VARIATIONAL PRINCIPLE FOR WEAKLY DEPENDENT RANDOM FIELDS

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ABSTRACT. Using an alternative notion of entropy introduced by Datta, the max-entropy, we present a new simplified framework to study the minimizers of the specific free energy for random fields which are weakly dependent in the sense of Lewis, Pfister, and Sullivan. The framework is then applied to derive the variational principle for the loop $O(n)$ model and the Ising model in a random percolation environment in the nonmagnetic phase, and we explain how to extend the variational principle to similar models. To demonstrate the generality of the framework, we indicate how to naturally fit into it the variational principle for models with an absolutely summable interaction potential, and for the random-cluster model.

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1. INTRODUCTION

1.1. Random fields with long-range interactions. One of the great results in statistical physics is the variational principle, which asserts that a shift-invariant infinite-volume measure is a Gibbs measure if and only if it minimizes the specific free energy. The class of models which fall under the scope of the variational principle is extremely broad. Models for which the interaction potential is absolutely summable were covered in \cite{Geo11}. There have been numerous attempts to extend or generalize the variational principle beyond, often in relation to a study of the points of continuity or quasilocality of the specification; a non-exhaustive list includes \cite{PV95, Sep98, MRV99a, EMSS00, FLR03, EV04, KLR04}. Further investigation into the variational principle was carried out in relation to renormalization \cite{EFS93, Lef99}, the large deviations principle \cite{Sep93a, Sep93b, Sep95}, and projections or restrictions of Gibbs measures \cite{MRV99b, Ver10}. Other works on the variational principle in the infinite-volume setting include \cite{SZ91, Zeg91, Fer06}. Despite those efforts there are still some interesting models for which it is not known if the variational principle holds true or not. Among those are various models of random fields in random environments: a noteworthy example is the Ising model on a random subgraph of the square lattice obtained from independent percolation. The inherent problem derives from the fact that the strength of the interactions between particles does not decay uniformly with the range.

This model belongs to a large, natural class of models known as weakly dependent: this term is due to Lewis, Pfister and Sullivan \cite{LPS95}. We develop a streamlined framework for studying the minimizers of the specific free energy within this class. The framework allows one to efficiently deduce the variational principle for many interesting weakly dependent models. Our discussion reviews the absolutely summable setting of \cite{Geo11}, and the random-cluster model \cite{Sep98} (see \cite{Gri06} for a general introduction). We break new ground by proving the variational principle for the Ising model in a random environment, in the nonmagnetic phase. This significantly extends the results of \cite{KLR04}. We furthermore deduce the variational principle for the loop $O(n)$ model (see \cite{PS17} for a general introduction) by extension of the discussion of the random-cluster model, and we explain how these models represent any model where the
nonvanishing long-range interaction is due to potential associated with clusters, level sets, paths, or other large geometrical objects that arise from the local structure.

1.2. The specific free energy. The specific free energy and a suitable characterization for it are of central importance to the study of the variational principle. A natural first question is thus to ask about restrictions on the model that guarantee that the specific free energy is well-behaved. Candidates are the previously mentioned weakly dependent \cite{LPS95}, and the more general asymptotically decoupled. The latter was introduced by Pfister \cite{Pfi02}. While either restriction guarantees a well-defined specific free energy, the former is more amenable to arguments involving regular conditional probability distributions, and is therefore better for studying the variational principle. Remark that we shall define the specific free energy in terms of the specification that characterizes the model, unlike in \cite{LPS95,Pfi02} where it is defined in terms of a reference random field. Our definition of weakly dependent is therefore differently.

There is a simple and natural definition of a weakly dependent specification once we introduce the max-entropy of two measures. The max-entropy of some measure \( \mu \) relative to another measure \( \nu \) equals
\[
\mathcal{H}^\text{max}(\mu|\nu) := \log \inf \{ \lambda \geq 0 : \mu \leq \lambda \nu \},
\]
and was introduced by Datta in \cite{Dat09}. We call a specification weakly dependent if the max-entropy between any two finite-volume Gibbs measures on a box \( \Lambda \subset \mathbb{Z}^d \) is of order \( o(|\Lambda|) \) as \( \Lambda \) grows large.

The class of weakly dependent models is rich, and it is not hard to prove that the various models that were mentioned are all weakly dependent. If the model of interest is weakly dependent, then the specific free energy has all the usual properties: its level sets (which are sets of shift-invariant random fields) are compact in the topology of local convergence, and there exist shift-invariant random fields that have zero specific free energy.

1.3. Main results. Consider a weakly dependent specification. We call a random field a minimizer if it is shift-invariant and has zero specific free energy with respect to this specification. It is a corollary of the definition of the specific free energy that shift-invariant Dobrushin-Lanford-Ruelle (DLR) states are minimizers. We show that a shift-invariant random field is a minimizer if and only if it is a limit of finite-volume Gibbs measures, where we allow mixed boundary conditions. If \( \mu \) is a minimizer, then we derive properties of the conditional probability distribution of \( \mu \) in a box \( \Lambda \), conditioned on what happens outside of \( \Lambda \). If \( \mu \) is supported on the points of continuity of the specification corresponding to the model, then we show that \( \mu \) is a DLR state, and almost Gibbs. In general, we demonstrate that all minimizers have finite energy in the sense of Burton and Keane, so that we are able to make their case for almost sure uniqueness of the infinite cluster (if this is relevant for the model under consideration).

The variational principle asserts that the minimizers of the specific free energy coincide with the shift-invariant almost Gibbs measures. The framework provides a clear route to demonstrating its validity for weakly dependent models: it is sufficient to prove that minimizers of the specific free energy are supported on the points of continuity of the specification, and in deriving this one may assume all the properties that minimizers of the specific free energy automatically have.

We apply the framework to all models that were previously mentioned. First, we show how to fit into our framework the known variational principles for models with an absolutely summable interaction potential \cite{Geo11}, and for the random-cluster model \cite{Sep98}. Then, we derive the variational principle for the loop \( O(n) \) model, and by extension we assert that the variational principle must hold true for a large class of models where the long-range interaction is due to weight on percolation clusters (such as for the random-cluster model), level sets, loops, or other large geometrical objects which arise from the local structure. Next, we derive the variational principle for the Ising model in a random percolation environment in the nonmagnetic phase. This complements the work of \cite{KLR04}, where it is shown that the variational principle fails in the nonmagnetic phase. The authors believe that for a large class of models in a random environment, the proposed framework significantly reduces the complexity of determining wether or not the variational principle holds true.

Finally, it should be remarked that in all our work we shall never require the state space to be finite; the framework works for any standard Borel space, much like the setting of Georgii \cite{Geo11}.

1.4. Structure. The article is organized as follows. In Section 2 we introduce the various mathematical objects necessary to define and study the specific free energy. In Section 3 we give a presentation of our main results. In Section 4 we show how to define the specific free energy for weakly dependent specifications, and we prove some of its properties. In Section 5 we give a characterization of the minimizers of the
specific free energy. In Section 4 we show how to derive easily from our framework various versions of the variational principle.

2. DEFINITIONS

If \((X, \mathcal{X})\) is any measurable space, then write \(\mathcal{P}(X, \mathcal{X})\) for the set of probability measures on \((X, \mathcal{X})\), and \(\mathcal{M}(X, \mathcal{X})\) for the set of \(\sigma\)-finite measures \(\mu\) with \(\mu(X) > 0\). In this paper we only consider measurable spaces that are standard Borel spaces. We shall follow the notation of Georgii [Geo11] wherever possible.

2.1. Random fields. We are concerned with the study of random fields. Fix a dimension \(d \in \mathbb{N}\) and a standard Borel space \((E, \mathcal{E})\) throughout this article. The set \(S := \mathbb{Z}^d\) is called the parameter set, and \((E, \mathcal{E})\) is called the state space. Elements of \(S\) are called sites. A configuration is a function \(\omega\) that assigns to each site \(x \in S\) a state \(\omega_x \in E\). Write \(\Omega := E^S\) for the set of configurations, and \(\mathcal{F}\) for the product \(\sigma\)-algebra \(\mathcal{B}^S\) on \(\Omega\). A random field is a probability measure on configurations: the set of random fields is \(\mathcal{P}(\Omega, \mathcal{F})\).

Define, for each site \(x \in S\), the measurable function \(\sigma_x : \Omega \to E\), \(\omega \mapsto \omega_x\). For any \(\Lambda \subseteq S\), we shall write \(\mathcal{F}_\Lambda := \sigma(\{\sigma_x : x \in \Lambda\}) \subset \mathcal{F}\). Write furthermore \(\sigma_\Lambda\) for the canonical projection map \(\Omega = E^S \to E^\Lambda\), and observe that \(\sigma_\Lambda\) extends canonically to a bijection from \(\mathcal{F}_\Lambda\) to \(\mathcal{B}(E^\Lambda, \mathcal{E}^\Lambda)\). Define \(\alpha_\Lambda := \sigma_\Lambda(\omega)\) for \(\omega \in \Omega\), and if \(\mu \in \mathcal{P}(\Omega, \mathcal{F})\) for some \(\mathcal{F}_\Lambda \subseteq \mathcal{F} \subseteq \mathcal{F}\), then write \(\mu_\Lambda := \sigma_\Lambda(\mu) \in \mathcal{P}(E^\Lambda, \mathcal{E}^\Lambda)\). If \(f\) is an \(\mathcal{F}_\Lambda\)-measurable function on \(\Omega\) and \(g\) an \(\mathcal{E}^\Lambda\)-measurable function on \(E^\Lambda\), then we shall without further notice write \(\int f \, d\mu = \int g \circ \sigma_\Lambda^{-1} \, d\nu\) on \(\Omega\) and \(g\) for the \(\mathcal{F}_\Lambda\)-measurable function \(f \circ \sigma_\Lambda^{-1}\) on \(E^\Lambda\). Finally, if \(\Lambda \subseteq \Delta \subseteq S\), then write also \(\sigma_\Delta\) for the canonical projection map \(E^S \to E^\Delta\), and if \(\omega \in E^\Delta\) and \(\zeta \in E^{S - \Delta}\), then write \(\omega \zeta\) for the unique element of \(E^S\) such that \(\sigma_\Delta(\omega \zeta) = \omega\) and \(\sigma_{S - \Delta}(\omega \zeta) = \zeta\).

Define, for every \(x \in \mathbb{Z}^d\), the map \(\theta_x : \mathbb{Z}^d \to \mathbb{Z}^d\), \(y \mapsto y + x\). Each map \(\theta_x\) is called a shift. Write \(\Theta\) for the set of shifts, that is, \(\Theta = \{\theta : x \in \mathbb{Z}^d\}\). If \(\omega \in \Omega\) and \(\theta \in \Theta\), then write \(\theta \omega\) for the configuration in \(\Omega\) satisfying \((\theta \omega)_x = \omega_{\theta x}\) for every \(x \in S\). Similarly, define \(\Theta A := \{\theta \omega : \omega \in A\}\) for \(A \subseteq \mathcal{F}\). A random field \(\mu \in \mathcal{P}(\Omega, \mathcal{F})\) is called shift-invariant if \(\mu(\theta A) = \mu(A)\) for any \(A \subseteq \mathcal{F}\) and \(\theta \in \Theta\). Write \(\mathcal{P}_\Theta(\Omega, \mathcal{F})\) for the collection of shift-invariant random fields.

2.2. Entropy and max-entropy. Consider two \(\sigma\)-finite measures \(\mu, \nu \in \mathcal{M}(X, \mathcal{X})\) on a standard Borel space \((X, \mathcal{X})\). The entropy of \(\mu\) relative to \(\nu\) is defined by

\[
H(\mu | \nu) := \begin{cases} 
\frac{\mu(\log f)}{\log \nu(f)} & \text{if } \mu \ll \nu \text{ where } f := d\mu/d\nu, \\
\infty & \text{otherwise.}
\end{cases}
\]

The max-entropy of \(\mu\) relative to \(\nu\) is defined by

\[
H^\infty(\mu | \nu) := \inf \{ \lambda \geq 0 : \mu \ll \lambda \nu \} = \sup \{ \log f : \text{if } \mu \ll \nu \text{ where } f := d\mu/d\nu, \}
\]

Note that both entropies are nonnegative when \(\mu\) and \(\nu\) are probability measures — if they are indeed probability measures, then each entropy equals zero if and only if \(\mu = \nu\). If \(\mathcal{F}\) is a sub-\(\sigma\)-algebra of \(\mathcal{X}\), then define \(H_\mathcal{F}(\mu | \nu) := H(\mu | | \nu| \mathcal{F})\). If \((X, \mathcal{X}) = (\Omega, \mathcal{F})\) and \(\Lambda \subseteq \mathcal{F}\), then abbreviate \(H_\mathcal{F}(\mu | \nu)\) to \(H_\Lambda(\mu | \nu)\). Finally, define the max-diameter of a nonempty set \(\mathcal{B} \subseteq \mathcal{M}(X, \mathcal{X})\) by

\[
\operatorname{Diam}^\infty \mathcal{B} := \sup_{\mu, \nu \in \mathcal{B}} H^\infty(\mu | \nu) \geq 0,
\]

where we observe equality if and only if \(\mathcal{B}\) contains exactly one measure.

2.3. Weakly dependent specifications. A specification is a family \(\gamma = (\gamma_\Lambda)_{\Lambda \in \mathcal{F}}\) with the following properties:

1. Each member \(\gamma_\Lambda\) is a probability kernel from \((\Omega, \mathcal{F}_{\mathcal{F}_{S - \Lambda}})\) to \((\Omega, \mathcal{F})\).
2. Each member \(\gamma_\Lambda\) satisfies \(\gamma_\Lambda(A, \omega) = 1(\omega \in A)\) whenever \(A \in \mathcal{F}_{S - \Lambda}\).
3. If \(\Lambda \subseteq \Delta \subseteq \mathcal{F}\), then \(\gamma_\Lambda = \gamma_\Delta\gamma_\Lambda\).

A member \(\gamma_\Lambda\) is called proper if it has the second property; the family \(\gamma\) is called consistent if it has the third property. We fix a specification \(\gamma\) throughout this article. The specification \(\gamma\) is called shift-invariant if \(\gamma_{\theta \Lambda}(A, \omega) = \gamma_\Lambda(A, \theta \omega)\) for any \(A \in \mathcal{F}\), \(\Lambda \in \mathcal{F}\), \(\omega \in \Omega\), \(\theta \in \Theta\).

Fix \(\Lambda \subseteq \mathcal{F}\), this is a probability kernel from \((\Omega, \mathcal{F}_{S - \Lambda})\) to \((\Omega, \mathcal{F})\). Write \(\gamma_\Lambda\) for the unique probability kernel from \((\Omega, \mathcal{F}_{S - \Lambda})\) to \((E^\Lambda, \mathcal{E}^\Lambda)\) such that \(\gamma_\Lambda(\cdot, \omega) = \sigma_\Lambda(\gamma_\Lambda(\cdot, \omega))\) for every \(\omega \in \Omega\). The measure \(\gamma_\Lambda(\cdot, \omega)\) is the finite-volume Gibbs measure on \((E^\Lambda, \mathcal{E}^\Lambda)\) with deterministic boundary.
conditions $\omega$. Of course, the original kernel $\gamma_A$ can be recovered from $\hat{\gamma}_A$ through the equation $\gamma_A(\cdot, \omega) = \hat{\gamma}_A(\cdot, \omega) \times \delta_{\omega_A}$ — this is because $\gamma_A$ is proper. It is often more convenient to define $\hat{\gamma}_A$ than $\gamma_A$ when describing a specific model.

Now fix a random field $\mu \in \mathcal{P}(\Omega, \mathcal{F})$, and consider the finite-volume measure $\mu \hat{\gamma}_A$. This is the finite-volume Gibbs measure on $(E^A, \mathcal{E}^A)$ with mixed boundary conditions $\mu$. Define

$$\mathcal{B}_A(\gamma) := \{ \mu \hat{\gamma}_A : \mu \in \mathcal{P}(\Omega, \mathcal{F}) \} \subset \mathcal{P}(E^A, \mathcal{E}^A):$$

the set of all such finite-volume Gibbs measures. This set is convex because the set of all random fields is convex. For each $n \in \mathbb{N}$, we use the notation $\Delta_n$ for the box

$$\Delta_n := \{-n, \ldots, n\}^d \in \mathcal{I}.$$

The specification $\gamma$ is called weakly dependent if $\gamma$ is shift-invariant and satisfies

$$\text{Diam}^\omega \mathcal{B}_A(\gamma) = o(|\Delta_n|)$$

as $n \to \infty$. For technical reasons we also require that $\text{Diam}^\omega \mathcal{B}_A(\gamma)$ is finite for any $\Lambda \in \mathcal{I}$; this additional condition is not restrictive. Write $\mathbb{S}$ for the collection of weakly dependent specifications.

Before proceeding, it is useful to remark that

$$\text{Diam}^\omega \mathcal{B}_A(\gamma) := \sup_{\mu, \nu} \mathcal{H}^\omega(\mu \hat{\gamma}_A | \nu \hat{\gamma}_A) = \sup_{\omega_0, \xi} \mathcal{H}^\omega(\hat{\gamma}_A(\cdot, \omega) | \hat{\gamma}_A(\cdot, \xi)).$$

it is sufficient to consider deterministic boundary conditions in calculating the max-diameter of $\mathcal{B}_A(\gamma)$. This can be deduced from Fubini’s theorem without effort.

### 2.4. The specific free energy

Consider a shift-invariant random field $\mu$ and a weakly dependent specification $\gamma$. The specific free energy (SFE) of $\mu$ relative to $\gamma$ is defined by

$$h(\mu | \gamma) := \lim_{n \to \infty} |\Delta_n|^{-1} \mathcal{H}_n(\mu | \nu \gamma_\Lambda) \in [0, \infty]$$

where $\nu \in \mathcal{P}(\Omega, \mathcal{F})$. Lemma 4.3 asserts that the limit exists for any $\nu$ and that this limit is independent of the choice of $\nu$. A shift-invariant random field $\mu$ with $h(\mu | \gamma) = 0$ is called a minimizer of $\gamma$. Write $h_0(\gamma)$ for the set of minimizers of $\gamma$.

Now take the perspective of a shift-invariant random field $\mu$. The random field $\mu$ is called weakly dependent if $\mu \in h_0(\gamma)$ for some weakly dependent specification $\gamma$. Write $\mathbb{S}$ for the collection of weakly dependent random fields. If $\mu$ is an arbitrary shift-invariant random field and $\nu$ a weakly dependent random field, then the specific free energy (SFE) of $\mu$ relative to $\nu$ is defined by

$$h(\mu | \nu) := \lim_{n \to \infty} |\Delta_n|^{-1} \mathcal{H}_n(\mu | \nu) \in [0, \infty].$$

Lemma 5.7 asserts that the limit converges for any choice of $\mu$ and $\nu$. The quantity $h(\mu | \nu)$ is also sometimes called the entropy density of $\mu$ with respect to $\nu$. Write $h_0(\nu)$ for the set of shift-invariant random fields $\mu$ with $h(\mu | \nu) = 0$. Measures $\mu \in h_0(\nu)$ are called minimizers of $\nu$.

### 2.5. DLR states

Now consider a random field $\mu$ and a finite set $\Lambda \in \mathcal{I}$. Write $\mu_\Lambda^\omega$ for the regular conditional probability distribution (r.c.p.d.) on $(E^\Lambda, \mathcal{E}^\Lambda)$ of $\mu$ corresponding to the projection map $\sigma_{\Lambda \setminus \omega} : \Omega \to E^{\Lambda \setminus \omega}$. Informally, this is the distribution of $\omega_A$ in $\mu$ given the states of $\omega$ outside $\Lambda$. Suppose that we are given an arbitrary specification $\gamma$. A Dobrushin-Lanford-Ruelle (DLR) state is a random field $\mu \in \mathcal{P}(\Omega, \mathcal{F})$ which satisfies the DLR equation $\mu = \mu \gamma_\Lambda$ for every $\Lambda \in \mathcal{I}$. In other words, $\mu$ is a DLR state if and only if $\mu^\omega = \hat{\gamma}_\Lambda(\cdot, \omega)$ for $\mu$-almost every $\omega \in \Omega$, for each $\Lambda \in \mathcal{I}$. Write $\mathcal{D}(\gamma)$ for the set of DLR states, and $\mathcal{D}_0(\gamma) := \mathcal{D}(\gamma) \cap \mathcal{P}_0(\Omega, \mathcal{F})$ for the set of shift-invariant DLR states.

### 2.6. Topologies

The topology of local convergence or $L$-topology on $\Omega$ is the coarsest topology on $\Omega$ that makes the map $\omega \mapsto \omega_A$ continuous for every $A \subset \mathbb{Z}^d$, with respect to the discrete topology on $E$. This means that $\omega^n \to \omega$ if and only if for any $\Lambda \in \mathcal{I}$, we have $\omega^n_A = \omega_A$ for $n$ sufficiently large.

Consider an arbitrary standard Borel space $(X, \mathcal{X})$. The strong topology on $\mathcal{H}(X, \mathcal{X})$ is the coarsest topology that makes the map $\mu \mapsto \mu(A)$ continuous for every $A \in \mathcal{X}$. If $\mathcal{B} \subset \mathcal{H}(X, \mathcal{X})$ is a convex set of probability measures subject to $\text{Diam}^\omega \mathcal{B}$ being finite, then write $\mathcal{C}(\mathcal{B})$ for the closure of $\mathcal{B}$ in the strong topology. In Lemma 6.5. we present an alternative definition for $\mathcal{C}(\mathcal{B})$, which we demonstrate is equivalent.

The topology of local convergence or $L$-topology on $\mathcal{P}(\Omega, \mathcal{F})$ that makes the map $\mu \mapsto \mu(A)$ continuous for every $A \in \bigcup_{\Lambda \in \mathcal{I}} \mathcal{F}$, this means that $\mu^n \to \mu$ in the $L$-topology if and only if $\sigma_\Lambda(\mu^n) \to \mu_\Lambda$ in the strong topology on $\mathcal{P}(E^\Lambda, \mathcal{E}^\Lambda)$ for every $\Lambda \in \mathcal{I}$.
2.7. Limits of finite-volume Gibbs measures. Let \( \gamma \) be a weakly dependent specification. Write \( \mathcal{W}(\gamma) \) for the set of limits of finite-volume Gibbs measures in the \( L \)-topology, that is,

\[
\mathcal{W}(\gamma) := \{ \mu \in \mathcal{P}(\Omega, \mathcal{F}) : \nu^n \gamma_{\Delta_n} \to \mu \text{ in the } L \text{-topology for some } (\nu^n)_{n \in \mathbb{N}} \subset \mathcal{P}(\Omega, \mathcal{F}) \}.
\]

We write \( \nu^n \gamma_{\Delta_n} \) in this definition and not \( \nu^n \tilde{\gamma}_{\Delta_n} \), so that all measures live in the same space and convergence in the \( L \)-topology makes sense. For simplicity the definition is in terms of the exhaustive sequence \( (\Delta_n)_{n \in \mathbb{N}} \); it is straightforward to verify that the definition is the same if we replace this sequence by any other increasing exhaustive sequence. Write \( \mathcal{W}_\Theta(\gamma) := \mathcal{W}(\gamma) \cap \mathcal{P}_\Theta(\Omega, \mathcal{F}) \). We shall later see that \( h_0(\gamma) = \mathcal{W}_\Theta(\gamma) \).

2.8. Continuity of the specification. Consider a weakly dependent specification \( \gamma \). We are going to define more sets of finite-volume Gibbs measures, now restricting the boundary conditions that are allowed. For any \( \Lambda, \Delta \in \mathcal{F} \) and \( \omega \in \Omega \), define

\[
\mathcal{B}_{\Lambda, \Delta, \omega}(\gamma) := \{ \mu \gamma_{\Delta} : \mu \in \mathcal{P}(\Omega, \mathcal{F}) \text{ such that } \mu_{\Delta} = \delta_{\omega_{\Delta}} \} \subset \mathcal{B}_\Lambda(\gamma).
\]

The sets \( \mathcal{B}_\Lambda(\gamma) \) and \( \mathcal{B}_{\Lambda, \Delta, \omega}(\gamma) \) are convex, and \( \mathcal{B}_{\Lambda, \Delta, \omega}(\gamma) \) is decreasing in \( \Delta \). Define

\[
\mathcal{B}_{\Lambda, \omega}(\gamma) := \cap_{\Delta \in \mathcal{F}} (\mathcal{B}_{\Lambda, \Delta, \omega}(\gamma)) = \cap_{n \in \mathbb{N}} (\mathcal{B}_{\Lambda, \Delta_n, \omega}(\gamma)).
\]

Consider a measure \( \mu \in \mathcal{P}(\mathcal{E}, \mathcal{F}) \). Then \( \mu \in \mathcal{B}_{\Lambda, \omega}(\gamma) \) if and only if \( \nu^n \gamma_{\Delta_n} \to \mu \) in the strong topology for some sequence of random fields \( (\nu^n)_{n \in \mathbb{N}} \) converging to \( \delta_{\omega} \) in the \( L \)-topology.

The alternative characterization of \( \mathcal{B}_{\Lambda, \omega}(\gamma) \) implies that \( \delta_{\omega} \gamma_{\Delta_n} = \hat{\gamma}_{\Lambda}(\cdot) \in \mathcal{B}_{\Lambda, \omega}(\gamma) \). Define

\[
\Omega_\gamma := \{ \omega \in \Omega : \mathcal{B}_{\Lambda, \omega}(\gamma) = \{ \hat{\gamma}_{\Lambda}(\cdot, \omega) \} \text{ for any } \Lambda \in \mathcal{F} \}
\]

\[
= \{ \omega \in \Omega : |\mathcal{B}_{\Lambda, \omega}(\gamma)| = 1 \text{ for any } \Lambda \in \mathcal{F} \}.
\]

In other words, \( \Omega_\gamma \) is the set of configurations \( \omega \in \Omega \) such that the map \( \zeta \mapsto \gamma_{\Lambda}(\cdot, \zeta) \) is continuous — both sides endowed with the \( L \)-topology — at \( \omega \) for any \( \Lambda \in \mathcal{F} \). If \( \omega \in \Omega_\gamma \), then we say that the specification \( \gamma \) is continuous or quasilocal at \( \omega \). If \( \Omega_\gamma = \Omega \), then each DLR state of \( \gamma \) is also called a Gibbs measure. If \( \mu \in \mathcal{B}(\gamma) \) and \( \mu(\Omega_\gamma) = 1 \), then \( \mu \) is called an almost Gibbs measure. This makes sense even if \( \Omega_\gamma \neq \Omega \).

3. Main results

3.1. The specific free energy. Consider a weakly dependent specification \( \gamma \). We prove that for any shift-invariant random field \( \mu \), the SFE

\[
h(\mu \mid \gamma) := \lim_{n \to \infty} |\Delta_n|^{-1} \mathcal{H}_{\Delta_n}(\mu \mid \nu^n \gamma_{\Delta_n}) \in [0, \infty]
\]

is well-defined, and independent of the choice of \( \nu \in \mathcal{P}(\Omega, \mathcal{F}) \) (Lemma 4.3). Moreover, we show that the level sets of the SFE — given by \( \{ h(\cdot \mid \gamma) \leq C \} \subset \mathcal{P}_\Theta(\Omega, \mathcal{F}) \) for \( C \in [0, \infty) \) — are compact in the topology of local convergence, and that \( h_0(\gamma) = \{ h(\cdot \mid \gamma) = 0 \} \) is nonempty (Lemma 4.5). We prove the first half of the variational principle, which asserts that \( \mathcal{B}_\Theta(\gamma) \subset h_0(\gamma) \) (Corollary 4.4).

3.2. Minimizers of the specific free energy. Next, we focus on the set of minimizers \( h_0(\gamma) \) of the weakly dependent specification \( \gamma \). We find some alternative characterizations for the set of minimizers. In particular, if \( \mu \) is a shift-invariant random field, then the following are equivalent:

1. \( \mu \) is a minimizer of \( \gamma \).
2. \( \mu \in \mathcal{W}(\gamma) \), that is, \( \mu \) is a limit of finite-volume Gibbs measures,
3. \( \mu_{\Delta_n} \in \mathcal{B}_{\Lambda, \omega}(\gamma) \) for each \( n \in \mathbb{N} \);

see Lemma 5.3 and Corollary 5.2. Moreover, if \( \mu \) is a minimizer, then we demonstrate that

1. \( \mu \) is almost Gibbs if \( \mu(\Omega_\gamma) = 1 \),
2. \( \mu_{\Delta_n} \in \mathcal{B}_{\Lambda, \omega} \) for \( \mu \)-almost every \( \omega \), for each \( \Lambda \in \mathcal{F} \),
3. \( \mu \) has finite energy, in the sense of Burton and Keane.

The first statement follows almost immediately from the second, see Lemma 5.4 and Corollary 5.5. The third statement requires a short argument, see Corollary 5.6.
3.3. **The relation between \( F \) and \( S \).** Now take a more abstract viewpoint, and consider the set of all weakly dependent random fields \( F \). Choose a weakly dependent specification \( \gamma \in S \) and a minimizer \( \nu \in F \) of \( \gamma \). First, we prove that \( h(\mu|\nu) \) is well-defined and equal to \( h(\mu|\gamma) \) for any shift-invariant random field \( \mu \) (Lemma 4.7). This implies in particular that \( h_0(\nu) = h_0(\gamma) \). For \( \mu, \nu \in F \), we declare \( \mu \sim \nu \) if \( h(\mu|\nu) = 0 \). We prove that \( \sim \) is an equivalence relation. Write \( F^\ast \) for the partition of \( F \) into equivalence classes. This provides a canonical way to partition the set of specifications \( S \) as well: define the map
\[
\Xi: S \rightarrow F^\ast, \gamma \mapsto h_0(\gamma),
\]
and write \( S^\ast \) for the partition of \( S \) into the level sets of \( \Xi \). This makes \( \Xi \) into a bijection from \( S^\ast \) to \( F^\ast \) — the original map \( \Xi \) was surjective by definition a weakly dependent random field.

3.4. **The variational principle in the weakly dependent setting.** Consider a weakly dependent specification \( \gamma \). The previous results provide efficient machinery for attacking the variational principle. Consider an arbitrary shift-invariant random field \( \mu \). The variational principle asserts that
\[
(1) \quad \mu \in h_0(\gamma) \iff \mu \text{ is almost Gibbs with respect to } \gamma.
\]

To prove the variational principle for the model of interest, we must always derive two results. First, we must show that the specification \( \gamma \) corresponding to the model is indeed weakly dependent. Second, one must show that \( h(\Omega_\gamma) = 1 \) for any minimizer \( \mu \) of \( \gamma \). The variational principle then follows from Corollaries 4.4 and 5.5.

Once weak dependence of the specification has been established, the systematic study of the minimizers of the SFE provides a number of useful properties that minimizers of the SFE automatically have — see Subsection 5.2. This usually makes it easier to prove that \( h(\Omega_\gamma) = 1 \) for arbitrary minimizers \( \mu \).

3.5. **Applications.** The weakly dependent setting is very general: it contains most nonpathological non-gradient models that do not have some form of combinatorial exclusion (such as for example the dimer models, which have a non-gradient interpretation but which are not weakly dependent). We start by showing how to naturally fit two known variational principles into our framework. Then we derive the variational principle for the loop \( O(n) \) model, and finally we derive new results for the Ising model in a random percolation environment.

In Subsection 6.1, we show how to efficiently derive the variational principle for models that are defined in terms of an absolutely summable interaction potential. This setting is treated in the classical work of Georgii [Geo11]. For such models we find that \( \Omega = \Omega_\gamma \), meaning that all almost Gibbs measures are in fact Gibbs. In Subsection 6.2, we show how to derive the variational principle for the random-cluster model. The original proof is due to Seppäläinen [Sep98]. The proofs (the one of Seppäläinen and the one presented here) rely on the finite energy of minimizers of the SFE, which implies that there is at most one infinite cluster almost surely with respect to such measures (see Burton and Keane [BK89]). In Subsection 6.3, we discuss how to derive the variational principle for the loop \( O(n) \) model, by analogy with the random-cluster model. We also discuss how to derive the variational principle for similar models. In Subsection 6.4, we prove the variational principle for the Ising model in a random percolation environment, in the nonmagnetic phase. Moreover, we demonstrate that the minimizer of the SFE is unique. These results are new. It implies a dichotomy: it is known that the variational principle fails when the Ising model does magnetize on the percolation clusters [KLR04]. Our new results thus complement that of [KLR04].

4. **The specific free energy**

This section has two main goals. The first goal is to prove Lemma 4.3 which asserts that the SFE is well-defined for weakly dependent specifications. It also provides some useful identities. As an immediate corollary we observe that DLR states minimize the SFE. The second goal is to prove Lemma 4.5 which asserts that the level sets of the SFE are compact in the \( \mathcal{L}^\prime \)-topology, and that there exist measures with zero SFE.

4.1. **Consistency of the definition.** The definition of the SFE relies on two key lemmas. Lemma 4.1 concerns superadditivity of a useful quantity. Lemma 4.2 bounds the difference of two relative entropies in terms of the max-entropy.
**Lemma 4.1.** Let $\gamma$ denote any specification and $\mu$ a random field. Consider a finite pairwise disjoint family of finite sets $(\Lambda_k)_{1 \leq k \leq n} \subseteq \mathcal{F}$, and write $\Lambda := \bigcup_k \Lambda_k \in \mathcal{F}$. Then

$$\inf_{\rho \in \mathcal{P}(\Omega, \mathcal{F})} \mathcal{H}_\Lambda(\mu | \rho \gamma_\Lambda) \geq \sum_k \inf_{\rho \in \mathcal{P}(\Omega, \mathcal{F})} \mathcal{H}_{\Lambda_k}(\mu | \rho \gamma_{\Lambda_k}).$$

**Proof.** Fix $v \in \mathcal{P}(\Omega, \mathcal{F})$, and replace $v$ by $v \gamma_\Lambda$ if the two are not equal. We must demonstrate that

$$\mathcal{H}_\Lambda(\mu | v) \geq \sum_k \inf_{\rho \in \mathcal{P}(\Omega, \mathcal{F})} \mathcal{H}_{\Lambda_k}(\mu | \rho \gamma_{\Lambda_k}).$$

By induction, it is sufficient to consider the case $n = 2$. We have

$$\mathcal{H}_\Lambda(\mu | v) = \mathcal{H}_{\Lambda_1}(\mu | v) + \int_{\mathcal{E}_{\Lambda_2}} \mathcal{H}_{\Lambda_2}(\mu^\xi | v^\xi) d\mu^\xi(\xi),$$

where $\mu^\xi$ and $v^\xi$ denote the r.c.p.d. on $(\Omega, \mathcal{F})$ of $\mu$ and $v$ respectively corresponding to the projection map $\Omega \to \mathcal{E}_{\Lambda_i}$. Recall that $v = v \gamma_{\Lambda_1}$. For the first term on the right in (2), consistency of $\gamma$ implies that $v = v \gamma_{\Lambda_1}$ and

$$\mathcal{H}_\Lambda(\mu | v) = \mathcal{H}_{\Lambda_1}(\mu | v) \geq \inf_{\rho \in \mathcal{P}(\Omega, \mathcal{F})} \mathcal{H}_{\Lambda_1}(\mu | \rho \gamma_{\Lambda_1}).$$

The goal is to obtain a similar lower bound for the integral in (2). Assume in the sequel that $\mathcal{H}_{\Lambda_1}(\mu | v)$ is finite; the lemma follows from (2) if it is not. This means in particular that $\mu_{\Lambda_1} \ll \nu_{\Lambda_1}$. Formally, $\mu^\xi$ and $v^\xi$ are probability kernels from $(\mathcal{E}_{\Lambda_1}, \mathcal{F}_{\Lambda_1})$ to $(\Omega, \mathcal{F})$, which may be measured by $\mu_{\Lambda_i}$. Moreover, these kernels satisfy $\sigma_{\Lambda_i}(\mu^\xi) = \sigma_{\Lambda_i}(v^\xi) = \delta_\xi$. First we assert that

$$\int_{\mathcal{E}_{\Lambda_2}} \mathcal{H}_{\Lambda_2}(\mu^\xi | v^\xi) d\mu^\xi(\xi) = \mathcal{H}(\mu_{\Lambda_2}, \mu^\xi | v^\xi).$$

It is straightforward to see that this holds true: an expansion of the expression on the right in this display similar to the expansion in (2) yields the integral on the left plus the entropy term $\mathcal{H}(\mu_{\Lambda_2}, \mu_{\Lambda_1}) = 0$. It is clear that $\mu_{\Lambda_2} \mu^\xi = \mu$. For the other kernel, we observe that $v^\xi = v^\xi \gamma_{\Lambda_2}$ by consistency for $v \gamma_{\Lambda_1}$-almost every $\xi$, and therefore also for $\mu_{\Lambda_2}$-almost every $\xi$. In particular, this means that

$$\mathcal{H}(\mu_{\Lambda_2}, \mu^\xi | v^\xi) = \mathcal{H}(\mu | \mu_{\Lambda_2} \gamma_{\Lambda_2}) \geq \mathcal{H}_{\Lambda_2}(\mu | \mu_{\Lambda_2} \gamma_{\Lambda_2}) \geq \inf_{\rho \in \mathcal{P}(\Omega, \mathcal{F})} \mathcal{H}_{\Lambda_2}(\mu | \rho \gamma_{\Lambda_2}).$$

**Lemma 4.2.** Let $(X, \mathcal{F})$ denote a measurable space, and consider $\mathcal{B} \subseteq \mathcal{H}(X, \mathcal{F})$ with $\text{Diam}^\infty \mathcal{B}$ finite. Then for any finite measure $\mu \in \mathcal{H}(X, \mathcal{F})$, and for any $\nu \in \mathcal{B}$, we have

$$|\mathcal{H}(\mu | \nu) - \mathcal{H}(\mu | \nu^\prime)| \leq \mu(X) \text{Diam}^\infty \mathcal{B},$$

where we interpret $|\infty - \infty|$ as $0$.

**Proof.** Note that $\mu \ll \nu$ if and only if $\mu \ll \nu^\prime$. Write $f := d\mu/d\nu$ and $f^\prime := d\mu/d\nu^\prime$. Then $\mu$-almost everywhere $df/df^\prime = f^\prime/f$, and $|\log f - \log f^\prime| \leq \text{Diam}^\infty \mathcal{B}$. In particular,

$$|\mathcal{H}(\mu | \nu) - \mathcal{H}(\mu | \nu^\prime)| = |\mu(\log f) - \mu(\log f^\prime)| \leq \mu(|\log f - \log f^\prime|) \leq \mu(X) \text{Diam}^\infty \mathcal{B}.$$

**Lemma 4.3.** The specific free energy functional $h(\cdot | \gamma) : \mathcal{P}(\Omega, \mathcal{F}) \to [0, \infty]$ satisfies

$$h(\mu | \gamma) := \lim_{\Delta_n \to 0} \Delta_n^{-1} \mathcal{H}_{\Lambda_0}(\mu | \gamma \Delta_0)$$

$$= \sup_{\rho \in \mathcal{P}(\Omega, \mathcal{F})} |\Delta_n|^{-1} (\mathcal{H}_{\Lambda_0}(\mu | \rho \gamma_{\Lambda_0}) - \text{Diam}^\infty \mathcal{B}_{\Lambda_0}(\gamma))$$

$$= \lim_{\Delta_n \to 0} \Delta_n^{-1} \inf_{\rho \in \mathcal{P}(\Omega, \mathcal{F})} \mathcal{H}_{\Lambda_0}(\mu | \rho \gamma_{\Lambda_0})$$

$$= \sup_{\rho \in \mathcal{P}(\Omega, \mathcal{F})} |\Delta_n|^{-1} \inf_{\rho \in \mathcal{P}(\Omega, \mathcal{F})} \mathcal{H}_{\Lambda_0}(\mu | \rho \gamma_{\Lambda_0})$$

for any weakly dependent specification $\gamma$ and for any $\nu \in \mathcal{P}(\Omega, \mathcal{F})$.

**Proof.** Together, Lemma 4.1 of the current paper and Lemma 15.11 of [Geo11] assert that the sequence in (3) converges, with limit (4). Lemma 4.2 and weak dependence of $\gamma$ imply that for any $\nu \in \mathcal{P}(\Omega, \mathcal{F})$,

$$|\mathcal{H}_{\Lambda_0}(\mu | \gamma \Delta_0) - \inf_{\rho \in \mathcal{P}(\Omega, \mathcal{F})} \mathcal{H}_{\Lambda_0}(\mu | \rho \gamma_{\Lambda_0})| \leq \text{Diam}^\infty \mathcal{B}_{\Lambda_0}(\gamma) = o(|\Delta_n|)$$

as $n \to \infty$. This means that (3) and (5) are the same. The inequality in the display implies that each term in the supremum in (4) is bounded from above by the corresponding term in (6), and therefore the supremum in (4) is bounded from above by the supremum in (6). However, the asymptotic bound on $\text{Diam}^\infty \mathcal{B}_{\Lambda_0}(\gamma)$ implies that the supremum in (6) equals at least the limit in (3). Conclude that (3), (4), (5) and (6) are all equal. \qed
Corollary 4.4. We have $\mathcal{G}_0(\gamma) \subset h_0(\gamma)$ whenever $\gamma$ is weakly dependent.

Proof. Consider $\mu \in \mathcal{G}_0(\gamma)$, and apply the previous lemma with $\nu = \mu$. 

4.2. Minimizers and level sets.

Lemma 4.5. Let $\gamma$ denote a weakly dependent specification. Then $\{h(\cdot | \gamma) \leq C\}$ is nonempty and compact in the $L^\infty$-topology for any $C \in [0, \infty)$. In particular, $h_0(\gamma)$ is nonempty and compact in the $L^\infty$-topology.

Proof. For this motivation for this lemma is standard; we include a proof for completeness. Fix a measure $\nu \in \mathcal{P}_0(\Omega, \mathcal{F})$ and a constant $C \in [0, \infty)$. Level sets of relative entropy are compact: in our setting

$$\mathcal{P}_{n,C} := \{ \mu \in \mathcal{P}(E^{\Delta_n}, \mathcal{F}^{\Delta_n}) : \mathcal{H}(\mu | \nu^{\Delta_n}) \leq |\Delta_n|C + \text{Diam}^\infty \mathcal{P}_{\Delta_n}(\gamma) \}$$

is compact in the strong topology on $\mathcal{P}(E^{\Delta_n}, \mathcal{F}^{\Delta_n})$ for any $n \in \mathbb{N}$. Equation 4 of Lemma 4.3 says that

$$\{h(\cdot | \gamma) \leq C\} = \cap_{n \in \mathbb{N}} \{ \mu \in \mathcal{P}_0(\Omega, \mathcal{F}) : \mu_{\Delta_n} \in \mathcal{P}_{n,C} \}.$$ 

Let $(\mu^m)_{m \in \mathbb{N}} \subset \mathcal{P}(\Omega, \mathcal{F})$ denote a sequence of random fields — not necessarily shift-invariant — such that for any fixed $n \in \mathbb{N}$, we have $\sigma_{\Delta_n}(\mu^m) \in \mathcal{P}_{n,C}$ for $m$ sufficiently large. By compactness of each set $\mathcal{P}_{n,C}$, a standard diagonalisation argument, and the Kolmogorov extension theorem, we obtain a subsequential limit $\mu \in \mathcal{P}(\Omega, \mathcal{F})$ of $(\mu^m)_{m \in \mathbb{N}}$ in the $L^\infty$-topology with the property that $\mu_{\Delta_n} \in \mathcal{P}_{n,C}$ for each $n \in \mathbb{N}$.

For the lemma, it suffices to prove that $\{h(\cdot | \gamma) \leq C\}$ is compact and that $h_0(\gamma)$ is nonempty. Start with the former. Suppose that $\{\mu^m\}_{m \in \mathbb{N}} \subset \{h(\cdot | \gamma) \leq C\}$. Then $\sigma_{\Delta_n}(\mu^m) \in \mathcal{P}_{n,C}$ for any $n, m \in \mathbb{N}$. Apply the previous argument to obtain a subsequential limit $\mu \in \mathcal{P}(\Omega, \mathcal{F})$. Then $\mu$ must be shift-invariant because each $\mu^m$ is shift-invariant. The argument says moreover that $\mu_{\Delta_n} \in \mathcal{P}_{n,C}$ for each $n \in \mathbb{N}$.

This proves that the level set $\{h(\cdot | \gamma) \leq C\}$ is compact. Finally, we prove that $h_0(\gamma)$ is nonempty. Set $C$ to 0, and define

$$\mu^m := \frac{1}{|\Delta_m|} \sum_{x \in \Delta_m} \psi_{\Delta_{2m} + x} = \frac{1}{|\Delta_m|} \sum_{x \in \Delta_n} \theta_{\Delta_n}^{\Delta_{2m}}.$$ 

The two measures are equal because $\nu$ is shift-invariant, and it is clear that any subsequential limit of $(\mu^m)_{m \in \mathbb{N}}$ is also shift-invariant. Moreover, $\mu^m_{\Delta_n} = \mu^m$ whenever $m \geq n$ because $\Delta_n \subset \Delta_m \subset \Delta_{2m} + x$ for any $x \in \Delta_m$. This means that $\sigma_{\Delta_n}(\mu^m) \in \mathcal{P}_{n,0}$ for $m$ sufficiently large, for each fixed $n \in \mathbb{N}$. The sequence thus has a subsequential limit $\mu$ in the $L^\infty$-topology. This limit $\mu$ must satisfy $\mu_{\Delta_n} \in \mathcal{P}_{n,0}$ for any $n$. Conclude that $\mu \in h_0(\gamma)$, that is, $h_0(\gamma)$ is nonempty. 

5. Minimizers of the specific free energy.

5.1. Mazur’s lemma.

Lemma 5.1. Let $(X, \mathcal{X})$ denote a standard Borel space and $\mathcal{B}$ a convex subset of $\mathcal{P}(X, \mathcal{X})$ subject to $\text{Diam}^\infty \mathcal{B}$ being finite. Then the set

$$\mathcal{C} := \mathcal{C}(\mathcal{B}) := \{ \mu \in \mathcal{P}(X, \mathcal{X}) : \inf_{\nu \in \mathcal{B}} \mathcal{H}(\mu | \nu) = 0 \}$$

is compact in the strong topology on $\mathcal{P}(X, \mathcal{X})$, satisfies $\text{Diam}^\infty \mathcal{C} = \text{Diam}^\infty \mathcal{B}$, and equals

1. The closure of $\mathcal{B}$ in the total variation topology;
2. The closure of $\mathcal{B}$ in the strong topology.

This lemma is close to trivial when $E$ is finite, which is the case for many, but certainly not all, interesting models. It is this lemma that makes the theory work also for models where $(E, \mathcal{E})$ is a general standard Borel space.

Proof of Lemma 5.1 Fix a measure $\lambda \in \mathcal{B}$; this measure will serve as reference measure. Write $f_\mu := d\mu / d\lambda$ for any $\sigma$-finite measure $\mu$ on $(X, \mathcal{X})$ that is absolutely continuous with respect to $\lambda$. For example, if $\mu \in \mathcal{B}$, then $\lambda$-almost everywhere $|\log f_\mu| \leq \text{Diam}^\infty \mathcal{B}$. In particular, the map $\mu \mapsto f_\mu$ injects $\mathcal{B}$ into $L^1(\lambda)$ — the image of $\mathcal{B}$ under this map is also convex. Write $f^-$ for the lattice infimum of the family $\{f_\mu : \mu \in \mathcal{B}\}$; this is the largest $\mathcal{X}$-measurable function such that $\lambda$-almost everywhere $f^- \leq f_\mu$ for each $\mu \in \mathcal{B}$. See Lemma 2.6 in [HM02] for existence and uniqueness of $f^- \in L^1(\lambda)$. Similarly, write $f^+$ for the lattice supremum of $\{f_\mu : \mu \in \mathcal{B}\}$. Observe that $\lambda$-almost everywhere $f^- \leq 1 \leq f^+$ and

$$0 \leq \frac{f^+}{f^-} \leq \text{Diam}^\infty \mathcal{B}.$$
the former because $\lambda \in B$, the latter follows from the definition of the diameter. In particular,

$$e^{-\text{Diam}^\infty B} \leq \text{ess inf}_x f^\pm \leq \text{ess sup}_x f^\pm \leq e^{\text{Diam}^\infty B}.$$ 

Define the measures $\mu^\pm := f^\pm \lambda$ — these should be considered the lattice innumera and supremum of the set $B$, and are independent of the choice of reference measure $\lambda \in B$. A measure $\mu \in \mathcal{P}(X, \mathcal{F})$ must satisfy $\lambda^- \leq \mu \leq \lambda^+$ if either $\mu \in C$, or if $\mu$ is in the closure of $B$ in the total variation topology, or if $\mu$ is in the closure of $B$ in the strong topology. This also implies that $\lambda$-almost everywhere $f^- \leq f_\mu \leq f^+$. 

We first show that $\text{Diam}^\infty C = \text{Diam}^\infty B$. The previous observation implies that

$$\text{Diam}^\infty C \leq \text{Diam}^\infty \{ \mu \in \mathcal{P}(X, \mathcal{F}) : \lambda^- \leq \mu \leq \lambda^+ \} = \mathcal{H}^\infty(\lambda^+ | \lambda^-) = \text{Diam}^\infty B.$$ 

Now $B \subset C$ and therefore $\text{Diam}^\infty B \leq \text{Diam}^\infty C$: we conclude that $\text{Diam}^\infty C = \text{Diam}^\infty B$. 

Fix a probability measure $\mu$ subject to $\lambda^- \leq \mu \leq \lambda^+$; the goal is to show that $\mu \in C$ if and only if $\mu$ is contained in the closure of $B$ in the total variation topology. Fix a sequence $(\nu_n)_{n \in \mathbb{N}} \subset B$. Observe that $d\mu/d\nu_n = f_\mu/f_{\nu_n}$, and that

$$\mathcal{H}(\mu | \nu_n) = \nu_n\left(\frac{d\mu}{f_{\nu_n}} \log \frac{d\mu}{f_{\nu_n}}\right) = \nu_n \left(\Xi\left(\frac{d\mu}{f_{\nu_n}}\right)\right),$$

where $\Xi : (0, \infty) \to (0, \infty)$ is defined by $\Xi(x) := 1 - x + x\log x$. The function $\Xi$ is convex and attains its minimum 0 at $x = 1$ only. We observe that, as $n \to \infty$,

$$\mathcal{H}(\mu | \nu_n) \to 0 \iff \nu_n(\Xi(f_\mu/f_{\nu_n})) \to 0$$

(7)

$$\iff \lambda(\Xi(f_\mu/f_{\nu_n})) \to 0$$

(8)

$$\iff f_{\nu_n} \to f_\mu \quad \text{in } L^1(\lambda)$$

(9)

$$\iff \nu_n \to \mu \quad \text{in total variation}.$$ 

The equivalence in (7) is due to the fact that $e^{-\text{Diam}^\infty B} \lambda \leq \nu_n \leq e^{\text{Diam}^\infty B} \lambda$ for each $n \in \mathbb{N}$, and nonnegativity of $\Xi$. Equivalence in (8) is due to said properties of the function $\Xi$, and the fact that all functions $f_\mu$ and $f_{\nu_n}$ are uniformly bounded away from zero and infinity. Equivalence in (9) is straightforward as $\lambda([f_{\nu_n} - f_\mu])$ equals the total variation distance from $\nu_n$ to $\mu$. We have now proven that $C$ equals the closure of $B$ in the total variation topology.

Claim that the closure of $B$ in the total variation topology equals the closure of $B$ in the strong topology. The map $\mu \mapsto f_\mu$ is a bijection from the closure of $B$ in the total variation topology to the closure of $\{f_\mu : \mu \in B\}$ in the norm topology on $L^1(\lambda)$. The map $\mu \mapsto f_\mu$ is also a bijection from the closure of $B$ in the strong topology to the closure of $\{f_\mu : \mu \in B\}$ in the weak topology on $L^1(\lambda)$. The set $\{f_\mu : \mu \in B\}$ is convex, and therefore Mazur’s lemma asserts that the closure of $\{f_\mu : \mu \in B\}$ in $L^1(\lambda)$ is the same for the norm topology and for the weak topology.

The set $C$ is compact in the strong topology because it is closed in the strong topology and has finite max-diameter: it is a subset of the compact set $\{\mu \in \mathcal{P}(X, \mathcal{F}) : \mathcal{H}(\mu | \lambda) \leq \text{Diam}^\infty C\}$. 

**Corollary 5.2.** Consider a weakly dependent specification $\gamma$, and a shift-invariant random field $\mu$. Then $\mu \in h_0(\gamma)$ if and only if $\mu_{\lambda_n} \in C(\mathcal{P}_{\lambda_n}(\gamma))$ for each $n \in \mathbb{N}$. 

**Proof.** This is due to (6) of Lemma 4.3 in combination with Lemma 5.1. 

**5.2. Limits of finite-volume Gibbs measures.**

**Lemma 5.3.** If $\gamma$ is a weakly dependent specification, then $h_0(\gamma) = \mathcal{W}_0(\gamma)$. 

**Proof.** If $\mu \in \mathcal{W}_0(\gamma)$, then $\mu_{\lambda_n} \in C(\mathcal{P}_{\lambda_n}(\gamma))$ by definition of $\mathcal{W}(\gamma)$, and therefore $\mu \in h_0(\gamma)$ by Corollary 5.2. Now consider $\mu \in h_0(\gamma)$. For the lemma, it suffices to prove that $\mu \in \mathcal{W}_0(\gamma)$. Again, Corollary 5.2 says that $\mu_{\lambda_n} \in C(\mathcal{P}_{\lambda_n}(\gamma))$ for each $n \in \mathbb{N}$. Write $d(\cdot, \cdot)$ for total variation distance. Lemma 5.1 implies that there exists a sequence of measures $(\nu^m)_{m \in \mathbb{N}} \subset \mathcal{P}(\Omega, \mathcal{F})$ such that

$$d(\mu_{\lambda_n}, \nu^m_{\lambda_n}) \leq 1/n$$

for each $n \in \mathbb{N}$. Now for any $m \geq n$, we observe that

$$d(\mu_{\lambda_n}, \sigma_{\lambda_m}(\nu^m_{\lambda_m})) \leq d(\mu_{\lambda_n}, \sigma_{\lambda_m}(\nu^m_{\lambda_m})) = d(\mu_{\lambda_n}, \nu^m_{\lambda_n}) \leq 1/m.$$ 

In particular, $\sigma_{\lambda_m}(\nu^m_{\lambda_m})$ approaches $\mu_{\lambda_n}$ in the total variation topology as $m \to \infty$, and therefore also in the strong topology. Conclude that $\nu^m_{\lambda_n} \to \mu$ in the topology of local convergence as $m \to \infty$. In other words, $\mu \in \mathcal{W}_0(\gamma)$. 


5.3. Regular conditional probability distributions. Recall that \( \mu^\omega \) denotes the r.c.p.d. on \((E^\Lambda, \mathcal{E}^\Lambda)\) of \( \mu \) corresponding to the projection map \( \sigma_{\Lambda} : \Omega \to E^{\Lambda} \), where \( \mu \in \mathcal{P}(\Omega, \mathcal{F}) \) is an arbitrary random field, and \( \Lambda \in \mathcal{I} \). Recall also that we use the notation \( \mathcal{R}_{\Lambda, \omega}(\gamma) \) for the set

\[
\mathcal{R}_{\Lambda, \omega}(\gamma) := \cap_{\Delta \in \mathcal{I} \setminus \{\Lambda\}} \mathcal{E}(\mathcal{R}_{\Lambda, \omega, \Delta}(\gamma)).
\]

Lemma 5.4. Let \( \gamma \) be a weakly dependent specification, and fix a minimizer \( \mu \in h_0(\gamma) \) and a finite set \( \Lambda \in \mathcal{I} \). Then the r.c.p.d. of \( \mu \) satisfies \( \mu^\omega \in \mathcal{R}_{\Lambda, \omega}(\gamma) \) for \( \mu \)-almost every \( \omega \).

Proof. Fix an arbitrary set \( \Delta \in \mathcal{I} \) that contains \( \Lambda \). For the lemma it suffices to show that \( \mu^\omega \in \mathcal{E}(\mathcal{R}_{\Lambda, \omega, \Delta}(\gamma)) \) for \( \mu \)-almost every \( \omega \). Write \( \mu^\omega \) for the r.c.p.d. of \( \mu \) on \((E^\Lambda, \mathcal{E}^\Lambda)\) corresponding to the natural projection map \( \sigma_{\Lambda} : \Omega \to E^{\Lambda \setminus \Lambda} \); we are only interested in \( n \) so large that \( \Delta_n \supset \Delta \). For such \( n \), we claim that

\[
\mu^\omega_n \in \mathcal{E}(\mathcal{R}_{\Lambda, \omega, \Delta}(\gamma))
\]

almost surely (by which we mean: for \( \mu \)-almost every \( \omega \)). Equation 6 of Lemma 5.3 implies that

\[
\inf_{\rho \in \mathcal{R}_{\Lambda}(\gamma)} \mathcal{H}(\mu_{\Lambda_n}|\rho) = 0.
\]

This implies that

\[
\inf_{\rho \in \mathcal{R}_{\Lambda}(\gamma)} \left( \mathcal{H}(\mu_{\Lambda_n - \Lambda}|\rho_{\Lambda_n - \Lambda}) + \int_{E^{\Lambda_n - \Lambda}} \mathcal{H}(\mu^\omega_n|\rho^\omega) d\mu_{\Lambda_n - \Lambda}(\omega) \right) = 0,
\]

where \( \rho^\omega \) is the r.c.p.d. of \( \rho \) on \((E^{\Lambda}, \mathcal{E}^{\Lambda})\) corresponding to the projection map \( E^{\Lambda_n} \to E^{\Lambda_n - \Lambda} \). Remark that \( \rho^\omega \in \mathcal{R}_{\Lambda_n, \omega, \Delta_n}(\gamma) \) almost surely because \( \rho \in \mathcal{R}_{\Lambda_n}(\gamma) \) and by consistency of \( \gamma \). This means that

\[
\inf_{\rho^\omega \in \mathcal{R}_{\Lambda_n, \omega, \Delta_n}(\gamma)} \mathcal{H}(\mu^\omega_n|\rho^\omega) = 0,
\]

and therefore \( \mu^\omega_n \in \mathcal{E}(\mathcal{R}_{\Lambda_n, \omega, \Delta_n}(\gamma)) \), almost surely. But \( \mathcal{E}(\mathcal{R}_{\Lambda_n, \omega, \Delta_n}(\gamma)) \subset \mathcal{E}(\mathcal{R}_{\Lambda, \omega, \Delta}(\gamma)) \) because \( \Delta_n \supset \Delta \), which proves the claim.

For any \( A \in \mathcal{E}^\Lambda \), the bounded martingale convergence theorem says that almost surely

\[
\mu^\omega_n(A) \to \mu^\omega(A).
\]

The set \( \mathcal{E}(\mathcal{R}_{\Lambda, \omega, \Delta}(\gamma)) \) is compact in the strong topology, hence \( \mu^\omega_n \to \mu^\omega_n \in \mathcal{E}(\mathcal{R}_{\Lambda, \omega, \Delta}(\gamma)) \) almost surely.

\[\square\]

Corollary 5.5. If \( \gamma \) is a weakly dependent specification and \( \mu \in h_0(\gamma) \) satisfies \( \mu(\Omega_\gamma) = 1 \), then \( \mu \) is almost Gibbs.

Proof. By the previous lemma, \( \mu^\omega_n \in \mathcal{R}_{\Lambda, \omega}(\gamma) = \{\gamma_\Lambda(\cdot, \omega)\} \) for \( \mu \)-a.e. \( \omega \), proving that \( \mu \) is a DLR state.

\[\square\]

Corollary 5.6. Let \( \gamma \) denote a weakly dependent specification, and fix a measure \( \lambda \in \mathcal{R}_{\{0\}}(\gamma) \). We pretend that \( \lambda \) is a probability measure on the state space \((E, \mathcal{E})\). Then there exists a constant \( \varepsilon > 0 \) such that, for any minimizer \( \mu \in h_0(\gamma) \) and for any \( \Lambda \in \mathcal{I} \), we have \( \mu^\Lambda \geq (\varepsilon \lambda)^\Lambda \) for \( \mu \)-almost every \( \omega \). In other words, \( \mu \) has finite energy.

In particular, if \( E \) is finite and every state \( \omega \in E \) has positive probability with respect to \( \lambda \), then one may replace \( \lambda \) by the counting measure on \( E \), which possibly has the effect of forcing us to take \( \varepsilon \) smaller. By doing so, we obtain the original finite energy formulation of Burton and Keane [BK89].

Proof of Corollary 5.6. Consider a weakly dependent specification \( \gamma \), and fix a probability measure \( \lambda \in \mathcal{R}_{\{0\}}(\gamma) \). The definition of a weakly dependent specification and Lemma 5.1 imply that \( \text{Diam}^{\omega}\mathcal{E}(\mathcal{R}_{\{0\}}(\gamma)) \) is finite, and therefore there exists an \( \varepsilon > 0 \) such that \( \mu \geq \varepsilon \lambda \) for any \( \mu \in \mathcal{E}(\mathcal{R}_{\{0\}}(\gamma)) \). (In fact, it is easy to see that the choice \( \varepsilon := \exp \text{Diam}^{\omega}\mathcal{E}(\mathcal{R}_{\{0\}}(\gamma)) \) suffices for this purpose.)

Claim that \( \mu \geq (\varepsilon \lambda)^\Lambda \) for any \( \mu \in \mathcal{R}_{\Lambda}(\gamma) \), for fixed \( \Lambda \in \mathcal{I} \). Write \( \mu = v\gamma_\Lambda \) for some \( v \in \mathcal{P}(\Omega, \mathcal{F}) \). Without loss of generality, we suppose that \( v = v\gamma_\Lambda \), so that \( \mu = v\Lambda \). We also have \( v = v\prod_{\Lambda \in \Lambda} \gamma_{\Lambda} \). By induction,

\[
v = v\prod_{\Lambda \in \Lambda} \gamma_{\Lambda} \geq (\varepsilon \lambda)^\Lambda \times v_{\Lambda - \Lambda}.
\]

This proves the claim. The claim also proves that \( \mu \geq (\varepsilon \lambda)^\Lambda \) for any \( \mu \in \mathcal{E}(\mathcal{R}_{\Lambda}(\gamma)) \), which implies the corollary due to Lemma 5.3.

\[\square\]
5.4. Duality between random fields and specifications.

**Lemma 5.7.** Let \( \gamma \) denote a weakly dependent specification and \( \nu \) a minimizer of \( \gamma \). Then for any shift-invariant random field \( \mu \), we have
\[
h(\mu|\gamma) = h(\mu|\nu) := \lim_{n \to \infty} |\Delta_n|^{-1} \mathcal{H}_n(\mu|\nu).
\]

**Proof.** We observe that \(|\mathcal{H}_n(\mu|\nu) - \mathcal{H}_n(\mu|\nu_{\gamma_n})| \leq \text{Diam}^\infty(\mathcal{F}_n(\gamma)) = o(|\Delta_n|) \) as \( n \to \infty \).

Let us now investigate the relation between \( S \) and \( F \). Define the relation \( \sim \) on \( F \) by declaring that \( \mu \sim \nu \) whenever \( \mu \in h_0(\nu) \).

**Lemma 5.8.** The relation \( \sim \) is an equivalence relation on \( F \) with \( h_0(\mu) \) the equivalence class of \( \mu \in F \).

**Proof.** Fix \( \nu \in F \). Clearly \( \nu \sim \nu \), because \( h(\nu|\nu) = 0 \). It suffices to show that \( h(\mu|\nu) = h(\nu|\nu) \) whenever \( \mu \sim \nu \). As \( \nu \in F \), there exists a specification \( \gamma \in S \) such that \( \nu \in h_0(\gamma) \). The previous lemma implies that \( h_0(\nu|\nu) = h_0(\gamma|\nu) \), that is, \( \mu \in h_0(\gamma) \), and therefore also \( h_0(\mu) = h_0(\gamma) \). This proves that \( h_0(\mu) = h_0(\nu) \).

This is sufficient for the conclusions that were drawn in Subsection 3.3.

6. Applications

Most of the classical results on the variational principle follow directly from our new setting. In this section we will give several examples of this fact. We derive new results for the loop \( O(n) \) model and for the Ising model in a random percolation environment, which is also called the Griffiths singularity random field.

6.1. Models with an absolutely summable interaction potential. In this subsection we show how to derive naturally from our work the variational principle for absolutely summable potentials as described in [Geo11] or [RS13]. The model of interest is described by a reference measure and a shift-invariant absolutely summable potential. Write \( \lambda \) for the reference measure, which is a probability measure on the state space \( (E, \mathcal{E}) \). This measure informs us of the most random distribution of the state of an isolated vertex in the absence of any interaction. Write \( \Phi = (\Phi_A)_{A \in \mathcal{S}} \) for the interaction potential. The potential encodes the interactions that exist between the states at different sites. Formally, an interaction potential \( \Phi = (\Phi_A)_{A \in \mathcal{S}} \) is a family of functions such that \( \Phi_A : \Omega \to \mathbb{R} \cup \{\infty\} \) is \( \mathcal{F}_A \)-measurable. The potential \( \Phi \) is called shift-invariant if \( \Phi_{\theta A}(\omega) = \Phi_A(\theta \omega) \) for any \( A \in \mathcal{S}, \theta \in \Theta, \omega \in \Omega \). The potential \( \Phi \) is called absolutely summable if
\[
\|\Phi\| := \sum_{A \in \mathcal{S}, 0 \in A} \|\Phi_A\|_{\infty} < \infty,
\]
where \( \|\cdot\|_{\infty} \) denotes the supremum norm. It is thus assumed that \( \Phi \) is shift-invariant and absolutely summable.

The potential induces a Hamiltonian. For \( \Lambda \in \mathcal{S} \) and \( \Delta \subset \mathbb{Z}^d \), define
\[
H_{\Lambda, \Delta} := \sum_{A \in \mathcal{S}, \Lambda \cap A \neq \emptyset, \Delta \subseteq A} \Phi_A.
\]

In particular, the Hamiltonians are the functions of the form \( H_{\Lambda} := H_{\Lambda, S} \), where \( \Lambda \in \mathcal{S} \). The reference measure \( \lambda \) and the potential \( \Phi \) generate a Gibbs specification \( \gamma = (\gamma_\Lambda)_{\Lambda \in \mathcal{S}} \) defined by
\[
\gamma_\Lambda(A, \omega) := \frac{1}{Z_\Lambda} \int_{E_A} 1_A(\xi_{\mathcal{O}_S - A}) e^{-H_{\Lambda}(\xi_{\mathcal{O}_S - A})} d\lambda^\Lambda(\xi)
\]
for any \( \Lambda \in \mathcal{S}, \omega \in \Omega, \) and \( A \in \mathcal{F}, \) where \( Z_\Lambda^\omega \) is the normalizing constant
\[
(10) Z_\Lambda^\omega := \int_{E_A} e^{-H_{\Lambda}(\xi_{\mathcal{O}_S - A})} d\lambda^\Lambda(\xi).
\]

The Hamiltonian \( H_{\Lambda} \) is always bounded by \( |\Lambda| \cdot \|\Phi\| \). Moreover, for absolutely summable potentials, the strength of the interaction decreases with the range. We aim to show two things: that the specification \( \gamma \) is weakly dependent, and that \( \Omega_\gamma = \Omega \). In that case, Corollary 5.2 and Corollary 5.3 prove the variational principle, where all almost Gibbs measures are Gibbs measures. For the analysis it is convenient to define, for \( \Lambda, \Delta \in \mathcal{S} \),
\[
\varepsilon_{\Lambda, \Delta} := \sum_{A \in \mathcal{S}, \Lambda \cap A \neq \emptyset, \Delta \subseteq A} \|\Phi_A\|_{\infty}.
\]
Compare this to the definition of $H_{\Lambda, \Delta}$ — the construction implies the inequality $\|H_{\Delta} - H_{\Lambda, \Delta}\|_\omega \leq \varepsilon_{\Lambda, \Delta}$. The constants $\varepsilon_{\Lambda, \Delta}$ contain precisely all the information that we need for proving weak dependence and that $\Omega_\gamma = \Omega$. To see this, we first prove the following lemma.

**Lemma 6.1.** For any $\omega \in \Omega$ and $\Lambda, \Delta \in \mathcal{F}$, we have $\text{Diam}^\omega \mathcal{C}(\mathcal{B}_{\Lambda, \Delta, \omega}) \leq 4\varepsilon_{\Lambda, \Delta}$.

**Proof.** Fix $\omega', \omega'' \in \Omega$ such that $\omega_0 = \omega'_0 = \omega''_0$. Choose $\zeta \in E^\Lambda$. Then $H_{\Lambda, \Delta}(\zeta \omega'_0 \omega''_\Delta) = H_{\Lambda, \Delta}(\zeta \omega''_0 \omega'_\Delta)$, and the triangular inequality implies that

$$|H_{\Lambda}(\zeta \omega'_0 \omega''_\Delta) - H_{\Lambda}(\zeta \omega''_0 \omega'_\Delta)| \leq |H_{\Lambda}(\zeta \omega'_0 \omega''_\Delta) - H_{\Lambda, \Delta}(\zeta \omega'_0 \omega''_\Delta)| + |H_{\Lambda}(\zeta \omega''_0 \omega'_\Delta) - H_{\Lambda, \Delta}(\zeta \omega''_0 \omega'_\Delta)| \leq 2\varepsilon_{\Lambda, \Delta}.$$

This inequality and (10) — the definition of $Z^\omega_\Lambda$ — imply that

$$|\log Z^\omega_\Lambda - \log Z'^\omega_\Lambda| \leq 2\varepsilon_{\Lambda, \Delta}.$$

The definition of the specification implies that $\hat{\gamma}_\Lambda(\cdot, \omega) = \frac{1}{\lambda} e^{-H_{\Lambda}(\omega_\Delta)} \lambda^\Lambda$, and therefore we deduce from the inequalities in the previous two displays that $\mathcal{H}^\omega(\hat{\gamma}_\Lambda(\cdot, \omega'), \hat{\gamma}_\Lambda(\cdot, \omega'')) \leq 4\varepsilon_{\Lambda, \Delta}$. Conclude that

$$\text{Diam}^\omega \mathcal{C}(\mathcal{B}_{\Lambda, \Delta, \omega}) = \text{Diam}^\omega \mathcal{B}_{\Lambda, \Delta, \omega} = \sup_{\omega', \omega'' \in \Omega, \omega_0 = \omega'_0 = \omega''_0} \mathcal{H}^\omega(\hat{\gamma}_\Lambda(\cdot, \omega'), \hat{\gamma}_\Lambda(\cdot, \omega'')) \leq 4\varepsilon_{\Lambda, \Delta}. \quad \square$$

We now simply employ the bound provided by the lemma, in order to arrive at the variational principle. To deduce the variational principle with Gibbs measures, we must prove that the specification $\gamma$ is weakly dependent, and that $\Omega_\gamma = \Omega$. By the lemma, we know that

1. $\text{Diam}^\omega \mathcal{B}_{\Lambda}(\gamma) \leq 4\varepsilon_{\Lambda, \Delta}$,
2. $\text{Diam}^\omega \mathcal{B}_{\Lambda, \omega}(\gamma) \leq 4\varepsilon_{\Lambda, \Delta}$ for any $\omega \in \Omega$.

To prove weak dependence, it is therefore sufficient to show that $\varepsilon_{\Lambda, \Delta} = o(|\Delta_\eta|)$ as $n \to \infty$. Similarly, to prove that $\Omega_\gamma = \Omega$, it is sufficient to show that $\varepsilon_{\Lambda, \Delta} \to 0$ as $n \to \infty$ for any $\Lambda \in \mathcal{F}$, as this would imply that

$$\text{Diam}^\omega \mathcal{B}_{\Lambda, \omega}(\gamma) \leq \varepsilon_{\Lambda, \Delta} \in \inf_{n \in \mathbb{N}} \text{Diam}^\omega \mathcal{C}(\mathcal{B}_{\Lambda, \omega, \gamma}(\cdot)) = 0.$$

Start with the latter. It is immediate from the definition of $\varepsilon_{\Lambda, \Delta}$, that

$$\varepsilon_{\Lambda, \Delta} \leq \sum_{x \in \Lambda} E(x)_{\Lambda, \Delta} = \sum_{x \in \Lambda} E(0)_{\Delta, \Delta-x} \to 0$$

as $n \to \infty$, because $|\Lambda|$ and $||\Phi||$ are both finite. This proves that $\Omega_\gamma = \Omega$. For weak dependence, decompose

$$\varepsilon_{\Lambda, \Delta} \leq \sum_{x \in \Lambda} E(x)_{\Lambda, \Delta} = \sum_{x \in \Lambda} E(0)_{\Delta, \Delta-x} = \sum_{x \in \Delta, \Delta-|x|} E(0)_{\Delta, \Delta-x} \sum_{x \in \Delta-\Delta-|x|} E(0)_{\Delta, \Delta-x} \leq |\Delta_n-|\log n|| \cdot E(0)_{\Delta, |\log n|} + |\Delta_n-|\log n|| \cdot ||\Phi|| = o(|\Delta_n|)$$

as $n \to \infty$.

### 6.2. The random-cluster model

Let us introduce the random-cluster model. Fix an edge-weight $p \in (0, 1)$ and a cluster-weight $q \in (0, \infty)$. The idea of the random-cluster model is to perform independent bond percolation (with parameter $p$) on (a subset of) the square lattice $\mathbb{Z}^d$, and subsequently weight each configuration by $q$ raised to the number of percolation clusters in the resulting random graph. To cast the random-cluster model into the formalism of this paper, we must first choose a suitable state space $(E, \mathcal{E})$ for the vertices $x \in \mathbb{Z}^d$, which allows us to encode for each edge if it is open or not. There exist a natural way to do this: with each vertex $x$ we associate the $d$ edges of the form $\{x, x+e_i\}$ with $1 \leq i \leq d$. The state space that we choose is

$$E = \{0, 1\}^{\{1, \ldots, d\}},$$

where for $a_i \in E$ the $i$-th coordinate is a 1 if the edge $\{x, x+e_i\}$ is open and 0 if it is closed. For $e \in E$ we define $|e| := |\{1 \leq i \leq d : e_i = 1\}|$, the number of open edges encoded in $e$. If $a \in E^\Lambda$ for some $\Lambda \in \mathcal{F}$, then define $||a|| := \sum_{x \in \Lambda} |a_x|$. If $\omega \in \Omega$ and $\Lambda \in \mathcal{F}$, then define

$$C(\omega, \Lambda) := \text{the number of open clusters of } \omega \text{ that intersect } \Lambda \text{ or contain a vertex adjacent to } \Lambda.$$

It is important to observe that

$$|C(\omega, \Lambda) - C(\zeta, \Lambda)| \leq 2|\partial \Lambda|$$

if $\omega_0 = \zeta_0$, where $\partial \Lambda$ denotes the edge boundary of $\Lambda$, that is, set of edges of the square lattice with exactly one endpoint in $\Lambda$. We now introduce the specification $\gamma = (\gamma_x)_{x \in \mathcal{F}}$ corresponding to the random-cluster model. For any $\omega \in \Omega$, $\Lambda \in \mathcal{F}$, and $\zeta \in E^\Lambda$, we define the weight function

$$w(\zeta, \omega, \Lambda) := p^{|\zeta|} (1-p)^{|\partial \Lambda| - |\zeta|} \frac{1}{d} C(\omega_\zeta \omega_{\partial \Lambda}).$$
The probability kernel $\hat{\gamma}_\Lambda$ corresponding to the random-cluster model is now defined by

$$\hat{\gamma}_\Lambda(\xi, \omega) := \frac{1}{Z^\omega_\Lambda} w(\xi, \omega, \Lambda),$$

where $Z^\omega_\Lambda$ is a suitable normalization constant. The complete, nonrestricted probability kernel $\gamma_\Lambda$ is given by $\gamma_\Lambda(\cdot, \omega) = \hat{\gamma}_\Lambda(\cdot, \omega) \times \delta_{\partial B_\infty, \Lambda}$. Let us now prove that the resulting specification $\gamma = (\gamma_\Lambda)_{\Lambda \in \mathcal{F}}$ is weakly dependent. From (11) and the definition of $\Lambda$ through the way that the percolation structure encoded in $\omega$ is invariant under changing $\omega$, we give a concise proof. Fix $\Lambda \in \mathcal{F}$ and claim that for some appropriate choice of $\Lambda$, the measure $\hat{\gamma}_\Lambda(\cdot, \omega)$ is invariant under changing $\omega$ on the complement of $\Lambda$. The point is that the dependence of $\hat{\gamma}_\Lambda(\cdot, \omega)$ on $\omega$ is through the way that the percolation structure encoded in $\omega$ connects the vertices in the boundary of $\Lambda$ with paths through the complement of $\Lambda$. Choose $\Delta \in \mathcal{F}$ such that

1. $\Delta$ contains $\Lambda$.
2. If $x$ is adjacent to $\Lambda$ and part of a finite $\omega$-cluster, then $\Delta$ contains that entire finite $\omega$-cluster and all vertices adjacent to it.
3. If $x$ and $y$ are adjacent to $\Lambda$ and contained in the infinite $\omega$-cluster, then $\Delta$ contains an open path from $x$ to $y$ through the complement of $\Lambda$.

The choice $\omega \in \Omega'$ guarantees that the open path from $x$ to $y$ through the complement of $\Lambda$ exists. The merit of this choice of $\Delta$ is of course that

$$C(\xi, \Lambda) = C(\xi', \Lambda),$$

whenever $\xi, \xi' \in \Omega$ are chosen such that $\xi_\Delta = \xi'_\Delta$ and $\xi_{\Lambda - \Delta} = \xi'_{\Lambda - \Delta} = \omega_{\Lambda - \Delta}$. In particular, this implies that

$$w(\xi, \omega, \Lambda) = w(\xi, \omega', \Lambda)$$

for any $\xi \in E^\Lambda$ and for any $\omega' \in \Omega$ such that $\omega'_\Delta = \omega_\Lambda$. Conclude that $\hat{\gamma}_\Lambda(\cdot, \omega') = \hat{\gamma}_\Lambda(\cdot, \omega)$ for such $\omega' \in \Omega$, which implies the claim.
6.3. The loop $O(n)$ model. The arguments for the variational principle for the random-cluster model work for any weakly dependent model in which the long-range interaction is due to weight on percolation clusters, level sets, paths, or other large geometrical objects which arise from the local structure (for the random-cluster model this was the cluster-weight $q$). The variational principle holds true for all such models. Consider, for example, the loop $O(n)$ model. In this model, one draws disjoint loops on the hexagonal lattice; the probability of a certain configuration depends on the number of loops and on the number of loop edges in that configuration. It is thus a two-parameter model, much like the random-cluster model. See the work of Peled and Spinka [PS17] for a detailed introduction. The loop $O(n)$ model may be formalized as follows: it is a model of random functions from the faces of the hexagonal lattice to $E = \{0, 1\}$. The number of level sets of these functions corresponds to the number of loops in the loop $O(n)$ model, and the number of edges on which the function is not constant corresponds to the number of edges that are contained in a loop. Remark that in this case the Burton and Keane argument tells us that there is at most one infinite level set on which the function equals 0, and at most one infinite level set on which the function equals 1. If both infinite level sets are present, then they are clearly distinguished by their type.

6.4. The Griffiths singularity random field. The Griffiths singularity random field was introduced by Van Enter, Maes, Schonmann, and Shlosman [EMSS00]. They study the model in relation to the phenomenon of so-called Griffiths singularities. The model depends on two parameters: the percolation parameter $p \in (0, 1)$, and the inverse temperature $\beta \in \mathbb{R}$; both are fixed throughout the discussion. We take $\beta \geq 0$ without loss of generality, which corresponds to the ferromagnetic setting. To draw from the Griffiths singularity random field $K_{p, \beta}$, one first samples independent site percolation with parameter $p$; then, on each percolation cluster, one samples an independent Ising model with parameter $\beta$. The Griffiths singularity random field is thus an Ising model in a random environment.

First, we introduce some notation. A natural choice for the state space is $E = \{-1, 0, 1\}$. The state 0 indicates a closed vertex, while the state $\pm 1$ indicates an open vertex of that spin. Write $\mathcal{E}$ for the powerset of $E$, a $\sigma$-algebra, and $\mathcal{E}^\infty$ for the $\sigma$-algebra on $E$ generated by the function $1_\emptyset$. Let $\mathcal{F}^0$ denote the product $\sigma$-algebra $\mathcal{E}^\infty$. If $\omega \in \Omega$ or $\omega \in E^\Lambda$ for some $\Lambda \subset S$, then write $\Pi(\omega) \subset \mathbb{Z}^d$ for the set of open vertices. We consider each configuration $\omega \in \Omega$ to be a function from $\mathbb{Z}^d$ to $\{-1, 0, 1\}$, and in that light we treat $|\omega|$, $-\omega$, and $1\omega$ as configurations in $\Omega$ for any $\omega \in \Omega$ or $\Lambda \subset \mathbb{Z}^d$. There is a natural ordering $\preceq$ on $\Omega$: write $\omega^1 \preceq \omega^2$ whenever $\omega^1_x \leq \omega^2_x$ for any $x \in \mathbb{Z}^d$. If $\mu_1, \mu_2 \in \mathcal{P}(\Omega, \mathcal{F})$, then write $\mu_1 \preceq \mu_2$ if $\mu_1$ is stochastically dominated by $\mu_2$, that is, if there exists a coupling between $\mu_1$ and $\mu_2$ such that $\omega^1 \preceq \omega^2$ almost surely. Finally, the square lattice $\mathbb{Z}^d$ has naturally associated to it an edge set; write $xy$ (juxtaposition) for an unordered pair of neighboring vertices $x, y \in \mathbb{Z}^d$ in this graph. Write $\partial \Lambda$ for the edge boundary of any set $\Lambda \in \mathcal{F}$, as in the analysis of the random-cluster model.

6.4.1. The Ising model on a finite graph. For finite sets $\Lambda \in \mathcal{F}$, the Ising model in $\Lambda$ is a probability measure on $E^\Lambda$ defined by

$$\alpha_\Lambda(\omega) \propto \prod_{xy \subseteq \Lambda} e^{-\beta \omega_x \omega_y},$$

if $\omega_x = \pm 1$ for every $x \in \Lambda$, and $\alpha_\Lambda(\omega) = 0$ otherwise. The following key identity is a corollary of the definition:

$$\alpha_\Lambda(\omega) = \frac{1}{Z} \cdot f_{\Lambda, \Lambda}(\omega) \cdot \alpha_{\Lambda \cup \Delta}(\omega_{\Lambda \cup \Delta}) \cdot \alpha_{\Lambda \cup \Delta}(\omega_{\Lambda \cup \Delta})$$

for any $\Lambda, \Delta \in \mathcal{F}$ and $\omega \in E^\Lambda$, where

$$f_{\Lambda, \Delta}(\omega) := \prod_{xy \subseteq \Lambda, xy \in \partial \Delta} e^{-\beta \omega_x \omega_y} \quad \text{and} \quad Z = \int_{E^\Lambda} f_{\Lambda, \Delta}(\omega) d(\omega_{\Lambda \cup \Delta} \times \omega_{\Lambda \cup \Delta}).$$

In particular, if $\Lambda \in \mathcal{F}$ and $\Delta$ a connected component of $\Lambda$, then (12) implies that $\alpha_\Lambda = \alpha_{\Lambda \cup \Delta} \times \alpha_{\Lambda \cup \Delta}$.

If $\Lambda \in \mathcal{F}$ and $\omega \in E^\Lambda$ for some $\Lambda \subseteq \Delta \subseteq S$, then we sometimes write $\alpha_\Lambda(\omega)$ for $\alpha_\Lambda(\omega_{\Lambda\Delta})$.

6.4.2. The Ising model on an infinite graph. The Ising model on infinite subgraphs of $\mathbb{Z}^d$ is introduced in terms of the associated specification, which is denoted by $\kappa = (\kappa_\Lambda)_{\Lambda \in \mathcal{F}}$. Consider arbitrary $\Lambda \in \mathcal{F}$ and $\omega \in \Omega$. Informally, the measure $\kappa_\Lambda(\cdot, \omega) \in \mathcal{P}(\Omega, \mathcal{F})$ is the Ising model in the graph $\Pi(\omega) \cap \Lambda$ — the edges inherited from the square lattice — subject to boundary conditions provided by the configuration $\omega$.

Formally, $\kappa_\Lambda(\cdot, \omega)$ is the unique random field such that

$$\kappa_\Lambda(\zeta, \omega) \propto \prod_{xy \subseteq \Lambda, xy \in \partial \Lambda} e^{-\beta \zeta_x \zeta_y},$$
for any \( \zeta \in \Omega \) such that \( \zeta_{S^-} = \omega_{S^-} \) and \( \Pi(\zeta) = \Pi(\omega) \), and \( \kappa_{\Lambda}(\zeta, \omega) = 0 \) for all other \( \zeta \). Of course, the only edges \( xy \) that contribute to the product in the display are the ones that are also contained in \( \Pi(\zeta) = \Pi(\omega) \). As per usual, we abbreviate \( \kappa_{\Lambda}(\cdot, \cdot) := \sigma_{\Lambda}(\kappa_{\Lambda}(\cdot, \cdot)) \), and we observe that \( \alpha_\beta = \kappa_{\Lambda}(1, \Lambda) \) in this notation.

The interest is however in the Ising model in the entire graph induced by \( \Pi(\omega) \). By monotonicity, the sequence of random fields \( (\kappa_{\Lambda}(\cdot, \cdot))_{\Lambda \in \mathbb{N}} \) is decreasing with respect to \( \preceq \), and therefore tends to a limit in the \( \mathcal{L}\)-topology as \( n \to \infty \). Write \( \kappa^+(\cdot, \cdot) \) for this limit, and similarly write \( \kappa^-(\cdot, \cdot) \) for the limit of the increasing sequence \( (\kappa_{\Lambda}(\cdot, -|\omega|))_{\Lambda \in \mathbb{N}} \). Remark that both \( \kappa^+(\cdot, \cdot) \) and \( \kappa^-(\cdot, \cdot) \) depend on the percolation structure \( \Pi(\omega) \) of \( \omega \) only, and not on the spins of the open sites. In other words, \( \kappa^+ \) and \( \kappa^- \) are probability kernels from \( (\Omega, F^0) \) to \( (\Omega, F) \). A monotonicity argument implies that \( \kappa^+(\cdot, \cdot) \preceq \kappa^-(\cdot, \cdot) \). If the two measures are distinct, then it is said that the Ising model magnetizes on \( \Pi(\omega) \). Write \( M \subset \Omega \) for the collection of configurations \( \omega \) such that the Ising model magnetizes on \( \Pi(\omega) \). The set \( M \) is measurable with respect to \( F^0 \). It is also measurable with respect to \( F^0 \) for any \( \Lambda \in \mathcal{S} \). In other words, \( M \) is tail measurable.

If \( \zeta \in \Omega - M \), then another monotonicity argument implies that \( \kappa^+(\cdot, \zeta) \) is the unique random field such that almost surely \( \Pi(\omega) = \Pi(\zeta) \) and which is invariant under each probability kernel \( \kappa_{\Lambda} \). We finally state an important proposition, which also follows from monotonicity.

**Proposition 6.2.** The map \( \omega \mapsto \kappa^+(\cdot, \omega) \) is continuous — both sides endowed with the \( \mathcal{L} \)-topology — at some \( \zeta \in \Omega \) if and only if \( \zeta \not\in M \).

6.4.3. The random percolation environment. Write \( P_{p} \) for the percolation measure with parameter \( p \), that is, the measure in which each vertex takes value 1 with probability \( p \), and value 0 with probability \( 1 - p \), independently of all other vertices. Note that we have a zero-one law for the tail-measurable event \( M \) in \( P_{p} \). We therefore distinguish three phases at most: one phase of subcritical percolation, one phase of supercritical percolation but with \( P_{p}(M) = 0 \), and one phase of supercritical percolation with \( P_{p}(M) = 1 \). Clearly \( P_{p}(M) = 0 \) in the subcritical percolation regime as there are no infinite clusters almost surely and therefore the infinite Ising model decomposes into the product of infinitely many finite cluster Ising models. The variational principle is known to fail in the magnetic phase, due to [KLR04]. Our goal is to prove the variational principle for the nonmagnetic phase — both in the subcritical and supercritical percolation regime.

6.4.4. Below critical percolation. Let us use now for assume that we are in the subcritical percolation regime \( p < p_c \), so that we avoid the presence of an infinite percolation cluster altogether. The Griffiths singularity random field \( K_{p, \beta} \) is simply defined by the equation \( K_{p, \beta} := P_{p} \kappa^+ \). To sample from \( K_{p, \beta} \), one first samples the percolation structure \( \zeta \) from \( P_{p} \), then one draws the final sample \( \omega \) from the Ising model \( \kappa^+(\cdot, \zeta) \), which decomposes into a product of Ising models on the finite clusters of \( \Pi(\zeta) \) almost surely.

Fix \( \Lambda \in \mathcal{S} \). Observe that \( K_{p, \beta} \) is invariant under the kernel which first resamples the percolation structure on \( \Lambda \), then resamples the Ising model on each percolation cluster that intersects \( \Lambda \). This motivates the definition of a natural specification associated to \( K_{p, \beta} \). First, consider those \( \omega \in \Omega \) for which there is no infinite percolation cluster. For any \( \Lambda \in \mathcal{S} \), write \( \Gamma(\omega, \Lambda) \subset \mathbb{Z}^d \) for the union of \( \omega \)-open clusters that contain a vertex that is in or adjacent to \( \Lambda \). Also write \( \|\omega_{\Lambda}\| \) for the number of \( \omega \)-open vertices in \( \Lambda \). For such \( \omega \) and \( \Lambda \), we define the probability measure \( \hat{\gamma}_\Lambda(\cdot, \omega) \) by

\[
\hat{\gamma}_\Lambda(\zeta, \omega) := \frac{1}{Z^\Lambda}\|\zeta\|(1 - p)^{|\Lambda| - \|\zeta\|}\alpha_{\gamma(\zeta, \omega_{S^-}\Lambda)}(\zeta \omega_{S^-}\Lambda),
\]

where \( Z^\Lambda \) is a suitable normalization constant, and \( \zeta \) ranges over \( E^\Lambda \). As per usual, the full kernel \( \gamma_\Lambda \) is recovered through the equation \( \gamma_\Lambda(\cdot, \omega) = \hat{\gamma}_\Lambda(\cdot, \omega) \times \delta_{\omega_{S^-}\Lambda} \). It follows from this definition and the intuitive picture that \( K_{p, \beta} = K_{p, \beta}\gamma_\Lambda \) for every \( \Lambda \in \mathcal{S} \), even though we have not yet defined \( \gamma_\Lambda(\cdot, \omega) \) for those \( \omega \) with an infinite percolation cluster.

Let us now rewrite the previous definition of \( \hat{\gamma}_\Lambda(\cdot, \omega) \) into an expression that is less intuitive but more useful for the analysis. First, write \( \xi := \zeta \omega_{S^-}\Lambda \) and \( \Gamma := \Gamma(\xi, \Lambda) \). Use \([12]\) to obtain

\[
\alpha_{\Gamma}(\xi) = \frac{f_{\Gamma}(\xi)(\alpha_{\Gamma}(\xi)) \cdot \alpha_{\Gamma}(\omega)}{(\alpha_{\Gamma}(\xi) \times \alpha_{\Gamma}(\omega))(f_{\Gamma}(\xi))}.
\]

Note that \( \Gamma \cap \Lambda = \Pi(\zeta) \). The set \( \Gamma(\xi \omega_{S^-}\Lambda, \Lambda) \) depends on \( \omega_{S^-}\Lambda \) only, and therefore \( \alpha_{\Gamma}(\omega) \) is independent of \( \xi \). We may therefore combine \( \alpha_{\Gamma}(\omega) \) with the normalization constant in \([13]\) to obtain

\[
\hat{\gamma}_\Lambda(\xi, \omega) = \frac{1}{Z^\Lambda}\|\xi\|(1 - p)^{|\Lambda| - \|\xi\|}\alpha_{\Gamma(\xi)}(\xi) \cdot \frac{f_{\Gamma}(\xi)}{(\alpha_{\Gamma}(\xi) \times \alpha_{\Gamma}(\omega))(f_{\Gamma}(\xi))}.
\]
now with a different normalization constant. If we write $f_\lambda$ for the function
\[
f_\lambda(\omega) := \prod_{x \in \partial \Lambda} e^{-\beta \omega_h \omega_h},
\]
then the previous equation simplifies to
\[
\hat{\gamma}_\lambda(\zeta, \omega) = \frac{1}{Z_\Lambda^\omega} e^{\beta \omega_0} (1 - p) |\partial \Lambda|^\beta \left( \frac{f_\lambda(\zeta)}{(\hat{\kappa}_\lambda(\cdot, 1)) \cdot \sigma_{\gamma} (\kappa^+ (\cdot, 1, \omega_0 - \lambda)) (f_\lambda(\zeta))} \right).
\]
This probability kernel is well-defined for any $\omega$, even if $\omega$ has infinite clusters or if the Ising model magnetizes on $\Pi(\omega)$. We shall take (14) as a definition for each kernel $\gamma$. The family $\gamma = (\gamma_\lambda)_{\lambda \in \mathcal{F}}$ so produced is a specification. The long-range interaction derives exclusively from the appearance of the measure $\kappa^+ (-1, 1, \omega_0 - \lambda)$ in the denominator in the fraction on the right in (14). Recall that $M$ is tall measurable: the Ising model magnetizes on $\Pi(\omega)$ if and only if the Ising model magnetizes on $\Pi(\omega) - \Lambda$. This leads to the following crucial observation.

**Proposition 6.3.** Consider $\zeta \in \Omega$. If $\zeta \notin M$, then the map $\omega \mapsto \gamma_\lambda (-1, \omega)$ is continuous — both sides endowed with the $\mathcal{L}$-topology — at $\zeta$ for any $\Lambda \in \mathcal{F}$. In other words, $\Omega_\gamma$ contains $\Omega - M$.

We claim that the specification $\gamma$ is weakly dependent. The reasoning is similar to the discussion of the random-cluster model. The dependence on $\omega$ in (14) is only through its appearance in the fraction on the right, and its effect on the normalization constant $Z_\Lambda^\omega$. But the definition of $f_\lambda$ implies that $|\log f_\lambda| \leq |\partial \Lambda| |\beta|$. The logarithm of the fraction in (14) is therefore bounded by $2 \partial \Lambda |\beta|$. Much like for the random-cluster model, this implies that
\[
\left| \log \frac{Z_\Lambda^\omega}{Z_\Lambda} \right| \leq 4 |\partial \Lambda| |\beta| \quad \text{and} \quad \left| \log \frac{\hat{\gamma}_\lambda(\zeta, \omega)}{\gamma_\lambda(\zeta, \omega')} \right| \leq 8 |\partial \Lambda| |\beta|,
\]
and we conclude with the asymptotic bound
\[
\text{Diam}^\omega \mathcal{R}_\Lambda(\gamma) \leq 8 |\partial \Lambda| |\beta| = o(|\Delta|)
\]
as $n \to \infty$; the specification $\gamma$ is weakly dependent. Note that the argument for weak dependence of $\gamma$ works for any choice of parameters $p \in (0, 1)$ and $\beta \geq 0$, regardless of the phase that we work in.

### 6.4.5. Below magnetization.

For the remainder of the theory, it is no longer necessary to require $p < p_c$. Instead, we fix the percolation parameter $p$ and inverse temperature $\beta$ subject only to $P_p(M) = 0$. Of course, the Griffiths singularity random field $K_{\cdot, \beta}$ is defined by the equation $K_{\gamma, \beta} := P_p \kappa^+ = P_p \kappa^-$. This measure is a DLR state of the specification $\gamma$ as defined in (14). Moreover, we observe that $K_{p, \beta}(\omega) = \omega$, and therefore $K_{p, \beta}$ is supported on $\Omega_p$. In other words, $K_{p, \beta}$ is almost Gibbs with respect to $\gamma$. Our final goal is to prove the following theorem.

**Theorem 6.4.** If the parameters $p$ and $\beta$ are such that $P_p(M) = 0$, then $h_0(\gamma) = \{K_{p, \beta}\}$.

This statement is stronger than the variational principle, it also implies that $K_{p, \beta}$ is the unique DLR state of $\gamma$, and that $K_{p, \beta}$ is the unique minimizer of $\gamma$.

**Proof of Theorem 6.4.** Fix $\mu \in h_0(\gamma)$. Then $\mu \in h_0(K_{p, \beta})$. Remark that $K_{p, \beta} |_{\mathcal{F}^0} = P_p |_{\mathcal{F}^0}$; sampling the Ising model on the percolation clusters alters the spins on those clusters, but not the percolation structure itself. Observe that
\[
h(\mu | K_{p, \beta}) = \lim_{n \to \infty} \frac{1}{\Delta_n} \mathcal{H}_{\mathcal{F}^0}(\mu_{\Delta_n} (K_{p, \beta})) \geq \lim_{n \to \infty} \frac{1}{\Delta_n} \mathcal{H}_{\mathcal{F}^0}(\mu_{\Delta_n} (K_{p, \beta})) = \lim_{n \to \infty} \frac{1}{\Delta_n} \mathcal{H}_{\mathcal{F}^0}(\mu_{\Delta_n} (P_p)).
\]
What we read on the last line in this display is exactly the SFE of $\mu |_{\mathcal{F}^0}$ with respect to $P_p |_{\mathcal{F}^0}$. But $P_p |_{\mathcal{F}^0}$ is a Gibbs measure with respect to an independent specification, which has a unique minimizer. We chose $\mu$ such that $h(\mu | K_{p, \beta}) = 0$, which now implies that $\mu |_{\mathcal{F}^0} = P_p |_{\mathcal{F}^0}$. We observe in particular that $\mu(M) = 0$, and consequently $\mu(\Omega_p) = 1$. Therefore $\mu$ is almost Gibbs with respect to $\gamma$. Finally, we observe that $\gamma_\lambda = \gamma_\lambda K_\lambda$. This implies that $\mu$ is also a DLR state of the specification $\kappa$. But the Ising model is nonmagnetizing on $\Pi(\omega)$ for $\mu$-almost every $\omega$, and therefore $\mu$ is also invariant under the probability kernel $\kappa^+$. This kernel is $\mathcal{F}^0$-measurable; conclude that $\mu = (\mu |_{\mathcal{F}^0}) \kappa^+ = P_p \kappa^+ = K_{p, \beta}$. \qed
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