ON THE FRACTIONAL STOCHASTIC NAVIER-STOKES EQUATIONS
ON THE TORUS AND ON BOUNDED DOMAINS.

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Abstract. In this work, we introduce and study the well-posedness of the multidimensional fractional stochastic Navier-Stokes equations on bounded domains and on the torus (Briefly dD-FSNSE). We prove the existence of a martingale solution for the general regime. We establish the uniqueness in the case a martingale solution enjoys a condition of Serrin’s type on the fractional Sobolev spaces. If an $L^2$— local weak (strong in probability) solution exists and enjoys conditions of Beale-Kato-Majda type, this solution is global and unique. These conditions are automatically satisfied for the 2D-FSNSE on the torus if the initial data has $H^1$—regularity and the diffusion term satisfies growth and Lipschitz conditions corresponding to $H^1$—spaces. The case of 2D-FSNSE on the torus is studied separately. In particular, we established thresholds for the global existence, uniqueness, space and time regularities of the weak (strong in probability) solutions in the subcritical regime.

Keywords: Fractional stochastic Navier-Stokes equation, classical Navier-Stokes equation, fractional stochastic vorticity Navier-Stokes equation, Q-Wiener process, trace class operators, subcritical, critical, supercritical, dissipative and hyperdissipative regimes, martingale, global and local weak-strong solutions, Riesz transform, Serrin’s condition, Beale-Kato-Majda condition, fractional Sobolev spaces, pseudo-differential operators, Skorokhod embedding theorem, Faedo-Galerkin approximation, compactness method, representation theorem.

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1. Introduction

The Navier-Stokes equation (briefly NSE) has been derived, more than one century ago, by the engineer C.L. Navier to describe the motion of an incompressible Newtonian fluid. Later, it has been reformulated by the mathematician-physicist G. H. Stokes. Since that time, this equation continues to attract a great deal of attention due to its mathematical and physical importance. This equation appears, alone or coupled with other active and passive scalar equations, in the study of many phenomena, see e.g. the list of references in this work and in [15]. The 3D-stochastic Navier-Stokes equation (briefly 3D-SNSE) is the most realistic model in fluid dynamics and for many other physical purposes, see e.g. [75]. The 2D-SNSE is used as an approximation of the 3D model when the velocity of the fluid belongs to a plane, as is the case for basins and oceans. The 2D-SNSE on a bounded domain $O \subset \mathbb{R}^2$ governs the flow of a fluid which fills in infinite cylinder of cross-section $O$ and moves parallel to the plane of $O$. Physically, NSE on the torus is not a realistic model, but it is used for some idealizations and for homogenization problems in turbulence see e.g. [26]. Mathematically, basic questions like the existence and the uniqueness of a global smooth solution of the dD-NSE is still an open problem for $d \geq 3$. In particular, for
$d = 3$, the statement above is one formulation of the so called the millennium problem of the Navier-Stokes equation. The main difficulty in the study of the dD-NSE is related to the nonlinear term. In particular, as this latter comes from kinematical considerations, i.e. deduced from a mathematical calculus, it is no possible to change or to replace it.

In this work, we deal with the d-dimensional fractional stochastic Navier-Stokes equation (dD-FSNSE) on bounded domains and on the torus. One of the benefits of the study of the fractional Navier-stokes equation is to contribute in the understanding of millennium problem. In fact, this last is regarded as a dimensional problem due to the fact that, contrarily to the 3D-NSE, the 2D-NSE is well-posed and well understood. However, it is also known that the 3D-hyperdissipative Navier-Stokes equation admits a global classical solution, provided that the order of dissipation is greater than or equal to $5$, see e.g. \[46, 85\].

In \[37\], the authors established a cheap Caffarelli-Kohn-Nirenberg inequality for the 3D-hyperdissipative Navier-Stokes equation. This latter is also used to regularize the classical NSE, see e.g. \[16, 19\]. Therefore, in addition to the dimension, the problem of the dD-NSE could also be regarded as a dissipative problem as well. Moreover, the author claims that the 2D-FSNSE behaves, i.e. exhibits difficulties, like the classical 3D-NSE, see the proof in Section \[7\]. The global existence of the solution for the 2D-FSNSE is obtained by using the vorticity regularization effect. This proves one of the classical conveniences stating that the main differences between the 2D & the 3D NSE appears in the vorticity, see e.g. \[10\]. To support more the authors’s claim above, we draw attention to the undimensional similarity between the 3D−vorticity NSE, see e.g. \[10, 29\] and the no-free divergence mode scalar active equation studied in \[15\]. For this latter, we are not able to prove the global existence.

Recently, the author studied a class of fractional stochastic active scalar equations generalizing, among many other equations, the 2D−fractional stochastic vorticity Navier-Stokes equation and the dD-stochastic quasi-geostrophic equation \[15\]. In particular, thresholds to ensure the existence, uniqueness and the regularities of several kinds of solutions have been established. The author characterized, among others, the following two intrinsic thresholds $\alpha_0(d, q) := 1 + \frac{d}{q}$, $q > d$ and $\alpha_0(d) := 1 + \frac{d-1}{3}$, for $d \in \{2, 3\}$, which guarantee the existence of the $L^q$—respectively the $L^\frac{3d}{d-1}$-mild solutions (weak-strong as well). Other critical dissipation values are also obtained according to the different Sobolev regularities required for the solutions.

Motivated by the results in \[15\], we try to make precise some balance relationships between the dissipation order, the dimension and the regularity of the solutions and establish dissipative thresholds for the well-posedness of the $dD$−FSNSE. In this work, we consider the Hilbert setting. To the best knowledge of the author, the present work is the first in the target to study the well-posedness of the $dD$−Navier-Stokes equation, fractional and classical, from this triple-view, i.e. simultaneously taking into account the dissipation, the regularity and the dimension, quantifying the balance between them and establishing optimal thresholds.

To further clarify what is new in the present work, it is of great importance to point out some features and some delicate problems related to the FSNSE. Some of these problems are inherited from the classical NSE. Other problems for general fractional stochastic partial differential equations have been discussed in \[15\] see also \[17\].
The energy method applied for the dissipative PDEs is based on the ability to control the kinetic energy $e(u)$ and the enstrophy energy $E(u)$ of the solution $u$. Recall that
\begin{equation}
(1.1) \quad e(u) := \frac{1}{2} \int_{O} |u|^2 dx, \quad E(u) := \int_{O} |\nabla u|^2 dx.
\end{equation}
The control of these quantities for the classical NSE emerges from the structure of the equation itself. However, for the FNSE, a priori, there is no guarantee about the control of the enstrophy energy. The structure of this equation guarantees only the control of a weaker Sobolev norm. This fact is again due to the weakness of the fractional dissipation. In some special cases like the 2D–FNSE on the torus, see Section 5, the control of the enstrophy energy emerges from the structure of the fractional equation. This improvement and also the improvement of the results in this case are consequences of the $H^{1/2}$-orthogonality. These facts generalize the classical features known for the classical NSE on the 2D-torus, see e.g. \cite{75}.

A delicate technical feature of FNSE is the estimation of the nonlinear term. In fact, as the structure of the equation cannot initially guarantee the boundedness of the enstrophy energy, mathematically, we are not allowed to estimate terms by $H^1$-norm. Moreover, contrarily to the classical NSE, where the $H^1$–space plays a common role for the linear and the nonlinear terms, the components of the Gelfand triple corresponding to the FNSE are not automatically coherent with respect to the two terms. More precisely, the nonlinear term is not bounded on the domain of definition of the fractional Stokes operator. Indeed, this latter is larger than $H^1$, see details in Section 4 and in Remark 1. Therefore, an extension of the nonlinear term is needed. In order to construct a coherent Gelfand triple, to extend and to estimate the nonlinear term, we have established more refined estimates via fractional Sobolev spaces of order less than one. These estimates are completely new.

We add to this enumeration of novelties and features the following two questions, which are simple to resolve but important to deal with. To introduce the FSNSE defined on $O = \mathbb{R}^d$ or $O = \mathbb{T}^d$, we take the fractional power of the Stokes operator, which is here equal to minus the Laplacian. Moreover, these equations take more advantage of the facts that the fractional power of the Stokes operator is defined as a pseudodifferential operator and commutes with the Helmholtz projection and with the partial differential operators $\partial_j, j = 1, \ldots, d$. Contrarily to these two cases, the situation for the FSNSE on a bounded domain, $O \subset \mathbb{R}^d$ is much more involved. In fact, it is well known that in this case, the Stokes operator is different than the Laplacian. To introduce the FNSE on a bounded domain we can use two approaches by taking either the fractional power of the Stokes operator or by taking the fractional power of minus the Dirichlet-boundary Laplacian operator and than apply Helmholtz projection. Initially, due to the effect of the boundaries and to the application of Helmholtz projection, we cannot conclude, a priori, whether or not the two approaches yield the same equation. In particular, it is intuitively seen that the fractional equation obtained by the first approach is more theoretical and the equation obtained by the second approach is more suitable for physical modeling, see more discussion in sections 2 & 3 and Appendix A. In this work, we introduce both equations and prove that they are well defined and equivalent. The author does not know any works considering deterministic or stochastic FNSE on bounded domains.

To prove the global existence of the weak solutions for the 2D-FSNSE on the torus, we use the regularization effect of the vorticity and the results from \cite{16}. For the classical\footnote{Enstrophy comes from Greek and means rotation.}
NSE, the evolution equation describing the vorticity is obtained by the application of the curl operator on the pathwise velocity equation, see e.g. [10, 47, 48]. As the fractional operator is nonlocal, it is of great importance to derive the vorticity equation corresponding to the FSNSE. We obtain, without difficulties, the 2D-fractal stochastic vorticity NSE by application of the curl operator to the abstract integral 2D-FSNSE. In particular, we investigate, in a rigorous way, the curl of the stochastic term and the composition of the curl and the fractional Stokes operators, see Appendix B. The study of the FSNSE on a bounded domain is more difficult in both classical and fractional cases. In fact, it is well known that, when boundaries are present for the classical NSE (either deterministic or stochastic), there is no simple boundary condition to impose on the vorticity in such a way that the velocity satisfies the right boundary conditions, see e.g. [28, 29, 47, 48]. In the fractional case, a new difficulty emerges due to the fact that the boundaries are also included in the definition of the fractional operator. Therefore, due to these multiple difficulties and to the fact that we need results already proved in the work in progress [10], we postpone the study of this case.

Recently, the deterministic fractional Navier-Stokes equation has been studied in some works using analytical and probabilistic tools, see e.g. [8, 18, 37, 70, 83, 85, 87]. The existence and the uniqueness of a local solution for the FNSE in Besov space in the subcritical regime and under conditions on the regularity of the initial data, have been proved in [55, Theorems 6.2 & 6.3]. If moreover, the Besov norm of the initial data is dominated by the viscosity, the solution is global [55, Theorems 6.1]. In [8], the authors studied the 2D-FNSE and proved the existence and the uniqueness of a global solution in some Besov spaces. They also proved that the family of viscosity fractional diffusion solutions converges in $L^q$—space (with $q$ depends on $\alpha$) to the unique solution of Euler equation. In particular, for the subcritical regime the convergence is obtained in Besov space. The convergence rates in both regimes have been established as well. In [18], the authors used the smoothing property of the fractional Oseen kernel, to establish the space analyticity and the decay estimates of the local mild solution of the FNSE in the subcritical regime. The results are proved in time weighted space. The stochastic Lagrangian particle approach has been used in [57] to prove the local existence and the uniqueness of the solution of the subcritical NSE driven by the infinitesimal generator of a Lévy semigroup. The authors assume that the real part of the Lévy-Khintchine formula behaves as a fractional power symbol and that the initial condition has $H^{1,q}$—regularity. The solution conserves the $H^{1,q}$—regularity, satisfies the nonlocal NSE in distribution sense and when the dimension $d=2$, the solution is global [57, Theorem 3.6 & 2.4]. In the periodic case and under the large viscosity condition, the author proved that the solution is global [57, Theorem 5.1]. In addition to the references about the hyperdissipative regime, [37, 46, 49] cited above, we mention here also the references [70, 84], where the authors treated the regularity properties of the solution of the hyperdissipative regime FNSE respectively of Magnetohydrodynamic equations with dissipation order $\alpha \geq 1 + d/2$.

As mentioned above, the aim of this work is to study the multi-dimesional fractional stochastic Navier-Stokes equation (dD-FSNSE) on bounded domains in $\mathbb{R}^d$ and on the torus $T^d$, with $d \geq 2$. We investigate the existence, the uniqueness and the regularity of weak (strong in probability) solution for the critical and subcritical 2D-FNSE on the torus, martingale solution for general regime dD-FNSE. In particular, we established, in the fractional framework, conditions of Serrin’s and of Beale-Kato-Majda type ensuring the global existence and the uniqueness of weak-strong solutions. The threshold $\alpha_0(d) := 1+\frac{d-1}{3}$
and the Sobolev order $\frac{d+2-a}{2}$ also emerge. We do not assume any restrictions neither on the viscosity nor on the initial condition (smallness or regularity). The local solutions can start from an $L^2$-initial data. The results obtained in this work cover not only our scopes of interest, which are the subcritical, critical and supercritical regimes and the stochastic case, but they are also valid for the deterministic case and for the dissipative and the hyperdissipative regimes. In some places, we need the condition $\alpha < 2$, but in these cases the same result can be proved for $\alpha \geq 2$ by using classical and simpler methods.

The paper is organized as follow, in Section 2, we introduce rigorously the FSNSE. We prove in Appendix A that the two approaches described above yield to the same equation. The main definitions and results are presented in Section 3. Section 4 is devoted to the prove in Appendix A that the two approaches described above yield to the same equation. Preliminary Notations & General Remarks

Let $\mathbb{N}_k := \{ j \in \mathbb{N}, \text{s.t. } j > k \}$ and $\mathbb{Z}_0^d := \mathbb{Z}^d - \{0\}$. For $d \in \mathbb{N}_0$, we denote by $\mathbb{T}^d$ the $d$-dimensional torus and by $D(\mathbb{T}^d)$ the set of infinitely differentiable scalar-valued (complex) functions on $\mathbb{T}^d$. By a domain $^*O^*$ we mean an open non empty set. For either $O = \mathbb{T}^d$ or $O \subset \mathbb{R}^d$ bounded, we define $H^{l,q}(O) := ((H^{\beta,q}(O))^l, l \in \mathbb{N}_0, \beta \in \mathbb{R}, 1 < q < \infty$, in particular for $\beta = 0, L^1(O) := (L^1(O))^l)$. Recall that $H^{l,q}(O)$, according to $O$, are either the Sobolev spaces on a bounded domain or the null average periodic Sobolev spaces on the torus. $C^\infty_0(O)$ is the set of infinitely differentiable real functions with compact support on the bounded domain $O \subset \mathbb{R}^d$, $H^{l,q}_0(O), \beta \in \mathbb{R}_+, 1 < q < \infty$ is the completion of $C^\infty_0(O)$ in $H^{l,q}_0(O)$, with $O \subset \mathbb{R}^d$ bounded. $\partial_{x_j}$ stands for the partial derivative with respect to the component $x_j$, sometimes we also use the notation $\partial_j$. We use the notation $| \cdot |_X$ to indicate the norm in $X$. For simplicity, we denote the norm of a matrix by the corresponding scalar space notation of the components or by a symbol of this space. The Sobolev norms used are those defined by Riesz-potential. The classification of the subcritical, critical and supercritical regimes corresponds to $\alpha \in (1,2), \alpha = 1$ and $\alpha \in (0,1)$ respectively. The dissipative (sometimes called also the Laplacian dissipation) and the hyperdissipative regimes correspond to $\alpha = 2$ respectively to $\alpha > 2$. The abbreviations (FSNSE), (SNSE) and (FNSE) are used respectively for fractional stochastic Navier-Stokes equation, the stochastic Navier-Stokes equation and the deterministic fractional stochastic Navier-Stokes equation. The abbreviation i.i.d means independent and identically distributed. $\{a_1,a_2\} \leq_k b$ (respectively $\{a_1,a_2\} \geq_k b$) means $a_k \leq b, a_j < b, j \neq k$ and $a_1 = a_2 < b$ (respectively $a_k \geq b, a_j > b, j \neq k$ and $a_1 = a_2 > b$). The expression $q \leq \infty \theta_0$ means $q \leq \theta_0 < \infty$ and $q < \theta_0 = \infty$. We say that $q^*$ is the conjugate of $q$, if for $1 < q < \infty$, $q^*$ satisfies the equation $\frac{1}{q} + \frac{1}{q^*} = 1$ and $q = 1$ respectively $q = \infty, q^* = \infty$ respectively $q^* = 1$. We define, in distribution sense, the curl of a vector field $v = (v_1,v_2)$ by $\text{curl}v := \partial_1v_2 - \partial_2v_1$. The vorticity matrix of a $dD$-vector field $v$ on $\mathbb{R}^d$ is the null diagonal, antisymmetric matrix defined by $\Omega(v) := ((\Omega(v))_{i,j})_{1 \leq i,j \leq d}$, where $\Omega(v)_{i,j} := \partial_jv_i - \partial_iv_j$. For $d = 2$, the vorticity $\Omega(v)$ is identified to the scalar function $\text{curl}v$ and for $d = 3$ to the transpose of the $3D$-vector function $(\partial_3v_2 - \partial_2v_3, \partial_1v_3 - \partial_3v_1, \partial_2v_1 - \partial_1v_2)$. In Appendix C.1 we have proved that if a Sobolev pointwise multiplication estimate is satisfied for Sobolev spaces on $\mathbb{R}^d$ and if $O \subset \mathbb{R}^d$ is a "good" bounded domain, then this pointwise multiplication estimate is also valid for Sobolev spaces on bounded domains. Therefore, in many cases, we referee directly to the source of the estimate on $\mathbb{R}^d$. We use the Einstein summation convention. Constants vary from line to line and we often delete their dependence on parameters.
2. Formulation of the problem.

To introduce the fractional stochastic Navier-Stokes equation we are interested in, let
us first recall the following classical deterministic Navier-Stokes equation on a bounded
domain \( O \subset \mathbb{R}^d \), \( d \in \mathbb{N}_1 \)

\[
\begin{aligned}
&\partial_t u = \nu \Delta u + (u, \nabla)u - \nabla \pi + f, \quad t > 0, \quad x \in O, \\
&\text{div} u = 0, \quad \text{(incompressible condition)},
\end{aligned}
\]

with no-slip boundary condition

\[
\begin{aligned}
&u/\partial O = 0 \\
&u(0) = u_0.
\end{aligned}
\]

The unknown quantity is the vector \((u, \pi)\). The vector \( u := (u_j(t, x))_{1 \leq j \leq d} \) and the scalar \( \pi := p(t, x) \) describe respectively the motion velocity and the pressure of an incompressible
fluid evaluated at time \( t \) and at point \( x \in O \). The positive constant \( \nu > 0 \) (later we take, for simplicity, \( \nu = 1 \)) is the viscosity of the fluid and \( f \) is an external force, which could be
random and could depend on the velocity \( u \). The notation \((u, \nabla)u\) stands for the product
of \( u \) and the gradient Matrix \((\partial_i u_j)_{1 \leq i, j \leq d}\). The no-slip boundary condition means that the
fluid is in a domain which is bounded by solid impermeable walls. For simplicity, we assume that

"\( O \) is an open bounded and connected set with a \( C^\infty \) boundary \( \partial O \) and
such that \( O \) is on only one side of \( \partial O \)."

It is well documented that to deal mathematically with Navier-Stokes equation, we have to
split up the problem \((2.1, 2.2, 2.3)\) in to \( u - \) respectively \( \pi - \) unkown problems. In this aim, we
introduce the following spaces

\[
L^q(O) := \text{completion in } L^q(O)^d \text{ of } \{u \in (C^\infty_0(O))^d; \text{div} u = 0\},
\]

\[
Y^q(O) := \{\nabla p, \quad p \in H^{1,q}(O)\},
\]

Then we get the Helmholtz decomposition

\[
L^q_d(O) = L^q(O) \oplus Y^q(O),
\]

where the notation \( \oplus \) stands the direct sum, see e.g. [3, 20, 23, 25, 30, 32]. In the case
\( q = 2 \), the sum above reduced to the orthogonal decomposition see e.g. [75, 77]. Explicitly,
\( L^q(O) \) is given by, see e.g. [14, 21, 25, 32, 67] and [76] p. 104,

\[
L^q(O) = \{u \in L^q_d(O); \text{div} u = 0, \text{on } O, \quad u \cdot \vec{n} = 0, \text{on } \partial O\},
\]

\[
\vec{n} \quad \text{is the unit interior normal vector to } \partial O.
\]

We denote by \( \Pi_q \) the continuous Helmholtz projection, see e.g. [1, 26, 32, 44],

\[
\Pi_q : L^q_q(O) \rightarrow L^q(O).
\]

It is easy to prove, using \((2.6)\) and the embedding property of the \( L^q \)-spaces on \( O \) that the
restriction of \( \Pi_{q'} \) on \( L^q_d(O) \), with \( q' \leq q \), coincides with \( \Pi_q \). Consequently, we will omit later
the dependence in $q$. The notations $-A_D^q$ and $A_S^q$ stand for the Laplacian with Dirichlet boundary condition respectively Stokes operator, i.e., see e.g. [21][23][24][30][31][75],

\[ A_D^q = -\Delta \quad \text{with} \quad D(A_D^q) = H^{2,q}_d(O) \cap H^{1,q}_d(O) \]

\[ \{ u \in H^{2,q}_d(O) : (H^{2,q}_d(O))^d; u/\partial O = 0 \}. \]

respectively

\[ A_S^q = -\Pi_q \Delta, \quad D(A_S^q) = D(A_D^q) \cap L^q(O). \]

Let us also recall, see e.g. [26][30], that $L^q(O)$ is a closed subspace of $L^q_d(O)$, the operator $\Pi_q$ defined on $L^q_d(O)$ (for simplicity we keep the same notation) is bounded and its dual is $(\Pi_q)^* = \Pi_{q^*}$, $(1/q + 1/q^* = 1)$ and

\[ (A^q_S)^* = A^q_S, \quad (L^q(O))^* = L^{q^*}(O), \quad (1/q + 1/q^* = 1). \]

Applying Helmholtz projection $\Pi$ on the two sides of Equation (2.1), we get on $L^q(O)$, 

\[ \left\{ \begin{array}{l} \partial_t u = -\nu A^q S u + B(u) + \hat{f}, \quad t > 0, \quad x \in O, \\ u(0) = \Pi u_0, \end{array} \right. \]

where $\hat{f} := \Pi f$ and

\[ B(u) := \Pi((u, \nabla) u). \]

If $\hat{f}$ is random, then Equation (2.12) is called stochastic Navier-Stokes equation.

For $O = \mathbb{T}^d$, $d \in \mathbb{N}_1$, we consider the Navier-Stokes problem [21] and [23] and we use the zero space average Lebesgue and Sobolev spaces. Physically, this condition is meaningful when the volume forces have zero space average. The above calculus remains also valid for $O = \mathbb{T}^d$ with

\[ L^q(\mathbb{T}^d) := \{ u \in L^q_d(\mathbb{T}^d) := (L^q(\mathbb{T}^d))^d; div u = 0 \}, \quad 1 < q < \infty, \]

\[ H^{\beta,q}(\mathbb{T}^d) := H^{\beta,q}_d(\mathbb{T}^d) \cap L^q(\mathbb{T}^d), \quad \beta \in \mathbb{R}_+, \quad 1 < q < \infty, \]

where $(L^q(\mathbb{T}^d))^d$ and $(H^{\beta,q}(\mathbb{T}^d))^d$, $\beta \in \mathbb{R}, 1 < q < \infty$ are the corresponding vectorial spaces of the following null average Lebesgue and periodic Riesz potential spaces, see e.g. [15][26][32][33][34][33],

\[ L^q(\mathbb{T}^d) := \{ f : \mathbb{T}^d \to \mathbb{C}; f(x) := \sum_{k \in \mathbb{Z}^d} c_k e^{ikx}, \text{ s.t. } c_0 = 0 \text{ and } |f|_{L^q} := |\sum_{k \in \mathbb{Z}^d} c_k e^{ik}|_{L^q} < \infty \}, \]

\[ H^{\beta,q}(\mathbb{T}^d) := \{ f \in D'(\mathbb{T}^d), \text{ s.t. } \hat{f}(0) = 0, \text{ and } |f|_{H^{\beta,q}} := |\sum_{k \in \mathbb{Z}^d} |k|^\beta \hat{f}(k)e^{ik}|_{L^q} < \infty \}, \]

\[ D'(\mathbb{T}^d) \text{ is the topological dual of } D(\mathbb{T}^d), \quad (c_k := \hat{f}(k))_{k \in \mathbb{Z}^d} \text{ is the sequence of Fourier coefficients corresponding to } f, \]

\[ c_k = \hat{f}(k) := (2\pi)^{-d} f(e^{ik}). \]

If $f \in L^q(\mathbb{T}^d) \subset D'(\mathbb{T}^d)$, then

\[ c_k = \hat{f}(k) := (2\pi)^{-d} \langle f, e^{ik} \rangle = (2\pi)^{-d} \int_{\mathbb{T}^d} f(x) e^{-ix_k} dx, \]
where the brackets in (2.19) stand for the duality, in particular, it also denotes the scalar product in the Hilbert space $L^2(\mathbb{T}^d)$. An equivalent definition to the spaces $H^{s,q}(\mathbb{T}^d)$ could be given by using the Bessel potential see [13]. Physically, as the velocity is a real function, one can add to the definition of $L^q(\mathbb{T}^d)$ and $H^{3,q}(\mathbb{T}^d)$ the condition $\hat{f}(−k) = \hat{f}(k)$, where the notation $\overline{\hat{f}(k)}$ stands for the complex conjugate, see e.g. [23, 75]. The techniques developed here and in [15] are valid for both the complex and the real cases. Moreover, the divergence free condition could be written as, see e.g. [25].

\begin{equation}
\text{divu} = 0 \iff \langle \hat{u}(k), k \rangle_{\mathbb{Z}^d} = 0, \ \forall k \in \mathbb{Z}^d.
\end{equation}

Recall also that in this case (i.e. $O = \mathbb{T}^d$) and thanks to (2.13) we have, see e.g. [25, 75] for $q = 2$

\begin{equation}
D(A_q^S) = H^{2,q}(\mathbb{T}^d).
\end{equation}

The equation characterizing the pressure $\pi$ is derived by applying the divergence operator on both sides of Equation (2.1), then we get

\begin{equation}
\Delta \pi = \text{div}(u \cdot \nabla u) + \text{div} f.
\end{equation}

For brevity reasons, we keep the study of the pressure $\pi$ beyond the scope of the present work. More discussions about the resolution of Equation (2.22) and the conditions ensuring the uniqueness of the solution, could be found e.g. in [10, 25, 29, 44].

We assume that $d \in \mathbb{N}_1$ and either $O = \mathbb{T}^d$ or $O \subset \mathbb{R}^d$ is a bounded domain. We define the d-dimensional fractional stochastic Navier-Stokes equation (dD-FSNSE) on $O$ by replacing the Stokes operator $A^S$ in Equation (2.12) by $A_\alpha := (A^S)^\frac{\alpha}{2}$, i.e. the d-D-FSNSE is then given by

\begin{equation}
\begin{cases}
\begin{aligned}
du(t) &= (-\nu A_\alpha u(t) + B(u(t))) dt + G(u(t))dW(t), \ 0 < t \leq T, \\
u(0) &= u_0,
\end{aligned}
\end{cases}
\end{equation}

where $B$ is given by (2.13), $W := (W(t), t \in [0,T])$ is a Wiener process, $G$ is a map from $L^q(O)$ to a set of bounded operators to be precise later and we assume that the initial data is of divergence free, i.e. $u_0 := \Pi_0(\cdot) = \Pi u_0(0, \cdot)$. To prove that Equation (2.23) with $A_\alpha$ being defined either by $(A^S)^\frac{\alpha}{2}$, for $O = \mathbb{T}^d$ and $O \subset \mathbb{R}^d$ bounded, or by $\Pi(A^D)^\frac{\alpha}{2}\Pi$ in the case $O \subset \mathbb{R}^d$ bounded, are well defined, we investigate simultaneously, some intrinsic properties of the Stokes operator $A^S$ and of the Laplacian operator with Dirichlet boundary condition $A^D$. Later on, we prove that the two equations are equivalent.

**Theorem 2.1.** [30, Lemma 1.1], [32, Lemma 2.1], [33, Theorem 2] and [31, 74].* The operators $A^S$ and $A^D$ are densely defined, have bounded inverse (0 is in the resolvent) and the corresponding semi groups $(e^{-tA^S})_{t \geq 0}$ respectively $(e^{-tA^D})_{t \geq 0}$ are analytic on $L^q(O)$ respectively $L^q_d(O)$, where $L^q(O)$ is defined by either (2.27) or by (2.21).

Consequently, as $A^S$ and $A^D$ are the infinitesimal generators of analytic semigroups, then we can define the fractional power of $A^\beta, \beta \in \mathbb{R}$, where $A$ stands either for $A^S$ or $A^D$, see e.g. [50, Definition 6.7], [50, Chap. IX] and [31].

**Definition 2.2.** For all $\beta > 0$, we define $A^\beta$, the fractional power of the operator $A$, as the inverse of

\begin{equation}
A^{-\beta} := \frac{1}{\Gamma(\beta)} \int_0^\infty z^{\beta-1} e^{-zA} dz,
\end{equation}

where the Dunford integral in RHS of (2.24) converges in the uniform operator topology.
Moreover, the domain of $A^S$ is given by the following complex interpolation, see e.g. \[1\] [24]  
\[26, 30, 31, 32, 74, 75, 78].

**Theorem 2.3.** For every $0 < \beta < 2$, we have

- For $O \subset \mathbb{R}^d$ bounded
  \[ D((A^D)_{\beta}) = [L_0^\beta(O), D(A^D)]_{\beta} = \hat{H}_d^{\beta,q}(O). \]  
  \[ (2.25) \]
  
- $D((A^S)_{\beta}) = [\mathbb{L}^\beta(O), D(A^S)]_{\beta} = D((A^D)_{\beta}) \cap \mathbb{L}^q(O) \hookrightarrow H_d^{\beta,q}(O) \cap \mathbb{L}^q(O).$
  \[ (2.26) \]

where $\hookrightarrow$ means continuously embedded.

- For $O = \mathbb{T}^d$,
  \[ D((A^S)_{\beta}) = [\mathbb{L}^\beta(\mathbb{T}^d), D(A^S)]_{\beta} = \mathbb{H}^{\beta,q}(\mathbb{T}^d). \]
  \[ (2.27) \]

Recall that for $O \subset \mathbb{R}^d$ bounded, see e.g. \[15],

\[ \hat{H}_d^{\beta,q}(O) = H_d^{\beta,q}(O), \text{ for } \beta \leq \frac{d}{q} \text{ and } \hat{H}_d^{\beta,q}(O) \subsetneq H_d^{\beta,q}(O), \text{ for } \beta > \frac{d}{q}. \]

To identify the notations in formulae \[2.26\] and \[2.27\] and the definition in \[2.15\], we define for $O \subset \mathbb{R}^d$ being bounded

\[ \mathbb{H}^{\beta,q}(O) := D((A^S)_{\beta}), \beta \in \mathbb{R}, 1 < q < \infty. \]

For $O = \mathbb{T}^d$, this notation has already been used for the Riesz potential Sobolev spaces \[2.15\]. It is important to mention that the Dirichlet boundary condition is included in the definition of $\mathbb{H}^{\beta,q}(O)$ in the case $O$ being bounded. Moreover, we have, see e.g. \[30],

\[ (\mathbb{H}^{\beta,q}(O))_{0}^q = \mathbb{H}^{-\beta,q^*}(O) \text{ and } (\mathbb{H}^{\beta,q}(O))_{L_d}^q = H_d^{-\beta,q^*}(O). \]

For further discussion see e.g. \[11, 26, 30, 31, 32\] and the references therein. Using a standard proof like in \[32\] Lemma 2.1 & Lemma 2.2, see also \[56\] Theorem 1.7.7 and \[58\], we infer that (bellow $A = A^S$ but the same result remains true for $A = A^D$ and $\mathbb{L}^q(O)$ replaced by $L_d^q(O)$),

**Lemma 2.4.** The operator $A^\beta := (A^S)_{\beta}$ is the infinitesimal generator of an analytic semi group $(e^{-tA^\beta})_{t \geq 0}$ on $\mathbb{L}^q(O)$. Moreover, we have for $\beta \geq 0$,

\[ |A^\beta e^{-tA^\beta}|_{L(\mathbb{L}^q)} \leq ct^{-\frac{n}{q}}. \]

Furthermore, we recall, see \[13, 15, 21, 23, 24, 27\] ps. 283, 303, \[70\] Chap. II], that $A_2 : D(A_2) \rightarrow \mathbb{L}^2(O)$ is an isomorphism, the inverse $A^{-1}$ is self adjoint and thanks to the compact embedding of $D(A)$ in $\mathbb{L}^2(O)$, we conclude that $A^{-1}$ is compact in $\mathbb{L}^2(O)$. Hence, there exists an orthonormal basis $(e_j)_{j \in \mathbb{N}} \subset D(A)$ consisting of eigenfunctions of $A^{-1}$ and such that the sequence of eigenvalues $(\lambda_j^{-1})_{j \in \mathbb{N}}$ with $\lambda_j > 0$, converges to zero. Consequently, $(e_j)_{j \in \mathbb{N}}$ is also a sequence of eigenfunctions of $A$ corresponding to the eigenvalues $(\lambda_j)_{j \in \mathbb{N}}$. The operator $A$ is positive, self adjoint on $\mathbb{L}^2(O)$ and densely defined. Using the spectral decomposition, we construct positive and negative fractional powers $A^\beta$, $\beta \in \mathbb{R}$.
In particular, as the spectrum of $A$ is reduced to the discrete one, we get an elegant representation for $(A^2, D(A^2))$. In fact, let $\beta \geq 0$, then, see e.g. [23],

$$\mathbb{H}^{\beta,2}(O) := D(A^2) = \{v \in L^2(O), \text{ s.t. } |v|^2_{D(A^2)} := \sum_{j \in \mathbb{N}} \lambda_j^\beta \langle v, e_j \rangle^2 < \infty\},$$

(2.31) \[ A^2 v = \sum_{j \in \mathbb{N}} \lambda_j^\beta \langle v, e_j \rangle e_j, \forall v \in D(A^2), \]

with $(\langle v, e_j \rangle := \hat{v}(j))_j$ is the sequence of Fourier coefficients in the case $O = \mathbb{T}^d$. Furthermore, it is easy to see that

$$\sum_{j \in \mathbb{N}} \lambda_j^\beta e_j_k := \lambda_k^\beta e_k, \quad k \in \mathbb{N}.$$ (2.32)

Now, we introduce the stochastic term. We fix the stochastic basis $(\Omega, \mathcal{F}, P, \mathbb{F}, W)$, where $(\Omega, \mathcal{F}, P)$ is a complete probability space, $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$ is a filtration satisfying the usual conditions, i.e. $(\mathcal{F}_t)_{t \geq 0}$ is an increasing right continuous filtration containing all null sets. The stochastic process $W := (W(t), t \in [0, T])$ is a Wiener process with covariance operator $Q$ being a positive symmetric trace class on $L^2(O)$. By a Wiener process on an abstract Hilbert space $H$, we mean, see e.g. [68, Definition 2.1] and [11, 14],

**Definition 2.5.** A stochastic process $W := (W(t), t \in [0, T])$ is said to be an $H$-valued $\mathcal{F}_t$-adapted Wiener process with covariance operator $Q$, if

- for all $0 \neq h \in H$, the process $(\langle Q^{1/2}h \rangle^{-1}(W(t), h), t \in [0, T])$ is a standard one dimensional Brownian motion,
- for all $h \in H$, the process $(\langle W(t), h \rangle, t \in [0, T])$ is a martingale adapted to $\mathbb{F}$.

Otherwise, the process $W := (W(t), t \in [0, T])$ is a mean zero Gaussian process defined on the filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F})$ with time stationary independent increments and covariance function given by:

$$\mathbb{E}[\langle W(t), f \rangle \langle W(s), g \rangle] = (t \wedge s) \langle Qf, g \rangle, \quad t, s \geq 0, f, g \in H.$$ (2.33)

Formally, we write $W$ as the sum of an infinite series

$$W(t) := \sum_{j \in \Sigma} \beta_j(t) Q^{1/2} e_j,$$ (2.34)

where $\Sigma = \mathbb{Z}_0^d$ if $O = \mathbb{T}^d$, or $\Sigma = \mathbb{N}_0$, if $O \subset \mathbb{R}^d$ bounded, $(\beta_j)_j \in \Sigma$ is an i.i.d. sequence of real Brownian motions and $(e_j)_j \in \Sigma$ is any orthonormal basis, here we consider the basis of the Stokes eigenvalues. For more illustration one can assume that the basis $(e_j)_j \in \Sigma$ diagonalizes simultaneously the Stokes operator $A$ and the Covariance $Q$ with $(q_j)_j \in \Sigma$ is the sequence of the eigenvalues of $Q$,

$$Qe_j = q_j e_j, \quad \text{and} \quad tr(Q) := \sum_{j \in \Sigma} q_j < \infty.$$ (2.35)

The following approximation result, see e.g. [11, 14], will be used later in some proofs,

$$W(t) = \lim_{n \to \infty} W_n(t) \quad \text{in } L^2(\Omega; H), \quad \text{where} \quad W_n(t) := \sum_{|j| \leq n} \beta_j(t) Q^{1/2} e_j.$$ (2.36)

In this work, we consider the stochastic Ito integral in Hilbert spaces. In particular, we define the following Hilbert spaces

$$H_0 := Q^{1/2}(H) \quad \text{endowed with the scalar product}$$

$$\langle \phi, \psi \rangle_{H_0} := \langle Q^{-1/2} \phi, Q^{-1/2} \psi \rangle_H, \quad \forall \phi, \psi \in H_0,$$ (2.37)
and the references therein, that for a separable UMD Banach space of type 2 
\[ H \]
and a Hilbert-Schmidt operator. Moreover, we have \( \iota^* = Q \). It is well known that the stochastic integral, \( \left( \int_0^t \sigma(s) dW(s), t \in [0, T] \right) \), is well defined for all \( \sigma \in \mathcal{P}_T(H) \), see e.g. [11, 54, 22, 24, 59, 60, 68]. In this setup, we consider in Section 4 the space \( \mathcal{P}_T(\mathbb{H}^{1,2}(O)) \) and in sections 7 and 8 the space \( \mathcal{P}_T(L^2(O)) \). As we shall refer to results from [15], where stochastic integrals have been considered in the Banach spaces \( \mathbb{H}^{\delta,q}(O) \) with \( 2 < q < \infty \) and \( \delta \geq 0 \) and we shall prove some results in the general framework of \( L^q \)-spaces, we give here some definitions about this notion. It is well known that the spaces \( \mathbb{H}^{\delta,q}(O), q \geq 2 \) are UMD Banach spaces of type 2. It is well known, see e.g. [80, 81, 82] and the references therein, that for a separable UMD Banach space of type 2 and a Hilbert space \( H \), the stochastic integral with respect to \( W \) is well defined provided that the integrator \( \sigma : [0, T] \times \Omega \rightarrow \mathcal{L}(H, X) \) is an \( H \)-strongly measurable, (i.e. \( \sigma \) is the pointwise limit of a sequence simple functions), \( \mathcal{F}_t \)-adapted process which takes values in the space of \( \gamma \)-radonifying operators \( R_{\gamma}(L^2, \mathbb{H}^{\delta,q}) \), see e.g. [82] Theorem 3.6 & Corollary 3.10],

\[ R_Q(H, X) := \{ S : H \rightarrow X, s.t. SQ^{\frac{1}{2}} \in R_{\gamma}(H, X) \} \]

(2.40)

\[ ||S||^2_{R_Q} := ||SQ^{\frac{1}{2}}||^2_{R_{\gamma}} := E\left[ \sum_{j \in \Sigma} |\gamma_j SQ^{\frac{1}{2}}h_j|^2_X \right], \forall S \in R_Q(H, X), \]

where \( (\gamma_j)_{j \in \Sigma} \) is a sequence of independent standard real-valued Gaussian random variables on a probability space \( (\Omega', F', \mathbb{P}') \) and \( (h_j)_{j \in \Sigma} \) is any orthonormal basis. Moreover, the necessary tools such as the Banach versions of the Itô isometry and the Burkholder-Davis-Gundy inequalities are also disposable, see for more details [82]. Similarly as above we can define

\[ \mathcal{P}_T(H, X) := \{ \sigma \in L^2(\Omega \times [0, T], R_Q(H, X)) \text{, H-strongly measurable } \mathcal{F}_t \text{-adapted processes } \}, \]

(2.41)

\[ ||\sigma||^2_{\mathcal{P}_T(H, X)} := E\left[ \int_0^T ||\sigma(s)||^2_{R_{\gamma}} ds \right] = E\left[ \int_0^T ||\sigma(s)Q^{\frac{1}{2}}||^2_{R_{\gamma}} ds \right]. \]

Recall that for \( X = H \) being a Hilbert space, \( R_Q(H, H) = L_Q(H) \) and \( \mathcal{P}_T(H, H) = \mathcal{P}_T(H) \). To simplify the notations, we use later on the subscript \( \mathcal{P}_T \). Furthermore for the same reason above, we introduce the set of diffusion terms we are dealing with in the general framework of Banach spaces, see for a comparison purpose the conditions in [11, 54, 22, 24, 59, 60, 68].

**Assumption (C)** For fixed \( 2 \leq q < \infty \) and \( \delta \geq 0 \), we assume that the operator

\[ G : L^2(O) \rightarrow R_{\gamma}(L^2, \mathbb{H}^{\delta,q}) \]

satisfies,

- Lipschitz condition: For all \( R > 0 \), there exists a constant \( C_R > 0 \), s.t.

\[ ||G(u) - G(v)||_{R_Q} := ||(G(u) - G(v))Q^{-\frac{1}{2}}||_{R_{\gamma}(L^2, \mathbb{H}^{\delta,q})} \leq C_R ||u - v||_{\mathbb{H}^{\delta,q}}, \forall u, v \in \mathbb{H}^{\delta,q}, ||u||_{\mathbb{H}^{\delta,q}} \leq R, \]

(2.42)
• Linear growth: There exists a constant $c > 0$, s.t.
\begin{equation}
\|G(u)\|_{R^q} := \|G(u)Q^\frac{1}{2}\|_{L^\infty,\mathbb{H}^q} \leq (1 + |u|_{\mathbb{H}^q}), \forall u \in \mathbb{H}^q(O).
\end{equation}

The parameters $q$ and $\delta$ are chosen independently for every result. It is of great interest to mention here that for simplicity reasons and without any loss of generality, we have assumed that the diffusion term $G(u)$ acts on $L^2(O)$. Otherwise, we can use $\Pi G(u)$.

Now, thanks to theorems 2.1-2.4 and to the calculus above, "equations" (2.23) with either $A := (\Delta)^{\frac{q}{2}}$ or $A := \Pi(-\Delta)^{\frac{q}{2}}$ on $L^q(O)$ are well defined. The main question is whether or not the two equations are equivalent. The answer is yes. The proof and further discussions are presented in Appendix A. We end this section by the following assumption on the initial condition

**Assumption (B):** Assume that the initial condition $u_0$ is an $F_0-$random variable satisfying
\begin{equation}
u_0 \in L^p(\Omega, F_0, P; \mathbb{H}^{\delta_0, p_0}(O)), \end{equation}
with $p \geq 2$ and either $2 \leq q_0 \leq \infty$ and $\delta_0 = 0$ or $2 \leq q_0 < \infty$ and $\delta_0 > 0$.

### 3. Definitions of solutions and Results.

In this section, we give the different definitions of solutions we are interested in.

**Definition 3.1.** Let $H$ be a separable Hilbert space and let $V, V_1, V_2$ be separable reflexive Banach spaces such that,
\begin{equation}V_2 \hookrightarrow V \hookrightarrow H \cong H^* \hookrightarrow V^* \hookrightarrow V_1,\end{equation}
with $V^*$ being the topological dual of $V$. Assume that $u_0 \in L^p(\Omega, F_0, P, H)$. A $\mathcal{F}_t-$adapted $H$-valued stochastic process $(u(t), t \in [0, T])$ is called a weak solution of Equation (2.23), if
\begin{equation}u(\cdot, \omega) \in L^\infty(0, T; H) \cap L^2(0, T; V) \cap C([0, T]; V_1) \quad P - a.s.
\end{equation}
and for all $t \in [0, T]$, the following identity holds $P - a.s.$, for all $\varphi \in V_2$,
\begin{align}
nu(t), \varphi) &= \langle u_0, \varphi \rangle_H + \int_0^t V_2^\ast \langle A^\frac{q}{2} u(s), \varphi \rangle_{V_2} ds + \int_0^t V_2^\ast \langle B(u(s)), \varphi \rangle_{V_2} ds \\
&\quad + \int_0^t G(u(s)) dW(s), \varphi \rangle_{V_2}.
\end{align}

**Definition 3.2.** Let $H$ be a separable Hilbert space and let $\tau$ be a stopping time, such that $P(\tau > 0) = 1$ and let $(u(t), t \in [0, T])$ be a time strongly continuous $H$-valued $\mathcal{F}_t-$adapted stochastic process. The couple $(u, \tau)$ is called a local weak solution of Equation (2.23) if
\begin{equation}u(t) = u(t \wedge \tau), \quad \forall t \in [0, T], \quad P - a.s.
\end{equation}
and the stopped process $(u(t \wedge \tau), t \in [0, T])$ is a weak solution of the stopped Equation (3.3) in the sense of Definition 3.1.

The local solution $(u, \tau)$ is said to be maximal if
\begin{equation}\limsup_{t \nearrow \tau} |u(t)|_H = \infty \quad \text{on} \quad \{ \tau < T \}.
\end{equation}

**Definition 3.3.** The multiple $(\Omega^*, \mathcal{F}^*, P^*, \mathbb{F}^*, W^*, u^*)$, where $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*, \mathbb{F}^*, W^*, u^*)$ is a stochastic basis with $W^*$ being a $Q-$Wiener process of trace class and $u^* := (u^*(t), t \in [0, T])$ being an $\mathcal{F}^*_T-$adapted stochastic process, is called a martingale solution of Equation (2.23), if $\theta^*$ is a solution of Equation (2.23) in the sense of Definition 3.1 on the basis $(\Omega^*, \mathcal{F}^*, P^*, \mathbb{F}^*, W^*)$. 

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Remark 1.  

• As a consequence of the condition (3.2), the trajectories of the weak solutions in the definition (3.1) are $H-$ weakly continuous and $P-$ a.s., $u(t) \in H$ for all $t \in [0,T]$, see e.g. [24], [77].

• Remark that for $V_2 = V$ in (3.1), we get the classical Gelfand triple and the well known classical definition of the weak solution (one can forget about $V_1$). We shall see that this classical formulation is also valued for the fractional case provided that $\alpha \geq 2$, or a weak-strong solution, see e.g. [15] for a similar definition and conditions for the $L^q$-spaces with $2 \leq q < \infty$. The main feature for the sub-super and critical regimes is that contrarily to the dissipative and the hyperdissipative regimes ($\alpha \geq 2$), the decrease of the values of $\alpha$ makes e.g. the spaces $H^{-\frac{\alpha}{2},2}(O)$ and their duals $H^{\frac{\alpha}{2},2}(O)$ approaching simultaneously the space $L^2(O)$ and thus approaching each other. Therefore the difficulty to give a sense to the fourth term in (3.3) arises. According to our calculus, the values $\alpha_0(d)$ makes a threshold which characterizes the two phenomena.

• The solution in Definition 3.1 is known in the literature either as a strong or a weak or a weak-strong solution, see e.g. [11], [59], [60], [68]. In fact, this solution is strong in probabilistic sense and weak in the analytic sense. In this work, we use initially the terminology weak. In some places, if there is need to recall, we also use the terminology weak-strong.

• The Definition 3.2 is used in [12] in more general framework, see also similar definitions in [12], [53], [54].

The main results of this work are

**Theorem 3.4.** Let $O = T^2$, $\alpha \in [1,2]$ and $u_0$ satisfying Assumption (B), with $\delta_0 \geq 1$, $q \geq 2$ and $p \geq 4$.

- **(3.4.1) Existence of weak-strong solution.** Assume that $G$ satisfies Assumption (C) with $q = 2$, $\delta \in \{0,1\}$ and $C_R$ being independent of $R$ (global Lipschitz). Then for $\alpha \in [\frac{1}{2}, 2]$, Equation (2.23) admits a weak solution (strong in probability) $(u(t), t \in [0,T])$ in the sense of Definition 7.1 with the corresponding Gelfand triple

$$H^{1+\frac{\alpha}{2},2}(T^2) \hookrightarrow H^{1,2}(T^2) \hookrightarrow (H^{1+\frac{\alpha}{2},2}(T^2))^* = H^{-1-\frac{\alpha}{2},2}(T^2),$$

satisfying

$$u(\cdot, \omega) \in L^\infty(0,T; H^{1/2}(T^2)) \cap L^2(0,T; H^{1+\frac{\alpha}{2},2}(T^2)) \cap C([0,T]; L^2(T^2)) \quad P-a.s.$$ and

$$E \left( \sup_{[0,T]} |u(t)|^p_{H^{1,2}} + \int_0^T |u(t)|^2_{H^{1+\frac{\alpha}{2},2}} dt \right) < \infty.$$  

- **(3.4.2) Uniqueness of the weak solution.** Assume that $G$ satisfies Assumption (C) with $q = 2$ and $\delta = 0$ (local Lipschitz). Then if for $\alpha \in [1,2]$, Equation (2.23) admits a weak solution in the sense of Definition 7.1 satisfying (3.7) and (3.8), pathwise uniqueness holds.

- **(3.4.3) Space regularity of the weak solution.** Assume that $\alpha \in (1,2]$, $G$ satisfies Assumption (C) with $\delta = 1$ and $\frac{2}{\alpha-1} < q < \infty$, $u_0$ satisfies

$$\text{curl} u_0 \in L^p(\Omega, \mathcal{F}_0, P; L^q(\mathbb{T}^d)),$$
and Equation (2.23) admits a weak solution \((u(t), t \in [0, T])\) in the sense of Definition (3.1) and (3.2). Then \((u(t), t \in [0, T])\) satisfies

\[
\begin{align*}
\mathbb{E} \sup_{[0,T]} |u(t)|^q_{L^q_\mathbb{F}} + \mathbb{E} \int_0^T |u(t)|^2_{\mathbb{H}^{1+\frac{\alpha}{2}}_\mathbb{F}} dt < \infty, \\
\end{align*}
\]

for \(\alpha, q\) and \(q_0\) follow one of the cases

- **case 1.** \(\alpha \in (\frac{d}{q}, \frac{3}{2}], 6 \leq q_0 \leq \infty\) and \(\frac{2}{\alpha-1} < q \leq 6\).
- **case 2.** \(6 < q \leq \min\{q_0, \frac{4}{2-\alpha}\}\) and \(2 - \frac{4}{q} < \alpha \leq 2\).
- **case 3.** \(\frac{2}{\alpha-1} < q \leq q_0 \leq 6\) and \(1 + \frac{2}{q} < \alpha \leq 2\).

**Theorem 3.5.** Let \(d \in \mathbb{N}_1\), \(\alpha \in (0, 2]\) and \(T > 0\) be fixed. Assume that \(u_0\) and \(G\) satisfy Assumption (B) respectively Assumption (C) (2.43) with \(\delta = 0, q = 2 \leq q_0 \leq \infty\) and \(p \geq 4\). Then

- **(3.3) 1) [Martingale solution.]** Equation (2.23) has a martingale solution,

  \((\Omega^*, \mathbb{F}^*, \mathbb{P}^*, \mathbb{F}^*, W^*, u^*)\), in the sense of Definition (3.2) satisfying (3.3), with \(V_2 = V = \mathbb{H}^{\frac{d}{2} + \delta}(O), H = \mathbb{L}^2(O), V_1 = \mathbb{H}^{-\delta, 2}(O)\), with \(\delta'> 1 + \frac{d}{2}\) and satisfies

\[
\mathbb{E} \sup_{[0,T]} |u(t)|^q_{\mathbb{H}^q_\mathbb{F}} + \mathbb{E} \int_0^T |u(t)|^2_{\mathbb{H}^{1+\frac{\alpha}{2}}_\mathbb{F}} dt \leq c < \infty.
\]

In particular, for \(1 + \frac{d-1}{2} < \alpha \leq 2\), we can take \(V_2 = \mathbb{H}^{\frac{d}{2} + \delta}(O), V_1 = \mathbb{H}^{-\delta, 2}(O)\).

- **(3.3) 2) [Uniqueness of the martingale solution.]** If \(G\) satisfies (2.42) with \(q = 2, \delta = 0\) and Equation (2.23) has a martingale solution \(u^*\) satisfying the following condition

\[
P^*\big(u^*(\cdot, \omega)\big) \in L^{\frac{4q}{2q - d}(0, T; \mathbb{H}^{\frac{d}{2} + \delta, 2}(O))} = 1,
\]

then pathwise uniqueness holds and consequently \(u^*\) is the unique global strong-weak solution.

**Theorem 3.6.** Let \(d \in \{2, 3\}\), \(O = \mathbb{T}^d\) or \(O \subset \mathbb{R}^d\) bounded, \(\alpha \in (1 + \frac{d-1}{2}, 2]\) and \(u_0\) satisfies Assumption (B), with \(q_0 = 2\), \(\delta_0 \geq 0\) and \(p \geq 4\). Assume that \(G\) satisfies Assumption (C) with \(C_R = c\) independent of \(R\), \(q = 2\) and \(\delta = 0\). Then

- **(3.4) 1) [Global weak solution for the 2D-FSNSE on the torus.]**

  for \(O = \mathbb{T}^2\), \(G\) satisfies, in addition, Assumption (C) for \(\delta\) replaced by 1 and \(u_0\) satisfies (3.5) and Equation (2.23) admits a local weak solution in the sense of Definition (3.3) with \(V_2 = V := \mathbb{H}^{\frac{d}{2} + \delta}(O), V_1 := \mathbb{H}^{-\delta, 2}(O), \delta'> 1 + \frac{d}{2}\) and \(H = \mathbb{L}^2(O)\), then this local solution becomes global in the sense of Definition (3.1) and satisfies (3.10), according to the values of \(p, q, \alpha\) in the cases (3.4) and \(q\) satisfies in addition that \(1 + \frac{2d}{\alpha} \leq q\).

- **(3.4) 2) [Global weak solution for the dD-FSNSE.]** If one of the maximal solutions \((u, \xi)\) enjoys either

\[
\begin{align*}
\mathbb{E} \int_0^T |\nabla u(t)|_{L^q_\mathbb{F}}^{\frac{1}{q}} dt \leq c < \infty, \\
\end{align*}
\]

or

\[
\begin{align*}
\mathbb{E} \int_0^T |u(t)|_{\mathbb{H}^{\frac{d}{2} + \delta, 2}_\mathbb{F}}^{\frac{4q}{2q - d}} dt \leq c < \infty
\end{align*}
\]

then \(T = \xi\) and the process \((u(t), t \in [0, T])\) is the unique global weak solution of Equation (2.23).
Remark 2.

- The conditions (3.13) and (3.14) are of Beale-Kato-Majda type, see e.g. [4, 17].
- The condition (4.2), we have assumed for the uniqueness of the martingale solution is of Serrin’s type on Sobolev spaces, see similar extension of the Serrin’s condition in [19] Theorem 5.2. In particular, our condition in this work is weaker in the time integrability than the condition in [19]. See also similar condition in [39] Theorem 2.8.
- Remark that thanks to [15] Theorem 2.6, Appendix B, the conditions in (3.6.1) we conclude that the condition (3.12) is satisfied in the case $O = \mathbb{T}^2$. Therefore, one can get the existence and the uniqueness of the global solution under these conditions for the 2D-fractional stochastic Navier-Stokes equation on the torus. In Section 5, the results are more stronger. The aim to develop (3.6.1) is to show that the conditions (3.13) and (3.14) sound natural.
- Similarly, [15] Theorem 2.6, Appendix B and the conditions in (3.6.1) ensure that the condition (3.12) is satisfied. Therefore a unique global strong-weak solution exists in sense of Definition [77] with $V_2 = V = \mathbb{H}^{2,2}(O)$, $H = L^2(O)$, $V_1 = \mathbb{H}^{-\delta,2}(O)$, with $\delta > 1 + \frac{d}{4}$ and satisfies (3.10) according to the cases in (3.4.3).
- The condition $q \geq 1 + \frac{2d}{\alpha}$ in (3.6.2) is not optimal.

4. Properties of the nonlinear term

Our aim in this section is to study the nonlinear operator $B$ defined by (2.15). Here $O$ denotes either the torus $\mathbb{T}^d$ or a bounded domain from $\mathbb{R}^d$ with smooth boundary as mentioned above. We define the bilinear operator $B : (\mathcal{D}(O))^2 \to L^2(O)$ and the tri-linear form $b : (\mathcal{D}(O))^3 \to \mathbb{R}$ by,

\begin{equation}
B(u, v) := \Pi((u \cdot \nabla)v), \quad \forall (u, v) \in (\mathcal{D}(O))^2
\end{equation}

respectively,

\begin{equation}
b(u, \theta, v) := \langle B(u, v), \theta \rangle, \quad \forall (u, \theta, v) \in (\mathcal{D}(O))^3,
\end{equation}

where the brackets in RHS of (4.2) stand for the scalar product in $L^2(O)$, see e.g. [3, 20].

\[
\mathcal{D}(O) := \begin{cases} 
(\mathcal{C}^\infty_0(O))^d, & \text{div} u = 0 \text{ and } u \text{ has a compact support when } O \text{ is a bounded domain} \\
(\mathcal{C}^\infty_0(O))^d \cap L^q(O), & \text{when } O \text{ is bounded}, \\
(\mathcal{C}^\infty_0(O))^d \cap L^q(O), & \text{when } O = \mathbb{T}^d.
\end{cases}
\]

The bilinear operator $B$ and the trilinear form $b$ have several extensions based on the $H^{\beta,q}$–norm, with $\beta \geq 1$, see e.g. [75] and [25] p. 97 for Hilbert spaces and [30] for Banach spaces and for a general survey. Unfortunately, due to the weakness of the fractional dissipation in our equation these extensions are useless for our case. Let us before dealing with the extensions we are interested in here, recall the following intrinsic properties

\begin{equation}
b(u, \theta, v) = -b(u, v, \theta), \quad \forall u, v, \theta \in \mathbb{H}^{1,2}(O).
\end{equation}

Hence

\begin{equation}
b(u, v, v) = 0, \quad \forall u, v \in \mathbb{H}^{1,2}(O).
\end{equation}

In particular, for $O = \mathbb{T}^2$, we have also, see e.g. either [76] Lemma 3.1 or [76] Lemma VI.3.1.

\begin{equation}
\langle B(u), u \rangle_{\mathbb{H}^{1,2}} = \langle B(u), u \rangle_{H^1_2} = 0, \quad \forall u \in D(A) := \mathbb{H}^{2,2}(\mathbb{T}^2).
\end{equation}

Now, we cite some basic lemmas.
Lemma 4.1. For all $1 \leq j \leq d$, $\eta \geq 0$ and $1 < q < \infty$, the operators $A^{-\frac{1}{2}}\Pi \partial_j$ extends uniquely to a bounded linear operator from $H^\alpha_{2,q}(O)$ to $H^\beta_{1,q}(O)$.

Proof. For $\eta \geq 0$ and $O = \mathbb{T}^d$, we use Marcinkiewicz’s theory for the pseudodifferential operator $A^{-\frac{1}{2}}\Pi \partial_j$. In fact, the symbol of this latter in Fourier modes is given by the matrix $i|k|^{-1}k_j(\delta_{m,n} - |k|^{-2}k_mk_n)_{mn}$. See also [15] and also [30] for similar calculus for the case $O = \mathbb{R}^d$. The case $O$ bounded and $\eta = 0$ has been proved in [30] Lemma 2.1]. We claim here that the method in [30] and also the proof below are also valid for $O = \mathbb{T}^d$. For $\eta \geq 1$ and $O$ is either a bounded domain of $\mathbb{R}^d$ or $O = \mathbb{T}^d$, thanks to the properties of Helmholtz projection, we prove for all $\beta \geq 0$ and $1 < q < \infty$, that $\Pi : H^\beta_{2,q}(O) \to H^\beta_{1,q}(O)$ is well defined and bounded. Using this statement and arguing as in the proof of [30] Lemma 2.1], we get the result for $\eta \geq 1$. The result for $0 < \eta < 1$ is a consequence of the interpolation for $\eta = 0$ and $\eta = 1$. \hfill \Box

The following Lemma has been proved in [30] for bounded domain. Our claim is that the same proof is also valid for $O = \mathbb{T}^d$, see similar calculus in [15] and for $q = 2$, see e.g. [75] p 13.

Lemma 4.2. [30] Lemma 2.2] Let $0 \leq \delta < \frac{1}{2} + \frac{d}{2}(1 - \frac{1}{q})$. Then
\begin{equation}
|A^{-\delta}\Pi(u,\nabla)v|_{L^q} \leq M|A^\nu u|_{L^q}|A^\rho v|_{L^q},
\end{equation}
with some constant $M := M_{\delta,\nu,\rho,q}$, provided that $\nu, \rho > 0$, $\delta + \nu > \frac{1}{2}$, $\delta + \nu + \rho \geq \frac{d}{2} + \frac{1}{2}$.

As a corollary of Lemma 4.2, we cite the following results, which will be generalized later on.

Corollary 4.3. Let either $O = \mathbb{T}^d$ or $O \subset \mathbb{R}^d$ be a bounded domain. Then

- For $\alpha \in (0, 2)$, there exists a constant $c := c(\alpha, d) > 0$ such that for all $(u, v) \in \mathbb{H}^{\frac{\alpha}{2} - \frac{d}{4}, 2}(O) \times \mathbb{H}^{1 + \frac{\alpha}{2}, 2}(O)$
\begin{equation}
|B(u, v)|_{L^2} \leq c|u|_{\mathbb{H}^{\frac{\alpha}{2} - \frac{d}{4}, 2}}|v|_{\mathbb{H}^{1 + \frac{\alpha}{2}, 2}}.
\end{equation}

- For $\alpha \in (0, 2]$ there exists a constant $c := c(\alpha, d) > 0$ such that for all $(u, v) \in (\mathbb{H}^{\frac{\alpha}{2} - \frac{d}{4}, 2}(O))^2$,
\begin{equation}
|B(u, v)|_{H^{-\frac{\alpha}{2}, 2}} \leq c|u|_{\mathbb{H}^{2 + d - \alpha, 2}}|v|_{\mathbb{H}^{2 + d - \alpha, 2}}.
\end{equation}

The following results generalize [30] Lemma 2.2] for $O \subset \mathbb{R}^d$ and [30] Lemma 1.4] for $O = \mathbb{R}^d$.

Proposition 4.4. Let $\epsilon > 0$, then the bilinear form $B$ extends uniquely $B : (L^2(O))^2 \to H^{-\epsilon - \frac{d}{4}, 2}(O)$ and there exists a constant $c := c_{\alpha, \epsilon, d}$ such that for all $(u, v) \in (L^2(O))^2$,
\begin{equation}
|B(u, v)|_{H^{-\epsilon - \frac{d}{4}, 2}} \leq c|u|_{L^2} |v|_{L^2}.
\end{equation}

We omit the proof here, as a more general one will be given in the proof of Lemma 6.1.

Proposition 4.5. Let $\eta \geq 0$ and
\begin{equation}
\alpha(d, \eta) := \begin{cases}
\max\left\{\frac{d+2-2\eta}{3}, 2\eta + 2 - d\right\}, & \text{if } \eta \in [0, \frac{d}{2}) \cap \left(\frac{d}{2} - 2, \frac{d}{2}\right), \\
1, & \text{if } \eta \geq \frac{d}{2}.
\end{cases}
\end{equation}
Then for either $\alpha \in [\alpha(d, \eta), 2]$ with $(\eta \in [0, d \frac{1}{2} - 2, (d \frac{1}{2} - 1)) \cup [d \frac{1}{2}, \infty))$ or $\alpha \in (\alpha(d, \eta), 2)$ with $\eta \in [d \frac{1}{2}, \frac{d}{4}]$, the bilinear operator $B$ extends uniquely

$$B : (H^{\eta + \frac{1}{4}, 2}(O))^2 \to H^{\eta - \frac{1}{2}, 2}(O)$$

and there exists a constant $c := c_{\alpha, \eta, d}$ such that for all $(u, v) \in (H^{\eta - \frac{1}{2}, 2}(O))^2$,

$$|B(u, v)|_{\eta - \frac{1}{2}, 2} \leq c|u|_{H^{\eta + \frac{1}{2}, 2}}|v|_{H^{\eta + \frac{1}{2}, 2}}.$$

**Proof.** Thanks to Lemma 4.1 there exists a constant $c > 0$ such that

$$|B(u, v)|_{\eta - \frac{1}{2}, 2} \leq c|u_jv|_{H^{\eta + 1 - \frac{1}{2}, 2}}.$$

First, let us suppose that $\eta \geq \frac{d}{2}$. Then $H^{\eta + 1 - \frac{2}{2}, 2}(O)$ is an algebra, therefore

$$|B(u, v)|_{\eta - \frac{1}{2}, 2} \leq c|u|_{\eta}d, \eta, \eta} \in \mathbb{R}^d$$. We introduce the following Gelfand triple $(\mathbb{H}^{\eta + \frac{1}{2}, 2}(O) \hookrightarrow H^{\eta - \frac{1}{2}, 2}(O))$. This last is guaranteed thanks to the condition $\eta > \frac{d}{2} - 2$ which guarantees that $\frac{d+2-2\eta}{3} < 2$ and to the equivalence $2 + 2\eta - d < \frac{d+2-2\eta}{3}$.

The investigation of the Gelfand triple corresponding to the fractional Navier-Stokes equation for which $B$ can be extended to a bounded operator, is one of the delicate questions of the theory of fractional nonlinear equations. To characterize this feature, let us first recall the following classical Gelfand triple

$$V_c = D((A^\frac{1}{2})) = H^{1, 2}(O) \hookrightarrow H := L^2(O) \hookrightarrow H^* \hookrightarrow V^*_c,$$

where $V^*_c$ is the dual of $V_c$. The operators $A : V_c \to V^*_c$ and $B : D(B) := V_c \times H \to V^*_c$ are bounded. For the fractional case, we have for $\alpha > 0$, $A_\alpha : V := D(A^\alpha) = H^{\frac{\alpha}{2}, 2}(O) \to V^* = (H^{\frac{\alpha}{2}, 2}(O))^*$ is bounded. However, for $\alpha < 2$, the space $V \times H$ is larger than $D(B)$. In particular, $H^1(O) \subset V$. Consequently, we need to extend uniquely the operator $B$ to a bounded operator from $V$ to $V^*$. This extension is not possible for all values of $\alpha \in (0, 2)$.

**Theorem 4.6.** Let $\alpha \in [\alpha(d, \eta), 2]$, with $\eta \in ([0, \frac{d-1}{2}) \cup ([\frac{d}{2}, \infty))$ or $\alpha \in (\alpha(d, \eta), 2]$ with $\eta \in [\frac{d-1}{2}, \frac{d}{2}]$ where $\alpha(d, \eta)$ is defined by (4.10). We introduce the following Gelfand triple

$$V_\eta := H^{\eta + \frac{1}{2}, 2}(O) \hookrightarrow H^{\eta, 2}(O) \hookrightarrow H^{\eta - \frac{1}{2}, 2}(O).$$

Then

$$B : V_\eta \times V_\eta := (D(A^{\eta + \frac{1}{2}}))^2 \hookrightarrow (H^{\eta + \frac{1}{2}, 2}(O))^2 \to V^*_\eta = H^{\eta - \frac{1}{2}, 2}(O).$$

is bounded. Moreover, there exists a constant $c := c_{\eta, \alpha, \eta} > 0$, such that

$$|B(u, v)|_{\eta - \frac{1}{2}, 2} \leq c|u|_{H^{\eta + \frac{1}{2}, 2}}|v|_{H^{\eta + \frac{1}{2}, 2}}.$$
In particular, we have the following useful cases,

- \( \eta = 0, \ d \in \{2, 3, 4\} \) and \( \alpha \in \left[ \frac{d+2}{3}, 2 \right] \).
- \( \eta = 1 \) and either \( d = 2 \) and \( \alpha \in [1, 2] \) or \( d = 3 \) and \( \alpha \in [1, 2] \) or \( d \in \{4, 5, 6\} \) and \( \alpha \in \left[ \frac{d}{3}, 2 \right] \).

Proof. The proof is a straightforward application of Proposition 4.5 and the classical case for \( \alpha = 2 \).

Further estimations for the bilinear operator \( B \) and the trilinear form \( b \) are summarized in the following lemma.

**Lemma 4.7.**

- (i) Assume \( 0 \leq \eta < \frac{d}{4} \), and \( \alpha \in (2\eta + 2 - d, 2] \). Then for all \( (u, v) \in (H^{\frac{d+2+2\eta-\alpha}{4}}) \), we have
  \[
  |B(u, v)|_{H^{\eta+1-\frac{\alpha}{2}}} \leq c|u|_{H^{\frac{d+2+2\eta-\alpha}{4}}} |v|_{H^{\frac{d+2+2\eta-\alpha}{4}}},
  \]

- (ii) For \( \eta \geq \frac{d}{2} \) and \( u, v \in H^{\eta+1-\frac{\alpha}{2}}(O) \), we have
  \[
  |B(u, v)|_{H^{\eta+1-\frac{\alpha}{2}}} \leq c|u|_{H^{\eta+1-\frac{\alpha}{2}}} |v|_{H^{\eta+1-\frac{\alpha}{2}}},
  \]

- (iii) Assume \( d \in \{2, 3, 4\} \) and \( \frac{d}{2} \leq \alpha \leq 2 \). For all \( (u, w) \in H^{1,2}(O) \times H^{\frac{d-\alpha}{2}}(O) \),
  \[
  |\langle B(w), u \rangle|_{L^2} \leq c|u|_{H^{1,2}} |w|_{H^{\frac{d-\alpha}{2}}} |w|_{L^2}^{\frac{2\alpha-d}{\alpha}}.
  \]

- (iv) Assume \( d \in \{2, \cdots, 5\} \) and \( \frac{d}{2} \leq \alpha < d \). For all \( (u, w) \in H^{1+\frac{d}{2}}(O) \times H^{\frac{d-\alpha}{2}}(O) \),
  \[
  |\langle B(w), u \rangle|_{L^2} \leq c|u|_{H^{1+\frac{d}{2}}} |w|_{H^{\frac{d-\alpha}{2}}} |w|_{L^2}^{\frac{2\alpha-d}{\alpha}}.
  \]

- (v) For all \( (u, v) \in H^{\frac{d}{2}}(O) \) with \( q > 2 \),
  \[
  |B(u, v)|_{H^{1-q}} \leq c|u|_{H^{\frac{d}{2}}} |v|_{H^{\frac{d}{2}}}.
  \]

- (vi) The following estimates is a classical result. For all \( u \in H^{1,2}(O) \),
  \[
  |B(u)|_{H^{-1,2}} \leq c|u|_{H^{1,2}} |u|_{L^2}.
  \]

Proof. (i)-(ii) Estimates (4.19) and (4.20) are copies of the estimates (4.14) respectively (4.15) proved above without restriction conditions. We only emphasize them here.

- (iii) Let \( d \in \{2, 3, 4\} \), \( \frac{d}{2} \leq \alpha \leq 2 \) and \( (u, w) \in (D(O))^2 \). Recall that \( (u, w) \in (D(O))^2 \) is dense in \( H^{1,2}(O) \) \( H^{\frac{d-\alpha}{2}}(O) \). Using the group property \( (A^\beta)_{\beta \in \mathbb{R}} \), Hölder inequality, Lemma 4.1 again Hölder inequality and Gagliardo-Nirenberg inequality in the case \( \alpha > \frac{d}{2} \) and Sobolev embedding in the case \( \alpha = \frac{d}{2} \), for bounded domain and \( \mathbb{R}^D \) Theorem 3.5.4. & Theorem 3.5.5. for the torus, we infer that
  \[
  |\langle B(w), u \rangle| \leq |u|_{H^{1,2}} |B(w)|_{H^{-1,2}} \leq c|u|_{H^{1,2}} |w_j w|_{L^2} \leq c|u|_{H^{1,2}} |w|_{L^4}^{\frac{d}{4}}
  \]

- (iv) Let \( d \in \{2, \cdots, 5\} \), \( \frac{d}{2} \leq \alpha < d \) and \( (u, w) \in (D(O))^2 \). Using the group property of \( (A^\beta)_{\beta \in \mathbb{R}} \), Hölder inequality, Lemma 4.1 [61] Theorem 4.6.1 for the bounded
domain and \([63, \text{Theorem iv.2. ii]}\) and \([62, \text{Remark 4p.164}]\) for \(O = \mathbb{T}^d\) and by interpolation, we infer that

\[
|B(w), u|_{L^2} \leq |u|_{H^{1.5}2} |B(w)|_{H^{−1.5}2} \leq c |u|_{H^{1.5}2} |w|_{H^{−1.5}2} \leq c |u|_{H^{1.5}2} |w|^2_{H^{−1.5}2}.
\]

(4.26)

- (v) We use Lemma 4.1, \([63, \text{Theorem IV.2.2 (ii)}], [62, \text{Theorem 3.5.4.ps.168-169}]\) and the monotonicity property in \([62, \text{Remark 4p.164}]\) for \(O = \mathbb{T}^d\) and \([61, \text{Theorem 4.6.1, p. 190 and Proposition Tr 6, 2.35, p 14}]\) and Theorem C.1 we infer that

\[
|B(u, v)|_{H^{−1.2}} \leq c |u_j v_j|_{L^q} \leq C |u|_{H^{1.5}2} |v|_{H^{−1.5}2}.
\]

The proof of \((4.23)\), follows from the first two estimates in \((4.27)\), with \(\eta = \frac{4}{9}\).

- (vi) We use Lemma 4.2, Hölder inequality and than Gagliardo-Nirenberg inequality, we get

\[
|B(u)|_{H^{1.2}} \leq c |u_j|_{L^2} \leq c |u|^2_{L^4} \leq |u|_{H^{1.2}} |u|_{L^2}.
\]

5. 2D-FSNSEs on the Torus with smooth data.

In this section, we assume \(O = \mathbb{T}^2\) and prove Theorem 3.4. As mentioned above, we have \((A^S)^{\frac{p}{2}} = (−Δ)^{\frac{p}{2}}, 0 < \alpha \leq 2\). Furthermore, \((A^S)^{\frac{p}{2}}\) can also be defined by \((2.31)\) and \((2.32)\) with the explicit orthonormal basis of eigenvectors \((e_k(·) := \frac{1}{\sqrt{|Ω|}} e^{ik})_{k \in \mathbb{Z}_d}\) and \((∥v, e_k∥ := \hat{v}_k)_{k \in \mathbb{Z}_d}\) being the sequence of Fourier coefficients, see e.g. \([73, \text{p316}]\).

For \(\alpha ∈ [1, 2]\), we fix the densely, continuous embedding Gelfand triple \((3.10)\) and we use the following Faedo-Galerkin approximation. Let us fix \(n \geq 1\) and introduce the projection \(P_n, n \geq 1\) on the finite space \(H_n \subset \mathbb{L}^2(\mathbb{T}^2)\) generated by \(\{e_k, k \in \mathbb{Z}_d^d, s.t. |k| \leq n\}\). The Faedo-Galerkin approximation scheme is defined for the process \((u_n(t), t ∈ [0, T]) \in H_n\) by

\[
\begin{cases}
du_n(t) = (−\alpha_n u_n(t)) + P_n B(u_n(t)) dt + P_n G(u_n(t)) dW_n(t), & 0 < t \leq T, \\
u(0) = P_n u_0 = u_0, &
\end{cases}
\]

(5.1)

where \(W_n(t) := P_n W(t) = \sum_{|j| \leq n} Q_{e_j}^2 \beta_j(t)\) (for example \((\sum_{|j| \leq n} \frac{1}{\sqrt{|j|}} e_j \beta_j(t))\)). Let us mention here that using a similar proof as in \([11, 22]\), we can prove that \(W_n(t)\) converges to \(W(t)\) in the space \(L^2(Ω, \mathbb{H}^{1.2}(\mathbb{T}^2))\), provided \((A^S)^{\frac{p}{2}}\) is a Hilbert-Schmidt in \(\mathbb{L}^2(\mathbb{T}^2)\). Since the finite dimensional space stochastic differential equation \((5.1)\) has locally Lipschitz and linear growth coefficients, then Equation \((5.1)\) admits a unique strong solution \((u_n(t), t ∈ [0, T]) \in L^2(Ω, \mathbb{C}([0, T]; H_n))\), see e.g. \([11, 40, 69]\) and the reference therien. Furthermore, we have the following result.

Lemma 5.1. Let \(α \in (0, 2)\) and \(u_0 \in L^p(Ω, \mathbb{H}^{1.2}(\mathbb{T}^d))\) with \(p \geq 4\). Then the solutions \((u_n(t), t ∈ [0, T])\) of equations \((5.1), n ∈ \mathbb{N}_0\), satisfy the following estimates

\[
\sup_n \mathbb{E} \left( \sup_{t \in [0, T]} |u_n(t)|_{\mathbb{H}^{1.2}}^p \right) + \int_0^T |u_n(t)|_{\mathbb{H}^{1.2}}^{p−2} \left( |u_n(t)|_{\mathbb{H}^{1.5}2}^2 + |u_n(t)|_{\mathbb{H}^{1.5}2}^2 \right) dt \\
+ \int_0^T |u_n(t)|_{\mathbb{H}^{1.2}}^4 dt + \int_0^T |u_n(t)|_{\mathbb{H}^{1.5}2}^{α} dt < \infty.
\]

(5.2)
where \( \beta \leq 1 + \frac{\alpha}{2} - \frac{d}{2} + \frac{d}{q_1}, \) \( 2 \leq q_1 < \infty \) and \( \frac{\alpha}{p} < \eta \leq \frac{\alpha}{2} \).

\[
\sup_n \left( \mathbb{E} \int_0^T \left( |P_n B(u_n(t))|^2_{\mathcal{H}_{1-\frac{\eta}{2}}} + |A^\frac{\alpha}{2} u_n(t)|^2_{\mathcal{H}_{1-\frac{\eta}{2}}} \right) dt \right) < \infty.
\]

**Proof.** First, we prove the following estimate

\[
\sup_n \mathbb{E} \left( \sup_{[0,T]} |u_n(t)|^2_{\mathcal{H}_{1+\frac{\eta}{2}}} + \int_0^T |u_n(t)|^2_{\mathcal{H}_{1+\frac{\eta}{2}}} dt \right) \leq C < \infty.
\]

By application of Ito’s formula, we have

\[
|u_n(t)|^2_{\mathcal{H}_{1+\frac{\eta}{2}}} = |u_n(0)|^2_{\mathcal{H}_{1+\frac{\eta}{2}}} - 2 \int_0^t \langle u_n(s), A^\frac{\alpha}{2} u_n(s) - P_n B(u_n(s)) \rangle_{\mathcal{H}_{1+\frac{\eta}{2}}} ds + 2 \int_0^t \langle u_n(s), P_n G(u_n(s)) dW_n(s) \rangle_{\mathcal{H}_{1+\frac{\eta}{2}}} + \int_0^t \sum_{|j| \leq n} |P_n G(u_n(s)) Q^\frac{\alpha}{2} e_j|^2_{\mathcal{H}_{1+\frac{\eta}{2}}} ds.
\]

Using the semigroup property of \((A^\beta)_{\beta \geq 0}\) and the definition of the Sobolev spaces in Section 2, we get

\[
\langle u_n(s), A_n u_n(s) \rangle_{\mathcal{H}_{1+\frac{\eta}{2}}} = |u_n(s)|^2_{\mathcal{H}_{1+\frac{\eta}{2}}}.
\]

The term \( \langle u_n(s), P_n B(u_n(s)) \rangle_{\mathcal{H}_{1+\frac{\eta}{2}}} \) in the RHS of (5.5) vanishes thanks to (4.5). To estimate the stochastic term in (5.5), we use the stochastic isometry, Minkowski and Hölder inequalities, the contraction property of \(P_n\) and Assumption (C) (2.43), with \(q = 2\) and \(\delta = 1\), we get

\[
\mathbb{E} \sup_{[0,T]} | \int_0^t \langle u_n(s), P_n G(u_n(s)) dW_n(s) \rangle_{\mathcal{H}_{1+\frac{\eta}{2}}} |
\leq c \mathbb{E} \left( \int_0^T \sum_{k \leq n} \left( \int_{\mathbb{R}^d} |A^\frac{\alpha}{2} u_n(s)||A^\frac{\alpha}{2} P_n G(u_n(s)) Q^\frac{\alpha}{2} e_k| dx \right)^2 ds \right)^{\frac{1}{2}}
\leq c \mathbb{E} \left( \int_0^T |u_n(s)|^2_{\mathcal{H}_{1+\frac{\eta}{2}}} |G(u_n(s)) Q^\frac{\alpha}{2}|^2_{H_{\mathcal{S}}(H_{1+\frac{\eta}{2}})} ds \right)^{\frac{1}{2}}
\leq c \mathbb{E} \left( \int_0^T (|u_n(s)|^2_{\mathcal{H}_{1+\frac{\eta}{2}}} + |u_n(s)|^4_{\mathcal{H}_{1+\frac{\eta}{2}}}) ds \right)^{\frac{1}{2}}.
\]

Than, we use Young and Hölder inequalities (\(\epsilon < 1\), we infer that

\[
\mathbb{E} \sup_{[0,T]} | \int_0^t \langle u_n(s), P_n G(u_n(s)) dW_n(s) \rangle_{\mathcal{H}_{1+\frac{\eta}{2}}} |
\leq c \mathbb{E} \left( \sup_{[0,T]} |u_n(s)|^2_{\mathcal{H}_{1+\frac{\eta}{2}}} \left[ \int_0^T (1 + |u_n(s)|^2_{\mathcal{H}_{1+\frac{\eta}{2}}}) ds \right]^{\frac{1}{2}} \right)
\leq c \mathbb{E} \sup_{[0,T]} |u_n(s)|^2_{\mathcal{H}_{1+\frac{\eta}{2}}} + c \mathbb{E} \int_0^T (1 + |u_n(s)|^2_{\mathcal{H}_{1+\frac{\eta}{2}}}) ds
\leq c \mathbb{E} \sup_{[0,T]} |u_n(s)|^2_{\mathcal{H}_{1+\frac{\eta}{2}}} + c \mathbb{E} \int_0^T \sup_{\tau \in [0,s]} |u_n(\tau)|^2_{\mathcal{H}_{1+\frac{\eta}{2}}} ds + C.
\]
For the last term in the RHS of (5.13), we use Assumption (C) (2.43, with \( q = 2 \) and \( \delta = 1 \)), we infer the existence of a positive constant \( c > 0 \) such that,

\[
\left| \int_0^t \int_{|x| \leq \tau} |A_{1/2}^* P_n G(u_n(s)) Q_{1/2} e_j|^2 \, dx \, ds \right| \leq \int_0^t \| G(u_n(s)) Q_{1/2}^2 \|_{HS(H^{1.2})} \, ds
\]

(5.9)

\[
\leq c \int_0^t (1 + \sup_{\tau \in [0,s]} |u_n(\tau)|^2_{H^{1.2}}) \, ds.
\]

Now, we replace (5.3), (5.8) and (5.9) in (5.10), we get

\[
\mathbb{E} \left[ \sup_{[0,T]} |u_n(t)|^2_{H^{1.2}} + \int_0^T |u_n(s)|^2_{H^{1.2}} + \int_0^T \mathbb{E} \sup_{[0,s]} |u_n(\tau)|^2_{H^{1.2}} \, ds \right] \leq C \left( 1 + \mathbb{E} |u_n(0)|^2_{H^{1.2}} + C \int_0^T \mathbb{E} \sup_{[0,s]} |u_n(\tau)|^2_{H^{1.2}} \, ds \right).
\]

(5.10)

By application of Gronwall’s lemma for the function \( \mathbb{E} \sup_{[0,T]} |u_n(t)|^2_{H^{1.2}} \), we get the estimation of the first term in the LHS of (5.11), (recall that \( \mathbb{E} |u_n(0)|^2_{H^{1.2}} \leq \mathbb{E} |u_0|^2_{H^{1.2}} \)). The second term in (5.11) is then deduced from (5.10) and the uniform boundedness of \( \mathbb{E} \sup_{[0,T]} |u_n(t)|^2_{H^{1.2}} \).

Now, we prove

\[
\mathbb{E} \sup_{[0,T]} |u_n(t)|^p_{H^{1.2}} + \mathbb{E} \int_0^T |u_n(t)|^{p-2}_{H^{1.2}} |u_n(s)|^{2}_{H^{1.2}} \, ds \leq c(1 + \mathbb{E} |u_n(0)|^p_{H^{1.2}}).
\]

(5.11)

By application of Ito’s formula to the process \( |u_n(t)|^2_{H^{1.2}}, t \in [0,T] \) given by (5.5), we get, see for similar calculus e.g [11, 24, 68],

\[
|u_n(t)|^p_{H^{1.2}} + \mathbb{E} \int_0^t |u_n(s)|^{p-2}_{H^{1.2}} |u_n(s)|^2_{H^{1.2}} \, ds
\]

\[
\leq |u_0|^p_{H^{1.2}} + \frac{p}{2} \mathbb{E} \int_0^t |u_n(s)|^{p-2}_{H^{1.2}} \| P_n G(u_n(s)) Q_{1/2} \|_{HS(H^{1.2})}^2 \, ds
\]

\[
+ \frac{p}{2} \frac{p}{2} - 1 \int_0^t |u_n(s)|^{p-4}_{H^{1.2}} Q_{1/2} G^*(u_n(s)) u_n(s)|^2_{H^{1.2}} \, ds
\]

(5.12)

We argue as above and use Assumption (C) (2.43, with \( q = 2 \) and \( \delta = 1 \)). In particular, for the third term in the RHS of (5.12), we follow a similar calculus as in (5.7) and (5.8), we infer that

\[
\mathbb{E} \sup_{[0,T]} \left| \int_0^t |u_n(s)|^{p-2}_{H^{1.2}} (u_n(s), P_n G(u_n(s)) u_n(s), dW(s))_{H^{1.2}} \right|
\]

\[
\leq C \mathbb{E} \left( \sup_{[0,T]} |u_n(s)|^p_{H^{1.2}} \left[ \int_0^T |u_n(s)|^{p-4}_{H^{1.2}} (1 + |u_n(s)|^2_{H^{1.2}}) \, ds \right] \right)^{\frac{1}{2}}
\]

\[
\leq C_1 \mathbb{E} \sup_{[0,T]} |u_n(s)|^p_{H^{1.2}} + C_2 \mathbb{E} \int_0^T |u_n(s)|^2_{H^{1.2}} (1 + |u_n(s)|^2_{H^{1.2}}) \, ds
\]

(5.13)

\[
\leq C \mathbb{E} \sup_{[0,T]} |u_n(s)|^p_{H^{1.2}} + C \int_0^T \mathbb{E} \sup_{[0,s]} |u_n(\tau)|^p_{H^{1.2}} \, ds.
\]
Hence,
\[ \mathbb{E} \sup_{[0,T]} |u_n(t)|^p_{H^{1,2}} \leq c \mathbb{E} \int_0^T |u_n(s)|^{p-2} |u_n(s)|^{2} \, ds \]
(5.14)

By application of Gronwall’s lemma, we conclude that the first term in the LHS of (5.11) is uniformly bounded in \( n \). The total estimate (5.11) follows easily from the above statement and from (5.14). The uniformity boundedness of the third, fourth and last terms in LHS of (5.2) is a consequence of the application of Sobolev embedding see e.g. [2, 61, 62, 64] and Appendix C respectively Hölder inequality respective Sobolev interpolation and Estimate (5.4).

Now we prove Estimate (5.3). Thanks to the contraction of \( P_n \) on \( H^{1-\frac{\alpha}{2},2}(\mathbb{T}^2) \), (5.13) with \( q = 1 \) and the interpolation, we infer that for all sequence \( (u_n)_n \) satisfying (5.2), we have
\[ \int_0^T |P_n B(u_n(t))|^{2}_{H^{1-\frac{\alpha}{2},2}} \, dt \leq C \int_0^T |u_n(t)|^{2}_{H^{2-\frac{\alpha}{2},2}} \, dt \]
(5.15)

provided \( \alpha \in [\frac{4}{3}, 2] \). Moreover, it is easy to see that
\[ \int_0^T |A^\frac{\alpha}{2} u_n(t)|^{2}_{H^{1-\frac{\alpha}{2},2}} \, dt \leq c \int_0^T |u_n(t)|^{2}_{H^{1+\frac{\alpha}{2},2}} \, dt \leq c < \infty. \]
(5.16)

\[ \square \]

**Proof of the existence.** We shall follow for this proof a quiet standard scheme, see e.g. [1], [2], [5], [8], [12], [15], but we shall use completely different estimates. These latter are of fractional type and have been developed in Section 4. We shall focus more on the key estimates and on the main features of our equation. In deed, thanks to Lemma 5.1 and Assumption (C), with \( q = 2 \) and \( \delta = 1 \), we conclude the existence of a subsequence (we keep the same notation \( (u_n)_n \)) that satisfies (5.17)
\[ u \in L^2(\Omega \times [0, T]; H^{1+\frac{\alpha}{2},2}(\mathbb{T}^2)) \cap L^p(\Omega, L^\infty([0, T]; H^{1,2}(\mathbb{T}^2))), \]
(5.17)
\[ F_1 \in L^2(\Omega \times [0, T]; L^2(\mathbb{T}^2)) \quad \text{and} \quad G_1 \in L^2(\Omega \times [0, T]; L_Q(H^{1,2}(\mathbb{T}^2))), \]
(5.18)

- (1) \( u_n \to u \) weakly in \( L^2(\Omega \times [0, T]; H^{1+\frac{\alpha}{2},2}(\mathbb{T}^2)) \).
- (2) \( u_n \to u \) weakly-star in \( L^p(\Omega, L^\infty([0, T]; H^{1,2}(\mathbb{T}^2))) \).
- (3) \( P_n(F(u_n) := (A^\frac{\alpha}{2} + B)(u_n)) \to F_1 \) weakly in \( L^2(\Omega \times [0, T]; H^{1-\frac{\alpha}{2},2}(\mathbb{T}^2)) \).
- (4) \( u_n \to u \) weakly in \( L^{\frac{8}{6-\gamma}}(\Omega \times [0, T]; H^{1+\eta,2}(\mathbb{T}^2)) \), for all \( \frac{\alpha}{2} < \eta \leq \frac{\alpha}{2} \).
- (5) \( P_n G(u_n) \to G_1 \) weakly in \( L^2(\Omega \times [0, T]; L_Q(H^{1,2}(\mathbb{T}^2))) \).

The statements (1) - (3) are straightforward consequence of Lemma 5.1. Statement (4) is a consequence of the combination of Lemma 5.1 and the Sobolev interpolation. Statement (5) holds thanks to the fact that \( P_n \) contracts the \( H^{1,2} \)-norm, Assumption (C) (with \( q = 2 \) and \( \delta = 1 \)) and the uniform boundedness of \( u_n \) in \( L^2(\Omega \times [0, T]; H^{1,2}(\mathbb{T}^2)) \).

Now, we construct a process \( (\tilde{u}(t), t \in [0, T]) \) as
\[ \tilde{u}(t) = u_0 + \int_0^t F_1(s) \, ds + \int_0^t G_1(s) \, dW(s) \]
(5.19)
and prove that $u = \tilde{u}, dt \times dP - a.e.$. Indeed, using Statement (1), Equation (5.1) and Fubini theorem, we infer that for all $\varphi \in L^\infty(\Omega \times [0, T], \mathbb{R})$ and $v \in \cup_n H_n$,

$$
\mathbb{E} \int_0^T \langle u(t), \varphi(t)v \rangle_{L^2} dt = \lim_{n \to +\infty} \mathbb{E} \int_0^T \langle u_n(t), \varphi(t)v \rangle_{L^2} dt
$$

$$
= \lim_{n \to +\infty} \left[ \mathbb{E} \int_0^T \left( \langle u_n(0), \varphi(t)v \rangle_{L^2} + \langle P_n F(u_n(t)), \int_t^T \varphi(s) ds \rangle_{L^2} + \langle \int_0^t P_n G(u_n(s)) dW_n(s), \varphi(t)v \rangle_{L^2} \right) dt \right].
$$

The convergence of the terms in the RHS of (5.20) to the terms in the RHS of (5.19) is as follows. The first term is a consequence of the convergence of $P_n \to I_{L^2}$ with respect to the bounded linear operator topology on $L^2(\mathbb{T}^2)$ and the application of Lebesgue dominated convergence theorem. The second term converges thanks to the stochastic isometry, Statement (5), Lebesgue dominated convergence theorem and Lemma 5.1, see e.g. [11]. Therefore, Statement (5.21)

$$
\mathbb{E} \int_0^T \left( u(t) - (u(0) + \int_0^t F_1(s) ds + \int_0^t G_1(s) dW(s)) \right) \varphi(t) v dt = 0.
$$

To achieve the proof of the existence, we have to prove that $F_1 = A^\frac{\alpha}{2} \tilde{u} + B(\tilde{u})$ and $G_1 = G(\tilde{u})$. First, we prove the following key estimates.

- \text{(K1) The local monotonicity property: There exists a constant $c > 0$ such that \forall u, v \in H^{1+\frac{\alpha}{2}}(\mathbb{T}^2),$

\begin{align}
-2 \langle A_\alpha (u - v), u - v \rangle_{L^2} + 2 (B(u) - B(v), u - v)_{L^2} + \|G(u) - G(v)\|^2_{L_Q(L^2)} & \leq c (1 + |v| |u|_{H^{1+\frac{\alpha}{2}} L^2}) |u - v|_{L^2}^2.
\end{align}

- \text{(K2) For all $\psi \in L^\infty([0, T], \mathbb{R}_+)$ and $v \in L^2(\Omega \times [0, T]; H^{1+\frac{\alpha}{2}}(\mathbb{T}^2)),$$

\begin{align}
Z_n := \int_0^T \psi(t) dt \mathbb{E} \left\{ \int_0^t e^{-r(s)} (-r'(s)|u_n(s) - v(s)|^2_{L^2} + \|P_n G(u_n(s)) - P_n G(v(s))\|^2_{L_Q(L^2)}) + 2 (F(u_n(s)) - F(v(s)), u_n(s) - v(s))_{L^2} \right\} ds \leq 0,
\end{align}

where $r'(t) := c (1 + |v(t)| |u|_{H^{1+\frac{\alpha}{2}} L^2})$ and $c > 0$ is a constant relevantly chosen. In fact, by using

$$
B(u_1, u_1) - B(u_2, u_2) = B(u_1, u_1 - u_2) + B(u_1 - u_2, u_2),
$$

Property (K2), Hölder inequality, Estimate (1.21), interpolation in the case $1 < \alpha < 2$ (recall that $1 \leq \alpha \leq 2$ \Rightarrow $H^{1+\frac{\alpha}{2}}(\mathbb{T}^2) \hookrightarrow H^{1+\frac{\alpha}{2}}(\mathbb{T}^2)$) and Young inequality, we infer that

$$
|\langle B(u) - B(v), u - v \rangle_{L^2}| = |\langle B(u - v, u - v) \rangle_{L^2}| \leq \|B(u - v, u - v)\|_{L^2} |u - v|_{L^2} \leq c |v| |u - v|_{H^{1+\frac{\alpha}{2}}} |u - v|_{H^{1+\frac{\alpha}{2}}} \leq c |v| |u - v|_{H^{1+\frac{\alpha}{2}}} |u - v|_{H^{1+\frac{\alpha}{2}}} \leq c |v|^{\frac{2\alpha}{\alpha+2}} |u - v|_{H^{1+\frac{\alpha}{2}}} + \frac{1}{2} |u - v|_{H^{\frac{\alpha}{2}}}^2 + 1.
$$

Moreover, thanks to the semigroup property of $(A^\beta)_{\beta \geq 0}$, we infer that

$$
- \langle A_\alpha (u - v), u - v \rangle_{L^2} = -|u - v|_{H^{1+\frac{\alpha}{2}}}^2.
$$


Therefore, there exists a constant $c > 0$ such that
\[ -\langle A_\alpha(u - v), u - v \rangle_{L^2} + \langle B(u) - B(v), u - v \rangle_{L^2} \leq -\frac{1}{2} |u - v|_{H^1 + \frac{2}{d}}^2 + c|v|_{H^1 + \frac{2}{d}}^2 |u - v|_{L^2}^2. \]  
(5.27)

Combining (5.27) and Assumption (C): (2.42) with $q = 2$, $\delta = 0$ and $C_R := C$), we easily get (5.22). In particular, thanks to the contraction of $P_n$ on $L^2(T^2)$, Estimate (5.22) is still valid when replacing $u, v$ and $G$ by $u_n$ (recall $u_n$ is the solution of Equation (5.1)), $v \in L^2(\Omega \times [0,T]; \mathbb{H}^{1+\frac{2}{d}}(T^2))$ and $P_n G$ respectively. Furthermore, estimating the LHS of (5.23) by (5.22) endowed with these latter variables, we get $Z_n \leq 0$. Consequently, $(K_1)$ and $(K_2)$ are proved. Let us also mention and recall the following two statements

(a) If $v \in L^2(\Omega \times [0,T]; \mathbb{H}^{1+\frac{2}{d}}(T^2))$, with $\alpha \in [1,2]$, then
\[ \mathbb{E} \int_0^T |v(t)|_{\tilde{H}^{1+\frac{2}{d}}(T^2)}^2 dt \leq \mathbb{E} \int_0^T (1 + |v(t)|_{\mathbb{H}^{1+\frac{2}{d}}(T^2)}^2) dt < \infty. \]  
(5.28)

(b) If a sequence $(f_n)_n$ in a Hilbert space $H$ converges weakly to $f$ then $|f|_H \leq \lim\inf_{n \to \infty} |f_n|_H$.

Now, we take $\psi$ and $\tau(t)$ as defined above. Thanks to the equality $u = \tilde{u}, dt \times dP - a.e.,$ statements (1) & (b) and Fubini’s theorem, we infer that
\[ \int_0^T \psi(t)dt\mathbb{E}[u(s)|_{\tilde{H}^{1+\frac{2}{d}}(T^2)}^2 e^{-\tau(t)} - \mathbb{E}[u_0|_{\tilde{H}^{1+\frac{2}{d}}(T^2)}^2] \leq \lim\inf_{n \to \infty} \int_0^T \psi(t)dt\mathbb{E}[u_n(s)|_{\tilde{H}^{1+\frac{2}{d}}(T^2)}^2 e^{-\tau(t)} - \mathbb{E}[u_0|_{\tilde{H}^{1+\frac{2}{d}}(T^2)}^2]. \]  
(5.29)

By application of the Ito formula to the Ito process $\tilde{u}$ given by (5.19) and using the equality $u = \tilde{u}, dt \times dP - a.e.$ and the elementary identity
\[ \forall f,g \in H, |f|_H^2 = |f - g|_H^2 + 2\langle f - g, g \rangle_H + |g|_H^2 \]  
with $f = u(s), g = v(s)$ and $v \in L^2(\Omega \times [0,T]; \mathbb{H}^{1+\frac{2}{d}}(T^2))$, we get
\[ \mathbb{E}[u(s)|_{\tilde{H}^{1+\frac{2}{d}}(T^2)}^2 e^{-\tau(t)} - \mathbb{E}[u_0|_{\tilde{H}^{1+\frac{2}{d}}(T^2)}^2] = \mathbb{E} \int_0^t e^{-\tau(s)}(F_1(s), u(s))_{L^2} + \mathbb{E} \int_0^t e^{-\tau(s)}|G_1(s)|_{L^2(\mathbb{H}^{1+\frac{2}{d}})}^2 ds \]  
(5.31)

Similarly, we get Identity (5.31) for $\mathbb{E}[u_n(s)|_{\tilde{H}^{1+\frac{2}{d}}(T^2)}^2 e^{-\tau(t)}$ with $u, F_1$ and $G_1$ in the RHS of (5.31) are respectively replaced by $u_n, F_n(u_n), P_n G_n$ (Recall that $F := A^{\frac{2}{d}} + B$). Replacing Identity (5.31) for $\mathbb{E}[u(s)|_{\tilde{H}^{1+\frac{2}{d}}(T^2)}^2 e^{-\tau(t)}$ in the LHS of the first equality in (5.29) and Identity (5.31) for $\mathbb{E}[u_n(s)|_{\tilde{H}^{1+\frac{2}{d}}(T^2)}^2 e^{-\tau(t)}$ in the RHS of the second Inequality (5.29) and arranging terms (in particular, we introduce the term $G(v(s))$ and use the elementary identity (5.30)), we infer that
\[ \mathcal{E} := \int_0^T \psi(t)dt\mathbb{E} \int_0^t e^{-\tau(s)} \left[ 2 \langle F_1(s), u(s) \rangle_{L^2} + ||G_1(s)||_{L^2(\mathbb{H}^{1+\frac{2}{d}})}^2 ds ight. 
- \left. r'(s) (|u(s) - v(s)|_{\tilde{H}^{1+\frac{2}{d}})^2 + 2(u(s) - v(s), v(s))_{L^2}) \right] ds \]  
\[ \leq \lim\inf_{n \to \infty} (Z_n + Y_n + X_n). \]  
(5.32)
Therefore, we get
\begin{align*}
E \&= \int_0^T \psi(t)dt \mathbb{E}\left\{ \int_0^t e^{-r(s)} \left( -2 r'(s) u_n(s) - v(s), v(s) \right)_{L^2} + 2 \langle P_n G(u_n(s)), G(v(s)) \rangle_{L_Q(L^2)} \right. \\
&\quad + 2 \langle F(u_n(s)), v(s) \rangle_{L^2} + 2 \langle F(v(s)), u_n(s) \rangle_{L^2} - 2 \langle F(v(s)), v(s) \rangle_{L^2} \right\}ds,
\end{align*}
(5.33)

and
\begin{align*}
X_n \&= \int_0^T \psi(t)dt \mathbb{E}\left\{ \int_0^t e^{-r(s)} \left( 2 \langle P_n G(u_n(s)), P_n G(v(s)) - G(v(s)) \rangle_{L^2} - \|P_n G(v(s))\|_{L_Q(L^2)}^2 \right)ds\right\},
\end{align*}
(5.34)

The sequences \((Y_n)_n\) and \((X_n)_n\) converge to \(Y\) and \(X\) respectively, thanks to statements (1)–(3) and (5), the convergence of \(P_n \to I_{L^2}\), Assumption \((C)\):\( (2.42) \) with \( q = 2, \delta = 0 \) and \( C_R := C \), Lemma \( [4,1] \) Estimate \( (1.7) \), similar calculus as in \( (5.15) \) and \( (5.16) \) and the Lebesgue dominated, where
\begin{align*}
Y \&= \int_0^T \psi(t)dt \mathbb{E}\left\{ \int_0^t e^{-r(s)} \left( -2 r'(s) u(s) - v(s), v(s) \right)_{L^2} + 2 \langle G(s), G(v(s)) \rangle_{L_Q(L^2)} \right. \\
&\quad + 2 \langle F(s), v(s) \rangle_{L^2} + 2 \langle F(v(s)), u(s) \rangle_{L^2} - 2 \langle F(v(s)), v(s) \rangle_{L^2} \right\}ds,
\end{align*}
(5.35)

and
\begin{align*}
X \&= - \int_0^T \psi(t)dt \mathbb{E}\left\{ \int_0^t e^{-r(s)} \|G(v(s))\|_{L_Q(L^2)}^2 \right\}ds.
\end{align*}
(5.36)

Replacing \(X\) and \(Y\) in \( (5.32) \) and taking into account \( (5.23) \), we conclude that
\begin{align*}
\mathcal{E} - X - Y \leq \liminf_{n \to \infty} Z_n \leq 0.
\end{align*}
(5.37)

Therefore, we get
\begin{align*}
\int_0^T \psi(t)dt \mathbb{E}\left\{ \int_0^t e^{-r(s)} \left( -2 \left( r'(s) u(s) - v(s) \right)_{L^2} + \left( F_1(s) - F(v(s)), u(s) - v(s) \right)_{L^2} \right) \\
&\quad + 2 \left\| G_1(s) - G(v(s)) \right\|_{L_Q(L^2)}^2 \right\}ds \leq 0.
\end{align*}
(5.38)

Now, we take \( v = u \in L^2([\Omega \times [0,T]), \mathbb{H}^{1+\frac{\alpha}{2},2}(\mathbb{T}^2)) \), we conclude from \( (5.38) \), that \( G(s) = G(u(s)), ds \times dP - a.e. \). To get the equality \( F(s) = F(u(s)), ds \times dP - a.e. \), we consider Estimate \( (5.38) \) without the last term and we introduce \( \tilde{v} \in L^\infty(\Omega \times [0,T], \mathbb{H}^{1+\frac{\alpha}{2},2}(\mathbb{T}^2)) \) and a parameter \( \lambda \in [-1, +1] \). Replacing \( v \) and \( r'(s) \) by \( u - \lambda \tilde{v} \) respectively \( r'_\lambda(s) \):= \( c(1 + |u - \lambda \tilde{v}|^{\frac{2\alpha+2}{\alpha+2}}) \), we get
\begin{align*}
\mathbb{E}\int_0^T e^{-r(s)} \left( -r'(s) \lambda^2 \tilde{v}(s) \right)_{L^2} + 2 \lambda \left( F(s) - F(u(s) - \lambda \tilde{v}(s)), \tilde{v}(s) \right)_{L^2} \right)ds \leq 0.
\end{align*}
(5.39)

Dividing on \( \lambda < 0 \) and on \( \lambda > 0 \), we conclude that, when \( \lambda \to 0 \), the limit of the LHS of \( (5.39) \) exists and vanishes. Moreover, using the fact that \( \tilde{v} \in L^\infty(\Omega \times [0,T], \mathbb{H}^{1+\frac{\alpha}{2},2}(\mathbb{T}^2)) \), the
continuity of $r_\lambda$ and $r_\lambda'$ with respect to $\lambda$ and the Lebesgue dominated convergence theorem, we conclude that the first term in \( L^1 \) of (5.39) vanishes and also

\[
\mathbb{E} \int_0^T e^{-r_\lambda(s)}(F(s) - F(u(s)), \tilde{v}(s))_{L^2} ds = 0.
\]

The justification of the use of the Lebesgue dominated convergence theorem is due to, the positivity of $r_\lambda(s)$, Inequality (1.24), Minikowski inequality, the conditions $0 < \alpha \leq 2$ and $|\lambda| \leq 1$, the statements (1) & (3) and the definition of $\tilde{v}$. In fact,

\[
e^{-r_\lambda(s)}|\langle F(s) - F(u(s) - \lambda \tilde{v}(s)), \tilde{v}(s) \rangle_{L^2}| \\
\leq |\tilde{v}(s)|_{H^{1,2}} |\langle F(s) \rangle_{L^2} + |u(s)|_{H^{0,2}} + |\tilde{v}(s)|_{H^{0,2}} + |B(u(s) - \lambda \tilde{v}(s))|_{H^{-1,2}}| \\
\leq c|\tilde{v}(s)|_{H^{1,2}} |\langle F(s) \rangle_{L^2} + |u(s)|_{H^{0,2}} + |\tilde{v}(s)|_{H^{0,2}} + |u(s)|_{H^{1,2}}|_{L^2} \\
+ |\tilde{v}(s)|_{H^{1,2}} |\langle \tilde{v}(s) \rangle_{L^2} + |u(s)|_{H^{1,2}}|_{L^2} + |\tilde{v}(s)|_{H^{1,2}}|_{L^2}.
\]

This ends the proof of the existence of a solution $(u(t), t \in [0, T])$ belonging to the first intersection in (3.7) and satisfying by construction (3.8).

**Proof of the time regularity.** To prove the continuity of the trajectories of the weak solution $(u(t), t \in [0, T])$, i.e. $u(\cdot, \omega) \in C([0, T], L^2(O))$, $P$-a.s., we apply [H1, Proposition VII.3.2.2], see also [68, Proposition 2.5]. We consider the dense Gelfand Triple

$$
\mathbb{H}^{1,2}(T^2) \hookrightarrow L^2(T^2) \hookrightarrow (\mathbb{H}^{1,2}(T^2))^* = \mathbb{H}^{-1,2}(T^2).
$$

Using (1.24), Sobolev Embedding and (3.8), we infer that $B(u(\cdot, \omega)) \in L^2(0, T; \mathbb{H}^{-1,2}(T^2))$, $P$-a.s.. In fact,

\[
\mathbb{E} \int_0^T |B(u(s))|_{H^{1,2}}^2 ds \leq c\mathbb{E} \int_0^T |u(s)|_{H^{1,2}}^2 ds \leq c(1 + \mathbb{E} \sup_{[0, T]} |u(s)|_{H^{1,2}}^p) < \infty.
\]

And that for $\alpha \leq 2$,

\[
\mathbb{E} \int_0^T |A_\alpha u(s)|_{H^{0,2}}^2 ds \leq c\mathbb{E} \int_0^T |u(s)|_{H^{0,2}}^2 ds \leq c(1 + \mathbb{E} \sup_{[0, T]} |u(s)|_{H^{1,2}}^p) < \infty.
\]

Moreover, we prove that the martingale $M(t) := \int_0^t G(u(s))dW(s)$ belongs to $L^2(\Omega, C(0, T; \mathbb{H}^{1,2}(T^2)))$. In fact, we use Burkholdy-Davis-Gandy inequality, Assumption (C): (2.43) with $q = 2$ and $\delta = 1)$, (3.8), we obtain

\[
\mathbb{E} \sup_{[0, T]} \left| \int_0^t G(u(s))dW(s) \right|_{L^2}^2 \leq c\mathbb{E} \int_0^T |G(u(s))|_{L^2(H^{1,2})}^2 ds \leq c\mathbb{E} \int_0^T (1 + |u(s)|_{L^2(H^{1,2})}^p) ds
\]

(5.44)

\[
\leq c(1 + \mathbb{E} \sup_{[0, T]} |u(s)|_{H^{1,2}}^p) < \infty.
\]

Hence from (3.8), (5.42), (5.43) and (5.44), we establish the existence of a subset $\Omega' \subset \Omega$ (independent of "t"), such that $P(\Omega') = 0$ and $F(u(\cdot, \omega)) \in L^2(0, T; \mathbb{H}^{1,2}(T^2)), u(\cdot, \omega)$ and $M(\cdot, \omega)$ are in $L^2(0, T; \mathbb{H}^{1,2}(T^2))$, $\forall \omega \in \Omega^c$. These ingredients are enough to apply [H1, Proposition VII.3.2.2], hence we get the result. It is important to mention that the property $u(\cdot, \omega)$ and $M(\cdot, \omega)$ is $L^2(0, T; \mathbb{H}^{1,2}(T^2))$ is more what we need here. In deed, it is sufficient to prove $u(\cdot, \omega)$ and $M(\cdot, \omega)$ is $L^2(0, T; \mathbb{H}^{1,2}(T^2))$. By the above two subsections, the proof of (3.7) is achieved.
Proof of the pathwise uniqueness. Let $u^1$ and $u^2$ be two weak solutions of Equation (2.23) satisfying (3.7) and (3.8). Let $w := u^1 - u^2$, then $w$ satisfies the following equation

$$w(t) = \int_0^t \left( -A_ww(s) + B(w(s), u^1(s)) + B(u^2(s), w(s)) \right) ds + \int_0^t \left( G(u^1(s)) - G(u^2(s)) \right) dW(s). \tag{5.45}$$

For $N > 0$, we define the stopping times, $\tau_N^i : \text{inf}\{t \in (0, T); |u^i(t)|_{L^2} > N\} \wedge T, i = 1, 2$, with the understanding that $\text{inf}(\emptyset) = +\infty$ and define $\tau_N := \text{min}_{i \in \{1, 2\}} \{\tau_N^i\}$. Using Ito formula for the product $e^{-r(t)}|w(t)|_{L^2}^2$, with $(r(t), t \in [0, T])$ being a positive real stochastic process to be defined later, Property (1.1), Assumption (C) (2.42) with $q = 2$ and $\delta = 0$, locally Lipschitz), Estimate (4.22), Young inequality and arguing as in the proof of (5.4) with the replacement of the spaces $H^{1, 2}(T^2)$ and $H^{1+\frac{2}{q}, 2}(T^2)$ by $L^2(T^2)$ respectively $H^{2, 2}(T^2)$, we infer that for $1 \leq \alpha < 2$ (here we omit to writ the proof for the dissipative regime, as it is classical.)

$$
\begin{align*}
\mathbb{E} \ e^{-r(t \wedge \tau_N)}|w(t \wedge \tau_N)|_{L^2}^2 &+ 2\mathbb{E} \int_0^{t \wedge \tau_N} \ e^{-r(s)}|w(s)|_{H^{\frac{\alpha}{2}, 2}}^2 ds \\
& \leq \mathbb{E} \int_0^{t \wedge \tau_N} \ e^{-r(s)}|G(u^1(s)) - G(u^2(s))|_{L^2}^2 ds \\
& - \mathbb{E} \int_0^{t \wedge \tau_N} \ e^{-r(s)}(2\|B(w(s), u^1(s)) - r'(s)w(s)\|_{L^2}^2) ds \\
& \leq c_N \mathbb{E} \int_0^{t \wedge \tau_N} \ e^{-r(s)}(\|w(s)\|_{L^2}^2 + |u^1|_{H^{1+\frac{2}{q}, 2}} |w|_{H^{\frac{\alpha}{2}, 2}} |w|_{L^2}^{2-\alpha} - r'(s)|w(s)|_{L^2}^2) ds \\
& \leq c_N \mathbb{E} \int_0^{t \wedge \tau_N} \ e^{-r(s)}(\|w(s)\|_{L^2}^2 + 2c|w(s)|_{H^{\frac{\alpha}{2}, 2}}^2 + 2c_1|u^1(s)|_{H^{1+\frac{2}{q}, 2}} |w(s)|_{H^{\frac{\alpha}{2}, 2}}^2 - r'(s)|w(s)|_{L^2}^2) ds.
\end{align*}
\tag{5.46}$$

We choose $c < 1$ and $r'(s) = 2c_1|u^1(s)|_{H^{1+\frac{2}{q}, 2}}^{2-\alpha}$ and replace in (5.46), we end up with the simple formula

$$
\begin{align*}
\mathbb{E} \ e^{-r(t \wedge \tau_N)}|w(t \wedge \tau_N)|_{L^2}^2 &+ 2(1 - c)\mathbb{E} \int_0^{t \wedge \tau_N} \ e^{-r(s)}|w(s)|_{H^{\frac{\alpha}{2}, 2}}^2 ds \\
& \leq c_N c \mathbb{E} \int_0^{t \wedge \tau_N} \ e^{-r(s \wedge \tau_N)}|w(s \wedge \tau_N)|_{L^2}^2 ds.
\end{align*}
\tag{5.47}$$

By application of Gronwall’s lemma, we get $\forall \tau \in [0, T] \text{ the random variable } |w(t \wedge \tau_N)|_{L^2}^2 = 0, P - a.s.$ as much as $P(\int_0^{t \wedge \tau_N} |u^1(s)|_{H^{1+\frac{2}{q}, 2}}^{2-\alpha} ds < \infty) = 1$. This last statement is confirmed thanks to (3.8) and the condition $1 \leq \alpha < 2$. The proof is then achieved once we remark that thanks to Chebyshev inequality and (5.8), we have $\lim_{N \to \infty} \tau_N = T, P - a.s.$

Proof of the space regularity. In the aim to get Estimate (5.10), we use the regularization effect of the vorticity. Let $(u(t), t \in [0, T])$ be a weak solution of Equation (2.23) in the sense of Definition (3.1). Thanks to Appendix B, the curlu is a weak solution of Equation (B.19). We know from [15] that this equation admits a unique global solution which is simultaneously weak and mild and satisfies

$$
\mathbb{E}\left( \text{sup}_{[0, T]} |\theta(t)|_{L^2}^2 + \int_0^T |\theta(t)|_{H^{\frac{\alpha}{2}, 2}}^2 dt \right) < \infty.
\tag{5.48}$$
for \( q_0, q \) and \( \alpha \) being characterized as in (6.3.2.) and provided that \( \text{curl} u_0 \) fulfills (3.9) and \( \tilde{G} \), defined by (5.10), satisfies the Lipschitz and the growth conditions, i.e. \( \tilde{G} \) satisfies (2.42) and (2.43), with \( R_Q(L^2, H^{1,q}) \) in the LHSs and \( H^{1,q} \) in the RHSs are replaced by \( R_Q(L^2, L^q) \) and \( L^q \) respectively. As (3.9) is fulfilled by assumption, we check that the two latter conditions are also satisfied. In fact, thanks to the definition of \( \tilde{G} \), Assumption (C), with \( \delta = 1 \) and Lemma B.1, we get

\[
\left\| \tilde{G}(\theta) \right\|_{R_n(L^2, L^q)} = \left\| \sum_{k \in \Sigma} \text{curl} G(\mathcal{R}_1(\theta)) Q^{\frac{1}{2}} e_k \right\|_{L^q} \leq c \left\| \sum_{k \in \Sigma} |\partial_j G(\mathcal{R}_1(\theta)) Q^{\frac{1}{2}} e_k |^2 \right\|_{L^q}^{\frac{1}{2}}
\]

(5.49)

By the same way, we prove the Lipschitz condition. Estimate (3.10) follows from (5.49) and Lemma B.2.

6. Martingale solution of the multi-dimensional FSNSEs.

In this section, we prove Theorem 3.5. The main ingredients are Faedo-Galerkin approximations, compactness, Skorokhod embedding theorem and the representation theorem. In particular, once we prove Lemma 6.1 below, we can follow the same scheme e.g. as in [9, 24], see also similar calculus for the fractional stochastic scalar active equation in [15]. Thus we omit to give full details.

Lemma 6.1. The sequence \((u_n)\) of solutions of the equations (5.1) is uniformly bounded in the space

\[
L^2(\Omega, W^{1,2}(0, T; \mathbb{R}^{2}\mathbb{H}^{1/2}(O)))
\]

where \( \delta' \geq 1 \max\{\alpha, 1 + \frac{\delta}{2}\} \) and \( \gamma < \frac{1}{2} \).

Proof. Thanks to Lemma B.1, it is sufficient to prove that \((u_n(t), t \in [0, T])\) is uniformly bounded in \(L^2(\Omega, W^{1,2}(0, T; \mathbb{R}^{2}\mathbb{H}^{1/2}(O)))\). We recall that the Besov-Slobodetski space \(W^{\gamma,p}(0, T; E)\), with \( E \) being a Banach space, \( \gamma \in (0, 1) \) and \( p \geq 1 \), is the space of all \( v \in L^P(0, T; E) \) such that

\[
\|v\|_{W^{\gamma,p}} := \left( \int_0^T \|v(t)\|_E^p dt + \int_0^T \int_0^T \frac{|v(t) - v(s)|^p}{|t-s|^{1+\gamma p}} dtds \right)^{\frac{1}{p}} < \infty.
\]

As \((u_n(t), t \in [0, T])\) is the strong solution of the finite dimensional stochastic differential equation (5.1), then \( u_n(t) \) is the solution of the stochastic integral equation

\[
u_n(t) = P_n u_0 + \int_0^t \left( -A_n u_n(r) + P_n B(u_n(r)) \right) dr + \int_0^t P_n G(u_n(r)) dW_n(r), \quad \text{a.s.,}
\]

for all \( t \in [0, T] \). We denote by

\[
I(t) := \int_0^t \left( -A_n u_n(r) + P_n B(u_n(r)) \right) dr
\]

and

\[
J(t) := \int_0^t P_n G(u_n(r)) dW_n(r).
\]

We prove that \( I(\cdot) \) is uniformly bounded in \( L^2(\Omega; W^{1,2}(0, T; \mathbb{R}^{2}\mathbb{H}^{1/2}(O))) \) and that the stochastic term \( J(\cdot) \) is uniformly bounded in \( L^2(\Omega; W^{1,2}(0, T; L^2(O))) \), for all \( \gamma < \frac{1}{2} \). Let
\( \phi \in \mathbb{H}^{\delta', 2}(O) \), using Identity (6.5), we get
\[
|_{\mathbb{H}^{-\delta', 2}}(P_n B(u_n(r)), \phi)_{\mathbb{H}^{\delta', 2}}| = |(u_n(r) \cdot \nabla P_n \phi, u_n(r))_{L^2}| \leq |\nabla P_n \phi|_{L^\infty} |u_n(r)|^2_{L^2}.
\]
(6.6)

Thanks to [62, Remark 4 p 164, Theorem 3.5.4, p.168-169 and Theorem 3.5.5 p 170] for \( O = \mathbb{T}^d \), to [2, Theorem 7.63 and point 7.66] for \( O \) being a bounded domain and to the condition \( \delta' > 1 + \frac{d}{2} \), we deduce for \( 0 < \epsilon < \delta' - 1 - \frac{d}{2} \),
\[
|\nabla P_n \phi|_{L^\infty} \leq c |\nabla P_n \phi|_{H^{\epsilon+\frac{d}{2}, 2}} \leq c |\phi|_{H^{1+\epsilon+\frac{d}{2}, 2}} \leq c |\phi|_{\mathbb{H}^{\delta', 2}}.
\]
Therefore,
\[
|P_n B(u_n(r))|_{\mathbb{H}^{-\delta', 2}} \leq c |u_n(r)|^2_{L^2}
\]
and
\[
\int_0^T |I(t)|^2_{\mathbb{H}^{-\delta', 2}} dt \leq c \int_0^T \int_0^t \left( |(-A_n u_n(r)|^2_{\mathbb{H}^{-\delta', 2}} + |P_n B(u_n(r)|^2_{\mathbb{H}^{-\delta', 2}} \right) drdt
\]
(6.8)

Moreover, using Hölder inequality and arguing as before, we get for \( t \geq s > 0 \),
\[
|I(t) - I(s)|^2_{\mathbb{H}^{-\delta', 2}} = \int_s^t (-A_n u_n(r) + P_n B(u_n(r)) dr|^2_{\mathbb{H}^{-\delta', 2}} \leq C(t-s) \left( \int_s^t (|u_n(r)|^2_{L^2} + |u_n(r)|^4_{L^2}) dr \right).
\]
(6.9)

From (6.8), (6.9) and (6.2), we have for \( \gamma < \frac{1}{2} \),
\[
\mathbb{E} \left( \int_0^T |I(t)|^2_{\mathbb{H}^{-\delta', 2}} dt \right) + \int_0^T \int_0^t \frac{|I(t) - I(s)|^2_{\mathbb{H}^{-\delta', 2}}}{|t-s|^{1+2\gamma}} dtds \right)^{\frac{1}{2}} \leq C \left( \int_0^T (|u_n(r)|^2_{L^2} + |u_n(r)|^4_{L^2}) dr \right)^{\frac{1}{2}} \leq C < \infty.
\]
(6.10)

Now, we estimate the stochastic term \( J \). Using the stochastic isometry, the contraction property of \( P_n \) and Assumption (C), (Condition 2.43) with \( q = 2 \) and \( \delta = 0 \), we get
\[
\int_0^T \mathbb{E} \left| \int_0^t P_n G(u_n(r))dW_n(r) \right|^2_{L^2} dt \leq C \int_0^T \mathbb{E} \left( \int_0^t ||G(u_n(r)||_{L^2}^2 dr \right) dt
\]
(6.11)

Moreover, for \( t \geq s > 0 \) and \( \gamma < \frac{1}{2} \), the same ingredients above yield to
\[
\mathbb{E} \int_0^T \int_s^t \frac{|J(t) - J(s)|^2_{L^2}}{|t-s|^{1+2\gamma}} dtds \leq C \mathbb{E} \int_0^T \int_0^t \frac{|G(u_n(r))|^2_{L^2} dr}{|t-s|^{1+2\gamma}} dtds
\]
(6.12)

The proof of the lemma is now completed. \( \square \)
To prove the existence of a martingale solution, we use the following compact embedding, see [24, Theorem 2.1],

\[(6.13)\quad W^{α,2}(0, T; \mathbb{H}^{-δ,2}(O)) \cap L^2(0, T; \mathbb{H}^{δ,2}(O)) \hookrightarrow L^2(0, T; \mathbb{L}^2(O)).\]

Therefore, we deduce that the sequence of laws \((L(u_n))_n\) is tight on \(L^2(0, T; \mathbb{L}^2(O))\). Thanks to Prokhorov’s theorem there exists a subsequence, still denoted \((u_n)_n\), for which the sequence of laws \((L(u_n))_n\) converges weakly on \(L^2(0, T; \mathbb{L}^2(O))\) to a probability measure \(μ\). By Skorokhod’s embedding theorem, we can construct a probability basis \((Ω, F, F, P, P)\) and a sequence of \(L^2(0, T; \mathbb{L}^2(O)) \cap C([0, T]; \mathbb{H}^{-δ,2}(O))\) random variables \((u_n^*)_n\) and \(u^*\) such that \(L(u_n^*) = L(u_n), \forall n \in \mathbb{N}_0\), \(L(u^*) = μ\) and \(u_n^* \to u^*\) a.s. in \(L^2(0, T; \mathbb{L}^2(O)) \cap C([0, T]; \mathbb{H}^{-δ,2}(O))\). Moreover, \(u_n^*(\cdot, ω) \in C([0, T]; H_n)\). Thanks to Lemma 7.1 and to the equality in law, we infer that the sequence \(u_n^*\) converges weakly in \(L^2(Ω \times [0, T]; \mathbb{H}^{δ,2}(O))\) and weakly-star in \(L^p(Ω, L^∞([0, T]; \mathbb{L}^2(O))\) to a limit \(u^{**}\). It is easy to see that \(u^* = u^{**}\), \(dt \times dP - a.e.\) and

\[(6.14)\quad \mathbb{E}_* \sup_{[0,T]} |u^*(s)|^2_{\mathbb{L}^2} + \mathbb{E}_s \int_0^T |u^*(s)|^2_{\mathbb{H}^{δ,2}} ds \leq c < \infty.\]

We introduce the filtration

\[(6.15)\quad (G^*_n)_t := σ\{u_n^*(s), s ≤ t\}\]

and construct (with respect to \((G^*_n)_t\)) the time continuous square integrable martingale \((M_n(t), t \in [0, T])\) with trajectories in \(C([0, T]; \mathbb{L}^2(O))\) by

\[(6.16)\quad M_n(t) := u_n^*(t) - P_n u_0 + \int_0^t A_n u_n^*(s) ds - \int_0^t P_n B(u_n^*(s)) ds.\]

The equality in law yields to the fact that the quadratic variation is given by

\[(6.17)\quad ⟨⟨M_n⟩⟩_t = \int_0^t P_n G(u_n^*(s)) Q G(u_n^*(s))^* ds,\]

where \(G(u_n^*(s))^*\) is the adjoint of \(G(u_n^*(s))\). We prove that, for a.s., \(M_n(t)\) converges weakly in \(\mathbb{H}^{-δ,2}(O)\) to the martingale \(M(t)\), for all \(t \in [0, T]\), where \(M(t)\) is given by

\[(6.18)\quad M(t) := u^*(t) - u_0 + \int_0^t A_n u^*(s) ds - \int_0^t B(u^*(s)) ds.\]

Some of the main ingredients are the a.s. convergence of \(u_n^*\) in \(L^2(0, T; \mathbb{L}^2)\), \(δ_1 φ \in C^0\) and therefore we can estimate \(\int_0^T |B(u_n^*(s), v)| ds\) by \(\int_0^T |B(u_n^*(s))|_{L_1}|v|_C ds\). Now we apply the representation theorem [14, Theorem 8.2], we infer that there exists a probability basis \((Ω^*, F^*, P^*, P, W^*)\) such that

\[(6.19)\quad M(t) = \int_0^t G(u^*(s)) W^*(ds).\]

If moreover, \(α \in [α_0(d) := 1 + \frac{α}{δ}, 2\], then thanks to Burkholder-Davis-Gundy inequality, Assumption 243, with \(q = 2, δ = 0\) and (6.14)

\[(6.20)\quad \mathbb{E} \sup_{[0,T]} |\int_0^T G(u^*(s)) dW^*(s)|^2_{\mathbb{L}^2} \leq c \mathbb{E} \int_0^T |G^*(u^*(s))|^2_{L_2(Q^2)} ds \leq c(1 + \mathbb{E} \sup_{[0,T]} |u^*(s)|^2_{\mathbb{L}^2}) < \infty.\]
Further more, using Estimate (4.19) with \( \eta \) in the P we use the contraction property of \( u \) under the condition (3.12), we follow the scheme of the uniqueness in Section 5 taking into account the changes of the norms. Let \( u^1 \) and \( u^2 \) be two martingale solutions on the same probability basis \( (\Omega^*, \mathbb{F}^*, P^*, W^*) \) and such that \( u^1 \) satisfies (3.12). We define \( \tau_N, \tau_N \) and \( w := u^1 - u^2 \) as in Section 5. Then \( w \) satisfies Equation (5.45) with \( W \) replaced by \( W^* \). We use Ito formula to the product \( e^{-r(t)}|w(t)|^2_{H^1} \), with \( (r(t):= c \int_0^t |u^1(s)|_{\frac{4}{2} + \alpha}^2 ds, t \in [0,T]) \), Identity (5.21), condition (2.42), with \( q = 2, \delta = 0 \), Estimate (4.19) with \( \eta = 0 \) and argue as around (4.46) we get the proof of the uniqueness. Combining this latter result with Yamada-Watanabe theorem [38, 55], the global existence of a unique weak-strong solution follows.

7. Similarity of the 2D-FSNSE and the 3D-NSE.

In this section, we illustrate the fact that the 2D-FSNSE exhibits the same difficulty to prove the existence of the global solution as the 3D-NSE. We follow a similar calculus as in Section 5, replacing Property (4.5) by Property (4.4) and considering the densely continuously embedding Gelfand triple

\[
V_2 = V := H_\alpha^2(O) \hookrightarrow L^2(O) \hookrightarrow H_\alpha^{-2}(O) =: V^*. 
\]

We obtain the following Lemma

**Lemma 7.1.** Let \( d \in \{2, 3\} \), \( a_0(d) := 1 + \frac{d-1}{3} \leq \alpha \leq 2 \) and \( u_0 \in L^p(\Omega, L^2(O)), p \geq 4 \) and let \( G \) satisfying Assumption (C) \( (2.43) \) with \( q = 2 \) and \( \delta = 0 \). Then the solutions \( (u_n(t), t \in [0,T]) \) of the equations (5.1), \( n \in \mathbb{N}_0 \), satisfy the following estimates

\[
\sup_n \mathbb{E} \left( \sup_{[0,T]} |u_n(t)|_{L^2}^p \right) + \int_0^T |u_n(t)|_{L^2}^{p-2} \left( |u_n(t)|_{H^\alpha}^2 + |u_n(t)|_{H^{\alpha,p,q}}^2 \right) dt < \infty, 
\]

where \( \beta \leq \frac{\alpha}{2} - \frac{d}{2} + \frac{d}{q_1}, 2 \leq q_1 < \infty \) and \( \frac{\alpha}{p} < \eta \leq \frac{\alpha}{2} \).

\[
\sup_n \left( \int_0^T \left( |P_n B(u_n(t))|_{H^{\alpha-\frac{d}{2}}} + |A_{\alpha}^{\frac{\alpha}{2}} u_n(t)|_{H^{\alpha-\frac{d}{2}}} \right)^{\frac{2p}{\alpha}} dt \right) < \infty. 
\]

**Proof.** The proof of (7.2) follows exactly as for (5.2) by replacing the spaces \( H^{1,2}(T^2) \) and \( H^{1+\frac{d}{2}}(T^3) \) respectively by \( L^2(O) \) and \( H_\alpha^{-2}(O) \). For the first term in the Estimate (7.3), we use the contraction property of \( P_n \), Estimate (4.8) and the Sobolev interpolation (recall
that thanks to the condition $1 + \frac{d-1}{3} \leq \alpha \leq 2$, we have the following embedding $\mathbb{H}^{\frac{3}{d+2-\alpha},2}(O) \hookrightarrow \mathbb{H}^{\frac{d+2-\alpha}{3},2}(O) \hookrightarrow L^2(O)$, we end up, for $1 + \frac{d-1}{3} < \alpha \leq 2$, with

$$
\mathbb{E} \int_0^T |P_nB(u_n(t))| \frac{2\alpha}{d+2-\alpha} dt \leq c\mathbb{E} \int_0^T |u_n(t)| \frac{4\alpha}{d+2-\alpha} dt
$$

$$
\leq c\mathbb{E} \int_0^T (|u_n(t)| \frac{d+2-\alpha}{2} |u_n(t)| \frac{3\alpha-d-2}{2\alpha})^\frac{4\alpha}{d+2-\alpha} dt
$$

(7.4)

The last term in the RHS of (7.4) is uniformly bounded thanks to (7.2) and the condition $\frac{3\alpha-d-2}{d+2-\alpha} \leq p$. But this last is guaranteed thanks to $\frac{3\alpha-d-2}{d+2-\alpha} \leq 4 \leq p$. The case $1 + \frac{d-1}{3} = \alpha$ is easily obtained by application of Estimation (4.8). The second term in the RHS of (7.3) is uniformly bounded thanks to the fact that $A : V \hookrightarrow D(A^{\frac{7}{4}}) \rightarrow V^*$ is bounded, the condition $\alpha \leq 1 + \frac{d}{2}$ which yields to $\frac{3\alpha-d-2}{d+2-\alpha} \leq 2$ and thus we get

$$
\mathbb{E} \int_0^T |A^{\frac{7}{4}}u_n(t)| \frac{2\alpha}{d+2-\alpha} dt \leq c\mathbb{E} \int_0^T |u_n(t)| \frac{4\alpha}{d+2-\alpha} dt \leq c\mathbb{E} \int_0^T (1 + |u_n(t)|^2 \frac{2\alpha}{d+2-\alpha} dt < \infty.
$$

(7.5)

Finally we apply Estimate (7.2).

**Existence of the solution.** Assume that $1 + \frac{d-1}{3} \leq \alpha \leq 2$. Thanks to (7.2) and (7.3), we conclude the existence of a subsequence, which is still denoted by $(u_n)_n$,

$$
u \in L^2(\Omega \times [0,T]; \mathbb{H}^{\frac{3}{d+2-\alpha},2}(O)) \cap L^p(\Omega, L^\infty([0,T]; \mathbb{L}^2(O))),
$$

(7.6)

$$
F_2 \in L^\frac{2\alpha}{d+2-\alpha}(\Omega \times [0,T]; \mathbb{H}^{-\frac{d}{2},2}(O)) \quad \text{and} \quad G_2 \in L^2(\Omega \times [0,T]; L_Q(\mathbb{L}^2(O)), \text{s.t.}
$$

- (1') $u_n \rightarrow u$ weakly in $L^2(\Omega \times [0,T]; \mathbb{H}^{\frac{3}{d+2-\alpha},2}(O))$.
- (2') $u_n \rightarrow u$ weakly-star in $L^p(\Omega, \mathbb{L}^\infty([0,T]; \mathbb{L}^2(O)))$.
- (3') $P_nF(u_n) := A^{\frac{7}{4}}u_n + P_nB(u_n) \rightarrow F_2$ weakly in $L^\frac{2\alpha}{d+2-\alpha}(\Omega \times [0,T]; \mathbb{H}^{-\frac{d}{2},2}(O))$.
- (4') $u_n \rightarrow u$ weakly in $L^p(\Omega \times [0,T]; \mathbb{H}^{\frac{3}{d+2-\alpha},2}(O))$, for all $\frac{d}{d+2} < \eta \leq \frac{d}{2}$.
- (5') $P_nG(u_n) \rightarrow G_2$ weakly in $L^2(\Omega \times [0,T]; L_Q(\mathbb{L}^2(O)))$.

To prove the existence of a weak-strong solution of (2.23), we can follow the same scheme as in Section 3 with the replacement of the spaces $\mathbb{H}^{1,2}(\mathbb{T}^d)$ and $\mathbb{H}^{1+\frac{7}{4},2}(\mathbb{T}^d)$ by $\mathbb{L}^2(O)$ and $\mathbb{H}^{\frac{3}{d+2-\alpha},2}(O)$ respectively. We construct a process $\tilde{u}$ as in (3.13), with $F_1$ and $G_1$ are replaced by $F_2$ respectively $G_2$. The proof of the statement $u = \tilde{u}$, $dt \times dP - a.e.$, can be done exactly as in Section 3 with the brackets now stand for the $V - V^*$-duality. To check the main key estimates, we use (3.21), (4.3), Hölder inequality, (4.4), Sobolev interpolation and Young inequality, we get

$$
|\nu \cdot (B(u) - B(v), u - v)| = |\nu \cdot (B(u - v, v, u - v)v| \leq |B(u - v, v)| \frac{\nu}{\nu} |u - v| \frac{\nu}{\nu}
$$

$$
\leq c |v| \frac{\nu}{\nu} |u - v| \frac{\nu}{\nu} |u - v| \frac{\nu}{\nu}
$$

$$
\leq c |v| \frac{\nu}{\nu} |u - v| \frac{\nu}{\nu} |u - v| \frac{\nu}{\nu}
$$

$$
(7.8)
$$
Using the semigroup property of \((A^\beta)_{\beta \geq 0}\) and Assumption (C) \((2.42)\), with \(\delta = 0, q = 2\) and \(C_R := c\), we confirm

- \((K'_1)\): The local monotoncity property: There exists a constant \(c > 0\) such that
  \[
  \forall u, v \in H^{\frac{4 \alpha}{d+2-\alpha}}(O),
  \]
  \[
  -2 \nu \langle A_\alpha (u - v), u - v \rangle_V + 2 \nu \langle B(u) - B(v), u - v \rangle_V + ||G(u) - G(v)||_{L^q(O)}
  \leq r'(t) |u - v|^2_{L^2}.
  \]
  (7.9)
  where \(r'(t) := c(1 + |v(t)|^{\frac{4 \alpha}{2d+2-\alpha}})\) and \(c > 0\) is a constant relevantly chosen.

The main obstacle which prevent us in this stage to follow the same steps as in Section 5 is the fact that we are unable to prove that the solution \(u \in L^{\frac{4 \alpha}{d+2-\alpha}}(\Omega \times [0, T]; H^{\frac{4 \alpha}{3\alpha-d-2}}(O))\), unless we suppose that \(\alpha \geq 1 + \frac{d}{2}\). In fact, under the condition \(2 \frac{d+2-\alpha}{3\alpha-d-2} \leq 2 \iff \alpha \geq 1 + \frac{d}{2}\) and using the interpolation and Estimate \((2.2)\), we conclude that

\[
\sup_n E \int_0^T |u_n(t)|^{\frac{4 \alpha}{d+2-\alpha}} dt \leq c \sup_n E \int_0^T |u_n(t)|^{\frac{4 \alpha}{3\alpha-d-2}} dt < \infty.
\]
(7.10)

Remark that under the condition \(\alpha \geq 1 + \frac{d}{2}\), the regime is either dissipative or hyperdissipative. The proof of the existence and the uniqueness of the global solution for the dD-FSNSE under these two regimes is classical. In particular, one can follow the same machinery as in Section 3 with the relevant changes mentioned above. The obstacle mentioned in \((7.10)\) is similar to the well known one for the classical 3D-NSE but not for the 2D-NSE. To support more our claim mentioned in the begining of this section and in Section 1, we emphasize that the 2D-SNSE is covered by our technique and this proves that this latter is optimal. Moreover, we can remark also that the values, \((\alpha \geq 1 + \frac{d}{2})\), \((d = 2, \alpha = 2)\) and \((d = 3, \alpha \geq \frac{5}{2})\), known in the literature for the dD-NSEs emerge in our setting in a natural way.

8. Global existence and uniqueness of the weak solution of the multi-dimensional FSNSES.

In this section, we prove the global existence and the uniqueness of the weak solution for the dD-FSNSE \((2.23)\). For \(O = \mathbb{T}^2\), we have thanks to the conditions in \((3.41)\) and arguing as in the proof of the regularity in Section 5 we infer that the maximal solution \((u, \xi)\) satisfies

\[
E \sup_{0 \leq t \leq T} |u(t)|^q_{H^{1,q}} + E \int_0^T |u(t)|^2_{H^{\frac{1}{2}+\frac{d}{2}}} dt \leq c < \infty.
\]
(8.1)

We denote by \(\mathcal{E}\) the set of predictable stochastic processes \((v(t), t \in [0, T])\) (or the extension of \(v\) in the case \(v\) is defined up to a stopping time) satisfying that there exists a stopping time \(\tau\) such that \(v \in L^2(\Omega \times [0, \tau]; H^{\frac{2}{3\alpha-d-2}}(\mathbb{T}^2))\) and the process \((\nabla v(t), t \in [0, \tau])\) can be extended (we keep the same notation) to \(\nabla v \in L^{(1+\frac{2d}{\alpha})^{-1}}(\Omega \times [0, T]; L^q(\mathbb{T}^2))\), with the norm of \(\nabla v\) in this space is uniformly bounded, i.e. independently of the extension. We claim that \(\mathcal{E} \neq \emptyset\).

In fact, let us define, \((\tilde{v}(t), t \in [0, T])\), by \(\tilde{v}(t) := u(t \wedge \xi), \forall t \in [0, T]\), where \((u, \xi)\) is our maximal local solution. We have, for \(q\) characterized as in \((3.43)\), (bellow \(d=2\))

\[
E \int_0^T |\nabla u(t \wedge \xi_N)|^q_{H^{\frac{2}{\alpha}}} dt \leq cE \int_0^T |\theta(t \wedge \xi_N)|^{\frac{1}{L^q}} dt \leq cE \int_0^T (1 + |\theta(t)|^q_{L^q}) dt < \infty.
\]
(8.2)
Therefore \( \tilde{v} \in \mathcal{E} \). Remark that the condition \( \frac{1}{1 - \frac{d}{\alpha q}} \leq q \Leftrightarrow 1 + \frac{2d}{\alpha} \leq q \), see Remark 2. Now, we shall look for a solution in the set \( \mathcal{E} \). We can go back to the first part of Section 4 and we repeat the same calculus until Estimate (7.8), which we treat now as follows. Using H"older twice \((1/q + 1/q' = 1/2)\), Gagliardo-Nirenberg and than Young inequalities, we get (recall \( V := \mathbb{H}^{2,2}(O) \))

\[
|v \cdot (B(u) - B(v), u - v)| \leq \|(u - v) \nabla v\|_{L^2}^2 \leq |u - v|_{L^2}^2 \leq |u - v|_{L^2}^2 \leq c|v \cdot \nabla v|_{L^2}^2 \leq c|v \cdot \nabla v|_{L^2}^2 |u - v|_{L^2}^2 \leq c|v \cdot \nabla v|_{L^2}^2 |u - v|_{L^2}^2 + c|u - v|_{L^2}^2.
\]

(8.3)

We take \( r'(t) := c(1 + |\nabla v(t)|_q) \) with relevant constant \( c > 0 \). Then, we apply the whole machinery as in Section 5 and the estimations as in Section 7 to get the existence of the global solution. To prove the uniqueness of the solution in the set \( \mathcal{E} \), we follow the steps as in Section 5. In particular, in Formula (5.46), we estimate the term \( \langle B(w(s)), w^1(s) \rangle = -\langle B(w(s), w^1(s)), w(s) \rangle \) using (8.3). The existence and the uniqueness hold, therefore the local solution is global and unique. The estimate (4.10) is obtained from (8.1).

For the general case (3.62), if a maximal local weak solution enjoys (3.13), then we have \( \mathcal{E} \neq \emptyset \) and thus we follow the proof above (for \( O = T^2 \)) to get the results. If a maximal local weak solution enjoys Condition (3.14), then the set \( \mathcal{E}_1 \neq \emptyset \), where \( \mathcal{E}_1 \) is the set of predictable stochastic processes \((v(t), t \in [0, T])\) (or the extension of \( v \) in the case \( v \) is defined up to a stopping time) satisfying that there exists a predictable stopping time \( \tau \) such that \( v \in L^2(O \times [0, \tau]; \mathbb{H}^{2,2}(T^2)) \) and can be extended (we keep the same notation) to \( v \in L^{2,2,2}(O \times [0, T]; \mathbb{H}^{2,2}(T^2)) \) uniformly, i.e. with the norm of \( v \) in this space is uniformly bounded independently of the extension. Now, we can continue from Estimate (7.8) and follow the proof as above and as in Section 5.

**Appendix A. Equivalence between FSNS and SFNS equations.**

Recall that we have proved in Section 2 that Equation (2.23) with \( A_{d := (A^S)^{\frac{\alpha}{\alpha}}} \) is well defined. This equation can be seen as the fractional version of the stochastic Navier-Stokes equation (FSNSE). A stochastic version of the fractional Navier-Stokes equation (SFNSE) can also be constructed by taking \( A_{d := (\Delta)^{\frac{\alpha}{\alpha}}} \) on \( L^2(O) \). For simplicity, let us keep in mind for a short time that the two equations, FSNSE and SFNSE, are different. Later on, we shall prove that they are equivalent. By a fractional Navier-Stokes equation (FNSE), we mean Equation (2.1), with \( \Delta \) replaced by \( (\Delta)^{\frac{\alpha}{\alpha}} \). The SFNSE is a stochastic perturbation of FNSE. Thanks to theorems 2.1 and to the calculus above, the SFNSE is also well defined. As the derivation of equations describing physical phenomena is mainly based on the deterministic case it is intuitively seen that the stochastic version of the fractional Navier-Stokes equation is more suitable for physical modeling, rather than the fractional version of the stochastic Navier-Stokes equation, see e.g. [43] and see also a direct proof in Section 5.

Combining this
statement with the results in Theorem 2.3 we conclude that
\[
D((A^S)^\frac{\alpha}{2}) = D(\Pi (\Delta)^\frac{\alpha}{2} \Pi) = H^\alpha_d(T^d) \cap L^q(T^d),
\]
(A.1)
\[
(A^S)^\alpha u = \Pi (\Delta)^\alpha u = (\Delta)^\alpha u, \quad \forall u \in D((A^S)^\alpha).
\]
This proves that the FSNSE and the SFNSE defined on the torus are "equivalent". In the case \( O \subset \mathbb{R}^d \) being bounded, the Stokes operator \( A^S \) is not equal to \(-\Delta\). In fact, as we can not in general expect that if \( u \in D(A^S) \) we also have \( \Delta u \cdot \vec{n} = 0 \) on \( \partial\Omega \) it is not obvious whether or not \( \Delta u \in L^q(O) \). Our claim here is that \( (A^S)^\alpha = \Pi (A^D)^\alpha \Pi \). In deed, thanks to (2.9) and (2.10), it is easy to deduce that \( A^S = \Pi A^D \Pi \), see also [26]. Using Theorem 2.3 we infer that
\[
D(\Pi (A^D)^\frac{\beta}{2} \Pi) = D((A^D)^\frac{\beta}{2}) \cap L^q(O) = D((A^S)^\frac{\beta}{2}).
\]
Moreover, using the definition of the negative power of \( A^S \) and \( A^D \) via the resolvent, see e.g. [31, 56] and the definition of the Helmholtz projection, we infer that
\[
(A^D)^{-\frac{\beta}{2}} \Pi^{-1} u = \frac{1}{2\pi i} \int_{\Gamma} z^{-\frac{\beta}{2}} (A^D - z I_{L^q})^{-1} \Pi^{-1} udz, \quad \forall u \in L^q(O),
\]
(A.2)
where \( \Gamma \) is the path running the resolvent set from \( \infty e^{-i\theta} \) to \( \infty e^{i\theta}, \; 0 < \theta < \pi \), avoiding the negative real axis and the origin and such that the branch \( z^{-\frac{\beta}{2}} \) is taken to be positive for real positive values of \( z \) and \( I_X \) is the identity on the space \( X \). The integral in the RHS of (A.3) converges in the uniform operator topology. Furthermore, we have for all \( u \in L^q(O),
\]
\[
(A^D)^{-\frac{\beta}{2}} \Pi^{-1} u = \frac{1}{2\pi i} \int_{\Gamma} z^{-\frac{\beta}{2}} (\Pi A^D - z \Pi)^{-1} udz = \frac{1}{2\pi i} \int_{\Gamma} z^{-\frac{\beta}{2}} (A^S - z I_{L^q})^{-1} udz =: (A^S)^{-\frac{\beta}{2}}.
\]
(A.3)
As the operators \( (A^S)^{-1} \) and \( (A^D)^{-1} \) are one-to-one this achieved the proof of the equivalence between the FSNSE and the SFNSE.

**Appendix B. The Biot-Savart’s Law and the Corresponding Fractional Stochastic Vorticity Equation.**

In this appendix we consider only the case \( d = 2 \), for the multidimensional case see e.g. [10, Chap.3], [17, Chap.2], [85] and [72]. The Biot-Savart law determines the velocity \( u \) from the vorticity \( \theta \). This law is given as a pseudo-differential operator of order \(-1\) in the cases \( O = \mathbb{R}^d \) and \( O = T^d \). The case \( O \subset \mathbb{R}^d \), as mentioned before is much involved. Here we give a survey and some results about this law in the cases \( O \subset \mathbb{R}^d \) and \( O = T^d \) than we move to the derivation of the stochastic vorticity equation for the case \( O = T^d \). A generalization of the Biot-Savart’s law to a nonlocal pseudo-differential operators of fractional order \( \gamma \leq 0 \) has been investigated in [15]. We define the operator "curl" as follow, see Preliminary Notations,
\[
curl : v \in H^q_2(O) \rightarrow H^{1-q}_1(O) \ni \theta = \text{curl} v := \partial_1 v_2 - \partial_2 v_1, \; \beta \in \mathbb{R}, \; 1 < q < \infty.
\]
(B.1)
We introduce the stream function \( \psi \), as the solution of the Poisson equation endowed with a relevant boundary condition in the case \( O \subset \mathbb{R}^2 \) being bounded. In deed, we conclude from Dirichlet boundary condition of the velocity \( u \) and the third equation in (B.2) bellow, that \( \psi \) should satisfy vanishing Neumann boundary condition. Therefore, \( \psi/\partial \partial O = \text{const.} \)
We suppose that this constant is null, for further discussion, see e.g. [48]. The problem is then formulated as follow,

\[
\begin{align*}
\Delta \psi &= \theta, \\
\psi/\partial O &= 0, \\
u &= \nabla^\perp \psi, \quad \text{and} \quad \nabla^\perp := (-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}).
\end{align*}
\]

The formulation (B.2) is still valid for \( O = \mathbb{T}^2 \) without the boundary condition. Let us denote by \( A_1 \) either the Laplacian on \( O = \mathbb{T}^2 \) or the Laplacian with Dirichlet boundary condition on \( O \subset \mathbb{R}^2 \), then we formulate the recuperation problem for both cases as

\[
\begin{cases}
A_1 \psi = \theta, \\
u &= \nabla^\perp \psi.
\end{cases}
\]

Problem (B.3) is well posed, see also Section 2. Recall that for \( O = \mathbb{T}^2 \), the wellposdness is guaranteed thanks to the vanishing average condition for the torus.

The velocity \( u \) is obtained by a direct calculus,

\[
u(t,x) = \nabla^\perp A_1^{-1} \theta(t,x) = \int_O \nabla^\perp g_O(x,y) \theta(t,y) dy,
\]

where \( g_O \) is the Green function corresponding to the Poisson equation with Dirichlet boundary conditions for \( O \) bounded. In the case \( O = \mathbb{T}^2 \), the Green function \( g_{\mathbb{T}^2} \) is explicitly given by

\[
g_{\mathbb{T}^2}(x,y) := -\frac{1}{(2\pi)^2} \sum_{k \in \mathbb{Z}^2} \frac{1}{|k|^2} e^{k(x-y)}, \quad x,y \in \mathbb{T}^2.
\]

Moreover,

**Lemma B.1.** The operator

\[
R^1 : H^{\beta,q}_1(O) \to H^{\beta+1,q}_2(O)
\]

\[
\theta \mapsto R^1 \theta := u = \nabla^\perp A_1^{-1} \theta = \int_O \nabla^\perp g_O(\cdot,y) \theta(y) dy
\]

is well defined and bounded for all \( 1 < q < \infty \) and \( \beta \in \mathbb{R} \).

**Proof.** In fact,

\[
|u|_{H^{\beta+1,q}_2} \leq c |\nabla^\perp A_1^{-1} \theta|_{H^{\beta+1,q}_2} \leq c |\partial_j A_1^{-1} \theta|_{H^{\beta+1,q}_2} \leq c |\theta|_{H^{\beta,q}}.
\]

One can also use the representation of \( u \) via Green function (recall that \( \Delta_x g_O(x,y) = \delta_x(y) \)). For \( O = \mathbb{T}^2 \), it is also convenient to remark that \( R^1 \) is a pseudo-differential operator of Calderon-Zygmund Reisz type, see definitions, results and further discussions in [15]. In deed, we can rewrite \( R^1 = -R^\perp (-\Delta)^{-\frac{1}{2}} \), where \( R \) is Riesz transform and \( R^\perp := (-R_2, R_1) \). The proof of the above statement for a larger class of operators on \( \mathbb{T}^d, d \in \mathbb{N}_0 \) which includes \( R^1 \) can be found in [15].

The following result characterizes an intrinsic property between \( \text{curl} v \) and the Sobolev regularity of \( v \).

**Lemma B.2.** Let \( O \subset \mathbb{R}^2 \) bounded or \( O = \mathbb{T}^2 \), \( 1 < q < \infty \) and \( \beta \in \mathbb{R} \). Then there exists a constant \( c > 0 \) such that for all \( v \in H^{\beta,1,q}_2(O) \)

\[
|\nabla v|_{H^{\beta,q}} \leq c |\text{curl} v|_{H^{\beta,q}} \leq |\nabla v|_{H^{\beta,q}}.
\]
Proof. Using the definition of the curl operator, the sobolev spaces and Lemma \[\text{(B.1)}\] we infer that there exists $c > 0$ such that
\[
|\nabla v|_{H^\beta,q} \leq |\partial_j v_i|_{H^{\beta,q}} \leq c|v|_{H^{\beta+1,q}} \leq c|\text{curl}v|_{H^\beta,q}.
\]
Moreover, we have,
\[
|\text{curl}v|_{H^\beta,q} \leq |\partial_j v_i|_{H^{\beta,q}} \leq |\nabla v|_{H^\beta,q}.
\]
\[
\square
\]

Remark 3. If we assume that $v$ is of divergence free, i.e. $v \in \mathbb{H}^\beta_q(T^d)$, $d \in \mathbb{N}_1$, $\Omega = T^2$, $1 < q < \infty$ and $\beta \in \mathbb{R}_+$, then it is easy to adapt the proof of \[\text{[36, Lemma 3.1]}\].

Now, we derive the stochastic vorticity equation. Let $(u, \tau)$ be a maximal weak solution of FSNSE satisfying \[\text{(3.8)}\], up to the stopping time $\tau$. First, we claim that for $P - a.s.$ the following stochastic integral \[\int_0^{\tau \wedge T} \sum_{k \in \Sigma} \text{curl}\sigma^k(u(s))d\beta_k(s)\], with $\sigma^k(u(s))$ is given by
\[
(\text{B.11}) \quad \sigma^k(u) := G(u)Q^k_\beta e_k = q^k_\beta G(u)e_k, \quad \text{for } k \in \Sigma,
\]
is well defined and
\[
(\text{B.12}) \quad \int_0^{\tau \wedge T} \sum_{k \in \Sigma} \text{curl}\sigma^k(u(s))d\beta_k(s) = \text{curl} \int_0^{\tau \wedge T} G(u(s))dW(s), \quad \forall t \in [o,T].
\]
In fact, using the stochastic isometry, Assumption (C) \[\text{(2.43)}\] with $2 \leq q < \infty$ and $\delta \in \{0,1\}$ and \[\text{[35]}\], we infer that for $\beta$ equals either 1 or 0,
\[
\begin{align*}
\mathbb{E} \left| \int_0^{\tau \wedge T} \sum_{k \in \Sigma} \text{curl}\sigma^k(u(s))d\beta_k(s) \right|^2_{H^{\beta-1,q}} & \leq c \mathbb{E} \int_0^{\tau \wedge T} \sum_{k \in \Sigma} |\partial_j \sigma^k_i(u(s))|^2_{H^{\beta-1,q}} ds \\
& \leq c \mathbb{E} \int_0^{\tau \wedge T} |\sigma^k(u(s))|^2_{H^{\beta,q}} ds \leq c \mathbb{E} \int_0^{\tau \wedge T} |G(u(s))|^2_{RQ(H^{\beta,q})} ds \\
& \leq c \int_0^{\tau \wedge T} (1 + \mathbb{E}|u(s)|^2_{\mathbb{H}^{\beta,q}}) ds < \infty.
\end{align*}
\]
Moreover, thanks to \[\text{[2.36]}\] and \[\text{(B.11)}\], we infer on one hand that
\[
\begin{align*}
\text{curl} \sum_{k \in \Sigma_n} \int_0^{\tau \wedge T} \sigma^k(u(s))d\beta_k(s) & \rightarrow \text{curl} \int_0^{\tau \wedge T} \sum_{k \in \Sigma} \sigma^k(u(s))d\beta_k(s), \quad \text{in } L^2(\Omega, \mathcal{D}'(O)),
\end{align*}
\]
where $(\Sigma_n)_n$ is a sequence of subsets converging to $\Sigma$ and $\mathcal{D}'(O)$ is the dual of $\mathcal{D}(O)$. On the other hand, using the linearity of the operator curl, the stochastic isometry identity, \[\text{(2.36)}\], Assumption (C), \[\text{[35]}\] and \[\text{(B.13)}\], we end up with
\[
\begin{align*}
\text{curl} \sum_{k \in \Sigma_n} \int_0^{\tau \wedge T} \sigma^k(u(s))d\beta_k(s) & = \sum_{k \in \Sigma_n} \int_0^{\tau \wedge T} \text{curl}\sigma^k(u(s))d\beta_k(s) \\
& \rightarrow \int_0^{\tau \wedge T} \sum_{k \in \Sigma} \text{curl}\sigma^k(u(s))d\beta_k(s), \quad \text{in } L^2(\Omega, \mathcal{D}'(O)).
\end{align*}
\]
The uniqueness of the limit confirm the result. We use the following notation
\[
(\text{B.16}) \quad \tilde{G}(\theta) := \text{curl}G(\mathcal{R}^1(\theta)).
\]
Using the definition of Helmholtz projection, in particular, the fact that $\mathcal{N}_\theta \subset \text{Ker}(\text{Curl})$, an elementary calculus yields to

\begin{equation}
\text{curl} B(u) = u \cdot \nabla \theta.
\end{equation}

Now, we assume that $O = \mathbb{T}^2$, using Fourier transform, it is easy to prove that

\begin{equation}
\text{curl} A_{\alpha} u = (-\Delta)\Phi \text{curl} u, \ \forall u \in D(A_{\alpha}).
\end{equation}

In fact the relation above is also true for all $u \in H^{\beta + \alpha}(\mathbb{T}^2), \beta \in \mathbb{R}$. Applying the operator curl on the integral representation of Equation (2.23) stopped at the stopping time $\tau$ and using the calculus above, we infer that if $(u, \tau)$ is a local weak solution of (2.23), then $\theta := \text{curl} u$ is a weak (strong in probability) solution of

\begin{equation}
\begin{cases}
d\theta(t) = (-A_{\alpha} \theta(t) + u(t) \cdot \nabla \theta(t)) \, dt + \tilde{G}(\theta(t)) \, dW(t), \ 0 < t \leq \tau.
\end{cases}
\end{equation}

By the same way, we can prove that if $(u, \tau)$ is a local mild solution of (2.23), then the same calculus above is still valid and $\theta := \text{curl} u$ is a mild solution to equation (1.13). In the proof of this case, we use the commutativity property between the operators $\partial_j$ and the semigroup $(e^{-t A_{\alpha}})_{t \geq 0}$. A general formula of Equation (B.19) has been studied in [15].

**APPENDIX C. SOME SOBOLEV INEQUALITIES.**

**C.1. Sobolev pointwise multiplication on bounded sets.** Assume that $O \subset \mathbb{R}^d$ is a bounded $C^\infty$ domain, (recall, domain means an open subset, see e.g. [79] 5.2 p43). The notation $A^s_{pq}(\mathbb{R}^d), s \in \mathbb{R}, 0 < q \leq \infty, 0 < p < \infty$, stands either for Triebel-Lizorkin spaces $F^s_{pq}(\mathbb{R}^d)$ or for Besov spaces $B^s_{pq}(\mathbb{R}^d)$, see the definition in [61] p.8. We know that, see e.g. [61] Proposition Tr.6, 2.3.5, p 14,

\begin{equation}
F^s_{p2}(\mathbb{R}^d) = H^s(\mathbb{R}^d), \ 1 < p < \infty, \ s \in \mathbb{R},
\end{equation}

\begin{equation}
F^s_{pp}(\mathbb{R}^d) = B^s_{pp}(\mathbb{R}^d) = W^s(\mathbb{R}^d), \ 1 \leq p < \infty, \ 0 < s \neq \text{integer},
\end{equation}

where $H^s(\mathbb{R}^d)$ is the Bessel potential spaces or called also Sobolev spaces of fractional order and $W^s(\mathbb{R}^d), \ 1 \leq p < \infty, \ 0 < s \neq \text{integer}$ is Sobolev spaces. We define Triebel-Lizorkin and Besov spaces $A^s_{pq}(O)$ on bounded sets by, see e.g. [79] Definition 5.3. p 44

\begin{equation}
A^s_{pq}(O) = \{ f \in D'(O); \ \text{there is a} \ g \in A^s_{pq}(\mathbb{R}^d), \ \text{with} \ g/O = f \ \text{in distribution sense}\},
\end{equation}

endowed with the norm

\begin{equation}
|f|_{A^s_{pq}(O)} = \inf_{g \in A^s_{pq}(\mathbb{R}^d), \ g/O = f} |g|_{A^s_{pq}(\mathbb{R}^d)}.
\end{equation}

The relations in (C.1) still also valid for bounded sets, see e.g. [15] 5.8 p 52]. Our main theorem is the following

**Theorem C.1.** Let $p, s, q, p_i, s_i, q_i, i = 1, 2$, such that the following pointwise multiplication is satisfied for $A^s_{pq_i}(\mathbb{R}^d)$

\begin{equation}
|f_1 f_2|_{A^s_{pq}} \leq c |f_1|_{A^{s_1}_{pq_1}} |f_2|_{A^{s_2}_{pq_2}}.
\end{equation}

Then Inequality (C.4) is also valid for $O \subset \mathbb{R}^d$ being a bounded open $C^\infty$ set.
Proof. Let \( f_i \in A_{r_i}^s(O) \), then
\[
(C.5) \quad |f_1f_2|_{A_{r_i}^s(O)} = \inf_{g \in A_{r_i}^{r_i}(\mathbb{R}^d), g/O = (f_1f_2)} |g|_{A_{r_i}^{r_i}(\mathbb{R}^d)} \lesssim \inf_{g_i \in A_{r_i}^{r_i}(\mathbb{R}^d), g_i/O = f_i} |g_1g_2|_{A_{r_i}^{r_i}(\mathbb{R}^d)}.
\]
Applying Estimate (C.4), we infer that
\[
(C.6) \quad |f_1f_2|_{A_{r_i}^s(O)} \leq c \inf_{g_i \in A_{r_i}^{r_i}(\mathbb{R}^d), g_i/O = f_i} (|g_1|_{A_{r_i}^{r_i}(\mathbb{R}^d)}|g_2|_{A_{r_i}^{r_i}(\mathbb{R}^d)}) \lesssim c |f_1|_{A_{r_i}^{r_i}(O)} |f_2|_{A_{r_i}^{r_i}(O)}.
\]

C.2. Sobolev embedding.

**Theorem C.2.** Let \( O \) be either the whole space \( \mathbb{R}^d \), or the torus \( \mathbb{T}^d \), or an arbitrary domain \( O \subset \mathbb{R}^d \). If \( t \leq s \) and \( 1 < p < q \leq \frac{dp}{d-sp} < \infty \), then
\[
(C.7) \quad H^{s,p}(O) \hookrightarrow H^{t,q}(O).
\]

**Proof.** For the proof see [2] Theorem 7.63. p221 + 7.66 p222]. For \( O = \mathbb{R}^d \) and \( q = \frac{dp}{d-sp} \), see [69] Theorem 1, p 119 or Theorem 2 p 124] and [74] Proposition 6.4. p 24]. For \( O = \mathbb{T}^d \), see e.g. [74] pp 23-24].

As a consequence, we have
\[
(C.8) \quad H^{\frac{d}{2}}(O) \hookrightarrow H^{\frac{d}{2} + \frac{d}{q},q}(O), \quad \forall q \geq 2.
\]
See also the above result for the Sobolev solenoidal spaces in [3 Theorem 3.10].

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