Scattering on two Aharonov–Bohm vortices

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Abstract
The problem of two Aharonov–Bohm (AB) vortices for the Helmholtz equation is examined in detail. It is demonstrated that the method proposed by Myers (1963 J. Math. Phys. 6 1839) for slit diffraction can be generalised to obtain an explicit solution for AB vortices. Due to the singular nature of AB interaction the Green function and scattering amplitude for two AB vortices obey a series of partial differential equations. Coefficients entering these equations, fulfil ordinary non-linear differential equations whose solutions can be obtained by solving the Painlevé III equation. The asymptotics of necessary functions for very large and very small vortex separations are calculated explicitly. Taken together, this means that the problem of two AB vortices is exactly solvable.

Keywords: Aharonov–Bohm effect, two vortices, exact solution

1. Introduction

The Aharonov–Bohm (AB) effect [1, 2] is one of the most striking distinguishers between quantum and classical works. In a nutshell, it states that a quantum particle feels electromagnetic potentials even though no classical forces exist. Its first description can be traced to the paper of Ehrenberg and Siday [1], but it is only after the seminal work of Aharonov and Bohm [2] that this subject attracts wide attention. The success of that paper can be attributed to the fact that in addition to a general discussion of the phenomenon, the authors presented a clear-cut analytic calculation of physical scattering on one singular AB vortex thus validating common arguments.

Today there exists substantial literature about this effect but, surprisingly, analytically results are rare. In [3–5] a diagrammatic-like series for the scattering amplitude on a few AB vortices had been proposed, but for real energy, it is similar to a formal multiple-scattering expansion and can hardly be used for calculations.

The purpose of this paper is to investigate the problem of scattering on two AB vortices. The principal result is that this problem is exactly solvable and the calculations of the Green
function and scattering amplitude can be reduced to a solution of a series of differential equations whose lowest level includes the Painlevé V (or III) equation. Our method is a generalisation of that proposed in [6], where the diffraction on a finite slit has been treated. It is based on the point-like nature of the AB potential, which permits fixing solutions by correcting its behaviour near vortex positions. This means that only a few constants uniquely determine the full solution. Using different transformations commuting with the Laplacian leads to a system of equations for these constants. Besides equations, it is necessary to know the values of different quantities at small and/or large distances between vortices. For large vortex separation this can be accomplished by perturbation series, while for small distances between vortices it is achieved using the Riemann–Hilbert method.

The remainder of this paper is organized as follows. Section 2 is devoted to a general discussion of the problem of scattering on AB vortices. Special attention is focused on the uniqueness of the solution and, in particular, on the fact that any solution obeying all boundary and radiation conditions but without incoming incident waves is identically zero. In section 3 we demonstrate how the arguments of [6] can be generalised to the case of two vortices. First, a set of auxiliary functions independent of incident fields with prescribed singularities at vortex positions are introduced. These functions are analogues of the Hankel functions for the one-vortex problem and play a prominent role in what follows. The main idea of [6] is that there exists a group of invariant differential operators that commute with the Lagrangian and cancel the incident field. The transformed solution is non-zero because the action of these group operators change boundary conditions near vortices. However, these changes can be compensated by a suitable linear combination of new functions. In this manner one obtains a set of equations for unknown functions. To illustrate the general scheme without tedious calculations, the well-investigated case of one AB vortex is briefly considered in section 4. Even in this simple example, the use of invariant operators permits obtaining certain non-trivial relations, which are difficult to obtain by other methods. In section 5 the construction of the Green function for two AB vortices is discussed in detail. Calculation of group commutators in section 6 permits us to find equations for all necessary functions, and in section 7 this procedure determines scattering amplitude. Section 8 is a summary of the obtained results. The relation of these results to the theory of holonomic quantum field [18] is in short discussed here. Appendix A is devoted to the proof of the uniqueness of the solution and to derivation of the reciprocity relation for the scattering on two AB vortices. To use the obtained equations one must obtain the asymptotical behaviour of correct solutions at small and/or large separation between vortices. This is achieved in appendix B, where explicit forms of the solution when the distance between vortices approaches zero and infinity are calculated. To diminish the size of this paper only the main steps of derivations are presented and details are often omitted.

2. General considerations

The AB vector potential, $A_\mu$, is a pure gauge potential, $A_\mu = \partial_\mu \phi$, and can be removed by a gauge transformation. Nevertheless, the existence of AB vortices manifests in non-zero circulation along any closed contour encircling only vortex $j$

$$\oint A_\mu \, dx_\mu = -2\pi \alpha_j,$$  \hspace{1cm} (1)

where $\alpha_j$ is the magnetic flux associated with the vortex (we assume that flux is not an integer).

The existence of a non-zero circulation implies that after removing potential from each vortex a line of phase discontinuity (the cut) denoted $C_j$ emanates such that the function and
its normal derivative on the both sides of the cut differ by a phase
\[ \Psi_L(x, 0) = e^{2\pi i \lambda x} \psi_L(x, 0), \quad \partial_y \Psi_L(x, 0) = e^{2\pi i \lambda y} \partial_y \psi_L(x, 0), \quad x \in \mathcal{C}_j. \] (2)

Here \( x \) and \( y \) are coordinates along and perpendicular to the cut, respectively. Each cut has two different sides that can be connected by a contour encircling one or more vortices. The cuts can be chosen arbitrarily and wave functions with different cuts are gauge equivalent. In Figure 1 a convenient choice of cuts for two AB vortices used throughout the paper is sketched. In this case the two cuts coincide along the negative \( x \)-axis, and boundary values of the wave function and its \( y \)-derivative on the cut are related as follows:

\[ \Psi_L(x, 0) = e^{2\pi i \lambda (x)} \psi_L(x, 0), \quad \partial_y \Psi_L(x, 0) = e^{2\pi i \lambda (y)} \partial_y \psi_L(x, 0), \] (3)

where piece-wise constant function \( \chi(x) \) is

\[ \chi(x) = \begin{cases} \alpha_1 + \alpha_2, & x < 0 \\ \alpha_1, & 0 < x < L, \\ 0, & x > L \end{cases} \] (4)

and subscripts \((\pm)\) correspond to the limit \( y \to 0 \) from positive and negative \( y \) values. The one-vortex problem has been solved in [2] (and is shortly reviewed in section 4).

As with any problem of diffraction, the scattering on the AB vortices corresponds to finding a wave function \( \Psi(\vec{x}) \) with the following properties:

(a) \( \Psi(\vec{x}) \) is the sum of an incident wave \( \psi^{inc}(\vec{x}) \), which includes all incoming waves, and a reflected outgoing wave \( \psi^{ref}(\vec{x}) \),

\[ \Psi(\vec{x}) = \psi^{inc}(\vec{x}) + \psi^{ref}(\vec{x}). \] (5)

The choice of the incident wave is dictated by the problem. When one is interested in the Green function, the incident wave is the Green function of the free Helmholtz equation \( (\Delta + k^2) \psi^{inc}(\vec{x}) = \delta(\vec{x} - \vec{x}') \).
\[ \Psi^{\text{inc}}(\vec{x}) = \frac{1}{4i} H_0^{(1)}(k|\vec{x} - \vec{x}'|) \]

with \( \vec{x}' \) being the source point. Here and below \( H_0^{(1)}(x) \) denotes the first Hankel function of zero order, and \( k \) is the momentum.

For the problem of plane wave scattering,

\[ \Psi^{\text{inc}}(\vec{x}) = e^{ikx} \cos \phi + y \sin \phi, \]

(7)

where \( \phi \) is the angle of incidence.

To simplify notations only the dependence on point \( \vec{x} \) is explicitly stated in \( \Psi^{\text{inc}}(\vec{x}) \). Other parameters (like source point coordinates in (6) and an incident angle in (7)) will be indicated when necessary.

(b) The reflected field obeys the Helmholtz equation

\[ (\partial_x^2 + \partial_y^2 + k^2) \Psi^{\text{ref}}(\vec{x}) = 0 \]

(8)

everywhere in the plane \((x, y)\) except the cuts.

(c) At both sides of the cuts the full wave function and its normal derivative are related as in equation (2) (or for the cuts in figure 1, as in equation (3)).

(d) At large \(|\vec{x}|\) the reflected field obeys the outgoing radiation condition

\[ \lim_{R \to \infty} \int_{C_R} \left| \partial_\tau \Psi^{\text{ref}} - i k \Psi^{\text{ref}} \right|^2 ds = 0 \]

(9)

where the integration is performed over a big circle \( C_R \) of radius \( R \), which includes all vortices, and \( s \) is the length along this circle (see dashed circle in figure 1).

(e) Vortices are considered impenetrable. This means that the full wave function approaches zero at vortex positions. More precisely, in a small vicinity of each vortex the full wave function should have the following behaviour:

\[ \Psi(\vec{x}) \xrightarrow[\vec{x} \to \vec{L}_j]{} a_j (x - L_j + iy)^{\alpha_j} + b_j (x - L_j - iy)^{1-\alpha_j}, \]

(10)

Since the integer part of the flux does not change boundary conditions, we consider fluxes in the interval \( 0 < \alpha_j < 1 \).

Conditions (a), (b), and (d) are valid for any scattering problem, but (c) and (e) are specific to the AB effect. According to the definition of the AB vortices (2) a wave function must get a phase factor \( e^{2\pi i \alpha} \) when going around a vortex with flux \( \alpha \). Therefore its Fourier expansion in a polar coordinate system \((r, \phi)\) centred at the vortex should have the form

\[ \Psi(r, \phi) = \sum_{n \in \mathbb{Z}} w_n(r) e^{i(n+\alpha)\phi} \]

(11)

In a small vicinity of the vortex, when \( r \to 0 \), this wave function obeys the Laplace equation \( \Delta \Psi(r, \phi) = 0 \), from which it follows that

\[ w_n(r) \xrightarrow[r \to 0]{} a_n r^{n+\alpha} + b_n r^{-(n+\alpha)} \]

(12)

The impenetrability condition means that wave function \( \Psi(r, \phi) \) approaches zero when \( r \) approaches the vortex position, which implies that \( \lim_{r \to 0} w_n(r) = 0 \) for all \( n \). Consequently, for small \( r \) (assuming that \( 0 < \alpha < 1 \)) \( w_n \sim r^{n+\alpha} \) for \( n \geq 0 \) and \( w_n \sim r^{-(n+\alpha)} \) for \( n \leq -1 \). Since \( re^{i\phi} = x + iy \) (where \( x \) and \( y \) are Cartesian coordinates centred at the vortex) one obtains the boundary conditions (e). More careful treatment of such boundary conditions can be found in [7, 8].

From physical considerations it is clear that the solution obeying all conditions (a)–(e) is unique, or if a function fulfils all these conditions with zero incoming incident field it is identically zero. This can be proved by a generalisation of usual arguments developed for
diffraction problems (see [12] and references therein). For completeness, we present in appendix A a brief demonstration of uniqueness stressing the necessity of all requirements (a)–(e).

The usual way of solving the two-vortices problem is to represent the reflected outgoing field as a sum of single and double layers along the cuts,

\[
\Psi_{\text{ref}}(x, y) = \int_{\text{cuts}} H_0^{(1)}(k\sqrt{(x-t)^2 + y^2}) \mu(t) dt \\
+ \partial_y \int_{\text{cuts}} H_0^{(1)}(k\sqrt{(x-t)^2 + y^2}) \nu(t) dt.
\]

(13)

Functions \(\mu(t)\) and \(\nu(t)\) are piece-wise functions on the cuts that must be determined from the boundary conditions (3). These calculations are standard (cf [11]) and lead to the following system of equations

\[
\nu(x) = \frac{1}{2} \tan \pi \chi(x) \int_{\text{cuts}} H_0^{(1)}(k|x-l|) \mu(t) dt + \mathcal{F}(x, 0),
\]

(14)

\[
\mu(x) = -\frac{1}{2} \tan \pi \chi(x)(\partial_x^2 + k^2) \int_{\text{cuts}} H_0^{(1)}(k|x-l|) \nu(t) dt + \partial_x \mathcal{F}(x, 0),
\]

(15)

where (provided that the incident wave \(\Psi_{\text{inc}}(\vec{x})\) has no phase jumps)

\[
\mathcal{F}(x, 0) = \frac{1}{2} \tan \pi \chi(x) \Psi_{\text{inc}}(x, 0), \quad \partial_x \mathcal{F}(x, 0) = \frac{1}{2} \tan \pi \chi(x) \partial_x \Psi_{\text{inc}}(x, 0).
\]

(16)

This approach is well suited for numerical calculations. To progress analytic treatment, we generalise in the next section the method of [6], which has been developed for diffraction on a finite slit.

3. Invariant operators

To calculate the Green function for the two-vortex problem it is necessary to fix the incident field as in equation (6). Due to dependence of the source coordinates, \(\vec{x}'\), the asymptotics of the Green function denoted by \(G(\vec{x}, \vec{x}')\) is

\[
G(\vec{x}, \vec{x}') \rightarrow_{x \rightarrow L_j} a_j(\vec{x}') \left( x - L_j + iy \right)^\nu_j + b_j(\vec{x}') \left( x - L_j - iy \right)^{1-\nu_j}.
\]

(17)

The choice of the fractional power branch is dictated by the choice of the cuts.

Let us introduce auxiliary functions \(A_j(\vec{x})\) and \(B_j(\vec{x})\) independent of the incident field, which obey all conditions (8)–(10) except that at one vortex indicated by \(j\) they have the following asymptotic behaviour:

\[
A_j(\vec{x}) \rightarrow_{x \rightarrow L_j} \partial_x \left( x - L_j + iy \right)^\nu_j = -\frac{\alpha_j}{(x - L_j + iy)^{1-\nu_j}},
\]

(18)

\[
B_j(\vec{x}) \rightarrow_{x \rightarrow L_j} \partial_x \left( x - L_j - iy \right)^{1-\nu_j} = -\frac{1 - \alpha_j}{(x - L_j - iy)^\nu_j}.
\]

(19)

These functions are uniquely fixed by these conditions and will play an important role below.

The uniqueness of the solution together with the point-like character of boundary conditions (17) permit finding various relations between functions \(a_j, b_j, A_j, B_j\) [6]. Indeed, assume that there exists an infinitesimal transformation \(\delta\) commuting with the Laplace operator such that the incident field is invariant, \(\delta \Psi_{\text{inc}}(\vec{x}) = 0\). Then the function \(\delta \Psi(\vec{x})\) is a solution with zero incident field. Nevertheless, such a transformed solution is not zero.
because, in general, the symmetry transformation changes the behaviour near one or many vortices. However, this change can be compensated for by a suitable chosen linear combination of functions $A_j$ and $B_j$. This means that the function

$$\hat{\delta}\Psi(x) + \sum_j [c_j A_j(x) + d_j B_j(x)]$$

(20)

where coefficients $c_j$ and $d_j$ are independent on $x$, obeys all conditions (8)–(10) but with zero internal field. By uniqueness it has to be identically zero,

$$\hat{\delta}\Psi(x) + \sum_j [c_j A_j(x) + d_j B_j(x)] \equiv 0.$$  

(21)

The explicit form of invariant operators depends on the quality considered. For the Green function $G(x, x')$, the incident field (6) (as any function dependant only on the distance between two points) has the following invariant operators:

- **Change of vortex positions**
  $$\delta = \partial_{L_i}.$$  

(22)

- **Translational invariance**
  $$\delta = \partial_x + \partial_{L_i}, \quad \hat{\delta} = \partial_y + \partial_{L_i}.$$  

(23)

- **Rotational invariance**
  $$\hat{\delta} = L^{(j)}_0 + L^{(j)}_{y'}, \quad L^{(j)}_0 = (x - L_j)\partial_x - y\partial_y.$$  

(24)

When $\Psi(x) = A_j(x)$ or $B_j(x)$ one invariant operator is evidently the rotation over the vortex position, $\hat{\delta} = L^{(j)}_0$. The second operator is different for functions $A_j(x)$ and $B_j(x)$.

- For functions $A_j(x)$, $\delta = \partial_x + i\partial_y$.
- For functions $B_j(x)$, $\delta = \partial_x - i\partial_y$.

These operators annihilate the singular parts in equations (18) and (19), and therefore can serve as invariant operators. Other linear combinations of the first derivatives are forbidden because they give too strong singularity at the vortex position, which cannot be compensated by functions $A_j(x)$ and $B_j(x)$.

For the scattering problem when the incident field is given by the plane wave (7), useful invariant operators have the form

$$\hat{\delta} = \partial_x - ik \cos \phi, \quad \hat{\delta} = \partial_y - ik \sin \phi$$

(25)

where $\phi$ is the angle of incidence.

Computing the action of such operators and their commutators leads to a sufficient number of equations, which permits reconstruction of the full solution.

The application of such an approach to the problem of two AB vortices unavoidably requires long and tedious calculations with many technical details. To acquaint the main ideas without unnecessary complications, the simple case of scattering on one AB vortex is discussed in the next section. In subsequent sections, the above transformations and their combinations are considered meticulously and equations for the Green function of two AB vortices are derived.
4. One-vortex solution and local constants

The purpose of this section is two-fold. First, the general formalism briefly discussed in the previous section is applied to the simple and well-investigated case of scattering on one AB vortex. This permits us to clarify the meaning of the above formulae before their application to much more difficult two-vortices cases. The obtained relations can be verified independently from known explicit expressions, thus increasing confidence for formal manipulations.

Second, the one-vortex case fixes certain constants required for two-vortex calculations.

It is straightforward to check that the Green function for the one-vortex AB problem with flux $\alpha$ situated at point $\vec{x}_0 = (L, 0)$ is

$$ G(\vec{x}, \vec{x}') = \frac{1}{4i} \sum_{n=-\infty}^{\infty} \left\{ J_{n+\alpha}^{(1)}(kr) e^{i(n+\alpha)\phi} H^{(1)}_{n+\alpha}(kR) e^{-i(n+\alpha)\phi}, \quad r < R \right\} _{\alpha=\frac{-\pi}{4}}$$

(26)

Here $\vec{x} = (L + r \cos \theta, r \sin \theta)$ and $\vec{x}' = (L + R \cos \phi, R \sin \phi)$ are coordinates of the observation point and source point, respectively. $J_n(x)$ is the Bessel function of the first kind and $H^{(1)}_{n+\alpha}(x)$ is the first Hankel function (cf [10]).

When $0 < \alpha < 1$, in the limit $|x - x_0| \to 0$ terms with $n = 0$ and $n = -1$ dominate. Due to the asymptotics of the Bessel functions [10]

$$ J_n(r) \to \left( \frac{L}{2} \right)^{\frac{n}{2}} \frac{1}{\Gamma(n+1)}$$

(27)

it follows that

$$ G(\vec{x}, \vec{x}') \to \frac{a(\vec{x}')(x - L + iy) + b(\vec{x}')(x - L - iy)^{1-\alpha}}{\rho_{x}}$$

(28)

Here

$$ a(\vec{x}') = c_1(\alpha) H^{(1)}_{\alpha}(kR) e^{-ik\phi}, \quad b(\vec{x}') = c_2(\alpha) H^{(1)}_{-\alpha}(kR) e^{-i(\alpha-1)\phi},$$

(29)

with

$$ c_1(\alpha) = -\frac{i k^\alpha}{2^{\alpha+2} \Gamma(1 + \alpha)}, \quad c_2(\alpha) = -\frac{i k^{1-\alpha}}{2^{3-\alpha} \Gamma(2 - \alpha)}.$$  

(30)

Notice that equation (28) exactly corresponds to general equations (10) and (17). The peculiarity of the one-vortex solution is that auxiliary functions $A(\vec{x})$ and $B(\vec{x})$, defined as solutions of the Helmholtz equation (8) with asymptotic behaviour at small $|\vec{x}|$ given by equations (18) and (19) (plus the requirement (9) at infinity), are known. Such conditions fix these functions uniquely

$$ A(\vec{x}) = c_3(\alpha) H^{(1)}_{-\alpha}(kR) e^{i(\alpha-1)\phi}, \quad B(\vec{x}) = c_4(\alpha) H^{(1)}_{\alpha}(kR) e^{i\alpha \phi},$$

(31)

with

$$ c_3(\alpha) = -i \sin \pi \alpha k^{1-\alpha} 2^{\alpha-1} \Gamma(1 + \alpha), \quad c_4(\alpha) = -i \sin \pi \alpha k^{2-\alpha} \Gamma(2 - \alpha).$$

(32)

For later purposes it is useful to know the asymptotic behaviour of functions $A(\vec{x})$ and $B(\vec{x})$ for large $|\vec{x}|$
According to the previous section, the first step of calculations consists of derivation of the Green function over vortex position (22). From (28) and definitions (18) and (19) it follows that the combination
\[
\partial_x G(x, x') = -a(x')A(x) - b(x')B(x)
\]
obey all conditions (a)–(d) and is zero at all vortices as required by (e). Therefore, as discussed above, it must be identically zero.

Consequently,
\[
\partial_x G(x, x') = a(x')A(x) + b(x')B(x).
\]

Using equations (29) and (31) one concludes that the following relation must be valid
\[
\partial_x G(x, x') = -\left(\partial_x + \partial_y\right)G(x, x') = -\frac{k \sin \pi \alpha}{8} [H^{(1)}_0(kr)H^{(1)}_{-\alpha}(kr)e^{i\phi} + H^{(1)}_{-\alpha}(kr)e^{-i\phi}H^{(1)}_0(kr)e^{i\phi - \pi \alpha}].
\]

This is the simplest non-trivial application of Myers’ formalism to the AB problem. It is difficult to anticipate such a relation from the explicit expression (26). Nevertheless, using recursive relations for an arbitrary Bessel function \(Z_n\) and the definition of the Hankel function \([10]\)
\[
Z'_\nu(r) \pm \frac{\nu}{r}Z_\nu(r) = \pm Z_{\nu\mp 1}(r), \quad H^{(1)}_\nu = \frac{1}{i \sin \pi \nu}(J_{\nu} - e^{-i\pi \nu}J_{\nu}), \quad H^{(1)}_{-\nu} = e^{i\pi \nu}H^{(1)}_\nu
\]
after tedious calculations, one can confirm its validity.

Using another invariant operator, \(\delta = \partial_y + \partial_y\), (23) leads to the following relation
\[
i(\partial_y + \partial_x)G(x, x') = a(x')A(x) - b(x')B(x)
\]
whose validity can also be confirmed by explicit calculations.

The third operator (24) leads to the evident result \((L_\theta + L_\phi')G(x, x') = 0\).

As aforementioned, the uniqueness theorem applies equally well to functions \(A(x)\) and \(B(x)\). From this simple observation it appears, without any calculations, that the following relations must be valid
\[
(\partial_x + i\partial_y)A(x) = g_1A(x) + g_2B(x), \quad (\partial_x - i\partial_y)B(x) = g_3A(x) + g_4B(x)
\]
for certain constants \(g_i\). Their values are calculated from their explicit forms given in equation (31)
\[
g_1 = 0, \quad g_2 = ke^{i\pi \alpha}c_3/c_4, \quad g_3 = -ke^{-i\pi \alpha}c_3/c_4, \quad g_4 = 0.
\]

With exactly the same reasoning, one concludes that
\[
L_\theta A(x) = mA(x), \quad L_\theta B(x) = nB(x), \quad m = 1 - \alpha, \quad n = -\alpha.
\]
According to the reciprocity relation (76) one should obtain
\[ A(\bar{x}) = t b(\bar{\hat{x}}), \quad B(\bar{x}) = t a(\bar{\hat{x}}). \] (43)

Here transformation \( \hat{S} \) changes the sign of the second coordinate, \( \hat{S}(x, y) = (x, -y) \).
Its explicit form in polar coordinates depends on the choice of cut direction. \( \hat{S}(r, \theta) = (r, 2\pi \xi - \theta) \) where \( \xi = 0 \) if the cut is along the negative x-axis, i.e. \(-\pi < \theta < \pi\), and \( \xi = 1 \) if the cut is chosen along the positive x-axis, i.e. \( 0 < \theta < 2\pi \).
Comparison with the above formulas gives \( |t| = c_3(\alpha)/c_2(\alpha) = 4\pi\alpha(1 - \alpha) \) and
\[ t = 4\pi\alpha(1 - \alpha)e^{-2\pi\alpha\xi}. \] (44)

It is also instructive to solve a one-vortex case using only the above arguments. Let us fix the incident field as \( \psi^{inc}(\bar{x}) = e^{ikx} \), which corresponds to incident angle \( \phi = 0 \). The general form of such a function is \[ \psi^{scat}(\bar{x}) = \sum_{n=-\infty}^{\infty} j^{n+\alpha} C_n e^{i(n+\alpha)\theta} \] (45)
where \( \bar{x} = (r \cos \theta, r \sin \theta) \). Here the cut is chosen along the positive x-axis, which implies that \( 0 \leq \theta \leq 2\pi \).

Our purpose is to find coefficients \( C_n \) to fulfill all requirements of the wave function. When \( 0 < \alpha < 1 \) only two coefficients, \( C_0 \) and \( C_{-1} \), fix the function as they determine wave function singularity at the position of the vortex
\[ \psi^{scat}(\bar{x}) \longrightarrow a(x + iy)^\alpha + b(x - iy)^{1-\alpha}, \]
\[ a = \frac{C_0 e^{i\pi\alpha/2k\alpha}}{2\alpha\Gamma(1 + \alpha)}, \quad b = \frac{C_{-1} e^{i\pi(1-\alpha)/2k^{1-\alpha}}}{2^{1-\alpha}\Gamma(2 - \alpha)}. \] (46)

Applying operators (25) (with \( \phi = 0 \)) and evoking now usual arguments of the uniqueness, one finds that
\[ (\partial_x - ik)\psi^{scat}(\bar{x}) = c H^{(1)}_{1-\alpha}(kr)e^{i\alpha-1}\theta + d H^{(1)}_{\alpha}(kr)e^{i\alpha\theta}, \]
\[ \partial_\theta \psi^{scat}(\bar{x}) = ic H^{(1)}_{1-\alpha}(kr)e^{i\alpha-1}\theta - id H^{(1)}_{\alpha}(kr)e^{i\alpha\theta}. \] (47)
where
\[ c = \frac{1}{2} \sin \pi\alpha C_0 e^{i\pi\alpha/2k}, \quad d = \frac{1}{2} i \sin \pi\alpha C_{-1} e^{i\pi(1-\alpha)/2k}. \] (48)

Then, \( \cos \theta \partial_x + \sin \theta \partial_\theta = \partial_\theta \) obtains the equation
\[ (\partial_\theta + ik)\psi^{scat}(\bar{x}) = c H^{(1)}_{1-\alpha}(kr)e^{i\theta} + d H^{(1)}_{\alpha}(kr)e^{i(\alpha-1)\theta} \] (49)
whose solution, which approaches zero at the vortex position, is as follows:
\[ \psi^{scat}(\bar{x}) = e^{ikr\cos \theta} F(r, \theta), \quad F(r, \theta) = \int_0^r V(t, \theta)dt = \left( \int_0^\infty - \int_r^\infty \right) V(t, \theta)dt \] (50)
where
\[ V(t, \theta) = e^{-ikr\cos \theta}[c H^{(1)}_{1-\alpha}(kr)e^{i\theta} + d H^{(1)}_{\alpha}(kr)e^{i(\alpha-1)\theta}]. \] (51)
Using (7.14.2.30) and (7.14.2.31) of [10] one finds that when \( 0 < \nu < 1 \)
\[ \int_0^\infty H^{(1)}_{\nu}(kr)e^{-ikt\cos \theta}dt = \frac{2e^{-ikt/2}}{k \sin \pi\nu|\sin \theta|} \sin \left[ \nu \left( \frac{\pi}{2} + \arcsin(\cos \theta) \right) \right] \] (52)
where the principal branch of arcsin(x) is assumed, \(-\pi/2 \leq \arcsin(x) \leq \pi/2\).
Since $0 < \theta < 2\pi$, $\frac{\theta}{2} + \arcsin(\cos \theta) = |\pi - \theta|$. Due to the modulus sign in the denominator of (52), all modulus can be removed. In the end one obtains

$$F(\infty, \theta) = \frac{2e^{-i\alpha \theta}}{k \sin \pi \alpha \sin \theta} \left[ c e^{i(\alpha-1)\pi/2} \sin[(1 - \alpha)(\pi - \theta)] + d e^{-i\pi/2 - i\theta} \sin[\alpha(\pi - \theta)] \right].$$

(53)

According to the chosen asymptotics at infinity, $F(\infty, \theta) = 1$. This condition gives equations for $c$ and $d$, whose solutions are

$$c = \frac{1}{2}ik \sin \pi \alpha e^{-\pi i \alpha /2}, \quad d = -\frac{1}{2}k \sin \pi \alpha e^{-\pi i \alpha /2}$$

(54)

which correspond to (remembering that $0 < \alpha < 1$)

$$C_0 = e^{-2\pi i \alpha}, \quad C_1 = 1.$$  

(55)

Fixing these constants determines the scattering functions uniquely as it follows from equation (50).

The main physical quantity of interest for a scattering problem is the scattering amplitude, $\mathcal{F}(\theta)$, obtained from the asymptotic behaviour of the wave function

$$\Psi_{\text{scat}}(\vec{x}) \underset{|\vec{x}| \to \infty}{\longrightarrow} e^{ik x} + \frac{2}{\pi i kr} e^{ikr} \mathcal{F}(\theta).$$

(56)

This is given by the asymptotic behaviour of the second integral in equation (50). Since

$$\int_{r}^{\infty} H_0^{(1)}(kr)e^{-ikr \cos \theta} dr \underset{r \to \infty}{\longrightarrow} \frac{2}{\pi i kr} e^{ikr(1 - \cos \theta)} \frac{i e^{-imr/2}}{k(1 - \cos \theta)}$$

(57)

it follows that the scattering amplitude is

$$\mathcal{F}(\theta) = \frac{\sin \pi \alpha}{2 \sin(\theta/2)} e^{i(\alpha-1/2)\theta - i\pi \alpha}.$$

(58)

The same calculations for the case incident waves coming from the right, $\Psi_{\text{inc}}(\vec{x}) = e^{-ik x}$, considered in [2], give

$$\mathcal{F}(\theta) = -\frac{\sin \pi \alpha}{\cos(\theta/2)} e^{i(\alpha-1/2)\theta}$$

(59)

which agrees with [2] and with a correction first noted in [22] (here $-\pi < \theta < \pi$).

5. Construction of the Green function for two vortices

It is clear that the case of two AB vortices is much more difficult than the one-vortex problem considered in the previous section. First, instead of two (known) auxiliary functions $A(\vec{x})$ and $B(\vec{x})$ given by equations (31), for two vortices one must determine four functions (two for each vortex). However, the main difficulty comes from the fact that coefficients in obtained relations are not numerical constants but unknown functions of the distance between vortices. All necessary equations can be found only after long and tedious calculations.

5.1. Derivatives over vortex positions

From (17), definitions (18) and (19), and the uniqueness theorem, it follows that the derivatives of the Green function over vortex positions are
The behaviour of the Green function $G(x, x')$ near vortex positions is fixed by equation (17). When $0 < \alpha_j < 1/2$ term $(x - L_j)^{\alpha_j}$ dominates when $x \to x_j$ and coefficients $a_j(x')$ and $b_j(x')$ can be formally calculated from the following limits:

\[
a_j(x') = \lim_{x \to x_j} (x - L_j)^{-\alpha_j} G(x, x'),
\]

\[
b_j(x') = \lim_{x \to x_j} (x - L_j)^{\alpha_j-1} [G(x, x') - a_j(x')(x - L_j)^{\alpha_j}].
\]

When $1/2 < \alpha_j < 1$ one calculates necessary limits in the inverse order

\[
b_j(x') = \lim_{x \to x_j} (x - L_j)^{\alpha_j-1} G(x, x'),
\]

\[
a_j(x') = \lim_{x \to x_j} (x - L_j)^{-\alpha_j} [G(x, x') - b_j(x')(x - L_j)^{1-\alpha_j}].
\]

Differentiating these limits over $L_2$ for $j = 1$, over $L_1$ for $j = 2$, and using equation (60), one concludes that

\[
\partial_{L_2} \begin{pmatrix} a_1(x') \\ b_1(x') \end{pmatrix} = \begin{pmatrix} \beta_1 & \epsilon_1 \\ \delta_1 & \zeta_1 \end{pmatrix} \begin{pmatrix} a_2(x') \\ b_2(x') \end{pmatrix},
\]

\[
\partial_{L_1} \begin{pmatrix} a_2(x') \\ b_2(x') \end{pmatrix} = \begin{pmatrix} \beta_2 & \epsilon_2 \\ \delta_2 & \zeta_2 \end{pmatrix} \begin{pmatrix} a_1(x') \\ b_1(x') \end{pmatrix}.
\]

where all matrix elements are independent on space coordinates.

Calculating the mixed derivatives $\partial_{L_2 L_1} G(x, x')$ one finds that the derivatives of $A_i$ and $B_i$ are expressed through the same constants as follows:

\[
\partial_{L_2} \begin{pmatrix} A_1(x) \\ B_1(x) \end{pmatrix} = \begin{pmatrix} \beta_2 & \epsilon_2 \\ \delta_2 & \zeta_2 \end{pmatrix} \begin{pmatrix} A_2(x) \\ B_2(x) \end{pmatrix},
\]

\[
\partial_{L_1} \begin{pmatrix} A_2(x) \\ B_2(x) \end{pmatrix} = \begin{pmatrix} \beta_1 & \epsilon_1 \\ \delta_1 & \zeta_1 \end{pmatrix} \begin{pmatrix} A_1(x) \\ B_1(x) \end{pmatrix}.
\]

5.2. Translational invariance

A few simple consequences of translational invariance exist. First, constants like $\beta_j, \epsilon_j, \delta_j, \zeta_j$ depend only on the difference $L = L_1 - L_2$. Second, functions dependant only on $x$ or $x'$, like $f_i = a_i, b_i, A_i, B_i$, obey the equation

\[
(\partial_{L_1} + \partial_{L_2} + \partial_i) f_i(x) = 0.
\]

This equation permits one to calculate derivatives $\partial_{L_i} f_i$ in contrast to equation (63), which determine derivatives $\partial_{L_2} f_i$ only when $j \neq i$.

Consider now the change of two coordinates simultaneously. Comparing the behaviour near two vortices (17) one concludes that

\[
i(\partial_x + \partial_{x'}) G(x, x') = a_1(x')A_1(x) - b_1(x')B_1(x) + a_2(x')A_2(x) - b_2(x')B_2(x).
\]
Differentiating this expression by $L_1$ and using equations (64) and (65) leads to the identity valid for all $\rho_x$ and $\rho_x'$:

$$A_1(\vec{x})(\partial_{x'} + i\partial_{y'})a_1(\vec{x}') + a_1(\vec{x}')(\partial_x + i\partial_y)A_1(\vec{x})$$

$$- B_1(\vec{x})(\partial_{x'} - i\partial_{y'})b_1(\vec{x}') - b_1(\vec{x}')(\partial_x - i\partial_y)B_1(\vec{x})$$

$$= A_1(\vec{x})[\beta_1 a_2(\vec{x}') + \epsilon_1 b_2(\vec{x}')] - a_1(\vec{x}')[\beta_2 A_2(\vec{x}) + \delta_2 B_2(\vec{x})]$$

$$+ A_2(\vec{x})[\beta_2 a_1(\vec{x}') + \epsilon_2 b_1(\vec{x}')] + a_2(\vec{x}')[\beta_1 A_1(\vec{x}) + \delta_1 b_1(\vec{x})]$$

$$- B_2(\vec{x})[\beta_2 a_1(\vec{x}') + \zeta_2 b_1(\vec{x}')] - b_2(\vec{x}')[\epsilon_1 A_1(\vec{x}) + \zeta_1 B_1(\vec{x})].$$  

(67)

Substituting into this expression the most general linear relations between derivatives and unknown functions

$$\partial_{x'} + \partial_x = \partial_{x'}' + \partial_x'$$

$$\partial_{x'} - \partial_x = \partial_{x'}' - \partial_x'$$

$$\partial_{x'} + \partial_{x'}' = \partial_{x'}' + \partial_{x'} = \partial_{x'} = \partial_{x'}'$$

$$\partial_{x'} - \partial_{x'}' = \partial_{x'}' - \partial_{x'} = \partial_{x'} = \partial_{x'}'$$

and collecting identical terms, proves the following formulae:

$$A_1(\vec{x})(\partial_{x'} + i\partial_{y'})a_1(\vec{x}') + a_1(\vec{x}')(\partial_x + i\partial_y)A_1(\vec{x}) = g_1 A_1(\vec{x}) + g_2 B_1(\vec{x}) + h_1 A_2(\vec{x}) + h_2 B_2(\vec{x}),$$

$$- B_1(\vec{x})(\partial_{x'} - i\partial_{y'})b_1(\vec{x}') - b_1(\vec{x}')(\partial_x - i\partial_y)B_1(\vec{x}) = g_2 A_1(\vec{x}) + g_2 B_1(\vec{x}) + h_2 A_2(\vec{x}) + h_2 B_2(\vec{x}),$$

$$\partial_{x'} + i\partial_{y'})a_1(\vec{x}') = G_1 a_1(\vec{x}') + G_2 b_1(\vec{x}') + H_1 a_1(\vec{x}') + H_2 b_1(\vec{x}'),$$

$$\partial_{x'} - i\partial_{y'})b_1(\vec{x}') = G_2 a_1(\vec{x}') + G_2 b_1(\vec{x}') + H_2 a_1(\vec{x}') + H_2 b_2(\vec{x}'),$$

(68)

Similarly, differentiating equation (66) by $L_2$ gives rise to the relations

$$A_1(\vec{x})(\partial_{x'} + i\partial_{y'})a_1(\vec{x}') + a_1(\vec{x}')(\partial_x + i\partial_y)A_1(\vec{x}) = f_1 A_2(\vec{x}) + f_2 B_2(\vec{x}) - 2\delta_2 B_2(\vec{x}),$$

$$- B_1(\vec{x})(\partial_{x'} - i\partial_{y'})b_1(\vec{x}') - b_1(\vec{x}')(\partial_x - i\partial_y)B_1(\vec{x}) = g_2 A_2(\vec{x}) + g_2 B_2(\vec{x}) - 2\epsilon_2 A_2(\vec{x}),$$

$$\partial_{x'} + i\partial_{y'})a_1(\vec{x}') = -g_1 a_1(\vec{x}') + g_1 b_1(\vec{x}') - 2\epsilon_1 b_1(\vec{x}'),$$

$$\partial_{x'} - i\partial_{y'})b_1(\vec{x}') = g_2 a_1(\vec{x}') - g_4 b_1(\vec{x}') - 2\delta_2 a_1(\vec{x}').$$

(69)

Here $g_j$ and $f_j$ with $j = 1, 2, 3, 4$ are undetermined constants dependant only on $L = L_4 - L_2$.

5.3. Rotational invariance

The mutual rotation of $\vec{x}$ and $\vec{x}'$ around any vortex is evidently the invariance transformation for the incident field (6). Such rotations are generated by operators

$$L_4^{(j)} = x\partial_y - y\partial_x - L_j\partial_{x'}, \quad L_4^{(j)} = x'\partial_y - y'\partial_x - L_j\partial_{x'}.$$  

(71)

Thus, one has

$$(L_4^{(j)} + L_4^{(j)})H_0^{(1)}(k\sqrt{(x-x')^2 + (y-y')^2}) = 0.$$  

(72)

Exactly as done above, one concludes that

$$i(L_4^{(2)} + L_4^{(2)})G(\vec{x}, \vec{x}') = L_a(\vec{x}')A_1(\vec{x}) - L b_1(\vec{x}')B_1(\vec{x}),$$

$$i(L_4^{(1)} + L_4^{(1)})G(\vec{x}, \vec{x}') = L_a(\vec{x}')A_2(\vec{x}) - L b_2(\vec{x}')B_2(\vec{x})$$

(73)

with $L = L_4 - L_2$. 

Differentiating these equations by \( L \), one finds the following relations (similar equations for functions \( a_i \) and \( b_i \) are not presented)

\[
iL^{(1)}_\theta \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = M_1 \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} + L \left( -\beta_2 \delta_1 - \epsilon_2 \zeta_2 \right) \begin{pmatrix} A_2 \\ B_2 \end{pmatrix},
\]

\[
iL^{(2)}_\theta \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = M_2 \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} + L \left( \beta_1 - \delta_1 \right) \begin{pmatrix} A_1 \\ B_1 \end{pmatrix},
\]

(74)

where \( M_j \) are \( 2 \times 2 \) matrices with undetermined coefficients.

6. Determination of coefficients

Equations (69), (70), and (74) are concise consequences of symmetries of the incident field (6). However, these equations contain many coefficients dependant on, in general, the distance between vortices, \( L \).

The first simplification comes from the fact that functions \( a_j \) and \( b_j \) for two vortices can be expressed through functions \( A_j \) and \( B_j \) by the reciprocity relation (A.14) proven in appendix A. It states that

\[
G(\vec{x}, \vec{x}') = G(\vec{Sx}', \vec{Sx}), \quad \vec{S}(x, y) = (x, -y).
\]

(75)

Using equations (60) and (73) and considering that \( L^{(1)}_\theta F(\vec{x}) = -\vec{S}L^{(1)}_\theta F(\vec{x}) \), one concludes that

\[
\begin{pmatrix} A_1(\vec{x}) \\ B_1(\vec{x}) \end{pmatrix} = \begin{pmatrix} 0 & t_1 \\ t_1 & 0 \end{pmatrix} \begin{pmatrix} a_1(\vec{x}) \\ b_1(\vec{x}) \end{pmatrix}, \quad \begin{pmatrix} A_2(\vec{x}) \\ B_2(\vec{x}) \end{pmatrix} = \begin{pmatrix} 0 & t_2 \\ t_2 & 0 \end{pmatrix} \begin{pmatrix} a_2(\vec{x}) \\ b_2(\vec{x}) \end{pmatrix}.
\]

(76)

Substituting these values to equation (64) and comparing with equation (63) gives

\[
\begin{pmatrix} \beta_2 \\ \delta_2 \\ \epsilon_2 \\ \zeta_2 \end{pmatrix} = \rho \begin{pmatrix} \epsilon_1 \\ -\beta_1 \\ \delta_1 \\ -\epsilon_1 \end{pmatrix}, \quad \rho = \frac{t_1}{t_2}.
\]

(77)

These relations mean that one can consider equations only for functions \( A_j \) and \( B_j \). Values of constants \( t_1 \) and \( \rho \) are calculated in section 4.

The first series of relations between these coefficients is obtained by differentiating both sides of equations (69) and (70) by \( L \). Since all derivatives are known through previous formulae, one obtains many interrelations between the constants. In particular, it follows that

\[
\partial_L g_3 = \partial_L g_4 = \partial_L f_1 = \partial_L f_4 = 0.
\]

(78)

This means that constants \( g_3, g_4, f_1, f_4 \) are independent on \( L \) therefore they can be calculated from one-vortex solution. From section 4 it follows that \( g_3 = g_4 = f_1 = f_4 = 0 \). We rewrite the above equations with these values as

\[
(\partial_x + i\partial_y) \begin{pmatrix} a_1(\vec{x}') \\ a_2(\vec{x}') \end{pmatrix} = \begin{pmatrix} g_1 & -2\epsilon_1 \\ -2\epsilon_2 & f_3 \end{pmatrix} \begin{pmatrix} b_1(\vec{x}') \\ b_2(\vec{x}') \end{pmatrix},
\]

\[
(\partial_x - i\partial_y) \begin{pmatrix} b_1(\vec{x}') \\ b_2(\vec{x}') \end{pmatrix} = \begin{pmatrix} g_2 & -2\delta_1 \\ -2\delta_2 & f_2 \end{pmatrix} \begin{pmatrix} a_1(\vec{x}') \\ a_2(\vec{x}') \end{pmatrix},
\]

(79)
Applying the \( \partial_{\epsilon} \pm i\partial_{\delta} \) operator to these equations and using the fact that each function \( f(\vec{x}) \) must obey the Helmholtz equation, \( \partial^2 + k^2 f(\vec{x}) = 0 \), one finds that
\[
\begin{align*}
g_2g_1 + 4\epsilon_2\delta_1 + k^2 = 0, \quad &\epsilon_2 f_2 + \delta_2 g_3 = 0, \\
\delta_2 f_3 + \epsilon_2 g_2 = 0, \quad &\epsilon_1 \delta_2 = \delta_1 \epsilon_2. \quad (81)
\end{align*}
\]

Differentiating (80) by \( L \) gives
\[
\begin{align*}
g_2 &= 2(\zeta_2 \delta_2 + \beta_2 \delta_1), \quad &g_3 &= 2(\epsilon_2 \beta_1 + \epsilon_1 \zeta_2), \\
f_2 &= -2(\delta_1 \zeta_2 + \delta_2 \beta_1), \quad &f_3 &= -2(\delta_1 \zeta_2 + \delta_2 \beta_1) \\
2\delta_2 &= f_2 \beta_2 - g_2 \zeta_2, \quad &2\epsilon_2 &= f_3 \zeta_2 - g_3 \beta_2, \\
2\delta_1 &= f_2 \zeta_1 - g_2 \beta_1, \quad &2\epsilon_1 &= f_3 \beta_1 - g_3 \zeta_1. \quad (82)
\end{align*}
\]

Here and below the dot indicates the derivative over \( L \).

In a similar way, the differentiation by \( L \) in the rotation equation (74) yields many other relations. In particular, both matrices \( M_1 \) and \( M_2 \) are constant. From the solution of well-separated vortices (see section 4) it follows that both matrices are diagonal. For simplicity we impose this condition from now on. Therefore, the rotations have the form
\[
\begin{align*}
i\gamma^{(1)}_{(L)} A_1 &= m_1 A_1 - L \beta_2 A_2 + L \delta_2 B_2, \quad &i\gamma^{(1)}_{(L)} B_1 &= n_1 B_1 - L \epsilon_2 A_2 + L \zeta_2 B_2, \\
i\gamma^{(2)}_{(L)} A_2 &= m_2 A_2 + L \beta_1 A_1 - L \delta_1 B_1, \quad &i\gamma^{(2)}_{(L)} B_2 &= n_2 B_2 + L \epsilon_1 A_1 - L \zeta_1 B_1. \quad (83)
\end{align*}
\]

By differentiating these expressions over \( L \) one obtains the following list of equations
\[
\begin{align*}
(L \epsilon_1) &= (m_1 - n_2) \epsilon_1 - L \zeta_1 g_3, \quad & (L \epsilon_2) &= (m_2 - n_1) \epsilon_2 + L \zeta_2 f_3, \\
(L \zeta_1) &= (n_1 - n_2) \epsilon_2 - L \epsilon_1 g_3, \quad & (L \zeta_2) &= (m_2 - n_1) \epsilon_2 + L \epsilon_2 f_3, \\
(L \beta_1) &= (m_1 - n_2) \beta_1 - L \delta_1 g_3, \quad & (L \beta_2) &= (m_2 - m_1) \beta_2 + L \delta_2 f_3, \\
(L \delta_1) &= (m_1 - n_1) \beta_1 + L \delta_1 g_3, \quad & (L \delta_2) &= (m_1 - n_2) \beta_1 + L \delta_2 f_3. \quad (84)
\end{align*}
\]

According to equations (42) in section 4 the values of \( m_i \) and \( n_i \) are
\[
\begin{align*}
m_1 &= 1 - \alpha_1, \quad & n_1 &= -\alpha_1, \quad & m_2 &= 1 - \alpha_2, \quad & n_2 &= -\alpha_2. \quad (85)
\end{align*}
\]

As discussed above, one must consider only equations related to one vortex (say \( L_1 \)). We rewrite them again as
\[
\begin{align*}
\dot{\epsilon}_1 &= \frac{\gamma}{L} \epsilon_1 - \zeta_1 g_3, \quad & \dot{\zeta}_1 &= -\frac{\gamma + 1}{L} \zeta_1 - \epsilon_1 g_2, \\
\dot{\beta}_1 &= \frac{\gamma - 1}{L} \beta_1 - \delta_1 g_3, \quad & \dot{\delta}_1 &= -\frac{\gamma}{L} \delta_1 - \beta_1 g_2 \quad (86)
\end{align*}
\]

with \( \gamma = \alpha_2 - \alpha_1 \).
To these equations one should add the following:

\[ g_2 g_3 + 4 \epsilon_3 \delta_1 + k^2 = 0, \quad \beta_1 \epsilon_2 g_2 - \beta_2 \delta_1 g_3 + \frac{2 \gamma}{L} \delta_2 \epsilon_1 = 0. \] (87)

The second of these equations is a consequence of equation (84) when relations (77) are imposed.

It is convenient to introduce the following notations

\[ y = \delta_2 \epsilon_1 = \delta_1 \epsilon_2, \quad v = f_3 \delta_2 = -g_2 \epsilon_2, \quad w = g_3 \delta_1 = -f_2 \epsilon_1. \] (88)

Direct checking proves that these variables obey the equations

\[
\begin{align*}
\dot{y} &= \beta_1 v - \beta_2 w, \quad \beta_1 = \frac{\gamma - 1}{L} \beta_1 - w, \quad \beta_2 = -\frac{\gamma + 1}{L} \beta_2 + v, \\
\dot{w} &= -\frac{\gamma}{L} w + \beta_1 (k^2 + 8y), \quad \dot{v} = \frac{\gamma}{L} v - \beta_2 (k^2 + 8y)
\end{align*}
\] (89)

with \( \gamma = m_1 - m_2 = \alpha_2 - \alpha_1 \).

Equation (87) are equivalent to the existence of two integrals of motion

\[ vw - y(4y + k^2) = 0, \quad \beta_1 v + \beta_2 w - \frac{2 \gamma}{L} y = 0. \] (90)

Introducing a new variable, \( z = \beta_1 \beta_2 \), one obtains the following equations:

\[
\begin{align*}
\dot{z} &= -\frac{2}{L} z + y, \quad \beta_1 v = \frac{\gamma}{L} y + \frac{1}{2} \dot{y}, \quad \beta_2 w = \frac{\gamma}{L} y - \frac{1}{2} \dot{y}.
\end{align*}
\] (91)

Combination with equation (90) leads to the final system of equations

\[ \dot{z} = -\frac{2}{L} z + y, \quad \frac{1}{4} \dot{y}^2 - \frac{\gamma^2}{L^2} y^2 + zy(4y + k^2) = 0. \] (92)

Substitutions

\[ y = -\frac{k^2}{4} Y, \quad z = -\frac{Z - \gamma^2}{4L^2}, \quad L = \frac{x}{k} \] (93)

transform equations (92) to the form

\[ \frac{x^2}{4} Y'' - \gamma^2 Y = ZY(Y - 1), \quad Z' = x^2 Y'. \] (94)

Removing \( Z \) gives a non-linear equation for \( Y \)

\[ Y'' = \left( \frac{1}{2Y} + \frac{1}{2(Y - 1)} \right) Y'^2 - \frac{Y'}{x} + 2Y(Y - 1) - \frac{2\gamma^2 Y}{x^2(Y - 1)}. \] (95)

Finally, one more change of variables

\[ Y = \frac{V}{V - 1}, \quad x = \sqrt{t} \] (96)

transforms this equation to the canonical form of the Painlevé V equation (see [17])

\[ \frac{d^2 V}{dt^2} = \left( \frac{1}{2V} + \frac{1}{V - 1} \right) \left( \frac{dV}{dt} \right)^2 - \frac{1}{V - 1} \frac{dV}{dt} + \frac{\gamma^2 V(V - 1)^2}{2t^2} - \frac{V}{2t}. \] (97)
The general Painlevé V equation [16] is

\[
y'' = y^2 \left( \frac{1}{2y} + \frac{1}{y - 1} \right) - \frac{y'}{t} + \left( \frac{y - 1}{t^2} \right) \left[ \alpha y + \frac{\beta}{y} \right] + \frac{\gamma}{t} + \delta \frac{(y + 1)}{y - 1}.
\]

Therefore, equation (97) is the Painlevé V equation with the parameters

\[
\alpha = \frac{1}{2} \gamma^2, \quad \epsilon = -\frac{1}{2}, \quad \beta = 0, \quad \delta = 0.
\]

The Painlevé V equation with \( \delta = 0 \) can be reduced to the Painlevé III equation [17]. Consider the following Bäcklund-type transformation for two functions \( Y \) and \( W \)

\[
\begin{align*}
\dot{Y} &= \frac{2\gamma}{x} Y - \frac{2Y(Y - 1)}{W}, \\
\dot{W} &= 1 - 2Y + \frac{1 + 2\gamma}{x} W + W^2.
\end{align*}
\]

Solving \( W \) from the first equation and substituting it into the second leads to equation (95) for \( Y \). Calculating \( Y \) from the second equation and substituting it into the first produces

\[
\dot{W} = \frac{W^2}{W} - \frac{W}{x} + W^3 + \frac{2(1 + \gamma)W^2 - 2\gamma}{x} - \frac{1}{W}.
\]

The standard form of the Painlevé III equation is (see [17])

\[
\dot{y} = \frac{y^2}{y} - \frac{y}{x} + \alpha y^3 + \frac{\beta y^2}{x} + \frac{\epsilon}{y} + \delta.
\]

Therefore, equation (101) is the Painlevé III equation with parameters

\[
\alpha = 1, \quad \beta = 2(1 + \gamma), \quad \epsilon = -2\gamma, \quad \delta = -1.
\]

The solution \( y = y(L) \) permits one to find all other quantities by simple integration. First, \( z(L) \) is determined directly from the second equation of (92). From equation (89) it follows that

\[
u = \sqrt{y(4y + k^2)} e^{-s}, \quad w = \sqrt{y(4y + k^2)} e^{s},
\]

\[
\beta_1 = \frac{\gamma y/L + \dot{y}/2}{\sqrt{y(4y + k^2)}} e^s, \quad \beta_2 = \frac{\gamma y/L - \dot{y}/2}{\sqrt{y(4y + k^2)}} e^{-s}
\]

where function \( S = S(L) \) is calculated from the equation

\[
\dot{S} = \frac{4\gamma y}{L(k^2 + 4y)}.
\]

Using equation (86) one demonstrates that

\[
\epsilon_1 = C \sqrt{y}, \quad \delta_1 = \frac{1}{\rho C \sqrt{y}}, \quad \epsilon_2 = \rho C \sqrt{y}, \quad \delta_2 = \frac{1}{C \sqrt{y}}
\]

where \( C = C(\alpha_1, \alpha_2) \) is a constant that can be calculated from limiting values found in appendix B

\[
C(\alpha_1, \alpha_2) = \left( \frac{k}{2} \right)^{\alpha_1+\alpha_2-1} \frac{\Gamma(2 - \alpha_2)}{\Gamma(1 + \alpha_1)} \sqrt{\frac{\sin \pi \alpha_2}{\sin \pi \alpha_1}} e^{-\pi i (3\alpha_1 + \alpha_2)/2}.
\]

Other quantities can be calculated from the definition and reciprocity relation (77)

\[
\zeta_1 = \frac{1}{\rho} \beta_2, \quad \zeta_2 = \rho \beta_1, \quad g_2 = -\frac{v}{\epsilon_2}, \quad g_3 = \frac{w}{\delta_1}.
\]
7. Scattering amplitude

A typical scattering problem consists of the determination of the wave function, \( \Psi^{\text{scat}}(\vec{x}) \), when the incident field is chosen as the plane wave (7). The main physical quantity of interest is the scattering amplitude, \( \mathcal{F}(\theta, \phi) \), obtained from the asymptotic behaviour of this function

\[
\Psi^{\text{scat}}(\vec{x}) \xrightarrow{|\vec{x}| \to \infty} e^{ikr \cos(\theta - \phi)} + \sqrt{\frac{2}{\pi ikr}} e^{ikr} \mathcal{F}(\theta, \phi),
\]

where \( \vec{x} = (r \cos \theta, r \sin \theta) \) and angle \( \phi \) determines the direction of the incident wave.

Let us consider the limit of the Green function, \( G(\vec{x}, \vec{x}') \), with \( \vec{x}' = (R \cos \phi, R \sin \phi) \) and \( R \to \infty \). As

\[
\lim_{R \to \infty} \frac{H_0^{(1)}(k|\vec{x}' - \vec{x}|)}{H_0^{(1)}(k|\vec{x}'|)} = e^{-ikr \cos(\theta - \phi)},
\]

the scattering wave function with asymptotics (110) can be extracted from the Green function as follows (cf [6])

\[
\Psi^{\text{scat}}(\vec{x}) = \lim_{R \to \infty} \frac{G(\vec{x}, -\vec{x}')}{\frac{1}{4i} H_0^{(1)}(k|\vec{x}'|)}.
\]

Functions \( A_j(\vec{x}) \) and \( B_j(\vec{x}) \) defined in the previous section (cf equations (18) and (19)) obey the Helmholtz equation, and due to the radiation condition (9), have the following asymptotic behaviours

\[
A_j(\vec{x}) \xrightarrow{|x| \to \infty} \frac{2}{\pi ikr} e^{ikr} F_j(\theta) e^{-ik \cos \theta L_j}, \quad B_j(\vec{x}) \xrightarrow{|x| \to \infty} \sqrt{\frac{2}{\pi ikr}} e^{ikr} G_j(\theta) e^{-ik \cos \theta L_j}
\]

with certain functions \( F_j(\theta) \) and \( G_j(\theta) \).

Due to translational invariance \( (\partial L_x + \partial L_y + \partial L_z)A_j = 0, (\partial L_x + \partial L_y + \partial L_z)B_j = 0 \), functions \( F_j(\theta) \equiv F_j(\theta, L) \) and \( G_j(\theta) \equiv G_j(\theta, L) \) depend only on the distance between vortices, \( L = L_x - L_y \) (and angle \( \theta \)). They are asymptotic (113) when the centre of polar coordinates is chosen at vortex \( j \). To simplify notations, the arguments of \( F_j \) and \( G_j \) are dropped when it will not lead to confusion.

For scattering on AB vortices the exact wave function \( \Psi(\vec{x}) \) approaches zero at the vortex positions (cf (10))

\[
\Psi^{\text{scat}}(\vec{x}) \xrightarrow{x \to L_j} \tilde{a}_j(x - L_j + iy)^{\alpha_j} + \tilde{b}_j(x - L_j - iy)^{1-\alpha_j}.
\]

It is clear that the operator \( \delta = \partial_x - ik \cos \phi \) yields zero when acting on the incident plane wave in equation (110). Therefore \( \delta \Psi^{\text{scat}}(\vec{x}) \) corresponds to the zero incident field. Comparing the behaviour near the vortices, one finds that

\[
(\partial_x - ik \cos \phi)\Psi^{\text{scat}}(\vec{x}) + \sum_j [\tilde{a}_j A_j(\vec{x}) + \tilde{b}_j B_j(\vec{x})] = 0.
\]

Applying this relation to asymptotic expression (110) one concludes that

\[
\delta (\cos \theta - \cos \phi) \mathcal{F}(\theta, \phi) = -\sum_j [\tilde{a}_j F_j(\theta) + \tilde{b}_j G_j(\theta)] e^{-ik \cos \theta L_j}.
\]

In the theory of diffraction, such expressions are called embedding formulæ [13].

The values of \( \tilde{a}_j \) and \( \tilde{b}_j \) can be calculated from the limit (112) together with (115) and the reciprocity conditions (76).
\[ \tilde{a}_j = \lim_{R \to \infty} \frac{a_j(-\dot{x}^i)}{4i \frac{H^{(1)}}{H^{(1)}}(k|\dot{x}^i|)} = \frac{4i}{t_j} G_j(\pi - \phi)e^{ik\cos \phi L_j}, \]

\[ \tilde{b}_j = \lim_{R \to \infty} \frac{b_j(-\dot{x}^i)}{4i \frac{H^{(1)}}{H^{(1)}}(k|\dot{x}^i|)} = \frac{4i}{t_j} F_j(\pi - \phi)e^{ik\cos \phi L_j}. \]  

(117)

For clarity, the argument of \( G_j \) and \( F_j \) functions is written here as \( \pi - \phi \). In general, the choice of the branch must be consistent with the position of the cut.

Finally one obtains that the AB scattering amplitude is

\[
\mathcal{F}(\theta, \phi) = \frac{4}{k(\cos \theta - \cos \phi)} \sum_j \frac{1}{t_j} |G_j(\pi - \phi)F_j(\theta)| + F_j(\pi - \phi)G_j(\theta)e^{ikL_j(\cos \phi - \cos \theta)}.
\]  

(118)

The relations derived in previous sections induce equations for \( F_j \) and \( G_j \). From equation (80) it follows that functions \( G_j \) are linear combinations of \( F_j \)

\[
\begin{pmatrix} G_1 \\ G_2 \end{pmatrix} = V_1 \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \quad \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = V_2 \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}
\]  

(119)

where

\[
V_1 = \frac{e^{i\theta}}{ik} \begin{pmatrix} g_3 & -2\epsilon_2 e^{ikL\cos \theta} \\ -2\epsilon_1 e^{-ikL\cos \theta} & f_3 \end{pmatrix},
\]

\[
V_2 = \frac{e^{-i\theta}}{ik} \begin{pmatrix} g_2 & -2\delta_2 e^{ikL\cos \theta} \\ -2\delta_1 e^{-ikL\cos \theta} & f_2 \end{pmatrix}.
\]  

(120)

Conditions (81) imply that \( V_1V_2 = 1 \).

Equations (64) and (83) signify that derivatives of \( F_j \) and \( G_j \) are over \( L \) and \( \theta \) obey equations

\[
\partial_L \begin{pmatrix} F_1 \\ G_1 \end{pmatrix} = -e^{ikL\cos \theta} \begin{pmatrix} \beta_2 & \delta_2 \\ \epsilon_2 & \zeta_2 \end{pmatrix} \begin{pmatrix} F_1 \\ G_1 \end{pmatrix},
\]

\[
\partial_L \begin{pmatrix} F_2 \\ G_2 \end{pmatrix} = e^{-ikL\cos \theta} \begin{pmatrix} \beta_1 & \delta_1 \\ \epsilon_1 & \zeta_1 \end{pmatrix} \begin{pmatrix} F_1 \\ G_1 \end{pmatrix}.
\]  

(121)

and

\[
i\partial_\theta \begin{pmatrix} F_1 \\ G_1 \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & n_1 \end{pmatrix} \begin{pmatrix} F_1 \\ G_1 \end{pmatrix} + L \begin{pmatrix} -\beta_2 & \delta_2 \\ -\epsilon_2 & \zeta_2 \end{pmatrix} \begin{pmatrix} F_2 \\ G_2 \end{pmatrix} e^{ikL\cos \theta},
\]

\[
i\partial_\theta \begin{pmatrix} F_2 \\ G_2 \end{pmatrix} = \begin{pmatrix} m_2 & 0 \\ 0 & n_2 \end{pmatrix} \begin{pmatrix} F_1 \\ G_1 \end{pmatrix} + L \begin{pmatrix} \beta_1 & -\delta_1 \\ \epsilon_1 & -\zeta_1 \end{pmatrix} \begin{pmatrix} F_2 \\ G_2 \end{pmatrix} e^{-ikL\cos \theta}.
\]

Expressing \( G_j \) through \( F_j \) by equation (119) one finds

\[
\partial_L \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = M \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \quad i\partial_\theta \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = N \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}.
\]  

(122)
where matrices $M$ and $N$ are

$$
M = \begin{bmatrix}
-2iy e^{i\theta} & \beta_2 - i\nu e^{i\theta} e^{ikL \cos \theta} \\
\beta_1 - i\nu e^{i\theta} e^{-ikL \cos \theta} & 2iy e^{i\theta}
\end{bmatrix},
$$

$$
N = \begin{bmatrix}
m_1 + 2iy e^{i\theta} & -L \beta_2 + i\nu e^{i\theta} e^{ikL \cos \theta} \\
L \beta_1 + i\nu e^{i\theta} e^{-ikL \cos \theta} & m_2 - 2iy e^{i\theta}
\end{bmatrix},
$$

(123)

(124)

Here the notations are the same as in equation (88).

The compatibility condition

$$
\partial_y N - i\partial_x M = MN - NM
$$

(125)

is equivalent to equations (89), which is another form of their derivation.

8. Conclusion

The main result of the paper is the demonstration that the problem of two AB vortices is exactly solvable. As often the case in integrable systems, the exact solution is lengthy and tedious. This solution has been obtained by a generalisation of the method used in [6] to solve the scalar diffraction problem on scattering on a finite slit in two dimensions. The principal steps of the solution are as follows.

- Due to the singular nature of AB interactions, wave functions are fixed uniquely by their behaviour in close vicinity to the vortices and only a finite number of coefficients is necessary to reconstruct wave functions.
- To find these coefficients it is useful to introduce auxiliary functions $A_j(\vec{x})$ and $B_j(\vec{x})$ independent on incident fields with prescribed singularities at vortex $j$ (see equations (18) and (19)).
- For the Helmholtz equation in the plane (and in other symmetric spaces as well) there exists a group of first-order differential transformations that commute with the Lagrangian and cancels the incident field.
- When any such transformation is applied to the exact wave function, the resulting function corresponds to a zero incident field. However, in general, the transformed function becomes singular in one or many vortices. Since all invariant operators are of the first order, these singularities can be compensated by a suitable linear combination of auxiliary singular functions $A_j(\vec{x})$ and $B_j(\vec{x})$. In this manner one obtains a large set of equations that express certain derivatives of the Green function, and the scattering amplitude through auxiliary functions $A_j(\vec{x})$ and $B_j(\vec{x})$ (see equations (60), (66), (73), and (118)).
- Specialising these relations within the vicinity of vortex positions proves that certain derivatives of functions $A_j(\vec{x})$ and $B_j(\vec{x})$ are linear combinations of the same functions (see equations (64), (69), (70), and (74)).
• Coefficients in these relations are functions of vortex separations, and by calculating commutators of different group transformations, one obtains a system of non-linear equations for them (see equations (86)–(89)).
• All necessary coefficients can be calculated by solving the Painlevé V equation (97) or (after a non-linear Bäcklund transformation) Painlevé III equation (101).
• Since all equations are differential, it is necessary to know the values of coefficients at a certain point in order to utilize them. Analytically, one can calculate the asymptotics of these coefficients in the limit \( L \to 0 \) (see equations (B.33)–(B.35), (B.66)–(B.67), and (B.70)–(B.72)) and/or \( L \to \infty \) (see equations (B.80)–(B.85)).

The method of [6] used throughout this paper is quite general and flexible. Originally it has been used for solving certain integral equations (see [6] and [21]). As demonstrated in this paper, it can also be adapted to the problem of scattering on two AB vortices. Its generalisations for similar problems for the Klein–Gordon and Dirac operators in the Minkowski and Euclidean spaces (and probably in other symmetry spaces as well) appears possible.

**Notes**

The principal ingredient of the above solution was the adaptation of the method of [6] to problems of singular AB vortices. After the paper was practically finished, O Lisovyy pointed out to the author that similar equations (even for an arbitrary number of vortices) had been derived by Sato, Miwa, and Jimbo in [18] in a different manner. That work is one in a long series of papers devoted to developments of the theory of holonomic quantum fields (see [19] and references therein). In [18], the authors constructed a wave function with prescribed monodromy around a finite number of points. In two dimensions, monodromy transformations for a scalar equation are reduced to the appearance of the phase factor \( e^{2\pi i j} \) after encircling a point \( j \) which corresponds exactly to a AB flux line at this point. To obtain the necessary equations, the authors of [18] wrote the most general behaviour of auxiliary functions \( A_j(\vec{x}) \) and \( B_j(\vec{x}) \) in small vicinities of vortex positions, which in our notations are as follows:

\[
\begin{align*}
A_1(\vec{x}) & \rightarrow L_1 \\
A_2(\vec{x}) & \rightarrow -L_1 \\
B_1(\vec{x}) & \rightarrow L_2 \\
B_2(\vec{x}) & \rightarrow -L_2
\end{align*}
\]  

\[
\begin{pmatrix}
A_1(\vec{x}) \\
A_2(\vec{x}) \\
B_1(\vec{x}) \\
B_2(\vec{x})
\end{pmatrix} \rightarrow \begin{pmatrix}
\partial_{L_1}(x - L_1 + iy)^{\alpha_1} - \frac{1}{2}g_2(x - L_1 - iy)^{1-\alpha_1} \\
\beta_1(x - L_1 + iy)^{\alpha_1} + \delta_1(x - L_1 - iy)^{1-\alpha_1} \\
\frac{1}{2}g_3(x - L_1 + iy)^{\alpha_1} + \partial_{L_1}(x - L_1 - iy)^{1-\alpha_1} \\
\epsilon_1(x - L_1 + iy)^{\alpha_1} + \zeta_1(x - L_1 - iy)^{1-\alpha_1}
\end{pmatrix},
\]  

(126)

and

\[
\begin{pmatrix}
A_1(\vec{x}) \\
A_2(\vec{x}) \\
B_1(\vec{x}) \\
B_2(\vec{x})
\end{pmatrix} \rightarrow \begin{pmatrix}
\beta_2(x - L_2 + iy)^{\alpha_2} + \delta_2(x - L_2 - iy)^{1-\alpha_2} \\
\partial_{L_2}(x - L_2 + iy)^{\alpha_2} - \frac{1}{2}g_2(x - L_2 - iy)^{1-\alpha_2} \\
\epsilon_2(x - L_2 + iy)^{\alpha_2} + \zeta_2(x - L_2 - iy)^{1-\alpha_2} \\
-\frac{1}{2}g_3(x - L_2 + iy)^{\alpha_2} + \partial_{L_2}(x - L_2 - iy)^{1-\alpha_2}
\end{pmatrix},
\]  

(127)

Computing the action of operators commuting with the Lagrangian, comparing the dominant singularities at all vortices, and using the uniqueness, the authors of [18] obtained equations (63), (64), (69), (70), and (74).

The main differences of this paper and [18] is that in the latter the Euclidean space has been considered. Therefore, instead of the Helmholtz equation (8), the Klein–Gordon...
equation (with reversed sign of $k^2$) has been used
\[(\partial_x^2 + \partial_y^2 - k^2)\Psi(\vec{x}) = 0.\] (128)
The analogue of the radiation condition (9) in such a case is a requirement of a decreasing wave function at infinity,
\[\Psi(\vec{x}) \xrightarrow{R \to \infty} \frac{1}{\sqrt{R}} e^{-iRF(\theta)}.\] (129)

Therefore, complex conjugation of the solution does not change the correct asymptotics at infinity, and the wave function with all opposite fluxes (necessary in section 6) is simply the complex conjugate of the initial wave function. In the Minkowski space used throughout the paper, the complex conjugate turns outgoing waves into incoming waves and is not an allowed transformation. For two vortices in Minkowski space, these two functions are related by inversion at the line connecting the vortices (see equation (A.12)). In general, wave functions with opposite fluxes appearing in the reciprocity relation should be calculated separately, which effectively doubles the number of unknown variables.

The interrelations of the theory of holonomic quantum fields and the AB problem are not widely known and fully understood (e.g. there is no reference in [18] to the paper of Aharonov and Bohm [2]) and further investigation of this subject appears to be of interest.

Another difference between this paper and [18] is that in the latter the question of correct limiting values of necessary variables has not been explicitly discussed. It seems that certain asymptotics can also be obtained using general relations between boundary conditions of Painlevé equations and their monodromy group, as in [20]. Even for two vortices, direct calculations of wave functions in the limit of small and large vortex separation is a complicated problem (cf appendix B). For larger numbers of vortices this remains an open question.

Acknowledgements

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Appendix A. Uniqueness of solution and reciprocity relation for scattering on AB vortices

The standard method for proving general statements about wave equation solutions is the use of current conservation. Let $\Psi_1$ and $\Psi_2$ be two solutions of the Helmholtz equation (8). Current conservation means that
\[\int \vec{J} \, d\vec{x} = 0\] (A.1)
where current $\vec{J}$ is
\[\vec{J} = \Psi_2 \partial_\vec{x} \Psi_1 - \Psi_1 \partial_\vec{x} \Psi_2,\] (A.2)
and the integration is performed along any closed contour inside which there are no singularities of $\Psi_{1,2}$. For two AB vortices possible contours of integration can be chosen as in figure 1.

To cancel the current along both sides of the cuts due to phase jumps, it is necessary to choose the boundary jumps of $\Psi_1$ and $\Psi_2$ differently. If $\Psi_1$ obeys conditions (2), then to have
zero current through the cuts function, $\Psi_2$ should obey the same conditions but with reversed signs of all fluxes

$$\Psi_2(x, 0) = e^{-2\pi i \nu} \Psi_2(x, 0), \quad \partial_\nu \Psi_2(x, 0) = e^{-2\pi i \nu} \partial_\nu \Psi_2(x, 0), \quad x \in C_j.$$ \hspace{1cm} (A.3)

When these conditions are fulfilled, current conservation must still be checked along two other types of contours. The first consists of small circles around each vortex. If both functions obey the regularity condition (10), the integral of the current over such circles approaches zero with decreasing radius. The last contour is the circle with a large radius encircling all vortices (cf figure 1). Its treatment depends on the problem considered.

If there exist two different solutions for scattering on the same set of AB vortices, then their difference, $\Phi$, obeys the Helmholtz equation and all conditions (a)–(e) in section 2 but with zero incident field. Choosing $\Psi_1 = \Phi$ and $\Psi_2 = \Phi^*$ in the above equations, one concludes that the current conservation implies that

$$\lim_{R \to \infty} R \int_0^{2\pi} (\Phi^* \partial_R \Phi - \Phi \partial_R \Phi^*) d\phi = 0.$$ \hspace{1cm} (A.4)

However,

$$|\partial_R \Phi - i k \Phi|^2 = |\partial_R \Phi|^2 + k^2 |\Phi|^2 + i k (\Phi^* \partial_R \Phi - \Phi \partial_R \Phi^*).$$ \hspace{1cm} (A.5)

The field difference, $\Phi$, by construction obeys the radiation condition (9). Therefore, from the previous expression one concludes that if $k = 0$

$$\lim_{R \to \infty} R \int_0^{2\pi} |\Phi|^2 d\phi = 0, \quad \lim_{R \to \infty} R \int_0^{2\pi} |\partial_R \Phi|^2 d\phi = 0.$$ \hspace{1cm} (A.6)

Let $\nu = \sum_\alpha \alpha_j$ be the total flux of the vortices. Outside the circle of radius $R$, which encircles all vortices, function $\Phi$ can be expanded in formal series on a Hankel function

$$\Phi = \sum_{n=-\infty}^{\infty} A_n H^{(1)}_{n+1}(kr) e^{i(\nu+1)n\phi}.$$ \hspace{1cm} (A.7)

Since Hankel functions decrease when $r \to \infty$ as $r^{-1/2}$, from (A.6) it follows that $\sum |A_n|^2 = 0$, which implies that $|A_n|^2 = 0$ for all $n$. Furthermore, since $A_n$ are coefficients of expansion over a complete set of functions, the only possibility is that $\Phi \equiv 0$. In other words, the only solution obeying all conditions (a)–(e) with zero incident field is identically zero.

Similar arguments are used to find the reciprocity relation that relates the Green functions with interchanged positions of the source point and observation point. For scalar diffraction problems these two functions are equal, but for scattering on AB vortices one must reverse all vortex fluxes. Let us denote the Green function for scattering on vortices $\alpha = \alpha_1, \ldots, \alpha_n$ by $G_{\alpha}(\vec{x}, \vec{x}')$. Then the reciprocity relation reads

$$G_{\alpha}(\vec{x}, \vec{x}') = G_{-\alpha}(\vec{x}', \vec{x})$$ \hspace{1cm} (A.8)

where $-\alpha = 1 - \alpha_1, \ldots, 1 - \alpha_n$.

The proof of this formula is as follows:

$$(\Delta + k^2) G_{\alpha}(\vec{x}, \vec{x}') = \delta(\vec{x} - \vec{x}'), \quad (\Delta + k^2) G_{-\alpha}(\vec{x}, \vec{x}'') = \delta(\vec{x} - \vec{x}'').$$ \hspace{1cm} (A.9)

By definition, both functions obey the Helmholtz equation and on any cut $C_j$ they have opposite phase jumps.
From equation (A.9) it follows that
\[ G_\alpha(x'', x') - G_{-\alpha}(x'', x') = \oint J d\xi \]  
(A.10)
where
\[ \bar{J}(x) = G_\alpha(x, x') \partial_x G_{-\alpha}(x, x'') - G_{-\alpha}(x, x'') \partial_x G_\alpha(x, x'), \]  
(A.11)
with the integration being taken over the same contour as above. Since both Green functions obey the same radiation conditions (9) the current over a big circle approaches zero, which proves the reciprocity relation (A.8).

In [18], the Euclidean case, \( k^2 < 0 \), has been considered. As a consequence, \( \bar{\Psi}_{-\alpha}(\vec{x}) = \Psi_\alpha^\dagger(\vec{x}) \), and as in this case, the complex conjugation does not change the asymptotics of the wave function on infinity (\( \psi \sim e^{ikr} \)). For real \( k \) when the \( \psi \sim e^{ikr} \) complex conjugation contradicts the radiation condition (9), \( \bar{\Psi}_{-\alpha}(\vec{x}) \) is not immediately related to \( \Psi_\alpha(\vec{x}) \). An additional symmetry exists for the problem of two vortices, namely the reflection in the line connecting the two vortices. Since such inversion interchanges the upper and lower parts of the cuts, conditions (2) are now fulfilled but with opposite fluxes (in other words, fluxes are pseudo-scalars). Another method to confirm this relation is to consider small-x behaviour of wave functions in equation (10). It is clear that the inversion of the \( y \)-coordinate is equivalent to reversing the flux, \( \alpha \to 1 - \alpha \). Therefore, up to a phase factor
\[ \bar{\Psi}_{-\alpha}(\vec{x}) = \Psi_\alpha(\hat{S}\vec{x}) \]  
(A.12)
where the inversion \( \hat{S} \) acts as follows:
\[ \hat{S}(x, y) = (x, -y). \]  
(A.13)
Combining it with equation (A.8), one concludes that the following form of the reciprocity is valid for the two-vortex problem
\[ G_\alpha(\vec{x}, \vec{x}') = G_\alpha(\hat{S}\vec{x}', \hat{S}\vec{x}). \]  
(A.14)
To determine constants \( \Gamma_\alpha \) in (76) one can proceed as follows. Closing a vortex with flux \( \alpha \) situated at 0, the Green function behaves as in (17)
\[ G(x, x') \xrightarrow{x \to 0} a(x') (x + iy)^\alpha + b(x') (x - iy)^{1-\alpha}. \]  
(A.15)
Assume for simplicity that the cut associated with this vortex lies on the negative \( x \)-axis. When the position of the vortex is shifted from 0 to \( \delta L \) along the \( x \)-axis, the new field close to the vortex remains practically unchanged
\[ G_{\delta L}(x, x') \xrightarrow{x \to \delta L} a(x') (x - \delta L + iy)^\alpha + b(x') (x - \delta L - iy)^{1-\alpha} \]  
(A.16)
but a new portion of the cut from 0 to \( \delta L \) appears. Implicitly it is assumed that \( \delta L > 0 \) so the length of the cut increases. From (A.16) it follows that the difference of the field on both sides of the new cut is (for \( x \in [0, \delta L] \))
\[ [G_{\delta L}^{(+)}(x, x') - G_{\delta L}^{(-)}(x, x')]_{\gamma=0} = 2i \sin \pi \alpha [a(x')\nu_1(x) - b(x')\nu_2(x)], \]
\[ \partial_x G_{\delta L}^{(+)}(x, x') - \partial_x G_{\delta L}^{(-)}(x, x')]_{\gamma=0} = 2i \sin \pi \alpha [a(x')i\nu_1'(x) + b(x')i\nu_2'(x)] \]
where
\[ \nu_1(x) = (\delta L - x)^\alpha, \quad \nu_2(x) = (\delta L - x)^{1-\alpha}. \]  
(A.17)
These expressions determine the field on both sides of the cut \([0, \delta L] \). According to the Green theorem the field everywhere is given by the integral
Here $\bar{z}$ denotes point $(z, r)$ with $r \to 0$.

Due to the reciprocity relation (A.14) the behaviour of the Green function $G(\bar{x}, \bar{x}')$ when $x' \to 0$ is as follows (cf (A.15))

$$G(\bar{x}, \bar{x}') \xrightarrow{x' \to 0} a(S\bar{x})(x' - iy')^\alpha + b(S\bar{x})(x' + iy')^{1-\alpha}, \quad x' = z, \ y' = r \to 0.$$ (A.19)

Therefore in the leading order in $\delta L$ the Green function is

$$G_{\delta L}(\bar{x}, \bar{x}') \approx G(\bar{x}, \bar{x}') = 2 \sin \pi \alpha \{ a(S\bar{x})q_1 + b(S\bar{x})q_2 \} + b(S\bar{x})[b(S\bar{x})q_2 + a(S\bar{x})r_2]$$ (A.20)

where

$$q_1 = \int_0^{\delta L} [z^{\alpha} \nu_1'(z) + \alpha z^{\alpha-1} \nu_1(z)] dz, \quad r_1 = \int_0^{\delta L} [z^{\alpha-1} \nu_2'(z) - \alpha z^{\alpha-1} \nu_2(z)] dz,$$

$$q_2 = \int_0^{\delta L} [z^{1-\alpha} \nu_2'(z) + (1 - \alpha) z^{1-\alpha} \nu_2(z)] dz,$$

$$r_2 = \int_0^{\delta L} [z^{1-\alpha} \nu_1'(z) - (1 - \alpha) z^{1-\alpha} \nu_1(z)] dz.$$

It is clear that $q_j = 0$ and

$$r_1 = r_2 = -\frac{2\pi \alpha (1 - \alpha)}{\sin \pi \alpha} \delta L.$$ (A.21)

This means that one obtains (76) with

$$t_f = 4\alpha_j(1 - \alpha_j).$$ (A.22)

The same result follows from the one-vortex solution (see section 4).

**Appendix B. Solution at small and large vortex separation**

To really utilize the equations derived in the previous sections, it is necessary to know the values of all variables at a certain point. In this section it is demonstrated how to find the wave function and scattering amplitude for two AB vortices when the separation between them, $L$, is small and large with respect to the wavelength. The cases of small vortex separation with opposite fluxes (i.e. $\alpha_2 = 1 - \alpha_1$) and those with two arbitrary fluxes, $\alpha_1$ and $\alpha_2$, require different arguments and are discussed separately.

**B.1. Two vortices with opposite fluxes at small distances**

The vortices with opposite fluxes is considered first. Let the vortex with flux $\alpha$ be in the point $L_1 = L$ and the second vortex with opposite flux be at $L_2 = 0$ ($0 < \alpha < 1$).
The full wave function is represented as in equation (13)

$$\Psi(x, y) = \Psi_{\text{inc}}(x, y) + \int_0^L H_0^{(1)}(k\sqrt{(x-t)^2 + y^2})\mu(t)dt$$

$$+ \partial_x \int_0^L H_0^{(1)}(k\sqrt{(x-t)^2 + y^2})\nu(t)dt.$$  \hspace{1cm} (B.1)

Equations (14) and (15) in this case take the form

$$\nu(x) = \frac{\tan \pi \alpha}{2} \left[ \Psi_{\text{inc}}(x, 0) + \int_0^L H_0^{(1)}(k|x-t|)\mu(t)dt \right],$$  \hspace{1cm} (B.2)

$$\mu(x) = \frac{\tan \pi \alpha}{2} \left[ \partial_x \Psi_{\text{inc}}(x, 0) - \left( \frac{d^2}{dx^2} + k^2 \right) \int_0^L H_0^{(1)}(k|x-t|)\nu(t)dt \right].$$  \hspace{1cm} (B.3)

The main simplification for small-distance vortices comes from the fact that when condition

$$\frac{KL}{\alpha} \ll 1$$  \hspace{1cm} (B.4)

is fulfilled, one can substitute into the above equations the asymptotics of the Hankel function at small arguments [10]

$$H_0^{(1)}(z) \sim -\frac{2i}{\pi} \left( \ln \frac{x}{2} + \gamma \right) + 1 + O(x^2 \ln x)$$  \hspace{1cm} (B.5)

where $\gamma = -\Psi(1)$ is the Euler constant and the term proportional to $k^2$ in equation (B.3) is dropped.

After these approximations the equations for $\nu(x)$ and $\mu(x)$ become

$$\nu(x) = \frac{\tan \pi \alpha}{2} \left[ \Psi_{\text{inc}}(x, 0) + \frac{2i}{\pi} \int_0^L \ln \left( \frac{|x-t|}{L} \right)\mu(t)dt \right.\right.$$

$$+ \left. \left( \frac{2i}{\pi} \ln \frac{KL}{2} + \gamma \right) + 1 \right] \int_0^L \mu(t)dt \right],$$  \hspace{1cm} (B.6)

$$\mu(x) = \frac{\tan \pi \alpha}{2} \left[ \partial_x \Psi_{\text{inc}}(x, 0) - \frac{2i}{\pi} \frac{d^2}{dx^2} \int_0^L \ln \left( \frac{|x-t|}{L} \right)\nu(t)dt \right].$$  \hspace{1cm} (B.7)

Deriving equation (B.6) on $x$, integrating part of equation (B.7), and considering that

$$\nu(0) = 0, \quad \nu(L) = 0,$$  \hspace{1cm} (B.8)

and

$$\partial_x \int \ln(|x-t|)g(t)dt = \mathcal{P} \int \frac{g(t)}{x-t}dt$$  \hspace{1cm} (B.9)

where $\mathcal{P}$ denotes the principal value of the integral, one transforms the above equations into the following system of equations

$$\nu'(x) = \frac{\tan \pi \alpha}{2} \partial_x \Psi_{\text{inc}}(x, 0) + \frac{\tan \pi \alpha}{\pi} \int_0^L \frac{\mu(t)}{t-x}dt,$$

$$\mu(x) = \frac{\tan \pi \alpha}{2} \partial_x \Psi_{\text{inc}}(x, 0) - \frac{\tan \pi \alpha}{\pi} \int_0^L \frac{\nu'(t)}{t-x}dt.$$  \hspace{1cm} (B.10)
These equations are decoupled by introducing new variables
\[ \zeta_n(x) = \nu'(x) \pm i \nu(x). \] (B.11)

This leads to
\[ \zeta_+(x) = \tan \pi \alpha f_+(x) = \frac{\tan \pi \alpha}{\pi} \int_0^L \frac{\zeta_+(t)}{t-x} dt, \]
\[ \zeta_-(x) = \tan \pi \alpha f_-(x) + \frac{\tan \pi \alpha}{\pi} \int_0^L \frac{\zeta_-(t)}{t-x} dt, \] (B.12)

where
\[ f_\pm(x) = \frac{1}{2} (\partial_x \pm i \partial_y) \Psi_{inc}(x, 0). \] (B.13)

These equations can be solved by the Riemann–Hilbert method (see [14]). Let us introduce the following functions of complex argument \( z \)
\[ \Phi_\pm(z) = \frac{1}{2\pi i} \int_0^L \frac{\zeta_\pm(t)}{t-z} dt. \] (B.14)

It is clear that
\[ \frac{1}{\pi i} \int_0^L \frac{\zeta_\pm(t)}{t-x} dt = \Phi_\pm^{up}(x) + \Phi_\pm^{down}(x), \quad \zeta_\pm(x) = \Phi_\pm^{up}(x) - \Phi_\pm^{down}(x) \] (B.15)

where \( \Phi_\pm^{up}(x) \) and \( \Phi_\pm^{down}(x) \) are the limiting values of functions (B.14) from, respectively, positive and negative \( y \).

After a little algebra one finds that equation (B.12) are equivalent to
\[ \Phi_\pm^{up}(x) = e^{\pm 2\pi i \alpha} \Phi_\pm^{down}(x) + e^{\mp \pi \alpha} \sin \pi \alpha f_\pm(x) \] (B.16)

whose general solutions are (here \( |z| > L \) [14]
\[ \Phi_+(z) = \frac{C_1}{(z-L)^{\alpha+1-\alpha} z^{\alpha}} + \frac{\sin \pi \alpha}{2\pi i (z-L)^{\alpha+1-\alpha} z^{\alpha}} \int_0^L \frac{f_+(t)(L-t)^{\alpha+1-\alpha}}{t-z} dt, \]
\[ \Phi_-(z) = \frac{C_2}{(z-L)^{\alpha+1-\alpha} z^{\alpha}} - \frac{\sin \pi \alpha}{2\pi i (z-L)^{\alpha+1-\alpha} z^{\alpha}} \int_0^L \frac{f_-(t)(L-t)^{1-\alpha} z^{\alpha}}{t-z} dt \] (B.17)

with arbitrary constants \( C_1 \) and \( C_2 \).

When condition (B.4) is valid, functions \( f_\pm(x) \) can be approximated by their values at 0 and
\[ \Phi_+(z) = \frac{C_1}{(z-L)^{\alpha+1-\alpha} z^{\alpha}} + \frac{f_+(0)}{2i} \left[ 1 + \frac{\alpha L - z}{(z-L)^{\alpha+1-\alpha}} \right], \]
\[ \Phi_-(z) = \frac{C_2}{(z-L)^{\alpha+1-\alpha} z^{\alpha}} - \frac{f_-(0)}{2i} \left[ 1 + \frac{(1-\alpha)L - z}{(z-L)^{\alpha+1-\alpha}} \right]. \] (B.18)

The same expressions can be obtained directly from (B.16) by imposing that \( \Phi_\pm(z) = \mathcal{O}(z^{-1}) \).

According to equation (B.15) \( \zeta_\pm(x) = \Phi_\pm^{up}(x) - \Phi_\pm^{down}(x) \), and therefore, when \( 0 < x < L \)

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The J. Phys. A: Math. Theor. 49 (2016) 485202 E Bogomolny
Constants $C_j$ must be determined from conditions (B.8). First, it is necessary that
\[ \int_0^L \nu'(x) dx = 0. \]  
(B.20)

Second, from equation (B.6) calculated at $x = 0$ it follows that
\[ \Psi_{\text{inc}}(0,0) + \frac{2i}{\pi} \int_0^L \ln \left( \frac{t}{L} \right) \mu(t) dt + \left( \frac{2i}{\pi} \left( \ln \frac{kL}{2} + \gamma \right) + 1 \right) \int_0^L \mu(t) dt = 0. \]  
(B.21)

Here $\Psi_{\text{inc}}(0,0)$ is the value of the chosen incident wave (5) at the position of the second vortex.

The Euler integral (see e.g. 1.5 of [9]) and its derivative
\[
\int_0^1 t^{y-1} (1-t)^{y-1} dt = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)},
\]
\[
\int_0^1 \ln t t^{y-1} (1-t)^{y-1} dt = [\Psi(x) - \Psi(x+y)] \frac{\Gamma'(x) \Gamma(y)}{\Gamma(x+y)}
\]  
(B.22)

permit one to calculate all necessary integrals analytically. Function $\Psi(x)$ (with a scalar argument), here and in a few places below, is the standard notation of the logarithmic derivatives of the $\Gamma$ function
\[ \Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}. \]  
(B.23)

Care must be taken not to confuse it with $\Psi(\vec{r})$ (with a vector argument), which denotes a wave function.

From (B.20) it follows that $C_2 = -C_1$ and then equation (B.21) gives
\[ C_1 = [\Psi_{\text{inc}}(0,0) + (1 - \alpha)f_+(0) L + \alpha f_-(0)L] \Delta, \]
\[ \Delta = \frac{1}{2\pi + 4i[\ln(kL/4) + \gamma + \beta(\alpha)]} \]  
(B.24)

with $\gamma = -\Psi(1)$ and
\[ \beta(\alpha) = \ln 2 + \frac{1}{2} \Psi(1 - \alpha) + \frac{1}{2} \Psi(\alpha) + \gamma. \]  
(B.25)

Knowing $\mu(t)$ and $\nu(t)$ permits one to reconstruct the full wave function. In particular, the scattering amplitude at small $kL$ is large only for the $s$-wave scattering and
\[ \mathcal{F} = \int_0^L \mu(t) dt = -2\pi C_1 = -\left[ 1 + \frac{2i}{\pi} \left( \ln \frac{kL}{4} + \gamma + \beta(\alpha) \right) \right]^{-1}. \]  
(B.26)
In [15] it has been obtained in a different manner that
\[ \beta(\alpha) = \ln 4 + \frac{1}{2} \psi\left( \frac{1 - \alpha}{2} \right) + \frac{1}{2} \psi\left( \frac{\alpha}{2} \right) + \gamma + \frac{\pi}{2 \sin \pi \alpha}. \quad (B.27) \]

Using Legendre’s duplication formula (see e.g. 1.2.15 in [9]) it is straightforward to confirm that
\[ \psi\left( \frac{1 - \alpha}{2} \right) + \psi\left( \frac{\alpha}{2} \right) + \frac{\pi}{\sin \pi \alpha} = \psi(1 - \alpha) + \psi(\alpha) - 2 \ln 2. \quad (B.28) \]

Therefore these two results are identical.

The same formulae permit one to calculate limiting values of other quantities discussed in the preceding section. First, one has \((z = x + iy, \bar{z} = x - iy)\)

\[ (\partial_x + i\partial_y)\Psi(x, y) = (\partial_x + i\partial_y)\Psi_{\text{inc}}(x, y) + \frac{2}{\pi} \int_0^\ell \frac{\zeta(t)}{z - t} dt \]

\[ = (\partial_x + i\partial_y)\Psi_{\text{inc}}(x, y) - 4i\Phi(z), \]

\[ (\partial_x - i\partial_y)\Psi(x, y) = (\partial_x - i\partial_y)\Psi_{\text{inc}}(x, y) - \frac{2}{\pi} \int_0^\ell \frac{\zeta(t)}{z - t} dt \]

\[ = (\partial_x - i\partial_y)\Psi_{\text{inc}}(x, y) + 4i\Phi(z). \quad (B.29) \]

From these expressions it follows (as it has been confirmed that \(\Psi(L_j) = 0\)) that
\[ \Psi(x, y) \xrightarrow{\ell \to L_j} a_1(x - L_1 + iy)^{\alpha} + b_1(x - L_1 - iy)^{1 - \alpha}, \]

\[ \Psi(x, y) \xrightarrow{\ell \to L_2} a_2(x - L_2 + iy)^{1 - \alpha} + b_2(x - L_2 - iy)^{\alpha} \quad (B.30) \]

with
\[ a_1 = \left[ \frac{2iC_1}{\alpha} - f_-(0)L \right] L^{-\alpha}, \quad b_1 = -\left[ \frac{2iC_1}{1 - \alpha} - f_+(0)L \right] L^{1 - \alpha}, \]

\[ a_2 = e^{i\pi \alpha} \left[ \frac{2iC_1}{1 - \alpha} + f_-(0)L \right] L^{1 - \alpha}, \quad b_2 = -e^{i\pi \alpha} \left[ \frac{2iC_1}{\alpha} + f_+(0)L \right] L^{-\alpha}. \quad (B.31) \]

For the Green function one has
\[ \Psi_{\text{inc}}(0, 0) = \frac{1}{4i} H_0^{(1)}(k|x' - \bar{L}_2|), \quad (\partial_x \pm i\partial_y)\Psi_{\text{inc}}(0, 0) = \frac{k}{4i} H_1^{(1)}(k|x' - \bar{L}_2|) e^{\pm ik|x'|}. \quad (B.32) \]

Differentiating these expressions over \(L_j\) and using definitions (63) one finds that
\[ \begin{pmatrix} \beta_2 & \epsilon_2 \\ \delta_2 & \zeta_2 \end{pmatrix} \xrightarrow{L \to 0} \begin{pmatrix} \alpha + \frac{2i\Delta}{1 - \alpha} L^{2\alpha - 1} & 2i\Delta L^{-1} \\ -2i\Delta L^{-1} & -(1 - \alpha) + \frac{2i\Delta}{\alpha} L^{-2\alpha} \end{pmatrix} e^{i\pi \alpha} \quad (B.33) \]

and
\[ \begin{pmatrix} \beta_1 & \epsilon_1 \\ \delta_1 & \zeta_1 \end{pmatrix} \xrightarrow{L \to 0} \begin{pmatrix} -(1 - \alpha) + \frac{2i\Delta}{1 - \alpha} L^{2\alpha - 1} & 2i\Delta L^{-1} \\ -2i\Delta L^{-1} & \alpha + \frac{2i\Delta}{1 - \alpha} L^{-2\alpha} \end{pmatrix} e^{-i\pi \alpha}. \quad (B.34) \]
The limiting behaviours of coefficients $g_j$ and $f_j$ follow from equations (69) and (70)

$$
\begin{align*}
  g_2 & \rightarrow \frac{4i\alpha}{L-0} \Delta L^{2\gamma-2}, & g_3 & \rightarrow \frac{4i(1-\alpha)}{L-0} \Delta L^{-2\alpha}, \\
  f_2 & \rightarrow \frac{4i(1-\alpha)}{\alpha} \Delta L^{-2\alpha}, & f_3 & \rightarrow \frac{4i\alpha}{L-0} \Delta L^{2\gamma-2}.
\end{align*}
$$

(B.35)

B.2. Two vortices with arbitrary fluxes at small distances

This general case consists of two vortices with fluxes $\alpha_1$ and $\alpha_2$ ($0 < \alpha_1 < 1$) separated by a distance, $L$, obeying (B.4). The principal difference with opposite flux vortices (i.e. with $\alpha_2 = 1 - \alpha_1$) considered above is the existence of an additional cut going from infinity to the vortex positions. For convenience we choose both cuts along the x-axis, as in figure 1, such that the function $\chi(x)$ is as in equation (4). The reflected field is chosen as in equation (13), which leads to equations (14) and (15) for unknown functions $\nu(x)$ and $\mu(x)$. As a consequence, one must know these functions along the whole negative x-axis and not only at the short cut between the two vortices. To consider the condition (B.4) explicitly, it is convenient to look for the wave function of this problem in a form slightly different from equation (13), namely

$$
\Psi(\vec{x}) = \Phi(\vec{x}) + \int_{-\infty}^{L} H^{(1)}_0(k\sqrt{(x-t)^2 + y^2}) \mu(t) \, dt \\
+ \partial_x \int_{-\infty}^{L} H^{(1)}_0(k\sqrt{(x-t)^2 + y^2}) \nu(t) \, dt.
$$

(B.36)

Here $\Phi(\vec{x})$ is the one-vortex solution generated by the desired incident field $\Phi^{inc}(\vec{x})$ (see 4) multiplied by $e^{i\pi\alpha}$ for the cuts in figure 1. It also corresponds to one vortex whose flux equals the total flux of two vortices

$$
\beta = \{\alpha_1 + \alpha_2\} = \alpha_1 + \alpha_2 - \eta, \quad \eta = \begin{cases} 0, & 0 < \alpha_1 + \alpha_2 < 1 \\ 1, & 1 < \alpha_1 + \alpha_2 < 2 \end{cases}
$$

(B.37)

situates at point $L_2 = 0$.

Functions $\nu(x)$ and $\mu(x)$ must fulfill equations (14) and (15), which we rewrite below for convenience

$$
\nu(x) = \frac{1}{2} \tan \pi \chi(x) \int_{-\infty}^{L} H^{(1)}_0(k|x-t|) \mu(t) \, dt + \mathcal{F}(x, 0),
$$

(B.38)

$$
\mu(x) = -\frac{1}{2} \tan \pi \chi(x)(\partial_x^2 + k^2) \int_{-\infty}^{L} H^{(1)}_0(k|x-t|) \nu(t) \, dt + \partial_x \mathcal{F}(x, 0).
$$

(B.39)

Since function $\Psi(\vec{x})$ obeys the correct boundary conditions at the cut $(-\infty, 0]$, functions $\mathcal{F}(x, 0)$ and $\partial_x \mathcal{F}(x, 0)$ have the form

$$
\mathcal{F}(x, 0) = \Theta(x) \frac{\tan \pi \alpha_1}{2} \psi(x, 0), \quad \partial_x \mathcal{F}(x, 0) = \Theta(x) \frac{\tan \pi \alpha_1}{2} \psi(x, 0)
$$

(B.40)

where $\Theta(x) = 0$ for $x < 0$, and $\Theta(x) = 1$ for $x > 0$.

Since the vortex separation is assumed to be small (cf equation (B.4)), functions $\nu(x)$ and $\mu(x)$ in (B.36) should decrease quickly from vortex positions so that all integrals are dominated by and within vicinity of the origin.
In such conditions one can (i) approximate the above equations using \((B.5)\) as in the previous section, and (ii) use the small-\(\rho\) asymptotics of function \(\Psi_0(x')\) given by \((10)\)

\[
\Psi_0(x') \xrightarrow{\epsilon \to 0} a(x + iy)^\beta + b(x - iy)^{1-\beta}
\]

(B.41)

with certain (known) quantities \(a\) and \(b\) (fixed by the quantity considered). For the Green functions this expansion is given by equation \((28)\)

\[
a \equiv a(x') = -\frac{i k^\beta e^{i\alpha_1}}{2^{\beta+1} \Gamma(1 + \beta)} H_0^{(1)}(kR) e^{-i\beta \phi},
\]

\[
b \equiv b(x') = -\frac{i k^{1-\beta} e^{i\alpha_1}}{2^{1-\beta} \Gamma(2 - \beta)} H_0^{(1)}(kR) e^{i(1-\beta) \phi}.
\]

(B.42)

In the small-distance approximation equations \((14)\) and \((15)\) take the form (for \(-\infty < x < L\))

\[
\nu(x) = \frac{i \tan \pi \chi(x)}{\pi} \int_{-\infty}^{L} \ln \left(\frac{|x - t|}{L}\right) \mu(t) dt + \left(\ln \frac{kL}{2} + \gamma + \frac{\pi}{2i}\right) \int_{-\infty}^{L} \mu(t) dt + \mathcal{F}(x),
\]

(B.43)

\[
\mu(x) = -\frac{i \tan \pi \chi(x)}{\pi} \left[ \frac{d^2}{dt^2} \int_{-\infty}^{L} \ln(|x - t|) \nu(t) dt \right] + \partial_x \mathcal{F}(x)
\]

(B.44)

with \(\chi(x)\) defined as in equation \((4)\).

For contributions from large negative values of \(t\) to be small, the following asymptotics are required:

\[
\mu(t) \sim |t|^{-\gamma_1}, \quad \nu(t) \sim |t|^{-\gamma_2}, \quad \gamma_j > 1.
\]

(B.45)

Differentiating equation \((B.6)\) and introducing functions \((B.11)\) one obtains the equations

\[
\zeta_+(x) = \tan \pi \alpha_1 \Theta(x) f_+(x) - \frac{\tan \pi \chi(x)}{\pi} \int_{-\infty}^{L} \frac{\zeta_+(t)}{t - x} dt,
\]

(B.46)

\[
\zeta_-(x) = \tan \pi \alpha_1 \Theta(x) f_-(x) + \frac{\tan \pi \chi(x)}{\pi} \int_{-\infty}^{L} \frac{\zeta_-(t)}{t - x} dt.
\]

(B.47)

For the incident field \((B.41)\)

\[
f_+(x) \equiv \frac{1}{2} (\partial_x + i \partial_y) \Psi_0(x, 0) = (1 - \beta) a x^{-\beta},
\]

\[
f_-(x) \equiv \frac{1}{2} (\partial_x - i \partial_y) \Psi_0(x, 0) = \beta a x^{3-1}.
\]

(B.48)

Introducing analytic functions similar to equation \((B.14)\),

\[
\Phi_\pm(z) = \frac{1}{2\pi i} \int_{-\infty}^{L} \frac{\zeta_\pm(t)}{t - z} dt
\]

(B.49)

permits one to find the general solution of equations \((B.46)\) and \((B.47)\) (cf equation \((B.17)\))

\[
\Phi_+(z) = \frac{C_1}{z^{1/2} (z - L)^{\alpha_1}} + \frac{\sin \pi \alpha_1}{2\pi i} \int_{0}^{L} \frac{f_+(L - t)^{\alpha_1}}{t - z} dt.
\]

(B.50)
Branches are fixed by requiring that fractional powers are real at real $z > L$. Imposing the correct behaviour at large negative $x$ (B.45), one concludes that for $0 < \alpha_1 + \alpha_2 < 1$ (i.e. $\eta = 0$) constant $C_1 = 0$ and for $0 < \alpha_1 + \alpha_2 < 1$ (i.e. $\eta = 1$) constant $C_0 = 0$. Notice that for opposite fluxes (i.e. when $\alpha_2 = 1 - \alpha_1$) both constants are non-zero.

The remaining integrals in equations (B.50) and (B.51) reduce to the following:

\[
\int_0^L \frac{t^\alpha (L-t)^{1-\alpha}}{t-z} dt = \frac{\pi}{\sin \pi \alpha} [z^\alpha (z-L)^{1-\alpha} - z + (1-\alpha)L],
\]

\[
\int_0^L \frac{t^{-\alpha} (L-t)^\alpha}{t-z} dt = \frac{\pi}{\sin \pi \alpha} [z^{-\alpha} (z-L)^\alpha - 1].
\]

For $\eta = 0$ one obtains

\[
\Phi_+^{(0)}(z) = \frac{(1-\alpha_1 - \alpha_2)b}{2i \alpha_2 (z-L)^{\alpha_1}} z^{-\alpha_1} (z-L)^{\alpha_1} - 1,
\]

\[
\Phi_-^{(0)}(z) = \frac{C_0}{\alpha_1} - \frac{(\alpha_1 + \alpha_2)a}{2i \alpha_1 (z-L)^{\alpha_1}} z^{\alpha_1} (z-L)^{1-\alpha_1} - z + (1-\alpha)L.
\]

For $\eta = 1$

\[
\Phi_+^{(1)}(z) = \frac{C_1}{\alpha_1} + \frac{(2 - \alpha_1 - \alpha_2)b}{2i \alpha_2 (z-L)^{\alpha_1}} z^{1-\alpha_1} (z-L)^{\alpha_1} - z + \alpha_1 L,
\]

\[
\Phi_-^{(1)}(z) = \frac{(\alpha_1 + \alpha_2 - 1)a}{2i \alpha_1 (z-L)^{\alpha_1}} [z^{\alpha_1-1} (z-L)^{1-\alpha_1} - 1].
\]

Functions $\zeta^{(0)}_{\pm}(t)$ are boundary jumps of these functions (cf (B.15)). They have different forms depending on the cuts

\[
\zeta^{(0)}_{\pm}(t) = \begin{cases} 
\sin \pi \alpha_1 F^{(0)}_{\pm}(t), & 0 < t < L, \\
\sin \pi (\alpha_1 + \alpha_2) G^{(0)}_{\pm}(t), & t < 0,
\end{cases}
\]

where

\bullet $\eta = 0$

\[
F_+^{(0)}(t) = \frac{(1-\alpha_1 - \alpha_2)b}{t^{\alpha_2} (L-t)^{\alpha_1}},
\]

\[
G_+^{(0)}(t) = -(1-\alpha_1 - \alpha_2)b \frac{(-t)^{-\alpha_1} (L-t)^{\alpha_1} - 1}{(t^{\alpha_2} (L-t)^{\alpha_1})},
\]

\[
F_-^{(0)}(t) = \frac{2i C_0 + (\alpha_1 + \alpha_2)a [t - (1-\alpha)L]}{t^{1-\alpha_2} (L-t)^{1-\alpha_1}},
\]

\[
G_-^{(0)}(t) = \frac{2i C_0 + (\alpha_1 + \alpha_2)a [(-t)^{\alpha_1} (L-t)^{1-\alpha_1} + t - (1-\alpha)L]}{(t^{1-\alpha_2} (L-t)^{1-\alpha_1})}.
\]
\* \( \eta = 1 \)

\[
F_+^{(1)}(t) = -\frac{2iC_1 - (2 - \alpha_1 - \alpha_2)b[t - \alpha_1L]}{t^\alpha_2(L - t)^\alpha_1},
\]

\[
G_+^{(1)}(t) = \frac{2iC_1 - (2 - \alpha_1 - \alpha_2)b[(-t)^{1-\alpha_1}(L - t)^\alpha_1 + t - \alpha_1L]}{(-t)^{1-\alpha_2}(L - t)^{1-\alpha_1}},
\]

\[
F_-^{(1)}(t) = \frac{(\alpha_1 + \alpha_2 - 1)a}{t^{1-\alpha_2}(L - t)^{1-\alpha_1}},
\]

\[
G_-^{(1)}(t) = \frac{(\alpha_1 + \alpha_2 - 1)a[(-t)^{\alpha_1-1}(L - t)^{1-\alpha_1} - 1]}{(-t)^{1-\alpha_2}(L - t)^{1-\alpha_1}}.
\]

Functions \( \mu(t) \) and \( \nu'(t) \) are

\[
\mu^{(\eta)}(t) = \frac{i}{2} \left( \zeta^{(\eta)}(t) - \zeta^{(\eta)}(t) \right), \quad \nu'^{(\eta)}(t) = \frac{1}{2} \left( \zeta^{(\eta)}(t) + \zeta^{(\eta)}(t) \right). \quad (B.56)
\]

For all values of \( \eta \) functions \( \Phi_\pm(z) \) have singularities at \( z = 0 \) and \( z = 1 \), and decay as \( z^{-\gamma} \) with \( \gamma > 1 \) at infinity.

Therefore (which can also be confirmed by direct calculations),

\[
\int_{-\infty}^{L} \zeta^{(\eta)}(t) dt = 0, \quad (B.57)
\]

and the only condition to fulfill is (as \( F(0) = 0 \))

\[
\int_{-\infty}^{L} \ln \left( \frac{|f|}{L} \right) \mu(t) dt = 0. \quad (B.58)
\]

The necessary integrals can be calculated by differentiation of the Euler integral as follows:

\[
\int_{-\infty}^{L} \zeta^{(\eta)}(t) \ln \left( \frac{|f|}{L} \right) dt = -ZL^{\alpha_1} + \alpha_2 - 1 \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)} R_0,
\]

\[
\int_{-\infty}^{L} \zeta^{(\eta)}(t) \ln \left( \frac{|f|}{L} \right) dt = -ZL^{1-\alpha_1-\alpha_2}b \frac{\Gamma(1 - \alpha_1)\Gamma(1 - \alpha_2)}{\Gamma(2 - \alpha_1 - \alpha_2)},
\]

\[
\int_{-\infty}^{L} \zeta^{(\eta)}(t) \ln \left( \frac{|f|}{L} \right) dt = -ZL^{\alpha_1+\alpha_2-1}a \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2 - 1)},
\]

\[
\int_{-\infty}^{L} \zeta^{(\eta)}(t) \ln \left( \frac{|f|}{L} \right) dt = ZL^{1-\alpha_1-\alpha_2} \frac{\Gamma(1 - \alpha_1)\Gamma(1 - \alpha_2)}{\Gamma(2 - \alpha_1 - \alpha_2)} R_1.
\]

Here

\[
Z = \frac{\pi \cos(\pi(\alpha_1 + \alpha_2))\sin \pi \alpha_1}{\sin(\pi(\alpha_1 + \alpha_2))}, \quad (B.59)
\]

and

\[
R_0 = 2iC_0 + \alpha_1(\alpha_1 - \alpha_2 - 1)a, \quad R_1 = 2iC_1 - (1 - \alpha_1)(1 - \alpha_1 - \alpha_2)b. \quad (B.60)
\]

Condition \( (B.58) \) fixes values of constants \( C_\eta \)

\[
2iC_0 = \alpha_1(1 - \alpha_1 - \alpha_2)aL + RbL^{2(1-\alpha_1-\alpha_2)}, \quad (B.61)
\]

\[
2iC_1 = -(1 - \alpha_1)(\alpha_1 + \alpha_2 - 1)bL - TaL^{2(\alpha_1+\alpha_2-1)}, \quad (B.62)
\]
where
\[
R \equiv R(\alpha_1, \alpha_2) = \frac{\Gamma(1 - \alpha_1)\Gamma(1 - \alpha_2)\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(1 - \alpha_1 - \alpha_2)}, \quad T = R(1 - \alpha_1, 1 - \alpha_2).
\]

(B.63)

Calculating the derivatives one obtains the following relations
\[
\frac{1}{2}(\partial_x + i\partial_y)\Psi(x, y) = \frac{1}{2}(\partial_x + i\partial_y)\Psi(x, y) - 2i\Phi_\nu(z),
\]
\[
\frac{1}{2}(\partial_x - i\partial_y)\Psi(x, y) = \frac{1}{2}(\partial_x - i\partial_y)\Psi(x, y) + 2i\Phi_\nu(z).
\]

From the above formulae it follows that in a small vicinity of the origin
\[
\frac{1}{2}(\partial_x + i\partial_y)\Psi^{(0)}(x, y) = \frac{(1 - \alpha_1 - \alpha_2)b}{\bar{\varphi}^{\alpha_1}(\bar{z} - L)^{\alpha_1}},
\]
\[
\frac{1}{2}(\partial_x - i\partial_y)\Psi^{(0)}(x, y) = \frac{2iC_0 + (\alpha_1 + \alpha_2)a(z - (1 - \alpha_1)L)}{\bar{z}^{1-\alpha_2}(z - L)^{1-\alpha_1}},
\]
\[
\frac{1}{2}(\partial_x + i\partial_y)\Psi^{(1)}(x, y) = \frac{-2iC_1 - (2 - \alpha_1 - \alpha_2)b(z - \alpha_1L)}{\bar{z}^{1-\alpha_1}(\bar{z} - L)^{\alpha_1}},
\]
\[
\frac{1}{2}(\partial_x - i\partial_y)\Psi^{(1)}(x, y) = \frac{(\alpha_1 + \alpha_2 - 1)a}{\bar{z}^{1-\alpha_2}(z - L)^{1-\alpha_1}}.
\]

From these expressions it is possible to calculate functions \(a_j\) and \(b_j\) from definition (17) and equations (B.61) and (B.62)

\[
a_1^{(0)} = \frac{2iC_0 + (\alpha_1 + \alpha_2)a}{aL_0L^{1-\alpha_2}} = aL^{\alpha_2} + \frac{Rb}{\alpha_1}L^{1-2\alpha_1-\alpha_2},
\]
\[
b_1^{(0)} = \frac{(1 - \alpha_1 - \alpha_2)b}{(1 - \alpha_1)L^{\alpha_2}},
\]
\[
a_2^{(0)} = -e^{i\pi\alpha_1}\frac{2iC_0 - (1 - \alpha_1)(\alpha_1 + \alpha_2)}{aL_0L^{1-\alpha_1}} = e^{i\pi\alpha_1}\left[aL^{\alpha_1} - \frac{Rb}{\alpha_2}L^{1-2\alpha_2-\alpha_1}\right],
\]
\[
b_2^{(0)} = e^{i\pi\alpha_1}\frac{(1 - \alpha_1 - \alpha_2)b}{(1 - \alpha_2)L^{\alpha_1}}.
\]

and
\[
a_1^{(1)} = \frac{(\alpha_1 + \alpha_2 - 1)a}{\alpha_1L^{1-\alpha_2}},
\]
\[
b_1^{(1)} = \frac{-2iC_1 + (1 - \alpha_1)(2 - \alpha_1 - \alpha_2)b}{(1 - \alpha_1)L^{\alpha_2}} = bL^{1-\alpha_2} + \frac{Ta}{1 - \alpha_1}L^{2a_1+\alpha_2-2},
\]
\[
a_2^{(1)} = -e^{i\pi\alpha_1}\frac{(\alpha_1 + \alpha_2 - 1)a}{\alpha_2L^{1-\alpha_1}},
\]
\[
b_2^{(1)} = -e^{i\pi\alpha_1}\frac{2iC_1 + (\alpha_1 + \alpha_2 - 1)bL}{(1 - \alpha_2)L^{\alpha_1}}
\]
\[
= -e^{i\pi\alpha_1}\left[bL^{1-\alpha_2} - \frac{Ta}{1 - \alpha_2}L^{2\alpha_1+2\alpha_2-2}\right].
\]

Differentiating these expressions on \(L_j\), considering that derivatives of \(a\) and \(b\) will contain additional smallness and, therefore, can be considered as constants, one finds
\[ h = 0 \]

\[
\begin{pmatrix}
\beta_1 \\
\delta_1
\end{pmatrix}
\xrightarrow{L \to 0}
\begin{pmatrix}
-\alpha_2 L^{\alpha_2 - \alpha_1 - 1} & 0 \\
\frac{1}{\alpha_1 (1 - \alpha_2) L^{\alpha_1 - \alpha_2 - 1}} \alpha_1 & \frac{1 - \alpha_2}{1 - \alpha_1} L^{\alpha_1 - \alpha_2 - 1}
\end{pmatrix} e^{-i \pi \alpha_1}, \quad (B.64)
\]

\[
\begin{pmatrix}
\beta_2 \\
\delta_2
\end{pmatrix}
\xrightarrow{L \to 0}
\begin{pmatrix}
\alpha_1 L^{\alpha_1 - \alpha_2 - 1} & 0 \\
0 & \frac{1}{\alpha_1 (1 - \alpha_1) L^{\alpha_2 - \alpha_1 - 1}} \alpha_2
\end{pmatrix} e^{i \pi \alpha_1}, \quad (B.65)
\]

\[ \eta = 1 \]

\[
\begin{pmatrix}
\beta_1 \\
\delta_1
\end{pmatrix}
\xrightarrow{L \to 0}
\begin{pmatrix}
\frac{\alpha_2 (1 - \alpha_2)}{\alpha_1} L^{\alpha_2 - \alpha_1 - 1} & 0 \\
\frac{\alpha_2}{1 - \alpha_1} T L^{\alpha_1 + \alpha_2 - 2} & (1 - \alpha_2) L^{\alpha_1 - \alpha_2 - 1}
\end{pmatrix} e^{-i \pi \alpha_1}, \quad (B.66)
\]

\[
\begin{pmatrix}
\beta_2 \\
\delta_2
\end{pmatrix}
\xrightarrow{L \to 0}
\begin{pmatrix}
\frac{\alpha_1 (1 - \alpha_1)}{\alpha_2} L^{\alpha_1 - \alpha_2 - 1} & 0 \\
\frac{\alpha_1}{1 - \alpha_2} T L^{\alpha_1 + \alpha_2 - 2} & (1 - \alpha_1) L^{\alpha_2 - \alpha_1 - 1}
\end{pmatrix} e^{i \pi \alpha_1}, \quad (B.67)
\]

In all cases relation (77) is fulfilled with

\[ \rho = \frac{\alpha_1 (1 - \alpha_1)}{\alpha_2 (1 - \alpha_2)} e^{2 \pi i \alpha_1}, \quad (B.68) \]

Coefficients \( \delta_1 \) and \( \delta_2 \) for \( \eta = 0 \), and \( \epsilon_1 \) and \( \epsilon_2 \) for \( \eta = 1 \), are zero in the leading order. Instead of explicit calculation for the next terms, it is convenient to use equation (79) by considering that

\[
(\partial_x + i \partial_y) a = \eta_1 b, \quad \eta_1 = -k^2 \sqrt{2} e^{-i \pi \beta} \frac{\Gamma(2 - \beta)}{\Gamma(1 + \beta)} (\partial_x - i \partial_y) b = r_2 a, \quad r_2 = \frac{k^2}{\eta_1}, \quad (B.69)
\]

Comparing coefficients in front of \( a \) and \( b \) one obtains the following limiting values

\[ \eta = 0 \]

\[ \delta_1 = \frac{\alpha_2 (1 - \alpha_1 - \alpha_2)}{2 (1 - \alpha_1) (\alpha_1 + \alpha_2)} r_2 e^{-i \pi \alpha_1} L^{-\alpha_1 - \alpha_2}, \]

\[ \delta_2 = \frac{\alpha_1 (1 - \alpha_1 - \alpha_2)}{2 (1 - \alpha_2) (\alpha_1 + \alpha_2)} r_2 e^{i \pi \alpha_1} L^{-\alpha_1 - \alpha_2}, \quad (B.70) \]

and
\[ g_2 = \frac{\alpha_1(1 - \alpha_1 - \alpha_2)}{2(1 - \alpha_1)(\alpha_1 + \alpha_2)} r_2 L^{-2\alpha}, \]
\[ g_1 = -\frac{2(1 - \alpha_1)}{\alpha_1} RL^{-2\alpha} + \frac{1 - \alpha_1}{1 - \alpha_1 - \alpha_2} r_1 L^{2\alpha}, \]
\[ f_2 = \frac{\alpha_2(1 - \alpha_1 - \alpha_2)}{2(1 - \alpha_2)(\alpha_1 + \alpha_2)} r_2 L^{-2\alpha}, \]
\[ f_1 = -\frac{2(1 - \alpha_2)}{\alpha_2} RL^{-2\alpha} + \frac{1 - \alpha_2}{1 - \alpha_1 - \alpha_2} r_1 L^{2\alpha}. \]

\[
\eta = 1
\]
\[ e_1 = \frac{(1 - \alpha_2)(\alpha_1 + \alpha_2 - 1)}{2\alpha_1(2 - \alpha_1 - \alpha_2)} r_1 e^{-i\alpha_1} L^{\alpha_1 + \alpha_2 - 2}, \]
\[ e_2 = \frac{(1 - \alpha_1)(\alpha_1 + \alpha_2 - 1)}{2\alpha_2(2 - \alpha_1 - \alpha_2)} r_2 e^{i\alpha_1} L^{\alpha_1 + \alpha_2 - 2}, \]  
\[ \text{(B.71)} \]

and
\[ g_2 = -\frac{2\alpha_1}{1 - \alpha_1} T L^{2\alpha_1 - 2} + \frac{\alpha_1}{\alpha_1 + \alpha_2 - 1} r_1 L^{2\alpha_1 - 2}, \]
\[ g_1 = \frac{(1 - \alpha_1)(\alpha_1 + \alpha_2 - 1)}{\alpha_1(2 - \alpha_1 - \alpha_2)} r_1 L^{2\alpha_1 - 2}, \]
\[ f_2 = -\frac{2\alpha_2}{1 - \alpha_2} T L^{2\alpha_1 - 2} + \frac{\alpha_2}{\alpha_1 + \alpha_2 - 1} r_2 L^{2\alpha_1 - 2}, \]
\[ f_1 = \frac{(1 - \alpha_2)(\alpha_1 + \alpha_2 - 1)}{\alpha_2(2 - \alpha_1 - \alpha_2)} r_2 L^{2\alpha_1 - 2}. \]

In these calculations it has been considered that \( (\partial_\mu - i\partial_\nu) a \) and \( (\partial_\mu + i\partial_\nu) b \) correspond to higher order terms in expansion (B.41), and therefore they were put to zero in the leading order.

The small-distance behaviour of \( y = \delta_1 \epsilon_2 \) follows directly from the above expressions. For \( \eta = 0 \)
\[ y \xrightarrow{L \to 0} \left( \frac{k}{2} \right)^2 \left( \frac{kL}{2} \right)^{-2(\alpha_1 + \alpha_2)} \frac{\sin^2 \pi(\alpha_1 + \alpha_2)}{\sin \pi \alpha_1 \sin \pi \alpha_2} \frac{\Gamma^4(\alpha_1 + \alpha_2)}{\Gamma^2(\alpha_1)\Gamma^2(\alpha_2)} e^{i\pi(\alpha_1 + \alpha_2)}. \]  
\[ \text{(B.72)} \]

For \( \eta = 1 \) the limiting behaviour of \( y \) is given by the same formula but with substitution \( \alpha_1 \to 1 - \alpha_1, \alpha_2 \to 1 - \alpha_2 \).

**B.3. Two vortices at large distances**

Knowledge of the one-vortex solution (see 4) permits one to calculate the two-vortex case within the perturbation series when the distance between vortices is large. Consider functions \( A_1(\vec{x}) \) and \( B_1(\vec{x}) \). In the lowest order, when the second vortex is absent, their asymptotics are determined by equations (113) and (34). The existence of the second vortex even at a very large distance modifies these expressions to the following:
Here $\theta_1$ is the polar angle around the first vortex, $\vec{x}_1 = (r \cos \theta_1, r \sin \theta_1)$, and $\theta_2$ indicates the polar angle with its centre in the second vortex, $\vec{x}_2 = (R \cos \theta_2, R \sin \theta_2)$. It is assumed that vortices and cuts are such as indicated in figure 1. Functions $f_j^{(0)}(\theta)$ and $g_j^{(0)}(\theta)$ with $j = 1, 2$ are obtained from equation (34) by extracting the factor $e^{i\varrho_{\theta}}$

$$f_j^{(0)}(\theta) = -ic_3(\alpha_j)e^{-i\varrho_{\theta}+i\varpi_{\theta}/2}, \quad g_j^{(0)}(\theta) = c_4(\alpha_j)e^{-i\varrho_{\theta}/2}. \quad (B.74)$$

Notice that at the position of the first vortex $\theta_1 = 0$, but at the second vortex $\theta_1 = \pi$. This choice of cuts has as a consequence such that functions $A_2(\vec{x}_2)$ and $B_2(\vec{x}_2)$, with $\vec{x}_2$ centred at the second vortex, are given by slightly different expressions

$$A_2(\vec{x}_2)_{\mid \vec{x}_1 \rightarrow \infty} \rightarrow \sqrt{\frac{2}{\pi ikR}} e^{i\varrho_{\theta}} f_2^{(0)}(\theta_2) e^{i\varrho_{\theta_1}(-\varpi_{\theta})+i\varpi_{\theta_2} } ,$$

$$B_2(\vec{x}_2)_{\mid \vec{x}_1 \rightarrow \infty} \rightarrow \sqrt{\frac{2}{\pi ikR}} e^{i\varrho_{\theta}} g_2^{(0)}(\theta_2) e^{i\varrho_{\theta_1}(-\varpi_{\theta})+i\varpi_{\theta_2} } . \quad (B.75)$$

The first order corrections correspond to re-scattering of these fields on the second vortex. When coordinates are calculated from the second vortex ($\vec{x}_2 = \vec{x}_1 - \vec{L}$) and $L \rightarrow \infty$, $A_1(\vec{r})$ has the following asymptotics

$$\lim_{L \rightarrow \infty} A_1(\vec{x}_2) = D(L) f_1^{(0)}(\pi) e^{i\varrho_{\theta_1}+i\varpi_{\theta_2} } e^{-i\varrho_{\theta}} , \quad D(L) = \frac{2}{\sqrt{\pi ikL}} e^{i\varrho_{\theta}} . \quad (B.76)$$

According to equation (110) the scattering function for this incident field is $F_2(\theta)$ in equation (59) without factor $e^{i\varpi_{\theta}}$, which is included in the above definition,

$$F_2(\theta) = \frac{\sin \pi \varrho_{\theta}}{2\cos(\varpi_{\theta}/2)} e^{-i\varrho_{\theta}/2} . \quad (B.77)$$

In (110) radius, $R$, is counted from the second vortex. To shift it to the first vortex requires $R \approx r + L \cos \vartheta$. Therefore, the full contribution to function $F_1(\theta, L)$ in two lowest orders is (when $r \rightarrow \infty$, $\theta_1 = \theta_2 = \theta$)

$$F_1(\theta, L) = (f_1^{(0)}(\theta) + D(L) f_1^{(0)}(\pi)) F_2(\theta) e^{i\varrho_{\theta} \cos \vartheta} e^{i(\alpha_1 + \alpha_2)\vartheta} . \quad (B.78)$$

In a similar manner

$$G_1(\theta, L) = (g_1^{(0)}(\theta) + D(L) g_1^{(0)}(\pi)) F_2(\theta) e^{i\varrho_{\theta} \cos \vartheta} e^{i(\alpha_1 + \alpha_2)\vartheta} . \quad (B.79)$$

Derivatives of functions $F_1(\theta, L)$ and $G_1(\theta, L)$ over $L$, according to equation (121), are (in the lowest order) linear combinations of $F_2^{(0)}(\theta) = f_2^{(0)}(\theta) e^{i(\alpha_1 + \alpha_2)\vartheta}$ and $G_2^{(0)}(\theta) = g_2^{(0)}(\theta) e^{i(\alpha_1 + \alpha_2)\vartheta}$.

Following calculations, one finds that

$$\left( \begin{array}{c} \beta_2 \\ \delta_2 \\ \epsilon_2 \\ \zeta_2 \\ L \rightarrow \infty \end{array} \right) \rightarrow \frac{k \sin \pi \varrho_{\theta}}{2} \cdot \begin{array}{cccc} c_1(\alpha_1) e^{i\varrho_{\varpi} \alpha_1} & c_2(\alpha_1) e^{i\varrho_{\varpi} \alpha_2} & c_1(\alpha_2) e^{i\varrho_{\varpi} \alpha_1} & c_2(\alpha_2) e^{i\varrho_{\varpi} \alpha_2} \\ \frac{c_1(\alpha_1)}{c_2(\alpha_2)} & \frac{c_2(\alpha_1)}{c_1(\alpha_2)} & \frac{c_1(\alpha_1)}{c_2(\alpha_2)} & \frac{c_2(\alpha_1)}{c_1(\alpha_2)} \\ \frac{c_2(\alpha_1)}{c_1(\alpha_2)} & \frac{c_1(\alpha_1)}{c_2(\alpha_2)} & \frac{c_2(\alpha_1)}{c_1(\alpha_2)} & \frac{c_1(\alpha_1)}{c_2(\alpha_2)} \end{array} \right) \cdot D(L) e^{i\varrho_{\varpi} \alpha_1} . \quad (B.80)$$
For the second vortex

\[
F_2(\theta, L) = (f_2^{(0)}(\theta) + D(L)f_2^{(0)}(0))G_1(\theta)e^{-iL\cos(\theta)}e^{i(\alpha_1 + \alpha_2)\theta - i\pi\alpha_1},
\]
\[
G_2(\theta, L) = (g_2^{(0)}(\theta) + D(L)g_2^{(0)}(0))G_1(\theta)e^{-iL\cos(\theta)}e^{i(\alpha_1 + \alpha_2)\theta - i\pi\alpha_1}, \quad (B.81)
\]

Here \(G_1(\theta)\) indicates the scattering amplitude \(F_1(\theta)\) in equation (58) with factor \(e^{i\pi(\theta - i\pi\alpha_1)}\) removed

\[
F_1(\theta) \equiv G_1(\theta)e^{i\pi\theta}, \quad G_1(\theta) = \frac{\sin\pi\alpha_1}{2\sin(\theta/2)}e^{-i\theta/2}. \quad (B.82)
\]

Differentiating them by \(L\) and using equation (121) one finds

\[
\begin{pmatrix}
\beta_1 \\
\beta_1
\end{pmatrix} \longrightarrow \frac{k\sin\pi\alpha_1}{2} L \rightarrow \infty \begin{pmatrix}
\frac{-c_2(\alpha_2)}{c_2(\alpha_1)}e^{-i\pi(\alpha_1 - \alpha_2)/2} - \frac{ic_2(\alpha_2)}{c_2(\alpha_1)}e^{i\pi(\alpha_1 + \alpha_2)/2} \\
\frac{-ic_2(\alpha_2)}{c_2(\alpha_1)}e^{-i\pi(\alpha_1 + \alpha_2)/2} + \frac{c_2(\alpha_2)}{c_2(\alpha_1)}e^{i\pi(\alpha_1 - \alpha_2)/2}
\end{pmatrix} D(L)e^{-i\pi\alpha_1}. \quad (B.83)
\]

These asymptotic values obey equation (77) with

\[
\rho = \frac{\alpha_1(1 - \alpha_1)}{\alpha_2(1 - \alpha_2)}e^{2i\pi\alpha_1} \quad (B.84)
\]
in agreement with (44).

The asymptotic behaviours of variables (88) are

\[
y \equiv \delta_2 \epsilon_1 \xrightarrow{L \rightarrow \infty} - \frac{1}{4} k^2 \sin\pi\alpha_1 \sin \pi\alpha_2 D^2(L),
\]
\[
z \equiv \beta_1 \beta_2 \xrightarrow{L \rightarrow \infty} - \frac{1}{4} k^2 \sin\pi\alpha_1 \sin \pi\alpha_2 D^2(L). \quad (B.85)
\]

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