A LIFTING THEOREM FOR 3-ISOMETRIES

SCOTT MCCULLOUGH* AND BENJAMIN RUSSO

Abstract. An operator $T$ on Hilbert space is a 3-isometry if there exists operators $B_1$ and $B_2$ such that $T^*T^n = I + nB_1 + n^2B_2$. An operator $J$ is a Jordan operator if it the sum of a unitary $U$ and nilpotent $N$ of order two which commute. If $T$ is a 3-isometry and $c > 0$, then $I - c^{-2}B_2 + sB_1 + s^2B_2$ is positive semidefinite for all real $s$ if and only if it is the restriction of a Jordan operator $J = U + N$ with the norm of $N$ at most $c$. As a corollary, an analogous result for 3-symmetric operators, due to Helton and Agler, is recovered.

1. Introduction

Let $B(H)$ denote the bounded operators on the (complex) Hilbert space $H$. An operator $T \in B(H)$ is a 3-isometry if

$$T^*T^3 - 3T^*T^2 + 3T^*T - I = 0.$$  

Equivalently $T$ is a 3-isometry if and only if there exist operators $B_1(T^*, T)$ and $B_2(T^*, T)$ such that for all natural numbers $n$

(1) \hspace{1cm} $T^*T^n = I + nB_1(T^*, T) + n^2B_2(T^*, T).$

In this case it is straightforward to verify that

(2) \hspace{1cm} $2B_2(T^*, T) = T^*T^2 - 2T^*T + I$

and

(3) \hspace{1cm} $2B_1(T^*, T) = -T^*T^2 + 4T^*T - 3.$

Evidently, each $B_j(T^*, T)$ is selfadjoint.

From Equation (1), it is evident that that $\|T^n\|^2$ is bounded by a quadratic in $n$. It follows from the spectral radius formula that the spectrum of $T$, denoted $\sigma(T)$, is a subset of $\overline{D}$, the closed unit disc. If $T$ is invertible, then Equation (1) holds for all integers $n$. In particular, $T^{-1}$ is also a 3-isometry and hence $\sigma(T^{-1}) \subseteq \overline{D}$. Thus, in this case, $\sigma(T)$ is a subset of the unit circle. As will be seen later, if $T$ is not invertible, then in fact $\sigma(T) = \overline{D}$ [3] (or see Lemma 5.2).

2010 Mathematics Subject Classification. 47A20 (Primary) 47B37, 47B38 (Secondary).

Key words and phrases. 3-symmetric operators, 3-isometric operators, Non-normal spectral theory.

* Partially supported by NSF grant DMS-1101137.
Likewise, an operator $T$ on a Hilbert space $H$ is a 3-symmetric operator if there exists operators $B_j(T^*, T)$ on $H$ such that

$$\exp(-isT^*) \exp(isT) = I + sB_1(T^*, T) + s^2B_2(T^*, T)$$

for all real $s$. Evidently, if $T$ is 3-symmetric, then $T = \exp(it)$ is a 3-isometry. Helton introduced 3-symmetric operators as both a generalization of selfadjoint operators and as a class of non-Normal operators for which a viable spectral theory exists. In a series of papers ([7] [9] [8]) Helton modeled these operators as multiplication $t$ on a Sobolev space and showed, under some additional hypotheses, that they are the restriction to an invariant subspace of a Jordan operator (of order two) as explained below. In [2] the connection between Jordan operators and 3-symmetric operators was established in general. See also the references [6] [3] [4] [5] [12] [10].

Given a positive number $c$, let $\mathfrak{F}_c$ denote those 3-isometries $T$ such that the quadratic

$$Q(T, s) = [I - \frac{1}{c^2}B_2(T^*, T)] + sB_1(T^*, T) + s^2B_2(T^*, T)$$

is positive semidefinite for all real numbers $s$. It turns out there are 3-isometries which do not belong to any of these classes. If $U$ is a unitary operator on the Hilbert space $\mathcal{F}$, then the Jordan operator

$$J = \begin{pmatrix} U & cU \\ 0 & U \end{pmatrix}$$

acting on $K = \mathcal{F} \oplus \mathcal{F}$ is a canonical member of the class $\mathfrak{F}_c$. Indeed, it is readily verified that

$$Q(J, s) = \begin{pmatrix} I & scI \\ scI & s^2c^2I \end{pmatrix} \succeq 0.$$

Here, for an operator $A$ on Hilbert space, $A \succeq 0$ means $A$ is positive semidefinite. Moreover, if $T$ is an operator acting on the Hilbert space $H$ and there is an isometry $V : H \to K$ such that $VT = JV$, then $T^nT^n = V^*J^nJV$ for all $n$. It follows that $T$ is a 3-isometry and further,

$$Q(T, s) = V^*Q(J, s)V \succeq 0$$

so that $T \in \mathfrak{F}_c$. The following is the main result of this article.

**Theorem 1.1 (3-isometric lifting theorem).** An operator $T$ on a Hilbert space $H$ is in the class $\mathfrak{F}_c$ if and only if there is an operator $J$ of the form in Equation (6) acting on a Hilbert space $K$ and an isometry $V : H \to K$ such that $VT = JV$.

If $T$ is invertible, then, necessarily, $VT^{-1} = J^{-1}V$. Moreover, in this case, the spectrum of $T$ is a subset of the unit circle, and $J$ can be chosen so that $\sigma(T) = \sigma(J)$.

The 3-symmetric lifting Theorem of Helton and Agler is fairly easily seen to be a consequence of Theorem 1.1. The details are in Section 6. The proof of Theorem 1.1 for the case that
T is invertible uses Arveson’s complete positivity machinery in the form of a version of the Arveson Extension Theorem and operator-valued Fejer-Riesz Factorization. The proof of the Arveson Extension Theorem along with the needed background on the theory of completely positive maps appears in Section 2 gives the The proof of the lifting theorem for invertible 3-isometries appears in Section 3. The reduction of the general case of Theorem 1.1 to the invertible case is the topic of Section 4. A functional calculus argument establishes the spectral condition, $\sigma(T) = \sigma(J)$, in Section 5.

2. Completely Positive Maps and the Arveson Extension Theorem

In this section some of Agler’s hereditary calculus machinery based upon the Arveson Extension Theorem is reviewed. Let $n$ and $N$ be given positive integers. An hereditary polynomial $p(x, y)$ of size $n$ and degree at most $N$ in noncommuting (invertible) variables $x$ and $y$ is a polynomial of the form

$$p(x, y) = \sum_{\alpha, \beta = -N}^{N} p_{\alpha, \beta} y^{\alpha} x^{\beta}. \tag{7}$$

Here the sum is finite and the $p_{\alpha, \beta}$ are $n \times n$ matrices over $\mathbb{C}$. Such a polynomial is evaluated at an invertible operator $T$ by

$$p(T^*, T) = \sum p_{\alpha, \beta} \otimes T^{*\alpha} T^{\beta}.$$ 

Let $\mathcal{P}_n$ denote the collection of hereditary polynomials of size $n$ and let $\mathcal{P} = (\mathcal{P}_n)_n$ denote the collection of all hereditary polynomials.

Given an operator $T$, let $\mathcal{H}(T)$ denote the span by $\{T^{*\alpha} T^{\beta} : \alpha, \beta \in \mathbb{Z}\}$. Given an operator $J$ on the Hilbert space $K$, if $p(J^*, J) \geq 0$ implies $p(T^*, T) \geq 0$, then the mapping $\rho : \mathcal{H}(J) \rightarrow \mathcal{H}(T)$ given by $\rho(p(J^*, J)) = p(T^*, T)$ is well defined. Let $M_n$ denote the $n \times n$ matrices. If in addition, for each $n$ the mapping $1_n \otimes \rho : M_n \otimes \mathcal{H}(J) \rightarrow M_n \otimes \mathcal{H}(T)$ obtained by applying $\rho$ entry-wise is positive, then $\tau$ is completely positive.

**Theorem 2.1** (Arveson Extension Theorem [1]). Suppose $T$ and $J$ are invertible operators on Hilbert spaces $H$ and $K$ respectively. There is a Hilbert space $\mathcal{K}$, a representation $\pi : B(K) \rightarrow B(\mathcal{K})$ and an isometry $V : H \rightarrow \mathcal{K}$ such that $VT^j = \pi(J^j)V$ for all $j \in \mathbb{Z}$ if and only if the mapping $\rho : \mathcal{H}(J) \rightarrow \mathcal{H}(T)$ is completely positive.

**Proof.** Since $\rho : \mathcal{H}(J) \rightarrow \mathcal{H}(T)$ determined by $\rho(J^{*\alpha} J^{\beta}) = T^{*\alpha} T^{\beta}$ is (well defined) and completely positive, by Theorem ?? in [11], there is a Hilbert space $\mathcal{K}$, a representation $\pi : B(K) \rightarrow B(\mathcal{K})$ and an isometry $V : H \rightarrow \mathcal{K}$ such that

$$T^{*\alpha} T^{\beta} = \rho(J^{*\alpha} J^{\beta}) = V^{*} \pi(J^{*\alpha} J^{\beta}) V.$$
For each $\beta \in \mathbb{Z}$,
\[ V^* \pi(J)^{\beta} \pi(J)^{\beta} V = T^{\beta} T^{\beta} \]
\[ = V^* \pi(J)^{\beta} V V^* \pi(J)^{\beta} V. \]

Thus, as $I - V V^*$ is a projection,
\[ V^* \pi(J)^{\beta} (I - V V^*)^2 \pi(J)^{\beta} V = 0 \]
and therefore $(I - V V^*) \pi(J)^{\beta} V = 0$. Consequently, $V V^* \pi(J)^{\beta} V = \pi(J)^{\beta} V$. It follows that, for each $\beta$,
\[ V T^{\beta} = V V^* \pi(J)^{\beta} V = \pi(J)^{\beta} V. \]

The proof of the converse is routine.

2.1. Symmetrization. In this section Agler’s symmetrization technique [2], which leads to a strong variant of Theorem 2.1, is reviewed. An operator $J$ is symmetric if exp$(it)J$ is unitarily equivalent to $J$ for each real number $t$. Given an invertible operator $T$, let $\mathcal{H}_s(T)$ denote the span of $\{T^{\alpha} T^{\beta} : \alpha \in \mathbb{Z}\}$.

**Proposition 2.2** (Agler). Suppose $T$ and $J$ are invertible operators on Hilbert spaces $H$ and $K$ respectively. If $J$ is symmetric and the mapping $\rho : \mathcal{H}_s(J) \to \mathcal{H}_s(T)$ determined by $\rho(J^{\alpha} J^{\beta}) = T^{\alpha} T^{\beta}$ is (well defined and) completely positive, then there is a Hilbert space $\mathcal{K}$, a representation $\pi : B(K) \to B(\mathcal{K})$ and an isometry $V$ such that $VT^j = \pi(J)^j V$ for all $j \in \mathbb{Z}$.

The proof of this Proposition occupies the remainder of this section. Let $S$ denote the **bilateral shift** operator on $L^2 = L^2(\mathbb{T})$. Since $S$ is symmetric, it is readily seen that, for any operator $T$, the operator $\tilde{T} = T \otimes S$ acting on $H \otimes L^2$ is also symmetric. Moreover, if $T \in \mathcal{S}_c$, then so is $\tilde{T}$. Given $p \in \mathcal{P}$ as in Equation (7) let $p^s$ denote its **symmetrization**,
\[ p^s = \sum p_{\alpha,\beta} x^{\alpha} y^{\beta}. \]

**Lemma 2.3.** If $J$ is symmetric, $q \in \mathcal{P}$ and $q(J) \geq 0$, then $q^s(J) \geq 0$.

Let $T$ be a given operator on the Hilbert space $H$ and let $W : H \to H \otimes L^2$ denote the isometry $Wh = h \otimes 1$. If $P \in \mathcal{P}_n$, then
\[ P^s(T^*, T) = (I_n \otimes W)^* P(\tilde{T}^*, \tilde{T})(I_n \otimes W). \]

**Proof.** For each $t$ there is a unitary operator $U_t$ such that $e^{it} J = U_t^* J U_t$. Letting $n$ denote the size of $q$ (so that $q \in \mathcal{P}_n$), it follows that
\[ q(e^{-it} J^*, e^{it} J) = (I_n \otimes U_t^*) q(J^*, J) (I_n \otimes U_t) \geq 0. \]

Hence,
\[ q^s(J^*, J) = \frac{1}{2\pi} \int_0^{2\pi} q(e^{-it} J^*, e^{it} J) dt \geq 0. \]
To prove the second part, write \( p \in \mathcal{P}_1 \) as in Equation (7) in which case,
\[
\langle p(\tilde{T}, \tilde{T}^*)Wh, Wh \rangle = \langle p(\tilde{T}, \tilde{T}^*)h \otimes 1, f \otimes 1 \rangle
\]
\[
= \sum_{\alpha, \beta} p_{\alpha, \beta} T^{\alpha} h \otimes z^\alpha, T^\beta f \otimes z^\beta \rangle
\]
\[
= \sum_{\alpha} p_{\alpha, \alpha} T^{\alpha} T^\alpha h, f \)
\]
\[
= \langle p^*(T, T^*)h, f \rangle.
\]
Applying the result for \( p \in \mathcal{P}_1 \) entry-wise to \( P \) completes the proof.

**Lemma 2.4.** Suppose \( T \) and \( J \) are invertible operators on Hilbert spaces \( H \) and \( K \) respectively. If \( J \) is symmetric and the mapping \( \rho : \mathcal{H}(J) \rightarrow \mathcal{H}(T) \) determined by \( \rho(J^\alpha J^\beta) = T^\alpha T^\beta \) is (well defined and) completely positive, then the mapping \( \tilde{\rho} : \mathcal{H}(J) \rightarrow \mathcal{H}(\tilde{T}) \) determined by
\[
\tilde{\rho}(J^\alpha J^\beta) = \tilde{T}^\alpha \tilde{T}^\beta
\]
is also (well defined and) completely positive.

**Proof.** Fix a positive integer \( n \) and a \( p \in \mathcal{P}_n \). In particular, \( p(T^*, T) \) acts on \( \mathbb{C}^n \otimes H \). Given a positive integer \( N \) consider the \((2N+1) \times (2N+1)\) matrix whose entries are \( n \times n \) matrix polynomials
\[
P = \left( (I_n \otimes y^j)p(x, y)(I_n \otimes x^k) \right)_{j, k = -N}^N.
\]
Here \( I_n \) is the \( n \times n \) identity matrix. Thus, the \((j, k)\) entry of \( P(T^*, T) \) is the operator on \( \mathbb{C}^n \otimes H \) given by
\[
(I_n \otimes T^*)p(T^*, T)(I_n \otimes T^k).
\]
Viewing \( P(T, T^*) \) as an operator on \( (\mathbb{C}^n \otimes H) \otimes \mathbb{C}^{2N+1} \), let \( \{e_{-N}, \ldots, e_0, \ldots, e_N\} \) denote the corresponding standard basis for \( \mathbb{C}^{2N+1} \). Given a vector \( h = \sum h_a \otimes e_a \in (\mathbb{C}^n \otimes H) \otimes \mathbb{C}^{2N+1} \), an application of Lemma 2.3 gives,
\[
\langle P^*(T^*, T)h, h \rangle = \langle p(\tilde{T}^*, \tilde{T}^*)h \otimes 1, 1 \otimes 1 \rangle
\]
\[
= \sum \langle (I \otimes \tilde{T}^*)p(\tilde{T}^*, \tilde{T}^*)h_k \otimes 1, h_j \otimes 1 \rangle
\]
\[
= \sum \langle p(\tilde{T}^*, \tilde{T}^*)h_k \otimes 1, (I \otimes \tilde{T}^*)h_j \otimes 1 \rangle
\]
\[
= \sum_{k = -N}^N \langle p(\tilde{T}^*, \tilde{T}^*)h_k \otimes z^k, (I \otimes T^j)h_j \otimes z^j \rangle
\]
\[
= \langle p(\tilde{T}^*, \tilde{T}^*) \sum_{k = -N}^N [(I \otimes T^k)h_k] \otimes z^k, \sum_{j = -N}^N [(I \otimes T^j)h_j] \otimes z^j \rangle.
\]
(8)

Now suppose that \( p(J^*, J) \succeq 0 \). It then follows that \( P(J^*, J) \succeq 0 \) and thus \( P^*(J^*, J) \succeq 0 \). The hypotheses imply \( P^*(T^*, T) \succeq 0 \). From Equation (8) and the fact that sums of the form \( \sum_{j = -N}^N T^j h_j \otimes z^j \) are dense in \( H \otimes L^2 \) (since \( T \) is invertible), it follows that \( p(\tilde{T}^*, \tilde{T}^*) \succeq 0 \).
Lemma 2.5. Suppose $T \in B(H)$ is invertible. If $p \in \mathcal{P}$ and $p(\tilde{T}^*, \tilde{T}) \succeq 0$, then $p(T^*, T) \succeq 0$. In particular, the canonical mapping $p(\tilde{T}^*, \tilde{T}) \mapsto p(T^*, T)$ is well defined.

Proof. Let

$$D_N = \frac{1}{2N + 1} \sum_{j=-N}^{N} e^{ij} \in L^2(\mathbb{T}).$$

If $h, f \in H$, then for $\alpha, \beta \in \mathbb{Z}$,

$$\langle \tilde{T}^\alpha h \otimes D_N, \tilde{T}^\beta f \otimes D_N \rangle = \langle T^\alpha h, T^\beta f \rangle \langle z^{\alpha-\beta} D_N, D_N \rangle = \langle T^\alpha h, T^\beta f \rangle \frac{2N + 1 - |\alpha - \beta|}{2N + 1}.$$  

Thus, if $p \in \mathcal{P}_1$, then

$$\lim_{N \to \infty} \langle p(\tilde{T}^*, \tilde{T}) h \otimes D_N, f \otimes D_N \rangle = \langle p(T^*, T) h, f \rangle.$$  

In particular, if $p(\tilde{T}^*, \tilde{T}) \succeq 0$, then also $p(T^*, T) \succeq 0$. The square matrix version of this implication is readily established and proves the lemma.  

Proof of Proposition 2.2. From Lemma 2.4, the mapping $\tilde{\rho} : \mathcal{H}(J) \to \mathcal{H}(\tilde{T})$ (as defined in Lemma 2.4) is completely positive. On the other hand, from Lemma 2.5, the canonical mapping $\tau : \mathcal{H}(\tilde{T}) \to \mathcal{H}(T)$ is also (well defined and) completely positive. Thus, the composition $\rho = \tau \circ \tilde{\rho}$ is also completely positive. The conclusion now follows from the Arveson Extension Theorem, Theorem 2.1.  

3. Lifting Invertible 3-Isometries

In this section Theorem 1.1 is established in the case that $T$ is invertible. The first step uses Proposition 2.2 to prove that if $T$ is invertible, then $T$ lifts to a $J$ of the form in Equation (6). A separate argument, found in Section 5, shows that the spectrum of $J$ can be chosen to be the same as that of $T$.

Given $T \in \mathcal{D}c$, let $B_0(T^*, T) = I - \frac{1}{c^2}B_2(T^*, T)$. The operator-valued quadratic

$$Q(T, s) = \sum_{j=0}^{2} B_j(T^*, T)s^j$$

takes positive semi-definite values. Hence, ([13]) there exists an auxiliary Hilbert space $\mathcal{Y}$ and operators $V_0, V_1 : H \to \mathcal{Y}$ such that as

$$Q(T, s) = (V_0 + sV_1)^*(V_0 + sV_1).$$
The following lemma validates the hypotheses of Proposition 2.2. As before, let $S$ denote the bilateral shift equal the operator of multiplication by $z = e^{it}$ on $L^2(\mathbb{T})$. In particular, $S$ is unitary and

\begin{equation}
\mathcal{J} = \begin{pmatrix} S & cS \\ 0 & S \end{pmatrix}
\end{equation}

has the form of Equation (6). Recalling the definitions of $B_j(\mathcal{J}^*, \mathcal{J})$, straightforward computation shows,

\begin{align*}
B_0(\mathcal{J}^*, \mathcal{J}) &= \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \\
B_1(\mathcal{J}^*, \mathcal{J}) &= \begin{pmatrix} 0 & cI \\ cI & 0 \end{pmatrix} \\
B_2(\mathcal{J}^*, \mathcal{J}) &= \begin{pmatrix} 0 & 0 \\ 0 & c^2 \end{pmatrix}.
\end{align*}

**Lemma 3.1.** If $T$ is in the class $\mathfrak{S}_{c}$, then the mapping $\rho : \mathcal{H}_s(\mathcal{J}) \to \mathcal{H}_s(T)$ determined by $\rho(\mathcal{J}^* \mathcal{J}^a) = T^a T^a$ is (well defined and) completely positive.

*Proof.* The spaces $\mathcal{H}_s(\mathcal{J})$ and $\mathcal{H}_s(T)$ are spanned by the triples $\{B_0(\mathcal{J}^*, \mathcal{J}), B_1(\mathcal{J}^*, \mathcal{J}), B_2(\mathcal{J}^*, \mathcal{J})\}$ and $\{B_0(T^*, T), B_1(T^*, T), B_2(T^*, T)\}$ respectively, since both $\mathcal{J}$ and $T$ are 3-isometries. In particular, for $n$ a positive integer and with $M_n$ denoting the $n \times n$ matrices, an element $X \in M_n \otimes \mathcal{H}_s(\mathcal{J})$ has the form,

\begin{align*}
X &= X_0 \otimes B_0(\mathcal{J}^*, \mathcal{J}) + X_1 \otimes B_1(\mathcal{J}^*, \mathcal{J}) + X_2 \otimes B_2(\mathcal{J}^*, \mathcal{J}) \\
&\quad \equiv \begin{pmatrix} X_0 & cX_1 \\ cX_1 & c^2 X_2 \end{pmatrix} \otimes I,
\end{align*}

where the $X_j$ are $n \times n$ matrices and $I$ is the identity on the space that $\mathcal{J}$ acts upon. In particular, if $X \geq 0$, then each $X_j$ is self adjoint. Further, $X \geq 0$ if and only if

$$
Y = \begin{pmatrix} X_0 & X_1 \\ X_1 & X_2 \end{pmatrix}
$$

is too in which case there exists $n \times 2n$ matrices $Y_0$ and $Y_1$ such that $Y^*_j Y_k = X_{j+k}$.

To see that $\rho$ is completely positive, recall Equation (9) and observe

\begin{align*}
1_n \otimes \rho(X) &= \sum X_j \otimes B_j(T^*, T) \\
&= X_0 \otimes V_0 V_0 + X_1 \otimes (V_0^* V_1 + V_1^* V_0) + X_2 \otimes V_1 V_1 \\
&= (Y_0 \otimes V_0 + Y_1 \otimes V_1)^* (Y_0 \otimes V_0 + Y_1 \otimes V_1).
\end{align*}
Lemma 3.2. Suppose \( \tilde{J} \) acts on the Hilbert space \( \tilde{E} \) and is of the form in Equation (6). If \( E \) is also a Hilbert space and \( \pi : B(\tilde{E}) \to B(E) \) is a unital *-representation, then \( J = \pi(\tilde{J}) \) has, up to unitary equivalence, the form in Equation (6) too.

Proof. The operator \( \tilde{J} \) can be written as \( \tilde{W} + c\tilde{N} \) where \( \tilde{W} \) is unitary, \( \tilde{N}^2 = 0, \tilde{W}\tilde{N} = \tilde{N}\tilde{W} \) and also \( \tilde{N}^*\tilde{N} + \tilde{N}\tilde{N}^* = I \). It follows that the same is true for \( J = \pi(\tilde{J}) \); i.e., \( J = W + cN \) where \( W \) is unitary, \( N \) is nilpotent of order two, \( W \) and \( N \) commute and \( N^*N + NN^* = I \). It is readily verified from these identities that \( NN^* \) and \( N^*N \) are pairwise orthogonal projections. For instance,

\[
NN^* = N(NN^* + N^*N)N^* = (NN^*)^2.
\]

With respect to the orthogonal decomposition of \( E \) determined by the projections \( NN^* \) and \( N^*N \) and up to unitary equivalence,

\[
N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

Since \( W \) commutes with \( N \) it must have the form

\[
W = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}.
\]

Since \( W \) is unitary, \( U \) is unitary. It now follows, that up to unitary equivalence, \( J \) has the desired form. \( \blacksquare \)

Proposition 3.3. If \( T \) in the class \( \mathcal{F}_c \) and \( T \) acts on the Hilbert space \( H \), then there exists a Hilbert space \( K \) an operator \( J \) acting on \( K \) with the form in Equation (6) and an isometry \( V : H \to K \) such that \( VT^n = JV \) for all natural numbers \( n \).

Proof. Choose \( J \) as in Equation (10). By Lemma 3.1, the mapping sending \( J^*\alpha J^\alpha \) to \( T^*\alpha T^\alpha \) is completely positive. Since \( J \) is symmetric and \( T \) is invertible, Proposition 2.2 implies there exists a Hilbert space \( K \) an isometry \( V : \mathcal{K} \to K \) and a representation \( \pi : B(\mathcal{K}) \to B(K) \) such that \( T^jV = V\pi(J)^j \) for all \( j \in \mathbb{Z} \). Finally, \( J = \pi(J) \) has the form of Equation (6) by Lemma 3.2. \( \blacksquare \)

4. Lifting to an invertible 3-isometry

Theorem 1.1, save for the equality of spectra, follows immediately from the following Proposition together with Proposition 3.3.

Proposition 4.1. If \( T \in B(H) \) is in the class \( \mathcal{F}_c \), then there is a Hilbert space \( K \), an operator \( Y \in B(K) \) such that \( Y \) is invertible and in the class \( \mathcal{F}_c \) and an isometry \( V : H \to K \) such that \( VT^n = Y^nV \) for all natural numbers \( n \).
The proof of the proposition occupies the remainder of this section. Given \( T \) in \( \mathfrak{H}_c \), let
\[
Q_+(T, s) = I + sB_1(T^*, T) + s^2B_2(T^*, T).
\]
Thus, \( Q_+(T, s) = Q(T, s) + c^{-2}B_2(T^*, T) \geq 0 \).

**Lemma 4.2.** If \( T \) is in the class \( \mathfrak{H}_c \), then

(i) \( \|B_2(T^*, T)\| \leq c^2 \);
(ii) \( \|B_1(T^*, T)\| \leq 2c \);
(iii) \( \|T\| \leq 1 + c \); and
(iv) \( 2(1 + c^2)Q_+(T, s) - Q_+(T, s \pm 1) \succeq 0 \).

**Proof.** Since \( Q(T, 0) \succeq 0 \), it follows that \( 0 \leq B_2(T^*, T) \leq c^2I \) and item (i) follows.

Since \( Q_+(T, s) \) is positive semidefinite for all \( s \), for each vector \( x \),
\[
|\langle B_1(T^*, T)x, x \rangle|^2 \leq 4\langle B_2(T^*, T)x, x \rangle \langle x, x \rangle.
\]
Since all the operators involved are selfadjoint it follows that
\[
\|B_1(T^*, T)\|^2 \leq 4\|B_2(T^*, T)\|.
\]
Hence, \( \|B_1(T^*, T)\| \leq 2c \) in view of (i).

To prove item (iii), observe,
\[
\|T\|^2 = \|T^*T\| \leq 1 + \|B_1(T^*, T)\| + \|B_2(T^*, T)\| \leq 1 + 2c + c^2 = (1 + c)^2.
\]

Straightforward computation reveals,
\[
2(1 + c^2)Q_+(T, s) - Q_+(T, s \pm 1) = Q_+(T, s \mp 1) + 2c^2Q(T, s) \succeq 0,
\]
proving item (iv).

**Lemma 4.3.** If \( T \) is a 3-isometry, then for all natural numbers \( j \) and integers \( n \),

(11) \( T^jQ_+(T, n)T^j = Q_+(T, n + j) \).

In particular,

(12) \( T^jB_2(T^*, T)T = B_2(T^*, T) \)

and

(13) \( T^jB_1(T^*, T)T = \frac{T^{*2}T^2 - I}{2} \).

**Proof.** Equation (11) is evident in the case that \( n \) is also a natural number. Equations (12) and (13) follow from Equation (1) and Equations (2) and (3) respectively. From here Equation (11) follows from the definition of \( Q_+(T, n) \).
Lemma 4.4. Suppose $T \in B(H)$. If for each $h \in H$ there exists scalars $b_j(h)$ such that for all natural numbers $\alpha$,

$$\langle T^{\alpha^2} T^n h, h \rangle = b_0(h) + \alpha b_1(h) + \alpha^2 b_2(h),$$

then $T$ is a 3-isometry. If moreover,

$$b_0(h) - b_2(h) + s b_1(h) + s^2 b_2(h) \geq 0$$

for each $h$ and all real $s$, then $T \in F_c$.

Proof. The first hypothesis imply that for each fixed $h \in H$,

$$\langle T^* T^3 - 3T^2 T^2 + 3T^* T - I)h, h \rangle = 0.$$

By polarization, it now follows that $T$ is a 3-isometry. The second hypothesis is easily seen to imply $T$ is in the class $F_c$. 

Proof of Proposition 4.1. Let $V$ denote a vector space (over $\mathbb{C}$) with countable basis $\{e_j : j \in \mathbb{Z}\}$ and let $K$ denote the vector space $V \otimes H$ and let define a sesquilinear form on $K$ by

$$[e_m \otimes h, e_n \otimes k] = \begin{cases} 
\langle Q_+(T, n) T^{m-n} h, k \rangle & \text{if } n \leq m; \\
\langle T^{n-m} Q_+(T, m) h, k \rangle & \text{if } m \leq n.
\end{cases}$$

To see that this form is positive semi-definite, fix positive integers $N$ and $M$ and let

$$h = \sum_{m=-N}^{M} c_m e_m \otimes h_m \in K$$

by given. Note that by Lemma 4.3, if $-N \leq n$, then

$$Q_+(n) = Q_+(-N + (n + N)) = T^{n(N)} Q_+(-N) T^{n+N}.$$ 

Thus

$$[h, h] = \sum_{m, n=-N}^{M} c_m c_n^* [e_m \otimes h_m, e_n \otimes h_n]$$

$$= \sum_{-N \leq m \leq n \leq M} c_m c_n^* \langle T^{n-m} Q_+(T, m) h_m, h_n \rangle + \sum_{-N \leq n < m \leq M} c_m c_n^* \langle Q_+(T, n) T^{m-n} h_m, h_n \rangle$$

$$= \sum_{-N \leq m, n \leq M} c_m c_n^* \langle T^{n+N} Q_+(T, -N) T^{m+N} h_m, h_n \rangle$$

$$= \langle Q_+(T, -N) g, g \rangle \geq 0,$$

where

$$g = \sum_n c_n T^{n+N} h_n = \sum_{j=0}^{N+M} c_{j-N} T^j h_{j-N}$$
and the very last inequality follows from the assumption that $Q(T, -N) \geq 0$. Now let $K$ denote the Hilbert space obtained from $\mathcal{K}$ by moding out null vectors and then forming the completion.

Define $Y : \mathcal{K} \to \mathcal{K}$ by

$$Yh = \sum_{m=-N}^{M} c_m e_{m+1} \otimes h_m = \sum_{m=-(N-1)}^{M+1} c_{m-1} e_m \otimes h_{m-1},$$

where $h \in \mathcal{K}$ as in Equation (14). From equation (15)

$$[Yh, Yh] = \langle Q_+(T, -N + 1)g, g \rangle.$$

Hence another applications of Equation (15), the definitions and Lemma 4.2 give

$$2(1 + c^2)[Yh, Yh] - [h, h] = \langle \left( (2 + c^2)Q_+(T, -N + 1) - Q_+(T, -N) \right) g, g \rangle \geq 0.$$ 

Thus $Y$ determines a bounded operator on $K$ (denoted also by $Y$). Similarly one finds

$$2(1 + c^2)[h, h] - [Yh, Yh] = \langle \left( (2 + c^2)Q_+(T, -N) - Q_+(T, -N + 1) \right) g, g \rangle \geq 0.$$ 

Hence $Y$ has a bounded inverse.

To see that $Y \in \mathcal{F}_c$, observe, for natural numbers $\alpha$, and with $h$ denoting the class of $h$ in $K$,

$$\langle Y^{\alpha} Y^{\alpha} h, h \rangle = \langle Y^{\alpha} h, Y^{\alpha} h \rangle$$

$$= \langle Q_+(T, -N + \alpha)g, g \rangle$$

$$= \langle \left( Q_+(T, -N) + \alpha(B_1(T^*, T) - 2NB_2(T^*, T)) + \alpha^2B_2(T^*, T) \right) g, g \rangle$$

and moreover,

$$\langle Q_+(T, -N)g, g \rangle + s\langle (B_1(T^*, T) - 2NB_2(T^*, T))g, g \rangle + s^2\langle B_2(T^*, T)g, g \rangle - \frac{\langle B_2(T^*, T)g, g \rangle}{c^2}$$

$$= \langle Q(T, -N + s)g, g \rangle \geq 0$$

and apply Lemma 4.4 to conclude $Y \in \mathcal{F}_c$.

Now suppose $n \in \mathbb{Z}$ and $h \in H$ and observe

(16) \quad $||e_n \otimes Th - (e_{n+1} \otimes h)||^2 = \langle T^* Q(n)Th, h \rangle - 2\langle T^* Q(n)Th, h \rangle + \langle Q(n + 1)h, h \rangle = 0.$

Thus, $e_n \otimes Th = e_{n+1} \otimes h$ in $\mathcal{K}$ (they represent the same equivalence class). To finish the proof, define $V : H \to K$ by $Vh = e_0 \otimes h$. From Equation (16)

$$VT = e_0 \otimes Th = e_1 \otimes h = YVh$$

and thus $VT = YV$. \hfill \blacksquare

**Corollary 4.5.** If $T$ is in the class $\overline{\mathcal{F}}_c$, then

$$||T||^2 \leq 1 + \frac{c^2}{2} + c \sqrt{1 + \frac{c^2}{4}}.$$
Proof. The norm of $J$ as in Equation (6) is easily seen to satisfy the inequality (with equality). The result then follows from Theorem 1.1.

5. Spectral Considerations

In this section it is shown that, in the setting of Proposition 3.3, the operator $J$ can be chosen to satisfy $\sigma(J) = \sigma(T)$.

**Proposition 5.1.** Suppose $T \in B(H)$ is in the class $F_c$. If $T$ is invertible, then there is a Hilbert space $\mathcal{E}$, unitary operator $W \in B(\mathcal{E})$ and an isometry $V : H \to \mathcal{E} \oplus \mathcal{E}$ such that $\sigma(W) = \sigma(T)$ and

$$VT = \begin{pmatrix} W & cW \\ 0 & W \end{pmatrix} V.$$  

If $T \in B(H)$ is not invertible, then $\sigma(T) = \overline{D}$.

Before turning to the proof of this proposition, we state the other main result of the section.

**Proposition 5.2 ([3]).** If $T$ is a non-invertible 3-isometry, then $\sigma(T) = \overline{D}$.

Proof. Recall from the introduction that for any three isometry $\sigma(T) \subseteq \overline{D}$ and the 3-isometry $T$ is invertible if and only if $\sigma(T) \subseteq \mathbb{T}$.

Suppose $\lambda \in \mathbb{D}$ and $T - \lambda$ is invertible. Let

$$S = (I - \overline{\lambda}T)(T - \lambda)^{-1}.$$  

That $S$ is a 3-isometry follows from directly calculating

$$= (I - \lambda T^*)(I - \overline{\lambda}T)^3 - 3(T^* - \overline{\lambda}T)(I - \lambda T^*)^2(I - \overline{\lambda}T)^2(T - \lambda)$$

$$+ 3(T^* - \overline{\lambda}T)(I - \lambda T^*)(T - \lambda)^2 - (T^* - \overline{\lambda}T)^3(T - \lambda)^3.$$  

By inspection, $S$ is invertible. Thus $\sigma(S) \subseteq \mathbb{T}$. By spectral mapping the $\phi(\sigma(T)) = \sigma(S)$ where

$$\phi(\zeta) = (\zeta - \lambda)(1 - \overline{\lambda} \zeta)^{-1}.$$  

Hence $\sigma(T) \subseteq \mathbb{T}$ too.

**Remark 5.3.** In the case that $T \in F_c$ (for some $c$) it is possible to use Theorem 1.1 to prove Proposition 5.2. Indeed, if $\phi$ is analytic in a neighborhood of $\overline{D}$ and is unimodular on the boundary of $\mathbb{D}$, then that $\phi(T)$ is a 3-isometry can be seen from $V\phi(T) = \phi(J)V$ and

$$\phi(J) = \begin{pmatrix} \phi(U) & U\phi'(U) \\ 0 & \phi(U) \end{pmatrix},$$

since in this case $\phi(J)$ is evidently a unitary plus commuting nilpotent of order two.
The remainder of this section contains a proof Proposition 5.1 and a brief subsection on the
functional calculus for use in the next section

5.1. A proof of Proposition 5.1. Assuming \( T \) is invertible, by Theorem 3.3, there is a unitary
operator \( U \) acting on a Hilbert space \( \mathcal{F} \) and an isometry \( V : H \to \mathcal{F} \oplus \mathcal{F} \) such that \( VT = JV \),
where
\[
J = \begin{pmatrix} U & cU \\ 0 & U \end{pmatrix}.
\]
The aim is to show that \( U \) can be replaced by \( W = (I - P)U(I - P) \), where \( P \) is the spectral
projection for the complement set \( \sigma(T) \) associated to the unitary (normal) operator \( U \) (so that \( I - P \)
is the spectral projection corresponding to \( \sigma(T) \)).

Of course if \( \sigma(T) = \mathbb{T} \), then there is nothing to prove. Otherwise, consider a nonempty closed
arc \( A \) in \( \mathbb{T} \setminus \sigma(T) \) and, for the purposes of this construction, suppose the end points \( \zeta' \) and \( \zeta'' \) of the
arc \( A \) are equidistant from \( \sigma(T) \). We shall call this a centered arc. Let \( \lambda \) be the midpoint of the arc,
and choose a \( t > 1 \). Consider the following diagram.

A geometric argument shows, for \( t > 1 \) fixed, that
\[
\inf_{\zeta \in \lambda} \left| \frac{1}{\zeta - t\lambda} \right| = \left| \frac{1}{\zeta' - t\lambda} \right| = \left| \frac{1}{\zeta'' - t\lambda} \right|.
\]
Choose an \( \alpha \) such that
\[
\left| \frac{\alpha}{\zeta - t\lambda} \right| = 1
\]
and let \( f(\zeta) = \alpha(\zeta - t\lambda)^{-1} \). Choose a contour \( \Gamma \) with \( \sigma(T) \) on its inside and the arc \( A \) on its outside
(the bounded and unbounded components determined by \( \Gamma \) respectively) and such that the modulus
of \( f \) is less than one on and inside \( \Gamma \).

Since \( f \) is analytic in a neighborhood of the closed unit disc and the spectrum of \( J \) is in \( \mathbb{T} \),
the expression \( f(J) \) can be defined as a convergent power series. On the other hand \( f(U) \) can be
defined in terms of power series or by the Borel functional calculus, since \( U \) is unitary (and hence
normal). Of course both give the same value for \( f(U) \). It is straightforward to verify

\[
f(J) = \begin{pmatrix} f(U) & f'(U) \\ 0 & f(U) \end{pmatrix}.
\]

Now \( f(T) \) can be defined as a convergent power series or by the Riesz functional calculus,

\[
f(T) = \frac{1}{2\pi i} \int_{\Gamma} f(z)(z - T)^{-1} \, dz.
\]

Write, with respect to the decomposition \( K = \mathcal{F} \oplus \mathcal{F} \),

\[
V = \begin{pmatrix} V_1 \\ V_0 \end{pmatrix}.
\]

Let \( E \) denote the spectral measure for the unitary operator \( U \). Thus, for any Borel set \( B \subseteq \mathbb{T} \) the projection \( E(B) \) and \( U \) commute and moreover,

\[
U = \int_{\mathbb{T}} \lambda \, dE(\lambda).
\]

**Lemma 5.4.** If \( A \) is a closed centered arc such that the \( A \cap \sigma(T) = \emptyset \), then \( E(A)V_\ell = 0 \) for \( \ell = 0, 1 \).

**Proof.** From the Riesz functional calculus, \( f^n(T) \) converges to 0 in the operator norm since \( f^n \) converges to 0 uniformly on \( \Gamma \). On the other hand,

\[
V f^n(T) = f^n(J) V
\]

and hence \( f^n(J)V \) also tends to 0.

Let \( P \) denote the spectral projection (for \( U \)) corresponding to the arc \( A \),

\[
P = \int_{A} \, dE = E(A).
\]

Consider, with respect to the decomposition \( K = \mathcal{F} \oplus \mathcal{F} \),

\[
0 \oplus P = \begin{pmatrix} 0 & 0 \\ 0 & P \end{pmatrix}
\]

and similarly \( P \oplus 0 \). Because \( f^n(J)V \) tends to 0 in operator norm, so do both

\[
V^* f^n(J)^* (0 \oplus P) B_2(J^*, J)(0 \oplus P) f^n(J)V
\]

and

\[
V^* f^n(J)^* (P \oplus 0) [I - \frac{1}{c^2} B_2(J^*, J)] (P \oplus 0) f^n(J)V.
\]
Straightforward computation shows
\[
\frac{1}{c^2} f^n(J)^* (0 \oplus P) B_2(J^*, J)(0 \oplus P) f^n(J) = \begin{pmatrix} f^n(U)^* & 0 & 0 \\ 0 & f^n(U)^* & 0 \\ 0 & 0 & f^n(U) \end{pmatrix} \begin{pmatrix} 0 & 0 & f^n(U) \\ 0 & P & 0 \\ 0 & 0 & f^n(U) \end{pmatrix} = \begin{pmatrix} 0 & 0 & f^n(U)P f^n(U) \\ 0 & f^n(U)^* P f^n(U) & 0 \\ f^n(U)^* P f^n(U) & 0 & 0 \end{pmatrix}.
\]

It follows that \( P f^n(U)V_0 \) tends to 0. On the other hand, \( P f^n(U) = f^n(U)P \) since \( P \) is a spectral projection. Consequently, using the Riesz functional calculus,
\[
V_0^* P |f^n|^2(U) PV_0 = V_0^* f^n(U)^* P f^n(U)V_0
\]
also tends to 0. On the other hand, \( P |f^n|^2 \geq P \) since \( |f^n| \geq 1 \) on the support \( A \) of \( P \). Thus \( PV_0 = 0 \); i.e., the range of \( V_0 \) lies in the range of \( I - P \).

Similarly,
\[
f^n(J)^* (P \oplus 0)(I - \frac{1}{c^2} B_2(J^*, J)(P \oplus 0)) f^n(J) = \begin{pmatrix} P & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f^n(U)^* f^n(U) & 0 & 0 \\ 0 & f^n(U)^* & 0 \\ 0 & 0 & f^n(U) \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Hence, using the already established \( PV_0 = 0 \),
\[
V^* f^n(J)^* (P \oplus 0)(I - \frac{1}{c^2}(P \oplus 0) B_2(J^*, J)) f^n(J)V = V_1^* P f^n(U)^* f^n(U)PV_1
\]
converges to 0. Thus \( PV_1 = 0 \).

Note that
\[
JV = \begin{pmatrix} W & cW \\ 0 & W \end{pmatrix} V,
\]
where \( W = (I - P)U(I - P) \) is a unitary operator on the Hilbert space \((I - P)\mathcal{F}\).

**Lemma 5.5.** If \( A \subseteq \mathbb{T} \) is a closed arc in the complement of \( \sigma(T) \), then \( E(A)V_\ell = 0 \) for \( \ell = 1, 2 \).

**Proof.** Any such arc \( A \) is contained in a closed centered arc \( I \) disjoint from \( \sigma(T) \). Since \( E(A) \leq E(I) \) and, by Lemma 5.4, \( E(I)V_\ell = 0 \), the conclusion of the lemma follows.

**Lemma 5.6.** Suppose \( A_1 \subseteq A_2 \subseteq \mathbb{T} \) is an increasing sequence of Borel subsets of \( \mathbb{T} \) and let \( A = \bigcup_j A_j \). If \( E(A_j)V_\ell = 0 \) for all \( j \) and \( \ell = 0, 1 \), then \( E(A)V_\ell = 0 \).

**Proof.** Consider the supremum \( P \) of the projections \( P_j = E(A_j) \). Because \( E \) is a spectral measure, \( P = E(A) \). Since \( P_j \) converges SOT (strong operator topology) to \( P \), it follows that \( P_jV_\ell \) converges to \( PV_\ell \). Thus \( PV_\ell = 0 \).

Now let \( A \) be an open arc in the complement of \( \sigma(T) \). Since \( A \) can be written as a disjoint union of closed arcs satisfying the hypothesis of Lemma 5.5 it follows that \( E(A)V_\ell = 0 \) for \( \ell = 1, 2 \) by Lemma 5.6.
Finally, \( B = \mathbb{T} \setminus \sigma(T) \) is the disjoint union of open arcs \( \{A_j\} \). In particular, \( E(A_j)V_\ell = 0 \) for each \( j \) and \( \ell \). Thus, another application of Lemma 5.6 gives

\[
E(B)V_\ell = 0.
\]

Let \( P = E(B) \). The operator \( \mathfrak{U} = (I - P)U(I - P) \) is unitary and

\[
\mathfrak{Z} = \begin{pmatrix}
\mathfrak{U} & c\mathfrak{U} \\
0 & \mathfrak{U}
\end{pmatrix}
\]

has the form in Equation (6). Moreover, \( \sigma(\mathfrak{U}) \subseteq \sigma(T) \) and hence \( \sigma(\mathfrak{Z}) \subseteq \sigma(T) \) also. Of course,

\[
VT = JV = \mathfrak{Z}V
\]

too.

It remains to show that \( \sigma(\mathfrak{Z}) \supset \sigma(T) \). To this end, suppose that \( \lambda \in \mathbb{T} \) and \( \mathfrak{Z} - \lambda \) is invertible. Let \( L = V^*(\mathfrak{Z} - \lambda)^{-1}V \) and observe that \( L(T - \lambda) = I \) so that \( T - \lambda \) is left invertible. On the other hand, choosing a sequence \( \lambda_n \) not on the circle \( \mathbb{T} \) but converging to \( \lambda \) it follows that \( (\mathfrak{Z} - \lambda_n)^{-1} \) is bounded and converges to \( (\mathfrak{Z} - \lambda)^{-1} \). Since \( (T - \lambda_n)^{-1} = V^* (\mathfrak{Z} - \lambda_n)^{-1} \), it follows that \( (T - \lambda_n)^{-1} \) converges to the left inverse \( L \). Thus \( T - \lambda \) is invertible (and its inverse is \( L \)).

5.2. More on the holomorphic functional calculus. Let \( \Omega \) denote an open simply connected subset of the plane. Given an operator \( T \) whose spectrum lies in \( \Omega \) and a function \( g \) holomorphic on \( \Omega \), the operator \( g(T) \) can be defined by the holomorphic (Riesz) functional calculus. Moreover, if \( T \) is normal, then so is \( g(T) \). Further, by Runge’s Theorem, there exists a sequence of polynomials \( p_n \) such that \( p_n \) converges to \( G \) uniformly on compact subsets of \( \Omega \). Thus by standard properties of the functional calculus, \( p_n(T) \) converges to \( g(T) \). Likewise, \( p'_n \) converges uniformly to \( g' \) and that \( p'_n(T) \) converges to \( g'(T) \).

In the special case that \( \sigma(T) \subseteq \overline{\Omega} \subseteq \Omega \), the function \( g \) has a power series expansion whose partial sums \( (s_n) \) converges uniformly on compact subsets of \( \Omega \). In particular, \( s_n(T) \) converges to \( g(T) \).

For \( J \) as in Equation (6) with \( \sigma(U) \) a subset of \( \Omega \) and \( p \) a polynomial, a simple calculation shows,

\[
p(J) = \begin{pmatrix}
p(U) & Up'(U) \\
0 & p(U)
\end{pmatrix}.
\]

In particular,

\[
g(J) = \lim p_n(J) = \begin{pmatrix}
g(U) & Ug'(U) \\
0 & g(U)
\end{pmatrix},
\]

(18)
Likewise, if \( A \) is selfadjoint with spectrum in \( \Omega \), then with

\[
J = \begin{pmatrix} A & cI \\ 0 & A \end{pmatrix},
\]

we have

\[
g(J) = \begin{pmatrix} g(A) & cg'(A) \\ 0 & g(A) \end{pmatrix}.
\]

Further, if \( g(\sigma(A)) \subseteq \mathbb{T} \), then \( g(A) \) is normal with spectrum in the unit circle and is thus unitary. Consequently, \( g(J) \) takes the form in Equation (6).

Let \( G \) denote the mapping \( G(z) = \exp(iz) \), suppose \([a, b]\) of length strictly less than \(2\pi\) and let \( S = G([a, b]) \). In particular, \( S \) is a proper subset of the unit circle \( \mathbb{T} \). There exists open simply connected sets \( \Omega \) and \( \Omega_* \) containing \([a, b]\) and \( S \) respectively such that \( G : \Omega \to \Omega_* \) is bianalytic. If \( T \) is an operator with spectrum in \([a, b]\), then \( G(T) \) is well defined and has, by the spectral mapping theorem, its spectrum in \( S \). Letting \( H \) denote the inverse of the mapping \( G : \Omega \to \Omega_* \), the composition property of the holomorphic functional calculus implies that \( H(G(T)) = T \).

Now suppose that \( T \) is a 3-symmetric operator with spectrum in \([a, b]\). In this case

\[
T = G(T) = \exp(itT)
\]
is a 3-isometry since \( T^n = \exp(inT) \) for natural numbers \( n \). Moreover, the spectrum of \( T \) is a proper subset of the unit circle.

6. 3-Symmetric Operators

Fix a 3-symmetric operator \( T \in B(H) \). For a real numbers \( s \) and \( t \),

\[
\exp(isT)^* \exp(isT) = I + s t B_1(T^*, T) + st^2 B_2(T^*, T).
\]

Thus \( tT \) is also a 3-symmetric operator and

\[
B_j((tT)^*, tT) = t^j B_j(T^*, T).
\]

Let \( c^2 = \|B_2(T^*, T)\| \).

Choose a \( t_0 > 0 \) such that \( \sigma(t_0T) \) is a subset of an interval \([a, b]\) of length less than \(2\pi\). Let

\[
S = \{\exp(is) : s \in [a, b]\} \subseteq \mathbb{T}.
\]

Let \( \Omega, \Omega_* \) and \( G \) be as at the end of Subsection 5.2. In this case

\[
T = \exp(it_0T) = G(T)
\]
is a 3-isometry with spectrum contained in \( S \). Moreover,

\[
B_2(T^*, T) = t_0^2 B_2(T^*, T)
\]
and thus,

\[(t_0c)^2 = \|B_2(T^*, T)\|.
\]

By Proposition 5.1, there is a Hilbert space \(E\) and a unitary operator \(W \in B(\mathcal{E})\), an isometry \(V : H \to \mathcal{E} \oplus \mathcal{E}\) such that \(\sigma(T) = \sigma(W)\) such that Equation (17) holds with \(t_0c\) in place of \(c\). Thus with

\[J = \begin{pmatrix} W & ct_0W \\ 0 & W \end{pmatrix},\]

\(VT = JV\). Hence, as \(G^{-1}\) is analytic in a neighborhood of the spectrum of \(J\),

\[t_0VT = VG^{-1}(T) = G^{-1}(J)V = \begin{pmatrix} G^{-1}(W) & ct_0W(G^{-1})'(W) \\ 0 & G^{-1}(W) \end{pmatrix}.
\]

Let \(A = G^{-1}(W)\) and note that \((G^{-1})'(W) = -iW^*\). Thus, with

\[J = \frac{1}{t_0} \begin{pmatrix} A & -ict_0 \\ 0 & A \end{pmatrix},\]

\(VT = JV\) and most of the following Theorem of Helton and Agler is established.

**Theorem 6.1** ([2][7][9][8][6]). If \(T \in B(H)\) is a 3-symmetric operator, but not selfadjoint, then \(B_2(T^*, T) \neq 0\). In this case, with \(c = \|B_2(T^*, T)\|\),

\[(19) \quad \exp(isT)^* \exp(isT) - \frac{B_2(T^*, T)}{c^2} \geq 0
\]

for all \(s\). Moreover, there exists a Hilbert space \(\mathcal{E}\), a selfadjoint operator \(A \in B(\mathcal{E})\) and an isometry \(V : H \to \mathcal{E} \oplus \mathcal{E}\) such that

\[VT = \begin{pmatrix} A & -ic \\ 0 & A \end{pmatrix}.
\]

All that remains to be proved is the inequality of Equation (19). To this end let \(B_j = B_j(T^*, T)\), and note

\[
\exp(isT)^*[I + tB_1 + t_2B_2] \exp(isT) = \exp(i(s + t)T)^* \exp(i(s + t)T) = I + (s + t)B_1 + (s + t)^2B_2,
\]

from which follows that

\[
\exp(isT)^*B_2 \exp(isT) = B_2.
\]

Thus, with \(c^2 = 2\|B_2(T^*, T)\|\),

\[
\exp(isT)^* \exp(isT) - \frac{B_2(T^*, T)}{c^2} = \exp(-isT^*)[I - \frac{B_2(T^*, T)}{c^2}] \exp(isT) \geq 0.
\]

Some of the results in this article were part of the first (alphabetically) listed authors research during his PhD studies at the University of California, San Diego, under the direction of Jim Agler.
References

[1] Agler, Jim, *The Arveson extension theorem and coanalytic models*, Integral Equations Operator Theory 5 (1982), no. 5, 608631.

[2] Agler, Jim, *SubJordan Operators*, PhD Thesis, Indiana University, 1980.

[3] Agler, Jim; Stankus, Mark, *m-isometric transformations of Hilbert space. II*, Integral Equations Operator Theory 23 (1995), no. 1, 148.

[4] Agler, Jim; Stankus, Mark, *m-isometric transformations of Hilbert space. I*, Integral Equations Operator Theory 21 (1995), no. 4, 383429.

[5] Agler, Jim; Stankus, Mark, *m-isometric transformations of Hilbert space. III*, Integral Equations Operator Theory 24 (1996), no. 4, 379421.

[6] Ball, Joseph A.; Helton, J. William, *Nonnormal dilations, disconjugacy and constrained spectral factorization*, Integral Equations Operator Theory 3 (1980), no. 2, 216309.

[7] Helton, J. William, *Jordan operators in infinite dimensions and Sturm Liouville conjugate point theory*, Bull. Amer. Math. Soc. 78 (1971) 5761.

[8] Helton, J. William, *Operators with a representation as multiplication by on a Sobolev space*, Hilbert space operators and operator algebras (Proc. Internat. Conf., Tihany, 1970), 279287.

[9] Helton, J. William, *Infinite dimensional Jordan operators and Sturm-Liouville conjugate point theory*, Trans. Amer. Math. Soc. 170 (1972), 305331.

[10] McCullough, Scott, *Sub-Brownian operators*, J. Operator Theory 22 (1989), no. 2, 291305.

[11] Paulsen, Vern, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics, 78. Cambridge University Press, Cambridge, 2002.

[12] Richter, Stefan, *A representation theorem for cyclic analytic two-isometries*, Trans. Amer. Math. Soc. 328 (1991), no. 1, 325349.

[13] Rosenblum, Marvin; Rovnyak, James, *Hardy classes and operator theory*, Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1985.

Scott McCullough, Department of Mathematics, University of Florida, Gainesville

E-mail address: sam@math.ufl.edu

Benjamin Russo, Department of Mathematics, University of Florida, Gainesville

E-mail address: russo5@math.ufl.edu