Covariance and Causality in the Transition Radiation of an Electron Bunch

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Abstract

A theoretical model of the transition radiation (TR) emission of an $N$ electron bunch must comply with the covariance and the temporal-causality principles. A charge-density-like covariance must indeed imprint the formal expression of the TR energy spectrum. A causality relation must constrain the emission phases of the radiation pulse to the temporal sequence of the $N$ electron collisions onto the metallic screen. Covariance and causality are the two faces of the same coin: failing in implementing one of the two constraints into the model necessarily implies betraying the other one. The main formal aspects of a covariance and causality consistent formulation of the TR energy spectrum of an $N$ electron beam will be here described with reference to the case of a radiator surface with an arbitrary size.

Keywords: Virtual Quanta, Coherence, Fourier Transform, Charge Form Factor

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1. Introduction

Transition radiation (TR) can be observed when a relativistic charge is crossing a dielectric interface, for instance, a vacuum-metal interface [1–8]. The dipolar oscillation of the conduction electrons induced on the metallic surface by the incident relativistic charge is responsible for the radiation emission. TR develops backward and forward from the metallic surface with a characteristic angular distribution scaling down with the inverse of the...
Lorentz $\gamma$ factor of the charge ($\gamma = E/mc^2$). The well-known Ginzburg-Frank formula accounts for the spectral and the angular distribution of the TR of a single electron colliding onto a flat metallic screen whose size is supposed to be infinite compared to the transverse extension of the electromagnetic field traveling with the electron.

The present paper deals with the case of a bunch of $N$ electrons colliding at a normal angle of incidence onto a flat metallic surface, which is supposed to be placed in vacuum in the plane $z = 0$. The $N$ electrons are supposed to travel along the $z$-axis of the laboratory reference frame with a rectilinear and uniform motion and a common velocity $\vec{w} = (0, 0, w)$. The radiation field is supposed to be observed at a given point of the $z$-axis. With reference to such a charge collision scenario, the roles played by the longitudinal and transverse coordinates of the $N$ electrons in determining the formal expression of the TR energy spectrum are different in relation to the covariance and the temporal-causality principles. The distribution of the $N$ electron longitudinal coordinates, defining indeed the temporal sequence of the $N$ electron collisions onto the metallic screen, determines the causality character of the emission phases of the radiation pulse from the radiator surface. The distribution of the $N$ electron transverse coordinates plays as well a role in determining the phase of the radiation pulse at the observation point. In fact, as a function of the distance of the given single electron of the bunch from the $z$-axis of the laboratory reference frame, a further phase delay depending on the transverse coordinate of the given electron adds up, at the observation point, to the related emission phase of the single electron contribution to the radiation pulse. In the formal expression of the TR energy spectrum, the role of the $N$ electron transverse coordinates goes beyond the simple contribution to the phase factor distribution of the radiation field at the observation point [9, 10]. The distribution of the $N$ electron transverse coordinates is indeed an invariant under a Lorentz transformation with respect to the direction of motion of the electron bunch. The signature of such a Lorentz invariance intrinsically affects the $N$ single electron amplitudes composing the radiation field. A covariant formulation of the TR energy spectrum of a $N$ electron bunch is expected to preserve the signature of such a Lorentz invariance characterizing the radiation field in both the temporal coherent and incoherent components of the spectrum [9, 10].

In conclusion, passing from a single electron to a $N$ electron bunch, the formal expression of the TR energy spectrum is expected to show a charge-density-like covariance and to be causality consistent [9, 10]. The failure
in implementing the causality in the formula of the TR energy spectrum necessarily implies the failure of the covariance and vice versa. This will be demonstrated in the following with reference to the general case of a round radiator with an arbitrary radius \( r \). For simplicity, ideal conductor properties for the metallic surface will be supposed in the following.

### 2. Transition Radiation Energy Spectrum

Under far-field observation conditions, with reference to the charge collision scenario considered in the present paper, the harmonic component of the TR field of a \( N \) electron bunch reads, see \([9, 11, 12]\):

\[
E_{x,y}^{tr}(\vec{\kappa}, \omega) = \sum_{j=1}^{N} H_{x,y}(\vec{\kappa}, \omega, \vec{\rho}_{0j}) e^{-i(\omega/w)z_{0j}}, \quad (1)
\]

where

\[
H_{\mu}(\vec{\kappa}, \omega, \vec{\rho}_{0j}) = H_{\mu,j} = \frac{i e k}{2 \pi^2 D w} \int_{S} d\vec{\rho} \int d\vec{\tau} \mu e^{-i\vec{\tau} \cdot \vec{\rho}_{0j}} e^{i(\vec{\kappa} - \vec{\rho}) \cdot \vec{\tau}} \quad (2)
\]

with \( \mu = x, y \). In previous equation, \( \vec{\rho}_{0j} = (x_{0j}, y_{0j}) \) and \( z_{0j} \ (j = 1, \ldots, N) \) are, respectively, the transverse and the longitudinal coordinates of the \( N \) electrons in the laboratory reference frame at the time \( t = 0 \) when the center of mass of the electron bunch is supposed to strike the metallic surface; \( D \) is the distance from the radiator surface to the observation point which, in the present context, is supposed to be on the \( z \)-axis of the laboratory reference frame; \( \vec{\rho} = (x, y) \) are the spatial coordinates of the radiator surface \( S \) which, in general, has an arbitrary shape and size (either infinite \( S = \infty \) or finite \( S < \infty \)); \( k = \omega/c \) is the radiation wave-number and \( \vec{\kappa} = (k_x, k_y) \) is the transverse component of the related wave-vector; finally, \( \alpha = \frac{\omega}{w \gamma} \) where \( \vec{w} = (0, 0, w) \) is the common velocity and \( \gamma \) the Lorentz factor of the electrons.

With reference to Eqs. (1,2), the TR energy spectrum of a \( N \) electron beam is obtainable as the flux of the Poynting vector:

\[
\frac{d^2 I}{d\Omega d\omega} = \frac{c D^2}{4 \pi^2} \left( |E_{x}^{tr}(k_x, k_y, \omega)|^2 + |E_{y}^{tr}(k_x, k_y, \omega)|^2 \right) = \sum_{\mu=x,y} \sum_{j=1}^{N} |H_{\mu,j}|^2 + \sum_{j,l(j \neq l)=1}^{N} e^{-i(\omega/w)(z_{0j}-z_{0l})} H_{\mu,j} H_{\mu,l}^* . \quad (3)
\]
The size and the shape of the radiator surface $S$ being arbitrary in Eqs. (12), Eq.(3) only states the TR energy spectrum in an implicit form. As already argued in [9], such an implicit formulation of the TR energy spectrum meets the covariance and the temporal causality constraints. The phase structure of the $N$ single electron amplitudes composing the radiation field - see Eqs. (12) - is indeed causality related to the temporal sequence of the $N$ electron collision onto the metallic screen. A causality constraint characterizes the reciprocal interference of the single electron radiation field amplitudes in Eq.(3) as well. About the covariance consistency of the formulation of the TR energy spectrum as given in Eqs.(123), it can be demonstrated [9, 10] that: (1) under a Lorentz transformation from the laboratory to the rest reference frame, the dependence of the charge electric field on the $N$ electron transverse coordinates is a Lorentz invariant; (2) the Lorentz invariant dependence of the charge electric field on the $N$ electron transverse coordinates transfers into the TR field leaving, on both the temporal incoherent and coherent components of the TR energy spectrum, a covariant imprinting whose observability is, in principle, a function of the Lorentz invariant itself.

In the following, it will be demonstrated how the signature of the causality and covariance principles may imprint the formula of the TR energy spectrum or be vanished in Eq.(3) depending on the way how - in the expression of the radiation field Eqs.(12) - the integral calculus with respect to the radiator surface $S$ is performed and, in particular, how the limit $S \rightarrow \infty$ is implemented.

2.1. Single Electron Ginzburg-Frank Formula

The well-known Ginzburg-Frank formula accounts for the TR energy spectrum of a single electron colliding onto a radiator surface whose transverse size is much larger - at the limit, infinite - compared to the transverse extension of the electromagnetic field traveling with the relativistic electron. Under the limit $S \rightarrow \infty$, in the case of a single electron $N = 1$ - and $\rho_0 = (0, 0)$ - the integral with respect to the spatial coordinates $\vec{\rho} = (x, y)$ of the radiator surface $S$ transforms into a Dirac delta function in the formal expression of the radiation field, see Eqs. (12). This makes easy the derivation of the Ginzburg-Frank formula via Eq.(3):

$$\frac{d^2I_e}{d\Omega d\omega} = \frac{(e\beta)^2}{\pi^2c} \frac{\sin^2\theta}{(1 - \beta^2\cos^2\theta)^2}.$$  \hspace{1cm} (4)
The single electron Ginzburg-Frank formula can be obtained from Eqs. (1,2,3) by applying the limit $S \to \infty$, equivalently, either before performing the integral calculus of the radiation field with respect to the radiator surface - see above - or after performing the integral calculus with respect to a screen surface with a finite size ($S < \infty$), see [13–18]. In order to calculate the TR energy spectrum of a single electron colliding onto an infinite metallic surface, the two mathematical procedures are equivalent and lead to the same result, see Eq. (4).

2.2. $N$ Electron Bunch Formula

Contrary to the case of a single electron, in the case of a $N$ electron bunch the above mentioned two mathematical procedures to implement the limit $S \to \infty$ in Eqs. (1,2) - either before or after the integral calculus - lead to two completely different results, as in the following described.

In the case of a $N$ electron bunch, if the limit to infinity of the metallic surface ($S \to \infty$) is performed in Eqs. (1,2) prior to the integral calculus of the radiation field, then the following expression of the TR field can be obtained:

$$E_{tr}^{x,y}(\vec{\kappa},\omega) = \sum_{j=1}^{N} E_{e}^{x,y}(\vec{\kappa},\omega) e^{-i(\omega/w)z_{0j}} e^{-i\vec{\kappa}\cdot\vec{\rho}_{0j}},$$

where

$$E_{e}^{x,y}(\vec{\kappa},\omega) = \frac{2ie\kappa_{x,y}}{Dw\kappa_{x,y}^{2} + \alpha^{2}}$$

is the harmonic component of the TR field produced by a single electron moving along the $z$-axis of the laboratory reference frame and $\vec{\kappa} = (\kappa_{x},\kappa_{y}) = k\sin\theta(\cos\phi,\sin\phi)$ is the transverse component of the wave-vector ($k = 2\pi/\lambda$). With reference to Eq. (3), taking into account Eqs. (5,6), the TR energy spectrum of a $N$ electron bunch results to be described by the following formula:

$$\frac{d^{2}I}{d\Omega d\omega} = \frac{d^{2}I_{e}}{d\Omega d\omega} \left( N + \sum_{j,l,(j\neq l)}^{N} e^{-i(\omega/w)(z_{0j}-z_{0l})} e^{-i\vec{\kappa}\cdot(\vec{\rho}_{0j}-\vec{\rho}_{0l})} \right)$$

where $\frac{d^{2}I_{e}}{d\Omega d\omega}$ is the Ginzburg-Frank formula of a single electron, see Eq. (4), and the double summation is proportional to the charge factor which, under the continuous limit approximation, reads as the square module of the
Fourier transformation of the distribution function of the $N$ electron spatial coordinates.

Conversely, if the integral calculus in Eqs. (112) is performed with respect to a metallic screen with a finite size - a round metallic screen with a finite radius $R$ ($R \gg \rho_{0j}, j = 1, ..., N$), for instance - then the formula of the radiation field of a $N$ electron bunch reads, see [10]:

$$E_{x,y}^{tr}(\vec{k}, \omega) = \sum_{j=1}^{N} H_{x,y}(\vec{k}, \omega, \vec{\rho}_{0j}) e^{-i(\omega/w)z_{0j}} = \sum_{j=1}^{N} \frac{2iek}{Dw} \frac{\kappa}{\kappa^2 + \alpha^2} e^{-i[(\omega/w)z_{0j} + \vec{k} \cdot \vec{\rho}_{0j}]} \times$$

$$\times \left( \begin{array}{c} \cos \phi \\ \sin \phi \end{array} \right) [\rho_{0j} \Phi(\kappa, \alpha, \rho_{0j}) - (R + \rho_{0j}) \Phi(\kappa, \alpha, R + \rho_{0j})],$$

(8)

where $\vec{k} = (\kappa_x, \kappa_y) = k \sin \theta (\cos \phi, \sin \phi)$ is the transverse component of the wave-vector with $k = 2\pi/\lambda$ and $\vec{\rho}_{0j} = (x_{0j}, y_{0j})$ ($j = 1, ..., N$) are the transverse coordinates of the $N$ electrons with $\rho_{0j} = \sqrt{x_{0j}^2 + y_{0j}^2}$. Furthermore,

$$\Phi(\kappa, \alpha, \rho_{0j}) = \alpha J_0(\kappa \rho_{0j}) K_1(\alpha \rho_{0j}) + \frac{\alpha^2}{\kappa} J_1(\kappa \rho_{0j}) K_0(\alpha \rho_{0j})$$

(9)

and

$$\Phi(\kappa, \alpha, R + \rho_{0j}) = \alpha J_0[\kappa(R + \rho_{0j})] K_1[\alpha(R + \rho_{0j})] + \frac{\alpha^2}{\kappa} J_1[\kappa(R + \rho_{0j})] K_0[\alpha(R + \rho_{0j})].$$

(10)

It can be demonstrated [10] that, in the limit $R \to \infty$,

$$(R + \rho_{0j}) \Phi(\kappa, \alpha, R + \rho_{0j}) \to 0.$$  

(11)

Moreover, in the case $N = 1$ and under the limit $\rho_{01} \to 0$, it can be demonstrated [10] that the model represented by Eqs. (8, 9, 10, 11) can reproduce the well known formula of the TR energy spectrum of a single electron colliding onto a round metallic screen with a finite radius $R$ [13–18]. See, for instance, the comparison of the results reported in [10] and in [18]. Furthermore, it can be demonstrated [10] that the Ginzburg-Frank formula - Eq. (4) - can be obtained from Eqs. (8, 9, 10, 11) under the limits $\rho_{01} \to 0$ and $R \to \infty$.

Finally, according to the model represented by Eqs. (8, 9, 10, 11), under the limit $R \to \infty$, the TR energy spectrum of a $N$ electron bunch colliding onto a radiator surface with an infinite size ($S = \infty$) can be finally formulated via
Eq. (3) as, see also [10]:

\[
\frac{d^2 I}{d\Omega d\omega} = \frac{d^2 I_e}{d\Omega d\omega} \left( \sum_{j=1}^{N} |A_j|^2 + \sum_{j,l (j \neq l)=1}^{N} A_j A^*_l e^{-i[(\omega/w)(z_{0j}-z_{0l})+\vec{\kappa} \cdot (\vec{\rho}_{0j}-\vec{\rho}_{0l})]} \right) \quad (12)
\]

where \( \frac{d^2 I_e}{d\Omega d\omega} \) is the Ginzburg-Frank formula, see Eq. (4), and

\[
A_j = \rho_{0j} \Phi(\kappa, \alpha, \rho_{0j}). \quad (13)
\]

More details on the derivation of the formulae above in [10]. Numerical results of the angular distribution of the spectral intensity of the TR, which are obtainable from Eq. (12) and Eq. (7) under an observation condition of temporal incoherence, are compared in Fig. (1). In Fig. (1), it can be observed that, for given values of the beam energy and size, the angular distribution of the TR undergoes a broadening with the increase of the observed wavelength. Such an angular broadening of the TR spectral intensity is consistent with a diffractive effect due to the finite transverse size of the electron beam in comparison with the observed wavelength.

Contrary to the case of a single electron, in the case of an \( N \) electron bunch different formulae of the TR energy spectrum - see Eq. (7) - follows from the two different mathematical methods to implement the limit to infinity of the metallic surface \( (S \to \infty) \) in the integral calculus of the radiation field, see Eqs. (1,2).

If the limit \( S \to \infty \) is implemented in Eqs. (12) before performing the integral calculus of the TR field - see Eq. (7) - then the causality role played by the longitudinal coordinates of the \( N \) electrons in determining the emission phases of the \( N \) single electron amplitudes composing the radiation field becomes indistinguishable from the role played by the \( N \) electron transverse coordinates which, in principle, are only expected to contribute to the relative phase distribution of the radiation field at the observation point with an additional phase delay adding up to the emission phase. In practice, looking at Eq. (5) and Eq. (7), the temporal causality feature characterizing the radiation emission cannot be univocally attributed to the distribution function of the \( N \) electron longitudinal coordinates \( z_{0j} \) \((j = 1, \ldots, N)\). Moreover, the expected Lorentz invariant signature of the \( N \) electron transverse coordinates on the radiation field is completely lost since, for a possible observer of the radiative mechanism under an observation condition of temporal incoherence,
the radiation field results to be the linear addition of \( N \) identical single electron radiation field contributions originated by an electron traveling exactly on the \( z \)-axis of the laboratory reference frame.

On the contrary, if the limit \( S \to \infty \) is implemented in Eqs.(11,12) after performing the integral calculus of the TR field then - in the formula of the TR energy spectrum, see Eq.(12) - the causality role played by the \( N \) electron longitudinal coordinates in determining the emission phases of the \( N \) single electron radiation field amplitudes maintains distinct from the role played by the \( N \) electron transverse coordinates which are only expected to contribute with a further phase delay to the observation phase distribution of the radiation field. Moreover, both the temporal coherent and incoherent components of the TR energy spectrum - see Eq.(12) - bear the signature of the Lorentz invariance of the distribution function of the \( N \) electron transverse coordinates. The \( N \) single electron radiation field amplitudes - see Eq.(8) - being indeed an intrinsic function of such a Lorentz invariant quantity characterize with a charge-density-like covariance the formula of the TR energy spectrum, see Eq.(12).

3. Conclusions

In the collision of a \( N \) electron bunch at a normal angle of incidence onto a flat metallic surface with an arbitrary size, the longitudinal and the transverse coordinates of the \( N \) electrons bring, respectively, into the formal expression of the TR energy spectrum the causality and the covariance marks characterizing the electromagnetic radiative mechanism. In the case of a \( N \) electron bunch, in the present paper it is demonstrated how an improper mathematical procedure to implement the limit of infinite surface in the integral calculus of the radiation field can lead to the non-physical result of mixing up and, in conclusion, losing the distinct roles which the longitudinal and transverse coordinates of the \( N \) electrons play, respectively, in determining the causality and the covariance features of the TR emission.

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Figure 1: Numerical simulation of the angular distribution of the TR originated by a bunch of $N = 10^4$ electrons with a Gaussian transverse distribution of standard deviation $\sigma = 50 \, \mu m$. Different beam energy are supposed: 250 MeV (a), 500 MeV (b) and 750 MeV (c). The Red, Green, and Blue curves are the numerical results that are obtainable from the first term of Eq. (12) with reference to Eqs. (4, 9, 13) for a wavelength $\lambda = 680 \, nm$, $\lambda = 530 \, nm$ and $\lambda = 400 \, nm$, respectively. The Black curves represent the numerical results obtainable from the first term of Eq. (7).