WFIRST Ultra-Precise Astrometry I: Kuiper Belt Objects

ABSTRACT

I show that the WFIRST microlensing survey will enable detection and precision orbit determination of Kuiper Belt Objects (KBOs) down to $H_{\text{vega}} = 28.2$ over an effective area of $\sim 17 \text{deg}^2$. Typical fractional period errors will be $\sim 1.5\% \times 10^{0.4(H-28.2)}$ with similar errors in other parameters for roughly 5000 KBOs. Binary companions to detected KBOs can be detected to even fainter limits, $H_{\text{vega}} = 29$, corresponding to $R \sim 31$ and effective diameters $D \sim 7.5 \text{ km}$. This will provide an unprecedented probe of orbital resonance and KBO mass measurements.

Subject headings: astrometry – Kuiper belt – gravitational microlensing

1. Introduction

Kuiper Belt Objects (KBOs) provide an extraordinary probe of the origin and history of the Solar System. When Pluto was discovered by Clyde Tombaugh (Slipher 1930a,b) and was then found to be in a 3:2 resonance with Neptune, it was hardly guessed that it was only the largest of a vast class of such objects. Subsequent discovery of KBOs in 2:1 resonance, in various kinematic and composition subclasses, of binary KBOs, and of a break in the size distribution at $R \sim 26$ (Bernstein et al. 2004) have placed extremely detailed constraints on early Solar System evolution, even leading to radical conjectures like the idea that Uranus and Neptune originally formed much closer to the orbits of Jupiter and Saturn (Gomes et al. 2005). I show that the WFIRST microlensing survey will, without any adjustment, yield a KBO survey that is both substantially deeper and three orders of magnitude wider and more precise than existing deep KBO surveys based on Hubble Space Telescope (HST) data.

2. Astrometry From A Microlensing Survey

A space-based microlensing survey will almost automatically return high-quality astrometric data. I discuss this potential within the framework of the proposed WFIRST survey, but the same principles could be applied to any mission of this type. I adopt the following parameters for the microlensing component of the proposed WFIRST mission.

1) 10 contiguous, 0.28 \text{deg}^2 \text{ Galactic bulge fields}
2) six $\Delta t = 72$ day continuous “campaigns”, each centered at quadrature (March or September)
3) 15 min cycle time consisting of ten 52s exposures
4) 90% of exposures in broad $H$ band, 10% in a narrower band, e.g., $Y$
5) 2.4m telescope, $\theta_{pix} = 110$ mas pixels,
6) one photon per second at $H_{vega} = 26.1$
7) $B = 341$ total “counts” per pixel in read noise, zodiacal light and dark current per 52s integration

The diffraction limited point spread function (PSF) has FWHM= 275 mas, implying an equivalent Gaussian width $\sigma_{psf} = \text{FWHM}/2.35 = 75$ mas. Due to the slightly undersampled PSF, I assume a total background light of $9B$, i.e., about 1.5 times larger than the $4\pi(\sigma_{psf}/p)^2B$ appropriate to the oversampled limit. This leads to an equivalent “sky” of $H_{sky} = 26.1 - 2.5 \times \log(9B/52) = 21.7$. Since this paper will concern only sources well below this limit, I restrict further consideration to “below sky” sources. For these the signal-to-noise ratio (SNR) is given by

$$\text{SNR} = 10^{0.4(H_{zero} - H)}; \quad H_{zero} = 26.1$$

The 15 min cadence implies a total of 6908 observations during a 72 day campaign. Of these 10% are not in the wide band and so contribute much less astrometric information. These will simply be ignored (thought they will provide important color information). I further assume that 10% of the remainder land on sources that are significantly above sky and therefore have substantial additional noise in the difference images. This leaves

$$N_{cam} = 5600$$

images, distributed roughly uniformly in time over each campaign. I assume that each of these has astrometric precision

$$\sigma_{ast} = \sqrt{2} \frac{\sigma_{psf}}{\text{SNR}} = \frac{106 \text{ mas}}{\text{SNR}}.$$  

The images will be dithered and the KBOs will move relative to the stellar background, so the astrometric and photometric errors of the $N_{cam} = 5600$ images will be essentially uncorrelated. Because of the very large number of images, the central limit theorem therefore implies that Gaussian statistics strictly apply.
3. Precision of Orbit Determination

Assuming that a KBO is identified in the data, how well can its orbit be measured? To answer this question, we must first ask how long it will stay within the field of view. The first point is that at semi-major \( a \sim 40 \text{ AU} \), and hence period \( P = (a/\text{AU})^{3/2} \text{yr} \sim 250 \text{ yr} \), a KBO will move about 1.4 deg from one year to the next, and hence will typically not return to the microlensing fields the next year. On the other hand, since the fields are observed at quadrature when Earth is accelerating transverse to the line of sight at \( A_\oplus = 4\pi^2 \text{AU yr}^{-2} \sim 0.5 \text{ km s}^{-1} \text{ day}^{-1} \) while the KBO will be moving of order \( v_\perp \sim 5 \text{ km s}^{-1} \), the net projected relative motion of Earth against the KBO will be

\[
\Delta x \sim \sin \left( \frac{2\pi(\Delta t/2 + v_\perp/\text{AU})}{\text{yr}} \right) \text{AU} \sim 0.71 \text{ AU},
\]

which corresponds to 1.0 deg. Hence, of order 60% of the KBOs will remain in the field for the duration of the campaign. I will initially restrict attention to this subgroup and reserve discussion regarding the remainder to Section 7.

To estimate the precision of the orbital parameters, I approximate Earth as being in a circular orbit and approximate the KBO physical motion during the period of observation as uniform. The deviations from both assumptions are slight and, what is more important, deterministic with respect to the adopted parameters. For example, the acceleration of the KBO is given directly by its distance and position on the sky. Thus, making these assumptions only slightly changes, but vastly simplifies the “trial functions” and hence renders tractable the error estimates while not significantly impacting these estimates. I follow Gould & Yee (2013) in making the initial estimates in Cartesian phase-space coordinates (instantaneous positions and velocities) rather than the traditional orbital invariants. Of course, actual fits to data will use Kepler invariants, but the Cartesian approach is more closely matched to short timescale observations and therefore facilitates both deeper understanding and simpler results. The implications for Kepler invariants are then easily derived.

The KBO then has motion \( x(t) = x_0 + vt \), described by six parameters \((x_0, v) = (r_0, x_\perp, v_r, v_\perp)\). I treat the origin of time as the midpoint of the campaign. For convenience, these six parameters can be re-expressed as \((\Pi, \theta_0, \nu_\perp, \nu_r)\),

\[
\theta_0 \equiv \frac{x}{r_0}; \quad \nu_\perp \equiv \frac{v}{\Omega r_0}; \quad \Pi \equiv \frac{\text{AU}}{r_0}; \quad \nu_r \equiv \frac{v_r}{\Omega r_0},
\]

where \( \Omega = 2\pi \text{ yr}^{-1} \) is Earth’s orbital frequency. The WFIRST fields are only a few degrees from the ecliptic, and for simplicity I assume that the KBO is directly on the ecliptic.

The first four of these parameters in Equation (5) are essentially direct observables, i.e., the position and instantaneous (normalized) proper motion of the KBO at the zero point
of time. The last two pose the main challenge. Since these are derived entirely from the motion of the KBO within the ecliptic plane, I restrict attention to these two dimensions (radial and 1-D transverse). Then the equation for the angular position \( \theta(t) \) is

\[
\theta(t) = \frac{x_0 + v_\perp t - AU(\cos \Omega t - 1)}{r + v_r t - AU \sin \Omega t} = \frac{\theta_0 + v_\perp \Omega t + 2\Pi \sin^2(\Omega t/2)}{1 + v_r \Omega t - \Pi \sin \Omega t}
\]

(6)

Since there are four parameters to be determined, we should expand to third order in time

\[
\theta(t) = \sum_{i=0}^{3} a_i(\Omega t)^i
\]

(7)

where

\[
a_0 = \theta_0; \quad a_1 = v_\perp + \theta_0 Z; \quad a_2 = 0.5\Pi + v_\perp Z + \theta_0 Z^2;
\]

\[
a_3 = -0.5\Pi \nu_r + 0.5\Pi^2 + \Pi^2 v_\perp - 2\Pi v_\perp \nu_r + \nu_r^2 v_\perp + \theta_0 (Z^3 - \Pi/6)
\]

(8)

and \( Z \equiv \Pi - \nu_r \). To a good approximation the four coefficients are well represented by their leading terms

\[
a_0 = \theta_0; \quad a_1 \rightarrow \nu_\perp; \quad a_2 \rightarrow 0.5\Pi; \quad a_3 \rightarrow -0.5\Pi \nu_r
\]

(9)

For a uniform set of \( N \) observations over time \( \Delta t \), the covariance matrix for these four coefficients is given by (e.g., Gould 2004),

\[
c_{ij} = \frac{\sigma_{\ast}^2}{N_{\text{cam}}} (\Omega \Delta t)^{-i-j} \tilde{c}_{ij}; \quad \tilde{c}_{ij} = \begin{pmatrix}
9/4 & 0 & -15 & 0 \\
0 & 75 & 0 & -420 \\
-15 & 0 & 180 & 0 \\
0 & -420 & 0 & 2800
\end{pmatrix}
\]

(10)

Therefore, the errors in the three angular variables of interest (i.e., excluding \( \theta_0 \)) are

\[
\frac{\sigma(\nu_\perp)}{\Pi^{3/2}} = \sqrt{\frac{75}{N}} (\Omega \Delta t)^{-1}\Pi^{-3/2} \sigma_{\ast},
\]

\[
\frac{\sigma(\Pi)}{\Pi} = \sqrt{\frac{720}{N}} (\Omega \Delta t)^{-2}\Pi^{-1} \sigma_{\ast},
\]

\[
\frac{\sigma(\nu_r)}{\Pi^{3/2}} = \sqrt{\frac{11200}{N}} (\Omega \Delta t)^{-3}\Pi^{-5/2} \sigma_{\ast},
\]

(11)

where in each case I have normalized to a relevant physical scale. For typical parameters \( (\Omega \Delta t \sim 1.25, \Pi \sim 1/40), \) the final expression in Equation (11) is larger than either of the others by a factor > 300. This is the justification for ignoring the correlations between these
levels embedded in Equation (8) and simply using Equation (9): the errors in $\nu_\perp$ completely dominate. Translating to physical variables, we obtain

$$\frac{\sigma(v_\perp)}{v_\perp \Pi^{1/2}} = \sqrt{\frac{75}{N}} (\Omega \Delta t)^{-1} \Pi^{-3/2} \sigma_{\text{ast}} \rightarrow \frac{1.2 \times 10^{-5}}{\text{SNR}} \left( \frac{N}{5600} \right)^{-1/2} \left( \frac{\Delta t}{72 \text{ d}} \right)^{-1} \left( \frac{r}{40 \text{ AU}} \right)^{3/2},$$

$$\frac{\sigma(r)}{r} = \sqrt{\frac{720}{N}} (\Omega \Delta t)^{-2} \Pi^{-1} \sigma_{\text{ast}} \rightarrow \frac{4.8 \times 10^{-6}}{\text{SNR}} \left( \frac{N}{5600} \right)^{-1/2} \left( \frac{\Delta t}{72 \text{ d}} \right)^{-2} \left( \frac{r}{40 \text{ AU}} \right),$$

$$\frac{\sigma(v_r)}{v_\oplus \Pi^{1/2}} = \sqrt{\frac{11200}{N}} (\Omega \Delta t)^{-3} \Pi^{-5/2} \sigma_{\text{ast}} \rightarrow \frac{3.8 \times 10^{-3}}{\text{SNR}} \left( \frac{N}{5600} \right)^{-1/2} \left( \frac{\Delta t}{72 \text{ d}} \right)^{-3} \left( \frac{r}{40 \text{ AU}} \right)^{5/2}. \quad (12)$$

These results imply that the orbit-parameter “error ellipse” is essentially a 1-dimensional structure. In Cartesian space, this one dimension is associated with a single parameter: $v_r$. After transforming to Kepler coordinates, all parameters inherit this error in $v_r$ but in a highly correlated way. For example, for roughly circular orbits, the period error $\sigma(P)$ is

$$\frac{\sigma(P)}{P} = \frac{3}{2} \frac{\sigma(a)}{a} \approx \frac{3}{2} \frac{\sigma(\text{KE})}{\text{KE}} \approx \frac{3v_r}{v_\oplus \Pi^{1/2}} \frac{\sigma(v_r)}{v_\oplus \Pi^{1/2}}, \quad (13)$$

where KE is the kinetic energy. Hence, for typical values $v_r \sim 0.2 v_\oplus \Pi^{1/2}$, the fractional period error is somewhat smaller than the last expression in Equation (12). In the next section, I will show that the theoretical limit for finding KBOs is near $\text{SNR} \sim 1/7$. Thus, even at this limit, the period precision is of the order of 1.5%.

4. Finding KBOs in the Data

WFIRST may well be in geosynchronous orbit. In principle, this would add information to the measurements of $r$ and $v_r$. However, as I show below, this added information plays an insignificant role except in the margins of parameter space and was therefore ignored in Section 3. A geosynchronous orbit would also somewhat complicate the search for KBOs in the data and so cannot be completely ignored in the present section. Nevertheless, I begin by ignoring it, partly to cover the case of non-geosynchronous (e.g., L2) orbit and partly to be able to show explicitly, further below, that the complications induced by geosynchronous orbit are not in fact significant.

The WFIRST field has $N_{\text{pix}} = 2.8 \text{ deg}^2/(110 \text{ mas})^2 = 3.0 \times 10^9$ pixels. Thus, for KBOs with

$$\text{SNR} \gtrsim \sqrt{2 \ln \frac{N_{\text{pix}}}{\text{SNR}}} = 6.3, \quad (14)$$

it is possible to comfortably identify KBO candidates without fear of massive contamination by noise spikes using a simple two-dimensional (2-D) search over the image. At this limit,
there may be some contamination, but this could easily be vetted by examining successive images. There are two points to make about Equation (14). First, it assumes Gaussian statistics. This may not be valid for the case of a 2-D search. However, Equation (14) primarily serves as an entry point to the much larger (4-D, 5-D, and 6-D) searches that I describe below, for which Gaussian statistics are valid. I therefore ignore this complication. Second, the SNR appears on both sides of the equation, meaning that the equation must be solved self-consistently. This poses no actual difficulties, since it appears inside a rather large logarithm factor on the rhs, but does call for an explicit remark.

Next, I consider 4-D searches over position (2-D, as above) and proper motion (2-D). I consider a search in a “proper motion” circle \( \mu_0 = 12'' \text{ day}^{-1} \) (relative to a KBO in circular motion at \( r = 40 \text{ AU} \). By conducting a 4-D search, I am implicitly assuming that the other two Cartesian phase-space coordinates (\( \Pi \) and \( \nu_r \)) are “not important”. Explicitly, this assumption means that the parallax differences among the KBOs being searched lead to angular displacements of \(< 1 \text{ pixel} \) during the time of the search. For definiteness, I assume that KBOs of interest have \( \Pi < \Pi_0 = 0.03 \). The acceleration of Earth then leads to a differential pixel displacement of

\[
\Delta \theta_\Pi = \frac{1}{2} (\Omega t)^2 \Pi_0 = 8 \theta_{\text{pix}} \left( \frac{\Delta t}{\text{day}} \right)^2
\]

(15)
due to parallax. Thus a 4-D search is restricted to \( \Delta t < 9 \text{ hr} \), and therefore requires a search radius of \( \mu_0 \Delta t \) and hence a total number

\[
N_\mu = \pi \left( \frac{\mu_0 \Delta t}{\theta_{\text{pix}}} \right)^2 = 5300 \left( \frac{\Delta t}{9 \text{ hr}} \right)^2
\]

(16)
of searches at each of \( N_{\text{pix}} = 3 \times 10^9 \) pixels, for a total of \( N_{\text{try}} = N_{\text{pix}} N_\mu = 1.6 \times 10^{13} \) searches. In nine hours, there are approximately \( N_{\text{im}} = 30 \) images. Hence this yields a SNR threshold of

\[
\text{SNR} \gtrsim N_{\text{im}}^{-1/2} \sqrt{2 \ln \frac{N_{\text{try}}}{\text{SNR}}} - \ln N_{\text{im}} = 1.38.
\]

(17)

To dig to lower SNR, one must probe over longer durations, which requires 5-D or 6-D searches. To find the boundary, I adopt a radial-velocity search range \( \Delta \nu_r = 1/300 \) corresponding to \( \pm 2 \text{ km s}^{-1} \) at \( r = 40 \text{ AU} \). Thus the radial velocity becomes important at \( \Delta t \sim \theta_{\text{pix}}/(\Delta \nu_r \Pi \Omega) \sim 9 \text{ hr} \), which is nearly identical to the onset of extra searches due to parallax. Combining all factors (\( (\Delta t)^2 \) for proper motions, \( (\Delta t)^2 \) for parallax, and \( (\Delta t)^1 \) for radial velocity), implies a search total of

\[
N_{\text{try}} = 7 \times 10^{14} \left( \frac{\Delta t}{\text{day}} \right)^5,
\]

(18)
implying a maximum search total for \( \Delta t = 72 \) day of \( N_{\text{try}} = 1.4 \times 10^{24} \). Applying Equation (17), with \( N_{\text{im}} = 5600 \), yields a threshold \( \text{SNR} \gtrsim 0.15 \), and so a theoretical limit of \( H_{\text{vega}} = 28.2 \), or roughly \( R \sim 30 \). Recall from Equation (13) that even such extreme below-sky KBOs would have orbital parameter errors of order 1%.

However, reaching this theoretical limit will be no picnic. One could convolve all the images with the PSF, so that each search would require only \( \sim 10 N_{\text{im}} \) floating point operations, and thus a total of \( \sim 8 \times 10^{28} \) operations. This should be compared to the \( \sim 10^{12} \) floating point operations per second (FLOPS) of a current Graphics Processing Unit (GPU). One might imagine assigning \( 10^4 \) GPUs to this task for a year, but this would only enable \( 3 \times 10^{23} \) operations. Since the data will not be available for at least a decade, we should fold in a “Moore’s Law” factor of 30 (assuming doubling time of 2 years), but this is still only \( \sim 10^{25} \) operations. Since the number of operations scales \( \propto (\Delta t)^6 \), this seems to permit analysis of data intervals \( \Delta t \lesssim 16 \) days, so only reaching \( \text{SNR} \gtrsim 0.32 \) and limiting magnitude \( H_{\text{vega}} = 27.4 \).

In fact, it should be possible to go deeper using search techniques that are more clever than blind trials. For example, one could begin by restricting the search to \( \Delta t = 16 \) days as above, but initially cull out trajectories with \( \Delta \chi^2 > 28 \). This would capture exactly half of all ultimately recoverable KBOs (i.e., those with \( \text{SNR} > 0.15 \)), since these would have \( \langle \Delta \chi^2 \rangle = 28 \), while at the same time suffering noise-spike contamination of “only” \( \sim \exp(-28/2)/\sqrt{28} \sim 10^{-7} \). Now, of course, this would still result in \( \sim 10^{14} \) noise spikes, but these could be vetted fairly efficiently. From Equation (11), \( \sigma(\Pi) = 4 \times 10^{-5} \) and \( \sigma_{\mu} = 11 \text{ mas day}^{-1} \). In fact, it is easily shown that the proper-motion error in the direction perpendicular to the ecliptic is smaller by \( \sqrt{12/75} = 0.4 \). Therefore, even allowing for a 3\( \sigma \) range for these two quantities, the total number of searches required for each such “preliminary candidate” is only \( \sim 10^7 \). If the procedure were repeated on 4 independent subsamples, it would recover \( 1 - 2^{-4} \sim 94\% \) of all with \( \text{SNR} > 0.15 \).

One could imagine yet more clever ideas. It is premature to work these out in detail because the real algorithms would have to take account of not only operation speed but also memory access for processors that have not even been designed. The point is that it is not unrealistic to think that the theoretical limit of \( \text{SNR} > 0.15 \) can be reached, or at least approached within a few tens of percent. To reiterate what was said above, this limit corresponds to \( H_{\text{vega}} < 28.2 \) or roughly \( R < 30 \), with period errors \( \sigma(P)/P \lesssim 1\% \).
5. Binary Companions and Mass Measurements

Regardless of the exact limit achievable for an ab initio search for KBOs, it is possible to reliably identify binary companions to all those that are found down to $H_{\text{vega}} \sim 29$, significantly fainter than the theoretical limit for isolated KBOs. Detection of such binary companions will lead to mass estimates and mass measurements of the parent KBO.

I parameterize the semi-major axis $a_c$, and hence the characteristic angular separation of the companion by

$$\theta_c \equiv \frac{a_c}{a} = \eta \left( \frac{M}{M_\odot} \right)^{1/3} = \eta \left( \frac{\rho}{\rho_\odot} \right)^{1/3} \frac{D}{D_\odot} \sim \eta \frac{D}{D_\odot} \quad (19)$$

where $D$ and $\rho$ are the primary KBO effective diameter and density respectively, and where I have approximated $(\rho/\rho_\odot)^{1/3} \sim 1$. To be bound (within the Hill sphere), $\eta \lesssim 1$. Hence, $\theta_c \sim \eta \times 1.4''(D/10\ \text{km})$ is constrained to be within a few arcsec near the magnitude limit. The relative proper motion of the primary and companion are then of order

$$\Delta \mu \sim \eta^{-3/2} \theta_c \Omega \Pi^{3/2} \sim \frac{7\ \text{mas}}{72\ \text{day}} \eta^{-1/2} \frac{D}{10\ \text{km}} \left( \frac{a}{40\ \text{AU}} \right)^{-3/2} \quad (20)$$

That is, near the detection limit (for primaries), one can search for companions simply by looking for non-moving objects (i.e., those that move much less than 1 pixel relative to the primary during a 72-day campaign) within a few arcsec of the primary, i.e., $\sim 10^3$ trials for each primary. If we estimate that there will be $\sim 10^4$ KBO-primary detections, then SNR $\gtrsim 0.07$ is required to avoid noise-spike contamination. This implies a flux limit $H_{\text{vega}} \sim 29.0$, plausibly corresponding to a diameter $D \sim 7.5\ \text{km}$.

For such extremely faint KBO companions, one would measure only their position. However, if we consider more generally a pair of (for simplicity) equal brightness KBOs at a given SNR, their relative proper motion can be measured to a precision

$$\sigma(\Delta \mu) = \sqrt{\frac{24}{N} \sigma^2_{\text{ast}}} \rightarrow \frac{35\ \text{mas yr}^{-1}}{\text{SNR}}. \quad (21)$$

Then adopting, again for simplicity, $D = 25\ \text{km(SNR)}^{1/2}$ and $a = 40\ \text{AU}$, we expect

$$\Delta \mu \sim 90\ (\text{SNR})^{1/2} \eta^{-1/2}\ \text{mas yr}^{-1}. \quad (22)$$

This implies, very roughly, that such proper motions can be detected for $\eta \lesssim (\text{SNR})^3$, i.e., for all bound companions at SNR $\gtrsim 1$ and for a rapidly declining fraction at fainter magnitudes.

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1 When converting from magnitudes to diameters, I adopt an albedo of 0.04 and a “typical” distance of 40 AU, which yields 9 km at $H_{\text{vega}} = 28.2$.
Such proper motion estimates would, by themselves, give crude mass estimates for individual KBOs. But the ensemble of such measurements could be studied statistically to give the mean mass as a function of (solar system) absolute magnitude.

Much more detailed orbital motion could be obtained by shallow follow-up surveys. The ensemble of KBOs could be expected to disperse (relative to mean motion) at roughly $\sim 2\, \text{km s}^{-1}$, corresponding to about $0.6^{\circ}\, \text{yr}^{-1}$. Thus, one could use WFIRST itself (in its survey mode) to make brief (e.g., one day) surveys of the fields to which the KBOs were drifting one to several years after and/or before the discovery campaign in order to better characterize the orbits. The utility and characteristics of such observations could be much better assessed after logging the discoveries from the first campaign.

6. Effects of Geosynchronous Orbit

If WFIRST is in geosynchronous orbit, this will have almost no effect on the calculations in this paper. However, it will have some modest benefits for the followup observations proposed just above in the previous section.

Geosynchronous orbit induces diurnal parallax of amplitude $\Pi_{\text{geo}} = \epsilon \Pi$ where $\epsilon \sim 1/4000$. Since the the target fields lie near the plane of this motion, diurnal parallax enables a fractional distance measurement

$$\sigma(\Pi) = \sqrt{\frac{2}{N}} \frac{\sigma_{\text{ast}}}{\epsilon}$$

where $N$ is the number of observations in some time that is at least one day. Equating this to the second expression in Equation (22), one finds that orbital parallax overtakes geosynchronous parallax at $(\Delta t)_{\Pi} = 360^{1/4} e^{1/2} \Omega^{-1} \sim 4.0 \, \text{day}$. A similar calculation for the radial velocity determination (now assuming at least several days of data) yields a crossover that is just slightly larger: $(\Delta t)_{v_r} = (35/27)^{1/4} (\Delta t)_{\Pi} \sim 4.3 \, \text{days}$. Thus, diurnal parallaxes are really only useful for the handful of KBOs that briefly cross the field, or for followup observations that are carried out over of order one day in non-campaign years.

Similarly, the diurnal parallaxes do not significantly complicate the search. Diurnal motion is $\sim \epsilon \Pi \sim 1.3^{\prime\prime}$. This implies an absolute minimum of about 20 pixels in the parallax “direction” of the search space, contrary to the assumption of the “position and proper motion” search described at the beginning of Section 4. However, this simplified search was actually presented only for didactic purposes. The real searches described further along in that section already have a much larger parallax footprint. Hence, diurnal parallax due to geosynchronous orbit has essentially no impact either on the precision of measurement or
the difficulty of searching for KBOs.

7. Edge Effects

Up to this point, I have assumed that the KBOs remain in the field for the entire 72 day campaign. However, KBOs that are initially near the ecliptic East (West) edge of the field for a Spring (Autumn) campaign will move off the field as Earth approaches the equinox and will return into the field later on, whereas other KBOs that initially lie just beyond the ecliptic West (East) edge will enter the field and then leave it. Now, as shown in Section 3, the majority of KBOs do in fact remain in the field for the full 72-day campaign, and so to zeroth order one could in principle just ignore these edge effects. However, here I show that these effects actually play only a small role even at first order.

Since the apparent motion of the KBOs is dominated by reflex motion of Earth, I calculate the time spent within in the field as though this were the only cause. Then the time spent inside (and outside) the chip is symmetric about the midpoint of the campaign. I first consider the KBOs that begin within the field and parameterize the time spent out of the field by $f$, i.e., they spend a time $f\Delta t$ outside the field. I focus on the precision of the radial velocity measurement because it is by far the weakest of the six phase space coordinates. Repeating the integral that led to Equation (10) but excluding observations during the time spent outside the field, $f\Delta t$, yields

$$Q_{\text{exit}}(f) \equiv \frac{c_{33}(\text{full})}{c_{33}(\text{partial})} = \frac{1 - (25/4)f^3(1 - 1.68f^2 + f^4) + f^{10}}{1 - f^3}$$

This function declines monotonically with $f$, but there is a compensating effect of KBOs entering the other side of the field and spending time $f\Delta t$ within the field. Since these entering KBOs are observed continuously, the corresponding calculation is trivial,

$$Q_{\text{enter}}(f) = f^{3.5}.$$ 

Finally, I note that one must account for the fact that the distribution of KBOs leaving (or entering) the field for a time $f\Delta t$ is not uniform in $f$ but rather in distance from the edge of the field at the midpoint of the campaign, which scales $\propto f^2$. That is, 75% of all KBOs that leave (or enter) the field do so for more than half the campaign.

To visualize these effects, I plot $Q_{\text{exit}}$ and $Q_{\text{enter}}$ versus $f^2$ in Figure [1]. When taking account of both effects, the maximum degradation factor is $Q \sim 0.365$ at $f^2 \sim 0.57$. This factor is modest given the huge range of KBO brightness being probed. Moreover, it is compensated by the fact that twice as many KBOs are probed at these distances from the
edge of the field. In brief, the total area of the fields in all six campaigns, \(6 \times 2.8 \text{deg}^2 \sim 17\text{deg}^2\) is a reasonable estimate of the effective area of the KBO survey.

8. Expectations from Previous Deep KBO Surveys

The \textit{WFIRST} microlensing survey will, without any modification, probe KBOs down to \(H_{\text{vega}} \sim 28.2\) over an area of \(\sim 17\text{deg}^2\) and yield orbits with period precision of 1.5\% at the magnitude limit (and much better at brighter magnitudes). For solar-colored KBOs, this corresponds to \(R \sim 30\). How does this compare to previous deep surveys?

Bernstein et al. (2004) searched 0.019\text{deg}^2 down to \(R \leq 29\) (strictly, \(m_{606} \leq 29.2\)) using 22 ks exposures with ACS on the \textit{HST}. They found three new objects. For the two of these with \(m_{606} > 28\), they obtained only crude (\(\sim 30\%\)) orbital parameters. They found no KBOs \(m_{606} > 28.3\) even though they had near-100\% completeness to \(m_{606} = 29\), implying that the distribution is flat (or falling) beyond \(R = 28.5\). Combining their results with previous work at brighter magnitudes, they derived a flat roughly flat distribution \(\sim 100\text{mag}^{-1}\text{deg}^{-2}\) for \(26 < R < 28.5\) (and possibly beyond). Based on this estimate, we can expect that \textit{WFIRST} would detect 4000–7000 KBOs.

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Fig. 1.— Degradation factor $Q$ for the error in the radial velocity measurement (by far the worst measured phase-space coordinate) as a function of $f$, the fraction of a campaign that is spent outside (inside) the microlensing field by a KBO that exits (enters) during the campaign. For low $f$, $Q_{\text{exit}}$ is modest for those exiting, and this is compensated at high $f$ by those entering. Only about 40% of all KBOs that are initially in the field exit during a campaign, so the net effect of exits/entrances is modest. Abscissa is $f^2$ because the distribution of field area is uniform in this quantity.