SHAPE OPTIMIZATION METHOD FOR AN INVERSE GEOMETRIC SOURCE PROBLEM AND STABILITY AT CRITICAL SHAPE

LEKBIR AFRAITES*, CHOROUK MASNAOUI AND MOURAD NACHAOUI

Laboratory of Mathematics and Applications (LMA)
Faculty of Sciences and Techniques, Sultan Moulay Slimane University
Beni Mellal, Morocco

Abstract. This work deals with a geometric inverse source problem. It consists in recovering inclusion in a fixed domain based on boundary measurements. The inverse problem is solved via a shape optimization formulation. Two cost functions are investigated, namely, the least squares fitting, and the Kohn-Vogelius function. In this case, the existence of the shape derivative is given via the first order material derivative of the state problems. Furthermore, using the adjoint approach, the shape gradient of both cost functions is characterized. Then, the stability is investigated by proving the compactness of the Hessian at the critical shape for both considered cases. Finally, based on the gradient method, a steepest descent algorithm is developed, and some numerical experiments for non-parametric shapes are discussed.

1. Introduction. Inverse problems have been encountered in many areas of applied science, engineering and bio-engineering [10, 16, 18, 26, 33, 40, 39]. Inverse source problems is a class of inverse problems that aims to find \((u, F)\) solution of the following equation

\[
Lu = F \quad \text{in } \Omega,
\]

based on a single pair of additional data on the boundary \(\partial \Omega\), where \(\Omega\) is an open bounded set with boundary \(\partial \Omega\), \(L\) is a real elliptic linear differential operator and \(F\) is the unknown support source term. It is well known that solving the inverse source problem is extremely difficult for both analytical and numerical solutions. Indeed, this is one of the highly ill-posed problems in Hadamard sense [25] and consequently, a general source function could not be identified uniquely from the boundary measurements see [37] and [20]. Several works have addressed the identification of particular source functions. For instance, point sources [20, 31], surface sources [9], source functions with harmonic part [5], and characteristic source functions of star-shaped sets [41]. In this case, several methods have been proposed. In [29], an iterative method was proposed via domain derivative of the corresponding observation operator, using the boundary element method (BEM) as a numerical

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* Corresponding author: lekbir.afraites@gmail.com.
solver but this latter study was limited to starlike domains. In [1], a direct algorithm is used to reconstruct $F$ in the case where it is a distributed function having compact support within a finite number of small subdomains. Recently in [11], an approach based on the method of fundamental solution (MFS) was proposed where the unknown source domain is assumed to be star-shaped.

In this work, we consider the problem (1) with the unknown source $F$ of the form $\chi_\omega$ ($\chi$ is the characteristic function). We point out that this problem has been studied by [32] using a topological optimization formulation, the unknown source was reconstructed using a level-set curve of the topological gradient. [34] presented an iterative solution method via boundary integral equations, by reformulating the inverse source problem as an inverse boundary value problem with a non-local Robin condition on the boundary of the source domain. In [42] a method for the reconstruction of star-shaped characteristic sources was developed, by reducing the problem to an algebraic system of equations. The paper [6] treated the inverse characteristic source problem, in the case of Helmholtz equations, from the determination of the barycenter of the characteristic source and the recovery of its geometry from a class of star-shaped characteristic sources, using an algorithm based on an equivalent reciprocity functional formulation. [21] have also investigated the inverse source problem of the Helmholtz equation, where the source consists of multiple point sources, an algebraic algorithm was proposed to identify the number, locations and intensities of the point sources from boundary measurements. The main idea of our approach is to reformulate the inverse source problem to a shape optimization one, where the unknown is the support of the characteristic source function.

In this case, we propose two cost functionals to solve this non-parametric shape optimization problem. The first one is the classical least squares fitting and the second one is the Kohn-Vogelius function. Thus, we investigate a shape derivative of the state equations, which allows calculating the shape gradient of both cost functionals. Then, using the second shape derivative of both cost functionals at the critical shape, we invoke the stability study.

The outline of the paper is organized as follows. Section 2, is devoted to the presentation of the inverse source problem and its two proposed shape optimization reformulations. In section 3, based on a rigorous proof for the existence of the material derivative, we deduce the shape derivative of the states. In section 4, we establish the shape gradient calculus and the second shape derivative at the critical shape of each one of the considered cost functions. In section 5, we investigate the stability of the optimal solution by proving the compactness of the Hessian at the critical shape for both considered cost functionals. In the last section, we present details of the developed numerical algorithm based on the shape gradient. Then, we present some numerical experiments for some non-parametric shapes.

2. The geometric inverse source problem and reformulation to shape optimization. Let $\Omega$ be an open, bounded, and connected subset of $\mathbb{R}^d$ ($d = 2$ or $3$) with $C^1$ boundary $\partial \Omega$. Let $d_0$ be a positive constant, we define the set of admissible domains denoted by $\Omega_{d_0}$ as the set of all open simply connected subdomains $\omega$ of $\Omega$ with a $C^{2,1}$ boundary (see [23] for definitions of $C^1$ and $C^{2,1}$ spaces), such that $d(x, \partial \Omega) > d_0$ for all $x \in \omega$. The notation $\chi_\omega$ denotes the characteristic function of $\omega$. Based on some knowledge on the boundary conditions, namely the voltage $f \in H^{1/2}(\partial \Omega)$ and the current measurement $g \in H^{-1/2}(\partial \Omega)$, we try to find $\omega$ and $u$ solution of the following overdetermined problem
SHAPE OPTIMIZATION METHOD

\[ \begin{aligned}
-\Delta u + \alpha u &= \chi_\omega \quad \text{in } \Omega, \\
u &= f \quad \text{on } \partial \Omega, \\
\partial_n u &= g \quad \text{on } \partial \Omega,
\end{aligned} \]  

(2)

where \( \partial_n u \) denotes the normal derivative of \( u \) at the boundary \( \partial \Omega \), and \( \alpha \) is a non-negative constant.

Assuming that there exists \( \omega^* \in \Omega_{d_0} \) such that (2) has a solution, then the considered shape inverse problem is defined as follows:

find \( w \in \Omega_{d_0} \) and \( u \in H^1(\Omega) \) which satisfy the problem (2).

(3)

To solve this inverse problem, we opt for the shape optimization approach where the unknown is the inclusion \( \omega \). The usual strategy to determine the shape of the inclusion \( \omega \) is to minimize an objective function. In this case, many choices are possible. One of the most popular cost functions is the least squares function, which is defined as

\[ J_{LS}(\omega) = \frac{1}{2} \int_{\partial \Omega} |u_n - f|^2, \]  

(4)

where \( f \) is the boundary measurement and \( u_n \) is the state function that solves

\[ \begin{aligned}
-\Delta u_n + \alpha u_n &= \chi_\omega \quad \text{in } \Omega, \\
\partial_n u_n &= g \quad \text{on } \partial \Omega.
\end{aligned} \]  

(5)

Recently, the use of Kohn-Vogelius criterion as a cost function has been increased (see [43] and [22], [2]), for the transmission problem [3, 4] and for the Stokes problem [15]. In fact, this type of cost functions gives more accurate optimization procedures. In the case of inverse source problems, the shape optimization formulation based on the Kohn-Vogelius functional is written as

\[ J_{KV}(\omega) = \frac{1}{2} \int_{\Omega} |\nabla (u_d - u_n)|^2 + \frac{1}{2} \int_{\partial \Omega} |u_d - u_n|^2, \]  

(6)

where \( u_n \) is the solution of the problem (5) and \( u_d \) is the solution of the following Dirichlet problem

\[ \begin{aligned}
-\Delta u_d + \alpha u_d &= \chi_\omega \quad \text{in } \Omega, \\
u_d &= f \quad \text{on } \partial \Omega.
\end{aligned} \]  

(7)

The cost function \( J_{KV} \) is a positive function that measures the difference between \( u_d \) the solution of Dirichlet problem and \( u_n \) the solution of Neumann problem. By integration by parts, the cost function (6) can be written in a new expression which involves the boundary \( \partial \Omega \), reads as

\[
J_{KV}(\omega) = \frac{1}{2} \int_{\Omega} |\nabla (u_d - u_n)|^2 + \frac{1}{2} \int_{\omega} |\nabla (u_d - u_n)|^2 + \frac{1}{2} \alpha \int_{\Omega \setminus \varnothing} |u_d - u_n|^2 + \frac{1}{2} \alpha \int_{\partial \Omega} |u_d - u_n|^2 + \frac{1}{2} \alpha \int_{\partial \Omega} |u_d - u_n|^2 \\
= \frac{1}{2} \int_{\partial \Omega} (\partial_n u_d - g)(f - u_n).
\]

The two considered shape formulations are defined by

\[ \omega^* = \arg\min_{w \in \Omega_{d_0}} J_{KV}(w), \]  

(8)

and

\[ \omega^* = \arg\min_{w \in \Omega_{d_0}} J_{LS}(w). \]  

(9)

In order to solve those shape optimization problems, the commonly used method is the steepest descent method using the gradient of the cost functions. Thus, based
on the work of Murat-Simon [38] we will compute the shape gradient. The details of differentiation rules with respect to the domain can be found in [28, 44, 45].

3. First order shape derivative. In this section, we present some preliminary notions concerning the shape derivative as velocity field deformation, material derivative, and shape derivative. Then, we give a rigorous proof of the existence of material derivative of the state and we deduce its shape derivative.

3.1. Elements of shape calculus. Before starting, we give some notations. We denote by \( n \) the outward unit normal to \( \partial \omega \) pointing into \( \Omega \setminus \omega \), thus \( \partial_n u^+ \) (resp. \( \partial_n u^- \)) is the normal derivative from the inside of \( \omega \) (resp. \( \Omega \setminus \omega \)) at interface \( \partial \omega \), and \([ \cdot ]\) denotes the jump across the same interface. Denote by \( V \) a smooth vector field with compact support in \( \Omega \), we denote by \( V \) the space of admissible deformations \( V \) and let \( V_n := \langle V, n \rangle \) its normal component. Consider the smooth transformation defined by \( T_t(x) = x + tV(x) \), which is invertible for small \( t \). We denote by \( J_t(x) = \det DT_t(x) \) and \( A_t(x) = DT_t(x) - 1 \).

Note that \( A_t(x) \) is symmetric positive and for \( t < t_0 \) we have \( y^\top A_t(x)y \geq \mu \| y \|^2 \) and \( 0 < \beta \leq J_t \).

Furthermore, the application \( A_t \) is smooth with \( A_0(x) = I \), \( J_0(x) = 1 \) and

\[
\frac{d}{dt}J_t(x)|_{t=0} = \text{div} (V), \quad A = \frac{d}{dt}A_t(x)|_{t=0} = \text{div} (V)I - (DV^\top + DV).
\]

We denote by \( u_t \) the solution of (5) with inclusion \( \omega_t = T_t(\omega) \) and \( u^t = u_t \circ T_t \). The function \( u^t \) is defined on the fixed domain \( \Omega \) and its material derivative (or Lagrangian derivative) is given by

\[
\dot{u} := \lim_{t \to 0} \frac{u^t - u}{t}, \quad \forall x \in \Omega.
\]

The shape derivative (or Eulerian derivative) is defined by

\[
u' := \dot{u} - \nabla u \cdot V.
\]

3.2. First order derivatives of the state.

**Theorem 3.1.** The Neumann solution \( u_n \) has a material derivative \( \dot{u}_n \) that satisfies

\[
\forall v \in H^1(\Omega), \quad \langle \nabla \dot{u}_n, \nabla v \rangle_\Omega + \alpha \langle \dot{u}_n, v \rangle_\Omega = - \langle A \nabla u_n, \nabla v \rangle_\Omega - \alpha \langle \nabla (V) u_n, v \rangle_\Omega + \langle \nabla (V), v \rangle_\omega.
\]  

(10)

The state \( u_n \) is shape differentiable and its shape derivative \( u'_n \) solves

\[
\begin{cases}
\Delta u'_n + \alpha u'_n &= 0 \quad \text{in } \Omega \setminus \omega \text{ and in } \omega, \\
[u'_n] &= 0 \quad \text{on } \partial \omega, \\
[V_n] &= V_n \quad \text{on } \partial \omega, \\
[\partial_n u'_n] &= 0 \quad \text{on } \partial \Omega.
\end{cases}
\]  

(11)

In [29] and [30], formula (11) was first derived. We think it is useful to show how, using the classical methods of shape optimization, we can prove both existence of the derivative and problem (11). Therefore, we give a rigorous proof of Theorem 3.1 and also show that material derivative is belonging to \( H^1(\Omega) \).
From Young’s inequality, we deduce that

\[ \forall v \in H^1(\Omega), \quad \int_{\Omega} \nabla u_t \cdot \nabla v + \alpha \int_{\Omega} u_tv = \int_{\partial \Omega} g v + \int_{\omega} v. \]

Then, the transported \( u^t = u_t \circ T_t \) solves the variational equation

\[ \int_{\Omega} A_t(x) \nabla u^t(x) \cdot \nabla v(x) + \alpha \int_{\Omega} u^t(x)v(x)J_t(x) = \int_{\partial \Omega} g(x)v(x) + \int_{\omega} v(x)J_t(x). \quad (12) \]

**First Step.** Let \( u_t \) denote the solution of problem (5) given in a domain with inclusion \( \omega_t = T_t(\omega) \), we have

\[ \forall v \in H^1(\Omega), \quad \int_{\Omega} \nabla u_t \cdot \nabla v + \alpha \int_{\Omega} u_tv = \int_{\partial \Omega} g v + \int_{\omega} v. \]

Then, the transported \( u^t = u_t \circ T_t \) solves the variational equation

\[ \int_{\Omega} A_t(x) \nabla u^t(x) \cdot \nabla v(x) + \alpha \int_{\Omega} u^t(x)v(x)J_t(x) = \int_{\partial \Omega} g(x)v(x) + \int_{\omega} v(x)J_t(x). \quad (12) \]

**Second Step.** Using the variational equation solved by \( u \) subtracted from the equation (12), and the fact that \( T_t(x) = x \) on \( \partial \Omega \), we obtain

\[ \langle A_t \frac{\nabla u^t - \nabla u}{t}, \nabla v \rangle_{\Omega} + \alpha \langle J_t \frac{u^t - u}{t}, v \rangle_{\Omega} = \langle \frac{I - A_t}{t} \nabla u, \nabla v \rangle_{\Omega} + \alpha \langle \frac{1 - J_t}{t} u, v \rangle_{\Omega} \]

\[ + \langle J_t - 1, v \rangle_{\omega}. \quad (13) \]

Using \( \frac{(u^t - u)}{t} \) as a test function, and from the properties of \( A_t \) and \( J_t \) we get

\[ \mu \left\| \frac{\nabla u^t - \nabla u}{t} \right\|^2_{L^2(\Omega)} + \beta \alpha \left\| \frac{u^t - u}{t} \right\|^2_{L^2(\Omega)} \leq \left\| \frac{A_t - I}{t} \right\|_{\infty} \left\| \nabla u \right\|_{L^2(\Omega)} \left\| \frac{\nabla u^t - \nabla u}{t} \right\|_{L^2(\Omega)} \]

\[ + \left\| \frac{1 - J_t}{t} \right\|_{\infty} \left\| u \right\|_{L^2(\Omega)} \left\| \frac{u^t - u}{t} \right\|_{L^2(\Omega)} \]

\[ + \left\| J_t - 1 \right\|_{\infty} \left\| \frac{u^t - u}{t} \right\|_{L^2(\Omega)} mes(\Omega_d) \right)^{1/2}. \]

From Young’s inequality, we deduce that

\[ \min(\mu, \beta\alpha) \left\| \frac{u^t - u}{t} \right\|_{H^1(\Omega)} \leq C \left( \left\| \frac{A_t - I}{t} \right\|_{\infty} \left\| \nabla u \right\|_{L^2(\Omega)} + \left\| \frac{1 - J_t}{t} \right\|_{\infty} \left\| u \right\|_{L^2(\Omega)} \]

\[ + \left\| J_t - 1 \right\|_{\infty} mes(\Omega_d) \right)^{1/2} \right), \]

where \( C \) is a positive constant. Therefore the sequence \( (u^t - u)/t \) is bounded in \( H^1(\Omega) \). Thus, we obtain the weak convergence of the sequence in \( H^1 \) and its weak limit is \( \dot{u} \) the material derivative of \( u \).

**Third Step.** We show the strong convergence of \( (u^t - u)/t \). We pass to the limit as \( t \to 0 \) in (13), we show that \( \dot{u} \) solves

\[ \langle \nabla \dot{u}, \nabla v \rangle + \alpha \langle \dot{u}, v \rangle = -\langle A \nabla u, \nabla v \rangle - \alpha \langle \text{div}(V)u, v \rangle + \langle \text{div}(V), v \rangle_{\omega}. \]
This was stated as (10) in Theorem 3.1. This equation allows us to prove the strong convergence in $H^1(\Omega)$, in fact, taking $v = (u^t - u)/t$ in (13), we get

$$\langle A_t \nabla v, \nabla v \rangle + \alpha \langle J_t v, v \rangle = \frac{I - A_t}{t} \langle \nabla u, \nabla v \rangle + \alpha \left( \frac{1 - J_t}{t} u, v \right) + \left( \frac{J_t - 1}{t} v, v \right)_\Omega$$

$$= \langle (A_t - I) \nabla v, \nabla v \rangle + \alpha \langle (J_t - 1) v, v \rangle$$

$$+ \left( \frac{I - A_t}{t} \nabla u^t, \nabla v \right) + \alpha \left( \frac{1 - J_t}{t} u^t, v \right) + \left( \frac{J_t - 1}{t} v, v \right)_\Omega$$

$$= E_{1,t} + E_{2,t},$$

we denote by

$$E_{1,t} = \langle (A(t) - I) \nabla v, \nabla v \rangle + \alpha \langle (J_t - 1) v, v \rangle,$$

$$E_{2,t} = \left( \frac{I - A(t)}{t} \right) \nabla u^t, \nabla v \rangle + \alpha \left( \frac{1 - J_t}{t} u^t, v \right) + \left( \frac{J_t - 1}{t} v, v \right)_\Omega.$$

Using the weak convergence of $(u^t - u)/t$, we get after straightforward calculations $E_{1,t} \to 0$ and $E_{2,t} \to \langle (A \nabla u, \nabla \hat{u}) - \alpha (\nabla (V \cdot u), \hat{u}) + (\nabla (V) \cdot \hat{u}) \rangle \omega$ when $t \to 0$.

By (10), we conclude that $E_{2,t} \to \langle (\nabla \hat{u}_d, \nabla \hat{u}_d) \rangle \omega$.

This shows that $\langle A_t \nabla v, \nabla v \rangle + \alpha \langle J_t v, v \rangle$ converges to $\|\nabla \hat{u}\|^2 + \alpha \|u\|^2$. Using the properties of $A_t$ and $J_t$ we deduce the strong convergence of $v = (u^t - u)/t$ to $\hat{u}$ in $H^1(\Omega)$.

**Fourth Step.** We obtain the equations satisfied by the shape derivative $u' = \hat{u} - V \nabla u$. Let $b = (V \nabla u) \nabla v + (V \nabla v) \nabla u - (\nabla u \cdot \nabla v)V$. Using the following classical identity

$$-\nabla u \cdot \nabla v = \text{div} \ (b) - (V \nabla u) \Delta v - (V \nabla v) \Delta u,$$

and equation (10) satisfied by $\hat{u}$ we get

$$\int_{\Omega} \nabla \hat{u} \cdot \nabla v + \alpha \int_{\Omega} \hat{u} v = \int_{\Omega} \text{div} \ (b) - \int_{\Omega} (V \nabla u) \Delta v - \int_{\Omega} (V \nabla v) \Delta u$$

$$- \alpha \int_{\Omega} \text{div} \ (V u) v + \int_{\omega} \text{div} \ (V v).$$

Using the divergence theorem and integration by parts (considering $-\Delta u = -\alpha u + \chi_\omega$), we obtain

$$\int_{\Omega} \nabla \hat{u} \cdot \nabla v + \alpha \int_{\Omega} \hat{u} v = - \int_{\partial \omega} [(V \nabla v) \partial_n u] + \int_{\partial \omega} [(\nabla u \cdot \nabla v) V_n] + \int_{\Omega} \nabla (V \nabla u) \nabla v$$

$$- \alpha \int_{\Omega} \text{div} \ (V u) v + \int_{\omega} \text{div} \ (V v) - \alpha \int_{\Omega} (V \nabla u) v + \int_{\omega} (V \nabla v).$$

Which implies that

$$\int_{\Omega} \nabla (\hat{u} - V \nabla u) \cdot \nabla v \ dx + \alpha \int_{\Omega} (\hat{u} - V \nabla u) v = - \alpha \int_{\Omega} (\text{div} \ (V u v) + \int_{\partial \omega} \text{div} \ (V v).$$

We obtain the following equation satisfied by the shape derivative $u' = \hat{u} - V \nabla u$

$$\int_{\Omega} \nabla u' \cdot \nabla v + \alpha \int_{\Omega} u' v = \int_{\partial \omega} V_n v,$$

and using Green’s formula we get

$$- \int_{\Omega} \Delta u' v + \alpha \int_{\Omega} u' v - \int_{\partial \omega} v [\partial_n u'] = - \int_{\partial \omega} V_n v.$$
We deduce that \( u' \) satisfies the equation 
\[-\Delta u' + \alpha u' = 0 \text{ on } \Omega \setminus \bar{\omega} \cup \omega \]
with the condition 
\[ [\partial_n u'] = V_n \text{ on } \partial \omega. \]
Then, we determine the jump of \( u' \). Since \( \dot{u} \in H^1(\Omega) \), we have 
\[ [u'] = -V_n \ [\partial_n u] = 0 \text{ on } \partial \omega. \]
Using the boundary condition \( \partial_n u = g \text{ on } \partial \Omega \) and the fact that \( V = 0 \text{ on } \partial \Omega \), we obtain 
\[ \partial_n u' = 0 \text{ on } \partial \Omega. \]
We came to the end of our proof of shape differentiability of \( u_n \).

Following the same steps, we obtain the shape derivative of \( u_d \) solution of the Dirichlet problem
\[
\begin{cases}
\Delta u'_d + \alpha u'_d = 0 & \text{in } \Omega \setminus \bar{\omega} \text{ and in } \omega, \\
[u'_d] = 0 & \text{on } \partial \omega, \\
[\partial_n u'_d] = V_n & \text{on } \partial \omega, \\
u'_d = 0 & \text{on } \partial \Omega.
\end{cases}
\] (14)

4. **Shape derivatives of the cost functionals.** In this section, we characterize the shape derivatives of the two cost functionals considered and then, we give the second order shape derivative at the critical shape in order to study the stability of our process in the next section.

4.1. **First order shape derivatives of the functionals.** Using the previous characterization of the state, we now compute the shape gradient of the cost functions \( J_{LS} \) and \( J_{KV} \) in the following lemmas.

**Lemma 4.1 (The least squares cost function).** Let \( V \in \mathcal{V} \), the least squares functional \( J_{LS} \) is differentiable with respect to the shape \( \omega \) in the direction of \( V \) and its shape gradient is given by
\[
DJ_{LS}(\omega).V = -\int_{\partial \omega} p V_n, \tag{15}
\]
where \( p \) solves the following adjoint problem
\[
\begin{cases}
-\Delta p + \alpha p = 0 & \text{in } \Omega, \\
\partial_n p = u_n - f & \text{on } \partial \Omega.
\end{cases}
\] (16)

**Proof.** Using the derivation rule (see Proposition 5.4.18 in [28]) and knowing that \( V = 0 \text{ on } \partial \Omega \), we obtain
\[
DJ_{LS}(\omega).V = \int_{\partial \Omega} (u_n - f) u'_n.
\]
Comparing variational formulation of (16) with \( u'_n \) as a test function and variational formulation of (11) with \( p \) as a test function in both \( \Omega \setminus \bar{\omega} \) and \( \omega \), we obtain
\[
\int_{\partial \Omega} \partial_n p u'_n = \int_{\partial \omega} (\partial_n p)^+ u'_n^+ - (\partial_n u'_n)^+ p^+, \tag{17}
\]
\[
\int_{\partial \omega} (\partial_n u'_n)^- p^- = \int_{\partial \omega} (\partial_n p)^-(u'_n)^-. \tag{18}
\]
using the above equations (17) and (18), and the jump conditions of \( p \) and \( u'_n \), we get
\[
\int_{\partial \Omega} \partial_n p u'_n = \int_{\partial \omega} (\partial_n u'_n)^- p - \int_{\partial \omega} (\partial_n u'_n)^+ p = - \int_{\partial \omega} [\partial_n u'_n] p = - \int_{\partial \omega} p V_n.
\]
Finally, using the boundary conditions in (16) we get the result. \( \square \)

**Lemma 4.2** (The Kohn-Vogelius cost function). Let \( V \in \mathcal{V} \), the Kohn-Vogelius functional \( J_{KV} \) is differentiable with respect to the shape \( \omega \) in the direction of \( V \) and its shape gradient is given by
\[
DJ_{KV}(\omega).V = \int_{\partial \omega} (u_d - u_n) V_n.
\]

**Proof.** We have already shown that state functions are differentiable, thus \( J_{KV} \) is also differentiable, and its shape derivative is obtained thanks to the chain rule
\[
DJ_{KV}(\omega).V = \frac{1}{2} \int_{\partial(\Omega \setminus \varnothing)} |\nabla (u^+_d - u^+_n)|^2 V. n_{\Omega \setminus \varnothing} + \int_{\Omega \setminus \varnothing} \nabla (u_d - u_n). \nabla (u'_d - u'_n) + \frac{1}{2} \int_{\partial \omega} |\nabla (u^-_d - u^-_n)|^2 V_n + \int_{\omega} \nabla (u_d - u_n). \nabla (u'_d - u'_n) + \frac{1}{2} \alpha \int_{\partial(\Omega \setminus \varnothing)} |(u^+_d - u^+_n)|^2 V. n_{\Omega \setminus \varnothing} + \alpha \int_{\Omega \setminus \varnothing} (u_d - u_n)(u'_d - u'_n) + \frac{1}{2} \alpha \int_{\partial \omega} |(u^-_d - u^-_n)|^2 V_n + \alpha \int_{\omega} (u_d - u_n)(u'_d - u'_n).
\]
For the boundary terms, by construction, we have \( V_n = 0 \) on \( \partial \Omega \), then the integral on \( \partial \Omega \) disappears. To investigate the term on \( \partial \omega \), we take into account the jump relations and we use the normal and tangential components of the gradient of state, we obtain
\[
\int_{\partial \omega} [|\nabla (u^-_d - u^-_n)|^2 - |\nabla (u^+_d - u^+_n)|^2] V_n = 0,
\]
\[
\int_{\partial \omega} [(u^-_d - u^-_n)^2 - (u^+_d - u^+_n)^2] V_n = 0.
\]
Based on the Gauss formula we define the volume term. According to (7) and (5), we get:
\[
\int_{\Omega \setminus \varnothing} \nabla (u_d - u_n). \nabla (u'_d - u'_n) + \alpha \int_{\Omega \setminus \varnothing} (u_d - u_n)(u'_d - u'_n) =
\]
\[- \int_{\partial \omega} \partial_n (u^+_d - u^+_n)(u'_d - u'_d) - \int_{\partial \omega} \partial_n (u_d - u_n) u'_n,
\]
and
\[
\int_{\omega} \nabla (u_d - u_n). \nabla (u'_d - u'_n) + \alpha \int_{\omega} (u_d - u_n)(u'_d - u'_n) = \int_{\partial \omega} \partial_n (u^-_d - u^-_n) ((u'_d)^- - (u'_n)^-).
\]
The use of the jump condition leads to
\[
I = \int_{\Omega \setminus \varnothing} \nabla (u_d - u_n). \nabla (u'_d - u'_n) + \alpha (u_d - u_n)(u'_d - u'_n)
\]
\[+
\int_{\omega} \nabla (u_d - u_n). \nabla (u'_d - u'_n) + \alpha (u_d - u_n)(u'_d - u'_n)
\]
\[= - \int_{\partial \omega} \partial_n (u_d - u_n) u'_n.
\]
The term on $\partial \Omega$ contains $u_n'$ defined by (11), using the Gauss formula, we transform it into an integral on $\partial \omega$. We have in the first place
\[
\int_{\partial \Omega} \partial_n (u_d - u_n) u_n' = \int_{\partial \omega} \partial_n (u_d - u_n) u_n' - (u_d - u_n)(\partial_n u_n')^+, \n\] and
\[
\int_{\partial \omega} (\partial_n u_n')^- (u_d - u_n) = \int_{\partial \omega} \partial_n (u_d - u_n) u_n'. \n\] Then using the jump relation, we obtain
\[
\int_{\partial \Omega} \partial_n (u_d - u_n) u_n' = \int_{\partial \omega} (\partial_n u_n')^- (u_d - u_n) + \int_{\partial \omega} (\partial_n u_n')^- (u_d - u_n)
= \int_{\partial \omega} [\partial_n u_n'] (u_d - u_n)
= \int_{\partial \omega} V_n (u_d - u_n). \n\] We get the new formula
\[
I = - \int_{\partial \Omega} \partial_n (u_d - u_n) u_n' = \int_{\partial \omega} (u_d - u_n)V_n. \n\] We obtain the desired expression by adding (20) and (21).

Remark. The Kohn-Vogelius functional has an important property lies in the fact that its gradient does not contain the derivative of the states. This is a typical behavior of energy-type shape functionals. For numerical simulation, this means that no adjoint state is needed to evaluate the gradients.

4.2. Second order shape derivatives of the cost functions at critical shape. Let us consider $\omega^* \in \Omega_{d_0}$ solution of the inverse problem (3), then $\omega^*$ is the critical shape of our cost functionals $J_{LS}$ and $J_{KV}$. In order to study the stability of the optimization problems (4) and (6) at $\omega^*$, we will characterize the second order shape derivative of $J_{LS}$ and $J_{KV}$, i.e., the shape Hessian. Notice that the existence of $p'$ the shape derivative of the adjoint state $p$, with respect to the shape $\omega \in \Omega_{d_0}$, can be shown in the same way as was done for the state $u$. Then, we have the following results

Proposition 1. (Characterization of the shape Hessian at a critical shape of cost function $J_{LS}$). For $V \in \mathcal{V}$, we have
\[
D^2 J_{LS}(\omega^*)[V,V] = \int_{\partial \omega^*} (p' - u_n')V_n, \tag{22} \n\] where $u_n'$ is the solution of (11) and $p'$ is the derivative of the following adjoint problem
\[
\begin{align*}
-\Delta p' + \alpha p' & = 0 \quad \text{in } \Omega \setminus \overline{\omega} \text{ and in } \omega^*, \n \left[ p' \right] & = 0 \quad \text{on } \partial \omega^*, \n \left[ \partial_n p' \right] & = V_n \quad \text{on } \partial \omega^*, \n \partial_n p' & = u_n' \quad \text{on } \partial \Omega. \n\end{align*} \tag{23} \n\]

Proof. Using Hadamard’s formula, the second derivative of cost function $J_{LS}$ is given by
\[
D^2 J_{LS}(\omega)[V,V] = \int_{\partial \Omega} (u_n')^2 + (u_n - f)u_n'. \n\]
We assume that there exists an admissible inclusion \( \omega^* \) such that \( J_{\text{LS}}(\omega^*) = 0 \) \((u_n = f)\). It realizes the absolute minimum of the criterion \( J_{\text{LS}} \). This is satisfied by the solution of the inverse problem, and the solution of adjoint problem \( p = 0 \) in \( \Omega \setminus \overline{\omega^*} \) and in \( \omega^* \). Then, Euler’s equation \( D J_{\text{LS}}(\omega^*)(V) = 0 \) holds, and we prove that

\[
D^2 J_{\text{LS}}(\omega^*)[V, V] = \int_{\partial \Omega} (u'_n)^2. \tag{24}
\]

The shape derivative of the adjoint state \( p' \) is characterized in the same way as \( u'_n \) which leads to the formula (23). Then, we use Green’s formula in (11) and (23) with the test functions \( p' \) and \( u'_n \) respectively, we obtain

\[
\int_{\partial \Omega} \partial_n p' u'_n = \int_{\partial \omega^*} [\partial_n u'_n] p' - \int_{\partial \omega^*} [\partial_n p'] u'_n.
\]

Using the jump conditions of (23) and (11), we get

\[
\int_{\partial \Omega} u'^2 = \int_{\partial \Omega} \partial_n p' u'_n = \int_{\partial \omega^*} (p' - u'_n) V_n,
\]

which concludes our proof. \( \square \)

**Proposition 2.** (Characterization of the shape Hessian at a critical shape of cost function \( J_{K^V} \)). For \( V \in \mathcal{V} \), we have

\[
D^2 J_{K^V}(\omega^*)[V, V] = \int_{\partial \omega^*} (u'_d - u'_n) V_n, \tag{25}
\]

where \( u'_d \) is the solution of problem (14) and \( u'_n \) is the solution of problem (11).

**Proof.** From Lemma 4.2, Divergence theorem, and \( V = 0 \) on \( \partial \Omega \), we have

\[
DJ_{K^V}(\omega).V = \int_{\partial \omega} (u_d - u_n) V_n
\]

\[
= \frac{1}{2} \int_{\partial \omega} (u'_d - u'_n) V_n + \frac{1}{2} \int_{\partial \omega} (u'_d - u'_n) V_n \tag{26}
\]

\[
= \frac{1}{2} \int_{\partial \omega} \text{div}((u_d - u_n) V) + \frac{1}{2} \int_{\Omega \setminus \omega} \text{div}((u_d - u_n) V).
\]

Using Hadamard’s formula (see [28], Theorem 5.2.2), we get

\[
D^2 J_{K^V}(\omega)[V, V] = \frac{1}{2} \int_{\partial \omega} \text{div}((u'_d - u'_n) V) + \frac{1}{2} \int_{\Omega \setminus \omega} \text{div}((u'_d - u'_n) V)
\]

\[
+ \frac{1}{2} \int_{\partial \omega} \text{div}(u'_d - u'_n) V) V_n + \frac{1}{2} \int_{\partial \omega} \text{div}(u'_d - u'_n) V)V_n.
\]

Using again the divergence theorem, we obtain

\[
D^2 J_{K^V}(\omega)[V, V] = \frac{1}{2} \int_{\partial \omega} ((u'_d - u'_n) V) V_n + \frac{1}{2} \int_{\partial \omega} (u'_d - u'_n) V_n
\]

\[
+ \frac{1}{2} \int_{\partial \omega} \text{div}(u'_d - u'_n) V) V_n + \frac{1}{2} \int_{\partial \omega} \text{div}(u'_d - u'_n) V)V_n.
\]

We consider an admissible shape \( \omega^* \in \Omega_{d_0} \), solution of the inverse problem (2). Then, \( \omega^* \) realizes the minimum of the cost function \( J_{K^V}, J_{K^V}(\omega^*) = 0 \) which
implies that $u_d = u_n$, therefore
\[
D^2 J_{KV}(\omega^*)[V,V] = \frac{1}{2} \int_{\partial \omega^*} (u'_d - u'_n) V_n + \frac{1}{2} \int_{\partial \omega^*} (u'^+_d - u'^+_n) V_n
\]
\[
= \int_{\partial \omega^*} (u'_d - u'_n) V_n,
\]
since $[u'_d] = 0$ and $[u'_n] = 0$ on $\partial \omega^*$. \qed

5. Instability of the problem. To prove the instability result of the inverse problem (3), we adopt the method already used in [8, 13, 15, 14]. Thus, a local regularity argument is used to show the compactness of the Riesz operator corresponding to the shape Hessian at a solution $\omega^* \in \Omega_0$ of the inverse problem. An alternative proof using the potential layers is developed in [22, 2, 4]. Here, we investigate the properties of stability of the cost functions. We will study the $J_{LS}$ cost function but we can use the same techniques for $J_{KV}$ cost function. We assume that there exists an admissible inclusion $\omega^*$ such that $J_{LS}(\omega^*) = 0$. It realizes the absolute minimum of the criterion $J_{LS}$. This is satisfied by solution of the inverse problem. Then, Euler’s equation $DJ_{LS}(\omega^*).V = 0$ holds, and we prove in (24) that
\[
D^2 J_{LS}(\omega^*)[V,V] = \int_{\partial \Omega} (u'_n)^2. \tag{27}
\]
Moreover, if $V_n \neq 0$, then $D^2 J_{LS}(\omega^*)[V,V] > 0$ holds. Nevertheless, (27) does not mean that the minimization problem is well-posed. In fact, the following proposition explains the instability of standard minimization algorithms.

**Proposition 3.** (Compactness at the critical shape) If $\omega^*$ is the critical shape of $J_{LS}$ and if $u_n = f$, then the Riesz operator associated to the quadratic shape Hessian
\[
D^2 J_{LS}(\omega^*) : H^{1/2}(\partial \omega^*) \rightarrow H^{-1/2}(\partial \omega^*)
\]
is compact.

The above proposition highlights the instability of the optimization problem (9). This compactness result implies that, in a neighborhood of $\omega^*$, the functional $J_{LS}$ acts as its second-order approximation, indeed, it cannot be estimated in the form $Ct \leq \sqrt{J_{LS}(\omega_t)}$ for small $t$, with a constant $C$ uniform in $V$. Thus, the gradient does not have a uniform sensitivity with respect to the deformation directions, hence, $J_{LS}$ is degenerate for highly oscillating perturbations (see, for example, [8] and [13]).

**Proof.** The proof is based on writing the shape Hessian as a composition of some linear continuous operators whose one is compact. The compactness is obtained using the compact embeddings of Sobolev Spaces.

We consider the formula of the characterization of the shape Hessian given in Proposition 1
\[
D^2 J_{LS}(\omega^*)[V,V] = \int_{\partial \omega^*} (p' - u'_n)V_n.
\]
To simplify the writing, we set $w = p' - u'_n$, therefore, $w$ satisfies the following equation
\[
\begin{cases}
-\Delta w + \alpha w = 0 & \text{in } \Omega, \\
\partial_n w = u'_n & \text{on } \partial \Omega,
\end{cases}
\]
If $⟨.,.⟩$ is the product of duality $H^{1/2}(\partial \omega^*) \times H^{-1/2}(\partial \omega^*)$, the Hessian is given by
\[ D^2 J_{LS}(\omega^*)[V,V] = \langle V_n, (p' - u'_n) \rangle. \]

Then, one introduces (spaces related to $\partial \omega^*$)
\[ T : H^{1/2} \rightarrow H^{1/2}, \quad M : H^{-1/2} \rightarrow H^{1/2} \]

\[ V \mapsto V_n \quad V \mapsto w \]

So, the Hessian can be written as
\[ D^2 J_{LS}(\omega^*)[V,V] = \langle T(V), M(V) \rangle. \]

The operator $T$ is clearly linear continuous [36], but the operator $M$ is compact. Indeed, according to the characterization of $w$, we decompose $M$ into $M = M_2 o M_1$ with
\[ M_1 : H^{1/2}(\partial \omega^*) \rightarrow H^{1/2}(\partial \Omega), \quad M_2 : H^{-1/2}(\partial \omega^*) \rightarrow H^{-1/2}(\partial \omega^*) \]

and $\phi$ is the trace on $\partial \omega^*$ of $\Phi$ solution of
\[ \begin{cases} -\Delta \Phi + \alpha \Phi = 0 & \text{in } \Omega, \\ \partial_n \Phi = \psi & \text{on } \partial \Omega, \end{cases} \quad (28) \]

$M_1$ is linear continuous but $M_2$ is compact. In order to prove this last property, we decompose $M_2$ as $M_2 = M_{2,3} o M_{2,2} o M_{2,1}$ with
\[ M_{2,1} : H^{1/2}(\partial \Omega) \rightarrow H^2(\Omega_{d_0}), \quad \psi \mapsto \Phi \]
\[ M_{2,2} : H^2(\Omega_{d_0}) \rightarrow H^{3/2}(\partial \omega^*), \quad \Phi \mapsto \phi \]
\[ M_{2,3} : H^{3/2}(\partial \omega^*) \rightarrow H^{-1/2}(\partial \omega^*), \quad \phi \mapsto \phi \]

The operators $M_{2,1}$ and $M_{2,2}$ are then linear continuous and the operator $M_{2,3}$ is the compact embedding of $H^{3/2}(\partial \omega^*)$ into $H^{-1/2}(\partial \omega^*)$. Hence, we obtain the compactness result.

Note that the regularity $H^2(\Omega_{d_0})$ is due to a local regularity argument (as the one used in [8, 13, 15, 14]), since the object $\omega^*$ has a $C^2$ boundary and since the condition on $\partial \omega^*$ is homogeneous (and then smooth), the solution of problem (28) is globally $H^1(\Omega)$, but locally $H^2(\Omega_{d_0})$.

In the same way, we have the following result corresponding to the Kohn-Vogelius cost function.

**Proposition 4.** (Compactness at a critical shape) If $\omega^*$ is the critical shape of $J_{KV}$ and if $u_n = u_d$, then the Riesz operator associated to the quadratic shape Hessian
\[ D^2 J_{KV}(\omega^*) : H^{1/2}(\partial \omega^*) \rightarrow H^{-1/2}(\partial \omega^*) \]
is compact.

**Remark 1.** The instability study of our shape optimization problem is very important and prove that this inverse geometric source problem is unstable in the following sens: the functionals $J_{LS}$ and $J_{KV}$ are degenerate for the highly oscillating perturbations (frequencies) (see [2, 14], Numerical Section). Hence, the choice of low frequency in measurements data is a regularization way of the cost functionals considered.
6. **Algorithm and numerical results.** In this section, we present some numerical simulations in order to confirm and complete our previous theoretical results with the comparison between the least squares $J_{LS}$ and Kohn-Vogelius $J_{KV}$ cost functions. In order to solve numerically the optimization problems (4) and (6), we opt for the classical shape variation descent algorithm without any regularization method. First of all, we describe the algorithm and the framework used, then, we present the numerical simulations and some comparisons.

6.1. **Algorithm.** The shape derivative of the cost function $J$ along a deformation field $V$ can be expressed as

$$\nabla J(\omega)[V] = \int_{\partial\omega}RV_n\,d\sigma,$$

where $R = p$ for $J_{LS}$ and $R = (u_d - u_n)$ for $J_{KV}$ with $p$ is solution of the adjoint problem (16) and $u_d, u_n$ are respectively solutions of the direct problems (7) and (5).

The deformation field $V$ is chosen to provide a descent direction of the cost function $J(\omega)$, thus $V = -R_n$ on $\partial\omega$ is a descent direction. In addition, it is well known that the shape gradient is defined on the boundary of the moving shape [19], using this approach, the direction of descent must be defined only on $\partial\omega$. However, if the boundary measurements $(f, g)$ is not sufficiently smooth, the surface expression of the shape gradient may not exist or the direction of descent $V$ may have a poor regularity. Therefore, it is interesting to derive a direction of descent $V$ on $\Omega$ from the volumetric expression of the shape gradient. Which requires solving another additional variational problem. Let $V$ be the Riesz representative of $-\nabla J(\omega)$, i.e. (see [7] and [24])

$$< V, \phi >_{H^1(\Omega)} = -\nabla J(\omega)[\phi] = -< R_n, \phi >_{L^2(\partial\omega)} \quad \forall \phi \in D^2,$$

where

$$D = \left\{ \phi \in H^1(\Omega), \phi = 0 \text{ in } \partial\Omega \right\},$$

and $<,>$ is the inner product on $D^2$ defined by

$$< V, \phi >_{H^1(\Omega)} = \int_{\Omega} \nabla V : \nabla \phi + V.\phi.$$

The equation (29) is the week formulation for the following system

$$\begin{cases}
-\Delta V + V &= 0 \quad \text{in } \Omega, \\
V &= 0 \quad \text{on } \partial\Omega, \\
[\partial_n V] &= -R_n \quad \text{on } \partial\omega.
\end{cases}$$

We give in the following algorithm the gradient method of our problem

**Algorithm 1** Gradient algorithm for shape optimization

1: Choose an initial shape $\omega_0$, set $k = 0$ and iterate:
2: Solve the state problem (5) and the adjoint problem (16) for $J_{LS}$ (or Dirichlet and Neumann problems (7) and (5) for $J_{KV}$), for $\omega = \omega_k$.
3: Compute the descent direction $V_k$ using (30).
4: Update the current boundary $\partial\omega_k$ by $V_k$ to obtain $\partial\omega_{k+1}$, i.e., set $\partial\omega_{k+1} := \{ x + t_kV_k(x) : x \in \partial\omega_k \}$, for some sufficiently small scalar $t_k > 0$.
5: While $\|\nabla J(\omega_k).V_k\| \geq \epsilon$, $k = k + 1$ and repeat.
6.2. Numerical results. For the numerical simulations, we consider the dimension two and we use the finite elements Software Freefem++ (see [27]). The exterior boundary \( \partial \Omega \) is assumed to be the unit circle or the square \([-1,1] \times [-1,1]\). We construct the synthetic data on \( \partial \Omega \), by fixing the shape \( \omega \) and choosing the Neumann boundary condition \( g(t) = \sin(t) \), \( t \in [0, 2\pi] \), then, we compute the trace of state \( u \) solution of (5) to extract the measurement \( f = u \) on \( \partial \Omega \). For the latter equation, we use a \( P_2 \) finite elements discretization to solve the direct problem. The examples with noisy data are generated by perturbing the Dirichlet data \( f \) using a fixed amplitude of Gaussian noise. The synthetic data has been chosen, in a way to avoid the so-called inverse crime phenomena. To this end, the size of discretization used to obtain the synthetic data is different from the one used for solving the inverse problem. In order to change the shape of the objects at each iteration (step 3 in the previous algorithm), we use the function \textit{movemesh} of Freefem++. The \textit{movemesh} applies a global diffeomorphism to the mesh. To avoid degeneracy of the triangles in the meshes we use the function \textit{adaptmesh} to refine at each step (see the tutorial of Freefem++ [27]).

As it is well known, the stopping criteria have a primordial role in iterative algorithms. However, in our case, it was not simple to find a good criteria for both synthetic and measured test data. Actually, based on interactive monitoring of the iterative processes, we stop the iteration when the change in the reconstruction obtained from an iteration step was no longer noticeable.

6.2.1. Results without noise. The boundary \( \partial \Omega \) is discretized by \( N_e := 100 \) discretization points and the interior boundary \( \partial \omega \) by \( N_i := 70 \) points and we try to identify the following shapes, starting from the simple shape to the complex one. We precise that in all tests, the exterior boundary is represented by the black line, the initial shape by the green line, the exact shape to identify by the blue line and the reconstructed shape by the dotted red line.

**First domain:** We assume the exterior boundary \( \partial \Omega \) to be the unit circle, and we consider the domain defined by

\[
\partial \omega = \left\{ \begin{pmatrix} 0.5 \cos(t) \\ 0.5 \sin(t) \end{pmatrix}, \ t \in [0, 2\pi] \right\},
\]

we present in Figure 1, the reconstruction result by two cost functionals, in the left, the identification by the Kohn-Vogelius cost function and in the right, the result obtained by the least squares. we notice that the results obtained by the two methods are effective and are similar.

In Figure 2, we plot the variation of the cost functionals according to the number of iterations and the evolution of their associated gradients also. The Kohn-Vogelius cost function converges quicker than the least squares, but at the end of the convergence, the two functionals behave in the same way.

**Second domain:** The exterior boundary \( \partial \Omega \) is assumed to be the unit circle, and the interior boundary is defined by

\[
\partial \omega = \left\{ \begin{pmatrix} (0.5 + 0.1 \sin(2t)) \cos(t) \\ (0.5 + 0.1 \sin(2t)) \sin(t) \end{pmatrix}, \ t \in [0, 2\pi] \right\},
\]

In Figure 3, we observe that the result obtained by the Kohn-Vogelius is more robust than that obtained by the least squares. In Figure 4, the convergence of the Kohn-Vogelius dominates that of the least squares.
Third domain: We represent again the exterior boundary $\partial \Omega$ by the unit circle, and we consider the domain defined by

$$\partial \omega = \left\{ \begin{array}{c} 1.2 \times (0.4 + 0.05 \times \cos(3t)) \times \cos(t) \\ 1.2 \times (0.4 + 0.05 \times \cos(3t)) \times \sin(t) \end{array} \right\}, \quad t \in [0, 2\pi].$$

In Figure 5, we even observe that, in the case of a more complex shape, the Kohn-Vogelius method is more robust than the least squares.

Fourth domain: This experiment was done considering the exterior boundary as the square $[-1, 1] \times [-1, 1]$, and the source support boundary is centered at $(0.3, 0.2)$ with the same parametrization as First domain. The reconstructions were again very efficient, as shown in Figure 6.

Fifth domain: In this experiment, the exterior boundary was again considered as the square $[-1, 1] \times [-1, 1]$, and the source support was considered the same as Second domain. The reconstructions obtained are shown in Figure 7. It can be observed that the results are very close to the exact shape.
6.2.2. Results with noise 2\%. In this subsection, we present the results obtained from the noisy data as follows

\[
\tilde{f}(x) = f(x)(1 + \tau \xi),
\]

where \(\xi\) is a uniformly distributed random variable in \([-1, -1]\) and \(\tau\) dictates the level of noise.

We present in Figures 8, 9, 11 and 10, the results with the noise. We observe that for simple geometries the two functionals behave in the same way, but in the case of more complex configuration the Kohn-Vogelius functional gives better results than the least squares.

7. Conclusion. The geometric inverse source problem is studied by the shape optimization approach, through the minimization of a cost functional, where the unknown is a non-parametric shape. Two cost functionals are considered, the least squares fitting, and the Kohn-Vogelius energy gap. Using the shape derivative approach, we show in a rigorous way the existence of the states derivatives, and we
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Figure 5. Reconstruction of the third domain by Kohn-Vogelius and least squares methods.

Figure 6. Reconstruction of the fourth domain by Kohn-Vogelius and least squares methods.

Figure 7. Reconstruction of the fifth domain by Kohn-Vogelius and least squares methods.
characterize the shape gradient of the cost functions in order to make a numerical resolution based on the descent method. For the stability study, the second derivative of the cost functions at the critical shape is investigated, which allows us to establish the compactness of the Riesz operator corresponding to the shape Hessian, and deduce the ill-posedness of the associated identification problem. Therefore, the choice of low frequency in measurements data is a regularization way of the cost functionals considered. Finally, the obtained results confirm that the Kohn-Vogelius method is more robust than the least squares fitting.

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FIGURE 10. Reconstruction of the third domain by Kohn-Vogelius and least squares methods with noise.

FIGURE 11. Reconstruction of the fourth domain by Kohn-Vogelius and least squares methods with noise.

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E-mail address: lekbir.afraites@gmail.com
E-mail address: chorouk.mas@gmail.com
E-mail address: nachaoui@gmail.com