Quantum fluctuations and random matrix theory

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1 Abstract

The random matrix ensembles are applied to the quantum statistical two-dimensional sys-
tems of electrons. The quantum systems are studied using the finite dimensional real, com-
plex and quaternion Hilbert spaces of the eigenfunctions. The linear operators describing
the systems act on these Hilbert spaces and they are treated as random matrices in generic
bases of the eigenfunctions. The random eigenproblems are presented and solved. Examples
of random operators are presented with connection to physical problems.

2 Introduction

Random Matrix Theory RMT studies random matrix variables [1, 2, 3, 4, 5, 6, 7, 8]. We
study generic quantum statistical systems with energy dissipation. Let us consider Hilbert’s
space $\mathcal{V}$ with some basis $\{ |\Psi_i\rangle \}$. The space of the linear bounded operators $\hat{X}$ acting on
Hilbert space $\mathcal{V}$ is called Liouville space and is denoted $\mathcal{L}(\mathcal{V})$. The Liouville space is again
Hilbert’s space with the scalar product: $\langle \hat{X}|\hat{Y}\rangle = \text{Tr}(\hat{X}^\dagger\hat{Y})$. Hence, it is a Banach space with
norm: $||\hat{X}|| = (\langle \hat{X}|\hat{X}\rangle)^{1/2}$, and it is metric space with distance: $\rho(\hat{X}, \hat{Y}) = ||\hat{X} - \hat{Y}||$. Finally,
it is topological space with the balls $B(\hat{X}, r) = \{ \hat{Y} | \rho(\hat{Y}, \hat{X}) < r \}$. The Liouville space is also a differentiable manifold. There exist a tangent space $T_{\hat{X}}\mathcal{L}(\mathcal{V})$, tangent bundle $T\mathcal{L}(\mathcal{V})$, cotangent space $T^*_{\hat{X}}\mathcal{L}(\mathcal{V})$, and cotangent bundle $T^*\mathcal{L}(\mathcal{V})$. The quantum operator $\hat{X} \in \mathcal{L}(\mathcal{V})$
can be represented in the given basis by a matrix $X \in \text{MATRICE}(N, N, \mathbf{F})$ with matrix
elements $X_{ij} \in \mathbf{F}$, where $\text{MATRICE}(N, N, \mathbf{F})$ is set of all $N \times N$ matrices with elements from
the field $\mathbf{F}$. We are allowed to define random operator variable $X : \Omega \ni \omega \rightarrow \hat{X}(\omega) \in \mathcal{L}(\mathcal{V})$,
where $\Omega$ is sample space, $\omega$ is sample point, and $\hat{X} = \hat{X}(\omega)$ is deterministic operator. It
has the analogical random matrix variable $X : \Omega \ni \omega \rightarrow X(\omega) \in \text{MATRICE}(N, N, \mathbf{F})$, where
$X = X(\omega)$ is deterministic matrix [1, 2, 3]. The generic nonhermitean quantum Hamiltonian
operator $\hat{K} \in \mathcal{L}(\mathcal{V})$ can be represented in the given basis by a matrix $K \in \text{MATRICE}(N, N, \mathbf{F})$
with matrix elements $K_{ij} \in \mathbb{F}$. Thus, we define random Hamiltonian operator variable
\[ \hat{K} : \Omega \ni \omega \rightarrow \hat{K}(\omega) \in \mathcal{L}(V), \]
and we define analogical random Hamiltonian matrix variable
\[ K : \Omega \ni \omega \rightarrow K(\omega) \in \text{MATRIX}(N, N, \mathbb{F}). \]
Here, $\hat{K} = \hat{K}(\omega)$ is deterministic nonhermitean Hamiltonian operator, and $K = K(\omega)$ is deterministic nonhermitean Hamiltonian matrix. The matrix elements $K_{ij}$ are independent random scalar variables:
\[ K_{ij} : \Omega \ni \omega \rightarrow K_{ij}(\omega) \in \mathbb{F}, \quad i, j = 1, \ldots, N. \]
Let us assume that the Hamiltonian operator $\hat{K} = \hat{K}(\omega)$ is not hermitean operator, hence the Hamiltonian matrix $K = K(\omega)$ is not hermitean, therefore quantum system is dissipative system. Ginibre ensemble of random matrices is one of many Gaussian Random Matrix ensembles GRME. The above approach is an example of Random Matrix theory RMT [1, 2, 3]. The other RMT ensembles are for example Gaussian orthogonal ensemble GOE, unitary GUE, symplectic GSE, as well as circular ensembles: orthogonal COE, unitary CUE, and symplectic CSE.

There were studied among others the following Gaussian Random Matrix ensembles GRME: orthogonal GOE, unitary GUE, symplectic GSE, as well as circular ensembles: orthogonal COE, unitary CUE, and symplectic CSE. The choice of ensemble is based on quantum symmetries ascribed to the hermitean Hamiltonian matrix $H$. The hermitean Hamiltonian matrix $H$ acts on space $W$ of eigenvectors. It is assumed that $W$ is $N$-dimensional Hilbert space $W = \mathbb{F}^N$, where the real, complex, or quaternion field $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, corresponds to GOE, GUE, or GSE, respectively. If the Hamiltonian matrix $H$ is hermitean $H = H^\dagger$, then the probability density function of random Hamiltonian matrix $\mathcal{H}$ reads:
\[ f_{\mathcal{H}}(H) = C_{H\beta} \exp \left[ -\beta \cdot \frac{1}{2} \cdot \text{Tr}(H^2) \right], \quad (1) \]
\[ C_{H\beta} = \left( \frac{\beta}{2\pi} \right)^{N_{H\beta}/2}, \]
\[ N_{H\beta} = N + \frac{1}{2}N(N-1)\beta, \]
\[ \int f_{\mathcal{H}}(H) dH = 1, \]
\[ dH = \prod_{i=1}^{N} \prod_{j \geq i}^{N} \prod_{\gamma=0}^{D-1} dH_{ij}^{(\gamma)}, \]
\[ H_{ij} = (H_{ij}^{(0)}, \ldots, H_{ij}^{(D-1)}) \in \mathbb{F}, \]
where the parameter $\beta$ assume values $\beta = 1, 2, 4$, for GOE($N$), GUE($N$), GSE($N$), respectively, and $N_{H\beta}$ is number of independent matrix elements of hermitean Hamiltonian matrix $H$. The Hamiltonian matrix $H$ is hermitean $N \times N \mathbb{F}$-matrix, and the matrix Haar’s measure $dH$ is invariant under transformations from the unitary group $U(N, \mathbb{F})$. The eigenenergies
\[ \mathcal{E}_i : \Omega \ni \omega \rightarrow \mathcal{E}_i(\omega) \in \mathbb{R} \] of \( \mathcal{H} \) are real-valued random variables, where \( E_i = \mathcal{E}_i(\omega) \) are deterministic eigenenergies of \( H \), and \( E_i = E_i^*, i = 1, ..., N \). It was Eugene Wigner who firstly dealt with eigenenergy level repulsion phenomenon studying nuclear spectra [1, 2, 3]. RMT is applicable now in many branches of physics: nuclear physics (slow neutron resonances, highly excited complex nuclei), condensed phase physics (fine metallic particles, random Ising model [spin glasses]), quantum chaos (quantum billiards, quantum dots), disordered mesoscopic systems (transport phenomena), quantum chromodynamics, quantum gravity, field theory.

3 The Ginibre ensembles

Jean Ginibre considered another example of GRME dropping the assumption of hermiticity of Hamiltonians thus defining generic \( F \)-valued Hamiltonian \( K \) [1, 2, 9, 10]. Hence, \( K \) belongs to \( \text{MATRIX}(N, N, F) \), and the matrix Haar’s measure \( dK \) is invariant under transformations form the general linear Lie group \( GL(N, F) \). The distribution of random nonhermitean Hamiltonian variable \( K \) is given by:

\[
 f_K(K) = C_{K\beta} \exp \left[ -\beta \cdot \frac{1}{2} \cdot \text{Tr}(K^\dagger K) \right],
\]

\[
 C_{K\beta} = \left( \frac{\beta}{2\pi} \right)^{N_K\beta/2},
\]

\[
 N_K\beta = N^2\beta,
\]

\[
 \int f_K(K)dK = 1,
\]

\[
 dK = \prod_{i=1}^{N} \prod_{j=1}^{N} \prod_{\gamma=0}^{D-1} dK_{ij}^{(\gamma)},
\]

\[
 K_{ij} = (K_{ij}^{(0)}, ..., K_{ij}^{(D-1)}) \in F,
\]

where \( \beta = 1, 2, 4 \), stands for real, complex, and quaternion Ginibre ensembles, respectively. Therefore, the eigenenergies \( Z_i \) of quantum system ascribed to Ginibre ensemble are complex-valued random variables. The eigenenergies \( Z_i, i = 1, ..., N \), of nonhermitean Hamiltonian \( K \) are not real-valued random variables \( Z_i \neq Z_i^* \). Jean Ginibre postulated the following joint probability density function of random vector of complex eigenvalues \( Z_1, ..., Z_N \) for \( N \times N \) Hamiltonian matrices \( K \) for \( \beta = 2 \) [1, 2, 9, 10]:

\[
 P(Z_1, ..., Z_N) = \prod_{j=1}^{N} \frac{1}{\pi \cdot j!} \cdot \prod_{i<j}^{N} |Z_i - Z_j|^2 \cdot \exp(-\sum_{j=1}^{N} |Z_j|^2),
\]

where \( Z_i \) are complex-valued sample points (\( Z_i \in \mathbb{C} \)).

We emphasize here Wigner and Dyson’s electrostatic analogy. A Coulomb gas of \( N \) unit charges moving on complex plane (Gauss’s plane) \( \mathbb{C} \) is considered. The vectors of positions
of charges are $Z_i$ and potential energy of the system is:

$$U(Z_1, ..., Z_N) = -\sum_{i<j} \ln |Z_i - Z_j| + \frac{1}{2} \sum_i |Z_i|^2.$$  \hspace{1cm} (4)$$

If gas is in thermodynamical equilibrium at temperature $T = \frac{1}{2k_B} \beta = \frac{1}{k_B}T = 2$, $k_B$ is Boltzmann's constant, then probability density function of vectors of positions is $P(Z_1, ..., Z_N)$ Eq. (3). Therefore, complex eigenenergies $Z_i$ of quantum system are analogous to vectors of positions of charges of Coulomb gas. Moreover, complex-valued spacings $\Delta^1 Z_i$ of complex eigenenergies of quantum system:

$$\Delta^1 Z_i = Z_{i+1} - Z_i, i = 1, ..., (N-1),$$  \hspace{1cm} (5)$$

are analogous to vectors of relative positions of electric charges. Finally, complex-valued second differences $\Delta^2 Z_i$ of complex eigenenergies:

$$\Delta^2 Z_i = Z_{i+2} - 2Z_{i+1} + Z_i, i = 1, ..., (N-2),$$  \hspace{1cm} (6)$$

are analogous to vectors of relative positions of vectors of relative positions of electric charges.

The eigenenergies $E_i = E(i)$ can be treated as values of function $Z$ of discrete parameter $i = 1, ..., N$. The "Jacobian" of $Z_i$ reads:

$$\text{Jac}Z_i = \frac{\partial Z_i}{\partial i} \approx \frac{\Delta^1 Z_i}{\Delta^1 i} = \Delta^1 Z_i.$$  \hspace{1cm} (7)$$

We readily have, that the spacing is an discrete analog of Jacobian, since the indexing parameter $i$ belongs to discrete space of indices $i \in I = \{1, ..., N\}$. Therefore, the first derivative with respect to $i$ reduces to the first differential quotient. The Hessian is a Jacobian applied to Jacobian. We immediately have the formula for the diagonal element of discrete "Hessian" for the eigenenergies $Z_i$:

$$\text{Hess}Z_i = \frac{\partial^2 Z_i}{\partial i^2} \approx \frac{\Delta^2 Z_i}{\Delta^1 i^2} = \Delta^2 Z_i.$$  \hspace{1cm} (8)$$

Thus, the second difference of $Z$ is discrete analog of Hessian of $Z$. One emphasizes that both "Jacobian" and "Hessian" work on discrete index space $I$ of indices $i$. The spacing is also a discrete analog of energy slope whereas the second difference corresponds to energy curvature with respect to external parameter $\lambda$ describing parametric “evolution” of energy levels [11, 12]. The finite differences of order higher than two are discrete analogs of compositions of "Jacobians" with "Hessians" of $Z$.

The eigenenergies $E_i, i \in I$, of the hermitean Hamiltonian $H$ are ordered increasingly real-valued random variables. They are values of discrete function $E_i = E(i)$. The first difference of adjacent eigenenergies is:

$$\Delta^1 E_i = E_{i+1} - E_i, i = 1, ..., (N-1),$$  \hspace{1cm} (9)$$
are analogous to vectors of relative positions of electric charges of one-dimensional Coulomb gas. It is simply the spacing of two adjacent energies. Real-valued second differences $\Delta^2 E_i$ of eigenenergies:

$$\Delta^2 E_i = E_{i+2} - 2E_{i+1} + E_i, \quad i = 1, \ldots, (N-2),$$  

are analogous to vectors of relative positions of vectors of relative positions of charges of one-dimensional Coulomb gas. The $\Delta^2 Z_i$ have their real parts Re$\Delta^2 Z_i$, and imaginary parts Im$\Delta^2 Z_i$, as well as radii (moduli) $|\Delta^2 Z_i|$, and main arguments (angles) Arg$\Delta^2 Z_i$. $\Delta^2 Z_i$ are extensions of real-valued second differences:

$$\Delta^2 E_i = E_{i+2} - 2E_{i+1} + E_i, \quad i = 1, \ldots, (N-2),$$

of adjacent ordered increasingly real-valued eigenenergies $E_i$ of Hamiltonian $H$ defined for GOE, GUE, GSE, and Poisson ensemble PE (where Poisson ensemble is composed of uncorrelated randomly distributed eigenenergies) $[13, 14, 15, 16, 17]$. The Jacobian and Hessian operators of energy function $E(i) = E_i$ for these ensembles read:

$$\text{Jac} E_i = \frac{\partial E_i}{\partial i} \approx \frac{\Delta^1 E_i}{\Delta^1 i} = \Delta^1 E_i,$$

and

$$\text{Hess} E_i = \frac{\partial^2 E_i}{\partial i^2} \approx \frac{\Delta^2 E_i}{\Delta^1 i^2} = \Delta^2 E_i.$$

The treatment of first and second differences of eigenenergies as discrete analogs of Jacobians and Hessians allows one to consider these eigenenergies as a magnitudes with statistical properties studied in discrete space of indices. The labelling index $i$ of the eigenenergies is an additional variable of "motion", hence the space of indices $I$ augments the space of dynamics of random magnitudes.

References

[1] F. Haake, *Quantum Signatures of Chaos*, Springer-Verlag, Berlin Heidelberg New York, 1990, Chapters 1, 3, 4, 8, pp. 1–11, pp. 33–77, pp. 202–213.

[2] T. Guhr, A. Müller-Groeling, H. A. Weidenmüller, “Random Matrix Theories in Quantum Physics: Common Concepts,” *Phys. Rept.* 299, pp. 189–425, 1998.

[3] M. L. Mehta, *Random matrices*, Academic Press, Boston 1990, Chapters 1, 2, 9, pp. 1–54, pp. 182–193.

[4] L. E. Reichl, *The Transition to Chaos In Conservative Classical Systems: Quantum Manifestations*, Springer-Verlag, New York, 1992, Chapter 6, pp. 248–286.

[5] O. Bohigas, in *Proceedings of the Les Houches Summer School on Chaos and Quantum Physics*, North-Holland, Amsterdam, 1991, p. 89.
[6] C.E. Porter, Statistical Theories of Spectra: Fluctuations, Academic Press, New York, 1965.

[7] T. A. Brody, J. Flores, J. B. French, P. A. Mello, A. Pandey, S. S. M. Wong, “Random-matrix physics: spectrum and strength fluctuations,” Rev. Mod. Phys. 53, pp. 385–479, 1981.

[8] C. W. J. Beenakker, “Random-matrix theory of quantum transport,” Rev. Mod. Phys. 69, pp. 731–808, 1997.

[9] J. Ginibre, J. Math. Phys. 6, p. 440, 1965.

[10] M. L. Mehta, Random matrices, Academic Press, Boston, 1990, Chapter 15, pp. 294–310.

[11] J. Zakrzewski, D. Delande, “Parametric motion of energy levels in quantum chaotic systems. I. Curvature distributions,” Phys. Rev. E 47, pp. 1650–1664, 1993.

[12] J. Zakrzewski, D. Delande, M. Kuś, “Parametric motion of energy levels in quantum chaotic systems. II. Avoided-crossing distributions,” Phys. Rev. E 47, pp. 1665–1676, 1993.

[13] M. M. Duras, K. Sokalski, “Higher-order finite-element distributions in statistical theory of nuclear spectra,” Phys. Rev. E 54, pp. 3142–3148, 1996.

[14] M. M. Duras, Finite difference and finite element distributions in statistical theory of energy levels in quantum systems, PhD thesis, Jagellonian University, Cracow, 1996.

[15] M. M. Duras, K. Sokalski, “Finite Element Distributions in Statistical Theory of Energy Levels in Quantum Systems,” Physica D125, pp. 260–274, 1999.

[16] M. M. Duras, “Description of Quantum Systems by Random Matrix Ensembles of Large Dimensions,” in Proceedings of the Sixth International Conference on Squeezed States and Uncertainty Relations, 24 May-29 May 1999, Naples, Italy, NASA, Greenbelt, Maryland, at press 2000.

[17] M. M. Duras, “Finite-difference distributions for the Ginibre ensemble,” J. Opt. B: Quantum Semiclass. Opt. 2, pp. 287–291, 2000.