TWISTED CONJUGACY SEPARABLE GROUPS

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Abstract. We study the notion of twisted conjugacy separability (essentially introduced in our previous paper for a proof of twisted version of Burnside-Frobenius theorem) and some related properties. We give examples of groups with and without this property and study its behavior under some extensions. An affirmative answer to the twisted Dehn conjugacy problem for polycyclic-by-finite group is obtained. Some problems for the further study are indicated.

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1. Introduction

Definition 1.1. Let $G$ be a countable discrete group and $\phi : G \to G$ an endomorphism. Two elements $x, x' \in G$ are said to be $\phi$-conjugate or twisted conjugate, iff there exists $g \in G$ with $x' = gx\phi(g^{-1})$. We shall write $\{x\}_\phi$ for the $\phi$-conjugacy or twisted conjugacy class of the element $x \in G$. The number of $\phi$-conjugacy classes is called the Reidemeister number of an endomorphism $\phi$ and is denoted by $R(\phi)$. If $\phi$ is the identity map then the $\phi$-conjugacy classes are the usual conjugacy classes in the group $G$.

If $G$ is a finite group, then the classical Burnside-Frobenius theorem (see e.g. [35], [23, p. 140]) says that the number of classes of irreducible representations is equal to the number of conjugacy classes of elements of $G$. Let $\hat{G}$ be the unitary dual of $G$, i.e. the set of equivalence classes of unitary irreducible representations of $G$. The attempts to
generalize this theorem to the case of non-identical automorphism and of non-finite group (i.e., to identify the Reidemeister number of $\phi$ and the number of fixed points of $\hat{\phi}$ on an appropriate dual object of $G$, provided that one of these numbers is finite) were inspired by the dynamical questions and were the subject of a series of papers [5, 6, 4, 9, 11, 8, 10].

In the present paper we study the following property for a countable discrete group $G$ and its automorphism $\phi$: we say that the group is $\phi$-\textit{conjugacy separable} if its Reidemeister classes can be distinguished by homomorphisms onto finite groups, and we say that it is $\textit{twisted conjugacy separable}$ if it is $\phi$-conjugacy separable for any automorphism $\phi$ with $R(\phi) < \infty$ (\textit{strongly twisted conjugacy separable}, if we remove this finiteness restriction) (Definitions 3.2 and 3.5). This notion was used in [10] to prove the twisted Burnside-Frobenius theorem for polycyclic-by-finite groups with the finite-dimensional part of the unitary dual $\hat{G}$ as an appropriate dual object. Related questions were studied in [2], [27].

After some preliminary considerations we prove the main results of the paper, namely

1. \textbf{Classes of twisted conjugacy separable groups:} Polycyclic-by-finite groups are strongly twisted conjugacy separable groups (Theorem 4.2).

2. \textbf{Twisted conjugacy separability respects some extensions:} Suppose, there is an extension $H \to G \to G/H$, where the group $H$ is a finitely generated characteristic twisted conjugacy separable group; $G/H$ is finitely generated FC-group (i.e., a group with finite conjugacy classes). Then $G$ is a twisted conjugacy separable group (a reformulation of Theorem 5.1).

3. \textbf{Examples of groups, which are not twisted conjugacy separable:} HNN, Ivanov and Osin groups (Section 6).

4. \textbf{The affirmative answer to the twisted Dehn conjugacy problem for polycyclic-by-finite groups} (Section 7).

5. \textbf{Residually finite finitely generated groups are twisted conjugacy separable, in particular twisted Burnside-Frobenius theorem is true for them} in the following formulation: Let $G$ be a finitely generated residually finite group and $\phi$ its automorphism with $R(\phi) < \infty$. Then $R(\phi) = S_f(\phi)$, where $S_f(\phi)$ is the number of fixed points of $\hat{\phi} : \hat{G}_f \to \hat{G}_f$, $\hat{\phi}(\rho) = \rho \circ \phi$, where $\hat{G}_f$ is the part of the unitary dual $\hat{G}$, which is formed by the finite-dimensional representations (Section 9).

A number of examples of groups and automorphisms with finite Reidemeister numbers was obtained and studied in [4, 14, 7, 11, 8].

The interest in twisted conjugacy relations has its origins, in particular, in the Nielsen-Reidemeister fixed point theory (see, e.g. [21, 4]), in Selberg theory (see, eg. [37, 1]), and Algebraic Geometry (see, e.g. [17]). The congruences give some necessary conditions for the realization problem for Reidemeister numbers in topological dynamics.

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2. Preliminary considerations

The following construction relates $\phi$-conjugacy classes and some conjugacy classes of another group. It was obtained in topological context by Boju Jiang and Laixiang Sun in [22]. Consider the action of $\mathbb{Z}$ on $G$, i.e. a homomorphism $\mathbb{Z} \to \text{Aut}(G), \ n \mapsto \phi^n$. Let $\Gamma$ be a corresponding semi-direct product $\Gamma = G \rtimes \mathbb{Z}$:

\[
\Gamma := \langle G, t \mid tgt^{-1} = \phi(g) \rangle
\]

in terms of generators and relations, where $t$ is a generator of $\mathbb{Z}$. The group $G$ is a normal subgroup of $\Gamma$. As a set, $\Gamma$ has the form

\[
\Gamma = \sqcup_{n \in \mathbb{Z}} G \cdot t^n,
\]

where $G \cdot t^n$ is the coset by $G$ containing $t^n$.

Remark 2.1. Any usual conjugacy class of $\Gamma$ is contained in some $G \cdot t^n$. Indeed, $gg't^n g^{-1} = gg' \phi^n(g^{-1})t^n$ and $tg't^n t^{-1} = \phi(g)t^n$.

Lemma 2.2. Two elements $x, y$ of $G$ are $\phi$-conjugate iff $xt$ and $yt$ are conjugate in the usual sense in $\Gamma$. Therefore $g \mapsto g \cdot t$ is a bijection from the set of $\phi$-conjugacy classes of $G$ onto the set of conjugacy classes of $\Gamma$ contained in $G \cdot t$.

Proof. If $x$ and $y$ are $\phi$-conjugate then there is a $g \in G$ such that $gx = y\phi(g)$. This implies $gx = yt g^{-1}$ and therefore $g(xt) = (yt)g$ so $xt$ and $yt$ are conjugate in the usual sense in $\Gamma$. Conversely, suppose $xt$ and $yt$ are conjugate in $\Gamma$. Then there is a $gt^n \in \Gamma$ with $gt^n xt = yt gtn$. From the relation $txt^{-1} = \phi(x)$ we obtain $g\phi^n(x)t^{n+1} = y\phi(g)t^{n+1}$ and therefore $g\phi^n(x) = y\phi(g)$. Hence, $y$ and $\phi^n(x)$ are $\phi$-conjugate. Thus, $y$ and $x$ are $\phi$-conjugate, because $x$ and $\phi(x)$ are always $\phi$-conjugate: $\phi(x) = x^{-1}x\phi(x)$. \qed

3. Twisted conjugacy separability

We would like to give a generalization of the following well known notion.

Definition 3.1. A group $G$ is conjugacy separable if any pair $g, h$ of non-conjugate elements of $G$ are non-conjugate in some finite quotient of $G$.

It was proved that polycyclic-by-finite groups are conjugacy separated ([32, 13], see also [34, Ch. 4]). Also, residually finite recursively presented Burnside $p$-groups constructed by R. I. Grigorchuk [16] and by N. Gupta and S. Sidki [18] are shown to be conjugacy separable when $p$ is an odd prime in [38].

We can introduce the following notion, which coincides with the previous definition in the case $\phi = \text{Id}$.

Definition 3.2. A group $G$ is $\phi$-conjugacy separable with respect to an automorphism $\phi : G \to G$ if any pair $g, h$ of non-$\phi$-conjugate elements of $G$ are non-$\phi^{-1}$-conjugate in some finite quotient of $G$ respecting $\phi$.

This notion is closely related to the notion $\text{RP}(\phi)$ introduced in [10].

Definition 3.3. We say that a group $G$ has the property $\text{RP}$ if for any automorphism $\phi$ with $R(\phi) < \infty$ the characteristic functions $f$ of Reidemeister classes (hence all $\phi$-central functions) are periodic in the following sense.

There exists a finite group $K$, its automorphism $\phi_K$, and epimorphism $F : G \to K$ such that
(1) The diagram
\[
\begin{array}{ccc}
G & \xrightarrow{\phi} & G \\
\downarrow F & & \downarrow F \\
K & \xrightarrow{\phi_K} & K
\end{array}
\]
commutes.

(2) \( f = F^* f_K \), where \( f_K \) is a characteristic function of a subset of \( K \).

If this property holds for a concrete automorphism \( \phi \), we will denote this by \( \text{RP}(\phi) \).

One gets immediately the following statement.

**Theorem 3.4.** Suppose, \( R(\phi) < \infty \). Then \( G \) is \( \phi \)-conjugacy separable if and only if \( G \) is \( \text{RP}(\phi) \).

**Proof.** Indeed, let \( F_{ij} : G \to K_{ij} \) distinguish \( i \)th and \( j \)th \( \phi \)-conjugacy classes, where \( K_{ij} \) are finite groups, \( i, j = 1, \ldots, R(\phi) \). Let \( F : G \to \bigoplus_{i,j} K_{ij}, \ F(g) = \sum_{i,j} F_{ij}(g), \) be the diagonal mapping and \( K \) its image. Then \( F : G \to K \) gives \( \text{RP}(\phi) \).

The opposite implication is evident. \( \square \)

**Definition 3.5.** A group \( G \) is **twisted conjugacy separable** if it is \( \phi \)-conjugacy separable for any \( \phi \) with \( R(\phi) < \infty \).

A group \( G \) is **strongly twisted conjugacy separable** if it is \( \phi \)-conjugacy separable for any \( \phi \).

From Theorem 3.4 one immediately obtains

**Corollary 3.6.** A group \( G \) is twisted conjugacy separable if and only if it is \( \text{RP} \).

**Theorem 3.7.** Let some class of conjugacy separable groups be closed under taking semidirect products by \( \mathbb{Z} \). Then this class consists of strongly twisted conjugacy separable groups.

**Proof.** This follows immediately from Theorem 3.7 and Theorem 3.4. \( \square \)
4. First examples: polycyclic-by-finite groups

As an application we obtain another proof of the main theorem for polycyclic-by-finite groups.

Let $G' = [G, G]$ be the commutator subgroup or derived group of $G$, i.e. the subgroup generated by commutators. $G'$ is invariant under any homomorphism, in particular it is normal. It is the smallest normal subgroup of $G$ with an abelian factor group. Denoting

$$G^{(0)} := G, \quad G^{(1)} := G', \quad G^{(n)} := (G^{(n-1)})', \quad n \geq 2,$$

one obtains derived series of $G$:

$$G = G^{(0)} \supset G' \supset G^{(2)} \supset \cdots \supset G^{(n)} \supset \cdots$$

If $G^{(n)} = e$ for some $n$, i.e. the series (3) stabilizes by trivial group, the group $G$ is solvable.

Definition 4.1. A solvable group with derived series with cyclic factors is called polycyclic group.

Theorem 4.2. Any polycyclic-by-finite group is a strongly twisted conjugacy separable group.

Proof. The class of polycyclic-by-finite groups is closed under taking semidirect products by $\mathbb{Z}$. Indeed, let $G$ be an polycyclic-by-finite group. Then there exists a characteristic (polycyclic) subgroup $P$ of finite index in $G$. Hence, $P \rtimes \mathbb{Z}$ is a polycyclic normal group of $G \rtimes \mathbb{Z}$ of the same finite index.

Polycyclic-by-finite groups are conjugacy separable ([32, 13], see also [34, Ch. 4]). It remains to apply Theorem 3.8. □

5. Twisted conjugacy separability and extensions

It is known that conjugacy separability does not respect extensions. In particular, in [15] an example of a group $G$ which is not conjugacy separable, but contains a subgroup $H$ of index 2 which is conjugacy separable, is given.

For twisted conjugacy separable groups the situation is much better under some finiteness conditions. More precisely one has the following statement.

Theorem 5.1. Suppose, there exists a commutative diagram

$$\begin{array}{ccc}
0 & \longrightarrow & H & \overset{i}{\longrightarrow} & G & \overset{p}{\longrightarrow} & G/H & \longrightarrow & 0 \\
& & \phi \downarrow & & \phi \downarrow & & \phi \downarrow & & \\
0 & \longrightarrow & H & \overset{i}{\longrightarrow} & G & \overset{p}{\longrightarrow} & G/H & \longrightarrow & 0,
\end{array}$$

where $H$ is a finitely generated normal subgroup of a finitely generated group $G$. Suppose, $R(\phi) < \infty$, $G/H$ is a FC group, i.e., all conjugacy classes are finite, and $H$ is a $\phi'$-conjugacy separable group. Then $G$ is a $\phi$-conjugacy separable group.

Another variant of finiteness is $|G/H| < \infty$ (without the property $R(\phi) < \infty$). In the relation to the next theorem, note that examples of a conjugacy separable subgroup of index 2 in a conjugacy non-separable group are known [15]. The proof of the next theorem is related to [27, Prop. 3.6].
Theorem 5.2. Let $H$ be a characteristic strongly twisted conjugacy separable subgroup of finite index in $G$. Then $G$ is a strongly twisted conjugacy separable group.

Proof. Obviously $G$ is strongly twisted conjugacy separable if and only if $\{g\}_\phi$ is closed in the profinite topology, where $\phi : G \to G$ is an arbitrary automorphism and $g \in G$ is an arbitrary element.

Suppose, $x_i$, $i = 1, \ldots, r$, are coset representatives, where $r$ is the index of $H$ in $G$, and $g_i := x_i \phi(x_i^{-1})$. Then

$$\{g\}_\phi = \{wg\phi(w^{-1}) \mid w \in G\} = \bigcup_i \{hx_i \phi(x_i^{-1}) \phi(h^{-1}) \mid h \in H\}$$

$$= \bigcup_i \{h \phi(h^{-1}) \mid h \in H\} = \bigcup_i \{(h \phi(h^{-1}) g_i^{-1})g_i \mid h \in H\}$$

$$= \bigcup_i \{e\}_{\tau_{g_i} \circ \phi'} \cdot g_i.$$  

Cosets by a normal subgroup of finite index are evidently closed and open in the profinite topology. Thus, by the supposition $\{e\}_{\tau_{g_i} \circ \phi'}$ is closed in $H$ and $G$, as well as their right translations, i.e., entries of the above union. Since the union is finite, $\{g\}_\phi$ is closed in $G$. \qed

6. Examples and counterexamples

Some of examples of groups, for which the twisted Burnside-Frobenius theorem in the above formulation is true, out of the class of polycyclic-by-finite groups were obtained by F. Indukaev [20]. Namely, it is proved that wreath products $A \wr \mathbb{Z}$ are RP groups, where $A$ is a finitely generated abelian group (these groups are residually finite).

Now let us present some counterexamples to the twisted Burnside-Frobenius theorem in the above formulation for some discrete groups with extreme properties. Suppose, an infinite discrete group $G$ has a finite number of conjugacy classes. Such examples can be found in [36] (HNN-group), [30, p. 471] (Ivanov group), and [31] (Osin group). Then evidently, the characteristic function of the unity element is not almost-periodic and the argument above is not valid. Moreover, let us show, that these groups give rise counterexamples to the above theorem.

In particular, they are not twisted conjugacy separable. Evidently, they are not conjugacy separable, because they are not residually finite.

Example 6.1. For the Osin group the Reidemeister number $R(\text{Id}) = 2$, while there is only trivial (1-dimensional) finite-dimensional representation. Indeed, Osin group is an infinite finitely generated group $G$ with exactly two conjugacy classes. All nontrivial elements of this group $G$ are conjugate. So, the group $G$ is simple, i.e. $G$ has no nontrivial normal subgroup. This implies that group $G$ is not residually finite (by definition of residually finite group). Hence, it is not linear (by Mal’cev theorem [25], [33, 15.1.6]) and has no finite-dimensional irreducible unitary representations with trivial kernel. Hence, by simplicity of $G$, it has no finite-dimensional irreducible unitary representation with nontrivial kernel, except of the trivial one.

Let us remark that Osin group is non-amenable, contains the free group in two generators $F_2$, and has exponential growth.
Example 6.2. For large enough prime numbers $p$, the first examples of finitely generated infinite periodic groups with exactly $p$ conjugacy classes were constructed by Ivanov as limits of hyperbolic groups (although hyperbolicity was not used explicitly) (see [30, Theorem 41.2]). Ivanov group $G$ is infinite periodic 2-generator group, in contrast to the Osin group, which is torsion free. The Ivanov group $G$ is also a simple group. The proof (kindly explained to us by M. Sapir) is the following. Denote by $a$ and $b$ the generators of $G$ described in [30, Theorem 41.2]. In the proof of Theorem 41.2 on [30] it was shown that each of elements of $G$ is conjugate in $G$ to a power of generator $a$ of order $s$. Let us consider any normal subgroup $N$ of $G$. Suppose $\gamma \in N$. Then $\gamma = ga^s g^{-1}$ for some $g \in G$ and some $s$. Hence, $a^s = g^{-1} \gamma g \in N$ and from periodicity of $a$, it follows that also $a \in N$ as well as $a^k \in N$ for any $k$, because $p$ is prime. Then any element $h$ of $G$ also belongs to $N$ being of the form $h = \tilde{h} a^k (\tilde{h})^{-1}$, for some $k$, i.e., $N = G$. Thus, the group $G$ is simple. The discussion can be completed in the same way as in the case of Osin group.

Example 6.3. In paper [19], Theorem III and its corollary, G. Higman, B. H. Neumann, and H. Neumann proved that any locally infinite countable group $G$ can be embedded into a countable group $G^*$ in which all elements except the unit element are conjugate to each other (see also [36]). The discussion above related Osin group remains valid for $G^*$ groups.

7. Twisted Dehn conjugacy problem

The subject is closely related to some decision problem. Recall that M. Dehn in 1912 [3] (see [24, Ch. 1, §2; Ch. 2, §1]) has formulated in particular

Conjugacy problem: Does there exists an algorithm to determine whether an arbitrary pair of group words $U$, $V$ in the generators of $G$ define conjugate elements of $G$?

The following question was posed by G. Makanin [28, Question 10.26(a)]:

Question: Does there exists an algorithm to determine whether for an arbitrary pair of group words $U$ and $V$ of a free group $G$ and an arbitrary automorphism $\phi$ of $G$ the equation $\phi(X) U = V X$ solvable in $G$?

In [2] the following problem, which generalizes the two above problems, was posed:

Twisted conjugacy problem: Does there exists an algorithm to determine whether for an arbitrary pair of group words $U$ and $V$ in the generators of $G$ the equality $\phi(X) U = V X$ holds for some $W \in G$ and $\phi \in H$, where $H$ is a fixed subset of Aut$(G)$?

In [2] a partial affirmative question to the Makanin’s question is obtained.

We will discuss the twisted conjugacy problem for $H = \{\phi\}$.

Theorem 7.1. The twisted conjugacy problem has the affirmative answer for $G$ being polycyclic-by-finite group and $H$ be equal to a unique automorphism $\phi$.

Proof. It follows immediately from Theorem 4.2 by the same argument as in the paper of Mal’cev [26] (see also [29], where the property of conjugacy separability was first formulated) for the (non-twisted) conjugacy problem.

In fact we have proved the following statement.

Theorem 7.2. If $G$ is strongly twisted conjugacy separable then the twisted Dehn conjugacy problem is solvable for any automorphism of $G$. 
Also one can study some more particular cases of this problem. In particular, one has

**Theorem 7.3.** Let $G$ be a $\phi$-conjugacy separable group. Then the twisted Dehn conjugacy problem is solvable for $\phi$.

The results of Section 9 imply

**Theorem 7.4.** If $G$ is a finitely generated residually finite group and $R(\phi) < \infty$, then the twisted Dehn conjugacy problem is solvable for $\phi$.

From Corollary 3.4 and Proposition 3.5 in [27] one can obtain

**Theorem 7.5.** Suppose $G$ is the fundamental group of a closed hyperbolic surface and $\phi : G \to G$ is virtually inner. Then the twisted Dehn conjugacy problem is solvable for $\phi$.

8. **Some questions**

It is evident, that any conjugacy separable group is residually finite (because the unity element is an entire conjugacy class). This argument does not work for general Reidemeister classes. In this relation we wish to formulate several questions:

**Question 1:** Does the $\phi$-conjugacy separability imply residually finiteness?

**Question 2:** Does the $\phi$-conjugacy separability imply residually finiteness, provided $R(\phi) < \infty$?

**Question 3:** Does the twisted conjugacy separability imply residually finiteness, provided the existence of $\phi$ with $R(\phi) < \infty$?

**Question 4:** Let $G$ be a residually finite group and $\phi$ its automorphism with $R(\phi) < \infty$. Is $G$ $\phi$-conjugacy separable?

The affirmative answer to the last question implies twisted Burnside-Frobenius theorem for $\phi$. This will be made in Section 9.

9. **Residually finite groups**

Recall that $G$ is called residually finite, if for any $g \in G$ there exists an (epi)morphism $F_g$ of $G$ onto a finite group $K_g$ such that $F_g(g) \neq e \in K_g$. In other words, $\{e\} \in G$ is closed in the profinite topology.

The following theorem is proved in [12].

**Theorem 9.1.** Let $G$ be a residually finite finitely generated group and $\phi$ its automorphism with $R(\phi) < \infty$. Then $G$ is $\phi$-conjugacy separable.

**Sketch of a proof.** It is clear, that it is sufficient to prove that $R(\phi) \leq S_f(\phi)$. $R(\phi)$ equals the dimension of the space of twisted invariant elements of $\ell^\infty(G)$, i.e. functionals on $\ell^1(G)$ such that their kernels contain the closure $K_1$ in $\ell^1(G)$ of the space of elements of the form $b - g[b], g[b](x) := b(g\phi(g^{-1})).$

Since $R(\phi) < \infty$, codim $K_1 = R(\phi)$, and $K_1$ has a Banach space complement of dimension $R(\phi)$. We can take it in a way such that it has a base $a_i \in \mathbb{C}[G], i = 1, \ldots, R(\phi)$, i.e., all $a_i$’s have a finite support. Let $p : G \to F = G/H$ be an epimorphism on a finite group $F$ such that it distinguishes all elements from the union of (finite) supports of $a_i$ and $H$ is characteristic. The image of $\ell^1(G)$ under the induced homomorphism $p_1$ is $\ell^1(F) = \mathbb{C}[F]$. Also $K_1$ maps epimorphically onto the space $K_p$ of elements
\beta - p(g)[\beta] = p_1(b) - p(g)[p_1(b)] = p_1(b - g[b]) \text{ in } \mathbb{C}[F]. \text{ Thus, } \{p_1(a_i)\} \text{ form a basis of a complement to } K_p \text{ in } \mathbb{C}[F]. \text{ Decompose this (finite dimensional) algebra } \mathbb{C}[F] \text{ into a direct sum of matrix algebras, i.e., decompose the left regular representation of } F \text{ into irreducible ones: } \lambda_F \cong \bigoplus_{i=1}^{N} V_i \otimes V_i^*. \text{ Let } K_i \text{ be formed by } x - p_i(g)[x] \text{ in } \text{End } V_i. \text{ Since } J \text{ is an algebra isomorphism, } R(\phi) = \text{codim } K_1 = \sum_i \text{codim } K_i. \text{ The last one is } 1 \text{ if } \hat{\phi}(p_i) = p_i \text{ and } 0 \text{ otherwise. Thus, } R(\phi) \leq \text{the number of finite dimensional fixed points of } \hat{\phi}. \quad \square

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