General higher—order breathers and rogue waves in the two-component long-wave–short-wave resonance interaction model

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Abstract
General higher order breather and rogue wave (RW) solutions to the two-component long wave–short wave resonance interaction (2-LSRI) model are derived via the bilinear Kadomtsev–Petviashvili hierarchy reduction method and are given in terms of determinants. Under particular parametric conditions, the breather solutions can reduce to homoclinic orbits, or a mixture of breathers and homoclinic orbits. There are three families of RW solutions, which correspond to a simple root, two simple roots, and a double root of an algebraic equation related to the dimension reduction procedure. The first family of RW solutions consists of \( \frac{N(N+1)}{2} \) bounded fundamental RWs, the second family is composed of \( \frac{N_1(N_1+1)}{2} \) bounded fundamental RWs coexisting with another \( \frac{N_2(N_2+1)}{2} \) fundamental RWs of different a
bounded state ($N, N_1, N_2$ being positive integers), while
the third one has $[\bar{N}_1^2 + \bar{N}_2^2 - \bar{N}_1(\bar{N}_2 - 1)]$ fundamental bounded RWs ($\bar{N}_1, \bar{N}_2$ being nonnegative integers).
The second family can be regarded as the superpositions of the first family, while the third family can be
the degenerate case of the first family under particular parameter choices. These diverse RW patterns are
illustrated graphically.

**KEYWORDS**
breathers, rogue waves, two-component long wave–short wave resonance interaction model

### 1 | INTRODUCTION

Resonant three-wave interactions play a crucially important role in various setups, which occur
in Bose–Einstein condensates, fluid mechanics, optics, and other areas of physics.\textsuperscript{1–5} Theoretical
studies of these phenomena provide an essential contribution to the nonlinear-wave dynamics.\textsuperscript{6–11} Generally, in nonlinear systems with linear dispersion relation $\omega = \omega(k)$ ($\omega$ and $k$ are, as usual,
the frequency and wavenumber, respectively), the resonant interaction takes place if the corre-
sponding frequencies and wavenumbers, which form a resonance triad,\textsuperscript{12} are mutually locked by
conditions

$$k_3 = k_1 + k_2, \omega_3 = \omega_1 + \omega_2. \tag{1}$$

The limit case of the triad amounts to the long-wave–short-wave (LW-SW) resonance, when one
wave is much longer than the other two,\textsuperscript{12–14} that is,

$$k_3 = k + (\Delta k)/2, k_2 = k - (\Delta k)/2, k_1 = \Delta k, |\Delta k| \ll |k|. \tag{2}$$

In this special case, condition (1) for the frequency amounts, in the first approximation, to

$$\frac{d\omega}{dk} \approx \omega(\Delta k)/\Delta k, \tag{3}$$

that is, the LW phase velocity must match the SW group velocity, at the special value of wavenum-
ber $k$. In this case, the nonlinear equation, governing a slowly varying, complex-valued SW packet
evelope ($S$) and a real-valued LW field ($L$), has been derived by means of the multiple-scale
asymptotic expansion\textsuperscript{15–17}:

$$iS_t - S_{xx} + LS = 0,$$

$$L_t = 2(\delta|S|^2)_x. \tag{4}$$
The sign of the real nonlinearity coefficient $\delta$ depends on the physical realization of the system. It is often called the LW–SW resonant interaction (LSRI) model, or the Yajima–Oikawa (YO) system. It applies to fluid mechanics and a number of other physical settings.

The LSRI model (4) is completely integrable as it admits a Lax pair, and was solved by dint of the inverse scattering transform. Additionally, Cheng had proposed the LSRI model (4) from the so-called $K$-constrained Kadomtsev–Petviashvili (KP) hierarchy with $K = 2$, while $K = 1$ corresponds to the classical nonlinear Schrödinger (NLS) equation. Bright- and dark-soliton solutions of Equation (4) were constructed in Refs. 16, 17. The first-order rogue wave (RW) solutions to the LSRI model (4) were studied by means of the Hirota’s bilinear method and Darboux transformation. The higher order breather and RW solutions were derived with the help of the bilinear KP hierarchy reduction method. Homoclinic connections of unstable plane waves were investigated by means of the Bäcklund transform.

RWs are large displacements from an otherwise tranquil (but usually unstable) background whose most striking features are unpredictability and localization in time and space. In the past decade, RWs have attracted a great deal of interest in the experimental and theoretical communities alike. Explicit solutions of integrable equations for RWs help to understand these phenomena in the physical systems. Such solutions have been found for many integrable models, such as the NLS equation and its multicomponent version, the Davey–Stewartson (DS), and Ablowitz-Ladik equations. The RW solutions can be regarded as the limit case of breathers, which are periodic in time or spatial coordinate. The breathers can also provide a model for the observation of RWs in experiments. An RW may be thought of as the consequence of modulation instability (MI), but conversely not all of MIs necessarily result in RW generation. Besides, Baronio et al. found that RW solutions in several physical systems exist in the subset of parameters where the MI is present if and only if the unstable sideband spectrum also contains continuous wave or zero-frequency perturbations as a limit case.

In this paper, we consider the following two-component LSRI (2-LSRI) model governing the resonance of two SW components with a common LW one:

\[
\begin{align*}
    iA_t - A_{xx} + LA &= 0, \\
    iB_t - B_{xx} + LB &= 0, \\
    L_t &= 2(\delta_1 |A|^2 + \delta_2 |B|^2)_x,
\end{align*}
\]

where $A$, $B$ are the SWs and $L$ is the LW, real nonlinear coefficients $\delta_1, \delta_2$ depend on the precise physical properties of the system, for example, the density stratification profile in fluid. In optical contexts, the two SW components $A$ and $B$ are called optical waves, while the LW component $L$ is regarded as the induced optical rectification or the low-frequency terahertz wave.

The 2-LSRI model is an integrable extension of the LSRI model (4), and it reduces to the LSRI model (4) for $A = B = S$. This system also admits soliton solutions of bright, dark, and mixed bright–dark types. The first-order RW solution was first constructed by the Darboux transformation and later by means of the Hirota’s bilinear method. The connection between the existence criterion of RWs and the onset of baseband MI in the LSRI model (4) and the 2-LSRI model (5) was confirmed in Ref. 72 and Ref. 80, respectively. Additionally, Chen et al. have obtained several RW solutions from the general rational solutions of the (2+1)-dimensional multicomponent LSRI system. Very recently, Li and Geng have constructed a family of higher order solutions.
bounded RW solutions consisting of \( N(N + 1)/2 \) fundamental RWs with the help of the Darboux transformation.\textsuperscript{82–84} That family of solutions of the 2-LSRI model (5) shares the same RW patterns with the higher order RW of LSRI model,\textsuperscript{23} and there do not exist new RW patterns for 2-LSRI model in contrast with LSRI model. It is known that the multicomponent NLS equation admits more RW patterns in contrast to the scalar NLS equation,\textsuperscript{47–52} such as the mixed bounded RWs consisting of RWs of different types, and the degenerate bounded RWs. Very recently, Yang and Yang derived a family of new bounded RW solutions to the three-wave resonant interaction system,\textsuperscript{61} which consists of \([\hat{N}_1^2 + \hat{N}_2^2 - \hat{N}_1(\hat{N}_2 - 1)]\) fundamental bounded RWs (\(\hat{N}_1, \hat{N}_2\) being nonnegative integers). By taking \(\hat{N}_1 = 0\) or \(\hat{N}_2 = 0\), this solution family amounts to one consisting of the degenerate bounded RW solutions. For \(\hat{N}_1 \hat{N}_2 \neq 0\), the family contains nondegenerate bounded RW solutions. Thus, we call this family of RW solutions as degradable bounded RW solutions in this paper. Up to now, the higher order mixed bounded and degradable bounded RWs were not reported for the 2-LSRI model (5), to the best of our knowledge. A natural motivation is to construct the higher order mixed bounded and degradable bounded RWs in the 2-LSRI model (5). Additionally, the first-order breather solutions were constructed by means of the Hirota’s bilinear method,\textsuperscript{80} and the first-order homoclinic orbits were derived by the Bäcklund transformation.\textsuperscript{85} However, higher order breather and homoclinic-orbit solutions have not been reported, as yet, for the 2-LSRI model (5), to the best of our knowledge. Very recently, we constructed general higher order breather solutions for the multicomponent two-dimensional LSRI model in terms of determinants via the bilinear KP-hierarchy reduction method,\textsuperscript{86} which can also be applied to derive the higher order breather and homoclinic-orbit solutions of the 2-LSRI model (5) in the form of determinants.

The main goal of the present paper is to construct the general higher order breather and RW solutions by employing the bilinear KP hierarchy reduction method. The objectives of our work are as follows:

- The general higher order breather solutions in terms of determinants will be constructed. Under particular parametric restrictions, the breather solutions can reduce to higher order homoclinic orbits or a mixture of homoclinic orbits and breathers.
- Three families of RW solutions, corresponding to a simple root, two simple roots, and a double root of an algebraic equation related to the dimension reduction procedure, will be derived. The first family is the bounded \( N \)-th-order RWs consisting of \( N(N + 1)/2 \) fundamental RWs. The second family is the mixed bounded \((N_1, N_2)\)-th order RWs comprising \( N_1(N_1 + 1) + N_2(N_2 + 1)/2 \) fundamental RWs, in which the \( N_1 \)-th-order bounded RWs and \( N_2 \)-th bounded RW can be included in different states. The third one is the degradable bounded \((\hat{N}_1, \hat{N}_2)\)-th-order RWs. Here \( N, N_1, N_2 \) are positive integers, and \( \hat{N}_1, \hat{N}_2 \) are nonnegative integers. When \( \hat{N}_1 = 0 \) or \( \hat{N}_2 = 0 \), the third family solutions are degenerate RWs.

The paper is organized as follows. In Section 2, we present the general higher order breather solutions for the 2-LSRI model in the form of Theorem 1, and then investigate dynamics of the breathers. In Section 3, we present three different families of RW solutions by three theorems (i.e., Theorems 2, 3, 4), then we exhibit dynamics of these three families of RWs, respectively. In Section 4, we give the derivation of the breather solutions in Theorem 1 and RW solutions in Theorems 2, 3, and 4, that is, the proofs of these four theorems. The discussion of the obtained results and the conclusions are presented in Section 5.
2 | DYNAMICS OF BREATHERS IN THE 2-LSRI MODEL

This section focuses on the dynamics of breathers in the 2-LSRI model (5). For this purpose, we first present the general breather solutions in forms of determinants for the 2-LSRI model by the following theorem. The proof for this theorem is in Section 4.

Theorem 1. The 2-LSRI model (5) admits the following breather solutions:

$$A = \rho_1 e^{(k_1 x + (\gamma + k_2^1) t)} g, B = \rho_2 e^{(k_2 x + (\gamma + k_2^2) t)} h, L = \gamma - 2 (\log f)_{xx},$$

(6)

where the real function $f$ and the complex functions $g, h$ are given by

$$f = \tau_{0,0}, g = \tau_{1,0}, h = \tau_{0,1}$$

(7)

and $\tau_{n,k}$ is defined as the following $N \times N$ determinant:

$$\tau_{n,k} = \det_{1 \leq s,j \leq N} \left( \frac{m_{s,j}^{(n,k)}}{m_{s,j}} \right),$$

(8)

with

$$\frac{m_{s,j}^{(n,k)}}{m_{s,j}} = \frac{1}{p_{s}^{[1]} + p_{j}^{[1]*}} \left( \frac{-p_{s}^{[1]} - ik_{1}}{p_{j}^{[1]*} + ik_{1}} \right)^{n} \left( \frac{-p_{s}^{[1]} - ik_{2}}{p_{j}^{[1]*} + ik_{2}} \right)^{k} e^{\xi_{j}^s} + \frac{1}{p_{s}^{[2]} + p_{j}^{[2]*}} \left( \frac{-p_{s}^{[2]} - ik_{1}}{p_{j}^{[2]*} + ik_{1}} \right)^{n} \left( \frac{-p_{s}^{[2]} - ik_{2}}{p_{j}^{[2]*} + ik_{2}} \right)^{k} e^{\xi_{j}^s},$$

(9)

and

$$\xi_{s} = \hat{\xi}_{s}^{[1]} - \hat{\xi}_{s}^{[2]},$$

$$\hat{\xi}_{s}^{[\alpha]} = p_{s}^{[\alpha]} x - i p_{s}^{[\alpha]^2} t + \frac{1}{\xi_{s}}, \alpha = 1, 2.$$ (10)

In the above expressions, $p_{s}^{[1]}, p_{s}^{[2]}, \xi_{s}^{[\alpha]}$ are arbitrary complex constants, $\rho_{\epsilon}, k_{\epsilon}, \gamma$ are freely real parameters, and the parameters $p_{s}^{[1]}, p_{s}^{[2]}$ have to satisfy the following constraints:

$$\frac{\delta_{1}^{\rho_{1}^2}}{(p_{s}^{[1]} - ik_{1})(p_{s}^{[2]} - ik_{2})} + \frac{\delta_{2}^{\rho_{2}^2}}{(p_{s}^{[1]} - ik_{2})(p_{s}^{[2]} - ik_{2})} - i (p_{s}^{[1]} + p_{s}^{[2]}) = 0.$$ (11)
Remark 1. If we assume $p_{sR}^{[\alpha]} > 0$, then $f$ in Equation (7) is positive, and the above breather solutions are nonsingular. Thus, hereafter we take $p_{sR}^{[\alpha]} > 0$ to avoid the singularities of the breathers. The subscripts $R$ and $I$ represent the real and imaginary parts of a given parameter or a function, respectively.

Remark 2. When $\delta_1 = \delta_2, k_1 = k_2, \rho_1 = k\rho_2$, the above breather solution reduces to the breather solutions of the scalar LSRI model (4) reported in Ref. 22. However, the breather solutions in Ref. 22 were given by $2N \times 2N$ determinants, whereas they are given by $N \times N$ determinants in Theorem 1.

Remark 3. If $p_{sR}^{[1]} - p_{sI}^{[1]} = p_{sR}^{[2]} - p_{sI}^{[2]}$, then the coefficients of $t$ in $\zeta_{sI}$ are zero, and the solutions (6) become the higher order homoclinic orbits. If $p_{sR}^{[1]} - p_{sI}^{[1]} \neq p_{sR}^{[2]} - p_{sI}^{[2]}$ $(1 \leq s \neq j \leq N)$, the solutions (6) are a mixture of breathers and homoclinic orbits.

By taking $N = 1$ in Equation (8), Theorem 1 yields the first-order breather solutions of the 2-LSRI model, which can be expressed in terms of hyperbolic and trigonometric functions as:

$$A = -\frac{e^{i\alpha} \cosh(\xi_{1R} + \theta_0 + i\beta_1)}{e^{i\alpha} \cosh(\xi_{1R} + \theta_0 + i\beta) + e^{i\beta}},$$

$$B = -\frac{e^{i\alpha} \cosh(\xi_{1R} + \theta_0 + i\beta_2)}{e^{i\alpha} \cosh(\xi_{1R} + \theta_0 + i\beta) + e^{i\beta} \cosh(\xi_{1I} + \hat{\beta})},$$

$$L = y - \frac{y_2 \cosh(\xi_{1R} + \theta_0) \cos(\xi_{1I} + \hat{\beta}) + y_1 \sinh(\xi_{1R} + \theta_0) \sin(\xi_{1I} + \hat{\beta}) + y_0}{e^{i\alpha} \cosh(\xi_{1R} + \theta_0) + e^{i\beta} \cosh(\xi_{1I} + \hat{\beta})}.$$

The auxiliary functions in the above expressions are defined by

$$\bar{\zeta}_1 = \zeta^{[1]}_1 - \zeta^{[2]}_1 = (p^{[1]}_1 - p^{[2]}_1)x - i(p^{[1]}_1 - p^{[2]}_1)t + \zeta^{[1]}_1 - \zeta^{[2]}_1,$$

$$\bar{A} = -\rho_1(p^{[2]}_1 + p^{[2]}_1^*) - \frac{p^{[2]}_1 + ik_1}{p^{[2]}_1 - ik_1} e^{i(k_1x + (\gamma + k_1^2)t + \bar{\beta})},$$

$$\bar{B} = -\rho_2(p^{[2]}_1 + p^{[2]}_1^*) - \frac{p^{[2]}_1 + ik_2}{p^{[2]}_1 - ik_2} e^{i(k_2x + (\gamma + k_2^2)t + \bar{\beta})},$$

$$e^{i\alpha} + i\beta_\ell = \frac{p^{[1]}_1 - ik_\ell}{p^{[2]}_1 - ik_\ell}, e^{i\beta} = \frac{p^{[2]}_1 + p^{[2]}_1^*}{p^{[2]}_1 + p^{[1]}_1}, \ell = 1, 2,$$

$$y_1 = 2e^{i\alpha} e^{i\beta}, y_2 = 4e^{i\alpha} e^{i\beta}, y_0 = 2(c^2 e^{2i\alpha} - e^2 e^{2i\beta}),$$

$$\bar{c} = \frac{1}{2}(p_1 - p_2 + p_1^* - p_2^*), \bar{c} = \frac{1}{2i}(p_1 - p_2 - (p_1^* - p_2^*)), e^{i\alpha} = \sqrt{\frac{p_2 + p_2^*}{p_1 + p_1^*}}.$$
The first-order breather in the 2-LSRI model (5), which is described by the solutions (12) with parameters

\[
\delta_1 = -1, \delta_2 = 1, \rho_1 = \frac{\sqrt{195}}{2}, \rho_2 = \sqrt{3}, k_1 = 0, k_2 = -1, \gamma = 0, p_1^{[1]} = 2 + i, p_1^{[2]} = 3 - 2i, \xi_1^{[1]} = 0, \xi_1^{[2]} = 0
\]

and \(\xi_1^{[1]}, \xi_1^{[2]}\) are defined in Equation (10).

From the above expressions of the first-order breather solution (12), we can obtain that the first-order breather propagates along the line \(\xi_{1R} + \vartheta_0 = 0\), and is periodic along the line \(\xi_{1L} + \hat{\beta} = 0\). Figure 1 shows the first-order breather in the 2-LSRI model (5).

When \(p_1^{[1][R]} - p_1^{[1][L]} = p_1^{[2][R]} - p_1^{[2][L]}\), the coefficients of \(t\) in \(\xi_{1L}\) are zero, hence the first-order breather solutions (12) reduce to the first-order homoclinic orbit solutions. Figure 2 shows the first-order homoclinic orbit in the 2-LSRI model (5) with parameters \(p_1^{[1]} = 1 + i, p_1^{[2]} = 1 - i\).

Here, we have to note that the first-order breather solutions (12) for the 2-LSRI model (5) have been reported in Ref. 80, and were derived by the Hirota’s direction method. By taking the LW limit of the obtained first-order breather solution, the first-order RW solutions were also derived for the 2-LSRI model (5). Additionally, as shown in Figures 1 and 2, the LW component \(L\) has wave structures similar to those of the SW components \(A\) and \(B\), so we only focus on the dynamics of breathers in the SW components \(A\) and \(B\), and in what follows, we will not show the LW component.

By taking \(N = 2\) in Equation (8), Theorem 1 generates the second-order breather solutions of the 2-LSRI model. The determinant forms of the functions \(f, g, \) and \(h\) of solutions (6) can be explicitly written as

\[
f = \begin{vmatrix}
m_{1,1}^(0,0) & m_{1,2}^(0,0) \\
m_{2,1}^(0,0) & m_{2,2}^(0,0) \\
\end{vmatrix}, \quad g = \begin{vmatrix}
m_{1,1}^(1,0) & m_{1,2}^(1,0) \\
m_{2,1}^(1,0) & m_{2,2}^(1,0) \\
\end{vmatrix}, \quad h = \begin{vmatrix}
m_{1,1}^(0,1) & m_{1,2}^(0,1) \\
m_{2,1}^(0,1) & m_{2,2}^(0,1) \\
\end{vmatrix},
\]

(14)
FIGURE 3 The leftmost panels: the second-order homoclinic orbits with parameters $\delta_1 = 1, \delta_2 = -1, \rho_1 = \sqrt{2}, \rho_2 = \sqrt{5}, k_1 = 0, k_2 = -1, \gamma = 0, p_{1[1]}^{[1]} = 1 + i, p_{1[2]}^{[1]} = 1 - i, p_{2[1]}^{[1]} = 1.78521 + 2i, p_{2[2]}^{[1]} = 0.53496 - 1.048432i$; the middle panels: a mixture of a first-order breather and a first-order homoclinic orbit with parameters $\delta_1 = 1, \delta_2 = -1, \rho_1 = \sqrt{2}, \rho_2 = \sqrt{5}, k_1 = 0, k_2 = -1, \gamma = 0, p_{1[1]}^{[1]} = 1 + i, p_{1[2]}^{[1]} = 1 - i, p_{2[1]}^{[1]} = 1 + 2i, p_{2[2]}^{[1]} = 0.6780 - 1.2539i$; the rightmost panels: the second-order breather with parameters $\delta_1 = -1, \delta_2 = -1, \rho_1 = \sqrt{\frac{5}{6}}, \rho_2 = \sqrt{\frac{55}{12}}, k_1 = 0, k_2 = -1, \gamma = 0, p_{1[1]}^{[1]} = 1 + i, p_{1[2]}^{[1]} = 1 - \frac{1}{2}i, p_{2[1]}^{[1]} = 1 - \frac{1}{3}i, p_{2[2]}^{[1]} = 1.0362 + 0.7905i$; where $\overline{m}_{s,j}^{(n,k)}$ are given by Equation (9). Because this second-order breather is a superposition of first-order breathers, thus these solutions have three different dynamical behaviors: (i) the second-order homoclinic orbits for $p_{sR}^{[1][2]} - p_{sl}^{[1][2]} = p_{sR}^{[2][2]} - p_{sl}^{[2][2]}, s = 1, 2$; (ii) the mixture of the first-order breather and first-order homoclinic orbit for $p_{sR}^{[1][2]} - p_{sl}^{[1][2]} = p_{sR}^{[2][2]} - p_{sl}^{[2][2]}$ and $p_{3-s R}^{[1][2]} - p_{3-s l}^{[1][2]} \neq p_{3-s R}^{[2][2]} - p_{3-s l}^{[2][2]}$; and (iii) the second-order breather for $p_{sR}^{[1][2]} - p_{sl}^{[1][2]} \neq p_{sR}^{[2][2]} - p_{sl}^{[2][2]}$. Figure 3 shows these three different types of periodic waves of the 2-LSRI model (5). In the leftmost panels of Figure 3, corresponding to the second-order homoclinic orbits, the two periodic waves are only periodic in space (i.e., $x$); in the middle panels, corresponding to a mixture of a first-order breather and a first-order homoclinic orbit, one periodic wave (namely, the one propagating along the $x$-axis) is only periodic along space $x$ and the other periodic wave is periodic along both space $x$ and time $t$; in the rightmost panels, both of the two periodic waves are periodic along space $x$ and time $t$.

For larger $N$ in Theorem 1, the higher order breathers, or higher order homoclinic orbits, or a mixture of them to the 2-LSRI model can be obtained, namely, a superposition of $N$ individual first-order solutions given by Equation (12).

3 | DYNAMICS OF RWs IN THE 2-LSRI MODEL

In this section, we study the dynamics of RWs in the 2-LSRI model (5). The RW solutions, which will be explicitly expressed in this section, depend on the root structure of the following algebraic equation:

$$\frac{\partial Q}{\partial p} = 0,$$

(15)
where

\[ Q(p) = \frac{\delta_1 p_1^2}{p - ik_1} + \frac{\delta_2 p_2^2}{p - ik_2} + ip^2. \]  

(16)

For different types of roots of Equation (15), the algebraic expressions of the corresponding RW solutions to the 2-LSRI model (5) are different. In what follows, we will mainly discuss the RW solutions associated to a nonimaginary simple root, two nonimaginary simple roots, and a nonimaginary double root of Equation (15), respectively. These three families of solutions are bounded RWs, mixed bounded RWs, and the degradable bounded RWs.

**Remark 4.** If \( p_0 \) is a root of Equation (15), then \(-p_0^*\) is also a root for Equation (15), thus hereafter we assume \( p_{0R} > 0 \) without loss of generality.

**Remark 5.** Equation (15) is a quintic equation having up to five roots. If \( p_0 \) is a triple root of Equation (15), then \(-p_0^*\) is also a triple root of Equation (15), thus \( p_0 = -p_0^* \) must hold, otherwise there are six roots for Equation (15), so \( p_0 \) is pure imaginary when it is a triple root of Equation (15). Besides, there is a factor \( \frac{1}{p_0^* + p_0} \) in RW solutions, hence \( p_0 \) cannot be pure imaginary. Thus, there do not exist RW solutions when \( p_0 \) is a triple root of Equation (15). That is also true for \( p_0 \) being a quadruple or a quintuple root of Equation (15), so we only consider the RW solutions associated with \( p_0 \) being a nonimaginary simple root, two nonimaginary simple roots, and a nonimaginary double root of Equation (15).

Below, we give three families of higher order RW solutions to the 2-LSRI model (5) and then consider their dynamics. These RW solutions are expressed through both differential operator forms and Schur polynomials. Before that, we have to review the definition of these Schur polynomials \( S_j(x) \) with \( x = (x_1, x_2, ...) \):

\[
\sum_{j=0}^{\infty} S_j(x) e^j = \exp \left( \sum_{k=1}^{\infty} x_k e^k \right),
\]

(17)

or more explicitly

\[
S_0(x) = 1, \\
S_1(x) = x_1, \\
S_2(x) = \frac{1}{2} x_1^2 + x_2, \\
\vdots \\
S_j(x) = \sum_{l_1 + 2l_2 + \cdots + m_{l_m} = j} \left( \prod_{k=1}^{m} \frac{x_k^{l_k}}{l_k!} \right).
\]

(18)
3.1 The bounded $N$-th-order RW solutions and their dynamics

3.1.1 The bounded $N$-th-order RW solutions

If $p_0$ is a nonimaginary simple root of Equation (15), the bounded $N$-th-order RW solutions of the 2-LSRI model are expressed as the following theorem.

**Theorem 2.** The 2-LSRI model (5) admits the following bounded $N$-th-order RW solutions

$$A = \rho_1 e^{i(k_1 x + (\gamma + k_1^2)t)} \frac{g}{f}, B = \rho_2 e^{i(k_2 x + (\gamma + k_2^2)t)} \frac{h}{f}, L = \gamma - 2(\log f)_{xx},$$  

where

$$f = \tau_{0,0}, g = \tau_{1,0}, h = \tau_{0,1}$$

and $\tau_{n,k}$ is defined as the following $N \times N$ determinant:

$$\tau_{n,k} = \det_{1 \leq s,j \leq N} \left(m^{(n,k)}_{s-1, j-1}\right),$$

and the matrix elements $m^{(n,k)}_{s,j}$ are given either (a) in differential operator form:

$$m^{(n,k)}_{s,j} = \left[\tau(p) \partial_p^s \right] \left[\tau(p^*) \partial_{p^*}^j \right] m \bigg|_{p = p_0},$$

or (b) through Schur polynomials:

$$\hat{m}^{(n,k)}_{s,j} = \sum_{\nu=0}^{\min(i,j)} \left[\frac{|\lambda_1|^2}{(p_0 + p_0^*)^2}\right] \nu^{s_j} (x^+(n, k) + \nu s) S_{j-\nu} (x^+(n, k) + \nu s^*).$$

Here, $N$ is an arbitrary positive integer. The auxiliary functions $\tau(p), \xi$ in Equation (22) are defined by

$$\xi = px - i p^2 t + \sum_{r=1}^{\infty} \hat{a}_r \ln^r \mathcal{W}(p),$$

$$\tau(p) = \sqrt{\frac{Q^2(p) - Q^2(p_0)}{Q^2(p)}},$$

$$\mathcal{W}(p) = \frac{Q(p) \pm \sqrt{Q^2(p) - Q^2(p_0)}}{Q(p_0)}.$$
The vectors $\mathbf{x}(n, k) = (x_1^+, x_2^+, \ldots)$ in Equation (23) are defined as:

\[
x_+(n, k) = \lambda_r x - i\beta_r t + n\delta_r^{(1)} + k\delta_r^{(2)} + a_r,
\]

\[
x_-(n, k) = \lambda_r^* x + i\beta_r^* t - n\delta_r^{(1)} - k\delta_r^{(2)} + a_r^*,
\]

where $\alpha_r, \beta_r, \delta_r^{(1)}$, and $\delta_r^{(2)}$ are coefficients from the expansions

\[
p(\kappa) - p_0 = \sum_{r=1}^{\infty} \lambda_r \kappa^r, \quad p^2(\kappa) - p_0^2 = \sum_{r=1}^{\infty} \beta_r \kappa^r, \quad \ln \left(\frac{p(\kappa) - i k_1}{p_0 - i k_1}\right) = \sum_{r=1}^{\infty} \delta_r^{(1)} \kappa^r, \quad \ln \left(\frac{p(\kappa) - i k_2}{p_0 - i k_2}\right) = \sum_{r=1}^{\infty} \delta_r^{(2)} \kappa^r,
\]

and the vector $\mathbf{s} = (s_1, s_2, s_3, \ldots)$ is defined by the expansion:

\[
\ln \left[\frac{1}{\kappa} \left(\frac{p_0 + p_0^*}{p_1 + p_1^*}\right) \left(\frac{p(\kappa) - p_0}{p(\kappa) + p_0^*}\right)\right] = \sum_{r=1}^{\infty} s_r \kappa^r.
\]

### 3.1.2 Dynamics of the bounded RWs

We first consider the fundamental (i.e., first-order) RW solutions of the 2-LSRI model (5), which can be regarded as the basic elements of high-order bounded RW solutions. For this purpose, we take $N = 1$ in Theorem 2, the first-order RW solutions of the 2-LSRI model (5) are explicitly expressed as

\[
A = \rho_1 e^{i(k_1 x + (\gamma + k_2) t)} \frac{g}{f}, \quad B = \rho_2 e^{i(k_2 x + (\gamma + k_2 t))} \frac{h}{f}, \quad L = \gamma - 2(\log f)_{xx},
\]

where

\[
f = \hat{m}^{(0,0)}_{1,1} g = \hat{m}^{(1,0)}_{1,1} h = \hat{m}^{(0,1)}_{1,1}, \quad \hat{m}^{(n,k)}_{1,1} = (\lambda_1 x - i\beta_1 t + n\delta_1^{(1)} + k\delta_1^{(2)})(\lambda_1^* x + i\beta_1^* t - n\delta_1^{(1)} - k\delta_1^{(2)}) + \xi_0.
\]
and

\[ \lambda_1 = \left. \frac{dp(x)}{dx} \right|_{x=0}, \hat{\beta}_1 = 2\lambda_1 p_0, \]

\[ \theta_1^{(1)} = \frac{\lambda_1}{p_0 - ik_1}, \theta_1^{(2)} = \frac{\lambda_1}{p_0 - ik_2}, \zeta_0 = \frac{|\lambda_1|^2}{(p_0 + p_0^*)^2}. \]  

(30)

Here, we have taken \( a_1 = 0 \) so that the center of the fundamental RW is located at \( x = 0, t = 0 \). After simple algebraic calculations, the above RW solutions can also be expressed as follows:

\[ A = \hat{\rho}_1 \left[ 1 - \frac{2i(\tilde{a}_1 \ell_2 - \tilde{b}_1 \ell_1) + (\tilde{a}_1^2 + \tilde{b}_1^2)}{\ell_1^2 + \ell_2^2 + \zeta_0} \right], \]

\[ B = \hat{\rho}_2 \left[ 1 - \frac{2i(\tilde{a}_2 \ell_2 - \tilde{b}_2 \ell_1) + (\tilde{a}_2^2 + \tilde{b}_2^2)}{\ell_1^2 + \ell_2^2 + \zeta_0} \right], \]

\[ L = \gamma + 4 \frac{\ell_1^2 - \ell_2^2 - \zeta_0}{(\ell_1^2 + \ell_2^2 + \zeta_0)^2}, \]

(31)

where \( \hat{\rho}_j = \rho_j e^{i(k_j x + (\gamma + k_j^2) t)}, \ell_1 = x + 2p_{01} t, \ell_2 = -2p_{02} t, \zeta_0 = \frac{1}{4p_{01}^2}, \tilde{a}_j = \frac{p_{01}}{p_{01}^2 + (p_{01} - k_1)^2}, \tilde{b}_j = -\frac{k_j - p_{01}}{p_{01}^2 + (p_{01} - k_1)^2} \) for \( j = 1, 2 \).

This fundamental RW in the \( A \) component is classified into three different types:

- a bright RW when \((p_{01} - k_1)^2 \leq \frac{1}{3} p_{01}^2;\)
- a four-petaled RW when \(\frac{1}{3} p_{01}^2 < (p_{01} - k_1)^2 < 3 p_{01}^2;\)
- a dark RW when \((p_{01} - k_1)^2 \geq 3 p_{01}^2;\)

By replacing \( k_1 \) with \( k_2 \) in the corresponding parameter condition, this classification is also valid for the fundamental RW in the \( B \) component. The LW component \( L \) is always a dark RW. Figure 4 displays these three different types of fundamental RWs in both \( A \) and \( B \) components.

These fundamental RWs admit the center amplitudes given below:

\[ |A^c| = |\rho_1| \left[ 1 - \frac{4 p_{01}^2}{p_{01}^2 + (p_{01} - k_1)^2} \right], \]

\[ |B^c| = |\rho_2| \left[ 1 - \frac{4 p_{02}^2}{p_{02}^2 + (p_{02} - k_2)^2} \right], \]

\[ L^c = \gamma - 16 p_{01}^2, \]  

(32)

which are calculated at the origin. The peak-to-background ratios of the fundamental RWs in the two SW components are \(|A^c|/|\rho_1|, |B^c|/|\rho_2| \leq 3\), which indicates that the fundamental RWs cannot reach a peak amplitude that exceeds three times the background level, as in the case of fundamental RWs in the 2-NLS equations. 68,69
The three different types of first-order RW solutions (31) in the LSRI model. The leftmost panels: 
\[ \delta_1 = -1, \delta_2 = -1, \rho_1 = \frac{5\sqrt{2}}{6}, \rho_2 = \frac{2\sqrt{6}}{3}, k_1 = \frac{1}{2}, k_2 = 0, \gamma = 0, p^{[1]}_1 = 1 + i; \]
the middle panels: 
\[ \delta_1 = -1, \delta_2 = 1, \rho_1 = \frac{2\sqrt{6}}{3}, \rho_2 = \frac{5\sqrt{2}}{6}, k_1 = 0, k_2 = -1; \]
the rightmost panels: 
\[ \delta_1 = -1, \delta_2 = -1, \rho_1 = \frac{5\sqrt{2}}{2}, \rho_2 = \frac{\sqrt{14}}{2}, k_1 = -1, k_2 = 1, \gamma = 0, p^{[1]}_1 = 1 + i \]

The higher order bounded RW solutions in Theorem 2 are superpositions of \(N(N+1)/2\) (\(N \geq 2\)) fundamental RWs (28). The coexistence of these \(N(N+1)/2\) fundamental RWs can generate diverse waveforms. For instance, taking \(N = 2\) and \(N = 3\) in Theorem 2, the second-order and third-order RW solutions can be derived, respectively. These second-order and third-order RWs are displayed in Figure 5. It is seen that the second-order RWs are composed of three fundamental RWs (the left panels of Figure 5), which form triangle patterns. The third-order RWs comprising six fundamental RWs exhibit ring waveforms (the right panels of Figure 5).
3.2  The mixed bounded \((N_1, N_2)\)-th order RW solutions and their dynamics

3.2.1  The mixed bounded \((N_1, N_2)\)-th order RW solutions

If \(p_0^{(1)}\) and \(p_0^{(2)}\) (\(p_0^{(1)} \neq p_0^{(2)}\)) are two nonimaginary simple roots of Equation (15), the mixed RW solutions comprising of bounded \(N_1\) th-order RWs and \(N_2\) th-order RWs are given by the following theorem.

**Theorem 3.** The 2-LSRI model (5) admits the following mixed bounded \((N_1, N_2)\)-th-order RW solutions:

\[
A = \rho_1 e^{i(k_1 x + (\gamma + k_1^2) t)} \frac{g}{f}, \quad B = \rho_2 e^{i(k_2 x + (\gamma + k_2^2) t)} \frac{h}{f}, \quad L = \gamma - 2 \log f_{xx},
\]

where

\[
f = \tau_{0,0}, \quad g = \tau_{1,0}, \quad h = \tau_{0,1}
\]

and \(\tau_{n,k}\) is defined as the following 2 \(\times\) 2 block determinant:

\[
\tau_{n,k} = \det \begin{pmatrix} \tau_{[1,1]}^{n,k} & \tau_{[1,2]}^{n,k} \\ \tau_{[2,1]}^{n,k} & \tau_{[2,2]}^{n,k} \end{pmatrix},
\]

\[
\tau_{[\alpha,\beta]}^{n,k} = \left( m^{(n,k,\alpha,\beta)}_{2s-1,2j-1} \right)_{1 \leq s \leq N_{\alpha}, 1 \leq j \leq N_{\beta}}
\]

for \(\alpha, \beta = 1, 2\), and the matrix elements \(m^{(n,k,\alpha,\beta)}_{s,j}\) are given either (a) in differential operator form:

\[
m^{(n,k,\alpha,\beta)}_{s,j} = \frac{[\tau (p) \partial_p]^s [\hat{\tau} (p^*) \partial_{p^*}]^j}{s! j!} m^{(n,k,\alpha,\beta)} \bigg|_{p = p_0^{(\alpha)}, p^* = p_0^{(\beta)}},
\]

\[
m^{(n,k,\alpha,\beta)} = \frac{1}{p + p^*} \left( \frac{p - ik_1}{p^* + ik_1} \right)^n \left( \frac{-p - ik_2}{p^* + ik_2} \right)^k e^{\xi_{\alpha} + \xi_{\beta}},
\]

or (b) through Schur polynomials:

\[
\hat{m}^{(n,k,\alpha,\beta)}_{s,j} = \sum_{\gamma=0}^{\min(i,j)} \left( \frac{1}{P^{(\alpha)}_0 + P^{(\beta)}_0} \right) \left( \frac{|\lambda_{1,\alpha}|^2}{(P^{(\alpha)}_0 + P^{(\beta)}_0)^2} \right)^\gamma S_{i-\gamma} (x^+_{\alpha,\beta} (n, k) + vs_{\alpha,\beta}) S_{j-\gamma} (x^-_{\alpha,\beta} (n, k) + vs_{\alpha,\beta}).
\]
Here $N_1, N_2$ are arbitrary positive integers. The auxiliary functions in Equation (37) are defined by

$$
\mathcal{T}(p) = \sqrt{\frac{Q^2(p) - Q^2(p_0^{(\overline{\alpha})})}{Q^2(p)}}, \hat{\mathcal{T}}(p) = \sqrt{\frac{\hat{Q}^2(p) - \hat{Q}^2(p_0^{(\overline{\beta})})}{\hat{Q}^2(p)}},
$$

$$
\xi_\alpha = px - ip^2 t + \sum_{r=1}^{\infty} \hat{a}_{r,\alpha} \ln \hat{\mathcal{W}}(p), \xi_\beta = \overline{p}x - i\overline{p}^2 \tau + \sum_{r=1}^{\infty} \hat{a}_{r,\beta} \ln \hat{\mathcal{W}}(p),
$$

$$
\mathcal{W}(\overline{\alpha})(p) = \frac{Q(p) \pm \sqrt{Q^2(p) - Q(p_0^{(\overline{\alpha})})}}{Q(p^{(\overline{\alpha})})}, \hat{\mathcal{W}}(\overline{\beta})(p) = \frac{\hat{Q}(p) \pm \sqrt{\hat{Q}^2(p) - \hat{Q}(p_0^{(\overline{\beta})})}}{\hat{Q}(p_0^{(\overline{\beta})})},
$$

$$
\hat{\mathcal{W}}(p) = \frac{\delta_1 \rho_1^2}{\overline{p} + i k_1} + \frac{\delta_2 \rho_2^2}{\overline{p} + i k_2} - i \overline{p}^2,
$$

where $Q$ is given by Equation (16). The vectors $x_{\alpha,\beta}^\pm(n, k) = (x_{\overline{\alpha},\overline{\beta}}^\pm, x_{\overline{\beta},\overline{\alpha}}^\pm, \ldots)$ in Equation (38) are defined as:

$$
\begin{align*}
x_{r,\alpha,\overline{\beta}}^+(n, k) &= \lambda_{r,\alpha} x + i \beta_{r,\alpha} t + n \theta_{r,\alpha}^{(1)} + k \theta_{r,\alpha}^{(2)} - b_{r,\alpha,\overline{\beta}} + a_{r,\alpha}, \\
x_{r,\alpha,\overline{\beta}}^-(n, k) &= \lambda^*_{r,\alpha} x - i \beta^*_{r,\alpha} t - n \theta_{r,\alpha}^{(1)} - k \theta_{r,\alpha}^{(2)} - b^*_{r,\alpha,\overline{\beta}} + a^*_{r,\alpha},
\end{align*}
$$

where $\lambda_{r,\alpha}, \beta_{r,\alpha}, \theta_{r,\alpha}^{(1)}, \theta_{r,\alpha}^{(2)}$ and $b_{r,\alpha,\overline{\beta}}$ are coefficients from the expansions

$$
\begin{align*}
p(x) - p_0^{(\overline{\alpha})} &= \sum_{r=1}^{\infty} \lambda_{r,\alpha} x^r, \\
p^2(x) - p_0^{(\overline{\alpha})^2} &= \sum_{r=1}^{\infty} \beta_{r,\alpha} x^r, \\
\ln \frac{p(x) - ik_1}{p_0^{(\overline{\alpha})} - ik_1} &= \sum_{r=1}^{\infty} \theta_{r,\alpha}^{(1)} x^r, \\
\ln \frac{p(x) - ik_2}{p_0^{(\overline{\alpha})} - ik_2} &= \sum_{r=1}^{\infty} \theta_{r,\alpha}^{(2)} x^r, \\
\ln \left[ \frac{p^{(\overline{\alpha})}(x) + p^{(\overline{\beta})}}{p_0^{(\overline{\alpha})} + p_0^{(\overline{\beta})}} \right] &= \sum_{r=1}^{\infty} b_{r,\alpha,\overline{\beta}} x^r,
\end{align*}
$$

and $Q$ are defined by Equation (24).
3.2.2 The dynamics of the mixed bounded \((N_1, N_2)\)-th order RWs

This family of RW solutions is a mixture of a bounded \(N_1\)-th-order RW and another different bounded \(N_2\)-th-order RW, thus it comprises of \(\frac{N_1(N_1+1)+N_2(N_2+1)}{2}\) fundamental RWs. To exhibit the dynamics of the mixed bounded RWs, here we take the two nonimaginary simple roots \(p_0^{(\alpha)}\) \((\alpha = 1, 2)\) of Equation (15) and the parameters \(\delta_\alpha, k_\alpha, \rho_\alpha\) as

\[
p_0^{(1)} = \frac{1}{2} + \frac{1}{3}i, \quad p_0^{(2)} \approx 0.5466 - 1.11i, \quad \delta_1 = 1, \delta_2 = 1,
\]

\[
k_1 = 1, k_2 = -1, \rho_1 = \frac{25\sqrt{4002}}{1656}, \quad \rho_2 = \frac{73\sqrt{46}}{552}, \quad h = 0.
\]

We first consider the simplest RWs in this solution family, which are composed of two different fundamental RWs. For this purpose, we take \(N_1 = N_2 = 1\) in Theorem 3. Then, the corresponding mixed bounded RW solution is

\[
A = \rho_1 e^{(k_1\alpha + (\gamma + k_1^2)\theta)} \frac{g}{f}, \quad B = \rho_2 e^{(k_2\alpha + (\gamma + k_2^2)\theta)} \frac{h}{f}, \quad L = \gamma - 2(\log f)_{xx},
\]

where

\[
f = \begin{bmatrix}
\hat{m}_{1,1}^{(0,0,1,1)} & \hat{m}_{1,1}^{(0,0,1,2)} \\
\hat{m}_{1,1}^{(0,0,2,1)} & \hat{m}_{1,1}^{(0,0,2,2)}
\end{bmatrix}, \quad g = \begin{bmatrix}
\hat{m}_{1,1}^{(1,0,1,1)} & \hat{m}_{1,1}^{(1,0,1,2)} \\
\hat{m}_{1,1}^{(1,0,2,1)} & \hat{m}_{1,1}^{(1,0,2,2)}
\end{bmatrix}, \quad h = \begin{bmatrix}
\hat{m}_{1,1}^{(0,1,1,1)} & \hat{m}_{1,1}^{(0,1,1,2)} \\
\hat{m}_{1,1}^{(0,1,2,1)} & \hat{m}_{1,1}^{(0,1,2,2)}
\end{bmatrix}, (44)
\]

and

\[
\hat{m}_{1,1}^{(n,k,\alpha,\beta)} = \frac{1}{p_0^{(\alpha)} + p_0^{(\beta)}} \left[ x^+_{\alpha,\beta}(n,k) x^-_{\beta,\alpha}(n,k) + \frac{\lambda_{1,\alpha} \lambda_{1,\beta} n_1 \theta_{1,\alpha} + n_2 \theta_{1,\beta} + b_{1,\alpha} + a_{1,\alpha}}{(p_0^{(\alpha)} + p_0^{(\beta)})^2} \right],
\]

\[
x^+_{\alpha,\beta}(n,k) = a^+_{1,\alpha} x + i \beta^+_{1,\alpha} t + \left( n + \frac{1}{2} \right) \lambda^+_{1,\alpha} + n_1 \theta_{1,\alpha} + n_2 \theta_{1,\beta} - b_{1,\alpha} - a_{1,\alpha},
\]

\[
x^-_{\alpha,\beta}(n,k) = a^-_{1,\beta} x - i \beta^-_{1,\beta} t - \left( n + \frac{1}{2} \right) \lambda^-_{1,\beta} - n_1 \theta_{1,\alpha} - n_2 \theta_{1,\beta} - b^+_{1,\alpha} + a^+_{1,\alpha},
\]

for \(\alpha, \beta = 1, 2\). Here, we have taken \(a_{0,1}^{(1)} = 1\) and \(a_{0,2}^{(2)} = 1\) for simplicity. The degree of polynomials in functions \(f, g, h\) is four in both \(x\) and \(t\).

According to the classifications of the first-order RW solution (31) discussed previously, because

\[
\frac{1}{3} p_0^{(1)} < (p_0^{(1)} - k_1)^2 < 3 p_0^{(1)} \quad \text{and} \quad (p_0^{(2)} - k_2)^2 > 3 p_0^{(2)}, \quad (p_0^{(1)} - k_1)^2 < \frac{1}{3} p_0^{(1)} \quad \text{and} \quad (p_0^{(2)} - k_2)^2 > 3 p_0^{(2)},
\]

thus the \(A\) component contains a dark RW and a four-petaled RW, while the \(B\) component comprises a bright RW and a dark RW. Figure 6 shows this mixed RW solution. It is seen that the component \(A\) features a four-petaled RW of larger shape coexisting with a dark RW of smaller shape, while the component \(B\) displays a dark RW of smaller shape mixing with a bright RW of bigger shape.

For larger \(N_1\), or \(N_2\), or both, the higher order mixed bounded RWs can be generated, which are composed of more fundamental RWs. For instance, by taking \(N_1 = 1, N_2 = 2\) or \(N_1 = 2, N_2 = 1\) in
The mixed RW solution (43) comprising of two first-order RW with parameters given by Equation (42) and $N_1 = 1, N_2 = 1, a_{1,1} = 0, a_{1,2} = 10$

The leftmost side panels: the mixed RW solution (33) comprising of a first-order RW and a bounded second-order RW with parameters given by Equation (42) and $N_1 = 1, N_2 = 2, a_{1,1} = 0, a_{1,2} = 0, a_{2,1} = 0, a_{3,2} = 400$. The middle panels: the mixed RW solution (33) comprising of a first-order RW and a second-order bounded RW with parameters given by Equation (42) and $N_1 = 2, N_2 = 1, a_{1,1} = 0, a_{2,1} = 0, a_{3,1} = 200i, a_{1,2} = a_{2,2} = 0$. The rightmost panels: the mixed RW solution (33) consisting of two second-order bounded RWs with with parameters (42) and $N_1 = 2, N_2 = 2, a_{1,1} = 0, a_{2,1} = 0, a_{3,1} = 500i, a_{1,2} = a_{2,2} = 0, a_{3,2} = 200$

Theorem 3, the corresponding solutions consist of a first-order RW and a second-order bounded RW (i.e., four first-order RWs), and the degree of the corresponding functions $f, g,$ and $h$ are eight in both $x$ and $t$. The leftmost and middle panels of Figure 7 display the mixed RWs with $N_1 = 1, N_2 = 2$ and $N_1 = 2, N_2 = 1$, respectively. It is seen that the component $B$ is composed of three bounded fundamental RWs of bright type and a fundamental dark RW when $N_1 = 1, N_2 = 2$ (see the leftmost panels of Figure 7), while it features three bounded fundamental RWs of dark type and a fundamental bright RW when $N_1 = 2, N_2 = 1$ (see the middle panels of Figure 7). With $N_1 = 2, N_2 = 2$, the corresponding RW solutions (33) are comprised of two bounded second-order RWs, which are displayed in the rightmost panels of Figure 7. It is seen that there are three bounded fundamental bright RWs and three bounded fundamental dark RWs in the $A$ component, and three bounded fundamental RWs of dark type mixing with three bounded RWs of four-petaled type in the $A$ component.
It is noted that such mixed RWs were studied for the two-component NLS equations, 47–52 but they have not been reported for the 2-LSRI model before, to the best of our knowledge. Additionally, in Theorem 3, the parameters $N_1, N_2$ cannot be zero, thus the $2 \times 2$ block determinant in Theorem 3 cannot degenerate into a single-block determinant. Therefore, the corresponding mixed bounded RWs cannot reduce to pure bounded higher order RWs, that is, they are always in mixed bounded states.

3.3 The degradable bounded $(\hat{N}_1, \hat{N}_2)$-th order RW solutions and their dynamics

3.3.1 The degradable bounded $(\hat{N}_1, \hat{N}_2)$-th order RW solutions

If $p_0$ is a nonimaginary double root of Equation (15), a family of the degradable bounded RW solutions to the 2-LSRI model (5) can be constructed, which are given by the following theorem.

**Theorem 4.** The 2-LSRI model (5) has the following degradable bounded $(\hat{N}_1, \hat{N}_2)$-th-order RW solutions:

$$A = \rho_1 e^{(k_1 x + (\gamma + k_2^2) t)} \frac{g}{f}, \quad B = \rho_2 e^{(k_2 x + (\gamma + k_2^2) t)} \frac{h}{f}, \quad L = \gamma - 2(\log f)_{xx},$$

(46)

where

$$f = \tau_{0,0}, g = \tau_{1,0}, h = \tau_{0,1}$$

(47)

and $\tau_{n,k}$ is defined as the following $2 \times 2$ block determinant:

$$\tau_{n,k} = \det \left( \begin{bmatrix} \tau_{1,1} & \tau_{1,2} \\ \tau_{2,1} & \tau_{2,2} \end{bmatrix} \right),$$

(48)

$$\tau_{n,k}^{[\alpha, \beta]} = \left( m_{[n,k,\alpha, \beta]}^{[\alpha, \beta]} \right)_{1 \leq s \leq \hat{N}_\alpha, 1 \leq j \leq \hat{N}_\beta},$$

(49)

for $\alpha, \beta = 1, 2$, and the matrix elements $m_{s,j}^{(n,k,\alpha, \beta)}$ are given either (a) in differential operator form:

$$m_{s,j}^{(n,k,\alpha, \beta)} = \sum_{\mu=1}^s \sum_{l=1}^j \left[ T(p) \bar{\sigma}_\alpha \right]^l \left[ T^*(p) \bar{\sigma}_\beta \right]^j \frac{m_{s,j}^{(n,k,\alpha, \beta)}}{s! j!},$$

$$p = p_0,$$

(50)

$$m_{s,j}^{(n,k,\alpha, \beta)} = \frac{1}{p + p^*} \left( \frac{p - ik_1}{p^* + ik_1} \right)^n \left( \frac{p - ik_2}{p^* + ik_2} \right)^k e^{i \pi + \xi p}.$$
or (b) through Schur polynomials:

$$
m_{s,j}^{(n,k,\alpha,\beta)} = \min(i,j) \sum_{\nu=0}^{\min(i,j)} \left[ \frac{|\lambda_1|^2}{(p_0 + p_0^*)^2} \right]^\nu S_{i-\nu}(x^+(n,k) + \nu s)S_{j-\nu}(x^-(n,k) + \nu s^*). \tag{51}$$

Here $N_1, N_2$ are arbitrary nonnegative integers. The auxiliary functions $\tau(p)$ in Equation (50) is given through $Q(p)$

$$(\tau(p)\partial_p)^3 Q(p) = Q(p), \tag{52}$$

and $\xi_\pi$ is defined by

$$\xi_\pi = px - ip^2t + \sum_{r=1}^{\infty} a_{r,\pi} \ln^r \mathcal{W}(p). \tag{53}$$

The vectors $x_{\alpha}^\pm(n,k) = (x_{1,\alpha}^\pm, x_{2,\alpha}^\pm, \ldots)$ in Equation (51) are defined as:

$$x_{r,\alpha}^+(n,k) = \lambda_{r,\alpha} x + i\beta_{r,\alpha} t + n\theta_{r,\alpha}^{(1)} + k\theta_{r,\alpha}^{(2)} + a_{r,\alpha},$$

$$x_{r,\alpha}^-(n,k) = \lambda_{r,\alpha}^* x - i\beta_{r,\alpha}^* t - n\theta_{r,\alpha}^{(1)} - k\theta_{r,\alpha}^{(2)} + a_{r,\alpha}^*, \tag{54}$$

where $\alpha, \beta, \theta_{r}^{(1)},$ and $\theta_{r}^{(2)}$ are defined in Equation (26) with $p_0$ replaced by $\beta_0$, $s = (s_1, s_2, s_3, \ldots)$ is given by Equation (27) with $p_0$ replaced by $\beta_0$, and the function $p(x)$ appearing in Equations (26) and (27) are defined by the following equation:

$$Q(p) = \frac{Q(p_0)}{3} \left[ e^x + 2e^{-x} \cos \left( \frac{\sqrt{3}}{2} x \right) \right], \tag{55}$$

where $Q$ is given by Equation (16).

Remark 6. By using the same method developed in Ref. 61, one can get that the polynomial degree of the tau functions $\tau_{n,k}$ in Theorems 2–4 are $N(N + 1), N_1(N_1 + 1) + N_2(N_2 + 1)$, and $2[N_1^2 + N_2^2 - N_1(N_2 - 1)]$ in both $x$ and $t$ variables, respectively, where $N, N_1, N_2$ are positive integers and $N_1, N_2$ are nonnegative integers.

3.3.2 Dynamics of the degradable bounded RWs

Similar to the mixed bounded RWs in Theorem 3, the RW solutions in the above theorem are also given by $2 \times 2$ block determinants. However, the $2 \times 2$ block determinants in this theorem can degenerate into single-block determinants if one takes $N_1 = 0$ or $N_2 = 0$. That is the reason we call this family of solutions degradable RW solutions. In what follows, we will reveal the dynamics of the degenerate single-block RW solutions and the nondegenerate $2 \times 2$ block solutions, respectively. For this purpose, we take the following parameters in the solutions (46) of Theorem 4:
The degenerate RW solutions (46) with $\hat{N}_1 = 0$ and parameters given by Equation (56). The left panels: $\hat{N}_2 = 1$ and $a_{1,2} = 0$, the corresponding solutions only comprise a single fundamental RW. The right panels: $\hat{N}_2 = 2$ and $a_{1,2} = 0, a_{2,2} = 0, a_{3,2} = 0, a_{4,2} = 100$, the corresponding solutions consist of four fundamental RWs

$$\bar{p}_0 = \frac{1}{4} (\sqrt{15} + 3i), \rho_1 = \frac{2\sqrt{6}}{3}, \rho_2 = \frac{5\sqrt{3}}{3}, \delta_1 = -1, \delta_2 = -1, k_1 = 1, k_2 = -\frac{1}{2}. \quad (56)$$

With this set of parameters, upon the classifications of the first-order RW solutions (31) discussed previously, the solutions in the components $A$ and $B$ are dark RWs and bright RWs, respectively.

We first consider the degenerate single-block RW solutions (i.e., $\hat{N}_1 = 0$ or $\hat{N}_2 = 0$). In the degenerate case of $\hat{N}_1 = 0$, $\hat{N}_2 \neq 0$ and the degenerate case of $\hat{N}_1 \neq 0, \hat{N}_2 = 0$, the dynamical features of the corresponding RWs are diverse. To more clearly show the difference between the two degenerate RWs, we next exhibit them.

When $\hat{N}_1 = 0, \hat{N}_2 \neq 0$, the tau solution in Equation (48) has the following form:

$$\tau_{n,k} = \det \left( \tau^{[2,2]}_{n,k} \right) = \det \left( \hat{m}^{(n,k,2,2)}_{3s-2,3j-2} \right)_{1 \leq s, j \leq \hat{N}_2}, \quad (57)$$

with $\hat{m}^{(n,k,\alpha,\beta)}_{s,j}$ being given by Equation (51). In this degenerate case, the parameter $\hat{N}_2$ determines the order of the degenerate RWs, the corresponding solutions consist of $\hat{N}_2^2$ fundamental RWs. Figure 8 shows the degenerate RWs for $\hat{N}_2 = 1$ and $\hat{N}_2 = 2$. It is seen that there is only one fundamental RW in the degenerate solutions when $\hat{N}_2 = 1$ (see the left panels). In this case, the corresponding degenerate RW solution is equivalent to the fundamental RW solutions given by Equation (28). However, in the case of $\hat{N}_2 = 2$, there are four fundamental RWs (see the right panels). As discussed previously, the mixed bounded RWs can also form an RW pattern consisting of four fundamental RWs (see the leftmost side and middle panels of Figure 7). However, the four fundamental RWs in the mixed bounded solutions are composed of three bounded RWs and a fundamental RW of different state (see the leftmost side and middle panels of Figure 7). In the degenerate solution, the four fundamental RWs are bounded (see the right panels of Figure 8).
When $\hat{N}_1 \neq 0, \hat{N}_2 = 0$, the tau solution in Equation (48) is written in the following form:

$$\tau_{n,k} = \det \left( \begin{array}{c} \tau_{n,k}^{[1,1]} \\ \tau_{n,k}^{[1,2]} \\ \tau_{n,k}^{[2,1]} \\ \tau_{n,k}^{[2,2]} \end{array} \right)_{1 \leq s, j \leq \hat{N}_1},$$

where $\hat{m}_{s,j}^{(n,k,\alpha,\beta)}$ is given by Equation (51). In this case, the parameter $\hat{N}_1$ controls the order of the corresponding degenerate RW, and there are $(\hat{N}_1^2 + \hat{N}_1)$ fundamental RWs in degenerate RW solutions. The RW solutions (46) with $\hat{N}_1 = 1$ and $\hat{N}_1 = 2$ are displayed in Figure 9. It is seen that there are two bounded fundamental RWs in the case $\hat{N}_1 = 1$ (see the left panels of Figure 9) while there are six bounded fundamental RWs when $\hat{N}_1 = 2$ (see the right panels of Figure 9). The solutions shown in Figure 6 are also composed of two fundamental RWs, but they are in different states.

Then we consider the nondegenerate RWs solutions when $\hat{N}_1 \hat{N}_2 \neq 0$, which are expressed by the $2 \times 2$ block determinants. To illustrate the dynamics of these nondegenerate RWs, we consider the case of $\hat{N}_1 = 2, \hat{N}_2 = 1$. In this case, the tau function is explicitly expressed as

$$\tau_{n,k} = \det \left( \begin{array}{cc} \tau_{n,k}^{[1,1]} & \tau_{n,k}^{[1,2]} \\ \tau_{n,k}^{[2,1]} & \tau_{n,k}^{[2,2]} \end{array} \right)$$

$$= \det \begin{pmatrix} \hat{m}_{2,2}^{(n,k,1,1)} & \hat{m}_{2,5}^{(n,k,1,1)} & \hat{m}_{2,1}^{(n,k,1,2)} \\ \hat{m}_{5,2}^{(n,k,1,1)} & \hat{m}_{5,5}^{(n,k,1,1)} & \hat{m}_{5,1}^{(n,k,1,2)} \\ \hat{m}_{1,2}^{(n,k,2,1)} & \hat{m}_{1,5}^{(n,k,2,1)} & \hat{m}_{1,1}^{(n,k,2,2)} \end{pmatrix},$$

where $\hat{m}_{s,j}^{(n,k,\alpha,\beta)}$ is given in Equation (51). The degree of these tau functions is 10 in both $x$ and $t$, thus the corresponding nondegenerate RW should be composed of five fundamental RWs, which is demonstrated in Figure 10. Because the solutions in Theorems 2 and 3 consist of $\frac{N(N+1)}{2}$ and...
FIGURE 10  The nondegenerate RW solutions (46) with $\tilde{N}_1 = 2, \tilde{N}_2 = 1$ and parameters (56) and $a_{1,1} = 0, a_{2,1} = 0, a_{3,1} = 0, a_{4,1} = 0, a_{5,1} = 500, a_{1,2} = 0$, which consist of five bounded fundamental RWs

\[
\frac{N_i(N_i+1)+N_j(N_j+1)}{2}
\]

fundamental RWs, respectively, which do not contain the solutions comprising five fundamental RWs, thus the RW solution displayed in Figure 10 is distinctive in contrast with the solutions in Theorems 2 and 3.

4  DERIVATION OF HIGHER ORDER BREATHER AND RW SOLUTIONS

In this section, we construct the general breather in Theorem 1 and RW solutions in Theorems 2, 3, and 4 to the 2-LSRI model (5) via the bilinear KP hierarchy reduction method. 87–89

4.1  Tau functions of the 2-LSRI model

The 2-LSRI model (5) is transformed into the bilinear form,

\[
\begin{align*}
(D_x^2 + 2ik_1D_x - iD_t)g \cdot f &= 0, \\
(D_x^2 + 2ik_2D_x - iD_t)h \cdot f &= 0, \\
(D_tD_x - 2\delta_1\rho_1^2 - 2\delta_2\rho_2^2)f \cdot f &= -2(\delta_1\rho_1^2|g|^2 + \delta_2\rho_2^2|h|^2),
\end{align*}
\]

through the variable transformation,

\[
A = \rho_1 e^{i(k_1x + (\gamma + k_1^2)t)} \frac{g}{f}, B = \rho_2 e^{i(k_2x + (\gamma + k_2^2)t)} \frac{h}{f}, L = \gamma - 2(\log f)_{xx},
\]

where $f$ is a real function, and $g$, $h$ are complex functions. Here, the operator $D$ is the Hirota’s bilinear differential operator 87 defined by

\[
P(D_x, D_y, D_t)F(x, y, t \cdots) \cdot G(x, y, t, \cdots) \equiv P(\partial_x - \partial_{x'}, \partial_y - \partial_{y'}, \partial_t - \partial_{t'}, \cdots)F(x', y', t', \cdots)G(x', y', t', \cdots)|_{x'=x, y'=y, t'=t},
\]

where $P$ is a polynomial of $D_x, D_y, D_t, \cdots$.

Then we start with the general tau functions for the multicomponent KP hierarchy expressed in the form of Gramian determinants. The following bilinear equation in the multicomponent KP
hierarchy,\(^90\)

\[
\begin{align*}
(D_{x_1}^2 + 2\alpha_1 D_{x_1} - D_{x_2}) \tau_{n+1,k} \cdot \tau_{n,k} &= 0, \\
(D_{x_1}^2 + 2\alpha_2 D_{x_1} - D_{x_2}) \tau_{n,k+1} \cdot \tau_{n,k} &= 0, \\
(D_{x_1} D_r - 2) \tau_{n,k} \cdot \tau_{n,k} &= -2\tau_{n+1,k} \tau_{n-1,k}, \\
(D_{x_1} D_s - 2) \tau_{n,k} \cdot \tau_{n,k} &= -2\tau_{n,k+1} \tau_{n,k-1},
\end{align*}
\]

(63)

has the following Gramian determinant tau functions:

\[
\tau_{n,k} = \text{det}_{1 \leq s, j \leq N} (m_{s,j}^{(n,k)}).
\]

(64)

Here the matrix element \(m_{ij}^{(n)}\) satisfies

\[
\begin{align*}
\frac{\partial}{\partial x_1} m_{s,j}^{(n,k)} &= \psi_s^{(n,k)} \phi_j^{(n,k)}, \\
\frac{\partial}{\partial x_2} m_{s,j}^{(n,k)} &= \psi_s^{(n+1,k)} \phi_j^{(n,k)} + \psi_s^{(n,k)} \phi_j^{(n+1,k)} - \frac{\partial^2}{\partial x_2} \phi_s^{(n,k)}, \\
\frac{\partial}{\partial x_2} \phi_s^{(n+1,k)} &= -(\partial_{x_1} + \alpha_1) \phi_s^{(n+1,k)} + \psi_s^{(n,k)} \phi_j^{(n+1,k)} - \frac{\partial^2}{\partial x_2} \phi_s^{(n,k)}, \\
\frac{\partial}{\partial x_2} \phi_s^{(n,k)} &= -(\partial_{x_1} + \alpha_1) \phi_s^{(n,k)} + \psi_s^{(n+1,k)} \phi_j^{(n,k)} - \frac{\partial^2}{\partial x_2} \phi_s^{(n,k)},
\end{align*}
\]

(65)

where \(m_{s,j}^{(n,k)}, \phi_s^{(n,k)}, \) and \(\phi_j^{(n,k)}\) are variables of \(x_1, x_2, r,\) and \(s.\)

If one restricts the above tau functions satisfying the following dimension-reduction condition:

\[
\mathcal{L}_0 \tau_{n,k} = (\delta_1 \rho_1^2 \partial_s + \delta_2 \rho_2^2 \partial_s + i \partial x_2) \tau_{n,k} = C \tau_{n,k},
\]

(66)

where

\[
\mathcal{L}_0 = (\delta_1 \rho_1^2 \partial_s + \delta_2 \rho_2^2 \partial_s + i \partial x_2),
\]

(67)

and \(C\) is some constant, then the third and fourth bilinear equations in Equation (64) would combine into the following bilinear equation:

\[
(D_{x_1} D_{x_2} - 2\delta_1 - 2\delta_2) \tau_{n,k} \cdot \tau_{n,k} = -2(\delta_1 \tau_{n+1,k} \tau_{n-1,k} + \delta_2 \tau_{n,k+1} \tau_{n,k-1}).
\]

(68)

Based on this dimension-reduction condition, the derivatives with respect to variables \(r\) and \(s\) are replaced by the derivative with respect to another variables \(x_1, x_2.\) Then in the above tau functions, \(s\) and \(r\) are just parameters that can be regarded as having any values, thus we take \(s = r = 0\) for
convenience. Applying the change of independent variables in the tau functions (65)

\[ x_1 = x, \, x_2 = -it, \, \alpha_1 = ik_1, \, \alpha_2 = ik_2, \]  

if the tau functions \( \tau_{n,k} \) satisfy the complex conjugacy condition:

\[ \tau^*_{n,k}(x, t) = \tau_{-n,-k}(x, t), \]  

then the bilinear equation (69) and the first and second bilinear equations in Equation (64) would become the bilinear equations (62) of the 2-LSRI model for

\[ f = \tau_{0,0}, \, g = \tau_{1,0}, \, g^* = \tau_{-1,0}, \, h = \tau_{0,1}, \, h^* = \tau_{0,-1}. \]  

In this way, we obtain the tau functions of the 2-LSRI model.

Choosing different forms of the matrix elements \( m_{s,j}^{(n,k)} \), we can construct the breather and RW solutions for the 2-LSRI model (5), the details of the derivations will be given in the following subsections.

4.2 Derivation of breather solutions to the 2-LSRI model

To construct the breather solutions of the 2-LSRI model (5), we take the matrix elements \( m_{s,j}^{(n,k)} \) of tau functions \( \tau_{n,k} \) given by Equation (65) in the following form:

\[ m_{s,j}^{(n,k)} = \sum_{\alpha,\beta=1}^{2} \frac{1}{p_s^{[\alpha]} + q_j^{[\beta]}} \left( -\frac{p_s^{[\alpha]} - \alpha_1}{q_j^{[\beta]} + \alpha_1} \right)^n \left( -\frac{p_s^{[\alpha]} - \alpha_2}{q_j^{[\beta]} + \alpha_2} \right)^k e^{\xi_{s}^{[\alpha]} + \eta_{s}^{[\beta]}}, \]  

\[ \psi_{s}^{(n,k)} = \sum_{\alpha=1}^{2} (p_s^{[\alpha]} - \alpha_1)^n (p_s^{[\alpha]} - \alpha_2) e^{\xi_{\alpha}}, \]  

\[ \phi_{j}^{(n,k)} = \sum_{\beta=1}^{2} (-q_j^{[\beta]} - \alpha_1)^{-n} (-q_j^{[\beta]} - \alpha_2)^{-k} e^{\eta_{\beta}}, \]  

and

\[ \xi_{s}^{[\alpha]} = \frac{1}{p_s^{[\alpha]} - \alpha_1} r + \frac{1}{p_s^{[\alpha]} - \alpha_2} s + p_s^{[\alpha]} x_1 + p_s^{[\alpha]} x_2 + \xi_{s}^{[\alpha]}, \]  

\[ \eta_{s}^{[\beta]} = -\frac{1}{q_j^{[\beta]} + \alpha_1} r - \frac{1}{q_j^{[\beta]} + \alpha_2} s + q_j^{[\beta]} x_1 - q_j^{[\beta]} x_2 + \eta_{j}^{[\beta]}, \]  

(73)
where \( p_s^{[\alpha]}, q_j^{[\beta]}, \xi_s^{[\alpha]} \), and \( \eta_j^{[\gamma]} \) are arbitrary complex parameters. Then the tau function in Equation (65) can be rewritten in the following form:

\[
\tau_{n,k} = \Lambda \det_{1 \leq s, j \leq N} \left( \frac{m_{(n,k)}}{m_{s,j}} \right),
\]

(74)

where \( \Lambda = \prod_{s=1}^{N} e^{\xi_s^{[\alpha]} + \eta_j^{[\gamma]}} \) and

\[
\frac{m_{(n,k)}}{m_{s,j}} = \frac{1}{p_s^{[1]} + q_j^{[1]}} \left( \frac{p_s^{[1]} - \alpha_1}{q_j^{[1]} + \alpha_1} \right)^{n} \left( \frac{p_s^{[1]} - \alpha_2}{q_j^{[1]} + \alpha_2} \right)^{k} e^{\xi_s^{[\alpha]} - \xi_j^{[\alpha]} - \eta_j^{[\gamma]} + \eta_j^{[\gamma]}} + \frac{1}{p_s^{[2]} + q_j^{[2]}} \left( \frac{p_s^{[2]} - \alpha_1}{q_j^{[2]} + \alpha_1} \right)^{n} \left( \frac{p_s^{[2]} - \alpha_2}{q_j^{[2]} + \alpha_2} \right)^{k} e^{\eta_j^{[\gamma]} - \eta_j^{[\gamma]}},
\]

(75)

If the parameters \( p_s^{[\alpha]}, q_j^{[\beta]} \) satisfy the constraints

\[
\frac{\delta_1 \rho_1^2}{(p_s^{[1]} - \alpha_1)(p_s^{[2]} - \alpha_2)} + \frac{\delta_2 \rho_2^2}{(p_s^{[1]} - \alpha_2)(p_s^{[2]} - \alpha_2)} - i(p_s^{[1]} + p_s^{[2]}) = 0, \tag{76}
\]

then from Equations (75), (76), we get the tau function in the form of Equation (75) satisfying the dimension-reduction condition (67). By taking the variable transformations (70) into the tau functions (75), and setting the following complex conjugacy conditions on the parameters:

\[
q_j^{[\beta]} = p_j^{[\beta]^*}, \eta_j^{[\gamma]} = \xi_j^{[\alpha]^*}, \tag{77}
\]

then we have

\[
\eta_j^{[1]} - \eta_j^{[2]} = \xi_s^{[\alpha]^*} - \xi_j^{[\alpha]^*}, m_{(n,k)}^{s,j} = m_{(n,k)}^{(-n,-k)}, \tag{78}
\]

which further yields the tau function (75) satisfying the complex conjugacy condition (71).

Thus, under the parameter constraint in Equation (77) and the variable transformations in Equation (70), the tau functions (75) reduce to the solutions of bilinear equations (62) of the 2-LSRI model (5) for \( f = \tau_{0,0}, g = \tau_{1,0}, g^* = \tau_{-1,0}, h = \tau_{0,1}, h^* = \tau_{0,-1} \). The multiplicative factor \( \Lambda \) in Equation (75) can be removed in \( \frac{\tau_{1,0}}{\tau_{0,0}} \) and \( \frac{\tau_{0,1}}{\tau_{0,0}} \), and the variables \( s, r \) are taken as zero, then we can obtain the breather solutions of the 2-LSRI model (5). Theorem 1 is then proved.
4.3 Derivation of RW solutions to the 2-LSRI model.

In this section, we construct the rational RW solutions to the 2-LSRI model by starting with matrix elements $m_{s,j}^{(n,k)}$ of tau functions $\tau_{n,k}$ (65) in the following form:

$$m_{s,j}^{(n,k)} = A_s B_j m^{(n,k)},$$

$$m^{(n,k)} = \frac{1}{p+q} \left( \frac{p-\alpha_1}{q+\alpha_1} \right)^n \left( \frac{p-\alpha_2}{q+\alpha_2} \right)^k e^{\xi+\eta},$$

$$\psi_s^{(n,k)} = A_s (p-\alpha_1)^n (p-\alpha_2)^k e^{\xi},$$

$$\phi_s^{(n,k)} = B_j (q+\alpha_1)^n (q+\alpha_2)^k e^{\eta},$$

where

$$\xi = \frac{1}{p-\alpha_2} s + \frac{1}{p-\alpha_1} r + px_1 + p^2 x_2 + \xi_0,$$

$$\eta = \frac{1}{q+\alpha_2} s + \frac{1}{q+\alpha_1} r + qx_1 - q^2 x_2 + \eta_0,$$

and $\tau(p)$ and $\hat{\tau}(q)$ are arbitrary functions of $p$ and $q$, respectively. It is easy to see that these functions also satisfy the differential and difference relations (66), thus $\tau_{n,k} = \det(m_{s,j}^{(n,k)})$ with (80) satisfies the bilinear equations (62).

Then, we constrain the tau functions $\tau_{n,k} = \det(m_{s,j}^{(n,k)})$ with (80) satisfying the dimension reduction condition (67). One can directly obtain that:

$$\mathcal{L}_0 m_{s,j}^{(n,k)} = A_s B_j m^{(n,k)} = A_s B_j \left( Q(p) + \hat{Q}(q) \right) m^{(n,k)},$$

where

$$Q(p) = \frac{\delta_1 p_1^2}{p - ik_1} + \frac{\delta_2 p_2^2}{p - ik_2} + ip^2,$$

$$\hat{Q}(q) = \frac{\delta_1 q_1^2}{q + ik_1} + \frac{\delta_2 q_2^2}{q + ik_2} - iq^2.$$

Following the works, 60, 61 if functions $\tau(p)$ and $\hat{\tau}(q)$ are taken as

$$\tau(p) = \frac{\mathcal{W}(p)}{\mathcal{W}'(p)}, \quad \hat{\tau}(q) = \frac{\hat{\mathcal{W}}(q)}{\hat{\mathcal{W}}'(q)},$$

(83)
we get $\tau(p)\partial_p = \partial_{\ln \mathcal{W}}, \hat{\tau}(q)\partial_q = \partial_{\ln \hat{\mathcal{W}}}$). Upon the Leibnitz rules, Equation (82) is rewritten as

$$\mathcal{L}_0 m_{s,j}^{(n,k)} = \sum_{\mu=0}^s \frac{1}{\mu!} \left[\left(\tau(p)\partial_p\right)^\mu \mathcal{Q}(p)\right] m_{s-\mu,j}^{(n,k)} + \sum_{l=0}^j \frac{1}{l!} \left[\left(\hat{\tau}(q)\partial_q\right)^l \hat{\mathcal{Q}}(q)\right] m_{s,j-l}^{(n,k)}.$$  

(84)

As in Ref. 61, selecting proper forms of functions $\tau(p), \hat{\tau}(q)$ can give rise to the coefficients of certain indices on the right-hand side of the above equation vanishing at some values of $p, q$. To this end, we will choose $p = p_0, q = q_0$ as the roots of the following algebraic equation:

$$\frac{\partial Q}{\partial p} = 0, \quad \frac{\partial \hat{Q}}{\partial q} = 0,$$

namely,

$$Q'(p_0) = 0, \quad \hat{Q}'(q_0) = 0.$$  

(85)

(86)

Here $p_0, q_0$ are not pure imaginary, and $p_{0R} > 0, q_{0R} > 0$.

(1) When $p_0, q_0$ are simple roots of the algebraic equation (86), we impose functions $\tau(p), \hat{\tau}(q)$ meeting the following condition:

$$\left(\tau(p)\partial_p\right)^2 Q(p) = Q(p), \quad \left(\hat{\tau}(q)\partial_q\right)^2 \hat{Q}(q) = \hat{Q}(q).$$  

(87)

From Equation (84), the above condition is rewritten as

$$\partial^2_{\ln \mathcal{W}} Q(p) = Q(p), \quad \partial^2_{\ln \hat{\mathcal{W}}} \hat{Q}(q) = \hat{Q}(q).$$  

(88)

With Equation (86) and the scaling $\mathcal{W}(p_0) = 1, \hat{\mathcal{W}}(q_0) = 1$, we obtain:

$$Q(p) = \frac{1}{2} Q(p_0) \left(\mathcal{W}(p) + \frac{1}{\mathcal{W}(p)}\right), \quad \hat{Q}(q) = \frac{1}{2} \hat{Q}(q_0) \left(\hat{\mathcal{W}}(q) + \frac{1}{\hat{\mathcal{W}}(q)}\right)$$  

(89)

and

$$\mathcal{W}(p) = \frac{Q(p) \pm \sqrt{Q^2(p) - Q^2(p_0)}}{Q(p_0)}, \quad \hat{\mathcal{W}}(q) = \frac{\hat{Q}(q) \pm \sqrt{\hat{Q}^2(q) - \hat{Q}^2(q_0)}}{\hat{Q}(q_0)}.$$  

(90)

From Equation (84), the explicit forms of $\tau(p), \hat{\tau}(q)$ are given through $Q(p), \hat{Q}(q)$ as:

$$\tau(p) = \sqrt{\frac{Q^2(p) - Q^2(p_0)}{Q^2(p)}}, \quad \hat{\tau}(q) = \sqrt{\frac{\hat{Q}^2(q) - \hat{Q}^2(q_0)}{\hat{Q}^2(q)}}.$$  

(91)
The Equation (85) becomes

\[
\mathcal{L}_0 m_{s,j}^{(n,k)} \bigg|_{p=p_0, q=q_0} = Q(p_0) \sum_{\mu=0}^{s} \frac{1}{\mu!} m_{s-\mu,j}^{(n,k)} \bigg|_{p=p_0, q=q_0} + \hat{Q}(q_0) \sum_{l=0}^{j} \frac{1}{l!} m_{s,j-l}^{(n,k)} \bigg|_{p=p_0, q=q_0}.
\]  

(92)

Then, for the restricted indices of the determinants (65) of the tau functions:

\[
\tau_{n,k} = \det_{1 \leq s, j \leq N} \left( m_{2s-1, 2j-1}^{(n,k)} \right),
\]  

(93)

upon the relation (85), the above tau functions satisfy the following relation:

\[
\mathcal{L}_0 \tau_{n,k} = \left[ Q(p_0) + \hat{Q}(q_0) \right] N \tau_{n,k},
\]  

(94)

which is nothing but the dimension reduction condition (67).

(2) When \( p_0^{(\alpha)} \), \( q_0^{(\alpha)} \) \((\alpha = 1, 2, p_0^{(1)} \neq \pm p_0^{(2)})\) are two simple roots of the algebraic equation (86), then the tau functions are expressed by the following \(2 \times 2\) block determinants:

\[
\tau_{n,k} = \det \begin{pmatrix} \tau_{[1,1]}^{(n,k)} & \tau_{[1,2]}^{(n,k)} \\ \tau_{[2,1]}^{(n,k)} & \tau_{[2,2]}^{(n,k)} \end{pmatrix},
\]  

(95)

where

\[
\tau_{[\alpha, \beta]}^{(n,k)} = \det_{1 \leq s, j \leq N_{\alpha, \beta}} \left( m_{2s-1, 2j-1}^{(n,k)} \bigg|_{p=p_0^{(\alpha)}, q=q_0^{(\beta)}} \right), 1 \leq \alpha, \beta \leq 2,
\]  

(96)

and \( m_{s,j}^{(n,k)} \) are given by Equation (80) with \([\tau(p), \hat{\tau}(q), \xi_0, \eta_0]\) replaced by \([\tau^{(\alpha)}(p), \hat{\tau}^{(\alpha)}(q), \xi_0^{(\alpha)}, \eta_0^{(\alpha)}]\), here the functions \(\tau^{(\alpha)}(p), \hat{\tau}^{(\alpha)}(q)\) are provided by (92) with \(p_0, q_0\) replaced by \(p_0^{(\alpha)}\) and \(q_0^{(\alpha)}\) for \(\alpha, \beta = 1, 2\), respectively. These tau functions (96) also satisfy the bilinear equations (64), which can be proved by the same method given in Appendix C of Ref. 61, thus the proof is omitted here.

For this \(2 \times 2\) block determinant forms of tau functions (96), the contiguity relation (85) is written as

\[
\mathcal{L}_0 m_{s,j}^{(n,k)} \bigg|_{p=p_0^{(\alpha)}, q=q_0^{(\beta)}} = Q(p_0^{(\alpha)}) \sum_{\mu=0}^{s} \frac{1}{\mu!} m_{s-\mu,j}^{(n,k)} \bigg|_{p=p_0^{(\alpha)}, q=q_0^{(\beta)}} + \hat{Q}(q_0^{(\beta)}) \sum_{l=0}^{j} \frac{1}{l!} m_{s,j-l}^{(n,k)} \bigg|_{p=p_0^{(\alpha)}, q=q_0^{(\beta)}}.
\]  

(97)

Similar to the results reported in Ref. 45, this contiguity relation can further yield:

\[
\mathcal{L}_0 \tau_{n,k} = \left\{ \left[ Q(p_0^{(1)}) + \hat{Q}(q_0^{(1)}) \right] N_1 + \left[ Q(p_0^{(2)}) + \hat{Q}(q_0^{(2)}) \right] N_2 \right\} \tau_{n,k},
\]  

(98)

namely, the tau functions (96) expressed by \(2 \times 2\) block determinant also satisfy the dimension reduction condition (67).
When \( p_0, q_0 \) are double roots of the algebraic equation (86), namely,

\[
\frac{\partial Q}{\partial p} \bigg|_{p=p_0} = \frac{\partial^2 Q}{\partial p^2} \bigg|_{p=p_0} = 0, \quad \frac{\partial \hat{Q}}{\partial q} \bigg|_{q=q_0} = \frac{\partial^2 \hat{Q}}{\partial q^2} \bigg|_{q=q_0} = 0,
\]

(99)

\( \mathcal{T}(p), \hat{\mathcal{T}}(p) \) have the same forms as in Equation (84), but they have to further meet the following condition:

\[
\left( \mathcal{T}(p) \partial_p \right)^3 Q(p) = Q(p), \left( \hat{\mathcal{T}}(q) \partial_q \right)^3 \hat{Q}(p) = \hat{Q}(q),
\]

(100)

namely,

\[
\partial^3_{\ln \mathcal{W}} Q(p) = Q(p), \partial^3_{\ln \hat{\mathcal{W}}} \hat{Q}(q) = \hat{Q}(q).
\]

(101)

Scaling \( \mathcal{W}(p_0) = \hat{\mathcal{W}}(q_0) = 1 \), a solution to above equation under conditions (100) is

\[
Q(p) = \frac{Q(p_0)}{3} \left( \mathcal{W}(p) + \frac{2}{\sqrt{\mathcal{W}(p)}} \cos \left[ \frac{\sqrt{3}}{2} \ln \mathcal{W}(p) \right] \right),
\]

(102)

\[
\hat{Q}(q) = \frac{\hat{Q}(q_0)}{3} \left( \hat{\mathcal{W}}(q) + \frac{2}{\sqrt{\hat{\mathcal{W}}(q)}} \cos \left[ \frac{\sqrt{3}}{2} \ln \hat{\mathcal{W}}(q) \right] \right).
\]

With \( Q(p), \hat{Q}(q) \) being of this form, the following tau functions expressed by block determinants:

\[
\tau_{n,k} = \det \begin{pmatrix} \tau_{1,1}^{[n,k]} & \tau_{1,2}^{[n,k]} \\ \tau_{2,1}^{[n,k]} & \tau_{2,2}^{[n,k]} \end{pmatrix},
\]

(103)

where

\[
\tau_{[\alpha, \beta]}^{[n,k]} = \text{mat} \left( m_{s, \alpha, \beta}^{(n,k)} \bigg|_{p=p_0^\alpha, q=q_0^\beta} \right), 1 \leq \alpha, \beta \leq 2,
\]

(104)

also satisfy the bilinear equations (64). This result can be proved using the method given in Appendix C of Ref. 61, thus we omit here its proof. Using the above relations, the Equation (85) becomes:

\[
\mathcal{L}_0 \mathcal{M}_{s,j}^{(n,k)} \bigg|_{p=p_0, q=q_0} = Q(p_0) \sum_{\mu=0}^s \frac{1}{\mu!} m_{s-\mu, j}^{(n,k)} \bigg|_{p=p_0, q=q_0} + \hat{Q}(q_0) \sum_{l=0}^j \frac{1}{l!} m_{s,j-l}^{(n,k)} \bigg|_{p=p_0, q=q_0},
\]

(105)
Upon this contiguity relation, then the tau function (96) expressed by 2×2 block determinant also satisfy the dimension reduction condition (67).

Finally, we impose the complex conjugacy condition

$$\tau_{n,k}^* = \tau_{-n,-k}.$$  \hfill (106)

For this purpose, we first take $\eta_0 = \xi_0^*$ in Equation (94) for a simple root, and $\eta_{0,\beta} = \xi_{0,\alpha}^*$ in Equation (97) and Equation (105) for two simple roots and a double root. Then, by applying the change of independent variables (70), it is easy to find that $\hat{Q}$ will be the conjugate of $Q$ in Equation (83) if $q = p^*$, thus $q_0^* = p_0$. When $p_0, q_0$ are single simple roots of Equation (86), then

$$m_{j,s}^{(-n,-k)}\bigg|_{p=p_0, q=q_0^*} = \left[m_{s,j}^{(n,k)}\right]^*\bigg|_{p=p_0, q=q_0^*},$$  \hfill (107)

thus the complex conjugacy condition (107) is realized. If $p_0^{[\alpha], q_0^{[\beta]}}$ are two simple single roots of Equation (86), because $q_0^{(\alpha)} = p_0^{(\alpha)*}$, thus

$$m_{j,s}^{(-n,-k)}\bigg|_{p=p_0^{(\alpha)}, q=q_0^{(\beta)*}} = \left[m_{s,j}^{(n,k)}\right]^*\bigg|_{p=p_0^{(\beta)}, q=q_0^{(\alpha)*}},$$  \hfill (108)

which can further imply:

$$\tau_{n,k}^{[\alpha,\beta]} = \tau_{-n,-k}^{[\beta,\alpha]},$$  \hfill (109)

then the complex conjugacy condition (107) is also satisfied. When $p_0$ is a double root of Equation (86), the complex conjugacy condition (107) can also be proved in a similar way.

To obtain the solutions in operator differential forms in Theorems 2–4, we take

$$\xi_0 = \sum_{r=1}^{\infty} \hat{a}_r \ln^r \mathcal{W}(p).$$  \hfill (110)

for a simple root $p_0$ of Equation (86), where $\mathcal{W}(p)$ is given by Equation (91), and

$$\xi_{0,\alpha} = \sum_{r=1}^{\infty} \hat{a}_{r,\alpha} \ln^r \mathcal{W}^{(\alpha)}(p), \alpha = 1, 2$$  \hfill (111)

for two simple roots $p_0^{(\alpha)}$ and $p_0^{(\beta)}$ of Equation (86), where $\mathcal{W}^{(\alpha)}(p)$ is defined in Equation (91) with $p_0$ being replaced by $p_0^{(\alpha)}$, and

$$\xi_{0,\alpha} = \sum_{r=1}^{\infty} \hat{a}_{r,\alpha} \ln^r \mathcal{W}(p), \alpha = 1, 2$$  \hfill (112)

for a double root of Equation (86), where $\mathcal{W}(p)$ is given by Equation (103). Here, $\hat{a}_r, \hat{a}_{r,\alpha}$ are arbitrary complex parameters. Then, we can obtain the solutions in differential operator forms in Theorems 2–4.
The last content of this section is to convert the solutions given through differential operator forms to Schur polynomials in Theorems 2, 3, and 4. For this purpose, we use the following generator $\mathcal{G}$ of differential operators $[\mathcal{T}(p)\partial_p]$ and $[\hat{\mathcal{T}}(q)\partial_q]$ introduced in Ref. 45,

$$
\mathcal{G} = \sum_{s=0}^{\infty} \sum_{j=0}^{\infty} \frac{x^s}{s!} \frac{\lambda^j}{j!} [\mathcal{T}(p)\partial_p]^s [\hat{\mathcal{T}}(q)\partial_q]^j.
$$

(113)

From the relations between $\mathcal{T}$, $\hat{\mathcal{T}}$ and $\mathcal{W}$, $\hat{\mathcal{W}}$ in Equation (84), the generator $\mathcal{G}$ can also be written as

$$
\mathcal{G} = \sum_{s=0}^{\infty} \sum_{j=0}^{\infty} \frac{x^s}{s!} \frac{\lambda^j}{j!} [\partial_{\mathcal{W}(p)}]^s [\partial_{\hat{\mathcal{W}}(q)}]^j = \exp \left( k \partial_{\ln \mathcal{W}(p)} + \lambda \partial_{\ln \hat{\mathcal{W}}(q)} \right).
$$

(114)

The operator $\mathcal{G}$ acting on an arbitrary function $\mathcal{F}(\mathcal{W}, \hat{\mathcal{W}})$ can result in the following relation:

$$
\mathcal{G} \mathcal{F}(\mathcal{W}, \hat{\mathcal{W}}) = \mathcal{F} \left( e^x \mathcal{W}, e^y \hat{\mathcal{W}} \right).
$$

(115)

Because the parameter $p$ and $q$ are related to $\mathcal{W}$ and $\hat{\mathcal{W}}$, respectively, thus one can regard $p$ as a function of $\mathcal{W}$, and $q$ as a function of $\hat{\mathcal{W}}$, namely, $p = p(\mathcal{W}), q = q(\hat{\mathcal{W}})$. Furthermore, $\mathcal{W}(p_0) = \hat{\mathcal{W}}(q_0) = 1$. For $m^{(n,k)}$ in Equation (22) of Theorem 2,

$$
\frac{1}{m^{(n,k)}} \mathcal{G} m^{(n,k)} \bigg|_{p=p_0, q=q_0} = \frac{p_0 + q_0}{p(x) + q(\lambda)} \left( \frac{p(x) - ik_1}{p_0 - ik_1} \right)^n \left( \frac{q(\lambda) + ik_2}{q_0 + ik_2} \right)^{-n} \left( \frac{p(x) - ik_1}{p_0 - ik_1} \right)^k \left( \frac{q(\lambda) + ik_2}{q_0 + ik_2} \right)^{-k}
$$

$$
\exp \left( \sum_{r=1}^{\infty} \hat{a}_{r'} x^{r'} + \hat{a}_{r''} \lambda^{r''} \right) \times \exp \left[ (p(x) - p_0 + q(\lambda) - q_0) x - i(p^2(x) - p_0^2 - q^2(\lambda) + q_0^2)t \right].
$$

(116)

Here, we have set $q = p^*$ in Equation (22). After expanding the right side of the above equation into power series of $x$ and $\lambda$, the first term can be written into the following form $^{60,61}$:

$$
\frac{p_0 + q_0}{p(x) + q(\lambda)} = \sum_{v=0}^{\infty} \left( \frac{|\lambda_1|^2}{(p_0 + q_0)^2} \lambda^{1v} \right) \exp \left( \sum_{r=1}^{\infty} (v s_r - b_r) x^{r'} + (v s_r^* - b_r^*) \lambda^{r''} \right),
$$

(117)

where $\lambda_1$ is given by Equation (26), and $b_r$ is given by the Taylor coefficient of $\lambda^{r''}$ in the expansion of

$$
\ln \left[ \frac{p(x) + q_0}{p_0 + q_0} \right] = \sum_{r=1}^{\infty} b_r x^{r'},
$$

(118)
and $s_r$ is defined in Equation (27). The rest term can be written as

$$\exp \left\{ \sum_{r=1}^{\infty} \kappa^r \left[ \lambda_r x - i \beta_r t + n \delta_r^{(1)} + k \delta_r^{(1)} \right] + \sum_{r=1}^{\infty} \lambda_r^r \left[ \lambda_r^x x + i \beta_r^x t \right. \right.$$ 

$$\left. + n \delta_r^{(1)} + k \delta_r^{(1)} \right] + \sum_{r=1}^{\infty} (\hat{a}_r^{(r)} + \hat{a}_r^{(r)} \kappa) \right\}.$$  

(119)

Combining these expansions, then Equation (117) becomes

$$\frac{1}{m(n,k)} \begin{vmatrix} m(n,k) \end{vmatrix}_{p=p_0, q=q_0}$$

$$= \sum_{v=0}^{\infty} \left( \frac{|\lambda_1|^2}{(p_0 + q_0)^2} \right)^v \exp \left( \sum_{r=1}^{\infty} (x_r^+ + vs_r) \kappa_r + \sum_{r=1}^{\infty} (x_r^- + vs^*) \lambda_r^r \right),$$

(120)

where $x_r^\pm(n, k)$ are as given by Equation (25) with

$$a_r = \hat{a}_r - b_r.$$

(121)

Taking the coefficients of $\kappa^s \lambda^j$ on both sides of Equation (121), we get

$$\frac{m^{(s,j)}_{n,k}}{m(n,k)}_{p=p_0, q=q_0} = \min(s,j) \sum_{v=0}^{\infty} \left( \frac{|\lambda_1|^2}{(p_0 + q_0)^2} \right)^v S_{s-v}(x^+(n, k) + vs)S_{j-v}(x^-(n, k) + vs^*),$$

(122)

where $m^{(s,j)}_{n,k}$ is the matrix element given by Equation (22). The right side of the above equation is the $m^{(s,j)}_{n,k}$ defined in Equation (23). Furthermore, from Equation (123) we can obtain the following relation:

$$\det_{1 \leq s, j \leq N} \left( m^{(s,j)}_{n,k} \right)_{2s-1, 2j-1} = \hat{H} \det_{1 \leq s, j \leq N} \left( m^{(s,j)}_{n,k} \right)_{2s-1, 2j-1},$$

(123)

where $\hat{H} = (m^{(s,j)}_{n,k})_{p=p_0, q=q_0}^N$. This relation indicates that the solutions given through differential operator form and through Schur polynomials in Theorem 2 are equivalent.

Similarly, we can also transform the solutions expressed by differential operator into Schur polynomials in this way. This completes the proof of Theorems 2, 3, and 4.

## 5 CONCLUSION AND DISCUSSION

In this paper, we have constructed general higher order breather and RW solutions of the 2-LSRI model (5) in the form of determinants, by means of the bilinear KP-hierarchy reduction method. We first studied the dynamics of the breather solutions. Under particular restrictions imposed on the parameters, the breather solutions can become the homoclinic orbits in the 2-LSRI model (5). It has been shown that the second-order breather solutions have three different dynamical behaviors: two breathers, second-order homoclinic orbits, and a mixture of a breather and a first-order
homoclinic orbit. We derived three families of RW solutions to the 2-LSRI model, which correspond to a simple root, two simple roots, and double roots of Equation (15) related to the dimension reduction condition. They are bounded RW, mixed bounded RWs, and degradable bounded RWs. The dynamics of these three families of RWs have been exhibited. The differences between these three families of RWs can be summarized as follows:

- The polynomial degree of the tau functions \( \tau_{n,k} \) in Theorems 2–4 are \( N(N + 1) \), \( N_1(N_1 + 1) + N_2(N_2 + 1) \) and \( 2[\tilde{N}_1^2 + \tilde{N}_2^2 - \tilde{N}_1(\tilde{N}_2 - 1)] \) in both \( x \) and \( t \) variables, respectively, where \( N, N_1, N_2 \) are positive integers and \( \tilde{N}_1, \tilde{N}_2 \) are nonnegative integers.
- The solutions in Theorems 2–4 comprise of \( \frac{N(N + 1)}{2} \), \( \frac{N_1(N_1 + 1) + N_2(N_2 + 1)}{2} \), and \( [\tilde{N}_1^2 + \tilde{N}_2^2 - \tilde{N}_1(\tilde{N}_2 - 1)] \) fundamental RWs, respectively. The RWs in Theorem 2 and Theorem 4 are bounded states, while they are a mixture of two different bounded states in Theorem 3.

We point out that the breathers and RW solutions to the 2-LSRI model have been derived earlier by using the bilinear method\(^8^0\) and the Darboux transformation.\(^7^9,8^2–8^4\) Comparing with those previously reported results, the main results obtained in this paper can be summarized as follows:

- There were only the first-order breather and first-order RW solutions given in Ref. \(^8^0\), while only the first-order RW solutions were studied in Ref. \(^7^9\). In this paper, we have constructed the general higher order breather and RW solutions in terms of determinants.
- The higher order RW solutions given in Refs. \(^8^2–8^4\) comprise \( N(N + 1)/2 \) fundamental RWs, which correspond to the family of RW solutions in Theorem 2 of this paper. The general higher order RW solutions in Theorems 3 and 4 are new RW solutions to the 2-LSRI model (5), which have not been reported before, to the best of our knowledge.

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