Physical interpretation of constants in the solutions to the Brans-Dicke equations

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Abstract

Using an energy-momentum complex we give a physical interpretation to the constants in the well-known static spherically symmetric asymptotically flat vacuum solution to the Brans-Dicke equations. The positivity of the tensor mass puts a bound on parameters in the solution.

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I. INTRODUCTION

It is well-known that scalar fields have been conjectured to give rise to long-range gravitational fields [1]. Several theories involving scalar fields are known (see [2]–[3] and references therein). Brans-Dicke (BD) gravity [4] involves a scalar field and is perhaps the most viable alternative theory to Einstein’s general theory of relativity.

This theory is the simplest of the scalar-tensor theories and is a modification of general relativity which accommodates Mach’s principle as well as Dirac’s large numbers hypothesis. BD theory contains a massless scalar field Φ and a dimensionless constant ω that describes the strength of the coupling between Φ and the matter. It is usually believed that the post-Newtonian expansions of BD theory and general relativity agree in the limit |ω| → ∞.

Whilst this is usually the case, Romero and Barros [4] worked out a number of examples when this theory does not always go over to general relativity as |ω| → ∞. Recently, Banerjee and Sen [5] showed that BD theory goes over to general relativity in the large ω limit only if the energy-momentum tensor of the source term in the BD field equations is traceless. This theory has satisfied all the known solar system experimental tests for |ω| ≥ 500 [6]. Though the theory allows both positive as well as negative values of ω, it is usually assumed that ω > 0 for certain physical reasons.

Though BD theory as well as the general theory of relativity “agree” in the post-Newtonian limit described above, it is imperative to study strong field cases in which the two theories could give different results. These studies could provide experimental tests that might support one of them and reject the other. One such avenue is gravitational waves (as opposed to general relativity theory, BD theory allows monopole as well as dipole radiation). The nature of black holes forming due to gravitational collapse could be another avenue to test these theories. The Hawking theorem [7] states that the Schwarzschild metric is the only spherically symmetric solution of the vacuum Brans-Dicke field equations. However, his theorem assumes the weak energy condition and considers the scalar field Φ to be constant outside the black hole.
Shortly after BD theory was proposed, one of the authors (Brans [1]) obtained the exact static vacuum solution to the BD equations. The Brans type I solution is the only one which is permitted for all values of \( \omega \) (the other three forms are allowed only for \( \omega \leq -3/2 \)). Recently Campanelli and Lousto [10] studied this solution and found that for a certain range of parameters, the Brans type I solutions represent black holes. It is possible that a realistic gravitational collapse in BD theory could lead to the formation of “Brans black holes”. However, not much (if any) studies have been done to give physical interpretations to the constant parameters appearing in the Brans solutions. Using an energy-momentum complex [8] in BD theory we investigate this problem. This is the aim of this paper. We use units in which the speed of light in vacuum \( c \) and \( \Phi_\infty \) (the value of the scalar field far from all sources) are unity. The metric has signature + − − −, and Latin (Greek) indices take values 0 . . . 3 (1 . . . 3).

II. SCALAR AND TENSOR MASS IN BD THEORY

The subject of energy-momentum localization has been debated since the beginning of general relativity. Bergqvist [12] investigated seven different coordinate-independent definitions of quasi-local mass and found that no two of them give the same result for the Reissner-Nordström (RN) and Kerr spacetimes. Virbhadra and his collaborators (Rosen, Aguirregabiria and Chamorro) ([3]-[6]) showed that several energy-momentum complexes coincide and give the same result (local values) for several solutions in the Einstein as well as the Einstein-Maxwell theories when calculations are carried out in “cartesian coordinates”. For any Kerr-Schild class solution, all the well-known energy-momentum complexes coincide [6]. Though this is an encouraging result, the long-standing problem cannot be considered to be settled and is still debatable. However, the total energy and momentum of asymptotically flat spacetimes are unambiguously accepted. The situation in BD theory is not better. The use of the energy-momentum complex in BD theory [8] is also restricted to “cartesian coordinates”, and only the total value (integrated over all space) is unambigu-
ously considered. In BD theory, orbiting test particles (far away from a bounded system) measure the total active gravitational mass (Keplerian mass) $M$ while orbiting test black holes measure the tensor mass $M_T$ [4]. The difference between the two $M - M_T$ is the scalar mass. The tensor mass is always positive definite. It decreases monotonically by emission of gravitational waves [8]. The tensor mass is the active gravitational mass measured by a test Schwarzschild black hole in the asymptotic region.

The tensor, scalar and the Keplerian masses are given by [8]

$$M_T = \frac{1}{16\pi} \int \left[ \Phi^2 \Theta^{0\alpha\beta} \right]_{\alpha} d^2\Sigma_{\beta}, \quad (1)$$

$$M_S = \frac{1}{16\pi} \int \left[ (\Phi^2 - 1) \Theta^{0\alpha\beta} \right]_{\alpha} d^2\Sigma_{\beta}, \quad (2)$$

and

$$M = \frac{1}{16\pi} \int \left[ (2\Phi^2 - 1) \Theta^{0\alpha\beta} \right]_{\alpha} d^2\Sigma_{\beta}, \quad (3)$$

respectively. The quantity $\Theta^{m\alpha\beta}$ in the above expressions is given by

$$\Theta^{m\alpha\beta} = -g \left( g^{mn} g^{jk} - g^{mk} g^{jn} \right), \quad (4)$$

which has the symmetries of the Riemann curvature tensor.

### III. THE BRANS SOLUTION

The Brans-Dicke vacuum field equations are

$$R_{ik} = \frac{\omega}{\Phi^2} \Phi,_{i} \Phi,_{k} + \frac{\Phi,_{ik}}{\Phi}, \quad \Box \Phi = 0. \quad (5)$$

There exist exact solutions to the above equations [10]-[11]. Static spherically symmetric and asymptotically flat exact solutions to the above equations were given by Brans (see in [10]), which are expressed by the line element.
\[ ds^2 = A(r)^{m+1} \, dt^2 - A(r)^{n-1} \, dr^2 - A(r)^n \, r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right), \quad (6) \]

and the scalar field

\[ \Phi(r) = A(r)^{-\frac{m+n}{2}}, \quad (7) \]

where

\[ A(r) = 1 - \frac{r_0}{r}. \quad (8) \]

The quantities \( m, n, \Phi_0 \) and \( r_0 \) are arbitrary constants.

Putting \( m = 0, n = 0 \) in the above solutions gives the Schwarzschild metric. Campanelli and Lousto \[10\] have demonstrated that the above solution represents black holes if \( n \leq -1 \).

As it is well-known that the use of the energy-momentum complex is restricted to quasi-cartesian coordinates (see \[13\]-\[16\] and references therein), one transforms the line element (6) to these coordinates, according to

\[ x = r \sin \theta \cos \phi, \]
\[ y = r \sin \theta \sin \phi, \]
\[ z = r \cos \theta, \quad (9) \]

and gets

\[ ds^2 = A(r)^{m+1} \, dt^2 - A(r)^n \left( dx^2 + dy^2 + dz^2 \right) - \frac{A(r)^{n-1} - A(r)^n}{r^2} \left( xdx + ydy + zdz \right)^2. \quad (10) \]

**IV. CALCULATIONS**

As the metric under consideration is static, the momentum components will vanish. Therefore, we evaluate only the masses associated with this spacetime. Using the line element given by Eq. (10), we calculate the determinant and the contravariant components of the metric tensor. Further we calculate the required components of \( \Theta^{mnjk} \), which are
$$
\Theta^{0101} = \left(1 - \frac{2m}{r}\right) \frac{2^{n-1}}{r^3} \left(-r^3 + 2r_0x^2\right), \\
\Theta^{0202} = \left(1 - \frac{2m}{r}\right) \frac{2^{n-1}}{r^3} \left(-r^3 + 2r_0y^2\right), \\
\Theta^{0303} = \left(1 - \frac{2m}{r}\right) \frac{2^{n-1}}{r^3} \left(-r^3 + 2r_0z^2\right), \\
\Theta^{0102} = 2 \left(1 - \frac{2m}{r}\right) \frac{2^{n-1}}{r^3} r_0xy, \\
\Theta^{0203} = 2 \left(1 - \frac{2m}{r}\right) \frac{2^{n-1}}{r^3} r_0yz, \\
\Theta^{0301} = 2 \left(1 - \frac{2m}{r}\right) \frac{2^{n-1}}{r^3} r_0zx. \quad (11)$$

Substituting the above in Eqs. (1−3) and taking the limit $r \to \infty$ we get

$$
M_T = \frac{r_0}{2} (m - n + 2), \\
M_S = \frac{r_0}{2} (m + n), \\
M = r_0 (m + 1). \quad (12)
$$

It is clear that $M = M_T + M_S$. Using Eqs. (6), (7), (8) and (12), the values of $g_{00}, g_{11}$ and $\Phi$ in the asymptotic region are

$$
g_{00} = 1 - \frac{2M}{r}, \\
g_{11} = -1 + \frac{2}{r} (M_S - M_T), \\
\Phi = 1 + \frac{2M_S}{r}. \quad (13)
$$

For the Schwarzschild metric ($m = 0$ and $n = 0$) Eq. (12) gives

$$
M_T = M = r_o, \\
M_S = 0, \quad (14)
$$

as expected. Now Eq. (12) can be expressed as

$$
r_o = M_{sch}. $$
\[ m = \frac{M_S + M_T}{M_{sch}} - 1, \]
\[ n = \frac{M_S - M_T}{M_{sch}} + 1, \]

where \( M_{sch} \) stands for the Schwarzschild mass. To have the total mass of a Schwarzschild black hole positive, one considers \( r_0 > 0 \). Further, to respect the positivity of the tensor mass associated with the Brans metric considered here, one has to put a restriction on the parameters \( m \) and \( n \) (i.e. \( m - n + 2 \geq 0 \)).

V. DISCUSSION

BD theory is the most viable alternative theory to Einstein’s general theory of relativity and it incorporates Mach’s principle as well as Dirac’s large numbers hypothesis. This theory has been supported by observational tests. In recent years there has been renewed interest in BD theory as its application to the cosmological models of the universe (during the inflationary era) makes bubble percolation more natural, and the low-energy limit to the theory of fundamental strings reduces to Brans-Dicke theory. The Oppenheimer-Snyder collapse in BD theory leading to the formation of black holes suggested the possibility of black holes in this theory. With the help of an energy-momentum complex we have obtained the tensor, scalar and Keplerian masses associated with the Brans spacetime. Though the spacetime is not flat for \( m = -n = -1 \), the scalar as well as the tensor masses are zero. Campanelli and Lousto [10] showed that the Brans type I solutions represent black holes if \( n \leq 0 \), but their investigations do not put any restriction on the parameter \( m \). However, we have shown that the positivity of tensor mass puts a bound \( m \geq n - 2 \). Thus, one gets an interesting result that a physically realistic Brans black hole must be given by \( n \leq 0 \) with \( m \geq n - 2 \). The physical interpretation of the constant parameters \( r_o, m \) and \( n \) in the Brans solution are clear from the Eq. (14), i.e., these can be expressed in terms the Schwarzschild, scalar and tensor masses.
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