New Geometric Framework for $SU(2)$ Gauge Theory
Zafar Ya. Turakulov

Institute of Nuclear Physics, Ulugbek, Tashkent, 702132 Uzbekistan (e-mail: zafar@suninp.tashkent.su)

Abstract

An explicit model of fiber bundle with local fibers being distinct copies of vector 3-space is introduced. They are endowed with frames which are used as local isotopic ones. The field local of isotopic frames is considered as gauge field itself while the form of gauge connections is derived from it. A covariant equation for the field of local frames is found. It is shown that Yang-Mills equation follows from it, but variety of solutions of the new equation is highly reduced in such that no ambiguities (Yang-Wu and vacuum ones) arise. It is shown that Lagrangian for the field gives non-zero trace for the stress-energy tensor and zero value for spin of the field of plane wave. Some new solutions for the fields of punctual source and spherical wave are found.
1 Introduction

Fiber bundles are known to provide adequate geometric models for classical gauge theories in which classical gauge fields are described by connections on the bundles whose sections are scalar fields with isospin (Wu and Yang, 1975; Trautman, 1979). Ordinarily, one introduces such a section as a column of three scalar fields and calls them isotopic components of the field (Actor, 1979, Ryder, 1985). Isotopic rotations are well-defined on the space of columns and this definition allows introducing the notion of gauge field as small rotations corresponding to small translations in the space-time. Though this approach is generally accepted it leads to the following difficulty.

First, since local isotopic spaces are vector ones they are to be endowed with a frame for each of them. Second, connection, which, by definition, connects the frames, can be derived in the same way as it is done in Riemannian geometry. Third, since some three fields are called isotopic components they are to be projections of an isovector onto elements of the frame. And, last, the connection must give in general non-zero curvature. In fact, these four requirements are inconsistent.

Indeed, if the frames are introduced as columns they belong to one and the same space $\mathbb{R}^3$, not to different copies of a vector space. And since they are elements of one and the same space the bundle obtained this way has by construction, identically zero curvature. Triplets of such columns forming local frames can be parallelized by some appropriate field of local isotopic rotations which gauges out the corresponding connection. The goal of this work is to construct an explicit model of fiber bundle such whose local fibers are different copies of a vector space each of which can be endowed with a frame, the connection can be obtained from these frames and gives non-zero curvature.

It is expected that if this is made the theory suffers some changes. Indeed, if a field of local isotopic frames is introduced and the connection can be derived from the local frames and, hence, it is not longer an independent variable. Since it is the field of local frames which predetermines explicitly all the rest, i.e. connection and curvature, and thus specifies the gauge field itself. Since it is not evident that some bundle of frames exists for any given 1-form accepted as connection, variety of connections derived from local frames may be highly reduced. As well-known, in Riemannian geometry connections specified arbitrarily may contain torsion while those found from local frames constitute a special class of torsion-free connections. If something similar takes place in other fiber bundles i.e. their connections contain in general two terms of different nature as connection and torsion are, there exists a problem to separate them and introduce a special field equation for each of them. Otherwise, either only one of them is to be considered in the theory or the extra one obeying no any equation will destroy univalence of field produced by a certain source.

If the field is represented by local frames and connection is derived from them there is no extra fields, but the field equation is now to be written for the frames and it obviously differs from the standard the Yang-Mills equation. Such an approach looks to fit C.N. Yang and T. T. Wu's claim that it is fiber bundle which represents the field itself (Wu and Yang, 1975; see also Socolowski, 1991). However the generally accepted point of
view reads that one needs fiber bundles ‘when studying topologically non-trivial objects’ (Trautman, 1979). In fact, one deals with fiber bundle whenever there is a gauge field regardless of its topological properties.

If, like all field equations of this sort, the equation for local isotropic frames is of the second order, it contains the curvature components not their derivatives. Thus, it is expected to be somewhat similar to the Einstein equation. As for the field properties like mass and spin, they are to be studied specially.

2 Natural models of fiber bundles

As was pointed out above it is not sufficient to write down a column of three numbers and call them isotopic vector in given spacetime point. To introduce genuine local frames for a fiber bundle on needs collection of spaces forming the bundle in which the frames can be introduced. Thus, the first problem is to find an explicit representation of a bundle in which all the fibers are really different copies of a space. It seems that the only way to obtain such a construction is to find among known ones.

Examples of bundles with fibers given as different copies of a vector space are well-known. One of them is co-tangent bundle \( \Lambda^1 M \) over a Riemannian manifold \( M \) whose sections are ordinary fields of 1-forms on \( M \) (Warner, 1983). If \( \{x^i\} \) is a coordinate system on \( M \) each fiber is endowed with a frame \( \{dx^i\} \) which can be normalized by appropriate local linear transformations. Geometry of the bundle is predetermined by that of its base \( M \) and its frames are consistent with the only non-trivial connection found from them by solving the first structure equation as a system of algebraic equations for the connection components. So are the bundles \( \Lambda^p M \) of \( p \)-forms on the manifold. All of them have one and the same structure group which is Lorentz group (here and below \( M \) is \((3+1)\)-dimensional).

In order to introduce bundles with other structure groups one can consider so-called subbundles of \( \Lambda^p M \), i.e. bundles whose typical fibers are subspaces of local fibers \( \Lambda^p_x, x \in M \). For example a certain choice of 3-dimensional spacelike hyperplanes in local co-tangent spaces \( \Lambda^1_x \) specifies a new fiber bundle with \( SO(3) \) as the structure group. Another model with this group occurs when choosing 3-dimensional subspaces in local spaces \( \Lambda^2_x \). An interesting example of a bundle with \( SU(3) \) as the structure group can, in principle, be composed by taking complex combinations of 2-forms in \( \Lambda^2_x \) and their duals and introducing Hermitean metric for the spaces of such combinations requiring the Lorentz transformations to enter as a \( SU(2) \) subgroup of the structure group. Perhaps, no other structure groups but \( U(1) \) can be obtained this way. In this work we restrict ourselves with considering one case with local fibers being 3-dimensional subspaces in local spaces \( \Lambda^2_x \) and \( SO(3) \) as the structure group.

Since local fibers constructed this way are subspaces of \( \Lambda^2_x \)’s the corresponding structure group is a subgroup of the Lorentz group. However, we seek to employ the subbundle to describe gauge fields and, hence, rotations of local fibers as local gauge transformation that are to be unaffected by any coordinate transformations in the spacetime. It is possible
because it suffices to have only the spaces over spacetime points as local fibers. It will be shown below how they may be used as models of local isotopic spaces.

3 Local frames, gauge transformations and connection

Consider the fiber bundle $\Lambda^2 M$ of 2-forms over the spacetime $M$ and select a 3-dimensional subspace $I_x$ in each local fiber $\Lambda^2_x$. As was pointed out above we use only the collection of spaces $I_x$ that constitute a vector bundle $IM$. As subspaces of $\Lambda^2_x$ they have a certain metric inherited from these spaces. However, it is unnecessary to accept all their properties like this metric. They themselves form the structure of fiber bundle as a whole, and metric on each of them can be introduced arbitrarily.

This may be done as follows. Let $\{\pi^a\}_x \in I_x$ be a triplet of 2-forms. A new Euclidean metric for spaces $I_x$ can be introduced simply by postulating that $\pi^a$'s constitute orthonormalized frames in the spaces. Thus, in general, $\pi^a$'s are neither orthogonal nor normalized with respect to the natural metric of $\Lambda^2_x$ but they are orthonormalized due to the new metric introduced thereby. Hereafter the spaces $I_x$ and frames $\pi^a$'s are considered as models of local isotopic spaces and as local isotopic frames respectively. The isotopic vectors $\pi^a$ are invariant under Lorentz transformations (though their coordinate components obey the usual transformation law). In other words, coordinate transformations in $M$ change their Lorentzian components, but the vectors themselves remain the same. Since isotopic vectors are referred to the local frames $\{\pi^a\}_x$ and, hence, they have no coordinate components, their components are unaffected by any coordinate transformations in the spacetime. Thus, we have composed a model of fiber bundle whose fibers are different copies of a vector space by construction.

Now, local gauge transformations can be introduced as linear transformations conserving orthogonality and normalization of the isotopic frames $\{\pi^a\}_x$. They are defined as follows. Let $S_x$ be a linear operator that changes the metric of space $I_x$ into the natural metric induced on it from $\Lambda^2 M$ and $R$ be a usual rotation in the space $\Lambda^2_x$. Then, linear operators of the form $S_x^{-1}R_xS_x$ local isotopic frames $\{\pi^a\}_x$ orthonormalized. At the same time they constitute the group $SO(3)$ which, therefore, can be considered as a model of the group of gauge transformations.

Infinitesimal parameters of a small gauge transformation $dg^a$ referred to the chosen isotropic frame can now be introduced by the standard procedure: if $\{\pi^a\}_x$ and $\{\pi'^a\}_x$ are two frames connected by a small gauge transformation then

$$\pi'^a_x = \pi^a_x + \varepsilon^a_{\ bc}dg^b\pi^c.$$ 

This definition allows introducing a connection that really connects isotopic frames established in two neighbouring spacetime points via the first structure equation

$$d\pi^a + \varepsilon^a_{\ bc}\alpha^b\pi^c = 0 \quad (1)$$
where the connection appeared as 1-form $\alpha^b$. Hereafter this equation is used as the definition of the connection. The definition of the connection form just proposed is univalent. Indeed, as was shown by R. Roskies (Roskies, 1977), M. Calvo (Calvo, 1977) and M. Halpern (Halpern, 1977) the Yang-Wu ambiguity of the vector potential does not arise if the algebraic system for components of $\alpha$ is well-defined, i.e. if it consists of 12 equations. As the gauge connection defined all the rest can be built in the ordinary manner (see Actor, 1979 or Ryder, 1985 for details).

4 The field equations and Lagrange formalism

Now compose the field equation for the field of local frames $\pi^a$ assuming it to be covariant and form-invariant with respect to gauge transformations as we introduced them i.e. ordinary orthogonal transformations of the frame. Therefore, the equation is to be composed of the frame itself and the curvature 2-form $K^a$ which is

$$K^a = d\alpha^a + \frac{1}{2} \varepsilon^{abc} \alpha^b \wedge \alpha^c \quad (2)$$

The only manifestly covariant equation of the second order on $\pi^a$’s composed of these two elements is

$$K^a - m^2 \pi^a = 0 \quad (3)$$

where $m$ is a constant. Apparently, such a field equation is consistent with Bianchi identities because $\pi^a$’s satisfy the first structure equation (1). Denoting, as usual, the operator acting on $\pi^a$ in the equation $D\pi^a$ and calling it covariant exterior derivative we obtain the standard Yang-Mills equation in the form

$$D^* \pi^a = I^a \quad (4)$$

with $I^a$ being the 3-form of conserving current

$${}^*I^a \equiv J^a_i dx^i, \quad DI = 0.$$  

It should be noted that normalization of the current as it is defined here differs from the standard one by the constant factor $m^2$. A similar approach to Yang-Mills equations has been proposed by L. Castillejo and M. Kugler in their work (Castillejo and Kugler, 1980).

Now we pass temporarily to tensorial denotions to describe Lagrangian formalism for the equations just proposed. All the equations can be derived in standard manner from the following Lagrangian density

$$L = (\partial_i A_j + \frac{1}{2} \varepsilon^{abc} A^b_i A^c_j) \pi^i_j - \frac{m^2}{2} \pi^a_i \pi^a_j + J^a_i A^a_i \quad (5)$$

where $\alpha^a \equiv A^a_i dx^i$, $\pi^a \equiv \pi^a_{ij} dx^i \wedge dx^j$ and $J^a_i$ stands for currents of an external source. Indeed, since the Lagrangian does not include derivatives of $\pi^a$’s one of the Euler-Lagrange equations reads simply

$$\frac{\partial L}{\partial \pi^a_{ij}} = 0$$
that is exactly the equations (2) and (3). Now the equation (1) follows from Bianchi identities for $A_i^a$’s. To derive another Euler-Lagrange equation we evaluate the following derivatives:

$$\frac{\partial L}{\partial A_i^a} = \frac{1}{2} \varepsilon^a_{bc} \left( \delta^b_d \delta_i^k A_j^c + \delta^c_d \delta_i^k A_j^b \right) \pi_i^{kj} + J_d^k,$$

where $\nabla_i$ denotes the usual Riemannian covariant derivative. It is seen that the Euler-Lagrange equation in question coincides with the Yang-Mills equation (4).

As the explicit form of the Lagrangian is known it is possible to determine the main characteristics of the field i.e. its mass, spin and isospin. Spin and isospin of the field will be evaluated for a plane wave in one of the following sections together with the corresponding solution and the field mass can be found regardless of the form of solution directly from that of Lagrangian. In the manifestly covariant form the field stress-energy tensor is

$$T_{mn} = \frac{\partial L}{\partial K_{mi}^a} K_{ni}^a - \delta^m_n L = \pi_{ni}^a K_{mi}^a - \delta^m_n L,$$

whereas, as follows from the equations (2) and (3) the source-free Lagrangian is equal to $\frac{1}{2} m^2 \pi_{ij}^a \pi_{ij}^a$. Therefore, the trace of the stress-energy tensor is non-zero. It is equal to $-m^2 \pi_{ij}^a \pi_{ij}^a$ that means that the field is massive. On the other hand, the factor $m$ cannot be put zero because this would erase mathematical structure of the theory, particularly, the links between the field of local isotopic frames and the Yang-Mills equation (4) as it appears in the above considerations.

Apart from univalence of links between the source and vector potential the mathematical framework for theory of $SU(2)$ gauge field proposed has another important feature. As seen from the equations (1) and (3) the Yang-Mills equation follows from them, thus, any solution of (3) satisfies the Yang-Mills equation. In fact, introducing the field of local frames is nothing but order lowering operation for the Yang-Mills equation. In the following sections it will be demonstrated that, unlike original Yang-Mills equation, the system (1-3) is soluble simply by the method of variables separation.

5 The field of a pointlike non-Abelian charge

The notion of spherical symmetry has been generalized to the case of Yang-Mills field by considering the $SU(2)$ group of combined transformations with $\mathbf{L} + \mathbf{T}$ as their generators where $\mathbf{L}$ is generator of spatial and $\mathbf{T}$ is that of local rotations of isotopic space (Wu and Wu, 1974; Wilkinson and Goldhaber, 1977). This symmetry has been used for obtaining an exact solution for a source-free field first by T. T. Wu and C. N. Yang (Wu and Yang, 1975). Spherically-symmetric solution they found has only spatial components and describes a pointlike magnetic monopole. It was extended by J. P. Hsu and E. Mac to the case of a field with non-trivial time-like components which are. Each of three time
components of the vector potential is proportional to one of Cartesian coordinates in the
space, thus, none of them is spherically symmetric (Hsu and Mac, 1977).

The approach proposed above gives other solutions. To obtain them we introduce the
field of local frames \( \{ \pi^a \} \) in spherical coordinates \( \{ t, r, \theta, \varphi \} \) in the form

\[
\pi^1 = -P(r) dt \wedge dr + Q(r) r^2 \sin \theta d\theta \wedge d\varphi
\]

\[
\pi^2 = p(r) dt \wedge d\theta + q(r) \sin \theta d\varphi \wedge dr
\]

\[
\pi^3 = p(r) \sin \theta dt \wedge d\varphi + q(r) dr \wedge d\theta.
\]

and it is seen that the conjugation acts as transmutation

\[
P \rightarrow -Q, \quad Q \rightarrow P, \quad p \rightarrow -q, \quad q \rightarrow p.
\]

To solve the first structure equation \((\ref{structure_equation})\) we evaluate exterior derivatives of \( \pi^a \):s:

\[
d\pi^1 = (r^2 Q)' \sin \theta dr \wedge d\theta \wedge d\varphi
\]

\[
d\pi^2 = -p' dt \wedge dr \wedge d\theta - q \cos \theta dr \wedge d\theta \wedge d\varphi
\]

\[
d\pi^3 = p' \sin \theta dt \wedge d\varphi \wedge dr - p \cos \theta dt \wedge d\theta \wedge d\varphi.
\]

Inserting this into the first structure equation \((\ref{structure_equation})\) and source-free Yang-Mills equation \((\ref{m_f})\) that can be rewritten as the latter in which the substitutions \((\ref{substitutions})\) are made, with the connection 1-form \( \alpha^a \):

\[
\alpha^1 = \Phi(r) dt - \cos \theta d\varphi; \quad \alpha^2 = A(r) \sin \theta d\varphi; \quad \alpha^3 = A(r) d\theta.
\]

gives the following:

\[
(r^2 Q)' = 2Aq, \quad p' = AP + \Phi q, \quad (r^2 P)' = 2Ap, \quad q' = AQ - \Phi p.
\]

The curvature 2-form corresponding to the connection \((\ref{connection})\) is

\[
K^1 = -\Phi' r^2 dt \wedge dr - r^{-2}(1 - A^2) \sin \theta d\theta \wedge d\varphi
\]

\[
K^2 = A\Phi dt \wedge d\theta - A' \sin \theta d\varphi \wedge dr;
\]

\[
K^3 = A\Phi \sin \theta dt \wedge d\varphi - A' dr \wedge d\theta.
\]

Apparently, if the 2-forms \( \pi^a \) constitute an orthonormal frame in the spacetime the curvature is zero. As seen from the form of \( \pi^a \) \((\ref{connection})\) such a frame corresponds to \( P = Q = 1, \quad p = q = r \) and then \( \Phi = 0 \quad A = 1 \). Therefore, if the field is asymptotically trivial, i.e. has no strength, we are to put \( A = 1 \) at infinity. Inserting this into the field equation \((\ref{field_equation})\) gives another quartet of equations:

\[
\Phi' = m^2 P, \quad m^2 Q = -r^{-2}(1 - A^2), \quad m^2 p = A\Phi, \quad m^2 q = A'.
\]
Now it is seen that the first two equations are not independent because they can be derived from the rest ones:

\[
(r^2Q)' - 2Aq = m^{-2}[(1 - A^2)' - AA'] \equiv 0
\]
\[
p' - AP - \Phi Q = m^{-2}[A\Phi' - A\Phi' - A'\Phi] \equiv 0.
\]

These two equations are satisfied automatically because they are nothing but Bianchi identities. Thus, there are only six equations which would be independent. However, the last quartet of equations specifies only the form of connection in terms of the coefficients in the definition of the field of local frame (7). Thus, it remains to solve the well-known system of ordinary differential equations (Hsu and Mac, 1977):

\[
A'' - r^{-2}A(1 - A^2) - A\Phi^2 = 0,
\]
\[
(r^2\Phi')' - 2A^2\Phi = 0
\]

which could also be derived from the Yang-Mills equation for the connection (9). In our case the connection is found from the first structure equation for localframes and the Yang-Mills form \(D^*K\) is annulled by the independent part of the system of equations (10). Due to univalence of our definition of connection the expressions obtained represent the unique form of field produced by pointlike non-Abelian charge. The equations are not solvable in analytical form however, asymptotical behaviour of the field in question at large \(r\) may be found easily. Indeed, since at infinity the function \(A\) is equal to unit the field behaves asymptotically as \(\Phi(r) = C_1r^{-2}, \ A(r) = 1 + C_2r^{-1}\). This solution was obtained in our work (Turakulov 1, 1995).

### 6 The field of plane wave

For the next example we consider the field equations in the standard coordinate system of circular cylinder \(\{t, z, \rho, \varphi\}\). It is convenient to introduce first a parameter of Lorentz transformations that allows transferring the solution to be found into an arbitrary frame of reference. Let \(S\) and \(T\) be new coordinates introduced as follows:

\[
S = z \cosh \psi + t \sinh \psi; \ T = t \cosh \psi - z \sinh \psi.
\]

Here the parameter \(\psi\) labels inertial reference frames moving along the \(z\)-axis.

Consider the following triplet of 2-forms:

\[
\pi^1 = P(T)dt \wedge dZ - Q(T)\rho d\rho \wedge d\varphi
\]
\[
\pi^2 = p(T)dZ \wedge d\rho + q(T)\rho dt \wedge d\varphi
\]
\[
\pi^3 = p(T)\rho d\varphi \wedge dZ + q(T)dt \wedge d\rho.
\]
Similarly to the previous case the Hodge conjugation acts as the substitution (8). Inserting this into the first structure equation (1) and source-free Yang-Mills equation (4) with the connection 1-form \( \alpha^a \):

\[
\alpha^1 = f(T) dZ - d\varphi; \quad \alpha^2 = g(T) \rho d\varphi; \quad \alpha^3 = g(T) d\rho.
\] (14)
gives the following:

\[-Q' + 2gq = 0, \quad p' = Pg + FQ, \quad -P' + 2gp = 0, \quad -q' - gQ + pf = 0.\] (15)

The curvature 2-form corresponding to the connection (14) is

\[
K^1 = f' dT \wedge dZ - g^2 \rho d\rho \wedge d\varphi, \quad K^2 = g' \rho dT \wedge d\varphi - f g dZ \wedge d\rho, \quad K^3 = g' dT \wedge d\rho - f g \rho d\varphi \wedge dZ.
\]

Inserting this and the expressions (13) into the equation (3) gives:

\[
f' = m^2 P, \quad g^2 = m^2 Q, \quad g' = m^2 q, \quad -fg = m^2 p.
\]

Now we introduce the new function \( h = f/\sqrt{2} \) and henceforth deal only with the functions \( g \) and \( h \) which are

\[
g = m\sqrt{Q}, \quad h = mp/2\sqrt{Q}.
\] (16)

The complete sextet of independent equations including (13) in this case is

\[
h'' + 2g^2 h = 0, \quad g'' + g^3 + 2gh^2 = 0, \quad h' = m^2 P \sqrt{2},
\]

\[g^2 = m^2 Q, \quad g' = m^2 q, \quad gh = m^2 p \sqrt{2}.
\]

Apparently, the first two of them are to solved separately and all the rest define the explicit form of the connection (14). The independent equations

\[
h'' + 2g^2 h = 0, \quad g'' + g^3 + 2gh^2 = 0
\] (17)
draw a classical mechanical problem with Hamiltonian

\[
H = \frac{1}{2} (g'^2 + h'^2) + \frac{g^4}{4} + \frac{g^2h^2}{2}.
\] (18)

In the case \( h = 0 \) the system is well studied and solutions in Jacobi elliptic functions with \( g = m \cdot \text{sd}(mT \mid 1/2) \) are found long ago (Actor, 1979). Note that in the case \( h = 0 \) there exists the only solution with mass \( m \), i.e., transformations (12) change the argument of the elliptic function \( mT \) into \( \omega t - k z \) where \( \omega = m \cosh \psi \) and \( k = m \sinh \psi \), thus, \( \omega^2 - k^2 = m^2 \). The field equations have been reduced to the system (17) and the Hamiltonian (18) in our work (Turakulov 3)
7 The field of spherical wave

The coordinate system used below is introduced in the framework of the standard spherical coordinates \( \{ t, r, \theta, \varphi \} \) by the following substitutions \( \{ \zeta, \eta, \theta, \varphi \} \):

\[
\begin{align*}
t &= \zeta \cdot \cosh \eta; \\
r &= \zeta \cdot \sinh \eta; \\
\zeta &= \sqrt{t^2-r^2}; \\
\eta &= \text{arctanh}(r/t).
\end{align*}
\]

It is seen that \( \zeta \) is equal to interval between a given point and the cone point \( t = r = 0 \), and the surface \( \zeta = 0 \) is the light cone. The coordinate \( \eta \) labels 3-dimensional cones with timelike generatrixes orthogonal to the surfaces \( \zeta = \text{const} \). The extremal values of \( \eta \), namely, \( \eta = 0 \) and \( \eta = \infty \) correspond to timelike straight line \( r = 0 \), which, meanwhile, could be chosen arbitrarily, and to the light cone itself respectively.

The metric of this coordinate system is

\[
g = d\zeta \otimes d\zeta - \zeta^2[d\eta \otimes d\eta + \sinh^2 \eta (d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi)]
\]

and as it is orthogonal the unit 4-form is \( [1 - (\eta^2/4)]^{-3/2} \zeta^3 \eta^2 \sin \theta d\zeta \wedge d\eta \wedge d\theta \wedge d\varphi \). Therefore, non-zero components of Levi-Civita symbol are \( \varepsilon_{\zeta \eta \theta \varphi} = \zeta^3 \sinh^2 \eta \sin \theta \) Levi-Civita symbol components will be used when completing the \( * \)-conjugating (Warner, 1983) of the 2-form of Yang-Mills field strength.

Consider the following triplet of 2-forms:

\[
\begin{align*}
\pi^1 &= p(\zeta) \zeta d\zeta \wedge d\eta - q(\zeta) \zeta^2 \sinh^2 \eta \sin \theta d\theta \wedge d\varphi \\
\pi^2 &= p(\zeta) \zeta \sinh \eta d\eta \wedge d\theta + q(\zeta) \zeta^2 \sinh \eta \sin \theta d\varphi \wedge d\eta \\
\pi^3 &= p(\zeta) \zeta \sinh \eta \theta d\zeta \wedge d\phi + q(\zeta) \zeta^2 \sinh \eta d\eta \wedge d\theta.
\end{align*}
\]

Here \( P = p \) and \( Q = q \) due to more wide symmetry group than that encountered in previous cases. In this case the field is not only spherically symmetric but also Lorentz-invariant. As usual, the Hodge conjugation acts as the substitution \( p \rightarrow q, \quad q \rightarrow q \). Now solving the first structure equation (1) in combination with the source-free Yang-Mills equation (4) with the connection 1-form \( \alpha^a \):

\[
\begin{align*}
\alpha^1 &= A(\zeta) d\eta + \cos \theta d\varphi; \\
\alpha^2 &= A(\zeta) \sinh \eta d\theta - \cosh \eta \sin \theta d\varphi; \\
\alpha^3 &= \cosh \eta d\theta + A(\zeta) \sinh \eta \sin \theta d\varphi,
\end{align*}
\]

and the corresponding curvature form which is

\[
\begin{align*}
K^1 &= A' \zeta d\zeta \wedge d\eta + (1 + A^2) \sinh^2 \eta \sin \theta d\theta \wedge d\varphi \\
K^2 &= A' \sinh \eta d\zeta \wedge d\theta + (1 + A^2) \sinh \eta \sin \theta d\varphi \wedge d\eta \\
K^3 &= A' \sinh \eta \sin \theta d\zeta \wedge d\varphi + (1 + A^2) \sinh \eta d\eta \wedge d\theta
\end{align*}
\]
gives the following (we put $m = 1$):

$$A' = \rho \zeta, \quad (1 + A^2) = q \zeta^2, \quad (\zeta^2 q)' = 2 \rho \zeta, \quad (\zeta^2 \rho)' = -2 A q \zeta.$$ 

Consequently, the function $A$ satisfies the equation

$$\zeta (\zeta A')' + 2 A (1 + A^2) = 0.$$ 

As seen from this equation the solutions are form invariant under Lorentz transformations as well as the equation itself. The substitutions $s = \ln \zeta, \ A(\zeta) = y(s)$ reduce this equation to the form of equations of motion of a point mass in classical mechanics, i.e., the well-known Newton equation:

$$y'' + 2 y(1 + y^2) = 0.$$ 

Now, applying the standard methods we find that the function $y$ satisfies the following equation:

$$y' = \sqrt{c_1^2 - (1 + y^2)^2}$$

whose solution is one of Jacobi elliptical functions (Abramowicz and Stegun, 1964):

$$y = s d \left( (s + c_2) \sqrt{\frac{2 c_1}{c_1^2 - 1}} \frac{c_1 - 1}{2 c_1} \right)$$

This result was obtained in our work (Turakulov 3, 1995).

## 8 Spin of plane wave

As a rule, when dealing with linear classical fields one determines spin of the field as the number of vectorial indices the field has accounting spinorial index as half of vectorial one. They are equal indeed if the field is linear. In fact, spin of a classical field is defined as integral of spin density over the space. In the case of plane waves the density is proportional to number of indicies. The difference between linear and non-linear fields arising in the case under consideration can be seen in the following example. The usual solutions of Maxwell equations that describe linearly polarized plane waves correspond zero spin density because they represent a mixed state with +1 and −1 helicities. To have the proper value with spin 1 To have the proper value with spin 1 one composes a complex linear combination of two plane waves which has circular polarization, and considers it as a pure state with certain helicity. But this is possible only for linear fields. If the field is non-linear no such linear combinations satisfy the original equation.

In this section we evaluate the spin density of plane wave. Applying the Nöther thorem gives the spin density in the form

$$S_{ij} = \partial L \delta_{k l} A_{i l}^{a j} - \delta_{k l} A_{i l}^{a j} \pi \delta_{k l} A_{i l}^{a j}$$
where we used the denotation \( \delta_{klij} \) for \((1/2)(\eta_{ki}\delta_{l}^{j} - \eta_{kj}\delta_{l}^{i})\). Thus,

\[
S_{ij}^t = (1/2)[(\pi_a)^{t}\cdot j A_t^a - (\pi_a)^{l}\cdot j A_l^a]
\]

Now, a straightforward evaluation for the plane wave solution (13) and (14) shows that the only non-zero component of the spin density is \( S_{x\phi}^t = -f' \). Apparently, this density gives zero total spin because locally it is everywhere pointed out from the axis. Consequently, plane waves of the field are spinless.

9 Acknowledgments

The author thanks Prof. H.-D. Doebner for hospitality in XXI International Colloquium on Group Theoretic Methods in Physics and Prof. B. F. Schutz for hospitality in Albert Einstein Institute, Potsdam and for interesting discussions.

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