The group of unimodular automorphisms of a principal bundle and the Euler-Yang-Mills equations

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Abstract

Given a principal bundle \( G \hookrightarrow P \to B \) (each being compact, connected and oriented) and a \( G \)-invariant metric \( h^P \) on \( P \) which induces a volume form \( \mu^P \), we consider the group of all unimodular automorphisms \( \text{SAut}(P, \mu^P) := \{ \varphi \in \text{Diff}(P) \mid \varphi^* \mu^P = \mu^P \text{ and } \varphi \text{ is } G\text{-equivariant} \} \) of \( P \), and determines its Euler equation à la Arnold. The resulting equations turn out to be (a particular case of) the Euler-Yang-Mills equations of an incompressible classical charged ideal fluid moving on \( B \).

It is also shown that the group \( \text{SAut}(P, \mu^P) \) is an extension of a certain volume preserving diffeomorphisms group of \( B \) by the gauge group \( \text{Gau}(P) \) of \( P \).

1 Introduction

Since \([4]\), it is well known that an appropriate configuration space for the study of equations of hydrodynamical type (more precisely, the incompressible Euler equations of an incompressible fluid) on a Riemannian manifold \((M,g)\) endowed with a volume form \( \mu \) (\( \mu \) being not necessarily induced by the metric \( g \)), is given by the group of all unimodular diffeomorphisms \( \text{SDiff}(M,\mu) := \{ \varphi \in \text{Diff}(M) \mid \varphi^* \mu = \mu \} \) of \( M \). This group is –in a suitably chosen sense– an infinite dimensional Lie group whose Lie algebra \( \mathfrak{X}(M,\mu) := \{ X \in \mathfrak{X}(M) \mid \text{div}_\mu(X) = 0 \} \) is the space of divergence free vector fields endowed with the opposite of the usual vector field bracket, and if \( X \in \mathfrak{X}(M,\mu) \) is a time-dependant divergence free vector field describing the velocity field of an incompressible fluid, then its dynamics is governed by the incompressible Euler equation \( \frac{d}{dt}X + \nabla_X X = \nabla p \), where \( p \) is the pressure of the fluid. It turns out that this equation characterizes geodesics on \( \text{SDiff}(M,\mu) \) with respect to the natural right-invariant \( L^2 \)-metric on \( \text{SDiff}(M,\mu) \) (see \([6]\)), and can be seen as an Euler equation (or Lie-Poisson equation) on the “regular dual” of \( \mathfrak{X}(M,\mu) \) (see \([2]\)).

In this paper, we propose another configuration space to study the Euler equation when some symmetries are involved. Our point of departure is to assume that the fluid evolves on the total space of a principal bundle \( G \hookrightarrow P \to B \) (\( P \) being connected and oriented). We assume also that the metric \( h^P \) on \( P \) is \( G \)-invariant. In particular, the volume form \( \mu^P \) on \( P \) induced by \( h^P \) is also \( G \)-invariant. This leads naturally to consider the group \( \text{SAut}(P,\mu^P) \) of automorphisms of \( P \) preserving the volume form \( \mu^P \) instead of the group \( \text{SDiff}(P,\mu^P) \). In other words, we assume the vector field describing the velocity of the fluid to be initially \( G \)-invariant. This approach allows us to describe the Euler equation (in the presence of symmetries), as a system of two coupled equations, one living on the space of free divergence (for a certain volume form) vector fields on \( B \), the other living on the Lie algebra of the gauge group \( \text{Gau}(P) \) of \( P \). In some cases, these equations are a particular case of the Euler-Yang-Mills equation of an incompressible classical charged ideal fluid moving on \( B \), and are physically relevant for the cases \( G = S^1 \) (super-conductivity...
equation, see [21], \( G = SU(2) \) and \( G = SU(3) \) (chromohydrodynamics, see [5, 7]). The terminology “Euler-Yang-Mills equation” comes from [7].

The second section of this paper describes the Lie group structure of the group \( \text{SDiff}(M, \mu)^G \) of all \( G \)-equivariant diffeomorphisms of a compact manifold \( M \) which preserve a volume form \( \mu \). The arguments are essentially those used by Hamilton in [9], Theorem 2.5.3, except that one has to check the constructions involving the Nash-Moser inverse function theorem to “respect symmetries”. In section 3, the careful study of the “structure” of a \( G \)-invariant volume form \( \mu^F \) on the total space \( P \) of a principal bundle \( G \leftarrow P \rightarrow B \), allows us to give an integration formula (Proposition 3.11) which is necessary to determine the Euler equation of the group \( \text{SAut}(P, \mu^F) \) (Theorem 4.19). Finally in section 5, we show, in the same spirit of [1], that \( \text{SAut}(P, \mu^F) \) is a \( \text{Gau}(P) \)-principal bundle whose base is a collection of connected components of \( \text{SDiff}(B, \nu_B^B) \), where \( \nu_B^B \) is a volume form on \( B \) related to the volume of the orbits of \( P \). In particular, \( \text{SAut}(P, \mu^F) \) is a non-abelian extension of this collection of connected components of \( \text{SDiff}(B, \nu_B^B) \) by the gauge group \( \text{Gau}(P) \).

2 The group \( \text{SDiff}(M, \mu)^G \) as a tame Lie group

This section deals with the differentiable and Lie group structure of some subgroups of the group of smooth diffeomorphisms of a compact manifold, using the infinite dimensional geometry point of view. For that purpose, we will use the category of tame Fréchet manifolds developed by Hamilton in [9], and not simply the usual category of Fréchet manifolds [1]. This choice is motivated by the necessity to use an inverse function theorem, which is available in Hamilton’s category contrary to the general Fréchet setting. For the convenience of the reader, we recall here the basic definitions relevant for Hamilton’s category:

**Definition 2.1.** (i) A graded Fréchet space \( (F, \{\|\cdot\|_n\}_{n \in \mathbb{N}}) \), is a Fréchet space \( F \) whose topology is defined by a collection of seminorms \( \{\|\cdot\|_n\}_{n \in \mathbb{N}} \) which are increasing in strength:

\[
\|x\|_0 \leq \|x\|_1 \leq \|x\|_2 \leq \cdots
\]

for all \( x \in F \).

(ii) A linear map \( L : F \rightarrow G \) between two graded Fréchet spaces \( F \) and \( G \) is tame (of degree \( r \) and base \( b \)) if for all \( n \geq b \), there exists a constant \( C_n > 0 \) such that for all \( x \in F \),

\[
\|L(x)\|_n \leq C_n \|x\|_{n+r}.
\]

(iii) If \( (B, \|\cdot\|_B) \) is a Banach space, then \( \Sigma(B) \) denotes the graded Fréchet space of all sequences \( \{x_k\}_{k \in \mathbb{N}} \) of \( B \) such that for all \( n \geq 0 \),

\[
\|\{x_k\}_{k \in \mathbb{N}}\|_n := \sum_{k=0}^{\infty} e^{nk}\|x_k\|_B < \infty.
\]

(iv) A graded Fréchet space \( F \) is tame if there exist a Banach space \( B \) and two tame linear maps \( i : F \rightarrow \Sigma(B) \) and \( p : \Sigma(B) \rightarrow F \) such that \( p \circ i \) is the identity on \( F \).

(v) Let \( F, G \) be two tame Fréchet spaces, \( U \) an open subset of \( F \) and \( f : U \rightarrow G \) a map. We say that \( f \) is a smooth tame map if \( f \) is smooth\(^1\) and if for every \( k \in \mathbb{N} \) and for every \( (x, u_1, \ldots, u_k) \in U \times F \times \cdots \times F \),

\(^1\)The reader should be aware that beyond the Banach case, several nonequivalent theories of infinite dimensional manifolds coexist (see [13]), but when the modelling spaces are Fréchet manifolds, then most of these theories coincide, and it is thus natural to talk, without any further references, of a Fréchet manifold (as defined in [9] for example).

\(^2\)By smooth we mean that \( f : U \subseteq F \rightarrow G \) is continuous and that for all \( k \in \mathbb{N} \), the kth derivative \( D^k f : U \times F \times \cdots \times F \rightarrow \) exists and is jointly continuous on the product space, such as described in [9].
there exist a neighborhood $V$ of $(x,u_1,\ldots,u_k)$ in $U \times F \times \cdots \times F$ and $b_k, r_0, \ldots, r_k \in \mathbb{N}$ such that for every $n \geq b_k$, there exists $C_{k,n}^V > 0$ such that

$$
\|D^k f(y)\{v_1,\ldots,v_k\}\|_n \leq C_{k,n}^V (1 + \|y\|_{n+r_0} + \|v_1\|_{n+r_1} + \cdots + \|v_k\|_{n+r_k}),
$$

for every $(y,v_1,\ldots,v_k) \in V$, where $D^k f : U \times F \times \cdots \times F \to G$ denotes the $k$th derivative of $f$.

**Remark 2.2.** In the sequel, we will use interchangeably the notation $(Df)(x)\{v\}$ or $f_* v$ for the first derivative of $f$ at a point $x$ in direction $v$.

As one may notice, tame Fréchet spaces and smooth tame maps form a category, and it is thus natural to define a tame Fréchet manifold as a Hausdorff topological space with an atlas of coordinates charts taking their value in tame Fréchet spaces, such that the coordinate transition functions are all smooth tame maps (see [9]). The definition of a tame smooth map between tame Fréchet manifolds is then straightforward, and we thus obtain a subcategory of the category of Fréchet manifolds.

In order to avoid confusion, let us also precise our notion of submanifold. We will say that a subset $M$ of a tame Fréchet manifold $N$, endowed with the trace topology, is a submanifold, if for every point $x \in M$, there exists a chart $(U, \varphi)$ of $N$ such that $x \in U$ and such that $\varphi(U \cap M) = U \times \{0\}$, where $\varphi(U) = U \times V$ is a product of two open subsets of tame Fréchet spaces. Note that a submanifold of a tame Fréchet manifold is also a tame Fréchet manifold.

Finally, we define a tame Lie group $G$ as a tame Fréchet manifold with a group structure such that the multiplication map $G \times G \to G$, $(g,h) \mapsto gh$ and the inverse map $G \to G$, $g \mapsto g^{-1}$ are smooth tame maps. A tame Lie subgroup is defined as being a subset of a tame Lie group which is a submanifold and a subgroup. A tame Lie subgroup is in particular a tame Lie group.

**Remark 2.3.** The above notions of submanifolds, Lie groups and Lie subgroups are stated in the framework of tame Fréchet manifolds, but of course, similar definitions –that we adopt– hold in the more general framework of Fréchet manifolds.

For the sake of completeness, let us state here the raison d’être of tame Fréchet spaces and tame Fréchet manifolds (see [9]):

**Theorem 2.4** (Nash-Moser inverse function Theorem). Let $F,G$ be two tame Fréchet spaces, $U$ an open subset of $F$ and $f : U \to G$ a smooth tame map. If there exists an open subset $V \subseteq U$ such that

(i) $Df(x) : F \to G$ is an linear isomorphism for all $x \in V$,

(ii) the map $V \times G \to F$, $(x,v) \mapsto (Df(x))^{-1}\{v\}$ is a smooth tame map,

then $f$ is locally invertible on $V$ and each local inverse is a smooth tame map.

**Remark 2.5.** The Nash-Moser inverse function Theorem is important in geometric hydrodynamics, since one of its most important geometric object, namely the group of all smooth volume preserving diffeomorphisms $\text{SDiff}(M,\mu) := \{\varphi \in \text{Diff}(M) \mid \varphi^* \mu = \mu\}$ of an oriented manifold $(M,\mu)$, can only be given a rigorous Fréchet Lie group structure by using an inverse function theorem (at least up to now). To our knowledge, only two authors succeeded in doing this. The first was Omori who showed and used an inverse function theorem in terms of ILB-spaces (“inverse limit of Banach spaces”, see [13]), and later on, Hamilton with his category of tame Fréchet spaces together with the Nash-Moser inverse function Theorem (see [9]). Nowadays, it is nevertheless not uncommon to find mistakes or big gaps in the literature when it comes to the differentiable structure of $\text{SDiff}(M,\mu)$, even in some specialized textbooks in infinite dimensional geometry. The case of $M$ being non-compact is even worse, and of course, no proof that $\text{SDiff}(M,\mu)$ is a “Lie group” is available in this case.
Now let $M$ be a compact manifold and $G$ a compact and connected Lie group acting on $M$. The action of $G$ is denoted by $\vartheta : G \times M \to M$ and for $g \in G$, we write $\vartheta_g : M \to M$, $x \mapsto \vartheta(g, x)$.

**Proposition 2.6.** The group $\text{Diff}(M)^G := \{ \varphi \in \text{Diff}(M) \mid \vartheta_g \circ \varphi = \varphi \circ \vartheta_g, \forall g \in G \}$ is a tame Lie subgroup of the group $\text{Diff}(M)$. Its Lie algebra is the space $\mathfrak{X}(M)^G := \{ X \in \mathfrak{X}(M) \mid \vartheta_{g*}X = X, \forall g \in G \}$.

**Proof.** Choose a $G$-invariant metric $h$ on $M$ and define a map $pr : \Omega^1(M) \to \Omega^1(M)$, $\theta \mapsto \theta^G$ by

$$\theta^G_x(X_x) := \frac{1}{\text{Vol}(G)} \int_G (\vartheta_g^*\theta)_x(X_x) \nu^G,$$

where $X_x \in T_x M$. Since $pr$ is a continuous projection, we have the following topological direct sum:

$$\Omega^1(M) = \Omega^1(M)^G \oplus \ker(pr),$$

and as $h$ is $G$-invariant,

$$\mathfrak{X}(M) = \mathfrak{X}(M)^G \oplus \ker(pr),$$

where $\tilde{pr} : \mathfrak{X}(M) \to \mathfrak{X}(M)^G$ is the projection obtained from $pr$ using the duality between $TM$ and $T^*M$ via the metric $h$. Notice that the decomposition (5) implies that $\mathfrak{X}(M)^G$ is a tame Fréchet space (it’s a Fréchet space because $\mathfrak{X}(M)^G$ is closed in $\mathfrak{X}(M)$ and it’s also a tame space because $\mathfrak{X}(M)$ is tame, see [3], Definition 1.3.1 and Corollary 1.3.9).

Let $(\mathcal{U}, \varphi)$ be the “standard” chart of $\text{Diff}(M)$ at the identity element $\text{Id}_M$ obtained using the metric $h$, i.e., $\varphi(\mathcal{U}) \subseteq \mathfrak{X}(M)$ and $\varphi^{-1}(X)(x) = \exp_x(X)$ for $X \in \varphi(\mathcal{U}) \subseteq \mathfrak{X}(M)$ and $x \in M$.

Restricting $\mathcal{U}$ if necessary, we may assume $\varphi(\mathcal{U}) = U_1 \times U_2$ where $U_1$ is an open subset of $\mathfrak{X}(M)^G$ and $U_2$ an open subset of $\ker(\tilde{pr})$. From the $G$-invariance of $h$, we also have:

(i) if $X \in U_1$, then $\varphi^{-1}(X) \in \text{Diff}(M)^G$,

(ii) $\exp_{\vartheta^{-1}_g}(\vartheta_{g*}X, X) = \vartheta_g(\exp_g(X))$ for all $x \in M$, for all $X_x \in T_x M$ and for all $g \in G$.

From (i) we get $\varphi^{-1}(U_1 \times \{0\}) \subseteq U \cap \text{Diff}(M)^G$.

On the other hand, if $X \in \varphi(\mathcal{U})$ is such that $\varphi^{-1}(X) \in U \cap \text{Diff}(M)^G$, then for all $g \in G$:

$$\vartheta_g \circ (\varphi^{-1}(X)) = ((\varphi^{-1}(X)) \circ \vartheta_g) \Rightarrow \vartheta_g(\exp_x(X)) = \exp_{\vartheta^{-1}_g(x)} \vartheta^{-1}_g(x) \forall x \in M.$$
A proof of Lemma 2.8 is available in [1], page 341 or [5]. Note that in the decomposition \( \mathcal{Q} \), the space \( \nabla \Omega^0(M) \) is isomorphic to \( C_0^\infty(M, \mathbb{R}) \) where \( C_0^\infty(M, \mathbb{R}) := \{ f \in C^\infty(M, \mathbb{R}) | \int_M f \mu = 0 \} \).

**Lemma 2.9.** Let \( G \) be a connected, compact Lie group which acts by isometries on a Riemannian manifold \((M, h)\). We assume \( M \) compact, connected and oriented, the orientation being given by \( \mu := d\text{vol}_h \).

If \( X = X^\mu + \nabla f \) is the Helmholtz-Hodge decomposition of a vector field \( X \in \mathfrak{X}(M) \) (i.e. \( X^\mu \in \mathfrak{X}(M, \mu) \) and \( f \in C^\infty_0(M, \mathbb{R}) \)), then we have the following equivalence:

\[
X \in \mathfrak{X}(M)^G \Leftrightarrow X^\mu \in \mathfrak{X}(M, \mu)^G \text{ and } f \in C^\infty_0(M, \mathbb{R})^G.
\]

In other words,

\[
\mathfrak{X}(M)^G = \mathfrak{X}(M, \mu)^G \oplus C^\infty_0(M, \mathbb{R})^G,
\]

where \( C^\infty_0(M, \mathbb{R})^G := \{ f \in C^\infty_0(M, \mathbb{R}) | f \circ \vartheta_g = f, \forall g \in G \} \) (we denote by \( \vartheta : G \times M \rightarrow M \) the action of \( G \) on \( M \)).

**Proof.** Let \( X = X^\mu + \nabla f \in \mathfrak{X}(M)^G \) be the Helmholtz-Hodge decomposition of a \( G \)-invariant vector field. For \( g \in G \), we have:

\[
\text{div}(X) = \text{div}(X^\mu) + \Delta f \Rightarrow \text{div}(X) = \Delta f \Rightarrow \text{div}(X) \circ \vartheta_g = \Delta f \circ \vartheta_g.
\]

On the other hand, as \( X \) and \( h \) are \( G \)-invariant,

\[
\text{div}(X) \circ \vartheta_g = \text{div}(X) \quad \text{and} \quad (\Delta f) \circ \vartheta_g = (\Delta f \circ \vartheta_g).
\]

From (8) together with (9), we get

\[
\text{div}(X) = \Delta(f \circ \vartheta_g).
\]

We deduce from (8) and (10) that \( f \) and \( f \circ \vartheta_g \) satisfy the same elliptic equation on a compact connected manifold, and it is well known (see for example [12]), that the kernel of the Laplacian \( \Delta \) on the space \( C^\infty(M, \mathbb{R}) \) is reduced to the space of constant functions. Hence \( f \circ \vartheta_g = f + c(g) \) where \( c(g) \in \mathbb{R} \), and as \( \int_M f \mu = 0 \), we must have \( c(g) = 0 \) for all \( g \in G \), i.e. \( f \in C^\infty_0(M, \mathbb{R})^G \). It follows that \( X^\mu = X - \nabla f \in \mathfrak{X}(M, \mu)^G \) since \( X \) and \( \nabla f \) are \( G \)-invariant.

The other implication being trivial, the lemma follows. \( \square \)

Let us introduce some terminology before the second lemma. Let \( (\mathcal{U}, \varphi) \) be the “standard” chart of \( \text{Diff}(M) \) near the identity element \( Id_M \) such as in the proof of Proposition 2.6, constructed from a \( G \)-invariant metric \( h \) (note that we can take \( h = \text{dvol}_h \)). For \( X \in \varphi(\mathcal{U}) \), define \( P(X) \in C^\infty(M, \mathbb{R}) \) by:

\[
(\varphi^{-1}(X))^* \mu = P(X) \cdot \mu.
\]

Without loss of generality, we may assume the volume form \( \mu \) to be normalized and take \( \mathcal{U} \) such that \( \int_M P(X) \mu = 1 \) for all \( X \in \mathcal{U} \). According to the Helmholtz-Hodge decomposition, we have the following direct sum

\[
\mathfrak{X}(M) = \mathfrak{X}(M, \mu) \oplus C^\infty_0(M, \mathbb{R})
\]

which allows us to define a map

\[
Q : \{ \varphi(\mathcal{U}) \subseteq \mathfrak{X}(M) = \mathfrak{X}(M, \mu) \oplus C^\infty_0(M, \mathbb{R}) \rightarrow \mathfrak{X}(M, \mu) \oplus C^\infty_0(M, \mathbb{R}); (X, f) \mapsto (X, P(X + \nabla f) - 1) \}
\]

It is shown in [9], Theorem 2.5.3, that \( Q \) is invertible in a neighborhood of 0 in \( \mathfrak{X}(M) \). The following lemma shows also that \( Q \) is compatible with the symmetries of \( M \).
Lemma 2.10. For all sufficiently small neighborhoods \( K \) of 0 in \( \mathfrak{X}(M) \), we have
\[
Q \left( K \cap \mathfrak{X}(M)^G \right) = Q(K) \cap \mathfrak{X}(M)^G. \tag{11}
\]

Proof. From the inverse function Theorem of Nash-Moser, there exists \( W \subseteq \mathfrak{X}(M) \), a neighborhood of 0 in \( \mathfrak{X}(M) \), \( V_1 \) a neighborhood of 0 in \( \mathfrak{X}(M, \mu) \) and \( V_2 \) a neighborhood of 0 in \( C^\infty_0(M, \mathbb{R}) \) such that
\[
Q|_{V_1 \times V_2} : V_1 \times V_2 \to W
\]
is a diffeomorphism. Let us make the following two observations:

- restricting \( \mathcal{U} \) if necessary, we may assume \( \varphi(\mathcal{U}) = V_1 \times V_2 \),
- by compactness of the group \( G \) and continuity of the map \( G \times V_2 \to C^\infty_0(M, \mathbb{R}), (g, f) \to f \circ \vartheta_g \), we can find \( \tilde{V}_2 \subseteq V_2 \) a neighborhood of 0 in \( C^\infty_0(M, \mathbb{R}) \) such that if \( f \in \tilde{V}_2 \), then \( f \circ \vartheta_g \in V_2 \) for all \( g \in G \).

Let us show that the map \( Q \) restricted to \( (V_1 \times \tilde{V}_2) \cap \mathfrak{X}(M)^G \) is a diffeomorphism from \( (V_1 \times \tilde{V}_2) \cap \mathfrak{X}(M)^G \) onto \( Q(V_1 \times \tilde{V}_2) \cap \mathfrak{X}(M)^G \). For that purpose, it is sufficient to show that
\[
Q((V_1 \times \tilde{V}_2) \cap \mathfrak{X}(M)^G) = Q(V_1 \times \tilde{V}_2) \cap \mathfrak{X}(M)^G. \tag{12}
\]

According to Lemma 2.9, and since \( h \) is \( G \)-invariant, the inclusion from the left-handside to the right-handside of (12) is clear.

Let us show the inverse inclusion. For \((X, P(X + \nabla f) - 1) \in Q(V_1 \times \tilde{V}_2) \cap \mathfrak{X}(M)^G \), we have according to Lemma 2.9,
\[
X \in \mathfrak{X}(M, \mu)^G \text{ and } P(X + \nabla f) - 1 \in C^\infty_0(M, \mathbb{R})^G.
\]

Thus, for \( g \in G \):
\[
\left( P(X + \nabla f) - 1 \right) \circ \vartheta_g = P(X + \nabla f) - 1
\Rightarrow P(X + \nabla f) \circ \vartheta_g - 1 = P(X + \nabla f) - 1
\Rightarrow P((\vartheta_g)^*(X + \nabla f)) = P(X + \nabla f) \quad (P \text{ is } G \text{-invariant})
\Rightarrow P((X + \nabla (f \circ \vartheta_g))) = P(X + \nabla f)
\Rightarrow Q(X, f \circ \vartheta_g) = Q(X, f)
\Rightarrow f \circ \vartheta_g = f \quad (Q \text{ is a diffeomorphism on } V_1 \times V_2).
\]

Hence, \((X, f) \in (V_1 \times \tilde{V}_2) \cap \mathfrak{X}(M)^G \) which implies (12). It follows that (11) holds for all sufficiently small neighborhoods \( K \) of 0 in \( \mathfrak{X}(M) \).

Proof of Proposition 2.7. Let us recall how to construct a chart centered at \( Id_M \) of the group \( \text{SDiff}(M, \mu) \) using the map \( Q \). According to the proof of Theorem 2.5.3. in [9] and restricting the domain \( \mathcal{U} \) of the chart \((\mathcal{U}, \varphi)\) if necessary, we can find \( K_1 \subseteq \mathfrak{X}(M, \mu) \) and \( K_2 \in C^\infty_0(M, \mathbb{R}) \), two neighborhoods of 0 in \( \mathfrak{X}(M, \mu) \) and \( C^\infty_0(M, \mathbb{R}) \) respectively, such that \( Q : \varphi(\mathcal{U}) \to K_1 \times K_2 \) becomes a diffeomorphism. Then, denoting \( \mathcal{U}^S := \mathcal{U} \cap \text{SDiff}(M, \mu) \), one can check that \((\mathcal{U}^S, (Q|_{\varphi(\mathcal{U}^S)}) \circ (\varphi|_{\mathcal{U}^S}))\) is a chart of \( \text{SDiff}(M, \mu) \), i.e.,
\[
((Q|_{\varphi(\mathcal{U}^S)}) \circ (\varphi|_{\mathcal{U}^S}))^{-1}(K_1 \times \{0\}) = \mathcal{U}^S. \]
Lemma 11 that we may also assume $Q(\varphi(U) \cap X(M)^G) = Q(\varphi(U)) \cap X(M)^G$. We then get the following commutative diagram:

notations being obvious, for example, $U^G := U \cap \text{Diff}(M)^G$. Clearly,

$$
\left( U^{S,G}, \left( Q \circ \varphi \right) \big|_{U^G} \right) \circ \left( \varphi \big|_{U^{S,G}} \right) = \left( \varphi \big|_{U^{S,G}} \right) \circ \left( Q \circ \varphi \right) \big|_{U^G}
$$

is a chart of $\text{SDiff}(M,\mu)^G$ (where $U^{S,G} := U^S \cap U^G$) and therefore $\text{SDiff}(M,\mu)^G$ is a submanifold of $\text{Diff}(M)^G$ in a neighborhood of the identity. By translations, $\text{SDiff}(M,\mu)^G$ becomes a tame Lie subgroup of $\text{Diff}(M)^G$.

The fact that $\text{SDiff}(M,\mu)^G$ is also a Lie subgroup of $\text{SDiff}(M,\mu)$ can be proved similarly using the same techniques appearing above and in Proposition (2.6).

\section{Some integration formulas for a principal bundle}

Let $G \rightarrow P \xrightarrow{\pi} B$ be a principal bundle and $h^P$ a $G$-invariant metric on $P$ (we assume that $G$ and $P$ are compact and connected). In this section, we shall use the following terminology:

• $\vartheta : P \times G \rightarrow P$ is the right action of the structure group $G$ on the total space $P$,

• $O_x \subseteq P$ is the orbit through the point $x \in P$ for the action $\vartheta$,

• given $g \in G$ and $x \in P$, we write $\vartheta_g : P \rightarrow P$, $x \mapsto \vartheta(x, g)$ and $\vartheta_x : G \xrightarrow{\cong} O_x \subseteq P$, $g \mapsto \vartheta(x, g)$ for the associated maps (note that $\vartheta_x$ is a diffeomorphism from $G$ onto $O_x$, thus, one can consider the map $\vartheta_x^{-1} : O_x \rightarrow G$),

• if $X_x \in T_xP$ for a given point $x \in P$, we denote by $X^v$ the orthogonal projection of $X_x$ on $T_xO_x$ and $X^h$ the component of $X_x$ perpendicular to $T_xO_x$,

• the Lie algebra of the group $G$ is denoted by $\mathfrak{g}$.

The metric $h^P$ being $G$-invariant, we naturally get an induced connection form $\theta \in \Omega^1(P, \mathfrak{g})$ which is defined, for $x \in P$ and $X_x \in T_xP$, by:

$$
\theta_x(X_x) := (\vartheta_x^{-1})^* X^v_x, \quad X^v_x \in \mathfrak{g}.
$$
In particular, one can check that

\[(\varphi_g)^* \theta = Ad(g^{-1}) \theta,\]  

for all \(g \in G\). Recall also that for any vector field \(Z \in \mathfrak{X}(B)\), there exists a unique horizontal lift \(Z^* \in \mathfrak{X}(P)^G\) satisfying \(\pi_* Z^*_x = Z_{\pi(x)}\) for all \(x \in P\) (see [13]). The following easy lemma describes more precisely the metric \(h^B\).

**Lemma 3.1.** There exists a metric \(h^B\) on \(B\) and an Euclidean structure \(h^\theta\) on the trivial bundle \(P \times \mathfrak{g}\) such that:

(i) \(h^B \circ (\pi \times \text{id}) = h^B \circ \pi \),

(ii) \(\pi : (P, h^P) \to (B, h^B)\) is a Riemannian submersion,

(iii) \(h^\theta \circ (\pi \times \text{id}) = h^\theta \circ \pi\) for all \(g \in G\), \(x \in P\) and \(\xi, \zeta \in \mathfrak{g}\).

**Remark 3.2.** The point (i) of Lemma 3.1 gives a decomposition of the metric \(h^P\) with respect to the horizontal and vertical tangent vectors of \(P\).

**Remark 3.3.** For \(x \in P\), \(\ker(\pi_* x) = T_x \mathcal{O}_x\) and the map \(\pi_* x \mid_{(T_x \mathcal{O}_x)^\perp} : (T_x \mathcal{O}_x)^\perp \to T_{\pi(x)} B\) is a linear isomorphism (see [15]). In particular, there exists a canonical isomorphism between the space of \(G\)-invariant horizontal vector fields on \(P\) and the space of vector fields on \(B\).

Now, let us assume that \(P\) and \(B\) are oriented and let us denote by \(\mu^P\) and \(\mu^B\) the natural volume forms induced respectively on \(P\) and \(B\) by the metrics \(h^P\) and \(h^B\). As for the metric \(h^P\), we want to give a precise description of the volume form \(\mu^P\).

**Lemma 3.4.** Let \((E, h)\) be an Euclidean oriented vector space of finite dimension. We assume that \(E = E_1 \oplus E_2\) and also that \(h = p_1^* h^{E_1} + p_2^* h^{E_2}\) where \(h^{E_1}\) is a metric on \(E_1\) and \(p_i : E_1 \oplus E_2 \to E_i\) the canonical projection.

If \(E_1\) is endowed with a given orientation, then

\[\mu^E = p_1^* \mu^{E_1} \wedge p_2^* \mu^{E_2},\]

where \(\mu^E, \mu^{E_1}\) are the volume forms associated to the metrics \(h, h^{E_1}\) respectively (we adopt the following convention: a basis \(\{f_1, \ldots, f_m\}\) of \(E_2\) is positive if and only if the family \(\{e_1, \ldots, e_n, f_1, \ldots, f_m\}\) is a positive basis of \(E\) whenever \(\{e_1, \ldots, e_n\}\) is a positive basis of \(E_1\)).

**Proof.** Let \(\{e_1, \ldots, e_n\}\) be a positive basis for \(E_1\) and \(\{f_1, \ldots, f_m\}\) a positive basis for \(E_2\), the corresponding dual basis being respectively \(\{e_1^*, \ldots, e_n^*\}\) and \(\{f_1^*, \ldots, f_m^*\}\). We introduce also \(h_{ij}^{E_1} := h^{E_1}(e_i, e_j)\) for \(i, j \in \{1, \ldots, n\}\) and \(h_{ij}^{E_2} := h^{E_2}(f_i, f_j)\) for \(i, j \in \{1, \ldots, m\}\).

From the definition of the volume form induced by a metric, we have

\[\mu^E = (\det (h_{ij}^{E_1}))^{\frac{1}{2}} (\det (h_{ij}^{E_2}))^{\frac{1}{2}} e_1^* \wedge \cdots \wedge e_n^* \wedge f_1^* \wedge \cdots \wedge f_m^*;\]

on the other hand,

\[\mu^{E_1} = \det (h_{ij}^{E_1})^{\frac{1}{2}} e_1^* \mid_{E_1} \wedge \cdots \wedge e_n^* \mid_{E_1} \Rightarrow p_1^* \mu^{E_1} = (\det (h_{ij}^{E_1}))^{\frac{1}{2}} e_1^* \wedge \cdots \wedge e_n^*\]

and similarly, \(p_2^* \mu^{E_2} = (\det (h_{ij}^{E_2}))^{\frac{1}{2}} f_1^* \wedge \cdots \wedge f_m^*\). Hence,

\[p_1^* \mu^{E_1} \wedge p_2^* \mu^{E_2} = (\det (h_{ij}^{E_1}))^{\frac{1}{2}} (\det (h_{ij}^{E_2}))^{\frac{1}{2}} e_1^* \wedge \cdots \wedge e_n^* \wedge f_1^* \wedge \cdots \wedge f_m^* = \mu^E.\]

This proves the lemma. \(\square\)

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For the above notations, we have Lemma 3.6.

For $i \in \{1, 2\}$, $h^{E_i}$ is a metric on $E_i$ and we have $h^P_x = h^{E_1}_x + h^{E_2}_x$ where $p_i : E_1 \oplus E_2 \to E_i$ is the canonical projection. Since we assume the manifold $B$ oriented, the space $E_1$ is also oriented by the isomorphism $\pi_*|_{E_1} \to T_{\pi(x)}B$. We fix on $E_2$ the orientation given by Lemma 3.3. We then have:

$$
\mu^P_x = p_1^* \mu^{E_1} \wedge p_2^* \mu^{E_2}.
$$

\textbf{Remark 3.5.} Orientations on $P$ and $B$ induce an orientation on $G$ in the following way: for $x \in P$, the spaces $T_\xi \mathcal{O}_x$ and $\mathfrak{g}$ are isomorphic via the map $\theta_x|_{T_\xi \mathcal{O}_x} : T_\xi \mathcal{O}_x \to \mathfrak{g}$. But, the space $T_\xi \mathcal{O}_x$ being oriented (see above), the Lie algebra $\mathfrak{g}$ is also oriented and induces an orientation on $G$. This orientation doesn’t depend on the point $x \in P$. In fact, if $\mu^B_x$ is the volume form on $\mathfrak{g}$ induced by the metric $h^B_x$, it is obvious that $p^*_1 \mu^B$ depends continuously of the point $x \in P$, and the orientation induced by $\mu^B_x$ cannot be reversed.

\textbf{Lemma 3.6.} With the above notations, we have

$$
p_1^* \mu^{E_1} = (\pi^* \mu^B)_x.
$$

\textbf{Proof.} Let $(U, \varphi)$ be a positive chart of $B$ containing $\pi(x)$ with local coordinates $\{x_1, \ldots, x_n\}$. This gives a positive basis for $E_1$:

$$
\left\{ \left. (\pi_*|_{E_1})^{-1} \frac{\partial}{\partial x_i} \right|_{\pi(x)} \right\}, \quad i = 1, \ldots, n.
$$

For $i \in \{1, \ldots, n\}$, define

$$
e_i := \left. (\pi_*|_{E_1})^{-1} \frac{\partial}{\partial x_i} \right|_{\pi(x)}.
$$

We have

$$
\mu^{E_1} = \det(h^{E_1}_{ij}) e_1^* \wedge \cdots \wedge e_n^*|_{E_1}
$$

with

$$
h^{E_1}_{ij} = h^{E_1}(e_i, e_j) = h^B_{\pi(x)}(\pi_* e_i, \pi_* e_j) = h^B_{\pi(x)} \left( \left. \frac{\partial}{\partial \pi_i} \right|_{\pi(x)}, \left. \frac{\partial}{\partial \pi_j} \right|_{\pi(x)} \right) = (h^B_{\pi} \circ \pi)(x).
$$

Hence,

$$
p_1^* \mu^{E_1} = (\det(h^{E_1}_{ij}) \circ \pi)(x) e_1^* \wedge \cdots \wedge e_n^*.
$$

On the other hand,

$$
(\pi^* \mu^B)_x(e_1, \ldots, e_n) = \mu^B_{\pi(x)}(\pi_* e_1, \ldots, \pi_* e_n) = \mu^B_{\pi(x)} \left( \left. \frac{\partial}{\partial x_1} \right|_{\pi(x)}, \ldots, \left. \frac{\partial}{\partial x_n} \right|_{\pi(x)} \right) = (\det(h^B_{\pi} \circ \pi)(x)
$$

which implies that

$$
(\pi^* \mu^B)_x = (\det(h^{E_1}_{ij}) \circ \pi)(x) e_1^* \wedge \cdots \wedge e_n^* = p_1^* \mu^B.
$$

Thus, $p_1^* \mu^{E_1} = (\pi^* \mu^B)_x$.

\textbf{Lemma 3.7.} With the notations introduced before Lemma 3.6, we have

$$
p_2^* \mu^{E_2} = \theta_x^* \mu^B_x,
$$

where $\mu^B_x$ is the volume form on $\mathfrak{g}$ induced by the metric $h^B_x$ (see Remark 3.5) and where $\theta_x^* \mu^B_x$ is the pullback of $\mu^B_x$ by the linear map $\theta_x : T_xP \to \mathfrak{g}$.
Proof. Let \{\xi_1, \ldots, \xi_m\} be a positive basis for g (see Remark 3.5 for the question of the orientation of g). The family \{(d_x)_* \xi_1, \ldots, (d_x)_* \xi_m\} is a positive basis for E_2 and we have the formula:
\[ \mu^{E_2} = \det (h^{E_2})^\frac{1}{2} ( (d_x)_* \xi_1)^b \wedge \cdots \wedge ( (d_x)_* \xi_1)^b \]  
(21)
where "\^b" denotes the “dualisation” operator with respect to the metric \(h^{E_2}\). But,
\[ h^{E_2}_{ij} = h^{E_2}((d_x)_* \xi_i, (d_x)_* \xi_j) = h^2_x((d_x)_* \xi_i, (d_x)_* \xi_j) = h^2_x(\xi_i, \xi_j) = (h^2_\gamma)_{ij} \]  
(22)
and one can check, for \(u \in E_2\), that
\[ ((d_x)_* \xi_j)^b u = \xi_j^b(\theta_x(u)) \Rightarrow ( (d_x)_* \xi_j)^b = ( (d_x^{-1})_* \xi_j)^b. \]  
(23)
From (22) and (23) applied to (24), it follows that:
\[ \mu^{E_2} = \det (h^\theta_{ij})^\frac{1}{2} ( (d_x^{-1})_* \xi_1)^b \wedge \cdots \wedge ( (d_x^{-1})_* \xi_m)^b \]
\[ = \det (h^\theta_{ij})^\frac{1}{2} ( (d_x^{-1})_* \xi_1)^b \wedge \cdots \wedge (d_x^{-1})_* \xi_m)^b \]
\[ = ( (d_x^{-1})_* \xi_m)^b \mu^\theta = ( (d_x^{-1})_* \xi_m)^b \nu^\theta. \]
Finally,
\[ p_2^* \mu^{E_2} = p_2^* ((d_x^{-1})_* \xi_1)^b \mu^\theta = ((d_x^{-1})_* \xi_1)^b \nu^\theta = \theta^x \nu^\theta \]
which is the desired formula.
\[ \square \]

From Lemma 3.6 and Lemma 3.7 it follows, using formula (16), that
\[ \mu^\theta_x = (\pi^* \mu^B)_x \wedge \theta^x \nu^\theta. \]  
(24)
Let us consider the unique normalized volume form \(\nu^G\) on G (note that \(\nu^G\) is bi-invariant since G is compact and connected). For \(x \in P\), let \(\tilde{V}(x)\) be the unique real number satisfying \(\tilde{V}(x) \cdot \nu^G_x = \mu^\theta_x\). The G-invariance of \(\mu^\theta\) implies the existence of a function \(V \in C^\infty(B, \mathbb{R}^*_+)\) such that \(\tilde{V} = V \circ \pi\). To summarize,

**Proposition 3.8.** There exists a function \(V \in C^\infty(B, \mathbb{R}^*_+)\) such that
\[ \mu^\theta = (\pi^* \mu^B)_x \wedge \theta^x \nu^G, \]
(25)
where \(\theta^x \nu^G \in \Omega^m(P) (m = \dim(G))\) is defined by
\[ (\theta^x \nu^G)_x(X_1, \ldots, X_m) = \nu^G(\theta_x(X_1), \ldots, \theta_x(X_m)) , \]
for any \(x \in P\) and \(X_1, \ldots, X_m \in T_x P\).

In order to give a geometrical interpretation to the function \(V\), let us make the following remark.

**Remark 3.9.** For \(x \in P\), the orbit \(O_x\) of \(P\) through the point \(x\) is canonically oriented via the orbit map \(d_x : G \to O_x\). This orientation on \(O_x\) doesn’t depend of the orbit map which is used because, for \(g \in G\), the connectedness of \(G\) implies that the map \(d_{\sigma_x(x)} : G \to O_x\) induces the same orientation. Thus, we can consider without ambiguities the volume form \(\mu^{O_x}\) of \(O_x\) induced by the restriction of the metric \(h^P\) on \(O_x\).
Lemma 3.10. For $x \in P$, we have the formula:

$$\mu^{O_x} = (V \circ \pi)(x) \cdot (\vartheta_x^{-1})^* \nu^G. \tag{26}$$

In particular, $V(\pi(x)) = \text{Vol}(O_x)$.

Proof. Let $f \in C^\infty(O_x, \mathbb{R})$ be the unique map satisfying

$$(V \circ \pi)(x) \cdot (\vartheta_x^{-1})^* \nu^G = f \cdot \mu^{O_x}. \tag{27}$$

Let $g \in G$ be arbitrary. The forms $\nu^G$ and $\mu^{O_x}$ being $G$-invariant, we have:

$$(V \circ \pi)(x) \cdot (\vartheta_x^{-1})^* \nu^G = f \cdot \mu^{O_x} \Rightarrow \vartheta_x \left( (V \circ \pi)(x) \cdot (\vartheta_x^{-1})^* \nu^G \right) = \vartheta_x \left( f \cdot \mu^{O_x} \right)$$

$$\Rightarrow (V \circ \pi)(x) \cdot (\vartheta_x^{-1} \circ \vartheta_x)^* \nu^G = f \circ \vartheta_x \cdot \mu^{O_x}$$

$$= L_g \circ \vartheta_x^{-1}$$

$$\Rightarrow (V \circ \pi)(x) \cdot (\vartheta_x^{-1})^* \nu^G = f \circ \vartheta_x \cdot \mu^{O_x}. \tag{28}$$

From (27) and (28), it follows that $f \circ \vartheta_x = f$ for all $g \in G$. This implies that $f$ is constant on $O_x$.

Let us show that $f(x) = 1$. Let $\{u_1, \ldots, u_m\}$ be an orthonormal positive basis for $T_xO_x$ (we assume that the dimension of $G$ is equal to $m$). Observe that the map $\theta_x|_{T_xO_x} = (\vartheta_x^{-1})_{\ast x} : (T_xO_x, h^\nu|_{T_xO_x}) \to (\mathfrak{g}, h^\nu_x)$ is:

- an isometry according to (i) in Lemma 3.1,
- an isomorphism which preserves the orientation according to Remark 3.5.

It follows that $\{(\vartheta_x^{-1})_{\ast x} u_1, \ldots, (\vartheta_x^{-1})_{\ast x} u_m\}$ is an orthonormal basis of $\mathfrak{g}$ and

$$\left( f(x) \cdot \mu^{O_x} \right)_{\ast x} (u_1, \ldots, u_m) = \left( (V \circ \pi)(x) \cdot (\vartheta_x^{-1})^* \nu^G \right)_{\ast x} (u_1, \ldots, u_m)$$

$$\Rightarrow f(x) = (V \circ \pi)(x) \cdot \nu^G \left( (\vartheta_x^{-1})_{\ast x} u_1, \ldots, (\vartheta_x^{-1})_{\ast x} u_m \right)$$

$$\Rightarrow f(x) = \mu^\nu_x \left( (\vartheta_x^{-1})_{\ast x} u_1, \ldots, (\vartheta_x^{-1})_{\ast x} u_m \right)$$

$$\Rightarrow f(x) = 1. \tag{29}$$

The lemma follows.

Before the end of this section, let us give an integration formula.

Proposition 3.11. For $f \in C^\infty(B, \mathbb{R})$, we have the following formula:

$$\int_P (f \circ \pi) \cdot \mu^P = \int_B f \cdot V \mu^B. \tag{29}$$

Proposition 3.11 can be shown using two lemmas.

Lemma 3.12. Let $E_1, E_2$ be two vector spaces of respective dimension $n$ and $m$, $\mu \in (\Lambda^n E^*_1) \setminus \{0\}$ and $p_i : E := E_1 \times E_2 \to E_i$ the canonical projection associated ($i = 1, 2$). For $\alpha \in \Lambda^m E^*$, we have:

$$p_i^\ast \mu \wedge \alpha = p_i^\ast \mu \wedge \alpha, \tag{30}$$

where $\alpha \in \Lambda^m E^*$ is defined, for $(u_1, v_1), \ldots, (u_m, v_m) \in E$, by:

$$\tilde{\alpha}(u_1, v_1), \ldots, (u_m, v_m) := \alpha(0, v_1), \ldots, (0, v_m). \tag{31}$$
Proof. Let \( \{x_1, \ldots, x_n\} \) be a basis of \( E_1 \), \( \{y_1, \ldots, y_m\} \) a basis of \( E_2 \) and let \( \{z_1, \ldots, z_{n+m}\} \) denote the basis of \( E \) canonically associated, i.e., \( \{z_1, \ldots, z_{n+m}\} := \{(x_1, 0), \ldots, (x_n, 0), (0, y_1), \ldots, (0, y_m)\} \). We can write

\[
\mu = \kappa \cdot x_1^* \wedge \cdots \wedge x_n^* \quad (\kappa \in \mathbb{R}^*) \quad \text{and} \quad \alpha = \sum_{1 \leq i_1 < \cdots < i_m \leq n+m} \alpha_{i_1 \ldots i_m} \cdot z_{i_1}^* \wedge \cdots \wedge z_{i_m}^*.
\]

We then have,

\[
p_i^* \mu \wedge \alpha = \kappa \cdot z_1^* \wedge \cdots \wedge z_n^* \wedge \sum_{1 \leq i_1 < \cdots < i_m \leq n+m} \alpha_{i_1 \ldots i_m} \cdot z_{i_1}^* \wedge \cdots \wedge z_{i_m}^*
\]

\[
= \kappa \sum_{1 \leq i_1 < \cdots < i_m \leq n+m} \alpha_{i_1 \ldots i_m} \cdot z_1^* \wedge \cdots \wedge z_n^* \wedge z_{i_1}^* \wedge \cdots \wedge z_{i_m}^*
\]

\[
= \kappa \alpha((0, y_1), \ldots, (0, y_m)) \cdot z_1^* \wedge \cdots \wedge z_{n+m}^*.
\]

(33)

On the other hand, if

\[
\tilde{\alpha} = \sum_{1 \leq i_1 < \cdots < i_m \leq n+m} \tilde{\alpha}_{i_1 \ldots i_m} \cdot z_{i_1}^* \wedge \cdots \wedge z_{i_m}^*,
\]

then, according to (31),

\[
\tilde{\alpha}_{i_1 \ldots i_m} = \tilde{\alpha}(z_{i_1}, \ldots, z_{i_m}) = \begin{cases} \alpha((0, y_1), \ldots, (0, y_m)), & \text{for } (i_1, \ldots, i_m) = (1, \ldots, m), \\ 0, & \text{otherwise}. \end{cases}
\]

(34)

The equality between (33) and \( p_i^* \mu \wedge \tilde{\alpha} \) now follows from (34).

For the second lemma, we fix a local trivialization \((U, \varphi)\) of \( B \):

\[
\pi^{-1}(U) \xrightarrow{\Psi} U \times G \xrightarrow{\pi} U \xrightarrow{p_2} p_1^n,
\]

(the map \( \Psi \) being \( G \)-equivariant).

Lemma 3.13. We have

\[
(\Psi^{-1})^* \mu^P = (V \circ pr_1) \cdot (pr_1^* \mu^B) \wedge (pr_2^* \nu^G).
\]

(35)

Proof. From (25),

\[
(\Psi^{-1})^* \mu^P = (\Psi^{-1})^* ((V \circ \pi) \cdot \pi^* \mu^B \wedge \theta^* \nu^G)
\]

\[
= (V \circ \pi \circ \Psi^{-1}) \cdot \left( (\Psi^{-1})^* \pi^* \mu^B \wedge (\Psi^{-1})^* \theta^* \nu^G \right)
\]

\[
= (V \circ pr_1) \cdot \left( pr_1^* \mu^B \wedge (\Psi^{-1})^* \theta^* \nu^G \right).
\]

(36)
For \((x, g) \in U \times G\), \(u_1, \ldots, u_m \in T_x B\) and \(\xi_1, \ldots, \xi_m \in T_g G\) (we assume the dimension of \(G\) equal to \(m\)), we have:

\[
\left( (\Psi^{-1})^* \theta^* \nu^G_{\psi} \right)_{(x, g)} (u_1, \xi_1), \ldots, (u_m, \xi_m) \\
= (\theta^* \nu^G_{\psi})_{(\Psi^{-1})(x, g)} \left( \Psi^{-1}_{\psi(x, g)} (u_1, \xi_1), \ldots, \Psi^{-1}_{\psi(x, g)} (u_m, \xi_m) \right) \\
= \nu^G_{\psi} \left( \theta_{(\Psi^{-1})(x, g)} \left( \Psi^{-1}_{\psi(x, g)} (u_1, \xi_1), \ldots, \Psi^{-1}_{\psi(x, g)} (u_m, \xi_m) \right) \right). \tag{37}
\]

Let \(s : U \to P\) be the local section which characterizes the trivialization \(\Psi\), i.e.,

\[
\Psi^{-1}(x, g) = \vartheta(s(x)) = \vartheta(s(x), g),
\]

for all \((x, g) \in U \times G\). For \(i \in \{1, \ldots, m\}\), we have

\[
\Psi^{-1}_{\psi(x, g)} (u_i, \xi_i) = \left[ (\vartheta_g)_{s_{\psi(x)}} \circ (\vartheta_{s(x)} x)_{s_{\psi(x)}} \right] u_i \xi_i \\
= (\vartheta_g)_{\psi(x)} u_i + (\vartheta_{s(x)} x)_{s_{\psi(x)} \xi_i},
\]

which yields, together with \(\Box\),

\[
\theta_{(\Psi^{-1})(x, g)} \left( \Psi^{-1}_{\psi(x, g)} (u_1, \xi_1), \ldots, \Psi^{-1}_{\psi(x, g)} (u_m, \xi_m) \right) \\
= \theta_{\vartheta_{s(x)}} \left( (\vartheta_g)_{s(x)} u_i \xi_i \right) \\
= \vartheta_g^{-1} \vartheta_{s(x)} \xi_i = (L_{g^{-1}})^* \xi_i. \tag{38}
\]

We can notice in formula \(\Box\), that

\[
\theta_{\vartheta_{s(x)}} \left( (\vartheta_{s(x)} x)_{s_{\psi(x)} \xi_i} \right) \\
= (\vartheta_{s_{\psi(x)}} \vartheta_{s(x)} x)_{s_{\psi(x)} \xi_i} \\
= (\vartheta_{s_{\psi(x)}} \vartheta_{s(x)} x)_{s_{\psi(x)} \xi_i}. \tag{39}
\]

It follows, taking \(u_i = 0\) in \(\Box\), that:

\[
\nu^G_{\psi} \left( \theta_{(\Psi^{-1})(x, g)} \left( \Psi^{-1}_{\psi(x, g)} (0, \xi_1), \ldots, \Psi^{-1}_{\psi(x, g)} (0, \xi_m) \right) \right) \\
= \nu^G_{\psi} \left( (L_{g^{-1}})^* \xi_1, \ldots, (L_{g^{-1}})^* \xi_m \right) \\
= \nu^G_{\psi} \left( \xi_1, \ldots, \xi_m \right). \\
\]

One concludes by applying Lemma 5.12. \(\Box\)

**Proof of Proposition 5.13** Let \(\{(U_i, \varphi_i) \mid i \in \{1, \ldots, s\}\}\) be an atlas of \(B\) whose charts \((U_i, \varphi_i)\) are positive and trivializing:

\[
\pi^{-1}(U_i) \xrightarrow{\Psi_i} U_i \times G \\
\pi \xrightarrow{pr_1} U_i \\
\]

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We also take \( \{(U_i, \varphi_i), \alpha_i \mid i \in \{1, \ldots, s\}\} \) a partition of unity of \( B \) subordinate to \( \{U_i \mid i \in \{1, \ldots, s\}\} \).

We take on \( U_i \times G \) the orientation induced by the volume form \( ((pr_1)^*(V\mu^B)) \land (pr_2)^*(\nu^G) \). For this orientation, \( \Psi_\text{reg} \) is a diffeomorphism which preserves orientation and from Lemma 3.13, we have

\[
\int_P (f \circ \pi) \cdot \mu^P = \sum_{i=1}^s \int_{U_i} (\alpha_i \circ \pi) \cdot (f \circ \pi) \cdot \mu^P = \sum_{i=1}^s \int_{\pi^{-1}(U_i)} (\alpha_i \circ \pi) \cdot (f \circ \pi) \cdot \mu^P
\]

This proves the proposition.

\[\square\]

4 The Euler equation of \( \text{SAut}(P, \mu^P) \)

For a Fréchet Lie algebra \( (g, [\cdot, \cdot]) \) endowed with a continuous, symmetric, weakly non-degenerate and positive-definite bilinear form \( <\cdot, \cdot> \), we define the regular dual \( g^*_\text{reg} \subseteq g^* \) of \( g \) as the range of the injective and continuous operator \( g \to g^*, \xi \to <\xi, \cdot> \). For \( \xi \in g \), we also define the operator \( ad^*(\xi) : g^*_\text{reg} \to g^* \) via the formula:

\[
(\text{ad}^*(\xi) \alpha, \xi') := - (\alpha, \text{ad}(\xi) \xi'),
\]

where \( \alpha \in g^*_\text{reg} \) and \( \xi' \in g \). Observe that the range of \( ad^*(\xi) \) is not necessarily included in \( g^*_\text{reg} \) (it is the case, for example if \( ad(\xi) \) possesses a transpose with respect to the metric \( <\cdot, \cdot> \).

**Definition 4.1.** If \( ad^*(\xi) \) takes values in \( g^*_\text{reg} \) for all \( \xi \in g \), we define the Euler equation associated to the Lie algebra \( g \) with respect to the metric \( <\cdot, \cdot> \) as:

\[
\frac{d}{dt}\eta = ad^*(\eta^*) \eta,
\]

where \( \eta \) is a smooth curve in \( g^*_\text{reg} \) and where \( "\overset{\circ}{\circ}\" : g^*_\text{reg} \to g \) denotes the canonical operator induced by the metric \( <\cdot, \cdot> \).

**Remark 4.2.** If \( g \) is a geodesic in a finite dimensional Lie group \( G \) with respect to a right-invariant metric \( <\cdot, \cdot> \), then the curve \( \eta := ([R_{g^{-1}}]_{o\theta})^\theta \), where \( R_{g^{-1}} : G \to G, h \mapsto h g^{-1} \), is a curve in \( g^* \) satisfying the Euler equation \( \overset{(1)}{\text{(i)}} \). Conversely, if \( \eta \) is a curve in \( g^* \) satisfying \( \overset{(1)}{\text{(i)}} \), then one may recover a geodesic in \( G \) via the “reconstruction procedure”, i.e., by solving a specific first order differential equation (see \( \overset{(1)}{\text{(1)}} \) for more details). The geodesic equation on a Lie group with respect to a right-invariant metric and the Euler equation \( \overset{(1)}{\text{(i)}} \) are thus equivalent.

We want next to determine the Euler equation of the Lie algebra of the group

\[
\text{SAut}(P, \mu^P) := \{ \varphi \in \text{Diff}(P) \mid \varphi^* \mu^P = \mu^P \text{ and } \varphi \text{ is } G\text{-equivariant} \},
\]

with respect to a natural \( L^2 \)-metric (see \( \overset{(1)}{\text{(2)}} \)). Note that \( \text{SAut}(P, \mu^P) = \text{SDiff}(P, \mu^P)^G \) and thus it is a tame Lie group by Proposition 2.7, and its Lie algebra is the space \( \mathfrak{X}(P, \mu^P)^G \) endowed with the opposite of the usual vector field bracket. Note also that \( \text{SAut}(P, \mu^P) = \text{Aut}(P) \cap \text{SDiff}(P, \mu^P) \) where

\[
\text{Aut}(P) := \{ \varphi \in \text{Diff}(P) \mid \varphi \text{ is } G\text{-equivariant} \}
\]

is the group of smooth automorphisms of \( P \).
4.1 The identification of $\mathfrak{X}(P,\mu^P)^G$ and $\mathfrak{X}(B,V\mu^B) \oplus C^\infty(P,\mathfrak{g})^G$

Set

$$C^\infty(P,\mathfrak{g})^G := \{ f \in C^\infty(P,\mathfrak{g}) \mid f \circ \theta_g = \text{Ad}(g^{-1}) f, \forall g \in G \}$$

and define $\Phi : \mathfrak{X}(P)^G \rightarrow \mathfrak{X}(B) \oplus C^\infty(P,\mathfrak{g})^G$ as:

$$\Phi(X) := \left( \pi_* X^h, \theta(X^v) \right), \quad (44)$$

where $X \in \mathfrak{X}(P)^G$ and where $\pi_* X^h \in \mathfrak{X}(B)$ denotes the vector field defined for $x = \pi(y) \in B$, by $(\pi_* X^h)_x := \pi_* X^h_y$. One can check using Remark 3.3 and (15) that $\Phi$ is well defined and invertible, the inverse being given by $(\Phi^{-1}(X,f))_x = X^v_x + (\vartheta_x)_* f(x)$, where $X \in \mathfrak{X}(B)$, $f \in C^\infty(P,\mathfrak{g})^G$ and $x \in P$. The space $\mathfrak{X}(P)^G$ being a Lie algebra, $\mathfrak{X}(B) \oplus C^\infty(P,\mathfrak{g})^G$ naturally inherits a Lie algebra structure. More precisely,

**Proposition 4.3.** The Lie bracket of the Lie algebra $\mathfrak{X}(B) \oplus C^\infty(P,\mathfrak{g})^G$ is given by:

$$[(Z,f),(Z',f')] = -\left( [Z,Z'], [f,f'] + Z^* (f') - (Z')^* (f) + \Omega(Z^*,(Z')^*) \right) \quad (45)$$

where $\Omega \in \Omega^2(P,\mathfrak{g})$ is the curvature of the connection $\theta$, i.e., $\Omega_x(X,Y) = \theta_x([X^h,Y^h])$ for $x \in P$ and $X,Y \in T_x P$.

**Remark 4.4.** The minus sign appearing in front of the term $[Z,Z']$ comes from the fact that we consider on $\mathfrak{X}(P)^G$ the Lie bracket induced by the Lie group structure of $\text{Aut}(P)$.

Let us give some lemmas to prove this result.

**Lemma 4.5.** Let $X,Y \in \mathfrak{X}(P)^G$ be $G$-invariant vector fields with $Y$ vertical. We have

$$[X,Y]_x = (\vartheta_x)_* X_x(\theta(Y)), \quad (46)$$

where $x \in P$.

**Proof.** We have

$$[X,Y]_x = \left. \frac{d}{dt} \right|_0 (\varphi^X_t)_* (\varphi^Y_t)_* Y_{\varphi^X_t(x)}. \quad (47)$$

Moreover, $Y$ being vertical,

$$Y_{\varphi^X_t(x)} = \left. \frac{d}{ds} \right|_0 \vartheta \left( \varphi^X_t(x), \exp(s \theta_{\varphi^X_t(x)}(Y)) \right). \quad (48)$$

Using the $G$-invariance of $X$ together with (18) in (17), we get:

$$[X,Y]_x = \left. \frac{d}{dt} \right|_0 \left. \frac{d}{ds} \right|_0 (\varphi^X_t)_* \left( \vartheta \left( \varphi^X_t(x), \exp(s \theta_{\varphi^X_t(x)}(Y)) \right) \right)$$

$$= \left. \frac{d}{dt} \right|_0 \left. \frac{d}{ds} \right|_0 \vartheta \left( x, \exp(s \theta_{\varphi^X_t(x)}(Y)) \right) = \left. \frac{d}{dt} \right|_0 (\vartheta_x)_* \theta_{\varphi^X_t(x)}(Y)$$

$$= (\vartheta_x)_* X_x(\theta(Y)), \quad (49)$$

which proves the lemma. $\square$

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The following lemma is proved in [15].

**Lemma 4.6.** Let $X, Y \in \mathfrak{X}(P)^G$ be $G$-invariant vector fields and $x \in P$. We have:

$$[X^h, Y^h]_x = [(\pi_* X^h), (\pi_* Y^h)]_x + (\partial_x)_x \Omega_x(X^h, Y^h). \tag{49}$$

**Proof of Proposition 4.3.** Let $X, Y \in \mathfrak{X}(P)^G$ be two $G$-invariant vector fields and $x \in P$. From Lemma 4.6 and Lemma 4.10, we get:

$$[X, Y]_x = [X^h, Y^h]_x + [X^v, Y^v]_x + [X^v, Y^h]_x + [X^h, Y^v]_x$$

$$= \Bigl( (\pi_* X^h), (\pi_* Y^h) \Bigr)_x + (\partial_x)_x \Omega_x(X^h, Y^h) + (\partial_x)_x X^h \Bigl( \theta(X^v) \Bigr)$$

$$- (\partial_x)_x Y^h \Bigl( \theta(X^v) \Bigr) + \Bigl( (\partial_x)_x \theta(X^v), (\partial_x)_x \theta(Y^v) \Bigr)_x$$

$$= \Phi^{-1} \left( \Bigl( (\pi_* X^h), (\pi_* Y^h) \Bigr), \left( \theta(X^v), \theta(Y^v) \right) + X^h \bigl( \theta(X^v) \bigr) - Y^h \bigl( \theta(X^v) \bigr) + \Omega_x(X^h, Y^h) \right)_x,$$

which is the desired formula. □

For $G$-invariant vector fields on $P$ with zero divergence with respect to the volume form $\mu^P$, we have the following proposition.

**Proposition 4.7.** The map $\Phi$ induces a $\mathbb{R}$-linear isomorphism:

$$\mathfrak{X}(P, \mu^P)^G \cong \mathfrak{X}(B, V \mu^B) \oplus C^\infty(P, g)^G, \tag{50}$$

i.e., if $X \in \mathfrak{X}(P)^G$, then $X \in \mathfrak{X}(P, \mu^P)^G$ if and only if $\Phi(X) \in \mathfrak{X}(B, V \mu^B) \oplus C^\infty(P, g)^G$.

To show Proposition 4.7, we need the following Lemma:

**Lemma 4.8.** For $X \in \mathfrak{X}(P)^G$, we have:

(i) $X(V \circ \pi) = \left( (\pi_* X)(V) \right) \circ \pi$,

(ii) $\mathcal{L}_X(\pi_* \mu^B) = \pi^* \left( \mathcal{L}_{\pi_* X}(\mu^B) \right) = \left( \text{div}_{\pi^* (\pi_* X) \circ \pi} \right) \cdot \pi^* \mu^B$,

(iii) $(\pi_* \mu^B) \wedge \mathcal{L}_X(\theta^G) = 0$.

**Proof.** The point (i) is obvious. Let us show (ii). Using the relation $\pi \circ \varphi^X = \varphi^X \circ \pi$, we see that

$$\mathcal{L}_X(\pi_* \mu^B) = \frac{d}{dt} \biggl|_0 (\varphi^X)^* \pi_* \mu^B = \frac{d}{dt} \biggl|_0 (\pi \circ \varphi^X)^* \mu^B = \frac{d}{dt} \biggl|_0 (\varphi^X \circ \pi)^* \mu^B$$

$$= \frac{d}{dt} \biggl|_0 \pi^* (\varphi^X)^* \mu^B = \pi^* \frac{d}{dt} \biggl|_0 (\varphi^X)^* \mu^B = \pi^* \left( \mathcal{L}_{\pi_* X}(\mu^B) \right).$$

For (iii), let us take $x \in P$ and $X, Y \in \mathfrak{X}(P)^G$ with $Y$ vertical. We have:

$$\left( \mathcal{L}_X \theta \right)_x Y_x = \frac{d}{dt} \biggl|_0 \left( (\varphi^X)^* \theta \right)_x (Y_x) = \frac{d}{dt} \biggl|_0 \theta_{\varphi^X(x)} \left( (\varphi^X)^* \varphi^X \right)_x Y_x = \frac{d}{dt} \biggl|_0 (\varphi^{-1})^* \varphi^X \varphi^X \varphi^X Y_x. \tag{51}$$
If in (51), we look at $Y_z$ as an element of $T_zO_x$ and $\tilde{\varphi}^X_{\alpha}$ as a diffeomorphism between $O_x$ and $O_{\varphi}^X(x)$, then,

$$(L_X\theta_x)(Y_z) = \frac{d}{dt}igg|_0 (\varphi^{-1}_{\beta} \circ \varphi^X_{\alpha})_* Y_z = \frac{d}{dt}igg|_0 (\varphi^{-1}_{\beta})_* Y_z = 0.$$ 

Hence, the form $L_X(\theta^* \nu^G)$ only depends on horizontal vector fields of $P$. Now (iii) follows from Lemma 3.12. \hfill $\square$

**Proof of Proposition 4.7.** Let $X \in \mathfrak{X}(P)^G$ be a $G$-invariant vector field. From (25) together with Lemma 4.8 we have:

$$L_X \mu^P = L_X((V \circ \pi) \cdot \pi^* \mu^G \wedge \theta^* \nu^G)$$

$$= X(V \circ \pi) \cdot \pi^* \mu^B \wedge \theta^* \nu^G + (V \circ \pi) \cdot L_X(\pi^* \mu^B) \wedge \theta^* \nu^G + (V \circ \pi) \cdot \pi^* \mu^B \wedge L_X(\theta^* \nu^G)$$

$$= \left((\pi_* X)(V)\right) \circ \pi \cdot \frac{1}{V \circ \pi} \cdot \mu^P + \left(\text{div}_{\mu^P} (\pi_* X) \circ \pi \right) \cdot \mu^P$$

$$= \left(\text{div}_{\mu^P} (\pi_* X)\right) \circ \pi \cdot \mu^P.$$ 

Hence,

$$\text{div}_{\mu^P} (X) = \left(\text{div}_{\mu^P} (\pi_* X)\right) \circ \pi,$$

which proves the proposition. \hfill $\square$

Finally, $\mathfrak{X}(B, \mu^B)$ and $C^\infty(P, \mathfrak{g})^G$ being closed subspaces of the Fréchet spaces $\mathfrak{X}(B)$ and $C^\infty(P, \mathfrak{g})$ respectively, we naturally get a structure of Fréchet space on $\mathfrak{X}(B, \mu^B) \oplus C^\infty(P, \mathfrak{g})^G$. We denote by $\tilde{\Phi} : \mathfrak{X}(P, \mu^P)^G \to \mathfrak{X}(B, \mu^B) \oplus C^\infty(P, \mathfrak{g})^G$ the restriction of $\Phi$ to $\mathfrak{X}(P, \mu^P)$.

**Lemma 4.9.** The map $\tilde{\Phi}$ is a continuous $\mathbb{R}$-linear isomorphism between Fréchet spaces.

**Proof.** From Proposition 4.7 we know that $\tilde{\Phi}$ is a bijection. Let us show that $\tilde{\Phi}$ is continuous. If $\alpha$ is a smooth curve of $\mathfrak{X}(P, \mu^P)^G$, then, according to the characterization of smooth curves in a space of sections (see [14], Lemma 30.8), and also from the definition of $\Phi$ (see [14]), it comes out that $\Phi \circ \alpha$ is a smooth curve of $\mathfrak{X}(B, \mu^B) \oplus C^\infty(P, \mathfrak{g})^G$. This implies that $\tilde{\Phi}$ is smooth, in particular, $\tilde{\Phi}$ is continuous. In a similar way, one can prove that $\tilde{\Phi}^{-1}$ is also continuous. \hfill $\square$

**Remark 4.10.** It follows from Proposition 4.7 and Lemma 4.9 that $\mathfrak{X}(P, \mu^P)^G$ and $\mathfrak{X}(B, \mu^B) \oplus C^\infty(P, \mathfrak{g})^G$ are isomorphic in the category of Fréchet Lie algebras.

### 4.2 The regular dual of $\mathfrak{X}(P, \mu^P)^G$

Let $\langle, \rangle$ be the scalar product on $\mathfrak{X}(P, \mu^P)^G$ defined as

$$\langle X, Y \rangle := \int_P h^P_x (X_x, Y_x) \cdot \mu^P,$$ 

(52)

where $X, Y \in \mathfrak{X}(P, \mu^P)^G$. This scalar product induces a metric on $\mathfrak{X}(B, \mu^B) \oplus C^\infty(P, \mathfrak{g})^G$ via the map $\Phi$ (see Proposition 4.7) :

$$\langle (X, f), (X', f') \rangle := \int_P h^P_x (\Phi^{-1}(X, f)_x, \Phi^{-1}(X', f')_x) \cdot \mu^P,$$
Proposition 4.12. We have an isomorphism of Fréchet spaces as the range of the operator dualisation operator defined as
\[ (X, f) \mapsto (X^\flat, f^\flat) \]
where \( (X^\flat, f^\flat) = (X_\pi(x)^\flat, f(x)) \cdot \mu^P + \int_p h^g(f(x), f'(x)) \cdot \mu^P \) for \( f \in C^\infty(P, g^*) \) and \( \varphi = \text{Ad}^*(g^{-1}) f, \forall g \in G \), we can rewrite (53) as:
\[
\langle (X, f), (X', f') \rangle = \int_B X^\flat(x) \cdot V^B + \int_p (f^\flat(x), f'(x)) \cdot \mu^P,
\]
where \( (\ldots, \ldots) \) denotes the pairing between \( g \) and \( g^* \).

Set \( A : \mathcal{F}(B, V^B) \oplus C^\infty(P, g)^G \rightarrow \big( \mathcal{F}(B, V^B) \oplus C^\infty(P, g)^G \big)^* \) to be the continuous and injective dualisation operator defined as
\[ A((X, f)) := \langle (X, f), \ldots \rangle \] ("*" being the topological dual).

**Definition 4.11.** We define the regular dual \( \big( \mathcal{F}(B, V^B) \oplus C^\infty(P, g)^G \big)^*_\text{reg} \) of \( \mathcal{F}(B, V^B) \oplus C^\infty(P, g)^G \) as the range of the operator \( A \) in the full topological dual of \( \mathcal{F}(B, V^B) \oplus C^\infty(P, g)^G \).

**Proposition 4.12.** We have an isomorphism of Fréchet spaces
\[
\big( \mathcal{F}(B, V^B) \oplus C^\infty(P, g)^G \big)^*_\text{reg} \overset{\Psi}{\cong} \Omega^1(B) / d\Omega^0(B) \oplus C^\infty(P, g^*)^G,
\]
where \( \Psi \) is defined for \( (X, f) \in \mathcal{F}(B, V^B) \oplus C^\infty(P, g)^G \) by
\[
\Psi(A((X, f))) := [X^\flat, f^\flat].
\]

We will show Proposition 4.12 using two lemmas. The first lemma is a slight generalization of the Helmholtz-Hodge decomposition (see Lemma 2.8).

**Lemma 4.13** (Helmholtz-Hodge decomposition). Let \((M, g)\) be a compact, connected, oriented Riemannian manifold without boundary, endowed with the volume form induced by \( g \). For \( f \in C^\infty(M, \mathbb{R}^*_+) \), we have the following decomposition:
\[
\mathcal{F}(M) = \mathcal{F}(M, \mu) \oplus f^\nabla \Omega^0(M).
\]

**Proof.** Let \( X \in \mathcal{F}(M) \) be a vector field and assume that the decomposition (57) exists. Thus, we can write \( X = X^\mu + f^\nabla p \) for \( X^\mu \in \mathcal{F}(M, \mu) \), \( p \in \Omega^0(M) \), and we have:
\[
\text{div}_\mu(X) = \text{div}_\mu(f^\nabla p) = (df)(\nabla p) + f \Delta p.
\]
Let \( f : [0, 1] \rightarrow C^\infty(M, \mathbb{R}^*_+) \) be a continuous path such that \( f_0 = 1 \) and \( f_1 = f \). For \( t \in [0, 1] \), we also denote \( I_t : C^\infty(M, \mathbb{R}) \rightarrow C^\infty_0(M, \mathbb{R}) := \{ h \in C^\infty(M, \mathbb{R}) | \int_M h \cdot \mu = 0 \} \) the operator defined for \( p \in C^\infty(M, \mathbb{R}) \), by \( I_t(p) := (df_t)(\nabla p) + f_t \Delta p \). It comes out that \( I_t \) is a continuous path of elliptic
operators (acting on a suitable Sobolev space), and for \( t \in [0,1] \), the kernel of \( I_t \) is 1-dimensional (this comes from the fact that locally, \( I_t \) is without constant terms, and for that kind of elliptic operators, the kernel reduces to constant functions, see [12]). Moreover, it is well known that on a compact orientable Riemannian manifold, the operator \( \Delta : C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R}) \) is surjective (see for example [11]). Thus \( \text{Ind}(I_1) = \text{Ind}(I_0) = 1 - 0 = 1 \). It follows that \( I_1 \) is surjective and in particular, equation (58) possesses a unique solution defined modulo a constant. If we take a function \( p \) as a solution of (58), it is straightforward to check that \( X = (X - f\nabla p) + f\nabla p \) is the desired decomposition. 

The second lemma concerns the topology we put on the space \((\Omega^1(B)/d\Omega^0(B)) \oplus C^\infty(P, g^*)^G\).

**Lemma 4.14.** The space \(d\Omega^0(B)\) is closed in \(\Omega^1(B)\). In particular, the quotient \(\Omega^1(B)/d\Omega^0(B)\) is a Fréchet space.

**Proof.** Let \((f_n)_{n \in \mathbb{N}}\) be a sequence of \(\Omega^0(B)\) such that \(df_n \to \alpha \in \Omega^1(B)\) for \(\alpha \in \Omega^1(B)\). We have to show that the form \(\alpha\) is exact. For that, it is sufficient to show that the integral of \(\alpha\) on any smooth closed curve of \(B\) is zero.

Let \(c : S^1 \to B\) be a smooth closed curve of \(B\). From the continuity of integration on \(\Omega^1(B)\), it follows that:

\[
\int_c \alpha = \int_{n \to \infty} \lim (df_n) = \lim_{n \to \infty} \int_c df_n = 0.
\]

This proves the lemma.

The set \(C^\infty(P, g^*)^G\) being closed in the Fréchet space \(C^\infty(P, g^*)\), it follows that \(C^\infty(P, g^*)^G\) is naturally a Fréchet space and, according to Lemma 4.14, the direct sum \((\Omega^1(B)/d\Omega^0(B)) \oplus C^\infty(P, g^*)^G\) is a Fréchet space.

**Remark 4.15.** Lemma 4.14 implies that the sum (57) is a topological sum.

**Proof of Proposition 4.12.** We will explicitly construct an inverse of \(\Psi \circ A\). First, observe that the relations \(\mathfrak{X}(B, V \mu^B) = \left(1/V\right)\mathfrak{X}(B, \mu^B)\) and \(\mathfrak{X}(B) = \mathfrak{X}(B, \mu^B) \oplus V\nabla\Omega^0(B)\) (see Lemma 4.13) imply the decomposition \(\mathfrak{X}(B) = \mathfrak{X}(B, V \mu^B) \oplus V\nabla\Omega^0(B)\). With respect to this decomposition, we define \(P : \mathfrak{X}(B) \to \mathfrak{X}(B, V \mu^B)\), the associated projection. One can check that the map \(\left(\Omega^1(B)/d\Omega^0(B)\right) \oplus C^\infty(P, g^*)^G \to \mathfrak{X}(B, V \mu^B) \oplus C^\infty(P, g)^G\) \(([\alpha], \xi) \mapsto (P(\alpha^*), \xi)\) is the inverse of \(\Psi \circ A\) (\(^*\) denotes the inverse of the dualisation operator). For the continuity of \(\Psi \circ A\) and its inverse, one can use arguments similar to those we used in Lemma 4.10.

**Remark 4.16.** Since the two vector spaces \(\left(\mathfrak{X}(B, V \mu^B) \oplus C^\infty(P, g)^G\right)_{\text{reg}}\) and \(\left(\Omega^1(B)/d\Omega^0(B)\right) \oplus C^\infty(P, g^*)^G\) are linearly isomorphic via \(\Psi\), it follows that the spaces \(\mathfrak{X}(B, V \mu^B) \oplus C^\infty(P, g)^G\) and \(\left(\Omega^1(B)/d\Omega^0(B)\right) \oplus C^\infty(P, g^*)^G\) are naturally in duality, the pairing, according to (54), being:

\[
([\alpha], \xi, (X, f)) := \int_B \alpha(X) \cdot V \mu^B + \int_P \xi(f) \cdot \mu^P,
\]

for \(\alpha \in \Omega^1(B)\), \(\xi \in C^\infty(P, g^*)^G\), \(X \in \mathfrak{X}(B, V \mu^B)\) and \(f \in C^\infty(P, g)^G\).
4.3 Determination of the Euler equation

With the above identifications of Fréchet spaces, namely \( \mathcal{X}(P, \mu^P)^G \cong \mathcal{X}(B, V \mu^B) \oplus C^\infty(P, g)^G \) and \((\mathcal{X}(P, \mu^P)^G)^{*}_{reg} \cong \left( \Omega^1(B)/d\Omega^0(B) \right) \oplus C^\infty(P, g^*)^G \), we can give a geometrical description of the map \( ad^* \) associated to the Lie algebra \( \mathcal{X}(P, \mu^P)^G \).

**Proposition 4.17.** For \( X \in \mathcal{X}(B, V \mu^B) \), \( f \in C^\infty(P, g)^G \), \( \alpha \in \Omega^1(B) \) and \( \xi \in C^\infty(P, g^*)^G \), we have

\[
\text{ad}^*(X, f)((\alpha, \xi), (X', f')) = \left( \left[ -\mathcal{L}_X \alpha - \langle \xi, df \rangle + \langle \xi, i_{X'} \Omega \rangle \right], -\text{ad}^*(f) \xi - X^*(\xi) \right),
\]

where \( \langle \xi, df \rangle \) and \( \langle \xi, i_{X'} \Omega \rangle \) are two 1-forms of \( B \) defined for \( b \in B \), \( Z \in \mathcal{X}(B) \) and \( x \in P \) such that \( \pi(x) = b \), by:

\[
\langle \xi, df \rangle_b(Z_b) := \left( \langle \xi(x), (df)_x Z_x \right) \quad \text{and} \quad \langle \xi, i_{X'} \Omega \rangle_b(Z_b) := \left( \langle \xi(x), \Omega(X'_x, Z'_x) \rangle \right).
\]

**Remark 4.18.** One can check that the forms defined in (61) are well defined.

**Proof of Proposition 4.17.** Let \( X, X' \in \mathcal{X}(B, V \mu^B) \) be vector fields with zero divergence on \( B \), \( f, f' \in C^\infty(P, g)^G \), \( \alpha \in \Omega^1(B) \) and \( \xi \in C^\infty(P, g^*)^G \). From (15), (51) and Remark 4.11, we have:

\[
\left( \text{ad}^*(X, f)((\alpha, \xi), (X', f')) \right) = \left( ([\alpha, \xi], \text{ad}(X, f)(X', f')) \right)
\]

\[
= \left( ([\alpha, \xi], (X', f') \right) + X'(f') - (X')^* (f) + \Omega(X^*, (X')^*) \right)
\]

\[
= \int_B \alpha([X, X']) \cdot V \mu^B + \int_P \left( \xi, [f, f'] + X^*(f') - (X')^* (f) + \Omega(X^*, (X')^*) \right) \cdot \mu^P.
\]

We now compute separately each term:

- \[
\int_B \alpha([X, X']) \cdot V \mu^B = \int_B \alpha \wedge i_{[X,X']}(V \mu^B) = \int_B \alpha \wedge [\mathcal{L}_X, i_{X'}](V \mu^B)
\]

\[
= \int_B \alpha \wedge \mathcal{L}_X i_{X'}(V \mu^B) - \int_B \alpha \wedge i_{X'} \mathcal{L}_X (V \mu^B) = - \int_B (\mathcal{L}_X \alpha)(X') \cdot V \mu^B.
\]

- \[
\int_P (\xi, [f, f']) \cdot \mu^P = \left[ \int_P \langle \xi, \text{ad}(f)(f') \rangle \cdot \mu^P = - \int_P (\text{ad}^*(f) \xi, f') \cdot \mu^P \right.
\]

- \[
\int_P (\xi, X^*(f')) \cdot \mu^P = \left[ \int_P X^*(\xi, f') \cdot \mu^P = - \int_P (X^*(\xi), f') \cdot \mu^P \right.
\]

- \[
\int_P (\xi, (X')^*(f)) \cdot \mu^P = \left[ \int_P \langle \xi, df(X'^*) \rangle \cdot \mu^P = - \int_B (\xi, df)(X') \cdot V \mu^B \right.
\]

- \[
\int_P (\xi, \Omega(X^*, (X')^*)) \cdot \mu^P = \left[ \int_P \langle \xi, (i_{X'} \Omega)(X'^*) \rangle \cdot \mu^P = \int_B (\xi, i_{X'} \Omega)(X') \cdot V \mu^B \right.
\]

Hence,

\[
\left( \text{ad}^*(X, f)((\alpha, \xi), (X', f')) \right) = \int_B \left( [ -\mathcal{L}_X \alpha - \langle \xi, df \rangle + \langle \xi, i_{X'} \Omega \rangle \right) (X') \cdot V \mu^B
\]

\[
+ \int_P \left( - \text{ad}^*(f) \xi - X^*(\xi), f' \right) \cdot \mu^P = \left( \left[ [ -\mathcal{L}_X \alpha - \langle \xi, df \rangle + \langle \xi, i_{X'} \Omega \rangle, -\text{ad}^*(f) \xi - X^*(\xi) \right] \right)(X', f').
\]

The proposition follows. \( \square \)
The Euler equation of the group \( \text{SAut}(P, \mu^P) \) on the regular dual of \( \mathfrak{X}(P, \mu^P)^G \) can be written:

\[
\begin{align*}
\frac{d}{dt}[\alpha] &= \begin{bmatrix} -\mathcal{L}_X \alpha - (\xi, df) + (\xi, i_X \cdot \Omega) \end{bmatrix}, \\
\frac{d}{dt}\xi &= -\text{ad}^* (f) \xi - X^*(\xi),
\end{align*}
\]

where \( X \in \mathfrak{X}(B, V\mu^B) \), \( \alpha \in \Omega^1(B) \), \( f \in C^\infty(P, \mathfrak{g})^G \), \( \xi \in C^\infty(P, \mathfrak{g}^*)^G \) (these quantities being time-dependant) and where

\[
\begin{align*}
\{ f^* = \xi, \text{ i.e., } \xi(x) := h^P_x(f(x), .) \text{ for } x \in P; \\
\{ [X^*] = [\alpha] \text{ where } X^*_x := h^P_x(X_x, .) \text{ for } x \in B. 
\end{align*}
\]

Remark 4.20. According to Remark 4.2, equations (67) describes –at least formally– geodesics in \( \text{SDiff}(P, \mu^P) \) with respect to the natural \( L \tilde{\cdot} \) -metric; a smooth curve \( \varphi \) in \( \text{SDiff}(P, \mu^P) \) is (formally) a geodesic in \( \text{SDiff}(P, \mu^P) \) if and only if the curve

\[
(\Psi \circ A \circ \Phi)((R_{\varphi}^{-1})_* \varphi) = (\Psi \circ A \circ \Phi)(\dot{\varphi} \circ \varphi^{-1})
\]

is a solution of equation (67) (see [20] and proposition 4.12 for the definitions of \( \Psi, A, \Phi \)).

Remark 4.21. If the Euclidean structure \( h^\mathfrak{g} \) on \( P \times \mathfrak{g} \) is constant (i.e. independent of the fibers), then:

- \( (\xi, df) = \frac{1}{2} d\left(\|f\|^2\right) \) , thus \( \|\xi, df\| = 0 \),

- the function \( V \in C^\infty(B, \mathbb{R}^+ \) is constant and \( \mathfrak{X}(B, V\mu^B) = \mathfrak{X}(B, \mu^B) \).

Remark 4.22. If the Euclidean structure \( h^\mathfrak{g} \) on \( P \times \mathfrak{g} \) is constant and if the curvature \( \Omega \) of the bundle \( G \hookrightarrow P \to B \) vanishes, then the first equation of (67) reduces to the autonomous equation:

\[
\frac{d}{dt}[\alpha] = \begin{bmatrix} -\mathcal{L}_X \alpha \end{bmatrix}.
\]

In this case, system (67) models the passive motion in ideal hydrodynamical flow (see [20], [10]).

Remark 4.23. Using the formula \( \mathcal{L}_X(X^*) = (\nabla_X X)^\flat + \frac{1}{2} d(h^B(X, X)) \) for \( X \in \mathfrak{X}(B) \) (see [2]), we can rewrite the first equation of (67) as:

\[
\begin{align*}
\frac{d}{dt}X &= -\nabla_X X - (\xi, df)^\flat + (\xi, i_X \cdot \Omega)^\flat + \nabla p,
\end{align*}
\]

where \( p \in C^\infty(B, \mathbb{R}) \) is determined by the condition \( \text{div}_{V\mu^B}(X) = 0 \).

If we specialize to the case of a \( S^1 \)-principal bundle with a 3-dimensional base manifold, and if \( h^\mathfrak{g} \) is given by the formula \( h^\mathfrak{g}(\rho, \varrho) := \rho \varrho \) for \( x \in P \) and \( \rho, \varrho \in \mathbb{R} \) (we identify the Lie algebra of \( S^1 \) with \( \mathbb{R} \)), then:

- the curvature \( \Omega \) projects itself on a 2-form \( \tilde{\Omega} \in \Omega^2(B) \). Similarly, any function \( f \in C^\infty(P, \mathbb{R})^{S^1} \) projects itself on a function \( \tilde{f} \in C^\infty(B, \mathbb{R}) \).

- One can define a vector field \( \mathfrak{B} \in \mathfrak{X}(B, \mu^B) \) via the relation \( i_{\mathfrak{B}} \mu^B = \tilde{\Omega} \),

- we have the formula \( X \times \mathfrak{B} = (i_X \tilde{\Omega})^\flat \) for all vector fields \( X \in \mathfrak{X}(B) \).

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In these conditions, it is easy to see that (71) is equivalent to:

\[
\begin{cases}
X = -\nabla_X X + \tilde{f} X \times \mathfrak{B} + \nabla p, \\
dt \tilde{f} = -X(\tilde{f}).
\end{cases}
\]  

(71)

These equations, known as the “superconductivity equations”, models the motion of an ideal charged fluid in a given magnetic field \( \mathfrak{B} \) where \( X \) represents the velocity field and \( \tilde{f} \) the charge density (see [21]).

**Remark 4.24.** The appearance of the magnetic term \( \mathfrak{B} \) in (71) is not surprising since classical electromagnetism is described in the language of gauge theories, where electromagnetic field is interpreted as the curvature of a connection form on a \( S^1 \)–principal bundle.

**Remark 4.25.** If the Euclidean structure \( h^g \) on \( P \times g \) is constant, then the metric \( h^P \) turns out to be a Kaluza-Klein metric on \( P \) (see formula (2.5) of [7]) and (67) becomes a particular case of the Euler-Yang-Mills equations of an incompressible homogeneous Yang-Mills ideal fluid (compare with formula (5.23) in [2]). The absence of an electric term in (67) seems to be due to the fact that the connection \( \theta \) is not a dynamical variable in our framework. This is not surprising since in the Yang-Mills formulation of electromagnetism, the configuration space is the space of all connections of the principal bundle describing the physical system.

5 The group \( \text{SAut} (P, \mu^P) \) as the total space of a \text{Gau}(P)-principal bundle

5.1 The principal fiber bundle structure of \( \text{SAut} (P, \mu^P) \)

For \( \varphi \in \text{Aut}(P) \), we denote by \( \tilde{\varphi} \in \text{Diff}(B) \) the unique diffeomorphism of \( B \) satisfying:

\[
\tilde{\varphi} \circ \pi = \pi \circ \varphi.
\]  

(72)

Note that the map \( \overline{\text{F}} : \text{Aut}(P) \rightarrow \text{Diff}(B), \varphi \mapsto \tilde{\varphi} \) is a group morphism.

**Proposition 5.1.** An automorphism \( \varphi \in \text{Aut}(P) \) belongs to \( \text{SAut} (P, \mu^P) \) if and only if \( \tilde{\varphi} \in \text{SDiff}(B, V \mu^B) \).

**Proof.** From (25), we have:

\[
\varphi^* \mu^P = \varphi^* (V \circ \pi \cdot \pi^* \mu^B \wedge \theta^* \nu_C^G) = (V \circ \pi \circ \varphi) \cdot \left( \varphi^* \pi^* \mu^B \right) \wedge \left( \varphi^* \theta^* \nu_C^G \right).
\]  

(73)

For \( \varphi \in \text{Aut}(P) \), we write \( f^\varphi \in C^\infty (B, \mathbb{R}^*) \) the unique function determined by the relation \( \tilde{\varphi}^* \mu^B = f^\varphi \cdot \mu^B \). We then have:

\[
(V \circ \pi \circ \varphi) \cdot \left( \varphi^* \pi^* \mu^B \right) = (V \circ \tilde{\varphi} \circ \pi \cdot (\pi \circ \varphi)^* \mu^B = (V \circ \tilde{\varphi} \circ \pi) \cdot (\tilde{\varphi} \circ \pi)^* \mu^B \]

\[
= (V \circ \tilde{\varphi} \circ \pi) \cdot (f^\varphi \circ \pi) \cdot \pi^* \mu^B.
\]  

(74)

On the other hand, for \( x \in P \), and for vertical tangent vectors \( u_1, ..., u_m \in T_x P \) (we assume \( \text{dim}(G) = m) \), we have:

\[
(\varphi^* (\theta^* \nu_C^G))_x (u_1, ..., u_m) = (\theta^* \nu_C^G)_{\varphi(x)} (\varphi_* u_1, ..., \varphi_* u_m)
\]

\[
= (\nu_C^G)_{\theta_{\varphi(x)} (\varphi_* u_1), ..., \theta_{\varphi(x)} (\varphi_* u_m)}
\]

\[
= (\nu_C^G)_{(\varphi^* \theta)_x (u_1), ..., (\varphi^* \theta)_x (u_m)}.
\]
The diffeomorphism \( \varphi \) being \( G \)-equivariant, one can show that \( \varphi^* \theta \) is a connection form. In particular, \( u_i \) being vertical, \( (\varphi^* \theta)_x(u_i) = \theta_x(u_i) \) for \( i \in \{1, \ldots, m\} \), and also,

\[
(\varphi^*(\theta^* \nu^G))_x(u_1, \ldots, u_m) = (\nu^G_x(\theta_x(u_1), \ldots, \theta_x(u_m))) = (\theta^* \nu^G)_x(u_1, \ldots, u_m).
\]

From Lemma 5.12 (75) and (74), we get

\[
\varphi^* \mu^P = (V \circ \tilde{\varphi} \circ \pi) \cdot (f^\varphi \circ \pi) \cdot \pi^* \mu^B \wedge \theta^* \nu^G = \frac{V \circ \tilde{\varphi} \circ \pi \cdot f^\varphi \circ \pi}{V \circ \pi} \cdot \mu^P.
\]

Thus,

\[
\varphi^* \mu^P = \mu^P \iff \frac{V \circ \tilde{\varphi} \circ \pi \cdot f^\varphi \circ \pi}{V \circ \pi} = 1 \iff f^\varphi \circ \pi = \left( \frac{V}{V \circ \tilde{\varphi}} \right) \circ \pi \iff f^\varphi = \frac{V}{V \circ \tilde{\varphi}}\bigg|_{\pi}
\]

\[
\iff \varphi^* \mu^B = \frac{V}{V \circ \tilde{\varphi}} \cdot \mu^B \iff \tilde{\varphi}(V \mu^B) = V \mu^B.
\]

This proves the proposition. \( \square \)

Before we show that \( \text{SAut}(P, \mu^P) \) is a \( \text{Gau}(P) \)-principal fiber bundle, where \( \text{Gau}(P) := \{ \varphi \in \text{Aut}(P) \mid \tilde{\varphi} = \text{Id}_B \} \), we will first prove that \( \text{Aut}(P) \) is a \( \text{Gau}(P) \)-principal fiber bundle and we will see how to use Proposition 5.1 to get a similar result for \( \text{SAut}(P, \mu^P) \).

Let us recall some basic facts about the group \( \text{Gau}(P) \) (see [14], [1]):

**Proposition 5.2** ([1]). We have :

(i) the group \( \text{Gau}(P) = \{ \varphi \in \text{Aut}(P) \mid \tilde{\varphi} = \text{Id}_B \} \) is a closed Fréchet Lie subgroup of \( \text{Aut}(P) \) whose Lie algebra can be identified with the space of vertical vector fields of \( P \) (see Theorem 3.1 and Theorem 3.7 in [1]),

(ii) The set \( \{ f \in C^\infty(P, G) \mid f \circ \vartheta_g = e_{g^{-1}} \circ f, \forall g \in G \} =: C^\infty(P, G)^G \) (where \( e_g : G \to G, h \to ghg^{-1} \)), is a closed Fréchet Lie subgroup of the current group \( C^\infty(P, G) \) endowed with the pointwise multiplication (see [12]), whose Lie algebra can be identified with the Fréchet space \( C^\infty(P, G)^G := \{ f \in C^\infty(P, g) \mid f \circ \vartheta_g = \text{Ad}(g^{-1}) f, \forall g \in G \} \),

(iii) we have an isomorphism of Fréchet Lie groups:

\[
C^\infty(P, G)^G \to \text{Gau}(P), \ f \mapsto \vartheta_{f(\cdot)}(\cdot).
\]

**Remark 5.3.** Note that the above proposition is expressed in the category of Fréchet Lie groups, and not in the category of tame Fréchet Lie groups of Hamilton (see [1]). This is not really burdensome since, in the rest of this paper, we will not have to use the inverse function Theorem of Nash-Moser. Consequently, we don’t need the subtle category of Hamilton anymore, and the rest of this paper should be –unless otherwise stated– understood within the framework of Fréchet Lie groups.

In the following, we will often identify \( \text{Gau}(P) \) and \( C^\infty(P, G)^G \) via the isomorphism defined in (76).

Let us introduce some terminology :

- let \( \lambda : \text{Aut}(P) \times \text{Gau}(P) \to \text{Aut}(P) \) be the right action of the group \( \text{Gau}(P) \) on \( \text{Aut}(P) \), defined by :

\[
\left( \lambda(\varphi, f) \right)(x) := \vartheta_{f(x)}(\varphi(x)),
\]

for \( \varphi \in \text{Aut}(P), f \in \text{Gau}(P) \) and \( x \in P \),
• for $X \in \mathfrak{X}(P)^G$, let $\tilde{X} \in \mathfrak{X}(B)$ be the vector field defined by $\tilde{X}_b := \pi_* X_x$ for $b \in B$ and where $x \in P$ is such that $\pi(x) = b$,
• $\text{Diff}^{-\sim}(B) := \{ \tilde{\varphi} \in \text{Diff}(B) \mid \varphi \in \text{Aut}(P) \}$ (according to (72), $\text{Diff}^{-\sim}(B)$ is a group).

**Lemma 5.4.** The group $\text{Diff}^{-\sim}(B)$ is a union of connected components of $\text{Diff}(B)$ containing $\text{Diff}^0(B)$. In particular, $\text{Diff}^{-\sim}(B)$ is naturally a tame Fréchet Lie group.

**Proof.** Let $\varphi \in \text{Aut}(P)$ be an automorphism of $P$ and $\psi$ an element of the connected component of $\text{Diff}(B)$ containing $\tilde{\varphi}$. To prove the lemma, it is sufficient to show that $\psi \in \text{Diff}^{-\sim}(B)$.

Let $\psi_t$ be a smooth curve of $\text{Diff}(B)$ joining $\tilde{\varphi}$ and $\psi$, i.e.:

$$\psi_0 = \tilde{\varphi} \quad \text{and} \quad \psi_1 = \psi.$$  

For $t_0 \in [0, 1]$ and $x_0 \in B$, we set

$$\left( X_{t_0} \right)_{x_0} := \left. \frac{d}{dt} \right|_{t_0} \psi_t \left( \psi_{t_0}^{-1}(x_0) \right).$$

It turns out that $X$ is a time-dependant vector field on $B$ with the property that the flow $\varphi_t^{X_t}$ of its horizontal lift $\varphi_t^{X_t}$ satisfies:

$$\left( \varphi_t^{X_t} \right) = \varphi_t^{X_t} = \varphi_t = \psi_t \circ \tilde{\varphi}^{-1}.$$  

Thus,

$$\left( \varphi_t^{X_t} \circ \varphi \right) = \varphi_t^{X_t} \circ \varphi = \psi_t \circ \tilde{\varphi}^{-1} \circ \tilde{\varphi} = \psi_t.$$  

It follows that $\psi = \psi_1 = \left( \varphi_t^{X_t} \circ \varphi \right)$ belongs to $\text{Diff}^{-\sim}(B)$.

**Lemma 5.5.** If $\varphi, \psi \in \text{Aut}(P)$ satisfy $\tilde{\varphi} = \tilde{\psi}$, then the map

$$\Lambda(\varphi, \psi) : P \to G, \ x \mapsto \left( \psi_{\varphi(x)}^{-1} \right)(\psi(x)),$$

is smooth.

**Proof.** Let $U$ and $V$ be the domains of two trivializing charts of $B$

$$
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\Psi_U} & U \times G \\
\pi \downarrow & & \downarrow \pi \\
U \quad & & V \\
\end{array}
\quad 
\begin{array}{ccc}
\pi^{-1}(V) & \xrightarrow{\Psi_V} & V \times G \\
\pi \downarrow & & \downarrow \pi \\
V \quad & & \Psi_U^{-1} \\
\end{array}
$$

such that $\tilde{\varphi}(U) \subseteq V$. As $\varphi$ and $\psi$ are $G$-equivariant, there exists $s^\varphi, s^\psi \in C^\infty(U, G)$ such that for all $(x, g) \in U \times G$, $\left( \Psi_V \circ \varphi \circ \Psi_U^{-1} \right)(x, g) = (\tilde{\varphi}(x), s^\varphi(x) \cdot g)$ and $\left( \Psi_V \circ \psi \circ \Psi_U^{-1} \right)(x, g) = (\tilde{\psi}(x), s^\psi(x) \cdot g)$. For
Let us show that we have a principal bundle. Let $(\Phi, \psi, \map)\xrightarrow{\sigma} X$ for all $\phi \in \Aut(P)$.

**Remark 5.9.** Theorem 5.8 means that the above sequence is a short exact sequence of Lie groups such that $\Aut(P)$ is a $\Aut(P)$-principal bundle over the group $\Diff^{-}(B)$ (see [17]).

**Proof.** The freeness of $\lambda$ is obvious.

Let us fix $\phi \in (\overline{\sigma})^{-1}(\overline{\psi}(\varphi))$. Since $\tilde{\varphi} = \tilde{\psi}$, there exists a unique map $f \in C^\infty(P, G)$ satisfying $f(x) := \varphi_{x(x)}(\phi(x))$ for all $x \in P$. According to Lemma 5.6, this map is smooth and one can check that $f \in \Aut(P)$ and also that $\phi = \lambda(\varphi, f)$. Thus, $\phi \in \Orb_{\varphi}$ and $(\overline{\psi})^{-1}(\overline{\psi}(\varphi)) \subseteq \Orb_{\varphi}$. The inverse inclusion being trivial, the lemma follows.

**Lemma 5.7.** The map $\Aut(P) \xrightarrow{\overline{\psi}} \Diff^{-}(B)$ is smooth and admits local smooth sections.

**Proof.** The map $\overline{\psi}$ being a morphism, it is sufficient to show that there exists a local smooth section of $\overline{\psi}$ in a neighbourhood of $Id_{B}$ in $\Diff(B)$.

Recall that $\pi : (P, h^P) \to (B, h^B)$ is a Riemannian submersions (Lemma 3.1). Therefore,

$$\pi(\exp(X_x)) = \exp_{\pi(x)}(\tilde{X}_{\pi(x)}),$$

(79)

for all $X \in \mathfrak{X}(P)^G$ and $x \in P$. For a $G$-invariant vector field $X \in \mathfrak{X}(P)^G$ sufficiently closed to 0, the map $\varphi : P \to P, x \mapsto \exp_x(X_x)$ is a diffeomorphism of $P$ (observe that $\varphi \in \Aut(P)$ since $X$ and $h^P$ are $G$-invariant). In view of (29), $\mathfrak{X}(\varphi)$ is simply the map $B \to B, x \mapsto \exp_x(\tilde{X}_x)$. It follows that the local expression of $\overline{\psi}$ in the standard charts of $\Aut(P)$ and $\Diff^{-}(B)$, are the projection $\mathfrak{X}(P)^G \equiv \mathfrak{X}(B) \oplus C^\infty(P, G)^G \to \mathfrak{X}(B)$ on the first factor (see [14]). Hence this map is smooth and admits local sections.

**Theorem 5.8 ([1]).** The group $\Aut(P)$ is an extension of the group $\Diff^{-}(B)$ by the gauge group $\Aut(P)$:

$$\{e\} \to \Aut(P) \to \Aut(P) \to \Diff^{-}(B) \to \{e\}. $$

(80)

**Remark 5.9.** Theorem 5.8 means that the above sequence is a short exact sequence of Lie groups such that $\Aut(P)$ is a $\Aut(P)$-principal bundle over the group $\Diff^{-}(B)$ (see [17]).

**Proof.** The sequence (80) is obviously exact.

Let us show that we have a principal bundle. Let $(\Phi(U), \Phi^{-1})$ be the standard chart of $\Diff^{-}(B)$, i.e., $U \subseteq \mathfrak{X}(B)$ and $\Phi^{-1}$ is defined by $\Phi^{-1}(X)(x) := \exp_x(X_x)$. We also take a section $\sigma : \Phi(U) \to \Aut(P)$.
One may recover Theorem 4.19 from Theorem 5.11 using the description of geodesics on Remark 5.12.

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