RECOLLEMENTS FROM GENERALIZED TILTING

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Abstract. Let \( A \) be a small dg category over a field \( k \) and let \( \mathcal{U} \) be a small full subcategory of the derived category \( \mathcal{D} A \) which generate all free dg \( A \)-modules. Let \((\mathcal{B}, X)\) be a standard lift of \( \mathcal{U} \). We show that there is a recollement such that its middle term is \( \mathcal{D} \mathcal{B} \), its right term is \( \mathcal{D} A \), and the three functors on its right side are constructed from \( X \). This applies to the pair \((A, T)\), where \( A \) is a \( k \)-algebra and \( T \) is a good \( n \)-tilting module, and we obtain a result of Bazzoni–Mantese–Tonolo. This also applies to the pair \((A, \mathcal{U})\), where \( A \) is an augmented dg category and \( \mathcal{U} \) is the category of ‘simple’ modules, e.g. \( A \) is a finite-dimensional algebra or the Kontsevich–Soibelman \( A_\infty \)-category associated to a quiver with potential.

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A recollement of triangulated categories is a diagram of triangulated categories and triangle functors

\[
\begin{array}{c}
\mathcal{T}'' \\
\downarrow_{i^*} \\
\mathcal{T} \\
\downarrow_{i_*} \\
\mathcal{T}' \\
\downarrow_{j_*} \\
\mathcal{T}'' \\
\end{array}
\]

such that

- \((i^*, i_*, i!)\) and \((j!, j^*, j_*)\) are adjoint triples;
- \(i_*, j_*, i!\) are fully faithful;
- \(j^* \circ i_* = 0\);
- for every object \( X \) of \( \mathcal{T} \) there are two triangles

\[
\begin{array}{c}
i_! i^* \\
\downarrow \\
X
\end{array} \longrightarrow
\begin{array}{c}
X
\end{array} \longrightarrow
\begin{array}{c}
j_* j^* X
\end{array} \longrightarrow
\begin{array}{c}
X
\end{array} \longrightarrow
\begin{array}{c}
j^* j_* X
\end{array} \longrightarrow
\begin{array}{c}
X
\end{array} \longrightarrow
\begin{array}{c}
i_! i^* X
\end{array}
\end{array}
\]

where the four morphisms are the units and counits.

We also say that this is a recollement of \( \mathcal{T} \) in terms of \( \mathcal{T}' \) and \( \mathcal{T}'' \). This notion was introduced by Beilinson–Bernstein–Deligne in [4] in geometric contexts, where stratifications of varieties induce recollements of derived categories of sheaves.

In algebraic contexts, recollements are closely related to tilting theory. Let \( A \) be a ring. Let \( \mathcal{D}(A) = \mathcal{D}(\text{Mod} A) \) denote the derived category of (right) \( A \)-modules, and \( \text{per} A \) denote the triangulated subcategory of \( \mathcal{D}(A) \) generated by the free module of rank 1. An object \( T \) of \( \text{per} A \) is called a partial tilting complex if \( \text{Hom}_{\mathcal{D}(A)}(T, \Sigma^n T) = 0 \), and a tilting complex if in
addition $\text{tria}(T) = \text{per} \ A$, where $\text{tria}(T)$ is the triangulated subcategory of $\mathcal{D}(A)$ generated by $T$. Rickard’s Morita theorem for derived categories states that the standard functors associated to a tilting complex $T$ over $A$ are triangle equivalences between $\mathcal{D}(A)$ and $\mathcal{D}(\text{End}_{\mathcal{D}(A)}(T))$, see [18]. Later in [12], Koenig proved that under certain conditions a partial tilting complex $T$ over $A$ yields a recollement of $\mathcal{D}(A)$ in terms of $\mathcal{D}(\text{End}_{\mathcal{D}(A)}(T))$ and a third derived category which measures how far the associated standard functors are from being equivalences (see also [8] [15]). In this sense, a recollement of derived categories can be viewed as a natural generalization of a derived equivalence. The relation between tilting theory and recollements of derived categories has been further studied in [1] [6]. The dg version of Rickard’s theorem was developed by Keller in [10], and the result of Koenig was generalized to the dg setting by Jørgensen [9] and Nicolás–Saorín [17], where the role of partial tilting complexes is played by compact objects.

In this paper we deal with a situation which is ‘dual’ to the one in [12] [9] [17]. Starting from a dg category $\mathcal{A}$ and a set of objects in the derived category $\mathcal{D}(\mathcal{A})$ which generates all the compact objects, we construct a dg category $\mathcal{B}$ together with a recollement of $\mathcal{D}(\mathcal{B})$ in terms of $\mathcal{D}(\mathcal{A})$ and another derived category, see Theorem 1. We identify this third derived category with a certain known category in the special case when $\mathcal{A}$ is the Kontsevich–Soibelman $A_{\infty}$-category associated to a quiver with potential (Corollary 3) or when $\mathcal{A}$ is a finite-dimensional self-injective algebra (Corollary 4). The motivation for our study was to have a better understanding of the ‘exterior’ case of the Koszul duality (Corollary 2) and a result of Bazzoni–Mantese–Tonolo which says that the right derived Hom-functor associated to an (infinitely generated) good tilting module is fully faithful (Corollary 1).

1. The main result

Let $k$ be a field and let $\mathcal{A}$ be a small dg $k$-category. Denote by $\text{Dif} \ A$ the dg category of (right) dg $\mathcal{A}$-modules. A dg $\mathcal{A}$-module $M$ is $\mathcal{K}$-projective if the dg functor $\text{Dif} \ A(M, ?)$ preserves acyclicity. For example, the free modules $A^\wedge = \text{Dif} \ A(?, A), A \in \mathcal{A}$, are $\mathcal{K}$-projective. Let $\mathcal{D}(\mathcal{A})$ denote the derived category of $\mathcal{A}$, which is triangulated with suspension functor $\Sigma$ the shift functor. For a set of objects or a subcategory $\mathcal{S}$ of $\mathcal{D}(\mathcal{A})$ we denote by $\text{tria} \mathcal{S}$ the smallest triangulated subcategory of $\mathcal{D}(\mathcal{A})$ containing all objects in $\mathcal{S}$ and closed under taking direct summands. Let $\text{per} \ A = \text{tria} (A^\wedge, A \in \mathcal{A})$. An object $M$ of $\mathcal{D}(\mathcal{A})$ is compact if the functor $\mathcal{D}(\mathcal{A})(M, ?)$ commutes with infinite (set-indexed) direct sums, or equivalently, if $M$ belongs to $\text{per} \ A$. See [10].

Let $\mathcal{U}$ be a full small subcategory of $\mathcal{D}(\mathcal{A})$ such that

$$\text{tria} \mathcal{U} \supseteq \text{per} \ A.$$

(1)
Let \((B, X)\) be a standard lift of \(U\) (\cite{10} Section 7). Precisely, \(B\) is a dg subcategory of \(\text{Dif} \ A\) consisting of \(K\)-projective resolutions over \(A\) of objects of \(U\) (to avoid confusion, for each object \(B\) of \(B\) we will denote by \(U_B\) the corresponding dg \(A\)-module) and \(X\) is the dg \(B^{op} \otimes A\)-module defined by \(X(B, A) = U_B(A)\). It induces a pair of adjoint dg functors and a pair of adjoint triangle functors

\[
\begin{array}{ccc}
\text{Dif} \ B & \overset{TX}{\longrightarrow} & \text{Dif} \ A, \\
\downarrow H_X & & \downarrow R_H X \\
\text{DB} & \overset{LT X}{\longrightarrow} & \text{DA}.
\end{array}
\]

When \(A\) and \(B\) are dg \(k\)-algebras (i.e. dg \(k\)-categories with one object), the functors \(LT X\) and \(R_H X\) are usually written as \(L \otimes X\) and \(R \text{Hom}(X, ?)\).

Let \(X^T\) be the dg \(A^{op} \otimes B\)-module defined by

\[
X^T(A, B) = \text{Dif} \ A(X^B, A^\vee),
\]

where for \(B \in B\), \(X^B\) is by definition the dg \(A\)-module \(X(B, ?)\). From the definition of \(X\) we see that \(X^B = U_B\). The main result of this paper is

**Theorem 1.** Assume notations as above. There is a dg \(k\)-category \(C\) and a recollement of triangulated categories

\[
\begin{array}{ccc}
\mathcal{D}C & \overset{i^*}{\longrightarrow} & \mathcal{D}B \\
\downarrow i_* & & \downarrow j^* \\
\mathcal{D}A & \overset{j_!}{\longrightarrow} & \mathcal{D}A,
\end{array}
\]

where the adjoint triple \((i^*, i_*, i^!\)) is defined by a dg functor \(F : B \rightarrow C\) (which is bijective on objects) such that \(i_* = F^* : \mathcal{D}C \rightarrow \mathcal{D}B\) is the pull-back functor, and the adjoint triple \((j_!, j^*, j_*\)) is given by

\[
\begin{align*}
\hat{j}! &= LT X^T, \\
\hat{j}^* &= RH X^T \simeq LT X, \\
\hat{j}_* &= RH X.
\end{align*}
\]

**Proof.** In view of \cite{17} Theorem 5, it suffices to prove

(a) \(LT X^T\) is fully faithful,

(b) \(RH X^T \simeq LT X\).

The proof for (a) is the same as the proof of \cite{10} Lemma 10.5 the ‘exterior’ case c)]. Since \((B, X)\) is a lift, the restriction of \(LT X\) on the perfect derived category \(\text{per} B\) is fully faithful, and its essential image is \(\text{tria}\ U\) (see \cite{10} Section 7.3):

\[
LT X|_{\text{per} B} : \text{per} B \sim \text{tria}\ U.
\]
It is clear that $\mathbf{R}H_X$ takes an object of $\text{tria} \mathcal{U}$ into $\text{per} \ B$. Therefore, the restriction $\mathbf{R}H_X|_{\text{tria} \mathcal{U}}$ is a quasi-inverse of $\mathbf{L}T_X|_{\text{per} \ B}$, and hence is fully faithful. It follows from [10, Lemma 6.2 a)] that the restriction $\mathbf{L}T_X|_{\text{per} \ A}$ is naturally isomorphic to the restriction of $\mathbf{R}H_X|_{\text{per} \ A}$, which is fully faithful by condition (i). Condition (i) also implies that $\mathbf{R}H_X(A^\wedge) = (X^T)^A \ (A \in A)$ belongs to $\text{per} \ B$, and hence is compact by [10, Theorem 5.3]. Now applying [10, Lemma 4.2 b)], we obtain that $\mathbf{L}T_X$ is fully faithful, finishing the proof of (a).

Let $Y \to X^T$ be a $\mathcal{K}$-projective resolution of $\text{dg} \ A^{\text{op}} \otimes B$-modules. Then the specialization $Y^A \to (X^T)^A$ is a $\mathcal{K}$-projective resolution of $\text{dg} \ B$-modules for any object $A$ of $\mathcal{A}$. Recall that $(X^T)^A$ is compact. It follows from [10, Lemma 6.2 a)] that $\mathbf{L}T_Y \simeq \mathbf{R}H_Y$. By [10, Lemma 6.1 b)], in order to prove $\mathbf{R}H_X \simeq \mathbf{L}T_X$, it suffices to prove that as $\text{dg} \ B^{\text{op}} \otimes A$-modules $Y^T$ and $X$ are quasi-isomorphic. Let $A \in A$ and $B \in B$. We have $H_X(U_B) = B^\wedge$, and hence

$$Y^T(A, B) = \text{Dif } B(Y^A, B^\wedge)$$

$$= \text{Dif } B(Y^A, H_X(U_B))$$

$$\cong \text{Dif } A(T_X(Y^A), U_B).$$

The composition $T_X(Y^A) \to T_X((X^T)^A) = T_X \circ H_X(A^\wedge) \to A^\wedge$ is exactly the counit $\mathbf{L}T_X \circ \mathbf{R}H_X(A^\wedge) \to A^\wedge$, which is an isomorphism in $\mathcal{D}A$ because the restriction of $\mathbf{R}H_X$ on $\text{per} \ A$ is fully faithful. Moreover, both $T_X(Y^A)$ and $A^\wedge$ are $\mathcal{K}$-projective $\text{dg} \ A$-modules. Therefore we have

$$Y^T(A, B) \cong \text{Dif } A(A^\wedge, U_B)$$

$$= U_B(A)$$

$$= X(A, B).$$

Further, every morphism in the above is functorial in both $A$ and $B$. This completes the proof of (b).

\[ \sqrt{ } \]

**Corollary 1** ([3]). Let $A$ be a $k$-algebra, and $n$ be a positive integer. Let $T$ be a good $n$-tilting module, i.e. $T$ is an $A$-module such that

(T1) the projective dimension of $T$ is less than or equal to $n$;

(T2) $\text{Ext}_A^i(T, T^{(\alpha)}) = 0$ for any integer $i > 0$ and for any cardinal $\alpha$;

(T3) there is an exact sequence

$$0 \to A \xrightarrow{T} T^0 \xrightarrow{T} T^1 \xrightarrow{\cdots} T^n \to 0,$$

where $T^0, \ldots, T^n$ are direct summands of direct sums of finite copies of $T$. 


Put $B = \operatorname{End}_A(T)$. Then the right derived functor $\mathcal{R} \operatorname{Hom}(T, ?) : \mathcal{D}(A) \to \mathcal{D}(B)$ is fully faithful, and $\mathcal{D}(A)$ is triangle equivalent to the triangle quotient of $\mathcal{D}(B)$ by the kernel of the left derived functor $? \otimes_B T$.

**Proof.** Let $\mathcal{U}$ be the full subcategory of $\mathcal{D}(A)$ consisting of one object $T$. Then the condition (T3) implies the condition (1). Let $X$ be a projective resolution of $T$ over $B^{\text{op}} \otimes_k A$, and let $\tilde{B}$ be the dg $k$-algebra $\operatorname{Dif} A(X, X)$. Then $X$ is $K$-projective over $A$, and $(\tilde{B}, X)$ is a standard lift of $T$. Thanks to (T2), the representation map $B \to \tilde{B}$ of the dg $B$-$A$-bimodule $X$ is a quasi-isomorphism, inducing mutually quasi-inverse triangle equivalences $? \otimes_B \tilde{B} \cong \mathcal{R} \operatorname{Hom}(\tilde{B}, ?) : \mathcal{D}(\tilde{B}) \to \mathcal{D}(B)$ and $? \otimes_B \tilde{B} : \mathcal{D}(B) \to \mathcal{D}(\tilde{B})$. Now applying Theorem 1 and composing the resulting recollement with the above triangle equivalences, we obtain a recollement

\[
\begin{array}{ccc}
\mathcal{D}(C) & & \mathcal{D}(B) \\
\mathcal{R} \operatorname{Hom}_B(X, ?) \otimes \mathcal{R} \operatorname{Hom}_A(T, ?) & & \mathcal{D}(A)
\end{array}
\]

where $C$ is a dg $k$-algebra. Since $X$ and $T$ are quasi-isomorphic as dg $B^{\text{op}} \otimes_k A$-modules, we have natural isomorphisms $? \otimes_B X \simeq ? \otimes_B T$ and $\mathcal{R} \operatorname{Hom}_A(X, ?) \simeq \mathcal{R} \operatorname{Hom}_A(T, ?)$ ([10, Lemma 6.1 b]). The desired result follows at once.

**Remark.**

a) This result is due to Bazzoni [2] for $n = 1$ and Bazzoni–Mantese–Tonolo [3] for general $n$ for all rings $A$.

b) By Theorem 1, the left half of the recollement in the proof is induced from a dg homomorphism $B \to C$. For the case $n = 1$ and for all rings $A$ Chen–Xi obtained in [6] such a recollement with $C$ being an ordinary ring (so that the map $B \to C$ becomes a homomorphism of rings). They used some results in [1] and many other results such as the homological properties of the tilting module $T$.

To state the next corollary, we need to introduce some notions. Let $\mathcal{A}$ be an augmented dg $k$-category ([10, Section 10.2]), i.e.

- distinct objects of $\mathcal{A}$ are non-isomorphic,
- for each $A \in \mathcal{A}$, a dg module $\hat{A}$ is given such that $H^0 \hat{A}(A) \cong k$ and $H^n \hat{A}(A')$ whenever $n \neq 0$ or $A' \neq A$.

Let $(\mathcal{A}^*, X)$ be a standard lift of $\mathcal{U} = \{ \hat{A} | A \in \mathcal{A} \} \subset \mathcal{D} \mathcal{A}$. By abuse of language, we call the dg $k$-category $\mathcal{A}^*$ the Koszul dual of $\mathcal{A}$. Assume that the condition (1) holds, e.g. this happens in the ‘exterior’ case in [10, Section 10.5].
Corollary 2. Assume notations as above. There is a recollement of derived categories of dg \(k\)-categories

\[
\begin{tikzcd}
DC & DA^* & DA \\
\end{tikzcd}
\]

Proof. This is a direct consequence of Theorem 1.

2. The left term

As in the preceding section, we let \(k\) be a field, \(A\) be a small dg \(k\)-category, \(U\) be a full small subcategory of the derived category \(DA\) such that \(\text{tria} U \supseteq \text{per} A\), and let \((B, X)\) be a standard lift of \(U\). Theorem 1 says that there is a recollement of \(DB\) in terms of \(DA\) and a third derived category \(DC\), where \(C\) is a dg \(k\)-category whose objects are in bijection with the objects of \(U\).

Let \(V = \{(X^A)|A \in A\} \subset DB\). From the proof of the theorem we obtain a commutative diagram

\[
\begin{tikzcd}
\text{RH}_X|_{\text{tria} U} & \text{tria} U \ar[r, swap, \sim] & \text{per} B \\
\text{RH}_X|_{\text{per} A} & \text{per} A \ar[u] \ar[r, swap, \sim] & \text{tria} V \ar[u]
\end{tikzcd}
\]

Therefore \(\text{RH}_X\) induces a triangle equivalence between the triangle quotient categories

\[
\begin{tikzcd}
\text{tria} U/\text{per} A \ar[r, swap, \sim] & \text{per} B/\text{tria} V.
\end{tikzcd}
\]

For a triangulated category \(T\), let \(T^c\) denote the subcategory of compact objects in \(T\). Let \(\text{Tria} V\) be the localizing subcategory of \(DB\) generated by the objects in \(V\). We have \((DB)^c = \text{per} B\), and \((\text{Tria} V)^c = \text{tria} V\). Thus by [10 Theorem 2.1], the category \((DB/\text{Tria} V)^c\) is triangle equivalent to the idempotent completion of \(\text{per} B/\text{tria} V\). Since the essential image of \(LT_{X^T}\) is exactly \(\text{Tria} V\), it follows that \(DC\) is triangle equivalent to the triangle quotient \(DB/\text{Tria} V\), and hence is an ‘unbounded version’ of \(\text{tria} U/\text{per} A \cong \text{per} B/\text{tria} V\). Apparently, \(DC\) vanishes if and only if so does \(\text{tria} U/\text{per} A\), in which case \(U\) consists of a set of compact generators for \(DA\).

In the following two special cases, we are able to identify \(DC\) with a certain known category (however, the dg category \(C\) is not easy to describe).

Corollary 3. Let \((Q, W)\) be a quiver with potential. Let \(A_{(Q,W)}\) be the Kontsevich–Soibelman \(A_\infty\)-category ([13 Section 3.3]) (or its enveloping dg category), let \(\hat{\Gamma}_{(Q,W)}\) be the complete
Ginzburg dg category ([7, Section 5]), and let $\tilde{C}_{(Q,W)}$ be the ‘unbounded version’ of the generalized cluster category ([11, Remark 4.1]). Then there is a recollement of triangulated categories

$$\tilde{C}_{(Q,W)} \longrightarrow D\hat{\Gamma}_{(Q,W)} \longrightarrow DA_{(Q,W)}.$$ 

Proof. Let $A = A_{(Q,W)}$ and let $U$ be the category of simple $A$-modules. Then condition (I) holds since $A$ is finite-dimensional, and there is a standard lift $(B, X)$ such that the dg category $\Gamma = \hat{\Gamma}_{(Q,W)}$ (as the Koszul dual of $A_{(Q,W)}$) is quasi-isomorphic to $B$. By Corollary 2, there is a recollement with the middle term being $D\Gamma$, the right term being $DA$, and the right upper functor being $LT_X T$. It remains to prove that the left term of this recollement is triangle equivalent to $\tilde{C}_{(Q,W)}$. Object sets of $A$, of $U$, and of $\Gamma$ can all be identified with the vertice set $Q_0$ of the quiver $Q$. For a vertex $i$ of $Q$, considered as an object of $A$, the right dg $\Gamma$-module $(X^T)^i$ is isomorphic in $D\Gamma$ to $\Sigma^{-3} S_i$, where $S_i$ the simple top of the free $\Gamma$-module $i^\wedge$. Thus the essential image of $LT_X T$ is the localizing subcategory $D_0 \Gamma = \tria(S_i, i \in Q_0)$ of $D\Gamma$ generated by the $S_i$, $i \in Q_0$. Thus the left term of the recollement is triangle equivalent to the triangle quotient $D\Gamma/D_0 \Gamma$, which is by definition $\tilde{C}_{(Q,W)}$. \hfill \checkmark

Let $A$ be a finite-dimensional basic $k$-algebra. Let $S$ be the direct sum of the objects in a set of representatives of isomorphism classes of simple $A$-modules, and let $X$ be a projective resolution of $S$. Then $A^* = \text{Diff}_A(X, X)$ is the Koszul dual of $A$.

**Corollary 4 ([14]).** Let $A$ be a finite-dimensional basic self-injective $k$-algebra, and $A^*$ its Koszul dual. Let $\text{Mod} A$ be the stable category of the category $\text{Mod} A$ of $A$-modules. Then there is a recollement of triangulated categories

$$\text{Mod} A \longrightarrow D(A^*) \longrightarrow D(A).$$

Proof. Let $\text{mod} A$ be the category of finite-dimensional $A$-modules, and $\text{mod} A$ its stable category. As a triangulated subcategory of $D(A)$, the bounded derived category $D^b(\text{mod} A)$ of $\text{mod} A$ coincides with $\tria S$. Recall that the essential image of $\otimes_A X^T$ is $\tria X^T$. Consider
the following commutative diagram

\[
\begin{array}{cccccc}
\text{Mod} A & \xrightarrow{\mathcal{R}\text{Hom}_A(X,\cdot)} & \mathcal{D}(A^*) & \xrightarrow{\mathcal{D}(A^*)/\text{Tria }X^T} \\
\text{mod } A & \xrightarrow{\mathcal{D}^b(\text{mod } A)} & \text{per } A^* & \xrightarrow{\text{per } A^*/\text{tria }X^T} \\
\text{per } A & \xrightarrow{\sim} & \text{tria }X^T
\end{array}
\]

where the leftmost horizontal functors are the canonical embeddings, and the rightmost horizontal functors are the canonical projections. The restriction of $\mathcal{R}\text{Hom}_A(X,\cdot)$ on $\text{Mod } A$ commutes with infinite direct sums, because $X$ can be chosen such that its component in each degree is a finitely generated projective $A$-module. Therefore the composition of the three functors in the first row, denoted by $F$, commutes with infinite direct sums. Since $\mathcal{R}\text{Hom}_A(X, A) \cong X^T$ belongs to $\text{Tria }X^T$, it follows that $F$ factors through the stable category $\text{Mod } A$. In this way, we obtain a triangle functor

\[\bar{F} : \text{Mod } A \to \mathcal{D}(A^*)/\text{Tria }X^T,\]

which commutes with infinite direct sums. It is known that $\text{Mod } A$ is compactly generated by $\text{mod } A$ and $(\text{Mod } A)^c = \text{mod } A$. Moreover, the restriction $\bar{F}|_{\text{mod } A}$ is the composition of the following three functors

\[\text{mod } A \xrightarrow{\sim} \mathcal{D}^b(\text{mod } A)/\text{per } A \xrightarrow{\sim} \text{per } A^*/\text{tria }X^T \xrightarrow{\sim} \mathcal{D}(A^*)/\text{Tria }X^T.\]

The first functor is also an equivalence ([19, Theorem 2.1]). Therefore $\bar{F}$ induces a triangle equivalence between $\text{mod } A = (\text{Mod } A)^c$ and $\text{per } A^*/\text{tria }X^T = (\mathcal{D}(A^*)/\text{Tria }X^T)^c$. By [10, Lemma 4.2], $\bar{F}$ itself is an equivalence. Now applying Corollary [2] we obtain the desired recollement.

**Remark.** Let $\mathcal{H}(\text{Inj } A)$ be the homotopy category of injective $A$-modules and $\mathcal{H}_{ac}(\text{Inj } A)$ be its full subcategory of acyclic complexes. Applying a result of Krause [14, Corollary 4.3] to the algebra $A$, we obtain a recollement of $\mathcal{H}(\text{Inj } A)$ in terms of $\mathcal{D}(A)$ and $\mathcal{H}_{ac}(\text{Inj } A)$ with the right middle functor being the canonical projection $Q : \mathcal{H}(\text{Inj } A) \to \mathcal{D}(A)$. We claim that this recollement is equivalent to the one in Corollary [4]. Indeed, Krause proved in [14] that $\mathcal{H}(\text{Inj } A)$ is compactly generated by (an injective resolution of) the $A$-module $S$, and that there is a triangle equivalence $\Theta : \mathcal{H}(\text{Inj } A) \to \mathcal{D}(A^*)$ taking $S$ to $A^*$. Since both $\Theta(?) \otimes_{A^*} X$ and $Q$ commute with infinite direct sums and $\Theta(S) \otimes_{A^*} X \cong X \cong S$, it follows that they are isomorphic. Namely, the right middle parts of the two recolllements are equivalent via the equivalence $\Theta$. Therefore the two recolllements are equivalent.
Now let us construct the equivalence $\Theta$ by sketching the proof of the assertion that $\mathcal{H}(\text{Inj}_A)$ and $\mathcal{D}(A^*)$ are triangle equivalent. Let $\text{Dif}_{\text{Inj}}A$ be the full dg subcategory of $\text{Diff} A$ consisting of complexes of injective $A$-modules, let $iS$ be an injective resolution of the $A$-module $S$, and put $B = \text{Dif} A(iS, iS)$. Then the dg $B^{op} \otimes A^*$-module $\text{Dif} A(X, iS)$ yields a triangle equivalence $\Phi : \mathcal{D}(B) \to \mathcal{D}(A^*)$ (see [10, Section 7.3]). Moreover, $\text{Dif}_{\text{Inj}} A$ is a dg enhancement of the triangulated category $\mathcal{H}(\text{Inj}_A)$ in the sense of Bondal–Kapranov [5], and there is a dg functor $\text{Dif}_{\text{Inj}} A(iS, ?) : \text{Dif}_{\text{Inj}} A \to \text{Dif} B$. Taking zeroth comhomologies gives us a triangle functor $\mathcal{H}(\text{Inj}_A) \to \mathcal{H}(B)$, and composing it with the canonical projection $\mathcal{H}(B) \to \mathcal{D}(B)$ we obtain a triangle equivalence $\Psi : \mathcal{H}(\text{Inj}_A) \to \mathcal{D}(B)$ (cf. the proof of [10, Theorem 4.3]). Now the composition $\Theta = \Phi \circ \Psi : \mathcal{H}(\text{Inj}_A) \to \mathcal{D}(A^*)$ is a triangle equivalence, which takes $iS$ to $A^*$ (up to isomorphism), as desired.

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