Scattering theory for the defocusing fourth-order Schrödinger equation

Changxing Miao\(^1,2\) and Jiqiang Zheng\(^3\)

\(^1\) Institute of Applied Physics and Computational Mathematics, PO Box 8009, Beijing 100088, People’s Republic of China
\(^2\) Beijing Center for Mathematics and Information Interdisciplinary Sciences, Beijing 100048, People’s Republic of China
\(^3\) Université Nice Sophia-Antipolis, 06108 Nice Cedex 02, France

E-mail: miao_changxing@iapcm.ac.cn, zhengjiqiang@gmail.com and zheng@unice.fr

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Abstract

In this paper, we study the global well-posedness and scattering theory for the defocusing fourth-order nonlinear Schrödinger equation (FNLS) \(iu_t + \Delta^2 u + |u|^p u = 0\) in dimensions \(d \geq 8\). We prove that if the solution \(u\) is apriorily bounded in the critical Sobolev space, that is, \(u \in L_T^\infty(I; H_x^{s_c}(\mathbb{R}^d))\) with all \(s_c := \frac{d - 4}{2} - \frac{4}{p} \geq 1\) if \(p\) is an even integer or \(s_c \in [1, 2 + p)\) otherwise, then \(u\) is global and scatters. We will give a uniform way to treat the energy-subcritical, energy-critical and energy-supercritical FNLS by making use of the strategy derived from concentration compactness ideas, and we are able to overcome the logarithmic blowup in the double Duhamel trick in dimension eight by exploiting the refined dispersive estimate which is in sharp contrast to the Schrödinger equation.

Keywords: fourth-order Schrödinger equation, scattering theory, Strichartz estimate, critical regularity, concentration compactness

Mathematics Subject Classification numbers: 35P25, 35Q55, 47J35
1. Introduction

This paper is mainly concerned with the Cauchy problem of the defocusing fourth-order nonlinear Schrödinger equation (FNLS)

\[
\begin{cases}
    iu_t + \Delta^2 u + f(u) = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \; d \geq 8, \\
    u(0,x) = u_0(x) \in H^{s_c}(\mathbb{R}^d),
\end{cases}
\]  

(1.1)

where \( f(u) = |u|^p u \), \( u \) is a complex-valued function defined in \( \mathbb{R}^{d+1} \), \( \Delta \) is the Laplacian in \( \mathbb{R}^d \), and \( s_c := \frac{d}{2} - \frac{4}{p} \).

If the solution \( u \) of (1.1) has sufficient decay at infinity and smoothness, it conserves mass

\[
M(u) = \int_{\mathbb{R}^d} |u(t,x)|^2 \, dx = M(u_0)
\]

(1.2)

and energy

\[
E(u) = \frac{1}{2} \int_{\mathbb{R}^d} |\Delta u|^2 \, dx + \frac{1}{p+2} \int_{\mathbb{R}^d} |u(t,x)|^{p+2} \, dx = E(u_0).
\]

(1.3)

As similarly explained in [12], the above quantities are also conserved for the energy solutions \( u \in C^1(\mathbb{R}, H^s(\mathbb{R}^d)) \). We call \( H^s(\mathbb{R}^d) \) the energy space.

The equation (1.1) has the scaling invariance symmetry:

\[
u(t,x) \mapsto \lambda^{\frac{d}{p}} u(\lambda^2 t, \lambda x), \; \forall \; \lambda > 0
\]

(1.4)

in the sense that both the equation and the \( H^s \)-norm are invariant under the scaling transformation:

\[
\|u_\lambda\|_{H^s(\mathbb{R}^d)} = \|u\|_{H^s(\mathbb{R}^d)}.
\]

We call FNLS (1.1) mass-critical when \( p = \frac{8}{d} \), which corresponds to \( s_c = 0 \), energy-subcritical when \( d \geq 5 \) and \( p < \frac{8}{d-4} \), which corresponds to \( s_c < 2 \), energy-critical when \( p = \frac{8}{d-4} \), which corresponds to \( s_c = 2 \), energy-supercritical when \( p > \frac{8}{d-4} \), which corresponds to \( s_c > 2 \).

Fourth-order Schrödinger equations have been introduced by Karpman [13] and Karpman and Shagalov [14] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. Such fourth-order Schrödinger equations are written as

\[
\begin{aligned}
i \partial_t u + \Delta^2 u + \varepsilon \Delta u + f(|u|^2 u &= 0, \\
\end{aligned}
\]

(1.5)

where \( \varepsilon \in \{ \pm 1, 0 \} \). Such equations have been studied from the mathematical viewpoint in Fibich, Ilan and Papanicolaou [11] who describe various properties of the equation in the subcritical regime, with part of their analysis relying on very interesting numerical developments. A related reference is [1] by Ben-Artzi, Koch, and Saut, which gives sharp dispersive estimates for the biharmonic Schrödinger operator which lead to the Strichartz estimates for the fourth-order Schrödinger equation, see also [30, 32, 33]. For other special FNLS, please refer to [38, 45, 46]. For FNLS (1.1), the defocusing energy-critical case with nonlinearity given by \( f(u) = |u|^4 u \) was handled by Pausader [32, 33] in dimension \( d = 8 \), in which case the nonlinearity is cubic, and Miao, Xu and Zhao [28] in dimension \( d \geq 9 \). We also refer to Miao, Xu and Zhao [27] and Pausader [34] for the focusing case with radially symmetrical
initial data. For the defocusing mass-critical case with nonlinearity given by \( f(u) = |u|^8 u \), we refer to Pausader and Shao [35], and Xia and Pausader [44].

On the other hand, the global well-posedness and scattering theory for the nonlinear Schrödinger equations (NLS)

\[
i \partial_t u - \Delta u \pm |u|^p u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d
\]

have been intensively studied recently, most notably by Bourgain [2], Colliander, Keel, Staffilani, Takaoka and Tao [4], Kenig and Merle [16] and Killip and Visan [20], Ryckman and Visan [37], Visan [42, 43] and Dodson [9] for the energy-critical case and Tao, Visan and Zhang [40], Killip, Tao and Visan [18], Killip, Visan and Zhang [24] and Dodson [5–8] for the mass-critical case.

So far, there is no technology for treating large-data NLS without some a priori control of a critical norm other than the energy-critical NLS and mass-critical NLS. In [17], Kenig–Merle first showed that if the radial solution \( u \) to NLS obeys \( u \in L^6_t(\mathbb{R}^+; \dot{H}^1(\mathbb{R}^d)) \) with \( s_c = \frac{1}{2} \), then \( u \) is global and scatters, where they utilized their concentration compactness technique as in [16], together with the Lin–Strauss Morawetz inequality which scales like \( \dot{H}^1 \) and is scaling-critical in this case. Thereafter, Killip–Visan [19] proved such a result for NLS in some energy-supercritical regimes. In particular, they deal with the case of a cubic nonlinearity for \( d \geq 5 \), along with some other cases for which \( s_c > 1 \) and \( d \geq 5 \), where the restriction to high dimensions comes from the double Duhamel trick. Using the long-time Strichartz estimate, the authors and Murphy [26] are able to overcome the logarithmic blowup in four dimensions. Recently, Murphy [31] considers the energy-subcritical NLS by making use of the tool ‘long time Strichartz estimate’ developed by Dodson [5] for almost periodic solutions in the mass-critical setting.

In this paper, we will give a uniform way to treat the energy-subcritical, energy-critical and energy-supercritical FNLS in dimension \( d \geq 8 \), where we utilize the strategy derived from concentration compactness ideas. In the proof, we want the decay rate faster than \( r^{-2} \) to make sense of the double-Duhamel formula. In high dimensions \( d \geq 9 \), the classical decay rate is not enough to make sense of the double-Duhamel formula; we should use a more refined decay estimate which would allow the \( L^\infty \)-norm to decay like \( t^{-d/2} \) after taking \( d \)-derivatives. We refer to section 4.2 for more details. This is in sharp contrast to NLS, where there is no such refined decay estimate.

Now we introduce some background materials.

**Definition 1.1 (solution).** A function \( u : I \times \mathbb{R}^d \to \mathbb{C} \) on a nonempty time interval \( I \subset \mathbb{R} \) is a strong solution to (1.1) if \( u \in C_t(K; \dot{H}^s(\mathbb{R}^d)) \cap L^4_{t,x} (K \times \mathbb{R}^d) \) for any compact interval \( K \subset I \) and for any \( t, t_0 \in I \), if it obeys the Duhamel formula:

\[
u(t, x) = e^{i(t-t_0)\Delta}u(t_0) - i \int_{t_0}^t e^{i(t-s)\Delta}f(u(s))ds.
\]

We say that the interval \( I \) is the lifespan of \( u \). We call \( u \) a maximal-lifespan solution if the solution cannot be extended to any strictly larger interval. In particular, if \( I = \mathbb{R} \), then we say that \( u \) is a global solution.

The assumption that the solution lies in the space \( L^4_{t,x} (I \times \mathbb{R}^d) \) locally in time is natural since by the Strichartz estimate (see proposition 2.1 below), the linear flow always lies in this
space. Also, if a solution \( u \) to (1.1) is global, with \( \| u \|_{L^{\infty}_t(\mathbb{R}^d \times \mathbb{R}^d)} < +\infty \), then it scatters in both time directions in the sense that there exist unique \( v_\pm \in \mathcal{H}_\pm(\mathbb{R}^d) \) such that

\[
\| u(t) - e^{i\sqrt{\varepsilon} \Delta} v_\pm \|_{L_t^\infty(\mathbb{R}^d)} \to 0, \quad as \quad t \to \pm \infty. \tag{1.8}
\]

In view of this, we define

\[
S_f(u) = \| u \|_{L^{\infty}_t(L^{\infty}_x(\mathbb{R}^d))}. \tag{1.9}
\]

as the scattering size of \( u \).

Closely associated with the notion of scattering is the notion of blow-up:

**Definition 1.2 (Blow-up).** Let \( u : I \times \mathbb{R}^d \to \mathbb{C} \) be a maximal-lifespan solution to (1.1). If there exists a time \( t_0 \in I \) such that \( S_{[t_0,\sup I]}(u) = +\infty \), then we say that the solution \( u \) blows up forward in time. Similarly, if there exists a time \( t_0 \in I \) such that \( S_{[\inf I, t_0]}(u) = +\infty \), then we say that \( u(t, x) \) blows up backward in time.

Now we state our main result.

**Theorem 1.1.** Assume that \( d \geq 8 \), and \( s_c \geq 1 \) if \( p \) is an even integer or \( s_c \in [1, 2 + p) \) otherwise. Assume in addition, \( s_c > 2 \) when \( d = 8 \). Let \( u : I \times \mathbb{R}^d \to \mathbb{C} \) be a maximal-lifespan solution to (1.1) such that

\[
\| u \|_{L^{\infty}_t(L^s_x(\mathbb{R}^d))} < +\infty. \tag{1.10}
\]

Then \( I = \mathbb{R} \), and the solution \( u \) scatters in the sense (1.8).

**Remark 1.1.**

(i) We remark that the bound required by theorem 1.3 is the source of our constraint to dimensions \( d \geq 8 \). One may refer to remark 1.3 and (4.45) for more details.

(ii) Our restriction \( s_c \geq 1 \) serves to simplify the analysis for the local theory, which still becomes a bit complicated. However, modifying the argument in the local theory, one may extend theorem 1.1 to \( s_c \geq \frac{1}{2} \) which enables us to adopt the interaction Morawetz inequality (see lemma 3.1 below).

(iii) We also need the nonlinearity to obey a certain smoothness condition; more precisely, we ask that \( s_c < 2 + p \) when \( p \) is not an even integer, which is only a restriction in large dimension \( d \geq 13 \). The role of this restriction is to allow us to take \( (s_c - 1) \)-many derivatives of the nonlinearity \( f(u) \). This is in sharp contrast to NLS, where the restriction for the regularity \( s_c < 1 + p \) when \( p \) is not an even integer. The main reason is the Strichartz estimate since there is the smoothing effect for all higher-order nonlinear Schrödinger equations, see proposition 2 in [30]. This enables us to consider \( s_c < 2 + p \) for \( p \) being not an even integer in FNLS. One may extend the limitation in regularity to be \( [s - 1] < p + 1 \) by using a better definition for \( S^0 \) in the Strichartz estimates, see remark 2.1 for more details.

(iv) For the focusing case (i.e. \( f(u) = -|u|^p u \)), by the same argument as the defocusing case, one can exclude the finite-time blowup solution and cascade in the sense of theorem 1.2. However, we can not preclude the soliton since the Morawetz estimate fails in the focusing case. One needs to explore some new methods to treat the focusing case. We refer to [10] which considers the radial bounded solutions of focusing supercritical wave equations in \( \mathbb{R}^3 \).
1.1. The outline of the proof of theorem 1.1

For each $E > 0$, let us define $\Lambda(E)$ to be the quantity
\[
\Lambda(E) := \sup \left\{ S(u) : u : I \times \mathbb{R}^d \to \mathbb{C} \text{ such that } \sup_{t \in I} \|u\|_{H^s(\mathbb{R}^d)} \leq E \right\},
\]
where $u$ ranges over all solutions to (1.1) on the spacetime slab $I \times \mathbb{R}^d$ with $\|u\|_{H^s(\mathbb{R}^d)} \leq E$. Thus, $\Lambda : [0, +\infty) \to [0, +\infty)$ is a non-decreasing function. Furthermore, from the small data theory, see proposition 2.2, one has
\[
\Lambda(E) \lesssim E \quad \text{for} \quad E \leq \eta_0,
\]
where $\eta_0 = \eta(d)$ is the threshold from the small data theory.

From the stability theory (see corollary 2.2 below), we know that $\Lambda$ is continuous. Thus, there is a unique critical $E_c \in (0, +\infty]$ such that $\Lambda(E) < +\infty$ for $E < E_c$ and $\Lambda(E) = +\infty$ for $E \geq E_c$. In particular, if $u : I \times \mathbb{R}^d \to \mathbb{C}$ is a maximal-lifespan solution to (1.1) satisfying $\sup_{t \in I} \|u\|_{H^s(\mathbb{R}^d)} < E_c$, then $u$ is global and moreover,
\[
S_\mathbb{R}(u) \leq L(\|u\|_{L^\infty(\mathbb{R}; H^s(\mathbb{R}^d))}).
\]
Therefore, the proof of theorem 1.1 is equivalent to show $E_c = +\infty$. We argue by contradiction. The failure of theorem 1.1 would imply the existence of a very special class of solutions; that is the almost periodicity modulo symmetries:

**Definition 1.3.** Let $s_c \geq 1$. A solution $u$ to (1.1) with maximal-lifespan $I$ is called almost periodic modulo symmetries if $u$ is bounded in $H^s(\mathbb{R}^d)$ and there exist functions $N(t) : I \to \mathbb{R}^+$, $\phi(t) : I \to \mathbb{R}^d$ and $c(\eta) : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $t \in I$ and $\eta > 0$,
\[
\int_{|x - \phi(t)| \geq \frac{c(\eta)}{N(t)}} |\nabla|^s u(t, x)|^2 \, dx \leq \eta \tag{1.11}
\]
and
\[
\int_{|\xi| \leq c(\eta)/N(t)} |\xi|^{2s_c - 1} |\hat{u}(t, \xi)|^2 \, d\xi \leq \eta. \tag{1.12}
\]

We refer to the function $N(t)$ as the frequency scale function for the solution $u$, to $\phi(t)$ as the spatial center function, and to $c(\eta)$ as the compactness modules function.

**Remark 1.2.** By the Ascoli–Arzela theorem, $u$ is almost periodic modulo symmetries if and only if the set
\[
\left\{ N(t)^s \frac{d}{dx} \left( t, x(t) + \frac{x}{N(t)} \right), \ t \in I \right\}
\]
falls in a compact set in $H^s(\mathbb{R}^d)$. The following are consequences of this statement. If $u$ is almost periodic modulo symmetries, then there exists $c(\eta) > 0$ such that
\[
\int_{|x - \phi(t)| \leq \frac{c(\eta)}{N(t)}} |\nabla|^s u(t, x)|^2 \, dx \leq \eta \tag{1.13}
\]
and
\[ \int_{\left| |\xi| \leq \tau(t) \right| N(t)} |\xi|^{2\gamma} : |\hat{u}(t, \xi)|^2 d\xi \leq \eta. \tag{1.14} \]

By the same argument as in [27, 34], we can show that if theorem 1.1 fails, then we will inevitably encounter at least one of the following three enemies.

**Theorem 1.2 (Three enemies, [27, 34]).** Suppose \( d \geq 7 \) is such that theorem 1.1 fails, that is, \( E_\xi < +\infty \). Then there exists a maximal-lifespan solution \( u : I \times \mathbb{R}^d \to \mathbb{C} \), which is almost periodic modulo symmetries, with \( S_0(u) = +\infty \). Furthermore, we can also ensure that the lifespan \( I \) and the frequency scale function \( N(t) : I \to \mathbb{R}^+ \) satisfy one of the following three scenarios:

1. **(Finite time blowup)** Either \( \inf(I) < +\infty \) or \( \sup(I) < +\infty \).
2. **(Soliton-like solution)** \( I = \mathbb{R} \) and \( N(t) = 1 \) for all \( t \in \mathbb{R} \).
3. **(Low-to-high frequency cascade)** \( I = \mathbb{R} \),
   \[ \inf N(t) \geq 1, \quad \text{and} \quad \lim_{t \to \infty} N(t) = +\infty. \]

One can adapt the proof of lemma 5.18 in [23] to prove that the almost periodic solutions satisfy the following local constancy property:

**Lemma 1.1 (Local constancy of \( N(t), x(t) \)).** Let \( u \) be an almost periodic solution to (1.1) on \( I \). Then there exists \( \delta \) such that for all \( t_0 \in I \),

\[ |t - t_0| \leq \delta N(t_0)^{-4}, t_0 + \delta N(t_0)^{-4} | \subset I, \tag{1.15} \]

and

\[ N(t) \sim N(t_0), \quad |x(t) - x(t_0)| \lesssim_{\omega} N(t_0), \]

whenever \( |t - t_0| \leq \delta N(t_0)^{-4} \). Moreover, for any \( t_1, t_2 \in I \), we have

\[ |x(t_1) - x(t_2)| \lesssim \int_{t_1}^{t_2} N(t)^2 dt. \tag{1.16} \]

We also need the following Duhamel formula, which is important for showing the additional decay and negative regularity in section 4. This is a robust consequence of almost periodicity modulo symmetries; see, for example, [4, 28, 34].

**Lemma 1.2 (No-waste Duhamel formula).** Let \( u : I \times \mathbb{R}^d \to \mathbb{C} \) be a maximal-lifespan solution which is almost periodic modulo symmetries. Then, for all \( t \in I \), there holds that

\[ u(t) = \lim_{T \to \sup(t)} \frac{1}{T} \int_{T}^{T} e^{i(t-s)\Delta} f(u(s)) ds = -\lim_{T \to \inf(t)} \frac{1}{T} \int_{T}^{T} e^{i(t-s)\Delta} f(u(s)) ds \tag{1.17} \]
as weak limits in \( \dot{H}^s_\omega(\mathbb{R}^d) \).

With this lemma in hand, we can deduce that the energy or mass of the finite time blow-up solution is zero and so get a contradiction. We refer to section 3 for more details.

In view of the no-waste Duhamel formula and noting that the minimal \( L^\infty_t \dot{H}^s_\omega \)-norm blowup solution is localized in both physical and frequency space, we will show that it admits lower regularity.
Theorem 1.3 (Lower regularity in the global case). Let \( u \) be a global solution to (1.1) which is almost periodic modulo symmetries in the sense of theorem 1.2. Assume that \( \inf_{t \in \mathbb{R}} N(t) \geq 1 \), then there exists a constant \( \alpha > 0 \) such that for any \( 0 < \varepsilon < \alpha \)

\[
u \in L_t^\infty(\mathbb{R}; \dot{H}^{-\varepsilon}(\mathbb{R}^d)).\tag{1.18}
\]

Remark 1.3. For \( d \leq 7 \), one cannot obtain that the above global solution \( \nu \) is \( \dot{H}^{\beta}(\mathbb{R}^d) \) with \( \beta < \frac{1}{2} \) but \( \beta > \frac{1}{2} \) by the same argument as \( d = 8 \) in section 4.2. Thus, one cannot exclude the soliton by the Morawetz estimate. This is the source of our constraint to dimensions \( d \geq 8 \) in theorem 1.1.

Combining theorem 1.3 with the Morawetz estimate and interpolation, we will get a contradiction for the global almost periodic solutions in the sense of theorem 1.2. Thus, we conclude theorem 1.1. We refer to section 3 for more details.

The paper is organized as follows. In section 2, we deal with the local theory for the equation (1.1). In section 3, we exclude three scenarios in the sense of theorem 1.2 under the assumption that theorem 1.3 holds. In section 4, we show the global solutions which are almost periodic modulus symmetries admit the lower regularity, that is, theorem 1.3. Hence we conclude the proof of theorem 1.1. Finally, we show the stability in the appendix.

1.2. Notations

Finally, we conclude the introduction by giving some notations which will be used throughout this paper. To simplify the expression of our inequalities, we introduce some symbols \( \lesssim, \sim, \ll \). If \( X, Y \) are nonnegative quantities, we use \( X \lesssim Y \) or \( X = O(Y) \) to denote the estimate \( X \leq cY \) for some \( C \) which may depend on the critical energy \( E_c \) but not on any parameter such as \( \eta \) and \( \rho \), and \( X \sim Y \) to denote the estimate \( X \lesssim Y \lesssim X \). We use \( X \ll Y \) to mean \( X \leq cY \) for some small constant \( c \) which is again allowed to depend on \( E_c \). We use \( \gg C \) to denote various large finite constants, and \( 0 < c \ll 1 \) to denote various small constants. Any summations over capitalized variables such as \( M_j \) are presumed to be dyadic, i.e. these variables range over numbers of the form \( 2^k \) for \( k \in \mathbb{Z} \). For any \( r, 1 \leq r \leq \infty \), we denote by \( \| \cdot \|_r \) the norm in \( L^r = L^r(\mathbb{R}^d) \) and by \( r' \) the conjugate exponent defined by \( \frac{1}{r} + \frac{1}{r'} = 1 \). We denote \( a \pm \epsilon \) to be any quantity of the form \( \pm \epsilon a \) for any \( \epsilon > 0 \).

The Fourier transform on \( \mathbb{R}^d \) is defined by

\[
\hat{f}(\xi) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx,
\]
giving rise to the fractional differentiation operators \( |\nabla|^\alpha \) and \( \langle \nabla \rangle^\alpha \), defined by

\[
|\nabla|^\alpha f(\xi) := |\xi|^\alpha \hat{f}(\xi), \quad \langle \nabla \rangle^\alpha f(\xi) := \langle \xi \rangle^\alpha \hat{f}(\xi),
\]
where \( \langle \xi \rangle := 1 + |\xi| \). This helps us to define the homogeneous and inhomogeneous Sobolev norms

\[
\| f \|_{\dot{H}^\alpha(\mathbb{R}^d)} := \| |\xi|^\alpha \hat{f} \|_{L^2(\mathbb{R}^d)}, \quad \| f \|_{H^\alpha(\mathbb{R}^d)} := \| \langle \xi \rangle^\alpha \hat{f} \|_{L^2(\mathbb{R}^d)}.
\]

We will also need the Littlewood–Paley projection operators. Specifically, let \( \varphi(\xi) \) be a smooth bump function adapted to the ball \( |\xi| \leq 2 \) which equals 1 on the ball \( |\xi| \leq 1 \). For each dyadic number \( N \in 2^\mathbb{Z} \), we define the Littlewood–Paley operators
\[ P_{\xi N} f(\xi) := \varphi\left(\frac{\xi}{N}\right) \widehat{f}(\xi), \]
\[ \widehat{P_{\xi N}} f(\xi) := \left(1 - \varphi\left(\frac{\xi}{N}\right)\right) \widehat{f}(\xi), \]
\[ \widehat{P_N} f(\xi) := \left(\varphi\left(\frac{\xi}{N}\right) - \varphi\left(\frac{2\xi}{N}\right)\right) \widehat{f}(\xi). \]

Similarly we can define \( P_{\xi N}, P_{\xi N}, \) and \( P_{\xi < \xi N} = P_{\xi N} - P_{\xi M}, \) whenever \( M \) and \( N \) are dyadic numbers. We will frequently write \( f_{\xi N} \) for \( P_{\xi N} f \) and similarly for the other operators.

The Littlewood–Paley operators commute with derivative operators, the free propagator, and the conjugation operation. They are self-adjoint and bounded on every \( L^p_x \) and \( \dot{H}^s_x \) space for \( 1 \leq p \leq \infty \) and \( s \geq 0, \) moreover, they also obey the following Bernstein estimates:

**Lemma 1.3 (Bernstein estimates).**
\[
\begin{align*}
\| P_{\xi N} f \|_{L^p_x} &\lesssim N^{-s} \| \nabla^s P_{\xi N} f \|_{L^q_x}, \\
\| \nabla^s P_{\xi N} f \|_{L^p_x} &\lesssim N^s \| P_{\xi N} f \|_{L^q_x}, \\
\| \nabla^s P_N f \|_{L^p_x} &\sim N^{s\frac{s}{2}} \| P_N f \|_{L^q_x}, \\
\| P_{\xi N} f \|_{L^2_x} &\lesssim N^{d-s} \| P_{\xi N} f \|_{L^2_x}, \\
\| P_N f \|_{L^2_x} &\lesssim N^{d-s} \| P_N f \|_{L^2_x},
\end{align*}
\]

where \( s \geq 0 \) and \( 1 \leq p \leq q \leq \infty. \)

### 2. Preliminaries

#### 2.1. Strichartz estimate and nonlinear estimates

In this section, we consider the Cauchy problem for the fourth-order Schrödinger equation
\[
\begin{cases}
iu_t + \Delta^2 u + f(u) = 0, \\
u(0) = u_0.
\end{cases}
\]

(2.1)

The integral equation for the Cauchy problem (2.1) can be written as
\[
u(t,x) = e^{it\Delta}u(t_0) - i \int_{t_0}^t e^{i(t-s)(\Delta)}f(u(s))ds.
\]

(2.2)

The biharmonic Schrödinger semigroup is defined for any tempered distribution \( g \) by
\[
\mathcal{F}^{-1}e^{it\Delta^2}g.
\]

Now we recall the dispersive estimate for the biharmonic Schrödinger operator.

**Lemma 2.1 (Dispersive estimate, [1]).** Let \( 2 \leq q \leq +\infty. \) Then, we have the following dispersive estimate
\[
\| e^{it\Delta^2}f \|_{L^2_x} \lesssim C|t|^{\frac{d}{4}\left(\frac{1}{q} - \frac{1}{2}\right)} \| f \|_{L^q_x(G^d)}
\]

(2.3)

for all \( t \neq 0 \) and \( 2 \leq q \leq +\infty, \) \( \frac{1}{q} + \frac{1}{q} = 1. \)
We will also make use of the following refined decay estimate which would allow the \( L^\infty \)-norm to decay like \( t^{-\frac{d}{2}} \), after taking \( d - 1 \) derivatives. This plays an important role in treating the case of lower dimensions.

**Lemma 2.2 (Refined decay estimate, [36]).** We have

\[
\| \nabla^d e^{it\Delta_j} f \|_{L^q_t L^r_x(\mathbb{R}^d)} \lesssim C |t|^{-\frac{d}{2}} \| f \|_{L^q_t L^r_x(\mathbb{R}^d)}
\]  

for all \( t \neq 0 \).

As a direct consequence, we have the following refined dispersive estimate.

**Lemma 2.3 (Refined dispersive estimate).** Let \( 2 \leq q \leq +\infty \). Then, we have the following refined dispersive estimate

\[
\| \nabla^d P_N e^{it\Delta} f \|_{L^q_t L^r_x(\mathbb{R}^d)} \lesssim C |t|^{-\frac{d}{2}} \left( \frac{1}{q} \right)^{\frac{d}{2}} \| P_N f \|_{L^q_t L^r_x(\mathbb{R}^d)}
\]

for all \( t \neq 0 \) and \( 2 \leq q \leq +\infty \), \( \frac{1}{q} + \frac{1}{r} = 1 \). This together with Bernstein’s inequality implies that

\[
\| P_N e^{it\Delta} f \|_{L^q_t L^r_x(\mathbb{R}^d)} \lesssim C |t|^{-\frac{d}{2}} \left( \frac{1}{q} \right)^{\frac{d}{2}} \| P_N f \|_{L^q_t L^r_x(\mathbb{R}^d)}.
\]

**Proof.** From (2.4), we know that

\[
\| \nabla^d P_N e^{it\Delta_j} f \|_{L^q_t L^r_x(\mathbb{R}^d)} \lesssim C |t|^{-\frac{d}{2}} \| P_N f \|_{L^q_t L^r_x(\mathbb{R}^d)},
\]

Interpolating this with

\[
\| \nabla^d P_N e^{it\Delta} f \|_{L^q_t L^r_x(\mathbb{R}^d)} \lesssim C \| P_N f \|_{L^q_t L^r_x(\mathbb{R}^d)}^{\frac{d}{2}}
\]

implies (2.5).

The Strichartz estimates involve the following definitions:

**Definition 2.1.** A pair of Lebesgue space exponents \((q, r)\) are called Schrödinger admissible for \( \mathbb{R}^{1+d} \), or denoted by \((q, r) \in \Lambda_0 \) when \( q, r \geq 2 \), \((q, r, d) = (2, \infty, 2)\), and

\[
\frac{2}{q} = d \left( \frac{1}{2} - \frac{1}{r} \right).
\]

**Definition 2.2.** In addition, a pair of Lebesgue space exponents \((\gamma, \rho)\) are called biharmonic admissible for \( \mathbb{R}^{1+d} \) or denoted by \((\gamma, \rho) \in \Lambda_1 \) when \( \gamma, \rho \geq 2 \), \((\gamma, \rho, d) = (2, \infty, 4)\), and

\[
\frac{4}{\gamma} = d \left( \frac{1}{2} - \frac{1}{\rho} \right).
\]

For a fixed spacetime slab \( I \times \mathbb{R}^d \), we define the Strichartz norm

\[
\| u \|_{\text{Str}(I)} := \begin{cases} 
\sup_{(q, r) \in \Lambda_0} \| u \|_{L^q_t L^r_x(I \times \mathbb{R}^d)} & \text{if } d \neq 4 \\
\sup_{(q, r) \in \Lambda_1, q \geq 2} \| u \|_{L^q_t L^r_x(I \times \mathbb{R}^d)} & \text{if } d = 4
\end{cases}
\]
where $0 < \epsilon \ll 1$. We denote $S^0(I)$ to be the closure of all test functions under this norm and write $N^0(I)$ for the dual of $S^0(I)$.

According to the above dispersive estimate, the abstract duality and interpolation argument (see [15]), we have the following Strichartz estimates.

**Proposition 2.1 (Strichartz estimates for fourth-order Schrödinger [30, 32]).** Let $s \geq 0$, suppose that $u(t, x)$ is a solution on $[0, T]$ to the initial value problem

$$
\begin{cases}
(i\partial_t + \Delta^2)u(t, x) = h, & (t, x) \in [0, T] \times \mathbb{R}^d \\
u(0) = u_0(x),
\end{cases}
$$

for some data $u_0$ and $T > 0$. Then we have the Strichartz estimate, for $(q, r), (a, b) \in \Lambda_0$

$$
|||\nabla^s u|||_{L^q_t L^r_x(qd\mathbb{R}^d)} \lesssim \left( |||\nabla^{s-1} h|||_{L^q_t L^r_x(qd\mathbb{R}^d)} + |||\nabla^{s-1} h|||_{L^q_t L^r_x(qd\mathbb{R}^d)} \right),
$$

and for $(\gamma, \rho), (c, d) \in \Lambda_1$

$$
|||\nabla^s u|||_{L^q_t L^r_x(qd\mathbb{R}^d)} \lesssim \left( |||\nabla^s u|||_{L^q_t L^r_x(qd\mathbb{R}^d)} + |||\nabla^s u|||_{L^q_t L^r_x(qd\mathbb{R}^d)} \right).
$$

In particular, we have

$$
|||\nabla^s u|||_{L^q_t L^r_x(qd\mathbb{R}^d)} \lesssim \left( |||\nabla^s u|||_{L^q_t L^r_x(qd\mathbb{R}^d)} + |||\nabla^s u|||_{L^q_t L^r_x(qd\mathbb{R}^d)} \right),
$$

where $h = h_1 + h_2$ and $I = [0, T]$.

**Remark 2.1.**

(i) The key feature of such a proposition is that the spacetime-norm of the $s$-derivative of $u$ can be estimated by $(s-1)$-derivative of the forcing term, which is the consequence of the smoothing effect for all higher-order nonlinear Schrödinger equations, see proposition 2 in [30]. This enables us to consider $s < 2 + p$ for $p$ being not an even integer. This is in sharp contrast with NLS, where the restriction for the regularity $s < 1 + p$ when $p$ is not an even integer.

(ii) If we replace $|||\nabla^s u|||_{L^q_t L^r_x(qd\mathbb{R}^d)}$ by

$$
|||\nabla^s u|||_{L^q_t L^r_x(qd\mathbb{R}^d)} := \sup_{(q, r) \in \Lambda_1} \left( \sum_{j \in \mathbb{N}} 2^{2j} \|P_N u\|^2_{L^q_t L^r_x(qd\mathbb{R}^d)} \right)^{1/2},
$$

then one can extend the limitation in regularity to be $|s - 1| < p + 1$.

Now we give a few nonlinear estimates which will be applied to show the local well-posedness which is the first step in obtaining the global time-space estimate that leads to the scattering.

**Lemma 2.4.**

(i) (Product rule) Let $s \geq 0$, and $1 < p_i, q_i < \infty$ such that $\frac{1}{s} = \frac{1}{p_1} + \frac{1}{p_2}$ (i = 1, 2). Then, we have

$$
|||\nabla^s (fg)|||_{L^q_t L^r_x(qd\mathbb{R}^d)} \lesssim \|f\|_{L^p_t L^q_x(qd\mathbb{R}^d)} \|g\|_{L^p_t L^q_x(qd\mathbb{R}^d)} + |||\nabla^{s-1} f|||_{L^q_t L^r_x(qd\mathbb{R}^d)} |||\nabla^{s-1} g|||_{L^q_t L^r_x(qd\mathbb{R}^d)}. 
$$

(ii) ($C^1$ continuous) Assume that $G \in C^1(\mathbb{C})$, $s \in (0, 1]$, $1 < p_2, p_2 < +\infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then, we have
\[
||\nabla f(u)||_p \lesssim ||G(u)||_{p_1} ||\nabla u||_{p_2}.
\] (2.15)

(iii) (Hölder continuous) Let \( G \in C^\alpha(\mathbb{C}) \) with \( 0 < \alpha < 1 \). Then, for every \( 0 < s < \alpha, 1 < p < +\infty \), \( \frac{1}{\alpha} < \sigma < 1 \), we have
\[
||\nabla f(u)||_p \lesssim ||u^{\alpha - \frac{s}{\sigma}}||_{\frac{p}{p_1}} ||\nabla^s u||_{\frac{p_1}{p_2}}^{\frac{s}{\sigma}};
\] (2.16)
provided \( \frac{1}{p} = \frac{1}{p_1} + \frac{s}{p_2} \), and \( \left( 1 - \frac{s}{\alpha \sigma} \right) p_1 > 1 \).

**Proof.** We refer to [3, 41] for the proof. \( \square \)

As a direct consequence, we obtain the following nonlinear estimate.

**Corollary 2.1.** Let \( f(u) = |u|^p u \), and let \( s \geq 0 \) if \( p \) is an even integer or \( 0 \leq s < 1 + p \) otherwise. Then, we have
\[
||\nabla f(u)||_{L^p} \lesssim ||\nabla^s u||_{L^p} ||u||_{L^p}^p,
\] (2.17)
where \( \frac{1}{q} = \frac{1}{q_1} + \frac{p}{q_2} \).

We will also make use of the following refinement of the fractional chain rule, which appears in [21]. This will be used in the proof of the perturbation for \( s_i \in [1, 2] \).

**Lemma 2.5 (Derivatives of differences, [21]).** For \( 0 < s < 1 \) and \( f(u) = |u|^p u \). Then for \( 1 < r, r_1, r_2 < +\infty \) such that \( \frac{1}{r} = \frac{1}{r_1} + \frac{p}{r_2} \), we have
\[
||\nabla^s \left[ f(u + v) - f(u) \right] \|_{L^r} \lesssim ||\nabla^s u||_{L^{r_1}} ||v||_{L^{r_2}} + ||\nabla^s v||_{L^{r_1}} ||u + v||_{L^p}^p.
\] (2.18)

Next, we give a nonlinear estimate in [19]. It is used in the proof of lemma 4.

**Lemma 2.6 ([19]).** Let \( G \in C^\alpha(\mathbb{C}) \) with \( 0 < \alpha \leq 1 \), and \( 0 < s \leq \sigma \alpha < \alpha \). For \( 1 < q, q_1, q_2, r_1, r_2, r_3 < +\infty \), such that \( \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \), we have
\[
||\nabla^s \left[ \omega \cdot (G(u + v) - G(u)) \right] \|_{L^q} \lesssim ||\nabla^s \omega||_{L^{q_1}} ||v||_{L^{q_2}} + ||\omega||_{L^{r_1}} ||v||_{L^{r_2}} \left( ||\nabla^s v||_{L^{r_3}} + ||\nabla^s u||_{L^{r_3}} \right)^{\frac{\sigma}{\sigma - \frac{s}{\alpha}}}.
\] (2.19)

where \( 1 - \alpha) r_3, \left( \alpha - \frac{s}{\sigma} \right) r_2 > 1 \).

We remark that one can extend lemma 2.6 to \( G(u) \simeq O(|u|^\alpha) \) with \( \alpha > 1 \), which will be used in the proof of (A.27) for \( p > 1 \).

We will also need the following lemma which is similar to lemma 2.15. It is useful to the proof of proposition 4.1 for \( p < 1 \).

**Lemma 2.7 (Nonlinear Bernstein inequality [23]).** Assume that \( G \in C^{\alpha}(\mathbb{C}) \) with \( 0 < \alpha \leq 1 \). Then, we have
\[
||P_{k} G(u) ||_{L^q} \lesssim N^{-\alpha} ||\nabla u||_{L^q}^{\alpha},
\] (2.20)
for all \( 1 \leq q < +\infty \).
2.2. Local well-posedness in inhomogeneous space

Now we can state the following standard local well-posedness result, where we assume that the initial data is in the inhomogeneous critical Sobolev space. This assumption simplifies the proof since one can use the \( L^q_tL^r_x \)-norm with \((q, r) \in \Lambda_t\) as the metric (that is in mass-critical spaces) when we prove the map is a contraction. This assumption can be removed by using the perturbation results proved in corollary 2.2 below, see proposition 2.2.

Theorem 2.1 (Local well-posedness). Assume \( u_0 \in H^s_x(\mathbb{R}^d) \), and let \( s_c \geq 1 \) if \( p \) is an even integer or \( s_c \in [1, 2 + p) \) otherwise. Then there exists \( \eta_0 = \eta_0(d) > 0 \) such that if \( I \) is a compact interval containing zero such that

\[
\|u\|_{Z(I)} \leq 2\eta
\]

where \( 0 < \eta \leq \eta_0 \), then there exists a unique solution \( u \) to (1.1) on \( I \times \mathbb{R}^d \). Furthermore, the solution \( u \) obeys

\[
\| u \|_{Z(I)} \leq 2\eta
\]

\[
\| \nabla^k u \|_{S(I)} \leq 2C \| \nabla^k u_0 \|_{L^2_t} + C\eta^{1+p}
\]

\[
\| u \|_{S(I)} \leq 2C\| u_0 \|_{L^2_t}
\]

where \( C \) is the Strichartz constant as in proposition 2.1.

Proof. We apply the Banach fixed point argument to prove this lemma. First we define the map

\[
\Phi(u(t)) = e^{it\Delta}u_0 - i \int_0^t e^{i(t-s)\Delta}f(u(s))ds
\]

(2.25)

on the complete metric space \( B \)

\[
B := \left\{ u \in C(I; H^s_x) : \| u \|_{L^q_tL^r_x(I \times \mathbb{R}^d)} \leq 2C\| u_0 \|_{H^s_x} + C\eta^{1+p}; \right. \\
\left. \| u \|_{L^q_tL^r_x(I \times \mathbb{R}^d)} \leq 2C\| u_0 \|_{L^2_t}; \| u \|_{Z(I)} \leq 2\eta \right\}
\]

with the metric \( d(u, v) = \| u - v \|_{L^q_tL^r_x(I \times \mathbb{R}^d)} \).

It suffices to prove that the operator defined by the right-hand side of (2.25) is a contraction map on \( B \) for \( I \). If \( u \in B \), then by the Strichartz estimate, corollary 2.1 and (2.21), we have

\[
\| \Phi(u) \|_{Z(I)} = \| \nabla^k \Phi(u) \|_{L^q_tL^r_x(I \times \mathbb{R}^d)} \leq \eta + C\| u \|_{Z(I)}
\]

Plugging the assumption \( \| u \|_{Z(I)} \leq 2\eta \), we see that for \( u \in B \),
\[ \| \Phi(u) \|_{Z(I)} \leq \eta + 8C \eta^{p+1} \leq 2\eta \]

provided we take \( \eta \) sufficiently small such that \( 8C \eta^p \leq 1 \). Similarly, if \( u \in B \), then

\[
\| \Phi(u) \|_{L^\infty_t H^s_x(I \times \mathbb{R}^d)} \leq C\| u_0 \|_{H^p_x(\mathbb{R}^d)} + C \| \nabla^s f(u) \|_{L^2_t L^2_x(I \times \mathbb{R}^d)} + C \| f(u) \|_{L^2_t L^\infty_x(I \times \mathbb{R}^d)}
\]

\[
\leq C\| u_0 \|_{H^p_x(\mathbb{R}^d)} + C\| u \|_{Z(I)} \| u \|_{L^2_t L^\infty_x(I \times \mathbb{R}^d)} + C\| u \|_{L^{2p+1}_t L^{2(p+1)}_x(I \times \mathbb{R}^d)} \| u \|_{Z(I)}^p
\]

\[
\leq C\| u_0 \|_{H^p_x(\mathbb{R}^d)} + (2\eta)^{p+1} + 2C\| u_0 \|_{L^{2}_t(2\eta)^p}
\]

\[ \leq 2C\| u_0 \|_{H^p_x(\mathbb{R}^d)} + C\eta^{p+1} \]

and

\[
\| \Phi(u) \|_{L^{2p+1}_t L^{2(p+1)}_x(I \times \mathbb{R}^d)} \leq C\| u_0 \|_{L^{2}_x(\mathbb{R}^d)} + C \| f(u) \|_{L^2_t L^{\infty}_x(I \times \mathbb{R}^d)}
\]

\[
\leq C\| u_0 \|_{L^{2}_x(\mathbb{R}^d)} + C\| u \|_{L^{2p+1}_t L^{2(p+1)}_x(I \times \mathbb{R}^d)} \| u \|_{Z(I)}^p
\]

\[
\leq C\| u_0 \|_{L^{2}_x(\mathbb{R}^d)} + C^2\| u_0 \|_{L^{2}_x(2\eta)^p}
\]

\[ \leq 2C\| u_0 \|_{L^{2}_x(\mathbb{R}^d)}.
\]

Hence \( \Phi(u) \in B \) for \( u \in B \). That is, the functional \( \Phi \) maps the set \( B \) back to itself.

On the other hand, by the same argument as before, we have for \( u, v \in B \),

\[ d(\Phi(u), \Phi(v)) \leq C \| f(u) - f(v) \|_{L^2_t L^{\infty}_x(I \times \mathbb{R}^d)}
\]

\[
\leq C\| u - v \|_{L^{2p+1}_t L^{2(p+1)}_x(I \times \mathbb{R}^d)} \| (u, v) \|_{Z(I)}^p
\]

\[ \leq C(4\eta)^p d(u, v)
\]

which allows us to derive

\[ d(\Phi(u), \Phi(v)) \leq \frac{1}{2} d(u, v),
\]

by taking \( \eta \) small such that

\[ C(4\eta)^p \leq \frac{1}{2}.
\]

A standard fixed point argument gives a unique solution \( u \) of (1.1) on \( I \times \mathbb{R}^d \) which satisfies the bound (2.22). The bounds (2.23) and (2.24) follow from another application of the Strichartz estimate.

\[ \square \]

2.3. Perturbation

Closely related to the continuous dependence on the data, an essential tool for concentration compactness arguments is the perturbation theory. We will show this perturbation theory in the appendix.
Lemma 2.8 (Perturbation lemma). Let $s_c \geq 1$. Assume in addition that $s_c < 2 + p$ if $p$ is not an even integer. Let $I$ be a compact time interval and $u$, $\tilde{u}$ satisfy

\begin{align*}
\left( i\partial_t + \Delta \right) u &= -f(u) + eq(u) \\
\left( i\partial_t + \Delta \right) \tilde{u} &= -f(\tilde{u}) + eq(\tilde{u})
\end{align*}

for some function $eq(u)$, $eq(\tilde{u})$, and $f(u) = |u|^p u$. Assume that for some constants $M, E > 0$, we have

\begin{equation}
\|u\|_{L_x^\infty(I; \dot{H}_x^s(\mathbb{R}^d))} + \|\tilde{u}\|_{L_x^\infty(I; \dot{H}_x^s(\mathbb{R}^d))} \leq E.
\end{equation}

(2.26)

Let $t_0 \in I$, and let $u(t_0)$ be close to $\tilde{u}(t_0)$ in the sense that

\begin{equation}
\|u_0 - \tilde{u}_0\|_{\dot{H}_x^s(\mathbb{R}^d)} \leq \varepsilon,
\end{equation}

(2.27)

where $0 < \varepsilon < \varepsilon_1(M, E)$ is a small constant. Assume also that we have smallness conditions

\begin{equation}
\|\nabla|^{k-1}(eq(u), eq(\tilde{u}))\|_{L_x(I; L_x^{p+1}(\mathbb{R}^d))} \leq \varepsilon,
\end{equation}

(2.29)

where $\varepsilon$ is as above.

Then we conclude that

\begin{align*}
S_I(u - \tilde{u}) &\leq C(M, E)\varepsilon^{c_1} \\
\|\nabla|^{k}(u - \tilde{u})\|_{\dot{H}_x^s(I)} &\leq C(M, E)\varepsilon^{c_2} \\
\|\nabla|^{k}u\|_{\dot{H}_x^s(I)} &\leq C(M, E),
\end{align*}

(2.30)

where $c_1, c_2$ are positive constants that depend on $d$, $p$, $E$ and $M$.

### 2.4. Local well-posedness in homogeneous space and stability

As stated in the section 2.2, the assumption is that the initial data in the inhomogeneous critical Sobolev space can be removed by the perturbation results. Now we give a detailed proof.

**Proposition 2.2 (Local well-posedness in homogeneous space).** Assume that $s_c \geq 1$ if $p$ is an even integer or $I \leq s_c < 2 + p$ otherwise. Let $u_0 \in \dot{H}_x^{s_c}(\mathbb{R}^d)$. Then, if $I$ is a compact interval containing zero such that

\begin{equation}
\|u_{\text{tot}}\|_{Z(I)} := \|\nabla|^{k-1}u_{\text{tot}}\|_{L_x^{p+1}(\mathbb{R}^d)} \leq \eta,
\end{equation}

(2.31)

where $\eta$ is as in theorem 2.1, then there exists a unique solution $u$ to (1.1) on $I \times \mathbb{R}^d$. Furthermore, the solution $u$ satisfies the bounds

\begin{align*}
\|u\|_{Z(I)} &\leq 2\eta \\
\|\nabla|^{k}u\|_{\dot{H}_x^s(I)} &\leq \|\nabla|^{k}u_0\|_{\dot{H}_x^s(I)} + 2C\eta^{1+p},
\end{align*}

(2.32)

(2.33)

where $C$ is the Strichartz constant as in proposition 2.1.
In particular, if \( \|u_0\|_{H^s(\mathbb{R}^d)} \leq \frac{\eta}{2} \), then the solution \( u \) is global and scatters.

**Proof.** Since \( H^s(\mathbb{R}^d) \) is dense in \( \dot{H}^s(\mathbb{R}^d) \), we know that for any \( u_0 \in \dot{H}^s(\mathbb{R}^d) \), there exists a sequence \( \{u_n(0)\} \subset H^s(\mathbb{R}^d) \) such that
\[
\|u_n(0) - u_0\|_{H^s(\mathbb{R}^d)} \to 0, \quad n \to +\infty.
\]

Hence, \( \forall \varepsilon > 0, \exists N > 0, s.t. \forall n > N, \)
\[
\|u_n(0) - u_0\|_{H^s(\mathbb{R}^d)} < \varepsilon.
\]

By the Strichartz estimate and (2.31), we get for \( 2\varepsilon < \eta \) and \( n > N \),
\[
\|e^{it\Delta}u_n(0)\|_{L_t^p(\mathbb{R}^d)} \leq C\|u_n(0) - u_0\|_{H^s(\mathbb{R}^d)} + \|e^{it\Delta}u_0\|_{L_t^p(\mathbb{R}^d)} \leq C\varepsilon + \frac{\eta}{2} \leq \eta.
\]

This together with \( u_n(0) \in H^s(\mathbb{R}^d) \), and theorem 2.1 yield that there exists a unique solution \( u_n(t, x) : I \times \mathbb{R}^d \to \mathbb{C} \) to (1.1) with initial data \( u_n(0) \) obeying (2.22)–(2.24). In particular, it satisfies
\[
\|\nabla^k u_n\|_{L_t^p(\mathbb{R}^d)} \lesssim \|\nabla^k u_n(0)\|_{L_x^2} + \eta^{1+p} \lesssim \|\nabla^k u_0\|_{L_x^2} + \eta^{1+p} + \varepsilon. \tag{2.34}
\]

Next we use lemma 2.8 to show the solution sequence \( \{u_n(t, x)\} \) is Cauchy in \( S^s(I) \), where
\[
\|u\|_{S^s(I)} := \|\nabla^s u\|_{L^p(I)}. \quad \text{In fact, it follows from lemma 2.8 if we set} \ u = u_n, \ u = u_0, \ \text{and} \ eq(u) = eq(\tilde{u}) = 0. \ \text{Thus, by (2.29), we get}
\]
\[
\|\nabla^k (u_n - u_m)\|_{S^s(I)} \leq C(E, M)\varepsilon,
\]

which means \( \{u_n(t, x)\} \) is Cauchy in \( S^s(I) \). So it is convergent to a solution \( u(t, x) \) with initial data \( u(0, x) = u_0 \) obeying \( \|\nabla^s u\|_{S^s(I)} \leq C(I) \).

Using the theorem 2.1 and lemma 2.8 as well as their proof, one easily derives the following local theory for (1.1). We refer the author to Pausader [32] for the special energy-critical case \( s_c = 2 \).

**Theorem 2.2.** Let \( s_c \geq 1 \). Assume in addition that \( s_c < 2 + p \) if \( p \) is not an even integer. Then, given \( u_0 \in H^s(\mathbb{R}^d) \) and \( t_0 \in \mathbb{R} \), there exists a unique maximal-lifespan solution \( u : I \times \mathbb{R}^d \to \mathbb{C} \) to (1.1) with initial data \( u(t_0) = u_0 \). This solution also has the following properties:

1. (Local existence) \( I \) is an open neighborhood of \( t_0 \).
2. (Blowup criterion) If \( \sup(I) \) is finite, then \( u \) blows up forward in time in the sense of definition 1.2. If \( \inf(I) \) is finite, then \( u \) blows up backward in time.
3. (Scattering) If \( \sup(I) = +\infty \) and \( u \) does not blow up forward in time, then \( u \) scatters forward in time in the sense (1.8). Conversely, given \( u \in H^s(\mathbb{R}^d) \), there is a unique solution to (1.1) in a neighborhood of infinity so that (1.8) holds.

It is easy to show the following stability result by proposition 2.2 and lemma 2.8 as well as their proof.
Corollary 2.2 (stability). Assume that $s_c \geq 1$ if $p$ is an even integer or $1 \leq s_c < 2 + p$ otherwise. Let $I$ be a compact time interval containing zero and $\tilde{u}$ be a near solution to (1.1) on $I \times \mathbb{R}^d$ in the sense that

$$i\tilde{u}_t + \Delta \tilde{u} - f(\tilde{u}) + e = 0$$

for some function $e$. Assume that for some constants $M, E > 0$, we have

$$\|\tilde{u}\|_{L^\infty(I; H^s_c(\mathbb{R}^d))} \leq E,$$  \hspace{1cm} (2.35)

$$S_\epsilon(\tilde{u}) \leq M.$$  \hspace{1cm} (2.36)

Let $u_0 \in H^s_c(\mathbb{R}^d)$ and assume the smallness conditions

$$\|u_0 - \tilde{u}_0\|_{H^s_c(\mathbb{R}^d)} + \|\nabla|^{s-1} e\|_{L^2_t L_x^{2s}(I \times \mathbb{R}^d)} \leq \epsilon$$  \hspace{1cm} (2.37)

where $0 < \epsilon < \epsilon_1 = \epsilon_1(M, E)$ is a small constant. Then there exists a unique solution $u : I \times \mathbb{R}^d \to \mathbb{C}$ to (1.1) with initial data $u_0$ at time $t = 0$ obeying

$$S_\epsilon(u - \tilde{u}) \leq C(M, E) \epsilon^{c_1}$$  \hspace{1cm} (2.38)

$$\|\nabla|^{s} (u - \tilde{u})\|_{L^1(I)} \leq C(M, E) \epsilon^{c_2},$$  \hspace{1cm} (2.39)

$$\|\nabla|^{s} |u|\|_{L^1(I)} \leq C(M, E).$$  \hspace{1cm} (2.40)

where $c_1, c_2$ are positive constants that depend on $d, p, E$ and $M$.

3. Extinction of three scenarios

In this section, we preclude three scenarios in the sense of theorem 1.2 under the assumption that theorem 1.3 holds. We will prove theorem 1.3 in the next section. First, we preclude the finite time blowup solution by making use of the No-waste Duhamel formula.

3.1. The finite blowup solution

We argue by contradiction. Assume that there exists a solution $u : I \times \mathbb{R}^d \to \mathbb{C}$ which is a finite time blowup solution in the sense of theorem 1.2. Assume also $T := \sup(I) < +\infty$, then, we have by (1.10) and Sobolev embedding

$$\|u\|_{L^\infty_t L^s_x(I \times \mathbb{R}^d)} \lesssim 1.$$  \hspace{1cm} (3.1)

First, we consider the energy-subcritical case.

Case 1: $1 \leq s_c < 2$. Using the Bernstein inequality, the Strichartz estimate, Sobolev embeddings, (3.1), (1.17) and Hölder’s inequality, we have for $N \in 2\mathbb{Z}, \beta \in \mathbb{R}$
\[ \|\nabla|^{\nu-\beta} P_{\nu} u(t)\|_{L^2} \lesssim N^{\nu-\beta} \|P_{\nu} u(t)\|_{L^2} \]
\[ \leq N^{\nu-\beta} \left\| \int_0^T e^{it(s-t)\Delta} P_{\nu} f(u(s)) ds \right\|_{L^2} \]
\[ \lesssim N^{\nu-\beta} \|P_{\nu} f(u(s))\|_{L^{2+\nu/2}(0,T) \times \mathbb{R}^d} \]
\[ \lesssim (T-t)^{1/2} N^{\nu-\beta} \|P_{\nu} f(u)\|_{L^{2+\nu/2}(0,T) \times \mathbb{R}^d} \]
\[ \lesssim (T-t)^{1/2} N^{\nu-\beta}. \]

This implies by taking \( s_0 < \beta_1 < 2 < \beta_2 \)
\[ \|\nabla|^{\nu-\beta} P_{\nu} u(t)\|_{L^2} + \|\nabla|^{\nu-\beta} P_{\nu} u(t)\|_{L^2} \lesssim (T-t)^{1/2}. \]

Interpolating this with \( u \in L^\infty_t H^\nu([0,T) \times \mathbb{R}^d) \), we deduce that for some \( \alpha > 0 \)
\[ \|u(t)\|_{L^2} \lesssim (T-t)^{\alpha} \rightarrow 0, \quad \text{as} \quad t \rightarrow T \]
which shows that \( u \in L^\infty_t L^2([0,T) \times \mathbb{R}^d) \) and also \( u \equiv 0 \) by the mass conservation. This contradicts the fact that \( u \) is a blowup solution.

Next, we consider the energy-critical and energy-supercritical case. Using the assumption (1.10) and Sobolev embedding, we have
\[ \|u\|_{L^2_{x,t} L^{2+\nu/2}_{x,t}(\mathbb{R}^d)} \lesssim \|\nabla|^{\nu-2} u\|_{L^{2+\nu/2}_{x,t}(\mathbb{R}^d)} \lesssim \|u\|_{L^{2+\nu}_{x,t}(\mathbb{R}^d)} \lesssim 1. \]  
(3.2)

Case 2: \( s_0 \in [2,4] \). Combining (3.2) with the No-waste Duhamel formula (1.17), the Strichartz estimate, Hölder’s inequality and corollary 2.1, one has
\[ \|\nabla|^{\nu-2} u(t)\|_{L^2_{x,t}} \leq \left\| \int_0^T e^{it(s-t)\Delta} \nabla|^{\nu-2} f(u(s)) ds \right\|_{L^2_{x,t}} \]
\[ \lesssim \|\nabla|^{\nu-2} f(u(s))\|_{L^{2+\nu/2}_{x,t}(0,T) \times \mathbb{R}^d} \]
\[ \lesssim (T-t)^{1/2} \|\nabla|^{\nu-2} u\|_{L^{2+\nu/2}_{x,t}(0,T) \times \mathbb{R}^d} \|u\|_{L^{2+\nu}_{x,t}(0,T) \times \mathbb{R}^d} \]
\[ \lesssim (T-t)^{1/2}. \]  
(3.3)

Interpolating this with (1.10), using \( L^{p+1}_{t,x} \subset L^p_{t,x} \cap L^{\nu/2}_{t,x} \) and Sobolev embedding \( L^{\nu/2}_{t,x} \hookrightarrow H^{\nu-2} \), we derive that for \( s_0 \in (2,4) \)
\[ E(u_0) = E(u(t)) \rightarrow 0, \quad \text{as} \quad t \rightarrow T, \]
and for \( s_0 = 2 \)
\[ M(u_0) = M(u(t)) \rightarrow 0, \quad \text{as} \quad t \rightarrow T. \]

These imply that \( u \equiv 0 \). This contradicts the fact that \( u \) is a blowup solution.
Case 3: $x \in (4, 6]$. It follows from (3.3) that $u \in \dot{L}^8_t \dot{H}^{4-2}_x(\mathbb{R}^d)$. Using the No-waste Duhamel formula (1.17), the Strichartz estimate, Hölder’s inequality and fractional chain rule, we obtain

$$
\|\nabla|^{d-4} u(t)\|_{L^2_x} \leq \left\| \int_0^T e^{i(t-s)\Delta} \nabla|^{d-4} f(u(s)) \, ds \right\|_{L^2_x} \\
\lesssim \|\nabla|^{d-4} f(u(s))\|_{\dot{L}^{\frac{d}{d-4}}_{t,x}(t,T) \times \mathbb{R}^d} \\
\lesssim (T-t)^{\frac{1}{2}} \|\nabla|^{d-4} u\|_{L^\infty_t L^2_x(t,T \times \mathbb{R}^d)} \|u\|_{L^\infty_t L^\infty_x(t,T \times \mathbb{R}^d)} \\
\lesssim (T-t)^{\frac{1}{2}},
$$

(3.4)

Interpolating this with (1.10) again, we also deduce that

$$
E(u_0) = E(u(t)) \to 0, \quad \text{as} \quad t \to T.
$$

This contradicts the fact that $u$ is a blowup solution.

Case 4: $x \in (6, +\infty)$. We can iterate the argument presented above to obtain the contradiction.

Hence, we exclude the finite time blowup solution in the sense of theorem 1.2.

3.2. The soliton-like solution

Next, we adopt the Morawetz estimate to kill the soliton-like solution. We argue by contradiction. Assume that there exists a solution $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ which is a soliton-like solution in the sense of theorem 1.2.

Assume also theorem 1.3 holds. In particular, we have

$$
u(t, x) \in \dot{L}^\infty_t L^\infty_x(\mathbb{R}; \mathbb{C}).
$$

(3.5)

Therefore, the solution $u$ satisfies the following interaction Morawetz estimate.

**Lemma 3.1 (Interaction Morawetz estimate, [29, 33]).** Assume that $d \geq 7$. Let $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ be the solution to (1.1), and $u \in \dot{L}^\infty_t \dot{H}^1_x(\mathbb{R}^d)$. Then, for any compact interval $I \subset \mathbb{R}$, we have

$$
\int_I \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\nabla u(t, x)|^2 |\nabla u(t, y)|^2}{|x-y|^6} \, dx \, dy \, dt \lesssim \|u\|_{L^\infty_t L^2_x(I \times \mathbb{R}^d)}^2 \left\| \nabla \right\|_{L^\infty_t L^2_x(I \times \mathbb{R}^d)}^2 \lesssim 1.
$$

(3.6)

From (3.6), we know that (see theorem 5.9 in [25])

$$
\left\| \nabla \frac{d-5}{2} |u|^2 \right\|_{L^\infty_t L^2_x(I \times \mathbb{R}^d)} \approx \int_I \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(t, x)|^2 |u(t, y)|^2}{|x-y|^6} \, dx \, dy \, dt \lesssim 1.
$$

So, it follows from [33] that

$$
\|\nabla|^{d-5} u\|_{L^\infty_t L^2_x} \lesssim \left\| \sum_{N \in \mathbb{Z}^d} N \frac{d-5}{2} |P_N u|^2 \right\|_{L^\infty_t L^2_x(I \times \mathbb{R}^d)} \\
\lesssim \left\| \nabla \frac{d-5}{2} |u|^2 \right\|_{L^\infty_t L^2_x(I \times \mathbb{R}^d)} \lesssim 1.
$$

(3.7)
Interpolating this with \( u \in L^\infty_t (\mathbb{R}; H^1_x (\mathbb{R}^d)) \), we obtain for all compact time intervals \( I \subset \mathbb{R} \)

\[
\| u \|_{L_s^{\frac{3d}{d-1}, \frac{3d}{d-1}} (I \times \mathbb{R}^d)} \lesssim 1. \tag{3.8}
\]

Now we claim that

\[
\| u \|_{L_s^{\frac{2d}{d-1}, \frac{2d}{d-1}} (\mathbb{R}^d)} \gtrsim 1, \quad \text{uniformly for } t \in \mathbb{R}. \tag{3.9}
\]

If this claim holds, then we derive a contradiction by taking the length of the interval \( I \) to be sufficiently large.

Hence it suffices to prove the claim (3.9). We argue by contradiction. Suppose that the claim fails, then there exists a time sequence \( \{t_n\} \) such that \( u(t_n) \) converges to zero in \( L_s^{\frac{2d}{d-1}} \). On the other hand, since \( u(t) \) is uniformly bounded in \( H^s (\mathbb{R}^d) \), \( u(t_n, x - x_n) \) converges weakly to zero in \( H^s (\mathbb{R}^d) \) for any sequence \( x_n \). This contradicts the fact that the orbit of \( u \) is precompact in \( H^s (\mathbb{R}^d) \) and \( u \) is not identically zero.

So the claim holds. This completes the proof of excluding the soliton-like solution in the sense of theorem 1.2.

### 3.3. Low to high frequency cascade

Finally, we turn to exclude the low to high frequency cascade solution.

We argue by contradiction. Assume that there exists a solution \( u : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C} \) which is a low to high frequency cascade solution in the sense of theorem 1.2.

Assume also that theorem 1.3 holds. In particular, there exists \( \varepsilon > 0 \) such that

\[
u(t, x) \in L^\infty_s (\mathbb{R}; \dot{H}^{s-\varepsilon} (\mathbb{R}^d)). \tag{3.10}
\]

From \( \lim_{n \rightarrow +\infty} N(t_n) = +\infty \), we can find a time sequence \( \{t_n\} \) such that

\[
\lim_{n \rightarrow +\infty} N(t_n) = +\infty. \tag{3.11}
\]

Using Bernstein’s inequality, interpolation, the compactness (1.14), the hypothesis (3.10), and the assumption (1.10), we have

\[
\| u(t_n, x) \|_{L^2_s} \lesssim \| P_{s<\varepsilon} u(t_n) \|_{L^2_s} + \| P_{s>\varepsilon} u(t_n) \|_{L^2_s} \lesssim \| u \|_{H^s} \| P_{s<\varepsilon} u(t_n) \|_{H^s} + (c(\eta) N(t_n))^{-\varepsilon} \| u \|_{H^s} \lesssim \eta s + (c(\eta) N(t_n))^{-\varepsilon}, \tag{3.12}
\]

Taking \( \eta \) small, and then \( n \) large, we have by (3.11) and (3.12)

\[
M(u_0) = M(u(t_n)) \rightarrow 0, \quad \text{as } n \rightarrow \infty,
\]

which implies that \( u \equiv 0 \). This contradicts the fact that \( u \) is a blowup solution. Therefore, we preclude the low to high frequency cascade solution in the sense of theorem 1.2.

In sum, it reduces to prove theorem 1.3.
4. Lower regularity

As stated in section 3, it remains to show theorem 1.3. That is, we need to prove that the global solutions to (1.1) which are almost periodic modulo symmetries enjoy the lower regularity. We will divide two steps to prove it. First, we show additional decay for the soliton-like and frequency-cascade solutions in the sense of theorem 1.2. Then, this together with the double Duhamel trick yields the lower regularity for the soliton-like and frequency-cascade solutions.

4.1. Additional decay

We first consider the energy-supercritical case.

**Proposition 4.1 (Additional decay I, energy-supercritical).** Let $d \geq 8$ and $s_c > 2$. Assume in addition that $s_c < 2 + p$ if $p$ is not an even integer, and let $u$ be a global solution to (1.1) that is almost periodic modulo symmetries. In particular:

$$\|u\|_{L_t^\infty(\mathbb{R}; H^s(\mathbb{R}^d))} < +\infty.$$  

(4.1)

And assume that $\inf_{t \in \mathbb{R}} N(t) \geq 1$. Then, we have

$$u \in L_t^\infty(\mathbb{R}; L_x^q(\mathbb{R}^d)), \quad q \in \left(\frac{2d}{d-4}, \frac{d-4}{4}\right].$$  

(4.2)

**Remark 4.1.**

(i) It is easy to see that we have by Sobolev embedding and (4.1)

$$u \in L_t^\infty L_x^\infty(\mathbb{R} \times \mathbb{R}^d).$$

(ii) (4.2) can be reduced to show that there exists $\alpha > 0$ and $N_0 \in \mathbb{Z}^d$ such that for all dyadic number $N \leq N_0$

$$\|u_N\|_{L_t^\infty L_x^q(\mathbb{R}^d)} \lesssim N^\alpha, \quad q \in \left(\frac{2d}{d-4}, \frac{d-4}{4}\right].$$

In fact, we have by Bernstein’s inequality and (4.1)

$$\|u\|_{L_t^\infty L_x^q(\mathbb{R}^d)} \lesssim \sum_{N \leq N_0} \|u_N\|_{L_x^q} + \|P_{N_0} u\|_{L_x^q}$$

$$\lesssim \sum_{N \leq N_0} N^\alpha + \|\nabla^\alpha u\|_{L_x^q}$$

$$\lesssim N_0^\alpha + N_0^\frac{d}{4} - \frac{d}{4} \|\nabla^\alpha u\|_{L_x^q} < +\infty.$$  

The proof of proposition 4.1. From (1.14), we know that

$$\|u_{\eta \leq c(\eta)}\|_{H^s} \leq \eta.$$  

Combining this with $\inf_{t \in \mathbb{R}} N(t) \geq 1$, we deduce that if we take $N_0$ such that $N_0 \leq c(\eta)$, then

$$\|u_{\eta \leq c(\eta)}\|_{H^s} \leq \eta.$$  

(4.3)
Now we define $A_q(N)$ by
\[
A_q(N) = N^{d - \frac{d - 4}{p}} \|u_N\|_{L^q([B_r(0); R^d])}, \quad q > \frac{2d}{d - 4}.
\] (4.4)

It is easy to see that $A_q(N) \lesssim 1$ by Bernstein’s inequality and (4.1).

We first consider that $p$ is an even integer.

**Case 1: p even.** We claim that $A_q(N)$ satisfies the following recurrence formula
\[
A_q(N) \lesssim \left( \frac{N}{N_0} \right)^{d - 4 - \frac{4}{p} - \frac{d}{q}} + \eta^p \sum_{M \in N_0} \left( \frac{N}{M} \right)^{d - 4 - \frac{4}{p} - \frac{d}{q}} A_q(M)
\] (4.5)

for any $q \in \left( \frac{2d}{d - 4}, \min\left\{ \frac{2d}{d - 8}, \frac{4pd}{d - 8p + 8} \right\} \right)$. Note that $d - 4 - \frac{4}{p} - \frac{d}{q} < \frac{d}{2} + 4 + \frac{d}{q} > 0$ whenever $d \geq 9$ and $q \in \left( \frac{2d}{d - 4}, \frac{2d}{d - 8} \right)$, or $d = 8$, $q \in \left( \frac{2d}{d - 4}, +\infty \right)$.

We postpone the proof of this claim, and we recall a acausal Gronwall inequality.

**Lemma 4.1 (Acausal Gronwall inequality [22]).** Given $\eta, C, \gamma, \gamma' > 0$, let $\{x_k\}_{k \geq 0}$ be a bounded nonnegative sequence obeying
\[
x_k \lesssim C \left( 2^{-\gamma} + \eta \sum_{l=0}^{k-1} 2^{-\gamma(k-l)}x_l + \eta \sum_{l \geq k} 2^{-\gamma(\eta^l)}x_l \right)
\] (4.6)

for all $k \geq 0$. If $\eta \lesssim \frac{1}{\min\{1 - 2^{-\gamma}, 1 - 2^{-\gamma'}, 1 - 2^{\rho - \gamma}\}}$ for some $0 < \rho < \gamma$, then
\[
x_k \lesssim (4C + \|x\|_{\infty}) 2^{-\gamma k}.
\] (4.7)

Now we use the claim (4.5) to prove proposition 4.1 for $p$ being an even integer. Applying lemma 4.1 with $x_k = A_q(2^{-k}N_0)$, we obtain by (4.5)
\[
x_k \lesssim C \left( 2^{-k\left(d - 4 - \frac{4}{p} - \frac{d}{q}\right)} + C \eta^p \sum_{l=0}^{k-1} 2^{-(k-l)\left(d - 4 - \frac{4}{p} - \frac{d}{q}\right)}x_l + C \eta^p \sum_{l \geq k} 2^{-(\gamma^l)\left(d - 4 - \frac{4}{p} - \frac{d}{q}\right)}x_l \right).
\]

Then $x_k \lesssim 2^{-k\rho} 0 < \rho < d - 4 - \frac{4}{p} - \frac{d}{q}$, that is,
\[
A_q(N) \lesssim N^{\left(d - 4 - \frac{4}{p} - \frac{d}{q}\right)^+}, \quad q \in \left( \frac{2d}{d - 4}, \min\left\{ \frac{2d}{d - 8}, \frac{4pd}{d - 8p + 8} \right\} \right), \quad d \geq 8,
\]

which means for $N \leq N_0$
\[
\|u_N\|_{L^q} \lesssim N^{\left(d - 4 - \frac{2d}{d - 4}\right)^+}, \quad q \in \left( \frac{2d}{d - 4}, \min\left\{ \frac{2d}{d - 8}, \frac{4pd}{d - 8p + 8} \right\} \right), \quad d \geq 8.
\] (4.8)
This together with remark 4.1 (ii) yields that

\[ u \in \mathcal{L}_d^\infty(\mathbb{R}^2), \quad q \in \left( \frac{2d}{d - 4}, \min \left\{ \frac{2d}{d - 8}, \frac{4pd}{(d - 8)p + 8} \right\} \right), \quad d \geq 8. \]

Interpolating this with \( u \in \mathcal{L}_d^\infty(\mathbb{R} \times \mathbb{R}^d) \), we conclude proposition 4.1 for \( p \) being an even integer.

Therefore, it suffices to prove the claim (4.5). By time-translation symmetry, we only need to estimate (4.4) at \( t = 0 \). Using the No-waste Duhamel formula (1.17), Bernstein’s inequality, dispersive estimate (2.3), we obtain for all \( q > \frac{2d}{d - 4} \)

\[
\|\alpha_0(0)\|_{L_x^q} \leq \left\| \int_0^{+\infty} e^{it\Delta} P\mathcal{F}(u)(t)dt \right\|_{L_x^q} \\
\lesssim N^{d(\frac{1}{2} - \frac{1}{q})} \int_0^{N^{\frac{1}{4}}} \|e^{it\Delta} P\mathcal{F}(u)\|_{L_x^{2}}dt + \int_0^{+\infty} t^{\frac{d}{2} - \frac{1}{2q}}\|P\mathcal{F}(u)\|_{L_x^q}dt \\
\lesssim N^{d - 4 - 2\frac{d}{q}}\|P\mathcal{F}(u)\|_{L_x^{2}}.
\]

Thus

\[
A_p(N) \lesssim N^{d - 4 - 4\frac{d}{p}}\|P\mathcal{F}(u)\|_{L_x^{2}}, \quad q > \frac{2d}{d - 4}. \tag{4.9}
\]

Decomposing \( u \) by

\[ u = u_{>N_0} + u_{\leq N_0} = u_{>N_0} + \sum_{N_0 \leq N \leq \frac{N}{10^p}} u_{>N} + u_{< \frac{N}{10^p}} \]

and using the fact that \( p \) is an even integer, we can write \( P\mathcal{F}(u) \) by

\[
P\mathcal{F}(u) = P_N \left[ O \left( u_{>N_0} \sum_{k=0}^{p} u_{>N_0}^k u_{\leq N_0}^p \right) + O \left( \sum_{k=0}^{p} u_{>N_0}^k u_{>N}^{p - 1 - k} \right) \right]. \tag{4.10}
\]

Here we use the notation \( O(X) \) to denote a quantity that resembles \( X \), that is, a finite linear combination of terms that look like those in \( X \), but possibly with some factors replaced by their complex conjugates.

We first consider the terms which contain at least one factor of \( u_{>N_0} \). By Hölder’s inequality, Bernstein’s inequality, Sobolev embedding: \( \mathcal{H}_x^s(\mathbb{R}^d) \hookrightarrow \mathcal{L}_x^{\frac{pd}{d - 8}}(\mathbb{R}^d) \) and the assumption (4.1), we get

\[
\|P_N O(u_{>N_0}, u^p)\|_{L_x^{2}} \lesssim \|u_{>N_0}\|_{L_x^{2}} \|u\|_{\mathcal{L}_x^{\frac{pd}{d - 8}}}^p \\
\lesssim N_0^{-d + 4 + \frac{d}{2} + \frac{d}{q}} \|u\|_{L_x^{H^s}}^p \\
\lesssim N_0^{-d + 4 + \frac{d}{2} + \frac{d}{q}}. \tag{4.11}
\]

where \( 1 - \frac{1}{q} = \frac{1}{r} + \frac{4}{d} \) and \( q < \frac{2d}{d - 8} \) guarantees that \( r > 2 \).
To estimate the contribution of the second term on the right-hand side of (4.10) to (4.9), we first note that
\[
\left\| P_2 O \left( \sum_{k=0}^{p} u^k < N \frac{u^{p+1-k}}{10^p} \leq N_0 \right) \right\|_{L^r_t L^s_x} \lesssim \left\| O \left( u^{p+1} \frac{N}{10^p} \leq N_0 \right) \right\|_{L^r_t L^s_x} + \left\| O \left( u^p \frac{N}{10^p} \leq N_0 \right) \right\|_{L^r_t L^s_x}.
\]
Using Hölder’s inequality, Bernstein’s inequality, the assumption (4.1) and compactness (4.3), we estimate
\[
\left\| O \left( u^{p+1} \frac{N}{10^p} \leq N_0 \right) \right\|_{L^r_t L^s_x} \lesssim \|u_N\|_{L^r_t L^s_x} \leq \sum_{M_0, \ldots, M_{N-1} \leq N_0} \frac{\|u_M\|_{L^r_t L^s_x}}{M^p} \leq \frac{1}{2} \eta_{p+1, \ldots, p}^{\frac{d}{2}} + \frac{\|u_M\|_{L^r_t L^s_x}}{M^{\frac{d}{2}}}
\]
where \(1 - \frac{1}{q} = \frac{d(p-1)}{pd} + \frac{1}{q} + \frac{1}{r} \), and we use \(q < \frac{4pd}{8d - 8p + 8} \) to guarantee \(r > 2 \), and the fact \(q > \frac{2d}{d-4} \) in the last inequality.

Similarly, we estimate
\[
\left\| O \left( u^p \frac{N}{10^p} \leq N_0 \right) \right\|_{L^r_t L^s_x} \lesssim \|u_N\|_{L^r_t L^s_x} \leq \sum_{M_0, \ldots, M_{N-1} \leq N_0} \frac{\|u_M\|_{L^r_t L^s_x}}{M^p} \leq \frac{1}{2} \eta_{p+1, \ldots, p}^{\frac{d}{2}} + \frac{\|u_M\|_{L^r_t L^s_x}}{M^{\frac{d}{2}}}
\]
where \(1 - \frac{1}{q} = \frac{d(p-1)}{pd} + \frac{1}{q} + \frac{1}{r} \), and we use \(q < \frac{4pd}{8d - 8p + 8} \) to guarantee \(r > 2 \), and the fact \(q > \frac{2d}{d-4} \) in the last inequality.
where \( \varepsilon \) is a sufficiently small positive constant, and we use \( q > \frac{2d}{d-4} \) and \( A_q(M) \leq \eta \) with \( M \leq N_0 \) in the above inequality. This together with (4.11), (4.12) and (4.9) imply the claim (4.5). Thus, we conclude proposition 4.1 for \( p \) being an even integer.

**Case 2: \( p \) not even.** Now we turn to consider that \( p \) is not an even integer and \( s_c \in (2, 2 + p) \).

By the same argument as above, we estimate

\[
A_q(N) \lesssim N^{d-4} \frac{4}{d-4} \| P_{N} f(u) \|_{L^q}, \quad q > \frac{2d}{d-4}.
\]  

(4.13)

For \( N \leq N_0 \), using the fundamental theorem of calculus, we decompose \( f(u) \) by

\[
f(u) = O(u_{\geq N_0} \cdot u_{\leq N_0}^p) + O(u_{\leq N_0}^{p+1}) + f(u_{\leq N_0}^{\infty})
\]  

(4.14)

\[
+ u_{\leq N_0} \int_0^1 f_1(u_{\leq N_0}^{\infty} + \theta u_{\leq N_0}^{\infty}) \, d\theta
\]  

(4.15)

\[
+ \frac{p - N_0}{10} \int_0^1 f_2(u_{\leq N_0}^{\infty} + \theta u_{\leq N_0}^{\infty}) \, d\theta.
\]  

(4.16)

The contribution to the right-hand side of (4.13) coming from that contains at least one copy of \( u_{\geq N_0} \) which can be estimated by the same argument as (4.11).

By a simple computation, we have the following equivalence for \( p \) being not an even integer

\[
s_c < 2 + p \iff 2p^2 - (d - 4)p + 8 > 0,
\]

\[
\begin{cases}
\frac{8}{d-4} < p \leq 1, \\
2 < s_c < 2 + p,
\end{cases}
\]

\[
\begin{cases}
d = 13, \ p \leq 1 < p_1 := \frac{(d - 4) - \sqrt{(d - 4)^2 - 64}}{4} \\
d \geq 14, \ p < p_1
\end{cases}
\]

\[
\begin{cases}
p > \max\left\{ 1, \frac{8}{d-4} \right\}, \\
2 < s_c < 2 + p
\end{cases}
\]

\[
\begin{cases}
d = 13: \ 1 < p < p_1 \text{ or } p > p_2, \\
d \geq 14: \ p > p_2 := \frac{(d - 4) + \sqrt{(d - 4)^2 - 64}}{4}
\end{cases}
\]

Next, we divide two cases to estimate the contribution coming from the remaining terms.

**Subcase 2(i):** \( p \leq 1 \). In this case we have only \( f_1(u) \in C^1(\mathbb{C}) \).

We first consider the contribution coming from the term \( f(u_{\leq N_0}^{\infty}) \). Using \( l^p \subset l^1 \), Hölder’s inequality, Bernstein’s inequality and compactness (4.3), we deduce that
\[
\left\| \left( u_{N} - \frac{N}{10} \right) \right\|_{L_{t}^{q} L_{x}^{r}}
\leq \sum_{N \in M_{i} \in M_{N}} \left\| u_{M_{i}} \left| u_{N} - \frac{N}{10} \right|^{p} \right\|_{L_{t}^{q} L_{x}^{r}}
\leq \sum_{N \in M_{i} \in M_{N}} \left\| u_{M_{i}} \right\|_{L_{t}^{q} L_{x}^{r}}
\leq \sum_{N \in M_{i} \in M_{N}} \left\| u_{M_{i}} \right\|_{L_{t}^{q} L_{x}^{r}}^{p} \left\| u_{M_{i}} \right\|_{L_{t}^{q} L_{x}^{r}}^{p}
+ \sum_{N \in M_{i} \in M_{N}} \left( \left\| u_{M_{i}} \right\|_{L_{t}^{q} L_{x}^{r}}^{p} \left\| u_{M_{i}} \right\|_{L_{t}^{q} L_{x}^{r}}^{p} \right)
\leq \left( \eta \right)^{p} \sum_{N \in M_{i} \in M_{N}} \left( M_{1} \right)^{2p} \left( M_{2} \right)^{p}
\leq \eta \left( N \right)^{d+4 + \frac{q}{q-p}} \sum_{N \in M_{i} \in M_{N}} \left( \frac{N}{M} \right)^{d+4 + \frac{q}{q-p}} \left( M_{1} \right)^{2p} \left( M_{2} \right)^{p}
\end{align*}

where we need the restriction \( q \leq \frac{2}{2-p} \) to guarantee \( \frac{p}{q} \geq 2 \).

Now we consider the contribution coming from (4.15) and (4.16). It suffices to consider (4.15), since similar arguments can be used to deal with (4.16). By Hölder’s inequality, we obtain

\[
\left\| P_{x} \left( u_{N} - \frac{N}{10} \right) \int_{0}^{1} f_{c} \left( u_{N} - \frac{N}{10} + \theta u_{N} \right) d\theta \right\|_{L_{t}^{q} L_{x}^{r}}
\leq \left\| u_{N} \right\|_{L_{t}^{q} L_{x}^{r}} \left\| P_{x} \left( \int_{0}^{1} f_{c} \left( u_{N} - \frac{N}{10} + \theta u_{N} \right) d\theta \right) \right\|_{L_{t}^{q} L_{x}^{r}}
\leq \left\| u_{N} \right\|_{L_{t}^{q} L_{x}^{r}} \left\| P_{x} f_{c} \left( \left( u_{N} \right) \right) \right\|_{L_{t}^{q} L_{x}^{r}}
\end{align*}

where \( 1 - \frac{1}{q} = \frac{2}{2-p} + \frac{d}{2-p} \), and we need the restriction \( q \leq \frac{2}{d-2(p+1)} \) to guarantee \( r \geq 2 \). On the other hand, it follows from the nonlinear Bernstein inequality (2.20) that

\[
\left\| P_{x} f_{c} \left( \left( u_{N} \right) \right) \right\|_{L_{t}^{q} L_{x}^{r}} \leq N^{-d} \left\| \nabla u_{N} \right\|_{L_{t}^{q} L_{x}^{r}}^{p} \leq N^{-d} \left\| \nabla \left( u_{N} \right) \right\|_{L_{t}^{q} L_{x}^{r}}^{p}
\end{align*}

Plugging this into (4.17), and by Bernstein’s inequality, compactness (4.3) we derive
Using the \(H^R\) method. Applying lemma 4.1 again, we obtain

\[
\left\| P_N \left( u_{\xi R} \int_0^1 f_\xi \left( \frac{a_N \xi \xi \epsilon e N + \theta u_{\xi e N} \right) \right) \right\| \lesssim N^{-p} \left\| \nabla u_{\xi e N} \right\|_{L^p_x L^2_t} \left\| u_{\xi e N} \right\|_{L^p_x L^2_t}
\]

\[
\lesssim \eta^p N^{-d+4+\frac{d}{p} + \frac{d}{2} - d} \sum_{M \leq N} \left( \frac{M}{N} \right)^{p+4+\frac{d}{p} + \frac{d}{2} - d} A_q(M).
\]

Putting everything together, we deduce that \(A_q(N)\) satisfies the following recurrence formula

\[
A_q(N) \lesssim \left( \frac{N}{N_0} \right)^{-d+4+\frac{d}{p} + \frac{d}{2} - d} + \eta^p \sum_{N \leq M \leq N_0} \left( \frac{M}{N} \right)^{-d+4+\frac{d}{p} + \frac{d}{2} - d} A_q(M)
\]

\[
+ \eta^p \sum_{N \leq M \leq N_0} \left( \frac{M}{N} \right)^{-d+4+\frac{d}{p} + \frac{d}{2} - d} A_q(M)
\]

(4.18)

for any \(q \in \left( \frac{2d}{d-a}, \min \left\{ \frac{4}{2-p}, \frac{d}{d-4-p - \frac{2}{p}} \right\} \right)\). Applying lemma 4.1 again, we obtain

\[
A_q(N) \lesssim N^{\left( d-4+\frac{4}{p} \right) \cdot \left( \frac{d}{d-4-p - \frac{2}{p}} \right)} \frac{c}{d - 4}, \quad q \in \left( \frac{2d}{d-4}, \min \left\{ \frac{4}{2-p}, \frac{d}{d-4-p - \frac{2}{p}} \right\} \right)
\]

(4.19)

which means for \(N \leq N_0\)

\[
\left\| u_{\xi e N} \right\|_{L^p_x L^2_t} \lesssim N^{\left( d-4+\frac{4}{p} \right) \cdot \left( \frac{d}{d-4-p - \frac{2}{p}} \right)} \frac{c}{d - 4}, \quad q \in \left( \frac{2d}{d-4}, \min \left\{ \frac{4}{2-p}, \frac{d}{d-4-p - \frac{2}{p}} \right\} \right)
\]

(4.20)

This together with remark 4.1 (ii) yields that

\[
u \in L^\infty_{\tau} L^2(R^d), \quad q \in \left( \frac{2d}{d-4}, \min \left\{ \frac{4}{2-p}, \frac{d}{d-4-p - \frac{2}{p}} \right\} \right)
\]

Interpolating this with \(u \in L^\infty_{\tau} L^2(R^d)\), we conclude proposition 4.1 for \(P \in \left( \frac{8}{d-4}, 1 \right)\).

**Subcase 2(ii):** \(p > \max \left\{ 1, \frac{8}{d-4} \right\} \). By the same argument as (4.12), we estimate

\[
\left\| f \left( \frac{u_{\xi e N}}{N} \right) \right\|_{L^p_x L^2_t} \lesssim \eta^p N^{-d+4+\frac{4}{p} + \frac{d}{2} - d} \sum_{N \leq M \leq N_0} \left( \frac{N}{M} \right)^{d-4+\frac{4}{p} + \frac{d}{2} - d} A_q(M).
\]

Next, we consider the contribution coming from (4.15) and (4.16). It suffices to consider (4.15), since similar arguments can be used to deal with (4.16). Given \(p\), there exists \(\epsilon > 0\) such that \(s_\epsilon < 2 + p - \epsilon\). Using the Hölder, Bernstein’s inequalities and compactness (4.3), we derive that
where $1 - \frac{1}{q} = \frac{1}{n_1} + \frac{1}{r_2}$ and $\frac{1}{r_2} = \frac{d - 4 + \varepsilon}{2d} + \frac{d(p - 1)}{pd}$. Thus, we derive that $A_q(N)$ satisfies the following recurrence formula

$$A_q(N) \lesssim \left( \frac{N}{N_0} \right)^{d - 4 + \frac{4}{q} - \frac{d}{q}} + \eta^p \sum_{M \in \mathbb{N}} \left( \frac{N}{M} \right)^{d - 4 + \frac{4}{q} - \frac{d}{q}} A_q(M)$$

$$+ \eta^p \sum_{M \in \mathbb{N}} \left( \frac{M}{N} \right)^{d - 4 + \frac{4}{q} - \frac{d}{q}} A_q(M)$$

(4.22)

for any $q \in \left( \frac{2d}{d - 4}, \frac{2d}{d - 4 + \varepsilon} \right)$. Applying lemma 4.1 again, we get

$$A_q(N) \lesssim N^{d - 4 + \frac{4}{q} - \frac{d}{q}}, \quad q \in \left( \frac{2d}{d - 4}, \frac{2d}{d - 4 + \varepsilon} \right)$$

(4.23)

which means for $N \leq N_0$

$$\|u_N\|_{L^q_t L^r_x} \lesssim N^{d - 4 + \frac{2d}{d - 4}}, \quad q \in \left( \frac{2d}{d - 4}, \frac{2d}{d - 4 + \varepsilon} \right)$$

(4.24)

This together with remark 4.1 (ii) yields that

$$u \in L^\infty_t L^q_x(\mathbb{R}; \mathbb{R}^d), \quad q \in \left( \frac{2d}{d - 4}, \frac{2d}{d - 4 + \varepsilon} \right)$$

Interpolating this with $u \in L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^d)$, we conclude proposition 4.1 for $p > \max \left\{ 1, \frac{8}{d - 2} \right\}$.

Therefore, we complete the proof of proposition 4.1.

The next result shows the additional decay for the energy-subcritical and energy-critical cases.

**Proposition 4.2 (Additional decay II).** Let $d \geq 9$ and $1 \leq s_0 \leq 2$. Let $u$ be a global solution to (1.1) that is almost periodic modulo symmetries. Assume also $\inf_{t \in \mathbb{R}} N(t) \geq 1$. Then we have $u \in L^\infty_t L^q_x(\mathbb{R}; \mathbb{R}^d)$ with
\[
q \in \begin{cases}
(r_1, \frac{pd}{4}], & r_1 = \frac{2d}{p} - d + 6 + \kappa \\
(r_2, \frac{pd}{4}], & r_2 = \frac{2d}{d-4} - d + 4 + \kappa
\end{cases}
\]

if \( p > 1 \),

\[
r_2 = \frac{2d}{d-4} - d + 4 + \kappa
\]

if \( p \leq 1 \).

In particular, if \( \kappa = 2 \); that is: \( p = \frac{8}{d-4} \), then

\[
u \in L^\infty_s(\mathbb{R}; L^d_0(\mathbb{R}^d)), \quad q \in \begin{cases}
\left( \frac{2d}{d-3}, \frac{2d}{d-4} \right], & \text{if } p > 1, \text{ i.e. } d < 12,
\left( \frac{2(d+4)}{d}, \frac{2d}{d-4} \right], & \text{if } p \leq 1, \text{ i.e. } d \geq 12.
\end{cases}
\]

Remark 4.2. It is easy to check that

\[ r_1, r_2 < \frac{pd}{4}, \text{ whenever } p > \frac{8}{d}. \]

The proof of proposition 4.2: Noting that \( \frac{2d}{d-4} \leq \frac{d}{d-4} - \frac{2}{d} \) in this case and by a similar argument as proposition 4.1, we have for \( N \leq N_0 \)

\[
\|u_N\|_{L^\infty_s L^d_0} \lesssim N^{(d-4)\frac{2d}{d}}\frac{d}{p}, \quad q \in \begin{cases}
\left( \frac{d}{d-4} - \frac{2}{d} \frac{d}{d-5} - \frac{2}{p} \right], & \text{if } p > 1,
\left( \frac{d}{d-4} - \frac{2}{d} \frac{d}{d-4} - \frac{2}{p} \right], & \text{if } p \leq 1.
\end{cases}
\]

We remark that one estimates the term (4.21) in the case \( p > 1 \) by a different way. Term (4.21) is estimated by

\[
\left\| P_N \left( u_{< N} \int_0^1 f_\xi \left( u_{< N} + \frac{\theta u_{< N}}{\eta} \right) d\theta \right) \right\|_{L^d_t L^d_x} \\
\lesssim \left\| u_{< N} \right\|_{L^\infty_t L^d_x} \left\| P_{> N} \int_0^1 f_\xi \left( u_{< N} + \frac{\theta u_{< N}}{\eta} \right) d\theta \right\|_{L^d_t L^d_x}
\lesssim N^{-1} \left\| u_{< N} \right\|_{L^\infty_t L^d_x} \left\| \nabla u_{< N} \right\|_{L^d_t L^\infty_x} \left\| u_{< N} \right\|^{d-1} \left\| \frac{\theta u_{< N}}{\eta} \right\|_{L^\infty_t L^d_x}
\lesssim \eta^d N^{-d+4} \frac{d-4}{\eta} + \frac{4}{d} \sum_{M < N} \left( \frac{M}{N} \right)^{-d+5} + \frac{4}{d} A_q(M).
\]
Therefore, we get (4.27).

**Case 1:** \( p > 1 \). We have by (4.27) with \( q = \frac{d}{d-5-s} \)

\[
\|u_N\|_{L_t^\infty L_x^{s+\frac{d-8}{d-5-s}}} \lesssim N^{\frac{8}{2p} - d + 6}.
\] (4.28)

On the other hand, from the assumption (1.10): \( u \in L_t^\infty \dot{H}^s_x(\mathbb{R} \times \mathbb{R}^d) \), we know that

\[
\|u_N\|_{L_t^\infty L_x^s} \lesssim N^{-\kappa}.
\]

Interpolating this with (4.28), we deduce that for \( N \leq N_0 \)

\[
\|u_N\|_{L_t^\infty L_x^s} \lesssim N^{0^+}.
\]

Hence we obtain \( u \in L_t^\infty L_x^s(\mathbb{R} \times \mathbb{R}^d) \) by remark 4.1 (ii). Interpolating this with \( u \in L_t^\infty \dot{H}^s_x \subset L_t^\infty L_x^s \) again, we derive that

\[
u \in L_t^\infty (\mathbb{R}; L_x^s(\mathbb{R}^d)), \quad q \in \left(r_1, \frac{pd}{4}\right), \quad r_1 = \frac{2d}{d} \left( \frac{5}{2d - d + 6} + \frac{s_i}{d} \right).
\] (4.29)

**Case 2:** \( p \leq 1 \). By (4.27), we get

\[
\|u_N\|_{L_t^\infty L_x^{s+\frac{d}{d-8}}} \lesssim N^{\frac{2p}{d} - d + 6}.
\] (4.29)

On the other hand, from the assumption (1.10): \( u \in L_t^\infty \dot{H}^s_x(\mathbb{R} \times \mathbb{R}^d) \), we know that

\[
\|u_N\|_{L_t^\infty L_x^s} \lesssim N^{-\kappa}.
\]

Combining this with (4.29), we obtain for \( N \leq N_0 \)

\[
\|u_N\|_{L_t^\infty L_x^{s+\frac{d}{d-8}}} \lesssim N^{0^+}.
\]

This implies \( u \in L_t^\infty L_x^{s+\frac{d}{d-8}} \) by remark 4.1 (ii). Interpolating this with \( u \in L_t^\infty L_x^s \) concludes the proof of this proposition.

### 4.2. Lower regularity

Now we utilize the double Duhamel trick to show theorem 1.3. First, we drive a preliminary lemma.

**Lemma 4.2.** Suppose that \( u \in L_t^\infty(\mathbb{R}; \dot{H}^s_x(\mathbb{R}^d)) \) for some \( s \in [0, s_i] \). Assume also that there exists a positive constant \( \alpha \) independent of \( s \) such that

\[
\| \| \nabla \|^s P_\alpha u \|_{L_t^\infty L_x^s} \lesssim 1, \quad N \leq 1.
\] (4.30)

Then, for any \( \beta \in [0, \alpha) \), we have \( u \in L_t^\infty(\mathbb{R}; \dot{H}^s_x(\mathbb{R}^d)) \).

**Proof.** Using Bernstein’s inequality and the assumption \( u \in L_t^\infty(\mathbb{R}; \dot{H}^s_x(\mathbb{R}^d)) \), we have

\[
\| \| \nabla \|^s P_\beta u \|_{L_t^\infty L_x^s} \lesssim 1, \quad N \leq 1.
\] (4.30)
\[ \|u\|_{L^{r}_{\infty,\omega}} \lesssim \sum_{N \leq 1} \|\nabla^{\|r-\beta\|}P_{\omega}u\|_{L^{r}_{\infty,\omega}} + \|\nabla^{\|r-\beta\|}P_{\omega-}\|_{L^{r}_{\infty,\omega}} \]
\[ \lesssim \sum_{N \leq 1} N^{-\beta} \|\nabla^{\|r\|}P_{\omega}u\|_{L^{r}_{\infty,\omega}} + \|\nabla^{\|r\|}u\|_{L^{r}_{\infty,\omega}} \]
\[ \lesssim \sum_{N \leq 1} N^{n-\beta} + 1 < +\infty. \]

This completes the proof of lemma 4.2. □

**The proof of theorem 1.3** From lemma 4.2, we know that the proof of theorem 1.3 with \(d \geq 9\) is reduced to show that for any \(s \in [0, s_c]\), there exists a positive constant \(\alpha\) independent of \(s\) such that

\[ \|\nabla^{\|s\|}u\|_{L^{q}(\mathbb{R};L^{p}_{\omega})} \lesssim N^{\alpha}, \quad \forall \ N \leq 1. \]  

(4.31)

Indeed, we first apply (4.31) with \(s = s_c\). Then we conclude that \(u \in L^{\infty}_{t}(\mathbb{R};H^{s_{c}-\alpha+}_{\omega})\) by lemma 4.2. Then we apply (4.31) with \(s = s_c - \alpha + \) and obtain \(u \in L^{\infty}_{t}(\mathbb{R};H^{s_{c}-2\alpha+}_{\omega})\). Iterating this procedure \([s_c/\alpha]+1\) times, we derive \(u \in L^{\infty}_{t}(\mathbb{R};H^{s}_{\omega})\) for some \(0 < \varepsilon < \alpha\). Interpolating this with \(u \in L^{\infty}_{t}(\mathbb{R},H^{s_{c}})\) implies that \(u \in L^{\infty}_{t}(\mathbb{R},L^{2}_{\omega})\). Applying (4.31) with \(s = 0\) and using lemma 4.2 again, we obtain \(u \in L^{\infty}_{t}(\mathbb{R};H^{s}_{\omega})\) for any \(0 < \varepsilon < \alpha\).

Hence it remains to prove the claim (4.31). We divide two cases to discuss. First, we consider the energy-supercritical case.

**Case 1: \(s_c > 2\) (energy-supercritical).** We first consider the higher dimensions: \(d \geq 9\).

**Subcase 1: \(d \geq 9\).** Assume that \(u \in L^{\infty}_{t}(\mathbb{R};H^{s}_{\omega}(\mathbb{R}^{d}))\) for some \(0 \leq s \leq s_c\). It follows from the additional decay (proposition 4.1) that

\[ u \in L^{\infty}_{t}(\mathbb{R};L^{q}_{\omega}(\mathbb{R}^{d})), \quad q \in \left(\frac{2d}{d-4}, \frac{d}{4}\right]. \]  

(4.32)

Thus, we obtain by (2.17)

\[ \|\nabla^{\|r\|}f(u)\|_{L^{p}_{\omega}} \lesssim \|\nabla^{\|r\|}u\|_{L^{p}_{\omega}} \|u\|_{L^{q}_{\omega}} \lesssim 1, \quad r = \frac{2q}{q+2p} < 2. \]  

(4.33)

Then the condition \(r \geq 1\) requires \(q \geq 2p\).

It follows from the No-waste Duhamel formula (1.17) that

\[ u(0) = - \int_{0}^{\infty} e^{iu \Delta t} f(u(t)) dt = \int_{-\infty}^{0} e^{i\tau \Delta t} f(u(\tau)) d\tau. \]

Thus

\[ \|\nabla^{\|r\|}u(0)\|_{L^{q}_{\omega}}^{2} = - \left( \int_{0}^{\infty} e^{iu \Delta t} \nabla^{\|r\|} P_{\omega} f(u(t)) dt, \int_{-\infty}^{0} e^{i\tau \Delta t} \nabla^{\|r\|} P_{\omega} f(u(\tau)) d\tau \right) \]
\[ = - \int_{0}^{\infty} \int_{-\infty}^{0} (e^{iu \tau} - e^{i\tau \tau}) \nabla^{\|r\|} P_{\omega} f(u(t), |\nabla^{\|r\|} P_{\omega} f(u(\tau))|) d\tau d\tau \]
\[ \Delta \int_{0}^{\infty} F(t, \tau) d\tau dt, \]  

(4.34)
On one hand, using the Hölder, Bernstein inequalities and (4.33), we get

\[
\begin{align*}
F(t, \tau) & \lesssim \|e^{it(\tau - t)}\|_{L^2_t} \|\nabla |P_N f(u(t))\|_{L^2_t}^2 \\
& \lesssim N^{2d(\frac{1}{4} - \frac{1}{2})} \|\nabla |P_N f(u(t))\|_{L^2_t}^2 \\
& \lesssim N^{2d(\frac{1}{4} - \frac{1}{2})}.
\end{align*}
\]  

(4.35)

On the other hand, by Hölder’s inequality and the dispersive estimate (2.3), we derive

\[
\begin{align*}
F(t, \tau) & \lesssim \|e^{it(\tau - t)}\|_{L^2_t} \|\nabla |P_N f(u(t))\|_{L^2_t}^2 \\
& \lesssim (t - \tau)^{-d(\frac{1}{4} - \frac{1}{2})} \|\nabla |P_N f(u(t))\|_{L^2_t}^2 \\
& \lesssim (t - \tau)^{-d(\frac{1}{4} - \frac{1}{2})}.
\end{align*}
\]  

(4.36)

Hence, plugging (4.35) and (4.36) into (4.34), we obtain

\[
\begin{align*}
\|\nabla |u_N(0)|\|_{L^2_r} & \lesssim \int_0^\infty \int_{-\infty}^0 \min \left\{ N^{2d(\frac{1}{4} - \frac{1}{2})}, (t - \tau)^{-d(\frac{1}{4} - \frac{1}{2})} \right\} \, d\tau \, dt \\
& \lesssim \frac{1}{4} \int_{\mathbb{R}} \int_{\mathbb{R}} \min \left\{ N^{2d(\frac{1}{4} - \frac{1}{2})}, (|\tau| + |\tau|)^{-d(\frac{1}{4} - \frac{1}{2})} \right\} \, d\tau \, dt \\
& \lesssim \int_{|\tau| + |\tau| \leq N^{-4}} N^{\frac{2d}{4}} \, d\tau \, dt + \int_{|\tau| + |\tau| > N^{-4}} (|\tau| + |\tau|)^{\frac{pd}{2}} \, d\tau \, dt \\
& \lesssim N^{-8 + \frac{2pd}{q}},
\end{align*}
\]  

where \( r = \frac{2q}{q + 2p} \) and we also need the restriction \( \frac{pd}{q} > 2 \) to guarantee the above integral converges. Therefore,

\[
\|\nabla |u_N(0)|\|_{L^2_r} \lesssim N^{-8 + \frac{pd}{q}}, \quad q \in \left[ 2p, \frac{pd}{4} \right) \cap \left( \frac{2d}{d - 4}, \frac{pd}{4} \right].
\]  

(4.38)

The condition \( 2p < \frac{pd}{4} \) requires the dimension \( d \) such that \( d \geq 9 \). Now if we take \( q = \max \left\{ 2p, \frac{2d}{d - 4} \right\} \), then we obtain

\[
\alpha = -4 + \frac{pd}{q} = \min \left\{ -4 + \frac{d}{2}, -4 + \frac{(d - 4)p}{2} \right\} > 0.
\]

Therefore we conclude (4.31), and so we complete the proof of theorem 1.3 for \( s_c > 2 \).

**Subcase 2:** \( d = 8 \). We will estimate (4.33) in a different way by using the refined dispersive estimate (2.6). First, we claim that

\[
u(t, x) \in L^\infty_t \left( \mathbb{R}, H^s_x(\mathbb{R}^d) \right), \quad \forall \ s \in (0, s_c].
\]  

(4.39)

Indeed, when \( s \geq \frac{1}{2} \), using (4.32) and \( u \in L^\infty_t H^s_x \), we estimate
\[
\|\nabla^\alpha P_f(\phi)\|_{L^2_t L^\infty_x} \lesssim N^{\frac{1}{2}} \left( \sum_{\ell = 0}^{\infty} \nabla_{x_\ell}^\alpha \phi \right)
\]

\[\lesssim N^{\frac{1}{2}} \left( \sum_{\ell = 0}^{\infty} \nabla_{x_\ell}^\alpha \phi \right) \lesssim N^{\frac{1}{2}} \left( \sum_{\ell = 0}^{\infty} \nabla_{x_\ell}^\alpha \phi \right) \lesssim N^{\frac{1}{2}} \left( \sum_{\ell = 0}^{\infty} \nabla_{x_\ell}^\alpha \phi \right) \lesssim \frac{2d}{d-4} \cdot \frac{pd}{4}.
\]

(4.40)

The condition \(1 > \frac{1}{r} = \frac{d-1}{2d} + \frac{p}{q} \geq \frac{1}{2}\) requires \(\frac{2pd}{d+1} \leq q \leq pd\).

On one hand, by the same argument as (4.35), (4.36) and (4.40), we have
\[
\|\nabla^\alpha u_0(0)\|_{L^2_x}^2 = \int_0^\infty \int_{-\infty}^\infty F(t, \tau) \, d\tau \, dt,
\]

and
\[
F(t, \tau) \lesssim N^{2d} \left( \frac{1}{r} \right)^{1+1}.
\]

(4.41)

On the other hand, by Hölder’s inequality and refined dispersive estimate (2.6), we derive
\[
F(t, \tau) \lesssim \left( t - \tau \right)^{-d(\frac{1}{r} - \frac{1}{2})} N^d \left( \frac{2d}{d-4} \cdot \frac{pd}{4} \right).
\]

(4.42)

Hence, plugging (4.41) and (4.42) into (4.34), we obtain
\[
\|\nabla^\alpha u_0(0)\|_{L^2_x}^2 \lesssim \int_0^\infty \int_{-\infty}^\infty \min \left\{ N^{d} \left( \frac{1}{r} \right)^{1+1}, N^{d} \left( 1 - \frac{2}{r} \right)^{1/2} \right\} \, d\tau \, dt \lesssim \frac{1}{4} \int_0^\infty \int_{-\infty}^\infty \min \left\{ N^{d} \left( \frac{1}{r} \right)^{1+1}, N^{d} \left( 1 - \frac{2}{r} \right)^{1/2} \right\} \, d\tau \, dt
\]

\[
\lesssim \int_{|t| + |\tau| \leq N} \frac{2pd}{d+1} \, d\tau \, dt + N^{2} \int_{|t| + |\tau| > N} \left( |t| + |\tau| \right) \frac{pd}{d+1} \, d\tau \lesssim N^{d} \cdot \frac{2pd}{d+1}.
\]

(4.43)

where \(\frac{1}{r} = \frac{d-1}{2d} + \frac{p}{q}\) and we also need the restriction \(\frac{pd}{q} - \frac{1}{2} > 2\) to guarantee the above integral converges. Therefore,
\[
\|\nabla^\alpha u_0(0)\|_{L^2_x}^2 \lesssim N^{d} \cdot \frac{2pd}{d+1}, \quad q \in \left[ \frac{2dp}{d+1}, \frac{2pd}{d-4} \right]
\]

(4.44)

Now if we take \(q = \max \left\{ \frac{2dp}{d+1}, \frac{2dp}{d-4} \right\}\), then we obtain
\[
\alpha = -4 + \frac{pd}{q} = \min \left\{ \frac{d - 7}{2}, -4 + \frac{(d - 4)p}{2} \right\}.
\]

(4.45)

The condition \(\frac{2dp}{d+1} < \frac{pd}{d-4}\) requires the dimension \(d\) such that \(d \geq 8\). Therefore we conclude (4.31) with \(s \in \left[ \frac{1}{2}, s_0 \right]\) and so we conclude the claim (4.39). By Sobolev embedding, we get
\[ u \in L^q_t(L^p_x(\mathbb{R}^d)), \quad \forall \ q \in (2, 2p]. \]

Hence,
\[ \|\nabla \left\| Pf(u) \right\|_{L^q_t(L^p_x(\mathbb{R}^d))} \lesssim N^{\frac{1}{2}} \|u\|_{L^q_t(L^p_x(\mathbb{R}^d))} \lesssim N^{\frac{1}{2}}.\]

By the same argument as (4.43) with \( q = \frac{8}{p} \), we obtain
\[ \|\nabla u_N(0)\|_{L^q_x(\mathbb{R}^d)} \lesssim N^{-1}.\]

Therefore, we complete the proof of theorem 1.3 in dimension \( d = 8 \).

**Case 2:** \( 1 \leq s_c \leq 2 \) (energy-subcritical and energy-critical). By the same argument as the energy-supercritical case, we have
\[ \|\nabla u_N(0)\|_2 \lesssim N^{-4 + \frac{pd}{q}}, \]

where \( q \) satisfies
\[ q \in \left[ 2p, \frac{pd}{4} \right) \bigcap \left\{ \begin{array}{ll}
(1, \frac{pd}{4}) & \text{if } p > 1, \\
(2, \frac{pd}{4}) & \text{if } p \leq 1.
\end{array} \right. \tag{4.46} \]

where \((r_1, r_2)\) is as in proposition 4.2. If we take \( q = \frac{pd}{4} \), then we obtain \( \alpha = -4 + \frac{pd}{q} = 0 + > 0 \). Hence we get (4.31). Therefore, we complete the proof of theorem 1.3 for \( s_c \in [1, 2] \). Therefore, we conclude theorem 1.1.

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**Appendix**

In this appendix, we show the perturbation theory. We first consider that \( p \) is an even integer.

**A.3. Perturbation I: \( p \) even**

Here we give a perturbation result under the weakest assumption on the difference of the initial data \((A.3)\), since it is easy to show the smallness assumption \((A.3)\) can be derived from (2.28) by the Strichartz estimate.

**Lemma A.1 (Perturbation lemma, \( p \) even).** Assume that \( s_c = \frac{d}{2} - \frac{4}{p} \geq 1 \), and \( p \) is an even integer. Let \( I \) be a compact time interval and \( u, \tilde{u} \) satisfy
\[(i\partial_t + \Delta^2)u = -f(u) + eq(u)\]
\[(i\partial_t + \Delta^2)\tilde{u} = -f(\tilde{u}) + eq(\tilde{u})\]

for some function \(eq(u), eq(\tilde{u})\), and \(f(u) = |u|^p u\). Assume that for some constants \(M, E > 0\), we have

\[\|u\|_{L^p_t(L^2_y)} + \|\tilde{u}\|_{L^p_t(L^2_y)} \leq E,\]

(A.1)

\[S_t(\tilde{u}) \leq M,\]

(A.2)

Let \(t_0 \in I\), and let \(u(t_0)\) be close to \(\tilde{u}(t_0)\) in the sense that

\[\|\nabla|^{-1}e^{it(t-t_0)\Delta^2}(u - \tilde{u})(t_0)\|_{L^2_t(L^{\frac{2(p+1)}{p}}(\mathbb{R}^n_+ \times \mathbb{R}^n))} \leq \varepsilon,\]

(A.3)

where \(0 < \varepsilon < \varepsilon_1 = \varepsilon_0(M, E)\) is a small constant. Assume also that we have smallness conditions

\[\|\nabla|^{-1}(eq(u), eq(\tilde{u}))\|_{L^2_t(L^{\frac{2(p+1)}{p}}(\mathbb{R}^n_+ \times \mathbb{R}^n))} \leq \varepsilon,\]

(A.4)

where \(\varepsilon\) is as above.

Then we conclude that

\[S_t(u - \tilde{u}) \leq C(M, E)\varepsilon\]
\[\|\nabla|^{-1}(u - \tilde{u})\|_{L^2_t(L^{\frac{2(p+1)}{p}}(\mathbb{R}^n_+ \times \mathbb{R}^n))} \leq C(M, E)\varepsilon\]
\[\|\nabla|^{-1}u\|_{L^2_t(L^{\frac{2(p+1)}{p}}(\mathbb{R}^n_+ \times \mathbb{R}^n))} \leq C(M, E)\varepsilon.\]

(A.5)

**Proof.** Since \(S_t(\tilde{u}) \leq M\), we may subdivide \(I\) into \((C(M, E)\varepsilon)\) time intervals \(I_j\) such that

\[S_t(\tilde{u}) \leq \varepsilon_0 \ll 1, \quad 1 \leq j \leq C(M, \varepsilon_0).\]

By the Strichartz estimate and standard bootstrap argument we have

\[\|\nabla|^{-1}u\|_{L^2_t(L^{\frac{2(p+1)}{p}}(\mathbb{R}^n_+ \times \mathbb{R}^n))} \leq C(E), \quad 1 \leq j \leq C(M, \varepsilon_0).\]

(A.6)

In particular, we have by Sobolev embedding

\[\|\tilde{u}\|_{L^2_t(L^{\frac{2(p+1)}{p}}(\mathbb{R}^n_+ \times \mathbb{R}^n))} \leq C(E, M),\]

(A.7)

which implies that there exists a partition of the right half of \(I\) at \(t_0:\)

\[t_0 < t_1 < \cdots < t_N, \quad I_j = (t_j, t_{j+1}), \quad I \cap (t_0, \infty) = (t_0, t_N),\]

such that \(N \leq C(L, \delta)\) and for any \(j = 0, 1, \cdots, N - 1\), we have

\[\|\tilde{u}\|_{L^2_t(L^{\frac{2(p+1)}{p}}(\mathbb{R}^n_+ \times \mathbb{R}^n))} \leq \delta \ll 1.\]

(A.8)

The estimate on the left half of \(I\) at \(t_0\) is analogue, and we omit it.

Let

\[\gamma(t) = u(t) - \tilde{u}(t),\]

(A.9)

and
\[ \gamma(t) = e^{i(t-t_0)\Delta^j} \gamma(t_j), \quad 0 \leq j \leq N-1, \quad (A.10) \]
then \( \gamma \) satisfies the following difference equation
\[
\begin{cases}
(i\partial_t + \Delta^j)\gamma = -f(\tilde{u} + \gamma) + f(\tilde{a}) + eq(u) - eq(\tilde{a}), \\
\gamma(t_j) = \gamma(t_j),
\end{cases}
\]
which implies that
\[
\begin{align*}
\gamma(t) &= \gamma(t_j) - i \int_{t_j}^{t} e^{i(t-s)\Delta^j}(-f(\tilde{u} + \gamma) + f(\tilde{a}) + eq(u) - eq(\tilde{a}))ds, \\
\gamma(t) &= \gamma(t_j) - i \int_{t_j}^{t} e^{i(t-s)\Delta^j}(-f(\tilde{u} + \gamma) + f(\tilde{a}) + eq(u) - eq(\tilde{a}))ds.
\end{align*}
\]
It follows from the Strichartz estimate and nonlinear estimate (2.17) that
\[
\begin{align*}
\|\gamma - \gamma\|_{L^2(I)} + \|\gamma_{j+1} - \gamma\|_{L^2(I)} &\lesssim \||\nabla|^k-1(f(\tilde{u} - \gamma) + f(\tilde{a}))\|_{L^{\frac{2k}{k+1}}(I \times \mathbb{R}^d)} + \||\nabla|^k-1(eq(u), eq(\tilde{a}))\|_{L^{\frac{2k}{k+1}}(I \times \mathbb{R}^d)} \\
&\lesssim \sum_{k=1}^{p+1} \|\gamma\|_{L^2(I)} \|\tilde{u}\|_{L^2(I)}^{1-k} + \||\nabla|^k-1(eq(u), eq(\tilde{a}))\|_{L^{\frac{2k}{k+1}}(I \times \mathbb{R}^d)} \\
&\lesssim \sum_{k=1}^{p+1} \|\gamma\|_{L^2(I)} \|\tilde{u}\|_{L^2(I)}^{1-k} + \varepsilon.
\end{align*}
(A.11)
\]
Therefore, assuming that
\[
\|\gamma\|_{L^2(I)} \lesssim \delta \ll 1, \quad \forall \ j = 0, 1, \ldots, N-1, \quad (A.12)
\]
then by (A.8) and (A.11), we have
\[
\|\gamma\|_{L^2(I)} + \|\gamma_{j+1}\|_{L^2(I_{j+1}, I_{j})} \leq C\|\gamma\|_{L^2(I_{j}, I_{j})} + \varepsilon, \quad (A.13)
\]
for some absolute constant \( C > 0 \). By (A.3) and iteration on \( j \), we obtain
\[
\|\gamma\|_{L^2(I)} \leq (2C)^j \varepsilon \ll \frac{\delta}{2}, \quad (A.14)
\]
if we choose \( \varepsilon_1 \) sufficiently small. Hence the assumption (A.12) is justified by continuity in \( t \) and induction on \( j \). Then repeating the estimate (A.11) once again, we can get the critical-norm estimate on \( \gamma \), which implies the Strichartz estimates on \( u \). This concludes the proof of this lemma. \( \square \)

### A.4. Perturbation II: \( p \) not even

In this section, we will establish the perturbation theory of the solution of (1.1) with \( p \) being not an even integer. We restate the perturbation lemma as follows.

**Lemma A.2 (Perturbation lemma, \( p \) not even).** Assume that \( p \) is not an even integer and \( 1 \leq s_0 < 2 + \frac{p}{2} \). Let \( I \) be a compact time interval and \( u, \tilde{u} \) satisfy
\[
\begin{align*}
(i\partial_t + \Delta^j)u &= -f(u) + eq(u) \\
(i\partial_t + \Delta^j)\tilde{u} &= -f(\tilde{u}) + eq(\tilde{u})
\end{align*}
\]
for some function $eq(u), eq(\tilde{u})$, and $f(u) = |u|^p u$. Assume that for some constants $M, E > 0$, we have
\[ \|u\|_{L^\infty_t(H^s_x(\mathbb{R}^3))} + \|\tilde{u}\|_{L^\infty_t(H^s_x(\mathbb{R}^3))} \leq E, \] (A.15)
\[ S_t(\tilde{u}) \leq M, \] (A.16)
Let $t_0 \in I$, and let $u(t_0)$ be close to $\tilde{u}$ in the sense that
\[ \|u(t_0) - \tilde{u}(t_0)\|_{H^s} \leq \varepsilon, \] (A.17)
where $0 < \varepsilon < \varepsilon(M, E)$ is a small constant. Assume also that we have smallness conditions
\[ \|\nabla|^{k-1}(eq(u), eq(\tilde{u}))\|_{L^2_tL^r_x(\mathbb{R}^3)} \leq \varepsilon, \] (A.18)
where $\varepsilon$ is as above.
Then we conclude that
\[ S_t(u - \tilde{u}) \leq C(M, E)\varepsilon^{c_1}, \]
\[ \|\nabla|^{k}(u - \tilde{u})\|_{S_t(I)} \leq C(M, E)\varepsilon^{c_2}, \]
\[ \|\nabla|^{k}u\|_{S_t(I)} \leq C(M, E), \] (A.19)
where $c_1, c_2$ are positive constants that depend on $d, p, E$ and $M$.

The proof of the above lemma with $p > s_c - 1$ is similar to lemma 1 based on the use of the standard Strichartz estimates. However this proof cannot be applied directly to $p \leq s_c - 1$. The main reason for this is that for $p \leq s_c - 1$ the derivative of the nonlinearity is no longer Lipschitz continuous in the standard Strichartz space. In [39], Tao and Visan first overcame this problem in the context of the energy-critical NLS in dimensions $d > 6$ by making use of certain ‘exotic Strichartz’ spaces which have the same scaling with standard Strichartz space but a lower derivative. Later, Killip and Visan simplified the proof in [23] where stability is established in Sobolev Strichartz spaces where they utilized the fractional chain rule.

Therefore, we always assume that $p \geq s_c - 1$. We give a sketch proof by a similar argument as in [19]. First, it is useful to define several spaces and give estimates of the nonlinearities in terms of these spaces. Given $s := \frac{d}{2}$, define
\[ \|u\|_{X(I)} := \|u\|_{L^2_tL^2_x(\mathbb{R}^3)} \triangleq \|u\|_{L^\infty_tL^\infty_x(\mathbb{R}^3)} \leq \|u\|_{L^2_tL^\infty_x(\mathbb{R}^3)} \]
\[ \|u\|_{Y(I)} := \|\nabla|^{k}u\|_{L^2_tL^\infty_x(\mathbb{R}^3)}, \quad r_1 = \frac{2rd}{2d - r_0(2s_c - p)} \]
\[ \|F\|_{Y(I)} := \|\nabla|^{k}F\|_{L^2_tL^\infty_x(\mathbb{R}^3)}, \quad \frac{1}{r_1} + \frac{1}{r_1} = 1, \] (A.20)
where $(q_0, r_0) = \left(\frac{4p(p + 2)}{p^2 - pd(d - 4)} + \frac{d(p + 2)}{d - p + s_c(p + 2)}, \frac{d(p + 2)}{d - p + s_c(p + 2)}\right)$, $2 < r_0 < \frac{d + 4}{4} - p < q_0 < +\infty$. It is easy to check that $(q_0, r_0, r_1, r_1)$ satisfies
(1) $(q_0, r)$: $s_c$-admissible pair, that is $\frac{4}{q_0} = d\left(\frac{1}{2} - \frac{1}{r}\right) - s_c = \frac{4}{p} - \frac{d}{r}$.
(2) $(q_0, r_1)$: $(s_c - s)$-admissible pair, that is $\frac{4}{q_0} = d\left(\frac{1}{2} - \frac{1}{r}\right) - (s_c - s)$.
(3) Nonlinear estimate

\[ \| f(u) \|_{V(I)} \lesssim \| u \|_{X(I)} \| u \|_{\dot{X}^p_t(I)} \lesssim \| u \|_{\dot{X}^{p+1}_t(I)} \]  \tag{A.21}

requires \( \frac{1}{r_1} = \frac{1}{r_2} + \frac{p}{r} \).

(4) ‘Exotic Strichartz estimate’ Hardy–Littlewood–Sobolev requires:

\[ 1 + \frac{1}{q_0} = d \left( \frac{1}{4} - \frac{1}{2r_1} \right) + \frac{p + 1}{q_0} . \]

It is easy to verify that the Sobolev embedding relations

\[ \| u \|_{\dot{X}^p_t(I)} \lesssim \| u \|_{X(I)} \lesssim \| \nabla^k u \|_{\dot{S}^0(I)} \]  \tag{A.22}

and interpolation implies that there exist \( 0 < \theta_1, \theta_2 < 1 \) such that

\[ \| u \|_{X(I)} \lesssim \| u \|_{L_{t,x}^b(I \times \mathbb{R}^d)} \| \nabla^k u \|_{\dot{S}^0(I)}^{1 - \theta_1} \]  \tag{A.23}

\[ \| u \|_{L_{t,x}^b(I \times \mathbb{R}^d)} \lesssim \| u \|_{X(I)} \| \nabla^k u \|_{\dot{S}^0(I)}^{-\theta_2} \]  \tag{A.24}

Also, as a direct consequence of Hardy–Littlewood–Sobolev inequality, we have the following ‘exotic Strichartz estimate’.

**Lemma A.3 (Exotic Strichartz estimate).** Let \( I \) be a compact time interval containing \( t_0 \), then

\[ \int_{t_0}^t e^{i(\theta - r)\Delta} F(s) \, ds \|_{X(I)} \lesssim \| F \|_{V(I)} . \]  \tag{A.25}

**Proof.** It follows from the dispersive estimate (2.3) that

\[ \| e^{i(\theta - r)\Delta} F(s) \|_{L_{t,x}^1} \lesssim |t - s|^{-\frac{pd - r \rho_0}{4r_0}} \| F(s) \|_{L_{t,x}^{p,d}} . \]

This together with the Hardy–Littlewood–Sobolev inequality yields that

\[ \int_{t_0}^t e^{i(\theta - r)\Delta} F(s) \, ds \|_{L_{t,x}^1(I \times \mathbb{R}^d)} \lesssim \int_{t_0}^t \| e^{i(\theta - r)\Delta} F(s) \|_{L_{t,x}^{p,d}} \, ds \|_{L^{p,d}(I)} \lesssim \int_{t_0}^t |t - s|^{-\frac{pd - r \rho_0}{4r_0}} \| F(s) \|_{L_{t,x}^{p,d}} \, ds \|_{L^{p,d}(I)} \lesssim \| F \|_{L_{t,x}^{p,d}(I \times \mathbb{R}^d)}. \]

**Lemma A.4 (Nonlinear estimates).** Let \( d \geq 9 \), \( 1 \leq \kappa < 2 + p \), and \( I \) be a time interval. Then

\[ \| f_1(u + v) \|_{Y(I)} + \| f_2(u + v) \|_{Y(I)} \lesssim \left( \| u \|_{X(I)} \| \nabla^k u \|_{\dot{S}^0(I)}^{\kappa} + \| v \|_{X(I)} \| \nabla^k v \|_{\dot{S}^0(I)}^{\kappa} \right) \| \omega \|_{X(I)} , \]  \tag{A.26}

and there exists \( \beta \in (0, p) \) such that...
\[ \left\| \nabla^\nu u \right\|_{L^2(\mathbb{R}^d)} \leq \left\| \nabla^\nu u \right\|_{L^2(\mathbb{R}^d)} \left[ \left\| u \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla^\nu u \right\|_{L^2(\mathbb{R}^d)} \left( \left\| u \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla^\nu u \right\|_{L^2(\mathbb{R}^d)} \right) \right] \]

\[ \left\| v \right\|_{L^2(\mathbb{R}^d)} \leq \left\| v \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla^\nu u \right\|_{L^2(\mathbb{R}^d)} \left( \left\| u \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla^\nu u \right\|_{L^2(\mathbb{R}^d)} \right) \]

\[ \left\| v \right\|_{L^2(\mathbb{R}^d)} \leq \left\| v \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla^\nu u \right\|_{L^2(\mathbb{R}^d)} \left( \left\| u \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla^\nu u \right\|_{L^2(\mathbb{R}^d)} \right) \]

(A.27)

**Proof.** The proof of (A.26): It suffices to prove the first term on the left-hand side, as the second term can be estimated in the same way. Using (2.14) and (A.22), we derive

\[ \left\| f(x) \nu u \right\|_{L^2(\mathbb{R}^d)} \leq \left\| f(x) \nu u \right\|_{L^2(\mathbb{R}^d)} \left( \left\| u \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla^\nu u \right\|_{L^2(\mathbb{R}^d)} \right) \]

\[ \left\| v \right\|_{L^2(\mathbb{R}^d)} \leq \left\| v \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla^\nu u \right\|_{L^2(\mathbb{R}^d)} \left( \left\| u \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla^\nu u \right\|_{L^2(\mathbb{R}^d)} \right) \]

\[ \left\| v \right\|_{L^2(\mathbb{R}^d)} \leq \left\| v \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla^\nu u \right\|_{L^2(\mathbb{R}^d)} \left( \left\| u \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla^\nu u \right\|_{L^2(\mathbb{R}^d)} \right) \]

(A.28)

Hence, from (A.22), we know that the proof of (A.26) can be reduced to prove

\[ \left\| \nabla^\nu u \right\|_{L^2(\mathbb{R}^d)} \leq \left\| \nabla^\nu u \right\|_{L^2(\mathbb{R}^d)} \left( \left\| u \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla^\nu u \right\|_{L^2(\mathbb{R}^d)} \right) \]

\[ \left\| v \right\|_{L^2(\mathbb{R}^d)} \leq \left\| v \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla^\nu u \right\|_{L^2(\mathbb{R}^d)} \left( \left\| u \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla^\nu u \right\|_{L^2(\mathbb{R}^d)} \right) \]

\[ \left\| v \right\|_{L^2(\mathbb{R}^d)} \leq \left\| v \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla^\nu u \right\|_{L^2(\mathbb{R}^d)} \left( \left\| u \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla^\nu u \right\|_{L^2(\mathbb{R}^d)} \right) \]

(A.27)

**Case 1:** \( p \geq 1 \). Using (2.17) and (A.22), we get

\[ \left\| \nabla^\nu u \right\|_{L^2(\mathbb{R}^d)} \leq \left\| \nabla^\nu u \right\|_{L^2(\mathbb{R}^d)} \left( \left\| u \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla^\nu u \right\|_{L^2(\mathbb{R}^d)} \right) \]

**Case 2:** \( p < 1 \). By the fractional chain rule (2.16) with \( s = \frac{p}{2} < 1 \), Hölder inequality, Sobolev embedding and interpolation, we obtain

\[ \left\| \nabla^\nu u \right\|_{L^2(\mathbb{R}^d)} \leq \left\| \nabla^\nu u \right\|_{L^2(\mathbb{R}^d)} \left( \left\| u \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla^\nu u \right\|_{L^2(\mathbb{R}^d)} \right) \]

\[ \left\| v \right\|_{L^2(\mathbb{R}^d)} \leq \left\| v \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla^\nu u \right\|_{L^2(\mathbb{R}^d)} \left( \left\| u \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla^\nu u \right\|_{L^2(\mathbb{R}^d)} \right) \]

(A.28)

The proof of (A.27). When \( s \in (1, 2) \), it is easy to show (A.27) by lemma 2.5, Hölder’s inequality and Sobolev embedding. Now we consider \( s \in (2, p + 2) \). For \( p \leq 1 \), using the fundamental theorem of calculus and triangle inequality, we deduce that
\[ \| \nabla^{k-1} [f(u + v) - f(u)] \|_{L^2_t L^2_x (\mathbb{R}^d)} ^{\frac{2}{r_2}} \lesssim \| \nabla^{k-2} [\nabla v \cdot f'(u + v)] \|_{L^2_t L^2_x (\mathbb{R}^d)} ^{\frac{2}{r_2}} + \| \nabla^{k-2} [\nabla u \cdot (f'(u + v) - f'(u))] \|_{L^2_t L^2_x (\mathbb{R}^d)} ^{\frac{2}{r_2}}, \]  \tag{A.29}

One one hand, by lemma 2.14, the fractional chain rule (2.16), Hölder inequality and interpolation, we obtain
\[
\| \nabla^{k-2} [\nabla v \cdot |v|^p] \|_{L^2_t L^2_x (\mathbb{R}^d)} ^{\frac{2}{r_2}} \lesssim \| \nabla^{k-1} v \|_{L^2_t L^2_x (\mathbb{R}^d)} \| v \|_{L^2_t L^2_x (\mathbb{R}^d)} ^{\frac{p}{r_2}} \lesssim \| \nabla^{k-1} v \|_{L^2_t L^2_x (\mathbb{R}^d)} \| v \|_{L^2_t L^2_x (\mathbb{R}^d)} ^{\frac{p}{r_2}},
\]
where
\[
\begin{cases}
\frac{1}{2} = \frac{1}{q_2} + \frac{p}{q_0}, \Rightarrow q_2 = \frac{2q_0}{q_0 - p}, \\
\frac{d + 2}{2} = \frac{d}{r_2} + \frac{d - p}{p + 2}, \\
\frac{4}{q_2} = d \left( \frac{1}{2} - \frac{1}{r_2} \right), \Rightarrow r_2 = \frac{2qd}{(d - 4)d_0 + 4p}, \\
1 - \frac{d}{r_2} = -\frac{d}{r_2}.
\end{cases}
\]

Hence
\[
\| \nabla^{k-2} [\nabla v \cdot f'(u + v)] \|_{L^2_t L^2_x (\mathbb{R}^d)} \lesssim \| \nabla^{k-1} v \|_{L^2_t L^2_x (\mathbb{R}^d)} (\| u \|_{L^2_t L^2_x (\mathbb{R}^d)} + \| v \|_{L^2_t L^2_x (\mathbb{R}^d)} ) \| \nabla^{k-1} v \|_{L^2_t L^2_x (\mathbb{R}^d)} \| v \|_{L^2_t L^2_x (\mathbb{R}^d)} ^{\frac{p}{r_2}}.
\]

On the other hand, by lemma 2.6, Hölder inequality, interpolation and (A.22), one has
\[
\| \nabla^{k-2} [\nabla u \cdot (f'(u + v) - f'(u))] \|_{L^2_t L^2_x (\mathbb{R}^d)} \lesssim \| \nabla^{k-1} u \|_{\mathcal{S}^{k-1} (\mathbb{R}^d)} \| v \|_{L^2_t L^2_x (\mathbb{R}^d)} ^{\frac{p-1}{r_2}} \| \nabla^{k-1} u \|_{\mathcal{S}^{k-1} (\mathbb{R}^d)} \| v \|_{L^2_t L^2_x (\mathbb{R}^d)} ^{\frac{p-1}{r_2}} + \| \nabla^{k-1} u \|_{\mathcal{S}^{k-1} (\mathbb{R}^d)} \| v \|_{L^2_t L^2_x (\mathbb{R}^d)} ^{\frac{p-1}{r_2}},
\]
where \(\kappa - 1 < \alpha < p\). Letting \(\beta := \frac{\kappa - 1}{\alpha}\), we derive (A.27) for \(p \leq 1\). We can iterate the argument presented above to obtain (A.27) for \(p > 1\). This concludes the proof of this lemma. \(\square\)

Before we prove the perturbation, we first show the short-time perturbations.

**Lemma A.5 (Short-time perturbation).** Let \(d \geq 9\), \(p\) be not an even integer and \(1 \leq \kappa < 2 + p\). Let \(I\) be a compact time interval and \(u, \bar{u}\) satisfy
\[
(i\partial_t + \Delta^2) u = -f(u) + eq(u) \tag{1} \]
\[
(i\partial_t + \Delta^2) \bar{u} = -f(\bar{u}) + eq(\bar{u}) \tag{2}
\]

Before we prove the perturbation, we first show the short-time perturbations.
for some function \(eq(u), eq(\tilde{u})\) and \(f(u) = |u|^p u\). Assume that for some constants \(E > 0\), we have
\[
\|u\|_{L_{t,x}^{\infty}(\mathbb{R}^n)} + \|\tilde{u}\|_{L_{t,x}^{\infty}(\mathbb{R}^n)} \leq E. \tag{A.30}
\]
Moreover, for \(t_0 \in I\), assume that smallness conditions
\[
\|\tilde{u}(t_0)\|_{H^s} \leq \delta, \tag{A.31}
\]
\[
\|
\|f(u) - f(\tilde{u})\|_{H^s} \leq \varepsilon \tag{A.32}
\]
\[
|||\nabla|^{k-1}(eq(u), eq(\tilde{u}))\|_{L_{t,x}^{\infty}(\mathbb{R}^n)} \leq \varepsilon \tag{A.33}
\]
for some small \(0 < \varepsilon < \varepsilon_0 = \varepsilon_0(M, E)\) and \(0 < \delta < \varepsilon_0(E)\). Then we conclude that
\[
\|u - \tilde{u}\|_{X(I)} \lesssim \varepsilon \tag{A.34}
\]
\[
\|\|\nabla|^k (u - \tilde{u})\|_{S(I)} \lesssim \varepsilon^{(d,p)} \tag{A.35}
\]
\[
|||\nabla|^p u\|_{S(I)} \lesssim E \tag{A.36}
\]
\[
|||\nabla|^k (f(u) - f(\tilde{u}))\|_{S(I)} \lesssim \varepsilon \tag{A.37}
\]
\[
|||\nabla|^{k-1}(f(u) - f(\tilde{u}))\|_{L_{t,x}^{\infty}(\mathbb{R}^n)} \lesssim \varepsilon^{(d,p)} \tag{A.38}
\]
for some positive constant \(c(d, p)\).

**Proof.** Step 1: We claim that \(|||\nabla|^k \tilde{u}\|_{S(I)} \lesssim E\).

Indeed, using the Strichartz estimate, corollary 2.1, (A.31) and (A.33), we get
\[
|||\nabla|^k \tilde{u}\|_{S(I)} \lesssim \|\tilde{u}\|_{L_{t,x}^{\infty}} + \|\nabla|^k f(\tilde{u})\|_{S(I)} + \|\nabla|^k e\|_{S(I)} \lesssim E + \|u\|_{L_{t,x}^{\infty}(\mathbb{R}^n)} + \|\nabla|^k \tilde{u}\|_{S(I)} + \|\nabla|^k e\|_{S(I)} \lesssim E + \delta^{\rho_d} \|\|\nabla|^k \tilde{u}\|_{S(I)}^{1 + p(1 - \theta_d)} + \varepsilon.
\]

Hence, by the standard bootstrap argument, and choosing \(\delta, \varepsilon_0\) to be sufficiently small, we obtain
\[
|||\nabla|^k \tilde{u}\|_{S(I)} \lesssim E \tag{A.39}
\]

Step 2: We claim that \(\|u\|_{X(I)} \lesssim \delta\).

By lemma 3, (A.21), (A.31) and (A.33), one has
\[
\|e^{it \Delta} u(t_0)\|_{X(I)} \lesssim \|\tilde{u}\|_{X(I)} + \|f(\tilde{u})\|_{X(I)} + \|\nabla|^k eq(\tilde{u})\|_{S(I)} \lesssim \delta + \delta^{p+1} + \varepsilon \lesssim \delta.
\]
Combining this with the triangle inequality, (A.22), the Strichartz estimate and (A.32), we derive
\[
\|e^{it \Delta} u(t_0)\|_{X(I)} \lesssim \|e^{it \Delta} \tilde{u}(t_0)\|_{X(I)} + \|e^{it \Delta} (u - \tilde{u})(t_0)\|_{X(I)} \lesssim \delta + \|u(t_0) - \tilde{u}(t_0)\|_{H^s} \lesssim \delta.
\]
On the other hand, by lemmas A.3 and A.4, we get
Thus, we obtain by the bootstrap argument
\[ \|u\|_{X(I)} \lesssim \delta. \]  
(A.40)

**Step 3:** Next we prove the following iteration formula
\[ \|\omega\|_{X(I)} \lesssim \varepsilon + \|\nabla^k \omega\|_{Y(\delta)}^{\frac{p}{k-\mu(1)}} \|\omega\|_{X(I)}^{\frac{p(k-1)}{k-\mu(1)}}, \]  
(A.41)
\[ \|\nabla^k \omega\|_{Y(\delta)} \lesssim \varepsilon + \|\omega\|_{X(I)}^{\frac{p(k-1)}{k}} + \|\nabla^k \omega\|_{Y(\delta)}^{\frac{p}{k-\mu(1)}} \|\omega\|_{X(I)}^{\frac{k-1}{k}}. \]  
(A.42)
where \( \omega = u - \tilde{u} \) satisfies the difference equation
\[ \begin{cases} \partial_t \omega + \Delta \omega = -f(\tilde{u} + \omega) + f(\tilde{u}) + eq(u) - eq(\tilde{u}), \\ \omega(t_0, x) = u(t_0, x) - \tilde{u}(t_0, x) \in H^s(\mathbb{R}^d). \end{cases} \]  
(A.43)
Using lemma A.3, the Strichartz estimate, (A.31) and (A.32), we get
\[ \|\omega\|_{X(I)} \lesssim \|u(t_0) - \tilde{u}(t_0)\|_{H^s} + \|\nabla^k \omega\|_{Y(\delta)} \|f(u) - f(\tilde{u})\|_{Y(I)} \lesssim \varepsilon + \|f(u) - f(\tilde{u})\|_{Y(I)}. \]  
(A.44)

**The estimate of** \( \|f(u) - f(\tilde{u})\|_{Y(I)} \): From lemma 4, (A.31) and step 1: (A.39), we know that
\[ \|f(u) - f(\tilde{u})\|_{Y(I)} \leq \left( \|\tilde{u}\|_{X(I)}^{\frac{p(k-1)}{k-\mu(1)}} \|\nabla^k \tilde{u}\|_{Y(\delta)}^{\frac{p}{k-\mu(1)}} \|\omega\|_{X(I)}^{\frac{p(k-1)}{k}} + \|\nabla^k \omega\|_{Y(\delta)}^{\frac{p}{k-\mu(1)}} \|\omega\|_{X(I)}^{\frac{k-1}{k}} \right) \|\omega\|_{X(I)} \lesssim \delta^{\frac{p(k-1)}{k}} \|\omega\|_{X(I)} + \|\nabla^k \omega\|_{Y(\delta)}^{\frac{p}{k-\mu(1)}} \|\omega\|_{X(I)}^{\frac{k-1}{k}}. \]  
(A.45)
Plugging this into (A.44), and taking \( \delta \) sufficiently small, we have
\[ \|\omega\|_{X(I)} \lesssim \varepsilon + \|\nabla^k \omega\|_{Y(\delta)}^{\frac{p}{k-\mu(1)}} \|\omega\|_{X(I)}^{\frac{k-1}{k}}. \]  
(A.46)
This is (A.41).
On the other hand, using the Strichartz estimate, (A.32) and (A.33), we obtain
\[ \|\nabla^k \omega\|_{Y(\delta)} \lesssim \|u(t_0) - \tilde{u}(t_0)\|_{H^s} + \|\nabla^k \omega\|_{Y(\delta)}^{\frac{p}{k-\mu(1)}} \|\omega\|_{X(I)}^{\frac{k-1}{k}} + \|\nabla^k \omega\|_{Y(\delta)}^{\frac{p}{k-\mu(1)}} \|\omega\|_{X(I)}^{\frac{k-1}{k}} \lesssim \varepsilon + \|\nabla^k \omega\|_{Y(\delta)}^{\frac{p}{k-\mu(1)}} \|\omega\|_{X(I)}^{\frac{k-1}{k}}. \]  
(A.47)

**The estimate of** \( \|\nabla^k \omega\|_{Y(\delta)} \): By (A.27), (A.31), step 1: (A.39) and step 2: (A.40), one has
\[
\|\nabla^\nu f(u) - f(\bar{u})\|_{L^2_t L^p_x(I \times \mathbb{R}^d)} \\
\lesssim \|\nabla^\nu \omega\|_{S^0(I)} \left(\delta + \delta^\nu \left(1 + \frac{1}{2}E^\nu\right)\right) + E\delta^\beta \|\omega\|_{\mathcal{X}(I)}^{p-\beta} + \delta^{1-\frac{1}{2}E^\nu} \|\nabla^\nu \omega\|_{S^0(I)}^{1-\frac{1}{2}E^\nu} \|\omega\|_{\mathcal{X}(I)}^{\nu-1+\frac{1}{2}E^\nu}.
\] (A.48)

Plugging this into (A.47), we have
\[
\|\nabla^\nu \omega\|_{S^0(I)} \lesssim \varepsilon + \frac{1}{E}\|\omega\|_{\mathcal{X}(I)}^{p-\beta} + \|\nabla^\nu \omega\|_{S^0(I)}^{1-\frac{1}{2}E^\nu} \|\omega\|_{\mathcal{X}(I)}^{\nu-1+\frac{1}{2}E^\nu}.
\] (A.49)

This is (A.42). Putting this into (A.46), and by the bootstrap argument, we conclude (A.34) and (A.35).

Then, it is easy to show (A.36) by the triangle inequality, (A.39) and (A.35).

Using (A.34), (A.35), (A.45) and (A.45), we obtain (A.37) and (A.38). We conclude the proof of this lemma. □

Now we turn to prove lemma A.2.

**The proof of lemma A.2:** First, we claim
\[
\|\nabla^{\mu} \tilde{u}\|_{S^0(J)} \leq C(E, M). \tag{A.50}
\]

In fact, from the hypothesis (A.15), we know that one can subdivide time interval \(I\) by \(I = \bigcup_j I_j, \ I_j = [t_j, t_{j+1}], \ 0 \leq j < J_0 = J_0(M, \eta)\), such that
\[
\|\tilde{u}\|_{L^{p_{\mu+\eta}}(t_j, t_{j+1} \times \mathbb{R}^d)} \leq \eta,
\]
where \(\eta > 0\) is sufficiently small to be determined. Using the Strichartz estimate, fractional chain rule corollary 2.1, (A.15) and (A.17), we get
\[
\|\nabla^{\nu} \tilde{u}\|_{S^0(I_j)} \lesssim \|\tilde{u}(t_j)\|_{H^\nu} + \|\nabla^{\nu-1} \bar{q}(\tilde{u})\|_{L^{p_{\mu+\eta}}(t_j, t_{j+1} \times \mathbb{R}^d)} + \|\nabla^{\nu-1} \bar{q}(\tilde{u})\|_{L^{\infty}(t_j, t_{j+1} \times \mathbb{R}^d)}
\lesssim E + \varepsilon + \|\tilde{u}\|_{L^{p_{\mu+\eta}}(t_j, t_{j+1} \times \mathbb{R}^d)} \|\nabla^{\nu} \tilde{u}\|_{S^0(I_j)}
\lesssim E + \varepsilon + \eta^p \|\nabla^{\nu} \tilde{u}\|_{S^0(I_j)}.
\]
Thus, by the bootstrap argument, we have
\[
\|\nabla^{\nu} \tilde{u}\|_{S^0(I_j)} \lesssim E + \varepsilon.
\]

Summing the above bound over all subintervals \(I_j\), we get the claim (A.50). In particular, from Sobolev embedding (A.22), we know that
\[
\|\tilde{u}\|_{K(I)} \leq C(E, M). \tag{A.51}
\]
Hence, we can subdivide time interval \(I\) by \(I = \bigcup_j I_j, \ I_j = [t_j, t_{j+1}], \ 0 \leq j < J_1 = J_1(M, \eta)\), such that
\[
\|\tilde{u}\|_{K(I_j)} \leq \delta,
\]
where \(\delta\) is as in lemma A.5.
Therefore, we can apply lemma A.5 to each $I_j$, and so, $\forall \ 0 \leq j < J$, $0 < \varepsilon < \varepsilon_1$,

\[
\begin{align*}
\|u - \tilde{u}\|_{X(I_j)} &\leq \varepsilon, \\
\|\nabla|^{k}(u - \tilde{u})\|_{S(I_j)} &\leq \varepsilon^{(d,p)}, \\
\|\nabla|^{k}u\|_{S(I_j)} &\leq E, \\
\|f(u) - f(\tilde{u})\|_{X(I_j)} &\leq \varepsilon,
\end{align*}
\]  
(A.52) provided that one can prove for any $0 \leq j < J$

\[
\|u(t_j) - \tilde{u}(t_j)\|_{H^s(I_j)} \leq C \varepsilon^{1+}\|u\|_{H^s}\|
\]  
(A.53)

Indeed, by the Strichartz estimate and the inductive hypothesis, one has

\[
\begin{align*}
\|u(t_j) - \tilde{u}(t_j)\|_{H^s(I_j)} &\leq \|u_0 - \tilde{u}_0\|_{C^{s}(I_j)} + \|\nabla|^{k-1}(eq(u), eq(\tilde{u}))\|_{L^2(I_j \times \mathbb{R}^d)} \\
&+ \|\nabla|^{k-1}[f(\tilde{u} + \omega) - f(\tilde{u})]\|_{L^2(I_j \times \mathbb{R}^d)} \\
&\leq \varepsilon + \sum_{k=0}^{j-1} C \varepsilon^{1+}\|u\|_{H^s}\|
\end{align*}
\]

Taking $\varepsilon_1$ sufficiently small compared to $\varepsilon_0$, we derive (A.53).

Summing the bounds in (A.52) over all subintervals $I_j$, we conclude lemma A.2.

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