A Universal Model
of D=4 Spinning Particle

S.L. Lyakhovich, A.Yu. Segal and A.A. Sharapov*

Department of Theoretical Physics, Tomsk State University,
Tomsk 634050, Russia

Abstract

A universal model for D=4 spinning particle is constructed with the configuration space chosen as \( R^{3,1} \times S^2 \), where the sphere corresponds to the spinning degrees of freedom. The Lagrangian includes all the possible world–line first order invariants of the manifold. Each combination of the four constant parameters entering the Lagrangian gives the model, which describes the proper irreducible Poincaré dynamics both at the classical and quantum levels, and thereby the construction uniformly embodies the massive, massless and continuous helicity cases depending upon the special choice of the parameters. For the massive case, the connection with the Souriau approach to elementary systems is established. Constrained Hamiltonian formulation is built and Dirac canonical quantization is performed for the model in the covariant form. An explicit realization is given for the physical wave functions in terms of smooth tensor fields on \( R^{3,1} \times S^2 \). One-parametric family of consistent interactions with general electromagnetic and gravitational fields is introduced in the massive case. The spin tensor is shown to satisfy the Frenkel-Nyborg equation with arbitrary fixed giromagnetic ratio in a limit of weakly varying electromagnetic field.

1 Introduction

The longstanding problem of the (higher)spin particle dynamics provokes today the new interest stimulated by the progress of the string theory which contains the infinite spectrum of higher spin massive excitations. Although the considerable number of the particle models has been proposed involving either even or odd spinning degrees of freedom [1]–[16], the universal Lagrangian for an arbitrary spin particle in arbitrary dimension is still in question.

Besides the pure academic interest, this topic seems to be closely related to the higher spin interaction problem, as for the sake of the consistency requirements, all the spins have

*e-mail: sharapov@phys.tsu.tomsk.su
usually to be involved all together each time when the attempt is undertaken to introduce the interaction both in string and field theoretical approaches [17, 18].

It is commonly accepted to construct the particle configuration space as a direct product of the Minkowski space $\mathbb{R}^{3,1}$ to an inner manifold presenting the spinning degrees of freedom. So the first question is to choose an appropriate space for spin. As a rule, a linear space is used for this end for the sake of explicit covariance. On the other hand, for the reasons of the Poincaré irreducibility the model should have a proper number of physical degrees of freedom, that requires to eliminate the extra variables from the linear space with the aid of constraints. The structure of the constraints may be rather complicated and thereby obstructs switching on the interaction or/and prevents to perform the quantization in the covariant form.

Recently the massive spinning particle model has been proposed with a two–sphere $S^2$ as the inner space [19]. Although $S^2$ has a minimal possible dimension among the homogeneous spaces of the Lorentz group, it does provide the uniform description for an arbitrary spin dynamics. It should be noted that the group acts on the space non-linearly, namely by the holomorphic fractional–linear transformations, i.e. it coincides with the group of complex automorphisms of $S^2$. The geometric construction of the Ref. [19] gives an explicitly invariant Lagrangian whose canonical quantization leads to the physical state subspace being transformed by an irreducible representation of the Poincaré group. This approach is naturally being generalized to the massive higher superspin superparticle case [20] as well as to the spinning particle in the constant curvature spaces [21]. However, the action of the Ref. [19] has not involved a possible geometric invariant of the configuration space, that breaks down the possibility to introduce a consistent interaction to a general external fields, although the free theory has been well-defined in itself.

The present paper contains a systematic study of the most general $d = 4$ spinning particle model which employs a two–sphere as a configuration space of the spinning degrees of freedom. All the possible $\mathcal{M}^6 = \mathbb{R}^{3,1} \times S^2$ world–line Poincaré group invariants without higher derivatives are found including the new Wess-Zumino–like one unnoticed in Ref. [19]. This invariant transforms under the Poincaré group action by a total derivative. However, its exact invariance will be recovered if one reformulates the model considering $S^2$ as the complex projective space $\mathbb{C}P^1$, wherein the projective transformations serve as a gauge symmetry of the model. This enables to construct the most general reparametrization invariant action functional on $\mathcal{M}^6$ describing the particle of an arbitrary fixed spin(helicity).

The action includes four constant parameters, two of which are identified with mass and spin, and the model is able to describe both massive and massless particle depending upon the parameters. The rest parameters are responsible for some arbitrariness in description of the spinning particle in the framework of the model. In particular, for the massive case the theory includes two–parametric family of the physically equivalent Lagrangians of the free particle, however, there is the only special case in the family, which admits a consistent interaction with general electromagnetic and gravitational external fields. In this case the spin tensor is shown to satisfy the Frenkel–Nyborg equation [30] with an arbitrary giromagnetic ratio in the approximation of a weak electromagnetic field.

Constrained Hamiltonian formulation of the theory is constructed and the phase space Casimir functions are shown to conserve identically on the constraint surface. The con-
straint structure drastically differs in the general case from the special one, although they describe the equivalent massive particle dynamics. Whereas the general family is characterized by two abelian first class constraints, the special case has two second class constraints and the only first class one. The latter constraint structure is stable with respect to the deformation of the model by an interaction, while the first one is not. It is the fact which explains why the only special case of the model admits the consistent interaction to an external fields.

The global geometry of the reduced phase space is studied. It is shown that all the independent gauge invariants (observables) of the model are exhausted by the Hamiltonian generators of the Poincaré group. On the constraint surface, they obey algebraic identities, being phase space counterparts of operatorial ones characterizing certain irreducible representation. In fact, the physical phase space of the model can be identified with the corresponding coadjoint orbit of the Poincaré group. Particularly, this identification has been explicitly established for the special massive case, where Hamiltonian reduction leads to the dynamics on $\mathbb{R}^6 \times S^2$, which is just the group coadjoint orbit being used to describe the spinning particle dynamics in the framework of Souriau approach [23]. The prequantization condition for the orbit [22, 23], being imposed in this framework, restricts the spin of the massive particle to be (half)integer. Thus, the special case of the model can be treated as a Lagrangian counterpart of the Souriau description of the spinning particles.

The basic technical tool applied to realize the Hilbert space of the model is the relativistic harmonic analysis on $S^2$ [19], which enables to provide the manifest covariance of the consideration. By these means, the canonical quantization of the model leads to the embedding of the physical state subspace into the space of some smooth tensor fields on $\mathcal{M}^6$. In the framework of the Dirac canonical quantization procedure [26], the physical wave function subspace is extracted by quantum constraint conditions which are proved to coincide with the equations determining a proper unitary irreducible representation of the Poincaré group.

The relevant Poincaré-invariant inner product is constructed to endow the physical subspace with the Hilbert space structure for all the cases under investigation: massive, massless with discrete helicity and massless with continuous helicity. The standard spin–tensor fields on $\mathbb{R}^{3,1}$ obeying relativistic wave equations appear as coefficients in the expansion of the physical wave functions via special “relativistic harmonics” being Poincaré-covariant generalization of ordinary spherical harmonics.

The paper is organized as follows. In Section 1 we study the geometry of the configuration space and derive the model in the Lagrangian framework. Section 2 is devoted to the Hamiltonian formulation of the model. We study the constraints’ structure and identify the parameters of the Lagrangian with the particle’s mass and spin. Also we consider the global geometry of the physical phase space for the special massive case and establish its connection to the Kirillov–Kostant–Souriau approach. This special case allows a consistent interaction to external electromagnetic and gravitational fields which is studied in Section 3. Section 4 concerns the quantization of the model. In Section 5 we make some comments and concluding remarks.

Our notations and conventions coincide mainly with those adopted in the book [25] with the exception that, in contrast to Ref. [23], two–component spinor indices are num-
bered by 0, 1 \((\varepsilon^{01} = \varepsilon^{01} = 1)\) and the spinning matrices \(\sigma_{ab}\) and \(\tilde{\sigma}_{ab}\) are defined with the additional minus sign.

2 General model of relativistic particle on \(\mathcal{M}^6\)

Similarly to the Minkowski space \(\mathbb{R}^{3,1}\), \(\mathcal{M}^6\) is a homogeneous transformation space for the Poincaré group, and therefore \(\mathcal{M}^6\) can serve as a configuration space where dynamics of relativistic particle is developed. The requirement of Poincaré invariance strongly restricts the possible choice of an action functional governing a particle dynamics in \(\mathcal{M}^6\).

In this Section, we classify all the Poincaré invariant action functionals under the following natural conditions:

i) The Lagrangian does not contain higher derivatives.

ii) The action is invariant under reparametrizations of the particle world line.

iii) The relativistic particle mass-shell condition \(p^2 = -m^2\) should arise in the theory.

Let us begin with description of \(\mathcal{M}^6\) geometry. In order to present the action of the Poincaré group on \(\mathcal{M}^6\) in the manifestly covariant form, it is suitable to treat sphere \(S^2\) as complex projective space \(\mathbb{C}P^1\). This space is spanned by nonzero complex two-vector \(\lambda^\alpha = (\lambda^0, \lambda^1)\) subject to the equivalence relation

\[\lambda^\alpha \sim \kappa \lambda^\alpha, \quad \forall \kappa \in \mathbb{C}_* = \mathbb{C}\setminus\{0\}\]  

Then, the relations

\[x^{\alpha \dot{\alpha}} \mapsto x^{\prime \alpha \dot{\alpha}} = x^{\beta \dot{\beta}}(N^{-1})_\beta^\alpha (N^{-1})_\dot{\beta}^{\dot{\alpha}} + f^{\alpha \dot{\alpha}}\]  

\[\lambda^\alpha \mapsto \lambda^\alpha = \lambda^\beta (N^{-1})_\beta^\alpha\]

define the action of the Poincaré group on \(\mathcal{M}^6\). Here \(x^a = -\frac{1}{2}(\sigma^a)_{\alpha \dot{\alpha}} x^{\alpha \dot{\alpha}}\) are the coordinates on \(\mathbb{R}^{3,1}\), \(f^a = -\frac{1}{2}(\sigma^a)_{\alpha \dot{\alpha}} f^{\alpha \dot{\alpha}}\) are the parameters of translations, and the transformations from the Lorentz group \(SO^+(3,1) \sim SL(2,\mathbb{C})/\pm 1\) are associated in the standard fashion with complex unimodular \(2 \times 2\) matrices

\[\|N^\alpha_\beta\| = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})\]  

In addition to the action of the connected Lorentz group \(SO^+(3,1)\), the discrete transformations \(P_1\) and \(P_2\), being associated with space-time reversions, can also be defined on \(\mathcal{M}^6\) by the relations

\[P_1 : (x^0, x^1, x^2, x^3) \mapsto (x^0, x^1, -x^2, x^3), \quad \lambda^\alpha \mapsto \bar{\lambda}^{\dot{\alpha}}\]  

\[P_2 : x^a \mapsto -x^a, \quad \lambda^\alpha \mapsto \lambda^\alpha\]

Alternatively, one can use the standard complex structure on \(S^2 = \mathbb{C} \cup \{\infty\}\) covered by two charts \(\mathbb{C}\) and \(\mathbb{C}_* \cup \{\infty\}\), with local complex coordinates \(z\) and \(w\), respectively, related by the transition function \(w = -1/z\) in the overlap of the charts. The projective coordinates \(\lambda^\alpha\) on \(S^2\) are simply connected with the local ones

\[z = \lambda^1/\lambda^0 \quad \text{if} \quad \lambda^0 \neq 0\]  

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and
\[ w = -\lambda^0/\lambda^1 \quad \text{if} \quad \lambda^1 \neq 0 \quad (5.b) \]

In terms of the local coordinates (5.a) the action of the Lorentz group \( SO^\uparrow(3,1) \) on \( S^2 \) is given by the holomorphic fractional-linear transformations
\[ z \mapsto z' = \frac{a z - b}{-c z + d} \quad (6) \]

where \( a, b, c, d \) are taken from (3), and for reversions (4) we have
\[ P_1 : z \mapsto \bar{z}, \quad P_2 : z \mapsto z \quad (7) \]

Thus we arrive to the nonlinear action of the Poincaré group on \( M^6 \). In spite of the nonlinearity, the forthcoming description, being preformed in the local coordinates, has an explicitly covariant form. This is achieved by making use of the following two-component objects
\[ z^\alpha \equiv z^\alpha = (1, z), \quad \bar{z}^\dot{\alpha} \equiv (z^\alpha)^* = (1, \bar{z}), \quad \alpha, \dot{\alpha} = 0, 1 \quad (8) \]

constructed in terms of the local complex coordinates \( z \) on \( S^2 \) and connected with the projective coordinates \( \lambda^\alpha \) by the rule \( z^\alpha = \lambda^\alpha / \lambda^0, \quad \alpha = 0, 1 \). Then the fractional–linear transformations (6) can be equivalently rewritten in the covariant form
\[ z^\alpha \mapsto z'^\alpha = \left( \frac{\partial z'}{\partial z} \right)^{1/2} z^\beta (N^{-1})^\beta_\alpha \quad (9) \]

This relation means that \( z^\alpha \) transforms as a left-handed Weyl spinor and, simultaneously, as a spinor field on \( S^2 \).

Describe the Poincaré invariants of the tangent bundle \( T(M^6) \). There are only three independent invariants of the world line :
\[ \dot{x}^2 = \dot{x}^\alpha \dot{x}^\alpha, \quad \zeta = \frac{|\lambda_\alpha \dot{\lambda}^\alpha|}{|\dot{x}^\beta \lambda^\beta \bar{\lambda}^\beta|} \quad (10.a) \]
\[ \Gamma = \frac{\dot{x}^\alpha \dot{\lambda}^\alpha \dot{\lambda}^\dot{\alpha}}{\dot{x}^\beta \lambda^\beta \bar{\lambda}^\beta} + \text{complex conjugate} \quad (10.b) \]

Here \( \rho \) is an arbitrary complex parameter and dot over variables stands for the derivative with respect to the evolution parameter \( \tau \) along \( M^6 \) trajectory \( \{ x^\alpha(\tau), \lambda^\alpha(\tau) \} \). It should be mentioned that the first two invariants (10.a) have already been exploited for the construction of the action functional of the massive spinning particle proposed earlier [19]. Let us make two remarks concerning the \( \Gamma \)-invariant which has been unnoticed in Ref [19].

**Remark 1.** Strictly speaking, only first two expressions (10.a), being the true projective invariants with respect to the equivalence relation \( \lambda^\alpha \sim \kappa \lambda^\alpha \), are well-defined on \( T(M^6) \), while the last one \( \Gamma \) is an invariant modulo a total derivative:
\[ \Gamma \mapsto \Gamma' = \Gamma + \frac{d}{d\tau}(\rho \ln \kappa + \bar{\rho} \ln \bar{\kappa}) \quad (11) \]
Remark 2. \( \Gamma \) is invariant under reversion \( P_1 \) only provided \( \rho \) is real. At the same time, for a pure imaginary \( \rho \), \( \Gamma \) remains unchanged under the composition \( P_1 \) with the reflection of the evolution parameter \( \tau \rightarrow -\tau \).

Even though \( \Gamma \) cannot be defined on \( T(\mathcal{M}^6) \) unambiguously, one can use it to construct the Poincaré invariant action functional for which the equivalence relation (1) has to serve as a gauge symmetry.

Using invariants (10.a-b) and taking into account the transformation property of \( \Gamma \), one arrives at the most general form of the Poincaré and reparametrization invariant action functional

\[
S = \int d\tau \mathcal{L} = \int d\tau (\sqrt{-\dot{x}^2 \mathcal{F}(\zeta) + \Gamma})
\]

(12)

Here \( \mathcal{F} \) is an arbitrary function of \( \zeta \)-invariant. Since the model of a relativistic particle is considered, the mass-shell condition

\[
p^2 \equiv \partial \mathcal{L} \partial \dot{x}_a = -m^2 c^2
\]

(13)

should be imposed. This leads to the following equation for \( \mathcal{F} \):

\[
- \mathcal{F} + \zeta \frac{d\mathcal{F}}{d\zeta} + 4|\rho|^2 \zeta^2 = -m^2 c^2
\]

(14)

The general solution to the equation is given by

\[
\mathcal{F} = m^2 c^2 - 4\Delta \zeta - 4|\rho|^2 \zeta^2
\]

(15)

with \( \Delta \) being an arbitrary real constant of integration.

So we are finally arriving at the most general admissible action

\[
S = \int d\tau \sqrt{-\dot{x}^2 \left(m^2 c^2 - 4\Delta \frac{|\lambda_\alpha \dot{\lambda}^\alpha|}{|\dot{x}_\beta \lambda^\beta \dot{\lambda}^\beta|} - 4|\rho|^2 \frac{|\lambda_\alpha \dot{\lambda}^\alpha|^2}{(|\dot{x}_\beta \lambda^\beta \dot{\lambda}^\beta|)^2} \right)} +
\]

\[
+ \int d\tau \left\{ \rho \frac{\dot{x}_{a\dot{a}} \lambda^\alpha \dot{\lambda}^\alpha}{\dot{x}_\beta \lambda^\beta \dot{\lambda}^\beta} + \rho \frac{\dot{x}_{a\dot{a}} \dot{\lambda}^\alpha \lambda^\alpha}{\dot{x}_\beta \lambda^\beta \dot{\lambda}^\beta} \right\}
\]

(16)

which depends on three arbitrary parameters : \( m^2, \Delta, \) and \( \rho \).

The action (16) is seen to be manifestly Poincaré invariant and possesses local symmetries. First of all, there are the world-line reparametrizations

\[
\delta_\epsilon x^a = \dot{x}^a \epsilon, \quad \delta_\epsilon \lambda^\alpha = \dot{\lambda}^\alpha \epsilon \quad (17)
\]

and local \( \lambda^\alpha \)-rescalings

\[
\delta_\kappa \lambda^\alpha = \kappa \lambda^\alpha, \quad \delta_\kappa x^a = 0 
\]

(18)

Then, for \( \Delta \neq 0 \) there exists one more gauge transformation of the form

\[
\delta_\mu x^a = \frac{\partial \mathcal{L}}{\partial \dot{x}_a} \mu, \quad \delta_\mu \lambda^\alpha = 0
\]

(19)
which directly follows from the mass-shell condition. Whereas for $\Delta = m(\rho - \bar{\rho}) = 0$, instead of (19), one finds two gauge transformations which can be presented in the manner

$$\delta_{\nu}\lambda^\alpha = \frac{1}{2}\bar{\lambda}^\dot{\alpha}x^{\dot{\alpha}}\sqrt{-\frac{F(\zeta)}{\dot{x}^2}}\nu, \quad \delta_{\nu}\bar{\lambda} = 0$$

(20)

$$\delta_{\nu}x^a = \left(\frac{\sigma^a}{\lambda^\alpha\bar{\lambda}^\dot{\alpha}}|\rho|^2\zeta^2 + \frac{-\dot{x}^2}{F(\zeta)} - \rho\frac{\dot{\sigma}^a}{\dot{x}_\gamma\lambda^\gamma\bar{\lambda}^\dot{\gamma}}\right)\nu$$

where $\nu(\tau)$ is a complex parameter.

Locally, the action (16) can be readily rewritten in terms of projective-invariant coordinates (5.a): $\lambda^\alpha$ is represented as

$$\lambda^\alpha = \kappa z^\alpha$$

(21.a)

then its substitution to (16) gives

$$S_U = \int d\tau \left[\frac{-\dot{x}^2}{m^2c^2 - 4\Delta} - 4|\rho|^2\frac{\dot{z}^2}{(\dot{x}, \xi)^2}\right] +$$

$$+ \int d\tau \left\{ \rho \dot{z} \partial_z \ln(\dot{x}, \xi) + \bar{\rho} \dot{\bar{z}} \partial_{\bar{z}} \ln(\dot{x}, \xi) \right\} +$$

(21.b)

$$+ \int d\tau \frac{d}{d\tau}(\rho \ln \kappa + \bar{\rho} \ln \bar{\kappa})$$

where

$$\xi_a = (\sigma_a)_{\alpha\dot{\alpha}} z^\alpha z^{\dot{\alpha}} = \bar{\xi}_a = (1 + z\bar{z}, z + \bar{z}, iz - i\bar{z}, 1 - z\bar{z}), \quad \xi^a \xi_a = 0.$$  

(22)

Here $U$ stands for an open domain in $\mathcal{M}^6$ parametrized by the coordinates $(x^a, z)$. In this form the action functional (21.b) is invariant under the Poincaré transformations where the action of the Lorentz group (6) is completed by the corresponding transformation for $\kappa$

$$\kappa \rightarrow \kappa' = \left(\frac{\partial z'}{\partial z}\right)^{-1/2} \kappa$$

(23)

In what follows, the last term in (21.b) containing $\kappa$ is being omitted as it represents the total derivative, but in so doing the Lagrangian becomes invariant under the Lorentz transformation up to a total derivative only. The remaining local symmetries of the action (21) can be read off from the Eqs. (17),(19),(20) by making formal replacement $\lambda^\alpha \rightarrow z^\alpha$.

It is interesting to mention that the local form of the invariants

$$\zeta^2 = \frac{\dot{z}^2}{(\dot{x}, \xi)^2}, \quad \Gamma = \rho \dot{z} \partial_z \ln(\dot{x}, \xi) + c.c.$$

admits a neat geometrical interpretation connected with the Kähler manifold structure on $S^2$. Defining in a domain $U$ the $\mathbb{R}^{1,3}$ velocity-dependent Kähler potential

$$\phi = \ln(\dot{x}, \xi)$$
one finds the metric
\[ 2g_{z\bar{z}}dzd\bar{z} = 4dzd\bar{z}\partial_z\partial_{\bar{z}}\phi, \quad -2\dot{x}^2\zeta^2 = g_{z\bar{z}}\frac{dz}{d\tau}d\bar{z} \]
and the 1-form potential
\[ \theta = \rho \partial_z \ln(\dot{x}, \xi)dz + c.c. \]
for the symplectic structure
\[ \omega = -2i\Im\rho \partial_z \partial_{\bar{z}}\phi dz \wedge d\bar{z}. \]
In the rest reference system for \( \dot{x}^a = (\dot{x}^0, 0, 0, 0) \) one comes to the standard metric on \( S^2 \)
\[ 2g_{z\bar{z}}dzd\bar{z} = 4\frac{dzd\bar{z}}{(1 + z\bar{z})^2}. \]
This geometrical background of the Lagrangian construction (21.b) will be some more discussed below when the Hamiltonian formalism for the model is being analyzed.

In the conclusion of this section it is worth to note that the parameters \( \Delta \) and \( \rho \) are dimensional, and they can not be made dimensionless with the help of the constants \( m \) and \( c \) only. Accounting, however, the Planck constant, \( \Delta \) and \( \rho \) may be represented as
\[ \Delta = \hbar mc\delta, \quad \rho = \hbar r \]
where \( \delta \) and \( r \) are dimensionless numbers. In what follows, the parameters \( \delta \) and \( r \) will be connected with spin or helicity of the particle. The appearance of \( \hbar \) in the classical action seems to be a quite natural phenomenon from the common standpoint that spin should manifest itself as a quantum effect disappearing in the classical limit \( \hbar \to 0 \). This fact is not being emphasised in the further consideration, where we put \( \hbar = c = 1 \).

3 Hamiltonian formalism

Now, we are going to construct a constrained Hamiltonian formulation for the model.

Beginning with the general Lagrangian (16), introduce conjugated momenta for the coordinates \( x^a, \lambda^a \) and \( \bar{\lambda}^{\bar{a}} \)
\[ p^a = \frac{\partial L}{\partial \dot{x}^a}, \quad \pi_\alpha = \frac{\partial L}{\partial \dot{\lambda}_\alpha}, \quad \bar{\pi}_{\dot{\alpha}} = \frac{\partial L}{\partial \dot{\bar{\lambda}}_{\dot{\alpha}}}, \]
subject to the canonical Poisson brackets relations
\[ \{x^a, p_b\} = \delta^a_b, \quad \{\lambda^\alpha, \pi_\beta\} = \delta^\alpha_\beta, \quad \{\bar{\lambda}^{\dot{\alpha}}, \bar{\pi}_{\dot{\beta}}\} = \delta^{\dot{\alpha}}_{\dot{\beta}} \]
Canonical Hamiltonian of the model vanishes identically by virtue of the reparametrization invariance. Therefore, the model is completely characterized by the Hamiltonian constraints. It turns out that number and structure of the constraints essentially
depends on the values of the model parameters. Moreover, as it has already been mentioned, the arbitrariness in the choice of parameters’ values proves to cover all Poincaré-irreducible particle dynamics: massive, massless (including continuous helicity case), and tachion$^1$.

First of all, the mass-shell constraint

$$T_1 = p^2 + m^2 \approx 0 \quad (27)$$

always appears for the action (16); then, the constraints corresponding to the equivalence relation (1) arise

$$d = \lambda^a \pi_a - \rho \approx 0 \quad , \quad \bar{d} = \bar{\lambda}^\alpha \bar{\pi}_\alpha - \bar{\rho} \approx 0 \quad ,$$

and at last in the case of $\Delta \neq 0$ there is one more constraint of the form

$$T_2 = \theta \bar{\theta} - \Delta^2 \approx 0 \quad (29.a)$$

where

$$\theta = \bar{\lambda}_b \tilde{p}^{\dot{\alpha}} \pi_\alpha \quad , \quad \bar{\theta} = \bar{\pi}_a \rho^{\dot{\alpha}} \lambda_\alpha \quad (29.b)$$

The complete set of the constraints: $T_1, T_2, d$ and $\bar{d}$ generates an Abelian algebra with respect to the Poisson brackets. Hence, for nonzero $\Delta$ we get four constraints of the first class.

In the special case of $\Delta = 0$ the situation changes drastically. Instead of one constraint $T_2$ there emerge two additional constraints

$$\theta \approx 0 \quad , \quad \bar{\theta} \approx 0 \quad (30)$$

They possess vanishing Poisson brackets with $T_1, d$ and $\bar{d}$, and nonzero bracket among themselves

$$\{ \theta, \bar{\theta} \} = p^2 (\lambda^a \pi_a - \bar{\lambda}^\alpha \bar{\pi}_\alpha) \approx -m^2 (\rho - \bar{\rho}) \quad (31)$$

This implies that, always, excepting $m^2 (\rho - \bar{\rho}) = 0$, the constraints $\theta$ and $\bar{\theta}$ are of the second-class and reduce to the first-class ones whenever combination $m^2 (\rho - \bar{\rho})$ vanishes.

Notice that the constraints $T_i, i = 1, 2$ have a simple group theoretical interpretation which may, in particular, assign the clear physical sense to the parameters entering the Lagrangian (16). Indeed, the Poincaré group acts on the phase space by canonical transformations, and the corresponding Hamiltonian group, being constructed as a Nöther generators, look like

$$\mathcal{P}_a = p_a \quad , \quad \mathcal{J}_{ab} = x_a p_b - x_b p_a + \mathcal{M}_{ab} \quad (32.a)$$

$$\mathcal{M}_{ab} = (\sigma_{ab})_{\beta}^{\dot{\gamma}} \lambda^{\beta} \pi_\alpha - (\bar{\sigma}_{ab})^{\dot{\gamma}}_{\dot{\beta}} \bar{\lambda}^{\dot{\alpha}} \bar{\pi}_{\dot{\beta}} \quad (32.b)$$

Then the Casimir functions for Poincaré generators (32) read

$$C_1 = \mathcal{P}^a \mathcal{P}_a \approx -m^2 \quad (33.a)$$

$$C_2 = \mathcal{W}_a \mathcal{W}^a = \theta \bar{\theta} + \frac{1}{4} p^2 (\lambda^a \pi_a - \bar{\lambda}^\alpha \bar{\pi}_\alpha)^2 \approx \Delta^2 + m^2 (3m \rho)^2 \quad (33.b)$$

$^1$In the present paper, we leave the tachion case aside.
where $W^a = \frac{1}{2} \varepsilon^{abcd} P_b J_{cd}$ is the classical Pauli-Lubanski vector. Thus, Eqs. (27-30) imply that $C_1$ and $C_2$ are identically conserved on the total constrained surface. Eq. (33.b) also shows that, for nonzero mass, the value

$$\frac{\Delta^2}{m^2} + (\Im m \rho)^2$$

has the meaning of the squared spin. At the same time, in the case of $m = \Delta = 0$ it is a simple exercise to check that the set of constraints (28),(30) is equivalent to the condition

$$W^a - (\Im m \rho) P^a \approx 0$$

and, reversely, Eq. (35) leads to the mentioned constraints.

From the above consideration it follows that there are four essentially different cases depending on the values of the parameters $\Delta, m$ and $\rho$. Each case is identified with some Poincaré-irreducible dynamics. They can be summarized as follows:

i) $\Delta \neq 0, m \neq 0$
In this case the model is described by the four first-class constraints and thereby the number of the physical degrees of freedom comes out to $8 - 4 = 4 = 3$ (position) + 1 (spin). So, one may see that this case corresponds to the massive spinning particle with squared spin magnitude $s^2$ given by (34). For the case $\Im m \rho = 0$ the model coincides with the one studied earlier, and it was demonstrated that the canonical quantization of the theory leads to the equations for the physical wave functions which prove to be equivalent to the relativistic wave equations for massive spin-$s$ field [19].

ii) $\Delta = 0, m(\rho - \bar{\rho}) \neq 0$
Despite the change of the constraints’ structure (there are three first and two second-class constraints), the number of physical degrees of freedom is the same as in the previous case. The model still describes massive spinning particle whose spin value, in accordance with (34), is equal to $|\Im m \rho|$. As it will be shown in the Sec. 5, the canonical quantization procedure, as applied to this case, leads to the irreducible mass $-m$, spin $-s$ representation of the Poincaré group. This case is been considered in the paper with most details since it is the only case which allows to introduce a consistent interaction both with electromagnetic and gravitational fields (see Sec.4).

iii) $\Delta = 0, m = 0$
In this case, there are five first-class constraints. Hence, the number of the physical degrees of freedom equals three. From the Rel. (35) it follows that the model describes the relativistic massless particle with helicity $\Im m \rho$. In the Sec. 5 we show that the canonical quantization of this model leads to the relativistic wave equations for massless, helicity $\Im m \rho$ field.

Also, it is evident from (16) that, provided $\rho$ is pure imaginary, the space reflection $P_1$ reverses the helicity’s sign, as it must happen in a massless particle case.

i\ve) $\Delta \neq 0, m = 0$
This situation is special: the mass is zero while the spin is not zero. The parameter $\Delta$ can not longer be made dimensionless (see Rel.(24)), and so new dimensional parameter has
to be introduced. The model possesses four physical degrees of freedom. As it becomes clear when the theory is being quantized (see Sec.5), this case corresponds to the massless particle with a continuous helicity.

In the remained case ($\Delta = 3m\rho = 0$) the model describes a spinless particle.

Let us now discuss the general structure of physical observables in the theory. Each physical observable $A(x^a, p_b, z, p_z, \bar{z}, p_{\bar{z}})$, being gauge invariant function on the phase space should meet the requirement

$$\{A_{phys}, (\text{first-class constraints})\} \approx 0 \quad (36)$$

It turns out that the basis of physical observables is formed by the generators of Poincaré group, i.e. any physical observable of the theory is a function of the Poincaré generators only

$$A_{phys} = A(P_a, J_{ab}) \quad (37)$$

To prove this assertion consider, first, the cases (i),(ii), and (i\textgreater). For all these cases the dimension of physical phase space is equal to 8. On the other hand, the constrained surface is obviously Poincaré invariant that provides all the Poncaré generators to be the physical observables. As a result, the physical phase space can be covariantly parametrized by 10 generators (32) subject to 2 Casimir conditions (33). As for the remained case (iii), the phase-space, being reduced by the constraints, is six-dimensional that is agreed well with 10 Poincaré generators subject to 4 first class constraints (35).

In the previous section we have described two equivalent Lagrange formulations for the model: in terms of projective (16) and local (21) coordinates on the sphere. In this regard it is pertinent to note that the Hamiltonian formalism for the second case, once constructed, leads to the essentially similar results to that just considered. The only difference is that the first-class constraints (28) are satisfied automatically now, so that the rest constraints are straightforwardly expressed in terms of the cotangent bundle $T^*(\mathcal{M}^6)$ variables. Let $p_z$ be a canonically conjugated momentum to the local complex coordinate on the sphere $z$

$$\{z, p_z\} = \{\bar{z}, p_{\bar{z}}\} = 1 \quad (38)$$

Then the constraints (27),(29),(30) can be rewritten as

$$T_1' = T_1 \approx 0 \quad (39.a)$$

$$T_2' = 2g^{zz}\nabla_z\nabla_{\bar{z}} - (\Delta^2 + m^2(3\rho)^2) \approx 0 \quad (39.b)$$

$$\theta' = (p, \xi)\nabla_z \approx 0, \quad \bar{\theta}' = (p, \xi)\nabla_{\bar{z}} \approx 0 \quad (39.c)$$

where

$$\nabla_z = p_z - \rho \partial_z \ln(p, \xi), \quad \nabla_{\bar{z}} = p_{\bar{z}} - \bar{\rho} \partial_{\bar{z}} \ln(p, \xi), \quad (40)$$

As before, the constraints $T_1'$ and $T_2'$ represent the Casimir functions for the Poincaré generators (32.a), wherein the spinning part of Lorentz transformations are now realized in the manner

$$M_{ab} = (\sigma_{ab})_{\alpha\beta}M^{\alpha\beta} - (\bar{\sigma}_{ab})_{\dot{\alpha}\dot{\beta}}\bar{M}^{\dot{\alpha}\dot{\beta}}, \quad (41.a)$$
\[ M^{\alpha\beta} = -z^\alpha z^\beta p_z + \frac{\rho}{2} \partial_z (z^\alpha z^\beta), \quad \bar{M}^{\dot{\alpha}\dot{\beta}} = -\bar{z}^\dot{\alpha} \bar{z}^\dot{\beta} \bar{p}_z + \frac{\bar{\rho}}{2} \partial_{\bar{z}} (\bar{z}^\dot{\alpha} \bar{z}^\dot{\beta}). \] (41.b)

Let us focus now upon the case (ii), where the model has only one independent gauge transformation corresponding to the reparametrization invariance, that makes possible to introduce the minimal coupling of the model to external gravitational and electromagnetic fields (see Sec. 4). Because of the physical importance of this fact, it seems instructive to consider the structure of reduced phase space corresponding to the free particle in more details.

The constraints (39.c) extract the 9-dimensional surface \( E \) in the cotangent bundle \( T^*(\mathcal{M}^6) \). Topologically, the manifold \( E \) is equivalent to the \( \mathbb{R}^7 \times S^2 \). The restriction of the canonical symplectic structure \( T^*(\mathcal{M}^6) \)

\[ 2(dp_a \wedge dx^a + dp_z \wedge dz + dp_{\bar{z}} \wedge d\bar{z}), \]

to the constraint surface induces on \( E \) the following 2-form

\[ \Omega^E = 2dp_a \wedge dx^a + (3m\rho) \left( -4im \frac{\bar{z}^\dot{\alpha} \bar{z}^\dot{\beta} p_z z^\alpha z^\beta}{(p, \xi)^2} dz \wedge dp_{\dot{\alpha}\dot{\beta}} - 2i \frac{p^\dot{\alpha} \bar{z}^\dot{\beta} z^\alpha z^\beta}{(p, \xi)^2} d\bar{z} \wedge dp_{\dot{\alpha}\dot{\beta}} \right) \]

where \( p_a \) obeys the constraint \( p^2 = -m^2 \). Note that the second term in the expression for \( \Omega^E \) may be regarded as a ‘relativistic’ generalization of the usual symplectic structure on the sphere. Indeed, let us restrict \( \Omega^E \) to a surface \( dp_a = 0 \), then in the rest reference system where \( p_a = \tilde{p}_a = (m, \vec{0}) \), the expression (42) coincides with the standard symplectic structure on a sphere

\[ m^2 \frac{4idz \wedge d\bar{z}}{(p, \xi)^2} \bigg|_{p=\tilde{p}} = \frac{4idz \wedge d\bar{z}}{(1 + zz)^2} \] (43)

Hence the sphere \( S^2 \) appears to be both the configuration and the phase space for the spin (after Hamiltonian reduction, for the latter).

It is evident that the 2-form \( \Omega^E \) is degenerate and the corresponding foliation \( \ker \Omega^E \) coincides with integral curves of the vector field

\[ \vec{v} = p^a \frac{\partial}{\partial x^a} \] (44)

The symplectic reduction procedure is completed through the factorization of \( E \) over the action of the vector field \( \vec{v} \). This can be explicitly done by means of a gauge fixing condition for the constraint \( p^2 = -m^2 \), for example

\[ x^0 = t \] (45)

where \( t \) is an arbitrary parameter. The resulting manifold \( \Sigma \sim E / \ker \Omega^E \), being topologically equivalent to \( \mathbb{R}^6 \times S^2 \), carries a nondegenerate symplectic structure \( \Omega^\Sigma \) obtained by the restriction of \( \Omega^E \) to the surface (45). So, we finally conclude that for the case (ii) the physical phase space of the model is the symplectic manifold \( (\Sigma, \Omega^\Sigma) \).

Let us now return to the presymplectic manifold \( E \). Since \( E \) possesses nontrivial 2-cycles associated with \( S^2 \), the closed 2-form \( \Omega^E \) is not exact. This, in particular, implies
that its potential can be defined only locally. In an open domain $U \subset E$ having a spherical part parametrized by the local complex coordinate $z$ the potential $\omega$ can be chosen as

$$\omega = 2(p_a dx^a + \rho dz \partial_z \varphi + \bar{\rho} d\bar{z} \partial_{\bar{z}} \varphi), \quad d\omega = \Omega^E$$

where

$$\varphi = \ln(p, \xi) \quad (47)$$

plays the role of the Kähler potential for the ‘relativistic’ symplectic structure (43) on $S^2$. Using the 1-form $\omega$ one can write down the Hamiltonian action of the model as

$$S^U = \frac{1}{2} \int \omega \quad (48)$$

Although functional $S^U$ is defined only for those trajectories which do not pass through the singular point $z = \infty \not\in U$, it leads to a proper classical dynamics on the whole $E$. The variational principle, as applied to $S^U$, determines the particle’s trajectories as the foliation $\text{ker} \Omega^E$.

$$\frac{\delta S^U}{\delta \Gamma^i} = \Omega^E_{ij} \dot{\Gamma}^j = 0 \quad (49)$$

where $\Gamma^i = (x^a, p_b, z, \bar{z})$. Note that the Eqs. (49) are well-defined on $E$ as well as $\Omega^E$ is. Taking into account that $\text{ker} \Omega$ is generated by the vector field (44) one can rewrite Eqs. (49) in the equivalent form

$$\dot{x}^a = \lambda p^a, \quad \dot{p}^a = 0, \quad \dot{z} = 0 \quad (50)$$

where $\lambda$ is an arbitrary Lagrange multiplier corresponding to the reparametrization invariance. It follows from (50) that the free massive particle (ii) keeps rest on the sphere and, in Minkowski space, the trajectories coincide with time-like geodesic lines\(^2\).

It is pertinent to discuss here the relationship between the above consideration and the well-known Kirillov-Kostant-Souriau approach \[22, 23, 24\] to classical elementary systems (the so-called orbit method). The basic objects in this approach are the coadjoint orbits $O$ of a Lee group $G$ for which a system is ‘elementary’. The coadjoint orbit is known to be a homogeneous symplectic manifold, so it may be naturally identified with the physical phase space of the system. Applied to the Poincaré group this method gives the Souriau classification of free relativistic particles \[23\]. In the case of mass-$m$ spin-$s$ particle the corresponding orbit $O_{m,s}$ exactly coincides with the manifold $(\Sigma, \Omega^E)$ which has just emerged from the action functional (21.b) in the case (ii) through the conventional Dirac analysis and the Hamiltonian reduction with the aid of constraints.

The geometrical quantization \[24\] of such ‘an elementary system’ implies that the symplectic structure, being associated to the orbit $O$, must satisfy the prequantization condition requiring that the integral of $\Omega$ over an arbitrary 2-cycle is a multiple $2\pi$:

$$\int_{2\text{-cycle}} \Omega = 2\pi n, \quad n \in \mathbb{Z} \quad (51)$$

\(^2\)So this case does not exhibit Zitterbewegung phenomenon which appears in the case (i) \[19\].
Owing to the topological structure of $\Sigma \sim \mathbb{R}^6 \times S^2$, the only nontrivial 2-cycle corresponds to the sphere in the case under consideration. This leads to the condition

$$\int_{2\text{-cycle}} \Omega^2 = -m^2 \Im \rho \int_{S^2} \frac{4i d\zeta \wedge d\bar{\zeta}}{(p, \xi)^2} = 2\pi n, \quad n \in \mathbb{Z}$$

(52)

Integrating over sphere and taking into account the constraint $p^2 = -m^2$ we come to the restriction on an admissible value of spin

$$s = |\Im \rho| = \frac{n}{2}, \quad n = 0, 1, 2, \ldots$$

(53)

It can be also shown that the quantization rule (53) for the spin parameter $s$ provides the expression $\exp iS U$ to be single-valued, being taken on the trajectories whose projections to $S^2$ are closed curves. The last fact makes possible to define path-integral correctly in this model along the lines of Ref. [29], where the pure $S^2$ case has particularly been described. In Sec. 5 we consider the Dirac canonical quantization uniformly in all the cases of the model in the operator approach, and in so doing the quantization rule for spin (53) emerges from the analysis of the spectrum of constraint operators.

4 Coupling to external fields

So far we have discussed the free relativistic spinning particle propagating in the flat spacetime and have seen that, for each certain choice of the parameters $m, \Delta, \rho$, the model describes a proper Poincaré-irreducible dynamics. Now we proceed to a generalization of the theory to the case of presence of external gravitational and electromagnetic fields.

It is well known that there exists a consistency problem in higher spin field dynamics coupled to gravity and electromagnetism [32]. One may expect to find a similar problem in the context of the particle mechanics too. We are going to show that this happens in the model (16) indeed, and there is the case (ii) only, where a consistent coupling can be introduced to the general external fields for the massive particle.

It is convenient to consider this problem starting with the Hamiltonian formulation for the free model described by the constraints (27–30). In what follows $e_m^a(x), \omega_{ab}(x)$ and $A_m(x)$ are the vierbein, spin connection and electromagnetic potential, respectively. Then the simplest way to introduce an interaction known as a minimal coupling consists in the replacement of the momentum $p_a$ by its covariant generalization $\Pi_a$:

$$p_a \mapsto \Pi_a = e_m^a (p_m + e A_m + \frac{1}{2} \omega_{m cd} M^{cd})$$

(54)

in all constraints (27–30). Here $e_m^a$ is the inverse vierbein, $e$ is electric charge and the Hamiltonian generators of the Lorentz transformations $M^{ab}$ are defined as in Rel. (32.b). Geometrically, the last expression assumes that we replace the ‘flat’ configuration space of the model $M^6 = \mathbb{R}^{3,1} \times S^2$ by the Lorentz bundle $M^4 \times S^2$ over curved spacetime $M^4$ with a Lorentz connection $\omega_{m ab}$. In so doing the obtained model turns out to be invariant under the gauge transformations of external fields since they may be induced by a canonical transformation.
Notice however that, contrary to $p_a$, the extended momenta $\Pi_a$ possess nonzero Poisson bracket among themselves
\[
\{\Pi_a, \Pi_b\} = eF_{ba} + T^c_{ba} \Pi_c + \frac{1}{2} R_{abcd} \mathcal{M}^{cd}
\]  
(55)
where $R_{abcd}$, $T^c_{ab}$ and $F_{ab}$ are, respectively, Riemann curvature, torsion and electromagnetic strength tensors. As a result, the Poisson brackets of the abelian first-class constraints $T^1$, $T^2$ corresponding to the cases (i),(i)$\lor$ and $\theta$, $\bar{\theta}$ of the case (iii) become proportional to the r.h.s. of Rel. (55) after substitution (54). This indicates that the coupling, being introduced in such a way, is inconsistent with the gauge invariance underlying the free model, and so the resulting theory proves to be contradictory. What is more, one may show that it is also impossible to preserve the involution relation of the first-class constraints even by adding nonminimal terms to them.

The only occurrence, when the construction (54) works well, is the case (ii). This fact lies mainly in the exceptional algebraic structure of the constraints in the case. Let us explain this in more details. First, all the constraints (27 – 30) transform homogeneously under the dilatations generated by $d$ and $\bar{d}$. Therefore, both for the free and coupled models, the constraints $d$ and $\bar{d}$ are of the first class and so may be omitted from the subsequent analysis. (Moreover, passing to the local parametrization these constraints can be explicitly resolved). Then consider the set of three rest Hamiltonian constraints $T^1$, $\theta$ and $\bar{\theta}$. From the general algebraic viewpoint there are only two different possibilities for the set of three arbitrary constraints: (I) all constraints are of the first class, (II) one constraint is of the first class and two are of the second class ones. The case under consideration corresponds to the last situation for the free particle. Notice that the coupling, being introduced into the constraints, may transform the constraints of type (I) to those of type (II), whereas the inverse transformation is highly impossible if a general configuration of external fields is considered\(^3\). In other words, the constraint structure of case (ii) is stable with respect to general deformations by external fields.

Let us next consider the following generalization of the Hamiltonian constraints (27–30):
\[
T^1_{\text{int}} = \Pi^2 + \frac{g}{2}(eF_{ab} \mathcal{M}^{ab} + \frac{1}{2} R_{abcd} \mathcal{M}^{ab} \mathcal{M}^{cd} + \Pi_c T^c_{ab} \mathcal{M}^{ab}) \approx 0
\]  
(56.6.a)
\[
\theta_{\text{int}} = \bar{\lambda}_a \Pi^{\dot{\alpha} \alpha} \bar{\pi}_\alpha \approx 0 \quad \bar{\theta}_{\text{int}} = \bar{\pi}_a \Pi^{\dot{\alpha} \alpha} \lambda_\alpha \approx 0
\]  
(56.6.b)
\[
\bar{\psi}_{\text{int}} = \bar{d} \approx 0 \quad \bar{\bar{d}}_{\text{int}} = \bar{d} \approx 0
\]  
(56.6.c)
where, apart from the minimal covariantization, the nonminimal term is added to the first constraint, being proportional to an arbitrary constant $g$.

In order to clarify the physical meaning of the parameter $g$, consider the equations of motion for the coupled model described by the constraints (56). To this end we, first, introduce the Dirac bracket associated with the pair of the second class constraints $\theta$ and $\bar{\theta}$ by the rule
\[
\{A, B\}_{DB} = \{A, B\} - \frac{1}{2i\hbar M^2} \left(\{A, \theta\}\{\bar{\theta}, B\} - \{A, \bar{\theta}\}\{\theta, B\}\right)
\]  
(57.6.a)
\(^3\)There may exist, however, some special configurations of background fields for which interaction (54) is consistent for another values of $\Delta$, $m$, $\rho$. In particular, $\Delta$ may be arbitrary real constant for anti-de Sitter (de Sitter) space–time without torsion and electromagnetism\(^2\) since the r.h.s. of Rel. (55) vanishes in the case.
where
\[
M^2 = -\frac{1}{2is}\{\theta, \bar{\theta}\} \approx m^2 + \frac{1+g}{2}(e\mathcal{F}_{ab}\mathcal{M}^{ab} + \frac{1}{2}\mathcal{R}_{abcd}\mathcal{M}^{ab}\mathcal{M}^{cd} + \Pi_d T^d_{ab}\mathcal{M}^{ab})
\]

(57.b)

may be regarded as an 'effective' mass of the particle. Here we choose the parameter \(\rho\) to be pure imaginary constant \(\rho = is\) as it leads to the dynamics being invariant under the space-time reversions. Then the dynamics of the system is completely determined by the evolution of extended momenta \(\Pi_a\), space–time coordinates \(x^a\) and spin tensor \(\mathcal{M}_{ab}\).

Note that the latter satisfies the standard conditions
\[
\Pi_a \mathcal{M}^{ab} \approx 0, \quad \mathcal{M}_{ab} \mathcal{M}^{ab} \approx 2s^2
\]

(58.a)

which allows to express this tensor via the vector of spin \(\tilde{W}^a\)
\[
\tilde{W}^a \equiv \frac{1}{2\sqrt{-\Pi^2}}\epsilon^{abcd}\mathcal{M}_{bc}\Pi_d, \quad \mathcal{M}_{ab} = \frac{1}{2\sqrt{-\Pi^2}}\epsilon_{abcd}\tilde{W}^c\Pi^d
\]

(58.b)

With the use of Dirac brackets (57) evolution of a phase space function \(A\) is given by the equation
\[
\dot{A} = \lambda\{A, T^{int}_{1}\}_{DB}
\]

(59)

\(\lambda\) being an arbitrary Lagrange multiplier corresponding to the reparametrization invariance. For the sake of simplicity we shall restrict our consideration to the case of pure electromagnetic field. Then the corresponding equations read
\[
(\lambda e)^{-1}\dot{\mathcal{M}}_{ab} = g\mathcal{F}_a^c\mathcal{M}_{cb} + \Pi_a K_b - (a \leftrightarrow b)
\]

(60.a)
\[
-(\lambda e)^{-1}\dot{\Pi}_a = 2\mathcal{F}_{ab}\Pi^b + \frac{g}{2}\partial_a \mathcal{F}^{bc}\mathcal{M}_{bc} + e\mathcal{F}_{ab}K^b
\]

(60.b)
\[
(\lambda)^{-1}\dot{x}^a = 2\Pi^a + eK^a
\]

(60.b)

where
\[
K_a = \frac{1}{M^2}\mathcal{M}_{ab}\left\{(2 - g)\mathcal{F}^{bc}\Pi_c + \frac{g}{2}\partial_b \mathcal{F}^{cd}\mathcal{M}_{cd}\right\}
\]

(60.d)
\[
M^2 = m^2 + \frac{1+g}{2}e\mathcal{F}_{ab}\mathcal{M}^{ab}
\]

(60.e)

Let us briefly discuss the obtained equations. First of all, one may see that for the case of weak (that justifies an approximation \(M^2 \approx m^2\)) and constant electromagnetic field the first equation reduces to the well known Frenkel-Nyborg equation for spin \([30,31]\). In this case, the introduced parameter \(g\) has the sense of Landé factor. As is seen there are three special values of \(g\) when the Eqs. (60) are considerably simplified, namely \(g = 0, 2, -1\). The first case corresponds to the minimal coupling. The second one leads to the equations which, in the case of constant electromagnetic field, describe the conventional precession of spin tensor and four-velocity \(\dot{x}^a = 2\lambda\Pi^a\). Finally, putting \(g = -1\) one finds that \(M^2 = m^2\) that guarantees the non-singularity of the Dirac bracket (57) on the whole phase space of the system regardless of background field configurations.

Being associated with the constraints (56), the first-order (Hamiltonian) action of the model looks like
\[ S_H = \int d\tau (p_m \dot{x}^m + \pi_\alpha \dot{\lambda}^\alpha + \bar{\pi}_{\dot{\alpha}} \ddot{\lambda}^{\dot{\alpha}} - \lambda T_1^{\text{int}} - \mu \theta^{\text{int}} - \bar{\mu} \bar{\theta}^{\text{int}} - \nu d - \bar{\nu} \bar{d}) , \]  

(61)

where \( \lambda, \mu, \bar{\mu}, \nu, \bar{\nu} \) are Lagrange multipliers. The action can be readily brought to a second-order form by eliminating the momenta \( p_a, \pi_\alpha, \bar{\pi}_{\dot{\alpha}} \) and Lagrange multipliers with the aid of their equations of motion

\[
\delta S \overline{\delta p_a} = \delta S \overline{\delta \pi_\alpha} = \delta S \overline{\delta \bar{\pi}_{\dot{\alpha}}} = \delta S \overline{\delta \mu} = \delta S \overline{\delta \nu} = 0
\]

It turns out, however, that in a general case this procedure leads to an action functional with a rather complicated dependence upon the curvature and torsion. Whereas, restricting to the case of minimal coupling in the gravitational sector (i.e. neglecting the terms containing the Riemann curvature and torsion in the constraints \( T_1^{\text{int}} \)), we come to the neat generalization of the free model action

\[ S = \int d\tau \mathcal{L} \]  

(62.a)

\[
\mathcal{L} = \frac{1}{4\lambda} \dot{g}_{mn} \dot{x}^m \dot{x}^n - \lambda \left\{ m^2 - 4|\rho|^2 \left| \dot{\lambda}_\alpha \lambda^\alpha \right|^2 \right\} + \rho \dot{x}^m e_{m\beta\dot{\beta}} \dot{{\lambda}}^\alpha \dot{\lambda}^\beta + \rho \dot{x}^m e_{m\beta\dot{\beta}} \dot{\bar{\lambda}}^\dot{\alpha} \dot{\bar{\lambda}}^{\dot{\beta}}
\]

(62.b)

\[
- e A_m \dot{x}^m
\]

where

\[
\dot{\lambda}^\alpha = \dot{\lambda}^\alpha - \frac{1}{2} (\lambda g e F_{ab} + \dot{x}^m \omega_{mab}) \sigma^a_{\beta\dot{\beta}} \dot{\lambda}^\beta \]  

(62.c)

is the extended Lorentz-covariant derivative. Further, the case \( g = 0 \) allows to exclude explicitly the Lagrange multiplier \( \lambda \) from the Lagrangian by means of the equation \( \delta S / \delta \lambda = 0 \):

\[
\mathcal{L} = \sqrt{-\dot{x}^2 \left( m^2 c^2 - 4|\rho|^2 \left| \dot{\lambda}_\alpha \lambda^\alpha \right|^2 \right)} + \rho \dot{x}^m e_{m\beta\dot{\beta}} \dot{\lambda}^\alpha \dot{\lambda}^\beta + \rho \dot{x}^m e_{m\beta\dot{\beta}} \dot{\bar{\lambda}}^\dot{\alpha} \dot{\bar{\lambda}}^{\dot{\beta}} - e A_m \dot{x}^m
\]

(63)

The Lagrangian (63), along with the one for the free model (16) possesses (for \( \Delta = 0 \)), besides \( \lambda^\alpha \)–rescalings, the only gauge symmetry – reparametrization invariance.
5 Quantization

In this section we consider the operatorial formulation for the quantum theory of the free model. We show that, for every case (i–i\(\vee\)), the Hilbert space of the physical states is identified with the space of corresponding unitary irreducible representation of the Poincaré group.

An explicit geometric construction underlying the model makes it possible to operate in a manifestly covariant fashion, namely, to embed the Hilbert space of physical states of the system into a space of smooth (spin)tensor fields on \(\mathcal{M}^6\). A complementary advantage of this approach is in its direct relationship with the conventional on-shell realization for unitary irreps of Poincaré group in terms of (spin)tensor fields on Minkowski space.

We will be interested in those tensor fields on \(\mathcal{M}^6\) which are scalars in Minkowski space and tensor fields of type \(\{k/2, l/2\}\) on the sphere. This is equivalent to the following transformation property under the action of Poincaré group (2.1,a),(6):

\[\Phi'(x', z', \bar{z}') = \left(\frac{\partial z'}{\partial z}\right)^{k/2} \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{l/2} \Phi(x, z, \bar{z}).\] (64)

The fields of this type will naturally arise if one starts with the particle wave functions \(\Psi(x^a, \lambda^\alpha, \bar{\lambda}^{\dot{\alpha}})\) defined over the original configuration space \(\mathbb{R}^{3,1} \times \mathbb{C}^2\) as scalar fields subject to the quantum kinematical constraints \(\hat{d}\) and \(\hat{d}\):

\[(i\lambda^\alpha \frac{\partial}{\partial \lambda^\alpha} + \rho)\Psi(x^a, \lambda^\alpha, \bar{\lambda}^{\dot{\alpha}}) = 0 \quad (65.a)\]
\[(i\bar{\lambda}^{\dot{\alpha}} \frac{\partial}{\partial \bar{\lambda}^{\dot{\alpha}}} + \bar{\rho})\Psi(x^a, \lambda^\alpha, \bar{\lambda}^{\dot{\alpha}}) = 0. \quad (65.b)\]

These equations imply that \(\Psi\) is a homogeneous function in \(\lambda^\alpha, \bar{\lambda}^{\dot{\alpha}}\) and locally it can be rewritten in terms of \(\Phi\{i\rho/2, i\bar{\rho}/2\}\) as

\[\Psi(x^a, \lambda^\alpha, \bar{\lambda}^{\dot{\alpha}}) = \kappa^{i\rho\bar{\rho}} \Phi\{i\rho, i\bar{\rho}\}(x^a, z, \bar{z})\] (66)

where the representation (21.a) for \(\lambda^\alpha\) is used. Then the transformation property (64) follows at once from (21.a) , (23) , (66).

It should be stressed that all the subsequent results could be obtained without use of local coordinates \(z\) by working directly in terms of homogeneous functions over \(\mathbb{R}^{3,1} \times \mathbb{C}^2\). However, the tensor fields on \(\mathcal{M}^6\) are convenient for our purpose due to their transparent structure and well-realized decomposition with respect to the Poincaré group \([19]\). Let us recall relevant facts concerning the structure of these fields.

The infinitesimal form of (64) is given by the generators

\[P_a = -i\partial_a, \quad J_{ab} = -i(x_a \partial_b - x_b \partial_a + M_{ab}),\] (67.a)

where the spinning part of \(J_{ab}\) is realized by spherical variables in the manner

\[M_{ab} = (\sigma_{ab})_{\alpha\dot{\beta}} M^{\alpha\dot{\beta}} - (\bar{\sigma}_{ab})_{\alpha\dot{\beta}} \bar{M}^{\alpha\dot{\beta}}, \quad (67.b)\]
\[ M^{\alpha \beta} = -z^\alpha z^\beta \partial_z + \frac{k}{2} \partial_z (z^\alpha z^\beta), \quad \bar{M}^{\alpha \beta} = -\bar{z}^\alpha \bar{z}^\beta \partial_{\bar{z}} + \frac{\bar{l}}{2} \partial_{\bar{z}} (\bar{z}^\alpha \bar{z}^\beta). \] (67.c)

Then, two Casimir operators: the squared mass and the squared spin \( C^{(k/2,l/2)} = W^a W_a \), \( W^a = \frac{i}{2} \varepsilon_{abcd} P_b J_{cd} \) being the Pauli—Lubanski vector, have the form
\[
P^a P_a = -\Box \quad (68.a)
\]
\[
C^{(k/2,l/2)} = - (p^\alpha \xi^\beta - \Pi_{\alpha \beta} (p, \xi) (p, \partial_z \xi) \partial_z + l (p^\alpha \partial_z \xi) (p, \partial_z \xi) \partial_z -
kl (p, \partial_z \xi) (p, \partial_z \xi) - p^2 \left( \frac{k-l}{2} \right)^2 \frac{k+l}{2} \right). \] (68.b)

Let us consider the space \( \mathcal{H}^{(k/2,l/2)}(M^6, m) \) of massive positive-frequency fields on \( M^6 \) of the spherical type \{k/2, l/2\} where \( k, l \) are integer. Such fields are defined to satisfy the mass-shell condition
\[
(\Box - m^2) \Phi^{(k/2,l/2)}(x, z, \bar{z}) = 0 \] (69.a)
and possess the Fourier decomposition
\[
\Phi^{(k/2,l/2)}(x, z, \bar{z}) = \int \frac{d^3 p}{p^0} e^{i(p \cdot x)} \Phi^{(k/2,l/2)}(p, z, \bar{z}), \] (69.b)
where
\[
p^2 + m^2 = 0, \quad p^0 > 0. \] (69.c)

After passing to momentum representation with respect to \((x^a, p_b)\) the second Casimir operator \( C^{(k/2,l/2)} \) admits a simple geometric interpretation. With every point of massive hyperboloid \( p^2 = -m^2 \) one can associate the smooth metric on \( S^2 \)
\[
ds^2 = \frac{4dzd\bar{z}}{(p^a \xi_a)^2} = 2g_{zz}dzd\bar{z}, \] (70)
where \( \xi_a \) is defined as in the Eq. (22). Since \( p_a \) is a time-like Lorentz vector, the Rel. (70) proves to define a smooth tensor field on \( S^2 \) of type \{-1, -1\}. This metric is covariant under the Lorentz transformations (6), (9) in the following sense
\[
\frac{dz'd\bar{z}'}{(p'_a z'^{\alpha} \bar{z}'^\beta)^2} = \frac{dzd\bar{z}}{(p_a z^{\alpha} \bar{z}^\beta)^2}, \] (71.a)
where
\[
p'_a = N^\alpha_\beta \bar{N}^\beta_\alpha p_\beta, \] (71.b)
and \( \bar{z} \) is transformed by the linear–fractional law related to the Lorentz transformation by Rel.(9). The corresponding covariant derivatives \( \hat{\nabla}_z \) and \( \hat{\nabla}_{\bar{z}} \) (acting on a tensor field of the spherical type \{k/2, l/2\}) have the form
\[
\hat{\nabla}_z = \partial_z + \frac{k}{2} \Gamma_z^{zz}, \quad \Gamma_z^{zz} = -2\partial_z \ln(p, \xi), \] (72.a)
\[
\hat{\nabla}_{\bar{z}} = \partial_{\bar{z}} + \frac{l}{2} \Gamma_{\bar{z}}^{\bar{z}z}, \quad \Gamma_{\bar{z}}^{\bar{z}z} = -2\partial_{\bar{z}} \ln(p, \xi), \] (72.b)
and satisfy the commutation relation
\[ [\hat{\nabla}_z, \hat{\nabla}_{\bar{z}}] = \frac{1}{2} (k - l) g_{z\bar{z}} R, \quad R = -p^2. \tag{73} \]

The last expression manifests that, for every point of massive hyperboloid \( p^2 = -m^2 \), the metric (70) is characterized by the constant positive curvature \( R = m^2 \). Now, the squared spin operator (68.b) can be rewritten as
\[ C^{(k/2,l/2)} \equiv \Delta^{(k/2,l/2)} = -2 g^{z\bar{z}} \hat{\nabla}_z \hat{\nabla}_{\bar{z}} + \frac{(k - l)}{2} \left( \frac{k - l}{2} + 1 \right) R. \tag{74} \]

In this form the Casimir operator \( C^{(k/2,l/2)} \) is identified with a special spherical Laplacian.

Employing the metric (70), the space \( \uparrow \mathcal{H}^{(k/2,l/2)}(\mathcal{M}^6; m) \) can be endowed with a Hilbert space structure where the corresponding inner product is introduced as
\[ \langle \Phi_1 | \Phi_2 \rangle_{(k/2,l/2)} = N \int \frac{d^3p}{p^0} \frac{dz d\bar{z}}{(p, \xi)^2} \frac{1}{(p, \xi)^{k+l}} \hat{\Phi}_1(p, z, \bar{z}) \hat{\Phi}_2(p, z, \bar{z}), \tag{75} \]
with \( N \) being some normalization constant.

It should be stressed that the inner product (75) is well defined on \( S^2 \) and is seen to be Poincaré invariant. As a result, the representation of Poincaré group in the space \( \uparrow \mathcal{H}^{(k/2,l/2)}(\mathcal{M}^6; m) \) is unitary. This representation can be readily decomposed into a direct sum of irreducible ones by accounting already mentioned fact that the spin operator \( C^{(k/2,l/2)} \) coincides with the Laplacian \( \Delta^{(k/2,l/2)} \), which proves to be the Hermitian operator with respect to the inner product (75). The spectrum of the Laplacian \( \Delta^{(k/2,l/2)} \) is given by the eigenvalues
\[ s(s + 1) R, \quad s = \frac{|k - l|}{2}, \frac{|k - l|}{2} + 1, \frac{|k - l|}{2} + 2, \ldots \tag{76} \]

The dimension of the eigenspace corresponding to an eigenvalue \( s(s + 1) R \) equals \( 2s + 1 \). This leads to the following decomposition
\[ \uparrow \mathcal{H}^{(k/2,l/2)}(\mathcal{M}^6; m) = \bigoplus_{s=\frac{|k-l|}{2}, \frac{|k-l|}{2}+1, \ldots} \uparrow \mathcal{H}_s^{(k/2,l/2)}(\mathcal{M}^6; m) \tag{77} \]
Here the invariant subspace \( \uparrow \mathcal{H}_s^{(k/2,l/2)}(\mathcal{M}^6; m) \) realizes unitary Poincaré representation of mass \( m \) and spin \( s \). The expansion of an arbitrary field from \( \uparrow \mathcal{H}^{(k/2,l/2)}(\mathcal{M}^6; m) \), which corresponds to the decomposition (77), reads as follows
\[ \Phi^{(k/2,l/2)}(p, z, \bar{z}) = \sum_{s=\frac{|k-l|}{2}, \frac{|k-l|}{2}+1, \ldots} F_{\alpha_1 \ldots \alpha_{s-(k-l)/2} \bar{\alpha}_1 \ldots \bar{\alpha}_{s+(k-l)/2}}(p) \times \tag{78.a} \]
\[ \times z^{\alpha_1} \ldots z^{\alpha_{s-(k-l)/2}} \bar{z}^{\bar{\alpha}_1} \ldots \bar{z}^{\bar{\alpha}_{s+(k-l)/2}} \]
\[ \frac{(p, \xi)^{s-(k+l)/2}}{(p, \xi)^{s-(k+l)/2}} \]
In this expansion, each coefficient $F_{\alpha(s-(k-l)/2)} \hat{\alpha}(s+(k-l)/2)(p)$ is symmetric in its undotted and, independently, in its dotted indices

$$F_{\alpha_1 \ldots \alpha_s-(k-l)/2} \hat{\alpha_1} \ldots \hat{\alpha_s}+(k-l)/2(p) = F_{\alpha_1 \ldots \alpha_s-(k-l)/2} \hat{\alpha_1} \ldots \hat{\alpha_s}+(k-l)/2(p),$$

(78.b)

and $p$-transversal,

$$p^{\beta\beta} F_{\beta \alpha(s-(k-l)/2-1)} \hat{\beta} \hat{\alpha}(s+(k-l)/2-1)(p) = 0.$$  

(78.c)

Together with the mass-shell condition $p^2 = -m^2$ Eqs. (78.b,c) constitute the set of relativistic wave equations for a massive field of spin $s$. Thus, the $F$’s are identified with the Fourier transformations of Poincaré-irreducible tensor fields on Minkowski space.

Now since we have recalled the necessary features of relativistic harmonic analysis on $\mathcal{M}^6$, we are in a position to define the correct operatorial formulation for the quantum theory of the model.

In Sec. 3, it has been established that all the physical observables of the model, being the gauge invariant values, are the functions of the Poincaré generators only. After quantization, every physical observable must be assigned with a Hermitian operator acting in a Hilbert space. From this point of view the quantization procedure for this model is equivalent to the construction of unitary irreducible representation of Poincaré group with the mass and spin (helicity for $m = 0$) fixed by the constraints (33), (35). As we have seen, for $m \neq 0$ such representations are naturally realized in the spaces of massive positive frequency fields $\Uparrow \mathcal{H}^{(k/2,l/2)}(\mathcal{M}^6; m)$. It seems instructive, however, to consider this problem from the standpoint of the conventional Dirac quantization.

In the ordinary coordinate representation, the quantization of original phase space variables is performed by the standard substitution

$$p_a \mapsto -i\partial_a, \quad p_z \mapsto -i\partial_z, \quad p_{\bar{z}} \mapsto -i\partial_{\bar{z}}$$

(79)

Then, the operators corresponding to Poincaré generators can be obtained by the substitution of (79) into Hamiltonian generators (32.a),(41). Notice, however, that there exists some inherent ambiguity in the ordering of $\hat{p}_z$, $\hat{z}$ and $\hat{p}_{\bar{z}}$, $\bar{z}$ due to the nonlinear character of Lorentz transformations. Fortunately, the different operator orderings in (41) result only in renormalization of the parameter $\rho$. So, in general, the quantum Poincaré generators has the form (67) wherein $k$ and $l$ are arbitrary complex numbers. In what follows, however, one has to restrict the parameters $k$ and $l$ to be integer, as for only this case the infinitesimal Lorentz transformations are globally integrable in the space of smooth tensor on $\mathcal{M}^6$ what is of the primary importance for existence of a Hilbert space structure.

Thus, the Hilbert space of the system is embedded into the space $\mathcal{H}^{(k/2,l/2)}$ of smooth tensor fields of spherical type $\{k/2,l/2\}$.

Consider now the constraints (39). Once the space $\mathcal{H}^{(k/2,l/2)}$ has been chosen, we must require the quantum analogues of Hamiltonian constraints to be well-defined operators on $\mathcal{H}^{(k/2,l/2)}$. Up to a constant, this leads to the unique expression for quantum constraints. Namely, the first class constraints $\hat{T}_1, \hat{T}_2$ are naturally associated with the Casimir operators for the Poincaré generators (67)

$$\hat{T}_1 = p^2 + m^2$$

(80.a)
\[ \hat{T}_2 = -2g^{zz}\hat{\nabla}_z\hat{\nabla}_z - (\Delta^2 + m^2(\Im m)^2) - \alpha m^2 \]  

(80.b)

while the constraints \( \hat{\theta} \) and \( \hat{\bar{\theta}} \) can be expressed in terms of covariant derivatives and metric

\[ \hat{\theta} = -i\sqrt{2}g^{zz}\hat{\nabla}_z \quad , \quad \hat{\bar{\theta}} = -i\sqrt{2}\bar{g}^{zz}\hat{\nabla}_{\bar{z}} \]  

(81)

(here we have passed to the momentum representation with respect to \( x^a \) and \( p_a \)). The arbitrary real constant \( \alpha \) is introduced to account the ambiguity in the ordering of covariant derivatives \( \hat{\nabla}_z \) and \( \hat{\nabla}_{\bar{z}} \) (as is seen from (73)), that can be thought about as a quantum correction to the constraint condition (39.b).

Consider now the quantization of all the special cases (i-iv) in turn.

Case (i)

There are two first class constraints \( \hat{T}_1 \) and \( \hat{T}_2 \). Hence, the physical subspace is extracted by the conditions

\[ (p^2 + m^2)\Phi_{phys}^{(k/2,l/2)}(p, z, \bar{z}) = 0 \]  

(82.a)

\[ (C_{phys}^{(k/2,l/2)} - \Delta^2 - m^2(\Im m)^2 + \alpha))\Phi_{phys}^{(k/2,l/2)}(p, z, \bar{z}) = 0 \]  

(82.b)

Upon fulfillment of the mass-shell condition (82.a) the eigenvalues of operators \( C_{phys}^{(k/2,l/2)} \) are given by (76). Hence, the equations (82) admit a nontrivial solution only provided

\[ \left( \frac{\Delta}{m} \right)^2 + (\Im m)^2 + \alpha = s(s + 1) \]

\[ s = \frac{|k - l|}{2}, \quad \frac{|k - l|}{2} + 1, \quad \frac{|k - l|}{2} + 2 \ldots \]  

(83.a)

Comparing (83.a) with the classical value for the squared spin (34) we get the final relations

\[ \left( \frac{\Delta}{m} \right)^2 + (\Im m)^2 = s^2, \quad \alpha = s \]  

(83.b)

Then, the positive-frequency solutions (i.e. \( p^0 > 0 \)) of Eqs. (82) form the Hilbert space \( \uparrow \mathcal{H}_s^{(k/2,l/2)}(M^6, m) \).

Case (ii)

Besides the mass-shell condition (80.a) there are two constraints (81) satisfying the algebra

\[ [\hat{\theta}, \hat{\bar{\theta}}] = (k - l)p^2 \]  

(84)

We restrict \( k, l \) to satisfy

\[ |k - l| = |\Im m| \]  

(85)

This condition implies that the classical and quantum spin values should coincide and equal \( |\Im m| \), as we show below.

Thus, we have the set of one first (80.a) and two second class constraints (81). A correct definition of physical states will be achieved if they are required to be annihilated by the first-class constraint operator

\[ (p^2 + m^2)\Phi_{phys}^{(k/2,l/2)}(p, z, \bar{z}) = 0 \]  

(86.a)
as well as by the half of the second class constraints

\[ \text{for } k \geq l, \quad \hat{\theta} \Phi_{\text{phys}}^{\{k/2,l/2\}}(p, z, \bar{z}) = 0 \quad (86.b) \]

\[ \text{for } k \leq l, \quad \hat{\theta} \Phi_{\text{phys}}^{\{k/2,l/2\}}(p, z, \bar{z}) = 0 \quad (86.c) \]

As a consequence of Eqs. (74, 86), the positive frequency solutions of Eqs. (86) \( \Phi \) satisfy

\[ \Phi_{\text{phys}}^{\{k/2,l/2\}} \in \Upsilon \mathcal{H}_z^{\{k/2,l/2\}}(\mathcal{M}; m) \quad , \quad s = \frac{1}{2}|k - l| \quad (87) \]

Reversely, each \( \Phi, \Phi \in \Upsilon \mathcal{H}_{|k-l|/2}^{\{k/2,l/2\}}(\mathcal{M}; m) \) satisfies to Eqs. (86) as it is seen from the following sequence (for a moment, we take \( k > l \) for definiteness)

\[ 0 = < \Phi | C^{\{k/2,l/2\}} - m^2 s(s + 1) | \Phi >_{\{k/2,l/2\}} = < \Phi | \hat{\theta} | \Phi >_{\{k/2,l/2\}} = ||\hat{\theta} \Phi||^2_{\{k/2,l/2\}} \Rightarrow \hat{\theta} \Phi = 0 \quad (88.a) \]

Thus the subspace of smooth positive-frequency solutions to Eqs. (86) is identified with the space of unitary irreducible representation of the Poincaré group \( \Upsilon \mathcal{H}_{|k-l|/2}^{\{k/2,l/2\}}(\mathcal{M}; m) \), being the lowest member of the direct sum (77).

Case (iii)

This case corresponds to the massless particle with discrete helicity \( \Im \rho = 0, \pm 1/2, \pm 1, \pm 3/2, \pm 2, \ldots \) It is characterized by the set of three first-class constraints \( \hat{T}_1, \hat{\theta} \) and \( \theta \). The physical wave functions are subject to the conditions

\[ \hat{\theta} \Phi_{\text{phys}}^{\{k/2,l/2\}}(p, z, \bar{z}) = 0 \quad (89.a) \]

\[ \hat{\theta} \Phi_{\text{phys}}^{\{k/2,l/2\}}(p, z, \bar{z}) = 0 \quad (89.b) \]

\[ p^2 \Phi_{\text{phys}}^{\{k/2,l/2\}}(p, z, \bar{z}) = 0 \quad (89.c) \]

Note that, for \( k \neq l \), the constraint (89.c) arises as a consistency condition for the Eqs. (89.a,b) by virtue of (84). The important fact is that Eqs. (89.a-c) do not possess smooth solutions for arbitrary \( k, l \), so one has to define correctly the functional space where the solution is sought. To formulate the right conditions let us introduce the mapping \( \hat{\pi} \) by the rule

\[ \text{for } k \geq l, \quad \hat{\pi} : \Phi_{\text{phys}}^{\{k/2,l/2\}} \rightarrow \Phi_{\text{phys}}^{\{k-l/2,0\}} = (p, \xi)^{-1} \Phi_{\text{phys}}^{\{k/2,l/2\}} \quad (90.a) \]

\[ \text{for } k \leq l, \quad \hat{\pi} : \Phi_{\text{phys}}^{\{k/2,l/2\}} \rightarrow \Phi_{\text{phys}}^{\{0,(l-k)/2\}} = (p, \xi)^{-k} \Phi_{\text{phys}}^{\{k/2,l/2\}} \quad (90.b) \]

Then the right restriction on the functional space is the requirement for \( \hat{\pi} \Phi_{\text{phys}}^{\{k/2,l/2\}} \) to be a smooth tensor field on \( S^2 \). Choose, for a moment, \( k \geq l \). Under the mapping \( \hat{\pi} \) any solution of Eqs. (89.a-c) transforms to that for

\[ (p, \xi) \partial_z (\hat{\pi} \Phi)^{\{(k-l)/2,0\}} = 0 \quad (91.a) \]

\[ \hat{\theta} (\hat{\pi} \Phi)^{\{(k-l)/2,0\}} = ((p, \xi) \partial_z - (k - l)(p, \partial_z \xi)) (\hat{\pi} \Phi)^{\{(k-l)/2,0\}} = 0 \quad (91.b) \]
According to the Riemann-Roch theorem [27], the space of smooth solutions for Eq. (91.a) (for every $p_a$) is $(k - l + 1)$-dimensional space of holomorphic tensor fields of the form

$$\Phi^{[(k-l)/2,0]}(p, z) = \Phi_{\alpha_1...\alpha_{k-l}}(p)z^{\alpha_1}...z^{\alpha_{k-l}}$$

(92)

Substituting (92) into (91.b) and making use of the identity

$$z^\alpha \partial_z z^\beta - z^\beta \partial_z z^\alpha = \epsilon^{\alpha\beta}$$

(93)

one gets

$$p^{\beta\alpha_1}\Phi_{\alpha_1...\alpha_{k-l}}(p) = 0$$

(94)

Here one immediately recognizes the relativistic wave equation for a massless field of helicity $(k - l)/2$.

The treatment of the case $k \leq l$, being analogous to the above, gives the result

$$\Phi^{[0,(l-k)/2]}(p, z) = \Phi_{\dot{\alpha}_1...\dot{\alpha}_{l-k}}(p)\bar{z}^{\dot{\alpha}_1}...\bar{z}^{\dot{\alpha}_{l-k}}$$

(95.a)

$$p^{\dot{\alpha}_1\beta}\Phi_{\dot{\alpha}_1...\dot{\alpha}_{l-k}}(p) = 0$$

(95.b)

presenting wave functions of helicity $-1/2(l - k)$ field.

Comparing the above formulas (94),(95.b) with the classical expression for helicity (35) we set

$$\frac{k - l}{2} = 3m\rho$$

(96)

Furthermore, one observes that a general solution to Eqs. (89.c),(94),(95.b) has the form

$$\Phi_{\alpha_1...\alpha_{k-l}}(p) = \eta_{\alpha_1}...\eta_{\alpha_{k-l}}\Phi(p)$$

(97.a)

$$\Phi_{\dot{\alpha}_1...\dot{\alpha}_{k-l}}(p) = \bar{\eta}_{\dot{\alpha}_1}...\bar{\eta}_{\dot{\alpha}_{k-l}}\dot{\Phi}(p)$$

(97.b)

where

$$p_{a\dot{a}} = \eta_{\alpha}\bar{\eta}_{\dot{\alpha}}$$

(97.c)

is the standard twistor representation of a light-like four vector $p_{a\dot{a}}$ through a commuting Weyl spinor $\eta_{\alpha}$ determined by the Eq. (97.c) modulo a phase: $\eta_{\alpha} \sim e^{i\chi}\eta_{\alpha}$. Accounting Eqs. (90),(92),(95),(97) as well as the equality $(p, \xi) = (\eta_{\alpha}\bar{z}^{\alpha})(\bar{\eta}_{\dot{\alpha}}\dot{z}^{\dot{\alpha}})$ one may come to the following general solution to the original Eqs. (89) as

$$\Phi_{\text{phys}}^{[k/2,l/2]}(p, z, \bar{z}) = (\eta_{\alpha}\bar{z}^{\alpha})^k(\bar{\eta}_{\dot{\alpha}}\dot{z}^{\dot{\alpha}})^l\Phi(p), \quad p^2 = 0$$

(98)

Then the relevant realization for the norm, providing positive-frequency wave functions (98) with a Hilbert space structure, is simply built as

$$\langle \Phi | \Phi \rangle = N \int \frac{d^3p}{p^0} \Phi^\dagger \Phi, \quad p^0 = \sqrt{\vec{p}^2}$$

(99)

where $N$ is a normalization constant. Note that the phase ambiguity for $\Phi$ arising from that for $\eta_{\alpha}$ does not contribute to this norm, as it should be.
This time the model describes a particle with continuous helicity \( \Delta, \ 0 < \Delta < \infty \). The dynamics is characterized by two abelian first-class constraints (39.a,b). The quantum constraints \( \hat{T}_1, \hat{T}_2 \) and the physical states \( \Phi_{\text{phys}}^{(k/2,l/2)} \) are defined for arbitrary \( k \) and \( l \) as in the case (i) provided that \( m = 0 \):

\[
p^2 \Phi_{\text{phys}}^{(k/2,l/2)}(p, z, \bar{z}) = 0 \tag{100.a}
\]

\[
(2g^{zz}\hat{\nabla}_z \hat{\nabla}_{\bar{z}} + \Delta^2) \Phi_{\text{phys}}^{(k/2,l/2)}(p, z, \bar{z}) = 0 \tag{100.b}
\]

Note that, as a consequence of Eq. (73),

\[
[\hat{\nabla}_z, \hat{\nabla}_{\bar{z}}] = 0
\]

Hence, the ordering ambiguity does not appear for the constraint \( \hat{T}_2 \). For the purposes of quantization it is enough to deal with the even \( k-l \). Then, the operators \( i\Delta \hat{\theta} = i\Delta(p, \xi)\hat{\nabla}_z \) and \( i\Delta \hat{\theta} = i\Delta(p, \xi)\hat{\nabla}_{\bar{z}} \) are inverse to each other on the space of solutions to Eqs. (91), that makes possible to define the smooth one-to-one mapping

\[
\Phi_{\text{phys}}^{(k/2,l/2)} \rightarrow \Phi_{\text{phys}}^{((k+l)/2, (k-l)/2)} = (i\Delta \hat{\theta})(k-l)/2 \Phi_{\text{phys}}^{(k/2,l/2)} \tag{102.a}
\]

\[
\Phi_{\text{phys}}^{((k+l)/2, (k-l)/2)} \rightarrow \Phi_{\text{phys}}^{(k/2,l/2)} = (i\Delta \hat{\theta})(k-l)/2 \Phi_{\text{phys}}^{((k+l)/2, (k-l)/2)} \tag{102.b}
\]

Furthermore, for \( k = l \) the physical subspaces are identified with \( \Phi_{\text{phys}}^{(0,0)} \) by means of the formula

\[
\Phi_{\text{phys}}^{(k/2,k/2)} \rightarrow \Phi_{\text{phys}}^{(0,0)} = (p, \xi)^{-1} \Phi_{\text{phys}}^{(k/2,k/2)} \tag{103}
\]

The last mapping is smooth except the point \( (p, \xi) = 0 \) and we restrict the functional space for arbitrary \( k, l \) by requiring the image of two consequent mappings (102.a), (103) to be a smooth function on \( S^2 \). For \( \Phi_{\text{phys}}^{(0,0)} = \Phi \), the Eqs. (100) take the form

\[
((p, \xi)^2 \partial_z \partial_{\bar{z}} + \Delta^2)\Phi = 0 \ , \ p^2 = 0 \tag{104}
\]

The smooth positive-frequency solutions to the Eq. (104) form a Hilbert space with respect to the measure (75) (wherein \( k = l = 0 \)) taken at \( m = 0 \):

\[
< \Phi | \Phi > = N \int \frac{d^3p}{p^0} \frac{dzd\bar{z}}{(p, \xi)^2} \Phi \bar{\Phi} , \ p^0 = \sqrt{p^2} \tag{105}
\]

Indeed, despite \( (p, \xi)^{-2} \) has the singular point, one may rewrite (105) using Eq. (94) as

\[
< \Phi | \Phi > = -\Delta^{-2} N \int \frac{d^3p}{p^0} dzd\bar{z} \bar{\Phi} \partial_z \partial_{\bar{z}} \Phi = \Delta^{-2} N \int \frac{d^3p}{p^0} dzd\bar{z} |\partial_z \Phi|^2 \tag{106}
\]

\[^4\text{In the distinction to the massive case the spherical metric } g_{zz} = 2(p, \xi)^{-2} \text{ is no longer smooth but has one singular point when } p_a \sim \xi_a. \text{ From Eq. (73) it follows that the curvature of the metric } g_{zz} \text{ equals zero. In the special reference system } p^a = (p^z, 0, 0, p^0) \text{ one finds the euclidean plane metric } g_{zz} = \frac{1}{p^0} \text{ and the corresponding singular point on } S^2 \text{ is identified with the infinitely removed point } z = \infty. \]

\[^5\text{It is also interesting to note that this case is characterized by the same constraints and possesses the same number of degrees of freedom as the massive one (i). Thus, it could be treated as a massless limit of the massive particle model (i).}\]
which is obviously well-defined since $\Phi$ is a smooth function on $S^2$. Hence, the Poincaré group representation, being realized on the $\Phi_{phys}^{(0,0)}$, is unitary. To prove the irreducibility, it is suitable to apply the Wigner method of a little group and to make sure that the representation of the momentum $p_a$ ($p^2 = 0$) stability subgroup (little group) is irreducible in the subspace $|p_a > \equiv \Phi_{phys}^{(0,0)}(p, z, \bar{z})$. The little group of a light-like vector is known to be isomorphic to the group $E(2)$ of motions of two-dimensional euclidean plane, and the apparent proof of the fact is that for the special $p_a$ of the form $p^a = (p^0, 0, 0, p^0)$ the metric becomes flat and the Eq. (104) takes the form of the ordinary one determining eigenfunctions for the Laplace operator on a plane,

$$ (\partial_z \partial_{\bar{z}} + \Delta^2)\Phi(p_a, z, \bar{z}) = 0 \quad (107) $$

The restriction of the norm (105) to the subspace $|p_a >$ yields the ordinary $L^2$-norm for the functions on a plane. The smooth solutions to Eq.(101), being normalizable with respect to $L^2$-norm, prove to form the $E_2$-irreducible infinite-dimensional Hilbert space and can be written in terms of the first-kind Bessel functions $J_\nu$, $\nu = 0, 1, 2, \ldots$, \cite{28} as follows

$$ \Phi(p_a, z, \bar{z}) = \sum_{\nu=0}^{\infty} c_\nu (-i)^\nu \left( \frac{z}{\bar{z}} \right)^\nu J_\nu(\Delta |z|) \quad (108.a) $$

where $c_\nu$ are complex numbers obeying the restriction

$$ \sum_{\nu=0}^{\infty} |c_\nu|^2 < \infty \quad (108.b) $$

This completes the proof that, in this case, the model describes the relativistic massless particle with continuous helicity $\Delta$. Note, that it is the same little group technique which reduces the proof of irreducibility and unitarity to the analogous proof for the stability subgroup representation, can be employed for all the cases (i),(ii),(iii), but we have preferred there to keep the manifest Poincaré–invariance for the reasons of aesthetics. As to the explicit Poincaré–covariant description for (iv), note, that, since the representation of the little group is infinite-dimensional, the description of the physical wave function in terms of Lorentz tensors is irrelevant and, thus, the approach used above seems to be the most appropriate to the case.

6 Conclusion

Let us briefly overview and comment the results. In this paper we have considered the most general Poincaré invariant mechanical system (without higher derivatives) with configuration space $\mathcal{M}^6$. We have shown, that the requirement of relativistic mass-shell conditions to arise in the theory, determines the reparametrization invariant action of the theory up to the four arbitrary constant parameters. The model can be conceptually treated as the universal model of relativistic spinning particle in $d = 4$ Minkowski space.

$^6$The stability subgroup of a momentum $p_a$ coincides with the invariance group of the Eq. (104) at fixed $p_a$. 

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Depending on a choice of the parameters, the model is characterized by a certain number of Hamiltonian constraints providing the identical (on the constraint surface) conservation of mass and spin (helicity) of the particle. On the other hand, the number and the structure of the arising constraints is precisely that which provides the correct number of physical degrees of freedom for the corresponding particle.

The remarkable feature of the model is the existence (in the massive case) of special points in the space of the model parameters where one of the first class constraints disappears, being substituted by two second class ones, that provides the different description of the same physical situation, i.e. the massive spinning particle. The significance of this special formulation becomes evident by the observation that there is the constraint structure in the case which can be noncontradictory deformed, and thereby the model allows to introduce a consistent interaction of the particle to arbitrary external electromagnetic and gravity fields. It should be mentioned that the proposed method leaves a wide freedom in the choice of possible consistent interactions. We have considered in more details the particular type of nonminimal interaction which leads to the description of a massive spinning particle with arbitrary fixed giromagnetic ratio.

It is interesting to note that for some special values of giromagnetic ratio \( g = 0, 2, -1 \) the interaction is simplified. The first case corresponds to the minimal coupling. The second one leads to the equations which, in the case of constant electromagnetic field, describe the conventional precession of spin tensor and four-velocity \( \dot{x}^a = 2\lambda \Pi^a \). Finally, putting \( g = -1 \) one finds that \( M^2 = m^2 \) that guarantees the non-singularity of the Dirac bracket (57) on the whole phase space of the system regardless of background field configurations. It should be noted that these values are special only from the standpoint of classical spinning particle dynamics and tell us nothing about the values of giromagnetic ratio in (unknown yet) consistent higher-spin field theories. In fact, a variety of interactions, being admissible in the classical regime for the particle, corresponds to the set of higher spin equations coupled to electromagnetic field. At the mean time, the free Lagrangian massive higher spin field theories contain a subtle auxiliary field structure \([32]\), that makes almost all these naive interactions to be contradictory. Therefore, the question of ‘the true’ giromagnetic ratio value may probably have another answer from the standpoint of the field theory.

We have thoroughly performed the covariant operatorial quantization of the model and have explicitly constructed the corresponding Hilbert spaces which are shown to coincide with the spaces of unitary irreducible representations of the Poincaré group.

The constructed model reveals the properties of universal and minimal model for \( D = 4 \) spinning particle. The model is universal since it’s configuration space is mass- and spin-independent; it is minimal since its configuration manifold \( \mathcal{M}^6 \) turns out to have the minimal possible dimension among the homogeneous spaces of the Poincaré group admitting a nontrivial dynamics for spin.

Perhaps, this universality can open a way to consider the joint dynamics of the higher-spin fields both at the classical and quantum levels. The promising example of such a description is that the relativistic harmonic analysis allows to treat a massive tensor field on \( \mathcal{M}^6 \) as a direct sum of massive Poincaré-irreducible spintensor fields on Minkowski space. If it is possible to find the analogous phenomena beyond the free level then one could try to construct the consistent interaction between higher spins in terms of tensor
The proposed model admits the generalization to arbitrary dimension. The basic problem is to choose an appropriate inner space. For example, for $D = 3$ the universal model of anyon has been derived with $S^1$ as the inner space [33]. For $D = 6$ the corresponding universal model could be constructed with the inner space being complex projective space $CP^3$ [34].

The possibility to introduce a consistent interaction in the massless case still remains in question within the proposed construction, since the constraint structure does not survive nontrivial deformations when the mass vanishes.

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