String Geometry and Non-perturbative Formulation of String Theory

Matsuo Sato

Department of Natural Science, Faculty of Education, Hirosaki University
Bunkyo-cho 1, Hirosaki, Aomori 036-8560, Japan

Abstract

We define string geometry: spaces of superstrings including the interactions, their topologies, charts, and metrics. Trajectories in asymptotic processes on the topological space of strings reproduce the right moduli space of the super Riemann surfaces in a target manifold. Based on the string geometry, we define Einstein-Hilbert action coupled with gauge fields, and formulate superstring theory non-perturbatively by summing over metrics and the gauge fields on the spaces of strings. This theory does not depend on backgrounds. The theory has a supersymmetry as a part of the diffeomorphisms symmetry on the superstring manifolds. We derive the all-order perturbative scattering amplitudes that possess the super moduli in type IIA, type IIB and SO(32) type I superstring theories from the single theory, by considering fluctuations around fixed backgrounds representing type IIA, type IIB and SO(32) type I perturbative vacua, respectively. The theory predicts that we can see a string if we microscopically observe not only a particle but also a point in the space-time. That is, this theory unifies particles and the space-time.
1 Introduction

In the T-duality and its generalization, the mirror symmetry, there is a coincidence between geometric invariants of two different manifolds. It is thought that the reason for this is that the spaces observed by the strings are the same although they are in the different target manifolds. Therefore, the space observed by the strings, which is invariant under the T-duality and mirror transformations, will be a geometric principle of string theory. A moduli space in a target manifold, which is a collection of embedding functions of the Riemann surfaces $X^\mu(\sigma, \tau)$, is invariant under the T-duality transformations. Actually, the T-duality rule $\partial_a X^\mu(\sigma, \tau) = i \epsilon_{ab} \partial^b X'^\mu(\sigma, \tau)$ gives an one-to-one correspondence between the embedding functions of the Riemann surfaces $X^\mu(\sigma, \tau)$ and $X'^\mu(\sigma, \tau)$. Moreover, the Riemann surfaces in the target manifold can be generated by trajectories in a space of strings. Therefore, a
space of strings will be the geometric principle.

Furthermore, string theory as a quantum gravity also suggests that a space of strings will be a geometric principle of string theory as follows. It has not succeeded to obtain ordinary relativistic quantum gravity that is defined by a path integral over metrics on a space representing the spacetime itself because of ultraviolet divergences. The reason would be impossibility to regard points as fundamental constituents of the spacetime because the spacetime itself fluctuates at the Plank scale. Thus, it is reasonable to define quantum gravity by a path integral over metrics on a space that consists of strings, by making a point have a structure of strings. In fact, perturbative strings are shown to suppress the ultraviolet divergences in quantum gravity.

In this paper, we geometrically define a space of superstrings including the effect of interactions. For this purpose, here we first review how such spaces of strings are defined in string field theories. In these theories, after a free loop space of strings are prepared, interaction terms of strings in actions are defined. In other words, the spaces of strings are defined by deforming the ring on the free loop space. Geometrically, the space of strings is defined by deformation quantization of the free loop space as a noncommutative geometry. Actually, in Witten’s cubic open string field theory \[5\], the interaction term is defined by using the $*$-product of noncommutative geometry. On the other hand, we adopt different approach, namely (infinite-dimensional) manifold theory\[2\]. We do not start with a free loop space, but we define a space of strings including the effect of interactions from the beginning. The criterion to define a topology, which represents how near the strings are, is that trajectories in asymptotic processes on the space of strings reproduce the right moduli space of the super Riemann surfaces in a target manifold. We need Riemannian geometry

\[1\] Recently, in the homological mirror symmetry \[1\], it is shown in \[2\] that the moduli space of the pseudo holomorphic curves in the A-model on a symplectic torus is homeomorphic to a moduli space of Feynman diagrams in the configuration space of the morphisms in the B-model on the corresponding elliptic curve. Therefore, a dynamical and non-perturbative generalization of the moduli space of the pseudo holomorphic curves will be the geometric principle of string theory. Here we discuss this generalization. First, a moduli space of pseudo holomorphic curves is defined even in closed string theory in \[3\]. Moreover, the moduli space can be defined by restricting a moduli space of curves, which is not necessarily holomorphic, to the holomorphic sector \[4\]. Because this is a restriction to the topological string theory, the moduli space of curves is the dynamical generalization. Furthermore, the curves can be generated by trajectories in a space of strings. Therefore, a space of strings will be the dynamical and non-perturbative generalization, that is the geometric principle.

\[2\] See \[6\] as an example of text books for infinite-dimensional manifolds
naturally for fields on the space of strings because it is not flat.\(^3\)

By adopting the space of superstrings as geometric principle, we formulate superstring theory non-perturbatively. That is, we formulate the theory by summing over metrics on the space of strings. As a result, the theory is independent of backgrounds.

The organization of the paper is as follows. In section 2, we define string geometry and its Einstein-Hilbert action coupled with gauge fields. In section 3, we solve the equations of motion and obtain a string geometry solution that represents perturbative string vacuum. We derive the propagator of the fluctuations around the solution. Then, we move to the first quantization formalism, and we derive the all-order perturbative scattering amplitudes that possess its moduli in string theory. We extend these results to a supersymmetric theory in section 4, to a theory including open strings in section 5, and to a supersymmetric theory including open superstrings in section 6. The theory in section 6 is a non-perturbatively formulated superstring theory. We derive the all-order perturbative scattering amplitudes that possess the super moduli in type IIA, type IIB and SO(32) type I superstring theories from the single theory, by considering fluctuations around fixed backgrounds representing type IIA, type IIB and SO(32) type I perturbative vacua, respectively. In section 7, we discuss a relation between the superstring geometry and supersymmetric matrix models of a new type. In section 8, based on superstring geometry, we formulate and study a theory that manifestly possesses the $SO(32)$ and $E_8 \times E_8$ heterotic perturbative vacua. We expect that this theory is equivalent to the theory in section 6, which manifestly possesses the type IIA, type IIB and $SO(32)$ type I perturbative vacua. We conclude in section 9 and discuss in section 10.

2 String geometry

Let us define unique global time on a Riemann surface $\bar{\Sigma}$ with punctures $P^i \ (i = 1, \cdots , N)$ in order to define string states by world-time constant lines. On $\bar{\Sigma}$, there exists an unique Abelian differential $dp$ that has simple poles with residues $f^i$ at $P^i$ where $\sum_i f^i = 0$, if it is appropriately normalized. An example of such normalizations is that $dp$ has purely

\(^3\)The spaces of strings in string field theories are different with those in string geometry because non-commutative geometry and Riemannian geometry describe different spaces.
imaginary and real periods with respect to A-cycles and B-cycles, respectively. The Abelian differentials are called 1st kind if there is no puncture on \( \bar{\Sigma} \), whereas they are called 3rd kind in general. Global coordinates are defined by \( \tilde{w} = \tilde{\tau} + i \tilde{\sigma} := \int P \, dp \) at any point \( P \) on \( \bar{\Sigma} \) \([7,8]\). In particular, \( \tilde{\tau} = -\infty \) at \( P^i \) with negative \( f^i \) and \( \tilde{\tau} = \infty \) at \( P^i \) with positive \( f^i \).

A contour integral on \( \tilde{\tau} \) constant line around \( P^i \): \( i \Delta \tilde{\sigma} = \oint \tilde{\sigma} \, dp = 2\pi i f^i \) indicates that the \( \tilde{\sigma} \) region around \( P^i \) is \( 2\pi f^i \). This means that \( \bar{\Sigma} \) around \( P^i \) represents a semi-infinite cylinder with radius \( f^i \). The condition \( \sum_i f^i = 0 \) means that the total \( \tilde{\sigma} \) region of incoming cylinders equals to that of outgoing ones if we choose the outgoing direction as positive. That is, the total \( \tilde{\sigma} \) region is conserved. In order to define the above global time uniquely, we fix the \( \tilde{\sigma} \) regions \( 2\pi f^i \) around \( P^i \).

We divide \( N^P \)'s to arbitrary two sets consist of \( N_- \) and \( N_+ \) \( P^i \)'s, respectively \((N_- + N_+ = N)\), then we divide -1 to \( N_- \) \( f^i \equiv \frac{\tilde{\tau}}{N_-} \) and 1 to \( N_+ \) \( f^i \equiv \frac{\tilde{\tau}}{N_+} \) equally for all \( i \).

Thus, under a conformal transformation, one obtains a Riemann surface \( \bar{\Sigma} \) that has coordinates composed of the global time \( \tilde{\tau} \) and the position \( \tilde{\sigma} \). Because \( \bar{\Sigma} \) can be a moduli of Riemann surfaces, any two-dimensional Riemannian manifold \( \Sigma \) can be obtained by \( \Sigma = \psi(\bar{\Sigma}) \) where \( \psi \) is a diffeomorphism times Weyl transformation.

Next, we will define a string state \( M_0 \). We consider a state \((\bar{\Sigma}, u, \tilde{\tau}_s)\) determined by an arbitrary map \( u \) from \( \bar{\Sigma} \) to a d-dimensional Riemannian manifold \( M \) and a \( \tilde{\tau} = \tilde{\tau}_s \) constant line. \( \bar{\Sigma} \) is a direct sum of \( N_\pm \) cylinders with radii \( f_i \) at \( \tilde{\tau} \equiv \mp \infty \). Thus, we define a string state as an equivalence class \([\Sigma, u, \tilde{\tau}_s \simeq \mp \infty] \) by a relation \((\Sigma, u, \tilde{\tau}_s \simeq \mp \infty) \sim (\Sigma', u', \tilde{\tau}_s \simeq \mp \infty) \) if \( N_\pm = N'_\pm \), \( f_i = f'_i \), and \( u|_{\tilde{\tau}_s \simeq \mp \infty} = u'|_{\tilde{\tau}_s \simeq \mp \infty} \) as in Fig. 1. Because \( \bar{\Sigma}|_{\tilde{\tau}_s} \simeq S_1 \times S_1 \times \cdots \times S_1 \) and \( u|_{\tilde{\tau}_s} : \Sigma|_{\tilde{\tau}_s} \rightarrow M \), \([\Sigma, u, \tilde{\tau}_s] \) represent many-body states of strings in \( M \) as in Fig. 2. A string space \( M_0 \) is defined by a collection of all the string states \( \{[\Sigma, u, \tilde{\tau}_s]\} \).

Here, we will define topologies of \( M_0 \). We define an \( \epsilon \)-open neighbourhood of \([\bar{\Sigma}, u, \tilde{\tau}_s] \) by

\[
U([\bar{\Sigma}, u, \tilde{\tau}_s], \epsilon) := \left\{ \bar{\Sigma}, \exp_u X, \tilde{\tau} \mid \sqrt{|\tilde{\tau} - \tilde{\tau}_s|^2 + \|X|_{\tilde{\tau}}^2 + \|X|_{\tilde{\tau}_s}^2} < \epsilon \right\},
\]

as in Fig. 3 \( X : \Sigma \rightarrow \mathbb{R}^{d-1,1} \) is a section of \( u^*TM \). The following statements do not depend on how to define the norm. An example is given by

\[
\|X|_{\tilde{\tau}}^2 := \int_{\bar{\Sigma}|_{\tilde{\tau}}} d\tilde{\sigma} \sqrt{h}_{\tilde{\tau}} \frac{1}{e^2_{\tilde{\tau}}} < \partial_{\theta} X|_{\tilde{\tau}}, \partial_{\theta} X|_{\tilde{\tau}}>,
\]

as in Fig. 3.
Figure 1: An equivalence class of a string state $[\Omega, u, \tau \simeq -\infty]$. If the cylinders and the embedding functions are the same at $\tau \simeq -\infty$, the states of strings at $\tau \simeq -\infty$ specified by the red lines $(\Sigma, u, \tau \simeq -\infty)$, $(\Sigma', u', \tau \simeq -\infty)$, and $(\Sigma'', u'', \tau \simeq -\infty)$ should be identified.

Figure 2: Various string states. The red and blue lines represent one-string and two-string states, respectively.
where $\bar{h}_{mn}(\bar{\sigma}, \bar{\tau})$ ($m, n = 0, 1$) is the worldsheet metric of $\bar{\Sigma}$. $\bar{e} := \sqrt{\bar{h}_{\bar{\sigma}\bar{\sigma}}}$. $\langle,\rangle$ is the inner product in $R^{d-1,1}$. $U([\bar{\Sigma}, u, \bar{\tau}_s \simeq \mp \infty], \epsilon) = U([\bar{\Sigma}', u', \bar{\tau}_s \simeq \mp \infty], \epsilon)$ consistently if $N_{\bar{\pi}} = N'_{\bar{\pi}}$, $f_i = f'_i$, and $u|_{\bar{\tau}_s \simeq \mp \infty} = u'|_{\bar{\tau}_s \simeq \mp \infty}$, and $\epsilon$ is small enough, because the $\bar{\tau}_s \simeq \mp \infty$ constant line traverses only cylinders overlapped by $\Sigma$ and $\Sigma'$. $U$ is defined to be an open set of $M_0$ if there exists $\epsilon$ such that $U([\bar{\Sigma}, u, \bar{\tau}_s], \epsilon) \subset U$ for an arbitrary point $[\bar{\Sigma}, u, \bar{\tau}_s] \in U$.

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{neighbourhood.png}
\caption{A neighbourhood. The green part represents a neighbourhood of the blue state.}
\end{figure}

Let $U$ be a collection of all the open sets $U$. The topology of $M_0$ satisfies the axiom of topology (i), (ii), and (iii).

(i) $\emptyset, M_0 \in U$

(ii) $U_1, U_2 \in U \Rightarrow U_1 \cap U_2 \in U$

(iii) $U_\lambda \in U \Rightarrow \cup_{\lambda \in \Lambda} U_\lambda \in U$.

**Proof.** (i) Clear.

(ii) If $U_1 \cap U_2 = \emptyset$, it is clear. Let us consider the case $U_1 \cap U_2 \neq \emptyset$. Because $U_1, U_2 \subset U$, there exist $\epsilon$ and $\epsilon'$ such that $U([\bar{\Sigma}, u, \bar{\tau}_s], \epsilon) \subset U_1$ and $U([\bar{\Sigma}, u, \bar{\tau}_s], \epsilon') \subset U_2$ for all $[\bar{\Sigma}, u, \bar{\tau}_s] \in U_1 \cap U_2$. Let $\epsilon'' := \min(\epsilon, \epsilon')$. Because $U([\bar{\Sigma}, u, \bar{\tau}_s], \epsilon'') \subset U([\bar{\Sigma}, u, \bar{\tau}_s], \epsilon) \subset U_1$ and $U([\bar{\Sigma}, u, \bar{\tau}_s], \epsilon'') \subset U([\bar{\Sigma}, u, \bar{\tau}_s], \epsilon') \subset U_2$, there exists $\epsilon''$ such that $U([\bar{\Sigma}, u, \bar{\tau}_s], \epsilon'') \subset U_1 \cap U_2$ for all $[\bar{\Sigma}, u, \bar{\tau}_s] \in U_1 \cap U_2$.

(iii) If $\cup_{\lambda \in \Lambda} U_\lambda = \emptyset$, it is clear. Let us consider the case $\cup_{\lambda \in \Lambda} U_\lambda \neq \emptyset$. For all $[\bar{\Sigma}, u, \bar{\tau}_s] \in \cup_{\lambda \in \Lambda} U_\lambda$, there exists $\lambda_0$ such that $[\bar{\Sigma}, u, \bar{\tau}_s] \in U_{\lambda_0}$. Because $U_{\lambda_0} \in U$, there is $\epsilon$ such that $U([\bar{\Sigma}, u, \bar{\tau}_s], \epsilon) \subset U_{\lambda_0}$. Then, $U([\bar{\Sigma}, u, \bar{\tau}_s], \epsilon) \subset \cup_{\lambda \in \Lambda} U_\lambda$ for all $[\bar{\Sigma}, u, \bar{\tau}_s] \in \cup_{\lambda \in \Lambda} U_\lambda$. \qed
By definition of the $\varepsilon$-open neighbourhood, arbitrary two string states on a connected Riemann surface are connected continuously. Thus, there is an one-to-one correspondence between a Riemann surface with punctures in $M$ and a curve parametrized by $\bar{\tau}$ from $\bar{\tau} = -\infty$ to $\bar{\tau} = \infty$ on $M_0$. That is, curves that represent asymptotic processes on $M_0$ reproduce the right moduli space of the Riemann surfaces in the target manifold.

By a general curve parametrized by $t$ on $M_0$, string states on different Riemann surfaces that have even different genera, can be connected continuously, for example see Fig. 4, whereas different Riemann surfaces that have different genera cannot be connected continuously in the moduli space of the Riemann surfaces in the target space. Therefore, the string geometry is expected to possess non-perturbative effects.

Next, we will define structures of manifold on the string topological space $M_0$. First, we define a model space $E$ such that

$$E := \{[\bar{\Sigma}, X, \bar{\tau}_s]\}.$$  

An $\varepsilon$-open neighbourhood of $[\bar{\Sigma}, X_s, \bar{\tau}_s]$ is defined by

$$U([\bar{\Sigma}, X_s, \bar{\tau}_s], \varepsilon) := \{[\bar{\Sigma}, X, \bar{\tau}] \mid \sqrt{|\bar{\tau} - \bar{\tau}_s|^2 + \|X|_{\bar{\tau}} - X_s|_{\bar{\tau}}\|^2 + \|X|_{\bar{\tau}_s} - X_s|_{\bar{\tau}_s}\|^2 < \varepsilon}\}. \quad (2.3)$$

Open sets of $E$ are defined as in the case of $M_0$. By these definitions, one can prove that the topology of $E$ satisfies the axiom of topology in the same way. Here, we define charts $(U([\bar{\Sigma}, u, \bar{s}_s], \varepsilon), \Psi)$ on $M_0$. First, the $\varepsilon$-open neighbourhoods $U([\bar{\Sigma}, u, \bar{s}_s], \varepsilon)$ cover $M_0$ because they are defined at any point $[\bar{\Sigma}, u, \bar{s}_s] \in M_0$. Next, $\Psi : U([\bar{\Sigma}, u, \bar{s}_s], \varepsilon) \rightarrow U([\bar{\Sigma}, 0, \bar{s}_s], \varepsilon)$ is a homeomorphism from an open set in $M_0$ to that in $E$. 

Figure 4: A continuous trajectory. In case of general $\bar{\tau}(t)$ as in the left graph, string states on different Riemann surfaces can be connected continuously in $M_0$ as $[\bar{\Sigma}, u, \bar{\tau}(t_1)]$ and $[\bar{\Sigma}', u', \bar{\tau}(t_3)]$ on the pictures.
Proof. For

\[
U([\bar{\Sigma}, u, \bar{s}], \epsilon) = \left\{ [\bar{\Sigma}, \exp_u X, \bar{\tau}] \mid \sqrt{|\bar{\tau} - \bar{s}|^2 + \|X\|^2} < \epsilon \right\} \quad (2.4)
\]

\[
U([\bar{\Sigma}, 0, \bar{s}], \epsilon) = \left\{ [\bar{\Sigma}, X, \bar{\tau}] \mid \sqrt{|\bar{\tau} - \bar{s}|^2 + \|X\|^2} < \epsilon \right\}, \quad (2.5)
\]

\(\Psi\) is bijective because the exponential map \(X \mapsto \exp_u X\) are bijective on \(\bar{\Sigma}\). Then, \(\Psi\) and \(\Psi^{-1}\) are continuous variables, because the \(\epsilon\)-open neighbourhoods and the open sets are defined by the same conditions (2.4) and (2.5).

If \(U([\bar{\Sigma}, u, \bar{s}], \epsilon) \cap U([\bar{\Sigma}, u', \bar{s}'], \epsilon') \neq \emptyset\), a transition function between \([\bar{\Sigma}, X, \bar{\tau}]\) and \([\bar{\Sigma}, X', \bar{\tau}']\) is given by \([\bar{\Sigma}, \exp_u X, \bar{\tau}] = [\bar{\Sigma}, \exp_{u'} X', \bar{\tau}']\). This condition is equivalent to \(\bar{\tau} = \bar{\tau}'\) and \(\exp_u X = \exp_{u'} X'\). In the following, we denote \([\bar{h}_{mn}, X, \bar{\tau}]\) instead of \([\bar{\Sigma}, X, \bar{\tau}]\) because giving a Riemann surface is equivalent to giving a metric up to diffeomorphism and Weyl transformations.

Let us consider how generally we can define general coordinate transformations between \([\bar{h}_{mn}, X, \bar{\tau}]\) and \([\bar{h}'_{mn}, X', \bar{\tau}']\) where \([\bar{h}_{mn}, X, \bar{\tau}] \in U \subset E\) and \([\bar{h}'_{mn}, X', \bar{\tau}'] \in U' \subset E\). \(\bar{h}_{mn}\) does not transform to \(\bar{\tau}\) and \(X\) and vice versa, because \(\bar{\tau}\) and \(X\) are continuous variables, whereas \(\bar{h}_{mn}\) is a discrete variable: \(\bar{\tau}\) and \(X\) vary continuously, whereas \(\bar{h}_{mn}\) varies discretely in a trajectory on \(E\) by definition of the neighbourhoods. \(\bar{\tau}\) and \(\bar{\sigma}\) do not transform to each other because the string states are defined by \(\bar{\tau}\) constant lines. Under these restrictions, the most general coordinate transformation is given by

\[
[\bar{h}_{mn}(\bar{\sigma}, \bar{\tau}), X^\mu(\bar{\sigma}, \bar{\tau}, \bar{\tau})] \mapsto [\bar{h}'_{mn}(\bar{\sigma}', \bar{\tau}', \bar{\tau}(\bar{\tau}, X)), X'^\mu(\bar{\sigma}', \bar{\tau}', (\bar{\tau}, X), \bar{\tau}'(\bar{\tau}, X))], \quad (2.6)
\]

where \(\mu = 0, 1, \ldots d - 1\). Because \(\bar{h}_{mn}, X^\mu\) and \(\bar{\tau}\) are all independent, where \(\bar{h}_{mn}\) and \(X^\mu\) are coordinates as functionals, \(\frac{\partial}{\partial \bar{\tau}}\) is an explicit derivative on functions over the string manifolds, especially, \(\frac{\partial}{\partial \bar{\tau}} \bar{h}_{mn} = 0\) and \(\frac{\partial}{\partial \bar{\tau}} X^\mu = 0\).

Here, we consider all the manifolds which are constructed by patching open sets of the model space \(E\) by the general coordinate transformations (2.6) and call them string manifolds \(\mathcal{M}\). We also call the topologies and spaces of the string manifolds, string topologies and string spaces, respectively. The string space \(\mathcal{M}_0\), the string topological space \(\mathcal{M}_0\), and the string manifold \(\mathcal{M}_0\) are ones of the string spaces, string topological spaces, and string manifolds, respectively.
The tangent space is spanned by $\frac{\partial}{\partial \tau}$ and $\frac{\partial}{\partial X^\nu(\sigma, \tau)}$ as one can see from the chart (2.5). We should note that $\frac{\partial}{\partial h_{mn}}$ cannot be a part of basis that span the tangent space, because $\bar{h}_{mn}$ is just a discrete variable in $E$. The index of $\frac{\partial}{\partial X^\nu(\sigma, \tau)}$ can be $(\mu \bar{\sigma})$. We define a summation over $\bar{\sigma}$ by $\int d\bar{\sigma} \bar{e}(\bar{\sigma}, \bar{\tau})$, where $\bar{e} := \sqrt{h_{\bar{\sigma} \bar{\tau}}}$. This summation is invariant under $\bar{\sigma} \mapsto \bar{\sigma}'(\bar{\sigma})$ and transformed as a scalar under $\bar{\tau} \mapsto \bar{\tau}'(\bar{\tau}, X)$.

Riemannian string manifold is obtained by defining a metric, which is a section of an inner product on the tangent space. The general form of a metric is given by

$$ds^2(\bar{h}, X, \bar{\tau}) = G(\bar{h}, X, \bar{\tau})_{dd}(d\bar{\tau})^2 + 2d\bar{\tau} \int d\bar{\sigma} \bar{e}(\bar{\sigma}, \bar{\tau}) \sum_\mu G(\bar{h}, X, \bar{\tau})_{d(\mu \bar{\sigma})} dX^\mu(\bar{\sigma}, \bar{\tau}) + \int d\bar{\sigma} \bar{e}(\bar{\sigma}, \bar{\tau}) \int d\sigma' \bar{e}(\sigma', \bar{\tau}) \sum_{\mu, \mu'} G(\bar{h}, X, \bar{\tau})_{(\mu \mu')}(\sigma', \bar{\tau}) dX^\mu(\sigma', \bar{\tau}) dX^{\mu'}(\sigma', \bar{\tau}).$$

We summarize the vectors as $dX^I$ ($I = d, (\mu \bar{\sigma})$), where $dX^d := d\bar{\tau}$ and $dX^{(\mu \bar{\sigma})} := dX^\mu(\bar{\sigma}, \bar{\tau})$. Then, the components of the metric are summarized as $G_{IJ}(\bar{h}, X, \bar{\tau})$. The inverse of the metric $G^{IJ}(\bar{h}, X, \bar{\tau})$ is defined by $G_{IJ}G^{JK} = G^{KJ}G_{JI} = \delta^J_K$, where $\delta^d_d = 1$ and $\delta^{\mu \bar{\sigma}}_{\mu \bar{\sigma}} = \frac{1}{\bar{e}(\bar{\sigma}, \bar{\tau})} \delta^\mu_{\mu} \delta(\bar{\sigma} - \bar{\sigma}')$. The components of the Riemannian curvature tensor are given by $R^d_{JKL}$ in the basis $\frac{\partial}{\partial X^\nu}$. The components of the Ricci tensor are $R_{IJ} := R^K_{IKJ} = R^d_{idJ} + \int d\bar{\sigma} \bar{e} R^{(\mu \bar{\sigma})}_{I(\mu \bar{\sigma}) J}$. The scalar curvature is

$$R := G^{IJ} R_{IJ} = G^{dd} R_{dd} + 2 \int d\bar{\sigma} \bar{e} G^{d(\mu \bar{\sigma})} R_{d(\mu \bar{\sigma})} + \int d\bar{\sigma} \int d\sigma' \bar{e} G^{(\mu \bar{\sigma}) (\mu' \bar{\sigma}')} R_{(\mu \bar{\sigma})(\mu' \bar{\sigma}')}.$$ 

The volume is $\sqrt{G}$, where $G = det(G_{IJ})$.

By using these geometrical objects, we formulate string theory non-perturbatively as

$$Z = \int \mathcal{D}G \mathcal{D}A e^{-S},$$

where

$$S = \frac{1}{G_N} \int \mathcal{D}h \mathcal{D}X \mathcal{D}\bar{\tau} \sqrt{G} (-R + \frac{1}{4} G_N G^{I_1 I_2} G^{J_1 J_2} F_{I_1 J_1} F_{I_2 J_2}).$$

As an example of sets of fields on the string manifolds, we consider the metric and an $u(1)$ gauge field $A_I$ whose field strength is given by $F_{IJ}$. The path integral is canonically
defined by summing over the metrics and gauge fields on $\mathcal{M}$. By definition, the theory is background independent. $\mathcal{D}h$ is the invariant measure of the metrics $h_{mn}$ on the two-dimensional Riemannian manifolds $\Sigma$, divided by the volume of the diffeomorphism and the Weyl transformations. $h_{mn}$ and $\bar{h}_{mn}$ are related to each others by the diffeomorphism and the Weyl transformations.

Under $$(\check{\tau}, X) \mapsto (\check{\tau}'(\check{\tau}, X), X'(\check{\tau}, X)),$$ (2.10) $G_{IJ}(\check{h}, X, \check{\tau})$ and $A_I(\check{h}, X, \check{\tau})$ are transformed as a symmetric tensor and a vector, respectively and the action is manifestly invariant.

We define $G_{IJ}(\check{h}, X, \check{\tau})$ and $A_I(\check{h}, X, \check{\tau})$ so as to transform as scalars under $\check{h}_{mn}(\check{\sigma}, \check{\tau}) \mapsto \check{h}'_{mn}(\check{\sigma}'(\check{\sigma}), \check{\tau})$. Under $\check{\sigma}$ diffeomorphisms: $\check{\sigma} \mapsto \check{\sigma}'(\check{\sigma})$, which are equivalent to

$$[\check{h}_{mn}(\check{\sigma}, \check{\tau}), X^\mu(\check{\sigma}, \check{\tau}), \check{\tau}] \mapsto [\check{h}'_{mn}(\check{\sigma}'(\check{\sigma}), \check{\tau}), X'^\mu(\check{\sigma}', \check{\tau})(X), \check{\tau}],$$

$$= [\check{h}'_{mn}(\check{\sigma}'(\check{\sigma}), \check{\tau}), X'^\mu(\check{\sigma}, \check{\tau}), \check{\tau}],$$

(2.11)

$G_{d(\mu\sigma)}$ is transformed as a scalar;

$$G'_{d(\mu\sigma')}(\check{h}', X', \check{\tau}) = G'_{d(\mu\sigma')}(\check{h}, X', \check{\tau}) = \frac{\partial X^I}{\partial X'(\mu\sigma')} \frac{\partial X^J}{\partial X'(\mu\sigma')} G_{IJ}(\check{h}, X, \check{\tau})$$

$$= \frac{\partial X^I}{\partial X'(\mu\sigma')} \frac{\partial X^J}{\partial X'(\mu\sigma')} G_{IJ}(\check{h}, X, \check{\tau}) = G_{d(\mu\sigma)}(\check{h}, X, \check{\tau}).$$

(2.12)

because (2.11) and (2.10). In the same way, the other fields are also transformed as

$$G'_{dd}(\check{h}', X', \check{\tau}) = G_{dd}(\check{h}, X, \check{\tau})$$

$$G'_{(\mu\sigma') (\nu\rho')}(\check{h}', X', \check{\tau}) = G_{(\mu\sigma') (\nu\rho')}(\check{h}, X, \check{\tau})$$

$$A'_d(\check{h}', X', \check{\tau}) = A_d(\check{h}, X, \check{\tau})$$

$$A'_{(\mu\sigma')}(\check{h}', X', \check{\tau}) = A_{(\mu\sigma')}(\check{h}, X, \check{\tau}).$$

(2.13)

Thus, the Lagrangian is invariant under $\check{\sigma}$ diffeomorphisms, because $\int d\check{\sigma}'e'(\check{\sigma'}, \check{\tau}) = \int d\check{\sigma}e(\check{\sigma}, \check{\tau})$. Therefore, $G_{IJ}(\check{h}, X, \check{\tau})$ and $A_I(\check{h}, X, \check{\tau})$ are transformed covariantly and the action (2.9) is invariant under the diffeomorphisms (2.6), including the $\check{\sigma}$ diffeomorphisms.
3 Perturbative string amplitudes from string geometry

The background that represents a perturbative vacuum is given by

$$\dd s^2 = 2\lambda \bar{\rho}(\bar{h}) N^2(X)(dX^d)^2 + \int d\bar{\sigma}e \int d\bar{\sigma}'e' N^{2D}(X) \frac{e^3(\bar{\sigma}, \bar{\tau})}{\sqrt{h(\bar{\sigma}, \bar{\tau})}} \delta_{(\mu\sigma)(\mu'\sigma')} dX^{(\mu\sigma)} dX^{(\mu'\sigma')}$$

$$\tilde{A}_d = i \sqrt{\frac{2-2D}{2-D}} \frac{\sqrt{2\lambda \bar{\rho}(\bar{h})}}{\sqrt{G_N}} N(X), \quad \tilde{A}_{(\mu\sigma)} = 0, \quad (3.1)$$

on $\mathcal{M}_0$ where the target metric is fixed to $\eta_{\mu\nu'}$. $\bar{\rho}(\bar{h}) := \frac{1}{4\pi} \int d\bar{\sigma} \sqrt{h} \bar{R}_{h}$, where $\bar{R}_h$ is the scalar curvature of $h_{mn}$. $D$ is a volume of the index $(\mu\sigma)$: $D := \int d\bar{\sigma} e(\delta_{(\mu\sigma)}(\mu\sigma)) = d2\pi \delta(0)$. $N(X) = \frac{1}{1+\v(X)}$, where $\v(X) = \frac{\alpha}{\sqrt{d-1}} \int d\bar{\sigma} \epsilon_{\mu
u} X^\mu \partial_\sigma X^\nu$. The inverse of the metric is given by

$$G^{dd} = \frac{1}{2\lambda \bar{\rho} N^2}$$

$$\bar{G}^{d(\mu\sigma)} = 0$$

$$\bar{G}^{(\mu\sigma)}(\mu'\sigma') = N \frac{2-2D}{D-2} \frac{\sqrt{h}}{\epsilon^2} \delta_{(\mu\sigma)(\mu'\sigma')} \quad (3.2)$$

because $\int d\bar{\sigma} \bar{\epsilon}^\mu G_{(\mu\sigma)(\mu'\sigma')} G^{(\mu'\sigma')} = \int d\bar{\sigma} \bar{\epsilon}^\mu \delta_{(\mu\sigma)(\mu'\sigma')} \delta_{(\mu'\sigma')(\mu'\sigma')} = \delta_{(\mu\sigma)(\mu'\sigma')}$. From the metric, we obtain

$$\sqrt{G} = N^{2-D} \sqrt{2\lambda \bar{\rho} \exp(\frac{D}{2\pi} \int d\bar{\sigma} \ln \frac{e^3}{\sqrt{h}})}$$

$$\bar{R}_{dd} = -2\lambda \bar{\rho} N^{2-D} \int d\bar{\sigma} \frac{\sqrt{h}}{e^2} \partial_{(\mu\sigma)} N \partial_{(\mu\sigma)} N$$

$$\bar{R}(\mu\sigma) = 0$$

$$\bar{R}_{(\mu\sigma)}(\mu'\sigma') = \frac{D-1}{2-D} N^{-2} \partial_{(\mu\sigma)} N \partial_{(\mu'\sigma')} N$$

$$+ \frac{1}{D-2} N^{-2} \int d\bar{\sigma} \frac{\sqrt{h}}{e^2} \partial_{(\mu'\sigma')} N \partial_{(\mu'\sigma')} N \frac{e^3}{\sqrt{h}} \delta_{(\mu\sigma)(\mu'\sigma')}$$

$$\bar{R} = \frac{D - 3}{2-D} N^{2D-6} \int d\bar{\sigma} \frac{\sqrt{h}}{e^2} \partial_{(\mu\sigma)} N \partial_{(\mu\sigma)} N \quad (3.3)$$

By using these quantities, one can show that the background (3.1) is a classical solution to the equations of motion of (2.9). We also need to use the fact that $\v(X)$ is a harmonic function with respect to $X^{(\mu\sigma)}$. Actually, $\partial_{(\mu\sigma)} \partial_{(\mu\sigma)} \v = 0$. In these calculations, we should

---

4This solution is a generalization of the Majumdar-Papapetrou solution \[9,10\] of the Einstein-Maxwell system.
note that $\tilde{h}_{mn}$, $X^\mu$ and $\tau$ are all independent. Because the equations of motion are differential equations with respect to $X^\mu$ and $\tau$, $\tilde{h}_{mn}$ is a constant in the solution \((3.1)\) to the differential equations. The dependence of $\tilde{h}_{mn}$ on the background \((3.1)\) is uniquely determined by the consistency of the quantum theory of the fluctuations around the background. Actually, we will find that all the perturbative string amplitudes are derived.

Let us consider fluctuations around the background \((3.1)\), $G_{IJ} = \tilde{G}_{IJ} + \tilde{\tilde{G}}_{IJ}$ and $A_I = \tilde{A}_I + \tilde{\tilde{A}}_I$. The action \((2.9)\) up to the quadratic order is given by,

$$S = \frac{1}{G_N} \int \mathcal{D}h \mathcal{D}X \mathcal{D}\tau \sqrt{G} \left( -\tilde{R} + \frac{1}{4} \tilde{F}'_{IJ} \tilde{F}'^{IJ} + \frac{1}{4} \tilde{\nabla}^I \tilde{G} \tilde{\nabla}^\tau \tilde{G} + \frac{1}{2} \tilde{\nabla}^I \tilde{G}_{IJ} \tilde{\nabla}^J \tilde{G} - \frac{1}{2} \tilde{\nabla}^I \tilde{G}_{IJ} \tilde{\nabla}^K \tilde{G} \tilde{\nabla}^{JK} - \frac{1}{4} (-\tilde{R} + \frac{1}{4} \tilde{F}'_{KL} \tilde{F}'^{KL}) (\tilde{G}_{IJ} \tilde{G}^{IJ} - \frac{1}{2} \tilde{G}^2) + (-\frac{1}{2} \tilde{R}'_{IJ} + \frac{1}{2} \tilde{F}'^{IK} \tilde{F}'_{JK}) \tilde{G}_{IJ} \tilde{G}^{KL} + (-\frac{1}{2} \tilde{R}'^{IJKL} + \frac{1}{4} \tilde{F}'^{IJ} \tilde{F}'^{KL}) \tilde{G}_{IK} \tilde{G}_{JL} + \frac{1}{4} G_N \tilde{F}_{IJ} \tilde{F}^{IJ} + \sqrt{G_N} \left( \frac{1}{4} \tilde{F}'^{IJ} \tilde{F}'_{IJ} \tilde{G} - \tilde{F}'^I \tilde{F}_{IK} \tilde{G}^{K} \right) \right), \quad(3.4)$$

where $\tilde{F}'_{IJ} := \sqrt{G_N} \tilde{F}_{IJ}$ is independent of $G_N$. $\tilde{G} := G^{IJ} \tilde{G}_{IJ}$. There is no first order term because the background satisfies the equations of motion. If we take $G_N \to 0$, we obtain

$$S' = \frac{1}{G_N} \int \mathcal{D}h \mathcal{D}X \mathcal{D}\tau \sqrt{G} \left( -\tilde{R} + \frac{1}{4} \tilde{F}'_{IJ} \tilde{F}'^{IJ} + \frac{1}{4} \tilde{\nabla}^I \tilde{G} \tilde{\nabla}^\tau \tilde{G} + \frac{1}{2} \tilde{\nabla}^I \tilde{G}_{IJ} \tilde{\nabla}^J \tilde{G} - \frac{1}{2} \tilde{\nabla}^I \tilde{G}_{IJ} \tilde{\nabla}^K \tilde{G} \tilde{\nabla}^{JK} - \frac{1}{4} (-\tilde{R} + \frac{1}{4} \tilde{F}'_{KL} \tilde{F}'^{KL}) (\tilde{G}_{IJ} \tilde{G}^{IJ} - \frac{1}{2} \tilde{G}^2) + (-\frac{1}{2} \tilde{R}'_{IJ} + \frac{1}{2} \tilde{F}'^{IK} \tilde{F}'_{JK}) \tilde{G}_{IJ} \tilde{G}^{KL} + (-\frac{1}{2} \tilde{R}'^{IJKL} + \frac{1}{4} \tilde{F}'^{IJ} \tilde{F}'^{KL}) \tilde{G}_{IK} \tilde{G}_{JL} \right), \quad(3.5)$$

where the fluctuation of the gauge field is suppressed. In order to fix the gauge symmetry \((2.10)\), we take the harmonic gauge. If we add the gauge fixing term

$$S_{\text{fix}} = \frac{1}{G_N} \int \mathcal{D}h \mathcal{D}X \mathcal{D}\tau \sqrt{G} \frac{1}{2} \left( \tilde{\nabla}^J (\tilde{G}_{IJ} - \frac{1}{2} \tilde{G}_{IJ} \tilde{G}) \right)^2, \quad(3.6)$$
we obtain

\[
S' + S_{fix} = \frac{1}{G_N} \int \mathcal{D}h \mathcal{D}X \mathcal{D}\bar{\tau} \sqrt{G} \left( -\bar{R} + \frac{1}{4} \tilde{F}^{ij}_I \tilde{F}^{ij}_J \right) + \frac{1}{4} \nabla_I \bar{G}_{JK} \nabla^I \tilde{G}^{JK} - \frac{1}{8} \nabla_I \tilde{G} \nabla^I \tilde{G} \\
- \frac{1}{4} \left( -\bar{R} + \frac{1}{4} \tilde{F}^{KL}_{IJ} \tilde{F}^{KL}_{IJ} \right) (\tilde{G}_{IJ} \tilde{G}^{IJ} - \frac{1}{2} \tilde{G}^2) + \left( -\frac{1}{2} \bar{R}^I_J + \frac{1}{2} \tilde{F}^{IK}_{IJ} \tilde{F}^{JK}_{IJ} \right) \tilde{G}_{II} \tilde{G}^{JL} \\
+ \left( \frac{1}{2} \bar{R}^{IL} - \frac{1}{4} \tilde{F}^{IK}_{IJ} \tilde{F}^{JK}_{I} \right) \tilde{G}_{IJ} \tilde{G} + \left( -\frac{1}{2} \bar{R}^{IK}_{KL} + \frac{1}{4} \tilde{F}^{IK}_{IJ} \tilde{F}^{KL}_{IJ} \right) \tilde{G}_{IK} \tilde{G}_{JL} \right). \tag{3.7}
\]

In order to obtain perturbative string amplitudes, we perform a derivative expansion of \( \tilde{G}_{IJ} \),

\[
\tilde{G}_{IJ} \to \frac{1}{\alpha} \tilde{G}_{IJ} \\
\partial_K \tilde{G}_{IJ} \to \partial_K \tilde{G}_{IJ} \\
\partial_K \partial_L \tilde{G}_{IJ} \to \alpha \partial_K \partial_L \tilde{G}_{IJ},
\]

and take

\[
\alpha \to 0, \tag{3.9}
\]

where \( \alpha \) is an arbitrary constant in the solution (3.1). We normalize the fields as \( \tilde{H}_{IJ} := Z_{IJ} \tilde{G}_{IJ} \), where \( Z_{IJ} := \frac{1}{\sqrt{G_N}} \tilde{G}^i_2 (\tilde{a}_I \tilde{a}_J)^{-\frac{1}{2}} \). \( \tilde{a}_I \) represent the background metric as \( \tilde{G}_{IJ} = \tilde{a}_I \delta_{IJ} \), where \( \tilde{a}_d = 2\lambda \tilde{\rho} \) and \( \tilde{a}_{(\mu \nu)} = \frac{\tilde{g}^3}{\sqrt{\tilde{h}}} \). Then, (3.7) with appropriate boundary conditions reduces to

\[
S' + S_{fix} \to S_0 + S_2, \tag{3.10}
\]

where

\[
S_0 = \int \mathcal{D}h \mathcal{D}X \mathcal{D}\bar{\tau} \left( \frac{1}{G_N} \sqrt{G} \left( -\bar{R} + \frac{1}{4} \tilde{F}^{ij}_I \tilde{F}^{ij}_J \right) \right), \tag{3.11}
\]

and

\[
S_2 = \int \mathcal{D}h \mathcal{D}X \mathcal{D}\bar{\tau} \frac{1}{8} \tilde{H}_{IJ} \tilde{H}_{IJ, KL} \tilde{H}_{KL}. \tag{3.12}
\]

The non-zero matrices are given by

\[
H_{dd, dd} = -\frac{1}{2\lambda \tilde{\rho}} \left( \frac{\partial}{\partial \bar{\tau}} \right)^2 - \int_0^{2\pi} d\bar{\sigma} \sqrt{\tilde{h}} \left( \frac{\partial}{\partial X^\mu} \right)^2 + \frac{18 - 4D}{2 - D} \int_0^{2\pi} d\bar{\sigma} \sqrt{\tilde{h}} \partial_{\bar{\sigma}} X^\mu \partial_{\bar{\sigma}} X^\mu, \tag{3.13}
\]
\[
\begin{align*}
H_{dd; (\mu, \sigma)(\mu', \sigma')} &= H_{(\mu, \sigma)(\mu', \sigma')dd} \\
&= \delta_{(\mu, \sigma)(\mu', \sigma')} \left( -\frac{1}{2\lambda^2} \frac{\partial}{\partial \sigma^2} - \int_0^{2\pi} d\sigma \frac{\sqrt{h}}{e^2} \left( \frac{\partial}{\partial X^\mu} \right)^2 + \frac{6D - 16}{(2 - D)^2} \int_0^{2\pi} d\sigma \frac{\sqrt{h}}{e^2} \partial_{\sigma} X^\mu \partial_{\sigma} X_\mu \right) \\
&\quad + \frac{44 - 12D}{2 - D} \frac{\sqrt{h}}{e^2} \left( \sigma \right) \epsilon_{\mu \nu \sigma} \partial_{\sigma} X^\nu \hat{h} \frac{1}{4} e^{-\frac{3}{2}}(\sigma') \epsilon_{\mu' \nu' \sigma} \partial_{\sigma} X^\nu', \\
&\quad + \int D\tilde{h} D\pi \frac{1}{8} \int_0^{2\pi} d\tilde{\sigma} e \tilde{h} \int_0^{2\pi} d\sigma' e' \tilde{H}_{d(\mu, \sigma)} H_{d(\mu, \sigma); d(\mu', \sigma')} \tilde{H}_{d(\mu', \sigma')},
\end{align*}
\]
decouples from the other modes. Here we use Einstein notation with respect to $(\mu, \bar{\sigma})$. By using projection of $\tilde{H}_{d(\mu)}$ on $\bar{h}^\mu e^{-\frac{i}{\hbar} (\bar{\sigma}) \epsilon_{\mu\nu} \partial_{\nu} X^\nu}$, we obtain $\tilde{H}^0_{d(\mu)}$. Then, $\tilde{H}_{d(\mu)} = \tilde{H}^0_{d(\mu)} + \tilde{H}^1_{d(\mu)}$ where

$$\tilde{H}^\mu_{d(\mu)} e^{-\frac{i}{\hbar} (\bar{\sigma}) \epsilon_{\mu\nu} \partial_{\nu} X^\nu} \tilde{H}_{d(\mu)} = \tilde{H}^\mu_{d(\mu)} e^{-\frac{i}{\hbar} (\bar{\sigma}) \epsilon_{\mu\nu} \partial_{\nu} X^\nu} \tilde{H}^\mu_{d(\mu)} \tilde{H}^1_{d(\mu)} = 0$$ \hspace{1cm} (3.18)

There exists a decomposition $\tilde{H}^1_{d(\mu)} = \tilde{H}^1_{d(\mu)} + \tilde{H}^1_{d(\mu)}$ such that

$$\tilde{H}^1_{d(\mu)} \tilde{H}^0_{d(\mu)} = 0$$

$$\partial_I \tilde{H}^1_{d(\mu)} \partial_I \tilde{H}^0_{d(\mu)} = 0$$

$$\tilde{H}^1_{d(\mu)} \tilde{H}^1_{d(\mu)} = 0$$

$$\partial_I \tilde{H}^1_{d(\mu)} \partial_I \tilde{H}^1_{d(\mu)} = 0.$$ \hspace{1cm} (3.19)

**Proof.**

$$\tilde{H}^1_{d(\mu)} (X^I_0) \tilde{H}^0_{d(\mu)} (X^I_0) = 0$$

$$\tilde{H}^1_{d(\mu)} (X^I_0) \tilde{H}^1_{d(\mu)} (X^I_0) = 0,$$ \hspace{1cm} (3.20)

and

$$\partial_I \tilde{H}^1_{d(\mu)} (X^I_0) \partial_I \tilde{H}^0_{d(\mu)} (X^I_0) = 0$$

$$\partial_I \tilde{H}^1_{d(\mu)} (X^I_0) \partial_I \tilde{H}^1_{d(\mu)} (X^I_0) = 0,$$ \hspace{1cm} (3.21)

are necessary so that (3.19) are satisfied at an arbitrary point $X^I = X^I_0$. (3.20) are the enough conditions for $\tilde{H}^1_{d(\mu)} (X^I_0)$, whereas there are necessary conditions for $\partial_I \tilde{H}^1_{d(\mu)} (X^I_0)$:

$$\tilde{H}^1_{d(\mu)} (X^I_0 + dX^I) \tilde{H}^0_{d(\mu)} (X^I_0 + dX^I) = 0$$

$$\tilde{H}^1_{d(\mu)} (X^I_0 + dX^I) \tilde{H}^1_{d(\mu)} (X^I_0 + dX^I) = 0,$$ \hspace{1cm} (3.22)

in addition to (3.21). (3.21) and (3.22) are the enough conditions for $\partial_I \tilde{H}^1_{d(\mu)} (X^I_0)$. Because (3.20) are 2 equations for D variables $\tilde{H}^1_{d(\mu)} (X^I_0)$, there exist solutions. Because (3.21) and (3.22) are 2+2(D+1) equations for D(D+1) variables $\partial_I \tilde{H}^1_{d(\mu)} (X^I_0)$, there also exist solutions. Thus, there exists $\tilde{H}^1_{d(\mu)}$ that satisfies (3.19) everywhere.
By using (3.18) and (3.19), we obtain

\[
\int \mathcal{D}h \mathcal{D}X \mathcal{D}\tau \frac{1}{8} \int_0^{2\pi} d\bar{\sigma} \int_0^{2\pi} d\sigma' e^i H_{d(\mu\sigma)} H_{d(\mu',\sigma')} \tilde{H}_{d(\mu\sigma)}
\]
\[
= \int \mathcal{D}h \mathcal{D}X \mathcal{D}\tau \left( \int_0^{2\pi} d\sigma \tilde{H}^{-1}_{d(\mu\sigma)} H \tilde{H}^{-1}_{d(\mu\sigma)} + \int_0^{2\pi} d\sigma (\tilde{H}^{0}_{d(\mu\sigma)} + \tilde{H}'_{d(\mu\sigma)}) H (\tilde{H}^{0}_{d(\mu\sigma)} + \tilde{H}'_{d(\mu\sigma)}) \right) - \frac{7 - D}{2 - D} \left( \int_0^{2\pi} d\bar{\sigma} \frac{1}{e^2} \epsilon_{\mu\nu} \partial_{\sigma} X' \tilde{H}^{-1}_{d(\mu\sigma)} \right)^2.
\]

(3.23)

where

\[
H = -\frac{1}{2} \frac{1}{2\lambda \tilde{\rho}} \left( \frac{\partial}{\partial \tau} \right)^2 - \frac{1}{2} \int_0^{2\pi} d\sigma \frac{\sqrt{h}}{e^2} \left( \frac{\partial}{\partial \tau} \right)^2 + \frac{1}{2} D^2 - 9D + 20 \int_0^{2\pi} d\sigma \frac{\sqrt{h}}{e^2} \partial_{\sigma} X' \partial_{\sigma} X_\mu.
\]

(3.24)

As a result, a part of the action

\[
\int \mathcal{D}h \mathcal{D}X \mathcal{D}\tau \frac{1}{4} \int_0^{2\pi} d\bar{\sigma} \tilde{H}^{-1}_{d(\mu\sigma)} H \tilde{H}^{-1}_{d(\mu\sigma)}
\]

(3.25)
decouples from the other modes.

By adding to (3.25)

\[
0 = \int \mathcal{D}h \mathcal{D}X \mathcal{D}\tau \left( \int_0^{2\pi} d\sigma' e^i \tilde{H}^{-1}_{d(\mu\sigma')} \left( \int_0^{2\pi} d\sigma \frac{1}{4} \tilde{n}^\sigma \partial_{\sigma} X_\mu \partial_{\sigma} \tilde{H}^{-1}_{d(\mu\sigma')} \right) \right),
\]

(3.26)

where \( \tilde{n}^\sigma (\bar{\sigma}, \bar{\tau}) \) is the shift vector in the ADM formalism, summarized in the appendix A, we obtain (3.25) with

\[
H(-i \frac{\partial}{\partial \tau}, -i \frac{1}{e} \frac{\partial}{\partial \chi}, X, \tilde{h}) = \frac{1}{2} \frac{1}{2\lambda \tilde{\rho}} \left( -i \frac{\partial}{\partial \tau} \right)^2 + \int_0^{2\pi} d\sigma \left( \frac{\sqrt{h}}{2} \left( -i \frac{1}{\tilde{\epsilon}} \frac{\partial}{\partial \chi} \right)^2 \right. + \frac{1}{2} \left( \frac{\partial}{\partial \sigma} X_\mu \right)^2 + i \tilde{n}^\sigma \partial_{\sigma} X_\mu \left( -i \frac{1}{\tilde{\epsilon}} \frac{\partial}{\partial \chi} \right) \right),
\]

(3.27)

where we have taken \( D \to \infty \). (3.20) is true because

\[
(r.h.s) = \int \mathcal{D}h \mathcal{D}X \mathcal{D}\tau \left( \int_0^{2\pi} d\sigma \frac{1}{8} \tilde{n}^\sigma \partial_{\sigma} X_\mu \partial_{\sigma} \tilde{H}^{1}_{d(\mu\sigma')} \right) \int_0^{2\pi} d\sigma'' \tilde{H}^{1}_{d(\mu\sigma'')} \tilde{H}^{1}_{d(\mu\sigma'')}
\]
\[
= - \int \mathcal{D}h \mathcal{D}X \mathcal{D}\tau \int_0^{2\pi} d\sigma \lim_{\sigma' \to \sigma} \frac{1}{8} \partial_{\sigma} X_\mu (\sigma') \tilde{n}^\sigma (\sigma) \int_0^{2\pi} d\sigma'' \tilde{H}^{1}_{d(\mu\sigma'')} \tilde{H}^{1}_{d(\mu\sigma'')}
\]
\[
= - \int \mathcal{D}h \mathcal{D}X \mathcal{D}\tau \int_0^{2\pi} d\sigma \lim_{\sigma' \to \sigma} \frac{1}{8} \partial_{\sigma} \delta (\sigma - \sigma') \tilde{n}^\sigma (\sigma) \int_0^{2\pi} d\sigma'' \tilde{H}^{1}_{d(\mu\sigma'')} \tilde{H}^{1}_{d(\mu\sigma'')}
\]
\[
= 0.
\]

(3.28)
The propagator \( \Delta_F(\hat{h}, X, \tau; \hat{h}', X', \tau') \) for \( \hat{H}_{d(\mu\sigma)} \) is defined by

\[
H(-i \frac{\partial}{\partial \tau}, -i \frac{1}{\epsilon} \frac{\partial}{\partial X}, X, \hat{h}) \Delta_F(\hat{h}, X, \tau; \hat{h}', X', \tau') = \delta(\hat{h} - \hat{h}') \delta(X - X') \delta(\tau - \tau').
\] (3.29)

In order to obtain a Schwinger representation of the propagator, we use the operator formalism \((\hat{h}, \hat{X}, \hat{\tau})\) of the first quantization, whereas the conjugate momentum is written as \((\hat{p}_h, \hat{p}_X, \hat{p}_\tau)\). The eigen state is given by \(|\hat{h}, X, \tau\rangle\). First,

\[
< \hat{h}, X, \tau | \hat{H}(\hat{p}_\tau, \hat{p}_X, \hat{X}, \hat{h}) | \hat{h}', X', \tau' >= H(-i \frac{\partial}{\partial \tau}, -i \frac{1}{\epsilon} \frac{\partial}{\partial X}, X, \hat{h}) \delta(\hat{h} - \hat{h}') \delta(X - X') \delta(\tau - \tau'),
\] (3.30)

because

\[
(l.h.s.) = \int \mathcal{D}p_h \mathcal{D}p_\tau \mathcal{D}p_X < \hat{h}, X, \tau | \hat{H}(\hat{p}_\tau, \hat{p}_X, \hat{X}, \hat{h}) | p_h, p_\tau, p_X > < p_h, p_\tau, p_X | \hat{h}', X', \tau' >
\]

\[
\quad = \int \mathcal{D}p_h \mathcal{D}p_\tau \mathcal{D}p_X H(p_\tau, p_X, X, \hat{h}) < \hat{h}, X, \tau | p_h, p_\tau, p_X > < p_h, p_\tau, p_X | \hat{h}', X', \tau' >
\]

\[
\quad = \int \mathcal{D}p_h \mathcal{D}p_\tau \mathcal{D}p_X (p_\tau, p_X, X, \hat{h}) e^{ip_h(\hat{h} - \hat{h}') + ip_\tau(\tau - \tau') + ip_X(X - X')}
\]

\[
\quad = (r.h.s.),
\] (3.31)

where \( p_X \cdot X := \int d\sigma \bar{e}_\mu X_\mu \). By using this, we obtain

\[
\Delta_F(\hat{h}, X, \tau; \hat{h}', X', \tau,) = < \hat{h}, X, \tau | \hat{H}^{-1}(\hat{p}_\tau, \hat{p}_X, \hat{X}, \hat{h}) | \hat{h}', X', \tau >,
\] (3.32)

because

\[
\delta(\hat{h} - \hat{h}') \delta(X - X') \delta(\tau - \tau')
\]

\[
= < \hat{h}, X, \tau | \hat{h}', X', \tau >
\]

\[
= < \hat{h}, X, \tau | \hat{H} \hat{H}^{-1} | \hat{h}', X', \tau >
\]

\[
= \int dh'' d\tau'' dX'' < \hat{h}, X, \tau | \hat{H} | h'', X'', \tau'' > < h'', X'', \tau'' | \hat{H}^{-1} | \hat{h}', X', \tau >
\]

\[
\quad = H(-i \frac{\partial}{\partial \tau}, -i \frac{1}{\epsilon} \frac{\partial}{\partial X}, X, \hat{h}) \int dh'' d\tau'' dX'' \delta(\hat{h} - \hat{h}'') \delta(\tau - \tau'') \delta(X - X'')
\]

\[
\quad < h'', X'', \tau'' | \hat{H}^{-1} | \hat{h}', X', \tau >
\]

\[
\quad = H(-i \frac{\partial}{\partial \tau}, -i \frac{1}{\epsilon} \frac{\partial}{\partial X}, X, \hat{h}) < \hat{h}, X, \tau | \hat{H}^{-1} | \hat{h}', X', \tau >.
\] (3.33)

On the other hand,

\[
\hat{H}^{-1} = \int_0^\infty dT e^{-T \hat{H}},
\] (3.34)
because
\[
\lim_{\epsilon \to 0^+} \int_0^\infty dT e^{-T(\hat{H}+\epsilon)} = \lim_{\epsilon \to 0^+} \left[ \frac{1}{-(\hat{H} + \epsilon)} e^{-T(\hat{H}+\epsilon)} \right]_0^\infty = \hat{H}^{-1}.
\] (3.35)

This fact and (3.32) imply
\[
\Delta_F(\bar{h}, X, \bar{\tau}; \bar{h}', X', \bar{\tau}') = \int_0^\infty dT < \bar{h}, X, \bar{\tau} | e^{-TH} | \bar{h}', X', \bar{\tau}' > .
\] (3.36)

We define in and out states as
\[
|| X_i | h_f ; h_i > := \int_{h_i}^{h_f} \mathcal{D}h' \bar{h}', X_i, \bar{\tau} = -\infty >
\]
\[
< X_f | h_f ; h_i |_{out} := \int_{h_i}^{h_f} \mathcal{D}h < \bar{h}, X_f, \bar{\tau} = \infty |,
\] (3.37)

where \( h_i \) and \( h_f \) represent the metrics of the cylinders at \( \bar{\tau} = \mp \infty \), respectively. The two-
point correlation function for these states is given by

\[
\Delta_F(X_f; X_i| h_f, ; h_i) := \int_0^\infty dT <X_f | h_f, ; h_i||_{out} e^{-\frac{1}{\hbar} \mathcal{H} t} ||X_i | h_f, ; h_i >_{in}
\]

\[
= \int_0^\infty \lim_{N \to \infty} dT_N \int_{h_i}^{h_f} \mathcal{D} \mathcal{H} \int_{h_i}^{h_f} \mathcal{D} \mathcal{H}' \prod_{n=1}^N \int d\bar{h}_n dX_n d\tilde{\tau}_n \prod_{n=0}^N \delta(\bar{h}_{m+1}, X_{m+1}, \tilde{\tau}_{m+1}) e^{-\frac{1}{\hbar} \mathcal{H} t} \delta(\bar{h}_m, X_m, \tilde{\tau}_m) \delta(T_m - T_{m+1})
\]

\[
= \int_0^\infty \lim_{N \to \infty} dT_N \int_{h_i}^{h_f} \mathcal{D} \mathcal{H} \prod_{n=1}^N \int dT_n dX_n d\tilde{\tau}_n \prod_{n=0}^N \int dp_{\tau_n} dp_{X_m} dp_{\tilde{\tau}_m} \exp \left( - \sum_{m=0}^N \Delta t \left( -ip_{T_m} T_m - T_{m+1} \right) - ip_{\tau_n} \bar{\tau}_m - \tilde{\tau}_{m+1} \right) - ip_{X_m} X_m - X_{m+1} \right)
\]

\[
+ T_m H(p_{\tau_m}, p_{X_m}, X_m, \bar{h}) \right)
\]

\[
= \int_{h_i, X_i, -\infty}^{h_f, X_f, \infty} \mathcal{D} h \mathcal{D} X \mathcal{D} \tilde{\tau} \int \mathcal{D} T \int \mathcal{D} p_T \mathcal{D} p_X \mathcal{D} p_{\tau} \exp \left( - \int_0^1 dt \left( -ip_{T}(t) \frac{d}{dt} T(t) - ip_{\tau}(t) \frac{d}{dt} \tilde{\tau}(t) - ip_{X}(t) \frac{d}{dt} X(t) \right) - T(t) H(p_{\tau}(t), p_X(t), X(t), \bar{h}) \right),
\]

(3.38)

where \( \bar{h}_0 = \bar{h}', X_0 = X_i, \bar{\tau}_0 = -\infty, \bar{h}_{N+1} = \bar{h}, X_{N+1} = X_f, \bar{\tau}_{N+1} = \infty, \) and \( \Delta t := \frac{1}{N}. \) A trajectory of points \([\bar{\Sigma}, X, \bar{\tau}]\) is necessarily continuous in \( \mathcal{M}_0 \) so that the kernel \( <\bar{h}_{m+1}, X_{m+1}, \tilde{\tau}_{m+1}| e^{-\frac{1}{\hbar} \mathcal{H} t} |\bar{h}_m, X_m, \tilde{\tau}_m > \) in the third line is non-zero when \( N \to \infty. \) If we integrate out \( p_{\tau}(t) \) and \( p_X(t) \) by using the relation of the ADM formalism in the appendix

\footnote{The correlation function is zero if \( h_i \) and \( h_f \) of the in state do not coincide with those of the out states, because of the delta functions in the fifth line.}
A, we obtain

\[
\begin{align*}
\Delta_F(X_f; X_i|h_f; h_i) &= \int_{h_i,X_i,-\infty}^{h_f,X_f,\infty} DT^f Dh DX D\tau Dp_T \exp \left( - \int_0^1 dt \left( -ip_T(t) \frac{dT(t)}{dt} + \lambda \bar{\rho} \frac{1}{T(t)} \left( \frac{d\tau(t)}{dt} \right)^2 \right) ight. \\
&\quad + \int d\bar{\sigma} \sqrt{\bar{h}} \left( \frac{1}{2} \bar{h}^{00} \frac{1}{T(t)} \partial_t X_\mu(\bar{\sigma}, \bar{\tau}, t) \partial_t X_\mu(\bar{\sigma}, \bar{\tau}, t) + \bar{h}^{01} \partial_t X_\mu(\bar{\sigma}, \bar{\tau}, t) \partial_b X_\mu(\bar{\sigma}, \bar{\tau}, t) \right. \\
&\quad \left. \left. + \frac{1}{2} \bar{h}^{11} T(t) \partial_{\bar{\sigma}} X_\mu(\bar{\sigma}, \bar{\tau}, t) \partial_{\bar{\sigma}} X_\mu(\bar{\sigma}, \bar{\tau}, t) \right) \right), \tag{3.39}
\end{align*}
\]

The path integral is defined over all possible trajectories \([\bar{h}, X(t), \bar{\tau}(t)] \in M_0\) with fixed boundary values as in Fig. 4. We should note that the time derivative in (3.39) is in terms of \(t\), not \(\bar{\tau}\) at this moment. In the following, we will see that \(t\) can be fixed to \(\bar{\tau}\) by using a reparametrization of \(t\) that parametrizes a trajectory.

By inserting \(\int Dc Db \bar{f}_0^t dt \left( \frac{db(t)}{dt} \frac{dc(t)}{dt} \right)\), where \(b(t)\) and \(c(t)\) are bc ghosts, we obtain

\[
\begin{align*}
\Delta_F(X_f; X_i|h_f; h_i) &= Z_0 \int_{h_i,X_i,-\infty}^{h_f,X_f,\infty} DT^f Dh DX D\tau Dp_T Dc Db \\
&\quad \exp \left( - \int_0^1 dt \left( -ip_T(t) \frac{dT(t)}{dt} + \lambda \bar{\rho} \frac{1}{T(t)} \left( \frac{d\tau(t)}{dt} \right)^2 \right) + \int d\bar{\sigma} \sqrt{\bar{h}} \left( \frac{1}{2} \bar{h}^{00} \frac{1}{T(t)} \partial_t X_\mu(\bar{\sigma}, \bar{\tau}, t) \partial_t X_\mu(\bar{\sigma}, \bar{\tau}, t) + \bar{h}^{01} \partial_t X_\mu(\bar{\sigma}, \bar{\tau}, t) \partial_b X_\mu(\bar{\sigma}, \bar{\tau}, t) \right. \\
&\quad \left. \left. + \frac{1}{2} \bar{h}^{11} T(t) \partial_{\bar{\sigma}} X_\mu(\bar{\sigma}, \bar{\tau}, t) \partial_{\bar{\sigma}} X_\mu(\bar{\sigma}, \bar{\tau}, t) \right) \right), \tag{3.40}
\end{align*}
\]

where we have redefined as \(c(t) \rightarrow T(t)c(t)\). \(Z_0\) represents an overall constant factor, and we will rename it \(Z_1, Z_2, \cdots\) when the factor changes in the following. This path integral is obtained if

\[
F_1(t) := \frac{d}{dt} T(t) = 0 \tag{3.41}
\]
gauge is chosen in
\[
\Delta_F (X_f; X_i | h_f; h_i) = Z_1 \int_{h_f, X_f, -\infty}^{h_f, X_f, \infty} \mathcal{D}h \mathcal{D}X \mathcal{D}\bar{\tau} \exp \left( - \int_0^1 dt \left( \lambda \bar{\rho} \frac{1}{T(t)} \left( \frac{d\bar{\tau}(t)}{dt} \right)^2 \right. \right. \\
+ \int d\bar{\sigma} \sqrt{\bar{h}} \left( \frac{1}{2} \bar{h}^{00} \frac{1}{T(t)} \partial_t X^\mu (\bar{\sigma}, \bar{\tau}, t) \partial_t X_\mu (\bar{\sigma}, \bar{\tau}, t) \right. \\
\left. + \bar{h}^{01} \partial_t X^\mu (\bar{\sigma}, \bar{\tau}, t) \partial_\sigma X_\mu (\bar{\sigma}, \bar{\tau}, t) \right) \left( \bar{h}^{11} T(t) \partial_\sigma X^\mu (\bar{\sigma}, \bar{\tau}, t) \partial_\sigma X_\mu (\bar{\sigma}, \bar{\tau}, t) \right) \right),
\]
which has a manifest one-dimensional diffeomorphism symmetry with respect to \( t \), where \( T(t) \) is transformed as an einbein \[11\].

Under a rescale \( \bar{\tau} = \bar{\tau}' T(t) \), which implies
\[
\bar{h}^{00} = T^2 \bar{h}^{00}, \\
\bar{h}^{01} = T \bar{h}^{01}, \\
\bar{h}^{11} = \bar{h}^{11}, \\
\sqrt{\bar{h}} = \frac{1}{T} \sqrt{\bar{h}'}, \\
\bar{\rho} = \frac{1}{T} \bar{\rho}', \\
\left( \frac{d\bar{\tau}(t)}{dt} \right)^2 = T^2 \left( \frac{d\bar{\tau}'(t)}{dt} \right)^2,
\]
\[ (3.43) \]
\( T(t) \) disappears in \[3.42\] and we obtain
\[
\Delta_F (X_f; X_i | h_f; h_i) = Z_2 \int_{h_f, X_f, -\infty}^{h_f, X_f, \infty} \mathcal{D}h \mathcal{D}X \mathcal{D}\bar{\tau} \exp \left( - \int_0^1 dt \left( \lambda \bar{\rho} \frac{1}{T(t)} \left( \frac{d\bar{\tau}(t)}{dt} \right)^2 \right. \right. \\
+ \int d\bar{\sigma} \sqrt{\bar{h}} \left( \frac{1}{2} \bar{h}^{00} \partial_t X^\mu (\bar{\sigma}, \bar{\tau}, t) \partial_t X_\mu (\bar{\sigma}, \bar{\tau}, t) \right. \\
\left. + \bar{h}^{01} \partial_t X^\mu (\bar{\sigma}, \bar{\tau}, t) \partial_\sigma X_\mu (\bar{\sigma}, \bar{\tau}, t) \right) \left( \bar{h}^{11} \partial_\sigma X^\mu (\bar{\sigma}, \bar{\tau}, t) \partial_\sigma X_\mu (\bar{\sigma}, \bar{\tau}, t) \right) \right),
\]
\[ (3.44) \]
This action is still invariant under the diffeomorphism with respect to \( t \) if \( \bar{\tau} \) transforms in the same way as \( \frac{1}{T(t)} \).

If we choose a different gauge
\[
F_2 (t) := \bar{\tau} - t = 0,
\]
\[ (3.45) \]
in (3.44), we obtain

\[
\Delta_F(X_f; X_i|h_f; h_i) = Z \int_{h_i, X_i}^{h_f, X_f} \mathcal{D}h \mathcal{D}X \mathcal{D}\tau \mathcal{D}\alpha \mathcal{D}c \mathcal{D}b \\
\exp \left( - \int_0^1 dt \left( + \alpha(t)(\bar{\tau} - t) + b(t)c(t)(1 - \frac{d\bar{\tau}(t)}{dt}) + \lambda \rho \left( \frac{d\bar{\tau}(t)}{dt} \right)^2 \right) \right. \\
+ \int d\sigma \sqrt{h} \left( \frac{1}{2} h^{00} \partial_\tau X^\mu (\bar{\sigma}, \bar{\tau}) \partial_t X_\mu (\bar{\sigma}, \bar{\tau}) + \bar{h}^{01} \partial_\tau X^\mu (\bar{\sigma}, \bar{\tau}) \partial_\rho X_\mu (\bar{\sigma}, \bar{\tau}) \right. \\
\left. + \frac{1}{2} \bar{h}^{11} \partial_\sigma X^\mu (\bar{\sigma}, \bar{\tau}) \partial_\sigma X_\mu (\bar{\sigma}, \bar{\tau}) \right) \right) \\
\left. \left. = Z \int_{h_i, X_i}^{h_f, X_f} \mathcal{D}h \mathcal{D}X \exp \left( - \int_{-\infty}^{\infty} d\bar{\tau} \int d\sigma \sqrt{h} \left( \frac{\lambda}{4\pi} R(\bar{\sigma}, \bar{\tau}) \right. \right. \right. \\
+ \frac{1}{2} \bar{h}^{00} \partial_\tau X^\mu (\bar{\sigma}, \bar{\tau}) \partial_t X_\mu (\bar{\sigma}, \bar{\tau}) + \bar{h}^{01} \partial_\tau X^\mu (\bar{\sigma}, \bar{\tau}) \partial_\rho X_\mu (\bar{\sigma}, \bar{\tau}) \right. \\
\left. \left. + \frac{1}{2} \bar{h}^{11} \partial_\sigma X^\mu (\bar{\sigma}, \bar{\tau}) \partial_\sigma X_\mu (\bar{\sigma}, \bar{\tau}) \right) \right) \right) . \tag{3.46}
\]

The path integral is defined over all possible two-dimensional Riemannian manifolds with fixed punctures in \( \mathbb{R}^{d-1,1} \) as in Fig. 3. The diffeomorphism times Weyl invariance of the action in (3.46) implies that the correlation function in the string manifold \( \mathcal{M}_0 \) is given by

\[
\Delta_F(X_f; X_i|h_f; h_i) = Z \int_{h_i, X_i}^{h_f, X_f} \mathcal{D}h \mathcal{D}X e^{-\lambda \chi} e^{-S_s}, \tag{3.47}
\]

where

\[
S_s = \int_{-\infty}^{\infty} d\tau \int d\sigma \sqrt{h(\sigma, \tau)} \left( \frac{1}{2} h^{mn}(\sigma, \tau) \partial_m X^\mu (\sigma, \tau) \partial_n X_\mu (\sigma, \tau) \right) , \tag{3.48}
\]

and \( \chi \) is the Euler number of the two-dimensional Riemannian manifold.

Here, we insert asymptotic states. Punctures exist only at \( \bar{\tau} = \pm \infty \). We represent as \( V_{j_l}(X_i)(k_l; h_i(l)) \), an incoming asymptotic state on the incoming \( l \)-th cylinder \( \Sigma_i(l) \) in \( \Sigma \) at \( \bar{\tau} \simeq -\infty \), where \( h_i(l) \) denotes the metric on \( \Sigma_i(l) \) and \( l = 1, 2, \ldots, m \). Similarly, an outgoing asymptotic state is denoted by \( V_{j'_l}(X_f)(k_l'(l')) \) at \( \bar{\tau} \simeq \infty \), where \( l' = m+1, m+2, \ldots, N \). \( j_l \) and \( j'_l \) are the levels, whereas \( k^\mu_l = -(E_l, k_l) \) and \( k^\mu_{l'} = (E_{l'}, k_{l'}) \) are the momenta. By the state-operator isomorphism, these states correspond to incoming and outgoing states of vertex operators, \( V_{j_l}(X)(k_l, \sigma_l) \) and \( V_{j'_l}(X)(k_{l'}, \sigma_{l'}) \). \( \sigma_l \) and \( \sigma_{l'} \) are points to which the
Figure 5: A path and a Riemann surface. The line on the left is a trajectory in the path integral. The trajectory parametrized by \( \bar{\tau} \) from \( \bar{\tau} = -\infty \) to \( \bar{\tau} = \infty \), represents a Riemann surface with fixed punctures in \( \mathbb{R}^{d-1,1} \) on the right.

cylinders \( \Sigma_i(l) \) and \( \Sigma_f(l') \) are conformally transformed, respectively. By inserting these asymptotic states into the propagator (3.47), we define scattering amplitudes,

\[
S_{j_1,j_2,\ldots,j_N}(k_1,k_2,\ldots,k_N) = \frac{\int dh_f dh_i dX_f dX_i \Delta_F(X_f; X_i| h_f; h_i)/Z}{\prod_{l,l'} V_{j_l}(X_i)(k_l; h_i(l))V_{j_l'}(X_f)(k_l'; h_f(l'))}.
\]

(3.49)

For regularization, we have divided the correlation function by \( Z \) by renormalizing \( \tilde{H}_{d(\mu \bar{\sigma})} \).

(3.49) are the all-order perturbative scattering amplitudes themselves that possess the moduli in the string theory [12]. Especially, in string geometry, the consistency of the perturbation theory around the background (3.1) determines \( d = 26 \) (the critical dimension).

4 Superstring geometry

In this section, we will define superstring geometry and derive perturbative superstring amplitudes.

First, let us prepare a moduli space\(^6\) of type II superstring worldsheet \( \Sigma \) [13–15] with

\(^6\)Strictly speaking, this should be called a parameter space of integration cycles [13–14] because superstring
punctures $P_i$ ($i = 1, \cdots, N$). We consider two super Riemann surfaces $\Sigma_L$ and $\Sigma_R$ with Neveu-Schwarz (NS) and Ramond (R) punctures whose reduced spaces $\Sigma_{L,\text{red}}$ and $\Sigma_{R,\text{red}}$ are complex conjugates. A reduced space is defined by setting odd variables to zero in a super Riemann surface. The complex conjugates means that they are complex conjugate spaces with punctures at the same points. There are four types of punctures: NS-NS, NS-R, R-NS, R-R because the punctures in $\Sigma_L$ and $\Sigma_R$ are not necessarily of the same type. A type II superstring worldsheet $\bar{\Sigma}$ is defined by the subspace of $\Sigma_L \times \Sigma_R$ whose reduced space $\bar{\Sigma}_{L,\text{red}} \times \Sigma_{R,\text{red}}$ is restricted to its diagonal $\bar{\Sigma}_{\text{red}}$.

Next, let us define global times uniquely on $\bar{\Sigma}$ in order to define string states by world-time constant hypersurfaces. If there are R punctures on a super Riemann surface $\Sigma_R$, the superconformal structures have singularities on the R divisors [13–15]. On a R divisor, a closed holomorphic 1-form takes the form, $\mu = \frac{w}{\sqrt{2\pi i}} d\theta$ (mod $z$), which is uniquely determined by an odd period $w$. On any other point, it takes the form, $\mu = b(z) dz + d(\theta\alpha(z))$. One can define an even period $\oint_S \gamma^*(\mu)$ on a cycle $S$ with dimension 1|0, where $\gamma^*$ is a pullback by a map $\gamma : S \to \Sigma_R$. A- and B-periods on $\Sigma_R$ are defined by those on the reduced space because $\oint_S \gamma^*(\mu) = \oint_{S,\text{red}} b(z) dz$. The periods do not depend on a choice of reduced space because they only depend on the homology class determined by the map $\gamma$ if $\mu$ is closed. Because the period of $d(\theta\alpha(z))$ vanishes, we take the quotient of the space of 1-forms by the subspace consisting of those whose periods vanish, and thus we have $\mu = b(z) dz$. As a result, on $\Sigma_R$ except for R-divisors, a closed holomorphic 1-form is uniquely determined by even periods on the complete basis of A- and B-cycles.

Therefore, on $\Sigma_R$, there exists an unique Abelian differential $dp$ that has simple poles with residues $f^i$ where $\sum_i f^i = 0$, at $P_i$ if it is appropriately normalized. An example of such normalizations is that $dp$ has purely imaginary and real periods with respect to A-cycles and B-cycles, respectively. The Abelian differentials are called 1st kind if there is no puncture on $\Sigma_R$, whereas they are called 3rd kind in general. A global time is defined by $\bar{w} = \bar{\tau} + i\bar{\sigma} := \int^P dp$ at any point $P$ on $\Sigma_R$. By setting the even coordinates to $\bar{w}$ under a

---

7 $P_i$ not necessarily represents a point, whereas the corresponding $P_{i,\text{red}}$ on a reduced space represents a point. A Ramond puncture is located over a R divisor.

8 The odd periods do not contribute to the residues because residues are defined around $P_i$ not on $P_i$.

9 We define the integral by avoiding the R punctures and define the global time on $P_i$ by a limit to $P_i$ in order that the odd periods do not contribute to the global time.
superconformal transformation, a reduced space $\Sigma_{R,\text{red}}$ is canonically defined. In particular, $\bar{\tau} = -\infty$ at $P^i$ with negative $f^i$ and $\bar{\tau} = \infty$ at $P^i$ with positive $f^i$. A contour integral on $\bar{\tau}$ constant line around $P^i$: $i\Delta\bar{\sigma} = \oint dp = 2\pi if^i$ indicates that the $\bar{\sigma}$ region around $P^i$ is $2\pi f^i$. This means that $\Sigma_R$ around $P^i$ represents a semi-infinite supercylinder with radius $f^i$. The condition $\sum_i f^i = 0$ means that the total $\bar{\sigma}$ region of incoming supercylinders equals to that of outgoing ones if we choose the outgoing direction as positive. That is, the total $\bar{\sigma}$ region is conserved. In order to define the above global time uniquely, we fix the $\bar{\sigma}$ regions $2\pi f^i$ around $P^i$. We divide $N$ $P^i$'s to arbitrary two sets consist of $N_-$ and $N_+$ $P^i$'s, respectively ($N_- + N_+ = N$), then we divide -1 to $N_-$ $f^i \equiv \frac{1}{N_-}$ and 1 to $N_+ f^i \equiv \frac{1}{N_+}$ equally for all $i$.

If we give residues $-f^i$ and the same normalization on $\Sigma_L$ as on $\Sigma_R$, we can set the even coordinates on $\Sigma_L$ to the complex conjugate $\bar{w} = \bar{\tau} - i\bar{\sigma} := \int^P d\bar{\sigma}$ by a superconformal transformation, because the Abelian differential is uniquely determined on $\Sigma_L$ and $\Sigma_{L,\text{red}}$ is complex conjugate to $\Sigma_{R,\text{red}}$. Therefore, we can define the global time $\bar{\tau}$ uniquely and reduced space canonically on the type II superstring worldsheet $\tilde{\Sigma}$.

Thus, under a superconformal transformation, one obtains a type II worldsheet $\Sigma$ that has even coordinates composed of the global time $\bar{\tau}$ and the position $\bar{\sigma}$ and $\Sigma_{\text{red}}$ is canonically defined. Because $\Sigma$ can be a moduli of type II worldsheets with punctures, any two-dimensional super Riemannian manifold with punctures $\Sigma$ can be obtained by $\Sigma = \psi(\tilde{\Sigma})$ where $\psi$ is a superdiffeomorphism times super Weyl transformation $[16, 17]$.

Next, we will define a superstring space $\mathcal{M}_0$. We consider a state $(\Sigma, u, \bar{\tau}_s)$ determined by an arbitrary map $u$ from $\Sigma$ to a $d$-dimensional Riemannian manifold $M$ and a $\bar{\tau} = \bar{\tau}_s$ constant hypersurface. $\tilde{\Sigma}$ is a direct sum of $N_\pm$ supercylinders with radii $f_i$ at $\bar{\tau} \simeq \mp \infty$. Thus, we define a superstring state as an equivalence class $[\Sigma, u, \bar{\tau}_s \simeq \mp \infty]$ by a relation $(\Sigma, u, \bar{\tau}_s \simeq \mp \infty) \sim (\Sigma', u', \bar{\tau}_s \simeq \mp \infty)$ if $N_\pm = N_\pm'$, $f_i = f_i'$, $u|_{\bar{\tau}_s \simeq \mp \infty} = u'|_{\bar{\tau}_s \simeq \mp \infty}$, and the corresponding supercylinders are the same type (NS-NS, NS-R, R-NS, or R-R) as in Fig. 1. Because the reduced space of $\Sigma|_{\bar{\tau}_s}$ is $S^1 \times S^1 \times \cdots \times S^1$ and $u|_{\bar{\tau}_s} : \Sigma|_{\bar{\tau}_s} \to M$, $[\Sigma, u, \bar{\tau}_s]$ represent many-body states of superstrings in $M$ as in Fig. 2. A superstring space $\mathcal{M}_0$ is defined by a collection of all the superstring states $\{[\Sigma, u, \bar{\tau}_s]\}$.

Here, we will define topologies of $\mathcal{M}_0$. We define an $\epsilon$-open neighbourhood of $[\Sigma, u, \bar{\tau}_s]$ by

$$U([\Sigma, u, \bar{\tau}_s], \epsilon) := \left\{ [\Sigma, \exp u X, \bar{\tau}] \mid \sqrt{|\bar{\tau} - \bar{\tau}_s|^2 + \|X|_{\bar{\tau}_s}\|^2 + \|X|_{\bar{\tau}_s}\|^2} < \epsilon \right\}, \quad (4.1)$$

25
as in Fig. 3. \( X : \Sigma \rightarrow \mathbb{R}^{d-1,1} \) is a section of \( u^*TM \). The following statements do not depend on how to define the norm. An example is given by

\[
\|X_\tau\|^2 := \int_{\bar{\Sigma}_\tau} d\bar{\sigma} d^2\bar{\theta} \bar{E}_\tau < \tilde{D}_\alpha X_\tau, \tilde{D}_\alpha X_\tau >,
\]

(4.2)

where \(<,>\) is the inner product in \( \mathbb{R}^{d-1,1} \). \( \tilde{D}_\alpha \) is a \( \bar{\tau} \) independent super derivative in the two-dimensional curved spacetime. \( \bar{E}_M A(\bar{\sigma}, \bar{\tau}, \bar{\theta}^\alpha) \) \( (M = (m, \alpha), A = (q, a), m, q = 0, 1, \alpha, a = 1, 2) \) is the worldsheet super vierbein on \( \bar{\Sigma} \). \( U(\bar{\Sigma}, u, \bar{\tau}_s \simeq \mp \infty, \epsilon) = U(\bar{\Sigma}', u', \bar{\tau}_s \simeq \mp \infty, \epsilon) \) consistently if \( N_\mp = N'_\mp, f_i = f'_i, u|_{\bar{\tau}_s \simeq \mp \infty} = u'|_{\bar{\tau}_s \simeq \mp \infty} \), the corresponding supercylinders are the same type (NS-NS, NS-R, R-NS, or R-R), and \( \epsilon \) is small enough, because the \( \bar{\tau}_s \simeq \mp \infty \) constant hypersurfaces traverses only supercylinders overlapped by \( \bar{\Sigma} \) and \( \bar{\Sigma}' \). \( U \) is defined to be an open set of \( \mathcal{M}_0 \) if there exists \( \epsilon \) such that \( U(\bar{\Sigma}, u, \bar{\tau}_s, \epsilon) \subset U \) for an arbitrary point \( \{\bar{\Sigma}, u, \bar{\tau}_s\} \in U \). In exactly the same way as in section 2, one can show that the topology of \( \mathcal{M}_0 \) satisfies the axiom of topology.

By definition of the \( \epsilon \)-open neighbourhood, arbitrary two superstring states on a connected super Riemann surface are connected continuously. Thus, there is an one-to-one correspondence between a super Riemann surface with punctures in \( M \) and a curve parametrized by \( \bar{\tau} \) from \( \bar{\tau} = -\infty \) to \( \bar{\tau} = \infty \) on \( \mathcal{M}_0 \). That is, curves that represent asymptotic processes on \( \mathcal{M}_0 \) reproduce the right moduli space of the super Riemann surfaces in the target manifold.

By a general curve parametrized by \( t \) on \( \mathcal{M}_0 \), superstring states on different super Riemann surfaces that have even different genera, can be connected continuously, for example see Fig. 4 whereas different super Riemann surfaces that have different genera cannot be connected continuously in the moduli space of the super Riemann surfaces in the target space. Therefore, the superstring geometry is expected to possess non-perturbative effects.

Next, we will define structures of manifold on the superstring topological space \( \mathcal{M}_0 \).

In this supersymmetric case, we define a model space \( E \) such that \( E := \cup_T \{[\Sigma, X_T, \bar{\tau}]\} \) where \( T \) runs IIA and IIB. IIA and IIB GSO projections are attached for \( T = \text{IIA} \) and \( \text{IIB} \), respectively. We can define the worldsheet fermion numbers of states in a Hilbert space because the states consist of the fields over the local coordinates \( X_T^\mu = X^\mu + \bar{\theta}^\alpha \psi^\mu_\alpha + \frac{1}{2} \bar{\theta}^2 F^\mu \), where \( \psi^\mu_\alpha \) is a Majorana fermion and \( F^\mu \) is an auxiliary field. We abbreviate \( T \) of \( X^\mu, \psi_\alpha^\mu \) and \( F^\mu \). We define the Hilbert space in these coordinates by the states only with \( e^{\pi i F} = 1 \) and \( e^{\pi i \bar{F}} = (-1)^{\bar{\alpha}} \) for \( T = \text{IIA} \), and \( e^{\pi i F} = e^{\pi i \bar{F}} = 1 \) for \( T = \text{IIB} \), where \( F \) and \( \bar{F} \) are
left- and right-handed fermion numbers respectively, and \( \bar{\alpha} \) is 1 or 0 when the right-handed fermion is periodic (R sector) or anti-periodic (NS sector), respectively. Although the model space is defined by using the coordinates \([ \bar{\Sigma}, X_T, \bar{T} ]\), the model space does not depend on the coordinates, because a model space is a topological space.

An \( \epsilon \)-open neighbourhood of \([ \bar{\Sigma}, X_{Ts}, \bar{T}_s ]\) is defined by

\[
U([\bar{\Sigma}, X_{Ts}, \bar{T}_s], \epsilon) := \left\{ [\bar{\Sigma}, X_T, \bar{T}] \mid \sqrt{|\bar{T} - \bar{T}_s|^2 + \|X_T|_T^2 + \|X_T|_{T_s}^2} < \epsilon \right\} \tag{4.3}
\]

Open sets of \( E \) are defined as in the case of \( \mathcal{M}_0 \). By these definitions, one can prove that the topology of \( E \) satisfies the axiom of topology in the same way. Here, we define charts \((U([\bar{\Sigma}, u, \bar{T}_s], \epsilon), \Psi)\) on \( \mathcal{M}_0 \). First, the \( \epsilon \)-open neighbourhoods \( U([\bar{\Sigma}, u, \bar{T}_s], \epsilon) \) cover \( \mathcal{M}_0 \) because they are defined at any point \([\bar{\Sigma}, u, \bar{T}_s] \in \mathcal{M}_0 \). Next,

\[
\Psi : U([\bar{\Sigma}, u, \bar{T}_s], \epsilon) \rightarrow U([\bar{\Sigma}, 0, \bar{T}_s], \epsilon)
\]

is a homeomorphism from an open set in \( \mathcal{M}_0 \) to that in \( E \).

**Proof.** For

\[
U([\bar{\Sigma}, u, \bar{T}_s], \epsilon) = \left\{ [\bar{\Sigma}, \exp u X_T, \bar{T}] \mid \sqrt{|\bar{T} - \bar{T}_s|^2 + \|X_T|_T^2 + \|X_T|_{T_s}^2} < \epsilon \right\} \tag{4.4}
\]

\[
U([\bar{\Sigma}, 0, \bar{T}_s], \epsilon) = \left\{ [\bar{\Sigma}, X_T, \bar{T}] \mid \sqrt{|\bar{T} - \bar{T}_s|^2 + \|X_T|_T^2 + \|X_T|_{T_s}^2} < \epsilon \right\} \tag{4.5}
\]

\( \Psi \) is bijective because the exponential map \( X_T \mapsto \exp u X_T \) are bijective on \( \Sigma \). Then, \( \Psi \) and \( \Psi^{-1} \) are continuous because the \( \epsilon \)-open neighbourhoods and the open sets are defined by the same conditions \((4.3)\) and \((4.5)\). \( \square \)

If \( U([\bar{\Sigma}, u, \bar{T}_s], \epsilon) \cap U([\bar{\Sigma}, u', \bar{T}'_s], \epsilon') \neq \emptyset \), a transition function between \([\bar{\Sigma}, X_T, \bar{T}]\) and \([\bar{\Sigma}, X'_T, \bar{T}']\) is given by \([\bar{\Sigma}, \exp u X_T, \bar{T}] = [\bar{\Sigma}, \exp u' X'_T, \bar{T}']\). This condition is equivalent to \( \bar{T} = \bar{T}' \) and \( \exp u X_T = \exp u' X'_T \). In the following, we denote \([E_M A, X_T, \bar{T}]\) instead of \([\bar{\Sigma}, X_T, \bar{T}]\) because giving a super Riemann surface is equivalent to giving a super vierbein up to super diffeomorphism and super Weyl transformations.

Let us consider how generally we can define general coordinate transformations between \([E_M A, X_T, \bar{T}]\) and \([E_M' A, X'_T, \bar{T}']\) where \([E_M A, X_T, \bar{T}] \in U \subset E \) and \([E_M A, X'_T, \bar{T}'] \in U' \subset E \). \( E_M A \) does not transform to \( \bar{T} \) and \( X_T \) and vice versa, because \( \bar{T} \) and \( X_T \) are continuous variables, whereas \( E_M A \) is a discrete variable: \( \bar{T} \) and \( X_T \) vary continuously, whereas \( E_M A \)
topologies and superstring spaces, respectively. The superstring space surfaces. Under these restrictions, the most general coordinate transformation is given by

\[ [\mathbf{E}_M^A(\bar{\sigma}, \bar{\tau}, \hat{\theta}^\alpha), \mathbf{X}_T^\mu(\bar{\sigma}, \bar{\tau}, \hat{\theta}^\alpha), \bar{\tau}] \]

\[ \mapsto [\mathbf{E}'_M^A(\bar{\sigma}'(\bar{\sigma}, \bar{\tau}), \bar{\tau}'(\bar{\tau}, X_T), \hat{\theta}'^\alpha(\bar{\sigma}, \bar{\theta})), \mathbf{X}'_T^\mu(\bar{\sigma}', \bar{\tau}', \hat{\theta}'^\alpha)(\bar{\tau}, X_T), \bar{\tau}'(\bar{\tau}, X_T)], \quad (4.6) \]

where \( \mu = 0, 1, \ldots, d-1 \). Because \( \mathbf{E}_M^A, \mathbf{X}_T^\mu \) and \( \bar{\tau} \) are all independent, where \( \mathbf{E}_M^A \) and \( \mathbf{X}_T^\mu \) are coordinates as functionals, \( \frac{\partial}{\partial \bar{\tau}} \) is an explicit derivative on functions over the superstring manifolds, especially, \( \frac{\partial}{\partial \bar{\tau}} \mathbf{E}_M^A = 0 \) and \( \frac{\partial}{\partial \bar{\tau}} \mathbf{X}_T^\mu = 0 \).

Here, we consider all the manifolds which are constructed by patching open sets of the model space \( \mathbf{E} \) by general coordinate transformations \((4.6)\) and call them superstring manifolds \( \mathcal{M} \). We also call the topologies and spaces of the superstring manifolds, superstring topologies and superstring spaces, respectively. The superstring space \( \mathcal{M}_0 \), the superstring topological space \( \mathcal{M}_0 \), and the superstring manifold \( \mathcal{M}_0 \) are ones of the superstring spaces, superstring topological spaces, and superstring manifolds, respectively.

The tangent space is spanned by \( \frac{\partial}{\partial \bar{\tau}} \) and \( \frac{\partial}{\partial X_T^\mu(\bar{\sigma}, \bar{\tau}, \hat{\theta})} \) as one can see from the chart \((4.5)\). We should note that \( \frac{\partial}{\partial \mathbf{E}_M^A} \) cannot be a part of basis that span the tangent space because \( \mathbf{E}_M^A \) is just a discrete variable in \( \mathbf{E} \). The index of \( \frac{\partial}{\partial X_T^\mu(\bar{\sigma}, \bar{\tau}, \hat{\theta})} \) can be \((\mu \bar{\sigma} \hat{\theta})\). Then, let us define a summation over \( \bar{\sigma} \) and \( \hat{\theta} \) that is invariant under \((\bar{\sigma}, \hat{\theta}^\alpha) \mapsto (\bar{\sigma}'(\bar{\sigma}, \bar{\tau}), \hat{\theta}'^\alpha(\bar{\sigma}, \bar{\theta})) \) and transformed as a scalar under \( \bar{\tau} \mapsto \bar{\tau}'(\bar{\tau}, X_T) \). First, \( \int d\bar{\tau} \int d\bar{\sigma} d^2 \hat{\theta} \mathbf{E}(\bar{\sigma}, \bar{\tau}, \hat{\theta}^\alpha) \) is invariant under \((\bar{\sigma}, \bar{\tau}, \hat{\theta}^\alpha) \mapsto (\bar{\sigma}'(\bar{\sigma}, \bar{\tau}), \bar{\tau}'(\bar{\tau}, X_T), \hat{\theta}'^\alpha(\bar{\sigma}, \bar{\theta})) \), where \( \mathbf{E}(\bar{\sigma}, \bar{\tau}, \hat{\theta}^\alpha) \) is the superdeterminant of \( \mathbf{E}_M^A(\bar{\sigma}, \bar{\tau}, \hat{\theta}^\alpha) \). A super analogue of the lapse function, \( \frac{1}{\sqrt{\mathbf{E}_A^0 \mathbf{E}_0^A}} \) transforms as an one-dimensional vector in the \( \bar{\tau} \) direction: \( \int d\bar{\tau} \frac{1}{\sqrt{\mathbf{E}_A^0 \mathbf{E}_0^A}} \) is invariant under \( \bar{\tau} \mapsto \bar{\tau}'(\bar{\tau}, X_T) \) and transformed as a superscalar under \((\bar{\sigma}, \hat{\theta}^\alpha) \mapsto (\bar{\sigma}'(\bar{\sigma}, \bar{\theta}), \hat{\theta}'^\alpha(\bar{\sigma}, \bar{\theta})) \). Therefore, \( \int d\bar{\sigma} d^2 \hat{\theta} \mathbf{E}(\bar{\sigma}, \bar{\tau}, \hat{\theta}^\alpha) \), where \( \mathbf{E}(\bar{\sigma}, \bar{\tau}, \hat{\theta}^\alpha) := \sqrt{\mathbf{E}_A^0 \mathbf{E}_0^A} \mathbf{E}(\bar{\sigma}, \bar{\tau}, \hat{\theta}^\alpha) \), is transformed as a scalar under \( \bar{\tau} \mapsto \bar{\tau}'(\bar{\tau}, X_T) \) and invariant under \((\bar{\sigma}, \hat{\theta}^\alpha) \mapsto (\bar{\sigma}'(\bar{\sigma}, \bar{\theta}), \hat{\theta}'^\alpha(\bar{\sigma}, \bar{\theta})) \).

Riemannian superstring manifold is obtained by defining a metric, which is a section of
an inner product on the tangent space. The general form of a metric is given by

\[ ds^2(\mathbf{E}, X_T, \bar{\tau}) = G(\mathbf{E}, X_T, \bar{\tau})_{dd}(d\bar{\tau})^2 + 2d\bar{\tau} \int d\bar{\sigma} d^2\theta \mathbf{E} \sum_{\mu} G(\mathbf{E}, X_T, \bar{\tau})_{d(\mu\bar{\sigma}\bar{\theta})} dX_T^\mu(\bar{\sigma}, \bar{\tau}, \bar{\theta}) + \int d\bar{\sigma} d^2\theta \mathbf{E} \int d\bar{\sigma'} d^2\theta' \mathbf{E}' \sum_{\mu, \mu'} G(\mathbf{E}, X_T, \bar{\tau})_{(\mu\bar{\sigma}\bar{\theta}) (\mu'\bar{\sigma'}\bar{\theta'})} dX_T^\mu(\bar{\sigma}, \bar{\tau}, \bar{\theta}) dX_T'^{\mu'}(\bar{\sigma'}, \bar{\tau}, \bar{\theta'}) \]

(4.7)

We summarize the vectors as \( dX_T^I (I = d, (\mu\bar{\sigma}\bar{\theta})) \), where \( dX_T^d := d\bar{\tau} \) and \( dX_T^{(\mu\bar{\sigma}\bar{\theta})} := dX_T^\mu(\bar{\sigma}, \bar{\tau}, \bar{\theta}) \). Then, the components of the metric are summarized as \( G_{IJ}(\mathbf{E}, X_T, \bar{\tau}) \). The inverse of the metric \( G^{IJ}(\mathbf{E}, X_T, \bar{\tau}) \) is defined by \( G_{IJ} \delta^{JK} = G^{KJ} G_{JI} = \delta^K_I \), where \( \delta^d_d = 1 \) and \( \delta^{(\mu'\bar{\sigma}'\bar{\theta}')}_{\mu\bar{\sigma}\bar{\theta}} = \frac{1}{\mathbf{E}_{\bar{\sigma}\bar{\theta}}} \delta(\bar{\sigma} - \bar{\sigma}') \delta^2(\bar{\theta} - \bar{\theta}') \). The components of the Riemannian curvature tensor are given by \( R_{dIJ} := R_{dIKJ} = R^{d}_{dIJ} + \int d\bar{\sigma} d^2\theta \mathbf{E} R_{1(\mu\bar{\sigma}\bar{\theta})}^{(\mu'\bar{\sigma}'\bar{\theta}')} \). The components of the Ricci tensor are \( R_{IJ} := R_{IKJ} \).

The scalar curvature is

\[ R := G^{IJ} R_{IJ} = G^{dd} R_{dd} + 2 \int d\bar{\sigma} d^2\theta \mathbf{E} G^{d(\mu\bar{\sigma}\bar{\theta})} R_{d(\mu\bar{\sigma}\bar{\theta})} + \int d\bar{\sigma} d^2\theta \mathbf{E} \int d\bar{\sigma'} d^2\theta' \mathbf{E}' G^{(\mu\bar{\sigma}\bar{\theta})(\mu'\bar{\sigma}'\bar{\theta}')} R_{(\mu\bar{\sigma}\bar{\theta})(\mu'\bar{\sigma}'\bar{\theta}')} \]

The volume is \( \text{vol} = \sqrt{G} \), where \( G = \text{det}(G_{IJ}) \).

By using these geometrical objects, we formulate superstring theory non-perturbatively as

\[ Z = \int \mathcal{D}G \mathcal{D}A e^{-S}, \]  

(4.8)

where

\[ S = \frac{1}{G_N} \int \mathcal{D}E \mathcal{D}X_T \mathcal{D}\bar{\tau} \sqrt{G} (-R + \frac{1}{4} G_{N} G^{I_1I_2} G^{J_1J_2} F_{1_1J_1} F_{1_2J_2}). \]  

(4.9)

As an example of sets of fields on the superstring manifolds, we consider the metric and an \( u(1) \) gauge field \( A_I \) whose field strength is given by \( F_{IJ} \). The path integral is canonically defined by summing over the metrics and gauge fields on \( \mathfrak{M} \). By definition, the theory is background independent. \( \mathcal{D}\mathbf{E} \) is the invariant measure of the super vierbeins \( \mathbf{E}_M^A \) on the two-dimensional super Riemannian manifolds \( \Sigma \), divided by the volume of the super
diffeomorphism and super Weyl transformations. \( E_M^A \) and \( \bar{E}_M^A \) are related to each others by the super diffeomorphism and super Weyl transformations.

Under
\[
(\bar{\tau}, X_T) \mapsto (\bar{\tau}', X_T), \quad X_T' = (\bar{\tau}', X_T), \tag{4.10}
\]
\( G_{1\bar{1}}(\bar{E}, X_T, \bar{\tau}) \) and \( A_1(\bar{E}, X_T, \bar{\tau}) \) are transformed as a symmetric tensor and a vector, respectively and the action is manifestly invariant.

We define \( G_{1\bar{1}}(\bar{E}, X_T, \bar{\tau}) \) and \( A_{1\bar{1}}(\bar{E}, X_T, \bar{\tau}) \) so as to transform as scalars under \( E_M^A(\bar{\sigma}, \bar{\tau}, \bar{\theta}^\alpha) \mapsto \bar{E}_M^A(\bar{\sigma}', \bar{\tau}, \bar{\theta}^\alpha(\bar{\sigma}, \bar{\theta})) \). Under \( (\bar{\sigma}, \bar{\theta}) \) superdiffeomorphisms: \( (\bar{\sigma}', \bar{\theta}') \mapsto (\bar{\sigma}'(\bar{\sigma}, \bar{\theta}), \bar{\theta}'(\bar{\sigma}, \bar{\theta})) \), which are equivalent to
\[
[\bar{E}_M^A(\bar{\sigma}, \bar{\tau}, \bar{\theta}^\alpha), X_T^\mu(\bar{\sigma}, \bar{\tau}, \bar{\theta}^\alpha), \bar{\tau}] \mapsto [\bar{E}_M^A(\bar{\sigma}, \bar{\tau}, \bar{\theta}^\alpha), X_T^\mu(\bar{\sigma}', \bar{\tau}, \bar{\theta}^\alpha(\bar{\sigma}, \bar{\theta})), \bar{\tau}]
\]
\[
= [\bar{E}_M^A(\bar{\sigma}', \bar{\tau}, \bar{\theta}^\alpha(\bar{\sigma}, \bar{\theta})), X_T^\mu(\bar{\sigma}, \bar{\tau}, \bar{\theta}^\alpha), \bar{\tau}], \tag{4.11}
\]
\( G_{d(\mu\alpha\bar{\theta})} \) is transformed as a superscalar;
\[
G_{d(\mu\alpha\bar{\theta})}'(\bar{E}', X_T', \bar{\tau}) = G_{d(\mu\alpha\bar{\theta})}'(\bar{E}, X_T, \bar{\tau}) = \frac{\partial X_T^I}{\partial \bar{X}_T^I} \frac{\partial X_T^J}{\partial \bar{X}_T^J} G_{1\bar{1}}(\bar{E}, X_T, \bar{\tau})
\]
\[
= \frac{\partial X_T^I}{\partial \bar{X}_T^I} \frac{\partial X_T^J}{\partial \bar{X}_T^J} G_{1\bar{1}}(\bar{E}, X_T, \bar{\tau}) = G_{d(\mu\alpha\bar{\theta})}(\bar{E}, X_T, \bar{\tau}), \tag{4.12}
\]
because (4.11) and (4.10). In the same way, the other fields are also transformed as
\[
G_{dd}'(\bar{E}', X_T', \bar{\tau}) = G_{dd}(\bar{E}, X_T, \bar{\tau})
\]
\[
G_{(\mu\alpha\bar{\theta})}'(\nu\bar{\nu}\bar{\theta})'(\bar{E}', X_T', \bar{\tau}) = G_{(\mu\alpha\bar{\theta})}(\nu\bar{\nu}\bar{\theta})(\bar{E}, X_T, \bar{\tau})
\]
\[
A_{d}'(\bar{E}', X_T', \bar{\tau}) = A_{d}(\bar{E}, X_T, \bar{\tau})
\]
\[
A_{(\mu\alpha\bar{\theta})}'(\bar{E}', X_T', \bar{\tau}) = A_{(\mu\alpha\bar{\theta})}(\bar{E}, X_T, \bar{\tau}). \tag{4.13}
\]
Thus, the Lagrangian is invariant under the \( (\bar{\sigma}, \bar{\theta}) \) superdiffeomorphisms, because
\[
\int d\sigma' d^2\bar{\theta}' \bar{E}'(\sigma', \bar{\tau}, \bar{\theta}') = \int d\sigma' d^2\bar{\theta}\bar{E}(\sigma, \bar{\tau}, \bar{\theta}). \tag{4.14}
\]
Therefore, \( G_{1\bar{1}}(\bar{E}, X_T, \bar{\tau}) \) and \( A_{1\bar{1}}(\bar{E}, X_T, \bar{\tau}) \) are transformed covariantly and the action (4.9) is invariant under the diffeomorphisms (4.6), including the \( (\bar{\sigma}, \bar{\theta}) \) superdiffeomorphisms, whose
infinitesimal transformations are given by

\[ \sigma^\xi = \bar{\sigma} + i \xi^\alpha (\bar{\sigma}) \gamma^1_{\alpha\beta} \bar{\theta}^\beta \]
\[ \bar{\theta}^\alpha (\bar{\sigma}) = \bar{\theta}^\alpha + \xi^\alpha (\bar{\sigma}). \] (4.15)

\[ (4.15) \] are dimensional reductions in \( \bar{\tau} \) direction of the two-dimensional \( \mathcal{N} = (1,1) \) local supersymmetry infinitesimal transformations. The number of supercharges

\[ \xi^\alpha Q_\alpha = \xi^\alpha (\frac{\partial}{\partial \theta^\alpha} + i \gamma^1_{\alpha\beta} \ddot{\theta}^\beta \frac{\partial}{\partial \bar{\sigma}}) \] (4.16)

of the transformations is the same as of the two-dimensional ones. The supersymmetry algebra closes in a field-independent sense as in ordinary supergravities.

The background that represents a perturbative vacuum is given by

\[ ds^2 = 2\lambda \rho(h) N^2(X_T)(dX^d_T)^2 + \int d\bar{\sigma} d\bar{\tau} \bar{E} \int d\sigma' d\bar{\tau}' \bar{E}' \nu \frac{\epsilon^2(\bar{\sigma}, \bar{\tau}, \bar{\theta}) \bar{E}(\bar{\sigma}, \bar{\tau}, \bar{\theta})}{\sqrt{h(\sigma, \tau)}} \delta(\mu \bar{\sigma}) (\mu' \bar{\sigma}') dX^\mu_T dX^\mu' T. \]
\[ A_d = i \sqrt{\frac{2 - 2D}{2 - D}} \frac{\sqrt{2\lambda \rho(h)}}{\sqrt{G_N}} N(X_T), \quad \bar{A}(\mu \bar{\sigma}) = 0, \] (4.17)
on \( \mathfrak{M}_0 \) where the target metric is fixed to \( \eta_{\mu\nu} \). \( \rho(h) := \frac{1}{4\pi} \int d\sigma \sqrt{h} \bar{R}_h \), where \( \bar{R}_h \) is the scalar curvature of \( h_{mn} \). \( D \) is a volume of the index \( (\mu \bar{\sigma}) \): \( D := \int d\sigma d\bar{\tau} \bar{E} \delta(\mu \bar{\sigma}) = d \int d\sigma d\bar{\tau} \bar{E} \delta(\sigma - \bar{\sigma}) \delta(\bar{\theta} - \bar{\theta}) \). \( N(X_T) = \frac{1}{1 + \nu(X_T)} \), where \( v(X_T) = \sqrt{\frac{\alpha}{a - 1}} \int d\sigma d\bar{\tau} \bar{E} \frac{\sqrt{E}}{\sqrt{h}} e^\mu X^\mu_T \sqrt{\bar{D}^2 \bar{X}_T}. \)

\( \bar{D}_\alpha \) is a \( \bar{\tau} \) independent super derivative that satisfies

\[ \int d\bar{\tau} d\bar{\sigma} d^2 \bar{\sigma} \bar{E} \frac{1}{2} (\bar{D}_\alpha X_T^\mu)^2 \]
\[ = \int d\bar{\tau} d\bar{\sigma} \sqrt{\frac{1}{2}} (\bar{h} X^\mu + \frac{1}{2} \bar{n}_m \bar{X}^\mu - \bar{\psi}_\mu \bar{E}^\gamma \gamma q \psi_\mu)^2 \]
\[ + \bar{n}_m \bar{E}^\gamma \gamma q \gamma q \psi_\mu \bar{X}^\mu - \frac{1}{8} \bar{\psi}_\mu \bar{\psi}_\mu \bar{X}^\mu \bar{E}^\gamma \gamma q \gamma q \gamma q \bar{X}_m), \] (4.18)

where \( \bar{E}^\mu_q, \bar{X}_m, \) and \( \gamma^q \) are a vierbein, a gravitino, and gamma matrices in the two dimensions, respectively. On the other hand, the ordinary super covariant derivative \( D_\alpha \) satisfies

\[ \int d\bar{\tau} d\bar{\sigma} d^2 \bar{\sigma} \bar{E} \frac{1}{2} (D_\alpha X_T^\mu)^2 \]
\[ = \int d\bar{\tau} d\bar{\sigma} \sqrt{\frac{1}{2}} (\bar{h} X^\mu \bar{D}_\mu \bar{X}_m - \bar{\psi}_\mu \bar{E}^\gamma \gamma q \gamma q \psi_\mu)^2 \]
\[ + \bar{n}_m \bar{E}^\gamma \gamma q \gamma q \psi_\mu \bar{D}_\mu \bar{X}^\mu - \frac{1}{8} \bar{\psi}_\mu \bar{\psi}_\mu \bar{X}^\mu \bar{E}^\gamma \gamma q \gamma q \gamma q \bar{X}_m). \] (4.19)
The inverse of the metric is given by

$$
\tilde{G}^{dd} = \frac{1}{2\tilde{\rho}} \frac{1}{N^2}
$$

$$
\tilde{G}^d(\mu\tilde{\sigma}) = 0
$$

$$
\tilde{G}^d(\mu\tilde{\sigma}) = \sqrt{\frac{\tilde{h}}{e^2E}} \delta(\mu\tilde{\sigma}), \quad (4.20)
$$

because

$$
\int d\tilde{\sigma} d^2\tilde{\sigma} E^n \tilde{G}^d(\mu\tilde{\sigma}) \tilde{G}^d(\mu\tilde{\sigma}) = \int d\tilde{\sigma} d^2\tilde{\sigma} E^n \delta(\mu\tilde{\sigma}) \delta(\mu\tilde{\sigma}) = \delta(\mu\tilde{\sigma})\delta(\mu\tilde{\sigma}).
$$

From the metric, we obtain

$$
\sqrt{\tilde{G}} = N^{\frac{2}{D-2}} \sqrt{2\tilde{\rho} \exp(\int d\tilde{\sigma} d^2\tilde{\sigma} E \delta(\mu\tilde{\sigma})) \ln \frac{e^2E}{\sqrt{\tilde{h}}}}.
$$

$$
\tilde{R}_{dd} = -2\tilde{\rho} \int d\tilde{\sigma} ^2 d\tilde{\sigma} \delta(\mu\tilde{\sigma}) N \partial(\mu\tilde{\sigma}) N 
$$

$$
\tilde{R}_{(\mu\tilde{\sigma})} = \frac{D - 1}{2 - D} N^{-2} \partial(\mu\tilde{\sigma}) N \partial(\mu\tilde{\sigma}) N
$$

$$
+ \frac{1}{D - 2} N^{-2} \int d\tilde{\sigma} d^2\tilde{\sigma} \delta(\mu\tilde{\sigma}) N \partial(\mu\tilde{\sigma}) N \frac{e^2E}{\sqrt{\tilde{h}}} \delta(\mu\tilde{\sigma}),
$$

$$
(\text{4.21})
$$

By using these quantities, one can show that the background \( \text{(4.17)} \) is a classical solution to the equations of motion of \( \text{(4.13)} \). We also need to use the fact that \( v(X_T) \) is a harmonic function with respect to \( X_T(\mu\tilde{\sigma}) \). Actually, \( \partial(\mu\tilde{\sigma}) v = 0 \). In these calculations, we should note that \( E_A^M, X_T^\mu, \text{ and } \bar{T} \) are all independent. Because the equations motion are differential equations with respect to \( X_T^\mu \) and \( \bar{T} \), \( E_A^M \) is a constant in the solution \( \text{(4.17)} \) to the differential equations. The dependence of \( \bar{E}_A^M \) on the background \( \text{(4.17)} \) is uniquely determined by the consistency of the quantum theory of the fluctuations around the background. Actually, we will find that all the perturbative superstring amplitudes are derived.

Let us consider fluctuations around the background \( \text{(4.17)} \), \( G_D = \tilde{G}_{D} + \tilde{G}_{D} \) and \( A_T = \tilde{A}_T + \tilde{A}_T \). Here we fix the charts, where we choose \( T=\text{IIA} \) or \( \text{IIB} \). The action \( \text{(4.9)} \) up to the
quadratic order is given by,

\[
S = \frac{1}{G_N} \int \mathcal{D}E \mathcal{D}X_T \mathcal{D}\bar{\tau} \sqrt{G} \left( -\bar{R} + \frac{1}{4} \bar{F}'_{IJ} \bar{F}^{IJ} \right ) \\
+ \frac{1}{4} \bar{\nabla}_I \bar{G}_{JK} \bar{\nabla}^I \bar{G}^{JK} - \frac{1}{8} \bar{\nabla}_I \bar{G} \bar{\nabla}^I \bar{G} + \frac{1}{2} \bar{\nabla}_I \bar{G}_{IJ} \bar{\nabla}^J \bar{G} - \frac{1}{2} \bar{\nabla}_I \bar{G}_{IJ} \bar{\nabla}_K \bar{G}^{JK} \\
- \frac{1}{4} (-\bar{R} + \frac{1}{4} \bar{F}'_{KL} \bar{F}^{KL}) (\bar{G}_{IJ} \bar{G}^{IJ} - \frac{1}{2} \bar{G}^2) + (-\frac{1}{2} \bar{R}_{IJ} + \frac{1}{2} \bar{F}'_{IK} \bar{F}^{IK} \bar{G}_{IL} \bar{G}^{JL} \\
+ \frac{1}{2} \bar{R}_{IJ} - \frac{1}{4} \bar{F}'_{IK} \bar{F}^{IK} \bar{G}_{IL} \bar{G}^{JL} + (-\frac{1}{2} \bar{R}_{LJ} \bar{G}^{LJ} + \frac{1}{2} \bar{F}'_{LJ} \bar{F}^{LJ} \bar{G}_{IK} \bar{G}^{JL} \\
+ \frac{1}{4} G_N \bar{F}_{IJ} \bar{F}^{IJ} + \sqrt{G_N} \left( \frac{1}{4} \bar{F}'_{IJ} \bar{F}_{IJ} \bar{G} - \bar{F}'_{LJ} \bar{F}_{IK} \bar{G}^{JK} \right ), \tag{4.22}
\]

where \( \bar{F}'_{IJ} := \sqrt{G_N} \bar{F}_{IJ} \) is independent of \( G_N \). \( \bar{G} := \bar{G}_{IJ} \bar{G}^{IJ} \). There is no first order term because the background satisfies the equations of motion. If we take \( G_N \to 0 \), we obtain

\[
S' = \frac{1}{G_N} \int \mathcal{D}E \mathcal{D}X_T \mathcal{D}\bar{\tau} \sqrt{G} \left( -\bar{R} + \frac{1}{4} \bar{F}'_{IJ} \bar{F}^{IJ} \right ) \\
+ \frac{1}{4} \bar{\nabla}_I \bar{G}_{JK} \bar{\nabla}^I \bar{G}^{JK} - \frac{1}{8} \bar{\nabla}_I \bar{G} \bar{\nabla}^I \bar{G} + \frac{1}{2} \bar{\nabla}_I \bar{G}_{IJ} \bar{\nabla}^J \bar{G} - \frac{1}{2} \bar{\nabla}_I \bar{G}_{IJ} \bar{\nabla}_K \bar{G}^{JK} \\
- \frac{1}{4} (-\bar{R} + \frac{1}{4} \bar{F}'_{KL} \bar{F}^{KL}) (\bar{G}_{IJ} \bar{G}^{IJ} - \frac{1}{2} \bar{G}^2) + (-\frac{1}{2} \bar{R}_{IJ} + \frac{1}{2} \bar{F}'_{IK} \bar{F}^{IK} \bar{G}_{IL} \bar{G}^{JL} \\
+ \frac{1}{2} \bar{R}_{IJ} - \frac{1}{4} \bar{F}'_{IK} \bar{F}^{IK} \bar{G}_{IL} \bar{G}^{JL} + (-\frac{1}{2} \bar{R}_{LJ} \bar{G}^{LJ} + \frac{1}{2} \bar{F}'_{LJ} \bar{F}^{LJ} \bar{G}_{IK} \bar{G}^{JL} \\
+ \frac{1}{4} G_N \bar{F}_{IJ} \bar{F}^{IJ} + \sqrt{G_N} \left( \frac{1}{4} \bar{F}'_{IJ} \bar{F}_{IJ} \bar{G} - \bar{F}'_{LJ} \bar{F}_{IK} \bar{G}^{JK} \right ), \tag{4.23}
\]

where the fluctuation of the gauge field is suppressed. In order to fix the gauge symmetry \([4.10]\), we take the harmonic gauge. If we add the gauge fixing term

\[
S_{fix} = \frac{1}{G_N} \int \mathcal{D}E \mathcal{D}X_T \mathcal{D}\bar{\tau} \sqrt{G} \frac{1}{2} \left( \bar{\nabla}^J (\bar{G}_{IJ} - \frac{1}{2} \bar{G} \bar{G}^{IJ} \right )^2, \tag{4.24}
\]

we obtain

\[
S' + S_{fix} = \frac{1}{G_N} \int \mathcal{D}E \mathcal{D}X_T \mathcal{D}\bar{\tau} \sqrt{G} \left( -\bar{R} + \frac{1}{4} \bar{F}'_{IJ} \bar{F}^{IJ} \right ) \\
+ \frac{1}{4} \bar{\nabla}_I \bar{G}_{JK} \bar{\nabla}^I \bar{G}^{JK} - \frac{1}{8} \bar{\nabla}_I \bar{G} \bar{\nabla}^I \bar{G} \\
- \frac{1}{4} (-\bar{R} + \frac{1}{4} \bar{F}'_{KL} \bar{F}^{KL}) (\bar{G}_{IJ} \bar{G}^{IJ} - \frac{1}{2} \bar{G}^2) + (-\frac{1}{2} \bar{R}_{IJ} + \frac{1}{2} \bar{F}'_{IK} \bar{F}^{IK} \bar{G}_{IL} \bar{G}^{JL} \\
+ \frac{1}{2} \bar{R}_{IJ} - \frac{1}{4} \bar{F}'_{IK} \bar{F}^{IK} \bar{G}_{IL} \bar{G}^{JL} + (-\frac{1}{2} \bar{R}_{LJ} \bar{G}^{LJ} + \frac{1}{2} \bar{F}'_{LJ} \bar{F}^{LJ} \bar{G}_{IK} \bar{G}^{JL} \right ) \tag{4.25}
\]

In order to obtain perturbative string amplitudes, we perform a derivative expansion of
\[ \tilde{G}_{IJ}, \]
\[ \tilde{G}_{IJ} \rightarrow \frac{1}{\alpha} \tilde{G}_{IJ} \]
\[ \partial_K \tilde{G}_{IJ} \rightarrow \partial_K \tilde{G}_{IJ} \]
\[ \partial_K \partial_L \tilde{G}_{IJ} \rightarrow \alpha \partial_K \partial_L \tilde{G}_{IJ}, \] (4.26)

and take
\[ \alpha \rightarrow 0, \] (4.27)

where \( \alpha \) is an arbitrary constant in the solution (4.17). We normalize the fields as \( \tilde{H}_{IJ} := \tilde{Z}_{IJ} \tilde{G}_{IJ} \), where \( \tilde{Z}_{IJ} := \frac{1}{\sqrt{G_{ij}}} |\tilde{a}_{(\mu \bar{\sigma} \bar{\theta})} + \frac{1}{2} \tilde{F}_{IJ}^\alpha \bar{\theta} \rangle \). Then, (4.25) with appropriate boundary conditions reduces to
\[ S' + S_{fix} \rightarrow S_0 + S_2, \] (4.28)

where
\[ S_0 = \frac{1}{G_N} \int DEDX_T D\tau \sqrt{G} \left( -\bar{R} + \frac{1}{4} \tilde{F}_{IJ} \tilde{F}^{IJ} \right), \] (4.29)

and
\[ S_2 = \int DEDX_T D\tau \frac{1}{8} \tilde{H}_{IJ} H_{KL} \tilde{H}_{KL}. \] (4.30)

In the same way as in the previous section, a part of the action
\[ \int DEDX_T D\tau \frac{1}{4} \int_0^{2\pi} d\bar{\sigma} d^2 \bar{\theta} \tilde{H}_{d(\mu \bar{\sigma} \bar{\theta})} H \tilde{H}_{d(\mu \bar{\sigma} \bar{\theta})} \] (4.31)

with
\[ H = \frac{1}{2} 2 \lambda \bar{\rho} \left( \frac{\partial}{\partial \bar{\sigma}} \right)^2 - \frac{1}{2} \int_0^{2\pi} d\bar{\sigma} \int d^2 \bar{\theta} \sqrt{\bar{h}} \left( \frac{\partial}{\partial \bar{X}_T^\mu} \right)^2 + \frac{1}{2} (D^2 - 9D + 20) \int_0^{2\pi} d\bar{\sigma} \int d^2 \bar{\theta} E (\tilde{D}_\alpha X_T^\mu)^2, \] (4.32)

decouples from the other modes.

Because \( X_T^\mu \) can be expanded as \( X_T^\mu = X^\mu + \tilde{\theta} \psi^\mu_\alpha + \frac{1}{2} \tilde{\theta}^2 F^\mu_\alpha \), we have
\[ \int d^2 \bar{\theta} \left( \frac{\partial}{\partial \bar{X}_T^\mu (\bar{\sigma}', \bar{\theta})} \right)^2 \tilde{H}_{d(\mu \bar{\sigma} \bar{\theta})} = \left( \frac{\partial}{\partial \bar{X}_T^\mu (\bar{\sigma}') \bar{\theta}} \right)^2 \tilde{H}_{d(\mu \bar{\sigma} \bar{\theta})}, \] (4.33)

Then, (4.31) can be simplified where
\[ H = \frac{1}{2} \lambda \bar{\rho} \left( \frac{\partial}{\partial \bar{\sigma}} \right)^2 - \frac{1}{2} \int_0^{2\pi} d\bar{\sigma} \int d^2 \bar{\theta} \sqrt{\bar{h}} \left( \frac{\partial}{\partial \bar{X}_T^\mu} \right)^2 + \frac{1}{2} (D^2 - 9D + 20) \int_0^{2\pi} d\bar{\sigma} \int d^2 \bar{\theta} E (\tilde{D}_\alpha X_T^\mu)^2. \] (4.34)
By adding to (4.31), a formula similar to the bosonic case

\[
0 = \int \mathcal{D}E \mathcal{D}X_T \mathcal{D}\bar{\tau} \left\{ \frac{1}{4} \right\} \left( \int_0^{2\pi} d\sigma' d^2\theta' \tilde{H}_{d(\mu\sigma')}^\dagger \left( \int_0^{2\pi} d\bar{\sigma} \bar{\gamma}^\mu \frac{\partial}{\partial \bar{X}^\mu} \tilde{H}_{d(\mu\sigma')}^\dagger \right) \right\}
\]

and

\[
0 = \int \mathcal{D}E \mathcal{D}X_T \mathcal{D}\bar{\tau} \left\{ \frac{1}{4} \right\} \left( \int_0^{2\pi} d\sigma' d^2\theta' \tilde{H}_{d(\mu\sigma')}^\dagger \right) \left( \int_0^{2\pi} d\bar{\sigma} \tilde{E} \left\{ -i \right\} \bar{\bar{\gamma}}^\mu \frac{\partial}{\partial X^\mu} \tilde{H}_{d(\mu\sigma')}^\dagger \right)
\]

we obtain (4.31) with

\[
H(-i \frac{\partial}{\partial \bar{\tau}}, -i \left\{ \frac{1}{\bar{\epsilon}} \right\} \frac{\partial}{\partial \tilde{X}}, \tilde{X}_T, \tilde{E}) = \frac{1}{2} 2\lambda \rho (-i \frac{\partial}{\partial \bar{\tau}})^2 + \int d\sigma \left( \sqrt{\hbar} \left\{ \frac{1}{2} \left\{ -i \right\} \frac{\partial}{\partial \tilde{X}^\mu} \right\}^2 - i \frac{\bar{\bar{\gamma}}^\mu \bar{\bar{\gamma}}^\sigma \psi^\dagger (-i \left\{ \frac{\partial}{\partial \tilde{X}^\mu} \right\}) \right) + \int d\sigma d^2\theta \bar{E} \frac{1}{2} (\tilde{D}_\alpha \tilde{X} \tilde{E}_\mu)^2
\]

(4.37)

where we have taken $D \rightarrow \infty$. (4.37) is true because the integrand of the right hand side is a total derivative with respect to $X^\mu$.

The propagator $\Delta_F(\tilde{E}, X_T, \bar{\tau}; \tilde{E}', \tilde{X}_T, \bar{\tau}')$ for $\tilde{H}_{d(\mu\sigma')}^\dagger$ is defined by

\[
\Delta_F(\tilde{E}, X_T, \bar{\tau}; \tilde{E}', \tilde{X}_T, \bar{\tau}') = \delta(\tilde{E} - \tilde{E}') \delta(\tilde{X}_T - \tilde{X}_T') \delta(\bar{\tau} - \bar{\tau}').
\]

(4.38)

In order to obtain a Schwinger representation of the propagator, we use the operator formalism $\hat{H}(\tilde{E}, \tilde{X}_T, \bar{\tau})$ of the first quantization. The eigenstate for $\hat{E}(\tilde{E}, \tilde{X}_T, \bar{\tau})$ is given by $|\tilde{E}, \tilde{X}, \bar{\tau} >$. The conjugate momentum is written as $\hat{p}(\tilde{E}, \tilde{X}, \bar{\tau})$. There is no conjugate momentum for the auxiliary field $F^\mu$, whereas the Majorana fermion $\tilde{\psi}^\mu$ is self-conjugate.

The renormalized operators $\tilde{\psi}^\mu$ satisfy $\tilde{\psi}^\mu(\bar{\sigma}), \tilde{\psi}^\mu(\bar{\sigma}')$ = $\frac{1}{\bar{\epsilon}} \delta_{\alpha\beta} \eta^{\mu\nu} \delta(\bar{\sigma} - \bar{\sigma}')$ as summarized in the appendix B. By defining creation and annihilation operators for $\tilde{\psi}^\mu$ as $\tilde{\psi}^\mu \tilde{\psi}^\mu := \frac{1}{\sqrt{2}}(\tilde{\psi}_1^\mu - i\tilde{\psi}_2^\mu)$ and $\tilde{\psi}^\mu := \frac{1}{\sqrt{2}}(\tilde{\psi}_1^\mu + i\tilde{\psi}_2^\mu)$, one obtains an algebra $\{\tilde{\psi}^\mu(\bar{\sigma}), \tilde{\psi}^\mu(\bar{\sigma}')\} = \frac{1}{\bar{\epsilon}} \eta^{\mu\nu} \delta(\bar{\sigma} - \bar{\sigma}')$, $\{\tilde{\psi}^\mu(\bar{\sigma}), \tilde{\psi}^\mu(\bar{\sigma}')\} = 0$, and $\{\tilde{\psi}^\mu(\bar{\sigma}), \tilde{\psi}^\mu(\bar{\sigma}')\} = 0$. The vacuum $|0 >$ for this algebra is defined by $\tilde{\psi}^\mu(\bar{\sigma})|0 > = 0$. The eigenstate $|\psi >$, which satisfies $\tilde{\psi}^\mu(\bar{\sigma})|\psi > = \tilde{\psi}^\mu(\bar{\sigma})|\psi >$, is given by $e^{-\tilde{\psi}^\mu |0 >} = e^{-i \int d\sigma \tilde{\psi}^\mu(\bar{\sigma})|\psi >} = e^{-\hat{E} \tilde{\psi}^\mu(\bar{\sigma})|\psi >} |0 >$. Then, the inner product is given by $< \tilde{\psi}^\mu(\bar{\sigma})|\psi >= e^{\hat{E} \tilde{\psi}^\mu(\bar{\sigma})} |\psi >$, whereas the completeness relation is $\int \mathcal{D}\tilde{\psi}^\dagger \mathcal{D}\tilde{\psi}^\mu |\psi > e^{-\hat{E} \tilde{\psi}^\dagger(\bar{\sigma})} |\psi > = 1$.  

35
Because (4.38) means that $\Delta F$ is an inverse of $H$, $\Delta F$ can be expressed by a matrix element of the operator $\hat{H}^{-1}$ as

$$\Delta F(\bar{E}, X_T, \bar{\tau}; \bar{E}', X'_T, \bar{\tau}') = \langle \bar{\tau}', X'_T, \hat{E}' | \hat{H}^{-1} (\hat{p}_{\bar{\tau}} \hat{p}_X, \hat{X}_T, \hat{E}) | \bar{\tau}, X_T, \bar{E} \rangle.$$  \hfill (4.39)

(3.34) implies that

$$\Delta F(\bar{E}, X_T, \bar{\tau}; \bar{E}', X'_T, \bar{\tau}') = \int_0^\infty dT < \bar{\tau}, X_T, \bar{E} | e^{-T\hat{H}} | \bar{\tau}', X'_T, \bar{E}' \rangle.$$ \hfill (4.40)

We define in and out states as

$$||X_{Ti} | E_f; E_i >_{in} := \int_{E_i}^{E_f} D\bar{E} | \bar{E}', X_{Ti}, \bar{\tau} = -\infty >$$

$$< X_{Tf} | E_f; E_i | >_{out} := \int_{E_i}^{E_f} D\bar{E} < \bar{\tau}, X_{Tf}, \bar{E} | X_{Ti}, \bar{\tau} = \infty |,$$ \hfill (4.41)

where $E_i$ and $E_f$ represent the super vierbeins of the supercylinders at $\bar{\tau} = \mp\infty$, respectively.

By inserting

$$1 = \int d\bar{E}_m d\bar{\tau}_m dX_{Tm} | \bar{E}_m, \bar{\tau}_m, X_{Tm} > e^{-\tilde{\psi}^\dagger_m \tilde{\psi}_m} < \bar{E}_m, \bar{\tau}_m, X_{Tm} |$$

$$1 = \int dp_{\bar{\tau}} dp_X | p_{\bar{\tau}}, p_X > < p_{\bar{\tau}}, p_X |,$$ \hfill (4.42)
the two-point correlation function for the in and out states is given by\(^{10}\)

\[
\Delta_F(X_{Tf}; X_{Ti}|E_f; ; E_i) := \int_0^\infty dT < X_{Tf} | E_f; ; E_i | \exp(-TH)| X_{Ti} | E_f; ; E_i >_{in}
\]

\[
= \int_0^\infty dT \lim_{N \to \infty} \int \frac{E_f}{E_i} \frac{DE}{DE'} \prod_{m=1}^N \prod_{i=0}^N \int d\bar{E}_m d\bar{\tau}_m dX_{Tm} e^{-\bar{\psi}_m \bar{\psi}_m}\nonumber
\]

\[
= \int_0^\infty dT_0 \lim_{N \to \infty} \int dT_{N+1} \int \frac{E_f}{E_i} \frac{DE}{DE'} \prod_{m=1}^N \prod_{i=0}^N \int dT_m d\bar{E}_m d\bar{\tau}_m dX_{Tm} e^{-\bar{\psi}_m \bar{\psi}_m}\nonumber
\]

\[
= \int_0^\infty dT_0 \lim_{N \to \infty} \int dT_{N+1} \int \frac{E_f}{E_i} \frac{DE}{DE'} \prod_{m=1}^N \prod_{i=0}^N \int dT_m d\bar{E}_m d\bar{\tau}_m dX_{Tm} e^{-\bar{\psi}_m \bar{\psi}_m}\nonumber
\]

\[
\exp \left( - \sum_{i=0}^N \Delta t \left( -ip_{T_i} \frac{T_{i+1} - T_i}{\Delta t} - ip_{F_i} \frac{F_{i+1} - F_i}{\Delta t} + \bar{\psi}_{i+1} \frac{\bar{\psi}_{i+1} - \bar{\psi}_i}{\Delta t} \right) \right)
\]

\[
e^{\psi_{N+1} \bar{\psi}_N} \Delta F(X_{Tf}; X_{Ti}, E_f, E_i)
\]

\[
= \int_{E_i; X_{Ti}, -\infty} DT \frac{DE \frac{d\tau}{dX_T} DX_T}{D \frac{Dp_T \frac{Dp_F \frac{Dp_X \frac{Dp_F}{Dp_F} X_{Tf}}{Dp_F} X_{Tf}}{Dp_F}}{D p_F} X_{Tf}} e^{-\int_0^1 dt (-ip_{\tau} \frac{d}{dT} - ip_{F} \frac{d}{dF} + \bar{\psi} \frac{d}{d\bar{\psi}} - ip_{\tau} \frac{d}{d\psi} X + TH(p_{\tau}, p_{F}, p_{X}, E_f, E_i, X_{Tf}), (4.43)}
\]

where \(E_0 = E'\), \(\tau_0 = -\infty\), \(X_{T0} = X_{Ti}\), \(E_{N+1} = E\), \(\tau_{N+1} = \infty\), and \(X_{TN+1} = X_{Tf}\).

\(^{10}\)The correlation function is zero if \(E_i\) and \(E_f\) of the in state do not coincide with those of the out states, because of the delta functions in the sixth line.
\( p_X \cdot \frac{dX}{dt} = \int d\bar{\sigma} \bar{p}_X^\mu \frac{dX}{dt} X_\mu \) and \( \Delta t = \frac{1}{k} \) as in the bosonic case. A trajectory of points \( [\Sigma, X_T, \tau] \) is necessarily continuous in \( M_0 \) so that the kernel \( <E_{t+1}, \bar{r}_{t+1}, X_{T_{t+1}}|e^{-\frac{k}{T}H}|E_t, \bar{r}_t, X_{T_t}> \) in the fourth line is non-zero when \( N \to \infty \). If we integrate out \( p_{r}(t), p_{X}(t) \) and \( p_{F}(t) \) by using the relation of the ADM formalism and the relation between \( \bar{\psi}^\mu \) and \( \psi^\mu \) in the appendix A and B, we obtain

\[
\Delta_F(X_{T_{t+1}}|E_f, ; E_i) = \int_{E_i, X_{T_i}, -\infty}^{E_f, X_{T_f}, \infty} DT\mathbb{D}E\mathbb{D}\bar{\tau}\mathbb{D}X_T \int Dp_T 
\]

\[
\exp \left( -\int_0^1 dt \left( -ip_T(t) \frac{d}{dt} T(t) + \lambda \bar{\rho} \left( \frac{d\bar{\tau}(t)}{dt} \right)^2 \right)
\]

\[
+ \int d\bar{\sigma} \sqrt{\gamma} T(t) \left( \frac{1}{2\bar{n}} \left( \frac{\partial}{\partial t} X^\mu - \bar{n}^\rho \partial_{\rho} X^\mu + \frac{1}{2} \bar{n}^2 \bar{X}_m \bar{E}_r^0 g^{\rho q} E_q \gamma^q \psi^\rho \right)^2
\]

\[
- \frac{1}{2} \left( \frac{1}{T(t)} \bar{\psi}^\mu \bar{E}_q g^{\rho q} \frac{\partial}{\partial t} \psi^\mu \right) + \int d\bar{\sigma} d^2 \bar{\psi} \left( \frac{1}{2} T(t) (\bar{D}_\alpha X^\mu) \right)^2 \right) \right)
\]

When the last equality is obtained, we use (4.19) and (4.18). In the last line, \( F^\mu \) is constant with respect to \( t \), and \( \bar{D}_\alpha' \) is given by replacing \( \frac{\partial}{\partial \tau} \) with \( \frac{1}{T(t)} \frac{\partial}{\partial t} \) in \( D_\alpha \). The path integral is defined over all possible trajectories with fixed boundary values, on the superstring manifold \( \mathbb{M}_0 \).

By inserting \( \int Dc\mathbb{D}b e^{\int_0^1 dt \left( \frac{db(t)}{dt} \frac{dc(t)}{dt} \right)} \), where \( b(t) \) and \( c(t) \) are bc ghosts, we obtain

\[
\Delta_F(X_{T_{t+1}}|E_f, ; E_i) = Z_0 \int_{E_i, X_{T_i}, -\infty}^{E_f, X_{T_f}, \infty} DT\mathbb{D}E\mathbb{D}\bar{\tau}\mathbb{D}X_T \mathbb{D}c \mathbb{D}b \int Dp_T \exp \left( -\int_0^1 dt \left( -ip_T(t) \frac{d}{dt} T(t)
\]

\[
+ \frac{db(t)}{dt} \frac{d(T(t) c(t))}{dt} + \lambda \bar{\rho} \left( \frac{d\bar{\tau}(t)}{dt} \right)^2 + \int d\bar{\sigma} d^2 \bar{\psi} \left( \frac{1}{2} T(t) (\bar{D}_\alpha' X^\mu) \right)^2 \right) \right).
\]

where we have redefined as \( c(t) \to T(t) c(t) \). \( Z_0 \) represents an overall constant factor, and we will rename it \( Z_1, Z_2, \ldots \) when the factor changes in the following. This path integral is
obtained if
\[ F_1(t) := \frac{d}{dt}T(t) = 0 \] (4.46)
gauge is chosen in
\[
\Delta_F(X_T; X_{T_i}|E_f,; E_i) = \int_{E_f, X_T, -\infty}^{E_f, X_T, \infty} \mathcal{D}T \mathcal{D}E \mathcal{D}T \mathcal{D}X_T \int \exp \left( - \int_0^1 dt \left( + \lambda \bar{\rho} \frac{1}{T(t)} \left( \frac{d\bar{\tau}(t)}{dt} \right)^2 + \int d\bar{\sigma} d^2 \bar{\theta} (E_1^2 \frac{1}{2} (D_{\alpha} X_{T_{\mu}})^2) \right) \right),
\] (4.47)
which has a manifest one-dimensional diffeomorphism symmetry with respect to \( t \), where \( T(t) \) is transformed as an einbein [11].

Under a rescale \( \bar{\tau} = \bar{\tau}' T(t) \), \( T(t) \) disappears in (4.47) as in the bosonic case, and we obtain
\[
\Delta_F(X_T; X_{T_i}|E_f,; E_i) = \int_{E_f, X_T, -\infty}^{E_f, X_T, \infty} \mathcal{D}E \mathcal{D}\bar{\tau} \mathcal{D}X_T \int \exp \left( - \int_0^1 dt \left( + \lambda \bar{\rho} \frac{1}{T(t)} (\frac{d\bar{\tau}(t)}{dt})^2 + \int d\bar{\sigma} d^2 \bar{\theta} (E_1^2 (D_{\alpha} X_{T_{\mu}})^2) \right) \right),
\] (4.48)
where \( D_{\alpha}' \) is given by replacing \( \frac{\partial}{\partial \bar{\tau}} \) with \( \frac{\partial}{\partial t} \) in \( D_{\alpha} \). This action is still invariant under the diffeomorphism with respect to \( t \) if \( \bar{\tau} \) transforms in the same way as \( \frac{1}{T(t)} \).

If we choose a different gauge
\[ F_2(t) := \bar{\tau} - t = 0, \] (4.49)
\[
\Delta_F(X_T^f; X_{Ti}|E_f, ; E_i) = Z_\beta \int_{E_i, X_{Ti}, -\infty}^{E_f, X_T^f, \infty} DED\bar{\tau}DX_T \int D\alpha Dc Db \\
\exp \left( -\int_0^1 dt \left( \alpha(t)(\bar{\tau} - t) + b(t)c(t)(1 - \frac{d\bar{\tau}(t)}{dt}) + \lambda\bar{\rho}\left(\frac{d\bar{\tau}(t)}{dt}\right)^2 \right) \\
+ \int d\bar{\sigma} d^2\theta (E_1^2(\bar{D}_\alpha X_{T\mu})^2) \right) \\
= Z \int_{E_i, X_{Ti}}^{E_f, X_T^f} DEDX_T \\
\exp \left( -\int_{-\infty}^\infty d\bar{\tau} \left( \frac{1}{4\pi} \int d\bar{\sigma} \sqrt{h}\lambda R(\bar{\sigma}, \bar{\tau}) + \int d\bar{\sigma} d^2\theta (E_1^2(\bar{D}_\alpha X_{T\mu})^2) \right) \right). \tag{4.50}
\]

In the second equality, we have redefined as \(c(t)(1 - \frac{d\bar{\tau}(t)}{dt}) \rightarrow c(t)\) and integrated out the ghosts. The path integral is defined over all possible two-dimensional super Riemannian manifolds with fixed punctures in \(\mathbb{R}^{d-1,1}\) as in Fig. 5. By using the two-dimensional superdiffeomorphism and super Weyl invariance of the action, we obtain

\[
\Delta_F(X_T^f; X_{Ti}|E_f, ; E_i) = Z \int_{E_i, X_{Ti}}^{E_f, X_T^f} DEDX_T e^{-\lambda x e^{-\int d^2\sigma d^2\theta E_2^2(\bar{D}_\alpha X_{T\mu})^2}}, \tag{4.51}
\]

where \(\chi\) is the Euler number of the two-dimensional super Riemannian manifold. By inserting asymptotic states to (6.21), we obtain the all-order perturbative scattering amplitudes that possess the supermoduli in the type IIA and IIB superstring theory for \(T = \text{IIA and IIB}\), respectively [12]. Especially, in superstring geometry, the consistency of the perturbation theory around the background (3.11) determines \(d = 10\) (the critical dimension).

5 Including open strings

Let us define unique global times on oriented open-closed string worldsheets \(\bar{\Sigma}\) with open and closed punctures in order to define string states by world-time constant lines. \(\bar{\Sigma}\) can be given by \(\bar{\Sigma} = \bar{\Sigma}_c/Z_2\) where \(Z_2\) is generated by an anti-holomorphic involution \(\rho\) and \(\bar{\Sigma}_c\) is an oriented Riemann surface with closed punctures that satisfies \(\rho(\bar{\Sigma}_c) = \bar{\Sigma}_c^* \cong \bar{\Sigma}_c\). That is,
\( \tilde{\Sigma} \) is an oriented closed double cover of \( \Sigma \). First of all, we define global coordinates on \( \tilde{\Sigma} \) in the same way as in section 2. The real part of the global coordinates \( \tilde{\tau} \) remains on \( \tilde{\Sigma} \) because \( \rho \) is an anti-holomorphic involution. If \( \rho \) maps a puncture to another puncture on \( \tilde{\Sigma} \), the discs around the punctures are identified and give a disk \( D_i \) around a closed puncture \( P^i \) on \( \tilde{\Sigma} \). On the other hand, if \( \rho \) maps a puncture to itself on \( \tilde{\Sigma} \), the disc around the puncture is identified with itself and gives an upper half disk \( \tilde{D}_j \) around an open puncture \( \tilde{P}^j \) on \( \tilde{\Sigma} \). The \( \tilde{\sigma} \) regions around \( P_i \) and \( \tilde{P}_j \) are \( 2\pi f^i \) and \( \pi \tilde{f}^j \), respectively where \( \sum_{i=1}^{N} 2f^i + \sum_{j=1}^{M} \tilde{f}^j = 0 \). This means that \( 2\pi f^i \) is the circumference of a cylinder from \( P^i \), whereas \( \pi \tilde{f}^j \) is the width of a strip from \( \tilde{P}^j \). \( \tilde{\tau} = -\infty \) at \( P_i \) and \( \tilde{P}_j \) with negative \( f^i \) and \( \tilde{f}^j \), respectively, whereas \( \tilde{\tau} = \infty \) at \( P_i \) and \( \tilde{P}_j \) with positive \( f^i \) and \( \tilde{f}^j \), respectively. The \( \tilde{\sigma} \) region of incoming cylinders and strips equals to that of outgoing ones if we choose the outgoing direction as positive. That is, the total \( \tilde{\sigma} \) region is conserved. In order to define the above global time uniquely, we need to fix the \( \tilde{\sigma} \) regions \( 2\pi f^i \) and \( \pi \tilde{f}^j \) around \( P_i \) and \( \tilde{P}_j \), respectively. We divide \( (NP_i, M_+ \tilde{P}_j) \) to arbitrary two sets consist of \( (N_- P^i, M_- \tilde{P}^j) \) for incoming punctures and \( (N_+ P^i, M_+ \tilde{P}^j) \) for outgoing punctures \( (N_- + N_+ = N, M_- + M_+ = M) \), then we divide \( -1 \) to \( N_- f^i \equiv -\frac{2}{2N_- + M_-} \) and \( N_+ \tilde{f}^j \equiv -\frac{1}{2N_- + M_-} \), and \( 1 \) to \( N_+ f^i \equiv \frac{2}{2N_+ + M_+} \) and \( M_+ \tilde{f}^j \equiv \frac{1}{2N_+ + M_+} \), equally for all \( i \) and \( j \).

Thus, under a conformal transformation, one obtains \( \tilde{\Sigma} \) that has coordinates composed of the global time \( \tilde{\tau} \) and the position \( \tilde{\sigma} \). Because \( \tilde{\Sigma} \) can be a moduli of oriented open-closed string worldsheets with open and closed punctures, any two-dimensional oriented Riemannian manifold with open and closed punctures and with or without boundaries \( \Sigma \) can be obtained by \( \Sigma = \psi(\tilde{\Sigma}) \) where \( \psi \) is a diffeomorphism times Weyl transformation.

Next, we will define a string space \( \mathcal{M}_D \). Here we fix not only a d-dimensional Riemannian manifold \( M \) but also a set of its submanifolds \( L \) and \( N \) dimensional vector bundles \( E \) with gauge connections on them, which we call D-submanifolds and D-bundles, respectively. We may also fix orientifold planes on \( M \) consistently. We consider a state \((\tilde{\Sigma}, u_D, \mathbf{I}, \tilde{\tau}_s)\) determined by a \( \tilde{\tau} = \tilde{\tau}_s \) constant line. \( u_D : \tilde{\Sigma} \rightarrow \tilde{M} \) is an arbitrary map that maps a boundary component into a D-submanifold, where \( D \) represents the above fixed background. \( \mathbf{I} = (i_1, \cdots, i_k, \cdots, i_b) \) represents a set of the Chan-Paton indices of the \( k \)-th boundary component on \( \tilde{\Sigma} \). \( i_k \) runs from 1 to \( N_k \) that represents the dimension of the D-bundle where
the $k$-th boundary component maps. The zero mode and the boundary conditions of $u_D$ are determined by the D-submanifolds $L$ and the gauge connections. $\Sigma$ is a direct sum of $N_\pi$ cylinders with radii $f_i$ and $M_\pi$ strips with width $\pi \hat{f}^j$ at $\hat{\tau} \simeq \hat{\tau}_\infty$. Thus, we define a string state as an equivalence class $[\Sigma, u_D, I, \hat{\tau}_s \simeq \hat{\tau}_\infty]$ by a relation $(\Sigma, u_D, I, \hat{\tau}_s \simeq \hat{\tau}_\infty) \sim (\Sigma', u'_D, I', \hat{\tau}_s \simeq \hat{\tau}_\infty)$ if $N_\pi = N'_\pi$, $M_\pi = M'_\pi$, $f^i = f'^i$, $\hat{f}^j = \hat{f}'^j$, $I|_{\hat{\tau}_s \simeq \hat{\tau}_\infty} = I'|_{\hat{\tau}_s \simeq \hat{\tau}_\infty}$ and $u_D|_{\hat{\tau}_s \simeq \hat{\tau}_\infty} = u'_D|_{\hat{\tau}_s \simeq \hat{\tau}_\infty}$. $[\Sigma, u_D, I, \hat{\tau}_s]$ represent many-body states of open and closed strings in $M$ because $\Sigma|_{\hat{\tau}_s} \simeq S^1 \times \cdots \times S^1 \times I^1 \times \cdots \times I^1$ where $I^1$ represents a line segment, and $u_D|_{\hat{\tau}_s} : \Sigma|_{\hat{\tau}_s} \to M$. A string space $M_D$ is defined by a collection of all the string states $\{[\Sigma, u_D, I, \hat{\tau}_s]\}$.

Here, we will define topologies of $M_D$. We define an $\epsilon$-open neighbourhood of $[\Sigma, u_D, I, \hat{\tau}_s]$ by

$$U([\Sigma, u_D, I, \hat{\tau}_s], \epsilon) := \left\{ [\Sigma, \exp_{u_D} X_D, I, \hat{\tau}] \mid \sqrt{\hat{\tau} - \hat{\tau}_s}^2 + \|X_D|_{\hat{\tau}}\|^2 + \|X_D|_{\hat{\tau}_s}\|^2 < \epsilon \right\},$$

$X_D : \Sigma \to \mathbb{R}^{d-1,1}$ is a section of $u_D^* TM$ and maps a boundary component into a D-submanifold $L_{loc}$ that is a submanifold of $\mathbb{R}^{d-1,1}$. The following statements do not depend on how to define the norm. An example is given by

$$\|X_D|_{\hat{\tau}}\|^2 := \int_{\Sigma|_{\hat{\tau}}} d\hat{\sigma} \sqrt{h}(\hat{\sigma}, \hat{\tau}) \frac{1}{\epsilon^2|_{\hat{\tau}}} < \partial_{\hat{\sigma}} X_D|_{\hat{\tau}}, \partial_{\hat{\sigma}} X_D|_{\hat{\tau}} >,$$

where $<,>$ is the inner product in $\mathbb{R}^{d-1,1}$, $\sqrt{h_{mn}}(\hat{\sigma}, \hat{\tau})$ is the worldsheet metric of $\Sigma$, and $\epsilon := \sqrt{h_{\hat{\sigma}\hat{\sigma}}}$. A D-submanifold $L$ in $M_D$ is an open set of $M_{d-1,1}$. D-bundles $E_{loc}$ with gauge connections are also induced. $\hat{D}$ represents the configuration of D-bundles $E_{loc}$ with gauge connections and the induced configuration of O-planes. $U([\Sigma, u_D, I, \hat{\tau}_s \simeq \hat{\tau}_\infty], \epsilon) = U([\Sigma', u'_D, \hat{\tau}_s \simeq \hat{\tau}_\infty], \epsilon)$ consistently if $N_\pi = N'_\pi$, $M_\pi = M'_\pi$, $f^i = f'^i$, $\hat{f}^j = \hat{f}'^j$, $I|_{\hat{\tau}_s \simeq \hat{\tau}_\infty} = I'|_{\hat{\tau}_s \simeq \hat{\tau}_\infty}$, $u_D|_{\hat{\tau}_s \simeq \hat{\tau}_\infty} = u'_D|_{\hat{\tau}_s \simeq \hat{\tau}_\infty}$, and $\epsilon$ is small enough, because the $\hat{\tau} = \hat{\tau}_s$ constant line traverses only propagators overlapped by $\Sigma$ and $\Sigma'$. $U$ is defined to be an open set of $M_D$ if there exists $\epsilon$ such that $U([\Sigma, u_D, I, \hat{\tau}_s], \epsilon) \subset U$ for an arbitrary point $[\Sigma, u_D, I, \hat{\tau}_s] \in U$. In exactly the same way as in section 2, one can show that the topology of $M_D$ satisfies the axiom of topology.

By definition of the $\epsilon$-open neighbourhood, on a connected Riemann surface with open and closed punctures and with or without boundaries in $M$, arbitrary two string states
with the same Chan-Paton indices are connected continuously. Thus, there is an one-to-one correspondence between such a Riemann surface that have Chan-Paton indices and a curve parametrized by $\bar{\tau}$ from $\bar{\tau} = -\infty$ to $\bar{\tau} = \infty$ on $M_D$. That is, curves that represent asymptotic processes on $M_D$ reproduce the right moduli space of the Riemann surfaces.

By a general curve parametrized by $t$ on $M_D$, string states on the different Riemann surfaces that have even different genera, can be connected continuously, whereas the different Riemann surfaces that have different genera cannot be connected continuously in the moduli space of the Riemann surfaces. Therefore, the string geometry is expected to possess non-perturbative effects.

Next, we will define structures of manifold on the string topological space $M_D$. First, we define a model space $E$ such that $E = \bigcup D \{ [\Sigma, X_{\hat{D}}, I, \bar{\tau}] \}$ where disjoint unions are taken over all the backgrounds except for the metric, that consist of a NS-NS B-field, a dilaton, all the configurations of D-bundles $E_{\hat{loc}}$ with gauge connections, including D-submanifolds $L_{\hat{loc}}$, and all the consistent configurations of O-planes. An $\epsilon$-open neighbourhood of $[\Sigma, X_{\hat{D}}, I, \bar{\tau}_s]$ is defined by

$$U([\Sigma, X_{\hat{D}}, I, \bar{\tau}_s], \epsilon) := \left\{ [\Sigma, X_{\hat{D}}, I, \bar{\tau}] \mid \sqrt{|\bar{\tau} - \bar{\tau}_s|^2 + \|X_{\hat{D}}|_{\bar{\tau}} - X_{\hat{D}}|_{\bar{\tau}_s}^2} + \|X_{\hat{D}}|_{\bar{\tau}_s} - X_{\hat{D}}|_{\bar{\tau}_s}^2 < \epsilon \right\}. \quad (5.3)$$

Open sets of $E$ are defined as in the case of $M_D$. By these definitions, one can prove that the topology of $E$ satisfies the axiom of topology in the same way. Here, we define charts $(U([\Sigma, u_D, I, \bar{\tau}_s], \epsilon), \Psi)$ on $M_D$. First, the $\epsilon$-open neighbourhoods $U([\Sigma, u_D, I, \bar{\tau}_s], \epsilon)$ cover $M_D$ because they are defined at any point $[\Sigma, u_D, I, \bar{\tau}_s] \in M_D$. Next, $\Psi : U([\Sigma, u_D, I, \bar{\tau}_s], \epsilon) \rightarrow U([\Sigma, 0, I, \bar{\tau}_s], \epsilon)$ is a homeomorphism from an open set in $M_D$ to that in $E$ with the corresponding $D$ that consists of the corresponding configurations of O-planes and D-bundles with gauge connections where all the backgrounds are turned off except for $\eta_{\mu\nu}$.

**Proof.** For

$$U([\Sigma, u_D, I, \bar{\tau}_s], \epsilon) = \left\{ [\Sigma, \exp_{u_D} X_{\hat{D}}, I, \bar{\tau}] \mid \sqrt{|\bar{\tau} - \bar{\tau}_s|^2 + \|X_{\hat{D}}|_{\bar{\tau}}^2} + \|X_{\hat{D}}|_{\bar{\tau}_s}^2 < \epsilon \right\}. \quad (5.4)$$
and

\[
U([\Sigma, 0, I, \tau_s], \epsilon) = \left\{ [\Sigma, X_D, I, \tau] \mid \sqrt{\tau - \tau_s^2} + \|X_D\tau]\| + \|X_D\tau_s\|^2 < \epsilon \right\},
\]  
(5.5)

\[\Psi\] is bijective because the exponential map \(X_D \mapsto \exp_{u_D} X_D\) are bijective on \([\Sigma].\) Then, \(\Psi\) and \(\Psi^{-1}\) are continuous because the \(\epsilon\)-open neighbourhoods and the open sets are defined by the same conditions (5.4) and (5.5).

If \(U([\Sigma, u_D, I, \tau_s], \epsilon) \cap U([\Sigma, u_D', I', \tau'_s], \epsilon') \neq \emptyset\), a transition function between \([\Sigma, X_D, I, \tau]\) and \([\Sigma, X'_D, I', \tau']\) is given by \([\Sigma, \exp_{u_D} X_D, I, \tau] = [\Sigma, \exp_{u_D'} X'_D, I', \tau']\). This condition is equivalent to \(\bar{\tau} = \bar{\tau}'\), \(\exp_{u_D} X_D = \exp_{u_D'} X'_D\), and \(I = I'\). In the following, we denote \([\bar{h}_{mn}, X_D, I, \tau]\) instead of \([\Sigma, X_D, I, \tau]\) because giving a Riemann surface is equivalent to giving a metric up to diffeomorphism and Weyl transformations.

Let us consider how generally we can define general coordinate transformations between \([\bar{h}_{mn}, X_D, I, \tau]\) and \([\bar{h}'_{mn}, X'_D, I', \tau']\) where \([\bar{h}_{mn}, X_D, I, \tau] \in E \subset U \subset E\) and \([\bar{h}'_{mn}, X'_D, I', \tau'] \in U' \subset E\). \(\bar{h}_{mn}\) and \(I\) do not transform to \(\tau\) and \(X_D\) and vice versa, because \(\bar{\tau}\) and \(X_D\) are continuous variables, whereas \(\bar{h}_{mn}\) and \(I\) are discrete variables: \(\bar{\tau}\) and \(X_D\) vary continuously, whereas \(\bar{h}_{mn}\) and \(I\) vary discretely in a trajectory on \(E\) by definition of the neighbourhoods. \(\bar{\tau}\) and \(\bar{\sigma}\) do not transform to each other because the string states are defined by \(\bar{\tau}\) constant lines. Under these restrictions, the most general coordinate transformation is given by \([\bar{h}_{mn}, X_D, I, \tau]\)

\[
[\bar{h}_{mn}(\bar{\sigma}, \bar{\tau}), X_D^\mu(\bar{\sigma}, \bar{\tau}), I, \bar{\tau}] \mapsto [\bar{h}'_{mn}(\bar{\sigma}'(\bar{\sigma}), \bar{\tau}'(\bar{\tau}, X_D)), X_D'^\mu(\bar{\sigma}', \tau')(\bar{\tau}, X_D), I, \bar{\tau}'(\bar{\tau}, X_D))],
\]  
(5.6)

where \(\mu = 0, 1, \ldots, d - 1\). Because \(\bar{h}_{mn}\), \(\bar{\tau}\), \(X_D^\mu\), and \(I\) are all independent, where \(\bar{h}_{mn}\) and \(X_D^\mu\) are coordinates as functionals, \(\frac{\partial}{\partial \bar{\tau}}\) is an explicit derivative on functions over the string manifolds, especially, \(\frac{\partial}{\partial \bar{\tau}} \bar{h}_{mn} = 0\), \(\frac{\partial}{\partial \bar{\tau}} X_D^\mu = 0\), and \(\frac{\partial}{\partial \bar{\tau}} I = 0\).

Here, we consider all the manifolds which are constructed by patching open sets of the model space \(E\) by the general coordinate transformations (5.6) and call them string manifolds \(\mathcal{M}\). We also call the topologies and spaces of the string manifolds, string topologies and string spaces, respectively. The string space \(\mathcal{M}_D\), the string topological space \(\mathcal{M}_D\), and
the string manifold $\mathcal{M}_D$ are ones of the string spaces, string topological spaces, and string manifolds, respectively.

The tangent space is spanned by $\frac{\partial}{\partial \sigma}$ and $\frac{\partial}{\partial X^\mu_D(\sigma, \tau)}$ as one can see from the chart (5.5). We should note that $\frac{\partial}{\partial h_{mn}}$ and $\frac{\partial}{\partial I}$ cannot be a part of basis that span the tangent space, because $h_{mn}$ and $I$ are just discrete variables in $E$. The index of $\frac{\partial}{\partial X^\mu_D(\sigma, \tau)}$ can be $(\mu, \sigma)$. We define a summation over $\tilde{\sigma}$ by $\int d\tilde{\sigma} \tilde{e}(\tilde{\sigma}, \tilde{\tau})$, where $\tilde{e} := \sqrt{h_{\tilde{\sigma} \tilde{\sigma}}}$. This summation is invariant under $\tilde{\sigma} \mapsto \tilde{\sigma}'(\tilde{\sigma})$ and transformed as a scalar under $\tilde{\tau} \mapsto \tilde{\tau}'(\tilde{\tau}, X_D)$.

Riemannian string manifold is obtained by defining a metric, which is a section of an inner product on the tangent space. The general form of a metric is given by

$$ds^2(\bar{h}, X_D, I, \tau) = G(\bar{h}, X_D, I, \tau)_{dd} (d\bar{\tau})^2 + 2 d\bar{\tau} \int d\tilde{\sigma} \tilde{e}(\tilde{\sigma}, \tilde{\tau}) \sum_{\mu} G(\bar{h}, X_D, I, \tau)_{d(\mu \sigma)} dX^\mu_D(\sigma, \tilde{\tau}) + \int d\tilde{\sigma} \tilde{e}(\tilde{\sigma}, \tilde{\tau}) \sum_{\mu, \mu'} G(\bar{h}, X_D, I, \tau)_{(\mu \sigma)(\mu' \sigma')} dX^\mu_D(\sigma, \tilde{\tau}) dX^\mu_D(\sigma', \tilde{\tau}).$$

We summarize the vectors as $dX^I_D (I = d, (\mu \sigma))$, where $dX^d_D := d\tau$ and $dX^{(\mu \sigma)}_D := dX^\mu_D(\sigma, \tilde{\tau})$. Then, the components of the metric are summarized as $G_{IJ}(\bar{h}, X_D, I, \tau)$. The inverse of the metric $G'^{IJ}(\bar{h}, X_D, I, \tau)$ is defined by $G_{IJ}G'^{JK} = \delta^K_I$, where $\delta^K_I = 1$ and $\delta^{\mu' \sigma'}_{\mu \sigma} = \frac{1}{\tilde{e}(\sigma, \tau)} \delta^{\mu' \sigma}$. The components of the Riemannian curvature tensor are given by $R^I_{JKL}$ in the basis $\frac{\partial}{\partial X^\mu_D}$. The components of the Ricci tensor are $R_{IJ} := R^K_{IKJ} = R^d_{Idd} + \int d\tilde{\sigma} \tilde{e} R^{(\mu \sigma)}_{I(\mu \sigma)}$. The scalar curvature is

$$R := G'^{IJ} R_{IJ} = G'^{dd} R_{dd} + 2 \int d\tilde{\sigma} \tilde{e} G^{(\mu \sigma)} R_{d(\mu \sigma)} + \int d\tilde{\sigma} \tilde{e}' \int d\tilde{\sigma}' \tilde{e} G^{(\mu \sigma)} G^{(\mu' \sigma')} R_{(\mu \sigma)(\mu' \sigma')}.$$

The volume is $\sqrt{G}$, where $G = \det(G_{IJ})$.

By using these geometrical objects, we formulate string theory non-perturbatively as

$$Z = \int \mathcal{D}G \mathcal{D}A e^{-S},$$

where

$$S = \frac{1}{G_N} \int D\tau DX^I_D D\tilde{\tau} D\tilde{I} \sqrt{G} (-R + \frac{1}{4} G_N G^{I_1 I_2} G^{J_1 J_2} F_{I_1 J_1} F_{I_2 J_2} ).$$

(5.9)
As an example of sets of fields on the string manifolds, we consider the metric and an $u(1)$ gauge field $A_I$ whose field strength is given by $F_{IJ}$. The path integral is canonically defined by summing over the metrics and gauge fields on $\mathcal{M}$. By definition, the theory is background independent. $D\bar{h}$ is the invariant measure of the metrics $h_{mn}$ on the two-dimensional Riemannian manifolds $\Sigma$, divided by the volume of the diffeomorphism and the Weyl transformations. $h_{mn}$ and $\bar{h}_{mn}$ are related to each others by the diffeomorphism and the Weyl transformations.

Under
\begin{equation}
(\bar{\tau}, X_D) \mapsto (\bar{\tau}', X_D'), X'_D(\bar{\tau}, X_D));
\end{equation}
$G_{IJ}(\bar{h}, X_D, I, \bar{\tau})$ and $A_I(\bar{h}, X_D, I, \bar{\tau})$ are transformed as a symmetric tensor and a vector, respectively and the action is manifestly invariant.

We define $G_{IJ}(\bar{h}, X_D, I, \bar{\tau})$ and $A_I(\bar{h}, X_D, I, \bar{\tau})$ so as to transform as scalars under $\bar{h}_{mn}(\bar{\sigma}, \bar{\tau}) \mapsto \bar{h}'_{mn}(\bar{\sigma}', \bar{\tau})$. Under $\bar{\sigma}$ diffeomorphisms: $\bar{\sigma} \mapsto \bar{\sigma}'(\bar{\sigma})$, which are equivalent to
\begin{equation}
[\bar{h}_{mn}(\bar{\sigma}, \bar{\tau}), X^\mu(\bar{\sigma}, \bar{\tau}), I, \bar{\tau}] \mapsto [\bar{h}'_{mn}(\bar{\sigma}', \bar{\tau}), X'^\mu(\bar{\sigma}', \bar{\tau})(X_D), I, \bar{\tau}],
\end{equation}
$G_{d,(\mu\sigma)}$ is transformed as a scalar;
\begin{equation}
G'_{d,(\mu\sigma)}(\bar{h}', X'_D, I, \bar{\tau}) = G_{d,(\mu\sigma)}(\bar{h}, X_D, I, \bar{\tau}) = \frac{\partial X_D^I}{\partial X_D^\nu} \frac{\partial X_D^J}{\partial X'_D^{(\mu\sigma)}} G_{IJ}(\bar{h}, X_D, I, \bar{\tau})
= \frac{\partial X_D^I}{\partial X_D^\nu} \frac{\partial X_D^J}{\partial X'_D^{(\mu\sigma)}} G_{d,(\mu\sigma)}(\bar{h}, X_D, I, \bar{\tau}),
\end{equation}
because (5.11) and (5.10). In the same way, the other fields are also transformed as
\begin{align*}
G'_{d}(\bar{h}', X'_D, I, \bar{\tau}) &= G_{d}(\bar{h}, X_D, I, \bar{\tau}) \\
G'_{(\mu\sigma)}(\bar{h}', X'_D, I, \bar{\tau}) &= G_{(\mu\sigma)}(\bar{h}, X_D, I, \bar{\tau}) \\
A'_{d}(\bar{h}', X'_D, I, \bar{\tau}) &= A_{d}(\bar{h}, X_D, I, \bar{\tau}) \\
A'_{(\mu\sigma)}(\bar{h}', X'_D, I, \bar{\tau}) &= A_{(\mu\sigma)}(\bar{h}, X_D, I, \bar{\tau}).
\end{align*}
Thus, the Lagrangian is invariant under $\bar{\sigma}$ diffeomorphisms, because $\int d\bar{\sigma}' \bar{e}'(\bar{\sigma}', \bar{\tau}) = \int d\bar{\sigma} \bar{e}(\bar{\sigma}, \bar{\tau})$. Therefore, $G_{IJ}(\bar{h}, X_D, I, \bar{\tau})$ and $A_I(\bar{h}, X_D, I, \bar{\tau})$ are transformed covariantly and the action (5.9) is invariant under the diffeomorphisms (5.6) including the $\bar{\sigma}$ diffeomorphisms.

46
The background that represents a perturbative vacuum is given by

\[ d\bar{s}^2 = 2\lambda\bar{\rho}(\bar{h})N^2(X_D)(dX_D^2)^2 \]

\[ + \int d\bar{\sigma}d\bar{\tau}' N^{2\nu}(X_D) \bar{\delta}^3(\bar{\sigma}, \bar{\tau}) \delta(\mu\nu) dX_D^\nu dX_D^\nu, \]

\[ \bar{A}_d = i\sqrt{\frac{2-2D}{2-D}} \sqrt{\frac{2\lambda\bar{\rho}(\bar{h})}{\sqrt{G_N}}} N(X_D), \quad \bar{A}(\mu\nu) = 0, \quad (5.14) \]

on \( \mathcal{M}_D \) where we fix the target metric to \( \eta_{\mu\nu} \), a set of D-submanifolds to arbitrary one, and the gauge connections to zero, respectively. \( \bar{\rho}(\bar{h}) := \frac{1}{4\pi} \int d\bar{\sigma}\sqrt{\bar{h}}\bar{R}_h + \frac{1}{2\pi}\bar{k}_h \), where \( \bar{R}_h \) is the scalar curvature of \( \bar{h}_{mn} \) and \( \bar{k}_h \) is the geodesic curvature of \( \bar{h}_{mn} \). \( D \) is a volume of the index \( (\mu\bar{\sigma}) : D := \int d\bar{\sigma}\bar{\delta}(\mu\bar{\sigma}) = d2\pi\delta(0) \). \( N(X_D) = \frac{1}{1+v(X_D)} \), where \( v(X_D) = \frac{2}{\sqrt{\bar{h}_{-1}}} \int d\bar{\sigma}\epsilon_{\mu\nu}X_D^\mu \partial_\nu X_D^\nu \). One can show that the background \( (5.14) \) is a classical solution to the equations of motion of \( (5.9) \) as in section 3. The dependence of \( \bar{h}_{mn} \) on the background \( (5.14) \) is uniquely determined by the consistency of the quantum theory of the fluctuations around the background. Actually, we will find that all the perturbative string amplitudes are derived as follows.

Let us consider fluctuations around the background \( (5.14) \), \( G_{IJ} = \tilde{G}_{IJ} + \hat{G}_{IJ} \) and \( A_I = \bar{A}_I + \hat{A}_I \). The action \( (5.9) \) up to the quadratic order is given by,

\[
S = \frac{1}{G_N} \int \mathcal{D}h \mathcal{D}X_D \mathcal{D}\bar{\tau} \sqrt{\bar{G}} \left(-\bar{R} + \frac{1}{4} \bar{F}_{IJ}^I \bar{F}_{IJ}^J \right) \\
+ \frac{1}{4} \bar{\nabla}_I \tilde{G}_{JK} \bar{\nabla}^I \bar{G}^{JK} - \frac{1}{2} \bar{\nabla}^I \tilde{G}_{IJ} \bar{\nabla}^J \bar{G} - \frac{1}{2} \bar{\nabla}^I \tilde{G}_{IJ} \bar{\nabla}_K \bar{G}^{JK} \\
- \frac{1}{4} \left(-\bar{R} + \frac{1}{4} \bar{F}_{KL}^I \bar{F}_{KL}^J \right) \left(\hat{G}_{IJ} \bar{G}^{IJ} - \frac{1}{2} \bar{G}^2 \right) + \left(-\frac{1}{2} \bar{R}^I_J + \frac{1}{2} \bar{F}_{IK}^I \bar{F}_{JK}^I \right) \tilde{G}_{IL} \tilde{G}^{JL} \\
+ \left(\frac{1}{2} \bar{R}^I_J - \frac{1}{4} \bar{F}_{IK}^I \bar{F}_{KL}^J \right) \tilde{G}_{IL} \tilde{G}_{JL} + \left(-\frac{1}{2} \bar{R}^I_J + \frac{1}{4} \bar{F}_{IK}^I \bar{F}_{KL}^J \right) \tilde{G}_{IL} \tilde{G}_{JL} \\
+ \frac{1}{4} G_N \tilde{F}^I_{IJ} \tilde{F}^I_{IJ} + \sqrt{G_N} \left(\frac{1}{4} \bar{F}^{I\prime IJ} \bar{F}_{IJ} \bar{G} - \bar{F}^{I\prime IJ} \bar{F}_{IK} \bar{G}^{K} \right). \quad (5.15) \]

The Lagrangian is independent of Chan-Paton indices because the background \( (5.14) \) is independent of them. \( \bar{F}_{IJ}^I := \sqrt{G_N} \hat{F}_{IJ}^I \) is independent of \( G_N \). \( \tilde{G} := \tilde{G}^{IJ} \tilde{G}_{IJ} \). There is no first order term because the background satisfies the equations of motion.

From these fluctuations, we obtain the correlation function in the string manifold \( \mathcal{M}_D \)

\footnote{This solution is a generalization of the Majumdar-Papapetrou solution \([9,10]\) of the Einstein-Maxwell system.}
in exactly the same way as in section 3,

$$\Delta_F(X_{Df}^f; X_{Di}^i|h_f^f; h_i^i) = Z \int_{h_i^i, X_{Di}^i} h_f^f, X_{Df}^f D h D X e^{-\lambda \chi e^{-S_s}}, \quad (5.16)$$

where

$$S_s = \int_{-\infty}^{\infty} d\tau \int d\sigma \sqrt{h(\sigma, \tau)} \left( \frac{1}{2} h^{mn}(\sigma, \tau) \partial_m X^\mu_d(\sigma, \tau) \partial_n X^{\mu_d}(\sigma, \tau) \right), \quad (5.17)$$

and $\chi$ is the Euler number of the two-dimensional Riemannian manifold with boundaries. By inserting asymptotic states with Chan-Paton matrices to (5.16), we obtain the all-order perturbative scattering amplitudes that possess the moduli in the open-closed string theory with Dirichlet and Neumann boundary conditions in the normal and tangential directions to the D-submanifolds, respectively [12]. Therefore, a set of D-submanifolds represents a D-brane background where back reactions from the D-branes are ignored. The consistency of the perturbation theory around the background (5.14) determines $d = 26$ (the critical dimension).

### 6 Non-perturbative formulation of superstring theory

In this section, we complete superstring geometry including open superstrings. Let us define unique global times on oriented open-closed superstring worldsheets with punctures $\Sigma$ [13–15] in order to define string states by world-time constant hypersurfaces. The worldsheets with boundaries can be constructed as orientifolds of ordinary type II superstring worldsheets with punctures $\bar{\Sigma}_c$ as follows. $\bar{\Sigma}_c$ are embedded in $\bar{\Sigma}_L \times \bar{\Sigma}_R$, which have reduced spaces $\bar{\Sigma}_{L,red} \times \bar{\Sigma}_{R,red}$. We restrict $\bar{\Sigma}_c$ such that $\bar{\Sigma}_{L,red} = \bar{\Sigma}_{R,red}$. Then, $\rho(\bar{\Sigma}_{L,red}) = \bar{\Sigma}_{L,red}$, where $\rho$ is the anti-holomorphic involution. An orbifold action $Z_2$ on $\bar{\Sigma}_c$ is defined in local coordinates by $(z, \theta_1, \rho(z), \theta_2) \mapsto (\rho(z), \theta_1, z, -\theta_2)$. $\Sigma_c/Z_2$ are topologically classified into open and/or unoriented super Riemann surfaces. By restricting them, we obtain oriented open-closed superstring worldsheets with punctures $\bar{\Sigma}$. Thus, for each $\bar{\Sigma}$, there exists a closed double cover $\Sigma_c$. Because $\bar{\Sigma}_{red}/Z_2 \cong \Sigma_{L,red}/Z_2 = \Sigma_{R,red}/Z_2$, reduced spaces $\bar{\Sigma}_{red}$ of $\bar{\Sigma}$ are oriented open-closed worldsheets with punctures. First of all, we define global coordinates on the double covers $\Sigma_c$ of $\bar{\Sigma}$ in the same way as in section 4. The real part of the global coordinates $\bar{\tau}$ remains on $\Sigma$ because $\rho$ is an anti-holomorphic involution. If $\rho$ maps a puncture to another puncture on $\Sigma_c$, the superdiscs around the punctures are identified and give a superdisk $D^i$. 
around a closed puncture $P^i$ on $\Sigma$. On the other hand, if $\rho$ maps a puncture to itself on $\Sigma$, the superdisk around the puncture is identified with itself and gives an upper half superdisk $\tilde{D}^j$ around an open puncture $\tilde{P}^j$ on $\Sigma$. The $\sigma$ regions around $P^i$ and $\tilde{P}^j$ are $2\pi f^i$ and $\pi \tilde{f}^j$, respectively where $\sum_{i=1}^n 2f^i + \sum_{j=1}^m \tilde{f}^j = 0$. $\tilde{\tau} = -\infty$ at $P^i$ and $\tilde{P}^j$ with negative $f^i$ and $\tilde{f}^j$, respectively, whereas $\tilde{\tau} = \infty$ at $P^i$ and $\tilde{P}^j$ with positive $f^i$ and $\tilde{f}^j$, respectively. In order to define the above global time uniquely, we fix the $\sigma$ regions $2\pi f^i$ and $\pi \tilde{f}^j$ around $P^i$ and $\tilde{P}^j$, respectively in exactly the same way as in section 5.

Thus, under a superconformal transformation, one obtains $\tilde{\Sigma}$ that has even coordinates composed of the global time $\tilde{\tau}$ and the position $\tilde{\sigma}$, and $\Sigma_{red}$ is canonically defined. Because $\Sigma$ can be a moduli of oriented open-closed superstring worldsheets with open and closed punctures, any two-dimensional oriented open-closed super Riemannian manifold with open and closed punctures $\Sigma$ can be obtained by $\Sigma = \psi(\Sigma)$ where $\psi$ is a superdiffeomorphism times super Weyl transformation.

Next, we will define a superstring space $\mathcal{M}^T_D$, where $T$ runs IIA, IIB and I. $\Omega$ projection is imposed for $T=I$. Here we fix not only a $d$-dimensional Riemannian manifold $M$ but also a set of its submanifolds $L$ and $N$ dimensional vector bundles $E$ with gauge connections on them, which we call D-submanifolds and D-bundles, respectively. We fix 32 D9-submanifolds for $T=I$. We also fix a consistent configuration of O-planes. We consider a state $(\tilde{\Sigma}, u_D, I, \tilde{\tau}_s)$ determined by a $\tilde{\tau} = \tilde{\tau}_s$ constant hypersurface. $u_D : \Sigma \rightarrow M$ is an arbitrary map that maps a boundary component of the reduced space into a D-submanifold, where $D$ represents the above fixed quantities. $I = (i_1, \cdots, i_k, \cdots, i_b)$ represents a set of the Chan-Paton indices where $i_k$ represents a Chan-Paton index on the $k$-th boundary component on $\Sigma$. $i_k$ runs from 1 to $N_k$ that represents the number of the D-branes where the $k$-th boundary component maps. The zero mode and the boundary conditions on $u_D$, which is defined by restricting $u_D$ on the reduced space $\Sigma$, are determined by the D-submanifolds $L$ and the gauge connections. $\Sigma$ is a direct sum of $N_+ \times$ supercylinders with radii $f_i$ and $M_\mp \times$ superstrips with width $\pi \tilde{f}^j$ at $\tilde{\tau} \simeq \mp \infty$. Thus, we define a superstring state as an equivalence class $[\Sigma, u_D, I, \tilde{\tau}_s \simeq \mp \infty]$ by a relation $(\Sigma, u_D, I, \tilde{\tau}_s \simeq \mp \infty) \sim (\Sigma', u'_D, I', \tilde{\tau}'_s \simeq \mp \infty)$ if $N_\mp = N'_\mp$, $M_\mp = M'_\mp$, $f^i = f'^i$, $\tilde{f}^j = \tilde{f}'^j$, $I|_{\tilde{\tau}_s \simeq \mp \infty} = I'|_{\tilde{\tau}'_s \simeq \mp \infty}$, $u_D|_{\tilde{\tau}_s \simeq \mp \infty} = u_D'|_{\tilde{\tau}'_s \simeq \mp \infty}$, and the corresponding supercylinders and superstrips are the same type (NS-NS, NS-R, R-NS, or R-R) and (NS or R), respectively as in Fig. [ Fig. 1] $[\Sigma, u_D, I, \tilde{\tau}_s]$ represent many-body states of open and closed superstrings in $M$
as in Fig. 2, because the reduced space of $\Sigma|_{\tau_s}$ is $S^1 \times \cdots \times S^1 \times I^1 \times \cdots \times I^1$ where $I^1$ represents a line segment, and $u_D|_{\tau_s} : \Sigma|_{\tau_s} \to M$. A superstring space $\mathcal{M}_D^T$ is defined by a collection of all the superstring states $\{[\Sigma, u_D, I, \tau_s]\}$.

Here, we will define topologies of $\mathcal{M}_D^T$. We define an $\epsilon$-open neighbourhood of $[\Sigma, u_D, I, \bar{\tau}_s]$ by

$$U([\Sigma, u_D, I, \bar{\tau}_s], \epsilon) := \left\{ [\Sigma, \exp_{u_D} X_D, I, \bar{\tau}] \mid \sqrt{|\bar{\tau} - \tau|^2 + \|X_D\|_\tau^2 + \|X_D|_{\tau_s}^2} < \epsilon \right\}, \quad (6.1)$$

as in Fig. 3. $X_D : \Sigma \to \mathbb{R}^{d-1,1}$ is a section of $u_D^*TM$ and maps a boundary component into a D-submanifold $L_{\text{loc}}$ that is a submanifold of $\mathbb{R}^{d-1,1}$. The following statements do not depend on how to define the norm. An example is given by

$$\|X_D|_{\tau}\|^2 := \int_{\Sigma_{\tau}} d\bar{\sigma} d^2\theta E|_{\tau} < \hat{D}_a X_D|_{\tau_s}, \hat{D}_a X_D|_{\tau_s}, \quad (6.2)$$

where $<,>$ is the inner product in $\mathbb{R}^{d-1,1}$, $\hat{D}_a$ is a $\bar{\tau}$ independent super derivative on $\Sigma$, and $E_M^{A(\bar{\sigma}, \bar{\tau}, \bar{\theta}^a)} (M = (m, \alpha), A = (q, a), m, q = 0, 1, \alpha, a = 1, 2)$ is the worldsheet super vierbein on $\Sigma$. A D-submanifold $L$ in $\mathcal{M}_D^T$ in the local Lorentz frame on $M$ is a D-submanifold $L_{\text{loc}}$ of $\mathbb{R}^{d-1,1}$. D-bundles $E_{\text{loc}}$ with gauge connections are also induced. $\hat{D}$ represents the configuration of D-bundles $E_{\text{loc}}$ with gauge connections, and the consistent configurations of O-planes. $U([\Sigma, u_D, I, \bar{\tau}_s \simeq \mp \infty], \epsilon) = U([\Sigma', u_D', I', \bar{\tau}_s \simeq \mp \infty], \epsilon)$ consistently if $N_+ = N'_+$, $M_+ = M'_+$, $f^i = f'^i$, $\bar{f}^i = \bar{f}'^i$, $I|_{\tau_s \simeq \mp \infty} = I'|_{\tau_s \simeq \mp \infty}$, $u_D|_{\tau_s \simeq \mp \infty} = u_D'|_{\tau_s \simeq \mp \infty}$, the corresponding supercylinders and superstrips are the same type (NS-NS, NS-R, R-NS, or R-R) and (NS or R), respectively, and $\epsilon$ is small enough, because the $\bar{\tau} = \bar{\tau}_s$ constant hypersurface traverses only propagators overlapped by $\Sigma$ and $\Sigma'$. $U$ is defined to be an open set of $\mathcal{M}_D^T$ if there exists $\epsilon$ such that $U([\Sigma, u_D, I, \bar{\tau}_s], \epsilon) \subset U$ for an arbitrary point $[\Sigma, u_D, I, \bar{\tau}_s] \in U$. In exactly the same way as in section 2, one can show that the topology of $\mathcal{M}_D^T$ satisfies the axiom of topology.

By definition of the $\epsilon$-open neighbourhood, on a connected super Riemann surface with open and closed punctures and with or without boundaries in $M$, arbitrary two superstring states with the same Chan-Paton indices are connected continuously. Thus, there is an one-to-one correspondence between such a super Riemann surface that have Chan-Paton indices and a curve parametrized by $\bar{\tau}$ from $\bar{\tau} = -\infty$ to $\bar{\tau} = \infty$ on $\mathcal{M}_D^T$. That is, curves
that represent asymptotic processes on $\mathcal{M}_D^T$ reproduce the right moduli space of the super Riemann surfaces.

By a general curve parametrized by $t$ on $\mathcal{M}_D^T$, superstring states on the different super Riemann surfaces that have even different genera, can be connected continuously, whereas the different super Riemann surfaces that have different genera cannot be connected continuously in the moduli space of the super Riemann surfaces. Therefore, the superstring geometry is expected to possess non-perturbative effects.

Next, we will define structures of manifold on the superstring topological space $\mathcal{M}_D^T$. In this supersymmetric case, we define a model space $E$ such that $E := \bigcup_{\hat{D}_T} \{[\bar{\Sigma}, X_{\hat{D}_T}, I, \bar{\tau}]\}$ where $T$ runs IIA, IIB and I. The IIA GSO projection is attached for $T = \text{IA}$, and the IIB GSO projection is attached for $T = \text{IB}$ and I. $\Omega$ projection is also imposed for $T = \text{I}$. $\hat{D}_T$ runs over all the backgrounds except for the metric, that consist of a NS-NS B-field, a dilaton, R-R fields $12$, all the configurations of D-bundles $E_{\text{loc}}$ with arbitrary gauge connections, including D-submanifolds $L_{\text{loc}}$, and consistent configurations of O-planes. We fix 32 D9-submanifolds for $T = \text{I}$. We can define the worldsheet fermion numbers of states in a Hilbert space because the states consist of the fields over the local coordinates $X^\mu_{\hat{D}_T} = X^\mu + \bar{\theta}^\alpha \psi^\mu_\alpha + \frac{1}{2} \bar{\theta} \bar{\theta} F^\mu$, where $\mu = 0, 1, \cdots d - 1$, $\psi^\mu_\alpha$ is a Majorana fermion and $F^\mu$ is an auxiliary field. We abbreviate $\hat{D}_T$ of $X^\mu$, $\psi^\mu_\alpha$ and $F^\mu$. We define the Hilbert space in these coordinates by the states only with $e^{\pi i F} = 1$ and $e^{\pi i \bar{F}} = (-1)^{\hat{\alpha}}$ for $T = \text{IA}$ and $e^{\pi i F} = e^{\pi i \bar{F}} = 1$ for $T = \text{IB}$ and I, where $F$ and $\bar{F}$ are left- and right-handed fermion numbers respectively, and $\hat{\alpha}$ is 1 or 0 when the right-handed fermion is periodic (R sector) or anti-periodic (NS sector), respectively. Although the model space is defined by using the coordinates $[\bar{\Sigma}, X_{\hat{D}_T}, I, \bar{\tau}]$, the model space does not depend on the coordinates, because a model space is a topological space.

An $\epsilon$-open neighbourhood of $[\bar{\Sigma}, X_{s_{\hat{D}_T}}, I, \bar{\tau}_s]$ is defined by

$$U([\bar{\Sigma}, X_{s_{\hat{D}_T}}, I, \bar{\tau}_s], \epsilon) := \left\{ [\bar{\Sigma}, X_{\hat{D}_T}, I, \bar{\tau}] \mid \sqrt{|\bar{\tau} - \bar{\tau}_s|^2 + \|X_{\hat{D}_T}|_{\tau} - X_{s_{\hat{D}_T}}|_{\tau_s}|^2 + \|X_{\hat{D}_T}|_{\tau_s} - X_{s_{\hat{D}_T}}|_{\tau_s}|^2 < \epsilon \right\}.$$  \hspace{1cm} (6.3)

Open sets of $E$ are defined as in the case of $\mathcal{M}_D^T$. By these definitions, one can prove that

---

12$X_{\hat{D}_T}$ do not depend on the R-R backgrounds because strings do not couple with them. However, open sets of the model space need to possess the R-R backgrounds (We may write $E := \bigcup_{\hat{D}_T} \{[\bar{\Sigma}, X_{\hat{D}_T}, I, \bar{\tau}]\}$ in order that D-brane states in the Hilbert space defined on the open sets couple with the R-R backgrounds.
the topology of $\mathbf{E}$ satisfies the axiom of topology in the same way. Here, we define charts $\left(U(\bar{\Sigma}, \mathbf{u}, \mathbf{I}, \bar{\tau}_s), \mathbf{\Psi}\right)$ on $\mathcal{M}_D^T$. First, the $\epsilon$-open neighbourhoods $U(\bar{\Sigma}, \mathbf{u}, \mathbf{I}, \bar{\tau}_s), \epsilon$ cover $\mathcal{M}_D^T$ because they are defined at any point $\bar{\Sigma}, \mathbf{u}, \mathbf{I}, \bar{\tau}_s \in \mathcal{M}_D^T$. Next, $\mathbf{\Psi} : U(\bar{\Sigma}, \mathbf{u}, \mathbf{I}, \bar{\tau}_s), \epsilon \rightarrow U(\bar{\Sigma}, \mathbf{0}, \mathbf{I}, \bar{\tau}_s), \epsilon$ is a homeomorphism from an open set in $\mathcal{M}_D^T$ to that in $\mathbf{E}$ with the corresponding $\hat{D}_T$ that consists of the corresponding configurations of O-planes and D-bundles with gauge connections where all the backgrounds are turned off except for $\eta_{\mu\nu}$.

**Proof.** For

$$U(\bar{\Sigma}, \mathbf{u}, \mathbf{I}, \bar{\tau}_s), \epsilon$$

$$= \left\{ \left[ \bar{\Sigma}, \exp_{\mathbf{u}} X_{\hat{D}_T}, \mathbf{I}, \bar{\tau} \right] \left| \sqrt{\|\bar{\tau} - \bar{\tau}_s\|^2 + \|X_{\hat{D}_T}\|^2 + \|X_{\hat{D}_T}\|_{\bar{\tau}_s}^2} < \epsilon \right\} \right. \quad (6.4)$$

and

$$U(\bar{\Sigma}, \mathbf{0}, \mathbf{I}, \bar{\tau}_s), \epsilon$$

$$= \left\{ \left[ \bar{\Sigma}, \mathbf{X}_{\hat{D}_T}, \mathbf{I}, \bar{\tau} \right] \left| \sqrt{\|\bar{\tau} - \bar{\tau}_s\|^2 + \|\mathbf{X}_{\hat{D}_T}\|^2 + \|\mathbf{X}_{\hat{D}_T}\|_{\bar{\tau}_s}^2} < \epsilon \right\} \right. \quad (6.5)$$

$\mathbf{\Psi}$ is bijective because the exponential map $X_{\hat{D}_T} \mapsto \exp_{\mathbf{u}} X_{\hat{D}_T}$ are bijective on $\bar{\Sigma}$. Then, $\mathbf{\Psi}$ and $\mathbf{\Psi}^{-1}$ are continuous because the $\epsilon$-open neighbourhoods and the open sets are defined by the same conditions (6.4) and (6.5). $\square$

If $U(\bar{\Sigma}, \mathbf{u}, \mathbf{I}, \bar{\tau}_s), \epsilon \cap U(\bar{\Sigma}, \mathbf{u}', \mathbf{I}', \bar{\tau}_s'), \epsilon' \neq \emptyset$, a transition function between $\left[ \bar{\Sigma}, \mathbf{X}_{\hat{D}_T}, \mathbf{I}, \bar{\tau} \right]$ and $\left[ \bar{\Sigma}, \mathbf{X}_{\hat{D}_T}', \mathbf{I}', \bar{\tau}' \right]$ is given by $\left[ \bar{\Sigma}, \exp_{\mathbf{u}} \mathbf{X}_{\hat{D}_T}, \mathbf{I}, \bar{\tau} \right] = \left[ \bar{\Sigma}, \exp_{\mathbf{u}'} \mathbf{X}_{\hat{D}_T}', \mathbf{I}, \bar{\tau}' \right]$. This condition is equivalent to $\bar{\tau} = \bar{\tau}'$, $\exp_{\mathbf{u}} \mathbf{X}_{\hat{D}_T} = \exp_{\mathbf{u}'} \mathbf{X}_{\hat{D}_T}'$, and $\mathbf{I} = \mathbf{I}'$. In the following, we denote $\left[ \bar{\Sigma}, \mathbf{X}_{\hat{D}_T}, \mathbf{I}, \bar{\tau} \right]$ instead of $\left[ \bar{\Sigma}, \mathbf{X}_{\hat{D}_T}, \mathbf{I}, \bar{\tau} \right]$ because giving a super Riemann surface is equivalent to giving a super vierbein up to super diffeomorphism and super Weyl transformations.

Let us consider how generally we can define general coordinate transformations between $\left[ E_M^A, \mathbf{X}_{\hat{D}_T}, \mathbf{I}, \bar{\tau} \right]$ and $\left[ E_M^A, \mathbf{X}_{\hat{D}_T}', \mathbf{I}', \bar{\tau}' \right]$ where $\left[ E_M^A, \mathbf{X}_{\hat{D}_T}, \mathbf{I}, \bar{\tau} \right] \in U \subset \mathbf{E}$ and $\left[ E_M^A, \mathbf{X}_{\hat{D}_T}', \mathbf{I}', \bar{\tau}' \right] \in U' \subset \mathbf{E}$. $E_M^A$ and $\mathbf{I}$ do not transform to $\bar{\tau}$ and $\mathbf{X}_{\hat{D}_T}$ and vice versa, because $\bar{\tau}$ and $\mathbf{X}_{\hat{D}_T}$ are continuous variables, whereas $\bar{\Sigma}$ and $\mathbf{I}$ are discrete variables: $\bar{\tau}$ and $\mathbf{X}_{\hat{D}_T}$ vary continuously, whereas $\bar{\Sigma}$ and $\mathbf{I}$ vary discretely in a trajectory on $\mathbf{E}$ by definition of the neighbourhoods. $\bar{\tau}$ does not transform to $\bar{\tau}$ and $\bar{\theta}$ and vice versa, because
the superstring states are defined by \( \bar{\tau} \) constant hypersurfaces. Under these restrictions, the most general coordinate transformation is given by

\[
[\bar{E}^A_M(\bar{\sigma}, \bar{\tau}, \bar{\theta}^\alpha), X^\mu_D(\bar{\sigma}, \bar{\tau}, \bar{\theta}^\alpha), I, \bar{\tau}]
\]

\[
\mapsto [\bar{E}^A_M(\bar{\sigma}', \bar{\theta}), \bar{X}^\mu_D(\bar{\sigma}', \bar{\theta}^\alpha)(\bar{\tau}, X_D), I, \bar{\tau}'(\bar{\tau}, X_D)],
\]

(6.6)

where \( \mu = 0, 1, \ldots, d-1 \). Because \( \bar{E}^A_M, \tau, X^\mu_D, I \) are all independent, where \( \bar{E}^A_M, X^\mu_D \) are coordinates as functionals, \( \partial_{\bar{E}^A_M} \) is an explicit derivative on functions over the superstring manifolds, especially, \( \partial_{\bar{E}^A_M} = 0, \partial_{\bar{X}^\mu_D} = 0, \text{ and } \partial_I = 0. \)

Here, we consider all the manifolds which are constructed by patching open sets of the model space \( E \) by general coordinate transformations (6.6) and call them superstring manifolds \( M \). We also call the topologies and spaces of the superstring manifolds, superstring manifolds, and superstring spaces, respectively. The superstring space \( \mathfrak{M}^T_D \), the superstring topological space \( \mathfrak{M}^T_D \), and the superstring manifold \( \mathfrak{M}^T_D \) are ones of the superstring spaces, superstring topological spaces, and superstring manifolds, respectively.

The tangent space is spanned by \( \partial_{\bar{E}^A_M} \) and \( \partial_{\bar{X}^\mu_D(\bar{\sigma}, \bar{\tau}, \bar{\theta})} \) as one can see from the chart (6.5). We should note that \( \partial_{\bar{E}^A_M(\bar{\sigma}, \bar{\tau}, \bar{\theta})} \) and \( \partial_I \) cannot be a part of basis that span the tangent space because \( \bar{E}^A_M \) and \( I \) are just discrete variables in \( E \). The index of \( \partial_{\bar{X}^\mu_D(\bar{\sigma}, \bar{\tau}, \bar{\theta})} \) can be \( (\mu \bar{\sigma} \bar{\theta}) \). We define a summation over \( \bar{\sigma} \) and \( \bar{\theta} \) by \( \int d\bar{\sigma} d^2 \bar{\theta} \bar{E}(\bar{\sigma}, \bar{\tau}, \bar{\theta}^\alpha) \), where \( \bar{E}(\bar{\sigma}, \bar{\tau}, \bar{\theta}^\alpha) := \sqrt{\bar{E}^A_M \bar{E}^A_M(\bar{\sigma}, \bar{\tau}, \bar{\theta}^\alpha)} \), because it is transformed as a scalar under \( \bar{\tau} \mapsto \bar{\tau}'(\bar{\tau}, X_D) \) and invariant under \( (\bar{\sigma}, \bar{\theta}^\alpha) \mapsto (\bar{\sigma}'(\bar{\sigma}, \bar{\theta}), \bar{\theta}^\alpha(\bar{\sigma}, \bar{\theta})) \) as one can see in section 4.

Riemannian superstring manifold is obtained by defining a metric, which is a section of an inner product on the tangent space. The general form of a metric is given by

\[
ds^2(\bar{E}, X_D, I, \bar{\tau}) \\
= G(\bar{E}, X_D, I, \bar{\tau}) d\bar{\tau}^2 \\
+ 2d\bar{\tau} \int d\bar{\sigma} d^2 \bar{\theta} \bar{E} \sum_{\mu} G(\bar{E}, X_D, I, \bar{\tau}) dX^\mu_D(\bar{\sigma}, \bar{\tau}, \bar{\theta}) \\
+ \int d\bar{\sigma} d^2 \bar{\theta} \bar{E} \int d\bar{\sigma}' d^2 \bar{\theta}' \bar{E}' \sum_{\mu, \mu'} G(\bar{E}, X_D, I, \bar{\tau}) dX^\mu_D(\bar{\sigma}, \bar{\tau}, \bar{\theta}) dX^\mu_D(\bar{\sigma}', \bar{\tau}, \bar{\theta}').
\]

(6.7)
We summarize the vectors as \(dX^I_{\bar{D}T}(\vec{I} = d, (\mu\bar{\sigma}\bar{\theta}))\), where \(dX^D_\bar{D} := d\bar{\tau}\) and \(dX^{(\mu\bar{\alpha}\bar{\theta})}_{\bar{D}T} := dX^\mu_{\bar{D}T}(\bar{\sigma}, \bar{\tau}, \bar{\theta})\). Then, the components of the metric are summarized as \(G^{IJ}(\vec{E}, X_{\bar{D}T}, I, \bar{\tau})\). The inverse of the metric \(G^{IJ}(\vec{E}, X_{\bar{D}T}, I, \bar{\tau})\) is defined by \(G_{IJ}G^{JK} = \delta^K_I\), where \(\delta^d_d = 1\) and 
\[
\delta^{\mu\bar{\alpha}\bar{\theta}}_{\mu\bar{\alpha}\bar{\theta}} = \frac{1}{E}\delta^\mu_{\mu}\delta(\bar{\sigma} - \bar{\sigma}')\delta^2(\bar{\theta} - \bar{\theta}')\).

The components of the Riemannian curvature tensor are given by \(R^I_{JKL}\) in the basis \(\frac{\partial}{\partial X^{\mu}_{\bar{D}T}}\). The components of the Ricci tensor are \(R^{K}_{IJ} = R^d_{IJKL} + \int d\bar{\sigma}d^2\bar{\theta}\vec{E}R_{IJ}^{(\mu\alpha\bar{\theta})}\). The scalar curvature is
\[
R = G^{IJ}R_{IJ} = G^{dd}R_{dd} + 2\int d\bar{\sigma}d^2\bar{\theta}\vec{E}G^{d(\mu\bar{\alpha}\bar{\theta})}R_{d(\mu\bar{\alpha}\bar{\theta})} + \int d\bar{\sigma}d^2\bar{\theta}\vec{E}\int d\bar{\sigma}'d^2\bar{\theta}'\vec{E}'G^{(\mu\bar{\alpha}\bar{\theta})}(\mu'\bar{\alpha}'\bar{\theta}')R_{(\mu\bar{\alpha}\bar{\theta})}(\mu'\bar{\alpha}'\bar{\theta}')\).

The volume is \(vol = \sqrt{G}\), where \(G = det(G^{IJ})\).

By using these geometrical objects, we formulate superstring theory non-perturbatively as
\[
Z = \int \mathcal{D}\vec{E}\mathcal{D}A e^{-S}, \quad (6.8)
\]
where
\[
S = \frac{1}{G_N} \int \mathcal{D}\bar{\tau}\mathcal{D}X_{\bar{D}T}\mathcal{D}I\sqrt{G}(-R + \frac{1}{4}G_{N}G^{IJ}G_{IJ}F_{IJ}F_{IJK}F_{IJK}). \quad (6.9)
\]
As an example of sets of fields on the superstring manifolds, we consider the metric and an \(u(1)\) gauge field \(A_I\) whose field strength is given by \(F_{IJ}\). The path integral is canonically defined by summing over the metrics and gauge fields on \(\mathfrak{M}\). By definition, the theory is background independent. \(\mathcal{D}\vec{E}\) is the invariant measure of the super vierbeins \(E_M^A\) on the two-dimensional super Riemannian manifolds \(\Sigma\), divided by the volume of the super diffeomorphism and super Weyl transformations. \(E_M^A\) and \(\bar{E}_M^A\) are related to each others by the super diffeomorphism and super Weyl transformations.

Under
\[
(\bar{\tau}, X_{\bar{D}T}) \mapsto (\bar{\tau}', X_{\bar{D}T}'), (\bar{\tau}, X_{\bar{D}T}) \mapsto (\bar{\tau}', X_{\bar{D}T}'), \quad (6.10)
\]
\(G^{IJ}(\vec{E}, X_{\bar{D}T}, I, \bar{\tau})\) and \(A_I(\vec{E}, X_{\bar{D}T}, I, \bar{\tau})\) are transformed as a symmetric tensor and a vector, respectively and the action is manifestly invariant.

We define \(G^{IJ}(\vec{E}, X_{\bar{D}T}, I, \bar{\tau})\) and \(A_I(\vec{E}, X_{\bar{D}T}, I, \bar{\tau})\) so as to transform as scalars under \(E_M^A(\bar{\sigma}, \bar{\tau}, \bar{\theta}^\alpha) \mapsto \bar{E}_M^A(\bar{\sigma}'(\bar{\sigma}, \bar{\theta}), \bar{\tau}, \bar{\theta}^\alpha(\bar{\sigma}, \bar{\theta}))\). Under \((\bar{\sigma}, \bar{\theta})\) superdiffeomorphisms: \((\bar{\sigma}, \bar{\theta}) \mapsto \)
(\bar{\sigma}', \bar{\theta}), \bar{\theta}^\alpha(\bar{\sigma}, \bar{\theta}))$, which are equivalent to

$$[ar{E}_M A(\bar{\sigma}, \bar{\tau}, \bar{\theta}^\alpha), X_\mu^D (\bar{\sigma}, \bar{\tau}, \bar{\theta}^\alpha), I, \bar{\tau}]$$

$$\mapsto [\bar{E}_M A'(\bar{\sigma}', \bar{\tau}, \bar{\theta}^\alpha(\bar{\sigma}, \bar{\theta})), X_\mu^D (\bar{\sigma}, \bar{\theta}, \bar{\theta}^\alpha(\bar{\sigma}, \bar{\theta}))(X_{D_T}), I, \bar{\tau}]$$

$$= [\bar{E}'_M A'(\bar{\sigma}', \bar{\tau}, \bar{\theta}^\alpha(\bar{\sigma}, \bar{\theta})), X_\mu^D (\bar{\sigma}, \bar{\tau}, \bar{\theta}^\alpha), I, \bar{\tau}], \quad (6.11)$$

$G_d(\mu\bar{\theta})$ is transformed as a superscalar;

$$G'_{d(\mu\bar{\sigma}'\bar{\theta})}(\bar{E}', X'_{D_T}, I, \bar{\tau}) = G_{d(\mu\bar{\sigma}\bar{\theta})}(\bar{E}, X_{D_T}, I, \bar{\tau}) = \frac{\partial X^I_{D_T}}{\partial X^I_{D_T}} \frac{\partial X^J_{D_T}}{\partial X^J_{D_T}} G_{I\bar{J}}(\bar{E}, X_{D_T}, I, \bar{\tau})$$

$$= \frac{\partial X^I_{D_T}}{\partial X^I_{D_T}} \frac{\partial X^J_{D_T}}{\partial X^J_{D_T}} G_{I\bar{J}}(\bar{E}, X_{D_T}, I, \bar{\tau}) = G_{d(\mu\bar{\sigma}\bar{\theta})}(\bar{E}, X_{D_T}, I, \bar{\tau}), \quad (6.12)$$

because $(6.11)$ and $(6.10)$. In the same way, the other fields are also transformed as

$$G'_{dd}(\bar{E}', X'_{D_T}, I, \bar{\tau}) = G_{dd}(\bar{E}, X_{D_T}, I, \bar{\tau})$$

$$G'_{(\mu\bar{\sigma}'\bar{\theta})(\nu\bar{\bar{\theta}})}(\bar{E}', X'_{D_T}, I, \bar{\tau}) = G_{(\mu\bar{\sigma}\bar{\theta})(\nu\bar{\bar{\theta}})}(\bar{E}, X_{D_T}, I, \bar{\tau})$$

$$A'_d(\bar{E}', X'_{D_T}, I, \bar{\tau}) = A_d(\bar{E}, X_{D_T}, I, \bar{\tau})$$

$$A'_{(\mu\bar{\sigma}'\bar{\theta})}(\bar{E}', X'_{D_T}, I, \bar{\tau}) = A_{(\mu\bar{\sigma}\bar{\theta})}(\bar{E}, X_{D_T}, I, \bar{\tau}). \quad (6.13)$$

Thus, the Lagrangian is invariant under the $(\bar{\sigma}, \bar{\theta})$ superdiffeomorphisms, because

$$\int d\bar{\sigma}' d^2\bar{\theta}' \bar{E}'(\bar{\sigma}', \bar{\tau}, \bar{\theta}') = \int d\bar{\sigma} d^2\bar{\theta} \bar{E}(\bar{\sigma}, \bar{\tau}, \bar{\theta}). \quad (6.14)$$

Therefore, $G_{I\bar{J}}(\bar{E}, X_{D_T}, I, \bar{\tau})$ and $A_d(\bar{E}, X_{D_T}, I, \bar{\tau})$ are transformed covariantly and the action $(6.9)$ is invariant under the diffeomorphisms $(6.6)$ including the $(\bar{\sigma}, \bar{\theta})$ superdiffeomorphisms, whose infinitesimal transformations are given by

$$\tilde{\sigma}^\xi = \bar{\sigma} + i \xi^\alpha(\bar{\sigma}) \gamma^1_{\alpha\beta} \bar{\theta}^\beta$$

$$\tilde{\bar{\theta}}^\alpha(\bar{\sigma}) = \bar{\theta}^\alpha + \xi^\alpha(\bar{\sigma}). \quad (15.16)$$

$(6.15)$ are dimensional reductions in $\bar{\tau}$ direction of the two-dimensional $N = (1,1)$ local supersymmetry infinitesimal transformations. The number of supercharges

$$\xi^\alpha Q_\alpha = \xi^\alpha (\frac{\partial}{\partial \theta^\alpha} + i \gamma^1_{\alpha\beta} \bar{\theta}^\beta \frac{\partial}{\partial \bar{\sigma}}) \quad (6.16)$$
of the transformations is the same as of the two-dimensional ones. The supersymmetry algebra closes in a field-independent sense as in ordinary supergravities.

The background that represents a perturbative vacuum is given by

\[
\begin{align*}
\mathcal{L} &= 2\lambda \bar{\rho}(\bar{h}) N^2(X_{D\tau}) (dX^d_{D\tau})^2 \\
&= \int d\sigma d^2\theta \hat{E} \int d\sigma' d^2\bar{\theta} \hat{E}' N \frac{\sqrt{\bar{h}(\bar{\sigma}, \bar{\tau})}}{\sqrt{h(\sigma, \tau)}} \delta_{(\mu\bar{\sigma}\bar{\theta})(\mu'\bar{\sigma}'\bar{\theta}')} dX_{D\tau}^{(\mu\bar{\sigma}\bar{\theta})} dX_{D\tau}^{(\mu'\bar{\sigma}'\bar{\theta}')}.
\end{align*}
\]

The background that represents a perturbative vacuum is given by

\[
\begin{align*}
\mathcal{L} &= 2\lambda \bar{\rho}(\bar{h}) N^2(X_{D\tau}) (dX^d_{D\tau})^2 \\
&= \int d\sigma d^2\theta \hat{E} \int d\sigma' d^2\bar{\theta} \hat{E}' N \frac{\sqrt{\bar{h}(\bar{\sigma}, \bar{\tau})}}{\sqrt{h(\sigma, \tau)}} \delta_{(\mu\bar{\sigma}\bar{\theta})(\mu'\bar{\sigma}'\bar{\theta}')} dX_{D\tau}^{(\mu\bar{\sigma}\bar{\theta})} dX_{D\tau}^{(\mu'\bar{\sigma}'\bar{\theta}')}.
\end{align*}
\]

\[
\mathcal{L}' = \frac{\sqrt{\bar{h}(\bar{\sigma}, \bar{\tau})}}{\sqrt{h(\sigma, \tau)}} \delta_{(\mu\bar{\sigma}\bar{\theta})(\mu'\bar{\sigma}'\bar{\theta}')} dX_{D\tau}^{(\mu\bar{\sigma}\bar{\theta})} dX_{D\tau}^{(\mu'\bar{\sigma}'\bar{\theta}')}.
\]

One can show that the background (6.17) is a classical solution of the equations of motion of (6.19) as in section 4. The dependence of \(\hat{E}^A_M\) on the background (6.17) is uniquely deter-

---

13This solution is a generalization of the Majumdar-Papapetrou solution [9,10] of the Einstein-Maxwell system.
minded by the consistency of the quantum theory of the fluctuations around the background. Actually, we will find that all the perturbative superstring amplitudes are derived as follows.

Let us consider fluctuations around the background \( (6.17) \), \( G_{IJ} = G^I_J + \tilde{G}_{IJ} \) and \( A_I = \hat{A}_I + \tilde{A}_I \). Here we fix the charts, where we choose \( T = \text{IIA, IIB or I} \). The action (6.9) up to the quadratic order is given by,

\[
S = \frac{1}{G_N} \int \mathcal{D}E \mathcal{D}X_{D_T} \mathcal{D}\tau \sqrt{\tilde{G}} \left( -\tilde{R} + \frac{1}{4} \tilde{F}_{IJ}' \tilde{F}^{IJ} \\
+ \frac{1}{4} \nabla^I \tilde{G}^J_K \nabla^J \tilde{G}^{JK} - \frac{1}{4} \nabla^I \tilde{G} \nabla^J \tilde{G} + \frac{1}{2} \nabla^I \tilde{G}^J_L \nabla^L \tilde{G} - \frac{1}{2} \nabla^I \tilde{G}^J_L \nabla^L \tilde{G}^{JK} - \frac{1}{2} \nabla^I \tilde{G}^J_L \nabla^L \tilde{G}^{JK} \\
- \frac{1}{4} (\tilde{R} + \frac{1}{4} \tilde{F}^I_{KL} \tilde{F}^{I, KL}) (\tilde{G}^I_{IJ} \tilde{G}^{IJ} - \frac{1}{2} \tilde{G}^2) + (-\frac{1}{2} \tilde{R}_I^{I, J} + \frac{1}{4} \tilde{F}^I_{KL} \tilde{F}^{I, KL}) \tilde{G}_{IJ} \tilde{G}_{KL} \\
+ \frac{1}{2} \tilde{G}^I_{KL} \tilde{F}^{I, KL} \tilde{G}_{IJ} \tilde{G} + (\frac{1}{2} \tilde{R}_I^{I, J} + \frac{1}{4} \tilde{F}^I_{KL} \tilde{F}^{I, KL}) \tilde{G}_{IKL} \tilde{G}_{JL} \\
+ \frac{1}{4} G_N \tilde{F}_{IJ} \tilde{F}^{IJ} + \sqrt{G_N} \left( \frac{1}{4} \tilde{F}^{I, J} \tilde{F}^{I, J} - \tilde{F}^{I, J} \tilde{F}^{I, J} \right) \right). \tag{6.20}
\]

The Lagrangian is independent of Chan-Paton indices because the background \( (6.17) \) is independent of them. \( \tilde{F}_{IJ} := \sqrt{G_N} \tilde{F}_{IJ} \) is independent of \( G_N \). \( \tilde{G} := \tilde{G}^I_J \tilde{G}^J_I \). There is no first order term because the background satisfies the equations of motion.

From these fluctuations, we obtain the correlation function in the string manifold \( \mathfrak{M}_D^T \) in exactly the same way as in section 4,

\[
\Delta \bar{F}(X_{D_T}, X_{D_T} \mid E_f, E_i) = Z \int_{E_f}^{E_f} \mathcal{D}E \mathcal{D}X_{D_T} \mathcal{D}\tau e^{-\lambda \chi} e^{-\int d^2 \sigma d^2 \theta E_i^1 (\mathcal{D}_\sigma X_{D_T})^2}, \tag{6.21}
\]

where \( \chi \) is the Euler number of the reduced space. By inserting asymptotic states with Chan-Paton matrices to \( (6.21) \) as in section 3, we obtain the all-order perturbative scattering amplitudes that possess the supermoduli in the type IIA, type IIB and SO(32) type I superstring theory for \( T = \text{IIA, IIB and I} \), respectively in the presence of D-branes with arbitrary configuration\(^{14}\). The open superstrings possess Dirichlet and Neumann boundary conditions in the normal and tangential directions to the D-submanifolds, respectively\(^{12}\). Therefore, a set of D-submanifolds represents a D-brane background where back reactions from the D-branes are ignored. Especially, in superstring geometry, the consistency of the perturbation theory around the background \( (6.17) \) determines \( d = 10 \) (the critical dimension).

\(^{14}\)This includes the case there is no D-brane.
7 Matrix models for superstring geometry

There are some types of supersymmetric matrix models such as, models that consist of super matrices [19–22], and dimensional reductions of supersymmetric Yang-Mills for examples, IKKT matrix model [23], BFSS matrix model [24] and matrix string [25–28]. In this section, we propose a new type of supersymmetric matrix models (7.3) and (7.5), namely, manifestly supersymmetric models that consist of matrices whose infinite dimensional indices include super coordinates.

It is shown in [29] that the equations of motions of the 10-dimensional gravity theory coupled with a $u(1)$ gauge field

$$S_e = \frac{1}{G_N} \int d^{10}x \sqrt{g}(-R + \frac{1}{4}G_{\mu\nu}F^{\mu\nu}), \quad (7.1)$$

are equivalent to the equations of motions of a matrix model

$$S_m = \text{tr}(-[A_\mu, A_\nu][A^\mu, A^\nu]), \quad (7.2)$$

if the matrices are mapped to the covariant derivatives on the superstring manifolds.

Moreover, it is interesting to study relations between the superstring geometry and a more simple supersymmetric matrix model

$$S_M = \int \mathcal{D}E\mathcal{D}\tilde{I}\text{tr}(-[A_{D_T, \mathbf{1}}(\mathbf{E}, \tilde{\mathbf{I}}), A_{\tilde{D}_T, \mathbf{j}}(\mathbf{E}, \tilde{\mathbf{I}})][A^1_{D_T}(\mathbf{E}, \tilde{\mathbf{I}}), A^J_{\tilde{D}_T}(\mathbf{E}, \tilde{\mathbf{I}})]), \quad (7.3)$$

which is decomposed as

$$S_M = \int \mathcal{D}E\mathcal{D}\tilde{I}\text{tr}\left(-2 \int d\sigma d^2\theta \bar{E}[A_{D_T, \mathbf{1}}(\mathbf{E}, \tilde{\mathbf{I}}), A_{\tilde{D}_T, (\mu\sigma\bar{\theta})}(\mathbf{E}, \tilde{\mathbf{I}})][A^1_{D_T}(\mathbf{E}, \tilde{\mathbf{I}}), A^J_{\tilde{D}_T, (\mu\sigma\bar{\theta})}(\mathbf{E}, \tilde{\mathbf{I}})]
- \int d\sigma d^2\theta \bar{E} \int d\sigma' d^2\bar{\theta} \bar{E}'[A_{D_T, (\mu\sigma\bar{\theta})}(\mathbf{E}, \tilde{\mathbf{I}}), A_{\tilde{D}_T, (\mu'\sigma'\bar{\theta}')}(\mathbf{E}, \tilde{\mathbf{I}})]
\left[A_{D_T, (\mu\sigma\bar{\theta})}(\mathbf{E}, \tilde{\mathbf{I}}), A_{\tilde{D}_T, (\mu'\sigma'\bar{\theta}')}(\mathbf{E}, \tilde{\mathbf{I}})\right]\right), \quad (7.4)$$

if the matrices are mapped to the covariant derivatives on the superstring manifolds.

Moreover, it is interesting to study relations between the superstring geometry and a more simple supersymmetric matrix model

$$S_{M_0} = \text{tr}(-[A_{\mathbf{1}}, A_{\mathbf{j}}][A^1, A^J]), \quad (7.5)$$

\[15\mathbf{I} = (d, (\mu\sigma\bar{\theta})), \text{ whereas } \tilde{\mathbf{I}} \text{ represent the Chan-Paton indices.} \]
decomposed as

\[ S_{M_0} = \text{tr} \left( -2 \int d\sigma d^2\bar{\theta} [A_d, A_{(\mu \bar{\sigma} \bar{\theta})}] [A_d, A_{(\mu \bar{\sigma} \bar{\theta})}] 
- \int d\sigma d^2\bar{\theta} \int d\sigma' d^2\bar{\theta}' [A_{(\mu \bar{\sigma} \bar{\theta})}, A_{(\mu' \bar{\sigma}' \bar{\theta}'})] [A_{(\mu \bar{\sigma} \bar{\theta})}, A_{(\mu' \bar{\sigma}' \bar{\theta}'})] \right). \]  

(7.6)

because topological expansions of worldsheets can be derived in general by perturbations of matrix models \[30\text{--}38\]. (7.5) may correspond to (7.3) by an extension of the large N reduction \[39\].

8 Heterotic construction

In this section, based on superstring geometry, we formulate and study a theory that manifestly possesses the \(SO(32)\) and \(E_8 \times E_8\) heterotic perturbative vacua. We expect that this theory is equivalent to the theory in section 6, which manifestly possesses the type IIA, type IIB and \(SO(32)\) type I perturbative vacua, because of the S-duality.

First, let us prepare a moduli space\(^{16}\) of heterotic superstring worldsheets \(\Sigma\) \[13\text{--}15\] with punctures \(P_i\) \((i = 1, \cdots, N)\).\(^{17}\) We consider a super Riemann surface \(\tilde{\Sigma}_R\) with Neveu-Schwarz (NS) and Ramond (R) punctures whose reduced space \(\tilde{\Sigma}_{R,\text{red}}\) is complex conjugate to a Riemann surface \(\Sigma_L\). A reduced space is defined by setting odd variables to zero in a super Riemann surface. The complex conjugates means that they are complex conjugate spaces with punctures at the same points. A heterotic superstring worldsheet \(\Sigma\) is defined by the subspace of \(\tilde{\Sigma}_L \times \tilde{\Sigma}_R\) whose reduced space \(\tilde{\Sigma}_L \times \tilde{\Sigma}_{R,\text{red}}\) is restricted to its diagonal \(\tilde{\Sigma}_{\text{red}}\).

Next, we define global times uniquely on \(\tilde{\Sigma}_R\) in exactly the same way as in section 4. If we give residues \(-f^i\) and the same normalization on \(\tilde{\Sigma}_L\) as on \(\tilde{\Sigma}_{R,\text{red}}\), we can set the coordinates on \(\tilde{\Sigma}_L\) to the complex conjugate \(\bar{w} = \bar{\tau} - i\bar{\sigma} := \int P d\bar{p}\) by a conformal transformation, because the Abelian differential is uniquely determined on \(\tilde{\Sigma}_L\) and \(\tilde{\Sigma}_L\) is complex conjugate to \(\tilde{\Sigma}_{R,\text{red}}\). Therefore, we can define the global time \(\bar{\tau}\) uniquely and reduced space canonically on \(\tilde{\Sigma}\).

---

\(^{16}\)Strictly speaking, this should be called a parameter space of integration cycles \[13\text{--}14\] because superstring worldsheets are defined up to homology.

\(^{17}\)\(P^i\) not necessarily represents a point, whereas the corresponding \(P_{\text{red}}^i\) on a reduced space represents a point. A Ramond puncture is located over a R divisor.
Thus, under a superconformal transformation, one obtains a heterotic worldsheet $\bar{\Sigma}$ that has even coordinates composed of the global time $\tau$ and the position $\sigma$ and $\bar{\Sigma}_{red}$ is canonically defined. Because $\bar{\Sigma}$ can be a moduli of heterotic worldsheets with punctures, any two-dimensional heterotic super Riemannian manifold with punctures $\Sigma$ can be obtained by $\Sigma = \psi(\bar{\Sigma})$ where $\psi$ is a superdiffeomorphism times super Weyl transformation.

Next, we will define a heterotic superstring space $\mathcal{M}_T^D$, where $T$ runs $SO(32)$ and $E_8 \times E_8$. We consider a state $(\bar{\Sigma}, u_{D_T}, \lambda_{D_T}, \bar{\tau}_s)$ determined by a $\bar{\tau} = \bar{\tau}_s$ constant hypersurface, an arbitrary map $u_{D_T}$ from $\Sigma$ to a d-dimensional Riemannian manifold $M$, and an arbitrary left-handed fermionic map $\lambda_{D_T}$ from $\bar{\Sigma}_{red}$ to a 32-dimensional internal vector bundle $V$ on $M$. In order to define $V$ globally, we need to patch local vector spaces. The transition functions among them require gauge connections. We also fix NS-NS B-field and dilaton backgrounds. $D_T$ runs all the fixed backgrounds that consist of a metric, a NS-NS B-field, a dilaton, and a gauge field. For $T = SO(32)$, we take periodicities

$$\lambda_{D_{SO(32)}}^A (\sigma + 2\pi) = \pm \lambda_{D_{SO(32)}}^A (\sigma) \quad (A = 1, \cdots 32) \quad (8.1)$$

with the same sign on all 32 components. For $T = E_8 \times E_8$, the periodicity is given by

$$\lambda_{D_{E_8 \times E_8}}^A (\bar{\sigma} + 2\pi) = \begin{cases} 
\eta \lambda_{D_{E_8 \times E_8}}^A (\bar{\sigma}) \quad (1 \leq A \leq 16), \\
\eta' \lambda_{D_{E_8 \times E_8}}^A (\bar{\sigma}) \quad (17 \leq A \leq 32),
\end{cases} \quad (8.2)$$

with the same sign $\eta(\pm 1)$ and $\eta'(\pm 1)$ on each 16 components. $\bar{\Sigma}$ is a direct sum of $N_{\pm}$ supercylinders with radii $f_i$ at $\bar{\tau} \simeq \mp \infty$. Thus, we define a superstring state as an equivalence class $[\Sigma, u_{D_T}, \lambda_{D_T}, \bar{\tau}_s \simeq \mp \infty]$ by a relation $(\Sigma, u_{D_T}, \lambda_{D_T}, \bar{\tau}_s \simeq \mp \infty) \sim (\Sigma', u'_{D_T}, \lambda'_{D_T}, \bar{\tau}_s \simeq \mp \infty)$ if $N_{\mp} = N'_{\mp}$, $f_i = f'_i$, $u_{D_T}|_{\bar{\tau}_s \simeq \mp \infty} = u'_{D_T}|_{\bar{\tau}_s \simeq \mp \infty}$, $\lambda_{D_T}|_{\bar{\tau}_s \simeq \mp \infty} = \lambda'_{D_T}|_{\bar{\tau}_s \simeq \mp \infty}$ and the corresponding supercylinders are the same type (NS or R). Because the reduced space of $\Sigma|_{\bar{\tau}_s}$ is $S^1 \times S^1 \times \cdots \times S^1$ and $u_{D_T}|_{\bar{\tau}_s} : \Sigma|_{\bar{\tau}_s} \rightarrow M$, $[\Sigma, u_{D_T}, \lambda_{D_T}, \bar{\tau}_s]$ represent many-body states of superstrings in $M$. A heterotic superstring space $\mathcal{M}_D^T$ is defined by a collection of all the heterotic superstring states $\{[\Sigma, u_{D_T}, \lambda_{D_T}, \bar{\tau}_s]\}$.

Here, we will define topologies of $\mathcal{M}_D^T$. We define an $\varepsilon$-open neighbourhood of
\[
\left[ \Sigma, u_{DT}, \lambda_{DT}, \bar{\tau}_s \right] \quad \text{by}
\]
\[
U\left([\Sigma, u_{DT}, \lambda_{DT}, \bar{\tau}_s], \epsilon\right)
\]
\[
:= \left\{ \left[ \Sigma, \exp_{u_{DT}} X_{DT}, \exp_{\lambda_{DT}} \lambda_{DT}, \bar{\tau}_s \right] \quad \sqrt{||\bar{\tau} - \bar{\tau}_s||^2 + ||X_{DT}||^2 + ||\lambda_{DT}||^2 + ||\lambda_{DT}||^2} < \epsilon \right\}.
\]

\(X_{DT} : \Sigma \to \mathbb{R}^{d-1,1}\) is a section of \(u_{DT}^* TM\). \(\lambda_{DT} : \Sigma_{red} \to V\) is a section of \(\lambda_{DT}^* TV\), where \(V\) is a tangent vector space of \(V\). \(\tilde{D}_T\) runs all the induced backgrounds that consist of a NS-NS B-field, a dilaton, and a gauge field. For \(T = SO(32), \) we take periodicities
\[
\lambda^A_{DSO(32)}(\sigma + 2\pi) = \pm \lambda^A_{DSO(32)}(\sigma) \quad (A = 1, \cdots 32)
\]
with the same sign on all 32 components. For \(T = E_8 \times E_8\), the periodicity is given by
\[
\lambda^A_{DE_8 \times E_8}(\sigma + 2\pi) = \left\{ \begin{array}{ll}
\eta \lambda^A_{DE_8 \times E_8}(\sigma) & (1 \leq A \leq 16) \\
\eta' \lambda^A_{DE_8 \times E_8}(\sigma) & (17 \leq A \leq 32),
\end{array} \right.
\]
with the same sign \(\eta(= \pm 1)\) and \(\eta'(= \pm 1)\) on each 16 components. The following statements do not depend on how to define the norm. An example is given by
\[
\|X_{DT}|_\tau\|^2 := -\int_{[\Sigma, \bar{\tau}, \bar{\theta}]} d\sigma d\theta E|_\tau \cdot \tilde{D}_\theta X_{DT}|_\tau >
\]
and
\[
\|\lambda_{DT}|_\tau\|^2 := \int_{[\Sigma, \bar{\tau}, \bar{\theta}]} d\sigma \sqrt{\bar{h}|_\tau \cdot \lambda_{DT}|_\tau} \cdot \bar{E}_|_\tau \theta \lambda_{DT}|_\tau >'.
\]
By definition of the $\epsilon$-open neighbourhood, arbitrary two superstring states on a connected heterotic super Riemann surface are connected continuously. Thus, there is an one-to-one correspondence between a heterotic super Riemann surface with punctures in $M$ and a curve parametrized by $\bar{\tau}$ from $\bar{\tau} = -\infty$ to $\bar{\tau} = \infty$ on $\mathcal{M}_D^T$. That is, curves that represent asymptotic processes on $\mathcal{M}_D^T$ reproduce the right moduli space of the heterotic super Riemann surfaces in the target manifold.

By a general curve parametrized by $t$ on $\mathcal{M}_D^T$, superstring states on different heterotic super Riemann surfaces that have even different genera, can be connected continuously, for example see Fig. 4, whereas different super Riemann surfaces that have different genera cannot be connected continuously in the moduli space of the heterotic super Riemann surfaces in the target space. Therefore, the superstring geometry is expected to possess non-perturbative effects.

Next, we will define structures of manifold on the superstring topological space $\mathcal{M}_D^T$. We define a model space $\mathbf{E} := \bigcup_{D_T} \{ [\Sigma, \mathbf{X}_{D_T}, \lambda_{D_T}, \bar{\tau}] \}$, where the $SO(32)$ and $E_8 \times E_8$ heterotic GSO projections are attached for $T = SO(32)$ and $E_8 \times E_8$, respectively. We can define the worldsheet fermion numbers of states in a Hilbert space because the states consist of the fields over the local coordinates $\mathbf{X}_\mu_{D_T} = X_\mu_{D_T} + \theta \psi^\mu_{D_T}$ and $\lambda^A_{D_T}$, where $\mu = 0, 1, \cdots, d - 1$ and $\psi^\mu_{D_T}$ is a Majorana fermion. For $T = SO(32)$, we define the Hilbert space in these coordinates by the states only with $e^{\pi i F} = 1$ and $e^{\pi i \tilde{F}} = 1$, where $F$ and $\tilde{F}$ are the numbers of left- and right-handed fermions $\lambda^A_{SO(32)}$ and $\psi^\mu_{SO(32)}$, respectively. For $T = E_8 \times E_8$, the GSO projection is given by $e^{\pi i F_1} = 1$, $e^{\pi i F_2} = 1$ and $e^{\pi i \tilde{F}} = 1$, where $F_1$, $F_2$ and $\tilde{F}$ are the numbers of $\lambda^A_{E_8 \times E_8}$ ($A_1 = 1, \cdots, 16$), $\lambda^A_{E_8 \times E_8}$ ($A_2 = 17, \cdots, 32$) and $\psi^\mu_{E_8 \times E_8}$, respectively. Although the model space is defined by using the coordinates $[\Sigma, \mathbf{X}_{D_T}, \lambda_{D_T}, \bar{\tau}]$, the model space does not depend on the coordinates, because a model space is a topological space.

An $\epsilon$-open neighbourhood of $[\Sigma, \mathbf{X}_{D_T S}, \lambda_{D_T S}, \bar{\tau}_S]$ is defined by

$$U([\Sigma, \mathbf{X}_{D_T S}, \lambda_{D_T S}, \bar{\tau}_S], \epsilon)$$

$$:= \left\{ [\Sigma, \mathbf{X}_{D_T}, \lambda_{D_T}, \bar{\tau}] \mid (||\bar{\tau} - \bar{\tau}_S||^2 + ||\mathbf{X}_{D_T}||^2 + ||\lambda_{D_T}||^2 + ||\mathbf{X}_{D_T S}||^2 + ||\lambda_{D_T S}||^2)^{\frac{1}{2}} < \epsilon \right\}.$$ 

(8.8)

Open sets of $\mathbf{E}$ are defined as in the case of $\mathcal{M}_D^T$. By these definitions, one can prove that the topology of $\mathbf{E}$ satisfies the axiom of topology in the same way. Here, we define charts
(U([\Sigma, u_{D_T}, \lambda_{D_T}, \bar{\tau}_s], \epsilon), \Psi) on M^T_D). First, the \epsilon-open neighbourhoods U([\Sigma, u_{D_T}, \lambda_{D_T}, \bar{\tau}_s], \epsilon) cover M^T_D because they are defined at any point [\Sigma, u_{D_T}, \lambda_{D_T}, \bar{\tau}_s] \in M^T_D. Next,

\[ \Psi : U([\Sigma, u_{D_T}, \lambda_{D_T}, \bar{\tau}_s], \epsilon) \rightarrow U([\Sigma, 0, 0, \bar{\tau}_s], \epsilon) \]

is a homeomorphism from an open set in M^T_D to that in E where the corresponding NS-NS B-field, dilaton and gauge fields are turned on.

Proof. For

\[ U([\Sigma, u_{D_T}, \lambda_{D_T}, \bar{\tau}_s], \epsilon) \]

\[ = \left\{ [\Sigma, \exp u_{D_T} X_{\hat{D}_T}, \exp \lambda_{D_T} \lambda_{D_T}, \bar{\tau}] \mid \sqrt{|\bar{\tau} - \bar{\tau}_s|^2 + \|X_{\hat{D}_T}\|^2 + \|X_{\bar{D}_T}\|^2 + \|\lambda_{\bar{D}_T}\|^2 + \|\lambda_{\hat{D}_T}\|^2} < \epsilon \right\}, \quad (8.9) \]

and

\[ U([\Sigma, 0, 0, \bar{\tau}_s], \epsilon) \]

\[ = \left\{ [\Sigma, X_{\hat{D}_T}, \lambda_{\hat{D}_T}, \bar{\tau}] \mid \sqrt{|\bar{\tau} - \bar{\tau}_s|^2 + \|X_{\hat{D}_T}\|^2 + \|X_{\bar{D}_T}\|^2 + \|\lambda_{\bar{D}_T}\|^2 + \|\lambda_{\hat{D}_T}\|^2} < \epsilon \right\}, \quad (8.10) \]

\( \Psi \) is bijective because the exponential maps \( X_{\hat{D}_T} \leftrightarrow \exp u_{D_T} X_{\hat{D}_T} \) and \( \lambda_{\hat{D}_T} \leftrightarrow \exp \lambda_{D_T} \lambda_{D_T} \) are bijective on \( \Sigma \). Then, \( \Psi \) and \( \Psi^{-1} \) are continuous because the \( \epsilon \)-open neighbourhoods and the open sets are defined by the same conditions (8.9) and (8.10).

In the following, we denote \([E'_M A'(\bar{\sigma}, \bar{\tau}, \bar{\theta}), X'_{\hat{D}_T}, \lambda'_{\hat{D}_T}, \bar{\tau}] \) instead of \([\Sigma, X_{\hat{D}_T}, \lambda_{\hat{D}_T}, \bar{\tau}] \) because giving a super Riemann surface is equivalent to giving a super vierbein up to super diffeomorphism and super Weyl transformations.

Let us consider how generally we can define general coordinate transformations between \([E'_M A', X'_{\hat{D}_T}, \lambda'_{\hat{D}_T}, \bar{\tau}] \) and \([E'_M A', X'_{\bar{D}_T}, \lambda'_{\bar{D}_T}, \bar{\tau}] \). where \([E'_M A', X'_{\hat{D}_T}, \lambda_{\hat{D}_T}, \bar{\tau}] \) \( \in U \subset E \) and \([E'_M A', X'_{\bar{D}_T}, \lambda'_{\bar{D}_T}, \bar{\tau}] \) \( \in U' \subset E \). \( E'_M A \) does not transform to \( \bar{\tau}, X_{\hat{D}_T} \) and \( \lambda_{\hat{D}_T} \), and vice versa, because \( \bar{\tau}, X_{\hat{D}_T} \) and \( \lambda_{\hat{D}_T} \) are continuous variables, whereas \( E'_M A \) is a discrete variable: \( \bar{\tau}, X_{\hat{D}_T} \) and \( \lambda_{\hat{D}_T} \) vary continuously, whereas \( E'_M A \) varies discretely in a trajectory on \( E \) by definition of the neighbourhoods. \( \bar{\tau} \) does not transform to \( \bar{\sigma} \) and \( \bar{\theta} \) and vice versa, because
the superstring states are defined by \( \tau \) constant surfaces. Under these restrictions, the most general coordinate transformation is given by

\[
[E_M^A (\bar{\sigma}, \bar{\tau}, \bar{\theta}), X^\mu_{D_T}, \lambda^A_{D_T}, \bar{\tau}]
\]

\[
\mapsto [E_M'^A (\bar{\sigma}'(\bar{\sigma}, \bar{\theta}), \bar{\tau}'(\bar{\tau}, X_{D_T}, \lambda_{D_T}, \bar{\theta}'(\bar{\sigma}, \bar{\theta})), X^\mu_{D_T} (\bar{\sigma}', \bar{\tau}', \bar{\theta}')(\bar{\tau}, X_{D_T}, \lambda_{D_T}), \lambda^A_{D_T}, \bar{\tau}'), X^\mu_{D_T} (\bar{\sigma}', \bar{\tau}', \bar{\theta}')(\bar{\tau}, X_{D_T}, \lambda_{D_T})],
\]

(8.11)

where \( \mu = 0, 1, \ldots, d - 1 \). Because \( E_{M^A}, \tau, X^\mu_{D_T} \) and \( \lambda^A_{D_T} \) are all independent, where \( E_{M^A}, X^\mu_{D_T} \) and \( \lambda^A_{D_T} \) are coordinates as functionals, \( \frac{\partial}{\partial \tau} \) is an explicit derivative on functions over the superstring manifolds, especially, \( \frac{\partial}{\partial \tau} E_{M^A} = 0, \frac{\partial}{\partial \tau} X^\mu_{D_T} = 0 \) and \( \frac{\partial}{\partial \tau} \lambda^A_{D_T} = 0 \).

Here, we consider all the manifolds which are constructed by patching open sets of the model space \( E \) by general coordinate transformations (8.11) and call them heterotic superstring manifolds \( \mathcal{M} \). We also call the topologies and spaces of the heterotic superstring manifolds, heterotic superstring topologies and heterotic superstring spaces, respectively. The heterotic superstring space \( \mathcal{M}_{T_D} \), the heterotic superstring topological space \( \mathcal{M}_{T_D}^T \), and the heterotic superstring manifold \( \mathcal{M}_{T_D}^T \) are ones of the heterotic superstring spaces, heterotic superstring topological spaces, and heterotic superstring manifolds, respectively. In the following, instead of the fermionic coordinate \( \lambda^A_{D_T} (\bar{\sigma}, \bar{\tau}) \), we use a bosonic coordinate \( X^A_{L_{D_T}} (\bar{\sigma}, \bar{\tau}, \bar{\theta}^-) := \bar{\theta}^- \lambda^A_{D_T} (\bar{\sigma}, \bar{\tau}) \) where \( \bar{\theta}^- \) has the opposite chirality to \( \bar{\theta}^+ \).

The tangent space is spanned by \( \frac{\partial}{\partial X^\mu_{D_T}(\bar{\sigma}, \bar{\tau}, \bar{\theta})}, \frac{\partial}{\partial \lambda^A_{L_{D_T}}(\bar{\sigma}, \bar{\tau}, \bar{\theta}^-)} \) and \( \frac{\partial}{\partial \tau} \) as one can see from the chart (8.10). We should note that \( \frac{\partial}{\partial E_{M_A}} \) cannot be a part of basis that span the tangent space because \( \bar{E}^A_{M_A} \) is just a discrete variable in \( E \). The indices of \( \frac{\partial}{\partial X^\mu_{D_T}(\bar{\sigma}, \bar{\tau}, \bar{\theta})} \) and \( \frac{\partial}{\partial \lambda^A_{L_{D_T}}(\bar{\sigma}, \bar{\tau}, \bar{\theta}^-)} \) can be \( (\mu \bar{\sigma} \bar{\theta}) \) and \( (A \bar{\sigma} \bar{\theta}^-) \), where \( \mu = 0, 1, \ldots, d - 1 \) and \( A = 1, \ldots, 32 \), respectively. Then, let us define a summation over \( \bar{\sigma} \) and \( \bar{\theta}^- \) that is invariant under \( (\bar{\sigma}, \bar{\theta}) \mapsto (\bar{\sigma}'(\bar{\sigma}, \bar{\theta}), \bar{\theta}'(\bar{\sigma}, \bar{\theta})) \) and transformed as a scalar under \( \bar{\tau} \mapsto \bar{\tau}'(\bar{\tau}, X_{D_T}, X_{L_{D_T}}) \). First, \( \int d\bar{\tau} \int d\bar{\sigma} d\bar{\theta} \bar{E}(\bar{\sigma}, \bar{\tau}, \bar{\theta}) \) is invariant under \( (\bar{\sigma}, \bar{\tau}, \bar{\theta}) \mapsto (\bar{\sigma}'(\bar{\sigma}, \bar{\theta}), \bar{\tau}'(\bar{\tau}, X_{D_T}, X_{L_{D_T}}), \bar{\theta}'(\bar{\sigma}, \bar{\theta})) \), where \( \bar{E}(\bar{\sigma}, \bar{\tau}, \bar{\theta}) \) is the superdeterminant of \( E_{M^A}(\bar{\sigma}, \bar{\tau}, \bar{\theta}) \). A super analogue of the lapse function, \( \frac{1}{\sqrt{E_{M^A} E_{L^A}}} \) transforms as an one-dimensional vector in the \( \bar{\tau} \) direction: \( \int d\bar{\tau} \frac{1}{\sqrt{E_{M^A} E_{L^A}}} \) is invariant under \( \bar{\tau} \mapsto \bar{\tau}'(\bar{\tau}, X_{D_T}, X_{L_{D_T}}) \) and transformed as a superscalar under \( (\bar{\sigma}, \bar{\theta}) \mapsto (\bar{\sigma}'(\bar{\sigma}, \bar{\theta}), \bar{\theta}'(\bar{\sigma}, \bar{\theta})) \).

Therefore, \( \int d\bar{\sigma} d\bar{\theta} \bar{E}(\bar{\sigma}, \bar{\tau}, \bar{\theta}) \), where \( \bar{E}(\bar{\sigma}, \bar{\tau}, \bar{\theta}) := \sqrt{E_{M^A} E_{L^A}} \bar{E}(\bar{\sigma}, \bar{\tau}, \bar{\theta}) \), is transformed as a scalar under \( \bar{\tau} \mapsto \bar{\tau}'(\bar{\tau}, X_{D_T}, X_{L_{D_T}}) \) and invariant under \( (\bar{\sigma}, \bar{\theta}) \mapsto (\bar{\sigma}'(\bar{\sigma}, \bar{\theta}), \bar{\theta}'(\bar{\sigma}, \bar{\theta})) \). The summation over \( \bar{\sigma} \) and \( \bar{\theta}^- \) is defined by \( \int d\bar{\sigma} d\bar{\theta}^- \bar{e}(\bar{\sigma}, \bar{\tau}) \), where \( \bar{e} := \sqrt{h_{\bar{\sigma}\bar{\theta}}} \). This summation...
tion is also invariant under \((\bar{\sigma}, \bar{\theta}) \mapsto (\bar{\sigma}'(\bar{\sigma}, \bar{\theta}), \bar{\theta}'(\bar{\sigma}, \bar{\theta}))\), where \(\bar{\theta}^-\) is not transformed, and transformed as a scalar under \(\bar{\tau} \mapsto \bar{\tau}'(\bar{\tau}, X_{D_T}, X_{LD_T})\).

Riemannian heterotic superstring manifold is obtained by defining a metric, which is a section of an inner product on the tangent space. The general form of a metric is given by

\[
d s^2(E, X_{D_T}, X_{LD_T}, \tau) = G(E, X_{D_T}, X_{LD_T}, \tau) d\bar{\tau},
\]

\[
d s^2(E, X_{D_T}, X_{LD_T}, \tau) = G(E, X_{D_T}, X_{LD_T}, \tau) d\bar{\tau}^2 + 2d\bar{\tau} \int d\sigma d\bar{\sigma} E_{I\mu} G(E, X_{D_T}, X_{LD_T}, \tau) (\mu \bar{\sigma}) dX_{D_T}^\mu (\bar{\sigma}, \bar{\tau}, \bar{\theta})
\]

\[
+ 2d\bar{\tau} \int d\bar{\sigma} d\bar{\theta} \bar{e} A_{\mu \nu} G(E, X_{D_T}, X_{LD_T}, \tau) (\mu \bar{\sigma} \bar{\theta}) dX_{D_T}^\mu (\bar{\sigma}, \bar{\tau}, \bar{\theta})
\]

\[
+ dX_{D_T}^A (\bar{\sigma}, \bar{\tau}, \bar{\theta}) dX_{LD_T}^A (\bar{\sigma}', \bar{\tau}, \bar{\theta}')
\]

\[
(8.12)
\]

We summarize the vectors as \(dX_{D_T}^1 (I = d, (\mu \bar{\sigma}), (A \bar{\sigma} \bar{\theta}))\), where \(dX_{D_T}^d := d\bar{\tau}, dX_{D_T}^{(\mu \bar{\sigma})} := dX_{D_T}^\mu (\bar{\sigma}, \bar{\tau}, \bar{\theta})\) and \(dX_{LD_T}^{(A \bar{\sigma} \bar{\theta})} := dX_{LD_T}^A (\bar{\sigma}, \bar{\tau}, \bar{\theta})\). Then, the components of the metric are summarized as \(G_{IJ} (E, X_{D_T}, X_{LD_T}, \tau)\). The inverse of the metric \(G^{IJ} (E, X_{D_T}, X_{LD_T}, \tau)\) is defined by \(G_{IJ} G^{JK} = \delta^K_I\), where \(\delta^I_J = 1\), \(\delta^{\mu' \bar{\sigma}' \bar{\theta}'} = 1\), \(\delta^{A' \bar{\sigma}' \bar{\theta}'} = 1\), \(\delta^{\mu \bar{\sigma} \bar{\theta}} = 1\). The components of the Riemannian curvature tensor are given by \(R_{IJ}^{KL}\) in the basis \(\frac{\partial}{\partial X_{D_T}^\mu}\). The Ricci tensor is \(R_{IJ} := R^{K}_{IKJ}\) and the scalar curvature is \(R := G^{IJ} R_{IJ}\). The volume is \(vol := \sqrt{G}\), where \(G = det(G_{IJ})\).

By using these geometrical objects, we define a superstring theory non-perturbatively as

\[
Z = \int \mathcal{D}G \mathcal{D}A e^{-S},
\]

\[
S = \frac{1}{G_N} \int \mathcal{D}E dX_{D_T} dX_{LD_T} d\bar{\tau} \sqrt{G} (-R + \frac{1}{4} G_{IJ} G^{I_1J_2} G^{J_1J_2} F_{I_1J_1} F_{I_2J_2}).
\]

(8.13)

(8.14)
As an example of sets of fields on the superstring manifolds, we consider the metric and an
$u(1)$ gauge field $A_I$ whose field strength is given by $F_{IJ}$. The path integral is canonically
defined by summing over the metrics and gauge fields on $\mathcal{M}$. By definition, the theory
is background independent. $\mathcal{D}\mathbf{E}$ is the invariant measure of the super vierbeins $\mathbf{E}_M{}^A$ on
the two-dimensional super Riemannian manifolds $\Sigma$, divided by the volume of the super
diffeomorphism and super Weyl transformations. $\mathbf{E}_M{}^A$ and $\bar{\mathbf{E}}_M{}^A$ are related to each others
by the super diffeomorphism and super Weyl transformations.

Under

$$(\bar{\tau}, \mathbf{X}_{D_T}, \mathbf{X}_{L\bar{D}_T}) \mapsto (\bar{\tau}'(\bar{\tau}, \mathbf{X}_{D_T}, \mathbf{X}_{L\bar{D}_T}), \mathbf{X}'_{D_T}(\bar{\tau}, \mathbf{X}_{D_T}, \mathbf{X}_{L\bar{D}_T}), \mathbf{X}'_{L\bar{D}_T}(\bar{\tau}, \mathbf{X}_{D_T}, \mathbf{X}_{L\bar{D}_T})), \quad (8.15)$$

$G_{IJ}(\mathbf{E}, \mathbf{X}_{D_T}, \mathbf{X}_{L\bar{D}_T}, \bar{\tau})$ and $A_I(\mathbf{E}, \mathbf{X}_{D_T}, \mathbf{X}_{L\bar{D}_T}, \bar{\tau})$ are transformed as a symmetric tensor and
a vector, respectively and the action is manifestly invariant.

We define $G_{IJ}(\mathbf{E}, \mathbf{X}_{D_T}, \mathbf{X}_{L\bar{D}_T}, \bar{\tau})$ and $A_I(\mathbf{E}, \mathbf{X}_{D_T}, \mathbf{X}_{L\bar{D}_T}, \bar{\tau})$ so as to transform as scalars
under $\mathbf{E}_M{}^A(\bar{\sigma}, \bar{\tau}, \bar{\theta}) \mapsto \mathbf{E}'_M{}^A(\bar{\sigma}'(\bar{\sigma}, \bar{\tau}, \bar{\theta}), \bar{\tau}', (\bar{\sigma}, \bar{\theta}))$. Under $(\bar{\sigma}, \bar{\theta})$ superdiffeomorphisms: $(\bar{\sigma}, \bar{\theta}) \mapsto (\bar{\sigma}'(\bar{\sigma}, \bar{\theta}), \bar{\theta}'(\bar{\sigma}, \bar{\theta}))$, which are equivalent to

$$[\mathbf{E}_M{}^A(\bar{\sigma}, \bar{\tau}, \bar{\theta}), \mathbf{X}^\mu_{D_T}(\bar{\sigma}, \bar{\tau}, \bar{\theta}), \mathbf{X}^A_{L\bar{D}_T}(\bar{\sigma}, \bar{\tau}, \bar{\theta}^-), \bar{\tau}]$$

$$\mapsto [\mathbf{E}'_M{}^A(\bar{\sigma}'(\bar{\sigma}, \bar{\tau}, \bar{\theta}), \bar{\tau}', \bar{\theta}'(\bar{\sigma}, \bar{\theta})), \mathbf{X}'^\mu_{D_T}(\bar{\sigma}'(\bar{\sigma}, \bar{\tau}, \bar{\theta}), \bar{\tau}', \bar{\theta}'(\bar{\sigma}, \bar{\theta}))(\mathbf{X}_{D_T}, \mathbf{X}'_{L\bar{D}_T}(\bar{\sigma}'(\bar{\sigma}, \bar{\tau}, \bar{\theta})), \bar{\tau}, \bar{\theta}^-)(\mathbf{X}_{L\bar{D}_T}, \bar{\tau})$$

$$= [\mathbf{E}'_M{}^A(\bar{\sigma}'(\bar{\sigma}, \bar{\tau}, \bar{\theta}), \bar{\tau}', \bar{\theta}'(\bar{\sigma}, \bar{\theta})), \mathbf{X}^\mu_{D_T}(\bar{\sigma}, \bar{\tau}, \bar{\theta}), \mathbf{X}^A_{L\bar{D}_T}(\bar{\sigma}, \bar{\tau}, \bar{\theta}^-), \bar{\tau}], \quad (8.16)$$

$G_{d(A\bar{a}\bar{b}^-)}$ is transformed as a scalar;

$$G'_{d(A\bar{a}\bar{b}^-)}(\mathbf{E}', \mathbf{X}'_{D_T}, \mathbf{X}'_{L\bar{D}_T}, \bar{\tau}) = G'_{d(A\bar{a}\bar{b}^-)}(\mathbf{E}, \mathbf{X}_{D_T}, \mathbf{X}_{L\bar{D}_T}, \bar{\tau})$$

$$= \frac{\partial \mathbf{X}^I_{D_T}}{\partial \mathbf{X}^J_{D_T}} \frac{\partial \mathbf{X}^{d(A\bar{a}\bar{b}^-)}}{\partial \mathbf{X}_{D_T}} G_{IJ}(\mathbf{E}, \mathbf{X}_{D_T}, \mathbf{X}_{L\bar{D}_T}, \bar{\tau})$$

$$= \frac{\partial \mathbf{X}^I_{D_T}}{\partial \mathbf{X}^d_{D_T}} \frac{\partial \mathbf{X}^{(A\bar{a}\bar{b}^-)}}{\partial \mathbf{X}_{D_T}} G_{IJ}(\mathbf{E}, \mathbf{X}_{D_T}, \mathbf{X}_{L\bar{D}_T}, \bar{\tau})$$

$$= G_{d(A\bar{a}\bar{b}^-)}(\mathbf{E}, \mathbf{X}_{D_T}, \mathbf{X}_{L\bar{D}_T}, \bar{\tau}), \quad (8.17)$$

66
because (8.16) and (8.15). In the same way, the other fields are also transformed as

\[
G'_{dd}(\mathbf{E}', \mathbf{X}'_{D\tau}, \mathbf{X}'_{L\tilde{D}\tau}, \bar{\tau}) = G_{dd}((\bar{\mathbf{E}}), \mathbf{X}_{D\tau}, \mathbf{X}_{L\tilde{D}\tau}, \bar{\tau})
\]

\[
G'_{d(\mu\sigma')}((\mu\sigma')_{D\tau}, \mathbf{X}'_{D\tau}, \mathbf{X}'_{L\tilde{D}\tau}, \bar{\tau}) = G_{d(\mu\sigma')}(\mathbf{E}, \mathbf{X}_{D\tau}, \mathbf{X}_{L\tilde{D}\tau}, \bar{\tau})
\]

\[
G'_{(\mu\sigma')(\nu\rho')}(\mathbf{E}', \mathbf{X}'_{D\tau}, \mathbf{X}'_{L\tilde{D}\tau}, \bar{\tau}) = G_{(\mu\sigma')(\nu\rho')}(\mathbf{E}, \mathbf{X}_{D\tau}, \mathbf{X}_{L\tilde{D}\tau}, \bar{\tau})
\]

\[
G'_{(\mu\sigma')(\nu\rho')}(\mathbf{E}', \mathbf{X}'_{D\tau}, \mathbf{X}'_{L\tilde{D}\tau}, \bar{\tau}) = G_{(\mu\sigma')(\nu\rho')}(\mathbf{E}, \mathbf{X}_{D\tau}, \mathbf{X}_{L\tilde{D}\tau}, \bar{\tau})
\]

\[
A'_{d}(\mathbf{E}', \mathbf{X}'_{D\tau}, \mathbf{X}'_{L\tilde{D}\tau}, \bar{\tau}) = A_{d}(\mathbf{E}, \mathbf{X}_{D\tau}, \mathbf{X}_{L\tilde{D}\tau}, \bar{\tau})
\]

\[
A'_{d}(\mathbf{E}', \mathbf{X}'_{D\tau}, \mathbf{X}'_{L\tilde{D}\tau}, \bar{\tau}) = A_{d}(\mathbf{E}, \mathbf{X}_{D\tau}, \mathbf{X}_{L\tilde{D}\tau}, \bar{\tau})
\]

\[
A'_{d}(\mathbf{E}', \mathbf{X}'_{D\tau}, \mathbf{X}'_{L\tilde{D}\tau}, \bar{\tau}) = A_{d}(\mathbf{E}, \mathbf{X}_{D\tau}, \mathbf{X}_{L\tilde{D}\tau}, \bar{\tau})
\]

Thus, the Lagrangian is invariant under the \((\bar{\sigma}, \bar{\theta})\) superdiffeomorphisms, because

\[
\int d\bar{\sigma}'d\bar{\theta}'\, \bar{E}'(\bar{\sigma}', \bar{\tau}, \bar{\theta}') = \int d\bar{\sigma}d\bar{\theta}\, \bar{E}(\bar{\sigma}, \bar{\tau}, \bar{\theta})
\]

\[
\int d\bar{\sigma}'d\bar{\theta}'\, \bar{E}'(\bar{\sigma}', \bar{\tau}) = \int d\bar{\sigma}d\bar{\theta}\, \bar{E}(\bar{\sigma}, \bar{\tau})
\]

Therefore, \(G_{1\mathbf{1}}(\mathbf{E}, \mathbf{X}_{D\tau}, \mathbf{X}_{L\tilde{D}\tau}, \bar{\tau})\) and \(A_{1}(\mathbf{E}, \mathbf{X}_{D\tau}, \mathbf{X}_{L\tilde{D}\tau}, \bar{\tau})\) are transformed covariantly and the action (8.14) is invariant under the diffeomorphisms (8.11) including the \((\bar{\sigma}, \bar{\theta})\) superdiffeomorphisms, whose infinitesimal transformations are given by

\[
\bar{\sigma}^{\xi} = \bar{\sigma} - \frac{i}{2} \bar{\theta} \bar{\xi}
\]

\[
\bar{\theta}^{\xi}(\bar{\sigma}) = \bar{\theta} + \xi(\bar{\sigma}),
\]

(8.20) are dimensional reductions in \(\bar{\tau}\) direction of the two-dimensional \(\mathcal{N} = (0, 1)\) local supersymmetry infinitesimal transformations. The number of supercharges

\[
\xi Q = \xi(\frac{\partial}{\partial \theta} - \frac{i}{2} \bar{\theta} \frac{\partial}{\partial \bar{\sigma}})
\]

of the transformations is the same as of the two-dimensional ones. The supersymmetry algebra closes in a field-independent sense as in ordinary supergravities.
The background that represents a perturbative vacuum is given by
\[ ds^2 = 2\zeta\bar{\rho}(\bar{h})N^2(X_{DT}, X_{LDT})(dX_{DT}^d)^2 \]
\[ + \int d\sigma d\bar{\eta}E \int d\sigma' d\bar{\eta}' E'N^{2/3}(X_{DT}, X_{LDT})\frac{e^2(\bar{\sigma}, \bar{\tau})E(\bar{\sigma}, \bar{\tau}, \bar{\eta})}{\sqrt{h(\bar{\sigma}, \bar{\tau})}} \delta(\mu \bar{\sigma}\bar{\eta})(\mu' \bar{\sigma}'\bar{\eta}')dX_{DT}^{(\mu \bar{\sigma}\bar{\eta})} dX_{DT}^{(\mu' \bar{\sigma}'\bar{\eta}')} \]
\[ + \int d\sigma d\bar{\eta} \bar{\epsilon} \int d\sigma' d\bar{\eta}' \bar{\epsilon}' N^{2/3}(X_{DT}, X_{LDT})\frac{\bar{e}^3(\bar{\sigma}, \bar{\tau})E(\bar{\sigma}, \bar{\tau})}{\sqrt{h(\bar{\sigma}, \bar{\tau})}} \delta(\bar{A}\bar{\sigma}\bar{\eta})(\bar{A}' \bar{\sigma}'\bar{\eta}')dX_{LDT}^{(\bar{A}\bar{\sigma}\bar{\eta})} dX_{LDT}^{(\bar{A}' \bar{\sigma}'\bar{\eta}')}, \]
\[ A_d = i\sqrt{\frac{2 - 2D}{2 - D}} \sqrt{\frac{2\zeta\bar{\rho}(\bar{h})}{2}} N(X_{DT}, X_{LDT}), \quad A_{(\mu \bar{\sigma}\bar{\eta})} = 0, \quad A_{(\bar{A}\bar{\sigma}\bar{\eta})} = 0, \tag{8.22} \]

on \( \mathcal{M}_D^T \) where we fix the target metric to \( \eta_{\mu \nu} \) and the other backgrounds to zero, respectively.

\[ \bar{\rho}(\bar{h}) := \frac{1}{4\pi} \int d\sigma \sqrt{\bar{h}} \bar{R}_h, \quad \bar{R}_h \text{ is the scalar curvature of } \bar{h}_{mn}. \quad D \text{ is a volume of the index } (\mu \bar{\sigma}\bar{\eta}) \text{ and } (\bar{A}\bar{\sigma}\bar{\eta}) : \quad D := \int d\sigma d\bar{\eta} \bar{E}\delta(\mu \bar{\sigma}\bar{\eta})(\mu \bar{\sigma}\bar{\eta}) + \int d\sigma d\bar{\eta} \bar{\epsilon} \delta(\bar{A}\bar{\sigma}\bar{\eta})(\bar{A}\bar{\sigma}\bar{\eta}) = (d + 32) \int d\sigma d\bar{\eta} \bar{\delta}(\bar{\sigma} - \bar{\sigma}) \delta(\bar{\sigma} - \bar{\eta}). \]

\[ N(X_{DT}, X_{LDT}) = \frac{1}{\sqrt{\text{det}(X_{DT}, X_{LDT})}}, \quad \text{where} \]
\[ v(X_{DT}, X_{LDT}) = \frac{\alpha}{\sqrt{d-1}} \int d\sigma d\bar{\eta} \bar{E} \frac{\sqrt{\bar{E}}}{(\bar{h})^{\frac{3}{2}}} \epsilon_{\mu \nu} X^\mu_{DT} \sqrt{\bar{D}}_{\theta} X^\nu_{DT} \]
\[ + \frac{\beta}{\sqrt{31}} \int d\sigma d\bar{\eta} \bar{\epsilon} \frac{\sqrt{\bar{E}^L}}{(\bar{h})^{\frac{3}{2}}} \epsilon_{AB} X^A_{LDT} \sqrt{\bar{D}}_{\theta} X^B_{LDT}. \tag{8.23} \]

\( \bar{D}_\theta \) is a \( \bar{\tau} \) independent operator that satisfies
\[ \int d\bar{\tau} d\sigma d\bar{\eta} \bar{E} \frac{1}{2} X^\mu_{DT} \bar{D}_\theta X_{DT \mu} \]
\[ = \int d\bar{\tau} d\sigma \sqrt{\bar{h}} \frac{1}{2} \left( e^{2}(\partial_{\theta} X^{\mu})^{2} + E_{z}^{\mu} \psi_{\mu} \chi_{z} \partial_{\theta} X^{\mu} + \psi^{\mu} E_{1}^{\mu} \partial_{\theta} \psi_{\mu} \right. \]
\[ \left. - \frac{1}{4} \bar{n}^{\mu} \bar{E}_{2}^{\mu} \psi_{\mu} \chi_{z} \bar{E}_{2}^{\mu} \psi_{\mu} \chi_{z} + \bar{n}^{\mu} \partial_{\theta} X^{\mu} \bar{E}_{2}^{\mu} \psi_{\mu} \chi_{z} \right), \tag{8.24} \]

where \( \bar{E}_{2}^{\mu} \) and \( \chi_{z} \) are a vierbein and a gravitino in the two dimensions, respectively. On the other hand, the ordinary super covariant derivative \( \bar{D}_{\theta} = \partial_{\theta} + \bar{\theta} \partial_{\bar{z}} \) satisfies \[ \int d\bar{\tau} d\sigma d\bar{\eta} \bar{E} \frac{1}{2} X^\mu_{DT} \bar{D}_{\theta} X_{DT \mu} \]
\[ = \int d\bar{\tau} d\sigma \sqrt{\bar{h}} \frac{1}{2} (\bar{h}^{mn} \bar{D}_{m} X^{\mu} \bar{D}_{n} X_{\mu} + \psi^{\mu} \bar{E}_{z}^{\mu} \bar{D}_{m} \psi_{\mu} + \bar{E}_{z}^{m} \bar{D}_{m} X^{\mu} \psi_{\mu} \chi_{z}). \tag{8.25} \]

\( \bar{E}^{L} \) is the chiral conjugate of \( \bar{E} \). \( \bar{D}_{\theta} := \bar{E}_{z}^{\mu} \partial_{\theta} \) and \( \bar{D}_{\theta}^{-} := \partial_{\theta}^{-} + \bar{\theta}^{-} \bar{E}_{1}^{\mu} \partial_{\theta} \) satisfy
\[ \int d\bar{\tau} d\sigma d\bar{\eta} \bar{E}^{L} \frac{1}{2} X^{A}_{LDT} \bar{D}_{\theta} X_{LDT A} = \int d\bar{\tau} d\sigma \sqrt{\bar{h}} \frac{1}{2} \lambda^{A} \bar{E}_{1}^{\mu} \partial_{\theta} \lambda_{A}. \tag{8.26} \]
whereas the ordinary super covariant derivative \( \bar{D}_\theta^- = \partial_\theta^- + \bar{\theta}^- \partial_z \) satisfies
\[
\int d\tau d\sigma d\bar{\theta}^- e^{L} \frac{1}{2} X_{L D_T}^A \partial_\theta^- X_{L D_T A} = \int d\tau d\sigma \sqrt{\hbar} \frac{1}{2} \lambda A E_{\bar{n}}^m \delta m \lambda_A. \tag{8.27}
\]
The inverse of the metric is given by
\[
\bar{G}^{dd} = \frac{1}{2\zeta \bar{\rho} N^2}, \quad \bar{G}^{(\mu \bar{a} \bar{b}) (\mu' \alpha' \bar{\theta}')} = N \frac{\hbar}{e^2 \bar{E}} \delta_{(\mu \bar{a} \bar{b}) (\mu' \alpha' \bar{\theta}')}, \quad \bar{G}^{(A \bar{a} \bar{b}^-) (A' \alpha' \bar{\theta}^-)} = N \frac{\hbar}{e^3} \delta_{(A \bar{a} \bar{b}^-) (A' \alpha' \bar{\theta}^-)}, \tag{8.28}
\]
where the other components are zero. From the metric, we obtain
\[
\sqrt{G} = N \frac{\hbar^2}{e^2} \sqrt{2\zeta \bar{\rho}} \exp \left( \int d\bar{\sigma} d\bar{\theta}^- \bar{E} \delta_{(\mu \bar{a} \bar{b}) (\mu \bar{a} \bar{b})} \ln \frac{\bar{E} e^2}{\hbar} + \int d\bar{\sigma} d\bar{\theta}^- \frac{\hbar}{e^2} \delta_{(A \bar{a} \bar{b}^-) (A \bar{a} \bar{b}^-)} \ln \frac{\bar{E} e^3}{\hbar} \right)
\]
\[
\bar{R}_{dd} = -2\zeta \bar{\rho} N \frac{\hbar^2}{e^2} \left( \int d\bar{\sigma} d\bar{\theta}^- \frac{\hbar}{e^2} \partial_{(\mu \bar{a} \bar{b})} N \partial_{(\mu \bar{a} \bar{b})} N + \int d\bar{\sigma} d\bar{\theta}^- \frac{\hbar}{e^2} \partial_{(A \bar{a} \bar{b}^-)} N \partial_{(A \bar{a} \bar{b}^-)} N \right)
\]
\[
\bar{R}_{(\mu \bar{a} \bar{b}) (\mu' \alpha' \bar{\theta}')} = \frac{D - 1}{2 - D} N^{-2} \partial_{(\mu \bar{a} \bar{b})} N \partial_{(\mu' \alpha' \bar{\theta}')} N
\]
\[
+ \frac{1}{D - 2} N^{-2} \left( \int d\bar{\sigma} d\bar{\tau}'' \frac{\hbar}{e^2} \partial_{(\mu' \alpha' \bar{\theta}'')} N \partial_{(\mu' \alpha' \bar{\theta}'')} N \right) \frac{\bar{E} e^2}{\hbar} \delta_{(\mu \bar{a} \bar{b}) (\mu' \alpha' \bar{\theta}')} \tag{8.29}
\]
\[
\bar{R}_{(A \bar{a} \bar{b}^-) (A' \alpha' \bar{\theta}^-)} = \frac{D - 1}{2 - D} N^{-2} \partial_{(A \bar{a} \bar{b}^-)} N \partial_{(A' \alpha' \bar{\theta}^-)} N
\]
\[
+ \frac{1}{D - 2} N^{-2} \left( \int d\bar{\sigma} d\bar{\tau}'' \frac{\hbar}{e^2} \partial_{(A' \alpha' \bar{\theta}'')} N \partial_{(A' \alpha' \bar{\theta}'')} N \right) \frac{\bar{E} e^3}{\hbar} \delta_{(A \bar{a} \bar{b}^-) (A' \alpha' \bar{\theta}^-)} \tag{8.29}
\]
\[
\bar{R} = \frac{D - 3}{2 - D} N^{2D - 6} \left( \int d\bar{\sigma} d\bar{\theta}^- \frac{\hbar}{e^2} \partial_{(\mu \bar{a} \bar{b})} N \partial_{(\mu \bar{a} \bar{b})} N + \int d\bar{\sigma} d\bar{\theta}^- \frac{\hbar}{e^2} \partial_{(A \bar{a} \bar{b}^-)} N \partial_{(A \bar{a} \bar{b}^-)} N \right).
\]
By using these quantities, one can show that the background \( \text{[8.22]} \) is a classical solution to the equations of motion of \( \text{[8.14]} \). We also need to use the fact that \( v(X_{\bar{D}T}, X_{L D_T}) \) is a harmonic function with respect to \( X_{\bar{D}T}^{(\mu \bar{a} \bar{b})} \) and \( X_{L D_T}^{(A \bar{a} \bar{b}^-)} \). Actually, \( \partial_{(\mu \bar{a} \bar{b})} \partial_{(\mu \bar{a} \bar{b})} v = \partial_{(A \bar{a} \bar{b}^-)} \partial_{(A \bar{a} \bar{b}^-)} v = \).
0. In these calculations, we should note that $\bar{E}_M A$, $\tau$, $X^\mu_{D_D}$, and $X^A_{L_D}$ are all independent. Because the equations of motion are differential equations with respect to $\bar{\tau}$, $X^\mu_{D_D}$, and $X^A_{L_D}$, $\bar{E}_M A$ is a constant in the solution (8.22) to the differential equations. The dependence of $\bar{E}_M A$ on the background (8.22) is uniquely determined by the consistency of the quantum theory of the fluctuations around the background. Actually, we will find that all the perturbative superstring amplitudes are derived.

Let us consider fluctuations around the background (8.22), $G_{IJ} = \bar{G}_{IJ} + \bar{G}_{IJ}$ and $A_I = \bar{A}_I + \bar{A}_I$. Here we fix the charts, where we choose $T = SO(32)$ or $E_8 \times E_8$. The action (8.14) up to the quadratic order is given by,

$$
S = \frac{1}{G_N} \int D\bar{E}D\bar{X}_{D_D}D\bar{X}_{L_D}D\bar{\tau} \sqrt{\bar{G}} \left( -\bar{R} + \frac{1}{4} \bar{F}^I_{IJ} \bar{F}^I_{IJ} 
+ \frac{1}{4} \nabla_1 \bar{G} \nabla^1 \bar{G}^{JK} - \frac{1}{4} \nabla_1 \bar{G} \nabla^1 \bar{G}^{IJ} + \frac{1}{2} \nabla^1 \tilde{G}_{IJ} \nabla^J \bar{G} - \frac{1}{2} \nabla^1 \bar{G}_{IJ} \nabla^J \tilde{G}^{JK}
- \frac{1}{4} (-\bar{R} + \frac{1}{4} \bar{F}^I_{KL} \bar{F}^I_{KL}) (\bar{G}_{IJ} \tilde{G}^{IJ} - \frac{1}{2} \bar{G}^2) + (-\frac{1}{2} \bar{R}^I_{IJ} + \frac{1}{2} \bar{F}^I_{KJL} \bar{F}^I_{KJL}) \tilde{G}_{IJ} \tilde{G}^{JL}
+ \frac{1}{2} \bar{R}^I_{IJ} - \frac{1}{4} \bar{F}^I_{KJL} \bar{F}^I_{KJL}) \tilde{G}_{IJ} \tilde{G}^{JL}
+ \frac{1}{4} G_N \bar{F}^I_{IJ} \bar{F}^I_{IJ} + \sqrt{G_N} \left( \frac{1}{4} \bar{F}^I_{IJ} \bar{F}^I_{IJ} \bar{G} - \bar{F}^I_{IJ} \bar{F}^I_{JL} \tilde{G}^{JL} \right) \right), \quad (8.30)
$$

where $\bar{F}^I_{IJ} := \sqrt{G_N} \bar{F}^I_{IJ}$ is independent of $G_N$. $\tilde{G} := \tilde{G}^{IJ} \bar{G}_{IJ}$. There is no first order term because the background satisfies the equations of motion. If we take $G_N \to 0$, we obtain

$$
S' = \frac{1}{G_N} \int D\bar{E}D\bar{X}_{D_D}D\bar{X}_{L_D}D\bar{\tau} \sqrt{\bar{G}} \left( -\bar{R} + \frac{1}{4} \bar{F}^I_{IJ} \bar{F}^I_{IJ} 
+ \frac{1}{4} \nabla_1 \bar{G} \nabla^1 \bar{G}^{JK} - \frac{1}{4} \nabla_1 \bar{G} \nabla^1 \bar{G}^{IJ} + \frac{1}{2} \nabla^1 \tilde{G}_{IJ} \nabla^J \bar{G} - \frac{1}{2} \nabla^1 \bar{G}_{IJ} \nabla^J \tilde{G}^{JK}
- \frac{1}{4} (-\bar{R} + \frac{1}{4} \bar{F}^I_{KL} \bar{F}^I_{KL}) (\bar{G}_{IJ} \tilde{G}^{IJ} - \frac{1}{2} \bar{G}^2) + (-\frac{1}{2} \bar{R}^I_{IJ} + \frac{1}{2} \bar{F}^I_{KJL} \bar{F}^I_{KJL}) \tilde{G}_{IJ} \tilde{G}^{JL}
+ (-\frac{1}{2} \bar{R}^I_{IJ} - \frac{1}{4} \bar{F}^I_{KJL} \bar{F}^I_{KJL}) \tilde{G}_{IJ} \tilde{G}^{JL} \right), \quad (8.31)
$$

where the fluctuation of the gauge field is suppressed. In order to fix the gauge symmetry (8.15), we take the harmonic gauge. If we add the gauge fixing term

$$
S_{fix} = \frac{1}{G_N} \int D\bar{E}D\bar{X}_{D_D}D\bar{X}_{L_D}D\bar{\tau} \sqrt{\bar{G}} \frac{1}{2} \left( \nabla^J (\tilde{G}_{IJ} - \frac{1}{2} \bar{G}_{IJ} \tilde{G}) \right)^2, \quad (8.32)
$$
we obtain

\[ S' + S_{\text{fix}} = \frac{1}{G_N} \int \mathcal{D}E \mathcal{D}X_{\hat{D}r} \mathcal{D}X_{L\hat{D}r} \mathcal{D}\tau \sqrt{G} \left( -\dddot{R} + \frac{1}{4} \dddot{F}_{IJ} \dddot{F}^{IJ} \right) \]

\[ + \frac{1}{4} \nabla_I \tilde{G} \nabla^I \tilde{G}^{JK} - \frac{1}{8} \nabla_I \tilde{G} \nabla^I \tilde{G} \]

\[ - \frac{1}{4} (-\dddot{R} + \frac{1}{4} \dddot{F}_{KL} \dddot{F}^{KL}) (\tilde{G}_{IJ} \tilde{G}^{IJ} - \frac{1}{2} \tilde{G}^2) + \left( -\frac{1}{2} \tilde{R}^I_J + \frac{1}{2} \tilde{F}'_{IK} \tilde{F}^{IK} \right) \tilde{G}_{IL} \tilde{G}^{JL} \]

\[ + \left( \frac{1}{2} \dddot{F}_{IJ} - \frac{1}{4} \tilde{F}'_{IK} \tilde{F}^{IK} \right) \tilde{G}_{IJ} \tilde{G} + \left( -\frac{1}{2} \tilde{R}_{JKL} + \frac{1}{4} \tilde{F}'_{IJ} \tilde{F}^{KL} \right) \tilde{G}_{IK} \tilde{G}_{JL} \].

In order to obtain perturbative string amplitudes, we perform a derivative expansion of

\[ \tilde{G}_{IJ} \rightarrow \frac{1}{\alpha} \tilde{G}_{IJ} \]

\[ \partial_K \tilde{G}_{IJ} \rightarrow \partial_K \tilde{G}_{IJ} \]

\[ \partial_K \partial_L \tilde{G}_{IJ} \rightarrow \alpha \partial_K \partial_L \tilde{G}_{IJ} \]

and take

\[ \alpha = \beta \rightarrow 0 \],

where \(\alpha\) and \(\beta\) are arbitrary constants in the solution (8.22). We normalize the fields as

\[ \tilde{H}_{IJ} := Z_{IJ} \tilde{G}_{IJ} \],

where \(Z_{IJ} := \frac{1}{\sqrt{G_N}} \tilde{G}^2 (a_I a_J)^{-\frac{1}{2}}\). \(a_I\) represent the background metric as

\[ \tilde{G}_{IJ} = a_I \delta_{IJ} \],

where \(a_d = 2\zeta \rho, a_{(\mu \theta)} = \frac{\sqrt{h}}{\sqrt{h}}\) and \(a_{(A \theta)} = \frac{\sqrt{h}}{\sqrt{h}}\). Then, (8.33) with appropriate boundary conditions reduces to

\[ S' + S_{\text{fix}} \rightarrow S_0 + S_2 \],

where

\[ S_0 = \frac{1}{G_N} \int \mathcal{D}E \mathcal{D}X_{\hat{D}r} \mathcal{D}X_{L\hat{D}r} \mathcal{D}\tau \sqrt{G} \left( -\dddot{R} + \frac{1}{4} \dddot{F}_{IJ} \dddot{F}^{IJ} \right) \],

and

\[ S_2 = \int \mathcal{D}E \mathcal{D}X_{\hat{D}r} \mathcal{D}X_{L\hat{D}r} \mathcal{D}\tau \frac{1}{8} \tilde{H}_{IJ} \tilde{H}_{IJ}^{KL} \tilde{H}_{KL} \].

In the same way as in section 3, a part of the action

\[ \int \mathcal{D}E \mathcal{D}X_{\hat{D}r} \mathcal{D}X_{L\hat{D}r} \mathcal{D}\tau \frac{1}{4} \int_0^{2\pi} d\bar{\sigma} \bar{d}\bar{\theta} H^{\perp}_{d(\mu \theta)} H \tilde{H}^{\perp}_{d(\mu \theta)} \]

with
We abbreviate \( \hat{H} \) we obtain (8.39) with decouples from the other modes. In (8.39), the term including \((\frac{\partial}{\partial X^\mu_{LT}})\) needs to be proportional to \((X^A_{LT})^2 = (\theta^\lambda - \lambda^\theta) = 0\) so as not to vanish.

Because \( X^\mu_{LT} \) can be expanded as \( X^\mu_{LT} = X^\mu + \hat{\theta} \psi^\mu \), we have

\[
\int d\bar{\theta} \left( \frac{\partial}{\partial X^\mu_{LT}(\bar{\sigma}', \theta')} \right)^2 \hat{H}_{d(\mu\theta)} = \left( \frac{\partial}{\partial X^\mu(\bar{\sigma}') \right)^2 \hat{H}_{d(\mu\theta)}.
\]

(8.41)

We abbreviate \( \hat{D}_T \) of \( X^\mu \) and \( \psi^\mu \). Then, (8.39) can be simplified with

\[
H(-i\frac{\partial}{\partial \tau}, -\frac{i}{\epsilon} \frac{\partial}{\partial X}, X_{LT}, X_{LDT}, E)
\]

\[
= \frac{1}{2} \frac{1}{2\zeta \rho} (-i\frac{\partial}{\partial \tau})^2 + \int d\bar{\sigma} \sqrt{\frac{\hbar}{2}} (-i\frac{\partial}{\partial X})^2 - \int d\bar{\sigma} d\bar{\theta} \frac{1}{2} X^\mu_{LT} \hat{D}_\theta X_{LT\mu}
\]

\[
- \int d\bar{\sigma} d\bar{\theta} - E^L \frac{1}{2} X^A_{LT} \hat{D}_\theta X_{LT\mu} - X_{LDT} A,
\]

(8.42)

where we have taken \( D \to \infty \). By adding to (8.39), a formula similar to the bosonic case

\[
0 = \int \mathcal{D}E \mathcal{D}X_{LT} \mathcal{D}X_{LDT} \mathcal{D}\tau \frac{1}{4} \int_0^{2\pi} d\sigma' d\bar{\theta} \hat{H}_{d(\mu\theta)}(\int_0^{2\pi} d\bar{\sigma} \bar{n}^a \bar{\partial}_\theta X^\mu \frac{\partial}{\partial X^\mu}) \hat{H}_{d(\mu\theta)}
\]

(8.43)

and

\[
0 = \int \mathcal{D}E \mathcal{D}X_{LT} \mathcal{D}X_{LDT} \mathcal{D}\tau \frac{1}{4} \int_0^{2\pi} d\sigma' d\bar{\theta} \hat{H}_{d(\mu\theta)}(\int_0^{2\pi} d\bar{\sigma} E \frac{i}{2} \bar{n} \bar{X} \bar{E}_z \psi^\mu(-i\frac{1}{\epsilon} \frac{\partial}{\partial X})) \hat{H}_{d(\mu\theta)}
\]

(8.44)

we obtain (8.39) with

\[
H(-i\frac{\partial}{\partial \tau}, -\frac{i}{\epsilon} \frac{\partial}{\partial X}, X_{LT}, \lambda_{LT}, E)
\]

\[
= \frac{1}{2} \frac{1}{2\zeta \rho} (-i\frac{\partial}{\partial \tau})^2 + \int d\bar{\sigma} \left( \sqrt{\hbar} \left( \frac{1}{2} (-i\frac{\partial}{\partial X})^2 - \frac{i}{2} \bar{n} \bar{X} \bar{E}_z \psi^\mu(-i\frac{1}{\epsilon} \frac{\partial}{\partial X}) \right) \right)
\]

\[
- \int d\bar{\sigma} d\bar{\theta} \frac{1}{2} X^\mu_{LT} \hat{D}_\theta X_{LT\mu} + \int d\bar{\sigma} \sqrt{\frac{\hbar}{2}} \lambda^A_{LT} \hat{E}_z \partial_\theta \lambda_{LT} A,
\]

(8.45)

72
where we have used (8.26). (8.44) is true because the integrand of the right hand side is a total derivative with respect to $X^\mu$.

The propagator $\Delta_F(\bar{E}, \bar{\tau}, \bar{X}_{D_T}, \lambda_{D_T}; \bar{E}', \bar{\tau}', \bar{X}'_{D_T}, \lambda'_{D_T})$ is defined by

$$
H(-i \frac{\partial}{\partial \bar{\tau}}, -i \frac{\partial}{\partial \bar{X}}, X_{D_T}, \lambda_{D_T}; \bar{E}) \Delta_F(\bar{E}, \bar{\tau}, X_{D_T}, \lambda_{D_T}; \bar{E}', \bar{\tau}', X'_{D_T}, \lambda'_{D_T}) = \delta(\bar{E} - \bar{E}') \delta(\bar{\tau} - \bar{\tau}') \delta(X_{D_T} - X'_{D_T}) \delta(\lambda_{D_T} - \lambda'_{D_T}).
$$

(8.46)

In order to obtain a Schwinger representation of the propagator, we use the operator formalism $(\bar{E}, \hat{\tau}, \hat{X}_{D_T}, \hat{\lambda}_{D_T})$ of the first quantization. The eigen state for $(\bar{E}, \hat{\tau}, \hat{X})$ is given by $|E, \tau, X\rangle$. The conjugate momentum is written as $(\hat{p}_E, \hat{p}_\tau, \hat{p}_X)$. The Majorana fermions $\psi^\mu$ and $\lambda^A_{D_T}$ are self-conjugate. Renormalized operators $\hat{\psi}^\mu : = \sqrt{E_2} \psi^\mu$ and $\hat{\lambda}^A_{D_T} : = \sqrt{E_2} \lambda^A_{D_T}$ satisfy $\{\hat{\psi}(\bar{\sigma}), \hat{\psi}^\nu(\bar{\sigma}')\} = \frac{1}{\bar{E}} \eta^{\mu \nu} \delta(\bar{\sigma} - \bar{\sigma}')$ and $\{\hat{\lambda}^A_{D_T}(\bar{\sigma}), \hat{\lambda}^B_{D_T}(\bar{\sigma}')\} = \frac{1}{\bar{E}} \delta^{AB} \delta(\bar{\sigma} - \bar{\sigma}')$, respectively. By defining creation and annihilation operators for $\hat{\psi}^\mu$ as $\hat{\psi}^{\mu\dagger} : = \frac{1}{\sqrt{2}}(\hat{\psi}^\mu - i \hat{\psi}^\dagger \hat{\psi})$ and $\hat{\psi}^\mu : = \frac{1}{\sqrt{2}}(\hat{\psi}^\mu + i \hat{\psi}^\dagger \hat{\psi})$ where $\mu = 0, \cdots, \frac{d}{2} - 1$, one obtains an algebra $\{\hat{\psi}^\mu(\bar{\sigma}), \hat{\psi}^{\mu\dagger}(\bar{\sigma}')\} = \frac{1}{\bar{E}} \eta^{\mu \nu} \delta(\bar{\sigma} - \bar{\sigma}')$, $\{\hat{\psi}^{\mu\dagger}(\bar{\sigma}), \hat{\psi}^\nu(\bar{\sigma}')\} = 0$, and $\{\hat{\psi}^{\mu\dagger}(\bar{\sigma}), \hat{\psi}^{\nu\dagger}(\bar{\sigma}')\} = 0$. The vacuum $|0\rangle$ for this algebra is defined by $\hat{\psi}^{\mu\dagger}(\bar{\sigma})|0\rangle = 0$. The eigen state $|\psi\rangle$, which satisfies $\hat{\psi}^\mu(\bar{\sigma})|\psi\rangle = \hat{\psi}^{\mu\dagger}(\bar{\sigma})|\psi\rangle$, is given by $e^{-\hat{\psi}^\mu \hat{\psi}^{\mu\dagger}}|0\rangle = e^{-\int d\bar{\sigma} \hat{\psi}^\mu(\bar{\sigma}) \hat{\psi}^{\mu\dagger}(\bar{\sigma})}|0\rangle$. Then, the inner product is given by $<\psi|\psi' = e^{\hat{\psi}^\mu \hat{\psi}^{\mu\dagger}}$, whereas the completeness relation is $\int \mathcal{D}\hat{\psi} \hat{\psi} \langle \psi|\psi > e^{-\hat{\psi}^\mu \hat{\psi}^{\mu\dagger}} < \psi| = 1$. The same is applied to $\hat{\lambda}^A_{D_T}$.

Because (8.46) means that $\Delta_F$ is an inverse of $H$, $\Delta_F$ can be expressed by a matrix element of the operator $H^{-1}$ as

$$
\Delta_F(\bar{E}, \bar{\tau}, \bar{X}_{D_T}, \lambda_{D_T}; \bar{E}', \bar{\tau}', \bar{X}'_{D_T}, \lambda'_{D_T}) = < \bar{E}, \bar{\tau}, X_{D_T}, \lambda_{D_T} | H^{-1}(\hat{p}_E, \hat{p}_\tau, \hat{p}_X, \hat{\lambda}_{D_T}, \hat{E}) | \bar{E}', \bar{\tau}', \bar{X}'_{D_T}, \lambda'_{D_T} >.
$$

(8.47)

(8.34) implies that

$$
\Delta_F(\bar{E}, \bar{\tau}, X_{D_T}, \lambda_{D_T}; \bar{E}', \bar{\tau}', \bar{X}'_{D_T}, \lambda'_{D_T}) = \int_0^\infty dT < \bar{E}, \bar{\tau}, X_{D_T}, \lambda_{D_T} | e^{-TH} | \bar{E}', \bar{\tau}', \bar{X}'_{D_T}, \lambda'_{D_T} >.
$$

(8.48)

We define in and out states as

$$
||X_{\hat{D}_T f}; \lambda_{\hat{D}_T f} | E_f; ; E_i >_{in} : = \int_{E_i}^{E_f} D\mathcal{E} E' \langle \bar{E}', \bar{\tau}, X_{\hat{D}_T f}, \lambda_{\hat{D}_T f} >
$$

$$
< X_{\hat{D}_T f}; \lambda_{\hat{D}_T f} | E_f; ; E_i ||_{out} : = \int_{E_i}^{E_f} D\mathcal{E} \langle \bar{E}, \bar{\tau}, X_{\hat{D}_T f}, \lambda_{\hat{D}_T f} >.
$$

(8.49)
where $\mathbf{E}_i$ and $\mathbf{E}_f$ represent the super vierbeins of the supercylinders at $\bar{\tau} = \mp \infty$, respectively. By inserting

\[
1 = \int d\mathbf{E}_m d\bar{\tau}_m d\mathbf{X}_{\hat{D}_T m} d\lambda_{\hat{D}_T m} \\
|\mathbf{E}_m, \bar{\tau}_m, \mathbf{X}_{\hat{D}_T m}, \lambda_{\hat{D}_T m} > e^{-\bar{\psi}_m \bar{\psi}_m - \bar{\lambda}_m \hat{\lambda}} < \mathbf{E}_m, \bar{\tau}_m, \mathbf{X}_{\hat{D}_T m}, \lambda_{\hat{D}_T m} | \\
1 = \int dp_i^i dp_X^i |p_i^i, p_X^i > < p_i^i, p_X^i |, \\
\]  

(8.50)
the two-point correlation function for the in and out states is given by\textsuperscript{18}

\[
\Delta_F(X_{D_T f}, \lambda_{D_T f}; X_{D_T i}, \lambda_{D_T i}; E_f; E_i) = \int_0^\infty dT \left< X_{D_T f}, \lambda_{D_T f} \right| X_{D_T i}, \lambda_{D_T i} \left| E_f; E_i \right|_\text{out} e^{-TH} \right| X_{D_T i}, \lambda_{D_T i} \left| E_f; E_i \right>_{\text{in}} > m
\]

\[
= \int_0^\infty dT \lim_{N \to \infty} \int \mathcal{D}E \int \mathcal{D}E' \prod_{m=1}^N \prod_{i=0}^N \int \mathcal{D}\bar{E}_m d\bar{\tau}_m dX_{D_T m} d\lambda_{D_T m} e^{-\tilde{\psi}_m \tilde{\psi}_m - \tilde{\lambda}_{D_T m} \tilde{\lambda}_{D_T m}} < \bar{E}_{i+1}, \bar{\tau}_{i+1}, X_{D_T i+1}, \lambda_{D_T i+1} \left| e^{-\frac{1}{N} T H} \right| \bar{E}_i, \bar{\tau}_i, X_{D_T i}, \lambda_{D_T i} > \delta(T_i - T_{i+1})
\]

\[
= \int_0^\infty dT_0 \lim_{N \to \infty} \int \mathcal{D}E \int \mathcal{D}E' \prod_{m=1}^N \prod_{i=0}^N \int \mathcal{D}\bar{E}_m d\bar{\tau}_m dX_{D_T m} d\lambda_{D_T m} e^{-\tilde{\psi}_m \tilde{\psi}_m - \tilde{\lambda}_{D_T m} \tilde{\lambda}_{D_T m}} < \bar{\tau}_{i+1}, X_{D_T i+1}, \lambda_{D_T i+1} \left| e^{-\frac{1}{N} T_0 H} \right| \bar{\tau}_i, X_{D_T i}, \lambda_{D_T i} > \delta(T_i - T_{i+1})
\]

\[
= \int_0^\infty dT_0 \lim_{N \to \infty} \int \mathcal{D}E \int \mathcal{D}E' \prod_{m=1}^N \prod_{i=0}^N \int \mathcal{D}\bar{E}_m d\bar{\tau}_m dX_{D_T m} d\lambda_{D_T m} e^{-\tilde{\psi}_m \tilde{\psi}_m - \tilde{\lambda}_{D_T m} \tilde{\lambda}_{D_T m}}
\]

\[
\int dp^i_p dp^i_X < \bar{\tau}_{i+1}, X_{i+1} \left| p^i_p, p^i_X \right>
\]

\[
\int dp^i_p dp^i_X e^{-\frac{1}{N} T_0 H(p^i_p, p^i_X, X_{D_T i+1}; \lambda_{D_T i+1}; E)} e^{\tilde{\psi}_i \tilde{\psi}_i + \tilde{\lambda}_{D_T i+1} \tilde{\lambda}_{D_T i+1}} \delta(T_i - T_{i+1})
\]

\[
\exp \left( -\sum_{i=0}^\infty \Delta t \left( -ip^j_p \frac{T_{i+1} - T_i}{\Delta t} + \tilde{\psi}^j_{i+1} \frac{\tilde{\psi}_{i+1} - \tilde{\psi}_i}{\Delta t} + \tilde{\lambda}^j_{D_T i+1} \frac{\tilde{\lambda}_{D_T i+1} - \tilde{\lambda}_{D_T i}}{\Delta t} \right) \ight)
\]

\[
e^\frac{T_{i+1} - T_i}{\Delta t} \psi_{N+1} + \tilde{\lambda}_{D_T N+1} \tilde{\lambda}_{D_T N+1}
\]

\[
= \int_{E_{i-\infty}, X_{D_T f}, \lambda_{D_T f}} \mathcal{D}T \mathcal{D}E \mathcal{D}\tau \mathcal{D}X \mathcal{D}\lambda \mathcal{D}p \int \mathcal{D}p_T \mathcal{D}p_f \mathcal{D}p_X e^{-\int_0^1 dt \left( -ip_T \frac{\partial}{\partial t} + \tilde{\psi}_T \frac{\partial}{\partial \tilde{\psi}_T} + \tilde{\lambda}_T \frac{\partial}{\partial \tilde{\lambda}_T} - ip_f \frac{\partial}{\partial p_f} - ip_X \frac{\partial}{\partial p_X} + T_H(p_T, p_f, X, X_{D_T f}, \lambda_{D_T f}, \tilde{\psi}_T, \tilde{\lambda}_T) \right)}
\]

\textsuperscript{18}The correlation function is zero if \( E_i \) and \( E_f \) of the in state do not coincide with those of the out states, because of the delta functions in the sixth line.
where $\bar{E}_0 = \bar{E}'$, $\bar{\tau}_0 = -\infty$, $X_{D_T} 0 = X_{D_T i}$, $\lambda_{D_T} 0 = \lambda_{D_T i}$, $\bar{E}_{N+1} = \bar{E}$, $\bar{\tau}_{N+1} = \infty$, $X_{D_T N+1} = X_{D_T f}$, and $\lambda_{D_T} N+1 = \lambda_{D_T f}$. $p_X \cdot \frac{d}{dt} \bar{X} = \int d\bar{\sigma} \bar{e}_m^\mu \frac{d}{dt} \bar{X}_\mu$ and $\Delta t = \frac{1}{\bar{N}}$ as in the bosonic case. A trajectory of points $[\bar{\sigma}, X_{D_T}, \lambda_{D_T}, \bar{\tau}]$ is necessarily continuous in $\mathcal{M}_{D_T}$ so that the kernel $< \bar{E}_{i+1}, \bar{\tau}_{i+1}, X_{D_T i+1}, \lambda_{D_T i+1} e^{-\frac{N}{\bar{N}} T_{D_T}} | \bar{E}_{i}, \bar{\tau}_{i}, X_{D_T i}, \lambda_{D_T i} >$ in the fourth line is non-zero when $N \to \infty$. If we integrate out $p_{\tau}(t)$ and $p_X(t)$ by using the relation of the ADM formalism, the relation between $\bar{\psi}^\mu$ and $\psi^\mu$ and the relation between $\bar{\lambda}_{D_T}^A$ and $\lambda_{D_T}^A$, we obtain

$$
\Delta_F(X_{D_T i+1}, \lambda_{D_T i+1} | E_f, E_i) = \int_{E_i, -\infty}^E_{E_f, \infty} X_{D_T i+1} | \mathcal{M}_{D_T} \int \mathcal{D}p_T \exp \left( -\int_0^1 dt \left( -ip_{\tau}(t) \frac{d}{dt} T(t) + \frac{1}{T(t)} \frac{d\bar{\tau}(t)}{dt} \right)^2 + \int d\bar{\sigma} \frac{\sqrt{h}}{T(t)} \left( \frac{1}{2\bar{n}^2} \left( \frac{1}{T(t)} \frac{\partial}{\partial \bar{\tau}} X^\mu - \bar{n}^2 \partial_\sigma X^\mu + \frac{1}{2} \bar{n}^2 \bar{E}_0^0 \bar{\psi}_\mu \bar{\chi}_z \right)^2 + \frac{1}{2} \frac{1}{T(t)} \psi^\mu \bar{E}_0^0 \frac{\partial}{\partial \psi^\mu} + \frac{1}{2} \frac{\lambda_{D_T}^A \bar{E}_0^1 \partial_x \lambda_{D_T A}}{T(t)} \right) \right)
$$

When the last equality is obtained, we use (8.25) and (8.24). In the last line, $\hat{D}_\theta$, $\hat{\partial}_z$ and $\hat{\partial}'_z$ are given by replacing $\frac{\partial}{\partial \tau}$ with $\frac{1}{T(t)} \frac{\partial}{\partial \tau}$ in $D_\theta$, $\partial_z$ and $\partial'_z$, respectively. The path integral is defined over all possible trajectories with fixed boundary values, on the heterotic superstring manifold $\mathcal{M}_{D_T}$.

By inserting $\int \mathcal{D}c \mathcal{D}b e^{\int_0^1 dt (\frac{db(t)}{dt} \frac{dc(t)}{dt})}$, where $b(t)$ and $c(t)$ are bc ghosts, we obtain

$$
\Delta_F(X_{D_T i+1}, \lambda_{D_T i+1} | E_f, E_i) = \mathcal{Z}_0 \int_{E_i, -\infty}^E_{E_f, \infty} X_{D_T i+1} | \mathcal{M}_{D_T} \int \mathcal{D}p_T \exp \left( -\int_0^1 dt \left( -ip_{\tau}(t) \frac{d}{dt} T(t) \right) + \frac{db(t)}{dt} \frac{d(T(t) c(t))}{dt} + \frac{d\bar{\tau}(t)}{dt} \right)^2 + \int d\bar{\sigma} \frac{\sqrt{h}}{T(t)} \left( \frac{1}{2} \frac{\lambda_{D_T}^A \partial_z \lambda_{D_T A}}{T(t)} \right) \right) \quad \text{(8.53)}
$$
where we have redefined as $c(t) \to T(t)c(t)$. $Z_0$ represents an overall constant factor, and we will rename it $Z_1, Z_2, \cdots$ when the factor changes in the following. This path integral is obtained if

$$F_1(t) := \frac{d}{dt}T(t) = 0$$

(8.54)

gauge is chosen in

$$\Delta_F(X_{\bar{D}_T f}, \lambda_{\bar{D}_T f}; X_{\bar{D}_T i}, \lambda_{\bar{D}_T i} | E_f ; E_i)
= Z_1 \int_{E_i, -\infty}^{E_f, \infty} \mathcal{D}T \mathcal{D} \bar{E} \mathcal{D} \bar{X} \mathcal{D} \lambda \int \exp \left( - \int_0^1 dt \left( \right) \right)
+ \int d\bar{\sigma} d\bar{E} T(t) \frac{1}{2} \partial \bar{\tau} X^\mu \bar{D} \bar{g} X_{\bar{D}_T \mu} + \int d\bar{\sigma} \sqrt{h} T(t) \frac{1}{2} \lambda^A \bar{\partial} \bar{\tau} \lambda_{\bar{D}_T A} \right),
(8.55)$$

which has a manifest one-dimensional diffeomorphism symmetry with respect to $t$, where $T(t)$ is transformed as an einbein [11].

Under a rescale $\bar{\tau} = \bar{\tau}'T(t)$, $T(t)$ disappears in (8.55) as in the bosonic case, and we obtain

$$\Delta_F(X_{\bar{D}_T f}, \lambda_{\bar{D}_T f}; X_{\bar{D}_T i}, \lambda_{\bar{D}_T i} | E_f ; E_i)
= Z_2 \int_{E_i, -\infty}^{E_f, \infty} \mathcal{D}T \mathcal{D} \bar{E} \mathcal{D} \bar{X} \mathcal{D} \lambda \int \exp \left( - \int_0^1 dt \left( \right) \right)
+ \int d\bar{\sigma} d\bar{E} T(t) \frac{1}{2} \partial \bar{\tau} X^\mu \bar{D} \bar{g} X_{\bar{D}_T \mu} + \int d\bar{\sigma} \sqrt{h} T(t) \frac{1}{2} \lambda^A \bar{\partial} \bar{\tau} \lambda_{\bar{D}_T A} \right),
(8.56)$$

where $\bar{D}_g, \bar{\partial} g$ and $\bar{\partial} g$ are given by replacing $\frac{\partial}{\partial \bar{\tau}}$ with $\bar{D}_g, \bar{\partial} g$ and $\bar{\partial} g$, respectively. This action is still invariant under the diffeomorphism with respect to $t$ if $\bar{\tau}$ transforms in the same way as $\frac{1}{T(t)}$.

If we choose a different gauge

$$F_2(t) := \bar{\tau} - t = 0,$$

(8.57)
in (8.56), we obtain

\[
\Delta_F(X_{\bar{\Delta}_T f}, \lambda_{\bar{\Delta}_T f}; X_{\bar{\Delta}_T i}, \lambda_{\bar{\Delta}_T i})|E_f; ;E_i) \\
= Z \int_{E_f, E_{\bar{\Delta}_T i}} \mathcal{D}E \mathcal{D}X \mathcal{D}\lambda \mathcal{D}b \exp \left( -\int_0^1 dt \left( \alpha(t)(\bar{\tau} - t) + b(t)c(t)(1 - \frac{d\bar{\tau}(t)}{dt}) + \zeta \rho \left( \frac{d\bar{\tau}(t)}{dt} \right)^2 \right) \\
+ \int d\bar{\sigma} d\bar{\theta} \frac{1}{2} \partial^\mu \partial^\nu X_{\bar{\Delta}_T} \mathcal{D}_{\bar{\Delta}_T} X_{\bar{\Delta}_T} + \int d\bar{\sigma} \sqrt{g} \frac{1}{2} \lambda^A_{\bar{\Delta}_T} \partial_{\bar{\Delta}_T A} \right) \\
= Z \int_{E_f, X_{\bar{\Delta}_T i}} \mathcal{D}E \mathcal{D}X \mathcal{D}\lambda \mathcal{D}b \exp \left( -\int_0^1 dt \left( \right) \\
+ \frac{1}{4\pi} \int d\sigma \sqrt{g} R(\sigma, \bar{\tau}) + \int d\sigma d\bar{\theta} \frac{1}{2} \partial^\mu \partial^\nu X_{\bar{\Delta}_T} \mathcal{D}_{\bar{\Delta}_T} X_{\bar{\Delta}_T} + \int d\sigma \sqrt{g} \frac{1}{2} \lambda^A_{\bar{\Delta}_T} \partial_{\bar{\Delta}_T A} \right). \\
(8.58)
\]

In the second equality, we have redefined as \(c(t)(1 - \frac{d\bar{\tau}(t)}{dt}) \to c(t)\) and integrated out the ghosts. The path integral is defined over all possible heterotic super Riemannian manifolds with fixed punctures in \(\mathbb{R}^{d-1,1}\). By using the two-dimensional superdiffeomorphism and super Weyl invariance of the action, we obtain

\[
\Delta_F(X_{\bar{\Delta}_T f}, \lambda_{\bar{\Delta}_T f}; X_{\bar{\Delta}_T i}, \lambda_{\bar{\Delta}_T i})|E_f; ;E_i) \\
= Z \int_{E_f, X_{\bar{\Delta}_T i}} \mathcal{D}E \mathcal{D}X \mathcal{D}\lambda \mathcal{D}e^{-\zeta E} e^{-\int d^2\sigma d\bar{\theta} \frac{1}{2} \partial^\mu \partial^\nu X_{\bar{\Delta}_T} \mathcal{D}_{\bar{\Delta}_T} X_{\bar{\Delta}_T} - \int d^2\sigma \sqrt{g} \frac{1}{2} \lambda^A_{\bar{\Delta}_T} \partial_{\bar{\Delta}_T A}, \\
(8.59)
\]

where \(\chi\) is the Euler number of the reduced space. By inserting asymptotic states to (8.59), we obtain the perturbative all-order scattering amplitudes that possess the supermoduli in the \(SO(32)\) and \(E_8 \times E_8\) heterotic superstring theory for \(T = SO(32)\) and \(E_8 \times E_8\), respectively [12]. Especially, in superstring geometry, the consistency of the perturbation theory around the background (8.22) determines \(d = 10\) (the critical dimension).

9 Conclusion

In this paper, we defined superstring geometry: spaces of superstrings including the interactions, their topologies, charts, and metrics. Especially, we can define topological spaces where
the trajectories in asymptotic processes reproduce the moduli spaces of the super Riemann surfaces in target spaces. Based on the superstring geometry, we defined Einstein-Hilbert action coupled with gauge fields, and formulated superstring theory non-perturbatively by summing over metrics, and the gauge fields on superstring manifolds. This theory does not depend on backgrounds. The theory has a supersymmetry, as a part of the diffeomorphisms symmetry.

We have derived the all-order perturbative scattering amplitudes that possess the super moduli in type IIA, type IIB and SO(32) type I superstring theory from the single theory, by expanding the action to the second order of the metric around fixed backgrounds representing type IIA, type IIB and SO(32) type I perturbative vacua, respectively. Here, we explain some reasons for this in the point of view of symmetry. Because this expansion corresponds to see only one string state, we can move to a formalism of the first quantization, where the state is described by a trajectory in the superstring manifold $\mathcal{M}_{D}^{T}$. By definition of the neighbourhood, the effective action becomes local on a worldsheet. The $(\bar{\sigma}, \bar{\theta})$ supersymmetry of the action are dimensional reductions in $\bar{\tau}$ direction of the two-dimensional $\mathcal{N} = (1, 1)$ local supersymmetry, where the number of supercharges of the transformations is the same as of the two-dimensional ones as in (6.15). Because we can choose a gauge where a trajectory $t$ coincides $\bar{\tau}$ by using an one-dimensional diffeomorphism transformation on the trajectory, the supersymmetry becomes the two-dimensional local $\mathcal{N} = (1, 1)$ supersymmetry of the perturbative superstring theory.

We have shown that a trajectory in an asymptotic process on $\mathcal{M}_{D}^{T}$ is a worldsheet of a superstring with punctures in $M$. Macroscopically, such a worldsheet becomes a worldline of a superparticle in $M$, namely a trajectory in an asymptotic process on $M$. By the way, one way to identify a background as $M$ is to observe all the trajectories in asymptotic processes on the background. Because all the trajectories in asymptotic processes on $\mathcal{M}_{D}^{T}$ become macroscopically those on $M$, we see that macroscopically, a superstring manifold $\mathcal{M}_{D}^{T}$ becomes the space-time manifold $M$. Conversely, this means that if we look at space-time $M$ in a microscopic way, we see a superstring manifold $\mathcal{M}_{D}^{T}$. On the other hand, we have shown that the effective theory of a part of fluctuations of the action on $\mathcal{M}_{D}^{T}$ reduces to the perturbative superstring theory. Macroscopically, the perturbative superstring theory describes all the matter and gauge particles including graviton. That is, macroscopically, the fluctuations
of $\mathcal{M}_D^T$ become these particles. Conversely, if we observe particles in a microscopic way, we see superstrings, which are the fluctuations of $\mathcal{M}_D^T$. Therefore, superstring manifolds unify matter and the space-time: macroscopically, the fluctuations of $\mathcal{M}_D^T$ are particles and $\mathcal{M}_D^T$ itself is the space-time.

10 Discussion

The superstring geometry solution to the equations of motion of the theory in this paper, has the most simple superstring background, that is, the flat metric and the other zero backgrounds. We need to find superstring geometry solutions that have more general superstring backgrounds, namely, a metric, a NS-NS B-field, a dilaton, R-R fields, and gauge fields on D-branes. We can identify superstring backgrounds of superstring geometry solutions by deriving superstring actions in the backgrounds from the fluctuations around the solutions, in the same way as in this paper.

We formulated the single theory that manifestly includes type IIA, type IIB, and $SO(32)$ type I superstrings. We also formulated another theory that manifestly includes the $SO(32)$ and $E_8 \times E_8$ heterotic superstrings based on superstring geometry. We expect that these two theories are equivalent. Because $SO(32)$ type I / hetero duality is a strong-weak duality, the above superstring geometry solutions that have non-zero dilaton may be useful for the proof.

We derived the propagator of the fluctuations around the superstring geometry solution. Then, we moved to the first quantization formalism, and we derived the path-integral of the perturbative superstring action. This implies that we also derived the string states and the D-brane boundary states in the first quantization formalism. Next task is to derive a whole Hilbert space of the theory. We can identify string states and D-brane states in the Hilbert space corresponding to the string states and the D-brane boundary states in the first quantization formalism, by using the correspondence between the first and the second quantizations.
Acknowledgements

We would like to thank H. Aoki, M. Fukuma, Y. Hamada, K. Hashimoto, Y. Hosotani, K. Hotta, Y. Hyakutake, N. Ishibashi, K. Ishikawa, G. Ishiki, Y. Ito, Y. Kaneko, N. Kawamoto, T. Kobayashi, H. Kitamoto, T. Kuroki, H. Kyono, K. Maruyoshi, Y. Matsuo, S. Matsuura, T. Morita, S. Moriyama, K. Ohta, N. Ohta, T. Onogi, M. Sakaguchi, Y. Sakatani, S. Seki, S. Shiba, H. Shimada, S. Shimasaki, K. Suehiro, S. Sugimoto, Y. Sugimoto, S. Sugishita, T. Suyama, H. Suzuki, T. Tada, T. Takahashi, T. Takayanagi, K. Tsunura, S. Yamaguchi, K. Yamashiro, Y. Yokokura, and especially S. Iso, H. Itoyama, H. Kawai, J. Nishimura, A. Tsuchiya, and T. Yoneya for long and valuable discussions.

Appendix A: ADM formalism

The ADM decomposition of a two-dimensional metric is given by

$$\tilde{h}_{mn} = \left( \frac{\bar{n}^2}{\bar{n}} + \frac{\bar{n}_\sigma \bar{n}^\bar{\sigma}}{\bar{n}} \frac{\bar{n}_\sigma}{\bar{\epsilon}} + \frac{\bar{n}_\sigma}{\bar{\epsilon}^2} \right) ,$$

(10.1)

where $\bar{n}$ is a lapse function and $\bar{n}_\sigma$ is a shift vector. $\bar{\epsilon}^2$ is a metric on $\bar{\sigma}$ direction. $\bar{n}^\bar{\sigma} := \bar{\epsilon}^{-2} \bar{n}_\sigma$.

As a result, we obtain $\sqrt{\tilde{h}} = \bar{n} \bar{\epsilon}$ and

$$\tilde{h}^{mn} = \left( \frac{1}{\bar{n}^2} \frac{\bar{n}_\sigma}{\bar{n}} \frac{\bar{n}^\bar{\sigma}}{\bar{\epsilon}^2} + \frac{\bar{n}_\sigma}{\bar{\epsilon}^2} \right) .$$

(10.2)

An action for scalar fields $X^\mu$ is decomposed as

$$S = \int d\bar{\tau} d\bar{\sigma} \sqrt{\tilde{h}} \left( \frac{1}{2} \tilde{h}^{mn} \partial_m X^\mu \partial_n X_\mu \right)$$

$$= \int d\bar{\tau} d\bar{\sigma} (-i\bar{\epsilon} p^\mu_X \partial_\bar{\sigma} X_\mu) + \int d\bar{\tau} H,$$

(10.3)

where

$$H = \int d\bar{\sigma} \left( i\bar{\epsilon} \left( \frac{1}{2} (p^\mu_X)^2 + \frac{1}{2} \bar{\epsilon}^{-2} (\partial_\sigma X^\mu)^2 \right) + i\bar{n}^\bar{\sigma} p^\mu_X \partial_\bar{\sigma} X_\mu \right) .$$

(10.4)

Actually, if $p^\mu_X$ is integrated out in the second line, we obtain the first line.
Appendix B: Canonical commutation relations of Majorana fermions

An action for Majorana fermions \( \psi_\mu \) in a two-dimensional curved space-time is given by

\[
S_F = \int d^2 \bar{\sigma} \bar{E} (-\frac{1}{2} \bar{\psi}_\mu \bar{E}_q^m \gamma^q \partial_m \psi_\mu),
\]  

(10.1)

where \( \bar{E}_q^m \) is a vierbein, whose determinant \( \bar{E} \) satisfies \( \bar{E} = \sqrt{h} = \bar{n} \bar{e} \). We use \( \gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \) and \( \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). \( \bar{\psi}_\mu := \psi_\mu \gamma^0 \) and \( p_\mu := \delta S_F / \delta \partial_0 \bar{\psi}_\mu = -\bar{E} \psi_\mu \gamma^0 \gamma^q \bar{E}_q = -\bar{E} \psi_\mu \text{diag}(\bar{E}_0^0 - i \bar{E}_1^0, \bar{E}_0^0 + i \bar{E}_1^0) \). The canonical commutation relations are given by

\[
\{ \hat{p}_\mu^\alpha, \hat{\psi}_\nu (\bar{\sigma}, \bar{\tau}) \} = \eta^{\mu \nu} \delta_\beta^\alpha \delta (\bar{\sigma} - \bar{\sigma}').
\]  

(10.2)

Then we obtain

\[
\{ (-\bar{E}_0^0 + i \bar{E}_1^0) \hat{\psi}_0 (\bar{\sigma}, \bar{\tau}), \hat{\psi}_\nu (\bar{\sigma}', \bar{\tau}) \} = \frac{1}{\bar{E}} \eta^{\mu \nu} \delta (\bar{\sigma} - \bar{\sigma}').
\]

(10.3)

\[
\{ (-\bar{E}_0^0 - i \bar{E}_1^0) \hat{\psi}_1 (\bar{\sigma}, \bar{\tau}), \hat{\psi}_\nu (\bar{\sigma}', \bar{\tau}) \} = \frac{1}{\bar{E}} \eta^{\mu \nu} \delta (\bar{\sigma} - \bar{\sigma}').
\]

\[
\{ \hat{\psi}_0 (\bar{\sigma}, \bar{\tau}), \hat{\psi}_1 (\bar{\sigma}', \bar{\tau}) \} = 0.
\]

(10.3)

If we normalize as \( \hat{\psi}_0 (\bar{\sigma}, \bar{\tau}) := \sqrt{-\bar{E}_0^0 + i \bar{E}_1^0} \psi_0 (\bar{\sigma}, \bar{\tau}) \), and \( \hat{\psi}_1 (\bar{\sigma}, \bar{\tau}) := \sqrt{-\bar{E}_0^0 - i \bar{E}_1^0} \psi_1 (\bar{\sigma}, \bar{\tau}) \), we obtain

\[
\{ \hat{\psi}_\alpha (\bar{\sigma}, \bar{\tau}), \hat{\psi}_\beta (\bar{\sigma}', \bar{\tau}) \} = \frac{1}{\bar{E}} \eta^{\mu \nu} \delta_\alpha^\beta \delta (\bar{\sigma} - \bar{\sigma}').
\]  

(10.4)

References

[1] M. Kontsevich, “Homological algebra of mirror symmetry,” In Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (1995) Birkhauser, [alg-geom/9411018]

[2] M. Sato, “Topological Space in Homological Mirror Symmetry,” [arXiv:1708.09181] [hep-th]

[3] M. Gromov, “Pseudo holomorphic curves in symplectic manifolds,” Invent. math. 82 (1985) 307

[4] H. Hofer, K. Wysocki, E. Zehnder, “Applications of Polyfold Theory I: The Polyfolds of Gromov-Witten Theory,” [arXiv:1107.2097] [math.SG]
[5] E. Witten, “Noncommutative Geometry and String Field Theory,” Nucl. Phys. B268 (1986) 253

[6] W. Klingenberg, “Riemannian Geometry,” De Gruyter Studies in Mathematics 1, de Gruyter; 2nd Rev ed. (April 6, 1995)

[7] I. M. Krichever, S. P. Novikov, “Algebras of Virasoro type, Riemann surfaces and the structure of soliton theory,” Funct. Anal. Appl. 21 (1987) 126

[8] I. M. Krichever, S. P. Novikov, “Virasoro-type algebras, Riemann surfaces and strings in Minkowsky space,” Funct. Anal. Appl. 21 (1987) 294

[9] S. D. Majumdar, “A class of exact solutions of Einstein’s field equations,” Phys. Rev. 72 (1947) 390

[10] A. Papapetrou, “A static solution of the equations of the gravitational field for an arbitrary charge distribution,” Proc. Roy. Irish Acad. A 51 (1948) 191

[11] E.S. Fradkin, D.M. Gitman, “Path integral representation for the relativistic particle propagators and BFV quantization,” Phys. Rev. D44 (1991) 3230

[12] J. Polchinski, “String Theory Vol. 1, 2” Cambridge University Press, Cambridge, UK, 1998

[13] E. Witten, “Notes On Supermanifolds and Integration,” arXiv:1209.2199 [hep-th]

[14] E. Witten, “Notes On Super Riemann Surfaces And Their Moduli,” arXiv:1209.2459 [hep-th]

[15] E. Witten, “The Super Period Matrix With Ramond Punctures,” J. Geom. Phys. 92 (2015) 210

[16] P. S. Howe, “Super Weyl Transformations in Two-Dimensions,” J. Phys. A12 (1979) 393

[17] Eric D’Hoker, D. H. Phong, “Conformal scalar fields and chiral splitting on super Riemann surfaces,” Comm. Math. Phys. 125 3 (1989) 469
[18] L. Brink, P. Di Vecchia, P. S. Howe, “A Locally Supersymmetric and Reparametrization Invariant Action for the Spinning String,” Phys. Lett. 65B (1976) 471

[19] L. Smolin, “M theory as a matrix extension of Chern-Simons theory,” Nucl. Phys. B591 (2000) 227

[20] T. Azuma, S. Iso, H. Kawai, Y. Ohwashi, “Supermatrix models,” Nucl. Phys. B610 (2001) 251

[21] T. Okuda, T. Takayanagi, “Ghost D-branes,” JHEP 0603 (2006) 062

[22] R. Dijkgraaf, B. Heidenreich, P. Jefferson, C. Vafa, “Negative Branes, Supergroups and the Signature of Spacetime,” arXiv:1603.05665 [hep-th]

[23] N. Ishibashi, H. Kawai, Y. Kitazawa, A. Tsuchiya, “A Large-N Reduced Model as Superstring,” Nucl. Phys. B498 (1997) 467

[24] T. Banks, W. Fischler, S.H. Shenker, L. Susskind, “M Theory As A Matrix Model: A Conjecture,” Phys. Rev. D55 (1997) 5112

[25] L. Motl, “Proposals on nonperturbative superstring interactions,” hep-th/9701025

[26] T. Banks and N. Seiberg, “Strings from Matrices,” Nucl. Phys. B497 (1997) 41

[27] R. Dijkgraaf, E. Verlinde and H. Verlinde, “Matrix String Theory,” Nucl. Phys. B500 (1997) 43

[28] R. Dijkgraaf and L. Motl, “Matrix string theory, contact terms, and superstring field theory,” hep-th/0309238

[29] M. Hanada, H. Kawai, Y. Kimura, “Describing curved spaces by matrices,” Prog. Theor. Phys. 114 (2006) 1295

[30] G. ’t Hooft, “A PLANAR DIAGRAM THEORY FOR STRONG INTERACTIONS,” Nucl. Phys. B72 (1974) 461

[31] E. Brezin, C. Itzykson, G. Parisi and J.B. Zuber, “Planar Diagrams,” Commun. Math. Phys. 59 (1978) 35
[32] V.A. Kazakov and A.A. Migdal, “RECENT PROGRESS IN THE THEORY OF NON-CRITICAL STRINGS,” Nucl. Phys. B311 (1988) 171

[33] J. Distler and H. Kawai, “Conformal Field Theory and 2-D Quantum Gravity or Who’s Afraid of Joseph Liouville?,” Nucl. Phys. B321 (1989) 509

[34] S. R. Das and A. Jevicki, “STRING FIELD THEORY AND PHYSICAL INTERPRETATION OF D = 1 STRINGS,” Mod. Phys. Lett A5 (1990) 1639

[35] D. J. Gross and N. Miljkovic, “A NONPERTURBATIVE SOLUTION OF D = 1 STRING THEORY,” Phys. Lett. B238 (1990) 217

[36] P. H. Ginsparg and J. Zinn-Justin, “2-D GRAVITY + 1-D MATTER,” Phys. Lett. B240 (1990) 333

[37] L. Alvarez-Gaume, H. Itoyama, J.L. Manes and A. Zadra, “Superloop Equations and Two Dimensional Supergravity,” Int. J. Mod. Phys. A7 (1992) 5337

[38] For a review,
   P. Ginsparg and G. Moore, “Lectures on 2D gravity and 2D string theory,” [hep-th/9304011]
   I. Klebanov, “String Theory in Two Dimensions,” [hep-th/9108019]

[39] T. Eguchi and H. Kawai, Phys. Rev. Lett. 48 (1982) 1063.
   G. Parisi, Phys. Lett. 112B (1982) 463.
   D. Gross and Y. Kitazawa, Nucl. Phys. B206 (1982) 440.
   G. Bhanot, U. Heller and H. Neuberger, Phys. Lett. 113B (1982) 47.
   S. Das and S. Wadia, Phys. Lett. 117B (1982) 228.

[40] R. Brooks, F. Muhammad, S.J. Gates, “Unidexterous D=2 Supersymmetry in Superspace,” Nucl. Phys. B268 (1986) 599

[41] C.M. Hull, E. Witten, “Supersymmetric Sigma Models and the Heterotic String,” Phys. Lett. 160B (1985) 398

[42] E. Bergshoeff, E. Sezgin, H. Nishino, “Heterotic Models and Conformal Supergravity in Two-dimensions,” Phys. Lett. 166B (1986) 141

85