Polyhedral representations of discrete differential manifolds

Roman R. Zapatrin

Department of Mathematics, SPb UEF, Gribovedova 30/32, 191023, St-Petersburg, Russia
and
Division of Mathematics,
Istituto per la Ricerca di Base,
I-86075, Monteroduni, Molise, Italy

Abstract

Any discrete differential manifold $M$ (finite set endowed with an algebraic differential calculus) can be represented by appropriate polyhedron $\mathcal{P}(M)$. This representation demonstrates the adequacy of the calculus of discrete differential manifolds and links this approach with that based on finitary substitutes of continuous spaces introduced by R.D. Sorkin.

1 Introduction

During long time it was the concept of differentiable manifold that was the arena on which physical theories took place due to its adequacy to the intuitive feeling what the physical space ought to be. However, the development of quantum theory gave rise to the idea to deprive the spacetime of its status of primordial object and inspired the development of the theories dealing with 'deeper' entities than spacetime in order to make the latter an observable in a more general theory. Reasoning about spacetime at very small scales gave rise to the theories where the manifold was replaced by a discrete structure. The environment of the present paper is formed by the following scope of ideas and techniques.

The first belongs to Geroch and asserts that even in classical general relativity the notion of the spacetime manifold is essentially used only once: to set up the algebra of smooth functions. We can, instead, start from this (commutative) algebra as the basic object of the theory. Although it remains nothing but a reformulation of the conventional theory and does not mean that the points (events) are effectively smeared off.
If we accept an algebra to be the starting object (called the basic algebra) we can try to go beyond the class of commutative Banach algebras representable by functions on manifolds. In particular, we can assume these algebras to be non-commutative which gives rise to the non-commutative geometry \[1\]. Another opportunity looking very attractive from the computational point of view is to assume the basic algebra to be finite-dimensional and commutative.

At first sight, these objects look very trivial and poor since any such algebra can be realized by the algebra of functions on a finite set with the discrete topology. However, even in classical differential geometry a differentiable manifold is a topological space plus a differential structure rather than simply a topological space. Being applied to finite dimensional commutative algebras this observation gives rise to the notion of discrete differential calculus and discrete differential manifold \[2, 3\]. This construction will be recalled in sections 2 and 3.

The important entity in the theory of discrete differential manifolds is that of the *generated topological space* being a finite or at most countable topological space associated with a discrete differential manifold (section 4). The generated topological space is intended to play the role of finitary substitute for the continuous spacetime manifold.

The finitary spacetime substitutes introduced in \[8\] serve to simulate the continuous topological spaces. These techniques (reviewed in section 4) may be treated as the generalization of the Regge calculus in general relativity \[7\] for the case when the metric is given up \[8\]. The substitutes are built from continuous manifolds by applying the coarse-graining procedure yielding finite or at most countable \(T_0\)-spaces \[10\].

So, there are two sources of finitary \(T_0\) spaces: the generated topological spaces of discrete differential manifolds \[3\] on one hand, and the results of coarse graining of continuous topological spaces on the other.

In this paper the following techniques are suggested. Given a discrete differential manifold, the polyhedron (being a continuous topological space) is built and the coarse graining procedure for it is specified so that the generated topological space of the discrete differential manifold and the finitary substitute of the polyhedron are isomorphic \(T_0\)-spaces (the diagram on Fig. 1 is commutative).

Technically, it is done in the following way. A discrete differential calculus is represented as the quotient of the universal differential algebra over an appropriate differential ideal. In section 3 the structure of differential ideals is studied. As a result, with any finite dimensional discrete differential manifold \((\mathcal{M}, \Omega)\) the abstract simplicial complex \(K = K(\Omega)\) is associated. In section 3 the polyhedral representations of simplicial complexes associated with discrete differential manifolds are considered, and the coarse-graining procedure \[8, 10\] described in section 4 is specified in order to produce the finitary substitutes isomorphic to the topological spaces generated by the discrete differential manifolds.
2 Basic notions and results

Let $M$ be a finite set. An algebraic differential calculus on $M$ is an extension of the algebra $A = \text{Fun}(M, \mathbb{C})$ (called basic algebra) of all $\mathbb{C}$-valued functions on $M$ to a graded differential algebra $(\Omega, d)$:

$$\Omega = \Omega^0 \oplus \Omega^1 \oplus \ldots \oplus \Omega^r \oplus \ldots$$

with $\Omega^0 = A$, $d: \Omega^r \to \Omega^{r+1}$ and $d\Omega^r$ being the generating set for the next $A$-bimodule $\Omega^{r+1}$. If $\Omega^r \neq 0$ for some $r$ and $\Omega^s = 0$ for every $s > r$, then the calculus $\Omega$ is said to have the dimension $r$. The universal object $\hat{\Omega} = \hat{\Omega}(M)$ of such type is the differential envelope of $A$. The set $M$ is assumed to be finite, and $\hat{\Omega}$ can be explicitly described by setting its natural basis:

$$\hat{\Omega}^0 = \text{span}\{e_i | i \in M\}$$
$$\hat{\Omega}^1 = \text{span}\{e_{ik} | i, k \in M, i \neq k\}$$
$$\ldots$$
$$\hat{\Omega}^r = \text{span}\{e_{i_0, i_1, \ldots, i_r} | i_0, i_1, \ldots, i_r \in M; \forall s = 1, \ldots, r \ i_s - 1 \neq i_s\}$$

where $e_i : M \to \mathbb{C}$ is defined as $e_i(k) = \delta_{ik}$. Then each $\hat{\Omega}^r$ is $A$-bimodule:

$$e_p \cdot e_{i_0, i_1, \ldots, i_r} \cdot e_q = \delta_{pi_0} \delta_{i, q} e_{i_0, i_1, \ldots, i_r}$$

and the graded product $\hat{\Omega}^r \times \hat{\Omega}^s \to \hat{\Omega}^{r+s}$ is defined as:

$$e_{i_0, i_1, \ldots, i_r} e_{j_0, j_1, \ldots, j_s} = \delta_{i_r, j_0} e_{i_0, i_1, \ldots, i_r, j_1, \ldots, j_s}$$

with the operator $d: \hat{\Omega}^r \to \hat{\Omega}^{r+1}$ having the form:

$$de_{i_0, i_1, \ldots, i_r} = \sum_{s=0}^{r} (-1)^s \sum_{k \neq i_s, i_{s+1}} e_{i_0, \ldots, i_{s-1}, k, i_s, \ldots, i_r}$$

Figure 1: The environment of the contents of the paper
The graded algebra $\tilde{\Omega}$ is universal in the sense that any particular differential calculus $\Omega$ can be covered by an epimorphism $\pi : \tilde{\Omega} \to \Omega$ of graded differential algebras over the basic algebra $\mathcal{A}$. The kernel of this mapping $\pi$ is said to be differential ideal in $\tilde{\Omega}$. So, every differential calculus $\Omega$ can be unambiguously characterized by appropriate differential ideal $I(\Omega)$ in $\tilde{\Omega}$. Let us dwell on this issue in more detail.

**Definition 1** A linear subspace $I \subseteq \tilde{\Omega}$ is called differential ideal in $\tilde{\Omega}$ if:

$$\tilde{\Omega}I \subseteq I \quad dI \subseteq I$$

Denote by $\Omega(I) = \tilde{\Omega}/I$ the differential calculus induced by the differential ideal $I$. The decomposition of the graded algebra $\tilde{\Omega}$ gives rise to the decomposition of $I$:

$$I = I^0 \oplus \ldots \oplus I^r \oplus \ldots$$

Then the definition of the differential calculus $\Omega(I)$ is reformulated in terms of $I$ as follows:

$$\dim \Omega(I) = r \quad \text{def} \quad I^r \neq \tilde{\Omega}^r \quad \forall s > r \quad I^s = \tilde{\Omega}^s$$

**Assumption.** We shall consider only the differential ideals with $I^0 = 0$.

In the sequel, to introduce combinatorial structures confine ourselves by basic differential ideals, namely, the ideals spanned on the elements of the natural basis.

**Lemma 1** Let $I$ be a basic differential ideal, $\alpha = i_0, \ldots, i_r$ and $e_\alpha \in I$. Let $\beta$ be such sequence of elements of $\mathcal{M}$ that $\alpha$ is its subsequence $\alpha \subseteq \beta$. Then $e_\beta \in I$.

**Proof.** Use the induction over $\delta = |\beta| - |\alpha|$ (the difference of the lengths of the sequences). Suppose $\delta = 1$, that means $\beta = i_0, \ldots, i_{t-1}, q, i_t, \ldots, i_r$. Due to (3), $de_\alpha \in I$ and equals to (3), being the sum of the basis elements of $I$ (assumed to be the basic ideal). Therefore each summand of (3), in particular, $e_{i_\beta}$, is the element of $I$. Now suppose the result is valid for all $\beta$ such that $|\beta| - |\alpha| = s$, and let $\beta$ be such supersequence of $\alpha$ that $|\beta| = |\alpha| + s + 1$. Consider a sequence $\beta'$ obtained from $\beta$ by deleting an element which does not belong to the sequence $\alpha$. Then $e_{\beta'} \in I$ (being the supersequence of $\alpha$ of the length $|\alpha| + s$), and therefore $e_\beta \in I$ since it occurs in the decomposition (3) for $de_\beta$. 

4
Corollary. Let $\mathcal{I}$ is a basic differential ideal, and $\alpha, \beta$ are such sequences of elements of $\mathcal{M}$ that $\alpha \subseteq \beta$. Then $e_\beta \notin \mathcal{I} \Rightarrow e_\alpha \notin \mathcal{I}$.

It is suitable to impose the scalar product on $\tilde{\Omega}$ by assuming the natural basis (2) to be orthonormal:

$$(e_\alpha, e_\beta) = \begin{cases} 1, & \text{if } \alpha = \beta \\ 0, & \text{otherwise} \end{cases} \quad (8)$$

Definition 2. A basic differential calculus is the quotient $\Omega = \Omega(\mathcal{I}) = \tilde{\Omega}/\mathcal{I}$ of the differential envelope over a basic differential ideal $\mathcal{I}$.

Using the scalar product (8) the following description of basic differential calculi was suggested in [2]. The quotient space $\Omega = \tilde{\Omega}/\mathcal{I}$ can be identified with the subspace of $\tilde{\Omega}$ being the orthocomplement to $\mathcal{I}$, denote it with the same symbol $\Omega$. Thus

$$\Omega = \text{span}\{e_{i_0, \ldots, i_r} \mid e_{i_0, \ldots, i_r} \notin \mathcal{I}\}$$

or, in other words, the basic differential forms constituting the ideal $\mathcal{I}$ are said to be vanishing in the differential calculus $\Omega$:

$$e_{i_0, \ldots, i_r} \in \mathcal{I} \iff e_{i_0, \ldots, i_r} = 0e_{i_0, \ldots, i_r} \notin \mathcal{I} \iff e_{i_0, \ldots, i_r} \neq 0 \quad (9)$$

So, the Lemma 1 can be reformulated as follows:

$$\forall \alpha \subseteq \beta \quad e_\beta \neq 0 \Rightarrow e_\alpha \neq 0 \quad (10)$$

3. Discrete differential manifolds

In classical differential geometry a differentiable manifold is a topological space equipped with a differential structure. Its finitary counterpart looks as follows.

Definition 3. A discrete differential manifold is a couple $(\mathcal{M}, \Omega)$ where $\mathcal{M}$ is a finite set and $\Omega$ is a basic differential calculus (see Definition 2) over the functional algebra $\mathcal{A} = \text{Fun}(\mathcal{M})$. The dimension of the discrete differential manifold is the greatest grade of its nonvanishing differential forms:

$$\dim \mathcal{M} = \max\{r \mid \Omega^r \neq 0\}$$

If this number does not exist, the discrete differential manifold is said to be infinite dimensional:

$$\dim \mathcal{M} = \infty \iff \forall r \quad \Omega^r \neq 0 \quad (11)$$
Taking into account the notation (9), discrete differential manifolds are unambiguously determined by the collection of their nonvanishing basic differential forms:

\[ M = (\mathcal{M}, \Omega) = (\mathcal{M}, \mathcal{K}) \]

where \( \mathcal{K} = \mathcal{K}(M) \) is the collection of the sequences \( \alpha \) of elements of the set \( \mathcal{M} \) such that

\[ \alpha \in \mathcal{K} \iff e_\alpha \neq 0 \quad (12) \]

The set of sequences \( \mathcal{K} \) can be decomposed in the following way:

\[ \mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2 \cup \ldots \cup \mathcal{K}_r \cup \ldots \]

\[ \alpha \in \mathcal{K}_r \iff \alpha \in \mathcal{K} \text{ and } |\alpha| = r \]

**Lemma 2** If \((i_0, \ldots, i_r) \in \mathcal{K}_r\) then for any \(s\) such that \(0 \leq s \leq r\) we have \((i_0, \ldots, i_s, \ldots, i_r) \in \mathcal{K}_{r-1}\)

**Proof.** Follows from the corollary from Lemma [1].

The component \( \mathcal{K}_1 \) of the collection \( \mathcal{K} \) gives rise to a binary relation on the set \( \mathcal{M} \), denote it \( \preceq \):

\[ i \preceq j \iff (i, j) \in \mathcal{K}_1 \iff e_{ij} \notin \mathcal{I} \quad (13) \]

Since no \( e_{ii} \) is the basic form, \( e_{ii} \notin \mathcal{I} \), hence the relation \( \preceq \) on \( \mathcal{M} \) is reflexive: \( \forall i \in \mathcal{M} \quad i \preceq i \).

It follows immediately from Lemma [2] that

\[ (i_0, \ldots, i_r) \in \mathcal{K}_r \Rightarrow \forall s, t \quad 0 \leq s \leq t \leq r \quad i_s \preceq i_t \quad (14) \]

**Definition 4** A discrete differential manifold \( M \) is called network manifold if its differential structure is completely defined on the level of its 1-forms, that is the reverse of the implication (14) holds:

\[ \mathcal{K}_r = \{(i_0, \ldots, i_r) \mid \forall s, t \quad 0 \leq s \leq t \leq r \quad i_s \preceq i_t \} \quad (15) \]

**Lemma 3** Let \( M = (\mathcal{M}, \Omega) \) be a discrete differential manifold such that the set \( \mathcal{M} \) is finite. If \( M \) is infinite dimensional, then the relation \( \preceq \) on the set \( \mathcal{M} \) is not antisymmetric:

\[ \dim \mathcal{M} = \infty \Rightarrow \exists i, j \in \mathcal{M}, i \neq j : \quad i \preceq j, j \preceq i \]

If \( M \) is the network manifold, the above implication holds in both directions.

**Proof.** Let \( m = \text{card}\mathcal{M} \). Consider a number \( r > m \) and let \( e_{i_0, \ldots, i_r} \neq 0 \): it exists due to (11). Then at least one element \( i \in \mathcal{M} \) occurs at least twice in the string \((i_0, \ldots, i_r)\), that is for some \( s, t \) such that \( s > t + 1 \) we have \( i = i_s = i_t \). Denote \( j = i_{s+1} \), then \( j \neq i \) by virtue of (2), and according to (14) \( i \preceq j \) and \( j \preceq i \).

Now let \( M \) be a network manifold. Let \( i \neq j \) and \( i \preceq j \preceq i \). For any \( r > 0 \) consider the sequence \((ij \ldots ij)\) of length \( 2r \). Then, according to (15) \( e_{ij \ldots ij} \neq 0 \), thus \( \dim \mathcal{M} = \infty \).
Corollary. Let \( \dim M < \infty \). If \( e_\alpha \neq 0 \) in \( M \), then all the elements of the string \( \alpha \) are different. Therefore

\[
\text{card}\{i_0, \ldots, i_r\} = \text{length}(i_0, \ldots, i_r) \tag{16}
\]

¿From now on confine ourselves by finite dimensional discrete differential manifolds and recall further definitions.

Definition 5 A set \( K \) with a relation \( \preceq \) on it is called full ordered if the relation \( \preceq \) has the following properties: for any \( i, j, k \in K \)

\[
i \preceq i \quad \text{(reflexivity)}
\]

\[
i \preceq j, j \preceq i \quad \Rightarrow \quad i = j \quad \text{(antisymmetry)}
\]

\[
i \preceq j, j \preceq k \quad \Rightarrow \quad i \preceq k \quad \text{(transitivity)}
\]

\[
i \npreceq j \quad \Rightarrow \quad j \preceq i \quad \text{(linearity)}
\] \tag{17}

Note that any subset of a fully ordered set is fully ordered as well. The following lemma shows the relevance of full orders in the theory of discrete differential manifolds.

Lemma 4 Let \( M = (\mathcal{M}, \Omega) \) be a finite dimensional discrete differential manifold. Consider a sequence \( \alpha \). Then the following implication holds:

\[
e_\alpha \neq 0 \quad \Rightarrow \quad (\alpha, \preceq) \text{ is the full order} \tag{18}
\]

Moreover, if \( M \) is a network manifold, the implication (18) holds in both directions.

Proof. The implication (18) follows from (14) and the definition (13). Now let \( M \) be a network manifold, and \( \alpha = \{i_0, \ldots, i_r\} \) be a fully ordered (with respect to \( \preceq \) ) subset of \( \mathcal{M} \). Arrange the elements of the set \( \alpha \) to form the sequence \( \alpha \) such that \( i_0 \preceq i_1 \preceq \ldots \preceq i_r \). Then the transitivity of the relation \( \preceq \) implies that for all \( s, t \) such that \( 0 \leq s < t \leq r \) we have \( e_{i_s, i_t} \neq 0 \), therefore \( e_\alpha \neq 0 \) by virtue of (14).

So, from the combinatorial point of view the discrete differential manifolds are characterized in the following way. We have a set \( \mathcal{M} \) with a reflexive antisymmetric (but not generally transitive) relation \( \preceq \) on it. Then we select a family \( \mathcal{K} \) of subsets of \( \mathcal{M} \) such that

- any element \( \alpha \in \mathcal{K} \) is fully ordered by the relation \( \preceq \)
- \( \mathcal{K} \) is hereditary: \( \alpha \in \mathcal{K}, \beta \subseteq \alpha \Rightarrow \beta \in \mathcal{K} \)
- \( \mathcal{K} \) contains all singletons (since \( \preceq \) is reflexive): \( \forall i \in \mathcal{M} \quad \{i\} \in \mathcal{K} \)
Then we can build the discrete differential calculus $\Omega$ on $\mathcal{A} = \text{Fun}(\mathcal{M})$ by putting
$$\Omega = \Omega(\mathcal{K}) = \text{span}\{e_\alpha \mid \alpha \in \mathcal{K}\}$$
In particular, when $M$ is a finite dimensional network discrete differential manifold, the appropriate family $\mathcal{K}(M)$ is the collection of all $\preceq$-fully ordered subsets of $\mathcal{M}$.
To conclude the section, note that the properties of the family $\mathcal{K}$ listed above coincide with the definition of ordered simplicial complex. We shall return to this issue in section 5.

4 Two sources of finite topological spaces

The first source described in this section is the procedure manufacturing finite $T_0$-topological spaces (called generated topological spaces [2]) from discrete differential manifolds. The second source is the coarse graining procedure applicable to arbitrary topological spaces which also yields the finite $T_0$-spaces [8, 10]. Begin with the first source.

Generated topological spaces. Let $M = (\mathcal{M}, \Omega)$ be a discrete differential manifold and $\mathcal{K} = \mathcal{K}(\mathcal{M})$ be the collection of its nonvanishing basic forms (12). Define the topology $\tau$ on the set $\mathcal{K}$ by setting its prebase of open sets $\{U_\alpha\}, \alpha \in \mathcal{K}$:
$$U_\alpha = \{\beta \in \mathcal{K} \mid \alpha \subseteq \beta\}$$

Definition 6 The topological space $(\mathcal{K}, \tau)$ defined above is called the generated topological space $(\mathcal{K}, \tau)$ of the discrete differential manifold $M$.

It was shown in [3] that the topology of any generated topological space is always $T_0$ (that is, for each pair of distinct points of $\mathcal{K}$ there is an open set containing one point but not the other).

Example. Let $\mathcal{M} = \{1, 2, 3\}$. Define the relation $\preceq$ on $\mathcal{M}$ as $1 \prec 2 \prec 3 \prec 1$ (not being transitive), the graph of the relation is on Fig. 1a. Let $M$ be the network manifold induced by the relation $\preceq$ on $\mathcal{M}$, then
$$\Omega = \text{span}\{e_{12}, e_{23}, e_{31}\}$$
and hence $\dim M = 1$. The generated space $\mathcal{K}$ is:
$$\mathcal{K} = \{1, 2, 3, 12, 23, 31\}$$
and the topology $\tau$ on $\mathcal{K}$ is depicted on Fig. 1b in terms of the appropriate Hasse graph (see [2, 3, 10] for details).

\footnote{In [3] the set $\mathcal{K}$ is denoted by $\hat{\mathcal{M}}$.}
Finitary substitutes. Let $V$ be a topological space and let $T = \{V_{\alpha}\}$ be its finite open covering: $V = \bigcup V_{\alpha}$. Define the new topology $\tau$ on $V$ as that generated by the collection $T$ considered prebase of open sets. The set $(V, \tau)$ is in general not even $T_0$ space therefore the theorem of the uniqueness of limits of sequences may not hold. Define the relation denoted $\rightarrow$ on the set $V$ as follows:

$$x \rightarrow y \iff y = \lim_{\tau} \{x, x, \ldots, x, \ldots\}$$

where $\lim_{\tau}$ denotes the limit with respect to the topology $\tau$ or, in a more transparent form

$$x \rightarrow y \iff (\forall \alpha \ y \in V_{\alpha} \Rightarrow x \in V_{\alpha})$$

In general, the relation $\rightarrow$ is a preorder on the set $V$ hence we can consider its quotient $K = V/\sim$ with respect to the equivalence $\sim$:

$$x \sim y \iff x \rightarrow y, y \rightarrow x \quad (19)$$

As a result, the set $K$ is partially ordered by the relation $\rightarrow$ and the topology $\tau$ induced on $K$ as the quotient set is already the $T_0$-topology. The detailed account of this procedure can be found in [8, 9].

Example. Let $V$ be a circle, $V = e^{i\phi}$. Consider the covering $T = \{V_{\alpha}, V_{\beta}, V_{\gamma}\}$:

$$V_{\alpha} = \{e^{i\phi} \mid -\pi/2 < \phi < +\pi\}$$
$$V_{\beta} = \{e^{i\phi} \mid \pi/2 < \phi < 3\pi/4\}$$
$$V_{\gamma} = \{e^{i\phi} \mid -\pi < \phi < +\pi/4\}$$

Then the equivalence classes (19) (that is, the elements of $K$ are:

$$1 = \{e^{i\phi} \mid -\pi/2 < \phi < +\pi/4\}$$
$$12 = \{e^{i\phi} \mid \pi/4 < \phi < \pi/2\}$$
$$2 = \{e^{i\phi} \mid \pi/2 < \phi < \pi\}$$
$$23 = \{e^{i\pi}\}$$
$$3 = \{e^{i\phi} \mid -\pi < \phi < -\pi/2\}$$
$$31 = \{e^{-i\pi/2}\}$$

and the induced partial order has the Hasse diagram the same as depicted on Fig. 1b.

It will be shown in section 5 how, starting from an arbitrary finite dimensional discrete differential manifold to build a continuous metrical space (namely, a polyhedron) and specify its open covering so that the resulting $T_0$ spaces would be the same. To do it, we have to introduce one more technical issue.
Simplicial coarse graining of polyhedra. Let $\mathcal{P}$ be a simplicial complex and $|\mathcal{P}|$ be its realization by a polyhedron in a Euclidean space $\mathcal{E}$. That means that $|\mathcal{P}|$ is the union of well-positioned geometrical simplices $\alpha$ in $\mathcal{E}$. Initially, $|\mathcal{P}|$ is the metrical space being the subset of the space $\mathcal{E}$ with the standard Euclidean metric and having the topology associated with this metric. For every point $x \in |\mathcal{P}|$ of the polyhedron consider its star:

$$\text{St}(x) = \cup\{\alpha \mid x \in \alpha\}$$

and then define the neighborhood of the point $x$ as the interior of $\text{St}(x)$ in $|\mathcal{P}|$:

$$V_x = \text{int}_{|\mathcal{P}|}\text{St}(x)$$

(note that for some $x \neq y$ the neighborhoods may coincide). Evidently $\mathcal{T} = \{V_x \mid x \in |\mathcal{P}|\}$ is the open covering of $|\mathcal{P}|$. Moreover, infinite as the set $|\mathcal{P}|$ is, the covering $\mathcal{T}$ is finite. The elements of $\mathcal{T}$ are in 1-1 correspondence with the simplices of $\mathcal{P}$.

$$\forall \alpha \in \mathcal{P} \quad x \in \alpha \leftrightarrow y \in \alpha \quad \text{iff} \quad V_x = v_y$$

Now consider the topology $\tau$ on $|\mathcal{P}|$ induced by the covering $\mathcal{T}$ thought of as open prebase, and the appropriate $T_0$-quotient. For every $V_{\alpha}, V_{\beta} \in \mathcal{T}$ the intersection $V_{\alpha} \cap V_{\beta} = V_{\alpha \cap \beta}$ is either $\emptyset$ or the element of $\mathcal{T}$, therefore $\mathcal{T}$ is the base (rather than prebase) of the topology $\tau$.

Lemma 5 There is the 1-1 correspondence between the simplices of $\mathcal{P}$ and the points of the quotient space:

$$\mathcal{P} = |\mathcal{P}|/\sim$$

Proof. Associate with every simplex $\alpha \in \mathcal{P}$ its 'local interior' $I(\alpha)$:

$$I(\alpha) = \{(\mu_0^\alpha, \ldots, \mu_r^\alpha) \mid \forall i = 1, \ldots, r \quad 0 < \mu_i^\alpha < 1\}$$

where $r$ is the dimension of the simplex $\alpha$ and $\mu_i^\alpha$ are its baricentric coordinates. For the vertices $v \in \mathcal{P}$ put:

$$I(v) = \{v\}$$

The collection $\{I(\alpha) \mid \alpha \in \mathcal{P}\}$ is the partition of the polyhedron $|\mathcal{P}|$. For any $\alpha, x, y \in I(\alpha)$ implies $V_x = V_y$, and vice versa $V_x = V_y$ implies $\exists \alpha x, y \in I(\alpha)$.

So, we may conclude that the resulting quotient $T_0$-space is the simplicial complex $\mathcal{P}$ itself (that is, the points of the finitary substitute are the simplices of $|\mathcal{P}|$). The open base of the topology $\tau$ on $\mathcal{P}$ is the collection of the stars of all simplices of $\mathcal{P}$.

The appropriate open covering of the polyhedron $|\mathcal{P}|$ is called simplicial since it does not depend on particular realization $|\mathcal{P}|$ of the complex $\mathcal{P}$. 

10
Example. Let the polyhedron $|P|$ be the triangle without interior whose vertices are labelled by 1,2,3. Then the appropriate simplicial complex is $P = \{1, 2, 3, 12, 13, 23\}$ (for brevity I denote $1 = \{1\}, 12 = \{1, 2\}$ and so on). The simplicial covering $\tau$ consists of 6 sets:

\[
\begin{align*}
V_{12} &= (1, 2) & V_1 &= (1, 2) \cup (1, 3) \\
V_{23} &= (2, 3) & V_2 &= (1, 2) \cup (2, 3) \\
V_{13} &= (1, 3) & V_3 &= (1, 3) \cup (2, 3)
\end{align*}
\]

where $(\cdot, \cdot)$ denotes the open interval between appropriate vertices. Then the finitary substitute induced by the simplicial covering $\mathcal{T}$ is finite topological space depicted on the Fig. 1b.

5 Polyhedral representations of discrete differential manifolds and the correspondence theorem

In this section the main result of the paper is formulated, namely the transition from discrete differential manifolds to polyhedra associated with the same $T_0$ spaces is described. Let $M = (\mathcal{M}, \Omega)$ be a finite dimensional discrete differential manifold.

Lemma 6 Let $\{i_0, \ldots, i_r\}$ be a subset of $\mathcal{M}$. Then there exists at most one basic differential form $e_\alpha$ such that

- $\alpha$ is a permutation of the elements $\{i_0, \ldots, i_r\}$.
- $e_\alpha \neq 0$

Proof. Suppose there are $e_\alpha, e_\beta$ such that $\beta$ is obtained from $\alpha$ by a nontrivial permutation $\sigma$. Let $\alpha = (i_0, \ldots, i_r)$. Since $\sigma \neq \text{id}$ there exists a pair of distinct elements $i, j \in \mathcal{M}$ such that $i$ precedes $j$ in $\alpha$ and $j$ precedes $i$ in $\beta$. Thus it follows from (14) that $i \preceq j$ and $j \preceq i$ which contradicts with the assumption $\dim M < \infty$ (lemma 3).

With any finite dimensional discrete differential manifold $M = (\mathcal{M}, \Omega)$ its generated topological space $\mathcal{K}(M)$ (definition 4) may be considered as the collection of subsets (rather than sequences) of the set $\mathcal{M}$: due to lemma 3 we can forget about the order and different ordered sets will become different sets (note that this does not work when $\dim M = \infty$ !)

Recalling the properties of $\mathcal{K}(M)$ established in the end of section 3 we see that $\mathcal{K}(M)$ is exactly the simplicial complex with the set of vertices $\mathcal{M}$.

Definition 7 A polyhedral representation of a discrete differential manifold $M$ is the polyhedron $|\mathcal{K}|$ being a geometrical realization of the simplicial complex $\mathcal{K}$.
Now the main result of the paper can be formulated as the following correspondence theorem:

**Theorem 7** Let $M = (\mathcal{M}, \Omega)$ be a finite dimensional discrete differential manifold, $\mathcal{K}(M)$ be its polyhedral realization. The following two $T_0$-spaces are isomorphic:

- The finitary substitute $|\mathcal{K}|/T$ with respect to the simplicial covering $T$ of $|\mathcal{K}|$.
- The generated topological space $\mathcal{K}$ of the discrete differential manifold $M$.

**Proof.** In fact, everything is already proved. The points of both $T_0$-spaces are in 1-1 correspondence with the elements of the complex $\mathcal{K}(M)$. The partial orders on both finite sets $\mathcal{K}$ and $|\mathcal{K}|/T$ are the same being the set inclusion of simplices. Thus $\mathcal{K}$ and $|\mathcal{K}|/T$ are homeomorphic.

**Remark.** In [3] the question which finite $T_0$-spaces are generated by a discrete differential manifold remained open. In this paper these $T_0$ spaces are characterized. Moreover, it is seen that simplicial complexes and related structures such as polyhedra or simplicial spaces are more adapted to be the topological realizations for discrete differential manifolds rather than finitary topological spaces of general form.

**Acknowledgments.** The author is grateful to A.Dimakis and the participants of the Efroimsky seminar (St.Petersburg), particularly, G.N. Parfionov, for helpful discussions and remarks.

**References**

[1] Connes A., Noncommutative differential geometry, Hermann, Paris, 1989

[2] Dimakis A., F. Müller-Hoissen, *Discrete differential calculus, graphs, topologies and gauge theory*, Journal of Mathematical Physics, 35, 6703, (1994)

[3] Dimakis A., F. Müller-Hoissen, F. Vanderseypen, *Discrete differential manifolds and dynamics on networks*, Journal of Mathematical Physics, 36, 3771, (1995) (eprint [hep-th/9408114](http://arxiv.org/abs/hep-th/9408114))

[4] Geroch, R. *Einstein Algebras*, Communications in Mathematical Physics, 26, 271, (1972)

[5] Kastler, D., *Cyclic cohomology within the differential envelope*, Hermann, Paris, 1988
[6] G.N.Parfionov, R.R.Zapatrin, *Pointless Spaces in General Relativity*, International Journal of Theoretical Physics, 34, 737, 1995 (eprint gr-qc/9503048)

[7] T. Regge, *General relativity without coordinates*, Nuovo Cimento, 19, 568, 1961

[8] Sorkin, R.D., *Finitary substitutes for continuous topology*, International Journal of Theoretical Physics, 30, 923, (1991)

[9] Zapatrin, R.R., *Pre-Regge Calculus: Topology Via Logic*, International Journal of Theoretical Physics, 32, 779, (1993)

[10] Zapatrin, R.R., *Matrix models for spacetime topodynamics*, In: Proceedings of the ICOMM’95 (Vienna, June 3-6, 1995), W. Kainz, Ed., TUV (1995), 1-19 (eprint gr-qc/9503066)
Figure 2: a). The graph of the relation $\preceq$. b). The Hasse graph of the topological space $\mathcal{K}$. A set $A \subseteq \mathcal{K}$ is open iff with every its element $a \in A$ it contains all elements lying below $a$ and linked with it.