The Schrödinger–Newton equation as a non-relativistic limit of self-gravitating Klein–Gordon and Dirac fields

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Abstract

In this paper, we show that the Schrödinger–Newton equation for spherically symmetric gravitational fields can be derived in a WKB-like expansion in $1/c$ from the Einstein–Klein–Gordon and Einstein–Dirac systems.

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(Some figures may appear in colour only in the online journal)

1. Introduction

This work is the sequel to a recent study [1] in which we analysed the qualitative and quantitative behaviour of Gaussian wave packets moving according to the time-dependent Schrödinger–Newton equation:

$$i\hbar \frac{\partial}{\partial t} \psi(t, \vec{x}) = \left( -\frac{\hbar^2}{2m} \Delta - Gm^2 \int \frac{|\psi(t, \vec{y})|^2}{\|\vec{x} - \vec{y}\|} \, d^3y \right) \psi(t, \vec{x}).$$

Here, we shall be concerned with the question of how (1) can be understood as a consequence of known principles and equations.

Let us point out right at the beginning that there is a rather obvious context in which (1) makes good sense, namely as a Hartree approximation for a large number of non-relativistic spinless bosons with pairwise Newtonian gravitational attraction. If initially all bosons start out in the same state $\psi(t = 0, \vec{x})$, and if there are no initial correlations so that the total initial state is separable, i.e. a multiple pure tensor product of this one-particle initial state, then the time evolved total state will again be approximately separable with factors obeying (1). This is well known and uncontroversial. The primary mathematical task is to control the build-up of correlations. Once this is achieved, we may characterize those systems to which (1) applies...
to any desired level of accuracy. Compare [2, 3] and references therein for more rigorous statements.

Logically disconnected from this Hartree interpretation, there are far more radical and speculative suggestions [4, 5] according to which the Schrödinger–Newton equation is a model for the dynamical localization (also called ‘state reduction’) of certain ‘quantum objects’, induced by its own gravitational field. In this context, no precise characterization of systems to which (1) actually applies was given, nor was it ever spelled out unambiguously how the single complex-valued function ψ of the four variables (t, x) relates to that system. It seems to have been tacitly agreed that |ψ(t, x)|² is the probability density for the centre-of-mass motion and that (1) also applies to few-body systems, like atoms and molecules, possibly even down to more elementary ones. The requirement of complexity, that is so crucial in the Hartree approximation, does not seem to play any role in this context. Hence (1) was taken as a new hypothesis to be put to test by experiment. Consequently, it seemed natural to ask whether it predicts any observable consequences, say in molecular interferometry or quantum opto-mechanics, that have any chance of being detected in the foreseeable future. It was also envisaged that it might give interesting hints concerning the interface between quantum physics and gravitation, and that it might even shed some light on the intricate question concerning the necessity of quantum gravity. Compare [6, 7] and references therein.

At this point our investigation [1] sets in. We showed that on the basis of the Schrödinger–Newton equation, inhibitions of the dispersion of wave packets due to their own gravitational field start to become significant at mass scales around 10¹⁰ u for width of 500 nm. This is more than six orders of magnitude above the current masses in molecular interferometry [8, 9] but only three orders beyond the masses already envisaged possible in future experiments.

This paper is complementary to these attempts and devoted to a better foundational understanding of the Schrödinger–Newton equation, which we shall view as a general model for gravitational self-interaction of ‘matter waves’. Slightly more precisely, we are interested in probing its range of applicability. To that end, we shall discuss its derivability from known principles, equations and approximation schemes. Needless to say that this does of course not exclude the logical possibility that (1) may capture new physics, as originally anticipated. But even then it would be good to know what aspects of it just represent known physics, possibly beyond that already contained in the Hartree interpretation.

The question we pose closely relates to that of how classical gravitational fields couple to ‘quantum matter’, i.e. matter that is described by the Schrödinger equation. Note that here ‘gravitational fields’ include those sourced by the quantum matter itself. For classical fields the coupling prescription that derives from the equivalence principle is the minimal coupling scheme, which proceeds in the following three steps.

(1) Formulate the theory of the field you wish to couple to gravity in a way that satisfies the principles of special relativity. In particular, write the field equations in a Poincaré invariant form.

(2) Replace the Minkowski metric by a general Lorentzian metric g and the Levi–Civita covariant derivative w.r.t. the Minkowski metric (which is just the partial derivative in global inertial coordinates in Minkowski space) by the Levi–Civita covariant derivative for the general metric g.

(3) Calculate the energy–momentum tensor T for the matter field from the result of step (ii) through variation of the action with respect to g and impose on g Einstein’s gravitational field equation with a right-hand side (energy–momentum tensor) including T (and possible external sources).
There are some ambiguities in step (ii), which are well known to result in differences in the couplings to the curvature tensor. These are not essential for our concern here and will be ignored. In contrast, far more serious for us are the difficulties posed by step (i). It requires the formulation of special-relativistic quantum mechanics, the ‘non-relativistic limit’ \((1/c \to 0)\) of which is Schrödinger’s theory. However, if ‘special-relativistic quantum mechanics’ is interpreted as ‘special-relativistic one-particle quantum mechanics’, then this is impossible as such a theory is known not to exist (particle creation and annihilation), except in the trivial free case. This seems to force us to interpret ‘special-relativistic quantum mechanics’ as relativistic quantum field theory’. In that case, step (ii) would presumably be achieved for external gravitational fields by considering quantum field theory in curved space, but the gravitational self-interaction could certainly not be captured in that way.

One might try to avoid all these difficulties associated with special relativity by starting with Cartan’s and Friedrichs’ geometric reformulation of Newtonian gravity and then deduce the coupling to Schrödinger’s equation from a suitable ‘non-relativistic’ adaptation of the equivalence principle. This has indeed been carefully done in [10] with the expected result. However, here we wish to stick to the well-tested traditional formulation of the equivalence principle in the formulation given above, and that requires invoking special relativity.

We conclude that, strictly speaking, there is no obvious way to deduce from the standard formulation of the equivalence principle the general coupling of a classical gravitational field to matter described by Schrödinger’s equation. For a spatially homogeneous gravitational field \(\mathbf{g}(t)\) with arbitrary time dependence, one could deduce the standard coupling, which consists of an addition of \(V(\mathbf{x}, t) = m \mathbf{g}(t) \cdot \mathbf{x}\) to the potential in the Hamiltonian, from the standard requirement of equivalence with rigidly accelerated frames (see, e.g., [11]). Recently, this has also been shown to work if one starts from the Klein–Gordon equation written in co-moving coordinates of a rigidly but time-dependent accelerating frame and taking the appropriate \(1/c \to 0\) limit [12]. But, to stress it once more, the general case does not seem to be deducible by standard procedures.

What we will do is to follow the example just mentioned and formally regard the Schrödinger equation as the non-relativistic limit of the Klein–Gordon or Dirac equation. To be sure, this is just what is often done in textbooks on quantum mechanics, and there is indeed nothing to be upset about as long as this is understood strictly as an approximation scheme for differential equations of complex-valued fields on spacetime. However, what remains to be clarified is the interpretation of both fields, and their relation in the non-relativistic limit. This, we feel, is an important issue that should be given due thought before rushing into suggestions to look for possible experimental signatures of the Schrödinger–Newton equation. In this paper, however, we concentrate on the formal problem and will briefly return to the interpretational issues in the final section.

2. The non-relativistic approximation scheme

We systematize the notion of ‘non-relativistic approximation’ by following the WKB-type procedure of Kiefer and Singh [13]. Recall that the traditional WKB approximation is a scheme to obtain the semi-classical limit of a quantum theory by means of a formal expansion

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\(^3\) In this paper, we follow standard practice in speaking of ‘non-relativistic’ approximations or limits, even though this is unfortunate terminology since it really means to approximate Poincaré symmetry by Galilei symmetry, both implementing the relativity principle.
in terms of a dimensionful parameter $\hbar$, which is Planck’s constant $\hbar$ divided by $2\pi$. It starts by inserting the ansatz

$$\psi(\vec{x}, t) \sim \exp \left( \frac{i}{\hbar} S(\vec{x}, t) \right)$$

(2)

for the wavefunction into a given linear partial differential equation, e.g. the Schrödinger or Klein–Gordon equation, and subsequently expanding the exponent $S$ in terms of $\hbar$ [14]. The equation is then required to be satisfied at each order in $\hbar$. At each order, one obtains a certain truncation of the original theory, which makes mathematical sense and may or may not lead to good approximations of the latter. In this sense, $\hbar$ should be viewed as a deformation parameter of the theory rather than an approximation parameter. In the latter case, we would have to worry about convergence of this expansion, where the degree of smallness assigned to $\hbar$ will depend on the context. The formal expansion, however, makes sense independent of any context.

It was shown by Pauli [15] how this scheme can be adopted to wavefunctions with multiple components, as for the Dirac equation. Pauli derived as the semi-classical limit of the Dirac equation a set of equations that he could not solve in general. This was completed by Rubinow and Keller [16] and later by Rafanelli and Schiller [17], as well as Pardy [18], to yield, e.g., the classical Bargmann–Michel–Telegdi (BMT) equation [19] of the appropriate order in $\hbar$. An alternative method to obtain the semiclassical limit of the Dirac equation using matrix-valued Wigner functions, which also leads to the BMT equation, was more recently developed by Spohn [20].

Similar to the traditional WKB method, the concept of a non-relativistic limit of a relativistic field theory can be understood as the appropriate order in an expansion of the dimensionful parameter $1/c$. Again $1/c$ should be viewed as a deformation parameter linking different theories. In particular, it allows us to contract the Poincaré symmetric theory to a Galilei symmetric theory as $1/c \to 0$. This results in a WKB-like scheme applied to the parameter $1/c$. Kiefer and Singh [13] showed that this can be used to derive the Schrödinger equation from the Klein–Gordon equation, and then generalized this method to derive quantum gravity corrections from the functional Wheeler–de Witt equation.

We argue in some detail that this method provides a universal scheme in which $\hbar$ as well as $1/c$ act as deformation parameters, and show that the Schrödinger–Newton equation occurs in the $1/c \to 0$ limit of the Klein–Gordon and Dirac equations coupled to Einstein gravity. This goes beyond a former analysis of de Oliveira and Tiomno [21], which uses the traditional method of Foldy–Wouthuysen [22] but does not recouple the Dirac field into Einstein’s equations.

Following the strategy just outlined, we can make the general ansatz

$$\psi(\vec{x}, t) = \exp \left( \frac{ic^2}{\hbar} S(\vec{x}, t) \right) \sum_{n=0}^{\infty} \left( \frac{\sqrt{\hbar}}{c} \right)^n a_n(\vec{x}, t),$$

(3)

where $\psi$ can be a scalar, vector or spinor field and so are the $a_n$, but $S$ is always a scalar function. The semi-classical and non-relativistic limits can then be derived by inserting this ansatz into the field equation considered and sorting by powers of either $\hbar$ or $1/c$.

We will perform both the semi-classical and the non-relativistic limits for the Klein–Gordon equation with an electromagnetic field in section 3. Coupling the Klein–Gordon equation to general relativity we show that the Schrödinger–Newton equation can be derived as the non-relativistic limit of the coupled Einstein–Klein–Gordon system.

In section 4, we will repeat this analysis for the Dirac equation with the electromagnetic field. We obtain the known results, namely the BMT equation as the semi-classical and the
Pauli equation as the non-relativistic limit, respectively. When coupled to Einstein’s equations, the non-relativistic limit of the Dirac equation yields the Schrödinger–Newton equation too.

Throughout our signature convention for the metric will be ‘mostly plus’, i.e. \((-, +, +, +)\).

3. Klein–Gordon fields

The free Klein–Gordon equation reads

\[
\left( \Box - \frac{m^2 c^2}{\hbar^2} \right) \psi = 0, \quad \Box = - \frac{1}{c^2} \partial_t^2 + \Delta,
\]

where \(\psi\) is a scalar field. Introducing an electromagnetic field with electric potential \(\phi\) and vector potential \(A\) by replacing

\[
\partial_t \rightarrow \partial_t + \frac{ie}{\hbar} \phi(\vec{x}, t), \quad (5a) \]

\[
\partial_k \rightarrow \partial_k - \frac{ie}{\hbar} A_k(\vec{x}, t), \quad (5b)
\]

\[
\Box \rightarrow \Box + \frac{e^2}{\hbar^2 c^2} \left( \phi^2 - c^2 A^2 \right) - \frac{2ie}{\hbar c^2} \left( \phi \partial_t + c^2 \vec{A} \cdot \vec{\nabla} \right) - \frac{ie}{\hbar c^2} \left( \phi + c^2 \vec{\nabla} \cdot \vec{A} \right), \quad (5c)
\]

the Klein–Gordon equation takes the form

\[
\left( \partial_t^2 - c^2 \Delta + \frac{2ie}{\hbar} \left( \phi \partial_t + c^2 \vec{A} \cdot \vec{\nabla} \right) - \frac{e^2}{\hbar^2} \left( \phi^2 - c^2 A^2 \right) + \frac{m^2 c^4}{\hbar^2} + \frac{ie}{\hbar} \left( \phi + c^2 \vec{\nabla} \cdot \vec{A} \right) \right) \psi = 0.
\]

(6)

Note that the last entries of equations (5c) and (6) could be cancelled using the Lorenz gauge \(\vec{\phi} + c^2 \vec{\nabla} \cdot \vec{A} = 0\), but we do not want to fix a gauge at this stage because it would be \(c\)-dependent.

We now make use of the ansatz (3) and calculate the first- and second-order temporal and spatial derivatives for the field \(\psi\). They are

\[
\partial_t \psi = e^{i\omega t/\hbar} \sum_{n=0}^{\infty} \sqrt{\frac{\hbar}{c}} \left( i\mathcal{S} a_n + \mathcal{S}^* a_{n-2} \right)
\]

\[
\vec{\nabla} \psi = e^{i\omega t/\hbar} \sum_{n=0}^{\infty} \sqrt{\frac{\hbar}{c}} \left( i(\vec{\nabla}\mathcal{S}) a_n + \vec{\nabla} a_{n-2} \right)
\]

\[
\partial_t^2 \psi = e^{i\omega t/\hbar} \sum_{n=0}^{\infty} \sqrt{\frac{\hbar}{c}} \left( -\mathcal{S}^2 a_n + 2i\mathcal{S} \mathcal{S}^* a_n + \mathcal{S}^* \mathcal{S} a_{n-2} + \mathcal{S}^* \mathcal{S}^* a_{n-4} \right)
\]

\[
\Delta \psi = e^{i\omega t/\hbar} \sum_{n=0}^{\infty} \sqrt{\frac{\hbar}{c}} \left( - (\vec{\nabla}\mathcal{S})^2 a_n + 2i(\vec{\nabla}\mathcal{S}) \cdot \vec{\nabla} a_{n-2} + i(\mathcal{S}\mathcal{S}^* a_{n-2} + \mathcal{S}^* \mathcal{S}^* a_{n-4}) \right)
\]

(7a)

(7b)

(7c)

(7d)

where we denote the time derivative \(\partial_t\) by a dot and define \(a_n \equiv 0\) for all \(n < 0\). Inserting this ansatz into the Klein–Gordon equation (6) yields

\[
0 = \exp \left( \frac{ic^2}{\hbar} \sum_{n=0}^{\infty} \left( \sqrt{\frac{\hbar}{c}} \right)^n \left[ -\mathcal{S}^2 a_n + 2i\mathcal{S} \mathcal{S}^* a_n + \mathcal{S}^* \mathcal{S} a_{n-2} + \mathcal{S}^* \mathcal{S}^* a_{n-4} \right.ight.
\]

\[
+ \left. c^2 (\vec{\nabla}\mathcal{S})^2 a_n - 2ie^2 (\vec{\nabla}\mathcal{S}) \cdot \vec{\nabla} a_{n-2} - ic^2 (\mathcal{S}\mathcal{S}^* a_{n-2} + \mathcal{S}^* \mathcal{S}^* a_{n-4} \right)
\]

\[
- \left. \frac{2e\phi}{\hbar} \mathcal{S} a_{n-2} + \frac{2ie\phi}{\hbar} \mathcal{S}^* a_{n-4} - 2e(\vec{A} \cdot \vec{\nabla}\mathcal{S}) a_n + 2ie\vec{A} \cdot \vec{\nabla} a_{n-2} \right]
\]

\[
- \left. \frac{e^2 \phi^2}{\hbar^2} a_{n-4} - \frac{e^2 A^2}{\hbar} a_{n-2} + m^2 a_n + \frac{ie}{\hbar} \phi a_{n-4} + ie(\vec{\nabla} \cdot \vec{A}) a_{n-2} \right].
\]

(8)
3.1. The semi-classical limit

We first rewrite (8) by eliminating the $\hbar$ dependence inside the square brackets through appropriately shifting the summation index of the terms containing $\hbar$:

$$0 = \exp \left( \frac{ic^2}{\hbar} S \right) \frac{ic^4}{\hbar^3} \sum_{n=0}^{\infty} \left( \frac{\sqrt{\hbar}}{c} \right)^n \left[ -S^2 a_n + 2iS\tilde{a}_{n-2} + i\tilde{S}a_{n-2} + \tilde{a}_{n-4} 
+ c^2 (\bar{\nabla} S)^2 a_n - 2ic^2 (\bar{\nabla} S) \cdot \bar{\nabla} a_{n-2} - ic^2 (\Delta S)a_{n-2} - c^2 \Delta a_{n-4} 
- \frac{2e\phi}{c^2} \tilde{S}a_n + \frac{2ie\phi}{c^2} \tilde{a}_{n-2} - 2e(\bar{A} \cdot \bar{\nabla} S)a_n + 2ie\bar{A} \cdot \bar{\nabla} a_{n-2} 
- \frac{e^2\phi^2}{c^4} a_n + \frac{e^2A^2}{c^4} + m^2a_n + \frac{ie}{c^2} \phi a_{n-2} + ie(\bar{\nabla} \cdot \bar{A})a_{n-2} \right].$$

(9)

Sorting by powers of $n$, we obtain equations

$$0 = \left( m^2 - \tilde{S}^2 + c^2 (\bar{\nabla} S)^2 - \frac{2e\phi}{c^2} \tilde{S} - 2e(\bar{A} \cdot \bar{\nabla} S) - \frac{e^2\phi^2}{c^4} + \frac{e^2A^2}{c^4} \right) a_n$$

$$+ i \left( \tilde{S} - c^2 (\Delta S) + \frac{c^2}{c^2} (\phi + c^2 \bar{\nabla} \cdot \bar{A}) \right) a_{n-2} + 2i \left( \tilde{S} + \frac{e\phi}{c^2} \right) \tilde{a}_{n-2}$$

$$- 2ic^2 \left( \bar{\nabla} S - \frac{c^2}{c^2} \right) \cdot \bar{\nabla} a_{n-2} + \tilde{a}_{n-4} - c^2 \Delta a_{n-4}.$$ (10)

one for each $n$. For $n = 0$, this yields

$$0 = (mc^2)^2 - (c^2\tilde{S} + e\phi)^2 + c^2 (c^2 \bar{\nabla} S - e\bar{A})^2,$$ (11)

which is the Hamilton–Jacobi equation for a relativistic particle. The equations can be simplified further if we introduce the four vector $\pi_\mu$ with $\pi_0 = -c\tilde{S} - e\phi/c$ and $\pi_k = -c^2\tilde{q}_k + eA_k$. We obtain

$$0 = m^2c^2 + \pi_\mu \pi^\mu.$$ (12)

At order $n = 2$, now making use of the Lorenz gauge, we obtain

$$0 = (c^2\tilde{S} - c^2\Delta S)a_0 + 2(c^2\tilde{S} + e\phi)a_0 - 2c^2(c^2 \bar{\nabla} S - e\bar{A}) \cdot \bar{\nabla} a_0,$$ (13)

which (with $\tilde{a}_0 = \tilde{a}_{\mu}/c$) can be written as

$$0 = (\partial_\mu \pi^\mu) a_0 + 2\pi_\mu \partial^\mu a_0.$$ (14)

3.2. The non-relativistic limit

Again we rewrite equation (8), but this time we eliminate the $c$ dependence inside the square brackets through appropriately shifting the summation index of the terms containing $c$. In order not to list terms with an index larger than $n$, we make an overall shift $n \rightarrow (n - 2)$ and compensate for this by an overall multiplication with $c^2/\hbar$:

$$0 = \exp \left( \frac{ic^2}{\hbar} S \right) \frac{ic^4}{\hbar^3} \sum_{n=0}^{\infty} \left( \frac{\sqrt{\hbar}}{c} \right)^n \left[ -\tilde{S}^2 a_n - 2\hbar(\bar{\nabla} S)^2 a_n - 2i\hbar(\bar{\nabla} S) \cdot \bar{\nabla} a_{n-2} - i\hbar(\Delta S)a_{n-2} - h\Delta a_{n-4} 
- \frac{2e\phi}{h} \tilde{S}a_n + \frac{2ie\phi}{h} \tilde{a}_{n-2} - 2e(\bar{A} \cdot \bar{\nabla} S)a_n + 2ie\bar{A} \cdot \bar{\nabla} a_{n-2} 
- \frac{e^2\phi^2}{h^2} a_n + \frac{e^2A^2}{h^2} a_{n-2} + m^2a_{n-2} + \frac{ie}{h} \phi a_{n-6} + ie(\bar{\nabla} \cdot \bar{A})a_{n-4} \right].$$

(15)
Sorting by powers of $n$, we now obtain the equations

\[
0 = \hbar (\nabla S)^2 a_n + (m^2 - S^2 - i\hbar \Delta S - 2e(\vec{A} \cdot \vec{S})) a_{n-2} - 2i\hbar (\nabla S) \cdot \vec{S} a_{n-2} \\
+ \frac{1}{\hbar} (-i\hbar \nabla - e\vec{A})^2 a_{n-4} + \left( i\dot{S} - \frac{2e\phi}{\hbar} \right) a_{n-4} + 2i\dot{S} a_{n-4} \\
- \frac{1}{\hbar^2} (i\hbar \partial_t - e\phi)^2 a_{n-6}.
\]  

(16)

At order $n = 0$, this yields simply $\nabla S = 0$, thus $S(\vec{x}, t) = S(t)$ depends only on time.

At order $n = 2$, we then obtain

\[
(m^2 - S^2) a_0 = 0 \quad \Rightarrow \quad S = \pm mt + \text{const.},
\]

(17)

where the constant term can be ignored and we choose the positive energy solution $S = -mt$.

Using these results at order $n = 4$ finally yields the Schrödinger equation

\[
(i\hbar \partial_t - e\phi) a_0 = \frac{1}{2m} (-i\hbar \nabla - e\vec{A})^2 a_0.
\]

(18)

At order $n = 6$, we obtain

\[
(i\hbar \partial_t - e\phi) a_2 = \frac{1}{2m} (-i\hbar \nabla - e\vec{A})^2 a_2 - \frac{1}{2m\hbar} (i\hbar \partial_t - e\phi)^2 a_0.
\]

(19)

Neglecting the vector potential $\vec{A}$, this reduces to the equation already found by Kiefer and Singh [13]. Without any electromagnetic potentials, we have

\[
i\hbar \dot{a}_2 = -\frac{\hbar^2}{2m} \Delta a_2 - \frac{\hbar^3}{8m^2} \Delta^2 a_0.
\]

(20)

### 3.3. Gravitating Klein–Gordon fields

Next we consider a Klein–Gordon field coupled to Einstein’s equations

\[
G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu},
\]

(21)

where

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R
\]

(22)

is the Einstein tensor and $T_{\mu\nu}$ is the energy–momentum tensor of the Klein–Gordon field, the expression of which will be given below (cf (38)). The set of equations (4) and (21) is also known as the Einstein–Klein–Gordon system. We specialize in spherically symmetric metrics which, upon choosing appropriate coordinates, we may write in the form [23]

\[
d\mathcal{S}^2 = -e^{2A(r,t)} c^2 dt^2 + e^{2B(r,t)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2),
\]

(23)

with determinant

\[
g = -c^2 e^{2(A+B)r} \sin^2 \theta.
\]

(24)

We expand $e^A$ and $e^B$ as

\[
e^{A(r,t)} = \sum_{n=0}^{\infty} \left( \frac{\sqrt{\hbar}}{c} \right)^n A_n (r, t); \quad A_0 \equiv 1
\]

(25)

\[
e^{B(r,t)} = \sum_{n=0}^{\infty} \left( \frac{\sqrt{\hbar}}{c} \right)^n B_n (r, t); \quad B_0 \equiv 1
\]

(26)
and make use of the further expansions

\[ e^{-A} = \sum_{n=0}^{\infty} \left( \frac{\sqrt{\hbar}}{c} \right)^n C_n, \]
\[ e^{-B} - 1 = \sum_{n=0}^{\infty} \left( \frac{\sqrt{\hbar}}{c} \right)^n D_n, \]
\[ e^{-A} \dot{B} = \sum_{n=0}^{\infty} \left( \frac{\sqrt{\hbar}}{c} \right)^n E_n, \]
\[ e^{-B} A' = \sum_{n=0}^{\infty} \left( \frac{\sqrt{\hbar}}{c} \right)^n F_n, \]
\[ e^{-2A} = \sum_{n=0}^{\infty} \left( \frac{\sqrt{\hbar}}{c} \right)^n G_n, \]
\[ e^{-2A} (A - \dot{B}) = \sum_{n=0}^{\infty} \left( \frac{\sqrt{\hbar}}{c} \right)^n J_n, \]
\[ e^{-2B} (A' - B') = \sum_{n=0}^{\infty} \left( \frac{\sqrt{\hbar}}{c} \right)^n K_n, \]

(27)

with the coefficients

\[ C_0 = 1, \quad C_1 = -A_1, \quad C_2 = \dot{A}_1^2 - A_2 \]
\[ D_0 = 0, \quad D_1 = -B_1, \quad D_2 = B_2^2 - B_2 \]
\[ E_0 = 0, \quad E_1 = \dot{B}_1, \quad E_2 = -(A_1 + B_1) \cdot B_1 + \dot{B}_2 \]
\[ F_0 = 0, \quad F_1 = A_1', \quad F_2 = -(A_1 + B_1)A_1' + A_2' \]
\[ G_0 = 1, \quad G_1 = -2A_1, \quad G_2 = 3A_1^2 - 2A_2 \]
\[ H_0 = 0, \quad H_1 = -2B_1, \quad H_2 = 3B_1^2 - 2B_2 \]
\[ J_0 = 0, \quad J_1 = \dot{A}_1 - \dot{B}_1, \]
\[ J_2 = -2A_1 (A_1 - \dot{B}_1) - A_1 \dot{A}_1 + B_1 \dot{B}_1 + \dot{A}_2 - \dot{B}_2 \]
\[ K_0 = 0, \quad K_1 = A_1' - B_1', \]
\[ K_2 = -2B_1 (A_1' - B_1') - A_1 A_1' + B_1 B_1' + A_2' - B_2'. \]

The d’Alembert operator \( \Box \) in a curved background is

\[ \Box = \frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} g^{\mu \nu} \partial_{\nu} \right) \]
\[ = e^{-2A} e^{-2} \left( (A - \dot{B}) \partial_t - \partial_t^2 \right) + e^{-2B} (A' - B') \partial_r + (e^{-2B} - 1) \left( \frac{2}{r} \partial_r + \partial_r^2 \right) + \Delta, \]

(28)

where the dot denotes derivatives with respect to \( t \), the prime denotes derivatives with respect to \( r \) and \( \Delta \) is the flat, three-dimensional Laplace operator.

The Klein–Gordon equation then takes the following form:

\[ 0 = e^{-2A} e^{-2} \left( (A - \dot{B}) \ddot{\psi} - \dot{\psi}^2 \right) + e^{-2B} (A' - B') \psi' \]
\[ + (e^{-2B} - 1) \left( \frac{2}{r} \psi' + \psi'' \right) + \Delta \psi = \frac{m^2 c^2}{\hbar^2} \psi. \]

(30)

Using the same expansion (3) for \( \psi \) as before, and denoting by \( \Delta r = 2/r \partial_r + \partial_r^2 \) the radial component of the spatial Laplacian, this becomes
0 = \exp \left( \frac{i c^2}{\hbar} S \right) \frac{c^4}{\hbar^2} \sum_{n=0}^{\infty} \left( \frac{\sqrt{\hbar}}{c} \right)^n \left[ \frac{1}{\hbar} e^{-2A} (A - B) (i \delta a_{n-4} + \dot{a}_{n-6}) 
+ S^2 a_{n-2} - 2iS\tilde{a}_{n-4} - i\tilde{a}_{n-6} - \dot{a}_{n-6}) + e^{-2B} (A' - B') (i \delta a_{n-2} + \dot{a}_{n-4}) 
+ (e^{-2B} - 1) (-S^2 a_n + 2iS' \dot{a}_{n-2} + i(\Delta_a S) a_{n-2} + \Delta a_{n-4}) 
- (\tilde{S} \tilde{S}) a_n + 2i(\tilde{S} \tilde{S}) \cdot \hat{\nabla} a_{n-2} + i(\Delta S a_{n-2} + \Delta a_{n-4} - \frac{m^2}{\hbar} a_{n-2}} \right] (31)

and with the expansion for the exponentials (27), we obtain

0 = \exp \left( \frac{i c^2}{\hbar} S \right) \frac{c^4}{\hbar^2} \sum_{n=0}^{\infty} \left( \frac{\sqrt{\hbar}}{c} \right)^n \left[ \sum_{k=0}^{n} \frac{1}{\hbar} J_k (i \delta a_{n-k-4} + \dot{a}_{n-k-6}) 
+ \frac{1}{\hbar} G_k (S^2 a_{n-k} - 2iS\tilde{a}_{n-k-4} - i\tilde{a}_{n-k-6} - \dot{a}_{n-k-6}) 
+ K_k (i \delta S a_{n-k-2} + \dot{a}_{n-k-4}) 
+ H_k (-S^2 a_{n-k} + 2iS' \dot{a}_{n-k-2} + i(\Delta_a S) a_{n-k-2} + \Delta a_{n-k-4}) \right] 
- (\tilde{S} \tilde{S}) a_n + 2i(\tilde{S} \tilde{S}) \cdot \hat{\nabla} a_{n-2} + i(\Delta S a_{n-2} + \Delta a_{n-4} - \frac{m^2}{\hbar} a_{n-2}} \right] (32)

As \( H_0 = J_0 = K_0 = 0 \), this can still be simplified and for each \( n \) we obtain

0 = \hbar(\tilde{V} \tilde{S}) a_n + m^2 a_{n-2} - 2i\hbar(\tilde{V} \tilde{S}) \cdot \hat{\nabla} a_{n-2} - i\hbar(\Delta S) a_{n-2} - \tilde{S}^2 a_{n-2} 
+ 2i\delta \tilde{a}_{n-4} + i\delta a_{n-6} - \hat{\nabla} \Delta a_{n-4} + \dot{a}_{n-6} - \sum_{k=1}^{n} J_k (i \delta a_{n-k-4} + \dot{a}_{n-k-6}) 
+ G_k (S^2 a_{n-k-2} - 2iS\tilde{a}_{n-k-4} - i\tilde{a}_{n-k-6} - \dot{a}_{n-k-6}) 
+ K_k (i \hbar S a_{n-k-2} + \hbar \dot{a}_{n-k-4}) 
+ H_k (-\hbar S^2 a_{n-k} + 2i\hbar S' \dot{a}_{n-k-2} + i(\Delta_a S) a_{n-k-2} + \hbar \Delta a_{n-k-4}) \right] (33)

At lowest order \( n = 0 \), (33) is again equivalent to \((\tilde{V} \tilde{S})^2 = 0\). Thus, \( S \) is a function of time only and (33) is trivially fulfilled at order \( n = 1 \).

At order \( n = 2 \) we obtain \( S^2 = m^2 \) and choose, as before, the positive energy solution \( S = -mt \). With these results, equation (33) reduces to

0 = -i\hbar \delta a_{n-4} - \frac{\hbar^2}{2m} \Delta a_{n-4} + \frac{\hbar}{2m} \tilde{a}_{n-6} - \frac{\hbar}{2m} \sum_{k=1}^{n} \left[ G_k (m^2 a_{n-k-2} + 2i\hbar \dot{a}_{n-k-4} - \dot{a}_{n-k-6}) 
+ \hbar (K_k \dot{a}_{n-k-4} + H_k \Delta a_{n-k-4} - J_k (i \hbar a_{n-k-4} - \dot{a}_{n-k-6})) \right] (34)

At order \( n = 3 \) we now obtain \( G_1 = 0 \) and therefore \( A_1 = 0 \), \( G_2 = -2A_2 \).

Considering (33) at order \( n = 4 \), we finally see that the Klein–Gordon equation is equivalent to the Schrödinger–Newton equation

\[ i\hbar \dot{a}_0 = -\frac{\hbar^2}{2m} \Delta a_0 + V a_0 \] (35)

with potential \( V = m\hbar A_2 \).
3.3.1. Einstein’s equations. Let us now consider Einstein’s equations to derive the Poisson equation for the potential \( V \).

The non-vanishing components of the Einstein tensor are

\[
G_{tt} = e^{2A} \left( \frac{1}{r^2} - e^{-2B} \left( \frac{1}{r^2} - \frac{2B'}{r} \right) \right) \tag{36a}
\]

\[
G_{rr} = -\frac{1}{r^2} e^{2B} + \frac{1}{r^2} + \frac{2A'}{r} \tag{36b}
\]

\[
G_{\theta\theta} = \frac{2B}{r} \tag{36c}
\]

\[
G_{\phi\phi} = r^2 e^{-2B} \left( A'^2 - A' B' + A'' + \frac{A' - B'}{r} \right) + \frac{r^2}{c^2} e^{-2A} (-B^2 + \dot{A} \dot{B} - \ddot{B}) \tag{36d}
\]

\[
G_{\psi\psi} = \sin^2 \theta G_{\theta\theta}. \tag{36e}
\]

From the Lagrangian for the Klein–Gordon field

\[
\mathcal{L} = -\frac{\hbar^2}{2m} \left( (\partial^\mu \psi)(\partial_\nu \psi^*) + \frac{m^2 c^2}{\hbar^2} |\psi|^2 \right) \sqrt{-g}, \tag{37}
\]

the stress–energy tensor can be derived as

\[
T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \delta \mathcal{L} = \frac{\hbar^2}{2m} \left[ (\partial_\mu \psi)(\partial_\nu \psi^*) + (\partial_\mu \psi^*)(\partial_\nu \psi) - g_{\mu\nu} \left( (\partial^\mu \psi)(\partial_\nu \psi^*) + \frac{m^2 c^2}{\hbar^2} |\psi|^2 \right) \right]. \tag{38}
\]

Its non-vanishing components are

\[
T_t = \frac{m c^2}{2} e^{2A} |\psi|^2 + \frac{\hbar^2}{2m} e^{2(A-B)} |\psi'|^2 + \frac{\hbar^2}{2m} |\psi|^2 \tag{39a}
\]

\[
T_r = -\frac{m c^2}{2} e^{2B} |\psi|^2 + \frac{\hbar^2}{2m} |\psi'|^2 + \frac{\hbar^2}{2mc^2} e^{2(B-A)} |\psi|^2 \tag{39b}
\]

\[
T_\theta = \frac{\hbar^2}{2m} (\psi \psi'^* + \psi^* \psi') \tag{39c}
\]

\[
T_\phi = \frac{m c^2}{2} r^2 |\psi|^2 - \frac{\hbar^2}{2m} e^{-2B} |\psi'|^2 + \frac{\hbar^2}{2mc^2} e^{-2A} |\psi|^2 \tag{39d}
\]

\[
T_\psi = \sin^2 \theta T_{\theta\theta}. \tag{39e}
\]

We now expand both the Einstein tensors \( G_{\mu\nu} \) and \( T_{\mu\nu}/c^4 \) using Mathematica and consider Einstein’s equations for each component order by order. We make use of the fact that \( A_1 = 0 \) from our analysis of the Klein–Gordon equation, and we use the lower order results to simplify the equations at higher order. The components that are not mentioned are trivially fulfilled at the given order.
\(n = 0:\) the \(tt\)-component yields \((rB_1)' = 0.\)

\(n = 1:\) the \(tt\)-component yields
\[
\frac{2\hbar}{\pi^2} \left( \frac{3}{2} B_1^2 + (rB_2)' \right) = 8\pi Gm|a_0|^2.
\]

\(n = 2:\) the \(tt\)-component yields
\[
\frac{2\hbar}{\pi^2} (-4B_1^3 - 3rB_1 B_2' + (rB_2)') = 8\pi Gm(a_1^* a_0 + a_0^* a_1).
\]

The \(rr\) - and \(\theta\theta\)-components both yield \(B_1 = 0\) and the \(tr\)-component \(B_1' = 0\) is then trivial as well as the order \(n = 0\) equation. Equations (40) and (41) then simplify to
\[
(rB_2)' = \frac{4\pi Gmr^2}{\hbar} |a_0|^2
\]
\[
(rB_1)' = \frac{4\pi Gmr^2}{\hbar} (a_1^* a_0 + a_0^* a_1).
\]

\(n = 3:\) the \(tt\)-component yields
\[
4(A_2 - B_2)(rB_2)' + B_2^2 - 2rB_2 B_2' + (rB_3)' = \frac{4\pi Gr^2}{\hbar^2} \left[ \frac{\hbar^2}{2m} |a_0|^2 + \frac{i\hbar}{2} (a_0^* a_0 - a_0^* a_0^*) + m\hbar(A_2 |a_0|^2 + |a_1|^2 + a_2 a_0 + a_0^* a_2) \right].
\]

The \(rr\)-component yields
\[
B_2 = rA_2^*,
\]
and the \(\theta\theta\)-component is just the derivative of the \(rr\)-component. The \(tr\)-component yields
\[
\frac{2}{r} B_3 = 4\pi i G(a_2^* a_0^* - a_0 a_0^*).
\]

If we define the potential \(V = m\hbar A_2\) as before and analogously \(U = m\hbar B_2\), and also introduce the (\(r\)-component of the) probability current
\[
j_{KG} = \frac{i\hbar}{2m} (a_0^* a_0 - a_0^* a_0^*),
\]
we are left with the following set of equations:
\[
U = rV\quad (48a)
\]
\[
(rU)' = 4\pi Gm^2 r^2 |a_0|^2
\]
\[
\dot{U} = -4\pi Gm^2 r j\quad (48c)
\]
which are equations for \(U, V\) and \(a_0\) only, together with equations (43) and (44) which constrain \(B_3, B_4\) and \(a_2\) in terms of \(U, V\) and \(a_0\).
3.3.2. Poisson equation. Let us further analyse the set of equations (48) to show that they are equivalent to the Poisson equation for the potential $V$.

$U$ is determined from (48b) to be

$$U(r, t) = \frac{4\pi G m^2}{r} \int_0^r \tilde{d}r' r'^2 |a_0(\tilde{r}', t)|^2.$$  \hfill (49)

Inserting (48a) into (48b) yields

$$4\pi G m^2 |a_0|^2 = V'' + \frac{2}{r} V' = \Delta V$$  \hfill (50)

and therefore the Laplace equation for $V$ that we were looking for.

We still have to check the consistency of equation (48c). Note that by the continuity equation, which directly follows from the Schrödinger equation, we obtain

$$\text{div} \vec{j} + \partial_t |a_0|^2 = 0,$$  \hfill (51)

where in the spherically symmetric case the divergence is given by

$$\text{div} \vec{j} = j' + \frac{2}{r} j.$$  \hfill (52)

Differentiating (49) by $t$ then yields

$$\dot{U} = 4\pi G m^2 r \int_0^r \tilde{d}r' \frac{r''}{r^2} |\dot{a}_0(\tilde{r}', t)|^2$$

$$= -4\pi G m^2 \int_0^1 \dd x^2 \text{div} \vec{j}(x, t)$$

$$= -G m^2 \int_{\text{unit sphere}} \dd V \text{ div} \vec{j}(x, t)$$

$$= -G m^2 \int_{\partial(\text{unit sphere})} \dd S \vec{j}(r, t)$$

$$= -4\pi G m^2 r j.$$  \hfill (53)

Hence, equation (48c) already follows from the other equations and the system is consistent.

4. Dirac fields

Let us now turn to the Dirac equation and repeat our analysis for this case. The free Dirac equation is

$$\left( i \gamma^\mu \partial_\mu - mc \hbar \right) \psi = 0, \quad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}.$$  \hfill (54)

with the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  \hfill (55)

$\psi$ here is a four-component spinor field.

Defining the matrices $\beta = \gamma^0$ and $\alpha^k = \gamma^0 \gamma^k$ and multiplying equation (54) by $\beta c$ yields

$$\left( i \partial_t + ic \alpha^k \partial_k - \frac{mc^2}{\hbar} \beta \right) \psi = 0.$$  \hfill (56)

Now we can, again, introduce an electromagnetic field with electric potential $\phi$ and vector potential $A$ by replacing

$$\partial_t \rightarrow \partial_t + \frac{ie}{\hbar} \phi(\vec{x}, t)$$  \hfill (57)
\[
\partial_k \rightarrow \partial_k - \frac{ie}{\hbar}A_k(\vec{x}, t).
\] (58)

The Dirac equation then takes the form
\[
\left( i\partial_t + ic\partial^k\partial_k \frac{e}{\hbar} + \frac{ec}{\hbar}\partial^kA_k - \frac{mc^2}{\hbar} \beta \right) \psi = 0.
\] (59)

As for the Klein–Gordon equation, we now make use of our ansatz (3) with the derivatives (7), where \( S \) now is a scalar function but \( a_n \) are four-component spinors. Inserting this ansatz into the Dirac equation (59) yields
\[
0 = \exp \left( \frac{ic^2}{\hbar} \right) \frac{c^2}{\hbar} \sum_{n=0}^{\infty} \left( \frac{\sqrt{\hbar}}{c} \right)^n \left[ -\hat{S}a_n + i\hat{a}_{n-2} - \frac{e\phi}{c^2}a_n - c\vec{\alpha} \cdot (\vec{\nabla} S)a_n + ic\vec{\alpha} \cdot \vec{\nabla} a_{n-2} + \frac{ec}{\hbar} \vec{\alpha} \cdot \vec{A}a_{n-2} - \frac{mc^2}{\hbar} \beta a_{n-2} \right].
\] (60)

4.1. The semi-classical limit

As for the Klein–Gordon equation, we rewrite equation (60) eliminating the \( \hbar \) terms:
\[
0 = \exp \left( \frac{ic^2}{\hbar} \right) \frac{c^2}{\hbar} \sum_{n=0}^{\infty} \left( \frac{\sqrt{\hbar}}{c} \right)^n \left[ -\hat{S}a_n + i\hat{a}_{n-2} - \frac{e\phi}{c^2}a_n - c\vec{\alpha} \cdot (\vec{\nabla} S)a_n + ic\vec{\alpha} \cdot \vec{\nabla} a_{n-2} + \frac{ec}{\hbar} \vec{\alpha} \cdot \vec{A}a_{n-2} - m\beta a_n \right].
\] (61)

Sorting by powers of \( n \), this yields
\[
\left( m\beta + \hat{S} + \frac{e\phi}{c^2} + c\vec{\alpha} \cdot \vec{\nabla} S - \frac{e}{c} \vec{\alpha} \cdot \vec{A} \right) a_n - i\hat{a}_{n-2} - ic\vec{\alpha} \cdot \vec{\nabla} a_{n-2} = 0.
\] (62)

Making use of the notations \( \pi_0 = -c\hat{S} - e\phi/c \) and \( \pi_k = -c^2\partial_k S + eA_k \) we obtain at order \( n = 0 \),
\[
0 = (mc\beta - \pi_0 - \vec{\sigma} \cdot \vec{\pi}) a_0
\]
\[
\Leftrightarrow 0 \Rightarrow (mc - \pi_0)\gamma^0 a_0 = \left( \begin{array}{cc} (mc - \pi_0) & -\vec{\sigma} \cdot \vec{\pi} \\ \vec{\sigma} \cdot \vec{\pi} & (mc + \pi_0) \end{array} \right) a_0,
\] (63)

which has non-trivial solutions if and only if the determinant
\[
\begin{vmatrix}
(mc - \pi_0) & -\vec{\sigma} \cdot \vec{\pi} \\
\vec{\sigma} \cdot \vec{\pi} & (mc + \pi_0)
\end{vmatrix} = m^2c^2 - \pi_0^2 + |(\vec{\sigma} \cdot \vec{\pi})|^2 = 0
\] (64)

vanishes. The Pauli matrices obey the algebra
\[
\sigma^i \sigma^j = \delta^{ij} + ie^{ijk} \sigma^k \Rightarrow (\sigma \cdot \vec{u})(\sigma \cdot \vec{v}) = \vec{u} \cdot \vec{v} + i\vec{\sigma} \cdot (\vec{u} \times \vec{v});
\] (65)

therefore \((\vec{\sigma} \cdot \vec{\pi})^2 = \vec{\pi} \cdot \vec{\pi}\) and (64) yields again, as for the Klein–Gordon equation, the Hamilton–Jacobi equation for a relativistic particle
\[
0 = m^2c^2 + \pi_\mu \pi^\mu.
\] (66)

At order \( n = 2 \), we obtain
\[
(mc - \pi_\mu \gamma^\mu) a_2 = i\gamma^\theta (\partial_\theta + c\vec{\alpha} \cdot \vec{\nabla}) a_0
\]
\[
= -i\gamma^\theta \partial_\mu a_0.
\] (67)
If we name the operators
\[ L := mc - \pi_\mu \gamma^\mu, \quad D := -i \gamma^\mu \partial_\mu, \]
(68)
at order \( n = 0 \) we have the condition that \( a_0 \in \text{Ker}(L) \). At second order, we now have \( Da_0 \in \text{Im}(L) \), which is equivalent to \( Da_0 \in \left( \text{Ker}(L^\dagger) \right)^\perp \). Note now that \( L^\dagger = \gamma^0 L \gamma^0 \) and therefore \( x \in \text{Ker}(L^\dagger) \leftrightarrow \gamma^0 x \in \text{Ker}(L) \).

The condition that \( Da_0 \) is in the image of \( L \) is therefore equivalent to the condition that for any two solutions \( a_0, \tilde{a}_0 \in \text{Ker}(L) \) to the first-order equation we have
\[ \tilde{a}_0 \gamma^\mu \partial_\mu a_0 = 0, \]
(69)
where \( \tilde{a}_0 = (a_0)^\dagger \gamma_0 \) is the adjoint spinor. Equations (63) and (69) together determine the solutions \( a_0 \) at first order.

4.1.1. Derivation of the BMT equation. We can use these results to obtain the BMT equation \cite{19} as a necessary condition. First multiply equation (67) by \((mc + \pi_\mu \gamma^\mu)\) from the left. Then the left-hand side vanishes and we obtain
\[ (mc + \pi_\mu \gamma^\mu) \gamma^\nu \partial_\nu a_0 = 0 \]
(70)
where \( \gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \) and we made use of (70) in the third and (63) in the fourth line. Performing the same calculation for the adjoint leads to
\[ -2\pi_\mu \partial_\mu \tilde{a}_0 = (\partial_\mu \pi_\mu) \tilde{a}_0 - \frac{e}{2} F_{\mu\nu} \gamma^\nu \gamma^\mu a_0, \]
(71)
where \( \gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \) and we made use of (70) in the third and (63) in the fourth line.

We follow \cite{17} and set
\[ S^\mu = \tilde{a}_0 \gamma^5 \gamma^\mu a_0, \]
(73)
the spin density; then we obtain from equations (71) and (72)
\[ \pi_\mu \partial_\mu S^\nu = \pi_\mu (\partial_\mu \tilde{a}_0) \gamma^5 \gamma^\nu a_0 + \tilde{a}_0 \gamma^5 \gamma^\nu \pi_\mu a_0 \]
\[ = - (\partial_\mu \pi_\mu) S^\nu + \frac{e}{4} F_{\mu\nu} \tilde{a}_0 \left( \eta^\nu \gamma^\rho \gamma^\sigma - \gamma^5 \gamma^\rho \gamma^\sigma \right) a_0 \]
\[ = - (\partial_\mu \pi_\mu) S^\nu + \frac{e}{4} F_{\mu\nu} \gamma^5 \gamma^{\rho} a_0 \left( 2 \eta^\rho \gamma^\sigma - 2 \eta^\sigma \gamma^\rho \right) a_0 \]
\[ = - (\partial_\mu \pi_\mu) S^\nu + e F_{\nu}^\rho S^\rho. \]
(74)

For the normalized spin density
\[ \hat{S}^\mu = \frac{1}{\sqrt{-S^\mu S^\nu}} S^\nu, \]
(75)
the first term vanishes and we obtain the BMT equation (for $g$-factor $g = 2$)

$$\frac{d}{d\tau} \mathcal{S}^0 = \frac{1}{m} \pi^\mu \partial_\mu \mathcal{S}^0 = \frac{e}{m} F^\mu_\rho \mathcal{S}^\rho,$$ (76)

where the first equality holds because $\pi^\mu = mv^\mu$ with $v^\mu$ being the 4-velocity of the relativistic particle in an electromagnetic field.

### 4.2. The non-relativistic limit

Again, we rewrite equation (60) eliminating the $c$ terms:

$$0 = \exp \left( \frac{ic^2}{\hbar} S \right) \frac{c^3}{\hbar^{3/2}} \sum_{n=0}^{\infty} \left( \frac{\sqrt{\hbar}}{c} \right)^n \left[ -\mathcal{S}a_{n-1} + i\mathcal{S}a_{n-3} - \frac{e\phi}{\hbar}a_{n-3} \right] - \sqrt{\hbar} \mathcal{S} \cdot (\vec{\nabla}\mathcal{S})a_n + i\sqrt{\hbar} \mathcal{S} \cdot \vec{\nabla}a_{n-2} + \frac{e}{\sqrt{\hbar}} \vec{a} \cdot \vec{A}a_{n-2} - m\beta a_{n-1} \right].$$ (77)

Sorting by powers of $n$ we obtain

$$\sqrt{\hbar} \mathcal{S} \cdot (\vec{\nabla}\mathcal{S})a_n + (\mathcal{S} + m\beta)a_{n-1} + \frac{1}{\sqrt{\hbar}} \mathcal{S} \cdot (-i\hbar \vec{V} - e\vec{A})a_{n-2} - \frac{1}{\hbar} (i\hbar \partial_t - e\phi)a_{n-3} = 0.$$ (78)

At order $n = 0$ this simply becomes $\vec{\nabla}\mathcal{S} = 0$, and therefore $S = S(t)$ is a function of time only.

Now, splitting the four-component spinors $a_n = (a_{n,1}, a_{n,2}, a_{n,3}, a_{n,4})$ into two two-component spinors $a^\pm_n = (a_{n,1}, a_{n,2})$ and $a^{\mp}_n = (a_{n,3}, a_{n,4})$, at order $n = 1$, we obtain the two equations

$$(m + \dot{S})a_0^+ = 0 \quad (79a)$$

$$(m - \dot{S})a_0^- = 0, \quad (79b)$$

which can be consistently fulfilled only if either $\dot{S} = -m$ and the negative energy component $a_0^-$ vanishes or $\dot{S} = +m$ and the positive energy component $a_0^+$ vanishes. From here on, we will choose the first and set $a_0^- = 0$. With this choice, the Dirac equation (78) simplifies to the following two equations:

$$0 = \sqrt{\hbar} \mathcal{S} \cdot (-i\hbar \vec{V} - e\vec{A})a_{0,2} - (i\hbar \partial_t - e\phi) a_{0,3}^- \quad (80a)$$

$$a_{n-1}^+ = \frac{1}{2m\sqrt{\hbar}} \mathcal{S} \cdot (-i\hbar \vec{V} - e\vec{A})a_{n-2} - \frac{1}{2m} (i\hbar \partial_t - e\phi) a_{n-3}^- \quad (80b)$$

At order $n = 2$, (80a) is trivially fulfilled. Equation (80b) determines $a_1^+$ to be

$$a_1^+ = \frac{1}{2m\sqrt{\hbar}} \mathcal{S} \cdot (-i\hbar \vec{V} - e\vec{A})a_0^+ \quad (81)$$

Using this at order $n = 3$ in equation (80a), we obtain

$$\left( \frac{1}{2m} (\vec{S} \cdot (-i\hbar \vec{V} - e\vec{A}))^2 - (i\hbar \partial_t - e\phi) \right) a_0^- = 0.$$ (82)

Making use of

$$( -i\hbar \vec{V} - e\vec{A}) \times (-i\hbar \vec{V} - e\vec{A})a_0^- = -\hbar^2 \vec{V} \times (\vec{V}a_0^-)$$

$$= \underbrace{a_0^- \vec{V} \times \vec{V}a_0^-}_{=0} + \underbrace{ie\hbar \vec{V} \times (a_0^- \vec{A})}_{=0} + \underbrace{ie\hbar \vec{A} \times \vec{V}a_0^-}_{=0} + \underbrace{e^2 \vec{A} \times \vec{A} = ie\hbar \vec{V}a_0^-}_{=0}$$ (83)
and the algebra (65) of the Pauli matrices we end up with the Pauli equation
\[
(iℏ \partial_t - e\phi)a_0^\gamma = \frac{1}{2m}( -i\vec{\nabla} - e\vec{A})^2a_0^\gamma - \frac{eℏ}{2m}\vec{σ} \cdot \vec{B}a_0^\gamma.
\] (84)
Equation (80b) at order \( n = 3 \) determines \( a_2^\gamma \) to be
\[
a_2^\gamma = \frac{1}{2m\sqrt{ℏ}}\vec{σ} \cdot ( -i\vec{\nabla} - e\vec{A})a_1^\gamma.
\] (85)
This is exactly the same relation as (81) with the indices shifted by 1. Thus, from equation (80a) at order \( n = 4 \), we will get the Pauli equation again for \( a_1^\gamma \). But equation (80b) at order \( n = 4 \) now has an additional term depending on \( a_1^\gamma \):
\[
a_3^\gamma = \frac{1}{2m\sqrt{ℏ}}\vec{σ} \cdot ( -i\vec{\nabla} - e\vec{A})a_2^\gamma + \frac{1}{2m}( -iℏ\partial_t + e\phi) a_1^\gamma.
\] (86)
This will make the equation for \( a_2^\gamma \) more complicated, and different from the Pauli equation, at order \( n = 5 \).

4.3. Gravitating Dirac fields

Now let us consider the Einstein–Dirac system, consisting of the Dirac equation (54) and Einstein’s equations (21).

As for the Einstein–Klein–Gordon system, we make an ansatz using a spherically symmetric metric tensor and therefore also need the stress–energy tensor to be spherically symmetric. Note that a single Dirac particle cannot have spherical symmetry, which is why we have to average over all spin directions. But we will only take this into account at the very end of our considerations.

We use the general relativistic formulation of the Dirac equation according to Finster [24] and Finster et al [25]:
\[
\left( \not{D} - \frac{mc}{ℏ} \right) ψ = 0.
\] (87)
The Dirac operator is defined as
\[
\not{D} = i\Gamma^\mu(x)\partial_\mu + Y(x) \tag{88a}
\]
\[
Y(x) = \Gamma^\mu(x)Z_\mu(x) \tag{88b}
\]
\[
Z_\mu = \frac{i}{2} ρ\partial_\mu ρ - \frac{i}{16} tr \Gamma^\nu\nabla_\mu \Gamma^\rho\Gamma^\gamma\Gamma_\rho + \frac{i}{8} tr ρ\Gamma_\mu \nabla_\nu \Gamma^\nu \rho \tag{88c}
\]
\[
ρ = \frac{i}{4!}ε_{\muνρσ}\Gamma^\mu\Gamma^\nu\Gamma^ρ\Gamma^\sigma, \tag{88d}
\]
where \( \nabla \) is the covariant derivative, \( ε_{\muνρσ} \) is the Levi–Civita symbol defined by
\[
ε_{\muνρσ} = (-1)^{\frac{ν + ρ + σ}{2}} \begin{cases} +1 & \text{if } (μνρσ) \text{ is an even permutation of } (τθφ) \\ -1 & \text{if } (μνρσ) \text{ is an odd permutation of } (τθφ) \\ 0 & \text{if two or more indices are equal} \end{cases}
\] (89)
and \( Γ^\mu \) is a representation of the spacetime-dependent Dirac matrices, satisfying the Clifford algebra
\[
[Γ^\mu, Γ^\nu] = -2g_{μν}. \tag{90}
\]
The additional part \( Y(x) \) will turn out not to contribute to the Schrödinger–Newton equation and is only relevant at higher order in \( 1/c \).
It is useful to represent these matrices in the basis where they become the linear combinations
\[
\frac{1}{\Gamma^1_\alpha} = e^{-A} e^{-1} \gamma^0
\]
(91)
\[
\frac{1}{\Gamma^1_\rho} = e^{-B} (\gamma^1 \cos \theta + \gamma^2 \sin \theta \cos \varphi + \gamma^3 \sin \theta \sin \varphi)
\]
(92)
\[
\frac{1}{\Gamma^1_\theta} = \frac{1}{r} (-\gamma^1 \sin \theta + \gamma^2 \cos \theta \cos \varphi + \gamma^3 \cos \theta \sin \varphi)
\]
(93)
\[
\frac{1}{\Gamma^1_\varphi} = \frac{1}{r \sin \theta} (-\gamma^2 \sin \varphi + \gamma^3 \cos \varphi)
\]
(94)
of the Dirac matrices \(\gamma^\mu\) as defined before. These matrices satisfy the anti-commutator algebra, and simplify the equations because in this representation \(\rho = \gamma^5\) and therefore the first term of \(Z_\mu\) vanishes because \(\rho\) is constant, and the third term vanishes because derivatives of the \(\Gamma^\mu\) as well as the \(\Gamma^\mu\) themselves are linear in the \(\gamma^\alpha\) and \(\text{tr} \gamma^5 \gamma^\mu \gamma^\nu = 0\). Therefore,
\[
Y = -\frac{i}{16} \text{tr} \Gamma^\nu \nabla_\mu \Gamma^\rho \Gamma^\mu \Gamma_\nu \Gamma_\rho
\]
(95)
\[
= -\frac{i}{16} \text{tr} \Gamma^\nu \nabla_\mu \Gamma^\rho \left( \delta^\rho_\mu \Gamma_\nu - \delta^\mu_\nu \Gamma_\rho - \Gamma^\mu g_{\nu\rho} + i \epsilon^{\mu\nu\rho\sigma} \Gamma_\sigma \gamma^5 \right)
\]
(96)
\[
= -\frac{i}{8} \text{tr} \Gamma^\nu \nabla_\rho \Gamma^\mu \Gamma_\nu + \frac{1}{16} \epsilon^{\mu\nu\rho\sigma} \text{tr} \Gamma_\nu \nabla_\rho \Gamma_\sigma \gamma^5
\]
(97)
\[
= -\frac{i}{8} \text{tr} \Gamma^\nu \nabla_\rho \Gamma^\mu \Gamma_\nu.
\]
(98)
All this is in agreement with [25] and can be straightforwardly verified. The third line follows, because the \(\delta^\mu_\nu\) terms in the second line are equal and the \(g_{\nu\rho}\) term vanishes. The second term in the third line vanishes, because the trace vanishes if all three indices are different.

Using that \(\nabla_\mu \Gamma^\rho = \alpha_\mu \Gamma^\rho\) is some linear combination of the gamma matrices, we obtain
\[
\text{tr} \Gamma^\nu \nabla_\rho \gamma^\nu = \alpha_\mu \text{tr} \Gamma^\nu \Gamma^\rho
\]
(99)
\[
= \frac{1}{2} \alpha_\mu \text{tr} \left[ \Gamma^\nu, \Gamma^\rho \right] + \left[ \Gamma^\nu, \Gamma^\rho \right]
\]
(100)
\[
= \frac{1}{2} \alpha_\mu \left( \text{tr} \left[ \Gamma^\nu, \Gamma^\rho \right] + \text{tr} \Gamma^\nu \Gamma^\rho - \text{tr} \Gamma^\rho \Gamma^\nu \right)
\]
(101)
\[
= \frac{1}{2} \alpha_\mu \left( \text{tr} - 2 g^{\nu\rho} \mathbb{1}_{4 \times 4} + \text{tr} \Gamma^\nu \Gamma^\rho - \text{tr} \Gamma^\rho \Gamma^\nu \right)
\]
(102)
\[
= -\alpha_\mu g^{\nu\rho} \text{tr} \mathbb{1}_{4 \times 4}
\]
(103)
\[
= -4 \alpha_\nu
\]
(104)
and therefore
\[
\mathcal{D} = i \Gamma^\mu \partial_\mu + \frac{i}{2} \nabla_\mu \Gamma^\mu.
\]
(105)
For our ansatz for the Dirac matrices we obtain
\[ \nabla \gamma \Gamma^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \Gamma^\mu) \] (106)
\[ = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \Gamma^\mu) + \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} \Gamma^\nu) \]
\[ + \frac{1}{\sqrt{-g}} \partial_0 (\sqrt{-g} \Gamma^0) + \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} \Gamma^\nu) \]
\[ = B \Gamma^r + \left( \frac{A'}{r} + \frac{2}{r} \right) \Gamma^r - \frac{e^\theta}{r} \Gamma^r \]
\[ - \frac{1}{r \sin \theta} (\gamma^1 \cos \theta \sin \theta - (1 + \cos^2 \theta)(\gamma^2 \cos \varphi + \varphi^3 \sin \varphi)) \] (107)
\[ = B \Gamma^r + \left( \frac{A'}{r} + \frac{2}{r} (1 - e^\theta) \right) \Gamma^r \] (108)
and therefore
\[ P = i \Gamma^r \left( \tilde{\partial}_r + \frac{\tilde{B}}{2} \right) + i \Gamma^\nu \left( \tilde{\partial}_r + \frac{A'}{2} \right) + \frac{1 - e^\theta}{r} \Gamma^r + i \Gamma^\nu \partial_\nu. \] (109)

Now we make use of expansion (3) for \( \psi \) as before and insert it into the Dirac equation (87). We introduce the matrices \( \tilde{\alpha} \) analogously to the previously defined \( \tilde{\alpha} \),
\[ \tilde{\alpha}^r = \gamma^0 \gamma^1 \cos \theta + \gamma^0 \gamma^2 \sin \theta \cos \varphi + \gamma^0 \gamma^3 \sin \theta \sin \varphi \] (110a)
\[ \tilde{\alpha}^\nu = -\gamma^0 \gamma^1 \sin \theta + \gamma^0 \gamma^2 \cos \theta \cos \varphi + \gamma^0 \gamma^3 \cos \theta \sin \varphi \] (110b)
\[ \tilde{\alpha}^\varphi = -\gamma^0 \gamma^2 \sin \varphi + \gamma^0 \gamma^3 \cos \varphi, \] (110c)
as well as the rotated Pauli matrices
\[ \tilde{\alpha}^r = \sigma^1 \cos \theta + \sigma^2 \sin \theta \cos \varphi + \sigma^3 \sin \theta \sin \varphi \] (111a)
\[ \tilde{\alpha}^\theta = -\sigma^1 \sin \theta + \sigma^2 \cos \theta \cos \varphi + \sigma^3 \cos \theta \sin \varphi \] (111b)
\[ \tilde{\alpha}^\varphi = -\sigma^2 \sin \varphi + \sigma^3 \cos \varphi. \] (111c)

Multiplying the Dirac equation (87) with \( \gamma^0 = \beta \) from the left, we then obtain
\[ 0 = \exp \left( \frac{i c^2}{\hbar} \right) \sum_{n=0}^\infty \left( \frac{\sqrt{-g}}{c} \right)^n \left[ - \frac{1}{\sqrt{\hbar}} e^{-A} \tilde{\partial}_n a_{n-1} + \frac{i}{\sqrt{\hbar}} e^{-A} a_n \right] \]
\[ + \frac{i}{2 \sqrt{\hbar}} e^{-A} \tilde{\partial}_n a_{n-2} - \tilde{\alpha} \cdot (\nabla S) a_n - (e^{-B} - 1) \tilde{\alpha}' \nabla S a_n + i \tilde{\alpha} \cdot \nabla a_{n-2} \]
\[ + i (e^{-B} - 1) \tilde{\alpha}' a_{n-2} + \frac{i}{2} e^{-B} \tilde{\alpha}' a_{n-2} + \frac{i}{r} (e^{-B} - 1) \tilde{\alpha}' a_{n-2} - \frac{m}{\sqrt{\hbar}} a_{n-1} \] (112)
and with the expansion for the exponentials (27), we have for each $n$,

$$0 = \sqrt{n}\vec{a} \cdot (\vec{\nabla}S)a_n + \beta ma_{n-1} - i\sqrt{n}\vec{a} \cdot \vec{v}a_{n-2}$$

$$+ \sum_{k=0}^{n} \left[ C_k \dot{S}a_{n-k-1} - iC_k a_{n-k-3} - \frac{i}{2}E_k a_{n-k-3} + \sqrt{n}D_k a^{\gamma}a_n - \frac{\sqrt{n}}{r}D_k a^\gamma a_{n-k-2} \right]$$

$$- i\sqrt{n}D_k \vec{a}^\gamma a_{n-k-2} - \frac{i\sqrt{n}}{2} \left( F_k + \frac{2}{r}D_k \right) a^\gamma a_{n-k-2} - iC_k a_{n-k-3} - \frac{i}{2}E_k a_{n-k-3} \right]$$

(113)

$$= \sqrt{n}\vec{a} \cdot (\vec{\nabla}S)a_n + \beta ma_{n-1} + \dot{S}a_{n-1} - i\sqrt{n}\vec{a} \cdot \vec{v}a_{n-2} - i\dot{a}_{n-3}$$

$$+ \sum_{k=0}^{n} \left[ \sqrt{n}D_k \vec{a}^\gamma a_{n-k} + C_k \dot{S}a_{n-k-1} - i\sqrt{n}D_k \vec{a}^\gamma a_{n-k-2} \right]$$

$$- \frac{i\sqrt{n}}{2} \left( F_k + \frac{2}{r}D_k \right) \vec{a}^\gamma a_{n-k-2} - iC_k a_{n-k-3} - \frac{i}{2}E_k a_{n-k-3} \right].$$

(114)

As before, at order $n = 0$ this yields $\vec{V}S = 0$, i.e. $S$ is a function of time only, and at order $n = 1$ we obtain $(\beta m + S)a_0 = 0$. Again, we split the four-component spinors $a_n = (a_{n1}, a_{n2}, a_{n3}, a_{n4})$ into two two-component spinors $a_n^\gamma = (a_{n1}, a_{n2})$ and $a_n^\sigma = (a_{n3}, a_{n4})$ and choose $a_n^\sigma \equiv 0$. We are then left with $S = -mt$ and from (114) we obtain the following two equations:

$$0 = i\sqrt{n}\vec{a} \cdot \vec{v}a_{n-2} + \dot{a}_{n-3} + \sum_{k=1}^{n} \left[ mC_k a_{n-k-1} + i\sqrt{n}D_k \vec{a}^\gamma a_{n-k-2} \right]$$

$$+ \frac{i\sqrt{n}}{2} \left( F_k + \frac{2}{r}D_k \right) \vec{a}^\gamma a_{n-k-2} + iC_k a_{n-k-3} + \frac{i}{2}E_k a_{n-k-3} \right]$$

(115)

$$a_{n-1}^\gamma = -\frac{i\sqrt{n}}{2m} \vec{\sigma} \cdot \vec{v}a_{n-2}^\gamma - \frac{i}{2m} \dot{a}_{n-3}^\gamma - \sum_{k=1}^{n} \left[ \frac{1}{2}C_k a_{n-k-1}^\gamma + \frac{i\sqrt{n}}{2m}D_k \vec{a}^\gamma a_{n-k-2}^\gamma \right]$$

$$+ \frac{i\sqrt{n}}{4m} \left( F_k + \frac{2}{r}D_k \right) \vec{a}^\gamma a_{n-k-2}^\gamma + \frac{i}{2m}C_k a_{n-k-3}^\gamma + \frac{i}{4m}E_k a_{n-k-3}^\gamma \right].$$

(116)

At order $n = 2$, the first equation yields $0 = mC_1 a_{n-2}^\gamma$ and therefore $C_1 = 0$. This means $A_1 = 0$, $C_2 = -A_2$ and $F_1 = 0$. The second equation yields

$$a_{n}^\gamma = -\frac{i\sqrt{n}}{2m} \vec{\sigma} \cdot \vec{v}a_{n-2}^\gamma.$$ 

(117)

Inserting this into the first equation at order $n = 3$, we obtain

$$0 = i\sqrt{n}\vec{a} \cdot \vec{v} \left( -\frac{i\sqrt{n}}{2m} \vec{\sigma} \cdot \vec{v}a_{n-2}^\gamma \right) + \dot{a}_{n-3}^\gamma - mA_1 a_{n-2}^\gamma.$$ 

(118)

As $(\vec{\sigma} \cdot \vec{u})^2 = \vec{u}^2$ for any vector $\vec{u}$ we obtain the Schrödinger–Newton equation

$$i\hbar a_{n}^\gamma = -\frac{\hbar^2}{2m} \Delta a_{n}^\gamma + Va_{n}^\gamma,$$ 

(119)

with the potential $V = mAh_2$ as we did for the Klein–Gordon equation.

The second equation at order $n = 3$ yields

$$a_{n}^\gamma = -\frac{i\sqrt{n}}{2m} \vec{\sigma} \cdot \vec{v}a_{n-2}^\gamma + \frac{i\sqrt{n}}{2m}B_2 \vec{a}^\gamma \left( a_{n-2}^\gamma + \frac{1}{r}a_{n-2}^\gamma \right).$$ 

(120)
This result could now again be inserted into the first equation (115) at order \( n = 4 \), which would result in the evolution equation for \( a_1^\gamma \), but in contrast to the pure Dirac equation (section 4.2) where we obtained the Pauli equation also for \( a_1^\gamma \), this evolution equation will be different from the Schrödinger–Newton equation.

### 4.3.1. Einstein’s equations

Let us first derive the stress–energy tensor for the Dirac field. The Dirac–Lagrangian is

\[
L = \hbar c \overline{\psi} \left( \slashed{D} - \frac{mc}{\hbar} \right) \psi \sqrt{-g}.
\]  

(121)

Considering the variation with respect to \( g^{\mu\nu} \) on-shell (i.e. assuming the Dirac equation to be satisfied), we have

\[
\delta L = \hbar c \text{Re} \left[ \overline{\psi} \left( i (\delta \Gamma^\mu) \partial_\mu + \delta Y \right) \psi \sqrt{-g} \right].
\]  

(122)

According to Finster et al [25], with our special choice (91) for the Dirac matrices

\[
\text{Re} \left( \overline{\psi} \delta Y \psi \right) = \frac{1}{16} \epsilon^{\alpha\lambda\rho\beta} \left( \delta g_{\mu\nu} \right) \text{tr} \Gamma^\alpha \partial_\mu \Gamma^\beta \Gamma^\rho \Gamma^\lambda \psi.
\]  

(123)

\[
\delta \Gamma^\mu = -\frac{1}{2} \epsilon^{\mu\nu} (\delta g_{\nu\lambda}) \Gamma^\lambda
\]  

(124)

and therefore the stress–energy tensor is

\[
T_{\mu\nu} = -\frac{\hbar c}{2} \text{Re} \left[ \overline{\psi} \left( i \Gamma_\mu \partial_\mu + i \Gamma_\nu \partial_\nu \right) \psi \right] + \frac{\hbar c}{8} \epsilon^{\alpha\lambda\rho\beta} \left[ g_{\mu\lambda} \text{tr} \Gamma^\alpha \partial_\mu \Gamma^\beta + g_{\nu\rho} \text{tr} \Gamma^\nu \partial_\nu \Gamma^\rho \right] \overline{\psi} \gamma^5 \Gamma_\beta \psi.
\]  

(126)

Note that (124) is determined up to local Lorentz transformation. \( \delta \Gamma^\mu \) has to obey

\[
\delta \{ \Gamma^\mu, \Gamma^\nu \} = -\delta g^{\mu\nu} \Rightarrow \{ \delta \Gamma^\mu, \Gamma^\nu \} = -\delta g^{\mu\nu}
\]  

(127)

and (124) is a special solution to this equation.

The non-vanishing components of the stress–energy tensor for a Dirac field \( \psi \) are then given by

\[
T_\nu = -\hbar c \text{Re} \left[ -ie^2 A^2 \overline{\psi} \Gamma^\nu \psi \right]
\]  

(128a)

\[
T_\rho = -\hbar c \text{Re} \left[ ie^2 B^2 \overline{\psi} \Gamma^\rho \psi \right]
\]  

(128b)

\[
T_\nu = -\frac{\hbar c}{2} \text{Re} \left[ -ie^2 A^2 \overline{\psi} \Gamma^\nu \psi + ie^2 B^2 \overline{\psi} \Gamma^\nu \psi \right]
\]  

(128c)

\[
T_\mu = -\frac{\hbar c^2}{2} \text{Re} \left[ i e^2 \overline{\psi} \Gamma^\mu \psi - ie^2 A^2 \overline{\psi} \Gamma^\mu \partial_\mu \psi \right] - \frac{\hbar c^2}{2} e^{A-B} \frac{r^2}{r} \sin \theta \left( A' - \frac{1-e^B}{r} \right) \overline{\psi} \gamma^5 \Gamma^\mu \psi
\]  

(128d)

\[
T_\nu = -\frac{\hbar c}{2} \text{Re} \left[ i e^2 \frac{r^2}{r} \sin^2 \theta \overline{\psi} \Gamma^\nu \psi - ie^2 A^2 \overline{\psi} \Gamma^\nu \partial_\nu \psi \right]
\]  

\[
+ \frac{\hbar c^2}{2} e^{A-B} \frac{r^2}{r} \sin \theta \left( A' - \frac{1-e^B}{r} \right) \overline{\psi} \gamma^5 \Gamma^\nu \psi
\]  

(128e)
\[ T_{\theta\theta} = -\frac{\hbar c}{2} \text{Re} \left[ i r^2 \psi \Gamma^\alpha \psi' + i e^2 B \psi \Gamma^\alpha \partial_\theta \psi \right] - \frac{\hbar}{2} e^{B^2 r^2} \sin \theta \bar{B} \psi \gamma^5 \Gamma^\alpha \psi \]  

\[ T_{\theta\psi} = -\frac{\hbar c}{2} \text{Re} \left[ i r^2 \sin^2 \theta \psi \Gamma^\alpha \psi' + i e^2 B \psi \Gamma^\alpha \partial_\theta \psi \right] + \frac{\hbar}{2} e^{B^2 r^2} \sin \theta \bar{B} \psi \gamma^5 \Gamma^\alpha \psi \]  

\[ T_{\phi\phi} = -\frac{\hbar}{2} \text{Re} \left[ i r^2 \psi \Gamma^\alpha \psi' + i e^2 B \psi \Gamma^\alpha \partial_\phi \psi \right] + \frac{\hbar}{2} e^{B^2 r^2} \sin \theta \bar{B} \psi \gamma^5 \Gamma^\alpha \psi \]  

\[ T_{\theta\phi} = -\frac{\hbar c}{2} \text{Re} \left[ i r^2 \sin^2 \theta \psi \Gamma^\alpha \psi' + i e^2 B \psi \Gamma^\alpha \partial_\phi \psi \right] + \frac{\hbar}{2} e^{B^2 r^2} \sin \theta \bar{B} \psi \gamma^5 \Gamma^\alpha \psi \]  

\[ T_{\psi\psi} = -\hbar \text{Re} \left[ i r^2 \sin^2 \theta \psi \Gamma^\alpha \psi' \right]. \]  

Up to order \( c^0 \) these can be simplified to

\[ T_\psi = -\hbar c \text{Re} \left[ \psi^\dagger \left( -i e^2 \right) \psi \right] \]  

\[ T_\theta = -\hbar c \text{Re} \left[ \psi^\dagger \left( i e^2 \right) \psi \right] \]  

\[ T_\phi = -\hbar c \text{Re} \left[ \psi^\dagger \left( -i e^2 \right) \psi' + \psi^\dagger \left( i e^2 \right) \psi \right] \]  

\[ T_{\psi\psi} = -\hbar \text{Re} \left[ i r^2 \sin^2 \theta \psi \Gamma^\alpha \psi' \right]. \]  

Again, as for the Klein–Gordon stress–energy tensor, we expand \( T_{\mu\nu}/c^4 \) using Mathematica and consider Einstein’s equations with the Einstein tensor given in equation (36) for each component order by order, making use of \( A_1 = 0 \). The result differs from the Klein–Gordon result only slightly.
\( n = 0: \) the \( tt \)-component yields \( (rB_1)' = 0 \).

\( n = 1: \) the \( tt \)-component yields
\[
\frac{2h}{r^2} \left( 3B_1^2 + (rB_2)' \right) = 8\pi Gm|a_0^t|^2.
\] (130)

\( n = 2: \) the \( tt \)-component yields
\[
\frac{2h}{r^2} \left( -4B_1^2 - 3rB_1B_2' + (rB_3)' \right) = 8\pi Gm \left( a_1^t a_0^t + a_0^t a_1^t \right).
\] (131)

The \( rr \)-component yields \( B_1 = 0 \), the \( tr \)-component \( B_1 = 0 \) and the \( \theta\theta \)-component \( B_1' = 0 \) are then trivial as well as the order \( n = 0 \) equation. Equations (130) and (131) then simplify to
\[
(rB_2)' = \frac{4\pi Gm r^3}{h}|a_0^t|^2
\] (132)
\[
(rB_3)' = \frac{4\pi Gm r^3}{h} \left( a_1^t a_0^t + a_0^t a_1^t \right)
\] (133)

\( n = 3: \) the \( tt \)-component yields
\[
4(A_2 - B_2)(rB_2)' + B_2' - 2rB_1B_2' + (rB_4)' = \frac{4\pi G r^2}{h^2} \left[ \frac{ih}{2} (a_0^t a_0^t - a_0^t a_0^t) + m h (A_2|a_0^t|^2 + |a_1|^2 + a_2^t a_0^t + a_0^t a_2^t) \right]
\] (134)

The \( rr \)-component yields
\[
B_2 = rA_2'
\] (135)

and the \( \theta\theta \)-component is just the derivative of the \( rr \)-component. The \( tr \)-component yields \( (\nabla \tilde{\sigma}' a_1^t) \) as defined in (111)
\[
\frac{2}{r} B_2 = 2\pi iG \left( a_0^t a_0^t - a_0^t a_0^t \right) - \frac{8\pi Gm}{\sqrt{h}} \text{Re} \left( \frac{a_0^t \tilde{\sigma} a_1^t}{h} \right)
\]
\[
= 2\pi iG \left( a_0^t a_0^t - a_0^t a_0^t \right) + 4\pi G \text{Re} \left[ a_0^t \left( i\tilde{\nabla} + \tilde{\sigma} \times \tilde{\nabla} \right) a_0^t \right]
\]
\[
= 4\pi iG \left( a_0^t a_0^t - a_0^t a_0^t \right) + 4\pi G \text{Re} \left[ a_0^t \left( \tilde{\sigma} \times \tilde{\nabla} \right) a_0^t \right]
\] (136)

where we made use of (117) to express \( a_1^t \) in terms of \( a_0^t \) in the second line. The \( \theta\theta \)- and \( t\psi \)-components yield
\[
0 = \text{Re} (a_0^t \tilde{\sigma} a_1^t) \quad \Leftrightarrow \quad 0 = \text{Re} \left[ a_0^t \left( i\tilde{\nabla} + \tilde{\sigma} \times \tilde{\nabla} \right) a_0^t \right]
\] (137a)
\[
0 = \text{Re} (a_0^t \tilde{\sigma} a_1^t) \quad \Leftrightarrow \quad 0 = \text{Re} \left[ a_0^t \left( i\tilde{\nabla} + \tilde{\sigma} \times \tilde{\nabla} \right) a_0^t \right]
\] (137b)

Now, because we consider the spherically symmetric Einstein’s equations we have to average over all spin orientations in order to get a symmetric stress–energy tensor. This implies that terms proportional to \( \tilde{\sigma} \times \tilde{\nabla} \) in equations (136) and (137) will vanish and that (137) is equivalent to vanishing \( \theta \)- and \( \psi \)-components of the probability current
\[
\tilde{j}_\text{Dew} = \frac{ih}{2m} (\tilde{\nabla} a_0^t) a_0^t - a_0^t \tilde{\nabla} a_0^t).
\] (138)
If then, again, we define the potentials $V = m \hbar A_2$ and $U = m \hbar B_2$ and define $j$ as the $r$-component of the probability current, we obtain the same set of equations as in (48)

$$U = rV'$$ (139)

$$(rU)' = 4\pi G m^2 r^2 |a_0|^2$$ (140)

$$\dot{U} = -4\pi G m^2 r j,$$ (141)

together with equations (133) and (134). As shown in subsection 3.3.2, this yields the Poisson equation for the potential $V$.

5. Summary

From our analysis of the Klein–Gordon and Dirac equations, we conclude that the expansion in either $\hbar$ or $1/c$ according to our ansatz (3) for the fields is a valid scheme to obtain both semi-classical and non-relativistic limits of field equations in an unambiguous way. Applying the same ansatz to the self-gravitating Klein–Gordon and Dirac fields, as mathematically represented by the Einstein–Klein–Gordon and Einstein–Dirac systems, leads to the Schrödinger–Newton equation. Hence we may say that the Schrödinger–Newton equation follows from the self-gravitating fields in the same way as the linear Schrödinger equation can be derived in flat space. Seen from that direction one concludes that the Schrödinger–Newton equation should provide a better description than the linear equation.

However, one may ask: Description of what? In order to arrive at our result we considered the classical (i.e. not second quantized) Klein–Gordon or Dirac field as source for the classical gravitational field. The Schrödinger function then merely appears as part of these fields in the $1/c$ expansion and with certain phases (due to the rest mass) subtracted. The central question is: Whether this is the right way to represent the gravitational field of quantum systems? Reading the classical fields as one-particle amplitudes, it amounts to assuming the validity of the semi-classical Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} (\Psi |\hat{T}_{\mu\nu}| \Psi),$$ (142)

where $\Psi$ now represents the full (second quantized) field. This equation has a long and controversial history, which we will not review here. The interested reader is referred to the book by Kiefer [26] and references therein, and the recent discussion in Carlip [6], and Salzman and Carlip [7]. Interesting early (1957) discussions between Feynman and others on that subject may be found in [27].

An obvious objection is this: if there were a gravitational self-coupling of the Schrödinger function, then, in an analogous fashion, there should also be an electromagnetic self-coupling. In a lowest $1/c$ expansion (neglecting magnetic fields) this would again result in equation (1) with $Gm^2$ replaced by $-e^2/(4\pi\epsilon_0)$, leading to enhanced dispersion due to electrostatic repulsion. Applied to the hydrogen atom, where the electron now not only ‘sees’ the electrostatic attraction of the proton but also the electrostatic repulsion of its own charge-cloud, it seems obvious that this cannot again reproduce the known energy spectrum. Interestingly, precisely this idea of implementing the electromagnetic self-interaction of the quantum-mechanical wavefunction occurred to Schrödinger immediately after he wrote his famous papers on wave mechanics. In 1927, he argued [28] that, as a matter of principle, the self-coupling was required by consistency in order to get closed systems of field equations, just like one derives the radiation reaction of charges through interaction with their own field in ordinary Maxwell theory. But being convinced that the result of this was incompatible
with observed facts, like the hydrogen energy levels, Schrödinger concludes that, from a classical field-theoretic point of view, there is a ‘strange violation of the closedness of the field equations’ 4.

Schrödinger’s idea was revived in the mid 1980s by Barut and collaborators [29], who wrote down a rather obvious nonlinear Dirac equation, which is obtained from the linear equation by first splitting the field into a self-part and external part and then eliminating the self-part by means of Green’s functions and the self-current, the latter then introducing nonlinearities. They claimed that without any further input from QED this suffices to account for effects (at least to leading order in \( \alpha \)) usually attributed to the quantization of the electromagnetic field, like Lamb shift, spontaneous emission and anomalous \( g - 2 \). This may sound less spectacular in view of the fact that the nonlinear Dirac equation they considered is just that obtained from the path integral after integrating out the photon field. There has been some controversy over these claims, but one point seems definitely worth further effort: in the traditional perturbative scheme of field quantization we first eliminate self-interaction through the normal-ordering prescription, which we have to add by hand. Later we add the self-interaction, photon by photon, in a loop expansion. What, physically and not formally speaking, renders this method more correct than that where self-interaction is taken into account already at the classical level, before quantization?

Another problem concerns the application of the Schrödinger–Newton equation to molecular interferometry [8] or even quantum opto-mechanics [30]. What aspect of the complex system (molecule, mirror, etc), comprising many degrees of freedom, does our \( \psi \) represent? It just depends on three coordinates, so one might be tempted to identify it with the centre-of-mass motion. But one would certainly not expect an influence of the centre-of-mass motion from gravitational pair interactions of many-particle systems [31]. Self-interactions are also absent in a manifest Galilei invariant (i.e. manifest ‘non-relativistic’) second-quantized theory of the Schrödinger field coupled to Newton gravity in a Cartan–Friedrichs reformulation [10], simply because of normal ordering. (This does not answer the question posed above, since the normal-order prescription is imposed in an \textit{ad hoc} fashion.) This still leaves open the possibility of thinking of the Schrödinger–Newton equation as a fundamental deviation from one-particle quantum mechanics.

Our results concerning the derivability of the Schrödinger–Newton equation by WKB-like methods from the Einstein–Klein–Gordon and Einstein–Dirac systems are, as such, independent of these questions. But, as stressed in the introduction, they clearly need to be addressed, in one form or another, in any attempt to make well-founded physical applications.

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