INTEGRALS ON HOPF ALGEBRAS AND APPLICATION TO REPRESENTATION THEORY OF QUANTUM GROUPS OF TYPE $A_{0\mid 0}$

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INTRODUCTION

In this work we study some properties of (non-cosemisimple) Hopf algebras, possessing integrals, which are also called co-Frobenius Hopf algebras. We apply the result obtained to the classification of representations of quantum groups of type $A_{0\mid 0}$.

The notion of integral on Hopf algebras is motivated by the Haar integral on compact groups. In fact, the axiom of the Haar integral on a compact group can be given in a pure algebraic way as a linear functional on the algebra of (regular) function on the group (which is a Hopf algebra), satisfying a certain axiom, which can be explained in terms of the coproduct on the (Hopf) algebra of functions. One takes this axiom for the definition of an integral on an arbitrary Hopf algebra over an arbitrary field.

Since the pioneering work of Sweedler [13], integrals on Hopf algebras were studied by several authors [17, 11, 2, 16]. Among others, Sweedler proved the existence and uniqueness up to a constant of a (non-zero) integral on any finite-dimensional Hopf algebra. A theorem of Sullivan states that if an integral exists on a Hopf algebra then it is uniquely determined up to a constant. Therefore we shall refer to the integral on a Hopf algebra when ever it exists. We shall also assume that integral means non-zero integral.

In representation theory of a compact group, one uses the Haar integral to deduce the semisimplicity of its representations. There is an analogue in comodule theory of Hopf algebras. If a Hopf algebra possesses an integral which does not vanish at the unit element, then it is cosemisimple, i.e., all its comodules are semisimple. However, the integral may vanish at the unit element, which is equivalent to the non-cosemisimplicity of the Hopf algebra. While there are many examples of non-cosemisimple
finite-dimensional Hopf algebras, not so many infinite-dimensional non-cosemisimple Hopf algebras with integral are known. Moreover, a theorem of Sullivan \[17\] states that a commutative Hopf algebra over a field of characteristic zero possesses an integral if and only if it is cosemisimple.

New examples of infinite-dimensional non-cosemisimple Hopf algebras with integral come from Lie supergroups and quantum (super)groups theory. In studying Haar measure on compact supergroups, F. Berezin found a remarkable fact that the Haar measure exists but the whole volume of the supergroup with respect to this measure may be zero (see e.g. \[1\]). In other words, the function algebra on a compact supergroup is a infinite-dimensional Hopf algebra with integral, which may not be cosemisimple.

In \[8\] the author showed that the Hopf algebras associated with certain (non-even) Hecke symmetries (i.e., quantum groups of type \(A\)) are non-cosemisimple infinite-dimensional Hopf algebras with integral.

In studying representations of simple Lie super-algebras of classical type, V. Kac found out that their irreducible representations split into two classes of typical and atypical representations (see, e.g., \[9\]). It turns out that there is an analogous notion for simple comodules over a Hopf algebra with integrals and the integral provides a necessary and sufficient condition for a simple comodule to be “typical” (called “splitting” in this work). This is the main result of the first part of this work. In the second part we apply this result to study representations of quantum groups of type \(A_{0|0}\), i.e., Hopf algebras associated to Hecke symmetries of birank \((1, 1)\). Using the classification result we are able to classify the symmetries themselves.

The work is briefly divided into two parts. In order to reach to the main result of the first part (Theorem \[3.2\]), we first recall some definitions and known facts on integrals on Hopf algebras (Section \[1\]). Then we define a convolution product on a Hopf algebra by means of the integral making the Hopf algebra into a non-unital associated algebra and derive some auxiliary result for Section \(3\) (Section \[2\]). In Section \(3\) we introduce the notion of splitting comodule, which means injective, projective simple comodule. In the terminology of V. Kac, splitting comodule splits in any comodule. We provide in Theorem \[3.2\] a necessary and sufficient condition of a simple comodule to be splitting.

In the second part of the work, Section \[4\], we apply the result of the first part to Hopf algebras associated to Hecke symmetries of birank \((1, 1)\), i.e., Hecke symmetries, the quantum exterior algebras associated to which have the Poincaré series equal to \((1 + t)(1 - t)^{-1}\). We show that simple comodules of these Hopf algebras can be labelled by pairs of integers \((k, l)\), where \((1, 0)\) is the fundamental comodule, \((-1, 0)\) is its dual, \((0, 0)\) is the trivial comodule and the comodule labelled by \((k, l)\) is splitting iff \(k + l \neq 0\).
We show that the dimension of a simple comodule is 2 or 1 depending on whether it is splitting or not. Using this we able to classify the Hecke symmetries of birank \((1,1)\). In turns out that there are no other then those found by Manin [15] and Takeuchi-Tambara [19].

1. Co-Frobenius Coalgebras and Hopf algebras

We work over a field \(k\). Every tensor product if not explicitly indicated means tensor product over \(k\).

Let \(C\) be a coalgebra and \(M\) be a right \(C\)–comodule, the coaction of \(C\) on \(M\) is denoted by \(\rho\), \(\rho : M \rightarrow M \otimes C\), \(\rho(v) = v_0 \otimes v_1\). Let \(C^* := \text{Hom}_k(C,k)\) be the dual of \(C\). Then \(C^*\) is an algebra, acting on \(M\) from the left in the following way

\[
\phi \mapsto v := v_0\phi(v_1).
\]

Analogously, if \(\lambda : N \rightarrow C \otimes N\), \(\lambda(v) = v_1 \otimes v_2\), is a left \(C\)–comodule, then it is a right \(C^*\)-module through the action \(v \leftarrow \phi := \phi(v_1) \otimes v_2\).

Thus, we have a functor from the category of right (left) \(C\)–comodules into the category of left (right) \(C^*\)-modules, which is full, faithfull and exact. The following statement is due to Doi.

1.1. [2] Let \(M\) be a right (left) \(C\)–comodule, which is finite dimensional. Then \(M\) is injective (projective) if and only if it is injective (projective) as left (right) \(C^*\)-module.

A \(C^*\)-module may not be a \(C\)–comodule by the above correspondence. Those \(C^*\)-module induced from \(C\)–comodules are called rational modules. Each left \(C^*\)-module \(M\) contains a unique maximal rational submodule, denoted by \(\text{rat}\, M\). Analogously, for a right \(C^*\)-module \(M\), its rational submodule will be denoted by \(M_{\text{rat}}\).

Let \(M\) be a \(C\)–comodule. The map \(\rho : M \rightarrow M \otimes C\) induces a map \(M^* \otimes M \rightarrow C\), which can be considered as a coalgebra homomorphism or a morphism of \(C\)–comodules, where \(C\) coacts on \(M^* \otimes M\) on the second tensor component. In the latter case, we shall use the notation \((M^*) \otimes M\) to indicate that \(C\) coacts only on \(M\). The image of \(M^* \otimes M\) is called the coefficient space of \(M\), denoted by \(\text{Cf}(M)\).

Let \(S\) be a simple (left or right) \(C\)–comodule. The fundamental theorem of comodule (saying that a finite generated comodule is finite dimensional) implies that \(S\) is finite dimensional. Let \(\mathcal{D} := \text{End}^C(S)\). Then, by Schur lemma, \(\mathcal{D}\) is a division algebra over \(k\) and \(S\) is a vector space over \(\mathcal{D}\). We have \(\text{Cf}(S) \cong V^* \otimes_{\mathcal{D}} V\), as coalgebras [3]. Let \(\{M_\alpha | \alpha \in \mathcal{A}\}\) be the set of all simple \(C\)–comodules. We define \(\mathcal{D}_\alpha := \text{End}_{\mathcal{D}}^C(M_\alpha)\) and \(m_\alpha := \dim_{\mathcal{D}_\alpha}(M_\alpha)\), \(d_\alpha := \sqrt{\dim_k \mathcal{D}_\alpha}\). Note that \(d_\alpha\) are positive integers and if \(k\) is algebraically closed then \(\mathcal{D}_\alpha = k\), \(\forall \alpha\), i.e., \(d_\alpha = 1\).
By definition, the socle of a comodule $M$ is the sum of all its simple subcomodule. The sum is direct and is denoted by $\sigma(M)$. The injective hull (or cover) of $M$, is by definition an injective comodule $J(M)$ together with a morphism $M \to J(M)$ inducing an isomorphism $\sigma(M) \to \sigma(J(M))$. It is easy to see that the injective hull of a simple comodule, if it exists, is indecomposable. The following results are due to J. Green.

1.2. (i) The injective hull of any comodule exists uniquely.

(ii) $C$ itself decomposes in to indecomposable injective subcomodules as follows

$$ C \cong \bigoplus_{\alpha \in A} J(M_\alpha)^{\oplus m_\alpha}. $$

(iii) If $C = \bigoplus_{\lambda \in \mathcal{L}} N_\lambda$ is another decomposition then for each $\alpha \in A$, the set $\{\lambda \in \mathcal{L} | N_\lambda \cong J(M_\alpha)\}$ contains exactly $m_\alpha$ elements.

A bilinear form $b$ on $C$ is called balanced if, for all $\phi \in C^*$,

$$ b(x \leftarrow \phi, y) = b(x, \phi \rightarrow y). $$

Balanced bilinear forms on $C$ are in 1-1 correspondence with right $C^*$-comodules homomorphism $r : C \to C^*$ by the formula $r(x)(y) = b(x, y)$ and in 1-1 correspondene with left $C^*$-comodules homomorphism $r : C \to C^*$ by the formula $l(x)(y) = b(y, x)$.

A coalgebra is called left (right) co-Frobenius if there exist a left (right) monomorphism of $C^*$-modules $C \to C^*$. The following results are due to B. Lin.

1.3. (i) If $C$ is a left co-Frobenius coalgebra then:

(ii) The injective cover of every finite dimensional right $C$-comodule is finite dimensional.

(iii) Every injective right $C$-comodule is projective.

(iii) $\text{rat}C^*$ is dense in $C^*$.

In the next section we shall need the following result.

**Lemma 1.4.** Let $M$ be a $C$-comodule of finite dimension. Then $M$ is projective (resp. injective) if and only if it is projective (resp. injective) in the category of finite dimensional $C$-comodules.

**Proof.** It is sufficient to show the “if” part. Assume that $M$ is projective in the category of finite dimensional $C$-comodule. Consider a diagram

$$
\begin{align*}
M & \not\to \n \xrightarrow{\nu} P \\
N & \xrightarrow{\nu^{-1}(\text{Im}(P))} P \to 0.
\end{align*}
$$

By replacing $P$ with $\text{Im}(P)$ and $N$ with $\nu^{-1}(\text{Im}(P))$, we can assume that $\pi$ is surjective, thus $P$ is finite dimensional. Let $\mathcal{P}$ be a basis of $P$ and $\mathcal{N}$
be a set of elements of \( N \) such that \( \nu(N) = \mathcal{P} \). The submodule \( N_1 \) of \( N \), generated by \( N \) is finite dimensional and we have \( \nu(N_1) = P \). Hence, by assumption, there exists a morphism \( \mu : M \rightarrow N_1 \) : \( \nu \circ \mu = \pi \).

Assume that \( M \) is injective in the category of finite dimensional \( C \)-comodule. Consider the diagram

\[
0 \rightarrow P \xrightarrow{\nu} N \xrightarrow{\pi} M.
\]

By replacing \( P, N \) with \( P/\text{Ker}(\pi), N/\text{Ker}(\pi) \), we can assume that \( \pi \) injective. Hence \( P \) has finite dimension. If \( N/P \) is finite dimensional, we are done by the assumption on \( M \). Otherwise, consider the set

\[
\mathcal{A} := \{(N_\alpha, \mu_\alpha) | N_\alpha \supset P, \mu_\alpha : N_\lambda \rightarrow M, \mu_\lambda \circ \nu = \pi \}.
\]

Define an order on this set, setting \( \alpha \prec \beta \) iff \( N_\alpha \subset N_\beta \) and \( \mu_\beta|_{N_\beta} = \mu_\alpha \).

The chain condition is satisfied hence there exists a maximal element, say, \( N_1 \). Since any submodule of \( N \), containing \( P \) and having finite dimension is contained in \( \mathcal{A} \), \( N_1 \) is strictly bigger then \( P \). Were \( N_1 \neq N \), repeating the above process, we would get a submodule \( N_1, N_2 \succ N_1 \), which contradicts the maximality of \( N_1 \). Hence \( N_1 = N \).

Let now \( H \) be a Hopf algebra. Then \( k \) is a left (right) \( H \)-comodule be means of the unit map. A left (right) integral on \( H \) is an \( H \)-comodule morphism \( H \rightarrow k \), where \( H \) is considered as left (right) comodule on itself by means of the coproduct. Let \( \int_r \) (resp. \( \int_l \)) denoted a left (resp. right) integral on \( H \), then we have

\[
a_1 \int_l (a_2) = \int_l (a), \tag{2}
\]

\[
\int_r (a_1) a_2 = \int_r (a), \tag{3}
\]

\( \forall a \in H \).

We need the following information on the integrals.

1.5. Let \( H \) be a Hopf algebra. The following conditions are equivalent

(i) \( H \) possesses a left integral.

(ii) \( H \) is left co-Frobenius as a coalgebra.

(iii) \( H \) possesses a right integral.

(iv) \( H \) is right co-Frobenius as a coalgebra.

(v) The injective hull of every left comodule is finite dimension.

(vi) The injective hull of every right comodule is finite dimension.

(vii) \( H \) possesses a finite dimension injective left comodule.

(viii) \( H \) possesses a finite dimension injective right comodule.
(ix) $H_{\text{rat}}^*$ is dense in $H^*$.

(x) $\text{rat}H^*$ is dense in $H^*$.

(xi) Every injective left $H-$comodule is projective.

(xii) Every injective right $H-$comodule is projective.

The first 6 conditions are due to Larson-Sweedler-Sullivan, the conditions (vii)-(xii) follow from 1.3

Define bilinear form $b$: $b(x, y) := \int_l (xS(y))$. Using the identity

$$h_1 \int_l (h_2 S(g)) = \int_l (hS(g_1)) g_2$$

which follows immediately from the definition of $\int_l$, we can easily show that $b$ is balanced. The following results are due to D. Ţeşfan.

1.6. [16] Let $H$ be a Hopf algebra with integral. Then the following facts hold.

(i) The bilinear form $b$ is non-degenerate.

(ii) For any finite dimensional $H-$comodule, $\dim_k(\text{Hom}^H(H, M)) = \dim_k M$.

In particular, we have

(i) the antipode is injective, and

(ii) there exists $h$ such that $\int_l (S(h)) \neq 0$. Since $\int_l \circ S$ satisfies (3), it is a right integral on $H$.

Assume for a moment that the field $k$ is algebraically closed. Let $R$ be the radical of $H$, i.e. $R = \bigoplus_\alpha \text{Cf}(M_\alpha)$, where $\{M_\alpha, \alpha \in \mathcal{A}\}$ is the set of all simple left (or right) $H-$comodules. As we have seen in the previous subsection, $\text{Cf}(M_\alpha) \cong M(m_\alpha)^*$, $m_\alpha := \dim_k(M_\alpha)$. Fix idempotents $\{e_{\alpha,i}|\alpha \in \mathcal{A}, 1 \leq i \leq m_\alpha\}$ of the algebras $M(m_\alpha)$ – the matrix ring of degree $m_\alpha$. They can be considered as linear functional on $R$ by defining $e_{\alpha,i}(\text{Cf}(M_\beta)) = 0$, whenever $\alpha \neq \beta$.

A theorem of Sweeder-Sullivan [17], stating that there exists a coalgebra projection $H \rightarrow R$, implies that $e_{\alpha,i}$ can be extended on the whole $H$ and that

$$H \leftarrow e_{\alpha,i} \quad (\text{resp. } e_{\alpha,i} \rightarrow H)$$

is a right (resp. left) $H-$comodules.

Consequently, we have a decomposition

$$H \cong \bigoplus_{\alpha \in \mathcal{A}, 1 \leq i \leq m_\alpha} e_{\alpha,i} \left( \text{resp. } \bigoplus_{\alpha \in \mathcal{A}, 1 \leq i \leq m_\alpha} e_{\alpha,i} \rightarrow H \right)$$

as right (resp. left) $H-$comodules.
On the other hand, it is easy to see that $M_\alpha \subset H \leftarrow e_{\alpha,i}$ as right $H$–comodules. Thus, comparing with the decomposition in [1.2], we have:

1.7. Assume that the field $k$ is algebraically closed. Then
\[
J(M_\alpha) \cong H \leftarrow e_{\alpha,i}
\] (5)
as right $H$–comodules.

2. The Convolution Product on $H$

We define a new product on $H$:
\[
g * h := h_1 \int (h_2 S(g)) = \int (h S(g_1)) g_2 \text{ (by (4))}.
\]
Using (4) we can easily check that $*$ is associative. $*$ is called the convolution product on $H$. Denote $\tilde{H} := (H, *)$. Then $\tilde{H}$ is a (non-unital) algebra. Let $V$ be a right $H$–comodule. Then $V$ is a left $H$–module by means of the action
\[
h * v := v_0 \int (v_1 S(h)).
\]
The verification again uses (4). Denote $\tilde{V} := (V, *)$.

Let $f : V \rightarrow W$ be a homomorphism of right $H$–comodules, i.e.,
\[
f(v_0) \otimes f(v)_1 = f(v_0) \otimes v_1.
\]
We have
\[
h * f(v) = f(v_0) \int (f(v)_1 S(h)) = f(v_0) \int (v_1 S(h)) = f(h * v).
\]
Thus $f$ is a homomorphism of left $\tilde{H}$–modules. Conversely, if $f$ is a homomorphism $\tilde{V} \rightarrow \tilde{W}$, then we have, for all $h \in H$,
\[
f(v_0) \int (f(v)_1 S(h)) = f(v_0) \int (v_1 S(h)).
\]
By the non-degeneracy of the integral [1.4, (i)], we have
\[
f(v_0) \otimes f(v)_1 = f(v_0) \otimes v_1,
\]
which means that $f$ is a homomorphism of right $H$–comodules. Thus we have
\[
\text{Hom}_{\tilde{H}}(\tilde{V}, \tilde{W}) = \text{Hom}^H(V, W).
\]
In particular we have proved

Lemma 2.1. If $M$ is a simple right $H$–comodule then $\tilde{M}$ is a simple left $H$–module.

Let now $V$ be a cyclic $\tilde{H}$–module, that is, there exists $\bar{v} \in V$, such that $V$ is generated by $\bar{v}$. We want to define a coaction of $H$ on $V$. Let $v \in V$. Then there exists (not uniquely) $h \in \tilde{H}$, such that $v = h * \bar{v}$. Set
\( \hat{\delta}(v) := h_1 \ast \bar{v} \otimes h_2 \). We show that \( \hat{\delta} \) is independent of the choice of \( \bar{v} \) and \( h \), and that it is in fact a coaction of \( H \) on \( V \).

The fact that \( \hat{\delta} \) does not change when \( \bar{v} \) is replaced by \( \tilde{v} \) is represented by the equation

\[
\int_t (h_2 S(g)) h_1 \ast \bar{v} = (g \ast h) \ast \bar{v} = g \ast (h \ast \bar{v}) = 0,
\]

for all \( g \in H \). By the non-degeneracy of \( \mathcal{I} \), we conclude that \( h_1 \ast \bar{v} \otimes h_2 = 0 \).

The coassociativity and counitary of \( \hat{\delta} \) are also checked directly using (4). Moreover, denoting by \( \hat{V} \) the resulting \( H \)-comodule, we also have \( \hat{V} \cong V \).

Since simple modules are cyclic, we have

**Lemma 2.2.** Let \( M \) be a simple \( \bar{H} \)-module, then there exists a right \( H \)-comodule \( \hat{M} \), such that \( M \cong \hat{M} \).

We also need another action of \( \bar{H} \) on a right comodule \( V \) of \( H \), given by

\[
h \circ v := v_0 \int_t (hS(v_1)).
\]

Indeed, we have

\[
g \circ (h \circ v) = g \circ v_0 \int_t (hS(v_1)) = v_0 \int_t (gS(v_1)) \int_t (hS(v_2)) = v_0 \int_t (hS(g_1)) \int_t (g_2S(v_1)) = (g \ast h) \circ v.
\]

It is again easy to check that \( \bar{V} := (V, \circ) \) is a left \( \bar{H} \)-module, and that, if \( V \) is simple then \( \bar{V} \) is simple. Lemma 2.2 holds only in case the antipode is bijective. In fact, if \( V \) is a cyclic \( H \)-module. Then we can define the following coaction of \( H \) on \( V \):

\[
\hat{\delta}(v) := h_1 \circ \bar{v} \otimes S^{-2}(h_2),
\]

where \( \bar{v} \) is a generating element and \( h \ast \bar{v} = v \). Denote by \( \hat{V} \) the comodule of \( H \) induced from \( V \), we have \( \hat{V} \cong V \).

Composing the operation \( \hat{\circ} \) and \( \hat{\circ} \) on a simple comodule \( V \) we obtain a new simple comodule \( \hat{V} \), denoted by \( V^\bullet \). The coaction of \( H \) on \( V^\bullet \) is given
by
\[ \delta^\bullet(v) = v_0 \int (h_1 S(v_1)) \otimes h_2, \]
with \( h \) given by condition \( v_0 \int (h S(v_1)) = v \).

Now, assume that \( k \) is algebraically closed. Let \( M_\alpha \) be a simple right \( H \)-comodule. Then \( \check{M}_\alpha \) is a simple left \( \check{H} \)-comodule. The action of \( H \) on \( M_\alpha \) induces a \( \check{H} \)-module homomorphism \( \pi : \check{H} \to \check{M}_\alpha \otimes (\check{M}_\alpha^\ast) \), where \( \check{M}_\alpha \otimes (\check{M}_\alpha^\ast) \cong \check{M}_\alpha \otimes \dim_k M_\alpha \) as \( \check{H} \)-modules. The isomorphism \( (\mathbf{8}) \) shows that \( \pi \) is a homomorphism of \( H \)-comodules.

On the other hand, \( H \) decomposes into the direct sum of its indecomposable injective subcomodules as in Lemma 1.2. For \( h \in H \leftarrow e_{\beta,j} \), i.e., \( h = e_{\beta,j}(g_1)g_2 \) for some \( g \in H \), and for \( v \in \check{M}_\alpha \), we have
\[ h \ast v = v_0 e_{\beta,j}(g_1) \int (g_2 S(v_1)) = v_0 \int (g(S(v_1)) e_{\beta,i}(v_2)). \]
Thus, if \( \alpha \neq \beta \), \( h \ast M_\alpha = 0 \), therefore, \( \pi(h) = 0 \). Thus
\[ \Hom^H(\mathcal{J}(M_\alpha), M_\alpha^\ast) \neq 0. \] (7)

According to 1.8,
\[ \dim_k (\Hom^H(H, M_\alpha^\ast)) = \dim_k M_\alpha^\ast. \] (8)
Since \( H \) contains precisely \( m_\alpha = \dim_k M_\alpha \) copies of \( \mathcal{J}(M_\alpha) \), we conclude that
\[ \dim_k \Hom^H(\mathcal{J}(M_\alpha), M_\alpha^\ast) = 1, \] (9)
\[ \dim_k \Hom^H(\mathcal{J}(M_\beta), M_\alpha^\ast) = 0 \text{ if } \alpha \neq \beta. \] (10)

**Theorem 2.3.** Let \( H \) be a Hopf algebra with integral. Then for any simple comodules \( M_\alpha, M_\beta \)
\[ \dim_k \Hom(\mathcal{J}(M_\alpha), M_\beta^\ast) = \delta_\alpha^\beta d_\beta^2, \]
where \( d_\beta^2 \) is the dimension over \( k \) of \( D_\beta = \text{End}^H(M_\beta) \).

**Proof.** The case \( k \) is algebraically closed is already proved. Assume that \( k \neq k \). Then \( D_\beta = \text{End}^H(M_\beta) \) splits over \( k \):
\[ D_\beta \otimes_k k \cong M_\beta \otimes_k (d_\beta). \]
For the extension \( \overline{H} := H \otimes_k k \), the comodule \( \overline{M}_\beta := M_\beta \otimes_k k \) decomposes into \( d_\beta \) exemplars of the simple \( \overline{H} \)-comodule \( M_\beta \). Since \( \mathcal{J}(M_\beta) := \mathcal{J}(M_\beta) \otimes_k k \) remains a direct summand of \( \overline{H} \), it is an injective \( \overline{H} \)-comodule. Therefore \( \mathcal{J}(M_\beta) \) is a direct sum of \( d_\beta \) exemplars of \( \mathcal{J}(M_\beta) \). Since, for \( \alpha \neq \beta \),
\[ \Hom_{\overline{H}}(\mathcal{J}(M_\alpha), M_\beta^\ast) = 0, \]
we have
\[ \Hom_H(\mathcal{J}(M_\alpha), M_\beta^\ast) = 0. \]
Therefore, by virtue of Equation (8) (which is valid on any field),
\[
\dim_k \text{Hom}_H(J(M_{\alpha})^{\oplus m_{\alpha}}, M_{\alpha}) = \dim_k M_{\alpha} = d_{\alpha}^2 m_{\alpha}.
\]
Consequently
\[
\dim_k \text{Hom}_H(J(M_{\alpha}), M_{\alpha}) = d_{\alpha}^2.
\]

Let \( M \) be a finite dimensional right \( H \)-comodule then \( M^* := \text{Hom}_k(M, k) \) is also a right comodule with the coaction given by the equation
\[
\rho(\phi)(x) := \phi_0(x)\phi_1 = \phi(x_0)S(x_1), \quad x \in M, \phi \in M^*.
\]
The map \( \text{ev} : M^* \otimes M \to k, \phi \otimes x \mapsto \phi(x) \) is a morphism of \( H \)-comodules. The pair \( (M^*, \text{ev}) \) is called left dual to \( M \), it is defined uniquely up to isomorphism. There exists a monomorphism \( \text{db} : k \to S \otimes S^* \), defined by the conditions \( (\text{ev} \otimes \text{id}_{S^*})(\text{id}_S \otimes \text{db}) = \text{id}_{S^*} \) and \( (\text{id}_S \otimes \text{ev})(\text{db} \otimes \text{id}_S) = \text{id}_S \), which is also a comodule morphism. Dually, \( (M, \text{ev}) \) is called the right dual to \( M^* \).

Thus, we see that the left dual to a finite dimensional comodule always exists. If the antipode is bijective then the right dual to any finite dimensional comodule also exists. We shall need the following isomorphism, given by manipulating the morphism \( \text{ev} \) and \( \text{db} \): for any finite dimensional comodule \( N \),
\[
\text{Hom}_H(M \otimes N, P) \cong \text{Hom}_H(M, P \otimes N^*) \quad (11)
\]
\[
\text{Hom}_H(M, N \otimes P) \cong \text{Hom}_H(N^* \otimes M, P). \quad (12)
\]
As an immediate corollary of Lemma 1.4 and these equations, we have

**Lemma 2.4.** Let \( M \) be a finite dimensional comodule. Then we have:
(i) If \( M^* \) is projective (resp. injective) then \( M \) is injective (resp. projective).
(ii) If the antipode is bijective, then \( M \) is injective (resp. projective) iff \( M^* \) is projective (resp. injective).

**Proof.** Equations (11) and (12) imply
\[
\text{Hom}_H(M^*, N^*) \cong \text{Hom}_H(N, M). \quad (13)
\]
Thus, if \( M \) is projective (resp. injective) then \( M^* \) is injective (resp. projective) in the category of finite dimensional comodules.

**Corollary 2.5.** Let \( H \) be a Hopf algebra with integral. Assume that the antipode is injective. Then for any simple comodule \( M \)
\[
\mathcal{J}((M^*)^*) \cong \mathcal{J}(M^*).
\]
Proposition 2.6. Assume that the Hopf algebra $H$ as a left-right integral. Then $V^* \cong V^{**}$. If $H$ is moreover coquasitriangular then $V^* \cong V$ and, consequently, $J(M^*) \cong J(M)^*$.

Proof. Assume that $\int$ is a left-right integral. Thus we can define $V^*$ as above. We want to show that
\[ v_0 \int (h_1 S(v_1)) \otimes h_2 = c \otimes v_0 \otimes S^2(v_1), \]
for certain constant $c$, depending only on $\int$. By the non-degeneracy of integral, this equation is equivalent to
\[ v_0 \int (v_1 S(h)) \int (h_2 S(g)) = c \otimes v_0 \int (S^2(v_1) S(g)). \]
We have
\[ v_0 \int (h_1 S(v_1)) \int (h_2 S(g)) = v_0 \int ((g \ast h) S(v_1)) = (g \ast h) \circ v = g \ast (h \circ v) = g \circ v = v_0 \int (g S(v_1)). \]
By the uniqueness of integral, we can choose $c$ such that
\[ c \otimes \int (S^2(v_1) S(g)) = \int (g S(v_1)). \]
If $H$ is coquasitriangular then $V \cong V^{**}$. 

Remark 2.7. If the left and the right integrals do not coincide then in general, $M^* \not\cong M^{**}$. An example is Sweedler’s Hopf algebras, see, e.g., [10].

3. Splitting Comodules

Let $S$ be a simple comodule over $H$. $S$ is called splitting comodule, or typical comodule, if $S = J(S)$. Since $J(S)$ is injective and hence projective, we see that $S$ splits in any comodule. This explain the name splitting. The name typical was used by V. Kac for modules over a Lie superalgebra [9].

By virtue of conditions in 1.5, if a Hopf algebra possesses a splitting comodule then it possesses a non-zero integral. The converse statement is not true. The aim of this section is to give a criteria for a simple comodule to be typical.

Let $M$ be a right $H$–comodule then the coaction of $H$ on $M^{**}$ – the double left dual to $M$ is given by (identifying $M^{**}$ with $M$ as vector spaces)
\[ \rho_{M^{**}}(v) = x_0 \otimes S^2(v_1). \]
Lemma 3.1. Let \( M \) be a simple \( H \)-comodule. Then \( M \) is splitting iff \( M^* \) is splitting.

Proof. Assume that \( M \) is a typical. Then \( M \) is injective. By a theorem of Doi \([2]\), \( M^* \otimes M \) is injective, too. By definition of \( M^* \), we have an epimorphism \( \text{ev} : M^* \otimes M \rightarrow k \). Among indecomposable injective subcomodule of \( M^* \otimes M \) there exists one, say \( J \), such that the restriction of \( \text{ev} \) on \( J \) is not zero. On the other hand, since \( J \) is indecomposable and injective, it should appear in the decomposition \((4)\), and by Theorem 2.3, the only comodule with this property is \( J(k) \), the injective hull of \( k \). Thus we show that \( M^* \otimes M \) contains \( J(k) \) as a subcomodule, consequently, it contains \( k \) as subcomodule, i.e. \( \text{Hom}^H(k, M^* \otimes M) \neq 0 \). According to \((12)\), we have

\[ \text{Hom}^H(M^**, M) \neq 0. \]

Therefore \( M^** \cong M \) and hence is splitting. Consequently \( M^* \) is also splitting, by Lemma 2.4.

Assume now that \( M^* \) is splitting. The discussion above shows that \( M^** \) is also splitting and \( M^** \cong M^{****} \). Since

\[ \text{Hom}^H(M^*, N^*) \cong \text{Hom}^H(N, M), \]

we conclude that \( M \cong M^{**} \). Thus \( M \) is splitting. \( \square \)

Theorem 3.2. Let \( M \) be a simple right \( H \)-comodule. Then \( M \) is splitting if and only if the bilinear form \( c \), \( c(x, y) = \int_r (yS(x)) \), is not identically zero on \( Cf(M) \). In this case, \( c \) is also non-degenerate on \( Cf(M) \).

Proof. “if”.

For each \( g \in H \), define a linear functional \( \phi_g \in \mathcal{H}^* : \phi_g(h) := \int_r (hS(g)) \).

By assumption, there exist \( g \in Cf(M) \) such that \( \phi_g \) is not identically zero on \( Cf(M) \). Since \( M \) is simple, the right coideal generated by \( g \) is isomorphic to \( M \). Define a linear map \( \eta = \eta_g \):

\[ \eta : Cf(M) \rightarrow M^{**}, \quad h \mapsto g_1 \int_r (hS(g_2)), h \in M. \]  

(14)

Since \( \varepsilon(\eta(h)) = \phi_q(h) \), \( \eta \) is not trivial.

We have the following identity, which is an immediate consequence of \((3)\) and the injectivity of the antipode

\[ \int_r (h_1S(g)) h_2 = S^2(g_1) \int_r (hS(g_2)). \]  

(15)

It follows from \((13)\) that \( \eta \) is a morphism of \( H \)-comodules. Since \( \eta \) is non-trivial on \( Cf(M) \), which is a direct sum of copies of \( M \), we conclude that
η should induce a morphism $M \rightarrow M^{**}$, which is non-trivial. Since $M$ is simple and $\dim_k M = \dim_k M^{**}$, this morphism is an isomorphism. As a consequence, $\mathcal{C}f(M^{**}) = \mathcal{S}(\mathcal{C}f(M)) = \mathcal{C}f(M)$ and $\phi_g$ is $*$-invertible on $\mathcal{C}f(M)$. Let $ψ$ be the $*$-inverse to $φ$, define on $\mathcal{C}f(M)$, thus $φ(h_1)ψ(h_2) = ψ(h_1)φ(h_2) = \varepsilon(h)$.

Let now $M \hookrightarrow N$ be an inclusion of $H$–comodules. Let $f : N \rightarrow M$ be a linear projection on $M$. We define a new map $F : N \rightarrow M$ as follows,

$$F(v) := f(v)_{0}\psi(f(v)_{1}) \int_r (v_1 S(f(v)_{2})).$$

$F$ is well defined by the assumption that $\text{Im}(f) = M$, which implies $f(v)_{1} \in \mathcal{C}f(M)$. For $v \in M$, $f(v) = v$, hence

$$F(v) = v_0 \psi(v_1) \int_r (v_3 S(v_2)) = v_0 \psi(v_1)q(v_2) = v.$$

Thus, $F$ is again a projection of $M$. If we show that $F$ is a morphism of $H$–comodule, then we will be done.

By definition of $φ$, we can consider $F$ as a composition of the map $g : N \rightarrow M^{**}$:

$$g(v) = f(v)_{0} \int_r (v_1 S(f(v)_{1}),$$

and the morphism $η^{-1} : M^{**} \rightarrow M$. Thus, it is sufficient to show that $g$ is a morphism of $H$–comodules, which means

$$f(v)_{0} \int_r (v_1 S(f(v)_{1}) \otimes v_2 = f(v)_{0} \int_r (v_1 S(f(v)_{2}) \otimes S^2(f(v)_{1}).$$

We have, according to (15),

the left-hand side = $f(v)_{0} \otimes S^2(f(v)_{1}) \int_r (v_1 S(f(v)_{2}))$

= the right-hand side.

Therefore, $F$ is a morphism of $H$–comodules, consequently, $M$ is injective and $\mathcal{J}(M) = M$.

"only if" Assume now that $M$ is splitting, then, by Lemma 3.1, $M^*$ is also splitting, hence $M \otimes M^*$ is injective. By definition of $M^*$, there exists a monomorphism $db : k \rightarrow M \otimes M^*$, inducing a monomorphism $J(k) \hookrightarrow M \otimes M^*$. The latter inclusion induces the following inclusion

$$J(k) \subset \mathcal{C}f(J(k)) \hookrightarrow \mathcal{C}f(M \otimes M^*) = \mathcal{C}f(M) \otimes \mathcal{C}f(M^*).$$

Since the right integral does not vanish identically on $J(k)$ (by 2.3), we conclude that the set $\int_r (a^i_j S(a'^i_j))$ is not identically zero, as $a^i_j S(a'^i_j)$ span $\mathcal{C}f(M \otimes M^*)$.
4. **Simple Representations of Quantum Groups of Type $A_{00}$**

Let $V$ be a finite dimensional vector space over $k$, a field of characteristic zero. An operator $R : V \otimes V \to V \otimes V$ is called a Hecke symmetry if $R$ satisfies the Yang-Baxter equation

$$(R \otimes \text{id}_V)(\text{id}_V \otimes R)(R \otimes \text{id}_V) = (\text{id}_V \otimes R)(R \otimes \text{id}_V)(\text{id}_V \otimes R),$$

the Hecke equation

$$(R - \text{id})(R - q \cdot \text{id}) = 0, \quad q \neq 0$$

and is closed, that is, the operator $P : V^* \otimes V \to V \otimes V^*$, half dual (half-adjoint) to $R$—given by

$$P = (\text{ev}_V \otimes \text{id}_{V \otimes V^*})(\text{id}_{V^*} \otimes R \otimes \text{id}_{V^*})(\text{id}_{V^*} \otimes \text{db}_V)$$

is invertible. We shall also assume that $q$ is not a root of unity of degree greater than 1.

Being given a Hecke symmetry, one can define the associated quantum symmetric and anti-symmetric tensor algebras as factor algebras of the tensor algebra over $V$ by the ideal, generated by $\text{Im}(R - q \cdot \text{id})$ and $\text{Im}(R + \text{id})$, respectively. It is shown that the Poincaré series of these algebras, i.e., the formal sums with coefficients being dimensions of homogeneous components of these algebras, are rational functions [6] having negative roots and positive poles.

A Hecke symmetry $R$ is said to have birank $(1, 1)$ if the Poincaré series of the associated quantum symmetric tensor algebra has one pole and one root, i.e., is of the form $(1 + at)(1 - bt)^{-1}$, $a, b > 0$.

The quantum group (quantum semi-group) associated to $R$ is defined to be the Hopf algebra (bialgebra) universally coacting on the mentioned above quantum tensor algebras [14]. They are denoted by $H$ and $E$, respectively. If $R$ has birank $(1, 1)$, the associated quantum group is called quantum group of type $A_{00}$. Simple $E$—comodules can be labelled by hook-partitions of the form $(m, 1^n)$, $m \geq 1, n \geq 0$ and the trivial partition $(0)$. For simplicity we shall use the pair $(m, n)$ to denote the partition $(m, 1^n)$ and the pair $(0, 0)$ to denote the trivial partition. The endomorphisms ring of a simple $E$—comodule is isomorphic to $k$. On the other hand, simple $E$—comodules are also simple as $H$—comodules with the natural action induced from the inclusion $E \to H$. The reader is referred to [7] for detail.

From now on, for simplicity we shall use a dot $\cdot$ to denote the tensor product and a plus $+$ to denote the direct sum, thus $V^n$ will means $V^\otimes n$ and $n \cdot V$ will means $V^\oplus n$. We shall also use the equal sign $=$ to denote an isomorphism.
Simple $E$–comodules $I_{m,n}$, associated to pairs $(m, n), m \geq 1, n \geq 0$, are given by the following rule. $I_{n,0} = S_n$ is the $n$–th component of the quantum symmetric tensor over $V$, $I_{1,n-1} = \Lambda_n$ is the $n$–th component of the quantum anti-symmetric tensor over $V$, $n \geq 1$, $I_{0,0} := k$.

$$I_{p,q} \cdot I_{m,n} = I_{m+p,n+q} + I_{m+p-1,n+q+1},$$

(16)

for $m, p \geq 1, n, q \geq 0$. Particularly, we have $V = I_{1,0}, V^* = I_{1,0}$.

Our aim is to associate to each pair $(m, n)$ of integers a simple $H$–comodule and show that they furnish all simple $H$–comodules.

According to a result of [8], if $\text{rank}_q R = 0$, where $\text{rank}_q R$ is the full trace of the half-dual operator $P$, then $H_R$ possesses an integral. Thus, in order to apply the results of the previous section, we have to show that $\text{rank}_q R = 0$. To do this we consider the Koszul complex of the second type introduced by Manin [13] (see also [4, 5, 12]). It is shown that, if $\text{rank}_q R \neq [k - l]$, then the complex

$$\cdots \longrightarrow \Lambda_k \cdot S^*_l \xrightarrow{d_{k,l}} \Lambda_{k+1} \cdot S^*_l \xrightarrow{d_{k+1,l+1}} \Lambda_{k+2} \cdot S^*_l \longrightarrow \cdots$$

with the differential induced from the dual basis map $\text{ev} : k \longrightarrow V \cdot V^*$, is exact [7, 2].

Notice that, according to (16), (11) and (12), for $m, p \geq 1$ and $n, q \geq 0$,

\[
\text{Hom}^H(I_{p,q}, I_{m+p,n+q} \cdot I_{m,n}^*) = k
\]

\[
\text{End}^H(I_{m+p,n+q} \cdot I_{m,n}) = \text{End}^H(I_{m+p,n+q} \cdot I_{m,n}) = 2 \cdot k.
\]

Therefore, denoting $I_{m,-n} := I_{m,n}^*$, we have, for $p, q \geq 1, m \geq 1, n \geq 0$,

$$I_{p,q} + I_{p+1,q-1} \subset I_{m+p,n+q} \cdot I_{m,-n},$$

(17)

and $I_{m+p,n+q} \cdot I_{m,-n}$ does not contain any other simple $E$–comodule.

Assume that $\text{rank}_q R \neq 0$. Then the complex

$$0 \rightarrow k \xrightarrow{d_{0,0}} I_{1,0} \cdot I_{-1,0} \xrightarrow{d_{1,1}} I_{1,1} \cdot I_{-2,0} \xrightarrow{d_{2,2}} \cdots$$

(18)

is exact. We have, for $n \geq m \geq 1$,

$$I_{n,1} \cdot I_{1,m-1} \cdot I_{m,0} = (I_{n+1,m} + I_{n,m+1}) \cdot I_{m,0}$$

$$\supset 2 \cdot I_{n-m+1,m} + I_{n-m,m+1} + I_{n-m+2,m-1}. \quad (19)$$

Thus, multiplying (18) with $I_{n,1}$, we have a diagram

$$0 \rightarrow I_{n,1} \rightarrow I_{n,1} \cdot I_{1,0} \cdot I_{-1,0} \rightarrow I_{n,1} \cdot I_{1,1} \cdot I_{-2,0} \rightarrow \cdots$$

(20)

The exactness of the lower complex and the remark following (17) imply that $I_{n+1,0} = 0$, contradiction. Thus we have $\text{rank}_q R = 0$. As a consequence, the Hopf algebra $H$ possesses an integral and the formula for the integral in [8] implies that $I_{m,n}$, and hence $I_{m,-n}, m \geq 1, n \geq 0$, are all
splitting, except for $I_{0,0} = k$. Therefore, by means of the two isomorphism preceding (17), the inclusion in (17) is in fact an isomorphism: for $p, q, m \geq 1, n \geq 0$,

$$I_{m+p,n+q} \cdot I_{m,n} = I_{p,q} + I_{p+1,q-1} \ldots \quad (21)$$

The next step is to define the comodules $I_{1,1}$ and $I_{1,-1}$.

Consider the sequence (18). Since $\text{rank}_q R = 0$, $I_{1,0} \cdot I_{-1,0} = V \cdot V^*$ contains two exemplars of $k$ in its composition series but only one as subcomodule. Let $M := (V \cdot V^*)/k$, then the map $V \cdot V^* \to k$ factorizes through $k$ to a map $M \to k$. Dualizing this we get a sequence $k \to M^* \to V \cdot V^*$. Since

$$\text{Hom}^H(k, I_{1,1}, I_{-2,0}) \cong \text{Hom}^H(I_{2,0}, I_{1,1}) = 0,$$

that is $k$ cannot be a subcomodule of $I_{-2,0}$, $\text{Ind}_{1,1} \not\subset k$. From (20), we see that $\text{Ker}d_{2,2} \not\supset k$. Let $N := (V \cdot V^*)/\text{Ker}d_{2,2}$. Then $N$ is a factor comodule of $M$, which is different from $k$ and $M$. Therefore $V \cdot V^*$ contains at least 4 simple comodules in its composition series. It cannot be larger then 4, as on the left hand side of (19), there are 4 simple comodules. Denote by $A$ and $B$ the two simple subcomodules, that are different from $k$. Since $V \cdot V^*$ is self dual, either $A$ and $B$ are self dual or $B$ is dual to $A$.

Using (21), we have

$$I_{2,1} \cdot I_{-1,0} \cdot I_{1,0} = 2 \cdot I_{2,1} + I_{1,2} + I_{3,0}.$$ 

Thus, we can assume that $I_{2,1} \cdot A = I_{1,2}$ and $I_{2,1} \cdot B = I_{3,0}$. Using induction we can easily show that

$$I_{m,n} \cdot A = I_{m-1,n+1} \quad I_{m,n} \cdot B = I_{m+1,n-1}, \quad (22)$$

for all $m \geq 2, n \geq 1$. Using the fact, that $M \cdot M^*$ contains $k$ as a subcomodule, for any comodule $M$, we deduce that $A = B^*$ and $A \cdot B = k$.

Thus

$$I_{1,-1} \cdot I_{-1,1} = k \quad (23)$$

$$I_{m,n} \cdot I_{1,-1} = I_{m+1,n-1} \quad (24)$$

$$I_{m,n} \cdot I_{-1,1} = I_{m-1,n+1} \quad (25)$$

for all $m \geq 2, n \geq 1$. Dualizing these equalities, we obtain

$$I_{-m,-n} \cdot I_{1,-1} = I_{-m+1,-n-1} \quad (26)$$

$$I_{-m,-n} \cdot I_{-1,1} = I_{-m-1,-n+1} \quad (27)$$

for all $m \geq 2, n \geq 1$.

Consider now

$$I_{1,1} \cdot I_{1,0} \cdot I_{-1,0} = (I_{2,1} + I_{1,2}) \cdot I_{-1,0}$$

$$= I_{1,1} + I_{2,0} + I_{1,2} \cdot I_{-1,0}.$$
The left-hand side contains simple comodule $I_{1,1} \cdot I_{1,-1}$ and $I_{1,1} \cdot I_{-1,1}$. We therefore conclude that

$$I_{1,1} \cdot I_{1,-1} = I_{2,0},$$

and thus

$$I_{2,0} \cdot I_{-1,1} = I_{1,1}.$$

We are now at the stage to associate to each pair $(m, n)$ of integers a simple comodule $I_{m,n}$. Note that for $m \neq 1, n \geq 0$ or $m \leq 1, n \leq 0$, we have already defined $I_{m,n}$. We call

$$s(m, n) := m + n$$

the total degree of the pair $(m, n)$. Thus, there can be three possibilities: $s(m, n) > 0$; $< 0$ or $= 0$. If $s(m, n) = 0$, i.e., $m = -n$, set

$$I_{m,-m} := I_{1,-1}^m.$$

If $s(m, n) \neq 0$, set

$$I_{m,n} := I_{m+n,0} \cdot I_{-n,n}.$$

Using (23-27), it is easy to see that the definition is compatible with the predefined comodules and that these comodules are all simple. We want to find the formula for the tensor product of these comodules and deduce from this formula that these comodules furnish all simple $H-$comodule.

Let $(m, n)$ and $(p, q)$ be pairs of integers. Our aim is to decompose $I_{m,n} \cdot I_{p,q}$. The main role here plays the total degree. There can be three possibilities

1. either $m + n$ or $p + q$ is equal to zero;
2. $m + n$ and $p + q$ are both different from zero but their sum is zero;
3. $m + n$ and $p + q$ and $m + n + p + q$ are all different from zero.

1. If $m + n = 0$ then $I_{m,n} = I_{1,-1}^m$, hence

$$I_{m,-m} \cdot I_{p,q} = I_{p+m,q-m}.$$  \hspace{1cm} (28)

2. If $m + n + p + q = 0$ and $m + n \neq 0$, using (23-27), we can assume $n = p = 0$. Thus $m = -p$. We claim that

$$I_{m,0} \cdot I_{-m,0} = I_{0,1} + I_{1,0}. $$  \hspace{1cm} (29)

Indeed, $I_{m,0}^* = I_{-m,0}$, hence $I_{m,0} \cdot I_{-m,0}$ contains $k$ as subcomodule. Moreover, this comodule is injective. On the other hand, $I_{1,0} \cdot I_{-1,0}$ is the injective envelope of $k$, therefore (cf. [3]), is a subcomodule of $I_{m,0} \cdot I_{-m,0}$, $\forall m \geq 0$. Multiplying these comodules with $I_{m,1}$ we get the same comodule. Whence we conclude (29).

3. If $m + n$, $p + q$, $m + n + p + q$ are all non-zero, dualizing if necessary, we can assume $m + n + p + q > 0$. Using (23-27), we can assume $n = q = 0$. Assume $m > p$, thus $m > 0$. One is led to computing $I_{m,0} \cdot I_{p,0}$. If $p > 0$, the formula is already known (cf. [16-17]). Assume $p < 0$ and set $k = -p,$
then $k > 0$ and $m > k$. We consider two case: $m - k \geq 2$ and $m - k = 1$. If $m - k \geq 2$, then, according to (23-27),
\[
I_{m,0} \cdot I_{-k,0} = I_{1,-1} \cdot I_{m-1,1} \cdot I_{-k,0} = I_{1,-1} \cdot (I_{m-k-1,1} + I_{m-k,0}) = I_{m-k,0} + I_{m-k+1,-1}.
\]

(30)

In the case $m - k = 1$, we show that
\[
I_{m,0} \cdot I_{-m,0} = I_{1,0} + I_{2,-1}.
\]

(31)

We have
\[
\text{Hom}(I_{1,0}, I_{m,0} \cdot I_{-m+1,0}) = \text{Hom}(I_{1,0} \cdot I_{m-1,0}, I_{m,0}) = k.
\]

Remember that $I_{2,-1} = I_{1,-1} \cdot I_{1,0}$ and that $I_{2,-1}$ is also simple. Since
\[
\text{Hom}(I_{m,0} \cdot I_{-m+1,0}, I_{0,1}) = \text{Hom}(I_{-1,1} \cdot I_{m,0}, I_{m-1,0} \cdot I_{1,0}) = \text{Hom}(I_{m-1,1} \cdot I_{m-1,1} \cdot I_{1,0}) = k.
\]

Thus $I_{m,0} \cdot I_{-m+1,0}$ contains $I_{1,0}$ and $I_{2,-1}$ as subcomodules. On the other hand, multiplying both sides of (31) with $I_{m+2,1}$, we get an equality. Therefore (31) is proven. We summarize the results obtained in a theorem.

**Theorem 4.1.** Simple representation of a quantum group of type $A_{0}^{0}$ are classified by pairs $(m, n)$ of integers with the following properties:

1. $I_{m,0}$ is the $n$-th symmetric tensor, $I_{1,n-1}$ is the $n$-th anti-symmetric tensor, $I_{0,0} = k$, $I_{m,n}^* = I_{-m,-n}$, $I_{1,-1}$ is the super determinant. $I_{m,n}$ is splitting iff $m + n \neq 0$.

2. We have the following rule for tensor product of simple comodules.
   (a) for any integers $m, n$,
   \[
   I_{m,n} \cdot I_{1,1} = I_{m-1,n+1},
   \]
   (b) for any $m > n > 0$,
   \[
   I_{m,0} \cdot I_{n,0} = I_{m+n,0} + I_{m+n-1,1},
   I_{m,0} \cdot I_{-n,0} = I_{m-n,0} + I_{m-n+1,-1}.
   \]
   (c) for $m \neq 0$, $I_{m,0} \cdot I_{-m,0} = I_{1,0} \cdot I_{-1,0}$, this comodule is injective and indecomposable. It contains two exemplars of $k$ and the comodules $I_{1,-1}$ and $I_{-1,1}$ in its decomposition series.

The classification obtained above also allows us to classify Hecke symmetries of birank $(1, 1)$. The crucial point here is to compute the dimension of simple comodules. Since $I_{1,-1} \cdot I_{1,1} = I_{0,0} = k$, $I_{1,-1}$ is one-dimensional.
On the other hand, assuming that the Poincaré series of the quantum anti-symmetric algebra $\wedge$ is $(1 + at)(1 - bt)^{-1}$ with $a, b > 0$, we can compute the dimension of polynomial comodules $I_{m,n}$, $m \geq 1, n \geq 0$,

$$\dim_k I_{m,n} = a^m b^n + a^{m-1} b^{n+1}.$$  

According to (24), we have $a = b$. On the other hand, computing the dimension of $I_{1,0} \cdot I_{1,0}$ in two ways we obtain $a + b = 2$. Therefore $a = b = 2$, that is $\dim_k V = 2$. That means, a Hecke symmetry of birank $(1,1)$ should be defined on a vector space of dimension 2. There are only two families of such operators. The first one is two-paramenteric, found by Manin [15], the second one is one-paramenteric, found by Tambara-Takeuchi [19].

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