Three-fluid cosmological model using Lie and Noether symmetries

Michael Tsamparlis and Andronikos Paliathanasis

Faculty of Physics, Department of Astronomy–Astrophysics–Mechanics, University of Athens, Panepistemiopolis, Athens 157 83, Greece

E-mail: mtsampa@phys.uoa.gr and anpaliat@phys.uoa.gr

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Abstract
We employ a three-fluid model in order to construct a cosmological model in the Friedmann–Robertson–Walker flat spacetime, which contains three types of matter: dark energy, dark matter and a perfect fluid with a linear equation of state. Dark matter is described by dust and dark energy with a scalar field with potential $V(\phi)$. In order to fix the scalar field potential, we demand Lie symmetry invariance of the field equations, which is a model-independent assumption. The requirement of an extra Lie symmetry selects the exponential scalar field potential. The further requirement that the analytic solution is invariant under the point transformation generated by the Lie symmetry eliminates dark matter and leads to a quintessence and a phantom cosmological model containing a perfect fluid and a scalar field. Next we assume that the Lagrangian of the system admits an extra Noether symmetry. This new assumption selects the scalar field potential to be exponential and forces the perfect fluid to be stiff. Furthermore, the existence of the Noether integral allows for the integration of the dynamical equations. We find new analytic solutions to quintessence and phantom cosmologies which contain all three fluids. Using these solutions, one is able to compute analytically all main cosmological functions, such as the scale factor, the scalar field, the Hubble expansion rate, the deceleration parameter, etc.

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1. Introduction
The recent cosmological data indicate that the Universe (a) is spatially flat and (b) has suffered two acceleration phases: an early acceleration phase (inflation), which occurred prior to the radiation-dominated era, and a recently initiated accelerated expansion. The source for the late time cosmic acceleration has been attributed to an unidentified type of matter, the dark energy (DE). DE contrary to the ordinary baryonic matter has a negative pressure which counteracts
the gravitational force and leads to the observed accelerated expansion. The nature of DE is still an open question.

The simplest DE probe is the cosmological constant $\Lambda$ (vacuum) leading to the $\Lambda$CDM cosmology [1–3]. However, it has been shown that $\Lambda$CDM cosmology suffers from two major drawbacks known as the fine-tuning problem and the coincidence problem [4]. Besides $\Lambda$CDM cosmology, many other candidates have been proposed in the literature, such as time-varying $\Lambda(t)$ cosmologies, quintessence, $k$-essence, tachyons, modifications of gravity, Chaplygin gas and others [5–10].

In addition to DE and the ordinary baryonic matter, it is believed that the Universe contains a third type of matter, the dark matter (DM). This type of matter is assumed to be pressureless (non-relativistic) and interacts very weakly with the standard baryonic matter. Therefore, its presence is mainly inferred from gravitational effects on visible matter.

In the following we consider a model of the Universe which contains three types of matter. The DM is modeled by a dust fluid, the DE by a scalar field and the rest of matter by a perfect fluid. Since a scalar field can be considered to be a perfect fluid (see below) in effect, we model the matter in the Universe in terms of two perfect fluids and a dust. All three fluids are assumed to be self-interacting and minimally coupled to gravity.

The problem with the above scenario is that there does not exist an underline principle which will specify uniquely the potential $V(\phi)$ of the scalar field. Indeed in the literature, one finds several potentials $V(\phi)$ such as exponential, power law, hyperbolic, etc. Therefore, it is not possible that one could find an analytic solution of this model even if an equation of state (EoS) has been assumed.

The aim of this work is twofold: (a) to propose a geometric principle (‘selection rule’) for specifying the potential $V(\phi)$ and (b) to solve analytically the system of the resulting field equations. Concerning (a), following a recent paper [11], we propose that the potential should be selected by the requirement that the dynamical system of the three fluids admits an additional Lie or Noether symmetry. This point of view has also been considered in [12–15].

The main reason for the consideration of this hypothesis is that the Lie/Noether point symmetries provide first/Noether integrals, which assist the integrability of the system. A fundamental approach to derive the Lie point and the Noether symmetries of a given dynamical system moving in a Riemannian space has recently been proposed in [16]. A similar analysis can be found in [17–21].

Concerning the analytic solutions, we restrict our considerations to a flat FRW spacetime. Although there is a great number of papers devoted to the dynamics of scalar field minimally coupled to matter, not much is known about analytic exact solutions of these models. Most of the solutions correspond to spatially flat FRW models with no other source but the scalar field [22–26]. Recently, in [11] the analytic solution for matter in the form of dust and a scalar field has been given. A solution with two scalar fields (where one of them is a kination, i.e. stiff matter) and an exponential potential is given in [27].

Few exact solutions are known with spatial curvature [28, 29]. Even less solutions are known for a perfect fluid and a scalar field [5, 30–32]. In particular, in [32] the authors consider a prefect fluid minimally interacting with a scalar field having an arbitrary potential $V(\phi)$ and, by considering as variable the scale factor $a$, they develop an approach which replaces the potential $V(\phi)$ with an auxiliary arbitrary function ($F(a)$ in their notation) and a constant ($C$) and give the implicit solution of the problem. They consider an arbitrary specification of the function $F(a)$ and the constant $C$, which leads to an exponential potential and (by making some further assumptions) they find a set of analytic solutions in terms of the adiabatic index $\gamma$ of the fluid. In a different approach to the same problem in [12], the authors employ ad hoc an exponential potential and assume that the resulting system of equations admits a Noether
symmetry. They are not able to find an analytic solution and continue with numerical solutions. This is not necessary according to the results of [32] and our results below.

In [33, 34], the authors in an attempt to unify the early time (phantom) inflation with the late time (phantom or not) acceleration derive specific solutions within which both the early and the late phases of evolution of the Universe occur. The proposed scenario is based on the introduction of an additional unspecified function $\omega(\phi)$ in the kinetic term of the scalar field Lagrangian. Although this function can be absorbed into the field during inflation and acceleration by a redefinition of the scalar field, this is not possible right on the transition point, at which this term connects the two phases of evolution smoothly. A similar scenario considers two scalar fields.

There does not appear to exist a solution concerning a perfect fluid, a dust and a scalar field, all minimally coupled to gravity and non-interacting in a spatially flat FRW spacetime. In the following using symmetry assumptions only, we obtain many of the above solutions and also a new solution which involves all three fluids. This solution contains the solutions of [27] as a special case.

The structure of the paper is as follows. The basic equations and the resulting dynamical system are presented in section 2. In section 3, we review briefly the basics of the theory of Lie and Noether symmetries. In section 4, we define the scalar field potential and subsequently we determine analytic solutions of the field equations using an extra Lie symmetry and the Lie invariance of the solutions with respect to this symmetry. In section 5, we determine a new solution of the three-fluid dynamical system assuming only an extra Noether symmetry. The main conclusions are summarized in section 6.

2. The dynamical system

We consider the flat FRW spacetime in canonical coordinates:

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2),$$

with comoving observers $u^a = \delta_i^a$. We also consider that the spacetime contains dark energy, dark matter and ‘other matter’. DM is assumed to be described by dust ($p_{DM} = 0$) with the energy–momentum tensor

$$T_{ab}^{DM} = \mu_{DM} \eta_{ab},$$

The ‘other matter’ is assumed to be described by a perfect fluid with the energy–momentum tensor

$$T_{ab}^{PF} = \mu_{B} \eta_{ab} + p_{B} h_{ab},$$

and a linear EoS

$$p_{B} = (\gamma - 1) \mu_{B}, \quad 1 < \gamma \leq 2,$$

where $\gamma$ is the adiabatic index of the fluid and $h_{ab} = g_{ab} + u_{a}u_{b}$ is the tensor projecting normal to the four-velocity $u^a$. For $\gamma = 1$, the fluid is dust; for $\gamma = 2$, it is stiff matter and for $\gamma = \frac{4}{3}$, it is a radiation fluid.

Finally DE is described by a scalar field $\phi$ (quintessence or phantom) rolling down a potential $V(\phi)$ whose energy–momentum tensor is

$$T_{ab}^{\phi} = \left(\frac{1}{2} \epsilon \dot{\phi}^2 + V(\phi)\right) \eta_{ab} + \left(\frac{1}{2} \epsilon \dot{\phi}^2 - V(\phi)\right) h_{ab}. $$

When $\epsilon = +1$, we have quintessence (real $\phi$) and when $\epsilon = -1$, we have a phantom field (imaginary $\phi$). It follows that (for the comoving observers) the scalar field may be considered as a perfect fluid with energy density and isotropic pressure:

$$\mu_{\phi} = \frac{1}{2} \epsilon \dot{\phi}^2 + V(\phi)$$

$$p_{\phi} = \frac{1}{2} \epsilon \dot{\phi}^2 - V(\phi)$$
\[ p_\phi = \frac{1}{2} \varepsilon \dot{\phi}^2 - V(\phi). \]  
Equations (6) and (7) imply the EoS
\[ \mu_\phi - p_\phi = 2V(\phi) \]  
from which follows that the definition/specification of a potential function for the scalar field \( \phi(t) \) essentially defines an EoS for the perfect fluid defined by the scalar field. It also follows that stiff matter may be considered as a homogeneous, massless, free (i.e. with vanishing potential) scalar field minimally coupled to gravity. Because \( \frac{\mu_\phi}{\mu_B} = 1 \), this scalar field is a kination.

All three fluids are assumed to interact minimally, which implies the three independent conservation equations
\[ \text{DM} T^a_{\;b} = 0, \quad \text{PP} T^a_{\;b} = 0, \quad \phi T^a_{\;b} = 0. \]  
There are six unknowns in the problem, i.e. the \( a(t), \mu_B, p_B, \mu_{\text{DM}}, \phi(t), V(\phi) \), and five field equations, i.e. Einstein’s equation
\[ G_{ab} = \phi T_{ab} + \text{DM} T_{ab} + \mu T_{ab}, \]  
the three conservation equations (9) (equation \( \phi T^a_{\;b} = 0 \) is the Klein–Gordon equation) and the EoS (4). We need one more equation which must be given by an independent assumption.

However, there is not a general consensus or accepted practice as to the nature of this assumption. In the literature, one finds the following types of assumptions/equations. (i) Equations which are based on the dynamical system approach: that is, in the case the field equations follow from a Lagrangian, one demands that the Lagrangian admit a Noether symmetry. This assumption results in an extra Noether integral, which provides the required extra equation. With this approach, there have been given the analytic solutions for various probes (UCDM, exponential) [11, 12, 14], and in other models e.g. modified gravity [35–42]. An alternative but similar approach is to demand that the field equations admit an extra Lie (not necessarily a Noether) symmetry and require that the solution is invariant under the action of the Lie symmetry. (ii) Equations which are defined by an ansatz on the scale factor [24–26]. (iii) Equations which are ad hoc statements on the scalar field potential.

In this paper, we follow the dynamical system approach and demand that the system admits an extra Lie symmetry and subsequently an extra Noether symmetry. The Lie symmetry requirement specifies the potential to be exponential and produces (with an additional invariance assumption) the perfect fluid solutions found in [32] and the empty space solution of [24, 26]. The requirement of Noether symmetry in addition to fixing the potential constrains the perfect fluid to be stiff and determines the analytic solution of the three-fluid problem.

2.1. The field equations

Einstein field equations for the metric (1), comoving observers and a perfect fluid of energy density \( \mu_B \) and isotropic pressure \( p_B \) minimally coupled to a scalar field \( \phi \) with scalar field potential \( V(\phi) \) and DM (dust) with the energy density \( \mu_{\text{DM}} \) are
\[ H^2 = \frac{k}{3} (\mu_{\text{DM}} + \mu_B + \mu_\phi) (\text{Hubble equation}) \]  
\[ \frac{\dot{a}}{a} + \frac{1}{2} H^2 = -\frac{k}{2} (p_B + p_\phi) (\text{deceleration equation}), \]  
\[ \begin{aligned} p_\phi &= \frac{1}{2} \varepsilon \dot{\phi}^2 - V(\phi), \\ \mu_\phi - p_\phi &= 2V(\phi), \end{aligned} \]  
\[ G_{ab} = \phi T_{ab} + \text{DM} T_{ab} + \mu T_{ab}, \]  
\[ \text{DM} T^a_{\;b} = 0, \quad \text{PP} T^a_{\;b} = 0, \quad \phi T^a_{\;b} = 0. \]  
\[ \frac{\dot{a}}{a} + \frac{1}{2} H^2 = -\frac{k}{2} (p_B + p_\phi) \]  
\[ H^2 = \frac{k}{3} (\mu_{\text{DM}} + \mu_B + \mu_\phi) \]  
\[ \frac{\dot{a}}{a} + \frac{1}{2} H^2 = -\frac{k}{2} (p_B + p_\phi). \]
where $k = 8\pi G$ is Einstein’s gravitational constant. The deceleration equation (12) can be written in the alternative forms

$$H + \frac{3}{2}H^2 = -\frac{k}{2}(p_B + p_\phi)$$

(13)

$$\dot{H} + \frac{k}{2}(\mu_B + p_B + \epsilon \dot{\phi}^2) = 0.$$  

(14)

Equations (11) and (12) are supplemented by the Klein–Gordon equation

$$\ddot{\phi} = -3H\dot{\phi} - \epsilon V_\phi$$

(15)

and the conservation equations

$$\dot{\mu}_B + 3(\mu_B + p_B)H = 0$$

(16)

$$\dot{\mu}_{DB} + 3\mu_DBH = 0.$$  

(17)

The conservation equation (16) and the EoS (4) imply

$$\mu_B = Ca^{-\gamma}, \quad p_B = (\gamma - 1)Ca^{-\gamma},$$

(18)

where $C \geq 0$ is an integration constant. For DM, the conservation equation (17) gives

$$\mu_{DM} = \frac{E}{a^3},$$

(19)

where $E$ is an integration constant. Replacing these results in the field equations (11), (12), (15), we obtain the system of equations

$$3a\ddot{a} - \frac{1}{2}\epsilon ka^2\dot{\phi}^2 - ka^3V(\phi) - kCa^{-3(\gamma - 1)} = \ddot{\bar{E}}$$

(20)

$$\ddot{a} + \frac{1}{2a}a^2 + \frac{k}{4}\epsilon a\dot{\phi}^2 - \frac{k}{2}aV + \frac{k}{2}(\gamma - 1)Ca^{-3\gamma} = 0$$

(21)

$$\ddot{\phi} + \frac{3}{a}\dot{a}\dot{\phi} + \epsilon V_\phi = 0,$$

(22)

where $\ddot{\bar{E}} = kE$. Equation (20) is written in the equivalent form

$$H^2 = \frac{k}{3}\left(Ea^{-3} + Ca^{-3\gamma} + \frac{1}{2}\epsilon \dot{\phi}^2 + V(\phi)\right).$$

(23)

2.2. Looking for analytic solutions

In order to find an analytic solution of the system of equations (20)–(22), we need either two independent assumptions, which will result in two independent conditions, or one stronger assumption which will result in two conditions. In the following, we formulate these assumptions in terms of Lie and Noether symmetries.

Assuming the unknown potential $V(\phi)$ to be an externally specified function, the differential equations (21) and (22) contain only the unknowns $a(t)$ and $\phi(t)$. Hence, it is possible to consider a new autonomous two-dimensional dynamical system with variables $(a(t), \phi(t))$ defined by equations (21), (22) and treat the third field equation (20) (equivalently (23)) as a constraint. Following this remark we look for a Lagrangian producing equations (21) and (22). It is easily seen that this Lagrangian is

$$L = \left(3a\ddot{a} - \frac{1}{2}k\epsilon a^3\dot{\phi}^2\right) + ka^3V(\phi) + kCa^{-3(\gamma - 1)}$$

(24)
with Hamiltonian
\[
\hat{E} = 3aa^2 - \frac{1}{4}k\epsilon a^2 \phi^2 - ka^3 V(\phi) - kCa^{-3}\gamma^{-1}.
\] (25)

We note that the Hamiltonian is the constraint condition (20), that is, \(\hat{E} = \text{const}\). It is possible to give a geometric interpretation of the Hamiltonian \(\hat{E}\). Indeed the dynamical system defined by Lagrangian (24) is autonomous, hence admits the Noether symmetry \(\partial_t\) whose Noether integral is precisely the Hamiltonian. We note that \(\hat{E} = 0\) only when DM is absent.

We arrive at the following conclusion:

The dynamical system defined by the three fluids considered above is equivalent to a new autonomous two-dimensional dynamical system \(\{a(t), \phi(t)\}\) with Lagrangian (24) whose Hamiltonian \(\hat{E}\) is a constant and vanishes iff \(\mu_{DM} = 0\).

In the following we use the Lie and the Noether symmetries of the two-dimensional dynamical system in order to produce analytic solutions.

3. Lie and Noether symmetries

Before we proceed we review briefly the basic definitions concerning Lie and Noether symmetries of systems of second-order ordinary differential equations (ODEs),

\[
\ddot{x}^j = \omega^j(t, x^i, \dot{x}^i).
\] (26)

A vector field \(X = \xi(t, x^i) \partial_t + \eta^i(t, x^j) \partial_i\) in the augmented space \(\{t, x^i\}\) is the generator of a Lie point symmetry of the system of ODEs (26) if the following condition is satisfied [17, 18]:

\[
X^{[2]}(\ddot{x}^j - \omega(t, x^i, \dot{x}^i)) = 0,
\] (27)

where \(X^{[2]}\) is the second prolongation of \(X\) defined by the formula

\[
X^{[2]} = \xi \partial_t + \eta^i \partial_i + (\dot{\eta}^i - \dot{x}^i \dot{\xi}) \partial_{\dot{x}^i} + (\dot{\xi} - \dot{x}^i \dot{\xi} - 2 \ddot{x}^i \dot{\xi}) \partial_{\ddot{x}^i}.
\] (28)

Condition (27) is equivalent to the relation

\[
[X^{[1]}, A] = \lambda(x^i)A,
\] (29)

where \(X^{[1]}\) is the first prolongation of \(X\) and \(A\) is the Hamiltonian vector field:

\[
A = \partial_t \dot{x} + x \partial_x + \omega^j(t, x^i, \dot{x}^i) \partial_{\dot{x}^i}.
\] (30)

If the system of ODEs results from a first-order Lagrangian \(L = L(t, x^i, \dot{x}^i)\), then a Lie symmetry \(X\) of the system is a Noether symmetry of the Lagrangian if the additional condition is satisfied

\[
X^{[1]}L + L \frac{d\xi^j}{dt} = \frac{df}{dt},
\] (31)

where \(f = f(t, x^i)\) is the gauge function. To every Noether symmetry, there corresponds a first integral (a Noether integral) of the system of equations (26) which is given by the formula

\[
I = \xi E_H - \frac{\partial L}{\partial \dot{x}^i} \eta^i + f,
\] (32)

where \(E_H\) is the Hamiltonian of \(L\).

The vector field \(X\) for the Lagrangian (24) is

\[
X = \xi(t, a, \phi) \partial_t + \eta_a(t, a, \phi) \partial_a + \eta_\phi(t, a, \phi) \partial_\phi
\] (33)

and the first prolongation

\[
X^{[1]} = \xi \partial_t + \eta_a \partial_a + \eta_\phi \partial_\phi + (\dot{\eta}_a - \dot{a} \dot{\xi}) \partial_{\dot{a}} + (\dot{\eta}_\phi - \dot{\phi} \dot{\xi}) \partial_{\dot{\phi}}.
\] (34)
Having given the basic formulae for the Lie and Noether symmetries, we look for analytic solutions of the system of equations (20)–(22) using Lie and Noether symmetries.

To simplify the calculations, we introduce a new variable \( r \) with the relation \( a = r^\frac{2}{3} \). In the new variables \( r, \phi \), Lagrangian (24) takes the form

\[
L = \frac{1}{2}\left(\frac{8}{3}r^2 - \varepsilon kr^2\dot{\phi}^2\right) + kr^2V(\phi) + kCr^{-2(\gamma - 1)}.
\]

(35)

For a general \( V(\phi) \), this Lagrangian admits only the standard Lie/Noether symmetry \( \partial_t \).

We are looking for the possibility of extra Lie and Noether symmetries for special forms of the potential \( V(\phi) \). To find these potentials, we apply the results of [16], which concern all autonomous two-dimensional dynamical systems moving in flat (Euclidian or Minkowskian) space and admit Lie and/or Noether symmetries.

In order to apply the results of [16], we need a flat metric and a potential function for the dynamical system. This is done by decomposing Lagrangian (35) into kinetic energy, which defines the kinetic energy metric, and potential energy, which defines the potential. We consider the metric to be

\[
d_s^2 = \frac{8}{3}\dot{r}^2 - \varepsilon kr^2\dot{\phi}^2
\]

(36)

and the potential function

\[
W(r, \phi) = -kr^2V(\phi) - kCr^{-2(\gamma - 1)}.
\]

(37)

It can be shown that the Ricci scalar of the metric \( ds^2 \) vanishes; hence, the space \( \{r, \phi\} \) is \( M^2 \) (Minkowski two-dimensional space in polar coordinates). Therefore, the results of [16] apply and we make use of the tables in that paper to read the appropriate potentials.

4. Lie symmetry

We start with the demand that the system of equations (21), (22) admits an extra Lie symmetry (apart of \( \partial_t \)). From table 10, line 5 of [16], we find that the system of equations admits an extra Lie symmetry for the potential \( V(\phi) = V_0 e^{-d\phi} \) where \( V_0 \) and \( d \) are constants with \( d \neq 0 \); furthermore, the Lie symmetry vector is

\[
X = \gamma t\partial_t + r\partial_r + \frac{2\gamma}{d}\partial_\phi
\]

(38)

or, in the original coordinates \( \alpha(t), \phi(t) \),

\[
X = \gamma t\partial_t + \frac{2}{3}\partial_\alpha + \frac{2\gamma}{d}\partial_\phi.
\]

(39)

For the scalar field potential \( V(\phi) = V_0 e^{-d\phi} \), the field equations become

\[
\left(3\alpha \ddot{a} - \frac{1}{2} k\varepsilon a^2 \dot{\phi}^2\right) - ka^2V(\phi) - kCa^{-3(\gamma - 1)} = \ddot{E}
\]

(40)

\[
\ddot{a} + \frac{1}{2a}\dddot{a} + \frac{3}{4}k\varepsilon a^2 \dot{\phi}^2 - \frac{k}{2}aV_0 e^{-d\phi} + \frac{k}{2}(\gamma - 1)Ca^{-3\gamma} = 0
\]

(41)

\[
\ddot{\phi} + \frac{3}{a}\dddot{\phi} - dV_0 e^{-d\phi} = 0,
\]

(42)

where

\[
\mu_B = Ca^{-3\gamma}, \quad p_B = (\gamma - 1)Ca^{-3\gamma}.
\]

(43)

\[
\mu_\phi = \frac{1}{2}\varepsilon \dot{\phi}^2 + V_0 e^{-d\phi}, \quad p_\phi = \frac{1}{2}\varepsilon \dot{\phi}^2 - V_0 e^{-d\phi}.
\]

(44)

We look for analytic solutions of these equations.
4.1. Solution using Lie invariance

To find a solution of the system of equations, we need one further condition. We require the solution to be invariant under the action of the extra Lie symmetry (39). To find this solution, we compute the zeroth-order invariants of $X$ using the associated Lagrange system (recall that $1 < \gamma \leq 2$ and $d \neq 0$),

$$\frac{dr}{\gamma t} = \frac{da}{a} = \frac{d\phi}{a}.$$ 

The solution of the system is the two-parameter family of functions:

$$a(t) = t^{\frac{2}{d^2}}, \quad \phi(t) = \frac{2}{d} \ln t. \quad (45)$$

Subsequently, we demand that the (invariant) $a(t)$, $\phi(t)$ we computed are solutions of the field equations (21) and (22). This fixes the constants $V_0$, $C$ in terms of $\gamma$, $d$ as follows:

$$V_0 = \frac{2\varepsilon (2 - \gamma)}{\gamma d^2}, \quad C = \frac{4(3^2 - 3\varepsilon k\gamma)}{3k\gamma^2 d^2}, \quad \bar{E} = 0. \quad (46)$$

We note that in both solutions the DM is eliminated because $\bar{E} = 0$. Concerning the properties of these solutions, we have the following.

(A) In the quintessence case, the scalar field $\phi$ is real. Because $\gamma \leq 2$, $V_0 > 0$, the $V(\phi) > 0$; therefore, the potential is ‘repulsive’. The constant $C$ is related to the matter energy $\mu_B$ via relation (43). For the perfect fluid, $\mu_B$ and $C$ are positive which requires that $d^2 > 3k\gamma$.

(B) In the case of phantom field $\varepsilon = -1$, $\phi$ is complex, $d$ is complex but $V(\phi)$ is real. Furthermore, $V_0 > 0$ and $C \geq 0$ provided $|d^2| \geq 3k\gamma$.

When $|d^2| = 3k\gamma$, $C = 0$ and we have the empty space solution with either a quintessence ($\varepsilon = +1$) or a phantom field ($\varepsilon = -1$). This solution has only the unspecified parameter $\gamma$. We note that when $\gamma = 2$ (stiff matter), the constant $V_0 = 0$ which is not acceptable. Therefore, the case $\gamma = 2$ must be treated separately.

Perfect fluid solutions with a linear EoS and a scalar field with exponential potential have been considered previously in [32]. It can be shown that our solution contains the solution of [32]. It is to be noted that we have derived the solution using only fundamental symmetry assumptions and not ad hoc statements. Furthermore, we have covered the case of the phantom field.

For all cases with $\gamma < 2$, we have a genuine perfect fluid solution minimally interacting with a scalar field. At late time both $\mu_B$, $p_B \to 0$, that is, the solution (45) for late time tends to the empty space solution with a scalar field. These results agree with known results (see [22, 24, 26]).

5. Noether symmetry

In an alternative approach, we consider a stronger symmetry assumption and require that the dynamical system defined by Lagrangian (35) admits a Noether symmetry. It is known that, contrary to the Lie symmetry, the Noether symmetry gives two conditions on the system of ODEs which suffice to produce the analytic solution of the dynamical system.

Because the Noether symmetries are Lie symmetries and we have only one extra Lie symmetry, the Noether symmetry must coincide with the Lie symmetry (39) we found above. This leads to the value of the parameter $\gamma = 2$. Therefore, the requirement of Noether symmetry has the following implications: (a) selects the EoS $p_B = \mu_B$; (b) the original field equations (21) and (22) are non-homogeneous; (c) selects the scalar field potential $V(\phi) = V_0 e^{-d\phi}$.
where \( V_0 \) is a constant, (d) fixes the Noether symmetry vector (38) and (e) provides the Noether integral
\[
I_2 = 2t \dot{E} - 4a^2 \dot{a} + \frac{4}{d} \varepsilon k \dot{a} \dot{\phi}.
\] (47)

In the original coordinates \( a, \phi \), we have
the Lagrangian
\[
L = (3a \dot{a}^2 - \frac{1}{2} k \varepsilon a^3 \dot{\phi}^2) + k V_0 a^3 e^{-d \phi} + k \tilde{C} a^{-3},
\] (48)

the Hamiltonian
\[
\tilde{E} = (3a \dot{a}^2 - \frac{1}{2} k \varepsilon a^3 \dot{\phi}^2) - k a^3 V_0 e^{-d \phi} - k \tilde{C} a^{-3},
\] (49)

and the equations of motion
\[
\ddot{a} + \frac{1}{2} \dot{a}^2 + \frac{k}{4} a \dot{\phi}^2 - \frac{k}{2} a V_0 e^{-d \phi} + \frac{k}{2} \tilde{C} a^{-5} = 0
\] (50)

\[
\ddot{\phi} + \frac{3}{2} \dot{\phi} - d \varepsilon V_0 e^{-d \phi} = 0.
\] (51)

We conclude that the requirement of the existence of an extra Noether symmetry results in a model consisting of a kination, a scalar field with an exponential potential and dust.

5.1. The analytic solution

We determine the analytic solution of the system of field equations (50), (51) under constraint (49). In the solution, we follow a procedure initiated in [22]. We introduce the new variables \( \{u, v\} \) with the relations
\[
u = \frac{\sqrt{6k \varepsilon}}{4} \dot{\phi} + \frac{1}{2} \ln (a^3)
\] (52)
\[
u = -\frac{\sqrt{6k \varepsilon}}{4} \dot{\phi} + \frac{1}{2} \ln (a^3).
\] (53)

In the new variables, Lagrangian (48) is
\[
L = e^{(u + v)} \left[ \frac{4}{3} \dot{u}^2 + \dot{v} e^{-2K(u - v)} + \tilde{C} e^{-2(1 + K)(u - v)} \right],
\] (54)

where \( K = \frac{d}{\sqrt{6k \varepsilon}} V_0 = k V_0 \). Next we consider a change in the time coordinate
\[
\frac{dt}{dv} = \sqrt{\frac{3k V_0}{4}} e^{-K(u - v)}.
\] (55)

Using the variation integral, the Lagrangian becomes
\[
L = e^{(1 - K)u} e^{(1 + K)v} (u' v' + 1 + \tilde{C} e^{-2(1 - K)u} e^{-2(1 + K)v}),
\] (56)

where \( \tilde{C} = \frac{1}{3k V_0} C \) and the Hamiltonian
\[
\tilde{E} = e^{(1 - K)u} e^{(1 + K)v} (u' v' - 1 - \tilde{C} e^{-2(1 - K)u} e^{-2(1 + K)v}).
\] (57)

The resulting equations of motion are
\[
u'' + (1 - K) u^2 - (1 + K)(1 - \tilde{C} e^{-2(1 - K)u} e^{-2(1 + K)v}) = 0
\] (58)
\[
u'' + (1 + K) v^2 - (1 - K)(1 - \tilde{C} e^{-2(1 - K)u} e^{-2(1 + K)v}) = 0.
\] (59)

We consider the cases \( K = 1 \) and \( K \neq 1 \).
5.1.1. The case $K = 1$. For $K = 1$, the system of equations (58), (59) becomes

$$u'' - 2(1 - \tilde{C}e^{-4\nu}) = 0 \quad (60)$$

$$v'' + 2v^2 = 0 \quad (61)$$

and the Hamiltonian (57)

$$\tilde{E} = e^{2\nu}(u'v' - 1 - \tilde{C}e^{-4\nu}). \quad (62)$$

The solution of the system (60)–(62) is

$$u(\tau) = \tau^2 + \frac{1}{2} \tilde{C} \ln (\tau + c) + \frac{1}{2} (\tilde{E} + 2c) \tau \quad (63)$$

$$v(\tau) = \frac{1}{2} \ln (2\tau + 2c). \quad (64)$$

Replacing in (52)–(54), we find the analytic solution

$$ds^2 = -N^2(\tau)d\tau^2 + a^2(\tau)\delta_{ij}dx^i dx^j, \quad (65)$$

where

$$a^3(\tau) = \sqrt{2(\tau + c)e^{\tilde{C}(\tau + c)} e^{2(\tilde{E} + 2c)\tau}} \quad (66)$$

$$\phi(\tau) = \frac{2}{\sqrt{6k \bar{E}}} \left[ \tau^2 + (\tilde{E} + 2c) \tau + \frac{1}{2} \ln \left( \frac{1}{2} (\tau + c)^{\tilde{C}-1} \right) \right] \quad (67)$$

$$N(\tau) = \frac{2}{\sqrt{3k V_0}} \exp \left( \frac{\sqrt{6k \bar{E}}}{2} \phi(\tau) \right). \quad (68)$$

5.1.2. The case $K \neq 1$. Subcase $K < 1$. In this case the analytic solution of the system of equations ((50) and (49)) is (for details, see the appendix)

$$a^3(\tau) = \frac{3}{8} (1 - K)^{(1-K)} (1 + K)^{(1+K)} w(\tau)^2 e^{-2Kp(\tau)} \quad (69)$$

$$\phi(\tau) = \frac{2}{\sqrt{6k \bar{E}}} \left[ \ln \left( \frac{(1 - K)^{1-K}}{(1 + K)^{1+K}} \right) + K \ln \frac{8}{3w(\tau)^2} + 2p(\tau) \right] \quad (70)$$

$$N(\tau) = \frac{2}{\sqrt{3k V_0}} \exp \left( \frac{\sqrt{6k \bar{E}}}{2} - K \phi(\tau) \right). \quad (71)$$

where

$$w(\tau) = \sqrt{A e^{2\omega t} + \frac{\bar{E}}{\omega^2} + B e^{-2\omega t}} \quad (72)$$

$$p(\tau) = \frac{p_0 \omega}{\sqrt{4\omega^4 AB - \bar{E}^2}} \arctan \left( \frac{2\omega^2 A e^{2\omega t} - \bar{E}}{\sqrt{4\omega^4 AB - \bar{E}^2}} \right) \quad (73)$$

and the constants $A, B, p_0, \tilde{C}, \tilde{E}, \omega$ are related to the constraint

$$\frac{2\tilde{C}}{\omega^2} = -2\omega^2 AB - \frac{p_0^2}{2} + \frac{\bar{E}^2}{2\omega^2}. \quad (74)$$
Subcase $K > 1$. In this case, the analytic solution of the system of equations (50), (49) is

$$a^3(\tau) = \frac{3}{8} (K - 1)^{(1-K)} (1 + K)^{(1+K)} \bar{w}(\tau)^2 e^{-2K\bar{p}(\tau)}$$

(75)

$$\phi(\tau) = \frac{2}{\sqrt{6k\epsilon}} \left( \ln \left( \frac{(K - 1)^{1-K}}{(1 + K)^{1+K}} \right) + K \ln \frac{8}{3\bar{w}(\tau)^2} + 2\bar{p}(\tau) \right)$$

(76)

$$N(\tau) = \frac{2}{\sqrt{3kV_0}} \exp\left( \frac{\sqrt{6k\epsilon}}{2} K\phi(\tau) \right),$$

(77)

where

$$\bar{w}(\tau) = \sqrt{\bar{A} e^{2i\bar{\omega}t} - \bar{E} e^{-2i\bar{\omega}t}}.$$ 

(78)

$$\bar{p}(\tau) = -\frac{i\bar{p}_0}{\sqrt{4\bar{\omega}^4AB - E^2}} \arctan \left( \frac{2\bar{\omega}^2\bar{A} e^{2i\bar{\omega}t} - \bar{E}}{\sqrt{4\bar{\omega}^4AB - E^2}} \right),$$

(79)

and the constants $\bar{A}, \bar{B}, \bar{p}_0, \bar{C}, \bar{E}, \bar{\omega}$ are related to the constraint

$$2\bar{C} \frac{\bar{C}}{\bar{\omega}^2} = -2\bar{\omega}^2\bar{A}\bar{B} + \frac{\bar{p}_0^2}{2} + \frac{\bar{E}^2}{2\bar{\omega}^2},$$

(80)

Our solution reduces to that of [27] if we set $\bar{E} = 0$, that is, $\mu_{DM} = 0$ and to that of [11] if we set $C = 0$, that is, $p_B = \mu_B = 0$. Finally if we set $C = 0, \bar{E} = 0$, we obtain the solution of [22].

6. Conclusion

In order to model all types of matter in the Universe, we have considered a mixture of three fluids, a perfect fluid with a linear EoS, a dust for DM and a scalar field for DE. All fluids are assumed to be minimally interacting and the background space is assumed to be the flat FRW spacetime. The available field equations do not suffice to determine the dynamical system. We need two more assumptions, one which will specify the scalar field potential and another which will make the system of dynamical equations solvable. In order to define these conditions, we follow two steps. First we show that the system of the three fluids is equivalent to a two-dimensional dynamical system moving in $M^2$ under the constraint $\bar{E} = \text{constant}$. Then we require that the two-dimensional system admits an extra Lie symmetry. This requirement fixes the potential to be exponential. Requiring further that the solution is invariant under the Lie symmetry, we obtain a two-parameter family of analytic solutions containing a perfect fluid and a (quintessence of phantom) scalar field with an exponential potential.

Requiring next that the Lagrangian defining the two-dimensional dynamical system admits an extra Noether symmetry, we obtain a new analytic solution which involves all three types of matter, that is, stiff matter, scalar field and dust. It is interesting (see the appendix) that in this case the dynamical system reduces to the Ermakov–Pinney dynamical system which is known to be integrable [20, 21]; hence, we are able to obtain the analytic solution. This solution includes previously found solutions in [11, 22, 27] as special cases.
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Appendix

A.1 Analytic solutions for $K \neq 1$

Subcase $K < 1$. We introduce new coordinates $w$, $p$ with the relations

$$u = \frac{\ln \left( \sqrt{\frac{3}{2}(1-K)} w e^p \right)}{1-K}, \quad v = \frac{\ln \left( \sqrt{\frac{3}{2}(1+K)} w e^{-p} \right)}{1+K}. \quad (A.1)$$

Then Lagrangian (54) becomes

$$L = \frac{1}{2} w'^2 - \frac{1}{2} w^2 p'^2 + \frac{1}{2} \omega^2 w^2 - \frac{2\tilde{C}}{\omega^2} \frac{1}{w^2},$$

where $\omega^2 = 1 - K^2$. This is the Lagrangian for the hyperbolic Ermakov–Pinney dynamical system which is known to be integrable [20, 21]. The equations of motion are

$$w'' + w p'^2 - \omega^2 w + \frac{4\tilde{C}}{\omega^2} \frac{1}{w^3} = 0 \quad (A.2)$$

$$p'' + \frac{2}{w} p' w' = 0. \quad (A.3)$$

The Hamiltonian is

$$\tilde{E} = \frac{1}{2} w'^2 - \frac{1}{2} w^2 p'^2 - \omega^2 w^2 - \frac{2\tilde{C}}{\omega^2} \frac{1}{w^2}. \quad (A.4)$$

The solution of the system of equations is

$$w(\tau) = \sqrt{A e^{2\omega \tau} - \frac{\tilde{E}}{\omega^2} + B e^{-2\omega \tau}} \quad (A.5)$$

$$p(\tau) = \frac{p_0 \omega}{\sqrt{4\omega^4 AB - \tilde{E}^2}} \arctan \left( \frac{2\omega^2 A e^{2\omega \tau} - \tilde{E}}{\sqrt{4\omega^4 AB - \tilde{E}^2}} \right). \quad (A.6)$$

Using the Hamiltonian, we find the constraint

$$\frac{2\tilde{C}}{\omega^2} = -2\omega^2 AB - \frac{p_0^2}{2} + \frac{\tilde{E}^2}{2\omega^2}, \quad (A.7)$$

where $A$, $B$, and $p_0$ are constants.

Subcase $K > 1$. In this case, we apply the transformation

$$u = \frac{\ln \left( \sqrt{\frac{3}{2}(K-1)} \bar{w} e^\tilde{p} \right)}{1-K}, \quad v = \frac{\ln \left( \sqrt{\frac{3}{2}(K+1)} \bar{w} e^{-\tilde{p}} \right)}{1+K} \quad (A.8)$$

and Lagrangian (54) becomes

$$L = -\frac{1}{2} \bar{w}'^2 + \frac{1}{2} \bar{w}^2 \tilde{p}'^2 + \frac{1}{2} \omega^2 \bar{w}'^2 + \frac{2\tilde{C}}{\omega^2} \frac{1}{\bar{w}^2}. \quad (A.9)$$
where $\tilde{\omega} = K^2 - 1$. This is the Lagrangian for the harmonic Ermakov–Pinney dynamical system.

The solution of the Euler–Lagrange equations is

$$\tilde{u}(\tau) = \sqrt{\tilde{A} e^{2i\tilde{\omega} \tau} - \tilde{E} + \tilde{B} e^{-2i\tilde{\omega} \tau}}$$  \hspace{1cm} (A.10)$$

$$\tilde{p}(\tau) = -\frac{i\tilde{\omega} \tilde{\rho}_0}{\sqrt{4\tilde{\omega}^2 \tilde{A} \tilde{B} - \tilde{E}}} \arctan \left( \frac{2\tilde{\omega}^2 \tilde{A} e^{2i\tilde{\omega} \tau} - \tilde{E}}{\sqrt{4\tilde{\omega}^2 \tilde{A} \tilde{B} - \tilde{E}^2}} \right)$$  \hspace{1cm} (A.11)$$

where $\tilde{A}, \tilde{B}$ and $\tilde{\rho}_0$ are constants which are related to the constraint equation

$$\frac{2\tilde{C}}{\tilde{\omega}^2} = -2\tilde{\omega}^2 \tilde{A} \tilde{B} + \frac{\tilde{\rho}_0^2}{2} + \frac{\tilde{E}^2}{2\tilde{\omega}^2}.$$  \hspace{1cm} (A.12)$$

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