The sufficient condition for inclusion properties of discrete weighted Lebesgue spaces

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Abstract. In this paper, we define the discrete weighted Lebesgue spaces as generalization of discrete Lebesgue spaces, and have proven sufficient condition for inclusion properties on those spaces. To get the result, we will compare some parameters on discrete weighted Lebesgue spaces. In addition, the weak type of the discrete Lebesgue spaces is discussed.

1. Introduction

In mathematics, Lebesgue spaces are one of important topics, particularly in real and functional analysis. There are two kinds of Lebesgue spaces which are ‘continuous’ Lebesgue spaces denoted by $L_p$ and discrete Lebesgue spaces denoted by $\ell_p$. Many researchers have studied Lebesgue spaces and its generalization over few decades (see [1-15], etc.). For example, Osançliol [13] has obtained the sufficient and necessary conditions for inclusion properties of weighted Orlicz spaces and of weighted ‘continuous’ Lebesgue spaces. Related results for discrete Lebesgue spaces can be found in [12]. Recently, Gunawan et al. [5] have proven the sufficient and necessary conditions for inclusion properties of discrete Morrey spaces and of generalized discrete Morrey spaces.

Motivated by these results, we are interested in discussing the discrete weighted Lebesgue spaces and weighted weak discrete Lebesgue spaces. In particular, we will prove the sufficient condition for inclusion properties on these spaces.

First, we recall the definition of discrete weighted Lebesgue spaces and weighted weak discrete Lebesgue spaces. Let $1 \leq p < \infty$ and $W = (w_n)$ be a positive sequence on real numbers, the discrete weighted Lebesgue space $\ell^w_p (\mathbb{R})$ is the set of sequences $X = (x_n) \subseteq \mathbb{R}$ such that

$$\|X\|_{\ell^w_p (\mathbb{R})} := \left( \sum_{n=1}^{\infty} |x_n w_n|^p \right)^{\frac{1}{p}} < \infty.$$ 

For $W = (1)$, the space $\ell^w_p (\mathbb{R})$ is the discrete Lebesgue space $\ell_p (\mathbb{R})$. 
Note that, since \( \|X\|_{\ell_p(\mathbb{R})} := \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} \) is a norm on \( \ell_p(\mathbb{R}) \) and \( w_n > 0 \) for every \( n \in \mathbb{N} \), we can obtain \( \|X\|_{\ell_p^W(\mathbb{R})} \) is also a norm on \( \ell_p^W(\mathbb{R}) \) space. Furthermore, \( \ell_p^W(\mathbb{R}) \) is a Banach space with respect to the norm \( \|X\|_{\ell_p^W(\mathbb{R})} \).

Next, for \( p = \infty \), \( \ell_p^W(\mathbb{R}) \) is defined as the set of sequences \( X = (x_n) \subseteq \mathbb{R} \) such that

\[
\|X\|_{\ell_p^W(\mathbb{R})} := \max_{n \in \mathbb{N}} |x_n w_n| < \infty.
\]

Notice that, \( \|X\|_{\ell_p^W(\mathbb{R})} \) defines a norm on \( \ell_p^W(\mathbb{R}) \), and makes the space complete.

For \( 1 \leq p < \infty \) and \( W = (w_n) \) is a positive sequence on real numbers, the weighted weak discrete Lebesgue space \( \ell_p^W(\mathbb{R}) \) is the set of sequences \( X = (x_n) \subseteq \mathbb{R} \) such that

\[
\|X\|_{\ell_p^W(\mathbb{R}^\mathbb{N})} := \sup_{\gamma > 0} \{ \{ n \in \mathbb{N} : |x_n w_n| > \gamma \} \}^{1/p} < \infty.
\]

We use the notation \( \{ \{ n \in \mathbb{N} : |x_n w_n| > \gamma \} \} \) to denote the cardinality of set \( \{ n \in \mathbb{N} : |x_n w_n| > \gamma \} \). Since \( w_n > 0 \), by using similar argument in the proof of Proposition 3.4 in [5] we can show that \( \ell^W_{p,weak}(\mathbb{R}) \) is a quasi-Banach space equipped with the quasi-norm \( \|X\|_{\ell^W_{p,weak}(\mathbb{R})} \).

The rest of this paper is organized as follows. The main results are presented in Sections 2 and 3. In Section 2, we state the inclusion property of discrete weighted weak Lebesgue space \( \ell_p^W(\mathbb{R}) \) as Theorem 7 containing a sufficient condition for the inclusion property. A corresponding result for the discrete weak Lebesgue space \( \ell^W_{p,weak}(\mathbb{R}) \) is stated as Theorem 11 in Section 3.

2. Methods

To obtain the sufficient condition for inclusion properties of discrete weighted weak Lebesgue space \( \ell_p^W(\mathbb{R}) \), it will be compared with respect to parameters \( p \) and \( W \). Furthermore, one of the keys to prove the necessary conditions for inclusion properties of \( \ell_p(\mathbb{R}) \) space, \( \ell^W_p(\mathbb{R}) \) space, and \( \ell^W_{p,weak}(\mathbb{R}) \) space is to estimate the norms of the characteristic sequences in \( \mathbb{R} \) as in the following lemmas.

**Lemma 1.** [5] Let \( m \in \mathbb{Z} \) and \( N \in \{0,1,2,3,\ldots\} \), write \( S_{m,N} := \{m-N, \ldots, m, \ldots, m+N\} \). Let

\[
\xi_k^{m,N} := \begin{cases} 1, & \text{if } k \in S_{m,N} \\ 0, & \text{otherwise} \end{cases},
\]

then there exists \( C > 0 \) (independent of \( m \) and \( N \)) such that

\[
(2N + 1)^{1/p} \leq \|\xi_k^{m,N}\|_{\ell_p(\mathbb{R})} \leq C(2N + 1)^{1/p}
\]

for every \( N \in \{0,1,2,3,\ldots\} \).

**Lemma 2.** Let \( m \in \mathbb{Z} \) and \( N \in \{0,1,2,3,\ldots\} \), write \( S_{m,N} := \{m-N, \ldots, m, \ldots, m+N\} \). Let

\[
\eta_k^{m,N} := \begin{cases} \frac{1}{w_k}, & \text{if } k \in S_{m,N} \\ 0, & \text{otherwise} \end{cases},
\]

then there exists \( C > 0 \) (independent of \( m \) and \( N \)) such that

\[
(2N + 1)^{1/p} \leq \|\eta_k^{m,N}\|_{\ell_p^W(\mathbb{R})} \leq C(2N + 1)^{1/p}
\]

for every \( N \in \{0,1,2,3,\ldots\} \).

We can prove the Lemma 2 by using similar arguments in the proof of Lemma 4.2 in [5]. On the other hand, for weighted weak discrete Lebesgue space, we have analog lemma in the following.

**Lemma 3.** Let \( m \in \mathbb{Z} \) and \( N \in \{0,1,2,3,\ldots\} \), write \( S_{m,N} := \{m-N, \ldots, m, \ldots, m+N\} \).
Let $\mu_k^{m,N} = \left\{ \begin{array}{ll} \frac{1}{w_k} , & \text{if } k \in S_{m,N}, \\
0 , & \text{otherwise} \end{array} \right.$, then there exists $C > 0$ (independent of $m$ and $N$) such that

$$(2N + 1)^{1/p} \leq \|\mu_k^{m,N}\|_{\ell_p \text{weak}(\mathbb{R})} \leq C(2N + 1)^{1/p}$$

for every $N \in \{0,1,2,3,\ldots\}$.

3. Results and Discussion

3.1. Inclusion properties of $\ell_p^W(\mathbb{R})$ space and $\ell_p^W(\mathbb{R})$ space

In this section, we will present the sufficient condition for inclusion properties of $\ell_p^W(\mathbb{R})$ space and $\ell_p^W(\mathbb{R})$ space after providing the same discussion on the discrete Lebesgue space $\ell_p(\mathbb{R})$. First, we recall the sufficient and necessary conditions for inclusion properties of $\ell_p(\mathbb{R})$ spaces in the following.

**Theorem 4. [5]** Let $1 \leq p_1, p_2 < \infty$. Then, the following statements are equivalent:

1. $p_1 \leq p_2$.
2. $\ell_{p_1}(\mathbb{R}) \subseteq \ell_{p_2}(\mathbb{R})$.
3. There exists a constant $C > 0$ such that $\|X\|_{\ell_{p_2}(\mathbb{R})} \leq C\|X\|_{\ell_{p_1}(\mathbb{R})}$ for every $X \in \ell_{p_1}(\mathbb{R})$.

Proof.

The fact (1) implies (2) is proven in [3, 5]. We rewrite the proof here for the reader. Let $X = (x_n)$ be an element of $\ell_{p_1}$ and $p_1 \leq p_2$. Since $\sum_{n=1}^{\infty} |x_n|^{p_1} < \infty$, then there exists $k \in \mathbb{N}$ such that for every $n \geq k$, we have $|x_n| < 1$. We know that $p_2 - p_1 \geq 0$, hence $|x_n|^{p_2-p_1} \leq 1$, for every $n \geq k$.

Observe that,

$$\sum_{n=1}^{\infty} |x_n|^{p_2} = \sum_{n=1}^{\infty} |x_n|^{p_1}|x_n|^{p_2-p_1} \leq M \sum_{n=1}^{\infty} |x_n|^{p_1} < \infty,$$

for $M := \max\{|x_1|^{p_2-p_1}, |x_2|^{p_2-p_1}, \ldots, |x_k|^{p_2-p_1}, 1\}$. This shows that $X \in \ell_{p_2}(\mathbb{R})$. Hence, we can conclude that $\ell_{p_1}(\mathbb{R}) \subseteq \ell_{p_2}(\mathbb{R})$.

Next, since $(\ell_{p_1}(\mathbb{R}), \ell_{p_2}(\mathbb{R}))$ is a Banach pair, it follows from [16, Lemma 3.3] that (2) and (3) are equivalent.

Now, assume that (3) holds. By using Lemma 1, we have

$$(2N + 1)^{1/p_2} \leq \|s_{k}^{m,N}\|_{\ell_{p_2}(\mathbb{R})} \leq \|s_{k}^{m,N}\|_{\ell_{p_1}(\mathbb{R})} \leq C(2N + 1)^{1/p_1}$$

or $(2N + 1)^{1/p_2} \leq C$ for every $N \in \{0,1,2,3,\ldots\}$. Hence, we can conclude that $p_1 \leq p_2$. \(\blacksquare\)

Remark. If $p_1 < p_2$, then we have $\ell_{p_1}(\mathbb{R})$ is proper subset $\ell_{p_2}(\mathbb{R})$. For the example, we give in the following.
Example 5. Define a sequence \( X = (x_n) = n^{-1} \), for every \( n \in \mathbb{N} \). Since \( p_1 < p_2 \), we have \( \sum_{n=1}^{\infty} |x_n|^p_1 = \sum_{n=1}^{\infty} \frac{1}{n^{pt_1}} < \infty \). So, we obtain \( X \in \ell_{p_2}(\mathbb{R}) \). On the other hand, \( \sum_{n=1}^{\infty} |x_n|^p_1 = \sum_{n=1}^{\infty} \frac{1}{n^{pt_1}} \) and we know that \( \sum_{n=1}^{\infty} \frac{1}{n} = \infty \), so \( X \notin \ell_{p_1}(\mathbb{R}) \).

Next, we will give the sufficient condition for inclusion properties of discrete weighted Lebesgue space \( \ell^W_p(\mathbb{R}) \) and discrete weighted weak Lebesgue space \( \ell^W_{p_0}(\mathbb{R}) \). Before that, we will give some lemmas first, as follows.

Lemma 6. Let \( 1 \leq p < \infty \) and \( W = (w_n) \) be a positive sequence, then we have \( \ell^W_p(\mathbb{R}) \subseteq \ell^W_{p_0}(\mathbb{R}) \).

Proof.

Take an arbitrary sequence \( X = (x_n) \in \ell^W_p(\mathbb{R}) \) we have \( \sum_{n=1}^{\infty} |x_n w_n|^p < \infty \). So, there exists \( k \in \mathbb{N} \) such that for every \( n \geq k \), we obtain \( |x_n w_n| < 1 \). Choose \( M = \max\{ |x_1 w_1|, |x_2 w_2|, \ldots, |x_{k-1} w_{k-1}|, 1 \} \). Thus, we have \( |x_n|^p \leq M \) for every \( n \in \mathbb{N} \). Hence, we can conclude that \( X = (x_n) \in \ell^W_{p_0}(\mathbb{R}) \) and \( \ell^W_p(\mathbb{R}) \subseteq \ell^W_{p_0}(\mathbb{R}) \).

3.1.1. Inclusion properties of \( \ell^W_p(\mathbb{R}) \) space. Now we come to the inclusion properties of \( \ell^W_p(\mathbb{R}) \) space. We present the sufficient condition for \( \ell^W_p(\mathbb{R}) \) space in the following theorem.

Theorem 7. Let \( 1 \leq p_1, p_2 < \infty \) and \( W = (w_n) \), \( U = (u_n) \) be positive sequences on real numbers. If we have the conditions:

1. \( p_1 \leq p_2 \) and \( u_n \leq w_n \) for every \( n \in \mathbb{N} \).
2. \( \ell^W_{p_1}(\mathbb{R}) \subseteq \ell^U_{p_2}(\mathbb{R}) \).
3. There exists a constant \( C > 0 \) such that \( \|X\|_\ell^W_{p_1}(\mathbb{R}) \leq C \|X\|_\ell^U_{p_2}(\mathbb{R}) \) for every \( X \in \ell^W_{p_1}(\mathbb{R}) \).

Then we have (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3).

Proof.

Suppose that (1) hold. If \( X \in \ell^W_{p_1}(\mathbb{R}) \), then \( \sum_{n=1}^{\infty} |x_n w_n|^p_1 < \infty \). So, there exists \( k \in \mathbb{N} \) such that for every \( n \geq k \), we have \( |x_n w_n| < 1 \). Knowing that \( p_2 - p_1 > 0 \), hence we obtain \( |x_n w_n|^{p_2-p_1} < 1 \), for every \( n \geq k \). Furthermore, we also obtain

\[
\sum_{n=1}^{\infty} |u_n x_n|^{p_2} \leq \sum_{n=1}^{\infty} |w_n x_n|^{p_2} = \sum_{n=1}^{\infty} |w_n x_n|^{p_1} |w_n x_n|^{p_2-p_1} < M \sum_{n=1}^{\infty} |w_n x_n|^{p_1} < \infty,
\]

for \( M = \max\{ |x_1 w_1|^{p_2-p_1}, |x_2 w_2|^{p_2-p_1}, \ldots, |x_k w_k|^{p_2-p_1}, 1 \} \). This shows that \( X \in \ell^U_{p_2}(\mathbb{R}) \), hence we conclude that \( \ell^W_{p_1}(\mathbb{R}) \subseteq \ell^U_{p_2}(\mathbb{R}) \). Next, since \( (\ell^W_{p_1}(\mathbb{R}), \ell^U_{p_2}(\mathbb{R})) \)

\( \ell^W_{p_2}(\mathbb{R}) \) is Banach pair, by using similar argument in the proof of Theorem 1, we have (2) and (3) are equivalent.

Theorem 8. Let \( 1 \leq p_1, p_2 < \infty \) and \( W = (w_n) \) be a positive sequence on real numbers. Then the following statements are equivalent:

1. \( p_1 \leq p_2 \).
2. \( \ell^W_{p_1}(\mathbb{R}) \subseteq \ell^W_{p_2}(\mathbb{R}) \).
3. There exists a constant \( C > 0 \) such that \( \|X\|_{\ell^W_{p_2}(\mathbb{R})} \leq C \|X\|_{\ell^W_{p_1}(\mathbb{R})} \) for every \( X \in \ell^W_{p_1}(\mathbb{R}) \).
Proof.
The fact (1) $\Rightarrow$ (2) $\iff$ (3) is proven in Theorem 5. Now, assume that (3) holds. By using Lemma 2, we have

$$(2N + 1)^{1/p_2} \leq \|x_{k/n}^{m,n}\|_{\ell^W_{p_2}(\mathbb{R})} \leq \|x_{k/n}^{m,n}\|_{\ell^W_{p_1}(\mathbb{R})} \leq C(2N + 1)^{1/p_1}$$

or $(2N + 1)^{1/p_2} \leq C$ for every $N \in \{0,1,2,3,\ldots\}$. Hence, we can conclude that $p_1 \leq p_2$. 

3.1.2. Inclusion properties of $\ell^W_{\infty}(\mathbb{R})$ space. We come to the discussion of the sufficient and necessary conditions for inclusion properties of $\ell^W_{\infty}(\mathbb{R})$ space presented in the following theorem.

**Theorem 9.** Let $W = (w_n)$, $U = (u_n)$ be positive sequences on real numbers. Then, the following statements are equivalent:

1. $u_n \leq w_n$ for every $n \in \mathbb{N}$.
2. $\ell^W_{\infty}(\mathbb{R}) \subseteq \ell^U_{\infty}(\mathbb{R})$.
3. There exists a constant $C > 0$ such that $\|X\|_{\ell^U_{\infty}(\mathbb{R})} \leq C \|X\|_{\ell^W_{\infty}(\mathbb{R})}$ for every $X \in \ell^W_{\infty}(\mathbb{R})$.

**Proof.**

Suppose that (1) holds. Let $X = (x_n)$ be an element of $\ell^W_{\infty}(\mathbb{R})$, then there exists $M > 0$ such that $|x_n w_n| \leq M$ for every $n \in \mathbb{N}$. Since $u_n \leq w_n$ for every $n \in \mathbb{N}$, we have $|x_n u_n| \leq |x_n w_n| \leq M$. So, we obtain $X = (x_n) \in \ell^U_{\infty}(\mathbb{R})$ and $\ell^W_{\infty}(\mathbb{R}) \subseteq \ell^U_{\infty}(\mathbb{R})$. As before, we can prove (2) and (3) are equivalent. Lastly, we will prove (3) implies (1). Take $U = \left(\frac{1}{w_n}\right) \in \ell^W_{\infty}(\mathbb{R})$, observe that

$$\max_{n \in \mathbb{N}} \left|\left(\frac{1}{w_n}\right) u_n\right| = \max_{n \in \mathbb{N}} \left|\left(\frac{1}{w_n}\right) w_n\right| = 1.$$

This shows that $\frac{u_n}{w_n} \leq 1$ for every $n \in \mathbb{N}$. So we obtain $u_n \leq w_n$ for every $n \in \mathbb{N}$. 

**Example 10.** If $W = (w_n) = (n)$ and $U = (u_n) = \left(\frac{1}{n+1}\right)$, then we have $\ell^W_{\infty}(\mathbb{R})$ proper subset $\ell^U_{\infty}(\mathbb{R})$.

By using Theorem 8 we have $\ell^U_{\infty}(\mathbb{R}) \subseteq \ell^U_{\infty}(\mathbb{R})$. Define $X = (x_n) = (n)$ for every $n \in \mathbb{N}$, we have $|x_n u_n| \leq 1$. So, we obtain $X \in \ell^U_{\infty}(\mathbb{R})$. On the other hand, $|x_n w_n| = n^2$ and we know that $(n^2)$ is divergent, so $X \notin \ell^W_{\infty}$.

We have been discussing the inclusion properties of $\ell^W_{p}(\mathbb{R})$ space and $\ell^W_{\infty}(\mathbb{R})$ space. Now, we will discuss the inclusion properties of weighted weak discrete Lebesgue space $\ell^W_{p,\text{weak}}(\mathbb{R})$ in the following section.

3.2. Inclusion properties of $\ell^W_{p,\text{weak}}(\mathbb{R})$ space

In this section we will discuss the sufficient and necessary conditions for the inclusion properties of $\ell^W_{p,\text{weak}}(\mathbb{R})$ space. We present the results in theorems 10 and 11 which provide the information for different weights and a fixed weight.

**Theorem 11.** Let $1 \leq p_1, p_2 < \infty$, $W = (w_n)$ and $U = (u_n)$ be positive sequences on real numbers. If we have the conditions:

1. $p_1 \leq p_2$ and $u_n \leq w_n$ for every $n \in \mathbb{N}$.
2. $\ell^W_{p_1,\text{weak}}(\mathbb{R}) \subseteq \ell^U_{p_2,\text{weak}}(\mathbb{R})$.


(3) \( \|X\|_{\ell_{p_2,weak}(\mathbb{R})} \leq \|X\|_{\ell_{p_1,weak}(\mathbb{R})} \) for every \( X \in \ell_{p_1,weak}(\mathbb{R}) \).

Then we have (1) \( \implies \) (2) \( \iff \) (3).

Proof.

Suppose that (1) holds. Let \( X \) be an element of \( \ell_{p_1,weak}(\mathbb{R}) \), then there exists \( M > 0 \) such that
\[
g\{n \in \mathbb{N} : |x_n w_n| > \gamma\}^{\frac{1}{p_1}} \leq M \quad \text{for every } \gamma > 0.
\]

Since \( u_n \leq w_n \) for every \( n \in \mathbb{N} \) and \( p_1 \leq p_2 \), we have
\[
g\{n \in \mathbb{N} : |x_n u_n| > \gamma\}^{\frac{1}{p_2}} \leq \|X\|_{\ell_{p_2,weak}(\mathbb{R})} \leq \|X\|_{\ell_{p_1,weak}(\mathbb{R})} \leq \|X\|_{\ell_{p_1,weak}(\mathbb{R})} \leq M
\]

For every \( \gamma > 0 \). This shows that \( X \in \ell_{p_2,weak}(\mathbb{R}) \). Hence we conclude that
\[
\ell_{p_1,weak}(\mathbb{R}) \subseteq \ell_{p_2,weak}(\mathbb{R}).
\]

Next, by using similar argument in the proof of Theorem 5.4 in [5] we have (2) and (3) are equivalent. \( \blacksquare \)

**Theorem 12.** Let \( 1 \leq p_1, p_2 < \infty \) and \( W = (w_n) \) be a positive sequence on real numbers. Then the following statements are equivalent:

(1) \( p_1 \leq p_2 \),
(2) \( \ell_{p_1,weak}(\mathbb{R}) \subseteq \ell_{p_2,weak}(\mathbb{R}) \),
(3) \( \|X\|_{\ell_{p_2,weak}(\mathbb{R})} \leq \|X\|_{\ell_{p_1,weak}(\mathbb{R})} \) for every \( X \in \ell_{p_1,weak}(\mathbb{R}) \).

Proof.

The fact (1) \( \implies \) (2) \( \iff \) (3) is proven in Theorem 10. To prove (3) implies (1), we have similar statement with Lemma 7 in the following lemma.

Now, assume that (3) holds. By using Lemma 3, we have
\[
\|X\|_{\ell_{p_2,weak}(\mathbb{R})} \leq \|X\|_{\ell_{p_1,weak}(\mathbb{R})} \leq C(2N + 1)^{1/p_1}
\]
or \( (2N + 1)^{1/p_2} \leq C \) for every \( N \in \{0, 1, 2, 3, \ldots \} \). Hence, we can conclude that \( p_1 \leq p_2 \). \( \blacksquare \)

**Example 13.** If \( W = (w_n) = (n) \) and \( U = (u_n) = \left( \frac{1}{n^{p_1}} \right) \) for every \( n \in \mathbb{N} \), then \( \ell_{p_1,weak}(\mathbb{R}) \) proper subset \( \ell_{p_2,weak}(\mathbb{R}) \).

By Theorem 11 we have \( \ell_{p_1,weak}(\mathbb{R}) \subseteq \ell_{p_2,weak}(\mathbb{R}) \). Define a sequence \( X = (x_n) = (n) \), observe that \( |x_n u_n|^{p_2} \leq 1 \). So, we obtain \( X \in \ell_{p_2,weak}(\mathbb{R}) \). On the other hand, \( |x_n u_n|^{p_1} = n^{2p_1} \) and \( p_1 \geq 1 \). Since \( n^{2p_1} \) is divergent, we can conclude that \( X \notin \ell_{p_1,weak}(\mathbb{R}) \).

4. Conclusion

4. Conclusion

We have shown the inclusion properties of discrete weighted Lebesgue space \( \ell_{p}^{W}(\mathbb{R}) \) and discrete weighted weak Lebesgue space \( \ell_{\infty}^{W}(\mathbb{R}) \). As our final conclusion, we can state that the comparison of parameters \( p \) and comparison of parameters \( W \) are sufficient conditions for inclusion properties of discrete weighted Lebesgue spaces \( \ell_{p}^{W}(\mathbb{R}) \) and of discrete weighted weak Lebesgue space \( \ell_{\infty}^{W}(\mathbb{R}) \).

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