An Experimentally Accessible Geometric Measure for Entanglement in 3-qubit Pure states

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Right from its inception, quantum information fraternity is confronted with two basic questions:

(i) Given a multipartite quantum state (possibly mixed), how to find out whether it is entangled or separable?

(ii) Given an entangled state, how to decide how much entangled it is?
Answers to both these questions are known for bipartite pure states.

(i) If $\rho_A^2 = \rho_A$ then $\rho$ is separable.

(ii) Typical measure is the entanglement entropy

$$E(|\psi\rangle) = S(\rho_A) = - \sum_i \lambda_i \ln \lambda_i$$

Zero for separable states, $\ln N$ for maximally entangled states.
Multipartite states

General answers to both these questions are not knowns. Different types of entanglement. Many separability criteria are proposed. Example: Generalizations of Peres-Horodecki criterion. The genuine entanglement of pure multipartite quantum state is established by checking whether it is entangled in all bipartite cuts, which can be tested using Peres-Horodecki criterion.
For mixed states this strategy does not work because there are mixed states which are separable in all bipartite cuts but are genuinely entangled [PRL 1999, 82, 5385]. A direct and independent detection of genuine multipartite entanglement is lacking.
In this talk I present a new measure of entanglement for 3-qubit pure states. I present all the results for N-qubit pure states except one which we could prove only for two and three qubit pure states.
Let $\rho$ act on $H$; $\text{dim}(H) = d$. $\rho \in L(H)$; scalar product $(A, B) = Tr(A^\dagger B)$. $\text{dim}(L(H)) = d^2$.

$\rho$ can be expanded in any orthonormal basis of $L(H)$.

The basis comprising $d^2 - 1$ generators of $SU(d)$ is particularly useful:

$\{I_d, \lambda_i; \ i = 1, 2, \cdots, d^2 - 1\}$. 
{λ_i} are traceless Hermition operators satisfying

\[ Tr(λ_iλ_j) = 2δ_{ij} \]

and

\[ λ_iλ_j = \frac{2}{d} δ_{ij} l_d + if_{ijk} λ_k + g_{ijk} λ_k \]

\( f_{ijk}, g_{ijk} \) are completely antisymmetric (symm.) tensors.

\( d = 2: \)

\[ \lambda_i \leftrightarrow σ_i; \quad f_{ijk} = ε_{ijk} \quad (\text{Levi-civita}) \quad g_{ijk} = 0. \]
\[ \rho \text{ expanded in this basis:} \]

\[ \rho = \frac{1}{d}(I_d + \sum_i s_i \lambda_i) \quad (A) \]

where \( s_i = \langle \lambda_i \rangle = Tr(\rho \lambda_i) \) is the average value of the \( i \)th generator \( \lambda_i \) in the state \( \rho \).
Bloch Vectors

The vector \( s = (s_1, s_2, \ldots , s_{d^2-1}) \); \( s_i = \langle \lambda_i \rangle \) is called the Bloch vector of state.

The correspondence \( s \leftrightarrow \rho \)

via the expansion of \( \rho \) in (A) is one-to-one. Thus we can use \( s \) to specify a quantum state.
Note that $\mathbf{s}$ is very easily accessible experimentally because all the averages can be directly computed using the outputs of measurements of $\{\lambda_i\}; \ i = 1, 2, \cdots, d^2 - 1$. In fact the Bloch vector $\mathbf{s}$ can be obtained experimentally even if the form of $\rho$ is not known.
Bloch vector space

Bloch vectors for a given system live in $\mathbb{R}^{d^2-1}$.

If we put an arbitrary vector $\in \mathbb{R}^{d^2-1}$ in equation (A) we may not get a valid density operator.

A density operator has to satisfy

(i) $Tr \rho = 1$ (ii) $\rho = \rho^{\dagger}$

(iii) $x^{\dagger} \rho x \geq 0 \ \forall x \in \mathbb{C}$
So the problem is to find the set of Bloch vectors in $\mathbb{R}^{d^2-1}$, called Bloch vector space $B(\mathbb{R}^{d^2-1})$.

This problem is solved only for $d = 2$:

The Bloch vector space is a ball of unit radius in $\mathbb{R}^3$, known as the Bloch ball.
For $d > 2$, the problem is still open. However for pure states ($\rho^2 = \rho$) the following relations hold:

$$\|s\|_2 = \sqrt{\frac{d(d-1)}{2}}; \quad s_is_jg_{ijk} = (d-2)s_k \quad (A')$$

It is known that

$$D_r(\mathbb{R}^{d^2-1}) \subseteq B(\mathbb{R}^{d^2-1}) \subseteq D_R(\mathbb{R}^{d^2-1})$$

$$r = \sqrt{\frac{d}{2(d-1)}} \quad R = \sqrt{\frac{d(d-1)}{2}}$$
Bloch Representation of a Multipartite state

We construct the basis of $L(H)$ which is the product of individual bases comprising generators of $SU(d_k)$; $k = 1, 2, \cdots, N$. $k, k_i$: a subsystem chosen from $N$ subsys. 

$$\{I_{d_{k_i}}, \lambda_{\alpha_{k_i}}\}; \quad \alpha_{k_i} = 1, 2, \cdots, d_{k_i}^2 - 1$$ is the basis of $\mathbb{C}^{d_{k-i}^2}$, comprising the generators of $SU(d_{d_{k_i}})$. 
Define, for subsystems $k_1$ and $k_2$

\[
\lambda_{\alpha_{k_1}}^{(k_1)} = (I_d \otimes I_d \otimes \cdots \otimes \lambda_{\alpha_{k_1}} \otimes I_{d_{k_1+1}} \otimes \cdots \otimes I_{d_N})
\]

\[
\lambda_{\alpha_{k_2}}^{(k_2)} = (I_d \otimes I_d \otimes \cdots \otimes \lambda_{\alpha_{k_2}} \otimes I_{d_{k_2+1}} \otimes \cdots \otimes I_{d_N})
\]

\[
\lambda_{\alpha_{k_1}}^{(k_1)} \lambda_{\alpha_{k_2}}^{(k_2)} = (I_d \otimes I_d \otimes \cdots \otimes \lambda_{\alpha_{k_1}} \otimes I_{d_{k_1+1}} \otimes \cdots \otimes \lambda_{\alpha_{k_2}} \otimes I_{d_{k_2+1}} \otimes \cdots \otimes I_{d_N})
\]

$\lambda_{\alpha_{k_1}}$ and $\lambda_{\alpha_{k_2}}$ occur at the $k_1$th and $k_2$th places and are the $\lambda_{\alpha_{k_1}}$th and $\lambda_{\alpha_{k_2}}$th generators of $SU(d_{k_1})$, $SU(d_{k_2})$ respectively.
In this basis we can expand $\rho$ as

$$\rho = \frac{1}{\prod_{k}^{N} d_k} \left\{ \bigotimes_{k}^{N} I_{d_k} + \sum_{k \in \mathcal{N}} \sum_{\alpha_k} s_{\alpha_k} \lambda^{(k)}_{\alpha_k} + \sum \{k_1, k_2\} \sum_{\alpha_{k_1} \alpha_{k_2}} t_{\alpha_{k_1} \alpha_{k_2}} \lambda^{(k_1)}_{\alpha_{k_1}} \lambda^{(k_2)}_{\alpha_{k_2}} + \cdots + \sum \{k_1, k_2, \ldots, k_M\} \sum_{\alpha_{k_1} \alpha_{k_2} \cdots \alpha_{k_M}} t_{\alpha_{k_1} \alpha_{k_2} \cdots \alpha_{k_M}} \lambda^{(k_1)}_{\alpha_{k_1}} \lambda^{(k_2)}_{\alpha_{k_2}} \cdots \lambda^{(k_M)}_{\alpha_{k_M}} + \cdots + \sum_{\alpha_1 \alpha_2 \cdots \alpha_N} t_{\alpha_1 \alpha_2 \cdots \alpha_N} \lambda^{(1)}_{\alpha_1} \lambda^{(2)}_{\alpha_2} \cdots \lambda^{(N)}_{\alpha_N} \right\}. \quad (B)$$

$B$ is called the Bloch representation of $\rho$.

$s^{(k)} = [s_{\alpha_k}]^d_{\alpha_k=1} :$ Bloch vector for $k$th subsystem.
\binom{N}{M} \) terms in the sum \( \sum \{k_1, k_2, \ldots, k_M\} \)

Each contains a tensor (\(M\)-way array) of order \(M\)

\[
\begin{align*}
t_{\alpha k_1 \alpha k_2 \ldots \alpha k_M} &= d_{k_1} d_{k_2} \ldots d_{k_M} \frac{1}{2^M} \text{Tr}[\rho \lambda_{\alpha k_1}^{(k_1)} \lambda_{\alpha k_2}^{(k_2)} \cdots \lambda_{\alpha k_M}^{(k_M)}] \\
&= d_{k_1} d_{k_2} \ldots d_{k_M} \frac{1}{2^M} \text{Tr}[\rho k_1 k_2 \ldots k_M (\lambda_{\alpha k_1} \otimes \lambda_{\alpha k_2} \otimes \cdots \otimes \lambda_{\alpha k_M})] \\
\mathcal{I}\{k_1, k_2, \ldots, k_M\} &= [t_{\alpha k_1 \alpha k_2 \ldots \alpha k_M}]
\end{align*}
\]

The tensor in the last term is \( \mathcal{I}^{(N)} \).
Outer product of vectors

Let \( u^{(1)} , u^{(2)} , \ldots , u^{(M)} \) be vectors in \( \mathbb{R}^{d_1^2-1} , \mathbb{R}^{d_2^2-1} , \ldots , \mathbb{R}^{d_M^2-1} \).

The outer product \( u^{(1)} \odot u^{(2)} \odot \cdots \odot u^{(M)} \) is a tensor of order \( M \), (\( M \)-way array), defined by

\[
t_{i_1 i_2 \cdots i_M} = u^{(1)}_{i_1} u^{(2)}_{i_2} \cdots u^{(M)}_{i_M} ; \quad 1 \leq i_k \leq d_k^2 - 1 , \quad k = 1, 2, \ldots , M.
\]
We need the following result

A pure $N$-partite state with Bloch representation $(B)$ is fully separable (product state) if and only if 
\[ T^{(N)} = s^{(1)} \circ s^{(2)} \circ \ldots \circ s^{(N)} \]
where $s^{(k)}$ is the Bloch vector of $k$th subsystem reduced density matrix.
We propose the following measure for $N$-qubit pure state entanglement.
$$E(\rho) = \frac{(||T^{(N)}|| - 1)}{R}$$
where the normalization constant $R$ is given by
$$R = (1 + \frac{1}{4}(1 + (-1)^N)^2 + \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} (\binom{N}{2k})^{1/2} - 1$$
where $R = ||T^{(N)}|| - 1$ calculated for $N$-qubit GHZ state as shown below.
The general $GHZ$ state is

\[ |\psi\rangle = \sqrt{p}|0\cdots0\rangle + \sqrt{1-p}|1\cdots1\rangle \]

For this state the elements of $\mathcal{T}^{(N)}$ are given by

\[ t_{i_1 i_2 \cdots i_N} = \langle \psi | \sigma_{i_1} \otimes \cdots \otimes \sigma_{i_N} | \psi \rangle \]

Using this, the norm of $\mathcal{T}^{(N)}$ for the state $|\psi\rangle\langle\psi|$ is given by

\[ ||\mathcal{T}^{(N)}||^2 = 4p(1-p) + (p + (-1)^N(1-p))^2 + 4p(1-p) \sum_{k=1}^{\lfloor N/2 \rfloor} \binom{N}{2k} \]
For maximally entangled state $p = \frac{1}{2}$

$$R = \| \mathcal{T}^{(N)} \| - 1$$

$$= (1 + \frac{1}{4}(1 + (-1)^N)^2 + \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \binom{N}{2k})^{1/2} - 1$$

$$E(\rho_{GHZ}) = \frac{1}{R} \left[ (4p(1-p) + (p + (-1)^N(1-p)))^2 + 4p(1-p) \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \binom{N}{2k})^{1/2} - 1 \right]$$

$E$ as a function of $p$ is plotted in the next slide. Note that $E(\rho) \geq 0$ for general $N$-qubit GHZ state.
The $|W\rangle$ state

\[
|W\rangle = \frac{1}{\sqrt{N}} \sum_j |0 \ldots 01_j 0 \ldots 0\rangle
\]

\[
|\tilde{W}\rangle = \frac{1}{\sqrt{N}} \sum_j |1 \ldots 10_j 1 \ldots 1\rangle
\]

where the $j$th summand has a single 1 for $|W\rangle$ and single 0 for $|\tilde{W}\rangle$ at the $j$th bit.
For both the states we get

$$||T^{(N)}||^2 = 1 + 4\frac{N-1}{N}$$

so that,

$$E(|W\rangle) = E(|\tilde{W}\rangle) = \frac{1}{R}(\sqrt{1 + 4\frac{N-1}{N}} - 1).$$

$E(|W\rangle) = E(|\tilde{W}\rangle)$ is to be expected as these are $LU$ equivalent.
Figure 2 shows the variation of $E$ with weight $s$ in the state

$$|\psi_s\rangle = \sqrt{s} |W\rangle + \sqrt{1-s} e^{i\phi} |\tilde{W}\rangle$$

Note that the entanglement is independent of $\phi$. 
figure 2
Figure 3 shows the variation of $E$ with weight $s$ in the state

$$|\chi_s\rangle = \sqrt{s}|GHZ\rangle + \sqrt{1-s} e^{i\phi}|W\rangle$$

Note again that the entanglement is independent of $\phi$. 
figure 3
An entanglement measure must have the following basic properties
(a) (i) \( E(\rho) \geq 0 \) (ii) \( E(\rho) = 0 \) if and only if \( \rho \) is separable
(b) Monotonicity under probabilistic LOCC.
(c) Convexity,
\[
E(p\rho + (1 - p)\sigma) \leq pE(\rho) + (1 - p)E(\sigma)
\]
with \( p \in [0, 1] \).
We prove these properties for our measure one by one.
Proposition 1: Let $\rho$ be a $N$-qubit pure state with Bloch representation (B). Then, $||T^{(N)}|| = 1$ if and only if $\rho$ is a product state.
By the result we have quoted
\[ T^{(N)} = s^{(1)} \circ s^{(2)} \circ \ldots \circ s^{(N)} \]

Taking norm on both sides
\[ \|T^{(N)}\|^2 = \langle T^{(N)}, T^{(N)} \rangle = \prod_i \langle s_i, s_i \rangle = \prod_i \|s_i\|^2 = 1 \]

Immediatly it follows that \( N \)-qubit pure state \( \rho \) has \( E(\rho) = 0 \) if and only if \( \rho \) is a product state.
Proposition 2: For two and three qubit states $\| \mathcal{T}^{(N)} \| \geq 1$
We prove this by directly computing $\| \mathcal{T}^{(N)} \|$ for the general two and three qubit states.
Consider, the general two qubit state

\[ |\psi\rangle = a_1|00\rangle + a_2|01\rangle + a_3|01\rangle + a_4|11\rangle, \]

\[ \sum_i |a_i|^2 = 1. \]

\[ ||T^{(2)}||^2 = 1 + 8(a_2a_3 - a_1a_4)^2 \geq 1 \]
Consider, the general Schmidt form of three qubit state

$$|\psi\rangle = \lambda_0 |000\rangle + \lambda_1 e^{i\phi} |100\rangle + \lambda_2 |101\rangle + \lambda_3 |110\rangle + \lambda_4 |111\rangle, \ \lambda_i \geq 0, \ \sum_i |\lambda_i|^2 = 1.$$ 

By direct calculation of $||\mathcal{T}^{(3)}||$ we get

$$||\mathcal{T}^{(3)}||^2 \geq 1 + 12\lambda_0^2 \lambda_4^2 + 8\lambda_0^2 \lambda_2^2 + 8\lambda_0^2 \lambda_3^2 + 8(\lambda_0^2 \lambda_3 - \lambda_1 \lambda_4)^2 \geq 1$$
For any two and three qubit pure states $\rho$

$$E(\rho) \geq 0.$$ 

We conjecture that $\|\mathcal{T}^{(N)}\| \geq 1$ for any $N$-qubit pure state.
Proposition 3: Let $U_i$ be a local unitray operator acting on the Hilbert space of $i$th subsystem.

If $\rho' = (\bigotimes_{i=1}^N U_i) \rho (\bigotimes_{i=1}^N U_i^\dagger)$

then $\|\mathcal{T}'^{(N)}\| = \|\mathcal{T}^{(N)}\|$. 
Proposition 4: $E(\rho)$ is LOCC invariant.

This follows from proposition 3 and the result due to Bennett et al. that $N$-partite pure state is LOCC invariant if and only if it is LU invariant [PRA 2000, 63 012307].
Convexity

\[ E(p|\psi\rangle\langle\psi| + (1 - p)|\phi\rangle\langle\phi|) \]

\[ = \frac{1}{R} (\|pT^{(N)}_{|\psi\rangle} + (1 - p)T^{(N)}_{|\phi\rangle}\| - 1) \]

\[ \leq \frac{1}{R} (p\|T^{(N)}_{|\psi\rangle}\| + (1 - p)\|T^{(N)}_{|\phi\rangle}\| - 1) \]

\[ = pE(|\psi\rangle) + (1 - p)E(|\phi\rangle) \]
Continuity

\[ \| (|\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|) \| \rightarrow 0 \]
\[ \Rightarrow \left| E(|\psi\rangle) - E(|\phi\rangle) \right| \rightarrow 0 \]

\[ \| (|\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|) \| \rightarrow 0 \]
\[ \Rightarrow \| T^{(N)}_\psi - T^{(N)}_\phi \| \rightarrow 0 \]

But \[ \| T^{(N)}_\psi - T^{(N)}_\phi \| \geq \| \| T^{(N)}_\psi \| - \| T^{(N)}_\phi \| \| \]
Therefore \( \| T^{(N)}_\psi - T^{(N)}_\phi \| \to 0 \)

\[ \Rightarrow \| T^{(N)}_\psi \| - \| T^{(N)}_\phi \| \to 0 \]

\[ \Rightarrow \left| E(|\psi\rangle) - E(|\phi\rangle) \right| \to 0 \]