Skew Hadamard difference sets from cyclotomic strongly regular graphs

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Abstract

We find new constructions of infinite families of skew Hadamard difference sets in elementary abelian groups under the assumption of the existence of cyclotomic strongly regular graphs. Our construction is based on choosing cyclotomic classes in finite fields.

Keywords: skew Hadamard difference set, cyclotomic strongly regular graph, Gauss sum

1 Introduction

We assume that the reader is familiar with the basic theories of difference sets and strongly regular graphs (srg) as can found in [4, 6].

A difference set $D$ in an (additively written) finite group $G$ is called skew Hadamard if $G$ is the disjoint union of $D$, $-D$, and $\{0\}$. The primary example (and for many years, the only known example in abelian groups) of skew Hadamard difference sets is the classical Paley (quadratic residue) difference set in $(\mathbb{F}_q, +)$ consisting of the nonzero squares of $\mathbb{F}_q$, where $\mathbb{F}_q$ is the finite field of order $q$, a prime power congruent to 3 modulo 4. Skew Hadamard difference sets are currently under intensive study, see [7, 8, 9, 11, 13, 14, 18, 23, 24]. There were two major conjectures in this area: (i) If an abelian group $G$ contains a skew Hadamard difference set, then $G$ is necessarily elementary abelian. (ii) Up to equivalence the Paley difference sets mentioned above are the only skew Hadamard difference sets in abelian groups. The former conjecture is still open in general. The latter conjecture turned out to be false: Ding and Yuan [8] constructed a family of skew Hadamard difference sets in $(\mathbb{F}_{3^m}, +)$, where $m \geq 3$ is odd, and showed that two examples in the family are inequivalent to the Paley difference sets. Very recently, Muzychuk [18] constructed infinitely many inequivalent skew Hadamard difference sets in an elementary abelian group of order $q^3$. The reader may check the introduction of [13] for a good short survey of known constructions of skew Hadamard difference sets and related problems.

A classical method for constructing both connection sets of strongly regular Cayley graphs (called partial difference sets) and ordinary difference sets in the additive groups of finite fields is to use cyclotomic classes of finite fields. Let $p$ be a prime, $f$ a positive integer, and let $q = p^f$. Let $k > 1$ be an integer such that $k|(q - 1)$, and $\gamma$ be a primitive root of $\mathbb{F}_q$. Then the cosets $C_i^{(k,q)} = \gamma^i(C_k)$, $0 \leq i \leq k - 1$, are called the cyclotomic classes of order $k$ of $\mathbb{F}_q$. Many authors have studied the problem of determining when a union $D$ of some cyclotomic classes forms a (partial) difference set. Especially, when $D$ consists of only a subgroup of $\mathbb{F}_q$, many authors have studied extensively $[1, 2, 12, 14, 15, 17, 20, 21, 22]$. We call such a strongly regular Cayley graph Cay$(\mathbb{F}_q, D)$ cyclotomic. The well known Paley graphs are primary examples of cyclotomic srgs. Also, if $D$ is

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the multiplicative group of a subfield of \( \mathbb{F}_q \), then it is clear that \( \text{Cay}(\mathbb{F}_q, D) \) is strongly regular. These cyclotomic srgs are usually called subfield examples. Next, if there exists a positive integer \( t \) such that \( p^t \equiv -1 \pmod{k} \), then \( \text{Cay}(\mathbb{F}_q, D) \) is strongly regular. This case is usually called semi-primitive. In [20], Schmidt and White conjectured that if \( k \mid q-1 \) and \( \text{Cay}(\mathbb{F}_p, C_0^{(k,q)}) \) is strongly regular, then one of the following holds:

1. (subfield case) \( C_0 = \mathbb{F}_p^* \) where \( d \mid f \),
2. (semi-primitive case) \(-1 \in \langle p \rangle \leq (\mathbb{Z}/k\mathbb{Z})^*\),
3. (exceptional case) \( \text{Cay}(\mathbb{F}_p, C_0) \) has one of the parameters given in Table 1.

Table 1: Eleven sporadic examples

| No. | \( k \) | \( p \) | \( f \) | \( e := (\mathbb{Z}/k\mathbb{Z})^* : \langle p \rangle \) |
|-----|-----|-----|-----|-----------------|
| 1   | 11  | 3   | 5   | 2               |
| 2   | 19  | 5   | 9   | 2               |
| 3   | 35  | 3   | 12  | 2               |
| 4   | 37  | 7   | 9   | 4               |
| 5   | 43  | 11  | 7   | 6               |
| 6   | 67  | 17  | 33  | 2               |
| 7   | 107 | 3   | 53  | 2               |
| 8   | 133 | 5   | 18  | 6               |
| 9   | 163 | 41  | 81  | 2               |
| 10  | 323 | 3   | 144 | 2               |
| 11  | 499 | 5   | 249 | 2               |

Recently, in [12, 14, 15, 19], it was succeeded to generalize the sporadic examples of Table 1 except for the srg of No. 1 and several subfield examples into infinite families using “index 2 or 4 Gauss sums” and “relative Gauss sums.” Also, Wu [25] gave a necessary and sufficient condition for \( \text{Cay}(\mathbb{F}_{p(k-1)/e}, C_0^{(p^1,p(k-1)/e)}) \) to be strongly regular by generalizing the method of [15] when \( k \) is a prime. On the other hand, in [13, 14], Feng, Xiang, and this author found new constructions of skew Hadamard difference sets via a computation of a character sum involving index 2 Gauss sums. In particular, in [13, 14], it was shown that \( D = \bigcup_{i \in \langle p \rangle \cup \langle p \rangle \cup \langle 2 \rangle} C_i^{(k,p)} \) is a skew Hadamard difference sets or a Paley type partial difference sets for the triples \((k,p,f)\) of Table 2 and these examples can be generalized into infinite families. (A partial difference set \( D \) in a group \( G \) is said to be of Paley type if the parameters of the corresponding strongly regular Cayley graph are \((v,(v-1)/2,(v-5)/4,(v-1)/4))\). Now, one may recognize an interesting interaction between

Table 2: Skew Hadamard difference sets and Paley type partial difference sets from index 2 case

| No. | \( k \) | \( p \) | \( f \) |
|-----|-----|-----|-----|
| 1   | 2 \cdot 11 | 3   | 5   |
| 2   | 2 \cdot 19  | 5   | 9   |
| 3   | 2 \cdot 67  | 17  | 33  |
| 4   | 2 \cdot 107 | 3   | 53  |
| 5   | 2 \cdot 163 | 41  | 81  |
| 6   | 2 \cdot 499 | 5   | 249 |
cyclo
tic srgs and skew Hadamard difference sets of Tables 1 and 2 for odd primes $p$ and $p_1$ such that $p$ is of index 2 modulo $p_1$, the graph $\text{Cay}(F_q, C^{(p, p_1^{-1/2})}_0)$ is strongly regular if and only if $D = \bigcup_{i \in \{0\} \cup \langle 2 \rangle} C^{(2p_1, p_1^{-1/2})}_{i}$ is a skew Hadamard difference set or a Paley type partial difference set in $F_q$.

In this paper, we investigate such a relation between cyclotomic srgs and skew Hadamard difference sets, and find new constructions of infinite families of skew Hadamard difference sets from known cyclotomic srgs.

2 Background

Let $p$ be a prime, $f$ a positive integer, and $q = p^f$. The canonical additive character $\psi$ of $F_q$ is defined by

$$\psi: F_q \rightarrow \mathbb{C}^*, \quad \psi(x) = \zeta_p^{\text{Tr}_{q/p}(x)},$$

where $\zeta_p = \exp(\frac{2\pi i}{p})$ and $\text{Tr}_{q/p}$ is the trace from $F_q$ to $F_p$. For a multiplicative character $\chi_k$ of order $k$ of $F_q$, we define the Gauss sum

$$G_f(\chi_k) = \sum_{x \in F_q} \chi_k(x)\psi(x),$$

which belongs to the ring $\mathbb{Z}[\zeta_{kp}]$ of integers in the cyclotomic field $\mathbb{Q}(\zeta_{kp})$. Let $\sigma_{a,b}$ be the automorphism of $Q(\zeta_{kp})$ determined by

$$\sigma_{a,b}(\zeta_k) = \zeta_k^a, \quad \sigma_{a,b}(\zeta_p) = \zeta_p^b$$

for $\gcd(a, k) = \gcd(b, p) = 1$. Below are several basic properties of Gauss sums [3]:

(i) $G_f(\chi_k)\overline{G_f(\chi_k)} = q$ if $\chi$ is nontrivial;
(ii) $G_f(\chi_k) = G_f(\chi_k)$, where $p$ is the characteristic of $F_q$;
(iii) $G_f(\chi_k^{-1}) = \chi_k(-1)\overline{G_f(\chi_k)}$;
(iv) $G_f(\chi_k) = -1$ if $\chi_k$ is trivial;
(v) $\sigma_{a,b}(G_f(\chi_k)) = \chi_k^{-a}(b)G_f(\chi_k^a)$.

In general, to explicitly evaluate Gauss sums is very difficult. There are only a few cases where the Gauss sums have been evaluated. The most well known case is quadratic case, in other words, the order of $\chi$ is two. In this case, as can found in [3, Theorem 11.5.4], it holds that

$$G_f(\chi_k) = (-1)^{f-1} \left( \sqrt{(-1)^{\frac{k}{2}}} \right)^f. \quad (2.1)$$

The next simple case is the so-called semi-primitive case (also referred to as uniform cyclotomoy or pure Gauss sum), where there exists an integer $j$ such that $p^j \equiv -1 \pmod k$, where $k$ is the order of the multiplicative character $\chi$ involved. The explicit evaluation of Gauss sums in this case is given in [3]. The next interesting case is the index 2 case where the subgroup $\langle p \rangle$ generated by $p \in (\mathbb{Z}/k\mathbb{Z})^*$ is of index 2 in $(\mathbb{Z}/k\mathbb{Z})^*$ and $-1 \not\in \langle p \rangle$. In this case, it is known that $k$ can have at most two odd prime divisors. Many authors have investigated this case, see [26] for the complete solution to the problem of evaluating index 2 Gauss sums. Recently, these index 2 Gauss sums were applied to show the existence of infinite families of new strongly regular graphs and skew Hadamard difference sets in $F_q$ [12, 13, 14].

Now we recall the following well-known lemmas in the theories of difference sets and strongly regular graphs (see e.g., [3, 14]).
Lemma 2.1. Let \((G, +)\) be an abelian group of odd order \(v\), \(D\) a subset of \(G\) of size \(v - 1\). Assume that \(D \cap -D = \emptyset\). Then, \(D\) is a skew Hadamard difference set in \(G\) if and only if
\[
\psi(D) = \frac{-1 \pm \sqrt{-v}}{2}
\]
for all nontrivial characters \(\psi\) of \(G\). On the other hand, assume that \(0 \notin D\) and \(-D = D\). Then \(D\) is a Paley type partial difference set in \(G\) if and only if
\[
\psi(D) = \frac{-1 \pm \sqrt{v}}{2}
\]
for all nontrivial characters \(\psi\) of \(G\).

Lemma 2.2. Let \((G, +)\) be an abelian group and \(D\) a subset of \(G\). Then, \(\text{Cay}(G, D)\) is a strongly regular graph if and only if the size of the set
\[
\{\psi(D) | \psi \in \hat{G} \setminus \{\psi_0\}\}
\]
is exactly two, where \(\hat{G}\) is the character group of \(G\) and \(\psi_0\) is the trivial character.

Let \(q\) be a prime power and let \(C^{(k,q)}_i = \gamma^i(\gamma^k), 0 \leq i \leq k - 1\), be the cyclotomic classes of order \(k\) of \(\mathbb{F}_q\), where \(\gamma\) is a fixed primitive root of \(\mathbb{F}_q\). In this paper, we assume that \(D\) is a union of cyclotomic classes of order \(k\) of \(\mathbb{F}_q\). In order to check whether a candidate subset \(D = \bigcup_{i \in I} C^{(k,q)}_i\) is a skew Hadamard difference set or a Paley type partial difference set, we will compute the sums
\[
\psi(aD) = \sum_{x \in D} \psi(ax)
\]
for all \(a \in \mathbb{F}_q^*\), where \(\psi\) is the canonical additive character of \(\mathbb{F}_q\), because of Lemma 2.1. Similarly, to check whether \(D\) is a connection set of a strongly regular Cayley graph, we should compute the sums \(\psi(aD)\) for all \(a \in \mathbb{F}_q^*\) by Lemma 2.2. Note that the sum \(\psi(aD)\) can be expressed as a linear combination of Gauss sums (cf. [25, Lemma 3.1]) by using the orthogonality of characters:
\[
\psi(aD) = \frac{1}{k} \sum_{\chi \in C^{(k,q)}_0} \sum_{i \in I} \chi(a \gamma^i),
\]
where \(C^{(k,q)}_0\) is the subgroup of \(\hat{\mathbb{F}}_q\) consisting of all \(\chi\) which are trivial on \(C^{(k,q)}_0\). Thus, the computation to know whether a candidate subset \(D = \bigcup_{i \in I} C^{(k,q)}_i\) is a skew Hadamard difference set or a Paley type partial difference set is essentially reduced to evaluating Gauss sums. In fact, in [13, 14], known evaluation of index 2 Gauss sums are used. However, as previously said, to explicitly evaluate Gauss sums is very difficult. In this paper, we will show the existence of skew Hadamard difference sets and Paley type partial difference sets without computing explicit values of Gauss sums. Instead, we use the following theorem, called the Davenport-Hasse product formula.

Theorem 2.3. Let \(\eta\) be a multiplicative character of order \(\ell > 1\) of \(\mathbb{F}_q = \mathbb{F}_{p^{f}}\). For every nontrivial character \(\chi\) on \(\mathbb{F}_q\),
\[
G_f(\chi) = \frac{G_f(\eta)}{\chi^{\ell}} \prod_{i=1}^{\ell-1} \frac{G_f(\eta^i)}{G_f(\chi \eta^i)}.
\]

3 Construction of skew Hadamard difference sets

To show our main theorem, we use the following result of [20]. (They gave this result in terms of irreducible cyclic codes.)
Proposition 3.1. ([29] Lemma 2.8, Corollary 3.2]) Let \( m \) be the order of \( p \) modulo \( k \) and set \( q = p^f = p^{sm} \). Assume that \( k \) is odd and \( \frac{p^f - 1}{p - 1} \) is odd and \( \text{Cay}(\mathbb{F}_{p^f}, C_0^{(k,q)}) \) is strongly regular. Then, for a system \( L \) of coset representatives of \( \mathbb{F}_{p^f}/C_0^{(k,p^f)} \), there exists a partition \( L_1 \cup L_2 = L \) such that

\[
G_f(\chi_k) = \epsilon p^{\theta} \sum_{x \in L_1} \chi_k(x) = -\epsilon p^{\theta} \sum_{x \in L_2} \chi_k(x),
\]

where \( \epsilon = \pm 1 \) and \( \theta \) is the integer such that \( p^\theta || G_m(\chi_k) \). (In this case, \( p^\theta || G_f(\chi_k) \) also holds.) Furthermore, if \( |L_1| = k - d \) and \( |L_2| = d \), then it holds that

\[
k \cdot \psi(\gamma^\theta C_0^{(k,q)}) + 1 = \sum_{\chi \in C_0^{(k,q)}} \chi(\gamma^\theta) G_f(\chi^{-1})
\]

\[
= p^\theta \epsilon d \quad \text{or} \quad p^\theta \epsilon (d - k).
\]

Remark 3.2. Note that \( L_1 \) and \( L_2 \) are cyclic difference sets in \( \mathbb{F}_{p^f}/C_0^{(k,p^f)} \) since \( \chi_k(L_i) \chi_k(L_i) = G_f(\chi_k)G_f(\chi_k)/p^{2\theta d} = p^{f(j - 2d)} \). As determined in [10] [29], the corresponding cyclic \((v, k, \lambda)\) difference sets with \( k \leq (v - 1)/2 \) to cyclotomic strongly regular graphs of the Schmidt-White conjecture are as follows:

1. (subfield case) the Singer \((\frac{p^{f-1}}{p-1}, \frac{p^{f-d}}{p-1}, \frac{p^{f-2d}}{p-1})\) difference set;
2. (semi-primitive case) the trivial \((v, 1, 0)\) difference set;
3. (exceptional case) see Table 3

| No. | \( v \) | \( k \) | \( \lambda \) | Name |
|-----|-----|-----|-----|-----|
| 1   | 11  | 5   | 2   | Quadratic residue difference set [5] Theorem 1.12 |
| 2   | 19  | 9   | 4   | Quadratic residue difference set |
| 3   | 35  | 17  | 8   | Twin-prime difference set [5] Theorem 8.2 |
| 4   | 37  | 9   | 2   | Biquadratic residue difference set [5] Theorem 8.11 |
| 5   | 43  | 21  | 10  | Hall’s sextic difference set [5] Theorem 8.3 |
| 6   | 67  | 33  | 16  | Quadratic residue difference set |
| 7   | 107 | 53  | 26  | Quadratic residue difference set |
| 8   | 133 | 33  | 8   | Quadratic residue difference set |
| 9   | 163 | 81  | 40  | Hall’s sporadic difference set [5] Remarks 8.21(b) |
| 10  | 323 | 161 | 80  | Twin-prime difference set |
| 11  | 499 | 249 | 124 | Quadratic residue difference set |

The following is our main theorem of this paper.

Theorem 3.3. We assume that \( L_i \cap C_0^{(k,q)} = \emptyset \). Let \( L' = \{ y \mod k \mid y \in L_i; 0 \leq y \leq q - 2 \} \) and let \( I \) be the \(|L_i|\)-element set of odd integers modulo \( 2k \) such that \( I \mod k = L' \). Set

\[
J = \{ 0 \} \cup I \cup 2(\mathbb{Z}/k\mathbb{Z}) \backslash 2^{-1} \cdot (L' \cup \{ 0 \}) \mod 2k.
\]

Then, \( D = \bigcup_{J \in J} C_j^{(2k,q)} \) in \( \mathbb{F}_q \) is a skew Hadamard difference set or a Paley type partial difference set according to \( q \equiv 3 \mod 4 \) or \( q \equiv 1 \mod 4 \), i.e., it holds that

\[
\psi(D) = \frac{-1 \pm \sqrt{q}}{2}.
\]
Proof: First of all, we observe the following facts:

(1) It is clear that $J \mod k = \{0, 1, \ldots, k-1\}$. In particular, if $q \equiv 3 \mod 4$, i.e., $-1 \in C_k^{(2k,q)}$, it follows that $\mathbb{F}_q = \{0\} \cup D \cup -D$.

(2) By the Davenport-Hasse product formula, it holds that

$$G_f(\chi_{2k}) = \frac{G_f(\chi_k)G_f(\chi_2)}{\chi_k(2)G_f(\chi_k^{2-1})}.$$ 

Then, by noting that $G_f(\chi_k^{2-1})G_f(\chi_k^{-2-1}) = \chi_k^{-1}(1-q)$ and the restriction of $\chi_k$ to $\mathbb{F}_p$ is trivial, it follows that

$$G_f(\chi_{2k}) = \frac{1}{q}G_f(\chi_2)G_f(\chi_k)G_f(\chi_k^{-2-1}). \quad (3.2)$$

(3) The sum $\sum_{y \in J} \chi_{2k}^x(\gamma^y)$ for any $x$ such that $2, k \not\mid x$ is computable by using Proposition 3.1 as follows:

$$\sum_{y \in J} \chi_{2k}^x(\gamma^y) = \sum_{y \in J} (-1)^y \chi_{2k}^{x2-1}(\gamma^y)$$

$$= 1 - \sum_{y \in L'} \chi_{2k}^{x2-1}(\gamma^y) + \sum_{y \in (\mathbb{Z}/k\mathbb{Z}) \setminus (L' \cup \{0\})} \chi_{2k}^{x2-1}(\gamma^y)$$

$$= -2 \sum_{y \in L'} \chi_{2k}^{x2-1}(\gamma^y)$$

$$= -2 \sum_{\omega \in L_1} \chi_{2k}^{-x2-1}(\omega)$$

$$= (-1)^2 2eG(\chi_{2k}^{-x2-1})/p^q. \quad (3.3)$$

Now, we compute the sum

$$T_a = \sum_{2, k \not\mid x} G_f(\chi_{2k}^{-x}) \sum_{y \in J} \chi_{2k}^x(\gamma^{a+y}).$$

By (3.2) and (3.3), we have

$$T_a = (-1)^{a+i} \frac{2}{pq^q} \sum_{2, k \not\mid x} G_f(\chi_{2k}^{-x})G(\chi_{2k}^{-x2-1})\chi_{2k}^{x2-1}(\gamma^a)$$

$$= (-1)^{a+i} \frac{2}{pq^q} \sum_{x=1}^{k-1} G_f(\chi_{2k}^{-x})G(\chi_{2k}^{-x2-1})G(\chi_{2k}^{-x2-1})\chi_{2k}^{x2-1}(\gamma^a)$$

$$= (-1)^{a+i} \frac{2}{pq^q} \sum_{x=1}^{k-1} G_f(\chi_{2k}^{-x})\chi_{2k}^{x2-1}(\gamma^{2-1}a)$$

$$= (-1)^{a+i} \frac{2}{pq^q} (k \cdot \psi(\gamma^{2-1}a C_0^{(k,q)}) + 1),$$

By Proposition 3.1, we have
where we used $G_f(x^{x^{-1}})G_f(x^{-x^{-1}}) = x^{x^{-1}}(-1)^g$. Then, by (3.1), we obtain

$$k(2 \cdot \psi(aD) + 1) = \sum_{\ell=1}^{2k-1} G_f(x^{\ell}) \sum_{y \in J} \chi_{2k}(\gamma^{a+y})$$

$$= G_f(x^2) \sum_{y \in J} \chi_2(\gamma^{a+y}) + T_a$$

$$= (-1)^sG_f(x^2)(k-2|L_i|)$$

$$+ 2(-1)^aG_f(x^2)(k \cdot \psi(\gamma^{2^{-1}aC_0(k,q)} + 1)$$

$$= \pm(-1)^skG_f(x^2).$$

By (2.1), we obtain

$$\psi(aD) = \frac{-1 \pm \sqrt{\delta q}}{2},$$

where $\delta = 1$ or $-1$ according to $q \equiv 3 \pmod{4}$ or $\equiv 1 \pmod{4}$. This completes the proof. \(\square\)

Applying our theorem to subfield examples of cyclotomic strongly regular graphs, we obtain skew Hadamard difference sets in $\mathbb{F}_q$ for any $q = p^st$ with $st \geq 3$ by a nontrivial and different cyclotomic construction from that of the Paley difference sets although we do not know the constructed difference sets are inequivalent or not.

Also, we may obtain an infinite family of skew Hadamard difference sets starting from each skew Hadamard difference set of Theorem 3.3 by applying the following theorem.

**Theorem 3.4.** (12) Let $h = 2p_1$ with an odd prime $p_1$ and let $p$ be a prime such that $(p)$ is of index $e$ modulo $h$. Furthermore, let $k = 2p_1^m$ and assume that $(p)$ is again of index $e$ modulo $k$. Put $q = p^{(p_1-1)/e}$ and $q' = p^{m^{-1}(p_1-1)/e}$. Define $J$ as any subset of $\{0, 1, \ldots, h-1\}$ such that $J$ (mod $p_1$) $\{0, 1, \ldots, p_1 - 1\}$. Let

$$D = \bigcup_{i \in J} C_i^{(h,q)} \quad \text{and} \quad D' = \bigcup_{i_1=0}^{p_1^{-1}} \bigcup_{i \in J} C_{2i_1+ik/h}^{(k,q')}.$$  

If $D$ is a skew Hadamard difference set or a Paley type partial difference set in $\mathbb{F}_q$, then so does $D'$ in $\mathbb{F}_{q'}$.

By combining Theorems 3.3 and 3.4, we immediately have the following corollary, which yields an infinite family of skew Hadamard difference sets from a cyclotomic strongly regular graph.

**Corollary 3.5.** Let $k = p_1^m$ and let $p$ be of index $e$ both of modulo $p_1$ and $k$. Put $q = p^{(p_1-1)/e}$, $q' = p^{m^{-1}(p_1-1)/e}$, and

$$D = \bigcup_{i=0}^{p_1^{m-1}-1} \bigcup_{j \in J} C_{2i+p_1^{m-1}j}^{(2k,q)}.$$  

where $J$ is defined as in Theorem 3.3. If $\text{Cay}(\mathbb{F}_q,C_0^{(p_1,q)})$ is strongly regular, then $D$ in $\mathbb{F}_q$ is a skew Hadamard difference set or a Paley type partial difference set according to $q \equiv 3 \pmod{4}$ or $\equiv 1 \pmod{4}$.

**Example 3.6.** By Corollary 3.5, we obtain new constructions of infinite families of skew Hadamard difference sets and Paley type partial difference sets for the quadruples $(p_1, p, f, e)$ of No. 2, 4, 5, 6, 7, 9, and 11 in Table 1. Note that we can not obtain an infinite family of skew Hadamard difference sets from the cyclotomic srg of No. 1 because $p$ is not of index 2 in $\mathbb{Z}/2p_1^m\mathbb{Z}$ for $m \geq 2$ while $\bigcup_{j \in \{0\} \cup \{p_1\} \cup \{2(p_1, p')\}} C_j^{(2p_1, p')} \times j$ forms a skew Hadamard difference set.
Also, there are a lot of subfield examples satisfying \([(\mathbb{Z}/p_1\mathbb{Z})^* : \langle p \rangle] = e\) and \(p_1 = \frac{\phi(p_1^t) - 1}{(p_1 - 1)/e}\) for some \(t \mid (p_1 - 1)/e\). We list ten examples satisfying these conditions in Table 4. From these examples, we obtain infinite families of skew Hadamard difference sets and Paley type partial difference sets by Corollary 3.3.

### 4 Concluding remarks and open problems

In this section, we give important remarks and open problems related to our results.

**Remark 4.1.** In [19], the author found two examples of skew Hadamard difference sets of index 4, those are, \(\bigcup_{j \in \{p_1\} \cup Q \cup 2Q} C^{(2p_1,p^f)}_j\) for \((p_1, p, f) = (13, 3, 3)\) and \(\bigcup_{j \in \{0\} \cup Q \cup 2Q} C^{(2p_1,p^f)}_j\) for \((p_1, p, f) = (29, 7, 7)\), where \(Q\) is the subgroup of index 2 of \((\mathbb{Z}/2p_1\mathbb{Z})^*\). These two examples are not covered by Theorem 3.3 i.e., there do not exist corresponding cyclotomic strongly regular graphs and cyclic difference sets. More generally, via a computation similar to [15] involving known evaluations of index 4 Gauss sums, one can prove that either of \(\bigcup_{j \in \{0\} \cup Q \cup 2Q} C^{(2p_1,p^{(p_1-1)/4})}_j\) or \(\bigcup_{j \in \{p_1\} \cup Q \cup 2Q} C^{(2p_1,p^{(p_1-1)/4})}_j\) is a skew Hadamard difference set or a Paley type partial difference set in \(\mathbb{F}^{p^{(p_1-1)/4}}\) if the following conditions are fulfilled:

(i) \(p\) is of index 4 modulo \(p_1\),

(ii) \(p_1 = 4p(p_1-1)/4-2b + 1\), where \(b\) is defined as

\[
b = \min \left\{ \frac{1}{p_1} \sum_{x \in S} x \mid S \in (\mathbb{Z}/p_1\mathbb{Z})^*/\langle p \rangle \right\},
\]

(iii) \(p_1 = A^2 + 4\) for some integer \(A \equiv 3 \pmod{4}\).

The author found only three examples satisfying these conditions, which are

\((p_1, p, f) = (13, 3, 3), (29, 7, 7), (53, 13, 13)\).

For each of these three examples, we obtain an infinite family of skew Hadamard difference sets or Paley type partial difference sets by applying Theorem 3.3. Here, we have the following natural question.

**Problem 4.2.** Determine for which \((p, p_1, e)\) either \(\bigcup_{j \in \{0\} \cup Q \cup 2Q} C^{(2p_1,p^{(p_1-1)/e})}_j\) or \(\bigcup_{j \in \{p_1\} \cup Q \cup 2Q} C^{(2p_1,p^{(p_1-1)/e})}_j\) forms a skew Hadamard difference set or a Paley type partial difference set.

Also, by computer, the author found an interesting example of a skew Hadamard difference set in the case where \((p, f, p_1) = (7, 3, 19)\) and \(e = 6\):

\[D = \bigcup_{x \in I} C^{(2p_1,p^f)}_i,\]
where
\[ I = \{ p_1 \} \cup \langle p \rangle \cup 3 \langle p \rangle \cup 3^3 \langle p \rangle \cup 2 \cdot 3 \langle p \rangle \cup 2 \cdot 3^3 \langle p \rangle \cup 2 \cdot 3^4 \langle p \rangle \pmod{2p_1}. \]

One can use a computer to find that the automorphism group of the symmetric design \( \text{Dev}(D) \) derived from \( D \) has size \( 3^4 \cdot 7^3 \). (We will write the size as \( \# \text{Aut}(\text{Dev}(D)) \).) On the other hand, \( \# \text{Aut}(\text{Dev}(P)) \) = \( 3^3 \cdot 7^3 \cdot 19 \) for the Paley difference set \( P \) with the same parameter. Thus, the skew Hadamard difference set \( D \) is inequivalent to the Paley difference set. Furthermore, since the size of the Sylow \( p \)-subgroup of the automorphism group of the design derived from a difference set constructed by Muzychuk [18] is strictly greater than \( q \), we conclude that \( D \) is also inequivalent to the corresponding skew Hadamard difference sets of [18]. Also, since the set \( I \) satisfies \( I \pmod{p_1} = \{ 0, 1, \ldots, p_1 - 1 \} \), we obtain an infinite family of skew Hadamard difference sets including this example by Theorem 3.4.

**Remark 4.3.** As described in Introduction, to check whether obtained skew Hadamard difference sets and Paley type partial difference sets are equivalent or not to the classical Paley (partial) difference sets is very important. Although the problem is in general difficult and the author could not prove that our construction always yields inequivalent skew Hadamard difference sets and Paley type partial difference sets to the Paley (partial) difference sets, the author still believes that our infinite families include inequivalent ones abundantly. As an evidence for my believe, we can see by computer that the skew Hadamard difference set \( D = \bigcup_{x \in J} C^{(2p_1, p')}_{i_x} \) with
\[
J = \{ 0 \} \cup \left( \bigcup_{i \in I} g^i \langle p \rangle \right) \cup \left( 2 \bigcup_{i \in (Z/eZ) \setminus I} g^{i-s} \langle p \rangle \right) \pmod{2p_1},
\]
where assume that \((Z/kZ)^*/\langle p \rangle\) is a cyclic group of order \( e \) and let \( g \) be a representative of a generator of \((Z/kZ)^*/\langle p \rangle\), is inequivalent to the Paley difference set in the following cases:

- \((p, f, p_1) = (3, 5, 11), (g, s) = (-1, 1) \) and \( I = \{ 0 \} \): In this case, \( \# \text{Aut}(\text{Dev}(D)) = 3^5 \cdot 5 \cdot 11 \) and \( \# \text{Aut}(\text{Dev}(P)) = 3^5 \cdot 5 \cdot 11^2 \) for the corresponding Paley difference set \( P \).
- \((p, f, p_1) = (3, 7, 1093), (g, s) = (5, 63) \) and take \( I \) as \( \bigcup_{i \in I} g^i \langle p \rangle = 5 \cdot (S + 948) \) for the Singer difference set \( S \) of \( \text{PG}(6,3) \): In this case, \( \# \text{Aut}(\text{Dev}(D)) = 3^7 \cdot 7 \) and \( \# \text{Aut}(\text{Dev}(P)) = 3^7 \cdot 7 \cdot 1093 \) for the corresponding Paley difference set \( P \).
- \((p, f, p_1) = (7, 5, 2801), (g, s) = (3, 58) \) and take \( I \) as \( \bigcup_{i \in I} g^i \langle p \rangle = 3^{58} \cdot (S + 292) \) for the Singer difference set \( S \) of \( \text{PG}(4,7) \): In this case, \( \# \text{Aut}(\text{Dev}(D)) = 3 \cdot 5 \cdot 7^5 \) and \( \# \text{Aut}(\text{Dev}(P)) = 3 \cdot 5 \cdot 7^5 \cdot 2801 \) for the corresponding Paley difference set \( P \).

Furthermore, the author checked by computer that the Paley type srgs with parameters \((p_1, p, f) = (31, 5, 3)\) and \((307, 17, 3)\) of Example 3.6 are not isomorphic (as graph isomorphism) to the classical Paley graphs. (Note that in these cases there is no factor \( m > 2 \) of \( p^f - 1 \) such that \( p \) is semi-primitive modulo \( m \).)

**Problem 4.4.** Determine whether or not skew Hadamard difference sets and Paley type partial difference sets obtained in this paper are equivalent to the classical Paley (partial) difference sets.

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