Persistence and almost periodic problems for Non-autonomous predator system with generalized functional response functions

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Abstract. The classical functional response function is generalized to a generalized functional response function for a three-group mixed non-autonomous predator system with both predator and competitive relationships. Through the constructor and the qualitative knowledge of ordinary differential equations, this paper studies the continuity and global asymptotic stability of the system and further discusses the existence and stability of the positive almost periodic solution of the system, and obtains sufficient conditions for the existence of unique, globally asymptotically stable positive almost periodic solutions of the system, and the existing conclusions has been extended to a large extent.

1. Introduction

In recent years, the periodic solution of the predator-prey system has been widely concerned by many scholars, and the sufficient conditions for the survival of the population and the existence of positive periodic solutions with global asymptotic stability are obtained [1-3]. In many cases, especially considering the effects of seasonality and population reproduction, a more realistic and more general predator model should be almost periodic. As we all know, the almost periodic phenomenon is a special case of the periodic phenomenon, and it is a kind of phenomenon which is more extensive than the periodic phenomenon, and thus has a high research value. Predator-prey system has a variety of predator functional response functions for predators, the most common of which are Holling II and Holling III [4], which has been studied in many literatures. Recently, a predator-prey system with Holling IV functional response function \( \frac{ax}{1+bx^2} \) and a general Holling functional response function \( \frac{ax^n}{1+bx^m} \) (\( n \in N \)) [5] has been proposed and studied, and the existence and unique conditions of the limit cycle are obtained. For the first time, this paper generalizes the Holling functional response function to a more general form: \( \frac{ax^n}{1+bx^m} \) (\( m,n \in N, m \geq n \)).

Considering the persistence and almost periodic solutions of a three-group hybrid system with such a functional response function, we can choose a non-autonomous system.
\[
\begin{align*}
\dot{x} &= x[b_1(t) - a_1(t)x - e_1(t)y - \frac{d_1(t)zx^{n-1}}{c_1(t) + x^m}], \\
\dot{y} &= y[b_2(t) - e_2(t)x - a_2(t)y - \frac{d_2(t)zy^{n-1}}{c_2(t) + y^m}], \\
\dot{z} &= z[-b_3(t) + \frac{k_1(t)d_1(t)x^n}{c_1(t) + x^m} + \frac{k_2(t)d_2(t)y^n}{c_2(t) + y^m} - a_3(t)z].
\end{align*}
\] (1)

Among them \(a_i(t), b_i(t), i = 1, 2, 3, e_i(t), k_j(t), c_j(t), d_j(t), j = 1, 2\) is a non-negative continuous almost periodic function, and \(a_i(t), b_i(t), i = 1, 2, 3, c_j(t), j = 1, 2\) is strictly positive. There is both a predation relationship and a competition relationship among the three populations of system (1). The population \(z\) takes the population \(x\) and the population \(y\) as the prey, and there is resource competition among the prey populations.

For continuous positive almost periodic functions \(g(t)\) on the interval \([0, +\infty)\), denoted by \(g^L = \min_{t \in (0, +\infty)} g(t)\), \(g^M = \max_{t \in (0, +\infty)} g(t)\). According to the system (1), we can get

\[
\begin{align*}
\begin{cases}
  x(t) &= x(0)\exp\left(\int_{0}^{t}[b_1(s) - a_1(s)x(s) - e_1(s)y(s) - \frac{d_1(s)zx^{n-1}}{c_1(s) + x^m}]ds\right), \\
y(t) &= y(0)\exp\left(\int_{0}^{t}[b_2(s) - e_2(s)x(s) - a_2(s)y(s) - \frac{d_2(s)zy^{n-1}}{c_2(s) + y^m}]ds\right), \\
z(t) &= z(0)\exp\left(\int_{0}^{t}[-b_3(s) + \frac{k_1(s)d_1(s)x^n}{c_1(s) + x^m} + \frac{k_2(s)d_2(s)y^n}{c_2(s) + y^m} - a_3(s)z(s)]ds\right).
\end{cases}
\end{align*}
\]

Therefore, if \(x(0) > 0, y(0) > 0, z(0) > 0\), and whenever \(t \geq 0\), we have \(x(t) > 0, y(t) > 0, z(t) > 0\), \(R^3 = \{(x, y, z) | x > 0, y > 0, z > 0\}\) is a positive invariant set of system (1), in any solution with positive initial conditions for system (1) is always positive.

2. Persistence

**Theorem 1** Assume that system (1) meets the following conditions

\[
(H_1) \quad b_1^L > e_1^M, y^M + \frac{d_1^M x^M}{c_1^L}, \quad (H_2) \quad b_2^L > e_2^M, x^M + \frac{d_2^M y^M}{c_2^L},
\]

\[
(H_3) \quad b_3^M < \frac{k_1^L d_1^L (x^L)^n}{c_1^M + (x^M)^m} + \frac{k_2^L d_2^L (y^L)^n}{c_2^M + (y^M)^m}.
\]

and \(x^L, x^M, y^L, y^M\) is seen by the values taken in the proof process below, then system (1) is permanently persistent\(^7\).

**Proof** By knowledge of \(\dot{x} \leq x(b_1^M - a_1^L x)\), if \(0 < x(0) \leq \frac{b_1^M}{a_1^L} = x^M\), whenever \(t \geq 0\), we have \(x(t) \leq x^M\). By knowledge of \(\dot{y} \leq y(b_2^M - a_2^L y)\), if \(0 < y(0) \leq \frac{b_2^M}{a_2^L} = y^M\), whenever \(t \geq 0\), we have \(y(t) \leq y^M\). By knowledge of \(\dot{z} \leq z[-b_3^L + \frac{k_1^M d_1^M (x^M)^n}{c_1^L} + \frac{k_2^M d_2^M (y^M)^n}{c_2^L} - a_3^L z]\), if
0 < z(0) \leq \frac{-b_1 c_1 L c_2 + c_2 k_L d_1 M (x^M)^n + c_1 k_L d_2 M (y^M)^n}{a_3 c_1 c_2} \Delta z^M, \text{ whenever } t \geq 0, \text{ we have } z(t) \leq z^M.

At the same time, because of \( \dot{x} \geq x[b_1 L - a_1 M x - e_1 M y - d_1 M z^M (x^M)^{n-1}] \),
\[
\dot{y} \geq y[b_2 L - e_2 M y - d_2 M z^M (y^M)^{n-1}],
\end{equation*}
\[
\dot{z} \geq [b_3 L + k_1 d_1 L (x^L)^n + k_2 d_2 L (y^L)^n - a_3 M z^L],
\end{equation*}
As long as \( x(0) \geq \frac{b_1 L c_1 L - e_1 M y c_1 L - d_1 M z^M (x^M)^{n-1}}{a_3 c_1 c_2} = x^L, \)
\[
y(0) \geq \frac{b_2 L c_2 L - e_2 M y c_2 L - d_2 M z^M (y^M)^{n-1}}{a_3 c_1 c_2} = y^L,
\]
\[
z(0) \geq \frac{1}{a_3} [b_3 L + k_1 d_1 L (x^L)^n + k_2 d_2 L (y^L)^n - d_2 M z^M (y^L)^{n-1}] = z^L,
\end{equation*}
Whenever \( t \geq 0, \) we have \( x(t) \geq x^L, \ z(t) \geq z^L, \ y(t) \geq y^L. \)

Under the assumption of the theorem, \( 0 < x^L < x^M, 0 < y^L < y^M, 0 < z^L < z^M \) can be easily got, as a result, set \( K_1 = \{(x, y, z) \in [0, x^L] \times [0, x^M] \times [0, y^L] \times [0, y^M] \times [0, z^L] \times [0, z^M] \} \) is the invariant set of system (1), and system (1) is permanently persistent.

3. Global asymptotic stability

The global asymptotic stability of system (1) is discussed below.

**Theorem 2** If system (1) satisfies the condition in theorem 1 \((H_1), (H_2), (H_3)\) and the following three conditions:

\begin{equation*}
(H_4) \quad a_1(t) > \frac{d_1 M (x^M)^{2(n-1)}}{c_1^2} f_2(x^M, x^M) + k_4 c_3 d_4 f_3(x^M, x^M) - \frac{k_4 d_4 (x^L)^{2n} f_5(x^L, x^L)}{(c_1 + x^M)^2} - e_2(t),
\end{equation*}
\begin{equation*}
(H_5) \quad a_2(t) > \frac{z^M f_4(y^M, y^M) + k_5 c_3 d_4 f_7(y^M, y^M)}{c_2^2} - \frac{d_2 c_3 z^L f_3(y^L, y^L) + k_5 d_4 (y^L)^{2n} f_8(y^L, y^L)}{(c_2 + y^M)^2} - e_2(t),
\end{equation*}
\begin{equation*}
(H_6) \quad a_3(t) > -\frac{c_3 d_4 (x^L)^{n-1} + d_4 (x^L)^{m+n-1}}{(c_1 + x^M)^2} - \frac{c_5 d_3 (y^L)^{n-1} + d_4 (y^L)^{m+n-1}}{(c_2 + y^M)^2}.
\end{equation*}

system (1) is globally asymptotically stable.

**Proof** Since system (1) satisfies the condition \((H_1), (H_2), (H_3)\), it can be known from theorem 1 that there is a region \( K_1 \), which is the final bounded region of the solution of system (1). Therefore, it can be assumed that \( U(t) = (u(t), v(t), w(t)) \) is a positive solution of system (1) in region \( K_1 \), and \( X(t) = (x(t), y(t), z(t)) \) any solution of the system (1) that has a positive initial condition.
\(x(t_0) > 0, y(t_0) > 0, z(t_0) > 0\), then it can be assumed that
\[
\vec{x}(t) = \ln x(t), \quad \vec{y}(t) = \ln y(t), \quad \vec{z}(t) = \ln z(t), \quad \vec{u}(t) = \ln u(t), \quad \vec{v}(t) = \ln v(t), \quad \vec{w}(t) = \ln w(t),
\]

(2)

It can be known from \(X(t), U(t) \in K_1\), that \((\vec{x}(t), \vec{y}(t), \vec{z}(t)), (\vec{u}(t), \vec{v}(t), \vec{w}(t)) \in R^3_+\), then
\[
\frac{d}{dt}(\vec{x} - \vec{u}) = -a_1(t)(u - x) - e_1(t)(v - y) + \frac{d_1c_1z^{n-1} - wu^{n-1}}{(c_1 + x^m)(c_1 + u^m)}
\]
\[
= -a_1(t)(u - x) - e_1(t)(v - y) + \frac{d_1wz^{n-1}u^{n-1}f_1(u, x) - d_1z^{n-1}u^{n-1}f_1(u, x)}{(c_1 + x^m)(c_1 + u^m)}
\]

Where \(f_1(u, x) = x^{n-1} + u^{n-1} + u^{n-2}\), \(f_1(u, x) = x^{n-1} + u^{n-1} + u^{n-2}\).

By a similar method we can get
\[
\frac{d}{dt}(\vec{y} - \vec{v}) = -a_2(t)(v - y) - e_2(t)(u - x)
\]
\[
\frac{d}{dt}(\vec{w} - \vec{z}) = -a_3(t)(w - z) + k_1d_1c_1f_1(u, x) - x^{n-1}f_1(u, x)(u - x) + k_2d_2c_2f_2(v, y) - v^{n-1}f_2(v, y)(v - y)
\]

With Liapunov function \(V(t) = \left|\vec{u}(t) - \vec{x}(t)\right| + \left|\vec{v}(t) - \vec{y}(t)\right| + \left|\vec{w}(t) - \vec{z}(t)\right|\), considering the upper right derivative of \(V(t)\), we can get
\[
D^+V(t) = D^+(\left|\vec{u}(t) - \vec{x}(t)\right| + \left|\vec{v}(t) - \vec{y}(t)\right| + \left|\vec{w}(t) - \vec{z}(t)\right|)
\]
\[
= \frac{\vec{u}(t) - \vec{x}(t)}{|\vec{u}(t) - \vec{x}(t)|}\left|\vec{u}(t) - \vec{x}(t)\right| + \frac{\vec{v}(t) - \vec{y}(t)}{|\vec{v}(t) - \vec{y}(t)|}\left|\vec{v}(t) - \vec{y}(t)\right| + \frac{\vec{w}(t) - \vec{z}(t)}{|\vec{w}(t) - \vec{z}(t)|}\left|\vec{w}(t) - \vec{z}(t)\right|
\]
\[
\leq -\alpha \left|x(t) - x(t)\right| + \left|v(t) - y(t)\right| + \left|w(t) - z(t)\right|.
\]

Here take
\[
\alpha = \min \{a_1(t) + e_1(t)
\]
\[
- \frac{d_1z^M(x^M)^{n-1}f_2(x^M, x^M) + k_1c_1d_1f_1(x^M, x^M) + k_2c_2d_2f_1(x^M, x^T)}{(c_1 + x^M)^2}
\]
\[
e_1(t) + a_2(t) - \frac{z^Mf_4(y^M, y^M) + k_1c_1d_1f_1(y^M, y^M) + k_2c_2d_2f_1(y^M, y^M)}{(c_2 + y^M)^2}
\]
\[
c_1d_1(x^T)^{n-1} + d_1(x^T)^{m+1} + d_2(y^M)^{n+1} + c_2d_2(y^M)^{n+1}
\]
\[
+ a_3(t) > 0.
\]

It can be seen that \(X(t) = (x(t), y(t), z(t))\) is stable in the sense of Liapunov.

From integral (3), it can be obtained that
\[
V(t) + \alpha \int (|u(s) - x(s)| + |v(s) - y(s)| + |w(s) - z(s)|)ds \leq V(0) < +\infty.
\]
\[ \lim_{t \to +\infty} \sup \left\{ \left( |u(s) - x(s)| + |v(s) - y(s)| + |w(s) - z(s)| \right) ds \right\} < \frac{V(0)}{\alpha} < +\infty. \]

So the positive solution \( X(t) = (x(t), y(t), z(t)) \) is globally asymptotically stable.

4. Existence and uniqueness of almost periodic solutions

In order to study the existence and uniqueness of almost periodic solutions of system (1), consider the following product system

\[
\begin{align*}
\dot{x} &= x[b_1(t) - a_1(t)x - e_1(t)y - \frac{d_1(t)zx^{n-1}}{c_1(t) + x^m}], \\
\dot{y} &= y[b_2(t) - e_2(t)x - a_2(t)y - \frac{d_2(t)zy^{n-1}}{c_1(t) + y^m}], \\
\dot{z} &= z[-b_3(t) + \frac{k_1(t)d_1(t)x^n}{c_1(t) + x^m} + \frac{k_2(t)d_2(t)y^n}{c_1(t) + y^m} - a_3(t)z], \\
\dot{u} &= u[b_1(t) - a_1(t)u - e_1(t)v - \frac{d_1(t)zu^{n-1}}{c_1(t) + u^m}], \\
\dot{v} &= v[b_2(t) - e_2(t)u - a_2(t)v - \frac{d_2(t)v^{n-1}}{c_1(t) + v^m}], \\
\dot{w} &= w[-b_3(t) + \frac{k_1(t)d_1(t)u^n}{c_1(t) + u^m} + \frac{k_2(t)d_2(t)v^n}{c_1(t) + v^m} - a_3(t)w].
\end{align*}
\]

Lemma 1 Let \( D \) be an open set of \( R^3 \), function \( V(t, x, y) \) is defined as \( R_+ \times D \times D \), which satisfies:

(i) \( a(\|x - y\|) \leq V(t, x, y) \leq b(\|x - y\|) \), where \( a(r), b(r) \) is a positive definite function that increases continuously;

(ii) \( \|V(t, x_1, y_1) - V(t, x_2, y_2)\| \leq K(\|x_1 - x_2\| + \|y_1 - y_2\|) \), \( K > 0 \) is a constant;

(iii) \( V_0(t, x, y) \leq -cV(t, x, y) \), where \( c > 0 \) is a constant.

Moreover, if the original system (1) has a solution located in the compact set \( S \subseteq D \) when \( t \geq t_0 > 0 \), then the system (1) has a unique and uniformly asymptotically stable almost periodic solution \( p(t) \) in \( S \), satisfying \( \text{mod}(p(t)) = \text{mod}(f) \).

Theorem 3 If the almost periodic system (1) satisfies the condition \((H_1), (H_2), (H_3), (H_4), (H_5), (H_6)\), then the system (1) has a unique positive almost periodic solution and is globally asymptotically stable.

Proof since the system (1) satisfies the condition \((H_1), (H_2), (H_5)\), it is known by theorem 1 that the solution of the system (1) is finally bounded.

\[ K_1 = \{(x, y, z)| 0 < x^L \leq x \leq x^M, 0 < y^L \leq y \leq y^M, 0 < z^L \leq z \leq z^M \} \]

For this final bounded area, obviously, \( K_1 \) is a compact set. Let \( X(t), U(t) \) be the solution of the product system (4) on \( K_1 \times K_1 \cdot X(t), U(t) \) as set by equation (2).

Considering function Liapunov
\[ V(t) = V(t, X(t), U(t)) = \left| \tilde{r}(t) - \overline{x}(t) \right| + \left| \tilde{v}(t) - \overline{y}(t) \right| + \left| \overline{w}(t) - \overline{z}(t) \right| , \]

take
\[ a(r) = b(r) = \left| \tilde{r}(t) - \overline{x}(t) \right| + \left| \tilde{v}(t) - \overline{y}(t) \right| + \left| \overline{w}(t) - \overline{z}(t) \right| , \]

and \( a(r), b(r) \) are continuously increasing positive definite functions, then \( V(t, X(t), U(t)) \) satisfies the conditions (i) of Lemma 1.

Also because
\[
\left| \tilde{r}_1(t) - \overline{x}_1(t) \right| + \left| \tilde{r}_2(t) - \overline{x}_2(t) \right| + \left| \tilde{v}_1(t) - \overline{y}_1(t) \right| + \left| \tilde{v}_2(t) - \overline{y}_2(t) \right| + \left| \overline{w}_1(t) - \overline{z}_1(t) \right| + \left| \overline{w}_2(t) - \overline{z}_2(t) \right| \leq
\]
\[
\left| \tilde{r}_1(t) - \overline{r}_1(t) \right| + \left| \tilde{r}_2(t) - \overline{r}_2(t) \right| + \left| \tilde{v}_1(t) - \overline{v}_1(t) \right| + \left| \tilde{v}_2(t) - \overline{v}_2(t) \right| + \left| \overline{w}_1(t) - \overline{w}_1(t) \right| + \left| \overline{w}_2(t) - \overline{w}_2(t) \right| \leq
\]
\[ V(t, X(t), U(t)) \text{ satisfies the conditions (ii) of Lemma 1.} \]

In order to verify condition (iii) of Lemma 1, calculate the upper right derivative of \( V(t, X(t), U(t)) \) with respect to system (4).

Since the system (1) meets the conditions \((H_1), (H_2), (H_3), (H_4), (H_5), (H_6)\), \( D^V V(t) \) can be obtained by equation (3)
\[
D^V V(t) \leq -\alpha \left( |u(t) - x(t)| + |v(t) - y(t)| + |w(t) - z(t)| \right)
\]
\[= -\alpha \left( \exp \{ \tilde{r}(t) \} - \exp \{ \tilde{x}(t) \} \right) + \left| \exp \{ \tilde{v}(t) \} - \exp \{ \tilde{y}(t) \} \right| + \left| \exp \{ \tilde{w}(t) \} - \exp \{ \tilde{z}(t) \} \right|, \]

By the mean value theorem of differential, and by \( X(t), U(t) \) being bounded on \( K_1 \), we know that \( \tilde{x}(t), \tilde{y}(t), \tilde{z}(t), \tilde{r}(t), \tilde{v}(t), \tilde{w}(t) \) is bounded, and \( D^V V(t) \leq -\alpha \overline{m} V(t) \), where
\[ \overline{m} = \exp \{-M_0\}, M_0 \geq \max \{ \left| \tilde{r}(t) \right|, \left| \tilde{v}(t) \right|, \left| \overline{w}(t) \right|, \left| \overline{z}(t) \right|, \left| \overline{w}(t) \right| \} , \]

That is \( V(t, X(t), U(t)) \), condition (iii) is established, so the almost periodic system (1) has a unique and uniformly asymptotically stable positive almost periodic solution in \( S \). From the theorem 2, the almost periodic system (1) is globally asymptotically stable, and the conclusion is established.

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