Pyramids and monomial blowing-ups

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March 29, 2022

Abstract
We show that a convex pyramid $\Gamma$ in $\mathbb{R}^n$ with apex at 0 can be brought to the first quadrant by a finite sequence of monomial blowing-ups if and only if $\Gamma \cap (-\mathbb{R}_0^n) = \{0\}$. The proof is non-trivially derived from the theorem of Farkas-Minkowski. Then, we apply this theorem to show how the Newton diagrams of the roots of any Weierstraß polynomial

$$P(x, z) = z^m + h_1(x)z^{m-1} + \cdots + h_{m-1}(x)z + h_m(x),$$

$h_i(x) \in k[x_1, \ldots, x_n][z]$, are contained in a pyramid of this type. Finally, if $n = 2$, this fact is equivalent to the Jung-Abhyankar theorem.

1 Introduction

We will operate in the euclidean space $\mathbb{R}^n$ with its affine structure. As it is classical, we will distinguish between the point-space $X = \mathbb{R}^n$ and the underlying vector space $V = \mathbb{R}^n$. The vector addition is the canonical action (translations) of $V$ on $X$.

A polyhedron is the convex hull of a finite set of points generating the affine space $X$. Equivalently ([?], page 30), a polyhedron is the compact intersection of a finite set of half-spaces. Moreover, if $E = \{A_0, A_1, \ldots, A_m\}$ is a finite set of points generating $X$ and, if $\mathcal{H} = \{H_1, \ldots, H_p\}$ is the set of all (different) hyperplanes passing through all possible subsets of $E$ consisting of $n$ affinely independent points, then the set of vertices of the convex hull $[E]$ of $E$ is the set of points of intersection of all the subsets of $\mathcal{H}$ consisting of $n$ hyperplanes whose intersection is an only point$^1$.

$^1$c.f., for instance, Vicente, J.L.: Notas sobre convexidad at http://www.us.es/da
A pyramid is the projection, from the origin $0 \in \mathbb{R}^n$, of a polyhedron contained in a hyperplane $H$ not passing through $0$. To be more precise: given a hyperplane $H$ such that $0 \not\in H$ and a finite set $E$ of points generating $H$, denoting by $\Delta$ the convex hull $[E]$ of $E$, then the corresponding pyramid is

$$\Gamma(\Delta) = \bigcup_{a \in \Delta} \langle a \rangle_+,$$

where $\langle a \rangle_+$ is the half-line of the non-negative multiples of $a$. Equivalently (c.f. Vicente, J.L., loc.cit.), a pyramid can be given by a polyhedron $\Delta'$ in $X$, one of whose vertices is $0$. In this case, the pyramid is nothing but

$$\Gamma = \bigcup_{a \in \Delta' \setminus \{0\}} \langle a \rangle_+.$$ 

Moreover, if $\{0, a_1, \ldots, a_m\}$ are the vertices of $\Delta'$, there is a hyperplane $H$ strictly separating $0$ from $[a_1, \ldots, a_m]$. Then $\Delta = H \cap \Delta'$ is a polyhedron in $H$ and $\Gamma = \Gamma(\Delta)$.

**Definition 1.1.**— We will call a monomial blowing-up (resp. a monomial blowing-down) any linear automorphism of $\mathbb{R}^n$ of the form

$$(a_1, \ldots, a_n) \rightarrow (a_1, \ldots, a_n)M_{ij} \quad (\text{resp.} \quad (a_1, \ldots, a_n) \rightarrow (a_1, \ldots, a_n)N_{ij})$$

where:

1. $i, j \in \mathbb{Z}, \ i \neq j, \ 1 \leq i, j \leq n$
2. $M_{ij}$ (resp. $N_{ij}$) is equal to the identity matrix in which the $(i, j)$-entry is set to $1$ (resp. to $-1$).

**Remark 1.2.**— With the notations of definition 1.1, the monomial blowing-up (resp. monomial blowing-down) acts in the following way:

$$(a_1, \ldots, a_n) \rightarrow (a_1, \ldots, a_i + a_j, \ldots, a_n)$$

resp. $$(a_1, \ldots, a_n) \rightarrow (a_1, \ldots, a_j - a_i, \ldots, a_n)$$

This corresponds to the behavior of the exponents of a monomial under the geometric monomial blowing-up $x_i \rightarrow x_i x_j$ or the monomial blowing-down $x_i \rightarrow x_i / x_j$. In fact, this geometric monomial blowing-up (resp. monomial blowing-down) acts on monomials in the following way:

$$x_1^{a_1} \cdots x_j^{a_j} \cdots x_n^{a_n} \rightarrow x_1^{a_1} \cdots x_j^{a_j+a_i} \cdots x_n^{a_n}$$

resp. $$x_1^{a_1} \cdots x_j^{a_j} \cdots x_n^{a_n} \rightarrow x_1^{a_1} \cdots x_j^{a_j-a_i} \cdots x_n^{a_n}.$$
This is the reason of the name for these linear automorphisms.

From now on, we will use the name of monomial blowing-up (resp. monomial blowing-down) indistinctly for the linear automorphisms defined in \[\mathcal{M}\] or for the polynomial substitutions \(x_i \rightarrow x_ix_j\) (resp. \(x_i \rightarrow x_i/x_j\)).

**Remark 1.3.**— Let \(E = \{a_1, \ldots, a_m\}\) be a finite set of points generating a hyperplane \(H\) which does not contain the origin, let \(\Delta = [E] \subset H\) be the corresponding polyhedron and \(\Gamma(\Delta)\) the pyramid; then

\[
\Gamma(\Delta) = \left\{ \sum_{i=1}^{m} \lambda_i a_i \bigg| \lambda_i \geq 0, \forall i = 1, \ldots, m \right\}.
\]

Equivalently, let \(A\) be the matrix whose row vectors are \(\{a_1, \ldots, a_m\}\); then

\[
\Gamma(\Delta) = \{ (\lambda_1, \ldots, \lambda_m)A \big| (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_0 \}.
\]

If \(\varphi_{ij}\) is the monomial blowing-up (resp. the monomial blowing-down) with matrix \(M_{ij}\) (resp. \(N_{ij}\)) and \(a'_i = \varphi_{ij}(a_i)\) then \(E' = \{a'_1, \ldots, a'_m\}\) generates a hyperplane \(H'\) not passing through \(0\). If \(\Delta' = [E']\), then \(\Gamma(\Delta') = \varphi(\Gamma(\Delta))\), so it makes sense to speak on the transform of a pyramid by a monomial blowing-up or a monomial blowing-down.

**Definition 1.4.**— The first quadrant of \(X\) is the set \(\mathbb{R}^n_0\). The opposite of the first quadrant of \(X\) is the set \(-\mathbb{R}^n_0\).

The main problem we deal with in this paper is whether, given a pyramid \(\Gamma(\Delta)\), it exists a finite sequence of monomial blowing-ups such that the transform of the pyramid by the sequence is contained in the first quadrant. We solve it by giving a geometrical criterion, from which we derive explicit computations using existing optimization algorithms. The criterion is the following:

**Theorem 1.5.**— Let \(E = \{a_1, \ldots, a_m\}\) be a finite set of points generating a hyperplane \(H\) not containing the origin, let \(\Delta = [E] \subset H\) be the corresponding polyhedron and \(\Gamma(\Delta)\) the pyramid; then the following conditions are equivalent:

1. There exists a finite sequence of monomial blowing-ups such that the transform of \(\Gamma(\Delta)\) by the sequence is contained in the first quadrant.
2. \(\Gamma(\Delta) \cap (-\mathbb{R}^n_0) = \{0\}\).

The second condition can be easily checked by the simplex method; in remark 4.2 we will show how. Let us observe that the first condition implies the second because \(-\mathbb{R}^n_0\) is stable by monomial blowing-ups. In
fact, if $\Gamma(\Delta)$ had a point $a \neq 0$ in common with $-\mathbb{R}^n_0$ then, no matter what sequence of monomial blowing-ups we apply, the transform of $a$ will stay in $-\mathbb{R}^n_0$. The point is then to prove that the second condition implies the first.

As an application, we show that the Newton diagrams of all the roots of a Weierstraß polynomial

$$P(x, z) = z^m + h_1(x)z^{m-1} + \cdots + h_{m-1}(x)z + h_m(x) \in k[[x]][z], \quad m > 1$$

are contained in a pyramid satisfying the equivalent conditions of theorem 1.5. Moreover, we show how, in dimension 2, this fact is in some sense equivalent to the Jung-Abhyankar theorem (c.f. [?]).

2 The proof

Remark 2.1.– Let us denote by $A$ the matrix whose row vectors are $\{a_1, \ldots, a_m\}$. We will speak of “bringing $A$ to the first quadrant” as equivalent to bringing $\Gamma(\Delta)$ to the first quadrant. It is obvious that, if $A$ has column with only positive entries, then it can be brought to the first quadrant by a finite sequence of monomial blowing-ups: it is enough to add this column to the others a suitable number of times.

We use linear optimization methods to prove theorem 1.5. We refer to [?] for the theorem of Farkas-Minkowski and its consequences. In particular, we take from it (pages 42-51) the following consequence of this theorem (which might be also taken as an alternative statement of it):

Corollary 2.2.– Let $A$ be a matrix $m \times n$; then one and only one of the following conditions hold:

1. There exists a non-zero vector $x \geq 0$ such that $xA \leq 0$.
2. The system of inequalities $Ay > 0$ has a non-negative solution.

Inequalities must be understood componentwise.

From this result we derive the following, which is the useful one:

Corollary 2.3.– Let $A$ be a matrix $m \times n$; then one and only one of the following conditions holds:

1. There exists a non-zero vector $x \geq 0$ such that $xA \leq 0$.
2. The system of inequalities $Ay > 0$ has a positive integer solution (that is, a vector of positive integers).
Proof: By corollary 2.2, we must only prove that the existence of a non-negative real solution of $Ay > 0$ implies that there is a positive integer one. Hence, it is enough to show the existence of a positive rational solution.

Let $y$ be a non-negative real column vector such that $Ay > 0$, let $a_i$ be the rows of $A$, $i = 1, \ldots, m$, and let $a_i y = \alpha_i > 0$. Let $0 < \varepsilon < \min\{\alpha_i\}$ and let $K$ be a polidisc centered at $y$ such that, for all $z \in K$, one has $| - \alpha_i + a_i z | < \varepsilon$. Then, $-\varepsilon < a_i z - \alpha_i < \varepsilon$, so $\alpha_i - \varepsilon < a_i z < \varepsilon + \alpha_i$, which implies that $a_i z > 0$. Since $K$ contains rational vectors greater than zero, the corollary is proven.

Proposition 2.4.– Let $n > 1$ and $y = (y_1, \ldots, y_n)$ be a vector of positive integers with greatest common divisor equal to 1. There exists a $n \times n$ matrix $Y$, whose entries are non-negative integers, with determinant equal to 1, one of whose columns is $y$.

Proof: Let $a \in \mathbb{Z}$ and denote by $E_{ij}(a)$, $i \neq j$, the $n \times n$ elementary matrix, which is equal to the identity matrix with its $(i, j)$-entry replaced by $a$. Let us write $y$ as a column vector.

Let us assume that all the $y_j$ are multiple of one of them, say $y_i$; then it must be $y_i = 1$. Left multiplications by matrices $E_{ji}(-y_j)$ allow us to transform the column vector $y$ into a column vector having all entries equal to zero, except the $i$-th one which is equal to 1. Remark that all the elementary matrices we have used have a negative integer entry out of the main diagonal.

Let us assume that no entry of the column vector $y$ divides all the others and let $y_i$ be the smallest of all these entries. By assumption, there must be a $j$ such that, in the euclidean division, $y_j = q_j y_i + r_j$ with $0 < r_j < y_i$. Left multiplication of $y$ by $E_{ji}(-q_j)$ puts $r_j$ at the position $j$, leaving the other entries unchanged. In this way, we get a new column vector such that the greatest common divisor of its entries is 1 and the minimum of these entries has strictly decreased. Remark that, again, we have used an elementary matrix with a negative integer entry out of the main diagonal.

Let us assume that no entry of the column vector $y$ divides all the others and let $y_i$ be the smallest of all these entries. By assumption, there must be a $j$ such that, in the euclidean division, $y_j = q_j y_i + r_j$ with $0 < r_j < y_i$. Left multiplication of $y$ by $E_{ji}(-q_j)$ puts $r_j$ at the position $j$, leaving the other entries unchanged. In this way, we get a new column vector such that the greatest common divisor of its entries is 1 and the minimum of these entries has strictly decreased. Remark that, again, we have used an elementary matrix with a negative integer entry out of the main diagonal. If we repeat this process, after a finite number of steps, we fall in the preceeding situation.

Summing-up: we have proven that, by left multiplication of the column vector $y$ by elementary matrices having negative integer entries out of the main diagonal, we arrive at a matrix which is a column of the identity matrix. If we denote by $Y$ the inverse of this product of elementary matrices, we see that $Y$ is a product of elementary matrices with positive integer entries out of the main diagonal, so $Y$ is a matrix with non-negative integer entries. Let $I_n$ be the $n \times n$ identity matrix; then it is clear that $Y = Y I_n$ contains a column equal to $y$ and, of course, $\det(Y) = 1$. This proves the proposition.
Proposition 2.5.– Let $Y$ be a square matrix with non-negative integer entries whose determinant is equal to $1$. Then $Y$ can be written as a product of (by order): a finite number of monomial blowing-up matrices, a permutation matrix and another finite number of monomial blowing-up matrices.

Proof: It is enough to show that $Y$ can be brought to a permutation matrix by left and right multiplicaction by monomial blowing-down matrices. If $q \in \mathbb{Z}_+$, then $E_{ij}(-q) = E_{ij}(-1)^q$; since $E_{ij}(-1)$ is a monomial blowing-down matrix, then $E_{ij}(-q)$ is a product of monomial blowing-down matrices.

Let $y_{ij}$ be the smallest non-zero entry in $Y$. If all the elements in the $j$-th column are multiple of $y_{ij}$, we can get zeros in all the positions of this column, except $(i,j)$, by left multiplication by elementary matrices with a negative entry out of the main diagonal. In this case, the fact that $\det(Y) = 1$ implies $y_{ij} = 1$. If some entry in the $j$-th column is not a multiple of $y_{ij}$, say $y_{lj}$, and if $y_{lj} = qy_{ij} + r_{lj}$ is the euclidean division, then left multiplication by $E_{jl}(-q)$ puts $r_{lj}$ at the position $(l,j)$ so the smallest non-zero entry of $Y$ has strictly decreased. If we repeat this process, it is clear that we must arrive to the first case after a finite number of steps. The end of this process is a matrix $Y_1'$ with a $1$ at one position (denote it again by $(i,j)$) and zeros everywhere else in the $j$-th column. Moreover, $Y_1'$ is the result of left multiplying $Y$ by monomial blowing-down matrices.

By symmetry, it is clear that we can get a new matrix $Y_1$, obtained from $Y_1'$ by right multiplication by monomial blowing-down matrices, and having $1$ at the $(i,j)$ position and zeros everywhere else in the $i$-th row and the $j$-th column. This is the basic argument of our proof.

We may repeat the argument for the submatrix of $Y_1$ obtained by deleting the $i$-th row and the $j$-th column, but seeing the operations in the whole $Y_1$. This does not affect the form of $Y_1$. The very end of the process is a matrix $Y_p$, which is a permutation of the rows of the identity matrix, and which is obtained from $Y$ by left and right multiplication by monomial blowing-down matrices. This proves the proposition.

Lemma 2.6.– In the situation of theorem 1.5 let $A$ be the matrix whose row vectors are $\{a_1, \ldots, a_m\}$. For all $x = (x_1, \ldots, x_m) \in \mathbb{R}_0^n \setminus \{0\}$ one has

$$0 \neq \sum_{i=1}^m x_i a_i.$$ 

Proof: Let $f: \mathbb{R}^n \to \mathbb{R}$ be a linear function such that $f(a_i) > 0, \forall i = 1, \ldots, m$. There always exists such a function: we give an example.
Let \( g \) be an affine function such that \( H \) has the equation \( g = 0 \) and \( g(0) < 0 \). Then, for all \( i = 1, \ldots, m \) one has \( 0 = g(a_i) = g(0) + \overrightarrow{g}(a_i) \), so \( \overrightarrow{g}(a_i) > 0 \) and we may take \( f = \overrightarrow{g} \).

Then,
\[
f\left( \sum_{i=1}^{m} x_i a_i \right) = \sum_{i=1}^{m} x_i f(a_i) > 0,
\]
which proves the lemma.

**Remark 2.7.– Proof of theorem 1.5**

Let us assume that \( 0 \) is the only point of \( \Gamma(\Delta) \) belonging to \( -\mathbb{R}^n_0 \). By lemma 2.6, for every \( x \in \mathbb{R}^m_0 \setminus \{0\} \), the vector \( xA \) is different from zero. By assumption, it cannot be \( xA \leq 0 \). By corollary 2.3 there must exist a vector \( y \in \mathbb{Z}^n_+ \) such that, written as a column vector, \( Ay > 0 \). We may assume that the greatest common divisor of the entries of \( y \) is 1. By proposition 2.4, there exists a matrix \( Y \) with non-negative integer entries and determinant equal to 1 such that \( y \) is one of its columns. Therefore, one of the columns of \( Ay \) is \( Ay > 0 \), which implies by remark 2.1 that \( Ay \) can be brought to the first quadrant by a finite sequence of monomial blowing-ups. By proposition 2.5, we obtain \( Ay \) from \( A \) by applying to \( A \) a finite sequence of monomial blowing-ups, then a permutation of the columns and then another finite sequence of monomial blowing-ups. It is evident that the permutation of the columns plays no role: if the matrix with the permuted columns can be brought to the first quadrant, also the original one. This proves the theorem.

### 3 Applications

The main application of theorem 1.5 we consider here lies in the resolution of equations of the form: a Weierstraß polynomial equal to zero. Let \( k \) be an algebraically closed field of characteristic zero, \( x = (x_1, \ldots, x_n) \) a collection of indeterminates, \( R = k[[x]] \) the corresponding ring of power series, and let
\[
P(x, z) = z^n + h_1(x)z^{n-1} + \cdots + h_{n-1}(x)z + h_n(x) \in k[[x]][z], \quad n > 1
\]
be an irreducible Weierstraß polynomial; the object to study is the equation \( P(x, z) = 0 \). Let \( D \in k[[x]] \) be the discriminant of \( P \) with respect to \( z \); the Jung-Abhyankar theorem (c.f. [?]) asserts that, if \( D \) is of the form \( x^aU(x) \) with \( U(x) \in k[[x]] \), \( U(0) \neq 0 \), then the roots of \( P(x, z) = 0 \) are a full set of conjugate Puiseux power series in the variables \( x \). Here, \( x^a \) means \( x^a = x_1^{a_1} \cdots x_n^{a_n} \), where \( a = (a_1, \ldots, a_n) \in \mathbb{N}^n \).
When the discriminant has this very special form, we say that it is a normal crossing divisor.

In general, the roots are not Puiseux power series in x. However, we can say something very important about them, namely

**Theorem 3.1.**— The roots of \( P(x, z) = 0 \) are power series belonging to a ring \( k((x_1^{1/p}) \cdots (x_i^{1/p}))[x_1^{1/p}, \ldots, x_i^{1/p}] \) whose Newton diagrams are contained in a pyramid \( \Gamma(\Delta) \) such that \( \Gamma(\Delta) \cap (-\mathbb{R}_0^n) = \{0\} \).

We will prove the theorem through several remarks.

**Remark 3.2.**— We take the lexicographic order in the sense (c.f. [?], page 50): if \( a, b \in \mathbb{R}_0^n \) then \( a <_{\text{lex}} b \) if and only if the first component (from left to right) of \( a \) which is different of the corresponding in \( b \) is strictly smaller.

Let us observe that a monomial blowing-up \( \varphi_{ij} \) of the type \( a \rightarrow a M_{ij} \) with \( i < j \) preserves the lexicographic order, so it is an ordered automorphism of \( \mathbb{R}_0^n \) (endowed with the lexicographic order). In fact,

\[
\varphi_{ij}(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n) = (a_1, \ldots, a_i, \ldots, a_i + a_j, \ldots, a_n),
\]

so, if \( a <_{\text{lex}} b \), then:

1. If \( <_{\text{lex}} \) is decided before the position \( j \), it is evident that \( \varphi(a) <_{\text{lex}} \varphi(b) \).

2. If \( <_{\text{lex}} \) is decided at the position \( j \), this means that \( a_l = b_l, \forall l, 1 \leq l \leq j - 1 \) and \( a_j < b_j \). Therefore, \( a_i + a_j < b_i + b_j \), hence \( \varphi(a) <_{\text{lex}} \varphi(b) \).

3. If \( <_{\text{lex}} \) is decided at a position \( l \) after \( j \), this means that all the components of \( a \) until the \((l-1)\)-th coincide with the corresponding in \( b \), so the same happens with \( \varphi(a) \) and \( \varphi(b) \). Since \( a_l < b_l \) then \( \varphi(a) <_{\text{lex}} \varphi(b) \).

We call this an order-preserving monomial blowing-up. Notice that the corresponding monomial blowing-down \( \varphi_{ij}^{-1} \) is also order-preserving.

**Remarks 3.3.**— Let \( \Lambda \subset \mathbb{Z}_0^n \) be a non-empty cloud of points; we call the transform of \( \Lambda \) by a monomial blowing-up (or a monomial blowing-down) the set of the transforms of all the points of \( \Lambda \).

3.3.1. Any monomial blowing-up \( \varphi \) keeps \( \mathbb{Z}_0^n \), that is, \( \varphi(\mathbb{Z}_0^n) \subset \mathbb{Z}_0^n \). Therefore, if \( a, b \in \mathbb{Z}_0^n \) are two points such that \( b \in a + \mathbb{Z}_0^n \) then \( \varphi(b) \in \varphi(a) + \mathbb{Z}_0^n \).

3.3.2. Let us assume that \( \varphi = \varphi_{pq}, p < q \), is an order-preserving monomial blowing-up. Let \( a, b \in \mathbb{Z}_0^n \) and write \( a' = \varphi(a), b' = \varphi(b) \). Let us assume that there exists an index \( j, 1 \leq j \leq n \) such that
3.3.3. For any \( j = 1, \ldots, n \) and any \( v \in \mathbb{Z}_n^0 \), we write \( \Lambda_j(v) = \{ v' \in \mathbb{Z}_n^0 \mid v'_i \geq v_i, \forall i = 1, \ldots, j \} \); then \( \Lambda_1(v) \supset \cdots \supset \Lambda_n(v) \). By 3.3.2 for every order-preserving monomial blowing-up \( \varphi \), every \( v \in \mathbb{Z}_n^0 \) and every \( j = 1, \ldots, n \), one has \( \varphi(\Lambda_j(v)) \subset \Lambda_j(\varphi(v)) \). Moreover, if \( \Phi \) is a composition of a finite number of order-preserving monomial blowing-ups, then \( \Phi(\Lambda_j(v)) \subset \Lambda_j(\Phi(v)) \).

3.3.4. Since \( \emptyset \neq \Lambda \subset \mathbb{Z}_n^0 \), there is a minimum-lex \( u \in \Lambda \), so \( \Lambda \subset \Lambda_1(u) \). We will now prove that there exists a finite sequence of order-preserving monomial blowing-ups such that, calling \( \Phi \) the composition of all of them, one has \( \Phi(\Lambda) \subset \Phi(u) + \mathbb{Z}_n^0 \). If \( \Lambda \subset \Lambda_n(u) \) there is nothing to prove, so we assume this is not the case. Let \( j \) be the smallest index such that \( \Lambda \not\subset \Lambda_j(u) \); then, necessarily \( j > 1 \). By the minimality of \( j \), for every \( u' \in \Lambda \) one must have \( u'_i \geq u_i, \forall i = 1, \ldots, j - 1 \).

Moreover, if \( u' \in \Lambda \setminus \Lambda_j(u) \) then \( u'_j < u_j \), so \( u_j > 0 \). Since \( u \in \Lambda \) is the minimum-lex, for every \( u' \in \Lambda \setminus \Lambda_j(u) \) there must exist an index \( i < j \) such that \( u'_i > u_i \). Let \( i_1 < j \) be the smallest index such that there exists \( u' \in \Lambda \setminus \Lambda_j(u) \) satisfying \( u'_{i_1} > u_{i_1} \). Let \( \Phi_1 \) be the composition of \( u_j \) monomial blowing-ups equal to \( \varphi_{i_1,j} \); then \( \Phi_1(u) = (u_1, \ldots, u_{i_1}, \ldots, u_j u_{i_1} + u_j, \ldots, u_n) \) and, for every \( u' \in \Lambda \setminus \Lambda_j(u) \) with \( u'_i > u_i \), \( \Phi_1(u') = (u'_1, \ldots, u'_{i_1}, \ldots, u_j u'_{i_1} + u'_j, \ldots, u'_n) \).

Since \( u'_{i_1} > u_{i_1} \), then \( v_j(u'_{i_1} - u_{i_1}) > 0 \), so \( u_j > u'_{i_1} > u_{i_1} \). Hence \( u_j u'_{i_1} + u'_j > u_j u_{i_1} + u_j \), so \( \Phi_1(u') \in \Lambda_j(\Phi_1(u)) \).

Now, \( \Phi_1(u) \) is the minimum-lex of \( \Phi_1(\Lambda) \) and, by 3.3.3 \( \Phi_1(\Lambda_i(u)) = \Lambda_i(\Phi_1(u)), \forall i = 1, \ldots, n \). By 3.3.2 \( \Phi_1(\Lambda) \subset \Lambda_i(\Phi_1(u)), \forall i = 1, \ldots, j - 1 \). One must not forget that \( \Phi_1 \) leaves invariant the first \( j - 1 \) components of every vector. Therefore, if \( \Phi_1(\Lambda) \not\subset \Lambda_j(\Phi_1(u)) \), there exists a smallest index \( i_2 \) such that there exists \( u' \in \Phi_1(\Lambda) \setminus \Lambda_j(\Phi_1(u)) \) satisfying \( u'_{i_2} > u_{i_2} \). Necessarily \( i_2 > i_1 \) and we proceed as before, and so on. It is then clear that there exists a finite sequence of order-preserving monomial blowing-ups, whose composition \( \Phi_1 \) is such that \( \Phi_1(\Lambda) \subset \Lambda_i(\Phi_1(u)), \forall i = 1, \ldots, j \). If there is still a \( j_1 \) such that \( \Phi_1(\Lambda) \not\subset \Lambda_{j_1}(\Phi_1(u)) \), then \( j_1 > j \) and we proceed as before, and so on. This proves our assertion.

**Remark 3.4.**— **Proof of theorem 3.1** Let \( \Lambda \) be the Newton
diagram of the discriminant $D$ of $P(x, z)$; by 3.3.4 there exists a finite sequence of order-preserving monomial blowing-ups such that, calling $\Phi$ their composition, $\Phi(\Lambda) \subset a + \mathbb{Z}_n^0$ where $a \in \Phi(\Lambda)$. We make these monomial blowing-ups to act upon $P(x, z)$ and denote by $Q(x, z)$ the transform of $P(x, z)$ by $\Phi$. The discriminant $D'$ of $Q(x, z)$ is just the transform of $D$ because $D$ is a polynomial in the coefficients of the equation. Moreover, $D'$ is a normal crossing divisor, hence the roots of $Q = 0$ are all ordinary Puiseux power series, say with common denominator $p$ of the exponents, because every irreducible factor of $Q(x, z)$ has a discriminant which is a normal crossing divisor. If we come back to the beginning by applying the corresponding sequence of monomial blowing-downs, the region containing the Newton diagram of the roots of $Q = 0$, namely the first quadrant, obviously goes to a pyramid $\Gamma(\Delta)$ such that $\Gamma(\Delta) \cap (-\mathbb{R}_0^n) = \{0\}$.

Since all the monomial blowing-ups are of the form $\phi_{lj}$ with $l < j$, we denote by $i$ the minimum of all the indices $l$ of these monomial blowing-ups, then $\Phi$ leaves invariant the first $i$ coordinates of every point, so the same happens with $\Phi^{-1}$. Therefore, every monomial $x_1^{a_1/p} \cdots x_i^{a_i/p} x_{i+1}^{a_{i+1}/p} \cdots x_n^{a_n/p}$ occurring in a root of $Q$ evolves in a way such that the exponents $a_1/p, \ldots, a_i/p$ remain unchanged. Therefore, if we fix a root $\bar{q}$ of $Q = 0$, fix $a_1/p, \ldots, a_i/p$ and write $\bar{q}' = x_1^{a_1/p} \cdots x_i^{a_i/p} \bar{q}''(x_{i+1}^{a_{i+1}/p}, \ldots, x_n^{a_n/p})$, with $\bar{q}''(x_{i+1}^{a_{i+1}/p}, \ldots, x_n^{a_n/p}) \in k[x_1^{1/p}, \ldots, x_i^{1/p}]$, for the sum of all the terms of the root whose monomials start by $x_1^{a_1/p} \cdots x_i^{a_i/p}$, the transform of $\bar{q}'$ by $\Phi^{-1}$ produces a power series $x_1^{a_1/p} \cdots x_i^{a_i/p} \bar{q}''(x_{i+1}^{a_{i+1}/p}, \ldots, x_n^{a_n/p})$ where $\bar{q}''$ has possibly negative exponents. Since $(1/p) \cdot \mathbb{Z}_n^0$ is lexicographically well-ordered, so it is $\Phi^{-1}((1/p) \cdot \mathbb{Z}_n^0)$, hence the transform of $\bar{q}$ by $\Phi^{-1}$ belongs to $k((x_n^{1/p}) \cdots ((x_{i+1}^{1/p})))[x_1^{1/p}, \ldots, x_i^{1/p}]$, which proves the theorem.

When $n = 2$ there is much more to say, namely:

**Remark 3.5.**– In our joint paper (cf. [?]), we prove the following for $n = 2$:

1. The theorem 3.1 without using the Jung-Abhyankar theorem.

2. The Jung-Abhyankar theorem from the fact that the Newton diagrams of the roots lie in a pyramid $\Gamma(\Delta)$ such that $\Gamma(\Delta) \cap (-\mathbb{R}_0^2) = \{0\}$.

This shows that the Jung-Abhyankar theorem in dimension 2 can be proven by linear algebra techniques, without having resource to more sophisticated algebraic material. Moreover, in this case, the Jung-Abhyankar theorem is equivalent to the fact that the roots of the equation lie in a pyramid satisfying the conditions of theorem 1.5.
4 Short remarks on computations

The explicit computations are a consequence, more or less obvious, of the convex calculus and the optimization of a linear function on a polyhedron by the simplex method.

The point of departure will be always the list of points \( \{a_1, \ldots, a_m\} \), all different from \( 0 \), generating, either a hyperplane not passing through the origin, or \( X = \mathbb{R}^n \). We add \( 0 \) to the list, and write \( E = \{0, a_1, \ldots, a_m\} \); in both cases \( E \) generates the whole affine space. We denote by \( A \) the matrix whose row vectors are \( \{a_1, \ldots, a_m\} \).

Remark 4.1.– By elementary linear calculus (c.f. Vicente, J.L., loc. cit.), the \((n-1)\)-dimensional faces of the polyhedron \([E]\) are produced by the following algorithm: we pick all the subsets of \( E \) consisting of \( n \) affinely independent points, and find the hyperplane determined by them; then we drop repetitions and keep only those hyperplanes leaving all the points of \( E \) in an only half-space. This algorithm is not the best possible, but improvements are out of the scope of this paper. If \( C = \{H_1, \ldots, H_p\} \) is the list of faces, then we get the vertices of \([E]\) by the following algorithm: we pick all the subsets of \( C \) consisting of \( n \) hyperplanes whose intersection is an only point, find the point, drop repetitions and the remaining ones are the vertices. It is clear that \( E \) defines a pyramid \( \Gamma \) if and only if \( 0 \) is a vertex of \([E]\). In this case, the faces of \( \Gamma \) are those \( H_i \) passing through \( 0 \). For instance, if we start from the points

\[
\begin{align*}
a_1 &= (4, 7, -9), & a_2 &= (5, 7, -8), & a_3 &= (3, 5, -9), & a_4 &= (4, 0, -1),
\end{align*}
\]

which generate \( \mathbb{R}^n_0 \), the faces are

\[
\begin{align*}
7x_1 - 13x_2 - 7x_3 &= 0 & 18x_1 - 9x_2 + x_3 &= 0 \\
-7x_1 - 27x_2 - 28x_3 &= 0 & 5x_1 + 33x_2 + 20x_3 &= 0 \\
19 - 2x_1 + x_2 + 2x_3 &= 0 & 32 - 7x_1 + 5x_2 + 4x_3 &= 0
\end{align*}
\]

and the vertices are \( E = \{0, a_1, a_2, a_3, a_4\} \); therefore the points define a pyramid. The faces are normalized in the sense that all the points of \( E \) make their linear equations \( \geq 0 \). The faces of the pyramid are the first four and the edges are the positive half-lines determined by the four given points.

Remark 4.2.– It is not difficult to know whether the pyramid \( \Gamma \) satisfies the condition \( \Gamma \cap (-\mathbb{R}^n_0) = \{0\} \) or not. Let \( \Lambda = (\lambda_1, \ldots, \lambda_m) \) be a row of variables and let \( m_i \) be the element in the \( i \)-th column of the matrix \( \Lambda A \); then \( \Gamma \cap (-\mathbb{R}^n_0) \neq \{0\} \) if and only if there is a feasible
solution to the set of linear constraints

\[ 1 + \sum_{i=1}^{m} m_i = 0, \quad m_i \leq 0, \quad \lambda_i \geq 0, \quad i = 1, \ldots, m. \]

The existence of a feasible solution can obviously be decided by the simplex method. In the preceding example, the feasible solution does not exist, so \( \Gamma \cap (-\mathbb{R}_0^n) = \{0\} \).

**Remark 4.3.**— It is also easy to find a positive solution of the system of inequalities \( Ay > 0 \), where \( y \) is a column of variables. If \( m_i \) is the element in the \( i \)-th row of \( Ay \), we can easily get a positive solution of \( Ay > 0 \) by minimizing any of the coordinate functions on the set of constraints \( m_i \geq 1, y_i \geq 1, i = 1, \ldots, m \). In our example, minimizing \( y_1 \) by the simplex method will produce the point \((1, 7/5, 1)\), so the integer solution \((5, 7, 5)\).

The remaining computations to bring \( A \) to the first quadrant are straightforward matrix operations.