Cycles over DGH-semicategories and pairings in categorical Hopf-cyclic cohomology

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Abstract

Let $H$ be a Hopf algebra and let $D_H$ be a Hopf-module category. We describe the cocycles and coboundaries for the Hopf cyclic cohomology of $D_H$, which correspond respectively to categorified cycles and vanishing cycles over $D_H$. An important role in our work is played by semicategories, which are categories that may not contain identity maps. In particular, a cycle over $D_H$ consists of a differential graded $H$-module semicategory equipped with a trace on endomorphism groups satisfying some conditions. Using a pairing on cycles, we obtain a pairing $HC^p(C) \otimes HC^q(C') \rightarrow HC^{p+q}(C \otimes C')$ on cyclic cohomology groups for small $k$-linear categories $C$ and $C'$.

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1 Introduction

In [13], [14], [15], Connes and Moscovici introduced Hopf-cyclic cohomology as a generalization of Lie algebra cohomology adapted to Noncommutative Geometry. Given a Hopf algebra $H$ that is equipped with a modular pair in involution $(\delta, \sigma)$ and acts on an algebra $A$, they constructed a characteristic map

$$\gamma^\bullet : HC^\bullet(\delta, \sigma)(H) \rightarrow HC^\bullet(A) \quad (1.1)$$

taking values in the cyclic cohomology $HC^\bullet(A)$ of $A$. Both Hochschild homology and the cyclic theory have since been studied extensively in several categorical contexts (see, for instance, [7], [8], [22], [30], [32], [34], [36]). The purpose of this paper is to categorify the formalism of cycles, traces and vanishing cycles of Connes in the context of Hopf cyclic cohomology.

Let $k$ be a field. A Hopf-module category consists of a $k$-linear category $D_H$ with $H$ acting on its morphism spaces in such a way that the composition on $D_H$ is well-behaved with respect to the coproduct on $H$. This notion was introduced by Cibils and Solotar in [10], where they constructed a Morita equivalence connecting the Galois coverings of a category to its smash extensions via a Hopf algebra. A small Hopf-module category may be treated as a “Hopf module algebra with several objects,” in the same way as a small preadditive category plays the role of a “ring with several objects” in the sense of Mitchell [35]. In fact, the replacement of rings by small preadditive categories has been widely studied in the literature (see, for instance, [6], [16], [33], [35], [43], [44]). Further, cyclic modules associated to Hopf-module categories have been studied by Kaygun and Khalkhali [26], while the Hochschild-Mitchell cohomology of a Hopf-comodule category has been studied by Herscovich and Solotar [21]. This paper is also part of our larger program of studying Hopf-module categories as objects of independent interest, begun in [4], [5].

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For a Hopf-module category $\mathcal{D}_H$, we describe in this paper the cocycles and coboundaries that determine its Hopf cyclic cohomology groups by extending Connes’ original construction of cyclic cohomology from \[11\] and \[12\] in terms of cycles and closed graded traces on differential graded algebras. An important role in our paper is played by “semicategories,” which are categories that may not contain identity maps. This notion, introduced by Mitchell [37], is precisely what we need in order to categorify non unital algebras. We work with the Hopf cyclic cohomology groups $HC^*_H(\mathcal{D}_H, M)$ having coefficients in $M$, where $M$ is a stable anti-Yetter Drinfeld module in the sense of \[17\]. Accordingly, we interpret the cocycles $Z^*_H(\mathcal{D}_H, M)$ and the coboundaries $B^*_H(\mathcal{D}_H, M)$ as characters of differential graded $H$-module semicategories equipped with closed graded traces with coefficients in $M$. We therefore feel that the present article is the first step towards “categorification” of the Noncommutative Differential Geometry of Connes \[12\]. We continue this program by developing categorifications of Fredholm modules and associated Chern characters in \[8\].

We now describe the paper in more detail. In Section 2, we introduce the notion of a $\delta$-invariant $\sigma$-trace on a left $H$-category $\mathcal{D}_H$ (see Definition \[22\]), where $(\delta, \sigma)$ is a modular pair in involution for the Hopf algebra $H$. Given such a trace, we prove (see Theorem \[24\]) that there is a characteristic map

$$\gamma^*: HC^*_{(\delta, \sigma)}(H) \longrightarrow HC^*(\mathcal{D}_H) \quad (1.2)$$

from the Hopf-cyclic cohomology $HC^*_{(\delta, \sigma)}(H)$ taking values in the ordinary cyclic cohomology $HC^*(\mathcal{D}_H)$ of the category $\mathcal{D}_H$. When $\mathcal{D}_H$ is a left $H$-category with a single object, i.e., an $H$-module algebra $A$, this recovers the characteristic map $HC^*_{(\delta, \sigma)}(H) \longrightarrow HC^*(A)$ in \[11\].

In Section 3, we study the $\delta$-invariant $\sigma$-traces on $\mathcal{D}_H$ in more detail. For this, we first take the Hopf-cyclic cohomology $HC^*_{(\delta, \sigma)}(\mathcal{D}_H, M)$ of an $H$-category $\mathcal{D}_H$ with coefficients in an SAYD module $M$. We then show that $\delta$-invariant $\sigma$-traces on $\mathcal{D}_H$ are precisely the 0-cocycles, i.e., the elements of $Z^0_H(\mathcal{D}_H, \sigma k_\delta) = HC^0_H(\mathcal{D}_H, \sigma k_\delta)$ (see Proposition \[30\]). Since a $\delta$-invariant $\sigma$-trace on $\mathcal{D}_H$ induces a map $\gamma^*: HC^*_{(\delta, \sigma)}(H) \longrightarrow HC^*(\mathcal{D}_H)$, we obtain a pairing

$$HC^n_{(\delta, \sigma)}(H) \otimes HC^0_H(\mathcal{D}_H, \sigma k_\delta) \longrightarrow HC^n(\mathcal{D}_H) \quad (1.3)$$

The pairing in \[1.3\] suggests that we look for a similar pairing when the Hopf algebra $H$ is replaced by an $H$-module coalgebra $C$ and the SAYD module $\sigma k_\delta$ is replaced by an arbitrary SAYD module $M$. In \[17\], it was conjectured that there is a general pairing between the Hopf-cyclic cohomology of a module coalgebra and the Hopf-cyclic cohomology of a module algebra, a fact that was proved later by Khalkhali and Rangipour in \[29\]. In Theorem \[4.3\] we use methods similar to Rangipour \[40\] to construct a pairing

$$HC^n_C(C, M) \otimes HC^0_H(\mathcal{D}_H, M) \longrightarrow HC^{n+q}(\mathcal{D}_H) \quad (1.4)$$

for $p, q \geq 0$. For related work on pairings and Hopf-cyclic cohomology, we refer the reader to \[1\], \[2\], \[18\], \[19\], \[20\], \[21\], \[22\], \[25\], \[40\].

In Section 5, we provide a description of the space $Z^*_H(\mathcal{D}_H, M)$ of cocycles, for which we extend the formalism of Connes \[12\]. We first describe in detail the construction of the universal differential graded (DG)-semicategory associated to a small $k$-linear category. We then consider DGH-SEM categories which may be treated as differential graded (not necessarily unital) $H$-module algebras with several objects. Since $\delta$-invariant $\sigma$-traces on $\mathcal{D}_H$ are precisely the 0-cocycles in the Hopf-cyclic cohomology $HC^*_{(\delta, \sigma)}(\mathcal{D}_H, \sigma k_\delta)$, we are motivated to define more generally the $n$-dimensional closed graded $(H, M)$-traces on a DGH-semicategory $S_H$ (see, Definition \[5.3\]). We then introduce cycles $(S_H, \hat{\partial}_H, \hat{\gamma}^H)$ over the $H$-category $\mathcal{D}_H$ using which we provide a description of $Z^*_H(\mathcal{D}_H, M)$ in Theorem \[5.11\]. This result is an $H$-linear categorical version of Connes’ \[12\] Proposition 1, p. 98. We show that an element $\phi \in Z^*_H(\mathcal{D}_H, M)$ if and only if it is the character of an $n$-dimensional cycle over $\mathcal{D}_H$. It also follows from Theorem \[5.11\] that there is a one to one correspondence between $Z^*_H(\mathcal{D}_H, M)$ and the collection of $n$-dimensional closed graded $(H, M)$-traces on the universal DGH-semicategory $\Omega(\mathcal{D}_H)$ associated to $\mathcal{D}_H$. We then proceed to obtain a description of the space $B^*_H(\mathcal{D}_H, M)$ of coboundaries.

In Section 6, we show that the Hopf-cyclic cohomology of an $H$-category $\mathcal{D}_H$ is the same as that of its linearization $\mathcal{D}_H \otimes M_r(k)$ by the matrix ring $M_r(k)$. For this, we first construct a para-cyclic module.
\[ C_\bullet(D_H, M) = \{ M \otimes CN_n(D_H) \}_{n \geq 0} \] using the cyclic nerve \( CN_\bullet(D_H) \). We also consider inclusion and trace maps

\[
\begin{align*}
(inc_1, M) : M \otimes CN_n(D_H) & \to M \otimes CN_n(D_H \otimes M_r(k)) \\
tr^M : M \otimes CN_n(D_H \otimes M_r(k)) & \to M \otimes CN_n(D_H)
\end{align*}
\]

Then, we show in Proposition 6.5 that the induced morphisms

\[
\begin{align*}
C_\bullet(inc_1, M)^{hoc} : C_\bullet(D_H, M)^{hoc} & \to C_\bullet(D_H \otimes M_r(k), M)^{hoc} \\
C_\bullet(tr^M)^{hoc} : C_\bullet(D_H \otimes M_r(k), M)^{hoc} & \to C_\bullet(D_H, M)^{hoc}
\end{align*}
\]

between the underlying Hochschild complexes are homotopy inverses of each other. Applying the functor \( Hom_H(-, k) \), we show in Proposition 6.6 that there are mutually inverse isomorphisms of Hopf-cyclic cohomologies:

\[
HC^*_H(D_H, M) \xrightarrow{HC^*_H(tr^M)} HC^*_H(D_H \otimes M_r(k), M) \xleftarrow{HC^*_H(inc_1, M)} HC^*_H(D_H, M)
\]

(1.5)

In Section 7, we provide a description of \( B^*_H(D_H, M) \). Throughout, we take \( k = \mathbb{C} \). We consider families \( \eta \) of automorphisms \( \eta = \{ \eta(X) \in Aut_{D_H}(X) \}_{X \in Ob(D_H)} \) such that

\[
h(\eta(X)) = \varepsilon(h)\eta(X) \quad \forall \ h \in H, X \in Ob(D_H)
\]

We show that these families form a group, which we denote by \( U_H(D_H) \). Further, we show that the inner automorphism of \( D_H \) induced by conjugating with an element \( \eta \in U_H(D_H) \), i.e., the functor which fixes objects and takes any morphism \( f \in Hom_{D_H}(X, Y) \to \eta(Y) \circ f \circ \eta(X)^{-1} \), induces the identity functor on \( HC^*_H(D_H, M) \). Using this and the isomorphisms in (1.5), we obtain in Proposition 7.5 a set of sufficient conditions for the Hopf cyclic cohomology of an \( H \)-category to be zero.

We say that a cycle \( (S_H, \Delta_H, \mathcal{F}^H) \) is vanishing if \( S^0_H \) is an \( H \)-category and \( S^0_H \) satisfies the assumptions in Proposition 7.5. We describe the elements of \( B^*_H(D_H, M) \) in Proposition 7.11 as the characters of vanishing cycles over \( D_H \). Finally, in Theorem 7.13 we use categorified cycles and vanishing cycles to construct a pairing

\[
HC^p(C) \otimes HC^q(C') \to HC^{p+q}(C \otimes C')
\]

for \( k \)-linear small categories \( C \) and \( C' \).

**Notations:** Throughout the paper, \( H \) is a Hopf algebra over the field \( k \) of characteristic zero, with co-multiplication \( \Delta \), counit \( \varepsilon \) and bijective antipode \( S \). We will use Sweedler’s notation for the coproduct \( \Delta(h) = h_1 \otimes h_2 \) and for a left \( H \)-coaction \( \rho : M \to H \otimes M, \rho(m) = m_{(-1)} \otimes m_{(0)} \) (with the summation sign suppressed). The small cyclic category introduced by Connes in [11] will be denoted by \( \hat{\Lambda} \). The Hochschild differential will always be denoted by \( b \).

### 2 Categorified characteristic in Hopf-cyclic cohomology

It is well known that a ring can be identified with a preadditive category having a single object (see, for instance [22]). Accordingly, any small preadditive category may be treated as a ring with several objects in the sense of Mitchell (see [28,39]). We now recall the notion of an \( H \)-category, introduced by Cibils and Solotar [10], which may be considered as an “\( H \)-module algebra with several objects.”

**Definition 2.1.** Let \( H \) be a Hopf algebra over a field \( k \). A \( k \)-linear category \( D_H \) is said to be a left \( H \)-module category if

\[
(i) \ Hom_{D_H}(X, Y) \text{ is a left } H \text{-module for all } X, Y \in Ob(D_H) \\
(ii) \ h(id_X) = \varepsilon(h)id_X \text{ for all } X \in Ob(D_H) \text{ and } h \in H
\]
(iii) the composition map is a morphism of $H$-modules, i.e.,
\[ h(gf) = (h_1g)(h_2f) \]
for any $h \in H$, $f \in \text{Hom}_{D_H}(X,Y)$ and $g \in \text{Hom}_{D_H}(Y,Z)$.

A small left $H$-module category will be called a left $H$-category.

We now let $H$ be a Hopf algebra with a modular pair in involution $(\delta, \sigma)$ (see \cite{15}) and let $D_H$ be a left $H$-category. In this section, we introduce the notion of a $\delta$-invariant $\sigma$-trace on the category $D_H$. Using this trace, we then construct a characteristic map from the cyclic cohomology of the Hopf algebra $H$ to that of the category $D_H$. We first recall from \cite{14, 15} the cyclic cohomology of a Hopf algebra.

Let $\delta \in H^*$ be a character and $\sigma \in H$ be a group-like element (i.e., $\Delta(\sigma) = \sigma \otimes \sigma$ and $\varepsilon(\sigma) = 1$) such that $\delta(\sigma) = 1$. Then, the pair $(\delta, \sigma)$ is said to be a modular pair on $H$. The character $\delta$ determines a $\delta$-twisted antipode $S_\delta : H \rightarrow H$ given by
\[ S_\delta(h) = \delta(h_1)S(h_2) \quad \forall h \in H \]
The pair $(\delta, \sigma)$ is said to be a modular pair in involution if $S_\delta^2(h) = \sigma \delta^{-1}$ for all $h \in H$.

Given a modular pair $(\delta, \sigma)$ in involution for a Hopf algebra $H$, one can associate a $\Lambda$-module $C^* (H(\delta, \sigma))$ by setting $C^n(H) := H^\otimes n$, $\forall n \geq 1$ and $C^0(H) = k$. For $n > 1$, the face maps $\delta_i : C^{n-1}(H) \rightarrow C^n(H)$ are given as follows:
\[
\delta_i(h^1 \otimes \ldots \otimes h^{n-1}) = \begin{cases}
1 \otimes h^1 \otimes \ldots \otimes h^{n-1} & i = 0 \\
h^1 \otimes \ldots \otimes \Delta(h^i) \otimes \ldots \otimes h^{n-1} & 1 \leq i \leq n - 1 \\
h^1 \otimes \ldots \otimes h^{n-1} \otimes \sigma & i = n
\end{cases}
\]

For $n = 1$, we have $\delta_0(1) = 1$ and $\delta_1(1) = \sigma$. For $n > 0$, the degeneracy maps $\sigma_i : C^{n+1}(H) \rightarrow C^n(H)$ for $0 \leq i \leq n$ are given by
\[ \sigma_i(h^1 \otimes \ldots \otimes h^{n+1}) = h^1 \otimes \ldots \otimes \varepsilon(h^{i+1}) \otimes \ldots \otimes h^{n+1} \]
For $n = 0$, we have $\sigma_0(h) = \varepsilon(h)$. The cyclic operator $\tau_n : C^n(H) \rightarrow C^n(H)$ is given by
\[ \tau_n(h^1 \otimes \ldots \otimes h^n) = S_\delta(h^1) \cdot (h^2 \otimes \ldots \otimes h^n \otimes \sigma) \]

The cyclic cohomology determined by the $\Lambda$-module $C^* (H(\delta, \sigma))$ is said to be the cyclic cohomology of the Hopf algebra $H$ with respect to the modular pair $(\delta, \sigma)$ and will be denoted by $HC^* (\delta, \sigma)(H)$.

We now recall the cyclic cohomology of a small $k$-linear category due to McCarthy \cite{36}. The additive cyclic nerve of a small $k$-linear category $\mathcal{C}$ is defined to be the cyclic module determined by
\[ CN_n(\mathcal{C}) := \bigoplus Hom_\mathcal{C}(X_1, X_0) \otimes Hom_\mathcal{C}(X_2, X_1) \otimes \ldots \otimes Hom_\mathcal{C}(X_0, X_n) \quad (2.1) \]
where the direct sum runs over all $(X_0, X_1, \ldots, X_n) \in Ob(\mathcal{C})^n$. The structure maps are given by
\[ d_i(f^0 \otimes \ldots \otimes f^n) = \begin{cases}
f^0 \otimes f^1 \otimes \ldots \otimes f^i f^{i+1} \otimes \ldots \otimes f^n & 0 \leq i \leq n - 1 \\
f^n f^0 \otimes f^1 \otimes \ldots \otimes f^{n-1} & i = n
\end{cases} \]
\[ s_i(f^0 \otimes \ldots \otimes f^n) = \begin{cases}
f^0 \otimes f^1 \otimes \ldots \otimes f^i \otimes id_{X_{i+1}} \otimes f^{i+1} \otimes \ldots \otimes f^n & 0 \leq i \leq n - 1 \\
f^0 \otimes f^1 \otimes \ldots \otimes f^n \otimes id_{X_0} & i = n
\end{cases} \]
\[ t_n(f^0 \otimes \ldots \otimes f^n) = f^n \otimes f^0 \otimes \ldots \otimes f^{n-1} \]
for any $f^0 \otimes \ldots \otimes f^n \in Hom_\mathcal{C}(X_1, X_0) \otimes Hom_\mathcal{C}(X_2, X_1) \otimes \ldots \otimes Hom_\mathcal{C}(X_0, X_n)$. The cyclic cohomology groups of $\mathcal{C}$ are determined by the $k$-spaces $CN^\mathcal{C}(k) := Hom_k(CN_n(\mathcal{C}), k)$ with the structure maps
\[
\delta_i : CN^{n-1}(\mathcal{C}) \rightarrow CN^n(\mathcal{C}), \quad \phi \mapsto \phi \circ d_i \\
\sigma_i : CN^{n+1}(\mathcal{C}) \rightarrow CN^n(\mathcal{C}), \quad \psi \mapsto \psi \circ s_i \\
\tau_n : CN^n(\mathcal{C}) \rightarrow CN^n(\mathcal{C}), \quad \varphi \mapsto \varphi \circ t_n.
\]
We will use the notation \( CN^\bullet(C) := \{CN^n(C)\}_{n \geq 0} \) to denote the cocyclic module associated to the category \( C \) and \( HC^\bullet(C) \) for the corresponding cyclic cohomology.

We now introduce the notion of \( \delta \)-invariant \( \sigma \)-traces on an \( H \)-category.

**Definition 2.2.** Let \((\delta, \sigma)\) be a modular pair for a Hopf algebra \( H \) and let \( D_H \) be a left \( H \)-category. Suppose that we have a collection \( T^H := \{T^H_X : \text{Hom}_{D_H}(X, X) \rightarrow k\}_{X \in \text{Ob}(D_H)} \) of \( k \)-linear maps. Then, we say that the collection \( T^H \) is a \( \sigma \)-trace on \( D_H \) if

\[
T^H_X(g \circ f) = T^H_X(f \circ (\sigma g))
\]

for any \( f \in \text{Hom}_{D_H}(X, Y) \) and \( g \in \text{Hom}_{D_H}(Y, X) \). Moreover, we say that the \( \sigma \)-trace \( T^H \) is \( \delta \)-invariant under the action of \( H \) if

\[
T^H_X(h f') = \delta(h) T^H_X(f') \quad \forall h \in H,
\]

for all \( f' \in \text{Hom}_{D_H}(X, X) \).

**Lemma 2.3.** Let \( T^H \) be a \( \sigma \)-trace on a left \( H \)-category \( D_H \). Then, \( T^H \) is \( \delta \)-invariant under the action of \( H \) iff the following holds:

\[
T^H_X((hg) \circ f) = T^H_X(g \circ (S(h) f))
\]

for any \( h \in H, f \in \text{Hom}_{D_H}(X, Y) \) and \( g \in \text{Hom}_{D_H}(Y, X) \).

**Proof.** Let \( T^H \) be \( \delta \)-invariant. Then, we have

\[
T^H_X(g(S(h) f)) = T^H_X(g(\delta(h_1) S(h_2) f)) = T^H_X(h_1 (g(S(h_2) f))) = T^H_X((h_1 g)(h_2 S(h_3) f)) = T^H_X((hg) f)
\]

Conversely, suppose that the collection \( T^H \) satisfies (2.4). Then, for any \( f' \in \text{Hom}_{D_H}(X, X) \), we have

\[
T^H_X(h f') = T^H_X(f'(S(h) id_X)) = T^H_X(\delta(h_1) f'(\varepsilon(S(h_2)) id_X)) = \delta(h) T^H_X(f')
\]

We are now ready to prove that there is a characteristic map from the cyclic cohomology of the Hopf algebra \( H \) taking values in the cyclic cohomology of the \( H \)-category \( D_H \).

**Theorem 2.4.** Let \( H \) be a Hopf algebra with a modular pair in involution \((\delta, \sigma)\). Let \( D_H \) be a left \( H \)-category and let \( T^H \) be a \( \delta \)-invariant \( \sigma \)-trace on \( D_H \). Then, we have a characteristic map \( \gamma^\bullet : C^\bullet(H(\delta, \sigma)) \rightarrow CN^\bullet(D_H) \) of \( \Lambda \)-modules given by

\[
(\gamma^n(h^1 \otimes \ldots \otimes h^n))(f^0 \otimes \ldots \otimes f^n) := T^H_{X_0}(f^0(h^1 f^1) \ldots (h^n f^n))
\]

for \( h^1 \otimes \ldots \otimes h^n \in H^{\otimes n} \) and \( f^0 \otimes \ldots \otimes f^n \in \text{Hom}_{D_H}(X_1, X_0) \otimes \text{Hom}_{D_H}(X_2, X_1) \otimes \ldots \otimes \text{Hom}_{D_H}(X_0, X_n) \).

This induces a homomorphism in cyclic cohomology

\[
\gamma^\bullet : HC^\bullet((\delta, \sigma)(H)) \rightarrow HC^\bullet(D_H)
\]

**Proof.** We need to show that the following identities hold:

\[
\gamma^n \delta_i = \delta_i \gamma^{n-1} \quad 0 \leq i \leq n
\]

\[
\gamma^n \sigma_i = \sigma_i \gamma^{n+1} \quad 0 \leq i \leq n
\]

\[
\gamma^n \tau_n = \tau_n \gamma^n
\]

We first verify (2.5). The case \( i = 0 \) is straightforward. For \( i = n \), we have

\[
(\gamma^n(\delta_n(h^1 \otimes \ldots \otimes h^{n-1}))(f^0 \otimes \ldots \otimes f^n)) = (\gamma^n(h^1 \otimes \ldots \otimes h^{n-1} \otimes \sigma))(f^0 \otimes \ldots \otimes f^n)
\]

\[
= T^H_{X_0}(f^0(h^1 f^1) \ldots (h^{n-1} f^{n-1})(\sigma f^n))
\]

\[
= T^H_{X_n}(f^n f^0(h^1 f^1) \ldots (h^{n-1} f^{n-1})) \quad \text{(by (2.2))}
\]

\[
= (\gamma^{n-1}(h^1 \otimes \ldots \otimes h^{n-1}))(f^n f^0) \otimes f^1 \otimes \ldots \otimes f^{n-1}
\]

\[
= (\delta_n(\gamma^{n-1}(h^1 \otimes \ldots \otimes h^{n-1}))(f^0 \otimes \ldots \otimes f^n)
\]

\[
(\delta_{n-1})
\]

\[
(\delta_{n-1})
\]

\[
(\delta_{n-1})
\]

\[
(\delta_{n-1})
\]
For $1 \leq i \leq n-1$, we have
\[
(\gamma^n(\delta_i(h^1 \otimes \cdots \otimes h^{n-1}))(f^0 \otimes \cdots \otimes f^n)) = (\gamma^n(h^1 \otimes \cdots \otimes h^n))(f^0 \otimes \cdots \otimes f^n)
\]
= $T_{X_0}^H(f^0)(f^1(h^1f^1) \cdots (h^n f^n))
\]
= $T_{X_0}^H(f^0(h^1 f^1) \cdots (h^n f^n))
\]
= $(\delta_i(\gamma^{-1}(h^1 \otimes \cdots \otimes h^{n-1}))(f^0 \otimes \cdots \otimes f^n)
\]

Next we verify (2.6). For $0 \leq i \leq n-1$, we have
\[
(\gamma^n(\sigma_i(h^1 \otimes \cdots \otimes h^{n+1}))(f^0 \otimes \cdots \otimes f^n)) = (\gamma^n(h^1 \otimes \cdots \otimes h^{n+1}))(f^0 \otimes \cdots \otimes f^n)
\]
= $T_{X_0}^H(f^0)(h^1 f^1) \cdots (h^n f^n)
\]
= $(\sigma_i(\gamma^{-1}(h^1 \otimes \cdots \otimes h^{n+1}))(f^0 \otimes \cdots \otimes f^n)
\]

It now remains to verify (2.7). For that, we have
\[
(\gamma^n(\tau_n(h^1 \otimes \cdots \otimes h^n))(f^0 \otimes \cdots \otimes f^n)) = (\gamma^n(\delta(h^n)S(h^n)(h^n) \otimes \cdots \otimes h^n))(f^0 \otimes \cdots \otimes f^n)
\]
= $T_{X_0}^H(f^0)(S(h^n)(h^n) \otimes \cdots \otimes h^n)(f^0 \otimes \cdots \otimes f^n)
\]
= $(\tau_n(\gamma^{-1}(h^1 \otimes \cdots \otimes h^n))(f^0 \otimes \cdots \otimes f^n)
\]

This completes the proof. □

3 Invariant traces as Hopf-cyclic cocycles

We continue with $H$ being a Hopf algebra equipped with a modular pair $(\delta, \sigma)$ in involution. The key to the construction of the characteristic map $\gamma^*: HC^*_H(\delta, \sigma)(H) \to HC^*_H(D_H)$ in Theorem 2.4 is the $\delta$-invariant $\sigma$-trace $T^H$ on the left $H$-category $D_H$. In this section, we will describe such traces as Hopf-cyclic cocycles for $D_H$ taking values in a certain SAYD module.

Since $D_H$ is a left $H$-category, it is clear from the definition in (2.4) that each $\{CN_n(D_H)\}_{n \geq 0}$ is a left $H$-module via the diagonal action of $H$.

**Lemma 3.1.** Let $M$ be a right $H$-module. For each $n \geq 0$, $M \otimes CN_n(D_H)$ is a right $H$-module with action determined by
\[
(m \otimes f^0 \otimes \cdots \otimes f^n)h := mh_1 \otimes S(h_2)(f^0 \otimes \cdots \otimes f^n)
\]
for any $m \in M$, $f^0 \otimes \cdots \otimes f^n \in CN_n(D_H)$ and $h \in H$.

**Proof.** This follows from the fact that the antipode $S$ is an anti-algebra homomorphism and $\Delta(1_H) = 1_H \otimes 1_H$. □

We now recall the notion of a SAYD module from [18, Definition 2.1].
Definition 3.2. Let $H$ be a Hopf algebra with a bijective antipode $S$. A $k$-vector space $M$ is said to be a right-left anti-Yetter-Drinfeld module over $H$ if $M$ is a right $H$-module and a left $H$-comodule such that

$$
\rho(mh) = (mh)_{(-1)} \otimes (mh)_{(0)} = S(h_3)m_{(-1)}h_1 \otimes m_{(0)}h_2
$$

(3.1)

for all $m \in M$ and $h \in H$, where $\rho : M \to H \otimes M$, $m \mapsto m_{(-1)} \otimes m_{(0)}$ is the coaction. Moreover, $M$ is said to be stable if $m_{(0)}m_{(-1)} = m$.

We now take the Hopf-cyclic cohomology $HC^*_H(D_H, M)$ of an $H$-category $D_H$ with coefficients in a stable anti-Yetter-Drinfeld (SAYD) module $M$ (see also [26]). This generalizes the construction of the Hopf-cyclic cohomology for $H$-module algebras with coefficients in a SAYD module (see [17]). For each $n \geq 0$, we set

$$
C^n(D_H, M) := \operatorname{Hom}_k(M \otimes CN_n(D_H), k)
$$

$$
C^n_H(D_H, M) := \operatorname{Hom}_H(M \otimes CN_n(D_H), k)
$$

where $k$ is considered as a right $H$-module via the counit. It is clear from the definition that an element in $C^n_H(D_H, M)$ is a $k$-linear map $\phi : M \otimes CN_n(D_H) \to k$ satisfying

$$
\phi(mh_1 \otimes S(h_2)(f^0 \otimes \ldots \otimes f^n)) = \varepsilon(h)\phi(m \otimes f^0 \otimes \ldots \otimes f^n)
$$

(3.2)

Lemma 3.3. Let $M$ be a right-left SAYD module over $H$ and let $\phi \in C^n_H(D_H, M)$. Then,

$$
\phi(m \otimes h(f^0 \otimes f^1 \otimes \ldots \otimes f^n)) = \phi(mh \otimes f^0 \otimes f^1 \otimes \ldots \otimes f^n)
$$

for any $m \in M$ and $f^0 \otimes f^1 \otimes \ldots \otimes f^n \in CN_n(D_H)$.

Proof. Using the stability of $M$, we have

$$
\phi(mh \otimes f^0 \otimes \ldots \otimes f^n) = \phi((mh)_{(0)}(mh)_{(-1)} \otimes f^0 \otimes \ldots \otimes f^n)
$$

$$
= \phi((m_{(0)}h_2)((S(h_3)m_{(-1)}h_1) \otimes f^0 \otimes \ldots \otimes f^n))
$$

$$
= \phi((m_{(0)}m_{(-1)}h_1 \otimes \varepsilon(h_2)(f^0 \otimes \ldots \otimes f^n))
$$

$$
= \phi(mh_1 \otimes S(h_2)(f^0 \otimes \ldots \otimes f^n))
$$

$$
= \varepsilon(h_1)\phi(m \otimes h_2(f^0 \otimes \ldots \otimes f^n))
$$

by (3.2)

$$
= \phi(m \otimes h(f^0 \otimes \ldots \otimes f^n))
$$

We now recall that a (co)simplicial module is said to be para-(co)cyclic if all the relations for a (co)cyclic module are satisfied except $\tau_n^{n+1} = id$ (see, for instance [26]).

Proposition 3.4. Let $D_H$ be a left $H$-category and let $M$ be a right-left SAYD module over $H$. Then,

(1) we have a para-cocyclic module $C^*(D_H, M) := \{C^n(D_H, M)\}_{n \geq 0}$ with the following structure maps

$$(\delta_i\phi)(m \otimes f^0 \otimes \ldots \otimes f^n) = \begin{cases} 
\phi(m \otimes f^0 \otimes \ldots \otimes f^{i+1} \otimes \ldots \otimes f^n) & 0 \leq i \leq n-1 \\
\phi(m_{(0)} \otimes (S^{-1}(m_{(-1)})) f^0 \otimes \ldots \otimes f^{n-1}) & i = n
\end{cases}
$$

$$(\sigma_i\psi)(m \otimes f^0 \otimes \ldots \otimes f^n) = \begin{cases} 
\psi(m \otimes f^0 \otimes \ldots \otimes f^i \otimes \operatorname{id}_{X_{n-i}} \otimes f^{i+1} \otimes \ldots \otimes f^n) & 0 \leq i \leq n-1 \\
\psi(m \otimes f^0 \otimes \ldots \otimes f^n \otimes \operatorname{id}_{X_0}) & i = n
\end{cases}
$$

$$(\tau_n\varphi)(m \otimes f^0 \otimes \ldots \otimes f^n) = \varphi(m_{(0)} \otimes S^{-1}(m_{(-1)})) f^n \otimes f^0 \otimes \ldots \otimes f^{n-1})
$$

for any $\phi \in C^n(D_H, M)$, $\psi \in C^{n+1}(D_H, M)$, $\varphi \in C^n(D_H, M)$, $m \in M$ and $f^0 \otimes \ldots \otimes f^n \in \operatorname{Hom}_{D_H}(X_1, X_0) \otimes \operatorname{Hom}_{D_H}(X_2, X_1) \otimes \ldots \operatorname{Hom}_{D_H}(X_n, X_{n-1})$.

(2) by restricting to right $H$-linear morphisms $C^*_H(D_H, M) = \operatorname{Hom}_H(M \otimes CN_n(D_H), k)$, we obtain a cocyclic module $C^*_H(D_H, M) := \{C^*_H(D_H, M)\}_{n \geq 0}$.
Proof. (1) It is easy to verify that $\delta_i$ and $\sigma_i$ for $0 \leq i \leq n$ define a cosimplicial structure on $C^* (\mathcal{D}_H, M)$. Therefore, it remains to check that the following identities hold:

\[
\begin{align*}
\tau_n \delta_i &= \delta_{i-1} \tau_{n-1} \quad 1 \leq i \leq n \\
\tau_n \delta_0 &= \delta_n \\
\tau_n \sigma_i &= \sigma_{i-1} \tau_{n+1} \quad 1 \leq i \leq n \\
\tau_n \sigma_0 &= \sigma_n \tau_{n+1}
\end{align*}
\]

For $1 \leq i < n$, we have

\[
(\tau_n \delta_i (\phi)) (m \otimes f^0 \otimes \ldots \otimes f^n) = \delta_i (\phi) (m_{(0)} \otimes S^{-1} (m_{(-1)}) f^n \otimes f^0 \otimes \ldots \otimes f^{n-1}) = \phi (m_{(0)} \otimes S^{-1} (m_{(-1)}) f^n \otimes f^0 \otimes \ldots \otimes f^{n-1}) = (\delta_{i-1} \tau_{n-1} (\phi)) (m \otimes f^0 \otimes \ldots \otimes f^n)
\]

and

\[
(\tau_n \delta_n (\phi)) (m \otimes f^0 \otimes \ldots \otimes f^n) = \delta_n (\phi) (m_{(0)} \otimes S^{-1} (m_{(-1)}) f^n \otimes f^0 \otimes \ldots \otimes f^{n-1}) = \phi (m_{(0)} \otimes S^{-1} (m_{(-1)}) f^n \otimes f^0 \otimes \ldots \otimes f^{n-2}) = \phi (m_{(0)} \otimes S^{-1} (m_{(-1)}) f^n \otimes f^0 \otimes \ldots \otimes f^{n-2}) = (\delta_{n-1} \tau_{n-1} (\phi)) (m \otimes f^0 \otimes \ldots \otimes f^n)
\]

It follows immediately by definition that $\tau_n \delta_0 = \delta_n$. Next, we verify the identities involving the degeneracies. For $i = n$, we have

\[
(\sigma_{n-1} \tau_{n+1} (\psi)) (m \otimes f^0 \otimes \ldots \otimes f^n) = \tau_{n+1} (\psi) (m \otimes f^0 \otimes \ldots \otimes f^{n-1} \otimes \text{id}_{X_n} \otimes f^n) = \psi (m_{(0)} \otimes S^{-1} (m_{(-1)}) f^n \otimes f^0 \otimes \ldots \otimes f^{n-1} \otimes \text{id}_{X_n}) = (\tau_n \sigma_n (\psi)) (m \otimes f^0 \otimes \ldots \otimes f^n)
\]

The case $1 \leq i < n$ is easy to verify. Further, we have

\[
(\sigma_n \tau^2_{n+1} (\varphi)) (m \otimes f^0 \otimes \ldots \otimes f^n) = \tau^2_{n+1} (\varphi) (m \otimes f^0 \otimes \ldots \otimes f^n \otimes \text{id}_{X_n}) = \tau_{n+1} (\varphi) (m_{(0)} \otimes S^{-1} (m_{(-1)}) \text{id}_{X_n} \otimes f^0 \otimes \ldots \otimes f^n) = \tau_{n+1} (\varphi) (m_{(0)} \otimes \text{id}_{X_n} \otimes f^0 \otimes \ldots \otimes f^n) = \tau_{n+1} (\varphi) (m_{(0)} \otimes f^0 \otimes \ldots \otimes f^n) = \text{id}_{X_n} \otimes f^0 \otimes \ldots \otimes f^n = (\tau_n \sigma_0 (\varphi)) (m \otimes f^0 \otimes \ldots \otimes f^n)
\]

(2) Using (1), it now remains to prove that the structure maps are well-defined and $\tau^{n+1} = \text{id}$. Let us first verify that the cyclic operator $\tau_n$ is well-defined, i.e., $\tau_n (\phi)$ is $H$-linear for each $\phi \in C^n_H (\mathcal{D}_H, M)$.

\[
(\tau_n (\phi)) (m_{h_1} \otimes S(h_2) (f^0 \otimes \ldots \otimes f^n)) = (\tau_n (\phi)) (m_{h_1} \otimes S(h_{n+2}) (f^0 \otimes \ldots \otimes f^n)) = \phi ((m_{h_1})_{(0)} \otimes S^{-1} (m_{(-1)}) f^n \otimes S(h_{n+2}) f^0 \otimes \ldots \otimes S(h_3) f^{n-1}) = \phi (m_{h_2} S^{-1} (h_3) f^n \otimes (S(h_3) h_2) f^0 \otimes \ldots \otimes S(h_5) f^{n-1}) = \phi (m_{h_2} h_3 f^n \otimes S^{-1} (h_3) f^0 \otimes \ldots \otimes S(h_5) f^{n-1}) = \phi (m_{h_2} h_3 f^n \otimes S^{-1} (h_3) f^0 \otimes \ldots \otimes S(h_5) f^{n-1}) = \phi (m_{h_2} h_3 f^n \otimes S^{-1} (h_3) f^0 \otimes \ldots \otimes S(h_5) f^{n-1}) = \phi (m_{h_2} h_3 f^n \otimes S^{-1} (h_3) f^0 \otimes \ldots \otimes S(h_5) f^{n-1}) = \text{by Lemma 3.3}
\]

\[
= \phi (m_{h_2} h_3 f^n \otimes S^{-1} (h_3) f^0 \otimes \ldots \otimes S(h_5) f^{n-1}) = \varepsilon (h) \phi (m_{h_2} h_3 f^n \otimes S^{-1} (h_3) f^0 \otimes \ldots \otimes S(h_5) f^{n-1}) = \varepsilon (h) (\tau_n (\phi)) (m \otimes f^0 \otimes \ldots \otimes f^n)
\]


Similarly, it may be verified that the degeneracies are also well-defined. Next, we verify that the face maps are well-defined. For \( 0 \leq i < n \), we have

\[
\begin{align*}
(\delta_i(\phi)) (m_{h_1} \otimes S(h_2)(f_0 \otimes \ldots \otimes f^n)) \\
= (\delta_i(\phi)) (m_{h_1} \otimes S(h_{n+2-i}) f_0 \otimes \ldots \otimes S(h_2) f^n) \\
= \phi (m_{h_1} \otimes S(h_{n+1}) f_0 \otimes \ldots \otimes S(h_{n+2-(i+1)}) f^{i+1} \otimes \ldots \otimes S(h_2) f^n) \\
= \phi (m_{h_1} \otimes S(h_2) (f_0 \otimes \ldots \otimes f^{i+1} \otimes \ldots \otimes f^n)) \\
= \varepsilon(h) \phi(m \otimes f^0 \otimes \ldots \otimes f^{i+1} \otimes \ldots \otimes f^n) = \varepsilon(h) (\delta_i(\phi)) (m \otimes f^0 \otimes \ldots \otimes f^n)
\end{align*}
\]

Since \( \delta_n = \tau_n \delta_0 \), the preceding computations show that \( \delta_n \) is also well-defined.

Further, using the stability of \( M \), we have

\[
\tau_n^{i+1}(\phi)(m \otimes f^0 \otimes \ldots \otimes f^n) = \phi(m_{(0)} \otimes S^{-1}((m_{(-1)})_{n+1}) f^0 \otimes \ldots \otimes S^{-1}(m_{(-1)})_1 f_n) = \phi(m_{(0)} \otimes S^{-1}(m_{(-1)})(f^0 \otimes \ldots \otimes f^n)) \\
= \phi(m_{(0)} S^{-1}(m_{(-1)}) \otimes f^0 \otimes \ldots \otimes f^n) \quad \text{(by Lemma 3.3)} \\
= \phi(m \otimes f^0 \otimes \ldots \otimes f^n) \quad \text{(by (3.3))}
\]

This completes the proof.

The cohomology of the cocyclic module \( C^\bullet_H(D_H, M) \) is referred to as the Hopf-cyclic cohomology of the \( H \)-category \( D_H \) with coefficients in the SAYD module \( M \). The corresponding cohomology groups are denoted by \( HC^\bullet_H(D_H, M) \).

**Remark 3.5.** (1) As \( k \) contains \( \mathbb{Q} \), we recall that the cohomology of a cocyclic module \( \mathcal{C} \) can be expressed alternatively as the cohomology of the following complex (see, for instance [18, 2.5.9]):

\[
\cdots \rightarrow C^n_\lambda(\mathcal{C}) \rightarrow C^{n+1}_\lambda(\mathcal{C}) \rightarrow \cdots
\]

where \( C^n_\lambda(\mathcal{C}) = \text{Ker}(1 - \lambda) \subseteq C^n(\mathcal{C}) \), \( b = \sum_{i=0}^{n+1} (-1)^i \delta_i \) and \( \lambda = (-1)^n \tau_n \). In particular, an element \( \phi \in C^n_H(D_H, M) \) is a cyclic cocycle if and only if

\[
b(\phi) = 0 \quad \text{and} \quad (1 - \lambda)(\phi) = 0 \quad \text{(3.4)}
\]

(2) The field \( k \) is trivially a Hopf algebra with \( \Delta(1) = 1 \otimes 1 \), \( S(1) = 1 = \varepsilon(1) \), and also a SAYD module over itself with coaction given by \( \Delta \). Substituting \( H = k = M \) in the construction of \( C^\bullet_H(D_H, M) \), we get back the ordinary cyclic cohomology \( HC^\bullet(D_H) \) of the \( k \)-linear category \( D_H \) as discussed in Section 4.

We now let \( \sigma k_\delta \) denote the SAYD module structure on \( k \) (see, for instance [18]) over \( H \) defined by setting

\[
\alpha \cdot h := \delta(h) \alpha, \quad \rho(\alpha) := \sigma \otimes \alpha \quad \forall \alpha \in k, h \in H
\]

**Proposition 3.6.** Let \( D_H \) be a left \( H \)-category and let \( (\delta, \sigma) \) be a modular pair in invasion for \( H \). Then, \( Z_H^0(D_H, \sigma k_\delta) = HC^0_H(D_H, \sigma k_\delta) \) is in bijection with the \( k \)-space consisting of \( \delta \)-invariant \( \sigma \)-traces on \( D_H \).

**Proof.** By definition, \( HC^0_H(D_H, \sigma k_\delta) = \text{Ker}(b_0) \), where \( b_0 = \delta_0 - \delta_1 : C^0_H(D_H, \sigma k_\delta) \rightarrow C^1_H(D_H, \sigma k_\delta) \). Let \( \phi \in \text{Ker}(b_0) \). We now define a collection \( T^H := \{ T_X^H : \text{Hom}_{D_H}(X, X) \rightarrow k \}_{X \in \text{Ob}(D_H)} \) of \( k \)-linear maps given by

\[
T^H_X(f) = \phi(1 \otimes f) \quad \forall f \in \text{Hom}_{D_H}(X, X)
\]

Using the fact that \( \sigma \) is a group-like element in \( H \) and \( \delta(\sigma) = 1 \), we have

\[
T^H_X(gf) = \phi(1 \otimes gf) = (\delta_0(\phi))(1 \otimes g \otimes f) = (\delta_0(\phi))(1 \otimes S^{-1}(\sigma) f g) = \phi(1 \otimes (S(\sigma) f)(\varepsilon(\sigma) g)) \\
= \phi(\delta(\sigma) \otimes (S(\sigma) f)(S(\sigma) g(\varepsilon(\sigma) g))) = \phi(1 \cdot \sigma \otimes (S(\sigma)(f(\varepsilon(\sigma) g))) \\
= \phi(1 \otimes f(\varepsilon(\sigma) g)) \varepsilon(\sigma) = \phi(1 \otimes f(\varepsilon(\sigma) g)) = T^H_X(f(\varepsilon(\sigma) g))
\]

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for any $f \in \text{Hom}_\mathcal{D}_H(X,Y)$ and $g \in \text{Hom}_\mathcal{D}_H(Y,X)$. This shows that the collection $T^H$ is a $\sigma$-trace on the category $\mathcal{D}_H$. We now verify that $T^H$ is $\delta$-invariant. For any $f' \in \text{Hom}_\mathcal{D}_H(X,X)$, we have

$$
\delta(h)T^H(f') = \delta(h_1)S(h_2)f' = \delta(h_1)S(h_2)f' = T^H_h(S_h(f'))
$$

Conversely, suppose that $T^H$ is a $\delta$-invariant $\sigma$-trace on $\mathcal{D}_H$. We consider the $k$-linear map $\phi \in C^0(\mathcal{D}_H, \sigma k_3)$ determined by $\phi(1 \otimes f') = T^H_h(f')$ for each $f' \in \text{Hom}_\mathcal{D}_H(X,X)$. Let us first verify that $\phi$ is $H$-linear. For any $h \in H$, we have

$$
\phi((1 \otimes f')h) = \phi(1 \cdot h_1 S(h_2)f') = \phi(1 \otimes (h_1) S(h_2)f') = T^H_h(S_h(f'))
$$

Next, we verify that $\phi \in \text{Ker}(b_0)$. For any $f \in \text{Hom}_\mathcal{D}_H(X,Y)$ and $g \in \text{Hom}_\mathcal{D}_H(Y,X)$, we have

$$
(b_0(\phi))(1 \otimes g \otimes f) = (\delta_0(\phi))(1 \otimes g \otimes f) - (\delta_1(\phi))(- \otimes g \otimes f) = \phi(1 \otimes gf) - \phi(1 \otimes (S^{-1}(\sigma))g)
$$

This shows that $\phi \in HC^0_H(\mathcal{D}_H, \sigma k_3)$. □

4 Characteristic map with SAYD coefficients

Let $\mathcal{D}_H$ be a left $H$-category and let $(\delta, \sigma)$ be a modular pair in involution for $H$. We have shown that the $\delta$-invariant $\sigma$-traces on $\mathcal{D}_H$ are in bijection with $HC^0_H(\mathcal{D}_H, \sigma k_3)$ (Proposition 3.6). Moreover, a $\delta$-invariant $\sigma$-trace on $\mathcal{D}_H$ induces a homomorphism $HC_{(\delta, \sigma)}(H) \to HC^*(\mathcal{D}_H)$ (Theorem 2.3). Thus, we obtain a pairing

$$
HC_{(\delta, \sigma)}(H) \otimes HC^0_H(\mathcal{D}_H, \sigma k_3) \to HC^*(\mathcal{D}_H)
$$

The pairing in (4.1) leads us to ask if there exists a similar pairing when the Hopf algebra $H$ is replaced by an $H$-module coalgebra $C$ and the trivial SAYD module $\sigma k_3$ is replaced by a general SAYD module $M$. In this section, we obtain the following pairing

$$
HC^*_H(C,M) \otimes HC^0_H(\mathcal{D}_H, M) \to HC^*(\mathcal{D}_H)
$$

A coalgebra $(C, \Delta_C, \varepsilon_C)$ which is also a left $H$-module such that

$$
\Delta_C(hc) = h_1 C_1 \otimes h_2 C_2, \quad \varepsilon_C(hc) = \varepsilon(h) \varepsilon_C(c) \quad \forall h \in H, c \in C
$$

is said to be a left $H$-module coalgebra.

We now recall the Hopf-cyclic cohomology of a left $H$-module coalgebra $C$ with coefficients in a right-left SAYD module $M$ (see [17]). Let $C^n_H(C,M) := M \otimes_H C^{n+1}$ for $n \geq 0$. Then, $C^*_H(C,M) = \{C^n_H(C,M)\}_{n \geq 0}$ is a $\Lambda$-module with the following structure maps:

$$
\delta^n_i^l(m \otimes_H c^0 \otimes \ldots \otimes c^{n-1}) = \begin{cases} m \otimes_H c^0 \otimes \ldots \otimes c_i^1 \otimes c^1 \otimes \ldots \otimes c^{n-1} \otimes m_{0(i)} & 0 \leq i \leq n - 1 \\
0 & i = n \end{cases}
$$

$$
\sigma^n_i^l(m \otimes_H c^0 \otimes \ldots \otimes c^{n-1}) = m \otimes_H c^0 \otimes \ldots \otimes \varepsilon_C(c^{i+1}) \otimes \ldots \otimes c^{n+1} & 0 \leq i \leq n
$$

The cohomology of the cocyclic module $C^n_H(C,M)$ is said to be the Hopf-cyclic cohomology of the $H$-module coalgebra $C$ with coefficients in the SAYD module $M$. The corresponding cohomology groups will be denoted by $HC^*_H(C,M)$.

In the construction of the pairing (4.1), the Hopf algebra $H$ acts on the category $\mathcal{D}_H$ in the sense of Definition 2.1. We will now define the action of an $H$-module coalgebra $C$ on a left $H$-category $\mathcal{D}_H$. 
Definition 4.1. Let $\mathcal{D}_H$ be a left $H$-category and $C$ be a left $H$-module coalgebra. We say that $C$ acts on $\mathcal{D}_H$ if we have $k$-linear maps $\{C \otimes \text{Hom}_{\mathcal{D}_H}(X,Y) \to \text{Hom}_{\mathcal{D}_H}(X,Y)\}_{(X,Y) \in \text{Obj}(\mathcal{D}_H)}$ satisfying

$$c(gf) = (c_1g)(c_2f), \quad c(id_X) = c \text{id}_X, \quad h(cf) = (hc)f$$

for any $f \in \text{Hom}_{\mathcal{D}_H}(X,Y), \ g \in \text{Hom}_{\mathcal{D}_H}(Y,Z), \ c \in C$ and $h \in H$.

We now show that there is a pairing between the Hopf-cyclic cohomology of an $H$-module coalgebra $C$ and $HC^0_H(\mathcal{D}_H, M)$. This pairing takes values in the usual cyclic cohomology of the $k$-linear category $\mathcal{D}_H$ (as described in Section 2).

Theorem 4.2. Let $\mathcal{D}_H$ be a left $H$-category and let $C$ be a left $H$-module coalgebra such that $C$ acts on $\mathcal{D}_H$. Let $M$ be a right-left SAYD module over $H$. Then, for each $\phi \in HC^0_H(\mathcal{D}_H, M)$, we obtain a morphism $\gamma^\bullet : C^\bullet(M, C) \to CN^\bullet(\mathcal{D}_H)$ of $\Lambda$-modules defined by

$$\left(\gamma^n_M(m \otimes_H c^0 \otimes \ldots \otimes c^n)\right)(f^0 \otimes \ldots \otimes f^n) := \phi\left(m \otimes (c^n f^n) \ldots (c^0 f^0)\right)$$

for any $m \in M$, $c^0 \otimes \ldots \otimes c^n \in C^{\otimes n+1}$ and $f^0 \otimes \ldots \otimes f^n \in \text{Hom}_{\mathcal{D}_H}(X_0, X_0) \otimes \text{Hom}_{\mathcal{D}_H}(X_2, X_1) \otimes \ldots \otimes \text{Hom}_{\mathcal{D}_H}(X_0, X_n)$. Thus, we get the following pairing

$$HC^0_H(M, C) \otimes HC^0_H(\mathcal{D}_H, M) \to HC^\bullet(\mathcal{D}_H)$$

Proof. Let $\phi \in HC^0_H(\mathcal{D}_H, M)$. Then, by definition, we have

$$\phi(m \otimes gf) = (\delta_0(\phi))(m \otimes g \otimes f) = (\delta_1(\phi))(m \otimes g \otimes f) = \phi\left(m(0) \otimes (S^{-1}(m(-1))f)g\right)$$

for any $m \in M$, $f \in \text{Hom}_{\mathcal{D}_H}(X, Y)$ and $g \in \text{Hom}_{\mathcal{D}_H}(Y, X)$. In order to show that (4.3) defines a map of $\Lambda$-modules, we need to prove that the following identities hold:

$$\gamma^n_M \delta^n_i = \delta^n_M \gamma^n_i = 0 \leq i \leq n$$

(4.5)

$$\gamma^n_M \sigma^n_i = \sigma^n_M \gamma^n_i = 0 \leq i \leq n$$

(4.6)

$$\gamma^n_M \tau^n_M = \tau^n_M \gamma^n_M$$

(4.7)

For $0 \leq i \leq n-1$, we have

$$\left(\gamma^n_M \delta^n_i\right)(m \otimes_H c^0 \otimes \ldots \otimes c^{n-1})(f^0 \otimes \ldots \otimes f^n)$$

$$= \left(\gamma^n_M (m \otimes_H c^0 \otimes \ldots \otimes c^n)\right)(f^0 \otimes \ldots \otimes f^n)$$

$$= \phi\left(m \otimes (c^n f^n) \ldots (c^0 f^0)\right)$$

(4.8)

Moreover,

$$\left(\gamma^n_M \delta^n_i\right)(m \otimes_H c^0 \otimes \ldots \otimes c^{n-1})(f^0 \otimes \ldots \otimes f^n)$$

$$= \left(\gamma^n_M (m(0) \otimes_H c^0 \otimes \ldots \otimes c^{n-1} \otimes m(-1) c^i)\right)(f^0 \otimes \ldots \otimes f^n)$$

$$= \phi\left(\left(m(0) \otimes (c^0 f^0)\right)\left(c^i f^1\right)\ldots \left(c^{n-1} f^{n-1}\right)(m(-1) c^i f^n)\right)$$

(4.9)
This proves (4.5). Next, we verify the identity (4.6). For $0 \leq i \leq n - 1$, we have
\[
\left((\sigma^i_M\gamma^0_M)(m \otimes_H c^0 \otimes \ldots \otimes c^{n+1})\right)(f^0 \otimes \ldots \otimes f^n)
= \left(\gamma^0_M(m \otimes_H c^0 \otimes \ldots \otimes c^{n+1})\right)(f^0 \otimes \ldots \otimes f^i \otimes id_{X_{i+1}} \otimes f^{i+1} \otimes \ldots \otimes f^n)
= \phi(m \otimes (c^0 f^0) \ldots (c^i f^i)(c^{i+1}id_{X_{i+1}}) \ldots (c^{n+1}f^n))
= \phi(m \otimes (c^0 f^0) \ldots (c^i f^i) (c^{i+2}f^{i+1}) \ldots (c^{n+1}f^n))
= (\gamma_M^0 (m \otimes_H c^0 \otimes \ldots \otimes \varepsilon(c^{i+1}) \otimes \ldots \otimes c^{n+1})) (f^0 \otimes \ldots \otimes f^i \otimes f^{i+1} \otimes \ldots \otimes f^n)
= ((\gamma_M^0 \sigma^i_M)(m \otimes_H c^0 \otimes \ldots \otimes c^{n+1}))(f^0 \otimes \ldots \otimes f^n)
\]
It may be verified similarly that $\sigma^i_M \gamma^{n+1}_M = \gamma^0_M \sigma^i_M$. It remains to verify (4.7). We have
\[
\left((\gamma^0_M \tau^i_M)(m \otimes_H c^0 \otimes \ldots \otimes c^n)\right)(f^0 \otimes \ldots \otimes f^n)
= \left(\gamma^0_M(m(0) \otimes_H c^0 \otimes \ldots \otimes c^n \otimes (m(-1)c^0))\right)(f^0 \otimes \ldots \otimes f^n)
= \phi(m(0) \otimes (c^0 f^0) \ldots (c^n f^n-1))(m(-1)c^0) (f^0) \ldots (c^n f^n-1))
= \phi(m(0) \otimes [S^{-1}(m(0)(-1))] (m(-1)c^0) (f^0) \ldots (c^n f^n-1))
= \phi(m \otimes (c^0 f^n)(c^1 f^0) \ldots (c^n f^n-1))
= (\gamma^0_M(m \otimes_H c^0 \otimes \ldots \otimes c^n)) (f^n \otimes f^0 \otimes \ldots \otimes f^{n-1})
= ((\tau^i_M \gamma^0_M)(m \otimes_H c^0 \otimes \ldots \otimes c^n)) (f^0 \otimes \ldots \otimes f^n)
\]
This completes the proof.

We will now extend the result in Theorem 4.2 to a general pairing
\[
HC^q_H(C, \mathcal{M}) \otimes HC^p_H(\mathcal{D}_H, \mathcal{M}) \rightarrow HC^{q+p}(\mathcal{D}_H)
\]
(4.8)
Let $\mathcal{D}_H$ be a left $H$-category and $C$ be a left $H$-module coalgebra. Let $C^n_{(C,M,D_H)}$ be the diagonal complex
\[
C^n_{(C,M,D_H)} := C^n_H(C, \mathcal{M}) \otimes_k C^n_H(\mathcal{D}_H, \mathcal{M}) = (M \otimes_H C^{n+1}) \otimes_k \text{Hom}_H(M \otimes C N_n(\mathcal{D}_H), k) \quad \forall \ n \geq 0
\]
which is a co cyclic module with structure maps $\{\delta_i^i \otimes \delta_i, \sigma^i \otimes \sigma, \tau^i \otimes \tau\}_{0 \leq i \leq n}$ (see [31, § 2.5.1.2]).
We consider the $k$-linear category $(C, \mathcal{D}_H)$ defined as follows:
\[
\text{Ob}(C, \mathcal{D}_H) = \text{Ob}(\mathcal{D}_H), \quad \text{Hom}_{C, \mathcal{D}_H}(X, Y) = \text{Hom}_H(C, \text{Hom}_{\mathcal{D}_H}(X, Y))
\]
The composition in $(C, \mathcal{D}_H)$ is given by $(f * g)(c) = f(c_1) \circ g(c_2)$ for any $g \in \text{Hom}_{C, \mathcal{D}_H}(X, Y)$, $f \in \text{Hom}_{\mathcal{D}_H}(Y, Z)$ and $c \in C$.

**Proposition 4.3.** Let $\mathcal{D}_H$ be a left $H$-category and $C$ be a left $H$-module coalgebra. Then, the map
\[
\Psi : C^n_{(C,M,D_H)} = (M \otimes_H C^{n+1}) \otimes \text{Hom}_H(M \otimes C N_n(\mathcal{D}_H), k) \rightarrow C^n(C, \mathcal{D}_H) = \text{Hom}_k(C N_n(\mathcal{C}, \mathcal{D}_H), k)
\]
given by
\[
(\Psi(m \otimes_H c^0 \otimes \ldots \otimes c^n)) (g^0 \otimes \ldots \otimes g^n) := \phi(m \otimes g^0(c^0) \otimes \ldots \otimes g^n(c^n))
\]
determines a morphism of co cyclic modules.

**Proof.** We first verify that $\Psi$ is well-defined. We have
\[
\Psi(mh \otimes_H c^0 \otimes \ldots \otimes c^n) (g^0 \otimes \ldots \otimes g^n)
= \phi(mh \otimes g^0(c^0) \otimes \ldots \otimes g^n(c^n))
= \phi(m \otimes h_1 c^0 \otimes \ldots \otimes h_{n+1} g^n(c^n)) (by \ Lemma \ 3.36)
= \phi(m \otimes g^0(h_1 c^0) \otimes \ldots \otimes g^n(h_{n+1} c^n))
= (\Psi(m \otimes_H h_1 c^0 \otimes \ldots \otimes h_{n+1} c^n) (g^0 \otimes \ldots \otimes g^n)
= \left(\Psi(m \otimes_H h(c^0 \otimes \ldots \otimes c^n)) (g^0 \otimes \ldots \otimes g^n)
\]

For $0 \leq i \leq n-1$, we have
\[
((\Psi (\delta'_i \otimes \delta_i))(m \otimes_H c^0 \otimes \ldots \otimes c^n \otimes \phi))(g^0 \otimes \ldots \otimes g^{n+1})
= \Psi(m \otimes_H c^0 \otimes \ldots \otimes c^n \otimes \delta_i(\phi))(g^0 \otimes \ldots \otimes g^{n+1})
= \delta_i(\phi)(m \otimes g^0(c^0) \otimes \ldots \otimes g^i(c^i) \otimes \ldots \otimes g^{n+1}(c^n))
= \delta_i(\phi)(m \otimes g^0(c^0) \otimes \ldots \otimes g^i(c^i) \otimes \ldots \otimes g^{n+1}(c^n))
= \delta_i(\phi)(m \otimes g^0(c^0) \otimes \ldots \otimes (g^i \star g^{n+1})(c^i) \otimes \ldots \otimes g^{n+1}(c^n))
= \left((\delta_i \otimes \Psi)(m \otimes_H c^0 \otimes \ldots \otimes c^n \otimes \phi)\right)(g^0 \otimes \ldots \otimes g^{n+1})
\]
The case $i = n$ can be verified similarly. Further, for $0 \leq i \leq n-1$, we have
\[
((\Psi (\sigma'_i \otimes \sigma_i))(m \otimes_H c^0 \otimes \ldots \otimes c^n \otimes \phi))(g^0 \otimes \ldots \otimes g^{n+1})
= \Psi(m \otimes_H c^0 \otimes \ldots \otimes c^n \otimes \sigma_i(\phi))
= \sigma_i(\phi)(m \otimes g^0(c^0) \otimes \ldots \otimes g^i(c^i) \otimes \ldots \otimes g^{n+1}(c^n))
= \sigma_i(\phi)(m \otimes g^0(c^0) \otimes \ldots \otimes (g^i \star g^{n+1})(c^i) \otimes \ldots \otimes g^{n+1}(c^n))
= \left((\sigma_i \otimes \Psi)(m \otimes_H c^0 \otimes \ldots \otimes c^n \otimes \phi)\right)(g^0 \otimes \ldots \otimes g^{n+1})
\]
The case $i = n$ can be verified similarly. Additionally, suppose that $C$ acts on $D_H$ in the sense of Definition 4.3. Then, we obtain a pairing:
\[
HC_H^q(C, M) \otimes HC_H^p(D_H, M) \rightarrow HC^{p+q}(D_H)
\]

**Theorem 4.4.** Let $M$ be a right-left SAYD module over $H$. Let $C$ be a left $H$-module coalgebra and let $D_H$ be a left $H$-category. Then, we have a pairing
\[
HC_H^q(C, M) \otimes HC_H^p(D_H, M) \rightarrow HC^{p+q}(D_H)
\]

Additionally, suppose that $C$ acts on $D_H$ in the sense of Definition 4.3. Then, we obtain a pairing:
\[
HC_H^q(C, M) \otimes HC_H^p(D_H, M) \rightarrow HC^{p+q}(D_H)
\]

Proof. Let $\mathcal{B}(C_H^q(C, M))$ and $\mathcal{B}(C_H^p(D_H, M))$ denote respectively the mixed complexes corresponding to the cocyclic modules $C_H^q(C, M)$ and $C_H^p(D_H, M)$. By definition, $HC_H^q(C, M) = H^q(Tot(\mathcal{B}(C_H^q(C, M))))$ and $HC_H^p(D_H, M) = H^p(Tot(\mathcal{B}(C_H^p(D_H, M))))$. We have a canonical morphism
\[
HC_H^q(C, M) \otimes HC_H^p(D_H, M) = H^q(Tot(\mathcal{B}(C_H^q(C, M)))) \otimes H^p(Tot(\mathcal{B}(C_H^p(D_H, M))))
\]

where the vertical isomorphism follows from Eilenberg-Zilber Theorem. The morphism of cocyclic modules in Proposition 4.3 induces a morphism $HC^{p+q}(C_H(C, M, D_H)) \rightarrow HC^{p+q}(C, D_H)$. Composing with the morphism in 4.11 gives us the pairing in 4.10.

Finally, when $C$ acts on $D_H$, we have an inclusion $i : D_H \rightarrow (C, D_H)$ given by $i(f)(c) := cf$ for any morphism $f \in Hom_{D_H}(X, Y)$ and $c \in C$. Then, $i$ induces a morphism $HC^{p+q}(C, D_H) \rightarrow HC^{p+q}(D_H)$. Composing with the pairing in 4.9 now gives us the pairing in 4.10. 

\[\square\]
5 Traces, cocycles and DGH-semicategories

Our purpose is to develop a formalism analogous to that of Connes [12] in order to interpret the cocycles $Z^*_H(D_H, M)$ and the coboundaries $B^*_H(D_H, M)$, $B^*(D_H)$ as characters of differential graded semicategories. In this section, we will describe $Z^*_H(D_H, M)$ and $Z^*(D_H)$, for which we will need the framework of DG-semicolonies. Let us first recall the notion of a semicategory introduced by Mitchell in [37] (for more on semicategories, see, for instance, [9]).

**Definition 5.1.** (see [37], Section 4) A semicategory $\mathcal{C}$ consists of a collection $\text{Ob}(\mathcal{C})$ of objects together with a set of morphisms $\text{Hom}_\mathcal{C}(X, Y)$ for each $X, Y \in \text{Ob}(\mathcal{C})$ and an associative composition. A semifunctor $F : \mathcal{C} \rightarrow \mathcal{C}'$ between semicategories assigns an object $F(X) \in \text{Ob}(\mathcal{C}')$ to each $X \in \text{Ob}(\mathcal{C})$ and a morphism $F(f) \in \text{Hom}_\mathcal{C}'(F(X), F(Y))$ to each $f \in \text{Hom}_\mathcal{C}(X, Y)$ and preserves composition.

A left $H$-semicategory is a small $k$-linear semicategory $\mathcal{S}_H$ such that

(i) $\text{Hom}_{\mathcal{S}_H}(X, Y)$ is a left $H$-module for all $X, Y \in \text{Ob}(\mathcal{S}_H)$

(ii) $h(gf) = (h_1)(gh_2)$ for any $h \in H$, $f \in \text{Hom}_{\mathcal{S}_H}(X, Y)$ and $g \in \text{Hom}_{\mathcal{S}_H}(Y, Z)$.

It is clear that any ordinary category may be treated as a semicategory. Conversely, to any $k$-semicategory $\mathcal{C}$, we can associate an ordinary $k$-category $\check{\mathcal{C}}$ by adjoining unit morphisms as follows:

$$\text{Ob}(\check{\mathcal{C}}) : = \text{Ob}(\mathcal{C})$$

$$\text{Hom}_{\check{\mathcal{C}}}(X, Y) : = \begin{cases} \text{Hom}_\mathcal{C}(X, X) \oplus k & \text{if } X = Y \\ \text{Hom}_\mathcal{C}(X, Y) & \text{if } X \neq Y \end{cases}$$

A morphism in $\text{Hom}_\mathcal{C}(X, Y)$ will be denoted by $\check{f} = f + \mu$, where $f \in \text{Hom}_\mathcal{C}(X, Y)$ and $\mu \in k$. It is understood that $\mu = 0$ whenever $X = Y$. Any semifunctor $F : \mathcal{C} \rightarrow \mathcal{D}$ where $\mathcal{D}$ is an ordinary category may be extended to an ordinary functor $\check{F} : \check{\mathcal{C}} \rightarrow \check{\mathcal{D}}$. If $\mathcal{S}_H$ is a left $H$-semicategory, we note that $\check{\mathcal{S}}_H$ is a left $H$-category in the sense of Definition 2.1.

Next we recall the notion of the tensor product of complexes. Let $(A^*, \partial_A) : = \ldots \rightarrow A^n \overset{\partial_A^n}{\rightarrow} A^{n+1} \rightarrow \ldots$ and $(B^*, \partial_B) : = \ldots \rightarrow B^n \overset{\partial_B^n}{\rightarrow} B^{n+1} \rightarrow \ldots$ be two cochain complexes. Then, their tensor product $A^* \otimes B^*$ also forms a cochain complex which is defined as follows:

$$(A^* \otimes B^*)^n : = \bigoplus_{i+j=n} A^i \otimes B^j$$

$$\partial^n_{A \otimes B} : = \bigoplus_{i+j=n} (\partial^i_A \otimes 1_{B_j} + (-1)^i 1_{A_i} \otimes \partial^j_B)$$

**Definition 5.2.** A differential graded semicategory (DG-semicategory) $(\mathcal{S}, \hat{\partial})$ is a $k$-linear semicategory $\mathcal{S}$ such that

(i) $\text{Hom}^*_\mathcal{S}(X, Y) = (\text{Hom}^*_\mathcal{S}(X, Y), \hat{\partial}^n_{XY})_{n \geq 0}$ is a cochain complex of $k$-spaces for each $X, Y \in \text{Ob}(\mathcal{S})$.

(ii) the composition map

$$\text{Hom}^*_\mathcal{S}(Y, Z) \otimes \text{Hom}^*_\mathcal{S}(X, Y) \rightarrow \text{Hom}^*_\mathcal{S}(X, Z)$$

is a morphism of complexes. Equivalently,

$$\hat{\partial}^n_{XZ}(gf) = \hat{\partial}^{n-r}_{YZ}(gf) + (-1)^{n-r} g \hat{\partial}^r_{XY}(f)$$

for any $f \in \text{Hom}_{\mathcal{S}}(X, Y)$ and $g \in \text{Hom}_{\mathcal{S}}(Y, Z)^{n-r}$.

Whenever the meaning is clear from context, we will drop the subscript and simply write $\hat{\partial}$ for the differential on any $\text{Hom}^*_\mathcal{S}(X, Y)$. 

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A DG-semicategory with a single object is the same as a differential graded (but not necessarily unital) \( k \)-algebra. Accordingly, any small DG-semicategory may be treated as a differential graded (but not necessarily unital) \( k \)-algebra with several objects. The DG-semicategories may be treated in a manner similar to DG-categories (see, for instance, [27, 28]).

**Definition 5.3.** A DG-semifunctor \( \alpha : (\mathcal{S}, \hat{\partial}) \rightarrow (\mathcal{S}', \hat{\partial}') \) between two DG-semicategories is a \( k \)-linear semifunctor \( \alpha : \mathcal{S} \rightarrow \mathcal{S}' \) such that the induced map \( \text{Hom}^*_{\mathcal{S}}(X,Y) \rightarrow \text{Hom}^*_{\mathcal{S}'}(\alpha X, \alpha Y) \), \( f \mapsto \alpha(f) \), is a morphism of complexes for each \( X, Y \in \text{Ob}(\mathcal{S}) \).

**Remark 5.4.** We observe that corresponding to any DG-semicategory \( \mathcal{S} \), there is a semicategory \( \mathcal{S}^0 \) defined as:
\[
\text{Ob}(\mathcal{S}^0) := \text{Ob}(\mathcal{S}) \\
\text{Hom}_{\mathcal{S}^0}(X,Y) := \text{Hom}_{\mathcal{S}}^0(X,Y)
\]
The composition in \( \mathcal{S} \) induces a well-defined composition \( \text{Hom}_{\mathcal{S}^0}(Y,Z) \otimes \text{Hom}_{\mathcal{S}^0}(X,Y) \rightarrow \text{Hom}_{\mathcal{S}^0}(X,Z) \).

We now construct a “universal DG-semicategory” associated to a given \( k \)-linear semicategory, similar to the construction of the universal differential graded algebra associated to a (not necessarily unital) \( k \)-algebra (see, for instance, [12 p. 315]).

Let \( \Omega \mathcal{C} \) be the semicategory with \( \text{Ob}(\Omega \mathcal{C}) := \text{Ob}(\mathcal{C}) \) and \( \text{Hom}_{\Omega \mathcal{C}}(X,Y) = \bigoplus_{n \geq 0} \text{Hom}^n_{\mathcal{C}}(X,Y) \), where
\[
\text{Hom}^n_{\Omega \mathcal{C}}(X,Y) := \begin{cases} 
\text{Hom}^n_{\mathcal{C}}(X,Y) & \text{if } n = 0 \\
\bigoplus_{(X_1, \ldots, X_n) \in \text{Ob}(\mathcal{C})^n} \text{Hom}^n_{\mathcal{C}}(X_1,Y) \otimes \text{Hom}^n_{\mathcal{C}}(X_2, X_1) \otimes \cdots \otimes \text{Hom}^n_{\mathcal{C}}(X_{n+1}, X_n) & \text{if } n \geq 1
\end{cases}
\] (5.2)

Here the sum runs over the ordered tuples \((X_1, \ldots, X_n) \in \text{Ob}(\mathcal{C})^n\). In particular, \((\Omega \mathcal{C})^0 = \mathcal{C}\). For \( n \geq 1 \), an element of the form \( f^0 \otimes f^1 \otimes \cdots \otimes f^n \) in \( \text{Hom}^n_{\mathcal{C}}(X,Y) \) will be denoted by \( f^0 f^1 \cdots f^n = (f^0 + \mu) f^1 \cdots f^n \) and said to be homogeneous of degree \( n \). By abuse of notation, we will continue to use \( f^0 f^1 \cdots f^n = (f^0 + \mu) f^1 \cdots f^n \) to denote an element of \( \text{Hom}^n_{\mathcal{C}}(X,Y) \) even when \( n = 0 \). In that case, it will be understood that \( \mu = 0 \).

The composition in \( \Omega \mathcal{C} \) is determined by
\[
f^0 \circ f^1 \circ \cdots \circ f^n = f^0 f^1 \cdots f^n \quad (df^0) \circ f^1 = df^0 f^1 - f^0 (df^1) \quad df^1 \circ \cdots \circ df^n = df^1 \cdots df^n
\] (5.3)

In particular, it follows that
\[
((f^0 + \mu) f^1 \cdots f^i) \cdot ((g^0 + \mu') g^1 \cdots g^j)
= (f^0 + \mu) \left( f^0 (f^1 \cdots f^{i-1}) (f^i g^0) g^1 \cdots g^j + \sum_{l=1}^{i-1} (-1)^{i-l} f^0 f^1 \cdots f^{i-1} f^i (f^l g^{l+1}) \cdots g^j df^0 g^1 \cdots dg^j \right)
+ (-1)^i (f^0 + \mu) f^1 df^2 \cdots df^i g^0 g^1 \cdots g^j + \mu (f^0 + \mu) f^1 \cdots f^i df^1 \cdots dg^j
\] (5.4)

For each \( X, Y \in \text{Ob}(\Omega \mathcal{C}) \), the differential \( \partial^0_{X,Y} : \text{Hom}^n_{\Omega \mathcal{C}}(X,Y) \rightarrow \text{Hom}^{n+1}_{\Omega \mathcal{C}}(X,Y) \) is determined by setting
\[
\partial^0_{X,Y}((f^0 + \mu) f^1 \cdots f^n) := df^0 f^1 \cdots f^n
\]
It follows from definition that \( \partial^1_{X,Y} \circ \partial^0_{X,Y} = 0 \). Therefore, \( \text{Hom}^*_{\Omega \mathcal{C}}(X,Y) := \left( \text{Hom}^n_{\Omega \mathcal{C}}(X,Y), \partial^n_{X,Y} \right)_{n \geq 0} \) is a cochain complex for each \( X, Y \in \text{Ob}(\Omega \mathcal{C}) \). It may also be verified that the composition in \( \Omega \mathcal{C} \) is a morphism of complexes. Thus, \( \Omega \mathcal{C} \) is a DG-semicategory.

**Proposition 5.5.** Let \( \mathcal{C} \) be a small \( k \)-linear semicategory. Then, the associated DG-semicategory \( (\Omega \mathcal{C}, \partial) \) is universal in the following sense: given
(i) any DG-semicolontry \((S, \hat{\partial})\) and

(ii) a \(k\)-linear semifunctor \(\rho : C \rightarrow S^0\),

there exists a unique DG-semicolontry \(\hat{\rho} : (\Omega C, \hat{\partial}) \rightarrow (S, \hat{\partial})\) such that the restriction of \(\hat{\rho}\) to the semicomcategory \(C\) is identical to \(\rho : C \rightarrow S^0\).

Proof. We extend \(\rho\) to obtain a DG-semicolontry \(\hat{\rho} : (\Omega C, \hat{\partial}) \rightarrow (S, \hat{\partial})\) as follows:

\[
\hat{\rho}(X) := \rho(X)
\]
\[
\hat{\rho}((f^0 + \mu)df^1 \ldots df^n) := \rho(f^0) \circ \hat{\partial}^0(\rho(f^1)) \circ \ldots \circ \hat{\partial}^0(\rho(f^n)) + \mu \hat{\rho}^0(\rho(f^1)) \circ \ldots \circ \hat{\partial}^0(\rho(f^n))
\]

for all \(X \in \text{Ob}(\Omega C) = \text{Ob}(C)\) and \((f^0 + \mu)df^1 \ldots df^n \in \text{Hom}_\Omega(X, Y)\), \(n \geq 1\). Since each \(\rho(f^i)\) is a morphism of degree 0 in \(S\), it follows from (5.1) and (5.4) that

\[
\hat{\rho}(((f^0 + \mu)df^1 \ldots df^n) \circ ((f^{n+1} + \mu')df^{n+2} \ldots df^m)) = \hat{\rho}(((f^0 + \mu)df^1 \ldots df^n) \circ \hat{\rho}((f^{n+1} + \mu')df^{n+2} \ldots df^m))
\]

It is also clear by construction that \(\hat{\rho}|_C = \rho\). Moreover, we have

\[
\hat{\partial}^n(\hat{\rho}((f^0 + \mu)df^1 \ldots df^n)) = \hat{\partial}^n(\rho(f^0)\hat{\partial}^0(\rho(f^1)) \ldots \hat{\partial}^0(\rho(f^n))) + \mu \hat{\rho}^n(\rho(f^1)) \circ \ldots \circ \hat{\partial}^0(\rho(f^n)) = \hat{\rho}^n(\rho(f^0)) \circ \hat{\partial}^0(\rho(f^1)) \circ \ldots \circ \hat{\partial}^0(\rho(f^n))
\]

The uniqueness of \(\hat{\rho}\) is also clear from (5.3) and (5.4). □

**Definition 5.6.** A left DGH-semicolontry is a left \(H\)-semicolontry \(S_H\) equipped with a DG-semicolontry \((S_H, \hat{\partial}_H)\) structure such that for all \(n \geq 0\):

(a) \(\text{Hom}^n_{S_H}(X, Y)\) is a left \(H\)-module for \(X, Y \in \text{Ob}(S_H)\).

(b) \(\hat{\partial}_H^n : \text{Hom}^n_{S_H}(X, Y) \rightarrow \text{Hom}^{n+1}_{S_H}(X, Y)\) is \(H\)-linear for \(X, Y \in \text{Ob}(S_H)\).

We can similarly define the notion of a DG-semicolontry between DGH-semicategories. If \((S_H, \hat{\partial}_H)\) is a left DGH-semicolontry, we note that \(S_H\) is a left \(H\)-semicolontry.

**Proposition 5.7.** Let \(D_H\) be a left \(H\)-category. Then, the universal DG-semicolontry \((\Omega(D_H), \hat{\partial}_H)\) associated to \(D_H\) is a left DG-semicolontry with the \(H\)-action determined by

\[
h \cdot ((f^0 + \mu)df^1 \ldots df^n) := (h_1 f^0 + \mu \varepsilon(h_1))d(h_2 f^1) \ldots d(h_{n+1} f^n)
\]
for all \(h \in H\) and \((f^0 + \mu)df^1 \ldots df^n \in \text{Hom}_{\Omega(D_H)}(X, Y)\).

Proof. This is immediate from the definitions in (5.4) and (5.7). □

**Definition 5.8.** Let \((S_H, \hat{\partial}_H)\) be a left DGH-semicolontry and \(M\) be a right-left SAYD module over \(H\). A closed graded \((H, M)\)-trace of dimension \(n\) on \(S_H\) is a collection of \(k\)-linear maps

\[
\hat{\mathcal{T}}^H := \{ \hat{\mathcal{T}}^H_X : M \otimes \text{Hom}^n_{S_H}(X, X) \rightarrow k \}_{X \in \text{Ob}(S_H)}
\]

such that

\[
\hat{\mathcal{T}}^H_X(mh_1 \otimes S(h_2)f) = \varepsilon(h)\hat{\mathcal{T}}^H_X(m \otimes f)
\]
\[
\hat{\mathcal{T}}^H_X(m \otimes \hat{\partial}_H^{-1}(f')) = 0
\]
\[
\hat{\mathcal{T}}^H_X(m \otimes g'g) = (-1)^{ij} \hat{\mathcal{T}}^H_Y(m_{(0)} \otimes (S^{-1}(m_{(-1)})g)g')
\]

for all \(h \in H, m \in M, f \in \text{Hom}^n_{S_H}(X, X), f' \in \text{Hom}^{n-1}_{S_H}(X, X), g \in \text{Hom}^1_{S_H}(X, Y), g' \in \text{Hom}^1_{S_H}(Y, X)\) and \(i + j = n\).
In particular, putting $H = k = M$, we get: a closed graded trace of dimension $n$ on a DG-semicategory $(S, \hat{\partial})$ is a collection of $k$-linear maps $\hat{T} := \{\hat{T}_X : \text{Hom}_S^n(X, X) \to k\}_{X \in \Omega(S)}$ such that

\begin{align*}
\hat{T}_X (f^{n-1}) &= 0 \quad \text{(5.11)} \\
\hat{T}_X (g') &= (\hat{\partial} g') \quad \text{(5.12)}
\end{align*}

for all $f \in \text{Hom}_S^{n-1}(X, X)$, $g \in \text{Hom}_S^n(Y, X)$, $g' \in \text{Hom}_S^1(Y, X)$ and $i + j = n$.

**Definition 5.9.** An $n$-dimensional $S_H$-cycle with coefficients in a SAYD module $M$ is a triple $(S_H, \hat{\partial}_H, \hat{T}^H)$ such that

(i) $(S_H, \hat{\partial}_H)$ is a left DGH-semicategory.

(ii) $\hat{T}^H$ is a closed graded $(H, M)$-trace of dimension $n$ on $S_H$.

Let $D_H$ be a left $H$-category. By an $n$-dimensional cycle over $D_H$, we mean a tuple $(S_H, \hat{\partial}_H, \hat{T}^H, \rho)$ such that

(i) $(S_H, \hat{\partial}_H, \hat{T}^H)$ is an $n$-dimensional $S_H$-cycle with coefficients in a SAYD module $M$.

(ii) $\rho : D_H \to S^0_H$ is an $H$-linear semifunctor.

In particular, putting $H = k = M$, we get:

**Definition 5.10.** An $n$-dimensional $S$-cycle is a triple $(S, \hat{\partial}, \hat{T})$ such that

(i) $(S, \hat{\partial})$ is a DG-semicategory.

(ii) $\hat{T}$ is a closed graded trace of dimension $n$ on $S$.

Let $C$ be a small $k$-linear category. By an $n$-dimensional cycle over $C$, we mean a tuple $(S, \hat{\partial}, \hat{T}, \rho)$ such that

(i) $(S, \hat{\partial}, \hat{T})$ is an $n$-dimensional $S$-cycle.

(ii) $\rho : C \to S^0$ is a $k$-linear semifunctor.

We will denote by $\text{Cat}_H$ the category of all left $H$-categories with $H$-linear functors between them. We fix a left $H$-category $D_H$. Given an $n$-dimensional cycle $(S_H, \hat{\partial}_H, \hat{T}^H, \rho)$ over $D_H$, we define its character $\phi \in C^n_H(D_H, M)$ by setting

$$
\phi : M \otimes \text{CN}_n(D_H) \to k \\
\phi (m \otimes f^0 \otimes \cdots \otimes f^n) := \hat{T}^H_{X_0} (m \otimes \rho (f^0) \hat{\partial}_H^0 (\rho (f^1)) \cdots \hat{\partial}_H^0 (\rho (f^n)))$
$$

for $m \in M$ and $f^0 \otimes \cdots \otimes f^n \in \text{Hom}_{D_H}(X_1, X_0) \otimes \text{Hom}_{D_H}(X_2, X_1) \otimes \cdots \otimes \text{Hom}_{D_H}(X_0, X_n)$. We will often suppress the semifunctor $\rho$ and refer to $\phi$ simply as the character of the $n$-dimensional cycle $(S_H, \hat{\partial}_H, \hat{T}^H)$. The next result provides a characterization of the space $Z^n_H(D_H, M)$ of $n$-cocycles in the Hopf-cyclic cohomology of the category $D_H$ with coefficients in the SAYD module $M$.

**Theorem 5.11.** Let $D_H$ be a left $H$-category and $M$ be a right-left SAYD module over $H$. Let $\phi \in C^n_H(D_H, M)$. Then, the following conditions are equivalent:

1. $\phi$ is the character of an $n$-dimensional cycle over $D_H$, i.e., there is an $n$-dimensional cycle $(S_H, \hat{\partial}_H, \hat{T}^H)$ with coefficients in $M$ and an $H$-linear semifunctor $\rho : D_H \to S^0_H$ such that

$$
\phi (m \otimes f^0 \otimes \cdots \otimes f^n) = \hat{T}^H_{X_0} ((\text{id}_M \otimes \rho)(m \otimes f^0 f^1 \otimes \cdots f^n)) \\
= \hat{T}^H_{X_0} (m \otimes \rho f^0 \hat{\partial}_H^0 (\rho (f^1)) \cdots \hat{\partial}_H^0 (\rho (f^n))) \tag{5.13}
$$

for any $m \in M$ and $f^0 \otimes \cdots \otimes f^n \in \text{Hom}_{D_H}(X_1, X_0) \otimes \text{Hom}_{D_H}(X_2, X_1) \otimes \cdots \otimes \text{Hom}_{D_H}(X_0, X_n)$. 

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(2) There exists a closed graded \((H, M)\)-trace \(\mathcal{T}^H\) of dimension \(n\) on \((\Omega(D_H), \partial_H)\) such that
\[
\phi(m \otimes f^0 \otimes \ldots \otimes f^n) = \mathcal{T}^H_X(m \otimes f^0 df^1 \ldots df^n) \tag{14.1}
\]
for any \(m \in M\) and \(f^0 \otimes \ldots \otimes f^n \in \text{Hom}_{D_H}(X_1, X_0) \otimes \text{Hom}_{D_H}(X_2, X_1) \otimes \ldots \otimes \text{Hom}_{D_H}(X_0, X_n)\).

(3) \(\phi \in Z^n_H(D_H, M)\).

**Proof.** (1) \(\Rightarrow\) (2). By the universal property of \(\Omega(D_H)\), the \(H\)-linear semifunctor \(\rho : D_H \to S^n_H\) can be extended to a DGH-semifunctor \(\hat{\rho} : \Omega(D_H) \to S^n_H\) as in (5.5). We define a collection \(\mathcal{T}^H := \{\mathcal{T}^H_X : M \otimes \text{Hom}^n_{\Omega(D_H)}(X, X) \to k\}_{X \in \text{Ob}(\Omega(D_H))}\) of \(k\)-linear maps given by
\[
\mathcal{T}^H_X(m \otimes (f^0 + \mu) df^1 \ldots df^n) := \mathcal{T}^H_X(m \otimes \hat{\rho}(f^0 + \mu) df^1 \ldots df^n) \tag{15.15}
\]
for any \(m \in M\) and \(f^0 \otimes \ldots \otimes f^n \in \text{Hom}_{D_H}(X_1, X_0) \otimes \text{Hom}_{D_H}(X_2, X_1) \otimes \ldots \otimes \text{Hom}_{D_H}(X_0, X_n)\). In particular, it follows from (15.15) that
\[
\phi(m \otimes f^0 \otimes \ldots \otimes f^n) = \mathcal{T}^H_X(m \otimes \rho(f^0) df^1 \ldots df^n) = \mathcal{T}^H_X(m \otimes f^0 df^1 \ldots df^n) \tag{16.16}
\]
We now verify that the collection \(\mathcal{T}^H\) is an \(n\)-dimensional closed graded \((H, M)\)-trace on \(\Omega(D_H)\). For any \(\alpha = (f^0 + \mu) df^1 \ldots df^n \in \text{Hom}^n_{\Omega(D_H)}(X, X)\) and \(h \in H\), we have
\[
\mathcal{T}^H_X(m h_1 \otimes S(h_2)\alpha) = \mathcal{T}^H_X(m h_1 \otimes \hat{\rho}(S(h_2)\alpha)) = \mathcal{T}^H_X(m h_1 \otimes (S(h_2)(\hat{\rho}(\alpha)))) = \varepsilon(h) \mathcal{T}^H_X(m \otimes \alpha)
\]
Hence, \(\mathcal{T}^H\) satisfies the condition (5.8). Further, for any \(\beta = (p^0 + \mu) dp^1 \ldots dp^{n-1} \in \text{Hom}^{n-1}_{\Omega(D_H)}(X, X)\), we have
\[
\mathcal{T}^H_X(m \otimes \partial_H^{-1}(\beta)) = \mathcal{T}^H_X(m \otimes \hat{\rho}(\partial_H^{-1}(\beta))) = \mathcal{T}^H_X(m \otimes \hat{\rho}(\beta)) = 0
\]
Hence, \(\mathcal{T}^H\) satisfies the condition (5.9). Finally, we see that
\[
\mathcal{T}^H_X(m \otimes g' g) = \mathcal{T}^H_X(m \otimes \hat{\rho}(g') \hat{\rho}(g)) = (-1)^{ij} \mathcal{T}^H_X(m_{ij(0)} \otimes (S^{-1}(m_{i(-1)})) \hat{\rho}(g) \hat{\rho}(g'))
\]
\[
= (-1)^{ij} \mathcal{T}^H_X(m_{ij(0)} \otimes (S^{-1}(m_{i(-1)})) g' g)
\]
for any \(g \in \text{Hom}^i_{\Omega(D_H)}(X, Y)\), \(g' \in \text{Hom}^j_{\Omega(D_H)}(Y, X)\) with \(i + j = n\). This proves the condition in (5.10).

(2) \(\Rightarrow\) (1). Suppose that we have a closed graded \((H, M)\)-trace \(\mathcal{T}^H\) of dimension \(n\) on \(\Omega(D_H)\) satisfying (5.11). Then, the triple \((\Omega(D_H), \partial_H, \mathcal{T}^H)\) forms an \(n\)-dimensional cycle over \(D_H\) with coefficients in \(M\). Further, by observing that \(\partial_H(f) = df\) for any \(f \in \text{Hom}_{D_H}(X, Y)\), we get (5.12).

(1) \(\Rightarrow\) (3). Let \((S_H, \partial_H, \mathcal{T}^H)\) be an \(n\)-dimensional cycle over \(D_H\) with coefficients in \(M\) and \(\rho : D_H \to S^n_H\) be an \(H\)-linear semifunctor satisfying
\[
\phi(m \otimes f^0 \otimes \ldots \otimes f^n) = \mathcal{T}^H_X(m \otimes \rho(f^0) df^1 \ldots df^n) \tag{13.13}
\]
for any \(m \in M\) and \(f^0 \otimes \ldots \otimes f^n \in \text{Hom}_{D_H}(X_1, X_0) \otimes \text{Hom}_{D_H}(X_2, X_1) \otimes \ldots \otimes \text{Hom}_{D_H}(X_0, X_n)\). For simplicity of notation, we will drop the functor \(\rho\). To show that \(\phi\) is an \(n\)-cocycle, it suffices to check that (see 3.4)
\[
b(\phi) = 0 \quad \text{and} \quad (1 - \lambda)(\phi) = 0
\]
where \(b = \sum_{i=0}^{n+1} (-1)^i \delta_i\) and \(\lambda = (-1)^n \tau_n\). For any \(p^0 \otimes \ldots \otimes p^{n+1} \in \text{Hom}_{D_H}(X_1, X_0) \otimes \text{Hom}_{D_H}(X_2, X_1) \otimes \ldots \otimes \text{Hom}_{D_H}(X_0, X_{n+1})\), we have

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\[
\sum_{i=0}^{n+1} (-1)^i \delta_i(\phi)(m \otimes p^0 \otimes \ldots \otimes p^{n+1})
\]

\[
= \sum_{i=0}^{n} (-1)^i \phi(m \otimes p^0 \otimes \ldots \otimes p^i \otimes \ldots \otimes p^{n+1}) + (-1)^{n+1} \phi(m(0) \otimes (S^{-1}(m(-1))p^{n+1}))p^0 \otimes p^1 \otimes \ldots \otimes p^n
\]

\[
= \dot{\mathcal{F}}_{X_0}^H(m \otimes p^0 \otimes p^1 \partial_H^0(p^2) \ldots \partial_H^0(p^{n+1})) + \sum_{i=1}^{n} (-1)^i \dot{\mathcal{F}}_{X_0}^H(m \otimes p^0 \partial_H^0(p^1) \ldots \partial_H^0(p^i \otimes \ldots \otimes \partial_H^0(p^{n+1})) + \notag
\]

\[
(-1)^{n+1} \dot{\mathcal{F}}_{X_0}^H(m(0) \otimes (S^{-1}(m(-1))p^{n+1}))p^0 \partial_H^0(p^1) \ldots \partial_H^0(p^n))
\]

Now using the equality \(\partial_H^0(fg) = \partial_H^0(f)g + f \partial_H^0(g)\) for any \(f\) and \(g\) of degree 0, we have

\[
(p^0 \partial_H^0(p^1) \ldots \partial_H^0(p^n))p^{n+1} \notag
\]

\[
= \sum_{i=1}^{n} (-1)^n p^0 \partial_H^0(p^1) \ldots \partial_H^0(p^i \otimes \ldots \otimes \partial_H^0(p^{n+1}) + (-1)^n p^0 \partial_H^0(p^2) \ldots \partial_H^0(p^{n+1})
\]

Thus, using the condition in (6.10), we obtain

\[
\sum_{i=0}^{n+1} (-1)^i \delta_i(\phi)(m \otimes p^0 \otimes \ldots \otimes p^{n+1}) = \notag
\]

\[
= (-1)^n \dot{\mathcal{F}}_{X_0}^H(m \otimes (p^0 \partial_H^0(p^1) \ldots \partial_H^0(p^n))p^{n+1}) + (-1)^{n+1} \dot{\mathcal{F}}_{X_0}^H(m(0) \otimes (S^{-1}(m(-1))p^{n+1}))p^0 \partial_H^0(p^1) \ldots \partial_H^0(p^n)) = 0
\]

Next, using (5.9), (5.10), and the \(H\)-linearity of \(\partial_H\), we have

\[
\left( (1 - (-1)^n \tau_n ) \phi \right)(m \otimes f^0 \otimes \ldots \otimes f^n)
\]

\[
= \phi(m \otimes f^0 \otimes \ldots \otimes f^n) - (-1)^n \phi(m(0) \otimes S^{-1}(m(-1))f^n \otimes f^0 \otimes \ldots \otimes f^{n-1})
\]

\[
= \dot{\mathcal{F}}_{X_0}^H(m \otimes f^0 \partial_H^0(f^1) \ldots \partial_H^0(f^n)) - (-1)^n \dot{\mathcal{F}}_{X_0}^H(m(0) \otimes (S^{-1}(m(-1))f^n)\partial_H^0(f^1) \ldots \partial_H^0(f^{n-1}))
\]

\[
= (-1)^{n-1} \dot{\mathcal{F}}_{X_0}^H(m(0) \otimes (S^{-1}(m(-1))\partial_H^0(f^1) \ldots \partial_H^0(f^{n-1}))+(\notag
\]

\[
(-1)^{n-1} \dot{\mathcal{F}}_{X_0}^H(m(0) \otimes (S^{-1}(m(-1))f^n)\partial_H^0(f^1) \ldots \partial_H^0(f^{n-1}))
\]

\[
= (-1)^{n-1} \dot{\mathcal{F}}_{X_0}^H(m(0) \otimes \partial_H^{-1}((S^{-1}(m(-1))f^n)\partial_H^0(f^1) \ldots \partial_H^0(f^{n-1}))) = 0
\]

(3) \(\Rightarrow\) (2). Let \(\phi \in Z^*_H(D_H, M)\). For each \(X \in Ob(\Omega(D_H))\), we define an \(H\)-linear map \(\mathcal{F}_X^H : M \otimes Hom^0_{\Omega(D_H)}(X, X) \rightarrow k\) given by

\[
\mathcal{F}_X^H(m \otimes (f^0 + \mu)df^1 \ldots df^n) := \phi(m \otimes f^0 \otimes \ldots \otimes f^n)
\]

for \(f^0 \otimes \ldots \otimes f^n \in Hom_{\Omega(D_H)}(X_1, X) \otimes Hom_{\Omega(D_H)}(X_2, X_1) \otimes \ldots \otimes Hom_{\Omega(D_H)}(X, X_n)\). We now verify that the collection \(\{ \mathcal{F}_X^H : M \otimes Hom^0_{\Omega(D_H)}(X, X) \rightarrow k \}_{X \in Ob(\Omega(D_H))}\) is a closed graded \((H, M)\)-trace on \((\Omega(D_H), \partial_H)\).

For any \((p^0 + \mu)dp^1 \ldots dp^{n-1} \in Hom^0_{\Omega(D_H)}(X, X)\), we have

\[
\mathcal{F}_X^H(m \otimes \partial_H^{-1}((p^0 + \mu)dp^1 \ldots dp^{n-1})) = \mathcal{F}_X^H(m \otimes 1dp^0dp^1 \ldots dp^{n-1}) = \phi(m \otimes 0 \otimes p^0 \otimes \ldots \otimes p^{n-1}) = 0
\]

This proves the condition in (5.9). Using (3.2), it is also clear that \(\{ \mathcal{F}_X^H : M \otimes Hom^0_{\Omega(D_H)}(X, X) \rightarrow k \}_{X \in Ob(\Omega(D_H))}\) satisfies condition (5.8). Finally, for any \(g' = (g^0 + \mu')dg^1 \ldots dg^r \in Hom^0_{\Omega(D_H)}(Y, X)\) and
\[ g = (g^{r+1} + \mu)dg^{r+2} \ldots dg^{n+1} \in \text{Hom}^{n-r}_{\Omega(D_H)}(X,Y), \]

we have

\[ \mathcal{T}^H_\ast(m \otimes g') \]

\[ = \sum_{j=1}^{r} (-1)^{r-j} \mathcal{T}^H_\ast(m \otimes (g^0 + \mu')dg^1 \ldots d(g^{j+1}) \ldots dg^{n+1}) + (-1)^r \mathcal{T}^H_\ast(m \otimes (g^0 + \mu')g^1dg^2 \ldots dg^{n+1}) \]

\[ + \mathcal{T}^H_\ast(m \otimes \mu g^0 + \mu')dg^1 \ldots d(g^r)dg^{r+2} \ldots dg^{n+1} \]

\[ = \sum_{j=1}^{r} (-1)^{r-j} \phi(m \otimes g^0 \otimes \ldots \otimes g^j g^{j+1} \ldots \otimes g^{n+1}) + (-1)^r \phi(m \otimes g^0 \otimes g^2 \otimes \ldots \otimes g^{n+1}) \]

\[ + (-1)^r \mu' \phi(m \otimes g^0 \otimes g^1 \otimes \ldots \otimes g^r g^{r+2} \ldots \otimes g^{n+1}) \]

\[ = \sum_{j=0}^{r} (-1)^{r+j} \phi(m \otimes g^0 \otimes \ldots \otimes g^j g^{j+1} \ldots \otimes g^{n+1}) + (-1)^r \mu' \phi(m \otimes g^0 \otimes g^2 \otimes \ldots \otimes g^{n+1}) \]

\[ + \mu \phi(m \otimes g^0 \otimes g^1 \otimes \ldots \otimes g^r g^{r+2} \ldots \otimes g^{n+1}) \]

On the other hand, we have

\[ (-1)^{(r-n)} \mathcal{T}^H_Y(m(0) \otimes (S^{-1}(m(1))g)g') \]

\[ = (-1)^{(r-n)} \mathcal{T}^H_Y(m(0) \otimes [S^{-1}((m(1))_{n-r}) (g^{r+1} + \mu)] [d(S^{-1}((m(1))_{n-r}) g^{r+2})] \ldots [d(S^{-1}((m(1))_{n-r}) g^{n+1})]) \circ \]

\[ ((g^0 + \mu')dg^1 \ldots dg^r) \]

\[ = (-1)^{(r-n)} \sum_{j=r+2}^{n} (-1)^{n-j+1} \mathcal{T}^H_Y(m(0) \otimes [S^{-1}((m(1))_{n-r}) (g^{r+1} + \mu)] \ldots \ldots [d(S^{-1}((m(1))_{n-r}) g^{n+1})]) \circ \]

\[ ((g^0 + \mu')g^1 \ldots g^r) \]

\[ = (-1)^{(r-n)} \sum_{j=r+2}^{n} (-1)^{n-j+1} \phi(m(0) \otimes S^{-1}((m(1))_{n-r}) g^{r+1} \otimes \ldots \otimes S^{-1}((m(1))_{n-r+1}) g^{r+1} \otimes \ldots \otimes g^r) + \]

\[ (-1)^{(r-n)} \phi(m(0) \otimes S^{-1}((m(1))_{n-r+1}) g^{r+1} \otimes \ldots \otimes S^{-1}((m(1))_{n-r+1}) g^{n+1} \circ \ldots \circ g^r) + \]

\[ (-1)^{(r-n)} (-1)^{n-r} \mu \phi(m(0) \otimes S^{-1}((m(1))_{n-r}) g^{r+2} \otimes \ldots \otimes S^{-1}((m(1))_{n-r}) g^{n+1} \circ \ldots \circ g^r) \]

\[ + (-1)^{(r-n)} \mu' \phi(m(0) \otimes S^{-1}((m(1))_{n-r}) g^{r+1} \circ S^{-1}((m(1))_{n-r}) g^{r+2} \otimes \ldots \otimes S^{-1}((m(1))_{n-r}) g^{n+1} \otimes \ldots \otimes g^r) \]

Using repeatedly the fact that \( \phi = (-1)^{n-r} \), we get

\[ (-1)^{(r-n)} \mathcal{T}^H_Y(m(0) \otimes (S^{-1}(m(1))g)g') \]

\[ = \sum_{j=r+1}^{n} (-1)^{r+j} \phi(m \otimes g^0 \otimes \ldots \otimes g^j g^{j+1} \ldots \otimes g^{n+1}) - (-1)^{n+r+1} \phi(m(0) \otimes (S^{-1}(m(1))g^{n+1})g^0 \otimes g^1 \otimes \ldots \otimes g^n) \]

\[ + (-1)^r \mu' \phi(m \otimes g^0 \otimes g^1 \otimes \ldots \otimes g^{n+1}) + \mu \phi(m \otimes g^0 \otimes g^1 \otimes \ldots \otimes g^r g^{r+2} \ldots \otimes g^{n+1}) \]

The condition (5.11) now follows using the fact that \( b(\phi) = 0 \). This proves the result.

**Remark 5.12.** From the statement and proof of Theorem [5.11], it is clear that there is a one to one correspondence between \( n \)-dimensional closed graded \((H,M)\)-traces on \( \Omega(D_H) \) and \( Z_H^b(D_H, M) \).
Substituting $H = k = M$ in Theorem 5.11 we obtain the following categorical version of [12 Proposition 1, p. 98]. This gives a characterization of $Z^n(C)$, the $n$-cocycles in the ordinary cyclic cohomology of a small $k$-linear category $C$.

**Proposition 5.13.** Let $C$ be a small $k$-linear category and $\phi \in CN^n(C)$. Then, the following conditions are equivalent:

1. $\phi$ is the character of an $n$-dimensional cycle over $C$, i.e., there exists an $n$-dimensional cycle $(S, \hat{\partial}, \hat{T})$ and a $k$-linear semifunctor $\rho : C \to \mathcal{S}^0$ such that
   \[ \phi(f^0 \otimes \ldots \otimes f^n) = \hat{T}_{X_0}(\rho(f^0) \hat{\partial}^0 (\rho(f^1)) \ldots \hat{\partial}^0 (\rho(f^n))) \]
   for any $f^0 \otimes \ldots \otimes f^n \in \text{Hom}_C(X_1, X_0) \otimes \text{Hom}_C(X_2, X_1) \otimes \ldots \otimes \text{Hom}_C(X_0, X_n)$.

2. There exists a closed graded trace $T$ of dimension $n$ on $(\mathcal{S}_C, \partial)$ such that
   \[ \phi(f^0 \otimes \ldots \otimes f^n) = T_{X_0}(f^0 \hat{\partial}^1 \ldots \hat{\partial}^n) \]
   for any $f^0 \otimes \ldots \otimes f^n \in \text{Hom}_C(X_1, X_0) \otimes \text{Hom}_C(X_2, X_1) \otimes \ldots \otimes \text{Hom}_C(X_0, X_n)$.

3. $\phi \in Z^n(C)$.

**Remark 5.14.** By Proposition 5.10 it is clear that there is a one to one correspondence between $n$-dimensional closed graded traces on $(\mathcal{S}_C, \partial)$ and $Z^n(C)$.

### 6 Linearization by matrices and Hopf-cyclic cohomology

We continue with $M$ being a right-left SAYD module over $H$. In Section 5 we described the spaces $Z^*_H(\mathcal{D}_H, M)$ and $Z^*(\mathcal{D}_H)$. The next aim is to find a characterization of $B^*_H(\mathcal{D}_H, M)$ and $B^*(\mathcal{D}_H)$ which will be done in several steps. For this, we will show in this section that the Hopf-cyclic cohomology of an $H$-category $\mathcal{D}_H$ is the same as that of its linearization $\mathcal{D}_H \otimes M_r(k)$. We denote by $\text{Cat}_H$ the category whose objects are left $H$-categories and whose morphisms are $H$-linear semifunctors.

Let $C$ be a $k$-linear category and $r \in \mathbb{N}$. Then, its linearization $C \otimes M_r(k)$ is the $k$-linear category defined as follows:

\[
\text{Ob}(C \otimes M_r(k)) := \text{Ob}(C) \\
\text{Hom}_{C \otimes M_r(k)}(X, Y) := \text{Hom}_C(X, Y) \otimes M_r(k)
\]

for any $X, Y \in \text{Ob}(C \otimes M_r(k))$.

**Lemma 6.1.** Let $\mathcal{D}_H$ be a left $H$-category and let $r \in \mathbb{N}$. Then, $\mathcal{D}_H \otimes M_r(k)$ is also a left $H$-category.

**Proof.** This follows easily by defining the $H$-action on each $\text{Hom}_{\mathcal{D}_H \otimes M_r(k)}(X, Y)$ by setting $h(f \otimes B) := hf \otimes B$ for a morphism $f \otimes B$ in $\mathcal{D}_H \otimes M_r(k)$.

**Proposition 6.2.** Let $\mathcal{D}_H$ be a left $H$-category and let $M$ be a right-left SAYD module. Then:

1. We obtain a para-cyclic module $C_\bullet(\mathcal{D}_H, M) := \{C_n(\mathcal{D}_H, M) := M \otimes CN_n(\mathcal{D}_H)\}_{n \geq 0}$ with the following structure maps

   \[
   d_i(m \otimes f^0 \otimes \ldots \otimes f^n) = \begin{cases} 
   m \otimes f^0 \otimes f^1 \otimes \ldots \otimes f^{i+1} \otimes \ldots \otimes f^n & 0 \leq i < n-1 \\
   m_{(0)} \otimes (S^{-1}(m_{(-1)}))f^n & i = n \end{cases} \\
   0 \leq i \leq n-1
   
   \]

   \[
   s_i(m \otimes f^0 \otimes \ldots \otimes f^n) = \begin{cases} 
   m \otimes f^0 \otimes f^1 \otimes \ldots \otimes f^{i+1} \otimes \ldots \otimes f^n & 0 \leq i < n-1 \\
   m \otimes f^0 \otimes f^1 \otimes \ldots \otimes f^n & i = n \end{cases} \\
   0 \leq i \leq n-1
   
   \]

   \[
   t_n(m \otimes f^0 \otimes \ldots \otimes f^n) = m_{(0)} \otimes S^{-1}(m_{(-1)}))f^n \otimes f^0 \otimes \ldots \otimes f^{n-1}
   
   \]

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for any $m \in M$ and $f^0 \otimes f^1 \otimes \ldots \otimes f^n \in \text{Hom}_H(X_1,X_0) \otimes \text{Hom}_H(X_2,X_1) \otimes \ldots \otimes \text{Hom}_H(X_0,X_n)$.

(2) By passing to the tensor product over $H$, we obtain a cyclic module $C^H_\bullet (\mathcal{D}_H, M) := \{ C^H_n(\mathcal{D}_H, M) = M \otimes H \text{CN}_n(\mathcal{D}_H) \}_{n \geq 0}$.

**Proof.** The proof of (1) follows as in Proposition [5.4](#5.4) (1). To prove (2), we first verify that the cyclic operator $t_n$ is well-defined. For any $m \otimes H f^0 \otimes \ldots \otimes f^n \in C^H_n(\mathcal{D}_H, M)$, we have

$$t_n (m h \otimes H f^0 \otimes \ldots \otimes f^n) = (m h)(0) \otimes H S^{-1}(m h)(-1) f^n \otimes f^0 \otimes \ldots \otimes f^{n-1}$$

$$= m(0) h_2 \otimes H S^{-1}(S(h_3)m(-1) h_1) f^n \otimes f^0 \otimes \ldots \otimes f^{n-1}$$

(by [6.3](#6.3))

It may be verified easily that the face maps and the degeneracies are also well-defined. Moreover,

$$t_{n+1}^n (m \otimes H f^0 \otimes \ldots \otimes f^n) = m(0) \otimes H S^{-1}(m(-1)) f^n \otimes \ldots \otimes f^n = m(0) S^{-1}(m(-1)) \otimes H f^0 \otimes \ldots \otimes f^n$$

(6.3)

The cyclic homology groups corresponding to the cyclic module $C^H_\bullet (\mathcal{D}_H, M)$ will be denoted by $HC^H_\bullet (\mathcal{D}_H, M)$. We fix $r \geq 1$. For $1 \leq i, j \leq r$ and $\alpha \in k$, we let $E_{ij}(\alpha)$ denote the elementary matrix in $M_r(k)$ having $\alpha$ at $(i,j)$-th position and 0 everywhere else. We will often use $E_{ij}$ for $E_{ij}(1)$. Let $\mathcal{D}_H$ be a left $H$-category. For each $1 \leq p \leq r$, we have an inclusion $\text{inc}_p : \mathcal{D}_H \rightarrow \mathcal{D}_H \otimes M_r(k)$ in $\text{CAlg}_H$ given by

$$\text{inc}_p(X) = X \quad \text{inc}_p(f) = f \otimes E_{pp} = f \otimes E_{pp}(1)$$

For any right-left SAYD-module $M$, the inclusion $\text{inc}_p : \mathcal{D}_H \rightarrow \mathcal{D}_H \otimes M_r(k)$ induces an inclusion map

$$(\text{inc}_p, M) : M \otimes \text{CN}_n(\mathcal{D}_H) \rightarrow M \otimes \text{CN}_n(\mathcal{D}_H \otimes M_r(k))$$

$m \otimes f^0 \otimes \ldots \otimes f^n \mapsto m \otimes (f^0 \otimes E_{pp}) \otimes \ldots \otimes (f^n \otimes E_{pp})$

This induces a morphism of Hochschild complexes

$$C_\bullet (\text{inc}_p, M)^{\text{hoc}} : C_\bullet (\mathcal{D}_H, M)^{\text{hoc}} \rightarrow C_\bullet (\mathcal{D}_H \otimes M_r(k), M)^{\text{hoc}}$$

(6.1)

as well as a morphism of double complexes computing cyclic homology

$$C_\bullet (\text{inc}_p, M)^{\text{cy}} : C_\bullet (\mathcal{D}_H, M)^{\text{cy}} \rightarrow C_\bullet (\mathcal{D}_H \otimes M_r(k), M)^{\text{cy}}$$

(6.2)

**Definition 6.3.** Let $\mathcal{D}_H$ be a left $H$-category, $M$ be a right-left SAYD module over $H$ and let $r \in \mathbb{N}$. Then, for each $n \geq 0$, we define an $H$-linear trace map

$$\text{tr}^M : M \otimes \text{CN}_n(\mathcal{D}_H \otimes M_r(k)) \rightarrow M \otimes \text{CN}_n(\mathcal{D}_H)$$

$$\text{tr}^M (m \otimes (f^0 \otimes B^0) \otimes \ldots \otimes (f^n \otimes B^n)) := (m \otimes f^0 \otimes \ldots \otimes f^n) \text{trace}(B^0 \ldots B^n)$$

(6.3)

for any $m \in M$ and $(f^0 \otimes B^0) \otimes \ldots \otimes (f^n \otimes B^n) \in \text{CN}_n(\mathcal{D}_H \otimes M_r(k))$.

**Lemma 6.4.** Let $\mathcal{D}_H$ be a left $H$-category, $M$ be a right-left SAYD module and let $r \in \mathbb{N}$. Then, the trace map as in (6.3) defines a morphism $C_\bullet (\text{tr}^M) : C_\bullet (\mathcal{D}_H \otimes M_r(k), M) \rightarrow C_\bullet (\mathcal{D}_H, M)$ of para-cyclic modules. In particular, we have an induced morphism between underlying Hochschild complexes

$$C_\bullet (\text{tr}^M)^{\text{hoc}} : C_\bullet (\mathcal{D}_H \otimes M_r(k), M)^{\text{hoc}} \rightarrow C_\bullet (\mathcal{D}_H, M)^{\text{hoc}}$$
Moreover,

We now verify that \( C \circ (tr^M) \circ C = \text{id} \). For any \( m \otimes f^0 \otimes \ldots \otimes f^n \in M \otimes \text{Hom}_D(X_1, X_0) \otimes \text{Hom}_D(X_2, X_1) \otimes \ldots \otimes \text{Hom}_D(X_0, X_n) \), we have

\[
(C \circ (tr^M) \circ C)(m \otimes f^0 \otimes \ldots \otimes f^n) = tr^M(m \otimes (f^0 \otimes E_{11}) \otimes \ldots \otimes (f^n \otimes E_{11})) = (m \otimes f^0 \otimes \ldots \otimes f^n) \text{trace}(E_{11} \ldots E_{11}) = m \otimes f^0 \otimes \ldots \otimes f^n
\]

Therefore, it remains to show that \( C \circ (tr^M) \circ C = \text{id} \). We define \( k \)-linear maps \( h_i : C_n(D_H \otimes M_r(k), M) \longrightarrow C_{n+1}(D_H \otimes M_r(k), M) \) by setting:

\[
h_i(m \otimes (f^0 \otimes B^0) \otimes \ldots \otimes (f^n \otimes B^n)) := m \otimes \sum_{i=0}^n B_i \otimes \ldots \otimes (f^i \otimes E_{11}(B^i_{11}) \otimes \ldots \otimes (f^n \otimes E_{11}(B^i_{n})) \otimes \text{id}_{X_{n+1}} \otimes E_{11}(1)) \otimes (f^i+1 \otimes B^{i+1}) \otimes \ldots \otimes (f^n \otimes B^n)
\]

for \( 0 \leq i < n \) and

\[
h_n(m \otimes (f^0 \otimes B^0) \otimes \ldots \otimes (f^n \otimes B^n)) := m \otimes \sum_{i=0}^n B_i \otimes \ldots \otimes (f^i \otimes E_{11}(B^i_{11}) \otimes \ldots \otimes (f^n \otimes E_{11}(B^i_{n})) \otimes \text{id}_{X_{n+1}} \otimes E_{11}(1))
\]

We now verify that \( h^i := \sum_{i=0}^n (-1)^i h_i \) is a pre-simplicial homotopy (see, for instance, [31, § 1.0.8]) between \( C \circ (inc_1, M) \circ C \circ (tr^M) = \text{id} \). For this, we need to verify the following identities:

\[
d_i h_{i'} = h_{i'-1} d_i \quad \text{for } i < i' \\
d_i h_{i'} = d_i h_{i'-1} \quad \text{for } 0 < i \leq n \\
d_i h_{i'} = h_{i'-1} d_i \quad \text{for } i > i' + 1
\]

(6.4)

where \( d_i : C_{n+1}(D_H \otimes M_r(k), M) \longrightarrow C_n(D_H \otimes M_r(k), M) \), \( 0 \leq i \leq n + 1 \) are the face maps.

For \( 0 < i < i' \), we have

\[
d_i h_{i'}(m \otimes (f^0 \otimes B^0) \otimes \ldots \otimes (f^n \otimes B^n)) = d_i(m \otimes \sum_{i=0}^n B_i \otimes \ldots \otimes (f^i \otimes E_{11}(B^i_{11}) \otimes \ldots \otimes (f^n \otimes E_{11}(B^i_{n})) \otimes \text{id}_{X_{n+1}} \otimes E_{11}(1)) \otimes (f^i+1 \otimes B^{i+1}) \otimes \ldots \otimes (f^n \otimes B^n)
\]

Moreover,

\[
d_0 h_{i'}(m \otimes (f^0 \otimes B^0) \otimes \ldots \otimes (f^n \otimes B^n)) = d_0(m \otimes \sum_{i=0}^n B_i \otimes \ldots \otimes (f^i \otimes E_{11}(B^i_{11}) \otimes \ldots \otimes (f^n \otimes E_{11}(B^i_{n})) \otimes \text{id}_{X_{n+1}} \otimes E_{11}(1)) \otimes (f^i+1 \otimes B^{i+1}) \otimes \ldots \otimes (f^n \otimes B^n)
\]

(6.4)
Next, using the equality \( \sum_{i=1}^{r} E_{11}(B_{kl})E_{12}(1) = E_{1k}(1)B \) for all \( B \in \mathcal{M}_e(k) \), \( 1 \leq k \leq r \), we have for \( 0 < i < n \):

\[
d_h \left( m \otimes (f^0 \otimes B^0) \otimes \ldots \otimes (f^n \otimes B^n) \right) \\
= d \left( m \otimes \sum_{i,j,k,l,...,p,q,r} f^0 \otimes E_{1j}(B_{jk}^0) \otimes (f^1 \otimes E_{11}(B_{kl}^1)) \otimes \ldots \otimes (f^i \otimes E_{11}(B_{pq}^i)) \otimes \right.
\left. (id_{X_{i-1}} \otimes E_{1q}(1)) \otimes (id_{X_{i+1}} \otimes E_{1q}(1)) \right) \\
= m \otimes \sum_{i,j,k,l,...,p,q,r} f^0 \otimes E_{1j}(B_{jk}^0) \otimes (f^1 \otimes E_{11}(B_{kl}^1)) \otimes \ldots \otimes (f^i \otimes E_{11}(B_{pq}^i)) \otimes \right.
\left. (id_{X_{i-1}} \otimes E_{1q}(1)) \otimes (id_{X_{i+1}} \otimes E_{1q}(1)) \right) \\
= h_i \left( m \otimes (f^0 \otimes B^0) \otimes \ldots \otimes (f^n \otimes B^n) \right) \\
= h_{i-1} \left( m \otimes (f^0 \otimes B^0) \otimes \ldots \otimes (f^n \otimes B^n) \right)
\]

The case \( i = n \) follows similarly. For \( i > i' + 1 \), we have

\[
d_{i'} h_i \left( m \otimes (f^0 \otimes B^0) \otimes \ldots \otimes (f^n \otimes B^n) \right) \\
= d_i (m \otimes \sum_{i,j,k,l,...,p,q,r} f^0 \otimes E_{1j}(B_{jk}^0) \otimes (f^1 \otimes E_{11}(B_{kl}^1)) \otimes \ldots \otimes (f^i \otimes E_{11}(B_{pq}^i)) \otimes \right.
\left. (id_{X_{i-1}} \otimes E_{1q}(1)) \otimes (id_{X_{i+1}} \otimes E_{1q}(1)) \right) \\
= m \otimes \sum_{i,j,k,l,...,p,q,r} f^0 \otimes E_{1j}(B_{jk}^0) \otimes (f^1 \otimes E_{11}(B_{kl}^1)) \otimes \ldots \otimes (f^i \otimes E_{11}(B_{pq}^i)) \otimes \right.
\left. (id_{X_{i-1}} \otimes E_{1q}(1)) \otimes (id_{X_{i+1}} \otimes E_{1q}(1)) \right) \\
= h_i \left( m \otimes (f^0 \otimes B^0) \otimes \ldots \otimes (f^n \otimes B^n) \right) \\
= h_{i-1} \left( m \otimes (f^0 \otimes B^0) \otimes \ldots \otimes (f^n \otimes B^n) \right)
\]

Further, using the equality \( \sum_{i,j,k,s} E_{1j}(B_{jk})E_{1k}(1) = B \), we have

\[
d_{i+1} h_i \left( m \otimes (f^0 \otimes B^0) \otimes \ldots \otimes (f^n \otimes B^n) \right) \\
= d_{i+1} (m \otimes \sum_{i,j,k,l,...,p,q,r} f^0 \otimes E_{1j}(B_{jk}^0) \otimes (f^1 \otimes E_{11}(B_{kl}^1)) \otimes \ldots \otimes (f^n \otimes E_{11}(B_{pq}^n)) \otimes \right.
\left. (id_{X_{0}} \otimes E_{1q}(1)) \otimes (id_{X_{1}} \otimes E_{1q}(1)) \right) \\
= m \otimes \sum_{i,j,k,l,...,p,q,r} f^0 \otimes E_{1j}(B_{jk}^0) \otimes (f^1 \otimes E_{11}(B_{kl}^1)) \otimes \ldots \otimes (f^n \otimes E_{11}(B_{pq}^n)) \otimes \right.
\left. (id_{X_{0}} \otimes E_{1q}(1)) \otimes (id_{X_{1}} \otimes E_{1q}(1)) \right) \\
= m \otimes (f^0 \otimes B^0) \otimes \ldots \otimes (f^n \otimes B^n)
\]

Finally, using the fact that \( E_{1q}(1)E_{1j}(B_{jk}) = 0 \) unless \( q = j \), we have

\[
d_{n+1} h_n \left( m \otimes (f^0 \otimes B^0) \otimes \ldots \otimes (f^n \otimes B^n) \right) \\
= d_{n+1} (m \otimes \sum_{i,j,k,l,...,p,q,r} f^0 \otimes E_{1j}(B_{jk}^0) \otimes (f^1 \otimes E_{11}(B_{kl}^1)) \otimes \ldots \otimes (f^n \otimes E_{11}(B_{pq}^n)) \otimes \right.
\left. (id_{X_{0}} \otimes E_{1q}(1)) \otimes (id_{X_{1}} \otimes E_{1q}(1)) \right) \\
= m \otimes \sum_{i,j,k,l,...,p,q,r} f^0 \otimes E_{1j}(B_{jk}^0) \otimes (f^1 \otimes E_{11}(B_{kl}^1)) \otimes \ldots \otimes (f^n \otimes E_{11}(B_{pq}^n)) \otimes \right.
\left. (id_{X_{0}} \otimes E_{1q}(1)) \otimes (id_{X_{1}} \otimes E_{1q}(1)) \right) \\
= m \otimes \sum_{i,j,k,l,...,p,q,r} f^0 \otimes E_{1j}(B_{jk}^0) \otimes (f^1 \otimes E_{11}(B_{kl}^1)) \otimes \ldots \otimes (f^n \otimes E_{11}(B_{pq}^n)) \otimes \right.
\left. (id_{X_{0}} \otimes E_{1q}(1)) \otimes (id_{X_{1}} \otimes E_{1q}(1)) \right) \\
= m \otimes (f^0 \otimes E_{11}(B_{kl}^1)) \otimes \ldots \otimes (f^n \otimes E_{11}(B_{pq}^n)) \otimes \right.
\left. (id_{X_{0}} \otimes E_{1q}(1)) \otimes (id_{X_{1}} \otimes E_{1q}(1)) \right) \\
= \left( C \ast (inc_1, M) \ast C \ast (tr^M) \ast (m \otimes (f^0 \otimes B^0) \otimes \ldots \otimes (f^n \otimes B^n)) \right)
\]

This proves the result.

We denote by \( Vect_k \) the category of all \( k \)-vector spaces and by \( H-Mod \) the category of all left \( H \)-modules. Let \( \text{Hom}_H (-, k) : H-Mod \to Vect_k \) be the functor that takes \( N \mapsto \text{Hom}_H(N, k) \).
Proposition 6.6. (1) Applying the functor \( \text{Hom}_H(-,k) \), we obtain morphisms of Hochschild complexes

\[
C^\bullet_H(\text{inc}_1,M)^{\text{hoc}}: C^\bullet_H(D_H \otimes M_r(k), M)^{\text{hoc}} \rightarrow C^\bullet_H(D_H, M)^{\text{hoc}}
\]

as well as a morphism of double complexes computing cyclic cohomology

\[
C^\bullet_H(\text{inc}_1,M)^{\text{cy}}: C^\bullet_H(D_H \otimes M_r(k), M)^{\text{cy}} \rightarrow C^\bullet_H(D_H, M)^{\text{cy}}
\]

(2) Applying the functor \( \text{Hom}_H(-,k) \), we obtain morphisms of cocyclic modules:

\[
C^\bullet_H(\text{tr}^M) := \text{Hom}_H(C^\bullet(\text{tr}^M), k): C^\bullet_H(D_H, M) \rightarrow C^\bullet_H(D_H \otimes M_r(k), M)
\]

(3) The morphisms

\[
HC^\bullet_H(\text{inc}_1,M)^{\text{hoc}}: HC^\bullet_H(D_H \otimes M_r(k), M)^{\text{hoc}} \rightarrow HC^\bullet_H(D_H, M)^{\text{hoc}}
\]

\[
HC^\bullet_H(\text{tr}^M)^{\text{hoc}}: HC^\bullet_H(D_H, M)^{\text{hoc}} \rightarrow HC^\bullet_H(D_H \otimes M_r(k), M)^{\text{hoc}}
\]

induced by \( C^\bullet_H(\text{inc}_1,M)^{\text{hoc}} \) and \( C^\bullet_H(\text{tr}^M)^{\text{hoc}} \) are mutually inverse isomorphisms of Hochschild cohomologies. (4) Applying the functor \( \text{Hom}_H(-,k) \) gives isomorphisms

\[
HC^\bullet_H(D_H, M) \xrightarrow{HC^\bullet_H(\text{tr}^M)} HC^\bullet_H(\text{inc}_1,M)^{\text{hoc}} \xrightarrow{HC^\bullet_H(\text{tr}^M)^{\text{hoc}}} HC^\bullet_H(D_H \otimes M_r(k), M)
\]

of Hopf-cyclic cohomologies.

Proof. (1) This follows by applying the functor \( \text{Hom}_H(-,k) \) to the morphisms in (1) and (2).

(2) This follows from Proposition 6.6.5. (1), Proposition 6.6.6. and the fact that \( C^\bullet(\text{tr}^M): C^\bullet(D_H \otimes M_r(k), M) \rightarrow C^\bullet(D_H, M) \) is a morphism of para-cyclic modules.

(3) By Proposition 6.6.5. we know that \( C^\bullet(\text{tr}^M)^{\text{hoc}} \circ C^\bullet(\text{inc}_1,M)^{\text{hoc}} = \text{id}_{C^\bullet(D_H,M)^{\text{hoc}}} \) and \( C^\bullet(\text{inc}_1,M)^{\text{hoc}} \circ C^\bullet(\text{tr}^M)^{\text{hoc}} \sim \text{id}_{C^\bullet(D_H \otimes M_r(k), M)^{\text{hoc}}} \). Thus, applying the functor \( \text{Hom}_H(-,k) \), we obtain

\[
C^\bullet_H(\text{inc}_1,M)^{\text{hoc}} \circ C^\bullet_H(\text{tr}^M)^{\text{hoc}} = \text{id}_{C^\bullet_H(D_H,M)^{\text{hoc}}}
\]

\[
C^\bullet_H(\text{tr}^M)^{\text{hoc}} \circ C^\bullet_H(\text{inc}_1,M)^{\text{hoc}} \sim \text{id}_{C^\bullet_H(D_H \otimes M_r(k), M)^{\text{hoc}}}
\]

Therefore, \( C^\bullet_H(\text{inc}_1,M)^{\text{hoc}} \) and \( C^\bullet_H(\text{tr}^M)^{\text{hoc}} \) are homotopy inverses of each other.

(4) This follows immediately from (3) and Hochschild to cyclic spectral sequence.

Corollary 6.7. Given a small \( k \)-linear category \( \mathcal{C} \), there is an isomorphism \( HC^\bullet(\mathcal{C}) \xrightarrow{\sim} HC^\bullet(\mathcal{C} \otimes M_r(k)) \) of cyclic cohomology groups.

Proof. This follows by taking \( H = k = M \) in Proposition 6.6. (4).

Corollary 6.8. For an \( n \)-cocycle \( \phi \in Z^n_H(D_H, M) \), the \( n \)-cocycle \( \tilde{\phi} = Hom_H(\text{tr}^M,k)(\phi) = \phi \circ \text{tr}^M \in Z^n_H(D_H \otimes M_r(k), M) \) may be described as follows

\[
\tilde{\phi}(m \otimes (f^0 \otimes B^0) \otimes \ldots \otimes (f^n \otimes B^n)) = \phi(m \otimes f^0 \otimes \ldots \otimes f^n)\text{trace}(B^0 \ldots B^n)
\]

7 Vanishing cycles on an \( H \)-category and coboundaries

From now onwards, we will always assume that \( k = \mathbb{C} \). In this section, we will describe the spaces \( B^\bullet_H(D_H, M) \) and \( B^\bullet(D_H) \). This will be done using the isomorphism \( HC^\bullet_H(D_H, M) \xrightarrow{\sim} HC^\bullet_H(D_H \otimes M_r(k), M) \) proved in Section 6. Finally, we will use the formalism of categorified cycles and vanishing cycles developed in this paper to obtain a pairing on the cyclic cohomology of a \( k \)-linear category.

We begin by recalling the notion of an inner automorphism of a category.
Definition 7.1. [see [11], p 24] Let $\mathcal{D}_H$ be a left $H$-category. An automorphism $\Phi \in \text{Hom}_{\text{Cat}_H}(\mathcal{D}_H, \mathcal{D}_H)$ is said to be inner if $\Phi$ is isomorphic to the identity functor $\text{id}_{\mathcal{D}_H}$. In particular, there exist isomorphisms \( \{\eta(X) : X \to \Phi(X)\}_{X \in \text{Ob}(\mathcal{D}_H)} \) such that $\Phi(f) = \eta(Y) \circ f \circ (\eta(X))^{-1}$ for any $f \in \text{Hom}_{\mathcal{D}_H}(X, Y)$.

We now set
\[
\mathbb{G}(\mathcal{D}_H) := \prod_{X \in \text{Ob}(\mathcal{D}_H)} \text{Aut}_{\mathcal{D}_H}(X) \tag{7.1}
\]
By definition, an element $\eta \in \mathbb{G}(\mathcal{D}_H)$ corresponds to a family of automorphisms \( \{\eta(X) : X \to X\}_{X \in \text{Ob}(\mathcal{D}_H)} \).

We now set
\[
\mathbb{U}_H(\mathcal{D}_H) := \{\eta \in \mathbb{G}(\mathcal{D}_H) \mid h(\eta(X)) = \varepsilon(h)\eta(X) \text{ for every } h \in H \text{ and } X \in \text{Ob}(\mathcal{D}_H)\} \tag{7.2}
\]

Lemma 7.2. $\mathbb{U}_H(\mathcal{D}_H)$ is a subgroup of $\mathbb{G}(\mathcal{D}_H)$.

Proof. The element $e = \prod_{X \in \text{Ob}(\mathcal{D}_H)} \text{id}_X$ is the identity of the group $\mathbb{G}(\mathcal{D}_H)$. By definition of an $H$-category, we know that $h \cdot \text{id}_X = \varepsilon(h) \cdot \text{id}_X$ for each $X \in \text{Ob}(\mathcal{D}_H)$ and $h \in H$. Thus, $e \in \mathbb{U}_H(\mathcal{D}_H)$. Now, suppose that $\eta, \eta' \in \mathbb{U}_H(\mathcal{D}_H)$. Then, for each $X \in \text{Ob}(\mathcal{D}_H)$ and $h \in H$,

\[
h((\eta \circ \eta')(X)) = h(\eta(X) \circ \eta'(X)) = (h_1 \eta(X)) \circ (h_2 \eta'(X)) = \varepsilon(h_1)(\eta(X)) \circ \varepsilon(h_2)(\eta'(X)) = \varepsilon(h)(\eta(X) \circ \eta'(X))
\]

Hence, $\eta \circ \eta' \in \mathbb{U}_H(\mathcal{D}_H)$.

Now, let $\eta \in \mathbb{U}_H(\mathcal{D}_H)$. Then, $\eta^{-1} \in \mathbb{G}(\mathcal{D}_H)$ corresponds to a family of morphisms \( \{\eta^{-1}(X) := \eta(X)^{-1} : X \to X\}_{X \in \text{Ob}(\mathcal{D}_H)} \). Then, for each $h \in H$ and $X \in \text{Ob}(\mathcal{D}_H)$,

\[
\varepsilon(h)\eta(X) = h(\eta(X)^{-1}) = \varepsilon(h_1)(\eta(X)) \circ (h_2 \eta^{-1}(X)) = \eta(X) \circ (h \eta^{-1}(X))
\]

which gives $\varepsilon(h)\eta^{-1}(X) = h\eta^{-1}(X)$. Therefore, $\eta^{-1} \in \mathbb{U}_H(\mathcal{D}_H)$.

Lemma 7.3. Let $\mathcal{D}_H$ be a left $H$-category and let $\eta \in \mathbb{U}_H(\mathcal{D}_H)$.

1. Consider $\Phi_\eta : \mathcal{D}_H \to \mathcal{D}_H$ defined by

\[
\Phi_\eta(X) = X \quad \Phi_\eta(f) := \eta(Y) \circ f \circ \eta(X)^{-1}
\]

for every $X \in \text{Ob}(\mathcal{D}_H)$ and $f \in \text{Hom}_{\mathcal{D}_H}(X, Y)$. Then, $\Phi_\eta : \mathcal{D}_H \to \mathcal{D}_H$ is an inner automorphism of $\mathcal{D}_H$.

2. Consider $\hat{\Phi}_\eta : \mathcal{D}_H \otimes M_2(k) \to \mathcal{D}_H \otimes M_2(k)$ defined by

\[
\hat{\Phi}_\eta(X) = X \quad \hat{\Phi}_\eta(f \otimes B) = (id_Y \otimes E_{11} + \eta(Y) \otimes E_{22}) \circ (f \otimes B) \circ (id_X \otimes E_{11} + \eta(X)^{-1} \otimes E_{22})
\]

for every $X \in \text{Ob}(\mathcal{D}_H \otimes M_2(k)) = \text{Ob}(\mathcal{D}_H)$ and $f \otimes B \in \text{Hom}_{\mathcal{D}_H \otimes M_2(k)}(X, Y)$. Then, $\hat{\Phi}_\eta : \mathcal{D}_H \otimes M_2(k) \to \mathcal{D}_H \otimes M_2(k)$ is an inner automorphism.

Proof. (1) Using the fact that $\eta, \eta^{-1} \in \mathbb{U}_H(\mathcal{D}_H)$, we have

\[
h(\Phi_\eta(f)) = (h_1 \eta(Y)) \circ (h_2 f) \circ (h_3 \eta(X)^{-1}) = \varepsilon(h_1)(\eta(Y)) \circ \varepsilon(h_2)(\eta(X)^{-1}) = \eta(Y) \circ (h f) \circ \eta(X)^{-1}
\]

for any $h \in H$ and $f \in \text{Hom}_{\mathcal{D}_H}(X, Y)$. By Definition 7.1 we now see that $\Phi_\eta$ is an inner automorphism.

(2) Setting $\tilde{\eta}(X) : X \to X$ in $\mathcal{D}_H \otimes M_2(k)$ as $\tilde{\eta}(X) = id_X \otimes E_{11} + \eta(X) \otimes E_{22}$, we see that

\[
\hat{\Phi}_\eta(f \otimes B) = (id_Y \otimes E_{11} + \eta(Y) \otimes E_{22}) \circ (f \otimes B) \circ (id_X \otimes E_{11} + \eta(X)^{-1} \otimes E_{22}) = \tilde{\eta}(Y) \circ (f \otimes B) \circ \tilde{\eta}(X)^{-1}
\]
for any \( f \otimes B \in \text{Hom}_{\mathcal{D}_H \otimes M_2(k)}(X, Y) \). Considering the \( H \)-action on the category \( \mathcal{D}_H \otimes M_2(k) \), we have

\[
\begin{align*}
    h(\Phi_\eta((f \otimes B))) &= h_1(id_Y \otimes E_{11} + \eta(Y) \otimes E_{22}) \circ h_2(f \otimes B) \circ h_3(id_X \otimes E_{11} + \eta(X)^{-1} \otimes E_{22}) \\
    &= (h_3(id_Y \otimes E_{11} + \eta(Y) \otimes E_{22}) \circ h_2(f \otimes B) \circ h_3(id_X \otimes E_{11} + \eta(X)^{-1} \otimes E_{22})) \\
    &= (id_Y \otimes E_{11} + \eta(Y) \otimes E_{22}) \circ \epsilon_1(h_3(id_X \otimes E_{11} + \eta(X)^{-1} \otimes E_{22})) \\
    &= (id_Y \otimes E_{11} + \eta(Y) \otimes E_{22}) \circ \Phi_\eta(h(f \otimes B)) \circ (id_X \otimes E_{11} + \eta(X)^{-1} \otimes E_{22}) \\
    &= \Phi_\eta(h(f \otimes B)) 
\end{align*}
\]

for any \( h \in H \) and \( f \otimes B \in \text{Hom}_{\mathcal{D}_H \otimes M_2(k)}(X, Y) \). By Definition 7.4, we now see that \( \Phi_\eta \) is an inner automorphism.

For \( \eta \in \mathcal{U}_H(\mathcal{D}_H) \), we will always denote by \( \Phi_\eta \) and \( \tilde{\Phi}_\eta \) the inner automorphisms defined in Lemma 7.3

**Lemma 7.4.** Let \( M \) be a right-left SAYD module over \( H \). Then,

1. A semifunctor \( \alpha \in \text{Hom}_{\text{Cat}_H}(\mathcal{D}_H, \mathcal{D}_H') \) induces a morphism (for all \( n \geq 0 \))

\[
C^\alpha_H(\alpha, M) : C^n_H(\mathcal{D}_H', M) = \text{Hom}_H(M \otimes C_n(\mathcal{D}_H'), k) \rightarrow C^n_H(\mathcal{D}_H, M) = \text{Hom}_H(M \otimes C_n(\mathcal{D}_H), k)
\]
determined by

\[
C^n_H(\alpha, M)(\phi)(m \otimes f^0 \cdots \otimes f^n) = \phi(m \otimes \alpha(f^0) \cdots \otimes \alpha(f^n))
\]

for any \( \phi \in C^n_H(\mathcal{D}_H', M) \), \( m \in M \) and \( f^0 \otimes \cdots \otimes f^n \in C_n(\mathcal{D}_H) \). This leads to a morphism \( C^*_H(\alpha, M)n : C^*_H(\mathcal{D}_H', M)n \rightarrow C^*_H(\mathcal{D}_H, M)n \) of double complexes and induces a functor \( HC^*_H(-, M) : \text{Cat}_H \rightarrow \text{Vect}_k \).

2. Let \( \eta \in \mathcal{U}_H(\mathcal{D}_H) \). Then, \( \Phi_\eta \) induces the identity map on \( HC^*_H(\mathcal{D}_H, M) \).

**Proof.** (1) Since \( \phi \) and \( \alpha \) are \( H \)-linear, the morphisms \( C^n_H(\alpha, M) \) are well-defined and well behaved with respect to the maps appearing in the Hochschild and cyclic complexes. The result follows.

(2) Let \( \eta \in \mathcal{U}_H(\mathcal{D}_H) \) and \( \Phi_\eta \in \text{Hom}_{\text{Cat}_H}(\mathcal{D}_H, \mathcal{D}_H) \) be the corresponding inner automorphism. By Proposition 6.6, the maps \( HC^*_H(inc_1, M) \) and \( HC^*_H(tr^M) \) are mutually inverse isomorphisms of Hopf-cyclic cohomology groups. Thus, we have

\[
HC^*_H(inc_2, M) \circ (HC^*_H(inc_1, M))^{-1} = HC^*_H(inc_2, M) \circ HC^*_H(tr^M) = HC^*_H(tr^M \circ (inc_2, M)) = id \tag{7.3}
\]

Further, we have the following commutative diagram in the category \( \text{Cat}_H \):

\[
\begin{array}{ccc}
\mathcal{D}_H & \xrightarrow{inc_1} & \mathcal{D}_H \otimes M_2(k) & \xleftarrow{inc_2} & \mathcal{D}_H \\
\text{id}_{\mathcal{D}_H} & & \downarrow \Phi_\eta & & \downarrow \Phi_\eta \\
\mathcal{D}_H & \xrightarrow{inc_1} & \mathcal{D}_H \otimes M_2(k) & \xleftarrow{inc_2} & \mathcal{D}_H \\
\end{array}
\tag{7.4}
\]

Thus, by applying the functor \( HC^*_H(-, M) \) to the commutative diagram (7.4) and using (7.3), we obtain

\[
HC^*_H(\Phi_\eta, M) = (HC^*_H(inc_2, M)) \circ HC^*_H(inc_1, M)^{-1} \circ HC^*_H(id_{\mathcal{D}_H}, M) \circ (HC^*_H(inc_1, M)) \circ HC^*_H(inc_2, M)^{-1} = id_{HC^*_H(\mathcal{D}_H, M)}
\]

\[
\square
\]

**Proposition 7.5.** Let \( \mathcal{D}_H \) be a left \( H \)-category. Suppose that there is a semifunctor \( \nu \in \text{Hom}_{\text{Cat}_H}(\mathcal{D}_H, \mathcal{D}_H) \) and an \( \eta \in \mathcal{U}_H(\mathcal{D}_H \otimes M_2(k)) \) such that

1. \( \nu(X) = X \ \ \forall X \in \text{Ob}(\mathcal{D}_H) \)

2. \( \Phi_\eta(f \otimes E_{11} + \nu(f) \otimes E_{22}) = \nu(f) \otimes E_{22} \)
Lemma 7.8.\hspace{1em} We now recall from \cite[p.103]{12} the algebra $C_k$ so that $\tilde{f}$ and an
\begin{equation}
\alpha(X) := X \quad \alpha(f) := f \otimes E_{11} + \nu(f) \otimes E_{22}
\end{equation}
for all $X \in \text{Ob}(D_H)$ and $f \in \text{Hom}_{\text{Cat}_H}(D_H, D_H \otimes M_2(k))$ be the semifunctors defined by

\begin{equation}
\alpha(X) := X \quad \alpha(f) := f \otimes E_{11} + \nu(f) \otimes E_{22}
\end{equation}

Proof.\hspace{1em}Let $\alpha, \alpha' \in \text{Hom}_{\text{Cat}_H}(D_H, D_H \otimes M_2(k))$ be the semifunctors defined by

\begin{equation}
\alpha(X) := X \quad \alpha(f) := f \otimes E_{11} + \nu(f) \otimes E_{22}
\end{equation}

for all $X \in \text{Ob}(D_H)$ and $f \in \text{Hom}_{\text{Cat}_H}(D_H, D_H \otimes M_2(k))$. Then, $HC^*_H(D_H, M) = 0$.

$\text{HC}_H^*(\alpha', M) = HC^*_H(\alpha, M) \circ HC^*_H(\Phi_\eta, M) = HC^*_H(\alpha, M) : HC^*_H(D_H \otimes M_2(k), M) \to HC^*_H(D_H, M)$ (7.5)

Let $\tilde{\phi} \in Z^0_H(D_H, M)$ and $\hat{\phi} = \text{Hom}_H(\text{tr}^M, k)(\phi) = \phi \circ \text{tr}^M \in Z^0_H(D_H \otimes M_2(k), M)$ as in Corollary 6.8.\hspace{1em}Let $[\hat{\phi}]$ denote the cohomology class of $\hat{\phi}$. Then, by (7.5), we have $HC^*_H(\alpha, M)([\hat{\phi}]) = HC^*_H(\alpha', M)([\hat{\phi}])$, i.e.,

\begin{equation}
\hat{\phi} \circ (id_M \otimes CN_n(\alpha)) + B^n_H(D_H, M) = \hat{\phi} \circ (id_M \otimes CN_n(\alpha')) + B^n_H(D_H, M)
\end{equation}

so that $\hat{\phi} \circ (id_M \otimes CN_n(\alpha)) - \hat{\phi} \circ (id_M \otimes CN_n(\alpha')) \in B^n_H(D_H, M)$. Applying the definition of $\hat{\phi}$, we now have

\begin{align*}
(\hat{\phi} \circ (id_M \otimes CN_n(\alpha)))(m \otimes f^0 \otimes \ldots \otimes f^n) \\
= \hat{\phi}(m \otimes (f^0 \otimes E_{11} + \nu(f^0) \otimes E_{22}) \otimes \ldots \otimes (f^n \otimes E_{11} + \nu(f^n) \otimes E_{22})) \\
= \phi(m \otimes f^0 \otimes \ldots \otimes f^n) + \phi(m \otimes \nu(f^0) \otimes \ldots \otimes \nu(f^n))
\end{align*}

Similarly, $(\hat{\phi} \circ (id_M \otimes CN_n(\alpha')))(m \otimes f^0 \otimes \ldots \otimes f^n) = \phi(m \otimes (f^0) \otimes \ldots \otimes \nu(f^n))$. Thus, $\phi = \hat{\phi} \circ (id_M \otimes CN_n(\alpha)) - \hat{\phi} \circ (id_M \otimes CN_n(\alpha')) \in B^n_H(D_H, M)$. This proves the result. \hfill \Box

In particular, substituting $M = k = H$ in Proposition 7.5, we obtain the following result:

Corollary 7.6.\hspace{1em}Let $C$ be a small $k$-linear category. Suppose that there is a $k$-linear semifunctor $\nu : C \to C$ and an $\eta \in U(C \otimes M_2(k))$ such that

1. $\nu(X) = X$\hspace{1em}$\forall X \in \text{Ob}(C)$

2. $\Phi_\eta(f \otimes E_{11} + \nu(f) \otimes E_{22}) = \nu(f) \otimes E_{22}$

for all $f \in \text{Hom}_C(X, Y)$ and $X, Y \in \text{Ob}(C)$. Then, $HC^*_H(C) = 0$.

Definition 7.7.\hspace{1em}Let $(S_H, \hat{\delta}_H, \hat{T}_H)$ be an $n$-dimensional $S_H$-cycle with coefficients in a SAYD module $M$ over $H$ (see, Definition 5.9).\hspace{1em}Then, we say that the cycle $(S_H, \hat{\delta}_H, \hat{T}_H)$ is vanishing if $S^0_H$ is a left $H$-category and $S^0_H$ satisfies the assumptions in Proposition 7.5.

By taking $H = k = M$ in Definition 7.7, we obtain the notion of a vanishing $S$-cycle $(S, \hat{\delta}, \hat{T})$ associated to a $k$-linear DG-semi-category $S$.

We now recall from \cite[p.103]{12} the algebra $C$ of infinite matrices $(a_{ij})_{i,j \in \mathbb{N}}$ with entries from $\mathbb{C}$ satisfying the following conditions (see also 23)

(i) the set $\{a_{ij} \mid i, j \in \mathbb{N}\}$ is finite,

(ii) the number of non-zero entries in each row or each column is bounded.

Identifying $M_2(C) = C \otimes M_2(C)$, we recall the following result from \cite[p.104]{12}:

Lemma 7.8.\hspace{1em}There exists an algebra homomorphism $\omega : C \to C$ and an invertible element $\hat{U} \in M_2(C)$ such that the corresponding inner automorphism $\Xi : M_2(C) \to M_2(C)$ satisfies

\begin{equation}
\Xi(B \otimes E_{11} + \omega(B) \otimes E_{22}) = \omega(B) \otimes E_{22} \quad \forall B \in C
\end{equation}

Then, $HC^*_H(C) = 0.$
Remark 7.9. We note that the condition in (7.7) ensures that $\omega(1) \neq 1$, where $1$ is the unit element of $C$.

For any $k$-algebra $A$, we may define a $k$-linear category $A \otimes D_H$ as follows:

$$\text{Ob}(A \otimes D_H) = \text{Ob}(D_H) \quad \text{Hom}_{A \otimes D_H}(X,Y) = A \otimes \text{Hom}_{D_H}(X,Y)$$

The category $A \otimes D_H$ is a left $H$-category via the action $h(a \otimes f) := a \otimes hf$ for any $h \in H$, $a \otimes f \in A \otimes \text{Hom}_{D_H}(X,Y)$.

**Lemma 7.10.** We have $HC_H^*(C \otimes D_H, M) = 0$.

**Proof.** We will verify that the category $C \otimes D_H$ satisfies the assumptions of Proposition 7.8. Let $\omega$ and $\tilde{U}$ be as in Lemma 7.8. We now define $\nu : C \otimes D_H \to C \otimes D_H$ given by

$$\nu(X) := X \quad \nu(B \otimes f) := \omega(B) \otimes f$$

for any $X \in \text{Ob}(C \otimes D_H)$ and $B \otimes f \in \text{Hom}_{C \otimes D_H}(X, Y)$. Since $\omega : C \to C$ is an algebra homomorphism, it follows that $\nu$ is a semifunctor. By the definition of the $H$-action on $C \otimes D_H$, it is also clear that $\nu$ is $H$-linear.

Using the identification $C \otimes D_H \otimes M_2(C) = M_2(C) \otimes D_H$, we now define an element $\eta \in \mathbb{G}(C \otimes D_H \otimes M_2(C)) = \mathbb{G}(M_2(C) \otimes D_H)$ given by the family of morphisms

$$(\eta(X)) := \tilde{U} \otimes id_X \in \text{Hom}_{M_2(C) \otimes D_H}(X, X) = M_2(C) \otimes \text{Hom}_{D_H}(X, X) \quad (7.8)$$

Since $\tilde{U}$ is a unit in $M_2(C)$, it follows that each $\eta(X)$ in (7.8) is an automorphism. Since $H$ acts trivially on $M_2(C)$, we see that $\eta \in \mathbb{U}_H(C \otimes D_H \otimes M_2(C))$. Moreover, for any $B \otimes f \in \text{Hom}_{M_2(C) \otimes D_H}(X, Y) = M_2(C) \otimes \text{Hom}_{D_H}(X, Y)$, we have

$$\Phi_\eta(\tilde{B} \otimes f) = \eta(Y) \circ (\tilde{B} \otimes f) \circ \eta(X)^{-1} = (\tilde{U} \otimes id_Y) \circ (\tilde{B} \otimes f) \circ (\tilde{U}^{-1} \otimes id_X) = \tilde{U}\tilde{B}\tilde{U}^{-1} \otimes f = \Xi(\tilde{B}) \otimes f$$

Therefore, for any $B \otimes f \in C \otimes \text{Hom}_{D_H}(X, Y)$, we have

$$\Phi_\eta((B \otimes f) \otimes E_{11} + \nu(B \otimes f) \otimes E_{22}) = \Phi_\eta(B \otimes f \otimes E_{11} + \omega(B) \otimes f \otimes E_{22}) = \Phi_\eta(B \otimes E_{11} \otimes f + \omega(B) \otimes E_{22} \otimes f) = \Xi(B \otimes E_{11} + \omega(B) \otimes E_{22}) \otimes f = \omega(B) \otimes E_{22} \otimes f = \nu(B \otimes f) \otimes E_{22}$$

This proves the result.

We now provide a useful interpretation of the space $B^*_H(D_H, M)$.

**Proposition 7.11.** An element $\phi \in C^*_H(D_H, M)$ is a coboundary iff $\phi$ is the character of an $n$-dimensional vanishing $S_H$-cycle $(S_H, \hat{\partial}^H, \hat{\mathcal{F}}^H, \rho)$ over $D_H$.

**Proof.** Let $\phi$ be the character of an $n$-dimensional vanishing $S_H$-cycle $(S_H, \hat{\partial}^H, \hat{\mathcal{F}}^H, \rho)$. By definition, $\hat{\mathcal{F}}^H$ is an $n$-dimensional closed graded $(H, M)$-trace on the $H$-semicategory $S_H$ and that $S^0_H$ is an ordinary $H$-category. We now define $\psi \in C^n_H(S^0_H, M)$ by setting

$$\psi(m \otimes g^0 \otimes \ldots \otimes g^n) := \hat{\mathcal{F}}^H_{X_0}(m \otimes g^0 \hat{\partial}^H_H(g^1) \ldots \hat{\partial}^H_H(g^n))$$

for $m \in M$ and $g^0 \otimes \ldots \otimes g^n \in \text{Hom}_{S^0_H}(X_1, X_0) \otimes \text{Hom}_{S^0_H}(X_2, X_1) \otimes \ldots \otimes \text{Hom}_{S^0_H}(X_0, X_n)$. Then, by the implication (1) $\Rightarrow$ (3) in Theorem 5.11, we have that $\psi \in Z^*_H(S^0_H, M)$. Since $HC^*_H(S^0_H, M) = 0$, we have that $\psi = b\psi'$ for some $\psi' \in C^{n-1}_H(S^0_H, M)$.

By Lemma 7.4 the semifunctor $\rho \in \text{Hom}_{\text{Cat}_H}(D_H, S^0_H)$ induces a map $C^{n-1}_H(\rho, M) : C^{n-1}_H(S^0_H, M) \to C^{n-1}_H(D_H, M)$. Setting $\psi'' := C^{n-1}_H(\rho, M)(\psi')$, we have

$$(\psi'')(m \otimes p^0 \otimes \ldots \otimes p^{n-1}) = \psi'(m \otimes \rho(p^0) \otimes \ldots \otimes \rho(p^{n-1}))$$

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for any \( m \in M \) and \( p^0 \otimes \ldots \otimes p^{n-1} \in CN_{n-1}(\mathcal{D}_H) \). Therefore,

\[
\phi(m \otimes f^0 \otimes \ldots \otimes f^n) = \mathcal{T}^H_{n_0}(m \otimes \rho(f^0) \otimes (\rho(f^1)) \ldots \otimes \rho(f^n)) = \psi(m \otimes (\rho(f^0) \otimes \ldots \otimes \rho(f^n)) = (b\psi)(m \otimes f^0 \otimes \ldots \otimes f^n)
\]

for any \( m \in M \) and \( f^0 \otimes \ldots \otimes f^n \in \text{Hom}_{\mathcal{D}_H}(X_1, X_0) \otimes \text{Hom}_{\mathcal{D}_H}(X_2, X_1) \otimes \ldots \otimes \text{Hom}_{\mathcal{D}_H}(X_0, X_n) \). Thus, \( \phi \in B^n_H(\mathcal{D}_H, M) \).

Conversely, suppose that \( \phi \in B^n_H(\mathcal{D}_H, M) \). Then, \( \phi = b\psi \) for some \( \psi \in C^n_H(\mathcal{D}_H, M) \). We now extend \( \psi \) to get an element \( \psi' \in C^{n-1}_H(C \otimes \mathcal{D}_H, M) \) as follows:

\[
\psi'(m \otimes (B^0 \otimes f^0) \otimes \ldots \otimes (B^{n-1} \otimes f^{n-1})) = \psi(m \otimes B^0 f^0 \otimes \ldots \otimes B^{n-1} f^{n-1})
\]

We now set \( \phi' = b\psi' \in Z^n_H(C \otimes \mathcal{D}_H, M) \). We now consider the \( H \)-linear semifunctor \( \rho : \mathcal{D}_H \rightarrow C \otimes \mathcal{D}_H \) which fixes objects and takes any morphism \( f \) to \( 1 \otimes f \). Then, we have

\[
(C^n_H(\rho, M)(\phi'))(m \otimes f^0 \otimes \ldots \otimes f^n) = \phi'(m \otimes (\rho(f^0) \otimes \ldots \otimes (\rho(f^n)) = (b\psi')(m \otimes f^0 \otimes \ldots \otimes f^n)
\]

Since \( \phi' \in Z^n_H(C \otimes \mathcal{D}_H, M) \), the implication (3) \( \Rightarrow \) (2) in Theorem 7.11 gives us a closed graded \( (H, M) \)-trace \( \mathcal{T}^H \) of dimension \( n \) on the DGH-semicategory \( (\Omega(C \otimes \mathcal{D}_H), \partial_H) \) such that

\[
\mathcal{T}^H_{n_0}(m \otimes \rho(f^0) \otimes (\rho(f^1)) \ldots \otimes \rho(f^n)) = \phi'(m \otimes (\rho(f^0) \otimes \ldots \otimes \rho(f^n)) = \phi(m \otimes f^0 \otimes \ldots \otimes f^n)
\]

(7.9)

Since \( (\Omega(C \otimes \mathcal{D}_H))^0 = C \otimes \mathcal{D}_H \) is a left \( H \)-category, we see that \( \phi \) is the character associated to the cycle \((\Omega(C \otimes \mathcal{D}_H), \partial_H, \mathcal{T}^H, \rho) \) over \( \mathcal{D}_H \).

From the proof of Lemma 7.10, we know that \( C \otimes \mathcal{D}_H \) satisfies the assumptions in Proposition 7.5. Hence, \((\Omega(C \otimes \mathcal{D}_H), \partial_H, \mathcal{T}^H, \rho) \) is a vanishing cycle over \( \mathcal{D}_H \). From this, the result follows.

Substituting \( H = k = M \), we have

**Corollary 7.12.** An element \( \phi \in C^n(C) \) is a coboundary iff \( \phi \) is the character of an \( n \)-dimensional vanishing \( S \)-cycle \((S, \hat{\partial}, \hat{T}, \rho) \) over \( C \).

Our final aim is to use the method of categorified cycles and categorified vanishing cycles to obtain a pairing

\[
HC^p(C) \otimes HC^q(C') \rightarrow HC^{p+q}(C \otimes C')
\]

for \( k \)-linear categories \( C \) and \( C' \). Let \((S, \hat{\partial}_S) \) and \((S', \hat{\partial}_{S'}) \) be DG-semicategories. Then, their tensor product \( S \otimes S' \) is the DG-semicategory defined as follows:

\[
\text{Ob}(S \otimes S') = \text{Ob}(S) \times \text{Ob}(S')
\]

\[
\text{Hom}^i_{S \otimes S'}((X, X'), (Y, Y')) = \bigoplus_{i+j=n} \text{Hom}^i_S(X, Y) \otimes_k \text{Hom}^j_{S'}(X', Y')
\]

The composition in \( S \otimes S' \) is given by the rule:

\[
(g \circ g') \circ (f \otimes f') = (-1)^{deg(g')deg(f)}(gf \otimes g'f')
\]

for homogeneous \( f : X \rightarrow Y, g : Y \rightarrow Z \) in \( S \) and \( f' : X' \rightarrow Y', g' : Y' \rightarrow Z' \) in \( S' \). The differential \( \hat{\partial}^n_{S \otimes S'} : \text{Hom}^n_{S \otimes S'}((X, X'), (Y, Y')) \rightarrow \text{Hom}^{n+1}_{S \otimes S'}((X, X'), (Y, Y')) \) is determined by

\[
\hat{\partial}^n_{S \otimes S'}(f_i \otimes g_j) = \hat{\partial}^n_S(f_i) \otimes g_j + (-1)^i f_i \otimes \hat{\partial}^n_{S'}(g_j)
\]

for any \( f_i \in \text{Hom}^i_S(X, Y) \) and \( g_j \in \text{Hom}^j_{S'}(X', Y') \) such that \( i + j = n \). Clearly, \((S \otimes S')^0 = S^0 \otimes S^0\).
Theorem 7.13. Let $\mathcal{C}$ and $\mathcal{C}'$ be small $k$-linear categories. Then, we have a pairing

$$HC^p(\mathcal{C}) \otimes HC^q(\mathcal{C}') \rightarrow HC^{p+q}(\mathcal{C} \otimes \mathcal{C}')$$

for $p, q \geq 0$.

Proof. Let $\phi \in Z^p(\mathcal{C})$ and $\phi' \in Z^q(\mathcal{C}')$. We may express $\phi$ and $\phi'$ respectively as the characters of $p$ and $q$-dimensional cycles $(S, \hat{\partial}, \hat{T}, \rho)$ and $(S', \hat{\partial}', \hat{T}', \rho')$ over $\mathcal{C}$ and $\mathcal{C}'$. We consider the collection $\hat{T} \# \hat{T}' := \{(\hat{T} \# \hat{T}')_{(X,Y)} : Hom_{S \otimes S'}((X,X'), (X',Y')) \rightarrow C\}_{X,Y} \cup \mathcal{O}(S \otimes S')$ of $\mathcal{C}$-linear maps defined by

$$(\hat{T} \# \hat{T}')_{(X,Y)}(f \otimes f') = \hat{T}_X(f_p) \hat{T}'_{X'}(f'_p)$$

for any $f \otimes f' = (f_i \otimes f'_i)_{i+j=p+q} \in Hom_{S \otimes S'}((X,X'), (X',Y'))$. We will now prove that $\hat{T} \# \hat{T}'$ is a $p+q$-dimensional closed graded trace on the DG-semicategory $S \otimes S'$. For any $g \otimes g' = (g_i \otimes g'_i)_{i+j=p+q-1} \in Hom_{S \otimes S'}^l((X,X'), (X,Y))$, we have

$$(\hat{T} \# \hat{T}')_{(X,Y)}((g \otimes g')(f \otimes f')) = \sum_{i+j=p+q-1} (\hat{T} \# \hat{T}')_{(X,Y)}(\hat{T}_X(g_i) \otimes \hat{T}'_{X'}(g'_i) + (-1)^j g_i \otimes \hat{T}'_X(g'_i))$$

This proves the condition in (5.11). Next for any homogeneous $f : X \rightarrow Y$, $g : Y \rightarrow X$ in $S$ and $f' : X' \rightarrow Y'$, $g' : Y' \rightarrow X'$ in $S'$, we have

$$(\hat{T} \# \hat{T}')_{(X,Y)}((g \otimes g')(f \otimes f')) = (-1)^{deg(g')deg(f')} T_X((g(f))_p) T'_{X'}((g'f')_q)$$

This proves the condition in (5.12). Thus, we obtain a $p+q$ dimensional cycle $(S \otimes S', \hat{T} \# \hat{T}', \rho \otimes \rho')$ over the category $\mathcal{C} \otimes \mathcal{C}'$. Then, the character of this cycle, denoted by $\phi \# \phi' \in Z^{p+q}(\mathcal{C} \otimes \mathcal{C}')$, gives a well defined map $\gamma : Z^p(\mathcal{C}) \otimes Z^q(\mathcal{C}') \rightarrow Z^{p+q}(\mathcal{C} \otimes \mathcal{C}')$.

We now verify that the map $\gamma$ restricts to a pairing

$$B^p(\mathcal{C}) \otimes Z^q(\mathcal{C}') \rightarrow B^{p+q}(\mathcal{C} \otimes \mathcal{C}')$$

For this, we let $\phi \in Z^p(\mathcal{C})$ be the character of a $p$-dimensional vanishing cycle $(S, \hat{\partial}, \hat{T}, \rho)$ over $\mathcal{C}$. In particular, it follows from Definition 7.7 that $S^0$ is an ordinary category. From the implication (1) $\Rightarrow$ (2) in Theorem 5.11 it follows that we might as well take $S^0$ to be an ordinary category. In fact, we could assume that $S^0 = \Omega \mathcal{C}'$. Then, $S^0 \otimes S^0$ is an ordinary category. It suffices to show that the tuple $(S \otimes S', \hat{\partial}_{S \otimes S'}, \hat{T} \# \hat{T}', \rho \otimes \rho')$ is a vanishing cycle.

Since $(S, \hat{\partial}, \hat{T})$ is a vanishing cycle, we have a $\mathbb{C}$-linear semifunctor $v : S^0 \rightarrow S^0$ and an $\eta \in \mathbb{U}(S^0 \otimes M_2(\mathbb{C}))$ satisfying the conditions in Corollary 7.6. Extending $v$, we get the semifunctor $v \otimes id : S^0 \otimes S^0 \rightarrow S^0 \otimes S^0$. Identifying, $S^0 \otimes S^0 = M_2(\mathbb{C}) \cong S^0 \otimes M_2(\mathbb{C}) \cong M_2(\mathbb{C}) \otimes S^0$, we obtain $\eta \in \mathbb{U}(S^0 \otimes M_2(\mathbb{C}) \otimes S^0)$ given by

$$\{\eta(X, X') = \eta(X) \otimes id_{X'} \in Hom_{S^0}((X,X'), (X, X')) = Hom_{S^0 \otimes M_2(\mathbb{C})}(X, X) \otimes Hom_{S^0}(X', X')\}$$

It may also be easily verified that

$$\Phi_\eta(f \otimes f' \otimes E_{11} + (v \otimes id)(f \otimes f') \otimes E_{22}) = (v \otimes id)(f \otimes f') \otimes E_{22}$$

Thus, we see that the category $(S \otimes S')^0 = S^0 \otimes S^0$ satisfies the conditions in Corollary 7.6. Therefore, the tuple $(S \otimes S', \hat{\partial}_{S \otimes S'}, \hat{T} \# \hat{T}', \rho \otimes \rho')$ is a vanishing cycle. This proves the result. □
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