NP vs PSPACE
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Abstract. We present a proof of the conjecture \( \mathcal{NP} = \mathcal{PSPACE} \) by showing that arbitrary tautologies of Johansson’s minimal propositional logic admit “small” polynomial-size dag-like natural deductions in Prawitz’s system for minimal propositional logic. These “small” deductions arise from standard “large” tree-like inputs by horizontal dag-like compression that is obtained by merging distinct nodes labeled with identical formulas occurring in horizontal sections of deductions involved. The underlying “geometric” idea: if the height, \( h(\partial) \), and the total number of distinct formulas, \( \phi(\partial) \), of a given tree-like deduction \( \partial \) of a minimal tautology \( \rho \) are both polynomial in the length of \( \rho \), \( |\rho| \), then the size of the horizontal dag-like compression \( \partial^c \) is at most \( h(\partial) \times \phi(\partial) \), and hence polynomial in \( |\rho| \). Moreover if maximal formula length in \( \partial \), \( \mu(\partial) \), is also polynomial in \( |\rho| \), then so is the weight of \( \partial^c \). That minimal tautologies \( \rho \) are derivable by natural deductions \( \partial \) with \( |\rho| \)-polynomial \( h(\partial), \phi(\partial) \) and \( \mu(\partial) \) follows via embedding from the known result that there are analogous sequent calculus deductions of sequent \( \Rightarrow \rho \). The attached proof is due to the first author, but it was the second author who proposed an initial idea to attack a weaker conjecture \( \mathcal{NP} = \mathcal{coNP} \) by reductions in diverse natural deduction formalisms for propositional logic. That idea included interactive use of minimal, intuitionistic and classical formalisms, so its practical implementation was too involved. On the contrary, the attached proof of \( \mathcal{NP} = \mathcal{PSPACE} \) runs inside the natural deduction interpretation of Hudelmaier’s cut-free sequent calculus for minimal logic.

Keywords: Complexity theory, propositional complexity, proof theory, digraphs.

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1 Introduction

Recall standard definitions of the complexity classes \(\mathsf{NP}\), \(\mathsf{coNP}\) and \(\mathsf{PSPACE}\). \(L \subseteq \{0, 1\}^*\) is in \(\mathsf{NP}\), resp. \(\mathsf{coNP}\), if there exists a polynomial \(p\) and a polytime TM \(M\) such that
\[
x \in L \iff \exists u \in \{0, 1\}^{p(|x|)} \ M(x, u) = 1.
\]
resp.
\[
x \in L \iff \forall u \in \{0, 1\}^{p(|x|)} \ M(x, u) = 1.
\]
holds for every \(x \in \{0, 1\}^*\). Now \(L \subseteq \{0, 1\}^*\) is in \(\mathsf{PSPACE}\) if there exists a polynomial \(p\) and a TM \(M\) such that for every input \(x \in \{0, 1\}^*\), the total number of non-blank locations that occur during \(M\)'s execution on \(x\) is at most \(p(|x|)\), and \(x \in L \iff M(x) = 1\). It is well-known that \(\mathsf{NP} \subseteq \mathsf{PSPACE}\) and \(\mathsf{coNP} \subseteq \mathsf{PSPACE}\). Moreover, if \(\mathsf{NP} = \mathsf{PSPACE}\) then \(\mathsf{NP} = \mathsf{coNP}\). The latter conjecture seems more natural and/or plausible, as it reflects an idea of logical equivalence between model theoretical (re: \(\mathsf{NP}\)) and proof theoretical (re: \(\mathsf{coNP}\)) interpretations of non-deterministic polytime computability. So according to familiar \(\mathsf{NP}\)-(\(\mathsf{coNP}\))-completeness of boolean satisfiability (resp. validity) problem, in order to prove \(\mathsf{NP} = \mathsf{coNP}\) it will suffice to show that arbitrary tautologies admit “small” polynomial-size (abbr.: \(\mathsf{polysize}\)) deductions in a natural propositional proof system. The former (stronger) conjecture \(\mathsf{NP} = \mathsf{PSPACE}\) is less intuitive than \(\mathsf{NP} = \mathsf{coNP}\), but our proof thereof follows the same pattern with respect to minimal logic, instead of classical one. This is legitimate, since the validity of minimal propositional logic is \(\mathsf{PSPACE}\)-complete.

2 Towards \(\mathsf{NP} = \mathsf{PSPACE}\)

2.1 Proof theoretic background

We consider two types of proof theoretic formalism: Gentzen-style Sequent Calculus (abbr.: \(\mathsf{SC}\)) and Prawitz’s Natural Deduction (abbr.: \(\mathsf{ND}\)). Both \(\mathsf{SC}\) and \(\mathsf{ND}\) admit standard tree-like interpretation, as well as generalized dag-like interpretation in which proofs (or deductions) are regarded as labeled rooted monoedge dags.\(^1\) Our desired “small” deductions will arise from “large” standard tree-like inputs by appropriate dag-like compressing techniques. The compression in question is obtained by merging distinct nodes with identical labels, i.e. sequents or single formulas in the corresponding case of \(\mathsf{SC}\) or \(\mathsf{ND}\), respectively.

In our earlier \(\mathsf{SC}\) related proof-compression research \([1], [2], [3]\) dealing with sequent calculi\(^2\) we obtained such basic result (et al):

\(\text{Any tree-like deduction } \partial \text{ of any given sequent } S \text{ is constructively compressible to a dag-like deduction } \partial' \text{ of } S \text{ in which sequents occur at most once. I.e., in } \partial', \text{ distinct nodes are supplied with distinct sequents (that occur in } \partial).\)

\(^1\)Recall that ‘dag’ stands for directed acyclic graph (edges directed upwards).

\(^2\)Also note \([7]\) that shows a mimp-like formalization of natural deductions that admits “explicit” and size-preserving strong normalization procedure.
However, even in the case of cutfree SC having good proof search and other nice properties (like Gentzen’s subformula property), this result still gives us no polynomial control over the size of $\partial^c$. The reason is that sequents occurring in $\partial^c$ can be viewed as collections of subformulas of $S$, which allows their total number to grow exponentially in the size of $S$, $|S|$. In contrast, ND deductions consist of single formulas, which gives hope to overcome this problem. On the other hand, in ND, full dag-like compression merging arbitrary nodes supplied with identical formulas is problematic, as there is a risk of confusion between deduced formulas and the same formulas used above as discharged assumptions. But we can try horizontal dag-like compression that should merge only the nodes occurring in horizontal sections of ND deductions involved. The underlying idea is explained in the abstract. Namely, if a tree-like input deduction $\varnothing$ of a given formula $\rho$ has $|\rho|$-polynomial height (= maximal thread length), $h(\varnothing)$, and the foundation (= the total number of distinct formulas occurring in $\varnothing$), $\phi(\varnothing)$, is also polynomial in $|\rho|$, then the size (= total number of formulas) of the corresponding horizontal dag-like compression $\varnothing^c$, $|\varnothing^c|$, will be at most $h(\varnothing) \times \phi(\varnothing)$. Moreover if maximal formula length in $\varnothing$, $\mu(\varnothing)$, is also polynomial in $|\rho|$, then the weight (= total number of characters occurring inside) of $\varnothing^c$, $\|\varnothing^c\|$, is bounded by $h(\varnothing) \times \phi(\varnothing) \times \mu(\varnothing)$. It remains to show that every formula $\rho$ that is valid in minimal logic admits a ND deduction $\varnothing$ with $|\rho|$-polynomial parameters $h(\varnothing)$, $\phi(\varnothing)$ and $\mu(\varnothing)$. But this follows by a natural SC $\rightarrow$ ND embedding from Hudelmaier’s result saying that there are analogous SC deductions of the corresponding sequent $\Rightarrow \rho$.

2.2 Overview of the proof

We argue as follows along the lines 1–4:

1. Formalize minimal propositional logic as fragment $LM_\rightarrow$ of Hudelmaier’s tree-like cutfree intuitionistic sequent calculus. For any $LM_\rightarrow$ proof $\varnothing$ of sequent $\Rightarrow \rho$:

   (a) $h(\varnothing)$ (= the height) is polynomial (actually linear) in $|\rho|$,
   (b) $\phi(\varnothing)$ (= total number of formulas) and $\mu(\varnothing)$ (= maximal formula length) are also polynomial in $|\rho|$.

2. Show that there exists a constructive (1)+(2) preserving embedding $F$ of $LM_\rightarrow$ into Prawitz’s tree-like natural deduction formalism $NM_\rightarrow$ for minimal logic.

3. Elaborate polytime verifiable dag-like deducibility in $NM_\rightarrow$.

4. Elaborate and apply horizontal tree-to-dag proof compression in $NM_\rightarrow$. For any tree-like $NM_\rightarrow$ input $\varnothing$, the weight of dag-like output $\varnothing^c$ is bounded by $h(\varnothing) \times \phi(\varnothing) \times \mu(\varnothing)$. Hence the weight of $(F(\varnothing))^c$ for any given tree-like $LM_\rightarrow$ proof $\varnothing$ of $\rho$ is polynomially bounded in $|\rho|$. Since minimal logic is PSPACE-complete, conclude that $NP = PSPACE$.  

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3 More detailed exposition

In the sequel we consider standard language $L\to$ of minimal logic whose formulas $(\alpha, \beta, \gamma, \rho \text{ etc.})$ are built up from propositional variables $(p, q, r \text{ etc.})$ using one propositional connective $\to$. The sequents are in the form $\Gamma \Rightarrow \alpha$ whose antecedents, $\Gamma$, are viewed as multisets of formulas; sequents $\Rightarrow \alpha$, i.e. $\emptyset \Rightarrow \alpha$, are identified with formulas $\alpha$.

3.1 Sequent calculus $LM\to$

$LM\to$ includes the following axioms ($MA$) and inference rules ($MI_1\to$), ($MI_2\to$), ($ME\to P$), ($ME\to\to$) in the language $L\to$ (the constraints are shown in square brackets).

\[\begin{align*}
(MA) : & \quad \Gamma, p \Rightarrow p \\
(MI_1\to) : & \quad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta} \quad (\hat{\beta} \gamma) : (\alpha \to \beta) \to \gamma \in \Gamma \\
(MI_2\to) : & \quad \frac{\Gamma, \alpha, \beta \Rightarrow \gamma \Rightarrow \beta}{\Gamma, (\alpha \to \beta) \Rightarrow \gamma \Rightarrow \alpha \to \beta} \\
(ME\to P) : & \quad \frac{\Gamma, p, \gamma \Rightarrow q}{\Gamma, p, p \Rightarrow \gamma \Rightarrow q} \quad [q \in \text{VAR}(\Gamma, \gamma), p \neq q] \\
(ME\to\to) : & \quad \frac{\Gamma, \alpha, \beta \Rightarrow \gamma \Rightarrow \beta}{\Gamma, (\alpha \to \beta) \Rightarrow \gamma \Rightarrow q} \quad [q \in \text{VAR}(\Gamma, \gamma)]
\end{align*}\]

Claim 1 $LM\to$ is sound and complete with respect to minimal propositional logic \[3\] and tree-like deducibility. Thus any given formula $\rho$ is valid in the minimal logic iff it (i.e. sequent $\Rightarrow \rho$) is tree-like deducible in $LM\to$.

Proof. Easily follows from \[4\].

Recall that for any (tree-like or dag-like) deduction $\partial$ we denote by $h(\partial)$ and $\phi(\partial)$ its height and foundation, respectively. Furthermore for any sequent (in particular, formula) $S$ we denote by $|S|$ the total number of $'\to'$-occurrences in $S$ and following \[4\] define the complexity degree $\text{deg}(S)$:

1. $\text{deg}(\Gamma, \alpha \to \beta \Rightarrow \alpha) := |\alpha \to \beta| + \sum_{\xi \in \Gamma} |\xi|,$
2. $\text{deg}(\Gamma \Rightarrow \alpha) := |\alpha| + \sum_{\xi \in \Gamma} |\xi|,$ if $(\hat{\beta} \gamma) : \alpha \to \beta \in \Gamma.$

Lemma 2

\[3\]This is a slightly modified, equivalent version of the corresponding purely implicational and $\bot$-free subsystem of Hudelmaier’s intuitionist calculus LG, cf. \[4\]. The constraints $q \in \text{VAR}(\Gamma, \gamma)$ are added just for the sake of transparency.
1. Tree-like $\text{LM}_{\rightarrow}$ deductions share the semi-subformula property, where semi-subformulas of $(\alpha \rightarrow \beta) \rightarrow \gamma$ include $\beta \rightarrow \gamma$ along with proper subformulas $\alpha \rightarrow \beta$, $\alpha$, $\beta$, $\gamma$. In particular, any $\alpha$ occurring in a $\text{LM}_{\rightarrow}$ deduction $\partial$ of $\Rightarrow \rho$ is a semi-subformula of $\rho$, and hence $|\alpha| \leq |\rho|$. Thus $\mu(\partial) \leq |\rho|$. \\
2. If $S'$ occurs strictly above $S$ in a given tree-like $\text{LM}_{\rightarrow}$ deduction $\partial$, then $\deg(S') < \deg(S)$. \\
3. The height of any tree-like $\text{LM}_{\rightarrow}$ deduction $\partial$ of $S$ is linear in $|S|$. In particular if $S$ is $\Rightarrow \rho$, then $h(\partial) \leq 3|\rho|$. \\
4. The foundation of any tree-like $\text{LM}_{\rightarrow}$ deduction $\partial$ of $S$ is at most quadratic in $|S|$. In particular if $S$ is $\Rightarrow \rho$, then $\phi(\partial) \leq (|\rho| + 1)^2$. \\

**Proof.** 1: Obvious. Note that $\beta \rightarrow \gamma$ occurring in premises of $(MI2 \rightarrow)$ and $(ME \rightarrow)$ are semi-subformulas of $(\alpha \rightarrow \beta) \rightarrow \gamma$ occurring in the conclusions. 
2–3: See [4].
4: Let $\text{ssf}(\alpha)$ be the total number of distinct occurrences of semi-subformulas in a given formula $\alpha$. It is readily seen that $\text{ssf}(-)$ satisfies the following three conditions.

1. $\text{ssf}(p) = 1$.
2. $\text{ssf}(p \rightarrow \alpha) = 2 + \text{ssf}(\alpha)$.
3. $\text{ssf}((\alpha \rightarrow \beta) \rightarrow \gamma) = 1 + \text{ssf}(\alpha \rightarrow \beta) + \text{ssf}(\beta \rightarrow \gamma) - \text{ssf}(\beta)$.

Moreover 1–3 can be viewed as recursive clauses defining $\text{ssf}(\alpha)$, for any $\alpha$. Having this we easily arrive at $\text{ssf}(\alpha) \leq (|\alpha| + 1)^2$ (see Appendix A), which by the assertion 1 yields $\phi(\partial) \leq \text{ssf}(\rho) \leq (|\rho| + 1)^2$, as required, provided that $\Rightarrow \rho$ is the endsequent of $\partial$. ■

### 3.2 ND calculus $\text{NM}_{\rightarrow}$ and embedding of $\text{LM}_{\rightarrow}$

Denote by $\text{NM}_{\rightarrow}$ a ND proof system for minimal logic that contains just two rules $(\rightarrow I)$, $(\rightarrow E)$ [6] (we write ‘$\rightarrow$’ instead of ‘$\Rightarrow$’).

| $(\rightarrow I)$: $\alpha \rightarrow \beta$ | $(\rightarrow E)$: $\alpha \rightarrow \beta$ |
|-----------------------------------------|-----------------------------------------|
| $\alpha$                               | $\alpha \rightarrow \beta$            |
| $\beta$                                 |                                          |

**Claim 3 (Prawitz)** $\text{NM}_{\rightarrow}$ is sound and complete with respect to minimal propositional logic and tree-like deducibility.

**Proof.** See [6]. ■
Theorem 4 There exists a recursive operator \( \mathcal{F} \) that transforms any given tree-like \( \text{LM}_\to \) deduction \( \partial \) of \( \Gamma \Rightarrow \rho \) into a tree-like \( \text{NM}_\to \) deduction \( \mathcal{F}(\partial) \) with root-formula \( \rho \) and assumptions occurring in \( \Gamma \). Moreover \( \partial \) and \( \mathcal{F}(\partial) \) share the semi-subformula property and linear (polynomial) upper bounds on the height (resp. foundation). If \( \Gamma = \emptyset \), then \( \mathcal{F}(\partial) \) is a tree-like \( \text{NM}_\to \) proof of \( \rho \) such that
\[
|h(\mathcal{F}(\partial))| \leq 18|\rho| \quad \text{and} \quad \phi(\mathcal{F}(\partial)) < (|\rho| + 1)^2(|\rho| + 2) \quad \text{and} \quad \mu(\mathcal{F}(\partial)) \leq 2|\rho|.
\]

Proof. \( \mathcal{F}(\partial) \) is defined by straightforward recursion on \( h(\partial) \) by standard pattern sequent deduction \( \Rightarrow \) natural deduction, where sequent deduction of \( \Gamma \Rightarrow \alpha \) is interpreted as a ND deduction of \( \alpha \) from open assumptions occurring in \( \Gamma \). The recursive clauses are as follows.

1. \[(\text{MA}) : \Gamma, \rho \Rightarrow \rho \xrightarrow{\mathcal{F}} \rho\]

2. \[(\text{MI1} \to) : \Gamma, \alpha \Rightarrow \beta \xrightarrow{\mathcal{F}} \beta \xrightarrow{\alpha \Rightarrow \beta} (\to I)\]
\[\begin{array}{l}
\Gamma \Rightarrow \alpha \to \beta \\
[(\exists \gamma) : (\alpha \to \beta) \to \gamma \in \Gamma]
\end{array}\]

3. \[(\text{MI2} \to) : \Gamma, \alpha, \beta \Rightarrow \gamma \Rightarrow \beta \xrightarrow{\mathcal{F}} \gamma \xrightarrow{\alpha \Rightarrow \beta} \]
\[\begin{array}{l}
\frac{[\beta]^2}{\alpha \Rightarrow \beta} \to (I) \\
\frac{[\alpha]^1}{\gamma} \xrightarrow{\gamma \gamma \to [\beta]} \xrightarrow{\beta \Rightarrow \gamma \Rightarrow [\beta]} \to (I)
\end{array}\]

4. \[(\text{ME} \to P) : \Gamma, p, q \Rightarrow q \xrightarrow{\mathcal{F}} \]
\[\begin{array}{l}
p \xrightarrow{p, p \Rightarrow \gamma} (\to E) \\
\downarrow \quad \downarrow \\
\gamma \quad q
\end{array}\]
5. \[(ME \rightarrow \rightarrow) : \frac{\Gamma, \alpha, \beta \rightarrow \gamma \Rightarrow \beta, \gamma \Rightarrow q}{\Gamma, (\alpha \rightarrow \beta) \rightarrow \gamma \Rightarrow q} [q \in \text{VAR} (\Gamma, \gamma)] \]

Note that each embedding clause increases the height at most by 6 (just as in the case \((ME \rightarrow \rightarrow)\)), which yields \(h(F(\vartheta)) \leq 6 \cdot h(\vartheta) \leq 18 |\varrho|\) according to Lemma 2 (3). By the same token, formulas occurring in \(F(\vartheta)\) include the ones occurring in \(\vartheta\) together with possibly new formulas \(\gamma \rightarrow q\) (with old \(\gamma\) and \(q\)) shown on the right-hand side in the case \((ME \rightarrow \rightarrow)\). There are at most \(\phi(\vartheta)\) and \(|\varrho| + 1\) such \(\gamma\) and \(q\), respectively. Hence by Lemma 2 (1, 4) we arrive at \(\phi(F(\vartheta)) < (|\varrho| + 1)^2 + (|\varrho| + 1)^2 (|\varrho| + 1) = (|\varrho| + 1)^2 (|\varrho| + 2)\) and \(\mu(F(\vartheta)) \leq 2 |\varrho|\), as required. \(\blacksquare\)

3.3 Horizontal tree-to-dag compression in NM→

We claim that any given tree-like NM→ deduction \(\vartheta\) with root formula \(\varrho\) can be compressed into a dag-like NM→ deduction \(\vartheta^c\) of the same conclusion \(\varrho\) such that the size of \(\vartheta^c\) is at most \(h(\vartheta) \times \phi(\vartheta)\). In particular, if \(\vartheta = F(\vartheta_0)\) for \(\vartheta_0\) being a tree-like LM→ deduction of \(\Rightarrow \varrho\) and \(F\) the embedding of Theorem 4, then \(\vartheta^c\) will be a desired dag-like \(|\varrho|\)-polysize NM→ deduction of \(\varrho\). The operation \(\vartheta \mapsto \vartheta^c\) (that we call horizontal compression) runs by bottom-up recursion on \(h(\vartheta)\) such that for any \(n \leq h(\vartheta)\), the \(n^{th}\) horizontal section of \(\vartheta^c\) is obtained by merging all nodes with identical formulas occurring in the \(n^{th}\) horizontal section of \(\vartheta\) (this operation we call horizontal collapsing). Thus the horizontal compression is obtained by bottom-up iteration of the horizontal collapsing. \(|\vartheta^c| \leq h(\vartheta) \times \phi(\vartheta)\) is obvious, as the size of every (compressed) \(n^{th}\) horizontal section of \(\vartheta^c\) can’t exceed \(\phi(\vartheta)\). It remains to show that horizontal compression preserves the discharged assumptions. This requires a more insightful consideration of dag-like deducibility that we elaborate below.
3.4 Dag-like deducibility in NM→

We wish to elaborate, and work in, the space of dag-like natural deductions. To begin with we observe that horizontal collapsing may extend the premises of the underlying inferences. So let us denote by NM∗→ a tree-like extension of NM→ that contains multipremise rules of inference of the form

\[(M): \frac{\Gamma}{\gamma}\]

instead of original NM→ rules (→ I), (→ E). Here \(\Gamma\) is a multiset containing \(\gamma\), and/or \(\beta\), if \(\gamma = \alpha \rightarrow \beta\), and/or arbitrary \(\delta_i\) together with \(\delta_i \rightarrow \gamma\) \((i \in [m])\). Thus in particular, \((M)\) includes repetition rules

\[(R): \frac{\gamma}{\gamma} \quad (R)^*: \frac{\gamma \cdots \gamma}{\gamma}\]

as well as following inferences

\[(\rightarrow I)^*: \frac{[\alpha] \cdots [\alpha]}{\delta_1 \cdots \delta_m \delta_1 \rightarrow \gamma \beta \alpha \rightarrow \beta}\]
\[(\rightarrow I, R)^*: \frac{[\alpha] \cdots [\alpha]}{\delta_1 \cdots \delta_m \delta_1 \rightarrow \gamma \beta \gamma \cdots \gamma}\]
\[(\rightarrow E)^*: \frac{[\alpha] \cdots [\alpha]}{\delta_1 \delta_1 \rightarrow \gamma \delta_1 \rightarrow \gamma \beta \alpha \rightarrow \beta}\]
\[(\rightarrow E, R)^*: \frac{[\alpha] \cdots [\alpha]}{\delta_1 \delta_1 \rightarrow \gamma \delta_1 \rightarrow \gamma \beta \gamma \cdots \gamma}\]
\[(\rightarrow I, E): \frac{[\alpha] \cdots [\alpha]}{\delta \beta \delta \rightarrow (\alpha \rightarrow \beta) \beta \alpha \rightarrow \beta}\]
\[(\rightarrow I, E, R): \frac{[\alpha] \cdots [\alpha]}{\delta \beta \delta \rightarrow (\alpha \rightarrow \beta) \beta \alpha \rightarrow \beta}\]

Discharging in NM∗→ is inherited from NM→ via sub-occurrences of (→ I).

**Lemma 5** Tree-like provability in NM∗→ is sound and complete with respect to minimal propositional logic.

**Proof.** Completeness follows from Claim 3, as NM→ is contained in NM∗→. Soundness is obvious, as each \((M)\) strengthens valid rules \((R), (\rightarrow I)\) and/or \((\rightarrow E)\).
Further on we upgrade $\text{NM}_\varrho^*$ to a desired dag-like extension, $\text{NM}_\varrho^*$. Let us start with informal description (cf. formal definitions below). We'll consider only regular dags (abbr.: redags), which are specified as rooted monoedge dags $\partial$ (the roots being the lowest vertices) whose vertices (also called nodes) admit universal (i.e. path-invariant) height assignment such that all leaves $x$ have the same height $h(x) = h(\partial)$. We regard $\text{NM}_\varrho^*$ deductions as labeled redags $\partial$ whose nodes can have arbitrary many children and parents (as usual the roots, $\rho(\partial)$, have no parents and the leaves have no children). Distinct children are either singletons or conjugate pairs (mutually separated by fixed partitions $\varrho$). Moreover, all nodes of $\partial$ are labeled with formulas by a fixed assignment $\ell^\varrho$. The inferences $(M)$ associated with $\partial$ are determined by standard local correctness conditions on $\ell^\varrho$ and $\varrho$, such that $\ell^\varrho(\rho(\partial)) = \varrho$, while children’s $\ell^\varrho$-formulas either coincide with the conclusion’s ones or are premises $\beta$ of the conclusion’s $\ell^\varrho$-formulas $\alpha \rightarrow \beta$, or else are conjugate premises $\delta_i, \delta_i \rightarrow \gamma$ of the conclusion’s $\ell^\varrho$-formulas $\gamma$. Besides, there is a fixed assignment $\ell^e$ that is defined for any edge $e = \langle u, v \rangle$ ($u$ being a parent of $v$) that admits inverse branching below $u$. To put it more precisely we consider descending chains $K(u) = \{u = x_0, \ldots, x_k\}$ ($k > 0$) in $\partial$ such that for all $0 < i < k, x_i$ has exactly one parent $x_{i+1}$, whereas $x_k$ has at least two parents (such $K(u)$ is uniquely determined by $u$). Having this we regard $\ell^e(e)$ as a chosen nonempty set of parents of $x_k$, called $\ell^e$-grandparents of $v$ with respect to $u$. It is assumed that $\ell^e(e) \subseteq \ell^e(\langle x_i, x_{i-1} \rangle)$ holds for all $1 < i \leq k$, while all parents of $x_k$ are $\ell^e$-grandparents of some $x_{k-1}$’s children (with respect to $x_k$). Descending deduction threads connecting leaves with the root are naturally determined by $\ell^e$-grandparents that are regarded as “road signs” showing allowed ways from the leaves down to the root, when passing from $v$ to $x_k$ through $u$, as specified above. These parameters determine ‘global’ discharging function on the set of top formulas (also called assumptions).

3.4.1 Formal definitions

Definition 6 Consider a rooted monoedge redag $D = \langle v(D), e(D) \rangle$, $e(D) \subseteq v(D)^2$. $v(D)$ and $e(D)$ are called the vertices (or nodes) and the edges (ordered), respectively; if $\langle u, v \rangle \in e(D)$, then $u$ and $v$ are called parents and children of each other, respectively. For any $u \in v(D)$ denote by $h(u, D) \geq 0$ the height of $u$ and let $h(D) := \max \{h(u, D) : u \in v(D)\}$ (the height of $D$). Any $u \in v(D)$ has $\text{deg}(u, D) \geq 0$ children $c(u, D) := \{u^{(1)}, \ldots, u^{(\text{deg}(u, D))}\}$ and $\overrightarrow{\text{deg}}(u, D) \geq 0$ parents $p(u, D) := \{u^{(1)}, \ldots, u^{(\overrightarrow{\text{deg}}(u, D))}\}$ (both ordered).

Let $l(D) := \{u \in v(D) : \overrightarrow{\text{deg}}(u, D) = 0\}$ (leaves), and $g(D) :=$ the root of $D$; thus $p(u, D) = \emptyset \iff u = g(D) \Leftrightarrow h(u, D) = 0$ and $c(u, D) = \emptyset \iff u \in l(D) \Leftrightarrow h(u, D) = h(D)$. With every $u \in v(D) \setminus l(D)$ we associate a fixed

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4That is, $\overrightarrow{\text{deg}}(u, D)$ (resp. $\overrightarrow{\text{deg}}(u, D)$) is the total number of targets with source $u$ (resp. total number of sources with target $u$), in $D$. 

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partition $\mathcal{S}(u, D) \subset \mathcal{C}(u, D) \cup \mathcal{C}(u, D)^2$ such that $\mathcal{C}(u, D) = (\mathcal{S}(u, D) \cap \mathcal{V}(D)) \cup \{ x, y : \langle x, y \rangle \in \mathcal{S}(u, D) \}$. Set $s(D) := \bigcup_{u \in \mathcal{V}(D) \setminus \mathcal{L}(D)} \mathcal{S}(u, D)$, to be abbreviated by $s(D)$.

By the same token, we’ll often drop ‘$D$’ in $h(u, D)$, $\overrightarrow{\deg}(u, D)$, $\overrightarrow{\deg}(u, D)$, $\mathcal{C}(u, D)$, $\mathcal{P}(u, D)$, $\varrho(D)$, $K(u, D)$, $U(u, D)$ (see below), if $D$ is clear from the context. We let $E_0(D) := \{ \langle u, v \rangle \in \mathcal{E}(D) : v \in \mathcal{L}(D) \}$ (top edges) and use abbreviations $x \prec_D y \iff \langle x \rangle$ occurs strictly below $y$, in $D$ and $x \preceq_D y \iff x \triangleq_D y \lor x = y$.

For any $u \in \mathcal{V}(D)$ we let $K(u, D) = [u = x_0, \ldots, x_k = U(u, D)]$ be the uniquely determined descending chain of maximal length such that either deg$(u) \neq 1$ and $k = 0$ or else $\langle x_{i+1}, x_i \rangle \in E(D)$ and deg$(x_i) = 1$, for all $i < k$. If Let $E_{\Delta}(D) := \{ e = \langle u, v \rangle \in \mathcal{E}(D) : U(u, D) \neq \varrho \}$. Thus deg$(U(u)) > 1$ holds for every $e = \langle u, v \rangle \in E_{\Delta}(D)$.

Note that $E_{\Delta}(D) = \emptyset$, if $D$ is a tree.

Let $\partial = (\mathcal{D}, \mathcal{S}, \ell^\partial, \ell^C)$ extend $(\mathcal{D}, \mathcal{S})$ by labeling functions $\ell^\partial : \mathcal{V}(D) \rightarrow \mathcal{F}(\mathcal{L}_{\rightarrow})$ and $\ell^C : E_{\Delta}(D) \rightarrow \mathcal{F}(\mathcal{L}_{\rightarrow})$, where $\mathcal{F}(\mathcal{L}_{\rightarrow})$ is the set of $\mathcal{L}_{\rightarrow}$ formulas. $\partial$ is called a plain (or unencoded) dag-like $\text{NM}_{\star}^*$ deduction iff the following local correctness conditions hold (along with standard ones with regard to $(\mathcal{D}, \mathcal{S})$).

1. For any $u \in \mathcal{V}(D)$ and $x, y \in \mathcal{C}(u)$ it holds:
   
   (a) $h(x) = h(y) = h(u) + 1$,
   
   (b) if $x \in S(u)$ then either $\ell^\partial(u) = \ell^\partial(x)$
       or $\ell^\partial(u) = \alpha \rightarrow \ell^\partial(x)$ \{abbr.: $\langle u, x \rangle \in (\rightarrow I)_\alpha$\}
       for a (uniquely determined) $\alpha \in \mathcal{F}(\mathcal{L}_{\rightarrow})$,
   
   (c) $\langle x, y \rangle \in S(u)$ implies $\ell^\partial(y) = \ell^\partial(x) \rightarrow \ell^\partial(u)$.

2. For any $e = \langle u, v \rangle \in E_{\Delta}(D)$ and $w \in \mathcal{C}(u)$ it holds:

   (a) $\emptyset \neq \ell^C(e) \subseteq \mathcal{P}(U(u))$,
   
   (b) $\langle v, w \rangle \in S(u)$ implies $\ell^C(e) = \ell^C(\langle u, w \rangle)$,
   
   (c) $\overline{\text{deg}}(v) = 1$ implies $\ell^C(e) = \bigcup_{z \in \mathcal{C}(u)} \ell^C(\langle v, z \rangle)$.

3. For any $u \in \mathcal{V}(D) \setminus \mathcal{L}(D)$,

   $\overline{\text{deg}}(u) > 1$ implies $\mathcal{P}(u) \subseteq \bigcup_{v \in \mathcal{C}(u)} \ell^C(\langle u, v \rangle)$.

Denote by $\mathcal{D}^\star$ the set of plain dag-like $\text{NM}_{\star}^*$ deductions.

**Definition 7** For any $\partial = (\mathcal{D}, \mathcal{S}, \ell^\partial, \ell^C) \in \mathcal{D}^\star$, $e = \langle u, v \rangle \in \mathcal{E}(D)$, $z \prec_D u$, let $\mathcal{T}(e, z, \partial)$ be the set of deduction threads $\Theta = [v = x_0, u = x_1, \ldots, x_n = z]$ connecting $e$ with $z$, where any $\Theta$ in question is a descending chain such that for every $i < n$, $\langle x_{i+1}, x_i \rangle \in \mathcal{E}(D)$ and either $\overline{\text{deg}}(x_i) = 1$ or else $\overline{\text{deg}}(x_i) > 1$ and $x_{i+1} \in \ell^C(\langle x_{j+1}, x_j \rangle)$, where $j := \max \{ k < i : k = 0 \lor \overline{\text{deg}}(x_k) > 1 \}$. Now

---

*not necessarily disjoint.*
\[\alpha \in \forall (L_{\sim}) \text{ is called an open (or undischarged) assumption in } \partial \text{ if there is a } \Theta \in \text{TH}(e, \varrho, \partial) \text{ for } e = \langle u, v \rangle \in E_0(D) \text{ and } \ell^c(v) = \alpha \text{ that contains no } (x_{i+1}, x_i) \in (\to I)_\alpha, \ i < n; \text{ such } \Theta \text{ is called an open thread, in } \partial. \text{ Denote by } \Gamma_\partial \text{ the set of open assumptions in } \partial. \text{ Call } \partial \text{ a dag-like } \text{NM}_\ast \text{ deduction of } \rho := \ell^c(\varrho) \text{ from } \Gamma_\partial. \text{ If } \Gamma_\partial = \emptyset, \text{ then is called a dag-like } \text{NM}_\ast \text{ proof of } \rho.\]

In the sequel \text{NM}_\ast \text{ deductions (proofs) are also called } \text{plain dag-like } \text{NM}_\ast \text{ deductions (proofs).} \text{ Note that in the tree-like domain such dag-like (actually redag-like) provability is equivalent to canonical tree-like } \text{NM}_\ast \text{ provability. Indeed, in any tree-like deduction, every leaf has exactly one deduction thread, and hence } \ell^c \text{ can be dropped entirely. Also note that } \text{NM}_\ast \text{ (and hence also } \text{NM}_\sim \text{) is tree-like embeddable into } \text{NM}_\ast \text{ by iterating the repetition rule (R), if necessary, in order to fulfill the redag height condition } h(x) = h(\partial), \text{ for all leaves } x. \text{ Obviously this operation preserves } h(\partial), \phi(\partial) \text{ and } \mu(\partial).\]

3.5 Horizontal compression continued

Let us go back to the horizontal compression \(\partial \rightarrow \partial^c\), where without loss of generality we assume that \(\partial\) is an arbitrary tree-like \(\text{NM}_\ast\) deduction of \(\rho\). To complete our recursive definition of \(\partial^c\) via horizontal collapsing (see 3.3 above) it remains to specify \(\ell^c\). So let us take a closer look at the structure of \(\partial^c\). For any \(n \leq h(\partial)\), denote by \(\partial^c_n = \langle D_n, s_n, \ell^c_n, c_n^c \rangle\) a deduction that is obtained after executing the \(n\)th recursive step in question. Note that \(\partial^c_n = \partial \text{ and } \partial^c_{n+1} = \partial^c\). Moreover, for any \(i \leq n < j\) we have \(L_i(D_n) = L_i(D_{h(\partial)})\), \(L_j(D_n) = L_j(D_0)\) and \(h(D_n) = h(D_0) = h(D_{h(\partial)})\), where \(L_k(D_m) := \{x \in v(D_m): h(x) = k\} = \text{the } k\text{th section of } \partial^c_m\). Besides, if \(n < h(\partial)\), then all \(x \in L_{n+1}(D_n)\) are the roots of the corresponding (maximal) tree-like subgraphs of \(\partial\), while \(\partial^c_{n+1}\) arises from \(\partial^c_n\) by merging distinct \(x \in L_{n+1}(D_n)\) labeled with identical formulas, \(\ell^c(x)\), and defining edges by the corresponding homomorphism. Thus \(L_{n+1}(D_{n+1}) \subseteq L_{n+1}(D_n)\), while \(x \neq y \in L_{n+1}(D_{n+1})\) implies \(\ell^c(x) \neq \ell^c(y)\).

(If \(L_{n+1}(D_{n+1}) = L_{n+1}(D_n)\), then \(\partial^c_{n+1} = \partial^c_n\) and \(\ell^c_{n+1} = \ell^c_n\).) Now suppose \(L_{n+1}(D_{n+1}) \neq L_{n+1}(D_n)\), \(n < h(\partial)\), and let \(M_{n+1} \subseteq L_{n+1}(D_{n+1})\) be the set of all merge points in \(\partial^c_{n+1}\). The \(\ell^c_{n+1}\)-grandparents are defined as follows. For any \(e = \langle u, v \rangle \in E_\Delta(D_{n+1}), \ u \in L_j(D_{n+1}), \ v \in L_{j+1}(D_{n+1}), \ j < h(D_0)\), consider \(K(u, D_{n+1}) = \{u = x_0, \cdots, x_k\}\). (Note that \(x_i \in L_j(D_n)\) for all but at most one \(x_i, i \leq k\).) We let \(\ell^c_{n+1}(e) := \ell^c_n(e)\) except for the following two cases.

1. Suppose \(j = n + 1\) and \(v \in M_{n+1}\). We let \(\ell^c_{n+1}(e)\) be the union of all \(\ell^c_n(\langle u, w \rangle)\) such that \(w \in c(u, D_n)\) and \(\ell^c(v) = \ell^c(w)\).

2. Suppose \(h(\partial) - 1 \geq j > n + 1, \ x_k \in M_{n+1}\) and \(p(x_k-1, D_n) = \{y\}\), while \(p(y, D_n) = \{y(1)\}\), i.e. \(y(1)\) is the only parent of \(y\) in \(\partial^c_n\). Then we let \(\ell^c_{n+1}(e) := \{y(1)\}\).

\(^6\)Here and below ‘plain’ means ‘unencoded’ (see 3.6.1, below)

\(^7\)That is, every node \(x \neq \varrho(\partial)\) has exactly one parent.
Having this we observe that $\partial_{\rho}^{n+1}$ preserves the open (resp. closed) assumptions of $\partial_{\rho}^{n}$. The same conclusion with regard to $\partial$ and $\partial^c$ follows immediately by induction on $n \leq h(D_0) = h(\partial)$. In particular, if $\partial$ is a tree-like $\text{NM}_\rightarrow$ proof of $\rho$, then $\partial^c$ is a plain dag-like $\text{NM}_\rightarrow$ proof of $\rho$. This completes our informal description of the required tree-to-dag horizontal compression $\partial \mapsto \partial^c$. Formal definitions are shown below.

### 3.5.1 Horizontal collapsing

Recall that horizontal compression $\partial \mapsto \partial^c$ is obtained by bottom-up iteration of the horizontal collapsing that merges distinct nodes labeled with identical formulas occurring in the same horizontal section of $\partial$. Our next definition will formalize the latter operation. In the sequel for any $D$ and $x \in \text{v}(D)$ we let $(D)_{x} := \langle \text{v}((D)_{y}), \text{e}((D)_{y}) \rangle$ for $\text{v}((D)_{y}) = \{y \in \text{v}(D) : x \leq_{D} y\}$ and $\text{e}((D)_{y}) = \text{e}(D) \cap \text{v}((D)_{y})^2$. For any $n > 0$ we let $L_n(D) := \{x \in \text{v}(D) : h(x) = n\}$ and denote by $D^*_n$ the set of dag-like deductions $\partial = \langle D, s, l, \ell^c \rangle \in D^*$ such that $(D)_{x}$ are pairwise disjoint (sub)trees, for all $x \in L_n(D)$. Note that $D^*_n = D^*$ for $n > h(D)$, while $D^*_1$ consists of all tree-like $\text{NM}_\rightarrow^*$ deductions (see above). So in the sequel we’ll rename $D^*_1$ to $T^*$ and denote its elements by $\langle T, s, l, \ell^c \rangle$, rather than $\langle D, s, l, \ell^c \rangle$ (recall that $\ell^c$ is irrelevant in the tree-like case).

**Definition 8 (horizontal collapsing)** Suppose $\partial = \langle D, s, l, \ell^c \rangle \in D^*_n$, $n \leq h(D)$, $\alpha \in \text{v}(\text{L}_\rightarrow)$ and $S_{n, \alpha} = \{y \in L_n(D) : \ell^c(y) = \alpha\}$, $|S_{n, \alpha}| > 1$. Moreover let $r \in S_{n, \alpha}$ be fixed. Let $C_{\alpha} = \bigcup_{y \in S_{n, \alpha}} \text{c}(y, D)$ and denote by $(D)_{\alpha,r}$ a tree extending upper subtrees $\bigcup_{z \in C_{\alpha}} (D)_{z}$ by a new root $r$. We construct a dag-like deduction $\partial^c_{n, \alpha} = \langle D_{n, \alpha}, s_{n, \alpha}, l^c_{n, \alpha}, \ell^c_{n, \alpha} \rangle$ by collapsing $S_{n, \alpha}$ to $\{r\}$. To put it more precisely, we stipulate:

1. $D_{n, \alpha}$ arises from $D$ by substituting $(D)_{\alpha,r}$ for $(D)_{r}$ and deleting $(D)_{y}$ for all $r \neq y \in S_{n, \alpha}$. That is, in the formal terms, we have

$$\text{v}(D_{n, \alpha}) = \left(\text{v}(D) \setminus \bigcup_{y \in S_{n, \alpha}} \text{v}((D)_{y})\right) \cup \text{v}((D)_{\alpha,r}) \quad \text{and} \quad \text{e}(D_{n, \alpha}) = \left(\text{e}(D) \cap \text{v}(D_{n, \alpha})\right)^2 \cup \left\{\langle r, v \rangle : v \in \bigcup_{y \in S_{n, \alpha}} \text{c}(y, D)\right\} \cup \left\{\langle u, r \rangle : u \in \bigcup_{y \in S_{n, \alpha}} \text{p}(y, D)\right\}.$$  

2. For any $u \in \text{v}(D_{n, \alpha})$ we define $s_{n, \alpha}(u, D_{n, \alpha})$ by cases as follows.

   (a) If $u \notin \{r\} \cup \bigcup_{y \in S_{n, \alpha}} \text{p}(y, D)$, then $s_{n, \alpha}(u, D_{n, \alpha}) := s(u, D)$.

   (b) $s_{n, \alpha}(u, D_{n, \alpha}) := \bigcup_{y \in S_{n, \alpha}} s(y, D)$. 

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Lemma 9
For any operation \( \partial \to \), the iteration of \( \partial \) successively to all \( e \).

3. For any \( u \in \mathbb{V}(D_{n,\alpha}) \) we let \( \ell^\circ_{n,\alpha}(u) := \ell^\circ(u) \).

4. For any \( e = (u, v) \in E_D(D_{n,\alpha}) \) and \( K(u, D_{n,\alpha}) = [u = x_0, \ldots, x_k] \) we define \( \ell^\circ_{n,\alpha}(e) \), where \( u \in L_j(D_{n,\alpha}), v \in L_{j+1}(D_{n,\alpha}) \) for \( j < h(D) \). We can just as well assume that \( v \in \mathbb{V}(D_{n,\alpha}) \) or \( \deg(v, D_{n,\alpha}) > 1 \) and define the rest according to clause 2 (c) of Definition 6 by induction on \( h(D) - j \).

So assuming \( v \in \mathbb{V}(D_{n,\alpha}) \), consider the following cases. (Note that (c) and (e) are the only cases with \( \ell^\circ_{n,\alpha}(e) \neq \ell^\circ(e) \).

(a) Suppose \( j + 1 < n \). Then \( \ell^\circ_{n,\alpha}(e) := \ell^\circ(e) \).
(b) Suppose \( j + 1 = n \) and \( v \neq r \). Then \( \ell^\circ_{n,\alpha}(e) := \ell^\circ(e) \).
(c) Suppose \( j + 1 = n \) and \( v = r \). Then \( \ell^\circ_{n,\alpha}(e) := \bigcup_{w \in \mathbb{V}(u, D \cap S_{n,\alpha})} \ell^\circ((u, w)) \).
(d) Suppose \( j + 1 > n \) (and hence \( v \in \mathbb{V}(D) \)) and \( r \notin K(u, D_{n,\alpha}) \). Then \( \ell^\circ_{n,\alpha}(e) := \ell^\circ(e) \).
(e) Suppose \( j + 1 > n \), \( x_k = r \) and \( \mathbb{V}(x_k - 1, D) = \{y\} \). Then:

i. if \( \deg(y, D) > 1 \), then \( \ell^\circ_{n,\alpha}(e) := \ell^\circ(e) \),
ii. if \( \mathbb{V}(y, D) = \{y(1)\} \) (thus \( \deg(y, D) = 1 \)), then \( \ell^\circ_{n,\alpha}(e) := \{y(1)\} \).

To complete the \((n, \alpha)\)-collapsing operation \( \partial \to \partial^\circ_{n,\alpha} \), let \( \partial^\circ_{n,\alpha} := \partial \) in the case \( |S_{n,\alpha}| = 1 \). Now let \( \partial^\circ_{n,\alpha} \) arise from \( \partial \) by applying \((n, \alpha)\)-collapsing successively to all \( \alpha = \ell^\circ_n(x), x \in L_n(D) \), and arbitrary \( r \in S_{n,\alpha} \). Thus \( \partial^\circ_{n,\alpha} \) is the iteration of \( \partial^\circ_{n,\alpha} \) with respect to all \( \alpha \) occurring in the \( n^{th} \) section of \( D \). The operation \( \partial \to \partial^\circ_{n,\alpha} \) is called the horizontal collapsing on level \( n \), in \( N^*_\alpha \).

Lemma 9
For any \( \partial = (D, s, \ell^\circ, \ell^\circ) \in \mathcal{D}^*_n \), \( n \leq h(D) \), and \( \partial^\circ_n = (D_n, s_n, \ell^\circ_n, \ell^\circ_n) \), the following conditions 1–5 hold.

1. \( \partial^\circ_n \in \mathcal{D}^*_n \).
2. \( \mathbb{V}(D_n) \subseteq \mathbb{V}(D), \emptyset(D_n) = \emptyset(D) \) and \( h(D_n) = h(D) \).
3. For any \( n \neq i \leq h(D), L_i(D_n) = L_i(D) \), while \( L_n(D_n) \subseteq L_n(D) \) and \( |L_n(D_n)| \leq \phi(\partial) \).
4. For any \( i \leq h(D) \), \( \ell^\circ(L_i(D_n)) = \ell^\circ(L_i(D)) \). Thus \( \partial^\circ_n \) and \( \partial \) have the same formulas, and hence \( \phi(\partial^\circ_n) = \phi(\partial) \).
5. \( E_0(D) \subseteq E_0(D) \) and \( \Gamma_{\partial^\circ_n} = \Gamma_{\partial} \).
Proof. By iteration, it will suffice to prove analogous assertions with respect to every \((n, \alpha)\)-collapsing involved. We skip trivial conditions 2–4 and verify 1: \(\partial_{n,\alpha}^c = \langle D_{n,\alpha}, \Theta_{n,\alpha}, \ell^c_{n,\alpha}, \ell_{n,\alpha} \rangle \in D^*_n\). Consider the only nontrivial clause 3 of Definition 6. It will suffice to show that \(P(x, D_{n,\alpha}) \subseteq \bigcup_{y \in \langle x, D_{n,\alpha} \rangle} \ell^c_{n,\alpha}(x, y)\) holds for any \(x \in V(D_{n,\alpha}) \subseteq V(D)\) such that \(\deg (x, D_{n,\alpha}) > 0\) and \(\deg (x, D_{n,\alpha}) \geq 1\).

If \(h(x, D) < n\) or \(h(x, D) = n\) for \(x \neq r\), then \(P(x, D_{n,\alpha}) = P(x, D)\) and we are done by the assumption \(P(x, D) \subseteq \bigcup_{y \in \langle x, D \rangle} \ell^c((x, y))\) together with clauses 4 (a), (b) of Definition 8. Otherwise we have \(h(x, D) = n\) for \(x = r\). Then every \(z \in P(x, D_{n,\alpha})\) determines a \(u \in c(z, D) \cap S_{n,\alpha}\), and hence \(z \in P(u, D)\). Consider two cases.

1. Suppose \(\overrightarrow{\deg} (u, D) > 1\). By the assumption \(P(u, D) \subseteq \bigcup_{y \in \langle u, D \rangle} \ell^c((u, y))\) together with 4 (e) of Definition 8 this yields a \(y \in c(u, D) \subseteq c(x, D_{n,\alpha})\) with \(z \in \ell^c((u, y)) \subseteq \langle (x, y) \rangle\). Hence \(P(x, D_{n,\alpha}) \subseteq \bigcup_{y \in \langle x, D_{n,\alpha} \rangle} \ell^c_{n,\alpha}(x, y)\).

2. Suppose \(\overrightarrow{\deg} (u, D) = 1\). Then \(z = u_{(1)} \in \ell^c((u, y)) \subseteq \ell^c_{n,\alpha}(x, y)\) holds for any chosen \(y \in c(u, D) \subseteq c(x, D_{n,\alpha})\) according to 4 (e) of Definition 8. Hence \(P(x, D_{n,\alpha}) \subseteq \bigcup_{y \in \langle x, D_{n,\alpha} \rangle} \ell^c_{n,\alpha}(x, y)\).

This completes the proof of condition 1. Now consider 5 (with respect to every \((n, \alpha)\)-collapsing involved). \(E_0(D_{n,\alpha}) \subseteq E_0(D)\) is obvious, so it remains to establish \(\Gamma_{\partial_{n,\alpha}^c} \subseteq \Gamma_{\partial^c}\). In order to prove the (more important) inclusion \(\Gamma_{\partial_{n,\alpha}^c} \subseteq \Gamma_{\partial^c}\), it will suffice to show that there is an assumption-preserving embedding of the open threads in \(\partial_{n,\alpha}^c\) into the open threads in \(\partial^c\). So let \(\Theta_{n,\alpha} = \{v = x_0, u = x_1, \cdots, x_{h(D)} = \vartheta(D)\} \in \Theta(c_0, \Theta, \partial_{n,\alpha}^c), e = \langle u, v \rangle \in E_0(D_{n,\alpha})\), be any given open thread in \(\partial_{n,\alpha}^c\). A desired open thread in \(\partial^c\), \(\Theta = \{v' = x'_{0}, u' = x'_{1}, \cdots, x'_{h(D')} = \vartheta(D)\} \in \Theta(c_0, \Theta, \partial^c), e' = \langle u', v' \rangle \in E_0(D)\) for \(\ell^c_{n,\alpha}(v') = \ell^c(v)\) is defined by cases as follows.

1. Suppose \(r \neq x_i\) for all \(i \leq h(D)\). Then \(\Theta := \Theta_{n,\alpha}\), i.e. \((\forall i \leq h(D)) x'_i := x_i\).

2. Otherwise, \(r = x_m\) and \(\overrightarrow{\deg} (x_m, D_{n,\alpha}) > 1\), where \(m := h(D) - n > 0\). Consider the following two subcases.

   (a) Suppose \(m > 0\), i.e. \(n < h(D)\), and note that \(x_{m-1} \in V(D)\) and \(\overrightarrow{\deg} (x_{m-1}, D) = 1\). Then let \(x'_m := y\) such that \(P(x_{m-1}, D) = \{y\}\). Note that \(\ell^c(x'_m) = \ell^c_{n,\alpha}(x_m)\). For all \(i \neq m\) let \(x'_i := x_i\).

   (b) Let \(m = 0\), i.e. \(n = h(D)\). If \(\overrightarrow{\deg} (x_i, D) = 1\) for all \(0 < i < h(D)\), then let \(\Theta := \Theta_{n,\alpha}\). Otherwise, let \(j := \min \{i > 0 : \overrightarrow{\deg} (x_i, D) > 1\}\). Then let \(x'_j\) be any \(v' \in c(u, D) \cap S_{n,\alpha}\) such that \(x_{j+1} \in \ell^c((u, v'))\). Clearly \(\ell^c(x'_0) = \ell^c_{n,\alpha}(x_0)\). For all \(i > 0\) let \(x'_i := x_i\).
This completes our definition of \( \Theta \). That \( \Theta \) is an open thread is easily verified using definition of \( \ell^\alpha \) (see Definition 8 (4)). Thus \( \Gamma_{\Theta n,\alpha} \subseteq \Gamma_{\partial} \). \( \Gamma_{\partial} \subseteq \Gamma_{\Theta n,\alpha} \) is proved analogously by inversion \( \Theta \mapsto \Theta_{n,\alpha} \) that is defined by substituting \( r \) for (at most one) \( x_m \in S_{n,\alpha} \setminus \{ r \} \). This completes the whole proof. ■

### 3.5.2 Horizontal compressing

As mentioned above, horizontal compression \( \partial \mapsto \partial^c \) is obtained by bottom-up iteration of horizontal collapsing \( \partial \mapsto \partial^c_n, n \leq h(\partial) \). For the sake of brevity we consider tree-like inputs \( \partial \in T^* \).

**Definition 10 (horizontal compressing)** For any given \( \partial \in T^* \) denote by \( \partial^c \in D^* \) the last deduction in the following iteration chain

\[
\partial = \partial^c_{(0)}, \partial^c_{(1)}, \ldots, \partial^c_{(h(\partial))} = \partial^c
\]

where for every \( i < h(\partial) \) we let \( \partial^c_{(i+1)} := \left( \partial^c_{(i)} \right)^c_{i+1} \). It is readily seen that all \( \partial^c \) in question are mutually isomorphic (actually equal up to the choice of \( r \in S_{n,\alpha} \)). The operation \( \partial \mapsto \partial^c \) is called the horizontal dag-like compression, in \( \text{NM}^* \).

**Theorem 11** For any tree-like deduction \( \partial \in T^* \) with root-formula \( \rho \), the horizontal compression \( \partial \mapsto \partial^c \) is a plain dag-like \( \text{NM} \rightarrow \) deduction of \( \rho \) from the same assumptions \( \Gamma_{\partial} = \Gamma_{\partial^c} \). Moreover \( |\partial^c| \leq h(\partial) \times \phi(\partial) \) and \( \mu(\partial^c) = \mu(\partial) \). In particular, if \( \Gamma_\partial = \emptyset \) and \( h(\partial), \phi(\partial), \mu(\partial) \) are polynomial in \( |\rho| \), then \( \partial^c \) is a plain dag-like \( \text{NM} \rightarrow \) proof of \( \rho \) whose size and weight are polynomial in \( |\rho| \).

**Proof.** Let \( \partial = (T, s, \ell^\rho, c^\rho) \in T^* \) and \( \partial^c_n = (D_n, s_n, \ell^c_{n}, c^c_{n}) \) for \( n \leq h(D) \). By Lemma 9 (2, 3) we have

\[
|\partial^c| = \bigcup_{n=0}^{h(T)} |L_n(D_n)| \leq 1 + 2 + \bigcup_{n=2}^{h(T)} |L_n(D_n)| \leq 3 + (h(T) - 1) \cdot \phi(\partial) < h(T) \cdot \phi(\partial) = h(\partial) \times \phi(\partial)
\]

as required. The rest immediately follows from Lemma 9 (1, 4, 5) by induction on \( n \leq h(T) \). ■

Together with Theorem 4 and Lemma 5 this yields

**Corollary 12** Any given minimal tautology \( \rho \) has a plain dag-like \( \text{NM} \rightarrow \) proof \( \partial^c \) whose size and weight are polynomial in \( |\rho| \). Actually the following holds.

\[
|\partial^c| < 18 |\rho| (|\rho| + 1)^2 (|\rho| + 2) = O \left( |\rho|^4 \right) \text{ and } \|\partial^c\| = O \left( |\rho|^5 \right)
\]
Example 13 Consider a following (tree-like) NM→ deduction \( \partial \).

\[
\begin{array}{c}
\alpha \rightarrow \rho \\
\rho \rightarrow \alpha \\
\alpha \\
\rho \\
\alpha \rightarrow \rho \\
\rho \\
\alpha \rightarrow \rho \\
\rho \\
\alpha \rightarrow \rho \\
\rho \\
\alpha \rightarrow \rho \\
\rho \\
\alpha \rightarrow \rho \\
\rho \\
\alpha \rightarrow \rho \\
\rho \\
\alpha \rightarrow \rho \\
\rho \\
\alpha \rightarrow \rho \\
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\alpha \rightarrow \rho \\
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\alpha \rightarrow \rho \\
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\alpha \rightarrow \rho \\
\rho \\
\alpha \rightarrow \rho \\
\rho \\
\alpha \rightarrow \rho \\
\rho \\
\alpha \rightarrow \rho \\
\rho \\
\alpha \rightarrow \rho \\
\rho \\
\alpha \rightarrow \rho \\
\rho \in T^*
\end{array}
\]

(As usual \([\alpha]\) indicates that the right-hand side assumption \(\alpha\) is discharged by \((\rightarrow I) : \rho \rightarrow \alpha\) occurring below.) Horizontally compressed (re)dag-like NM*→ deduction \(\partial^c\) arises by successively merging two nodes with label \(\rho\) and two identical pairs of assumptions \(\alpha, \alpha \rightarrow \rho\).

\[
\begin{array}{c}
\rho \rightarrow \alpha \\
\rho \rightarrow \alpha \\
\alpha \\
\rho \\
\rho \rightarrow \alpha \\
\alpha \rightarrow \rho \\
\rho \\
\alpha \rightarrow \rho \\
\rho \\
\alpha \rightarrow \rho \\
\rho \\
\alpha \rightarrow \rho \\
\rho \\
\alpha \rightarrow \rho \\
\rho \\
\alpha \rightarrow \rho \\
\rho \\
\alpha \rightarrow \rho \\
\rho \in \text{NM}\* \rightarrow
\end{array}
\]

Clearly \(\partial^c\) is not a tree, as it contains a “diamond” \(\alpha \rightarrow \rho \).

Note that \(\Gamma_\partial = \{\alpha, \alpha \rightarrow \rho, \rho \rightarrow \alpha\}\), as the left-hand side assumption \(\alpha\) is open in a tree-like deduction \(\partial\). No consider the compressed redag \(\partial^c\). We have

\[
\begin{array}{c}
u_1 : \rho \rightarrow \alpha \\
v_2 : \alpha \\
u_3 : \alpha \rightarrow \rho \\
w_1 : \alpha \\
w_2 : \alpha \rightarrow \rho \\
z : \rho
\end{array}
\]

(\(u_i, v_j, w_j\) and \(z = q(\partial^c)\) being the underlying nodes). Except for \(v_2\) and \(z\), all nodes have exactly one parent, while \(\ell^\partial ((v_2, u_2)) = \ell^\partial ((v_2, u_3)) = \{w_1, w_2\}\), i.e. both leaves \(u_2\) and \(u_3\) have two \(\ell^\partial\)-grandparents \(w_1\) and \(w_2\) (which are inherited from standard tree-like grandparents of the first and the last top nodes in \(\partial\)). This yields 5 deduction threads in \(\partial^c\): \(\{u_1, v_1, w_1, z\}, \{u_2, v_2, w_1, z\}, \{u_2, v_2, w_2, z\}, \{u_3, v_2, w_1, z\}, \{u_3, v_2, w_2, z\}\) and \(\ell^\partial\)-threads \(\{\rho \rightarrow \alpha, \rho \rightarrow \alpha, \alpha, \rho\}, \{\alpha, \rho, \alpha \rightarrow \rho, \rho\}, \{\alpha \rightarrow \rho, \rho, \alpha, \rho\}, \{\alpha \rightarrow \rho, \rho, \alpha, \rho\}\), while \(\alpha\) is open in \(\Theta = \{u_2, v_2, w_1, z\}\) due to \(\ell^\partial (\Theta) = \{\alpha, \rho, \alpha, \rho\}\) (that other assumptions \(\alpha \rightarrow \rho, \rho \rightarrow \alpha\) are open in \(\partial^c\) is readily seen). Hence \(\Gamma_{\partial^c} = \{\alpha, \alpha \rightarrow \rho, \rho \rightarrow \alpha\} = \)


\^See Appendices B, C for more sophisticated examples.
\[ \Gamma_\theta \text{, i.e. } \vartheta \text{ and } \vartheta^c \text{ are deductions of } \rho \text{ from the assumptions } \{ \alpha, \alpha \to \rho, \rho \to \alpha \}, \text{ although at the first glance } \alpha \text{ seems to be discharged in } \vartheta^c. \]

3.6 Dag-to-tree unfolding in NM\(_n\)

We learned that all minimal propositional tautologies are provable by plain dag-like NM\(_n\) deductions of “small” size, but at the moment we don’t know whether underlying NM\(_n^*\) provability infers validity in minimal logic. The affirmative answer follows by dag-to-tree unfolding, to be thought of as inversion of the tree-to-dag compression under consideration. The unfolded tree-like deduction \( \vartheta^c \) is defined by descending recursion on the height of a given NM\(_n^*\) deduction \( \vartheta \) such that for any \( n \leq h(\vartheta) \), the \( n^{th} \) horizontal section of \( \vartheta^c \) is obtained by splitting previously obtained nodes \( v, h(v) = n \), having \( p \) parents, \( u_1, \ldots, u_p \), \( p > 1 \), into \( p \) new copies \( v_1, \ldots, v_p \). Previously obtained (tree-like!) successors of \( v \) are separated according to the underlying assignment \( \ell^\varepsilon \) such that for every \( 0 < i < p \), \( v_i \) becomes the only parent of \( v_i \). Moreover, if \( T \) is the old tree rooted in \( v \), then every \( v_i \) (\( 0 < i < p \)) becomes the root of a maximal subtree of \( T \) whose leaves are \( \ell^c \)-grandchildren of \( u_i \) (i.e. \( u_i \) is a \( \ell^c \)-grandparent of every leaf in question). Except for the \( \ell^c \)-related separation this is just standard graph theoretic dag-to-tree unfolding (see below a precise definition).

**Definition 14** Any \( \vartheta = \langle D, s, \ell^c, \ell^0 \rangle \in \mathcal{D}^*_n \), \( n \leq h(D) \) and a fixed \( r \in L_n(D) \) with \( p := |\varphi(r,D)| > 1 \). We define \((n,r)\)-unfolded deduction \( \vartheta^c_{n,r} = \langle D_{n,r}, s_{n,r}, \ell^c_{n,r}, \ell^0_{n,r} \rangle \in \mathcal{D}^*_n \) that arises by tree-like unfolding of \( r \), as follows. Let \( r_1, \ldots, r_p \notin \varphi(D) \) be a fixed collection of new vertices and \( \langle D \rangle_r_1, \ldots, \langle D \rangle_r_p \) the corresponding collection of disjoint copies of \( \langle D \rangle_r \). Let \( \varepsilon : [p] \to \varphi(r,D) \) be a fixed 1–1 enumeration of \( \varphi(r,D) \). Then for any \( i \in [p] \) we denote by \( \langle D \rangle^\varepsilon_i \) a subtree of \( \langle D \rangle_r \) that is obtained by deleting the (copies of) subtrees \( \langle D \rangle_{\varepsilon(i)} \) for all \( (x,y) \in \varepsilon^{-1}(\langle D \rangle_{\varepsilon(i)}) \) such that \( \varepsilon(i) \notin \ell^c(\langle x,y \rangle) \). Furthermore, we denote by \( \langle D \rangle^\varepsilon \) a tree that extends \( \langle D \rangle^\varepsilon_i \) by a new root \( \varepsilon(i) \); thus \( \varepsilon \langle D \rangle^\varepsilon_i = r_i \) and \( \langle D \rangle^\varepsilon = \langle D \rangle^\varepsilon \) with \( \varepsilon(i) = \varphi(r_i, \langle D \rangle^\varepsilon_i) \). Having this we stipulate:

1. \( D_{n,r} \) arises from \( D \) by deleting \( \langle D \rangle_r \) and replacing every remaining node \( \varepsilon(i) \in \varphi(r,D) \) by the whole subtree \( \langle D \rangle^\varepsilon_i \).

That is, \( \varphi(D_{n,r}) := \left( \varphi(D) \setminus \bigvee \langle D \rangle_r \right) \cup \bigcup_{i=1}^p \varphi \langle D \rangle^\varepsilon_i \). The edges are given by \( E(D_{n,r}) := (E(D) \setminus E(\langle D \rangle_r)) \cup \bigcup_{i=1}^p (E(\langle D \rangle^\varepsilon_i) \cup \langle \varepsilon(i), r_i \rangle) \).

2. For any \( u \in \varphi(D_{n,r}) \) we define \( s_{n,r}(u, D_{n,r}) \) by cases as follows.

(a) If \( u \notin \bigcup_{i=1}^p \varphi(\langle D \rangle^\varepsilon_i) \cup \varphi(r,D) \), then \( s_{n,r}(u, D_{n,r}) := s(u, D) \).
Lemma 15 For any horizontal unfolding on level \( n, r \) successively to all nodes \( n, r \) to every (\( \mathcal{NM}^* \)) (\( \mathcal{D} \)) (\( \mathcal{S} \)) (\( \mathcal{D} \)).

(c) For any \( i \in [1, p] \) we let \( \mathcal{S}_{n,r}(\varepsilon(i), D_{n,r}) := X_i \cup Y_i \), where

\[
X_i = \left\{ y \in L_n(D_{n,r}) : (\exists x \in S(\varepsilon(i), D)) (x = y \vee (x = r \land y = r_i)) \right\}
\]

\[
Y_i = \left\{ (y_0, y_1) \in L_n(D_{n,r}) : (\exists \langle x_0, x_1 \rangle \in S(\varepsilon(i), D)) (\forall j \leq 1) \right\}
\]

3. For any \( u \in \mathcal{V}(D_{n,r}) \) we let \( \ell_{n,r}^u(u) := \ell^u(\hat{u}) \), where \( \hat{u} \in \mathcal{V}(D) \) is a (uniquely determined) preimage of \( u \) in \( D \).

4. For any \( e = \langle u, v \rangle \in \mathcal{E}_\Delta(D_{n,r}) \) we define \( \ell_{n,r}^e(e) \) by cases as follows, without loss of generality assuming that \( v \in L(D_{n,r}) \) or \( \deg(v, D_{n,r}) > 1 \) (cf. analogous passage in Definition 8).

(a) If \( h(u, D_{n,r}) \in [0, n - 2] \), or \( h(u, D_{n,r}) = n - 1 \) and \( v \notin \{r_1, \ldots, r_p\} \), or else \( e \in E_0(D_{n,r}) \) with \( h(u, D_{n,r}) \geq n \) and \( (\forall i \in [p]) r_i \nsubseteq D_{n,r} \), then \( \ell_{n,r}^e(u) := \ell^e(u) \).

(b) Otherwise, if \( v = r_i \) (hence \( u = \varepsilon(i) \)), or else \( e \in E_0(D_{n,r}) \) with \( h(u, D_{n,r}) \geq n \) and \( r_i \nsubseteq D_{n,r} \), then \( \ell_{n,r}^e(e) := \ell^e((\varepsilon(i), r_i)) \).

To complete the \( (n, r) \)-unfolding operation \( \partial \mapsto \partial_{n,r}^u \), we let \( \partial_{n,r}^u := \partial \) in the case \( |\partial(r, D)| = 1 \). Now let \( \partial_{n}^u \) arise from \( \partial \) by applying \( (n, r) \)-unfolding successively to all nodes \( r \) occurring in the \( n \)th horizontal section of \( D \). The operation \( \partial \mapsto \partial_{n}^u \) is called the horizontal unfolding on level \( n \), in \( \mathcal{NM}^*_n \).

Lemma 15 For any \( \partial = \langle D, s, \ell^r, \ell^c \rangle \in \mathcal{D}^*_n \) and \( \partial_{n}^u = \langle D_n, s_n, \ell_{n,r}^u, \ell_{n}^u \rangle \), \( n \leq h(D) \), the following conditions 1–5 hold.

1. \( \partial_{n}^u \in \mathcal{D}^*_n \).
2. \( \mathcal{Q}(D_n) = \mathcal{Q}(D) \) and \( h(D_n) = h(D) \).
3. For any \( i < n \), \( L_i(D_n) = L_i(D) \), while \( L_n(D_n) \supseteq L_n(D) \).
4. For any \( i < n < j \), \( \ell^r(L_i(D_n)) = \ell^r(L_i(D)) \) and \( \ell^u(L_j(D_n)) \subseteq \ell^u(L_j(D)) \), while \( \ell^c(L_n(D_n)) = \ell^c(L_n(D)) \). Hence \( \phi(\partial_{n}^u) \subseteq \phi(\partial) \).
5. \( \Gamma_{\partial_{n}^u} \subseteq \Gamma_{\partial} \).

Proof. By iteration, it will suffice to prove analogous assertions with respect to every \( (n, r) \)-unfolding involved. We skip trivial conditions 2–4 and verify 1: \( \partial_{n,r}^u = \langle D_{n,r}, s_{n,r}, \ell_{n,r}^u, \ell_{n,r}^c \rangle \in \mathcal{D}^*_{n-1} \). First of all we observe that every subtree \( \mathcal{T}_{i}^{(D)} \) that replaced \( \varepsilon(i) \in \mathcal{P}(r, D) \) according to clause 1 of Definition 14 represents a (tree-like) \( \mathcal{NM}^*_n \) deduction of \( \ell^c(\varepsilon(i)) \) such that \( h(\mathcal{T}_{i}^{(D)}) = \mathcal{T}_{i}^{(D_{n,r})} \).
1 + h \left( (D)_i \right) = 1 + h \left( (D)_r \right). This easily follows by induction on \( h \left( (D)_r \right) \) using clause 2 of Definition 6 with respect to \( \partial \). So \( \partial_{n,r}^v \) is structurally well-defined.

To complete the proof of local correctness consider the only nontrivial clause 3 of Definition 6 with respect to \( \partial_{n,r}^v \). It will suffice to show that \( p(x,D_{n,r}) \subseteq \bigcup_{y \in c(x,D_{n,r})} \ell^v_{n,r}((x,y)) \) holds for any \( x \in v(D_{n,r}) \) such that \( \deg(x,D_{n,r}) > 0 \) and \( \deg(x,D_{n,r}) > 1 \). If \( h(x,D) < n \) for \( r \notin c(x,D) \), or \( h(x,D) = n \) for \( x \neq r \), then \( p(x,D_{n,r}) = p(x,D) \) and we are done by the assumption \( p(x,D) \subseteq \bigcup_{y \in c(x,D)} \ell^v((x,y)) \). Consider the remaining cases.

1. Suppose \( x = r_i \) (1 \( \leq i \leq p \)). We have \( p(x,D_{n,r}) = \{ \varepsilon(i) \} \subseteq p(r,D) \). Moreover, by the assumption \( p(r,D) \subseteq \bigcup_{y \in c(r,D)} \ell^v((r,y)) \), there exists a \( y \in c(r,D) \) with \( \varepsilon(i) \in \ell^v((r,y)) \). From this, by the definition of \( (D)_i \) and clauses 2 (b), (c) of Definition 6 with respect to \( \partial_{n,r}^v \), we arrive at \( y \in c(x,D_{n,r}) \) and \( \varepsilon(i) \in \ell^v_{n,r}((x,y)) \). Thus \( p(x,D_{n,r}) \subseteq \bigcup_{y \in c(x,D_{n,r})} \ell^v_{n,r}((x,y)) \).

2. Suppose \( r \in c(x,D) \), and hence \( x = u_i = \varepsilon(i) \) for some \( 1 \leq i \leq p \). Consider any \( z \in p(x,D_{n,r}) = p(x,D) \subseteq \bigcup_{y \in c(x,D)} \ell^v((x,y)) \) and let \( z \in \ell^v((x,y)) \) for some \( y \in c(x,D) \). If \( y \neq r \) then \( y \in c(x,D_{n,r}) \) and we are done. Otherwise \( y = r \), and then by clause 4 (b) of Definition 14 we arrive at \( x = \varepsilon(i) \in \ell^v((\varepsilon(i),r)) = \ell^v_{n,r}((\varepsilon(i),r)) \) with \( r_i \in c(x,D_{n,r}) \). Hence \( p(x,D_{n,r}) \subseteq \bigcup_{y \in c(x,D_{n,r})} \ell^v_{n,r}((x,y)) \).

This completes the proof of condition 1. Now consider 5 (with respect to every \( (n,r) \)-unfolding involved). In order to prove \( \Gamma_{\partial_{n,r}^v} \subseteq \Gamma_{\partial} \), it will suffice to show that there is an assumption-preserving embedding of the open threads in \( \partial_{n,r}^v \) into the open threads in \( \partial \). Let \( \Theta_{n,r} = [v = x_0, u = x_1, \ldots, x_{h(D)} = g(D)] \in \Theta(e, g(D), \partial_{n,r}^v), e = \langle u, v \rangle \in E_0(D_{n,r}) \), be any given open thread in \( \partial_{n,r}^v \).

(We consider only the proper case \( \deg(r,D) > 1 \).) A desired open thread in \( \partial \), \( \Theta = [v' = x'_0, u' = x'_1, \ldots, x'_{h(D)} = g(D)] \in \Theta(e', g(D), \partial) \), \( e' = \langle u', v' \rangle \in E_0(D) \) for \( \ell^v_{n,\alpha}(v') = \ell^v(v) \) is obtained by substituting \( r \) for any \( r_i \) occurring in \( \Theta \). That is, for any \( j \leq h(D) \) we let \( x'_j := r \), if \( j = n \) and \( x_j \neq r \), else \( x'_j := x_j \). That \( \Theta \in \Theta(e', g(D), \partial) \) and \( e' = \langle x'_1, x'_n \rangle \in E_0(D) \) easily follows by the definition of \( \ell^v_{n,\alpha} \). Hence \( \Gamma_{\partial_{n,r}^v} \subseteq \Gamma_{\partial} \). This completes the whole proof by iteration with respect to all \( r \in L_n(D), \deg(r,D) > 1 \) involved. ■

**Definition 16 (horizontal unfolding)** For any given \( \partial \in \mathcal{D}^* \) denote by \( \partial^v \in \mathcal{T}^* \) the last deduction in the following iteration chain

\[
\partial = \partial_{(h(\partial))}^v, \partial_{(h(\partial) - 1)}^v, \ldots, \partial_{(0)}^v = \partial^v
\]
where for every \( i < h(\partial) \) we let \( \partial_{(i-1)}^v := \left( \partial_{(i)}^v \right)^{v}_{i-1} \). It is readily seen that all \( \partial^v \) in question are mutually isomorphic (actually equal up to the enumerations \( \varepsilon \)). The operation \( \partial \rightarrow \partial^v \) is called the horizontal unfolding, in \( \text{NM}^*_\star \).

**Theorem 17** For any dag-like \( \text{NM}^*_\star \) deduction \( \partial \) with root-formula \( \rho \), the horizontal unfolding \( \partial^v \) is a tree-like \( \text{NM}^*_\star \) deduction of \( \rho \) such that \( \Gamma_{\partial^v} \subseteq \Gamma_\partial \). In particular, if \( \partial \) is a plain dag-like \( \text{NM}_\rightarrow \) proof of \( \rho \), then \( \partial^v \) is a tree-like \( \text{NM}^*_\star \) proof of \( \rho \).

**Proof.** The assertions follow by iteration from Lemma 15, as \( \Gamma_{\partial^v} \subseteq \Gamma_\partial = \emptyset \) obviously implies \( \Gamma_{\partial^v} = \emptyset \). Together with Lemma 5 the latter assertion yields

**Corollary 18** Plain dag-like \( \text{NM}_\rightarrow \) provability is sound and complete with respect to minimal propositional logic.

Together with Corollary 12 this yields

**Conclusion 19** A given formula \( \rho \) is a tautology in minimal propositional logic iff there exists a plain dag-like \( \text{NM}_\rightarrow \) proof of \( \rho \) whose size and weight are \( O\left(|\rho|^4\right) \) and \( O\left(|\rho|^5\right) \), respectively.

### 3.6.1 Local correctness and complexity of verification

Our definition of plain dag-like provability via ‘global’ discharging function (Definition 5) is inappropriate for polytime verification. This is because ‘\( \alpha \) is an open (resp. closed) assumption’ refers to potentially exponential set of threads \( \text{TH}(e, \varrho, \partial) \) for \( e = \langle u, v \rangle \) with \( \ell^\rho(v) = \alpha \) in a given plain redag \( \partial = \langle D, s, \ell^\rho, \ell^\varepsilon \rangle \), thus being merely a NP (resp. coNP) problem, unless \( \partial \) is a tree. To overcome this obstacle we upgrade basic (standard) conditions of local correctness of a given (re)dag-like deduction \( \partial \) (cf. Definition 6) by adding a new labeling function \( \ell^\partial \) that assigns boolean values 0 or 1 to all pairs \( (e, \alpha) \), where \( e = \langle u, v \rangle \) is an edge and \( \alpha \) an assumption, in \( \partial \). Informally, \( \ell^\partial(e, \alpha) = 1 \) says that in every \( \Theta \in \text{TH}(e, \varrho, \partial) \), \( \alpha \) is discharged at \( e \), or below, by (sub)occurrences of \( (\rightarrow I) \) with premise \( \alpha \). The corresponding new condition \( \bullet \) of the local correctness in question is shown below, where \( K(u) = \{x_0, \ldots, x_k\} \) for \( x_0 = u \) and \( x_k = U(u) \).

- \( \ell^\partial(e, \alpha) = 1 \) iff one of the following holds.
  1. \( u = \varrho \) and \( \ell^\rho(u) = \alpha \rightarrow \ell^\rho(v) \).
  2. \( u \neq \varrho \) and
    - (a) either \( \ell^\rho(x_{i+1}) = \alpha \rightarrow \ell^\rho(x_i) \) holds for some \( 0 \leq i < k \),
    - (b) or \( U(u) \neq \varrho \) and \( \prod_{w \in \ell^\rho(e)} \ell^\partial((w, U(u)), \alpha) = 1 \).
Keeping this in mind we can present “plain” assertion $\Gamma_\emptyset = \emptyset$ in a simplified “encoded” form $(\forall \langle u, v \rangle \in E_0(D)) \ell^0(\langle u, v \rangle, \ell^0(v)) = 1$.

**Definition 20** $\text{NM}_\emptyset^*$, deductions $\partial$ enriched by $\ell^0$ and satisfying all conditions of the upgraded local correctness (including $\bullet$), are called encoded dag-like $\text{NM}_\emptyset$ deductions. A given assumption $\alpha$ in an encoded dag-like $\text{NM}_\emptyset$ deduction $\partial$ of $\rho$ is called closed (or discharged) if for every leaf $v$ with $\ell^0(v) = \alpha$ and every edge $e = \langle u, v \rangle$ we have $\ell^0(e, \alpha) = 1$. Otherwise $\alpha$ is called open (or undischarged).

Furthermore, as in the case of plain dag-like $\text{NM}_\emptyset$ deductions, we denote by $\Gamma_\emptyset$ the set of open assumptions and call $\partial$ an encoded dag-like $\text{NM}_\emptyset$ deduction of $\rho$ from the assumptions $\Gamma_\emptyset$. If $\Gamma_\emptyset = \emptyset$, then $\partial$ is called an encoded dag-like $\text{NM}_\emptyset$ proof of $\rho$.

**Lemma 21 (plain = encoded)** The notions of plain and encoded deducibility and/or provability are equivalent, while $\ell^0$ is uniquely determined by $\ell^*$ and $\ell^\emptyset$.

In particular, any plain dag-like $\text{NM}_\emptyset$ proof of $\rho$ can be both upgraded to and degraded from an encoded dag-like $\text{NM}_\emptyset$ proof of $\rho$ of the same size. Moreover, a statement $\partial$ is an encoded dag-like $\text{NM}_\emptyset$ proof of $\rho^*$ is verifiable by a TM in $||\partial||$-polynomial time.

**Proof.** To prove first two assertions it will suffice to show that for any $e = \langle u, v \rangle \in E_0(D)$, $\ell^0(e, \ell^0(v)) = 0$ holds iff there exists an open thread $\Theta \in \text{TH}(e, g, \partial)$. Actually we observe that a stronger equivalence stating that for any $e = \langle u, v \rangle \in E(D)$ and $\alpha \in \ell^0(\forall (D))$, $\ell^0(e, \alpha) = 0$ iff there exists a $(\rightarrow)_\alpha$-free thread $\Theta \in \text{TH}(e, g, \partial)$, is provable by induction on $h(v)$. The corresponding induction step easily follows from clause 2 (b) of the local correctness condition $\bullet$.

To establish the last assertion we’ll specify standard encoding of an encoded $\partial = (D, s, \ell^*, \ell^\emptyset, \ell^0)$ in the alphabet of $\mathcal{L}_\emptyset$ extended by $0, 1$ and $v_0, \ldots, v_{|\forall (D)|-1}$ (encoded vertices). Let $N := \{0, \ldots, h(D) - 1\}$, $V := \{v_0, \ldots, v_{|\forall (D)|-1}\}$, $F := \ell^* (\forall (D))$ and consider the following sets/relations

\[
H \subseteq V \times N, \ E \subseteq V^2, \ D_1 \subseteq V, \ S \subseteq V^2 \cup V^3, \ K \subseteq V^2, \ L^F \subseteq V \times F, \ L^E \subseteq E \times V, \ L^p \subseteq E \times F
\]

representing respectively

\[
H \cong \{\langle u, h(u) \rangle\}_u \in V(D), \ E \cong E(D), \ D_1 \cong \{u \in \forall (D) : \deg (u) = 1\}, \ S \cong \{\langle u, z \rangle : z \in s(u)\}_u \in V(D), \ K \cong \{\langle u, x_i \rangle : K (u) = [x_0, \ldots, x_k], i \leq k\}_u \in V(D), \ L^F \cong \{\langle u, \ell^F(u) \rangle\}_u \in V(D), \ L^E \cong \{\langle x, \ell^E(e) \rangle : x \in \ell^E(e)\}_e \in E(D), \ L^p \cong \{\langle e, \alpha \rangle : \ell^p(e, \alpha) = 1\}_e \in E(D)
\]

(cf. Definition 6)\(^9\) Note that for any nontrivial $\partial$ we have:

---

\(^9\)Recall that $K(u)$ are uniquely determined by $u$ and $U(u) = \emptyset$. In particular this shows that in standard tree-like case the entire verification is trivial.

\(^{10}\)For brevity we assume that $v_0$ corresponds to $\emptyset$. Note that $C(u)$ and $P(u)$ are easily parametrizable in $E$. 

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\[ \|V\| = |V| = |\partial| \leq \|\partial\|, \]
\[ \|D_1\| = |D_1| \leq |\partial| \leq \|\partial\|, \]
\[ \|H\| \leq |V| \log h(D) \leq |V| \log |V(D)| = |\partial| \log |\partial| < \|\partial\|^2, \]
\[ \max \{\|E\|, \|K\|\} = 2 \max \{|E|, |K|\} \leq 2 |\partial|^2 < \|\partial\|^3, \]
\[ \|S\| = |S| < 2 |\partial|^3 \leq \|\partial\|^3, \]
\[ \|L^p\| \leq |V| \times \mu(\partial) = |\partial| \times \mu(\partial) \leq \|\partial\|, \]
\[ \|L^o\| = |L^o| \leq |E| \times |V| = |\partial|^3 \leq \|\partial\|^3, \]
\[ \|L^o\| \leq |E| \times |F| \times \mu(\partial) \leq |\partial|^2 \times \phi(\partial) \times \mu(\partial) \leq \|\partial\|^3. \]

Hence a tuple \( t = \langle H, E, L, D_1, S, K, L^p, L^o, L^d \rangle \) can be represented in the extended language by a string \( s \) of the length \( \leq O(\|\partial\|^3) \). Having this we observe that upgraded local correctness of any given encoded redag \( \partial = \langle D, S, \ell^p, \ell^o, \ell^d \rangle \) is a boolean combination of at most \( O(|\partial|^9) \) many elementary equations and queries over components of \( s \). To put it more exactly, the upgraded local correctness of \( \partial \) is the conjunction of the following boolean assertions 1 – 24 for \( x, y, z, u, v, w \) and \( i, j \) and \( \alpha, \beta \) ranging over \( V \) and \( N \) and \( F \), respectively, where we use abbreviations:

1. \( \bigvee_{i \in N} \langle u, i \rangle \in H \)
2. \( \langle u, 0 \rangle \in H \Leftrightarrow u = q \)
3. \( u \notin L \lor \langle u, h(D) - 1 \rangle \in H \)
4. \( \langle u, i \rangle \notin E \lor \langle u, j \rangle \notin E \lor i = j \)
5. $\langle u, x \rangle \notin E \lor \langle u, y \rangle \notin E \lor \langle u, i \rangle \notin H \lor \langle x, i + 1 \rangle \in H \land \langle y, i + 1 \rangle \in H$

6. $\langle u, \langle x, y \rangle \rangle \notin S \lor (\langle u, x \rangle \in E \land \langle u, y \rangle \in E)$

7. $\langle u, x \rangle \in E \iff \langle u, x \rangle \in S \lor \bigvee_{y \in V} (\langle u, \langle x, y \rangle \rangle \in S \lor \langle u, \langle y, x \rangle \rangle \in S)$

8. $x \notin D_1 \lor \bigvee_{v \in V} \langle v, x \rangle \in E$

9. $x \notin D_1 \lor \langle y, x \rangle \notin E \lor \langle z, x \rangle \notin E \lor y = z$

10. $\langle u, u \rangle \in K$

11. $u = x \lor (\langle u, x \rangle \in K \iff \bigvee_{y \in V} (\langle x, y \rangle \in E \land y \in D_1 \land \langle u, y \rangle \in K))$

12. $\langle \rho, \rho \rangle \in L^f$

13. $\langle u, \alpha \rangle \notin L^f \lor \langle u, \beta \rangle \notin L^f \lor \alpha = \beta$

14. $\langle u, x \rangle \notin S \lor \langle u, \gamma \rangle \notin L^f \lor \langle x, \gamma \rangle \in L^f \lor (\gamma = \alpha \rightarrow \beta \land \langle x, \beta \rangle \in L^f)$

15. $\langle u, \langle x, y \rangle \rangle \notin S \lor \langle u, \beta \rangle \notin L^f \lor \bigvee_{a \in F} \langle x, \alpha \rangle \in L^f \land \langle y, \alpha \rightarrow \beta \rangle \in L^f$

16. $\langle \langle y, x \rangle, x \rangle \notin L^a \lor \langle u, y \rangle \in E$

17. $\langle u, y \rangle \notin U \lor \langle (u, v) \rangle \notin L^a \lor \langle x, y \rangle \in E$

18. $\langle u, v \rangle \notin E \lor \langle u, y \rangle \notin U \lor y = q \lor \bigvee_{x \in V} \langle (u, v) \rangle \in L^a$

19. $\langle u, \langle v, w \rangle \rangle \notin S \lor (\langle (u, v) \rangle \in L^a \iff (\langle u, w \rangle \rangle \in L^a)$

20. $v \notin D_1 \lor (\langle (u, v) \rangle \in L^a \iff \bigvee_{z \in V} \langle (v, z) \rangle \in L^a)$

21. $u \in D_1 \lor \langle x, u \rangle \notin E \lor \bigvee_{v \in V} \langle (u, v) \rangle \in L^a$

22. $\langle (u, v) \rangle \notin L^b \lor \langle u, v \rangle \in E$

23. $\langle (\rho, v) \rangle \in L^b \iff \langle v, \rho \rangle \in (\rightarrow I)_\alpha$
24. \[ u = q \lor \langle u, z \rangle \notin U \lor (\langle \langle u, v \rangle, \alpha \rangle \notin L^D \iff R(u, v, z, \alpha)) \]

It is easily provable by induction on \( h(D) \) that the required statement ‘\( \partial \) is an encoded dag-like NM→ proof of \( \rho \)’ is equivalent to universal conjunction
\[
\left( \bigwedge_{(u,v) \in E} \langle v, h(D) - 1 \rangle \notin H \lor \langle v, \alpha \rangle \notin L^F \lor \langle \langle u, v \rangle, \alpha \rangle \in L^D \right)
\]
corresponds to \( \Gamma_{\partial} = \emptyset \).

The longest conjunct 24 includes \( \leq O(|\partial|^9) \) many equations \( \chi = \xi \) for \( \chi, \xi \in X \) and queries \( \chi \in X, \chi \notin X \) for \( X \in \{H, E, L, D, S, K, L^F, L^G, L^D\} \), while every query in question is verifiable (say, by binary search algorithm) by a deterministic TM in \( \leq O(\log |s|) \) time. Hence by any chosen polytime search and verification algorithm the whole conjunction \( \left( \bigwedge \right) 1 \land \cdots \land 24 \land 25 \) corresponding to ‘\( \partial \) is an encoded dag-like NM→ proof of \( \rho \)’ is verifiable by a deterministic TM in \( \|\partial\|\)-polynomial time.

**Corollary 22** \( \mathcal{NP} = \mathcal{PSPACE} \), and hence \( \mathcal{NP} = \text{coNP} = \mathcal{PSPACE} \).

**Proof.** Recall that the validity problem for both intuitionistic and minimal propositional logics is PSPACE-complete, cf. [9], [10], [8]. It will suffice to show that it is a NP problem. So consider any given \( L \rightarrow \) formula \( \rho \). By Conclusion 19, \( \rho \) is valid in the minimal logic iff there exists an encoded dag-like NM→ proof \( \partial \) of \( \rho \) of the size \( |\partial| = O(|\rho|^4) \) and weight \( \|\partial\| = O(|\rho|^5) \). Moreover, by Lemma 21, the assertion ‘\( \partial \) is an encoded dag-like NM→ proof of \( \rho \)’ is verifiable by a deterministic TM \( M \) in polynomial time with respect to \( \|\partial\| \), and hence also \( |\rho| \). Hence there exists a polytime TM \( M \) such that \( \rho \) is valid in the minimal logic iff we can “guess” an encoded dag-like NM→ proof \( \partial \) of the weight \( O(|\rho|^5) \) and confirm its local correctness by \( M \) in \( |\rho|\)-polynomial time. This shows that the underlying problem of minimal validity belongs to \( \mathcal{NP} \), as desired. The rest follows from Sawitch’s theorem [8].

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Appendix A: proof of Lemma 2 (4)

A required loose upper bound $ssf(\xi) \leq (|\xi| + 1)^2$ is proved by induction on $|\xi|$, as follows. Recall the recursive clauses 1–3:

1. $ssf(p) := 1$.
2. $ssf(p \to \alpha) := 2 + ssf(\alpha)$.
3. $ssf((\alpha \to \beta) \to \gamma) := 1 + ssf(\alpha \to \beta) + ssf(\beta \to \gamma) - ssf(\beta)$.

- Basis of induction. Suppose $|\xi| = 0$. Hence $\xi = p$ and $ssf(\xi) = 1 = (|\xi| + 1)^2$, since $|p| = 0$.

- Induction step. Suppose $|\xi| > 0$. Hence $\xi = \alpha \to \beta$.

  - If $|\alpha| = 0$, then $\alpha = p$ and $ssf(\xi) = 2 + ssf(\beta) \leq 2 + (|\beta| + 1)^2 \leq (|\beta| + 2)^2 = (|\xi| + 1)^2$.

  - Otherwise $\alpha = \gamma \to \delta$ and $\xi = (\gamma \to \delta) \to \beta$. If $|\delta| = 0$, then $\delta = p$ and $ssf(\xi) = 1 + ssf(\alpha) + ssf(p \to \beta) - ssf(p) = 2 + ssf(\alpha) + ssf(\beta) \leq 2 + (|\alpha| + 1)^2 + (|\beta| + 1)^2 < (|\alpha| + |\beta| + 1)^2 = (|\xi| + 1)^2$.

  - Otherwise $\delta = \zeta \to \eta$ and $\xi = (\gamma \to (\zeta \to \eta)) \to \beta$. If $|\eta| = 0$, then $\eta = p$ and $ssf(\xi) = 1 + ssf(\alpha) + ssf((\zeta \to p) \to \beta) - ssf(\zeta \to p) = 2 + ssf(\alpha) + ssf(p \to \beta) - ssf(p) = 3 + ssf(\alpha) + ssf(\beta) \leq 3 + (|\alpha| + 1)^2 + (|\beta| + 1)^2 < (|\alpha| + |\beta| + 1)^2 = (|\xi| + 1)^2$.

- \[ \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \]
Eventually we arrive at $\alpha = \gamma_1 \to \cdots \to \gamma_n \to p$ (right-associative) and $\text{ssf } (\xi) = \text{ssf } (\alpha \to \beta) = n + 1 + \text{ssf } (\alpha) + \text{ssf } (\beta) \leq n + 1 + (|\alpha| + 1)^2 + (|\beta| + 1)^2 < (|\alpha| + |\beta| + 1)^2 = (|\xi| + 1)^2$.

This completes the proof of Lemma 2 (4).

## 5 Appendix B: Gilbert’s example

Let $\varnothing$ be a following tree-like $\text{NM}_\to$ proof of $\xi \to s$, where

$$\xi = (p_2 \to (p_2 \to r) \to q \to r) \to (p_1 \to p_1 \to (p_1 \to r) \to q \to r) \to s$$

for arbitrary formulas $p_i, q, r, s$ of basic minimal language $\mathcal{L}_\to$ and $v_i, u_i, x_i$ are crucial nodes with formula-labels $\ell^\varnothing (v_1) = \ell^\varnothing (v_2) = r, \ell^\varnothing (u_1) = \ell^\varnothing (u_2) = q \to r$ and $\ell^\varnothing (x_1) = (p_1 \to r) \to q \to r \neq \ell^\varnothing (x_2) = (p_2 \to r) \to q \to r$.

$$\begin{align*}
(\varnothing) & : y_1 : [p_1] z_1 : [p_1 \to r] \\
(\varnothing) & \quad (\varnothing) : v_1 : r \\
(\varnothing) & \quad (\varnothing) : u_1 : q \to r \\
(\varnothing) & \quad (\varnothing) : x_1 : (p_1 \to r) \to q \to r \\
(\varnothing) & \quad (\varnothing) : p_1 \to (p_1 \to r) \to q \to r \\
(\varnothing) & \quad (\varnothing) : p_1 \to p_1 \to (p_1 \to r) \to q \to r \\
\xi & \quad (\varnothing) : s \quad (\varnothing) : \xi \to s
\end{align*}$$

Obviously all five assumptions $p_1, p_2, p_1 \to r, p_2 \to r, \alpha$ are closed, while $\ell^\varnothing (e) = \varnothing$ for every edge $e$, since $\varnothing$ is a tree. Moreover, for any $i \in \{1, 2\}$ and assumption $\alpha \in \{\xi, p_i, (p_i \to r)\}$ we have (note that $q$ is not an assumption in $\varnothing$):

1. $\ell^\varnothing ((x_i, u_i), \alpha) = 1 \Leftrightarrow \alpha \in \{\xi, p_i, (p_i \to r)\}$,
2. $\ell^\varnothing ((u_i, v_i), \alpha) = 1 \Leftrightarrow \alpha \in \{\xi, p_i, (p_i \to r)\}$.

Hence all assumptions are discharged in $\varnothing$. Now consider the compressed dag $\varnothing^c$ (for the sake of brevity we drop redag-like repetitions of $\xi$):

$$\begin{align*}
(\varnothing) & : y_1 : [p_1] z_1 : [p_1 \to r] \\
(\varnothing) & \quad (\varnothing) : v_1 : r \\
(\varnothing) & \quad (\varnothing) : u_1 : q \to r \\
(\varnothing) & \quad (\varnothing) : x_1 : (p_1 \to r) \to q \to r \\
(\varnothing) & \quad (\varnothing) : p_1 \to (p_1 \to r) \to q \to r \\
(\varnothing) & \quad (\varnothing) : p_1 \to p_1 \to (p_1 \to r) \to q \to r \\
\xi & \quad (\varnothing) : s \quad (\varnothing) : \xi \to s
\end{align*}$$
This time we have:

1. \( \ell^g(\langle v, y_i \rangle) = \ell^g(\langle v, z_i \rangle) = \{x_i\} \),
2. \( \ell^g(\langle u, v \rangle) = \{x_1, x_2\} \),
3. \( \ell^d(\langle x_i, u \rangle, \alpha) = 1 \iff \alpha \in \{\xi, p_i, (p_i \rightarrow r)\} \),
4. \( \ell^d(\langle u, v \rangle, \alpha) = 1 \iff \alpha = \xi \),
5. \( \ell^d(\langle v, y_i \rangle, \alpha) = \ell^d(\langle x_i, u \rangle, \alpha) \),
6. \( \ell^d(\langle v, z_i \rangle, \alpha) = \ell^d(\langle x_i, u \rangle, \alpha) \).

Thus \( \ell^d(\langle v, y_i \rangle, \alpha) = \ell^d(\langle v, z_i \rangle, \alpha) = 1 \) holds for every \( i \in \{1, 2\} \) and \( \alpha \in \{\xi, p_i, (p_i \rightarrow r)\} \). Hence \( \partial^c \) is a dag-like NM- proof of \( \xi \rightarrow s \), as expected.

As compared to analogous compression from Example 8, this current \( \partial^c \) allows only two pairs of maximal deduction threads, which are separated at the lowest mutual merge point \( u \).

6 Appendix C: Haeusler’s example

Consider the formulas: 1) \( \eta = \alpha_1 \rightarrow \alpha_2 \), and 2) \( \sigma_k = \alpha_{k-2} \rightarrow (\alpha_{k-1} \rightarrow \alpha_k) \) for \( k > 2 \). Note that \( \alpha_1 \rightarrow \alpha_n \) follows from \( \eta, \sigma_3, \ldots, \sigma_n \) and the size of standard tree-like normal proof of this statement exceeds \( \text{Fibonacci}(n) \). For \( n = 5 \) we have the derivation that is shown in Fig. 1

Generally, for each \( 5 \leq n \) we arrive at

\[
\begin{array}{c}
| \hline \\
[\alpha_1] & [\alpha_1] \\
\eta & \eta \\
\sigma_3, \ldots, \sigma_{n-1} & \sigma_3, \ldots, \sigma_{n-1} \\
\Pi_{n-1} & \Pi_{n-1} \\
\alpha_{n-1} & \alpha_{n-1} \\
\hline \\
\alpha_n & \alpha_n \\
\hline \\
\alpha_1 \rightarrow \alpha_n \\
\hline \\
\end{array}
\]

\( l(\Pi_2) = 1 \)

\( l(\Pi_3) = l(\Pi_2) + 1 \)

\( l(\Pi_k) = l(\Pi_{k-2}) + l(\Pi_{k-1}) + 2 \)

\( \text{Fibonacci}(n) \leq l(\Pi_n) \)

Towards polynomial representation.
Using (re)dags we compress our tree-like proofs by merging distinct occurrences of identical formulas \( \alpha_3, \alpha_2, \alpha_1 \) as shown in Fig. 2, 3 and 4.
Figure 1: A huge ND proof

Figure 2: Horizontal compression (1)

Figure 3: Horizontal compression (2)

Figure 4: Horizontal compression (3)
This procedure results in the plain dag-like proof shown in figure 5 (afterwards encoded in Fig. 6, see below), where we assume that for every non-leaf node $x$, $s(x)$ contains all $x$’s children available, while for every downward-branching node $u$, every $\ell^G((u, v))$ contains all parents of $u$. Obviously this dag-like deduction is smaller than its tree-like original. Generally, we obtain dag-like deductions (not encoded yet) of $\alpha_1 \rightarrow \alpha_n$, whose size is smaller than $\sum_{i=1, n} i$, i.e. $O(n^2)$. The corresponding encoded dag-like deduction of $\alpha_1 \rightarrow \alpha_5$ is shown in Fig. 6, where a string of bits $b_1 b_2 \ldots b_5$ represents the discharging function $\ell^d$. Namely, for any $e$ we let $\ell^d(e, \xi_i) = b_i$ iff $\xi_i$ is the $i^{th}$ assumption with respect to lexicographical order $\alpha_1 < \alpha_1 \rightarrow \alpha_2 < \alpha_1 \rightarrow (\alpha_2 \rightarrow \alpha_3) < \alpha_2 \rightarrow (\alpha_3 \rightarrow \alpha_4) < \alpha_3 \rightarrow (\alpha_4 \rightarrow \alpha_5)$. Actually we always arrive at $b_1 b_2 \ldots b_5 = 10000$, as $\xi_i = \alpha_1$ is the only closed assumption and it is discharged by the root inference ($\rightarrow I$). (For brevity we don’t expose labeling functions $\ell^f$ and $\ell^G$.)
Figure 6: Horizontal compression (2)