ON FLUCTUATIONS OF MATRIX ENTRIES OF REGULAR FUNCTIONS OF WIGNER MATRICES WITH NON-IDENTICALLY DISTRIBUTED ENTRIES

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Abstract. In this note, we extend the results about the fluctuations of the matrix entries of regular functions of Wigner random matrices obtained in [23] to Wigner matrices with non-i.i.d. entries provided certain Lindeberg type conditions for the fourth moments are satisfied. In addition, we relax our conditions on the test functions and require that for some \( s > 3 \)
\[
\int_k (1 + 2|k|)^s |f(k)|^2 \, dk < \infty.
\]

1. Introduction and Formulation of Main Results

Let \( X_N = \frac{1}{\sqrt{N}} W_N \) be a random Wigner real symmetric (Hermitian) matrix. In the real symmetric case, we assume that the entries
\[
(W_N)_{jk}, \ 1 \leq j \leq k \leq N,
\]
are independent random variables such that the off-diagonal entries satisfy
\[
E(W_N)_{jk} = 0, \ \forall 1 \leq j < k \leq N, \ m_4 := \sup_{j \neq k, N} E(W_N)_{jk}^4 < \infty, \quad (1.1)
\]
and the Lindeberg type condition for the fourth moments takes place,
\[
L_N(\epsilon) \to 0, \ \text{as} \ N \to \infty, \ \forall \epsilon > 0, \quad (1.2)
\]
where
\[
L_N(\epsilon) = \frac{1}{N^2} \sum_{1 \leq i < j \leq N} E \left( |(W_N)_{ij}|^4 1_{\{|(W_N)_{ij}| \geq \epsilon \sqrt{N}\}} \right). \quad (1.3)
\]

Here and throughout the paper, \( E \xi \) denotes the mathematical expectation and \( V \xi \) the variance of a random variable \( \xi \).

In addition, we assume that the diagonal entries satisfy
\[
E(W_N)_{ii} = 0, \ 1 \leq i \leq N, \ \sigma_1^2 := \sup_{i, N} E|(W_N)_{ii}|^2 < \infty, \quad (1.4)
\]
\[
l_N(\epsilon) \to 0, \ \text{as} \ N \to \infty, \ \forall \epsilon > 0, \ \text{where} \quad (1.5)
\]
\[
l_N(\epsilon) = \frac{1}{N} \sum_{1 \leq i \leq N} E \left( |(W_N)_{ii}|^2 1_{\{|(W_N)_{ii}| \geq \epsilon \sqrt{N}\}} \right). \quad (1.6)
\]

We note that (1.2) and (1.5) are satisfied if
\[
\sup_{i \neq j, N} E|\mathbb{E}(W_N)_{ij}|^{4+\epsilon} < \infty, \ \sup_{i, N} E|\mathbb{E}(W_N)_{ii}|^{2+\epsilon} < \infty. \quad (1.7)
\]

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If \( \{ \frac{1}{\sqrt{N}} (W_N)_{ii}, 1 \leq i \leq N, (W_N)_{jk}, 1 \leq j < k \leq N, \} \) are i.i.d. \( N(0, \sigma^2) \) random variables, \( X_N \) belongs to the Gaussian Orthogonal Ensemble (GOE).

In the Hermitian case, we assume that the entries are independent random variables such that the off-diagonal entries satisfy

\[
E \Re(W_N)_{jk} = E \Im(W_N)_{jk} = 0, \quad 1 \leq j < k \leq N,
\]

and the Lindeberg type condition (1.2) for the fourth moments of the off-diagonal entries takes place.

In addition, we assume that the diagonal entries satisfy

\[
E(W_N)_{ii} = 0, \quad 1 \leq i \leq N, \quad \sigma_1^2 := \sup_{i,N} E|W_N)_{ii}|^2 < \infty,
\]

and the Lindeberg type condition (1.5) for the second moments of the diagonal entries takes place.

If \( \{ \frac{1}{\sqrt{N}} (W_N)_{ii}, 1 \leq i \leq N, (W_N)_{jk}, 1 \leq j < k \leq N, \} \) are i.i.d. \( N(0, \frac{2}{\sqrt{N}}) \) random variables, \( X_N \) belongs to the Gaussian Unitary Ensemble (GUE).

We define the empirical distribution of the eigenvalues of \( X_N \) as

\[
\mu_{X_N} = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i},
\]

where \( \lambda_1 \leq \ldots \leq \lambda_N \) are the (ordered) eigenvalues of \( X_N \).

Wigner semicircle law (see e.g. [28], [6], [1], [2]) states that the random measure \( \mu_{X_N}(dx, \omega) \) converges almost surely in distribution to the (non-random) Wigner semicircle distribution \( \mu_{sc} \). The limiting distribution is absolutely continuous with respect to the Lebesgue measure and its density is given by

\[
d\mu_{sc}(x) = \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - x^2} 1_{[-2\sigma, 2\sigma]}(x).
\]

Its Stieltjes transform

\[
g_{\sigma}(z) := \int \frac{d\mu_{sc}(x)}{z-x} = \frac{z - \sqrt{z^2 - 4\sigma^2}}{2\sigma^2}, \quad z \in \mathbb{C}\setminus[-2\sigma, 2\sigma].
\]

is the solution to

\[
\sigma^2 g_{\sigma}^2(z) - zg_{\sigma}(z) + 1 = 0
\]

that decays to 0 as \( |z| \to \infty \).

This paper is devoted to the question of the fluctuations of matrix entries of \( f(X_N) \) for regular test functions \( f \). Lytova and Pastur ([13]) considered the GOE/GUE case and proved that

\[
\sqrt{N} (f(X_N)_{ij} - E(f(X_N)_{ij})) \to N(0, \frac{1 + \delta_{ij}}{\beta} \omega^2(f)),
\]
with $\beta = 1(2)$ in the GOE (GUE) case,

$$\omega^2(f) := \nu(f(\eta)) = \frac{1}{2} \int_{-2\sigma}^{2\sigma} \int_{-2\sigma}^{2\sigma} (f(x) - f(y))^2 \frac{1}{4\pi^2 \sigma^4} \sqrt{4\sigma^2 - x^2} \sqrt{4\sigma^2 - y^2} dx dy,$$

(1.16)

where $\eta$ is distributed according to the Wigner semicircle law (1.12).

In [23], Pizzo, Renfrew, and Soshnikov considered the non-Gaussian case and proved the following theorems.

**Theorem 1.1** (Theorem 1.3 in [23]). Let $X_N = \frac{1}{\sqrt{N}} W_N$ be a random real symmetric Wigner matrix (1.7), (1.4) such that the off-diagonal entries $(W_N)_{jk}, 1 \leq j < k \leq N$, are i.i.d. random variables with probability distribution $\mu$ and the diagonal entries $(W_N)_{ii}, 1 \leq i \leq N$, are i.i.d. random variables with probability distribution $\mu_1$.

Let $f : \mathbb{R} \to \mathbb{R}$ be four times continuously differentiable on $[-2\sigma - \delta, 2\sigma + \delta]$ for some $\delta > 0$ and $h(x)$ be a $C^\infty(\mathbb{R})$ function with compact support such that

$$h(x) \equiv 1 \text{ for } x \in [-2\sigma - \delta, 2\sigma + \delta], \ \delta > 0.$$  

(1.17)

Then the following holds.

(i) For $i = j$,

$$\sqrt{N} \left( f(X_N)_{ii} - \mathbb{E}((fh)(X_N)_{ii}) \right) - \frac{\alpha(f)}{\sigma}(W_N)_{ii} \to N(0, \nu^2(f)), \quad (1.18)$$

in distribution as $N \to \infty$, where

$$\nu^2(f) := 2 \left( \omega^2(f) - \alpha^2(f) + \frac{\kappa_4(\mu)}{2\sigma^4} \beta^2(f) \right), \quad (1.19)$$

$$\alpha(f) := \mathbb{E} \left( f(\eta) \frac{\eta}{\sigma} \right) = \frac{1}{\sigma} \int_{-2\sigma}^{2\sigma} x f(x) \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - x^2} dx, \quad (1.20)$$

$$\beta(f) := \mathbb{E} \left( f(\eta) \frac{\eta^2 - \sigma^2}{\sigma^2} \right) = \frac{1}{\sigma^2} \int_{-2\sigma}^{2\sigma} f(x)(x^2 - \sigma^2) \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - x^2}, \quad (1.21)$$

$\omega^2(f)$ defined in (1.16), and $\kappa_4(\mu)$ is the fourth cumulant of $\mu$,

$$\kappa_4(\mu) = \int u^4 \mu(dx) - 3 \left( \int u^2 \mu(dx) \right)^2 = \mathbb{E} |(W_N)_{12}|^4 - 3\sigma^4.$$

If $f$ is seven times continuously differentiable on $[-2\sigma - \delta, 2\sigma + \delta]$, then one can replace $\mathbb{E}((fh)(X_N)_{ii})$ in (1.18) by

$$\int_{-2\sigma}^{2\sigma} f(x) \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - x^2} dx. \quad (1.22)$$

(ii) For $i \neq j$,

$$\sqrt{N} \left( f(X_N)_{ij} - \mathbb{E}((fh)(X_N)_{ij}) \right) - \frac{\alpha(f)}{\sigma}(W_N)_{ij} \to N(0, d^2(f)) \quad (1.23)$$

in distribution as $N \to \infty$, where

$$d^2(f) := \omega^2(f) - \alpha^2(f).$$

(1.24)

If $f$ is six times continuously differentiable on $[-2\sigma - \delta, 2\sigma + \delta]$, then one can replace $\mathbb{E}((fh)(X_N)_{ij})$ in (1.23) by 0.

(iii) For any finite $m$, the normalized matrix entries

$$\sqrt{N} \left( f(X_N)_{ij} - \mathbb{E}((fh)(X_N)_{ij}) \right), \ 1 \leq i \leq j \leq m,$$

(1.25)
are independent in the limit $N \to \infty$.

**Remark 1.1.** If $f \in C^4(\mathbb{R})$ and $\|f\|_{4,1} < \infty$, where
\[
\|f\|_{n,1} := \max_{0 \leq k \leq n} \left( \int_{-\infty}^{\infty} |d^kf/dx^k(x)|dx \right) < \infty,
\]  
then one can replace $\mathbb{E}((fh)(X_N)_{ij})$ in (1.18) by $\mathbb{E}(f(X_N))_{ij}$.

In the Hermitian case, the analogue of Theorem 1.1 was proved in Theorem 1.7 of [23].

**Theorem 1.2.** (Theorem 1.7 in [23]) Let $X_N = \frac{1}{\sqrt{N}}W_N$ be a random Hermitian Wigner matrix (1.8-1.10), such that the off-diagonal entries $(W_N)_{jk}, 1 \leq j < k \leq N$, are i.i.d. complex random variables with probability distribution $\mu$ and the diagonal entries $(W_N)_{ii}, 1 \leq i \leq N$, are i.i.d. random variables with probability distribution $\mu_1$.

Let $f : \mathbb{R} \to \mathbb{R}$ be four times continuously differentiable on $[-2\sigma - \delta, 2\sigma + \delta]$ for some $\delta > 0$, and $h(x)$ be a $C^\infty(\mathbb{R})$ function with compact support satisfying (1.17). Then the following holds.

(i) For $i = j$,
\[
\sqrt{N} (f(X_N)_{ii} - \mathbb{E}((fh)(X_N)_{ii})) - \frac{\alpha(f)}{\sigma}(W_N)_{ii} \to N(0, \nu^2_2(f))
\]  
in distribution as $N \to \infty$, where
\[
\nu^2_2(f) := \omega^2(f) - \alpha^2(f) + \frac{\kappa_4(\mu)}{\sigma^4} \beta^2(f),
\]
\[
\omega^2(f), \alpha(f), \text{ and } \beta(f) \text{ are defined in (1.16), (1.20), and (1.21), and } \kappa_4(\mu) \text{ is given by}
\]
\[
\kappa_4(\mu) := \mathbb{E}[(W_N)_{12}]^4 - 2\sigma^4.
\]

If $f$ is seven times continuously differentiable on $[-2\sigma - \delta, 2\sigma + \delta]$, then one can replace $\mathbb{E}((fh)(X_N)_{ii})$ in (1.27) by (1.28).

(ii) For $i \neq j$,
\[
\sqrt{N} (f(X_N)_{ij} - \mathbb{E}((fh)(X_N)_{ij})) - \frac{\alpha(f)}{\sigma}(W_N)_{ij} \to N(0, d^2(f)),
\]  
in distribution as $N \to \infty$, where $N(0, d^2(f))$ stands for the complex Gaussian random variable with with i.i.d real and imaginary parts $N(0, \frac{1}{2}d^2(f))$, and $d^2(f)$ defined in (1.24).

If $f$ is six times continuously differentiable on $[-2\sigma - \delta, 2\sigma + \delta]$, then one can replace $\mathbb{E}((fh)(X_N)_{ij})$ in (1.29) by 0.

(iii) For any finite $m$, the normalized matrix entries
\[
\sqrt{N} (f(X_N)_{ij} - \mathbb{E}((fh)(X_N)_{ij})) \text{, } 1 \leq i \leq j \leq m,
\]  
are independent in the limit $N \to \infty$.

Almost simultaneously with [23], Pastur and Lytova (see Theorem 3.4 in [22]) extended the technique of [18] and proved the convergence in distribution for the normalized diagonal entries $\sqrt{N} (f(X_N)_{ii} - \mathbb{E}(f(X_N)_{ii}))$, $1 \leq i \leq N$, when the real symmetric Wigner matrix $X_N$ has i.i.d. entries up from the diagonal and, in
addition to the requirements of Theorem 1.1, the cumulant generating functions \( \log(\mathbb{E}e^{zW}) \) is entire. The results of [22] hold provided the test function satisfies

\[
\int_{\mathbb{R}} (1 + 2|k|)^3 |\hat{f}(k)| dk < \infty,
\]

where \( \hat{f}(k) \) is the Fourier transform

\[
\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ikx} f(x) dx.
\]  

The approaches of [23] and [22] are independent from each other. In particular, Pastur and Lytova prove the convergence of the characteristic function of \( \sqrt{N}(f(X_{N})_{ij} - \mathbb{E}(f(X_{N})_{ii})) \).

In addition, in the non-i.i.d. case, Theorem 3.2 of [22] proves that

\[
\mathbb{V} \left[ \sqrt{N}(f(X_{N})_{ij} - \mathbb{E}(f(X_{N})_{ii})) \right] \to 2\nu^2(f)
\]

provided the matrix entries \((W_{N})_{ij}\) are independent up to the diagonal and satisfy

\[
\mathbb{E}(W_{N})_{jk} = 0, \quad \mathbb{V}(W_{N})_{jk} = \sigma^2, \quad \mathbb{E}(W_{N})_{jk}^3 = m_3, \quad \mathbb{E}(W_{N})_{jk}^4 = m_4 < \infty, \quad (1.32)
\]

\[
\sup_{j,k,N} \mathbb{E}|(W_{N})_{jk}|^6 < \infty. \quad (1.33)
\]

In this paper, we extend Theorems 1.1 and 1.2 to the non-i.i.d. setting provided the matrix entries satisfy the fourth moment Lindeberg type conditions (1.2) and (1.36) for the off-diagonal entries and the second moment Lindeberg type condition (1.5) for the diagonal entries. Moreover, we relax the smoothness condition imposed in [23] on the test function.

Consider the space \( \mathcal{H}_s \) consisting of the functions \( \phi : \mathbb{R} \to \mathbb{R} \) that satisfy

\[
\|\phi\|_{s}^2 := \int_{\mathbb{R}} (1 + 2|k|)^{2s} |\hat{\phi}(k)|^2 dk < \infty. \quad (1.34)
\]

The result below is valid (see Remark 1.3) provided a test function \( f \) coincides on the interval \([-2\sigma - \delta, 2\sigma + \delta]\) with some function from \( \mathcal{H}_s \) for some \( s > 3 \), \( \delta > 0 \). Thus, roughly speaking, we require that \( f \) has \( 3 + \epsilon \) derivatives on \([-2\sigma - \delta, 2\sigma + \delta]\).

We recall that \( C^n(\mathbb{R}) \) and \( C^n([-L, L]) \) denote the spaces of \( n \) times continuously differentiable functions on \( \mathbb{R} \) and \([-L, L] \), respectively. We define the norm on \( C^n([-L, L]) \) as

\[
\|f\|_{C^n([-L, L])} := \max \left( \left| \frac{d^l f}{dx^l}(x) \right|, \ x \in [-L, L], \ 0 \leq l \leq n \right). \quad (1.35)
\]

**Theorem 1.3.** Let \( X_N = \frac{1}{\sqrt{N}}W_N \) be a random real symmetric (Hermitian) Wigner matrix (1.1), (1.4) (respectively (1.8-1.10) such that the Lindeberg type condition (1.2) for the fourth moments of the off-diagonal entries and the Lindeberg type condition (1.5) for the second moments of the diagonal entries are satisfied. Let \( f \in \mathcal{H}_s \), for some \( s > 3 \). Let \( m \) be a fixed positive integer, and for \( 1 \leq i \leq m \), assume that the following two conditions hold:

\( (A_1) \)

\( \mathcal{L}_{i,N}(\epsilon) \to 0, \text{ as } N \to \infty, \forall \epsilon > 0, \)  

(1.36)
where
\[ \mathcal{L}_{1,N}(\epsilon) = \frac{1}{N} \sum_{j \neq i} \mathbb{E} \left( |(W_N)_{ij}|^4 1_{\{|(W_N)_{ij}| \geq \epsilon N^{1/4}\}} \right); \quad (1.37) \]

\[ (A_2) \hspace{1cm} m_4(i) := \lim_{N \to \infty} \frac{1}{N} \sum_{j \neq i} \mathbb{E}|(W_N)_{ij}|^4 \] (1.38)

exists.

Then the results (i)-(iii) of Theorem 1.3 (respectively Theorem 1.2) hold for the joint distribution of the matrix entries \( \{\sqrt{N}(f(X_N)_{ij} - \mathbb{E}(f(X_N)_{ij}))\} \), 1 ≤ i ≤ j ≤ m, where \( \kappa_4(\mu) \) must be replaced in (1.16) by
\[ \kappa_4(i) := m_4(i) - 3\sigma^4, \quad 1 \leq i \leq m, \] (1.39)
in the real symmetric case and by
\[ \kappa_4(i) := m_4(i) - 2\sigma^4, \quad 1 \leq i \leq m, \] (1.40)
in the Hermitian case.

In addition, the following estimates for \( \mathbb{E}(f(X_N)_{ij}) \) take place.

(iv) Let \( f : \mathbb{R} \to \mathbb{R} \) belong to \( C^7_{\text{c}}(\mathbb{R}) \), the space of seven times continuously differentiable functions with compact support, and \( \text{supp}(f) \in [-L, L] \) for some \( L > 0 \). Then there exists a constant \( \text{Const}_1(L, \sigma, \sigma_1, m_4) \) depending on \( L, \sigma, \sigma_1, m_4 \), such that for \( 1 \leq i \leq N \),
\[ |\mathbb{E}(f(X_N)_{ii}) - \int_{-2\sigma}^{2\sigma} f(x) \frac{1}{2\pi \sigma^2} \sqrt{4\sigma^2 - x^2} \, dx| \leq \frac{\text{Const}_1(L, \sigma, \sigma_1, m_4)}{N} \left\| f \right\|_{C^7([-L, L])} \] (1.41)

(v) Let \( f \in C^8(\mathbb{R}) \), then there exists a constant \( \text{Const}_2(\sigma, \sigma_1, m_4) \) such that
\[ |\mathbb{E}(f(X_N)_{ii}) - \int_{-2\sigma}^{2\sigma} f(x) \frac{1}{2\pi \sigma^2} \sqrt{4\sigma^2 - x^2} \, dx| \leq \frac{\text{Const}_2(\sigma, \sigma_1, m_4)}{N} \left\| f \right\|_{8,1,+, 1 \leq i \leq N}, \] (1.42)

where
\[ \left\| f \right\|_{8,1,+:} := \max \left( \int_{\mathbb{R}} \{ |x| + 1 \} \frac{d^l f}{dx^l}(x) \, dx, \quad 0 \leq l \leq n \right). \] (1.43)

(vi) Let \( f \in C^6(\mathbb{R}) \), then there exists a constant \( \text{Const}_3(\sigma, \sigma_1, m_4) \) such that
\[ \left| \mathbb{E}(f(X_N)_{jk}) \right| \leq \frac{\text{Const}_3(\sigma, \sigma_1, m_4)}{N} \left\| f \right\|_{6,1}, \quad 1 \leq j < k \leq N, \] (1.44)
where \( \left\| f \right\|_{6,1} \) is defined in (1.20).

**Remark 1.2.** If the distribution of the entries of \( W_N \) does not depend on \( N \), Theorem 1.3 proves that \( \sqrt{N}(f(X_N)_{ii} - \mathbb{E}(f(X_N)_{ii})) \) converges in distribution to the sum of two independent random variables \( \frac{\alpha(f)}{\sqrt{2}} W_{ii} \) and \( N(0, 2\nu^2(f)) \) (in the Hermitian case, the second term is \( N(0, \nu^2(f)) \)), and for \( i \neq j \), \( \sqrt{N}(f(X_N)_{ij} - \mathbb{E}(f(X_N)_{ij})) \) converges in distribution to the sum of two independent random variables \( \frac{\alpha(f)}{\sqrt{2}} W_{ij} \) and \( N(0, d^2(f)) \), where in the Hermitian case \( N(0, \nu^2(f)) \) stands for the complex Gaussian random variable with i.i.d real and imaginary parts \( N(0, \frac{1}{2}d^2(f)) \). This is exactly the way Theorems 1.1 and 1.2 were formulated and proven in the i.i.d. case in [23].
Remark 1.3. If \( f : \mathbb{R} \rightarrow \mathbb{R} \) coincides on \([-2\sigma - \delta, 2\sigma + \delta]\) with a function \( \phi \in H_s \), for some \( \delta > 0 \) and \( s > 3 \), then Theorem 1.3 holds for \((f(X_N))_{ij} - \mathbb{E}(fh(X_N))_{ij}, 1 \leq i, j \leq m\), where \( h \in C_c^\infty(\mathbb{R}) \) is defined in (1.17).

If one requires that the test function \( f \) satisfies the same smoothness assumptions as in [23], then the extension of the results of [23] to the non-i.i.d. setting mostly follows the outline of the proof in [23]. To relax the conditions of Theorems 1.1 and 1.2 on the test functions, we improve the estimate on the variance of the resolvent entries (see Proposition 3.2), and employ Proposition 2.2.

We will denote throughout the paper by \( \text{const}_i, \text{Const}_i \), various positive constants that may change from line to line. Occasionally, we will drop the dependence on \( N \) in the notations for the matrix entries. Typically, we consider in detail only the real symmetric case as the proofs in the Hermitian case are very similar. Some parts of the proofs that are almost identical to the arguments in the i.i.d. case will be only sketched.

The rest of the paper is organized as follows. We prove several preliminary results in Section 2, including Proposition 2.2. Section 3 is devoted to the bounds on the mathematical expectation and variance of the resolvent entries. Theorem 1.3 is proved in Section 4. Finally, we discuss Central Limit Theorem for quadratic forms in the Appendix.

2. Preliminary Results

We start with the following lemma.

Lemma 2.1. Let \( X_N = \frac{1}{\sqrt{N}}W_N \) be a random real symmetric Wigner matrix (1.1), (1.4) such that the Lindeberg condition (1.2) for the fourth moments of the off-diagonal entries and the Lindeberg condition (1.5) for the second moments of the diagonal entries are satisfied. Then there exists a random real symmetric Wigner matrix \( \tilde{W}_N \) and a non-random positive sequence \( \epsilon_N \to 0 \) as \( N \to \infty \) such that

\[
\mathbb{E}((\tilde{W}_N)_{jk}) = 0, \quad \forall (\tilde{W}_N)_{jk} = \sigma^2, \quad 1 \leq j < k \leq N,
\]

(2.1)

\[
\sup_{N,j \neq k} \mathbb{E}((\tilde{W}_N)_{jk}^4) < \infty,
\]

(2.2)

\[
\mathbb{E}((\tilde{W}_N)_{ii}) = 0, \quad 1 \leq i \leq N,
\]

(2.3)

\[
\sup_{i, N} \mathbb{E}((\tilde{W}_N)_{ii}) < \infty,
\]

(2.4)

\[
\sup_{i,j} \left( |(\tilde{W}_N)_{ij}|, 1 \leq i,j \leq N \right) \leq \epsilon_N \sqrt{N}, \quad \mathbb{P}(\tilde{W}_N \neq \tilde{W}_N) \to 0, \quad \text{as} \quad N \to \infty.
\]

(2.5)

(2.6)

An equivalent result holds in the Hermitian case.

Proof. It follows from (1.2) and (1.3) that there exists a non-random positive sequence \( \epsilon_N \to 0 \) as \( N \to \infty \), such that

\[
\frac{1}{N^2 \epsilon_N} \sum_{1 \leq i < j \leq N} \mathbb{E}\left( |(W_N)_{ij}|^4 1_{\{|(W_N)_{ij}| \geq \epsilon_N \sqrt{N}\}} \right) \to 0.
\]

(2.7)

\[
\frac{1}{N \epsilon_N} \sum_{1 \leq i \leq N} \mathbb{E}\left( |(W_N)_{ii}|^2 1_{\{|(W_N)_{ii}| \geq \epsilon_N \sqrt{N}\}} \right) \to 0.
\]

(2.8)
One can always choose $\epsilon_N$ in such a way that it goes to zero sufficiently slow. Define $\tilde{W}_N$ by truncating the entries of $W_N$ at the level $\epsilon_N \sqrt{N}$, i.e.

$$ (\tilde{W}_N)_{ij} = (W_N)_{ij} 1_{\{|(W_N)_{ij}| \leq \epsilon_N \sqrt{N}\}}. $$

It follows from (2.7) and (2.8) that

$$ \mathbb{P}(W_N \neq \tilde{W}_N) \to 0, \text{ as } N \to \infty. \tag{2.10} $$

Let us now fix $i < j$ and consider the off-diagonal entry $(\tilde{W}_N)_{ij}$. We note that

$$ \tau_{i,j,N} := |\mathbb{E}(\tilde{W}_N)_{ij}| \leq \mathbb{E} \left(|(W_N)_{ij}| 1_{\{|(W_N)_{ij}| \geq \epsilon_N \sqrt{N}\}}\right) \tag{2.11} $$

$$ \leq \frac{1}{N^{3/2} \epsilon_N^3} \mathbb{E} \left(|(W_N)_{ij}|^4 1_{\{|(W_N)_{ij}| \geq \epsilon_N \sqrt{N}\}}\right), \tag{2.12} $$

$$ \gamma_{i,j,N}^2 := \mathbb{E}|\tilde{W}_N|_{ij}^2 - \sigma^2 = \mathbb{E} \left(|(W_N)_{ij}|^2 1_{\{|(W_N)_{ij}| \geq \epsilon_N \sqrt{N}\}}\right), \tag{2.13} $$

$$ \leq \frac{1}{N \epsilon_N^3} \mathbb{E} \left(|(W_N)_{ij}|^4 1_{\{|(W_N)_{ij}| \geq \epsilon_N \sqrt{N}\}}\right). \tag{2.14} $$

Then we can construct $(\tilde{W}_N)_{ij}$ as a mixture of the random variable $(\tilde{W}_N)_{ij}$ with weight $1 - \frac{\tau_{i,j,N}}{\sqrt{N} \epsilon_N} - \frac{\gamma_{i,j,N}^2}{N \epsilon_N}$ and some random variable $a_{i,j,N}$ with weight $\frac{\tau_{i,j,N}}{\sqrt{N} \epsilon_N} + \frac{\gamma_{i,j,N}^2}{N \epsilon_N}$ chosen so that

$$ |a_{i,j,N}| \leq \epsilon_N \sqrt{N}, \tag{2.15} $$

$$ \mathbb{E}(\tilde{W}_N)_{ij} = 0, \tag{2.16} $$

$$ \mathbb{E}(\tilde{W}_N)_{ij}^2 = \sigma^2. \tag{2.17} $$

It follows from our construction and (2.8) that

$$ \sum_{1 \leq i < j \leq N} \mathbb{P} \left((\tilde{W}_N)_{ij} \neq (\tilde{W}_N)_{ij}\right) \leq \frac{2}{N^2 \epsilon_N^3} \sum_{1 \leq i < j \leq N} \mathbb{E} \left(|(W_N)_{ij}|^4 1_{\{|(W_N)_{ij}| \geq \epsilon_N \sqrt{N}\}}\right) \to 0. \tag{2.18} $$

The diagonal case $i = j$ can be treated in a similar way. We write

$$ \tau_{i,i,N} := |\mathbb{E}(\tilde{W}_N)_{ii}| \leq \mathbb{E} \left(|(W_N)_{ii}| 1_{\{|(W_N)_{ii}| \geq \epsilon_N \sqrt{N}\}}\right) \tag{2.19} $$

$$ \leq \frac{1}{\sqrt{N} \epsilon_N} \mathbb{E} \left(|(W_N)_{ii}|^2 1_{\{|(W_N)_{ii}| \geq \epsilon_N \sqrt{N}\}}\right). \tag{2.20} $$

One then constructs $(\tilde{W}_N)_{ii}$ as a mixture of the random variable $(\tilde{W}_N)_{ii}$ with weight $1 - \frac{\tau_{i,i,N}}{\sqrt{N} \epsilon_N}$ and some random variable $a_{i,i,N}$ with weight $\frac{\tau_{i,i,N}}{\sqrt{N} \epsilon_N}$ chosen so that

$$ |a_{i,i,N}| \leq \epsilon_N \sqrt{N}, \tag{2.21} $$

$$ \mathbb{E}(\tilde{W}_N)_{ii} = 0. \tag{2.22} $$

Then

$$ \sum_{1 \leq i \leq N} \mathbb{P} \left((\tilde{W}_N)_{ii} \neq (\tilde{W}_N)_{ii}\right) \leq \frac{1}{N \epsilon_N^2} \mathbb{E} \left(|(W_N)_{ii}|^2 1_{\{|(W_N)_{ii}| \geq \epsilon_N \sqrt{N}\}}\right) \to 0, \tag{2.23} $$

as $N \to \infty$.

It follows from (2.10), (2.18), and (2.23) that (2.1) is satisfied. The equations (2.1) and (2.2) follow from (2.16), (2.17), and (2.22). The estimates (2.22) and (2.2) follow from the construction. \hfill \square
The proof of the next result is very similar to the proof of Lemma 2.1 and is left to the reader.

**Lemma 2.2.** Let \( W_N \) be a random real symmetric Wigner matrix (1.4), (1.8) and let (1.36) is satisfied for \( 1 \leq i \leq m \), where \( m \) is some fixed positive integer. Then there exists a random real symmetric Wigner matrix \( T_N \) and a non-random positive sequence \( \epsilon_N \to 0 \) as \( N \to \infty \) such that

\[
(T_N)_{jk} = (W_N)_{jk}, \quad m + 1 \leq j, k \leq N, \quad (2.24)
\]
\[
P((T_N)_{ik} = (W_N)_{ik}, 1 \leq i \leq m, \ 1 \leq k \leq N) \to 1, \quad (2.25)
\]
\[
\mathbb{E}(T_N)_{ik} = 0, \quad 1 \leq i \leq m, \ 1 \leq k \leq N, \quad (2.26)
\]
\[
\mathbb{V}(T_N)_{ik} = \sigma^2, \quad i \neq k, \ 1 \leq i \leq m, \ 1 \leq k \leq N, \quad \sup_{1 \leq i \leq m, \ 1 \leq k \leq N} \mathbb{V}(T_N)_{ii} < \infty, \quad (2.27)
\]
\[
\sup_{1 \leq i \leq m, \ 1 \leq k \leq N} |(T_N)_{ik}| \leq \epsilon_N N^{1/4}. \quad (2.28)
\]

The next Proposition is essentially due to Bai and Yin (see e.g. [5], [2]).

**Proposition 2.1.** Let \( X_N = \frac{1}{\sqrt{N}} W_N \) be a random real symmetric (Hermitian) Wigner matrix (1.4), (1.8) (respectively (1.8), (1.10)) such that the Lindeberg type condition (1.2) for the fourth moments of the off-diagonal entries and the Lindeberg type condition (1.3) for the second moments of the diagonal entries are satisfied. Then

\[
\|X_N\| \to 2\sigma \quad (2.30)
\]

in probability as \( N \to \infty \).

**Remark 2.1.** Bai and Yin ([5], [2]) considered the i.i.d. case and proved the almost sure convergence. However, convergence in probability is enough for our purposes.

**Proof.** Because of Lemma 2.1, it is enough to prove (2.30) for \( W_N \). Moreover, we can modify \( W_N \) by making all diagonal entries equal to zero. Clearly this changes the norm of \( W_N \) at most by \( \epsilon_N \). The proof uses the Method of Moments. It is enough to show that there exists a sequence \( k_N, \ N \geq 1, \) such that

\[
\frac{k_N}{\log N} \to \infty, \quad \frac{\epsilon_N^{1/3} k_N}{\log N} \to 0, \quad \text{as } N \to \infty, \quad (2.31)
\]

where \( \epsilon_N \) is the same as in Lemma 2.1 and for any constant \( z > 2\sigma \)

\[
\sum_N \text{Tr} \left( \left( W_N / \sqrt{N} \right)^{2k_N} \right) / z^{2k_N} < \infty. \quad (2.32)
\]

The proof of (2.31) in ([5]) is combinatorial in nature and does not use the fact that the entries are identically distributed. By Markov inequality, it follows from (2.32) that

\[
\sum_N \mathbb{P}(\|W_N/\sqrt{N}\| \geq z) < \infty,
\]

for any fixed \( z > 2\sigma \). Therefore, by Borel-Cantelli lemma, we have

\[
\mathbb{P}(\|W_N/\sqrt{N}\| \geq z \text{ i.o.}) = 0,
\]
Proposition 2.2. Let \( X \) be a random variable, if \( F \) is the family of bounded random variables, one can control \( V \) that \( \Omega \) is close to those in \([26]\), where \( \mu(dx, \omega) \) was taken to be the empirical spectral distribution of a random matrix.

The rest of this section is devoted to the bounds on \( V[f(x)\mu(dx, \omega)] \), where \( \mu(dx, \omega) \) is a random measure on \((\mathbb{R}, \mathcal{B})\) and \( \mathcal{B} \) is the Borel \( \sigma \)-algebra on \( \mathbb{R} \), provided one can control \( \mathbb{V}[\int_{-\infty}^{\infty} 3m \frac{1}{z} \mu(dx, \omega)] \) for \( 3m \neq 0 \). We follow the ideas of Proposition 1 in \([26]\) and Proposition 3.5 in \([13]\). In particular, our computations below are close to those in \([26]\), where \( \mu(dx, \omega) \) was taken to be the empirical spectral distribution of a random matrix.

Let \((\Omega, \mathcal{F})\) be a measurable space, and \((\Omega', \mathcal{F}', \mathcal{P})\) be a probability space such that \( \Omega' = \mathbb{R} \times \Omega \), and \( \mathcal{F}' \) is generated by \( B \times \mathcal{F} \). We denote an elementary outcome by \( \omega' = (x, \omega) \in \mathbb{R} \times \Omega \), and consider a random variable \( X(\omega') = x \). When it does not lead to ambiguity, we will denote the sub-algebra \( \{ \mathbb{R} \times D, \ D \in \mathcal{F} \} \) by \( \mathcal{F} \). Let us denote by \( \mu(B, \omega) \), \( B \in \mathcal{B}, \omega \in \Omega \), a regular conditional distribution for \( X \) given \( \mathcal{F} \), i.e.

For each \( B \subset \mathbb{R} \), \( B \in \mathcal{B} \), \( \omega \rightarrow \mu(B, \omega) \) is a version of \( \mathcal{P}(X \in B | \mathcal{F}) \). \hspace{1cm} (2.33)

For a.e. \( \omega \), \( B \rightarrow \mu(B, \omega) \) is a probability measure on \((\mathbb{R}, \mathcal{B})\). \hspace{1cm} (2.34)

Such regular conditional distribution for \( X \) always exists (see e.g. \([11]\)). In particular, if \( f : \mathbb{R} \rightarrow \mathbb{C} \) is such that

\[ \mathbb{E}|f(X)| < \infty, \]

then

\[ \mathbb{E}(f(X)|\mathcal{F}) = \int_{-\infty}^{+\infty} f(x)\mu(dx, \omega) \text{ a.s.} \]

The following proposition holds.

**Proposition 2.2.** Let \( E|X| < \infty \), \( s > \frac{1}{2} \), and \( f \in \mathcal{H}_s \), where \( \mathcal{H}_s \) is defined in \([1,3]\). Then

\[ \mathbb{V}[\int f(x)\mu(dx, \omega)] = \mathbb{V}[\mathbb{E}(f(X)|\mathcal{F})] \leq \text{Const}_s \| f \|_{\mathcal{L}}^2 \int_{0}^{\infty} dy e^{-y} y^{2s-1} \int_{-\infty}^{+\infty} dx V[\int_{-\infty}^{+\infty} 3m \frac{1}{t-x-iy} \mu(dt, \omega)]. \]

where \( \text{Const}_s \) is some absolute constant that depends only on \( s \).

**Proof.** Since \( s > \frac{1}{2} \), it follows from \([1,3]\) that \( \hat{f} \in L^1(\mathbb{R}) \) which implies that \( f \in C_0(\mathbb{R}) \), the space of continuous functions vanishing at infinity. In particular, \((2.35)\) holds and \( \mathbb{E}(f(X)|\mathcal{F}) \) is well defined. Since \( \mathbb{E}(e^{ikX}|\mathcal{F}), \ k \in \mathbb{R} \), is \( L^1 \) continuous family of bounded random variables, one can write

\[ \mathbb{E}(f(X)|\mathcal{F}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k)\mathbb{E}(e^{ikX}|\mathcal{F})dk. \hspace{1cm} (2.36) \]

Then

\[ \mathbb{V}[\mathbb{E}(f(X)|\mathcal{F})] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(k_1)\hat{f}(k_2)C(k_1, k_2)dk_1dk_2, \hspace{1cm} (2.37) \]

where

\[ C(k_1, k_2) = \text{Cov}(\mathbb{E}(e^{ik_1X}|\mathcal{F}), \mathbb{E}(e^{ik_2X}|\mathcal{F})). \hspace{1cm} (2.38) \]
Thus, it follows from (2.38) and (2.40) that
\[ K(k_1, k_2) = C(k_1, k_2)(1 + 2|k_1|)^{-s}(1 + 2|k_2|)^{-s}. \] (2.40)

Therefore,
\[ \mathbb{V}[\mathbb{E}(f(X)\mathcal{F})] \leq \frac{1}{2\pi} \|f\|^2 \|K\|, \] (2.41)

where \( \|K\| \) denotes the operator norm of the integral operator
\[ K : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad (Kg)(x) = \int_{-\infty}^{\infty} K(x, y)g(y)dy. \]

It follows from (2.38) and (2.40) that \( K \) is a non-negative definite operator. Since \( C(k_1, k_2) \) is a bounded continuous function on \( \mathbb{R}^2 \), the operator \( K \) is trace class and
\[ \|K\| \leq \text{Tr}K = \int_{-\infty}^{\infty} K(u, u)du. \] (2.42)

Thus,
\[ \mathbb{V}[\mathbb{E}(f(X)\mathcal{F})] \leq \frac{1}{2\pi} \|f\|^2 \int_{-\infty}^{\infty} C(k, k)(1 + 2|k|)^{-2s} dk. \] (2.43)

Let us fix \( z = x + iy, \ y \neq 0 \), and consider \( \mathfrak{Im} \frac{1}{X-z} \) as a function of \( \lambda \). Its Fourier transform is given by \( \frac{\sqrt{\pi}}{\lambda} e^{-|y| - i\lambda x} \). Therefore,
\[ V[\mathbb{E}(\mathfrak{Im}(X - x - iy)^{-1} \mathcal{F})] = \frac{1}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(|k_1|+|k_2|)|y|} e^{i(k_2 - k_1)x} C(k_1, k_2) dk_1 dk_2. \] (2.44)

Formally, taking into account
\[ \int_{-\infty}^{\infty} e^{i(k_2 - k_1)x} dx = 2\pi \delta(k_2 - k_1), \]
we obtain
\[ \int_{-\infty}^{\infty} V[\mathbb{E}(\mathfrak{Im}(X - x - iy)^{-1} \mathcal{F})] dx = \frac{\pi}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(|k_1|+|k_2|)|y|} \delta(k_2 - k_1) C(k_1, k_2) dk_1 dk_2. \] (2.45)

Since
\[ \int_{0}^{+\infty} dy e^{-y} y^{2s-1} e^{-2|k||y|} = \Gamma(2s)(1 + 2|k|)^{-2s}, \]
we conclude that
\[ \int_{0}^{\infty} dy e^{-y} y^{2s-1} \int_{-\infty}^{\infty} V[\mathbb{E}(\mathfrak{Im}(X - x - iy)^{-1} \mathcal{F})] dx = \frac{\pi}{2} \Gamma(2s) \int_{-\infty}^{\infty} C(k, k)(1 + 2|k|)^{-2s} dk. \] (2.46)

The bound on \( \mathbb{V}[\mathbb{E}(f(X)\mathcal{F})] \) in Proposition 2.2 now follows from (2.43) and (2.45).
Therefore, integrating by parts with respect to \( t \), we obtain
\[
\int_0^\infty \int_{-A}^A V[\mathbb{E}(3m(X-x-iy)^{-1}|F)]dx = \frac{1}{2} \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(|k_1|+|k_2|)|y|} \sin(A(k_2-k_1)) \frac{C(k_1,k_2)dk_1dk_2}{k_2-k_1}.
\] (2.48)

Multiplying (2.48) by \( e^{-y2s-1} \) and integrating over \( y \in (0, +\infty) \), we obtain
\[
\int_0^{+\infty} dy e^{-y2s-1} \int_{-A}^A dx V[\mathbb{E}(3m(X-x-iy)^{-1}|F)] = \frac{1}{2} \Gamma(2s) \int_{-\infty}^\infty \int_{-\infty}^\infty (1+|k_1|+|k_2|)^{-2s} \sin(A(k_2-k_1)) \frac{C(k_1,k_2)dk_1dk_2}{k_2-k_1}. \] (2.49)

We note that the integrand in (2.49) is absolutely integrable over \( \mathbb{R}^2 \) for \( s > \frac{1}{2} \), so the last step is justified by the Fubini theorem. Since \( E[X] < \infty \), it follows from (2.38) that the kernel \( C(k_1,k_2) \) has bounded continuous first partial derivatives (see Lemma 2.3 below). We split the integral in (2.50) into two, over

\[
S := \{(k_1,k_2) : |k_2-k_1| < A^{-\epsilon}\}
\]

and over \( \mathbb{R}^2 \setminus S \). For \((k_1,k_2) \in S\), we use
\[
|C(k_1,k_2)(1+|k_1|+|k_2|)^{-2s} - C(k_1,k_1)(1+2|k_1|)^{-2s}| \leq const|k_2-k_1|(1+2|k_1|)^{-2s}
\]
which implies that the integral over \( S \) equals to
\[
\frac{1}{2} \Gamma(2s) \int_{-A^{-\epsilon}}^{A^{-\epsilon}} dt \frac{\sin(At)}{t} \int_{-\infty}^\infty dk C(k,k)(1+2|k|)^{-2s} + o(1). \] (2.51)

where we made the change of variables \((k_1,k_2) \rightarrow (t = k_2 - k_1, k = k_1)\).

To estimate the integral over \( \mathbb{R}^2 \setminus S \), we restrict our attention to the quadrant \( k_1 \geq 0, k_2 \geq 0 \). The other three cases are similar. Denote \( C_1(t,u) = C(k_1,k_2) \), where \( u = k_1 + k_2 \) and \( t = k_2 - k_1 \). We have to estimate
\[
\int_0^\infty du \int_{A^{-\epsilon}}^{u} dt (1+u)^{-2s} \frac{\sin(At)}{t} C_1(t,u). \] (2.52)

Integrating by parts with respect to \( t \), we obtain
\[
\int_{A^{-\epsilon}}^{u} \frac{\sin(At)}{t} C_1(t,u) dt = \int_{A^{-\epsilon}}^{u} \frac{\cos(At)}{A} \left( \frac{\partial C_1(t,u)}{\partial t} - \frac{C_1(t,u)}{t} \right) + \frac{\cos(At)}{A} C_1(t,u) {u \atop A^{-\epsilon}}.
\] (2.53)

It is not difficult to see that the r.h.s. of (2.53) is bounded in absolute value by \( const \arg \log |u| + A^s \) and (2.51) is bounded in absolute value by \( constA^{1+s} \). Therefore, the integral over \( \mathbb{R}^2 \setminus S \) goes to zero as \( A \rightarrow \infty \).

Finally, we note that the term in (2.51) converges to
\[
\frac{\Gamma(2s)}{2} \int_{-\infty}^\infty C(k,k)(1+2|k|)^{-2s}dk.
\] (2.55)

This finishes the proof of Proposition 2.2 modulo Lemma 2.3 below. \( \square \)
In the proof of Proposition 2.2, we used the fact that \( C(k_1, k_2) \), defined in (2.38), has continuous bounded first partial derivatives. This is the statement of the following lemma.

**Lemma 2.3.** Let \( \mathbb{E}|X| < \infty \), and \( C(k_1, k_2) \) be defined as in (2.38). Then \( C(k_1, k_2) \) has continuous bounded first partial derivatives.

**Proof.** We recall that \( C(k_1, k_2) = \text{Cov}(\mathbb{E}(e^{ik_1X}|\mathcal{F}), \mathbb{E}(e^{ik_2X}|\mathcal{F})) \)

\[
= \mathbb{E}(\mathbb{E}(e^{ik_1X}|\mathcal{F})\mathbb{E}(e^{ik_2X}|\mathcal{F})) - \mathbb{E}(e^{ik_1X})\mathbb{E}(e^{ik_2X}).
\]

It follows from the Lebesgue dominated convergence theorem (for conditional expectations), that

\[
\frac{\partial C(k_1, k_2)}{\partial k_1} = i\text{Cov}(\mathbb{E}(Xe^{ik_1X}|\mathcal{F}), \mathbb{E}(e^{ik_2X}|\mathcal{F})).
\]  

(2.56)

Applying the Lebesgue dominated convergence theorem one more time, we obtain that \( \frac{\partial C(k_1, k_2)}{\partial k_1} \) is a bounded continuous function. \( \square \)

3. **Mathematical Expectation and Variance of Resolvent Entries**

This section is devoted to the estimates of the mathematical expectation and the variance of the resolvent entries. For \( z \in \mathbb{C} \setminus \mathbb{R} \), we denote the resolvent of \( X_N \) by

\[
R_N(z) := (zI_N - X_N)^{-1}.
\]  

(3.1)

If it does not lead to ambiguity, we will use the shorthand notation \( R_{ij}(z) \) for \( (R_N(z))_{ij} \), \( 1 \leq i, j \leq N \). We start with the following proposition.

**Proposition 3.1.** Let \( X_N = \frac{1}{\sqrt{N}}W_N \) be a random real symmetric (Hermitian) Wigner matrix \( \{L\} \), \( \{A\} \) (respectively \( \{S\}, \{I\} \)). Then

\[
\mathbb{E}R_{ii}(z) = g_\sigma(z) + O \left( \frac{1}{|\Im z|^{3/2}} \right),
\]

uniformly on bounded subsets of \( \mathbb{C} \setminus \mathbb{R} \),

(3.2)

\[
\mathbb{E}R_{ij}(z) = O \left( \frac{P_6(|\Im z|^{-1})}{N} \right), \quad 1 \leq i \neq j \leq N, \quad \text{uniformly on } \mathbb{C} \setminus \mathbb{R},
\]

(3.3)

\[
\forall R_{ij}(z) = O \left( \frac{P_6(|\Im z|^{-1})}{N} \right), \quad 1 \leq i, j \leq N, \quad \text{uniformly on } \mathbb{C} \setminus \mathbb{R}.
\]

(3.4)

where we denote by \( P_l(x) \), \( l \geq 1 \), a polynomial of degree \( l \) with fixed positive coefficients.

If, in addition,

\[
\sup_{i \neq j, N} \mathbb{E}|(W_N)_{ij}|^5 < \infty, \quad \sup_{i, N} \mathbb{E}|(W_N)_{ii}|^3 < \infty,
\]

then

\[
\mathbb{E}R_{ij}(z) = O \left( \frac{P_6(|\Im z|^{-1})}{N^{3/2}} \right), \quad 1 \leq i \neq j \leq N, \quad \text{uniformly on } \mathbb{C} \setminus \mathbb{R}.
\]  

(3.5)
This proposition is the extension of Proposition 3.1 in [23] to the non-i.i.d. case. Since the proofs of (3.2-3.5) are very similar to the proofs given in Proposition 3.1 in Section 2 of [23], we leave the details to the reader.

The next proposition is instrumental in extending Theorem 1.3 to the test functions from $H_s$ for $s > 3$. Our goal is to obtain an upper bound on $\mathbb{V}[(R_N)_{ij}(z)]$ which is integrable with respect to $x = \Re z$ over the real line for $\Im z \neq 0$.

Proposition 3.2. Let $X_N = \frac{1}{\sqrt{N}}W_N$ be a random real symmetric Wigner matrix (1.1), (1.4) such that the condition (1.36) is satisfied for some fixed $m \geq 1$. Then there exists a random real symmetric Wigner matrix $T_N$ and a non-random positive sequence $\epsilon_N \to 0$ as $N \to \infty$ such that the properties (2.24-2.29) from Lemma 2.2 are satisfied and, in addition,

$$\mathbb{V}[(G_N)_{ij}(z)] = O \left( \frac{(\mathbb{E}\|G_N(z)\|^2)P_4(\|\Im z\|^{-1})}{N} \right),$$

$$1 \leq i \leq m, \ 1 \leq j \leq N, \ \text{uniformly on } \mathbb{C} \setminus \mathbb{R}, \ \text{where } G_N(z) := \left(zI_N - \frac{1}{\sqrt{N}}T_N\right)^{-1} \text{ is the resolvent of } \frac{1}{\sqrt{N}}T_N.$$

An equivalent result holds in the Hermitian case.

Proof. The existence of a Wigner random matrix $T_N$ that satisfies (2.24-2.29) follows from Lemma 2.2. All is left to us is to show that (3.6) holds. Since $\mathbb{P}(X_N = T_N) \to 1$ as $N \to \infty$, we can assume, without loss of generality, that $T_N = X_N$.

Let $L$ be a positive constant that will be later chosen to be sufficiently large depending on $\sigma, \sigma_1, \text{ and } m_4$. We note that if

$$\frac{1}{|\Im z|^4 N} \geq \frac{1}{L} \quad \text{(3.7)}$$

then

$$\mathbb{V}[(R_N)_{ij}(z)] \leq \mathbb{E}\|R_N(z)\|^2 \leq \frac{L\mathbb{E}\|R_N(z)\|^2}{|\Im z|^4 N}. \quad \text{(3.8)}$$

Thus, (3.7) implies (3.6).

Now, let us assume that

$$\frac{1}{|\Im z|^4 N} < \frac{1}{L} \quad \text{(3.9)}$$

One can rewrite (3.9) as

$$|\Im z| > \frac{L^{1/4}}{N^{1/4}}. \quad \text{(3.10)}$$

Let us fix $1 \leq i, j \leq m$. Then

$$z\mathbb{E}R_{ij}(z) = \delta_{ij} + \sum_{k=1}^{N} \mathbb{E}(X_{ik}R_{kj}(z)). \quad \text{(3.11)}$$

To estimate $\mathbb{E}(X_{ik}R_{kj}(z))$, we use the decoupling formula (see e.g. (i) in Section 2 in [16] and Proposition 3.1 in [19]). Let $\xi$ be a real random variable with $p + 2$
finite moments, and \( \phi \) a real-valued function with \( p + 1 \) continuous and bounded derivatives. Then

\[
\mathbb{E}(\xi \phi(\xi)) = \sum_{a=0}^{p} \frac{\kappa_{a+1}}{a!} \mathbb{E}(\phi^{(a)}(\xi)) + \epsilon_{p+1},
\]

(3.12)

where \( \kappa_{a} \) are the cumulants of \( \xi \),

\[
|\epsilon_{p+1}| \leq C \sup_{t} |\phi^{(p+1)}(t)| |\mathbb{E}(|\xi|^{p+2})|,
\]

(3.13)

and \( C \) depends only on \( p \). Moreover, as follows from the proof of Proposition 3.1 in [19], if \( \text{supp}(\xi) \subset [-K, K] \) then the supremum on the r.h.s. of (3.13) can be taken over \( t \in [-K, K] \).

The derivative of \( R_{kl} \) with respect to \( X_{pq} \), for \( p \neq q \) is given by

\[
\frac{\partial R_{kl}}{\partial X_{pq}} = R_{kp}R_{ql} + R_{kq}R_{pl}.
\]

(3.14)

For \( p = q \) the derivative is given by

\[
\frac{\partial R_{kl}}{\partial X_{pp}} = R_{kp}R_{pl}.
\]

(3.15)

Applying (3.12–3.15) to the term \( \mathbb{E}(X_{ik}R_{kj}) \) in (3.11), we obtain the following Master equation

\[
\frac{z}{\mathbb{E}} R_{ij}(z) = \delta_{ij} + \sigma^2 \mathbb{E}[R_{ij}(z)\text{tr}_N R_N(z)] + \frac{\sigma^2}{N} \mathbb{E}[(R_N(z)^2)_{ij}]
\]

(3.16)

\[
- \frac{2\sigma^2}{N} \mathbb{E}[R_{ij}(z)R_{ij}(z)] + r_N
\]

(3.17)

\[
= \delta_{ij} + \sigma^2 \mathbb{E}[R_{ij}(z)\text{tr}_N R_N(z)] + r_N + O\left(\frac{\mathbb{E}[\|R_N(z)\|^2]}{N}\right),
\]

(3.18)

where \( r_N \) contains the third cumulant term corresponding to \( p = 2 \) in (3.12), and the error due to the truncation of the decoupling formula (3.12) at \( p = 2 \). For \( k = i \), we truncate the decoupling formula (3.12) at \( p = 0 \).

We will need the following lemma.

**Lemma 3.1.** The following two bounds hold.

\[
\text{Cov}(R_{ij}(z), \text{tr}_N R_N(z)) = O\left(\frac{P_2(|3m z|^{-1})\mathbb{E}[\|R_N(z)\|^{3/2}]}{N}\right),
\]

(3.19)

uniformly in \( z \in \mathbb{C} \setminus \mathbb{R} \).

\[
r_N = O\left(\frac{P_2(|3m z|^{-1})\mathbb{E}[\|R_N(z)\|^2]}{N}\right),
\]

(3.20)

uniformly in \( z \) satisfying (3.10), where \( L \) is an arbitrary fixed positive number.

**Proof.** The bound (3.19) follows from the first of the two bounds on the variance of the trace of the resolvent in Proposition 2 of [20]. It should be mentioned that the bound is valid provided the second moments of the diagonal entries are uniformly bounded and the fourth moments of the off-diagonal entries are also uniformly bounded ([24]).

To prove the bound (3.20), one has to study the third cumulant term that corresponds to \( p = 2 \) in the decoupling formula (3.12) for \( k \neq i \) and the error terms due to the truncation of (3.12) at \( p = 2 \) for \( k \neq i \) and at \( p = 0 \) for \( k = i \).
The third cumulant term gives
\[
\frac{1}{2! N^{3/2}} [4\mathbb{E}( \sum_{k:k \neq i} \kappa_3((W_N)_{ik}) R_{ik} R_{ik} R_{ik}) + 2\mathbb{E}( \sum_{k: k \neq i} \kappa_3((W_N)_{ik}) R_{ik} R_{kk} R_{kj}) + 2\mathbb{E}( \sum_{k: k \neq i} \kappa_3((W_N)_{ik}) (R_{ik})^2 R_{jk})],
\]
where \(\kappa_3((W_N)_{ik})\) denotes the third cumulant of \((W_N)_{ik}\). Since \(|\kappa_3((W_N)_{ik})| \leq \text{const}(m_4)\),
\[
\sum_k |R_{ik}|^2 \leq \|R_N(z)\|^2, \quad \text{and} \quad |R_{pq}|(z) \leq \|R_N(z)\| \leq \frac{1}{|\Im(z)|},
\]
(3.21) one observes that the third cumulant term can be bounded in absolute value by
\[
O \left( \frac{\mathbb{E}\|R_N(z)\|^2}{|\Im(z)| N} \right).
\]

To estimate the error term due to the truncation of (3.12) at \(p = 2\) for \(k \neq i\), we have to consider finitely many sums of the following form
\[
N^{-2} \mathbb{E} \left( \sum_{k:k \neq i} \sup_{|x| \leq \epsilon_N N^{-1/4}} |R_{ab}^{(1)}| |R_{cd}^{(2)}| |R_{ef}^{(3)}| |R_{pq}^{(4)}| \right),
\]
(3.22) where \(a, b, c, d, e, f, p, q, s \in \{i, k, j\}\), the supremum in (3.22) is considered over all possible resolvents \(R^{(l)} = (z - X_N^{(l)})^{-1}, \ l = 1, \ldots, 4\) of rank two perturbations \(X_N^{(l)} = X_N + x E_{ik}\) of \(X_N\) with \((E_{ik})_{jh} = \delta_{ij} \delta_{kh} + \delta_{ih} \delta_{kj}\). Since
\[
|X_{ik}| \leq \epsilon_N N^{-1/4}, \ k \neq i, \ \epsilon_N \to 0 \text{ as } N \to \infty,
\]
by (2.29), we can restrict \(x\) in the supremum in (3.22) to \(|x| \leq \epsilon_N N^{-1/4}\). Then
\[
R_{N}^{(l)}(z) = (z I_N - X_N)_{ik}^{-1} = (z I_N - X_N + x E_{ik})^{-1} = (I_N + R_N(z)x E_{ik})^{-1} R_N(z).
\]
Since by taking into account (3.10)
\[
\|R_N(z)x E_{ik}\| \leq \frac{1}{|\Im(z)|} \epsilon_N N^{-1/4} \leq \frac{N^{1/4}}{L^{1/4}} \epsilon_N N^{-1/4} = \epsilon_N L^{-1/4} = o(1),
\]
we have
\[
\|R_N^{(l)}(z)\| \leq \|R_N(z)\|(1 + o(1)),
\]
and we obtain that the expression in (3.22) can be bounded from above by \(O \left( \frac{\mathbb{E}\|R_N(z)\|^4}{N} \right)\).

It follows from
\[
\|R_N(z)\| = \frac{1}{\text{dist}(z, S_P(X_N))} \leq |\Im(z)|^{-1}.
\]
(3.23) that one can write the upper bound as \(O \left( \frac{\mathbb{E}\|R_N(z)\|^2}{|\Im(z)|^{1/2}} \right)\).

To estimate the error term due to the truncation of (3.12) at \(p = 0\) for \(k = i\), one proceeds in a similar manner. Lemma 3.1 is proven.

The rest of the proof of Proposition 3.2 is similar to the proof of (3.3) in [23]. The details are left to the reader.
4. Proof of Theorem 1.3

The goal of this Section is to prove Theorem 1.3.

First, we extend the estimates of Proposition 3.1 to a sufficiently wide class of test function by using Helffer-Sj"ostrand functional calculus ([14], [10]) as in [23]. Let \( f \in C^{l+1}(\mathbb{R}) \) decay at infinity sufficiently fast. Then, one can write

\[
f(X_N) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial \hat{f}}{\partial \bar{z}} R_N(z) \, dx \, dy, \quad \frac{\partial \hat{f}}{\partial \bar{z}} := \frac{1}{2} \left( \frac{\partial \hat{f}}{\partial x} + i \frac{\partial \hat{f}}{\partial y} \right)
\]

where:

i) \( z = x + iy \) with \( x, y \in \mathbb{R} \);

ii) \( \hat{f}(z) \) is the extension of the function \( f \) defined as follows

\[
\hat{f}(z) := \left( \sum_{n=0}^{l} \frac{f^{(n)}(x)(iy)^n}{n!} \right) \sigma(y);
\]

here \( \sigma \in C^\infty(\mathbb{R}) \) is a nonnegative function equal to 1 for \( |y| \leq 1/2 \) and equal to zero for \( |y| \geq 1 \).

Using the definition of \( \hat{f} \) (see (4.2)) one can calculate

\[
\frac{\partial \hat{f}}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial \hat{f}}{\partial x} + i \frac{\partial \hat{f}}{\partial y} \right) = \frac{1}{2} \left( \sum_{n=0}^{l} \frac{f^{(n)}(x)(iy)^n}{n!} \right) \frac{d\sigma}{dy} + \frac{1}{2} f^{(l+1)}(x)(iy)^l \frac{\sigma(y)}{l!}
\]

and derive the crucial bound

\[
\left| \frac{\partial \hat{f}}{\partial \bar{z}}(x + iy) \right| \leq \text{Const} \max \left( \frac{d^j f}{dx^j}(x), \ 1 \leq j \leq l + 1 \right) |y|^l.
\]

Directly following the calculations in Section 3 of [23], one obtains the following extension to a non-i.i.d. setting of Proposition 1.1 in [23].

**Proposition 4.1.** Let \( X_N = \frac{1}{\sqrt{N}} W_N \) be a random real symmetric (Hermitian) Wigner matrix \((1.1), (1.4)\) (respectively \(1.8, 1.10\)). Then the following holds.

(i) Let \( L \) be some positive number, \( f \in C^7(\mathbb{R}) \) with compact support, and \( \text{supp}(f) \subset [-L, +L] \). Then there exists a constant \( \text{Const}(L, \sigma, \sigma_1, m_4) \) such that

\[
|\mathbb{E}(f(X_N)_{ii}) - \int_{-2\sigma}^{2\sigma} f(x) \frac{1}{2\pi \sigma^2} \sqrt{4\sigma^2 - x^2} \, dx| \leq \text{Const}(L, \sigma, \sigma_1, m_4) \frac{\|f\|_{C^7([-L, +L])}}{N},
\]

\( 1 \leq i \leq N \).

(ii) Let \( f \in C^8(\mathbb{R}) \), then there exists a constant \( \text{Const}(\sigma, \sigma_1, m_4) \) such that

\[
|\mathbb{E}(f(X_N)_{ii}) - \int_{-2\sigma}^{2\sigma} f(x) \frac{1}{2\pi \sigma^2} \sqrt{4\sigma^2 - x^2} \, dx| \leq \text{Const}(\sigma, \sigma_1, m_4) \frac{\|f\|_{C^8([-L, +L])}}{N}, \quad 1 \leq i \leq N.
\]

where \( \|f\|_{n,1,+} \) is defined in \((1.43)\).
(iii) Let \( f \in C^6(\mathbb{R}) \), then
\[
|\mathbb{E}(f(X_N)_{jk})| \leq \text{Const}(\sigma, \sigma_1, m_4) \frac{\|f\|_{6,1}}{N}, \quad 1 \leq j < k \leq N,
\]  
where \( \|f\|_{n,1} \) is defined in (1.20).

(iv) Let \( f \in C^4(\mathbb{R}) \), then
\[
\nabla(f(X_N)_{ij}) \leq \text{Const}(\sigma, \sigma_1, m_4) \frac{\|f\|_{2,1}}{N}, \quad 1 \leq i, j \leq N.
\]  

(v) If

\[
\sup_{i \neq j, N} \mathbb{E}|(W_N)_{ij}|^5 < \infty, \quad \sup_{i, N} \mathbb{E}|(W_N)_{ii}|^3 < \infty,
\]
and \( f \in C^{10}(\mathbb{R}) \), then one can improve (4.8), namely
\[
|\mathbb{E}(f(X_N)_{jk})| \leq \text{Const} \frac{\|f\|_{10,1}}{N^{3/2}}, \quad 1 \leq j < k \leq N,
\]  
where \( \text{Const} \) depends on \( \sup_{i \neq j, N} \mathbb{E}|(W_N)_{ij}|^5 \), and \( \sup_{i, N} \mathbb{E}|(W_N)_{ii}|^3 \).

The next proposition is a corollary of Propositions 2.2 and 3.2.

**Proposition 4.2.** Let \( X_N = \frac{1}{\sqrt{N}} W_N \) be a random real symmetric Wigner matrix (1.1), (1.4) such that (1.36) is satisfied for some fixed \( m \geq 1 \). Then there exists a random real symmetric Wigner matrix \( T_N \) and a non-random positive sequence \( \varepsilon_N \to 0 \) as \( N \to \infty \) such that the properties (2.24-2.29) from Lemma 2.2 are satisfied. In addition, for \( s > 3 \), there exists a constant \( \text{const}_s \) that depends on \( s, \sigma, \sigma_1, \) and \( m_4 \) such that for \( f \in H_s \)
\[
\nabla[f(T_N/\sqrt{N})_{ij}] \leq \text{const}_s \frac{\|f\|^2}{N}, \quad 1 \leq i \leq m, \quad 1 \leq j \leq N.
\]  

**Proof.** The existence of random real symmetric Wigner matrix \( T_N \) satisfying (2.24, 2.29) has been proven in Lemma 2.2. Since \( P(X_N = T_N) \to 1 \) as \( N \to \infty \), we can assume without loss of generality that \( T_N = X_N \).

Let us first consider the diagonal case \( i = j \). Without loss of generality, one can assume \( i = 1 \). Define a random spectral measure
\[
\mu(dx, \omega) := \sum_{l=1}^{N} \delta(x - \lambda_l)|\phi_l(1)|^2,
\]
where \( \lambda_l, 1 \leq l \leq N \), are the eigenvalues of \( X_N \) and \( \phi_l, 1 \leq l \leq N \), are the corresponding normalized eigenvectors. Since by the result by Latala [17]
\[
\sup_N \mathbb{E}\|X_N\| < \infty,
\]
we have
\[
\mathbb{E} \int |x| \mu(dx, \omega) = \mathbb{E}(\|X_N\|_{11}) < \infty,
\]
one can apply Proposition 2.2 and obtain
\[
\nabla[f(X_N)_{11}] \leq \text{Const}_s \frac{\|f\|_{s}^2}{N} \int_0^\infty dy y^{2s-1} \int_{-\infty}^\infty V[(R_N(x + iy))_{11}] dx.
\]  

(4.12)
To estimate the integral \( \int_{-\infty}^{\infty} V[(R_N(x + iy))_{11}] \, dx \) in (4.12), one uses the upper bound (3.6) in Proposition 3.2 to obtain
\[
\frac{P_4(y^{-1})}{N} \int_{-\infty}^{\infty} ||R_N(x + iy)||^2 \, dx \quad (4.13)
\]
\[
+ \frac{P_4(y^{-1})}{N} \int_{-\infty}^{\infty} ||R_N(x + iy)||^{3/2} \, dx. \quad (4.14)
\]
We will treat the first term (4.13). The second term (4.14) can be treated in a similar fashion. For \( x \in [-\|X_N\|, +\|X_N\|] \), we use the trivial bound
\[
||R_N(x + iy)||^2 \leq \frac{1}{y^2}.
\]
For \( |x| > \|X_N\| \), we write
\[
||R_N(x + iy)||^2 \leq \frac{1}{(x - \|X_N\|^2 + y^2}.
\]
Thus,
\[
\int_{-\infty}^{\infty} ||R_N(x + iy)||^2 \, dx \leq \frac{2\|X_N\|}{y^2} + \frac{\pi}{y}. \quad (4.15)
\]
Since (17)
\[
\sup_N \mathbb{E}\|X_N\| < \infty,
\]
we obtain
\[
\forall [f(X_N)_{11}] \leq \text{Const}_s \frac{\|f\|^2}{N} \int_0^\infty dy e^{-y} y^{2s-1} P_4(y^{-1}) \left( \frac{\text{const}_1}{y^2} + \frac{\text{const}_2}{y^{1/2}} \right). \quad (4.16)
\]
If \( s > 3 \), the integral in (4.10) converges.

In the off-diagonal case \( i \neq j \), one can consider the (complex-valued) measure
\[
\mu(dx, \omega) := \sum_{l=1}^N \delta(x - \lambda_l) \overline{\phi_l(i) \phi_l(j)},
\]
write it as a linear combination of probability measures, and apply Proposition 2.2 to each probability measure in the linear combination. Proposition 4.2 is proven.

Now, we are ready to prove Theorem 1.3. Let \( m \) be a fixed positive integer. Denote by \( W_N^{(m)} \) the \( m \times m \) upper-left corner submatrix of \( W_N \), and by \( R_N^{(m)}(z) \) the \( m \times m \) upper-left corner of the resolvent matrix \( R_N(z) \). Our next step is to compute the limiting distribution of the normalized entries of \( R_N^{(m)}(z) \) in the limit \( N \to \infty \). In the i.i.d. setting, this was done in Theorem 1.1 (real symmetric case) and Theorem 1.5 (Hermitian case) in [23]. Below, we extend these results to the non-i.i.d. setting. We start with the real symmetric case. Define
\[
\Upsilon_N(z) := \sqrt{N} \left( R_N^{(m)}(z) - g_\sigma(z) I_m \right), \quad z \in \mathbb{C} \setminus [-2\sigma, 2\sigma], \quad (4.17)
\]
\[
\Psi_N(z) := \Upsilon_N(z) - g_\sigma^2(z) W_N^{(m)} = \sqrt{N} \left( R_N^{(m)}(z) - g_\sigma(z) I_m \right) - g_\sigma^2(z) W_N^{(m)}. \quad (4.18)
\]
Clerally, \( \Psi_N(z) \) and \( \Psi_N(z) \) are random function with values in the space complex symmetric \( m \times m \) matrices. (real symmetric \( m \times m \) matrices for real \( x \)). Define

\[
\varphi(z, w) := \int_{-2\sigma}^{2\sigma} \frac{1}{z-x} \frac{1}{w-x} \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - x^2} dx = \left\{ \begin{array}{ll}
-\frac{g_o(w)-g_o(z)}{w-z} : & \text{if } w \neq z, \\
-g_o'(z) : & \text{if } w = z.
\end{array} \right.
\]

for \( z, w \in \mathbb{C} \setminus [-2\sigma, 2\sigma] \). One can write \( \varphi(z, w) = \mathbb{E} \left( \frac{1}{z-\eta} \frac{1}{w-\eta} \right) \), where \( \eta \) is a Wigner semicircle law \((1.12)\) random variable. Let

\[
\varphi_+ (z, w) = \int_{-2\sigma}^{2\sigma} \Re \frac{1}{z-x} \frac{1}{w-x} \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - x^2} dx
\]

(4.20)

\[
= \frac{1}{4} \left( \varphi(z, w) + \varphi(\bar{z}, \bar{w}) + \varphi(\bar{z}, w) + \varphi(z, \bar{w}) \right),
\]

\[
\varphi_- (z, w) = \int_{-2\sigma}^{2\sigma} \Im \frac{1}{z-x} \frac{1}{w-x} \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - x^2} dx
\]

(4.21)

\[
= \frac{1}{4} \left( \varphi(z, w) + \varphi(\bar{z}, \bar{w}) - \varphi(\bar{z}, w) - \varphi(z, \bar{w}) \right),
\]

(4.22)

\[
\varphi_{++} (z, w) := \int_{-2\sigma}^{2\sigma} \Re \frac{1}{z-x} \frac{1}{w-x} \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - x^2} dx
\]

Theorem 4.1. Let \( X_N = \frac{1}{\sqrt{N}} W_N \) be a random real symmetric Wigner matrix \((1.1), \,(1.4)\). Let \( m \) be a fixed positive integer and assume that for \( 1 \leq i \leq m \) the conditions \((1.36)\) and \((1.38)\) are satisfied. Also assume that the Lindeberg type condition \((1.3)\) for the second moments of the off-diagonal entries and the Lindeberg type condition \((1.3)\) for the second moments of the diagonal entries are satisfied.

Then the random field \( \Psi_N(z) \) in \((4.19)\) converges in finite-dimensional distributions to the random field

\[
\Psi(z) = g_o^2(z) Y(z),
\]

(4.23)

where \( Y(z) = (Y_{ij}(z)) \), \( Y_{ij}(z) = Y_{ji}(z) \), \( 1 \leq i, j \leq m \), is the Gaussian random field such that

\[
\text{Cov}(\Re Y_{ii}(z), \Re Y_{ij}(w)) = \kappa(i) \Re g_o(z) \Re g_o(w) + 2\sigma^4 \varphi_{++}(z, w),
\]

(4.24)

\[
\text{Cov}(\Im Y_{ii}(z), \Im Y_{ij}(w)) = \kappa(i) \Im g_o(z) \Im g_o(w) + 2\sigma^4 \varphi_{--}(z, w),
\]

(4.25)

\[
\text{Cov}(\Re Y_{ij}(z), \Re Y_{ij}(w)) = \kappa(i) \Re g_o(z) \Re g_o(w) + 2\sigma^4 \varphi_{++}(z, w),
\]

(4.26)

\[
\text{Cov}(\Im Y_{ij}(z), \Im Y_{ij}(w)) = \sigma^4 \varphi_{++}(z, w), \ i \neq j,
\]

(4.27)

\[
\text{Cov}(\Re Y_{ij}(z), \Im Y_{ij}(w)) = \sigma^4 \varphi_{--}(z, w), \ i \neq j,
\]

(4.28)

\[
\text{Cov}(\Im Y_{ij}(z), \Re Y_{ij}(w)) = \sigma^4 \varphi_{--}(z, w), \ i \neq j,
\]

(4.29)

where \( \kappa(i) = m_4(i) - 3\sigma^4 \), \( 1 \leq i \leq m \), and \( m_4(i) \) is defined in \((1.38)\). In addition, for any finite \( r \geq 1 \), the entries \( Y_{i_l j_l}(z_l) \), \( 1 \leq i_l \leq j_l \leq m \), \( 1 \leq l \leq r \), are independent if for any \( 1 \leq l \leq r \) one has \((i_{l_1}, j_{l_1}) \neq (i_{l_2}, j_{l_2})\).

Now, we consider the Hermitian case. As before, we define by \((1.18)\) the matrix-valued random field \(\Psi_N(z)\), \( z \in \mathbb{C} \setminus [-2\sigma, 2\sigma] \). \(\Psi_N(x)\) is Hermitian for real \( x \) and, more generally, \(\Psi_N(z) = \Psi_N(\bar{z})^*\).
Theorem 4.2. Let \( X_N = \frac{1}{\sqrt{N}} W_N \) be a random real Hermitian Wigner matrix \((1.8)\). Let \( m \) be a fixed positive integer and assume that for \( 1 \leq i \leq m \) the conditions \((1.36)\) and \((1.38)\) are satisfied. Also assume that the Lindeberg type condition \((1.2)\) for the fourth moments of the off-diagonal entries and the Lindeberg type condition \((1.3)\) for the second moments of the diagonal entries are satisfied.

Then the random field \( \Psi_N(z) \) converges in finite-dimensional distributions to the random field

\[
\Psi(z) = g_2^2(z) Y(z),
\]

where \( Y(z) = (Y_{ij}(z)) \), \( 1 \leq i, j \leq m \), is the Gaussian random field such that

\[
\begin{align*}
\text{Cov}(\Re Y_{ii}(z), \Re Y_{ii}(w)) &= \kappa_4(i) \Re g_\sigma(z) \Re g_\sigma(w) + \sigma^4 \varphi_{++}(z, w), \\
\text{Cov}(\Im Y_{ii}(z), \Im Y_{ii}(w)) &= \kappa_4(i) \Im g_\sigma(z) \Im g_\sigma(w) + \sigma^4 \varphi_{--}(z, w), \\
\text{Cov}(\Re Y_{ij}(z), \Re Y_{ij}(w)) &= \frac{1}{2} \sigma^4 (\varphi_{++}(z, w) + \varphi_{--}(z, w)), \quad i \neq j, \\
\text{Cov}(\Im Y_{ij}(z), \Im Y_{ij}(w)) &= \frac{1}{2} \sigma^4 (\varphi_{--}(z, w) - \varphi_{++}(w, z)), \quad i \neq j,
\end{align*}
\]

where \( \kappa_4(i) = m_4(i) - 2 \sigma^4 \), \( 1 \leq i \leq m \), and \( m_4(i) \) is defined in \((1.38)\).

In addition, for any finite \( r \geq 1 \), the entries \( Y_{ii}(z_l) \), \( 1 \leq i \leq m \), \( 1 \leq l \leq r \), are independent provided \( (i_1, j_1) \neq (i_2, j_2) \) for \( 1 \leq l_1 \neq l_2 \leq r \).

Remark 4.1. If the distribution of the entries of \( W_N \) does not depend on \( N \), the random field

\[
\Upsilon_N(z) = \sqrt{N} \left( R^{(m)}(z) - g_\sigma(z) I_m \right), \quad z \in \mathbb{C} \setminus [-2\sigma, 2\sigma]
\]

converges in finite-dimensional distributions to \( g_2^2(z)(Y(z) + W^{(m)}) \), where \( Y(z) \) is independent from \( W^{(m)} \).

Below, we sketch the proof of Theorem 4.1. The proof in the Hermitian case is very similar.

Proof. As in \([23]\), one can write

\[
R_N^{(m)}(z) = \left( zI_m - X^{(m)} - M^* \tilde{R} M \right)^{-1} = \left( zI_m - \frac{1}{\sqrt{N}} W_N^{(m)} - M^* \tilde{R} M \right)^{-1},
\]

where \( X_N^{(m)} \) is the \( m \times m \) upper-left corner submatrix of \( X_N \), \( \tilde{X}^{(N-m)} \) is the \( (N-m) \times (N-m) \) lower-right corner submatrix of \( X_N \),

\[
\tilde{R}_N(z) = \left( zI_{N-m} - \tilde{X}^{(N-m)} \right)^{-1},
\]

is the resolvent of \( \tilde{X}^{(N-m)} \), and \( M \) is the the \( (N-m) \times m \) lower-left corner submatrix of \( X_N \). We will denote by \( x^{(1)}, \ldots, x^{(m)} \in \mathbb{R}^{N-m} \) the (column) vectors that form \( M \), and by \( M^* \) the adjoint matrix of \( M \).

It follows from Proposition \([23]\) that \( \tilde{R}_N(z) \) is well defined for any fixed \( z \in \mathbb{C} \setminus [-2\sigma, 2\sigma] \) with probability going to 1.
Define the $m \times m$ matrix $\Gamma_N(z)$ as
\[
(\Gamma_N)_{ij}(z) = (W_N)_{ij} + \sqrt{N} \left( (x^{(i)}, \tilde{R}(z)x^{(j)}) - \sigma^2 g_\sigma(z) \delta_{ij} \right), \quad 1 \leq i, j \leq m. \tag{4.38}
\]

Then
\[
\Gamma_N(z) = W_N^{(m)} + Y_N(z), \tag{4.39}
\]
where
\[
(Y_N(z))_{ij} = Y_{ij}(z) = \sqrt{N} \left( (x^{(i)}, \tilde{R}(z)x^{(j)}) - \sigma^2 g_\sigma(z) \delta_{ij} \right), \quad 1 \leq i, j \leq m. \tag{4.40}
\]

Equations (4.37) and (4.38) imply
\[
\tilde{R}^{(m)}(z) = \left( \frac{1}{g_\sigma(z)} I_m - \frac{1}{\sqrt{N}} \Gamma_N(z) \right)^{-1}. \tag{4.41}
\]

It will follow from the Central Limit Theorem for Quadratic Forms (see discussion below and the Appendix) that $\|\Gamma_N(z)\|$ is bounded in probability. This would imply that
\[
Y_N(z) = \sqrt{N} \left( \tilde{R}^{(m)}(z) - g_\sigma(z) I_m \right) = g_\sigma(z) \Gamma_N(z) + o(1), \tag{4.42}
\]
in probability (meaning that the error term goes to zero in probability), and
\[
\Psi_N(z) = \sqrt{N} \left( \tilde{R}^{(m)}(z) - g_\sigma(z) I_m \right) - g_\sigma(z) W_N^{(m)} = g_\sigma^2(z) Y_N(z) + o(1), \tag{4.43}
\]
in probability.

To estimate $\|\Gamma_N(z)\|$, where $\Gamma_N(z) = W_N^{(m)} + Y_N(z)$, we note that for fixed $m$, $\|W_N^{(m)}\|$ is bounded in probability. Let us consider in more detail $Y_N(z)$. Assume that $z$ is fixed and $\Im z \neq 0$. It follows from
\[
\mathbb{E}Y_N(z) = \sqrt{N} \sigma^2 (g_\sigma(z) - g_\sigma(z)) I_m,
\]
and Proposition 3.1 that $\mathbb{E}Y_N(z) \to 0$. Thus,
\[
Y_N(z)_{ij} = \sqrt{N} \left( (x^{(i)}, \tilde{R}(z)x^{(j)}) - \mathbb{E}(x^{(i)}, \tilde{R}(z)x^{(j)}) \right) + o(1), \quad 1 \leq i, j \leq m. \tag{4.44}
\]

We note that the vectors $x^{(i)}$, $1 \leq i \leq m$, are independent from $\tilde{R}(z)$. In the Appendix, we point out that the Central Limit Theorem for Quadratic Forms also holds in the non-i.i.d. case under the conditions on the entries of $x^{(i)}$, $1 \leq i \leq m$, that are equivalent to (1.36). This implies that $\|Y_N(z)\|$ is bounded in probability, and therefore $\|\Gamma_N(z)\|$ is bounded as well, which implies (4.2) (4.3).

To study the finite-dimensional distributions of $Y_N(z)$, we fix a positive integer $p \geq 1$, and consider $z_1, \ldots, z_p \in \mathbb{C} \setminus \mathbb{R}$. Taking into account (4.44), the problem is reduced to the question about the joint distribution of the entries
\[
\sqrt{N} \left( (R_N(z))_{i_l, j_l} - \mathbb{E}(R_N(z))_{i_l, j_l} \right), \quad 1 \leq i_l \leq j_l \leq m, \quad 1 \leq l \leq p.
\]
To this end, we apply Theorem A.4 in the Appendix with $r = m$, and
\[
B_N^{s,t} = \sum_{l=1}^{p} \left( a_{s,t}^{(l)} \mathbb{R}(\tilde{R}(z_l)) + b_{s,t}^{(l)} \mathbb{I}(\tilde{R}(z_l)) \right), \quad 1 \leq s \leq t \leq m, \tag{4.45}
\]
where $a_{s,t}^{(l)}$, $b_{s,t}^{(l)}$, $1 \leq s \leq t \leq m$, $1 \leq l \leq p$, are arbitrary real numbers, and
\[
y_N^{(s)} = \frac{\sqrt{N}}{\sigma} x^{(s)}, \quad 1 \leq s \leq m.
\]
The condition (i) of Theorem A.4 is equivalent to \(1.30\). The condition (ii) is automatically satisfied as long as \(\Im z_l \neq 0\), \(1 \leq l \leq m\). Conditions (iii) and (iv) are equivalent to

\[
\frac{1}{N-m} \text{Tr} \left( \Re(\tilde{R}(z)) \Re(\tilde{R}(w)) \right) \to \varphi_{++}(z, w),
\]

\[
\frac{1}{N-m} \text{Tr} \left( \Im(\tilde{R}(z)) \Im(\tilde{R}(w)) \right) \to \varphi_{--}(z, w),
\]

\[
\frac{1}{N-m} \text{Tr} \left( \Re(\tilde{R}(z)) \Im(\tilde{R}(w)) \right) \to \varphi_{+-}(z, w),
\]

\[
\frac{1}{N-m} \sum_{j=m+1}^{N} \kappa_4((W_N)_{ij})(\Re(\tilde{R}(z)))_{jj}(\Re(\tilde{R}(w)))_{jj} \to \kappa_4(i) \Re(g_0(z)) \Re(g_0(w)),
\]

\[
\frac{1}{N-m} \sum_{j=m+1}^{N} \kappa_4((W_N)_{ij})(\Im(\tilde{R}(z)))_{jj}(\Im(\tilde{R}(z)))_{jj} \to \kappa_4(i) \Im(g_0(z)) \Im(g_0(w)),
\]

\[
\frac{1}{N-m} \sum_{j=m+1}^{N} \kappa_4((W_N)_{ij})(\Re(\tilde{R}(z)))_{jj}(\Im(\tilde{R}(z)))_{jj} \to \kappa_4(i) \Re(g_0(z)) \Im(g_0(w)),
\]

\[
\frac{1}{N-m} \sum_{j=m+1}^{N} \kappa_4((W_N)_{ij})(\Im(\tilde{R}(z)))_{jj}(\Im(\tilde{R}(z)))_{jj} \to \kappa_4(i) \Re(g_0(z)) \Im(g_0(w)),
\]

for \(z, w \in \mathbb{C} \setminus [-2\sigma, 2\sigma]\), \(1 \leq i \leq m\), where \(\varphi_{++}(z, w), \varphi_{--}(z, w), \varphi_{+-}(z, w)\) are defined in \(4.49\), \(4.50\), and \(4.51\), and the convergence is in probability. To make the formulas \(4.49\), \(4.50\), \(4.51\) look less cumbersome, we label the diagonal entries of the \((N-m) \times (N-m)\) matrices \(\Re(\tilde{R}(z)), \Im(\tilde{R}(z))\) by index \(j = m+1, \ldots, N\).

The conditions \(4.46\), \(4.47\), \(4.48\), \(4.49\), \(4.50\), \(4.51\) follow from the semicircle law, and \(4.49\), \(4.51\) follow from the estimates \(3.42\) and \(3.44\) in Proposition 3.1. The details are left to the reader. Theorem A.4 now implies that \(Y_N(z)\) converges in finite-dimensional distributions to \(Y(z)\) for \(\Im z \neq 0\). For \(z \in \mathbb{R} \setminus [-2\sigma, 2\sigma]\), one can replace \(\tilde{R}(z)\) by \(h(X_N)\tilde{R}(z)\), where \(h\) satisfies \(1.17\) and repeat the arguments above since \(\mathbb{P}(\tilde{R}(z) \neq h(X_N)\tilde{R}(z)) \to 0\) as \(N \to \infty\).

To complete the proof of Theorem A.3 we first restrict our attention to the four time continuously differentiable test functions with compact support. Let \(f \in C^4_c(\mathbb{R})\). It follows from Theorem 1.1 and Proposition 2.1 that the result of Theorem A.3 holds for finite linear combinations

\[
\sum_{i=1}^{k} a_i h_i(x) \frac{1}{z_i-x}, \quad z_i \notin [-2\sigma, 2\sigma], \quad 1 \leq l \leq k,
\]

where \(h_i \in C^\infty_c(\mathbb{R}), 1 \leq l \leq k\), satisfies \(1.17\). By Stone-Weierstrass theorem (see e.g. \(23\)), one can approximate an arbitrary \(C^4_c(\mathbb{R})\) by functions of the form \(4.52\). Moreover, if \(\text{supp}(f) \subset [-A, A]\), one can choose the approximating sequence in such a way that \(\text{supp}(h_i) \subset [-A - 1, A + 1]\). Applying the bound \(1.9\) in Proposition 4.1 we show that

\[
\mathbb{V}[\sqrt{N}(f(X_N))_{ij} - \sum_{l=1}^{k} a_l (h_l(X_N))_{ij}]
\]
can be made arbitrary small uniformly in $N$, which finishes the proof for $f \in C^4_c(\mathbb{R})$.

To extend the proofs to the case of $f \in \mathcal{H}_s$, for some $s > 3$, we use the estimate (4.11) in Proposition 4.2 and approximate such $f$ by a sequence $\{f_n\}_{n \geq 1}$ such that

$$\|f - f_n\|_s \to 0, \text{ as } n \to \infty, \quad f_n \in C^4_c(\mathbb{R}), \quad n \geq 1.$$  

(4.53)

This finishes the proof of Theorem 1.3.

**Appendix A. Central Limit Theorem for Quadratic Forms**

The appendix is devoted to the formulation of the CLT type results for the quadratic forms $y^* N B y_N$ where $y_N$ is a random $N$-vector that contains independent entries with finite fourth moment and $B$ is a random $N \times N$ Hermitian matrix. The formulated results and their proofs are similar to the results in [4], [9] (see the appendix by Baik and Silverstein), and [7] since the arguments presented there work with small changes in the non-i.i.d. setting as well.

First we present the case where the entries of $Y_N$ are complex and then the case where the entries are real.

**Theorem A.1** (Central Limit Theorem for Quadratic Forms). Let $B = (b_{ij})_{1 \leq i,j \leq N}$ be a $N \times N$ random Hermitian matrix and $y_N = (y_{Nj})_{1 \leq j \leq N}$ be an independent vector of size $N$ which contains independent complex standardized entries such that $\sup_{N,j} \mathbb{E}|y_{Nj}|^4 = m_4 < \infty$ and $\mathbb{E}(y_{Nj}^2) = 0$. Assume that

(i) for all $\epsilon > 0$,

$$\frac{1}{N} \sum_{j=1}^{N} \mathbb{E}\left[|y_{Nj}|^2 - 1\right]^2 1_{\{|y_{Nj}|^2 - 1| > \epsilon \sqrt{N}\}} \longrightarrow 0$$  

(A.1)

as $N \to \infty$,

(ii) there exists a constant $a > 0$ (not depending on $N$) such that $\|B\| \leq a$,

(iii) $\frac{1}{N} \text{Tr} B^2$ converges in probability to a number $a_2$,

(iv) $\frac{1}{N} \sum_{i=1}^{N} b_{ii}^2 \kappa_4(y_{Ni})$ converges in probability to a number $a_1$,

where

$$\kappa_4(y_{Ni}) := \mathbb{E}|y_{Ni}|^4 - 2, \quad 1 \leq i \leq N.$$  

(A.2)

Then the random variable $\frac{1}{\sqrt{N}}(y_N^* B y_N - \text{Tr} B)$ converges in distribution to a Gaussian random variable with mean zero and variance

$$v^2 = a_1 + a_2.$$  

**Theorem A.2** (Central Limit Theorem for Real Quadratic Forms). Let $B = (b_{ij})_{1 \leq i,j \leq N}$ be a $N \times N$ random real symmetric matrix and $y_N = (y_{Nj})_{1 \leq j \leq N}$ be an independent vector of size $N$ which contains independent real standardized entries with $\sup_{N,j} \mathbb{E}|y_{Nj}|^4 = m_4 < \infty$ and $\mathbb{E}(y_{Nj}^2) = 0$. Assume that conditions (i)-(iv) hold as in Theorem A.1 with

$$\kappa_4(y_{Ni}) := \mathbb{E}|y_{Ni}|^4 - 3, \quad 1 \leq i \leq N.$$  

(A.3)

Then the random variable $\frac{1}{\sqrt{N}}(y_N^* B y_N - \text{Tr} B)$ converges in distribution to a Gaussian random variable with mean zero and variance

$$v^2 = a_1 + 2a_2.$$  

Finally, we formulate the multidimensional versions of Theorems A.1 and A.2. We again consider the real and complex cases separately.
Theorem A.3. Let \( \{B^{s,t} : 1 \leq s, t \leq r\} \) be a family of \( N \times N \) random matrices with the property that \( (B^{s,t})^\ast = B^{t,s} \). Let \( \{y_N^{(s)} : 1 \leq s \leq r\} \) be a family of independent \( N \)-vectors with independent complex standardized entries where \( y_N^{(s)} = (y_{N,j}^{(s)})_{1 \leq j \leq N} \), \( \sup_{N,j} E|y_{N,j}^{(s)}|^2 = m_4 < \infty \), and \( E[(y_{N,j}^{(s)})^2] = 0 \). Further assume that

1. for all \( \epsilon > 0 \),
   \[
   \frac{1}{N} \sum_{j=1}^{N} E \left[ |y_{N,j}^{(s)}|^2 - 1 \right] \left[ \frac{2}{N} \left| \frac{1}{\varepsilon \sqrt{N}} \right| \to 0 \right.
   \]
   as \( N \to \infty \) for each \( 1 \leq s \leq r \),
2. there exists a constant \( a > 0 \) (not depending on \( N \)) such that \( \max_{1 \leq s,t \leq r} \|B^{s,t}\| \leq a \),
3. \( \frac{1}{N} \text{Tr}((B^{s,t})^\ast B^{s,t}) \) converges in probability to a number \( a_2(s,t) \),
4. \( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (B^{s,s})_{i,i} \kappa_4(y_{N,i}^{(s)}) \) converges in probability to a number \( a_1(s) \), where
   \[
   \kappa_4(y_{N,i}^{(s)}) = E|y_{N,i}^{(s)}|^4 - 2, \quad 1 \leq i \leq N. \]

Then the \( r \times r \) matrix
\[
G_N = \frac{1}{\sqrt{N}} \left( (y_N^{(s)})^\ast B^{s,t} y_N^{(t)} - \delta_{s,t} \text{Tr}B^{s,t} \right)_{1 \leq s,t \leq r}
\]
converges in distribution to an \( r \times r \) Hermitian matrix \( G \) such that the linearly independent entries are statistically independent and \( \mathcal{N}(G_{s,t}), \Im(G_{s,t}) \sim \mathcal{N}(0, \frac{1}{2} a_2(s,t)) \) for \( s \neq t \) and \( G_{s,s} \sim \mathcal{N}(0, a_1(s) + 2 a_2(s,s)) \).

Theorem A.4. Let \( \{B^{s,t} : 1 \leq s, t \leq r\} \) be a family of \( N \times N \) real random matrices with the property that \( (B^{s,t})^\ast = B^{t,s} \). Let \( \{y_N^{(s)} : 1 \leq s \leq r\} \) be a family of independent \( N \)-vectors with independent real standardized entries where \( y_N^{(s)} = (y_{N,j}^{(s)})_{1 \leq j \leq N} \) and \( \sup_{N,j} E|y_{N,j}^{(s)}|^2 = m_4 < \infty \). Further assume that the conditions (i)-(iv) from Theorem A.3 hold with
\[
\kappa_4(y_{N,i}^{(s)}) = E|y_{N,i}^{(s)}|^4 - 3, \quad 1 \leq i \leq N. \]

Then the \( r \times r \) matrix
\[
G_N = \frac{1}{\sqrt{N}} \left( (y_N^{(s)})^\ast B^{s,t} y_N^{(t)} - \delta_{s,t} \text{Tr}B^{s,t} \right)_{1 \leq s,t \leq r}
\]
converges in distribution to an \( r \times r \) symmetric matrix \( G \) such that the linearly independent entries are statistically independent and \( G_{s,t} \sim \mathcal{N}(0, a_2(s,t)) \) for \( s \neq t \) and \( G_{s,s} \sim \mathcal{N}(0, a_1(s) + 2 a_2(s,s)) \).

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