Experimental Design under Network Interference *

Davide Viviano †

First Version: March, 2020
This version: July, 2022

Abstract

This paper studies the design of two-wave experiments in the presence of spillover effects when the researcher aims to conduct precise inference on treatment effects. We consider units connected through a single network, local dependence among individuals, and a general class of estimands encompassing average treatment and average spillover effects. We introduce a statistical framework for designing two-wave experiments with networks, where the researcher optimizes over participants and treatment assignments to minimize the variance of the estimators of interest, using a first-wave (pilot) experiment to estimate the variance. We derive guarantees for inference on treatment effects and regret guarantees on the variance obtained from the proposed design mechanism. Our results illustrate the existence of a trade-off in the choice of the pilot study and formally characterize the pilot’s size relative to the main experiment. Simulations using simulated and real-world networks illustrate the advantages of the method.

Keywords: Experimental Design, Spillovers, Two-wave experimentation, Causal Inference.

*Previous versions are available at https://arxiv.org/abs/2003.08421. I thank Jelena Bradic, Graham Elliott, James Fowler, Brian Karrer, Karthik Muralidharan, Yixiao Sun, and Kaspar Wüthrich for helpful comments and discussion. I acknowledge support of the Social Science Computing Facilities at UC San Diego, HPC@UC 2020 awards from the San Diego Super Computer Center. All mistakes are mine.

†Stanford Graduate School of Business. Correspondence: dviviano@stanford.edu.
1 Introduction

This paper studies the design of experiments for inference on treatment effects with network interference. Network interference induces (i) spillovers across units in the experiment and (ii) statistical dependence. Our goal is to obtain precise estimates of treatment and spillover effects. We consider a setting where individuals are connected in a single network, and interact locally (i.e., between neighbors) in the network.\(^1\) Differently from typical settings for clustered or saturation design experiments (e.g., Baird et al., 2018), no independent clusters are necessarily available. Instead, here researchers have access to a single network; they run a pilot study (*first-wave experiment*) to estimate the variance and covariances between units and to optimally select participants and treatments in the main experiment (*second-wave experiment*).\(^2\) Relevant applications include online and field experiments where network interference naturally occurs (e.g., Karrer et al., 2021; Muralidharan et al., 2017).

We consider a class of estimands of interest, which include as the main ones the (i) overall effect of treatment, the (ii) direct effect, the (iii) spillover effects and interactions of the latter two. For example, in the presence of a cash transfer program (Barrera-Osorio et al., 2011), we may be interested in the effect on recipients (i.e., direct effects), the effects on those non-recipients living close to the recipients (i.e., spillovers), and on the sum of these effects (i.e., overall effect). We consider a class of estimators linear in the observed outcomes.

We propose the following protocol: (1) researchers select a small sub-sample of individuals and conduct a pilot study; (2) using information from the pilot, they select *participants* and *treatment assignments* in the main experiment; (3) researchers collect information on the outcomes of the participating units. The selection of treatments and participants plays an important role in variance reduction: for example, selecting highly connected individuals with the same probability of peripherical nodes

---

\(^1\) This assumption is known as local interference (Manski, 2013), and it can be tested using, for instance, the framework in Athey et al. (2018). This is often assumed in practice (Egger et al., 2019; Dupas, 2014; Miguel and Kremer, 2004; Bhattacharya et al., 2013; Duflo et al., 2011) as well as in theoretical analysis (Forastiere et al., 2021; Leung, 2020; Sinclair et al., 2012).

\(^2\) Pilot studies are common practice, and some examples include Karlan and Appel (2018); Karlan and Zinman (2008); DellaVigna and Pope (2018).
may be suboptimal in the presence of positive correlations.

We contribute to the literature with the first statistical framework for the design and inference of a two-wave experiment (pilot and main experiment) under network interference. The selection of participants in the two (network) experiments is a contribution to the independent interest of this paper, which naturally arose as a fundamental step in the presence of networks.

Specifically, in the presence of interference, we show that the main experiment is unconfounded if individuals in the pilot and their neighbors are not participants in the main experiment, but not necessarily otherwise. This restriction on the choice of the main experiment induces the following trade-off: a larger pilot guarantees more precise variance estimators, which are useful to design a better second-stage experiment, but it imposes stricter restrictions on the design of the main experiment. We select participants in the pilot study by solving a variation of the minimum cut problem in the network to guarantee that the pilot is “well separated” from the main experiment. We then assign treatments and select participants in the main experiment to minimize the estimated variance of treatment effects estimators.

The trade-off in the pilot’s choice induces a novel characterization of the pilot size relative to the main experiment in our theoretical analysis. First, we characterize the rate of convergence of the difference between the variance of the two-wave experiment and the variance of the “oracle” experiments (regret). The oracle experiment selects participants and treatment assignments to minimize the true variance of the estimator without necessitating a pilot study and without restrictions on the main experiment’s participants. The regret converges to zero as a function of the inverse pilot’s size and the ratio of the pilot and the main experiment’s size. Such a result allows us to characterize the regret-minimizer pilot size as a function of the main experiment’s size. A key step in our proof consists in deriving lower bounds to the oracle solution under stricter constraints on the decision space of the experimenter.

The optimization problem naturally leads to arbitrary dependence among treatment assignments. Motivated by this consideration, we derive asymptotic properties of the estimator under the proposed design, conditional on the assignments.

Our mechanism imposes the following conditions: (i) interference, and depen-
dence is local and anonymous; (ii) effects may be heterogeneous in summary statistics of the network structure, such as the number of neighbors or centrality measures. These conditions encompass a large number of economic examples from the literature. We refer to our design as Experiment under Local Interference (ELI).

We conclude our discussion with a set of simulation results. We show that the proposed method significantly outperforms state-of-art competitors for estimating overall treatment effects as well as spillover and direct effects, especially in the presence of heteroskedastic variances and covariances. In the Appendix, we study extensions in the presence of partially observed networks, where only information on participants’ neighbors is available, but not necessarily the entire network.

This paper connects to the recent literature in statistics and econometrics, which studies design mechanisms in the presence of interference, but without using information from a pilot study for inference on treatment effects. Here, we show that the presence of the pilot study can be useful in improving precision. References include clustered experiments (Eckles et al., 2017; Taylor and Eckles, 2018; Ugander et al., 2013) and saturation design experiments (Baird et al., 2018; Basse and Feller, 2016; Pouget-Abadie, 2018), which often assume clustered observations. Additional references are Basse and Airoldi (2018b), who also assume Gaussian outcomes and lack of spillover effects; Wager and Xu (2021) who study sequential randomization for optimal pricing strategies under global interference, without discussing the problem of inference on treatment effects; Kang and Imbens (2016) who study encouragement designs, without focusing on the problem of variance-optimal design. Basse and Airoldi (2018a) discuss limitations of design-based causal inference under interference; Jagadeesan et al. (2020) and Sussman and Airoldi (2017) study the design of experiments for estimating direct treatment effects only, while this paper considers a more general class of estimands, which may include overall and spillover effects. Viviano (2020) studies inference on policies and welfare maximization in the presence of unobserved networks and clusters. None of these papers study variance-optimal

\footnote{Some examples include: spillovers in public policy programs (Muralidharan et al., 2017), cash transfer programs (Egger et al., 2019), health programs (Dupas, 2014), educational program (Duflo et al., 2011).}
designs with two-wave experiments.

We relate to a large literature on experimental design in the $i.i.d.$ setting for batch experiments, which can be divided into one-stage procedures (Harshaw et al., 2019; Kasy, 2016; Kallus, 2018; Barrios, 2014), and two-stage procedures (Bai, 2019; Tabord-Meehan, 2018). However, none of the above references study the problem under network interference. We more broadly connect to the literature on treatment effects under network interference, which include Aronow and Samii (2017), Hudgens and Halloran (2008), Forastiere et al. (2021), Manski (2013), Leung (2020), Vazquez-Bare (2017), Athey et al. (2018), Goldsmith-Pinkham and Imbens (2013), Sävje et al. (2021), Ogburn et al. (2017), Kitagawa and Wang (2021), Viviano (2019) among others. None of these references study the problem of experimental design.

The remainder of the paper is organized as follows: Section 2 introduces the problem; in Section 3, we discuss the design mechanism; in Section 4, we derive theoretical guarantees; Section 5 contains the numerical results and Section 6 concludes. The Appendix contains extensions and proofs.

2 Setup

In this section, we discuss the setup, model, and estimands.

2.1 Notation

We consider the following setting: $N$ units are connected by a binary symmetric adjacency matrix $A$, $A_{i,j} \in \{0, 1\}$, with $N_i = \{j : A_{i,j} = 1\}$ denoting the neighbors of individual $i$. The adjacency matrix is observed by the researcher. Throughout our discussion, we fix $A$ (i.e., $A$ is non-random), unless otherwise specified.\footnote{Fixing $A$ is equivalent to consider exposure mappings which are non-random conditional on the treatment assignments as in Aronow and Samii (2017).} The researcher conducts two experiments, a pilot and the main experiment. For each
unit $i \in \{1, \ldots, N\}$ we denote

$$R_i = 1 \{ i \text{ is in the main experiment} \}, \quad P_i = 1 \{ i \text{ is in the pilot experiment} \},$$

respectively the participation indicator variable in the main experiment, and in the pilot, with $\sum_{i=1}^{N} R_i = n \in (n_1, n_2), \sum_{i=1}^{N} P_i = m$, where $n_1, n_2$ denote the upper and lower bounds on the main experiment’s size. Each unit $i$ is associated with an outcome, pre-treatment observables, and binary assignment $(Y_i, T_i, D_i)$, respectively. Here, $T_i$ may depend on the network information (e.g., $T_i$ can be a function of $A$ and covariates such as $T_i = |\mathcal{N}_i|$), and is discrete for expositional convenience. The binary assignment is not reversible, i.e., whenever researchers assign $D_i = 1$ to pilot’s units, their treatment status cannot be changed in the main experiment. See Section 3 for a comprehensive discussion. We denote

$$\mathbf{R} = \{ R_i, i \in \{1, \ldots, N\} \}, \quad \mathbf{T}^{R} = \{ T_i : R_i = 1 \}, \quad \mathbf{D}^{R} = \{ D_i : R_i = 1 \text{ or } R_N_i = 1 \},$$

the vector of selection indicators of each individual, observable statistics of participants, and the vector of treatment assignments of all individuals and their neighbors, respectively. Similarly, $\mathbf{P}$ denote the selection indicators of individuals in the pilot, while $\mathbf{T} = (T_1, \cdots, T_N)$ denotes the covariates of all the individuals in the population, and similarly $\mathbf{D}$.

### 2.2 Model and Dependence

We let $Y_i(d)$ denote the potential outcome, as a function of the treatment assignment of all other units $d \in \{0, 1\}^N$, with $Y_i = Y_i(D)$.

**Assumption 2.1** (Potential outcomes).

$$Y_i(d) = r(d, \sum_{k \in \mathcal{N}_i} d_k, T_i, \varepsilon_i(d)) \mid A, \mathbf{T} \sim \mathcal{P}, \forall i \in \{1, \ldots, N\}, d \in \{0, 1\}^N, \quad (1)$$

where $r(\cdot)$ and $\mathcal{P}$ are potentially unknown, and $\varepsilon_i(d) = \varepsilon_i(d')$ for all $d, d' \in \{0, 1\}^N$.

The above model assumes that (i) individuals only depend on neighbors’ treatment assignments (and hence $\varepsilon_i(d)$ is a constant function in $d$);\(^5\) (ii) the network

\(^5\)Note that we can also allow $\varepsilon_i$ to depend on $T_i$, omitted for expositional convenience only.
affects the outcome variable through arbitrary observables $T_i$. The assumption does not impose restrictions on the distribution of $D$ which is left unspecified. Since $\varepsilon_i(d)$ is a constant function in $d$, throughout or discussion, we will write unobservables as $\varepsilon_i$, omitting its argument. Assumption 2.1 is consistent with local interference assumptions often documented in practice (e.g., Cai et al., 2015) where, for example, individuals depend on the share of treated friends, or the number of treated friends, in the spirit of Leung (2020). Local interference is testable (Athey et al., 2018).

Throughout the rest of our discussion, we denote
\[
E\left[ r\left(d, s, l, \varepsilon_i\right) \mid T_i = l \right] = m(d, s, l),
\tag{2}
\]
the potential outcomes’ conditional mean with $T_i = l$, fixing the individual and neighbors’ treatment assignments to be $(d, s)$.

**Example 2.1.** Sinclair et al. (2012) study spillover effects for political decisions within households. The authors propose a model of the form
\[
Y_i = \mu + \tau_1 D_i + \tau_2 \sum_{j \in N_i} D_j \geq 1 + \tau_3 \sum_{j \in N_i} D_j \geq |N_i|/2 + \tau_4 \sum_{j \in N_i} D_j = |N_i| + \varepsilon_i,
\tag{3}
\]
where $N_i$ denotes the element in the same household of individual $i$. The model satisfies Assumption 2.1, with $T_i = |N_i|$.

**Example 2.2.** Consider the following equation (similarly to Muralidharan et al., 2017)
\[
Y_i = \mu + \tau_1 D_i + \tau_2 \sum_{k \in N_i} D_k / |N_i| + \varepsilon_i.
\]
Then Assumption 2.1 holds with $T_i = |N_i|$.

We allow the observables $(T_i, D_i)$ to have arbitrary dependence. We instead impose restrictions on the dependence structure of unobservables $\varepsilon_i$. Namely, we assume that unobservables exhibit one-degree dependence for *expositional convenience only*. We leave extensions to arbitrary higher-order degree dependence to Appendix A.1.
Assumption 2.2 (One-degree dependence). Assume that for all $i \in \{1, \ldots, N\}$,

$$\{\varepsilon_i, \{\varepsilon_k\}_{k \notin \mathcal{N}_j, j \in \mathcal{N}_i}\} \perp \{\varepsilon_j\}_{j \notin \mathcal{N}_i} \mid \mathbf{A}, \mathbf{T}$$

$$(\varepsilon_i, \varepsilon_j) =_{d} (\varepsilon_{i'}, \varepsilon_{j'}) \mid \mathbf{A}, \mathbf{T} \text{ for all } (i, j, i', j') : i \in \mathcal{N}_j, i' \in \mathcal{N}_{j'}, T_i = T_{i'}, T_j = T_{j'}.$$

Assumption 2.2 states that unobservables of non-adjacent neighbors are mutually independent. Individuals are dependent if $(i, j)$ are neighbors, and allows the joint distribution of unobservables to depend on observables. Note that such an assumption may be implausible in applications where higher order dependence are more plausible (e.g., two-degree dependence as in Leung, 2020). This is allowed in our framework, as discussed below.

Remark 1 (Higher order dependence). Extensions to higher order dependence of degree $M$, reads as follows

$$\{\varepsilon_i, \{\varepsilon_k\}_{k \notin \cup_{u=1}^{M} \mathcal{N}_j, j \in \cup_{u=1}^{M} \mathcal{N}_i}\} \perp \{\varepsilon_j\}_{j \notin \cup_{u=1}^{M} \mathcal{N}_i} \mid \mathbf{A}, \mathbf{T}, \quad (4)$$

where $\mathcal{N}_i^u$ denotes the set of neighbors of degree $u$. In this case, unobservables that are not adjacent by at least $M$ edges are independent. Our results extend to this setting, as discussed in Appendix A.1.

2.3 Class of estimands

We consider a general class of estimands of interest which may capture the direct, spillover, or overall effects of the treatments. Such class consists of arbitrary weighted averages of the conditional expectations functions.

Definition 2.1 (Estimands). For some known weights $v(l)$, consider a class of estimands

$$\mathcal{C} = \left\{ \tau_v(d, s, d', s') \text{ for some } d, d' \in \{0, 1\}, s, s' \in \mathbb{Z}, v : \mathbb{Z} \mapsto [0, 1] \right\},$$

where $\tau_v(d, s, d', s') = \sum_{l \in T} v(l) \left[ m(d, s, l) - m(d', s', l) \right]$. 

8
Definition 2.1 defines a class of estimands of interest (which may contain more than one estimand). These estimands define the difference of potential outcomes’ conditional expectations, averaged by some weights \( v(l) \). Here, potential outcomes are exposed to individual treatments \((d, d')\) and neighbors’ treatments \((s, s')\). We average over the heterogeneity \( T_i = l \) using some arbitrary weights \( v(l) \), which are known to the researcher. We provide two examples below.

**Example 2.3** (Direct and Overall Treatment Effect). Let \( T_i = |N_i| \) denote the number of neighbors. Then \[ \left[ m(1, l, l) - m(0, 0, l), m(1, 0, l) - m(1, 0, l) \right] \] define the effect of treating all individuals compared to the one of treating none, and the direct effect of treating one single unit for those individuals with \( l \) many friends, respectively.

By letting \( v(l) = \sum_{i=1}^{N} 1\{|N_i| = l\}/N, \)

\[
\sum_{v=0}^{\infty} v(l) \left[ m(1, l, l) - m(0, 0, l) \right], \quad \sum_{v=0}^{\infty} v(l) \left[ m(1, 0, l) - m(0, 0, l) \right]
\]
denote their expected value, averaged over the number of friends. □

In the following condition, we impose a restriction on the estimands and assume that \( \tau_v(d, s, d', s') \) takes a linear representation in the outcomes’ conditional expectations. This is formalized below.

**Assumption 2.3.** For any \((R, D^R, T^R), \tau_v(d, s, d', s') \in C\), there exists a known function \( w_N \in \mathcal{W}_N \) (which may depend on \( v, d, s, d', s' \)), such that

\[
\tau_v(d, s, d', s') = \Gamma_n(w_N),
\]

where \( \Gamma_n(w_N) = \frac{1}{n} \sum_{i=1}^{N} R_i w_N(i, D^R, R, T^R, A) m(D_i, \sum_{k \in N_i} D_k, T_i) \). \hspace{1cm} (5)

Assumption 2.3 states that we can write each target estimand as a weighted average of individual potential outcomes’ conditional expectations \( m(\cdot) \), where the weights depend on treatment assignments, selection indicators, covariates, and adjacency matrix. Note that Assumption 2.3 does not require restrictions on unobserv-
ables $\varepsilon$. Throughout the rest of our discussion we leave implicit the dependence of $w_N$ with $A$. Two leading examples illustrate settings where the assumption holds.

**Example 2.4** (Difference in conditional expectations/Inverse-probability weights). Let $T_i = |N_i|$ and consider the following class of weights:

$$w_N(i, D^R, R, T^R, A) = \begin{cases} \frac{I_i(d, s, l)}{\sum_{j: R_j = 1} I_j(d, s, l)/n} & \text{if } R_i = 1 \\ 0 & \text{otherwise} \end{cases}$$

where $I_i(d, s, l) = 1\{D_i = d, \sum_{k \in N_i} D_k = s, T_i = l\}$. Then $\Gamma_n(w_N) = m(d, s, l) - m(d', s', l)$, and similarly for any weighted combination, weighted by $v(l)$.

**Example 2.5** (Linear model). Let $T_i = |N_i|$. Consider the following vector of weights:

$$w_N(i, .) = \begin{cases} \left(\frac{1}{n} \sum_{i: R_i = 1} X_i X_i'\right)^{-1} X_i, & \text{if } R_i = 1 \\ 0 & \text{otherwise} \end{cases}$$

where $X_i = \left(1, D_i, \sum_{k \in N_i} D_k/|N_i|\right)$. By further assuming that $m(d, s, l) = \mu + d\beta + \gamma s/l$, we have $\left[\Gamma_n(w_N^{(2)}), \Gamma_n(w_N^{(3)})\right] = (\beta, \gamma)$, where $\beta$ is the direct effect of the treatment and $\gamma$ the spillover effect of treating all the neighbors.

Under Assumption 2.3, we will refer to the class of estimands indexed by the weights $w_N$, as $\{\Gamma_n(w_n), w_n \in W_N\}$, where $W_N$ denotes the set of estimands.

Assumption 2.3 shows that our setting applies to (generic) estimands from nonlinear (Example 2.4), and linear models (Example 2.5). For some of our theoretical results, we will impose stability restrictions on the weights, which is an equivalent of strict overlap in our setting (see Assumption 4.3 and discussion therein).

## 3 Two-wave Experiment

In this section, we discuss the experimental protocol in Algorithm 1 (pilot) and Algorithm 2 (main experiment). We can summarize the algorithm as follows
• Researchers observe $A$ and $T_1, \ldots, T_N$ for all individuals. This may be obtained from survey data, or pre-experimental information.

• Researchers select a subset of individuals and collect information

$$\left\{ i \in \{1, \ldots, N\} : P_i = 1 \right\}, \left[ P_i(Y_i, D_i, T_i, D_N) \right]_{i=1}^N$$

who participate in the pilot study. They collect outcomes and treatment assignments and neighbors assignments of all individuals in the pilot. The treatments assigned to the neighbors of the individuals in the pilot who are not in the pilot are constant at zero. Researchers choose individuals in the pilot with the least number of connections to the remaining units, under the constraint that some individuals also have their neighbors selected in the pilot study (see below).

• Researchers use the pilot study to select the participants (i.e., indicators $R_i$) and the treatment assignments $D_i$ for individuals in the main experiment and their neighbors. Researchers do not select in the main experiment the participants in the pilot and their neighbors. The treatment $D_i$ for those units who do not participate in the experiment (and pilot) is assumed to be constant at zero, and the treatment assignment to the pilot units remains unchanged.

• Researchers collect information

$$\left[ R_i(Y_i, D_i, T_i, D_j \in N_i, N_i) \right]_{i=1}^N.$$

• For each weight (and estimand) $w_N \in W_N$ as in Assumption 2.3, researchers estimate treatment effects as follows:

$$\widehat{\Gamma}_n(w_N) = \frac{1}{n} \sum_{i=1}^N R_i w_N(i, D^R, R, T^R) Y_i. \quad (7)$$

The weights $w_N(.)$ are functions of treatment assignments, selection indicators, individual observables $T_i$.  

11
The complete algorithm is illustrated in Algorithms 1, 2, discussed below.

**Algorithm 1** Pilot study

**Require:** $A$, $T_1, \cdots, T_N$ observed by researchers, constant $\delta$.

1: Select the participants in the pilot study

$$P \in \arg \min_{(p_1, \cdots, p_N) \in \{0, 1\}^N} \sum_{i=1}^N \sum_{j \in \mathcal{N}_i} p_i (1 - p_j),$$

such that $\sum_{i=1}^N p_i = m$, $\sum_{i=1}^N p_i \sum_{j \in \mathcal{N}_i} p_j \geq \delta$. \hfill (8)

2: Assign treatments to the units in the pilot as

$$D_i \sim \begin{cases} \text{Bern}(1/2) & \text{if } P_i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

3: Collect information from the pilot: $\left[ P_i \left( Y_i, D_i, T_i, D_{\mathcal{N}_i} \right) \right]_{i=1}^N$.

4: Estimate $\left( \sigma^2(\cdot), \eta(\cdot) \right)$ and return the estimator $\left( \hat{\sigma}_p(\cdot), \hat{\eta}_p(\cdot) \right)$.

return $P, \left( \hat{\sigma}_p(\cdot), \hat{\eta}_p(\cdot) \right)$.

3.1 Unconfoundedness and Selection of the Pilot

Next, we discuss the selection of the pilot study. Our first observation is that if we were to select pilot units or their neighbors in the main experiment, we may lead to confounded experimentation. To gain further intuition, consider Figure 1. In the figure, the set of pilot units includes the vertices N4, N5, N6. Researchers may use their outcomes for the design of the main experiment. As a result, the treatment assignment mechanism is dependent on the unobservables of the pilot units. However, since the pilot units are statistically dependent on their neighbors (N7), selecting N7 may depend on the treatment assignment mechanism and unobservables in the main experiment and confound the experiment. We formalize this intuition below.
We define
\[ J = \left\{ i \text{ and } N_i : P_i = 1 \right\} \] (9)
the set of units after excluding individuals in the pilot study and the corresponding neighbors. The experiment is unconfounded if it satisfies the following restrictions.

**Proposition 3.1** (Conditions for Unconfounded Experiment). Suppose that
\[ \varepsilon_{i \in \{1, \ldots, N\} \setminus J} \perp \left( R^R, R \right) \mid A, T, P; \quad \varepsilon_{i \in \{1, \ldots, N\} } \perp P \mid A, T; \quad R_i = 0 \text{ for all } i \in J. \]
Then under Assumption 2.1, 2.2
\[ \mathbb{E} \left[ \hat{\Gamma}_n(w_N) \left| D^R, R, A, T, P \right. \right] = \Gamma_n(w_N). \]

The proof is in the Appendix. Proposition 3.1 provides sufficient conditions that guarantee that the estimator is unbiased, conditional on the treatment assignments.

The first condition states that unobservables in the set \( \{1, \ldots, N\} \setminus J \) (all units except the pilot units and their neighbors) are independent of treatment assignments and selection indicators. The second condition states that the choice of the pilot is randomized, as a function of \( A, T \) only. The third condition states that the units in the main experiment are not in the pilot and are not the pilot units’ neighbors.

Proposition 3.1 provides insights on the pilot’s design. In particular, (i) participants in the pilot study can be selected based on network information only; (ii) the larger the set \( J \) (i.e., pilot participants and their neighbors), the stricter the constraint imposed on the second-wave experiment. Therefore, Algorithm 1 chooses units in the pilot to minimize \( |J| \). It also requires that participants and some of their neighbors must be in the pilot study to identify and estimate covariances between participants. We then randomize treatments in the pilot exogenously.

The optimization problem in Equation (8) reads as a min-cut problem in a graph: we find a set of units that are “well” separated from the rest under constraints on the number of units and their neighbors be included in the pilot study. The optimization can be solved using mixed-integer quadratic programming (MIQP).

**Corollary.** Let Assumption 2.1 hold. Then the two-wave experiment constructed with Algorithm 1 and Algorithm 2 satisfies the conditions in Proposition 3.1.
Figure 1: Example of network. In such a setting, under one-degree dependence, N7 does not satisfy the validity condition since it connects to the pilot study, which is used for the randomization of treatments and indicators.

The above corollary illustrates that the proposed algorithm returns unbiased estimators of treatment effects.

We use information from the pilot study to estimate the variance of the estimator of interest, namely

$$\text{Var}(Y_i \mid A, D_i, T_i, D_{N_i}, P), \quad \text{Cov}(Y_i, Y_j \mid A, D_i, D_j, D_{N_i}, D_{N_j}, T_i, T_j, P).$$

The following lemma permits us to identify and estimate the above functions.

**Lemma 3.2.** Suppose that Assumption 2.1, 2.2 hold and the pilot is chosen as in Algorithm 1. Then for all individuals \((i, j) \in P\),

$$\text{Var}(Y_i \mid A, D_i, T_i, D_{N_i}, P) = \sigma^2(T_i, D_i, \sum_{k \in N_i} D_k)$$

$$\text{Cov}(Y_i, Y_j \mid A, D_i, D_j, D_{N_i}, D_{N_j}, T_i, T_j, P) = \begin{cases} \eta(T_i, D_i, \sum_{k \in N_i} D_k, T_j, D_j, \sum_{k \in N_j} D_k) & \text{if } i \in N_j \\ 0 & \text{otherwise} \end{cases}$$

for some functions \(\sigma^2(\cdot), \eta(\cdot)\).

The proof is in the Appendix. Lemma 3.2 guarantees identification (and estimation) of the variance and covariance pilot’s participants. We denote \(\left(\hat{\sigma}^2_p, \hat{\eta}_p\right)\), the variance and covariance function estimated from the pilot study. We impose high-level conditions on the convergence rate of these functions in Assumption 4.4.
3.2 Design of the Main Experiment

Next, we discuss the main experiment (Algorithm 2). We define the conditional variance of the estimators of interest, conditional on the treatment assignment and the underlying network below.

\[
V_N\left(w_N; A, D^R, R, T^R, P\right) = \text{Var}\left(\frac{1}{1^T R} \sum_{i: R_i = 1} w_N(i, D^R, R, T^R) Y_i | A, D^R, R, T^R, P\right).
\]

(10)

Given the selection of the pilot study \( P \), the design of the main experiment (i.e., second-wave experiment) minimizes the worst-case variance over the class of estimands (indexed by \( w_N \in \mathcal{W}_N \)) estimated variance \( \hat{V}_N, \hat{\sigma}_p, \hat{\eta}_p \). Here, \( \hat{V}_N, \hat{\sigma}_p, \hat{\eta}_p \) denotes the plug-in estimator of the variance using the estimated functions \( \hat{\sigma}_p, \hat{\eta}_p \) obtained from the pilot study. Formally, by letting \( n = 1^T R \),

\[
\hat{V}_{N, \hat{\sigma}_p, \hat{\eta}_p}(w_N) = \frac{1}{n^2} \sum_{i: R_i = 1} w_N^2(i, D^R, R, T^R) \hat{\sigma}_p^2 \left( T_i, D_i, \sum_{k \in N_i} D_k \right)
+ \frac{1}{n^2} \sum_{i: R_i = 1} \sum_{j \in N_i} R_j w_N(i, D^R, R, T^R) w_N(j, D^R, R, T^R) \hat{\eta}_p \left( T_i, D_i, \sum_{k \in N_i} D_k, T_j, D_j, \sum_{k \in N_j} D_k \right).
\]

(11)

The estimated variance depends on the (heteroskedastic) component \( \hat{\sigma}_p^2 \), estimated using information from the pilot, and the covariances \( \hat{\eta}_p \) between neighbors.\(^6\)

The optimization problem is in Equation (13). The minimization is with respect to the participation indicators and the treatment assignments. The optimization problem selects between \( n_1 \) and \( n_2 \) individuals. We also restrict the weights to be finite and smaller than some arbitrary constant \( \bar{C} \). This guarantees that the optimization problem does not return ill-posed solutions. For example, for the weights corresponding to the difference in means as in Example 2.4, the bound on the weights implies that at least some individuals are treated, and some are not.

Additional constraints may be included: for example, only some units can participate in the experiments, corresponding to constraints on \( R_i = 0 \) for some of the units. An alternative constraint is to impose \( D_i \times R_i \geq D_i \). This constraint imposes

\(^6\)In the presence of higher-order interference, we also add additional components which depend on higher-degree neighbors as discussed in Appendix.
that those units which are not selected as participants have treatment assignments equal to zero. This is omitted for brevity only, and our results extend in these scenarios. Also, note that an alternative specification of the objective function consists of minimizing a weighted combination of the variances of each estimator.\footnote{Formally, minimizing $\sum_{w_N \in W_N} u(w_N) V_N(w_N; A, D^R, R, T, P)$ for some given weights $u(w_N)$.}

The proposed mechanism and all our results directly extend also to this setting.

The constraint in Equation (13) illustrates the \textit{trade-off} in the selection of the pilot study: the larger the pilot study, the more precise the estimator of the variance. However, the larger the pilot study, the larger the set $\mathcal{J}$ and therefore, the more stringent the constraint imposed in the above optimization procedure.

Finally, Algorithm 2 also returns estimates of the variance and covariances $\hat{\sigma}, \hat{\eta}$ using information from the main experiment only. Such estimates will be used for inference on treatment effects. Variance and covariances can be identified as below.

\textbf{Lemma 3.3.} Suppose that Assumption 2.1, 2.2 hold. Consider an experimental design in Algorithm 2, with pilot chosen as in Algorithm 1. Then for all units participating in the main experiment:

\begin{align*}
\text{Var}(Y_i | A, D^R, R, T, P) &= \sigma^2(T_i, D_i, \sum_{k \in N_i} D_k) , \\
\text{Cov}(Y_i, Y_j | A, D^R, R, T, P) &= \begin{cases}
\eta(T_i, D_i, \sum_{k \in N_i} D_k, T_j, D_j, \sum_{k \in N_j} D_k) & \text{if } i \in \mathcal{N}_j \\
0 & \text{otherwise}
\end{cases}
\end{align*}

(12)

for some functions $\sigma^2(\cdot), \eta(\cdot)$.

The proof is in the Appendix. Given Lemma 3.3, we can identify and estimate the variance and covariance between units parametrically or non-parametrically. The estimators from the main experiment were defined as $\tilde{\sigma}, \tilde{\eta}$. Inference uses the plug-in estimator of the variance which uses the estimated $(\tilde{\sigma}, \tilde{\eta})$ obtained from the main experiment $\hat{V}_{N, \tilde{\sigma}^2, \tilde{\eta}}(w_N)$. Asymptotic inference is discussed in Section 4.2. Note that we can also consider a dual formulation of the optimization problem in Equation (13), choosing the minimum sample size with the variance not being larger than a pre-specified threshold.
Remark 2 (Partial Network Information). In Appendix A.2, we design an experiment that only uses partial information on the network structure. In particular, instead of observing the entire adjacency matrix, it observes some entries of A and imputes the remaining entries with a model, assuming that the pilot belongs to a cluster separated from the main experiment.

Remark 3 (Randomization). In some applications, we may want to introduce in Equation (13) additional constraints on \( D_i \) to guarantee randomization according to a given rule.\(^8\) This is possible, and inference in Section 4.2 is valid also in this setting.

Algorithm 2 Main Experiment

| Require: A, T observed by researchers, \( \left( P, \hat{\sigma}_p(\cdot), \hat{\eta}_p(\cdot) \right) \), \( n_1, n_2 \), finite constant \( \bar{C} \). |
|---|
| 1: Choose (with \( J \) as in Equation (9)) |
| \( (D^R, R) \in \arg\min_{r, d^r} \max_{w_N \in \mathcal{W}_N} \hat{V}_N, \hat{\sigma}_p, \hat{\eta}_p \left( w_N; A, d^r, r, T^r, P \right) \), |
| s.t. \( 1^T r \in [n_1, n_2], r_j = 0 \) for all \( j \in J, \max_{w_N \in \mathcal{W}_N} \max_i \left| w_N \left( i, d^r, r, T^r \right) \right| \leq \bar{C}, \) |
| where \( \hat{V}_N, \hat{\sigma}_p, \hat{\eta}_p(\cdot) \) is the plug-in estimator of the variance in Equation (11). |
| 2: Collect information \( \left[ R_i \left( Y_i, D_i, T_i, D_j \in \mathcal{N}_i, \mathcal{N}_i \right) \right]_{i=1}^N \); |
| 3: Estimate \( \Gamma_n(w_N) \) as in Equation (7). |
| 4: Estimate \( \hat{\sigma}, \hat{\eta} \) using observations in the main experiment, and construct the variance of the estimator as \( \hat{V}_{N, \hat{\sigma}, \hat{\eta}}(w_n; \cdot) \) |
| return \( \hat{\Gamma}_n(w_N), \hat{V}_{N, \hat{\sigma}, \hat{\eta}}(w_n; \cdot) \) for each \( w_n \in \mathcal{W}_N \). |

4 Theoretical Analysis and Inference

In this section, we study theoretical guarantees of the variance of the proposed design and asymptotic inference. First, we introduce additional notation. Recall that \( m, n \)

\(^8\)For example, we might randomize treatments independently across individuals with different probabilities, conditional on the number of treated units, and imposing that the number of treated individuals is neither zero nor \( n \).
denote the size of the pilot and main experiment. Let \( N_{\text{max}} = \max_{i \in \{1, \ldots, N\}} |\mathcal{N}_i| + 1. \)

### 4.1 Regret Analysis and Pilot’s Size

We study how the worst-case variance of the estimator obtained from the two-stage experiment in Section 3 compares to the variance obtained from the oracle experiment, where the oracle has access to the true (population) variance and covariances. The oracle selects the participants and treatments arbitrarily. Formally, the variance of the oracle method reads as follows:

\[
\mathcal{V}_N = \min_{d, r} \max_{w_N \in \mathcal{W}_N} \text{Var} \left( \frac{1}{n} \sum_{i=1}^{N} r_i w_N(i, d^r, r, r^T) \mathcal{Y}_i \left| A, D^r = d^r, R = r, T \right., \right)
\]

such that \( 1^\top r = n_2, \)

where \( D^r \) denotes the subvector of assignments for individuals with \( r_i = 1 \) or their friends. The oracle experiment minimizes the true variance and it does not impose any condition on the units in the pilot and their neighbors not participating in the main experiment. We assume that the oracle experiment treats exactly \( n_2 \) many individuals, which is the largest number of participants allowed in Equation (13) (instead of a number between \( (n_1, n_2) \)). This can be relaxed.\(^9\)

The regret is defined as follows.

\[
\mathcal{R}_N = \max_{w_N \in \mathcal{W}_N} \text{Var} \left( \frac{1}{n} \sum_{i=1}^{N} R_i w_N(i, D^R, R, T^R) \mathcal{Y}_i \left| A, D^R, R, T \right., \right) - \mathcal{V}_N \quad (15)
\]

where \( R, D^R \) solve the two-wave experiment in Equation (13). We study the behavior of \( n_2 \mathcal{R}_N \), after appropriately rescaling by the maximum sample size, since, otherwise

\(^9\)From an inspection of the proof, we observe that, we may also impose a lower bound equal to \( n_1 + |\mathcal{J}| \) and assume that \( 1^\top r \in [n_1 + |\mathcal{J}|, n_2] \) without affecting our results. In this alternative scenario, the upper bound matches the upper bound in the empirical design discussed in Equation (13), while we impose a stricter lower bound \( \sum_{i=1}^{N} r_i / n_1 \geq 1 + |\mathcal{J}| / n_1 \) for the oracle experiment by a factor \( |\mathcal{J}| / n_1 \). In the asymptotic regime, where the size of the pilot experiment is assumed to grow at a slower rate than the number of participants in the main experiment, \( |\mathcal{J}| / n_1 \lesssim N_{\text{max}}^m / n_1 = o(1), \) for the sample size \( n_1 \) being large enough and larger than the maximum degree and pilot’s size.
each component in the right-hand side of Equation (15) might converge to zero.

Before stating our first theorem, we impose the following conditions.

**Assumption 4.1.** Assume that for some $\xi > 0$, the following hold:

\[
\sup_{d,s,l} \left| \hat{\sigma}_p(d,s,l) - \sigma(d,s,l) \right| = O_p(m^{-\xi}), \\
\sup_{d,s,l,d',s',l'} \left| \hat{\eta}_p(d,s,l,d',s',l') - \eta(d,s,l,d',s',l') \right| = O_p(m^{-\xi}).
\]

Assumption 4.1 characterizes the convergence rate of the variance and covariance functions. For instance, the rate of convergence is $m^{-1/2}$ for parametric estimators of these functions. In the following assumption we impose moment conditions.

**Assumption 4.2.** Let (a) $Y_i \in [-M,M]$ where $M < \infty$; (b) $N_{\text{max}}^2/n_1^{1/2} = o(1)$.

Assumption 4.2 assumes that the outcome is bounded\textsuperscript{10}, and that the maximum degree grows at a rate slower than $n_1^{1/4}$.

The last assumption that we impose is a stability assumption on the weights.

**Assumption 4.3 (Weights’ stability).** Assume that for any $d \in \{0,1\}^N, i \in \{1,\cdots, n\}$,

\[
\left| w_N(i, d', r, T^r) - w_N(i, d''', r', T'r') \right| \leq \bar{C} \frac{1^{T}r-1^{T}r'}{\min(1^{T}r,1^{T}r')},
\]

for a finite constant $\bar{C} < \infty$, for any $r, r'$ such that $\mathbf{r}_i = r'_i, r_{N_i} = r'_{N_i}$.

Assumption 4.3 states that for two selector indicators, the difference between the weights for unit $i$, cannot differ by at most a finite constant $\bar{C}$ times the difference in the number of participants between the two vectors, divided by the minimum number of participants, if both vectors select unit $i$, and the same friends of unit $i$ (otherwise the difference can be arbitrary). Such as assumption is a stability condition.

**Example 2.4 cont’d** Consider the weights in Example 2.4, with estimand $m(d, s, l) - m(d', s', l')$. Then Assumption 4.3 holds if weights are bounded by a universal constant (i.e., $\min\{\sum_j I_j(d, s, l)/n, \sum_j I_j(d', s', l')/n\} > 1/\bar{C}$, with $I_j$ as defined in Example 2.4, for a finite $\bar{C} > 0$ independent of $n$).

\textsuperscript{10}This is a common assumption in econometrics, e.g., Kitagawa and Tetenov (2018). Less restrictive conditions would require sub-gaussianity.
Assumption 4.3 is a stability condition and holds for a difference in means estimator under the restriction that weights must be uniformly bounded, which can be imposed by design. We can now state the following theorem.

**Theorem 4.1.** Under Assumption 2.1, 2.2, 4.1, 4.2, 4.3 assuming that \( n_2/n_1 = \alpha \in (1, C) \), \( C < \infty \) for \((n_1, n_2)\) such that \( n_1 \geq \mathcal{N}_{\text{max}} m \left/ (\alpha - 1) \right. \),

\[ n_2 R_N = \mathcal{O}_p \left( \mathcal{N}_{\text{max}} m^{-\xi} + \left( \frac{\mathcal{N}_{\text{max}}^2 m}{n_1} \right)^2 \right). \tag{16} \]

The proof is in the Appendix. Theorem 4.1 characterizes the difference between the variance of the experiment with a pilot study against the variance of the oracle experiment with known variance and covariance functions. The theorem illustrates a trade-off: (i) the larger the size of the pilot experiment, the smaller the estimation error; (ii) the larger the size of the pilot, the stronger the constraints imposed in the optimization algorithm, and therefore the larger the regret with respect to the oracle assignment mechanism. A key challenge in the proof is to compare the solution of the design with its oracle counterpart, which assigns treatments without restrictions on the pilot and main experiment. We can now state the following corollary.

**Corollary.** Suppose that the conditions in Theorem 4.1 hold, with \( \xi = 1/2 \) (i.e., parametric rate). Then for \( m \approx (n_1/\mathcal{N}_{\text{max}})^{2/3} \), we have \( n_2 R_N = \mathcal{O}_p (\mathcal{N}_{\text{max}}^{4/3} n_1^{-1/3}) \). Therefore \( n_2 R_N \to_p 0 \).

To our knowledge, the above corollary is the first result that formally characterizes the pilot’s size with respect to the main experiment.

### 4.2 Asymptotic Inference

In the following lines, we derive the asymptotic properties of the estimator conditional on the treatment assignments. Inference depends on the estimated variance. The first two conditions that we impose guarantees consistency of the estimated variance \( \hat{V} \), whenever \( \hat{\sigma}, \hat{\eta} \) are consistent.

**Assumption 4.4.** Assume that the following holds:
(i) there exists a $L_N > 0$ such that
\[
\left| \{ i : |N_i| > L_N \} \right| = O(n^{3/4}), \quad \sup_{l,d,s,l',d',s'} \left| \eta(l, d, s, l', d', s') - \hat{\eta}(l, d, s, l', d', s') \right| = o_p(L_N^{-1}).
\]

(ii) $\sup_{l,d,s} |\sigma^2(l,d,s) - \hat{\sigma}^2(l,d,s)| = o_p(1)$;

(iii) $\mathcal{W}_N$ is finite dimensional;

(iv) for all $w_N \in \mathcal{W}_N$, $n_1 V_N(w_N) > 0$ almost surely.

Condition (i) states that the number of “influential nodes”, namely the number of individuals with a large degree (larger than some arbitrary large $L_N$), grows at a slower rate than the sample size. It also assumes that the rate of convergence of the estimated covariances is faster $L_N^{-1}$. The condition trivially holds whenever only a finite number of individuals have a growing degree (i.e., a network with few large hubs), and consistent covariance functions (at arbitrary rate), but it also allows for a growing number of highly connected nodes. For example, in a scale free network (Caldarelli, 2007), $\left| \{ i : |N_i| > L_N \} \right| = NL_N^{-\gamma}$, for a given positive constant $\gamma$, typically between two and three. For $\gamma = 2$, $n \propto N$, the assumption would hold for $L_N = n^{1/8}$, as long as the rate of convergence of the covariance is faster than $n^{-1/8}$.

Condition (ii) assumes a consistent variance at an arbitrary rate. Condition (iii) assumes that the number of estimands is finite dimensional, and (iv) that the variance is larger than zero after appropriately rescaling (to avoid degenerate convergence in distribution). We can state the next theorem.

**Theorem 4.2.** Suppose that Assumption 2.1, 2.2, 4.2, 4.4 hold. Then for all $w_N \in \mathcal{W}_N$,
\[
\frac{\sqrt{n}(\bar{\Gamma}_n(w_N) - \Gamma_n(w_N))}{\sqrt{nV_{N,\hat{\sigma},\hat{\eta}}(w_N)}} \to_d \mathcal{N}(0, 1). \tag{17}
\]

The proof of the theorem is contained in the Appendix. The above theorem establishes asymptotic normality for a general class of linear estimators. The rate of convergence of the estimator depends on the variance component $V_N(w_N)$. Whenever $nV_N(w_N) = O(1)$, the estimator achieves the parametric $\sqrt{n}$ convergence rate.
Asymptotic properties of estimator for network data have been discussed in a variety of contexts (e.g., Ogburn et al., 2017; Chin, 2018; Sävje et al., 2021). Here, differently, we derive the asymptotic result conditionally on the entire assignment mechanism. To achieve this goal, we leverage restrictions on the number of highly connected units, and properties of the estimated variance function.

5 Numerical Studies

In this section, we collect simulation results. Throughout this section, we set $T_i = |N_i|$. We consider the following functional form for the variance and covariance:

$$\sigma(l, d, s) = \mu + \beta_1 d + \frac{s \beta_2}{\max\{l, 1\}}, \quad \eta(l, d, s, l', d', s') = \sqrt{\sigma(l, d, s) \times \sigma(l', d', s') \alpha}.$$  \hspace{0.2cm} (18)

The variance depends on the individual treatment status and on the percentage of treated neighbors. The covariance instead is chosen in the spirit of the Cauchy-Swartz inequality with $\alpha$ being the equivalent of the intra-cluster correlation in the presence of clustered networks (Baird et al., 2018). Similarly to simulations in Baird et al. (2018), we choose $\alpha = 0.1$. We choose $\mu = 0.5$ and we collect results for parameters $\beta_1$ and $\beta_2$ in $(0, 0)$, $(0.5, 0.5)$, $(0.5, 1)$. We denote each case respectively homoskedastic, “small heteroskedasticity”, “large heteroskedasticity”. In the Appendix we discuss results for a broader choice of parameters. Using the same exposure mapping in the simulations in Eckles et al. (2017), we draw

$$Y_i = D_i \gamma_1 + \sum_{k \in N_i} \frac{D_k}{|N_i|} \gamma_2 + \epsilon_i,$$  \hspace{0.2cm} (19)

with $\epsilon_i$ Gaussian centered around zero. We choose $\gamma_1 = 0.5, \gamma_2 = 1$ (the choice of such coefficients does not affect the conditional variance of the estimator given $D_R$).

In the first set of simulations, we generate data from an Erdős-Rényi graph with $P(A_{i,j} = 1) = 2/n$ and an Albert-Barabasi graph.\footnote{For the latter, we first draw $n/5$ edges according to Erdős-Rényi graph with probabilities $p = 2/n$, and second, we draw connections of the new nodes sequentially to the existing ones with} We evaluate the methods over
200 replications, \( N = 800 \), and constraint on the main experiment having less than four-hundred participants \((1^\top R \leq 400)\).

In the second set of simulations, we evaluate results using the adjacency matrix from Cai et al. (2015). We consider two different adjacency matrices obtained from this study: the “weak” network, where two individuals are connected if either indicates the other as a friend, and the “strong” network, where two individuals are connected if both individuals indicate the other as a friend. The weak network presents a dense structure, while the strong network presents a sparse structure. We use the adjacency matrix from the first five villages, which counts in total \( N = 832 \), with no more than 416 participants in the main experiment.

### 5.1 Methods

We evaluate the proposed method (ELI), with complete knowledge of the adjacency matrix and with a pilot study containing 70 units. Estimation of the variance and covariances is performed using a quadratic program with a positivity constraint on the variance function. In the estimation, we impose constraints on the estimated parameter for \( \alpha \) being in \([0, 0.3]\). Such estimation problem reflects correct prior but imperfect knowledge of researchers on a positive correlation among neighbors, which often occurs in applications (Baird et al., 2018), and full incomplete knowledge of the parameters of the variance function. We solve the optimization problem over treatment and participation assignments using non-linear mixed-integer programming.

For the real-world network, we also consider the proposed method with partially observed network (see Appendix A.2). We estimate the variance and covariances, selecting 70 units for a pilot study from the sixth village. For such a method, only the network of the first two-hundred units in the main village is observed before the randomization of the experiment. We impute missing edges using a simple Erdős-Rényi model, with a uniform prior to the probability of connections. The model is clearly wrongly specified in the real-world scenario. However, it shows that even such a sim-probability equal to the number of connection of each pre-existing node divided by the overall number of connections in the graph.
ple specification may lead to improvements. We solve the optimization problem by alternating a Monte-Carlo step for estimating the variance over the unobserved edges and the optimization step over treatment assignments and participation indicators.

We consider competitors, where the number of participants either equals the number of participants in the main experiment or it equals the sum of the number of participants in the main experiments and the number of units used in the pilot study. We consider the following competing methods: (ii) the 3-$\epsilon$ net graph clustering method with 400 participants discussed in Ugander et al. (2013); (iii) the 3-$\epsilon$ net graph clustering method with 470 participants, denoted as Clustering+, and three different saturation designs. Since saturation design methods are not directly applicable in the presence of a fully connected network, we consider extensions of saturation designs, where we combine the $\epsilon$-net clustering discussed in Ugander et al. (2013), with the saturation design mechanism (Baird et al., 2018). Namely, (iv) Saturation1, with 400 participants, with uniform probability assignment across the estimated clusters; (v) the Saturation1+, having 470 participants and being as Saturation1; (vi) Saturation2+, with 470 participants, selects the saturation probabilities and the percentage of clusters for each probability of minimizing the sum of the standard errors of the treatment and spillover effect, with intracluster correlation equals to the true $\alpha$ and with the variance of the individual error set to be homoskedastic; (vi) Saturation3+, with 470 participants, instead minimizes the sum of the standard errors of treatment effects, spillover effects as well as on the slope effects as defined in Baird et al. (2018). However, we remark that saturation designs may have a poor performance in this particular case since they are not directly applicable in scenarios where (i) the network is not clustered; (ii) the variance is unknown to the researcher. Finally, we consider Random Assignment +, which selects at random 470 participants and assigns equal probabilities treatments. All the competitors, with the exception of the random assignment mechanism, use complete information of the network structure.

\[^{12}\text{Since the method in Jagadeesan et al. (2020) is only valid for direct effects, but not spillovers and overall effects, such method is not a suitable competitor in these simulations.}\]
5.2 Results

We collect results for the real-world network in Table 1, where we report the estimator’s variance. Each column corresponds to different values of the coefficients \((\beta_1, \beta_2)\). The left-hand side panel collects results for the network with strong ties, and the right-hand side panel collects results for the weak ties. Results show that the proposed method outperforms uniformly any competitor. The improvement is significantly larger as the values of the coefficients increase, i.e., in the presence of heteroskedasticity. In Figure B.1 in the Appendix, we show that the bias is zero.

In the presence of the partially observed network, the only valid competitor to the proposed method is the random allocation. In such a case, we observe that the proposed method outperforms the random allocation strategy uniformly.

In the right-hand side panel of Figure 2 (simulated networks), we report the variance in the log-scale of the proposed method against the competitor with the lowest median variance, which randomizes using the sum of participants and units in the pilot study. In the heteroskedastic case, we observe that the proposed method outperforms uniformly, and the improvement with respect to the competitors increases for a larger degree of heteroskedasticity. In the homoskedastic case (i.e., \((\beta_1, \beta_2) = (0, 0))\), we also observe uniform dominance with one exception, where the method is slightly worst than graph clustering with seventy additional participants.\(^{13}\) Additional results are contained in Appendix B.

The left-hand panel of Figure 2 (real-world networks) complements Table 1 and reports the percentage decrease in the sample size necessary to achieve the same level of variance for the overall effect which would make the best competitor equivalent to our method.\(^{14}\) The figure illustrates substantial improvements in terms of sample size, up to forty percentage points. Note that we count the number of units used by our method as the sum of participants, and the size of the pilot study.

\(^{13}\)Since, in this setting, we do not consider the presence of a separate cluster, as in the real-world network analysis, results for the partially observed network are not computed for simulated networks.

\(^{14}\)The “unobserved network” case in the panel compares the ELI method with the partially observed network to the random allocation only.
Table 1: Variance for estimating the overall effect, using data originated from Cai et al. (2015), using the first five villages as the population of interest ($N = 832$). Each column corresponds to a different design, for different values of the coefficients ($\beta_1, \beta_2$). “ELI” corresponds to the proposed method, where 416 participants from the 832 potential participants are sampled in the main experiment, and a pilot study with 70 units is used. The second row corresponds to the proposed method, where only the first sub-block with the first 200 observations is observable from the main experiment, and a pilot of 70 units from the sixth village is used. Methods with a + use 416+70 participants in the main experiment, and without a +, such methods use 416 participants in the main experiment. All competitors, with the exception of the random allocation (Random All+), exploit full knowledge of the network structure.

| Overall Effect                        | Strong (0,0) | (0.5,0.5) | (0.5,1) | Weak (0,0) | (0.5,0.5) | (0.5,1) |
|---------------------------------------|-------------|-----------|---------|------------|-----------|---------|
| ELI                                   | 0.551       | 1.134     | 1.404   | 0.769      | 1.442     | 1.665   |
| ELI - Unobserved Net                  | 0.914       | 1.829     | 2.183   | 2.018      | 4.139     | 5.067   |
| Random All+                           | 1.107       | 2.249     | 2.876   | 2.430      | 4.827     | 6.127   |
| Graph Clustering+                     | 0.694       | 1.591     | 2.038   | 0.874      | 1.830     | 2.345   |
| Saturation1+                          | 0.913       | 1.985     | 2.513   | 1.523      | 3.143     | 3.866   |
| Graph Clustering                      | 0.793       | 1.847     | 2.420   | 0.989      | 2.104     | 2.623   |
| Saturation1                           | 1.059       | 2.259     | 2.940   | 1.736      | 3.603     | 4.482   |
| Saturation2+                          | 0.719       | 1.669     | 2.104   | 0.944      | 1.973     | 2.418   |
| Saturation3+                          | 0.931       | 2.171     | 2.772   | 1.700      | 3.844     | 4.829   |

| Treatment and Spill                   | Strong (0,0) | (0.5,0.5) | (0.5,1) | Weak (0,0) | (0.5,0.5) | (0.5,1) |
|---------------------------------------|-------------|-----------|---------|------------|-----------|---------|
| ELI                                   | 0.491       | 1.028     | 1.299   | 0.790      | 1.525     | 1.825   |
| ELI - Unobserved Net                  | 0.596       | 1.286     | 1.639   | 1.613      | 3.133     | 4.165   |
| Random All+                           | 0.641       | 1.431     | 1.882   | 1.813      | 3.580     | 4.477   |
| Graph Clustering+                     | 0.864       | 2.147     | 2.600   | 1.838      | 3.528     | 4.431   |
| Saturation1+                          | 0.652       | 1.500     | 1.942   | 1.403      | 2.807     | 3.569   |
| Graph Clustering                      | 0.999       | 2.491     | 3.001   | 2.165      | 4.022     | 5.302   |
| Saturation1                           | 0.760       | 1.755     | 2.286   | 1.654      | 3.283     | 4.068   |
| Saturation2+                          | 0.773       | 1.900     | 2.371   | 1.516      | 2.986     | 3.724   |
| Saturation3+                          | 0.801       | 1.910     | 2.449   | 2.231      | 4.202     | 5.155   |
Figure 2: In the left panel, we report the percentage decrease in the number of units necessary to achieve the same level of variance between the best performing competitor and the ELI method for the overall treatment effect, using the simulations with the real-world network. The case denoted as “Unobserved network” compares the random allocation to the ELI method with the partially observed network. In the right panel of Figure 2, we report the variance in the log-scale of the proposed method (in blue) against the competitor with the lowest median variance, which randomizes using the sum of participants selected by ELI and units in the pilot study.

6 Conclusions

In this paper, we have introduced a novel method for designing experiments under interference. We propose a design that selects treatment assignments and participation indicators to minimize the variance of the final estimator. We propose the first statistical framework for two-wave experiments with interference and derive regret guarantees.

We considered designs where the complete network information is available to the researchers. In the Appendix and simulations, we show how our setting extends to partially observed networks. Our numeric findings suggest that imputing the network may lead to improvements in the variance of the estimators. We leave for
future research a comprehensive theoretical analysis of the scenario with the partially observed network and model selection in this case.

This paper makes two key assumptions: interactions are anonymous, and interference propagates to the neighbors only. Future research should address the question of design under general interactions and interference propagating on the entire network. Exploring the effect of the network topology and different exposure mappings on the performance of the design mechanisms remains an open research question.

References

Aronow, P. M. and C. Samii (2017). Estimating average causal effects under general interference, with application to a social network experiment. *The Annals of Applied Statistics* 11(4), 1912–1947.

Athey, S., D. Eckles, and G. W. Imbens (2018). Exact p-values for network interference. *Journal of the American Statistical Association* 113(521), 230–240.

Bai, Y. (2019). Optimality of matched-pair designs in randomized controlled trials. *Available at SSRN 3483834*.

Baird, S., J. A. Bohren, C. McIntosh, and B. Özler (2018). Optimal design of experiments in the presence of interference. *Review of Economics and Statistics* 100(5), 844–860.

Banerjee, A., A. G. Chandrasekhar, E. Duflo, and M. O. Jackson (2013). The diffusion of microfinance. *Science* 341(6144), 1236498.

Barrera-Osorio, F., M. Bertrand, L. L. Linden, and F. Perez-Calle (2011). Improving the design of conditional transfer programs: Evidence from a randomized education experiment in colombia. *American Economic Journal: Applied Economics* 3(2), 167–95.

Barrios, T. (2014). Optimal stratification in randomized experiments. *Manuscript, Harvard University*.

Basse, G. and A. Feller (2016). Analyzing multilevel experiments in the presence of peer effects. *arXiv preprint arXiv 1608*. 

28
Basse, G. W. and E. M. Airoldi (2018a). Limitations of design-based causal inference and a/b testing under arbitrary and network interference. *Sociological Methodology* 48(1), 136–151.

Basse, G. W. and E. M. Airoldi (2018b). Model-assisted design of experiments in the presence of network-correlated outcomes. *Biometrika* 105(4), 849–858.

Bhattacharya, D., P. Dupas, and S. Kanaya (2013). Estimating the impact of means-tested subsidies under treatment externalities with application to anti-malarial bednets. Technical report, National Bureau of Economic Research.

Breza, E., A. G. Chandrasekhar, T. H. McCormick, and M. Pan (2020). Using aggregated relational data to feasibly identify network structure without network data. *American Economic Review* 110(8), 2454–84.

Cai, J., A. De Janvry, and E. Sadoulet (2015). Social networks and the decision to insure. *American Economic Journal: Applied Economics* 7(2), 81–108.

Caldarelli, G. (2007). *Scale-free networks: complex webs in nature and technology*. Oxford University Press.

Charnes, A. and W. W. Cooper (1962). Programming with linear fractional functionals. *Naval Research logistics quarterly* 9(3-4), 181–186.

Chin, A. (2018). Central limit theorems via stein’s method for randomized experiments under interference. *arXiv preprint arXiv:1804.03105*.

DellaVigna, S. and D. Pope (2018). Predicting experimental results: who knows what? *Journal of Political Economy* 126(6), 2410–2456.

Duflo, E., P. Dupas, and M. Kremer (2011). Peer effects, teacher incentives, and the impact of tracking: Evidence from a randomized evaluation in kenya. *American Economic Review* 101(5), 1739–74.

Dupas, P. (2014). Short-run subsidies and long-run adoption of new health products: Evidence from a field experiment. *Econometrica* 82(1), 197–228.

Eckles, D., B. Karrer, and J. Ugander (2017). Design and analysis of experiments in networks: Reducing bias from interference. *Journal of Causal Inference* 5(1).

Egger, D., J. Haushofer, E. Miguel, P. Niehaus, and M. W. Walker (2019). General equilibrium effects of cash transfers: experimental evidence from kenya. Technical report, National Bureau of Economic Research.
Forastiere, L., E. M. Airoldi, and F. Mealli (2021). Identification and estimation of treatment and interference effects in observational studies on networks. *Journal of the American Statistical Association* 116(534), 901–918.

Goldsmith-Pinkham, P. and G. W. Imbens (2013). Social networks and the identification of peer effects. *Journal of Business & Economic Statistics* 31(3), 253–264.

Harshaw, C., F. Sävje, D. Spielman, and P. Zhang (2019). Balancing covariates in randomized experiments using the gram-schmidt walk. *arXiv preprint arXiv:1911.03071*

Hudgens, M. G. and M. E. Halloran (2008). Toward causal inference with interference. *Journal of the American Statistical Association* 103(482), 832–842.

Jagadeesan, R., N. S. Pillai, and A. Volfovsky (2020). Designs for estimating the treatment effect in networks with interference. *The Annals of Statistics* 48(2), 679–712.

Kallus, N. (2018). Optimal a priori balance in the design of controlled experiments. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 80(1), 85–112.

Kang, H. and G. Imbens (2016). Peer encouragement designs in causal inference with partial interference and identification of local average network effects. *arXiv preprint arXiv:1609.04464*

Karlan, D. and J. Appel (2018). *Failing in the field: What we can learn when field research goes wrong*. Princeton University Press.

Karlan, D. S. and J. Zinman (2008). Credit elasticities in less-developed economies: Implications for microfinance. *American Economic Review* 98(3), 1040–68.

Karrer, B., L. Shi, M. Bhole, M. Goldman, T. Palmer, C. Gelman, M. Konutgan, and F. Sun (2021). Network experimentation at scale. In *Proceedings of the 27th ACM SIGKDD Conference on Knowledge Discovery & Data Mining*, pp. 3106–3116.

Kasy, M. (2016). Why experimenters might not always want to randomize, and what they could do instead. *Political Analysis* 24(3), 324–338.

Kitagawa, T. and A. Tetenov (2018). Who should be treated? Empirical welfare maximization methods for treatment choice. *Econometrica* 86(2), 591–616.
Kitagawa, T. and G. Wang (2021). Who should get vaccinated? individualized allocation of vaccines over sir network. *Journal of Econometrics*.

Leung, M. P. (2020). Treatment and spillover effects under network interference. *Review of Economics and Statistics* 102(2), 368–380.

Manski, C. F. (2013). Identification of treatment response with social interactions. *The Econometrics Journal* 16(1), S1–S23.

Miguel, E. and M. Kremer (2004). Worms: identifying impacts on education and health in the presence of treatment externalities. *Econometrica* 72(1), 159–217.

Muralidharan, K., P. Niehaus, and S. Sukhtankar (2017). General equilibrium effects of (improving) public employment programs: Experimental evidence from India. Technical report, National Bureau of Economic Research.

Ogburn, E. L., O. Sofrygin, I. Diaz, and M. J. van der Laan (2017). Causal inference for social network data. *arXiv preprint arXiv:1705.08527*.

Paluck, E. L., H. Shepherd, and P. M. Aronow (2016). Changing climates of conflict: A social network experiment in 56 schools. *Proceedings of the National Academy of Sciences* 113(3), 566–571.

Pouget-Abadie, J. (2018). *Dealing with Interference on Experimentation Platforms*. Ph. D. thesis.

Ross, N. et al. (2011). Fundamentals of stein’s method. *Probability Surveys* 8, 210–293.

Sävje, F., P. Aronow, and M. Hudgens (2021). Average treatment effects in the presence of unknown interference. *Annals of statistics* 49(2), 673.

Sinclair, B., M. McConnell, and D. P. Green (2012). Detecting spillover effects: Design and analysis of multilevel experiments. *American Journal of Political Science* 56(4), 1055–1069.

Sussman, D. L. and E. M. Airoldi (2017). Elements of estimation theory for causal effects in the presence of network interference. *arXiv preprint arXiv:1702.03578*.

Tabord-Meehan, M. (2018). Stratification trees for adaptive randomization in randomized controlled trials. *arXiv preprint arXiv:1806.05127*. 

31
Taylor, S. J. and D. Eckles (2018). Randomized experiments to detect and estimate social influence in networks. In Complex Spreading Phenomena in Social Systems, pp. 289–322. Springer.

Ugander, J., B. Karrer, L. Backstrom, and J. Kleinberg (2013). Graph cluster randomization: Network exposure to multiple universes. In Proceedings of the 19th ACM SIGKDD international conference on Knowledge discovery and data mining, pp. 329–337. ACM.

Vazquez-Bare, G. (2017). Identification and estimation of spillover effects in randomized experiments. arXiv preprint arXiv:1711.02745.

Viviano, D. (2019). Policy targeting under network interference. arXiv preprint arXiv:1906.10258.

Viviano, D. (2020). Policy choice in experiments with unknown interference. arXiv preprint arXiv:2011.08174.

Wager, S. and K. Xu (2021). Experimenting in equilibrium. Management Science 67(11), 6694–6715.
A Extensions

A.1 Higher-Order Dependence

In this section, we relax the local dependence assumption and consider the general case where unobservables exhibit $M$-dependence. We denote $N_i^M$ the set of individuals connected to individual $i$ by exactly $M$ edges.

We replace Assumption 2.1 with the following conditions.

**Assumption A.1 (Model under Higher-Order Dependence).** Assume in addition that for all $i \in \{1, ..., N\}$,

$$\{\varepsilon_i, \{\varepsilon_k\}_{k \notin \bigcup_{u=1}^{M} N_i^M} \} \perp \{\varepsilon_j\}_{j \notin \bigcup_{u=1}^{M} N_i^M} \bigg| A, T$$

$$(\varepsilon_i, \varepsilon_j) =_{d} (\varepsilon_{i'}, \varepsilon_{j'}) \bigg| A, T$$

for all $(i, j, i', j') : i \in N_j, i' \in N_j', T_i = T_{i'}, T_j = T_{j'}, (A.1)$

$N_{\max} < c$,

for a universal constant $c < \infty$.

Assumption A.1 states the following: (i) unobservables are independent whenever they are distant by more than $M$ edges; (ii) the joint distribution of two unobservables given the adjacency matrix is the same, whenever covariates are the same, and these unobservables are connected by the same number of edges. In addition, the assumption states that the maximum degree is uniformly bounded, which can be relaxed.\(^{15}\) The second condition is the experimental restriction. We define

$$\hat{H} = \{1, \cdots, N\} \setminus \bigg\{ i : P_i = 1, \bigcup_{u=1}^{M} N_i^M \bigg\}. \quad (A.2)$$

The set $\hat{H}$ denotes all individuals in the population of interest, after excluding the pilot units and the neighbors of the pilot units up to the $M$th degree. The following restriction is imposed.

\(^{15}\)After a quick inspection of the derivations contained in Appendix E, the reader may observe that such condition can be replaced by assuming that the maximum degree of the sampled units and their neighbors up to order $M$ scales at a rate slower than $n^{1/4}$. 

33
Assumption A.2 (Experimental Restriction). Let the following hold:

(A) \( \varepsilon_{i \in \tilde{H}} \perp (D^R, R) \mid A, T, P; \)

(B) \( \varepsilon_{1, \ldots, N} \perp P \mid A, T; \)

(C) \( R_j = 0 \) for all \( j \in \{ i : P_i = 1, \cup_{u=1}^M N_i^u \}. \)

The following theorem guarantees unbiased of the estimator for any design satisfying the above restrictions.

**Theorem A.1.** Under Assumption 2.1, Assumption A.1, A.2

\[
E \left[ \hat{\Gamma}_n(w_n) \mid P, A, D^R, R, T \right] = \Gamma_n(w_n).
\]

The design minimizes the estimated variance from the pilot study under the restrictions in Assumption A.2. The variance takes the following form:

\[
nV_N(w_N) = \frac{1}{n} \sum_{i:R_i=1} w_N(i, D^R, R, T) \text{Var}(Y_i \mid A, D^R, R, T)
\]

\[
+ \frac{1}{n} \sum_{i:R_i=1} \sum_{j \in N_i^1} R_j w_N(i, D^R, R, T) w_N(j, D^R, R, T) \text{Cov}(Y_i, Y_j \mid A, D^R, R, T)
\]

\[
+ \frac{1}{n} \sum_{i:R_i=1} \sum_{j \in N_i^2} R_j w_N(i, D^R, R, T) w_N(j, D^R, R, T) \text{Cov}(Y_i, Y_j \mid A, D^R, R, T)
\]

\[
+ \cdots
\]

\[
+ \frac{1}{n} \sum_{i:R_i=1} \sum_{j \in N_i^M} R_j w_N(i, D^R, R, T) w_N(j, D^R, R, T) \text{Cov}(Y_i, Y_j \mid A, D^R, R, T).
\]

Therefore, the variance sums over the covariances of each individual and her neighbors up to the \( M \)th degree. Notice now that the variance and each covariance component is identified with information containing neighbors’ assignments and outcomes up to the \( M \)th degree, where each covariance component depends on the distance of
unit $i$ from element $j$. Formally, we obtain that the following holds.

\[ \text{Var}(Y_i|A, D^R, R, T) = \text{Var}(r(D_i, \sum_{k \in N_i} D_k, T_i, \varepsilon_i)|D_i, \sum_{k \in N_i} D_k, T_i) = \sigma^2(T_i, D_i, \sum_{k \in N_i} D_k), \tag{A.4} \]

which guarantees identifiability of the variance function. Similarly, for a given $j \in N_i^u$ we have

\[ \text{Cov}(Y_i, Y_j|A, D^R, R, T) = \text{Cov}(r(D_i, \sum_{k \in N_i} D_k, T_i, \varepsilon_i), r(D_j, \sum_{k \in N_j} D_k, T_j, \varepsilon_j)|A, D^R, T) \]

\[ = \text{Cov}(r(D_i, \sum_{k \in N_i} D_k, T_i, \varepsilon_i), r(D_j, \sum_{k \in N_j} D_k, T_j, \varepsilon_j)|j \in N_i^u, D^R) \]

\[ = \eta_u(T_i, D_i, \sum_{k \in N_i} D_k, T_j, D_j, \sum_{k \in N_j} D_k). \tag{A.5} \]

The above expression states that under the above condition the covariance between two individuals, whose shortest path between such two individual is of length $u$ is a function which only depends on (a) the length of the path, (b) the treatment assignment of each of these two individuals, (c) the treatment assignments of the corresponding neighbors, (d) the network statistics $T_i$ and $T_j$ of these two individuals.

The design of the experiment consists in minimizing the variance under the restriction in Assumption A.2.

The following theorem allows for inference.

**Theorem A.2.** Suppose that Assumption 2.1, A.1, A.2 hold. Then for $V_N(w_n)$ as defined in Equation (A.3), with $n_1V_N(w_n) > 0$,

\[ \frac{\sqrt{n}(\hat{\Gamma}_n(w_N) - \Gamma_n(w_n))}{\sqrt{n}V_N(w_n)} \to_d \mathcal{N}(0, 1). \tag{A.6} \]

The proof of the theorem is contained in Appendix E. The variance can then be estimated consistently via plug in of each estimated covariance function. Similarly, the regret analysis follows similarly under consistency of each estimated covariance component. This is omitted for brevity.
A.2 Design with Partial Network Information

In this section, we consider the case where the researcher has access to partial network information only. The main assumption is that individuals are organized into at least two disconnected components (clusters), and the pilot study is constructed using information from one of the two clusters only. The experimental protocol is discussed below.

Experimental Protocol:

1. **Pilot study**: Researchers select a random sample of individuals $P$, which is assumed to be disconnected from all other eligible units, and assign treatments $D^P$ to the units in the pilot. This sample may be collected from a disconnected component of the network, which we denote as $H$. Researchers observe

$$\left[P_i\left(Y_i, D_i, T_i, D_{N_i}\right)\right], \quad i \in H : (i, N_i) \cap \{1, \cdots, N\} = \emptyset,$$

where $\{1, \cdots, N\}$ denotes the eligible sample for the main experiment. Researchers estimate $\hat{\sigma}_p, \hat{\eta}_p$ from the pilot study (note that the pilot study must contain for some individuals also their neighbors to be able to estimate $\eta(\cdot)$).

2. **Survey**: researchers collect network information of a random subset of individuals $i \in \{1, ..., N\}$, namely

$$\tilde{A} = \left( N_i, i \in S \subset \{1, \cdots, N\} \right), \quad T = \left( T_1, \cdots, T_N \right).$$

---

16Examples include a village (Banerjee et al., 2013), school (Paluck et al., 2016) or region (Muralidharan et al., 2017).
3. Main Experiment: researchers select the participants and the corresponding treatment assignments in the main experiment. They minimize the posterior variance

\[
\min_{r,d^r, w_N \in W_N} \max_{W} \mathbb{E} \left[ \hat{V}_{N, \sigma_p, \eta_p}(w_N; A, d^r, r, T, P) \mid \tilde{A}, T \right], \quad s.t.: \sum_{i=1}^{N} r_i = n, \quad r_i = 0 \text{ for all } i \in C.
\]

The posterior variance is estimated with respect to a prior distribution of \((A, T_1, \cdots, T_N) \sim \mathcal{P}\).

4. Second survey and analysis: researchers collect information

\[
\left[ R_i \left( Y_i, D_i, D_j \in N_i, N_i \right) \right]
\]

for each participant, and construct \(\hat{\Gamma}_n(w_N)\) as in Equation (7). We also estimate \(\hat{\sigma}, \hat{\eta}\) using information from the units sampled in the main experiment and their neighbors. Inference follows similarly to Section 4.2.

The experiment consists of four steps: a pilot study, where network information is available to the researcher, a first survey that collects partial network information, the design, and the analysis. Note that the estimator \(\hat{\Gamma}_n(w_N)\) and the variance and covariance functions \(\hat{\eta}, \hat{\sigma}\) can be estimated without observing the entire network, but it suffices the outcomes, covariates and treatment assignments of individuals in the main experiment and their neighbors. In the main experiment, we optimize the posterior expectation of the variance, fixing the estimated variance and covariance functions obtained from the main pilot, and averaging over the edges’ distributions. The optimization can be solved by alternating a Monte Carlo simulation, and the optimization through non-linear mixed-integer programming, which we show in Section 5, performs well in practice.\(^{17}\) Under the above protocol, the following holds.

\(^{17}\)The imputation problem can also be solved in a fully Bayesian fashion by imposing a prior distribution also on potential outcomes and taking the posterior density over the outcomes. However, a full derivation of a hierarchical model goes beyond the scope of this paper, and we leave this for
**Proposition A.3.** Let Assumption 2.1, 2.2 hold. Then the experimental design in the current section satisfies the restrictions in Proposition 3.1.

**Example A.1.** Consider the following Erdős-Rényi model:

\[
\{A_{i,j}\}_{j>i} \sim_{i.i.d} \text{Bern}(p), \quad p \sim U(0,1).
\]  

Assume in addition that \(A_{i,j} = A_{j,i}\) and \(A_{i,i} = 0\). The model assumes that each individual connects with independent probabilities. Such probabilities are modeled based on a uniform prior. Suppose we observe edges of a subset of individuals \(\tilde{n}\). Then we obtain that

\[
P(A_{i,j} = 1 | \tilde{A}) \sim \begin{cases} 
\delta_1 & \text{if } \tilde{A}_{i,j} = 1 \\
\delta_0 & \text{if } \tilde{A}_{i,j} = 0 \\
\text{Beta}(\alpha, \beta) & \text{if } \tilde{A}_{i,j} \text{ is missing}
\end{cases}
\]  

where \(\delta_c\) denotes a point-mass distribution at \(c\) and

\[
\alpha = \sum_{u>v} \tilde{A}_{u,v} 1\{\tilde{A}_{u,v} \in \{0, 1\}\} + 1, \quad \beta = \tilde{N} - \sum_{u>v} \tilde{A}_{u,v} 1\{\tilde{A}_{u,v} \in \{0, 1\}\} + 1
\]  

\(\tilde{N}\) is the number of observed connections.

**Example A.2.** Following Breza et al. (2020), we can consider a model of the form

\[
P(A_{i,j} = 1 | \nu_i, \nu_j, z_i, z_j, \delta) \propto \exp\left(\nu_i + \nu_j + \delta \text{dist}(z_i, z_j)\right),
\]  

where \(\nu_i\) denotes individual fixed effect, \(z_i\) denotes a position in some latent space and \(\delta\) is an hyper-parameter of interest.
A.3 Minimax Design in the Absence of Pilots

Whenever the variance and covariance functions are unavailable to the researcher, we devise an optimization algorithm over the identity of participants, treatment assignments, and a number of participating units under a maximal constraint on the variance function.

Suppose that the researcher has prior knowledge on

\[ \sigma \in \mathcal{S}, \quad \eta \in \mathcal{E}(\mathcal{S}), \]  

where for some given \( B_\sigma \in (0, \infty), \) \( L_\eta, U_\eta \in [0, 1], \)

\[ \mathcal{S} = \{ f : \{0, 1\} \times \mathbb{Z}^2 \mapsto \mathbb{R}_+, \quad ||f||_\infty \leq B_\sigma \} \]
\[ \mathcal{E}(\mathcal{S}) = \{ g(f_1, f_2) \in [-L_\eta f_1 f_2, U_\eta f_1 f_2], \quad f_1, f_2 \in \mathcal{S} \}. \]  

The function class encodes upper and lower bounds on the variance and covariance function.

Then in such a case, for a given threshold \( \beta_\alpha(w_n), \) the min-max optimization problem can be written as follows:

\[ \min_{r, d'} \sum_{i=1}^{N} r_i \tag{A.14} \]
subject to

\[ (i) \sup_{w_N \in \mathcal{W}_n, \eta \in \mathcal{E}(\mathcal{S}), \sigma \in \mathcal{S}} \tilde{V}_{n,p}(w_n; \cdot) - \beta_\alpha(w_N) \leq 0. \tag{A.15} \]

The optimization can be written with respect to additional parameters \( \sigma_{i}^2 \) which denote the variance of each element \( i \) and the parameters \( \eta_{i,j} \) which denote the covariance between \( i, j. \) The supremum is taken over a finite set of such parameters, under the constraint that \( \sigma_{i}^2 = \sigma_{j}^2 \) whenever \( i \) and \( j \) have the same treatment status, number of treated neighbors and \( T_i = T_j. \) Similarly for any pair \( (\eta_{i,j}, \eta_{u,v}). \)
B Additional Tables and Figures

We collect results of the simulated network in Table B.1, and Table B.2. Each table reports the variance averaged over two-hundred replications. Each design corresponds to a different set of parameters \((\beta_1, \beta_2)\), which can be found at the top of the table. In Figure B.1 we report the box plot for the bias.

Table B.1: Simulated network. Variance of the overall effect (sum of spillover and treatment effects). 200 replications. Each column corresponds to different values of the coefficient. Panel at the top collects results for the Erdős-Rényi graph and at the bottom for the Albert-Barabasi graph.

| ER     | (0,0) | (0, 0.5) | (0,1) | (0, 1.5) | (0.5,0.5) | (0.5,1) | (0.5,1.5) | (1,1.5) |
|--------|-------|----------|-------|----------|-----------|---------|-----------|---------|
| ELI    | 0.624 | 0.927    | 1.194 | 1.415    | 1.199     | 1.414   | 1.637     | 1.853   |
| Random All+ | 1.162 | 1.739    | 2.315 | 2.891    | 2.329     | 2.905   | 3.479     | 4.068   |
| Graph Clust+ | 0.640 | 0.991    | 1.343 | 1.694    | 1.361     | 1.713   | 2.063     | 2.434   |
| Saturation1+ | 0.908 | 1.378    | 1.849 | 2.316    | 1.859     | 2.330   | 2.801     | 3.282   |
| Graph Clust | 0.767 | 1.188    | 1.607 | 2.029    | 1.631     | 2.051   | 2.471     | 2.916   |
| Saturation1 | 1.090 | 1.654    | 2.217 | 2.781    | 2.231     | 2.794   | 3.358     | 3.932   |
| Saturation2+ | 0.679 | 1.047    | 1.416 | 1.783    | 1.430     | 1.800   | 2.169     | 2.550   |
| Saturation3+ | 0.993 | 1.587    | 2.177 | 2.771    | 2.178     | 2.771   | 3.364     | 3.954   |

| AB     | (0,0) | (0, 0.5) | (0,1) | (0, 1.5) | (0.5,0.5) | (0.5,1) | (0.5,1.5) | (1,1.5) |
|--------|-------|----------|-------|----------|-----------|---------|-----------|---------|
| ELI    | 0.714 | 0.909    | 1.278 | 1.566    | 1.294     | 1.548   | 1.595     | 2.031   |
| Random All+ | 1.144 | 1.714    | 2.284 | 2.851    | 2.299     | 2.874   | 3.482     | 4.028   |
| Graph Clust+ | 0.693 | 1.098    | 1.503 | 1.908    | 1.531     | 1.938   | 2.060     | 2.773   |
| Saturation1+ | 0.936 | 1.435    | 1.936 | 2.434    | 1.950     | 2.451   | 2.800     | 3.464   |
| Graph Clust | 0.837 | 1.325    | 1.811 | 2.299    | 1.845     | 2.333   | 2.471     | 3.338   |
| Saturation1 | 1.132 | 1.733    | 2.336 | 2.934    | 2.354     | 2.955   | 3.358     | 4.179   |
| Saturation2+ | 0.732 | 1.152    | 1.572 | 1.992    | 1.594     | 2.015   | 2.169     | 2.882   |
| Saturation3+ | 1.091 | 1.762    | 2.433 | 3.103    | 2.425     | 3.096   | 3.364     | 4.425   |
Table B.2: Simulated network. Maximum variance between estimator of the direct treatment and spillover effect. 200 replications. Each column corresponds to different values of the coefficient. Panel at the top collects results for the Erdős-Rényi graph and at the bottom for the Albert-Barabasi graph.

|       | ER     | (0,0) | (0, 0.5) | (0, 1) | (0.5,0.5) | (0,5) | (0.5,1.5) | (1,1.5) |
|-------|--------|-------|----------|--------|-----------|-------|-----------|---------|
|       | ELI    | 0.545 | 0.782    | 1.091  | 1.308     | 1.102 | 1.321     | 1.580   | 1.864   |
|       | Random All+ | 0.676 | 1.037    | 1.400  | 1.763     | 1.379 | 1.740     | 2.101   | 2.441   |
|       | Graph Clust+ | 1.224 | 1.674    | 2.120  | 2.568     | 2.601 | 3.046     | 3.497   | 4.424   |
|       | Saturation1+ | 0.678 | 1.036    | 1.395  | 1.756     | 1.409 | 1.769     | 2.128   | 2.501   |
|       | Graph Clust | 1.496 | 2.038    | 2.585  | 3.129     | 3.173 | 3.717     | 4.259   | 5.397   |
|       | Saturation1 | 0.825 | 1.262    | 1.698  | 2.136     | 1.715 | 2.150     | 2.588   | 3.037   |
|       | Saturation2+ | 0.969 | 1.393    | 1.820  | 2.247     | 2.053 | 2.478     | 2.901   | 3.564   |
|       | Saturation3+ | 0.930 | 1.474    | 2.016  | 2.562     | 1.930 | 2.473     | 3.013   | 3.474   |

|       | AB     | (0,0) | (0, 0.5) | (0, 1) | (0.5,0.5) | (0,5) | (0.5,1.5) | (1,1.5) |
|-------|--------|-------|----------|--------|-----------|-------|-----------|---------|
|       | ELI    | 0.571 | 0.792    | 1.081  | 1.359     | 1.059 | 1.399     | 1.574   | 1.879   |
|       | Random All+ | 0.672 | 1.057    | 1.443  | 1.830     | 1.398 | 1.782     | 2.101   | 2.510   |
|       | Graph Clust+ | 0.984 | 1.383    | 1.784  | 2.184     | 2.192 | 2.594     | 3.495   | 3.809   |
|       | Saturation1+ | 0.676 | 1.060    | 1.444  | 1.829     | 1.453 | 1.837     | 2.127   | 2.613   |
|       | Graph Clust | 1.204 | 1.689    | 2.175  | 2.661     | 2.678 | 3.163     | 4.261   | 4.638   |
|       | Saturation1 | 0.827 | 1.294    | 1.763  | 2.233     | 1.773 | 2.239     | 2.587   | 3.189   |
|       | Saturation2+ | 0.859 | 1.262    | 1.665  | 2.066     | 1.902 | 2.307     | 2.904   | 3.350   |
|       | Saturation3+ | 0.984 | 1.590    | 2.196  | 2.805     | 2.107 | 2.713     | 3.015   | 3.834   |
Figure B.1: Box-plot of the difference between point estimate and true value for Albert-Barabasi and Erdos-Renyi networks, with \((\beta_1, \beta_2) = (0.5, 0.5)\). The box plot reports the results for two-hundred replications.

C Identification

Proof of Proposition 3.1

Consider all \(D^R\) such that \(D_i = d, \sum_{k \in N_i} D_k = s\), and all \(A, T\) such that \(T_i = l\). To derive the result we want to show that

\[
\mathbb{E}[Y_i \mid D_i = d, \sum_{k \in N_i} D_k = s, T_i = l, D^R, A, R, T, P] = m(d, s, l) \quad (C.1)
\]

for all those units in the sample (i.e., \(R_i = 1\)).

Notice first that under Assumption 2.1,

\[
\mathbb{E}[Y_i \mid D_i = d, \sum_{k \in N_i} D_k = s, T_i = l, D^R, A, R, T, P] = \\
\mathbb{E}
\left[
r(d, s, l, \varepsilon_i) \mid D_i = d, \sum_{k \in N_i} D_k = s, T_i = l, D^R, A, R, T, P
\right]. \quad (C.2)
\]
Observe now that under the conditions in Proposition 3.1, since participants are not units in the pilot study, and since $\varepsilon_i(d)$ is a constant function in $d$, we have that the following holds:

$$\mathbb{E}\left[ r\left(d, s, l, \varepsilon_i\right) \middle| D_i = d, \sum_{k \in N_i} D_k = s, T_i = l, D^R, A, R, T, P \right] = \mathbb{E}\left[ r\left(d, s, l, \varepsilon_i\right) \middle| T_i = l, A, T, P \right].$$

Since $P$ is exogenous conditional on $(A, T)$, we have

$$\mathbb{E}\left[ r\left(d, s, l, \varepsilon_i\right) \middle| T_i = l, A, T \right] = \mathbb{E}\left[ r\left(d, s, l, \varepsilon_i\right) \middle| T_i = l, A, T \right].$$

Under Assumption 2.1, since $\varepsilon_i \perp (A, T)$, the proof completes.

**Proof of Theorem A.1**

The proof follows similarly to the previous proof. Consider all $D^R$ such that $D_i = d$, $\sum_{k \in N_i} D_k = s$, and all $A, T$ such that $T_i = l$. Notice first that under Equation (1),

$$\mathbb{E}\left[ Y_i \middle| D_i = d, \sum_{k \in N_i} D_k = s, T_i = l, D^R, A, R, T, P \right] = \mathbb{E}\left[ r\left(d, s, l, \varepsilon_i\right) \middle| D_i = d, \sum_{k \in N_i} D_k = s, T_i = l, D^R, A, R, T, P \right].$$

Observe now that under Assumption A.2, since participants are not units in the pilot study and their neighbors up to the $Mth$ degree, we have that the following holds:

$$\mathbb{E}\left[ r\left(d, s, l, \varepsilon_i\right) \middle| D_i = d, \sum_{k \in N_i} D_k = s, T_i = l, D^R, A, R, T, P \right] = \mathbb{E}\left[ r\left(d, s, l, \varepsilon_i\right) \middle| A, T, T_i = l \right].$$

Under Equation (1), since $\varepsilon_i \perp (A, T)$, the proof completes.
Proof of Lemma 3.2

For all individuals $i$ selected in the pilot, we can write for $T_i$ such that $T_i = l$,

\[
\text{Var}(Y_i \mid A, T, D_i = d, T_i = l, \mathcal{D}_N = s, P) = \text{Var}(r(D_i, \sum_{k \in \mathcal{N}_i} D_k, T_i, \varepsilon_i) \mid A, T, D_i = d, T_i = l, \mathcal{D}_N = s, P)
\]

\[
= \text{Var}(r(d, 1^\top s, l, \varepsilon_i) \mid A, T_i = l, T, P)
\]

\[
= \text{Var}(r(d, 1^\top s, l, \varepsilon_i) \mid A, T_i = l)
\]

\[
= \text{Var}(r(d, 1^\top s, l, \varepsilon_i)) = \sigma^2(d, 1^\top s, l).
\]

This complete the claim for the variance. The first equation follows from Assumption 2.1. The second equation follows from the fact that treatments are randomized exogenously and $\varepsilon_i(D)$ is constant in $D$. The third equation follows from the fact that $P$ is exogenous conditional on $T, A$ only. The last equation follows from the fact that $\varepsilon_i \perp (T, A)$. The analysis of the covariances follows similarly. Namely, following the same steps as before, for $T$ such that $T_i = l, T_j = l'$

\[
\text{Cov}(Y_i, Y_j \mid A, T, D_i = d, D_j = d', D_N = s, D_N = s', T_i = l, T_j = l', P) = \text{Cov}(r(d, 1^\top s, l, \varepsilon_i), r(d', 1^\top s', l', \varepsilon_j) \mid A, T_i = l, T_j = l', P)
\]

where we used the exogeneity of the treatment assignment and the fact that $\varepsilon_i(d)$ is a constant function in $d$. Using Assumption 2.2,

\[
\text{(C.6)} = \begin{cases} 
\text{Cov}(r(d, 1^\top s, l, \varepsilon_i), r(d', 1^\top s', l', \varepsilon_j)) & \text{if } j \in \mathcal{N}_i, P, T, A, T_i = l, T_j = l', \\
0 & \text{otherwise.} 
\end{cases}
\]

Using the second condition in Assumption 2.2, and the fact that $P$ is exogenous conditional on $A, T$, we can write

\[
\text{Cov}(r(d, 1^\top s, l, \varepsilon_i), r(d', 1^\top s', l', \varepsilon_j)) \mid j \in \mathcal{N}_i, P, T, A, T_i = l, T_j = l', j \in \mathcal{N}_i
\]

\[
= \text{Cov}(r(d, 1^\top s, l, \varepsilon_i), r(d', 1^\top s', l', \varepsilon_j)) \mid j \in \mathcal{N}_i, T_i = l, T_j = l', A, T
\]

\[
= \text{Cov}(r(d, 1^\top s, l, \varepsilon_i), r(d', 1^\top s', l', \varepsilon_j)) \mid j \in \mathcal{N}_i, T_i = l, T_j = l'),
\]

44
where the last equation follows from Assumption 2.2. The proof completes under the second condition in Assumption 2.2.

**Proof of Lemma 3.3**

The proof follows similarly to the one of Lemma 3.2.

Consider all $D, R$ such that $D_i = d, \sum_{k \in N_i} D_k = s$, and all $A, T$ such that $T_i = l, R : R_i = 1$. Under Assumption 2.1,

\[
\text{Var}\left(Y_i \mid D_i = d, \sum_{k \in N_i} D_k = s, D, R, A, T_i = l, R_i = 1, T, P\right) \\
= \text{Var}\left(r(d, s, l, \varepsilon_i) \mid D_i = d, \sum_{k \in N_i} D_k = s, D, R, A, T_i = l, R_i = 1, T, P\right). \tag{C.7}
\]

Under Assumption 2.2, since $R_i = 0$ for all those units not being in the pilot study and their neighbors (and the algorithm satisfies the restrictions in Proposition 3.1), it follows that

\[
\text{Var}\left(r(d, s, l, \varepsilon_i) \mid D_i = d, \sum_{k \in N_i} D_k = s, D, R, A, T_i = l, R_i = 1, P, T\right) \\
= \text{Var}\left(r(d, s, l, \varepsilon_i) \mid T, A, P\right) = \text{Var}\left(r(d, s, l, \varepsilon_i)\right), \tag{C.8}
\]

where the first equality follows from the fact that $D, R$ are assigned deterministically based on $(T, A, P)$ and unobservables $\varepsilon_j$ which do not share common neighbors with $\varepsilon_i$. The second equality follows from Assumption 2.1, since $\varepsilon_i \perp (A, T)$, and the fact that $P$ is conditional independent of $\varepsilon$ given $A, T$.

For the covariance component, the same reasoning follows. Consider $(i, j) : R_i =
\( R_j = 1, \) and all \( A, T \) such that \( T_i = l, T_j = l' \). Then we write
\[
\text{Cov}\left( Y_i, Y_j \bigg| D_i = d, D_j = d', \sum_{k \in \mathcal{N}_i} D_k = s, \sum_{k \in \mathcal{N}_j} D_k = s', D^R, R, A, T_i = l, T_j = l', R_i = 1, R_j = 1, T, P \right) = \\
\text{Cov}\left( r(d, s, l, \varepsilon_i), r(d', s', l', \varepsilon_j) \bigg| D_i = d, D_j = d', \sum_{k \in \mathcal{N}_i} D_k = s, \sum_{k \in \mathcal{N}_j} D_k = s', D^R, R, A, T_i = l, T_j = l', R_i = 1, R_j = 1, T, P \right).
\] (C.9)

By Assumption 2.2, since \( D^R, R \) depends on \( (A, T, P) \) and unobservables which are not connected to \((i, j)\), and \( \varepsilon_i(d) \) is a constant function in \( d \) we obtain that Equation \( (C.9) \) equals
\[
\text{Cov}\left( r(d, s, l, \varepsilon_i), r(d', s', l', \varepsilon_j) \bigg| T_i = l, T_j = l', R_i = 1, R_j = 1, T, P \right). 
\] (C.10)

The covariance is zero if two individuals are not neighbors. In such a case the lemma trivially holds. Therefore, consider the case where individuals are neighbors. Then the above equation equals (under Assumption 2.2)
\[
(C.10) = \begin{cases} 
\text{Cov}\left( r(d, s, l, \varepsilon_i), r(d', s', l', \varepsilon_j) \bigg| T_i = l, T_j = l', j \in \mathcal{N}_i, A, T \right), & j \in \mathcal{N}_i \\
0 & \text{otherwise}
\end{cases}
\]

which follows by conditional exogeneity of \( P \) given \( (A, T) \) (see Algorithm 1). The proof completes under the second condition in Assumption 2.2.

**D Proof of Theorem 4.1**

**Preliminaries**  We say that \( a \preceq b \) if \( a \leq \bar{C}b \) for a finite constant \( \bar{C} \). Recall that \( D \) denotes the vector of treatments assigned to the pilot and main experiment as in Algorithm 1, 2. Denote \( R \) the participation indicators obtained after solving the experimenter problem in Equation (13). We denote \( D^R \) the subvector of assignments

46
for those units with \( R_i = 1 \) and their friends. For an arbitrary \((D^*, R^*)\), we denote

\[
\hat{V}_{n,p}(D^*, R^*) = \max_{w_N \in \mathcal{W}_N} \hat{V}_{n,\hat{\sigma}_{p, \hat{\eta}_p} (w_N; D^{* R^*}, R^*, T, A),}
\]

the maximum conditional variance over \( \mathcal{W}_N \) with estimated covariance and variance function obtained from the pilot experiment. Similarly, \( V_N(D^*, R^*) \) corresponds to the population counterpart. For notational convenience we refer to \( w_N(i, D, R) \), as the weight for unit \( i \) evaluated at the values \( D^R, R \) denoting the assignments of those in the experiments and their friends (either in or not in the experiment), and the participation indicators, and omitting the other arguments, whenever clear from the context. Let

\[
(\hat{D}, \hat{R}) \in \arg \min_{d, r, n_1 \leq \sum_{i=1}^N r_i \leq n_2, r_j = 0 \forall j \not\in J} V_N(d', r),
\]

(D.1)

the optimal participants’ selection for known variance and covariance function and constraint on the pilot units as in Algorithm 2. We define

\[
\sigma(i, D) = \sigma(T_i, D_i, \sum_{k \in N_i} D_k)
\]

and similarly for \( \eta(i, j, D) \). Recall that the population variance and covariance functions in Lemma 3.3 only depend on treatment assignments (and \( T, A \)), but not on the experimental selector indicator \( R \). Therefore, we can define \( \sigma(i, \cdot), \eta(i, j, \cdot) \) as a function of the vector of treatments \( D \) in the population of \( N \) units, leaving implicit its dependence with \( T, A \). Note that under Assumption 4.2, since \( Y \) is uniformly bounded, also \( \sigma^2(\cdot), \eta(\cdot) \) are uniformly bounded.

Finally, note that in Algorithm 2 we can assign treatments optimally to individuals who are not in the pilot, but are their friends. This implies that although the treatment assigned to the pilot units cannot be changed, the vector of treatments for all units in the main experiment and their friends can be chosen arbitrary in Algorithm 2, provided that participants in the main experiment are not in the pilot. Therefore, we can consider the optimization problem in Algorithm 2 unconstrained with respect to treatment assignments \( D \), given that individuals in the pilot and
their friends are not participants, since in this case the overall variance does not depend on the treatments of the pilot units. We can now discuss the proof.

**Preliminary upper bound**  First observe that $|J| \leq N_{\text{max}}$. Since $n_1 > N_{\text{max}}m/(n_2/n_1 - 1) \geq |J|/(n_2/n_1 - 1)$ we have that the constraint $1^\top r = n_2$ is a stricter constraint than $n_1 + |J| \leq 1^\top r \leq n_2$. We can therefore write

$$
\mathcal{R}_N = V_N(D, R) - \min_{d, r, \sum_{i=1}^N r_i = n_2} V_N(d, r)
\leq V_N(D, R) - \min_{d, r, n_1 + |J| \leq \sum_{i=1}^N r_i \leq n_2} V_N(d, r)
\leq V_N(D, R) - \min_{d, r, n_1 + |J| \leq \sum_{i=1}^N r_i \leq n_2} V_N(d, r) + V_N(\tilde{D}, \tilde{R}) - \hat{V}_{n, p}(D, R)
+ \hat{V}_{n, p}(D, R) - V_N(\tilde{D}, \tilde{R})
\leq \left( V_N(D, R) - \hat{V}_{n, p}(D, R) \right) + \left( \hat{V}_{n, p}(\tilde{D}, \tilde{R}) - V_N(\tilde{D}, \tilde{R}) \right)
+ V_N(\tilde{D}, \tilde{R}) - \min_{d, r, n_1 + |J| \leq \sum_{i=1}^N r_i \leq n_2} V_N(d, r).
$$

(D.3)

The last bound follows from the fact that $\hat{V}_{n, p}(\tilde{D}, \tilde{R}) \geq \hat{V}_{n, p}(D, R)$, since $\tilde{R}$ and $R$ satisfy the same set of constraints, and $(D^R, R)$ minimizes $\hat{V}_{n, p}(D, R)$. We study each component separately.

**Component (i) and (ii)**  We can write

$$
(i) \leq \max_{w_N \in W_N} \left| \frac{1}{(1 \top R)^2} \sum_{i=1}^N w_N^2(i, D, R) R_i \left( \sigma^2(i, D) - \hat{\sigma}^2_p(i, D) \right) \right|
+ \frac{1}{(1 \top R)^2} \sum_{i=1}^N \sum_{j \in N_i} w_N(i, D, R) w_N(j, D, R) R_i R_j \left( \eta(i, j, D) - \hat{\eta}_p(i, j, D) \right).
$$

(D.4)
Therefore, we obtain

\[
(i) \leq \max_{w_N \in W_N} \left\{ \frac{1}{(1^T R)^2} \sum_{i=1}^{N} w_N(i, D, R) R_i \left( \sigma^2(i, D) - \hat{\sigma}^2_p(i, D) \right) \right\}
\]

\[
+ \max_{w_N \in W_N} \left\{ \frac{1}{(1^T R)^2} \sum_{i=1}^{N} \sum_{j \in N_i} w_N(i, D, R) R_i R_j (\eta(i, j, D) - \hat{\eta}_p(i, j, D)) \right\}.
\]

(D.5)

The above term satisfies (since \(w(i, \cdot)\) is uniformly bounded by design, see Algorithm 2)

\[
(D.5) \lesssim N_{\max} \sup_{d,s,l,d',s',l'} \left| \eta(d, s, l, d', s', l') - \hat{\eta}_p(d, s, l', d', s', l') \right| / n_1
\]

\[
+ \sup_{d,s,l} \left| \sigma(l, d, s) - \hat{\sigma}_p(l, d, s) \right| / n_1.
\]

(D.6)

The same reasoning also applies to the term \((ii)\) in Equation (D.3). Therefore, we can write under Assumption 4.1

\[(i) + (ii) = O_p(N_{\max} m^{-\xi} / n_1).
\]

(D.7)

(iii) **Part 1: Lower bound for** \(\min V_n(d, r)\)  Finally, consider the term \((iii)\). As a first step, we provide a lower bound to \(\min_{d, r, n_1 + |J| \leq \sum_{i=1}^{N} r_i \leq n_2} V_N(d, r)\).

We can write

\[
\min_{d, r, n_1 + |J| \leq \sum_{i=1}^{N} r_i \leq n_2} V_N(d, r) = \min_{d, r, n_1 + |J| \leq \sum_{i=1}^{N} r_i \leq n_2} \max_{w_N \in W_N}
\]

\[
\left( \frac{1}{(\sum_{i=1}^{N} r_i)^2} \sum_{i \in J} r_i w_N^2(i, d, r) \sigma^2(i, d) + \sum_{j \in N_i} r_i r_j w_N(j, d, r) \eta(i, j, d) \right)
\]

\[
+ \frac{1}{(\sum_{i=1}^{N} r_i)^2} \sum_{i \in J} r_i w_N^2(i, d', r) \sigma^2(i, d') + \sum_{j \in N_i} r_i r_j w_N(j, d, r) \eta(i, j, d') \right)
\]

\[
\geq (A) + (B)
\]

(D.7)
where

\[(A) = \min_{d, r, n_1 + |\mathcal{J}| \leq \sum_{i=1}^{N} r_i \leq n_2} \max_{w_N \in W_N} \frac{1}{(\sum_{i=1}^{N} r_i)^2} \left( \sum_{i \in \mathcal{J}_c} r_i w_N^2(i, d, r) \sigma^2(i, d) \right) + \sum_{j \in N_i} r_i r_j w_N(i, d, r) w_N(j, d, r) \eta(i, j, d) \)

\[(B) = \min_{d, r, n_1 + |\mathcal{J}| \leq \sum_{i=1}^{N} r_i \leq n_2} \min_{w_N \in W_N} \frac{1}{(\sum_{i=1}^{N} r_i)^2} \left( \sum_{i \in \mathcal{J}} r_i w_N^2(i, d, r) \sigma^2(i, d) \right) + \sum_{j \in N_i} r_i r_j w_N(i, d, r) w_N(j, d, r) \eta(i, j, d) \)

where we decomposed the sum into the sum over two sets and flip the \(\max_{w_N}\) with the \(\min_{w_N}\) for one of the two sets to obtain a lower bound. Such sets are defined as

\[\mathcal{J}_c = \{1, \cdots, N\} \setminus \{i, N_i : P_i = 1\}, \quad \mathcal{J} = \{i, N_i : P_i = 1\}.\]

(iii) **Part two: lower bound decomposed into two groups \(\mathcal{J}_c, \mathcal{J}\)** We now analyze each component in the right hand side of Equation (D.7). Notice now that the following term

\[(B) \geq \min_{d, r, n_1 + |\mathcal{J}| \leq \sum_{i=1}^{N} r_i \leq n_2} \min_{w_N \in W_N} \frac{1}{(\sum_{i=1}^{N} r_i)^2} \sum_{i \in \mathcal{J}} \sum_{j \in \mathcal{N}_i} r_i r_j w_N(i, d, r) w_N(j, d, r) \eta(i, j, d) \]

\[\geq -\bar{C}|\mathcal{J}| \max_{i \in \mathcal{J}} |N_i| / (n_1 + |\mathcal{J}|)^2 \]

(D.8)

since the second moment and weights are bounded by Assumption 4.2, for a universal constant \(\bar{C} < \infty\). Therefore, the following holds:

\[(D.7) \geq (A) - \bar{C}|\mathcal{J}| \max_{i \in \mathcal{J}} |N_i| / (n_1 + |\mathcal{J}|)^2. \quad (D.9)\]
(iii) Part 3: Lower bound to (A)  
Next, we provide a lower bound to (A). We have

\[
(A) \geq \min_{d,r,n_1 + |\mathcal{J}| \leq \sum_i r_i \leq n_2} \max_{w_N \in \mathcal{W}_N} \left( \frac{1}{\sum_{i \in \mathcal{J}^c} r_i + |\mathcal{J}|} \right)^2 \sum_{i \in \mathcal{J}^c} r_i w_N^2(i, d, r) \sigma^2(i, d) \\
+ \sum_{j \in \mathcal{N}_i} r_i r_j w_N(i, d, r) w_N(j, d, r) \eta(i, j, d) := (J),
\]

where we replaced in the denominator \( \sum_{i \in \mathcal{J}} r_i \) with \(|\mathcal{J}|\). Therefore, we can write

\[
(iii) \leq V_N(\tilde{D}, \tilde{R}) - \min_{d,r,n_1 + |\mathcal{J}| \leq \sum_i r_i \leq n_2} \max_{w_N \in \mathcal{W}_N} \left( \frac{1}{\sum_{i \in \mathcal{J}^c} r_i + |\mathcal{J}|} \right)^2 \left( \sum_{i \in \mathcal{J}^c} r_i w_N^2(i, d, r) \sigma^2(i, d) \\
+ \sum_{j \in \mathcal{N}_i} r_i r_j w_N(i, d, r) w_N(j, d, r) \eta(i, j, d) \right) + \tilde{C} |\mathcal{J}| \max_{i \in \mathcal{J}} |\mathcal{N}_i| / (n_1 + |\mathcal{J}|)^2.
\]

(D.11)

(iii) Part 4: Upper bound to \( V_N(\tilde{D}, \tilde{R}) \)  
Consider now the right-hand side in Equation (D.10), defined as \((J)\). Observe, that we can write

\[
(J) = (L) + (M)
\]

where

\[
(L) = \min_{d,r,n_1 + |\mathcal{J}| \leq \sum_i r_i \leq n_2} \max_{w_N \in \mathcal{W}_N} \left( \frac{1}{\sum_{i \in \mathcal{J}^c} r_i + |\mathcal{J}|} \right)^2 \left( \sum_{i \in \mathcal{J}^c} r_i w_N^2(i, d, r) \sigma^2(i, d) \\
+ \sum_{j \in \mathcal{N}_i} r_i r_j w_N(i, d, r) w_N(j, d, r) \eta(i, j, d) \right)
\]

\[
(M) = \min_{r,n_1 + |\mathcal{J}| \leq \sum_i r_i \leq n_2} \max_{w_N \in \mathcal{W}_N} \left( \frac{1}{\sum_{i \in \mathcal{J}^c} r_i + |\mathcal{J}|} \right)^2 \left( \sum_{i \in \mathcal{J}^c} r_i w_N^2(i, d, r) \sigma^2(i, d) \\
+ \sum_{j \in \mathcal{N}_i} r_i r_j w_N(i, d, r) w_N(j, d, r) \eta(i, j, d) \right) - (L).
\]

(D.12)
where we added and substracted \((L)\) which contains the condition \(r_i = 0\) \(\forall i \in \mathcal{J}\), and \(n_1 \leq \sum_{i \in \mathcal{J}} r_i \leq n_2\). We study \((L)\) first. Define

\[
(D^{**}, R^{**}) \in \arg \min_{d, r, n_1 \leq \sum_{i \in \mathcal{J}} r_i \leq n_2, r_i = 0} \max_{w_N \in W_N} \frac{1}{(\sum_{i \in \mathcal{J}} r_i + |\mathcal{J}|)^2} \left( \sum_{i \in \mathcal{J}} r_i w_N^2(i, d, r) \sigma^2(i, d) + \sum_{j \in \mathcal{N}_i} r_j w_N(i, d, r) w_N(j, d, r) \eta(i, j, d) \right).
\]

Observe that by construction \(V_N(\tilde{D}, \tilde{R}) \leq V_N(D^{**}, R^{**})\), since \(\tilde{D}, \tilde{R}\) minimize \(V_N(\cdot)\), and since \(R^{**}\) satisfy the constraints in Equation (D.1). Therefore, we can write

\[
(D.11) \leq V_N(D^{**}, R^{**}) - \max_{w_N \in W_N} \frac{1}{(\sum_{i \in \mathcal{J}} R_i^{**} + |\mathcal{J}|)^2} \left( \sum_{i \in \mathcal{J}} R_i^{**} w_N^2(i, D^{**}, R^{**}) \sigma^2(i, D^{**}) + \sum_{j \in \mathcal{N}_i} R_j^{**} w_N(i, D^{**}, R^{**}) w_N(j, D^{**}, R^{**}) \eta(i, j, D^{**}) \right) + \bar{C}|\mathcal{J}| \max_{i \in \mathcal{J}} |\mathcal{N}_i|/(n_1 + |\mathcal{J}|)^2 - (M).
\]

\[(D.13)\]

(iii) Part 4: first Term in the right-hand side of Equation (D.13) By simple algebra, and using the same argument for the weights used for \((i)\), we obtain,

\[
V_N(D^{**}, R^{**}) - \max_{w_N \in W_N} \frac{1}{(\sum_{i \in \mathcal{J}} R_i^{**} + |\mathcal{J}|)^2} \left( \sum_{i \in \mathcal{J}} R_i^{**} w_N^2(i, D^{**}, R^{**}) \sigma^2(i, D^{**}) + \sum_{j \in \mathcal{N}_i} R_j^{**} w_N(i, D^{**}, R^{**}) w_N(j, D^{**}, R^{**}) \eta(i, j, D^{**}) \right) \leq \max_{w_N \in W_n} \left| \left( \frac{1}{(\sum_{i \in \mathcal{J}} R_i^{**} + |\mathcal{J}|)^2} - \frac{1}{(\sum_{i \in \mathcal{J}} R_i^{**})^2} \right) \left( \sum_{i \in \mathcal{J}} R_i^{**} w_N^2(i, D^{**}, R^{**}) \sigma^2(i, D^{**}) + \sum_{j \in \mathcal{N}_i} R_j^{**} w_N(i, D^{**}, R^{**}) w_N(j, D^{**}, R^{**}) \eta(i, j, D^{**}) \right) \right|.
\]

\[(D.14)\]

(iii) Part 5: bound on (iii) which also depends on \((M)\) By Assumption 4.2 (bounded outcome), for (iii) as in Equation (D.3), using Equation (D.14) we can
write

\[(iii) \leq \tilde{C}nN_{\text{max}} \frac{n|\mathcal{J}| + |\mathcal{J}|^2}{(\sum_{i=1}^{N} R_i^*)^4} + \tilde{C}|\mathcal{J}| \max_{i \in \mathcal{J}} |\mathcal{N}_i|/(n_1 + |\mathcal{J}|)^2 - (M)\]

\[\leq \tilde{C}N_{\text{max}} \frac{n^2|\mathcal{J}| + n|\mathcal{J}|^2}{\alpha^4 n^4} + \tilde{C}|\mathcal{J}| N_{\text{max}}/(n_1 + |\mathcal{J}|)^2 - (M)\]

for a finite constant \(\tilde{C} < \infty\). Notice now that \(|\mathcal{J}| \leq |N_{\text{max}}| \times m\) which implies that the above term is \(\mathcal{O}(N_{\text{max}}^2 m/n_1^2 + N_{\text{max}}^3 m^2/n_1^3) + \mathcal{O}(|(M)|)\). Here \(N_{\text{max}}^3 m^2/n_1^3 = \mathcal{O}(N_{\text{max}}^2 m/n_1^2)\) since under the assumptions \(n_1 \geq N_{\text{max}} m/(\alpha - 1)\) for \(\alpha > 1\).

**Bound on \((M)\)** We are left to provide a bound for \((M)\). The bound on \((M)\) follows from the stability assumption (Assumption 4.3). In particular, let

\[(d^*, r^*) \in \arg \min_{r, n_1 + |\mathcal{J}| \leq \sum_{i \in \mathcal{J}_c} r_i \leq n_2} \max_{w_N \in \mathcal{W}_N} \frac{1}{(\sum_{i \in \mathcal{J}_c} r_i + |\mathcal{J}|)^2} \left( \sum_{i \in \mathcal{J}_c} r_i w_N^2(i, d, r) \sigma^2(i, d) + \sum_{j \in \mathcal{N}_i} r_i r_j w_N(i, d, r) w_N(j, d, r) \eta(i, j, d) \right),\]

and \(\tilde{r}^*\) be such that \(r_j^* = \tilde{r}_j^*, j \not\in \mathcal{J}\) and \(r_j^* = 0\) otherwise. We note that \((d^*, \tilde{r}^*)\) is a feasible solution to minimize \((L)\), since \((L)\) contains a slacker constraint \(n_1 \leq \sum_{j \in \mathcal{J}_c} r_i \leq n_2\), while \(n_1 + |\mathcal{J}| \leq \sum_{i} r_j \leq n_2\). Define \(\mathcal{J}_3 = \mathcal{J}_c \setminus \{j : N_j \cap \mathcal{J} \neq \emptyset\}\) the set of individuals in \(\mathcal{J}_c\) without friends in \(\mathcal{J}\). We can then write

\[|\mathcal{M}| \leq \max_{w_N \in \mathcal{W}_N} \frac{1}{(\sum_{i \in \mathcal{J}_c} r_i^* + |\mathcal{J}|)^2} \left( \sum_{i \in \mathcal{J}_3} r_i^* (w_N^2(i, d^*, r^*) - w_N^2(i, d^*, \tilde{r}^*)) \sigma^2(i, d^*) + \sum_{j \in \mathcal{N}_i} r_i^* r_j^* w_N(i, d^*, r^*) w_N(j, d^*, \tilde{r}^*) \eta(i, j, d^*) \right) + \max_{w_N \in \mathcal{W}_N} \frac{1}{(\sum_{i \in \mathcal{J}_c} r_i^* + |\mathcal{J}|)^2} \sum_{i \in \mathcal{J}_c, j \in \mathcal{N}_i \cap \mathcal{J}} r_i^* r_j^* w_N(i, d^*, r^*) w_N(j, d^*, r^*) \eta(i, j, d^*) \]

\[+ \max_{w_N \in \mathcal{W}_N} \frac{1}{(\sum_{i \in \mathcal{J}_c} r_i^* + |\mathcal{J}|)^2} \sum_{i \in \mathcal{J}_c \setminus \mathcal{J}_3} r_i^* \left( w_N^2(i, d^*, r^*) - w_N^2(i, d^*, \tilde{r}^*) \right) \sigma^2(i, j, d^*).\]

(D.16)

The first term sums over variances and covariances of each individual in \(\mathcal{J}_3\) and excludes the individuals in \(\mathcal{J}_c\) which are friends with individuals in \(\mathcal{J}\). The second term
sums over the covariances of individuals in $J_c$ and individuals in $J$ friends with individuals in $J_c$. Such covariances multiply by $r_i^* r_j^* w_N(i, d^*, r^*) w_N(j, d^*, r^*) \eta(i, j, d^*)$ only (and not also $\tilde{r}_i^* \tilde{r}_j^* w_N(i, d^*, \tilde{r}^*) w_N(j, d^*, \tilde{r}^*) \eta(i, j, d^*)$) since $\tilde{r}_j^*$ is zero for individuals in $J$. The last term sums over the variances of individuals in $J_c$ which are not in $J_3$, for which $r_i^* = \tilde{r}_i^*$ by construction.

Using Assumption 4.3 (since $r^*$ and $\tilde{r}^*$ only differ by $N_{\text{max}} m$ units at most) we obtain that the first component in the bound in Equation (D.16) is $O(N_{\text{max}}^2 m/n^2)$. For the second component, note that individuals in $J$ have at most $N_{\text{max}} |J| \leq N_{\text{max}}^2 m$ many connections. Since $J_c$ and $J$ are disjoint sets (and by symmetry of $A$), the second term is at most $O(N_{\text{max}}^2 m/n^2)$. Similarly, for the third term, there are at most $N_{\text{max}} |J| \leq |J| N_{\text{max}} \leq m N_{\text{max}}^2$ which completes the proof, since weights and variances are uniformly bounded.

### E Inference

**Lemma E.1.** *(Ross et al., 2011)* Let $X_1, \ldots, X_n$ be random variables such that $\mathbb{E}[X_i^4] < \infty$, $\mathbb{E}[X_i] = 0$, $\sigma^2 = \text{Var}(\sum_{i=1}^n X_i)$ and define $W = \sum_{i=1}^n X_i/\sigma$. Let the collection $(X_1, \ldots, X_n)$ have dependency neighborhoods $N_i$, $i = 1, \ldots, n$ and also define $D = \max_{1 \leq i \leq n} |N_i|$. Then for $Z$ a standard normal random variable, we obtain

$$d_W(W, Z) \leq \frac{D^2}{\sigma^3} \sum_{i=1}^n \mathbb{E}[X_i]^3 + \frac{\sqrt{28} D^{3/2}}{\sqrt{\pi} \sigma^2} \sqrt{\sum_{i=1}^n \mathbb{E}[X_i^4]},$$

where $d_W$ denotes the Wasserstein metric.

**Theorem E.2.** Suppose that Assumption 2.1, 4.2 hold, and for all $w_N \in \mathcal{W}_N$, $n V_N(w_N) > 0$. Then for all $w_N \in \mathcal{W}_N$,

$$\frac{\sqrt{n} (\bar{\Gamma}_n(w_N) - \Gamma_n(w_N))}{\sqrt{n V_N(w_N)}} \Rightarrow_{d} \mathcal{N}(0, 1).$$

(E.2)
Proof of Theorem E.2. We prove asymptotic normality after conditioning on the sigma algebra \( \sigma(\mathbf{D}^R, \mathbf{A}, \mathbf{R}, \mathbf{T}, \mathbf{P}) \). Notice that unbiasedness holds by Proposition 3.1. Next, we show that \( Y_i \) for all \( i : R_i = 1 \) are locally dependent, given \( \sigma(\mathbf{A}, \mathbf{D}^R, \mathbf{R}, \mathbf{T}, \mathbf{P}) \). To show this, it suffices to show that

\[
\{\varepsilon_i\}_{i : R_i = 1} \mid \sigma(\mathbf{A}, \mathbf{D}^R, \mathbf{R}, \mathbf{T}, \mathbf{P})
\]

are locally dependent, i.e., form a local dependency graph as described in Ross et al. (2011). Define \( \mathcal{H} = \{1, \cdots, N\} \setminus \mathcal{J} \).

The argument is the following. Under Assumption 2.2, unobservables are locally dependent given the adjacency matrix \( \mathbf{A} \) and covariates \( \mathbf{T} \). Since \( \mathbf{P} \) is exogenous conditional on \( \mathbf{A} \), it follows that unobservables are locally dependent given \( (\mathbf{A}, \mathbf{T}, \mathbf{P}) \). That is,

\[
\varepsilon_{1, \cdots, N} \mid \sigma(\mathbf{A}, \mathbf{P}, \mathbf{T})
\]

are locally dependent. Consider now the distribution of all unobservables in the set \( \mathcal{H} \), given \( \mathbf{A}, \mathbf{P}, \mathbf{T} \). Here, unobservables are mutually independent on \( \mathbf{D}^R, \mathbf{R} \), given \( \sigma(\mathbf{A}, \mathbf{P}, \mathbf{T}) \) and \( \mathcal{H} \) is measurable with respect to \( \mathbf{P} \). Therefore,

\[
\varepsilon_{i \in \mathcal{H}} \mid \sigma(\mathbf{A}, \mathbf{P}, \mathbf{D}^R, \mathbf{R}, \mathbf{T})
\]

are locally dependent. Since \( \{i : R_i = 1\} \subseteq \mathcal{H} \) the local dependence assumption of unobservables in such a set holds conditional on \( \mathbf{A}, \mathbf{P}, \mathbf{D}^R, \mathbf{R}, \mathbf{T} \) for such units.

Recall that by Assumption 2.1

\[
Y_i = r\left(D_i, \sum_{k \in N_i} D_k, T_i, \varepsilon_i\right). \quad (E.3)
\]

Therefore, given \( \sigma(\mathbf{A}, \mathbf{P}, \mathbf{D}^R, \mathbf{R}, \mathbf{T}) \) outcomes \( Y_{\{1, \cdots, n\}} \) are locally dependent. Let

\[
X_i := \frac{1}{n \sqrt{V_N(w_N)}} w_N(i, \mathbf{D}^R, \mathbf{R}, \mathbf{T}) \left(Y_i - m(D_i, \sum_{k \in N_i} D_k, T_i)\right), \quad (E.4)
\]

where, recall, \( V_N(w_N) \) denotes the variance of the average in Equation (10). Notice
that by Proposition 3.1, we have

$$\mathbb{E}[X_i | \sigma(D^R, A, R, P, T)] = 0.$$  \hspace{1cm} (E.5)

To prove the theorem we invoke Lemma E.1. In particular, we observe that for $Z \sim \mathcal{N}(0, 1)$, we have

$$\sup_{x \in \mathbb{R}} \left| P \left( \sum_{i: R_i = 1} X_i \leq x \right| \sigma(D^R, A, R, P, T) \right) - \Phi(x) \right| \leq c \sqrt{d_{W | \sigma(D^R, A, R, P, T)} \left( \sum_{i: R_i = 1} X_i, Z \right)}.$$  \hspace{1cm} (E.6)

where $d_{W | \sigma(D^R, A, R, P, T)} \left( \sum_{i: R_i = 1} X_i, Z \right)$ denotes the Wesserstein metric taken with respect to the conditional marginal distribution of $\sum_{i: R_i = 1} X_i$ given $\sigma(D^R, A, R, P, T)$ and $\Phi(x)$ is the CDF of a standard normal distribution, and $c < \infty$ is a universal constant. To apply Lemma E.1 we take $\sigma^2 = 1$ since $X_i$ already contains the rescaling factor defined in Lemma E.1. In addition, since $nV_N(w_N)$ is strictly bounded away from zero we obtain under Assumption 4.2

$$\mathbb{E}[X_i^4 | \sigma(D^R, A, R, P, T)] \leq C \frac{1}{n^2}, \hspace{0.5cm} \mathbb{E}[X_i^3 | \sigma(D^R, A, R, P, T)] \leq C \frac{1}{n^{3/2}}.$$  \hspace{1cm} (E.7)

Therefore, the condition in Lemma E.1 are satisfied. Then we obtain

$$d_{W | \sigma(D^R, A, R, P, T)} \left( \sum_{i: R_i = 1} X_i, Z \right) \leq \mathcal{N}^2_{\max} \sum_{i: R_i = 1} \mathbb{E}[|X_i|^3 | D^R, R, A, P, T]$$

$$+ \sqrt{28 \mathcal{N}^3_{\max}} \sqrt{\frac{n^{3/2}}{\pi}} \mathbb{E}[|X_i|^4 | R, A, D^R, P, T]$$

$$\leq \frac{\mathcal{N}^2_{\max}}{n^{1/2}} C + \sqrt{28 \mathcal{N}^3_{\max}} \frac{n^{3/2}}{\sqrt{\pi n}} C.$$  \hspace{1cm} (E.8)
for a universal constant $C < \infty$. Since $N^2_{\max}/n^{1/2} = o(1)$, we obtain

$$
\sup_{x \in \mathbb{R}} \left| P \left( \sum_{i : R_i = 1} X_i \leq x \right \vert \sigma(D^R, A, R, P, T) \right) - \Phi(x) \right| \leq \sqrt{\frac{N^2_{\max}}{n^{1/2}}} \overline{C} + \frac{\sqrt{28} N^3_{\max}}{\sqrt{\pi n}} \overline{C} = o(1)
$$

(E.9)

where the latter result is true since the conditions in Lemma E.1 are satisfied pointwise for any $w_N \in \mathcal{W}_N$ and by the property of the Wesserstein metric. To prove that the result also holds unconditionally, we may notice that for some arbitrary measure $\mu_N$,

$$
\sup_{x \in \mathbb{R}} \left| \int P \left( \sum_{i : R_i = 1} X_i \leq x \right \vert \sigma(D^R, A, R, P, T) \right) - \Phi(x) \right| d\mu_N - \Phi(x)
\leq \sup_{x \in \mathbb{R}} \int \left| P \left( \sum_{i : R_i = 1} X_i \leq x \right \vert \sigma(D^R, A, R, P, T) \right) - \Phi(x) \right| d\mu_N
\leq \int \sup_{x \in \mathbb{R}} \left| P \left( \sum_{i : R_i = 1} X_i \leq x \right \vert \sigma(D^R, A, R, P, T) \right) - \Phi(x) \right| d\mu_N = o(1). \tag{E.10}
$$

This concludes the proof. \qed

**Corollary.** Theorem A.2 holds.

**Proof.** The proof follows similarly to the above theorem with an important modification. We observe that the variables $X_i$ in Equation (E.4) do not follow a dependence graph since they exhibit $M$ degree dependence. Instead, we construct a graph where two individuals are connected if they are connected by at least $M$ edges in the original graph. In such a graph, the variables $X_i$ as defined in Equation (E.4) satisfy the local dependence assumption in Lemma E.1. In order for the lemma to apply, we need to show that the maximum degree of such a graph, denoted as $\overline{N}^2_{M}/n^{1/2} = o(1)$. This follows under Assumption A.1, since the maximum degree is uniformly bounded. This completes the proof. \qed

**Theorem E.3.** Let Assumptions 2.1, 4.2, 4.4 hold. Then for all $w_N \in \mathcal{W}_N$,

$$
\frac{V_N(w_N)}{V_n(w_N)} - 1 \rightarrow_p 0. \tag{E.11}
$$

57
**Proof of Theorem E.3.** First, notice that under Assumption 2.1, 2.2, Lemma 3.3 holds, and therefore, the conditional variance can be written as a function of $\sigma(\cdot), \eta(\cdot)$.

With an abuse of notation, we will refer to $n = 1^T R$.

Next, we prove consistency pointwise for each element in $W_n$. Throughout the proof we denote

$$\eta(i, j) = \eta(T_i, D_i, \sum_{k \in N_i} D_k, T_j, D_j, \sum_{k \in N_j} D_k)$$

$$\sigma^2(i) = \sigma^2(T_i, D_i, \sum_{k \in N_i} D_k).$$

For notational convenience, we denote $w_N(i, \cdot)$ omitting the last arguments when clear from the context. Recall that under Assumption 4.1, $\hat{\sigma}$ and $\hat{\eta}$ converge uniformly to $\sigma, \eta$ respectively.

We have

$$|nV_N(w_N) - n\hat{V}_n(w_N)| \leq \left| \frac{1}{n} \sum_{i \in R_i = 1} w_N^2(i) (\hat{\sigma}^2(i) - \sigma^2(i)) \right|$$

$$+ \left| \frac{1}{n} \sum_{i \in R_i = 1} \sum_{j \in N_i} w_N(i, D^R, R, T) w_N(j) (\hat{\eta}(i, j) - \eta(i, j)) \right|. \quad (E.12)$$

Consider first term $(a)$. Then we can write (since $w_N(\cdot)$ are uniformly bounded by design, see Algorithm 2),

$$(a) \leq \max_{o \in \{1, \ldots, n\}} w_N(o) \frac{1}{n} \sum_{i \in R_i = 1} \left| (\hat{\sigma}^2(i) - \sigma^2(i)) \right| = o_p(1). \quad (E.13)$$

Consider now the covariance component. We have

$$(b) \leq \max_{o \in \{1, \ldots, n\}} \left| w_N(o) \frac{1}{n} \sum_{i \in R_i = 1} \left| \sum_{j \in N_i} w_N(j) (\hat{\eta}(i, j) - \eta(i, j)) \right| \right|. \quad (E.14)$$
We have

\[
(J) \leq \max_{o \in \{1, \ldots, n\}} \left| w_N(o) \frac{1}{n} \sum_{i:\|N_i\| \leq L_N} R_i \left| \sum_{j \in N_i} w_N(j) (\hat{\eta}(i, j) - \eta(i, j)) \right| \right. + \left. \max_{o \in \{1, \ldots, n\}} \left| w_N(o) \frac{1}{n} \sum_{i:\|N_i\| \geq L_N} R_i \left| \sum_{j \in N_i} w_N(j) (\hat{\eta}(i, j) - \eta(i, j)) \right| \right|
\]

(E.15)

We have by Holder’s inequality and Assumption 4.2,

\[
\max_{o \in \{1, \ldots, n\}} \left| w_N(o) \frac{1}{n} \sum_{i:\|N_i\| \leq L_N} R_i \left| \sum_{j \in N_i} w_N(j) (\hat{\eta}(i, j) - \eta(i, j)) \right| \right. \leq L_N \hat{C} \frac{1}{n} \sum_{i:\|N_i\| \leq L_N} R_i \left| \max_{j} |\hat{\eta}(i, j) - \eta(i, j)| \right| \leq L_N \max_{i,j} |\hat{\eta}(i, j) - \eta(i, j)| = o_p(1)
\]

where the last equality follows by Assumption 4.2, for a constant \( \hat{C} \). The second component reads as follows:

\[
\max_{o \in \{1, \ldots, n\}} \left| w_N(o) \frac{1}{n} \sum_{i:\|N_i\| \geq L_N} R_i \left| \sum_{j \in N_i} w_N(j) (\hat{\eta}(i, j) - \eta(i, j)) \right| \right. \leq \hat{C} \mathcal{N}_{\text{max}} \frac{1}{n} \sum_{i:\|N_i\| \geq L_N} R_i \left| \max_{j} |\hat{\eta}(i, j) - \eta(i, j)| \right|
\]

(E.17)

By Assumption 4.4, we have that (note that \( n \propto n_2 \), since \( n_1 \propto n_2 \))

\[
\hat{C} \mathcal{N}_{\text{max}} \frac{1}{n} \sum_{i:\|N_i\| \geq L_N} R_i \left| \max_{j} |\hat{\eta}(i, j) - \eta(i, j)| \right| \leq O_p(1) \mathcal{N}_{\text{max}} n^{3/4}/n = o_p(1).
\]

(E.18)

Here \( \max_{j} |\hat{\eta}(i, j) - \eta(i, j)| = O_p(1) \) (and also \( o_p(1) \)) since \( \hat{\eta} \) converges uniformly to \( \eta \), and \( \mathcal{N}_{\text{max}} n^{3/4}/n = o(1) \) by Assumption 4.2. Uniform consistency over \( w_N \in \mathcal{W}_N \) follows from the union bound, since \( |\mathcal{W}_n| \) is finite dimensional. The proof is complete by the fact that \( nV_N(w_N) > 0 \) and the continuous mapping theorem.

**Corollary.** Theorem 4.2 holds.

**Proof.** The proof follows from Theorem E.2 and Theorem E.3 by Slutsky theorem.

\( \square \)
F Optimization: MILP for Difference in Means Estimators

In this sub-section we discuss the optimization algorithm for difference in means estimators.

To show that the optimization problem admits a mixed-integer linear program formulation, we first introduce the following proposition, which follows similarly to what discussed in Viviano (2019).

Lemma F.1. (Viviano, 2019) Any function $g_i$ that depends on $D_i$ and $\sum_{k \in N_i} D_k$ can be written as

$$g_i(D_i, \sum_{k \in N_i} D_k) = \sum_{h=0}^{N_i} (g_i(1, h) - g_i(0, h)) u_{i,h} + (t_{i,h,1} + t_{i,h,2} - 1) g_i(0, h), \quad (F.1)$$

where $u_{i,h}, t_{i,h,1}, t_{i,h,2}$ are defined by the following linear inequalities.

\begin{align*}
(A) \quad & \frac{D_i + t_{i,h,1} + t_{i,h,2}}{3} - 1 < u_{i,h} \leq \frac{D_i + t_{i,h,1} + t_{i,h,2}}{3}, \quad u_{i,h} \in \{0, 1\} \quad \forall h \in \{0, ..., |N_i|\}, \\
(B) \quad & \frac{\sum_k A_{i,k} D_k - h}{|N_i| + 1} < t_{i,h,1} \leq \frac{\sum_k A_{i,k} D_k - h}{|N_i| + 1} + 1, \quad t_{i,h,1} \in \{0, 1\}, \quad \forall h \in \{0, ..., |N_i|\} \\
(C) \quad & \frac{h - \sum_k A_{i,k} D_k}{|N_i| + 1} < t_{i,h,2} \leq \frac{h - \sum_k A_{i,k} D_k}{|N_i| + 1} + 1, \quad t_{i,h,2} \in \{0, 1\}, \quad \forall h \in \{0, ..., |N_i|\}. \quad (F.2)
\end{align*}

Proof. We define the following variables:

$$t_{i,h,1} = 1\{\sum_{k \in N_i} D_k \geq h\}, \quad t_{i,h,2} = 1\{\sum_{k \in N_i} D_k \leq h\}, \quad h \in \{0, ..., |N_i|\}. \quad \text{(F.2)}$$

The first variable is one if at least $h$ neighbors are treated, and the second variable is one if at most $h$ neighbors are treated.

Since each unit has $|N_i|$ neighbors and zero to $|N_i|$ neighbors can be treated, there are in total $\sum_{i=1}^{n} (2|N_i| + 2)$ of such variables.
The variable \( t_{i,h,1} \) can be equivalently be defined as

\[
\frac{\left( \sum_k A_{i,k} D_k - h \right)}{|\mathcal{N}_i| + 1} < t_{i,h,1} \leq \frac{\left( \sum_k A_{i,k} D_k - h \right)}{|\mathcal{N}_i| + 1} + 1, \quad t_{i,h,1} \in \{0, 1\}. \tag{F.3}
\]

The above equation holds for the following reason. Suppose that \( h < \sum_k A_{i,k} D_k \). Since \( \frac{\left( \sum_k A_{i,k} D_k - h \right)}{|\mathcal{N}_i| + 1} < 0 \), the left-hand side of the inequality is negative and the right-hand side is positive and strictly smaller than one. Since \( t_{i,h,1} \) is constrained to be either zero or one, in such case, it is set to be zero. Suppose now that \( h \geq \sum_k A_{i,k} D_k \). Then the left-hand side is bounded from below by zero, and the right-hand side is bounded from below by one. Therefore \( t_{i,h,1} \) is set to be one. Similarly, we can write

\[
\frac{\left( h - \sum_k A_{i,k} D_k \right)}{|\mathcal{N}_i| + 1} < t_{i,h,2} \leq \frac{\left( h - \sum_k A_{i,k} D_k \right)}{|\mathcal{N}_i| + 1} + 1, \quad t_{i,h,2} \in \{0, 1\}. \tag{F.4}
\]

By definition,

\[
t_{i,h,1} + t_{i,h,2} = \begin{cases} 1 & \text{if and only if } \sum_{k \in \mathcal{N}_i} D_k \neq h \\ 2 & \text{otherwise} \end{cases}. \tag{F.5}
\]

Therefore, we can write

\[
\frac{1}{n} \sum_{i=1}^{n} \sum_{h=0}^{|\mathcal{N}_i|} (g_i(1,h) - g_i(0,h)) D_i(t_{i,h,1} + t_{i,h,2} - 1) + (t_{i,h,1} + t_{i,h,2} - 1) g_i(0,h). \tag{F.6}
\]

Finally, we introduce the variable \( u_{i,h} = D_i(t_{i,h,1} + t_{i,h,2} - 1) \). Since \( D_i, t_{i,h,1}, t_{i,h,2} \in \{0, 1\} \) it is easy to show that such variable is completely determined by the above constraint. This completes the proof.

We first start from the case where \(|W_N| = 1\) and then we extend to the case of multiple estimators. By Lemma F.1, we showcase that each function of the individual and neighbors’ treatment assignment can be written as a linear function of the decision variables under linear constraints.
We define
\[
\widetilde{\sigma}^2_i(D_i, \sum_{k \in N_i} D_k) = \sigma(T_i, D_i, \sum_{k \in N_i} D_k)
\]
\[
\tilde{\eta}_{i,j}(D_i, \sum_{k \in N_i} D_k, D_j, \sum_{k \in N_j} D_k) = \eta(T_i, D_i, \sum_{k \in N_i} D_k, T_j, D_j, \sum_{k \in N_j} D_k)
\]
the variance function and \(\tilde{\eta}_{i,j}(\cdot)\) the covariance for unit \(i\) and \(j\), given their number of neighbors and the observed treatment assignments.

We define
\[
v^1_i(D_i, \sum_{k \in N_i} D_k) = 1\{D_i = d_1, \sum_{k \in N_i} D_k = s_1, T_i = l\},
\]
\[
v^0_i(D_i, \sum_{k \in N_i} D_k) = 1\{D_i = d_0, \sum_{k \in N_i} D_k = s_0, T_i = l\}.
\]

The objective function reads as follows.
\[
\sum_{i : R_i=1} R_i \left( \frac{v^1_i(D_i, \sum_{k \in N_i} D_k) \eta_i(D_i, \sum_{k \in N_i} D_k)}{\sum_{i : R_i=1} R_i v^0_i(D_i, \sum_{k \in N_i} D_k)/n} \right)^2 + \sum_{i : R_i=1} R_i \left( \frac{v^0_i(D_i, \sum_{k \in N_i} D_k) \eta_i(D_i, \sum_{k \in N_i} D_k)}{\sum_{i : R_i=1} R_i v^0_i(D_i, \sum_{k \in N_i} D_k)/n} \right)^2
\]
\[
+ \sum_{j \in N_i} R_j \left( \frac{v^1_j(D_j, \sum_{k \in N_j} D_k)}{\sum_{i : R_i=1} R_i v^0_i(D_i, \sum_{k \in N_i} D_k)/n} \right) \tilde{\eta}_{i,j}(D_i, \sum_{k \in N_i} D_k, \sum_{k \in N_j} D_k)
\]
\[
- \sum_{j \in N_i} R_j \left( \frac{v^0_j(D_j, \sum_{k \in N_j} D_k)}{\sum_{i : R_i=1} R_i v^0_i(D_i, \sum_{k \in N_i} D_k)/n} \right) \tilde{\eta}_{i,j}(D_i, \sum_{k \in N_i} D_k, \sum_{k \in N_j} D_k).
\]

We now introduce the following auxiliary variables: \(n \times \sum_{i : R_i=1} |N_i|\) variables \(t_{i,h,1} = 1\{\sum_{k \in N_i} D_k \geq h\}\) and \(n \times \sum_{i : R_i=1} |N_i|\) variables \(t_{i,h,2} = 1\{\sum_{k \in N_i} D_k \leq h\}\).

We define \(\tilde{t}_{i,h} = t_{i,h,1} + t_{i,h,2} - 1\) and we define \(u_{i,h} = D_i \times \tilde{t}_{i,h}\). Such variables are fully characterize by the two linear constraints for each variable as discussed in Lemma F.1 and the 0-1 constraint for each variable. By Lemma F.1, each function or product of functions of the variables \((D_i, \sum_{k \in N_i} D_k)\) can now be described as a linear function of these new decision variables. Consider for example,
\[(v_i^1(D_i, \sum_{k \in \mathcal{N}_i} D_k) \tilde{\sigma}_i(D_i, \sum_{k \in \mathcal{N}_i} D_k))^2\] first. Then such function is rewritten as
\[
(v_i^1(D_i, \sum_{k \in \mathcal{N}_i} D_k) \tilde{\sigma}_i(D_i, \sum_{k \in \mathcal{N}_i} D_k))^2 = \\
\sum_{h=1}^{\mathcal{|N|}} (v_i^1(1, h)^2 \tilde{\sigma}_i(1, h)^2 - v_i^1(0, h)^2 \tilde{\sigma}_i(0, h)^2) u_{i,h} \quad + \quad v_i^1(0, h)^2 \tilde{\sigma}_i(0, h)^2 \tilde{t}_{i,h}.
\]  

Similarly, consider the following function
\[
K(D_i, D_j, \sum_{k \in \mathcal{N}_i} D_k, \sum_{k \in \mathcal{N}_j} D_k) := v_i^1(D_i, \sum_{k \in \mathcal{N}_i} D_k) v_j^1(D_j, \sum_{k \in \mathcal{N}_j} D_k) \tilde{\eta}_{i,j}(D_i, \sum_{k \in \mathcal{N}_i} D_k, D_j, \sum_{k \in \mathcal{N}_j} D_k).
\]  

By Lemma F.1, the function can be written as
\[
\sum_{h=0}^{\mathcal{|N_i|}} \left( K(1, D_j, h, \sum_{k \in \mathcal{N}_j} D_k) - K(0, D_j, h, \sum_{k \in \mathcal{N}_j} D_k) \right) u_{i,h} + \tilde{t}_{i,h} K(0, D_j, h, \sum_{k \in \mathcal{N}_j} D_k).
\]  

We can now linearize the function and obtain the following equivalent formulation
\[
\sum_{h'=0}^{\mathcal{|N_j|}} \left( \sum_{h=0}^{\mathcal{|N_i|}} \left( K(1, 1, h, h') - K(0, 1, h, h) \right) u_{i,h} + \tilde{t}_{i,h} K(0, 1, h, h')
\right.
\quad - \left( K(1, 0, h, h') - K(0, 0, h, h) \right) u_{i,h} + \tilde{t}_{i,h} K(0, 0, h, h')
\left. \right) u_{j,h'}
\quad + \left( K(1, 0, h', h') - K(0, 0, h, h) \right) u_{i,h} \tilde{t}_{j,h'} + \tilde{t}_{i,h} K(0, 0, h', h') \tilde{t}_{j,h'}.
\]  

which is quadratic in the decision variables, as defined in Lemma F.1. Therefore, each function in the numerators and denominators of Equation (F.8) can be written as a linear or quadratic function in the decision variables \(D_i, u_{i,h}, \tilde{t}_{i,h}\). We now linearize the quadratic expressions in the numerator and denominators, to show that also quadratic expression have a linear formulation. To do so we introduce a new set of variables that we denote as
\[
A_{i,j,h',h'} = u_{i,h} u_{j,h'}, \quad B_{i,j,h',h'} = u_{i,h} \tilde{t}_{j,h'}, \quad C_{i,h,h',h} = \tilde{t}_{i,h} \tilde{t}_{j,h'}.
\]
Since each of the above variable takes values in \( \{0, 1\} \), such variables can be expressed with linear constraints. For instance, \( A_{i,j,h',h'} \) is defined as follows.

\[
\frac{u_{i,h} + u_{j,h'}}{2} - 1 < A_{i,j,h',h'} \leq \frac{u_{i,h} + u_{j,h'}}{2}, \quad A_{i,j,h',h'} \in \{0, 1\}. \tag{F.14}
\]

In fact, if both \( u_{i,h}, u_{j,h'} \) are both equal to one, the left hand size is zero, and under the 0-1 constraint, the resulting variable is equal to one. This follows similarly also for the other variables. Finally, notice that since also \( R_i \in \{0, 1\} \), the product of \( R_i \) for any other 0-1 variable can be similarly linearized. Therefore, the above problem reads as a mixed-integer fractional linear program. By the linear representation of fractional linear programming discussed in Charnes and Cooper (1962), the proof completes for the case where \( |\mathcal{W}_N| = 1 \).

To solve the optimization problem over multiple weights \( \mathcal{W}_N \), we can add an auxiliary variables \( \lambda \), and solve the following program

\[
\min \lambda, \quad \lambda \geq f_{w_N} \forall w_N \in \mathcal{W}_N \tag{F.15}
\]

where \( f_{w_n} \) denotes the linearized objective function for each \( w_N \in \mathcal{W}_N \).