Bounds on the roots of the Steiner Polynomial

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February 2, 2009

Abstract
We consider the Steiner polynomial of a $C^2$ convex body $K \subset \mathbb{R}^n$. Denote by $\rho_{\min}$ the minimum value of the principal radii of curvature of $\partial K$ and by $\rho_{\max}$ their maximum. When $n \leq 5$, the real parts of the roots are bounded above by $-\rho_{\min}$ and below by $-\rho_{\max}$. These bounds are valid for any $n$ such that all of the roots of the Steiner polynomial of every convex body in $\mathbb{R}^n$ lie in the left half-plane.

1 Introduction
Let $K \subset \mathbb{R}^n$ be a (compact) convex body and let $B$ denote the unit ball in $\mathbb{R}^n$. We form the outer parallel body $K + tB$ by taking the Minkowski sum of $K$ and a ball of radius $t > 0$, that is:

$$K + tB = \{ \bar{v} + t\bar{u} \mid \bar{v} \in K, \bar{u} \in B \}. $$

Thinking of the outer parallel body as the result of the unit-speed outward normal flow applied to $K$ at time $t$ makes it relevant to applied problems such as combustion [1]. The volume of $K + tB$ can be written as a polynomial of degree $n$, the Steiner polynomial [4]:

$$V_{K+tB} = \sum_{i=0}^{n} \binom{n}{i} V(K^{n-i}, B^i) t^i$$

where the coefficient $V(K^{n-i}, B^i)$ is the mixed volume of $n - i$ copies of $K$ and $i$ copies of the unit ball. We will adopt the notation $S_K(t) = V_{K+tB}$ for the Steiner polynomial of $K$ in the variable $t$.

In two dimensions, consideration of the roots of the Steiner polynomial leads to a Bonnesen-style inequality. When $K$ is a convex planar region, with area $A_K$ and perimeter $L_K$, we have

$$S_K(t) = A_K + L_K t + \pi t^2.$$ 

Since the discriminant of the Steiner polynomial in two dimensions is $L_K^2 - 4\pi A_K$, we see that the isoperimetric inequality for $K$ is equivalent to the fact
that \( S_K(t) = 0 \) has (one double or two single) real roots. Moreover, since \( S_K \) gives the area of the region \( K + tB \), the roots must also be negative when \( A_K > 0 \).

Furthermore, it is known that

**Theorem 1.1.** Let \( K \) be a strictly convex region which is not a disc. Let \( R_i = \sup \{ r \mid \text{a translate of } rB \subset K \} \) be the inradius of \( K \), and let \( R_e = \inf \{ r \mid \text{a translate of } K \subset rB \} \) be the outradius. Let \( \rho_{\min} \) and \( \rho_{\max} \) denote the minimum and maximum values of the radius of curvature of \( K \). If the roots of \( S_K \) are \( t_1 < t_2 \), then

\[
-\rho_{\max} < t_1 < -R_e < -\frac{L_K}{2\pi} < -R_i < t_2 < -\rho_{\min}. \tag{1}
\]

When \( K \) is a disc, then all of the above quantities are equal, giving a version of Bonnesen’s inequality. Green and Osher provide a proof in [1].

Teissier [5], working in the setting of ample divisors on algebraic varieties, posed the following problems aimed at generalizing the appealing state of affairs in the planar case. Suppose a convex body \( K \subset \mathbb{R}^n \) is given and that the roots of \( S_K \) have real parts \( r_1 \leq r_2 \leq \cdots \leq r_n \).

P1. Is \( S_K \) stable (i.e. do all the roots lie in the left half-plane)?

P2. Let \( R_i \) indicate the inradius of \( K \), that is, the largest real number \( s \) such that a translate of \( sB \) is contained in \( K \). Does the inequality \(-R_i \leq r_n\) hold?

By the Routh-Hurwitz stability criterion and the Aleksandrov-Fenchel inequalities [4], we know that \( S_K \) is stable for all convex bodies \( K \subset \mathbb{R}^n \) provided that \( n \leq 5 \). On the other hand, Cifre and Henk construct an example in [2] to show that \( S_K \) need not be stable when \( K \subset \mathbb{R}^{15} \). Less is known about the inradius bound. However, in those cases where Teissier’s first problem has an affirmative answer, we can prove a generalization of the extreme upper and lower bounds in inequality (1) relatively easily.

**Theorem 1.2.** Assume that in \( \mathbb{R}^n \), \( S_K \) is stable for every convex \( K \subset \mathbb{R}^n \). Let \( K \subset \mathbb{R}^n \) be a \( C^2 \) convex body, and suppose that the roots of \( S_K \) have real parts \( r_1 \leq r_2 \leq \cdots \leq r_n \). Denote by \( \rho_{\min} \) and \( \rho_{\max} \) the minimum and maximum values of the principal radii of curvature of \( K \). Then

\[
-\rho_{\max} \leq r_1 \leq \cdots \leq r_n \leq -\rho_{\min}.
\]

### 2 Technical Background

#### 2.1 The Steiner Polynomial

A general reference for this section is Schneider’s volume [4]. The fundamental tool for what follows is the support function \( p_K : \mathbb{R}^n \to \mathbb{R} \) of a convex body \( K \subset \mathbb{R}^n \), defined as follows:
\( p_K(\vec{x}) = \sup\{\vec{x} \cdot \vec{v} \mid \vec{v} \in K\}, \)

where \( \cdot \) denotes the standard inner product. Because of the homogeneity of the support function, \( p_K \) is determined by its restriction to the unit sphere. Thus we frequently treat \( p_K \) as a function on \( S^{n-1} \).

A particularly important feature of \( p_K \) is the way in which it carries information about the curvature of \( \partial K \) when the boundary satisfies certain smoothness conditions. When \( K \) (and thus \( p_K \)) is \( C^2 \), we consider the Hessian matrix \( H(p_K) \). Given \( \omega \in S^{n-1} \), we choose a basis \( \{e_1, \ldots, e_n\} \) where \( \{e_1, \ldots, e_{n-1}\} \) is an orthonormal basis for \( TS^{n-1} \) and \( e_n = \omega \). One can show using homogeneity [4] that the eigenvalues of \( H(p_K(\omega)) \) computed with respect to this basis are 0 and the principal radii of curvature of \( K \) at \( \omega \), which we denote \( \rho_1, \ldots, \rho_{n-1} \).

Since the infinitesimal element of area on \( \partial K \) is the product of the principal radii of curvature, we may write the volume of \( K \) equivalently as

\[
V_K = \frac{1}{n} \int_{S^{n-1}} p_K \rho_1 \cdots \rho_{n-1} \, d\omega
\]

or

\[
V_K = \frac{1}{n} \int_{S^{n-1}} p_K \det H(p_K) \, d\omega
\]

where \( H \) denotes the Hessian matrix computed with respect to an orthonormal frame for \( TS^{n-1} \).

Applying this formula to \( K + tB \), noting that \( p_{K+tB} = p_K + t \), we have

\[
S_K = V_{K+tB} = \frac{1}{n} \int_{S^{n-1}} (p_K + t) \det (H(p_K) + tI) \, d\omega. \tag{2}
\]

The integrand above is a polynomial of degree \( n \) in \( t \). We can isolate the coefficient of each \( t^i \) using the Minkowski integral formulas ([4], p. 291) to obtain an integral expression for \( V(K^{n-i}, B^i) \).

\[
V(K^{n-i}, B^i) = \frac{1}{n} \int_{S^{n-1}} s_{n-i}(\rho_1, \ldots, \rho_{n-1}) \, d\omega
\]

\[
= \frac{1}{n} \int_{S^{n-1}} p_K s_{n-i-1}(\rho_1, \ldots, \rho_{n-1}) \, d\omega,
\]

where \( s_j \) is the normalized \( j^{th} \) elementary symmetric function in \( \rho_1, \ldots, \rho_{n-1} \) (ie. \( \binom{n-1}{j} s_j \) is the usual \( j^{th} \) elementary symmetric function).

### 2.2 Minkowski Subtraction

The proof of theorem 1.2 will also rely on the concept of Minkowski subtraction. Given convex bodies \( K, L \subset \mathbb{R}^n \), the Minkowski difference of \( K \) and \( L \) is

\[
K \sim L = \{ \vec{v} \in \mathbb{R}^n \mid L + \vec{v} \subset K \}.
\]
We may think of \( K \sim L \) as the intersection of all translates of \( K \) by opposites of vectors in \( L \). If \( K \) and \( L \) are both convex, then \( K \sim L \) is as well, but the operations of Minkowski sum and difference are not inverse to one another. Although \( (K+L) \sim L = K \) holds for any convex bodies \( K \) and \( L \), \((K \sim L)+L = K\) only when there exists a convex body \( M \) such that \( L+M = K \). In this case we say that \( L \) is a Minkowski summand of \( K \), and \((K \sim L)+L = M+L = K\).

Specializing to a situation relevant to the proof, when we know that \( cB \) is a Minkowski summand of \( K \), we have that \((K \sim cB) + cB = K \) and it follows that \( p_{K \sim cB} = p_K - c \). This allows us to compute \( S_{K \sim cB} \) fairly easily using Equation (2).

We will make use of the following lemma appearing in [3] and [4], which gives a condition under which \( L \) is a Minkowski summand of \( K \).

**Lemma 2.1.** Suppose \( K, L \subset \mathbb{R}^n \) are convex. If the maximum of all the principal radii of curvature of \( L \) is bounded above by the minimum of the principal radii of curvature of \( K \) at each \( \omega \in S^{n-1} \), then \( L \) is a Minkowski summand of \( K \) — i.e. there is a convex body \( M \) such that \( L + M = K \).

## 3 Proof of Theorem 1.2

We first establish the upper bound, which is the easier of the two. Since \( K \) is convex, each \( \rho_i \geq 0 \). We may assume that \( K \) is \( C^2_+ \), (in other words the principal radii of curvature are all strictly positive and hence \( \rho_{\text{min}} > 0 \)) since otherwise there is nothing to prove. If \( 0 \leq c \leq \rho_{\text{min}} \), then let \( K' = K \sim cB \). \( cB \) is a Minkowski summand of \( K \) by Lemma 2.1, so

\[
S_{K'}(t) = \frac{1}{n} \int_{S^{n-1}} (p_K - c + t) \det(\overline{\Pi}(p_K) + (-c + t)I) \, d\omega = S_K(t - c).
\]

The roots of \( S_{K'} \) have real parts \( r_i + c \), so the stability assumption implies \( r_i + c < 0 \), hence \( r_i < -c \) for any \( c \leq \rho_{\text{min}} \). Letting \( c = \rho_{\text{min}} \) yields the claimed upper bound.

Turning to the lower bound, let \( c \geq \rho_{\text{max}} \). Then \( K \) is a Minkowski summand of \( cB \) and we write \( K' = cB \sim K \). Writing \( p_K \) for the support function of \( K \), we have \( p_{K'} = c - p_K \). Expanding the Steiner polynomial of \( K' \),

\[
\frac{1}{n} \int_{S^{n-1}} (-p_K + c + t) \det(\overline{\Pi}(-p_K) + (c + t)I) \, d\omega,
\]

in the case \( n = 3 \) we have \( S_{K'} = - (V_K - A_K(c + t) + H_K(c + t)^2 - V_B(c + t)^3) \), and in general \( S_{K'} = (-1)^n S_K(-t-c) \). The roots of \( S_{K'} \) have real parts \( -(r+c) \), so by stability \( -r-c < 0 \) and we conclude that \( -c < r \). The lower bound follows by taking \( c = \rho_{\text{max}} \).

**Corollary 3.1.** The real parts of the roots of \( S_K \) are bounded by \( -\rho_{\text{min}} \) and \(-\rho_{\text{max}} \) for any \( C^2 \) convex body \( K \subset \mathbb{R}^n \) where \( n \leq 5 \).
Proof. It is known [5] that for \( n \leq 5 \), \( S_K \) is stable for every convex body \( K \subset \mathbb{R}^n \). This follows from the Routh-Hurwitz stability criterion and the Aleksandrov-Fenchel inequalities.

\[ \square \]

References

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