ON THE MEAN VALUE PROPERTY
OF FRACTIONAL HARMONIC FUNCTIONS

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ABSTRACT. As well known, harmonic functions satisfy the mean value property, namely the average of the function over a ball is equal to its value at the center. This fact naturally raises the question on whether this is a characterizing feature of balls, namely whether a set for which all harmonic functions satisfy the mean value property is necessarily a ball.

This question was investigated by several authors, including Bernard Epstein [Proc. Amer. Math. Soc., 1962], Bernard Epstein and Menahem Max Schiffer [J. Anal. Math., 1965], Myron Goldstein and Wellington H. Ow, [Proc. Amer. Math. Soc., 1971], who obtained a positive answer to this question under suitable additional assumptions.

The problem was finally elegantly, completely and positively settled by ¨Ulk¨u Kuran [Bull. London Math. Soc., 1972], with an artful use of elementary techniques.

This classical problem has been recently fleshed out by Giovanni Cupini, Nicola Fusco, Ermanno Lanconelli and Xiao Zhong [J. Anal. Math., in press] who proved a quantitative stability result for the mean value formula, showing that a suitable “mean value gap” (measuring the normalized difference between the average of harmonic functions on a given set and their pointwise value) is bounded from below by the Lebesgue measure of the “gap” between the set and the ball (and, consequently, by the Fraenkel asymmetry of the set). That is, if a domain “almost” satisfies the mean value property, then it must be necessarily close to a ball.

The goal of this note is to investigate the nonlocal counterparts of these results. In particular we will prove a classification result and a stability result, establishing that:
(i) if fractional harmonic functions enjoy a suitable exterior average property for a given domain, then the domain is necessarily a ball,
(ii) a suitable “nonlocal mean value gap” is bounded from below by an appropriate measure of the difference between the set and the ball.

Differently from the classical case, some of our arguments rely on purely nonlocal properties, with no classical counterpart, such as the fact that “all functions are locally fractional harmonic up to a small error”.

2010 Mathematics Subject Classification. 35R11, 34A08, 35B05.

Key words and phrases. Mean value formulas, fractional harmonic functions, inverse problems, classification results.

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The authors are members of INdAM. The first author is supported by the INdAM Starting Grant “PDEs, free boundaries, nonlocal equations and applications”. The second and third authors are supported by the Australian Research Council Discovery Project DP170104880 NEW “Nonlocal Equations at Work”. The second author is supported by the Australian Research Council DECRA DE180100957 “PDEs, free boundaries and applications”. Part of this work was carried out during a very pleasant and fruitful visit of the first author to the University of Western Australia, which we thank for the warm hospitality.
1. Introduction

1.1. A fractional version of Kur'an's Theorem. A classical question, dating back to the works [15, 16, 18], is to determine under which conditions a domain providing a mean value property at a given point which is valid for every harmonic function needs to be necessarily a ball.

More precisely, it is well known that if $u$ is harmonic in a domain, then $u$ satisfies the mean value property on every ball compactly contained in that domain. In particular, say the closure of a ball $B_r$ (centered at the origin) is contained in the domain, then

$$\tag{1.1} u(0) = \int_{B_r} u(y) \, dy,$$

where, as usual, the “dashed” integral symbol stands for the average.

The mean value property is certainly remarkable, and of great importance in the classical theory of harmonic functions. Looking at (1.1), a natural question is then to consider an “inverse problem” and try to classify all domains for which a mean value formula can hold: namely, if $\Omega$ is a given domain of $\mathbb{R}^n$ containing the origin and with the property that

$$\tag{1.2} u(0) = \int_{\Omega} u(y) \, dy$$

for all functions $u$ that are harmonic in $\Omega$, is it possible to say anything about $\Omega$? That is, how “special” are the domains satisfying (1.2)?

This problem was definitely settled by Ülkü Kuran in [20], who established, with a concise and very elegant proof, that if $\Omega$ is a bounded domain, containing the origin, such that (1.2) holds for every harmonic, integrable function $u$ in $\Omega$, then $\Omega$ is a ball centered at the origin.

As a matter of fact, the work in [20] was the climax of a rather intense research in the sixties and seventies, that started with [15], in which the classification result for domains satisfying (1.2) was obtained under the additional assumption that $\Omega$ was simply connected. The simple connectivity assumption was later replaced in [16] by the hypothesis that the complement of $\Omega$ possesses a nonempty interior. Also, in [18] the classification result was obtained for planar domains with at least one boundary component which is a continuum.

Interestingly, not only the result in [20] completed the previous works in [15, 16, 18], but it also presented an elementary\(^1\) approach to the question based on the Poisson Kernel of the ball.

\(^1\)For completeness, let us briefly recall the proof in [20]: up to a dilation, we can suppose that $B_1 \subset \Omega$, with $\tilde{x} \in (\partial \Omega) \cap (\partial B_1)$. Then, let

$$h(x) := \frac{|x|^2 - 1}{|x - \tilde{x}|^n} + 1.$$ 

Since $h(0) = 0$, $h \geq 1$ in $\mathbb{R}^n \setminus B_1$, and $h$ is harmonic in $\mathbb{R}^n \setminus \{\tilde{x}\}$, using (1.2) twice (once for $\Omega$ and once for $B_1$), it follows that

$$0 = |\Omega| h(0) = \int_{\Omega} h(y) \, dy = \int_{B_1} h(y) \, dy + \int_{\Omega \setminus B_1} h(y) \, dy = |B_1| h(0) + \int_{\Omega \setminus B_1} h(y) \, dy = \int_{\Omega \setminus B_1} h(y) \, dy \geq |\Omega \setminus B_1|,$$

therefore $|\Omega \setminus B_1| = 0$ and thus $\Omega = B_1$. 
Besides its theoretical interest, the result in [20] has also natural consequences in game theory, since the expected payoff of a random walk with prizes placed at the boundary of a domain is clearly related to harmonic functions, and thus the mean value property of harmonic functions in this context translates into the possibility of exchanging the average expected payoff in a given region with the pointwise expected payoff calculated at a special point of that region (concretely, the center of the ball, and Kuran’s result states that this reduction is not possible either with other regions, or with other points of the ball).

For other type of classification results concerning different averages, see [17], and also [6, 19, 26].

This paper deals with a fractional counterpart of this classical result. Namely, we prove that if the value of a function at a given point equals the (fractional) mean value on a domain, and this property holds for any fractional harmonic function, then the domain has to be a ball, centered at the given point. Our result thus provides the natural counterpart of Kuran’s result.

We also mention that a fractional version of the expected payoff game with Lévy processes in a given domain and prizes set in the complement of the domain are described in details, for instance, in Chapter 2.2 of [10].

It is also interesting to observe that the classical proof of [20] relies on the Poisson Kernel of a ball, which has the interesting properties to be defined and harmonic also outside the ball, and to be changing sign on the boundary of the ball. In the fractional case, the analogue of the Poisson Kernel has been widely employed in the literature (see formula (1.14) in [9] and the references therein), but, differently from the classical case, the fractional Poisson Kernel can not be naturally extended outside the ball — hence, a simple generalization of the proof presented in footnote 1 is not available in the fractional case.

We will completely circumvent such a difficulty in our proof by taking advantage of the particular structure of nonlocal equations. In particular, we will exploit the main result of [13] to construct a fractional harmonic function with the desired properties, and we think that this is a nice example of how, in some occasions, the nonlocal setting not only permits to overcome a series of difficulties that arise in the fractional framework, but it also provides a technical and conceptual simplification with respect to the classical case (specifically, we do not rely on special functions).

To state our fractional version of Kuran’s result, we introduce some notations and preliminary notions. Here and in the rest of the paper $\Omega \subset \mathbb{R}^n$ is a bounded open set and $s \in (0, 1)$ is a fixed number. Moreover, as customary, we use the notation $C\Omega := \mathbb{R}^n \setminus \Omega$.

We recall that a function $u : \mathbb{R}^n \to \mathbb{R}$ (say, for simplicity, sufficiently smooth in a given domain $\Omega \subset \mathbb{R}^n$), satisfying

\begin{equation}
\int_{\mathbb{R}^n} \frac{|u(y)|}{1 + |y|^{n+2s}} \, dy < +\infty
\end{equation}

is $s$-harmonic in $\Omega$ if

\[ (-\Delta)^s u = 0 \quad \text{in} \quad \Omega, \]

where

\[ (-\Delta)^s u(x) := P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(x - y)}{|y|^{n+2s}} \, dy \]
is the fractional Laplace operator (see, for instance, \([2, 10, 21]\)). It is known, as in the classical case, that a function \(u\) is \(s\)-harmonic in \(\Omega\) if and only if \(u\) possesses the following mean value property:

\[
(1.4) \quad u(0) = c(n, s) \int_{CB_r} \frac{r^{2s} u(y)}{(|y|^2 - r^2)^s |y|^n} dy,
\]

for any \(r > 0\) such that \(B_r \subset \subset \Omega\) (see [1, Theorem 2.1], [9, Lemma A.6] or [23, Chapter 1.6]). In (1.4), the notation \(c(n, s)\) stands for a positive, normalizing constant. In particular, taking \(u := 1\) in (1.4), it follows that, setting

\[
(1.5) \quad d\mu_r(y) := \frac{c(n, s) r^{2s} dy}{(|y|^2 - r^2)^s |y|^n},
\]

then \(\mu_r\) is a measure on \(CB_r\), with

\[
(1.6) \quad \mu_r(CB_r) = 1.
\]

In this framework, we can write (1.4) in the form

\[
(1.7) \quad u(0) = \int_{CB_r} u(y) d\mu_r(y) = \frac{1}{\mu_r(CB_r)} \int_{CB_r} u(y) d\mu_r(y),
\]

for any \(r > 0\) such that

\[
(1.8) \quad B_r \subset \subset \Omega.
\]

As a matter of fact, if in addition \(u \in C(\mathbb{R}^n)\), then (1.8) can be replaced by the weaker condition that

\[
(1.9) \quad B_r \subset \Omega,
\]

see Lemma A.1 in the appendix.

Our fractional version of Kuran’s result is thus related to a suitable inverse problem for (1.7) — to state it, we define

\[
(1.10) \quad \mathcal{H}^s(\Omega) := \{u \in C(\mathbb{R}^n) \text{ s.t. (1.3) holds true, and } (-\Delta)^s u = 0 \text{ in } \Omega\}.
\]

In this setting, we can express our fractional version of Kuran’s result in the following way:

**Theorem 1.1.** Let \(\Omega \subset \mathbb{R}^n\) be a bounded open set, containing the origin, and define

\[
(1.11) \quad r := \text{dist}(0, \partial \Omega).
\]

Suppose that

\[
(1.12) \quad u(0) = \frac{1}{\mu_r(C\Omega)} \int_{C\Omega} u(y) d\mu_r(y)
\]

for all functions \(u \in \mathcal{H}^s(\Omega)\).

Then

\[
(1.13) \quad \Omega = B_r.
\]
In short, Theorem 1.1 says that if Ω satisfies a fractional mean value property (compare (1.7) and (1.12)) with respect to a suitable measure, then Ω is necessarily a ball.

Moreover, we stress that if r is as in (1.11), then (1.9) is satisfied (but (1.8) does not hold, and this makes the result in Lemma A.1 technically important for our goals).

We remark that the situation in Theorem 1.1 would be completely different if one replaces (1.12) with a similar formula holding for a suitable measure μ, of the type

\begin{equation}
    u(0) = \int_{\partial \Omega} u(y) d\mu(y).
\end{equation}

Indeed, the problem in (1.14) is structurally very different from the setting in (1.12), since identities as in (1.14) are related to the “balayage” problems for fractional harmonic functions and hold true by taking μ as the fractional harmonic measure (see e.g. [23, formula (4.5.9) and Theorem 4.16], [8, Lemma 17], [22, Theorem 7.2], [7, Section 2.2], [11, Remark 3.1], and also [28] and the references therein). The fractional harmonic measure μ in (1.14) has also a probabilistic interpretation, being the distribution of a Lévy process started at the origin and stopped when exiting the domain Ω. In general, these considerations highlight the importance of carefully choosing the measure μ_r in (1.12) if one is interested in classification results for the domain Ω, which would not be valid for other types of measures (e.g., for the fractional harmonic measure).

We also remark that, denoting by \( \mathcal{H}^{n-1} \) the \((n-1)\)-dimensional Hausdorff measure, an interesting variant of Kuran’s result is that if Ω is a given domain of \( \mathbb{R}^n \), containing the origin and with the property that

\begin{equation}
    u(0) = \int_{\partial \Omega} u(y) d\mathcal{H}^{n-1}(y)
\end{equation}

for all functions \( u \) that are harmonic in \( \Omega \), then \( \Omega \) is necessarily a ball: this was proved in Theorem III.2 of [26] (see also [5, 14, 27]). Interestingly, the condition in (1.15) can also be classically dealt with with “dual” formulations involving a prescription on the normal derivative of the Green function of \( \Omega \) (see Theorem III.1 in [26], Section 7 in [24], and the references therein). Related results are contained in [3, 4]. See also [25] for a classical survey on the spherical and volume averages of harmonic functions.

In this sense, we can consider the setting in (1.12) as a nonlocal transposition of that in (1.15), in which the classical averages along the boundary of the domain (corresponding to classical Dirichlet conditions) are replaced by suitable fractional averages in the exterior of the domain (corresponding to fractional Dirichlet conditions, which are indeed external, and not boundary, prescriptions).

Another natural question is whether a quantitative version of Theorem 1.1 is possible, namely, in view of the stability results in [12] for the classical case, whether it is possible to bound from below a suitable “fractional mean value gap” of a given set by an appropriate quantity measuring the difference between the set and the corresponding ball, thus showing that sets that “almost” satisfy the fractional mean value property must be necessarily “close” to the ball. Our answer to this question occupies the forthcoming subsection.
1.2. A stability result for the fractional mean value theorem. A second argument of this note is a stability result for the mean value property. We want to obtain a fractional counterpart of [12, Theorem 1.1], by proving that if \( u \) is “close” to being its mean value on a domain for any \( s \)-harmonic function, then the domain is “close” to being a ball.

To this end, we consider \( \Omega \subset \mathbb{R}^n \) to be a bounded and open set containing the origin, and we denote \( r \) as in (1.11) and \( \mu_r \) as in (1.5). Taking inspiration from in [12, formula (1.2)], we define the rescaled fractional Gauss mean value gap

\[
G_r(\Omega) := \sup_{u \in \mathcal{H}_s(\Omega)} \left| \frac{u(0) - \frac{1}{\mu_r(C\Omega)} \int_{C\Omega} u(y) \, d\mu_r(y)}{\int_{C\Omega} |u(y)| \, d\mu_r(y)} \right|.
\]

In light of (1.7), we know that balls make the fractional Gauss mean value gap vanish. Our goal is to provide a stability result stating that if the fractional Gauss mean value gap is small, then the set \( \Omega \) is “close to a ball”. The precise quantitative result that we have is the following:

**Theorem 1.2.** Let

\[
R := \max_{y \in \Omega} |y|.
\]

Then, there exists a positive constant \( C \), depending only on \( n, s \) and \( R/r \) such that

\[
\mu_r(\Omega \setminus B_r) \leq C G_r(\Omega).
\]

Let us point out the classical counterpart of (1.18) here is given in [12, formula (1.3)]. Interestingly, normalizing the volume, the constant in [12] depends only on the dimension, whereas in our case and with our techniques, we have a dependence of the constant in (1.18) also on \( R/r \). We think that it is an interesting problem to decide whether this structural dependence can be ameliorated.

The rest of this note contains the proofs of Theorems 1.1 and 1.2. An interesting feature of these proofs is the important conceptual difference with respect to the case of the classical Laplacian, since they rely on a structural property of fractional harmonic functions discovered in [13] (i.e., “all functions are locally fractional harmonic up to a small error”) which does not have any analogue in the classical case.

2. Proofs of the main results

In this section, we provide the proofs of the main results of this note, namely the classification result in Theorem 1.1 and the stability result in Theorem 1.2. We start with the classification result:

**Proof of Theorem 1.1.** We argue towards a contradiction, assuming that (1.13) does not hold true. Namely, we suppose that \( \Omega \setminus B_r \neq \emptyset \). Then, let \( p \in \Omega \setminus B_r \). Since \( \Omega \) is open, there exists \( \rho > 0 \) such that \( B_\rho(p) \subset \Omega \). Furthermore, one sees that, since \( p \notin B_r \), it holds that \( B_\rho(p) \setminus \overline{B_r} \neq \emptyset \). These observations give that

\[
\emptyset \neq B_\rho(p) \setminus \overline{B_r} \subset \Omega \setminus B_r.
\]
and therefore, by (1.5),

\begin{equation}
\mu_r(\Omega \setminus B_r) > 0.
\end{equation}

Moreover, according to (1.7) and (1.12), for any \( u \in \mathcal{H}^s(\Omega) \) with \( u(0) = 0 \) we have that

\begin{equation}
0 = \mu_r(C\Omega) u(0) = \int_{C\Omega} u(y) d\mu_r(y)
= \int_{CB_r} u(y) d\mu_r(y) - \int_{\Omega \setminus B_r} u(y) d\mu_r(y)
= \mu_r(CB_r) u(0) - \int_{\Omega \setminus B_r} u(y) d\mu_r(y)
= - \int_{\Omega \setminus B_r} u(y) d\mu_r(y).
\end{equation}

Now we consider \( \varepsilon > 0 \) suitably small, possibly in dependence of \( r \). For concreteness, we can take

\begin{equation}
\varepsilon := \frac{r^2}{4}.
\end{equation}

For every \( x \in \mathbb{R}^n \), we also let \( f(x) := |x|^2 \), and we define

\[ R := \max_{y \in \Omega} |y|. \]

We exploit Theorem 1.1 in [13] for \( \varepsilon \) as in (2.3) to obtain the existence of a function \( f_{r,R} \in C^s_c(\mathbb{R}^n) \) such that

\[ (-\Delta)^s f_{r,R} = 0 \quad \text{in } B_R, \]

and \[ \|f_{r,R} - f\|_{L^\infty(B_R)} \leq \varepsilon = \frac{r^2}{4}. \]

Then, we define

\begin{equation}
\begin{aligned}
& u^*(x) := -f_{r,R}(x) + f_{r,R}(0).
\end{aligned}
\end{equation}

We remark that, for all \( x \in B_R \),

\[ u^*(x) = -f_{r,R}(x) + f(x) + f_{r,R}(0) - f(0) - f(x) + f(0) \leq - f(x) + f(0) + |f(x) - f_{r,R}(x)| + |f_{r,R}(0) - f(0)| \leq - |x|^2 + \frac{r^2}{2}. \]

Hence

\[ -u^*(x) \geq |x|^2 - \frac{r^2}{2} \geq \frac{r^2}{2} \quad \text{for all } x \in B_R \setminus B_r. \]

Since \( \Omega \subset B_R \), it follows from this and (2.1) that

\begin{equation}
\begin{aligned}
& \int_{\Omega \setminus B_r} -u^*(y) d\mu_r(y) \geq \frac{r^2}{2} \mu_r(\Omega \setminus B_r) > 0.
\end{aligned}
\end{equation}
We also point out that $u^*(0) = 0$ and $(-\Delta)^s u^*(x) = (-\Delta)^s f_{r,R}(x) = 0$ for all $x \in B_R$, consequently $u^* \in \mathcal{H}^s(B_R) \subset \mathcal{H}^s(\Omega)$.

Hence, we can exploit (2.2) with $u := u^*$, obtaining a contradiction with (2.5). \qed

Now, we focus on the proof of the stability result in Theorem 1.2.

**Proof of Theorem 1.2.** The argument that we use here is a suitable quantification of the one in the proof of Theorem 1.1, combined with some rescaling methods.

We denote

(2.6) \[ \Omega_r := \frac{\Omega}{r} \quad \text{and} \quad u_r(x) := u(rx). \]

By (1.17), we know that $\Omega \subset B_R$, and consequently

(2.7) \[ \Omega_r \subset B_{R/r}. \]

Moreover, in view of (1.10), we remark that

(2.8) \[ u \in \mathcal{H}^s(\Omega) \text{ if and only if } u_r \in \mathcal{H}^s(\Omega_r). \]

Furthermore, for every $\Omega'$ which contains $B_r$, by (1.5), and using the substitution $z := y/r$, we see that

\[
\int_{\mathcal{C}\Omega'} u(y) d\mu_r(y) = c(n, s) r^{2s} \int_{\mathcal{C}\Omega} \frac{u(y) dy}{(|y|^2 - r^2)^{s}|y|^n}
\]

(2.9)

\[
= c(n, s) \int_{\mathcal{C}\Omega_r} \frac{u(rz) dz}{(|z|^2 - 1)^s|z|^n}
\]

\[
= \int_{\mathcal{C}\Omega_r} u_r(z) d\mu_1(z),
\]

where the notation in (2.6) has been used for $\Omega'$ as well.

In particular, taking $\Omega' := B_r$ in (2.9),

(2.10) \[ \int_{\mathcal{C}B_r} u(y) d\mu_r(y) = \int_{\mathcal{C}B_1} u_r(z) d\mu_1(z). \]

Also, taking $u := 1$ in (2.9),

(2.11) \[ \mu_r(\mathcal{C}\Omega') = \mu_1(\mathcal{C}\Omega_r'). \]

Choosing $\Omega' := \Omega$ in (2.11), we have that

\[
\mu_r(\mathcal{C}\Omega) = \mu_1(\mathcal{C}\Omega_r).
\]

Making use of this, (2.9) and (2.10), we get that

\[
\left| \frac{u(0) - 1}{\mu_r(\mathcal{C}\Omega)} \int_{\mathcal{C}\Omega} u(y) d\mu_r(y) \right| = \left| \frac{u_r(0) - 1}{\mu_1(\mathcal{C}\Omega_r)} \int_{\mathcal{C}\Omega_r} u_r(y) d\mu_1(y) \right|
\]

Gathering this, (1.16) and (2.8), we obtain that

(2.12) \[ G_r(\Omega) = G_1(\Omega_r). \]
Now, we define $f(x) := |x|^2$, and fix $\varepsilon > 0$ sufficiently small — for concreteness, we can take

$$\varepsilon := \frac{1}{4}.$$  

Then, we exploit Theorem 1.1 in [13] for this value of $\varepsilon$ and see that there exists $f_{R/r} \in C_c^s(\mathbb{R}^n)$ such that

$$(-\Delta)^s f_{R/r} = 0 \quad \text{in } B_{R/r},$$  

(2.13)

and

$$\|f_{R/r} - f\|_{L_\infty(B_{R/r})} \leq \frac{1}{4}.$$  

We let

$$u^*(x) := -f_{R/r}(x) + f_{R/r}(0).$$  

(2.14)

We observe that

$$u^*(0) = 0.$$  

(2.15)

In addition, we have that

$$u^* \in \mathcal{H}^s(B_{R/r}).$$  

(2.16)

This and (2.7) give that $u^* \in \mathcal{H}^s(\Omega_r)$. As a consequence of this and (1.16) we know that

$$G_1(\Omega_r) \geq \frac{\left| u^*(0) - \frac{1}{\mu_1(C\Omega_r)} \int_{C\Omega_r} u^*(y) \, d\mu_1(y) \right|}{\int_{CB_1} |u^*(y)| \, d\mu_1(y)}.$$  

From this, (2.12) and (2.15), we deduce that

$$G_r(\Omega) \geq \frac{\left| \frac{1}{\mu_1(C\Omega_r)} \int_{C\Omega_r} u^*(y) \, d\mu_1(y) \right|}{\int_{CB_1} |u^*(y)| \, d\mu_1(y)}.$$  

(2.17)

Besides, in light of (2.16), we also know that $u^* \in \mathcal{H}^s(B_1)$. As a result, by (1.7) and (2.15),

$$0 = u^*(0) = \int_{CB_1} u^*(y) \, d\mu_1(y).$$

For this reason,

$$\int_{C\Omega_r} u^*(y) \, d\mu_1(y) = \int_{CB_1} u^*(y) \, d\mu_1(y) - \int_{\Omega_r \setminus B_1} u^*(y) \, d\mu_1(y)$$  

(2.18)

$$= -\int_{\Omega_r \setminus B_1} u^*(y) \, d\mu_1(y).$$

Furthermore, exploiting (2.13) and (2.14), we find that, for all $x \in \Omega_r$,

$$-u^*(x) = f_{R/r}(x) - f_{R/r}(0) \geq f(x) - f(0) - 2\|f_{R/r} - f\|_{L_\infty(\Omega_r)}$$

$$\geq |x|^2 - 2\|f_{R/r} - f\|_{L_\infty(B_{r/2})} \geq |x|^2 - \frac{1}{2}.$$
As a result, for all $x \in \Omega_r \setminus B_1$, 

$$-u^*(x) \geq \frac{1}{2}.$$ 

Combining this with (2.18), we conclude that 

$$\int_{\Omega_r} u^*(y) \, d\mu_1(y) \geq \frac{\mu_1(\Omega_r \setminus B_1)}{2}. $$ 

Hence, by plugging this information into (2.17), we see that 

(2.19) 

$$G_r(\Omega) \int_{\Omega_r} u^*(y) \, d\mu_1(y) \geq \left| \frac{1}{\mu_1(\Omega_r)} \int_{\Omega_r} u^*(y) \, d\mu_1(y) \right| \geq \frac{\mu_1(\Omega_r \setminus B_1)}{2\mu_1(C\Omega_r)}. $$

Now, using (2.14), we know that, for every $x \in \mathbb{R}^n$, 

$$|u^*(x)| \leq |f_{R/r}(x)| + |f_{R/r}(0)| \leq 2\|f_{R/r}\|_{L^\infty(\mathbb{R}^n)}. $$

Consequently, recalling (1.6), 

$$\int_{B_1} |u^*(y)| \, d\mu_1(y) \leq 2\|f_{R/r}\|_{L^\infty(\mathbb{R}^n)} \mu_1(CB_1) = 2\|f_{R/r}\|_{L^\infty(\mathbb{R}^n)}. $$

This and (2.19) yield that 

$$4\|f_{R/r}\|_{L^\infty(\mathbb{R}^n)} G_r(\Omega) \geq \frac{\mu_1(\Omega_r \setminus B_1)}{\mu_1(C\Omega_r)}. $$

In addition, using (1.6) and the fact that $B_1 \subset \Omega_r$, 

$$\mu_1(C\Omega_r) \leq \mu_1(CB_1) = 1, $$

we find that 

(2.20) 

$$4\|f_{R/r}\|_{L^\infty(\mathbb{R}^n)} G_r(\Omega) \geq \mu_1(\Omega_r \setminus B_1). $$

Now, we exploit (2.11) with $\Omega' := B_r \cup (C\Omega)$. Since in this case 

$$C\Omega' = \Omega \cap (CB_r) = \Omega \setminus B_r, $$

and thus 

$$C\Omega'_r = \Omega_r \setminus B_1, $$

we deduce from (2.11) that 

$$\mu_r(\Omega \setminus B_r) = \mu_r(C\Omega') = \mu_1(C\Omega'_r) = \mu_1(\Omega_r \setminus B_1). $$

Then, we insert this information into (2.20), finding that 

$$4\|f_{R/r}\|_{L^\infty(\mathbb{R}^n)} G_r(\Omega) \geq \mu_r(\Omega \setminus B_r). $$

This establishes (1.18), as desired. $\square$
Appendix A. The fractional mean value formula for balls touching the boundary from inside

In this appendix, we present an auxiliary result that shows that continuous functions that are \(s\)-harmonic in a given domain satisfy the mean value formula for every ball contained in the domain (and not only for the balls that are compactly contained in the domain):

**Lemma A.1.** Assume that \(u\) satisfies (1.3) and is \(s\)-harmonic inside a domain \(\Omega \subset \mathbb{R}^n\). Assume also that \(u \in C(\mathbb{R}^n)\). Suppose that \(B_r \subset \Omega\). Then (1.4) holds true.

**Proof.** Let \(c = \left(\frac{r}{2}, r\right)\). Then \(B_c \subset B_r \subset \Omega\). Therefore, in view of (1.8), we can employ (1.4) with respect to the ball \(B_c\), hence

\[
u(0) = c(n, s) \int_{cB_c} \frac{\rho^{2s} u(y)}{|y|^2 - \rho^{2s}|y|^n} \, dy.
\]

Furthermore, if \(R \geq 2r\) and \(y \in cB_c\), we have that

\[|y|^2 - \rho^2 = (|y| + \rho)(|y| - \rho) \geq |y| \left(\frac{|y|}{2} + \frac{R}{2} - \rho\right) \geq \frac{|y|^2}{2}.
\]

As a result, given any \(\varepsilon \in (0, 1)\), taking a suitable \(R \geq 2r\) to be chosen sufficiently large, possibly in dependence of \(\varepsilon, u, r, n\) and \(s\), but independent of \(\rho\), and exploiting (1.3), we see that

\[
c(n, s) \left| \int_{cB_c} \frac{\rho^{2s} u(y)}{|y|^2 - \rho^{2s}|y|^n} \, dy \right| \leq 2^s c(n, s) \int_{cB_c} \frac{r^{2s} |u(y)|}{|y|^{n+2s}} \, dy \leq \varepsilon,
\]

and similarly

\[
c(n, s) \left| \int_{cB_c} \frac{r^{2s} u(y)}{|y|^2 - r^{2s}|y|^n} \, dy \right| \leq \varepsilon.
\]

This and (A.1) give that

\[
\left| u(0) - c(n, s) \int_{cB_c} \frac{r^{2s} u(y)}{|y|^2 - r^{2s}|y|^n} \, dy \right| = \left| c(n, s) \int_{cB_c} \frac{\rho^{2s} u(y)}{|y|^2 - \rho^{2s}|y|^n} \, dy - c(n, s) \int_{cB_c} \frac{r^{2s} u(y)}{|y|^2 - r^{2s}|y|^n} \, dy \right|
\]

\[
\leq 2\varepsilon + c(n, s) \left| \int_{B_R \setminus B_r} \frac{\rho^{2s} u(y)}{|y|^2 - \rho^{2s}|y|^n} \, dy - \int_{B_R \setminus B_r} \frac{r^{2s} u(y)}{|y|^2 - r^{2s}|y|^n} \, dy \right|.
\]

Hence, after the change of variable \(z := ry/\rho\) in one integral, we obtain that

\[
\left| u(0) - c(n, s) \int_{cB_c} \frac{r^{2s} u(y)}{|y|^2 - r^{2s}|y|^n} \, dy \right| \leq 2\varepsilon + c(n, s) \left| \int_{B_R \setminus B_r} \frac{r^{2s} u(\rho z/r)}{|z|^2 - r^{2s}|z|^n} \, dz - \int_{B_R \setminus B_r} \frac{r^{2s} u(y)}{|y|^2 - r^{2s}|y|^n} \, dy \right|.
\]

Moreover, since \(u\) is continuous, we have that

\[
\chi_{B_{R \setminus B_r}}(z) \frac{r^{2s} |u(\rho z/r)|}{|z|^2 - r^{2s}|z|^n} \leq \frac{r^{2s} \|u\|_{L^\infty(B_R)}}{|z|^2 - r^{2s}|z|^n} \in L^1(\mathbb{R}^n).
\]
Consequently, by the Dominated Convergence Theorem and the continuity of $u$, we can take the limit as $\rho \to r$ in (A.2), with $\varepsilon$ fixed, concluding that

$$
\left| u(0) - c(n, s) \int_{C_B} \frac{r^{2s} u(y)}{|y|^2 - r^2} \frac{1}{|y|^n} \, dy \right| \\
\leq 2\varepsilon + c(n, s) \left( \int_{B_{2r}\setminus B_r} \frac{r^{2s} u(z)}{|z|^2 - r^2} \frac{1}{|z|^n} \, dz - \int_{B_{2r}\setminus B_r} \frac{r^{2s} u(y)}{|y|^2 - r^2} \frac{1}{|y|^n} \, dy \right)
= 2\varepsilon.
$$

Since $\varepsilon$ can now be taken arbitrarily small, we conclude that

$$
u(0) = c(n, s) \int_{C_B} \frac{r^{2s} u(y)}{|y|^2 - r^2} \frac{1}{|y|^n} \, dy,$$

as desired. \qed

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