QUANTUM ALGEBRAIC APPROACH TO REFINED TOPOLOGICAL VERTEX

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1. Introduction

The aim of the present paper is to study the refined topological vertex $C_{\lambda \mu \nu}(t, q)$ of Iqbal-Kozcaz-Vafa [IKV] from the point of view of the quantum algebra of type $W_{1+\infty}$ introduced by Miki [Mi]. We also treat the vertex $C_{\lambda \mu \nu}(q, t)$ considered by Awata and Kanno [AK2] in the same footing.

Let us first recall briefly the notion of the topological vertex [AKMV], [I]. A trivalent graph plays an important role, since it encodes the information where the cycles of a $T^2$ fibration of a toric 3-fold degenerate. The Calabi-Yau threefold is then mapped to a Feynman graph with fixed Schwinger terms (Kähler classes of the threefold), and the topological vertex is associated to the quantum algebra of type $A$. The gluing factors appear correctly when we consider any compositions of $\Phi$ and $\Phi^*$.

The spectral parameters attached to Fock spaces $(\nu, \rho)$ considered by Awata and Kanno [AK2] in the same footing.

The refined topological vertex $C_{\lambda \mu \nu}$ is represented by Okounkov, Reshetikhin and Vafa using the skew Schur functions [ORV], [OR].

\begin{equation}
C_{\lambda \mu \nu}(q) = q^{\frac{\kappa(\lambda)}{2}} s_{\nu}(q^{\rho}) \sum_{\eta} s_{\lambda'/\eta}(q^{-\nu'-\rho}) s_{\mu/\eta}(q^{-\nu'-\rho}),
\end{equation}

where $\lambda, \mu, \nu$ are partitions labeling the states in the threefold tensor of the Fock spaces, $\lambda'$ denotes the transpose of $\lambda$, $\rho = (-1/2, -3/2, -5/2, \cdots)$, and $\kappa(\lambda) = \sum_i \lambda_i(\lambda_i + 1 - 2i)$.

In [IKV] a refined version of the topological vertex was introduced, based on the arguments of geometric engineering concerning the $K$-theoretic lift of the Nekrasov partition functions [N], [FP]. See also [NY1], [NY2]. In this refined version, one more parameter $t$ comes in and the theory seems to be deeply relate with the Macdonald functions $P_{\lambda}(x; q, t)$ [Ma]. The formula is

\begin{equation}
C_{\lambda \mu \nu}^{(IKV)}(t, q) = \left( \frac{q}{t} \right)^{\frac{|\mu|+|\nu|}{2}} t^{\frac{|\nu|}{2}} q^{\frac{|\mu|}{2}} \bar{Z}_\nu(t, q) \sum_{\eta} \left( \frac{q}{t} \right)^{\frac{|\mu|+|\nu|+|\eta|}{2}} s_{\lambda'/\eta}(t^{-\nu} q^{-\nu}) s_{\mu/\eta}(t^{-\nu'} q^{-\rho}),
\end{equation}

\begin{equation}
\bar{Z}_\nu(t, q) = \prod_{s \in \nu} \left( 1 - q^{a_{\nu}(s)} t^{\ell_{\nu}(s)+1} \right)^{-1} = t^{-\frac{|\nu|}{2}} P_{\nu}(t^{-\rho}; q, t),
\end{equation}

where $\lambda, \mu, \nu$ are partitions labeling the states in the threefold tensor of the Fock spaces, $\lambda'$ denotes the transpose of $\lambda$, $\rho = (-1/2, -3/2, -5/2, \cdots)$, and $\kappa(\lambda) = \sum_i \lambda_i(\lambda_i + 1 - 2i)$.
where \( \| \lambda \|^2 = \sum \lambda_i^2 \). See [IK] for recent development, and a remark on their notational
convention.

There is another approach by Awata and Kanno [AK2], where Macdonald functions are
used in some symmetric way
\[
C_{\mu \lambda} (q, t) = P_\lambda (t^\rho; q, t) \sum_{\sigma} \tau P_{\mu / \sigma} (-t^\lambda q^\rho; q, t) P_{\nu / \sigma} (q^\lambda t^\rho; q, t) (q^{1/2} / t^{1/2})^{\sigma - |\rho|} f_\nu (q, t)^{-1}. \tag{1.4}
\]
(See Section 4.1 as for the notations.) Here they incorporated the ‘framing factor’
\[
f_\lambda (q, t) = (-1)^{|\lambda|} q^{n(\lambda') + |\lambda|/2} 2t^{-n(\lambda) - |\lambda|/2}, \tag{1.5}
\]
which was introduced by Taki [T]. It has been recognized that these two different formulas give
us essentially the same result, and the difference should be superficial. As for the preliminary
version of the formula (1.5), see [AK1].

Now we turn to the quantum algebra side. The algebra we consider was first introduced
by Miki in his study on the \( W_{1+\infty} \) algebra. After the first discovery by Miki, essentially the
same algebraic structure has been rediscovered by several authors. See [FT], [FHHSY], [SV1],
[SV2], [Sc1], [Sc2], [FFJMM1], [FFJMM2], [FJMM]. This verifies the naturalness and the
richness of the algebra. Because of this, it has been called by several different names, and
there is no good choice at this moment than waiting for well established terminologies. In this
paper, we denote the algebra by \( \mathcal{U} \).

Motivated by the construction in (refined) topological vertex, we study a representation
theory of the quantum algebra \( \mathcal{U} \) which includes the following ingredients:

1. triple of the Fock spaces and associated intertwining operators,
2. trivalent vertex with edges labeled by vectors \( \in \mathbb{Z}^2 \) satisfying the Calabi-Yau and
   smoothness conditions,
3. spectral parameters playing the role of the Kähler parameters.

It has been recognized that the quantum algebra \( \mathcal{U} \) has two central elements, and they
obey a certain transformation formula with respect to the \( SL(2, \mathbb{Z}) \) action [Mi], [SV1] [SV2].
Namely, the \( SL(2, \mathbb{Z}) \) action preserves the structure of the algebra up to the shift in the
central elements. As a consequence of the \( SL(2, \mathbb{Z}) \) action, we have two types of the Fock
representations of \( \mathcal{U} \), one in [FT] and the other in [FHHSY]. After fixing convention suitably,
one can say that the former has level \((0, 1)\) (vertical), and the latter has level \((1, 0)\) (horizontal).
The action of the \( T \) generator of the \( SL(2, \mathbb{Z}) \) can be easily treated, and we can modify the
‘horizontal’ Fock representation to level \((1, N)\) with \( N \in \mathbb{Z} \). We restrict ourselves only to the
family of the Fock modules \( \mathcal{F}_u^{(1, N)} \) and \( \mathcal{F}_u^{(1, N)} (N \in \mathbb{Z}) \), where \( u \) is the spectral parameter. (See
Sections 2.3, 2.4.) Note if one of the edges (the preferred edge) is labeled by \((0, 1)\), then from
the Calabi-Yau and the smoothness condition the rest should be \((1, N)\) and \((-1, -N - 1)\)
where \( N \in \mathbb{Z} \).

Consider the intertwining operators of \( \mathcal{U} \)-modules associated with three Fock modules of
the forms \( \Phi = \Phi \left[ \begin{array}{c} v_1, u_1, v_2, u_2 \end{array} \right] : \mathcal{F}_{u_1}^{v_1} \otimes \mathcal{F}_{u_2}^{v_2} \rightarrow \mathcal{F}_{u_3}^{v_3} \) and \( \Phi^* = \Phi^* \left[ \begin{array}{c} v_2, u_2, v_1, u_1 \end{array} \right] : \mathcal{F}_{u_3}^{v_3} \rightarrow \mathcal{F}_{u_2}^{v_2} \otimes \mathcal{F}_{u_1}^{v_1} \). The following particular cases are essential in our construction:
\[
\Phi : \mathcal{F}_u^{(0,1)} \otimes \mathcal{F}_u^{(1,N)} \rightarrow \mathcal{F}_{-v_u}^{(1,N+1)}, \quad a\Phi = \Phi \Delta (a) \quad (\forall a \in \mathcal{U}), \tag{1.6}
\]
\[
\Phi_\lambda (\alpha) = \Phi (P_\lambda \otimes \alpha) \quad (\forall P_\lambda \otimes \alpha \in \mathcal{F}_u^{(0,1)} \otimes \mathcal{F}_u^{(1,N)}), \quad \Phi_0 (1) = 1 + \cdots, \tag{1.7}
\]
\[
\Phi^* : \mathcal{F}_{-v_u}^{(1,N+1)} \rightarrow \mathcal{F}_u^{(1,N)} \otimes \mathcal{F}_u^{(0,1)}, \quad \Delta (a) \Phi^* = \Phi^* a \quad (\forall a \in \mathcal{U}), \tag{1.8}
\]
\[
\Phi^* (\alpha) = \sum_{\lambda} \Phi^*_\lambda (\alpha) \otimes Q_\lambda \quad (\forall \alpha \in \mathcal{F}_{-v_u}^{(1,N+1)}), \quad \Phi^*_0 (1) = 1 + \cdots. \tag{1.9}
\]
where $\Phi_\lambda$ and $\Phi_\lambda^*$ are normalized components of $\Phi$ and $\Phi^*$. We prove that such (normalized) intertwining operators $\Phi$ and $\Phi^*$ exist uniquely. (Theorems 3.3, 3.6)

Let $S_\lambda(q, t)$'s be the dual of the Schur function $s_\mu$'s with respect to the Macdonald scalar product in (2.5) satisfying $\langle S_\lambda(q, t), s_\mu(q, t) \rangle = \delta_{\lambda, \mu}$. We show that the refined topological vertex $C^{(KIV)}(t, q)$ coincides with the matrix element $\langle S_\mu(q, t) \mid \Phi_\mu^* \mid s_\lambda \rangle$ ut to a simple factor. (Proposition 4.4) If we use the bases $(\iota P_\mu)$ and $(\iota Q_\mu)$, then the refined topological vertex $C^{(\text{IV})}_{\mu, \nu}$ arises as the matrix element $\langle \iota P_\nu \mid \Phi_\lambda^* \mid \iota Q_\mu \rangle$. (Proposition 4.7)

We check that any types of the compositions of the intertwining operators $\Phi$ and $\Phi^*$ produces contractions of topological vertices involving correct gluing factors (see Definition 4.9). Thereby proving the equivalence of the topological vertex and our representation theory. (Theorem 4.10) Since the discovery of Alday, Gaiotto and Tachikawa [AGT], it has been intensively studied that we have the representation theory of the Virasoro and $\text{K}_{\text{IV}}$ algebras playing a profound role in the theory of the Virasoro and $W$ algebras.

For this purpose, we calculate $\langle \iota P_\nu \mid \Phi_\lambda^* \mid \iota Q_\mu \rangle$. (Propositions 4.11, 4.12, 4.13) Our main theorem is stated in Theorem 4.10. In Section 5, we present theorems of the refinement of the spectral parameters. Proofs of Theorems 3.3 and 3.6 are given in Section 6.

A proof of Proposition 5.1 is stated in Section 7.

2. Preliminaries

2.1. Algebra $\mathcal{U}$. Let $q$ and $t$ be independent indeterminates, and set $\mathbb{F} = \mathbb{Q}(q, t)$. We follow the notation in [FHHSY] which is based on [DI]. Let

$$g(z) = \frac{G^+(z)}{G^-(z)} \in \mathbb{Q}(q, t)[[z]], \quad G^\pm(z) = (1 - q^{\pm 1}z)(1 - t^{\mp 1}z)(1 - q^{\mp 1}t^{\pm 1}z).$$  \hspace{1cm} (2.1)$$

**Definition 2.1.** Let $\mathcal{U}$ be a unital associative algebra over $\mathbb{F}$ generated by the Drinfeld currents $x^\pm(z) = \sum_{n \in \mathbb{Z}} x_n^\pm z^{-n}$, $\psi^\pm(z) = \sum_{n \in \mathbb{N}} \psi_n^\pm z^{-n}$, and the central element $\gamma^{\pm 1/2}$, satisfying the defining relations

$$\psi^\pm(z)\psi^\pm(w) = \psi^\pm(w)\psi^\pm(z), \quad \psi^+(z)\psi^-(w) = \frac{g(\gamma^{+1}w/z)}{g(\gamma^{-1}w/z)} \psi^-(w)\psi^+(z),$$  \hspace{1cm} (2.2)$$

$$\psi^+(z)x^+(w) = g(\gamma^{1/2}w/z)^{+1} x^+(w)\psi^+(z),$$  \hspace{1cm} (2.3)$$

$$\psi^-(z)x^+(w) = g(\gamma^{1/2}z/w)^{-1} x^+(w)\psi^-(z),$$  \hspace{1cm} (2.4)$$
\[ [x^+(z), x^-(w)] = \frac{(1 - q)(1 - 1/t)}{1 - q/t} \left( \delta(\gamma^{-1}z/w)\psi^+(\gamma^{1/2}w) - \delta(\gamma z/w)\psi^-(\gamma^{-1/2}w) \right), \quad (2.5) \]
\[ G^\pm(z/w)x^\pm(z)x^\pm(w) = G^\pm(z/w)x^\pm(w)x^\pm(z), \quad (2.6) \]
where \( \delta(z) = \sum_{n \in \mathbb{Z}} z^n \).

**Proposition 2.2.** The algebra \( \mathcal{U} \) has a Hopf algebra structure defined by the coproduct \( \Delta \):
\[ \Delta(\gamma^{\pm1/2}) = \gamma^{\pm1/2} \otimes \gamma^{\pm1/2}, \]
\[ \Delta(x^+(z)) = x^+(z) \otimes 1 + \psi^-(\gamma^{1/2}z) \otimes x^+(\gamma(1)z), \]
\[ \Delta(x^-(z)) = x^-(\gamma(2)z) \otimes \psi^+(\gamma^{1/2}z) + 1 \otimes x^-(z), \]
\[ \Delta(\psi^\pm(z)) = \psi^\pm(\gamma^{\pm1/2}z) \otimes \psi^\pm(\gamma^{\mp1/2}z), \]
where \( \gamma^{\pm1/2} = \gamma^{\pm1/2} \otimes 1 \) and \( \gamma^{\mp1/2} = 1 \otimes \gamma^{\pm1/2} \). We omit the counit \( \varepsilon \) and the antipode \( a \) since we do not need them here.

**Remark 2.3.** The \( \psi^\pm \) are central elements in \( \mathcal{U} \).

**Definition 2.4.** Let \( M \) be a left \( \mathcal{U} \)-module over \( \overline{\mathbb{F}} \). If we have
\[ \gamma^{1/2}\alpha = (t/q)^{l/4}\alpha, \quad (\psi_0^\pm)^{-1}\psi_0^\pm\alpha = (t/q)^{l\alpha} \quad (2.7) \]
for any \( \alpha \in M \) and for some fixed \( l_1, l_2 \in \mathbb{Z} \), we call \( M \) of level \( (l_1, l_2) \).

2.2. **Macdonald symmetric functions and Fock space \( \mathcal{F} \).** We basically follow [Ma] for the notations. A partition \( \lambda \) is a series of nonnegative integers \( \lambda = (\lambda_1, \lambda_2, \ldots) \) such that \( \lambda_1 \geq \lambda_2 \geq \cdots \) with finitely many nonzero entries. We use the following symbols: \( |\lambda| := \sum_{i \geq 1} \lambda_i \), \( n(\lambda) := \sum_{i \geq 1} (i - 1)\lambda_i \). If \( \lambda_1 > 0 \) and \( \lambda_{i+1} = 0 \), we write \( l(\lambda) := l \) and call it the length of \( \lambda \). The conjugate partition of \( \lambda \) is denoted by \( \lambda' \) which corresponds to the transpose of the diagram \( \lambda \). The empty sequence is denoted by \( \emptyset \). The dominance ordering is defined by \( \lambda \geq \mu \) \( \iff |\lambda| = |\mu| \) and \( \sum_{i = 1}^k \lambda_i \geq \sum_{i = 1}^k \mu_i \) for all \( i = 1, 2, \ldots \).

We also follow [Ma] for the convention of the Young diagram. Namely, the first coordinate \( i \) (the row index) increases as one goes downwards, and the second coordinate \( j \) (the column index) increases as one goes rightwards. We denote by \( \square = (i, j) \) the box located at the coordinate \( (i, j) \). For a box \( \square = (i, j) \) and a partition \( \lambda \), we use the following notations:
\[ i(\square) := i, \quad j(\square) := j, \quad a_\lambda(\square) := \lambda_i - j, \quad \ell_\lambda(\square) := \lambda'_j - i. \]

Let \( \Lambda \) be the ring of symmetric functions in \( x = (x_1, x_2, \ldots) \) over \( \mathbb{Z} \), and let \( \Lambda_{Q(q,t)} := \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}(q,t) \). Let \( m_\lambda \) be the monomial symmetric functions. Denote the power sum function by \( p_n = \sum_{i \geq 1} x_i^n \). For a partition \( \lambda \), we write \( p_\lambda = \prod_i p_{\lambda_i} \). Macdonald’s scalar product on \( \Lambda_{Q} \) is
\[ \langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda,\mu} z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}, \quad z_\lambda = \prod_{i \geq 1} t^{m_i} \cdot m_i!, \quad (2.8) \]
Here we denote by \( m_i \) the number of entries in \( \lambda \) equal to \( i \).

**Fact 2.5.** The Macdonald symmetric function \( P_\lambda(x; q, t) \) is uniquely characterized by the conditions [Ma, Chap. VI, (4.7)].
\[ P_\lambda = m_\lambda + \sum_{\mu < \lambda} u_{\lambda\mu} m_\mu \quad (u_{\lambda\mu} \in \mathbb{Q}(q, t)), \]
\[ \langle P_\lambda, P_\mu \rangle_{q,t} = 0 \quad (\lambda \neq \mu). \]
Denote \( Q_\lambda := P_\lambda / \langle P_\lambda, P_\lambda \rangle_{q,t} \). Then \((Q_\lambda)\) and \((P_\lambda)\) are dual bases of \( \Lambda_\mathbb{F} \). We have \( \langle P_\lambda, P_\lambda \rangle_{q,t} = c_\lambda^*/c_\lambda \) where
\[
c_\lambda := \prod_{\square \in \lambda} (1 - q^{\alpha_\lambda(\square)} t^{\ell_\lambda(\square)}), \quad c_\lambda^* := \prod_{\square \in \lambda} (1 - q^{\alpha_\lambda(\square) + 1} t^{\ell_\lambda(\square)}).
\]

Let \( x = (x_1, x_2, \ldots) \) and \( y = (y_1, y_2, \ldots) \) be two infinite sets of independent indeterminates. The skew Macdonald polynomials \( P_{\lambda/\mu} \) satisfy \( P_\lambda(x, y) = \sum_\mu P_\mu(x) P_{\lambda/\mu}(y) \) [Ma, Chap. VI, (7.9')].

Let \( \mathcal{H} \) be the Heisenberg algebra over \( \mathbb{F} \) with generators \( \{a_n \mid n \in \mathbb{Z}\} \) satisfying
\[
[a_m, a_n] = m \frac{1 - q^{\langle m \rangle}}{1 - t^{\langle m \rangle}} \delta_{m+n,0} a_0.
\]
Let \( |0\rangle \) be the vacuum state satisfying the annihilation conditions for the positive Fourier modes \( a_n |0\rangle = 0 \) (\( n \in \mathbb{Z}_{>0} \)). For a partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \), we denote \( |\lambda\rangle = a_{-\lambda_1} a_{-\lambda_2} \cdots |0\rangle \) for short. Denote by \( \mathcal{F} \) the Fock space having the basis \( \{ |a_\lambda\rangle \} \).

As graded vector spaces, the space of the symmetric functions \( \Lambda_\mathbb{F} \) and the Fock space \( \mathcal{F} \) are isomorphic, and we may identify them:
\[
\mathcal{F} \xrightarrow{\sim} \Lambda_\mathbb{F}, \quad |a_\lambda\rangle \mapsto p_\lambda.
\]
We give an \( \mathcal{H} \)-module structure on \( \Lambda_\mathbb{F} \) by setting \( a_0 v = v \) and
\[
a_{-n} v = p_n v, \quad a_n v = \frac{1 - q^n}{1 - t^n} \frac{\partial v}{\partial p_n}, \quad (n > 0, v \in \Lambda_\mathbb{F}).
\]
Let \( \langle 0 \rangle \) be the dual vacuum satisfying \( \langle 0 | a_n \rangle = 0 \) (\( n \in \mathbb{Z}_{<0} \)), and \( \langle a_\lambda \rangle = \langle 0 | a_{\lambda_1} a_{\lambda_2} \cdots \rangle \). The dual Fock space \( \mathcal{F}^* \) has the basis \( \{ \langle a_\lambda \rangle \} \). We identify symmetric functions with states in \( \mathcal{F} \) (or \( \mathcal{F}^* \)) when it is convenient, and write \( \langle P_\lambda \rangle \) (or \( \langle P_\lambda \rangle \)) for \( P_\lambda \) for example. With this notation we have \( \langle P_\lambda \rangle \mathcal{O} \langle P_\mu \rangle = \langle P_\lambda \mathcal{O} P_\mu \rangle_{q,t} \) for any \( \mathcal{O} \in \mathcal{U}(\mathcal{H}) \).

### 2.3. Level \((0, 1)\) module \( \mathcal{F}_{u^{(0,1)}} \).

**Definition 2.6.** Let \( \lambda = (\lambda_1, \lambda_2, \ldots) \) be a partition, and \( i \in \mathbb{Z}_{\geq 0} \). Set \( A_{\lambda,i}^+ \in \mathbb{Q}(q, t), B_{\lambda,i}^+(z) \in \mathbb{Q}(q, t)[[z]] \) by
\[
A_{\lambda,i}^+ = (1 - t) \prod_{j=1}^{i-1} \frac{1 - q^{\lambda_j - \lambda_{j+1}} t^{-j+1} + 1 (1 - q^{\lambda_j - \lambda_{j+1}} t^{-j+1})}{(1 - q^{\lambda_j - \lambda_{j+1}} t^{-j+1}) (1 - q^{\lambda_j - \lambda_{j+1}} t^{-j+1})},
\]
\[
\text{for } i = 1, \ldots, \infty \cdot
\]

Note that if \( \lambda_i = \lambda_{i-1} \) then \( A_{\lambda,i}^+ = 0 \), and if \( \lambda_i = \lambda_{i+1} \) then \( A_{\lambda,i}^- = 0 \). If \( \lambda_i < \lambda_{i-1} \), we may obtain a new partition by adding one box to the \( i \)-th row, and we denote it by \( \lambda + 1_i = (\lambda_1, \lambda_2, \ldots, \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1}, \ldots) \) for simplicity. If \( \lambda_i > \lambda_{i+1} \), we may obtain a new partition by removing one box from the \( i \)-th row, and we write \( \lambda - 1_i = (\lambda_1, \lambda_2, \ldots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \ldots) \).

**Proposition 2.7.** Let \( u \) be an indeterminate. We can endow a left \( \mathcal{U} \)-module structure over \( \mathbb{F} \) to \( \mathcal{F} \) by setting
\[
\gamma^{1/2} P_\lambda = P_\lambda,
\]
Definition 2.8. Let \(\Phi = \Phi_{\lambda}^{N}\) be the trivalent intertwining operator satisfying

\[
\varphi^+(z) = \exp \left( - \sum_{n=1}^{\infty} \frac{1-t^n}{n} q^{n/4} t^{-n/2} a_n z^n \right), \\
\varphi^-(z) = \exp \left( \sum_{n=1}^{\infty} \frac{1-t^n}{n} q^{n/4} t^{-n/2} a_n z^n \right).
\]

Proposition 2.9. Let \(u\) be an indeterminate, and \(N \in \mathbb{Z}\). We can endow a left \(\mathcal{U}\)-module structure over \(\overline{\mathcal{F}}\) to \(\mathcal{F}\) by setting

\[
\gamma^{1/2} P_\lambda = (t/q)^{1/4} P_\lambda, \\
x^+(z) P_\lambda = uz^{-N} q^{-N/2} t^{N/2} \eta(z) P_\lambda, \\
x^-(z) P_\lambda = u^{-1} z^{-N} q^{N/2} t^{-N/2} \xi(z) P_\lambda, \\
\psi^+(z) P_\lambda = q^{N/2} t^{-N/2} \varphi^+(z) P_\lambda, \\
\psi^-(z) P_\lambda = q^{-N/2} t^{N/2} \varphi^-(z) P_\lambda.
\]

This is a level \((0, 1)\) module. We denote this \(\mathcal{U}\)-module by \(\mathcal{F}_u^{(0,1)}\).

This was obtained in \([FT]\), \([FFJMM1]\).

2.4. Level \((1, N)\) module \(\mathcal{F}_u^{(1,N)}\).

Definition 3.1. Let \(\Phi = \Phi_{\lambda}^{N}\) be the trivalent intertwining operator satisfying

\[
\varphi^+(z) = \exp \left( - \sum_{n=1}^{\infty} \frac{1-t^n}{n} q^{n/4} t^{-n/2} a_n z^n \right), \\
\varphi^-(z) = \exp \left( \sum_{n=1}^{\infty} \frac{1-t^n}{n} q^{n/4} t^{-n/2} a_n z^n \right).
\]

This was obtained as an easy modification of the representation constructed in \([FFJMM1]\).

3. Trivalent Intertwining Operators \(\Phi\) and \(\Phi^*\)

3.1. Intertwining operator \(\Phi\). Let \(N \in \mathbb{Z}\) and \(u, v, w\) be independent indeterminates.

Definition 3.1. Let \(\Phi = \Phi_{(1,N+1),w}^{(1,N),v; (1,N),u}\) be the trivalent intertwining operator satisfying the conditions

\[
\Phi : \mathcal{F}_v^{(0,1)} \otimes \mathcal{F}_u^{(1,N)} \rightarrow \mathcal{F}_w^{(1,N+1)}, \\
a \Phi = \Phi \Delta(a) \quad (\forall a \in \mathcal{U}).
\]

Introduce the components \(\Phi_\lambda\) by setting

\[
\Phi_\lambda(\alpha) = \Phi(P_\lambda \otimes \alpha) \quad (\forall P_\lambda \otimes \alpha \in \mathcal{F}_v^{(0,1)} \otimes \mathcal{F}_u^{(1,N)}).
\]

We normalize \(\Phi\) by requiring \(\Phi_{\emptyset}(1) = 1 + \ldots\).
Lemma 3.2. The intertwining relations (3.2) read
\[
\sum_{i=1}^{\ell(\lambda)} A_{\lambda,i}^+ \delta(q^{\lambda} t^{-i+1} v/z) \Phi_{\lambda+1,i} + q^{-1/2} t^{1/2} B_{\lambda}^- (z/v) x_+^+ (z) \Phi_{\lambda} = x_+^+ (z) \Phi_{\lambda},
\]
(3.4)
\[
q^{1/2} t^{-1/2} \sum_{i=1}^{\ell(\lambda)} A_{\lambda,i}^- \delta(q^{\lambda} t^{-i+1} v/z) \Phi_{\lambda-1,i} + \Phi_{\lambda} \psi^+ (q^{1/4} t^{-1/4} z) + x^- (q^{1/2} t^{-1/2} z) = x^- (q^{1/2} t^{-1/2} z) \Phi_{\lambda},
\]
(3.5)
\[
q^{1/2} t^{-1/2} B_{\lambda}^+ (v/z) \Phi_{\lambda} \psi^+ (q^{1/4} t^{-1/4} z) = \psi^+ (q^{1/4} t^{-1/4} z) \Phi_{\lambda},
\]
(3.6)
\[
q^{-1/2} t^{1/2} B_{\lambda}^- (z/v) \Phi_{\lambda} \psi^- (q^{-1/4} t^{1/4} z) = \psi^- (q^{-1/4} t^{1/4} z) \Phi_{\lambda}.
\]
(3.7)

Theorem 3.3. The normalized intertwining operator \( \Phi \) exists only when \( w = -vu \). In this case, it is determined uniquely and written in terms of the Heisenberg generators as
\[
\Phi_{\lambda} \begin{pmatrix} (1, N + 1), -vu \\ (0, 1), v; (1, N), u \end{pmatrix} = t(\lambda, u, v, N) \tilde{\Phi}_{\lambda}(v),
\]
(3.8)
\[
t(\lambda, u, v, N) = (-vu)^{|\lambda|} (-v)^{-(N+1)|\lambda|} f_{\lambda}^{-N-1}.
\]
(3.9)
Here we have used the notations
\[
\tilde{\Phi}_{\lambda}(v) = \frac{q^{n(\lambda)}}{c_{\lambda}} : \Phi_{\theta} (v) \eta_{\lambda}(v) ;,
\]
(3.10)
\[
\tilde{\Phi}_{\theta}(v) = \exp \left( -\sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{q^n a_n v^n} \right) \exp \left( -\sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1 - q^n a_n v^n} \right),
\]
(3.11)
\[
\eta_{\lambda}(v) = : \prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{n_{\lambda,i}} \eta (q^{j-1} t^{-i+1} v) ;.
\]
(3.12)
where the symbol \( : \cdots : \) denotes the usual normal ordering, and \( f_{\lambda} \) is Taki’s framing factor (1.3).

A proof of this will be given in Section 6.

3.2. Intertwining operator \( \Phi^* \).

Definition 3.4. Let \( \Phi^* = \Phi^* \begin{pmatrix} (1,N),v; (0,1),u \\ (1,N+1),w \end{pmatrix} \) be the trivalent intertwining operator satisfying the conditions
\[
\Phi^* : F^{(1,N+1)}_w \longrightarrow F^{(1,N)}_v \otimes F^{(0,1)}_u,
\]
(3.13)
\[
\Delta(a) \Phi^* = \Phi^* a \quad (\forall a \in U).
\]
(3.14)
Introduce the components \( \Phi^*_{\lambda} \) by setting
\[
\Phi^* (\alpha) = \sum_{\lambda} \Phi^*_{\lambda}(\alpha) \otimes Q_{\lambda} \quad (\forall \alpha \in F^{(1,N+1)}_w).
\]
(3.15)
We normalize \( \Phi^* \) by requiring \( \Phi^*_0 (1) = 1 + \cdots \).

Lemma 3.5. The intertwining relations (3.14) read
\[
\Phi^*_\lambda (q^{1/2} t^{-1/2} z) = x^+ (q^{1/2} t^{-1/2} z) \Phi^*_\lambda - \psi^+ (q^{1/4} t^{-1/4} z) \sum_{i=1}^{\ell(\lambda)} q A_{\lambda,i}^- \delta(q^{\lambda-1} t^{-i+1} u/z) \Phi_{\lambda-1,i},
\]
(3.16)
Φ_λ^* x^{-}(z) = q^{1/2}t^{-1/2} B_λ^*(u/z)x^{-}(z)\Phi_λ^* - q^{1/2}t^{-1/2} \sum_{i=1}^{\ell(\lambda)+1} q^{-1} A_{\lambda,i}^+ \delta(q^{\lambda_i}t^{-i+1}u/z)\Phi_{\lambda+1,i}^*, \quad (3.17)

Φ_λ^+(q^{-1/4}t^{1/4}z) = q^{1/2}t^{-1/2} B_λ^*(u/z)\psi^{+}(q^{-1/4}t^{1/4}z)\Phi_λ^*, \quad (3.18)

Φ_λ^-(q^{-1/4}t^{-1/4}z) = q^{-1/2}t^{1/2} B_λ^*(z/u)\psi^{-}(q^{-1/4}t^{-1/4}z)\Phi_λ^*. \quad (3.19)

**Theorem 3.6.** The intertwining operator Φ^* exists uniquely only when w = −vu. In this case, it is written in terms of the Heisenberg generators as

\[ \Phi_λ^*[\begin{pmatrix} (1, N), v; (0, 1), u \end{pmatrix}] = t^* (\lambda, u, v, N) \Phi_λ^*(u), \quad \] (3.20)

\[ t^* (\lambda, u, v, N) = (q^{-1}v)^{|\lambda|} (-u)^N f(N)^N. \quad (3.21) \]

Here, f_λ is given in (4.5), and

\[ \tilde{\Phi}_λ^* (u) = \frac{q^n(\lambda)}{c_λ} : \tilde{\Phi}_λ^0 (u) \xi_λ (u) :, \quad (3.22) \]

\[ \tilde{\Phi}_λ^0 (u) = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{q^n} q^{-n/2} \ell^{n/2} a_n u^n \right) \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1-q^n} q^{-n/2} \ell^{n/2} a_n u^{-n} \right), \quad (3.23) \]

\[ \xi_λ (u) = \prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (q^{-1}t^{-i+1}u) :. \quad (3.24) \]

In Section 4, we give a proof of this.

4. Identification with refined topological vertex

4.1. **Notations.** Let s_λ(x) ∈ Λ_2 be the Schur function, and c_μ^ν be the Littlewood-Richardson coefficient determined by s_λ s_μ = ∑_ν c_μ^ν s_ν. The skew Schur function is defined by s_λ/μ = ∑_ν c_μ^ν s_ν, and we have s_λ(x, y) = ∑_μ s_λ/μ(x) s_μ(y) [Mac, Chap. I. (5.9)]. Let S_λ(x; q, t) ∈ Λ_2 be the dual of s_λ with respect to the scalar product (2.8), namely <S_λ(q, t), s_μ(q, t)> = δ_λ,μ. Set S_λ/μ(x; q, t) = ∑_ν c_μ^ν S_ν(x; q, t). We have S_λ(x, y; q, t) = ∑_μ S_λ/μ(x; q, t) S_μ(y; q, t).

Recall the F-algebra endomorphism ω_{u,v} of Macdonald [Mac, Chap. VI. (2.14)].

\[ \omega_{u,v}(p_n) = (-1)^n \frac{1-u^n}{1-v^n} p_n. \quad (4.1) \]

It is convenient to have two operations ι and ε_λ^± acting on Λ_2 introduced in [AK2]. The ι is defined to be the involution on Λ_2 given by

\[ \iota : \Lambda_2 \rightarrow \Lambda_2, \quad \iota(p_n) = -p_n \quad (n \in \mathbb{Z}_{>0}). \quad (4.2) \]

The ε_λ^± = ε_λ^±_{q,t} is defined to be the algebra homomorphism

\[ \varepsilon^\pm_\lambda : \Lambda_2 \rightarrow \mathbb{F}, \quad \varepsilon^\pm_\lambda (p_n) = \sum_{i=1}^{\infty} (q^{\pm \lambda i n} - 1) t^{\pm (i-1/2)n} + \frac{t^{\pm n}}{1-t^{\pm n}}, \quad (4.3) \]

For any symmetric functions, we shall use the shorthand notations such as

\[ \varepsilon^\pm_\lambda (s_\mu) = s_\mu (q^{\pm \lambda \mp \rho}), \quad \varepsilon^\pm_\lambda (t s_\mu) = t s_\mu (q^{\pm \lambda \mp \rho}), \quad (4.4) \]

since we may have the interpretation ρ = (−1/2, −3/2, −5/2, ...) in mind.

We have

\[ \iota s_\lambda (x) = s_{\lambda'} (-x) = (-1)^{|\lambda|} s_\lambda (x), \quad (4.5) \]

\[ S_\lambda (x; q, t) = \iota \omega_{t,q} s_\lambda (-x), \quad (4.6) \]
Corollary 4.2. We have
\[ \varepsilon_{\lambda,q,t}^+(p_n(x)) = \varepsilon_{\lambda',t,q}^- \omega_{\alpha,t}(p_n(-q^{-1/2}t^{1/2}x)). \] (4.7)
Hence
\[ \varepsilon_{\lambda,q,t}^+ S_\mu(x; q, t) = (q^{1/2} t^{-1/2})^{-|\mu|} \varepsilon_{\lambda',t,q}^- S_\mu(x). \] (4.8)
In the shorthand notation this is written as \( \iota S \mu(q^{-1/2}t^{-1/2})^{-|\mu|} S_\mu(t^{-\lambda'}q^{-\rho}). \)

4.2. Matrix elements of \( \Phi \) and \( \Phi^* \). We have simple but important formulas which essentially control the property of our intertwining operators.

Proposition 4.1. We have
\[ \Phi_\emptyset(q^{1/2}u) \eta_\lambda(q^{1/2}v) : \]
\[ = \exp \left( \sum_{i=1}^\infty \frac{1}{n} \frac{1}{1 - q^n} (q^{1/2} t^{-1/2})^n q^n \varepsilon_\lambda^+(p_n) \right) \exp \left( - \sum_{i=1}^\infty \frac{1}{n} \frac{1}{1 - q^n} a_n (q^{1/2} t^{-1/2})^n q^{n-\varepsilon_\lambda^-}(p_n) \right), \] (4.9)
\[ : \Phi_\emptyset^*(q^{1/2}u) \xi_\lambda(q^{1/2}v) : \]
\[ = \exp \left( - \sum_{i=1}^\infty \frac{1}{n} \frac{1}{1 - q^n} a_n u^n \varepsilon_\lambda^+(p_n) \right) \exp \left( \sum_{i=1}^\infty \frac{1}{n} \frac{1}{1 - q^n} a_n u^n \varepsilon_\lambda^- (p_n) \right). \] (4.10)

Corollary 4.2. We have
\[ \langle S_\nu(q, t) | : \Phi_\emptyset(q^{1/2}v) \eta_\lambda(q^{1/2}v) : | s_\mu \rangle \]
\[ = u^{\nu|\nu|} (q^{1/2} t^{-1/2})^{\nu|\nu|} \sum_\sigma S_\nu/\sigma(q^{\lambda t}t^0; q, t) \iota S_\mu/\sigma(q^{-\lambda t}t^{-1/2}) (q^{1/2} t^{-1/2})^{-2|\sigma|}, \] (4.11)
\[ \langle S_\nu(q, t) | : \Phi_\emptyset^*(q^{1/2}u) \xi_\lambda(q^{1/2}u) : | s_\mu \rangle \]
\[ = u^{\nu|\nu|} \sum_\sigma tS_\nu/\sigma(q^{\lambda t}t^0; q, t) s_\mu/\sigma(q^{-\lambda t}t^{-1/2}), \] (4.12)
and
\[ \langle P_\nu | : \Phi_\emptyset(q^{1/2}v) \eta_\lambda(q^{1/2}v) : | P_\mu \rangle \]
\[ = u^{\nu|\nu|} (q^{1/2} t^{-1/2})^{\nu|\nu|} \sum_\sigma P_\nu/\sigma(q^{\lambda t}t^0) tP_\mu/\sigma(q^{-\lambda t}t^{-1/2}) \langle P_\sigma, P_\sigma \rangle_{q,t} (q^{1/2} t^{-1/2})^{-2|\sigma|}, \] (4.13)
\[ \langle P_\nu | : \Phi_\emptyset^*(q^{1/2}u) \xi_\lambda(q^{1/2}u) : | P_\mu \rangle \]
\[ = u^{\nu|\nu|} \sum_\sigma tP_\nu/\sigma(q^{\lambda t}t^0) P_\mu/\sigma(q^{-\lambda t}t^{-1/2}) \langle P_\sigma, P_\sigma \rangle_{q,t}. \] (4.14)

Proof. From Proposition 4.1 we have
\[ \langle 0 | \prod_i a_{v_i} : \Phi_\emptyset(q^{1/2}v) \eta_\lambda(q^{1/2}v) : \prod_j a_{-\mu_j} | 0 \rangle \]
\[ = \langle 0 | \prod_i (a_{v_i} + (q^{1/2} t^{-1/2})^{v_i} \varepsilon_\lambda^+(p_{v_i})) \prod_j (a_{-\mu_j} - (q^{1/2} t^{-1/2} v^{-1})^{\mu_j} \varepsilon_\lambda^-(p_{\mu_j})) | 0 \rangle, \]
\[ \langle 0 | \prod_i a_{v_i} : \Phi_\emptyset^*(q^{1/2}u) \xi_\lambda(q^{1/2}u) : \prod_j a_{-\mu_j} | 0 \rangle \]
\[ = \langle 0 | \prod_i (a_{v_i} - u^{v_i} \varepsilon_\lambda^+(p_{v_i})) \prod_j (a_{-\mu_j} + u^{\mu_j} \varepsilon_\lambda^-(p_{\mu_j})) | 0 \rangle. \]

Then (4.11), (4.12), (4.13) and (4.14) follow from the property of the skew functions. \( \square \)
4.3. Topological vertex of Iqbal-Kozcaz-Vafa.

**Definition 4.3** (Iqbal-Kozcaz-Vafa). The refined topological vertex is defined by

\[
C_{\lambda\mu\nu}^{(IKV)}(t, q) = \left( \frac{q}{t} \right)^{\frac{||\mu||^2}{2} + \frac{||\nu||^2}{2}} \frac{1}{c_\lambda} \sum_\eta \left( \frac{q}{t} \right)^{\frac{|\mu| + |\nu| - |\lambda|}{2}} s_\lambda \eta (t^{-\rho} q^{-\mu} s_\mu \eta (t^{-\nu} q^{-\rho})),
\]

where \(c_\lambda\) is defined in (2.9), \(||\lambda||^2 = \sum_i \lambda_i^2\), \(\kappa(\lambda) = \sum_i \lambda_i (\lambda_i + 1 - 2i)\).

**Proposition 4.4.** The matrix elements of the intertwining operators \(\Phi\) and \(\Phi^*\) are written in terms of the refined topological vertex as

\[
\langle S_\nu(q, t) | \Phi_\lambda \left[ \begin{array}{c} (1, N + 1), -v u \\ (0, 1), v; (1, N), u \end{array} \right] | s_\mu \rangle = \left( \frac{q}{t} \right)^{\frac{|\mu|}{2} - |\nu|} \cdot \left( \frac{q}{t} \right)^{\frac{|\nu|}{2} - |\mu|} f_\lambda f_\nu \cdot (t^{-1/2} q^{-1/2})^{|\mu|} \cdot (t^{-1/2} q^{-1/2})^{|\nu|} \cdot (1)^{|\mu| + |\nu| + |\lambda|} C_{\mu\nu}^{(IKV)}(t, q),
\]

\[
\langle S_\nu(q, t) | \Phi_\lambda \left[ \begin{array}{c} (1, N), v; (0, 1), u \\ (1, N + 1), -v u \end{array} \right] | s_\mu \rangle = \left( \frac{q}{t} \right)^{\frac{|\mu|}{2} - |\nu|} \cdot \left( \frac{q}{t} \right)^{\frac{|\nu|}{2} - |\mu|} f_\mu f_\nu^{-1} \cdot (t^{-1/2} q^{-1/2})^{|\mu|} \cdot (t^{-1/2} q^{-1/2})^{|\nu|} \cdot C_{\mu\nu}^{(IKV)}(t, q).
\]

**Proof.** Using Corollary 4.2 and (4.8) we have the results. \(\square\)

**Remark 4.5.** Note that in the formulation of Iqbal-Kozcaz-Vafa, the transpose of the partition is assigned to each outgoing edge. To identify the refined topological vertices with vertices for \(\Phi, \Phi^*\), all the arrows should be reversed as shown in Fig. 1.

4.4. Topological vertex of Awata-Kanno.

**Definition 4.6** (Awata-Kanno). The refined topological vertices \(C_{\mu\nu}(q, t)\), \(C_{\mu\nu}^{(IKV)}(q, t)\) are defined by

\[
C_{\mu\nu}(q, t) = P_\lambda (t^\rho; q, t) \sum_\sigma \lambda P_{\rho/\sigma}(q^{1/2} t^{1/2})^{|\rho| - |\sigma|} f_\nu(q, t)^{-1},
\]

\[
C_{\mu\nu}^{(IKV)}(q, t) = (-1)^{|\mu| + |\nu| + |\lambda|} C_{\mu\nu}^{(IKV)}(t, q)
\]

where \(f_\lambda\) being defined in (1.5).

**Proposition 4.7.** The matrix elements of the intertwining operators \(\Phi\) and \(\Phi^*\) are written in terms of the refined topological vertices as

\[
\frac{1}{\langle P_\lambda, P_\lambda \rangle_{q, t}} \langle t P_\mu \Phi_\lambda \left[ \begin{array}{c} (1, N + 1), -v u \\ (0, 1), v; (1, N), u \end{array} \right] | t P_\nu \rangle = \left( \frac{q}{t} \right)^{\frac{|\mu|}{2} - |\nu|} \cdot \left( \frac{q}{t} \right)^{\frac{|\nu|}{2} - |\mu|} f_\lambda^{-N} (t^{-1/2} q^{-1/2})^{|\mu| - |\nu|} f_\nu^{-1} C_{\mu\nu}^{(IKV)}(q, t),
\]

where \(C_{\mu\nu}^{(IKV)}(q, t)\) is defined in (2.9).
Remark 4.8. We note that all the vertical arrows should be get reversed to est ablish a correspondence between $\Phi$, $\Phi^*$ and $C_{\mu\lambda}^\nu$, $C_{\mu\lambda}'^\nu$ as seen in Fig. 2.

Proof. Note that we have

$$C_{\mu\lambda}^\nu(q,t) = \frac{(-1)^{|\lambda|}q^{\lambda}(q^2)^{1/2}}{c_\lambda} (q^{1/2}t^{-1/2})^{(|\mu|-|\nu|)} f^\nu_{\mu} \frac{1}{(P_{\mu}, P_{\mu}')_{q,t}}$$

(4.22)

$$C_{\mu\lambda}'^\nu(q,t) = \frac{q^{1/2}t^{-1/2}}{c_\lambda} (q^{1/2}t^{-1/2})^{2|\nu|} f^\nu_{\mu} \frac{1}{(P_{\mu}, P_{\mu}')_{q,t}}$$

(4.23)

Using Corollary 4.2 we have the results.

Remark 4.8. We note that all the vertical arrows should be get reversed to establish a correspondence between $\Phi$, $\Phi^*$ and $C_{\mu\lambda}^\nu$, $C_{\mu\lambda}'^\nu$ as seen in Fig. 2.

4.5. Gluing rules. Consider a trivalent vertex with edges, say, $i$, $j$ and $k$, with two component vectors $v_i$, $v_j$, $v_k \in \mathbb{Z}^2$ attached respectively (see Fig. 3 (a)). Here we regard all the vectors being outgoing. We assume that they satisfy the (Calabi-Yau and smoothness) conditions

$$v_i + v_j + v_k = 0, \quad v_i \land v_j = 1,$$

(4.24)

where we have used the notation $(a, b) \land (c, d) = ad - bc$. Note that these mean $v_j \land v_k = v_k \land v_i = 1$.

Definition 4.9. Let $v_i$, $v_j$, $v_k$, $v_{i'}$, $v_{j'} \in \mathbb{Z}^2$, and consider a graph as in Fig. 3 (b). Let $\lambda_k$ be a partition, and $Q_k$ be a parameter (Kähler parameter). To the internal edge, with the data $v_k$, $\lambda_k$, $Q_k$ attached, we associate the ‘gluing factor’

$$Q_k^{\lambda_k}(f_{\lambda_k})^{v_k \land v_{i'}}.$$

(4.25)

The refined topological vertices are contracted by multiplying the gluing factor and making summation with respect to the repeated indices.

4.6. Check of gluing rules. Consider any intertwining operator of the $U$ modules $F_{u_i}^\nu \otimes \cdots \otimes F_{u_m}^\nu \rightarrow F_{u_1}^{\nu_1} \otimes \cdots \otimes F_{u_n}^{\nu_n}$ obtained by composing the trivalent intertwining operators $\Phi$ and $\Phi^*$ in a certain way. The matrix elements can be evaluated by virtue of Proposition 4.4 or Proposition 4.7. Then we have a (not necessarily connected) graph with trivalent vertices, with the following structure associated:
(1) a spectral parameter and a vector $\in \mathbb{Z}^2$ is attached to each edge,
(2) the condition (4.23) is satisfied with respect to every vertex,
(3) to each vertex a refined topological vertex is associated (Propositions 4.4, 4.7),
(4) each internal edge gives a contraction of refined topological vertices.

Hence if it is shown that the correct gluing factor (4.25) appears to every internal edge in the matrix element, our approach from the representation theory of the algebra $\mathcal{U}$ precisely reproduces the quantity derived from the refined topological vertex, up to a factor depending on the data attached to the external edges. One can show that this is the case by checking it for all the possible (local) processes stated in Propositions 4.11, 4.12, 4.13, 4.14, 4.15 below. We demonstrate these for the case of Awata-Kanno construction, since our notation gets a little simpler. The calculations for the topological vertex of Iqbal-Kozcaz-Vafa goes exactly the same way, and we omit them.

**Theorem 4.10.** Suppose we choose the preferred direction to be vertical $(0,1)$ in the web diagram. The matrix element of the composition of the trivalent intertwining operators $\Phi$ and $\Phi^*$, and the corresponding quantity derived from the theory of the refined topological vertex of Iqbal-Kozcaz-Vafa or Awata-Kanno coincide, up to a factor depending on the data attached to external edges.

**Proposition 4.11.** The matrix element of the composition (see Fig 4)

$$
\mathcal{F}_{u}^{(1,L)} \otimes \mathcal{F}_{v}^{(1,M)} \Phi^* \otimes \text{id} \rightarrow \mathcal{F}_{u/w}^{(1,L-1)} \otimes \mathcal{F}_{v}^{(0,1)} \otimes \mathcal{F}_{u/w}^{(1,M)} \text{id} \otimes \Phi \mathcal{F}_{w}^{(1,L-1)} \otimes \mathcal{F}_{vw}^{(1,M+1)},
$$

with respect to $\langle tP_{\rho} \otimes tP_{\rho} \rangle$ and $\langle tQ_{\mu} \otimes tQ_{\rho} \rangle$ is

$$
(-t^{-1/2}w)^{-|\rho|+|\nu|+|\mu|+|\sigma|} f_{\rho} f_{\sigma}^{-1} \sum_{\lambda} \left( w^{L-M} \nu / u \right)^{|\lambda|} f_{\lambda}^{L-M-1} C_{\mu \lambda} \nu C_{\sigma \lambda} \rho.
$$

Recall we should reverse the vertical arrow, and apply the rule for calculating the gluing factor (4.25). We have $(1, M) \wedge (1, L - 1) = L - M - 1$, and thus the factor $\left( w^{L-M} \nu / u \right)^{|\lambda|} f_{\lambda}^{L-M-1}$ agrees with the gluing factor (4.25).
Proof. We have
\[ \iota Q_\mu \otimes \iota Q_\rho \mapsto \sum_{\lambda} \Phi^*_\lambda(\iota Q_\mu) \otimes Q_\lambda \otimes \iota Q_\rho \mapsto \frac{1}{\langle P_\lambda, P_\lambda \rangle_{q,t}} \sum_{\lambda} \Phi^*_\lambda(\iota Q_\mu) \otimes \Phi_\lambda(\iota Q_\rho). \]
Hence the matrix element is
\[ \sum_{\lambda} \frac{1}{\langle P_\lambda, P_\lambda \rangle_{q,t}} \langle \iota P_\nu | \Phi^*_\lambda | \iota Q_\mu \rangle \langle \iota P_\sigma | \Phi_\lambda | \iota Q_\rho \rangle. \]

Proposition 4.12. The matrix element of the composition (see Fig. 5)
\[ F_{(0,1)} \otimes F_{(1,L)} \rightarrow F_{(0,1)} \otimes F_{(1,L-1)} \rightarrow F_{(0,1)} \otimes F_{(1,L)} \rightarrow F_{(0,1)}, \]
with respect to \( \langle \iota P_\sigma \otimes P_\lambda \rangle \) and \( |Q_\lambda \otimes \iota Q_\mu\rangle \) is
\[ \left( \frac{q x^{L-1}}{-t^{1/2} u / x} \right)^{|\lambda|} f^L_\lambda \left( \frac{-t^{1/2} u / x}{q y^{L-1}} \right)^{|\rho|} f^{L-1}_\rho \times (-t^{-1/2} y)^{|\nu|} \sum_{\sigma} (x/y)^{|\nu|} C_{\mu \nu}^\gamma \lambda^\sigma \rho^\nu. \] (4.26)
Note that we have \((1, L) \wedge (1, L) = 0\), and the factor \((x/y)^{|\nu|}\) agrees with (4.25).

Proof. We have
\[ Q_\rho \otimes \iota Q_\mu \mapsto \sum_{\nu} Q_\rho \otimes \Phi^*_\nu(\iota Q_\mu) \otimes Q_\nu \mapsto \frac{1}{\langle P_\rho, P_\rho \rangle_{q,t}} \sum_{\nu} \Phi_\rho \Phi^*_\nu(\iota Q_\mu) \otimes Q_\nu. \]
Hence the matrix element is
\[ \frac{1}{\langle P_\rho, P_\rho \rangle_{q,t}} \langle \iota P_\nu | \Phi_\rho \Phi^*_\nu | \iota Q_\mu \rangle = \sum_{\nu} \frac{1}{\langle P_\rho, P_\rho \rangle_{q,t}} \langle \iota P_\nu | \Phi_\rho | \iota Q_\nu \rangle <\iota P_\nu | \Phi^*_\nu | \iota Q_\mu \rangle. \]

Proposition 4.13. The matrix element of the composition (see Fig. 6)
\[ F_{(0,1)} \otimes F_{(1,M)} \rightarrow F_{(1,M+1)} \rightarrow F_{(1,M)} \rightarrow F_{(0,1)}, \]
with respect to \( \langle \iota P_\sigma \otimes P_\rho \rangle \) and \( |Q_\lambda \otimes \iota Q_\mu\rangle \) is
\[ \left( \frac{-t^{1/2} u/1}{q x^{M}} \right)^{|\lambda|} f^M_\lambda \left( \frac{q y^{M}}{-t^{1/2} u x / y} \right)^{|\rho|} f^{-1}_\rho \times (-t^{-1/2} x)^{-|\nu|} \sum_{\sigma} (x/y)^{|\nu|} C_{\nu \rho}^\gamma \lambda^\sigma \mu. \] (4.27)
Note that we have \((1, M) \wedge (1, M) = 0\), and the factor \((x/y)^{|\nu|}\) agrees with (4.25).
Proof. We have
\[ Q_\lambda \otimes t Q_\mu \mapsto \frac{1}{\langle P_\lambda, P_\lambda \rangle_{q,t}} \Phi_\lambda(t Q_\mu) \mapsto \frac{1}{\langle P_\lambda, P_\lambda \rangle_{q,t}} \sum_\nu \Phi_\nu^* \Phi_\lambda(t Q_\mu) \otimes Q_\nu. \]
Hence the matrix element is
\[ \frac{1}{\langle P_\lambda, P_\lambda \rangle_{q,t}} \langle t P_\sigma | \Phi_\rho^* \Phi_\lambda | t Q_\mu \rangle = \sum_\nu \frac{1}{\langle P_\lambda, P_\lambda \rangle_{q,t}} \langle t P_\sigma | \Phi_\rho^* | t Q_\nu \rangle \langle t P_\nu | \Phi_\lambda | t Q_\mu \rangle. \]
\[ \square \]

Proposition 4.14. The matrix element of the composition (see Fig. 7)
\[ F_{(0,1)} \otimes F_{(0,1)} \otimes F_{(1,M)} \mathop{\mapsto}^{id \otimes \Phi} F_{(0,1)} \otimes F_{(1,M+1)} \mathop{\mapsto}^{\Phi} F_{(1,M+2)}, \]
with respect to \( \langle t P_\sigma \rangle \) and \( |Q_\rho \otimes Q_\lambda \otimes t Q_\mu \rangle \) is
\[ \left( \frac{-t^{1/2} v_x}{q x^{M+1}} \right)^{|\lambda|} f_\lambda^{-M} \left( \frac{-t^{1/2} v_x}{q y^{M+1}} \right)^{|\rho|} f_\rho^{-M-1} \times (-t^{-1/2} x)^{-|\lambda|} (-t^{-1/2} y)^{|\rho|} f_\mu^{-1} \sum_\nu (x/y)^{|\nu|} f_\nu^{-1} C^\sigma_\rho \sigma_\nu^\lambda \mu. \]
(4.28)
Note that we have \((1, M) \wedge (0, -1) = -1\), and the factor \((x/y)^{|\nu|} f_\nu^{-1}\) agrees with (4.25).

Proof. We have
\[ Q_\rho \otimes Q_\lambda \otimes t Q_\mu \mapsto \frac{1}{\langle P_\lambda, P_\lambda \rangle_{q,t}} Q_\rho \otimes \Phi_\lambda(t Q_\mu) \mapsto \frac{1}{\langle P_\lambda, P_\lambda \rangle_{q,t} \langle P_\rho, P_\rho \rangle_{q,t}} \Phi_\rho \Phi_\lambda(t Q_\mu). \]
Hence the matrix element is
\[ \frac{1}{\langle P_\lambda, P_\lambda \rangle_{q,t}} \frac{1}{\langle P_\rho, P_\rho \rangle_{q,t}} \langle t P_\sigma | \Phi_\rho \Phi_\lambda | t Q_\mu \rangle \]
\[ = \sum_\nu \frac{1}{\langle P_\rho, P_\rho \rangle_{q,t}} \langle t P_\sigma | \Phi_\rho | t Q_\nu \rangle \frac{1}{\langle P_\lambda, P_\lambda \rangle_{q,t}} \langle t P_\nu | \Phi_\lambda | t Q_\mu \rangle. \]
\[ \square \]
In this section, we try to have an interpretation of the spectral parameters attached to our

\[ \Phi \]

Hence the matrix element is

\[ \langle \Phi^* \rangle \]

We have

\[ \text{Proof.} \]

The matrix element of the composition (see Fig. 8)

\[ \text{Proposition 4.15.} \]

Based on the findings in \[ \text{IKV, T, AK2} \], it explains clearly the reason why the Nekrasov

\[ \Phi \]

\[ L \]

\[ \mu \]

\[ (1, L-1), u/x \]

\[ (1, L), u \]

\[ \rho \]

\[ (1, L-2), u/xy \]

\[ (0,1), -x \]

\[ (0,1), -y \]

\[ (1, L), u \]

\[ (1, L-1), u/x \]

\[ (1, L-2), u/xy \]

**Figure 8. Case 5**

\[ \Phi^* \]

\[ \Phi^* \]

\[ \sigma \]

\[ \text{with respect to } \langle tP_\sigma \otimes P_\rho \otimes P_\lambda | \text{ and } |tQ_\mu \rangle \text{ is} \]

\[ \left( \frac{q^{L-1}}{-t^{1/2} u/x} \right)^{[\lambda]} f^{L-1}_\lambda \left( \frac{q y^{L-2}}{-t^{1/2} u/y} \right)^{|\rho|} f^{L-2}_\rho \]

\[ \times (-t^{-1/2} x)^{|\sigma|} (-t^{-1/2} y)^{|\nu|} f_\nu \sum_{\nu} (x/y)^{|\nu|} f_\nu C_{\mu \lambda}^\nu C_{\nu \sigma}^\mu. \]

Note that we have \((1, L) \land (0, 1) = 1\), and the factor \((x/y)^{|\nu|} f_\nu\) agrees with \((4.25)\).

\[ \text{Proof.} \]

We have

\[ tQ_\mu \leftrightarrow \sum_\nu \Phi^*_\nu (tQ_\mu) \otimes Q_\nu \leftrightarrow \sum_{\nu, \rho} \Phi^*_\rho \Phi^*_\nu (tQ_\mu) \otimes Q_\rho \otimes Q_\nu. \]

Hence the matrix element is

\[ \langle tP_\sigma | \Phi^*_\rho \Phi^*_\lambda | tQ_\mu \rangle = \sum_\nu \langle tP_\sigma | \Phi^*_\rho | tQ_\nu \rangle \langle tP_\nu | \Phi^*_\lambda | tQ_\mu \rangle. \]

\[ \square \]

5. Examples of Compositions of Intertwining Operators

We have shown in Theorem 4.10 that our construction based on the intertwining operators \( \Phi, \Phi^* \) derives the same result as the one from the theory of the refined topological vertex. Based on the findings in \[ \text{IKV, T, AK2} \], it explains clearly the reason why the Nekrasov partition functions appear from matrix elements of intertwining operators of the algebra \( \mathcal{U} \).

In this section, we try to have an interpretation of the spectral parameters attached to our Fock modules by looking at two examples of the Nekrasov partition functions [N, FP].

5.1. Pure \( SU(N_c) \) Partition Function. Recall the formula of the instanton part of the \((K\text{-theoretic})\) partition function \( Z_{\text{inst}}^m \) of the pure \( SU(N_c) \) gauge theory on \( \mathbb{R}^4 \times S^1 \) with eight supercharges, associated with the \( m \)-th power \( L^\otimes m \) of the line bundle \( L \) over the instanton moduli space \( M(N_c, k) \)

\[ Z_m^{\text{inst}}(e_1, \cdots, e_{N_c}, \Lambda; q, t) = \sum_{\lambda(1), \cdots, \lambda(N_c)} \prod_{\alpha=1}^{N_c} \left((q^{1/2} t^{-1/2})^{N_c} \Lambda^2 N_c (-e_\alpha)^{-m} |(\lambda)_\alpha| f_{\lambda}^{m} \right) \prod_{\alpha, \beta=1}^{N_c} N_{\lambda(\alpha)}^{-1} \Lambda^2 N_c (e_\alpha/e_\beta) \]

where the notation

\[ N_{\lambda, \mu}(u) = \prod_{(i,j) \in \lambda} (1 - uq^{-\mu_i + j - 1 - \lambda_j + i}) \prod_{(k,l) \in \mu} (1 - uq^{\lambda_k - l \mu_l' - k + 1}) \]

\[ (5.2) \]

\[ \prod_{\square \in \lambda} (1 - uq^{-a_{\square}(\square)} - 1 - \ell_{\lambda}(\square)) \prod_{\bullet \in \mu} (1 - uq^{a_{\lambda}(\bullet)} f_{\mu(\bullet)} + 1), \]

\[ (5.3) \]
has been used. We demonstrate how $Z^{\text{inst}}_m$ appears from our construction. Let $L, M \in \mathbb{Z}$ and $u, v, w$ be indeterminates. Consider the four point operator (see Fig. 9)

$$
\Phi \left[ (1, L - 1), u/w; (1, M + 1), vw \right] \colon \mathcal{F}^{(1, L)}_u \otimes \mathcal{F}^{(1, M)}_v \rightarrow \mathcal{F}^{(1, M-1)}_{u/w} \otimes \mathcal{F}^{(1, M+1)}_{vw},
$$

defined by the composition of the trivalent intertwining operators (which we already considered in Proposition 4)

$$
\mathcal{F}^{(1, L)}_u \otimes \mathcal{F}^{(1, M)}_v \Phi^* \otimes \text{id} \rightarrow \mathcal{F}^{(1, L-1)}_{u/w} \otimes \mathcal{F}^{(1, M)}_v \Phi \otimes \text{id} \Phi \rightarrow \mathcal{F}^{(1, L-1)}_{u/w} \otimes \mathcal{F}^{(1, M+1)}_{vw}.
$$

For any $\alpha \otimes \beta \in \mathcal{F}^{(1, L)}_u \otimes \mathcal{F}^{(1, M)}_v$, we have

$$
\Phi \left[ (1, L - 1), u/w; (1, M + 1), vw \right] (\alpha \otimes \beta) = \sum_{\lambda} \frac{1}{(P_\lambda, P\lambda_{\alpha}, t)} \Phi^*_\lambda \left[ (1, L - 1), u/w; (0, 1), -w \right] \Phi_\lambda \left[ (1, M + 1), vw \right] \langle (0, 1), -w; (1, M), v \rangle (\beta)
$$

from Theorems 3.3, 3.6 and the formula (7.3).

Let $u_1, u_2, \ldots, u_{N_c}$ be a set of indeterminates. Set

$$
u_i = u \prod_{k=1}^{i-1} w_k^{-1}, \quad v_i = v \prod_{k=1}^{i-1} w_k, \quad (i = 1, 2, \ldots, N_c + 1),
$$

for simplicity. Define the four point operator (see Fig. 10)

$$
\Phi \left[ (1, L - N_c), u_{N_c+1}; (1, M + N_c), v_{N_c+1} \right] (1, L), u_1; (1, M), v_1 \colon \mathcal{F}^{(1, L)}_{u_{N_c+1}} \otimes \mathcal{F}^{(1, M)}_{v_{N_c+1}} \rightarrow \mathcal{F}^{(1, M-N_c)}_{u_{N_c+1}} \otimes \mathcal{F}^{(1, M+N_c)}_{v_{N_c+1}},
$$

as the composition

$$
\Phi \left[ (1, L - N_c), u_{N_c+1}; (1, M + N_c), v_{N_c+1} \right] (1, L), u_1; (1, M), v_1 \rightarrow \Phi \left[ (1, L - N_c + 1), u_{N_c+1}; (1, M + N_c), v_{N_c+1} \right] \cdots \Phi \left[ (1, L), u_2; (1, M + 1), v_2 \right].
$$

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{four_point_operator.png}
\caption{Four point operator.}
\end{figure}
When we identify parameters as
\[ e_i = -w_i, \quad \Lambda^{2N_c} = \frac{v}{u} \prod_{i=1}^{N_c} w_i, \quad m = -L + M + N_c, \]  
(5.10)

we have
\[ \langle P_0 \otimes P_0 | \Phi \left[ \frac{(1, L - N_c), u_{N_c+1}; (1, M + N_c), v_{N_c+1}}{1, L; u_1; (1, M), v_1} \right] | P_0 \otimes P_0 \rangle = \prod_{1 \leq i < j \leq N} G(e_i/e_j) G(q^{-1} e_i/e_j) \cdot Z^\text{inst}_m(e_1, \cdots, e_{N_c}, \Lambda; q, t), \]  
(5.11)

where
\[ G(u) = \exp \left( -\sum_{n>0} \frac{1}{n} \frac{1}{(1 - q^n)(1 - t^{-n})} u^n \right) \in \mathbb{Q}(q, t)[[u]]. \]  
(5.12)

A proof of this will be given in Section 7.

5.2. \( SU(N_c) \) with \( N_f = 2N_c \). Next, we turn to the case with \( N_f = 2N_c \) fundamental matters. Let \( u, v, w, x, y \) be indeterminates. Consider the six point operator
\[ \Phi \]  
(5.13)

defined by the composition of the intertwining operators
\[ \mathcal{F}^{(0,1)}_{-x} \otimes \mathcal{F}^{(1,0)}_u \otimes \mathcal{F}^{(1,0)}_v \rightarrow \mathcal{F}^{(1,0)}_{ux/w} \otimes \mathcal{F}^{(1,0)}_{vy/w} \otimes \mathcal{F}^{(0,1)}_y, \]  
(5.14)

For any \( P_\lambda \otimes \alpha \otimes \beta \in \mathcal{F}^{(0,1)}_{-x} \otimes \mathcal{F}^{(1,0)}_u \otimes \mathcal{F}^{(1,0)}_v \), we have
\[ \Phi \left[ \frac{(1, 0), ux/w; (1, 0), vy/y; (0, 1), -y}{(0, 1), -x; (1, 0), w; (1, 0), v} \right] (P_\lambda \otimes \alpha \otimes \beta) = \sum_{\mu, \nu} \frac{1}{\langle P_\mu, P_\nu \rangle_{q, t}} \times \Phi_\lambda \left[ \frac{(1, 0), ux/w}{(0, 1), -x; (1, -1), u/w} \right] \Phi^* \left[ \frac{(1, -1), u/w; (0, 1), -w}{(1, 0), u} \right] (\alpha) \]  
(5.15)
\[
= \sum_{\mu, \nu} q^{n(\lambda)}(ux/w)^{3|q^{n(\nu')}}(qy/vw)^{|\mu|} (q^{1/2}t^{-1/2})^{-|\mu|} (v/u)^{|\nu|} f^{-1}_\mu
\]
\[
\times \left( \Phi_\emptyset(-x)\eta_\lambda(-x) \right) \Phi^*_\emptyset(-w)\xi_\mu(-w) : \alpha \right) \\
\otimes \left( \Phi^*_\emptyset(-y)\xi_\nu(-y) \right) \Phi_\emptyset(-w)\eta_\mu(-w) : \beta) \right) \otimes Q_\nu. 
\]

Restricting this six point operator, introduce the four point operator
\[
\Phi \left[ (1, 0), uw/w; (1, 0), vw/y \right] \otimes F^{(1,0)}_u \otimes F^{(1,0)}_v \rightarrow F_{uw/w} \otimes F_{vw/y}, 
\]
defined by specifying the action on any \( \alpha \otimes \beta \in F^{(1,0)}_u \otimes F^{(1,0)}_v \) as
\[
\Phi \left[ (1, 0), uw/w; (1, 0), vw/y \right] (\alpha \otimes \beta) = \text{id} \otimes \text{id} \otimes \langle \Phi_\emptyset, \bullet \rangle \otimes \Phi \left[ (1, 0), uw/w; (1, 0), vw/y; (0, 1), -y \right] (\alpha \otimes \beta) \\
= \sum_{\mu} \frac{(q^{1/2}t^{-1/2})^{-|\mu|} (v/u)^{|\nu|} f^{-1}_\mu}{N_{\mu, \mu}(1)} \\
\times \left( \Phi_\emptyset(-x) \Phi^*_\emptyset(-w)\xi_\mu(-w) : \alpha \right) \otimes \left( \Phi^*_\emptyset(-y) \Phi_\emptyset(-w)\eta_\mu(-w) : \beta \right), 
\]
where we have used the shorthand notation \( \langle \Phi_\emptyset, \bullet \rangle \otimes Q_\nu = \langle \Phi_\emptyset, Q_\nu \rangle_{q,t}. \)

Let \( u, v, w_1, \ldots, w_{N_c}, x_1, \ldots, x_{N_c}, y_1, \ldots, y_{N_c} \) be a set of indeterminates. Set
\[
u_i = v \prod_{k=1}^{i-1} w_k / y_k, \quad (i = 1, 2, \ldots, N_c + 1), 
\]
for simplicity.

**Proposition 5.2.** Set
\[
e_i = -w_i, \quad e_i' = -q^{1/2}t^{-1/2}y_i, \quad e_i'' = -q^{-1/2}t^{1/2}x_i, \quad \Lambda^{2N_c} = (q^{1/2}t^{-1/2})^{-N} \frac{\nu}{u} \prod_{i=1}^{N_c} \frac{w_i}{y_i}. 
\]
We have
\[
\langle P_\emptyset \otimes P_\emptyset | \Phi \left[ (1, 0, u_{N+1}; (1, 0), v_{N+1}) \right] \ldots \Phi \left[ (1, 0, u_2; (1, 0), v_2) \right] | P_\emptyset \otimes P_\emptyset \rangle
\]
\[
= \prod_{k=1}^{N_c} \frac{1}{G(e_k/e_k')G(qt^{-1}e_k/e_k')} \cdot \prod_{1 \leq i < j \leq N_c} G(e_i/e_j)G(qt^{-1}e_i'/e_j')G(e_i'/e_j)
\]
\[
\times \sum_{\lambda(1), \ldots, \lambda(N_c)} \prod_{k=1}^{N_c} \prod_{1 \leq i,j \leq N_c} \frac{N_{\lambda(j)}(e_i/e_j)N_{\lambda(i)}(e_i/e_j)}{\Lambda(\lambda(k)) \Lambda_{\lambda(k)}(e_i/e_j)}.
\]
Since our proof of this goes in a parallel way as the one for Proposition 5.3, we omit it.

6. PROOFS OF THEOREMS 3.3 AND 3.6

6.1. Some formulas concerning \( A_{\lambda,i}^+, B_{\lambda}^\pm(z) \).

**Lemma 6.1.** Let \( c_\lambda, c'_\lambda, A_{\lambda,i}^+, A_{\lambda,i}^- \) be as in (2.10), (2.11), (2.12). We have
\[
c'_{\lambda+1,k} c_{\lambda} A_{\lambda,i}^+ = -q A_{\lambda+1,i}^+, \\
c_{\lambda-1} c_{\lambda} A_{\lambda,i}^- = -q^{-1} A_{\lambda-1,i}^+,
\]
(6.1)

Hence the action of \( \mathcal{U} \) is written in terms of the basis \( (Q_\lambda) \) as
\[
\gamma Q_\lambda = Q_\lambda,
\]
(6.2)
\[
x^+(z)Q_\lambda = -q A_{\lambda+1,i}^- \delta(q^{\lambda_i t^{-i+1} u/z})Q_{\lambda+1,i},
\]
(6.3)
\[
x^-(z)Q_\lambda = -q^{1/2} t^{-1/2} \sum_{i=1}^{\ell(\lambda)} q A_{\lambda-1,i}^+ \delta(q^{\lambda_i t^{-i+1} u/z})Q_{\lambda-1,i},
\]
(6.4)
\[
\psi^+(z)Q_\lambda = q^{1/2} t^{1/2} B_{\lambda}^+(u/z)Q_\lambda,
\]
(6.5)
\[
\psi^-(z)Q_\lambda = q^{-1/2} t^{-1/2} B_{\lambda}^-(z/u)Q_\lambda.
\]
(6.6)

**Proof.** From the definitions of \( c_\lambda, c'_\lambda \), it immediately follows that
\[
\frac{c_{\lambda+1,k}}{c_{\lambda}} = (1 - q^{\lambda_k t^{\ell(\lambda)-k+1}}) \prod_{i=1}^{k-1} \frac{1 - q^{\lambda_i - \lambda_k t^{i-k+i}}}{1 - q^{\lambda_i - \lambda_k t^{i-k}}},
\]
\[
\frac{c'_{\lambda+1,k}}{c'_{\lambda}} = (1 - q^{\lambda_k + t^{\ell(\lambda)-k}}) \prod_{i=1}^{k-1} \frac{1 - q^{\lambda_i - \lambda_k t^{i-k}}} {1 - q^{\lambda_i - \lambda_k t^{i-k-1}}}.
\]
Noting that
\[
A_{\lambda,k}^- = (1 - t^{-1}) \prod_{j=k+1}^{\ell(\lambda)} \frac{1 - q^{-\lambda_k t^{-\ell(\lambda)+k}}}{1 - q^{-\lambda_k + t^{-\ell(\lambda)+k-1}}} \prod_{j=k+1}^{\ell(\lambda)} \frac{1 - q^{-\lambda_k + t^{-j+k+1}}}{1 - q^{-\lambda_k + t^{-j+k}}},
\]
one obtains (6.1). \( \square \)

**Lemma 6.2.** Let \( B_{\lambda}^\pm(z) \) be as in (2.13), (2.14). We have
\[
\prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda} q^{(j-1) t^{i-1} u/z} = \frac{1 - v/z}{1 - q^{-1} t v/z} B^+(v/z),
\]
(6.7)
\[
\prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\ell(\lambda)} g(q^{-j}t^{-i+1}v/z) = \frac{1 - z/v}{1 - qt^{-1}z/v} B_\lambda^-(z/v), \tag{6.8}
\]

where \( g(z) \) is given in (2.1).

The following will also be needed.

**Lemma 6.3.** We have
\[
\prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\ell(\lambda)} f(q^{-j}t^{-i+1}v/z) = \frac{1 - \lambda_i t^{-i+1}v/z}{1 - t^{-\ell(\lambda)}v/z} \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i} t^{-i+1}v/z}{1 - q^{t^i} t^{-i+1}v/z}. \tag{6.9}
\]

**Lemma 6.4.** We have
\[
\frac{q^{n(\lambda')}}{c_\lambda} \left( \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i} t^{-i+1}v/z}{1 - q^{\lambda_i} t^{-i+1}v/z} + \frac{z}{v} \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i} t^{-i+1}v/z}{1 - q^{\lambda_i} t^{-i+1}v/z} \right) = \sum_{i=1}^{\ell(\lambda)+1} \frac{q^{n((\lambda+1)i')}}{c_{\lambda+1,i}} A_{\lambda,i}^+ \delta(q^{\lambda_i} t^{-i+1}v/z). \tag{6.10}
\]

**Proof.** It follows from
\[
\frac{1}{1 - t^{\ell(\lambda)}v/z} \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i} t^{-i+1}v/z}{1 - q^{\lambda_i} t^{-i+1}v/z} + \frac{z}{v} \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i} t^{-i+1}v/z}{1 - q^{\lambda_i} t^{-i+1}v/z} = \sum_{i=1}^{\ell(\lambda)+1} q^{\lambda_i} t^{-i+1} \delta(q^{\lambda_i} t^{-i+1}v/z) \prod_{j=1}^{\ell(\lambda) - i + 1} \frac{1 - q^{\lambda_i} t^{-i+1}v/z}{1 - q^{\lambda_i} t^{-i+1}v/z} \prod_{j=i+1}^{\ell(\lambda)} \frac{1 - q^{t^j} t^{-i+1}v/z}{1 - q^{t^j} t^{-i+1}v/z},
\]

and
\[
\frac{c_{\lambda+1,i}}{c_\lambda} = q^{t^{-1}}(1 - q^{\lambda_i} t^{\ell(\lambda)-i+1}) \prod_{j=1}^{i-1} \frac{1 - q^{\lambda_i} t^{j+1} t^{-i-1}}{1 - q^{\lambda_i} t^{j+1} t^{-i-1}} \prod_{j=i+1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i} t^{j-i}}{1 - q^{\lambda_i} t^{j-i+1}},
\]

with
\[
n(\lambda') = \sum_{i \geq 0} \lambda_i (\lambda_i - 1) = \frac{1}{2}, \quad \frac{q^{n((\lambda+1)i')}}{q^{n(\lambda')}} = q^{\lambda_i}.
\]

**Lemma 6.5.** We have
\[
\frac{q^{n(\lambda')}}{c_\lambda} \left( (1 - q^{-1} t^{\ell(\lambda)+1}v/z) \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i} t^{-i+2}v/z}{1 - \lambda_i t^{-i+1}v/z} + \frac{1}{v} \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i} t^{-i+1}v/z}{1 - q^{\lambda_i} t^{-i+1}v/z} \right) = \sum_{i=1}^{\ell(\lambda)} \frac{q^{n((\lambda+1)i')}}{c_{\lambda-1,i}} A_{\lambda,i}^- \delta(q^{\lambda_i-1} t^{-i+1}v/z).
\tag{6.11}
\]
6.2. Operator product formulas for $\tilde{\Phi}_\lambda(v)$.

Lemma 6.6. The operator product formulas between $\tilde{\Phi}_\theta(v)$ and the generators of $\mathcal{U}$ are

$$\eta(z)\tilde{\Phi}_\theta(v) = \frac{1}{1 - v/z} : \eta(z)\tilde{\Phi}_\theta(v) :,$$

(6.14)

$$\tilde{\Phi}_\theta(v)\eta(z) = \frac{1}{1 - vt^{-1}z/v} : \eta(z)\tilde{\Phi}_\theta(v) :,$$

(6.15)

$$\xi(z)\tilde{\Phi}_\theta(v) = (1 - q^{-1/2}t^{1/2}v/z) : \xi(z)\tilde{\Phi}_\theta(v) :,$$

(6.16)

$$\tilde{\Phi}_\theta(v)\xi(z) = (1 - q^{1/2}t^{-1/2}z/v) : \xi(z)\tilde{\Phi}_\theta(v) :,$$

(6.17)

$$\varphi^+(q^{1/4}t^{-1/4}z)\tilde{\Phi}_\theta(v) = \frac{1 - q^{-1}tv/z}{1 - u/z} \tilde{\Phi}_\theta(v)\varphi^+(q^{1/4}t^{-1/4}z),$$

(6.18)

$$\varphi^-(q^{-1/4}t^{1/4}z)\tilde{\Phi}_\theta(v) = \frac{1 - qt^{-1}z/v}{1 - z/v} \tilde{\Phi}_\theta(v)\varphi^-(q^{-1/4}t^{1/4}z).$$

(6.19)

Proposition 6.7. We have

$$\varphi^+(q^{1/4}t^{-1/4}z)\tilde{\Phi}_\lambda(v)\varphi^+(q^{1/4}t^{-1/4}z)^{-1} = B^+_\lambda(z/v)\tilde{\Phi}_\lambda(v),$$

(6.20)

$$\varphi^-(q^{-1/4}t^{1/4}z)\tilde{\Phi}_\lambda(v)\varphi^-(q^{-1/4}t^{1/4}z)^{-1} = B^-\lambda(z/v)\tilde{\Phi}_\lambda(v).$$

(6.21)

Proof. Note that

$$\varphi^+(q^{1/4}t^{-1/4}z)\eta(v)\varphi^+(q^{1/4}t^{-1/4}z)^{-1} = g(v/z)^{-1}\eta(v),$$

$$\varphi^-(q^{-1/4}t^{1/4}z)\eta(v)\varphi^-(q^{-1/4}t^{1/4}z)^{-1} = g(v/z)\eta(v).$$

Then (6.20), (6.21) follow from Lemma 6.2 and (6.18), (6.19) in Lemma 6.6 □

Lemma 6.8. We have

$$\eta(z)\tilde{\Phi}_\lambda(v) = \frac{1}{1 - t^{-\ell(\lambda)}v/z} \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}t^{-i-1}v/z}{1 - q^{\lambda_i}t^{-i+1}v/z} : \eta(z)\tilde{\Phi}_\lambda(v) :,$$

(6.22)

$$\tilde{\Phi}_\lambda(v)\eta(z) = \frac{1}{1 - qt^{\ell(\lambda)}-1z/v} \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{-\lambda_i+1}t^{-i-2}z/v}{1 - q^{-\lambda_i+1}t^{-i-1}z/v} : \eta(z)\tilde{\Phi}_\lambda(v) :,$$

(6.23)

$$B^-\lambda(z/v)\tilde{\Phi}_\lambda(v)\eta(z) = \frac{1}{1 - q^{t^{\ell(\lambda)}-1}z/v} \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{-\lambda_i}t^{-i}z/v}{1 - q^{-\lambda_i}t^{-i-1}z/v} : \eta(z)\tilde{\Phi}_\lambda(v) :.$$ 

(6.24)

Proof. We have $\eta(z)\eta(v) = f(v/z) : \eta(z)\eta(v) :$. Hence from Lemma 6.3

$$\eta(z)\eta_\lambda(v) = \frac{1 - v/z}{1 - t^{-\ell(\lambda)}v/z} \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}t^{-i+1}v/z}{1 - q^{\lambda_i}t^{-i+1}v/z} : \eta(z)\eta_\lambda(v) :,$$

$$\eta_\lambda(v)\eta(z) = \frac{1 - qt^{-1}z/v}{1 - qt^{\ell(\lambda)}-1z/v} \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{-\lambda_i+1}t^{-i-2}z/v}{1 - q^{-\lambda_i+1}t^{-i-1}z/v} : \eta(z)\eta_\lambda(v) :.$$ 

Then (6.22), (6.23) follow from (6.14), (6.15) in Lemma 6.6 □

Proposition 6.9. We have

$$\eta(z)\tilde{\Phi}_\lambda(v) + \tilde{B}^-\lambda(z/v)\tilde{\Phi}_\lambda(v)\eta(z) = \sum_{i=1}^{\ell(\lambda)+1} A^+_{\lambda,\lambda} \tilde{\Phi}_{\lambda+1,\lambda}(v)\delta(q^{\lambda_i}t^{-i+1}v/z).$$ 

(6.25)

Proof. It follows from Lemma 6.4 and (6.22), (6.24) in Lemma 6.8 □
Lemma 6.10. We have
\[
\xi(q^{1/2}t^{-1/2}z)\tilde{\Phi}_\lambda(v) = (1 - q^{-1}t^{-\ell(\lambda)+1}v/z) \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{-\lambda_i - 1}t^{-i+1}v/z}{1 - q^{-\lambda_i - 1}t^{-i+1}v/z} : \xi(q^{1/2}t^{-1/2}z)\tilde{\Phi}_\lambda(v) :, \quad (6.26)
\]
\[
\tilde{\Phi}_\lambda(v)\xi(q^{1/2}t^{-1/2}z) = (1 - qt^{\ell(\lambda)-1}z/v) \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{-\lambda_i + 1}t^{-2}z/v}{1 - q^{-\lambda_i + 1}t^{-1}z/v} : \tilde{\Phi}_\lambda(v)\xi(q^{1/2}t^{-1/2}z) : . \quad (6.27)
\]
Proof. Note that \(\xi(q^{1/2}t^{-1/2}z)\eta(v) = f(q^{-1}v/z)^{-1} : \xi(q^{1/2}t^{-1/2}z)\eta(v) :,\) and \(\eta(v)\xi(q^{1/2}t^{-1/2}z) = f(z/v)^{-1} : \xi(q^{1/2}t^{-1/2}z)\eta(v) :,\) Thus from Lemma 6.3 we have
\[
\xi(q^{1/2}t^{-1/2}z)\eta(v) = \frac{1 - q^{-1}t^{-\ell(\lambda)+1}v/z}{1 - q^{-1}t} \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{-\lambda_i - 1}t^{-i+1}v/z}{1 - q^{-\lambda_i - 1}t^{-i+1}v/z} : \xi(q^{1/2}t^{-1/2}z)\eta(v) :,
\]
\[
\eta(v)\xi(q^{1/2}t^{-1/2}z) = \frac{1 - qt^{\ell(\lambda)-1}z/v}{1 - qt^{-1}} \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{-\lambda_i + 1}t^{-2}z/v}{1 - q^{-\lambda_i + 1}t^{-1}z/v} : \xi(q^{1/2}t^{-1/2}z)\eta(v) : .
\]
Then (6.26), (6.27) follow from (6.16), (6.17) in Lemma 6.6 \(\square\)

Proposition 6.11. We have
\[
\xi(q^{1/2}t^{-1/2}z)\tilde{\Phi}_\lambda(v) + q^{-1}\frac{v}{z}\tilde{\Phi}_\lambda(v)\xi(q^{1/2}t^{-1/2}z)
= \sum_{i=1}^{\ell(\lambda)} A_{\lambda,i}^{-1}(v)\delta(q^{\lambda_i - 1}t^{-i+1}v/z)\varphi^+(q^{1/4}t^{-1/4}z). \quad (6.28)
\]
Proof. It follows from Lemmas 6.5 6.10 and \(\xi(q^{1/2}t^{-1/2}z)\eta(z) := \varphi^+(q^{1/4}t^{-1/4}z).\) \(\square\)

6.3. Operator product formulas for \(\tilde{\Phi}_\lambda^+(u).\)

Lemma 6.12. We have
\[
\eta(z)\tilde{\Phi}_\lambda^+(u) = (1 - q^{-1/2}t^{1/2}u/z) : \eta(z)\tilde{\Phi}_\lambda^+(u) :, \quad (6.29)
\]
\[
\tilde{\Phi}_\lambda^+(u)\eta(z) = (1 - q^{-1/2}t^{1/2}z/u) : \eta(z)\tilde{\Phi}_\lambda^+(u) :, \quad (6.30)
\]
\[
\xi(z)\tilde{\Phi}_\lambda^+(u) = \frac{1}{1 - q^{-1}tu/z} : \xi(z)\tilde{\Phi}_\lambda^+(u) :, \quad (6.31)
\]
\[
\tilde{\Phi}_\lambda^+(u)\xi(z) = \frac{1}{1 - z/u} : \xi(z)\tilde{\Phi}_\lambda^+(u) :, \quad (6.32)
\]
\[
\varphi^+(q^{-1/4}t^{1/4}z)^{-1}\tilde{\Phi}_\lambda^+(u) = \frac{1 - q^{-1}tu/z}{1 - u/z}\tilde{\Phi}_\lambda^+(u)\varphi^+(q^{-1/4}t^{1/4}z)^{-1}, \quad (6.33)
\]
\[
\varphi^-(q^{1/4}t^{-1/4}z)^{-1}\tilde{\Phi}_\lambda^+(u) = \frac{1 - qt^{-1}z/u}{1 - z/u}\tilde{\Phi}_\lambda^+(u)\varphi^-(q^{1/4}t^{-1/4}z)^{-1}. \quad (6.34)
\]

Proposition 6.13. We have
\[
\varphi^+(q^{-1/4}t^{1/4}z)^{-1}\tilde{\Phi}_\lambda^+(u)\varphi^+(q^{-1/4}t^{1/4}z) = B^+_\lambda(u/z)\tilde{\Phi}_\lambda^+(u), \quad (6.35)
\]
\[
\varphi^-(q^{1/4}t^{-1/4}z)^{-1}\tilde{\Phi}_\lambda^+(u)\varphi^-(q^{1/4}t^{-1/4}z) = B^-_\lambda(z/u)\tilde{\Phi}_\lambda^+(u). \quad (6.36)
\]
Proof. Note that
\[
\varphi^+(q^{-1/4}t^{1/4}z)^{-1}\xi(u)\varphi^+(q^{-1/4}t^{1/4}z) = g(u/z)^{-1}\xi(u),
\]
\[
\varphi^-(q^{1/4}t^{-1/4}z)^{-1}\xi(u)\varphi^-(q^{1/4}t^{-1/4}z) = g(z/u)\xi(u).
\]
Then (6.35), (6.36) follow from Lemmas 6.2 and (6.33), (6.34) in Lemma 6.12 \(\square\)
Lemma 6.14. We have

$$\xi(z)\bar{\Phi}_\lambda^*(u) = \frac{1}{1-q^{-1}\ell(t)^{-1}u/z} \prod_{i=1}^{\ell(\lambda)} \frac{1-q^\lambda t^{-\ell(t)+1}u/z}{1-q^\lambda t^{-\ell(t)+i+1}u/z} : \xi(z)\bar{\Phi}_\lambda^*(u) : \,.$$  \hfill (6.37)

$$B_\lambda^+(u/z)\xi(z)\bar{\Phi}_\lambda^*(u) = \frac{1}{1-t^{-1}u/z} \prod_{i=1}^{\ell(\lambda)} \frac{1-q^\lambda t^{-i}u/z}{1-q^\lambda t^{-i+1}u/z} : \xi(z)\bar{\Phi}_\lambda^*(u) : \,.$$  \hfill (6.38)

$$\bar{\Phi}_\lambda^*(u)\xi(z) = \frac{1}{1-t^{-1}u/z} \prod_{i=1}^{\ell(\lambda)} \frac{1-q^\lambda t^{-i}z/u}{1-q^\lambda t^{-1}z/u} : \xi(z)\bar{\Phi}_\lambda^*(u) : \,.$$  \hfill (6.39)

Proof. From $\xi(z)\xi(u) = f(q^{-1}tu/z) : \xi(z)\xi(u) :$, and Lemma 6.4 we have

$$\xi(z)\xi_\lambda(u) = \frac{1-q^{-1}tu/z}{1-q^{-1}t^{-1}(\lambda)+1u/z} \prod_{i=1}^{\ell(\lambda)} \frac{1-q^\lambda t^{-\ell(t)+1}u/z}{1-q^\lambda t^{-\ell(t)+i+1}u/z} : \xi(z)\xi_\lambda(u) : \,.$$  \hfill (6.40)

Then (6.37), (6.39) follow from (6.31), (6.32) in Lemma 6.12

Proposition 6.15. We have

$$B_\lambda^+(u/z)\xi(z)\bar{\Phi}_\lambda^*(u) + \frac{z}{u}\bar{\Phi}_\lambda^*(u)\xi(z) = \sum_{i=1}^{\ell(\lambda)+1} A_{i,\lambda}^+ \Phi_{\lambda+1}^*(u) \delta(q^\lambda t^{-i+1}u/z).$$  \hfill (6.41)

Proof. It follows from Lemma 6.4 and (6.38), (6.39) in Lemma 6.14

Lemma 6.16. We have

$$\eta(q^{1/2}t^{-1/2}z)\bar{\Phi}_\lambda^*(u) = (1-q^{-1}t^{-\ell(\lambda)+1}u/z) \prod_{i=1}^{\ell(\lambda)} \frac{1-q^\lambda t^{-\ell(t)+1}u/z}{1-q^\lambda t^{-\ell(t)+i+1}u/z} : \eta(q^{1/2}t^{-1/2}z)\bar{\Phi}_\lambda^*(u) : \,.$$  \hfill (6.42)

$$\bar{\Phi}_\lambda^*(u)\eta(q^{1/2}t^{-1/2}z) = (1-q^\ell(t)^{-1}u/z) \prod_{i=1}^{\ell(\lambda)} \frac{1-q^\lambda t^{-i}z/u}{1-q^\lambda t^{-i+1}z/u} : \bar{\Phi}_\lambda^*(u)\eta(q^{1/2}t^{-1/2}z) : \,.$$  \hfill (6.43)

Proof. Note that $\eta(q^{1/2}t^{-1/2}z)\xi(u) = f(q^{-1}tu/z)^{-1} : \eta(q^{1/2}t^{-1/2}z)\xi(u) :$, and $\xi(u)\eta(q^{1/2}t^{-1/2}z) = f(z/u)^{-1} : \eta(q^{1/2}t^{-1/2}z)\xi(u) :$. From Lemma 6.3 we have

$$\eta(q^{1/2}t^{-1/2}z)\xi_\lambda(u) = \frac{1-q^{-1}t^{-\ell(\lambda)+1}u/z}{1-q^{-1}t^{-1}u/z} \prod_{i=1}^{\ell(\lambda)} \frac{1-q^\lambda t^{-\ell(t)+1}u/z}{1-q^\lambda t^{-\ell(t)+i+1}u/z} : \eta(q^{1/2}t^{-1/2}z)\xi_\lambda(u) : \,.$$  \hfill (6.44)

$$\xi_\lambda(u)\eta(q^{1/2}t^{-1/2}z) = \frac{1-q^\ell(t)^{-1}z/u}{1-q^\ell(t)^{-1}z/u} \prod_{i=1}^{\ell(\lambda)} \frac{1-q^\lambda t^{-i}z/u}{1-q^\lambda t^{-i+1}z/u} : \xi_\lambda(u)\eta(q^{1/2}t^{-1/2}z) : \,.$$  \hfill (6.45)

Then (6.41), (6.42) follow from (6.29), (6.30) in Lemma 6.12

Proposition 6.17. We have

$$\eta(q^{1/2}t^{-1/2}z)\bar{\Phi}_\lambda^*(u) + q^{-1}\frac{u}{z}\bar{\Phi}_\lambda^*(u)\eta(q^{1/2}t^{-1/2}z)$$

$$= \sum_{i=1}^{\ell(\lambda)} A_{i,\lambda}^+ \bar{\Phi}_{\lambda+1}^*(v) \delta(q^\lambda t^{-i+1}v/z) \varphi(q^{1/4}t^{-1/4}z) \,.$$  \hfill (6.46)
Proof. It follows from Lemmas 6.5 6.16 and : \( \eta(q^{1/2}t^{-1/2}z)\xi(z) := \varphi^{-(q^{1/4}t^{-1/4}z)}. \)

6.4. Final step of proofs. The intertwining relations in Lemma 3.2 are rewritten in terms of \( \eta, \xi, \varphi^{\pm} \) as

\[
\varphi^{+}(q^{-1/4}t^{-1/4}z)\Phi_{\lambda}\varphi^{+}(q^{-1/4}t^{-1/4}z)^{-1} = B^{+}_{\lambda}(u/z)\Phi_{\lambda},
\]

(6.44)

\[
\varphi^{-}(q^{-1/4}t^{-1/4}z)\Phi_{\lambda}\varphi^{-}(q^{-1/4}t^{-1/4}z)^{-1} = B^{-}_{\lambda}(z/u)\Phi_{\lambda},
\]

(6.45)

\[
\eta(z)\Phi_{\lambda} - \frac{u}{w}B^{-}_{\lambda}(z/v)\Phi_{\lambda}\eta(z)
\]

(6.46)

\[
= \sum_{i=1}^{\ell(\lambda)+1} q^{-1}(q^{1/2}t^{-1/2}q^{\lambda_{i}}t^{-i+1}u)^{N+1}A_{\lambda_{i}}^{+}\delta(q^{\lambda_{i}}t^{-i+1}u/z)\Phi_{\lambda+1,1},
\]

\[
\xi(q^{1/2}t^{-1/2}z)\Phi_{\lambda} - q^{-1}(w/z)\Phi_{\lambda}\xi(q^{1/2}t^{-1/2}z)
\]

(6.47)

\[
= \sum_{i=1}^{\ell(\lambda)} q(w^{1/2}t^{-1/2}q^{\lambda_{i}}t^{-i+1}u)^{-N-1}A_{\lambda_{i}}^{-}\delta(q^{\lambda_{i}}t^{-i+1}u/z)\Phi_{\lambda-1,1},
\]

Proof of Theorem 3.3. From (6.44) and (6.45), we must have that \( \Phi_{\lambda} \) be proportional to \( \widetilde{\Phi}_{\lambda}(v) \) by virtue of Proposition 6.7. Write \( \Phi_{\lambda} = t(\lambda, v, u, N)\widetilde{\Phi}_{\lambda}(v) \). Then in view of Propositions 6.9 6.11 we find that (6.46) and (6.47) may hold only in the case \( w = -vu \) and when \( t(\lambda, v, u, N) \) is given by (3.2).

The intertwining relations in Lemma 3.2 are rewritten in terms of \( \eta, \xi, \varphi^{\pm} \) as

\[
\varphi^{+}(q^{-1/4}t^{-1/4}z)\Phi_{\lambda}\varphi^{+}(q^{-1/4}t^{-1/4}z)^{-1} = B^{+}_{\lambda}(u/z)\Phi_{\lambda},
\]

(6.48)

\[
\varphi^{-}(q^{-1/4}t^{-1/4}z)\Phi_{\lambda}\varphi^{-}(q^{-1/4}t^{-1/4}z)^{-1} = B^{-}_{\lambda}(z/u)\Phi_{\lambda},
\]

(6.49)

\[
B^{+}(u/z)\xi(z)\Phi_{\lambda} - \frac{u}{vz}\Phi_{\lambda}\xi(z)
\]

(6.50)

\[
= \sum_{i=1}^{\ell(\lambda)+1} q^{-1}(q^{1/2}t^{-1/2}q^{\lambda_{i}}t^{-i+1}u)^{-N}A_{\lambda_{i}}^{+}\delta(q^{\lambda_{i}}t^{-i+1}u/z)\Phi_{\lambda+1,1},
\]

\[
\eta(q^{1/2}t^{-1/2}z)\Phi_{\lambda} - q^{-1}(w/z)\Phi_{\lambda}\eta(q^{1/2}t^{-1/2}z)
\]

(6.51)

\[
= \varphi^{-}(q^{1/4}t^{-1/4}z)\sum_{i=1}^{\ell(\lambda)} qv^{-1}(q^{1/2}t^{-1/2}q^{\lambda_{i}}t^{-i+1}u)^{N}A_{\lambda_{i}}^{-}\delta(q^{\lambda_{i}}t^{-i+1}u/z)\Phi_{\lambda-1,1}.
\]

Proof of Theorem 3.6. From (6.48) and (6.49), we must have that \( \Phi_{\lambda}^{*} \) be proportional to \( \widetilde{\Phi}_{\lambda}(v) \) by virtue of Proposition 6.13. Write \( \Phi_{\lambda}^{*} = t^{*}(\lambda, v, u, N)\Phi_{\lambda}^{*}(v) \). Then in view of Propositions 6.15 6.17 we find that (6.50) and (6.51) may hold only in the case \( w = -vu \) and when \( t^{*}(\lambda, v, u, N) \) is given by (3.2).

7. Proof of Proposition 5.1

7.1. Some formulas concerning \( N_{\lambda,\mu}(u) \).

Lemma 7.1. We have

\[
N_{\lambda,\mu}(u) = \prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\ell(\lambda)} (uq^{-\mu_{i}+\lambda_{j}+1}t^{-i-j}; q)_{\lambda_{i}-\lambda_{j}+1} \cdot \prod_{\alpha=1}^{\ell(\mu)} \prod_{\beta=1}^{\ell(\mu)} (uq^{\lambda_{\alpha}-\mu_{\beta}+1}t^{-\alpha+\beta+1}; q)_{\mu_{\beta}-\mu_{\beta}+1},
\]

(7.1)
Proposition 7.3. These follow from (4.9), (4.10) and (7.4).

Proof. Fix an integer \( \lambda, \mu \) where \( \lambda, \mu \neq 0 \). Let \( \xi, \eta \) be as in (5.12). We have

\[
\text{LHS of (5.11)} = \frac{1}{\tfrac{1}{n} + 1} \sum_{n \geq 0} \frac{1}{n} \frac{t^n}{1 - t^n} \left( \frac{t^n}{1 + t^n} + \sum_{i=1}^\ell (q^{\lambda_i t^{-i}} - 1) t^{i n} \right)
\]

\[
= \frac{\mathcal{G}(u)^{-1} \sum_{n \geq 0} \frac{1}{n} \frac{t^n}{1 - t^n} u^n}{\mathcal{G}(u)^{-1} \sum_{n \geq 0} \frac{1}{n} \frac{t^n}{1 - t^n} u^n \left( \sum_{i=1}^\ell (q^{\lambda_i t^{-i}} - 1) t^{i n} \right) + \sum_{j=1}^\ell (q^{\mu_j t^{j}} - 1) t^{j n}}
\]

\[
= \frac{\mathcal{G}(u)^{-1} \prod_{i=1}^\ell \prod_{j=1}^\ell \left( \frac{u q^{\mu_i + \lambda_i t^{-i} - j - 1}}{u q^{\mu_i + \lambda_i t^{-i}}} \right) \prod_{k=1}^\ell \left( \frac{u q^{\mu_k t^{-k} - \ell + 1}}{u q^{\mu_k t^{-k}}} \right)}{\mathcal{G}(u)^{-1} \sum_{n \geq 0} \frac{1}{n} \frac{t^n}{1 - t^n} u^n}
\]

where \( \mathcal{G}(u) \) being as in (5.12).

Lemma 7.2. Let \( \xi^\dagger \) be the algebra homomorphism in (4.3). We have

\[
\exp \left( \sum_{n \geq 0} \frac{1}{n} \frac{t^n}{1 - t^n} (\xi^\dagger p_n) (\xi^\dagger p_n) u^n \right) = \mathcal{G}(u)^{-1} N_{\lambda, \mu}(u),
\]

where \( \mathcal{G}(u) \) being as in (5.12).

Proof. Fix an integer \( \ell \) such that \( \ell \geq \max(\ell(\lambda), \ell(\mu)) \). We have

\[
\text{LHS} = \exp \left( \sum_{n \geq 0} \frac{1}{n} \frac{t^n}{1 - t^n} u^n \left( \frac{t^n}{1 - t^n} + \sum_{i=1}^\ell (q^{\lambda_i t^{-i}} - 1) t^{i n} \right) \right)
\]

\[
= \mathcal{G}(u)^{-1} \exp \left( \sum_{n \geq 0} \frac{1}{n} \frac{t^n}{1 - t^n} u^n \left( \sum_{i=1}^\ell (q^{\lambda_i t^{-i}} - 1) t^{i n} \right) + \sum_{j=1}^\ell (q^{\mu_j t^{j}} - 1) t^{j n} \right)
\]

\[
= \mathcal{G}(u)^{-1} \prod_{i=1}^\ell \prod_{j=1}^\ell \left( \frac{u q^{\mu_i + \lambda_i t^{-i} - j - 1}}{u q^{\mu_i + \lambda_i t^{-i}}} \right) \prod_{k=1}^\ell \left( \frac{u q^{\mu_k t^{-k} - \ell + 1}}{u q^{\mu_k t^{-k}}} \right),
\]

were we have used the notation

\[
(u; q)_\infty = \exp \left( - \sum_{n=1}^\infty \frac{1}{1 - q^n} u^n \right) \in \mathbb{Q}(q)[[u]].
\]

Note that \( (u; q)_\infty/(q^n u; q)_\infty = (u; q)_n \) \( (n = 0, 1, 2, \ldots) \), and use (7.1), then we have (7.2).

Proposition 7.3. We have the operator product formulas

\[
: \tilde{\Phi}_\theta(z) \xi_\lambda(z) : \tilde{\Phi}_\theta(w) \xi_\mu(w) : = \frac{\mathcal{G}(w/z)}{N_{\mu, \lambda}(w/z)} : \tilde{\Phi}_\theta(z) \xi_\lambda(z) \tilde{\Phi}_\theta(w) \xi_\mu(w) :,
\]

\[
: \tilde{\Phi}_\theta(z) \eta_\lambda(z) : \tilde{\Phi}_\theta(w) \eta_\mu(w) : = \frac{q f_{\lambda}^{L-M-2t+2}}{f_{\lambda}^{L-M-2t+2}} : \tilde{\Phi}_\theta(z) \xi_\lambda(z) \tilde{\Phi}_\theta(w) \eta_\mu(w) :,
\]

\[
: \tilde{\Phi}_\theta(z) \eta_\lambda(z) : \tilde{\Phi}_\theta(w) \xi_\mu(w) : = \frac{q f_{\lambda}^{L-M-2t+2}}{f_{\lambda}^{L-M-2t+2}} : \tilde{\Phi}_\theta(z) \xi_\lambda(z) \tilde{\Phi}_\theta(w) \eta_\mu(w) :,
\]

\[
: \tilde{\Phi}_\theta(z) \eta_\lambda(z) : \tilde{\Phi}_\theta(w) \eta_\mu(w) : = \frac{q f_{\lambda}^{L-M-2t+2}}{f_{\lambda}^{L-M-2t+2}} : \tilde{\Phi}_\theta(z) \eta_\lambda(z) \tilde{\Phi}_\theta(w) \eta_\mu(w) :.
\]

Proof. These follow from (4.9), (4.10) and (7.4).

7.2. Proof of Proposition 5.4 Using Lemma 7.1 and Proposition 7.3 we have

\[
\text{LHS of (5.11)} = \sum_{\lambda^{(1)}, \ldots, \lambda^{(N_c)}} \prod_{k=1}^{N_c} \frac{(q^{1/2} t^{1/2} u_i w_i u_i^{-1} w_i^{L-M-2t+2})^{1/2}}{N_{\lambda^{(1)}, \lambda^{(2)}}(1)} f_{\lambda^{(1)}, \lambda^{(2)}}^{L-M-2t+1}
\]

\[
\times (0) : \tilde{\Phi}_\theta(-w_{N_c}) \xi_{\lambda(N_c)}(-w_{N_c}) : \cdots : \tilde{\Phi}_\theta(-w_1) \xi_{\lambda(1)}(-w_1) : |0\rangle
\]

\[
\times (0) : \tilde{\Phi}_\theta(-w_{N_c}) \eta_{\lambda(N_c)}(-w_{N_c}) : \cdots : \tilde{\Phi}_\theta(-w_1) \eta_{\lambda(1)}(-w_1) : |0\rangle.
\]
\[ = \sum_{\lambda^{(i)}, \ldots, \lambda^{(N_c)}} \prod_{k=1}^{N_c} \left( \frac{(q^{-1/2} t^{1/2} v_i^{-1} w_i^{L-M-2i+2})^{\lambda^{(i)}}}{N_{\lambda^{(i)}}^{\lambda^{(i)}(1)}} f_{\lambda^{(i)}}^{L-M-2i+1} \right) \]

\[ \times \prod_{1 \leq i < j \leq N_c} \frac{G(w_i/w_j)}{N_{\lambda^{(i)}, \lambda^{(j)}}^{w_i/w_j} N_{\lambda^{(i)}, \lambda^{(j)}}^{qt^{-1} w_i/w_j}}. \]

Simplifying the factors by using Lemma 7.4 below, we have the result. \[ \square \]

**Lemma 7.4.** We have

\[ \prod_{1 \leq i < j \leq N} (q^{1/2} t^{-1/2} - |\lambda^{(i)}| - |\lambda^{(j)}|) = \prod_{i=1}^N (q^{1/2} t^{-1/2} - (N-1)|\lambda^{(i)}|), \]

\[ \prod_{1 \leq i < j \leq N} w_i^{-|\lambda^{(i)}|} w_j^{|\lambda^{(j)}|} = \prod_{i=1}^N w_i^{-(N+2i-1)|\lambda^{(i)}|}, \]

\[ \prod_{1 \leq i < j \leq N} w_i^{-|\lambda^{(i)}|} w_j^{|\lambda^{(j)}|} = \prod_{i=1}^N (w_1 w_2 \cdots w_N)^{|\lambda^{(i)}|} w_i^{-|\lambda^{(i)}|} (w_1 w_2 \cdots w_{i-1})^{-2|\lambda^{(i)}|}. \]

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