Noncommutative Geometry

and

Gauge Theory on Fuzzy Sphere

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Abstract

The differential algebra on the fuzzy sphere is constructed by applying Connes’ scheme. The $U(1)$ gauge theory on the fuzzy sphere based on this differential algebra is defined. The local $U(1)$ gauge transformation on the fuzzy sphere is identified with the left $U(N+1)$ transformation of the field, where a field is a bimodule over the quantized algebra $A_N$. The interaction with a complex scalar field is also given.

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1 Introduction

The concept of quantized spaces is discussed in a variety of fields in physics and mathematics. From the physicists’ viewpoint, the main motivation for investigating noncommutative spaces stems from the need of an appropriate framework to describe the quantum theory of gravity. Recently quantized spaces are also discussed in connection with M(atrix) theory which has been proposed as a nonperturbative formulation of string theory[1, 2]. This development in string theory supports the idea that the noncommutative structure of spacetime becomes relevant when constructing the theory of gravitation at Planck scale.

To describe noncommutative spaces, the noncommutative geometry is now investigated by many authors and using this framework one can even consider the differential geometry of singular spaces like, for example, a 2-point space which has been shown to provide a geometrical interpretation of the Higgs mechanism[3].

On the other hand, in order to describe gravity we have to know the theory of a wider class of noncommutative geometry. In this context, the class of noncommutative spaces which can be considered as deformations of continuous spaces is especially interesting. In general, such noncommutative spaces can be obtained by quantizing a given space with its Poisson structure. Furthermore, if the original space is compact one obtains a finite dimensional matrix algebra as a quantized algebra of functions over this space. In this case, we may consider the deformation as a kind of regularization with the special property that we can keep track of the geometric structure, a feature which is missing in the conventional regularization schemes.

In physics the algebra of the fuzzy sphere is well known and has been investigated in a variety of contexts: as an example for a general quantization procedure [4, 5] (see also for example [6, 7, 8, 9, 10, 11] and references therein) and in relation with geometric quantization. It is also discussed as the algebra appearing in membranes [12, 13], in relation with coherent states [14, 15], and recently in connection with noncommutative geometry [16, 17, 18]. The same structure also appears in the context of the quantum Hall effect [19, 20]. In this paper, we investigate the differential geometry of the fuzzy sphere and the field theory on it. We formulate the $U(1)$ gauge theory on the fuzzy sphere. The fuzzy sphere is one example in the above mentioned class of noncommutative geometry and thus the field theory on this space is a very
instructive model to examine the ideas of noncommutative geometry. Besides that, it is a deformation of the sphere obtained by quantization based on the Poisson structure on $S^2$, and the resulting algebra $A_N$ is a finite dimensional matrix algebra. Thus, what we obtain is a regularized field theory on the sphere. From this point of view, we are also interested in the gauge theory on this noncommutative space.

In order to formulate the local $U(1)$ gauge theory on the fuzzy sphere, we first have to define the differential algebra based on the above algebra $A_N$. We apply Connes’ framework of noncommutative differential geometry [9] by using a spectral triple $(A_N, \mathcal{H}_N, D)$ proposed recently by the authors[25], where $D$ is the Dirac operator and $\mathcal{H}_N$ is the corresponding Hilbert space of spinors. We analyze the space of 1-forms which corresponds to the gauge potential and give the 2-forms to define the field strength.

This paper is organized as follows. In section 2, we summarize the definitions of the Dirac operator, the chirality operator and the spectral triple. We give a complete derivation of the spectrum of the Dirac operator and discuss its properties in detail. Then we define the differential algebra on the fuzzy sphere. In section 3, the gauge field and the field strength are defined using this differential algebra. We examine the structure of the $U(1)$ gauge transformation of the charged scalar field. Then the corresponding invariant actions are formulated. Section 4 contains the discussion. We also discuss the commutative limit.

2 Noncommutative Differential Algebra

2.1 Algebra of Fuzzy Sphere

The algebra of the fuzzy sphere can be obtained by quantizing the function algebra over the sphere by using its Poisson structure. For this end we adopt the Berezin-Toeplitz quantization which gives the quantization procedure for a Kähler manifold [4, 5]. Applying this method to the function algebra over the sphere we obtain the algebra $A_N$. $A_N$ can be represented by operators acting on a $(N + 1)$ dimensional Hilbert space $\mathcal{F}_N$. The algebra $A_N$ can thus be identified with the algebra of the complex $(N + 1) \times (N + 1)$ matrices.

The basic algebra to be quantized is the function algebra $A_\infty$ of the square integrable functions over a 2-sphere. The basis of this algebra is given by
the spherical harmonics \( Y_{lm} \) and the multiplication of the algebra is a usual pointwise product of functions. The fuzzy sphere may also be introduced as an approximation of the function algebra over the sphere by taking a finite number \( N \) of spherical harmonics, where this number \( N \) is limited by the maximal angular momentum \( \{ Y_{lm}; l \leq N \} \). However with respect to the usual multiplication this set of functions does not form a closed algebra since the product of two spherical harmonics \( Y_{lm} \) and \( Y_{l'm'} \) contains \( Y_{l+l',m} \). It is a new multiplication rule that solves the above described situation and gives a closed function algebra with a finite number of basis elements. The resulting algebra \( A_N \) is noncommutative. We can identify the algebra of the fuzzy sphere with the algebra of complex matrices \( M_{N+1}(\mathbb{C}) \) and thus we can consider it as a special case of matrix geometry \([21, 22, 23, 24]\).

The operator algebra \( A_N \) and the Hilbert space \( \mathcal{F}_N \) can be formulated keeping the symmetry properties under the rotation group. We introduce a pair of creation-annihilation operators \( a^b, a_b (b = 1, 2) \) which transforms as a fundamental representation under the \( SU(2) \) action of rotation.

\[
[a^a, a^b_\dagger] = \delta^a_b. \tag{1}
\]

Define the number operator by \( N = a^\dagger_1 a^1 \), then the set of states \( |v> \) in the Fock space associated with the creation-annihilation operators satisfying

\[
N|v> = N|v>, \tag{2}
\]

provides an \( N + 1 \) dimensional Hilbert space \( \mathcal{F}_N \). The orthogonal basis \( |k> \) of \( \mathcal{F}_N \) can be defined as

\[
|k> = \frac{1}{\sqrt{k!(N-k)!}}(a^\dagger_1)^k(a^\dagger_2)^{N-k}|0>, \tag{3}
\]

where \( k = 0, \ldots, N \) and \( |0> \) is the vacuum.

The operator algebra \( \mathcal{A}_N \) acting on \( \mathcal{F}_N \) is unital and given by operators \( \{ O; [N, O] = 0 \} \). The generators of the algebra \( \mathcal{A}_N \) are defined by

\[
x_i = \frac{1}{2} \alpha \sigma_i^a a^\dagger_1 a^b, \tag{4}
\]

where the normalization factor \( \alpha \) is a central element \( [\alpha, x_i] = 0 \) and is defined by the constraint

\[
x_i x_i = \frac{\alpha^2}{4} N(N + 2) = \ell^2. \tag{5}
\]
The above equation means that $\ell > 0$ is the radius of the 2-sphere and we get for $\alpha$

$$\alpha = \frac{2\ell}{\sqrt{N(N+2)}}. \quad (6)$$

The algebra of the fuzzy sphere is generated by $x^i$ and the basic relation is

$$[x_i, x_j] = i\alpha\epsilon_{ijk}x_k. \quad (7)$$

On the Hilbert space $F_N$, $\alpha$ is constant and plays the role of the "Planck constant". The commutative limit corresponds to $\alpha \to 0$, i.e., $N \to \infty$.

Now let us consider the derivations of $A_N$. Among them, the derivative operator $L_i$ is defined by the adjoint action of $x^i$

$$\frac{1}{\alpha}ad_{x^i}a = \frac{1}{\alpha}[x_i, a] \equiv L_i a, \quad (8)$$

where $a \in A_N$. These objects are the noncommutative analogue of the Killing vector fields on the sphere, and the algebra of $L_i$ closes. We obtain thus

$$[L_i, x_j] = i\epsilon_{ijk}x_k, \quad [L_i, L_j] = i\epsilon_{ijk}L_k. \quad (9)$$

Finally, the integration is given by the trace over the Hilbert space $F_N$. The integration over the fuzzy sphere which corresponds to the standard integration over the sphere in the commutative limit is defined by

$$\langle O \rangle = \frac{1}{N+1}\mathrm{Tr}\{O\} = \frac{1}{N+1}\sum_k \langle k|O|k \rangle. \quad (10)$$

where $O \in A_N$.

**2.2 Chirality Operator and Dirac Operator**

We introduce the spinor field $\Psi$ as an $A_N$-bimodule $\Gamma A_N \equiv \mathbb{C}^2 \otimes A_N$, which is the noncommutative analogue of the space of sections of a spin bundle. $\Psi$ is represented by 2-component spinors $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ where each entry is an

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3Another possible choice is to take $\alpha = \frac{2}{N}$ as in ref. With this choice, the radius of the fuzzy sphere depends on $N$. 4
element of $A_N$ and we require that it transforms as a spinor under rotation of the sphere.

Since left multiplication and right multiplication commute, the $A_N$-bimodule can be considered as a left module over the algebra $A_N \otimes A_N^\alpha$, where $A_N^\alpha$ denotes the opposite algebra which is defined by:

$$x_i^o x_j^o \equiv (x_j x_i)^o, \ x_i \in A_N.$$  \hspace{1cm} (11)

The action of $a, b \in A_N$ onto the $A_N$-bimodule $\Psi \in \Gamma A_N$ is

$$ab^o \Psi \equiv a \Psi b.$$  \hspace{1cm} (12)

We define the Dirac operator and the chirality operator in the algebra $A_N \otimes A_N^\alpha$ \cite{25}, i.e. as $2 \times 2$ matrices the entries of which are elements in the algebra $A_N \otimes A_N^\alpha$. The construction of the Dirac operator is performed by the following steps:

(a) Define a chirality operator which commutes with the elements of $A_N$ and which has a standard commutative limit.

(b) Define the Dirac operator by requiring that it anticommutes with the chirality operator and, in the commutative limit it reproduces the standard Dirac operator on the sphere.

Requiring the above condition (a) we obtain for the chirality operator \cite{25}

$$\gamma_\chi = \frac{1}{\mathcal{N}} (\sigma_i x_i^o - \frac{\alpha}{2}).$$  \hspace{1cm} (13)

$\mathcal{N}$ is a normalization constant defined by the condition

$$(\gamma_\chi)^2 = 1,$$  \hspace{1cm} (14)

as $\mathcal{N} = \frac{\alpha}{2}(\mathcal{N} + 1)$ and $\sigma_i$ ($i = 1, 2, 3$) are the Pauli matrices. In the commutative limit, the operator $x_i$ can be identified with the homogeneous coordinate $x_i$ of sphere and the chirality operator given in eq.(5) becomes $\frac{1}{\ell} \sigma_i x_i$ which is the standard chirality operator invariant under rotation. \cite{26}

The chirality operator (13) defines a $Z_2$ grading of the differential algebra and it commutes with the algebra $A_N$. 

5
Proposition 1:

The Dirac operator $D$ satisfying the condition (b), i.e., $\{\gamma_x, D\} = 0$, is given by

$$D = \frac{i}{\ell \alpha} \gamma_x \epsilon_{ijk} \sigma_i x^j x_k.$$  \hspace{1cm} (15)

Proof: See ref.[25].

Note that this Dirac operator is selfadjoint, $D^\dagger = D$.

Acting with this operator on a spinor $\Psi \in \Gamma A_N$, we obtain

$$D\Psi = \frac{i}{\ell} \gamma_x \chi_i J_i \Psi,$$  \hspace{1cm} (16)

where

$$J_i = L_i + \frac{1}{2} \sigma_i,$$  \hspace{1cm} (17)

and

$$\chi_i \equiv \epsilon_{ijk} x_j \sigma_k.$$  \hspace{1cm} (18)

The action of the angular momentum operator on the bimodule is defined by

$$L_i \Psi \equiv \frac{1}{\alpha} [x_i, \Psi] = \frac{1}{\alpha} (x_i \Psi - \Psi x_i).$$  \hspace{1cm} (19)

The second condition of (b) concerning the commutative limit of the Dirac operator is also satisfied. If we replace each operator $\chi_i, J_i$ and $\gamma_x$ in eq.(14) by the corresponding quantity which is obtained in the commutative limit, we get

$$D_\infty = \frac{i}{\ell} \gamma_x \chi_i J_i = \frac{i}{\ell^2} (\sigma_i x_i) \epsilon_{ijk} x_i \sigma_j (iK_k + \frac{1}{2} \sigma_k) = -(i \sigma_i K_i + 1),$$  \hspace{1cm} (20)

where $x_i$ is the homogeneous coordinate of $S^2$ and $K_i$ is the Killing vector.

Therefore, in the commutative limit this Dirac operator is equivalent to the standard Dirac operator.

\footnote{Note that this Dirac operator is different from the one given in ref.[17]. The difference is that the operator in ref.[17] contains a product of Pauli matrix and angular momentum operator, whereas the operator defined here contains a product of $\chi_i$ and angular momentum operator as in eq.(16), i.e. it also contains $x_i$. Consequently, the spectra are not the same.}
2.3 Spectral Triple

In order to establish Connes’ triple we have to identify the Hilbert space. The space of the fermions $\Psi \in A_N \otimes \mathbb{C}^2$ defines a Hilbert space $\mathcal{H}_N$ with norm

$$\langle \Psi | \Psi \rangle = \text{Tr}_F (\Psi^\dagger \Psi) = \sum_{\rho=1}^2 \text{Tr}_F \{ (\psi^\rho)^* \psi^\rho \} , \quad (21)$$

where $\text{Tr}_F$ is the trace over the $(N + 1)$ dimensional Hilbert space $\mathcal{F}_N$.

The dimension of the Hilbert space $\mathcal{H}_N$ is $2(N + 1)^2$ and the trace over $\mathcal{H}_N$ is the trace over the spin suffices and over the $(N + 1)^2$ dimensional space of the matrices. Since the Dirac operator is defined in the algebra $A_N \otimes A_o^N$, the trace must be taken for operators of the form $ab^o$, with $a, b \in A_N$, and it is given by

$$\text{Tr}_N \{ ab^o \} = \sum_{K=1}^{2(N+1)^2} \langle \Psi_K | ab^o \Psi_K \rangle = 2 \text{Tr}_F \{ a \} \text{Tr}_F \{ b \} . \quad (22)$$

Here $\Psi_K$ is an appropriate basis in $\mathcal{H}_N$ labeled by an integer $K \in \{ 1, ..., 2(N + 1)^2 \}$. The factor 2 on the r.h.s. comes from the trace over the spin suffices.

To examine the structure of the Hilbert space we compute the spectrum $\lambda_j$ of the Dirac operator:

$$D^2 \Psi_{jm} = \lambda_j^2 \Psi_{jm} . \quad (23)$$

$\Psi_{jm}$ is a state with total angular momentum $j$, $J^2 \Psi_{jm} = j(j + 1) \Psi_{jm}$ and $J_3 \Psi_{jm} = m \Psi_{jm}$ is the $x_3$ component of the total angular momentum operator $J_i$ in eq.(17). $j$ and $m$ are half integers and run $\frac{1}{2} \leq j \leq N + \frac{1}{2}$ and $-j \leq m \leq j$.

**Proposition 2:**

The spectrum of the Dirac operator is given by

$$\lambda_j^2 = (j + \frac{1}{2})^2 [1 + \frac{1 - (j + \frac{1}{2})^2}{N(N + 2)}] . \quad (24)$$

**Proof:**

$$\frac{\ell^2}{\alpha^2} D^2 = (\epsilon_{ijk} \sigma^i \mathbf{X}^j \mathbf{Y}^k)(\epsilon_{i'j'k'} \sigma^{i'} \mathbf{X}^{j'} \mathbf{Y}^{k'})$$

\[7\]
\[ = X^2Y^2 - (XY)(XY) + 1 + (X\sigma) + (Y\sigma) \]  \hspace{1cm} (25)

where \( X_i = \frac{1}{\alpha}x_i \), \( Y_i = -\frac{1}{\alpha}x_i^o \) and \( (XY) = \sum_i X_i Y_i \). Using the relations

\[ L_i = X_i + Y_i \quad \text{and} \quad J_i = L_i + \frac{1}{2}\sigma_i , \]  \hspace{1cm} (26)

we obtain \( (XY) = \frac{1}{2}[L^2 - X^2 - Y^2] \) and \( (\sigma X) + (\sigma Y) = J^2 - L^2 - \frac{3}{4} \).

In order to evaluate the spectrum we use the representation of the spinor and substitute

\[ J^2 = j(j + 1) \quad \text{and} \quad L^2 = (j + s)(j + s + 1) , \]  \hspace{1cm} (27)

where \( j \leq N + \frac{1}{2} \) is a half integer and \( s = \pm \frac{1}{2} \). With this value we get

\[ (XY) = \frac{1}{2}[j(j + 1) + s(2j + 1) + \frac{1}{4} - X^2 - Y^2] \]

\[ (\sigma X) + (\sigma Y) = -s(2j + 1) - 1 . \]  \hspace{1cm} (28)

Thus, the eigenvalue is

\[ \frac{\ell^2}{\alpha^2} \lambda_j^2 = -\frac{1}{4}(j + \frac{1}{2})^2[(j + \frac{1}{2})^2 - 2(X^2 + Y^2) - 1] - \frac{1}{4}(X^2 - Y^2)^2 . \]  \hspace{1cm} (29)

If we substitute \( X^2 = Y^2 = \frac{N}{2}(\frac{N}{2} + 1) \) we obtain the relation (24). \hspace{1cm} q.e.d.

This spectrum coincides with the classical spectrum of the Dirac operator in the limit \( N \to \infty \). For finite \( N \), it contains zeromodes. When the angular momentum takes its maximal value we see that \( \lambda_{N + \frac{1}{2}} = 0 \). This happens since there is no chiral pair for the spin \( N + \frac{1}{2} \) state and therefore this part must be a zeromode for consistency. We can also confirm this property by computing: \( \text{Tr}_H(\gamma\chi) = 2(N + 1) \). Since these zeromodes have no classical analogue, one way to treat them is to project them out from the Hilbert space. On the other hand, the contribution of the zeromodes in the integration is of order \( \frac{1}{N} \) and thus their contribution vanishes in the limit \( N \to \infty \). Therefore, considering the differential algebra on the fuzzy sphere as a kind of regularization of the differential algebra on the sphere, it is sufficient to take the full Hilbert space \( \mathcal{H}_N \).

In this way we obtain Connes’ triple \( (\mathcal{A}_N, \mathcal{D}, \mathcal{H}_N) \). We thus can apply the construction of the differential algebra.
2.4 Differential Algebra

In this section we construct the differential algebra associated with \((A_N, D, H_N)\) by using Connes’ method [9]. See also [27].

We define the universal differential algebra \(\Omega^*(A_N)\) over \(A_N\). An element \(\omega \in \Omega^*(A_N)\) is in general given by

\[
\omega = \sum_{\lambda \in I} a^{(0)}_\lambda da^{(1)}_\lambda da^{(2)}_\lambda \cdots da^{(p)}_\lambda ,
\]

where \(p\) is an integer, \(a^{(k)}_\lambda \in A_N (k = 0 \cdots p)\) and \(I\) is an appropriate set labeling the elements. \(da\) is a symbol defined by the operation of the differential \(d\) on \(a \in A_N\), which satisfies Leibnitz rule \(d(ab) = (da)b + a(db)\) for \(a, b \in A_N\), and \(d\mathbf{1} = 0\) for the identity \(\mathbf{1} \in A_N\). We also require \((da)^* = -da^*\). The Leibnitz rule provides a natural product among the elements in \(\Omega^*(A_N)\) and the differential \(d\) on \(\Omega^*(A_N)\) is defined by

\[
d(\sum_{\lambda \in I} a^{(0)}_\lambda da^{(1)}_\lambda da^{(2)}_\lambda \cdots da^{(p)}_\lambda ) = \sum_{\lambda \in I} da^{(0)}_\lambda da^{(1)}_\lambda da^{(2)}_\lambda \cdots da^{(p)}_\lambda .
\]  

Then, it follows \(d^2 \omega = 0\) and the graded Leibnitz rule.

In order to define the \(p\)-forms as operators on \(H_N\), a representation \(\pi\) is defined by

\[
\pi(\sum_{\lambda \in I} a^{(0)}_\lambda da^{(1)}_\lambda da^{(2)}_\lambda \cdots da^{(p)}_\lambda ) = \sum_{\lambda \in I} a^{(0)}_\lambda[D, a^{(1)}_\lambda][D, a^{(2)}_\lambda] \cdots [D, a^{(p)}_\lambda] .
\]

Recall that \(A_N\) is defined as an algebra of operators in \(H_N\). Then the graded differential algebra is defined by

\[
\Omega^*_D(A_N) = \Omega^*(A_N)/J ,
\]

where \(J = ker\pi + dker\pi\) is the differential ideal of \(\Omega^*(A_N)\).

In order to establish the differential calculus on the fuzzy sphere, we have to examine the structure of the differential kernel \(J\). For this we denote the kernel of each level as

\[
ker\pi^{(p)} \equiv \Omega^p(A_N) \cap ker\pi ,
\]

then the differential kernel \(J^{(p)}\) for the \(p\)-form is

\[
J^{(p)} = ker\pi^{(p)} + dker\pi^{(p-1)} .
\]
Since the elements of the algebra $A_N$ are defined as operators in $H_N$, $\ker \pi^{(0)} = \{0\}$, i.e., $J^{(0)} = \{0\}$. It means that $\Omega^0_D(A_N) = A_N$. The differential kernel of the 1-form is $J^{(1)} = \ker \pi^{(1)} + d \ker \pi^{(0)} = \ker \pi^{(1)}$, and thus for any element $a \in A_N$ the derivative is defined by

$$\pi(da) = [D, a].$$

(36)

The space of 1-forms $\omega \in \Omega^1_D(A_N)$ can be identified with the operators $\pi(\omega)$ in $H_N$:

$$\pi(\Omega^1_D(A_N)) = \{ \pi(\omega) \mid \pi(\omega) = \sum_{\lambda \in I} a_\lambda [D, b_\lambda] ; \ a_\lambda, b_\lambda \in A_N \}.$$

(37)

Thus, with the above identification, the exterior derivative $d$ defines a map:

$$d : A_N \rightarrow M_2(\mathbb{C}) \otimes (A_N \otimes A_N^o),$$

(38)

where $M_2(\mathbb{C})$ is the algebra of $2 \times 2$ complex matrices.

Using the definition of the Dirac operator (15), a 1-form is expressed as follows: Take a 1-form $\pi(\omega) \in \pi(\Omega^1_D(A_N))$ in eq.(37). Using eq.(15) we obtain

$$\pi(\omega) = \sum_{\lambda} i \epsilon_{i\ell\gamma} \epsilon_{ijk} \sigma_i x^o_{j \lambda} a_\lambda [x_k, b_\lambda] = i \ell \chi_{\lambda k}^o \omega_k,$$

(39)

where

$$\chi_k^o \equiv \epsilon_{i\ell j} \sigma_i x_j^o;$$

(40)

and the components $\omega_k$ of $\pi(\omega)$ can be rewritten by using the definition (8) of $L$ as:

$$\omega_k \equiv \frac{1}{\alpha} \sum_{\lambda} a_\lambda [x_k, b_\lambda] = \sum_{\lambda} a_\lambda (L_k b_\lambda).$$

(41)

Here, $\omega_k \in A_N$ may be considered as the component of a vector field.

In order to write the gauge field action, we have to define the 2-form. A 2-form $\eta \in \Omega^2_D(A_N)$ can be given in general as

$$\pi(\eta) = \sum_{\lambda} a^{(1)}_\lambda [D, a^{(2)}_\lambda] [D, a^{(3)}_\lambda]$$

$$= \frac{1}{e^2} \chi_i^o \chi_j^o \sum_{\lambda} a^{(1)}_\lambda (L_i a^{(2)}_\lambda) (L_j a^{(3)}_\lambda).$$

(42)
where $a^{(i)}_\lambda \in \mathcal{A}_N$. Since the 2-form in eq. (42) is defined up to the differential kernel $\pi(d\text{ker}\pi^{(1)})$, $\pi(\eta)$ contains redundant components.

Note that when we perform the calculation, we do not use the $\Omega^2_D(\mathcal{A}_N)$, but its representation $\pi(\Omega^2_D(\mathcal{A}_N))$, thus it is sufficient to compute $\pi(d\text{ker}\pi^{(1)})$, since $\pi(\Omega^*(\mathcal{A}_N))$ is isomorphic to $\pi(\Omega^*(\mathcal{A}_N))/\pi(d\text{ker}\pi)$.

The nontrivial contribution of $\pi(d\text{ker}\pi^{(1)})$ is proportional to the traceless part of the symmetric product $\chi_o^i \chi_o^j$ as we shall see in the following.

The exterior derivative of a general 1-form $\omega$ defined in eq.(37) is
\[
\pi(d\omega) = \sum_{\lambda} [D, a_\lambda][D, b_\lambda].
\]
Using the Dirac operator we obtain
\[
\pi(d\omega) = \frac{1}{\ell^2} \sum_{\lambda} \chi_o^i \chi_o^j \left[ L_i (a_\lambda L_i b_\lambda) - \frac{1}{2} \epsilon_{i'k} a_\lambda L_k b_\lambda + \frac{2}{3\alpha} \delta_{i',i} (a_\lambda L_i b_\lambda) x_i 
- a_\lambda \left[ \frac{1}{2} (L_i, L_{i'}) - \frac{1}{3} \delta_{i',i'} L^2 \right] b_\lambda \right].
\]
The first three terms vanish for $\omega \in \text{ker}\pi^{(1)}$. Only the last term gives a nontrivial contribution for the differential kernel and thus $\pi(d\text{ker}\pi^{(1)})$ is proportional to the symmetric traceless product of $\chi_o^i \chi_o^j$.

The proof of the existence of the nontrivial 1-forms which contribute to $d\text{ker}\pi^{(1)}$ is given in the appendix. Using the explicit expression $\omega_{p,q} = x_A^p dx_A^q$ of $\text{ker}\pi^{(1)}$ obtained in the appendix (see eq.(37)) we compute $\pi(d\text{ker}\pi^{(1)})$. We find that $d\omega_{p,q}$ gives an element of $d\text{ker}\pi^{(1)}$:

**Proposition 3:** $\pi(d\omega_{p,q}) \neq 0$, for $p + q = N + 2$ and $p, q > 1$.

**Proof:**
\[
\pi(d\omega_{p,q}) = [D, x_A^p][D, x_A^q] = -\frac{1}{\ell^2} \gamma^i \gamma^o \chi^j \chi_o^o [-2p x_A^{p-1}(x_3 + \frac{\alpha}{2}(p - 1))] [-2q x_A^{q-1}(x_3 + \frac{\alpha}{2}(q - 1))]
= \frac{1}{\ell^2} \chi_o^i \chi_o^j 4pq x_A^{p+q-2} \left( x_3^2 + x_3 \left( \frac{\alpha}{2} p + \frac{3\alpha}{2} q - 2\alpha \right) + \frac{\alpha^2}{4} (p + 2q - 3)(q - 1) \right).
\]
Using the identity $x_+ x_- = \ell^2 + \alpha x_3 - x_3^2$, this expression can be simplified to
\[
\pi(d\omega_{p,q}) = 4pq\frac{1}{\ell^2}x_+^o x_+^{p+q-2}[A(q)x_3 + B(q)]
\]  
(46)

where
\[
A(q) = \frac{\alpha}{2}p + \frac{3\alpha}{2}q - \alpha ,
\]
(47)
\[
B(q) = \ell^2 + \frac{\alpha^2}{4}(p + 2q - 3)(q - 1) - x_+ x_-.
\]
(48)

This means that \(\pi(d\omega_{p,q})\) does not vanish for \(p + q = N + 2\) although \(\omega_{p,q} \in ker\pi^{(1)}\) for \(p + q = N + 2\).

The result of the proposition 3 and the corresponding contribution from the kernels of the other directions show that there exist nontrivial differential kernel elements \(\pi(dker\pi^{(1)})\) proportional to the symmetric traceless product of \(\chi_o^i\chi_o^{i'}\).

With this result we can prove the following proposition.

**Proposition 4**:

\[
\pi(dker\pi^{(1)}) = \{ \Lambda | \Lambda = \frac{1}{\ell^2}x_+^o \chi_+^o a_{ij} \text{ where } a_{ij} \in \mathcal{A}_N, a_{ij} = a_{ji} \text{ and } \sum_{i=1}^{3} a_{ii} = 0 \}.
\]

(49)

**Proof**: Using Proposition 3, we obtain a nontrivial element by multiplying \(a'_\lambda, b'_\lambda \in \mathcal{A}_N\) and

\[
\pi\left(d\sum_\lambda a'_\lambda \omega_{p,q} b'_\lambda\right) = \sum_\lambda \pi\left(a'_\lambda (d\omega_{p,q}) b'_\lambda\right)
\]
\[
= \sum_\lambda \frac{1}{\ell^2}x_+^o x_+^o a'_\lambda (L_-x_+^p)(L_-x_+^q) b'_\lambda
\]
\[
= 4pq\frac{1}{\ell^2}x_+^o x_+^o \sum_\lambda a'_\lambda x_+^N [A(q)x_3 + B(q)] b'_\lambda
\]
(50)

where we have used \(p + q = N + 2\). Choosing appropriate elements \(a'_\lambda, b'_\lambda \in \mathcal{A}_N\), the factor \(\sum_\lambda a'_\lambda x_+^N [A(q)x_3 + B(q)] b'_\lambda\) can become any element in \(\mathcal{A}_N\). We have six independent directions for \(\omega_{p,q}\) and combining the results from them we get the traceless symmetric combinations of suffices \(i, i'\) in \(\chi_i^o \chi_{i'}^o\).
Identifying the $\Omega^2_D(\mathcal{A}_N)$ with its representation $\pi(\Omega^2_D(\mathcal{A}_N))$, a general 2-form $\tilde{\eta} \in \pi(\Omega^2_D(\mathcal{A}_N))$ is given by

$$\tilde{\eta} = \sum_{\lambda} a^{(i)}_{\lambda}[D, a^{(2)}_{\lambda}][D, a^{(3)}_{\lambda}],$$

(51)

where $a^{(i)}_{\lambda} \in \mathcal{A}_N$ up to $\pi(d \text{ker} \pi^{(1)})$.

Combining eqs.(43) and (44) we can compute the operation of the derivative $d$ on a general 1-form in eq.(37) and we obtain

$$\pi(d\omega) = \frac{1}{2\ell^2} \chi^0_k \chi^0_k \left( \{ L_k \omega_{k'} - L_{k'} \omega_k \} - i\epsilon_{kk'kk''} \omega_{k''} + \frac{2}{3\alpha} [x_i \omega_i + \omega_i x_i] \right),$$

(52)

where we have used the definition of the components $\omega_k$ in eq.(39).

Since the trace part does not belong to the differential kernel, the last term in the above equation is not removed by dividing differential kernels. We continue here our construction of the gauge field action with this definition of the differential algebra and we shall obtain a kind of mass term in the gauge theory. The commutative limit $\alpha \rightarrow 0$ becomes singular, as can be seen from eq.(39). However, as we discuss in the following, we can still interpret the resulting theory as a regularization of the corresponding commutative theory.

An alternative strategy to the one taken here would be to restrict the above defined 2-form. With the above 2-form as it stands the naive commutative limit does not give the standard differential calculus. One possibility to handle this situation is to use the property of the trace: $\chi^0_k \chi^0_k = 2N^2 - \alpha N \gamma_k$. It turns out that the trace part $J_T$ is an ideal of the $\pi(\Omega^2(\mathcal{A}_N))$. Furthermore, in each $p$-form space $\pi(\Omega^p(\mathcal{A}_N))$, the set $J_T \pi(\Omega^{p-2}(\mathcal{A}_N)) \cup \pi(\Omega^{p-2}(\mathcal{A}_N))J_T$ is an ideal and thus there is a possibility to divide the differential algebra so that we can take the commutative limit and obtain the standard differential calculus. This procedure will be discussed elsewhere.

3 \textbf{U(1) Gauge Field Theory}

3.1 Vector Field

Using the geometric notions defined in the previous sections, we formulate the $U(1)$ gauge theory on the fuzzy sphere. We identify the differential algebra
\( \Omega_D(\mathcal{A}_N) \) with its representation \( \pi(\Omega_D(\mathcal{A}_N)) \) and do not write the map \( \pi \) explicitly.

First, to formulate the gauge field theory we define the real vector field \( \mathbf{A} \) which is a 1-form on the fuzzy sphere. We impose the reality condition for this 1-form by

\[
\mathbf{A}^\dagger = \mathbf{A} .
\]

Using the general definition of a 1-form, \( \mathbf{A} \) can be written as

\[
\mathbf{A} = \sum_\lambda a_\lambda [\mathbf{D}, b_\lambda] ,
\]

where \( a_\lambda, b_\lambda \in \mathcal{A}_N \) are appropriate elements. According to the general discussion about 1-forms in the previous section we can write

\[
\mathbf{A} = \frac{i}{\ell} \gamma_\lambda \chi^0 \mathbf{A}_k ,
\]

where \( \mathbf{A}_k \) is the component field of \( \mathbf{A} \) given by

\[
\mathbf{A}_k = \sum_\lambda a_\lambda (\mathbf{L}_k b_\lambda) .
\]

For the component field the reality condition gives

\[
\mathbf{A}^*_k = \mathbf{A}_k .
\]

Thus each component of the gauge field is represented by an \((N+1) \times (N+1)\) hermitian matrix.

Note that, in the commutative case, the 1-form satisfies the constraint \( x_i A_i = 0 \), which shows the reduction of the degrees of freedom. However in the noncommutative case the 1-form defined by the equation (56) does not satisfy the similar constraint on \( \mathbf{A}_k \) in general. Further discussion on the treatment of this property is given in section 4.

In the remaining part, let us push forward the construction of the gauge theory on the noncommutative sphere. In the commutative case, we obtain

\[
\text{The hermiticity condition requires the form } A = \sum_\rho a_\rho [\mathbf{D}, b_\rho] + b_\rho^* [\mathbf{D}, a_\rho^*] - \frac{1}{2} [\mathbf{D}, a_\rho b_\rho] + b_\rho^* a_\rho^* . \text{ This can be again written in the form (54).}
\]
the field strength of the $U(1)$ gauge theory by taking the exterior derivative of the 1-form. In the noncommutative case, the exterior derivative gives

$$dA = \sum_{\lambda} [D, a_{\lambda}][D, b_{\lambda}] .$$  \hspace{1cm} (58)$$

Applying the result of the previous section we obtain

$$dA = \frac{i}{2\ell^2} \chi_{k}^{\alpha} \chi_{k'}^{\alpha} F_{kk'} ,$$  \hspace{1cm} (59)

with

$$F_{kk'} = -i\{L_k A_{k'} - L_{k'} A_k\} - \epsilon_{kk'}^{k''} A_{k''} - i\delta_{kk'} \frac{2}{3\alpha} [A_i x_i + x_i A_i] .$$  \hspace{1cm} (60)$$

We can show that the above $F_{kk'}$ corresponds to the field strength for the abelian gauge field in the commutative limit. To see this, we use the following correspondence which holds in the commutative limit:

$$A_k = K^\mu_k A_\mu \quad \text{and} \quad L_k = iK_k .$$  \hspace{1cm} (61)$$

Here $A_\mu$ is a gauge field and $K^\mu_k$ ($k = 1, 2, 3, \mu = 1, 2$) is the Killing vector on the sphere with appropriate coordinates $\rho^\mu$, and $K_k = K^\mu_k \partial_\mu$. With the above identification we get

$$F_{kk'} = K^\mu_k K^\nu_{k'} F_{\mu\nu} ,$$  \hspace{1cm} (62)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Here we have used the relation $A_i x_i = 0$ which holds in the commutative case.

In the noncommutative case, however, the exterior derivative of the 1-form $dA$ does not give the field strength.

### 3.2 $U(1)$ Gauge Transformation

For the formulation of the $U(1)$ gauge theory on the fuzzy sphere, let us consider the $U(1)$ gauge transformation of a charged scalar field, i.e., a complex scalar field \[25\]. The algebraic object corresponding to the complex scalar field on the fuzzy sphere is the $A_N$-bimodule $\Phi \in A_N$. Its action is given by...
\[ S = \frac{1}{2(N + 1)^2} \text{Tr}_N \{ (d\Phi)^\dagger d\Phi \} . \]  

(63)

Apparently, the above action is invariant under global \( U(1) \) transformation of the phase

\[ \Phi' = e^{i\phi} \Phi . \]  

(64)

Following the standard approach, the local \( U(1) \) gauge transformation can be defined if we let the phase \( e^{i\phi} \) be a function on the fuzzy sphere. In the present algebraic formulation this means we multiply an element \( u \in \mathcal{A}_N \) on the field \( \Phi \), where unitarity is implemented by

\[ u^* u = 1 . \]  

(65)

When we generalize the transformation, we may take either left or right multiplication of \( u \) on the field \( \Phi \) due to the ordering ambiguity. Here we take the left multiplication as the \( U(1) \) gauge transformation for \( \Phi \):

\[ \Phi' = u \Phi . \]  

(66)

The transformation of the conjugate field \( \bar{\Phi} = \Phi^* \) is given by \( \bar{\Phi}' = \bar{\Phi} u^* \).

Since the algebra \( \mathcal{A}_N \) is isomorphic to the algebra of \( (N + 1) \times (N + 1) \) matrices, the condition (65) shows that, as a matrix, \( u \) is an element of \( U(N + 1) \). In other words, the local \( U(1) \) gauge transformation on the fuzzy sphere in matrix representation is defined as the left \( U(N + 1) \) transformation.

Therefore, we define the covariant derivative \( \nabla_A \) as

\[ \nabla_A \Phi = d\Phi + A \Phi . \]  

(67)

Then the gauge transformation of the gauge field can be defined by requiring the covariance of \( \nabla_A \Phi \):

\[ \nabla_{A'} (u\Phi) = u \nabla_A \Phi . \]  

(68)

This defines the standard form of the gauge transformation

\[ A' = u d u^* + u A u^* . \]  

(69)

In components it reads

\[ A_k' = u (L_k u^*) + u A_k u^* . \]  

(70)
The above transformation keeps the hermiticity condition (53) and may
be interpreted as the transformation of the $U(N+1)$ gauge theory on a one-
point space, and thus the covariant field strength is given by the standard
curvature form (71)

$$\Theta = dA + AA.$$  

In components the curvature 2-form is

$$\Theta = \frac{-i}{2\ell^2} \chi^o_k \chi^o_{k'} \Theta_{kk'},$$

where the component of the field strength is

$$\Theta_{kk'} = i\{L_k A_{k'} - L_{k'} A_k\} + \epsilon_{kk'kk''} A_{k''} + i[A_k, A_{k'}]$$

$$+ \frac{1}{3\alpha} \delta_{kk'} [A_i x_i + x_i A_i + \alpha A_i A_i].$$  

### 3.3 The Action of Gauge Field and Matter

With the above results we define the noncommutative analogue of the gauge
invariant action.

The action of the charged scalar is

$$S_M = \frac{1}{2(N+1)^2} \text{Tr}_H \{(\nabla A \Phi)^\dagger \nabla A \Phi\}.$$  

The action of the gauge field is given by

$$S_G \equiv \frac{1}{2(N+1)^2} \text{Tr}_H \{\Theta^2\}.$$  

Both actions are invariant under local $U(1)$ gauge transformation. Thus,
combining these two actions, we obtain the action of the $U(1)$ gauge theory
with scalar matter on the fuzzy sphere. Note that we may introduce the
gauge coupling constant $g$ by rescaling the gauge field $A$ to $g A$.

In order to see the detailed structure of the above actions, we take a part
of the trace. We perform the trace relating to the opposite algebra and the
spin suffixes. Then we obtain the action which contains only the fields $A$ and
$\Phi$ and the trace of this action is taken over the Hilbert space $F_N$.  

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Then the matter action (74) can be reduced as
\[ S_M = \frac{2}{3(N+1)} \text{Tr}_F \{ (L_i \Phi + A_i \Phi)^* (L_i \Phi + A_i \Phi) \} . \]  
(76)

Similarly, the gauge field action (75) is reduced to
\[ S_G = \frac{C_A}{(N+1)} \text{Tr}_F \{ \Theta^A_{ii'} \Theta^A_{ii'} \} + \frac{C_S}{(N+1)} \text{Tr}_F \{ \Theta^S_{ii'} \Theta^S_{ii'} \} , \]  
(77)

where
\[ C_A = \frac{N^2}{2\ell^2} \left( \frac{-\alpha^2}{3N^2} + \frac{2}{3} \right) , \]
\[ C_S = 1 + \frac{1}{N(N+2)} \]  
(78)
and \( \Theta^S_{ij} (\Theta^A) \) is the (anti)symmetric part of the field strength given in eq.(73).

Since the trace over the Hilbert space \( F_N \) corresponds to the volume integration in the commutative limit, the actions \( S_G \) and \( S_M \) given in eqs.(76), and (77) respectively, should correspond to the standard action on the sphere in the limit \( N \to \infty \).

Apparently the \( \Theta^S \) in the gauge action does not have a classical correspondence. Furthermore, as we see below this term is singular in the naive \( N \to \infty \) limit. This is unavoidable since our differential algebra is singular in this limit.

However, under certain conditions we may consider the above action as a regularized theory of the commutative case as follows: The symmetric part of the action is
\[ (\Theta^S)^2 \sim \frac{1}{\alpha^2} [(A_i x_i + x_i A_i) + \alpha A_i A_i]^2 . \]  
(79)

The above combination is gauge invariant under the gauge transformation given in eq.(69). This term can be understood as the gauge invariant mass term of the radial component of the gauge field. Thus, physically we can understand the effect of the symmetric part as follows: When we consider the quantization of the above regularized theory using the path integral which respects the gauge symmetry, then in the \( \alpha \to 0 \) limit the symmetric term behaves like a (gauge invariant) delta function which drops the radial component.
Furthermore, from the point of view of gauge theory it is not necessary to take $\Theta^2$ as an action. Instead, we can simply take any linear combination of the gauge invariant terms. This means that we can take $C_A$ and $C_S$ as independent parameters.

Thus, we obtain in general the following action for the gauge field.

$$S = \frac{1}{(N+1)} \text{Tr}_F \{ C_1 G_{kk'} G_{kk'} + C_2 G'^2 \}$$  \hspace{1cm} (80)

where $C_1$ and $C_2$ are c-numbers and

$$G_{kk'} = iL_k A_{k'} - iL_{k'} A_k + \epsilon_{kk'k''} A'_{k''} + i[A_k, A_{k'}]$$

$$G' = x_i A_i + A_i x_i + \alpha A_i A_i .$$  \hspace{1cm} (81)

The above action (77) is a special case of the general form given here.

### 4 Discussions and Conclusion

In this paper we have formulated the $U(1)$ gauge theory on the fuzzy sphere, following Connes’ framework of noncommutative differential geometry. The differential algebra on the fuzzy sphere has been constructed by applying the chirality operator and Dirac operator proposed in ref.\[25\]. This chirality operator anticommutes with the Dirac operator and the structure of the differential algebra becomes simple. Then we analyzed the structure of the 1-forms and 2-forms which are necessary to construct the gauge field action. In ref.\[25\], the action of a complex scalar field on the fuzzy sphere which is invariant under the global $U(1)$ transformation of the phase of the complex scalar field has been formulated. Here, the local $U(1)$ gauge transformation on the fuzzy sphere is introduced by making the global phase transformation into a local transformation, i.e. the phase becomes a function over the fuzzy sphere. By construction, a function over the fuzzy sphere is simply given by elements of the algebra $\mathcal{A}_N$. Thus, the local $U(1)$ gauge transformation is defined by multiplication of an element $u \in \mathcal{A}_N$, satisfying unitarity $u^* u = 1$.

Since the algebra $\mathcal{A}_N$ is noncommutative, there is an ambiguity of operator ordering when replacing the global phase by the algebra elements $u$. We have chosen here the left multiplication. Thus, when we represent the algebra $\mathcal{A}_N$ by matrices, the local $U(1)$ gauge transformation is identified with
the left transformation by a unitary \((N + 1) \times (N + 1)\) matrix. Therefore, the gauge field action is analogous to the Yang-Mills action.

Once we know the Dirac operator, the construction of the differential calculus is rather straightforward, however, as we have seen when defining the 1-forms, their components \(A_i\) do not satisfy \(A_i x^i = 0\) in general. In the commutative case this relation holds since the Killing vector is perpendicular to the normal direction of the sphere. However in the noncommutative case \(x_i L^i\) is not necessarily zero. Since the relation \[\tag{82}\]

\[x_i A_i + A_i x_i = \frac{1}{\alpha} [x_i, a][x_i, b],\]

holds, this property is related with the trace part of the 2-form as follows:

As we have seen in the construction of the 2-forms performed here, the differential kernel \(\pi^{(2)}\) does not contain a trace part, i.e., the part proportional to \(\chi^a \chi^c\). In the course of deriving eq.\((\text{[1]})\) we get \([x_i, a][x_i, b]\) as a coefficient of \(\chi^a \chi^c\). Up to the kernel condition this product of commutators is equivalent to \(a L^2 b\). The reason why the trace part drops from the differential kernel is due to the relation \(a L^2 b = -\frac{2}{\alpha} a (L_i b) x_i\). This relation is a direct consequence of the condition that \(\ell^2\) is central. This type of problem relating to the reduction of degrees of freedom as well as to the structure of the differential kernel is a rather general feature when defining the differential forms by the adjoint action \(L_i\). \[\footnote{This relation follows from definition (56).}\]

Thus, in the noncommutative case the construction gives 1-forms which have three independent components. One possibility to drop the trace part (which is proportional to the third component) in the present approach has been indicated in section 2.4.

On the other hand, although the 2-form is singular in the \(N \to \infty\) limit, the action given in eq.\((77)\) still allows the interpretation as a regularized theory of the gauge theory on the sphere.

It is easy to check that both terms in the action eq.\((77)\) are invariant under the gauge transformation \((\text{[70]})\). Thus, the most general gauge action can be written as in eq.\((81)\). The first term corresponds to the standard

\[6\text{This relation follows from definition (56).}\]

\[7\text{The structure of the Dirac operator depends on the choice of the fermion, but on the other hand if the Dirac operator has the form }\theta^i x_i, \text{ and if }\theta^i \text{ commutes with } x_i, \text{ where }\theta^i \sim \gamma^c x^c \text{ in our case, then the derivative } d \text{ is always given by } da = \theta^i (L_i a) \text{ with } L_i \text{ being the adjoint action.}\]
gauge action in the commutative limit. This term is usually taken as the action for the gauge field in the fuzzy sphere. The second term approaches simply \((2x_iA_i)^2\) in this limit.

As we mentioned, the symmetric part of the action can be understood as a gauge invariant mass for the radial component of the gauge field. Furthermore, in the action (75), this mass is diverging in the limit of \(N \to \infty\) and can be treated as delta function constraint under the path integral. Thus by taking a limit which respects the gauge symmetry, the freedom corresponding to \(x_iA_i + A_i x_i + \alpha A^2\) is freezeed and thus effectively drops from the theory. Since in this limit this procedure is equivalent to the constraint \(x_iA_i = 0\), it reduces the freedom of the vector potential in the commutative theory properly.

From the point of view of constructing a gauge theory on the fuzzy sphere, we have an even simpler choice to treat the degrees of freedom of the theory. If we require only the gauge invariance under the gauge transformation (69), we can take the symmetric term as a constraint for the gauge field from the begining on. Then the action contains only the antisymmetric part, i.e., \(C_2 = 0\) in eq.(81) and the gauge field is constrained by

\[
G' = (A_i x_i + x_i A_i) + \alpha A_i A_i = 0.
\]  

Then in this construction, the gauge field has correct degrees of freedom, even in the noncommutative case. Apparently, this theory also gives the correct commutative limit.

To complete our discussion, we want to mention that the use of the constraint \(G' = 0\) to restrict the differential calculus is not straightforward, since \(dG'\) does not automatically vanish. The treatment of this constraint within the differential calculus needs more investigation.

The fuzzy sphere is one of the easiest examples of a noncommutative space. We can consider the \(U(1)\) gauge theory on the fuzzy sphere formulated in this paper as a regularized version of a gauge theory on the sphere. The gauge theory on the noncommutative sphere is also investigated in ref.[28]. The differential calculus there is based on the supersymmetric fuzzy sphere and the structure of the fermion is different from the one discussed here. Thus the structure of the differential algebra is also different. However, this is not a contradiction since, in principle, there are many types of differential algebra associated with the fuzzy sphere algebra, depending on the choice of the spectral triple.
In the formulation given here we can also see an interesting analogue with the M(atrix) theory. If we introduce a new field
\[ \nabla_i = \frac{1}{\alpha} x_i + A_i , \] (84)
then the field strength \( \Theta^A_{ij} \) is given by
\[ \Theta^A_{ij} = i[\nabla_i, \nabla_j] - \epsilon_{ijk} \nabla_k . \] (85)

Using the same replacement for the symmetric part, the action is
\[ S_G = \frac{1}{(N + 1)} \text{Tr}_F \{ C_1 (i[\nabla_i, \nabla_j] - \epsilon_{ijk} \nabla_k)^2 + C_2 (\nabla_i \nabla_i - \ell^2/\alpha^2) \} . \] (86)

After rewriting the gauge field action in the above form, we can make the following reinterpretation: There is a general theory defined by the matrix \( \nabla_i \) and the action (86). The geometry of the base space is then defined by the vacuum expectation value of the field \( \nabla_i \) given by \( \langle \nabla_i \rangle = \frac{x_i}{\alpha} \). Then the original gauge field action can be obtained by expanding the field around this vacuum expectation value.

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5 Appendix

5.1 One form kernels

We show the existence of nontrivial elements of \( J^{(1)} \) which contribute to \( \pi(dker\pi^{(1)}) \). Consider the 1-forms:
\[ (x_A)^p d(x_B)^q \] (87)
with \( A, B = +, -, 3 \), where we have used the coordinates \( x_\pm = x_1 \pm ix_2 \). Since \( A_N \) is the algebra of \((N + 1) \times (N + 1)\) matrices, corresponding to the
$(N+1)$ dimensional representation of the algebra of the angular momentum up to the normalization, the identity $(x_\pm)^{N+1}=0$ holds, and thus one easily finds that elements of the differential kernel appear for $A=B=\pm$. Here, we give the proof for $A=B=+$. (The proof for $A=B=-$ works correspondingly.)

Let us define the 1-forms $\omega_{p,q}$ as

$$\omega_{p,q} = x_+^p d x_+^q .$$

then the following proposition holds.

**Proposition 5 :** $\omega_{p,q}$ is an element of $ker \pi^{(1)}$, for integers $p, q$ satisfying $1 < p, q < N + 1$ and $p + q \geq N + 2$.

**Proof :** Using the Dirac operator given in (15) we obtain

$$\pi(\omega_{p,q}) = x_+^p [D, x_+^q] = i \sum_{A=+,3,=} \gamma_\chi x_+^{p} L_A x_+^{q}$$

A straightforward calculation yields

$$L_+ x_+^q = 0 ,$$
$$L_3 x_+^q = qx_+^q ,$$
$$L_- x_+^q = -2q x_+^{q-1} \left(x_3 + \frac{\alpha}{2} (q-1)\right) .$$

Substituting the above relations, and using that $x_+^{N+1} = 0$, the r.h.s. of eq.(89) vanishes.

Note that there are six different elements $x_\lambda$ which correspond to the raising (lowering) operators of the three different directions $x_\lambda = x_j \pm i x_k$, where $j < k$ and $j, k \in \{1, 2, 3\}$, satisfying

$$(x_\lambda)^{N+1} = 0 .$$

For each direction $x_\lambda$ we can obtain kernels of the type $\omega_{p,q} = x_\lambda^p dx_\lambda^q$. These one forms as well as all one forms obtained by multiplying elements $a \in A_N$ onto them, belong to the kernel $J^{(1)} = ker \pi^{(1)}$.

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8 In fact we have a whole 'tower' of kernels

$$\prod_{k=0}^{m} (x_3 - \frac{(N-2k)(N-2k)}{2}) \alpha x_{\lambda} x_+^{N-m} = 0.$$ since $\prod_{k=0}^{m} (x_3 - \frac{(N-2k)(N-2k)}{2}) x_+^{N-m} = 0$. However the above kernel $\omega_{p,q}$ is enough for the following discussions.
We may still find other elements of $\ker\pi(1)$. However, the above kernel $\omega_{p,q}$ is sufficient to prove that $\pi(d\ker\pi(1))$ is not empty and contains the symmetric traceless part of $\chi^i_0\chi^j_0$.

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