TRIANGULATED MATLIS EQUIVALENCE

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Abstract. This paper is a sequel to [24] and [25]. We extend the classical Harrison–Matlis module category equivalences to a triangulated equivalence between the derived categories of the abelian categories of torsion modules and contramodules over a Matlis domain. This generalizes to the case of any commutative ring \( R \) with a fixed multiplicative system \( S \) such that the \( R \)-module \( S^{-1}R \) has projective dimension 1. The latter equivalence connects complexes of \( R \)-modules with \( S \)-torsion and \( S \)-contramodule cohomology modules. It takes a nicer form of an equivalence between the derived categories of abelian categories when \( S \) consists of nonzero-divisors or the \( S \)-torsion in \( R \) is bounded.

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Introduction

The category of torsion abelian groups is abelian. So is the category of reduced cotorsion abelian groups [25]; in fact, this observation was made, in a much greater generality, already in [19, Remarks in §2] (the reader should be warned that Matlis, following Harrison [15], calls such groups simply “cotorsion”, including the “reduced” condition into the definition of “cotorsion”). The derived categories of these two abelian categories are equivalent: one has

\[
D^\ast(\text{mod}_{\text{tors}}) \simeq D^\ast(\text{mod}_{\text{rcot}})
\]

for any derived category symbol \( \ast = b, +, - \), or \( \emptyset \).

Furthermore, the abelian category of torsion abelian groups is the Cartesian product of the abelian categories of \( p \)-primary torsion abelian groups, while the abelian category of reduced cotorsion abelian groups is the Cartesian product of the abelian
categories of $p$-contramodule abelian groups. The equivalence of categories (1) is the Cartesian product of the equivalences

\[ D^*(\mathbb{Z-mod}_{p\text{-tors}}) \simeq D^*(\mathbb{Z-mod}_{p\text{-tra}}). \]

The latter equivalence, in turn, sits in the intersection of two classes of triangulated equivalences. On the one hand, there is an equivalence between the coderived category of discrete modules and the contraderived category of contramodules over a pro-coherent topological ring $\mathfrak{R}$ with a dualizing complex $D^*$ [23, Section D.2]. In particular, given a Noetherian commutative ring $R$ with a fixed ideal $I \subset R$ and a dualizing complex of $I$-torsion $R$-modules $D^*$, there is an equivalence between the coderived category of $I$-torsion $R$-modules modules and the contraderived category of $I$-contramodule $R$-modules [23, Section C.1]

\[ D_{\mathbb{R}}(\mathbb{Z-mod}_{I\text{-tors}}) \simeq D_{\mathbb{R}}(\mathbb{Z-mod}_{I\text{-tra}}). \]

The equivalence is provided by the derived functors of Hom and tensor product with the dualizing complex $D^*$.

The simplest example of the equivalence (3) occurs when $I$ is a maximal ideal in $R$. Then an injective envelope $C$ of the irreducible $R$-module $R/I$, viewed as a one-term complex of $R$-modules, is a dualizing complex of $I$-torsion $R$-modules (see the discussions in [28, Section 5] and [24, Remark 4.10]). Taking the Hom from $C$ and the tensor product with $C$ establishes a (covariant) equivalence between the additive categories of injective $I$-torsion $R$-modules and projective $I$-contramodule $R$-modules,

\[ \text{Hom}_R(C, -): R\text{-mod}^{\text{inj}}_{I\text{-tors}} \simeq R\text{-mod}^{\text{proj}}_{I\text{-tra}}: C \otimes_R - . \]

For any Noetherian commutative ring $R$ with an ideal $I$, the coderived category $D^0(R\text{-mod}_{I\text{-tors}})$ is equivalent to the homotopy category of unbounded complexes of injective $I$-torsion $R$-modules and the contraderived category $D^0(R\text{-mod}_{I\text{-tra}})$ is equivalent to the homotopy category of unbounded complexes of projective $I$-contramodule $R$-modules,

\[ D^0(R\text{-mod}_{I\text{-tors}}) \simeq \text{Hot}(R\text{-mod}^{\text{inj}}_{I\text{-tors}}) \quad \text{and} \quad D^0(R\text{-mod}_{I\text{-tra}}) \simeq \text{Hot}(R\text{-mod}^{\text{proj}}_{I\text{-tra}}). \]

Combining (5) with (4), we obtain the equivalence of exotic derived categories (3) in the case of a maximal ideal $I \subset R$.

On the other hand, for any commutative ring $R$ with a finitely generated ideal $I \subset R$ there is a natural equivalence between the full subcategories of the derived category $D^*(R\text{-mod})$ formed by the complexes with $I$-torsion and $I$-contramodule cohomology modules [24, Section 3]

\[ D^*_{I\text{-tors}}(R\text{-mod}) \simeq D^*_{I\text{-tra}}(R\text{-mod}). \]

When the finitely generated ideal $I \subset R$ is weakly proregular (in particular, when the ring $R$ is Noetherian), the equivalence (6) takes a nicer form of an equivalence between the derived categories of the abelian categories of $I$-torsion and $I$-contramodule
\( R \)-modules [24, Sections 1–2]

\[ (7) \quad D^+(R-\text{mod}_{I-\text{tors}}) \simeq D^+(R-\text{mod}_{I-\text{tra}}). \]

Notice the difference between the equivalences (3) and (7) in that the former is an equivalence between the coderived and the contraderived categories, while the latter connects the (bounded or unbounded) conventional derived categories. The results of [24, Sections 4–5] extend the equivalence (7) to the case of the absolute derived categories \( D^{\text{abs}}, D^{\text{abs}+}, \) or \( D^{\text{abs}}. \)

The equivalence (6) and, especially, (7) is provided by the functors of Hom and tensor product with a dedualizing complex of \( R \)-modules \( B^\bullet. \) The simplest example occurs when \( I \) is a principal ideal in \( R. \) Then the dedualizing complex \( B^\bullet \) for the ideal \( I \) is

\[ (8) \quad R \longrightarrow R[s^{-1}], \]

where \( s \) is a generating element of an ideal \( I. \) More precisely, the weak proregularity condition in the case of a principal ideal \( I \) simply says that the \( s \)-torsion in \( R \) should be bounded. Assuming this, one can use a complex of \( I \)-torsion \( R \)-modules quasi-isomorphic to (8) in the role of a dedualizing complex \( B^\bullet. \)

To compare the dualizing and dedualizing complexes, one can express them in terms of the derived functor \( \mathbb{R}\Gamma_I \) of the functor \( \Gamma_I(M) = \lim_{\to n} \text{Hom}_R(R/I^n, M) \) of maximal \( I \)-torsion submodule of an \( R \)-module \( M. \) Let \( D^*_R \) be a dualizing complex of \( R \)-modules. Then one can take

\[ D^* = \mathbb{R}\Gamma_I(D^*_R) \quad \text{and} \quad B^* = \mathbb{R}\Gamma_I(R). \]

In the case of a regular Noetherian ring \( R \) (of finite Krull dimension), the abelian categories \( R-\text{mod}_{I-\text{tors}} \) and \( R-\text{mod}_{I-\text{tra}} \) have finite homological dimension, so there is no difference between the conventional derived, co/contraderived, and the absolute derived category for them. There is also no difference between a dualizing and a dedualizing complex. So the two triangulated equivalences (3) and (7) become one and the same. We have explained that the equivalence of derived categories (2) is a common particular case of (3) and (7).

Back in the 1950–60’s, when the derived categories were not yet known, a version of the equivalence (1) was first observed by Harrison [15]. In fact, there are two equivalences of additive categories in [15, Section 2], both of which we now see as related to the triangulated equivalence (1). One of them is an equivalence between what we would now call the additive categories of injective objects in \( \mathbb{Z}-\text{mod}_{\text{tors}} \) and projective objects in \( \mathbb{Z}-\text{mod}_{\text{cot}} \) [15, Proposition 2.1]. It is provided by the functors \( \text{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, -) \) and \( \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} -. \) The other one is an equivalence between the full subcategory of objects having no injective direct summands in \( \mathbb{Z}-\text{mod}_{\text{tors}} \) and the full subcategory of objects having no projective direct summands in \( \mathbb{Z}-\text{mod}_{\text{cot}} \) [15, Proposition 2.3] (see also [8, Theorem 55.6]). This one is provided by the functors \( \text{Ext}^{1}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, -) \) and \( \text{Tor}^{1}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, -) \).

Harrison then remarks that one can obtain a correspondence between arbitrary objects in \( \mathbb{Z}-\text{mod}_{\text{tors}} \) and \( \mathbb{Z}-\text{mod}_{\text{cot}} \) by taking a direct sum of the Hom and Ext from
\(\mathbb{Q}/\mathbb{Z}\) on the one side and a direct sum of the \(\otimes\) and \(\text{Tor}_1\) with \(\mathbb{Q}/\mathbb{Z}\) on the other side. This is no longer an equivalence of categories, of course, but only a bijection between the isomorphism classes of objects. The discussion of this “nonnatural isomorphism” continues in Matlis’ [19, Remarks in §3], where he observes that such a bijection between the torsion and reduced cotorsion modules holds over any Dedekind domain, but not over other domains. One feels pained by reading today these discussions which would be so much illuminated and clarified by an introduction of the derived category point of view.

In the modern homological language, we say that the equivalence of derived categories (1) is provided by the derived functors \(\mathbb{R}\text{Hom}_\mathbb{Z}(\mathbb{Q}/\mathbb{Z}, -)\) and \(\mathbb{Q}/\mathbb{Z} \otimes_\mathbb{Z} -\), which have homological dimension 1. So, generally speaking, acting in each direction, they take a group into a two-term complex of groups. Restricting the equivalence (1) to those the complexes concentrated in the cohomological degree 0 on each side which are taken by this equivalence to complexes concentrated in the cohomological degree 0 on the other side, one obtains the equivalence of categories \(\mathbb{Z}\text{-mod}_{\text{tors}} \approx \mathbb{Z}\text{-mod}_{\text{proj}}\) [15, Proposition 2.1]. Restricting the equivalence (1) to those complexes of reduced cotorsion groups that are concentrated in the cohomological degree 0 and are taken to complexes of torsion groups concentrated in the cohomological degree \(-1\) (and vice versa), one obtains the second Harrison’s equivalence [15, Proposition 2.3].

Furthermore, the abelian categories \(\mathbb{Z}\text{-mod}_{\text{tors}}\) and \(\mathbb{Z}\text{-mod}_{\text{rcot}}\) also have homological dimension 1, hence every complex in these categories is noncanonically isomorphic to the direct sum of its cohomology groups. Decomposing the two-term complexes of torsion groups into the direct sums of their cohomology groups, one recovers Harrison’s direct sum decomposition of reduced cotorsion groups into the “torsion-free” and “adjusted” parts [15, Proposition 2.2].

Matlis [19] extended Harrison’s theory to modules over arbitrary commutative domains. The two Matlis’ equivalences of categories, generalizing the two Harrison’s equivalences, are [19, Theorems 3.4 and 3.8] (see also [10, Theorem VIII.2.8]). The aim of this paper is to interpret the two Matlis equivalences of additive categories of modules as a single triangulated equivalence between the derived categories.

There is one caveat: our generality level differs from Matlis’. On the one hand, Matlis’ paper [19] only deals with integral domains \(R\). In the language of [19], an element \(x\) in an \(R\)-module \(M\) is said to be torsion if there exists \(r \in R, r \neq 0\) such that \(rx = 0\). Subsequently, in the book [20, Chapters I–II], Matlis extends his theory to arbitrary commutative rings. In the context of [20], an element \(x \in M\) is said to be torsion if there exists a nonzero-divisor \(r \in R\) such that \(rx = 0\).

This generalizes naturally much further: let \(R\) be an arbitrary commutative ring and \(S \subset R\) be a multiplicative set. We say that an element \(x\) in an \(R\)-module \(M\) is \(S\)-torsion if there exists \(s \in S\) such that \(sx = 0\). For the beginning, one can assume that all the elements of \(S\) are nonzero-divisors in \(R\) (cf. [12] and [1]). Then this restriction can be relaxed to the condition that the \(S\)-torsion in \(R\) is bounded, or even dropped altogether. One of the aims of this paper is to explain how to extend the classical theory to the situation when \(S\) contains some zero-divisors.
On the other hand, the category of what Matlis calls “cotorsion $R$-modules” (and we call $S$-contramodule $R$-modules) is only well-behaved homologically (i.e., abelian with an exact embedding functor $R\text{-mod}_{S,\text{tra}} \to R\text{-mod}$) when the projective dimension of the $R$-module $S^{-1}R$ does not exceed 1. Matlis observes in [19, §10] that “A remarkable smoothing of the whole theory takes place under the assumption” of the projective dimension of the field of fractions $Q$ of his domain $R$ being equal to 1. Still, he formulates the main results, including the equivalences of categories in [19, §3], for an arbitrary commutative domain.

Commutative domains $R$ for which $\text{pd}_R Q = 1$ are now called Matlis domains [18, Section 2], [10, Section IV.4]. In this paper, we pay tribute to Matlis’ (and now traditional) generality preferences by discussing the $S$-topology for an arbitrary multiplicative subset $S$ in a commutative ring $R$, but then make the assumption $\text{pd}_R S^{-1}R \leq 1$ in order to formulate and prove our homological results.

This assumption holds, in particular, for every countable multiplicative subset $S \subset R$. It also holds under certain more complicated countability conditions ([9, Theorem 3.2] and [1, Theorem 1.1]), and under Noetherian Krull one-dimensionality conditions ([20, Theorem 4.2] and [25, Section 13]). In fact, one has $\text{pd}_R S^{-1}R \leq 1$ for every multiplicative subset $S$ in a Noetherian commutative ring of Krull dimension 1 ([27, Corollaire II.3.3.2] and [25, Remark 13.9]).

It remains to explain the connection between the MGM (Matlis–Greenlees–May) duality of the paper [24] and the triangulated Matlis equivalence/duality of the present paper. To pass from the former to the latter, one first restricts generality in the situation of a finitely generated ideal $I$ in a commutative ring $R$, by assuming that $I$ is a principal ideal generated by an element $s \in R$. Then one expands generality in a different direction, by replacing the multiplicative set $\{s^n \mid n \in \mathbb{Z}_{\geq 0}\} \subset R$ by an arbitrary multiplicative subset $S \subset R$.

Returning to the above discussion of dedualizing complexes, let us point out that, in our present context, the two-term complex

\begin{equation}
R \longrightarrow S^{-1}R,
\end{equation}

or a complex of $S$-torsion $R$-modules quasi-isomorphic to it, plays the role of a dedualizing complex (cf. the more elementary (8)). When all the elements of $S$ are nonzero-divisors in $R$, the complex (9) is quasi-isomorphic to the $R$-module $(S^{-1}R)/R$. In this connection, it is worth mentioning that there is a long tradition of considering the functor $\text{Ext}_{R}^{1}(Q/R, -)$ in the homological algebra of commutative domains $R$, going back to the papers of Nunke, Harrison, and Matlis [21, 15, 18, 19].

To conclude this introduction, let us explain, in the most simple homological terms, why the equivalence of triangulated categories (1) holds. First of all, both $D^*(\mathbb{Z}\text{-mod}_{\text{tors}})$ and $D^*(\mathbb{Z}\text{-mod}_{\text{cot}})$ are full triangulated subcategories in $D^*(\mathbb{Z}\text{-mod})$. Furthermore, the derived category of $\mathbb{Q}$-vector spaces $D^*(\mathbb{Q}\text{-vect})$, which is also a full triangulated subcategory in $D^*(\mathbb{Z}\text{-mod})$, takes part in two semiorthogonal decompositions of the category $D^*(\mathbb{Z}\text{-mod})$. The left orthogonal complement to $D^*(\mathbb{Q}\text{-vect})$ in $D^*(\mathbb{Z}\text{-mod})$ is $D^*(\mathbb{Z}\text{-mod}_{\text{tors}})$, while the right orthogonal complement to $D^*(\mathbb{Q}\text{-vect})$ in
the same ambient category is $D^\star(Z-\text{mod}_{\text{rcot}})$. As a corollary to these two semiorthogonal decompositions, one has

$$D^\star(Z-\text{mod}_{\text{tors}}) \simeq D^\star(Z-\text{mod})/D^\star(Q-\text{vect}) \simeq D^\star(Z-\text{mod}_{\text{rcot}}).$$

Similarly in the equivalence (2), both $D^\star(Z-\text{mod}_{p\text{-tors}})$ and $D^\star(Z-\text{mod}_{p\text{-stra}})$ are full triangulated subcategories in $D^\star(Z-\text{mod})$. So is the derived category $D^\star(Z[p^{-1}]-\text{mod})$ of abelian groups with invertible action of $p$. Furthermore, the full triangulated subcategory $D^\star(Z[p^{-1}]-\text{mod}) \subset D^\star(Z-\text{mod})$ takes part in two semiorthogonal decompositions of the ambient category $D^\star(Z-\text{mod})$. The left orthogonal complement to $D^\star(Z[p^{-1}]-\text{mod})$ in $D^\star(Z-\text{mod})$ is $D^\star(Z-\text{mod}_{p\text{-tors}})$, while the right orthogonal complement is $D^\star(Z-\text{mod}_{p\text{-stra}})$. In the result, one obtains [24, Corollary 3.5]

$$D^\star(Z-\text{mod}_{p\text{-tors}}) \simeq D^\star(Z-\text{mod})/D^\star(Z[p^{-1}]-\text{mod}) \simeq D^\star(Z-\text{mod}_{p\text{-stra}}).$$

As we show in this paper, these results generalize to an arbitrary commutative ring $R$ with a multiplicative subset $S \subset R$ such that the projective dimension of the $R$-module $S^{-1}R$ does not exceed 1. In this setting, the full subcategory of $S$-contramodule $R$-modules $R-\text{mod}_{S\text{-stra}} \subset R-\text{mod}$ is an abelian category with an exact embedding functor $R-\text{mod}_{S\text{-stra}} \rightarrow R-\text{mod}$. The full subcategory of $S$-torsion $R$-modules $R-\text{mod}_{I\text{-tors}} \subset R-\text{mod}$ is always so (in fact, $R-\text{mod}_{I\text{-tors}}$ is even a Serre subcategory, which $R-\text{mod}_{S\text{-stra}}$ is not [25, Section 1]).

Let $D^\star_{S\text{-tors}}(R-\text{mod})$ and $D^\star_{S\text{-stra}}(R-\text{mod})$ denote the full subcategories of complexes with $S$-torsion and, respectively, $S$-contramodule cohomology modules. The derived category of $(S^{-1}R)$-modules $D^\star((S^{-1}R)-\text{mod})$ is a full triangulated subcategory in $D^\star(R-\text{mod})$ taking part in two semiorthogonal decompositions of $D^\star(R-\text{mod})$. The left orthogonal complement to $D^\star((S^{-1}R)-\text{mod})$ in $D^\star(R-\text{mod})$ coincides with $D^\star_{S\text{-tors}}(R-\text{mod})$, while the right orthogonal complement is $D^\star_{S\text{-stra}}(R-\text{mod})$. Hence the triangulated equivalences

$$D^\star_{S\text{-tors}}(R-\text{mod}) \simeq D^\star(R-\text{mod})/D^\star((S^{-1}R)-\text{mod}) \simeq D^\star_{S\text{-stra}}(R-\text{mod}). \tag{10}$$

Furthermore, when the $S$-torsion in the ring $R$ is bounded, the triangulated functors $D^\star(R-\text{mod}_{S\text{-tors}}) \rightarrow D^\star(R-\text{mod})$ and $D^\star(R-\text{mod}_{S\text{-stra}}) \rightarrow D^\star(R-\text{mod})$ are fully faithful. Their essential images coincide with the full subcategories $D^\star_{S\text{-tors}}(R-\text{mod})$ and $D^\star_{S\text{-stra}}(R-\text{mod}) \subset D^\star(R-\text{mod})$. So we have

$$D^\star_{S\text{-tors}}(R-\text{mod}) \simeq D^\star(R-\text{mod}_{S\text{-tors}}) \quad \text{and} \quad D^\star_{S\text{-stra}}(R-\text{mod}) \simeq D^\star(R-\text{mod}_{S\text{-stra}}) \tag{11}$$

Comparing (10) with (11), we obtain a triangulated equivalence between the derived categories of the abelian categories $R-\text{mod}_{S\text{-tors}}$ and $R-\text{mod}_{S\text{-stra}}$

$$D^\star(R-\text{mod}_{S\text{-tors}}) \simeq D^\star(R-\text{mod}_{S\text{-stra}}). \tag{12}$$

In the last section of this paper, we show that the triangulated equivalences (12) hold for the absolute derived categories $D^{\text{abs}+}$, $D^{\text{abs}–}$, and $D^{\text{abs}}$ of $S$-torsion modules and $S$-contramodule modules over a commutative ring $R$ with bounded $S$-torsion and $\text{pd}_S S^{-1}R \leq 1$ as well as for the conventional derived categories $D^+$, $D^–$, and $D$. It should be mentioned that a very different generalization of the Harrison–Matlis additive category equivalences was developed many years ago by Facchini in [6, 7] (see
also [13, Example 13.4]). This was further generalized and formulated in terms of fully faithful Verdier quotient functors between triangulated categories by Bazzoni [4]. Let us briefly point out one of the differences between our approaches. In Facchini’s papers, the aim was to study the additive categories of divisible and reduced modules. The restriction to torsion modules was viewed as undesirable and successfully removed. In the present paper, our aim is to study the abelian categories of $S$-torsion modules and their covariantly dual counterparts, the $S$-contramodules.

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1. Preliminaries

Let $R$ be an associative ring. We denote by $R\text{-mod}$ the abelian category of left $R$-modules. A pair of full subcategories $(T, F)$ in $R\text{-mod}$ is called a torsion theory [5] if one has $\text{Hom}_R(T, F) = 0$ for all $T \in T$ and $F \in F$, and for every left $R$-module $M$ there exists a short exact sequence $0 \rightarrow T \rightarrow M \rightarrow F \rightarrow 0$ with $T \in T$ and $F \in F$. In this case, such a short exact sequence is unique and functorial. Given an $R$-module $M$, the $R$-module $T$ is the maximal submodule of $M$ belonging to $T$, and the $R$-module $F$ is the maximal quotient module of $M$ belonging to $F$.

The full subcategory $T \subset R\text{-mod}$ is called the torsion class of a torsion theory $(T, F)$, and the full subcategory $F \subset R\text{-mod}$ is called the torsion-free class. For any torsion theory $(T, F)$ in $R\text{-mod}$, the torsion class $T$ is closed under the passages to arbitrary quotient objects, extensions, and infinite direct sums, while the torsion-free class $F$ is closed under the passages to subobjects, extensions, and infinite products in $R\text{-mod}$. Since the direct sum of a family of modules is a submodule of their product, it follows that the class $F$ is closed under infinite direct sums, too.

Conversely, any full subcategory $T \subset R\text{-mod}$ closed under quotient objects, extensions, and infinite direct sums is the torsion class of a certain torsion theory $(T, F)$, and any full subcategory $F \subset R\text{-mod}$ closed under subobjects, extensions, and infinite products is the torsion-free class of a torsion theory $(T, F)$. The complementary class can be uniquely recovered by the rules that $F$ consists of all the $R$-modules $F$ such that $\text{Hom}_R(T, F) = 0$ for all $T \in T$, and $T$ consists of all the $R$-modules $T$ such $\text{Hom}_R(T, F) = 0$ for all $F \in F$.

A torsion theory $(T, F)$ is called hereditary if the class $T \subset R\text{-mod}$ is closed under quotient objects. In this case, $T$ is a Serre subcategory in $R\text{-mod}$; so, in particular, it is an abelian category with an exact embedding functor $T \rightarrow R\text{-mod}$.

From now on and for the rest of this paper, let $R$ be a commutative ring and $S \subset R$ be a multiplicative subset. An element $x \in M$ in an $R$-module $M$ is said to be $S$-torsion if there exists $s \in S$ such that $sx = 0$ in $M$. The submodule of all
Let $S$-torsion elements in $M$ be denoted by $\Gamma_S(M) \subset M$ and the embedding morphism $\Gamma_S(M) \rightarrow M$ is denoted by $\gamma_{S,M}$.

We will use the notation $S^{-1}M = S^{-1}R \otimes_R M$ for the $S$-localization of an $R$-module $M$. An $R$-module $M$ is said to be $S$-torsion if $\Gamma_S(M) = M$, or equivalently, if $S^{-1}M = 0$. An $R$-module $M$ is said to be $S$-torsion-free if $\Gamma_S(M) = 0$. The full subcategories of $S$-torsion $R$-modules and $S$-torsion free $R$-modules form a hereditary torsion theory in $R\mathbf{-mod}$; the related canonical short exact sequence is $0 \rightarrow \Gamma_S(M) \rightarrow M \rightarrow M/\Gamma_S(M) \rightarrow 0$ for any $R$-module $M$. In particular, the full subcategory $R\mathbf{-mod}_{S\text{-tors}}$ of all $S$-torsion $R$-modules is an abelian category with an exact embedding functor $R\mathbf{-mod}_{S\text{-tors}} \rightarrow R\mathbf{-mod}$.

An $R$-module $M$ is said to be $S$-divisible if for every element $s \in S$ the action map $s: M \rightarrow M$ is surjective. An $R$-module $M$ is said to be $S$-reduced if it has no $S$-divisible $R$-submodules. The full subcategories of $S$-divisible and $S$-reduced $R$-modules form a torsion theory in $R\mathbf{-mod}$; the $S$-divisible modules are the torsion class and the $S$-reduced $R$-modules are the torsion-free class. In addition to the general closure properties of such classes, the class of all $S$-divisible $R$-modules is also closed under infinite products. An $R$-module $M$ is both $S$-torsion-free and $S$-divisible if and only if it is an $(S^{-1}R)$-module.

An $R$-module $M$ is said to be $S$-h-divisible if is a quotient $R$-module of an $(S^{-1}R)$-module. Clearly, every $S$-h-divisible $R$-module is $S$-divisible. The class of all $S$-h-divisible $R$-modules is closed under quotient objects, infinite direct sums, and infinite products in $R\mathbf{-mod}$, but it is not always closed under extensions. Every $R$-module $M$ has a unique maximal $S$-h-divisible submodule $h_S(M) \subset M$, which can be constructed as the image of the natural map $\text{Hom}_R(S^{-1}R, M) \rightarrow M$.

An $R$-module $M$ is said to be $S$-h-reduced if it has no $S$-h-divisible submodules, or equivalently, if $\text{Hom}_R(S^{-1}R, M) = 0$. An $R$-module $M$ is $S$-h-reduced if and only if $h_S(M) = 0$, but the quotient module $M/h_S(M)$ for an arbitrary $R$-module $M$ is not always $S$-h-reduced. The class of all $S$-h-reduced $R$-modules is closed under subobjects, extensions, infinite direct sums, and infinite products in $R\mathbf{-mod}$. Every $S$-reduced $R$-module is $S$-h-reduced. Every $S$-h-reduced $S$-torsion-free $R$-module is $S$-reduced (because every $S$-divisible $S$-torsion-free $R$-module is $S$-h-divisible).

So the full subcategories of $S$-h-divisible and $S$-h-reduced $R$-modules do not form a torsion theory in $R\mathbf{-mod}$ in general. However, when $\text{pd}_R S^{-1}R \leq 1$, the problem disappears and these two classes do form a torsion theory, as we will see below in Lemmas 1.8 and 5.1. Furthermore, when all the elements of $S$ are nonzero-divisors in $R$ and $\text{pd}_R S^{-1}R \leq 1$, the classes of $S$-divisible and $S$-h-divisible $R$-modules coincide [14, Theorem 2.6], [9, Theorem 3.2], [1, Proposition 6.4]. Hence the classes of $S$-reduced and $S$-h-reduced $R$-modules also coincide.

**Lemma 1.1.** Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence of $R$-modules such that the $R$-module $N$ is $S$-h-divisible, while the $R$-module $L$ is $S$-torsion-free and $S$-h-divisible. Then the $R$-module $M$ is $S$-h-divisible.

**Proof.** Let $N$ be the quotient $R$-module of an $(S^{-1}R)$-module $D$. Pulling back the short exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ with respect to the morphism
Lemma 1.2. Let $M$ and $Ext^0_D$, where $D 	o Ext^t$ is provided by Lemma 1.3(a). Now let $f$ Closedness with respect to infinite products is obvious, and closedness under ex-

$\text{Proof.}$ Arguing by increasing induction in $S$, this is essentially a particular case of the first assertion of [11, Proposition 1.1].

Lemma 1.3. $kernels, extensions, infinite products, and projective lim-

$\text{Proof.}$ This is [19, Lemma 1.1] or [20, Theorem 1.5]. The proof is obvious. □

Lemma 1.4. The full subcategory of $S$-contramodule $R$-modules is closed under the kernels, extensions, infinite products, and projective limits in $R-mod$.

$\text{Proof.}$ This is essentially a particular case of the first assertion of [11, Proposition 1.1]. Closedness with respect to infinite products is obvious, and closedness under ex-

$D \to N$, we obtain a short exact sequence of $R$-modules $0 \to L \to E \to D \to 0$, where $D$ and $L$ are $(S^{-1}R)$-modules. It follows that $E$ is an $(S^{-1}R)$-module, too; and $M$ is a quotient $R$-module of $E$. □

Lemma 1.2. Let $C$ be an $R$-module and $D$ be an $(S^{-1}R)$-module. Assume that $\text{Ext}^i_R((S^{-1}R), C) = 0$ for all $0 \leq i \leq n$, where $n \geq 0$ is a fixed integer. Then $\text{Ext}^i_R(D, C) = 0$ for all $0 \leq i \leq n$.

$\text{Proof.}$ Arguing by increasing induction in $i$, pick a short exact sequence of $(S^{-1}R)$-modules $0 \to B \to F \to D \to 0$ with a free $(S^{-1}R)$-module $F$. Then we have an exact sequence

$\cdots \to \text{Ext}^{i-1}_R(B, C) \to \text{Ext}^i_R(D, C) \to \text{Ext}^i_R(F, C) \to \cdots$

where $\text{Ext}^i_R(F, C) = 0$ since $\text{Ext}^i_R((S^{-1}R), C) = 0$, and $\text{Ext}^{i-1}_R(B, C) = 0$ by the induction assumption. □

An $R$-module $C$ is said to be $S$-cotorsion if $\text{Ext}^1_R((S^{-1}R), C) = 0$, and strongly $S$-cotorsion if $\text{Ext}^n_R((S^{-1}R), C) = 0$ for all $n \geq 1$. An $R$-module $C$ is called an $S$-contramodule if it is $S$-h-reduced and $S$-cotorsion, that is $\text{Ext}^n_R((S^{-1}R), C) = 0$ for $n = 0$ and 1. An $R$-module $C$ is called a strong $S$-contramodule if it is $S$-h-reduced and strongly $S$-cotorsion, that is $\text{Ext}^n_R((S^{-1}R), C) = 0$ for all $n \geq 0$.

When $R$ is a domain and $S = R \setminus \{0\}$, our (strong) $S$-contramodules are what are called “(strongly) cotorsion $R$-modules” in [19]. For any commutative ring $R$ with a multiplicative subset $S$, the full subcategory of $S$-contramodules in $R-mod$ is the “right perpendicular category” to the $R$-module $S^{-1}R$, as defined by Geigle and Lenzing in [11, Section 1] (cf. Theorem 3.4 below).

Lemma 1.3. Let $0 \to C' \to C \to C'' \to 0$ be a short exact sequence of $R$-modules. Then

(a) if $C'$ and $C''$ are $S$-contramodules, then $C$ is an $S$-contramodule;

(b) if $C$ is an $S$-contramodule, then $C''$ is $S$-h-reduced if and only if $C'$ is an

S-contramodule;

(c) if $C$ is an $S$-contramodule and $C'$ is a strong $S$-contramodule, then $C''$ is an

$S$-contramodule.

$\text{Proof.}$ This is [19, Lemma 1.1] or [20, Theorem 1.5]. The proof is obvious. □

Lemma 1.4. The full subcategory of $S$-contramodule $R$-modules is closed under the kernels, extensions, infinite products, and projective limits in $R-mod$.

$\text{Proof.}$ This is essentially a particular case of the first assertion of [11, Proposition 1.1]. Closedness with respect to infinite products is obvious, and closedness under ex-

$\text{Lemma 1.3(a).}$ Now let $f: C \to D$ be a morphism of $S$-contramodule $R$-modules; set $I = \text{im}(f)$ and $E = \text{ker}(f)$. Then the $R$-module $I$ is $S$-h-reduced as a submodule of an $S$-h-reduced $R$-module $D$. Applying Lemma 1.3(b) to the short exact sequence $0 \to E \to C \to I \to 0$, we conclude that $E$ is an $S$-contramodule $R$-module. Finally, the projective limit of any diagram is the kernel of a certain morphism between infinite products. □
Following the traditional notation of $K = Q/R$, where $Q$ is the field of fractions of a commutative domain $R$, we denote by $K^\bullet$ the two-term complex (9)

$$R \longrightarrow S^{-1}R,$$

where the term $R$ sits in the cohomological degree $-1$ and the term $S^{-1}R$ in the cohomological degree 0. The cokernel $H^0(K^\bullet)$ of the morphism $R \longrightarrow S^{-1}R$ will be denoted simply by $S^{-1}R/R$.

When all the elements of $S$ are nonzero-divisors in $R$, so the morphism $R \longrightarrow S^{-1}R$ is injective, one can use the quotient module $S^{-1}R/R$ in lieu of the two-term complex $K^\bullet$.

We will use the special notation

$$\text{Tor}_n^R(K^\bullet, M) = H^{-n}(K^\bullet \otimes_R M) = H^{-n}(K^\bullet \otimes_R M)$$

and

$$\text{Ext}_n^R(K^\bullet, M) = \text{Hom}_{\mathcal{D}^b(R-\text{mod})}(K^\bullet, M[n])$$

for an $R$-module $M$ (treating $K^\bullet$ as if it were a module rather than a complex).

**Lemma 1.5.** For every $R$-module $M$, one has

(a) $\text{Tor}_n^R(K^\bullet, M) = 0 = \text{Ext}_n^R(K^\bullet, M)$ for all $n < 0$;

(b) $\text{Tor}_n^R(K^\bullet, M) = 0$ and $\text{Ext}_n^R(K^\bullet, M) = \text{Ext}_n^R(S^{-1}R, M)$ for all $n > 1$;

(c) $\text{Tor}_0^R(K^\bullet, M) = (S^{-1}R/R) \otimes_R M$ and $\text{Ext}_0^R(K^\bullet, M) = \text{Hom}_R(S^{-1}R/R, M)$;

(d) $\text{Tor}_1^R(K^\bullet, M) = \Gamma_S(M)$.

**Proof.** All the assertions follow easily from the (co)homology long exact sequences related to the distinguished triangle

$$R \longrightarrow S^{-1}R \longrightarrow K^\bullet \longrightarrow R[1]$$

in $\mathcal{D}^b(R-\text{mod})$. \hfill \Box

Warning: it may well happen that $\text{Tor}_n^R(K^\bullet, F) \neq 0$ for a flat $R$-module $F$. Similarly, one may have $\text{Ext}_1^R(K^\bullet, J) \neq 0$ for an injective $R$-module $J$.

Following the exposition in [19] (see also [20, Theorem 1.1]), we introduce special indexing for the three short exact sequences of low-dimensional Tor and Ext related to the distinguished triangle (13). Concerning the Tor, for any $R$-module $M$ we have

(I) \hspace{1cm} 0 \longrightarrow M/\Gamma_S(M) \longrightarrow S^{-1}R \otimes_R M \longrightarrow \text{Tor}_0^R(K^\bullet, M) \longrightarrow 0.

Concerning the Ext, for any $R$-module $C$ we have an exact sequence

$$0 \longrightarrow \text{Ext}_0^R(K^\bullet, C) \longrightarrow \text{Hom}_R(S^{-1}R, C) \longrightarrow C \longrightarrow \text{Ext}_1^R(K^\bullet, C) \longrightarrow \text{Ext}_1^R(S^{-1}R, C) \longrightarrow 0,$$

which can be rewritten in the form of two short exact sequences

(II) \hspace{1cm} 0 \longrightarrow \text{Ext}_0^R(K^\bullet, C) \longrightarrow \text{Hom}_R(S^{-1}R, C) \longrightarrow h_S(C) \longrightarrow 0,

(III) \hspace{1cm} 0 \longrightarrow C/h_S(C) \longrightarrow \text{Ext}_1^R(K^\bullet, C) \longrightarrow \text{Ext}_1^R(S^{-1}R, C) \longrightarrow 0.
Let us introduce the notation $\Delta_S(C) = \text{Ext}^1_R(K^\bullet, C)$, and denote by $\delta_{S,C}$ the natural map $C \rightarrow \Delta_S(C)$.

**Lemma 1.6.** (a) An $R$-module $C$ is an $S$-contramodule if and only if the map $\delta_{S,C}: C \rightarrow \text{Ext}^1_R(K^\bullet, C)$ is an isomorphism.

(b) Any $R$-module annihilated by the action of an element from $S$ is a strong $S$-contramodule.

**Proof.** Part (a): from the short exact sequences (II–III) we see that the equations $\text{Hom}_R(S^{-1}R, C) = 0 = \text{Ext}^1_R(S^{-1}R, C)$ imply that the map $\delta_{S,C}$ is an isomorphism. Conversely, if $\delta_{S,C}$ is an isomorphism, then $\text{Ext}^1_R(S^{-1}R, C) = 0$ and $h_S(C) = 0$. The latter implies that $\text{Hom}_R(S^{-1}R, C) = 0$.

Part (b): if $rC = 0$ for some $r \in R$, then $r\text{Ext}^*_R(M, C) = 0$ for every $R$-module $M$. If $M$ is an $(S^{-1}R)$-module, then the $R$-modules $\text{Ext}^*_R(M, C)$ are $(S^{-1}R)$-modules annihilated by the action of $r$. When $r \in S$, these can only be zero modules. \qed

We denote by $\text{pd}_R M$ the projective dimension of an $R$-module $M$.

An $R$-module $M$ is said to have bounded $S$-torsion if there exists $r \in S$ such that $r\Gamma_S(M) = 0$. An $R$-module $M$ is said to have no $S$-$h$-divisible $S$-torsion if the $R$-module $\Gamma_S(M)$ is $S$-$h$-reduced. Clearly, every $R$-module with bounded $S$-torsion has no $S$-divisible $S$-torsion. The following lemma is our version (of the most important special case) of [19, Theorem 2.1]; see also [1, Lemma 6.1 and Proposition 6.3] and [3, Theorem 3.5].

**Lemma 1.7.** (a) For any $R$-module $C$, the $R$-module $\text{Ext}^0_R(K^\bullet, C)$ is an $S$-contramodule.

(b) If there is no $S$-$h$-divisible $S$-torsion in $C$, then the $R$-module $\text{Ext}^1_R(K^\bullet, C)$ is an $S$-contramodule.

(c) If $\text{pd}_R S^{-1}R \leq 1$, then for any $R$-module $C$ the $R$-module $\text{Ext}^1_R(K^\bullet, C)$ is an $S$-contramodule.

**Proof.** There is a spectral sequence

$$E_2^{pq} = \text{Ext}_R^p(S^{-1}R, \text{Ext}_R^q(K^\bullet, C)) \Rightarrow E_\infty^{pq} = \text{gr}^p \text{Ext}_R^{p+q}(S^{-1}R \otimes_R K^\bullet, C) = 0,$$

where $\text{Ext}_R^q(S^{-1}R \otimes_R K^\bullet, C) = \text{Hom}_{\text{D}^b(R:\text{mod})}(S^{-1}R \otimes_R K^\bullet, C[n]) = 0$ for all $n \in \mathbb{Z}$ and all $C \in R$-mod, because the complex $S^{-1}R \otimes_R K^\bullet$ is contractible. The differentials are $d_r^{pq}: E_r^{pq} \rightarrow E_r^{p+q-r, q+r+1}$, $r \geq 2$. Now all the differentials going through $E_r^{0,0}$ and $E_r^{1,0}$ vanish for the dimension reasons, so $E_\infty^{0,0} = 0 = E_\infty^{1,0}$ implies $E_2^{0,0} = 0 = E_2^{1,0}$. This proves part (a). Furthermore, the only possibly nontrivial differentials going through $E_2^{0,1}$ and $E_2^{1,1}$ are

$$d_2^{0,1}: E_2^{0,1} \rightarrow E_2^{2,0} \text{ and } d_2^{1,1}: E_2^{1,1} \rightarrow E_2^{3,0}.$$

When $\text{pd}_R S^{-1}R \leq 1$, one has $E_2^{pq} = 0$ for $p \geq 2$. When there is no $S$-$h$-divisible $S$-torsion in $C$, one has $\text{Ext}_R^0(K^\bullet, C) = 0$ by Lemma 1.5(c), because $S^{-1}R/R$ is an $S$-$h$-divisible $S$-torsion $R$-module. Hence $E_2^{0,0} = 0$ for all $p \geq 0$. In both cases, $E_\infty^{0,1} = 0 = E_\infty^{1,1}$ implies $E_2^{0,1} = 0 = E_2^{1,1}$, proving parts (b) and (c). \qed
The following lemma is our version of [20, Lemma 1.8 and Theorem 1.9].

**Lemma 1.8.** (a) If \( \text{pd}_R S^{-1}R \leq 1 \), then the class of all \( S \)-h-divisible \( R \)-modules is closed under extensions.

(b) If all the elements of \( S \) are nonzero-divisors in \( R \) and the class of all \( S \)-h-divisible \( R \)-modules is closed under extensions, then \( \text{pd}_R S^{-1}R \leq 1 \).

**Proof.** Part (a): it suffices to check that \( h_S(M/h_S(M)) = 0 \) for every \( R \)-module \( M \). Indeed, according to the short exact sequence (III), the quotient module \( M/h_S(M) \) is a submodule in \( \text{Ext}^1_R(K^\bullet, M) \), and by Lemma 1.7(c), the \( R \)-module \( \text{Ext}^1_R(K^\bullet, M) \) is \( S \)-h-reduced. (For another argument, see the paragraph after Lemma 5.1 below.)

Part (b): let \( E \) be an \( R \)-module; we have to prove that \( \text{Ext}^2_R(S^{-1}R, E) = 0 \). Pick an injective \( R \)-module \( J \) such that \( E \) is a submodule in \( J \). Then \( \text{Ext}^2_R(S^{-1}R, E) = \text{Ext}^1_R(S^{-1}R, J/E) \). As any \( R \)-module morphism \( R \to J \) can be extended to an \( R \)-module morphism \( S^{-1}R \to J \), the \( R \)-module \( J \) is \( S \)-h-divisible. Hence so is the \( R \)-module \( L = J/E \). Let \( 0 \to L \to M \to S^{-1}R \to 0 \) be a short exact sequence of \( R \)-modules. By assumption, the \( R \)-module \( M \) has to be \( S \)-h-divisible; so it is a quotient module of an \( (S^{-1}R) \)-module \( D \). The composition of surjective morphisms \( D \to M \to S^{-1}R \) is a morphism of \( (S^{-1}R) \)-modules; so it is a split surjection. Composing a splitting \( S^{-1}R \to D \) with the morphism \( D \to M \), we obtain a splitting of the surjection \( M \to S^{-1}R \). Thus \( \text{Ext}^1_R(S^{-1}R, L) = 0 \). \( \square \)

In other words, when \( \text{pd}_R S^{-1}R \leq 1 \), the full subcategories of \( S \)-h-divisible and \( S \)-h-reduced \( R \)-modules form a torsion theory in \( R\text{-mod} \). The \( S \)-h-divisible modules are the torsion class and the \( S \)-h-reduced modules are the torsion-free class.

The next lemma is quite standard. We include it here for the sake of completeness of the exposition.

**Lemma 1.9.** If a multiplicative subset \( S \) in a commutative ring \( R \) is countable, then \( \text{pd}_R S^{-1}R \leq 1 \).

**Proof.** Let \( s_1, s_2, s_3, \ldots \) be a sequence of elements of \( S \) such that every element of \( S \) appears at least once in this sequence. Denote by \( M \) the inductive limit of the sequence of \( R \)-module morphisms

\[
R \xrightarrow{s_1} R \xrightarrow{s_1s_2} R \xrightarrow{s_1s_2s_3} R \to \cdots \to R \xrightarrow{s_1\cdots s_i} R \to \cdots
\]

The natural map \( R \to M \) has the property that its kernel and cokernel are \( S \)-torsion \( R \)-modules. Furthermore, the \( R \)-module \( M \) is \( S \)-torsion-free and \( S \)-divisible. Hence \( M \simeq S^{-1}R \). Now the telescope construction of countable filtered inductive limits provides a two-term free \( R \)-module resolution of the \( R \)-module \( S^{-1}R \),

\[
0 \to \bigoplus_{n=1}^\infty R \to \bigoplus_{n=1}^\infty R \to S^{-1}R \to 0.
\]

\( \square \)

The following important lemma is our version of [19, Proposition 2.4].
Lemma 1.10. Let \( b: A \rightarrow B \) and \( c: A \rightarrow C \) be two \( R \)-module morphisms such that \( C \) is an \( S \)-contramodule, while \( \ker(b) \) is an \( S \)-h-divisible \( R \)-module and \( \coker(b) \) is an \( (S^{-1}R) \)-module. Then there exists a unique morphism \( f: B \rightarrow C \) such that \( c = fb \), i. e., the triangle diagram \( A \rightarrow B \rightarrow C \) is commutative.

Proof. Any morphism \( f': B \rightarrow C \) such that \( f'b = 0 \) would factorize as \( B \rightarrow \coker(b) \rightarrow C \), and any morphism \( \coker(b) \rightarrow C \) vanishes when \( \coker(b) \) is \( S \)-h-divisible and \( C \) is \( S \)-h-reduced. This proves uniqueness. To check existence, notice that the composition \( \ker(b) \rightarrow A \rightarrow C \) vanishes if \( \ker(b) \) is \( S \)-h-divisible and \( C \) is \( S \)-h-reduced. Hence we have a short exact sequence \( 0 \rightarrow A/\ker(b) \rightarrow B \rightarrow \coker(b) \rightarrow 0 \) and a morphism \( \bar{b}: A/\ker(b) \rightarrow C \). The obstruction to extending the morphism \( \bar{b} \) to an \( R \)-module morphism \( B \rightarrow C \) lies in the group \( \text{Ext}^1_R(\coker(b), C) \). Applying Lemma 1.2 to the \( R \)-modules \( C \) and \( D = \coker(b) \) and the integer \( n = 1 \), we conclude that this Ext group vanishes. \( \Box \)

Lemma 1.11. For any \( R \)-module \( A \) and every element \( s \in S \), the map \( \bar{\delta}_{S,A}: A/sA \rightarrow \Delta_S(A)/s\Delta_S(A) \) is an isomorphism.

Proof. Both \( A/sA \) and \( \Delta_S(A)/s\Delta_S(A) \) are \( R/sR \)-modules. Furthermore, any \( R/sR \)-module \( D \) is an \( S \)-contramodule by Lemma 1.6(b). In view of the short exact sequence (III), the morphism \( \bar{\delta}_{S,A}: A \rightarrow \Delta_S(A) \) and any morphism \( A \rightarrow D \) satisfy the conditions of Lemma 1.10, so there exists a unique morphism \( \Delta_S(A) \rightarrow D \) making the triangle diagram \( A \rightarrow \Delta_S(A) \rightarrow D \) commutative. Hence the morphism \( \bar{\delta}_{S,A} \) induces an isomorphism

\[
\text{Hom}_{R/sR}(\Delta_S(A)/s\Delta_S(A), D) \simeq \text{Hom}_{R/sR}(A/sA, D),
\]

and it follows that \( \bar{\delta}_{S,A} \) is an isomorphism. \( \Box \)

2. The \( S \)-Topology

The \( R \)-topology was introduced by Nunke in [21, Section 6] and studied by Matlis in [19, §6] (see also [20, Chapter II]). The \( S \)-topology is discussed in [12] and [10, Section VIII.4] in the case of a countable multiplicative subset \( S \) consisting of nonzero-divisors (the discussion in [13, Chapter 1] partly avoids the countability assumption; cf. the counterexample in [13, Section 1.2]).

As in Section 1, let \( R \) be a commutative ring and \( S \subset R \) be a multiplicative subset. By the definition, the \( S \)-topology on an \( R \)-module \( A \) is the topology with a base of neighborhoods of zero formed by the submodules \( sA \subset A \), where \( s \in S \) are arbitrary elements. The \( S \)-completion of an \( R \)-module \( A \) is defined as

\[
\Lambda_S(A) = \lim_{\leftarrow\, s \in S} A/sA,
\]

where the partial (pre)order on the set \( S \) is defined by the rule that \( s \leq t \) for \( s, t \in S \) if there exists \( r \in R \) for which \( t = rs \) (this is the reverse inclusion order on
the principal ideals in $R$ generated by elements from $S$). There is an obvious natural $R$-module morphism

$$\lambda_{S:A}: A \longrightarrow \Lambda_S(A).$$

An $R$-module $A$ is said to be $S$-separated if the morphism $\lambda_{S:A}$ is injective. An $R$-module $A$ is said to be $S$-complete if the morphism $\lambda_{S:A}$ is surjective. We will denote the full subcategory of $S$-separated and $S$-complete $R$-modules by $R-\text{mod}_{S,\text{secmp}} \subset R-\text{mod}$.

**Lemma 2.1.** For every $R$-module $A$

(a) the $R$-module $\Lambda_S(A)$ is an $S$-contramodule;

(b) there exists a unique $R$-module morphism $\beta_{S:A}: \Delta_S(A) \longrightarrow \Lambda_S(A)$ making the triangle diagram $A \longrightarrow \Delta_S(A) \longrightarrow \Lambda_S(A)$ commutative. Taken together for all the $R$-modules $A$, the morphisms $\beta_{S:A}$ form a morphism of functors $\beta_S: \Delta_S \longrightarrow \Lambda_S$.

**Proof.** Part (a) follows from Lemmas 1.6(b) and 1.4. Part (b) is provided by Lemma 1.10 together with part (a) and the short exact sequence (III). The second assertion in part (b) (commutativity of the diagram in the definition of a natural transformation of functors) follows from the uniqueness claim in Lemma 1.10. \qed

The following proposition, which is a direct generalization of [10, Lemma VIII.4.1] provable by essentially the same method, may help the reader feel more comfortable. We will not use its part (b) in this paper.

**Proposition 2.2.** (a) The $R$-module $\Lambda_S(A)$ is $S$-separated for every $R$-module $A$.

(b) Suppose that the multiplicative set $S$ is countable. Then the $R$-module $\Lambda_S(A)$ is $S$-separated and $S$-complete for every $R$-module $A$. Moreover, the functor $\Lambda_S$ is left adjoint to the fully faithful embedding functor $R-\text{mod}_{S,\text{secmp}} \longrightarrow R-\text{mod}$.

**Proof.** By the definition, the $R$-module $\Lambda_S(A)$ is separated and complete in the projective limit topology, where a base of neighborhoods of zero is formed the kernel submodules $U_s \subset \Lambda_S(A)$, $s \in S$ of the projection maps $\Lambda_S(A) \longrightarrow A/sA$. In particular, one has $\bigcap_{s \in S} U_s = 0$ in $\Lambda_S(A)$. Clearly, the inclusions $s\Lambda_S(A) \subset U_s$ hold for all $s \in S$. Hence we have $\bigcap_{s \in S} s\Lambda_S(A) = 0$, and part (a) follows.

To prove part (b), let us show that under its assumption $U_s = s\Lambda_S(A)$ for every $s \in S$. Let $1 = s_0 \leq s_1 \leq s_2 \leq \cdots$ be an increasing chain of elements from $S$ such that for every $r \in S$ there exists $n \geq 1$ for which $r \leq s_n$. For every $R$-module $A$ and every sequence of elements $c_0, c_1, c_2, \ldots$ in $\Lambda_S(A)$ define the expression

$$\sum_{n=0}^{\infty} s_n c_n \in \Lambda_S(A)$$

as the limit of finite partial sums in the projective limit topology of $\Lambda_S(A)$. In other words, if the element $c_n \in \Lambda_S(A)$ is represented by a family of cosets $(c_{n,r} + rA \in A/rA)_{r \in S}$, $c_{n,r} \in A$, then the element $\sum_{n=0}^{\infty} s_n c_n$ is given by $(\sum_{n=0}^{\infty} s_n c_{n,r} + rA)_{r \in S}$, where the sum is essentially finite modulo $rA$ for every $r \in S$ due to our condition on the sequence of elements $s_n$.

Now set $t_0 = 1$ and $t_n = s_1 \cdots s_n$ for every $n \geq 1$; this sequence of elements in $S$ also satisfies our conditions. Let $c \in \Lambda_S(A)$ be an element represented by a family of
cosets \((c_r + rA)_{r \in S}\). Then \(c_{n+1} - c_n \in t_n A\) for every \(n \geq 0\). Set \(a_0 = c_1 \in A\) and choose \(a_n \in A\) such that \(c_{n+1} - c_n = t_n a_n\) for every \(n \geq 1\). Then we have

\[
c = \sum_{n=0}^{\infty} t_n \lambda_{S/A}(a_n) \in \Lambda_S(A).
\]

Assume that \(c \in U_s\), and choose \(n_0 \geq 1\) such that \(s \leq t_{n_0}\) in \(S\). Then \(\lambda_{S/A}(\sum_{n=0}^{n_0-1} t_n a_n) = \sum_{n=0}^{n_0-1} t_n \lambda_{S/A}(a_n) \in U_s\), hence \(\sum_{n=0}^{n_0-1} t_n a_n \in sA\). At last, we have

\[
\sum_{n=n_0}^{\infty} t_n \lambda_{S/A}(a_n) = t_{n_0} \sum_{n=0}^{\infty} s_{n_0+1} \cdots s_{n_0+n} \lambda_{S/A}(a_{n_0+n}) \in s\Lambda_S(A),
\]
hence \(c \in s\Lambda_S(A)\) (notice that the sequence of elements \(1, s_{n_0+1}, s_{n_0+1}s_{n_0+2}, \ldots \in S\) also satisfies our conditions, so the sum in the right-hand side converges).

We have shown that the \(S\)-topology on \(\Lambda_S(A)\) coincides with the projective limit topology; so it is separated and complete. Finally, we have \(\Lambda_S(A)/s\Lambda_S(A) = \Lambda_S(A)/U_s = A/sA\) for every \(s \in A\). Let \(D\) be an \(R\)-module belonging to \(R\text{-}\text{mod}_{S\text{-secmp}}\). Then \(D = \lim_{s \in S} D/sD\). So, for any \(R\)-module \(A\), morphisms of \(R\)-modules \(A \rightarrow D\) correspond bijectively to compatible systems of \(R\)-module morphisms \(A \rightarrow D/sD\), or, which is the same, \(A/sA \rightarrow D/sD\). The isomorphism \(A/sA \cong \Lambda_S(A)/s\Lambda_S(A)\) now implies an isomorphism of the Hom modules

\[
\text{Hom}_R(A, D) \cong \text{Hom}_R(\Lambda_S(A), D),
\]

providing the desired adjunction (cf. the proof of [25, Theorem 5.8]).

Without the countability assumption, a long list of conditions equivalent to \(S\)-completeness of the \(R\)-module \(\Lambda_S(A)\) for a given \(R\)-module \(A\), in the case of a domain \(R\) and \(S = R \setminus \{0\}\), is presented in [19, Theorem 6.8]. All of them appear to be also equivalent in our generality. In the next theorem, we list only some of these equivalent conditions from [19], and add a couple of our own.

As in the proof of Proposition 2.2, for every element \(s \in S\) we denote by \(U_s \subset \Lambda_S(A)\) the kernel of the projection \(\Lambda_S(A) \rightarrow A/sA\). Following [19], we denote for brevity by \(\Pi_A\) the infinite product \(\prod_{s \in S} A/sA\). The \(R\)-module \(\Lambda_S(A)\) is naturally a submodule in \(\Pi_A\). Given a submodule \(N \subset M\) in an \(R\)-module \(M\), we say that \(N\) is \(S\)-pure in \(M\) if for every \(s \in S\) one has \(N \cap sM = sN\). One easily notices that \(\text{im}(\lambda_{S,A})\) is always an \(S\)-pure submodule in \(\Lambda_S(A)\); moreover, \(\lambda^{-1}_{S,A}(s\Lambda_S(A)) = sA\) for every \(s \in S\).

**Theorem 2.3.** For any commutative ring \(R\), multiplicative subset \(S \subset R\), and an \(R\)-module \(A\), the following conditions are equivalent:

(i) the \(R\)-module \(\Lambda_S(A)\) is (\(S\)-separated and) \(S\)-complete;

(ii) the \(S\)-topology on \(\Lambda_S(A)\) coincides with the projective limit topology;

(iii) for every element \(s \in S\), the submodule \(U_s \subset \Lambda_S(A)\) coincides with \(s\Lambda_S(A)\);

(iv) for every element \(s \in S\), the map \(\lambda_{S,A}: A/sA \rightarrow \Lambda_S(A)/s\Lambda_S(A)\) is an isomorphism;

(v) for every module \(D \in R\text{-}\text{mod}_{S\text{-secmp}}\) and every \(R\)-module morphism \(A \rightarrow D\) there exists a unique \(R\)-module morphism \(\Lambda_S(A) \rightarrow D\) making the triangle diagram \(A \rightarrow \Lambda_S(A) \rightarrow D\) commutative;
(vi) the $R$-module $\text{coker}(\lambda_{S,A})$ is $S$-divisible;
(vii) $\Lambda_S(A)$ is an $S$-pure submodule in $\Pi_A$.

Proof. The condition (ii) implies (i), because a projective limit is always complete in its projective limit topology. The implication (iii) $\implies$ (ii) is clear (see the discussion in the proof of Proposition 2.2(a)).

To compare (iii) with (iv), notice that we have in fact two natural maps

$$\begin{array}{c}
A/sA \longrightarrow \Lambda_S(A)/s\Lambda_S(A) \longrightarrow A/sA
\end{array}$$

with the composition equal to the identity map. The condition (iv) says that the leftmost arrow is an isomorphism (or surjective), while (iii) means that the rightmost arrow is an isomorphism (or injective). These are clearly two equivalent conditions.

To see that (i) implies (iii) and (iv), we notice that (14) implies that $\Lambda_S(A)$ is a direct summand in $\Lambda_S(\Lambda_S(A))$ (this observation comes from [29, Theorem 1.5 and Corollary 1.7]). When the complementary direct summand vanishes, one easily concludes that the maps (14) are isomorphisms. To make this argument more explicit, one can follow [19] in considering the two morphisms

$$\lambda_{S,\Lambda_S(A)} \text{ and } \Lambda_S(\lambda_{S,A}) : \Lambda_S(A) \longrightarrow \Lambda_S(\Lambda_S(A)).$$

Assume that $\Lambda_S(A)$ is $S$-complete, that is the map $\lambda_{S,\Lambda_S(A)}$ is surjective. Let $x$ be an element in $\Lambda_S(A)$; then there exists $y \in \Lambda_S(A)$ such that $\lambda_{S,\Lambda_S(A)}(y) = \Lambda_S(\lambda_{S,A})(x)$. Let the components of $x$ be $(x_t + tA)_{t \in S}$ and the components of $y$ be $(y_t + tA)_{t \in S}$. Then we have

$$\lambda_{S,\Lambda_S(A)}(y) = (y + t\Lambda_S(A))_{t \in S} \quad \text{and} \quad \Lambda_S(\lambda_{S,A})(x) = (\lambda_{S,A}(x_t) + t\Lambda_S(A))_{t \in S}.$$

Hence $y - \lambda_{S,A}(x_t) \in t\Lambda_S(A)$. Comparing the $t$-components, we obtain $y_t - x_t \in tA$, hence $y = x$ in $\Lambda_S(A)$. We have shown that

$$x - \lambda_{S,A}(x_s) \in s\Lambda_S(A) \quad \text{for all } s \in S,$$

and it follows that $U_s = s\Lambda_S(A)$.

Applying (v) in the case of an $R$-module $D$ annihilated by the action of $s$, one can see that (v) implies (iv). Computing $\text{Hom}_R(A, D)$ as $\lim_{\leftarrow s \in S} \text{Hom}_R(A, D/sD)$ and similarly for $\text{Hom}_R(\Lambda_S(A), D)$, one concludes that (iv) implies (v) (see the last paragraph of the proof of Proposition 2.2(b)).

To prove that (vi) implies (iii), suppose that $x$ is an element of $U_s$. Using (vi), we can present it in the form $x = \lambda_{S,A}(a) + sy$ with $a \in A$ and $y \in \Lambda_S(A)$. Now the $s$-component of $\lambda_{S,A}(a)$ vanishes in $A/sA$, because the $s$-components of $x$ and $sy$ vanish. Hence $a \in sA$ and $x \in s\Lambda_S(A)$. Conversely, let $x$ be an element in $\Lambda_S(A)$ with the components $(x_t + tA)_{t \in S}$. Set $y = x - \lambda_{S,A}(x_s)$. Then $y \in U_s$, and by (iii) we can conclude that $y \in s\Lambda_S(A)$. So $x \in \lambda_{S,A}(x_s) + s\Lambda_S(A)$, providing (vi).

To check that (vii) is equivalent to (iii), let us show that $U_s = \Lambda_S(A) \cap s\Pi_A \subset \Pi_A$ for every $R$-module $A$ and every $s \in S$. Indeed, the inclusion $\Lambda_S(A) \cap s\Pi_A \subset U_s$ is clear. Conversely, let $x$ be an element of $U_s$ with the components $(x_t + tA)_{t \in S}$. Then $x_s \in sA$. By the definition of the projective limit, we have $x_st - x_t \in tA$ and...
Lemma 2.4. For every $S$-torsion-free $R$-module $A$, the equivalent conditions of Theorem 2.3 hold.

Proof. Let us check the condition (vii). Let $x$ be an element in $\Lambda_S(A) \cap s\Pi_A$. Then $x = (x_t + tA)_{t \in S}$ with $x_t = sz_t$ for every $t \in S$. Set $y_t = z_{st}$. Then for every $r \in S$ with $r \geq t$ we have $s(y_r - y_t) = sz_{sr} - sz_{st} = x_{sr} - x_{st} \in stA$, because $sr \geq st$. Since there is no $s$-torsion in $A$, we can conclude that $y_r - y_t \in tA$. So there is an element $y \in \Lambda_S(A)$ with the components $(y_t + tA)_{t \in S}$. Finally, $sy_t = sz_{st} = x_t \in x_t + tA$ for every $t \in S$, hence $x = sy$ in $\Lambda_S(A)$. □

The following theorem is our version of [19, Theorem 6.10].

Theorem 2.5. Let $B$ be an $R$-module with bounded $S$-torsion. Then

(a) the $R$-module $\Lambda_S(B)$ is $S$-complete;
(b) the kernel and cokernel of the morphism $\lambda_{S,B} : B \to \Lambda_S(B)$ are $(S^{-1}R)$-modules;
(c) the natural morphism $\beta_{S,B} : \Delta_S(B) \to \Lambda_S(B)$ is an isomorphism;
(d) the morphism $\lambda_{S,B}$ induces an isomorphism $\Gamma_S(B) \cong \Gamma_S(\Lambda_S(B))$;
(e) there is a short exact sequence

$$0 \longrightarrow \Gamma_S(B) \longrightarrow \Lambda_S(B) \longrightarrow \Lambda_S(B/\Gamma_S(B)) \longrightarrow 0.$$ 

Proof. Set $A = B/\Gamma_S(B)$; and let us first prove the assertions of the theorem for an $S$-torsion-free $R$-module $A$ (“Case I” in [19]). For the $R$-module $A$ in place of $B$, the assertion (a) holds by Lemma 2.4; and from Theorem 2.3(vi) we also see that $\ker(\lambda_{S,A})$ is an $S$-divisible $R$-module. Furthermore, $\ker(\lambda_{S,A}) = \bigcap_{s \in R} sA$ is $S$-torsion-free (as a submodule in $A$) and $S$-divisible, because $a \in \ker(\lambda_{S,A})$ implies that for every $s, t \in S$ there exist $b, c \in A$ with $a = sb = stc$. Since $A$ is $s$-torsion-free, one has $b = tc$, that is $b \in \ker(\lambda_{S,A})$.

To check that $\ker(\lambda_{S,A})$ is $S$-torsion-free, consider an element $x = (x_t + tA) \in \Lambda_S(A)$ such that $sx = \lambda_{S,A}(a)$ for some $a \in A$. Then $a \in sA$, so there is $b \in A$ such that $a = sb$. For every $t \in S$ we have $sx_t - sb \in tA$, hence $sx_t \equiv sx_{st} \equiv sb$ mod $stA$. Since $A$ has no $s$-torsion, it follows that $x_t \equiv b$ mod $tA$ and $x = \lambda_{S,A}(b)$. This proves part (b) for the $R$-module $A$, as we recall that an $R$-module is $S$-torsion-free and $S$-divisible if and only if it is an $(S^{-1}R)$-module.

In addition, the $R$-module $A/\ker(\lambda_{S,A})$ is $S$-torsion-free, because $sa \in \ker(\lambda_{S,A})$ for $s \in S$ and $a \in A$ implies that there is $b \in \ker(\lambda_{S,A})$ such that $sb = sa$, according to the above. Hence $b = a$ and $a \in \ker(\lambda_{S,A})$. The $R$-modules im($\lambda_{S,A}$) and coker($\lambda_{S,A}$) being $S$-torsion-free, it follows that the $R$-module $\Lambda_S(A)$ is $S$-torsion-free, too. So part (d) holds for $A$; and part (e) is trivial in this case.

To prove part (c), one applies Lemma 1.10. In view of part (b) and Lemma 1.7(b), there exists a unique $R$-module morphism $\Lambda_S(A) \to \Delta_S(A)$ making the triangle
diagram \( A \to \Lambda_S(A) \to \Delta_S(A) \) commutative. Using the uniqueness assertion of Lemma 1.10, one shows that the compositions \( \Delta_S(A) \to \Lambda_S(A) \to \Delta_S(A) \) and \( \Lambda_S(A) \to \Delta_S(A) \to \Lambda_S(A) \) are the identity maps. This finishes the proof of the theorem in the case of an \( S \)-torsion-free \( R \)-module \( A \).

Returning to the general case of an \( R \)-module \( B \) with bounded \( S \)-torsion ("Case II" in [19]), consider the short exact sequence of \( R \)-modules

\[
0 \to \Gamma_S(B) \to B \to A \to 0.
\]

Applying the functors \( \Delta_S \) and \( \Lambda_S \) and the natural transformation \( \beta_S \), we obtain a commutative diagram

\[
\begin{array}{cccccc}
0 & \to & \Delta_S(\Gamma_S(B)) & \to & \Delta_S(B) & \to & \Delta_S(A) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Lambda_S(\Gamma_S(B)) & \to & \Lambda_S(B) & \to & \Lambda_S(A) & &
\end{array}
\]

The upper line is exact, since \( \text{Ext}^0_R(K^\bullet, A) = 0 \) (by Lemma 1.5(c), as \( S^{-1}R/R \) is \( S \)-torsion and \( A \) is \( S \)-torsion-free) and \( \text{Ext}^2_R(K^\bullet, \Gamma_S(B)) = 0 \) (by Lemma 1.6(b)).

The lower line is exact (at the leftmost and the middle terms) because \( \Gamma_S(B) \) is an \( S \)-pure submodule in \( B \) and the functor of projective limit is left exact.

Furthermore, the morphism \( \Delta_S(A) \to \Lambda_S(A) \) is an isomorphism, as we have already proven in part (c) of "Case I" above. It follows that the morphism \( \Lambda_S(B) \to \Lambda_S(A) \) is surjective and the lower line of the diagram is a short exact sequence, too. Finally, the morphism \( \Delta_S(\Gamma_S(B)) \to \Lambda_S(\Gamma_S(B)) \) is an isomorphism, because the morphism \( \Gamma_S(B) \to \Delta_S(\Gamma_S(B)) \) is an isomorphism (by Lemma 1.6(a-b)) and the morphism \( \Gamma_S(B) \to \Lambda_S(\Gamma_S(B)) \) is (as it is clear from the construction of the \( S \)-completion functor \( \Lambda_S \)). We have proved parts (c) and (e) for the \( R \)-module \( B \).

As the \( R \)-module \( \Lambda_S(B/\Gamma_S(B)) = \Lambda_S(A) \) is \( S \)-torsion-free, part (d) follows from (e).

Very little remains to be done. Since \( \lambda_{S, \Gamma_S(B)} \) is an isomorphism, the kernel and cokernel of the morphism \( \lambda_{S, B} \) are isomorphic to, respectively, the kernel and cokernel of the morphism \( \lambda_{S, A} \). This proves part (b). Now we also know that \( \text{coker}(\lambda_{S, B}) \simeq \text{coker}(\lambda_{S, A}) \) is an \( S \)-divisible \( R \)-module. Applying Theorem 2.3 (vi) \( \implies \) (i), we deduce the assertion (a) for the \( R \)-module \( B \). \qed

From our point of view, the functor \( \Delta_S = \text{Ext}^1_R(K^\bullet, -) \) is of primary importance, while the significance of the \( S \)-completion functor \( \Lambda_S \) lies in the fact that, according to Theorem 2.5(c), it sometimes allows to compute the functor \( \Delta_S \).

**Corollary 2.6.** An \( R \)-module \( B \) with bounded \( S \)-torsion is an \( S \)-contramodule if and only if it is \( S \)-separated and \( S \)-complete.

**Proof.** Any \( S \)-separated \( S \)-complete \( R \)-module is an \( S \)-contramodule by Lemma 2.1(a). Conversely, for an \( R \)-module \( B \) with bounded \( S \)-torsion the natural map \( \beta_{S, B} : \Delta_S(B) \to \Lambda_S(B) \) is an isomorphism by Theorem 2.5(c). Suppose that \( B \) is an \( S \)-contramodule; then, by Lemma 1.6(a), the map \( \delta_{S, B} : B \to \Delta_S(B) \) is an isomorphism.
Hence the map $\lambda_{S,B}: B \longrightarrow \Lambda_S(B)$ is an isomorphism. By Proposition 2.2(a) and Theorem 2.5(a), the $R$-module $\Lambda_S(B)$ is $S$-separated and $S$-complete. Thus $B$ is $S$-separated and $S$-complete.

The following corollary is to be compared with [24, Lemma 2.5].

**Corollary 2.7.** Assume that the $R$-module $R$ has bounded $S$-torsion. Then for every flat $R$-module $F$ the conclusions of Theorem 2.5 hold. In particular, the morphism $\beta_{S,F}: \Delta_S(F) \longrightarrow \Lambda_S(F)$ is an isomorphism.

**Proof.** Using the Govorov–Lazard description of flat $R$-modules as filtered inductive limits of (finitely generated) projective $R$-modules, one easily shows that if $r\Gamma_S(R) = 0$ for some $r \in S$ then $r\Gamma_S(F) = 0$ for every flat $R$-module $F$.

\[ \square \]

3. **Projective $S$-contramodule $R$-modules**

Let $S$ be a multiplicative subset in a commutative ring $R$. For every injective $R$-module $J$, the $S$-torsion $R$-module $E = \Gamma_S(J)$ is an injective object of the abelian category of $S$-torsion $R$-modules $R:\text{mod}_{S\text{-tors}}$. There are enough injective objects of this form in $R:\text{mod}_{S\text{-tors}}$, so an $S$-torsion $R$-module is an injective object in $R:\text{mod}_{S\text{-tors}}$ if and only if it is a direct summand of an $R$-module of the form $\Gamma_S(J)$, where $J$ is an injective $R$-module.

Our aim in this section is to describe the projective objects in the category of $S$-contramodule $R$-modules, under suitable assumptions. We also discuss complexes of $R$-modules with $S$-torsion or $S$-contramodule cohomology modules generally. The following proposition, which is our version of [19, Proposition 5.1] and [9, Proposition 4.1], holds for any multiplicative subset $S$ in a commutative ring $R$.

**Proposition 3.1.** The ring $\mathcal{R} = \text{Hom}_{\text{D}^b(R\text{-mod})}(K^\bullet, K^\bullet)$ is commutative.

**Proof.** The point is that the complex $K^\bullet[-1]$ is a unit object in a certain tensor (monoidal) triangulated category structure. One can consider the unbounded derived category of $R$-modules $\text{D}(R\text{-mod})$ with the tensor product functor $\otimes_R^L$ defined in terms of homotopy flat $R$-module resolutions; or one can restrict oneself to the bounded above derived category $\text{D}^-(R\text{-mod})$, where the tensor product can be defined in terms of the conventional resolutions by complexes of flat $R$-modules. One can even consider the derived category $\text{D}^b(R\text{-mod}^{fl}) \subset \text{D}^b(R\text{-mod})$ of bounded complexes of flat $R$-modules, with the obvious tensor product structure on it.

In any event, the full subcategory of complexes with $S$-torsion cohomology modules $\text{D}_{S\text{-tors}}(R\text{-mod}) \subset \text{D}(R\text{-mod})$ is a tensor ideal in $\text{D}(R\text{-mod})$, and similarly in the bounded situations. Indeed, a complex of $R$-modules $M^\bullet \in \text{D}(R\text{-mod})$ belongs to $\text{D}_{S\text{-tors}}(R\text{-mod})$ if and only if $S^{-1}R \otimes_R M^\bullet$ is a zero object in $\text{D}(R\text{-mod})$; and $S^{-1}R \otimes_R (M^\bullet \otimes_R N^\bullet) \simeq (S^{-1}R \otimes_R M^\bullet) \otimes_R^L N^\bullet$ for every $M^\bullet, N^\bullet \in \text{D}(R\text{-mod})$. Furthermore, the tensor ideal $\text{D}_{S\text{-tors}}(R\text{-mod})$ is itself a tensor category, and as such it has its own unit object, which is $K^\bullet[-1] \in \text{D}_{S\text{-tors}}(R\text{-mod})$. 

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Now, the endomorphism semigroup of the unit object of any monoidal

\begin{equation}
\text{Proposition 3.2. (a) The natural map } \beta_{S,R} : \mathfrak{R} \to \Lambda_S(R) \text{ is a ring homomorphism.}
\end{equation}

\begin{proof}
\text{(a): by construction, } \Lambda_S(R) \text{ is a subring in the ring } \Pi_R = \prod_{s \in S} R/sR; \text{ so it suffices to check that the maps } \mathfrak{R} \to R/sR \text{ are ring homomorphisms. The latter maps are uniquely defined by the condition of commutativity of the triangle diagrams } R \to \mathfrak{R} \to R/sR \text{ (see Lemmas 1.11 and 2.1(b)). Finally, there is a functor of reduction modulo the ideal } sR \subset R, \text{ acting, e. g., between the bounded derived categories of flat modules } \text{D}^b(R\text{-mod}^R) \to \text{D}^b((R/sR)\text{-mod}^R), \text{ or even between the unbounded derived categories } \text{D}(R\text{-mod}) \to \text{D}((R/sR)\text{-mod}), \text{ etc., and taking the complex } K^*[-1] \text{ to } R/sR. \text{ This functor induces a map } \mathfrak{R} \to R/sR = \text{Hom}_{\text{D}^b((R/sR)\text{-mod})}(R/sR, R/sR), \text{ which is a ring homomorphism forming a commutative triangle diagram with the maps } R \to \mathfrak{R} \text{ and } R \to R/sR.
\end{proof}

\begin{proof}
\text{(b) Denoting by } I \subset \mathfrak{R} \text{ the kernel ideal of the ring homomorphism } \beta_{S,R}, \text{ one has } I^2 = 0 \text{ in } \mathfrak{R}.
\end{proof}

\begin{proof}
\text{(c) When the } S\text{-torsion in the ring } R \text{ is bounded, the map } \beta_{S,R} \text{ is an isomorphism.}
\end{proof}

\begin{proof}
\text{Part (b): for every } S\text{-torsion } R\text{-module } M, \text{ the isomorphism } M \simeq K^*[-1] \otimes_R M \text{ in the derived category } \text{D}^b(R\text{-mod}) \text{ (as discussed in the proof of Proposition 3.1) endows } M \text{ with a natural } \mathfrak{R}\text{-module structure. Specifically, an endomorphism } f : K^*[-1] \to
\end{proof}
Lemma 1.11, one can see that this \( R \)-module structure on \( M \) comes from the (obvious) \( \Lambda_S(R) \)-module structure on \( M \) via the ring homomorphism \( \beta_{S,R} \).

Furthermore, the isomorphism \( M^* \cong K^*[-1] \otimes_R M^* \) in \( D(\mathbb{R}-\text{mod}) \) holds for every complex of \( R \)-modules \( M^* \in D(\mathbb{R}-\text{mod}) \). This isomorphism provides an action of the ring \( \mathbb{R} \), and in fact even of the graded ring \( (16) \), by (graded) endomorphisms of the object \( M^* \in D(\mathbb{R}-\text{mod}) \). This action already does not necessarily factorize through an action of the ring \( \Lambda_S(R) \). In fact, the action of the ring \( \mathbb{R} \) by endomorphisms of the object \( K^*[-1] \) constructed in this way, i.e., in terms of the isomorphism \( K^*[-1] \cong K^*[-1] \otimes_R K^*[-1] \) in \( D(\mathbb{R}-\text{mod}) \), coincides with the action of \( \mathbb{R} \) by endomorphisms of \( K^*[-1] \) coming from the definition of \( \mathbb{R} \) as the endomorphism ring of the object \( K^*[-1] \), because we have \( f \otimes \text{id} = f \) for every morphism \( f : K^*[-1] \rightarrow K^*[-1] \) in \( D(\mathbb{R}-\text{mod}) \).

By the definition, the natural action of \( \mathbb{R} \) by endomorphisms of objects of the category \( D_{S-tors}(\mathbb{R}-\text{mod}) \) commutes with all the morphisms \( M^* \rightarrow N^* \) between the objects of \( D_{S-tors}(\mathbb{R}-\text{mod}) \). Set \( L^* = K^*[-1] \). We have shown that for every morphism \( f : L^* \rightarrow L^* \) in \( D(\mathbb{R}-\text{mod}) \) belonging to the ideal \( I \subset \mathbb{R} \) the induced cohomology module endomorphisms \( H^*(f) : H^*(L^*) \rightarrow H^*(L^*) \) vanish.

Now the point is that \( L^* \) is a two-term complex whose only possibly nontrivial cohomology modules are \( H^0(L^*) \) and \( H^1(L^*) \). Considering the distinguished triangle

\[
H^0(L^*) \longrightarrow L^* \longrightarrow H^1(L^*)[-1] \longrightarrow H^0(L^*)[1]
\]

in \( D(\mathbb{R}-\text{mod}) \), one easily comes to the conclusion that the morphism \( f \) is the composition of the morphism \( L^* \rightarrow H^1(L^*)[-1] \) on one side, the morphism \( H^0(L^*) \rightarrow L^* \) on the other side, and an \( \text{Ext}^1_R \)-extension class

\[
H^1(L^*)[-1] \longrightarrow H^0(L^*)
\]

in the middle. The composition \( H^0(L^*) \rightarrow L^* \rightarrow H^1(L^*)[-1] \) being a zero morphism in \( D(\mathbb{R}-\text{mod}) \), it follows immediately that \( f^2 = 0 \) in \( \mathbb{R} \).

Part (c) is a particular case of Theorem 2.5(c) or Corollary 2.7.

The following lemma is our version of [19, first assertion of Theorem 2.1] or [20, Theorem 1.6].

**Lemma 3.3.** The \( R \)-module \( \text{Hom}_R(M,C) \) is an \( S \)-contramodule whenever either \( M \) is an \( S \)-torsion \( R \)-module or \( C \) is an \( S \)-contramodule \( R \)-module.

**Proof.** The case when \( M \) is an \( S \)-torsion \( R \)-module can be dealt with similarly to the proof of Lemma 1.7(a), with the complex \( K^* \) replaced by a module \( M \). When \( C \) is an \( S \)-contramodule, one can consider the same spectral sequence

\[
E_2^{pq} = \text{Ext}_R^p(S^{-1}R, \text{Ext}_R^q(M,C)) \Rightarrow E_\infty^{pq} = \text{gr}^p \text{Ext}_R^{p+q}(S^{-1}R \otimes_R M, C),
\]

where \( \text{Ext}_R^n(S^{-1}M, C) = 0 \) for \( n \leq 1 \) by Lemma 1.2, hence \( E_\infty^{pq} = 0 \) for \( p+q \leq 1 \). Now \( E_\infty^{0,0} = 0 = E_\infty^{1,0} \) implies \( E_2^{0,0} = 0 = E_2^{1,0} \), since no nontrivial differentials go through \( E_2^{0,0} \) or \( E_2^{1,0} \). Alternatively, the argument from [19, Theorem 2.1] also works. \( \square \)
From this point on and for the rest of this paper we assume that $\text{pd}_R S^{-1} R \leq 1$. In this assumption, the classes of $S$-cotorsion $R$-modules and strongly $S$-cotorsion $R$-modules coincide, as do the classes of $S$-contramodule $R$-modules and strong $S$-contramodule $R$-modules. We denote the full subcategory of $S$-contramodule $R$-modules by $R\text{-mod}_{S\text{-ctra}} \subset R\text{-mod}$.

**Theorem 3.4.** (a) The full subcategory $R\text{-mod}_{S\text{-ctra}}$ is closed under the kernels, cokernels, extensions, and infinite products in $R\text{-mod}$. Therefore, the category $R\text{-mod}_{S\text{-ctra}}$ is abelian and its embedding functor $R\text{-mod}_{S\text{-ctra}} \longrightarrow R\text{-mod}$ is exact.

(b) For every $R$-module $C$, the $R$-module $\Delta_S(C) = \text{Ext}^1_R(K^\bullet, C)$ is an $S$-contramodule. The functor $\Delta_S: R\text{-mod} \longrightarrow R\text{-mod}_{S\text{-ctra}}$ is left adjoint to the fully faithful embedding functor $R\text{-mod}_{S\text{-ctra}} \longrightarrow R\text{-mod}$.

**Proof.** Part (a) is [11, Proposition 1.1] or [25, Theorem 1.2(a)] applied to the $R$-module $S^{-1} R$. The first assertion of part (b) is Lemma 1.7(c). The second assertion of part (b) means that for every $R$-module $C$, every $S$-contramodule $R$-module $D$, and an $R$-module morphism $C \longrightarrow D$ there exists a unique $R$-module morphism $\Delta_S(C) \longrightarrow D$ making the triangle diagram $C \longrightarrow \Delta_S(C) \longrightarrow D$ commutative. This follows from Lemma 1.10 together with the short exact sequence (III). \qed

Using homotopy projective and homotopy injective $R$-module resolutions, one can endow the monoidal triangulated category $D(R\text{-mod})$ with a closed monoidal structure provided by the functor

$$\mathbb{R}\text{Hom}_R: D(R\text{-mod})^{op} \times D(R\text{-mod}) \longrightarrow D(R\text{-mod}).$$

Restricting oneself to bounded above complexes $L^\bullet$ in the first argument, one construct the complex $\mathbb{R}\text{Hom}_R(L^\bullet, M^\bullet)$ using a conventional resolution of $L^\bullet$ by a (bounded above) complex of projective $R$-modules. One can even ask $L^\bullet$ to belong to the homotopy category of bounded complexes of projective $R$-modules $\text{Hot}^b(R\text{-mod}^{proj}) \subset D^b(R\text{-mod})$. Using any of these points of view, one notices the natural isomorphism $\mathbb{R}\text{Hom}_R(K^\bullet[-1], C) \simeq C$ in $D^b(R\text{-mod})$ for every $S$-contramodule $R$-module $C$. This isomorphism endows $C$ with a natural structure of $\mathfrak{R}$-module (cf. [20, Theorem 2.7], where it is also explained how to show that such an $\mathfrak{R}$-module structure on $C$ is unique).

Similarly one can construct an action of the ring $\mathfrak{R}$, and even of the graded ring (16), in every object of the full subcategory $D_{S\text{-ctra}}(R\text{-mod}) \subset D(R\text{-mod})$ of complexes with $S$-contramodule cohomology modules in $D(R\text{-mod})$. The following proposition provides some details.

**Proposition 3.5.** (a) A complex of $R$-modules $A^\bullet \in D(R\text{-mod})$ has $S$-contramodule cohomology modules if and only if $\mathbb{R}\text{Hom}_R(S^{-1} R, A^\bullet) = 0$. Hence one has $A^\bullet \simeq \mathbb{R}\text{Hom}_R(K^\bullet[-1], A^\bullet)$ in $D(R\text{-mod})$ for every $A^\bullet \in D_{S\text{-ctra}}(R\text{-mod})$.

(b) The complex of $R$-modules $\mathbb{R}\text{Hom}_R(M^\bullet, A^\bullet)$ has $S$-contramodule cohomology modules whenever either a complex $M^\bullet$ has $S$-torsion cohomology modules, or a complex $A^\bullet$ has $S$-contramodule cohomology modules.
Proof. For every complex of $R$-modules $A^\bullet$, there are natural short exact sequences

$$0 \longrightarrow \text{Ext}^1_R(S^{-1}R, H^{n-1}(A^\bullet)) \longrightarrow H^n(\mathbb{R}\text{Hom}_R(S^{-1}R, A^\bullet)) \longrightarrow \text{Hom}_R(S^{-1}R, H^n(A^\bullet)) \longrightarrow 0$$

for all $n \in \mathbb{Z}$. This proves part (a); and part (b) follows from the isomorphisms

$$\mathbb{R}\text{Hom}_R(S^{-1}R, \mathbb{R}\text{Hom}_R(M^\bullet, A^\bullet)) \simeq \mathbb{R}\text{Hom}_R(S^{-1}R \otimes_R M^\bullet, A^\bullet) \simeq \mathbb{R}\text{Hom}_R(M^\bullet, \mathbb{R}\text{Hom}_R(S^{-1}R, A^\bullet)).$$

For any set $X$ and an $R$-module $M$, we denote by $M[X]$ the direct sum of $X$ copies of the $R$-module $M$. We set $\mathfrak{R}[[X]] = \Delta_S(R[X])$. The $R$-module $\mathfrak{R}[[X]]$ is called the free $S$-contramodule $R$-module generated by the set $X$. According to Theorem 3.4(b), we have

$$\text{Hom}_R(\mathfrak{R}[[X]], D) \simeq \text{Hom}_R(R[X], D) \simeq D^X$$

for every $S$-contramodule $R$-module $D$. It follows that free $S$-contramodule $R$-modules are projective objects of the category $R\text{-mod}_{S\text{-contra}}$. Furthermore, there are enough of them, as every $S$-contramodule $R$-module $D$ is a quotient module of the free $S$-contramodule $R$-module $\mathfrak{R}[[X]]$ with the set of generators $X = D$. It follows that an $S$-contramodule $R$-module is a projective object of $R\text{-mod}_{S\text{-contra}}$ if and only if it is a direct summand of a free $S$-contramodule $R$-module.

According to Corollary 2.7, when the $S$-torsion in the ring $R$ is bounded, one has

$$\mathfrak{R}[[X]] = \Delta_S(R[X]) \simeq \Lambda_S(R[X]) = \lim_{s \in S} R/sR[X].$$

4. The Triangulated Equivalence

Let $S$ be a multiplicative subset in a commutative ring $R$. Since the middle of the previous section, we keep assuming that the projective dimension of the $R$-module $S^{-1}R$ does not exceed 1.

Following the notation in [24, Section 1] and [25, Section 7], for every $R$-module $M$ we denote by $\check{C}^\bullet(M)$ the two-term complex $M \longrightarrow S^{-1}M$, with the term $M$ sitting in cohomological degree 0 and the term $S^{-1}M$ sitting in degree 1. In other words, $\check{C}^\bullet(M) = K[1] \otimes_R M$. Given a complex of $R$-modules $M^\bullet$, the notation $\check{C}^\bullet(M^\bullet)$ stands for the total complex of the corresponding bicomplex with two rows.

The next two lemmas are almost obvious.

**Lemma 4.1.** (a) For every $R$-module $M$, the cohomology modules $H^\bullet \check{C}^\bullet(M)$ of the complex $\check{C}^\bullet(M)$ are $S$-torsion $R$-modules.

(b) For every $R$-module $M$, one has an isomorphism $H^0 \check{C}^\bullet(M) \simeq \Gamma_S(M)$.

(c) For every $S$-torsion $R$-module $M$, the natural projection $\check{C}^\bullet(M) \longrightarrow M$ is an isomorphism of complexes.
Proof. Part (a) holds, because the complex $S^{-1}R \otimes_R \tilde{C}^\bullet(M)$ is contractible. Part (b) is Lemma 1.5(d). Part (c) holds, because $S^{-1}M = 0$ when $M$ is $S$-torsion.

Following the notation in [24, Section 2] and [25, Section 7], we choose a left projective resolution $\underline{T}^\bullet = (\underline{T}^0 \to \underline{T}^1)$ of the $R$-module $S^{-1}R$ (notice the unusual indexing/cohomological grading: it is presumed that $H^0(\underline{T}^\bullet) = 0$ and $H^1(\underline{T}^\bullet) = S^{-1}R$). Furthermore, we lift the morphism $R \to S^{-1}R$ to an $R$-module morphism $R \to T^1$ and denote by $T^\bullet = (T^0 \to T^1)$ the two-term complex $(R \oplus \underline{T}^0 \to \underline{T}^1)$ quasi-isomorphic to $K^\bullet[-1]$. The aim of the grading shift is to have a natural (projection) morphism $T^\bullet \to R$. Given an $R$-module $C$, we will sometimes view the complex $\text{Hom}_R(T^\bullet, C)$ as a homological complex sitting in the homological degrees 0 and 1, which are denoted by the lower indices.

Lemma 4.2. (a) For every $R$-module $C$, the homology modules $H_* \text{Hom}_R(T^\bullet, C)$ of the complex $\text{Hom}_R(T^\bullet, C)$ are $S$-contramodule $R$-modules.

(b) For every $R$-module $C$, one has $H_0 \text{Hom}_R(T^\bullet, C) \simeq \Delta_S(C)$.

(c) For every $S$-contramodule $R$-module $C$, the morphism of complexes $T^\bullet \to R$ induces a quasi-isomorphism of complexes of $R$-modules $C \to \text{Hom}_R(T^\bullet, C)$.

Proof. Part (a) is Lemma 1.7(a,c). Part (b) is the definition of $\Delta_S = \text{Ext}^1_R(K^\bullet, -)$. Part (c) means that the complex $\text{Hom}_R(T^\bullet, C)$ is acyclic, which is the definition of an $S$-contramodule $R$-module.

The following proposition is essentially proved in [24, Proposition 3.3]. In the paper [24], it was being applied to the case of the Čech DG-algebra of a finite sequence of elements $s$ in a commutative ring $R$ (which was denoted by $C^\bullet_s(R)$ in [24] and would be denoted by $\tilde{C}^\bullet_s(R)$ in the present paper’s notation system). In this paper, we will apply this result to the $R$-algebra $\tilde{C} = S^{-1}R$ (viewed as a DG-algebra concentrated in the cohomological degree 0).

Let $\tilde{C}^\bullet$ be a finite complex of $R$-modules whose terms $\tilde{C}^n$ are flat $R$-modules of finite projective dimension. Suppose that $\tilde{C}^\bullet$ is endowed with the structure of an (associative and unital, not necessarily commutative) DG-algebra over the ring $R$, and that the following condition is satisfied: the three morphisms of complexes

(17) $\tilde{C}^\bullet \Rightarrow \tilde{C}^\bullet \otimes_R \tilde{C}^\bullet \to \tilde{C}^\bullet$

provided by the unit and multiplication in the DG-algebra $\tilde{C}^\bullet$ are quasi-isomorphisms of complexes of $R$-modules.

Let $*$ be one of the derived category symbols $b$, $+$, $-$, or $\emptyset$. By the definition, the derived category $D^*(\tilde{C}^\bullet-\text{mod})$ is constructed by inverting the class of quasi-isomorphisms in the homotopy category of $*$-bounded left DG-modules over the DG-ring $\tilde{C}^\bullet$. Denote by

$k_*: D^*(\tilde{C}^\bullet-\text{mod}) \to D^*(R-\text{mod})$

the functor of restriction of scalars with respect to the morphism of DG-rings $k: R \to \tilde{C}^\bullet$.

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Proposition 4.3. (a) The triangulated functor \( k_* \) has a left adjoint functor \( k^* \) and a right adjoint functor \( \mathbb{R}k_! \),

\[ k^*, \mathbb{R}k_! : D^*(R\text{-mod}) \rightarrow D^*(\mathcal{C}^*\text{-mod}). \]

(b) The compositions \( k^* \circ k_* \) and \( \mathbb{R}k_! \circ k_* \) are isomorphic to the identity functors on the category \( D^*(\mathcal{C}^*\text{-mod}) \), the functor \( k_* \) is fully faithful, and the functors \( k^* \) and \( \mathbb{R}k_! \) are Verdier quotient functors.

(c) The passage to the quotient category by the image of the functor \( k_* \) establishes an equivalence between the kernels of the functors \( k^* \) and \( \mathbb{R}k_! \),

\[ \ker(k^*) \simeq D^*(R\text{-mod})/\text{im}\,k_* \simeq \ker(\mathbb{R}k_!). \]

In other words, there are two semiorthogonal decompositions in the triangulated category \( D^*(R\text{-mod}) \), one of them formed by the two full subcategories \( \text{im}(k_*) \) and \( \ker(k^*) \), and the other one by the two full subcategories \( \ker(\mathbb{R}k_!) \) and \( \text{im}(k_*) \). In particular, \( \text{im}(k_*) \) is a thick subcategory in \( D^*(R\text{-mod}) \).

Proof. Part (a) does not depend on the assumption that the maps (17) are quasi-isomorphisms; parts (b) and (c) do. The assumption of finite projective dimension of the \( R \)-modules \( \mathcal{C}^n \) is irrelevant in the case of the derived category symbol \( * = + \) or \( \varnothing \), and relevant only for \( * = b \) or \( - \) (and insofar as the functor \( \mathbb{R}k_! \) is concerned). All the assertions in part (b) are equivalent to each other for purely formal reasons applicable to triangulated functors generally, and part (c) is a purely formal restatement of part (b). We refer to [24, proof of Proposition 3.3] for the details of the argument. \( \square \)

In other words, Proposition 4.3 says that the DG-algebra morphism \( R \rightarrow \mathcal{C} \) gives rise to a “recollement” of triangulated categories for every symbol \( * = b, +, - \), or \( \varnothing \).

In the case of \( \mathcal{C} = S^{-1}R \), the ring homomorphism \( k: R \rightarrow S^{-1}R \) is a homological ring epimorphism in the sense of [11, Section 4], which is also a sufficient condition for the conclusions of the proposition to hold (for \( * = \varnothing \)). For a generalization to arbitrary morphisms of associative DG-rings, see [22, Theorem 3.9].

Generalizing our previous notation from the case \( \mathcal{C} = S^{-1}R \) to a DG-algebra \( \mathcal{C}^* \) as above, denote by \( \mathcal{C}^*\sim \) the cocone (that is, the cone shifted by \([-1]\)) of the morphism of complexes \( R \rightarrow \mathcal{C}^* \). Choose a finite complex of projective \( R \)-modules \( T \) mapping quasi-isomorphically onto \( \mathcal{C}^* \), lift the morphism \( R \rightarrow \mathcal{C}^* \) to a morphism of complexes \( R \rightarrow T \), and denote by \( T^* \) the cocone of the latter morphism of complexes. The complexes \( \mathcal{C}^*\sim \) and \( T^* \) are, by construction, quasi-isomorphic to each other and endowed with morphisms of complexes \( \mathcal{C}^*\sim \rightarrow R \) and \( T^* \rightarrow R \).

The following proposition is essentially proved in [24, first half of the proof of Theorem 3.4].

Proposition 4.4. (a) The functor \( D^*(R\text{-mod}) \rightarrow \ker(k^*) \) right adjoint to the embedding \( \ker(k^*) \rightarrow D^*(R\text{-mod}) \) takes a complex of \( R \)-modules \( M^* \) to the complex \( \mathcal{C}^*\sim \otimes_R M^* \). The full subcategory \( \ker(k^*) \subset D^*(R\text{-mod}) \) consists precisely of all the complexes of \( R \)-modules \( M^* \) for which the morphism of complexes \( \mathcal{C}^*\sim \otimes_R M^* \rightarrow M^* \) induced by the morphism \( \mathcal{C}^*\sim \rightarrow R \) is a quasi-isomorphism.
(a) The functor \( D^*(R\text{-mod}) \to \ker(\mathbb{R}k^\triangleright) \) left adjoint to the embedding \( \ker(\mathbb{R}k^\triangleright) \to D^*(R\text{-mod}) \) takes a complex of \( R \)-modules \( A^\bullet \) to the complex \( \text{Hom}_R(T^\bullet, A^\bullet) \). The full subcategory \( \ker(\mathbb{R}k^\triangleright) \subset D^*(R\text{-mod}) \) consists precisely of all the complexes of \( R \)-modules \( A^\bullet \) for which the morphism of complexes \( A^\bullet \to \text{Hom}_R(T^\bullet, A^\bullet) \) induced by the morphism \( T^\bullet \to R \) is a quasi-isomorphism.

Now we return to the situation with the \( R \)-algebra \( C = S^{-1}R \). As in the previous section, we denote by \( D^\bullet_{S\text{-tors}}(R\text{-mod}) \subset D^\bullet(R\text{-mod}) \) the full triangulated subcategory in the derived category \( D^\bullet(R\text{-mod}) \) formed by all the complexes of \( R \)-modules \( M^\bullet \) whose cohomology modules \( H^\bullet(M^\bullet) \) are \( S \)-torsion \( R \)-modules. Similarly, \( D^\bullet_{S\text{-ctr}}(R\text{-mod}) \subset D^\bullet(R\text{-mod}) \) denotes the full triangulated subcategory in \( D^\bullet(R\text{-mod}) \) formed by all the complexes of \( R \)-modules \( A^\bullet \) whose cohomology modules \( H^\bullet(A^\bullet) \) are \( S \)-contramodule \( R \)-modules.

**Lemma 4.5.** Let \( k \) be the \( R \)-algebra homomorphism \( R \to S^{-1}R \). Then

(a) the full subcategory \( \ker(k^\triangleright) \subset D^\bullet(R\text{-mod}) \) coincides with \( D^\bullet_{S\text{-tors}}(R\text{-mod}) \subset D^\bullet(R\text{-mod}) \);

(b) the full subcategory \( \ker(\mathbb{R}k^\triangleright) \subset D^\bullet(R\text{-mod}) \) coincides with \( D^\bullet_{S\text{-ctr}}(R\text{-mod}) \subset D^\bullet(R\text{-mod}) \).

**Proof.** This is similar to [24, second half of the proof of Theorem 3.4]. The argument is based on Proposition 4.4 and Lemmas 4.1–4.2.

To prove that \( \ker(k^\triangleright) \subset D^\bullet_{S\text{-tors}}(R\text{-mod}) \), we have to show that the cohomology modules of a complex of \( R \)-modules \( M^\bullet \) are \( S \)-torsion \( R \)-modules whenever the morphism of complexes \( \check{C}^\bullet(M^\bullet)^\sim \to M^\bullet \) is a quasi-isomorphism. It suffices to check that the cohomology modules of the complex \( \check{C}^\bullet(M^\bullet)^\sim \) are \( S \)-torsion \( R \)-modules for every complex of \( R \)-modules \( M^\bullet \). This is so because the complex \( S^{-1}R \otimes_R \check{C}^\bullet(M^\bullet)^\sim \) is contractible (cf. Lemma 4.1(a)).

To prove that \( D^\bullet_{S\text{-tors}}(R\text{-mod}) \subset \ker(k^\triangleright) \), we have to check that the morphism \( \check{C}^\bullet(M^\bullet)^\sim \to M^\bullet \) is a quasi-isomorphism for every complex of \( R \)-modules \( M^\bullet \) with \( S \)-torsion cohomology modules. This is so because the complex \( S^{-1}R \otimes_R M^\bullet \) is acyclic (cf. Lemma 4.1(c)).

To prove that \( \ker(\mathbb{R}k^\triangleright) \subset D^\bullet_{S\text{-ctr}}(R\text{-mod}) \), it suffices to check that the cohomology modules of the complex \( \text{Hom}_R(T^\bullet, A^\bullet) \) are \( S \)-contramodule \( R \)-modules for every complex of \( R \)-modules \( A^\bullet \). Every cohomology module of the complex \( \text{Hom}_R(T^\bullet, A^\bullet) \) only depends on a finite number of terms of the complex \( A^\bullet \). This reduces the question to the case of a finite complex \( A^\bullet \). Since the full subcategory \( D^\bullet_{S\text{-ctr}}(R\text{-mod}) \) is closed under shifts and cones in \( D^\bullet(R\text{-mod}) \), the question further reduces to the case of a one-term complex \( A^\bullet = A \). Here it remains to apply Lemma 4.2(a).

To prove that \( D^\bullet_{S\text{-ctr}}(R\text{-mod}) \subset \ker(\mathbb{R}k^\triangleright) \), we have to check that the morphism \( A^\bullet \to \text{Hom}_R(T^\bullet, A^\bullet) \) is a quasi-isomorphism for every complex of \( R \)-modules \( A^\bullet \) with \( S \)-contramodule cohomology modules. It suffices to consider the case when the complex \( A^\bullet \) is finite, and the question reduces further to the case of a one-term complex \( A^\bullet = A \) (see Hartshorne’s lemma on way-out functors [16, Proposition I.7.1]). It remains to use Lemma 4.2(c).
The following theorem is the most general version of a triangulated Matlis equivalence that we are able to obtain.

**Theorem 4.6.** For any commutative ring $R$ and a multiplicative subset $S \subset R$ such that $\text{pd}_R S^{-1}R \leq 1$, and any conventional derived category symbol $\star = b, +, -,\,$ or $\emptyset$, the functor $k_* : D^*((S^{-1}R)\text{-mod}) \to D^*(R\text{-mod})$ is fully faithful and its image is a thick subcategory in $D^*(R\text{-mod})$. Furthermore, there are natural equivalences of triangulated categories (10)

$$D^s_{S\text{-tors}}(R\text{-mod}) \simeq D^s(R\text{-mod})/D^s((S^{-1}R)\text{-mod}) \simeq D^s_{S\text{-cta}}(R\text{-mod}).$$

The resulting triangulated Matlis equivalence

$$D^s_{S\text{-tors}}(R\text{-mod}) \simeq D^s_{S\text{-cta}}(R\text{-mod})$$

is provided by the functors taking a complex $M^\star \in D^s_{S\text{-tors}}(R\text{-mod})$ to the complex $\text{Hom}_R(T^\star, M^\star) \in D^s_{S\text{-cta}}(R\text{-mod})$ and a complex $A^\star \in D^s_{S\text{-cta}}(R\text{-mod})$ to the complex $\tilde{C}^\star(A^\star) \in D^s_{S\text{-tors}}(R\text{-mod}).$

Notice that, of course, the complex $\tilde{C}^\star(A^\star)$ is isomorphic to $T^\star \otimes_R A^\star$ as an object of $D^s_{S\text{-tors}}(R\text{-mod})$ for every complex $A^\star \in D^s(R\text{-mod})$.

**Proof.** Follows from Proposition 4.3 and Lemma 4.5. The first assertion of the theorem is explainable by saying that $k : R \to S^{-1}R$ is a homological ring epimorphism [11, Theorem 4.4 and Corollary 4.7(2)], [22, Theorem 3.7]. Both this and the leftmost one of the two triangulated equivalences do not depend on the projective dimension assumption $\text{pd}_R S^{-1}R \leq 1$ (as the construction and the properties of the functor $k^*$ in Proposition 4.3 do not require it, and neither does Lemma 4.5(a)). The rightmost triangulated equivalence needs $\text{pd}_R S^{-1}R \leq 1$. \hfill $\Box$

**Remark 4.7.** The following observations, the most part of which the author learned from the anonymous referee, point out a connection between our exposition and the infinitely generated tilting/silting theory. When all the elements of $S$ are nonzero-divisors in $R$, the ring homomorphism $k : R \to S^{-1}R$ is an injective homological ring epimorphism. Hence, according to [13, Definition 4.46 and Theorem 14.59] or [3, Theorem 3.5], the direct sum $S^{-1}R \oplus S^{-1}R/R$ is a 1-tilting $R$-module. More generally, according to [17, Example 6.5], the direct sum $S^{-1}R \oplus (S^{-1}R)/k(R)$ is a silting $R$-module, and a 2-term projective resolution of the complex $S^{-1}R \oplus K^\star$ is a 2-silting complex of $R$-modules in the sense of [2, Remark 2.7, Proposition 4.2, and Theorem 4.9]. Equivalently, the complex $S^{-1}R \oplus K^\star$ is a bounded silting object of the derived category $D(R\text{-mod})$ in the sense of [26, Propositions 4.13 and 4.17]. When there is no $S$-h-divisible $S$-torsion in the $R$-module $R$, the complex $S^{-1}R \oplus K^\star$ is even a tilting object in $D(R\text{-mod})$ in the sense of [26, Definition 1.1].
5. Two Exact Category Equivalences

In this section we deduce from Theorem 4.6 our versions of the Matlis additive category equivalences of [19, §3]. In fact, we will even obtain equivalences of exact categories (in Quillen’s sense). As in Section 4, our setting is that of a commutative ring $R$ with a multiplicative subset $S \subset R$ such that the projective dimension of the $R$-module $S^{-1}R$ does not exceed 1.

**Lemma 5.1.** (a) An $R$-module $N$ is $S$-torsion-free if and only if $\text{Tor}^R_1(K^\bullet, N) = 0$.

(b) An $R$-module $C$ is $S$-h-divisible if and only if $\text{Ext}^1_R(K^\bullet, C) = 0$.

**Proof.** Part (a) is Lemma 1.5(d). In part (b), the “if” claim follows from the short exact sequence (III). To prove the “only if”, assume that categories (in Quillen’s sense). As in Section 4, our setting is that of a commutative ring $R$.

It follows from Lemma 5.1(b) that the full subcategory of $S$-h-divisible $R$-modules is closed under extensions in $R\text{-mod}$. This provides another proof of Lemma 1.8(a).

In particular, the full subcategory of $S$-h-divisible $S$-torsion $R$-modules in $R\text{-mod}$ is closed under extensions, quotients and infinite direct sums. So it inherits an exact category structure from the abelian category $R\text{-mod}$ or $R\text{-mod}_{S\text{-tors}}$.

The full subcategory of $S$-torsion-free $S$-contramodule $R$-modules is closed under extensions, kernels, and infinite products in $R\text{-mod}$; it is also closed under extensions and subobjects in the abelian category $R\text{-mod}_{S\text{-ctra}}$. So this full subcategory inherits an exact category structure from the abelian category $R\text{-mod}$ or $R\text{-mod}_{S\text{-ctra}}$.

The following corollary is our version of [19, Theorem 3.4] and [20, Corollary 2.4].

**Corollary 5.2.** The functors $M \mapsto \text{Ext}^b_R(K^\bullet, M)$ and $A \mapsto \text{Tor}^b_R(K^\bullet, A)$ establish an equivalence between the exact categories of $S$-h-divisible $S$-torsion $R$-modules $M$ and $S$-torsion-free $S$-contramodule $R$-modules $A$.

**Proof.** Clearly, the equivalence of triangulated categories $D^b_{S\text{-tors}}(R\text{-mod}) \simeq D^b_{S\text{-ctra}}(R\text{-mod})$ from Theorem 4.6 restricts to an equivalence between the exact categories of

- those complexes in $D^b_{S\text{-tors}}(R\text{-mod})$ whose cohomology are concentrated in the cohomological degree 0 and whose images in $D^b_{S\text{-ctra}}(R\text{-mod})$ have cohomology concentrated in degree 1, and
- those complexes in $D^b_{S\text{-ctra}}(R\text{-mod})$ whose cohomology are concentrated in the cohomological degree 0 and whose images in $D^b_{S\text{-tors}}(R\text{-mod})$ have cohomology concentrated in degree 0.

Since the complexes $\check{C}^\bullet(R)^\sim$ and $T^\bullet$ are quasi-isomorphic to $K^\bullet[-1]$, the former category consists of those $S$-torsion $R$-modules $M$ for which $\text{Ext}^1_R(K^\bullet, M) = 0$, while the latter category consists of those $S$-contramodule $R$-modules $A$ for which $\text{Tor}^1_R(K^\bullet, M) = 0$. It remains to recall Lemma 5.1. \(\square\)
Notice that by Corollary 2.6 an $S$-torsion-free $R$-module $A$ is an $S$-contramodule if and only if it is $S$-separated and $S$-complete. Thus Corollary 5.2 can be formulated as an equivalence between the additive categories of $S$-h-divisible $S$-torsion $R$-modules and $S$-torsion-free $S$-separated $S$-complete $R$-modules (cf. [10, Theorem VIII.2.8]).

Our last corollary in this section will involve a class of $R$-modules that were called “adjusted co-torsion” in [15] and “special cotorsion” in [19]. Following the terminology in [19], we say that an $R$-module $N$ is $S$-special if the quotient module $N/\Gamma_S(N)$ is $S$-divisible. If this is the case, $N/\Gamma_S(N)$, being an $S$-torsion-free $S$-divisible $R$-module, is an $(S^{-1}R)$-module.

The next lemma is our version of [19, Lemma 3.6] and [20, Corollary 1.3].

**Lemma 5.3.** (a) An $R$-module $N$ is $S$-special if and only if $\text{Tor}_0^R(K^\bullet, N) = 0$.

(b) An $R$-module $C$ has no $S$-h-divisible $S$-torsion if and only if $\text{Ext}_R^0(K^\bullet, C) = 0$.

**Proof.** Part (a): for an $S$-torsion $R$-module $M$, one has $S^{-1}R \otimes_R M = 0$, hence $\text{Tor}_0^R(K^\bullet, M) = 0$ in view of the short exact sequence (I). For an $(S^{-1}R)$-module $D$ one has $\text{Tor}_0^R(K^\bullet, D) = H^0(K^\bullet \otimes_R D) = 0$. Hence one has $\text{Tor}_0^R(K^\bullet, N) = 0$ for every $S$-special $R$-module $N$. Conversely, if $\text{Tor}_0^R(K^\bullet, N) = 0$, then $N/\Gamma_R(N)$ is an $(S^{-1}R)$-module according to the exact sequence (I).

Part (b): the $R$-module $S^{-1}R/R$ is $S$-torsion and $S$-h-divisible, hence by Lemma 1.5(c) one has $\text{Ext}_R^0(K^\bullet, C) = 0$ for every $R$-module $C$ without $S$-h-divisible $S$-torsion. Conversely, if $M = \Gamma_S(C)$ then for every morphism $f : S^{-1}R \rightarrow M$ there exists an element $s \in S$ such that $sf(1) = 0$. So $\text{Hom}_R(S^{-1}R/R, C) = 0$ implies $sf = 0$ and, since $S^{-1}R$ is $S$-divisible, $f = 0$. Thus $\Gamma_S(C)$ is $S$-h-reduced whenever $\text{Ext}_0^R(K^\bullet, C) = 0$. Unlike the proof of Lemma 5.1(b), the proof of the present lemma does not depend on the assumption $\text{pd}_R S^{-1}R \leq 1$. \quad \Box

It follows from Lemma 5.3(a) that the full subcategory of $S$-special $R$-modules is closed under extensions, quotients, and infinite direct sums in $R\text{-mod}$. Hence the full subcategory of $S$-special $S$-contramodule $R$-modules is closed under extensions in $R\text{-mod}$; it is also closed under extensions and quotients in $R\text{-mod}_{S,stra}$. So this full subcategory inherits an exact category structure from the abelian category $R\text{-mod}$ or $R\text{-mod}_{S,stra}$.

The full subcategory of $S$-h-reduced $S$-torsion $R$-modules is closed under extensions, subobjects, and infinite direct sums in $R\text{-mod}_{S,tors}$ and $R\text{-mod}$. So it inherits an exact category structure.

The following corollary is our version of [19, Theorem 3.8].

**Corollary 5.4.** The functors $M \mapsto \text{Ext}_R^1(K^\bullet, M)$ and $A \mapsto \text{Tor}_1^R(K^\bullet, A)$ establish an equivalence between the exact categories of $S$-h-reduced $S$-torsion $R$-modules $M$ and $S$-special $S$-contramodule $R$-modules $A$.

We recall that, according to Lemma 1.5(d), one has $\text{Tor}_1^R(K^\bullet, A) \cong \Gamma_S(A)$ and, by the definition, $\text{Ext}_R^1(K^\bullet, M) = \Delta_S(M)$.

**Proof.** The equivalence of triangulated categories $D^b_{S,tors}(R\text{-mod}) \cong D^b_{S,stra}(R\text{-mod})$ from Theorem 4.6 restricts to an equivalence between the exact categories of
• those complexes in $D_{S\text{-tors}}^b(R\text{-mod})$ whose cohomology are concentrated in the cohomological degree 0 and whose images in $D_{S\text{-ctra}}^b(R\text{-mod})$ have cohomology concentrated in degree 0, and

• those complexes in $D_{S\text{-ctra}}^b(R\text{-mod})$ whose cohomology are concentrated in the cohomological degree 0 and whose images in $D_{S\text{-tors}}^b$ have cohomology concentrated in degree 0.

Due to the quasi-isomorphisms $\hat{C}^\bullet(R) \sim = K^\bullet[-1] \cong T^\bullet$, the former category consists of those $S$-torsion $R$-modules $M$ for which $\text{Ext}^0_R(K^\bullet, M) = 0$, while the latter category consists of those $S$-contramodule $R$-modules $A$ for which $\text{Tor}^0_R(K^\bullet, A) = 0$. It remains to take account of Lemma 5.3.

In any $R$-module $M$, there is a unique maximal $S$-h-divisible $S$-torsion submodule $\theta_S(M) \subset M$, which can be constructed as the sum of all the $S$-module morphisms $S^{-1}R/R \rightarrow M$. The quotient module $M/\theta_S(M)$ is the (unique) maximal quotient $R$-module of $M$ having no $S$-h-divisible $S$-torsion. When $M$ is an $S$-torsion $R$-module, its submodule $\theta_S(M)$ belongs to the exact subcategory appearing in Corollary 5.2 and the quotient module $M/\theta_S(M)$ belongs to the exact subcategory appearing in Corollary 5.4.

In any $R$-module $C$, there is a unique maximal $S$-special $R$-submodule $\sigma_S(C) \subset C$, which can be constructed as the sum of all the $S$-special $R$-submodules in $C$. The quotient module $C/\sigma_S(C)$ is the (unique) maximal $S$-h-reduced (or $S$-reduced) $S$-torsion-free quotient $R$-module of $C$. When $C$ is an $S$-contramodule, the $R$-module $\sigma_S(C)$ is an $S$-contramodule by Lemma 1.3(b), and the $R$-module $C/\sigma_S(C)$ is an $S$-contramodule by Lemma 1.3(c) (this is our version of [19, Theorem 3.7]). So the submodule $\sigma_S(C) \subset C$ belongs to the subcategory appearing in Corollary 5.4 and the quotient module $C/\sigma_S(C)$ belongs to the subcategory appearing in Corollary 5.2.

**Remark 5.5.** Let us emphasize that our results in this section are both more and less general than in Matlis’ [19, §3] (also in [10, Section VIII.2]). On the one hand, in place of a commutative domain $R$ with the multiplicative subset $R \setminus \{0\}$, we have a rather arbitrary commutative ring $R$ and a multiplicative subset $S \subset R$. The ease with which replacing the quotient module $K = Q/R$ by a two-term complex $K^\bullet$ allows to work with zero-divisors in this theory is remarkable. On the other hand, our triangulated equivalence seems to be unable to avoid the assumption $\text{pd}_R S^{-1}R \leq 1$, which was not made in the classical approach.

### 6. Two Fully Faithful Triangulated Functors

Let $R$ be a commutative ring and $S \subset R$ be a multiplicative subset. In addition to our running assumption that $\text{pd}_S S^{-1}R \leq 1$, in this section we will also assume that the $S$-torsion in $R$ is bounded, that is there exists $r \in S$ such that $r \Gamma_S(R) = 0$. Our aim is to rewrite the triangulated equivalence of Theorem 4.6 as an equivalence between the derived categories of the abelian categories $R\text{-mod}_{S\text{-tors}}$ and $R\text{-mod}_{S\text{-ctra}}$ (cf. [24, Section 1 and 2]).
Lemma 6.1. (a) For every injective $R$-module $J$ one has $\text{Tor}_0^R(K^\bullet, J) = 0$.
(b) For every flat $R$-module $F$ one has $\text{Ext}_R^0(K^\bullet, F) = 0$.

Proof. We assume that $r\Gamma_S(R) = 0$, where $r \in S$. Part (a): let us first show that for every $s \in S$ one has $srJ = rJ$. Indeed, for every $t \in R$ elements of the submodule $tJ \subset J$ correspond to those $R$-module morphisms $R \to J$ that factorize through the surjection $t: R \to tR$ (since every morphism $R \to tR \to J$ can be extended to a morphism $R \to J$). By assumption, every element annihilated by $sr$ in $R$ is also annihilated by $r$. Therefore, the restriction $s: rR \to R$ of the map $s: R \to R$ to the submodule $rR \subset R$ is injective, and the map $s: rR \to srR$ is an isomorphism. It follows that every morphism $R \to J$ that factorizes through the surjection $r: R \to rR$ also factorizes through the surjection $sr: R \to srR$, that is $srJ = rJ$.

Now for every $j \in J$ there exists $j' \in J$ such that $rj = rsj'$, hence $j/s = rsj'/rs = j'$ in $S^{-1}J$. We have shown that the map $J \to S^{-1}J$ is surjective, and the vanishing of $\text{Tor}_0^R(K^\bullet, J)$ follows by means of the exact sequence (I).

Part (b): as it was explained in the proof of Corollary 2.7, one has $r\Gamma_S(F) = 0$. Hence the $R$-module $\Gamma_S(F)$ is $S$-$h$-reduced, and it remains to use Lemma 5.3(b). □

The next two lemmas follow straightforwardly from Lemmas 4.1(b)–4.2(b) and Lemma 6.1.

Lemma 6.2. (a) The complex $\check{C}^\bullet(M)$ assigned to an $R$-module $M$ computes the right derived functor $R^\bullet\Gamma_S(M)$ of the left exact functor $\Gamma_S: R^\text{mod} \to R^\text{mod}_{S\text{-tors}}$ viewed as taking values in the ambient category $R^\text{mod}$.
(b) The right derived functor $R^\bullet\Gamma_S(M)$ of the left exact functor $\Gamma_S: R^\text{mod} \to R^\text{mod}_{S\text{-tors}}$ has homological dimension $\leq 1$.

Lemma 6.3. (a) The complex $\text{Hom}_R(T^\bullet, C)$ assigned to an $R$-module $C$ computes the left derived functor $\mathbb{L}\Delta_S(C)$ of the right exact functor $\Delta_S: R^\text{mod} \to R^\text{mod}_{S\text{-ltra}}$ viewed as taking values in the ambient category $R^\text{mod}$.
(b) The left derived functor $\mathbb{L}\Delta_S(C)$ of the right exact functor $\Delta_S: R^\text{mod} \to R^\text{mod}_{S\text{-ltra}}$ has homological dimension $\leq 1$.

Proof. The proofs of Lemmas 6.2 and 6.3 are very similar. Let us explain Lemma 6.3. First of all, the functor $\Delta_S = \text{Ext}_R^1(K^\bullet, -)$ is right exact, because $\text{Ext}_R^2(K^\bullet, -) = 0$ by Lemma 1.5(b) (or because $\Delta_S: R^\text{mod} \to R^\text{mod}_{S\text{-ltra}}$ is a left adjoint functor to the exact embedding functor $R^\text{mod}_{S\text{-ltra}} \to R^\text{mod}$). Notice that $\text{Hom}_R(T^\bullet, C)$ is a complex in $R^\text{mod}$ and not in $R^\text{mod}_{S\text{-utra}}$; hence the caveat about it computing the derived functor $\mathbb{L}\Delta(C)$ viewed as taking values in $R^\text{mod}$ rather than $R^\text{mod}_{S\text{-utra}}$.

The embedding functor $R^\text{mod}_{S\text{-utra}} \to R^\text{mod}$ being, however, fully faithful and, in particular, taking nonzero objects to nonzero objects, part (b) follows from part (a).

To prove part (a), consider a left projective $R$-module resolution $F_\bullet$ of the $R$-module $C$. We have to show that the complexes $\Delta_S(F_\bullet)$ and $\text{Hom}_R(T^\bullet, C)$ are connected by a chain of quasi-isomorphisms of complexes of $R$-modules. Indeed, by Lemma 4.2(b) we have a natural isomorphism $H_0\text{Hom}_R(T^\bullet, C) = \Delta_S(C)$. Consider the total complex of the bicomplex $\text{Hom}_R(T^\bullet, F_\bullet)$. Then there are natural morphisms
of complexes of $R$-modules

$$
\Delta_S(F_\bullet) \leftarrow \Hom_R(T^\bullet, F_\bullet) \rightarrow \Hom_R(T^\bullet, C).
$$

Now $T^\bullet$ is a finite complex of projective $R$-modules, so the quasi-isomorphism $F_\bullet \rightarrow C$ induces a quasi-isomorphism $\Hom_R(T^\bullet, F_\bullet) \rightarrow \Hom_R(T^\bullet, C)$. On the other hand, for any $R$-module $C$ one has $H_1 \Hom_R(T^\bullet, C) = \Ext^0_R(K^\bullet, C)$, hence the morphism $\Hom_R(T^\bullet, F_\bullet) \rightarrow \Delta_S(F_\bullet)$ is a quasi-isomorphism by Lemma 6.1(b).

The following theorem is essentially proved in [24, Theorem 1.3 or Theorem 2.9].

**Theorem 6.4.** Assume that the left derived functor $L_\bullet \Delta$ of the right exact functor $\Delta$ has finite homological dimension, that is there exists $d \geq 0$ such that $L_n \Delta(A) = 0$ for all $A \in A$ and all $n > d$. Assume further that $L_n \Delta(C) = 0$ for every object $C \in C \subset A$ and all $n > 0$. Then for every derived category symbol $\ast = b, +, -, \emptyset, \text{abs}+, \text{abs}–$, or $\text{abs}$, the exact embedding functor $C \rightarrow A$ induces a fully faithful triangulated functor

$$
D^\ast(C) \longrightarrow D^\ast(A).
$$

**Sketch of proof.** Denote by $A_{\Delta-\text{adj}} \subset A$ the full subcategory of all objects $A \in A$ such that $L_n \Delta(A) = 0$ for all $n > 0$. Then the full subcategory $A_{\Delta-\text{adj}}$ is closed under extensions and the kernels of epimorphisms in $A$; in particular, $A_{\Delta-\text{adj}}$ inherits an exact category structure from the abelian category $A$. Furthermore, every object of $A$ admits a finite left resolution of length $d$ by objects of $A_{\Delta-\text{adj}}$. It follows that the functor $D^\ast(A_{\Delta-\text{adj}}) \rightarrow D^\ast(A)$ induced by the exact embedding $A_{\Delta-\text{adj}} \rightarrow A$ is an equivalence of triangulated categories [23, Proposition A.5.6]. The functor $\Delta$ restricted to the exact subcategory $A_{\Delta-\text{adj}} \subset A$ is exact. Applying $\Delta$ to complexes of objects from $A_{\Delta-\text{adj}}$, one constructs a triangulated functor

$$
L \Delta: D^\ast(A) \longrightarrow D^\ast(C),
$$

which is left adjoint to the functor (18). Now the composition $D^\ast(C) \rightarrow D^\ast(A) \rightarrow D^\ast(C)$ is the identity functor, since $C \subset A_{\Delta-\text{adj}}$ by an assumption of the theorem and the composition $C \rightarrow A \rightarrow C$ is the identity (as the functor $C \rightarrow A$ is fully faithful). It follows that the functor (18) is fully faithful.

Furthermore, the following proposition is essentially proved in [24, proof of Corollary 1.4 or Corollary 2.10].

**Proposition 6.5.** In the assumptions of Theorem 6.4, for every conventional derived category symbol $\ast = b, +, -, \emptyset$, or $\text{abs}$, the essential image of the triangulated functor (18) coincides with the full triangulated subcategory

$$
D^\ast_C(A) \subset D^\ast(A)
$$

of all complexes in $D^\ast(A)$ with the cohomology objects belonging to $C$. □
**Theorem 6.6.** Let $R$ be a commutative ring and $S \subset R$ be a multiplicative subset such that $\text{pd}_R S^{-1}R \leq 1$ and the $S$-torsion in $R$ is bounded. Then for every conventional derived category symbol $\ast = b, +, -, \text{ or } \emptyset$

(a) the triangulated functor between the derived categories

$$D^+(R-\text{mod}_{S,\text{tors}}) \longrightarrow D^+(R-\text{mod})$$

induced by the exact embedding functor $R-\text{mod}_{S,\text{tors}} \longrightarrow R-\text{mod}$ is fully faithful, and its essential image coincides with the full subcategory

$$D^+_{S,\text{tors}}(R-\text{mod}) \subset D^+(R-\text{mod}),$$

providing an equivalence of triangulated categories (11)

$$D^+(R-\text{mod}_{S,\text{tors}}) \simeq D^+_{S,\text{tors}}(R-\text{mod});$$

(b) the triangulated functor between the derived categories

$$D^*(R-\text{mod}_{S,\text{ctra}}) \longrightarrow D^*(R-\text{mod})$$

induced by the exact embedding functor $R-\text{mod}_{S,\text{ctra}} \longrightarrow R-\text{mod}$ is fully faithful, and its essential image coincides with the full subcategory

$$D^*_{S,\text{ctra}}(R-\text{mod}) \subset D^*(R-\text{mod}),$$

providing an equivalence of triangulated categories (11)

$$D^*(R-\text{mod}_{S,\text{ctra}}) \simeq D^*_{S,\text{ctra}}(R-\text{mod}).$$

**Proof.** To prove part (b), one applies Theorem 6.4 and Proposition 6.5. To prove part (a), one applies the opposite versions of Theorem 6.4 and Proposition 6.5. In both cases, it remains to check that the assumptions of Theorem 6.4 (or the opposite assumptions) are satisfied.

The condition of finite homological dimension of the derived functor $L_* \Delta_S$ (in the case of part (b)) or of the derived functor $R^* \Gamma_S$ (in the case of part (a)) holds by Lemmas 6.2(b)–6.3(b). To show that $L_n \Delta_S(C) = 0$ for every $S$-contramodule $R$-module $C$ and all $n > 0$ (which means, actually, $n = 1$), one compares Lemma 6.3(a) with Lemma 4.2(c). Similarly, to show that $R^n \Gamma_S(M) = 0$ for every $S$-torsion $R$-module $M$ and all $n > 0$ (i.e., $n = 1$) one compares Lemma 6.2(a) with Lemma 4.1(c). \(\square\)

The next corollary is the main result of this section.

**Corollary 6.7.** For every commutative ring $R$ with a multiplicative subset $S \subset R$ such that $\text{pd}_R S^{-1}R \leq 1$ and the $S$-torsion in $R$ is bounded, and for every conventional derived category symbol $\ast = b, +, -, \text{ or } \emptyset$, there is a natural triangulated equivalence (12) between the derived categories of the abelian categories $R-\text{mod}_{S,\text{tors}}$ and $R-\text{mod}_{S,\text{ctra}}$ of $S$-torsion and $S$-contramodule $R$-modules

$$D^*(R-\text{mod}_{S,\text{tors}}) \simeq D^*(R-\text{mod}_{S,\text{ctra}}).$$

**Proof.** Compare Theorem 4.6 with Theorem 6.6. \(\square\)
Remark 6.8. Both parts (a) and (b) of Theorem 6.6 remain true under somewhat weaker assumptions. Namely, the assumption that \( \text{pd}_R S^{-1}R \leq 1 \) was not used in the proof of Theorem 6.6(a), so it is not relevant for its validity. On the other hand, the full strength of the assumption that the \( S \)-torsion in \( R \) is bounded is not necessary for the proof of Theorem 6.6(b), as it suffices to require that there be no \( S \)-h-divisible \( S \)-torsion in the \( R \)-module \( R \) for the purposes of part (b). Indeed, the only place where the restriction on the \( S \)-torsion in \( R \) was used in the proof of Theorem 6.6(b) was in Lemma 6.1(b); and one easily observes that \( \text{Ext}^0_R(K^\bullet, F) = 0 \) for all projective \( R \)-modules \( F \) whenever there is no \( S \)-h-divisible \( S \)-torsion in \( R \).

The assumption that there be no \( S \)-h-divisible \( S \)-torsion in \( R \) is not sufficient for the validity of Theorem 6.6(a), though, as one can see using the following argument. Let \( R \) be a commutative ring and \( S \subset R \) be a multiplicative subset. Then, for any injective \( R \)-module \( J \), the functor right adjoint to the triangulated functor \( \text{D}^b(S\text{-mod}_{S\text{-tors}}) \longrightarrow \text{D}^b(R\text{-mod}) \) is defined on the object \( J \in \text{D}^b(R\text{-mod}) \) and takes it to the object \( \Gamma_S(J) \in \text{D}^b(R\text{-mod}_{S\text{-tors}}) \). On the other hand, the functor right adjoint to the embedding functor \( \text{D}^b(S\text{-tors})(R\text{-mod}) \longrightarrow \text{D}^b(R\text{-mod}) \) takes \( J \) to the object \( \tilde{C}^\bullet(M)^\sim \in \text{D}^b(S\text{-tors})(R\text{-mod}) \). Now if the functor \( \text{D}^b(R\text{-mod}_{S\text{-tors}}) \longrightarrow \text{D}^b(R\text{-mod}) \) is fully faithful, then it is clear that its essential image coincides with the full subcategory \( \text{D}^b(S\text{-tors})(R\text{-mod}) \subset \text{D}^b(R\text{-mod}) \), and it follows that the objects \( \Gamma_S(J) \) and \( \tilde{C}^\bullet(M)^\sim \) must be isomorphic in \( \text{D}^b(S\text{-tors})(R\text{-mod}) \), that is the assertion of Lemma 6.1(a) has to hold. When the multiplicative subset \( S \subset R \) is generated by one element \( s \in R \), this condition is equivalent to boundedness of \( S \)-torsion in \( R \).

7. The Dedualizing Complex

The proof of Corollary 6.7 still leaves something to be desired. Given a complex of \( S \)-torsion \( R \)-modules \( M^\bullet \), in order to obtain the related complex of \( S \)-contramodule \( R \)-modules \( C^\bullet \) following this proof, one would have to consider the complex of \( R \)-modules

\[
A^\bullet = \text{Hom}_R(T^\bullet, M^\bullet) \in \text{D}^\bullet_{S\text{-ctra}}(R\text{-mod})
\]

with \( S \)-contramodule cohomology \( R \)-modules, and then pass from it to a complex of \( S \)-contramodule \( R \)-modules \( C^\bullet \) by applying the functor \( \mathbb{L}\Delta_S \), that is

\[
C^\bullet = \mathbb{L}\Delta_S(A^\bullet).
\]

The terms of the complex \( A^\bullet \) themselves are not \( S \)-contramodules. One can avoid the preliminary step of this two-step procedure by setting directly

\[
C^\bullet = \mathbb{L}\Delta_S(M^\bullet),
\]

but in order to apply the derived functor \( \mathbb{L}\Delta_S \) to an \( S \)-torsion \( R \)-module or a complex of \( S \)-torsion \( R \)-modules one would still have to use resolutions in the category of arbitrary \( R \)-modules \( R\text{-mod} \).
Similarly, given a complex of $S$-contramodule $R$-modules $C^\bullet$, in order to obtain the related complex of $S$-torsion $R$-modules $M^\bullet$ one would have to consider the complex of $R$-modules

$$N^\bullet = \check{C}^\bullet(C^\bullet)^\sim \text{ or } T^\bullet \otimes_R C^\bullet \in D_{S\text{-tors}}(R\text{-mod})$$

with $S$-torsion cohomology $R$-modules, and then pass from it to a complex of $S$-torsion $R$-modules $M^\bullet$ by applying the functor $\mathbb{R}\Gamma_S$, that is

$$M^\bullet = \mathbb{R}\Gamma_S(N^\bullet).$$

The terms of the complex $N^\bullet$ themselves are not $S$-torsion $R$-modules. One can avoid the preliminary step of this procedure by setting

$$M^\bullet = \mathbb{R}\Gamma_S(C^\bullet),$$

but in order to apply the derived functor $\mathbb{R}\Gamma_S$ to an $S$-contramodule $R$-module or a complex of $S$-contramodule $R$-modules one would still have to use resolutions in the category of arbitrary $R$-modules.

We would like to have direct constructions of the mutually inverse triangulated functors

$$D^\bullet(R\text{-mod}_{S\text{-tors}}) \longrightarrow D^\bullet(R\text{-mod}_{S\text{-contra}}) \text{ and } D^\bullet(R\text{-mod}_{S\text{-contra}}) \longrightarrow D^\bullet(R\text{-mod}_{S\text{-tors}})$$

staying entirely inside the two covariantly dual worlds of $S$-torsion and $S$-contramodule $R$-modules and never involving $R$-modules of more general nature. The test assertions or constructions we want to be able to demonstrate with such a technique are the equivalences of the absolute derived categories of $S$-torsion and $S$-contramodules $R$-modules (12) with $\star = \text{abs}^+, \text{abs}^-$, or $\text{abs}$.

In this section, a partial solution for this problem is obtained: we indeed construct mutually inverse functors acting directly between the derived categories of $S$-torsion and $S$-contramodule $R$-modules without going through complexes of arbitrary $R$-modules, and we indeed obtain the equivalences (12) for absolute derived categories. However, the proof of the assertion that our functors are mutually inverse equivalences involves $R$-modules not belonging to $R\text{-mod}_{S\text{-tors}}$ or $R\text{-mod}_{S\text{-contra}}$.

Let $S$ be a multiplicative subset in a commutative ring $R$. We keep assuming that $\text{pd}_R S^{-1}R \leq 1$ and the $S$-torsion in $R$ is bounded. Lemma 7.1 and Proposition 7.2 will not depend on these assumptions yet.

For every element $s \in S$, we denote by $T^\bullet_s$ the two-term complex $R \xrightarrow{s} R$, with the two terms $R$ sitting in the cohomological degrees 0 and 1. Notice that the complex $T^\bullet_s$ is almost self-dual: one has $\text{Hom}_R(T^\bullet_s, R) \simeq T^\bullet_s[1]$. Hence for every complex of $R$-modules $A^\bullet$ there is an isomorphism $\text{Hom}_R(T^\bullet_s, A^\bullet) \simeq T^\bullet_s[1] \otimes_R A^\bullet$.

**Lemma 7.1.** Let $A^\bullet$ be a complex of $R$-modules such that either

(a) the cohomology modules $H^\bullet(A^\bullet)$ are $S$-torsion $R$-modules, or

(b) the cohomology modules $H^\bullet(A^\bullet)$ are $S$-contramodules.

Suppose that the complex $T^\bullet_s \otimes_R A^\bullet$ is acyclic for every $s \in S$. Then the complex $A^\bullet$ is acyclic.
Proof. For every complex of $R$-modules $A^\bullet$, there are natural short exact sequences

$$0 \rightarrow H^{n-1}(A^\bullet)/sH^{n-1}(A^\bullet) \rightarrow H^n(T^\bullet \otimes_R A^\bullet) \rightarrow sH^n(A^\bullet) \rightarrow 0,$$

where we denote by $sM \subset M$ the submodule of elements annihilated by $s$ in an $R$-module $M$. So $H^n(T^\bullet \otimes_R A^\bullet) = 0$ means that $s$ acts invertibly in $H^n(A^\bullet)$. If this holds for every $s \in S$ then the cohomology modules $H^n(A^\bullet)$ are $(S^{-1}R)$-modules. Now any $S$-torsion $R$-module that is simultaneously an $(S^{-1}R)$-module vanishes, and so does any $S$-contramodule $R$-module that is simultaneously an $(S^{-1}R)$-module. □

The exposition in this section follows the lines of [24, Section 5]. In order to make the analogy more explicit, we formulate the following version of [24, Proposition 5.1].

**Proposition 7.2.** The collection of all complexes $T_s^\bullet$, $s \in S$ is a set of compact generators of the triangulated category $D_{S\text{-tors}}(R\text{-mod})$. A complex of $R$-modules with $S$-torsion cohomology modules is a compact object of $D_{S\text{-tors}}(R\text{-mod})$ if and only if it is a compact object of $D(R\text{-mod})$.

**Proof.** Both assertions follows from Lemma 7.1(a) together with the observations that the complexes $T_s^\bullet$ belong to $D_{S\text{-tors}}(R\text{-mod})$ and are compact in $D(R\text{-mod})$. □

The following theorem is our version of [24, Remark 5.6]. The assumption of boundedness of $S$-torsion in $R$ is essential for its validity.

**Theorem 7.3.** (a) Let $M$ be an $S$-torsion $R$-module and $E$ be an injective object of $R\text{-mod}_{S\text{-tors}}$. Then $\text{Ext}_R^n(M, E) = 0$ for all $n > 0$.

(b) Let $C$ be an $S$-contramodule $R$-module and $P$ be a projective object of $R\text{-mod}_{S\text{-contra}}$. Then $\text{Ext}_R^n(P, C) = 0$ for all $n > 0$.

(c) Let $M$ be an $S$-torsion $R$-module and $P$ be a projective object of $R\text{-mod}_{S\text{-contra}}$. Then $\text{Tor}_n^R(M, P) = 0$ for all $n > 0$.

**Proof.** Part (a) is a particular case of the assertion that the functor $D^b(R\text{-mod}_{S\text{-tors}}) \rightarrow D^b(R\text{-mod})$ is fully faithful (see Theorem 6.6(a)). Part (b) is a particular case of the assertion that the functor $D^b(R\text{-mod}_{S\text{-contra}}) \rightarrow D^b(R\text{-mod})$ is fully faithful (see Theorem 6.6(b)). To deduce part (c) from part (b), choose an injective $R$-module $J$. Then $\text{Hom}_R(M, J)$ is an $S$-contramodule $R$-module by Lemma 3.3, hence $\text{Hom}_R(\text{Tor}_n^R(M, P), J) \cong \text{Ext}_R^n(P, \text{Hom}_R(M, J)) = 0$ for $n > 0$. □

One can define a dedualizing complex of $S$-torsion $R$-modules by the list of three conditions very similar to the conditions (i-iii) of [24, Section 5]. In this section, we follow a simpler path. The complex $K^\bullet[-1]$ has $S$-torsion cohomology modules, hence, according to Theorem 6.6(a), there exists a finite complex of $S$-torsion $R$-modules $B^\bullet$ quasi-isomorphic to $K^\bullet[-1]$. Obviously, one can assume $B^\bullet$ to be a two-term complex concentrated in the cohomological degrees 0 and 1. We choose such a complex of $S$-torsion $R$-modules $B^\bullet$, and call it the dedualizing complex for the ring $R$ and the multiplicative subset $S \subset R$.

Furthermore, we set $B_s^\bullet = T_s^\bullet \otimes_R B^\bullet$. Clearly, $B_s^\bullet$ is a finite complex of $S$-torsion $R$-modules quasi-isomorphic to the complex of $R$-modules $T_s^\bullet \otimes_R K^\bullet[-1]$, which is
The morphism quasi-isomorphic to $T_s^\bullet$. So we can assume to have chosen a quasi-isomorphism of complexes of $R$-modules $b_s: T_s^\bullet \rightarrow B_s^\bullet$.

The following lemma is our version of [24, Lemma 5.4(b-c)].

**Lemma 7.4.** (a) For every projective $R$-module $F$ and every element $s \in S$, the morphism of complexes of $R$-modules $T_s^\bullet \otimes_R F \rightarrow T_s^\bullet \otimes_R \Delta_S(F)$ induced by the morphism of $R$-modules $\delta_{S,F}: F \rightarrow \Delta_S(F)$ is a quasi-isomorphism.

(b) For every projective object $P \in R\text{-mod}_{S\text{-ctra}}$ and every element $s \in S$, the morphism of complexes of $R$-modules $T_s^\bullet \otimes_R P \rightarrow B_s^\bullet \otimes_R P$ induced by the morphism $b_s: T^\bullet \rightarrow B_s^\bullet$ is a quasi-isomorphism.

**Proof.** Part (a): according to the short exact sequences (II–III) and Lemma 5.3(b), for every $R$-module $F$ without $S$-torsion the cohomology modules of the two-term complex $F \rightarrow \Delta_S(F)$ are $(S^{-1}R)$-modules. Hence the tensor product complex $T_s^\bullet \otimes_R F \rightarrow \Delta_S(F)$ is acyclic.

Part (b): according to Theorem 7.3(c), both the complexes $T_s^\bullet \otimes_R P$ and $B_s^\bullet \otimes_R P$ compute the derived tensor product $T_s^\bullet \otimes_R^L P = B_s^\bullet \otimes_R^L P$. Alternatively, it suffices to consider the case of $P = \Delta_S(F)$, where $F$ is a projective $R$-module (as all the projective objects in $R\text{-mod}_{S\text{-ctra}}$ are direct summands of modules of this form). We have a commutative diagram

$$
\begin{array}{ccc}
T_s^\bullet \otimes_R F & \rightarrow & B_s^\bullet \otimes_R F \\
\downarrow & & \downarrow \\
T_s^\bullet \otimes_R \Delta_S(F) & \rightarrow & B_s^\bullet \otimes_R \Delta_S(F)
\end{array}
$$

The morphism $T_s^\bullet \otimes_R F \rightarrow B_s^\bullet \otimes_R F$ is a quasi-isomorphism since $F$ is a flat $R$-module. The morphism $B_s^\bullet \otimes_R F \rightarrow B_s^\bullet \otimes_R \Delta_S(F)$ is an isomorphism of complexes by Lemma 1.11, because $B_s^\bullet$ is a complex of $S$-torsion $R$-modules. The morphism $T_s^\bullet \otimes_R F \rightarrow T_s^\bullet \otimes_R \Delta_S(F)$ is a quasi-isomorphism by part (a). It follows that the fourth morphism in the diagram is also a quasi-isomorphism. □

The next lemma is our version of [24, Lemma 5.5].

**Lemma 7.5.** (a) For every injective $R$-module $J$ and every element $s \in S$, the morphism of complexes of $R$-modules $T_s^\bullet \otimes_R \Gamma_S(J) \rightarrow T_s^\bullet \otimes_R J$ induced by the embedding of $R$-modules $\gamma_{S,J}: \Gamma_S(J) \rightarrow J$ is a quasi-isomorphism.

(b) For every injective object $E \in R\text{-mod}_{S\text{-tors}}$ and every element $s \in S$, the morphism of complexes of $R$-modules $\text{Hom}_R(B_s^\bullet, E) \rightarrow \text{Hom}_R(T_s^\bullet, E)$ induced by the morphism $b_s: T_s^\bullet \rightarrow B_s^\bullet$ is a quasi-isomorphism.

**Proof.** Part (a): According to Lemma 6.1(a) and the short exact sequence (I), for every injective $R$-module $J$ the quotient module $J/\Gamma_S(J)$ is an $(S^{-1}R)$-module. Hence the complex $T_s^\bullet \otimes_R J/\Gamma_S(J)$ is acyclic.

Part (b): according to Theorem 7.3(a), both the complexes $\text{Hom}_R(B_s^\bullet, E)$ and $\text{Hom}_R(T_s^\bullet, E)$ compute the derived category object $\mathbb{R}\text{Hom}_R(B_s^\bullet, E) = \mathbb{R}\text{Hom}_R(T_s^\bullet, E)$. 37
Theorem 7.6. Let $S$ be a multiplicative subset in a commutative ring $R$. Assume that $\text{pd}_R S^{-1}R \leq 1$ and the $S$-torsion in $R$ is bounded. Then for every derived category symbol $\star = b$, $+$, $-$, $\emptyset$, $\text{abs+}$, $\text{abs-}$, or $\text{abs}$ there is an equivalence of derived categories $\mathcal{D}^*(\text{R-mod}_{S,\text{tors}}) \simeq \mathcal{D}^*(\text{R-mod}_{S,\text{ctra}})$ provided by mutually inverse functors $\mathbb{R}\text{Hom}_R(B^\bullet, -)$ and $B^\bullet \otimes^L_R -$.

Proof. Notice that the functor $\text{Hom}_R(B^\bullet, -)$ takes complexes of $S$-torsion $R$-modules to complexes of $S$-contramodule $R$-modules (by Lemma 3.3), and the functor $B^\bullet \otimes_R -$ takes complexes of $S$-contramodule $R$-modules to complexes of $S$-torsion $R$-modules (obviously). The construction of the two derived functors

$$\mathbb{R}\text{Hom}_R(B^\bullet, -): \mathcal{D}^*(\text{R-mod}_{S,\text{tors}}) \longrightarrow \mathcal{D}^*(\text{R-mod}_{S,\text{ctra}})$$

and

$$B^\bullet \otimes^L_R -: \mathcal{D}^*(\text{R-mod}_{S,\text{ctra}}) \longrightarrow \mathcal{D}^*(\text{R-mod}_{S,\text{tors}})$$

is explained in [24, Appendix B]; see also [24, proofs of Theorems 4.9 and 5.10]. They form a pair of adjoint triangulated functors between the derived categories $\mathcal{D}^*(\text{R-mod}_{S,\text{tors}})$ and $\mathcal{D}^*(\text{R-mod}_{S,\text{ctra}})$ [24, Appendix B].

As explained further in [24, proofs of Theorems 4.9 and 5.10], showing that the adjunction morphisms for this pair of adjoint functors are isomorphisms in $\mathcal{D}^*(\text{R-mod}_{S,\text{tors}})$ and $\mathcal{D}^*(\text{R-mod}_{S,\text{ctra}})$ reduces to the cases of a single injective object $E \in \text{R-mod}_{S,\text{tors}}$ or a single projective object $P \in \text{R-mod}_{S,\text{ctra}}$. The next proposition claims that the required morphisms are quasi-isomorphisms. $\square$

Proposition 7.7. (a) Let $E$ be an injective object in $\text{R-mod}_{S,\text{tors}}$ and let $P^\bullet$ be a bounded above complex of projective objects in $\text{R-mod}_{S,\text{ctra}}$ endowed with a quasi-isomorphism of complexes of $S$-contramodule $R$-modules $P^\bullet \longrightarrow \text{Hom}_R(B^\bullet, E)$. Then the natural morphism of complexes of $S$-torsion $R$-modules $B^\bullet \otimes_R P^\bullet \longrightarrow E$ is a quasi-isomorphism.
(b) Let $P$ be a projective object in $\text{mod}_{S\text{-tra}}$ and let $E^\bullet$ be a bounded below complex of injective objects in $\text{mod}_{S\text{-tors}}$ endowed with a quasi-isomorphism of complexes of $S$-torsion $R$-modules $B^\bullet \otimes_R P \longrightarrow E^\bullet$. Then the natural morphism of complexes of $S$-contramodule $R$-modules $P \longrightarrow \text{Hom}_R(B^\bullet, E^\bullet)$ is a quasi-isomorphism.

Proof. Part (a): by Lemma 7.1(a), it suffices to show that the morphism of complexes of $S$-torsion $R$-modules

$$B^\bullet \otimes_R P^\bullet = T^\bullet \otimes_R B^\bullet \otimes_R P^\bullet \longrightarrow T^\bullet \otimes_R E$$

is a quasi-isomorphism for every element $s \in S$. Notice that a morphism of complexes of $S$-torsion $R$-modules is a quasi-isomorphism of complexes in the abelian category $\text{mod}_{S\text{-tors}}$ if and only if it is a quasi-isomorphism of complexes of $R$-modules. According to Lemma 7.4(b), the morphism $T^\bullet \otimes_R P^\bullet \longrightarrow T^\bullet \otimes_R \text{Hom}_R(B^\bullet, E)$ induced by the morphism $b_s: T^\bullet \longrightarrow B^\bullet$ is a quasi-isomorphism of complexes of $R$-modules. Since $T^\bullet$ is a finite complex of (finitely generated) projective $R$-modules, the morphism of complexes of $(S\text{-contramodule})$ $R$-modules

$$T^\bullet \otimes_R P^\bullet \longrightarrow T^\bullet \otimes_R \text{Hom}_R(B^\bullet, E)$$

induced by the quasi-isomorphism $P^\bullet \longrightarrow \text{Hom}_R(B^\bullet, E)$ is also a quasi-isomorphism. We have a commutative diagram

$$
\begin{array}{ccc}
T^\bullet \otimes_R P^\bullet & \longrightarrow & T^\bullet \otimes_R \text{Hom}_R(B^\bullet, E) \\
\downarrow & & \downarrow \\
T^\bullet \otimes_R B^\bullet \otimes_R P^\bullet & \longrightarrow & T^\bullet \otimes_R E
\end{array}
$$

Hence it remains to show that the morphism of complexes of $R$-modules

$$T^\bullet \otimes_R \text{Hom}_R(B^\bullet, E) \longrightarrow T^\bullet \otimes_R E$$

induced by the morphism $b_s: T^\bullet \longrightarrow B^\bullet = T^\bullet \otimes_R B^\bullet$ is a quasi-isomorphism. Since $T^\bullet \simeq \text{Hom}_R(T^\bullet, R)[-1]$, this is equivalent to saying that the morphism

$$\text{Hom}_R(b_s, E): \text{Hom}_R(T^\bullet \otimes_R B^\bullet, E) \longrightarrow \text{Hom}_R(T^\bullet, E)$$

is a quasi-isomorphism, which is the result of Lemma 7.5(b).

Part (b): by Lemma 7.1(b), it suffices to show that the morphism of complexes of $S\text{-contramodule} R$-modules

$$\text{Hom}_R(T^\bullet, P) \longrightarrow \text{Hom}_R(T^\bullet \otimes_R B^\bullet, E^\bullet) = \text{Hom}_R(B^\bullet, E^\bullet)$$

is a quasi-isomorphism for every element $s \in S$. Notice that a morphism of complexes of $S\text{-contramodule} R$-modules is a quasi-isomorphism of complexes in the abelian category $\text{mod}_{S\text{-tra}}$ if and only if it is a quasi-isomorphism of complexes of $R$-modules. According to Lemma 7.5(b), the morphism

$$\text{Hom}_R(B^\bullet, E^\bullet) \longrightarrow \text{Hom}_R(T^\bullet, E^\bullet)$$
induced by the morphism $b_s: T_s^\bullet \to B_s^\bullet$ is a quasi-isomorphism of complexes of $R$-modules. Since $T_s^\bullet$ is a finite complex of (finitely generated) projective $R$-modules, the morphism of complexes of $(S$-torsion) $R$-modules

$$\text{Hom}_R(T_s^\bullet, B^\bullet \otimes_R P) \longrightarrow \text{Hom}_R(T_s^\bullet, E^\bullet)$$

induced by the quasi-isomorphism $B^\bullet \otimes_R P \to E^\bullet$ is also a quasi-isomorphism. We have a commutative diagram

$$\begin{array}{ccc}
\text{Hom}_R(T_s^\bullet, P) & \longrightarrow & \text{Hom}_R(T_s^\bullet \otimes_R B^\bullet, E^\bullet) \\
\downarrow & & \downarrow \\
\text{Hom}_R(T_s^\bullet, B^\bullet \otimes_R P) & \longrightarrow & \text{Hom}_R(T_s^\bullet, E^\bullet)
\end{array}$$

Hence it remains to show that the morphism of complexes of $R$-modules

$$\text{Hom}_R(T_s^\bullet, P) \longrightarrow \text{Hom}_R(T_s^\bullet, B^\bullet \otimes_R P)$$

induced by the morphism $b_s: T_s^\bullet \to B_s^\bullet = T_s^\bullet \otimes_R B^\bullet$ is a quasi-isomorphism. Since $\text{Hom}_R(T_s^\bullet, R) \cong T_s^\bullet[1]$, this is equivalent to saying that the morphism

$$T_s^\bullet \otimes_R P \longrightarrow T_s^\bullet \otimes_R B^\bullet \otimes_R P$$

is a quasi-isomorphism, which is the result of Lemma 7.4(b). $\square$

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