Abstract. Motivated by the recent result of [EJT10] showing that MSO properties are Logspace computable on graphs of bounded tree-width, we consider the complexity of computing the determinant of the adjacency matrix of a bounded tree-width graph and prove that it is L-complete. It is important to notice that the determinant is neither an MSO-property nor counts the number of solutions of an MSO-predicate.

We extend this technique to count the number of spanning arborescences and directed Euler tours in bounded tree-width digraphs, and further to counting the number of spanning trees and the number of Euler tours in undirected graphs, all in L. Notice that undirected Euler tours are not known to be MSO-expressible and the corresponding counting problem is in fact #P-hard for general graphs. Counting undirected Euler tours in bounded tree-width graphs was not known to be polynomial time computable till very recently Chebolu et al [CCM13] gave a polynomial time algorithm for this problem (concurrently and independent of this work).

Finally, we also show some linear algebraic extensions of the determinant algorithm to show how to compute the characteristic polynomial and trace of the powers of a bounded tree-width graph in L.

1 Introduction

The determinant is a fundamental algebraic invariant of a matrix. For a $n \times n$ matrix $A$ the determinant is given by the expression

$$\text{Det}(A) = \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i \in [n]} a_{i, \sigma(i)}$$

where $S_n$ is the symmetric group on $n$ elements, $\sigma$ is a permutation from $S_n$ and sign($\sigma$) is the number of inversions in $\sigma$. Even though the summation in the definition runs over $n!$ many terms, there are many efficient sequential [vzGG13] and parallel [Ber84] algorithms for computing the determinant.

Apart from the inherently algebraic methods to compute the determinant there is also a combinatorial algorithm (see, for instance, Mahajan and Vinay [MV97]) which interprets the determinant as a signed sum of cycle covers in the weighted adjacency matrix of a graph. [MV97] are thus able to give another proof of the GapL-completeness of the determinant, a result first proved by Toda [Tod91]. For a more complete discussion on the known algorithms for the determinant, see [MV97].

Armed with this combinatorial interpretation (and the fact that the determinant is GapL-complete), one can ask if the determinant is any easier if the underlying matrix represents simpler classes of graphs. Datta, Kulkarni, Limaye, Mahajan [DKLM10] study the complexity of the determinant and permanent respectively, when the underlying graph is planar and...
show that they are as hard as the general case, i.e., respectively \textbf{GapL} and \#P-hard. We revisit these questions in the context of bounded tree-width graphs.

Many \textbf{NP}-complete graph problems become tractable when restricted to graphs of bounded treewidth. In an influential paper, Courcelle [Cou90] proved that any property of graphs expressible in Monadic Second Order logic can be decided in linear time on bounded treewidth graphs. For example, Hamiltonicity is an \textbf{MSO} property and hence deciding if a bounded treewidth graph has a Hamiltonian cycle can be done in linear time. More recently Elberfeld, Jakoby, Tantau [EJT10] showed that in fact, \textbf{MSO} properties on bounded treewidth graphs can be decided in \textbf{L}.

We study the Determinant problem when the underlying directed graph has bounded treewidth and show that the determinant computation for this case is in logspace. As a corollary we also compute other linear algebraic invariants of matrix, viz. the characteristic polynomial, rank and trace of powers of the matrix in logspace. We also give a tight bound on the complexity of the problem by showing that it is \textbf{L}-hard via a reduction from directed reachability is paths.

1.1 Spanning Trees

The parallel complexity of the problem for planar graphs was investigated by Braverman, Kulkarni and Roy [BKR09], who give tight bounds on the complexity of the problem, both in general and in the modular setting. They show that the problem is \textbf{L}-complete when the modulus is $2^k$, for constant $k$ and for any other modulus and in the non-modular case, the problem is shown to be as hard in the planar case as for the case of arbitrary graphs. We consider the bounded treewidth case and show that the problem is in \textbf{L} for exactly counting the number of spanning trees in contrast to [BKR09].

1.2 Euler Tours

An Euler tour of a graph is a walk on the graph that traverses every edge in the graph exactly once. Given a graph, deciding if there is an Euler tour of the graph is quite simple. Indeed, the famous Konisberg bridges problem that founded graph theory is just a question of existence of Euler tours on these bridges. Euler settled in the negative and in the process gave a necessary and sufficient condition for a graph to be \textit{Eulerian} (A graph is Eulerian if and only if all the vertices are of even degree). This gives a simple algorithm to check if a graph is Eulerian.

An equally natural question is to ask for the number of distinct Euler tours in a graph. For the case of directed graphs, the \textbf{BEST} theorem due to De Bruijn, Ehrenfest, Smith and Tutte gives an exact formula that gives the number of Euler tours in a directed graph (see Fact 4) which yields a polynomial time algorithm via a determinant computation. For undirected graphs, no such closed form expression is known and the computational problem is \#P-complete [BW05]. In fact, the problem is \#P-complete even when restricted to 4-regular planar graphs [GS12]. So exactly computing the number of Euler tours is not in polynomial time unless \#P = \textbf{P}. 

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This is an instance of an interesting phenomenon in the complexity of counting problems: the decision version of the problem is tractable and the counting version is intractable. The flagship example of this phenomenon is the Perfect Matching problem for which there is a polynomial time algorithm \cite{Edm65}, whereas the counting version is \#P-complete via a reduction from the permanent. Faced with this adversity, traditionally there are two directions that have been pursued in the community:

1. Can one find a good approximate solution in polynomial time? This problem is wide open for counting Euler Tours.
2. Find restricted classes of graphs for which one can count exactly the number of Euler tours in polynomial time.

Previously Chebolu, Cryan, Martin have given a polynomial time algorithm for counting Euler tours in undirected series-parallel graphs\cite{CCMI10}.

### 1.3 Our Techniques and Results

We show that the following can be computed in \(L\):

1. The Determinant of an \((n \times n)\) matrix \(A\) whose underlying undirected graph has bounded treewidth. As a corollary we can also compute the coefficients of the characteristic polynomial of a matrix.
2. Given an \((n \times n)\) matrix \(A\) whose underlying undirected graph has bounded treewidth, compute the trace of \(A^k\).
3. Counting the number of Spanning Trees in graphs of bounded treewidth.
4. Counting the number of Euler tours in a directed graph where the underlying undirected graph is bounded treewidth.
5. Counting the number of Euler tours in an undirected bounded treewidth graph.

At the core of our result is our algorithm to compute the determinant by writing down an MSO\(_2\) formula that evaluates to true on every valid cycle cover of the bounded treewidth graph underlying \(A\). The crucial point being that the cycle covers are parameterised on a quantity closely related to the sign of the cycle covers. This makes it possible to invoke the cardinality version of Courcelle’s theorem (for logspace) due to \cite{EJT10} to compute the determinant. We use this determinant algorithm and the BEST theorem to count directed Euler tours. Using the algorithm for directed Eulerian tours, we get an algorithm to count the number of Euler tours in a undirected bounded treewidth graph. Concurrently and independent of this work, using different techniques, Chebolu, Cryan, Martin \cite{CCMI13} have given a polynomial time algorithm for counting Euler tours in undirected bounded treewidth graphs.

\footnote{For the Perfect Matching problem, a long line of work culminating in the beautiful algorithm due to Jerrum and Sinclair \cite{JS89,SJ89} gives a Fully Polynomial Randomized Approximation Scheme (FPRAS). But these techniques have not yielded an FPRAS for counting Euler tours.}
1.4 Organization of the paper

Section 2 introduces some notation and terminology required for the rest of the paper. In Section 3, we give a Logspace algorithm to compute the Determinant of matrices of bounded treewidth. In Section 4, we give some applications of the determinant algorithm to computing certain linear algebraic invariants of a matrix. In Section 5, we give some applications of the determinant algorithm to some counting problems on bounded treewidth graphs. In Section 6, we summarize our contributions and mention some interesting open questions raised by this paper.

2 Preliminaries

Definition 1. Given an undirected graph \( G = (V_G, E_G) \) a tree decomposition of \( G \) is a tree \( T = (V_T, E_T) \) (the vertices in \( V_T \subseteq 2^{V_G} \) are called bags), such that

1. Every vertex \( v \in V_G \) is present in at least one bag, i.e., \( \cup_{X \in V_T} X = V_G \).
2. If \( v \in V_G \) is present in bags \( X_i, X_j \in V_T \), then \( v \) is present in every bag \( X_k \) in the unique path between \( X_i \) and \( X_j \) in the tree \( T \).
3. For every edge \( (u, v) \in E_G \), there is a bag \( X_r \in V_T \) such that \( u, v \in X_r \).

The width of a tree decomposition is the \( \max_{X \in V_T} (|X| - 1) \). The tree width of a graph is the minimum width over all possible tree decompositions of the graph.

Definition 2. Given a weighted directed graph \( G = (V, E) \) by its adjacency matrix \([a_{ij}]_{i,j \in [n]}\), a cycle cover \( C \) of \( G \) is a set of cycles that cover \( G \). I.e., \( C = \{C_1, C_2, \ldots, C_k\} \), where \( C_i = \{c_{i_1}, \ldots, c_{i_r}\} \subseteq V \) such that \((c_{i_1}, c_{i_2}), (c_{i_2}, c_{i_3}), \ldots, (c_{i_{r-1}}, c_{i_r}), (c_{i_r}, c_{i_1}) \in E \) and \( \bigcup_{i=1}^k C_i = V \). The weight of the cycle \( C_i = \prod_{j \in [r]} wt(a_{ij}) \) and the weight of the cycle cover \( wt(C) = \prod_{i \in [k]} wt(C_i) \). The sign of the cycle cover \( C \) is \((-1)^{n+k}\).

Every permutation \( \sigma \in S_n \) can be written as a union of disjoint cycles. Hence a permutation corresponds to a cycle cover of a graph on \( n \) vertices. In this light, the determinant of an \((n \times n)\) matrix \( A \) can be seen as a signed sum of cycle covers:

\[
\text{Det}(A) = \sum_{\text{cycle cover } C} \text{sign}(C) \text{wt}(C)
\]

For the definitions of Monadic Second Order logic (MSO) and related terminology we refer the reader to Section 2 of [EJT10].

Theorem 1. [EJT10] For every \( k \geq 1 \), there is a logspace DTM that on input of any graph \( G \) of treewidth at most \( k \) outputs a width-\( k \) tree decomposition of \( G \).

Theorem 2. [EJT10] For every \( k \geq 1 \) and every MSO-formula \( \phi \), there is a logspace DTM that on input of any logical structure \( A \) of treewidth at most \( k \) decides whether \( A \models \phi \) holds.

Theorem 3. [EJT10] Let \( k \geq 1 \) and let \( \phi(X_1, \ldots, X_d) \) be an MSO – \( \tau \)-formula. Then there are an \( s \geq 1 \), an MSO – \( \tau_s \)-tree-formula \( \psi(X_1, \ldots, X_d) \) and a logspace DTM that on input of any \( \tau \)-structure \( A \) with universe \( A \) and \( \text{tw}(A) \leq k \) outputs a balanced binary \( s \)-tree structure \( T \) such that for all indices \( i \in \{0, \ldots, |A|\}^d \) we have \( \text{histogram}(A, \phi)[i] = \text{histogram}(T, \psi)[i] \).
3 Determinants in Bounded Treewidth Matrices

Given a square \( \{0, 1\} \)-matrix \( A \), we can view it as the bipartite adjacency matrix of a bipartite graph \( G(A) \). The permanent of this matrix \( A \) counts the number of perfect matchings in \( G(A) \), while the determinant counts the signed sum of perfect matchings in \( G(A) \).

If \( G \) is a bounded treewidth graph then we can count the number of perfect matchings in \( G \) in \( L \) [EJT10] (see also [DDN13]). Hence the complexity of the permanent of \( A \), above is well understood in this case while the complexity of computing the determinant is not clear.

On the other hand the determinant of a \( \{0, 1\} \)-matrix reduces (say by a reduction \( g_{MV} \) to counting the number of paths in another graph (see e.g. [MV97]). Also counting \( s,t \)-paths in a bounded treewidth graph is again in \( L \) via [EJT10] (see also [DDN13]). But the problem with this approach is that the graph \( g_{MV}(G) \) obtained by reducing a bounded treewidth \( G \) is not bounded treewidth.

However, we can also view \( A \) as the adjacency matrix of a directed graph \( H(A) \). If \( H(A) \) has bounded treewidth (which implies that \( G(A) \) also has bounded treewidth) then we have a way of computing the determinant of \( A \). To see this, consider the following lemma:

**Lemma 1.** There is an MSO\(_2\)-formula \( \theta(X) \) with a free variable \( X \) that takes values from the set of subsets of vertices such that \( \theta(S) \) is true exactly when \( S \) is the set of heads of a cycle cover of the given graph.

Note that the heads of a cycle cover are the least numbered vertices from each of the cycles of the cycle cover.

Let \( T \) be a tree decomposition of \( G \) and let \( C : V_G \to \{1, \ldots, 2k+2\} \) be a coloring of the vertices of \( G \) such that if \( x \neq y \) are two vertices of \( G \) in the same bag of \( T \) then \( C(x) \neq C(y) \). [EJT10] proves that such a coloring exists and is computable in logspace (in fact, they prove this for the slightly stronger property that if \( C(x) = C(y) \), the two vertices \( x, y \) cannot even be in neighboring bags).

We define the relation \( ION(x,y) \) which is essentially the “in-order-next” relation for the tree decomposition where ties are broken according to the coloring above. More precisely, we define \( ION(x,y) \) to be true if precisely one of the following holds:

1. \( x, y \) are in the same bag for some bag and \( C(x) < C(y) \)
2. \( x, y \) do not occur in the same bag but \( x \in B_x, y \in B_y \) where:
   (a) Bag \( B_x \) is the left child of bag \( B_y \) in the tree decomposition \( T \)
   (b) Bag \( B_y \) is the right child of bag \( B_x \) in the tree decomposition \( T \)

The following is well known and follows from Euler Traversal of a tree in logspace: (see [CMS87]).

**Proposition 1.** \( ION \) is computable in logspace.

From the definition of the coloring defined above the following proposition about the transitive closure \( ION^* \) of \( ION \) is also clear:

**Proposition 2.** \( ION^* \) is a total order on the vertices of \( G \).
Finally, the local nature of ION yields that:

**Proposition 3.** If we add ION to the structure representing the tree decomposition $T$, its tree width goes up only by a constant factor.

Since we just need to replace each bag by the union of it with its neighbors to obtain a tree decomposition such that two vertices related by ION are in the same bag.

**Proof.** (of Lemma 1) We guess a set of edges $Y$ and vertices (heads of a cycle cover) $H$ and verify that the subgraph induced by $Y$ indeed forms a cycle cover of $G$. Our MSO$_2$ formula $\theta(X)$ is of the form:

$$(\exists Y \subseteq E) \phi(X, Y)$$

where,

$\phi(X, Y) \equiv (\forall v \in V(Y))(\exists h \in H) (\text{DEG}(v)) \land (\text{PATH}(h, v) \implies \text{ION}^*(h, v))$$

Here $V(Y) = \cup_{y \in Y} y$ is the set of vertices on which edges in $Y$ are incident.

1. $\text{DEG}(v)$ is the predicate that asserts that the degree of $v$ (in the subgraph induced by the edges in $Y$) is exactly 2.
2. $\text{PATH}(x, y)$ is the predicate that asserts that there is a path from $x$ to $y$.

One can check that all the predicates above are MSO$_2$ definable. \hfill \Box

Now we are done with the help of the following fact (see e.g. [MV97]):

**Fact 1** The sign of a cycle cover consisting of $k$ cycles is $(-1)^{n+k}$ where $n$ is the number of vertices in the graph.

Thus, using the histogram version of Courcelle’s theorem from [EJT10] we get that:

**Theorem 4.** The determinant of a matrix $A$, which can be viewed as the adjacency matrix of a directed graph of bounded treewidth, is in L.

**Proof.** The histogram version of Courcelle’s theorem as described in [EJT10] when applied to the formula $\theta(X)$ above yields the number of cycle covers of $H(A)$ parametrized on $|X|$. But in the notation of the fact above, $|X| = k$ so we can easily compute the determinant as the alternating sum of these counts. \hfill \Box

Our final algorithm for computing the determinant of bounded treewidth matrices is the following: On input 0−1 matrix $A$:

1. Compute the tree decomposition $T_A$ of the underlying directed graph $G_A$.
2. Compute an ordering of the vertices of $G_A$ using the labels $(\text{DFS}(T_A), \text{B}(T_A))$.
3. Implement the cardinality version of [EJT10] via Theorem 4

**Lemma 2.** For all constant $k \geq 1$, computing the determinant of an $(n \times n)$ matrix $A$ whose underlying undirected graph has treewidth at most $k$ is L-hard.
Proof. We reduce the problem ORD of deciding for a directed path $P$ and two vertices $s, t \in V(P)$ if there is a path from $s$ to $t$ (known to be $L$-complete via [Ete97]) to computing the determinant of bounded treewidth matrices (Note that $P$ is a path and hence it has treewidth 1). Since ORD is $L$-hard, we get that the determinant of bounded treewidth matrices is $L$-hard.

Our reduction is as follows: Given a directed path $P$ with source $a$, sink $b$ and distinguished vertices $s$ and $t$, we construct a new graph $P'$ as follows: Add edges $(a, s'), (s', t), (t, s), (s, a)$ and $(b, t')$ and remove edges $(s', s), (t, t')$ where $s'$ and $t'$ are vertices in $P$ such that $(s', s), (t, t') \in E(P)$ (See Figure 1).

We claim that there is a directed path between $s$ and $t$ if and only if the determinant of the adjacency matrix of $P'$ is zero. To see this, notice that if there is a directed path from $s$ to $t$ in $P$, then there are exactly two cycle covers in $P' - (a, s')(s, t)(t', b)$ with three cycles and $(a, s', t, s), (t', b)$ with two cycles. Using Fact 1, the signed sum of these cycle covers is $(-1)^{n+3} + (-1)^{n+2} = 0$, which is exactly the determinant of $P'$.

In the case that $P$ has a directed path from $t$ to $s$, then there is exactly one cycle namely $(a, t, s', b, t', s)$. We argue as follows: The edges $(t, s), (s, b), (b, t'), (t', s')$ are in the cycle cover since they are the only incoming edges to $s, b, t', s'$ respectively. So $(t, s, b, t', s')$ is a part of any cycle cover of the graph. This forces one to pick the edge $(s', a)$ and hence we have exactly one cycle in the cycle cover for $P'$.

4 Applications to Linear Algebraic problems on Bounded Treewidth Matrices

4.1 Characteristic Polynomial

Corollary 1. (of Theorem 4) There is a logspace machine that takes as input a $(n \times n)$ matrix $A$, $1^m$ where $1 \leq m \leq n$ and computes the coefficient of $x^m$ in characteristic polynomial ($\chi_A(x) = \det(xI - A)$) of $A$.

Proof. While counting the number of cycle covers with $k$ cycles, we can keep track of the number of self-loops occurring in a cycle cover. Hence we can also compute the characteristic polynomial in $L$. \qed

4.2 Trace of matrix powers

Theorem 5. There is a logspace algorithm that on input a $(n \times n)$ matrix $A \in \{0, 1\}^{n^2}$ $1^k$ and $1^i$ computes the $i$-th bit of trace of $A^k$.

We first introduce some notation:

Definition 3. Let $S_k^n$ denote the elementary symmetric polynomials (for $n > 0, 0 \leq k \leq n$) i.e.

$$S_k^n(X_1, \ldots, X_n) = \sum_{1 \leq j_1 < j_2 \leq \ldots < j_k \leq n} X_{j_1}X_{j_2} \ldots X_{j_k}$$
**Fig. 1.** L-hardness for bounded treewidth determinant: Above - \( s \) occurs before \( t \); Below - \( t \) occurs before \( s \)
By convention, \( S_0^n(X_1, \ldots, X_n) = 1 \) and if \( k > n \), the \( S_k^n \) is identically zero. Let \( P_k^n \) denote the \( k \)-th power sums (for \( k > 0, n > 0 \)) i.e.

\[
P_k^n(X_1, \ldots, X_n) = \sum_{i=1}^{n} X_i^k
\]

Note that \( P_0^n = \sum_{i=1}^{n} X_i^0 = n \). Also, let \( S_n(t) \) be the following univariate polynomial, (where the variables \( X_i \) are implicit):

\[
S_n(t) = \sum_{i=0}^{n} S_i^n(X_1, \ldots, X_n)t^k
\]

Similarly, let,

\[
P_n(t) = \sum_{i=1}^{\infty} (-1)^k P_k^n(X_1, \ldots, X_n)t^k
\]

We will often write \( S_k^n, P_k^n \) where \( X_1, \ldots, X_n \) are understood.

The following is immediate:

**Fact 2**

\[
S_n(t) = \prod_{i=1}^{n} (1 + X_i t)
\]

\[
P_n(t) = \sum_{i=1}^{n} \frac{1}{1 + X_i t}
\]

**Proof.** Follows straightaway by expanding out the series and comparing coefficients. \( \square \)

These identities are closely related to Newton’s identities [Mac98].

**Proposition 4.**

\[
t S'_n(t) = S_n(t)(n - P_n(t))
\]

where \( S'_n(t) \) denotes the derivative of \( S_n(t) \) wrt \( t \).

**Proof.** Immediate from expanding out the series and comparing coefficients. \( \square \)

**Proof.** (of Theorem 5) We can compute \( P_k^n(X_1, \ldots, X_n) = \sum_{i=1}^{n} X_i^k \) where \( X_1, \ldots, X_n \) are the eigenvalues of \( A^k \) by Proposition 4. This is achieved by oracle access to the characteristic polynomial of \( A \) which is computed by a logspace machine as in Corollary 1. The trace of \( A^k \) is the number of closed walks of length exactly \( k \). That is \( \text{Trace}(A^k) = \sum_{i=1}^{n} w_{ii}^k \) where \( w_{ii}^k \) is the number of closed walks of length \( k \) that start and end at vertex \( i \). \( P_k^n \) itself is computed by polynomial division as a ratio \( \frac{n S_n(t) - t S'_n(t)}{S_n(t)} \). This can be done in \( L \) : Substitute \( t = 2^{n^2} \), reducing the problem of polynomial division to integer division which can be done in \( L \) via [HAB02]. Now the \( i \)-th bit of the coefficient of \( t^k \) of \( P_n(t) \) can be read off the quotient of this integer division via scaling by a suitable power of 2, in this case \( 2^{n^2(n-k)+i} \). \( \square \)
5 Applications to Counting problems on Bounded Treewidth graphs

We now document some consequences of Theorem 4 to some counting problems on bounded treewidth graphs. We show that counting the number of spanning trees and Euler tours of directed graphs and the Euler tours of undirected graphs is in L.

5.1 Counting the number of Spanning Trees in Bounded Treewidth graphs

Counting the number of spanning trees of a graph (both directed and undirected) was long known to be in polynomial time via a determinant computation by Kirchoff’s2 matrix tree theorem[Sta13]:

**Fact 3** The number of arborescences of a directed graph equals any cofactor of its Laplacian.

where the Laplacian of a directed graph $G$ is $D - A$ where $D$ is the diagonal matrix with the $D_{ii}$ being the out-degree of vertex $i$ and $A$ is the adjacency matrix of the underlying undirected graph.

However, Theorem 4 holds only for $\{0, 1\}$-matrices while Laplacians contains entries which are from $\{0, -1\} \cup \mathbb{N}$ because the diagonal entries are the out-degrees of the various vertices and the off-diagonal non-zero terms are all $-1$'s.

We first consider a generalisation of the determinant of $\{0, 1\}$-matrices of bounded treewidth viz. the determinant of matrices where the entries are from a set whose size is a fixed universal constant and the underlying graph consisting of the non-zero entries of $A$ is of bounded tree-width.

**Lemma 3.** Let $A$ be a matrix whose entries belong to a set $S$ of fixed size independent of the input or its length. If the underlying digraph with adjacency matrix $A'$, where $A'_{ij} = 1$ iff $A_{ij} \neq 0$, is of bounded tree-width then the determinant of $A$ can be computed in L.

**Proof.** Let $s = |S|$ be a universal constant, $S = \{c_1, \ldots, c_s\}$ and let $\text{val}_i$ be the predicates that partitions the edges of $G$ according to their values i.e. $\text{val}_i(e)$ is true iff the edge $e$ has value $c_i \in S$. Our modified formula $\psi(X, Y_1, \ldots, Y_s)$ will contain $s$ unquantified new edge-set variables $Y_1, \ldots, Y_s$ along with the old vertex variable $X$, and is given by:

$$\forall e \in E((e \in Y_i \rightarrow \text{val}_i(e))) \land (e \in Y \leftrightarrow \lor_{i=1}^{s} (e \in Y_i) \land \phi(X, Y))$$

Notice that we verify that the edges in the set $Y_i$ belong to the $i^{th}$ partition and each edge in $Y$ is in one of the $Y_i$'s. The fact that the $Y_i$'s form a partition of $Y$ follows from the assumption that $\text{val}_i(e)$ is true for exactly one $i \in [s]$ for any edge $e$.

To obtain the determinant we consider the histogram parameterised on the $s$ variables $Y_1, \ldots, Y_s$ and the heads $X$. For an entry indexed by $x, y_1, \ldots, y_s$, we multiply the entry by $(-1)^{n+s} \prod_{i=1}^{s} c_i^{y_i}$ and take a sum over all entries.

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2 See Section 1.2 of [http://math.mit.edu/~levine/18.312/alg-comb-lecture-19.pdf](http://math.mit.edu/~levine/18.312/alg-comb-lecture-19.pdf) for a nice proof of the undirected case via the directed case
Now the proof is completed by observing the following:

**Lemma 4.** Given a digraph $G$ there is an easily constructible super-digraph $G'$ such that the number of rooted spanning trees in both the digraphs is the same and the outdegree of every vertex in $G$ is either 1 or $|V(G)|$. In addition if $G$ is bounded tree-width so is $G'$ and the latter has a tree-decomposition easily obtainable from that of $G$.

**Proof.** The construction of $G'$ is simple for every vertex $v$ of $G$ which has out-degree $od(v)$, add $k = n - od(v)$ many new vertices $v_1, v_2, \ldots, v_k$ in $G'$ and ensure that the edges $(v_i, v), (v, v_i)$ are added also. Then $od_{G'}(v_i) = 1$ and $od_{G'}(v) = n$ in $G'$. The tree-width of $G'$ is clearly same as that of $G$ merely by introducing new bags containing $v, v_i$ for each $i \in [k]$. □

The previous fact implies that the Laplacian matrix of $G'$ has entries from the set $S = \{0, 1, -1, |V(G)|, -|V(G)|\}$.

### 5.2 Counting Euler Tours in Directed Bounded Treewidth Graphs

The BEST Theorem states:

**Fact 4** [AEB87][TS41] The number of Euler Tours in a directed Eulerian graph $K$ is exactly:

$$t(K) \prod_{v \in V} (\deg(v) - 1)!$$

where $t(K)$ is the number of arborescences in $K$ rooted at an arbitrary vertex of $K$ and $\deg(v)$ is the indegree as well as the outdegree of the vertex $v$.

We combine Facts 3 and 4 with Theorem 4 to compute the number of directed Euler Tours in a directed Eulerian graph. Combining Lemmas 3 and 4 with the Theorem 4 and the fact that iterated integer sum and product is in $L$ [HAB02], we get that:

**Theorem 6.** Counting the number of directed Euler Tours in a directed Eulerian graph $G$ is in $L$.

### 5.3 Counting Euler tours in undirected bounded treewidth graphs

There is no closed form expression for the number of Euler Tours in undirected graphs unlike the BEST formula for directed graphs. However, notice that every undirected tour corresponds to an orientation of the undirected graph in the following sense. Suppose we orient the edges of the tour so that the first edge is the lex-least edge (i.e. from the smallest numbered vertex to its least numbered neighbour), then this induces an orientation of the edges of the undirected graph. Also given any directed tour of any orientation of the undirected graph the corresponding undirected tour imposes an orientation which is either the given orientation (if the lex-least edge is oriented according to the orientation) or its reverse (if the lex-leasts edge is oriented according tot he reverse of the orientation).

The long and short of it is that
Observation 1 The number of undirected tours is half of the sum of the directed tours over all Eulerian-orientations of the graph.

Let $G = (V,E)$ be an undirected graph and let $G'$ denote the graph obtained by subdividing every edge by introducing a new vertex $v_e$ for every $e \in E$ and replacing the edge $e = \{u,v\}$ by the two edges $\{u,v_e\}, \{v,v_e\}$. Notice the following bijection:

Observation 2 The orientations of $G$ are in bijection with functions that pick for each vertex $v_e$ of $V(G')$ one of the two edges incident on it. In particular, Eulerian orientations correspond to functions such that the edges picked at a vertex $v \in V$ are exactly equal to the edges not picked at $v$.

To see this we just associate the direction $(u,v)$ to $\{u,v\}$ iff the edge $\{u,v_e\}$ is picked in the set. Under this bijection, a directed cycle in an oriented version of $G$ consists of a cycle in $G'$ in which alternating edges are picked by the function above.

Also notice that $G'$ has the same number of Euler tours as $G$ and also has bounded treewidth.

With all this in mind we can write an MSO-formula, $\phi(X,Z)$ where $Z$ is the set of edges picked above to indicate an orientation. The formula is similar to the one in Lemma ??, replacing directed cycles/paths and alternating ones (in the sense above). Notice we don’t have to stipulate anything special such as per-vertex cardinality constraints on $Z$ - this is guaranteed by the rest of the formula.

Now, the following theorem is obvious in light of Observation 1:

**Theorem 7.** Counting the number of Euler Tours in an undirected bounded tree-width Eulerian graph $K$ is in L.

6 Conclusion and Open Ends

We show that the Logspace version of Courcelle’s theorem can be used to compute the Determinant of a $\{0,1\}$-matrix of bounded tree-width in L. With some more work we are able to show some problems in linear algebra such as characteristic polynomial, rank and trace of powers of binary matrices is also in L. We are also able to show that counting graph theoretic structures such as spanning trees and Euler tours in both graphs and digraphs is in L.

Many open questions remain:

- What is the complexity of the determinant of a matrix with arbitrary weights whose support is a bounded tree-width graph?
- What is the complexity of powering a matrix of bounded tree-width?
- More generally, what is the complexity of iterated product of a bounded tree-width matrix?
- What is the complexity of other linear algebraic invariants such as minimal polynomial of a bounded tree-width matrix?
- Can the algorithms we design be made less “galactic” in terms of dependence on the tree-width? In other words, can we eliminate the use of Courcelle’s theorem in these algorithms (CCM13 is a step in this direction)?
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