Perfect and Maximum Randomness in Stratified Sampling over Joins

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ABSTRACT
Supporting sampling in the presence of joins is an important problem in data analysis. Pushing down the sampling operator through both sides of the join is inherently challenging due to data skew and correlation issues between output tuples. Joining simple random samples of base relations typically leads to results that are non-random. Current solutions to this problem perform biased sampling of one (and not both) of the base relations to obtain a simple random sample. These techniques are not always practical since they may result in the sample size being greater than the size of the relations due to sample inflation, rendering sampling counter-productive.

This paper presents a unified strategy towards sampling over joins, comprising two key contributions. First, in the case that perfect sampling is a requirement, we introduce techniques to generate a perfect random sample from both sides of a join. We show that the challenges faced in sampling over joins are ameliorated in the context of stratified random sampling as opposed to simple random sampling. Our technique minimizes the sample size while maintaining perfect randomness.

Second, in the case that random sampling is not a requirement but is still preferred, we provide a novel sampling heuristic to maximize randomness of the join. It allows us to allocate a fixed sample size between multiple relations consisting of multiple strata to maximize the join randomness. We validate our techniques theoretically and empirically using synthetic datasets and a standard benchmark.

1. INTRODUCTION
Data sampling has greatly increased in popularity, as evidenced by inclusion of the TABLESAMPLE keyword into the SQL standard, with implementations in most mainstream databases. In cases where the growth of data volume outpaces the hardware or software’s ability to process it in time, sampling presents a pragmatic approach towards providing insights at scale. Incorporating sampling into analytics not only speeds up the analysis cycle but also potentially reduces the resource requirements by orders of magnitude.

Due to the non-exact nature of results, the quality and method of sampling is a key consideration. A random probability sample as compared to a non-probability sample is usually necessary in order to obtain meaningful results. While the use of non-random samples has been considered in aggregation queries [15], the use of random samples allows the user to utilize statistical methods over the data in a principled manner. The nature of the sample also impacts visual representations – a visualization of a sample may look very different from the original if it is not sampled correctly.

Techniques such as simple random sampling and stratified random sampling can be used sample the data. Stratified random sampling is usually preferred since it provides a better representation of the data, yielding higher efficiency (lower variance) of the measure [7]. Stratified sampling is especially suited for joins, since the data can be considered to be stratified by the groups in the join column.

In the context of JOIN queries, sampling is a far more compelling, yet challenging task. It is generally preferred for the sampling techniques to preserve the randomness of the join output. Performing a join can be expensive – in a join between relations $R_1$ and $R_2$, the minimum time complexity of using a simple hash join would be $O(|R_1| + |R_2|)$, and
$O(|R_1| \times |R_2|)$ for a simple nested loop join. Therefore, performing sampling after the join result is materialized, may not be a pragmatic approach.

Another alternative approach is to produce the sampled join result by sampling the base relations themselves. Consider the potential rewrite:

\[
\text{TABLESAMPLE}(A \bowtie B) \quad \text{to} \quad \text{TABLESAMPLE}(A) \bowtie \text{TABLESAMPLE}(B)
\]

where $\bowtie$ denotes a new join algorithm. If possible, this approach would have significant benefits – in the case of visual data exploration tools over large dataset, a visualization of the join result can be approximated from samples available in the local cache. In the case of distributed join execution, the sampling of the base relations can be performed at the \textit{Map} step, allowing for the entire join to be performed as a single \textit{MapReduce} job.

However, obtaining random samples of a join by sampling base relations (also termed \textit{join sampling}) is an inherently difficult problem from a correctness perspective \cite{Gibbons2012, Chaudhari2016}. Common solutions include sampling from only one side of the join, or presenting a non-representative sample to the user. In order to highlight the challenges involved, we now walk through an example to motivate our line of inquiry.

\subsection{Motivating Example}

Excellent examples have been provided by Gibbons et al. \cite{Gibbons2012} and Chaudhari et al. \cite{Chaudhari2016} that demonstrate the inherent difficulty of join sampling. We briefly revisit the example given in the latter, since it motivates our algorithms.

Consider joining two relations $R_1$ and $R_2$ with the values in the join column being $\{1, 2, \ldots, 2\}$ for $R_1$ and $\{2, 1, \ldots, 1\}$ for $R_2$, such that there are 99 tuples that have the value 2 in $R_1$ and have the value 1 in $R_2$. In a simple random sample of the join between $R_1$ and $R_2$, on average, half the tuples would be joined on 1 and the other half on 2. However, it is clear that if both relations are sampled randomly in an unweighted fashion, their resultant join would more likely be an empty set. This motivates us towards using weighted sampling and guides us towards the reasons behind simple random sampling of joins being difficult.

\subsection{Why Stratified Random Sampling?}

Using the motivating example, we enumerate the difficulties involved in obtaining \textit{simple} random samples of joins. We then show how they can be alleviated using \textit{stratified} random sampling.

\textbf{Sample Inflation:} The key reason behind infeasibility of using simple random sampling in joins is sample inflation. In order to avoid correlation in output tuples, a sampled tuple can be joined only once with another tuple. Therefore, it is possible for the size of the samples to exceed size of the input relations, resulting in sampling being counter-productive.

In stratified sampling, the data is divided into strata and simple random sampling is then applied within each stratum. Thus, the different strata can be sampled independently of one another. If the sample size needed for a stratum exceeds the stratum size, we can choose to not sample it. The entire base stratum can be added to the sample. While performing the join, tuples can be chosen from the stratum randomly. In this way, we are able to handle sample inflation and minimize the sample size. A detailed explanation is provided in Section 4.4.

\textbf{Smaller Output Size & Sample Tuple Loss:} In sparse or skewed data, using simple random sampling, it is possible for the join of the samples to be smaller than intended. If both the relations involved in the join are sampled without a smart sampling strategy, there may be an absence of joinable tuples on both sides, yielding a smaller output size than expected. This also renders some of the sampled tuples non-productive.

This challenge can be handled by using stratified random sampling by choosing equal number of tuples from both relations for each stratum. This makes sure that all sample tuples are utilized and the sample size is as expected.

\textbf{Non-representative Output:} Using simple random sampling, while samples will be random, and representative in the average case, for any non-trivial sampling fraction (lesser than 1), there is a non-zero likelihood of some groups being left out of the join. This issue is resolved using stratified random sampling since join will contain tuples from all strata.

\section{Overview & Contributions}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{overview.png}
\caption{Overview}
\end{figure}

Figure 2 presents an overview of our \textit{perfect} and \textit{maximum} randomness solutions to the join sampling problem. Sampling in the presence of primary key and foreign key relationship has been investigated by the \textit{Aqua} system, $\text{STRATJOIN\_IN}$ (Section 4.1) presents a stratified version of their join algorithm. If the join has a low level of sample inflation (Section 4.4), we can use $\text{STRATJOIN\_OVERALL}$ (Section 5) to obtain a \textit{perfectly} random output. If the reduction in sample size is low due to sampling inflation, we provide techniques to \textit{maximize} the sample randomness and use a correlated output sample (Section 6). The decision to use a perfectly random or maximally random sample is ultimately user-dependent.

We now describe the two main contributions of this paper in greater detail. First, we provide means to obtain stratified random samples of equi-joins using samples of either one or both relations. The use of stratified random sampling alleviates some of the issues involved in simple random sampling. We introduce a generalized algorithm, $\text{STRATJOIN\_OVERALL}$, that gives us control over deciding the ideal sampling as well as join strategies across different strata independently of each other.

Our second contribution is in the case where random samples are not mandated but preferred nonetheless. Here, a novel algorithm is presented that maximizes the \textit{randomness} of a sample given the constraint of fixed sample size. Towards this end, it provides a strategy for allocating samples to the different strata of the different relations involved in the join. These results have been proven theoretically as well as validated experimentally using well defined metrics.
2. RELATED WORK

Join sampling has been incorporated in systems such as Microsoft SQL Server, IBM DB2 and more recently BlinkDB. While these approaches involve techniques implemented at various layers of the database such as table scans, offline catalogs, and aggregation, there still exist several open opportunities. In order to obtain random samples of join, the typical approach is to sample a single relation [2,3,6,24] and use the entire second relation in the join.

Work by Olken et al. shows that it is easy to commute selection with sampling but projections and joins are harder [24]. They sample a single relation.

Chaudhari et al. provide better algorithms for joins using sample of a single relation [6] and consider different availabilities of statistics and indexes. They describe reasons behind the inherent difficulty of obtaining a simple random sample of join while extending sampling towards multiple relations.

The Aqua system [1] provides techniques to obtain a simple random sample of the join involving relations having primary key and foreign key relationships. This is an easier case to handle since sampled tuples do not result in correlation between the output tuples.

Research in obtaining random samples of joins has stalled since inherent problems involved in simple random sampling of joins were presented [6]. However, a lot of effort has been devoted towards finding meaningful aggregate estimates in presence of correlation. Online aggregation and ripple joins provided join evaluation techniques to obtain meaningful error estimation during query execution [11–15,22].

In the SciBORQ system, Sidirourgos et al. [25] use join sampling to maintain correlation between the join attributes in order to provide the query results with greater precision. It plays an important role in their design of impressions, which consist of multi-level, workload-adaptive query-focused samples.

Jermaine et al. remove the dependence of the ripple join algorithms on the data residing in memory in order to provide statistical guarantees for the estimates [16,17]. The DBO [18] and Turbo-DBO [9] systems elevated the approach towards processing multiple relations in a scalable fashion. Nirkhiwale et al. present the sampling algebra inherent in these techniques [23].

3. PRELIMINARIES

In this section, we introduce preliminaries and the terminology used in this paper. We enumerate the criteria that need to be satisfied for samples to be termed as simple random sample [7].

3.1 Random Sampling Criteria

We can show that a sample is a simple random sample or a stratified random sample by showing that they satisfy the criteria given below respectively.

3.1.1 Simple Random Sampling Criteria

We briefly present the two criteria that need to be satisfied for a sample to be termed as a simple random sample [7].

**SRS_1**: All tuples in the population should have the same probability of occurrence.

**SRS_2**: All samples of a given size should be equally likely, i.e., a tuple being present in the sample should not affect the probability of another.

3.1.2 Stratified Random Sampling Criteria

In stratified random sampling, the population is first divided into strata such that the strata are mutually exclusive and the union of the strata covers the entire population. Samples are then chosen from each stratum using simple random sampling. The criteria can be given as follows.

**SRS_1**: Every sample stratum should have the intended size depending on the allocation type (proportional, optimal, cost-based, etc.).

**SRS_2**: All tuples belonging to a stratum should have the same probability of occurrence.

**SRS_3**: Within a stratum, a tuple being present in the sample should not affect the probability of another.

These criteria clearly show the ease of obtaining stratified random samples over simple random samples since randomness only needs to be achieved at the stratum level.

3.2 The Mini-Join Operator

If all tuples within a sample stratum are joined with all tuples in the corresponding stratum of the other, the output tuples will be correlated. The Mini-Join operator, denoted by $\Join$, is designed to avoid this correlation. Instead of producing all $|m_1(a) \times m_2(a)|$ tuples of the cartesian product for a stratum $a$, random tuples from both samples are selected and joined. A sample tuple is used only in a single output tuple. Each stratum sample will be a sample without replacement of the non-sampled join between the strata. The complexity of this operator is $\sum_{Strata} \min(m_1(a), m_2(a))$, where Strata is the set of all strata in $S_1 \Join S_2$. This operator is influenced by similar actions recurring in [6].

**Algorithm:**

For every stratum $a \in \text{Strata}$:

Repeat $\min(m_1(a), m_2(a))$ times:

Choose hitherto non-selected random tuples from $m_1(a)$ and $m_2(a)$, join them, and output the resulting tuple.
Example: Consider two relations given below that are specified as
\[(stratum\_value : stratum\_count)\]:
\[R_1 = (a : 2, b : 5) \text{ and } R_2 = (a : 3, b : 3)\].
The output counts after applying the \(Mini-Join\) operator will be \(R_1 \bowtie R_2 = (a : 2, b : 3)\). Thus, projecting the joined tuples on any relation will result in distinct tuples. For example, for the first stratum, for each tuple from \(R_1\) we will select a different tuple from \(R_2\).

3.3 Simple Random Sampling Over Joins

In this section, we review the existing techniques used to obtain simple random samples over joins by sampling a single relation – \textit{Simplified Aqua} and \textit{Stream-Sample}. We provide a simple extension to \textit{Stream-Sample} that enables sampling both sides of the join to obtain a simple random sample over join – \textit{Srs\_Both}. Finally, we provide the distribution of strata counts in a simple random sample over joins to better understand sampling.

3.3.1 \textit{Simplified AQUA Algorithm}

We briefly describe a simplified version of the basic join sampling algorithm used in the AQUA system [1] as we provide a stratified version of it in Section 4.1.

Algorithm:
1. Perform simple random sampling over the foreign key table.
2. Join the sample with the primary key table.

Theorem 1: \textit{Simplified Aqua} results in a without-replacement simple random sample of \(R_1 \bowtie R_2\), while sampling a \textit{single} relation.

3.3.2 \textit{Strategy Stream-Sample}

We briefly describe the \textit{Strategy Stream-Sample} algorithm given by Chaudhari et al. [6], since our algorithms are based on the same problem formulation. The \textit{Stream-Sample} algorithm is designed for obtaining a simple random sample of the equi-join between \(R_1\) and \(R_2\) by sampling a single relation for the general case of multiple tuples from a relation joining with multiple tuples from another.

Algorithm:
1. Obtain a with-replacement sample \(S_1\) of \(R_1\), where the sampling weight \(w(t_1)\) for a tuple \(t_1\) is set to \(m_2(t_1.A)\).
2. For each tuple from \(S_1\) in a streaming fashion do:
   (a) Sample a random tuple \(t_2\) from amongst all tuples \(t \in R_2\) that satisfy \(t.A = t_1.A\).
   (b) Add \(t_2\) to the sample \(S_2\), where the insertion order of tuples is preserved.
3. For each tuple from \(S_2\) in a streaming fashion do:
   (a) Sample a random tuple \(t_3\) from amongst all tuples \(t \in R_3\) that satisfy \(t.A = t_2.A\).
   (b) Add \(t_3\) to the sample \(S_3\), where the insertion order of tuples is preserved.

Theorem 2: \textit{Stream-Sample} gives a with-replacement sample of \(R_1 \bowtie R_2\), while sampling a \textit{single} relation.

Discussion: The proof of correctness is provided in Section 10.1. Since the sample size is \(f \times |R_1 \bowtie R_2|\), the join complexity of \(R_1 \bowtie R_2\). \textit{Group-Sample} and \textit{Frequency-Partition-Sample} deal with the availability of indexes and statistics [6]. In this paper, we have considered the \textit{Stream-Sample} algorithm because it is the simplest amongst them. It helps illustrate the basic ideas behind join sampling clearly. Further, the sample size inflation issue which hurts \textit{Stream-Sample} affects the other two algorithms as well.

3.3.3 \textit{Srs\_Both}

We provide a method to sample both sides of a join in order to obtain a simple random sample of the join by making a \textit{minor} modification to \textit{Stream-Sample}. If the relations are sampled independently, the number of output tuples, using the \textit{Mini-Join} operator, will be \(\sum_{a \in \text{Stratum}} \min(m_1(a), m_2(a))\). Therefore, in order to avoid loss of sampled tuples, after sampling a relation, the cardinality information of each sampled stratum should be used to sample the other relation. It is easy to notice that \textit{Stream-Sample} can be modified so that instead of selecting and joining tuples from \(R_2\) while tuples in \(S_1\) are streaming by, the selected tuples are added to \(S_2\). Once the entire \(S_2\) is materialized, tuples that would have joined in \textit{Stream-Sample} are joined now instead.

Algorithm:
1. Obtain a with-replacement sample \(S_1\) of \(R_1\), where the sampling weight \(w(t_1)\) for a tuple \(t_1\) is set to \(m_2(t_1.A)\).
2. For each tuple from \(S_1\) in a streaming fashion do:
   (a) Sample a random tuple \(t_2\) from amongst all tuples \(t \in R_2\) that satisfy \(t.A = t_1.A\).
   (b) Add \(t_2\) to the sample \(S_2\) while preserving the insertion order of tuples.
3. For each tuple from \(S_1\) in a streaming fashion do:
   (a) Choose the next tuple \(t_3\) from \(S_2\), and output \(t_1 \bowtie t_2\).

Theorem 3: \textit{Srs\_Both} results in a with-replacement simple random sample of \(R_1 \bowtie R_2\), sampling \textit{both} relations.

Proof of Correctness:
\textit{Srs\_Both} can be considered as a \textit{minor} modification of \textit{Stream-Sample}, where instead of choosing tuples from \(R_2\) as tuples from \(S_1\) are streaming by, the chosen tuples from \(R_2\) are used to materialize \(S_2\) and joined later. Both algorithms result in the same join output. A detailed proof of correctness for \textit{Stream-Sample} in Appendix 10.1.

3.3.4 \textit{Strata Counts in SRS over Joins}

The strata counts follow a multinomial distribution with \(p_i = \frac{m_i}{|R_1 \bowtie R_2|}\), where \(p_i\) is the probability of any chosen sample tuple belonging to stratum \(i\). Thus, the number of tuples in the output belonging to a stratum might vary in successive trials. It is possible for smaller strata to not have any tuples in the output. Stratified random sampling based algorithms resolve these issues.

4. STRATIFIED RANDOM SAMPLING OVER JOINS

The join sampling problem for obtaining perfectly random samples can be formally defined as follows:

\textit{Given relations to be joined, obtain a stratified random sample of the overall join by sampling one or both relations.}

First, we provide techniques for sampling a single side of the join (Sections 4.1 and 4.2). Then, we extend our methods towards sampling both the sides of the join (Section 4.3). We note that in order to avoid correlated samples in the output, sampling in the general case of non-primary key based join can become expensive (Section 4.4). We provide means to alleviate this problem as well. The combination of these methods forms our overall algorithm (Section 5).

It should be noted that our contribution is towards the overall algorithm and its correctness; efficient implementations, performance optimizations, and index constraints are
stratified

Theorem 4
join complexity will equal
Since each tuple in
Satisfaction of
Algorithm:
1. From each stratum in base relation \( R_1 \), sample \( f \) fraction of the
tuples without replacement giving \( S_1 \).
2. Join \( S_1 \) with \( R_2 \).

Complexity:
Sampling complexity is equal to the size of \( S_1 \), \( n_1 = f \times |R_1| \).
Since each tuple in \( S_1 \) will join with a single tuple from \( R_2 \),
join complexity will equal \( n_1 \).

Theorem 4 \textsc{StratJoin\_1n} results in a without replacement
\textit{stratified} random sample of \( R_1 \bowtie R_2 \), while sampling a
\textit{single} relation.

Proof of Correctness:
\textbf{Satisfaction of \textit{STRS}1:} A stratum \( a \) has the correct number of tuples, \( f \times m_1(a) \), since in the absence of sampling, it
would have \( m_1(a) \) tuples.

\textbf{Satisfaction of \textit{STRS}2 \& \textit{STRS}3:} Since each stratum in \( S_1 \)
is a sample without replacement and will join with a single tuple from \( R_2 \),
each output stratum will be a sample without replacement and will therefore satisfy \( \text{STRS}_2 \) \& \( \text{STRS}_3 \).

Discussion:
Note that \( R_2 \) is usually smaller (dimension table) than \( R_1 \)
(fact table). Further, sampling from \( R_2 \) wastes tuples that are sampled from both relations which do not join. The
therefore, we can see that sampling from both relations will be counter-productive. \textsc{StratJoin\_1n} can be extended to
join multiple foreign key based tables by simply performing a normal join of the other tables with \( S_1 \bowtie R_2 \).

4.2 \textsc{StratJoin\_nn}

We now provide an algorithm for obtaining stratified random samples of joins by sampling a single relation, based on \textit{Stream-Sample}. This algorithm handles the general case where multiple tuples from a relation can join with multiple tuples from another. We will then build upon these concepts to design algorithms to sample both sides (\textsc{StratJoin\_both} and \textsc{StratJoin\_overall}).

Algorithm:
1. For a stratum \( a \), use sampling with replacement to sample \( n(a) = f \times m_1(a) \times m_2(a) \) tuples from \( m_1(a) \) tuples.
2. For each sample tuple, join with a randomly chosen tuple from amongst the \( m_2(a) \) tuples of \( R_2 \) that can join with it.
3. Output the joined tuple.

Complexity:
The sampling complexity is determined by the size of the sample \( S_1 \), \( O(f \times |R_1| \bowtie R_2|) \). Each tuple in \( S_1 \) is joined with a single tuple from \( R_2 \) chosen randomly and, thus, the join complexity will be defined by the number of join operations, \( O(f \times |R_1| \bowtie R_2|) \).

Theorem 5 \textsc{StratJoin\_nn} results in a with-replacement
\textit{stratified} random sample of \( R_1 \bowtie R_2 \), while sampling a
\textit{single} relation.

Proof of Correctness:
\textbf{Satisfaction of \textit{STRS}1:} A stratum \( a \) has the correct number of tuples, \( f \times m_1(a) \times m_2(a) \).

\textbf{Satisfaction of \textit{STRS}2 \& \textit{STRS}3:} Any stratum \( a \) in \( S_1 \) is a sample with replacement from the corresponding stratum in \( R_1 \). Thus, every tuple in a sample stratum is equally likely as every other tuple in the same sample stratum. Similarly, while selecting a tuple from \( R_2 \) to join with it, each tuple amongst the \( m_2(a) \) tuples is equally likely, since it is randomly chosen. Therefore, for every joined output tuple, every
tuple amongst the \( m_1(a) \times m_2(a) \) tuples is equally likely, satisfying \( \text{STRS}_2 \). Further, selecting a tuple in \( S_1 \) does not influence the selection of another tuple in the same sampled stratum and similarly, since tuples from \( R_2 \) are randomly chosen to join with it, selecting a tuple from \( R_2 \) does not affect selection of another, thus satisfying \( \text{STRS}_3 \). Thus, every
sampled stratum is a sample with replacement from the corresponding non-sampled stratum.

Discussion:
We note that the \textsc{StratJoin\_nn} algorithm is a stratified version of \textit{Stream-Sample} and suffers from its sample inflation issue. Techniques given in Section 5 can be used to improve \textsc{StratJoin\_nn} by handling sample inflation. Note that BlinkDB [3] supports stratified random sampling over a single relation and does not consider the both-sides approach discussed in this paper and does not handle sample inflation either.

4.3 \textsc{StratJoin\_both}
The central idea used in sampling a single relation presented in \textsc{StratJoin\_nn} extends easily for sampling both relations, by sampling \( n(a) = f \times m_1(a) \times m_2(a) \) tuples from both the relations for a stratum \( a \). This algorithm handles the general case of sampling both sides of a join.

Algorithm:
1. Generate samples \( S_1 \) and \( S_2 \) as follows:
(a) For each stratum \( a \) in \( R_1 \), select \( f \times m_1(a) \times m_2(a) \) tuples from amongst \( m_1(a) \) tuples, using sampling with replacement giving \( S_1 \).
(b) For each stratum \( a \) in \( R_2 \), select \( f \times m_1(a) \times m_2(a) \) tuples from amongst \( m_2(a) \) tuples, using sampling with replacement giving \( S_2 \).
2. Apply the \textit{Mini-Join} operator between \( S_1 \) and \( S_2 \).

Complexity:
The complexity of constructing the samples is equal to the sample size, \( O(f \times |R_1| \bowtie R_2|) \). Since the \textit{Mini-Join} operator is used to join the two samples, and the two samples have identical sizes in every stratum, the join complexity is equal to \( O(f \times |R_1| \bowtie R_2|) \).

Theorem 6 \textsc{StratJoin\_both} results in a stratified random sample with replacement of \( R_1 \bowtie R_2 \), while sampling \textit{both} the relations.

5
Proof of Correctness:

**Satisfaction of SIRS1:** A stratum \( a \) has the correct number of tuples, \( f \times m_1(a) \times m_2(a) \), since in the absence of sampling, it would have consisted of \( m_1(a) \times m_2(a) \) tuples.

**Satisfaction of SIRS2 & SIRS3:** Each sampled stratum is a sample with replacement of the corresponding relation stratum. Selecting tuples from each sample stratum to join using the Mini-Join operator does not introduce any correlation (Section 3.2). Thus, output for stratum \( a \) is a sample with replacement from amongst \( m_1(a) \times m_2(a) \) tuples.

### 4.4 Avoiding Sample Inflation

The algorithms presented so far result in stratified random samples of the joins. However, the non-primary key based algorithms still suffer from sample inflation due to avoidance of correlation amongst output tuples. The techniques used in this section and the next can be used to rectify this issue.

It is well-known that in order to obtain simple random samples over joins, the sampling rates \( f_1 \) and \( f_2 \) over relations \( R_1 \) and \( R_2 \) respectively need to be chosen such that \( f_1 \times f_2 \geq f \) [6]. In our algorithms StratJoin_Both and StratJoin_NN, since we work at the level of strata, the sampling fraction of a stratum \( a \) in \( R_1 \) can be given by \( f_1(a) = \frac{f \times m_1(a) \times m_2(a)}{m_1(a)} = f \times m_2(a) \). This constraint is more flexible since the sampling fraction is dependent only on the size of the corresponding stratum in the other relation.

Even so, the sampling fraction can still be greater than 1, which is counter-productive. Therefore, depending on the strata sizes and the overall sampling rate, we can switch between different algorithms (even choosing to turn off sampling), since as long as each output stratum is a simple random sample of the desired size, the overall output will be a stratified random sample.

This ability to choose to sample the different relations at the stratum level is important in order to minimize the sample size and possible only in stratified random sampling as opposed to simple random sampling. This guides us towards the StratJoin_Overall algorithm provided below.

### 5. STRATJOIN_OVERALL

**StratJoin_Overall** provides a stratified random sample of a join while sampling both relations and minimizes the sample size by handling sample inflation. Using the fact that the sampling rate of a stratum \( a \) for the relation \( R_1 \) can be given by \( f \times m_2(a) \), we present a new algorithm that takes the cardinalities of individual strata and the sampling rate into consideration to decide the best sampling and join strategies in order to minimize the sample size. This simple algorithm provides the best of both worlds – it reduces the sample size when sampling is beneficial and avoids sampling when it is counter-productive.

**Algorithm:**

1. For both relations, sample a stratum \( a \) with replacement if \( f \times m_{mother}(a) < 1 \). Otherwise, use the entire stratum in the sample.
2. For a stratum, depending on whether none of the relations, one of the relations, or both of the relations are sampled, join the samples as explained below.
   - (a) If neither relation is sampled, choose random tuples with replacement from both the strata and join them till the desired sampling rate is achieved.
   - (b) If a single relation is sampled, while tuples from the sampled stratum are streaming by, join them with a randomly chosen tuple from the non-sampled stratum. This builds upon the StratJoin_NN algorithm, which samples a single relation (Section 4.2).
   - (c) If both strata are sampled, use Mini-Join operator between the two samples. This builds upon the StratJoin_Both algorithm, which samples both relations (Section 4.3).

Figure 3 helps explain the algorithm using a walkthrough of a stratum participating in the join.

**Complexity:**

In the worst case, \( f \times m_1 \) and \( f \times m_2 \) will be greater than 1 for every stratum in both the relations and as a result, none of the strata will be sampled with the resultant sample size being \(|R_1| + |R_2|\). In the best case, where all the strata from both the relations can be sampled, the complexity will be \( O(f \times |R_1 \bowtie R_2|) \). The join complexity in each case will be \( f \times |R_1 \bowtie R_2| \).

**Theorem 7** StratJoin_Overall results in a with-replacement stratified random sample of \( R_1 \bowtie R_2 \), while sampling both relations and minimizes the sample size.

**Proof of Correctness:**

**Satisfaction of SIRS1:** The algorithm ensures that in each of the three scenarios, the output stratum is of correct size.

**Satisfaction of SIRS2 & SIRS3:** For case (a), since we are repeatedly choosing samples with replacement from both sides to join, it is clear that the output stratum will be a sample with replacement from the join of size \( m_1(a) \times m_2(a) \). For case (b), the proof of correctness at the level of stratum for StratJoin_NN will be applicable. Similarly, for case (c), the proof of correctness at the level of stratum for StratJoin_Both will be applicable. It is clear that using smaller size samples will induce correlation between output tuples.

### 5.1 Handling Sample Inflation in StratJoin_Overall

| \( i \) | \( m_1 \) | \( m_2 \) | No Sampling | StratJoin_Overall | SRS_Both | Stream-Sample |
|---|---|---|---|---|---|---|
| 1 1000 | 5 | 1005 | 500+5 | 500+5 | 500+5 |
| 2 1000 | 15 | 1015 | 1000+15 | =1015 | 1500+15 | 1500+15 |
| 3 5 | 1000 | 1005 | 5+500 | =505 | 500+500 | 500+500 |
| 4 15 | 1000 | 1015 | 15+1000 | =1015 | 1500+1500 | 1500+1500 |

Table 2: StratJoin_Overall tackling Sample Inflation, \( f = 0.1 \)
5.2 Reduction in Sample Size

We provide below the reduction in the number of tuples that need to participate in the join.

\[ \sum_{i=1}^{2} \sum_{j=1}^{\text{Strata Count}} \max(m_i - f \times m_j \times m_2, 0) \]

(1)

Whenever sampling is feasible, it can result in large savings if the size of the stratum being sampled is large. We can also see that when sampling is feasible, the fraction of tuples saved can be given by \(1 - f \times m_{\text{other}}\).

5.3 Comparison with Stream-Sample

Stream-Sample is more efficient compared with the then previously available technique of Strategy Olken-Sample and has fewer restrictions regarding availability of indexes [6]. One of the major goals of sampling is being able to process smaller sized data. However, as we have seen, due to sample inflation the sample sizes using Stream-Sample can get exceedingly large, thereby, negating the benefits of sampling. We will see later in Figure 9, that bigger sample size might result in having to read the samples from the disk – resulting in a substantial increase in execution time.

Therefore, knowing when to sample and when not to is critically important. Our heuristics result in the sample size never exceeding the size of the relations, thereby, making join sampling a feasible proposition. Further, we enable sampling from both sides of a join. We reduce constraints on sampling from the sizes of relations to those of strata. As Table 2 shows, sampling can be greatly beneficial when a stratum size is large and the corresponding \(f \times m_{\text{other}}\) is low (\(m_1\) and \(m_2\)).

5.4 Sample Size Optimality

As we have seen, with a sampling rate of \(f\) and sizes of \(m_1(a)\) and \(m_2(a)\) for the stratum \(a\), the output needs to have \(f \times m_1(a) \times m_2(a)\) tuples. However, if \(f \times m_1(a) \times m_2(a)\) is greater than \(m_1(a)\), it is clearly counter-productive to sample the stratum \(a\) of \(R_1\). Therefore, in order to demonstrate minimization of sample size, we need to show that it is not possible to have a sample of size smaller than \(f \times m_1(a) \times m_2(a)\) when \(f \times m_1(a) \times m_2(a) < m_1(a)\).

5.5 Usefulness of Non-sampling

Joining random tuples from relations till the required sampling rate is met can be a better alternative to both Stream-Sample and Naive-Sample as given by Chaudhari et al. [6]. It is easy to see that in Naive-Sample, the entire join is materialized and then sample is generated – allowing Stream-Sample to perform join faster. As we have seen in Sections 4.4 and 5.1, Stream-Sample can result in the sample size being multiple orders of magnitude larger than the relation size and thus not sampling can actually result in the sample size being comparatively smaller.

6. MAXIMIZING JOIN RANDOMNESS

Although obtaining a stratified random sample of join is more feasible than obtaining a simple random sample using our techniques, join sampling is inherently expensive due to sample inflation. In the cases where random samples are preferred but not required, we present sample allocation techniques to increase the randomness of the sample given the constraint of fixed sample size. In order to do so, we introduce a novel concept of join randomness. It not only helps quantify the randomness of the join but also provides basis for maximizing it. As mentioned before, non-random sampling in joins has been researched in the context of aggregation queries. This is the first work to look at the degree of randomness in non-random samples.

6.1 Uniformity Confidence

Uniformity Confidence (UC) has been used in order to quantify randomness of a sample [4] and can be given by

\[ UC = \frac{\text{The number of different samples of the same size possible statistically}}{\text{The number of different samples of the same size possible with the algorithm}} \times 100 \]

We use this metric for two reasons. First, we wish to provide techniques to increase the randomness of the sample. Second, we wish to present the value of the UC metric to the user so that he can increase the sample size if needed. In Sections 6.2 and 6.3, we consider strategies to increase the number of possible samples (numerator of UC), since the number of maximum samples possible statistically (denominator of UC) is independent of any allocation strategy. We first find the optimal allocation in the case of multiple relations joining on a single stratum and then extend it towards the case of multiple relations joining on multiple strata, in the context of sampling without replacement. Section 7.3 shows that our heuristics yield high quality approximations to the best possible allocation found using brute force. We prove the validity of our techniques theoretically in Section 10.2.

6.2 Maximizing Randomness for Single Stratum

Consider having to allocate \(k^i\) tuples amongst relations \(R_1, R_2, \ldots, R_s\), each having a single stratum in order to increase the number of possible samples. Thus, \(\sum_{i=1}^{s} mm_i = k^i\).
The number of possible samples can be given by \( \prod_{i=1}^{z} C_{m \times n_i}^{n_i} \). We use the following allocation strategy to maximize it.

\[
mm_i' = \text{round}\left(\frac{k \times m_i}{\sum_{i=1}^{n} m_i}\right).
\]

Since we are considering a single stratum, \( j \) is constrained to be 1. The resultant allocation has been shown by our experiments in Section 7.3.1 to have low error with the maximum difference of 2 from the optimal allocation found by searching through all possible allocations. Section 10.2.1 presents its proof.

### 6.3 Maximizing Randomness for Multiple Strata

We now provide the technique for allocating a given sample size amongst the different strata. After allocating samples to a given stratum, the samples will be divided amongst the same stratum of different relations using Equation 2.

Let \( k_1, k_2, \ldots, k^n \) samples be allocated amongst the different strata with \( \sum_{i=1}^{z} k_i = k \). Further, an allocation \( k_i' \) is divided amongst the different relations \( R_1, R_2, \ldots, R_n \) as \( mm_1', mm_2', \ldots, mm_n' \) respectively, so that \( k_i' = \sum_{j=1}^{n} mm_i' \). Given these preliminaries, we aim to increase the number of possible samples given by \( \prod_{j=1}^{n} \prod_{i=1}^{z} C_{m \times n_i}^{n_i} \). We do so using the following allocation strategy:

\[
k_i' = \text{round}\left(\frac{k \times \sum_{j=1}^{n} m_j}{\sum_{i=1}^{z} \sum_{j=1}^{n} m_i}\right)
\]

This result has been verified in Section 7.3.2 to provide allocation close to perfect allocation with the maximum difference of 1. Section 10.2.2 presents its proof. It should be noted that only those strata having non-zero tuples in all relations should be considered.

### 6.4 Discussion

Using Lagrange multipliers is a popular method in the sampling community to find approximate optimal strata allocation in closed-form under space or cost constraints [8,19–21,26]. This includes important allocation topics of Neyman allocation and cost-based allocation [21] for finding strata sizes in stratified sampling. We build upon this framework to derive the above optimal strata allocation techniques (proofs in Section 10.2). Sample sizes suggested by these closed-form allocation strategies are close to the optimal solution in practice – with a maximum difference of 2 in the case of single stratum allocation and 1 for multiple strata allocation.

### 7. EXPERIMENTAL EVALUATION

The algorithms were implemented using Java 8 (Sections 3, 4, 5 and 6) and Matlab R2014b (Section 6). Experiments were performed on an Ubuntu Linux 14.04.1 LTS system with a 24-core 2.4GHz Intel Xeon CPU, 256GB DDR3 @ 1866 MHz memory, and a 500GB @ 7200 RPM disk.

In Sections 7.1 and 7.2, we study different aspects of \textsc{StratJoin} \_ \textsc{Overall} since it is our central algorithm for obtaining perfectly random samples. We also compare it with the state of the art, \textsc{Stream-Sample}. We use the TPC-DS benchmark and a similar experimental setup to that used by Chaudhari et al. [6]. The results have conformed to our prior expectations. Section 7.3 studies effectiveness of our sample allocation techniques to maximize join randomness.

#### 7.1 TPC-DS Based Evaluation of \textsc{StratJoin} \_ \textsc{Overall}

\textsc{StratJoin} \_ \textsc{Overall} is designed for the case where multiple tuples from a relation can join with multiple tuples from the other. 8 out of 99 queries in the TPC-DS benchmark were found to be well-suited for this use case. We ran these queries on the TPC-DS dataset, having sizes ranging from 2 GB to 10 GB, for different sampling rates, using Postgres 9.3. Different metrics were used to study various features of \textsc{StratJoin} \_ \textsc{Overall}. Unless otherwise stated, each measurement represents the average across queries, unless otherwise stated.

##### 7.1.1 Ratio of \textsc{StratJoin} \_ \textsc{Overall} Sample Size and Relation Size

As the sampling rate increases, fewer strata can be sampled due to the increasing sample size, resulting in the benefits of \textsc{StratJoin} \_ \textsc{Overall} decreasing (Figure 4). As expected, as the sampling fraction approaches 1, the ratio of \textsc{StratJoin} \_ \textsc{Overall} sample size and relation size approaches 1.

![Figure 4: As sampling becomes less feasible with increasing rate, \textsc{StratJoin} \_ \textsc{Overall} sample size approaches the relation size.](image)

##### 7.1.2 Ratio of Sample Sizes of \textsc{Stream-Sample} and \textsc{StratJoin} \_ \textsc{Overall}

\textsc{StratJoin} \_ \textsc{Overall} explicitly samples both sides of the join while \textsc{Stream-Sample} samples a single relation. At low sampling rates, it is more feasible to sample from both the relations. Therefore, the ratio of the sample sizes is more pronounced. The benefit of sampling decreases with increasing rate (Figure 5) as the stratum sample sizes approach the stratum sizes. However, \textsc{Stream-Sample} continues sampling even when infeasible (around sampling rate of 0.4) resulting in \textsc{Stream-Sample} sample size being greater than the \textsc{StratJoin} \_ \textsc{Overall} sample size.
7.1.3 Ratio of Sample Sizes of Stream-Sample and StratJoin_Overall During Infeasibility

Figure 5 has demonstrated the increase in sample size of Stream-Sample due to sampling during infeasibility caused by sample inflation. Figure 6 shows the benefit of our gating condition which detects this infeasibility in greater detail. (The median sample size ratio is provided since the average ratio is greater by multiple orders of magnitude due to a couple of outlying query results.) As expected, the ratio increases approximately linearly with the sampling rate, which causes the sample size to increase linearly as well.

Figure 6: With increasing sampling rate, size of infeasible samples using Stream-Sample increases.

7.1.4 Fraction of Strata Sampled

We have seen that the gating condition for sampling feasibility employed by StratJoin_Overall, which although can be restrictive, always results in the smallest possible sample size. Here, we look at its applicability by finding out the fraction of strata where sampling is feasible (Figure 7). There exist a couple of queries with large strata sizes where none of the strata can be sampled resulting in the fraction being around 0.6, even for sampling rate of 0.1. As is obvious, we can see that with increasing sampling rate, the number of strata that can be sampled decreases, thereby, reducing the applicability of sampling.

Figure 7: Fraction of strata sampled decreases with increasing sampling rate.

7.1.5 Fraction of Queries Partially Sampled

Here we look at the queries having at least a single stratum being sampled. We see that with increasing sampling rate, the fraction of queries having at least 1 stratum that can be sampled starts decreasing (Figure 8). However, even for high sampling rates, most queries contain some strata that can be sampled. The line plot for the 2 GB case is not visible since it overlaps with that of the 5 GB one.

Figure 8: Fraction of queries that are sampled decreases with increasing sampling rate.

7.2 In-Depth Comparison with Stream-Sample

In this section, we compare execution times of StratJoin_Overall and Stream-Sample. We use a similar experimental setup to that used by them [6]. Four tables were generated with 100K tuples each. The join column in each table had strata counts derived from a zipfian distribution. The parameter z of the zipfian distributin varied from 0 to 3. Additional four tables with 1M tuples were generated similarly. Each row in the table consists of three columns – RID (integer), JoinKey (integer), and Padding (integer). LHS refers to the relation with 100K tuples while RHS consists of 1M tuples.

7.2.1 Varying Sampling Rate

The join complexity of StratJoin_Overall and Stream-Sample is $f \times |R_1 \bowtie R_2|$. Therefore, StratJoin_Overall
and Stream-Sample require nearly equal time when the samples can be stored in memory, \( z = 0 \) for both relations (Figure 9). The execution duration increases nearly proportionally with the sampling rate. However, due to sample inflation, the sample size in Stream-Sample can get exceedingly large (\( f = 0.01 \) and \( 0.1 \), \( z = 3 \) for both relations). This results in the sample needing to be written to disk for Stream-Sample after sampling and then read from the disk during the join causing slowdown by an order of magnitude, illustrating the need for the gating condition.

7.2.2 Varying Skew

With increased skew, the join size and thereby join duration increases (Figure 10). With the RHS \( z \) values of 2 and 3, and LHS \( z \) value of 3, samples need to read from disk for Stream-Sample resulting again in significant increase in execution duration.

7.3 Maximizing Join Randomness

In this section, we study the effectiveness of the sample allocation techniques. We first validate our allocation strategy for a single stratum and then for multiple strata. Since finding the optimal solution by considering all possible combinations is computationally expensive, we have restricted the size of our datasets.

The effectiveness of our sample allocation techniques was studied by comparing their results with those of the best possible allocation found by searching all possible allocations and providing Mean-Squared Error, Mean-Squared Relative Error\(^1\), and maximum difference.

### 7.3.1 Single Stratum Partitioning

![Figure 9: Sample Inflation causes samples to be read from disk increasing the execution time for Stream-Sample.](image)

![Figure 10: With increased skew, the join size increases causing samples to be read from disk for Stream-Sample.](image)

| \( mm^2 \) | \( k = 6 \) | \( k = 12 \) | \( k = 18 \) |
|-------|--------|--------|--------|
| 1     | 155040 | 1679600| 11400  |
| 2     | 218025 | 8341020| 218025 |
| 3     | 136800 | 20155200| 1860480|
| 4     | 39900  | 26453700| 8139600|
| 5     | 5040   | 19535040| 19535040|
| 6     | 210    | 8139600 | 26453700|
| 7     | -      | 1860480 | 20155200|
| 8     | -      | 218025  | 8341020 |
| 9     | -      | 11400   | 1679600 |
| 10    | -      | 190     | 125970 |

Table 4: Number of Possible Samples

We first provide the allocation for an illustrative use case. It enables an intuitive understanding of the behavior of the numerator of \( UC \). Table 4 shows the number of possible samples when \( k^i \) tuples need to be assigned between two relations having a single stratum. Here, \( m_1^2 = 10 \) and \( m_2^2 = 20 \). It can be easily verified that Equation 2 results in the maximum number of possible samples.

**Approximation Error:**

![Table 5: Error in Single Stratum Partitioning](image)

However, our technique might not result in ideal allocation for all possible sample size configurations. Hence, here we study their accuracy. We used 3 relations. The population, varying from 150 to 1000, was divided in all combinations between the 3 relations such that each relation at least had 5 tuples. The sample size was varied from 15 to 100. Table 5 shows that all error metrics are low – validating the effectiveness of our formula in practice for the single stratum use case.

### 7.3.2 Multiple Strata Partitioning

We proceed with an illustrative use case as before. Figure 11 shows the number of possible samples for two relations having two strata each, with the configuration of \( m_1^1 = 10 \), \( m_2^1 = 20 \), \( m_1^2 = 5 \), \( m_2^2 = 15 \), and \( k = 20 \). We can see that our heuristic resulted in the maximum number of samples as \( k^i = \text{round} \left( \frac{10 \times (10 \times 20 + 5 \times 15)}{10 \times (10 \times 5 + 20 \times 15)} \right) = 12 \).

\[ 1 \text{defined as } \frac{\sum_{i=1}^{n} (\text{actual}(i) - \text{predicted}(i))^2}{\text{sampleSize}} \]
saw that our techniques to maximize the join randomness were effective over varying population and sample sizes.

Going forward, several interesting avenues of research are enabled by our findings. First, there are opportunities into improving the efficiency of our algorithms. A second appealing area of investigation is to compare and contrast our work in the context of online aggregation research. Specifically, it would be interesting to confirm if our techniques on maximizing randomness of the sample can benefit aggregate estimation using correlated samples. We would also like to investigate the applicability of a search-based solution for maximizing randomness [27]. Third, we plan to look at the performance aspects of our approach by considering hashing, or auxiliary data structures such as indexes (inspired by outlier indexing [5]) to store exact or approximate distribution information. Fourth, we would also like to develop algorithms that consider different levels of availabilities of indexes and statistics since high degree of presence of indexes in numerous use cases such as OLAP is uncommon. Finally, we hope that our work enables applications of join sampling that were so far infeasible, especially in the areas of approximate data mining, exploratory visualization, and distributed query processing, and that the necessity or lack thereof of randomness is taken into consideration while choosing the algorithms.

8. CONCLUSION & FUTURE WORK

Sampling using both sides of a join is an important area of work. In this paper, we have presented techniques to sample joins, in the context of perfect as well as maximum randomness – illustrating the benefits and drawbacks in doing so. In the context of perfect randomness, we have provided algorithms for obtaining stratified random samples of a join by sampling a single relation or both relations of the join. We show how the sample inflation issue can be handled using stratified random sampling resulting in minimization of the sample size. We provided novel techniques for increasing randomness of joins when randomness is preferable but not mandatory. Our experiments confirmed that the benefits of sampling reduce with increasing sampling rate due to increasing sample size. Further, we saw that not handling sample inflation results in the sample size far exceeding the relation sizes. In some cases, this had a detrimental effect of sampled tuples being needed to be read from the disk – further compounding the ill-effects of sample inflation. We also

7.3.3 Heuristic Discussion

Our heuristics maximize the number of samples in the continuous domain whereas the allocation exists in the discrete domain, in order to derive the allocation strategies. Thus, the heuristics provide optimal solution when the rounding function does not have any effect. The approximation errors being low further affirms their usefulness in practice.

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The number of possible samples under sampling without replacement for an equi-join can be given by

\[ f(mm_1, mm_2, \ldots, mm_z) = \prod_{i=1}^{z} \frac{m_i!}{mm_i!(m_i - mm_i)!} \times \frac{1}{m_i!} \times \frac{1}{1 - \sum_{a=1}^{m_i 1\over a}} \]  

(4)

The notation used in Tables 1 and 3 is used in this subsection and the next. Note that in that notation the superscript denotes the stratum index and the subscript denotes the relation index. The superscript has been omitted from the usual notation of \( m_i \) and \( mm_i \) since a single stratum is under consideration. The constraint of fixed sample size can be given by

\[ g(mm_1, mm_2, \ldots, mm_z) = \sum_{i=1}^{z} mm_i = k^1 \]  

(5)

\( k^1 \) denotes the number of samples being allocated to the first stratum and \( g \) denotes the constraint function. Lagrange multipliers gives us a system of \( z \) equations. The \( i^{th} \) \((i \in [1, z])\) equation is obtained by taking partial derivative of equation 4 with respect to \( mm_i \)

\[ f_{mm_i} = \left( \prod_{j=1}^{z} \frac{m_j!}{mm_j!(m_j - mm_j)!} \right) \times \frac{1}{m_i!} \times \frac{1}{1 - \sum_{a=1}^{m_i 1\over a}} \times \frac{\partial}{\partial mm_i} \left( \frac{1}{mm_i! \times (m_j - mm_j)!} \right) \]  

(6)

Using the basic Lagrange multipliers equation of \( f_{mm_i} = \lambda \times g_{mm_i} \) and equation 5, we get

\[ f_{mm_i} = \lambda \times g_{mm_i} = \lambda \times \frac{\partial}{\partial mm_i} \left( \sum_{i=1}^{z} mm_i \right) = \lambda \]  

(7)

Using the Gamma function to represent factorial, its derivative can be given by

\[ (n!)^\gamma = \Gamma(n + 1) = n! \times (-\gamma + \sum_{a=1}^{n} 1\over a) \]  

(8)

where \( \gamma \) is Euler-Mascheroni constant. Using equation 8, we can simplify

\[ \frac{\partial}{\partial mm_i} \left( \frac{1}{mm_i! \times (m_j - mm_j)!} \right) = \frac{\left( \frac{1}{mm_i! \times (m_j - mm_j)!} \right) \times \left( \frac{1}{(m_i - mm_i)!} \right) \times \left( \frac{1}{(m_j - mm_j)!} \right)}{\left( \frac{1}{mm_i! \times (m_j - mm_j)!} \right)^2} \]  

\[ = \frac{mm_i! \times (m_i - mm_i)! \times (-\gamma + \sum_{a=1}^{mm_i 1\over a} 1\over a)}{\left( \frac{1}{mm_i! \times (m_j - mm_j)!} \right)^2} \]  

\[ = \frac{\left( \frac{1}{mm_i! \times (m_j - mm_j)!} \right) \times (\gamma - \sum_{a=1}^{mm_i 1\over a} 1\over a)}{\left( \frac{1}{mm_i! \times (m_j - mm_j)!} \right)^2} \]  

\[ = \frac{\left( \sum_{a=1}^{mm_i 1\over a} 1\over a - \sum_{a=1}^{mm_i 1\over a} 1\over a \right) \times (\gamma - \sum_{a=1}^{mm_i 1\over a} 1\over a)}{\left( \frac{1}{mm_i! \times (m_j - mm_j)!} \right)^2} \]  

(9)

Plugging above into equation 6 and using equation 7,

\[ \lambda = \left( \prod_{i=1}^{z} \frac{m_i!}{mm_i! \times (m_j - mm_j)!} \right) \times \frac{1}{m_i!} \times \frac{1}{1 - \sum_{a=1}^{m_i 1\over a}} \]  

\[ = \left( \prod_{j=1}^{z} \frac{m_j!}{mm_j! \times (m_j - mm_j)!} \right) \times \left( \frac{1}{\sum_{a=1}^{1\over a} 1\over a - \sum_{a=1}^{1\over a} 1\over a} \right) \]  

(10)
The sum of harmonic series is approximated by
\[
\sum_{a=1}^{\frac{m_m}{a}} \frac{1}{a} = \ln (m_m) + \gamma + \frac{1}{2 \times m_m} - \frac{1}{12 \times m_m^2} + \frac{1}{120 \times m_m^3} \approx \ln (m_m) + \gamma
\]
for large \( m_m \), where \( \gamma \) is the Euler-Mascheroni constant.

Using equations 10 and 11,
\[
-\lambda = \left( \prod_{j=1}^{z} \frac{m_j!}{m_m! \times (m_j - m_m)!} \right) \times \\
\left( \ln (m_m) + \gamma - \ln (m_i - m_m) - \gamma \right) = \left( \prod_{j=1}^{z} \frac{m_j!}{m_m! \times (m_j - m_m)!} \right) \times \ln \left( \frac{m_m}{m_i - m_m} \right)
\]
(12)

Note that the term \( \left( \prod_{j=1}^{z} \frac{m_j!}{m_m! \times (m_j - m_m)!} \right) \) is common for all \( i \in [1, z] \) equations and we denote it by \( C \). The above system of equations can thus be simplified as
\[
-\frac{\lambda}{C} = \ln \left( \frac{m_m}{m_i - m_m} \right)
\]
(13)

Consider the allocation \( mm_p \) for another relation \( p \) and its corresponding equation given by
\[
-\frac{\lambda}{C} = \ln \left( \frac{m_m}{m_p - m_m} \right)
\]
(14)

From the above two equations,
\[
\frac{m_m}{m_i - m_m} = \frac{m_m}{m_p - m_m} = A
\]
(15)

for a constant \( A \). Therefore,
\[
m_m = \frac{A \times m_i}{1 + A}
\]
(16)

Using equations 5 and 16
\[
k^i = \sum_{i=1}^{z} mm_i = \sum_{i=1}^{z} \frac{A \times m_i}{1 + A} = \frac{A}{1 + A} \sum_{i=1}^{z} m_i
\]
(17)

Using equations 16 and 17
\[
m_m = \left( \frac{A}{1 + A} \right) \times m_i = \frac{k^i \times m_i}{\sum_{a=1}^{z} m_a}
\]
(18)

As we have seen from the experiments in Section 7.3.1, this is close to the optimal allocation found using brute force.

10.2.2 Maximizing Randomness for Multiple Strata

This proof is similar to the one in the previous section. In this case, our goal is to maximize the number of possible samples under sampling without replacement. The constraint of fixed sample size is given by
\[
k = \sum_{j=1}^{n} k^j = g(k^1, k^2 \ldots k^n)
\]
(19)

where \( g \) represents the constraint function. Also,
\[
k^j = \sum_{i=1}^{z} mm_i
\]
(20)

The number of possible samples can be given by
\[
f(k^1, k^2 \ldots k^n) = \prod_{j=1}^{n} \prod_{i=1}^{z} C_{mm_i}^{m_i}
\]

We can simplify
\[
\frac{\partial}{\partial k_j} \left( \frac{k^j - \sum_{i=1}^{z-1} m_m^j}{m^j - (k^j - \sum_{i=1}^{z-1} m_m^j)} \right) \times \\
\frac{1}{(k^j - \sum_{i=1}^{z-1} m_m^j) \times (m_m^j - (k^j - \sum_{i=1}^{z-1} m_m^j))} \\
\left( (k^j - \sum_{i=1}^{z-1} m_m^j) \times (m_m^j - (k^j - \sum_{i=1}^{z-1} m_m^j)) \right)^2
\]
(22)

We can simplify
\[
\frac{\partial}{\partial k_j} \left( \frac{m_m^j - (k^j - \sum_{i=1}^{z-1} m_m^j)}{m^j - (k^j - \sum_{i=1}^{z-1} m_m^j)} \right) \\
\left( m_m^j - (k^j - \sum_{i=1}^{z-1} m_m^j) \times (k^j - \sum_{i=1}^{z-1} m_m^j) \right) \\
\left( m^j - (k^j - \sum_{i=1}^{z-1} m_m^j) \right) \\
\left( m_m^j - (k^j - \sum_{i=1}^{z-1} m_m^j) \right) \\
\left( m^j - (k^j - \sum_{i=1}^{z-1} m_m^j) \right)
\]
(23)
Substituting above equation in equation 22,

\[ f_{k_j} = \left( \prod_{j=1}^{n} \prod_{i=1}^{z} \frac{m_i^j!}{m_i^j \times (m_i^j - mm_i^j)!} \right) \ln \left( \frac{m_i^j - k_j}{k_j - \sum_{i=1}^{z-1} mm_i^j} \right) \]

From the standard Lagrange multipliers equation of \( f_{k_j} = \lambda \times g_{k_j} \) and equation 19,

\[ f_{k_j} = \lambda \times g_{k_j} = \lambda \times \frac{\partial (\sum_{j=1}^{n} k_j)}{\partial k_j} = \lambda \]  
(24)

From the above 2 equations,

\[ \lambda = \left( \prod_{j=1}^{n} \prod_{i=1}^{z} \frac{m_i^j!}{m_i^j \times (m_i^j - mm_i^j)!} \right) \ln \left( \frac{m_i^j}{k_j - \sum_{i=1}^{z-1} mm_i^j} - 1 \right) \]

Consider another stratum \( p \) and its corresponding equation

\[ \lambda = \left( \prod_{j=1}^{n} \prod_{i=1}^{z} \frac{m_i^p!}{m_i^p \times (m_i^p - mm_i^p)!} \right) \ln \left( \frac{m_i^p}{k^p - \sum_{i=1}^{z-1} mm_i^p} - 1 \right) \]

From the above 2 equations, given a constant \( A \), we get

\[ \frac{k_j - \sum_{i=1}^{z-1} mm_i^j}{m_i^j} = \frac{k^p - \sum_{i=1}^{z-1} mm_i^p}{m_i^p} = A \]  
(25)

Using equation 18 and the above equation,

\[ A = \frac{k_j - \sum_{i=1}^{z-1} \frac{k_j \times m_i^j}{\sum_{a=1}^{n} m_i^a}}{m_i^j} = \frac{k^p \times (\sum_{a=1}^{z} m_i^a - \sum_{i=1}^{z-1} m_i^j)}{m_i^j \times \sum_{a=1}^{n} m_i^a} \]  
(26)

\[ k^j = A \times \sum_{a=1}^{z} m_i^a \]  
(27)

Using above equation and equation 19

\[ k = \sum_{j=1}^{n} (A \times \sum_{a=1}^{z} m_i^a) \]  
(28)

\[ A = \frac{k}{\sum_{j=1}^{n} \sum_{a=1}^{z} m_i^j} \]  
(29)

Using equations 27 and 29

\[ k^j = \frac{k \times \sum_{i=1}^{z} m_i^j}{\sum_{j=1}^{n} \sum_{i=1}^{z} m_i^j} \]  
(30)

Section 7.3.2 shows that this approximation gives allocation close to the optimal allocation found using brute force.