EXISTENCE OF POSITIVE SOLUTION FOR A CLASS OF NONLOCAL ELLIPTIC PROBLEMS IN THE HALF SPACE WITH A HOLE

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Abstract This work concerns with the existence of solutions for the following class of nonlocal elliptic problems

\[
\begin{aligned}
(-\Delta)^s u + u &= |u|^{p-2}u \text{ in } \Omega_r \\
\geq 0 \text{ in } \Omega_r \text{ and } u \neq 0 \\
u &= 0 \quad \mathbb{R}^N \setminus \Omega_r
\end{aligned}
\]

involving the fractional Laplacian operator \((-\Delta)^s\), where \(s \in (0, 1), N > 2s, \Omega_r\) is the half space with a hole in \(\mathbb{R}^N\) and \(p \in (2, 2^*_s)\). The main technical approach is based on variational and topological methods.

Keywords: Nonlocal elliptic problems, Positive high energy solution, Half space with a hole

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1. Introduction

Let \(\mathbb{R}^N_+ = \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid 0 < x_N < \infty\}\) be the upper half space. \(\Omega_r\) is an unbounded smooth domain such that

\[\Omega_r \subset \mathbb{R}^N_+\]

and

\[\mathbb{R}^N_+ \setminus \Omega_r \subset B_r(a_r) \subset \mathbb{R}^N_+\]

with \(a_r = (a, r) \in \mathbb{R}^N_+\). Indeed, \(\Omega_r\) is the upper half space with a hole.

We consider the following fractional elliptic problem:

\[
\begin{aligned}
(-\Delta)^s u + u &= |u|^{p-2}u \text{ in } \Omega \\
\geq 0 \text{ in } \Omega_r \text{ and } u \neq 0 \\
u &= 0 \quad \mathbb{R}^N \setminus \Omega
\end{aligned}
\]

where \(\Omega = \Omega_r, s \in (0, 1), N > 2s, p \in (2, 2^*_s), \) where \(2^*_s = \frac{2N}{N-2s}\) is the fractional critical Sobolev exponent and \((-\Delta)^s\) is the classical fractional Laplace operator.

When \(s \nearrow 1^-\), problem (1.1) is related to the following elliptic problem

\[-\Delta u + u = |u|^{p-1}u, \quad x \in \Omega, \quad u \in H^1_0(\Omega)\]

. When \(\Omega\) is a bounded domain, by applying the compactness of the embedding \(H^1_0(\Omega) \hookrightarrow L^p(\Omega), 1 < p < \frac{2N}{N-2}\), there is a positive solution of (1.2). If \(\Omega\) is an unbounded domain, we can not obtain a solution for problem (1.2) by using Mountain-Pass Theorem directly

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because the embedding \( H^1_0(\Omega) \to L^p(\Omega), 1 < p < \frac{2N}{N-2} \) is not compactness. However, if \( \Omega = \mathbb{R}^N \), Berestycki-Lions [1], proved that there is a radial positive solution of equation (1.2) by applying the compactness of the embedding \( H^1_0(\mathbb{R}^N) \to L^p(\mathbb{R}^N), 2 < p < \frac{2N}{N-2} \), where \( H^1_0(\mathbb{R}^N) \) consists of the radially symmetric functions in \( H^1(\mathbb{R}^N) \). By the P.L.Lions’s Concentration-Compactness Principle [14], there exists an unique positive solution for problem (1.2) in \( \mathbb{R}^N \). By moving Plane method, Gidas-Ni-Nirenberg [12] also proved that every positive solution of equation

\[-\Delta u + u = |u|^{p-1}u, \quad x \in \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N)\]  

(1.3)

is radially symmetric with respect to some point in \( \mathbb{R}^N \) satisfying

\[u(r)r^\gamma = \gamma + o(1) \text{ as } r \to \infty.\]  

(1.4)

Kwong [15] proved that the positive solution of (1.3) is unique up to translations.

In fact, Esteban and Lions [8] proved that there is not any nontrivial solution of equation (1.2) when \( \Omega \) is an Esteban-Lions domain (for example \( \mathbb{R}^3_+ \)). Thus, we want to change the topological property of the domain \( \Omega \) to look for a solution of problem (1.2). Wang [18] proved that if \( \rho \) is sufficiently small and \( z_{0N} \to \infty \), then Eq.(1.2) admits a positive higher energy solution in \( \mathbb{R}^N \setminus B_\rho(z_{01}', z_{0N}) \). Such problem has been extensively studied in recent years, see for instance, [11, 13] and references therein. From the above researches, we believed that the existence of the solution to the equation (1.2) will be affected by the topological property of the domain \( \Omega \).

Recently, the case \( s \in (0, 1) \) has received a special attention, because involves the fractional Laplacian operator \((-\Delta)^s\), which arises in a quite natural way in many different contexts, such as, among the others, the thin obstacle problem, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasigeostrophic flows, multiple scattering, minimal surfaces, materials science and water waves, for more detail see [4, 6, 7, 16, 17].

When \( \Omega \subset \mathbb{R}^N \) is an exterior domain, i.e. an unbounded domain with smooth boundary \( \partial \Omega \neq \emptyset \) such that \( \mathbb{R}^N \setminus \Omega \) is bounded, \( s \in (0, 1), N > 2s, p \in (2, 2^*_s) \), the above problem has been studied by O. Alves, Giovanni Molica Bisci, César E. Torres Ledesma in [11] proving that (1.1) does not have a ground state solution. This fact represents a serious difficulty when dealing with this kind of nonlinear fractional elliptic phenomena. More precisely, the authors analyzed the behavior of Palais-Smale sequences and showed a precise estimate of the energy levels where the Palais-Smale condition fails, which made it possible to show that the problem (1.1) has at least one positive solution, for \( \mathbb{R}^N \setminus \Omega \) small enough. A key point in the approach explored in [10, 11] is the existence and uniqueness, up to a translation, of a positive solution \( Q \) of the limit problem associated with (1.1) given by

\[(-\Delta)^s u + u = |u|^{p-2}u \text{ in } \mathbb{R}^N,\]  

(1.5)

for every \( p \in (2, 2^*_s) \). Moreover, \( Q \) is radially symmetric about the origin and monotonically decreasing in \( |x| \). On the contrary of the classical elliptic case, the exponential decay at infinity is not used in order to prove the existence of a nonnegative solution for (1.1).

When \( \Omega \subset \mathbb{R}^N \) is \( \mathbb{R}^N_+ \), by moving Plane method, Wenhong Chen, Yan Li and Pei Ma [5](p123 Theorem 6.8.3) proved that there is no nontrivial solution of problem (1.1). It is interesting in considering the existence of the high energy equation for the problem (1.1) in the half space with a hole in \( \mathbb{R}^N \).
Theorem 1.1. There is \( \rho_0 > 0, r_0 > 0 \) such that if \( 0 < \rho \leq \rho_0 \) and \( r \geq r_0 \), then there is a positive solution of equation (0.1).

The paper is organized as follows. In section 2, we give some preliminary results. The Compactness lemma will be given in Section 3. At last, we give the proof of Theorem 1.1.

2. SOME PRELIMINARY RESULTS

For \( s \in (0, 1) \) and \( N > 2s \), the fractional Sobolev space of order \( s \) on \( \mathbb{R}^N \) is defined by

\[
H^s (\mathbb{R}^N) := \left\{ u \in L^2 (\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(z)|^2}{|x - z|^{N + 2s}} \, dx \, dz < \infty \right\}
\]

endowed with the norm

\[
\| u \|_s := \left( \int_{\mathbb{R}^N} |u(x)|^2 \, dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(z)|^2}{|x - z|^{N + 2s}} \, dx \, dz \right)^{1/2}.
\]

We recall the fractional version of the Sobolev embeddings (see [9]).

Theorem 2.1. Let \( s \in (0, 1) \), then there exists a positive constant \( C = C(N, s) > 0 \) such that

\[
\| u \|_{L^{2s} (\mathbb{R}^N)} \leq C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s}} \, dy \, dx
\]

and then \( H^s (\mathbb{R}^N) \leftrightarrow L^q (\mathbb{R}^N) \) is continuous for all \( q \in [2, 2^*_c] \). Moreover, if \( \Theta \subset \mathbb{R}^N \) is a bounded domain, we have that the embedding \( H^s (\mathbb{R}^N) \hookrightarrow L^q (\Theta) \) is compact for any \( q \in [2, 2^*_c) \).

Hereafter, we denote by \( X_0^s \subset H^s (\mathbb{R}^N) \) the subspace defined by

\[
X_0^s := \left\{ u \in H^s (\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}.
\]

We endow \( X_0^s \) with the norm \( \| \cdot \|_s \). Moreover we introduce the following norm

\[
\| u \| := \left( \int_{\Omega_r^c} |u(x)|^2 \, dx + \iint_{\mathbb{Q}} \frac{|u(x) - u(z)|^2}{|x - z|^{N + 2s}} \, dx \, dz \right)^{1/2}
\]

where \( \mathbb{Q} := \mathbb{R}^{2N} \setminus (\Omega_r^c \times \Omega_r^c) \). We point out that \( \| u \|_s = \| u \| \) for any \( u \in X_0^s \). Since \( \partial \Omega \) is bounded and smooth, by [16, Theorem 2.6], we have the following result.

Theorem 2.2. The space \( C^\infty_c (\Omega) \) is dense in \( (X_0^s, \| \cdot \|) \).

In what follows, we denote by \( H^s (\Omega) \) the usual fractional Sobolev space endowed with the norm

\[
\| u \|_{H^s} := \left( \int_{\Omega} |u(x)|^2 \, dx + \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(z)|^2}{|x - z|^{N + 2s}} \, dx \, dz \right)^{1/2}.
\]

Related to these fractional spaces, we have the following properties

Proposition 2.3. The following assertions hold true:

(i) If \( v \in X_0^s \), we have that \( v \in H^s (\Omega) \) and

\[
\| v \|_{H^s} \leq \| v \|_s = \| v \|.
\]
(ii) Let $\Theta$ an open set with continuous boundary. Then, there exists a positive constant $C = C(N, s)$, such that
\[
\|v\|_{L^2_s(\Theta)} = \|v\|_{L^2_s(\mathbb{R}^N)} \leq C \int_{\mathbb{R}^2N} \frac{|v(x) - v(z)|^2}{|x - z|^{N+2s}} \, dz \, dx
\]
for every $v \in X^s_0$; see [3], Theorem 6.5.

From now on, $M_\infty$ denotes the following constant
\[
M_\infty := \inf \left\{ \|u\|_s^2 : u \in H^s(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} |u(x)|^p \, dx = 1 \right\},
\]
which is positive by Theorem 2.1. Furthermore, for any $v \in H^s(\mathbb{R}^N)$ and $z \in \mathbb{R}^N$, we set the function
\[
v^z(x) := v(x + z).
\]
Then, by doing the change of variable $\tilde{x} = x + z$ and $\tilde{y} = y + z$, it is easily seen that
\[
\|v^z\|_s^2 = \|v\|_s^2 \text{ as well as } \|v^z\|_{L^p(\mathbb{R}^N)} = \|v\|_{L^p(\mathbb{R}^N)}.
\]
Arguing as in [3] the following result holds true.

**Theorem 2.3.** Let $\{u_n\} \subset H^s(\mathbb{R}^N)$ be a minimizing sequence such that
\[
\|u_n\|_{L^p(\mathbb{R}^N)} = 1 \text{ and } \|u_n\|_s^2 \to M_\infty \text{ as } n \to +\infty.
\]
Then, there is a sequence $\{y_n\} \subset \mathbb{R}^N$ such that $\{u_{y_n}^n\}$ has a convergent subsequence, and so, $M_\infty$ is attained.

As a byproduct of the above result the next corollary is obtained.

**Corollary 1** There is $v \in H^s(\mathbb{R}^N)$ such that $\|v\|_s = M_\infty$ and $\|v\|_{L^p(\mathbb{R}^N)} = 1$.

Let $\varphi$ be a minimizer of (2.1), that is
\[
\varphi \in H^s(\mathbb{R}^N), \quad \int |\varphi|^p \, dx = 1 \text{ and } M_\infty = \|\varphi\|_s^2.
\]
Take
\[
\xi \in C^\infty(\mathbb{R}^+, \mathbb{R}), \eta \in C^\infty(\mathbb{R}, \mathbb{R}),
\]
such that
\[
\xi(t) = \begin{cases} 0, & 0 \leq t \leq \rho, \\ 1, & t \geq 2\rho, \end{cases}
\]
\[
\eta(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t \geq 1, \end{cases}
\]
and
\[
0 \leq \xi \leq 1, 0 \leq \eta \leq 1.
\]
Now, we define
\[
f_y(x) = \xi(|x - a_r|)\eta(x_N)\varphi(x - y),
\]
and
\[
\Psi_y(x) = \frac{f_y(x)}{\|f_y\|_{L^p(\mathbb{R}^N)}} = c_y f_y(x) \text{ where } c_y = \frac{1}{\|f_y\|_{L^p(\mathbb{R}^N)}}.
\]
Throughout this section we endow $X_0^s$ with the norm
\[
\|u\| := \left( \int_Q \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dy dx + \int_{\Omega_r} |u|^2 dx \right)^{1/2}
\]
and denote by $M > 0$ the number
\[
M := \inf \left\{ \|u\|^2 : u \in X_0^s, \int_{\Omega_r} |u(x)|^p dx = 1 \right\}.
\]

**Lemma 2.1.** Let $y = (y', y_N)$, we have
1. $\|f_y - \varphi(x - y)\|_{L^p(\mathbb{R}^n)} = o(1)$, $|y - a_r| \rightarrow \infty$, and $y_N \rightarrow +\infty$, or $y_N \rightarrow \infty$ and $\rho \rightarrow 0$;
2. $\|f_y - \varphi(x - y)\|_{L^p(\mathbb{R}^n)} = o(1)$, $|y - a_r| \rightarrow \infty$, and $y_N \rightarrow +\infty$, or $y_N \rightarrow +\infty$ and $\rho \rightarrow 0$.

**Proof.** Similarly as [11] [18], we have
(i) After the change of variables $z = x - y$, one has
\[
\|f_y - \varphi(x - y)\|_{L^p(\mathbb{R}^n)} = \int_{\mathbb{R}^N} |\xi(|x - a_N|)\eta(x_N) - 1|^p |\varphi(x - y)|^p dx
\]
\[
= \int_{\mathbb{R}^N} |\xi(|x + y - a_r|)\eta(x_N + y) - 1|^p |\varphi(z)|^p dz
\]
Let $g_y(z) = |\xi(|x + y - a_r|)\eta(x_N + y) - 1|^p |\varphi(z)|^p$. Since $|y - a_r| \rightarrow \infty$, and $y_N \rightarrow +\infty$, it follows that
\[
g_y(z) \rightarrow 0 \text{ a.e. in } \mathbb{R}^N
\]
Now, taking into account that
\[
g_y(z) = |\xi(|x + y - a_r|)\eta(x_N + y) - 1|^p |\varphi(z)|^p \leq 2^p |\varphi(z)|^p \in L^1(\mathbb{R}^N),
\]
the Lebesgue’s dominated convergence theorem yields
\[
\int_{\mathbb{R}^N} g_y(z) dz \rightarrow 0 \text{ as } |y - a_r| \rightarrow \infty, \ y_N \rightarrow +\infty.
\]
Therefore
\[
\|f_y - \varphi(x - y)\|_{L^p(\mathbb{R}^n)} = o(1), \ |y - a_r| \rightarrow \infty, \ y_N \rightarrow +\infty
\]
\[
\|f_y - \varphi(x - y)\|_{L^p(\mathbb{R}^n)} = \int_{B_{2^p(a)}(a)} \frac{\xi(|x - a_N|)\eta(x_N) - 1}{p} |\varphi(x - y)|^p dx
\]
\[
= \int_{B_{2^p(a)}(a)} \frac{\xi(|x - a_N|)\eta(x_N) - 1}{p} |\varphi(x - y)|^p dx
\]
and
\[
\int_{\mathbb{R}^N} \frac{\xi(|x - a_N|)\eta(x_N) - 1}{p} |\varphi(x - y)|^p dx \leq C mes B_{2^p(a)}(a) \max_{x \in \mathbb{R}^N} \varphi(x) \rightarrow 0 \text{ as } \rho \rightarrow 0,
\]
\[
\int_{\mathbb{R}^N} \frac{\xi(|x - a_N|)\eta(x_N) - 1}{p} |\varphi(x - y)|^p dx = \int_{\mathbb{R}^N} \frac{\eta(x_N) - 1}{p} |\varphi(x - y)|^p dx
\]
\[
= \int_{\mathbb{R}^N} \frac{\eta(x_N + y_N) - 1}{p} |\varphi(z)|^p dz \rightarrow 0 \text{ as } y_N \rightarrow +\infty, \ \rho \rightarrow 0.
\]
Therefore
\[
\|f_y - \varphi(x - y)\|_{L^p(\mathbb{R}^n)} = o(1), \ y_N \rightarrow \infty \text{ and } \rho \rightarrow 0.
\]
(ii) Now, we claim that
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\left(\xi(|x - a_N|)\eta(x_N) - 1\right)\varphi(x - y) - \left(\xi(|z - a_N|)\eta(z_N) - 1\right)\varphi(z - y)}{|x - z|^{N+2s}}^2 dz dx = o_n(1)
\]
Indeed, let
\[ \Upsilon_n(x, y) := \frac{u(x) - u(z)}{|x - z|^\frac{N}{2} + s}. \]

Then, after the change of variables \( \tilde{x} = x - y \) and \( \tilde{y} = z - y \), one has
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|(\xi(|x-a_r|)\eta(x_N) - \xi(|z-a_r|)\eta(z_N) - 1)\varphi(z)|}{|x-z|^\frac{N}{2} + s} d\mu^N \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\Upsilon_n(x, z)|^2 d\mu^N.
\]

where
\[
\Phi_n(x, z) := \frac{(\xi(|x + y - a_r|)\eta(x_N + y_N) - 1)\varphi(x) - (\xi(|z + y - a_r|)\eta(z_N + y_N) - 1)\varphi(z)}{|x - z|^\frac{N}{2} + s}
\]

Recalling that \(|y - a_r| \to \infty\), \(\Phi_n \to 0\) a.e. in \(\mathbb{R}^N \times \mathbb{R}^N\).

On the other hand, a direct application of the mean value theorem yields
\[
|\Upsilon_n(x, z)| \leq \left| \frac{(\xi(|x + y - a_r|)\eta(x_N + y_N) - 1)\varphi(x) - (\xi(|z + y - a_r|)\eta(z_N + y_N) - 1)\varphi(z)}{|x - z|^\frac{N}{2} + s} \right|
\]

for almost every \((x, z) \in \mathbb{R}^N \times \mathbb{R}^N\). Now, it is easily seen that the right hand side in (2.4) is \(L^2\)-integrable. Thus, By the Lebesgue’s dominated convergence theorem and i, it follows that

\[ \|f_y - \varphi(x - y)\| = o(1), |y - a_r| \to \infty, \text{ and } y_N \to +\infty \]

By i, we have \(\|f_y - \varphi(x - y)\|_{L^2(\mathbb{R}^N)} = o(1), |y - a_r| \to \infty, \text{ and } y_N \to +\infty, \text{ or } y_N \to \infty \text{ and } \rho \to 0;\)

\[
\|f_y - \varphi(x - y)\|^2 = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f_y(x) - \varphi(x - y)|^2}{|x - z|^\frac{N}{2} + s} d\mu^N
\]

Setting
\[
I_1 := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\xi(|x - a_r|)\eta(x_N) - 1)(\xi(|z - a_r|)\eta(z_N) - 1)\varphi(x - y)}{|x - z|^\frac{N}{2} + s} d\mu^N
\]

and
\[
I_2 := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|(\xi(|z - a_r|)\eta(z_N) - 1)\varphi(x - y) - (\xi(|z - a_r|)\eta(z_N) - 1)\varphi(z - y)|^2}{|x - z|^\frac{N}{2} + s} d\mu^N
\]

the following inequality holds
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|(f_y - \varphi(y - z))(x) - (f_y - \varphi(y - z))(z)|^2}{|x - z|^\frac{N}{2} + s} d\mu^N \leq I_1 + I_2.
\]

Moreover, by definition of \(\xi\), we also have
\[
(\xi(|z + y - a_r|)\eta(z + y_N) - 1)^2 \left| \frac{|\varphi(x - y) - \varphi(z)|^2}{|x - z|^\frac{N}{2} + s} \right| \leq 4 \left| \frac{|\varphi(x) - \varphi(z)|^2}{|x - z|^\frac{N}{2} + s} \right| \in L^1(\mathbb{R}^N \times \mathbb{R}^N)
\]

and
\[
(\xi(|z + y - a_r|)\eta(z + y_N) - 1)^2 \left| \frac{|\varphi(x) - \varphi(z)|^2}{|x - y|^\frac{N}{2} + s} \right| \to 0 \text{ a.e. in } \mathbb{R}^N \times \mathbb{R}^N
\]
as \(y_N \to \infty\) and \(\rho \to 0\). Hence, the Lebesgue’s theorem ensures that
\[ I_2 \to 0 \text{ as } \rho \to 0. \]
Now, by \([19],\) Lemma 2.3, for every \(y \in \mathbb{R}^N\), one has
\[
I_1 = \int_{\mathbb{R}^{2N}} \frac{(\xi(|x - a_r|)\eta(x_N) - \xi(|z - a_r|)\eta(z_N))|\varphi(x - y)|^2}{|x - z|^{N+2s}} dz dx \to 0 \text{ as } \rho \to 0.
\]
Therefore
\[
\|f_y - \varphi(x - y)\| = o(1), y_N \to +\infty, \rho \to 0.
\]

Lemma 2.2. The equality \(M_\infty = M\) holds true. Hence, there is no \(u \in X_0^s\) such that \(\|u\|^2 = M\) and \(\|u\|_{L^p(\mathbb{R}^N)} = 1\), and so, the minimization problem (2.3) does not have solution.

Proof. The proof is similar to \([18]\), and we only give a sketch here. By Proposition 2.3 - part (i) it follows that
\[
M_\infty \leq M
\]
Take a sequence \(y^n\) in \(\Omega_r\) such that
\[
|y^n - a_r| \to \infty, \text{ and } y^n_3 \to +\infty \text{ as } n \to \infty.
\]
Then by lemma 2.1 we have
\[
\|f_{y^n} - \varphi(x - y^n)\|_{L^p(\mathbb{R}^N)} = o(1), |y^n - a_r| \to \infty, \text{ and } y^n_3 \to +\infty,
\]
\[
\|f_{y^n} - \varphi(x - y^n)\|_s = o(1), |y^n - a_r| \to \infty, \text{ and } y^n_3 \to +\infty,
\]
\[
c_{y^n} = \frac{1}{\|f_{y^n}\|_{L^p(\mathbb{R}^N)}} \to 1, |y^n - a_r| \to \infty, \text{ and } y^n_3 \to +\infty
\]
Now, since \(\varphi\) is a minimizer of (2.1), one has
\[
\|f_{y^n}\|^2_s = \|\varphi(\cdot - y_n)\|^2_s + o_n(1) = \|\varphi\|^2_s + o_n(1) = M_\infty + o_n(1)
\]
Similar arguments ensure that
\[
\|\Psi_n\|^2_s = \|\Psi_n\|^2 = M_\infty + o_n(1)
\]
and
\[
\|\Psi_n\|_{L^p(\mathbb{R}^N)} = 1
\]
So
\[
M \leq M_\infty
\]
We then conclude that \(M = M_\infty\). Now, suppose by contradiction that there is \(v_0 \in X_0^s\) satisfying
\[
\|v_0\| = M \text{ and } \|v_0\|_{L^p(\Omega)} = 1.
\]
Without loss of generality, we can assume that \(v_0 \geq 0\) in \(\Omega\). Note that by \(M = M_\infty\), since \(v_0 \in H^s(\mathbb{R}^N)\) and \(\|v_0\| = \|v_0\|_s\), it follows that \(v_0\) is a minimizer for (2.1), and so, a solution of roblem
\[
\begin{aligned}
(-\Delta)^s u + u &= M_\infty u^{p-1} \text{ in } \mathbb{R}^N \\
u &\in H^s(\mathbb{R}^N).
\end{aligned}
\]
(2.5)
Therefore, by the maximum principle we get that \(v_0 > 0\) in \(\mathbb{R}^N\), which is impossible, because \(v_0 = 0\) in \(\mathbb{R}^N \setminus \Omega_r\). This completes the proof. □
3. A Compactness Lemma

In this section we prove a compactness result involving the energy functional \( I : X^s_0 \to \mathbb{R} \) associated to the main problem (0.1) and given by

\[
I(u) := \frac{1}{2} \left( \int_{\Omega_r} |u(x) - u(y)|^2 \frac{dy}{|x - y|^{N+2s}} + \int_{\Omega_r} |u|^2 \right) - \frac{1}{p} \int_{\Omega_r} |u|^p dx.
\]

In order to do this, we consider the problem

\[
\begin{align*}
\left\{ (-\Delta)^s u + u = |u|^{p-2} u \text{ in } \mathbb{R}^N \\
u \in H^s(\mathbb{R}^N)
\end{align*}
\]

whose energy functional \( I_\infty : H^s(\mathbb{R}^N) \to \mathbb{R} \) has the form

\[
I_\infty(u) := \frac{1}{2} \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x) - u(y)|^2 \frac{dy}{|x - y|^{N+2s}} + \int_{\mathbb{R}^N} |u|^2 \right) - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p dx.
\]

With the above notations we are able to prove the following compactness result.

**Lemma 3.1.** Let \( \{u_n\} \subset X^s_0 \) be a sequence such that

\[
I(u_n) \to c \text{ and } I'(u_n) \to 0 \text{ as } n \to \infty.
\]

Then there are a nonnegative integer \( k \), \( k \) sequences \( \{y^i_n\} \) of points of the form \((x^i_n, m_n + 1/2)\) for integers \( m_n, i = 1, 2, \ldots, k, \) \( u_0 \in X^s_0 \) solving equation (0.1) and nontrivial functions \( u^1, \ldots, u^k \) in \( H^s(\mathbb{R}^N) \) solving equation (3.1). Moreover there is a subsequence \( \{u_n\} \) satisfying

1. \( u_n(x) = u^0(x) + u^1(x - x^1_n) + \cdots + u^k(x - x^n_n) + o(1) \) strongly, where \( x^i_n = y^i_n + \cdots + y^i_n \to \infty, i = 1, 2, \ldots, k \)
2. \( \|u_n\|^2 = \|u^0\|^2_{\Omega_r} + \|u^1\|^2 + \cdots + \|u^k\|^2 + o(1) \)
3. \( I(u_n) = I(u^0) + I_\infty(u^1) + \cdots + I_\infty(u^k) + o(1) \)

If \( u_n \geq 0 \) for \( n = 1, 2, \ldots \), then \( u^1, \ldots, u^k \) can be chosen as positive solutions, and \( u^0 \geq 0 \)

**Proof.** See [1] [18].

**COROLLARY 2** Let \( \{u_n\} \subset M_{\Omega_r} \) satisfy \( \|u_n\|^2_{\Omega_r} = c + o(1) \) and \( M < c < 2^{(p-2)/p} M. \) Then \( \{u_n\} \) contains a strongly convergent subsequence.

**Proof.** See [1] [18].

4. Proof of Theorem 1

Set

\[
\chi(t) = \begin{cases} 
1 & \text{if } 0 \leq t \leq 1 \\
\frac{1}{t} & \text{if } 1 \leq t < \infty
\end{cases}
\]

and define \( \beta : H^s(\mathbb{R}^N) \to \mathbb{R}^N [18] \) by

\[
\beta(u) = \int_{\mathbb{R}^N} u^2(x) \chi(|x|) dx.
\]
For $r \geq r_1$, let
\[
V_r = \left\{ u \in H_0^1(\Omega_r) \mid \int_{\Omega_r} |u|^p = 1, \beta(u) = a_r \right\}
\]
and
\[
c_r = \inf_{u \in V_r} \|u\|_{\Omega_r}^2.
\]

**Lemma 4.1.** $c_r > M$.

Proof: It is easy to see that $c_r \geq M$. Suppose $c_r = \alpha$. Take a sequence $\{v_m\} \subset X_0^s$ such that for $m = 1, 2, \cdots$,
\[
\|v_m\|_{H^p(\Omega_r)} = 1, \beta(v_m) = a_r
\]
and
\[
\|v_m\|^2 = M + o(1).
\]
Let $u_m = M^{1/(p-2)} v_m$ for $m = 1, 2, \cdots$. Then
\[
I'(u_m) = o_n(1) \quad \text{in} \quad (X_0^s)^*
\]
and
\[
I(u_m) = \left( \frac{1}{2} - \frac{1}{p} \right) M^{\frac{p}{p-2}} + o_n(1).
\]
By the maximum principle, $\{u_m\}$ does not contain any convergent subsequence. By lemma (3.1) there is a sequence $\{x_m\}$ of the form $(x'_m, m + \frac{1}{2})$ for integers $m$ such that
\[
|x_m| \rightarrow \infty
\]
and
\[
u_m(x) = \varphi(x - x_m) + o(1) \quad \text{strongly}.
\]
Since $\varphi$ is radially symmetric, we may take $m$ to be positive. Next, we consider the following sets
\[
\mathbb{R}^N_+ := \left\{ x \in \mathbb{R}^N : \langle x, x_m \rangle > 0 \right\} \quad \text{and} \quad \mathbb{R}^N_- := \mathbb{R}^N \setminus \mathbb{R}^N_+.
\]
We may assume that $|x_m| \geq 4$ from $m = 1, 2, \cdots$. Now
\[
\langle \beta(\varphi(x - x_m)), x_m \rangle = \int_{\mathbb{R}^N} \varphi^2(x - x_m) \chi(|x|) \langle x, x_m \rangle \, dx
\]
\[
= \int_{\mathbb{R}^N_+} \varphi^2(x - x_m) \chi(|x|) \langle x, x_m \rangle \, dx
\]
\[
+ \int_{\mathbb{R}^N_-} \varphi^2(x - x_m) \chi(|x|) \langle x, x_m \rangle \, dx
\]
\[
\geq \int_{B_1(x_m)} \varphi^2(x - x_m) \chi(|x|) \langle x, x_m \rangle \, dx
\]
\[
+ \int_{\mathbb{R}^N_-} \varphi^2(x - x_m) \chi(|x|) \langle x, x_m \rangle \, dx.
\]
Note that there are $c_1 > 0, c_2 > 0$ such that for $x \in B_1(x_m)$, we have
\[
\varphi^2(x - x_m) \geq c_1
\]
\[
\langle x, x_m \rangle \geq c_2 |x| |x_m| \quad \text{for} \quad m = 1, 2, \cdots.
\]
Thus
\[
\int_{B_1(x_m)} \varphi^2(x - x_m) \chi(|x|) \langle x, x_m \rangle \, dx \geq c_1 c_2 \int_{B_1(x_m)} \chi(|x|) |x| |x_m| \, dx
\]
\[
\geq c_3 |x_m|^{N+1}, \quad c_3 > 0 \quad \text{a constant}.
\]
Recalling that for each \( x \in \mathbb{R}^N \), 
\[
|x - y_n| \geq |x|
\]
it follows that 
\[
|u(x - y_n)|^2 \chi(|x|)|x| \leq R|u(|x|)|^2 \in L^1(\mathbb{R}^N) (R > 0)
\]
(see \( \text{[II]} \) lemma 4.3).

This fact, combined with the limit 
\[
u(x - y_n) \to 0 \text{ as } |y_n| \to +\infty
\]
implies that 
\[
\int_{\mathbb{R}^N} |u(x - y_n)|^2 \chi(|x|)|x| dx = o_n(1).
\]
We conclude that 
\[
M^{1/(p-2)} |a_r| \geq \left< \beta(\varphi_m), \frac{x_m}{|x_m|} \right>
\]
\[
eq \left< \beta(\varphi(x - x_m)), \frac{x_m}{|x_m|} \right> + o(1)
\]
\[
\geq c_3 |x_m|^N + o(1)
\]
a contradiction. Thus \( c_r > M \).

**REMARK 1** By Lemma 2.1(1), there is \( r_1 > 0 \) such that
\[
\frac{1}{2} \leq \|f_y\|_{L^p(\Omega_r)} \leq \frac{3}{2}
\]
where \( r \geq r_1 \) and \( |y - a_r| \geq r/2 \) and \( y_N \geq r/2 \).

**REMARK 2.** By Lemma 2.1(2), there is \( r_2 \geq r_1 \) such that \( M < \|\Psi_y\|^2 < \frac{c_4 + M}{2} \) where \( r \geq r_2 \) and \( |y - a_r| \geq r/2 \) and \( y_N \geq r/2 \).

**Lemma 4.2.** There is \( r_3 \geq r_2 \) such that if \( r \geq r_3 \), then
\[
\left< \beta(\varphi_y), y \right> > 0 \quad \text{for} \quad y \in \partial(B_{r/2}(a_r))
\]

**Proof.** By lemma 2.1, we have \( 2/3 \leq c_y \leq 2 \). For \( r \geq r_2 \), let 
\[
A_{(3/8)r, (5/8)r} = \left\{ x \in \mathbb{R}^N \left| \frac{3}{8} r \leq |x - a_r| \leq \frac{5}{8} r \right\}
\]
\[
\mathbb{R}^N_x(y) = \left\{ x \in \mathbb{R}^N \left| \langle x, y \rangle > 0 \right\}
\]
\[
\mathbb{R}^N_y(y) = \left\{ x \in \mathbb{R}^N \left| \langle x, y \rangle < 0 \right\}
\]
\[
\left< \beta(\varphi_y), y \right> = c_y \left[ \int_{\mathbb{R}^N_y(y)} \xi^2(|x - a_r|) \eta^2(x_N) \varphi^2(x - y) \chi(|x|) \langle x, y \rangle dx \right.
\]
\[
\left. + \int_{\mathbb{R}^N_y(y)} \xi^2(|x - a_r|) \eta^2(x_N) \varphi^2(x - y) \chi(|x|) \langle x, y \rangle dx \right]
\]
\[
\geq \frac{2}{3} \left[ \int_{A_{(3/8)r, (5/8)r}} \varphi^2(x - y) \chi(|x|) \langle x, y \rangle dx \right.
\]
\[
\left. + \int_{\mathbb{R}^N_y(y)} \varphi^2(x - y) \chi(|x|) \langle x, y \rangle dx \right]
\]
\[
\int_{A((3/8)r,(5/8)r)} \varphi^2(x-y)\chi(|x|)(x,y)dx \geq c_6 \int_{A((3/8)r,(5/8)r)} \chi(|x||y|)dx \text{ for } c_6 > 0
\]
\[
\geq c_6|y| \left[ \left( \frac{5}{8}r \right)^N - \left( \frac{3}{8}r \right)^N \right] \]
\[
\geq c_7r^{N+1} \text{ for } c_7 > 0.
\]
\[
\int_{R^N(y)} \varphi^2(x-y)\chi(|x|)(x,y)dx = o_n(1).
\]
Therefore, there is \( r_3 \geq r_2 \), such that if \( r \geq r_3 \), \( |y - a_r| = r/2 \)
\[\langle \beta (\Psi_y), y \rangle \geq c_7r^{N+1} - o_n(1) > 0.\]
This completes the proof. 

By Lemma 2.1 and Lemma 4.2 fix \( \rho_0 > 0, r_0 \geq r_3 \) such that if \( 0 < \rho \leq \rho_0, r \geq r_0 \) then \( \| \varphi_y \|_{\Omega_r}^2 < 2^{(p-2)/p} \alpha \) for \( y \in B_{r/2}(a_r) \). From now on, fix \( \rho_0, r_0 \), for \( r \geq r_0 \). Let
\[
B = \left\{ \Psi_y \| y - a_r \leq \frac{r}{2} \right\}
\]
\[
\Gamma = \left\{ h \in C(V_r, V_r) \mid h(u) = u \text{ if } \| u \|^2 < \frac{\alpha + \alpha}{2} \right\}
\]

**Lemma 4.3.**

\( h(B) \cap V_r \neq \emptyset \) for each \( h \in \Gamma \).

**Proof.** Let \( h \in \Gamma \) and \( H(x) = \beta \circ h \circ \varphi_x : \mathbb{R}^N \to \mathbb{R}^N \). Consider the homotopy, for \( 0 \leq t \leq 1 \)

\[
F(t, x) = (1 - t)H(x) + tI(x) \text{ for } x \in \mathbb{R}^N.
\]

If \( x \in \partial \left( B_{r/2}(a_r) \right) \), then, by Remark 8 and Lemma 9,
\[
\langle \beta (\Psi_x), x \rangle > 0
\]
\[
\alpha < \| \Psi_x \|^2 < \frac{\alpha + \alpha}{2}.
\]

Then
\[
\langle F(t, x), x \rangle = \langle (1 - t)H(x), x \rangle + \langle tx, x \rangle
\]
\[
= (1 - t) \langle \beta (\Psi_x), x \rangle + t \langle x, x \rangle
\]
\[
> 0.
\]

Thus \( F(t, x) \neq 0 \) for \( x \in \partial \left( B_{r/2}(a_r) \right) \). By the homotopic invariance of the degree
\[
d(H(x), B_{r/2}(a_r), a_r) = d(I, B_{r/2}(a_r), a_r) = 1.
\]

There is \( x \in B_{r/2}(a_r) \) such that
\[
a_r = H(x) = \beta (h \circ \Psi_x).
\]

Thus \( h(B) \cap V_r \neq \emptyset \) for each \( h \in \Gamma \). Now we are in the position to prove Theorem A: Consider the class of mappings
\[
F = \left\{ h \in C \left( \overline{B_{r/2}(a_r)} \right), H^1 (R_N) : h|_{\partial B_{r/2}(a_r)} = \Psi_y \right\}
\]
and set
\[
c = \inf_{h \in F} \sup_{y \in B_{r/2}(a_r)} \| h(y) \|^2_{\Omega_r}.
\]
It follows from the above Lemmas, with the appropriate choice of \( r \) that
\[
\alpha < c_r = \inf_{u \in V_\gamma} \| u \|^2_{\Omega_r} \leq c < 2^{(p-2)/p} \alpha
\]
and
\[
\max_{\partial B_{r/2}(\alpha_r)} \| h(y) \|^2_{\Omega_r} < \max_{B_{r/2}(\alpha_r)} \| h(y) \|^2_{\Omega_r}.
\]

Theorem 1.1 then follows by applying the version of the mountain pass theorem from Brezis-Nirenberg \[2\].

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