BLACK-HOLE SOLUTION WITHOUT CURVATURE SINGULARITY

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An exact solution of the vacuum Einstein field equations over a nonsimply-connected manifold is presented. This solution is spherically symmetric and has no curvature singularity. It can be considered to be a regularization of the Schwarzschild solution over a simply-connected manifold, which has a curvature singularity at the center. Spherically symmetric collapse of matter in $\mathbb{R}^4$ may result in this nonsingular black-hole solution, if quantum-gravity effects allow for topology change near the center.

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1. Introduction

The vacuum Einstein field equations over $\mathcal{M}_4 = \mathbb{R}^4$ have a spherically symmetric solution, the Schwarzschild black-hole solution $^1$ with its maximal extension $^2$, $^3$, $^4$.

Recently, a related solution has been found for the vacuum Einstein field equations over $\tilde{\mathcal{M}}_4$, a particular nonsimply-connected manifold $^5$.

The goal of the present article is to discuss the connection between these two vacuum solutions. The comparison of these two solutions will, in fact, suggest a black-hole-type solution, where the curvature singularity is eliminated by a space-time defect, i.e., a “hole” in spacetime.

2. Singular black-hole solution

The extended Schwarzschild–Kruskal–Szekeres solution $^6$ has a metric given by the following line element ($G_N = c = 1$):

$$ds^2 \bigg|_{SKS}^{(r > 0)} = 32 M^3 \exp \left( -\frac{r}{2M} \right) \left( -dv^2 + du^2 \right) + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right), \quad (1a)$$

with $\theta \in [0, \pi], \phi \in [0, 2\pi)$, and $r$ implicitly given in terms of the coordinates $u \in \mathbb{R}$ and $v \in \mathbb{R}$ by the relation

$$\left( \frac{r}{2M} - 1 \right) \exp \left[ \frac{r}{2M} \right] \bigg|_{SKS} = u^2 - v^2 > -1, \quad (1b)$$

for $r > 0$. The explicit expression for $r$ from $(1b)$ is

$$r \bigg|_{SKS} = 2M \left( 1 + W_0 \left[ \frac{u^2 - v^2}{e} \right] \right), \quad (1c)$$
in terms of the principal branch of the Lambert $W$–function, $W_0[z]$, which gives the principal solution for $w$ in $z = w \exp[w]$. Recall that $W_0[x]$ is real for $x \geq -1/e$.

The solution (1) has $M > 0$ as a free parameter and the corresponding topology is

$$\mathcal{M}_{\text{SKS}} = \mathbb{R}^2 \times S^2.$$  \hspace{1cm} (2)

The solution approaches a physical singularity for $r \to 0$, as shown by the divergence of the Kretschmann scalar (defined in terms of the Riemann curvature tensor),

$$K \bigg|_{\text{SKS}} = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \bigg|_{\text{SKS}} = 48 \frac{M^2}{r^6}. \hspace{1cm} (3)$$

Further details can be found in, e.g., Ref. 4.

### 3. Nonsingular black-hole solution

Consider, now, the vacuum Einstein field equations over

$$\tilde{\mathcal{M}}_4 = \mathbb{R} \times \tilde{\mathcal{M}}_3,$$  \hspace{1cm} (4a)

where the 3-space $\tilde{\mathcal{M}}_3$ is a noncompact, orientable, nonsimply-connected manifold without boundary. Specifically, there is the following homeomorphism:

$$\tilde{\mathcal{M}}_3 \simeq \mathbb{R}P^3 - \{\text{point}\}, \hspace{1cm} (4b)$$

with $\mathbb{R}P^3$ the 3-dimensional real projective plane, which is topologically equivalent to a 3-sphere with antipodal points identified. Here, and in the following, $\tilde{\mathcal{M}}$ with tilde indicates a nonsimply-connected manifold [having a nontrivial first homotopy group, $\pi_1(\tilde{\mathcal{M}}) \neq 0$], whereas $\mathcal{M}$ without tilde stands for a simply-connected manifold [having $\pi_1(\mathcal{M}) = 0$].

The explicit construction of $\tilde{\mathcal{M}}_3$ has been given in Ref. 5: start from $\mathbb{R}^3$ with the standard Cartesian coordinates and the Euclidean metric and, then, perform surgery on $\mathbb{R}^3$ by removing the interior of a ball $B_b$ with radius $b$ and identifying antipodal points on the boundary of the ball, $\partial B_b = S^2_b$ (2-sphere with radius $b$). In Ref. 5 also appropriate coordinate charts are reviewed. There is, for example, one chart (labeled $n = 2$) with ‘radial’ coordinate $Y_2 \in (\infty, +\infty)$ instead of the standard radial coordinate $r \in [b, \infty)$ and angular coordinates $(X_2, Z_2)$ having restricted ranges. (The fundamentally different nature of the coordinate $Y_2$ compared to that of the coordinate $r$ will play an important role later on.) For the moment, the focus will be on these chart-2 coordinates and the subscript ‘2’ will simply be omitted.

The new exact vacuum solution over $\tilde{\mathcal{M}}_4$ involves two parameters, $b$ and $M$, which, for the present purpose, are taken to be related as follows:

$$0 < b < 2M.$$  \hspace{1cm} (5)

The defect solution for $2M < b$ has been given by Eqs. (4.1), (5.1b), and (5.1c) in Ref. 5 for chart-2 coordinates and inspection shows it to be precisely of the form of
the standard exterior-region Schwarzschild solution if $Y^2 + b^2$ and $r^2$ are identified. This observation agrees with Birkhoff’s theorem[3] and, moreover, allows us to use some of the techniques of the Kruskal–Szekeres procedure in our search for a solution with parameters (5).

It turns out that the Kruskal–Szekeres procedure is useful to eliminate the apparent singularities at the Schwarzschild horizon but is not suited to deal with the spacetime defect. For this reason, we will present our black-hole-type solution without curvature singularity in terms of two sets of coordinates, one set of coordinates appropriate to the spacetime defect and another set of coordinates further out.

Start, then, with an Ansatz in terms of Kruskal–Szekeres-type coordinates, valid for $\zeta > b$ and having a metric given by the following line element:

$$ds^2\bigg|_{(\zeta > b)} = 32 M^3 \frac{\exp\left[-\zeta/(2M)\right]}{\zeta} \left(-dV^2 + dU^2\right) + \zeta^2 \left(dZ^2 + \sin^2 Z \ dX^2\right), \quad (6a)$$

with $\zeta$ given in terms of $U \in \mathbb{R}$ and $V \in \mathbb{R}$ by

$$\zeta = 2M \left(1 + W_0 \left[\frac{U^2 - V^2}{e}\right]\right) = \sqrt{b^2 + Y^2}; \quad (6b)$$

where the last equality follows from the relations

$$U = \begin{cases} \left[\sqrt{Y^2 + b^2/(2M)} - 1\right]^{1/2} \exp\left[\sqrt{Y^2 + b^2/(4M)}\right] \cosh[T/(4M)], \\ \left[1 - \sqrt{Y^2 + b^2/(2M)}\right]^{1/2} \exp\left[\sqrt{Y^2 + b^2/(4M)}\right] \sinh[T/(4M)], \end{cases} \quad (6c)$$

$$V = \begin{cases} \left[\sqrt{Y^2 + b^2/(2M)} - 1\right]^{1/2} \exp\left[\sqrt{Y^2 + b^2/(4M)}\right] \sinh[T/(4M)], \\ \left[1 - \sqrt{Y^2 + b^2/(2M)}\right]^{1/2} \exp\left[\sqrt{Y^2 + b^2/(4M)}\right] \cosh[T/(4M)], \end{cases} \quad (6d)$$

taking the top/bottom entries in (6c) and (6d) for the exterior/interior regions (labeled I/II in Sec. 31.5 of Ref. 4).

Next, write the Ansatz (6a) in terms of the chart-2 coordinates $\{Y, T\} \in (-\infty, +\infty)$ by using (6b) and the bottom-row expressions on the right-hand sides of (6c) and (6d). In this way, it is possible to obtain the interior metric with the defect at $Y = 0$ included,

$$ds^2\bigg|_{\text{chart-2}}^{(0 \leq Y^2 < 4M^2 - b^2)} = + \left(\frac{2M}{\sqrt{b^2 + Y^2}} - 1\right) dT^2 - \left(\frac{2M}{\sqrt{b^2 + Y^2}} - 1\right)^{-1} \frac{Y^2}{b^2 + Y^2} dY^2 + (b^2 + Y^2) \left(dZ^2 + \sin^2 Z \ dX^2\right). \quad (7a)$$

Similarly, one obtains from the identical Ansatz (6a), but now using the top-row
expressions on the right-hand sides of (6c) and (6d), the exterior metric,
\[ ds^2 \big|_{\text{chart-2}} = -\left(1 - \frac{2M}{\sqrt{b^2 + Y^2}}\right) dT^2 + \left(1 - \frac{2M}{\sqrt{b^2 + Y^2}}\right)^{-1} \frac{Y^2}{b^2 + Y^2} dY^2 + (b^2 + Y^2) \left(dZ^2 + \sin^2 Z \, dX^2\right), \quad (7b) \]
which takes the same form as (7a). For the \( \zeta = 2M \) boundary between the exterior and interior spacetime regions mentioned below (6d), the Ansatz (6a) in terms of \((V, U)\) coordinates can be kept as it is,
\[ ds^2 \big|_{\text{chart-2}} = \frac{32}{M^3} \exp\left[-\frac{\zeta}{(2M)}\right] \left(-dV^2 + dU^2\right) + \zeta^2 \left(dZ^2 + \sin^2 Z \, dX^2\right), \quad (7c) \]
where \( \zeta \) is now given by (6b) and \( \Delta L \) is a small enough positive length, so that \( b < 2M - \Delta L \). The coordinate transformations for the overlap regions between (7c) and (7a) or (7b) are given by (6c) and (6d). Note that, strictly speaking, (7b) could be omitted if the range of (7c) is extended to \( \zeta \in (b, \infty) \), with the case \( \zeta = b \) covered by (7a). But, for now, we continue with (7) as it stands.

The timelike coordinate \( T \) and spacelike coordinate \( Y \) in the exterior metric (7b) become, respectively, spacelike and timelike in the interior metric (7a). This behavior is analogous to what happens for the standard Schwarzschild solution (cf. Fig. 31.1 in Ref. 4). Note, however, that the timelike coordinate \( Y \) of the interior metric (7a) ranges from \(-\infty\) to \(+\infty\), unlike the usual radial coordinate \( r \). Moreover, this timelike coordinate \( Y \) is part of a topologically nontrivial manifold (see below), which may have important physics implications as will be discussed in Sec. 4.

The Riemann curvature tensor \( R^\kappa{}_{\lambda\mu\nu}(T, X, Y, Z) \) from (7a) is found to be even in \( Y \) and finite at \( Y = 0 \); see the explicit expressions given by Eq. (5.4) in Ref. 5. The Ricci tensor \( R_{\mu\nu}(T, X, Y, Z) \) from (7) vanishes identically and the same holds for the Ricci scalar \( R(T, X, Y, Z) \). Hence, the vacuum Einstein equations are solved.

With the Riemann tensor of the vacuum solution (7), the Kretschmann scalar is found to be given by
\[ K \equiv R^{\kappa}{}_{\lambda\mu\nu} R_{\mu\nu\rho\sigma} = 48 \frac{M^2}{\zeta^6}, \quad (8) \]
with \( \zeta^2 = b^2 + Y^2 \) for the interior metric (7a) or the exterior metric (7b) and \( \zeta \) given by (6b) for the boundary metric (7c). For all three metrics from (7), the Kretschmann scalar remains finite because \( b > 0 \).

As discussed in Sec. 2 of Ref. 5, the proper description of \( \tilde{M}_3 \) requires three coordinate charts. In the present context, there are, then, the coordinates
(V_n, U_n, X_n, Z_n) and (T_n, X_n, Y_n, Z_n), for n = 1, 2, 3, with (V_2, U_2, X_2, Z_2) = (V, U, X, Z) as used in (7c) and (T_2, X_2, Y_2, Z_2) ≡ (T, X, Y, Z) as used in (7b). Hence, the interior spacetime manifold from (7a) extended to all charts and the exterior spacetime manifold from (7b) extended to all charts have the same topology, namely, \( \tilde{M}_4 \) from (4a) in terms of \( \tilde{M}_3 \) from (4b).

4. Discussion

The solution (7) of the vacuum Einstein field equations for the topology (4) is the main result of this article. The metrics (7b) and (7c) are, of course, not new, as these metrics correspond to the standard Schwarzschild–Kruskal–Szekeres metric. New is the interior metric (7a) for the coordinate \( Y \in (-\infty, \infty) \) of the nonsimply-connected manifold \( \tilde{M}_3 \), as this metric describes the spacetime defect at \( \zeta = \sqrt{b^2 + Y^2} = b \), which effectively eliminates the curvature singularity at \( \zeta = 0 \).

Radial geodesics \( Y(T) \) passing through \( Y = 0 \) have been obtained numerically and analytically for the metric (7a) with parameters \( 0 < b < 2M \). (The leading functional behavior of these radial geodesics at \( Y = 0 \) is found to be the same as that of the radial geodesics for the flat defect metric with parameters \( M = 0 \) and \( b > 0 \).) For the spacetime region inside the Schwarzschild event horizon, \( Y^2 < 4M^2 - b^2 \), the coordinate \( Y \) from the metric (7a) is timelike and the coordinate \( T \) spacelike, similar to what happens for the standard Schwarzschild metric. Now, however, this timelike coordinate \( Y \) is part of the manifold \( \tilde{M}_3 \) from (4b), which gives rise to closed timelike curves (CTCs). These CTCs imply all possible horrors, but, classically, these horrors remain confined within the Schwarzschild horizon. Whether or not CTCs in the interior region are physically acceptable depends on the behavior of the matter fields. Further discussion of this issue can be found in Ref. 6. At this moment, let us simply return to the vacuum solution (7).

Purely mathematically, the nonsingular vacuum solution (7) with parameter \( b > 0 \) can be considered to be a “regularization” of part of the singular vacuum solution (1), with all the surprises this may entail. It is, however, also possible that this nonsingular solution appears in a physical context.

Start from a nearly flat spacetime (metric approximately equal to the Minkowski metric and trivial topology \( \mathbb{R}^4 \)), where a large amount of matter with total mass \( M \) is arranged to collapse in a spherically symmetric way. Within the realm of classical Einstein gravity, one expects to end up with part of the singular solution (1); see, for example, Fig. 32.1b of Ref. 4. But, very close to the final curvature singularity, something else may happen due to quantum mechanics.

Considering a precursor mass \( \Delta M \sim \hbar/(bc) \ll M \) and using typical curvature values from (5) and (8), the local spacetime integral of the action density related to (1) differs from that related to (7) by an amount \( \lesssim \hbar \). Then, as argued by Wheeler (cf. Secs. 34.6, 43.4, and 44.3 of Ref. 4 and references therein), the local
topology of the manifold may change by a quantum jump if $b$ is sufficiently close to $L_{\text{Planck}} \equiv (\hbar G_N/c^3)^{1/2}$, resulting in a transition from a simply-connected manifold to a nonsimply-connected manifold. If the transition amplitude between the different topologies is indeed nonzero for appropriate matter content, quantum mechanics can operate a change between the classical solution (1) and the classical solution (7), thereby removing the curvature singularity.

**Note Added**

The flat spacetime metric (7a) for the case $M = 0$ and $b \neq 0$ does not describe a smooth Lorentzian manifold. There is no diffeomorphism (invertible one-to-one $C^\infty$ function) which maps the spacetime near $Y = 0$ to a part of Minkowski spacetime. The same holds for the general case, $M \in \mathbb{R}$ and $b \neq 0$. This violation of the elementary-flatness condition appears to be one of the surprises from the “regularization,” as mentioned in the third paragraph of Sec. 4.

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