Analytic determination of the asymptotic quasi-normal mode spectrum of small Schwarzschild-de Sitter black holes

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Abstract

Following the monodromy technique performed by Motl and Neitzke, we consider the analytic determination of the highly damped (asymptotic) quasi-normal modes of small Schwarzschild-de Sitter (SdS) black holes. We comment the result as compared to the recent numerical data of Konoplya and Zhidenko.

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For intermediate times, perturbations of a black hole are dominated by characteristic, damped oscillations - the so-called quasi-normal modes (QNMs), the pure tones it emits in the final stage of its perturbation (for reviews, see [1]).

The search for large imaginary, highly damped QNM frequencies has recently attracted a great deal of interest. This has been motivated by the work of Hod, who, based on the Bohr correspondence principle, proposed that the real part of the asymptotic, highly damped QNM frequencies of a black hole can be used to determine the spacing (fundamental unit) of its quantum area spectrum [2]. The quantization of black holes horizon area has been proposed by Bekenstein thirty years ago [3], based on the adiabatic invariance of the horizon area and the analogy with the quantum mechanics of adiabatic systems (for a review, see [4]). In [2], Hod has used the highly damped spectrum of QNMs of a Schwarzschild black hole, numerically obtained by Nollert [5] and later confirmed by Andersson [6] and Liu [7]. Recently, Dreyer [8] used Hod’s conjecture in the context of Loop Quantum Gravity to fix a free-parameter (the Barbero-Immirzi parameter) which appears in the area spectrum provided by this theory (for reviews, see [9]). In this way, he could naturally obtain the Bekenstein-Hawking entropy (see also [10]). Thereafter, several works concerning the highly damped QNMs and their role in the black hole area quantization have appeared [11] (for a review and more references, see [12]). There have also been studies along the same lines, but concerning non-asymptotically flat black holes, namely, black holes in asymptotically de Sitter (dS) and anti-de Sitter (AdS) space-times [13]. Furthermore, the study of QNMs of dS [14] and AdS [15] black holes has been under intensive study. Finally, we mention that the works of Motl [16] and Motl and Neitzke [17] deserve special attention, since they first provided an analytic determination of the large imaginary QNMs of a Schwarzschild black hole, in agreement with the numerical formula of Nollert [5]. Recently, following the approach of Motl and Neitzke in [17], Krasnov and Solodukhin [18] rederived the real part of the highly damped QNMs of a Schwarzschild black hole by means of a more detailed study, on the complex plane, of the relationship between the monodromy associated to the type of the differential equation which appears in black hole perturbation problem and the related Riemann surfaces.

The Schwarzschild-de Sitter (SdS) black hole metric in Schwarzschild coordinates \((t, r, \theta, \phi)\) is given by [19] (we will be using natural units, \(G = c = k = \hbar = 1\), throughout the text) \(ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\), where \(f(r) = 1 - \frac{2M}{r} - \frac{r^2}{a^2}\). \(M\)
is the black hole mass and $a$ the "cosmological radius", related to the cosmological constant $\Lambda > 0$ by $a^2 = 3/\Lambda$. Describing a black hole in a cosmological background, $M$ has to be limited to the interval $0 < M < M_N = a/3\sqrt{3}$. In this interval, the function $f(r)$ has three distinct real zeros: at $r = r_e$, the black hole event horizon, $r_c (\geq r_e)$, the cosmological event horizon, and $r_0 = -(r_e + r_c)$. The extreme value $M_N$ gives the Nariai limit [20], for which the event and cosmological horizons coincide.

Associated to each horizon $r_i$ there is a surface gravity $\kappa_i$, defined by $\kappa_i = \frac{1}{2} \left| \frac{df}{dr} \right|_{r=r_i}$. Explicitly, we have $\kappa_e = (r_c - r_e)(r_e - r_0)/2a^2 r_e$, $\kappa_c = (r_c - r_e)(r_c - r_0)/2a^2 r_c$, and $\kappa_0 = (r_e - r_0)(r_c - r_0)/2a^2(r_0)$. The metric parameters $M$ and $a$ can be written in terms of $r_e$ and $r_c$ as $a^2 = r_e^2 + r_c^2 + r_e r_c$ and $2Ma^2 = r_e r_c(r_e + r_c)$.

Assuming a time dependence $e^{i\omega t}$, expanding the perturbating field into multipoles and defining the tortoise coordinate $x$ by $dx/dr = 1/f(r)$, one can show that the equations for scalar, gravitational and electromagnetic perturbations of a SdS black hole are summarized by [21]

$$\frac{d^2\psi(r)}{dx^2} + \left[ \omega^2 - V(r(x)) \right] \psi(r) = 0 \quad ,$$

where the potential $V(r)$ is

$$V(r) = f(r) \left[ \frac{\ell(\ell+1)}{r^2} - \frac{2M(j^2-1)}{r^3} \right] \quad ,$$

for gravitational ($j = 2$) and electromagnetic ($j = 1$) perturbations. $\ell$ and $j$ are the multipole index and the spin of the perturbating field, respectively. We will consider here only electromagnetic and axial gravitational perturbations (in four dimensions, axial and polar gravitational perturbations are isospectral).

Explicitly, the tortoise coordinate is given by

$$x = \frac{1}{2\kappa_e} \ln \left( \frac{r}{r_e} - 1 \right) - \frac{1}{2\kappa_c} \ln \left( 1 - \frac{r}{r_c} \right) + \frac{1}{2\kappa_0} \ln \left( 1 - \frac{r}{r_0} \right) \quad .$$

Since $V(x) \equiv V(r(x))$ goes to zero exponentially for $x \to \pm \infty$, in terms of the tortoise coordinate the boundary conditions for the QNMs of a SdS black hole are typically the same as those of an asymptotically Schwarzschild black hole, namely, ingoing waves at the black hole horizon ($x \to -\infty$) and outgoing waves at the cosmological horizon ($x \to +\infty$), i.e., in view of the dependence $e^{i\omega t}$ we have

$$\psi \sim e^{\mp i\omega x} \quad , \quad x \to \pm \infty \quad .$$
Motl and Neitzke [17] have considered the analytic extension to the complex $r$-plane of the perturbation equation (similar to (1)) for the Schwarzschild metric. Since in this case the equation has (regular) singular points at the origin ($r = 0$) and at the black hole horizon ($r = r_e$), its solution $\psi(r)$ is multivalued around these points. Such multivaluedness is a crucial feature in the analysis of [17], for the key idea is to use the QNM boundary conditions to compute the monodromy of $\psi(r)$ as we run along a conveniently chosen contour in the $r$-plane. Comparison of the monodromy calculated locally (by solving the perturbation equation directly) and globally (by direct application of the QNM boundary conditions) allowed the determination of the highly damped quasi-normal modes, i.e., those with $\text{Im}\omega > \text{Re}\omega$.

Although the physical region of interest for the problem of QNMs of a SdS black hole is $r_e < r < r_c$, an analytic extension to the region $0 < r < r_e$ is essential for a “monodromic analysis”. As we will verify later, the Schwarzschild and SdS cases have similar features which concern the behaviour of the potential near the black hole singularity ($r = 0$).

Due to the logarithmic terms in (3), the analytic continuation to complex-$r$ implies that $x$ is a multivalued function of $r$. But $\text{Re}x$ is free of ambiguity and we can determine its sign in the $r$-plane. The multivaluedness of $x(r)$ is also fundamental in this approach. Together with the multivaluedness of $\psi(r)$ around the (regular) singular point $r_e$, it will lead us to the results we look for.

As in [17], we also conveniently introduce the variable $z = x + \text{const}$. Choosing the constant such that $z = 0$ for $r = 0$, and the branch $n = 0$ for $\ln(\pm1)$, from (3) we have $z = x - \pi i/2\kappa e$.

By analytically continuing $r$ we can distinguish the functions $e^{\mp i\omega x}$ by their monodromy at $r = r_e$. This enables us to define the boundary condition at $r = r_e$ by simply requiring that $\psi(r)$ have monodromy $e^{2\pi i(\omega/2\kappa e)} = e^{\pi i\omega/\kappa e}$ on a clockwise contour around this point. We note that we cannot do the same at $r = r_c$, since as $r$ is now complex, $x$ does not tend to $+\infty$ for $r = r_c$. Nevertheless, if we consider the special case of small black holes $(r_c >> r_e)$, we can approximately consider that $x >> 1$ for $r = r_c$, as an expansion of $x(r)$ for $(r_c/r_e) >> 1$ shows.

Now we can closely follow [17] and in order to define the boundary condition at $r_c$, we simply analytically continue $\psi(r)$ via rotation to the line $\text{Im}(\omega x) = 0$. Hence we will have a purely oscillatory asymptotic behaviour on the real line $\omega x$, what implies that it
is possible to select a particular solution by specifying its asymptotics. We will match the asymptotics along the line $Re \, x = 0$ (corresponding to the left part of the contour $\gamma$ between $A$ and $B$ in the figure) and use the boundary conditions to determine the QNMs. Taking the limit where $\omega$ is almost purely imaginary (highly damped modes), we see that the line $Im \, (\omega x) = 0$ is slightly sloped off the line $Re \, x = 0$. Since there are two possible directions for the rotation, we can choose the one corresponding to an angle smaller than $\pi/2$.

From (4), we note that for QNMs we have $Im \, \omega > 0$. Assuming initially that $Re \, \omega > 0$, in view of the condition $Im \, (\omega x) = 0$, we see that $x = +\infty$ is rotated to $\omega x = +\infty$, and on this line the boundary condition at $x = +\infty$ is

\[ \psi(r) \sim e^{-i\omega x}, \quad \omega x \to +\infty. \]  

Let us first compute the monodromy locally. This will be done by matching the asymptotics along the line $Im \, (\omega x) = 0$ (or along $Im \, (\omega z) = 0$, for $\omega$ (almost) purely imaginary). We will start at $A$ and move along the contour $\gamma$ towards the origin (see figure). The asymptotics at $A$ can be matched to that near the origin. First, to obtain the solution near the origin, we expand $x$ near $r = 0$ and thus obtain for small $z$

\[ z \approx \beta r^2, \]  

where $\beta = \frac{1}{4} \left( \frac{1}{\kappa c_r^2} - \frac{1}{\kappa c_r^2} - \frac{1}{\kappa c_r^0} \right)$.

Therefore, near the origin the potential behaves as

\[ V(z) \approx \frac{4M^2 \beta^2 (j^2 - 1)}{z^2}. \]  

We remark that (6) has been deduced without using the approximation of small black holes. Hence, the near origin behaviour (7) remains valid in general.

In view of (7), (1) can be reduced to a Bessel equation. Its solution near the origin is then given by

\[ \psi(z) = A_+ c_+ \sqrt{\omega z} J_{\nu}(\omega z) + A_- c_- \sqrt{\omega z} J_{-\nu}(\omega z), \]  

with the index $\nu$ being given by

\[ \nu = \frac{1}{2} \sqrt{1 + (4M\beta)^2(j^2 - 1)}. \]  

We will now match the solution (8) to the asymptotics away from the origin. Taking the asymptotic behaviour of $J_{\pm \nu}(\omega z)$ as $\omega z \to \infty$, we can choose the constants $c_\pm$ in (8) such
that
\[ c_\pm \sqrt{\omega z} J_{\pm \nu}(\omega z) \sim 2 \cos(\omega z - \alpha_\pm) \quad \text{as} \quad \omega z \to \infty \] ,

where \( \alpha_\pm = \frac{\pi}{4}(1 \pm 2\nu) \).

From (10) and (8), and making use of the boundary condition (5), we can write for the asymptotics at \( A^+ \)
\[ \psi(z) \sim (A^+ e^{i\alpha^+} + A^- e^{i\alpha^-}) e^{-i\omega z} \quad \text{as} \quad \omega z \to \infty \] ,

since
\[ A^+ e^{-i\alpha^+} + A^- e^{-i\alpha^-} = 0 \] .

In order to follow the contour until \( B \) we have to turn around the origin. Thus we must perform a rotation of \( 3\pi/2 \) in the \( r \)-plane, corresponding to \( 3\pi \) in the \( z \)-plane. Using the asymptotic behaviour of the Bessel function near the origin, we can match it to the asymptotic solution. In fact, using for the Bessel function the behaviour \( J_{\pm \nu}(z) \sim \left(\frac{z}{2}\right)^{\pm \nu} \) near the origin, after the \( 3\pi \) rotation we have
\[ c_\pm \sqrt{\omega z} J_{\pm \nu}(\omega z) \sim e^{6i\alpha^\pm} 2 \cos(\omega z - \alpha_\pm) \quad \text{as} \quad \omega z \to -\infty \] ,

which combined with (8) gives for the asymptotics at \( B \)
\[ \psi(z) \sim (A^+ e^{5i\alpha^+} + A^- e^{5i\alpha^-}) e^{-i\omega z} + (A^+ e^{7i\alpha^+} + A^- e^{7i\alpha^-}) e^{i\omega z} \quad \text{as} \quad \omega z \to -\infty . \]
We can go from $B$ back to $A$ along the large semi-circle ($|r| \sim |r_c|$). In this region, taking into account our approximation of small black holes, we have $x >> 1$, and thus $V(x)$ goes to zero, such that $\omega^2$ dominates the potential, and we can approximate the solution of the wave equation there by plane waves. Hence, the coefficient of $e^{-i\omega z}$ remains essentially unchanged as we return to $A$, whereas the same cannot be said for the coefficient of $e^{i\omega z}$, which gives an exponentially small contribution to $\psi(r)$ in the region where $Re x > 0$. Therefore, from (11) and (13), we see that the monodromy around the contour $\gamma$ is simply given by

$$\frac{A_+ e^{5i\alpha_+} + A_- e^{5i\alpha_-}}{A_+ e^{i\alpha_+} + A_- e^{i\alpha_-}} = -(1 + 2 \cos 2\pi \nu)$$

where we have taken into account (12).

The global computation of the monodromy around $\gamma$ can be obtained by making use of the QNM boundary condition at $r_e$, since the only singularity of $\psi(r)$ or $e^{-i\omega z}$ inside the contour $\gamma$ occurs at this point. The boundary condition at $r_e$ implies that after a full trip around the contour clockwise, $\psi(r)$ acquires a phase $e^{\pi \omega / \kappa_e}$, while $e^{-i\omega z}$ acquires a phase $e^{-\pi \omega / \kappa_e}$. Therefore, we must multiply the coefficient of $e^{-i\omega z}$ in the asymptotics of $\psi(r)$ by $e^{2\pi \omega / \kappa_e}$.

Comparing the local monodromy (14) with the global one, we find

$$e^{2\pi \omega_n / \kappa_e} = -(1 + 2 \cos 2\pi \nu)$$

For the choice $Re \omega < 0$ we would have to consider the rotation in the opposite direction and then reverse considerations on the points $A$ and $B$, as well as to run the contour $\gamma$ in the counter-clockwise direction. We thus conclude that the asymptotic, highly damped quasi-normal mode spectrum of small SdS black holes is given by

$$\omega_n / \kappa_e = (n + \frac{1}{2})i \pm \frac{1}{2\pi} \ln |1 + 2 \cos 2\pi \nu|$$

with $n \to \infty$.

From the relation $4M\beta = -\left[1 + \frac{r_e r_c}{(r_+ r_c)^2}\right]$, if we take the limit as $r_c \to \infty$, which corresponds to $\Lambda = 0$, we find $4M\beta = -1$, such that we recover the result of Motl and Neitzke for asymptotically flat Schwarzschild black holes [17],

$$8\pi M \omega_n = (2n + 1)\pi i \pm \ln |1 + 2 \cos \pi j|$$

The result (16) shows that the real part of the asymptotic modes has a constant value and then it fails to reproduce the oscillatory behaviour recently found by Konoplya and
Zhidenko in the case of gravitational perturbations (j = 2). Nevertheless, for the electromagnetic case (j = 1), \(Re \omega_n = 0\), what agrees with the result of those authors, but the behaviour of the imaginary part of the modes, as given by (16), only partially agrees with that found in [22], since it does not show an oscillatory behaviour in its spacing, found for both gravitational and electromagnetic cases in [22]. According to [22], an oscillatory term should appear in \(Im \omega_n\), in addition to \((n + \frac{1}{2})\kappa_e\).

Summarizing, by applying the monodromy technique as performed by Motl and Neitzke [17], we have considered the determination of the asymptotic, highly damped quasi-normal mode spectrum of small \((r_e << r_c)\) Schwarzschild-de Sitter black holes. However, we note that before we have restricted the analysis for small black holes, we verified that the dominant contribution for potential near the black hole singularity is exactly of the same type as found for the asymptotically flat case, i.e., the solution is a Bessel function around the origin. This result is important in applying the monodromy approach to find the asymptotic QNMs for general SdS black holes. Compared to the numerical results of Konoplya and Zhidenko [22], it seems our results do not give the correct asymptotic QNM frequencies for small SdS black holes. Consequently, in the context of the monodromy technique, it is important to search for the general analytic solution and compare it with the available numerical data in [22]. In fact, this has already been done very recently by Cardoso et al. and it will appear soon [23].

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