1. Introduction

At Chiral Dynamics 2009, I presented a talk on *Kaons on the Lattice* in which the main topics were:

1. Chiral Behaviour
2. \( V_{us} \) from \( K_{\ell 2} \) and \( K_{\ell 3} \) Decays
3. The \( B_K \) parameter of neutral kaon mixing
4. \( K \rightarrow \pi\pi \) Decays

Recent progress in lattice simulations \( \Rightarrow \) quantities such as \( V_{us} \) and \( B_K \) are known with impressive precision, e.g. using the experimental leptonic and semileptonic widths

\[
|V_{us}| = 0.2254(9) \quad \text{and} \quad |V_{ud}| = 0.97427(21) \quad \text{within the Standard Model}
\]

while using also \( |V_{ud}| = 0.97425(22) \)

\[
|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1.0000(7) \quad \text{using lattice semileptonic form factor } f_+(0)
\]

\[
= 0.9999(6) \quad \text{using lattice ratio of decay constants } f_K/f_\pi
\]

For \( B_K \) FLAG quotes:

\[
\hat{B}_K = 0.738(20).
\]

For \( K \rightarrow \pi\pi \) decays I discussed the prospects for the evaluation of the amplitudes.
At Chiral Dynamics 2012, I continue the story. As the precision of lattice calculations continues to improve, it becomes both possible and necessary to extend the range of physical quantities being studied.

Outline of Talk:

1. Introduction
2. $K \rightarrow (\pi\pi)_{I=2}$ decay amplitudes (benchmark calculation completed)
3. $K \rightarrow (\pi\pi)_{I=0}$ decay amplitudes (advanced exploratory work done)
4. $\Delta m_K \equiv m_{K_L} - m_{K_S}$ (significant exploratory work done)
5. Rare kaon decays (manifesto being prepared)
6. Summary and Conclusions

Help from the CD community very much needed.
2. $K \to (\pi\pi)_{I=2}$ decay amplitudes

In this section I demonstrate that we can calculate $\text{Re} A_2$ and $\text{Im} A_2$ and to present our results.

Of course we would also like to determine the $K \to (\pi\pi)_{I=0}$ amplitude $A_0$ and to understand the $\Delta I = 1/2$ rule and the value of $\varepsilon'/\varepsilon$ and I review our significant progress towards achieving this in the following section.

In the meantime however, we know $\text{Re} A_0$ and $\text{Re} A_2$ from experiment. The experimental value of $\varepsilon'/\varepsilon$ gives us one relation between $\text{Im} A_0$ and $\text{Im} A_2$, thus if we evaluate $\text{Im} A_2$ then within the standard model we know $\text{Im} A_0$ to some precision.

I stress again that soon of course, we wish to do better than this.

Before discussing the direct evaluation of $K \to (\pi\pi)_{I=2}$ decay amplitudes, I need to introduce the ensembles we use.
RBC-UKQCD Ensembles used in this study

We have three datasets of $N_f = 2 + 1$ DWF with the Iwasaki Gauge Action:

1. $a \simeq 0.114 \text{fm}$ \hspace{1cm} \{24^3 \times 64 \times 16 \ (L \simeq 2.74 \text{fm}) \text{ and } 16^3 \times 32 \times 16 \ (L \simeq 1.83 \text{fm})\} \hspace{1cm} \text{arXiv:0804.0473, hep-lat/0701013}
   - Four light-quark masses corresponding to $m_\pi \simeq 330, 415, 555, \text{ and } 670 \text{ MeV.}$
   - The lightest partially quenched pion has a mass of about 240 MeV.
   - Only data from masses with $m_\pi \lesssim 420 \text{ MeV}$ are used in the analyses.

2. $a \simeq 0.086 \text{fm}$ \hspace{1cm} \{32^3 \times 64 \times 16 \ (L \simeq 2.765 \text{fm})\} \hspace{1cm} \text{arXiv:1011.0892}
   - Three light-quark masses corresponding to $m_\pi \simeq 290, 343 \text{ and } 390 \text{ MeV}$
   - The lightest partially quenched pion has a mass of about 223 MeV.

For $K \rightarrow \pi\pi$ decays we require pions with a physical mass and hence a large volume $\Rightarrow$ coarse lattice.

3. $a \simeq 0.14 \text{fm}, \ (\text{DWF+IDSDR})$ \hspace{1cm} \{32^3 \times 64 \times 32 \ (L \simeq 4.58 \text{fm})\}
   - Two light-quark masses corresponding to pions with $m_\pi \simeq 170 \text{ and } 250 \text{ MeV.}$
   - The lightest partially quenched pion has a mass of about 142 MeV.
   - The goal was to have a physical $K \rightarrow \pi\pi$ decay, with $|p_\pi| = \sqrt{2\pi/L}$.
   - With this coarse lattice, it is not surprising that lattice artefacts are the largest source of systematic error. We mitigate against this in a number of ways.
\( K \rightarrow (\pi \pi)_{I=2} \) Decays and the Wigner-Eckart Theorem

- The operators whose matrix elements have to be calculated are:
  
  \[
  O_{(27,1)}^{3/2} = (\bar{s}^i d^i)_L \left\{ (\bar{u}^j u^j)_L - (\bar{d}^j d^j)_L \right\} + (\bar{s}^i u^i)_L (\bar{u}^j d^j)_L \\
  O_7^{3/2} = (\bar{s}^i d^i)_L \left\{ (\bar{u}^j u^j)_R - (\bar{d}^j d^j)_R \right\} + (\bar{s}^i u^i)_L (\bar{u}^j d^j)_R \\
  O_8^{3/2} = (\bar{s}^i d^i)_L \left\{ (\bar{u}^j u^j)_R - (\bar{d}^j d^j)_R \right\} + (\bar{s}^i u^i)_L (\bar{u}^j d^j)_R
  \]

- It is convenient to use the Wigner-Eckart Theorem: (Notation - \( O_{\Delta I \Delta I_z}^{\Delta I} \))

  \[
  I=2 \langle \pi^+(p_1) \pi^0(p_2) | O_{1/2}^{3/2} | K^+ \rangle = \frac{\sqrt{3}}{2} \langle \pi^+(p_1) \pi^+(p_2) | O_{3/2}^{3/2} | K^+ \rangle,
  \]

  where
  
  - \( O_{3/2}^{3/2} \) has the flavour structure \((\bar{s}d)(\bar{u}d)\).
  - \( O_{1/2}^{3/2} \) has the flavour structure \((\bar{s}d)(\bar{uu} - (\bar{dd})) + (\bar{s}u)(\bar{ud})\).

- We can then use antiperiodic boundary conditions for the \(u\)-quark say, so that the \(\pi \pi\) ground-state is \( \langle \pi^+(\pi/L) \pi^+(-\pi/L) | . \)
  
  - Do not have to isolate an excited state.
  - Size \((L)\) needed for physical \( K \rightarrow \pi \pi \) decay halved.

C-h Kim, Ph.D. Thesis
Kinematics

Chris Sachrajda
Jefferson Lab, 7th August 2012 7

| units   | \( m_\pi \)       | \( m_K \)       | \( E_{\pi,2} \) | \( E_{\pi\pi,0} \) | \( E_{\pi\pi,2} \) | \( m_K - E_{\pi\pi,2} \) |
|---------|-------------------|-----------------|-----------------|-------------------|-------------------|-----------------------------|
| lattice | 0.1042(2)         | 0.3707(7)       | 0.1739(9)       | 0.2100(4)         | 0.356(2)          | 0.015(2)                    |
| MeV     | 142.1(9)          | 505.5(3.4)      | 237(2)          | 286(2)            | 486(4)            | 20.0(3.1)                   |

- The subscripts 0 and 2 refer to \(|p_\pi| = 0\) and \(\sqrt{2}\pi/L\) respectively.
\[ \Delta I = 3/2 \] Matrix Elements

\[
\begin{align*}
\text{Source} & & \text{Re}(A_2) \ (10^{-8} \text{ GeV}) \\
 t_K = 20 & & 1.41(6) \\
 t_K = 24 & & 1.35(6) \\
 t_K = 28 & & 1.43(7) \\
 t_K = 32 & & 1.30(9) \\
 \text{Weighted Average} & & 1.38(4) \\
 \text{Experiment} & & 1.5
\end{align*}
\]

Stat. error only

\[ O'_{(27,1)}^{3/2} = (\bar{s}d)_L (\bar{u}d)_L \]

\[ O'_{7}^{3/2} = (\bar{s}d)_L (\bar{u}d)_R \]

\[ O'_{8}^{3/2} = (\bar{s}d^i)_L (\bar{u}d^i)_R \]

Sample plateaus for the matrix elements at matched kinematics \((p_\pi = \sqrt{2p_{\text{min}}})\).
The dominant error is due to lattice artefacts and the fact that our lattice is coarse. This will be eliminated when the calculation is repeated at a second lattice spacing.

The 15% estimate, intended to be conservative, is obtained by

1. Studying the dependence on $a$ of quantities which have been calculated at several lattice spacings.
2. In particular by determining the $a$ dependence of $B_K$, which is also given by the matrix element of a $(27,1)$ operator.

| Error Source                          | Re $A_2$ | Im $A_2$ |
|---------------------------------------|----------|----------|
| lattice artefacts                    | 15%      | 15%      |
| finite-volume corrections            | 6.0%     | 6.5%     |
| partial quenching                    | 3.5%     | 1.7%     |
| renormalization                      | 1.8%     | 5.6%     |
| unphysical kinematics                | 0.4%     | 0.8%     |
| derivative of the phase shift        | 0.97%    | 0.97%    |
| Wilson coefficients                  | 6.6%     | 6.6%     |
| **Total**                            | **18%**  | **19%**  |
Our results for the amplitude $A_2$ are:

$$\text{Re}A_2 = (1.381 \pm 0.046_{\text{stat}} \pm 0.258_{\text{syst}}) \times 10^{-8} \text{ GeV}$$
$$\text{Im}A_2 = -(6.54 \pm 0.46_{\text{stat}} \pm 1.20_{\text{syst}}) \times 10^{-13} \text{ GeV}.\$$

- The result for $\text{Re}A_2$ agrees well with the experimental value of $1.479(4) \times 10^{-8}$ GeV obtained from $K^+$ decays and $1.573(57) \times 10^{-8}$ GeV obtained from $K_S$ decays.
- $\text{Im}A_2$ is unknown so that our result provides its first direct determination.
- For the phase of $A_2$ we find $\text{Im}A_2/\text{Re}A_2 = -4.42(31)_{\text{stat}}(89)_{\text{syst}} 10^{-5}$.
- Combining our result for $\text{Im}A_2$ with the experimental results for $\text{Re}A_2$, $\text{Re}A_0 = 3.3201(18) \times 10^{-7}$ GeV and $\varepsilon'/\varepsilon$ we obtain:

$$\frac{\text{Im}A_0}{\text{Re}A_0} = -1.61(19)_{\text{stat}}(20)_{\text{syst}} \times 10^{-4}.$$

(Of course, we wish to confirm this directly.)

$$\frac{\text{Im}A_0}{\text{Re}A_0} = \frac{\text{Im}A_2}{\text{Re}A_2} - \frac{\sqrt{2} |\varepsilon|}{\omega} \frac{\varepsilon'}{\varepsilon}$$

$$-1.61(19)_{\text{stat}}(20)_{\text{syst}} \times 10^{-4} = -4.42(31)_{\text{stat}}(89)_{\text{syst}} \times 10^{-5} - 1.16(18) \times 10^{-4}.$$
Contributions from the 3 Matrix Elements

Writing $A_2$ in terms of the matrix elements for the physical $K^+ \rightarrow \pi^+ \pi^0$ decay:

$$A_2 = \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \frac{1}{\sqrt{3}} \sum_i C_i(3 \text{ GeV}) \mathcal{A}_{i}^{\overline{\text{MS}}-\text{NDR}}(3 \text{ GeV}).$$

we find the matrix elements to be

\begin{align*}
\mathcal{A}_{(27,1)}^{\overline{\text{MS}}-\text{NDR}}(3 \text{ GeV}) &= 0.0461(14) \text{ GeV}^3 \\
\mathcal{A}_{(8,8)}^{\overline{\text{MS}}-\text{NDR}}(3 \text{ GeV}) &= 0.874(49) \text{ GeV}^3 \\
\mathcal{A}_{(8,8)_{\text{mix}}}^{\overline{\text{MS}}-\text{NDR}}(3 \text{ GeV}) &= 3.96(23) \text{ GeV}^3.
\end{align*}

Their contributions to $A_2$ are:

\begin{align*}
\text{Re}(A_2)_{(27,1)} &= (1.398 \pm 0.044) \times 10^{-8} \text{ GeV} \\
\text{Re}(A_2)_{(8,8)} &= (4.29 \pm 0.24) \times 10^{-11} \text{ GeV} \\
\text{Re}(A_2)_{(8,8)_{\text{mix}}} &= (-2.14 \pm 0.12) \times 10^{-10} \text{ GeV} \\
\text{Im}(A_2)_{(27,1)} &= (1.55 \pm 0.36) \times 10^{-13} \text{ GeV} \\
\text{Im}(A_2)_{(8,8)} &= (4.47 \pm 0.25) \times 10^{-14} \text{ GeV} \\
\text{Im}(A_2)_{(8,8)_{\text{mix}}} &= (-8.14 \pm 0.47) \times 10^{-13} \text{ GeV}.
\end{align*}
The *ab initio* calculation of $A_2$ described above builds upon substantial theoretical advances, achieved over many years.

The agreement we find for $\text{Re} A_2$ with the experimental result is very satisfying.

We are also able to determine $\text{Im} A_2$ for the first time.

It will be important to repeat this calculation using a second lattice spacing so that a continuum extrapolation can be performed thus eliminating the dominant contribution to the error, reducing the total uncertainty to about 5%.

We expect that the dominant remaining errors in $A_2$ will then come from the omission of electromagnetic and other isospin breaking mixing between the large amplitude $A_0$ and $A_2$.

We now turn to the evaluation of the amplitude $A_0$. 
For this work we received the 2012 Ken Wilson Lattice award at Lattice 2012.

Criteria: The paper must be important research beyond the existing state of the art. ⋮
3. $K \rightarrow (\pi\pi)_{I=0}$ Decays

The $I = 0$ final state has vacuum quantum numbers.

Vacuum contribution must be subtracted; disconnected diagrams require statistical cancelations to obtain the $e^{-2m_\pi t}$ behaviour.

Consider first the two-pion correlation functions, which are an important ingredient in the evaluation of $K \rightarrow \pi\pi$ amplitudes.

For $I=2$ $\pi\pi$ states the correlation function is proportional to $D-C$.

For $I=0$ $\pi\pi$ states the correlation function is proportional to $2D+C-6R+3V$.

The major practical difficulty is to subtract the vacuum contribution with sufficient precision.

In the paper we report on high-statistics experiments on a $16^3 \times 32$ lattice, $a^{-1} = 1.73$ GeV, $m_\pi = 420$ MeV, with the propagators evaluated from each time-slice.
For $I=2$ $\pi\pi$ states the correlation function is proportional to $D-C$.
For $I=0$ $\pi\pi$ states the correlation function is proportional to $2D+C-6R+3V$. 

![Graph showing correlation function over time](image-url)
Two-pion Correlation Functions

\[ M_{\text{eff}} = \log \frac{C(t)}{C(t+1)}. \]
$K \rightarrow (\pi\pi)_{I=0}$ Decays

- There are 48 different contractions and we classify the contributions into the 6 different types illustrated above.
- Mix3 and Mix4 are needed to subtract the power divergences which are proportional to matrix elements of $\bar{s}\gamma_5d$. 
These results are for the $K \to \pi\pi$ (almost) on-shell amplitudes with 420 MeV pions at rest:

\[
\begin{align*}
\text{Re } A_0 &= (3.80 \pm 0.82) \times 10^{-7} \text{ GeV} \\
\text{Im } A_0 &= -(2.5 \pm 2.2) \times 10^{-11} \text{ GeV} \\
\text{Re } A_2 &= (4.911 \pm 0.031) \times 10^{-8} \text{ GeV} \\
\text{Im } A_2 &= -(5.502 \pm 0.0040) \times 10^{-13} \text{ GeV}
\end{align*}
\]

This is an exploratory exercise in which we are learning how to do the calculation.

Since this work was finished we have been developing techniques which seem to enhance the signal considerably.

The exploratory results for $K \to (\pi\pi)_{I=0}$ decays encourage us to proceed to physical kinematics.

⇒ an understanding of the $\Delta I = 1/2$ rule and the value of $\epsilon'/\epsilon$.

The evaluation of disconnected diagram has allowed us to study the $\eta$ and $\eta'$ mesons and their mixing.
Speculations on the Origin of the $\Delta I = 1/2$ Rule

Q. Liu, Columbia Univ. Thesis 2012; RBC-UKQCD (in preparation)

In his thesis Qi Liu extended the above study to the $24^3 \times 64$ ensembles.

1. $16^3 \times 32$ ensembles; 877 MeV kaon decaying into two 422 MeV pions at rest:

$$\frac{\text{Re}A_0}{\text{Re}A_2} = 9.1 (21).$$

2. $24^3 \times 64$ ensembles; 662 MeV kaon decaying into two 329 MeV pions at rest:

$$\frac{\text{Re}A_0}{\text{Re}A_2} = 12.0 (17).$$

Whilst both these results are obtained at unphysical kinematics and are different from the physical value of 22.5, it is nevertheless interesting to understand the origin of these enhancements.

99% of the contribution to the real part of $A_0$ and $A_2$ come from the matrix elements of the current-current operators.
Speculations on the Origin of the $\Delta I = 1/2$ Rule (Cont.)

We find that approximately

$$\text{Re} A_2 \propto \sqrt{\frac{2}{3}} (C_1 + C_2) \quad \text{and} \quad \text{Re} A_0 \propto \sqrt{\frac{1}{3}} (2C_1 - C_2).$$

Colour counting might suggest that $C_2 \simeq \frac{1}{3} C_1$ so that the two amplitudes are approximately equal.

We find instead that $C_2 \simeq -C_1$ so that $A_2$ is significantly suppressed!

$A_2$ has a larger kinematic dependence than $A_0$.

We are now preparing this calculation for publication and moving on to the next generation calculation with advanced techniques such as G-parity boundary conditions, all-to-all propagators etc.
More recently we have began to consider long-distance contributions to physical quantities. These are not given in terms of matrix elements of local operators, but require the evaluation for example of:

\[ \int d^4 x \int d^4 y \langle h_2 | T\{O_1(x) O_2(y)\} | h_1 \rangle. \]

The most advanced of our projects is on the evaluation of long-distance contributions to the \( K_L - K_S \) mass difference.

Jianglei Yu, arXiv:1111.6953; paper in preparation.

\[ \int d^4 x \int d^4 y \langle \bar{K}^0 | T\{H_W(x) H_W(y)\} | K^0 \rangle. \]

In the following section I will present some preliminary thoughts about the rare kaon decays \( K \to \pi \nu \bar{\nu} \) and \( K \to \pi \ell^+ \ell^- \):

\[ \int d^4 x e^{-i q \cdot x} \int d^4 y \langle \pi | T\{J^\mu(x) H_W(y)\} | K^0 \rangle. \]

Up to now, the main theoretical tool for these processes has been Chiral Perturbation Theory with its many limitations and uncertainties.
How do you prepare the states $h_{1,2}$ in

$$\int d^4x \int d^4y \langle h_2 | T\{O_1(x) O_2(y)\} | h_1 \rangle,$$

when the time of the operators is integrated.

The practical solution is to integrate over a large subinterval in time $t_A \leq t_{x,y} \leq t_B$, but to create $h_1$ and to annihilate $h_2$ well outside of this region:

This is the natural modification of standard field theory for which the asymptotic states are prepared at $t \rightarrow \pm \infty$ and then the operators are integrated over all time.

This approach has been successfully implemented in the $\Delta M^K$ project.

N.Christ arXiv:1012.6034; Jianglei Yu arXiv:1111.6953; paper in preparation
\[ \Delta m_{K} \] is given by

\[ \Delta m_{K} \equiv m_{K_{L}} - m_{K_{S}} = \frac{1}{2m_{K}} 2 \mathcal{P} \sum_{\alpha} \frac{\langle \bar{K}^0 | \mathcal{H}_W | \alpha \rangle \langle \alpha | \mathcal{H}_W | K^0 \rangle}{m_{K} - E_{\alpha}} = 3.483(6) \times 10^{-12} \text{ MeV}. \]

The above correlation function gives \((T = t_B - t_A + 1)\)

\[ C_4(t_A, t_B; t_i, t_f) = |Z_K|^2 e^{-m_{K}(t_f-t_i)} \sum_n \frac{\langle \bar{K}^0 | \mathcal{H}_W | n \rangle \langle n | \mathcal{H}_W | K^0 \rangle}{(m_K - E_n)^2} \times \left\{ e^{(M_K-E_n)T} - (m_K - E_n)T - 1 \right\}. \]

From the coefficient of \(T\) we can, (in principle at least), obtain

\[ \Delta m_{K}^{\text{FV}} \equiv 2 \sum_n \frac{\langle \bar{K}^0 | \mathcal{H}_W | n \rangle \langle n | \mathcal{H}_W | K^0 \rangle}{(m_K - E_n)}. \]
In order to evaluate $\Delta m_K$ we need to be able to:

- Relate $\Delta m_K$ and $\Delta m_{K}^{\text{FV}}$. ✓
  RBC-UKQCD; N.H.Christ, G.Martinelli, CTS (in preparation)
  This is a significant extension of the theory of finite-volume effects for two-pion states: the Lüscher quantization condition, Lellouch-Lüscher factor, ⋯.

- Control the additional ultraviolet divergences when the weak Hamiltonians are close together. ✓
  J.Yu, arXiv:1111.6953; RBC-UKQCD (in preparation)
  This is facilitated by the GIM mechanism which requires the presence of charm quarks.

- $\Delta S = 1$ effective weak Hamiltonian including four flavours:

\[
H_W = \frac{G_F}{\sqrt{2}} \sum_{q,q'=u,c} V_{qd} V_{q's}^* (C_1 Q_{1}^{qq'} + C_2 Q_{2}^{qq'})
\]

where

\[
Q_{1}^{qq'} = (\bar{s}_i d_i)_{V-A} (\bar{q}_j q'_j)_{V-A} \quad \text{and} \quad Q_{2}^{qq'} = (\bar{s}_i d_j)_{V-A} (\bar{q}_j q'_i)_{V-A}.
\]
Evaluating $\Delta m_K$ (cont.)

- Evaluation of graphs such as:
  
  ![Type 1 Graphs](image1)
  ![Type 3 Graphs](image2)
  ![Type 2 Graphs](image3)
  ![Type 4 Graphs](image4)

- In our exploratory study on the $16^3$ ensembles with $m_\pi = 420$ MeV, we only evaluate Type 1 and Type 2 graphs.

- We obtain $\Delta m_K$ in the range \{5.81(28) – 10.58(75)\} $\times$ $10^{-12}$ MeV as $m_K$ is varied from 563 to 839 MeV. (The physical value is $3.483(6) \times 10^{-12}$ MeV.)
As an example of our investigations consider the behaviour of the integrated $Q_1 - Q_1$ correlation function without GIM subtraction but with an artificial cut-off, $R = \sqrt{\{(t_2 - t_1)^2 + (\vec{x}_2 - \vec{x}_1)^2\}}$ on the coordinates of the two $Q_1$ insertions.

The plot exhibits the quadratic divergence as the two operators come together. The quadratic divergence is cancelled by the GIM mechanism.
Rare kaon decays which are dominated by short-distance FCNC processes, $K \rightarrow \pi \nu \bar{\nu}$ in particular, provide a potentially valuable window on new physics at high-energy scales.

The decays $K_L \rightarrow \pi^0 e^+ e^-$ and $K_L \rightarrow \pi^0 \mu^+ \mu^-$ are also considered promising because the long-distance effects are reasonably under control using ChPT.

- They are sensitive to different combinations of short-distance FCNC effects and hence in principle provide additional discrimination to the neutrino modes.
- A challenge for the lattice community is therefore either to calculate the long-distance effects reliably or at least to determine the Low Energy Constants of ChPT.
There are three main contributions to the amplitude:

1. **Short distance contributions:**
   
   \[ H_{\text{eff}} = - \frac{G_F \alpha}{\sqrt{2}} V_{ts}^* V_{td} \left\{ y_7 V_s (\bar{s} \gamma_\mu d) (\bar{\ell} \gamma^\mu \ell) + y_7 A (\bar{s} \gamma_\mu d) (\bar{\ell} \gamma^\mu \gamma_5 \ell) \right\} + \text{h.c.} \]

   - Direct CP-violating contribution.
   - In BSM theories other effective interactions are possible.

2. **Long-distance indirect CP-violating contribution**

   \[ A_{\text{ICPV}}(K_L \to \pi^0 \ell^+ \ell^-) = \varepsilon A(K_1 \to \pi^0 \ell^+ \ell^-). \]

3. **The two-photon CP-conserving contribution**

   \[ K_L \to \pi^0 (\gamma^* \gamma^* \to \ell^+ \ell^-). \]
The current phenomenological status for the SM predictions is nicely summarised by:

\[
\text{Br}(K_L \to \pi^0 e^+ e^-)_{\text{CPV}} = 10^{-12} \times \left\{ 15.7|a_S|^2 \pm 6.2|a_S| \left( \frac{\text{Im} \lambda_t}{10^{-4}} \right) + 2.4 \left( \frac{\text{Im} \lambda_t}{10^{-4}} \right)^2 \right\}
\]

\[
\text{Br}(K_L \to \pi^0 \mu^+ \mu^-)_{\text{CPV}} = 10^{-12} \times \left\{ 3.7|a_S|^2 \pm 1.6|a_S| \left( \frac{\text{Im} \lambda_t}{10^{-4}} \right) + 1.0 \left( \frac{\text{Im} \lambda_t}{10^{-4}} \right)^2 \right\}
\]

\[\lambda_t = V_{td} V_{ts}^*\] and \(\text{Im} \lambda_t \simeq 1.35 \times 10^{-4}\).

\(|a_S|\), the amplitude for \(K_S \to \pi^0 \ell^+ \ell^-\) at \(q^2 = 0\) as defined below, is expected to be \(O(1)\) but the sign of \(a_S\) is unknown. \(|a_S| = 1.06^{+0.26}_{-0.21}\).

For \(\ell = e\) the two-photon contribution is negligible.

Taking the positive sign (??) the prediction is

\[
\text{Br}(K_L \to \pi^0 e^+ e^-)_{\text{CPV}} = (3.1 \pm 0.9) \times 10^{-11}
\]

\[
\text{Br}(K_L \to \pi^0 \mu^+ \mu^-)_{\text{CPV}} = (1.4 \pm 0.5) \times 10^{-11}
\]

\[
\text{Br}(K_L \to \pi^0 \mu^+ \mu^-)_{\text{CPC}} = (5.2 \pm 1.6) \times 10^{-12}.
\]

The current experimental limits (KTeV) are:

\[
\text{Br}(K_L \to \pi^0 e^+ e^-) < 2.8 \times 10^{-10}\] and \(\text{Br}(K_L \to \pi^0 \mu^+ \mu^-) < 3.8 \times 10^{-10}\).

\[\]
CPC Decays: $K_S \rightarrow \pi^0 \ell^+ \ell^-$ and $K^+ \rightarrow \pi^+ \ell^+ \ell^-$

We now turn to the CPC decays $K_S \rightarrow \pi^0 \ell^+ \ell^-$ and $K^+ \rightarrow \pi^+ \ell^+ \ell^-$ and consider

$$T_i^\mu = \int d^4xe^{-iq\cdot x}\langle \pi(p)|T\left\{J_{\text{em}}^\mu(x)Q_i(0)\right\}|K(k)\rangle,$$

where $Q_i$ is an operator from the effective Hamiltonian.

Gauge invariance implies that

$$T_i^\mu = \frac{\omega_i(q^2)}{(4\pi)^2}\left\{q^2(p+k)^\mu - (m_K^2 - m_\pi^2)q^\mu\right\}.$$

Within ChPT the Low energy constants $a_+$ and $a_S$ are defined by

$$a = \frac{1}{\sqrt{2}}V_{us}^*V_{ud}\left\{C_1\omega_1(0) + C_2\omega_2(0) + \frac{2N}{\sin^2\theta_W}f_+(0)C_{7V}\right\}$$

where $Q_{1,2}$ are the two current-current GIM subtracted operators and the $C_i$ are the Wilson coefficients. ($C_{7V}$ is proportional to $y_{7V}$ above).

Phenomenological values: $a_+ = -0.578 \pm 0.016$ and $|a_S| = 1.06^{+0.26}_{-0.21}$.

Can we do better in lattice simulations?
Minkowski and Euclidean Correlation Functions

- The generic non-local matrix elements which we wish to evaluate is

\[ X \equiv \int_{-\infty}^{\infty} dt_x d^3x \langle \pi(p) | T[J(0)H(x)] | K \rangle \]

\[ = i \sum_n \frac{\langle \pi(p) | J(0) | n \rangle \langle n | H(0) | K \rangle}{m_K - E_n + i\varepsilon} - i \sum_{n_s} \frac{\langle \pi(p) | H(0) | n_s \rangle \langle n_s | J(0) | K \rangle}{E_{n_s} - E_\pi + i\varepsilon}, \]

where \( \{|n\} \) and \( \{|n_s\} \) represent complete sets of non-strange and strange sets.

- In Euclidean space we envisage calculating correlation functions of the form

\[ C \equiv \int_{-T_a}^{T_b} dt_x \langle \phi_\pi(\vec{p}, t_\pi) T[J(0)H(t_x)] \phi_K^+(t_K) \rangle \equiv \sqrt{Z_K} \frac{e^{-m_K|t_K|}}{2m_K} X_E \sqrt{Z_\pi} \frac{e^{-E_\pi t_\pi}}{2E_\pi}, \]

where

\[ X_{E_-} = - \sum_n \frac{\langle \pi(p) | J(0) | n \rangle \langle n | H(0) | K \rangle}{m_K - E_n} \left( 1 - e^{(m_K-E_n)T_a} \right) \] and

\[ X_{E_+} = \sum_{n_s} \frac{\langle \pi(p) | H(0) | n_s \rangle \langle n_s | J(0) | K \rangle}{E_{n_s} - E_\pi} \left( 1 - e^{-E_{n_s}-E_\pi)T_b} \right). \]
Removing the single-pion intermediate state

- Chiral ward identities imply that we can add a term proportional to the scalar density $\bar{s}d$ to the Hamiltonian without changing physical results. We can therefore subtract the single pion intermediate state by imposing $\langle \pi | H + c_S \bar{s}d | K \rangle = 0$.

- It is instructive to see how this works in the present case at lowest order in chiral perturbation theory. The scalar density in the effective theory can be written as

\[
S^{sd} = \text{Tr} \left[ \lambda^{sd} \left( \Sigma + \Sigma^\dagger \right) \right] \quad \text{where} \quad \lambda^{sd} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\]

- The em current is of the form

\[
J^\mu = i\frac{f^2}{4} \text{Tr} \left[ Q \left( \Sigma \partial^\mu \Sigma^\dagger + \Sigma \partial^\mu \Sigma \right) \right]
\]

- The $c_S$ term leads to additional diagrams:

which are proportional to

\[
\frac{(p_\pi + p_K)^\mu}{p_K^2 - m_\pi^2} + \frac{(p_\pi + p_K)^\mu}{p_\pi^2 - m_K^2}.
\]

On shell, when $p_K^2 = m_K^2$ and $p_\pi^2 = m_\pi^2$, the sum of the two terms indeed gives zero.
In the $\Delta M_K$ computation, there is, of course, a two-pion intermediate state and we have had to control the corresponding finite-volume effects.

This has been done on the assumption that the dominant intermediate states below $m_K$ are the two-pion states.
We have seen that we can remove the single pion intermediate state.

Which intermediate states contribute?

- Are there any states below $M_K$?
- We can control $q^2$ and stay below the two-pion threshold.

- Are there two-pion intermediate states as a result of the Wess-Zumino term?
- Do we need to consider three-pion intermediate states?
- Answers to the above questions will affect what the finite-volume corrections are?
- The ChPT-based phenomenology community neglect such possibilities.

All to be investigated further!

It looks as though the FV corrections are much simpler than for $\Delta M_K$ and may be exponentially small?
Short Distance Effects

\[ T_i^\mu = \int d^4xe^{-i\mathbf{q} \cdot \mathbf{x}} \langle \pi(p) | T\{J^\mu(x)Q_i(0)\} | K(k) \rangle , \]

- Each of the two local \( Q_i \) operators can be normalized in the standard way and \( J \) can be normalized.
- Calculation of long-distance effects \( \Rightarrow \) must treat additional divergences as \( x \rightarrow 0 \).

Quadratic divergence is absent by gauge invariance \( \Rightarrow \) Logarithmic divergence.
- Checked explicitly for Wilson and Clover at one-loop order.
  
  G.Isidori, G.Martinelli and P.Turchetti, hep-lat/0506026

- Absence of power divergences does not require GIM.
- Logarithmic divergence cancelled by GIM.
- For DWF the same applies for the axial current.

Control of short-distance effects also appears to be much simpler than for \( \Delta M_K \).

To be investigated further!
Lots of diagrams to evaluate!

- Sample diagrams:

  
  
  + lots more

- The last two diagrams are examples of \textit{disconnected} diagrams.
6. Summary, Conclusions and Prospects

Goal is wide-ranging precision flavour physics

- *Standard* quantities, such as quark masses, decay constants, $B_K$, formfactors, are now calculated with excellent precision.

- We have performed the first direct calculation of the $K \rightarrow (\pi\pi)_{I=2}$ decay amplitude $A_2$. We believe that this will serve as an important benchmark for future improved calculations.

- Although significant technical problems remain, we are well on our way towards calculating $A_0$.
  (I did not talk about our exploratory studies refining all-to-all propagators or using G-parity.)

- We are beginning to tackle the calculation of long-distance effects in $\Delta m_K$ and rare kaon decays.

- As the precision improves we are having to think about electromagnetic and other isospin breaking effects.

- As we extend the range of quantities which are studied in lattice simulations we will need the continued help of the CD community to organise our projects most effectively.