A new approach to the vakonomic mechanics

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Abstract The aim of this paper was to show that the Lagrange–d’Alembert and its equivalent the Gauss and Appel principle are not the only way to deduce the equations of motion of the nonholonomic systems. Instead of them we consider the generalization of the Hamiltonian principle for nonholonomic systems with non-zero transpositional relations. We apply this variational principle, which takes into the account transpositional relations different from the classical ones, and we deduce the equations of motion for the nonholonomic systems with constraints that in general are nonlinear in the velocity. These equations of motion coincide, except perhaps in a zero Lebesgue measure set, with the classical differential equations deduced with the d’Alembert–Lagrange principle. We provide a new point of view on the transpositional relations for the constrained mechanical systems: the virtual variations can produce zero or non-zero transpositional relations. In particular, the independent virtual variations can produce non-zero transpositional relations. For the unconstrained mechanical systems, the virtual variations always produce zero transpositional relations. We conjecture that the existence of the nonlinear constraints in the velocity must be sought outside of the Newtonian mechanics. We illustrate our results with examples.

Keywords Variational principle · Generalized Hamiltonian principle · d’Alembert–Lagrange principle · Constrained Lagrangian system · Transpositional relations · Vakonomic mechanic · Equation of motion · Vorones system · Chapligyn system · Newtonian model

1 Introduction

The history of nonholonomic mechanical systems is long and complex and goes back to the 19 century, with important contribution by Hertz [17] (1894), Ferrers [11] (1871), Vierkandt [51] (1892) and Chaplygin [7] (1897).

The nonholonomic mechanic is a remarkable generalization of the classical Lagrangian and Hamiltonian mechanic. The birth of the theory of dynamics of nonholonomic systems occurred when Lagrangian-Euler formalism was found to be inapplicable for studying
the simple mechanical problem of a rigid body rolling without slipping on a plane.

A long period of time has been needed for finding the correct equations of motion of the nonholonomic mechanical systems and the study of the deeper questions associated with the geometry and the analysis of these equations. In particular, the integration theory of equations of motion for nonholonomic mechanical systems is not so complete as in the case of holonomic systems. This is due to several reasons. First, the equations of motion of nonholonomic systems have more complex structure than the Lagrange ones, which describes the behavior of holonomic systems. Indeed, a holonomic system can be described by a unique function of its state and time, the Lagrangian function. For the nonholonomic systems, this is not possible. Second, in general, the equations of motion of nonholonomic systems have no invariant measures, by the contrary the equations of motion of holonomic systems always have an invariant measure (see [6, 26, 30, 50]).

One of the most important directions in the development of the nonholonomic mechanics is the research connected with the general mathematical formalism to describe the behavior of such systems which differs from the Lagrangian and Hamiltonian formalism. The main problem with the equations of motion of the nonholonomic mechanics has been centered on whether or not these equations can be derived from the Hamiltonian principle in the usual sense, such as for the holonomic systems (see for instance [33]). But there is not doubt that the correct equations of motion for nonholonomic systems are given by the d’Alembert–Lagrange principle.

The general understanding of inapplicability of Lagrange equations and variational Hamiltonian principles to the nonholonomic systems is due to Hertz, who expressed it in his fundamental work *Die Prinzipien der Mechanik in neuem Zusammenhang dargestellt* [17]. Hertz’s ideas were developed by Poincaré in [39]. At the same time, various aspects of nonholonomic systems need to be studied such as

(a) The problem of the realization of nonholonomic constraints (see for instance [25, 27]).

(b) The stability of nonholonomic systems (see for instance [35, 43]).

(c) The role of the so called transpositional relations (see [20, 34, 35, 42])

$$\delta \frac{dx}{dt} = \frac{d}{dt} \delta x = \left(\delta \frac{dx_1}{dt}, \ldots, \delta \frac{dx_N}{dt} \frac{d}{dt} \delta x_1, \ldots, \delta \frac{dx_N}{dt} \delta x_N\right),$$

where \(\frac{d}{dt}\) denotes the differentiation with respect to the time, \(\delta\) is the virtual variation, and \(x = (x_1, \ldots, x_N)\) is the vector of the generalized coordinates.

Indeed the most general formulation of the Hamiltonian principle is the Hamilton–Suslov principle

$$\int_{t_0}^{t_1} \left(\delta \tilde{L} - \sum_{j=1}^{N} \frac{\partial \tilde{L}}{\partial \dot{x}_j} \left(\delta \frac{dx_j}{dt} - \frac{d}{dt} \delta x_j\right)\right) dt = 0,$$

suitable for constrained and unconstrained Lagrangian systems, where \(\tilde{L}\) is the Lagrangian of the mechanical system. Clearly, the equations of motion obtained from the Hamilton–Suslov principle depend on the point of view on the transpositional relations. This fact shows the importance of these relations.

(d) The relation between nonholonomic mechanical systems and vakonomic mechanical systems.

There was some confusion in the literature between nonholonomic mechanical systems and variational nonholonomic mechanical systems also called vakonomic mechanical systems. Both kinds of systems have the same mathematical “ingredients”: a Lagrangian function and a set of constraints. But the way in which the equations of motion are derived differs. As we observe the equations of motion in nonholonomic mechanics are deduced using the d’Alembert–Lagrange’s principle. In the case of vakonomic mechanics, the equations of motion are obtained through the application of a constrained variational principle. The term vakonomic (“variational axiomatic kind”) is due to Kozlov (see [22–24]), who proposed this mechanics as an alternative set of equations of motion for a constrained Lagrangian system.

The distinction between the classical differential equations of motion and the equations of motion of variational nonholonomic mechanical systems has a long history going back to Korteweg’s survey (1899) [21], and discussed in a more modern context in [10, 19, 29, 49]. In these papers, the authors have analyzed the domain of the vakonomic and nonholonomic
mechanics. In the paper Critics of some mathematical model to describe the behavior of mechanical systems with differential constraints [19], Kharlamov studied the Kozlov model and in a concrete example showed that the subset of solutions of the studied nonholonomic systems is not included in the set of vakonomic model and proved that the principle of determinacy is not valid in the Kozlov model. In [28], the authors put in evidence the main differences between the d’Alembertian and the vakonomic approaches. From the results obtained in several papers, it follows that in general the vakonomic model is not applicable to the nonholonomic constrained Lagrangian systems.

The equations of motion for the constrained mechanical systems deduced by Kozlov (see for instance [2]) from the Hamiltonian principle with the Lagrangian
\[ L = L_0 - \sum_{j=1}^{M} \lambda_j L_j, \]
where \( L_j = 0 \) for \( j = 1, \ldots, M < N \) are the given constraints, and \( L_0 \) is the classical Lagrangian. These equations are
\[
E_k L = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_k} - \frac{\partial L}{\partial x_k} = 0 \iff E_k L_0 = \sum_{j=1}^{M} \left( \lambda_j E_k L_j + \frac{d\lambda_j}{dt} \frac{\partial L_j}{\partial \dot{x}_k} \right),
\]
for \( k = 1, \ldots, N \), see for more details [2]. Clearly, Eq. (3) differ from the classical equations by the presence of the terms \( \lambda_j E_k L_j \). If the constraints are integrable, i.e., \( L_j = \frac{d}{dt} g_j(t, \dot{x}) \), then the vakonomic mechanics reduces to the holonomic one.

In this paper, we give a modification of the vakonomic mechanics. This modification is valid for the holonomic and nonholonomic constrained Lagrangian systems. We apply the generalized constrained Hamiltonian principle with non-zero transpositional relations. By applying this constrained variational principle, we deduce the equations of motion for the nonholonomic systems with constraints which in general are nonlinear in the velocity. These equations coincide, except perhaps in a zero Lebesgue measure set, with the classical differential equations deduced from the d’Alembert–Lagrange principle.

2 Statement of the main results

In this paper, we state and solve the following inverse problem of the constrained Lagrangian systems (see [31])

We consider the constrained Lagrangian systems with configuration space \( Q \) and phase space \( TQ \).

Let \( L : \mathbb{R} \times TQ \times \mathbb{R}^M \rightarrow \mathbb{R} \) be a smooth function such that
\[
L(t, x, \dot{x}, \Lambda) = L_0(t, x, \dot{x}) - \sum_{j=1}^{M} \lambda_j L_j(t, x, \dot{x}) - \sum_{j=M+1}^{N} \lambda_j^0 L_j(t, x, \dot{x}),
\]
where \( \Lambda = (\lambda_1, \ldots, \lambda_M) \) are the additional coordinates (Lagrange multipliers), \( L_j : \mathbb{R} \times TQ \rightarrow \mathbb{R}, \ (t, x, \dot{x}) \mapsto L_j(t, x, \dot{x}) \), be smooth functions for \( j = 0, \ldots, N \), where \( L_0 \) is the nonsingular function i.e., \( \det \left( \frac{\partial^2 L_0}{\partial x_j \partial x_j} \right) \neq 0 \), and \( L_j = 0 \), for \( j = 1, \ldots, M \), are the constraints satisfying
\[
\text{rank} \left( \frac{\partial (L_1, \ldots, L_M)}{\partial (\dot{x}_1, \ldots, \dot{x}_N)} \right) = M
\]
in all the points of \( \mathbb{R} \times TQ \), except perhaps in a zero Lebesgue measure set, \( L_j \) and \( \lambda_j^0 \) are arbitrary functions and constants, respectively, for \( j = M + 1, \ldots, N \).

We must determine the smooth functions \( L_j \), constants \( \lambda_j^0 \) for \( j = M + 1, \ldots, N \) and the matrix \( A \) in such a way that the differential equations describing the behavior of the constrained Lagrangian systems are obtained from the Hamiltonian principle
\[
\int_{t_0}^{t_1} \delta L = \int_{t_0}^{t_1} \left( \frac{\partial L}{\partial x_j} \delta x_j + \frac{\partial L}{\partial \dot{x}_j} \frac{d}{dt} \delta x_j \right) + \sum_{j=1}^{N} \frac{\partial L}{\partial \dot{x}_j} \left( \delta \frac{d x_j}{d t} - \frac{d}{d t} \delta \dot{x}_j \right) \right) dt = 0,
\]
with transpositional relation given by
\[
\frac{d}{dt} \delta x_j - \frac{d}{dt} \delta \dot{x}_j = A(t, x, \dot{x}, \ddot{x}) \delta x_j,
\]
where \( A = A(t, x, \dot{x}, \ddot{x}) = \left( A_{ij}(t, x, \dot{x}, \ddot{x}) \right) \) is a \( N \times N \) matrix.

We give the solutions of this problem in two steps. First we obtain the differential equations along the solutions satisfying (6). Second we shall contrast the obtained equations and classical differential equations which described the behavior of the constrained mechanical systems. The solution of this inverse problem is presented in Sect. 4.
Note that the function \( L \) is singular, due to the absence of \( \lambda \).

We observe that the arbitrariness of the functions \( L_j \), of the constants \( \lambda_j^0 \) for \( j = M + 1, \ldots, N \), and of the matrix \( A \) will play a fundamental role in the construction of the mathematical model which we propose in this paper.

Our main results are the following:

**Theorem 1** We assume that \( \delta x_v(t), v = 1, \ldots, N \), are arbitrary functions defined in the interval \([t_0, t_1]\), smooth in the interior of \([t_0, t_1]\) and vanishing at its endpoints, i.e. \( \delta x_v(t_0) = \delta x_v(t_1) = 0 \). If (7) holds then the path \( \gamma(t) = (x_1(t), \ldots, x_N(t)) \) compatible with the constraints \( L_j(t, x, \dot{x}) = 0, \) for \( j = 1, \ldots, M \) satisfies (6) with \( L \) given by the formula (4) if and only if it is a solution of the differential equations

\[
D_v L := E_v L - \sum_{j=1}^{N} A_{vj} \frac{\partial L}{\partial x_j} = 0, \quad \frac{\partial L}{\partial \lambda_k} = -L_k = 0, \quad (8)
\]

for \( v = 1, \ldots, N, \) and \( k = 1, \ldots, M, \) where \( E_v = \frac{d}{dt} \frac{\partial}{\partial \dot{x}_v} \). System (8) is equivalent to the following two differential systems:

\[
D_v L_0 = \sum_{j=1}^{M} \left( \lambda_j D_v L_j + \frac{d}{dt} \frac{\partial L_j}{\partial \dot{x}_v} \right) + \sum_{j=M+1}^{N} \lambda_j^0 D_v L_j, \quad L_k = 0 \iff \quad (9)
\]

\[
E_v L_0 = \sum_{k=1}^{N} A_{jk} \frac{\partial L_0}{\partial x_k} + \sum_{j=1}^{M} \left( \lambda_j D_v L_j + \frac{d}{dt} \frac{\partial L_j}{\partial \dot{x}_v} \right) + \sum_{j=M+1}^{N} \lambda_j^0 D_v L_j, \quad L_k = 0.
\]

for \( v = 1, \ldots, N \) and \( k = 1, \ldots, M \).

**Theorem 2** Using the notation of Theorem 1 let

\[
L = L(t, x, \dot{x}, \Lambda) = L_0(t, x, \dot{x}) - \sum_{j=1}^{M} \lambda_j L_j(t, x, \dot{x}) - \sum_{j=M+1}^{N} \lambda_j^0 L_j(t, x, \dot{x}) \quad (10)
\]

be the Lagrangian and let \( L_j(t, x, \dot{x}) = 0 \) be the independent constraints for \( j = 1, \ldots, M \), and let \( \lambda_j^0 \) be the arbitrary constants for \( k = M + 1, \ldots, N \),

\[
L_k : \mathbb{R} \times TQ \to \mathbb{R} \text{ for } k = M + 1, \ldots, N \text{ arbitrary functions such that}
\]

\[
|W_1| = \det W_1 = \det \left( \frac{\partial (L_1, \ldots, L_N)}{\partial (\dot{x}_1, \ldots, \dot{x}_N)} \right) \neq 0,
\]

except perhaps in a zero Lebesgue measure set \( |W_1| = 0 \). We determine the matrix \( A \) satisfying

\[
W_1 A = \Omega_1 := \begin{pmatrix} E_1 L_1 & \cdots & E_N L_1 \\ \vdots & \ddots & \vdots \\ E_1 L_N & \cdots & E_N L_N \end{pmatrix}.
\]

Then the differential equations (9) become

\[
D_v L_0 = \sum_{\alpha=1}^{M} \lambda_\alpha \frac{\partial L_\alpha}{\partial \dot{x}_v} \quad \text{for } v = 1, \ldots, N
\]

\[
\iff \quad \frac{d}{dt} \frac{\partial L_0}{\partial \dot{x}_v} - \frac{\partial L_0}{\partial x_v} = \left( W_1^{-1} \Omega_1 \right)^T \frac{\partial L_0}{\partial \dot{x}} + W_1 \frac{d}{dt} \frac{\partial \lambda}{\partial \dot{x}},
\]

where

\[
\frac{\partial}{\partial \dot{x}_v} = \left( \frac{\partial}{\partial \dot{x}_1}, \ldots, \frac{\partial}{\partial \dot{x}_N} \right)^T, \quad \frac{\partial}{\partial x_v} = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_N} \right)^T, \lambda = (\lambda_1, \ldots, \lambda_M, 0, \ldots, 0)^T, \quad \text{and the transposition relation (7) becomes}
\]

\[
\delta \frac{d}{dt} \frac{\partial}{\partial \dot{x}} = \left( W_1^{-1} \Omega_1 \right) \delta \dot{x}.
\]

**Theorem 3** Using the notation of Theorem 1 let

\[
L(t, x, \dot{x}, \Lambda) = L_0(t, x, \dot{x}) - \sum_{j=1}^{M} \lambda_j L_j(t, x, \dot{x}) - \sum_{j=M+1}^{N} \lambda_j^0 L_j(t, x, \dot{x})
\]

be the Lagrangian and \( L_j(t, x, \dot{x}) = 0 \) be the independent constraints for \( j = 1, \ldots, M < N \), and let \( \lambda_j^0 \) be arbitrary constants, for \( j = M + 1, \ldots, N - 1 \) and \( \lambda_N^0 = 0 \). Let \( L_j : \mathbb{R} \times TQ \to \mathbb{R} \) for \( j = M + 1, \ldots, N - 1 \) arbitrary functions, and \( L_N = L_0 \) such that

\[
|W_2| = \det W_2 = \det \left( \frac{\partial (L_1, \ldots, L_{N-1}, L_0)}{\partial (\dot{x}_1, \ldots, \dot{x}_N)} \right) \neq 0,
\]
except perhaps in a zero Lebesgue measure set \(|W_2| = 0\). We determine the matrix \(A\) satisfying

\[
W_2 A = \Omega_2 := \begin{pmatrix}
E_1 L_1 & \ldots & E_N L_1 \\
E_1 L_{N-1} & \ldots & E_N L_{N-1} \\
0 & \ldots & 0
\end{pmatrix}.
\]

Then the differential equations (9) become

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = W_2 T \frac{d}{dt} \lambda,
\]

where \(\lambda := \tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_M, 0, \ldots, 0)^T\), and the transpositional relation (7) becomes

\[
\frac{d}{dt} x - \frac{d}{dt} \dot{x} = \left( W_2^{-1} \Omega_2 \right) \dot{\lambda}.
\]

The proofs of Theorems 1, 2, and 3 are given in Sect. 5.

**Theorem 4** Under the assumptions of Theorem 2 and assuming that

\[
x_\alpha = x_\alpha, \quad x_\beta = y_\beta, \quad x = (x_1, \ldots, x_s), \quad y = (y_1, \ldots, y_s), \quad L_\alpha = \dot{x}_\alpha - \Phi_\alpha (x, y, \dot{x}, \dot{y}) = 0, \quad L_\beta = \dot{y}_\beta,
\]

for \(\alpha = 1, \ldots, s_1 = M\) and \(\beta = s_1 + 1, \ldots, s_1 + s_2 = N\).

Then \(|W_1| = 1\) and the differential equations (12) take the form

\[
E_j L_0 = \sum_{\alpha=1}^{s_1} \left( E_j L_\alpha \frac{\partial L_0}{\partial \dot{x}_\alpha} + \lambda_j \right), \quad j = 1, \ldots, s_1,
\]

\[
E_k L_0 = \sum_{\alpha=1}^{s_1} \left( E_k L_\alpha \frac{\partial L_0}{\partial \dot{x}_\alpha} + \lambda_k \frac{\partial L_\alpha}{\partial \dot{y}_k} \right), \quad k = 1, \ldots, s_2.
\]

or, equivalently (excluding the Lagrange multipliers)

\[
E_k L_0 = \sum_{\alpha=1}^{s_1} \left( E_k L_\alpha \frac{\partial L_0}{\partial \dot{x}_\alpha} \right) + (E_\alpha L_0)
\]

\[
- \sum_{\beta=1}^{s_1} \left( E_\alpha L_\beta \frac{\partial L_0}{\partial \dot{x}_\beta} \right) \frac{\partial L_\alpha}{\partial \dot{y}_k}, \quad k = 1, \ldots, s_2.
\]

In particular if we choose \(L_0 = \tilde{L} (x, y, \dot{x}, \dot{y}) - \tilde{L} (x, y, \dot{x}, \dot{y}) = \tilde{L} - L^s\), where \(\Phi = (\Phi_1, \ldots, \Phi_{s_1})\), then (19) holds if

\[
E_k \tilde{L} = \sum_{\alpha=1}^{s_1} E_\alpha \tilde{L} \frac{\partial L_\alpha}{\partial \dot{y}_k}, \quad k = 1, \ldots, s_2,
\]

and

\[
E_k (L^*) = \sum_{\alpha=1}^{s_1} \left( \frac{d}{dr} \left( \frac{\partial \Phi_\alpha}{\partial y_k} \right) - \left( \frac{\partial \Phi_\alpha}{\partial y_k} + \sum_{\nu=1}^{s_1} \frac{\partial \Phi_\alpha}{\partial x_\nu} \frac{\partial L_\nu}{\partial \dot{y}_k} \right) \right) \Psi_\alpha
\]

\[
+ \sum_{\nu=1}^{s_1} \frac{\partial L^*}{\partial x_\nu} \frac{\partial \Psi_\nu}{\partial \dot{y}_k},
\]

where \(\Psi_\alpha = \frac{\partial \tilde{L}}{\partial \dot{x}_\alpha} \bigg|_{\dot{x}_1 = \Phi_1, \ldots, \dot{x}_s = \Phi_s}\).

The transpositional relations (13) in this case are

\[
\delta \frac{dx_\alpha}{dt} - \frac{d}{dt} \delta x_\alpha = \sum_{k=1}^{s_2} \left( \sum_{j=1}^{s_1} E_j (L_\alpha) \frac{\partial L_j}{\partial \dot{y}_k} + E_k (L_\alpha) \right) \delta y_k,
\]

\[
\delta \frac{dy_m}{dt} - \frac{d}{dt} \delta y_m = 0, \quad m = 1, \ldots, s_2.
\]

**Proposition 5** Differential equations (20) describe the motion of the nonholonomic systems with the constraints \(L_\alpha = \dot{x}_\alpha - \Phi_\alpha (x, y, \dot{x}, \dot{y}) = 0\) for \(\alpha = 1, \ldots, s_1\).

In particular if the constraints are given by the formula

\[
\dot{x}_j = \sum_{k=1}^{s_2} a_{jk} (t, x, y) \dot{y}_k + a_j (t, x), \quad j = 1, \ldots, s_1,
\]

then systems (20) become

\[
E_k (L^*) = \sum_{\alpha=1}^{s_1} \left( \frac{d a_{\alpha k}}{dt} - \left( \frac{\partial a_{\alpha m}}{\partial y_k} \right) \right) \dot{y}_m \Psi_\alpha + \sum_{\nu=1}^{s_1} \frac{\partial L^*}{\partial x_\nu} a_{\nu k},
\]

which are the classical Voronets differential equations.

Consequently, Eq. (20) are an extension of the Voronets differential equations for the case when the constraints are nonlinear in the velocities.

**Proposition 6** Differential equations (20) describe the motion of the constrained Lagrangian systems with the
constraints $L_\alpha = \dot{x}_\alpha - \Phi_\alpha(y, \dot{y}) = 0$ and Lagrangian $L^* = L^*(y, \dot{y})$. Under these assumptions Eq. (20) take the form

$$E_k(L^*) = \sum_{\alpha=1}^{s_1} \left( \frac{d}{dt} \left( \frac{\partial \Phi_\alpha}{\partial \dot{y}_k} - \frac{\partial \Phi_\alpha}{\partial y_k} \right) \right) \Psi_\alpha. \tag{23}$$

In particular if the constraints are given by the formula

$$\dot{x}_\alpha = \sum_{k=1}^{s_2} a_{\alpha k}(y) \dot{y}_k, \quad \alpha = 1, \ldots, s_1, \tag{24}$$

then systems (23) become

$$E_k L^* = \sum_{j=1}^{s_1} \sum_{k=1}^{s_2} \left( \frac{\partial a_{jk}}{\partial y_r} - \frac{\partial a_{jk}}{\partial y_k} \right) \dot{y}_r \psi_j, \tag{25}$$

for $k = 1, \ldots, s_2$, which are the equations which Chaplygin published in the Proceeding of the Society of the Friends of Natural Science in 1897.

Consequently, Eq. (23) are an extension of the classical Chaplygin equations for the case when the constraints are nonlinear.

From (5) and in view of the Implicit Function Theorem, we can locally express the constraints (reordering coordinates if is necessary) as

$$\dot{x}_\alpha = \Phi_\alpha(x, \dot{x}_{M+1}, \ldots, \dot{x}_N) \tag{26}$$

for $\alpha = 1, \ldots, M$. We note that Propositions 5 and 6 are also valid for every constrained mechanical systems with constraints locally given by (26), this follows from Theorem 4 changing the notations, see Corollary 22.

The proofs of Theorem 4 and Propositions 5 and 6 are given in Sect. 8.

The next result is the third point of view on the transpositional relations.

**Corollary 7** For the constrained mechanical systems the virtual variations can produce zero or non-zero transpositional relations. For the unconstrained mechanical systems the virtual variations always produce zero transpositional relations.

The proof of this corollary is given in Sect. 9.

We have the following conjecture.

**Conjecture 8** The existence of mechanical systems with nonlinear constraints in the velocity must be sought outside of the Newtonian model.

This conjecture is supported by several facts see Sect. 9.

The results are illustrated with precise examples.

### 3 Variational principles. Transpositional relations

#### 3.1 Hamiltonian principle

We introduce the following results, notations, and definitions which we will use later on (see [2]).

A Lagrangian system is a pair $(Q, L)$ consisting of a smooth manifold $Q$, and a smooth function $\bar{L} : \mathbb{R} \times TQ \rightarrow \mathbb{R}$, where $TQ$ is the tangent bundle of $Q$. The point $x = (x_1, \ldots, x_N) \in Q$ denotes the position (usually its components are called generalized coordinates) of the system and we call each tangent vector $\dot{x} = (\dot{x}_1, \ldots, \dot{x}_N) \in T_x Q$ the velocity (usually called generalized velocity) of the system at the point $x$. A pair $(x, \dot{x})$ is called a state of the system. In Lagrangian mechanics it is usual to call $Q$, the configuration space, the tangent bundle $TQ$ is called the phase space, $\bar{L}$ is the Lagrange function or Lagrangian and the dimension $N$ of $Q$ is the number of degrees of freedom.

Let $a_0$ and $a_1$ be two points of $Q$. The map

$$\gamma : [t_0, t_1] \subset \mathbb{R} \rightarrow Q,$$

$$t \rightarrow \gamma(t) = (x_1(t), \ldots, x_N(t)),$$

such that $\gamma(t_0) = a_0$, $\gamma(t_1) = a_1$ is called a path from $a_0$ to $a_1$. We denote the set of all these path by $\Omega(Q, a_0, a_1, t_0, t_1) := \Omega$.

We shall derive one of the most simplest and general variational principles the Hamiltonian principle (see [40]).

The functional $F : \Omega \rightarrow \mathbb{R}$ defined by

$$F(\gamma(t)) = \int_{\gamma(t)} \bar{L} dt = \int_{t_0}^{t_1} \bar{L}(t, x(t), \dot{x}(t)) dt$$

is called the action.

We consider the path $\gamma(t) = x(t) = (x_1(t), \ldots, x_N(t)) \in \Omega$.

Let the variation of the path $\gamma(t)$ be defined as a smooth mapping

$$\gamma^* : [t_0, t_1] \times [-\delta, \delta] \rightarrow Q,$$

$$(t, \delta) \rightarrow \gamma^*(t, \delta) = x^*(t, \delta)$$

$$= (x_1(t) + \delta \delta x_1(t), \ldots, x_N(t) + \delta \delta x_N(t)),$$
satisfying
\[ x^* (t_0, \varepsilon) = a_0, \quad x^* (t_1, \varepsilon) = a_1, \quad x^* (t, 0) = x(t). \]
By definition we have
\[ \delta x(t) = \left. \frac{\partial x^*(t, \varepsilon)}{\partial \varepsilon} \right|_{\varepsilon=0}. \]
This function is called the virtual displacement or virtual variation corresponding to the variation of \( \gamma(t) \) and it is a function of time, all its components are functions of \( t \) of class \( C^2 (t_0, t_1) \) and vanish at \( t_0 \) and \( t_1 \) i.e., \( \delta x(t_0) = \delta x(t_1) = 0. \)
A varied path is a path which can be obtained as a variation path.

The first variation of the functional \( F \) at \( \gamma(t) \) is
\[ \delta F := \left. \frac{\partial F (x^*(t, \varepsilon))}{\partial \varepsilon} \right|_{\varepsilon=0}, \]
and it is called the differential of the functional \( F \) (see [2]). The path \( \gamma(t) \in \Omega \) is called the critical point of \( F \) if \( \delta F (\gamma(t)) = 0. \)

Let \( \mathbb{L} \) be the space of all smooth functions \( g : \mathbb{R} \times TQ \rightarrow \mathbb{R} \). The operator
\[ E_v : \mathbb{L} \rightarrow \mathbb{R}, \quad g \mapsto E_v g = \frac{d}{dt} \frac{\partial g}{\partial x_v} - \frac{\partial g}{\partial x_v}, \quad \text{for} \quad v = 1, \ldots, N, \]
is known as the Lagrangian derivative.

It is easy to show the following property of the Lagrangian derivative
\[ E_v \frac{df}{dt} = 0, \quad (27) \]
for arbitrary smooth function \( f = f(t, x) \). We observe that in view of (27) we obtain that the Lagrangian derivative is unchanged if we replace the function \( g \) by \( g + \frac{df}{dt} \) for any function \( f = f(t, x) \). This reflects the gauge invariance. We shall say that the functions \( g = g(t, x, \hat{x}) \) and \( \hat{g} = \hat{g}(t, x, \hat{x}) \) are equivalent if \( g - \hat{g} = \frac{df(t, x)}{dt} \), and we shall write \( g \simeq \hat{g} \).

**Proposition 9** The differential of the action can be calculated as follows:
\[ \delta F = - \int_{t_0}^{t_1} \sum_{k=1}^{N} \left( E_k \tilde{L} \delta x_k - \frac{\partial \tilde{L}}{\partial x_k} \left( \delta \frac{dx_k}{dt} - \frac{d}{dt} \delta x_k \right) \right) dt, \]
where \( x = x(t), \quad \dot{x} = \frac{dx}{dt}, \quad \text{and} \quad \tilde{L} = \tilde{L} \left( t, x, \frac{dx}{dt} \right). \]

**Proof** We have that
\[ \delta F = \left. \frac{\partial F (x^*(t, \varepsilon))}{\partial \varepsilon} \right|_{\varepsilon=0} \]
\[ = \int_{t_0}^{t_1} \sum_{k=1}^{N} \left( \frac{\partial \tilde{L}}{\partial x_k} \delta x_k + \frac{\partial \tilde{L}}{\partial \dot{x}_k} \delta \dot{x}_k \right) dt \]
\[ = \int_{t_0}^{t_1} \sum_{k=1}^{N} \left( \frac{\partial \tilde{L}}{\partial x_k} \delta x_k + \frac{\partial \tilde{L}}{\partial \dot{x}_k} \delta \dot{x}_k \right) dt \]
\[ = \int_{t_0}^{t_1} \sum_{k=1}^{N} \left( \frac{\partial \tilde{L}}{\partial x_k} \delta x_k + \frac{\partial \tilde{L}}{\partial \dot{x}_k} \delta \dot{x}_k \right) dt \]
\[ = \sum_{k=1}^{N} \int_{t_0}^{t_1} \left( \frac{\partial L}{\partial x_k} \delta x_k + \frac{\partial L}{\partial \dot{x}_k} \delta \dot{x}_k \right) dt. \]

Hence, by considering that the virtual variation vanishes at the points \( t = t_0 \) and \( t = t_1 \) we obtain the proof of the proposition. \( \square \)

**Corollary 10** The differential of the action for a Lagrangian system \( (Q, \tilde{L}) \) can be calculated as follows:
\[ \delta F = - \int_{t_0}^{t_1} \sum_{k=1}^{N} E_k \tilde{L} \left( t, x, \frac{dx}{dt} \right) \delta x_k dt. \]

**Proof** Indeed, for the Lagrangian system the transpositional relation is equal to zero (see for instance [32] page 29), i.e.,
\[ \frac{d}{dt} \frac{dx}{dt} - \frac{d}{dt} \delta x = 0. \]
Thus, from Proposition 9, it follows the proof of the corollary. \( \square \)

The path \( \gamma(t) \in \Omega \) is called a motion of the Lagrangian systems \( (Q, \tilde{L}) \) if \( \gamma(t) \) is a critical point of the action \( F \), i.e.,
\[ \delta F (\gamma(t)) = 0 \iff \int_{t_0}^{t_1} \delta \tilde{L} dt = 0. \]

This definition is known as the Hamiltonian variational principle or Hamilton variational principle of least action or simple Hamilton principle.

Now we need the Lagrange lemma or fundamental lemma of calculus of variations (see for instance [1])
Let \( f \) be a continuous function of the interval \([t_0, t_1]\) satisfying the equation
\[
\int_{t_0}^{t_1} f(t) \zeta(t) dt = 0,
\]
for arbitrary continuous function \( \zeta(t) \) such that \( \zeta(t_0) = \zeta(t_1) = 0 \). Then \( f(t) \equiv 0 \).

**Corollary 12** The Hamiltonian principle for Lagrangian systems is equivalent to the Lagrangian equations
\[
E_v \dot{L} = \frac{d}{dt} \left( \frac{\partial \tilde{L}}{\partial \dot{x}_v} \right) - \frac{\partial \tilde{L}}{\partial x_v} = 0,
\]
for \( v = 1, \ldots, N \).

**Proof** Clearly, if (30) holds, by Corollary 2, \( \delta F = 0 \). The reciprocal result follows from Lemma 11. \( \square \)

3.2 D’Alembert–Lagrange principle

Let \( L_j : \mathbb{R} \times TQ \rightarrow \mathbb{R} \) be smooth functions for \( j = 1, \ldots, M \). The equations
\[
L_j = L_j(t, x, \dot{x}) = 0, \quad \text{for} \quad j = 1, \ldots, M < N,
\]
with rank \( \left( \frac{\partial (L_1, \ldots, L_M)}{\partial (\dot{x}_1, \ldots, \dot{x}_N)} \right) = M \) in all the points of \( \mathbb{R} \times TQ \), except perhaps in a zero Lebesgue measure set, define \( M \) independent constraints for the Lagrangian systems \((Q, L)\).

Let \( \mathcal{M}^* \) be the submanifold of \( \mathbb{R} \times TQ \) defined by the equations (??), i.e.,
\[
\mathcal{M}^* = \{(t, x, \dot{x}) \in \mathbb{R} \times TQ : L_j(t, x, \dot{x}) = 0, \quad \text{for} \quad j = 1, \ldots, M \}.
\]
A constrained Lagrangian system is a triplet \((Q, \tilde{L}, \mathcal{M}^*)\). The number of degree of freedom is \( \kappa = dimQ - M = N - M \).

The constraint is called integrable if it can be written in the form \( L_j = \frac{d}{dt} (G_j(t, x)) = 0 \), for a convenient function \( G_j \). Otherwise the constraint is called nonintegrable. According to Hertz [17] the nonintegrable constraints are also called nonholonomic.

The Lagrangian systems with nonintegrable constraints are usually called (also following to Hertz) the nonholonomic mechanical systems or nonholonomic constrained mechanical systems, and with integrable constraints are called the holonomic constrained mechanical systems or holonomic constrained Lagrangian systems. The systems free of constraints are called Lagrangian systems or holonomic systems.

Sometimes it is also useful to distinguish between constraints that are dependent on or independent of time. Those that are independent of time are called scleronomic, and those that depend on time are called rheonomic. This terminology can also be applied to the mechanical systems themselves. Thus, we say that the constrained Lagrangian systems are scleronomic (reonomic) if the constraints and Lagrangian are time independent (dependent).

The constraints
\[
L_k = \sum_{j=1}^{N} a_{kj} \dot{x}_j + a_k = 0, \quad \text{for} \quad k = 1, \ldots, M, \quad (31)
\]
where \( a_{kj} = a_{kj}(t, x) \), \( a_k = a_k(t, x) \), are called linear constraints with respect to the velocity. For simplicity we shall call linear constraints.

We observe that (31) admits an equivalent representation as a Pfaffian equations (for more details see [38])
\[
\omega_k := \sum_{j=1}^{N} a_{kj} dx_j + a_k dt = 0.
\]

We shall consider only two classes of systems of equations, the equations of constraints linear with respect to the velocity \((\dot{x}_1, \ldots, \dot{x}_N)\), or linear with respect to the differential \((dx_1, \ldots, dx_N, dt)\). In order to study the integrability or nonintegrability problem of the constraints the last representation of a Pfaffian system is
the more useful. This is related with the fact that for the
given 1-forms we have the Frobenius theorem which
provides the necessary and sufficient conditions under
which the 1-forms are closed and consequently the
given set of constraints is integrable.

The constraints \( L_j(t, \mathbf{x}, \dot{\mathbf{x}}) = 0 \) are called perfect
constraints or ideal if they satisfy the Chetaev condi-
tions (see [8])

\[
\sum_{k=1}^{N} \frac{\partial L_\alpha}{\partial \dot{x}_k} \delta x_k = 0, \tag{32}
\]

for \( \alpha = 1, \ldots, M \).

In what follows, we shall consider only perfect con-
straints.

If the constraints admit the representation (26) then
the Chetaev conditions take the form

\[
\delta x_\alpha = \sum_{k=M+1}^{N} \frac{\partial \Phi_\alpha}{\partial \dot{x}_k} \delta x_k.
\]

The virtual variations of the variables \( x_\alpha \) for \( \alpha = 1, \ldots, M \) are called dependent variations and of the
variable \( x_\beta \) for \( \beta = M + 1, \ldots, N \) are called indepen-
dent variations.

We say that the path \( \gamma(t) = \mathbf{x}(t) \) is admissible with
the perfect constraint if \( L_j(t, \mathbf{x}(t), \dot{\mathbf{x}}(t)) = 0 \).

The admissible path is called the motion of the
constrained Lagrangian systems \((\mathbf{Q}, \bar{L}, \mathcal{M}^\mathbf{a})\) if for all
\( \mathbf{x}(t) \in \mathcal{M}^\mathbf{a} \)

\[
\sum_{i=1}^{N} E_i \bar{L}_i(t, \mathbf{x}(t), \dot{\mathbf{x}}(t)) \delta x_i(t) = 0,
\]

for all virtual displacement \( \delta \mathbf{x}(t) \) of the path \( \gamma(t) \). This
definition is known as d’Alembert–Lagrange principle.

The following result is well known (see for instance
\([2,5,15,35]\)).

**Proposition 13** The d’Alembert–Lagrange principle
for constrained Lagrangian systems is equivalent to
the Lagrangian differential equations with multipliers

\[
E_j \bar{L}_j = \frac{d}{dt} \frac{\partial \bar{L}}{\partial \dot{x}_j} - \frac{\partial \bar{L}}{\partial x_j}
= \sum_{a=1}^{M} \mu_\alpha \frac{\partial L_\alpha}{\partial x_j}, \quad \text{for } j = 1, \ldots, N,
\]

\( L_j(t, \mathbf{x}, \dot{\mathbf{x}}) = 0 \), \quad \text{for } j = 1, \ldots, M, \tag{33}

where \( \mu_\alpha \) for \( \alpha = 1, \ldots, M \) are the Lagrangian mul-
tipliers.

3.3 The varied path

The varied path produced in Hamiltonian’s principle
is not in general an admissible path if the perfect con-
straints are nonholonomic, i.e., the mechanical systems
cannot travel along the varied path without violating the
constraints. We prove the following result, which shall
play an important role in the all assertions below.

**Proposition 14** If the varied path is an admissible path
then the following relations hold

\[
\sum_{k=1}^{N} \frac{\partial L_\alpha}{\partial \dot{x}_k} \left( \delta \frac{dx_k}{dr} - \frac{d}{dr} \delta x_k \right) = \sum_{k=1}^{N} E_k L_\alpha \delta x_k, \tag{34}
\]

for \( \alpha = 1, \ldots, M \).

**Proof** Indeed, the original path \( \gamma(t) = \mathbf{x}(t) \) by definition
satisfies the Chetaev conditions and constraints, i.e.,
\( L_j(t, \mathbf{x}(t), \dot{\mathbf{x}}(t)) = 0 \). If we suppose that the vari-
ated path \( \gamma^\varepsilon(t) = \mathbf{x}(t) + \varepsilon \delta \mathbf{x}(t) \) also satisfies the con-
straints i.e.,

\[
L_j(t, \mathbf{x} + \varepsilon \delta \mathbf{x}, \dot{\mathbf{x}} + \varepsilon \delta \dot{\mathbf{x}}) = L_j(t, \mathbf{x}(t), \dot{\mathbf{x}}(t))
+ \varepsilon \delta L_\alpha (t, \mathbf{x}(t), \dot{\mathbf{x}}(t)) + \ldots = 0.
\]

Thus, restricting only to the terms of first order with
respect to \( \varepsilon \) and by the Chetaev conditions we have
(for simplicity we omitted the argument)

\[
0 = \delta L_\alpha = \sum_{k=1}^{N} \left( \frac{\partial L_\alpha}{\partial x_k} \delta x_k + \frac{\partial L_\alpha}{\partial \dot{x}_k} \delta \dot{x}_k \right), \tag{35}
\]

for \( \alpha = 1, \ldots, M \). The Chetaev conditions are satisfied
at each instant, so

\[
\frac{d}{dr} \left( \sum_{k=1}^{N} \frac{\partial L_\alpha}{\partial \dot{x}_k} \delta x_k \right) = \sum_{k=1}^{N} \frac{d}{dr} \left( \frac{\partial L_\alpha}{\partial \dot{x}_k} \right) \delta x_k
+ \sum_{k=1}^{N} \frac{\partial L_\alpha}{\partial \dot{x}_k} \frac{d}{dr} \delta x_k = 0.
\]

Subtracting these relations from (35) we obtain (34).
Consequently, if the varied path is an admissible path,
then relations (34) must hold. □
From (34) and (7) it follows that the elements of the matrix $A$ satisfy
\[
\sum_{m=1}^{N} \delta x_m \left( E_m L_\alpha - \sum_{k=1}^{N} A_{km} \frac{\partial L_\alpha}{\partial \dot{x}_k} \right) \\
= \sum_{m=1}^{N} \delta x_m D_m L_\alpha = 0, \quad \text{for } \alpha = 1, \ldots, M. \quad (36)
\]
This property will be used below.

**Corollary 15** For the holonomic constrained Lagrangian systems the relations (34) hold if and only if
\[
\sum_{k=1}^{N} \frac{\partial L_\alpha}{\partial \dot{x}_k} \left( \delta \frac{dx_k}{dt} - \frac{d}{dt} \delta x_k \right) = 0, \quad \text{for } \alpha = 1, \ldots, M.
\]
(37)

**Proof** Indeed, for holonomic constrained Lagrangian systems the constraints are integrable, consequently in view of (27) we have $E_k L_\alpha = 0$ for $k = 1, \ldots, N$ and $\alpha = 1, \ldots, M$. Thus, from (34), we obtain (37).

Clearly, the equalities (37) are satisfied if (29) holds. We observe that in general for holonomic constrained Lagrangian systems relation (29) cannot hold (see example 2).

### 3.4 Transpositional relations

As we observe in the previous subsection for nonholonomic constrained Lagrangian systems the curves, obtained doing a virtual variation in the motion of the systems, in general are not kinematical possible trajectories when (29) is not fulfilled. This leads to the conclusion that the Hamiltonian principle cannot be applied to nonholonomic systems, as it is usually employed for holonomic systems. The essence of the problem of the applicability of this principle for nonholonomic systems remains unclarified (see [35]). In order to clarify this situation, it is sufficient to note that the question of the applicability of the principle of stationary action to nonholonomic systems is intimately related to the question of transpositional relation.

The key point is that the Hamiltonian principle assumes that the operations of differentiation with respect to the time $\frac{d}{dt}$ and the virtual variation $\delta$ commute in all the generalized coordinate systems.

For the holonomic constrained Lagrangian systems relations (29) cannot hold (see Corollary 3). For a nonholonomic systems the form of the Hamiltonian principle will depend on the point of view adopted with respect to the transpositional relations.

What are then the correct transpositional relations?

Until now, there does not exist a common point of view concerning to the commutativity of the operation of differentiation with respect to the time and the virtual variation when there are nonintegrable constraints. Two points of view have been maintained. According to one (supported, for example, by Volterra, Hamel, Hölder, Lurie, Pars,...), the operations $\frac{d}{dt}$ and $\delta$ commute for all the generalized coordinates, independently if the systems are holonomic or nonholonomic, i.e.,
\[
\delta \frac{dx_k}{dt} = \frac{d}{dt} \delta x_k = 0, \quad \text{for } k = 1, \ldots, N.
\]

According to the other point of view (supported by Suslov, Voronets, Levi-Civita, Amaldi,...) the operations $\frac{d}{dt}$ and $\delta$ commute always for holonomic systems, and for nonholonomic systems with the constraints
\[
\dot{x}_\alpha = \sum_{j=M+1}^{N} a_{\alpha j}(t, x) \dot{x}_j + a_\alpha(t, x), \quad \text{for } \alpha = 1, \ldots, M.
\]
the transpositional relations are equal to zero only for the generalized coordinates $x_{M+1}, \ldots, x_N$, (for which their virtual variations are independent). For the remaining coordinates $x_1, \ldots, x_M$, (for which their virtual variations are dependent), the transpositional relations must be derived on the basis of the equations of the nonholonomic constraints, and cannot be identically zero, i.e.,
\[
\delta \frac{dx_k}{dt} = \frac{d}{dt} \delta x_k = 0, \quad \text{for } k = M + 1, \ldots, N
\]
\[
\delta \frac{dx_k}{dt} = \frac{d}{dt} \delta x_k \neq 0, \quad \text{for } k = 1, \ldots, M.
\]

The second point of view acquired general acceptance and the first point of view was considered erroneous (for more details see [35]). The meaning of the transpositional relations (1) can be found in [20,32,34,35].

In the results given in the following section the equalities play a key role (34). From these equalities and from the examples it will be possible to observe that the second point of view is correct only for the so called Voronets–Chaplygin systems, and in general for locally nonholonomic systems. There exist many examples for which the independent virtual variations...
generated non-zero transpositional relations. Thus, we propose a third point of view on the transpositional relations: the virtual variations can generate the transpositional relations given by the formula (7) where the elements of the matrix \( A \) satisfy the conditions (see formula (36))

\[
D_vL_\alpha = E_vL_\alpha - \sum_{k=1}^{N} A_{kv} \frac{\partial L_\alpha}{\partial \dot{x}_k} = 0,
\]

for \( v = 1, \ldots, M, \ \alpha = 1, \ldots, M. \) (38)

we observe that here the \( L_\alpha = 0 \) are constraints which in general are nonlinear in the velocity.

3.5 Hamiltonian–Suslov principle

After the introduction of the nonholonomic mechanics by Hertz, it appeared the question of extending to the nonholonomic mechanics the results of the holonomic mechanics. Hertz [17] was the first in studying the problem of applying the Hamiltonian principle to systems with nonintegrable constraints. In [17] Hertz wrote: “Application of Hamilton’s principle to any material systems does not exclude that between selected coordinates of the systems rigid constraints exist, but it still requires that these relations could be expressed by integrable constraints. The appearance of nonintegrable constraints is unacceptable. In this case the Hamilton’s principle is not valid.” Appell [3] in correspondence with Hertz’s ideas affirmed that it is not possible to apply the Hamiltonian principle for systems with nonintegrable constraints.

Suslov [48] claimed that “Hamilton’s principle is not applied to systems with nonintegrable constraints, as derived based on this equation are different from the corresponding equations of Newtonian mechanics”.

The applications of the most general differential principle, i.e., the d’Alembert–Lagrange and their equivalent Gauss and Appel principle, is complicated due to the presence of the terms containing the second-order derivative. On the other hand the most general variational integral principle of Hamilton is not valid for nonholonomic constrained Lagrangian systems. The generalization of the Hamiltonian principle for nonholonomic mechanical systems was deduced by Voronets and Suslov (see for instance [48,53]). As we can observe later on from this principle it follows the importance of the transpositional relations to determine the correct equations of motion for nonholonomic constrained Lagrangian systems.

**Proposition 16** The d’Alembert–Lagrangian principle for the constrained Lagrangian systems \( \sum_{k=1}^{N} \delta x_k E_k \tilde{L} = 0 \) is equivalent to the Hamilton–Suslov principle (2) where we assume that \( \delta x_v(t), \ v = 1, \ldots, N, \) are arbitrary smooth functions defined in the interior of the interval \([t_0, t_1]\) and vanishing at its endpoints, i.e., \( \delta x_v(t_0) = \delta x_v(t_1) = 0.\)

**Proof** From the d’Alembert–Lagrangian principle we obtain the identity

\[
0 = -\sum_{k=1}^{N} \delta x_k E_k \tilde{L} = \sum_{k=1}^{N} \delta x_k \frac{\partial \tilde{L}}{\partial \dot{x}_k} - \sum_{k=1}^{N} \delta x_k \frac{d}{dt} \frac{\partial \tilde{L}}{\partial \dot{x}_k}
\]

\[
= \sum_{k=1}^{N} \left( \delta x_k \frac{\partial \tilde{L}}{\partial \dot{x}_k} + \delta \dot{x}_k \frac{\partial \tilde{L}}{\partial \dot{x}_k} \right)
\]

\[
- \sum_{k=1}^{N} \left( \left( \frac{d}{dt} \delta x_k \right) \frac{\partial \tilde{L}}{\partial \dot{x}_k} - \frac{d}{dt} \left( \frac{\partial \tilde{L}}{\partial \dot{x}_k} \delta \dot{x}_k \right) \right)
\]

\[
= \delta \tilde{L} - \sum_{k=1}^{N} \left( \left( \frac{d}{dt} \delta x_k \right) \frac{\partial \tilde{L}}{\partial \dot{x}_k} - \frac{d}{dt} \left( \frac{\partial \tilde{L}}{\partial \dot{x}_k} \delta \dot{x}_k \right) \right),
\]

where \( \delta \tilde{L} \) is a variation of the Lagrangian \( \tilde{L} \). After the integration and assuming that \( \delta x_k(t_0) = 0, \delta x_k(t_1) = 0 \) we easily obtain (2), which represent the most general formulation of the Hamiltonian principle (Hamilton–Suslov principle) suitable for constrained and unconstrained Lagrangian systems. \( \square \)

Suslov determines the transpositional relations only for the case when the constraints are of Voronets type, i.e., given by the formula (22). Assume that

\[
\delta \frac{dy_k}{dt} - \frac{d}{dt} \delta y_k = 0, \ \text{for} \ k = M + 1, \ldots, N,
\]

Voronets and Suslov deduced that

\[
\delta \frac{dx_k}{dr} - \frac{d}{dr} \delta x_k = \sum_{k=1}^{N} B_{kr} \delta y_r - \delta a_k
\]

for convenient functions \( B_{kr} = B_{kr}(t, x, y, \dot{x}, \dot{y}) \), for \( r = M + 1, \ldots, N \) and \( k = 1, \ldots, M. \)

Thus, we obtain

\[
\int_{t_0}^{t_1} \left( \delta \tilde{L} - \sum_{k=1}^{N} \frac{\partial \tilde{L}}{\partial \dot{x}_j} \left( \sum_{k=1}^{N} B_{kr} \delta y_r - \delta a_k \right) \right) dt = 0,
\]
This is the Hamiltonian principle for nonholonomic systems in the Suslov form (see for instance [48]). We observe that the same result was deduced by Voronets in [53].

It is important to observe that Suslov and Voronets require a priori that the independent virtual variations produce the zero transpositional relations. Sometimes these authors consider only linear constraints with respect to the velocity of the type (22).

3.6 Modification of the vakonomic mechanics (MVM)

As we observe in the introduction, the main objective of this paper was to construct the variational equations of motion describing the behavior of the constrained Lagrangian systems in which the equalities (34) take place in the most general possible way. We shall show that the d’Alembert–Lagrange principle is not the only way to deduce the equations of motion for the constrained Lagrangian systems. Instead of it we consider that (C) is always fulfilled.

We introduce the additional coordinates \( \lambda_1, \ldots, \lambda_M \), and Lagrangian \( \hat{L} : \mathbb{R} \times TQ \times \mathbb{R}^M \rightarrow \mathbb{R} \) given by

\[
\hat{L} (t, \mathbf{x}, \dot{\mathbf{x}}, \hat{\Lambda}) = L_0 (t, \mathbf{x}, \dot{\mathbf{x}}) - \sum_{j=1}^{M} \lambda_j L_j (t, \mathbf{x}, \dot{\mathbf{x}}),
\]

where \( L_0 = 0 \) for \( \alpha = 1, \ldots, \lambda_M \) are the constraints.

A lot of authors consider that (C) is always fulfilled (see for instance [32,38]), together with the conditions (A) and (B). However, these conditions are incompatible in the case of the nonintegradable constrains. We observe that these authors deduced that the Hamiltonian principle is not applicable to the nonholonomic systems.

To obtain a generalization of the Hamiltonian principle for the nonholonomic mechanical systems, some of these three conditions must be excluded.

In particular for the Hölder principle conditions (A) is excluded and keep (B) and (C) (see [18]). For the Hamiltonian–Suslov principle condition (A) and (B) hold, and (C) only holds for the independent variations.

In this paper we extend the Hamiltonian principle by supposing that conditions (A) and (B) hold and (C) does not hold. Instead of (C) we consider that (7) holds where elements of matrix \( A \) satisfy the relations (38).

4 Solution of the inverse problem of the constrained Lagrangian systems

We shall determine the equations of motion of the constrained Lagrangian systems using the Hamiltonian principle with non-zero transpositional relations, whereby the motions of the systems are extremals of the variational Lagrange’s problem (see for instance [13]), i.e., are the critical points of the action functional

\[
\int_{t_0}^{t_1} L_0 (t, \mathbf{x}, \dot{\mathbf{x}}) \, dt,
\]

in the class of path with fixed endpoints satisfying the independent constraints

\( L_j (t, \mathbf{x}, \dot{\mathbf{x}}) = 0, \text{ for } j = 1, \ldots, M. \)

In classical solution of the Lagrange problem usually we apply the Lagrange multipliers method which consists in the following. We introduce the additional coordinates \( \Lambda = (\lambda_1, \ldots, \lambda_M) \), and Lagrangian \( \hat{L} : \mathbb{R} \times TQ \times \mathbb{R}^M \rightarrow \mathbb{R} \) given by

\[
\hat{L} (t, \mathbf{x}, \dot{\mathbf{x}}, \hat{\Lambda}) = L_0 (t, \mathbf{x}, \dot{\mathbf{x}}) - \sum_{j=1}^{M} \lambda_j L_j (t, \mathbf{x}, \dot{\mathbf{x}}),
\]

Under this choice we reduce the Lagrange problem to a variational problem without constraints, i.e., we must determine the extremal of the action functional

\[
\int_{t_0}^{t_1} \hat{L} \, dt.\]

We shall study a slight modification of the Lagrange multipliers method. We introduce the additional coordinates \( \Lambda = (\lambda_1, \ldots, \lambda_M) \), and the
Lagrangian systems.

Now we determine the critical points of the action functional $\int_{t_0}^{t_1} L (t, \dot{x}, x, \Lambda) \, dt$, i.e., we determine the path $\gamma(t)$ such that $\int_{t_0}^{t_1} \delta (L (t, \dot{x}, x, \Lambda)) \, dt = 0$ under the additional condition that the transpositional relations are given by the formula (7).

The solution of the inverse problem stated in Sect. 2 is the following. Differential equations obtained from (6) are given by the formula (8) (see Theorem 1). We choose the arbitrary functions $L_j$ in such a way that the matrices $W_1$ and $W_2$ given in Theorems 2 and 3 are nonsingular, except perhaps in a zero Lebesgue measure set. The constants $\lambda^0_j$ for $j = M + 1, \ldots, N$ are arbitrary in Theorem 2, and $\lambda^0_j$ for $j = 1, \ldots, N - 1$ are arbitrary and $\lambda^0_N = 0$ in Theorem 3. The matrix $A$ is determined from the equalities (11) and (15) of Theorems 2 and 3, respectively.

**Remark 17** It is interesting to observe that from the solutions of the inverse problem, the constants $\lambda^0_j$ for $j = M + 1, \ldots, N$ are arbitrary except in Theorem 3 in which $\lambda^0_N = 0$. Clearly, if $L_j (t, x, \dot{x}, x) = \frac{d}{dt} f_j (t, x)$ for $j = M + 1, \ldots, N$, then the $L \simeq \tilde{L}$. Using the arbitrariness of the constants $\lambda^0_j$ we can always determine that $\lambda_k = 0$ if $L_k (t, x, \dot{x}) \neq \frac{d}{dt} f_k (t, x)$. Consequently, we can always suppose that $L \simeq \tilde{L}$. Thus, the only difference between the classical and the modified Lagrangian multipliers method consists only on the transpositional relations: for the classical method the virtual variations produce zero transpositional relations (i.e., the matrix $A$ is the zero matrix) and for the modified method in general it is determined by the formulae (7) and (36).

A very important subcase is obtained when the constraints are given in the form (Voronets-Chapliguin constraints type) $\dot{x}_\alpha - \Phi_\alpha (t, x, \dot{x}_{M+1}, \ldots, \dot{x}_N) = 0$, for $\alpha = 1, \ldots, M$. As we shall show under these assumptions the arbitrary functions are determined as follows: $L_j = \dot{x}_j$ for $j = M + 1, \ldots, N$. Consequently, the action of the modified Lagrangian multipliers method and the action of the classical Lagrangian multipliers method are equivalently. In view of (26) this equivalence always locally holds for any constrained Lagrangian systems.

### 5 Proof of Theorems 1, 2 and 3

**Proof of Theorem 1** In view of the equalities

$$\int_{t_0}^{t_1} \delta L \, dt = \int_{t_0}^{t_1} \sum_{k=1}^{M} \left( \frac{\partial L}{\partial \lambda_k} \delta \lambda_k \right) \, dt + \int_{t_0}^{t_1} \sum_{j=1}^{N} \left( \frac{\partial L}{\partial x_j} \delta x_j + \frac{\partial L}{\partial \dot{x}_j} \delta \dot{x}_j \right) \, dt$$

$$= \int_{t_0}^{t_1} \sum_{k=1}^{M} (-L_k \delta \lambda_k) \, dt + \int_{t_0}^{t_1} \sum_{j=1}^{N} \left( \frac{\partial L}{\partial x_j} \delta x_j + \frac{\partial L}{\partial \dot{x}_j} \right) \delta \dot{x}_j \, dt$$

$$= \int_{t_0}^{t_1} \sum_{k=1}^{M} (-L_k \delta \lambda_k) \, dt + \int_{t_0}^{t_1} \sum_{j=1}^{N} \left( \frac{\partial T}{\partial \dot{x}_j} \right) \delta x_j \, dt$$

Consequently,

$$\int_{t_0}^{t_1} \delta L \, dt \bigg|_{L'=0} = \int_{t_0}^{t_1} \sum_{j=1}^{N} \left( \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}_j} \right) \delta x_j \right) \, dt$$

$$= \sum_{j=1}^{N} \frac{\partial T}{\partial \dot{x}_j} \delta x_j \bigg|_{t_0}^{t_1} - \int_{t_0}^{t_1} \sum_{j=1}^{N} \left( \frac{\partial E_j L}{\partial \dot{x}_j} - \sum_{k=1}^{N} A_{jk} \frac{\partial L}{\partial \dot{x}_k} \right) \delta x_j \, dt$$

$$= - \int_{t_0}^{t_1} \sum_{j=1}^{N} \left( E_j L - \sum_{k=1}^{N} A_{jk} \frac{\partial L}{\partial \dot{x}_k} \right) \delta x_j \, dt = 0,$$

where $\nu = 1, \ldots, M$. Here we use the equalities $\delta x(t_0) = \delta x(t_1) = 0$. Hence if (8) holds then (6) is satisfied. The reciprocal result is proved by choosing

$$\delta x_k (t) = \begin{cases} \zeta (t) & \text{if } k = 1, \\ 0 & \text{otherwise}, \end{cases}$$

where $\zeta (t)$ is a positive function in the interval $(t_0^*, t_1^*)$, and it is equal to zero in the intervals $[t_0, t_0^*]$ and $[t_1^*, t_1]$, and applying Corollary.
From the definition (8) we have that
\[ D_v(fg) = D_v f \, g + f \, D_v g + \frac{df}{d\dot{x}_v} \, \frac{dg}{dt} \]
\[ + \frac{df}{dt} \, \frac{dg}{d\dot{x}_v}, \quad D_v a = 0, \]
where \( a \) is a constant.

Now we shall write (8) in a more convenient way
\[ 0 = D_v L = D_v \left( L_0 - \sum_{j=1}^{M} \alpha_j L_j - \sum_{j=M+1}^{N} \lambda_j^0 L_j \right) \]
\[ = D_v L_0 - \sum_{j=1}^{M} D_v (\alpha_j L_j) - \sum_{j=M+1}^{N} \lambda_j^0 D_v L_j \]
\[ = D_v L_0 - \sum_{j=M+1}^{N} \lambda_j^0 D_v L_j - \sum_{j=1}^{M} \lambda_j D_v L_j \]
\[ + \sum_{j=1}^{M} \left( D_v \lambda_j L_j + \lambda_j D_v L_j + \frac{d\lambda_j}{dt} \frac{\partial L_j}{\partial \dot{x}_v} \right). \]

From these relations and since the constraints \( L_j = 0 \) for \( j = 1, \ldots, M \), we easily obtain Eq. (9) or equivalently
\[ E_v L_0 = \sum_{k=1}^{N} A_{kj} \frac{\partial L_0}{\partial \dot{x}_k} + \sum_{j=1}^{M} \left( \lambda_j D_v L_j + \frac{d\lambda_j}{dt} \frac{\partial L_j}{\partial \dot{x}_v} \right) \]
\[ + \sum_{j=M+1}^{N} \lambda_j^0 D_v L_j. \quad (39) \]

Thus, the theorem is proved. \( \square \)

Now we show that the differential Eq. (39) for convenient functions \( L_j \) constants \( \lambda_j^0 \) for \( j = M + 1, \ldots, N \) and for convenient matrix \( A \) describe the motion of the constrained Lagrangian systems.

**Proof of Theorem 2** The matrix Eq. (11) can be rewritten in components as follows:
\[ \sum_{j=1}^{N} A_{kj} \frac{\partial L_0}{\partial \dot{x}_k} = E_k L_0 \iff D_k L_0 = 0, \quad (40) \]
for \( \alpha, k = 1, \ldots, N \). Consequently, the differential Eq. (39) become
\[ E_v L_0 = \sum_{k=1}^{N} \left( A_{vk} \frac{\partial L_0}{\partial \dot{x}_k} + \frac{d\lambda_k}{dt} \frac{\partial L_k}{\partial \dot{x}_v} \right) \]
\[ \iff D_v L_0 = \sum_{j=1}^{M} \frac{d\lambda_j}{dt} \frac{\partial L_j}{\partial \dot{x}_v}, \quad (41) \]
which coincide with the first systems (12).

In view of the condition \(|W_1| \neq 0\) we can solve Eq. (11) with respect to \( A \) and obtain \( A = W_1^{-1} \Omega_1 \). Hence, by considering (40) we obtain the second systems from (12) and the transpositional relation (13). \( \square \)

**Proof of Theorem 3** The matrix Eq. (15) is equivalent to the systems
\[ \sum_{j=1}^{N} A_{kj} \frac{\partial L_0}{\partial \dot{x}_k} = E_k L_0 \iff D_k L_0 = 0, \]
\[ \sum_{j=1}^{N} A_{kj} \frac{\partial L_0}{\partial \dot{x}_k} = 0, \]
for \( k = 1, \ldots, N \), and \( \alpha = 1, \ldots, N - 1 \). Thus, by considering that \( \lambda_0^0 = 0 \) we deduce that systems (39) take the form
\[ E_v L_0 = \sum_{j=1}^{M} \frac{d\lambda_j}{dt} \frac{\partial L_j}{\partial \dot{x}_v}. \]

Hence we obtain systems (16). On the other hand from (15) we have that \( A = W_2^{-1} \Omega_2 \). Hence we deduce that the transpositional relation (7) can be rewritten in the form (17). \( \square \)

The mechanics based on the Hamiltonian principle with non-zero transpositional relations given by formula (7), Lagrangian (4), and equations of motion (8) are called here the modification of the vakonomic mechanics and we shortly write MVM.

From the proofs of Theorems 2 and 3 it follows that the relations (36) hold identically in MVM.

**Corollary 18** Differential equations (12) are invariant under the change
\[ L_0 \longleftrightarrow L_0 - \sum_{j=1}^{N} a_j L_j, \]
where the \( a_j \)s are constants for \( j = 1, \ldots, N \).

**Proof** Indeed, from (41) and (40) it follows that
\[ D_v \left( L_0 - \sum_{j=1}^{N} a_j L_j \right) = D_v L_0 - \sum_{j=1}^{N} a_j D_v L_j \]
\[ = D_v L_0 = \sum_{j=1}^{M} \frac{d\lambda_j}{dt} \frac{\partial L_j}{\partial \dot{x}_v}. \]
\( \square \)
The following interesting facts follow from Theorems 2 and 3.

1. The equations of motion obtained from Theorem 2 are more general than the equations obtained from Theorem 3. Indeed in (12) there are \( N - M \) arbitrary functions, while in (16) there are \( N - M - 1 \) arbitrary functions.

2. If the constraints are linear in the velocity then between the Lagrangian multipliers \( \mu \), \( \frac{d\lambda}{dt} \) and \( \frac{d\tilde{\lambda}}{dt} \) there is the following relation

\[
\mu = \frac{d\tilde{\lambda}}{dt} = \left( W_2^{-1} \right)^T \left( W_1^T \frac{d\lambda}{dt} + W_2^{-1} \Omega_1^T W_1^{-T} \frac{dL_0}{\partial \Omega} \right).
\]

where \( W_1 \) and \( W_2 \) are the matrixes defined in Theorems 2 and 3.

3. If the constraints are linear in the velocity then one of the important question which appears in MVM is related with the arbitrariness functions \( L_j \) for \( j = M + 1, \ldots, N \). The following question arises: Is it possible to determine these functions in such a way that \( |W_1| \) or \( |W_2| \) is non-zero everywhere in \( M^* \)? If we have a positive answer to this question, then the equations of motion of the MVM give a global behavior of the constrained Lagrangian systems, i.e., the obtained motions completely coincide with the motions obtained from the classical mathematical models. Thus, if \( |W_1| \neq 0 \) and \( |W_2| \neq 0 \) everywhere in \( M^* \) then we have the equivalence

\[
D_v L_0 = \sum_{j=1}^{M} \frac{d\lambda_j}{dt} \frac{\partial L_j}{\partial \dot{x}_v} \iff E_v L_0 = \sum_{j=1}^{M} \frac{d\tilde{\lambda}_j}{dt} \frac{\partial L_j}{\partial \dot{x}_v} \iff E_v L_0 = \sum_{j=1}^{M} \mu_j \frac{\partial L_j}{\partial \dot{x}_v}.
\]

If the constraints are nonlinear in the velocity and \( |W_2| \neq 0 \) everywhere in \( M^* \) then we have the equivalence

\[
E_v L_0 = \sum_{j=1}^{M} \frac{d\tilde{\lambda}_j}{dt} \frac{\partial L_j}{\partial \dot{x}_v} \iff E_v L_0 = \sum_{j=1}^{M} \mu_j \frac{\partial L_j}{\partial \dot{x}_v}.
\]

The equivalence with respect to the equations \( D_v L_0 = \sum_{j=1}^{M} \frac{d\lambda_j}{dt} \frac{\partial L_j}{\partial \dot{x}_v} \) in general is not valid in this case because the term \( \Omega_1^T W_1^{-T} \frac{dL_0}{\partial \Omega} \) depends on \( \dot{x} \).

5.1 Application of Theorems 2 and 3 to the Appell–Hamel mechanical systems

As a general rule the constraints studied in classical mechanics are linear with respect to the velocities, i.e., \( L_j \) can be written as (31). However, Appell and Hamel (see [3,16]), in 1911, considered an artificial example of nonlinear nonholonomic constrains. A big number of investigations have been devoted to the derivation of the equations of motion of mechanical systems with nonlinear nonholonomic constraints see for instance [9,16,35,36]. The works of these authors do not contain examples of systems with nonlinear nonholonomic constraints differing essentially from the example given by Appell and Hamel.

**Corollary 20** The equivalence (42) also holds for the Appell–Hamel system i.e., for the constrained Lagrangian systems

\[
\begin{align*}
\mathbb{R}^3, & \quad \tilde{L} = \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - g z, \quad \{\dot{z} - a \sqrt{\dot{x}^2 + \dot{y}^2} = 0\},
\end{align*}
\]

where \( a \) and \( g \) are positive constants.

**Proof** The classical equations (33) for the Appell-Hamel system are

\[
\dot{x} = -\frac{a \dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \mu, \quad \dot{y} = -\frac{a \dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \mu, \quad \dot{z} = -g + \mu,
\]

where \( \mu \) is the Lagrangian multiplier.

Now we apply Theorem 3. Hence, in order to obtain that \( |W_2| \neq 0 \) everywhere we choose the functions \( L_j \) for \( j = 1, 2, 3 \) as follows:

\[
L_1 = \dot{z} - a \sqrt{\dot{x}^2 + \dot{y}^2} = 0, \quad L_2 = \arctan \frac{\dot{x}}{\dot{y}}, \quad L_3 = L_0 = \tilde{L}.
\]
In this case the matrices $W_2$, $\Omega_2$, and $A$ are
\[
W_2 = \begin{pmatrix}
-\frac{a\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} & -\frac{a\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} & 1 \\
\frac{a\dot{x}}{\dot{x}^2 + \dot{y}^2} & -\frac{a\dot{y}}{\dot{x}^2 + \dot{y}^2} & 0 \\
\frac{\dot{y}}{\dot{x}^2 + \dot{y}^2} & -\frac{\dot{x}}{\dot{x}^2 + \dot{y}^2} & 0
\end{pmatrix},
\]
and the matrix $A|_{L_1=0}$ is
\[
A = \begin{pmatrix}
-\frac{\dot{y}(a^2\dot{x}^2 + (a^2 + 1)\dot{y}^2 + \dot{x}^2)}{(1 + a^2)(\dot{x}^2 + \dot{y}^2)} & 0 & -a^2\dot{y}^2 \\
(1 + a^2)(\dot{x}^2 + \dot{y}^2) & 0 & 0 \\
\frac{\dot{y}a(\dot{x}^2 + \dot{y}^2)}{(1 + a^2)(\dot{x}^2 + \dot{y}^2)^{3/2}} & -a^2\dot{y}^2 & 0
\end{pmatrix}.
\]

Under the condition $L_1 = 0$ we obtain that the transpositional relations are
\[
\delta \frac{dx}{dt} - \frac{d}{dt}\delta x = \frac{a(\dot{x}\dot{y} - \dot{x}\dot{y})}{(1 + a^2)(\dot{x}^2 + \dot{y}^2)^{3/2}}.
\]

From this example we obtain that the independent virtual variations $\delta x$ and $\delta y$ produce non-zero transpositional relations. This result is not in accordance with the Suslov point on view on the transpositional relations.

Now we apply Theorem 2. The functions $L_0$, $L_1$, $L_2$, and $L_3$ are determined as follows:
\[
L_0 = L, \quad L_1 = \dot{z} - a\sqrt{\dot{x}^2 + \dot{y}^2}, \quad L_2 = \dot{y}, \quad L_3 = \dot{x}.
\]

Thus, the matrices $W_1$ and $\Omega_1$ are
\[
W_1 = \begin{pmatrix}
-\frac{a\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} & -\frac{a\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \Omega_1 = \begin{pmatrix}
\dot{y}q & -\dot{x}q & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

where $q = a(\dot{x}\dot{y} - \dot{y}\dot{x})$. Therefore, $|W_1| = -1$.

Hence, after some computations from (11) we have that
\[
A = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\dot{y}q & -\dot{x}q & 0
\end{pmatrix}.
\]
The equations of motion (12) become

\[
\begin{align*}
\ddot{x} &= -\frac{a^2 \dot{y}}{x^2 + y^2} (\dot{y} \ddot{x} - \dot{x} \ddot{y}) - \frac{a \dot{\lambda}}{\sqrt{x^2 + y^2}} \dot{x}, \\
\ddot{y} &= -\frac{a^2 \dot{x}}{x^2 + y^2} (\dot{x} \ddot{y} - \dot{y} \ddot{x}) - \frac{a \dot{\lambda}}{\sqrt{x^2 + y^2}} \dot{y}, \\
\ddot{z} &= -g + \dot{\lambda}.
\end{align*}
\] (48)

By solving these equations with respect to \(\dot{x}, \dot{y},\) and \(\ddot{z}\) we obtain the equations

\[
\ddot{x} = -\frac{a \dot{x}}{\sqrt{x^2 + y^2}} \dot{\lambda}, \quad \ddot{y} = -\frac{a \dot{y}}{\sqrt{x^2 + y^2}} \dot{\lambda}, \quad \ddot{z} = -g + \dot{\lambda}.
\]

We observe in this case that \(|W_1| = -1\), consequently these equations, obtained from Theorem 2, give a global behavior of the Appel–Hamel systems, i.e., coincide with the classical equations (44) with

\[
\dot{\lambda} = \ddot{\lambda} = \mu = \frac{g}{1 + a^2}.
\]

The transpositional relations (13) can be written as

\[
\begin{align*}
\delta \frac{dx}{dt} - \frac{d}{dt} \delta x &= 0, \quad \delta \frac{dy}{dt} - \frac{d}{dt} \delta y = 0, \quad \delta \frac{dz}{dt} - \frac{d}{dt} \delta z = q (\dot{y} \delta x - \dot{x} \delta y).
\end{align*}
\] (49)

From this corollary we observe that the independent virtual variations \(\delta x\) and \(\delta y\) produce non-zero transpositional relations (47) and zero transpositional relations (49).

The Lagrangian (10) in this case takes the form

\[
L = \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - gz - \frac{g}{1 + a^2} (\dot{z} - a \sqrt{x^2 + y^2}) - \lambda_0 \dot{\lambda}.
\]

From (34) it follows that

\[
\begin{align*}
\delta \frac{dz}{dt} - \frac{d}{dt} \delta z &= q (\dot{y} \delta x - \dot{x} \delta y) \\
&+ \frac{a \dot{x}}{\sqrt{x^2 + y^2}} \left( \delta \frac{dx}{dt} - \frac{d}{dt} \delta x \right) \\
&+ \frac{a \dot{y}}{\sqrt{x^2 + y^2}} \left( \delta \frac{dy}{dt} - \frac{d}{dt} \delta y \right).
\end{align*}
\]

Therefore, this relation holds identically for (47) and (49).

In the next sections we show the importance of the equations of motion (12) and (16) contrasting them with the classical differential equations of nonholonomic mechanics.

6 Modified vakonomic mechanics versus vakonomic mechanics

Now we show that the equations of the vakonomic mechanics (3) can be obtained from Eq. (9). More precisely, if in (7) we require that all the virtual variations of the coordinates produce the zero transpositional relations, i.e., the matrix \(A\) is the zero matrix and we require that \(\lambda_j^0 = 0\) for \(j = M + 1, \ldots, N\), then from (9) by considering that \(D_k L = E_k L\), we obtain the vakonomic equations (3), i.e.,

\[
D_\nu L_0 = \sum_{j=1}^{M} \left( \lambda_j D_\nu L_j + \frac{d\lambda_j}{dt} \frac{\partial L_j}{\partial \dot{x}_\nu} \right) + \sum_{j=M+1}^{N} \lambda_j D_\nu L_j \implies \\
E_\nu L_0 = \sum_{j=1}^{M} \left( \lambda_j E_\nu L_j + \frac{d\lambda_j}{dt} \frac{\partial L_j}{\partial \dot{x}_\nu} \right), \quad \nu = 1, \ldots, N
\]

In the following example in order to contrast Theorem 2 with the vakonomic model we study the skate or knife edge on an inclined plane.

Example 1 To set up the problem, consider a plane \(\Sigma\) with cartesian coordinates \(x\) and \(y\), slanted at an angle \(\alpha\). We assume that the \(y\)-axis is horizontal, while the \(x\)-axis is directed downward from the horizontal and let \((x, y)\) be the coordinates of the point of contact of the skate with the plane. The angle \(\varphi\) represents the orientation of the skate measured from the \(x\)-axis. The skate is moving under the influence of the gravity. Here the acceleration due to gravity is denoted by \(g\). It also has mass \(m\), and the moment inertia of the skate about a vertical axis through its contact point is denoted by \(J\). (see page 108 of [35] for a picture). The equation of nonintegrable constraint is

\[
L_1 = \dot{x} \sin \varphi - \dot{y} \cos \varphi = 0.
\] (50)

With these notations the Lagrangian function of the skate is

\[
\hat{L} = \frac{m}{2} \left( \dot{x}^2 + \dot{y}^2 \right) + \frac{J}{2} \dot{\varphi}^2 + mg x \sin \alpha.
\]
Thus, we have the constrained mechanical systems
\[
\left( \mathbb{R}^2 \times S^1, \quad \hat{\mathcal{L}} = \frac{m}{2} \left( \dot{x}^2 + \dot{y}^2 \right) + J \dot{\phi}^2 + mg x \sin \alpha, \quad \{ \dot{x} \sin \varphi - \dot{y} \cos \varphi = 0 \} \right).
\]

For appropriate choice of mass, length, and time units, we reduce the Lagrangian \( \mathcal{L} \) to
\[
L_0 = \frac{1}{2} \left( \dot{x}^2 + \dot{y}^2 + \dot{\phi}^2 \right) + x g \sin \alpha,
\]
here for simplicity we leave the same notations for the all variables. The question is, what is the motion of the point of contact? To answer this question we shall use the vakonomic equations (3) and the Eq. (12) proposed in Theorem 2.

6.1 The study of the skate applying Theorem 2

We determine the motion of the point of contact of the skate using Theorem 2. We choose the arbitrary functions \( L_2 \) and \( L_3 \) as follows:
\[
L_2 = \dot{x} \cos \varphi + \dot{y} \sin \varphi, \quad L_3 = \dot{\phi},
\]
in order that the determinant \(|W_1| \neq 0\) everywhere in the configuration space.

The Lagrangian (10) becomes
\[
L(x, y, \varphi, \dot{x}, \dot{y}, \dot{\phi}, \Lambda) = \frac{1}{2} \left( \dot{x}^2 + \dot{y}^2 + \dot{\phi}^2 \right) + g \sin \alpha x \\
- \lambda (\dot{x} \sin \varphi - \dot{y} \cos \varphi) - \lambda \dot{\phi} \\
\simeq \frac{1}{2} \left( \dot{x}^2 + \dot{y}^2 + \dot{\phi}^2 \right) + g \sin \alpha x - \lambda (\dot{x} \sin \varphi - \dot{y} \cos \varphi),
\]
where \( \lambda := \lambda_1 \).

The matrices \( W_1 \) and \( \Omega_1 \) are
\[
W_1 = \begin{pmatrix}
\sin \varphi & -\cos \varphi & 0 \\
\cos \varphi & \sin \varphi & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad |W_1| = 1,
\]
\[
\Omega_1 = \begin{pmatrix}
\phi \cos \varphi & \phi \sin \varphi & -L_2 \\
-\phi \sin \varphi & \phi \cos \varphi & -L_1 \\
0 & 0 & 0
\end{pmatrix}.
\]

The matrix \( A = W_1^{-1} \Omega_1 \) becomes
\[
A = \begin{pmatrix}
0 & \phi & -\sin \varphi L_2 - \cos \varphi L_1 \\
-\phi & 0 & \cos \varphi L_2 - \sin \varphi L_1 \\
0 & 0 & 0
\end{pmatrix}_{L_1=0}
\]
\[
= \begin{pmatrix}
0 & \phi - \dot{\varphi} & -\dot{y} \\
-\phi & 0 & \dot{x} \\
0 & 0 & 0
\end{pmatrix}.
\]

Hence the Eq. (12) and transpositional relations (13) take the form
\[
\ddot{x} + \dot{\varphi} \ddot{y} = g \sin \alpha + \lambda \sin \varphi, \quad \ddot{y} - \dot{\varphi} \ddot{x} = -\lambda \cos \varphi, \quad \ddot{\varphi} = 0,
\]
and
\[
\delta \frac{dx}{dt} - \frac{d\delta x}{dt} = \dot{y} \delta \varphi - \dot{\varphi} \delta y,
\]
\[
\delta \frac{dy}{dt} - \frac{d\delta y}{dt} = \dot{\varphi} \delta x - \dot{x} \delta \varphi,
\]
\[
\delta \frac{d\varphi}{dt} - \frac{d\delta \varphi}{dt} = -L_2 (\delta x \sin \varphi - \delta y \cos \varphi) = 0,
\]
respectively, here we have applied the Lagrange–Chetaev’s condition \( \sin \varphi \delta x - \cos \varphi \delta y = 0 \).

The initial conditions
\[
x_0 = x|_{t=0}, \quad y_0 = y|_{t=0}, \quad \varphi_0 = \varphi|_{t=0}, \quad \dot{x}_0 = \dot{x}|_{t=0}, \quad \dot{y}_0 = \dot{y}|_{t=0}, \quad \dot{\varphi}_0 = \dot{\varphi}|_{t=0},
\]
satisfy the constraint, i.e.,
\[
\sin \varphi_0 \dot{x}_0 - \cos \varphi_0 \dot{y}_0 = 0.
\]

After the derivation of the constraint along the solutions of the equation of motion (51), and using (50) we obtain
\[
0 = \sin \varphi \ddot{x} - \cos \varphi \ddot{y} + \ddot{\varphi} (\cos \varphi \dot{x} + \sin \varphi \dot{y})
\]
\[
= \sin \varphi \left( g \sin \alpha + \dot{\lambda} \sin \varphi - \dot{\varphi} \right) - \cos \varphi \left( -\dot{\lambda} \cos \varphi + \dot{\varphi} \dot{x} \right) + \ddot{\varphi} (\cos \varphi \dot{x} + \sin \varphi \dot{y}).
\]

Hence \( \dot{\lambda} = -g \sin \alpha \sin \varphi \). Therefore, the differential equations (51) can be written as
\[
\ddot{x} + \dot{\varphi} \ddot{y} = g \sin \alpha \cos^2 \varphi, \quad \ddot{\varphi} = g \sin \alpha \\
\sin \varphi \cos \varphi, \quad \ddot{\varphi} = 0.
\]

We study the motion of the skate in the following three cases:

(i) \( \dot{\varphi}|_{t=0} = \omega = 0 \).

(ii) \( \dot{\varphi}|_{t=0} = \omega \neq 0 \).

(iii) \( \alpha = 0 \).

For the first case (\( \omega = 0 \)), after the change of variables
\[
X = \cos \varphi_0 x - \sin \varphi_0 y, \quad Y = \cos \varphi_0 x + \sin \varphi_0 y,
\]
the differential equations (9) and the constraint become
\[
\ddot{X} = 0, \quad \ddot{Y} = g \sin \alpha \cos \varphi_0, \quad \varphi = \varphi_0, \quad \dot{X} = 0,
\]
respectively. Consequently,
\[
X = X_0, \quad Y = g \sin \alpha \cos \varphi_0 t^2 + \dot{Y}_0 t + Y_0, \quad \varphi = \varphi_0,
\]
thus the trajectories are straight lines.

For the second case (\( \omega \neq 0 \)), we take \( \varphi_0 = \dot{y}_0 = \dot{x}_0 = x_0 = y_0 = 0 \) in order to simplify the computations. In view of the equality \( \dot{\varphi} = \dot{\varphi}|_{t=0} = \omega \) and denoting by \( ' \) the derivation with respect \( \varphi \) we get that (54) becomes

\[
\begin{align*}
x'' + y' &= \frac{g \sin \alpha}{\omega^2} \cos^2 \varphi, \\
x'' - x' &= \frac{g \sin \alpha}{\omega^2} \sin \varphi \cos \varphi, \quad \varphi' = 1,
\end{align*}
\]

(55) which are easy to integrate and we obtain

\[
x = -\frac{g \sin \alpha}{4 \omega^2} \cos (2\varphi), \quad y = -\frac{g \sin \alpha}{4 \omega^2} \sin (2\varphi) + \frac{g}{2 \omega} \varphi, \quad \varphi = \omega t,
\]

which correspond to the equation of the cycloid. Hence the point of contact of the skate follows a cycloid along the plane, but does not slide down the plane.

For the third case (\( \alpha = 0 \)), if \( \varphi_0 = 0, \omega \neq 0 \) we obtain that the solutions of the given differential systems (54) are

\[
x = \dot{y}_0 \cos \varphi + \dot{x}_0 \sin \varphi + a, \quad y = \dot{y}_0 \sin \varphi + \dot{x}_0 \cos \varphi + b, \quad \varphi = \varphi_0 + \omega t,
\]

where \( a = x_0 - \frac{\dot{y}_0}{\omega}, \quad b = y_0 + \frac{\dot{x}_0}{\omega} \), which correspond to the equation of the circle with center at \((a, b)\) and radius \( \frac{\dot{x}_0^2 + \dot{y}_0^2}{\omega^2} \).

If \( \alpha = 0 \) and \( \varphi_0 = 0, \omega = 0 \) then we obtain that the solutions are

\[
x = \dot{x}_0 t + x_0, \quad y = \dot{y}_0 t + y_0.
\]

All these solutions coincide with the solutions obtained from the Lagrangian equations (33) with multipliers (see [2])

\[
\ddot{x} = g \sin \alpha + \mu \sin \varphi, \quad \ddot{y} - \dot{\varphi} \dot{x} = -\mu \cos \varphi, \quad \ddot{\varphi} = 0,
\]

with \( \mu = \dot{\lambda} = -g \sin \alpha \sin \varphi \).

6.2 The study of the skate applying vakonomic model

Now we consider instead of Theorem 2 the vakonomic model for studying the motion of the skate.

We consider the Lagrangian

\[
L(x, y, \varphi, \dot{x}, \dot{y}, \dot{\varphi}, \lambda) = \frac{1}{2} \left( \dot{x}^2 + \dot{y}^2 + \dot{\varphi}^2 \right) + g x \sin \alpha - \lambda (\dot{x} \sin \varphi - \dot{y} \cos \varphi).
\]

The equations of motion (3) for the skate are

\[
\begin{align*}
\frac{d}{dt} (\dot{x} - \lambda \sin \varphi) &= 0, \\
\frac{d}{dt} (\dot{y} + \lambda \cos \varphi) &= 0
\end{align*}
\]

\[
\ddot{\varphi} = -\lambda (\dot{x} \cos \varphi + \dot{y} \sin \varphi).
\]

We shall study only the case when \( \alpha = 0 \). After integration we obtain the differential systems

\[
\begin{align*}
\dot{x} &= \lambda \sin \varphi + a = \cos \varphi (a \cos \varphi + b \sin \varphi), \\
\dot{y} &= -\lambda \cos \varphi + b = \sin \varphi (a \cos \varphi + b \sin \varphi), \\
\ddot{\varphi} &= (b \cos \varphi - a \sin \varphi) (a \cos \varphi + b \sin \varphi), \\
\lambda &= b \cos \varphi - a \sin \varphi,
\end{align*}
\]

where \( a = \dot{x}_0 - \lambda_0 \sin \varphi_0, b = \dot{y}_0 + \lambda_0 \cos \varphi_0 \) and \( \lambda_0 = \lambda|_{t=0} \) is an arbitrary parameter. After the integration of the third equation we obtain that

\[
\int_{\varphi_0}^{\varphi} \frac{d\varphi}{\sqrt{1 - \kappa^2 \sin^2 \varphi}} = t \sqrt{\frac{h + a^2 + b^2}{2}},
\]

(57) where \( h \) is an arbitrary constant which we choose in such a way that \( \kappa^2 = \frac{2(a^2 + b^2)}{h + a^2 + b^2} < 1 \).

From (57) we get \( \sin \varphi = \text{sn} \left( t \sqrt{\frac{h + a^2 + b^2}{2}} \right) \),

\[
\cos \varphi = \text{cn} \left( t \sqrt{\frac{h + a^2 + b^2}{2}} \right),
\]

where \( \text{sn} \) and \( \text{cn} \) are the Jacobi elliptic functions. Hence, if we take \( \dot{x}_0 = 1, \dot{y}_0 = \varphi_0 = 0 \), then the solutions of the differential equations (56) are

\[
\begin{align*}
x &= x_0 + \int_{t_0}^{t} \left( \text{cn} \left( t \sqrt{\frac{h + 1 + \lambda_0^2}{2}} \right) \text{sn} \left( t \sqrt{\frac{h + 1 + \lambda_0^2}{2}} \right) \right. \\
+ &\left. \lambda_0 \text{sn} \left( t \sqrt{\frac{h + 1 + \lambda_0^2}{2}} \right) \right) \lambda_0 \text{dn} \left( t \sqrt{\frac{h + 1 + \lambda_0^2}{2}} \right) \text{dn} \left( t \sqrt{\frac{h + 1 + \lambda_0^2}{2}} \right) dt, \\
y &= y_0 + \int_{t_0}^{t} \left( \text{sn} \left( t \sqrt{\frac{h + 1 + \lambda_0^2}{2}} \right) \right. \\
&\left. \lambda_0 \text{sn} \left( t \sqrt{\frac{h + 1 + \lambda_0^2}{2}} \right) \right) \lambda_0 \text{dn} \left( t \sqrt{\frac{h + 1 + \lambda_0^2}{2}} \right) \text{dn} \left( t \sqrt{\frac{h + 1 + \lambda_0^2}{2}} \right) dt, \\
\varphi &= \lambda \text{sn} \left( t \sqrt{\frac{h + 1 + \lambda_0^2}{2}} \right).
\end{align*}
\]

(58)

It is interesting to compare this amazing motions with the motions that we obtained above. For the same initial conditions the skate moves sideways along the circles. By considering that the solutions (58) depend on the
arbitrary parameter \( \lambda_0 \) we obtain that for the given initial conditions, there do not exist a unique solution of the differential equations in the vakonomic model. Consequently, the principle of determinacy is not valid for vakonomic mechanics with nonintegrable constraints (see the Corollary of page 36 in [2]).

7 Modified vakonomic mechanics versus Lagrangian and constrained Lagrangian mechanics

7.1 MVM versus Lagrangian mechanics

The Lagrangian equations which describe the motion of the Lagrangian systems can be obtained from Theorem 2 by supposing that \( M = 0 \), i.e., there is no constraint. We choose the arbitrary functions \( L \) for \( \alpha = 1, \ldots, N \) as follows:

\[
L_\alpha = \frac{d x_\alpha}{dt}, \quad \alpha = 1, \ldots, N.
\]

Hence the Lagrangian (10) takes the form

\[
L = L_0 - \sum_{j=1}^{N} \lambda_j \frac{dx_j}{dt} \simeq L_0.
\]

In this case we have that \( |W_1| = 1 \).

By considering the property of the Lagrangian derivative (see (27)) we obtain that \( \Omega_1 \) is a zero matrix. Hence the matrix \( A_1 \) is the zero matrix. As a consequence the Eq. (12) become

\[
D_v L = E_v L = E_v \left( L_0 - \sum_{j=1}^{N} \lambda_j \frac{dx_j}{dt} \right) = E_v L_0 = 0
\]

because \( L \simeq L_0 \). The transpositional relations (13) in this case are \( \frac{dx}{dt} - \frac{d \delta x}{dt} = 0 \), which are the well-known relations in the Lagrangian mechanics (see formula (29)).

7.2 MVM versus constrained Lagrangian systems

From the equivalences (42) we have that in the case when the constraints are linear in the velocity the equations of motions of the MVM coincide with the Lagrangian equations with multipliers (33) except perhaps in a zero Lebesgue measure set \( |W_2| = 0 \) or \( |W_1| = 0 \). When the constraints are nonlinear in the velocity, we have the equivalence (43). Consequently, equations of motions of the MVM coincide with the Lagrangian equations with multipliers (33) except perhaps in a zero Lebesgue measure set \( |W_2| = 0 \).

We illustrate this result in the following example.

Example 2 Let

\[
\begin{pmatrix}
\mathbb{R}^2, & L_0 = \frac{1}{2} (\dot{x}^2 + \dot{y}^2) - U(x, y), \\
{2 (x \dot{x} + y \dot{y}) = 0}
\end{pmatrix},
\]

be the constrained Lagrangian systems.

In order to apply Theorem 2 we choose the arbitrary function \( L_1 \) and \( L_2 \) as follows:

(a) \( L_1 = 2 (x \dot{x} + y \dot{y}) \), \( L_2 = -y \dot{x} + x \dot{y} \).

Thus, the matrices \( W_1 \) and \( \Omega_1 \) are

\[
W_1 = \begin{pmatrix}
2x & 2y \\
-y & x
\end{pmatrix}, \quad |W_1| = 2x^2 + 2y^2 = 2,
\]

\[
\Omega_1 = \begin{pmatrix}
0 & 0 \\
-2y & 2x
\end{pmatrix}.
\]

Consequently, Eq. (12) describe the motion everywhere for the constrained Lagrangian systems. Equations (12) become

\[
\ddot{x} = -\frac{\partial U}{\partial x} + 2y (y \ddot{x} - x \ddot{y}) + 2x \dot{\lambda} \bigg|_{L_1=0} = -\frac{\partial U}{\partial x} + x (\dot{\lambda} - 2(\dot{x}^2 + \dot{y}^2)),
\]

\[
\ddot{y} = -\frac{\partial U}{\partial y} + 2x (y \ddot{x} - x \ddot{y}) + 2y \dot{\lambda} \bigg|_{L_1=0} = -\frac{\partial U}{\partial y} + y (\dot{\lambda} - 2(\dot{x}^2 + \dot{y}^2)),
\]

Transpositional relations take the form

\[
\frac{dx}{dt} - \frac{d \delta x}{dt} = 2y (\dot{y} \delta x - \dot{x} \delta y), \quad \frac{dy}{dt} - \frac{d \delta y}{dt} = -2x (\dot{y} \delta x - \dot{x} \delta y).
\]

(b) If we choose \( L_2 = \frac{y \dot{x}}{x^2 + y^2} - \frac{x \dot{y}}{x^2 + y^2} = \frac{d}{dt} \arctan \frac{x}{y} \), then

\[
W_i = \begin{pmatrix}
\frac{2x}{x^2 + y^2} & \frac{2y}{x^2 + y^2} \\
\frac{2y}{x^2 + y^2} & \frac{-2x}{x^2 + y^2}
\end{pmatrix}, \quad |W_i| = -2, \quad \Omega_i = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}.
\]

Equations (12) and transpositional relations become

\[
\ddot{x} = -\frac{\partial U}{\partial x} + 2x \dot{\lambda}, \quad \ddot{y} = -\frac{\partial U}{\partial y} + 2y \dot{\lambda},
\]

\[
\frac{dx}{dt} - \frac{d \delta x}{dt} = 0, \quad \frac{dy}{dt} - \frac{d \delta y}{dt} = 0.
\]
From this example we obtain that for the holonomic constrained Lagrangian systems the transpositional relations can be non-zero (see (59)), or can be zero (see (60)). We observe that from condition (34) it follows the relation

\[ x \left( \delta \frac{dx}{dt} - \frac{\delta x}{\delta \dot{x}} \right) + y \left( \delta \frac{dy}{dt} - \frac{\delta y}{\delta \dot{y}} \right) = 0. \]

This equality holds identically if (60) and (59) take place.

The equations of motions (33) in this case are

\[ \ddot{x} = -\frac{\partial U}{\partial x} + 2x \mu, \quad \ddot{y} = -\frac{\partial U}{\partial y} + 2y \mu, \]

with \( \mu = \dot{\lambda} - 2(\dot{x}^2 + \dot{y}^2) \).

**Example 3** To contrast the MVM with the classical model we apply Theorems 2 to the Gantmacher's systems (see for more details [12,45]).

Two material points \( m_1 \) and \( m_2 \) with equal masses are linked by a metal rod with fixed length \( l \) and small mass. The systems can move only in the vertical plane and so the speed of the midpoint of the rod is directed along the rod. It is necessary to determine the trajectories of the material points \( m_1 \) and \( m_2 \).

Let \((q_1, r_1)\) and \((q_2, r_2)\) be the coordinates of the points \( m_1 \) and \( m_2 \), respectively. Clearly, \((q_1 - q_2)^2 + (r_1 - r_2)^2 = l^2 \). Thus, we have a constrained Lagrangian system in the configuration space \( \mathbb{R}^4 \) with the Lagrangian function \( L = \frac{1}{2} (\dot{q}_1^2 + \dot{q}_2^2 + \dot{r}_1^2 + \dot{r}_2^2) - \frac{g}{2}(r_1 + r_2) \), and with the linear constraints

\[ (q_2 - q_1)(\dot{q}_2 - \dot{q}_1) + (r_2 - r_1)(\dot{r}_2 - \dot{r}_1) = 0, \]

\[ (q_2 - q_1)(\dot{r}_2 + r_1) - (r_2 - r_1)(\dot{q}_2 + q_1) = 0. \]

Introducing the following change of coordinates:

\[ x_1 = \frac{q_2 - q_1}{2}, \quad x_2 = \frac{r_1 - r_2}{2}, \]

\[ x_3 = \frac{r_2 + r_1}{2}, \quad x_4 = \frac{q_1 + q_2}{2}, \]

we obtain

\[ x_1^2 + x_2^2 = \frac{1}{4} \left( (q_2 - q_1)^2 + (r_2 - r_1)^2 \right) = \frac{l^2}{4}. \]

Hence we have the constrained Lagrangian mechanical systems

\[ \mathbb{R}^4, \quad \tilde{L} = \frac{1}{2} \sum_{j=1}^{4} x_j^2 - gx_3, \quad \{ x_1 \dot{x}_1 + x_2 \dot{x}_2 = 0, x_1 \dot{x}_3 - x_2 \dot{x}_4 = 0 \}. \]

The equations of motion (33) obtained from the d’Alembert–Lagrange principle are

\[ \ddot{x}_1 = \mu_1 x_1, \quad \ddot{x}_2 = \mu_2 x_2, \quad \ddot{x}_3 = -g + \mu_2 x_1, \quad \ddot{x}_4 = -\mu_2 x_2, \]

\[ \text{(61)} \]

where \( \mu_1, \mu_2 \) are the Lagrangian multipliers such that

\[ \mu_1 = -\frac{\dot{x}_1^2 + \dot{x}_2^2}{x_1^2 + x_2^2}, \quad \mu_2 = \frac{\dot{x}_2 x_4 - \dot{x}_1 x_3 + g x_1}{x_1^2 + x_2^2}. \]

\[ \text{(62)} \]

For applying Theorem 2 we have the constraints

\[ L_1 = x_1 \ddot{x}_1 + x_2 \ddot{x}_2 = 0, \quad L_2 = x_1 \dot{x}_3 - x_2 \dot{x}_4 = 0, \]

and we choose the arbitrary functions \( L_3 \) and \( L_4 \) as follows:

\[ L_3 = -x_1 \dot{x}_2 + x_2 \dot{x}_1, \quad L_4 = x_2 \dot{x}_3 + x_1 \dot{x}_4. \]

For the given functions we obtain that

\[ W_1 = \begin{pmatrix} x_1 & x_2 & 0 & 0 \\ 0 & 0 & x_1 & -x_2 \\ x_2 & -x_1 & 0 & 0 \end{pmatrix}, \quad \Omega_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -x_3 & \dot{x}_4 & \ddot{x}_1 & -x_2 \\ -x_2 \dot{x}_1 & 0 & 0 & -x_4 + x_3 \dot{x}_2 \end{pmatrix}. \]

Therefore, \( |W_1| = (x_1^2 + x_2^2)^2 = l^2 \frac{4}{l} \neq 0 \). The matrix \( A \) in this case is

\[ \begin{pmatrix} \frac{2x_2 \dot{x}_2}{x_1^2 + x_2^2} & -\frac{2x_2 \dot{x}_3}{x_1^2 + x_2^2} & 0 & 0 \\ -\frac{2x_2 \dot{x}_4}{x_1^2 + x_2^2} & \frac{2x_1 \dot{x}_1}{x_1^2 + x_2^2} & 0 & 0 \\ -\frac{x_1 \dot{x}_3 + x_2 \dot{x}_4}{x_1^2 + x_2^2} & \frac{x_1 \dot{x}_4 - x_2 \dot{x}_3}{x_1^2 + x_2^2} & \frac{x_2 \dot{x}_1 - x_1 \dot{x}_2}{x_1^2 + x_2^2} & \frac{x_2 \dot{x}_1 - x_1 \dot{x}_2}{x_1^2 + x_2^2} \\ -\frac{x_1 \dot{x}_3 - x_2 \dot{x}_4}{x_1^2 + x_2^2} & \frac{x_1 \dot{x}_4 + x_2 \dot{x}_3}{x_1^2 + x_2^2} & \frac{x_2 \dot{x}_1 + x_1 \dot{x}_2}{x_1^2 + x_2^2} & \frac{x_2 \dot{x}_1 + x_1 \dot{x}_2}{x_1^2 + x_2^2} \end{pmatrix}. \]

Consequently, differential equations (12) take the form

\[ \ddot{x}_1 = \left( \frac{2x_2 \ddot{x}_2 - 2x_1 \ddot{x}_1 - x_1 \ddot{x}_2 - x_2 \ddot{x}_1}{x_1^2 + x_2^2} + x_1 \dot{x}_1 \right) \bigg|_{L_1 = L_2 = 0} \]

\[ = x_1 \left( \frac{2\dot{x}_1^2 + 2\dot{x}_2^2 + x_1^2 + x_2^2}{x_1^2 + x_2^2} \right). \]

\[ \ddot{x}_2 = -\left( \frac{2x_1 \ddot{x}_1 + 2x_2 \ddot{x}_2 + x_2 \ddot{x}_1 + x_1 \ddot{x}_2}{x_1^2 + x_2^2} + x_2 \dot{x}_1 \right) \bigg|_{L_1 = L_2 = 0} \]

\[ = x_2 \left( \frac{2\dot{x}_1^2 + 2\dot{x}_2^2 + x_1^2 + x_2^2}{x_1^2 + x_2^2} \right). \]

\[ \ddot{x}_3 = \left( \frac{x_1 \dot{x}_1 + x_2 \dot{x}_2 - x_1 \dot{x}_1 - x_2 \dot{x}_2}{x_1^2 + x_2^2} + x_1 \dot{x}_2 - g \right) \bigg|_{L_1 = L_2 = 0} \]

\[ = \frac{x_4 (x_2 \dot{x}_1 - x_1 \dot{x}_2)}{x_1^2 + x_2^2} + x_1 \dot{x}_2 - g, \]

\[ \ddot{x}_4 = \left( \frac{x_4 (x_1 \dot{x}_2 + x_2 \dot{x}_1) - x_2 (x_2 \dot{x}_1 - x_1 \dot{x}_2)}{x_1^2 + x_2^2} - x_2 \dot{x}_2 \right) \bigg|_{L_1 = L_2 = 0}. \]
These equations coincide with equations (61) everywhere because \(|W_1| = \frac{l^2}{4}\), where \(l\) is the length of the rod.

The transpositional relations in this case are

\[
\dot{x}_1 = -\frac{x_1 \left(\dot{x}_1^2 + \dot{x}_2^2\right)}{x_1^2 + x_2^2}, \quad \dot{x}_2 = \frac{x_2 \left(\dot{x}_1^2 + \dot{x}_2^2\right)}{x_1^2 + x_2^2}, \\
\dot{x}_3 = -g + \frac{x_1^2 + x_2^2}{x_1 (x_2 \dot{x}_4 - \dot{x}_1 \dot{x}_3 + g x_1)}, \\
\dot{x}_4 = -\frac{x_2 (x_2 \dot{x}_4 - \dot{x}_1 \dot{x}_3 + g x_1)}{x_1^2 + x_2^2}.
\]

These equations coincide with equations (61) everywhere because \(|W_1| = \frac{l^2}{4}\), where \(l\) is the length of the rod.

The transpositional relations in this case are

\[
\frac{d x_1}{d t} - \frac{d \delta x_1}{d t} = -\frac{2 x_2}{x_1^2 + x_2^2} (\dot{x}_1 \delta x_2 - \dot{x}_2 \delta x_1), \\
\frac{d x_2}{d t} - \frac{d \delta x_2}{d t} = \frac{2 x_1}{x_1^2 + x_2^2} (\dot{x}_1 \delta x_2 - \dot{x}_2 \delta x_1), \\
\frac{d x_3}{d t} - \frac{d \delta x_3}{d t} = \frac{x_1}{x_1^2 + x_2^2} (\dot{x}_1 \delta x_3 - \dot{x}_3 \delta x_1 + \dot{x}_4 \delta x_2 - \dot{x}_2 \delta x_4), \\
\frac{d x_4}{d t} - \frac{d \delta x_4}{d t} = -\frac{x_2}{x_1^2 + x_2^2} (\dot{x}_1 \delta x_3 - \dot{x}_3 \delta x_1 + \dot{x}_4 \delta x_2 - \dot{x}_2 \delta x_4).
\]

Remark 21 From the previous example we observe that the virtual variations produce zero or non-zero transpositional relations, depending on the arbitrary functions which appear in the construction of the proposed mathematical model. Thus, the following question arises: Can be chosen the arbitrary functions \(L_j\) for \(j = M + 1, \ldots, N\) in such a way that for the nonholonomic systems only the independent virtual variations would generate zero transpositional relations?

The positive answer to this question is obtained locally for any constrained Lagrangian systems and globally for the Chaplygin–Voronets mechanical systems and for the generalization of these systems studied in the next section.

8 MVM and nonholonomic generalized Voronets–Chaplygin systems. Proofs of Theorem 4 and Proposition 5 and 6.

It was pointed out by Chaplygin [7] that in many conservative nonholonomic systems the generalized coordinates

\[
(\mathbf{x}, \mathbf{y}) := (x_1, \ldots, x_s, y_1, \ldots, y_s), \quad s_1 + s_2 = N,
\]

can be chosen in such a way that the Lagrangian function and the constraints take the simplest form. In particular Voronets in [53] studied the constrained Lagrangian systems with Lagrangian \(\tilde{L} = \tilde{L}(\mathbf{x}, \mathbf{y}, \dot{\mathbf{x}}, \dot{\mathbf{y}})\) and constraints (22). This system is called the Voronets mechanical systems.

We shall apply Eq. (12) to study the generalization of the Voronets systems, which we define now.

The constrained Lagrangian mechanical systems

\[
(\mathbf{Q}, \quad \tilde{L}(t, \mathbf{x}, \mathbf{y}, \dot{\mathbf{x}}, \dot{\mathbf{y}}), \quad \{\dot{x}_\alpha - \Phi_\alpha(t, \mathbf{x}, \mathbf{y}) = 0, \quad \alpha = 1, \ldots, s_1})
\]

are called the generalized Voronets mechanical systems.

An example of generalized Voronets systems is Appell–Hamel systems analyzed in the previous subsection.
Corollary 22 Every Nonholonomic constrained Lagrangian mechanical systems locally is a generalized Voronets mechanical systems.

Proof Indeed, the independent constraints can be locally represented in the form (26). Thus, by introducing the coordinates

\[ x_j = x_j, \quad \text{for} \quad j = 1, \ldots, M, \quad x_{M+k} = y_k, \quad \text{for} \quad k = 1, \ldots, N - M, \]

then we have that any constrained Lagrangian mechanical systems is locally a generalized Voronets mechanical systems. □

Proof of Theorem 4 For simplicity we shall study only scleronomic generalized Voronets systems.

To determine Eq. (12) we suppose that

\[ L_\alpha = \dot{x}_\alpha - \Phi_\alpha (x, y, \dot{y}) = 0, \quad \alpha = 1, \ldots, s_1. \]  

(66)

It is evident from the form of the constraint equations that the virtual variations, \( \delta y \), are independent by definition. The remaining variations, \( \delta x \), can be expressed in terms of them by the relations (Chetaev’s conditions)

\[ \delta x_\alpha - \sum_{j=1}^{s_2} \frac{\partial L_\alpha}{\partial y_j} \delta y_j = 0, \quad \alpha = 1, \ldots, s_1. \]  

(67)

We shall apply Theorem 2. To construct the matrix \( W_1 \). We first determine \( L_{s_1+1}, \ldots, L_{s_1+s_2} = L_N \) as follows:

\[ L_{s_1+j} = \dot{y}_j, \quad j = 1, \ldots, s_2. \]

Hence, the Lagrangian (4) becomes

\[ L = L_0 - \sum_{j=1}^{s_1} \lambda_j (\dot{x}_\alpha - \Phi_\alpha (x, y, \dot{y})) \]

\[- \sum_{j=s_1+1}^{N} \lambda_j \dot{y}_j \approx L_0 \]

\[- \sum_{j=1}^{s_1} \lambda_j (\dot{x}_\alpha - \Phi_\alpha (x, y, \dot{y})). \]  

(68)

The matrices \( W_1 \) and \( W_1^{-1} \) are

\[ \begin{pmatrix} 
1 & \ldots & 0 & a_{11} & \ldots & a_{1z} \\
0 & \ldots & 0 & a_{21} & \ldots & a_{2z} \\
\vdots & \ldots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 1 & a_{i1} & \ldots & a_{iz} \\
0 & 0 & 0 & 1 & \ldots & 0 \\
\vdots & \ldots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1 
\end{pmatrix} \]

\[ \begin{pmatrix} 
1 & \ldots & 0 & -a_{11} & \ldots & -a_{1z} \\
0 & \ldots & 0 & -a_{21} & \ldots & -a_{2z} \\
\vdots & \ldots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 1 & -a_{i1} & \ldots & -a_{iz} \\
0 & 0 & 0 & 1 & \ldots & 0 \\
\vdots & \ldots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1 
\end{pmatrix} \]  

(69)

respectively, where \( a_{\alpha j} = \frac{\partial L_\alpha}{\partial y_j} \), and the matrices \( \Omega_1 \) and \( A \) are

\[ A = \Omega_1 := \begin{pmatrix} 
E_1(L_1) & \ldots & E_{i_1}(L_1) & \ldots & E_{i_N}(L_1) \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
E_1(L_i) & \ldots & E_{i_1}(L_i) & \ldots & E_{i_N}(L_i) \\
0 & \ldots & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & \ldots & 0 
\end{pmatrix}. \]  

(70)

respectively. Consequently, the differential equations (12) take the form (18).

The transpositional relations (13) in view of (67) take the form (21). As we can observe from (21) the independent virtual variations \( \delta y \) for the systems with the constraints (66) produce the zero transpositional relations. The fact that the transpositional relations are zero follows automatically and it is not necessary to assume it a priori, and it is valid in general for the constraints which are nonlinear in the velocity variables.

We observe that the relations (34) in this case take the form

\[ \delta \frac{dx_\alpha}{dt} - \frac{dx_\alpha}{dt} + \sum_{m=1}^{s_2} \frac{\partial L_\alpha}{\partial y_m} \left( \delta \frac{dy_m}{dt} - \frac{dy_m}{dt} \right) = \sum_{k=1}^{s_1} E_k(L_\alpha) \delta x_k + \sum_{k=1}^{s_2} E_k(L_\alpha) \delta y_k. \]

for \( \alpha = 1, \ldots, s_1 \). Clearly, from (21) these relations hold identically.
From differential equations (18), eliminating the Lagrangian multipliers we obtain Eq. (19). After some computations we obtain

\[
\frac{d}{dt} \left( \sum_{\alpha=1}^{s_1} \frac{\partial L_0}{\partial \dot{x}_\alpha} - \sum_{\alpha=1}^{s_1} \frac{\partial L_0}{\partial \ddot{x}_\alpha} \right) - \left( \sum_{\alpha=1}^{s_1} \frac{\partial L_0}{\partial \dot{x}_\alpha} \right) = 0,
\]

for \( k = 1, \ldots, s_2 \).

By introducing the function \( \Theta = L_0|_{L_1=\ldots=L_{s_1}=0} \), Eq. (71) can be written as

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\gamma}_k} \right) - \sum_{\alpha=1}^{s_1} \frac{\partial L_0}{\partial \dot{x}_\alpha} \frac{\partial L}{\partial \dot{\gamma}_k} = 0,
\]

for \( k = 1, \ldots, s_2 \). Here we consider that \( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\gamma}_k} \right) = 0 \), for \( \alpha, \beta = 1, \ldots, s_1 \).

We shall study the case when Eq. (72) hold identically, i.e., \( \Theta = 0 \). We choose

\[
L_0 = \tilde{L}(x, y, \dot{x}, \dot{y}) - \tilde{L}(x, y, \Phi, \dot{y}) = \tilde{L} - L^*,
\]

being \( \tilde{L} \) the Lagrangian of (65). Now we establish the relations between Eq. (18) and the classical Voronets differential equations with the Lagrangian function \( L^* = \tilde{L}|_{L_1=\ldots=L_{s_1}=0} \). The functions \( \tilde{L} \) and \( L^* \) are determined in such a way that Eq. (19) take place in view of the equalities

\[
E_k \tilde{L} = \sum_{\alpha=1}^{s_1} E_\alpha \frac{\partial L_0}{\partial \dot{x}_\alpha},
\]

and

\[
E_k L^* = -\sum_{\alpha=1}^{s_1} \left( -E_k (L_\alpha) + \sum_{v=1}^{s_1} E_v (L_\alpha) \frac{\partial L_\alpha}{\partial \dot{x}_v} \right) \frac{\partial \tilde{L}}{\partial \dot{x}_\alpha},
\]

for \( k = 1, \ldots, s_2 \), which in view of equalities

\[
\frac{d}{dt} \left( \frac{\partial L^*}{\partial \dot{x}_v} \right) = 0 \text{ for } v = 1, \ldots, s_1, \text{ take the form } (20).
\]

**Proof of Proposition 5** Equations (20) describe the motion of the constrained generalized Voronets systems with Lagrangian \( L^* \) and constraints (66). The classical Voronets equations for scleronomic systems are easy to obtain from (20) with \( \Phi_\alpha = \sum_{k=1}^{s_2} a_k(x, y) \dot{y}_k \).

Finally by considering Corollary 4 we get that differential equations (20) describe locally the motions of any constrained Lagrangian systems.

### 8.1 Generalized Chaplygin systems

The constrained Lagrangian mechanical systems with Lagrangian \( \tilde{L} = \tilde{L}(x, \dot{x}, \dot{y}) \), and constraints (24) are called the *Chaplygin mechanical systems*.

The constrained Lagrangian systems

\[
\left\{ \begin{array}{l}
Q, \tilde{L}(x, \dot{x}, \dot{y}), \quad \{ \dot{x}_\alpha - \Phi_\alpha(y) = 0, \quad \alpha = 1, \ldots, s_1 \}
\end{array} \right.
\]

are called the *generalized Chaplygin systems*. Note that now the Lagrangian does not depend on \( x \) and the constraints do not depend on \( x \) and \( \dot{x} \). So, the generalized Chaplygin systems are a particular case of the generalized Voronets system.

**Proof of Proposition 6** To determine the differential equations which describe the behavior of the generalized Chaplygin systems we apply Theorem 2, with

\[
L_0 = L_0(x, \dot{x}, \dot{y}), \quad L_\alpha = \dot{x}_\alpha - \Phi_\alpha(y), \quad L_\beta = \dot{y}_\beta,
\]

for \( \alpha = 1, \ldots, s_1 \) and \( \beta = s_1 + 1, \ldots, s_2 \) and consequently the matrix \( W_1 \) is given by the formula (69) and

\[
A = \Omega_1 :=
\begin{pmatrix}
E_1(L_1) & \cdots & E_{s_1}(L_1) & E_{s_1+1}(L_1) & \cdots & E_{s_2}(L_1) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
E_1(L_{s_1}) & \cdots & E_{s_1}(L_{s_1}) & E_{s_1+1}(L_{s_1}) & \cdots & E_{s_2}(L_{s_1}) \\
0 & \cdots & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & E_{s_1+1}(L_{s_1}) & \cdots & E_{s_1}(L_{s_1}) \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & \cdots & 0
\end{pmatrix}
\]

(74)

Therefore, the differential equations (12) take the form

\[
E_j L_0 = \frac{d}{dt} \left( \frac{\partial L_0}{\partial \dot{x}_\alpha} \right) = \dot{\lambda}_j, \quad j = 1, \ldots, s_1,
\]

\[
E_k L_0 = \sum_{\alpha=1}^{s_1} \left( E_k L_\alpha \frac{\partial L_\alpha}{\partial \dot{x}_\alpha} + \dot{\lambda}_\alpha \frac{\partial L_\alpha}{\partial \dot{y}_k} \right) \quad k = 1, \ldots, s_2.
\]

(75)
The transpositional relations are
\[ \delta \frac{dx_\alpha}{dt} - \frac{d}{dt} \delta x_\alpha = \sum_{k=1}^{s_1} E_k(L_\alpha) \delta y_k, \quad \alpha = 1, \ldots, s_1, \]
\[ \delta \frac{dy_m}{dt} - \frac{d}{dt} \delta y_m = 0, \quad m = 1, \ldots, s_2. \] (76)

By excluding the Lagrangian multipliers from (75) we obtain the equations
\[ E_k L_0 = \sum_{\alpha=1}^{s_1} \left( E_k(L_\alpha) \frac{\partial L_0}{\partial \dot{x}_\alpha} + \frac{d}{dt} \left( \frac{\partial L_0}{\partial \dot{x}_\alpha} \right) \frac{\partial L_\alpha}{\partial y_k} \right), \]
for \( k = 1, \ldots, s_2. \)

In this case Eq. (73) take the form
\[ \frac{d}{dt} \left( \frac{\partial \Theta}{\partial \dot{y}_k} \right) - \left( \frac{\partial \Theta}{\partial y_k} \right) = 0, \] (77)

Analogously to the Voronets case we study the sub-case when \( \Theta = 0. \) We choose \( L_0 = \tilde{L}(y, \dot{x}, \dot{y}) - \tilde{L}(y, \Phi, \dot{y}) := \tilde{L} - L^*. \) We assume that the functions \( \tilde{L} \) and \( L^* \) are such that
\[ E_k \tilde{L}^* = -\sum_{\alpha=1}^{s_1} E_k(L_\alpha) \frac{\partial \tilde{L}}{\partial \dot{x}_\alpha} \Psi_\alpha, \] (78)
where \( \Psi_\alpha = \frac{\partial \tilde{L}}{\partial \dot{x}_\alpha} \bigg|_{L_1=\ldots=L_{s_1}=0} \)
and
\[ E_k(\tilde{L}) = \sum_{\alpha=1}^{s_1} \frac{d}{dt} \left( \frac{\partial \tilde{L}}{\partial \dot{x}_\alpha} \right) \frac{\partial L_\alpha}{\partial y_k}, \]
for \( k = 1, \ldots, s_2. \)

By inserting \( \dot{x}_j = \sum_{k=1}^{s_2} a_{jk}(y) \dot{y}_k, \quad j = 1, \ldots, s_1, \)
into Eq. (78) we obtain system (25). Consequently, system (78) is an extension of the classical Chaplygin equations when the constraints are nonlinear. \( \square \)

For the generalized Chaplygin systems the Lagrangian \( L \) takes the form
\[ L = \tilde{L}(y, \dot{x}, \dot{y}) - \tilde{L}(y, \Phi, \dot{y}) - \sum_{j=1}^{s_1} \left( \frac{\partial L^*_j}{\partial \dot{x}_j} + C_j \right) \dot{x}_j \]
\[ -\Phi_j(y, \dot{y}) - \sum_{j=1}^{s_2} \Lambda_j \dot{y}_j, \] (79)
for \( j = 1, \ldots, s_1 \) where the constants \( C_j \) for \( j = 1, \ldots, s_1 \) are arbitrary. Indeed, from (75) it follows that
\[ \lambda_j = \frac{\partial L_0}{\partial \dot{x}_j} + C_j = \frac{\partial L^*_j}{\partial \dot{x}_j} + C_j. \]

By inserting in (4) \( L_0 = \tilde{L} - L^* \) and \( \lambda_j \) for \( j = 1, \ldots, s_1 \) we obtain function \( L \) of (79).

We note that Vorones and Chaplygin equations with nonlinear constraints in the velocity were also obtained by Rumiansk and Sumbatov (see [44,47]).

Example 4 We shall illustrate the above results in the following example.

In the Appel’s and Hamel’s investigations the following mechanical system was analyzed. A weight of mass \( m \) hangs on a thread which passes around the pulleys and is wound round the drum of radius \( a \). The drum is fixed to a wheel of radius \( b \) which rolls without sliding on a horizontal plane, touching it at the point \( B \) with the coordinates \((x_B, y_B)\). The legs of the frame support the pulleys and keep the plane of the wheel vertical slide on the horizontal plane without friction. Let \( \theta \) be the angle between the plane of the wheel and the \( Ox \) axis; \( \varphi \) the angle of the rotation of the wheel in its own plane; and \((x, y, z)\) the coordinates of the mass \( m \). Clearly,
\[ \dot{z} = b \dot{\varphi}, \quad b > 0. \]

The coordinates of the point \( B \) and the coordinates of the mass are related as follows (see page 223 of [35] for a picture):
\[ x = x_B + \rho \cos \theta, \quad y = y_B + \rho \sin \theta. \]

The condition of rolling without sliding leads to the equations of nonholonomic constraints:
\[ \dot{x}_B = a \cos \theta \dot{\varphi}, \quad \dot{y}_B = a \sin \theta \dot{\varphi}, \quad b > 0. \]

We observe that the constraints \( \dot{z} = b \dot{\varphi} \) admit the representation
\[ \dot{z} = \frac{b}{a} \sqrt{\dot{x}^2 + \dot{y}^2 - \rho^2 \dot{\varphi}^2}. \]

Denoting by \( m_1, A \) and \( C \) the mass and the moments of inertia of the wheel and neglecting the mass of the frame, we obtain the following expression for the Lagrangian function
\[ \tilde{L} = \frac{m + m_1}{2} \left( \dot{x}^2 + \dot{y}^2 \right) + \frac{m_1}{2} \dot{\varphi}^2 + m_1 \rho \dot{\theta} (\sin \theta \dot{x} - \cos \theta \dot{y}) + \frac{A + m_1 \rho^2}{2} \dot{\varphi}^2 + C \dot{\varphi}^2 - mgz. \]
The equations of the constraints are
\[ \dot{x} - a \cos \theta \dot{\phi} + \rho \sin \theta \dot{\theta} = 0, \quad \dot{y} - a \sin \theta \dot{\phi} - \rho \cos \theta \dot{\theta} = 0, \quad \dot{z} - b \dot{\phi} = 0, \]

Now we shall study the motion of this constrained Lagrangian in the coordinates
\[ x_1 = x, \quad x_2 = y, \quad x_3 = \dot{\phi}, \quad y_1 = \theta, \quad y_2 = \dot{\theta}. \]
i.e., we shall study the nonholonomic system with Lagrangian
\[ L = L (y_1, y_2, \dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{\gamma}_1, \dot{\gamma}_2) = \frac{m + m_1}{2} (\dot{x}_1^2 + \dot{x}_2^2) + \frac{C}{2} \dot{x}_3^2 + \frac{J}{2} \dot{\gamma}_1^2 + \frac{m}{2} \dot{\gamma}_2^2 + m_1 \rho \dot{\gamma}_1 (\sin y_1 \dot{x}_1 - \cos y_1 \dot{x}_2) - \frac{mg}{b} y_2, \]

and with the constraints
\[ l_1 = \dot{x}_1 - \frac{a}{b} \dot{y}_2 \cos y_1 - \rho \dot{\gamma}_1 \sin y_1 = 0, \]
\[ l_2 = \dot{x}_2 - \frac{b}{l_2} \dot{y}_2 \sin y_1 + \rho \dot{\gamma}_1 \cos y_1 = 0, \]
\[ l_3 = \dot{x}_3 - \frac{a}{b} \dot{y}_1 = 0. \]

Thus, we have a classical Chaplygin system. To determine differential equations (78) and the transpositional relations (76) we define the functions
\[ L^* = -\dot{L}|_{l_1=l_2=l_3=0} = \frac{m(a^2 + b^2)m + a^2m_1 + C}{2b^2} \dot{y}_2^2 + \frac{mp^2 + J}{2} \dot{y}_1^2 - \frac{mg}{b} \dot{y}_2, \]
\[ L_1 = l_1, \quad L_2 = l_2, \quad L_3 = l_3, \quad L_4 = \dot{y}_1, \quad L_5 = \dot{y}_2. \]

After some computations we obtain that the matrix A (see formulae (74)) in this case becomes
\[ A = \begin{pmatrix} 0 & 0 & -\frac{a}{b} & 0 & 0 \\ 0 & 0 & \frac{a}{b} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \]

thus differential equations (78) take the form
\[ (mp^2 + J) \dot{y}_1 + \frac{a \rho m}{b} \dot{y}_1 \dot{y}_2 = 0, \]
\[ (m + m_1)a^2 + mb^2 \dot{y}_2 - mb \rho y_1 \dot{y}_2 = -mg \dot{y}_2. \]

Assuming that \((m + 2m_1)p^2 + J \neq 0\) and by considering the existence of the first integrals
\[ C_2 = \dot{y}_1 \exp \left( -\frac{a \rho m y_2}{b (mp^2 + J)} \right), \]
\[ h = \frac{(m + m_1)a^2 + mb^2}{2} \dot{y}_2^2 + \frac{b}{2} \left( mp^2 + J \right) \dot{y}_1^2 + mg \dot{y}_2, \]

after the integration of these first integrals we obtain
\[ \int \frac{\sqrt{(m + m_1)a^2 + mb^2 \dot{y}_2}}{\sqrt{2h - 2mg \dot{y}_2 - b^2 (mp^2 + J)}} \frac{dy_2}{C_3} = t + C_1, \]
\[ y_1(t) = C + C_2 \int \exp \left( \frac{2a \rho m y_2(t)}{bmp^2 + J} \right) dt. \]

Consequently, if \(\rho = 0\) then
\[ y_1 = C + C_2 t, \quad \int \frac{\sqrt{(m + m_1)a^2 + mb^2 \dot{y}_2}}{\sqrt{2h - 2mg \dot{y}_2 - b^2 (mp^2 + J)}} = t + C_1. \]

Hamel in [16] neglects the mass of the wheel \((m_1 = J = C = 0)\). Under these conditions the previous equations become
\[ \rho^2 \dot{y}_1 + \frac{a \rho}{b} \dot{y}_1 \dot{y}_2 = 0, \]
\[ (a^2 + b^2) \dot{y}_2 - ab \rho \dot{y}_2 = -gb \]

Appell and Hamel obtained the example of nonholonomic system with nonlinear constraints by means of the passage to the limit \(\rho \to 0\). However, as a result of this limiting process, the order of the system of differential equations is reduced, i.e., they become degenerate. In [35] the authors study the motion of the nondegenerate system for \(\rho > 0\) and \(\rho < 0\). From these studies it follows that the motion of the nondegenerate system \((\rho \neq 0)\) and degenerate system \((\rho \to 0)\) differ essentially. Thus, the Appell-Hamel example with nonlinear constraints is incorrect.

The transpositional relations (76) become
\[ \frac{\delta}{\delta y_1} \frac{dx_1}{dt} - \frac{\delta}{\delta y_1} \frac{dx_2}{dt} = a \sin y_1 \left( \frac{dy_1}{dt} \delta y_2 - \frac{dy_2}{dt} \delta y_1 \right), \]
\[ \frac{\delta}{\delta y_1} \frac{dx_2}{dt} = b \cos y_1 \left( \frac{dy_1}{dt} \delta y_2 - \frac{dy_2}{dt} \delta y_1 \right), \]
\[ \frac{\delta}{\delta y_1} \frac{dx_3}{dt} = 0, \quad \frac{\delta}{\delta y_1} \frac{dy_1}{dt} = 0, \quad \frac{\delta}{\delta y_2} \frac{dy_2}{dt} = 0. \]

Clearly, these relations are independent of \(\varphi, A, C,\) and \(m_1.\)

9 Consequences of Theorems 2 and 3 and the proof of Corollary 1.

We observe the following important aspects from Theorems 2 and 3.

(I) Conjecture 8 is supported by the following facts.
(a) As a general rule the constraints studied in classical
mechanics are linear in the velocities. However Appell and Hamel, in 1911, considered an artificial example with a constraint nonlinear in the velocity. As it follows from [35] (see example 4) this constraint does not exist in the Newtonian mechanics.

(b) The idea developed for some authors (see for instance [4]) to construct a theory in Newtonian mechanics, by allowing that the field of force depends on the acceleration, i.e., function of \( \dot{x} \) as well as of the position \( x \), velocity \( \dot{x} \), and the time \( t \), is inconsistent with one of the fundamental postulates of the Newtonian mechanics: when two forces act simultaneously on a particle the effect is the same as that of a single force equal to the resultant of both forces (for more details see [38] pages 11–12). Consequently, the forces depending on the acceleration are not admissible in Newtonian dynamics. This does not preclude their appearance in electrodynamics, where this postulate does not hold.

(c) Let \( T \) be the kinetic energy of the constrained Lagrangian systems. We consider the generalization of the Newton law: the acceleration (see [37,46])

\[
\frac{dT}{dt} \frac{\partial T}{\partial \dot{x}} - \frac{\partial T}{\partial x} = \text{equal to the force } F.
\]

Then in the differential equations (12) with \( L_0 = T \) we obtain that the field of force \( F \) generated by the constraints is

\[
F = \left(W_1^{-1} \Omega_1 \right)^T \frac{dT}{d\dot{x}} + W_1^T \frac{d}{dt} \lambda : = F_1 + F_2.
\]

The field of force \( F_2 = W_1^T \frac{d}{dt} \lambda = (F_21, \ldots, F_{2N}) \) is called the reaction force of the constraints. What is the meaning of the force

\[
F_1 = \left(W_1^{-1} \Omega_1 \right)^T \frac{dT}{d\dot{x}}
\]

If the constraints are nonlinear in the velocity, then \( F_1 \) depends on \( \dot{x} \). Consequently, in Newtonian mechanics does not exist a such field of force. Therefore, the existence of nonlinear constraints in the velocity and the meaning of force \( F_1 \) must be sought outside of the Newtonian model.

For example, for the Appel-Hamel constrained Lagrangian systems studied in the previous subsection we have that

\[
F_1 = \left(-\frac{a^2 \dot{x}}{\dot{x}^2 + \dot{y}^2} (\dot{x} \ddot{y} - \ddot{x} \dot{y}), \frac{a^2 \dot{y}}{\dot{x}^2 + \dot{y}^2} (\dot{x} \ddot{y} - \ddot{x} \dot{y}), 0 \right).
\]

For the generalized Voronets systems and locally for any nonholonomic constrained Lagrangian systems from the Eq. (18) we obtain that the field of force \( F_1 \) has the following components:

\[
F_{k1} = \sum_{a=1}^{s_1} E_k L_\alpha \frac{\partial L_0}{\partial \dot{x}_\alpha}
= \sum_{j=1}^{N} \sum_{a=1}^{s_1} \left( \frac{\partial^2 L_\alpha}{\partial \dot{x}_j \dot{x}_j} \frac{\partial L_0}{\partial \dot{x}_j} + \frac{\partial^2 L_\alpha}{\partial \dot{x}_k \partial \dot{x}_j} \frac{\partial L_0}{\partial \dot{x}_j} \right) + \sum_{a=1}^{s_1} \frac{\partial^2 L_\alpha}{\partial \dot{x}_k \partial \dot{x}_j},
\]

for \( k = 1 \ldots, N \) and \( s_1 = M \). Consequently, such field of force does not exist in Newtonian mechanics if the constraints are nonlinear in the velocity.

(II) Equations (12) can be rewritten in the form

\[
G \dot{x} + f(t, x, \dot{x}) = 0,
\]

where \( G = G(t, x, \dot{x}) \) is the matrix \((G_{j,k})\) given by

\[
G_{jk} = \frac{\partial^2 L_0}{\partial \dot{x}_j \partial \dot{x}_k} - \sum_{n=1}^{N} \frac{\partial A_{nk}}{\partial \dot{x}_j} \frac{\partial L_0}{\partial \dot{x}_n}, j, k = 1, \ldots, N,
\]

and \( f(t, x, \dot{x}) \) is a convenient vector function. If \( \det G \neq 0 \) then Eq. (82) can be solved with respect to \( \dot{x} \). This implies, in particular that the motion of the mechanical system at time \( t \in \{t_0, t_1\} \) is uniquely determined, i.e., the principle of determinacy (see for instance [2]) holds for the mechanical systems with equation of motion given in (12).

In particular for the Appel-Hamel constrained Lagrangian systems we have (see formula (48)) that

\[
\dot{x} = \left( x, y, z \right)^T,
\]

\[
\dot{y} = \left( \frac{a^2 \dot{x}}{\dot{x}^2 + \dot{y}^2}, \frac{a^2 \dot{y}}{\dot{x}^2 + \dot{y}^2}, g - \dot{z} \right)^T
\]

\[
G = \begin{pmatrix}
1 + \frac{a^2 \dot{x}^2}{\dot{x}^2 + \dot{y}^2} & -\frac{a^2 \dot{y}^2}{\dot{x}^2 + \dot{y}^2} & 0 \\
\frac{a^2 \dot{y}^2}{\dot{x}^2 + \dot{y}^2} & 1 + \frac{a^2 \dot{x}^2}{\dot{x}^2 + \dot{y}^2} & 0 \\
0 & 0 & 1
\end{pmatrix}, |G| = 1 + a^2.
\]

So in the Appel–Hamel system the principle of determinacy holds.

(III)

**Proof of Corollary 1** From Theorems 2 and 3 (see formulas (13) and (17)) and from all examples which we gave in the previous sections we see that are examples with zero transpositional relations and examples where all they are not zero. By contrasting the MVM
with the Lagrangian mechanics we obtain that for the unconstrained Lagrangian systems the transpositional relations are always zero. Thus, we have the proof of the corollary.

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