Abstract

We present an improved orderly algorithm for constructing all unlabelled lattices up to a given size, that is, an algorithm that constructs the minimal element of each isomorphism class relative to some total order.

Our algorithm employs a stabiliser chain approach for cutting branches of the search space that cannot contain a minimal lattice; to make this work, we grow lattices by adding a new layer at a time, as opposed to adding one new element at a time, and we use a total order that is compatible with this modified strategy.

The gain in speed is between one and two orders of magnitude. As an application, we compute the number of unlabelled lattices on 20 elements.

1 Introduction

Enumerating all isomorphism classes of unlabelled lattices, in the sense of systematically constructing a complete list of isomorphism classes up to a certain size threshold, is a difficult combinatorial problem. The number \( u_n \) of isomorphism classes of unlabelled lattices on \( n \) elements grows faster than exponentially in \( n \) [KL71, KW80], as does the number of (labelled) representatives of each isomorphism class. Indeed, the largest value of \( n \) for which \( u_n \) has been published previously is \( n = 19 \) [JL15, Slo].

When trying to enumerate combinatorial objects modulo isomorphism, one typically faces the problem that the number and the size of the isomorphism classes are so large that trying to weed out isomorphic objects through explicit isomorphism tests is out of the question. Instead, an orderly algorithm is needed, that is, an algorithm that traverses the search space in such a way that every isomorphism class is encountered exactly once.

A general strategy for the construction of isomorphism classes of combinatorial objects using canonical construction paths was described in [McK98]. Orderly algorithms for enumerating isomorphism classes of unlabelled lattices, as well as special subclasses of unlabelled lattices, were given in [HR02, JL15]. The fastest published method currently is the one described in [JL15]; the computations of \( u_{18} \) and \( u_{19} \) reported in [JL15] took 26 hours respectively 19 days on 64 CPUs.

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The algorithms described in [HR02, JL15] follow a similar strategy: (i) A total order \(<_{\text{wt}}\) on all labelled lattices of a given size is defined. (ii) Starting from the (unique) \(<_{\text{wt}}\)-minimal lattice on 2 elements, the \(<_{\text{wt}}\)-minimal labelled representative of each isomorphism class of unlabelled lattices with at most \(n\) elements is constructed using a depth first search, where the children of a parent lattice are obtained by adding a single new element covering the minimal element of the parent lattice. (iii) For each parent lattice, one child is obtained for every choice for the covering set of the added element that yields a labelled lattice that is \(<_{\text{wt}}\)-minimal in its isomorphism class of unlabelled lattices.

It is the test for \(<_{\text{wt}}\)-minimality in step (iii) that takes most of the time: While there are some necessary conditions that are easy to verify, ensuring that the newly constructed labelled lattice is indeed \(<_{\text{wt}}\)-minimal in its isomorphism class requires checking the candidate covering set of the added element against all possible relabellings of the elements of the existing lattice; details are given in Section 2. Basically, one has a certain permutation group that acts on a configuration space of covering sets, and one must verify that a given candidate is minimal in its orbit.

It turns out that the elements of a \(<_{\text{wt}}\)-minimal labelled lattice are arranged by levels (cf. Section 2), and thus it is tempting to construct and test candidate covering sets of a new element level by level, exploiting the levelised structure for a divide-and-conquer approach; such an approach promises two advantages: (i) The orbit of the restriction of a candidate covering set to a given level is potentially much smaller than the orbit of the complete covering set. (ii) The entire branch of the search space that corresponds to the candidate configuration of covers on the given level can potentially be discarded in a single test.

However, we shall see that the constructions from [HR02, JL15] do not adapt well to this levelised approach: In order to make the levelised approach work, we need to modify the depth-first-search to add one level at a time as opposed to one element at a time, and we need to modify the total order to be level-major.

The structure of the paper is as follows: In Section 2, we recall some results from [HR02, JL15] that are needed later, and we interpret the total order used in [HR02, JL15] as row-major. In Section 3, we describe our new construction using a level-major order and prove the results required to establish its correctness. In Section 4, we remark on implementation details and compare the performance of our new approach to that of those published in [HR02, JL15].

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2 Background

We start by giving a brief summary of the algorithms from [HR02, JL15]. We refer to these sources for details.
Definition 1. A finite bounded poset \( L \) is an \( n \)-poset, if the elements of \( L \) are labelled 0, 1, \ldots, \( n-1 \), where 0 is a lower bound of \( L \) and 1 is an upper bound of \( L \). An \( n \)-poset that is a lattice is called an \( n \)-lattice. To avoid confusion with the numerical order of integers, we denote the partial order of a \( n \)-poset \( L \) by \( \sqsubseteq_L \) and \( \sqsupseteq_L \), or simply \( \sqsubseteq \) and \( \sqsupseteq \) if the poset is obvious.

Notation 2. Assume that \( L \) is an \( n \)-poset.

For \( a \in L \), we define the shadow of \( a \) as \( \downarrow a = \downarrow_L a = \{ x \in L : x \sqsubseteq_L a \} \) and the shade of \( a \) as \( \uparrow a = \uparrow_L a = \{ x \in L : x \sqsupseteq_L a \} \). For \( A \subseteq L \), we define \( \uparrow A = \bigcup a \in A \uparrow_L a \) as well as \( \downarrow A = \bigcup a \in A \downarrow_L a \).

We say that \( a \in L \) has depth \( \text{dep}(a) = \text{dep}_L(a) = p \), if the maximum length of any chain from 1 to \( a \) in \( L \) is \( p+1 \). Given a non-negative integer \( k \), we call \( \text{lev}_k(L) = \{ a \in L : \text{dep}_L(a) = k \} \) the \( k \)-th level of \( L \). We say that \( L \) is levelwise, if \( \text{dep}_L(i) \leq \text{dep}_L(j) \) holds for all \( 0 < i \leq j < n \).

For \( a, b \in L \), we write \( a \prec_L b \) (or simply \( a \prec b \)) if \( a \) is covered by \( b \) in \( L \), that is, \( a \sqsubseteq_L b \) holds and \( a \sqsubseteq_L x \sqsubseteq_L b \) implies \( x = a \) or \( x = b \). We denote the covering set of \( a \in L \) by \( \bigwedge_L a = \{ x \in L : a \prec x \} \), and we say that \( a \in L \) is an atom in \( L \) if \( 0 \prec_L a \) holds.

If \( L \) is a lattice and \( a, b \in L \), we denote the least common upper bound of \( a \) and \( b \) in \( L \) by \( a \lor_L b \) (or simply \( a \lor b \)), and the greatest common lower bound of \( a \) and \( b \) in \( L \) by \( a \land_L b \) (or simply \( a \land b \)).

2.1 Canonical representatives

The idea of an orderly algorithm is to construct all those lattices that are minimal, with respect to a suitable total order, in their isomorphism class. In this section, we recall the total order used in [HR02, JL15] and some of its properties.

Definition 3. Let \( L \) be an \( n \)-poset.

(a) For \( A \subseteq L \), we define \( \text{wt}_L(A) = \sum_{j \in A} 2^j \).

(b) For \( i \in L \), we define \( \text{wt}_L(i) = \text{wt}_L(\bigvee_L i) \).

(c) We define \( \text{wt}(L) = (\text{wt}_L(2), \text{wt}_L(3), \ldots, \text{wt}_L(n-1)) \).

Ordering \( n \)-lattices lexicographically with respect to \( \text{wt}(L) \), we obtain a total order \( \prec_{\text{wt}} \) on the set of all \( n \)-lattices. We call an \( n \)-lattice \( \prec_{\text{wt}} \)-minimal, if it is minimal with respect to \( \prec_{\text{wt}} \) in its isomorphism class.

Remark 4. An \( n \)-poset \( L \) is completely defined by its covering relation. Indeed, the upper bound 1 is not covered by any element, and it covers precisely those elements that are not covered by any other element. Similarly, the lower bound 0 covers no element, and it is covered precisely by those elements that do not cover any other element. Thus, \( L \) is completely described by specifying the pairs \( (i, j) \), for \( 1 < i < j < n \), for which \( i \prec j \) holds.

The latter information can be interpreted as an \((n-2) \times (n-2)\) matrix over \( \mathbb{F}_2 \), where rows and columns are numbered 2, \ldots, \( n-1 \), and the entry in row \( i \) and column \( j \) indicates whether or not \( i \prec j \) holds. The total order \( \prec_{\text{wt}} \) from Definition 3 then amounts to a right-to-left row-major lexicographic order on the associated matrices; cf. Figure 1.
Theorem 5 ([HR02, Theorem 1]). If $L$ is a $<_{\text{wt}}$-minimal $n$-lattice and one has $0 < i \leq j < n$, then $\text{dep}_L(i) \leq \text{dep}_L(j)$ holds.

Corollary 6. If $L$ is a $<_{\text{wt}}$-minimal $n$-lattice and one has $0 < i$ and $i \prec j$, then $j < i$ holds.

Remark 7. Theorem 5 and Corollary 6 say that a $<_{\text{wt}}$-minimal $n$-lattice $L$ is levellised, that is, that the non-minimal levels of $L$ are filled by elements labelled in their numerical order; cf. Figure 2.

Remark 8. Any relabelling $\pi \in \text{Sym}(\{0, \ldots, n-1\})$ of the elements of an $n$-lattice $L$ preserves the levels, in the sense that $\text{dep}_{\pi(L)}(\pi(i)) = \text{dep}_L(i)$ holds for every $i \in L$. In particular, if $L$ is a levellised lattice with the elements labelled as in Figure 2, then the relabelled lattice $\pi(L)$ is levellised if and only if

$$\pi \in \text{Sym}(\{0\}) \times \text{Sym}(\{1\}) \times \text{Sym}(\{2, \ldots, a_2 - 1\}) \times \text{Sym}(\{a_2, \ldots, a_3 - 1\}) \times \cdots \times \text{Sym}(\{a_k, \ldots, n-1\})$$

holds.

Theorem 9 ([HR02, Theorem 2]). If $L$ is a $<_{\text{wt}}$-minimal $n$-lattice, one has $\text{wt}_L(2) \leq \text{wt}_L(3) \leq \ldots \leq \text{wt}_L(n-1)$.

Remark 10. Theorem 9 says that the rows of the matrices describing the covering relation of a $<_{\text{wt}}$-minimal $n$-lattice $L$ (cf. Figure 1) are sorted in non-decreasing order with respect to a (right-to-left) lexicographic order on the rows.

2.2 Incremental construction

The algorithms from [HR02, JL15] work by traversing a tree of $<_{\text{wt}}$-minimal $n$-lattices in a depth-first manner; the root of the tree is the unique 2-lattice, and an $(n+1)$-lattice $\tilde{L}$ is a descendant of the $<_{\text{wt}}$-minimal $n$-lattice $L$, if $\tilde{L}$ is obtained from $L$ by adding a new cover of 0 (labelled $n$) and $\tilde{L}$ is $<_{\text{wt}}$-minimal.

The lattice $\tilde{L}$ is determined by $L$ and the covering set of the new element $n$; the possible choices for the latter can be characterised effectively.

Definition 11 ([HR02]). If $L$ is an $n$-lattice, a non-empty antichain $A \subseteq L \setminus \{0\}$ is called a lattice-antichain for $L$, if $a \land_L b \in \{0\} \cup (\uparrow_L A)$ holds for any $a, b \in \uparrow_L A$.

Figure 1: Interpreting the order $<_{\text{wt}}$ as right-to-left row-major lexicographic order on the matrices specifying the covering relation. Note that adding a row and a column for 1 does not affect the order: All entries in the added row are 0, and each entry in the added column is determined by the other entries in the same row, and it is checked after those in the lexicographic comparison.
Remark 12. To test the condition in Definition 11, it is clearly sufficient to verify that \( a \land_L b \in \uparrow_L A \) holds for those pairs \((a, b)\) that are minimal in the set

\[ \{(a, b) \in (\uparrow_L A) \times (\uparrow_L A) : a \land_L b \neq 0\} \]

with respect to the product partial order in \( L \times L \).

Theorem 13 ([HR02, Lemma 2]). Let \( L \) be an \( n \)-lattice. A subset \( A \subseteq L \setminus \{0\} \) is a lattice-antichain for \( L \), if and only if \( L \) is a subposet of an \((n+1)\)-lattice \( L_A \) in which \( 0 \prec_L n \) (that is, \( n \) is an atom in \( L_A \)) and \( \land_L A n = A \) hold.

Remark 14. In the situation of Theorem 13, it is clear that the pair \((L, A)\) uniquely determines \( L_A \) and vice versa. Moreover, the covering relation of \( L_A \) is obtained from the covering relation of \( L \) by

(i) adding the pair \((0, n)\);
(ii) adding all pairs of the form \((n, a)\) for \( a \in A \); and
(iii) removing all pairs of the form \((0, a)\) for \( a \in A \) that are present.

As mentioned in Remark 4, the covers of 0 need not be stored explicitly; in this case, only step (ii) is needed.

Indeed, this definition of \( L_A \) makes sense for any \( n \)-poset \( L \) and any \( A \subseteq L \).

It is obvious from the definitions that one has \( wt_{L_A}(n) = wt_L(A) \). Moreover, for \( 1 \leq i < n \), we have \( i \not\prec_L A n \) and thus \( wt_{L_A}(i) = wt_L(i) \) and \( \text{dep}_{L_A}(i) = \text{dep}_L(i) \).

The following two results are consequences of Remark 14 and Theorem 9.

Corollary 15 ([HR02, §3]). If \( L \) is an \( n \)-lattice and \( A \) is a lattice-antichain for \( L \) such that \( L_A \) is \(<_{wt}\)-minimal, then \( L \) is \(<_{wt}\)-minimal.

Corollary 16 ([HR02, §5]). If \( L \) is an \( n \)-lattice and \( A \) is a lattice-antichain for \( L \) such that \( L_A \) is \(<_{wt}\)-minimal, then one has \( A \cap (\text{lev}_{k-1}(L) \cup \text{lev}_k(L)) \neq \emptyset \) for \( k = \text{dep}_L(n-1) \).

2.3 Testing for canonicity

In the light of Corollary 16, there are two cases to consider for testing whether the descendant \( L_A \) of a \(<_{wt}\)-minimal \( n \)-lattice \( L \) defined by a lattice-antichain \( A \) for \( L \) with \( \text{dep}_L(n-1) = k \) is \(<_{wt}\)-minimal:

(A) \( A \cap \text{lev}_k(L) \neq \emptyset \), that is, \( \text{dep}_{L_A}(n) = k + 1 \)

(B) \( A \cap \text{lev}_k(L) = \emptyset \neq A \cap \text{lev}_{k-1}(L) \), that is, \( \text{dep}_{L_A}(n) = k \)

![Figure 2: A levellised \( n \)-lattice; the dashed lines separate the levels.](image)
Case (A):

In this case, the new element $n$ forms a separate level of $L_A$; cf. Figure 3(a). Since $L_A$ is levellised by construction and the non-minimal elements of $L$ correspond to the levels $0, \ldots, k$ of $L_A$, any relabelling $\pi$ of $L_A$ for which $\pi(L_A)$ is levellised must fix $n$ and induce a relabelling of $L$ by Remark 8. By Remark 14 and the definition of the lexicographic order, $\text{wt}(\pi(L_A)) < \text{wt}(L_A)$ implies $\text{wt}(\pi(L)) \leq \text{wt}(L)$. By Corollary 15, the latter implies $\text{wt}(\pi(L)) = \text{wt}(L)$ and thus $\pi(L) = L$.

To test whether $L_A$ is $<$-minimal, it is thus sufficient to check that

$$\text{wt}_L(A) = \min \{ \text{wt}_L(\pi(A)) : \pi \in \text{Stab}(L) \times \text{Sym}\{n\} \cong \text{Stab}(L) \}$$

(1)

holds, again using Remark 14. The latter condition can, for instance, be verified by computing the orbit $L_A^{\text{Stab}(L)}$ of $A$ under the action of $\text{Stab}(L)$ as the closure of the set $\{A\}$ under the action of a generating set for $\text{Stab}(L)$; this is the approach taken in [HR02]. Alternatively, one can compute a canonical labelling of the lattice $L_A$ and use it to test whether $L_A$ lies on a canonical construction path; this is the approach taken in [JL15].

Case (B):

In this case, the new element $n$ is added to the lowest existing non-trivial level of $L$; cf. Figure 3(b). Thus, a relabelling $\pi$ of $L_A$ for which $\pi(L_A)$ is levellised need not fix $n$ and induce a relabelling of $L$. It will, however, induce a relabelling of the lattice $L'$ induced by the levels $0, \ldots, k-1, k+1$ of $L$ (or $L_A$), and one has $\pi(L') = L'$ by the same arguments as in the previous case.

If the lowest non-trivial level of $L_A$ contains the elements $a_k, \ldots, n$, checking whether $L_A$ is $<$-minimal means testing that $(\text{wt}_{L_A}(a_k), \ldots, \text{wt}_{L_A}(n))$ is lexicographically minimal in its orbit under the group $\text{Stab}(L') \times \text{Sym}\{a_k, \ldots, n\}$; the latter can again be done by an explicit computation of the orbit, or by using a canonical labelling.

Observe that one has $\bigwedge_{L_A} i \subseteq L'$ for $a_k \leq i \leq n$, and that the action of $\pi \in \text{Stab}(L') \times \text{Sym}\{a_k, \ldots, n\}$ not only modifies the individual weights $\text{wt}_{L_A}(i)$ (by relabelling the elements of $L'$), but also permutes their positions in the sequence (by acting on $\{a_k, \ldots, n\}$).

2.4 Vertically indecomposable lattices

Definition 17. An $n$-lattice $L$ is vertically decomposable, if there exists an element $i \in L \setminus \{0, 1\}$ that is comparable to every other element of $L$. Otherwise, $L$ is vertically indecomposable.

![Figure 3: The lattice $L_A$ obtained by adding a new cover $n$ of 0 to the $n$-lattice $L$.](image)
One can speed up the construction by restricting to lattices that are vertically indecomposable; a straightforward recursion makes it possible to recover all lattices from the vertically indecomposable ones. We refer to [HR02, §5] for details.

3 An improved algorithm

The test for minimality of $\text{wt}_L(A)$ or $(\text{wt}_{L_{A_k}}(a_k), \ldots, \text{wt}_{L_A}(n))$ in their orbit under the acting permutation group is the most time consuming part of the construction and thus an obvious target for improvement.

In Section 3.1, we sketch the basic idea for a more efficient algorithm, but we will see that the construction of [HR02, JL15] has to be modified to make this idea work. We describe our modified construction in Section 3.2.

3.1 Stabiliser chain approach

Case (A) from Section 2.3 suggests a possible approach, namely the use of a standard technique from computational group theory: stabiliser chains.

Since $\text{Stab}(L)$ preserves each level of $L$ by Remark 8 and $L$ is levelised, it is tempting to construct and test lattice-antichains level by level: Defining $S_k := \text{Stab}(L)$ as well as $A_d := A \cap \text{lev}_d(L)$ and $S_{d-1} := S_d \cap \text{Stab}(A_d)$ for $d = k, \ldots, 1$, condition (1) is equivalent to the following condition:

$$\text{wt}_L(A_d) = \min \{ \text{wt}_L(\pi(A_d)) : \pi \in S_d \} \quad \text{for } d = k, \ldots, 1$$

(2)

The sets $A_d$ for $d = k, \ldots, 1$ can be constructed and tested one at a time, which offers two advantages: Firstly, if the test at level $d$ fails, the sets for the levels $d-1, \ldots, 1$ don’t have to be constructed; an entire branch of the search space is discarded in one step. Secondly, even if the test succeeds on all levels, the cost of testing condition (2) by computing the orbits $A_{d}^{S_d}$ for $d = k, \ldots, 1$ is proportional to $\sum_{d=1}^{k} |A_{d}^{S_d}|$, and thus in general much smaller than $|A_{d}^{\text{Stab}(L)|} = \prod_{d=1}^{k} |A_{d}^{S_d}|$, which is the cost of testing condition (1) directly by computing the orbit $A_{d}^{\text{Stab}(L)}$.

However, when trying to use a similar approach for Case (B), we run into problems: We must compare

$$(\text{wt}_{L_{A_k}}(a_k), \ldots, \text{wt}_{L_A}(n)) = (\text{wt}_L(a_k), \ldots, \text{wt}_L(n-1), \text{wt}_L(A))$$

lexicographically to its images under the elements of the acting permutation group $\text{Stab}(L') \times \text{Sym}(\{a_k, \ldots, n\})$, but if $A$ has only been constructed partially, $\text{wt}_{L_A}(n) = \text{wt}_L(A)$ is not completely determined; in the interpretation of Remark 4 and Figure 1, the leftmost entries of the last row of the binary matrix corresponding to $L_A$ are undefined.

The elements of the group $\text{Stab}(L') \times \text{Sym}(\{a_k, \ldots, n\})$ can permute the rows of this matrix, so the position of the undefined entries will vary. Clearly, the lexicographic comparison of the two matrices must stop once it reaches an entry that is undefined in one of the matrices being compared; in this situation, the order of the two matrices cannot be decided on the current level.
The problem is that the position in the matrix at which the lexicographic comparison must stop depends on the relabelling that is applied (cf. Figure 4). A consequence of this is that the subset of elements of Stab($L'$) × Sym(\{a_k, \ldots, n\}) for which the parts of the matrices that can be compared are equal does not form a subgroup, so applying a stabiliser chain approach is not possible.

### 3.2 Levellised construction

The analysis at the end of the preceding section indicates that the problem is that possible relabellings can swap an element whose covering set is only partially determined with an element whose covering set is completely determined, or in other words, that we add a new element to an existing level of $L$.

The idea for solving this problem is simple: Rather than adding one element at a time, possibly to an already existing level, we only ever add an entire level at a time; that way, the problem of adding elements to an existing level is avoided.

To make the stabiliser chain approach work in this setting, we must use a total order that compares parts of the covering sets in the same order in which they are constructed; that is, we have to compare the entries of the matrices describing the covering relations in level-major order.

**Notation 18.** For a levellised $n$-lattice $L$ with $n > 2$ and dep$_L(n - 1) = k$, let $L'$ denote the lattice induced by the levels $0, \ldots, k - 1, k + 1$ of $L$, that is, the lattice obtained from $L$ by removing its last non-trivial level. Note that $L'$ is a levellised $n'$-lattice for some $n' < n$.

The total order we are about to define uses the partition of the covering set of each element according to levels.

**Definition 19.** Given a levellised $n$-lattice $L$ with dep$_L(n - 1) = k$, an element $i \in L \setminus \{0\}$ and an integer $d \in \{1, \ldots, k - 1\}$, we define $\lambda^d_L i = (\lambda_L i) \cap$ lev$_d(L)$.

**Definition 20.** Using induction on $n$, we define a relation $<$ on the set of levellised $n$-lattices that are isomorphic as unlabelled lattices as follows:

- $L_1 < L_2$ if one of the following holds:
  - $L_1' < L_2'$

Figure 4: Lexicographic comparison of a lattice $L_A$ obtained from a partially constructed lattice-antichain $A$ (left) and a relabelling (right); only the relevant parts of the matrices are shown. Thick lines indicate the boundaries between levels. The parts of the lattice-antichain not yet constructed are shown in grey.
\[ L_1' = L_2' = L' \] and, denoting \( \text{dep}_{L_1}(n-1) = \text{dep}_{L_2}(n-1) = k \) and \( \text{lev}_k(L_1) = \text{lev}_k(L_2) = \{a_k, \ldots, n-1\} \), there exist \( \ell \in \{1, \ldots, k-1\} \) such that both of the following hold:

- \( \text{wt}_{L'}(\uplus_{d=1}^{D-1} L_{s,i}^d) = \text{wt}_{L'}(\uplus_{d=1}^{D-1} L_{s,i}^d) \) if \( d > \ell \), or \( d = \ell \) and \( j \in \{a_k, \ldots, i-1\} \)
- \( \text{wt}_{L'}(\uplus_{d=1}^{D-1} L_{s,i}^d) < \text{wt}_{L'}(\uplus_{d=1}^{D-1} L_{s,i}^d) \) for \( d = 1, \ldots, D-1 \).

**Remark 21.** The relation from **Definition 20** corresponds to a level-major lexicographic comparison of the binary matrices describing the covering relations of \( L_1 \) and \( L_2 \) as illustrated in **Figure 5**.

As \( L_1 \) and \( L_2 \) are levellised, both matrices are lower block triagonal, with the blocks defined by the levels. Moreover, as \( L_1 \) and \( L_2 \) are isomorphic as unlabelled lattices, the block structures of both matrices are identical.

Notice also that the matrix describing the covering relation of \( L_1' \) (respectively \( L_2' \)) is obtained from that of \( L_1 \) (respectively \( L_2 \)) by removing the lowest row and the rightmost column of blocks.

**Lemma 22.** The relation \(<\) from **Definition 20** is a total order on the set of levellised \( n \)-lattices that are isomorphic as unlabelled lattices.

**Proof.** Verifying transitivity is routine. Trichotomy holds, as for \( s \in \{1, 2\} \) and \( i \in \{2, \ldots, n-1\} \) with \( \text{dep}_{L_1}(i) = \text{dep}_{L_2}(i) = D \), one has

\[
\uplus_{d=1}^{D-1} L_{s,i}^d = \begin{cases} \displaystyle \bigcup_{d=1}^{D-1} L_{s,i}^d & \text{if } \left( \bigcup_{d=1}^{D-1} L_{s,i}^d \right) \neq \emptyset \\ \{1\} & \text{if } \left( \bigcup_{d=1}^{D-1} L_{s,i}^d \right) = \emptyset \end{cases}
\]

whence \( \uplus_{L_1,i} = \uplus_{L_2,i} \) holds in the situation \( L_1' = L_2' = L' \) if and only if one has \( \text{wt}_{L'}(\uplus_{L_1,i}^d) = \text{wt}_{L'}(\uplus_{L_2,i}^d) \) for \( d = 1, \ldots, D-1 \).

**Definition 23.** An \( n \)-lattice \( L \) is **canonical** if \( L \) is levellised and \(<\)-minimal among all levellised \( n \)-lattices that are isomorphic to \( L \) as unlabelled lattices.

**Lemma 24.** Every isomorphism class of unlabelled lattices on \( n \) elements contains a unique canonical \( n \)-lattice.

![Figure 5: Interpreting the order \(<\) as level-major lexicographic order on the matrices specifying the covering relation. Thick lines indicate the boundaries between levels. The entries of the matrix shown in grey are zero.](image-url)
Proof. The set of representatives that are levellised \( n \)-lattices is clearly non-empty and finite, and \(<\) is a total order on this set by Lemma 22. □

The following Theorem 26 is an analogue of Corollary 15.

**Lemma 25.** If \( L \) is a levellised \((n+1)\)-lattice for \( n > 1 \), then \( \overline{L} = L \setminus \{n\} \) is a levellised \( n \)-lattice and \( \dep_{\overline{L}}(i) = \dep_L(i) \) holds for \( i = 1, \ldots, n-1 \). Moreover, for \( a, b \in \overline{L} \) one has the following identities:

\[
  a \lor_{\overline{L}} b = a \lor_L b \quad \text{and} \quad a \land_{\overline{L}} b = \begin{cases} a \land_L b & \text{if } a \land_L b \neq n \\
0 & \text{if } a \land_L b = n
\end{cases}
\]

Proof. As \( L \) is levellised, \( 1 \leq i < n \) implies \( \dep_L(i) \leq \dep_L(n) \), so \( i \not< L n \), and hence \( \dep_{\overline{L}}(i) = \dep_L(i) \). As \( L \) is levellised, so is \( \overline{L} \).

A routine verification shows the identities for \( a \lor_{\overline{L}} b \) and \( a \land_{\overline{L}} b \); in particular, \( \overline{L} \) is an \( n \)-lattice.

□

**Theorem 26.** If \( L \) is a canonical \( n \)-lattice, then \( L' \) is canonical.

Proof. Iterated application of Lemma 25 shows that \( L' \) is levellised.

Assume that \( \pi' \) is a relabelling of \( L' \) such that \( \pi'(L') \) is levellised and one has \( \pi'(L') < L' \). We can trivially extend \( \pi' \) to a relabelling \( \pi \) of \( L \) such that \( (\pi(L))' = \pi'(L') < L' \), contradicting the assumption that \( L \) is canonical. □

Theorem 26 means that we can again construct a tree of canonical \( n \)-lattices in a depth-first manner: the root of the tree is the unique 2-lattice, and an \((n+m)\)-lattice \( \tilde{L} \) is a descendant of a canonical \( n \)-lattice \( L \), if \( \tilde{L} \) is obtained from \( L \) by adding a new level consisting of \( m \) new covers of 0 (labelled \( n, \ldots, n+m-1 \)) and \( \tilde{L} \) is canonical.

The lattice \( \tilde{L} \) is determined by \( L \) and the covering sets of the new elements \( n, \ldots, n+m-1 \); the possible choices for the latter can again be characterised effectively using lattice-antichains, although this time, extra compatibility conditions are needed. The following Theorem 29, a generalisation of Theorem 13, makes this precise.

**Notation 27.** Given \( m \in \mathbb{N}^+ \), an \( n \)-poset \( L \), and \( A_n, \ldots, A_{n+m-1} \subseteq L \setminus \{0\} \), let \( L_{A_n,\ldots,A_{n+m-1}} = \langle (\cdots (L_{A_0}) \cdots)_{A_{n+m-1}} \rangle_{A_{n+m-1}} \) denote the \((n+m)\)-poset obtained from \( L \) by adding \( m \) new atoms \( n, \ldots, n+m-1 \) with \( \bigwedge L_{A_0,\ldots,A_{n+m-1}} = i = A_i \) for \( i = n, \ldots, n+m-1 \). (See Remark 14.)

**Lemma 28.** Let \( L \) be a levellised \( n \)-poset with \( \dep_L(n-1) = k \), let \( m \in \mathbb{N}^+ \), and let \( A_i \subseteq L \setminus \{0\} \) for \( i = n, \ldots, n+m-1 \). The following are equivalent:

(A) \( \tilde{L} = L_{A_0,\ldots,A_{n+m-1}} \) is a levellised \((n+m)\)-poset with \( \dep_{\tilde{L}}(a) = \dep_L(a) \leq k \) for \( 1 \leq a < n \) and \( \dep_{\tilde{L}}(n) = \ldots = \dep_{\tilde{L}}(n+m-1) = k+1 \).

(B) \( A_i \cap \lev_k(L) \neq \emptyset \) holds for \( n \leq i < n+m \).

Proof. As \( L \) is levellised, one has \( \dep_L(a) \leq \dep_L(n-1) = k \) for all \( a \in L \setminus \{0\} \), and induction using Remark 14 shows that \( \dep_{\tilde{L}}(a) = \dep_L(a) \leq k \) holds for all \( a \in L \setminus \{0\} \), which implies \( \dep_{\tilde{L}}(i) \leq k+1 \) for \( i = n, \ldots, n+m-1 \) by construction. Thus, for any \( i = n, \ldots, n+m-1 \), one has \( \dep_{\tilde{L}}(i) = k+1 \) if
Theorem 13. We can assume dep\(_L\)(n) = \ldots = dep\(_L\)(n + m - 1) = k + 1 implies that \(\tilde{L}\) is levellised.

**Theorem 29.** Let \(L\) be a levellised \(n\)-lattice with dep\(_L\)(n - 1) = k, let \(m \in \mathbb{N}^+\), and let \(A_i \subseteq L \setminus \{0\}\) for \(i = n, \ldots, n + m - 1\). The following are equivalent:

(A) \(\tilde{L} = L_{A_n, \ldots, A_{n+m-1}}\) is a levellised \((n + m)\)-lattice, dep\(_L\)(a) = dep\(_L\)(\(a\)) \(\leq k\) for \(1 \leq a < n\), and dep\(_L\)(n) = \ldots = dep\(_L\)(n + m - 1) = k + 1.

(B) (i) \(A_i \cap \text{lev}_L(\tilde{L}) \neq \emptyset\) holds for \(n \leq i < n + m\);

(ii) \(A_i\) is a lattice-antichain for \(L\) if \(n \leq i < n + m\); and

(iii) if \(a, b \in (\uparrow_L A_i) \cap (\uparrow_L A_j)\) for \(n \leq i < j < n + m\), then \(a \wedge_L b \neq 0\).

**Proof.** We use induction on \(m\). In the case \(m = 1\), condition (B)(iii) is vacuous and, by Theorem 13, the poset \(\tilde{L} = L_{A_n}\) is a lattice if and only if \(A_n\) is a lattice-antichain for \(L\). Together with Lemma 28, the claim for \(m = 1\) is shown.

Let \(m > 1\) and consider \(L^\circ = L_{A_n, \ldots, A_{n+m-2}}\) and \(A = A_{n+m-1}\); we have \(\tilde{L} = (L^\circ)^\circ\). By Lemma 28, we can assume dep\(_L\)(a) = dep\(_L\)(\(a\)) \(\leq k\) for \(1 \leq a < n\) and dep\(_L\)(n) = \ldots = dep\(_L\)(n + m - 1) = k + 1. In particular, \(\uparrow_L A = \uparrow_L A\). Also, for \(i = n, \ldots, n + m - 1\) and \(a \in L\), we have \(i \subseteq_L a\) if and only if \(a \in \uparrow_L A\).

First assume that (A) holds.

Induction using Lemma 25 shows that the sets \(A_n, \ldots, A_{n+m-2}\) are lattice-antichains for \(L\), and \(a, b \in (\uparrow_L A_i) \cap (\uparrow_L A_j)\) for \(n \leq i < j < n + m - 1\) implies \(a \wedge_L b \neq 0\). Further, by Theorem 13, the set \(A\) is a lattice-antichain for \(L^\circ\), which means that for \(a, b \in \uparrow_L A\), one has \(a \wedge_L b \neq 0\).

Let \(a, b \in \uparrow_L A\). If \(a \wedge_L b \neq 0\), we have \(a \wedge_L b = a \wedge_L b\) by repeated application of Lemma 25, whence \(a \wedge_L b \in \uparrow_L A\). Thus, \(A\) is a lattice-antichain for \(L\).

On the other hand, if \(a \wedge_L b = 0\), then \(n + m - 1\) is a maximal common lower bound of \(a\) and \(b\) in \(L\), and the lattice property of \(L\) then implies that \(i \subseteq_L a\) or \(i \subseteq_L b\), that is, \(a \notin \uparrow_L A_i\) or \(b \notin \uparrow_L A_i\), holds for all \(n \leq i < n + m - 1\). In particular, condition (B)(iii) holds. Together with Lemma 28, we have thus shown (B).

Now assume that (B) holds.

By induction, \(L^\circ\) is a levellised \((n + m - 1)\)-lattice. Let \(a, b \in \uparrow_L A\). If we have \(a \wedge_L b \neq 0\), then \(a \wedge_L b = a \wedge_L b\) holds, using Lemma 25 and the fact that \(A\) is a lattice-antichain for \(L\) by assumption. On the other hand, \(a \wedge_L b = 0\) implies \(a \wedge_L b = 0\), as by assumption, there is no \(i \in \{n, \ldots, n+m-2\}\) such that \(a, b \in \uparrow_L A_i\), holds. Thus, \(A\) is a lattice-antichain for \(L^\circ\), whence \(L\) is a lattice by Theorem 13. Together with Lemma 28, we have thus shown (A).

**Notation 30.** Let \(L\) be a levellised \(n\)-lattice with dep\(_L\)(n - 1) = k, let \(m \in \mathbb{N}^+\), and let \(A_i \subseteq L \setminus \{0\}\) for \(i = n, \ldots, n + m - 1\). For \(d = 1, \ldots, k\) we define \(\mathcal{LW}_L^d(A_n, \ldots, A_{n+m-1})\) as the sequence

\[
\left(\text{wt}_L(A_n \cap \text{lev}_L(L)), \ldots, \text{wt}_L(A_{n+m-1} \cap \text{lev}_L(L))\right)
\]

and we define the sequence \(\mathcal{LW}_L(A_n, \ldots, A_{n+m-1})\) as the concatenation of \(\mathcal{LW}_L^1(A_n, \ldots, A_{n+m-1}), \ldots, \mathcal{LW}_L^k(A_n, \ldots, A_{n+m-1})\) in this order.
Theorem 31. Let $L$ be a levellised $n$-lattice with $\text{dep}_{L}(n-1) = k$, let $m \in \mathbb{N}^+$, and assume that $A_i \subseteq L \setminus \{0\}$ for $i = n, \ldots, n+m-1$ satisfy condition (B) from Theorem 29. Then $L_{A_{n}, \ldots, A_{n+m-1}}$ is a canonical $(n+m)$-lattice if and only if:

(i) $L$ is canonical; and

(ii) the sequence $\mathcal{LW}_L(A_n, \ldots, A_{n+m-1})$ is lexicographically minimal under the action of $\text{Stab}(L) \times \text{Sym}\{n, \ldots, n+m-1\}$ given by

$$\pi\left(\mathcal{LW}_L(A_n, \ldots, A_{n+m-1})\right) = \mathcal{LW}_L\left(\pi(A_{\pi^{-1}(n)}), \ldots, \pi(A_{\pi^{-1}(n+m-1)})\right)$$

for $\pi \in \text{Stab}(L) \times \text{Sym}\{n, \ldots, n+m-1\}$.

Proof. As any $\pi \in \text{Stab}(L) \times \text{Sym}\{n, \ldots, n+m-1\}$ induces a relabelling of the elements of $L$, the action is well-defined. Moreover, defining $\tilde{L} = L_{A_{n}, \ldots, A_{n+m-1}}$, Theorem 29 implies $\mathcal{L}_{\tilde{L}}^d i = A_i \cap \text{lev}_d(L)$ for $i = n, \ldots, n+m-1$ and $1 \leq d \leq k$.

First assume that $\tilde{L}$ is canonical. By Theorem 26, $L = \tilde{L}'$ is canonical, so (i) holds. If (ii) does not hold, there exist $\pi \in \text{Stab}(L) \times \text{Sym}\{n, \ldots, n+m-1\}$ such as well as $\ell \in \{1, \ldots, k\}$ and $i \in \{n, \ldots, n+m-1\}$ such that one has

- $\text{wt}_L\left(\pi(A_{\pi^{-1}(j)}) \cap \text{lev}_d(L)\right) = \text{wt}_L(A_j \cap \text{lev}_d(L))$ if $d > \ell$, or $d = \ell$ and $j \in \{n, \ldots, i-1\}$; and
- $\text{wt}_L\left(\pi(A_{\pi^{-1}(j)}) \cap \text{lev}_\ell(L)\right) < \text{wt}_L(A_j \cap \text{lev}_\ell(L))$.

By Theorem 29, $\tilde{L} = L_{\pi(A_{\pi^{-1}(n)}), \ldots, \pi(A_{\pi^{-1}(n+m-1)})}$ is a levellised $(n+m)$-lattice, and $\tilde{L}' = L = \tilde{L}'$ holds by construction. Hence, the above conditions mean that $\tilde{L}$ is not canonical, contradicting the assumption. Thus (ii) holds.

Conversely, if $\tilde{L}$ is not canonical, there is a relabelling $\pi$ of $\tilde{L}$ such that $\tilde{L} = \pi(\tilde{L})$ is levellised and $\tilde{L} \subset \tilde{L}$ holds. As $\pi$ acts on the levels of $\tilde{L}$ it induces a relabelling of $L$, and we have $\tilde{L}' = (\pi(\tilde{L}))' = \pi(L)' = \pi(L)$, which is levellised as well. If (i) holds, we cannot have $\pi(L) < L$, so $\tilde{L} \not\subset \tilde{L}$ implies that one has $\tilde{L}' = \tilde{L}' = L$ and there exist $\ell \in \{1, \ldots, k\}$ as well as $i \in \{n, \ldots, n+m-1\}$ such that both of the following hold:

- $\text{wt}_L\left(\mathcal{L}_{\tilde{L}}^d i\right) = \text{wt}_L\left(\mathcal{L}_{\tilde{L}}^d j\right)$ if $d > \ell$, or $d = \ell$ and $j \in \{n, \ldots, i-1\}$
- $\text{wt}_L\left(\mathcal{L}_{\tilde{L}}^\ell i\right) < \text{wt}_L\left(\mathcal{L}_{\tilde{L}}^\ell j\right)$.

As we have $\mathcal{L}_{\tilde{L}}^d i = \pi(A_{\pi^{-1}(i)}) \cap \text{lev}_d(L)$ for $j = n, \ldots, n+m-1$ and $1 \leq d \leq k$, the above conditions imply that

$$\mathcal{LW}_L\left(\pi(A_{\pi^{-1}(n)}), \ldots, \pi(A_{\pi^{-1}(n+m-1)})\right) = \pi\left(\mathcal{LW}_L(A_n, \ldots, A_{n+m-1})\right)$$

is lexicographically smaller than $\mathcal{LW}_L(A_n, \ldots, A_{n+m-1})$. Since $\pi(L) = \tilde{L}' = L$, we have $\pi \in \text{Stab}(L) \times \text{Sym}\{n, \ldots, n+m-1\}$, so (ii) does not hold.

Corollary 32. Let $L$ be a levellised $n$-lattice with $\text{dep}_{L}(n-1) = k$, let $m \in \mathbb{N}^+$, and assume that $A_i \subseteq L \setminus \{0\}$ for $i = n, \ldots, n+m-1$ satisfy condition (B) from Theorem 29. Then $L_{A_{n}, \ldots, A_{n+m-1}}$ is a canonical $(n+m)$-lattice if and only if:
(i) $L$ is canonical; and

(ii) for $d = k, \ldots, 1$, the sequence $\mathcal{LW}_L^d(A_n, \ldots, A_{n+m-1})$ is lexicographically minimal under the action of $S_d$ given by

$$\pi \left( \mathcal{LW}_L^d(A_n, \ldots, A_{n+m-1}) \right) = \mathcal{LW}_L^d \left( \pi(A_{n-1(m)}), \ldots, \pi(A_{n-1(n+m-1)}) \right)$$

for $\pi \in S_d$, where we define $S_k = \text{Stab}(L) \times \text{Sym}(\{n, \ldots, n + m - 1\})$ and $S_{d-1} = S_d \cap \text{Stab}(\mathcal{LW}_L^d(A_n, \ldots, A_{n+m-1}))$ for $d = k, \ldots, 2$.

Proof. Since the sequence $\mathcal{LW}_L(A_n, \ldots, A_{n+m-1})$ is the concatenation of the sequences $\mathcal{LW}_L^d(A_n, \ldots, A_{n+m-1}), \ldots, \mathcal{LW}_L^1(A_n, \ldots, A_{n+m-1})$ in this order, condition (ii) from Theorem 31 is equivalent to condition (ii) of this corollary. □

Remark 33. Corollary 32 makes it possible to construct the lattice-antichains $A_n, \ldots, A_{n+m-1}$ level by level: The comparison at step $d$ in condition (ii) only involves the elements of $A_n, \ldots, A_{n+m-1}$ that live on the level $d$ of $L$. In particular, the benefits of using stabiliser chains mentioned in Section 3.1 apply:

(a) If the test at level $d$ fails, the levels $d-1, \ldots, 1$ do not have to be constructed; an entire branch of the search space is discarded in one step.

(b) The cost of testing condition (ii) of Corollary 32 is in general much smaller than the cost of testing condition (ii) of Theorem 31: The former is proportional to

$$\sum_{d=1}^{k} \left| \mathcal{LW}_L^d(A_n, \ldots, A_{n+m-1}) \right| S_d$$

while the latter is proportional to

$$\left| \mathcal{LW}_L(A_n, \ldots, A_{n+m-1}) \right| S_k = \prod_{d=1}^{k} \left| \mathcal{LW}_L^d(A_n, \ldots, A_{n+m-1}) \right| S_d.$$

Figure 6 shows the comparisons that are made when testing one step of condition (ii) of Corollary 32. Note that a reordering of the rows of the matrix does not change the position at which the lexicographic comparison stops; this property is necessary for the stabiliser chain approach to work.

3.3 Vertically indecomposable lattices

Restricting the construction to vertically indecomposable lattices is very easy: By the following lemma, the only modification required is to avoid adding a new level that contains a single element whose covering set equals the lowest non-trivial level of the given lattice in any step of the construction.

Lemma 34. Let $L$ be a levelised $n$-lattice, let $m \in \mathbb{N}^+$, and let $A_i \subseteq L \setminus \{0\}$ for $i = n, \ldots, n + m - 1$ satisfy condition (B) from Theorem 29.

(a) If $L$ is vertically decomposable, then $L_{A_n, \ldots, A_{n+m-1}}$ is vertically decomposable.

(b) If $L$ is vertically indecomposable, then $L_{A_n, \ldots, A_{n+m-1}}$ is vertically decomposable if and only if $m = 1$ and $\uparrow_L A_n = L \setminus \{0\}$ hold.
Proof. Let \( k = \text{dep}_L(n - 1) \) and let \( \tilde{L} = L_{A_n, \ldots, A_{n+m-1}} \).

(a) As \( L \) is vertically decomposable, there exists \( i \in L \setminus \{0,1\} \) such that \( i \) is comparable to every element of \( L \). In particular, \( j \sqsubseteq_L i \) holds for any \( j \in \text{lev}_k(L) \), since \( L \) is levelised. For any \( a \in \{n, \ldots, n + m - 1\} \), one has \( A_a \cap \text{lev}_k(L) \neq \emptyset \), and thus \( a \sqsubseteq_L i \), so \( i \) is comparable to every element of \( \tilde{L} \).

(b) This is obvious, as \( \tilde{L} \) is vertically decomposable if and only if there exists \( a \in \{n, \ldots, n + m - 1\} \), such that one has \( a \sqsubseteq_L i \) for all \( i \in \tilde{L} \setminus \{0\} \).

3.4 Graded lattices

We finish this section with a brief comment in relation to graded lattices.

Definition 35. A lattice \( L \) is called graded with rank function \( \rho : L \to \mathbb{N} \) if one has \( \rho(b) = \rho(a) + 1 \) for any \( a, b \in L \) satisfying \( a \prec_L b \).

Remark 36. It is clear from the definitions that a lattice \( L \) is graded if and only if max\( \{\text{dep}_L(i) : i \in L\} - \text{dep}_L \) is a rank function for \( L \), that is, if and only if \( a \prec_L b \) implies \( \text{dep}_L(a) = \text{dep}_L(b) + 1 \); the latter is equivalent to \( \bigcup_{\ell \in \text{lev}(L)} \bigwedge_{L} a = \text{lev}_{\ell-1}(L) \) for \( \ell = 1, \ldots, \text{dep}_L(0) \).

Lemma 37. Let \( L \) be a levelised \( n \)-lattice with \( \text{dep}_L(n - 1) = k \), let \( m \in \mathbb{N}^+ \), and assume that \( A_i \subseteq L \setminus \{0\} \) for \( i = n, \ldots, n + m - 1 \) satisfy condition (B) from Theorem 29. Then \( L_{A_n, \ldots, A_{n+m-1}} \) is graded if and only if

(a) \( L \) is graded; and

(b) \( \bigcup_{i=n}^{n+m-1} A_i = \text{lev}_k(L) \) holds.

Proof. Let \( \tilde{L} = L_{A_n, \ldots, A_{n+m-1}} \). By Theorem 29, one has \( \text{dep}_{\tilde{L}}(a) = \text{dep}_L(a) \) for \( a \in L \setminus \{0\} \), and \( \text{dep}_{\tilde{L}}(i) = k + 1 = \text{dep}_L(0) \) for \( n \leq i < n + m \). By construction, \( \text{dep}_{\tilde{L}}(0) = k + 2 \) holds. Thus, one has \( \text{lev}_{\ell}(L) = \text{lev}_{\ell}(\tilde{L}) \) for \( \ell = 0, \ldots, k \), as well as \( \text{lev}_{k+1}(\tilde{L}) = \{n, \ldots, n + m - 1\} \) and \( \text{lev}_k(\tilde{L}) = \{0\} = \text{lev}_{k+2}(\tilde{L}) \).

As \( a \prec_L b \) is equivalent to \( a \prec_{\tilde{L}} b \) for \( a, b \in L \setminus \{0\} \) and \( a \in L \setminus \{0\} \) (resp. \( a \in L \setminus \{0\} \)) covers 0 in \( L \) (resp. in \( \tilde{L} \)) if and only if \( a \) covers no other element of \( L \) (resp. of \( \tilde{L} \)), the claim follows with Remark 36, noting that one has \( \bigwedge_{\tilde{L}} i = A_i \) for \( i = n, \ldots, n + m - 1 \) by construction.

Figure 6: Lexicographic comparison of a lattice \( L_{A_n, \ldots, A_{n+m-1}} \) obtained from a sequence of partially constructed lattice-antichains and a relabelling; only the relevant part of the matrix is shown. Thick lines indicate the boundaries between levels. Parts of the lattice-antichains not yet constructed are shown in grey. Parts of the lattice-antichains known to coincide are shown in black.
In order to restrict the construction to graded lattices, it is thus sufficient to enforce condition (b) of Lemma 37 in every step of the construction. In the light of Lemma 34, the construction can be restricted to vertically indecomposable graded lattices by avoiding to add a new level that contains a single element in any step of the construction.

Note, however, that the stabiliser chain approach does not yield any benefit for graded lattices compared to [HR02, JL15], since by condition (b) of Lemma 37, the construction of covering sets only involves a single level anyway.

4 Implementation and results

4.1 Implementation notes

This section sketches some ideas that are crucial for an efficient implementation of the algorithm presented in the preceding sections as well as the validation methods employed. We also compare our approach to testing canonicity to that used in [JL15].

4.1.1 Representing antichains using up-closed sets

While the theoretical results of Section 3 are formulated in terms of antichains, it is easier and computationally more efficient to work with sets $S$ that are up-closed, meaning that $\uparrow S = S$ holds. (For instance, testing whether $A \subseteq L$ is a lattice-antichain for $L$ only involves $\uparrow_L A$.)

Clearly, if $A$ is an antichain, then $\uparrow_L A$ is up-closed and the set of minimal elements of $\uparrow_L A$ is equal to $A$.

**Lemma 38.** Let $L$ be a levellised $n$-lattice with $\text{dep}_L(n-1) = k$, let $A$ and $B$ be antichains in $L$, and let $\ell \in \{1, \ldots, k\}$. The following are equivalent:

(i) One has $\text{wt}_L(A \cap \text{lev}_d(L)) = \text{wt}_L(B \cap \text{lev}_d(L))$ for $d = k, \ldots, \ell + 1$, and $\text{wt}_L(A \cap \text{lev}_\ell(L)) < \text{wt}_L(B \cap \text{lev}_\ell(L))$.

(ii) One has $\text{wt}_L(\uparrow_L A \cap \text{lev}_d(L)) = \text{wt}_L(\uparrow_L B \cap \text{lev}_d(L))$ for $d = k, \ldots, \ell + 1$, and $\text{wt}_L(\uparrow_L A \cap \text{lev}_\ell(L)) < \text{wt}_L(\uparrow_L B \cap \text{lev}_\ell(L))$.

**Proof.** If (i) holds, one has $((\uparrow_L A) \setminus A) \cap \text{lev}_d(L) = ((\uparrow_L B) \setminus B) \cap \text{lev}_d(L)$ for $d = k, \ldots, \ell$, since $L$ is levellised. In particular, one has

$$((\uparrow_L B) \setminus (\uparrow_L A)) \cap \text{lev}_\ell(L) = (B \setminus A) \cap \text{lev}_\ell(L)$$

and

$$((\uparrow_L A) \setminus (\uparrow_L B)) \cap \text{lev}_\ell(L) = (A \setminus B) \cap \text{lev}_\ell(L),$$

which together with (i) imply (ii).

As $A$ and $B$ are the sets of minimal elements of $\uparrow_L A$ respectively $\uparrow_L B$, the converse implication is obvious. □

**Corollary 39.** Let $L$ be a levellised $n$-lattice with $\text{dep}_L(n-1) = k$, let $m \in \mathbb{N}^+$, and assume that $A_i \subseteq L \setminus \{0\}$ for $i = n, \ldots, n + m - 1$ satisfy condition (B) from Theorem 29. Then $L_{A_n, \ldots, A_{n+m-1}}$ is a canonical $(n + m)$-lattice if and only if:

(i) $L$ is canonical; and
(ii) for \( d = k, \ldots, 1 \), the sequence \( \mathcal{L}W_f^d(\uparrow_L A_n, \ldots, \uparrow_L A_{n+m-1}) \) is lexicographically minimal under the action of \( S_d \) as in Corollary 32.

Proof. The claim follows from Corollary 32 with Lemma 38 and the observation that one has \( \uparrow_L (\pi(A)) = \pi(\uparrow_L A) \) for any \( A \subseteq L \) and \( \pi \in S_d \).

4.1.2 Packed representation of antichains and Beneš networks

Let \( L \) be a canonical \( n \)-lattice with \( \text{dep}_L(n - 1) = k \), and let \( m \in \mathbb{N}^+ \). To generate the descendants of \( L \) with \( m \) elements on level \( k+1 \), we use a backtrack search to construct the sets \( (\uparrow_L A_i) \cap \text{lev}_d(L) \) for \( d = k, \ldots, 1 \) (outer loop) and \( i = n, \ldots, n + m - 1 \) (inner loop).

Every time a candidate set \((\uparrow_L A_i) \cap \text{lev}_d(L)\) has been chosen, we use condition (B) from Theorem 29 to check for possible contradictions (backtracking if there are any), and to keep track of any elements whose presence in \((\uparrow_L A_i) \cap \text{lev}_{d'}(L)\) for some \( d > d' \geq 1 \) is forced by the choices made so far (restricting the possible choices later in the backtrack search if there are any).

Once all candidate sets on the current level have been chosen, we check for minimality under the action of the appropriate stabiliser \( S_d \) (cf. Corollary 39) by explicit computation of the orbit, backtracking if necessary.

Given the large number of configurations that have to be generated and tested for canonicity, it is critical to use an efficient data structure to store a configuration of antichains.

The sets \((\uparrow_L A_n) \cap \text{lev}_d(L), \ldots, (\uparrow_L A_{n+m-1}) \cap \text{lev}_d(L)\) are encoded as a single \((m \cdot |\text{lev}_d(L)|)\)-bit integer. That way, a lexicographic comparison of two configurations reduces to a single comparison of two \((m \cdot |\text{lev}_d(L)|)\)-bit integers.

When constructing lattices with up to 18 elements, \( m \cdot |\text{lev}_d(L)| \) is at most 64; when constructing lattices with up to 23 elements, \( m \cdot |\text{lev}_d(L)| \) is at most 128. Thus, on a 64-bit CPU, a lexicographic comparison of two configurations costs only very few clock cycles.

To be able to apply permutations to a packed representation as described above effectively, we pre-compute a Beneš network [Knu09, §7.1.3] for each generator of the stabiliser \( S_d \). That way, the application of the generator to the configuration \((\uparrow_L A_n) \cap \text{lev}_d(L), \ldots, (\uparrow_L A_{n+m-1}) \cap \text{lev}_d(L)\) is realised by a sequence of bitwise operations (XOR and shift operations) on the \((m \cdot |\text{lev}_d(L)|)\)-bit integer representation.

If the sequence \((\uparrow_L A_n) \cap \text{lev}_d(L), \ldots, (\uparrow_L A_{n+m-1}) \cap \text{lev}_d(L)\) is lexicographically minimal in its orbit under the action of \( S_d \), then the computation of this orbit also yields generators of \( S_{d-1} \) [Cam99, §1.13]; we limit the number of generators by applying a technique known as Jerrum’s filter [Cam99, §1.14].

4.1.3 Canonicity testing through canonical construction paths

Recall that the algorithm in [JL15] constructs lattices by adding one element at a time, hence only a single lattice-antichain is used in each step; to ensure that only canonical lattices are constructed, the authors use a different approach:

1. The set of all lattice-antichains of the parent lattice \( L \) is computed and, starting from the partition of this set into singletons, the action of generators of the relevant permutation group \( G \) (cf. Section 2.3) is used to coarsen
this partition until the set of $G$-orbits on the set of lattice-antichains is obtained.

2. For each such $G$-orbit, an arbitrary representative $A$ is chosen, and the programme `nauty` [MP14] is used to compute a canonical labelling for the corresponding descendant $L_A$ of $L$ as well as the automorphism group of $L_A$; these data can be used to decide whether $L_A$ is canonical.

**Remark 40.** Comparing to our approach, we note the following points:

(a) The use of `nauty` for testing canonicity in [JL15] significantly reduces the complexity of the implementation: without counting the `nauty` code, the implementation used in [JL15] consists of approximately 1400 lines of code, compared to approximately 8300 lines of code for our implementation.

(b) Computing representatives of the $G$-orbits on the set of lattice-antichains as described above means that all lattice-antichains have to be kept in memory at the same time; our implementation avoids the latter, but it does so at the expense of having to compute (parts of) each orbit repeatedly during canonicity tests for different representatives of the same orbit.

Storing all lattice-antichains simultaneously is not a concern for the approach from [JL15], since the parent of a lattice with $n$ elements has at most $2^{n-1}$ lattice-antichains; this bound is reached when adding an element to the 2-fan with $n-1$ elements. (See Figure 7.) For $n = 20$, the number of lattice-antichains is at most 131 072, which presents no difficulty.

However, the situation would be very different for our approach which works with $m$-tuples of lattice-antichains when adding a new level with $m$ elements to a lattice: When constructing a lattice with $n$ elements, the worst case arises when adding $\lceil \frac{n}{2} \rceil - 1$ elements to a 2-fan with $\lfloor \frac{n}{2} \rfloor + 1$ elements; the number of tuples of lattice-antichains in this case is approximately $2^{(\lfloor \frac{n}{2} \rfloor - 1)(\lceil \frac{n}{2} \rceil - 1)}$. Thus, storing all $m$-tuples of lattice-antichains simultaneously becomes intractable for $n \gtrsim 14$.

(c) While the method of canonical construction paths described in [McK98] can be applied to a levellised construction as described here, and while it should in principle be possible to make use of `nauty` for this purpose, it seems that it would be far from easy to make this work for lattices of size 20; at the very least, one would have to resort to a different way of constructing representatives of the $G$-orbits of $m$-tuples of lattice-antichains. In any case, an implementation of a levellised construction using `nauty` would be significantly more complex than the implementation used in [JL15].

![Figure 7: The 2-fan with $k + 1$ elements. Its lattice-antichains are $\{1\}$ and the non-empty subsets of $\{2, \ldots, k\}$.](image-url)
It should be emphasised that both the theoretical approach used for testing canonicity and the actual implementation used in [JL15] are entirely independent from the ones used in this paper.

4.1.4 Validation

Given the complexity of both our implementation and the actual computations, a brief summary of the validation methods we employed is warranted.

During the implementation phase, the following measures were taken:

- We followed best-practice software engineering methods; in particular, the implementation was modularised as much as possible, with each module being subjected to thorough unit testing.
- For lattice sizes \( n \leq 13 \), we validated the implementation using a debug build of our code that enabled additional consistency checks for intermediate results. For instance, we used a naive implementation of permutations to check images obtained from Beneš networks, and we performed a naive iteration over all permutations in the acting permutation group to validate our test for canonicity.
- The low-level architecture dependent constructs, specifically those relating to packed representations of antichains and Beneš networks, were validated with code compiled for register sizes 32 and 16 (in addition to the 64-bit production code).
- All tests listed above were rerun under Valgrind\(^1\), checking for memory errors such as out-of-bound access and read-before-write.

All lattice counts reported in the paper were obtained repeatedly, using different hardware (as well as different versions of our code); the hardware configurations mentioned in Section 4.2 both used ECC RAM.

Except for the case \( n = 20 \), where our results are new, our lattice counts agree with the previously published results [HR02, JL15]. In the light of Remark 40 (d), this constitutes an independent verification of the results.

4.2 Results and performance

Table 1 shows the number \( i_n \) of isomorphism classes of vertically indecomposable unlabelled lattices on \( n \) elements, and the number \( u_n \) of isomorphism classes of unlabelled lattices on \( n \) elements for \( n \leq 20 \); the values \( i_{20} \) and \( u_{20} \) are new.

Table 2, Figure 8 and Figure 9 show the total CPU time and the real time taken by the computations for \( n \geq 14 \) using the algorithm described in this paper and the algorithm from [JL15] for two hardware configurations:

(A) 4 threads on a system with one 4-core Intel Xeon E5-1620 v2 CPU (clock frequency 3.70 GHz; 10 MB L3 cache) with DDR3-1600 RAM (single thread bandwidth\(^2\) 14.9 GB/s; total bandwidth 35.3 GB/s). The system load was just over 4 during the tests.

\(^1\)http://valgrind.org
\(^2\)https://zsmith.co/bandwidth.html
(B) 20 threads on a system with two 10-core Intel Xeon E5-2640 v4 CPUs (clock frequency 2.60 GHz; 25 MB L3 cache) with DDR4-2400 RAM (single thread bandwidth 9.9 GB/s; total bandwidth 89.2 GB/s). The system load was just over 20 during the tests.

The algorithm described in this paper was implemented in C using threads by the first author. The authors of [JL15] kindly provided C-code using MPI implementing their algorithm. All code was compiled using GCC with maximal optimisations for the respective architecture. The compiler version was 4.8.1 for hardware configuration (A) and 4.8.5 for hardware configuration (B).

Remark 41. We conclude by making some observations regarding the performance of the implementation of the algorithm presented.

(a) For longer enumerations, the speedup compared to the algorithm from [JL15] is between a factor of 13 (hardware configuration (A), \( n = 15 \)) and a factor of 30 (hardware configuration (B), \( n = 18 \)).

   On both hardware configurations, the speedup increases with \( n \). This is expected in the light of Remark 33, as the benefits of the stabiliser chain approach are the more pronounced, the more levels a lattice has.

(b) Obtaining a meaningful complexity analysis seems out of reach, as estimating the average case complexity would require a detailed understanding of the tree of canonical lattices. Experimentally, the algorithm seems to be close to optimal in the sense that the computation time grows roughly linearly in the number of lattices constructed.

(c) On hardware configuration (A), the L2 and L3 cache hit rates during the computations were on average around 55-60% respectively 80-85%; on hardware configuration (B), this information could not be obtained. The high L3 cache hit rate suggests that DRAM bandwidth is not a significant limiting factor for the overall performance of the algorithm.

(d) The throughput corresponds to roughly 1800 CPU clock cycles per lattice on hardware configuration (A) and to roughly 1750 CPU clock cycles per lattice on hardware configuration (B); these figures include pre-computations and inter-thread communication. This similarity of these values in spite of the different DRAM speeds on hardware configuration (A) and hardware configuration (B) is consistent with DRAM bandwidth not being a significant limiting factor for the overall performance.

Resources

The C source code implementing the described algorithm that was used for the computations reported in this paper is available under the GNU GPL v.3+ licence at https://bitbucket.org/vgebhardt/unlabelled-lattices.

Data describing the unlabelled lattices on \( n \) elements for \( n \leq 16 \) can be retrieved from http://doi.org/10.26183/5bb57347b10a0. The data are provided in the form of \( \text{xz} \)-compressed plain text files, in which each line describes the covering relation of the canonical (labelled) representative of one isomorphism class of unlabelled lattices.
| $n$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 |
|-----|----|----|----|----|----|----|----|----|----|----|----|
| $i_n$ | 1  | 1  | 0  | 1  | 2  | 7  | 27 | 126 | 664 | 3 954 | 26 190 |
| $u_n$ | 1  | 1  | 1  | 2  | 5  | 15 | 53 | 222 | 1 078 | 5 994 | 37 622 |
| $n$ | 12 | 13 | 14 | 15 | 16 |
| $i_n$ | 190 754 | 1 514 332 | 12 998 035 | 119 803 771 | 1 178 740 932 |
| $u_n$ | 262 776 | 2 018 305 | 16 873 364 | 152 233 518 | 1 471 613 387 |
| $n$ | 17 | 18 | 19 | 20 |
| $i_n$ | 12 316 480 222 | 136 060 611 189 | 1 582 930 919 092 | 19 328 253 734 491 |
| $u_n$ | 15 150 569 446 | 165 269 824 761 | 1 901 910 625 578 | 23 003 059 864 006 |

Table 1: Numbers $i_n$ and $u_n$ of isomorphism classes of vertically indecomposable unlabelled lattices, respectively arbitrary unlabelled lattices, on $n$ elements.

|       | $n$ | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|-------|----|----|----|----|----|----|----|----|
| CPU   | 8.69 | 58.1 | 550 | 5 840 | 66 600 | —  | —  |
| real  | 4.07 | 16.8 | 142 | 1 490 | 16 900 | —  | —  |
| CPU   | 73.4 | 728 | 7 830 | 90 900 | —  | —  | —  |
| real  | 18.4 | 183 | 1 990 | 23 000 | —  | —  | —  |

Table 2: Total CPU time and real time for the longer computations from Table 1 using the algorithm presented in this paper (labelled § 3-4) and the algorithm from [JL15] (labelled [JL]). Times are given in seconds.
Figure 8: Growth of the number $i_n$ of vertically indecomposable lattices with $n$ elements, as well as of the CPU time and real time taken for their enumeration on hardware configuration (A), in terms of $n$.

Figure 9: Growth of the number $i_n$ of vertically indecomposable lattices with $n$ elements, as well as of the CPU time and real time taken for their enumeration on hardware configuration (B), in terms of $n$. 

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The main obstacle to providing the lattices on more than 16 elements is the amount of data: already the compressed file containing the unlabelled lattices on 16 elements has a size of 3.7 GB.

References

[Cam99] Peter J. Cameron. *Permutation groups*, volume 45 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1999. MR1721031

[HR02] Jobst Heitzig and Jürgen Reinhold. Counting finite lattices. *Algebra Universalis*, 48(1):43–53, 2002. MR1930032

[JL15] Peter Jipsen and Nathan Lawless. Generating all finite modular lattices of a given size. *Algebra Universalis*, 74(3-4):253–264, 2015. MR3397437

[KL71] Walter Klotz and Lutz Lucht. Endliche Verbände. *J. Reine Angew. Math.*, 247:58–68, 1971. MR0276146

[Knu09] Donald E. Knuth. *The art of computer programming. Vol. 4, Fasc. 1*. Addison-Wesley, Upper Saddle River, NJ, 2009. Bitwise tricks & techniques; Binary decision diagrams.

[KW80] D. J. Kleitman and K. J. Winston. The asymptotic number of lattices. *Ann. Discrete Math.*, 6:243–249, 1980. Combinatorial mathematics, optimal designs and their applications (Proc. Sympos. Combin. Math. and Optimal Design, Colorado State Univ., Fort Collins, Colo., 1978). MR593536

[McK98] Brendan D. McKay. Isomorph-free exhaustive generation. *J. Algorithms*, 26(2):306–324, 1998. MR1606516

[MP14] Brendan D. McKay and Adolfo Piperno. Practical graph isomorphism, II. *J. Symbolic Comput.*, 60:94–112, 2014. MR3131381

[Slo] Neil J. A. Sloane. The on-line encyclopedia of integer sequences. [https://oeis.org](https://oeis.org). Sequences A006966 and A0588000.

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