Some examples of aspherical 4-manifolds that are homology 4-spheres

John G. Ratcliffe and Steven T. Tschantz

Department of Mathematics, Vanderbilt University,
Nashville, TN 37240, USA

Abstract
In this paper, Problem 4.17 on R. Kirby’s problem list is solved by constructing infinitely many aspherical 4-manifolds that are homology 4-spheres.

1 Introduction
Problem 4.17, attributed to W. Thurston, on R. Kirby’s 1978 problem list [7] asks:

Can a homology 4-sphere ever be a $K(\pi, 1)$? Who knows an example of a rational homology 4-sphere which is a $K(\pi, 1)$?

Examples of aspherical rational homology 4-spheres were constructed by F. Luo [8] in 1988. In this paper we affirmatively answer the main question of Problem 4.17 by constructing infinitely many aspherical 4-manifolds that are homology 4-spheres.

In §1, we construct our examples by Dehn surgery on the complement of five linked 2-tori in the 4-sphere recently constructed by D. Ivanšić [6]. In §2, we reprove the main results of §1 by a direct homology calculation which is independent of Ivanšić’s paper [6]. In §3, we derive geometric information that determines precisely which Dehn surgeries on Ivanšić’s link complement yield aspherical homology 4-spheres. In §4, we describe some nice properties of our examples. In particular, we discuss how recent work of M. Anderson [1] implies that many of our examples are Einstein 4-manifolds.

2 Dehn surgery on Ivanšić’s linked 2-tori in the 4-sphere
Using Jørgensen-Thurston’s hyperbolic Dehn surgery theory [11, 14], it is easy to construct infinitely many aspherical 3-manifolds that are homology 3-spheres.
Simply perform Dehn surgery on the complement $M$ of a hyperbolic knot in $S^3$ with surgery coefficient of the form $1/k$, with $k$ an integer. The 3-manifold $M(k)$ obtained by Dehn surgery is a homology 3-sphere, and for all but finitely many $k$, the 3-manifold $M(k)$ supports a hyperbolic metric and therefore is aspherical. The volume of the hyperbolic 3-manifold $M(k)$ is less than the volume of the hyperbolic manifold $M$, and the volume of $M(k)$ converges to the volume of $M$ as $k$ goes to infinity. Hence there are infinitely many homotopy types of 3-manifolds of the form $M(k)$ by Mostow’s rigidity theorem. Thus there are infinitely many aspherical 3-manifolds that are homology 3-sphere.

We construct our examples by performing Dehn surgery on the hyperbolic complement $M$ of five linked 2-tori in $S^3$ recently constructed by D. Ivanšič [6]. The hyperbolic 4-manifold $M$ is the orientable double cover of the most symmetric hyperbolic 4-manifold $N$ on our list [10] of hyperbolic 4-manifolds of minimum volume. The Euler characteristic of $N$ is one, and so the Euler characteristic of $M$ is two. The hyperbolic 4-manifold $M$ has five cusps each of which is homeomorphic to $T^3 \times [0, \infty)$ where $T^3$ is the 3-torus.

Here is an outline of our argument. We show that for infinitely many Dehn surgeries on each cusp of $M$ the closed 4-manifold $\hat{M}$ obtained by Dehn surgery is a homology 4-sphere. By the Gromov-Thurston $2\pi$ theorem, for all but finitely many surgeries on each cusp of $M$, the 4-manifold $\hat{M}$ supports a Riemannian metric of nonpositive curvature, and therefore is aspherical. There are infinitely many 4-manifolds of this form by recent work of M. Anderson.

Let $\overline{M}$ be the compact 4-manifold with boundary obtained by removing disjoint horoball neighborhoods of the ideal cusp points of $M$. Then $\overline{M}$ is a strong deformation retract of $M$. The boundary of $\overline{M}$ is the disjoint union of five flat 3-tori. According to Ivanšič [6], the manifold $\overline{M}$ is homeomorphic to the complement of the interior of a closed tubular neighborhood $V$ of five disjoint 2-tori $T^2_1, \ldots, T^2_5$ in $S^3$. The 4-manifold $V$ is the disjoint union of closed disjoint tubular neighborhoods $V_1, \ldots, V_5$ of $T^2_1, \ldots, T^2_5$, respectively. Each 4-manifold $V_i$ is homeomorphic to $D^2 \times T^2$. The boundary of $V$ is the disjoint union of the boundary components $T^3_1, \ldots, T^3_5$ of $\overline{M}$ with $T^3_i = \partial V_i$ for each $i$. Let $m_i$ be a meridian of $V_i$ represented by an oriented circle for each $i$, and let $\kappa_i = [m_i]$ be the class of $m_i$ in $H_1(T^3_i)$ for each $i$. By Alexander duality, $H_1(M) \cong \mathbb{Z}^5$, and so $H_1(\overline{M}) \cong \mathbb{Z}^5$. Let

$$\ell_i : H_1(T^3_i) \to H_1(\overline{M})$$

be the homomorphism induced by inclusion for each $i$, and let $\epsilon_i = \ell_i(\kappa_i)$ for each $i$. By a Mayer-Vietoris sequence argument, $\epsilon_1, \ldots, \epsilon_5$ generate $H_1(\overline{M}) \cong \mathbb{Z}^5$.

Let $h_i : T^3_i \to T^3_i$ be an affine homeomorphism for $i = 1, \ldots, 5$. Then $h_1, \ldots, h_5$ determine an affine homeomorphism $h : \partial V \to \partial V$. The closed 4-manifold obtained by Dehn filling $\overline{M}$ according to $h_1, \ldots, h_5$ is the attaching space

$$\hat{M} = V \cup_5 \overline{M}.$$ 

Let $m'_i = h(m_i)$ for each $i$. Then $\hat{M}$ is an orientable smooth 4-manifold whose diffeomorphism type depends only on $\pm [m'_i]$ in $H_1(T^3_i)$ for each $i$. We also say
that $\hat{M}$ is the closed 4-manifold obtained by Dehn surgery on $M$ determined by the circles $m'_1, \ldots, m'_5$.

We now prove that for infinitely Dehn surgeries on each cusp of $M$, we obtain a homology 4-sphere. We use a general argument which will be simplified in §2 by explicit calculations of the homomorphisms, $\ell_i : H_1(T_i^3) \to H_1(\hat{M})$, for $i = 1, \ldots, 5$. Let $W_1 = \langle \kappa_2, \ldots, \kappa_5 \rangle$ and let

$$q_1 : H_1(\hat{M}) \to H_1(\hat{M})/W_1$$

be the quotient homomorphism. Then $q_1\ell_1 : H_1(T_1^3) \to H_1(\hat{M})/W_1$ is an epimorphism. As $H_1(\hat{M})/W_1 \cong \mathbb{Z}$, we have a split short exact sequence

$$0 \to \ker(q_1\ell_1) \to H_1(T_1^3) \to \text{Im}(q_1\ell_1) \to 0.$$

Hence there are generators $\kappa_{11}, \kappa_{12}, \kappa_{13}$ of $H_1(T_1^3)$ such that $\kappa_{11} = \kappa_1$ and $\kappa_{12}, \kappa_{13}$ generate $\ker(q_1\ell_1)$.

Let $b_1$ and $c_1$ be arbitrary integers. Then $\kappa_1 + b_1\kappa_{12} + c_1\kappa_{13}$ is a primitive element of $H_1(T_1^3)$. Hence, there is an affine homeomorphism $h_1 : T_1^3 \to T_1^3$ such that

$$[h_1(m_1)] = \kappa_1 + b_1\kappa_{12} + c_1\kappa_{13}.$$ 

Let $m'_1 = h_1(m_1)$, $\kappa'_1 = [m'_1]$, and $\ell'_1 = \ell_1(\kappa'_1)$. Then $\ell'_1 = \ell_1 + \delta_1$ with $\delta_1$ in $W_1$. Hence $\ell'_1, \ell'_2, \ldots, \ell'_5$ also generate $H_1(\hat{M})$.

Let $W_2 = \langle \ell'_1, \ell'_3, \ldots, \ell'_5 \rangle$ and let $q_2 : H_1(\hat{M}) \to H_1(\hat{M})/W_2$ be the quotient homomorphism. By the above argument, there are generators $\kappa_{21}, \kappa_{22}, \kappa_{23}$ of $H_1(T_2^3)$ such that $\kappa_{21} = \kappa_2$ and $\kappa_{22}, \kappa_{23}$ generate $\ker(q_2\ell_2)$. Let $b_2$ and $c_2$ be arbitrary integers. Then there is an affine homeomorphism $h_2 : T_2^3 \to T_2^3$ such that

$$[h_2(m_2)] = \kappa_2 + b_2\kappa_{22} + c_2\kappa_{23}.$$ 

Let $m'_2 = h_2(m_2)$, $\kappa'_2 = [m'_2]$, and $\ell'_2 = \ell_2(\kappa'_2)$. Then $\ell'_2 = \ell_2 + \delta_2$ with $\delta_2$ in $W_2$. Hence $\ell'_1, \ell'_2, \ell'_3, \ell'_4, \ell'_5$ also generate $H_1(\hat{M})$.

Continuing in this way, we obtain affine homeomorphisms $h_i : T_i^3 \to T_i^3$ such that if $m'_i = h_i(m_i)$, $\kappa'_i = [m'_i]$, and $\ell'_i = \ell_i(\kappa'_i)$, then $\ell'_1, \ldots, \ell'_5$ generate $H_1(\hat{M})$. Moreover, there are infinitely many choices for $h_1$, and for each choice of $h_1, \ldots, h_i$, there are infinitely many choices for $h_{i+1}$ for each $i = 1, \ldots, 4$. The closed 4-manifold $\hat{M}$ obtained by Dehn filling $\hat{M}$ according to $h_1, \ldots, h_5$ has $H_1(\hat{M}) = 0$ by a Mayer-Vietoris sequence argument, since $\ell'_1, \ldots, \ell'_5$ generate $H_1(\hat{M})$. The 4-manifold $\hat{M}$ is orientable, since $\hat{M}$ and $D^2 \times T^2$ are orientable. By Poincaré duality, $H_3(\hat{M}) = 0$. By a Mayer-Vietoris sequence argument, $\chi(\hat{M}) = \chi(\hat{M}) = 2$. Hence we have $H_3(\hat{M}) = 0$. Therefore $\hat{M}$ is a homology 4-sphere for infinitely many Dehn surgeries on each cusp of $M$.

As discussed by M. Anderson in §2.1 of his paper [1], the Gromov-Thurston $2\pi$ theorem [4] implies that the hyperbolic metric in the interior of $\hat{M}$ extends to a Riemannian metric on $\hat{M}$ of nonpositive curvature when $\text{length}(m'_i) \geq 2\pi$ for each $i$, in which case, $\hat{M}$ is aspherical by Cartan’s theorem. There are only finitely many homology classes of oriented circles of length less than $2\pi$ on the flat 3-torus $T_i^3$ for each $i$. Therefore $\hat{M}$ is an aspherical homology 4-sphere.
for infinitely many Dehn surgeries on each cusp of \( M \). By Proposition 3.8 of Anderson [1], at most finitely many of the 4-manifolds \( \hat{M} \), with \( \text{length}(m'_i) \geq 2\pi \) for each \( i \), are homotopically equivalent. Thus there are infinitely many homotopy types of aspherical homology 4-spheres of the form \( \hat{M} \).

3 Explicit Homology Calculations

The most symmetric manifold on our list [10] of 1171 minimum volume complete hyperbolic 4-manifolds is the nonorientable manifold, number 1011, which is denoted by \( N \) in §1. The hyperbolic 4-manifold \( N \) has symmetry group of order 320. In particular, \( N \) has a symmetry of order five that cyclically permutes its five cusps. Each cusp of \( N \) is homeomorphic to \( G \times [0, \infty) \) where \( G \) is the first nonorientable flat 3-manifold in Theorem 3.5.9 of Wolf [12].

Let \( Q \) be the regular ideal 24-cell in the conformal ball model \( B^4 \) of hyperbolic 4-space with vertices \( \pm e_i \) for \( i = 1, \ldots, 4 \) and \((\pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2)\). The 24-cell \( Q \) has 24 sides each of which is a regular ideal octahedra. The hyperbolic 4-manifold \( N \) was constructed in [10] by gluing together pairs of sides of the 24-cell \( Q \) according to the side-pairing code 14FF28.

The regular ideal 24-cell \( Q \) has 24 ideal vertices. The side-pairing of \( Q \) induces an equivalence relation on the ideal vertices of \( Q \) whose equivalence classes are called cycles. The pair of vertices \( \pm e_i \) form a cycle for each \( i = 1, \ldots, 4 \) and \((\pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2)\) is the cycle corresponding to the fifth ideal cusp point of \( N \).

Let \( \overline{N} \) be the compact 4-manifold with boundary obtained by removing disjoint horocusp neighborhoods of the ideal cusp points of \( N \) which are invariant under the group of symmetries of \( N \). The compact 4-manifold \( \overline{N} \) is a strong deformation retract of \( N \). The five components of \( \partial \overline{N} \) are isometric copies of the flat 3-manifold \( G \).

Let \( \overline{Q} \) be the truncated regular ideal 24-cell obtained by removing from \( Q \) the disjoint horoball neighborhoods of the ideal vertices of \( Q \) corresponding to the horocusp neighborhoods of the ideal cusp points of \( N \) which are removed from \( N \) to form \( \overline{N} \). Then \( \overline{Q} \) is a compact 4-dimensional polytope with 48 sides, 24 cubical sides and 24 truncated octahedral sides. The side-pairing of \( Q \) determines a side-pairing of the truncated octahedral sides of \( \overline{Q} \) whose quotient space is the 4-manifold \( \overline{N} \) with boundary.

The side-pairing of the octahedral sides of \( \overline{Q} \) determines a side-pairing of the 24 cubical sides of \( \overline{Q} \) whose quotient space is the boundary of \( \overline{N} \). Pairs of antipodal cubes, corresponding to \( \pm e_i \), for \( i = 1, \ldots, 4 \), are glued together along their sides to form the first four components of \( \partial \overline{N} \), and the remaining 16 cubes, corresponding to the vertices \((\pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2)\), are glued together along their sides to form the fifth component of \( \partial \overline{N} \).

The cell structure of the truncated 24-cell \( \overline{Q} \) together with the side-pairing
of the octahedral sides of $\overline{Q}$ determines a cell structure for $\overline{N}$. From this cell structure, we computed in [10] the homology groups, $H_1(N) \cong \mathbb{Z}_2^5$, $H_2(N) \cong \mathbb{Z}_2^2$, and $H_3(N) = 0$.

Ivanšić’s link complement is homeomorphic to the orientable double cover $M$ of $N$. All the symmetries of $N$ lift to symmetries of $M$. In particular, $M$ has a symmetry of order five that cyclically permutes its five cusps. Each cusp of $M$ is homeomorphic to $T^3 \times [0, \infty)$ where $T^3$ is a 3-torus.

Let $Q'$ be a regular ideal 24-cell that is obtained from $Q$ by reflecting in a side of $Q$. Then the hyperbolic 4-manifold $M$ can be constructed by gluing together pairs of sides of $Q$ and $Q'$. The side-pairing of $Q$ and $Q'$ determines a side-pairing of the octahedral sides of the two corresponding truncated 24-cells $Q$ and $Q'$ whose quotient space is a compact 4-manifold $\overline{M}$ with boundary which is the orientable double cover of $N$. The compact 4-manifold $\overline{M}$ is a strong deformation retract of $M$. The cell structure of the two truncated 24-cells $\overline{Q}$ and $\overline{Q}'$ together with the side-pairing of the octahedral sides of $\overline{Q}$ and $\overline{Q}'$ determines a cell structure for $\overline{M}$. From this cell structure, we computed the homology groups, $H_1(M) \cong \mathbb{Z}_2^5$, $H_2(M) \cong \mathbb{Z}^{10}$, and $H_3(M) \cong \mathbb{Z}^4$. It is worth noting that this homology calculation agrees with calculation of the homology groups of $M$, as the complement of five disjoint 2-tori in $S^4$, by Alexander duality.

Let $k_i : \pi_1(T^3_i) \to \pi_1(\overline{M})$ be the homomorphism induced by inclusion for each $i = 1, \ldots, 5$. Some care must be taken with choices of base points but we will suppress base points to simplify notation. We explicitly compute the homomorphisms $k_i : \pi_1(T^3_i) \to \pi_1(\overline{M})$ in terms of the side-pairing transformations of the convex fundamental domain $Q \cup Q'$ for the hyperbolic 4-manifold $M$. We then explicitly compute the homomorphisms $\ell_i : H_1(T^3_i) \to H_1(\overline{M})$ by abelianizing the homomorphisms $k_i : \pi_1(T^3_i) \to \pi_1(\overline{M})$.

We derive a group presentation for $\pi_1(\overline{M})$ with generators the side-pairing transformations of the convex fundamental domain $Q \cup Q'$ for the hyperbolic 4-manifold $M$ by Poincaré’s fundamental polyhedron theorem. For each $i = 1, \ldots, 4$, we form a fundamental domain for $T^3_i$ from the cubical side of $Q$ corresponding to $e_i$ and transforms of three cubical sides of $\overline{Q}$ and $\overline{Q}'$. We derive a group presentation for $\pi_1(T^3_i)$ by Poincaré’s fundamental polyhedron theorem. We then explicitly compute the homomorphism $k_i : \pi_1(T^3_i) \to \pi_1(\overline{M})$ by rewriting the generators of the presentation for $\pi_1(T^3_i)$ in terms of the generators of the presentation for $\pi_1(M)$ for each $i = 1, \ldots, 4$. The last homomorphism $k_5 : \pi_1(T^3_5) \to \pi_1(\overline{M})$ is computed in the same way except that we require 32 cubes to assemble a fundamental domain for $T^3_5$.

We found generators $\kappa_{i2}, \kappa_{i3}$ for $H_1(T^3_i)$ for $i = 1, \ldots, 5$ and generators $\epsilon_1, \ldots, \epsilon_5$ for $H_1(\overline{M})$ such that $\ell_i(\kappa_{i1}) = \epsilon_i$ for each $i$ and $\kappa_{i2}, \kappa_{i3}$ generate $\ker(\ell_i)$ for each $i$.

Let $b_i, c_i$ be arbitrary integers for $i = 1, \ldots, 5$. By the argument at the end of §2, the closed 4-manifold $\hat{M}$ obtained by Dehn filling $\overline{M}$ according to the affine homeomorphisms $h_i : T^3_i \to T^3_i$ so that

$$[h_i(m_i)] = \kappa_{i1} + b_i\kappa_{i2} + c_i\kappa_{i3}.$$
for each $i$, is a homology 4-sphere; moreover, $\hat{M}$ is aspherical if the length of the circle $m'_i = h_i(m_i)$ is at least $2\pi$ for each $i$.

4 The Geometry of Ivanšić’s link complement

This section will appear in version 2.

5 Properties of our Aspherical Homology 4-spheres

Smooth 4-manifolds that are homology 4-spheres have some nice topological properties. Every smooth homology 4-sphere $X$ has zero signature and is a spin manifold with a unique spin structure. By a Theorem of T. Cochran [5], every smooth homology 4-sphere $X$ smoothly embeds in $\mathbb{R}^5$.

Let $M$ be the closed 4-manifold obtained by Dehn surgery on $M$ determined by the circles $m'_1, \ldots, m'_5$. If $\text{length}(m'_i) \geq 2\pi$ for each $i$, then $\hat{M}$ admits a Riemannian metric of nonpositive curvature. This implies that the universal cover of $\hat{M}$ is diffeomorphic to $\mathbb{R}^4$ by Cartan’s theorem; in which case $\hat{M}$ is aspherical, and so $\hat{M}$ is a $K(\pi, 1)$. If $\hat{M}$ is an aspherical homology 4-sphere, then $\pi_1(M)$ is an ultra super perfect group, since $H_i(\pi) = 0$ for $i = 1, 2, 3$.

M. Anderson [1] has proved that if the length of $m'_i$ is sufficiently large for each $i$ and the lengths of $m'_1, \ldots, m'_5$ are weakly balanced, in the sense that

$$\max_i(\text{length}(m'_i)) \leq \exp(c \min_i(\text{length}(m'_i))^3)$$

for some small constant $c$, then the closed 4-manifold $\hat{M}$ admits an Einstein metric $g$ so that $\text{Ric}_g = -3g$. Moreover the volume of $(\hat{M}, g)$ is less than the hyperbolic volume of $M$ and the volume of $(\hat{M}, g)$ approaches the volume of $M$ as $\min_i(m'_i)$ goes to infinity. Thus infinitely many of the aspherical homology 4-spheres of the form $\hat{M}$ admit an Einstein metric $g$ so that $\text{Ric}_g = -3g$.

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References

[1] M.T. Anderson, Dehn filling and Einstein metrics in higher dimensions, arXiv:math.DG /0303260 v3. 17 Oct 2003.

[2] M. Anderson, S. Carlip, J.G. Ratcliffe, S. Surya, S.T. Tschantz, Peaks in the Hartle-Hawking wave function from sums over topologies, arXiv:gr-qc /0310002 v3. 19 Nov 2003.

[3] R. Benedetti and C. Petronio, Lectures on Hyperbolic Geometry, Springer-Verlag, Berlin, 1992.
[4] S.A. Bleiler and C.D. Hodgson, Spherical space forms and Dehn filling, *Topology* **35** (1996), 809-833.

[5] T. Cochran, Embedding 4-manifolds in $S^5$, *Topology* **23** (1984), 257-269.

[6] D. Ivanšić, Hyperbolic structure on a complement of tori in the 4-sphere, *Adv. Geom.* **4** (2004).

[7] R. Kirby, Problems in low dimensional manifold theory, *Proc. Symposia Pure Math.* **32** (1978), 273-312.

[8] F. Luo, The existence of $K(\pi, 1)$ 4-manifolds which are rational homology 4-spheres, *Proc. Amer. Math. Soc.* **104** (1988), 1315-1321.

[9] J.G. Ratcliffe *Foundations of Hyperbolic Manifolds*, Graduate Texts in Math., vol. **149**, Springer-Verlag, Berlin, Heidelberg, and New York, 1994.

[10] J.G. Ratcliffe and S.T. Tschantz, The volume spectrum of hyperbolic 4-manifolds, *Experiment. Math.* **9** (2000), 101-125.

[11] W.P. Thurston, *The Geometry and Topology of 3-Manifolds*, Princeton Univ. (1979), www.msri.org/publications/books/gt3m/

[12] J.A. Wolf, *Spaces of Constant Curvature*, Publish or Perish, Houston, 1974.