HANKEL AND TOEPLITZ OPERATORS OF FINITE RANK
AND PRONY’S PROBLEM IN SEVERAL VARIABLES

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ABSTRACT. Prony’s problem in several variables has attracted some attention recently and provides an interesting combination of polynomial ideal theory with analytic and numeric computations. This note points out further connections to Hankel operators of finite rank as they appear in multidimensional moment problems, shift invariance signal spaces, annihilating ideals of filters and factorization of the Hankel matrices and operators by means of Vandermonde matrices. In fact, it turns out that these concepts are essentially equivalent.

1. Introduction

In 1795, Prony [12] gave an ingenious trick to recover an exponential sum

\[ f(x) := \sum_{j=1}^{n} f_j \rho_j^x, \quad f_j, \rho_j \in \mathbb{C}, \]

defined for 2n + 1 consecutive integer samples \( f(0), \ldots, f(2n) \). His method consists of finding a nontrivial solution \( p \in \mathbb{C}^{n+1} \) of the homogeneous problem

\[
\begin{pmatrix}
    f(0) & \cdots & f(n) \\
    \vdots & \ddots & \vdots \\
    f(n) & \cdots & f(2n)
\end{pmatrix}
\begin{pmatrix}
P_0 \\
P_1 \\
\vdots \\
P_n
\end{pmatrix} = 0,
\]

whose associated Prony polynomial \( p(x) = p_0 + p_1 x + \cdots + p_n x^n \) has the zeros \( \rho_1, \ldots, \rho_n \), which recovers the nonlinear part of \( f \); the coefficients \( f_j \) can be found by solving a linear system, cf. [10].

Emerging from the classical MUSIC [18] and ESPRIT [13] algorithms, the numerical behavior of Prony and Prony-like methods and their relationship to techniques from Numerical Linear Algebra have been studied carefully, see, for example, [11] and the references there.

The multivariate version of Prony’s problem has been considered only recently, with first attempts given in [7], mostly motivated by the connections to super-resolution. For the formulation of Prony’s problem in \( s \) variables, we follow the nowadays popular fashion to write it as an exponential reconstruction problem, i.e., the reconstruction of a function of the form

\[
f(x) = \sum_{\omega \in \Omega} f_\omega e^{\omega^T x}, \quad f_\omega \in \mathbb{C}, \quad \Omega \subset (\mathbb{R} + i\mathbb{T})^s, \quad \#\Omega < \infty,
\]

where \( \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z}) \) denotes the torus. We will see later that for a complete theory the coefficients \( f_j \) have to be chosen from \( \Pi = \mathbb{C}[z] \), the ring of all polynomials in \( s \) variables.
The relation to Hankel operators is obvious: the matrix in (1.1) is a Hankel matrix. Since the situation is more intricate in the multivariate case, we will define (generalized) Hankel matrices in a more generous way. To that end, we let $\ell_0(\mathbb{Z}^s)$ denote the space of all sequences $f : \mathbb{Z}^s \to \mathbb{C}$ such that the “0–norm”

$$
\|f\|_0 := \#\{\alpha \in \mathbb{Z}^s : f(\alpha) \neq 0\}
$$

is finite. Then, for $A, B \subset \mathbb{Z}^s$, $\#A, \#B < \infty$, the (generalized) Hankel matrix is defined as

$$(1.3) \quad H_{A,B}(f) := \left( f(\alpha + \beta) : \begin{array}{c} \alpha \in A \\ \beta \in B \end{array} \right) \in \mathbb{C}^{A \times B}, \quad f \in \ell_0(\mathbb{Z}^s).$$

It is common to only admit $A, B \in \mathbb{N}_0^s$ for Hankel matrices and we will see that for our purpose here this makes no difference. In the same way, a Toeplitz matrix can be defined as

$$(1.4) \quad T_{A,B}(f) := \left( f(\alpha - \beta) : \begin{array}{c} \alpha \in A \\ \beta \in B \end{array} \right) \in \mathbb{C}^{A \times B}, \quad f \in \ell_0(\mathbb{Z}^s).$$

Both matrices depend on finitely many values of $f$ on the subsets $A + B$ and $A - B$ of $\mathbb{Z}^s$.

Remark 1.1. (1.3) and (1.4) are a slightly nonstandard way to index matrices, but it is the one that captures the structure of this matrix. Clearly, $H_{A,B}(f)$ can be written as a conventional matrix by ordering the multiindices, for example with respect to the graded lexicographical ordering, but the resulting matrix is neither a Hankel matrix nor does it have any visible structure at all.

The most prominent occurrence of Hankel matrices is probably in the context of moment problems, cf. [19], where the Hankel matrix formed from the moment sequence $\mu(\alpha) = \int x^\alpha \, d\mu$ reveals information about the underlying measure $d\mu$. Also in this case, multivariate Hankel matrices are naturally multiindexed.

In [15] it has been shown that Prony’s method generalizes naturally to several variables if one takes into account two major points: one has to choose a set $A \subset \mathbb{Z}^s$ such that $(\cdot)^A := \text{span}\{(\cdot)^\alpha : \alpha \in A\}$ allows for interpolation at $e^{\Omega} = \{e^\omega : \omega \in \Omega\} \subset \mathbb{C}^s$ and find some $k$ such that

$$
\text{rank } H_{A,\Gamma_k}(f) = \text{rank } H_{A,\Gamma_{k+1}}(f), \quad \Gamma_k := \{\alpha \in \mathbb{N}_0^s : |\alpha| \leq k\}.
$$

Then the solutions $h \in C^{\Gamma_{k+1}}$ of $H_{A,\Gamma_{k+1}}(f) h = 0$ generate, interpreted as polynomials, the so called Prony ideal, a zero dimensional ideal whose associated variety equals $e^{\Omega}$. Moreover, it is shown in [15] how this set can be determined using methods from Numerical Linear Algebra; [17] gives a symbolic method and a canonical way to determine $A$ if only $\#\Omega$ is known.

In the context of this paper, we consider $H_{A,B}(f)$ as the restriction of the Hankel operator $H(f) : \ell_0(\mathbb{Z}^s) \to \ell(\mathbb{Z}^s)$ defined as the correlation

$$(1.5) \quad H(f) g = \sum_{\beta \in \mathbb{Z}^s} f(\cdot + \beta) g(\beta) =: f \ast g, \quad g \in \ell_0(\mathbb{Z}^s),$$

while the respective Toeplitz operator $T(f) g = f \ast g$ represents the convolution. The requirement $g \in \ell_0(\mathbb{Z}^s)$ serves the purpose of making (1.5) well–defined. We relate to $f \in \ell_0(\mathbb{Z}^s)$ its symbol

$$
\hat{f}(z) := \sum_{\alpha \in \mathbb{Z}^s} f(\alpha) z^\alpha \in \Lambda := \mathbb{C}[z, z^{-1}],
$$
which is a *Laurent polynomial* since the support of $f$ is finite. By means of the translation operator $\tau$, defined by $\tau_j f = f(\cdot + \epsilon_j)$, $j = 1, \ldots, s$, where $\epsilon_j$ are the unit multiindices, and $\tau^\alpha = \tau_1^{\alpha_1} \cdots \tau_s^{\alpha_s}$, we can rewrite (1.5) as

$$f \ast g = \sum_{\beta \in \mathbb{Z}^s} \tau^{\beta} f(\cdot) g(\beta) = \hat{g}(\tau) f =: (f, \hat{g}),$$

introducing the bilinear mapping $(\cdot, \cdot) : \ell(\mathbb{Z}^s) \times \Lambda \to \ell(\mathbb{Z}^s)$. In the same way we get

$$f \ast g = \sum_{\beta \in \mathbb{Z}^s} \tau^{-\beta} f(\cdot) g(\beta) = (f, \hat{\hat{g}} (\cdot \cdot^{-1})).$$

The simple computation

$$\langle \tau^\alpha f, \hat{g} \rangle = \sum_{\beta \in \mathbb{Z}^s} \tau^{\alpha+\beta} f(\cdot) g(\beta) = (\cdot^\alpha \hat{g}) f$$

leads to the almost trivial but very useful duality

(1.6) $$(\tau^\alpha f, \hat{g}) = (f, (\cdot)^\alpha \hat{g}),$$

which immediately results in the following observation.

**Proposition 1.2.** Let $f \in \ell(\mathbb{Z}^s)$ and $\hat{g} \in \Lambda$. 

1. The linear space $\ker(\cdot, \cdot) := \{ c \in \ell(\mathbb{Z}^s) : (c, \hat{g}) = 0 \}$ is shift invariant, i.e., closed under translations.
2. The linear space $\ker(f, \cdot) := \{ q \in \Lambda : (f, q) = 0 \}$ is a Laurent ideal.

**Definition 1.3.** A subspace $\mathcal{F} \subseteq \ell(\mathbb{Z}^s)$ is called *shift invariant* if $f \in \mathcal{F}$ implies that $\tau^\alpha f \in \mathcal{F}$, $\alpha \in \mathbb{N}_0^s$. Moreover, for $f \in \ell(\mathbb{Z}^s)$ the *shift invariant space* $\mathcal{S}(f)$ generated by $f$ is defined as

$$\mathcal{S}(f) := \text{span} \{ \tau^\alpha f : \alpha \in \mathbb{Z}^s \}.$$

We end this section by defining the rank of Hankel and Toeplitz operators, setting

(1.7) $\quad \text{rank} H(f) := \sup \{ \text{rank} H_{A,B}(f) : A, B \subset \mathbb{Z}^s, \#A, \#B < \infty \},$

(1.8) $\quad \text{rank} T(f) := \sup \{ \text{rank} T_{A,B}(f) : A, B \subset \mathbb{Z}^s, \#A, \#B < \infty \},$

(1.9) $\quad \text{rank}^+ H(f) := \sup \{ \text{rank} H_{A,B}(f) : A, B \subset \mathbb{N}_0^s, \#A, \#B < \infty \}.$

A Hankel or Toeplitz operator is said to be of *finite rank* if $\text{rank} H(f) < \infty$ or $\text{rank} T(f) < \infty$, respectively.

In the rest of the paper, we will study properties of of multivariate finite rank Hankel operators and relate them to shift invariant spaces and zero dimensional ideals. To that end, Section 2 will present the main results and the concepts needed to understand these results. The proofs and further background material will then be provided in Section 3. Finally, Section 4 will provide a short conclusion.

2. Main results

We begin by noting that the seemingly different ways of defining the rank of the operators in (1.7)–(1.9) lead all to the same number and can even be obtained by the square symmetric matrices $H_k(f) := H_{\Gamma_k, \Gamma_k}(f)$.

**Theorem 2.1.** For $f \in \ell(\mathbb{Z}^s)$ we have that

(2.1) $\quad \text{rank} H(f) = \text{rank} T(f) = \text{rank}^+ H(f) = \lim_{k \to \infty} H_k(f).$
In fact, this number is also directly connected to the shift invariant spaces.

**Theorem 2.2.** For \( f \in \ell(\mathbb{Z}^s) \) we have that \( \text{rank} \, H(f) = \dim S(f) \).

Since shifts of finitely supported sequences are linearly independent if the shifts are so large that the supports are disjoint, Theorem 2.2 has an immediate consequence.

**Corollary 2.3.** If \( f \in \ell_0(\mathbb{Z}^s) \) then \( \text{rank} \, H(f) = \infty \).

It is even possible to give a slightly more “quantitative” version of Theorem 2.1 for Hankel operators of finite rank. To formulate it, recall the positive part of the hyperbolic cross,

\[
\Upsilon_n := \left\{ \alpha \in \mathbb{N}^s : \prod_{j=1}^s (1 + \alpha_j) \leq n \right\},
\]

which is a canonical and to some extent minimal choice for the set \( A \) in Prony’s problem as \( (\cdot)^T \) allows for interpolation at arbitrary \( n + 1 \) points in \( \mathbb{C}^s \), cf. [17].

The next statement tells us when the ranks stabilize.

**Theorem 2.4.** If \( \text{rank} \, H(f) < \infty \), then

\[
\text{rank} \, H_k(f) = \text{rank} \, H_{\Upsilon_k}, \quad k \geq \text{rank} \, H(f).
\]

The next theorem connects finite rank Hankel and Toeplitz operators to ideals and Prony’s problem. To that end, recall that an ideal \( I \) in \( \Lambda \) or \( \Pi \) is a subset that is closed under addition and multiplication with arbitrary elements of \( \Lambda \) and \( \Pi \), respectively. Laurent ideals are somewhat intricate since they are only well-defined on \( (\mathbb{C} \setminus \{0\})^s \) and since there exists only a trivial grading on \( \Lambda \); but already the proofs in [15] showed that we can easily restrict ourselves to polynomial ideals \( I \subset \Pi \).

An ideal is called zero dimensional if \( \Pi / I \) is finite dimensional which also implies that the associated variety, \( V(I) = \{ z \in \mathbb{C}^s : f(z) = 0, f \in I \} \), is finite. Any polynomial ideal has a finite basis \( G \), i.e., a finite subset \( G \subset I \) such that

\[
I = \langle G \rangle = \left\{ \sum_{g \in G} q_g g : q_g \in \Pi, g \in G \right\}.
\]

A special choice for such a basis are the well-known Gröbner bases which can be computed efficiently and allow for a well-defined computation of division with unique remainder. cf. [3]. The ideal theoretic approach to solve Prony’s problem then leads to the following result.

**Theorem 2.5.** For \( f \in \ell(\mathbb{Z}^s) \) the following statements are equivalent.

1. \( \text{rank} \, H(f) < \infty \).
2. There exists an ideal \( I \subset \Pi \) with a Gröbner basis \( G \) such that
   \[
   S(f) = \bigcap_{q \in I} \ker (\cdot, q) = \bigcap_{g \in G} \ker (\cdot, \hat{g})
   \]
   and \( \text{rank} \, H(f) = \dim \Pi / I \).
3. There exists \( \Omega \subset (\mathbb{R} + i\mathbb{T})^s \), \( \# \Omega < \infty \) and shift invariant subspaces \( Q_\omega \subset \Pi \), \( \omega \in \Omega \), such that
   \[
   S(f) = \bigoplus_{\omega \in \Omega} Q_\omega \quad \text{and} \quad \text{rank} \, H(f) = \sum_{\omega \in \Omega} \dim Q_\omega.
   \]
(4) $f$ is of the form

$$f(x) = \sum_{\omega \in \Omega} f_{\omega}(x) e^{\omega^T x}, \quad f_{\omega} \in \mathbb{Q}_{\omega}, \quad \omega \in \Omega,$$

where $\Omega$ and $\mathbb{Q}_{\omega}$ are as in (3).

Remark 2.6. The ideal $I$ of statement (2) is the annihilating filter ideal of the shift invariant space $S(f)$. Strictly speaking, filters are usually defined as convolutions, but since a convolution is just a correlation with the reflection of the filter, this makes no difference. Alternatively, one could also consider the Gröbner basis $G$ as a system of partial difference equations whose homogeneous solution space is again $S(f)$.

The simplest case of the representation given in statement (4) of Theorem 2.5 is that all spaces $\mathbb{Q}_{\omega}$ are simplest possible, i.e., $\mathbb{Q}_{\omega} = \Pi_0 = \mathbb{C}$. This corresponds to the generic situation that all the common zero are simple, see Theorem 3.4 and the discussion following it, and deserves to be distinguished.

Definition 2.7. A Hankel operator $H(f)$ of finite rank is called simple if $\dim \mathbb{Q}_{\omega} = 1$, $\omega \in \Omega$.

Simple Hankel operators, i.e., Hankel operators formed from multiinteger samples of functions of the form

$$f(x) = \sum_{\omega \in \Omega} f_{\omega} e^{\omega^T x}, \quad f_{\omega} \in \mathbb{C} \setminus \{0\},$$

admit a particularly simple factorization that is obtained very easily. To that end, recall the concept of the Vandermonde matrix to a Lagrange interpolation problem at $\Theta$,

$$V(\Theta; A) = \begin{pmatrix} \theta^{\alpha} : \theta \in \Theta \\ \alpha \in A \end{pmatrix} \in \mathbb{C}^{\Theta \times A}, \quad \Theta \subset \mathbb{C}^s, \quad A \subseteq \mathbb{N}_0^s,$$

which allows us to write the interpolation problem

$$y_\theta = \sum_{\alpha \in A} a_\alpha \theta^{\alpha}, \quad \theta \in \Theta,$$

as the linear system $y = V(\Theta; A) a$. If $f$ is of the form (2.4), we get for $\alpha, \beta \in \mathbb{N}_0^s$ that

$$H(f)_{\alpha,\beta} = (H_{A,B}(f))_{\alpha,\beta} = \sum_{\omega \in \Omega} f_{\omega} e^{\omega^T (\alpha + \beta)} = (V(e^{\Omega}; A)^T F_{\Omega} V(e^{\Omega}; B))_{\alpha,\beta},$$

where $F_{\Omega} = \text{diag} \left\{ f_{\omega} : \omega \in \Omega \right\}$. This already proves the following result.

Corollary 2.8. $H(f)$ is a simple Hankel operator of finite rank if and only if there exists a nonsingular diagonal matrix $F_{\Omega} \in \mathbb{C}^{\text{rank } H(f) \times \text{rank } H(f)}$ such that

$$H_{A,B}(f) = V(e^{\Omega}; A)^T F_{\Omega} V(e^{\Omega}; B), \quad H(f) = V(e^{\Omega}; \mathbb{N}_0^s)^T F_{\Omega} V(e^{\Omega}; \mathbb{N}_0^s).$$

The factorizations (2.5) are known in various instances and play a fundamental role in the multidimensional (truncated) moment problem, cf. [19].

In the general situation, the analogy of (2.5) is slightly more intricate since now multiple zeros have to be considered. In the context of Prony’s problem this has been first done in [9]; a different approach has been studied in [16]. To recall the latter, let $\Theta \subset \mathbb{C}^s$ be a finite set of nodes, let $Q_\Theta = (Q_\theta : \theta \in \Theta)$ be a vector of
$D$-invariant multiplicity spaces which will be defined precisely in Definition 3.3 and let $Q_\theta \subset \Pi$, $\#Q_\theta = \dim Q_\theta$, $\theta \in \Theta$, be bases of these multiplicity spaces. Then the Vandermonde matrix

$$V(\Theta, Q_\theta; A) := \left( (q(D)(\cdot)^\alpha)(\theta) : \quad \alpha \in A \right)$$

encodes the Hermite interpolation problem (3.3) which will be discussed later as well. Moreover, it allows us to give the general factorization of finite rank Hankel operators.

**Corollary 2.9.** $H(f)$ is a finite rank Hankel operator if and only if there exists a finite set $\Omega \subset (\mathbb{R} + i\mathbb{T})^t$, finite dimensional $D$-invariant spaces $Q_\theta \subset \Pi$ with basis $Q_\theta$, $\theta \in \Theta$, and a nonsingular block diagonal matrix

$$F_{e^\alpha, Q_\theta} = (F_{e^\alpha} \in \mathbb{C}^{\dim Q_\theta \times \dim Q_\theta} : \omega \in \Omega)$$

such that

$$H_{A,B}(f) = V(\Theta, Q_\theta; A)^T F_{e^\alpha, Q_\theta} V(\Theta, Q_\theta; B), \quad A, B \subseteq \mathbb{N}_0^n,$$

and, in particular,

$$H(f) = V(\Theta, Q_\theta; \mathbb{N}_0^n)^T F_{e^\alpha, Q_\theta} V(\Theta, Q_\theta; \mathbb{N}_0^n).$$

3. Proofs, background and auxiliary results

We first note that since $T_{A,B}(f) = H_{A,B}(f)$, the first two numbers in (2.1) coincide trivially. However, we begin with the proof of Theorem 2.2.

**Proof of Theorem 2.2.** For $n \leq \dim S(f)$ and let $f_j = \tau^{\alpha^j} f$, $\alpha^j \in \mathbb{N}_0^n$, $j = 1, \ldots, n$, be linearly independent elements of $S(f)$. For $c \in \mathbb{C}^n$ define $g \in \ell(\mathbb{Z}^n)$ as $g := c_1 f_1 + \cdots + c_n f_n$, choose $B$ such that $g(B) \neq 0$ and $A$ such that $\bigcup_{j=1}^n \alpha^j \subseteq A$. Then

$$\sum_{j=1}^n c_j (H_{A,B}(f) g)_{\alpha^j} = \sum_{j=1}^n c_j \sum_{\beta \in B} f(\alpha^j + \beta) \sum_{k=1}^n c_k f_k(\beta) = \sum_{j,k=1}^n c_{j,k} \sum_{\beta \in B} f(\alpha^j + \beta) f(\alpha^k + \beta) = \|g(B)\|_2^2 > 0.$$

In other words, $H_{A,B}(f) g \neq 0$ for any $g \in \text{span}\{f_j(B) : j = 1, \ldots, n\}$. This shows that

$$n \leq \dim S(f) \quad \Rightarrow \quad n \leq \text{rank } H_{A,B}(f) \leq \text{rank } f,$$

implying $\dim S(f) \leq \text{rank } H(f)$, also in the case $\dim S(f) = \infty$.

Conversely, suppose first that $\text{rank } H(f) < \infty$ and choose $A, B$ so large that $\text{rank } H_{A,B}(f) = \text{rank } H(f) =: n$. Then $H_{A,B}(f)$ contains $n$ linearly independent rows with indices $\alpha^j$, $j = 1, \ldots, n$, so that the sequences $\tau^{\alpha^j} f$, $j = 1, \ldots, n$, are linearly independent even on $B$, and $\dim S(f) \geq \text{rank } H(f)$. If $\text{rank } H(f) = \infty$, the same argument applied to sequences $A_j, B_j$ such that

$$\infty = \text{rank } H(f) = \lim_{j \to \infty} H_{A_j, B_j}(f)$$

shows that $\dim S(f) = \infty$. \hfill \Box

The shift invariant spaces allow us to complete the proof of Theorem 2.1.
Proof of Theorem 2.1. Since \( \text{rank } H(f) \leq \text{rank } H(f) \) and \( \text{rank } H_k(f) \leq \text{rank } H(f) \), it suffices to prove that \( \text{rank } H(f) \geq \text{dim } S(f) \) and that
\[
\lim_{k \to \infty} H_k(f) \geq \text{dim } S(f).
\]
This will be done by choosing the \( \alpha^j \) in the preceding proof appropriately by taking into account that the \( \tau^\alpha \) are linearly independent if and only if \( \tau^\alpha + \beta f, \ j = 1, \ldots, n \), are linearly independent for any \( \beta \in \mathbb{Z}^* \). Moreover, \( S(f) = S(\tau^\gamma f) \) for any \( \gamma \in \mathbb{Z}^* \). We now only have to choose \( \gamma \in \mathbb{Z}^* \) such that \( g(B) \neq 0 \) for some \( B \subset \mathbb{N}_0^k \) and then \( \beta \in \mathbb{N}_0^k \) such that \( \alpha^j + \beta \in \mathbb{N}_0^k \). The same argument also works for \( H_k \).

The first step of the proof of Theorem 2.5 uses the algebraic solution method for Prony’s method: the kernel of a sufficiently large Hankel submatrix of \( H(f) \) defines a polynomial ideal, the so-called Prony ideal whose common zeros are \( e^\Omega \) and thus yield the frequencies.

Proof of Theorem 2.6. \( \{1\} \Rightarrow \{2\} \). Since \( H(f) \) is of finite rank, there exists, by Theorem 2.1, a minimal \( n \in \mathbb{N} \) such that \( \text{rank } H(f) = \text{rank } H_k(f) \) for any \( k \geq n \). If \( H_k(p) = 0 \) for some \( p \in \Gamma_k \), then \( \hat{p} \) is a polynomial such that \( (f, \hat{p}) = 0 \). By Proposition 1.2, the set of all \( \hat{p} \) such that \( H_k(p) = 0 \) for some \( k \geq n \) forms an ideal \( I \) which has a finite Gröbner basis \( G \) such that \( I = \langle G \rangle \). Hence,
\[
f \in \bigcap_{\hat{g} \in G} \ker \langle \cdot, \hat{g} \rangle
\]
and since, again by Proposition 1.2, these kernels are shift invariant, it follows that \( S(f) \subseteq \bigcap_{\hat{g} \in G} \ker \langle \cdot, \hat{g} \rangle \). Since, in addition
\[
\text{dim ker } \langle \cdot, \hat{g} \rangle = \text{dim } \Pi/I = \binom{k+s}{k} - \text{dim ker } H_k(f) = \text{rank } H_k(f)
\]
by Theorem 2.1 we can finally conclude that \( S(f) = \bigcap_{\hat{g} \in G} \ker \langle \cdot, \hat{g} \rangle. \)

Corollary 3.1. For the ideal \( I \) we have that \( z \in V(I) \) implies \( z_j \neq 0, \ j = 1, \ldots, s \).

Proof. Suppose that there exists \( z \in V(I) \) with \( z_j = 0 \) for some \( j \in \{1, \ldots, s\} \). Then \( z \not\in V(I : \langle \cdot, z_j \rangle) \). Choose any \( q \in I : \langle \cdot, z_j \rangle \), i.e., \( \langle \cdot, z \rangle q \in I \), then \( (S(f), I) = 0 \) yields that
\[
0 = \langle S(f), \cdot, z \rangle q = \langle \tau^{\alpha} S(f), q \rangle = \langle S(f), q \rangle.
\]
By the preceding proof of Theorem 2.6 \( \{1\} \Rightarrow \{2\} \), this then yields the contradiction
\[
\text{dim } S(f) = \text{dim } \Pi/I > \text{dim } \Pi/(I : \langle \cdot, z_j \rangle) = \text{dim } S(f).
\]
Hence, \( V(I) \subset (\mathbb{C} \setminus \{0\})^s \).

Remark 3.2. The requirements
\[
0 = \langle f, \hat{g} \rangle = f \ast g, \quad \hat{g} \in G
\]
yield a system of homogeneous difference equations to determine \( f \), or more precisely a shift invariant space of solutions. In this respect determining the Prony ideal can be formulated in the language of signal processing as determining a system of annihilating filters for the signal \( f \).
The next step in the proof of Theorem 2.5 requires some more background. To that end, recall that any zero dimensional ideal has finitely many zeros, say $\Theta \subset \mathbb{C}^s$, $\#\Theta < \infty$; as shown by Gröbner [5, 6], the multiplicities of these zeros are not mere numbers any more, but structural quantities.

**Definition 3.3.** A subspace $Q \subseteq \Pi$ of polynomials is called $D$–invariant, if $q \in Q$ also implies that $p(D)q \in Q$ for any $p \in \Pi$, where, as usually,

$$p(D) = \sum_{\alpha \in \mathbb{N}_0^s} p_{\alpha} \frac{\partial^{\alpha}}{\partial x^\alpha}, \quad p(z) = \sum_{\alpha \in \mathbb{N}_0^s} p_{\alpha} z^\alpha,$$

denotes the differential operator induced by $p$.

This notion allows us to formulate Gröbner’s result on multiple common zeros; more recent work on multiplicities in polynomial system solving can be found in [8].

**Theorem 3.4** (Gröbner, [5]). $I \subset \Pi$ is a zero dimensional ideal if and only if there exist a finite set $\Theta \subset \mathbb{C}^s$ and $D$–invariant subspaces $Q'_\theta \subset \Pi$ such that

\begin{equation}
I = \bigcap_{\theta \in \Theta} \{ f \in \Pi : q(D)f(\theta) = 0, \quad q \in Q'_\theta \}, \tag{3.1}
\end{equation}

Moreover,

\begin{equation}
\dim \Pi/I = \sum_{\theta \in \Theta} \dim Q'_\theta. \tag{3.2}
\end{equation}

As a consequence, it can easily be shown that the Hermite interpolation problem

\begin{equation}
q(D)f(\theta) = 0, \quad q \in Q'_\theta, \quad \theta \in \Theta, \tag{3.3}
\end{equation}

where $Q'_\theta$ is a basis of $Q'_\theta$, has a unique solution in $\Pi/I$, hence the functionals in (3.3) are the natural dual functionals for $\Pi/I$. Also note that this Hermite interpolation problem is an ideal interpolation in the sense of [1, 2].

Finally, we recall the operator

\begin{equation}
L : \Pi \rightarrow \Pi, \quad f \mapsto \sum_{\alpha \in \mathbb{N}_0^s} \frac{1}{\alpha!} (\tau - I)^{\alpha} f(0) (\cdot)^{\alpha}, \tag{3.4}
\end{equation}

from [14, 16]. With the Pochhammer symbols or falling factorials [4], defined as

$$\langle \cdot \rangle_\alpha := \prod_{j=1}^{\alpha} \prod_{\beta_j=0}^{\alpha_j} (\langle \cdot \rangle - \beta_j), \quad \alpha \in \mathbb{N}_0^s,$$

its inverse can be written explicitly as

\begin{equation}
L^{-1}f = \sum_{\alpha \in \mathbb{N}_0^s} \frac{1}{\alpha!} \frac{\partial^{\alpha}}{\partial x^\alpha} (0) (\cdot)_\alpha. \tag{3.5}
\end{equation}

The operator allows to switch between shift invariant and $D$–invariant polynomial subspaces.

**Proposition 3.5** ([16], Proposition 1). A subspace $Q$ of $\Pi$ is shift invariant if and only if $LQ$ is $D$–invariant.
formed by \( \Theta \sigma \) where (3.6) \( f \ast g = \sum_{\beta \in \mathbb{Z}^s} f(\cdot + \beta) g(\beta) = \sum_{\beta \in \mathbb{Z}^s} f(\cdot - \beta) g(-\beta) = f \ast g(-\cdot), \quad \hat{g} \in G, \)

which is solved by \( S(f) \) Let \( \Theta \) and \( Q_0 \subset \Pi, \theta \in \Theta \), denote the common zeros of \( I \) and their multiplicities where Corollary 3.1 ensures that \( \Theta \subset (\mathbb{C} \setminus \{0\})^\times \). It has been shown in [14] that all solutions of the homogeneous difference equation (3.6), or, equivalently, all common kernels of the convolution operators defined by \( g(\cdot) \), \( g \in G \), are of the form

\[
f = \sum_{\theta \in \Theta} q_0 \theta^{(\cdot)}, \quad q_0 \in Q_0 := L^{-1} \sigma_0 Q'_0, \quad \theta \in \Theta,
\]

where

\[
\Theta = V \left( \left\{ g(\cdot) : \hat{g} \in G \right\} \right)^{-1} = V \left( (G) \right),
\]

and \( \sigma_0 : f = f(\theta_1(\cdot)_1, \ldots, \theta_s(\cdot)_s) \) denotes the dilation by the diagonal matrix formed by \( \theta \). Writing \( \Theta = e^{\Omega} \) and taking into account that \( Q_0 \) is shift invariant due to Proposition 3.5, gives the desired representation. \( \Rightarrow \) is a direct consequence.

To complete the proof of Theorem 2.5, we recall from [16] the factorization theorem for Hankel operators associated to functions of “Prony form”.

**Theorem 3.6** ([16], Theorem 5). If \( f \) is of the form

\[
f(x) = \sum_{\omega \in \Omega} f_\omega(x) e^{\omega^T x}, \quad f_\omega \in \Pi \setminus \{0\}, \quad \omega \in (\mathbb{R} + iT)^s,
\]

then there exist \( D \)-invariant spaces \( Q_\omega \subset \Pi \) and a nonsingular block diagonal matrix

\[
F_{\Omega, Q_\omega} := \text{diag} \left( F_\omega \in \mathbb{C}^{\dim Q_\omega \times \dim Q_\omega} : \omega \in \Omega \right)
\]

such that

\[
H_{A,B}(f) = V \left( e^\Omega, Q_\Omega; A \right)^T F_{\Omega, Q_\omega} V \left( e^\Omega, Q_\Omega; B \right), \quad A,B \subset \mathbb{N}_0^s.
\]

This allows us to eventually complete the proof of Theorem 2.5.

**Proof of Theorem 2.5** ([4] \( \Rightarrow \) [1]). Statement (4) means that \( f \) is of the form (3.7) and therefore \( H_{A,B}(f) \) factorizes as in (3.9) for any choice of \( A,B \subset \mathbb{N}_0^s \). This implies that

\[
\text{rank} H_{A,B}(f) \leq \max \left\{ \text{rank} V \left( e^\Omega, Q_\Omega; A \right), \text{rank} V \left( e^\Omega, Q_\Omega; B \right), \text{rank} F_{\Omega, Q_\omega} \right\},
\]

and since

\[
\text{rank} V \left( e^\Omega, Q_\Omega; A \right) \leq \sum_{\omega \in \Omega} \dim Q_\omega = \text{rank} F_{\Omega, Q_\omega}, \quad A \subseteq \mathbb{N}_0^s,
\]

with equality if and only if \((\cdot)^A\) is an interpolation space for the Hermite interpolation problem (3.3), it follows that \( H_{A,B}(f) \leq \text{rank} F_{\Omega, Q_\omega} \), again with equality iff \((\cdot)^A\) and \((\cdot)^B\) are interpolation spaces. Consequently,

\[
\text{rank} H(f) = \text{rank} F_{\Omega, Q_\omega} = \sum_{\omega \in \Omega} \dim Q_\omega < \infty.
\]

\( \square \)
This also proves the factorization theorem, Corollary 2.9: the necessity of the factorization follows for a finite rank follows from Theorem 2.5 (4) and Theorem 3.6, its sufficiency was exactly the point in the proof above.

And we can prove our last remaining result of Section 2.

Proof of Theorem 2.4. Since $H(f)$ defines a Hermite interpolation problem with rank $H(f)$ conditions, it follows that $(\cdot)^{T_k}$ and $(\cdot)^{T_k}$ admit Hermite interpolation, cf. [10]. This yields that

$$\text{rank } V(e^{\Omega}, Q_{\Omega}; \Upsilon_k) = \text{rank } V(e^{\Omega}, Q_{\Omega}; \Gamma_k) = \text{rank } H(f),$$

hence (2.3). □

4. Conclusion

We have seen that Hankel operators of finite rank defined on sequences are practically equivalent to shift invariant subspaces and to zero dimensional annihilating ideals where even the rank of the operator, the dimension of the shift invariant space and the codimension of the ideal coincide. The connection between these notions is Prony’s problem in its generalized form with polynomial coefficients.

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