LARGE-TIME REGULAR SOLUTIONS TO THE MODIFIED QUASI-GEOSTROPHIC EQUATION IN BESOV SPACES

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Abstract. This paper is devoted to the study of the modified quasi-geostrophic equation

\[ \begin{align*}
\partial_t \theta + u \cdot \nabla \theta + \nu \Lambda^\alpha \theta &= 0, \\
u = \Lambda^\beta \mathcal{R}^\perp \theta, \quad \theta|_{t=0} = \theta_0
\end{align*} \]

in \( \mathbb{R}^2 \). By the Littlewood-Paley theory, we obtain the local well-posedness and the smoothing effect of the equation in critical Besov spaces. These results are applied to show the global existence of regular solutions for the critical case \( \beta = \alpha - 1 \) and the existence of regular solutions for large time \( t > T \) with respect to the supercritical case \( \beta > \alpha - 1 \) in Besov spaces. Earlier results for the equation in Hilbert spaces \( H^s \) spaces are improved.

1. Introduction. Consider the following modified quasi-geostrophic equation [21]

\[ \begin{align*}
\partial_t \theta + u \cdot \nabla \theta + \nu \Lambda^\alpha \theta &= 0, \\
u = \Lambda^\beta \mathcal{R}^\perp \theta, \quad \theta|_{t=0} = \theta_0
\end{align*} \]

in \( \mathbb{R}^2 \) for the parameters \( \nu > 0, \alpha \in (0, 2) \) and \( \beta \in (0, 1) \) and the operators \( \Lambda^\alpha = (-\Delta)^{\alpha/2} \) and \( \mathcal{R}^\perp = (-\partial_{x_2} \Lambda^{-1}, \partial_{x_1} \Lambda^{-1}) \). The analysis on (1) in Besov spaces is essentially based on the invariance property of (1) with respect to the scaling transformations

\[ \theta_\lambda = \lambda^{\alpha-\beta-1} \theta(\lambda x, \lambda^\alpha t) \quad \text{and} \quad u_\lambda = \lambda^{\alpha-1} (\lambda x, \lambda^\alpha t). \]

Equation (1) with \( \beta = 0 \) reduces to the celebrated quasi-geostrophic equation

\[ \begin{align*}
\partial_t \theta + u \cdot \nabla \theta + \nu \Lambda^\alpha \theta &= 0, \\
u = \mathcal{R}^\perp \theta, \quad \theta|_{t=0} = \theta_0
\end{align*} \]

Equation (3) at the critical case \( \alpha = 1 \) shares a dimensionally correct analogue with the three-dimensional Navier-Stokes equations [8, 20], although Leray’s global regularity problem [26] with respect to the Navier-Stokes equations still remains unsolved due to the absence of a maximum principle. In the past two decades, the global regularity problem of (3) has been extensively studied [1, 4, 6, 8, 10, 13, 14, 16, 17, 20, 30, 31]. In particular, for the subcritical case \( \alpha > \frac{1}{2} \), the existence of global regular solutions to (3) has been obtained [10, 31]. For the critical case \( \alpha = \frac{1}{2} \), it was obtained independently by Caffarelli and Vasseur [3] and Kiselev

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et al. [24]. The proof of [3] relies on the classical De Giorgi iteration, while the technique in [24] is known as the nonlocal maximum principle. Later, Kiselev and Nazarov [23] introduced the Hardy molecules method to bridge the examinations of [9] and [24]. Further approach called the nonlinear maximum principle was recently derived by Constantin and Vicol [9] in the understanding of the global regularity problem. For the supercritical case $0 < \alpha < 1$, the balance between the nonlinear convective term $u \cdot \nabla \theta$ and the dissipative term $\Lambda^\alpha \theta$ in the critical case $\alpha = 1$ is no longer valid. The latter becomes weaker than the former and thus all the methods on the critical case cannot be extended to the supercritical case. However, it has been shown that the supercritical equation of (3) admits the eventual regularity property (see [21, 22, 32] and references therein). That is, the solution initially smooth becomes regular after a large time $T > 0$ but the regularity behavior of the solution for moderate $t < T$ remains unsolved.

The proof of Caffarelli and Vasseur [3] in the critical case $\alpha = 1$ can be divided into three steps. Firstly, Leray-Hopf weak solution with $\theta_0 \in L_2(\mathbb{R}^2)$ is in $L_\infty(\mathbb{R}^2 \times (0, \infty))$; secondly, the $L_\infty$ solution is shown to be Hölder continuous by using the scaling invariance of $\|u_\lambda\|_{BMO}$; finally, the Hölder continuous solution is proved to be regular. In 2007, Constantin and Wu [12] examined whether the approach of [3] can be extended to (3) for $\alpha < 1$. They found that the first step is still valid. However, the second step failed because the BMO-norm of $u_\lambda$ is not preserved in the scaling process. In order to use the second step approach, they assumed additionally $u \in L_{\infty,loc}(0, \infty; C^{1-\alpha}(\mathbb{R}^2))$, and then proved that the Leray-Hopf weak solution $\theta \in C_{loc}^0(0, \infty; C^\delta(\mathbb{R}^2))$ for some $\delta > 0$ [12, Theorem 4.1]. Almost at the same time, Constantin and Wu [11] proved that for $\delta > 1 - \alpha$, the Leray-Hopf weak solution $\theta$ of (3) with $\alpha < 1$ belongs to $C^\infty((t_0, t) \times \mathbb{R}^2)$ if $\theta \in L_\infty([t_0, t]; C^\delta(\mathbb{R}^2))$. This result was later improved by Dong and Pavlović [18, 19] for the case $\delta \geq 1 - \alpha$. To remove the assumption $u \in L_{\infty,loc}(0, \infty; C^{1-\alpha}(\mathbb{R}^2))$, Constantin et al. [7] introduced the modified quasi-geostrophic equations (1) with $\beta = \alpha - 1$. Therefore, when the dissipation $\Lambda^\beta \theta$ of (1) with $\beta = \alpha - 1$ reduces from $\alpha = 1$ to $\alpha < 1$, the regularity of $u$ increase due to the following $L_p$ estimate

$$
\|\Lambda^{1-\alpha}u\|_{L_p} \leq c\|\theta\|_{L_p}, \quad 1 < p < \infty.
$$

Hence the velocity $u = \Lambda^{1-\alpha}R^{+}\theta$ is in $L_\infty(0, \infty; C^{1-\alpha}(\mathbb{R}^2))$ for the Leray-Hopf weak solution $\theta \in L_\infty(\mathbb{R}^2 \times (0, \infty))$ obtained by following the first step examination of [3]. Hence Constantin and Wu [12] derived that $\theta \in C_{loc}^\delta(0, \infty; C^\delta(\mathbb{R}^2))$ for some $\delta > 0$. Moreover, Constantin et al. [7] proved that $\theta \in L_{\infty,loc}(0, \infty; C^{\delta+\eta}(\mathbb{R}^2))$ for some $\eta > 0$ and hence the solution $\theta \in C^\infty(\mathbb{R}^2 \times (0, \infty))$ by induction [7, Theorems 2.2 and 3.5].

An interesting question arises as to whether the result of [7] still holds true for $\alpha = 1, 2$. This choice of $\alpha$ implies that the velocity $u = \Lambda^{\alpha-1}R^{+}\theta$ is more singular than $\theta$. Therefore, $u$ is not Hölder continuous when $\theta \in L_\infty(\mathbb{R}^2 \times (0, \infty))$. What is more, the regularity of the solution cannot be improved by the bootstrap argument presented in [7]. To answer this question, Miao and Xue [28, 29] proved the global existence of regular solutions to the critical and supercritical modified quasi-geostrophic equation respectively in the following sense.

**Theorem 1.1** (Miao and Xue [29]). Let $\alpha \in (1, 2)$ and $\theta_0 \in H^m(\mathbb{R}^2)$ with $m > 1$. Then (1) with the critical situation $\beta = \alpha - 1$ admits a unique solution

$$
\theta \in C([0, \infty); H^m(\mathbb{R}^2)) \cap L_{2,loc}([0, \infty); H^{m+\frac{2}{p}}(\mathbb{R}^2)) \cap C^\infty((0, \infty) \times \mathbb{R}^2).
$$
Theorem 1.2 (Miao and Xue [28]). For $\theta_0 \in H^m(\mathbb{R}^2)$ with $m > 2$ and the supercritical situation with respect to the parameter $\beta \in (0, 1)$ and $\alpha \in (2\beta, \beta + 1)$, assume that $\theta$ is a Leray-Hopf weak solution of (1). Then there exist positive constants $T_1 < T_2$ dependent upon the quantities $\alpha$, $\beta$, $\nu$ and $\theta_0$ such that

$$\theta \in C^\infty((0, T_1) \times \mathbb{R}^2) \cap C^\infty((T_2, \infty) \times \mathbb{R}^2).$$

Theorem 1.1 was improved by May [27] by assuming $\theta_0$ in the critical Hilbert space $H^1(\mathbb{R}^2)$. The demonstration of these theorems are based on the nonlocal maximum principle introduced by Kiselev et al. [24].

The purpose of the present paper is to investigate the existence of regular solutions to (1) in critical homogeneous Besov spaces with respect to the invariance of (1) under the scaling transformation (2). The well-posedness and the regularity of fluid flow motions in scaling critical spaces have been extensively studied (see, for example, [5, 25, 34] for incompressible Navier-Stokes equations and [1, 4, 6, 17, 20, 30] for (3). This is partially due to the fact that the solutions are expected to be regular in scaling critical spaces [15, 25].

For the readers’ convenience for the comparison with the existing results, the main results of the present paper read respectively as follows.

Theorem 1.3. Let $p, q \in [2, \infty)$, $\alpha \in (1, \frac{3}{2} + \frac{1}{p})$ and $\theta_0 \in B^\beta_{p,q}(\mathbb{R}^2)$. Then (1) with the critical case $\beta = \alpha - 1$ admits a unique global solution $\theta$ satisfying

$$\theta \in C([0, \infty); B^\beta_{p,q}(\mathbb{R}^2)) \cap L_{q,loc}([0, \infty); B^\beta_{p,q}(\mathbb{R}^2)) \cap C^\infty((0, \infty) \times \mathbb{R}^2).$$

Theorem 1.4. For $p, q \in [2, \infty)$, $\theta_0 \in B^{\frac{2}{p} + 1 + \beta - \alpha}_{p,q}(\mathbb{R}^2)$ and the supercritical situation $\beta \in (0, \frac{1}{2} + \frac{1}{p})$ and $\alpha \in (2\beta, \beta + 1)$, assume that $\theta$ is a Leray-Hopf weak solution of (1). Then there exist positive constants $T_1 < T_2$ dependent upon the quantities $p$, $q$, $\alpha$, $\beta$, $\nu$ and $\theta_0$ such that

$$\theta \in C^\infty((0, T_1) \times \mathbb{R}^2) \cap C^\infty((T_2, \infty) \times \mathbb{R}^2).$$

For the significance of the present study, Theorem 1.1 and the improved result of May [27] are extended by Theorem 1.3, while Theorem 1.2 is covered by Theorem 1.4, since, for $m > 1$,

$$H^m(\mathbb{R}^2) \subset H^1(\mathbb{R}^2) = B^2_{2,2}(\mathbb{R}^2), \quad H^{1+m}(\mathbb{R}^2) \subset H^{2+\beta - \alpha}(\mathbb{R}^2) = B^{2+\beta - \alpha}_{2,2}(\mathbb{R}^2).$$

On the other hand, Theorem 1.3 also improves the result of Yamazaki [35], where the global well-posedness of Equation (1) with $\beta = \alpha - 1$ is proved by a different method, which is independent of the work of May [27].

This paper is organized as follows. Section 2 contains basic properties of Littlewood-Paley theory and Besov spaces. In Section 3, we develop techniques from [6, 29, 30, 34] to derive local existence and uniqueness of solutions to (1) in critical Besov spaces. This local solution result generalizes the counterpart of [28, 29] in Hilbert spaces. In Section 4, by following the techniques of [29] with more careful manipulation, we show that the local solutions initially in Besov spaces given in Section 3 are smooth for $t > 0$. This result is an extension of the smoothing effect result of [16] in Hilbert spaces. Moreover, compared with the examination of Dong and Li [17] on the global existence of regular solutions to (3), our proof is delivered in a more simplified manner by skipping the use of an $L_\infty$ estimate of $\nabla \theta$ and $\nabla u$. This estimate plays a crucial role in [17]. Finally, in Section 5, we first prove Theorems 1.3; then Theorem 1.4 is derived in a straight-forward manner.
from Theorem 1.2 and the smoothing effect result obtained in Section 4 because the local solution in the Besov space \( B_{p,q}^s(\mathbb{R}^2) \) is smooth for \( t > 0 \) and hence is in the Hilbert spaces \( H^m(\mathbb{R}^2) \).

2. Preliminaries. Throughout this paper \( c \) represents a generic positive constant which is independent of the quantities \( t, T, x, f, g, u, \theta, j, k \) and \( q_1 \). For simplicity, \( A \lesssim B \) denotes the inequality \( A \leq cB \), and \( A \approx B \) stands for the combination of \( B \lesssim A \) and \( A \lesssim B \).

\( \mathcal{S}(\mathbb{R}^2) \) denotes the Schwartz space, \( \mathcal{S}'(\mathbb{R}^2) \) represents the space of tempered distributions, \( F \) is the Fourier transform and \( \Lambda^\alpha = F^{-1}[|\xi|^\alpha F] \). To define Besov spaces, we use the Littlewood-Paley dyadic decomposition (see, for example, [2]) by taking a positive function \( \phi \in \mathcal{S}(\mathbb{R}^2) \) such that

\[
\phi(\xi) = 0 \quad \text{for} \quad |\xi| > \frac{4}{3} \quad \text{and} \quad \phi(\xi) = 1 \quad \text{for} \quad |\xi| < \frac{3}{4}
\]

and the dyadic block symbols

\[
\phi_j(\xi) = \phi(2^{-j}\xi), \quad \psi_j(\xi) = \phi_{j+1}(\xi) - \phi_j(\xi) \quad \text{for} \quad j \in \mathbb{Z}.
\]

Thus Littlewood-Paley dyadic blocks are defined as

\[
S_j = F^{-1}\phi_jF, \quad \Delta_j = F^{-1}\psi_jF = S_{j+1} - S_j \quad \text{for} \quad j \in \mathbb{Z}.
\]

Thus we have Littlewood-Paley unit decompositions

\[
f = \sum_{j \geq 0} \Delta_jf + S_0f \quad \text{for} \quad f \in \mathcal{S}'(\mathbb{R}^2),
\]

\[
f = \sum_{j \in \mathbb{Z}} \Delta_j f \quad \text{for} \quad f \in \mathcal{S}'(\mathbb{R}^2)/\mathcal{P}(\mathbb{R}^2),
\]

where \( \mathcal{P}(\mathbb{R}^2) \) denotes the set of all polynomials over \( \mathbb{R}^2 \).

Definition 2.1. For \( s \in \mathbb{R} \) and \( p, q \in [1, \infty] \), the homogeneous Besov space \( \dot{B}_{p,q}^s(\mathbb{R}^2) \) is defined by

\[
\dot{B}_{p,q}^s(\mathbb{R}^2) = \left\{ f \in \mathcal{S}'(\mathbb{R}^2)/\mathcal{P}(\mathbb{R}^2); \quad \| f \|_{\dot{B}_{p,q}^s} < \infty \right\},
\]

where

\[
\| f \|_{\dot{B}_{p,q}^s} = \begin{cases} 
\left( \sum_{j \in \mathbb{Z}} 2^{jsq}\| \Delta_jf \|_{L_p}^q \right)^{\frac{1}{q}}, & \text{if} \quad q < \infty, \\
\sup_{j \in \mathbb{Z}} 2^{js}\| \Delta_jf \|_{L_p}, & \text{if} \quad q = \infty.
\end{cases}
\] (6)

Definition 2.2. For \( s \in \mathbb{R} \) and \( p, q \in [1, \infty] \), the Besov space \( B_{p,q}^s(\mathbb{R}^2) \) is defined as

\[
B_{p,q}^s(\mathbb{R}^2) = \left\{ f \in \mathcal{S}'(\mathbb{R}^2); \quad \| f \|_{B_{p,q}^s} < \infty \right\},
\]

where

\[
\| f \|_{B_{p,q}^s} = \begin{cases} 
\left( \sum_{j \geq 0} 2^{jsq}\| \Delta_jf \|_{L_p}^q \right)^{\frac{1}{q}} + \| S_0f \|_{L_p}, & \text{if} \quad q < \infty, \\
\sup_{j \geq 0} 2^{js}\| \Delta_jf \|_{L_p} + \| S_0f \|_{L_p}, & \text{if} \quad q = \infty.
\end{cases}
\] (7)

Note that \( B_{p,q}^s(\mathbb{R}^2) = \dot{B}_{p,q}^s(\mathbb{R}^2) \cap L_p(\mathbb{R}^2) \) for \( s > 0 \). For this case, the norm \( \| \cdot \|_{B_{p,q}^s} \) is equivalent to \( \| \cdot \|_{B_{p,q}^s} + \| \cdot \|_{L_p} \).
Definition 2.3. For $T > 0$, $s \in \mathbb{R}$ and $1 \leq p, q, \rho \leq \infty$. The homogeneous time-space Besov space is defined by

$$
\tilde{L}_\rho(0, T; \dot{B}_{p,q}^s(\mathbb{R}^2)) = \left\{ f \in \mathcal{D}(0, T; \mathcal{S}'(\mathbb{R}^2)/\mathcal{P}(\mathbb{R}^2)); \| f \|_{\tilde{L}_\rho(0, T; \dot{B}_{p,q}^s(\mathbb{R}^2))} < \infty \right\},
$$

where

$$
\| f \|_{\tilde{L}_\rho(0, T; \dot{B}_{p,q}^s(\mathbb{R}^2))} = \left( \sum_{j \in \mathbb{Z}} (2^j_s \| \Delta_j f \|_{L_\rho(0, T; L_p)})^q \right)^{\frac{1}{q}},
$$

and $\mathcal{D}(0, T; \mathcal{S}'(\mathbb{R}^2)/\mathcal{P}(\mathbb{R}^2))$ is the dual of $C_0^\infty((0, T); \mathcal{S}_0)$. Here $\mathcal{S}_0 = \{ f \in \mathcal{S} : \partial^\gamma F f(0) = 0$ for any multi-index $\gamma \in \mathbb{N}^2 \}$. It can be verified that the dual of $\mathcal{S}_0$ equals to $\mathcal{S}'(\mathbb{R}^2)/\mathcal{P}(\mathbb{R}^2)$.

By the Minkowski inequality, it is easy to clarify the embedding relations:

$$
\tilde{L}_\rho(0, T; \dot{B}_{p,q}^s(\mathbb{R}^2)) \hookrightarrow L_\rho(0, T; \dot{B}_{p,q}^s(\mathbb{R}^2)) \quad \text{if} \ \rho \geq q,
$$

$$
L_\rho(0, T; \dot{B}_{p,q}^s(\mathbb{R}^2)) \hookrightarrow \tilde{L}_\rho(0, T; \dot{B}_{p,q}^s(\mathbb{R}^2)) \quad \text{if} \ \rho \leq q.
$$

Similarly, we have the time-space Besov space $\tilde{L}_\rho(0, T; B_{p,q}^s(\mathbb{R}^2))$ and

$$
\| f \|_{\tilde{L}_\rho(0, T; B_{p,q}^s(\mathbb{R}^2))} \approx \| f \|_{\tilde{L}_\rho(0, T; \dot{B}_{p,q}^s(\mathbb{R}^2))} + \| f \|_{L_\rho(0, T; L_p)} \quad \text{if} \ s > 0. \quad (8)
$$

Proposition 2.4. (see, for example, [11, 34]) For $p, q \in [1, \infty]$, $s \in \mathbb{R}$ and $W_p^s(\mathbb{R}^2) \equiv (I - \Delta)^{\frac{s}{2}} L_p(\mathbb{R}^2)$, there hold the following embedding inequalities

$$
B_{p,q}^{s_1}(\mathbb{R}^2) \hookrightarrow B_{p,q}^{s_2}(\mathbb{R}^2), \quad s_1 \leq s_2,
$$

$$
B_{p,\rho}^s(\mathbb{R}^2) \hookrightarrow B_{p,q}^s(\mathbb{R}^2), \quad 1 \leq q_1 \leq q_2 \leq \infty,
$$

$$
B_{p,\rho}^{s_1}(\mathbb{R}^2) \hookrightarrow B_{p,\rho}^{s_2}(\mathbb{R}^2), \quad 1 \leq p_1 \leq p_2 \leq \infty, 1 \leq q_1, q_2 \leq \infty, s_1 > s_2 + \frac{2}{p_1} - \frac{2}{p_2},
$$

$$
B_{p,\min(p,2)}^s(\mathbb{R}^2) \hookrightarrow W_p^s(\mathbb{R}^2) \hookrightarrow B_{p,\max(p,2)}^s(\mathbb{R}^2), \quad 1 < p < \infty.
$$

We will also use the following Bernstein inequalities.

Proposition 2.5. For $s \geq 0$, $1 \leq p \leq q \leq \infty$, $j \in \mathbb{Z}$ and constants $K > 0$, $K_2 > K_1 > 0$, there hold the inequalities:

$$
\| \Lambda^s f \|_{L_q} \lesssim 2^{js + j(\frac{s}{2} - \frac{1}{4})} \| f \|_{L_p}, \quad (9)
$$

$$
2^{js} \| g \|_{L_q} \lesssim \| \Lambda^s g \|_{L_q} \lesssim 2^{js + j(\frac{s}{2} - \frac{1}{4})} \| g \|_{L_p}, \quad (10)
$$

if

$$
supp F f \subset \{ \xi \in \mathbb{R}^2; |\xi| \leq K2^j \} \text{ and supp } F g \subset \{ \xi \in \mathbb{R}^2; K_1 2^j \leq |\xi| \leq K_2 2^j \}.\n$$

To treat the fractional Laplacian in $L_p$ spaces, we need the following modified Bernstein’s inequalities.

Proposition 2.6. ([6]) Let $p \in [2, \infty)$, $s \in [0, 1]$, $f \in \mathcal{S}'(\mathbb{R}^2)$ and $j \in \mathbb{Z}$. Then we have

$$
2^{\frac{js}{p}} \| \Delta_j f \|_{L_p} \approx \| \Lambda^s (|\Delta_j f|^\frac{p}{2}) \|_{L_g^\frac{p}{2}}. \quad (11)
$$

The following pointwise multiplier estimate is a consequence of Bony’s decomposition (see, for example, [6]).

1

Proposition 2.7. For $s > \frac{1}{2}$, $2 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $f, g \in B_{p,q}^{\frac{1}{2} + s}(\mathbb{R}^2)$, we have

$$
\| fg \|_{B_{p,q}^s} \lesssim \| f \|_{B_{p,q}^s} \| g \|_{L_p} + \| f \|_{B_{p,q}^s} \| g \|_{L_p}. \quad (12)
$$
3. Local well-posedness in critical Besov spaces. This section is devoted to the following existence and uniqueness of local solutions to (1). Our analysis is developed from [6, 29, 30, 34].

**Theorem 3.1.** Let \( p \in [2, \infty), q \in [1, \infty), q_1 \in [1, \infty], \beta \in (0, \frac{1}{3} + \frac{1}{p}), \alpha \in (2\beta, \frac{2}{p} + 1) \) and \( \theta_0 \in B^\sigma_{p,q}(\mathbb{R}^2) \). Then there exists a positive constant \( T_0 \) such that (1) admits a unique solution \( \theta \) satisfying

\[
\theta \in C([0, T_0); B^\sigma_{p,q}(\mathbb{R}^2)) \cap \tilde{L}^{q_1}_{q_1}(0, T_0; B^{\sigma + \frac{3}{p}}_{p,q}(\mathbb{R}^2)) \quad \text{for} \quad \sigma = \frac{2}{p} + 1 + \beta - \alpha. \tag{13}
\]

**Proof.** This theorem is shown through four steps.

Firstly, we show an a priori estimate for the solution \( \theta \) in the space \( \tilde{L}^{q_1}_{q_1}(0, T; \dot{B}^{\tau + \frac{3}{p}}_{p,q}(\mathbb{R}^2)) \) for \( T > 0 \) and \( \tau > \beta - \sigma - \frac{2\beta}{q_1} \).

For \( \theta_0 \in B^\sigma_{p,q}(\mathbb{R}^2) \cap \dot{B}^\tau_{p,q}(\mathbb{R}^2) \), applying the operator \( \Delta_j \) to (1) and using Bony’s decomposition of the nonlinear term, we have

\[
\partial_t \Delta_j \theta + \nu \Lambda^n \Delta_j \theta = - \Delta_j (u \cdot \nabla \theta) = - \sum_{|k-j| \leq 2} (\Delta_j (S_{k-1} u \cdot \Delta_k \nabla \theta) - S_{k-1} u \cdot \Delta_j \Delta_k \nabla \theta) - \sum_{k \geq j-3} \Delta_j (\Delta_k u \cdot \Delta_k \nabla \theta) - \sum_{|k-j| \leq 2} (S_{k-1} u - S_j u) \cdot \Delta_j \Delta_k \nabla \theta - S_j u \cdot \Delta_j \nabla \theta = \sum_{i=1}^5 J_i. \tag{14}
\]

Multiplying (14) by \( p |\Delta_j \theta|^{p-2} \Delta_j \theta \) and then integrating in \( \mathbb{R}^2 \), we use Proposition 2.6 and the divergence free condition \( \nabla \cdot u = 0 \) to produce

\[
\frac{d}{dt} ||\Delta_j \theta||^p_{L^p} + c2^j \nu \||\Delta_j \theta||^p_{L^p} \leq \left| \int_{\mathbb{R}^2} (J_1 + J_2 + J_3 + J_4) |\Delta_j \theta|^{p-2} \Delta_j \theta dx \right|. \tag{15}
\]

By Hölder inequality, (15) becomes the inequality with the common factor \( ||\Delta_j \theta||^{p-1}_{L^p} \), which can be removed. Thus (15) reduces to

\[
\frac{d}{dt} ||\Delta_j \theta||_{L^p} + c2^{j\nu} ||\Delta_j \theta||_{L^p} \leq ||J_1||_{L^p} + ||J_2||_{L^p} + ||J_3||_{L^p} + ||J_4||_{L^p}. \tag{16}
\]

By the \( L^p \) commutator estimate [34], Hölder and Bernstein inequalities, the right-hand side of (16) is bounded by

\[
||\Delta_j \theta||_{L^p} \sum_{k \leq j-2} 2^{k(\frac{3}{p} + 1 + \beta)} ||\Delta_k \theta||_{L^p} + 2^{j\beta} ||\Delta_j \theta||_{L^p} \sum_{k \leq j-2} 2^{k(\frac{3}{p} + 1)} ||\Delta_k \theta||_{L^p} + 2^{j(\frac{3}{p} + 1 + \beta)} ||\Delta_j \theta||^2_{L^p}. \tag{17}
\]

Since the last term can be absorbed by the third term in (17), thus (16) is formulated as

\[
\frac{d}{dt} ||\Delta_j \theta||_{L^p} + c2^{j\nu} ||\Delta_j \theta||_{L^p} \leq ||\Delta_j \theta||_{L^p} \sum_{k \leq j-2} 2^{k(\frac{3}{p} + 1 + \beta)} ||\Delta_k \theta||_{L^p} + 2^{j(\frac{3}{p} + 1)} ||\Delta_j \theta||_{L^p} + 2^{j(\frac{3}{p} + 1)} \sum_{k \leq j-3} 2^{k\beta} ||\Delta_k \theta||^2_{L^p}. \]
By Gronwall inequality, the previous equation yields

$$
\|\Delta_j \theta\|_{L^p} \lesssim \int_0^t e^{-c2^{j\alpha}(t-s)} \|\Delta_j \theta(s)\|_{L^p} \sum_{k \leq j-2} 2^{k(\frac{\beta}{q}+1)} \|\Delta_k \theta(s)\|_{L^p} ds 
+ \int_0^t e^{-c2^{j\alpha}(t-s)} 2^{j\beta} \|\Delta_j \theta(s)\|_{L^p} \sum_{k \leq j-2} 2^{k(\frac{\beta}{q}+1)} \|\Delta_k \theta(s)\|_{L^p} ds 
+ \int_0^t e^{-c2^{j\alpha}(t-s)} 2^{j(\frac{\beta}{q}+1)} \sum_{k \geq j-3} 2^{k\beta} \|\Delta_k \theta(s)\|_{L^p}^2 ds + e^{-c2^{j\alpha}t} \|\Delta_j \theta(0)\|_{L^p}.
$$

(18)

Taking the $L_{q_1}(0,T)$ norm of (18), multiplying by $2^{j(\tau + \frac{\alpha}{q})}$ to the resultant equation and then taking $l^q(\mathbb{Z})$ norm, we obtain the required estimate in the space $\tilde{L}_{q_1}(0,T; \dot{B}^{s+\frac{\alpha}{q}}_{p,q} (\mathbb{R}^2))$:

$$
\|\theta\|_{\tilde{L}_{q_1}(0,T; \dot{B}^{s+\frac{\alpha}{q}}_{p,q})} \lesssim \sum_{j \in \mathbb{Z}} 2^{j(\tau + \frac{\alpha}{q})} \|\Delta_j \theta(0)\|_{L^p} \|e^{-c2^{j\alpha}t}\|_{L^q_{q_1}(0,T)}^q 
+ \sum_{j \in \mathbb{Z}} 2^{j(\tau + \frac{\alpha}{q})} I_j + \sum_{j \in \mathbb{Z}} 2^{j(\tau + \frac{\alpha}{q})} II_j + \sum_{j \in \mathbb{Z}} 2^{j(\tau + \frac{\alpha}{q})} III_j,
$$

(19)

with

$$
I_j = \left\| \int_0^t e^{-c2^{j\alpha}(t-s)} \|\Delta_j \theta(s)\|_{L^p} \sum_{k \leq j-2} 2^{k(\frac{\beta}{q}+1)} \|\Delta_k \theta(s)\|_{L^p} ds \right\|_{L^q_{q_1}(0,T)},
$$

$$
II_j = \left\| \int_0^t e^{-c2^{j\alpha}(t-s)} 2^{j\beta} \|\Delta_j \theta(s)\|_{L^p} \sum_{k \leq j-2} 2^{k(\frac{\beta}{q}+1)} \|\Delta_k \theta(s)\|_{L^p} ds \right\|_{L^q_{q_1}(0,T)},
$$

$$
III_j = \left\| \int_0^t e^{-c2^{j\alpha}(t-s)} 2^{j(\frac{\beta}{q}+1)} \sum_{k \geq j-3} 2^{k\beta} \|\Delta_k \theta(s)\|_{L^p}^2 ds \right\|_{L^q_{q_1}(0,T)}.
$$

To complete the estimate of (19), it is convenient to examine the cases $q_1 \in [2, \infty)$ and $q_1 \in [1, \infty)$ separately.

Secondly, we consider the case $q_1 \in [2, \infty)$ to obtain the estimate

$$
\|\theta\|_{\tilde{L}_{q_1}(0,T; \dot{B}^{s+\frac{\alpha}{q}}_{p,q})} \lesssim \sum_{j \in \mathbb{Z}} (1 - e^{-c2^{j\alpha}q_1T})^\frac{\alpha}{q_1} 2^{j\tau q_1} \|\Delta_j \theta(0)\|_{L^p}^q 
+ \|\theta\|_{\tilde{L}_{q_1}(0,T; \dot{B}^{s+\frac{\alpha}{q}}_{p,q})} \|\theta\|_{L^q_{q_1}(0,T; \dot{B}^{s+\frac{\alpha}{q}}_{p,q})}^q
$$

(20)

for $\tau > \beta - \sigma - \frac{2\alpha}{q_1}$ given in the first step.

It is readily seen that the first term on the righthand side of (19) is estimated as

$$
\sum_{j \in \mathbb{Z}} 2^{j(\tau + \frac{\alpha}{q_1})} \|\Delta_j \theta(0)\|_{L^p} \|e^{-c2^{j\alpha}t}\|_{L^q_{q_1}(0,T)}^q \lesssim \sum_{j \in \mathbb{Z}} (1 - e^{-c2^{j\alpha}q_1T})^\frac{\alpha}{q_1} 2^{j\tau} \|\Delta_j \theta(0)\|_{L^p}^q.
$$

(21)

To consider the other terms on the right-hand side of (19), we use Young inequality and the condition $q_1 \geq 2$ to obtain

$$
I_j \leq \|e^{-c2^{j\alpha}t}\|_{L^\infty_{q_1}(0,T)} \|\Delta_j \theta\|_{L^p} \sum_{k \leq j-2} 2^{k(\frac{\beta}{q}+1)} \|\Delta_k \theta\|_{L^p} 
\lesssim 2^{j\alpha(1 - \frac{1}{q_1})q} (1 - e^{-c2^{j\alpha}T}) (1 - \frac{1}{q_1})q.
$$
\[
\cdot \| \Delta \theta \|_{L_q(0,T;L_p)}^q \left( \sum_{k \geq j-2} 2^{k(\frac{\sigma + 2\alpha}{\alpha})^q} \| \Delta_k \theta \|_{L_q(0,T;L_p)} \right)^{q}
\lesssim \| \Delta \theta \|_{L_q(0,T;L_p)}^q \left( \sum_{k \geq j-2} 2^{k(\sigma + 2\alpha)} \| \Delta_k \theta \|_{L_q(0,T;L_p)} \right)^{q} \lesssim \| \Delta \theta \|_{L_q(0,T;L_p)}^q \sum_{k \geq j-2} 2^{k(\sigma + \frac{2\alpha}{\alpha})^q} \| \Delta_k \theta \|_{L_q(0,T;L_p)}^q \lesssim \| \Delta \theta \|_{L_q(0,T;L_p)}^q \| \theta \|_{L_q(0,T;B^{s+\frac{\alpha}{\alpha}})}^q.
\]

This implies
\[
\sum_{j \in \mathbb{Z}} 2^{j(\tau + \frac{2\alpha}{\alpha})^q} I_j \lesssim \| \theta \|_{L_q(0,T;B^{\tau + \frac{2\alpha}{\alpha}})}^q \| \theta \|_{L_q(0,T;B^{s+\frac{\alpha}{\alpha}})}^q. \tag{22}
\]

Note that the condition \( \alpha > 2 \beta > 0 \) and \( q_1 \geq 2 \) implies that \( \alpha > (1 - \frac{1}{q_1})^{-1} \beta \).

Similarly to estimate of \( I_j \), applying Young's inequality and Hölder inequality, we deduce that
\[
II_j \lesssim 2^{j(\tau + 2\alpha(T - 1 - \frac{1}{q_1})-\beta)}(1 - \frac{1}{q_1})^{\frac{1}{q_1}} \sum_{k \geq j-2} 2^{k(\frac{\sigma + 2\alpha}{\alpha})^q} \| \Delta_k \theta \|_{L_q(0,T;L_p)}^q \lesssim \| \Delta \theta \|_{L_q(0,T;B^{s+\frac{\alpha}{\alpha}})}^q \sum_{k \geq j-2} 2^{k(\sigma + \frac{2\alpha}{\alpha})^q} \| \Delta_k \theta \|_{L_q(0,T;L_p)}^q \lesssim \| \Delta \theta \|_{L_q(0,T;L_p)}^q \| \theta \|_{L_q(0,T;B^{s+\frac{\alpha}{\alpha}})}^q.
\]

This yields that
\[
\sum_{j \in \mathbb{Z}} 2^{j(\tau + \frac{2\alpha}{\alpha})^q} II_j \lesssim \| \theta \|_{L_q(0,T;B^{\tau + \frac{2\alpha}{\alpha}})}^q \| \theta \|_{L_q(0,T;B^{s+\frac{\alpha}{\alpha}})}^q \tag{23}
\]

The last term on the right-hand side of (19) can also be estimated in a similar way.

By Young inequality, Hölder inequality and the condition \( q_1 \geq 2 \), we find that
\[
III_j \lesssim 2^{j(\tau + 2\alpha(T - 1 - \frac{1}{q_1})-\beta)}(1 - \frac{1}{q_1})^{\frac{1}{q_1}} \sum_{k \geq j-3} 2^{k(\sigma + \frac{2\alpha}{\alpha})^q} \| \Delta_k \theta \|_{L_q(0,T;L_p)}^q \lesssim 2^{j(\tau + \sigma + \beta + \frac{2\alpha}{\alpha})^q} \left( \sum_{k \geq j-3} 2^{k(\sigma + \frac{2\alpha}{\alpha})^q} \| \Delta_k \theta \|_{L_q(0,T;L_p)}^q \right)^{\frac{1}{q}} \lesssim 2^{j(\tau + \sigma + \beta + \frac{2\alpha}{\alpha})^q} \left( \sup_{k \geq j-3} 2^{k(\tau + \sigma + \beta + \frac{2\alpha}{\alpha})^q} \| \Delta_k \theta \|_{L_q(0,T;L_p)} \right). \]

This together with Young inequality gives that
\[
\sum_{j \in \mathbb{Z}} 2^{j(\tau + \sigma + \beta + \frac{2\alpha}{\alpha})^q} III_j \lesssim \| \theta \|_{L_q(0,T;B^{s+\frac{\alpha}{\alpha}})}^q \cdot \sum_{j \in \mathbb{Z}} 2^{j(\tau + \sigma + \beta + \frac{2\alpha}{\alpha})^q} \left( \sum_{k \geq j-3} 2^{k(\sigma + \frac{2\alpha}{\alpha})^q} \| \Delta_k \theta \|_{L_q(0,T;L_p)} \right)^{\frac{1}{q}} \lesssim \| \theta \|_{L_q(0,T;B^{s+\frac{\alpha}{\alpha}})}^q \| \theta \|_{L_q(0,T;B^{s+\frac{\alpha}{\alpha}})}^q \tag{24}
\]

where we have used the condition \( \tau + \sigma + \beta + \frac{2\alpha}{q_1} > 0 \).
Hence we obtain (20) by combining (19) and (21)-(24).

Equation (20) with $q_1 \to \infty$ implies the estimate:

$$\|\theta\|_{L_2(0,T;B^{\tau+\alpha}_{p,q})}^q \lesssim \|\theta_0\|_{B^{\tau+\alpha}_{p,q}}^q + \|\theta\|_{L_2(0,T;B^{\tau+\alpha}_{p,q})}^q$$

However, the second term on the right-hand side of (25) does not vanish as $T$ tends to 0. Thus, to show the local existence result of (1) in the space $\tilde{L}_\infty(T;\tilde{B}^{\tau}_{p,q})$, it is necessary to derive an additional estimate in the next step analysis.

Thirdly, for $q_1 \in [1, \infty]$ and $\tau + \sigma - \beta + \alpha > 0$, we show the a priori estimate

$$\|\theta\|_{L_4(0,T;B^{\tau+\alpha}_{p,q})}^q \lesssim \|\theta_0\|_{B^{\tau+\alpha}_{p,q}}^q + \|\theta\|_{L_2(0,T;B^{\tau+\alpha}_{p,q})}^q.$$

We begin with $q_1 = 1$. Similar to the derivation of (20), we use Young inequality, Hölder inequality and the condition $\alpha > 2\beta$ to obtain

$$I_j \leq \|e^{-c2^{j\alpha}t}\|_{L_1(0,T)}^q \|\Delta_j \theta(t)\|_{L_p}^q \sum_{k \leq j-2} 2^{k(\tau+1+\beta)} \|\Delta_k \theta(t)\|_{L_p}^q$$

$$\lesssim 2^{-j\alpha q}(1 - e^{-c2^{j\alpha}T})^q \|\Delta_j \theta\|_{L_2(0,T;L_p)}^q \left( \sum_{k \leq j-2} 2^{k(\tau+1+\beta)} \|\Delta_k \theta\|_{L_2(0,T;L_p)}^q \right)^{q}$$

$$\lesssim 2^{-j\alpha q} \|\Delta_j \theta\|_{L_2(0,T;L_p)}^q \|\theta\|_{L_2(0,T;B^{\tau+\alpha}_{p,q})}^q.$$

$$II_j \lesssim 2^{2(\beta-\alpha)q}(1 - e^{-c2^{j\alpha}T})^q \|\Delta_j \theta\|_{L_2(0,T;L_p)}^q \left( \sum_{k \leq j-2} 2^{k(\tau+1)} \|\Delta_k \theta\|_{L_2(0,T;L_p)}^q \right)^{q}$$

$$\lesssim 2^{-j\alpha q} \|\Delta_j \theta\|_{L_2(0,T;L_p)}^q \|\theta\|_{L_2(0,T;B^{\tau+\alpha}_{p,q})}^q.$$

$$III_j \lesssim 2^{2(\tau+1-\alpha)q}(1 - e^{-c2^{j\alpha}T})^q \left( \sum_{k \geq j-3} 2^{k\beta} \|\Delta_k \theta\|_{L_2(0,T;L_p)}^2 \right)^{q}$$

$$\lesssim 2^{2(\sigma-\beta)q} \left( \sup_{k \in \mathbb{Z}} 2^{k(\tau+\frac{1}{2})} \|\Delta_k \theta\|_{L_2(0,T;L_p)} \sum_{k \geq j-3} 2^{k(\beta-\tau+\frac{1}{2})} \|\Delta_k \theta\|_{L_2(0,T;L_p)} \right)^{q}.$$

Collecting the above terms and (21) with $q_1 = 1$, we obtain from (19) that

$$\|\theta\|_{L_1(0,T;B^{\tau+\alpha}_{p,q})}^q \lesssim \sum_{j \in \mathbb{Z}} (1 - e^{-c2^{j\alpha}T})^q 2^{j\tau q} \|\Delta_j \theta_0\|_{L_p}^q + \sum_{j \in \mathbb{Z}} 2^{j(\tau+\alpha)q}(I_j + II_j + III_j)$$

$$\lesssim \sum_{j \in \mathbb{Z}} (1 - e^{-c2^{j\alpha}T})^q 2^{j\tau q} \|\Delta_j \theta_0\|_{L_p}^q + \|\theta\|_{L_2(0,T;B^{\tau+\alpha}_{p,q})}^q \|\theta\|_{L_2(0,T;B^{\tau+\alpha}_{p,q})}^q$$

$$\lesssim \|\theta_0\|_{B^{\tau+\alpha}_{p,q}}^q + \|\theta\|_{L_2(0,T;B^{\tau+\alpha}_{p,q})}^q \|\theta\|_{L_2(0,T;B^{\tau+\alpha}_{p,q})}^q.$$

(27)

In the same way, we have , for $q_1 = \infty$,

$$\|\theta\|_{L_\infty(0,T;B^{\tau+\alpha}_{p,q})}^q \lesssim \|\theta_0\|_{B^{\tau+\alpha}_{p,q}}^q + \|\theta\|_{L_2(0,T;B^{\tau+\alpha}_{p,q})}^q \|\theta\|_{L_2(0,T;B^{\tau+\alpha}_{p,q})}^q.$$

Therefore, (26) is obtained after the use of the interpolation inequality between $L_1$ and $L_\infty$.

Finally, we use the Banach contraction mapping principle to show the local existence and the uniqueness assertion for $1 \leq q \leq \infty$ and $\theta_0 \in B^{\tau+\alpha}_{p,q}(\mathbb{R}^2)$. 

We first consider $2 \leq q_1 < \infty$. Define the operator $M_{\theta_0}$ so that $M_{\theta_0}\theta$ solves the linear equation
\[
\begin{align*}
\partial_t M_{\theta_0}\theta + \nu \Lambda^\alpha M_{\theta_0}\theta &= -u \cdot \nabla \theta, \\
u = \Lambda^\beta R^\perp \theta, \quad M_{\theta_0}\theta|_{t=0} = \theta_0.
\end{align*}
\]
Note that $\beta > 0$, $\alpha < \frac{2}{p} + 1$ and $\sigma = \frac{2}{p} + 1 + \beta - \alpha$. It is easy to see that $2\sigma - \beta + \frac{2\alpha}{q_1} > \beta + \frac{2\alpha}{q_1}$ by setting $\tau = \sigma$. Therefore, Equation (20) together with its derivation shows the existence of a constant $c_0$ such that
\[
\|M_{\theta_0}\theta\|_{L^q(0,T;\tilde{B}^\sigma_{p,q})} \leq c_0 \sum_{j \in \mathbb{Z}} (1 - e^{-c_2^{2j\sigma} q_1 T}) \frac{n}{\pi} 2^{j\sigma q} \|\Delta_j \theta_0\|_{L^p} + c_0 \|\theta\|_{L^q(0,T;\tilde{B}^\sigma_{p,q})}.
\]
(28)

Now we define the complete metric space
\[
X_{T,\theta_0,q_1} = \{ \theta \in \tilde{L}_{q_1}(0,T;\tilde{B}^\sigma_{p,q}) (\mathbb{R}^2); \|\theta\|_{L^q(0,T;\tilde{B}^\sigma_{p,q})} \leq (c_0 + 1) \sum_{j \in \mathbb{Z}} (1 - e^{-c_2^{2j\sigma} q_1 T}) \frac{n}{\pi} 2^{j\sigma q} \|\Delta_j \theta_0\|_{L^p} \}.
\]
Therefore, for $\theta \in X_{T,\theta_0,q_1}$, it follows from (28) that
\[
\|M_{\theta_0}\theta\|_{L^q(0,T;\tilde{B}^\sigma_{p,q})} \leq c_0 \sum_{j \in \mathbb{Z}} (1 - e^{-c_2^{2j\sigma} q_1 T}) \frac{n}{\pi} 2^{j\sigma q} \|\Delta_j \theta_0\|_{L^p} + c_0 (c_0 + 1)^2 \left( \sum_{j \in \mathbb{Z}} (1 - e^{-c_2^{2j\sigma} q_1 T}) \frac{n}{\pi} 2^{j\sigma q} \|\Delta_j \theta_0\|_{L^p} \right)^2 \leq (c_0 + 1) \sum_{j \in \mathbb{Z}} (1 - e^{-c_2^{2j\sigma} q_1 T}) \frac{n}{\pi} 2^{j\sigma q} \|\Delta_j \theta_0\|_{L^p},
\]
(29)
provided that $T > 0$ is sufficient small. This gives the injection property
\[
M_{\theta_0} : X_{T,\theta_0,q_1} \hookrightarrow X_{T,\theta_0,q_1}.
\]
Moreover, for $\theta, \theta' \in X_{T,\theta_0,q_1}$, the difference $M_{\theta_0}\theta - M_{\theta_0}\theta'$ solves the linear equation
\[
\begin{align*}
(\partial_t + \nu \Lambda^\alpha)(M_{\theta_0}\theta - M_{\theta_0}\theta') &= -(u - u') \cdot \nabla \theta - u' \cdot \nabla (\theta - \theta'), \\
u = \Lambda^\beta R^\perp \theta, \quad u = \Lambda^\beta R^\perp \theta', \quad (M_{\theta_0}\theta - M_{\theta_0}\theta')|_{t=0} = 0.
\end{align*}
\]
Equation (20) with $\tau = \sigma$ together with its derivation shows
\[
\|M_{\theta_0}\theta - M_{\theta_0}\theta'\|_{L^q(0,T;\tilde{B}^\sigma_{p,q})} \leq 2c_0 (c_0 + 1) \sum_{j \in \mathbb{Z}} (1 - e^{-c_2^{2j\sigma} q_1 T}) \frac{n}{\pi} 2^{j\sigma q} \|\Delta_j \theta_0\|_{L^p} \|\theta - \theta'\|_{L^q(0,T;\tilde{B}^\sigma_{p,q})} \leq \frac{1}{2} \|\theta - \theta'\|_{L^q(0,T;\tilde{B}^\sigma_{p,q})},
\]
provided that the constant $T > 0$ is sufficiently small. Therefore, there exits a constant $T > 0$ so that $M_{\theta_0} : X_{T,\theta_0,q_1} \hookrightarrow X_{T,\theta_0,q_1}$ is a contraction mapping. By the
Banach contraction mapping principle, $M_{\theta_0}$ admits a unique fixed point $\theta \in X_{T,\theta_0,q_1}$ and hence $\theta$ solves (1) and satisfies the estimate

$$
\|\theta\|_{L^q_{q_1}(0,T;B_{p,q}^{s+\alpha})} \leq (c_0 + 1) \sum_{j \in \mathbb{Z}} (1 - e^{-c_2\alpha^q_{q_1} q T}) \frac{\tilde{\pi}}{2} 2^{j\sigma q} \|\Delta_j \theta_0\|_{L^q_p}, \quad 2 \leq q_1 < \infty. \quad (30)
$$

For any $1 \leq q_1 \leq \infty$, we know from (30) that (1) admits a unique solution $\theta \in X_{T,\theta_0,2}$. Moreover, by (26),

$$
\|\theta\|_{L^q_{q_1}(0,T;B_{p,q}^{s+\alpha})} \lesssim \|\theta_0\|_{B_{p,q}^{s+\alpha}}, \quad 1 \leq q_1 \leq \infty.
$$

This, together with the $L_p$ estimate of (1) (see [13]), implies

$$
\|\theta\|_{L^q_{q_1}(0,T;B_{p,q}^{s+\alpha})} \lesssim \|\theta_0\|_{B_{p,q}^{s+\alpha}}, \quad 1 \leq q_1 \leq \infty. \quad (32)
$$

To prove that $\theta \in C([0,T); B_{p,q}^\sigma(\mathbb{R}^2))$, we first show that $\Delta_j \theta \in C([0,T); B_{p,q}^\sigma(\mathbb{R}^2))$ for all $j_0 \in \mathbb{Z}$. By the equation

$$
\partial_t \Delta_j \theta - \Delta_j (u \cdot \nabla \theta) - \nu \Lambda^\sigma \Delta_j \theta = -\Delta_j \theta,
$$

Bernstein inequality, Young inequality and the embedding

$$
B_{p,q}^\sigma(\mathbb{R}^2) \hookrightarrow W_p^\beta(\mathbb{R}^2) \quad \text{as} \quad \sigma = \frac{2}{p} + 1 + \beta - \alpha,
$$

we deduce that

$$
\|\partial_t \Delta_j \theta\|_{L^\infty(0,T;B_{p,q}^\sigma)} \leq \|\Delta_j \theta\|_{L^\infty(0,T;B_{p,q}^\sigma)} + \nu \|\Lambda^\sigma \Delta_j \theta\|_{L^\infty(0,T;B_{p,q}^\sigma)}
$$

$$
\lesssim \left( \sum_{k = j+2}^{j+2} 2^{k\sigma q} \|\Delta_k \nabla \theta\|_{L^q_{q_1}(0,T;L^p)} \right)^{1\over 2} + \|\Lambda^\sigma \Delta_j \theta\|_{L^\infty(0,T;L^p)}
$$

$$
\lesssim \|\Delta_j \theta\|_{L^\infty(0,T;L^p)} + \|\Delta_j \theta\|_{L^\infty(0,T;L^{p/2})} + \|\Delta_j \theta\|_{L^\infty(0,T;L^p)}
$$

$$
\lesssim \|\theta\|_{L^\infty(0,T;W_{p/q}^\sigma)} + \|\theta\|_{L^\infty(0,T;L^p)} + \|\theta\|_{L^\infty(0,T;L^p)}
$$

$$
\lesssim \|\theta\|_{L^\infty(0,T;B_{p,q}^\sigma)} + \|\theta\|_{L^\infty(0,T;B_{p,q}^\sigma)} < \infty.
$$

Since $L^\infty(0,T;B_{p,q}^\sigma(\mathbb{R}^2)) \hookrightarrow L^\infty(0,T;B_{p,q}^\sigma(\mathbb{R}^2))$, by Sobolev embedding theorem, we have $\Delta_j \theta \in C([0,T); B_{p,q}^\sigma(\mathbb{R}^2))$.

Now we prove that $\theta \in C([0,T); B_{p,q}^\sigma(\mathbb{R}^2))$. In view of (32), we deduce that

$$
\lim_{j_0 \to \infty} \|\theta - \sum_{|j| \leq j_0} \Delta_j \theta\|_{L^\infty(0,T;B_{p,q}^\sigma)} = \lim_{j_0 \to \infty} \|\sum_{|j| > j_0} \Delta_j \theta\|_{L^\infty(0,T;B_{p,q}^\sigma)} = 0.
$$

Therefore, we obtain $\theta \in C([0,T); B_{p,q}^\sigma(\mathbb{R}^2))$ and hence the validity of (13) due to (32). The proof of Theorem 3.1 is completed. □
4. The smoothing effect of the local solution. The purpose of this section is to show the following regularity of local solutions to (1). Our proof is based on the induction that developed from [29].

**Theorem 4.1.** Let \( p, q \in [2, \infty), \gamma > 0, \beta \in (0, \frac{1}{2} \left( \frac{2}{p} + 1 \right)), \alpha \in (2\beta, \frac{3}{2} + 1), \sigma = \frac{\alpha}{p} + 1 + \beta - \alpha \) and \( \theta_0 \in B_{p,q}^\beta(\mathbb{R}^2) \). Assume that \( \theta \in C([0,T_0]; B_{p,q}^\beta(\mathbb{R}^2)) \cap L_q(0,T_0; B_{p,q}^{\frac{\alpha}{p} + 2}(\mathbb{R}^2)) \) is the local solution of (1) derived from Theorem 3.1. Then \( t^\gamma \theta \in L_\infty(0,T_0; B_{p,q}^{\frac{\alpha}{p} + \gamma}(\mathbb{R}^2)) \) satisfies the bound

\[
\sup_{t \in (0,T_0)} \|t^\gamma \theta(t)\|_{B_{p,q}^{\frac{\alpha}{p} + \gamma}} + \int_0^{T_0} \|s^\gamma \theta(s)\|_{B_{p,q}^{\frac{\alpha}{p} + \gamma}} ds \lesssim e^{\gamma \theta_0} \|B_{p,q}^{\gamma} \|.
\]  

(33)

The result stated in Theorem 4.1 is also valid for \( \alpha = 0 \). For this case, the index \( q \) can be extended to \( q \in [1, \infty) \).

**Proof.** It is easy to see that \( t^\gamma \theta \) is a solution of the following equation:

\[
\partial_t(t^\gamma \theta) + u \cdot \nabla (t^\gamma \theta) + \nu \Delta^\alpha (t^\gamma \theta) - \gamma t^{\gamma - 1} \theta = 0.
\]  

(34)

Applying the operator \( \Delta_j \) to the both sides of (34) and using Bony decomposition, we have

\[
\partial_t \Delta_j (t^\gamma \theta) + u \cdot \nabla \Delta_j (t^\gamma \theta) - \gamma \Delta_j (t^{\gamma - 1} \theta) + S_j u \cdot \Delta_j \nabla (t^\gamma \theta)
\]

\[
= - \sum_{|k-j| \leq 2} \left( \Delta_j (S_{k-1} u \cdot \Delta_k \nabla (t^\gamma \theta)) - S_{k-1} u \cdot \Delta_j \Delta_k \nabla (t^\gamma \theta) \right)
\]

\[
- \sum_{|k-j| \leq 2} \Delta_j (\Delta_k u \cdot S_{k-1} \nabla (t^\gamma \theta)) - \sum_{k > j - 3} \Delta_j (\Delta_k u \cdot \Delta_k \nabla (t^\gamma \theta))
\]

\[
- \sum_{|k-j| \leq 2} (S_{k-1} u - S_j u) \cdot \Delta_j \Delta_k \nabla (t^\gamma \theta) \equiv \sum_{i=1}^4 K_i.
\]  

(35)

Similar to the estimate of the Bony decomposition in the proof of Theorem 3.1, multiplying (35) by \( p |\Delta_j (t^\gamma \theta)|^{p-2} \Delta_j (t^\gamma \theta) \) and then integrating over \( \mathbb{R}^2 \), we obtain

\[
\frac{d}{dt} \|\Delta_j (t^\gamma \theta)\|_{L_p}^{(p-2)} \|\Delta_j (t^\gamma \theta)\|_{L_p} \leq \gamma \|\Delta_j (t^{\gamma - 1} \theta)\|_{L_p}^{(p-1)} + \sum_{i=1}^4 \|K_i\|_{L_p},
\]

after the use of Proposition 2.6 and the divergence free condition. Then multiplying the above inequality by \( q 2^{j(\sigma + \gamma \alpha)} \|\Delta_j (t^\gamma \theta)\|_{L_p}^{q-1} \) and summing over \( j \), we find that

\[
\frac{d}{dt} \|t^\gamma \theta\|_{B_{p,q}^{\frac{\alpha}{p} + \gamma}} + c \|t^\gamma \theta\|_{B_{p,q}^{\frac{\alpha}{p} + \gamma}} + \sum_{i=1}^4 \|K_i\|_{L_p} \lesssim \gamma \|\Delta_j (t^{\gamma - 1} \theta)\|_{L_p}^{(p-1)} + \sum_{i=1}^4 \|K_i\|_{L_p} \|2^{j(\sigma + \gamma \alpha)} \|\Delta_j (t^\gamma \theta)\|_{L_p}^{q-1}.
\]

After the use of Young inequality, we have

\[
\frac{d}{dt} \|t^\gamma \theta\|_{B_{p,q}^{\frac{\alpha}{p} + \gamma}} + c \|t^\gamma \theta\|_{B_{p,q}^{\frac{\alpha}{p} + \gamma}} \lesssim \gamma \|\Delta_j (t^{\gamma - 1} \theta)\|_{L_p}^{(p-1)} + \sum_{i=1}^4 \|K_i\|_{L_p} 2^{j(\sigma + \gamma \alpha)} \|\Delta_j (t^\gamma \theta)\|_{L_p}^{q-1}.
\]
\[
\|t^{\gamma-1}\theta\|^q_{B_{p,q}^{\sigma+\frac{5}{7}+(\gamma-1)\alpha}} + \sum_{j \in \mathbb{Z}} 2^j |(\sigma+\frac{5}{7}+(\gamma-1)\alpha)q| \sum_{\ell=1}^4 \|K_{\ell}\|^q_{L_p}.
\] (36)

To estimate the previous equation, we use the \(L_p\) commutator estimate [34] and Bernstein inequality to obtain

\[
\|K_1\|_{L_p} \lesssim \|\Delta_j(t^\gamma \theta)\|_{L_p} \sum_{k \leq j-2} 2^{k(\frac{5}{3}+1)+\beta} \|\Delta_k \theta\|_{L_p},
\]

\[
\|K_2\|_{L_p} \lesssim 2^j \|\Delta_j(t^\gamma \theta)\|_{L_p} \sum_{k \leq j-2} 2^{k(\frac{5}{3}+1)} \|\Delta_k \theta\|_{L_p},
\]

\[
\|K_3\|_{L_p} \lesssim 2^{j+1} \sum_{k \geq j-3} 2^{k\beta} \|\Delta_k \theta\|_{L_p} \|\Delta_j(t^\gamma \theta)\|_{L_p},
\]

\[
\|K_4\|_{L_p} \lesssim 2^{j+1}\|\Delta_j(t^\gamma \theta)\|_{L_p} \|\Delta_j(t^\gamma \theta)\|_{L_p}.
\]

Hence, due to \(q > 1\) and \(\alpha > 0\), we have

\[
\sum_{j \in \mathbb{Z}} 2^j |(\sigma+\frac{5}{7}+(\gamma-1)\alpha)q| \|K_1\|^q_{L_p} \lesssim \sum_{j \in \mathbb{Z}} 2^j |(\sigma+\frac{5}{7}+(\gamma-1)\alpha)q| \|\Delta_j(t^\gamma \theta)\|^q_{L_p} \left( \sum_{k \leq j-2} 2^{k(\frac{5}{3}+1)+\beta} \|\Delta_k \theta\|_{L_p} \right)^q
\]

\[
\lesssim \sup_{j \in \mathbb{Z}} \left( 2^j |(\sigma+\gamma)q| \|\Delta_j(t^\gamma \theta)\|_{L_p} \right)^q \sum_{k \leq j-2} 2^{k(\sigma+\frac{5}{7})} \|\Delta_k \theta\|_{L_p} 2^{(k-j)(\alpha-\frac{5}{7})} \right)^q
\]

\[
\lesssim \|\theta\|^q_{B_{p,q}^{\sigma+\frac{5}{7}+(\gamma-1)\alpha}}\|t^\gamma \theta\|^q_{B_{p,q}^{\sigma+\gamma}}. \tag{37}
\]

Since \(\alpha > 2\beta\) and \(q \geq 2\) implies \(\alpha > (1 - \frac{1}{q})^{-1}\beta\), by Young inequality, we obtain

\[
\sum_{j \in \mathbb{Z}} 2^j |(\sigma+\frac{5}{7}+(\gamma-1)\alpha)q| \|K_2\|^q_{L_p} \lesssim \sum_{j \in \mathbb{Z}} 2^j |(\sigma+\frac{5}{7}+(\gamma-1)\alpha+\beta)q| \|\Delta_j(t^\gamma \theta)\|^q_{L_p} \left( \sum_{k \leq j-2} 2^{k(\frac{5}{3}+1)} \|\Delta_k \theta\|_{L_p} \right)^q
\]

\[
\lesssim \sum_{j \in \mathbb{Z}} 2^j |(\sigma+\frac{5}{7}+(\gamma-1)\alpha+\beta)q| \|\Delta_j(t^\gamma \theta)\|^q_{L_p} \left( \sum_{k \leq j-2} 2^{k(\frac{5}{3}+1)} \|\Delta_k \theta\|_{L_p} \right)^q
\]

\[
\lesssim \sup_{j \in \mathbb{Z}} \left( 2^j |(\sigma+\gamma)q| \|\Delta_j(t^\gamma \theta)\|_{L_p} \right)^q \sum_{k \leq j-2} 2^{k(\sigma+\frac{5}{7})} \|\Delta_k \theta\|_{L_p} 2^{(j-k)(-\alpha+\frac{5}{7}+\beta)} \right)^q
\]

\[
\lesssim \|\theta\|^q_{B_{p,q}^{\sigma+\frac{5}{7}+(\gamma-1)\alpha}}\|t^\gamma \theta\|^q_{B_{p,q}^{\sigma+\gamma}}. \tag{38}
\]

Furthermore, observe that \(q > 1\), \(\beta > 0\), \(\alpha < \frac{3}{2} + 1\) and \(\gamma > 0\) yield \(\alpha < (2 - \frac{1}{q})^{-1}(\beta + 2(\frac{2}{p} + 1) + \gamma\alpha)\). Hence by Young inequality, we find that

\[
\sum_{j \in \mathbb{Z}} 2^j |(\sigma+\frac{5}{7}+(\gamma-1)\alpha)q| \|K_3\|^q_{L_p} \lesssim \sum_{j \in \mathbb{Z}} 2^j |(\sigma+\frac{5}{7}+(\gamma-1)\alpha+\frac{5}{7}+1)q| \left( \sum_{k \geq j-3} 2^{k\beta} \|\Delta_k \theta\|_{L_p} \|\Delta_j(t^\gamma \theta)\|_{L_p} \right)^q
\]

\[
\lesssim \sup_{j \in \mathbb{Z}} \left( 2^j |(\sigma+\gamma)q| \|\Delta_j(t^\gamma \theta)\|_{L_p} \right)^q
\]
Similarly, we have
\[
\sum_{j \in \mathbb{Z}} 2^{j(\sigma + \frac{q}{p} + (\gamma - 1)\alpha)} \| t^{\gamma} \theta \|_{B_{p,q}^{\sigma + \frac{q}{p} + \gamma}}^q \lesssim \| t^{\gamma} \theta \|_{B_{p,q}^{\sigma + \frac{q}{p} + \gamma}.}
\] (39)

Similarly, we have
\[
\sum_{j \in \mathbb{Z}} 2^{j(\sigma + \frac{q}{p} + (\gamma - 1)\alpha)} \| t^{\gamma} \theta \|_{B_{p,q}^{\sigma + \frac{q}{p} + \gamma}}^q \lesssim \| t^{\gamma} \theta \|_{B_{p,q}^{\sigma + \frac{q}{p} + \gamma}}.
\] (40)

Therefore, inserting (37)-(40) into (36), we have
\[
\frac{d}{dt} \| t^{\gamma} \theta \|_{B_{p,q}^{\sigma + \frac{q}{p} + \gamma}} + \| t^{\gamma} \theta \|_{B_{p,q}^{\sigma + \frac{q}{p} + \gamma}} \lesssim \| t^{\gamma - 1} \theta \|_{B_{p,q}^{\sigma + \frac{q}{p} + (\gamma - 1)\alpha}} + \| t^{\gamma} \theta \|_{B_{p,q}^{\sigma + \frac{q}{p} + \gamma}}.
\]

Let \( t \in (0, T_0) \). Integrating the previous inequality over the interval \((0, t)\) yields
\[
\| t^{\gamma} \theta(t) \|_{B_{p,q}^{\sigma + \frac{q}{p} + \gamma}} + \int_0^t \| s^{\gamma} \theta(s) \|_{B_{p,q}^{\sigma + \frac{q}{p} + \gamma}} ds \lesssim \int_0^t \| s^{\gamma - 1} \theta(s) \|_{B_{p,q}^{\sigma + \frac{q}{p} + (\gamma - 1)\alpha}} ds + \int_0^t \| t^{\gamma} \theta(s) \|_{B_{p,q}^{\sigma + \frac{q}{p} + \gamma}} ds.
\]

Then using Gronwall’s inequality, we have
\[
\sup_{t \in [0, T_0]} \| t^{\gamma} \theta(t) \|_{B_{p,q}^{\sigma + \frac{q}{p} + \gamma}} + \int_0^{T_0} \| s^{\gamma} \theta(s) \|_{B_{p,q}^{\sigma + \frac{q}{p} + \gamma}} ds \lesssim \int_0^{T_0} \| s^{\gamma - 1} \theta(s) \|_{B_{p,q}^{\sigma + \frac{q}{p} + (\gamma - 1)\alpha}} ds.
\] (41)

It remains to estimate the term \( t^{\gamma} \theta \) in time-space Besov spaces. In order to do this, it is sufficient to bound the \( L_p \)-norm of \( t^{\gamma} \theta \). Multiplying (34) by \( p[t^{\gamma} \theta]^{p-2} t^{\gamma} \theta \), applying the divergence free condition and using the fact that
\[
\nu \int_{\mathbb{R}^2} \Lambda^\alpha(t^{\gamma} \theta) |t^{\gamma} \theta|^{p-2} t^{\gamma} \theta dx \geq 0,
\]
we deduce that
\[
\| t^{\gamma} \theta \|_{L^{\infty}(0, T_0; L_p)} \leq \int_0^{T_0} \| s^{\gamma - 1} \theta(s) \|_{L_p} ds.
\] (42)

Thus, combining (41) and (42), we conclude that
\[
\sup_{t \in [0, T_0]} \| t^{\gamma} \theta(t) \|_{B_{p,q}^{\sigma + \frac{q}{p} + \gamma}} + \int_0^{T_0} \| s^{\gamma} \theta(s) \|_{B_{p,q}^{\sigma + \frac{q}{p} + \gamma}} ds \lesssim \int_0^{T_0} \| s^{\gamma - 1} \theta(s) \|_{B_{p,q}^{\sigma + \frac{q}{p} + (\gamma - 1)\alpha}} ds.
\] (43)

Now we are going to prove that the right-hand side of (43) is finite. By Theorem 3.1, we have
\[
\| \theta \|_{L_q(0, T_0; B_{p,q}^{\sigma + \frac{q}{p}})} \lesssim \| \theta_0 \|_{B_{p,q}^{\sigma + \frac{q}{p}}},
\] (44)

Thus for \( \gamma \) a positive integer, by induction on \( \gamma \), we have
\[
\sup_{t \in (0, T_0)} \| t^{\gamma} \theta(t) \|_{B_{p,q}^{\sigma + \frac{q}{p} + \gamma}} + \int_0^{T_0} \| s^{\gamma} \theta(s) \|_{B_{p,q}^{\sigma + \frac{q}{p} + \gamma}} ds \lesssim e^{\gamma} \| \theta_0 \|_{B_{p,q}^{\sigma + \frac{q}{p}}}. \] (45)

For the general case of \( \gamma > 0 \), it can be shown that (45) remains true by interpolation.
The proof of Theorem 4.1 is completed.

Remark 4.2. Dong and Li [17] studied the smoothing effect of the solution \( \theta' \) of (1) with \( p \in [2, \infty), \ q \in [1, \infty), \ \beta = 0 \) and \( \alpha \in (0, 1] \). More precisely, they first proved that there exists a \( T_1' \in (0, \infty) \) such that, for \( \gamma \in (0, 1/2) \),
\[
\| \theta' \|_{L^q(0, T_1'; B_{p,q}^{\gamma+\alpha+\omega_1})} \leq c \|	heta_0\|_{B_{p,q}^\gamma},
\]
for some constant \( c \) independent of \( \gamma \) and \( q_1 \). Then they showed that for any \( \delta \in (0, T_1') \), \( q_1 \in [1, \infty) \) and \( \gamma > 0 \), \( \| \theta' \|_{L^\infty(\delta, T_1'; B_{p,q}^\gamma)} < \infty \). Their proof is divided into these two steps because of the need of the \( L^\infty \) estimate for \( \nabla \theta \). In view of the work of [17], it is an interesting question whether or not the inequality (33') holds for the modified quasi-geostrophic equation (1). Similar to the proofs of Theorems 3.1 and 4.1, we point out that a result which is even stronger than (33') can be obtained for the equation (1). That is, there exists a constant \( T_1 \in (0, T_0) \) such that for any \( \gamma > 0 \) and \( q_1 \in [1, \infty] \),
\[
\| \theta' \|_{L^q(0, T_1; B_{p,q}^{\gamma+\omega_1})} \leq \|\theta_0\|_{B_{p,q}^\gamma},
\]
However, to avoid a tedious computation, we chose to state and prove (33), which is sufficient for us to deduce that \( \theta \) is smooth on any closed interval contained in \( (0, T_0) \).

We end this remark by pointing out that our proof of the infinite differentiability of (1) with \( \beta = 0 \) is simpler than the method presented in [17] because of the skip of the \( L^\infty \) estimate for \( \nabla \theta \).

5. Proofs of Theorems 1.3 and 1.4. A function \( \omega(\xi) : (0, \infty) \mapsto (0, \infty) \) is called a modulus of continuity if \( \omega \) is increasing, continuous on \( (0, \infty) \), concave, and piecewise \( C^2 \) with one sided derivatives (possibly infinite at \( \xi = 0 \)) defined at every point in \( [0, \infty) \); we say that a function \( f(x) : \mathbb{R} \mapsto \mathbb{R} \) obeys \( \omega \) if \( |f(x) - f(y)| < \omega(|x-y|) \) for all \( x \neq y \), see [22, Definition 2.1].

Proposition 5.1. Let \( \alpha \in (1, 2), \ \beta = \alpha - 1 \) and let \( 0 < \gamma < \delta < 1 \) be two small constants. For \( \alpha, \gamma \) and \( \delta \), we define the continuous function \( \omega \) as follows:
\[
\omega(\xi) = \begin{cases} \xi - \xi^{1+\beta} & \text{if } 0 \leq \xi \leq \delta, \\ \delta - \delta^{1+\beta} + \frac{\gamma}{4} \log \frac{\xi}{\delta} & \text{if } \xi > \delta. \end{cases}
\]
Let \( \omega \in C^\infty((0, T_0) \times \mathbb{R}^2) \) be the unique smooth solution of (1). If \( \theta_0(x) \) obeys \( \omega(\xi) \), then \( \theta(x, t) \) obeys \( \omega(\xi) \) for any \( t \geq 0 \). Moreover, for any \( \lambda > 0 \), if \( \theta_0(x) \) obeys \( \omega(\xi) \), then \( \theta(x, T_0) \) obeys \( \omega(\xi) \). Here \( \omega(\xi) = \omega(\xi) \).

Proposition 5.1 has been proved by Miao and Xue [29]. The conclusion of this proposition implies that \( \nabla \theta \in L^\infty((0, T_0) \times \mathbb{R}^2) \). Thus we can extend \( \theta(t) \) beyond \( T_0 \) and this ensures the global existence of smooth solution.

Proof of Theorem 1.3. For any \( \theta_0 \in B_{p,q}^{\frac{\beta}{\gamma}}(\mathbb{R}^2) \), applying Theorem 3.1 for \( \beta = \alpha - 1 \) and \( q = q_1 \), we deduce that (1) admits a unique local solution \( \theta \) satisfying
\[
\theta \in C((0, T_0); B_{p,q}^{\frac{\beta}{\gamma}}(\mathbb{R}^2)) \cap L^q(0, T_0; B_{p,q}^{\frac{\beta}{\gamma}+\frac{\omega_1}{\gamma}}(\mathbb{R}^2)).
\]
Moreover, Theorem 4.1 implies that \( \theta \in C^\infty((0, T_0) \times \mathbb{R}^2) \). Thus, for any \( t' \in (0, T_0) \), \( \theta(t') \in L^\infty(\mathbb{R}^2) \) and \( \nabla \theta(t') \in L^\infty(\mathbb{R}^2) \). Setting
\[
\lambda = \max \left\{ 2\|\nabla \theta(t')\|_{L^\infty}, \ \delta \exp \frac{16\|\theta(t')\|_{L^\infty}}{\gamma} \right\},
\]
Then $\theta(t', x)$ obeys $\omega_\lambda(\xi)$, here $\omega$ is defined by (47). Considering $\theta(t')$ as the initial data and applying Proposition 5.1, we deduce that

$$\|\nabla \theta\|_{L^\infty(t', T_0; L^\infty)} \leq \lambda.$$ 

Thus, $\theta(t)$ can be extended beyond $T_0$. This means that Equation (1) admits a unique global solution $\theta$ of (1) satisfying

$$\theta \in C([t', \infty); B^\frac{2}{p}\sigma_p(\mathbb{R}^2)) \cap L_q, \text{loc}([t', \infty); B^\frac{2}{p}+\frac{\alpha}{q}\sigma_p(\mathbb{R}^2)) \cap C^\infty((t', \infty) \times \mathbb{R}^2).$$

Using the uniqueness of $\theta$ on $[0, T_0)$, we conclude that

$$\theta \in C([0, \infty); B^\frac{2}{p}\sigma_p(\mathbb{R}^2)) \cap L_q, \text{loc}([0, \infty); B^\frac{2}{p}+\frac{\alpha}{q}\sigma_p(\mathbb{R}^2)) \cap C^\infty((0, \infty) \times \mathbb{R}^2).$$

The proof of Theorem 1.3 is completed.

**Proof of Theorem 1.4.** Let $\sigma = \frac{2}{p} + 1 + \beta - \alpha$. For $\theta_0 \in B^\sigma_{p,q}(\mathbb{R}^2)$, using Theorem 3.1 for $q_1 = q$, we find that (1) admits the local existence of the unique solution $\theta$ of (1) satisfying

$$\theta \in C([0, T_0); B^\sigma_{p,q}(\mathbb{R}^2)) \cap L_q(0, T_0; B^\frac{2}{p}+\frac{\alpha}{q}\sigma_p(\mathbb{R}^2)).$$

Theorem 4.1 implies $\theta(t') \in H^1(\mathbb{R}^2)$ for $t' \in (0, T_0)$. Considering $\theta(t')$ as the initial data, we use Theorem 1.2 to obtain that there exist positive constants $T_1 < T_2$ which depend on $p$, $q$, $\alpha$, $\beta$, $\nu$ and $\theta_0$ such that

$$\theta \in C^\infty((t', T_1) \times \mathbb{R}^2) \cap C^\infty((T_2, \infty) \times \mathbb{R}^2).$$

This, together with the uniqueness of the local smooth solution of (1) and Theorem 4.1, we deduce that $C^\infty((0, T_1) \times \mathbb{R}^2) \cap C^\infty((T_2, \infty) \times \mathbb{R}^2)$. The proof is completed.

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