ON THE DEGREE OF GLOBAL SMOOTHING MAPPINGS FOR SUBANALYTIC SETS

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ABSTRACT. Let $X \subset \mathbb{R}^n$ be a subanalytic set of dimension $k$, let $U$ be an open subset of the smooth part of $X$ of dimension $k$ and let $W$ be a connected component of $U$. In this work we present a criterion for any global smoothing section $\Gamma := (X', \varphi, U)$ of $X$ to have even degree over $W$.

1. Introduction

In [BP18] Bierstone and Parusiński proved the following two remarkable global smoothing results for subanalytic sets. The term ‘analytic’ means ‘real analytic’. Let $V$ be an analytic manifold of dimension $n$, and let $X$ be a closed subanalytic subset of $V$ of dimension $k$.

**Theorem A** ([BP18, Thm. 1.1] Non-embedded global smoothing). There exist an analytic manifold $X'$ of pure dimension $k$, a proper analytic mapping $\varphi : X' \to V$, and a smooth open subanalytic subset $U$ of $X$ such that:

1. $\varphi(X') \subset X$.
2. $\dim(X' \setminus U) < k$ and $\varphi^{-1}(X' \setminus U)$ is a simple normal crossings hypersurface $B'$ of $X'$.
3. For each connected component $W$ of $U$, $\varphi^{-1}(W)$ is a finite union of subsets open and closed in $\varphi^{-1}(U)$, each mapped isomorphically onto $W$ by $\varphi$.

**Theorem B** ([BP18, Thm. 1.2] Embedded global smoothing). There exist an analytic manifold $V'$, a smooth closed analytic subset $X' \subset V'$ of dimension $k$, a simple normal crossings hypersurface $B' \subset V'$ transverse to $X'$, and a proper analytic mapping $\varphi : V' \to V$ such that:

1. $\dim(\varphi(B')) < k$.
2. The restriction $\varphi|_{V' \setminus B'}$ is finite-to-one and of constant rank $n$;
3. $\varphi$ induces an isomorphism from a union of connected components of $X' \setminus B'$ to a smooth open subanalytic subset $U \subset X$ such that $\dim(X' \setminus U) < k$.

We give a couple of remarks motivating our study. Although the previous results are global, the techniques involved in their prove are local. Indeed, in [BP18, Sec. 2.3] the authors provide a partition of the analytic manifold $V$ into a countable number of semianalytic cells in general position with respect to $X$ and then they develop explicit desingularization techniques for these cells with respect to the global behaviour of $V$. More in detail, in [BP18, Sec. 2.2] the authors develop a desingularizing procedure for a semianalytic $n$-cell $C$ of $V$ by explicitly finding an analytic subset $Z_C$ of $V \times \mathbb{R}^m$, for some $m \in \mathbb{N}$ depending on the number of inequalities defining $C$, a map $\varphi_C : Z_C \to C$ and an open semianalytic subset $U_C$ of $C$ such that $\varphi_C^{-1}(U_C)$ is a $2^m$ covering of $U_C$ and $\dim(C \setminus U_C) < k$. Then the authors apply desingularization techniques in the sense of [BM97] to $Z_C$ finding a smoothing of the cell $C$. Thus, we see that the smoothing map $\varphi_C$ of a single cell $C$ of $V$ is even-to-one over $U_C$. Since the global maps $\varphi$ in Theorem A and Theorem B are constructed in terms of the local maps $\varphi_C$, we deduce that $\varphi$ is even-to-one over each open set $U_C$, hence, in particular, $\varphi$ is even-to-one over each intersection $U_C \cap X$.

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Let us give a definition.

**Definition 1.1.** Let $X'$, $\varphi$, $U$ and $W$ be as in the previous Theorem A, that is, $X'$ is an analytic manifold of pure dimension $k$, $\varphi : X' \to V$ is a proper analytic mapping, $U$ is an open subset of the smooth part of $X$ of dimension $k$ and $W$ is a connected component of $U$ such that $\varphi(X') \subset X$, $\dim(X' \setminus U) \leq k$, $\varphi^{-1}(X' \setminus U)$ is a simple normal crossings hypersurface of $X'$ and $\varphi^{-1}(W)$ is a finite union of subsets open and closed in $\varphi^{-1}(U)$, each mapped isomorphically onto $W$ by $\varphi$. We call the triple $\Gamma := (X', \varphi, U)$ global smoothing section of $X \subset V$ and the finite positive number of subsets open and closed in $\varphi^{-1}(U)$, each mapped isomorphically onto $W$ by $\varphi$, as the degree of $\Gamma$ over $W$.

Theorem A asserts that global smoothing sections of $X \subset V$ always exist.

Thanks to [BP18, Rmk. 2.6], if $V = \mathbb{R}^n$ and $X$ is a closed semialgebraic subset of $\mathbb{R}^n$, then Theorem B can be strengthened by requiring the mapping in (2) to be injective. On the other hand, in the setting of Theorem A, it is not possible in general to choose a global smoothing section $(X', \varphi, U)$ of $X \subset \mathbb{R}^n$ whose degree in each connected component of $U$ is equal to 1, as it happens in the case of Hironaka’s resolution of singularities, see Example 2.5 below.

The aim of this note is to give a criterion for the evenness of the degree of global smoothing sections on the connected components over an arbitrary open subset $U$ of the smooth part of $X$ of dimension $k$. This criterion aims to be useful, somehow, in producing counterexamples about the existence of a global smoothing section with $U$ to be the entire smooth part of $X$ of dimension $k$, as Bierstone and Parusiński assert to believe in [BP18, p. 3117] without explicit examples.

2. **The evenness criterion, consequences and examples**

By Whitney’s embedding theorem we can assume that the analytic manifold $V$ coincides with $\mathbb{R}^n$. Let $X$ be a subanalytic subset of $\mathbb{R}^n$ and let $k \in \mathbb{N}$. Recall that a point $x \in X$ is smooth of dimension $k$ if there exists an open neighborhood $N$ of $x$ in $\mathbb{R}^n$ such that $X \cap N$ is an analytic submanifold of $\mathbb{R}^n$ of dimension $k$, see [BM88, Def. 3.3]. The set of all points of $X$ of dimension $k$ is an open subset of $X$ and an analytic submanifold of $\mathbb{R}^n$ of pure dimension $k$.

Let us introduce the concept of nonbounding equator for subanalytic sets.

**Definition 2.1.** Let $X$ be a closed subanalytic subset of $\mathbb{R}^n$ of dimension $k$, let $W$ be an open subset of the smooth part of $X$ of dimension $k$ and let $Y$ be a subset of $W$. We say that $Y$ is a nonbounding equator of $W$ in $X$ if it satisfies the following properties:

(i) $Y$ is a compact $C^\infty$ submanifold of $\mathbb{R}^n$ of dimension $k - 1$.

(ii) $Y$ does not bound, that is, it is not the boundary of a compact $C^\infty$ manifold with boundary.

(iii) $Y$ has a collar in $W$, that is, there exists a $C^\infty$ map $\psi : Y \times (-1, 1) \to W$ such that the image $T := \psi(Y \times (-1, 1))$ of $\psi$ is an open neighborhood of $Y$ in $W$, the restriction $\psi : Y \times (-1, 1) \to T$ is a $C^\infty$ diffeomorphism and $\psi(Y \times \{0\}) = Y$.

(iv) There exists a relatively compact open subset $K$ of $X$ such that $\partial K := \overline{K} \setminus K = Y$ and $\overline{K} \cap T = \psi(Y \times (-1, 0))$. Here $\overline{K}$ denotes the closure of $K$ in $X$.

If such a $Y$ exists, we say that $W$ has a nonbounding equator in $X$.

The next lemma gives an alternative description of the notion of nonbounding equator. We keep the notations of Definition 2.1.

**Lemma 2.2.** The set $Y$ is a nonbounding equator of $W$ in $X$ if and only if there exists a continuous function $h : X \to \mathbb{R}$ with the following properties:

(1) There exist an open neighborhood $Z$ of $Y$ in $W$ and $\epsilon > 0$ such that the restriction $h' := h|_Z : Z \to \mathbb{R}$ is a $C^\infty$ function, $h^{-1}([-\epsilon, \epsilon])$ is a compact neighborhood of $Y$ in $Z$ containing no critical points of $h'$ and $h^{-1}(0) = Y$. 


(2) $Y$ does not bound.
(3) The subset $h^{-1}((-\infty, 0])$ of $X$ is compact.

Proof. Let $X$, $k$, $W$, $Y$, $\psi : Y \times (-1,1) \to W$ and $K$ be as in Definition 2.1 and let $\pi : Y \times (-1,1) \to (-1,1)$ be the projection onto the second factor. Let us prove that (1)-(3) of Lemma 2.2 are satisfied. Define $Z := \psi(Y \times (-1/2, 1/2))$ and $h' : Z \to \mathbb{R}$ as $h'(x) := (\pi \circ \psi)^{-1}(x)$. Then extend $h'$ to the whole $X$ as follows: define $h : X \to \mathbb{R}$ as $h(x) := -1/2$ if $x \in K \setminus Z$, $h(x) := h'(x)$ if $x \in Z$ and $h(x) := 1/2$ otherwise. Fix $\epsilon > 0$. Observe that $h|_Z = (\pi \circ \psi^{-1})|_Z$, thus $h|_Z$ has no critical points, $h^{-1}([-1/4, 1/4]) = \psi(Y \times [-1/4, 1/4])$, which is compact and contains $Y$, and $h^{-1}((-\infty, 0]) = K \cup Y = \overline{K}$.

On the other hand, assume that $X$, $Y$, $W$, $Z$ and $h$ satisfy conditions (1)-(3) of Lemma 2.2. By (1) of Lemma 2.2 and [Hir76, Cor. 2.3, p. 154], $h|_{h^{-1}([-\epsilon, \epsilon])}$ induces the existence of a collar of $Y$ in $W$, as in (iii) of Definition 2.1. Moreover, by (1) and (3) of Lemma 2.2, $K := h^{-1}((-\infty, 0))$ satisfies (iv).

Our evenness criterion reads as follows.

**Theorem 2.3.** Let $X$ be a closed subanalytic subset of $\mathbb{R}^n$, let $\Gamma := (X', \varphi, U)$ be a global smoothing section of $X \subset \mathbb{R}^n$ and let $W$ be a connected component of $U$. If $W$ has a nonbounding equator in $X$ then the degree of $\Gamma$ over $W$ is even.

Proof. Let $Y \subset W$ be a nonbounding equator of $W$ in $X$. By Definition 2.1, there is an open neighborhood $T$ of $Y$ in $W$, a diffeomorphism $\psi : Y \times (-1,1) \to T$ such that $\psi(Y \times \{0\}) = Y$ and a relatively compact open subset $K$ of $X$ such that $\partial K = Y$ and $K \cap T = \psi(Y \times (-1,0))$. Since $\Gamma$ is a global smoothing section, $\varphi^{-1}(W)$ consists of a finite disjoint union of open and closed subsets of $\varphi^{-1}(U)$, each mapped isomorphically onto $W$. Hence, each connected component of $\varphi^{-1}(W)$ contains a copy of $Y$ and a copy of the collar $T$ of $Y$ in $W$. By Definition 1.1, the map $\varphi$ is proper, hence $\varphi^{-1}(K)$ is a compact subset of $X'$. Moreover, since $\partial K = Y$, $K \cap T = \psi(Y \times (-1,0))$ and $\varphi$ is a diffeomorphism when restricted to each connected component of $\varphi^{-1}(W)$, we have that $\varphi^{-1}(K)$ is a manifold with boundary whose boundary is the disjoint union of $d$ copies of $Y$, where $d$ denotes the degree of $\Gamma$ over $W$. Since $Y$ is nonbounding, we deduce that $d$ is even since the Stiefel-Whitney numbers of $\bigcup Y$ must be all zero [MS74, Theorem 4.9, p. 52].

As a consequence, the nonexistence of nonbounding equators of the smooth part of $X$ of dimension $k$ is a necessary condition to have global one-to-one smoothings similar to Hironaka’s resolution of singularities.

**Corollary 2.4.** Let $X$ be a closed subanalytic subset of $\mathbb{R}^n$. If the degree of a global smoothing section of $X \subset \mathbb{R}^n$ over $W$ is 1, then $W$ does not have any nonbounding equator in $X$.

Here we present some examples of semialgebraic sets concerning our Theorem 2.3.

**Example 2.5.** Let $X := \mathbb{R}_{\geq 0} := \{x \in \mathbb{R} | x \geq 0\}$. There is a global smoothing section of the whole smooth part of $X$, that is $\Gamma := (X', \varphi, U)$ with $U := \mathbb{R}_{> 0} = \{x \in \mathbb{R} | x > 0\}$, $X' := \{(x,y) \in \mathbb{R}^2 | x = y^2\}$ and $\varphi : X' \to X$ defined as the projection onto the first factor. According to our Theorem 2.3, the degree of the above smoothing section over the whole smooth part of $X$ is 2. But our result says something more, indeed any global smoothing section $\Gamma := (X', \varphi, U)$ of $X$, with $U$ any open subset of the smooth part of $X$, has even degree over any connected component of $U$. Indeed, since $U$ is an open subset of $\mathbb{R}_{> 0}$, every connected component of $U$ has a nonbounding equator $Y$ consisting of a singleton $\{p\}$, with $K := [0, p)$ and the collar $(p - \epsilon, p + \epsilon) \subset U$ of $p$ in $W$, for $\epsilon > 0$ sufficiently small.

**Examples 2.6.** Let $M$ be a connected compact $C^\infty$ manifold of dimension $k - 1$, which does not bound (so $k - 1 \geq 2$): for instance, the real projective plane $\mathbb{P}^2(\mathbb{R})$. By the Nash-Tognoli theorem, [Nas52] and [Tog73], we can assume that $M$ is a compact nonsingular real algebraic subset of some $\mathbb{R}^n$. 

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(1) Consider the standard circumference $S^1 := \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 = 1\}$, the compact nonsingular real algebraic set $X' := M \times S^1 \subset \mathbb{R}^{n+2}$, and the polynomial maps $\pi_1 : X' \to \mathbb{R}^{n+2}$ and $\pi_2 : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$ defined as follows:

$$\pi_1(x, a, b) := (bx, a, b) \quad \text{and} \quad \pi_2(x, a, b) := (x, a, b^2),$$

where $x = (x_1, x_2, \ldots, x_n)$. The set $\pi_1(X')$ is equal to $X'$ with $M \times \{-1, 0\}$ crushed to the point $p := (0, \ldots, 0, -1, 0)$ and $M \times \{1, 0\}$ crushed to the point $q := (0, \ldots, 0, 1, 0)$. The set $X := \pi_2(\pi_1(X'))$ is a semialgebraic subset of $\mathbb{R}^{n+2}$ homeomorphic to the suspension of $M$. Define $X'_\pm := X' \cap \{\pm b > 0\}$ and the polynomial map $\varphi : X' \to \mathbb{R}^{n+2}$ by $\varphi(x, a, b) := \pi_2(\pi_1(x, a, b))$. Observe that $\varphi(X') = X$, $\varphi^{-1}(p) = M \times \{-1, 0\}$, $\varphi^{-1}(q) = M \times \{1, 0\}$, and the restriction of $\varphi$ from $X'_\pm$ to $U := X \setminus \{p, q\}$, namely to the whole smooth part of $X$, is a Nash diffeomorphism between connected Nash manifolds. For more details about Nash functions and Nash manifolds we refer to [BCR98, Sec. 8]. The triple $\Gamma := (X', \varphi, U)$ is a global smoothing section of $X \subset \mathbb{R}^{n+2}$ and $\varphi(M \times \{0, 1\})$ is a nonbounding equator of $W := U$ in $X$. The degree of $\Gamma$ over $W$ is two, in accordance with our Theorem 2.3.

(2) Let $Z' := M \times [-1, 1] \subset \mathbb{R}^{n+1}$, let $\phi : Z' \to \mathbb{R}^{n+1}$ be the polynomial map

$$\phi(x, a) := (x(1 - a^2), a)$$

and let $X$ be the semialgebraic subset $\phi(Z')$ of $\mathbb{R}^{n+1}$. Observe that $X$ is homeomorphic to the suspension of $M$, $\phi^{-1}(z_\pm) = M \times \{\pm 1\}$, where $z_\pm := (0, \ldots, 0, \pm 1)$, the restriction of $\phi$ from $Z'(M \times \{-1, 1\}) = M \times (-1, 1)$ to $U := X \setminus \{z_-, z_+\}$ is a Nash diffeomorphism between connected Nash manifolds (so $\phi$ has degree one over $U$), and $\phi(M \times \{0\})$ is a nonbounding equator of $W := U$ in $X$. However, the triple $(Z', \phi, U)$ is not a global smoothing section of $X \subset \mathbb{R}^{n+1}$, because $Z'$ is not an analytic manifold: it has the nonempty boundary $M \times \{-1, 1\}$.

Nevertheless, the previous construction arises as an explicit case of Theorem B. Let $V := \mathbb{R}^{n+1}$. By [AK92, Cor. 2.5.14, p. 50] we may assume in addition that $M$ is projectively closed, that is $M$ is the zero set $Z_{\mathbb{R}^n}(p)$ in $\mathbb{R}^n$ of some overt polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$. Write $p$ as follows: $p = \sum_{i=0}^d p_i$, where $p_i$ is an homogeneous polynomial of degree $i$. Recall that $Z_{\mathbb{R}^n}(p_d) = \{0\}$ as $p$ is overt. Thus, if $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is the polynomial map $(x, a) \mapsto ((1 - a^2)x, a)$, $Z := \varphi(M \times \mathbb{R})$ and $q(x, a) \in \mathbb{R}[x_1, \ldots, x_n, a]$ is the polynomial $q(x, a) := \sum_{i=0}^d (1 - a^2)^d i p_i(x)$, then $M \times \mathbb{R} = Z_{\mathbb{R}^{n+1}}(p_d)$ and $\varphi(q(x, a)) = (1 - a^2)^d p(x) = 0$ for all $(x, a) \in M \times \mathbb{R}$. It follows that

$$Z = Z_{\mathbb{R}^{n+1}}(q).$$

This proves that $Z$ is algebraic and irreducible, so $Z$ is the Zariski closure of $X$ in $\mathbb{R}^{n+1}$. Thus, we deduce that $X, Z, Y := \{z_-, z_+\}, U, Z'$ and $X'$ constitute an explicit embedded global smoothing as in [BP18, Rmk. 2.6].

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