Lagrangian formalism and its auxiliary conditions: special function equations and Bateman oscillators

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Abstract
Lagrangian formalism is established for differential equations whose solutions are given by special functions of mathematical physics, and for differential equations that describe the Bateman damped harmonic oscillators. The basis for the formalism are standard and non-standard Lagrangians. In this paper, it is shown that for non-standard Lagrangians the calculus of variations must be amended by auxiliary conditions whose existence have profound implications on the validity of the Helmholtz conditions. The obtained results are applied to the Bessel, Legendre and Hermit equations, and also to the Bateman oscillators.

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1. Introduction

There are numerous applications of linear second-order ordinary differential equations (ODEs) in applied mathematics, physics and engineering [1,2]. The most commonly used ODEs are those whose solutions are given by the special functions (SF) [3], and the ODEs that are equations of motion for damped oscillators [4]. In this paper, we concentrate on these two families of ODEs, and refer to the former as the special function ODEs, and to the latter as the Bateman damped harmonic oscillators [5]. We introduce $\mathcal{Q}_{sf}$ to be a set of special function ODEs, and $\mathcal{Q}_{Ba}$ to be a set of all ODEs that describe different Bateman damped harmonic oscillators, and use this notation throughout the paper. The sets are different as the ODEs of $\mathcal{Q}_{sf}$ have non-constant coefficients, however, the ODEs of $\mathcal{Q}_{Ba}$ have constant coefficients.

Let $\hat{D} \equiv d^2/dx^2 + B(x)d/dx + C(x)$ be an operator with $B(x)$ and $C(x)$ being ordinary (with the maps, $B: \mathcal{R} \to \mathcal{R}$ and $C: \mathcal{R} \to \mathcal{R}$) and smooth (that is, $C^\infty$) functions that depend on an independent variable $x$, and let $\hat{D}y(x) = 0$ be a linear second-order ODE with non-constant coefficients. The functions $B(x)$ and $C(x)$ can be selected so that the resulting equations represent either the ODEs of $\mathcal{Q}_{sf}$ or the ODEs of $\mathcal{Q}_{Ba}$. Moreover, for the ODEs of $\mathcal{Q}_{sf}$, it is expected that $B(x) \neq 0$ or $B(x) = 0$, and for the ODEs of $\mathcal{Q}_{Ba}$, $B(x) = b = \text{const}$ and $C(x) = c = \text{const}$. General solutions of the ODEs of $\mathcal{Q}_{sf}$ and $\mathcal{Q}_{Ba}$ are given by $y(x) = c_1y_1(x) + c_2y_2(x)$, where $c_1$ and $c_2$ are integration constants [1,2]. For the ODEs of $\mathcal{Q}_{sf}$, the solutions are the SF, while for the ODEs of $\mathcal{Q}_{Ba}$, they are elementary functions [6]. These differences in the solutions can also be used to separate the considered ODEs into the two sets $\mathcal{Q}_{sf}$ and $\mathcal{Q}_{Ba}$.

Typically, the ODEs of $\mathcal{Q}_{sf}$ are obtained by separation of variables in hyperbolic, parabolic and elliptic partial differential equations (PDEs) [1-3]. Another (less known) method is based on Lie groups, whose irreducible representations (irreps) are used to find the SF as well as their corresponding ODEs [1,7]. There have also been some attempts to establish the Lagrangian formalism for the ODEs of $\mathcal{Q}_{sf}$ (e.g., [8,9]), however, so far the problem has not yet been solved. Therefore, the main objective of this paper is to fully establish the Lagrangian formalism for the ODEs of $\mathcal{Q}_{sf}$ and derive new Lagrangians for these equations.

In theories of dynamical systems, the Lagrangian formalism has always played an important role in obtaining equations of motion [4,5]. For conservative dynamical systems, the existence of Lagrangians is guaranteed by the
Helmholtz conditions [10], which can also be used to derive the Lagrangians. However, for dissipative dynamical systems, the Helmholtz conditions prevent the existence of Lagrangians (e.g., [11,5]). The procedure of finding the Lagrangians is called the inverse (or Helmholtz) problem of the calculus of variations and there are different methods to solve this problem (e.g., [12,13]). A specific emphasis is given to the validity of the Helmholtz conditions for the Lagrangian formalism for dissipative systems, such as the Bateman damped harmonic oscillators.

We derive new families of Lagrangians for the ODEs of $Q_{sf}$ and $Q_{Ba}$ by solving the inverse calculus of variations problem for these equations. The obtained results show that some Lagrangians depend explicitly on the solutions of these ODEs, which requires that the calculus of variations is amended by auxiliary conditions. This is a new phenomenon in the Lagrangian formalism and it has profound implications on validity of the Helmholtz conditions. We explore applications of the obtained results to several selected ODEs, namely, the Bessel, Legendre and Hermite equations, as well as the equations describing the Bateman oscillators.

The outline of the paper is as follows: in Section 2, the Lagrangian formalism for the ODEs of $Q_{sf}$ and $Q_{Ba}$ is established using standard and non-standard Lagrangians, and validity of the Helmholtz conditions is also explored; in Section 3, applications of the obtained results to some selected ODEs of $Q_{sf}$ and $Q_{Ba}$ are presented and discussed; and our conclusions are given in Section 4.

2. Lagrangian formalism

2.1. Hamilton’s principle and existence of Lagrangians

From a mathematical point of view, in the Lagrange formalism we are provided with a functional $S[y(x)]$, which depends on an ordinary and smooth function $y(x)$ that must be determined. The functional is a map $S : C^\infty(\mathcal{R}) \to \mathcal{R}$, with $C^\infty(\mathcal{R})$ being a space of smooth functions. Typically $S[y(x)]$ is defined by an integral over a smooth function $L$ that depends on $y'(x) = dy/dx$, $y$ and on $x$, and the function $L(y', y, x)$ is called the Lagrangian function or simply Lagrangian. The functional $S[y(x)]$ defined in this way is known as the functional action, or simple action, and the Principle of Least Action or Hamilton’s Principle [12,14] requires that $\delta S = 0$, where $\delta$ is the variation.
defined as the functional (Fréchet) derivative of $S[y(x)]$ with respect to $y(x)$. Using $\delta S = 0$, the Euler-Lagrange (E-L) equation is obtained, and this equation becomes a necessary condition for the action to be stationary (to have either a minimum or maximum or saddle point).

In general, the E-L equation leads to a second-order ODE that can be further solved to obtain $y(x)$ that makes the action stationary. The described procedure is the basis of the classical calculus of variations and it works well when the Lagrangian $L(y', y, x)$ is already given. Deriving the second-order ODE from the E-L equation is known as the Lagrangian formalism, and in this paper we deal exclusively with this formalism. Our main goal is to establish the Lagrangian formalism for all ODEs of $Q_{af}$ and $Q_{Ba}$, and to find new Lagrangians.

There are two main families of Lagrangians, the so-called standard and non-standard Lagrangians. The standard Lagrangian denoted here as SLs are typically expressed as the difference between terms that can be identified as the kinetic and potential energy [14]. The non-standard (or not-natural) Lagrangians (NSLs) are such for which identification of kinetic and potential energy terms cannot be done and therefore these NSLs are simply the generating functions for the original equations as first pointed out by Arnold [15]. A broad range of different methods exists, and these methods were developed to obtain the SLs for both linear and non-linear ODEs as well as PDEs. Some methods involve the concept of Jacobi Last Multiplier [16-18], or used fractional derivatives [19], and others are based on different transformations that allow deriving the SLs for conservative and non-conservative physical systems described by either linear or non-linear ODEs [20]. There are also methods for finding the SLs for linear second-order PDEs, including the wave, Laplace and Tricomi-like equations [21].

There have been many attempts to obtain the NSLs for different ODEs. One of the first application of the NSLs to physics was done by Alekseev and Arbusov [22], who formulated the Yang-Mills field theory using NSLs, and thus demonstrated the usefulness of the NSLs for the fundamental theories of modern physics. Different forms of the NSLs have been recently proposed and applied to different physical problems [8,9,23,24], including a new NSL introduced by El-Nabulsi [25], who published a number of papers with interesting applications ranging from quantum fields and particle physics to general relativity and cosmology. Moreover, some NSLs were used by other authors, who established the Lagrangian formalism for Riccati [20] and Liénard [26] equations. Despite the efforts described above, the NSLs do not have
yet a well-established space in the theory of inverse variational problems.

To fully establish the Lagrangian formalism for the ODEs of \( Q_{sf} \) and \( Q_{Ba} \), we must know how to construct the SLs and NSLs for these equations. In the following, we describe new families of Lagrangians whose existence has profound implications on the validity of the Helmholtz conditions and on the calculus of variations as discussed later in this section.

2.2. Standard and null-Lagrangians

In the previous work [8], a very effective method of finding SLs for \( \hat{\mathcal{D}}y(x) = 0 \), which includes the ODEs of \( Q_{sf} \) and \( Q_{Ba} \), was proposed. The constructed SLs, denoted here as \( L_s \) are of the following form:

\[
L_s[y'(x), y(x), x] = G_s[y'(x), y(x), x] E_s(x) ,
\]

where

\[
G_s[y'(x), y(x), x] = \frac{1}{2} \left( (y'(x))^2 - C(x)y^2(x) \right) ,
\]

and \( E_s(x) = \exp \left[ \int^x B(\tilde{x}) d\tilde{x} \right] \).

As shown in [9], the equation \( \hat{\mathcal{D}}y(x) = 0 \) can also be derived from the following Lagrangian

\[
L_{so}[y'(x), y(x), x] = L_s[y'(x), y(x), x] + L_o[y'(x), y(x), x] ,
\]

where

\[
L_o[y'(x), y(x), x] = G_o[y'(x), y(x), x] E_s(x) ,
\]

is the null-Lagrangian (NL), and

\[
G_o[y'(x), y(x), x] = \frac{1}{2} B(x)y(x)y'(x) + \frac{1}{4} \left[ B'(x) + B^2(x) \right] y^2(x) .
\]

Having defined the Lagrangians \( L_s \), \( L_{so} \) and \( L_o \), we now state our main result concerning these Lagrangians in the following proposition.

**Proposition 1.** Let \( L_s \) and \( L_{so} \) be the standard Lagrangians given respectively by Eqs (1) and (3), and let \( L_o \) be defined by Eq. (4). Both \( L_s \) and \( L_{so} \) give the same \( \hat{\mathcal{D}}y(x) = 0 \) if, and only if, \( L_o \) is a full solution of the E-L equation that does not allow obtaining the original ODE.
Proof. The proof is straightforward as substitution of $L_s[y'(x), y(x), x]$ or $L_{so}[y'(x), y(x), x]$ into the E-L equation shows that these standard Lagrangians give exactly the same original equation $\hat{D}y(x) = 0$. It is also seen that

$$\frac{d}{dx} \left( \frac{\partial L_o}{\partial y'} \right) = \frac{1}{2} \left[ B'(x)y(x) + B(x)y'(x) + B^2(x)y(x) \right] E_s(x), \quad (6)$$

and

$$\frac{\partial L_o}{\partial y} = \frac{1}{2} \left[ B'(x)y(x) + B(x)y'(x) + B^2(x)y(x) \right] E_s(x), \quad (7)$$

are exactly equal, thus their substitution into the E-L equation gives

$$\frac{d}{dx} \left( \frac{\partial L_o}{\partial y'} \right) = \frac{\partial L_o}{\partial y}. \quad (8)$$

Thus, $L_o$ is a solution of the E-L equation, however, it makes null contributions to the Lagrange formalism because it does not allow obtaining the original equation $\hat{D}y(x) = 0$. This concludes the proof.

Important results that are consequences of Proposition 1 are now presented in the following three corollaries.

**Corollary 1.** The Lagrangian $L_s[y'(x), y(x), x]$ given by Eq. (1) is the simplest standard Lagrangian for all ODEs of $Q_{sf}$ and $Q_{Ba}$.

**Corollary 2.** The null-Lagrangian $L_o[y'(x), y(x), x]$ given by Eq. (4) can be added to any known Lagrangian without making any changes in the resulting original equation.

**Corollary 3.** By being the solutions of the E-L equation, $L_o[y'(x), y(x), x]$ obtained for different $B(x)$ form a new family of null-Lagrangians.

Our main result here is that the null-Lagrangian, $L_o[y'(x), y(x), x]$, cannot be used in the Lagrangian formalism because of its null effects on this formalism. In other words, the null-Lagrangian does not allow deriving any ODE for $y(x)$ because $L_o[y'(x), y(x), x]$ is a solution of the E-L equation for any differentiable $y(x)$. Nevertheless, the Lagrangian $L_o[y'(x), y(x), x]$ can be determined for any ODE of $Q_{sf}$ or $Q_{Ba}$.
2.3. **New non-standard Lagrangians**

Having established the Lagrangian formalism based on the SLs, we now develop the Lagrangian formalism based on the NSLs. Our new results are presented by the following two propositions.

**Proposition 2.** Let $L_{ns}[y'(x), y(x), x] = 1/[f(x)y'(x) + g(x)y(x)]$ be a non-standard Lagrangian with $f(x)$ and $g(x)$ being ordinary and smooth functions to be determined. The Lagrange formalism can be established for any ODE of $Q_{sf}$ and $Q_{Ba}$ if, and only if, the non-standard Lagrangian takes the following form

$$L_{ns}[y'(x), y(x), x] = H[y'(x), y(x), x] E_{ns}(x) ,$$

where

$$H[y'(x), y(x), x] = \frac{1}{[y'(x)\bar{v}(x) - y(x)\bar{v}'(x)] \bar{v}'(x)}$$

and $E_{ns}(x) = \exp\left[-2 \int B(\bar{x}) d\bar{x}\right]$, with the necessary auxiliary condition $\hat{D}\bar{v}(x) = 0$.

**Proof.** Substituting $L_{ns}[y'(x), y(x), x] = 1/[f(x)y'(x) + g(x)y(x)]$ into the Euler-Lagrange equations, we obtain

$$y'' + \frac{1}{2} \left( \frac{f'}{f} + \frac{3g}{f} \right) y' + \left( \frac{g'}{f} - \frac{f'g}{2f^2} + \frac{g^2}{2f^2} \right) y = 0 .$$

By comparing this equation to $\hat{D}y(x) = 0$, the following two conditions that allow finding $f(x)$ and $g(x)$ are obtained

$$\frac{1}{2} \frac{f'}{f} + \frac{3g}{2f} = B(x) ,$$

and

$$\frac{g'}{f} - \frac{1}{2} \frac{f'g}{f^2} + \frac{1}{2} \frac{g^2}{f^2} = C(x) ,$$

with $f(x) \neq 0$. Combining Eqs (12) and (13), and introducing $u(x) = f'(x)/f(x)$, the following Riccati equation for $u(x)$ is obtained

$$u' + \frac{1}{3} u^2 - \frac{1}{3} uB(x) - \left[ \frac{2}{3} B^2(x) + 2B'(x) - 3C(x) \right] = 0 .$$
With \( f(x) = \exp \left[ \int x u(\bar{x}) \, d\bar{x} \right] \) and \( g(x) = 2 [B(x) - u(x)/2] f(x)/3 \), it is seen that finding \( u(x) \), which satisfies the Ricatti equation, allows us to determine the functions \( f(x) \) and \( g(x) \).

We now transform Eq. (14) by introducing a new variable \( v(x) \), which is related to \( u(x) \) by \( u(x) = 3v'(x)/v(x) \) with \( v(x) \neq 0 \), and obtain

\[
v'' + B(x)v' + C(x)v = F(v', v, x),
\]

where \( F(v', v, x) = 2 \left[ -2B(x)v' + B'(x)v + B^2(x)v/3 \right]/3 \).

The homogeneous equation of Eq. (15) is \( \hat{D}\bar{v} = 0 \), or more explicitly

\[
\bar{v}''(x) + B(x)\bar{v}'(x) + C(x)\bar{v}(x) = 0.
\]

Let us now transform Eq. (15) by using

\[
v(x) = \bar{v}(x) \exp \left[ \int x \chi(\bar{x}) \, d\bar{x} \right],
\]

and obtain \( \chi(x) = 2B(x)/3 \), which allows writing the solutions for \( u(x) \) in the following form

\[
u(x) = 3\frac{\bar{v}'(x)}{\bar{v}(x)} + 2B(x).
\]

It is easy to verify that Eq. (18) is the solution of the Riccati equation given by Eq. (14), if Eq. (16) is taken into account. Having obtained \( u(x) \), the functions \( f(x) \) and \( g(x) \) can be calculated

\[
f(x) = \bar{v}^3(x) \exp \left[ 2 \int x B(x) \, d\bar{x} \right],
\]

and

\[
g(x) = -\frac{\bar{v}'(x)}{\bar{v}(x)} \bar{v}^3(x) \exp \left[ 2 \int x B(x) \, d\bar{x} \right],
\]

and the resulting non-standard Lagrangian is given by

\[
L_{ns}[y'(x), y(x), x] = H[y'(x), y(x), x] E_{ns}(x)
\]

where

\[
H[y'(x), y(x), x] = \frac{1}{[y'(x)\bar{v}(x) - y(x)\bar{v}'(x)] \bar{v}^2(x)},
\]
which is the same as that given by Eqs (11) and (12). Since the derived non-standard Lagrangian depends explicitly on $\bar{v}(x)$, the auxiliary condition $\bar{D}\bar{v}(x) = 0$ (see Eq. 16) must supplement $L_{ns}[y'(x), y(x), x]$. This concludes the proof.

Having derived the NSLs for the ODEs of $Q_{sf}$ and $Q_{Ba}$, we must now verify that the original ODEs can be obtained from the non-standard Lagrangian and its auxiliary condition. The following proposition and corollaries present our results.

**Proposition 3.** The Lagrange formalism for the ODEs of $Q_{sf}$ and $Q_{Ba}$ is established if, and only if, $L_{ns}[y'(x), y(x), x]$ of Proposition 2 is used together with the auxiliary condition $\bar{D}\bar{v}(x) = 0$.

**Proof.** Substituting $L_{ns}[y'(x), y(x), x]$ given by Eq. (9) into the E-L equation, we obtain
\begin{equation}
\frac{y''(x)\bar{v}(x) - y(x)\bar{v}''(x)}{y'(x)\bar{v}(x) - y(x)\bar{v}'(x)} + B(x) = 0 ,
\end{equation}
which can also be written as
\begin{equation}
[y''(x) + B(x)y'(x)]\bar{v}(x) = [\bar{v}''(x) + B(x)\bar{v}'(x)]y(x) .
\end{equation}
This is the result of substituting $L_{ns}[y'(x), y(x), x]$ obtained in Proposition 1 into the E-L equation, and it is seen that Eq. (24) is not the same as the original equation $\bar{D}y(x) = 0$. In order to derive the original equation, we must now use the auxiliary condition (see Eq. 16) that gives
\begin{equation}
\bar{v}''(x) + B(x)\bar{v}'(x) = -C(x)\bar{v}(x) .
\end{equation}
This shows that by applying the auxiliary condition $\bar{D}\bar{v}(x) = 0$, the E-L equation gives the original equation $\bar{D}y(x) = 0$, which concludes the proof.

**Corollary 4.** The Lagrangian $L_{ns}[y'(x), y(x), x]$ given by Eq. (9) is the non-standard Lagrangian for all ODEs of $Q_{sf}$ and $Q_{Ba}$.

**Corollary 5.** All non-standard Lagrangians $L_{ns}[y'(x), y(x), x]$ given by Eq. (9) form a new and separate family among all known non-standard Lagrangians.
Since in the case considered in this paper, the solutions to all ODEs of $Q_{sf}$ and $Q_{Ba}$ are known, then the solutions to the auxiliary condition $\hat{D}\bar{v}(x) = 0$ are also known. The implications of this are shown by the following corollary.

**Corollary 6.** Let $y(x) = c_1y_1(x) + c_2y_2(x)$ and $\bar{v}(x) = \bar{c}_1y_1(x) + \bar{c}_2y_2(x)$ be the superpositions of linearly independent solutions of $\hat{D}y(x) = 0$ and $\hat{D}\bar{v}(x) = 0$, respectively, with $c_1, c_2, \bar{c}_1$ and $\bar{c}_2$ being the integration constants. Then, the function $H[y'(x), y(x), x]$ of Proposition 2 becomes

$$H[y'_1(x), y'_2(x), y_1(x), y_2(x), x]$$

$$= (c_1\bar{c}_2 - c_1c_2)^{-1} [y'_1(x)y_2(x) - y_1(x)y'_2(x)]^{-1} [\bar{c}_1y_1(x) + \bar{c}_2y_2(x)]^{-2}, \quad (26)$$

where $c_1\bar{c}_2 \neq c_1c_2$, the term $[y'_1(x)y_2(x) - y_1(x)y'_2(x)]$ is the non-zero Wronskian, and the term $[\bar{c}_1y_1(x) + \bar{c}_2y_2(x)]^2$ is also non-zero for both oscillatory and non-oscillatory solutions.

Let us point out that the solutions $y_1(x)$ and $y_2(x)$ are the same for both $y(x)$ and $\bar{v}(x)$, which means that their dependence on $x$ is identical. However, the integration constants are different because $y(x)$ and $\bar{v}(x)$ are not the same variables and therefore they must obey different boundary conditions.

The requirement of Corollary 6 that $[\bar{c}_1y_1(x) + \bar{c}_2y_2(x)]^2 \neq 0$ needs additional explanation. Clearly, the requirement is non-zero when both $y_1(x)$ and $y_2(x)$ are non-oscillatory. Moreover, if both $y_1(x)$ and $y_2(x)$ are oscillatory, then the requirement still remains non-zero because the locations of the zeros of the two linearly independent solutions given by the SFs are never the same. The exception could be the point $x = 0$, thus, we consider $x \in (0, \infty)$ in our applications (see Section 3).

### 2.4. Helmholtz conditions and their validity

The existence of Lagrangians is guaranteed by the Helmholtz conditions (HCs), which are necessary and sufficient conditions [10,27]. Let $F(y'', y', y, x) = 0$ be a second-order ODE, and let $F(y'', y', y, x) \equiv \hat{D}y(x)$. Among the three HCs, the first and second conditions are trivially satisfied for all ODEs of $Q_{sf}$ and $Q_{Ba}$. However, the third HC given by Eq. (27) is not satisfied because the LHS $\neq$ RHS in this equation. The explicit form of this equation is

$$\frac{\partial F}{\partial y'} = \frac{d}{dx} \left( \frac{\partial F}{\partial y''} \right) \quad (27)$$
and it is seen that the LHS = \( B(x) \) and the RHS = 0. Since the third HC fails to be valid, the HCs imply that no Lagrangian can be constructed for any ODEs of \( Q_{sf} \) and \( Q_{Ba} \) with \( B(x) \neq 0 \). Despite this strong negative statement, the Lagrangians obtained in this paper (as well as in some previous papers, notable in [5]) seem to contradict the HCs requirement.

To explain this contradiction, we substitute \( L_s \) and \( L_{so} \) (see Eqs [1] and [4]) into the E-L equation, and obtain

\[
[y'' + B(x)y' + C(x)y] E_s(x) = 0,
\]

and it is easy to verify that this equation satisfies the third HC, and therefore the existence of \( L_s \) and \( L_{so} \) is justified by the HCs. Thus, there is no contradiction.

However, when the original equation \( \dot{y}(x) = 0 \) is obtained from Eq. (28), typically this is done by dividing the equation by \( E_s(x) \) with \( E_s(x) > 0 \), then the resulting original ODE violates the third HC as already shown above. The credit for discovering and explaining this phenomenon goes back to Bateman [5].

Surprisingly, for the non-standard Lagrangians the results are different than those obtained above for the standard Lagrangians. This can be shown by substituting \( L_{ns} \) given by Eqs (9) and (10) into the E-L equation, and find

\[
[y'' + B(x)y' + C(x)y] \bar{v}(x) E_{ns}(x) = 0,
\]

where \( \bar{v}(x) \) is a solution to the auxiliary condition, which is introduced in Proposition 2 together with \( E_{ns}(x) \). Clearly, this equation does not satisfy the third HC because of the presence of \( E_{ns}(x) \).

Nevertheless, taking \( \bar{v}(x) = \bar{v}_o [E_s(x)]^3 \), where \( \bar{v}_o \) is an integration constant, allows making Eq. (29) to be the same as Eq. (28). This is important because Eq. (28) already satisfies the third HC. The only problem that remains to be resolved is whether the imposed solution on \( \bar{v}(x) \) is also a solution of the auxiliary condition. In general, this will not be the case, however, the ODEs whose coefficients \( B(x) \) and \( C(x) \) are related by \( B'(x) + 4B^2(x) = -C(x)/3 \) will be the exception; for the Bateman oscillators, the relationship becomes \( 4b^2(x) = -c/3 \), where \( B(x) = b = \text{const} \) and \( C(x) = c = \text{const} \). Thus, the presented results show that there are only small subsets of ODEs in \( Q_{sf} \) and \( Q_{Ba} \) for which the existence of \( L_{ns} \) can be justified by the HCs. A new phenomenon is that the remaining ODEs have their non-standard Lagrangians despite the fact that the resulting ODEs violate
2.5. Implications of our results on calculus of variations

We have formally established the Lagrangian formalism for all ODEs of \( Q_{sf} \) and \( Q_{Ba} \), and demonstrated that this can be achieved by using either standard or non-standard Lagrangians. Our results have profound implications on the calculus of variations.

First, we have shown that there is a special family of null-Lagrangians that fully solves the E-L equation but cannot be used to derive the original ODEs. This problem is known in the calculus of variations and when it appears for certain dynamical systems it is suggested to evaluate the action for null-Lagrangian. Moreover, our results indicate that the set of \( Q_{sf} \) and the set of \( Q_{Ba} \) are unique as the Lagrangian formalism can be established by using either the SLs or the NSLs, and, in addition, each ODE has also its null-Lagrangian.

Second, in order to obtain the NSLs for the ODEs of \( Q_{sf} \) and \( Q_{Ba} \), we had to solve the non-linear Riccati equation (see Eq. 14) whose solutions led to the auxiliary condition that must be considered as the amendment to the E-L equation. The existence of the auxiliary condition in the Lagrangian formalism based on the NSLs is a new phenomenon in the calculus of variations, which has a profound implications on the Helmholtz conditions and their validity (see Section 2.4).

Third, the obtained NSLs for the ODEs of \( Q_{sf} \) and \( Q_{Ba} \) depend explicitly on the SFs. This is again a new phenomenon in the calculus of variations. It is also interesting that this dependence on the SFs is removed by the auxiliary conditions once the original ODE is derived.

Fourth, the dependence of the SLs \( (L_s) \) and the NSLs \( (L_{ns}) \) on \( B(x) \) and \( C(x) \) is significantly different. Moreover, the form of SLs changes for different ODEs, however, the basic form NSLs remains practically the same for all ODEs of \( Q_{sf} \) and \( Q_{Ba} \). This is a novel property of the NSLs derived here, which has not yet been observed in other NSLs previously obtained.

Finally, let us point out that our results clearly demonstrate that for non-standard Lagrangians there are only small subsets of ODEs of \( Q_{sf} \) and \( Q_{Ba} \) with \( B(x) \neq 0 \), for which the Helmholtz conditions are satisfied. However, other ODEs of \( Q_{sf} \) and \( Q_{Ba} \) have non-standard Lagrangians but do not obey the Helmholtz conditions. The effects of this novel discovery may become important in establishing the Lagrangian formalism based on non-standard
Table 1: Values of the constants $\alpha$, $\beta$, $\gamma$ and $\mu$ in Eq. (30) corresponding to the four types of Bessel equations.

| Bessel Equations | $\alpha$ | $\beta$ | $\gamma$ | $\mu$          |
|------------------|--------|--------|--------|---------------|
| Regular          | 1      | 1      | -1     | real or integer |
| Modified         | 1      | -1     | 1      | real or integer |
| Spherical        | 2      | 1      | -1     | $\mu^2 = l(l + 1)$ |
| Modified spherical| 2     | -1     | 1      | $\mu^2 = l(l + 1)$ |

3. Applications to selected equations

3.1. Bessel equations

Let us consider a general form of Bessel equations and write $\hat{D}y(x) = 0$ as

$$y''(x) + \frac{\alpha}{x} y'(x) + \beta \left( 1 + \gamma \frac{\mu^2}{x^2} \right) y(x) = 0$$

where $B(x) = \alpha/x$ and $C(x) = \beta(1 + \gamma \mu^2 / x^2)$. In addition, $\alpha$, $\beta$, $\gamma$ and $\mu$ are constants and their specific values determine the four different types of Bessel equations (see Table 1).

All Bessel equations have the same singularities at $x = 0$ (regular) and $x = \infty$ (irregular), thus, $x \in (0, \infty)$. The solutions to the Bessel equations are given as the power series expansions around the regular singular point $x = 0$. The obtained solutions to different Bessel equations are summarized in Table 2. All solutions are converging when $x \to \infty$, however, only some are finite in the entire range $x \in (0, \infty)$ but others become infinite when $x \to 0$ [3].

Using the results of Propositions 1, 2 and 3, we find the standard and non-standard Lagrangians for the Bessel equations are

$$L_s[y'(x), y(x), x] = \frac{1}{2} \left[ (y'(x))^2 - \beta \left( 1 + \gamma \frac{\mu^2}{x^2} \right) y^2(x) \right] x^{\alpha}$$

$$L_{ns}[y'(x), y(x), x] = H[y'(x), y(x), x] x^{-2\alpha}$$
where $H[y'(x), y(x), x]$ is given by Eq. (10), and the Lagrangian $L_o$ is defined as

$$L_o[y'(x), y(x), x] = \frac{\alpha}{2} \left[ y'(x) + \frac{(\alpha - 1)}{2x} y(x) \right] y(x) x^{\alpha - 1},$$

with $\alpha$ being either 1 or 2. According to Table 1, $\alpha = 1$ corresponds to regular and modified Bessel equations for which $L_o[y'(x), y(x), x] = y(x) y'(x) / 2$, however, for spherical and modified spherical Bessel equations $\alpha = 2$ and $L_o[y'(x), y(x), x] = y(x) y'(x) x + y^2(x) / 2$. It is easy to verify that both Lagrangians $L_o$ are the solutions to the E-L equation.

The auxiliary condition that must supplement $L_{ns}[y'(x), y(x), x]$ is given by

$$\ddot{\bar{v}}(x) + \alpha \frac{\ddot{v}}{x} = -\beta \left( 1 + \gamma \frac{\mu^2}{x^2} \right) \bar{v}(x),$$

and this condition is required in order to derive the original Bessel equations from the E-L equation (see Proposition 3).

The solutions presented in Table 2 are the two linearly independent solutions for the Bessel equations given as the special functions of mathematical physics. The notation used for these solutions is standard (e.g., [1-3]), which means that $J_\mu(x)$ is the Bessel function of the first kind, $J_{-\mu}(x)$ is the independent second solution, $Y_\mu(x)$ is the Bessel function of the second kind or the Neumann function, $I_\mu(x)$ is the modified Bessel function of the first kind, $I_{-\mu}(x)$ is the independent second solution, and $K_\mu(x)$ is the modified Bessel function of the second kind or the modified Neumann function. In addition, $j_l(x)$ and $y_l(x)$ are the spherical Bessel functions, and $i_l(x)$ and $k_l(x)$ are the modified spherical Bessel functions. According to the results of Corollary 6, the derived NSLs can be expressed in terms of the pairs of these solutions, and such explicit dependence of the NSLs on the solutions of the original ODEs is a new phenomenon in the calculus of variations.

Among the solutions listed in Table 2, the SFs $I_\mu(x)$, $I_{-\mu}(x)$, $K_\mu(x)$, $i_l(x)$ and $k_l(x)$ are non-oscillatory, however, all other special functions listed above are oscillatory. The superpositions of the solutions $y(x)$ and $\ddot{v}(x)$ for each Bessel equation do not lead to any discontinuity in the above SLs and NSLs. However, it must be noted that some of the solutions given in Table 2 become zero at $x = 0$ but this also does not result in discontinuities for the derived NSL because $x \in (0, \infty)$. 

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Table 2: The linearly independent solutions of the four Bessel equations (see Table 1). The standard notation commonly adopted in textbooks and monographs of mathematical physics is used for these solutions.

| Bessel Equations | Solutions | \( \mu \) or \( l \) |
|------------------|-----------|-----------------|
| Regular          | \( J_\mu(x) \), \( J_{-\mu}(x) \) | real           |
| Regular          | \( J_\mu(x) \), \( Y_\mu(x) \) | integer        |
| Modified         | \( I_\mu(x) \), \( I_{-\mu}(x) \) | real           |
| Modified         | \( I_\mu(x) \), \( K_\mu(x) \) | integer        |
| Spherical        | \( j_l(x) \), \( y_l(x) \) | integer        |
| Modified Spherical | \( i_l(x) \), \( k_l(x) \) | integer        |

3.2. Legendre equations

There are the regular and associated Legendre equations, and the latter can be written as

\[
y''(x) - \frac{2x}{(1-x^2)}y'(x) + \left[ \frac{l(l+1)}{(1-x^2)} - \frac{m^2}{(1-x^2)^2} \right] y(x) = 0 ,
\]

where \( l \) and \( m \) are constants, and when \( m = 0 \) the above equation becomes the regular Legendre equation (see Table 3). Comparing the above Legendre equation to \( \hat{D}y(x) = 0 \), we obtain \( B(x) = -2x/(1-x^2) \) and \( C(x) = l(l+1)/(1-x^2) - m^2/(1-x^2)^2 \).

For the regular Legendre equation, the power series solutions are calculated either at one of the regular singular points \( x = \pm 1 \) [1-3]. The two linearly independent solutions are the Legendre functions of the first kind or the Legendre polynomials \( P_l(x) \), which are oscillatory within \( x \in (-1,+1) \), and the Legendre functions of the second kind \( Q_l(x) \) that are singular at \( x = \pm 1 \) (see Table 3); note that \( Q_l(x) \) can be expressed in terms of \( P_l(x) \), nevertheless, the solutions remain linearly independent\(^1\). The power series solutions calculated at one of the regular singular point diverge at \( x = \pm 1 \) unless \( l \) is chosen to be an integer, which terminates the series and finite Legendre polynomials of order \( l \) are obtained [3]; therefore, in physical applications \( l \) is a positive integer.

We now present the SL only for the associated Legendre equations because the SL corresponding to the regular Legendre equation is obtained by taking \( m = 0 \). However, the NSL and \( L_o \) have the same forms for both regular and associated Legendre equations. Following Propositions 1 and 2, we find the
Table 3: The linearly independent solutions of the regular and associated Legendre equations. The standard notation commonly adopted in textbooks and monographs of mathematical physics is used for these solutions.

| Legendre Equations | Solutions | m |
|--------------------|-----------|---|
| Regular            | $P_l(x)$, $Q_l(x)$ | 0 |
| Associated         | $P_l^m(x)$, $Q_l^m(x)$ | -1, 0, ..., l |

following Lagrangians

$$L_s[y'(x), y(x), x] = \frac{1}{2} [y'(x)]^2 (1 - x^2) \quad - \left[ \frac{l(l+1)}{(1-x^2)} - \frac{m^2}{(1-x^2)^2} \right] y^2(x) (1 - x^2),$$  \hspace{1cm} (36)

$$L_{ns}[y'(x), y(x), x] = H[y'(x), y(x), x] \frac{1-x^2}{2},$$  \hspace{1cm} (37)

where $H[y'(x), y(x), x]$ is given by Eq. (10), and

$$L_o[y'(x), y(x), x] = - \left[ x y'(x) + \frac{1}{2} y(x) \right] y(x).$$  \hspace{1cm} (38)

It is interesting to note that the Lagrangians $L_o$ for the spherical Bessel and Legendre equations have $L_o[y'(x), y(x), x]$ that differ only by the sign.

Proposition 3 shows that substitution of $L_{ns}[y'(x), y(x), x]$ into the E-L equation does not result in the original Legendre equations unless the auxiliary condition

$$\tilde{v}''(x) - \left[ \frac{2x}{(1-x^2)} \right] \tilde{v}'(x) = - \left[ \frac{l(l+1)}{(1-x^2)} - \frac{m^2}{(1-x^2)^2} \right] \tilde{v}(x),$$  \hspace{1cm} (39)

is taken into account.

There are two linearly independent solutions of the associated Legendre equation, the associated Legendre functions of the first kind $P_l^m(x)$, which are related to $P_l(x)$, and the associated Legendre functions of the second kind $Q_l^m(x)$, which are related to $Q_l(x)$; in the case when $l$ is an integer, $P_l^m(x)$ is called the associated Legendre polynomials. The well-known property of the Legendre functions is the fact that all $P_l^m(x)$ with $m > 0$ can be generated from $P_l(x)$, which can be built recursively from $P_0(x) = 1$ and $P_1(x) = x$; the
same is true for \( Q_l^m(x) \). It must be pointed out that \( P_l^m(x) \) are oscillatory within \( x \in (-1, +1) \) and \( Q_l^m(x) \) are non-oscillatory. These properties of the solutions are important for expressing the SL, NSL and \( L_o \) in terms of the superpositions of linearly independent solutions \( y_1(x) \) and \( y_2(x) \) as shown by Corollary 6.

3.3. Hermite equation

The Hermite equation can be written as

\[
y''(x) - xy'(x) + ny(x) = 0,
\]

where \( n \) is any integer. Comparing this equation to \( \hat{D}y(x) = 0 \), we find \( B(x) = -x \) and \( C(x) = n \). The range of validity of the Hermite equation is \( x \in (0, \infty) \).

The Hermite equation has only one singular point at infinity and this point is the irregular singular point. Despite being irregular, the power series solutions can still be obtained about this point because of its location at the end of the interval for \( x \). Another possibility is to construct the power series solutions about any other point, say \( x = 0 \), which would be an ordinary point. The two power series solutions obtained around the irregular singular point located at \( x = \infty \) give two linearly independent solutions: the Hermite functions of the first kind or the Hermite polynomials \( H_n(x) \) if, and only if, \( n \geq 0 \) is an integer, and the Hermite functions of the second type \( h_n(x) \).

The explicit forms of the standard, non-standard and null-Lagrangian for the Hermite equation are

\[
L_s[y'(x), y(x), x] = \frac{1}{2} \left[ (y'(x))^2 - ny^2(x) \right] e^{-x^2/2},
\]
\[
L_{ns}[y'(x), y(x), x] = H[y'(x), y(x), x] e^{x^2},
\]
where \( H[y'(x), y(x), x] \) is given by Eq. (10), and

\[
L_o[y'(x), y(x), x] = -\frac{1}{2} \left[ xy'(x) + \frac{1}{2}(1 - x^2)y(x) \right] y(x) e^{-x^2/2}.
\]

It is seen that the terms \( xy(x)y'(x) + y^2(x)/2 \) that appeared in the special Lagrangians for the spherical Bessel and Legendre equations are also present in the special Lagrangians for the Hermite equation.
To obtain the original Hermite equation, the Lagrangian must be substituted into the E-L equation, and according to Proposition 3 the following auxiliary condition

$$\ddot{v}(x) + x \dot{v}(x) = -n \dot{v}(x), \quad (44)$$

must be used.

Note also that the superposition of the linearly independent solutions $y_1(x) = H_n(x)$ and $y_2(x) = h_n(x)$ can be used to replace $y(x)$ and $\dot{v}(x)$ in the derived NSL with the results presented in Corollary 6.

### 3.4. Bateman oscillators

Damped harmonic oscillators are called here the Bateman oscillators (BOs), and they are described by the following ODE [5]

$$y''(x) + b y'(x) + c y(x) = 0, \quad (45)$$

where $b = \text{const}$ and $c = \text{const}$. The results presented in Propositions 1, 2 and 3 are directly applicable to the BOs, thus, the standard, non-standard and null Lagrangians, $L_s$ and $L_{ns}$ and $L_o$, respectively, are given by

$$L_s[y'(x), y(x), x] = \frac{1}{2} \left[ (y'(x))^2 - cy^2(x) \right] e^{bx}, \quad (46)$$

$$L_{ns}[y'(x), y(x), x] = H[y'(x), y(x), x] e^{-2bx}, \quad (47)$$

where $H[y'(x), y(x), x]$ is given by Eq. (10), and the Lagrangian $L_o$ is defined as

$$L_o[y'(x), y(x), x] = \frac{1}{2} \left[ by'(x)y(x) + \frac{1}{4} b^2 y^2(x) \right] e^{bx}. \quad (48)$$

The auxiliary condition that must supplement $L_{ns}[y'(x), y(x), x]$ is given by

$$\ddot{v}(x) + \dot{v}(x) = -c \dot{v}(x), \quad (49)$$

and this condition is required in order to derive the original ODE that describes the BOs from the E-L equation (see Proposition 3).
3.5. Discussion of applications

Having obtained the standard and non-standard Lagrangians for the Bessel, Legendre and Hermite equations as well as for ODEs describing the Bateman oscillators, is equivalent of showing that the equations can be derived from the Lagrangian formalism based on these Lagrangians. Similarly, we may find the SLs and NSLs for all other ODEs of $Q_{sf}$ and $Q_{Ba}$, and establish the Lagrangian formalism for all considered ODEs. The obtained results are important in theoretical physics, since their main equations are typically derived from given Lagrangians. Our results may also be useful in applied mathematics and engineering where the ODEs $Q_{sf}$ and $Q_{Ba}$ are commonly used.

There are advantages to having the Lagrange formalism for the ODEs of $Q_{sf}$ and $Q_{Ba}$, and they include using the derived Lagrangians to study the group structure underlying the SLs and NSLs, and their symmetries (possibly some new ones), as well as finding relationships between the Lagrange formalism and the Lie group approach (e.g., [1,7]), which introduces the special functions by using irreducible representations of some Lie groups; these topics are out the scope of this paper but they will be investigated in the future.

We also derived the null-Lagrangians for the three selected ODEs and for the Bateman oscillators, and demonstrated that these Lagrangians depend the function $B(x)$ but are independent of $C(x)$. Our results can be easily applied to all ODEs of $Q_{sf}$ and $Q_{Ba}$, and the corresponding $L_0[y'(x), y(x), x]$ can be derived. However, since the null-Lagrangians depend on $B(x)$, they become non-zero only for the ODEs with $B(x) \neq 0$. Other special cases of $L_0[y'(x), y(x), x] = 0$ are also possible, and we now list the required conditions: (i) $y(x) = 0$ and $B(x) = 0$; (ii) $y(x) = 0$ but $B(x) \neq 0$; (iii) $y(x) \neq 0$ but $B(x) = 0$; and $G_{SS}[y(x), y(x), x] = 0$, which is only satisfied when $y(x)$ and $B(x)$ are related by $y(x) = C_0 \exp\left[-\left(\int x B(\tilde{x}) \, d\tilde{x}\right)/2\right]/\sqrt{B(x)}$, and with $C_0$ being the integration constant. The existence of null-Lagrangians for the ODEs of $Q_{sf}$ and $Q_{Ba}$ is an interesting result, which shows that there is a family of Lagrangians that gives null contributions to the calculus of variations but on the other hand, fully solves the E-L equation. The meaning of these null-Lagrangian solutions and their possible applications will be subjects of future explorations.
4. Conclusions

We considered linear second-order ODEs with non-constant and constant coefficients that are commonly used in applied mathematics, physics and engineering. We selected the ODEs with non-constant coefficients whose solutions are the special functions of mathematical physics, and denoted them as $Q_{sf}$. The ODEs with constant coefficients describe the Bateman oscillators and they are denoted by $Q_{Ba}$. We established the Lagrange formalism for these equations. Since the original ODEs were known, our main objective was to derive the Lagrangians corresponding to these equations. This required solving the inverse calculus of variations problem and we developed novel methods for solving it. These methods allowed for the derivation of standard and non-standard Lagrangians, and also the null-Lagrangians that make null contributions to the Lagrangian formalism but become full solutions of the Euler-Lagrange equation. The derived non-standard Lagrangians form a new family and they show an explicit dependence on the special function solutions. This is clearly an emergent phenomenon in the calculus of variations.

The dependence of non-standard Lagrangians on the special function solutions requires that the E-L equation is amended by the auxiliary condition, which is again a new phenomenon in the calculus of variations. To present the effects of these new phenomena, we considered the Helmholtz conditions and investigated their validity. The obtained results clearly showed that there ODEs in both sets $Q_{sf}$ and $Q_{Ba}$ for which the non-standard Lagrangians can be found despite the fact that the Helmholtz conditions are violated. This result may have profound implications on development of the Lagrangian formalism based on non-standard Lagrangians.

We considered specific applications of our results to the Bessel, Legendre and Hermite equations, and also to ODEs describing the Bateman oscillators. In these applications, we constructed the standard, non-standard and null-Lagrangians for each one of these equations, and discussed the similarities and differences between the resulting Lagrangians. The presented results demonstrate that the Lagrangian formalism is well-established for all special function equations, which means that there is a powerful and robust method of obtaining all ODEs of mathematical physics. Moreover, the presented Lagrangian formalism based on the non-standard Lagrangians for the Bateman oscillators significantly extends his original method [5].

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