Dynamics of a Charged Particle Around a Slowly Rotating Kerr Black Hole Immersed in Magnetic Field

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Abstract: The dynamics of a charged particle moving around a slowly rotating Kerr black hole in the presence of an external magnetic field is investigated. We are interested to explore the conditions under which the charged particle can escape from the gravitational field of the black hole after colliding with another particle. The escape velocity of the charged particle in the innermost stable circular orbit is calculated. The effective potential and escape velocity of the charged particle with angular momentum in the presence of magnetic field is analyzed. This work serves as an extension of a preceding paper dealing with the Schwarzschild black hole [Zahrani et al, Phys. Rev. D 87, 084043 (2013)].

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I. INTRODUCTION

The dynamics of particles (massive or massless, charged or neutral) around a black hole is among the most important and interesting problems of black hole astrophysics. These studies not only help us to understand the geometrical structure of spacetimes but also shed light on the high energy phenomenon occurring near the black hole such as formation of jets (which involve particles to escape) and accretion disks (particles orbiting in circular orbits). Due to the presence of strong gravitational and electromagnetic fields, charged particles in general do not follow stable orbits and inter-particle collisions are most common. The aftermath of these collisions among numerous particles lead to various interesting astrophysical phenomenon.

There are numerous astrophysical evidence that magnetic field might be present in the nearby surrounding of black holes [1, 2] which support the large scale jets. These jets are most likely the source of cosmic rays and high energy particles coming from nearby galaxies. The origin of this magnetic field is the probable existence of plasma in the vicinity of a black hole in the form of an accretion disk or a charged gas cloud [3, 4]. The relativistic motion of particles in the conducting matter in the accretion disk can generate the regular magnetic field inside the disk. Therefore near the event horizon of a black hole, it is expected that there exists much strong magnetic field. To an approximation, it is presumed that this field does not effect the geometry of the black hole but it does effect the motion of the charged particles moving around the black hole [5, 6].

More importantly, a rotating black hole may provide sufficient energy to the particle moving around it due to which the particle may escape to spatial infinity. This physical effect appears to play a crucial role in the ejection of high energy particles from accretion disks around black holes. In the process of ejection of high energy particles, besides the rotation of black hole, the magnetic field plays an important role [7, 8]. Note that if the black hole is carrying electric charge producing the static electric field (also called Coulomb field), then the mere rotation of black hole itself induces the magnetic field. Acceleration of the particle by the black hole is generally explained in [9]. Other interesting processes around black holes may include evaporation and phantom energy accretion onto black holes [10].

During the motion of a charged particle around a magnetized black hole, it remains under the influence of both gravitational and electromagnetic forces which makes the situation complicated [11, 12]. In the present article, it is considered that a charged particle is orbiting in the innermost stable circular orbit (ISCO) of a slowly rotating Kerr black hole and is suddenly hit by a radially incoming neutral particle. The aftermath of collision will depend on the energy of the incoming
particle which may result one of the three possible outcomes: charged particle may escape to
infinity; being captured by the black hole or keep orbiting in ISCO. However predicting the nature
of outcome is compounded by the facts that particle is charged and interacts with the magnetic field
and is frame dragged by the Kerr black hole. It should be noted that the present work is altogether
different from the BSW mechanism where two particles (with non-zero angular momentum and
high energies) arrive from spatial infinity and collide near the event horizon to generate surplus
energy in the center of mass frame \[13\]. In literature, motion of charged particles in ISCO around
various black holes has been studied \([14]\) and see therein).

Here we consider a slowly rotating Kerr black hole which is surrounded by an axially symmetric
magnetic field homogeneous at infinity. Almost similar problem was studied for weakly charged
rotating black holes in \([15]\). Their main conclusion is that, if the magnetic field is present than the
ISCO is located closer to the black hole horizon. In general, the effect of the black hole rotation on
the motion of a neutral particle is same as the effect of magnetic field on the motion of a charged
particle \([16, 17]\).

To study the escape velocity of a particle from the vicinity of black hole, in this paper we
first consider a neutral particle moving around a slowly rotating Kerr black hole in the absence
of magnetic field and collides with another particle. For simplicity we consider the motion in
the equatorial plane only. Then we consider the same problem for a charged particle in the
presence of magnetic field. We focus under what circumstances the particle can escape from
the strong gravitational field to infinity. Magnetic field is homogeneous far from the black hole
and gravitational field is ignorable. Thus, far from the black hole charged particle moves in a
homogeneous magnetic field. If the magnetic field is absent then the equations of motion are
simple a little and can be solved analytically. When a particle moving in a non uniform magnetic
field in the absence of black hole its motion is chaotic \([18, 19]\). We are extending a previous work
\([16]\) for the slowly rotating Kerr black hole.

The outline of the paper is as follows: In section II we explain our model and derive an expression
for escape velocity of the neutral particle. In section III we derive the equations of motion of the
charged particle moving around a slowly rotating weakly magnetized Kerr black hole. In section
IV we give the dimensionless form of the equations. Trajectories for escape energy and escape
velocity of the particle are discussed in section V and VI respectively and their graphs are given
in the appendix. Summery and conclusion are presented in section VII. Throughout we use sign
convention \((+, -, -)\) and units where \(c = 1, G = 1\).
II. ESCAPE VELOCITY FOR A NEUTRAL PARTICLE

We start with the simple case of calculating the escape velocity when the particle is neutral and magnetic field is absent. The Kerr metric is given by \[ ds^2 = \frac{\Delta - a^2 \sin^2 \theta}{\rho^2} dt^2 + \frac{4Mar \sin^2 \theta}{\rho^2} d\phi dt - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - \frac{A \sin^2 \theta}{\rho^2} d\phi^2, \]
where \( \Delta \equiv r^2 - 2Mr + a^2, \quad \rho^2 \equiv r^2 + a^2 \cos^2 \theta, \quad A \equiv (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta, \]

Here \( M \) is the mass and \( a \) is the spin of the black hole and interpreted as the angular momentum per unit mass of the black hole \( a = \frac{L}{M} \). The horizons of Kerr metric are obtained by solving \[ \Delta(r) = r^2 + a^2 - 2Mr = 0. \]

From the above equation we get two values of \( r \):

\[ r_+ = M + \sqrt{M^2 - a^2}, \quad r_- = M - \sqrt{M^2 - a^2}. \]

Note that \( \Delta > 0 \) for \( r > r_+ \) and \( r < r_- \) and \( \Delta < 0 \) for \( r_- < r < r_+ \) \[23\]. The region \( r = r_+ \) represents the event horizon while \( r_- \) is termed as the Cauchy horizon. Further \( r = 0 \) and \( \theta = \frac{\pi}{2} \) is the location of a curvature ring-like singularity in the Kerr spacetime.

In literature, slowly rotating Kerr black holes have been investigated for numerous astrophysical processes including as retro-MACHOS \[24\], particle acceleration via BSW mechanism \[25\], thin accretion disk and accretion rates \[26\], to list a few. Hence we consider the slowly rotating black hole and neglect the terms involving \( a^2 \). The line element in (1) becomes

\[ ds^2 = (1 - \frac{r_g}{r}) dt^2 + \frac{4aM \sin^2 \theta}{r} d\phi dt - \frac{1}{1 - \frac{r_g}{r}} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2. \]

Here \( r_g = 2M \), is the gravitational radius of the slowly rotating Kerr black hole just like Schwarzschild black hole (Note that for a slowly rotating Kerr and Schwarzschild black hole the horizon occurs at \( r = r_g \) ). Clearly the metric \[4\] is stationary but non-static since \( dt \rightarrow -dt \), changes the signature of metric. The metric is also axially symmetric (invariance under \( d\theta \rightarrow -d\theta \)).

In terms of Lagrangian mechanics \( (L = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu) \), the \( t \) and \( \phi \) coordinates are cyclic which lead to two conserved quantities namely energy and angular momentum with the corresponding Noether symmetry generators

\[ \xi_{(t)} = \xi^\mu_{(t)} \partial_\mu = \frac{\partial}{\partial t}, \quad \xi_{(\phi)} = \xi^\mu_{(\phi)} \partial_\mu = \frac{\partial}{\partial \phi}. \]
This shows that the black hole metric is invariant under time translation and rotation around symmetry axis. The corresponding conserved quantities are the energy $\mathcal{E}$ per unit mass and azimuthal angular momentum $L_z$ per unit mass \(^1\)

\[
\dot{t} = \frac{r^3 \mathcal{E} + a L_z r_g}{r^2(r - r_g)},
\]

\[
\dot{\phi} = \frac{1}{r^2} \left( \frac{a r_g \mathcal{E}}{(r - r_g)} + \frac{L_z}{\sin^2 \theta} \right). \tag{6}
\]

From the astrophysical perspective, it is known that particles orbit a rotating black hole in the equatorial plane \(^27\). Therefore we choose $\theta = \frac{\pi}{2}$ to get

\[
\dot{t} = \frac{r^3 \mathcal{E} + a L_z r_g}{r^2(r - r_g)},
\]

\[
\dot{\phi} = \frac{1}{r^2} \left( \frac{a r_g \mathcal{E}}{(r - r_g)} + L_z \right). \tag{7}
\]

Throughout in this paper the over dot represents differentiation with respect to proper time $\tau$.

Using the normalization condition, $u^\mu u_\mu = 1$, we get the equation of motion

\[
\dot{r}^2 = \frac{(\mathcal{E} r^2 - a L_z)^2}{r^4} - \frac{r^2 - r_g r}{r^4} (r^2 + L_z^2 - 2a \mathcal{E} L_z). \tag{8}
\]

At the turning points $\dot{r} = 0$, the equation (8) is quadratic in $\mathcal{E}$ whose solution is

\[
\mathcal{E} = \frac{a L_z r_g \pm \sqrt{r^5 (r - r_g) + L_z^2 (r^4 - r^3 r_g + a^2 r_g^2)}}{r^3}, \tag{9}
\]

which gives $\mathcal{E} = V_{\text{eff}}$, the effective potential. The condition $\dot{r} = 0$ is termed as the turning point because it gives the location at which an incoming particle turns around from the neighborhood of the gravitating source \(^28\). As we are considering only the positive energy therefore we will consider only the positive sign before the square root in equation (9) for all the further calculation.

Equations (8) and (9) hold for equatorial plane only. It can be seen from (9) that $\mathcal{E} \to 1$ for $r \to \infty$. Therefore the minimum energy for the particle to escape from the vicinity of black hole is 1.

Consider a particle in ISCO, where $r_o$ is the local minima (which is also the convolution point) of the effective potential \(^23\). The corresponding energy and azimuthal angular momentum are given by \(^23, 29\) after neglecting terms which involving $a^2$ we have

\[
L_{zo} = \frac{\sqrt{r_o} \left( r_o \pm a \sqrt{\frac{2r_o}{r_o}} \right)}{\sqrt{2r_o - 3r_g + 2a \sqrt{\frac{2r_o}{r_o}}}}. \tag{10}
\]

\(^1\) Given a Lagrangian $\mathcal{L} = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$, one can calculate the conserved quantities corresponding to cyclic coordinates $t$ and $\phi$ as $\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{t}} = 0$, and $\frac{d}{d\phi} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 0$, yielding $\frac{\partial \mathcal{L}}{\partial \dot{t}} = \mathcal{E} \equiv -p_t \xi^t /m$, and $\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = L_z \equiv p_\phi \xi^\phi /m$. Solving these equations simultaneously, one can obtain (6).
\[ E_o = \frac{1 - \frac{r_o}{r} \pm \frac{a}{r} \sqrt{\frac{r_o}{2r}}}{\sqrt{1 - \frac{3r_o}{2r} \pm \frac{a}{r} \sqrt{\frac{r_o}{2r}}}}. \]  

(11)

Now consider the particle in the ISCO which collides with another incoming particle. After collision between these particles, three cases are possible for the motion of the particle: (i) bound motion (ii) capture by the black hole (iii) escape to infinity. The result will depend on the collision process. For small change in energy and momentum, orbit of the particle will be slightly perturbed. While for large change in energy and angular momentum, the particle can either be captured by black hole or escape to infinity.

After the collision particle should have new values of energy and momentum \( \mathcal{E} \), \( L_z \) and the total angular momentum \( L^2 \). We simplify the problem by applying the following conditions (i) the azimuthal angular momentum is fixed (ii) initial radial velocity remains same after the collision. Under these conditions only energy of the particle can determine its motion. After collision particle acquires an escape velocity \( v_\perp \) in orthogonal direction of the equatorial plane \([21]\). The square of total angular momentum of the particle after collision is given by

\[ L^2 = r^4 \dot{\theta}^2 + r^4 \sin^2 \theta \dot{\phi}^2. \]  

(12)

Putting the value of \( \dot{\phi} \) from equation (6) in equation (12) we have

\[ L^2 = r^2 v_\perp^2 + \sin^2 \theta \left( \frac{a r_g E_o}{r - r_g} + \frac{L_{zo}}{\sin^2 \theta} \right)^2. \]  

(13)

Here we denote \( v \equiv -r \dot{\theta}_o \). Note that \( L^2 \) is not the integral of motion. It is conserved for \( a = 0 \) i.e. in the spherically symmetric case. However now the metric is axially symmetric, therefore only \( L_z \) component is conserved. In a flat spacetime, all three components \( L_x, L_y, L_z \) are conserved, and so is the square of the total angular momentum. The angular momentum \( L_{zo} \) and energy \( E_o \) appearing in (13) are given by (10) and (11) which provide the necessary corrections due to spin of the black hole.

From equations (9) and (13), the angular momentum and the energy of the particle after the collision becomes

\[ L^2 = r_o^2 v_\perp^2 + \left( \frac{a r_g E_o}{r_o - r_g} + L_{zo} \right)^2, \]  

(14)

\[ \mathcal{E}_{\text{new}} = \frac{a L r_g + \sqrt{r_o^5 (r_o - r_g) + L^2 (r_o^4 - r_o^3 r_g + a^2 r_g^2)}}{r_o^3}. \]  

(15)

These values of angular momentum and energy are greater than their values before the collision. Physically it means that the energy of the particle exceeds its rest mass energy. We have mentioned
above all the orbits with \( \mathcal{E}_{\text{new}} \geq 1 \) are unbounded in the sense that particle escapes to infinity. Conversely for \( \mathcal{E}_{\text{new}} < 1 \), particle cannot escape to infinity (the orbits are always bounded).

Therefore particle escapes to infinity if \( \mathcal{E}_{\text{new}} \geq 1 \), or
\[
v_\perp \geq \pm \frac{r(r_g - r)(L_z(r - r_g) + ar_g(\mathcal{E}_o - 1)) + \sqrt{r^2r_g(r - r_g)^2(r^4 + r_g(a^2 - r^2 - 2a^2\mathcal{E}_o))}}{r^2(r - r_g)^2}. \tag{16}\]
Particle escape condition is \( |v| \geq v_\perp \) i.e. the magnitude of velocity should be greater than any orthogonal velocity.

### III. CHARGED PARTICLE AROUND THE SLOWLY ROTATING MAGNETIZED KERR BLACK HOLE

Here we investigate the motion of a charged particle (electric charge \( q \)) in the presence of magnetic field in the exterior of the slowly rotating Kerr black hole. The Killing equation is
\[
\Box \xi^\mu = 0, \tag{17}\]
where \( \xi^\mu \) is a Killing vector. Note that (17) follows from the result: a Killing vector in a vacuum spacetime generates a solution of Maxwell equations i.e. \( F_{\mu\nu} = -2\xi_{\mu;\nu} \). From \( F_{\mu\nu} = 0 \), it follows that \( -2\xi_{\mu;\nu} = 0 \). Thus (17) coincides with the Maxwell equation for 4-potential \( A^\mu \) in the Lorentz gauge \( A^\mu ;\mu = 0 \). The special choice for \( A^\mu \) is \[ A^\mu = \left( aB, 0, 0, \frac{B}{2} \right), \tag{18}\]
where \( B \) is the magnetic field strength. The 4-potential is invariant under the symmetries which correspond to the Killing vectors, i.e.,
\[
L_\xi A^\mu = A^\mu ,\nu \xi^\nu + A_\nu \xi^\nu \xi_{,\mu} = 0. \tag{19}\]
A magnetic field vector is defined as
\[
B^\mu = -\frac{1}{2} e^{\mu\nu\lambda\sigma} F_{\lambda\sigma} u_{,\nu}, \tag{20}\]
where
\[
e^{\mu\nu\lambda\sigma} = \frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-g}}, \quad \epsilon_{0123} = 1, \quad g = \det(g_{\mu\nu}). \tag{21}\]
In (21) \( \epsilon^{\mu\nu\lambda\sigma} \) is the Levi Civita symbol and the Maxwell tensor is defined as
\[
F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu}. \tag{22}\]
For a local observer at rest we have

\[ u^\mu = \left( \frac{1}{\sqrt{1 - \frac{r_g}{r}}} + \frac{4aM}{r^2 \sin \theta \sqrt{(1 - \frac{r_g}{r})}}, 0, 0, \frac{1}{r \sin \theta \sqrt{1 + \frac{4aM}{r^2 \sin \theta \sqrt{(1 - \frac{r_g}{r})}}} \right). \]  

(23)

From (20) – (23) we can obtain the components of magnetic field

\[ B^\mu = B' \left( 0, \cos \theta \left( \frac{1 - \frac{r_g}{r}}{\sqrt{1 - \frac{r_g}{r}}} \right) + \frac{r ga \sin \theta \cos \theta}{r \sin \theta \sqrt{1 + \frac{2rga}{r^2 \sin \theta \sqrt{(1 - \frac{r_g}{r})}}}, \frac{1}{r} \right), \right. \]  

(24)

For the equatorial plane only, the third component of the magnetic field will survive. Hence equation (24) becomes

\[ B^\mu = B' \left( 0, 0, -\sqrt{\frac{(1 - \frac{r_g}{r})}{r \sqrt{1 - \frac{r_g}{r} + \frac{2rga}{r^2 \sin \theta}}}}, 0 \right). \]  

(25)

The Lagrangian of the particle of mass \( m \) and charge \( q \) moving in an external magnetic field in a curved spacetime is [22]

\[ L = \frac{1}{2} g_{\mu \nu} \dot{x}^\mu \dot{x}^\nu + \frac{qA^\mu}{m} \dot{x}^\mu, \]  

(26)

and generalized 4-momentum of the particle is \( P_\mu = mu_\mu + qA_\mu \). The constants of motion are

\[ \dot{t} = \frac{r^3 E + a L_z r_g}{r^2 (r - r_g)} - 2aB, \]

\[ \dot{\phi} = \frac{1}{r^2} \left( \frac{argE}{r - r_g} + L_z \right) - B. \]  

(27)

For the equatorial plane \( \theta = \frac{\pi}{2} \) the above integrals of motion become

\[ \dot{t} = \frac{r^3 E + a L_z r_g}{r^2 (r - r_g)} - 2aB, \]

\[ \dot{\phi} = \frac{1}{r^2} \left( \frac{argE}{r - r_g} + L_z \right) - B. \]  

(28)

Here we denote

\[ B \equiv \frac{qB}{2m}. \]  

(29)

After putting the value of \( \dot{t} \) and \( \dot{\phi} \) and neglecting the terms involving \( a^2 \), Eq. (26) yields

\[ L = \frac{1}{2r^2 (r - r_g)^2} \left[ 4Br^2 L_z (r_g - r) + L_z^2 (r_g - r) + r^2 Br (3Br^2 + 2aE)(E^2 - 3B^2 r^2 - \dot{r}^2) \right]. \]  

(30)
By using the above Lagrangian in Euler-Lagrange equation which is defined as
\[ \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0, \tag{31} \]
we get
\[ \ddot{r} = \frac{BaErg}{r(r - rg)} + \frac{1}{2r^4(r - rg)} \left[ 6B^2r^6 - 2L^2(r - rg)^2 
+ r^3g(-E^2 + 6B^2rg + \dot{r}^2 - 12B^2r^2) \right]. \tag{32} \]

Following the procedure of section II, using the normalization condition, \( u^\mu u_\mu = 1 \) and putting the value of new constants of motion \( \text{(28)} \), we obtain
\[ \mathcal{E} = \frac{1}{\rho_o^6(r_o - rg)} \left[ 2aBr_o^7 + ar_o^3 \left( 2Br_o^2(r_o - 2r_o) + L_z(r_o - r_o) \right) \right.
\pm \left( a^2r_o^6(r_o - rg)^2 \left( r_o(L_z + 2Br_o^2 - 2Br_o^2)^2 + r_o^2(1 - r_o)^2 \left( r_o^2 + (L_z + Br_o^2)^2 \right) \right) \right]^\frac{1}{2}. \tag{33} \]

If (33) is satisfied initially (at the time of collision), then it is always valid (throughout the motion), provided that \( r(\tau) \) is controlled by (32).

The system (26) – (33) is invariant with respect to reflection \((\theta \rightarrow \pi - \theta)\). This transformation retains the initial position of the particle and changes \((v_\perp \rightarrow -v_\perp)\) as it is defined, \((v_\perp \equiv -r\dot{\theta}_o)\). Therefore, it is sufficient to consider only the positive value of \((v_\perp)\).

\[ \text{IV. DIMENSIONLESS FORM OF THE DYNAMICAL EQUATIONS} \]

To perform the numerical analysis, it is convenient to convert equations (32) and (33) to dimensionless form by introducing the following dimensionless quantities
\[ \sigma = \frac{\tau}{rg}, \quad \rho = \frac{r}{rg}, \quad \ell = \frac{L_z}{rg}, \quad b = Br_g. \tag{34} \]

Equation (33) now becomes
\[ \mathcal{E}_o = \frac{1}{\rho_o^6(\rho_o - 1)} \left[ a\rho_o^3(1 - \rho_o)(\ell - 2b\rho_o^2(\rho_o - 1)) \right.
+ \left( \rho_o^6(\rho_o - 1)^2(a^2(\ell - 2b\rho_o^2(\rho_o - 1))^2) \right.
+ \rho_o^3(\rho_o - 1) \left( \rho_o^2 + (\ell + b\rho_o^2)^2 \right) \right]^\frac{1}{2}. \tag{35} \]
The magnetic field is zero at \( \rho \to \infty \). Therefore from the equation (35) as \( \rho \to \infty \) then \( \mathcal{E} \to 1 \).

Dimensionless form of equation (32) is
\[
\frac{d^2 \rho}{d \sigma^2} = \frac{1}{2\rho^4(\rho - 1)} [\rho^3 (2aE b + 6\mathcal{E}^2 b^2 \rho(\rho - 1)^2) - 2\ell(\rho - 1)^2 + \rho^3 \frac{d\rho}{d\sigma}].
\] (36)

We solved the equation (36) numerically by using the built in command NDSolve of Mathematica. As ISCO exists at \( r = 3r_g \), and using \( \rho = \frac{r}{r_g} \) and \( \sigma = \frac{\tau}{r_g} \), our initial conditions for solving (36) become \( \rho(1) = 3 \) and \( \dot{\rho}(1) = 3 \). We get the interpolating function \( \rho(\sigma) \) as the solution of the equation (36) which we plotted in figure 1 against \( \sigma \). In figure 2 we have plotted the radial velocity (derivative of the interpolating function) vs \( \sigma \) which shows that the particle will escape to infinity according to the initial conditions.

As is the case of a neutral particle, we assume that the collision does not change the azimuthal angular momentum of the particle but it changes the transverse velocity \( v > 0 \). Due to this, the angular momentum and the energy of the particle will change as \( \ell \to \ell_t \) and \( \mathcal{E}_o \to \mathcal{E} \) respectively which is given by
\[
\ell_t^2 = \rho^2 v_\perp^2 + \rho^4 \left[ \frac{1}{\rho^2} \left( \frac{a\mathcal{E}_o}{2(\rho - 1)} + \ell \right) - b \right]^2,
\] (37)
\[
\mathcal{E} = \frac{1}{\rho^6_0(\rho_0 - 1)} \left[ a\rho_0^3(1 - \rho_0)(\ell_t - 2b\rho_0^2(\rho_0 - 1)) \right.
+ \left( \rho_0^6(\rho_0 - 1)^2(a^2(\ell_t - 2b\rho_0^2(\rho_0 - 1))^2) \right.
+ \left. \rho_0^3(\rho_0 - 1)(\rho_0^2 + (\ell_t + b\rho_0^2)^2) \right]^2. \] (38)

Here \( \ell_t \) is the dimensionless form of \( L \) given by equation (13). For the unbound motion \( \mathcal{E} \geq 1 \). By solving (38) and putting \( \mathcal{E} = 1 \), we get escape velocity of the particle as given below
\[
v_\perp = \pm \frac{1}{4\rho^2(\rho - 1)} \left[ 4(\rho - 1) \left[ \sqrt{a^2\rho^2 + \rho^4(\rho - 1) - 2ab\rho^4(\rho - 1)(2\rho - 1)} \right. \\
- a\rho + \rho(\rho - 1) (b\rho^2 + (\ell - b\rho^2)^2) \right] + a\rho\mathcal{E}_o \left[ 4(\rho - 1)(\ell - b\rho^2) \right] \] (39)

We now discuss the behavior of the particle when it escapes to asymptotic infinity. For simplicity we consider the particle initially in ISCO. The parameter \( \ell \) and \( b \) are defined in term of \( \rho_0 \) and only \( \mathcal{E} \) specifies the motion of the particle. We can express the parameters \( \ell \) and \( b \) in term of \( \rho_0 \) by simultaneously solving the equations \( \frac{d\mathcal{E}}{d\rho} = 0 \), and \( \frac{d^2\mathcal{E}}{d\rho^2} = 0 \), for \( \ell \) and \( b \). But the first derivative and second derivative of effective potential are very complicated and we cannot find the explicit expression for \( \ell \) and \( b \) in term of \( \rho \).
V. TRAJECTORIES FOR ESCAPE ENERGY

Here we investigate the dynamics of particle for the positive energy $\mathcal{E}_+$. Particles with negative energy exist only inside the static limit surface ($r_{st} = 2m$) orbiting in the retrograde orbits and do not have the chance to escape. The equation for the rotational (angular) variable $\phi$ is

$$\frac{d\phi}{d\sigma} = \frac{\ell}{\rho^2} - b + \frac{a\mathcal{E}}{\rho^4(1 - \rho)}. \quad (40)$$

The Lorentz force acting on the massive charged particle is attractive when $d\phi/d\sigma < 0$ and vice versa. All the figures correspond to Eq. (35). In figure 3, the shaded region corresponds to unbound motion while the unshaded region refers to bounded trajectories of the particle. The curved line represents the minimum energy required for the particle to escape form the vicinity of the black hole. It can be seen from figure 4 that for large values of angular momentum, the plot is similar to the effective potential of Schwarzschild black hole [16]. In figure 4, $\mathcal{E}_{\text{max}}$ corresponds to unstable circular orbit and $\mathcal{E}_{\text{min}}$ refers to ISCO.

The effective potential $\mathcal{E}$ of a particle moving in a slowly rotating Kerr spacetime is plotted as a function of radial coordinate $\rho$ for different values of angular momentum $\ell$ in figure 5. We can see from figure 5 that for large value of angular momentum, the maxima is shifting upward. For a particle to be captured by the black hole it is required that the energy which should be greater than this maxima. If its energy is less than this maxima there are two possibilities for a particle either it will escape to infinity or it might start moving in ISCO. If energy of the particle $\mathcal{E} < 1$ then it will stay in some stable orbit and if $\mathcal{E} > 1$ then it will escape to infinity. In figure 5 we plotted effective potential against $\rho$ for different value of angular momentum. For $\ell > 0$ the Lorentz force is repulsive. Hence it can be concluded from figure 5 that the possibility of a particle to escape after collision from the vicinity of the black hole is greater for larger value of $\ell_+$ as compared to the lesser value of it. For $\ell < 0$ the lorentz force is attractive. Therefore, the possibility of a particle to escape after collision is less for larger value of $\ell_-$ as compared to smaller value of $\ell_-$, represented in figure 6. The graph for $\ell = 0$ and $b = 0$ in figure 6 corresponds to photon as there is no stable region. Moreover, we compare the effective potential for $\ell = 10$ and $\ell = -10$ in figure 7. It can be seen that the stability is larger for $\ell = -10$. Therefore it is concluded that for the attractive Lorentz force ($\ell = -10$), particle required more energy to escape. It can be seen from figure 8 that with the increase in the strength of magnetic field, the local minima of the effective potential is shifting toward the horizon. This local minima corresponds to ISCO, which is in agreement with the result of [15].
VI. TRAJECTORIES FOR ESCAPE VELOCITY

For all the figures of escape velocity we have denoted \( v_\perp \equiv v_{\text{esc}} \). From Eq. (38) we calculate the escape velocity by substituting \( E = 1 \). Figures 9-12 correspond to Eq. (39). In figure 9 the shaded region corresponds to escape velocity of the particle and the solid curve represents the minimum velocity required to escape from the vicinity of the black hole to infinity. The unshaded region represents the bound motion around the black hole. In figure 10 the shaded region corresponds to escape velocity of the particle and the solid curve represents the minimum velocity required to escape from the vicinity of the black hole. The unshaded region represents the bound motion around the black hole.

In figure 11 we plotted escape velocity of a particle moving in ISCO as a function of radial coordinate \( \rho \) for different values of magnetic field \( b \). It can be seen from figure 11 that due to the presence of magnetic field in the vicinity of black hole escape velocity of the particle increases. Therefore we can say that in the presence of magnetic field \( b \), the possibilities of the particle to escape is greater then the case when magnetic field is absent i.e. \( b = 0 \). We plotted the escape velocity against \( \rho \) in figure 12 for different values of angular momentum \( \ell \). We can see from the figure 12 that the escape velocity is increasing for large value of \( \ell \). Hence we can conclude that if particle has larger value of angular momentum \( \ell \) then it can easily escape to infinity as compared to the particle with smaller value of angular momentum \( \ell \) regardless of the magnetic field.

VII. DISCUSSION

We have studied the dynamics of a neutral and a charged particle around the slowly rotating Kerr black hole which is immersed in a magnetic field. Therefore the particle is under the influence of both gravitational and electromagnetic forces. We have obtained equations of motion by using Lagrangian formalism. We have derived the expression for magnetic field present in the vicinity of slowly rotating Kerr black hole. We have calculated the minimum energy for a particle to escape from ISCO to infinity. With zero spin i.e. \( a = 0 \), our results reduce to the case of the Schwarzschild black hole [12].

The behavior of effective potential and escape velocity against magnetic field and angular momentum are discussed in detail. It is shown in figures 4, 9 and 10 under what conditions particle can escape from the vicinity of the black hole to spatial infinity. For larger values of the angular momentum, behavior of the effective potential is similar to that of the Schwarzschild black hole.
It is concluded that magnetic field largely effects the motion of the particle in the vicinity of the black hole. This effect decreases far away from the black hole. It is found that as the value of magnetic field parameter is increased, the local minima of effective potential shifted towards the horizon, as shown in figure 5. This indicates that the ISCO shrinks as strength of magnetic field increases. It is concluded from the figures 5 and 12 that if particle has large value angular momentum $\ell_+$ then it can escape easily as compare to particle with smaller angular momentum $\ell_+$. Figure 7 shows that for attractive Lorentz force ($\ell_-$) the stability is larger in comparison with repulsive Lorentz force ($\ell_+$).

Escape velocity $v_{\text{esc}}$, for different values of magnetic field $b$ is plotted in figure 11. It is found that due to the presence of magnetic field in the vicinity of black hole escape velocity of the particle increases. Therefore we found that the possibility of the particle to escape from the vicinity of black hole to infinity is greater in the presence of magnetic field as compared to the case when magnetic field is absent $b = 0$.

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FIG. 1: Figure shows the graph for $\rho(\sigma)$ vs $\sigma$. Here $\mathcal{E} = 1, q = 1, b = 0.5, \ell = 2,$ and $a = 0.1$.

FIG. 2: Figure shows the graph for $\rho'(\sigma)$ (radial velocity) vs $\sigma$. Here $\mathcal{E} = 1, q = 1, b = 0.5, \ell = 2,$ and $a = 0.1$. 
FIG. 3: In this figure we plot Effective potential $\mathcal{E}$ as a function of $\rho$ for $\ell = 5$, $b = 0.5$ and $a = 0.1$.

FIG. 4: Here we plot the effective potential against $\rho$ for $\ell = 20$, $b = 0.5$, and $a = 0.1$. In this figure $\mathcal{E}_{\text{max}}$ corresponds to unstable circular orbit and $\mathcal{E}_{\text{min}}$ corresponds to stable circular orbit.
FIG. 5: The effective potential $E$ is plotted as a function of radial coordinate $\rho$ for different values of angular momentum $\ell$.

FIG. 6: The effective potential $E$ is plotted against radial coordinate $\rho$ for different values of negative angular momentum $\ell$. 
FIG. 7: In this figure we have plotted the effective potential $\mathcal{E}$ vs $\rho$ for $\ell = -10$ and $\ell = 10$.

FIG. 8: The effective potential $\mathcal{E}$ against $\rho$ for different values of magnetic field.
FIG. 9: Here we have plotted the escape velocity against $\rho$ for $\ell = 5$, $b = 0.5$ and $a = 0.1$.

FIG. 10: In this figure we have plotted the escape velocity against $\rho$ for $\ell = 5$, $b = 0.5$ and $a = 0.1$.

FIG. 11: Escape velocity $v_{\text{esc}}$ against $\rho$ for different values of magnetic field $b$. 
FIG. 12: Escape velocity $v_{\text{esc}}$ against $\rho$ for different values of angular momentum $\ell$. 