coefficients \(\alpha_j, \beta_j, \gamma_j, \delta_j (j = 1, 2)\) are calculated from the constraints proposed in [16].

In the rest of this section we review some results that we use in our study.

Firstly, let us recall some well known power series, such as:

\[
\ln(1 + x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}, \quad (-1 < x \leq 1)
\]
(3)

and

\[
\ln(1 - x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}, \quad (-1 \leq x < 1).
\]
(4)

Also, in accordance with [11], the following expansions hold:

\[
\ln \frac{\sin x}{x} = -\sum_{k=1}^{\infty} \frac{\alpha_{2k-1} |B_{2k}|}{k(2k)!} x^{2k}, \quad (0 < x < \pi),
\]
(5)

\[
\ln \cos x = -\sum_{k=1}^{\infty} \frac{\alpha_{2k-1} (2^{2k} - 1) |B_{2k}|}{k(2k)!} x^{2k}, \quad (-\pi/2 < x < \pi/2),
\]
(6)

where \(B_i (i \in N)\) are Bernoulli’s numbers.

The following Statement, which is a consequence of Theorem 2 from [13], was proved in [20].

**Statement 1.** For the function \(f : (a, b) \rightarrow \mathbb{R}\) let there exist the power series expansion:

\[
f(x) = \sum_{k=0}^{\infty} c_k (x - a)^k,
\]
(7)
for every \(x \in (a, b)\), where \(\{c_k\}_{k \in \mathbb{N}_0}\) is the sequence of non-negative coefficients. Then the following holds:

\[
\sum_{k=0}^{n-1} c_k (x - a)^k + \frac{1}{(b - a)^n} \left( f(b) - \sum_{k=0}^{n-1} c_k (b - a)^k \right) (x - a)^n \\
\geq f(x) \geq \sum_{k=0}^{n} c_k (x - a)^k,
\]

(8)

for every \(x \in (a, b)\) and \(n \in \mathbb{N}\).

**Remark 1.** Note that in (8) strict inequalities hold except for some special cases of polynomial function.

### 2. Main results

We start this Section by deriving some double-sided inequalities that are consequences of the power series expansions and Statement 1, needed in the proofs of Theorems 1 and 2.

#### 2.1. Some important double-sided inequalities

Using Statement 1 from (5) we get the following double-sided inequality:

\[
- \sum_{k=1}^{m-1} \frac{2^{2k-1}|B_{2k}|}{k(2k)!} x^{2k} > \ln \frac{\sin x}{x} > \\
> - \sum_{k=1}^{m-1} \frac{2^{2k-1}|B_{2k}|}{k(2k)!} x^{2k} + \left( \frac{2}{\pi} \right)^{2m} \left( \ln 2 - \sum_{k=1}^{m-1} \frac{2^{2k-1}|B_{2k}|}{k(2k)!} \left( \frac{\pi}{2} \right)^{2k} \right) x^{2m},
\]

(9)

for \(x \in (0, \frac{\pi}{2})\), where \(n, m \in \mathbb{N}\).

Based on Statement 1 from (6) we have:

\[
- \sum_{k=1}^{n} \frac{2^{2k-1}(2^{2k-1}-1)|B_{2k}|}{k(2k)!} x^{2k} > \ln \cos x > \\
> - \sum_{k=1}^{n} \frac{2^{2k-1}(2^{2k-1}-1)|B_{2k}|}{k(2k)!} x^{2k} + \left( \frac{1}{c^2} \right)^{2m} \left( \ln \cos c - \sum_{k=1}^{m-1} \frac{2^{2k-1}(2^{2k-1}-1)|B_{2k}|}{k(2k)!} c^{2k} x^{2m} \right),
\]

(10)

for \(x \in (0, c)\), where \(0 < c < \frac{\pi}{2}\), \(n, m \in \mathbb{N}\).

Using **Leibniz**’s theorem applied to (3) we obtain the following double-sided inequality:

\[
\sum_{k=1}^{2\ell_1-1} (-1)^{k-1} \frac{x^k}{k} > \ln (1 + x) > \sum_{k=1}^{2\ell_2} (-1)^{k-1} \frac{x^k}{k},
\]

(11)

for \(x \in (0, 1)\) and \(\ell_1, \ell_2 \in \mathbb{N}\).
Based on Statement 1 from (4) we get:

\[- \sum_{k=1}^{n} \frac{x^k}{k} > \ln(1 - x) > - \sum_{k=1}^{m-1} \frac{x^k}{k} + \frac{1}{c^m} \left( \ln(1 - c) + \sum_{k=1}^{m-1} \frac{c^k}{k} \right) x^m, \tag{12}\]

for \( x \in (0, c) \), where \( 0 < c < 1 \), and \( n, m \in \mathbb{N} \).

2.2. On some constraints and their consequences

Now we show that the constraints proposed in [16] induce linear relations among the coefficients \( \alpha_j, \beta_j, \gamma_j, \delta_j \) of the polynomials in the exponents of the functions on the left-hand side in (1) and (2).

Let us consider function \( f \) defined by:

\[ f(x) = \left( 1 - \frac{4(\pi - 2)}{\pi^3} x^2 \right)^{\alpha x^3 + \beta x^2 + \gamma x + \delta} - \frac{\sin x}{x}, \tag{13}\]

for some real coefficients \( \alpha, \beta, \gamma, \delta \), where \( x \in \left( 0, \frac{\pi}{2} \right] \).

Let us notice that for function \( f \) the following two conditions are always satisfied:

\[ f(0+) = 0 \quad i \quad f'(0+) = 0. \tag{14}\]

Next, the following holds for function \( f \):

\[ f''(0+) = 0 \iff \delta = \frac{\pi^3}{24(\pi - 2)}. \tag{15}\]

Also, for function \( f \) we have:

\[ f\left( \frac{\pi}{2} \right) = 0 \iff \delta = -\frac{\pi^3}{8} \alpha - \frac{\pi^2}{4} \beta - \frac{\pi}{2} \gamma. \tag{16}\]

Adding the condition \( f'\left( \frac{\pi}{2} \right) = 0 \) we get:

\[ f\left( \frac{\pi}{2} \right) = 0 \land f'\left( \frac{\pi}{2} \right) = 0 \iff \begin{cases} 
\delta = -\frac{\pi^3}{8} \alpha - \frac{\pi^2}{4} \beta - \frac{\pi}{2} \gamma \\
\gamma = -\frac{\pi^2(\ln \pi - \ln 2)(3 \pi \alpha + 4 \beta) + 8(\pi - 3)}{4 \pi (\ln \pi - \ln 2)}
\end{cases}. \tag{17}\]
Further, adding the condition \( f''(\frac{\pi}{2}) = 0 \) we get:

\[
f\left(\frac{\pi}{2}\right) = 0 \land f'(\frac{\pi}{2}) = 0 \land f''(\frac{\pi}{2}) = 0
\]

\[
\iff \left\{
\begin{array}{l}
\delta = \frac{\pi^3}{8} \alpha - \frac{\pi^2}{4} \beta - \frac{\pi}{2} \gamma \\
\gamma = \frac{\pi^2 (\ln \pi - \ln 2)(3 \pi \alpha + 4 \beta) + 8(\pi - 3)}{4 \pi (\ln \pi - \ln 2)} \\
\beta = \frac{-3 \pi^3 (\ln 2 - \ln \pi)^2 \alpha + (\pi - 2)(3 \pi - 6)(\ln 2 - \ln \pi) + 8(\pi - 3)}{2 \pi^2 (\ln 2 - \ln \pi)^2}
\end{array}
\right.
\]

Let us note that the conditions connecting the coefficients \( \alpha, \beta, \gamma, \delta \) are linear.

2.3. Proofs of exponential inequalities (1) and (2)

In this section we determine the real coefficients \( \alpha_j, \beta_j, \gamma_j, \delta_j, (j = 1, 2) \) from the constraints proposed in [16], and in Theorem 1 and Theorem 2 we prove the corresponding inequalities.

**Theorem 1.** Let function \( f_1 \) be defined in the interval \((0, \frac{\pi}{2}]\) by

\[
f_1(x) = \left(1 - \frac{4(\pi - 2)}{\pi^3} x^2\right)^{\alpha_1 x^2 + \beta_1 x^2 + \gamma_1 x + \delta_1} - \frac{\sin x}{x}
\]

and let the following conditions hold:

\[
f_1(0+) = f_1'(0+) = f_1''(0+) = 0, \quad f_1\left(\frac{\pi}{2}\right) = f_1'\left(\frac{\pi}{2}\right) = f_1''\left(\frac{\pi}{2}\right) = 0.
\]

Then:

\[
\alpha_1 = \frac{(-\pi^3 + 24\pi - 48)\ln \frac{\pi}{2} - 3(\pi - 2)(3\pi^2 - 20\pi + 36) \ln \frac{\pi}{2} + 24(\pi - 3)(\pi - 2)^2}{3(\pi - 2) \pi^3 \ln \frac{\pi}{2}},
\]

\[
\beta_1 = \frac{(\pi^3 - 24\pi + 48)\ln \frac{\pi}{2} + 6(\pi - 2)(\pi - 4)^2 \ln \frac{\pi}{2} - 16(\pi - 3)(\pi - 2)^2}{2(\pi - 2) \pi^2 \ln \frac{\pi}{2}},
\]

\[
\gamma_1 = \frac{(-\pi^3 + 24\pi - 48)\ln \frac{\pi}{2} + (3\pi - 10)(\pi - 6)(\pi - 2) \ln \frac{\pi}{2} + 8(\pi - 3)(\pi - 2)^2}{4\pi(\pi - 2) \ln \frac{\pi}{2}},
\]

\[
\delta_1 = \frac{\pi^3}{24(\pi - 2)}
\]

and

\[
\forall x \in \left(0, \frac{\pi}{2}\right], \quad f_1(x) > 0.
\]

**Proof.** As discussed in Subsection [22], the system derived from the conditions in (20) can be reduced to the system of linear algebraic equations from (14) and (15), in variables \( \alpha_1, \beta_1, \gamma_1, \delta_1 \). The solution to this system, i.e. the
coefficients of the polynomial in the exponent of the function $f_1$, are shown in \textsuperscript{[21]}. The corresponding numerical values are: $\alpha_1 = -0.0277933961 \ldots$, $\beta_1 = 0.0136111520 \ldots$, $\gamma_1 = -0.0366389131 \ldots$, $\delta_1 = 1.13168930 \ldots$.

We consider inequality \textsuperscript{(22)} in its equivalent form:

\[
(\alpha_1 x^3 + \beta_1 x^2 + \gamma_1 x + \delta_1) \ln \left( 1 - \frac{4(\pi - 2)}{\pi^3} x^2 \right) > \ln \frac{\sin x}{x},
\]

for $x \in \left( 0, \frac{\pi}{2} \right)$.

We will prove that the corresponding function of the difference:

\[
g(x) = (\alpha_1 x^3 + \beta_1 x^2 + \gamma_1 x + \delta_1) \ln \left( 1 - \frac{4(\pi - 2)}{\pi^3} x^2 \right) - \ln \frac{\sin x}{x}
\]

is positive for every $x \in \left( 0, \frac{\pi}{2} \right)$.

It is easy to verify that the polynomial of the third degree

\[
P_3(x) = \alpha_1 x^3 + \beta_1 x^2 + \gamma_1 x + \delta_1
\]

is positive in the interval $\left( 0, \frac{\pi}{2} \right)$.

The proof of positivity of function $g(x)$ is divided into two parts. In the first part we will consider the function around zero, and in the second part we focus on the remaining part of the interval $\left( 0, \frac{\pi}{2} \right)$.

1. Let us notice that

\[
\frac{4(\pi - 2)}{\pi^3} x^2 \leq \frac{\pi - 2}{\pi} < 1,
\]

for $x \in \left( 0, \frac{\pi}{2} \right)$. Therefore, we can use inequality \textsuperscript{(12)} in the form:

\[
\ln \left( 1 - \frac{4(\pi - 2)}{\pi^3} x^2 \right) >
\]

\[
> - \sum_{k=1}^{n-1} \frac{4^k(\pi - 2)^k}{\pi^{3k} k} x^{2k} + \left( \frac{2}{\pi} \right)^{2n} \ln \frac{2}{\pi} + \sum_{k=1}^{n-1} \frac{4^k(\pi - 2)^k}{\pi^{3k} k} \left( \frac{\pi}{2} \right)^{2k} x^{2n},
\]

for $x \in \left( 0, \frac{\pi}{2} \right)$ i $n \in \mathbb{N}$, $n \geq 2$. Also, from \textsuperscript{(11)} the following holds:

\[
- \ln \frac{\sin x}{x} > \sum_{k=1}^{m} \frac{2^{2k-1}|B_{2k}|}{k(2k)!} x^{2k},
\]

for $x \in \left( 0, \frac{\pi}{2} \right)$ and $m \in \mathbb{N}$. 

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Now let us construct the approximating polynomial:

\[
G_{n,m}(x) = P_3(x) \left( -\sum_{k=1}^{n-1} \frac{4^k(\pi - 2)^k}{\pi^{3k}} x^{2k} + \left( \frac{2}{\pi} \right)^{2n} \left( \ln \frac{2}{\pi} + \sum_{k=1}^{n-1} \frac{4^k(\pi - 2)^k}{\pi^{3k}} \left( \frac{\pi}{2} \right)^{2k} \right) x^{2n} \right) + \sum_{k=1}^{m} \frac{\ln 2^{2k-1} |B_{2k}|}{k(2k)!} x^{2k},
\]

for \( x \in \left[0, \frac{\pi}{2}\right] \).

We conclude that the following inequality holds:

\[
g(x) > G_{n,m}(x),
\]

for \( x \in \left(0, \frac{\pi}{2}\right] \) and \( m, n \in \mathbb{N}, \, n \geq 2 \).

If we specify the values \( m, n \in \mathbb{N} \), we can determine the zeros and the sign of the polynomial \( G_{n,m} \). For example, for \( n=3 \) and \( m=2 \) it is easy to verify that \( G_{3,2}(x) > 0 \) for every \( x \in (0, c_1) \), where \( c_1 \) is a positive number smaller than the first positive root of the considered polynomial, and that is \( x = 0.925930 \ldots \). Let us set: \( c_1 = 0.92 \). Let us notice that the polynomial \( G_{3,2} \) is of degree 9.

As \( g(x) > G_{3,2}(x) \) holds for every \( x \in \left(0, \frac{\pi}{2}\right] \), we conclude that \( g(x) > 0 \) for every \( x \in (0, c_1) \), i.e. inequality (22) holds in the interval \((0, c_1)\).

2. To prove that inequality (22) holds in the remaining part of the interval \( \left(0, \frac{\pi}{2}\right] \), we will introduce a change of variable \( x = \frac{\pi}{2} - t \) in inequality (23), \( t \in \left[0, \frac{\pi}{2}\right) \). It is sufficient to prove that the inequality holds for \( t \in [0, c_2] \), where \( c_2 = 0.65 > \frac{\pi}{2} - 0.92 = 0.650796327 \ldots \).

Now let us consider the equivalent inequality:

\[
Q_3(t) \ln \left( 1 - \frac{4(\pi - 2)}{\pi^3} \left( \frac{\pi}{2} - t \right)^2 \right) > \ln \frac{\sin \left( \frac{\pi}{2} - t \right)}{\frac{\pi}{2} - t} = \ln \cos t - \ln \left( \frac{\pi}{2} - t \right),
\]

for \( t \in [0, c_2] \), where \( Q_3(t) = P_3 \left( \frac{\pi}{2} - t \right) \).

We will prove that the corresponding function of the difference:

\[
h(t) = Q_3(t) \ln \left( 1 - \frac{4(\pi - 2)}{\pi^3} \left( \frac{\pi}{2} - t \right)^2 \right) - \ln \cos t + \ln \left( \frac{\pi}{2} - t \right),
\]

is positive for every \( t \in [0, c_2] \).

The positivity of the polynomial \( P_3(x) \) in the interval \( \left(0, \frac{\pi}{2}\right] \) yields positivity of the polynomial \( Q_3(t) \).
Since the following holds:
\[
\ln\left(1 - \frac{4(\pi - 2)}{\pi^3}\left(\frac{\pi}{2} - t\right)^2\right) = \ln\frac{2}{\pi} + \ln\left(1 + \frac{2(\pi - 2)t(\pi - t)}{\pi^2}\right)
\] (32)
and
\[
\frac{2(\pi - 2)t(\pi - t)}{\pi^2} < 1,
\] (33)
for every \(t \in (0, c_2]\), we can use inequality (12) in the following form:
\[
\ln\left(1 - \frac{4(\pi - 2)}{\pi^3}\left(\frac{\pi}{2} - t\right)^2\right) \geq \ln\frac{2}{\pi} + \sum_{k=1}^{n} \frac{(-1)^{k-1} 2^{k}(\pi - 2)}{k\pi^{2k}} t^k (\pi - t)^k.
\] (34)
for \(t \in [0, c_2]\) and \(n = 2\ell \in N\).

Next, we can use inequality (10) in the following form:
\[
-\ln \cos t \geq m_1 \sum_{k=1}^{m_1} \frac{2^{2k-1}(2^{2k-1})|B_{2k}|}{k(2k)!} t^{2k}
\] (35)
for \(t \in (0, c_2]\) and \(m_1 \in N\).

Finally, let us notice that the following holds true:
\[
\ln\left(\frac{\pi}{2} - t\right) = \ln\frac{\pi}{2} + \ln\left(1 - \frac{2t}{\pi}\right)
\] (36)
and
\[
\frac{2t}{\pi} < 1,
\] (37)
for \(t \in [0, c_2]\).

We conclude that:
\[
\ln\left(\frac{\pi}{2} - t\right) = \ln\frac{\pi}{2} - \ln\left(1 - \frac{2t}{\pi}\right) \geq \ln\frac{\pi}{2} - \sum_{k=1}^{m_2} \frac{2^{k}(\pi - 2k - 2)}{\pi^{2k}k^k} \left(\frac{1}{c_2}\right)^{m_2} \ln\left(1 - \frac{2c_2}{\pi}\right) + \sum_{k=1}^{m_2} \frac{2^{k}(c_2^{2k} - 2)}{\pi^{2k}k^k} \] (38)
for \(t \in [0, c_2]\) and \(m_2 \in N\).

Let us now consider the approximating polynomial:
\[
H_{n,m_1,m_2}(t) = Q_3(t)\left(\ln\frac{2}{\pi} + \sum_{k=1}^{n} \frac{(-1)^{k-1} 2^{k}(\pi - 2)}{k\pi^{2k}} t^k (\pi - t)^k\right)
+ \sum_{k=1}^{m_1} \frac{2^{2k-1}(2^{2k-1})|B_{2k}|}{k(2k)!} t^{2k}
\] (39)
\[
+ \ln\frac{\pi}{2} - \sum_{k=1}^{m_2} \frac{2^{k}(\pi - 2k - 2)}{\pi^{2k}k^k} t^k \left(\frac{1}{c_2}\right)^{m_2} \ln\left(1 - \frac{2c_2}{\pi}\right) + \sum_{k=1}^{m_2} \frac{2^{k}(c_2^{2k} - 2)}{\pi^{2k}k^k} t^{m_2},
\] for \(t \in [0, c_2]\), \(m, n \in N\).
As before, for the specified values of \( n = 6 \), \( m_1 = 4 \) and \( m_2 = 8 \), we can determine the zeros and the sign of the above-mentioned polynomial (for example applying STURM’s theorem). Let us notice that the degree of the polynomial \( H_{6,4,8}(t) \) is equal to 15 and that \( t = 0.789165 \ldots \) is the first positive root of the polynomial \( H_{6,4,8}(x) \).

Therefore,
\[
h(t) > H_{6,4,8}(t) > 0,
\]
for every \( t \in [0, c_2] \).

This proves inequality (30) in the interval \([0, c_2]\), and hence \( g(x) > 0 \) holds for every \( x \in \left[ \frac{\pi}{2} - c_2, \frac{\pi}{2} \right] \).

Finally, it follows from 1. and 2. that inequality (22) holds for every \( x \in \left[ 0, \frac{\pi}{2} \right] \).

**Theorem 2.** Let the function
\[
f_2(x) = \left( 1 - \frac{4(\pi - 2)}{\pi^3} x^3 \right)^{\alpha_2 x^3 + \beta_2 x^2 + \delta_2} x - \sin x,
\]
for \( x \in \left( 0, \frac{\pi}{2} \right] \) satisfy the following conditions:
\[
f_2(0+) = f_2'(0+) = f_2''(0+) = 0, \quad f_2 \left( \frac{\pi}{2} \right) = f_2' \left( \frac{\pi}{2} \right) = 0. \tag{40}
\]

Then:
\[
\alpha_2 = -\frac{2(\pi - 2)(\pi - 3)}{3} \left( \frac{(48 - 24\pi + \pi^3) \ln \frac{\pi}{2}}{\pi^3 - (48 - 24\pi + \pi^3) \ln \frac{\pi}{2}} \right),
\]
\[
\beta_2 = \frac{8(\pi - 2)(\pi - 3)}{2\pi^3 - (48 - 24\pi + \pi^3) \ln \frac{\pi}{2}}, \tag{41}
\]
\[
\delta_2 = \frac{\pi^3}{24(\pi - 2)},
\]
and
\[
f_2(x) < 0, \tag{42}
\]
for every \( x \in \left( 0, \frac{\pi}{2} \right] \).

**Proof.** As discussed in Subsection 2.2, the conditions in (40) yield a system of linear equations (shown in (15) and (17)) in variables \( \alpha_2, \beta_2 \) and \( \delta_2 \). The symbolic values (41) of \( \alpha_2, \beta_2 \) and \( \delta_2 \) are obtained by solving this system.

Notice that it is easy to get numeric values: \( \alpha_2 = -0.0129442047 \ldots \), \( \beta_2 = -0.0330389552 \ldots \), and \( \delta_2 = 1.13168930 \ldots \).
The exponential inequality (42) is equivalent to the following inequality:

\[ F_2(x) < 0 \quad (43) \]

for every \( x \in \left(0, \frac{\pi}{2}\right) \), where

\[ F_2(x) = (\alpha_2 x^3 + \beta_2 x^2 + \delta_2) \ln \left(1 - \frac{4(\pi - 2)}{\pi^3} \frac{x^3}{x^2}\right) - \ln \frac{\sin x}{x}. \]

It is not difficult to check that

\[ P_3(x) = \alpha_2 x^3 + \beta_2 x^2 + \delta_2 > 0 \quad (44) \]

for every \( x \in \left(0, \frac{\pi}{2}\right) \).

Based on (9) and (12), for \( x \in \left(0, \frac{\pi}{2}\right) \) and \( n, m \in \mathbb{N}, n \geq 1, m \geq 2 \) the following inequalities hold:

\[- \ln \frac{\sin x}{x} < \sum_{k=1}^{m-1} \frac{2^{2k-1}|B_{2k}|}{k(2k)!} x^{2k} - \left(\frac{2}{\pi}\right)^{2m} \left(\ln \frac{2}{\pi} - \sum_{k=1}^{m-1} \frac{2^{2k-1}|B_{2k}|}{k(2k)!} \left(\frac{2}{\pi}\right)^{2k}\right) x^{2m}\]

and

\[ \ln \left(1 - \frac{4(\pi - 2)}{\pi^3} \frac{x^2}{x}\right) < - \sum_{k=1}^{n} \frac{4^k(\pi - 2)^k}{\pi^{3k} k} x^{2k}. \]

Finally, for \( x \in \left(0, \frac{\pi}{2}\right) \), we have:

\[ F_2(x) < -\sum_{k=1}^{m} \frac{4^k(\pi - 2)^k}{\pi^{3k} k} x^{2k} + \sum_{k=1}^{m-1} \frac{2^{2k-1}|B_{2k}|}{k(2k)!} x^{2k} - \left(\frac{2}{\pi}\right)^{2m} \left(\ln \frac{2}{\pi} - \sum_{k=1}^{m-1} \frac{2^{2k-1}|B_{2k}|}{2k(2k)!} \left(\frac{2}{\pi}\right)^{2k}\right) x^{2m}. \]

Let us denote by \( H_{n,m}(x) \) the polynomial on the right-hand side of the above inequality. Thus, for \( x \in \left(0, \frac{\pi}{2}\right) \) and \( n, m \in \mathbb{N}, n \geq 1, m \geq 2 \) we have:

\[ F_2(x) < H_{n,m}(x). \quad (45) \]

Hence, to prove inequality (43) it is sufficient to prove the following polynomial inequality:

\[ H_{n,m}(x) < 0 \quad (46) \]

for \( x \in \left(0, \frac{\pi}{2}\right), n, m \in \mathbb{N}, m \geq 2 \).

Let us consider the polynomial \( H_{n,m}(x) \) for \( n = 3 \) and \( m = 3 \).

For the polynomial \( H_{3,3}(x) \) it is not difficult to find its smallest positive root \( x_1 = 1.0959152... \), and to determine the sign of \( H_{3,3}(x) \) for \( x \in (0, x_1) \):

\[ H_{3,3}(x) < 0. \]
Thus, inequality (46) holds for $n = 3$, $m = 3$ and every $x \in (0, x_1)$. Therefore, inequality (43) holds true for every $x \in (0, x_1)$.

Further, our goal is to prove inequality (43) for every $x \in [x_1, \frac{\pi}{2}]$. By the following change of variables:

$$x = \frac{\pi}{2} - t$$

for $x \in [x_1, \frac{\pi}{2}]$ inequality (43) becomes

$$F_2\left(\frac{\pi}{2} - t\right) < 0$$

for $t \in \left[0, \frac{\pi}{2} - x_1\right]$. We prove inequality (47) for

$$t \in [0, c)$$

where $\frac{\pi}{2} - x_1 < c < \frac{\pi}{2}$. Let us, for example, select $c = 0.6$.

We have:

$$F_2\left(\frac{\pi}{2} - t\right) = P_3\left(\frac{\pi}{2} - t\right) \ln\left(1 - \frac{4(\pi - 2)}{\pi^3} \left(\frac{2}{\pi} - t\right)^2\right) - \ln \cos t - \ln \frac{2}{\pi} - t$$

$$= P_3\left(\frac{\pi}{2} - t\right) \left(\ln \frac{2}{\pi} + \ln\left(1 + \frac{2(\pi - 2)t(\pi - t)}{\pi^2}\right)\right) - \ln \cos t - \ln \frac{\pi}{2} + \ln\left(1 - \frac{2t}{\pi}\right).$$

From (44) we conclude:

$$P_3\left(\frac{\pi}{2} - t\right) = \alpha_2 \left(\frac{\pi}{2} - t\right)^3 + \beta_2 \left(\frac{\pi}{2} - t\right)^2 + \delta_2 > 0$$

for every $t \in [0, c)$.

As

$$\frac{2(\pi - 2)t(\pi - t)}{\pi^2} < 1$$

and

$$\frac{2t}{\pi} < 1$$

for every $t \in \left[0, \frac{\pi}{2}\right)$ i.e. for every $t \in [0, c)$, and based on (10), (11) and (12), we can conclude that for $t \in [0, c)$ and $n = 2l - 1$, $l \in \mathbb{N}$, $m_1, m_2 \in \mathbb{N}$, $m_1 \geq 2$ the following inequalities hold:

$$- \ln \cos t \leq \sum_{k=1}^{m_1-1} \frac{2^{2k-1}(2^k - 1) B_{2k} | \pi^2 l^2 - \left(\frac{1}{c}\right)^{2m_1} \left(\ln \cos c - \sum_{k=1}^{m_1-1} \frac{2^{2k-1}(2^k - 1) B_{2k}}{k(2k)!} \right) t^{2m_1},$$

$$\ln\left(1 + \frac{2(\pi - 2)t(\pi - t)}{\pi^2}\right) \leq \sum_{k=1}^{n} (-1)^{k-1} \frac{2^k(\pi - 2)^k}{k\pi^{2k}} t^k (\pi - t)^k.$$
Finally, based on the above results we have:

\[ F_2\left(\frac{\pi}{2} - t\right) < P_3\left(\frac{\pi}{2} - t\right) \left(\ln \frac{\pi}{2} + \sum_{k=1}^{n} (-1)^{k-1} \frac{2^k (\pi - 2)^k}{k \pi^{2k}} t^k (\pi - t)^k\right) + \]

\[ + \sum_{k=1}^{m_1-1} \frac{2^{2k-1} (2^{2k} - 1) |B_{2k}|}{k (2k)!} t^{2m_1} \left(\ln \cos \frac{c}{2} - \sum_{k=1}^{m_1-1} \frac{2^{2k-1} (2^{2k} - 1) B_{2k}}{k (2k)!} c \right) + \sum_{k=1}^{m_2} \frac{2^k t^k}{k \pi^k}. \]

Let us denote by \( T_{n,m_1,m_2}(t) \) the polynomial on the right-hand side of the above inequality. Thus, for \( t \in [0, c) \) we have:

\[ F_2\left(\frac{\pi}{2} - t\right) < T_{n,m_1,m_2}(t). \] (48)

Hence, for the proof of inequality (47) it is sufficient to prove the following polynomial inequality:

\[ T_{n,m_1,m_2}(t) < 0 \] (49)

for every \( t \in [0, c) \), \( n,m_1,m_2 \in \mathbb{N}, m_1 \geq 2 \).

Let us consider the polynomial \( T_{n,m_1,m_2}(t) \) for \( n = 5, m_1 = 7 \) and \( m_2 = 9 \).

For the polynomial \( T_{5,7,9}(t) \) it is not difficult to find its smallest positive root \( t_1 = 0.6257524 \ldots \), and to determine the sign of \( T_{5,7,9}(t) \) for \( t \in [0, t_1) \):

\[ T_{5,7,9}(t) < 0. \]

Thus, inequality (49) holds for \( n = 5, m_1 = 7, m_2 = 9 \) and every \( t \in [0, c) \).

Therefore, inequality (47) holds true for every \( t \in [0, \frac{\pi}{2} - x_1] \subset [0, c) \), and inequality (48) holds for every \( x \in [x_1, \frac{\pi}{2}] \).

The proof of Theorem 2 is now complete. \( \square \)

The following theorem, stated as a conjecture in [13], was proved in [24] using the approximations and methods from [7, 14, 15, 17, 21] and [22]. In this paper we give another proof of this conjecture. In particular, we show that this conjecture is a consequence of Theorem 2.

**Theorem 3.** Let the function

\[ f_3(x) = \left(1 - \frac{4(\pi - 2)}{\pi^3} x^2 \right)^{\alpha_3 x^3 + \delta_3} - \frac{\sin x}{x^3} \]
for \( x \in \left(0, \frac{\pi}{2}\right) \) satisfy the following conditions:

\[
\begin{align*}
f_3(0^+) &= f_3'(0^+) = f_3''(0^+) = 0, \\
f_3\left(\frac{\pi}{2}\right) &= 0.
\end{align*}
\]  

(50)

Then:

\[
\begin{align*}
\alpha_3 &= -\frac{\pi^3 - 24\pi + 48}{3(\pi - 2)\pi^3}, \\
\delta_3 &= \frac{\pi^3}{24(\pi - 2)}.
\end{align*}
\]

(51)

and

\[f_3(x) < 0,\]

(52)

for every \( x \in \left(0, \frac{\pi}{2}\right) \).

Proof. As shown in Subsection 2.2, the conditions in (50) yield a system of linear equations (shown in (15) and (16)) in variables \( \alpha_3 \) and \( \delta_3 \). The symbolic values (51) of \( \alpha_3 \) and \( \delta_3 \) are obtained by solving this system.

Based on Theorem 2, it is enough to prove that for every \( x \in \left(0, \frac{\pi}{2}\right) \)

\[
\left(1 - \frac{4(\pi - 2)}{\pi^3} x^2\right)^{\alpha_3 x^3 + \delta_3} < \left(1 - \frac{4(\pi - 2)}{\pi^3} x^2\right)^{\alpha_2 x^3 + \beta_2 x^2 + \delta_2}.
\]

The above inequality is equivalent to the following inequalities:

\[
\alpha_3 x^3 + \delta_3 > \alpha_2 x^3 + \beta_2 x^2 + \delta_2 \iff x^2 (\alpha_3 - \alpha_2) x - \beta_2 > 0.
\]

It is not hard to check that the polynomial inequality on the right-hand side holds true for every \( x \in \left(0, \frac{\pi}{2}\right) \). The proof of Theorem 3 is therefore complete.

\[\square\]

3. Conclusion

In this paper we presented a new approach to proving some exponential inequalities. We illustrated our ideas in the proofs of Theorems 1 and 2, i.e. in the proofs of some exponential inequalities connected with the sinc function. Using particular approximations based on the power series expansions, proving of the exponential inequalities was reduced to proving of the corresponding polynomial inequalities.

Our approach can be applied more broadly than just to exponential inequalities. Another potential application of our technique is in the area of establishing new polynomial bounds as shown, for example, in inequalities (29) and (45) in the proofs of Theorem 1 and 2, respectively.

Competing Interests. The authors would like to state that they do not have any competing interests in the subject of this research.

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