Approximate Solutions and Error Bounds for Continuous-Time Separated Linear Programming Problem and Its Extensions

Ahmed Rasheed Khlefh¹, Rasheed Al-Salih¹, and Watheq Laith¹.
¹Department of Statistics, College of Administration and Economics, University of Sumer, Iraq
rbahhd@mst.edu

Abstract. Continuous-time separated linear programming problems have a wide range of real-world applications such as in business, economics, finances, communications, manufacturing, and so on. In this paper, we extend the technique that is presented by Wen et al. [1] for two classes of these problems. We introduce both primal and dual models for separated problems. In addition, by using discrete problems we obtain approximate solutions with error bounds. Moreover, we establish a computational procedure, to solve any separated continuous-time model and any state-constrained separated model. Furthermore, after we put the separated problem in the form that is presented by Wen et al. [1], we conclude that approximate solutions converge to an optimal solution for continuous-time separated problems.

Keywords. Continuous time linear programming, feasible solution, primal and dual model, optima solution

1. Introduction
Continuous-time linear programming problems have a wide range of real-world applications such as in economics, finances, communications, manufacturing, transportation, and so on. These problems were first studied by Bellman [2] as a bottleneck process. He established duality theorems. Levinson [3] and Tyndall [4] studied the strong duality theorem for these classes of problems. Grinold [5] has presented abstract proof for the strong duality theorem for continuous-time problems. A discrete approach to continuous-time linear programming presented by Buie and Abrahim [6]. Wen et al. [7] have established an approximation approach to solve continuous-time problems. Anderson and Nash [7][8] and Pullan [9][10][11] introduced a new class called separated continuous-time linear programming problems.

These types of problems used to model a wide range of real-world applications such as, job-shop scheduling problems. Weiss [12] and Wang et al. [13] presented new approaches for solving separated models. Luo and Bertsimas [14] presented an extension of the continuous-time separated model called state-constrained separated model. Duality theorems for separated continuous-time linear program and its extensions studied by Xiaoqing [15]. Some real-world applications of separated continuous-time linear programs presented by Wang [16] [17]. Al-Salih and Bohner [18][19][20] presented time scales...
formulation for these types of problems. A quantum calculus formulation for different classes of programming problems introduced by Al-Salih et al. [21][22][23] .

In this paper, we present a computational procedure for continuous-time separated linear programming problem and its extensions. The new approach is an application of the technique that is presented by Wen et al. [1]. The paper is organized as follows: In Section 2, we recall some recent results by Wen et al. [1] about continuous-time linear programming problems. In Section 3, the primal and the dual continuous-time separated linear programming models are formulated. In Section 4, we present state-constrained separated primal and dual models. Computational procedure for solving the separated problems is presented in Section 5. In Section 6, conclusions are given.

2. Continuous-time linear programming problems

In this section, we recall the continuous-time problem \((\text{CLP})\) that is presented in [1,2,3,4].

The primal problem is formulated as

\[
\begin{align*}
\text{Max } U(x) &= \int_{0}^{T} f'(t)x(t)dt \\
\text{S.T. } B(t)x(t) &\leq g(t) + \int_{0}^{t} K(t,s)x(s)ds \\
x(t) &\geq 0, t \in [0,T]
\end{align*}
\]

The dual problem \((\text{CLD})\) is formulated as

\[
\begin{align*}
\text{Min } V(z) &= \int_{0}^{T} g'(t)z(t)dt \\
\text{S.T. } B'(t)z(t) &\geq f(t) + \int_{0}^{t} K'(s,t)z(s)ds \\
z(t) &\geq 0, t \in [0,T]
\end{align*}
\]

\(f \in C([0,T],\mathbb{R}^n)\), \(g \in C([0,T],\mathbb{R}^m)\), \(B(t)\) and \(K(t,s)\) are \(m \times n\) matrices. “′” is the transpose of the matrix. The components of \(x(t), z(t), B(t), K(t,s), g(t), \) and \(f(t)\) are bounded measurable functions.

Wen in [1] presented the following computational procedure to solve (CLP).

1- Set the error tolerance \(\varepsilon\) and the initial value of natural number \(n \in \mathbb{N}\).
2- Evaluate the values of the following \(\varepsilon_n, \varepsilon_n, c, \sigma, \mathcal{K}\)
\[
\varepsilon_n := \max_{1 \leq j \leq n} \sup_{t \in [0,1]} \{|f_j(t) - (f_j)^n(t)|\}
\]
\[
\varepsilon_n := \max_{1 \leq j \leq m} \sup_{t \in [0,1]} \{|g_j(x) - (g_j)^n(t)|\}
\]
\[
c := \max_{1 \leq j \leq m} \{|f_j(t)|, j = 1,2,\ldots, n\}
\]
\[
\sigma := \min \{B_{ij} : B_{ij} > 0\}
\]
\[
\mathcal{K} := \max_{1 \leq j \leq n} \left\{\sum_{k=0}^{n} K_{ij}\right\}
\]
3- Evaluate \(\theta_n\) as follows.
\[
\theta_n := \varepsilon_n^\sigma \left(\frac{Tc}{\sigma}\right)^{\frac{\sigma}{c}} e^\sigma (n + e^{\sigma n} T_{\sigma} - 1) + (\varepsilon_n^\sigma \left(\frac{Tc}{\sigma}\right)^{\frac{\sigma}{c}} e^\sigma) \int_{0}^{T} e^\sigma (T-t) \, g(t) \, U \, dt
\]
4- If \(\theta_n > \varepsilon\), then update \(n\) and go to step(2); otherwise go to (5).
5- Evaluate the values of the following.
\[
b_{l\ell}^{(n)} := (b_{l\ell 1}^{(n)}, b_{l\ell 2}^{(n)}, \ldots, b_{l\ell p_l}^{(n)})^T \in \mathbb{R}^p
\]
\[
c_{i\ell}^{(n)} := (c_{i\ell 1}^{(n)}, c_{i\ell 2}^{(n)}, \ldots, c_{i\ell p_l}^{(n)})^T \in \mathbb{R}^q
\]
Where
\[
\sigma := \min \{B_{ij} : B_{ij} > 0\}
\]
\[
b_{l\ell}^{(n)} = \min \{g_{l(x)} : x \in \left\{\frac{(i-1)}{n}, \frac{1}{n}, \frac{1}{n}, \ldots, \frac{i}{n}, \frac{1}{n}, T\right\}\}
\]
\[
b_{\ell i}^{(n)} = \min \{f_j(x) : x \in \left\{\frac{(i-1)}{n}, \frac{1}{n}, \frac{1}{n}, \ldots, \frac{i}{n}, \frac{1}{n}, T\right\}\}
\]
and \(P_n = \left\{0, \frac{1}{n}, T, \frac{2}{n}, T, \ldots, \frac{n-1}{n}, T, T\right\}\) be a partition of \([0,1]\) in to \(n\) subintervals with equal length \(\frac{T}{n}\).
6- Formulate the finite dimensional \(P_n\) and \(D_n\) using the values obtained in step (5) as follows.

\[
\text{Max } U(x) = \frac{T}{n} \sum_{i=1}^{n} (c_{i\ell}^{(n)})^T \, x_i
\]
3.  Continuous-time separated linear programming problems (CSLP)

The primal problem is formulated as

\[
\begin{align*}
& \text{Max} \ U(x) = \int_0^T f'(t)x(t)dt \\
& \text{S.T.} \quad \int_0^T K(t,s)z(t)ds \leq a(t) \quad \text{(CSLP)} \\
& \quad B(t)x(t) \leq b(t) \\
& \quad x(t) \geq 0, \ t \in [0,T] \\
& \text{Rewriting the two inequalities in (CSLP) as one inequality} \\
& \quad \begin{pmatrix} 0 \\ B(t) \\ 0 \end{pmatrix} x(t) \leq \begin{pmatrix} a(t) \\ b(t) \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} -K(t,s) \\ 0 \\ 0 \end{pmatrix} x(s)ds,
\end{align*}
\]

So, we can put (CSLP) in (CLP), and then rewrite the dual as

\[
\begin{align*}
& \text{Min} \ V(y,z) = \int_0^T \begin{pmatrix} a'(t) \\ b'(t) \\ c'(t) \end{pmatrix} y(t) + \begin{pmatrix} -K(t,s) \\ 0 \\ 0 \end{pmatrix} z(t) + \int_0^t \begin{pmatrix} g' \end{pmatrix} U dt \\
& \quad \text{S.T.} \quad \int_0^T K(t,s)y(t)ds + B'(t)z(t) \geq f(t) \quad \text{(CSLDP)} \\
& \quad y(t), \ z(t) \geq 0, \ t \in [0,T]
\end{align*}
\]

4. Continuous-time state-constrained separated problems (CSCSLP)

The primal problem is formulated as

\[
\begin{align*}
& \text{Max} \ U(u, x) = \int_0^T g'(t)u(t) + f'(t)x(t)dt \\
& \text{S.T.} \quad \int_0^T K(t,s)u(s)ds + B(t)x(t) \leq a(t) \quad \text{(CSCSLP)} \\
& \quad G(t)u(t) \leq b(t) \\
& \quad H(t)x(t) \leq c(t) \\
& \quad x(t) \geq 0, \ t \in [0,T] \\
& \text{Rewriting the three inequalities in (CSCSLP) as one inequality} \\
& \quad \begin{pmatrix} G(t) \\ 0 \\ 0 \\ 0 \\ H(t) \end{pmatrix} \begin{pmatrix} u(t) \\ x(t) \\ x(t) \\ c(t) \end{pmatrix} \leq \begin{pmatrix} a(t) \\ b(t) \\ 0 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} -K(t,s) \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} u(s) \\ x(s) \end{pmatrix} ds,
\end{align*}
\]

so, we can put (CSCSLP) in (CLP), and then rewrite the dual as

\[
\begin{align*}
& \text{Min} \ V(y, z, w) = \int_0^T \begin{pmatrix} a'(t) \\ b'(t) \\ c'(t) \end{pmatrix} y(t) + \begin{pmatrix} -K(t,s) \\ 0 \\ 0 \end{pmatrix} z(t) + \int_0^t \begin{pmatrix} g' \end{pmatrix} U dt \\
& \quad \text{S.T.} \quad \int_0^T K(t,s)y(t)ds + G'(t)w(t) + B'(t)z(t) \geq f(t) \quad \text{(CSCSLD)} \\
& \quad y(t), \ w(t), \ z(t) \geq 0, \ t \in [0,T]
\end{align*}
\]
5. Computational procedure for solving continuous-time separated problems

In this section, we present a computational procedure to solve any separated problem. In fact, this procedure is an extension to the one that is presented by Wen [1].

1. Set the error tolerance \( \epsilon \) and the initial value of natural number \( n \in \mathbb{N} \).

2. Evaluate the values of the following

\[
\epsilon_n = \max_{1 \leq j \leq n} \sup_{t \in [0,1]} \{ f_j(t) - (f_j)^n(t) \}
\]

3. Set the function as follows.

\[
\theta_n = \epsilon_n^\sigma \left( n + e^{T \sigma} - 1 \right)^+ \epsilon_n^\sigma \int_0^T e^{\rho(T-t)} g^T(t) U dt
\]

4. If \( \theta_n < \epsilon \), then update \( n \) and go to step 2; otherwise, go to 5.

5. Evaluate the values of the following.

\[
b_l^{(n)} := (b_{l1}^{(n)}, b_{l2}^{(n)}, \ldots, b_{lp}^{(n)})^T \in \mathbb{R}^p
\]

\[
c_l^{(n)} := (c_{l1}^{(n)}, c_{l2}^{(n)}, \ldots, c_{lp}^{(n)})^T \in \mathbb{R}^q
\]

Where

\[
b_{l1}^{(n)} = \min \{ g_i(x) : x \in \left\{ \frac{1}{n} T, \frac{1}{n} T, \ldots, \frac{n-1}{n} T \right\} \}
\]

\[
b_{l2}^{(n)} = \min \{ f_j(x) : x \in \left\{ \frac{1}{n} T, \frac{1}{n} T, \ldots, \frac{n-1}{n} T \right\} \}
\]

6. Formulate the finite dimensional continuous-time state-constrained separated primal problem (CSCSLP) or the finite dimensional continuous-time separated primal problem (CSP) and the finite dimensional continuous-time state-constrained separated dual problem (CSCSDL) or the finite dimensional continuous-time separated dual problem (CSDL) using the values obtained in step 5.

7. Use the efficient algorithm to obtain optimal solutions for CSCSLP(CSP) and CSCSDL(CSDL).

8. Evaluate \( \epsilon_n \) as follows.

\[
\epsilon_n = \epsilon_n^\sigma \left( n + e^{T \sigma} - 1 \right)^+ \epsilon_n^\sigma \int_0^T e^{\rho(T-t)} g^T(t) U dt
\]

9. Set the step function defined as follows which will be the approximate solution of CSCSLP (CSP).

\[
(x_l)_{i}^{(n)(t)} = \begin{cases} (x_i)_{l}^{-n}, & \text{if } \frac{(i-1)T}{n} \leq t \leq \frac{iT}{n} \text{ for some } 1 \leq l \leq n \\ (x_i)_{l}^{n}, & \text{if } t = T \\ \end{cases}
\]

6. Conclusions

Computational procedure is presented for two classes of continuous-time separated problems. Our approach is an extension for the approach that has been presented by Wen et al. [1]. We formulate the primal separated model and the dual separated model. Also, we obtain approximate solutions with error bounds for continuous-time separated model and continuous-time state constrained model. Furthermore, after we rewrite separated problems in the form that has been presented by Wen et al. [1] and using the same theory, we conclude that approximate solutions converge to an optimal solution for continuous-time separated linear programming problems and state constrained continuous-time separated linear programming problems.

References
[1] H.-C. Wen, Ching-Feng and Lur, Yung-Yih and Lai, “Approximate solutions and error bounds for a class of continuous-time linear programming problems,” Optim. A J. Math. Program. Oper. Res., vol. 61, no. 2, pp. 163–185, 2012.

[2] R. E. Bellman, Dynamic programming. Princeton University Press, Princeton, NJ, 2010.

[3] N. Levinson, “A class of continuous linear programming problems,” J. Math. Anal. Appl., vol. 16, pp. 73–83, 1966.

[4] WILLIAM and F. TYNDALL, “On Two Duality Theorems for Continuous Programming Problems,” J. Math. Anal. Appl., vol. 14, pp. 6–14, 1970.

[5] C. Richard, “Continuous Programming Part I: Linear Objectives,” J. Math. Anal. Appl., 28:32–51, 1969.

[6] R. N. Buie and J. Abraham, “Some remarks concerning duality for continuous-time programming problems,” J. Math. Anal. Appl., vol. 114, no. 2, pp. 468–489, 1986.

[7] E. J. Anderson, “A continuous model for job-shop scheduling,” University of Cambridge, 1978.

[8] P. Anderson, Edward J. and Nash, Linear programming in infinite-dimensional spaces. John Wiley & Sons, Ltd., Chichester, 1987.

[9] M. C. Pullan, “An algorithm for a class of continuous linear programs,” SIAM J. Control Optim., vol. 31, no. 6, pp. 1558–1577, 1993.

[10] M. C. Pullan, “Existence and duality theory for separated continuous linear programs,” Math. Model. Syst., vol. 3, no. 3, pp. 219–245, 1997.

[11] M. C. Pullan, “An extended algorithm for separated continuous linear programs,” Math. Program. Ser. B, vol. 93, no. 3, pp. 415–451, 2002.

[12] G. Weiss, “A simplex based algorithm to solve separated continuous linear programs,” Math. Program., vol. 115, no. 1, pp. 151–198, 2008.

[13] X. Wang, S. Zhang, and D. D. Yao, “Separated continuous conic programming: strong duality and an approximation algorithm.” SIAM J. Control Optim., 48(4):2118–2138, 2009.

[14] X. Luo and D. Bertsimas, “A New algorithm for state-constrained separated,” vol. 37, no. 1, pp. 177–210, 1998.

[15] X. Wang, “Theory and algorithms for separated linear programming and its extensions,” The Chinese University of Hong Kong, 2005.

[16] X. Wang, “Review on the research for separated continuous linear programming: With applications on service operations,” Math. Probl. Eng., vol. 2013, no. 2, 2013.

[17] X. Wang, “Emergency department staffing: A separated continuous linear programming approach,” Math. Probl. Eng., vol. 2013.

[18] R. Al-Salih and M. Bohner, “Linear programming on time scales,” Applicable Analysis and Discrete Mathematics, vol. 12, pp. 192–204, 2018.

[19] R. Al-Salih and M. Bohner, “Linear fractional programming problems on time scales,” Journal of Numerical Mathematics and Stochastics, vol. 11, no. 1, pp. 1–18, 2019.

[20] R. Al-Salih and M. Bohner, “Separated and state-constrained separated linear programming problems on time scales,” Boletim da Sociedade Paranaense de Matemática, vol. 4, pp. 181–195, 2020.

[21] R. Al-salih, A. Habeeb, and W. Laith, “A Quantum calculus analogue of dynamic Leontief production model with linear objective function,” Journal of Physics conference series, Vol. 1234, 2019.

[22] R. Al-salih “Dynamic network flows in quantum calculus,” Journal of Mathematical analysis and applications, vol. 18, no. 1, pp. 53–66, 2020.

[23] R. Al-salih, A. Habeeb, and W. Laith, “A Quantum calculus analogue of dynamic Leontief production model with quadratic objective function,” Journal of Engineering and Applied Sciences,14:6415-6418,2019.