Abstract

One has believed that low energy effective theories of the Higgs branch of gauged linear sigma models correspond to supersymmetric nonlinear sigma models, which have been already investigated by many works. In this paper we discuss an explicit derivation of supersymmetric nonlinear sigma models from gauged linear sigma models. In this process we construct Kähler potentials of some two-dimensional toric varieties explicitly. Thus we will be able to study some algebraic varieties in the language of differential geometry.
1 Introduction

Investigations of string theory on curved backgrounds become more and more important tasks to understand the dynamics and dualities of string theory and M-theory. One of the most important strategies is studying and understanding (supersymmetric) nonlinear sigma models with conformal symmetries whose target spaces are non-trivially curved. But it is so difficult to discuss a direct investigation of nonlinear sigma models that we consider alternative models which become nonlinear sigma models at some specific limits. One of the powerful model is a gauged linear sigma model (GLSM).

GLSM is defined as one class of $\mathcal{N} = 2$ supersymmetric gauge theory in two-dimensional spacetime. This is one of the most important tools to investigate and understand the string theory dynamics and topological aspects of string theory. GLSM was introduced by Witten in order to analyze supersymmetric field theories whose target spaces are represented by the local descriptions of toric varieties [1].

Even though the formulation of GLSM is very simple, this gauge theory has various important properties. First, in the Higgs branch of GLSM, we believe that the low energy effective theory becomes $\mathcal{N} = 2$ supersymmetric nonlinear sigma model (NLSM) whose target space is a toric variety. In the Coulomb branch we can trace an effect of worldsheet instantons [2]. Moreover we can investigate some phases of GLSM by changing Fayet-Iliopoulos (FI) parameters. If we set FI parameters as positive under one specific condition (i.e., the Calabi-Yau condition or the chiral anomaly free condition), the low energy effective theory becomes the conformal sigma model on Calabi-Yau manifold. If we set them as negative, Landau-Ginzbrug model is realized in the low energy limit. Due to this phenomenon, we can discuss the Calabi-Yau/Landau-Ginzbrug correspondence easily.

GLSMs are applied to a lot of topics in field theory and string theory: topological field theory and topological string [1 3], conifold transitions [4], closed string tachyon condensations [5], mirror symmetry [6] and mirror symmetry with D-branes [7 8], and more.

$\mathcal{N} = 2$ supersymmetric NLSMs in two-dimensional spacetime and their gauge theory extensions have been investigated by many works for a long time because these models have various important phenomena such as asymptotic freedom, confinements, mass gaps in the spectra, and so on. First, supersymmetric NLSM on projective space $\mathbb{C}P^N$ was studied in order to understand chiral symmetry breaking and quark confinements [9]. The non-abelian extension to Grassmannian was considered by [10]. In recent years these models were generalized to supersymmetric NLSMs on hermitian symmetric spaces by [11].

It is important to understand the dynamics of $\mathcal{N} = 2$ supersymmetric NLSMs on (non)compact Calabi-Yau manifolds in order to understand the dynamics of superstring theories on curved back-
grounds. Thus we constructed Kähler potentials and metrics of $\mathcal{O}(-N)$ on $\mathbb{C}P^{N-1}$ as noncompact Calabi-Yau $N$-folds [12], and canonical line bundles on Einstein-Kähler manifolds [13] in the language of supersymmetric NLSMs. These potentials will be applied to classical and quantum supergravity and superstring theories on these curved backgrounds and we will understand some new properties of dynamics on curved spacetime. Moreover, we should discuss dynamics of supergravity and string theories on other curved spacetime in order to understand string dualities.

In this paper, we will construct Kähler potentials of some two-dimensional toric varieties, which are not Calabi-Yau manifolds. First we will review the projective space $\mathbb{C}P^2$, which is the basic varieties of other two-dimensional toric varieties. Second, we will construct Hirzebruch surfaces $\mathbb{F}_k$ ($k = 0, 1, 2$), which are $\mathbb{C}P^1$ bundles on weighted projective space $W\mathbb{C}P^{1,1,k}$. Third, del Pezzo surfaces $\mathbb{B}_k$ ($k = 0, 1, 2$), which is $\mathbb{C}P^2$ blown up at $k$ points, will be introduced. Since these toric varieties are powerful tools to investigate string duality, it is of important for us to understand their Kähler potentials and some objects derived from Kähler potentials in the language of differential geometry as well as algebraic geometry. If we understand some aspects of toric varieties in the language of differential geometry, we will be able to investigate the dynamics of supergravity and string/M-theory beyond the discussions of topological or non-dynamical aspects of these theories.

The organization of this paper is as follows: In section two we review gauged linear sigma models and supersymmetric NLSMs. And we consider a relation between these models. In section three we construct Kähler potentials of Hirzebruch surfaces and del Pezzo surfaces in detail. We give some comments on metrics and other geometrical objects. In section four we discuss some applications of these Kähler potentials. In appendix A we discuss an introduction of two-dimensional toric varieties.

2 Gauged Linear Sigma Model

Let us construct GLSM [1]. This is one class of $\mathcal{N} = 2$ supersymmetric gauge theories in two-dimensional spacetime. GLSM consists of chiral superfields $\Phi^i$, anti-chiral superfields $\overline{\Phi}^i$, and vector superfields $V_a$ with complexified $U(1)_a$ symmetry. Lagrangian is written as

$$L_{\text{GLSM}} = \int d^4\theta \left\{ \sum_a -\frac{1}{e^2_a} \Sigma_a \Delta_a + \sum_i \overline{\Phi}^i e \Sigma_a 2Q^i_a V_a \Phi^i \right\} + \left( \frac{1}{2\sqrt{2}} \int d\theta^+ d\overline{\theta}^\rightarrow \sum_a \tau_a \Sigma_a + (\text{h.c.}) \right),$$

(2.1)

where $e_a$ are $U(1)_a$ gauge coupling constants, which are dimensionful parameters. $Q^i_a$ are $U(1)_a$ charges of chiral superfields. We denote a kinetic term of a vector superfield in terms of a twisted chiral superfield, whose definition is $\Sigma_a = \frac{1}{\sqrt{2}} \overline{\mathcal{D}}_+ D_- V_a$. $\tau_a$ are complexified FI parameters: $\tau_a = r_a - i\theta_a/2\pi$, where $r_a$ are the FI parameters and $\theta_a$ are $U(1)_a$ theta angles.

Witten introduced GLSM as a field theory realization of a toric variety. In this context, chiral
superfields are regarded as homogeneous coordinates of toric variety, complexified $U(1)$ gauge symmetries as symmetries of algebraic torus $\mathbb{C}^*$. And $U(1)$ charges of chiral superfields correspond to Mori vectors of toric variety.

We would like to make a low energy effective theory of GLSM. In order to find this, we should take some vacua of GLSM to be supersymmetric, and consider the Higgs branch. Now let us discuss supersymmetric vacua. First, we investigate the potential of GLSM. Under the Wess-Zumino gauge the potential is written as

$$U = \sum_a \left\{ \frac{e_a^2}{2} \left( r_a - \sum_i Q_i^a |\phi^i|^2 \right)^2 \right\} + 2 \sum_{a,b} \sigma_a \sigma_b \left( \sum_i Q_i^a Q_i^b |\phi^i|^2 \right),$$

where $\phi^i$ and $\sigma$ are the scalar components of the chiral superfields and vector superfields, respectively. $D_a$ are auxiliary fields of $V_a$ whose equations of motion are given as

$$\frac{1}{e_a^2} D_a = r_a - \sum_i Q_i^a |\phi^i|^2.$$

In order to obtain the supersymmetric vacua, we set the potential and auxiliary fields as

$$U = 0, \quad D_a = 0. \quad (2.2)$$

This condition makes the moduli space of GLSM (2.1). Next let us consider the Higgs branch of this model. In general, one believe that the low energy effective theory of GLSM become a (supersymmetric) nonlinear sigma model (NLSM) whose target space is a toric variety. In the Higgs branch, some chiral superfields have vacuum expectation values. Thus gauge multiplets become massive by the Higgs mechanism. Theta angles $\theta_a$ vanish because the energy of the supersymmetric ground state must be zero. In the IR limit ($e_a \to \infty$), kinetic terms of gauge multiplets $\frac{1}{e_a} \Sigma_a \Sigma_a$ decouple from other terms. So the gauge multiplets do not propagate and freeze the dynamics: the gauge multiplets in (2.1) become auxiliary fields.

In the context of the above discussions, we start from a “frozen” Lagrangian

$$\mathcal{L}_F = \int d^4 \theta \left\{ \sum_i \Phi^i e^{\sum_a 2Q_i^a V_a} \Phi^i + \sum_a ( - 2r_a V_a) \right\} \equiv \int d^4 \theta K, \quad (2.3)$$

which does not contain the kinetic terms of vector superfields. Note that $K$ is a Kähler potential. Though we integrate out the auxiliary vector superfields from (2.3), the imaginary part of the complexified gauge symmetry is not fixed. So we choose one gauge to fix this residual gauge symmetry. In this process we obtain the supersymmetric NLSM [9, 10, 11]. By choosing one gauge we can normalize some components of homogeneous coordinates. This process corresponds to the constraint on the homogeneous coordinates (2.2) and then we obtain the supersymmetric NLSM in terms of local coordinates of toric variety.
3 Kähler Potentials

In this section we construct Kähler potentials of two-dimensional toric varieties. First, we review the construction of two-dimensional projective space in subsection 3.1. In subsection 3.2 we apply the technique to construction of Kähler potentials of Hirzebruch surfaces. Moreover in subsection 3.3 we construct Kähler potentials of del Pezzo surfaces in the same way.

Subsection 3.2 and 3.3 are the main parts of this paper.

3.1 Projective Space

Let us first review a construction of Kähler potential of two-dimensional projective space $\mathbb{C}P^2$. The toric data of $\mathbb{C}P^2$ are described in appendix A.1.

GLSM for $\mathbb{C}P^2$ consists of chiral superfields $\Phi^i$ ($i = 1, 2, 3$), anti-chiral superfields $\bar{\Phi}^i$ and a vector superfield $V$ with the complexified $U(1)$ gauge symmetry. We assign $U(1)$ charges of chiral superfields as in Table 1:

| $U(1)$ | $\Phi^1$ | $\Phi^2$ | $\Phi^3$ |
|--------|---------|---------|---------|
|        | 1       | 1       | 1       |

Table 1: $U(1)$ charges.

In the IR limit of GLSM for $\mathbb{C}P^2$, the Kähler potential of the frozen Lagrangian \(^\text{23}\) is written as

$$K = \left( |\Phi^1|^2 + |\Phi^2|^2 + |\Phi^3|^2 \right) e^{2V} - 2r V, \quad (3.1)$$

where $V$ is now an auxiliary field. In order to obtain the Kähler potential of supersymmetric NLSM, we solve the equation of motion of this auxiliary field:

$$\frac{\partial L_F}{\partial V} = 0 : \quad r = \left( |\Phi^1|^2 + |\Phi^2|^2 + |\Phi^3|^2 \right),$$

$$\therefore e^{2V} = \frac{r}{|\Phi^1|^2 + |\Phi^2|^2 + |\Phi^3|^2}.$$  

Note that we can divide $r$ by $\sum_i |\Phi^i|^2$ since all chiral superfields do not vanish simultaneously (due to the $D$-flatness condition of GLSM). Substituting this solution into Kähler potential \(^\text{23}\), we obtain the potential of supersymmetric NLSM:

$$K = r \log \left( |\Phi^1|^2 + |\Phi^2|^2 + |\Phi^3|^2 \right). \quad (3.2)$$

Note that we neglect a constant term which does not contribute to the Lagrangian. Now we should fix a residual gauge symmetry, i.e., an imaginary part of complex $U(1)$ gauge symmetry. This process
corresponds to dividing other chiral superfields \((\Phi^1, \Phi^3)\) by \(\Phi^2\) and we normalize \(\Phi^2\) to a constant as follows:

\[
K(X^a, \overline{X}^a) = r \log \left(1 + |X^1|^2 + |X^2|^2\right),
\]

(3.3)

where \(X^1 = \Phi^1/\Phi^2\) and \(X^2 = \Phi^3/\Phi^2\) and we neglect a “constant” term \(r \log |\Phi^2|^2\) as well as other constant terms. Note that this dividing process corresponds to choosing a local patch \(U_{\sigma_1}\) of \(\mathbb{CP}^2\) (see appendix A.1). Then we obtain the already known formulation of the Kähler potential of \(\mathbb{CP}^2\). It is easy to construct the Kähler metric of this space. The metric is defined as

\[
g_{ab} = \partial_a \overline{\partial}_b K(x, \overline{x}),
\]

(3.4)

where \(x^a\) are scalar components of chiral superfields \(X^a\). This metric is the Fubini-Study metric:

\[
g_{ab} = \frac{r}{(1 + |x^1|^2 + |x^2|^2)^2} \left\{ \left( \delta_a^1 \delta_b^1 + \delta_a^2 \delta_b^2 \right) \left( 1 + |x^1|^2 + |x^2|^2 \right) - \left( \delta_a^1 \overline{x}^1 + \delta_a^2 \overline{x}^2 \right) \left( \delta_b^1 x^1 + \delta_b^2 x^2 \right) \right\}.
\]

3.2 Hirzebruch Surfaces

Let us construct GLSMs for the Hirzebruch surfaces \(\mathbb{F}_k\) \((k = 0, 1, \ldots)\) by using the toric data (see appendix A.2). In terms of GLSMs, scalar components of chiral superfields represent homogeneous coordinates of toric varieties. Thus we sometimes call these fields “homogeneous coordinates.” In addition, two abelian vector superfields \(V_1\) and \(V_2\) are introduced in order to make the algebraic torus \((\mathbb{C}^*)^2\). We denote complexified gauge groups as \(U(1)_1^C \times U(1)_2^C\). Chiral superfields \(\Phi^i\) \((i = 1, 2, \ldots, 4)\) have charges of these gauge groups as in Table 2

|       | \(\Phi^1\) | \(\Phi^2\) | \(\Phi^3\) | \(\Phi^4\) |
|-------|----------|----------|----------|----------|
| \(U(1)_1\) | 1        | 1        | \(k\)    | 0        |
| \(U(1)_2\) | 0        | 0        | 1        | 1        |

Table 2: \(U(1) \times U(1)\) charges.

Substituting this data into (2.3) we obtain the frozen GLSM for this variety, whose Kähler potential is written as

\[
K = |\Phi^1|^2 e^{2V_1} + |\Phi^2|^2 e^{2V_2} + |\Phi^3|^2 e^{2kV_1+2V_2} + |\Phi^4|^2 e^{2V_2} - 2r_1 V_1 - 2r_2 V_2,
\]

(3.5)

where \(r_i\) are FI parameters and we assume \(r_1 > r_2\) for the positive definiteness of the volume of target space manifold.

\(k = 0\) case: \(\mathbb{F}_0 = \mathbb{CP}^1 \times \mathbb{CP}^1\)
The simplest Hirzebruch surface is $F_0$, which is globally equal to the direct product of projective spaces: $F_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$. The Kähler potential of this surface is well known and we can obtain from (2.3) by integrating out the vector superfields as

$$K(X^a, \overline{X}^a) = r_1 \log (1 + |X^1|^2) + r_2 \log (1 + |X^2|^2), \quad (3.6)$$

where chiral superfields $X^1 = \Phi^1/\Phi^2$ and $X^2 = \Phi^3/\Phi^4$ correspond to the local coordinates of $U_{\sigma_1}$ in $F_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$. Note that we identify the local patch $U_{\sigma_i}$ to the patches on toric cone $\sigma_i$ in Figure 3. Since the local coordinates of other patches and transition rules among them are described in appendix A.2, it is sufficient for us to consider the Kähler potential and metric only in one local patch, for example, as (3.6) in $U_{\sigma_1}$. We obtain the metric of $F_0$ by differentiating the Kähler potential in terms of the local coordinates $x^a$ ($a = 1, 2$):

$$g_{a\overline{b}} = \frac{\partial}{\partial x^a} \frac{\partial}{\partial \overline{X}^b} K(x, \overline{X}) = \delta^1_a \delta^1_{\overline{b}} \frac{r_1}{(1 + |x^1|^2)^2} + \delta^2_a \delta^2_{\overline{b}} \frac{r_2}{(1 + |x^2|^2)^2}. \quad (3.7)$$

Note that $x^a$ are scalar components of chiral superfields $X^a$, which denote the local coordinates of the target space of NLSM. This metric (3.7) corresponds to the sum of the Fubini-Study metrics, which has been already known.

$k = 1$ case: $F_1$

Here we construct the metric of $F_1$, which is the simplest variety that one $\mathbb{CP}^1$ is non-trivially fibered by the other $\mathbb{CP}^1$.

The equations of motion for the auxiliary vector superfields are obtained from (2.3) as:

$$\frac{\partial L_F}{\partial V_1} = 0 : \quad r_1 = (|\Phi^1|^2 + |\Phi^2|^2) e^{2V_1} + |\Phi^3|^2 e^{2V_1+2V_2}, \quad (3.8a)$$

$$\frac{\partial L_F}{\partial V_2} = 0 : \quad r_2 = |\Phi^3|^2 e^{2V_1+2V_2} + |\Phi^4|^2 e^{2V_2}. \quad (3.8b)$$

The solutions are obtained as

$$e^{2V_1} = \frac{1}{2A} \left\{ -B + \sqrt{B^2 - 4AC} \right\}, \quad (3.9a)$$

$$e^{2V_2} = \frac{2r_2(|\Phi^1|^2 + |\Phi^2|^2)}{2|\Phi^3|^2(|\Phi^1|^2 + |\Phi^2|^2) - B + \sqrt{B^2 - 4AC}}, \quad (3.9b)$$

where

$$A = (|\Phi^1|^2 + |\Phi^2|^2)|\Phi^3|^2,$$

$$B = (|\Phi^1|^2 + |\Phi^2|^2)|\Phi^4|^2 - (r_1 - r_2)|\Phi^3|^2,$$

$$C = -r_1|\Phi^4|^2.$$
Note that these solutions include all region of the Hirzebruch surface $\mathbb{F}_1$, i.e., some zero points $\Phi^3 = 0$ and $\Phi^4 = 0$ are also included. These zero points are the origin and the infinity point of the $\mathbb{C}P^1$ fiber.

Substituting these solutions (3.9) into the Kähler potential (3.5), we obtain the Kähler potential of the NLSM of $\mathbb{F}_1$:

$$
K = -r_1 \log \left\{ \frac{1}{2A} \left( -B + \sqrt{B^2 - 4AC} \right) \right\} - r_2 \log \left\{ \frac{2(\Phi^1)^2 + (\Phi^2)^2}{2(\Phi^1)^2 + (\Phi^2)^2(\Phi^4)^2 - B + \sqrt{B^2 - 4AC}} \right\} - r_2 \frac{2(\Phi^1)^2 + (\Phi^2)^2}{2(\Phi^1)^2 + (\Phi^2)^2(\Phi^4)^2 - B + \sqrt{B^2 - 4AC}}.
$$

(3.10)

Note that the third term comes from the non-trivial fibration of $\mathbb{C}P^1$. Chiral superfields $\Phi^i$ are related to each other because we should fix the residual gauge symmetry (imaginary part of $U(1)^C \times U(1)^C$).

As we will discuss in appendix A.2, we represent Hirzebruch surfaces using four local patches $U_{\sigma_i}$ ($i = 1, 2, \cdots, 4$). Thus we will define local coordinates on these patches and consider relationships among them where some patches overlap.

Now let us define the local coordinates of various patches in terms of homogeneous coordinates $\Phi^i$. In the same way as $\mathbb{F}_0$, we describe local coordinates as follows:

$$(X^1, X^2) = \left( \frac{\Phi^1}{\Phi^2}, \frac{\Phi^3}{\Phi^2\Phi^4} \right) \in U_{\sigma_1}, \quad \text{where } \Phi^2, \Phi^4 \neq 0.$$ Since transition rules of coordinates among local patches are described in appendix A.2, we denote the Kähler potential and metric of $\mathbb{F}_1$ in the local patch $U_{\sigma_1}$ only.

Let us write down the Kähler potential of $\mathbb{F}_1$ in terms of the local coordinates $X^a$ as

$$K(X^a, \bar{X}^a) = r_1 \log \bar{D}_+ + r_2 \log \left\{ 1 + \frac{\bar{D}_-}{2(1 + |X^1|^2)} \right\} - r_2 \frac{\bar{D}_-}{2(1 + |X^1|^2) + \bar{D}_-}.$$ (3.11)

Note that we neglect constant terms because they do not appear in the Lagrangian. In 3.11 we use the following function:

$$\bar{B} = (1 + |X^1|^2) - (r_1 - r_2)|X^2|^2, \quad \bar{D}_\pm = \pm \bar{B} + \sqrt{\bar{B}^2 + 4(1 + |X^1|^2)|X^2|^2}.$$ Thus we have obtained the Lagrangian of the supersymmetric NLSM on the Hirzebruch surface $\mathbb{F}_1$.

Let us give some comments on the geometrical objects. We obtain the Kähler metric as

$$g_{\alpha \bar{\alpha}} = \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial \bar{x}^\bar{\alpha}} K(x, \bar{x})$$

$$= -\delta_1^a \delta_1^{\bar{\alpha}} \frac{r_2}{(1 + |x|^2)^2} + \frac{r_1}{D_+} \partial_a \bar{D}_+ - \frac{r_1}{(D_+)^2} \partial_a \bar{D}_+ \partial_{\bar{b}} \bar{D}_+ + \delta_1^a \delta_1^{\bar{\alpha}} 2r_2 \frac{2r_2}{2(1 + |x|^2) + D_-}.$$
Table 2, the Kähler potential of the frozen Lagrangian (2.3) is written as

\[ K = \sum_{a=1}^{3} \frac{r_a}{2(1 + |x|^2) - D_+} \left\{ 2\delta_a^1 \delta_a^2 x^1 + \partial_a \bar{\partial}_a \bar{D}_+ + \partial_a \bar{\partial}_a \bar{D}_- \right\} \left\{ 2\delta_b^1 \delta_b^2 x^1 + \bar{\partial}_b \bar{\partial}_b \bar{D}_- \right\}, \quad (3.12) \]

where

\[
\begin{align*}
\partial_a \bar{D}_+ &= \delta_a^1 \bar{x}^1 - (r_1 - r_2) \delta_a^2 \bar{x}^2, \\
\bar{\partial}_a \bar{D}_+ &= \delta_a^1 \bar{x}^1 - (r_1 - r_2) \delta_a^2 \bar{x}^2, \\
\partial_a \bar{\partial}_b \bar{D}_+ &= \delta_a^1 \delta_b^1 \bar{x}^1 - (r_1 - r_2) \delta_a^2 \delta_b^1 \bar{x}^2, \\
\bar{\partial}_a \bar{\partial}_b \bar{D}_+ &= \delta_a^1 \delta_b^1 \bar{x}^1 - (r_1 - r_2) \delta_a^2 \delta_b^1 \bar{x}^2,
\end{align*}
\]

In principle, we can explicitly obtain curvature tensors, Ricci tensors and the Euler number of the Hirzebruch surface \( F_1 \) from the Kähler potential (3.13).

**k = 2 case:** \( F_2 \)

Now we consider another Hirzebruch surface, i.e., \( F_2 (k = 2) \). In terms of the toric data written in Table 2, the Kähler potential of the frozen Lagrangian (2.3) is written as

\[ K = |\Phi|^2 e^{2V_1} + |\Phi|^2 e^{2V_1} + |\Phi|^2 e^{2V_1} + 2|\Phi|^2 e^{2V_2} + 2|\Phi|^2 e^{2V_2} + 2|\Phi|^2 e^{2V_1} - 2r_1 V_1 - 2r_2 V_2. \quad (3.13) \]

Then the equations of motion for \( V_1 \) and \( V_2 \) are obtained as

\[
\begin{align*}
\frac{\partial L}{\partial V_1} &= 0 : \quad r_1 = \left( |\Phi|^2 + |\Phi|^2 \right) e^{2V_1} + 2|\Phi|^2 e^{4V_1 + 2V_2}, \\
\frac{\partial L}{\partial V_2} &= 0 : \quad r_2 = |\Phi|^2 e^{4V_1 + 2V_2} + |\Phi|^2 e^{2V_1}.
\end{align*}
\]

The solutions are very complicated:

\[
e^{2V_1} = \frac{E^{2/3} - 4(3AC - B^2)}{6AE^{1/3}}, \quad (3.15a)
\]
Then one finds the metric is smooth in all region.

In terms of these local coordinates, we can describe the Kähler potential in
coordinates $\Phi^i$. In the same way as the previous discussions, we describe local coordinates as follows:

$$X^1, X^2 \in U_{\sigma_1}, \quad \text{where} \quad \Phi^2, \Phi^4 \neq 0.$$ 

In terms of these local coordinates, we can describe the Kähler potential in $U_{\sigma_1}$ as follows:

$$K(X^a, \bar{X}^a) = r_1 \log \left( 1 + |X^1|^2 \right) + r_1 \log |X^2|^2 + \frac{r_1}{3} \log \tilde{E} - r_1 \log \left\{ \frac{\tilde{E}^{2/3} - 4(3A\tilde{C} - \tilde{B}^2) - 2\tilde{B}E^{1/3}}{6AE^{1/3}} \right\}$$

$$- 2r_2 \log \left\{ \frac{\tilde{E}^{2/3} - 4(3A\tilde{C} - \tilde{B}^2) - 2\tilde{B}E^{1/3}}{6AE^{1/3}} \right\}$$

$$+ \left(\text{constant terms} \right),$$

where

$$\tilde{A} = (1 + |X^2|^2)|X^2|^2, \quad \tilde{B} = -(r_1 - 2r_2)|X^2|^2, \quad \tilde{C} = 1 + |X^1|^2,$$

$$\tilde{E} = 36A\tilde{C} + 108r_1\tilde{A}^2 - 8\tilde{B}^3$$

$$+ 12\sqrt{3} \tilde{A}(4A\tilde{C}^3 - \tilde{B}^2\tilde{C}^2) + 18r_1\tilde{A}\tilde{B}\tilde{C} + 27(r_1)^2\tilde{A}^2 - 4r_1\tilde{B}^3)^{1/2}.$$

The Kähler potential (3.17) seems to give a singular metric at $x^2 = 0$ because of the existence of $r_1 \log |X^2|^2$. But if one combines this term with others in order to absorb some divergence at $x^2 = 0$, then one finds the metric is smooth in all region.
3.3 del Pezzo Surfaces

In this subsection we consider the Kähler potential of the del Pezzo surface $\mathbb{B}_k$ which corresponds to $\mathbb{C} \mathbb{P}^2$ blown up at $k$ points. For simplicity we consider only $k = 0, 1, 2$ case. It is very difficult to obtain $\mathbb{B}_3$ and we cannot construct Kähler potentials $\mathbb{B}_{k \geq 4}$ in the way we discussed in the previous sections.

Since the simplest del Pezzo surface $\mathbb{B}_0$ is not blown up by $\mathbb{C} \mathbb{P}^1$, this is the two-dimensional projective space $\mathbb{C} \mathbb{P}^2$ itself, which we have already discussed in subsection 3.1. It is well known that the del Pezzo surface $\mathbb{B}_1$ corresponds to the Hirzebruch surface $F_1$ because the toric fan of the former corresponds to the one of the latter. So we discuss only $\mathbb{B}_2$ in this subsection.

From the toric fan of the del Pezzo surface $\mathbb{B}_2$ (see appendix A.3), we construct GLSM for $\mathbb{B}_2$ in terms of chiral superfields $\Phi^i (i = 1, 2, \cdots, 5)$ and complexified $U(1)^3$ vector superfields $V_a (a = 1, 2, 3)$. The $U(1)^3$ charges are assigned as in Table 3:

|       | $\Phi^1$ | $\Phi^2$ | $\Phi^3$ | $\Phi^4$ | $\Phi^5$ |
|-------|---------|---------|---------|---------|---------|
| $U(1)_1$ | 1       | 1       | 1       | 0       | 0       |
| $U(1)_2$ | 0       | 0       | 1       | 1       | 0       |
| $U(1)_3$ | 1       | 0       | 0       | 0       | 1       |

Table 3: $U(1)^3$ charges.

Substituting the toric data of Table 3 into (2.3), we give the Kähler potential of the supersymmetric Lagrangian:

$$K = |\Phi^1|^2 e^{2V_1+2V_3} + |\Phi^2|^2 e^{2V_1} + |\Phi^3|^2 e^{2V_1+2V_2} + |\Phi^4|^2 e^{2V_2} + |\Phi^5|^2 e^{2V_3} - 2r_1 V_1 - 2r_2 V_2 - 2r_3 V_3. \tag{3.18}$$

Note that we assume $r_1 \geq r_2$ and $r_1 \geq r_3$ in order for the positive definiteness of the metric. We obtain the equations of motion for these vector superfields as

$$\frac{\partial L_F}{\partial V_1} = 0 : \quad r_1 = |\Phi^1|^2 e^{2V_1+2V_3} + |\Phi^2|^2 e^{2V_1} + |\Phi^3|^2 e^{2V_1+2V_2},$$
$$\frac{\partial L_F}{\partial V_2} = 0 : \quad r_2 = |\Phi^3|^2 e^{2V_1+2V_2} + |\Phi^4|^2 e^{2V_2},$$
$$\frac{\partial L_F}{\partial V_3} = 0 : \quad r_3 = |\Phi^1|^2 e^{2V_1+2V_3} + |\Phi^5|^2 e^{2V_3}.$$

The solutions are as follows:

$$e^{2V_1} = \frac{E^{2/3} - 4(3AC - B^2) - 2BE^{1/3}}{6AE^{1/3}}, \tag{3.19a}$$
$$e^{2V_2} = \frac{6r_2 AE^{1/3}}{6AE^{1/3}|\Phi^4|^2 + |\Phi^3|^2[E^{2/3} - 4(3AC - B^2) - 2BE^{1/3}]}, \tag{3.19b}$$
\[ e^{2Y_3} = \frac{6r_3AE^{1/3}}{6AE^{1/3}|\Phi^5|^2 + |\Phi|^2[E^{2/3} - 4(3AC - B^2) - 2BE^{1/3}]}, \] (3.19c)

where

\begin{align*}
A &= |\Phi|^2|\Phi^2|^2|\Phi^3|^2, \\
B &= |\Phi|^2(|\Phi|^2|\Phi^4|^2 + |\Phi|^2|\Phi^5|^2| + (-r_1 + r_2 + r_3)|\Phi|^2|\Phi^3|^2, \\
C &= |\Phi|^2|\Phi^4|^2|\Phi^5|^2 + (-r_1 + r_2)|\Phi|^2|\Phi^5|^2 + (-r_1 + r_3)|\Phi|^2|\Phi^2|^2, \\
D &= -r_1|\Phi|^2|\Phi^5|^2, \\
E &= 36ABC - 108A^2D - 8B^3 \\
&+ 12\sqrt{3}A(4AC^3 - B^2C^2 - 18ABCD + 27A^2D^2 + 4B^3D)^{1/2}.
\end{align*}

Substituting (3.19) into the Kähler potential (3.18), we obtain

\[ K = -r_1 \log \left\{ \frac{E^{2/3} - 4(3AC - B^2) - 2BE^{1/3}}{6AE^{1/3}} \right\} \]
\[ - r_2 \log \left\{ \frac{6r_2AE^{1/3}}{6AE^{1/3}|\Phi|^2 + |\Phi|^2[E^{2/3} - 4(3AC - B^2) - 2BE^{1/3}]} \right\} \]
\[ - r_3 \log \left\{ \frac{6r_3AE^{1/3}}{6AE^{1/3}|\Phi^5|^2 + |\Phi|^2[E^{2/3} - 4(3AC - B^2) - 2BE^{1/3}]} \right\} \]
\[ - r_2 \frac{\Phi^2}{6AE^{1/3} |\Phi|^2 |\Phi^3|^2 |\Phi^5|^2} \left\{ \frac{E^{2/3} - 4(3AC - B^2) - 2BE^{1/3}}{E^{2/3} - 4(3AC - B^2) - 2BE^{1/3}} \right\} \]
\[ - r_3 \frac{\Phi^2}{6AE^{1/3} |\Phi|^2 |\Phi^3|^2 |\Phi^5|^2} \left\{ \frac{E^{2/3} - 4(3AC - B^2) - 2BE^{1/3}}{E^{2/3} - 4(3AC - B^2) - 2BE^{1/3}} \right\}. \] (3.20)

Note that chiral superfields are related to each other under the gauge-fixing. Now let us discuss the local coordinate systems and the relations among them. We define the local coordinates of various patches in terms of homogeneous coordinates \( \Phi^i \). We describe local coordinates as follows:

\[ (X^1, X^2) = \left( \frac{\Phi^1}{\Phi^2 \Phi^5}, \frac{\Phi^3}{\Phi^2 \Phi^4} \right) \in U_{\sigma_1}, \quad \text{where} \quad \Phi^2, \Phi^4, \Phi^5 \neq 0. \]

As we will discussed as Figure 4 in appendix A.3 we describe transition rules of coordinates among local patches. Thus it is sufficient for us to describe the Kähler potential of \( \mathbb{B}_2 \) in \( U_{\sigma_1} \) as

\[ K(X^a, \overline{X}^a) = (r_1 - r_2 - r_3) \log \tilde{A} + \frac{1}{3}(r_1 - r_2 - r_3) \log \tilde{E} \]
\[ - r_1 \log \left\{ \frac{E^{2/3} - 4(3\tilde{A}C - \tilde{B}^2) - 2\tilde{B}E^{1/3}}{6AE^{1/3} + |X|^2[E^{2/3} - 4(3\tilde{A}C - \tilde{B}^2) - 2\tilde{B}E^{1/3}]} \right\} \]
\[ + r_2 \log \left\{ \frac{6AE^{1/3} + |X|^2[E^{2/3} - 4(3\tilde{A}C - \tilde{B}^2) - 2\tilde{B}E^{1/3}]}{6AE^{1/3} + |X|^2[E^{2/3} - 4(3\tilde{A}C - \tilde{B}^2) - 2\tilde{B}E^{1/3}]} \right\} \]
\[ - r_2 \frac{|X|^2[E^{2/3} - 4(3\tilde{A}C - \tilde{B}^2) - 2\tilde{B}E^{1/3}]}{6AE^{1/3} + |X|^2[E^{2/3} - 4(3\tilde{A}C - \tilde{B}^2) - 2\tilde{B}E^{1/3}]} \].
\[
- r_3 \frac{|X^1|^2 [\tilde{E}^{2/3} - 4(3\tilde{A}\tilde{C} - \tilde{B}^2) - 2\tilde{B}\tilde{E}^{2/3}]}{6\tilde{A}\tilde{E}^{1/3} + |X^1|^2 [\tilde{E}^{2/3} - 4(3\tilde{A}\tilde{C} - \tilde{B}^2) - 2\tilde{B}\tilde{E}^{2/3}]}
+ \text{(constant terms)},
\]

(3.21)

where
\[
\tilde{A} = |X^1|^2 |X^2|^2 ,
\]
\[
\tilde{B} = (-r_1 + r_2 + r_3) + |X^1|^2 + |X^2|^2 ,
\]
\[
\tilde{C} = 1 + (-r_1 + r_3)|X^1|^2 + (-r_1 + r_2)|X^2|^2 ,
\]
\[
\tilde{E} = 36\tilde{A}\tilde{B}\tilde{C} + 108r_1\tilde{A}^2 - 8\tilde{B}^3
+ 12\sqrt{3}\tilde{A}(4\tilde{A}\tilde{C}^3 - \tilde{B}^2\tilde{C}^2 + 18r_1\tilde{A}\tilde{B}\tilde{C} + 27(r_1)^2\tilde{A}^2 - 4r_1\tilde{B}^3)^{1/2}.
\]

Thus we have obtained the Lagrangian of the supersymmetric NLSM on \( \mathbb{B}_2 \).

Let us give one comment on the metric from the Kähler potential (3.21). The metric from (3.21) also seems to be singular at some points, but one can arrange (3.21) in order that delta functions (for example, \( \delta^2(x^2, \pi^2) \)) do not appear in the metric. Thus we have constructed a well-defined Kähler potential of \( \mathbb{B}_2 \).

4 Discussion

In this paper we have explicitly constructed Kähler potentials of some two-dimensional toric varieties, i.e., Hirzebruch surfaces \( \mathbb{F}_k \) \( (k = 1, 2) \) and del Pezzo surface \( \mathbb{B}_2 \). In principle we can construct the Kähler metrics, curvature tensors and more objects on these surfaces from the Kähler potentials.

In the language of algebraic geometry, one have discussed only topological aspects of string theory on nontrivial backgrounds, i.e., one have not been able to investigate some dynamics of theories. Let us give some examples. String theory on toric varieties have been considered in the context of string duality. Iqbal, Neitzke and Vafa investigated a correspondence between toroidal compactifications of M-theory and del Pezzo surfaces [16]. They called this correspondence a mysterious duality. But they considered them only in terms of cohomologies and exceptional curves in del Pezzo. There is another example: String theory with D-branes on curved backgrounds are discussed by Hori, Iqbal and Vafa in terms of GLSMs and its mirror dual models [7]. They also investigated string theory only in terms of algebro-geometric properties. By using the Kähler potentials of del Pezzo which we have constructed, we will directly check this correspondence at supergravity level and will be able to obtain deeper insights of the mysterious duality, and string theory dynamics with D-branes wrapped on non-trivial cycles.

We are able to give one comment on the geometric engineering. In [13], we have constructed Kähler potentials and metrics of the canonical line bundles on Einstein-Kähler manifolds as noncompact
Calabi-Yau manifolds. Thus we will be able to construct Kähler potentials on canonical bundle over toric varieties with vanishing first Chern classes.

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Appendix

A Toric Fans

In this appendix we consider complex two-dimensional toric varieties, which are the projective space $\mathbb{C}P^2$ (a basic example of toric varieties), Hirzebruch surfaces $\mathbb{F}_k$ and del Pezzo surfaces $\mathbb{B}_k$. We introduce toric fans of these varieties, and discuss local patches and their transition rules. The arguments presented in this appendix are given by the famous TASI lecture by Greene [15] in detail.

Now let us introduce some definitions. Let $N = \mathbb{Z}^d$ be an $d$-dimensional lattice and $N_\mathbb{R} = N \otimes \mathbb{Z} \mathbb{R}$ an extension of it. The dual lattice of $N$ and its $\mathbb{R}$ extension are defined by $M = \mathbb{Z}^d = \text{Hom}(N, \mathbb{Z})$ and $M_\mathbb{R} = M \otimes \mathbb{Z} \mathbb{R}$, where Hom means homomorphism. We denote the inner product between two vectors $n \in N_\mathbb{R}$ and $m \in M_\mathbb{R}$ as $\langle m, n \rangle$.

We define a $d_i$-dimensional strongly convex rational polyhedral cone $\sigma_i$ in $N_\mathbb{R}$ as follows:

$$\sigma_i \cap N_\mathbb{R} = \mathbb{R}_{\geq 0} n_{i1} + \mathbb{R}_{\geq 0} n_{i2} + \cdots + \mathbb{R}_{\geq 0} n_{id_i},$$

(A.1)

where $n_{ij}$ $(j = 1, 2, \cdots, d_i)$ are elements of $N$ such that

$$3^j m \in M_\mathbb{R}, \quad \langle m, n_{ij} \rangle > 0 \quad \text{for} \quad \forall j.$$

We refer to strongly convex rational polyhedral cone as simply cone.

A fan $\Delta$ is defined as a collection of cones $\sigma_i$ which satisfy the requirement that the face of any cone in $\Delta$ is also in $\Delta$.

The dual cones $\check{\sigma}_i$ are defined as

$$\check{\sigma}_i = \{ m_i \in M ; \quad \langle m_i, n_{jk} \rangle \geq 0 \quad \text{for} \quad \forall n_{jk} \in \sigma_i \}.$$

(A.2)
Now we choose a finite set of elements \( m_{ij} \in M \) such that
\[
\tilde{\sigma}_i \cap M_R = \mathbb{R}_{\geq 0} m_{i1} + \mathbb{R}_{\geq 0} m_{i2} + \cdots + \mathbb{R}_{\geq 0} m_{id_i}.
\] (A.3)

We then find a finite set of some relations as
\[
\sum_{j=1}^{d_i} p_{s,j} m_{ij} = 0, \quad p_{s,j} \in \mathbb{Z}, \quad s = 1, 2, \ldots, R,
\]
\[
\sum_{j=1}^{d_i} p_j m_{ij} = 0, \quad p_j = \sum_{s=1}^{R} \mu_s p_{s,j}, \quad \mu_s \in \mathbb{Z},
\]
where \( R \) and \( \mu_s \) are some finite integers. We associate a local coordinate patch \( U_{\sigma_i} \) to the cone \( \sigma_i \) by
\[
U_{\sigma_i} = \left\{ (u_{i1}, u_{i2}, \cdots, u_{id_i}) \in \mathbb{C}^{d_i}; \prod_{j=1}^{d_i} (u_{ij})^{p_{s,j}} = 1 \quad \text{for} \quad \forall s \right\},
\]
where \( u_{ij} \) are local coordinates in \( U_{\sigma_i} \).

Now we consider the coordinate transformations where some local patches overlap such as \( U_{\sigma_i} \cap U_{\sigma_j} \).

We prepare a complete set of relations of the form
\[
\sum_{\ell=1}^{d_i} q_{\ell} m_{i\ell} + \sum_{\ell'=1}^{d_j} q'_{\ell'} m_{j\ell'} = 0, \quad q_{\ell}, q'_{\ell'} \in \mathbb{Z}.
\] (A.4a)

For each of these relations we impose the coordinate transition relations as
\[
\prod_{\ell=1}^{d_i} (u_{i\ell})^{q_{\ell}} \prod_{\ell'=1}^{d_j} (u_{j\ell'})^{q'_{\ell'}} = 1.
\] (A.4b)

Using these relations we give some local descriptions of toric varieties.

The \( k \)-dimensional toric variety is also viewed as the Kähler quotient space as
\[
\frac{\mathbb{C}^d - F_\Delta}{(\mathbb{C}^*)^{d-k}},
\] (A.5)
where \( d \) is the number of one-dimensional cones in the fan \( \Delta \), and \( F_\Delta \) a subspace of \( \mathbb{C}^d \) determined by \( \Delta \). We parametrize \( \mathbb{C}^d \) in terms of \( d \) homogeneous coordinates \( z_i \), which are associated with one-dimensional cones \( v_i \) in \( \Delta \). Here we introduce the primitive generators \( n(v_i) \) of \( v_i \), i.e., one-dimensional cones are described as \( v_i = \mathbb{R}_{\geq 0} n(v_i) \).

Now let us define the action of \((\mathbb{C}^*)^{d-k}\) as follows:
\[
(\mathbb{C}^*)^{d-k} : z_i \mapsto \prod_{a=1}^{d-n} (\lambda_a)^{q_{a}} z_i, \quad (\lambda_1, \lambda_2, \cdots, \lambda_{d-k}) \in (\mathbb{C}^*)^{d-k},
\] (A.6a)
where \( q^a_i \) are integers which satisfy the following linear relations among \( \{n(v_i)\} \):

\[
\sum_{v_i \in \Delta} q^a_i n(v_i) = 0 \quad \text{for} \quad a = 1, 2, \cdots, d-k .
\] (A.6b)

Local coordinates \( \{u_{ij}\} \) in the local patch \( U_{\sigma_i} \) are represented by homogeneous coordinates as

\[
u_{ij} = \prod_{v_\ell \in \Delta} z^{{(m_{ij}, n(v_\ell))}}_{\ell}.
\] (A.7)

Note that local coordinates are invariant under the \((\mathbb{C}^*)^{d-k}\) action given by (A.6).

### A.1 Projective space

Let us consider the toric fan which represents the two-dimensional projective space \( \mathbb{C}P^2 \). This toric fan consists of various dimensional cones. Zero-dimensional cone is the origin. We define one-dimensional cones \( v_i \) and two-dimensional cones \( \sigma_i \) as

\[
\begin{align*}
  v_1 \cap N_R &= \mathbb{R}_{\geq 0} \vec{n}_1 \\
  v_2 \cap N_R &= \mathbb{R}_{\geq 0} (-\vec{n}_1 - \vec{n}_2) \\
  v_3 \cap N_R &= \mathbb{R}_{\geq 0} \vec{n}_2
\end{align*}
\]

\[
\begin{align*}
  \sigma_1 \cap N_R &= \mathbb{R}_{\geq 0} \vec{n}_1 + \mathbb{R}_{\geq 0} \vec{n}_2 \\
  \sigma_2 \cap N_R &= \mathbb{R}_{\geq 0} \vec{n}_2 + \mathbb{R}_{\geq 0} (-\vec{n}_1 - \vec{n}_2) \\
  \sigma_3 \cap N_R &= \mathbb{R}_{\geq 0} \vec{n}_1 + \mathbb{R}_{\geq 0} (-\vec{n}_1 - \vec{n}_2)
\end{align*}
\]

where \( \vec{n}_1 = (1, 0) \) and \( \vec{n}_2 = (0, 1) \) are the basis vectors in the space \( N \). Note that some one-dimensional cones \( v_i \) are elements of some two-dimensional cones \( \sigma_i \). For example, \( v_1 \) and \( v_3 \) are the elements of \( \sigma_1 \). The fan \( \Delta \) consists of zero-, one- and two-dimensional cones. It is described as Figure 1.

![Toric fan of the projective space \( \mathbb{C}P^2 \) in \( N_R \).](image)

From the definition of dual cones in (A.2), we describe two-dimensional dual cones \( \check{\sigma}_i \) as follows:

\[
\begin{align*}
  \check{\sigma}_1 \cap M_R &= \mathbb{R}_{\geq 0} m_{11} + \mathbb{R}_{\geq 0} m_{12} , \\
  m_{11} &= \vec{m}_1 , \quad m_{12} = \vec{m}_2 , \\
  \check{\sigma}_2 \cap M_R &= \mathbb{R}_{\geq 0} m_{21} + \mathbb{R}_{\geq 0} m_{22} , \\
  m_{21} &= -\vec{m}_1 , \quad m_{22} = -\vec{m}_1 + \vec{m}_2 , \\
  \check{\sigma}_3 \cap M_R &= \mathbb{R}_{\geq 0} m_{31} + \mathbb{R}_{\geq 0} m_{32} , \\
  m_{31} &= \vec{m}_1 - \vec{m}_2 , \quad m_{32} = -\vec{m}_2 ,
\end{align*}
\]
where $m_{ij}$ are elements of dual cones $\bar{\sigma}_i$. Note that $\bar{m}_1 = (1, 0)$ and $\bar{m}_2 = (0, 1)$ are basis vectors in dual lattice $M$. Since there are some relations among these elements represented in (A.4), we obtain transition rules among local coordinates $u_{ij}$ in local patches $U_{\sigma_i}$:

\[
\begin{align*}
\begin{cases}
m_{21} + m_{11} = 0 \\
m_{22} + m_{11} - m_{12} = 0
\end{cases} \rightarrow \begin{cases}
u_{21} = (u_{11})^{-1} \\
u_{22} = (u_{11})^{-1}u_{12}
\end{cases} \text{ in } U_{\sigma_1} \cap U_{\sigma_2},
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
m_{31} - m_{11} + m_{12} = 0 \\
m_{32} + m_{12} = 0
\end{cases} \rightarrow \begin{cases}
u_{31} = u_{11}(u_{12})^{-1} \\
u_{32} = (u_{12})^{-1}
\end{cases} \text{ in } U_{\sigma_1} \cap U_{\sigma_3},
\end{align*}
\]

where $(u_{i1}, u_{i2})$ are local coordinates in $U_{\sigma_i}$. We summarize these transition rules in Figure 2:

```
(x, y) ∈ U_{σ_1}
```

```
(x^{-1}, x^{-1}y) ∈ U_{σ_2}
```

```
(xy^{-1}, y^{-1}) ∈ U_{σ_3}
```

**Figure 2: Transition rules among local patches of $\mathbb{CP}^2$.**

Let us define the homogeneous coordinates $\{z_i\}$ of $\mathbb{CP}^2$ and consider the relations with local coordinates $\{u_{ij}\}$ in the local patch $U_{\sigma_i}$. In terms of (A.6) we find the following relation among primitive generators of one-dimensional cones:

\[
n(v_1) + n(v_2) + n(v_3) = 0 \quad \text{where} \quad \begin{cases}
n(v_1) = \bar{n}_1 \\
n(v_2) = -\bar{n}_1 - \bar{n}_2 \\
n(v_3) = \bar{n}_2
\end{cases}
\]

Thus we define the homogeneous coordinates $z_i$ and the integers $q_i$ of $\mathbb{CP}^2$ as Table 4:

| $q_i$ | $z_1$ | $z_2$ | $z_3$ |
|------|------|------|------|
| 1    | 1    | 1    | 1    |

**Table 4: Homogeneous coordinates and integers of $\mathbb{CP}^2$.**

Note that we regard homogeneous coordinates $z_i$ and integers $q_i$ as chiral superfields $\Phi^i$ and their $U(1)$ charges $Q_i$ in the GLSM (see Table 1).
We describe the relationships between homogeneous coordinates $z_i$ and local coordinates $u_{ij}$ in $U_{\sigma_i}$ in terms of (A.7):

$$
(u_{11}, u_{12}) = \left( \frac{z_1}{z_2}, \frac{z_3}{z_2} \right) \quad \text{in} \quad U_{\sigma_1},
$$

$$
(u_{21}, u_{22}) = \left( \frac{z_2}{z_1}, \frac{z_3}{z_1} \right) \quad \text{in} \quad U_{\sigma_2},
$$

$$
(u_{31}, u_{32}) = \left( \frac{z_1}{z_3}, \frac{z_2}{z_3} \right) \quad \text{in} \quad U_{\sigma_3}.
$$

### A.2 Hirzebruch surfaces

Let us consider a toric fan of a Hirzebruch surface $F_k$. It consists of a zero-dimensional cone, one-dimensional cones $v_i$ and two-dimensional cones $\sigma_i$ which are defined as

$$
\begin{align*}
\sigma_1 \cap N_{\mathbb{R}} &= \mathbb{R}_{\geq 0} \vec{n}_1 + \mathbb{R}_{\geq 0} \vec{n}_2 \\
\sigma_2 \cap N_{\mathbb{R}} &= \mathbb{R}_{\geq 0} \vec{n}_2 + \mathbb{R}_{\geq 0} (-\vec{n}_1 - \vec{n}_2) \\
\sigma_3 \cap N_{\mathbb{R}} &= \mathbb{R}_{\geq 0} (-\vec{n}_2) + \mathbb{R}_{\geq 0} (-\vec{n}_1 - \vec{n}_2) \\
\sigma_4 \cap N_{\mathbb{R}} &= \mathbb{R}_{\geq 0} \vec{n}_2 + \mathbb{R}_{\geq 0} (-\vec{n}_2)
\end{align*}
$$

The fan is a collection of these cones. We describe the fan in Figure 3. As we discussed in (A.2), two-dimensional dual cones $\tilde{\sigma}_i$ are defined in terms of basis vectors $\vec{m}_i$ of dual lattice $M$:

$$
\begin{align*}
\tilde{\sigma}_1 \cap M_{\mathbb{R}} &= \mathbb{R}_{\geq 0} m_{11} + \mathbb{R}_{\geq 0} m_{12}, \quad m_{11} = m_1, \quad m_{12} = m_2, \\
\tilde{\sigma}_2 \cap M_{\mathbb{R}} &= \mathbb{R}_{\geq 0} m_{21} + \mathbb{R}_{\geq 0} m_{22}, \quad m_{21} = -m_1, \quad m_{22} = -km_1 + m_2, \\
\tilde{\sigma}_3 \cap M_{\mathbb{R}} &= \mathbb{R}_{\geq 0} m_{31} + \mathbb{R}_{\geq 0} m_{32}, \quad m_{31} = -m_1, \quad m_{32} = km_1 - m_2, \\
\tilde{\sigma}_4 \cap M_{\mathbb{R}} &= \mathbb{R}_{\geq 0} m_{41} + \mathbb{R}_{\geq 0} m_{42}, \quad m_{41} = m_1, \quad m_{42} = -m_2.
\end{align*}
$$
There are some relations among the elements $m_{ij}$. Thus we construct transition rules from them as

\[
\begin{align*}
\begin{cases}
m_{21} + m_{11} = 0 \\
m_{22} + km_{11} - m_{12} = 0
\end{cases} & \quad \rightarrow \quad \begin{cases}
u_{21} = (u_{11})^{-1} \\
u_{22} = (u_{11})^{-k} u_{12}
\end{cases} \quad \text{in } U_{\sigma_1} \cap U_{\sigma_2}, \\
\begin{cases}
m_{31} + m_{11} = 0 \\
m_{32} - km_{11} + m_{12} = 0
\end{cases} & \quad \rightarrow \quad \begin{cases}
u_{31} = (u_{11})^{-1} \\
u_{32} = (u_{11})^k (u_{12})^{-1}
\end{cases} \quad \text{in } U_{\sigma_1} \cap U_{\sigma_3}, \\
\begin{cases}
m_{41} - m_{11} = 0 \\
m_{42} + m_{12} = 0
\end{cases} & \quad \rightarrow \quad \begin{cases}
u_{41} = u_{11} \\
u_{42} = (u_{12})^{-1}
\end{cases} \quad \text{in } U_{\sigma_1} \cap U_{\sigma_4}.
\end{align*}
\]

We denote these transition rules in Figure 4:

\[
(x, y) \in U_{\sigma_1} \quad \Rightarrow \quad (x, y^{-1}) \in U_{\sigma_4},
\]

\[
(x^{-1}, x^{-k} y) \in U_{\sigma_2} \quad \Rightarrow \quad (x^{-1}, x^k y^{-1}) \in U_{\sigma_3}.
\]

Figure 4: Transition rules among local patches of $\mathbb{F}_k$.

The relationships among the primitive generators of one-dimensional cones are described as

\[
n(v_1) + n(v_2) + n(v_3) = 0, \quad n(v_3) + n(v_4) = 0,
\]

where

\[
n(v_1) = \vec{n}_1, \quad n(v_2) = -\vec{n}_1 - \vec{n}_2, \quad n(v_3) = \vec{n}_2, \quad n(v_4) = -\vec{n}_2.
\]

Due to these relations we write down homogeneous coordinates and integers as Table 5.

| $q_1^1$ | $q_2^1$ | $q_1^2$ | $q_2^2$ |
|--------|--------|--------|--------|
| 1      | 1      | $k$    | 0      |
| 0      | 0      | 1      | 1      |

Table 5: Homogeneous coordinates $z_i$ and integers $q_i^a$ of $\mathbb{F}_k$.

Note that we regard homogeneous coordinates $z_i$ and integers $q_i^a$ as chiral superfields $\Phi^i$ and their $U(1) \times U(1)$ charges $Q_i^a$ in the GLSM (see Table 2).

We denote the relationships between homogeneous coordinates and local coordinates:

\[
(u_{11}, u_{12}) = \left( \frac{z_1}{z_2}, \frac{z_3}{(z_2)^k z_4} \right) \quad \text{in } U_{\sigma_1}, \quad (u_{21}, u_{22}) = \left( \frac{z_2}{z_1}, \frac{z_3}{(z_1)^k z_4} \right) \quad \text{in } U_{\sigma_2}, \\
(u_{31}, u_{32}) = \left( \frac{z_2}{z_1}, \frac{(z_1)^k z_4}{z_3} \right) \quad \text{in } U_{\sigma_3}, \quad (u_{41}, u_{42}) = \left( \frac{z_1}{z_2}, \frac{(z_2)^k z_3}{z_4} \right) \quad \text{in } U_{\sigma_4}.
\]

18
A.3 del Pezzo surfaces

Here we consider a toric data of del Pezzo surface $\mathbb{B}_2$. We define one- and two-dimensional cones as follows:

\[
\begin{align*}
\sigma_1 \cap N_\mathbb{R} &= \mathbb{R}_{\geq 0} \bar{n}_1 \\
\sigma_2 \cap N_\mathbb{R} &= \mathbb{R}_{\geq 0} \bar{n}_2 \\
\sigma_3 \cap N_\mathbb{R} &= \mathbb{R}_{\geq 0} (-\bar{n}_1 - \bar{n}_2) \\
\sigma_4 \cap N_\mathbb{R} &= \mathbb{R}_{\geq 0} \bar{n}_2 \\
\sigma_5 \cap N_\mathbb{R} &= \mathbb{R}_{\geq 0} (-\bar{n}_1)
\end{align*}
\]

The toric fan $\Delta$ consists of these cones. We describe $\Delta$ in Figure 5.

![Figure 5: Toric fan of the del Pezzo surface $\mathbb{B}_2$ in $N_\mathbb{R}$](image)

The dual cones $\check{\sigma}_i$ are defined in terms of basis vectors $\check{m}_i$:

\[
\begin{align*}
\check{\sigma}_1 \cap M_\mathbb{R} &= \mathbb{R}_{\geq 0} m_{11} + \mathbb{R}_{\geq 0} m_{12} \\
\check{\sigma}_2 \cap M_\mathbb{R} &= \mathbb{R}_{\geq 0} m_{21} + \mathbb{R}_{\geq 0} m_{22} \\
\check{\sigma}_3 \cap M_\mathbb{R} &= \mathbb{R}_{\geq 0} m_{31} + \mathbb{R}_{\geq 0} m_{32} \\
\check{\sigma}_4 \cap M_\mathbb{R} &= \mathbb{R}_{\geq 0} m_{41} + \mathbb{R}_{\geq 0} m_{42} \\
\check{\sigma}_5 \cap M_\mathbb{R} &= \mathbb{R}_{\geq 0} m_{51} + \mathbb{R}_{\geq 0} m_{52}
\end{align*}
\]

In terms of these data we determine the transition rules of local coordinates among local patches:

\[
\begin{align*}
\begin{cases}
m_{21} + m_{11} = 0 \\
m_{22} - m_{12} = 0
\end{cases} &\quad \rightarrow \quad \begin{cases}
u_{21} = (u_{11})^{-1} \\
u_{22} = u_{12}
\end{cases} \quad \text{in } U_{\sigma_1 \cap U_{\sigma_2}}, \\
\begin{cases}
m_{31} + m_{11} - m_{12} = 0 \\
m_{32} + m_{12} = 0
\end{cases} &\quad \rightarrow \quad \begin{cases}
u_{31} = (u_{11})^{-1} u_{12} \\
u_{32} = (u_{12})^{-1}
\end{cases} \quad \text{in } U_{\sigma_1 \cap U_{\sigma_3}},
\end{align*}
\]

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\[
\begin{align*}
\left\{ \begin{array}{c}
m_{41} - m_{11} + m_{12} = 0 \\
m_{42} + m_{11} = 0 \\
m_{51} - m_{11} = 0 \\
m_{52} + m_{12} = 0 
\end{array} \right. \\
\rightarrow \\
\left\{ \begin{array}{c}
u_{41} = u_{11}(u_{12})^{-1} \\
u_{42} = (u_{11})^{-1} \\
u_{51} = u_{11} \\
u_{52} = (u_{12})^{-1} 
\end{array} \right. \quad \text{in } U_{\sigma_1} \cap U_{\sigma_4},
\end{align*}
\]

where \((u_{11}, u_{12})\) are local coordinates in \(U_{\sigma_i}\). We summarize these transition rules in Figure 6:

![Diagram showing transition rules among local patches of \(B_2\).](image)

Figure 6: Transition rules among local patches of \(B_2\).

There are some relationships among the primitive generators:

\[
\begin{align*}
\left\{ \begin{array}{c}
n(v_1) + n(v_2) + n(v_3) = 0 \\
n(v_3) + n(v_4) = 0 \\
n(v_1) + n(v_5) = 0 
\end{array} \right. \\
\rightarrow \\
\left\{ \begin{array}{c}
n(v_1) = \bar{n}_1 \\
n(v_2) = -\bar{n}_1 - \bar{n}_2 \\
n(v_3) = \bar{n}_2 \\
n(v_4) = -\bar{n}_2 \\
n(v_5) = -\bar{n}_1 
\end{array} \right. \quad \text{where}
\end{align*}
\]

Due to these relations we define the homogeneous coordinates and integers as Table 6:

| \(q_i^1\) | \(q_i^2\) | \(q_i^3\) |
|---|---|---|
| 1 | 0 | 1 |
| 1 | 1 | 0 |
| 0 | 0 | 1 |
| 0 | 1 | 0 |
| 0 | 0 | 1 |

Table 6: Homogeneous coordinates \(z_i\) and integers \(q_i^a\) of \(B_2\).

Note that we regard homogeneous coordinates \(z_i\) and integers \(q_i^a\) as chiral superfields \(\Phi_i\) and their \(U(1)^3\) charges \(Q_i^a\) in the GLSM (see Table 3).
We discuss the relationships between local coordinates and homogeneous coordinates in terms of (A.7) as follows:

\[
(u_{11}, u_{12}) = \left( \frac{z_1}{z_2}, \frac{z_3}{z_4} \right) \quad \text{in} \quad U_{\sigma_1}, \quad (u_{21}, u_{22}) = \left( \frac{z_2 z_5}{z_1}, \frac{z_3}{z_4} \right) \quad \text{in} \quad U_{\sigma_2}, \\
(u_{31}, u_{32}) = \left( \frac{z_3 z_5}{z_1 z_4}, \frac{z_2}{z_3} \right) \quad \text{in} \quad U_{\sigma_3}, \quad (u_{41}, u_{42}) = \left( \frac{z_1 z_4}{z_3 z_5}, \frac{z_2}{z_3} \right) \quad \text{in} \quad U_{\sigma_4}, \\
(u_{51}, u_{52}) = \left( \frac{z_1}{z_2}, \frac{z_3}{z_4} \right) \quad \text{in} \quad U_{\sigma_5}.
\]

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