Bowen topological of generic point for fixed-point free flows

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Abstract

In this paper, we consider reparametrizations of the flows, and study the relationship between Bowen topological entropy and measure entropy for flows. We prove following variational principle: for an ergodic \( \phi \)-invariant measure \( \mu \),

\[
h_{top}^B(\phi, G_\mu(\phi)) = h_\mu(\phi),
\]

where \( G_\mu(\phi) \) is the set of generic points for \( \mu \) and \( h_{top}^B(\phi, G_\mu(\phi)) \) is the Bowen topological entropy on \( G_\mu(\phi) \). This generalizes the classical result of Bowen in 1973.

1 Introduction

Let \( X \) be a compact space with metric \( d \) and Borel \( \sigma \)-algebra \( B \), \( \phi : X \times \mathbb{R} \to X \) a continuous flow on \( X \), that is, \( \phi_t : X \to X \) is a homeomorphism given by \( \phi_t(x) = \phi(x, t) \) for each \( t \in \mathbb{R} \) and satisfies \( \phi_t \circ \phi_s = \phi_{t+s} \) for each \( t, s \in \mathbb{R} \). A Borel probability measure \( \mu \) on \( X \) is called \( \phi \)-invariant if for any Borel set \( B \in B \), it holds \( \mu(\phi_t(B)) = \mu(B) \) for all \( t \in \mathbb{R} \). It is called ergodic if any \( \phi \)-invariant Borel set has measure 0 or 1. The set of all Borel probability measures, all \( \phi \)-invariant Borel probability measures and all ergodic \( \phi \)-invariant Borel probability measures on \( X \) are denoted by \( M(X) \), \( M_\phi(X) \) and \( E_\phi(X) \), respectively. We shall assume throughout the paper that \( \phi \) is a continuous real flow on a compact metric space \( X \) without fixed points.

Assume that \( I \) is a closed interval which contains the origin, a continuous map \( \alpha : I \to \mathbb{R} \) is said to be a reparametrization if it is a homeomorphism onto its image and \( \alpha(0) = 0 \). Denote the set of all such reparametrizations on \( I \) by \( Rep(I) \). For a flow \( \phi \) on \( X \), given \( x \in X, t \in \mathbb{R}^+ \) and \( \epsilon > 0 \), we put

\[
B(x, t, \epsilon, \phi) = \{ y \in X : \exists \alpha \in Rep[0, t] \text{ s.t. } d(\phi_{\alpha(s)}(x), \phi_{\alpha(s)} y) < \epsilon, \forall 0 \leq s \leq t \},
\]
and call such set a \((t, \epsilon, \phi)\)-ball or a reparametrization ball. Clearly, all the reparametrization balls are open sets.

For a flow, it is sometimes useful to represent measure-theoretic entropy of time one map by the whole flow itself. However, an invariant measure for time one map is not, in general, flow-invariant and similarly — an ergodic measure for a flow is not necessarily ergodic for time one map. Thus there exist differences between flows and their time one maps. There are some nice works attempting to give a proper definition of the entropy of a flow. For a flow \((X, \phi)\), Abramov\cite{1} proved that \(h^\mu(\phi_t) = |t|h^\mu(\phi_1)\) for all \(t \in \mathbb{R}\) which is called the Abramov entropy formula. Bowen \cite{4} gave the definition of topological entropy for one parameter flows on compact metric spaces. To investigate the topological entropies of mutually conjugate expansive flows, Thomas \cite{14} defined the entropy for flows raised from allowing reparametrizations of orbits. Subsequently, he showed that his definition of entropy is equivalent to Bowen’s definition for any flow without fixed points on compact metric spaces in \cite{15}. Sun and Vergas studied measure-theoretic entropy of flows in \cite{12, 11}. Dou etc. \cite{6} generalized the result in \cite{7} and established a variational principle.

In 1973, Bowen\cite{4} introduced a definition of topological entropy of subset for \(\mathbb{Z}\)- or \(\mathbb{N}\)-actions systems, which later on was known as the Bowen topological entropy. Moreover, he proved the following theorems:

- If \(\mu\) is an invariant Borel probability measure and \(Y\) is a subset with \(\mu(Y) = 1\), then the Bowen topological entropy of \(Y\) is bigger than the measure-theoretic entropy with respect to \(\mu\);
- If in addition \(\mu\) is ergodic, then the Bowen topological entropy of the set of generic points of \(\mu\) is equal to the measure-theoretic entropy with respect to \(\mu\).

In this paper we will show that above theorems also hold to fixed-point free flows. The statement are the following. The idea of proof is inspired by Zheng and Chen \cite{16}.

**Theorem 1.1.** Let \((X, \phi)\) be a compact metric flow without fixed points. For \(\mu \in \mathcal{M}_\phi(X)\), if \(Y \subset X\) and \(\mu(Y) = 1\), then \(h^\mu(\phi_1) \leq h^B_{\text{top}}(\phi, Y)\).

**Theorem 1.2.** Let \((X, \phi)\) be a compact metric flow without fixed points and \(\mu \in \mathcal{E}_\phi(X)\). Suppose

\[
G^\mu(\phi) = \{x \in X : \lim_{t \to \infty} \frac{1}{t} \int_0^t f(\phi_\tau x) d\tau = \int_X f d\mu, \text{ for any } f \in C(X)\},
\]

(1)

the set of generic points for \(\mu\), then \(h^B_{\text{top}}(\phi, G^\mu(\phi)) = h^\mu(\phi_1)\).

Moreover, to obtain above theorems, we establish the Brin-Katok’s entropy formula for fixed-point free flow in non-ergodic case.

**Theorem 1.3.** Let \((X, \phi)\) be a compact metric flow without fixed points. For any \(\mu \in \mathcal{M}_\phi(X)\), then \(\mu\)-a.e. \(x \in X\),

\[
\overline{h}_\mu(\phi, x) = h^\mu(\phi, x) \triangleq h^\mu(x)
\]

and \(\int_X h^\mu(x) d\mu = h^\mu(\phi_1)\).
2 Preliminaries

Let $(X, \phi)$ be a flow and $Z \subset X$. For $s \geq 0$, $N \in \mathbb{N}$ and $\varepsilon > 0$, set
\[ M_{N,\varepsilon}^s(\phi, Z) = \inf \sum_i \exp(-s t_i), \]
where the infimum is taken over all finite or countable families of reparametrization balls \( \{ B(x_i, t_i, \varepsilon, \phi) \} \), $x_i \in X$ and $t_i \geq N$ such that $Z \subset \cup B(x_i, t_i, \varepsilon, \phi)$. Then set
\[ M_{\varepsilon}(\phi, Z) = \lim_{N \to \infty} M_{N,\varepsilon}^s(\phi, Z), \quad M^s(\phi, Z) = \lim_{\varepsilon \to 0} M_{\varepsilon}^s(\phi, Z). \]

The Bowen topological entropy $h_{top}^B(\phi, Z)$ is defined as critical value of the parameter $s$, where $M^s(\phi, Z)$ jumps from $\infty$ to $0$, i.e.,
\[ h_{top}^B(\phi, Z) = \inf \{ s : M^s(\phi, Z) = 0 \} = \sup \{ s : M^s(\phi, Z) = \infty \}. \]

Proposition 2.1.

1. If $Z_1 \subset Z_2 \subset X$, then $h_{top}^B(\phi, Z_1) \leq h_{top}^B(\phi, Z_2)$.
2. If $Y_i \subset X$ for $i = 1, 2, \ldots$, then $h_{top}^B(\phi, \bigcup_{i=1}^\infty Y_i) \leq \sup_i h_{top}^B(\phi, Z_i)$.

Moreover they defined lower and upper measure-theoretic entropy for any Borel probability measure $\mu$,
\[ h_\mu(\phi) = \int h_\mu(\phi, x) d\mu \quad \text{and} \quad \overline{h}_\mu(\phi) = \int \overline{h}_\mu(\phi, x) d\mu, \]
where
\[ h_\mu(\phi, x) = \lim_{\varepsilon \to 0} \liminf_{t \to \infty} -\frac{1}{t} \log \mu(B(x, t, \varepsilon, \phi)) \]
and
\[ \overline{h}_\mu(\phi, x) = \lim_{\varepsilon \to 0} \limsup_{t \to \infty} -\frac{1}{t} \log \mu(B(x, t, \varepsilon, \phi)). \]

3 Proof of Theorem 1.3

This section gives the proof of Theorem 1.3. The following lemma plays a key role in our proof.

Lemma 3.1. Let $(X, \phi)$ be a compact metric flow without fixed points. For any $\varepsilon_1 > 0$, there exists $\varepsilon > 0$ such that for any $x, y \in X$, and any closed interval $I$ containing the orig, and any reparametrization $\alpha \in Rep(I)$, if $d(\phi_{\alpha(s)}(x), \phi_s(y)) < \varepsilon$ for all $s \in I$, then it holds that
\[ |\alpha(s) - s| < \begin{cases} \varepsilon_1 |s|, & \text{if } |s| > 1; \\ \varepsilon_1, & \text{if } |s| \leq 1. \end{cases} \]
Now we give the proof of Theorem 1.3, which can be obtained form the following Proposition 3.1 and 3.2. For the proof, we follow the proof originally due to Brin and Katok for $\mathbb{Z}$-actions.

**Proposition 3.1.** Let $(X, \phi)$ be a compact metric flow. For any $\mu \in \mathcal{M}_\phi(X)$, then $\mu$-a.e.

$$\int_X R_\mu(\phi, x)d\mu \leq \frac{1}{|\tau|} h_\mu(\phi_\tau),$$

for all $\tau \in \mathbb{R} \setminus \{0\}$.

**Proof.** Case 1: We only need to show that $t = n\tau$, $\tau > 0$ and $n \in \mathbb{N}$. Otherwise, we can choose $n_t \in \mathbb{N}$ such that $n_t \tau < t < (n_t + 1)\tau$. Then we have

$$B(x, (n_t + 1)\tau, \varepsilon, \phi) \subset B(x, t, \varepsilon, \phi) \subset B(x, n_t\tau, \varepsilon, \phi).$$

Therefore,

$$\frac{1}{t} \int_X \lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\log \mu(B(x, t, \varepsilon, \phi)) d\mu \leq \frac{1}{n_t\tau} \int_X \lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\log \mu(B(x, (n_t + 1)\tau, \varepsilon, \phi)) d\mu \leq \frac{1}{\tau} h_\mu(\phi_\tau).$$

For any $\varepsilon > 0$, choose $\eta > 0$, such that $d(\phi_s x, \phi_s y) < \varepsilon$, $s \in [0, \tau]$ if $d(x, y) < \eta$. For any $x \in X$, we define

$$D(x, n, \eta, \phi_\tau) = \{y : d(\phi_\tau x, \phi_\tau y) < \eta, i = 0, 1, \ldots, n - 1\},$$

$$B_t(x, \varepsilon, \phi) = \{y : d(\phi_s x, \phi_s y) < \varepsilon, \forall 0 \leq s \leq t\}.$$

Then

$$D(x, n, \eta, \phi_\tau) \subset B_t(x, \varepsilon, \phi) \subset B(x, t, \varepsilon, \phi).$$

Choose a finite a partition $\xi$ of $X$, with diam($\xi$) < $\frac{\eta}{2}$. Then by Theorem SMB theorem, for $\mu$-a.e. $x \in X$,

$$\int_X \lim_{n \to \infty} -\frac{\log \mu(\xi_n(x))}{n} d\mu = h_\mu(\phi_\tau, \xi) \leq h_\mu(\phi_\tau)$$

where $\xi_n = \xi \vee \phi_\tau^{-1}\xi \vee \cdots \vee \phi_\tau^{-(n-1)}\xi$. Since $\xi_n(x) \subset D(x, n, \eta, \phi_\tau) \subset B(x, n\tau, \varepsilon, \phi)$,

$$\int_X \lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\frac{\log \mu(B(x, n\tau, \eta, \phi))}{n} d\mu \leq h_\mu(\phi_\tau).$$

So we have

$$\int_X H_\mu(\phi, x)d\mu \leq \frac{1}{\tau} h_\mu(\phi_\tau).$$
Case 2. Consider $\tau < 0, t > 0$.
Then we have $-\tau > 0$, by Case 1 we have
\[
\int_X \bar{h}_\mu(\phi, x) d\mu \leq -\frac{1}{\tau} h_\mu(\phi_{-\tau}) = -\frac{1}{\tau} h_\mu(\phi_{\tau}) = \frac{1}{|\tau|} h_\mu(\phi_{\tau}).
\]
\[\square\]

**Proposition 3.2.** Let $(X, \phi)$ be a compact metric flow without fixed points. For any $\mu \in \mathcal{M}_g(X)$, then $\mu$-a.e. $x \in X$\[
\int_X h_\mu(\phi, x) d\mu \geq \frac{1}{|\tau|} h_\mu(\phi_{\tau}),
\]
for all $\tau \in \mathbb{R} \setminus \{0\}$.

**Proof.** Now we claim that for any $p > 0$, we have
\[
p + \int_X h_\mu(\phi, x) d\mu \geq \frac{1}{|\tau|} h_\mu(\phi_{\tau}).
\]
To prove the above claim, we choose $L \in \mathbb{N}$ such that $L \geq \frac{2 \log 6 + p}{p|\tau|}$. We divide the proof into the following three cases.

**Case 1.** Let $\tau > 0, n \in \mathbb{N}, t = n L \tau$.
Consider the $\sigma$-algebra $\mathcal{T} = \{ A \in \mathcal{B} : \mu(A \triangle \phi_{-L\tau} A) = 0 \}$. Let $\rho : X \rightarrow X/\mathcal{T} := Y$ be the associated projection and $\mu = \int_Y \mu_y d\pi(y)$ be the $\phi_{L\tau}$-ergodic decomposition of $\mu$.
For each $y \in Y$, $\rho^{-1}(y)$ is $\phi_{L\tau}$-invariant and $(\rho^{-1}(y), \phi_{L\tau}, \mu_y)$ is a $\phi_{L\tau}$-ergodic dynamical systems, set $h(y) = h_{\mu_y}(\rho^{-1}(y), \phi_{L\tau})$ is the measure theoretic entropy restricted to the system $(\rho^{-1}(y), \phi_{L\tau}, \mu_y)$. For any $M > 0$, denote by $X_M = \rho^{-1}(h^{-1}([0, M]))$ and $X'_M = \rho^{-1}(h^{-1}([M, \infty)))$. Let $X_\infty = \rho^{-1}(h^{-1}(\infty))$. Then $X = X_M \cup X'_M \cup X_\infty$.

**Lemma 3.2.** (1) For any $M > 0$,
\[
\int_{X_M} p + h_\mu(\phi, x) du \geq \int_{h^{-1}([0, M])} h(y) d\pi(y) L \tau.
\]
(2) For $\mu$ almost every $x \in X_\infty$.
\[
p + \lim_{\delta \to 0} \liminf_{t \to \infty} -\frac{1}{t} \log \mu(B(x, t, \delta, \phi)) = \infty.
\]

**Proof.** If $\mu(X_M) = 0$ and $\mu(X_\infty) = 0$, then (2) and (3) hold respectively. So we may assume that both $\mu(X_M)$ and $\mu(X_\infty)$ are positive.
Take $K \in \mathbb{N}$ to be sufficiently larger and let $\gamma = \frac{M}{K}$. For $k = 0, 1, \cdots, K - 1$, let $B_k = \rho^{-1}(h^{-1}([k \gamma, (k + 1) \gamma]))$ and let $B_\infty = X_\infty$.
For a finite measurable partition of $X$, say $\beta$, denote by $diam(\beta) = \max_{B \in \beta} diam(B)$.
Let $\xi_m = \{ A_1^m, A_2^m, \cdots, A_{c_m}^m \}$ be a sequence of finite measurable partition of $X$ such that
\[ \cdot \quad A^m_1, \ldots, A^m_{e_m - 1} \text{ are piecewise disjoint compact sets;} \]

\[ \cdot \quad A_{e_m} = X \setminus \bigcup_{i=1}^{e_m-1} A_i. \]

\[ \cdot \quad \lim_{m \to \infty} \text{diam}(\xi_m) = 0. \]

Then

\[ \lim_{m \to \infty} h_\nu(\phi_{LT}, \xi_m) = h_\nu(X, \phi_{LT}), \text{ for } \nu \in M(X, \phi_{LT}). \]

By the SMB theorem, for \( \mu \)-a.e. \( x \in X \),

\[ \lim_{n \to \infty} -\frac{1}{n} \log \mu(\bigvee_{i=0}^{n-1} \phi_{LT}^i(x)) = h_{\mu}(\phi_{LT}, \xi_m) = h_{\mu}(\phi_{LT}, \xi_m), \]

where \(\rho^{-1}(y)\) is the ergodic component that contain \( x \), i.e. \( \rho(x) = y \). Hence for \( \mu \)-a.e \( x \in X \), \( \lim_{m \to \infty} h(x, \xi_m) = h(\rho^{-1}(y), \phi_{LT}) = h(y) \), where \( y = \rho(x) \).

For any \( \epsilon > 0 \), such that \( \epsilon < \gamma \), by Egorov’s Theorem, we then can choose \( \xi = \xi_m \) for \( m \) sufficiently large such that up to a subset of \( X \) with small \( \mu \) measure (say, less than \( \epsilon \)), it holds that

\[ h(x, \xi) > \min\left\{ \frac{1}{\epsilon}, h(\rho(x)) - \epsilon \right\} \]

Hence there exists sufficiently large \( N \), whence for each \( k = 0, 1, \ldots, K - 1 \),

\[ \mu(\{ x \in B_k : \forall n' \geq n, -\frac{1}{n} \log \mu(\xi_n(x)) > k\gamma - 2\epsilon \}) > \mu(B_k) - 2\epsilon, \quad (4) \]

and

\[ \mu(\{ x \in B_\infty : \forall n' \geq n, -\frac{1}{n} \log \mu(\xi_n(x)) > \frac{1}{\epsilon} - 2\epsilon \}) > \mu(B_\infty) - 2\epsilon. \quad (5) \]

Let \( \xi = \{A_1, A_2, \ldots, A_q, A_{q+1}\} \) satisfying

\[ \cdot \quad A_1, \ldots, A_q \text{ are piecewise disjoint compact sets;} \]

\[ \cdot \quad A_{q+1} = X \setminus \bigcup_{i=1}^{q} A_i. \]

Let \( \eta_0 = \min\{d(A_i, A_j) : 1 \leq i \neq j \leq q\} \). Fix \( \eta \in (0, \eta_0) \), we choose \( \theta > 0 \) such that

\[ d(\phi_s(z), z) < \frac{\eta}{3} \]

for all \( z \in X, |s| \leq \theta \). By Lemma 3.1 for \( \epsilon_1 = \frac{\theta}{4LT} \), we choose \( \epsilon \in (0, \frac{\eta}{3}) \). For any \( x \in X \), we define

\[ W_n := \left\{ A \in \bigvee_{i=0}^{n-1} \phi_{LT}^{-i}: A \cap B(x, t, \epsilon, \phi) \neq \emptyset \right\}. \]
Next we will estimate the numbers of $W_n$. Let

$$B(x, t, \epsilon, \phi) \subset \bigcup_{A \in W_n} A$$

In fact, $y \in A \cap B(x, t, \epsilon, \phi)$, $A \in \bigvee_{i=0}^{n-1} \phi_{L \tau}^{-i} \xi$, there exists $\alpha \in \text{Rep}([0, t])$, such that

$$d(\phi_{\alpha(s)} x, \phi_{\beta(s)} y) < \epsilon, \ 0 \leq s \leq t.$$ 

Let $u = s - s_1, \gamma(u) = \alpha(s) - \alpha(s_1)$, then $\gamma \in \text{Rep}([-s_1, t - s_1])$, satisfying

$$d(\phi_{\gamma(u)} \phi_{\alpha(s_1)} x, \phi_{\alpha(s_1)} y) < \epsilon, -s_1 \leq u \leq t - s_1.$$ 

By Lemma 3.1 one obtain $|\gamma(u) - u| < \epsilon_1 |u| = \frac{\theta}{4 L \tau} |u|$. Choose $u = s_2 - s_1$, where $|s_2 - s_1| < L \tau$, then we have

$$|\alpha(s_1) - \alpha(s_2)| \leq \frac{\theta}{4}.$$ 

We consider the following sequence:

$$S_\alpha = \left\{ \frac{\alpha(kL\tau) - kL\tau}{\theta/4} \right\}, \ k = 0, 1, \cdots, n - 1,$$

where $\lfloor z \rfloor$ denote the largest integer less or equal $z$. If for some $\tilde{A} \in W_n$ with $\tilde{A} \neq A$, there exists $z \in \tilde{A} \cap B(x, t, \epsilon, \phi)$, we can choose $\beta \in \text{Rep}[0, t]$ such that $d(\phi_{\beta(s)} x, \phi_{\alpha(s)} z) < \epsilon, 0 \leq s \leq t$. Similarly, we can define the sequence $S_\beta$. If $S_\alpha = S_\beta$, for any $s \in [0, t]$, we have

$$|\alpha(s) - \beta(s)| \leq \left| (\alpha(s) - s) - (\alpha(\frac{s}{L \tau}) L \tau) - (\frac{s}{L \tau} L \tau) \right|$$
$$\quad + \left| (\alpha(\frac{s}{L \tau} L \tau) - \frac{s}{L \tau} L \tau) - (\beta(\frac{s}{L \tau} L \tau) - \frac{s}{L \tau} L \tau) \right|$$
$$\quad + \left| (\beta(s) - s) - (\beta(\frac{s}{L \tau} L \tau) - \frac{s}{L \tau} L \tau) \right|$$
$$\leq \frac{\theta}{4} + \frac{\theta}{4} \left| \frac{\alpha(\frac{s}{\theta/4}) L \tau} {\theta/4} - (\frac{s}{\theta/4} L \tau) \right| + \frac{\theta}{4}$$
$$\leq \theta.$$ 

Moreover, for any $s \in [0, t]$, we have $d(\phi_{\alpha(s)} x, \phi_{\beta(s)} x) < \frac{\eta}{3}$. Since for any $0 \leq s \leq t$, one has

$$d(\phi_s y, \phi_s z) \leq d(\phi_s y, \phi_{\alpha(s)} x) + d(\phi_{\alpha(s)} x, \phi_{\beta(s)} x) + d(\phi_{\beta(s)} x, \phi_s z)$$
$$\leq \epsilon + \frac{\eta}{3} + \epsilon < \eta.$$ 

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Especially, \( d(\phi_{L,n}^{i}y, \phi_{L,n}^{i}z) \leq \eta, i = 0, 1, \cdots, n - 1 \). For any \( A \in \bigvee_{i=0}^{n-1} \phi_{L,n}^{-i} \xi \), there exists \( i_0, i_1, \cdots, i_{n-1} \in \{1, 2, \cdots, m + 1\} \) such that

\[
A = A_{i_0} \cap \phi_{L,n}^{-1}(A_{i_2}) \cap \cdots \phi_{L,n}^{-(n-1)}(A_{i_{n-1}}).
\]

By the chosen of \( \eta \), for the fixed \( S_{\alpha} \), there exist at most \( 2^n \) \( A \) such that \( A \cap B(x, t, \delta, \phi) \neq \emptyset \). Furthermore, the number of \( S_{\alpha} \) is at most \( 3^n - 1 \). Hence, \( \#W_n \leq 6^n \).

For \( k = 0, 1, \cdots, K - 1 \), let

\[
D_{k,n} = \{ x \in B_k : \mu(B(x, t, \delta, \phi)) > 6^{2n} \exp(-(k\gamma - p - 3\varepsilon)n) \}.
\]

In addition, let

\[
D_{\infty,n} = \{ x \in B_{\infty} : \mu(B(x, t, \delta, \phi)) > 6^{2n} \exp(-(1\frac{1}{\varepsilon} - p - 3\varepsilon)n) \}.
\]

To prove (2), we consider the case for \( k = 0, 1, \cdots, K - 1 \).

If we can prove that \( \sum_{n=N}^{\infty} \mu(D_{k,n}) < \infty \), then apply the Borel-Cantelli Lemma: for a.e. \( x \in B_k \),

\[
h_{\mu}(\phi, x) + p \geq \frac{k\gamma}{L\tau}.
\]

Hence, we can obtain that

\[
\int_{X_M} p + h_{\mu}(\phi, x) d\mu \geq \frac{\sum_{k=1}^{K-1} k\gamma \mu(B_k)}{L\tau} \geq \frac{1}{L\tau} \int_{h^{-1}(\{0,M\})} h(y) d\pi(y) - \frac{\gamma}{L\tau}.
\]

Let \( \gamma \to 0 \) (by letting \( K \) tend to infinity),

\[
\int_{X_M} p + h_{\mu}(\phi, x) d\mu \geq \frac{1}{L\tau} \int_{h^{-1}(\{0,M\})} h(y) d\pi(y). \tag{6}
\]

Now we estimate the measures of \( D'_{k,n,s} \).

For any \( x \in D_{k,n} \), in those \( 6^n \) atoms of \( \bigvee_{i=0}^{n-1} \phi_{L,n}^{-i} \xi \) such that \( A \cap B(x, t, \delta, \phi) \), there exists at least one corresponding atom of \( \bigvee_{i=0}^{n-1} \phi_{L,n}^{-i} \xi \) whose measure is greater than \( 6^n \exp(-(k\gamma - p - 3\varepsilon)n) \). The total number of such atoms will not exceed \( 6^{-n} \exp((k\gamma - 3\varepsilon)n) \). Hence \( Q_{k,n} \), the total number of elements of \( \bigvee_{i=0}^{n-1} \phi_{L,n}^{-i} \xi \) that intersect \( D_{k,n} \), satisfies:

\[
Q_{k,n} \leq 6^n 6^{-n} \exp((k\gamma - p - 3\varepsilon)n) = \exp((k\gamma - p - 3\varepsilon)n).
\]

Let \( S_{k,n} \) denote the total measure of such \( Q_{k,n} \) elements of \( \bigvee_{i=0}^{n-1} \phi_{L,n}^{-i} \xi \) whose intersections with \( B_k \) have positive measure. Then from (4),

\[
S_{k,n} \leq Q_{k,n} \exp((-k\gamma + 2\varepsilon)n) \leq \exp((-p - \varepsilon)n),
\]
which follows that
\[ \mu(D_{k,n}) \leq S_{k,n} \leq \exp((-p-\epsilon)n). \]

To prove (3), we need estimate the measures of \( D'_{\infty,n} \).

In the above treatment for \( D'_{k,n} \), replacing \( k \gamma \) (resp. \( D'_{k,n} \), \( Q'_{k,n} \), and \( S'_{k,n} \)) by \( \frac{1}{\epsilon} \) (resp. \( D'_{\infty,n} \), \( Q'_{\infty,n} \), and \( S'_{\infty,n} \)), it also holds that \( \sum_{n=1}^{\infty} D_{\infty,n} < \infty \). Then apply the Borel-Cantelli Lemma again: for a.e. \( x \in B_{\infty} \),
\[ p + \limsup_{n \to \infty} \frac{1}{nL\tau} \log \mu(B(x,nL\tau, \delta, \phi)) \geq \frac{1}{\epsilon} - 5\epsilon. \]

Letting \( \epsilon \) go to 0, we then have for \( \mu \) almost every \( x \in X_{\infty} \),
\[ p + \frac{h_{\mu}(\phi,x)}{\tau} = \infty. \]

Now we can finish the proof of Proposition 3.2. By Lemma 3.2, we have
\[ \int_{X} p + \frac{h_{\mu}(\phi,x)}{\tau} d\mu \geq \frac{1}{L\tau} \int_{h^{-1}([0,M])} h(y) d\pi(y), \quad \forall M > 0. \]

Let \( M \) tend to \( \infty \), then \( \mu(X_{M} \cup X_{\infty}) \) tend to 1. Hence for \( \mu \)-a.e. \( x \in X \),
\[ \int_{X} p + \frac{h_{\mu}(\phi,x)}{\tau} d\mu \geq \int_{Y} h_{\mu}(\phi_{L\tau}, \rho^{-1}(y)) d\pi(y) = \frac{h_{\mu}(\phi_{L\tau})}{L\tau} = \frac{h_{\mu}(\phi_{\tau})}{\tau}. \]

Let \( p \to 0 \), it follows that
\[ \int_{X} \frac{h_{\mu}(\phi,x)}{\tau} d\mu \geq \frac{h_{\mu}(\phi_{\tau})}{\tau}. \]

**Case 2.** Consider \( \tau > 0, t > 0 \). Choose \( n_{t} \in \mathbb{N} \) such that \( n_{t}L\tau < t < (n_{t}+1)L\tau \).

Then we have
\[ B(x,(n_{t}+1)L\tau, \epsilon, \phi) \subset B(x,t, \epsilon, \phi) \subset B(x,n_{t}L\tau, \epsilon, \phi). \]

Then,
\[ \int_{X} \lim_{\epsilon \to 0} \lim_{t \to \infty} -\frac{\log \mu(B(x,t, \delta, \phi))}{t} d\mu \leq \int_{X} \lim_{\epsilon \to 0} \lim_{t \to \infty} -\frac{\log \mu(B(x,(n_{t}+1)L\tau, \epsilon, \phi))}{t} d\mu \]
\[ \leq \int_{X} \lim_{\epsilon \to 0} \lim_{t \to \infty} -\frac{\log \mu(B(x,(n_{t}+1)L\tau, \epsilon, \phi))}{n_{t}L\tau} d\mu \]
\[ = \int_{X} \lim_{\epsilon \to 0} \lim_{t \to \infty} -\frac{\log \mu(B(x,(n_{t}+1)L\tau, \epsilon, \phi))}{(n_{t}+1)L\tau} d\mu \]
\[ = \frac{1}{\tau} h_{\mu}(\phi_{\tau}). \]

**Case 3.** Consider \( \tau < 0, t > 0 \). Then we have \( -\tau > 0 \), by **Case 2** we have
\[ \int_{X} h_{\mu}(\phi,x) d\mu \leq \frac{1}{\tau} h_{\mu}(\phi_{-\tau}) = \frac{1}{\tau} h_{\mu}(\phi_{\tau}) = \frac{1}{|\tau|} h_{\mu}(\phi_{\tau}). \]
4 Proof of Theorem 1.1

Theorem 4.1. ([8] Birkhoff ergodic theorem for flow) Let \((X, \phi)\) be a compact metric flow, \(\mu \in \mathcal{M}(X)\) and \(f \in L^1(\mu)\), then the limits

\[
\tilde{f}(x) = \lim_{t \to -\infty} \frac{1}{t} \int_0^t f(\phi_\tau x) d\tau = \lim_{t \to +\infty} \frac{1}{t} \int_0^t f(\phi_\tau x) d\tau,
\]

exist for almost every point \(x \in X\). Also \(\tilde{f}(x) = \tilde{f}(\phi_t x)\) for a.e \(x \in X\) and \(\int \tilde{f} d\mu = \int f d\mu\). Particularly, if \(\mu \in E(\mathcal{E}(X))\), then \(\tilde{f}(x) = \int f d\mu\) for \(\mu\)-a.e \(x \in X\).

Lemma 4.2. Let \((X, \phi)\) be compact metric flow and \(\mu \in E(\mathcal{E}(X))\), then \(\mu(\mathcal{G}(\mu, \phi)) = 1\).

Proof. The space \(C(X)\) of continuous functions admits some countable dense subset \(\{f_k\}_{k=1}^\infty\).

By Theorem 4.1 there exists \(Q_k \in \mathcal{B}, \mu(Q_k) = 1\) such that

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t f_k(\phi_\tau x) d\tau = \int f_k d\mu\]

for almost every \(x \in Q_k\). It suffices to show that \(G_\mu(\phi) = \bigcap_{k=1}^\infty Q_k\). In fact, on the one hand, for every \(k \geq 1\) and \(x \in G_\mu(\phi)\), we have

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t f_k(\phi_\tau x) d\tau = \int f_k d\mu.
\]

On the other hand, since each \(f \in C(X)\) can be approximated by \(\{f_k\}_{k=1}^\infty\), thus \(G_\mu(\phi)\) contains \(\bigcap_{k=1}^\infty Q_k\). \(\square\)

In [6], the authors proved the following variational principle between the lower local entropy and Bowen entropy of compact subsets.

Theorem 4.3. Let \((X, \phi)\) be a compact metric flow without fixed points. If \(K\) is a non-empty compact subsets of \(X\), then

\[
h_{top}^B(\phi, K) = \sup\{h_\mu(\phi) : \mu \in \mathcal{M}(X), \mu(K) = 1\}
\]

With the help of the above theorem, we can now give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let \(\mu \in \mathcal{M}(X)\) and \(Y\) a subset of \(X\) with \(\mu(Y) = 1\). Let \(\{Y_n\}_{n \in \mathbb{N}}\) be an increasing sequence of compact subsets of \(Y\) such that \(\mu(Y_n) > 1 - \frac{1}{n}\) for each \(n \in \mathbb{N}\). Clearly

\[
h_{top}^B(\phi, Y) \geq h_{top}^B(\phi, \bigcup_{n \in \mathbb{N}} Y_n) = \sup_n h_{top}^B(Y_n, \phi) = \lim_{n \to \infty} h_{top}^B(Y_n, \phi)
\]

Denote by \(\mu_n\) the restriction of \(\mu\) on \(Y_n\), i.e. for any \(\mu\)-measurable set \(A \subset X\),

\[
\mu_n(A) = \frac{\mu(A \cap Y_n)}{\mu(Y_n)}
\]
Applying Theorem 4.3,

\[ h^B_{\text{top}}(\phi, Y_n) = \sup\{ h_{\nu}(\phi) : \nu \in M(X), \nu(Y_n) = 1 \} \geq h_{\mu_n}(\phi) \quad (10) \]

Note that

\[ h_{\mu_n}(\phi) = \int_{x \in X} \lim_{t \to \infty} \liminf_{\varepsilon \to 0} -\frac{1}{t} \log \mu_n(B(x, t, \varepsilon, \phi)) d\mu_n \]

By Theorem 1.3,

\[ \int_{Y} \lim_{\varepsilon \to 0} \liminf_{t \to \infty} -\frac{1}{t} \log \mu(B(x, t, \varepsilon, \phi)) d\mu = h_{\mu}(\phi) = h_{\mu}(\phi_1). \]

Hence \( \lim_{n \to \infty} h_{\mu_n}(\phi) \geq h_{\mu}(\phi_1) \).

Together with (9) and (10),

\[ \lim_{n \to \infty} h^B_{\text{top}}(\phi, Y_n) \geq h_{\mu}(\phi_1) \]

Noticing that \( \mu(G_{\mu}(\phi)) = 1 \) for \( \mu \in E_\phi \), by Theorem 1.1, we have the following corollary.

**Corollary 4.4.** Let \((X, \phi)\) be a compact metric flow without fixed points and \( \mu \in E_\phi(X) \), then \( h_{\mu}(\phi_1) \leq h^B_{\text{top}}(\phi, G_{\mu}(\phi)) \)

**5 Proof of Theorem 1.2**

In this section, we will show the proof of Theorem 1.2. Corollary 4.4 gives the lower bound. For the upper bound, we use the ideas of Pfister and Sullivan [10].

For \( \mu \in E_\phi(X) \), let \( \{K_m\}_{m \in \mathbb{N}} \) be a decreasing sequence of closed convex neighborhoods of \( \mu \) in \( M(X) \) and set

\[ A_{n,m} = \left\{ x \in X : \frac{1}{n} \int_0^n \delta_x \circ \phi_{-s} \, ds \in K_m \right\}, \text{ for } m, n \in \mathbb{N}. \]

Then for any \( m, N \geq 1 \), \( G_{\mu}(\phi) \subset \bigcup_{n \geq N} A_{n,m} \). For \( E \subset X \) and \( \varepsilon > 0 \), we say that \( E \) is a \((t, \varepsilon)\)-strongly separated set in \( X \) if for every \( x, y \in E, x \neq y \) and for every \( \alpha, \beta \in \text{Rep}[0, t] \),

\[ d(\phi_{\alpha(s)}x, \phi_{s}y) > \varepsilon \quad \text{for some } s \in [0, t] \]
or
\[ d(\phi_\beta(s)y, \phi_s x) > \varepsilon \text{ for some } s \in [0, t]. \]

Let \( S_t(X, \varepsilon) = S_t(X, \varepsilon, \phi) \) be the largest cardinality of any \((t, \varepsilon)\)-strongly separated subset of \( X \).

Denote by \( S_n(A_{n,m}, \varepsilon, \phi) \) the maximal cardinality of any \((n, \varepsilon)\)-strongly separated subset of \( A_{n,m} \).

**Claim.**
\[ \lim_{\varepsilon \to 0} \lim_{m \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log S_n(A_{n,m}, \varepsilon) \leq h_\mu(\phi_1). \]

**Proof of the claim.** If not, suppose that
\[ \lim_{\varepsilon \to 0} \lim_{m \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log S_n(A_{n,m}, \varepsilon) > h_\mu(\phi_1) + \delta \]
for some \( \delta > 0 \). Then there exist \( \varepsilon_0 > 0 \) and \( M \in \mathbb{N} \) such that for any \( 0 < \varepsilon < \varepsilon_0 \) and any \( m > M \), it holds that
\[ \limsup_{n \to \infty} \frac{1}{n} \log S_n(A_{n,m}, \varepsilon) > h_\mu(\phi_1) + \delta. \]
Hence we can find a sequence \( \{m(n)\} \) with \( m(n) \to \infty \) such that
\[ \limsup_{n \to \infty} \frac{1}{n} \log S_n(A_{n,m(n)}, \varepsilon) \geq h_\mu(\phi_1) + \delta. \]

Now let \( E_n \) be a \((n, \varepsilon)\)-separated set of \( A_{n,m(n)} \) with maximal cardinality and define
\[ \delta_n = \frac{1}{\sharp E_n} \sum_{x \in E_n} \delta_x \text{ and } \mu_n = \frac{1}{n} \int_0^n \delta_n \circ \phi_{-\tau} d\tau. \]
Since
\[ \frac{1}{n} \int_0^n \delta_x \circ \phi_{-\tau} d\tau \in K_m, \text{ for any } x \in E_n \]
and
\[ \mu_n = \frac{1}{\sharp E_n} \sum_{x \in E_n} \frac{1}{n} \int_0^n \delta_x \circ \phi_{-\tau} d\tau. \]
by the convexity of \( K_m \), \( \mu_n \in K_{m(n)} \). And hence \( \mu_n \to \mu \) as \( n \) goes to infinity. Let \( \beta \) be a finite Borel partition of \( X \) such that \( \text{diam}(\beta) < \eta < \varepsilon \) and \( \mu(\partial \beta) = 0 \). Since \( X \) is compact and \( \phi \) is continuous. Then each element of \( \bigvee_{i=0}^{n-1} \phi_{-i} \beta \) contains at most one point in \( E_n \).
Hence \( S_n(A_{n,m(n)}, \varepsilon) \) members of \( \bigvee_{i=0}^{n-1} \phi_{-i} \beta \) each have \( \delta_n \)-measure \( S_n(A_{n,m(n)}, \varepsilon) \) and the
It is easily seen that
\[
\sum_{i=0}^{n-1} \phi_{-i} \beta = \sum_{r=0}^{a(j)-1} \phi_{-(rq+j)} \bigvee_{i=0}^{q-1} \phi_{-i} \beta \bigvee_{i \in S} \phi_{-i} \beta
\]
and S has cardinality at most 2q. Therefore
\[
\log S_n(A_{n,m(n)}, \varepsilon) = H_{\delta_n} \left( \bigvee_{i=0}^{n-1} \phi_{-i} \beta \right).
\]
Given 0 < \theta < 1. Since each element of \( \bigvee_{i=0}^{n-1} \phi_{-i} \beta \) contains at most one point in \( E_n \), then
\[
\log S_n(A_{n,m(n)}, \varepsilon) = H_{\delta_n} \left( \bigvee_{i=0}^{n-1} \phi_{-i} \beta \right) \leq \sum_{r=0}^{a(j)-1} H_{\delta_n \circ \phi_{-(\theta q+j)}} \left( \bigvee_{i=0}^{q-1} \phi_{-i} \beta \right) + \sum_{k \in S} H_{\delta_n \circ \phi_{-k}} \left( \phi_{-k} \beta \right) \leq \sum_{r=0}^{a(j)-1} H_{\delta_n \circ \phi_{-(\theta q+j)}} \left( \bigvee_{i=0}^{q-1} \phi_{-i} \beta \right) + 2q \log \# \beta.
\]
It is easily seen that
\[
\int_0^1 \log S_n(A_{n,m(n)}, \varepsilon) d\theta \leq \sum_{r=0}^{a(j)-1} \int_0^1 H_{\delta_n \circ \phi_{-(\theta q+j)}} \left( \bigvee_{i=0}^{q-1} \phi_{-i} \beta \right) d\theta + 2q \log \# \beta \leq \sum_{r=0}^{a(j)-1} \int_0^1 \delta_n \circ \phi_{-(\theta q+j)} \bigvee_{i=0}^{q-1} \phi_{-i} \beta + 2q \log \# \beta.
\]
Sum this inequality over \( j \) from to \( q-1 \), we can get
\[
q \log S_n(A_{n,m(n)}, \varepsilon) = \int_0^1 q \log S_n(A_{n,m(n)}, \varepsilon) d\theta \leq \sum_{r=0}^{a(j)-1} \int_0^1 H_{\delta_n \circ \phi_{-(\theta q+j)}} \bigvee_{i=0}^{q-1} \phi_{-i} \beta + 2q^2 \log \# \beta.
\]
If we divide by \( n \), we obtain
\[
\frac{q}{n} \log S_n(A_{n,m(n)}, \varepsilon) \leq \frac{1}{n} \sum_{r=0}^{a(j)-1} \int_0^1 H_{\delta_n \circ \phi_{-(\theta q+j)}} \bigvee_{i=0}^{q-1} \phi_{-i} \beta + 2q^2 \frac{\log \# \beta}{n} \leq H \frac{1}{n} \sum_{r=0}^{a(j)-1} \int_0^1 \delta_n \circ \phi_{-(\theta q+j)} \bigvee_{i=0}^{q-1} \phi_{-i} \beta + 2q^2 \frac{\log \# \beta}{n} \leq H \frac{1}{n} \sum_{r=0}^{a(j)-1} \int_0^1 \delta_n \circ \phi_{-(\theta q+j)} d\theta \bigvee_{i=0}^{q-1} \phi_{-i} \beta + 2q^2 \frac{\log \# \beta}{n} \leq H \frac{1}{n} \sum_{r=0}^{a(j)-1} \int_0^1 \delta_n \circ \phi_{-(\theta q+j)} d\theta \bigvee_{i=0}^{q-1} \phi_{-i} \beta + 2q^2 \frac{\log \# \beta}{n}.
\]
We know the members of $\bigvee_{i=0}^{q-1} \phi_{-i}\beta$ have boundaries of $\mu$-measure zero, so $\lim_{j \to \infty} \mu_{nj}(B) = \mu(B)$ for each member $B$ of $\bigvee_{i=0}^{q-1} \phi_{-i}\beta$ and therefore $\lim_{j \to \infty} H_{\mu_{nj}}(\bigvee_{i=0}^{q-1} \phi_{-i}\beta) = H_{\mu}(\bigvee_{i=0}^{q-1} \phi_{-i}\beta)$.

Therefore replacing $n$ by $n_j$ in (11) and letting $j$ go to $\infty$ we have

$$\limsup_{n \to \infty} \frac{1}{n} \log S_n(A_{n,m}(n), \varepsilon) \leq \frac{1}{q} h_{\mu}(\bigvee_{i=0}^{q-1} \phi_{-i}\beta).$$

This lead to

$$\limsup_{n \to \infty} \frac{1}{n} \log S_n(A_{n,m}(n), \varepsilon) \leq h_{\mu}(\phi_1),$$

a contradiction.

By the claim, for each $\delta > 0$, there exists $\varepsilon_0 > 0$ satisfying that for any $0 < \varepsilon < \varepsilon_0$, there exists $M \in \mathbb{N}$ such that whenever $m > M$, it holds that

$$\limsup_{n \to \infty} \frac{1}{n} \log S_n(A_{n,m}(n), \varepsilon) \leq h_{\mu}(\phi_1) + \frac{\delta}{2}.$$

Let $E_{n,m}$ be a $(n, \varepsilon)$-separated set of $A_{n,m}$ with maximal cardinality, then

$$A_{n,m} \subset \bigcup_{x \in E_{n,m}} B(x, n, 2\varepsilon, \phi).$$

Hence for $s = h_{\mu}(\phi_1) + 2\delta$,

$$\mathcal{M}_{N,2\varepsilon}(\phi, \mu_{\mu}(\phi)) \leq \mathcal{M}_{N,2\varepsilon}(\phi, \bigcup_{n \geq N} A_{n,m})$$

$$\leq \sum_{n \geq N} \mathcal{M}_{n,2\varepsilon}(\phi, A_{n,m})$$

$$\leq \sum_{n \geq N} \exp((h_{\mu}(\phi_1) + \delta - s)n)$$

$$= \sum_{n \geq N} \exp(-\delta n).$$

Letting $n \to \infty$,

$$\mathcal{M}_{2\varepsilon}(\mu_{\mu}(\phi), \phi) \leq \lim_{N \to \infty} \sum_{n \geq N} \exp(-\delta n) = 0,$$

which implies that $h_{\text{top}}(\mu_{\mu}(\phi), \phi) \leq h_{\mu}(\phi_1)$.
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