Symplectic quantization of multi-field Generalized Proca electrodynamics

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Abstract

We explicitly carry out the symplectic quantization of a family of multi-field Generalized Proca (GP) electrodynamics theories. In the process, we provide an independent derivation of the so-called secondary constraint enforcing relations — consistency conditions that significantly restrict the allowed interactions in multi-field settings already at the classical level. Additionally, we unveil the existence of quantum consistency conditions, which apply in both single- and multi-field GP scenarios. Our newly found conditions imply that not all classically well-defined (multi-)GP theories are amenable to quantization. The extension of our results to the most general multi-GP class is conceptually straightforward, albeit algebraically cumbersome.

1 Introduction

Quantum electrodynamics (QED) is the commonly employed relativistic quantum field theory of the electromagnetic force. Even so, generalizations of QED are relevant in many branches of physics, including condensed matter, cosmology, optics, particle physics and string theory, e.g. \cite{1–5}. Here, we derive the partition function of some recently proposed extensions of QED, which comprise an arbitrary number of massive photons with derivative (self-)interactions. The renowned quantization procedure put forward by Dirac, repeatedly refined and extended since its inception, would be the standard approach to achieve this goal. However, owing to the noteworthy difficulty of its implementation in our targeted class of theories, we resort to the distinct yet physically equivalent symplectic quantization methodology instead.

It was almost 160 years ago that Maxwell laid the foundations of classical electromagnetism \cite{6}. Viewed as a field theory, this describes an abelian massless vector field and its linear interactions with sources. The quantization of Maxwell’s theory took several decades, earned some of its key developers a Nobel Prize in 1965 and yielded what arguably remains the most successful theory to date: QED. For a historical review, we refer the reader to \cite{7}.

As is well-known and was nicely recapped in \cite{7}, early attempts at quantizing electromagnetism met with a divergent self-energy for any static point particle, such as the electron, placed in an electromagnetic
Table 1: Classification of single-field electromagnetic theories, whose Lagrangian density is manifestly first-order. Both NLE and GP stand for populous classes of such theories. In this work, we shall consider the non-trivial multi-field extension of the GP class, constructed in [20, 21].

|             | Linear                      | Non-linear                  |
|-------------|-----------------------------|-----------------------------|
| Massless    | Maxwell                     | Non-linear electrodynamics (NLE) |
| Massive     | Proca                       | Generalized Proca (GP)      |

field. In order to overcome this problem, two fundamentally different modifications to Maxwell’s theory were introduced. In 1934, Born and Infeld proposed a certain non-linear extension, which is gauge-invariant and contains a single free parameter [8]. On the other hand, in the period of 1936-1938, Proca constructed a massive version of Maxwell’s electrodynamics [9, 10], which explicitly breaks the gauge symmetry. The Born-Infeld (BI) model is a concrete realization of what ultimately became a large class of theories [11–13], collectively known as non-linear electrodynamics (NLE). Proca electrodynamics rapidly became and remains cornerstone to optics in its original form [14–17].

It is only comparatively recently, in 2014, that classical, non-linear extensions of Proca’s massive electromagnetism, containing derivative self-interactions of the vector field, were put forward [18, 19]. These conform a vast class of theories, usually referred to as Generalized Proca (GP) or Vector Galileon. The axiomatization and non-trivial extension to multiple fields of GP electrodynamics was carried out in [20, 21]. It is this class of theories, (multi-)GP electrodynamics, whose quantization we shall focus on. For the ease of the reader, we note that GP can be understood as the massive counterpart to the more familiar class of NLE theories\(^1\), see table 1. We highlight the relevance of the multi-field settings: they allow for non-abelian augmentations of GP, upon imposing the desired group structure in the field space.

All the theories mentioned so far are singular or constrained. Further examples are non-abelian gauge field theories, gravitational theories and supersymmetric theories. The systematic study of such systems was initiated by Dirac in 1950 [27], whose work was promptly and abundantly followed upon [28–32], including recent advancements [33–37]. In particular, the path integral formulation of Dirac’s canonical quantization procedure has been known for over four decades [38, 39].

The formalism instituted by Dirac is ubiquitous but not unique. In the present manuscript, we will employ the distinct quantization scheme introduced by Faddeev and Jackiw in 1988 [40]. This method is conceptually simpler and, for some theories, it is algebraically easier to implement as well. The main reason for the conceptual simplicity lies in the fact that Faddeev and Jackiw’s approach does not require to classify the constraints present in the theory into first and second class\(^2\). The algebraic ease is particularly prominent when considering systems with only second class constraints, as is the case of (multi-)GP electrodynamics. Last but not least, we note that Dirac’s method is a Hamiltonian based one, while

\(^1\)Our lightning review of extensions of classical electromagnetism is limited to theories described by first-order Lagrangian densities. Higher-order generalizations are of course possible. On the massless side, the most renowned example is that of Podolsky electrodynamics [22, 23]. On the massive side, there exists a single proposal so far: Proca-Nuevo [24, 25], which can also be extended through some GP interaction terms [26].

\(^2\)As a reminder, first/second class constraints are those which do/don’t have a weakly vanishing Poisson bracket with all constraints.
Faddeev and Jackiw’s is Lagrangian based. This makes the Faddeev-Jackiw prescription particularly befitting for dealing with (multi-)GP theories, which have been formulated and are almost exclusively employed in their Lagrangian formulation.

As with Dirac’s original work [27], Faddeev and Jackiw’s proposal [40] has been extensively followed upon [41–46]. Of particular interest for this work is the path integral formulation of their approach, established in [47,48]. Here, we refer as symplectic quantization to the quantization procedure derived from the cumulative consideration of [40,42,43,47,48], nicely summarized in section 2 of [48]. The outcome of this method is the central object of any quantum field theory: the partition function.

The paper is organized as follows. We begin with a technical review of multi-GP in section 2.1. For clarity, we focus on a particular subset of multi-GP in section 2.2 and perform its symplectic quantization in detail in sections 2.3-2.6. We thus identify two distinct sets of consistency conditions:

1. The already known conditions [20,21], which severely restrict classical, multi-field settings.
2. New conditions, which apply in the quantum realm and affect both single- and multi-field settings.

We exemplify the resulting quantization procedure in section 2.7. Section 3 is devoted to the elucidation of the novel quantum consistency conditions. We conclude with section 4, summarizing the results and pointing out possibilities for future work.

Conventions.
We work on a $d$-dimensional Minkowski spacetime manifold $\mathcal{M}$, with $d \geq 2$ and the mostly positive metric signature. Spacetime indices are denoted by the Greek letters ($\mu, \nu, \rho \ldots$) and raised/lowered with the metric $\eta_{\mu\nu} = \text{diag}(-1,1,1,\ldots,1)$ and its inverse $\eta^{\mu\nu}$. Space indices are denoted by the Latin letters ($i,j,k\ldots$) and are trivially raised/lowered. The alphabets ($\alpha, \beta, \ldots$) label different vector fields. These vector field labels are trivially raised/lowered. We employ the standard short-hand notations $\partial_{\mu} f := \partial f / \partial x^{\mu}$ and $\partial_{i} f := \partial f / \partial x^{i}$, where $x^{\mu}$ and $x^{i}$ are spacetime and space local coordinates in $\mathcal{M}$, respectively. The dot stands for derivation with respect to time: $\dot{f} := \partial_{0} f$ and $\ddot{f} := \partial_{0}^{2} f$. Here, $f$ is any local function $f : \mathcal{M} \rightarrow \mathbb{R}$. Einstein summation convention applies for all repeated indices and labels throughout the text.

2 Symplectic quantization

In this section, we perform the detailed symplectic quantization of a family of electrodynamics theories, all of which describe the dynamics of an arbitrary number $N \in \mathbb{N}$ of GP fields coupled through derivative (self-)interactions. By definition, the theories here considered describe multi-field, generalized massive electrodynamics, whose Lagrangian is manifestly first-order.

2.1 Review of multi-GP electrodynamics

In order to set the notation and contextualize the results obtained in this work, we start with a brief review of our previous work on multi-GP electrodynamics [20,21]. Let $N$ be the number of GP fields $A^{\alpha} = A^{\alpha}_{\mu} dx^{\mu}$, with $\alpha = 1, 2, \ldots N$. The most general first-order Lagrangian density, encoding the dynamics of these GP fields can be written as

$$L_{\text{gen}} = L_{\text{kin}} + L_{\text{int}},$$

(2.1)
where the kinetic piece is canonically normalized
\[ \mathcal{L}_{\text{kin}} = -\frac{1}{4} A_\mu^\alpha A^\mu_\alpha, \quad A_\mu^\alpha := \partial_\mu A^\alpha - \partial_\nu A^\alpha_\mu, \] (2.2)
and the (self-)interaction piece is given by
\[ \mathcal{L}_{\text{int}} = \mathcal{L}_{(0)} + \sum_{n=1}^{\infty} \mathcal{L}_{(n)}. \] (2.3)

Here, \( \mathcal{L}_{(0)} \) is an arbitrary real smooth function of the GP fields and their field strengths,
\[ \mathcal{L}_{(0)} = \mathcal{L}_{(0)}(A_\mu^\alpha, A^{\alpha}_{\mu\nu}), \] (2.4)
while the factors \( \mathcal{L}_{(n)} \) are of the general form
\[ \mathcal{L}_{(n)} = T_{\alpha_1...\alpha_n} A^{\alpha_1}_{\nu_1} \cdots A^{\alpha_n}_{\nu_n}, \] (2.5)
where the above \( T \) objects are real and smooth and can depend on the GP fields but not on their derivatives:
\[ T_{\alpha_1...\alpha_n} = T_{\alpha_1...\alpha_n}(A^{\alpha}_{\mu}), \quad T_{\alpha_1...\alpha_n} \neq T_{\alpha_1...\alpha_n}(\partial_{\mu} A^{\alpha}_{\nu}). \] (2.6)

Therefore, \( n \) counts the number of derivative terms of the GP fields present in \( \mathcal{L}_{(n \geq 1)} \). Notice that the Lagrangian \( \mathcal{L}_{\text{gen}} \) is manifestly first-order. Namely, it explicitly depends on the GP fields and (powers of) their first derivatives only. No second- or higher-order derivatives appear. This feature guarantees that the equations of motion are second-order at most.

In order for the above Lagrangian \( \mathcal{L}_{\text{gen}} \) to be mathematically well-defined at the classical level, it must fulfill two necessary and sufficient sets of constraints: (2.7) and (2.9) below. The initial GP works \([18,19]\) identified (2.7). The mathematical procedure was completed in \([20,21]\), with an outcome of (2.9). In more detail, (2.7) enforces the existence of a second class constraint for every GP field considered. Such constraints are preserved under time evolution iff (2.9) is fulfilled, which ensures the existence of another second class constraint per GP field. The trivialization of (2.9) for a single GP field implies the automatic existence of the latter second class constraint in this case. Contrastively, multi-field (and therefore non-abelian) settings are severely restricted by (2.9).

The first set of constraints has been referred to as primary constraint enforcing relations and is given by
\[ \frac{\partial^2 \mathcal{L}_{\text{gen}}}{\partial A_\mu^\alpha \partial A^\mu_\alpha} \equiv 0. \] (2.7)

This has two drastic consequences on \( \mathcal{L}_{\text{gen}} \). On the one hand, it truncates the sum over \( n \) in (2.3) at \( n = d \), so that the interaction piece reduces to
\[ \mathcal{L}_{\text{int}} = \mathcal{L}_{(0)} + \sum_{n=1}^{d} \mathcal{L}_{(n)}. \] (2.8)

On the other hand, it forces a certain form on the \( T \) objects in \( \mathcal{L}_{(n \geq 2)} \), albeit without fully fixing them. The interested reader can consult the form of such \( T \)’s, for the particular case when \( d = 4 \), in equations (21)-(23) of [20].
The second set of constraints, the so-called secondary constraint enforcing relations, is

\[
\frac{\partial^2 \mathcal{L}_{\text{gen}}}{\partial A_0^\alpha \partial A_0^\beta} - \frac{\partial^2 \mathcal{L}_{\text{gen}}}{\partial A_0^\beta \partial A_0^\alpha} = 0.
\]

(2.9)

The above further restricts the form of the \( T \) objects in all \( \mathcal{L}_{(n \geq 1)} \), although it still does not completely determine them. Owing to the very significant complexity of both \( \mathcal{L}_{\text{gen}} \) and (2.9), the latter has not yet been exhaustively implemented, even in the particular \( d = 4 \) case. Namely, to date there is no complete list of \( T \)'s that simultaneously satisfy (2.7) and (2.9). Therefore, for the time being, (2.9) is to be viewed as an essential classical consistency condition that must be fulfilled in any multi-GP electrodynamics one may wish to consider. Particular examples of \( T \)'s in \( \mathcal{L}_{(1)} \) and \( \mathcal{L}_{(2)} \) that satisfy both (2.7) and (2.9) have been proposed in \( d = 4 \) [20,21].

2.2 The targeted multi-GP electrodynamics

In the present work, we will restrict computational attention to the following subset of interactions within the above described \( \mathcal{L}_{\text{gen}} \) massive electrodynamics theory:

\[
\mathcal{L}(0) = -\frac{1}{4} m^2 A_\mu^\alpha A_\mu^\alpha + c A_\mu^\alpha A_\mu^\beta A_\nu^\alpha A_\nu^\beta, \quad \mathcal{L}(n \geq 2) = 0,
\]

(2.10)

with \( m \in \mathbb{R}^+ \) the (hard) mass of the GP fields (chosen to be the same for all GP fields for simplicity) and \( c \in \mathbb{R} \) a dimensionless constant. We will consider all the terms in \( \mathcal{L}_{(1)} \). Specifying a certain \( \mathcal{L}(0) \) is necessary in order to explicitly (as opposed to formally) carry out the symplectic quantization procedure. We here choose to consider the standard mass term for the GP fields, originally proposed in [9,10], as well as the quartic interactions among the GP fields. The latter are the simplest (self-)interactions for the massive vector fields, yet they are interesting in their own right. Remarkably, they have been shown to admit a non-Wilsonian ultraviolet completion [49]. They lead to time-dependent solitonic solutions [50]. Recently, such terms have attracted attention in the context of Proca Stars as well [51,52]. Here, we introduced the constant \( c \) to straightforwardly keep track of subsequent contributions stemming from these quartic interactions. We will comment on the non-trivialities involved in the extensions of (2.10) with \( \mathcal{L}(n \geq 2) \neq 0 \) shortly. At last, we will include external sources \( J_\mu^\alpha \). All in all, we shall consider the particular multi-GP electrodynamics theories encoded in

\[
\mathcal{L}_{\text{par}} = -\frac{1}{4} A_\mu^\alpha A_\mu^{\alpha\nu} - \frac{1}{2} m^2 A_\mu^\alpha A_\mu^\alpha + c A_\mu^\alpha A_\mu^\beta A_\nu^\alpha A_\nu^\beta + \mathcal{T}_\alpha^{\mu\nu} \partial_\mu A_\nu^\alpha + A_\mu^\alpha J_\mu^\alpha,
\]

(2.11)

where the objects \( \mathcal{T}_\alpha^{\mu\nu} \) are required to satisfy the classical consistency condition (2.9), with \( \mathcal{L}_{\text{gen}} \) replaced by \( \mathcal{L}_{\text{par}} \).

Here, the \( A_\mu^\alpha \)'s are the generalized coordinates (that is, the a priori independent degrees of freedom in terms of which the electrodynamics theories of our interest are described):

\[
Q = \{ A_\mu^\alpha \}.
\]

(2.12)

The generalized coordinates span the configuration space of the theories, which in this case is \( dN \)-dimensional. The time derivatives of the generalized coordinates are the generalized velocities:

\[
\dot{Q} = \{ \dot{A}_\mu^\alpha \}.
\]

(2.13)
Upon a space-time decomposition, (2.11) becomes

\[
L_{\text{par}} = \frac{1}{2} \dot{A}_i^0 A_i^0 + \dot{A}_i^0 \partial^j A_0^0 - \frac{1}{2} (\partial^i A_0^0) \partial_i A_0^0 - \frac{1}{4} A_0^0 A_0^0 A_j^j \\
- \frac{1}{4} m^2 (A_0^0 A_0^0 + A_i^0 A_i^0) + c \left( A_0^0 A_0^0 A_0^0 + 2 A_0^0 A_0^0 A_i^0 + A_i^0 A_i^0 A_j^j \right) \\
+ T_0^0 A_0^0 + T_0^i \dot{A}_i^0 + T_0^0 \partial_0 A_0^0 + T_0^i \partial_i A_0^0 + A_0^0 J_0^0 + A_i^0 j_i^j, 
\]

(2.14)

where, for the convenience of the reader, we have placed the terms coming from \( L_{\text{kin}} \), \( L^{(0)} \) and \( L^{(1)} \) (plus the coupling to the external sources) in the first, second and third lines, respectively. The classical consistency condition for the above explicitly reads

\[
\partial_\beta T_0^0 - \partial_0 T_0^\beta = 0, 
\]

(2.15)

where we have introduced the short-hand

\[
\partial_\alpha^\mu := \partial \partial A_\alpha^\mu. 
\]

(2.16)

2.3 Input for the iterative procedure

The symplectic quantization method can only be employed on Lagrangian densities which are linear in the generalized velocities. Namely, Lagrangian densities of the form

\[
L = \theta \cdot \dot{Q} + \tilde{L}, 
\]

(2.17)

where \( \theta \) and \( \tilde{L} \) are functions of the generalized coordinates \( Q \) but not of the generalized velocities \( \dot{Q} \). \( \theta \) is known as the canonical one-form. Upon termination of the symplectic quantization iterative procedure, \( \tilde{L} \) is minus the Hamiltonian density.

Clearly, (2.14) is not of the above form. Indeed, \( L_{\text{par}} \) contains quadratic terms in the generalized velocities. These stem from \( L_{\text{kin}} \). In order to bring (2.14) to the desired form (2.17), we will extend the configuration space of our theory, by declaring the canonical momenta \( p_\alpha^\mu \) (with respect to \( A_\alpha^\mu \)) generalized coordinates as well:

\[
Q = \{ A_\alpha^\mu, p_\alpha^\mu \}. 
\]

(2.18)

At this point, we thus consider a configuration space that is \( 2dN \)-dimensional, with the canonical momenta given by

\[
p_0^\alpha := \frac{\partial L_{\text{par}}}{\partial \dot{A}_0^0} = T_0^0, \quad p_i^\alpha := \frac{\partial L_{\text{par}}}{\partial \dot{A}_i^\alpha} = \dot{A}_i^\alpha + \partial_i A_0^0 + T_0^i. 
\]

(2.19)

It is of outmost importance to make the following two observations. First, the canonical momenta \( p_\alpha^i \) depend on (some of) the generalized velocities \( \dot{Q} \), while the canonical momenta \( p_\alpha^0 \) do not. The fact that \( p_0^0 \neq p_0^0 (\dot{Q}) \) is a direct consequence of the primary constraint enforcing relations (2.7) and it implies that we must view

\[
\varphi_\alpha := p_0^\alpha - T_0^0 = 0 
\]

(2.20)
as a set of $N$ number of (functionally independent) constraints that must be appropriately accounted for in our considered theories. This can be readily done via Lagrange multipliers $\lambda^\alpha$, which we must regard as further generalized coordinates:

$$Q = \{A^\alpha_\mu, p^\alpha_\mu, \lambda^\alpha\}. \quad (2.21)$$

We thus settle for a $(2d+1)N$-dimensional configuration space associated to (2.14) with views to performing the symplectic quantization of the theories.

Second, we notice that the second set of equalities in (2.19) forms a system of $(d-1)N$ number of linearly independent equations. Such linear independence is guaranteed by construction [20] for all electrodynamics theories reviewed in the previous section 2.1. Further, in the particular case at hand, it is straightforward to solve this system for $\dot{A}^i_\alpha$ in terms of $(p^i_\alpha, A^\alpha_\mu)$:

$$\dot{A}^i_\alpha = p^i_\alpha - \partial^i A^0_\alpha - T^{0i}_\alpha. \quad (2.22)$$

The situation becomes more involved if $\mathcal{L}_{(n\geq2)} \neq 0$. When $\mathcal{L}_{(2)} \neq 0$ with $\mathcal{L}_{(n\geq3)} = 0$, the aforementioned linear independence ensures a unique solution $\dot{A}^i_\alpha = \dot{A}^i_\alpha(p^i_\alpha, A^\alpha_\mu)$ exists. Then, the difficulty amounts to the algebraic effort required for its explicit determination. Whenever $\mathcal{L}_{(n\geq3)} \neq 0$, we encounter a polynomial in $\dot{A}^i_\alpha$ of order $(n-1)$ on the right-hand side of the the second set of equalities in (2.19). We are thus confronted with a setting where the inversion of the generalized velocities in terms of the canonical momenta (and the generalized coordinates) is multivalued. This looks like a worse problem than it actually is: the complication is a technical — as opposed to a fundamental — one and was elegantly resolved in [53] by defining a generalized notion for the Legendre transform. The increased algebraic effort associated with choosing $\mathcal{L}_{(n\geq2)} \neq 0$ is notorious, but certainly not insurmountable, and would obscure the transcendence of our results. For this reason, we have opted to set $\mathcal{L}_{(n\geq2)} = 0$ in this work.

Overall, the reconsideration of (2.14) such that (2.21) are the generalized coordinates yields, upon minor algebraic effort employing (2.22), a Lagrangian density of the desired form (2.17), with $\theta = \{p^\mu_\alpha, 0, \varphi_\alpha\}$ and

$$\tilde{\mathcal{L}} = -\frac{1}{2}p^i_\alpha p^i_\alpha - p^i_\alpha \partial^i A^0_\alpha - \frac{1}{4}A_{ij}A^{ij}$$

$$- \frac{1}{2}m^2 (A^0_\alpha A^0_\alpha + A_\alpha^i A^i_\alpha) + c \left(A^0_\alpha A^0_j A^0_\alpha A^j_\alpha + 2A^0_\alpha A^0_j A^j_\alpha A^i_\alpha + A^i_\alpha A^j_\alpha A^i_\alpha A^j_\alpha \right) \quad (2.23)$$

$$(p^\alpha_\mu + \partial^\mu A^0_\alpha)T^{0i}_\alpha + \frac{1}{2}T^{0i}_\alpha T^{0i}_\alpha + \mathcal{T}^{\alpha\beta}_{\alpha\beta} \partial^\alpha A^\alpha_0 + \mathcal{T}^{\alpha\beta}_{\alpha\beta} \partial^\alpha A^\beta_j + A^\alpha_0 \bar{J}^\alpha_0 + A^\alpha_i \bar{J}^i_\alpha,$$

where, once more for the convenience of the reader, we have placed the terms coming from $\mathcal{L}_{\text{kin}}$, $\mathcal{L}_{(0)}$ and $\mathcal{L}_{(1)}$ (plus the coupling to the external sources) in the first, second and third lines, respectively. Of course, the classical consistency conditions (2.15) must be fulfilled in this rewriting as well.

Here, it is important to note note that we have viewed the essential terms enforcing the constraints (2.20) via Lagrange multipliers as belonging within the symplectic part of the Lagrangian, i.e. the first term in (2.17). This is because the Lagrange multipliers are arbitrary, so we can enforce (2.20) via their time derivatives just as well. In other words, we can incorporate the constraints (2.20) to our electrodynamics theories in two physically equivalent ways: adding either $\lambda^\alpha \varphi_\alpha$ or $\bar{\lambda}^\alpha \varphi_\alpha$ to (2.14). The first way is followed in Dirac-based standard quantization procedures, whereas the second way is cornerstone to the symplectic quantization methodology. The interested reader can consult [43] for a detailed exploration of the said two manners to incorporate constraints, as well as a proof of their physical equivalence. In the present work, we have of course elected the second option.
An important technical remark is as follows. The expert reader may here worry that we are overlooking the prescription in [54] for field theories. Namely, that we may be missing out on unveiling purely spatial consistency conditions, since these can only be found by introducing $d$ number of Lagrange multipliers per constraint, in the form $\partial_\mu \lambda^{\mu \alpha} \varphi_\alpha$. We have explicitly checked that no such spatial conditions apply to our considered settings (2.11) and, a posteriori, have opted for alleviating the algebraic presentation throughout the text by only introducing one Lagrange multiplier per constraint: $\lambda^\alpha \varphi_\alpha$. The inclusion of all $d$ Lagrange multipliers leads to the generation of functionally dependent $(d - 1)$ number of constraints at the first iteration, given by $\partial_i \varphi_\alpha \dagger = 0$, which are simply redundant.

For completeness, we point out that our above manipulation of (2.11), or equivalently of (2.14), to bring it into the FJ form (2.17) is not the only possible one. It is the one employed in [48] for the symplectic quantization of Proca electrodynamics and therefore our forthcoming results are most easily compared to this reference, in the appropriate limit. It is worth noting that [46] also promotes canonical momenta and Lagrange multipliers to additional generalized coordinates for the quantization of Proca electrodynamics. However, this work is primarily concerned with the introduction of a distinct, albeit Faddeev-Jackiw-based, quantization procedure. Therefore, a step-wise comparison of our work to [46] is not possible. There is another possibility, which was exploited in [55], also in the context of the symplectic quantization of Proca electrodynamics. In this reference, the theory is first manipulated to enjoy a $U(1)$ gauge symmetry. This is achieved through the suitable inclusion of an additional scalar field, in a procedure that in some contexts is referred to as the St¨ uckelberg mechanism, originally proposed in [56]. (We refer the interested reader to [57] for a compelling modern review of this mechanism.) Afterwards, the Proca and scalar fields, together with their canonical momenta are regarded as the generalized coordinates and the symplectic quantization method is employed. While it is possible to proceed in an analogous manner for our considered electrodynamics theories (2.11), this is algebraically more cumbersome. With simplicity in mind, we have opted for quantizing the theories as they are, with no gauge symmetry at all. We stress that the said two distinct manners in which a Lagrangian can be brought into the form (2.17) are explicitly shown to yield the same physics in [58] for the non-trivial case of Podolsky electrodynamics [22,23]. For clarity, we point out that the authors of [58] refer to the aforementioned enlargement of the configuration space and to the Stückelberg mechanism as reduced order formalism and Ostrogradsky prescription, respectively. The latter name alludes to the original paper [59], but employs the modern understanding developed in [60,61].

2.4 First iteration

The first step in the symplectic quantization prescription amounts to the calculation of the so-called symplectic two-form $\Omega$, a totally anti-symmetric square matrix, whose components are given by

$$
\Omega_{mn} := \frac{\delta \theta'_n}{\delta Q^m} - \frac{\delta \theta'_m}{\delta Q^n},
$$

where $m, n = 1, 2, \ldots, (2d + 1)N$ label the individual elements in $\theta = \{p^\mu_\alpha, 0, \varphi_\alpha\}$ and $Q$ in (2.21). The symplectic two-form is defined on a constant time hypersurface $\Sigma \subset \mathcal{M}$. The non-primed quantities ($\theta, Q$) are to be understood as evaluated at some point $x = (t^*, x^i) \in \Sigma$, with $t^*$ an arbitrary but fixed time; while their primed counterparts ($\theta', Q'$) are to be understood as evaluated at some other point $x' = (t^*, x'^i) \in \Sigma$. 8
We can succinctly spell out \( \Omega \) as

\[
\Omega = \begin{pmatrix}
0 & -\delta^\mu_\nu \delta^\beta_\alpha & -\overline{\mathcal{J}}_0 \mathcal{T}^{00}_\beta \\
\delta^\nu_\mu \delta^\beta_\alpha & 0 & \delta^0_\mu \delta^0_\beta \\
\overline{\mathcal{J}}_0 \mathcal{T}^{00}_\alpha & -\delta^0_\nu \delta^0_\alpha & 0
\end{pmatrix} \delta^{d-1}(x^i - x'^i).
\] (2.25)

Next, we need to determine whether the above symplectic two-form is singular or not. The calculation of the determinant is subtle, so we will carry it out explicitly. To this aim, we will make use of Schur’s identity. Namely, given any square matrix \( M \) that admits a block decomposition of the form

\[
M = \begin{pmatrix}
M_1 & M_2 \\
M_3 & M_4
\end{pmatrix},
\] (2.26)

such that \( M_1 \) and \( M_4 \) are square and \( M_1 \) is invertible, its determinant can be computed as

\[
\det(M) = \det(M_1) \det(M_4 - M_3 M_1^{-1} M_2).
\] (2.27)

Notice that Schur’s identity does not require \( M_2 \) and \( M_3 \) to be square. Upon the identifications \( M = \Omega \),

\[
M_1 = \begin{pmatrix}
0 & -\delta^\mu_\nu \delta^\beta_\alpha \\
\delta^\nu_\mu \delta^\beta_\alpha & 0
\end{pmatrix} \delta^{d-1}(x^i - x'^i), \quad M_2 = \begin{pmatrix}
-\overline{\mathcal{J}}_0 \mathcal{T}^{00}_\alpha \\
\delta^0_\mu \delta^0_\beta
\end{pmatrix} \delta^{d-1}(x^i - x'^i),
\] (2.28)

\[
M_3 = \begin{pmatrix}
\overline{\mathcal{J}}_0 \mathcal{T}^{00}_\alpha \\
-\delta^0_\nu \delta^0_\alpha
\end{pmatrix} \delta^{d-1}(x^i - x'^i), \quad M_4 = 0
\]

and noting that

\[
\det(M_1) = 1, \quad M_1^{-1} = M_1,
\] (2.29)

we easily arrive at

\[
\det(\Omega) = \det(-M_3 M_1 M_2) = \det \left[ \left( \overline{\mathcal{J}}_0 \mathcal{T}^{00}_\alpha - \overline{\mathcal{J}}_0 \mathcal{T}^{00}_\beta \right) \delta^{d-1}(x^i - x'^i) \right].
\] (2.30)

By virtue of the classical consistency conditions (2.15), the above determinant vanishes. The symplectic two-form \( \Omega \) in (2.25) is therefore singular. Its singularity implies the existence of further constraints, beyond the already unveiled ones in (2.20). Before calculating these additional constraints, we reflect upon (2.30).

For just a moment, suppose that we would not have been aware of the classical consistency conditions (2.15) from the very beginning. In such a case, at this point we would have derived (2.15) from (2.30). This is because the singularity of the symplectic two-form is indispensable for the correct postulation of any electrodynamics theory and thus for our considered particular theory (2.11) too. For instance, it is well known that Proca electrodynamics is associated with two (second-class) constraints. The first such constraint amounts to the independence of the action from \( p^0 \) or, equivalently, from \( \dot{A}_0 \). The second constraint exists iff the symplectic two-form vanishes. In the Proca case, this vanishing is automatic. We now turn to the more general GP case. Since all (multi-)GP are non-linear extensions of Proca electrodynamics, they must have its same constraint algebraic structure: each GP field must be associated

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This statement will become clear shortly, in (2.35).
with two (second-class) constraints. The first set is that in (2.20). The second set exists iff (2.30) is zero, which uniquely and straightforwardly implies (2.15). Therefore, at this point we have obtained the following important side-result: an independent derivation of the classical consistency conditions applying to all multi-GP electrodynamics theories, which were originally disclosed in [20, 21] following a different approach, à la Dirac.

As a first step in the determination of the necessary additional constraints in our considered generalized massive electrodynamics theories, we compute the zero modes of Ω in (2.25). The number of linearly independent zero modes that Ω admits is equal to

$$\dim(\Omega) - \text{rank}(\Omega) = (2d + 1)N - 2dN = N.$$  \hfill (2.31)

The above rank readily follows from the observation that (2.30) identically vanishes for a single GP field, together with (2.29). The N linearly independent zero modes of Ω are of the generic form $\gamma_\alpha = (u^\alpha_\mu, v^\alpha_\mu, w^\alpha)$ and fulfil that their left multiplication with Ω vanishes. This vanishing implies

$$u^\alpha_0 = -w^\alpha, \quad u^\alpha_i = 0, \quad v^\alpha_0 = -w^\beta \overleftrightarrow{\partial}_\beta J^0_{\alpha 0}, \quad v^\alpha_i = -w^\beta \overleftrightarrow{\partial}_\beta J^0_{\alpha i}$$ \hfill (2.32)

and we have the freedom to choose the $w^\alpha$ components. A simple consistent choice amounts to setting

$$w^\alpha = (0, 0, \ldots, 0, -1, 0, 0, \ldots, 0) =: -\iota^\alpha,$$ \hfill (2.33)

where the non-zero entry is in the $\alpha$-th position. All in all, we shall consider the following zero modes of Ω:

$$\gamma_\alpha = \left(\delta_\alpha^0 \iota^\alpha, \iota^\alpha, \delta_\alpha^\beta \overleftrightarrow{\partial}_\beta J^0_{\alpha 0} + \delta_\alpha^\beta \overleftrightarrow{\partial}_\beta J^0_{\alpha i}, -\iota^\alpha\right).$$ \hfill (2.34)

There are as many new constraints as linearly independent zero modes. These additional constraints $\varphi_\alpha$ can be determined employing the above zero modes according to the formula

$$\varphi := \gamma \cdot \frac{\delta \widehat{L}}{\delta Q} = 0,$$ \hfill (2.35)

with the generalized coordinates $Q$, the non-symplectic part of the Lagrangian density $\widehat{L}$ and the zero modes $\gamma_\alpha$ as given in (2.21), (2.23) and (2.34), respectively. It is easy to verify that the above constraints are explicitly given by

$$\varphi_\alpha = -m^2 A^0_{\alpha 0} + 2c A^0_{\alpha 0} \left(A^0_{\alpha 0} A^0_{\alpha 0} + 2 A^0_{\alpha 0} A^i_{\alpha i}\right) + \left(p^\beta_0 + 2 \partial_0 A^\beta_0 + \tau^\beta_{00}\right) \overleftrightarrow{\partial}_\alpha T^0_{\beta 0} + \left(\partial_0 A^\beta_0 \right) \overleftrightarrow{\partial}_\alpha T^0_{\beta 0} + \left(\partial_1 A^\beta_{00} + \partial_0 \right) \overleftrightarrow{\partial}_\alpha T^0_{\beta 0} + \partial_0 A^\mu_{\alpha 0} - \partial_1 \left(\tau^0_{\alpha 0} + \tau^0_{\alpha i}\right) + J^0_{\alpha 0} = 0.$$ \hfill (2.36)

Henceforth, it is essential to only consider the functionally independent constraints. As was the case for (2.20) earlier on, the functional independence of the above constraints is also ensured by construction [20]. Therefore, all $N$ number of constraints in (2.36) must be taken into account. We redirect the interested reader to section IID in [62] for an astute methodology to deal with (almost all) scenarios where there is no

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4This is but a harmless choice. It is also possible to choose to define the zero modes as those column vector, whose right multiplication with Ω yields zero. Here, we opt for the left multiplication convention also employed in [48], with the constant goal to make our results easily comparable to the limiting scenario of Proca electrodynamics worked out in this reference.
functional independence among the constraints. It is worth noting that this reference contains enlightening examples as well.

The (functionally independent) constraints \((2.36)\) are to be incorporated through new Lagrange multipliers \(\tilde{\lambda}^{\alpha}\). The novel Lagrange multipliers must be viewed as further generalized coordinates, so that the configuration space of our electrodynamics theories is now spanned by

\[ Q = \{ A_\mu^{\alpha}, p^\alpha, \lambda^\alpha, \tilde{\lambda}^\alpha \} \quad (2.37) \]

and is \(2(d + 1)N\)-dimensional. Following our remarks below \((2.23)\), we include the terms \(\tilde{\varphi}_{\alpha}\tilde{\lambda}^\alpha\) to our Lagrangian density. This is of the required form \((2.17)\), with

\[ \theta = \{ p_\alpha^\mu, 0, \varphi_\alpha, \tilde{\varphi}_\alpha \} \quad (2.38) \]

and \(\tilde{\mathcal{L}}\) as in \((2.23)\). Once again, there is no need to include \(d\) number of Lagrange multipliers per constraint \(\partial_\mu\tilde{\lambda}^{\mu\alpha}\tilde{\varphi}_\alpha\), as generally required for field theories \([54]\). This is because no constraints arise at the second iteration, as we shall immediately show.

### 2.5 Second iteration

We proceed to calculate the symplectic two-form \(\Omega\) associated to the above obtained Lagrangian density. We do so according to the definition in \((2.24)\), but this time with \(n, m = 1, 2, \ldots, 2(d + 1)N\) referring to the components of \(Q\) and \(\theta\) in \((2.37)\) and \((2.38)\), respectively. The result is

\[ \Omega = \begin{pmatrix} 0 & -\delta_\nu^\alpha \delta_\beta^\gamma & -\overline{\partial}_{\alpha} \delta^\nu_\beta \theta^0_\alpha & X^\mu_{\alpha\beta} \\ \frac{\partial^\nu_{\beta}}{\partial^\mu_{\alpha}} & 0 & \delta^\mu_\beta \delta_\alpha^\nu & -Y^\alpha_{

and we have introduced

\[ X^0_{\alpha\beta} := m^2 \delta^\beta_\alpha - 2c \left( A_0^{\gamma} A_0^{\alpha} \delta^\beta_\alpha + 2A_{\mu} A_{\mu}^{\alpha} \right) + \left( \overline{\partial}_{\beta} T_{\gamma_0}^0 + \overline{\partial}_{\gamma} T_{\beta_0}^0 \right) \overline{\partial}_{\alpha} T_{\beta_0}^0 + \left( p^i_j + 2\delta^i_j A_0^\gamma + T_{0j}^\gamma \right) \overline{\partial}_{\alpha} T_{\gamma_0}^0 - \delta^i_{\alpha} \left( T_{\gamma_0}^0 + \beta^0_\gamma \right) - 2\partial_{\beta} T_{\alpha}^0 - \partial_{\alpha} T_{\beta}^0, \]

\[ X^i_{\alpha\beta} := 4c \left( A_0^{\gamma} A_0^i \delta^\beta_\alpha + \overline{\partial}_{\beta} T_{\gamma_0}^i + \overline{\partial}_{\gamma} T_{\beta_0}^i \right) \overline{\partial}_{\alpha} T_{\beta_0}^i + \left( p^i_j + 2\delta^i_j A_0^\gamma + T_{0j}^\gamma \right) \overline{\partial}_{\alpha} T_{\gamma_0}^i - \delta^i_{\alpha} \left( T_{\gamma_0}^i + \gamma^i_0 \right), \]

\[ Y^\alpha_{\mu\beta} := \delta^\alpha_{\mu} \delta_{\beta} \partial_i + \delta^i_{\alpha} T_{\beta_0}^0 + \overline{\partial}_{\alpha} T_{\gamma_0}^0, \]

The primed counterparts \(X'^\mu_{\alpha\beta}\) and \(Y'^\alpha_{\mu\beta}\) follow from replacing \(\partial_i \leftrightarrow (-)\partial'_i\) everywhere in the above expressions, with \(\partial'_i\) the short-hand for derivation with respect to \(x'^i\). The minus sign applies only for those partial derivatives \(\partial_i^{(\ast)}\) that act on the Dirac delta \(\delta^{d-1}(x^i - x'^i)\). Namely, the first term on the right-hand
side of \( Y_{\mu\beta}^\alpha \). We note the additional components in (2.39), as compared to (2.25) before. These stem directly from the newly found constraints in (2.36). We stress that, generically, \( X_{\alpha\beta}^\mu \neq X_{\beta\alpha}^\mu \) and \( Y_{\mu\beta}^\alpha \neq Y_{\mu\alpha}^\beta \); which holds true for \( X_{\mu\beta}^\alpha \) and \( Y_{\mu\beta}^\alpha \) as well.

As in the first iteration earlier on, we now calculate the determinant of the above symplectic two-form, with views to establishing whether it is singular or not. Once more, we employ Schur’s identity (2.27), with \( M = \Omega \) in (2.39), \( M_1 \) and \( M_4 \) as in (2.28) and

\[
M_2 = \begin{pmatrix}
-\delta_\mu^\alpha \partial_0 T_0^0 & X_{\alpha\beta}^\mu \\
\delta_\mu^\alpha \delta_\beta^\gamma & -Y_{\mu\beta}^\alpha
\end{pmatrix} \delta^{d-1}(x^i - x'^i), \quad M_3 = \begin{pmatrix}
\partial_\alpha T_0^0 & -\delta_\mu^0 \delta_\alpha^3 \\
-X_{\beta\alpha}^\mu & Y_{\nu\alpha}^\beta
\end{pmatrix} \delta^{d-1}(x^i - x'^i).
\]

(2.41)

As an intermediate step, we note that

\[
(M_3 M_1 M_2)_{\alpha\beta} = \begin{pmatrix}
0 & Z_{\alpha\beta} \\
-Z_{\beta\alpha}^\gamma & -X_{\gamma\alpha}^\mu Y_{\mu\beta}^\gamma + Y_{\mu\beta}^\gamma X_{\gamma\alpha}^\mu
\end{pmatrix} \delta^{d-1}(x^i - x'^i),
\]

(2.42)

where the vanishing components are a direct consequence of the classical consistency conditions (2.15) and where we have introduced

\[
Z_{\alpha\beta} := (\partial_\gamma T_0^0)_{\mu\beta} Y_{\mu\alpha}^\beta - X_{\alpha\beta}^0.
\]

(2.43)

As explained below (2.40), the primed analogue \( Z'_{\alpha\beta} \) stands for \( Z_{\alpha\beta} \) under the replacements \( \partial_t \leftrightarrow (-)\partial'_t \). From (2.42), it readily follows that the determinant of \( \Omega \) in (2.39), which we denote by \( \varrho \) henceforth, is not zero in general:

\[
\varrho = -\det \left[ (Z' \cdot Z) \delta^{d-1}(x^i - x'^i) \right] \neq 0.
\]

(2.44)

From (2.42), it is clear that this is a direct consequence of

\[
Z_{\alpha\beta} \neq 0.
\]

(2.45)

The fact that the above determinant \( \varrho \) does not vanish signals the closure of the symplectic quantization iterative method.

Upon recalling our discussion below (2.30), a crucial observation follows:

\[
\varrho \neq 0 \implies Z_{\alpha\beta} \neq 0.
\]

(2.46)

This is an essential self-consistency condition for the targeted family of massive electrodynamics theories (2.11). Indeed, if \( \varrho = 0 \), then more than \( 2N \) constraints would be present. These can be determined in a third iteration of the symplectic quantization procedure and would over-constrain the theories, which would no longer enjoy the same constraint algebraic structure of \( N \) copies of Proca electrodynamics\(^5\). We therefore name (2.46) as quantum consistency conditions for (2.11). This complements the classical consistency condition in (2.15). Remarkably and unlike (2.15), the new conditions (2.46) apply to both single and multiple GP field settings. We regard the unveiling of the quantum consistency conditions as another important result in this paper, which will be elaborated upon in section 3.

\(^5\)At this point, the attentive and expert reader may well develop an educated (yet unfounded) suspicion. Namely, that perhaps \( \varrho = 0 \) is possible, as long as each and every of the additional constraints that follow are functionally dependent on the already found \( 2N \) constraints. However, this is not possible in the targeted theories. The reason is that a closure of the iterative procedure through functional dependence of the constraints implies the presence of a (gauge) symmetry. Clearly, our considered massive electrodynamics theories explicitly break the \( U(1)^N \) gauge invariance of \( N \)-field massless electrodynamics theories and thus enjoy no symmetry at all. Therefore, the iterative algorithm cannot close in such a manner for these theories; for them, \( \varrho = 0 \) is necessary. We refer the interested reader to [37] for further details.
2.6 Output: the partition function

The above non-singular symplectic two-form is central to symplectic quantization. Indeed, the commutation relations between the generalized coordinates (2.37) are given by

\[ \{ Q^n, Q^m \} = (\Omega_{mn})^{-1}, \quad (2.47) \]

with the right-hand side denoting the inverse of \( \Omega \) in (2.39). It is convenient to make two observations at this point. First, it is easy to deduce that, in our case, (2.47) is not of the standard canonical form. This is because in the symplectic piece \( (\theta \cdot \dot{Q}) \) of our second iterated Lagrangian density — where \( \theta \) and \( Q \) are given by (2.38) and (2.37), respectively — the set \( (\theta, Q) \) is not formed by independent fields: recall (2.20) and (2.36). By construction [40], it is guaranteed that there exists a Darboux transformation that brings \( (\theta, Q) \) to a canonical set of variables, whose commutation relations will then be of the standard canonical form. In general, finding the said Darboux transformation is tedious, if not difficult as well. Its calculation is a pivotal point in [42, 43] and finds in [46] what could well be the most complicated worked out example available to date. In our persistent aim for a quantization without tears, amenable to extrapolation to more cumbersome Lagrangian densities and aligned with the very essence of the employed method [45], we omit the determination of such a Darboux transformation. Second, as a direct consequence of our first observation, the explicit computation of the inverse matrix in (2.47) is operationally lengthy and prone to error. In fact, it can become quite a mathematical feat to do so, depending on the theory under consideration. We therefore refrain from its calculation and instead will promptly follow [47, 48], which will lead to the path integral formulation of the partition function for the theories of our interest (2.11). For completeness, we note that yet another way around this technical complication was put forward in [46], which proposes a quantization methodology that markedly departs from the symplectic prescription à la Faddeev and Jackiw.

As just anticipated and adhering to [47, 48], our prior analysis readily yields the sought partition function [47]:

\[ Z = \int d\sigma \exp \left( i \int_{\mathcal{M}} d^4 x \mathcal{L} \right). \]

(2.48)

Here, the Lagrangian density \( \mathcal{L} \) is of the FJ form (2.17), with \( \tilde{\mathcal{L}}, Q \) and \( \theta \) as in (2.23), (2.37) and (2.38), respectively. The measure is

\[ d\sigma = J \left( \prod_{\mu,\alpha} [dA^\alpha_\mu] \right) \left( \prod_{\nu,\beta} [dp^\nu_\beta] \right) \left( \prod_{\gamma} [d\lambda^\gamma] \right) \left( \prod_{\delta} [d\tilde{\lambda}^\delta] \right), \quad (2.49) \]

where \( J \) stands for the Jacobian of the aforementioned Darboux transformation. It is the main result of [48] to prove the identification

\[ J = \varrho^{1/2}, \quad (2.50) \]

with \( \varrho \) as in (2.44) for the theories of our present interest. The transcendence of (2.50) is clear, given our above observation that obtaining the Darboux transformation is generically complicated: it fully specifies the path integral measure in terms of the central object of the symplectic quantization method — the (possibly iterated) non-singular symplectic two-form \( \Omega \) — in a computationally simple manner. Therefore and recalling Schwartz’s appreciation that “if you have an exact closed-form expression for \( Z \) for a particular
theory, you have solved it completely” [63], we have now concluded the symplectic quantization of (2.11) in the path integral formulation.  

2.7 Examples

With the main goal of neatly illustrating our above analysis, we proceed to examine two simple, massive extensions of QED. We will first contemplate the well-known Proca electrodynamics case and explicitly ensure we reproduce its familiar results. We then use the developed approach to quantize a single-field GP scenario, where the mass of the GP field is realized through a derivative self-interaction term.

**Proca electrodynamics.**

We begin by considering the renowned Lagrangian density

\[
L_P = -\frac{1}{4} A_{\mu\nu} A^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu, \quad m^2 \in \mathbb{R}_{>0},
\]

(2.51)
dating back to [9,10] and subjected to symplectic quantization in e.g. [46,48,55,65]. The theoretical appeal of the above singular theory is largely due to the fact that it is the simplest field theory with only second class constraints. As such, over the years it has been recurrently used in diverse contexts as a representative of the subtleties this class of theories displays during quantization; for instance, see [66–71].

Proca electrodynamics is a subcase of our targeted family of theories. It is obtained by considering the single field limit \( N = 1 \) in (2.11), along with the choices \( c = 0 = T^{\mu\nu} \) and in the absence of external sources \( J^\mu = 0 \).

It is immediate to see that the first iterated symplectic two-form (2.25) has a zero determinant in this case. Therefore, the classical consistency condition (2.15) is automatically satisfied. The second iterated symplectic two-form (2.39) always has a non-zero determinant \( \varrho \), with

\[
\varrho^{1/2} = \text{det} \left[ m^2 \delta^{d-1}(x^i - x'^i) \right].
\]

(2.52)

Consequently, the quantum consistency conditions (2.46) are also automatically satisfied.

The partition function of Proca electrodynamics in the symplectic quantization is of the form in (2.48), where the path integral measure is

\[
d\sigma = \varrho^{1/2} \left( \prod_{\mu} [dA_\mu] \right) \left( \prod_\nu [dp^\nu] \right) [d\lambda] [d\tilde{\lambda}],
\]

(2.53)

with \( \varrho^{1/2} \) as in (2.52), and where the Lagrangian density \( \mathcal{L} \) therein is explicitly given by

\[
\mathcal{L} = p^\mu \dot{A}_\mu + p^0 \dot{\lambda} + \left( \partial_i p^i - m^2 A^0 \right) \dot{\lambda} - \frac{1}{2} p^i p_i - p^i \partial_i A_0 - \frac{1}{4} A_{ij} A^{ij} - \frac{1}{4} m^2 A_\mu A^\mu.
\]

(2.54)

\[\text{As a side remark, we point out that the authors of [47] built upon their own work in [64], which seems to be a reference that [48] is unaware of. Here, they introduced the so-called equivalently extended Lagrangian, which does not contain the Jacobian } J, \text{ as a means to resolve the ambiguity in their prescribed measure for those cases where the Darboux transformation is such that } J \neq 1. \text{ We find this unillustrated proposal rather obscure and unnecessarily involved and therefore favor the neat resolution of [48].}\]
Our above (limiting) result is in agreement with the relevant literature. We restate that this can be most easily verified by direct comparison to \[48\].

A simple GP electrodynamics.

We proceed to consider the Lagrangian density

\[ \mathcal{L}_{\text{GP1}} = -\frac{1}{4} A_{\mu\nu} A^{\mu\nu} + f \partial_{\mu} A^{\mu}, \quad f = f(A_{\mu}), \quad (2.55) \]

which is a subcase of the original GP proposal in \[18,19\]. There exist preliminary results regarding the quantum behaviour of (single field) GP theories \[72–75\], also in a curved background \[76\]. All of these works are concerned with tree-level and one-loop observables. However, to our knowledge, no complete and rigorous quantization scheme had been proposed for GP theories prior to this paper.

The above \((2.55)\) follows from the single field limit \(N = 1\) of \((2.11)\), with

\[ m^2 = 0 = c, \quad \mathcal{T}^{\mu\nu} = f \eta^{\mu\nu}, \quad (2.56) \]

for no external sources: \(J^\mu = 0\).

As for Proca electrodynamics earlier on, the first iterated symplectic two-form \((2.25)\) has a zero determinant. This is because the classical consistency condition \((2.15)\) does not restrict single-GP theories. Minor algebraic effort yields the following determinant \(\varrho\) for the second iterated symplectic two-form \((2.39)\):

\[ \varrho^{1/2} = \det \left[ F \delta^{d-1}(x^i - x'^i) \right], \quad F = (\overline{D} f) (\partial_i f - \partial_i) - (\partial_i A^i) \overline{D} \partial^i f + (p_i + \partial_i A_0) \overline{D} \partial^i f - \partial_i \overline{D} f \quad (2.57) \]

which is non-zero for any \(f\) that is genuinely a function of the GP field \(A_\mu\). However, the quantum consistency conditions rule out the classical possibility that \(f\) be a constant (of suitable length dimension \(-2\)). For the simple case here studied, choosing \(f\) to be a constant in \((2.55)\) renders the mass-like derivative self-interaction into a boundary term, a case that is obviously of no interest from the very onset. Therefore, the quantum consistency conditions \((2.46)\) are also automatically satisfied in our second simple example.

Symplectic quantization gives rise to partition function of \((2.55)\) in the form \((2.48)\), where the measure is as in \((2.53)\), with \(\varrho^{1/2}\) given by \((2.57)\), and where

\[ \mathcal{L} = p^\mu \dot{A}_\mu + (p^0 + f) \dot{\lambda} + \left[ \partial_i p^i + (p_i + \partial_i A_0) \overline{D} f + (\partial_i A^i) \overline{D} f \right] \dot{\lambda} - \frac{1}{2} p^i p_i - \frac{1}{4} A_{ij} A^{ij} + f \partial_i A^i. \quad (2.58) \]

3 Quantum consistency conditions

Our above symplectic quantization of \((2.11)\) has revealed two consequential facts. On the one hand, the necessarily singular character of the first iterated symplectic two-form \((2.25)\) implies the (already known) classical consistency conditions \((2.15)\). On the other hand, the necessarily non-singular character of the second iterated symplectic two-form \((2.39)\) implies the (newly found) quantum consistency conditions \((2.46)\). If any given theory within \((2.11)\) fails to fulfill \((2.15)\), then this theory is ill-defined at the classical level. Specifically, it will be prone to Ostrogradski instabilities \[59\]. If any given theory within \((2.11)\) fulfills \((2.15)\) but not \((2.46)\), then this theory does not admit quantization. Namely, such a theory must be exclusively viewed as a classical effective field theory (EFT); it cannot be employed as a quantum EFT.
The violation of the quantum consistency conditions (2.46) should not be interpreted as an anomaly, i.e. the quantum breaking of a classical symmetry. This is because, in any (multi-)GP electrodynamics theory, the gauge symmetry is explicitly broken already at the classical level. Moreover, the violation of (2.46) should not be regarded as related to a symmetry enhancement, wherein multi-GP (partially) restores the $U(1)^N$ gauge symmetry of $N$ copies of Maxwell electrodynamics or its massless non-linear extensions. (Multi-)GP explicitly breaks the gauge symmetry, regardless of whether the quantum consistency conditions are satisfied or not. An easy way to see this is as follows. Consider the example (2.55). This Lagrangian density enjoys a $U(1)$ gauge symmetry when either $f = 0$ or $\partial_\mu A^\mu = 0$. The quantum consistency conditions for this theory imply that $F$ in (2.57) cannot vanish. Since $f = 0$, $\partial_\mu A^\mu = 0$ and $F = 0$ are functionally independent formulae, we readily deduce that there exists no relation between the violation of the quantum consistency conditions and the restoration of a gauge symmetry in the theory.

In full generality, the class of multi-GP electrodynamics theories in section 2.1 can be reasonably expected to reproduce the above described structure. Namely, the necessarily singular character of their first iterated symplectic two-form presumably implies the classical consistency conditions (2.9). Additionally, the necessarily non-singular character of their second iterated symplectic two-form presumably implies the suitable generalization of the quantum consistency conditions (2.46) to

$$P \neq 0 \implies Z_{\alpha\beta} \neq 0,$$

with $P$ the determinant of the second iterated symplectic two-form and $Z_{\alpha\beta}$ the appropriate extension of $Z_{\alpha\beta}$ in (2.43). We emphasize that (3.1) affects a large class of theories. For instance, it restricts in an unprecedented manner any GP electrodynamics theory wherein the mass of the GP field is realized exclusively through derivative self-interactions. This means considering a single-field $N = 1$ and setting $\mathcal{L}_{(0)} = 0$ with $\mathcal{L}_{(n \geq 1)} \neq 0$ in (2.8), for one or more such $n \geq 1$. In this case, (3.1) rules out the classically consistent possibility of constant $T$ objects for $\mathcal{L}_{(n \geq 2)}$, since (3.1) necessarily involves at least one derivative with respect to the GP field of these $T$’s. All in all, we conclude that, for the general multi-GP electrodynamics theories reviewed in section 2.1, the $T$’s are non-trivially constrained by (2.9), as well as by our newly found (3.1).

### 4 Conclusions and outlook

In this work, we have carried out the symplectic quantization of the family of multi-field Generalized-Proca (GP) electrodynamics theories in (2.11). Specifically, we have determined the partition function (2.48). As a by-product, we have obtained an independent derivation of the classical consistency conditions (2.15) that apply to these theories. Moreover, we have unveiled a necessary additional set of restrictions for (multi-)GP theories in the quantum regime, which we call quantum consistency conditions (2.46). Remarkably, these affect both single- and multi-field scenarios and imply that (most but) not all generalizations of massive electrodynamics considered here can be quantized.

It is possible that our newly found quantum consistency conditions, even when generically fulfilled for a given Lagrangian, are dynamically violated. For the family of theories (2.11), this would mean that there exists one or more points in the moduli space for which (2.46) does not hold true. In the second example considered in section 2.7, this is realized when the generalized coordinates $A_\mu$ and $P^i$ take on-shell values that result in the vanishing of $F$ in (2.57). This type of singularities in the second iterated symplectic two-form would imply the existence of further constraints in the theory, which, if functionally independent,
would lead to a reduction of the local number of physical modes. We have explicitly checked that such reduction of the local degrees of freedom indeed takes place in the example (2.55), for the particular choice $f = -A_{\mu}A^{\mu}/2$. Phenomena like shock wave propagation and birefringence are then expected to occur.

Indeed, similar degenerate behavior is theoretically well-known to happen in the massless sector: in the family of theories known as non-linear electrodynamics (NLE) — recall table 1. For instance, shock waves have been studied in the particular NLE cases of Born electrodynamics [77, 78], and of Euler-Heisenberg electrodynamics [79], as well as generically in the Plebanski formulation of the full NLE family [80] (see also references therein). Born-Infeld electrodynamics constitutes the only sensible exception within NLE: this theory displays no shock waves and no birefringence [81].

The massless scenarios in NLE are currently pending experimental verification. Our work suggests that analogue massive settings in (multi-)GP should be phenomenologically studied and confronted with the outcome of the relevant future experiments, such as PVLAS [82] and LUXE [83]. A particularly appealing question to be addressed is the examination of whether the class of multi-GP electrodynamics theories contains a subset which, like Born-Infeld, completely avoids degenerate behavior.

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References

[1] M. Tajmar, “Electrodynamics in Superconductors Explained by Proca Equations,” Phys. Lett. A 372, 3289 (2008) doi:10.1016/j.physleta.2007.10.070 [arXiv:0803.3080 [cond-mat.supr-con]].

[2] A. De Felice, L. Heisenberg, R. Kase, S. Mukohyama, S. Tsujikawa and Y. I. Zhang, “Cosmology in generalized Proca theories,” JCAP 06, 048 (2016) doi:10.1088/1475-7516/2016/06/048 [arXiv:1603.05806 [gr-qc]].

[3] V. S. Gorelik, “Effective mass of photons and the existence of heavy photons in photonic crystals,” Phys. Scr. 2010, 014046 (2010) doi:10.1088/0031-8949/2010/t140/014046.

[4] J. Ellis, N. E. Mavromatos and T. You, “Light-by-Light Scattering Constraint on Born-Infeld Theory,” Phys. Rev. Lett. 118, no. 26, 261802 (2017) doi:10.1103/PhysRevLett.118.261802 [arXiv:1703.08450 [hep-ph]].

[5] E. S. Fradkin and A. A. Tseytlin, “Non-linear electrodynamics from quantized strings,” Phys. Lett. B 163, 123 (1985) doi:10.1016/0370-2693(85)90205-9.

[6] J. C. Maxwell, “A dynamical theory of the electromagnetic field,” Philosophical Transactions of the Royal Society 155, 459-512 (1865) doi:10.1098/rstl.1865.0008.

[7] J. Bovy, “The self-energy of the electron: a quintessential problem in the development of QED,” [arXiv:physics/0608108 [physics.hist-ph]].

[8] M. Born and L. Infeld, “Foundations of the new field theory,” Proc. Roy. Soc. Lond. A 144, no. 852, 425-451 (1934) doi:10.1098/rspa.1934.0059.
[9] A. Proca, “Sur la théorie ondulatoire des électrons positifs et négatifs,” J. Phys. Radium 7, 347 (1936) doi:10.1051/jphysrad:0193600708034700.

[10] A. Proca, “Théorie non relativiste des particules à spin entier”, J. Phys. Radium 9, 61 (1938) doi:10.1051/jphysrad:019380090206100.

[11] J. Schwinger, “On Gauge Invariance and Vacuum Polarization,” Phys. Rev. 82, 664-479 (1951) doi:10.1103/PhysRev.82.664.

[12] J. Schwinger, “A note on the quantum dynamical principle,” Philos. Mag. 44 (357), 1171-1179 (1953) doi:10.1080/14766441008520377.

[13] J. Plebanski, “Lectures on non-linear electrodynamics,” NORDITA, Copenhagen (1968).

[14] N. Bloembergen, “Nonlinear Optics,” W. A. Benjamin Inc., New York (1965).

[15] M. Partanen, T. Häyrynen, J. Oksanen and J. Tulkki, “Photon mass drag and the momentum of light in a medium,” Phys. Rev. A 95, 2469-9934 (2017) doi:10.1103/physreva.95.063850 [arXiv:1603.07224 [physics.optics]].

[16] P. D. García, G. Kirsanskié, A. Javadi, S. Stobbe and P. Lodahl, “Two mechanisms of disorder-induced localization in photonic-crystal waveguides,” Phys. Rev. B 96, 2469-9969 (2017) doi:10.1103/PhysRevB.96.144201 [arXiv:1709.10310 [physics.optics]].

[17] K. Rechcińska1, M. Król, R. Mazur, P. Morawiak, R. Mirek, K. Lempicka, W. Bardyszewski, M. Matuszewski, P. Kula, W. Piecek, P. G. Lagoudakis, B. Piótko and J. Szczylko, “Engineering spin-orbit synthetic Hamiltonians in liquid-crystal optical cavities,” Science 366, 727-730 (2019) doi:10.1126/science.aay4182.

[18] G. Tasinato, “Cosmic Acceleration from Abelian Symmetry Breaking,” JHEP 04, 067 (2014) doi:10.1007/JHEP04(2014)067 [arXiv:1402.6450 [hep-th]].

[19] L. Heisenberg, “Generalization of the Proca Action,” JCAP 05, 015 (2014) doi:10.1088/1475-7516/2014/05/015 [arXiv:1402.7026 [hep-th]].

[20] V. Errasti Díez, B. Gording, J. A. Méndez-Zavaleta and A. Schmidt-May, “Maxwell-Proca theory: Definition and construction,” Phys. Rev. D 101, no.4, 045009 (2020) doi:10.1103/PhysRevD.101.045009 [arXiv:1905.06968 [hep-th]].

[21] V. Errasti Díez, B. Gording, J. A. Méndez-Zavaleta and A. Schmidt-May, “Complete theory of Maxwell and Proca fields,” Phys. Rev. D 101, no.4, 045008 (2020) doi:10.1103/PhysRevD.101.045008 [arXiv:1905.06967 [hep-th]].

[22] B. Podolsky, “A Generalized Electrodynamics Part I: Non-Quantum,” Phys. Rev. 62, 68-71 (1942) doi:10.1103/PhysRev.62.68.

[23] B. Podolsky and C. Kikuchi, “A Generalized Electrodynamics Part II: Quantum,” Phys. Rev. 65, 228-235 (1944) doi:10.1103/PhysRev.65.228.
[24] C. de Rham and V. Pozsgay, “New class of Proca interactions,” Phys. Rev. D 102, no.8, 083508 (2020) doi:10.1103/PhysRevD.102.083508 [arXiv:2003.13773 [hep-th]].

[25] C. de Rham, L. Heisenberg, A. Kumar and J. Zosso, “Quantum stability of Proca-Nuevo,” [arXiv:2108.12892 [hep-th]].

[26] C. de Rham, S. Garcia-Saenz, L. Heisenberg and V. Pozsgay, “Cosmology of Extended Proca-Nuevo,” [arXiv:2110.14327 [hep-th]].

[27] P. A. M. Dirac, “Generalized Hamiltonian dynamics,” Can. J. Math. 2, 129 (1950) doi:10.4153/CJM-1950-012-1.

[28] J. L. Anderson and P. G. Bergmann, “Constraints in covariant field theories,” Phys. Rev. 83, 1018-1025 (1951) doi:10.1103/PhysRev.83.1018.

[29] P. G. Bergmann and I. Goldberg, “Dirac Bracket Transformations in Phase Space,” Phys. Rev. 98, 531(1955) doi:10.1103/PhysRev.98.531.

[30] M. J. Gotay, J. M. Nester and G. Hinds, “Presymplectic manifolds and the Dirac-Bergmann theory of constraints,” J. Math. Phys. 19, 2388 (1978) doi:10.1063/1.523597.

[31] C. Batlle, J. Gomis, J. M. Pons and N. Román-Roy, “Equivalence between the Lagrangian and Hamiltonian formalism for constrained systems,” J. Math. Phys. 27, 2953 (1986) doi:10.1063/1.527274.

[32] J. Lee and R. M. Wald, “Local symmetries and constraints,” J. Math. Phys. 31, 725 (1990) doi:10.1063/1.528801.

[33] L. Vitagliano, “The Lagrangian-Hamiltonian Formalism for Higher Order Field Theories,” J. Geom. Phys. 60, 857-873 (2010) doi:10.1016/j.geomphys.2010.02.003 [arXiv:0905.4580 [math.DG]].

[34] J. B. Pitts, “A First Class Constraint Generates Not a Gauge Transformation, But a Bad Physical Change: The Case of Electromagnetism,” Annals Phys. 351, 382-406 (2014) doi:10.1016/j.aop.2014.08.014 [arXiv:1310.2756 [gr-qc]].

[35] P. D. Prieto-Martínez and N. Román-Roy, “Variational Principles for multisymplectic second-order classical field theories,” Int. J. Geom. Methods Mod. Phys. 12, 1560019 (2015) doi:10.1142/S0219887815600191 [arXiv:1412.1451 [math-ph]].

[36] M. Crisostomi, R. Klein and D. Roest, “Higher Derivative Field Theories: Degeneracy Conditions and Classes,” JHEP 06, 124 (2017) doi:10.1007/JHEP06(2017)124 [arXiv:1703.01623 [hep-th]].

[37] V. Errasti Díez, M. Maier, J. A. Méndez-Zavaleta and M. Taslimi Tehrani, “Lagrangian constraint analysis of first-order classical field theories with an application to gravity,” Phys. Rev. D 102, 065015 (2020) doi:10.1103/PhysRevD.102.065015 [arXiv:2007.11020 [hep-th]].

[38] L. D. Faddeev, “The Feynman integral for singular Lagrangians,” Theor. Math. Phys. 1, 1 (1969) doi:10.1007/BF01028566.
[39] P. Senjanovic, “Path Integral Quantization of Field Theories with Second Class Constraints,” Annals Phys. 100, 227-261 (1976) [erratum: Annals Phys. 209, 248 (1991)] doi:10.1016/0003-4916(76)90062-2.

[40] L. D. Faddeev and R. Jackiw, “Hamiltonian Reduction of Unconstrained and Constrained Systems,” Phys. Rev. Lett. 60, 1692-1694 (1988) doi:10.1103/PhysRevLett.60.1692.

[41] J. Barcelos-Neto and E. S. Cheb-Terrab, “Faddeev-Jackiw quantization in superspace,” Z. Phys. C 54, 133-138 (1992) doi:10.1007/BF01881716.

[42] J. Barcelos-Neto and C. Wotzasek, “Symplectic Quantization of Constrained Systems,” Mod. Phys. Lett. A 7, 1737 (1992) doi:10.1142/S0217732392001439.

[43] J. Barcelos-Neto and C. Wotzasek, “Faddeev-Jackiw quantization and constraints,” Int. J. Mod. Phys. A 7, 4981 (1992) doi:10.1142/S0217751X9200226X.

[44] H. Montani and C. Wotzasek, “Faddeev-Jackiw quantization of non-abelian systems,” Mod. Phys. Lett. A 8, 3387 (1993) doi:10.1142/S0217732393003810.

[45] R. Jackiw, “(Constrained) quantization without tears,” [arXiv:hep-th/9306075 [hep-th]].

[46] C. Prescod-Weinstein and E. Bertschinger, “An extension of the Faddeev–Jackiw technique to fields in curved spacetimes,” Class. Quant. Grav. 32, no.7, 075011 (2015) doi:10.1088/0264-9381/32/7/075011 [arXiv:1404.0382 [hep-th]].

[47] L. Liao and Y. C. Huang, “Path integral quantization corresponding to Faddeev-Jackiw canonical quantization,” Phys. Rev. D 75, 025025 (2007) doi:10.1103/PhysRevD.75.025025.

[48] D. J. Toms, “Faddeev-Jackiw quantization and the path integral,” Phys. Rev. D 92, no.10, 105026 (2015) doi:10.1103/PhysRevD.92.105026 [arXiv:1508.07432 [hep-th]].

[49] G. Dvali, G. F. Giudice, C. Gomez and A. Kehagias, “UV-Completion by Classicalization,” JHEP 08, 108 (2011) doi:10.1007/JHEP08(2011)108 [arXiv:1010.1415 [hep-ph]].

[50] F. D. Nobre and A. R. Plastino, “Generalized Nonlinear Proca Equation and its Free-Particle Solutions,” Eur. Phys. J. C 76, no.6, 1434 (2016) doi:10.1140/epjc/s10052-016-4196-4 [arXiv:1603.06126 [physics.gen-ph]].

[51] M. Minamitsuji, “Vector boson star solutions with a quartic order self-interaction,” Phys. Rev. D 97, no.10, 104023 (2018) doi:10.1103/PhysRevD.97.104023 [arXiv:1805.09867 [gr-qc]].

[52] C. A. R. Herdeiro, A. M. Pombo, E. Radu, P. V. P. Cunha and N. Sanchís-Gual, “The imitation game: Proca stars that can mimic the Schwarzschild shadow,” JCAP 04, 051 (2021) doi:10.1088/1475-7516/2021/04/051 [arXiv:2102.01703 [gr-qc]].

[53] E. Avraham and R. Brustein, “Canonical structure of higher derivative theories,” Phys. Rev. D 90, no.2, 024003 (2014) doi:10.1103/PhysRevD.90.024003 [arXiv:1401.4921 [hep-th]].

[54] W. M. Seiler, “Involution and constrained dynamics. II. The Faddeev-Jackiw approach,” J. Phys. A: Math. Gen. 28, 7315 (1995) doi:10.1088/0305-4470/28/24/026.
[55] B. M. Pimentel and G. E. R. Zambrano, “Faddeev-Jackiw quantization of Proca Electrodynamics,” Nucl. Part. Phys. Proc. 267-269, 183-185 (2015) doi:10.1016/j.nuclphysbps.2015.10.100

[56] E. C. G. Stueckelberg, “Interaction energy in electrodynamics and in the field theory of nuclear forces,” Helv. Phys. Acta 11, 225-244 (1938) doi:10.5169/seals-110852.

[57] H. Ruegg and M. Ruiz-Altaba, “The Stueckelberg field,” Int. J. Mod. Phys. A 19, 3265-3348 (2004) doi:10.1142/S0217751X04019755 [arXiv:hep-th/0304245 [hep-th]].

[58] A. A. Nogueira, C. Palechor and A. F. Ferrari, “Reduction of order and Fadeev–Jackiw formalism in generalized electrodynamics,” Nucl. Phys. B 939, 372-390 (2019) doi:10.1016/j.nuclphysb.2018.12.026 [arXiv:1806.08438 [hep-th]].

[59] M. Ostrogradsky, “Mémoires sur les équations différentielles, relatives au problème des isopérièmes,” Mem. Acad. St. Petersbourg 6, no.4, 385-517 (1850).

[60] T. S. Chang, “Field theories with high derivatives,” Proc. Camb. Philos. Soc. 44, 76 (1948) doi:10.1017/S0305004100024014.

[61] A. Pais and G. E. Uhlenbeck, “On Field theories with nonlocalized action,” Phys. Rev. 79, 145-165 (1950) doi:10.1103/PhysRev.79.145.

[62] B. Díaz and M. Montesinos, “Geometric Lagrangian approach to the physical degree of freedom count in field theory,” J. Math. Phys. 59, no.5, 052901 (2018) doi:10.1063/1.5008740 [arXiv:1710.01371 [gr-qc]].

[63] M. D. Schwartz, “Quantum Field Theory and the Standard Model,” Cambridge University Press, New York (2014).

[64] Y. C. Huang, L. Liao and X. G. Lee, “Faddeev-Jackiw canonical path integral quantization for a general scenario, its proper vertices and generating functionals,” Eur. Phys. J. C 60, 481-487 (2009) doi:10.1140/epjc/s10052-009-0922-5.

[65] J. Ramos, “On the equivalence and non-equivalence of Dirac and Faddeev-Jackiw formalisms for constrained systems,” Can. J. Phys. 95, 3 (2016) doi:10.1139/CJP-2015-0547.

[66] W. Greiner and J. Reinhardt, Chapter 6 “Spin-1 Fields: The Maxwell and Proca Equations,” “Field Quantization,” Springer, Berlin, Heidelberg (1996) doi:10.1007/978-3-642-61485-9_6.

[67] Y. W. Kim, M. I. Park, Y. J. Park and S. J. Yoon, “BRST quantization of the Proca model based on the BFT and the BFV formalism,” Int. J. Mod. Phys. A 12, 4217-4239 (1997) doi:10.1142/S0217751X97002309 [arXiv:hep-th/9702002 [hep-th]].

[68] F. Zamani and A. Mostafazadeh, “Quantum Mechanics of Proca Fields,” J. Math. Phys. 50, 052302 (2009) doi:10.1063/1.3116164 [arXiv:0805.1651 [quant-ph]].

[69] A. J. Silenko, “Relativistic quantum mechanics of a Proca particle in Riemannian spacetimes,” Phys. Rev. D 98, no.2, 025014 (2018) doi:10.1103/PhysRevD.98.025014 [arXiv:1712.08625 [gr-qc]].
[70] M. Schambach and K. Sanders, “The Proca Field in Curved Spacetimes and its Zero Mass Limit,” Rep. Math. Phys. 82, 203 (2018) doi:10.1016/S0034-4877(18)30086-7.

[71] H. Park and T. Lee, “Canonical Quantization of Massive Symmetric Rank-Two Tensor in String Theory,” Nucl. Phys. B 954, 115006 (2020) doi:10.1016/j.nuclphysb.2020.115006 [arXiv:1908.03704 [hep-th]].

[72] F. Charmchi, Z. Haghani, S. Shahidi and L. Shahkarami, “One-loop corrections to vector Galileon theory,” Phys. Rev. D 93, no.12, 124044 (2016) doi:10.1103/PhysRevD.93.124044 [arXiv:1511.07034 [hep-th]].

[73] A. Amado, Z. Haghani, A. Mohammadi and S. Shahidi, “Quantum corrections to the generalized Proca theory via a matter field,” Phys. Lett. B 772, 141-151 (2017) doi:10.1016/j.physletb.2017.06.040 [arXiv:1612.06938 [hep-th]].

[74] C. de Rham, S. Melville, A. J. Tolley and S. Y. Zhou, “Positivity Bounds for Massive Spin-1 and Spin-2 Fields,” JHEP 03, 182 (2019) doi:10.1007/JHEP03(2019)182 [arXiv:1804.10624 [hep-th]].

[75] L. Heisenberg and J. Zosso, “Quantum Stability of Generalized Proca Theories,” Class. Quant. Grav. 38, no.6, 065001 (2021) doi:10.1088/1361-6382/abd680 [arXiv:2005.01639 [hep-th]].

[76] S. Panda, A. Tinwala and A. Vidyarthi, “Covariant Effective Action for Generalized Proca Theories,” [arXiv:2112.04391 [hep-th]].

[77] C. Minz, H. von Borzeszkowski, T. Chrobok and G. Schellestede, “Shock Wave Polarizations and Optical Metrics in the Born and the Born-Infeld Electrodynamics,” Annals Phys. 364, 248-260 (2016) doi:10.1016/j.aop.2015.11.005 [arXiv:1411.3163 [math-ph]].

[78] H. Kadlecová, “Electromagnetic waves in Born Electrodynamics,” [arXiv:2103.03575 [hep-th]].

[79] H. Kadlecová, G. Korn and S. V. Bulanov, “Electromagnetic shocks in the quantum vacuum,” Phys. Rev. D 99, no.3, 036002 (2019) doi:10.1103/PhysRevD.99.036002 [arXiv:1807.11365 [physics.plasm-ph]].

[80] C. A. Escobar and R. Potting, “Degenerate behavior in nonlinear vacuum electrodynamics,” Phys. Scripta 95, no.6, 065218 (2020) doi:10.1088/1402-4866/ab842d [arXiv:2004.01852 [hep-ph]].

[81] G. Boillat, “Nonlinear Electrodynamics: Lagrangians and Equations of Motion,” J. Math. Phys. 11, 941 (1970) doi:10.1063/1.1665231.

[82] A. Ejlli, F. Della Valle, U. Gastaldi, G. Messineo, R. Pengo, G. Ruoso and G. Zavattini, “The PVLAS experiment: A 25 year effort to measure vacuum magnetic birefringence,” Phys. Rept. 871, 1-74 (2020) doi:10.1016/j.physrep.2020.06.001 [arXiv:2005.12913 [physics.optics]].

[83] H. Abramowicz, U. Acosta, M. Altarelli, R. Aßmann, Z. Bai, T. Behnke, Y. Benhammou, T. Blackburn, S. Boogert and O. Borysov, et al. “Conceptual design report for the LUXE experiment,” Eur. Phys. J. ST 230, no.11, 2445-2560 (2021) doi:10.1140/epjs/s11734-021-00249-z [arXiv:2102.02032 [hep-ex]].