Another presentation for symplectic Steinberg groups

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$T_{ij}(a) : w \mapsto w + e_i aw_j - e_j aw_i \varepsilon_i \varepsilon_j, \quad \varepsilon_k = \text{sign}(k)$
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$T_{ij}(a) : w \mapsto w + e_i aw_j - e_j aw_i \varepsilon_i \varepsilon_j$, $\varepsilon_k = \text{sign}(k)$
Definition

Proof

New result

R — any commutative unital ring, \( n \geq 3 \)
\( \text{Sp}(2n, R) \cong R^{2n} \)
\( T_{ij}(a) : w \mapsto w + e_iaw_j - e_{-j}aw_{-i}\varepsilon_i\varepsilon_j, \ \varepsilon_k = \text{sign}(k) \)
\( T_{i,-i}(a) : w \mapsto w + e_iaw_{-i} \)
$R$ — any commutative unital ring, $n \geq 3$

$\text{Sp}(2n, R) \cong R^{2n}$

$T_{ij}(a) : w \mapsto w + e_i aw_j - e_{-j} aw_{-i} \varepsilon_i \varepsilon_j, \; \varepsilon_k = \text{sign}(k)$

$T_{i,-i}(a) : w \mapsto w + e_i aw_{-i}$

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Definition

\[ \text{StSp}(2n, R) = \langle \{ X_{ij}(a), \ i \neq j, \ a \in R \} \mid \]

\[
\begin{align*}
X_{ij}(a) &= X_{-j,-i}(-a \epsilon_i \epsilon_j), & (R0) \\
X_{ij}(a)X_{ij}(b) &= X_{ij}(a+b), & (R1) \\
[X_{ij}(a), X_{hk}(b)] &= 1, \text{ for } h \notin \{ j, -i \}, \ k \notin \{ i, -j \}, & (R2) \\
[X_{ij}(a), X_{jk}(b)] &= X_{ik}(ab), \text{ for } i \notin \{ -j, -k \}, \ j \neq -k, & (R3) \\
[X_{i,-i}(a), X_{-i,j}(b)] &= X_{ij}(ab \epsilon_i)X_{-j,j}(-ab^2), & (R4) \\
[X_{ij}(a), X_{j,-i}(b)] &= X_{i,-i}(2ab \epsilon_i) & (R5)
\end{align*}
\]
Definition

\[ \text{StSp}(2n, R) = \langle \{ X_{ij}(a), \; i \neq j, \; a \in R \} | \]

\[ X_{ij}(a) = X_{-j,-i}(-a\varepsilon_i\varepsilon_j), \quad (R0) \]

\[ X_{ij}(a)X_{ij}(b) = X_{ij}(a+b), \quad (R1) \]

\[ [X_{ij}(a), X_{hk}(b)] = 1, \text{ for } h \notin \{ j, -i \}, \; k \notin \{ i, -j \}, \quad (R2) \]

\[ [X_{ij}(a), X_{jk}(b)] = X_{ik}(ab), \text{ for } i \notin \{ -j, -k \}, \; j \neq -k, \quad (R3) \]

\[ [X_{i,-i}(a), X_{-i,j}(b)] = X_{ij}(ab\varepsilon_i)X_{-j,j}(-ab^2), \quad (R4) \]

\[ [X_{ij}(a), X_{j,-i}(b)] = X_{i,-i}(2ab\varepsilon_i) \quad (R5) \]

\[ \phi : \quad \text{StSp}(2n, R) \quad \mapsto \quad \text{Sp}(2n, R) \]

\[ X_{ij}(a) \quad \mapsto \quad T_{ij}(a) \]
Theorem (Suslin–Kopeiko–Taddei)

\[ \text{Im}(\phi) \leq \text{Sp}(2n, R), \quad n \geq 2 \]
Theorem (Suslin–Kopeiko–Taddei)

\[ \text{Im}(\phi) \trianglelefteq \text{Sp}(2n, R), \ n \geq 2 \]

\[ T(u, v, a) : w \mapsto w + u\left( (v, w) + a(u, w) \right) + v(u, w) \]

symplectic form on \( R^{2n} \)
Theorem (Suslin–Kopeiko–Taddei)

\[ \text{Im}(\phi) \subseteq \text{Sp}(2n, R), \ n \geq 2 \]

\[ T(u, \nu, a) : w \mapsto w + u(v, w) + a(u, w) + \nu(u, w) \]

symplectic form on \( R^{2n} \)

\[ T_{ij}(a) = T(e_i, e_{-j} \alpha_{-j}, 0), \quad T_{i, -i}(a) = T(e_i, 0, a) \]
Theorem (Suslin–Kopeiko–Taddei)

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\[ T_{ij}(a) = T(e_i, e_{-j}a\varepsilon_{-j}, 0), \quad T_{i, -i}(a) = T(e_i, 0, a) \]

\[ g T(u, v, a)g^{-1} = T(gu, gv, a) \quad \forall g \in \text{Sp}(2n, R) \]
Theorem (Suslin–Kopeiko–Taddei)

\[ \text{Im}(\phi) \leq \text{Sp}(2n, R), \quad n \geq 2 \]

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Symplectic form on \( \mathbb{R}^{2n} \)

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\[ \text{Im}(\phi) \leq \langle T(u, v, a) \rangle \leq \text{Sp}(2n, R) \]
**Theorem (Suslin–Kopeiko–Taddei)**

\( \text{Im}(\phi) \subseteq \text{Sp}(2n, R), \ n \geq 2 \)

\[ T(u, v, a) : w \mapsto w + u \left( (v, w) + a (u, w) \right) + v (u, w) \]

symplectic form on \( R^{2n} \)

\[ T_{ij}(a) = T(e_i, e_{-j} a e_{-j}, 0), \quad T_{i, -i}(a) = T(e_i, 0, a) \]

\[ g T(u, v, a) g^{-1} = T(g u, g v, a) \quad \forall g \in \text{Sp}(2n, R) \]

\[ \text{Im}(\phi) \leq \langle T(u, v, a) \rangle \leq \text{Sp}(2n, R) \]
Theorem

$\text{Ker}(\phi) \subseteq \text{Cent StSp}(2n, R), \ n \geq 3$
Theorem

\[ \text{Ker}(\phi) \subseteq \text{Cent StSp}(2n, R), \quad n \geq 3 \]

How to define \( X(u, v, a) \in \text{StSp}(2n, R) \)
Theorem

\[ \text{Ker}(\phi) \subseteq \text{Cent StSp}(2n, R), \quad n \geq 3 \]

How to define \( X(u, v, a) \in \text{StSp}(2n, R) \) such that
\[ g \cdot X(u, v, a) \cdot g^{-1} = X(\phi(g)u, \phi(g)v, a) \quad \forall g \in \text{StSp}(2n, R) \]?
New result

Theorem

\[ \text{Ker}(\phi) \subseteq \text{Cent StSp}(2n, R), \quad n \geq 3 \]

How to define \( X(u, v, a) \in \text{StSp}(2n, R) \) such that
\[ g X(u, v, a) g^{-1} = X(\phi(g)u, \phi(g)v, a) \quad \forall g \in \text{StSp}(2n, R) \]?

W. van der Kallen, Another presentation for Steinberg groups, 1976
New result

\[ X(u, 0, a) = X(\tilde{u}, 0, a)X(u', 0, a)X(u', \tilde{u}a, 0) \]
Corollary

\[ H_2(\text{Im}(\phi), \mathbb{Z}) \hookrightarrow \ker(\phi) \rightarrow H_2(\text{StSp}(2l, R), \mathbb{Z}) \]

In particular, this map is bijective for \( l \geq 4 \) or for \( l = 3 \) and \( R \) having no residue field isomorphic to \( \mathbb{F}_2 \).

Corollary

\[ K_2\text{Sp}^Q(2l, R) \hookrightarrow \ker(\phi) \rightarrow H_2(\text{StSp}(2l, R), \mathbb{Z}) \]

In particular, this map is bijective for \( l \geq 4 \) or for \( l = 3 \) and \( R \) having no residue field isomorphic to \( \mathbb{F}_2 \).

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