Stack and register complexity of radix conversions

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Abstract
We investigate the question of computational resources (such as stacks and counters) necessary to perform radix conversions. To this end it is shown that no PDA can compute the significand of the best \(n\)-digit floating point approximation of a power of an incommensurable radix. This extends the results of W. Clinger. We also prove that a two counter machine with input is capable of such conversions. On the other hand we note a curious asymmetry with respect to the order in which the digits are input by showing that a two counter machine can decode its input online if the digits are presented in the most-to-least significant order while no such machine can decode its input in this manner if the digits are presented in the least-to-most significant order. Some structural results about two counter machines (with input) are also established.

Keywords: floating point arithmetic, radix conversions, push-down automata, two counter machines

1 Introduction
Among D. Matula’s pioneering papers that laid the foundation of modern floating-point arithmetic is [13], that investigates the subject of radix conversions. A number of different authors produced a variety of results dealing

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with the efficiency and precision issues of conversions of floating-point numbers between different radices. The two papers that formed the basis for subsequent work in this area are [18] and [5].

While the subject of principal concern for most authors working in the field of computer arithmetic is the efficiency (both time and space) of the algorithms performing such conversions, the question of minimal ‘resources’ needed for such computations was raised already by D. Matula in [13] and further investigated by W. Clinger in [5].

W. Clinger’s results in [5] have been used to justify the use of infinite precision arithmetic by all known algorithms dealing with radix conversions. One of his lemmas states that there is no finite automaton that consumes a string of digits representing an exponent and outputs the first digit of the best approximation of the corresponding power of some \( D \) in a radix that is not commensurable with \( D \). He provides a separate proof for each direction of the input (i.e. least or most significant digit first) and then points out that his proof of the former is somewhat incomplete in the sense that it does not work for all possible radix combinations, although it does succeed for the most common case of base 10 being used for the exponent encoding and 2 for the new radix. Somewhat less important, the proof only deals with case of the precision of the converted result being \( \geq 4 \) and not \( \geq 2 \) as would seem intuitively sufficient.

W. Clinger’s results can be restated by borrowing the language of automatic sequences (see [1] for a reference) as follows. Given positive integers \( D \) and \( d \) let \( S_N(d,D) \) be defined so that for some \( k \), \( S_N(d,D) \times d^k \) is the best 1-digit (pick rounding to even to settle ambiguous cases) approximation of \( D^N \) if \( d > 2 \); otherwise, it represents the best 2-digit approximation. Then Lemma 8 of [5] implies that \( \{ S_N(d,D) \mid N \in \mathbb{N} \} \) is not \( b \)-automatic for any integer \( b \). Using the robustness of \( b \)-automaticity (see [1], Theorem 5.2.3), one can see that the case of the least-significant digit first exponent input now follows from the general properties of \( b \)-automaticity, and does not require a separate number theoretic argument such as Lemma 10 of [5].

The simple argument above shows that W. Clinger’s results indeed imply that a fixed amount of memory is not enough to implement basic radix conversions. From the computability perspective though, it is still interesting to investigate how complex radix conversion algorithms must be. Here, we use complex in a naive sense, as a measure of the sophistication of the computational ‘machinery’ involved in implementing an algorithm. To be somewhat more precise, having seen that a finite automaton cannot perform
the computations we require, one can ask whether a push-down automaton is
enough. An automaton with two stacks? Two counters? Questions like these
have been posed before. As an example, see [8], whose authors investigate
the register complexity of programs composed of various looping constructs.

A push-down automaton (PDA for short, see [9]) can produce output
using a function that decodes its final state in a manner similar to a deter-
mindistic finite automaton with output (DFAO, see [1]).

In this paper we aim at establishing a ‘computability boundary’ for the
task of radix conversions in the sense just outlined. We show that the addition
of a stack is not enough to carry out the required computations.

In the second part of the paper we turn our attention to automata with
two counters (i.e. two stacks whose stack alphabets consist of a single sym-
bol). We define a two counter Minsky machine with input (TCMI) by analogy
with DFAO and provide a simple proof that such a machine can compute any
radix conversions. The subject of two counter machines has a long history
dating back to the original paper by M. Minsky [14]. A curious phenomen-
on was noted early on (see [17], [3], and [10]) that the full power of a two counter
machine can only be ‘tapped’ via an exponential encoding of its input and
output. Without such an encoding, even the simplest functions like $n^2$ are
not computable by a TCM (see [17], [3], [10] for this and many other results
of the same flavor).

The addition of an input brings about a new level of complexity because
an encoding is supplied automatically. As an example, to the best of the
authors’ knowledge, it is still unknown whether a TCM can compute $n$ when
given $2^n$ in one of the counters (see [17]). On the other hand, a TCMI can
simply count the zeros in its input to output $n$ when the input is the digits
of $2^n$ in radix 2 (the subject of using a different radix is a separate problem).

We show (see Proposition 2) that whenever the digits of $n$ (in an arbitrary
positive integer radix $b$) are input starting with the most significant digit, a
TCMI can compute the value of $n$ online, i.e. the value of $n$ is available in
one of the counters as soon as the input is stopped (i.e. no stop marker is
necessary). Somewhat surprisingly, it can be shown that no TCMI is capable
of such a feat if the digits of $n$ are input starting with the least significant
one (see Theorem 3). It is unknown to the authors whether a TCMI can
compute the value of $n$ if a stop marker is a part of the input alphabet
along with the radix $b$ digits. We also present some evidence that a TCMI
may in some sense be computationally weaker than TCM when the input
to the TCMI is presented least significant digit first. Namely we show (see
Theorem 4 and Theorem 5) that for an unbounded function $f$ computable by a TCM (in the sense that such a TCM halts with the value of $f(n)$ after having been ‘loaded’ $n$ in one of its counters) a TCMI that outputs $f(n)$ upon being presented the digits of $n$ in least to most significant order exists if and only if a TCM exists that can compute $n$ using the same input (in which case a TCMI can obviously compute $f(n)$). This asymmetry with respect to the order in which the input is presented is rather unexpected in light of the result about the robustness of automatic sequences mentioned above and the intuitive perception that a DFAO has a very limited ‘memory’ compared to a TCMI.

Before proceeding with the formal definitions and statements of the main results of this work, it is instructive to take another look at the arguments in [5]. The core of W. Clinger’s proof is formed by his Lemma 9 (see [5] or Lemma 4 below for a slightly weaker statement) and Kronecker’s lemma each highlighting different aspects of the dynamic behavior of irrational numbers such as $\log_d D$ for an incommensurable pair $(d,D)$ (the irrationality of $\log_d D$ can be taken as the definition of incommensurability of such a pair).

Such dynamic behavior manifests itself in many areas of science and mathematics. For some interesting connections to other areas of mathematics see, for example, [2], Ch. 3, Exercise 4 that illustrates a curious relationship between Poincaré’s recurrence theorem (similar in spirit to Kronecker’s lemma) and the digits of powers of 2. Also closely related to this subject is Benford’s law of digit distribution (see [4] and [15], as well as [7]; [16] provides a curious application of Benford’s law to economic forensics) which emphasizes a statistical facet of log type dynamics.

On the other hand, a simple but clever Lemma 9 of [5] quickly leads to deep number theoretic questions such as the normality and automaticity of $\log 2$ and similar numbers, automaticity of the digit sequence of $\sqrt{2}$, etc. (see [1]) if one wishes to obtain a stronger inequality.

2 Basic definitions and notation

To provide some motivation for the results in this section let us begin by restating the problem of computing a conversion as a problem of recognizing the digits of the result of the conversions. Lemma 10 of [5], declared ‘redundant’ above takes on unexpected significance as it seems that the part of the proof in [5] based on Kronecker’s lemma (see [6]) does not lend itself to a
similar generalization.

Below we use the notation \( a\{p\} \), where \( p \in \mathbb{N} \), and \( a \) is a letter in some alphabet \( \Gamma \), to mean a string in \( \Gamma^p \) that is a concatenation of \( p \) copies of \( a \).

Following the established tradition, we also use \( \{\theta\} \) to denote the fractional part of \( \theta \) (i.e. \( \{\theta\} = \theta \mod 1 \)). This should not cause any confusion with the use of \( a\{k\} \) as a regular expression for a string of \( a \)'s. Note that \( \{x + y\} \) is continuous at every \( (x, y) \) such that \( (x + y) \mod 1 \neq 0 \).

Given the ‘input radix’ \( D \), the ‘output radix’ \( d \), and the ‘exponent radix’ \( b \), consider the following language 
\[ P = \{10\{p\} \mid p \in \mathbb{N}\} \]
where \( 1 \) and \( 0 \) are \( b \)-digits. Let \( n \geq 1 \) be fixed. This language can be partitioned into \( P_{m,n} \), 
\[ d^{n-1} \leq m < d^n, \]
where
\[ P_{m,n} = \{10\{p\} \mid m \times d^k \text{ is the best } n \text{-digit approximation of } D^p\} \]

We again assume that \( n \geq 2 \) if \( d = 2 \). It is easy to see that if one of \( P_{m,n} \) is not a regular language, then the sequence \( S_N(D, d) \) above is not \( b \)-automatic thus proving that radix conversions cannot be computed using finite automata. For some combinations of \( b \) and \( d \) we have the following stronger statement:

**Lemma 1.** Suppose that \( d \) and \( D \) are incommensurable. Provided \( b^2/\left( (b-1) \right) < 2d^{n-1} \log d \), some of \( P_{m,n} \) are not context-free.

**Proof.** Suppose \( P_{m,n} \) is context free for each \( d^{n-1} \leq m < d^n \). Then Lemma 2 implies that for each such \( m \) there exist \( p, q \in \mathbb{N} \) such that for any \( k \in \mathbb{N} \)
\[ 10\{p + kq\} \in P_{m,n} \]. Putting \( Q \) to be the product of all \( q \)'s we conclude that for a large enough \( p \), if \( 10\{p\} \in P_{m,n} \) then so is \( 10\{p + Q\} \). Using Lemma 9 of [5] there is an arbitrarily large \( k \in \mathbb{N} \) such that
\[ \frac{b - 1}{b^2} \leq \left\{ b^k(b^{p+Q} - b^p) \log_d D \right\} < \frac{b^2 - b + 1}{b^2} \]
The same argument as that of Lemma 10 of [5] shows that this contradicts \( 10\{p + k\} \) and \( 10\{p + k + Q\} \) both being in \( P_{m,n} \). \( \square \)

A slightly surprising feature of the proof above is its dependence on a particular relationship between the different radices. Intuitively, no such dependence should exist. The trivial nature of the languages \( P_{m,n} \) also suggests that the inequality of Lemma 9 of [5] could be improved if more had been known about the distribution of \( b \)-digits of \( \log_d D \) for different radices \( b \). It seems, however, that even the most basic questions of this kind (such as, how
often, if at all, a certain digit appears in the decimal expansion of $\log_d D$) are rather hard (see [1] or [6] for some examples).

We use standard definitions for most concepts appearing in this paper as well as their natural extensions. If $z$ is a string of $b$-digits for some radix $b$, we write $\lceil z \rceil$ to indicate the value of the corresponding number in radix $b$ where we assume that the least significant digit of $z$ is the leftmost one. If the least significant digit of $z$ is the rightmost one we let $\lfloor z \rfloor$ stand for the value of $z$ in radix $b$.

In the proofs below we only use the natural correspondence between PDAs and CFLs (see [9]) and thus do not need the definition of a PDA. Since a deterministic PDA provides a good introduction to two counter machines treated later let us define this narrower concept.

A deterministic push-down automaton (see [9], we assume for simplicity that the stack is changed one symbol at a time) $M$ is defined as a 7-tuple $(Q, \Sigma, \Gamma, q_0, Z_0, F, \delta)$ where $Q$ is a finite set of states, $\Sigma$ is the input alphabet, $\Gamma$ is the stack alphabet, $q_0 \in Q$ is the initial state, $Z_0$ is the start symbol, $F \subseteq Q$ is a set of final states, and the partial function $\delta : Q \times (\Sigma \cup \{ \epsilon \}) \times \Gamma \to Q \times \Gamma$ is a collection of moves. Some additional restrictions are placed on $\delta$: each move $\delta(q, a, x) = (p, \cdot)$ is either a pop move (where $\cdot = \epsilon$), i.e. “if the input symbol is $a$, the top stack symbol is $x$ and the current state is $q$, remove $x$ from the stack and go to state $p$”, a push move with a similar natural meaning, namely, “under the circumstances as above, push $\cdot$ on the stack”, or a no change move. If $a = \epsilon$, the interpretation of $\delta(q, \epsilon, x)$ is “ignore the input for the moment, do something to the stack and go to the next state . . . ”. We require that whenever $\delta(q, \epsilon, x)$ is defined, no other $\delta(q, \cdot, x)$ is defined (i.e. the DPDA is never asked to choose whether or not to consume the input). Such moves are called $\epsilon$-moves and can be thought of as the post- or preprocessing performed by the DPDA. Naturally, when the stack is empty, no pop moves are possible. Empty stack can be recognized when the special symbol $Z_0$ is on top of the stack. $Z_0$ cannot be popped or pushed.

The input alphabet is $\Sigma = \{ 0, 1, \ldots, b - 1, \diamond \}$ everywhere below, where $b$ is some fixed radix. The existence of $\epsilon$-moves is the reason the input alphabet includes a stop marker, $\diamond$, to give $M$ one more chance at processing the stack it has accumulated. We will assume that $\diamond$ means the input is finished and that it appears only once at the end of the input. This is not part of a standard DPDA definition but is assumed everywhere below. If the sequence of $\epsilon$-moves following the appearance of $\diamond$ in the input does not affect the stack, we say that $M$ processes its input online (this concept becomes much
more important for two counter machines with input defined later).

One can view $M$ as a machine (which is, indeed, the terminology often used in this context) that starts in $q_0$ with only $Z_0$ on the stack then reads and processes its input one symbol at a time, until it sees $\diamond$ (in the general case this is unnecessarily restrictive but we will always follow this convention) upon which it enters the final phase of processing consisting of some $\epsilon$-moves until it ends up in one of the states in $F$. The particular state of $M$ at the end of the computation is $M$'s output and can be thought of as a finite encoding of the result $M$ is built to produce.

3 Radix conversions and PDAs

To show that PDAs cannot compute radix conversions, we again restate the computation problem as a recognition problem for the languages defined below.

**Definition 1.** Let $b$, $d$, and $D$ be fixed radices. Let $n = 1$ if $d > 2$ and $n = 2$ if $d = 2$. Define $L_d$ to consist of all sequences of $b$-digits $z$ such that the best $n$-digit approximation in radix $d$ of $D^\zeta$ is $d$ if $d > 2$ or $1d$ if $d = 2$. $M_d$ can be defined similarly with $D^\lceil z \rceil$ instead.

Just as before, it is immediate that if one of the $L_d$'s or $M_d$'s is not context-free there is no PDA that computes the best $n$-digit floating point approximation of $D$ where $e$ is presented in radix $b$ in the appropriate order.

To show that one of the $L_d$'s as well as one of the $M_d$'s is not context-free we modify the standard pumping lemma (see e.g., Theorem 7.18 of [9]) to pump without disrupting the prefix and the suffix. The result of the lemma is also a corollary of a very powerful **Multiple Pumping Lemma** (see [11], Theorem 1.82). The proof of the Multiple Pumping Lemma has not been published, however, so we present the following direct proof instead.

**Lemma 2.** Suppose that $L$ is a CFL. Let $a \in \Sigma$ and $u, w \in \Sigma^*$ be fixed. Then there exists a number $N$ such that if $ua \{n_1\}w \in L$ with $n_1 \geq N$ then we can find $1 \leq \Delta \leq N$ satisfying $ua \{n_1 + i\Delta\}w \in L$ for $i \geq -1$.

**Proof.** If $|uw| = 0$ then the claim is obvious from the standard pumping lemma. Thus, we assume $|uw| \geq 1$. The proof follows that of Theorem 7.18 of [9] almost exactly, using a Chomsky normal form (CNF) grammar which expresses $L$. 7
Let the CNF grammar have $m$ variables, and set $m' = (|uw| + 1)m$ and $N = 2^{m'}$. Suppose that $ua\{n_1\} w \in L$ with $n_1 \geq N$. Then the longest path of a parsing binary tree for $ua\{n_1\} w$ has $(m'' + 2)$ edges with $m'' \geq m'$, containing $m''$ productions of the form $A_i \rightarrow A_{i+1}B_{i+1}$ or $A_i \rightarrow B_{i+1}A_{i+1}$ for $i = 1, \ldots, m''$, starting from the root $A_1$. We can find a list of consecutive variables $A_{i_1}, A_{i_1+1}, \ldots, A_{i_2}$ of size at least $m$ such that the subtree $B_{i+1}$ generates a substring of $a\{n_1\}$ for every $i_1 \leq i \leq i_2$, and consequently we can find $A_{j_1} = A_{j_2}$ such that $i_2 - m + 1 \leq j_1 < j_2 \leq i_2 + 1$.

We write the yields of the subtrees $A_{j_1}$ and $A_{j_2}$ as $xvy$ and $v$ respectively.

Next we proceed with the number theoretic results that take advantage of the combinatorial lemma above. The first statement is the famous Kronecker’s lemma in a slightly weaker form than the original (see [6] for the full version and the proof).

**Lemma 3** (Kronecker’s lemma, see [6]). The set $\{\{n\theta\} | n \in \mathbb{N}\}$ is dense in $(0, 1)$ for every irrational $\theta > 0$.

The following lemma is an easy corollary of [5], Lemma 9 whose proof is based on the analysis of the fractional part of $\theta$ presented in radix $C$.

**Lemma 4** ([5]). Let $\theta > 0$ be irrational, $C > 1$ be a natural number. Then there exist infinitely many $m > 0$ such that for $\alpha = \{C^m \theta\}$, $\beta = \{C^{m+1} \theta\}$:

$$C^{-2} < |\alpha - \beta| < 1 - C^{-2}$$

Unfortunately, the inequality above is too weak for our goals and has to be amended. At present we do not have the number theoretic tools to produce a ‘clean’ proof of a better inequality and have to take an indirect approach, instead, by modifying a few digits of the number. The first modification puts the iterate of $\theta$ in one of the two ranges. It seems that it should be possible to ensure that it ends up in a specific range, however it is unclear how to do that at the moment.
Lemma 5. Let $\theta > 0$ be irrational, $C > 1$ be an integer. Then there exists an integer $K > 0$ and an infinite sequence $m_1, \ldots, m_i, \ldots$ of integers such that either

$$1/3 \leq \lim_{i \to \infty} |\{KC^{m_i}\theta\} - \{KC^{m_i+1}\theta\}| \leq 1/2$$

or

$$1/2 \leq \lim_{i \to \infty} |\{KC^{m_i}\theta\} - \{KC^{m_i+1}\theta\}| \leq 2/3$$

Proof. Using Lemma 4 find an infinite sequence $m_1, \ldots, m_i, \ldots$ such that $C^{-2} < |\{C^{m_i}\theta\} - \{C^{m_i+1}\theta\}| < 1 - C^{-2}$. Using the sequential compactness of $[C^{-2}, 1 - C^{-2}]$, and picking a convergent subsequence if necessary, assume that $L = \lim_{i \to \infty} |\{C^{m_i}\theta\} - \{C^{m_i+1}\theta\}|$ exists and $C^{-2} \leq L \leq 1 - C^{-2}$. If $L$ is irrational, use Kronecker’s lemma to find $K > 0$ such that $1/3 < \{KL\} < 1/2$. Otherwise, $L = p/q$ where $p$ and $q$ are relatively prime and $q \geq 2$. Let integers $m$ and $n$ be chosen so that $np + mq = 1$; then $nL = 1/q - m$. Thus $\{|n|L\} = 1/q$ or $\{|n|L\} = 1 - 1/q$ depending on the signs of $m$ and $n$. After multiplying $|n|$ by $q - 1$ if necessary, we can assume $\{|n|L\} = 1/q$. Putting $K = |n|\lfloor q/2 \rfloor$, we have $1/3 \leq \{KL\} \leq 1/2$.

Now, for each $i$, either $|\{KC^{m_i}\theta\} - \{KC^{m_i+1}\theta\}| = \{K|\{C^{m_i}\theta\} - \{C^{m_i+1}\theta\}|\}$ or $|\{KC^{m_i}\theta\} - \{KC^{m_i+1}\theta\}| = 1 - \{K|\{C^{m_i}\theta\} - \{C^{m_i+1}\theta\}|\}$. Thus, after possibly choosing a proper subsequence, either $\lim_{i \to \infty} |\{KC^{m_i}\theta\} - \{KC^{m_i+1}\theta\}| = \{KL\}$ or $\lim_{i \to \infty} |\{KC^{m_i}\theta\} - \{KC^{m_i+1}\theta\}| = 1 - \{KL\}$.

The second modification establishes the desired inequality.

Lemma 6. Let $\theta > 0$ be irrational, $\alpha_i$ and $\beta_i$, $i = 1, 2, \ldots$ be such that $\alpha_i, \beta_i \in (0, 1)$ and either $1/3 \leq \lim |\alpha_i - \beta_i| \leq 1/2$ or $1/2 \leq \lim |\alpha_i - \beta_i| \leq 2/3$. Then there is an increasing subsequence $i(j)$, $j = 1, 2, \ldots$ and an integer $q \geq 0$ such that $1/3 \leq \lim |\{\alpha_{i(j)} + q\theta\} - \{\beta_{i(j)} + q\theta\}| \leq 1/2$.

Proof. The only nontrivial case is $1/2 \leq \lim |\alpha_i - \beta_i| \leq 2/3$. Picking a subsequence if necessary, assume $\alpha_i > \beta_i$ (the other case is similar) and $\lim \beta_i$ exists. Using Kronecker’s lemma, pick an integer $q \geq 0$ so that $\{\lim \beta_i + q\theta\} > 1 - 1/4$. Then $m - 1/4 < \lim \beta_i + q\theta < m$ for some integer $m > 0$ so $m + 2/3 > \lim \alpha_i + q\theta > m + 1/4$. Put $\alpha = \{\lim \alpha_i + q\theta\}$ and $\beta = \{\lim \beta_i + q\theta\}$.

Now $\alpha < 2/3 < 3/4 < \beta$ and $\lim |\{\beta_i + q\theta\} - \{\alpha_i + q\theta\}| = |\{\lim \beta_i + q\theta\} - \lim \{\alpha_i + q\theta\}| = |\{\lim \beta_i + q\theta\} - \lim \alpha_i + q\theta| = \beta - \alpha$. Thus $\lim \beta_i + q\theta = (m + 1) + \beta$ and $\lim \alpha_i + q\theta = m + \alpha$ imply $\lim |\alpha_i - \beta_i| = \lim (\alpha_i + q\theta) - (\beta_i + q\theta) = (m + \alpha) - (m - 1 + \beta) = 1 - (\beta - \alpha)$. Hence $1/3 \leq \beta - \alpha \leq 1/2$. □
The lemma below deals with pairs of irrationals. It establishes a finite bound on the number of iterations required to put each in a desired ‘slot’.

**Lemma 7.** Let \( \theta \in \mathbb{R} \) be irrational, 0 < \( a < b < 1 \), \( \epsilon > 0 \). Then there exists an \( n(\theta, \epsilon, a, b) \in \mathbb{N} \) such that for any \( \alpha, \beta > 0 \) satisfying \( \{\beta\} < \{\alpha\} \) and \( b - a + \epsilon < \{\alpha\} - \{\beta\} < 1 - \epsilon \) there exists a \( k \leq n(\theta, \epsilon, a, b) \) such that \( \{\beta + k\theta\} < a \) and \( \{\alpha + k\theta\} > b \).

**Proof.** Kronecker’s lemma implies the existence of \( n(\theta, \epsilon, a, b) \) such that for any \( \tau \geq 0 \) the set \( P = \{\{\tau + k\theta\} \mid k < n(\theta, \epsilon, a, b)\} \) has the property \( P \cap (\max\{0, a - \epsilon\}, a) \neq \emptyset \) and \( P \cap (\max\{b, 1 - \epsilon\}, 1) \neq \emptyset \). Now consider two cases.

1. \( \{\alpha\} - \{\beta\} \leq 1 - a \). Let \( k < n(\theta, \epsilon, a, b) \) be such that \( \{\beta + k\theta\} \in (\max\{0, a - \epsilon\}, a) \). Let \( \{\beta\} + k\theta = m + \{\beta + k\theta\} \) for some integer \( m \geq 0 \). Then \( \{\alpha\} + k\theta \geq m + \{\beta + k\theta\} + (1 - a) < m + a + (1 - a) = m + 1 \). Also \( \{\alpha\} + k\theta \geq m + \{\beta + k\theta\} + b - a + \epsilon > m + a - \epsilon + b - a + \epsilon = m + b \). Thus \( \{\alpha + k\theta\} = \{\alpha\} + k\theta - m > b \).

2. \( \{\alpha\} - \{\beta\} > 1 - a \). Let \( k < n(\theta, \epsilon, a, b) \) be such that \( \{\alpha + k\theta\} \) is in \( (\max\{b, 1 - \epsilon\}, 1) \) and \( \{\alpha\} + k\theta = m + \{\alpha + k\theta\} \) for some integer \( m \geq 0 \). Then \( \{\beta\} + k\theta \geq m + \{\alpha + k\theta\} - (1 - \epsilon) > m + (1 - \epsilon) - (1 - \epsilon) = m \) and \( \{\beta\} + k\theta \leq m + \{\alpha + k\theta\} - (1 - a) < m + 1 - (1 - a) = m + a \). Hence \( \{\beta + k\theta\} = \{\beta\} + k\theta - m < a \). \( \square \)

Note that the choice of \( k \) in the lemma above implies that \( \{\alpha + k\theta\} - \{\beta + k\theta\} = \{\alpha\} - \{\beta\} \).

The results above are put to use in the following theorem. Recall that an \( n \)-digit best floating point approximation in radix \( d \) to a real \( r \) is a floating point number \( m \times d^\Delta \) such that \( r = (m + \epsilon) \times d^\theta \) where \( d^{n-1} \leq m < d^n \), \( |\epsilon| \leq 1/2 \) and where \( m = d^{n-1} \) only if \(-1/(2d) \leq \epsilon \). Such an approximation is unique whenever \( \epsilon < 1/2 \) which is the only case we need below.

**Theorem 1.** If \( d \) and \( D \) are incommensurable, \( d > 2 \) or \( n > 1 \) then there are \( d_1, d_2 < d \) such that \( L_{d_1} \) and \( M_{d_2} \) are not context-free.

**Proof.** Put \( C = b^\Delta \) and \( \theta = \log_d D \) where \( b \) is the radix in which the exponent is presented and \( \Delta \) is as in Lemma [2]. Use Lemma [5] to find a sequence of \( m_i \)'s and a \( K > 0 \) such that either \( 1/3 \leq \lim_{i \to \infty} |\{KC^{m_i}\theta\} - \{KC^{m_i+1}\theta\}| \leq 1/2 \) or \( 1/2 \leq \lim_{i \to \infty} |\{KC^{m_i}\theta\} - \{KC^{m_i+1}\theta\}| \leq 2/3 \) holds. Now apply Lemma [6] and possibly rename the terms of the sequence to find \( q \) so that \( 1/3 \leq \lim |\{KC^{m_i+1}\theta + q\theta\} - \{KC^{m_i}\theta + q\theta\}| \leq 1/2 \).
Lemma 2 implies that both \( \{K^\alpha \theta + q \theta \} \) and \( \epsilon \). Choose \( \log_{D_\epsilon} \) to find an \( n(\theta, \epsilon, a, b) \) such that for any \( i \) there exists a \( k < n(\theta, \epsilon, a, b) \) with the property that \( \{K^\alpha \theta + q \theta + k \theta \} < a \) and \( \{K^\alpha \theta + q \theta + k \theta \} > b \). Thinning out the sequence again one can assume that there is an integer \( p \geq 0 \) such that for every \( i \) \( \{K^\alpha \theta + q \theta + p \theta \} < a \) and \( \{K^\alpha \theta + q \theta + p \theta \} > b \).

Let \( u \) be the \( b \)-digits of \( p + q \), and \( w \) be \( b \)-digits of \( K \). Use Lemma 2 and choose \( n_0 \) so that \( n_1 = n_0 \Delta - |u| \geq N \). Then pick \( m_i \) large enough so that \( m_i > n_0 \), and \( m_i \) satisfies an additional property mentioned below. Lemma 2 implies that both

\[
0 < |m_i - n_0 \Delta| \leq 1
\]

are in the same \( L_d \) (or \( M_d \) after the strings have been reversed accordingly).

Thus the \( d \)-significands of \( D^\Delta_{Kb} \) and \( D^\Delta_{Kb} \) are the same.

First suppose \( d > 2 \), \( D^\Delta_{Kb} = \sum_{j=0}^{N-1} a_j d^j \), and \( D^\Delta_{Kb} = \sum_{j=0}^{N-1} a_j d^j \) where \( a_j, b_j < d, a_N > 0 \), and \( b_M > 0 \). Now \( \{K^\alpha + q + p \theta \} = \{(Kb^\alpha \theta + q + p \theta)\} = \log_d(a_N + \sum_{j=0}^{N-1} a_j d^j - N) \) therefore picking \( a = \log_d(1 + \eta) \) with \( \eta \leq 1/2 \) implies that the first digit of the significand of \( D^\Delta_{Kb} \) is 1. Let \( b = \log_d 2 \) and suppose it is possible to choose \( \eta \leq 1/2 \) so that \( \log_d(1 + \eta) + 1/2 < \log_d(d - 1/2) \). Then for some \( e' > 0 \) \( \log_d(1 + \eta) + 1/2 + e' < \log_d(d - 1/2) \). Let \( m_i \) be chosen large enough so that \( \{K^\alpha \theta + q \theta \} - \{K^\alpha \theta + q \theta \} < 1/2 + e' \). Now \( \log_d(b_M + \sum_{j=0}^{M-1} b_j d^j) = \{(K^\alpha \theta + q \theta)\} \) and \( \{K^\alpha \theta + q \theta \} = \{(K^\alpha \theta + q \theta)\} = \{(K^\alpha \theta + q \theta)\} < \log_d(1 + \eta) + e' < \log_d(d - 1/2) \). Thus the first digit of the significand of \( D^\Delta_{Kb} \) is between 2 and \( d - 1 \) contradicting our assumption.

The case of \( d = 2 \) is somewhat special since the first digit of the significand is always 1. In this case we pick \( a = \log_2(1 + \eta) \) for \( \eta < 1/4 \), \( b = \log_2(3/2) \) and \( e' > 0 \) so that \( \log_2(1 + \eta) + 1/2 + e' < \log_2(1 + 1/2 + 1/4) \). An argument
similar to the one for $d > 2$ shows that the second digit of the significand of $D^{k^*} b^{(m_1 + 1) \Delta + q + p}$ is 0 while the second digit of the significand of $D^{k^*} b^m \Delta + q + p$ is 1.

To complete the proof it is sufficient, for every $d > 1$, to pick $\eta \leq 1/2$ ($\eta < 1/4$ in the case $d = 2$) so that $\log_d (1 + \eta) + 1/2 < \log_d (d - 1/2)$ and $\log_d 2 - \log_d (1 + \eta) < 1/3$ (respectively $\log_2 (1 + \eta) + 1/2 < \log_2 (1 + 1/2 + 1/4)$ and $\log_2 3/2 - \log_2 (1 + \eta) < 1/3$ for $d = 2$).

Consider three cases.

1. $d \geq 4$. Put $\eta = 1/2$. Now it can be easily verified that $\log_4 2 - 1/2 < \log_4 (3/2) < 1/3$. Since $\log_2 r$ decreases as $x > 0$ increases for any $r > 1$ it follows that $\log_d 2 - \log_d (3/2) < 1/3$. A simple computation also shows that $\log_d (3/2) + 1/2 < \log_d (d - 1/2)$ for $d \geq 4$.

2. $d = 3$. We must pick an $\eta \leq 1/2$ so that $\log_3 2 - 1/3 < \log_3 (1 + \eta) < \log_3 (5/2) - 1/2$. That such $\eta$ exists follows from $23^{-1/3} - 1 < 1/2$ and $52^{-1/3} - 1/2 > 23^{-1/3}$.

3. $d = 2$. We must pick $\eta \leq 1/4$ so that $\log_2 (3/2) - 1/3 < \log_2 (1 + \eta) < \log_2 (3/2 + 1/4) - 1/2$. The existence of $\eta$ is a consequence of $32^{-1/4} - 1 < 1/4$ and $74^{-1/2} - 1/2 > 32^{-1/3}$.

\[\square\]

4 Clinger’s problem for two counter machines

Implicit in [5] is the following problem.

**Definition 2.** Consider the problem of determining whether a specific class of machines can compute the best $n$-digit approximation in radix $d$ to $f \times D^e$, where $f$ and $e$ are integers and $f$ is positive. If a machine from a given class can compute the significand of the best approximation we say that such a machine “solves Clinger’s problem of the floating point arithmetic”.

D. Matula (see [15] and [5]) demonstrated that this problem can be solved by a deterministic finite automaton (DFA) if $d$ and $D$ are commensurable. By Theorem 1 we now know precisely that it is the only affirmative result for PDA (or DFA), which we state as the following corollary.

**Corollary 1.** Clinger’s problem of the floating point arithmetic can be solved by a PDA (or DFA) if and only if $d$ and $D$ are commensurable.

Having seen that a DPDA is too limited to compute radix conversions, it is natural to seek a more powerful machine to accomplish the task. A
straightforward modification of the DPDA concept immediately leads to the definition of a deterministic push-down automaton with two stacks (DPDA2S for short). Such automata, however, are too powerful for our purposes. To limit their computational power we can consider DPDA2S whose stack alphabets are limited to two symbols each (two, because we need a special $Z_0$ that indicates the bottom of each stack). It is easy to see that the configuration of such an automaton can be described by its current state and the two values representing the number of symbols other than $Z_0$ currently on the stack. Hence, each stack acts simply as a counter and the construction just presented defines a two counter machine with input or TCMI for short.

A TCMI is capable of carrying out very sophisticated computations even when no input is present. Indeed, when the only moves a TCMI is allowed are $\varepsilon$-moves, we arrive at the concept of a two register Minsky machine (TCM). As M. Minsky showed in [14], for any recursive function $f$ there is a TCM that computes $f(n)$ in the sense that a computation that starts with $2^n$ in one register (counter) and zero in the other terminates with the value of $2^{f(n)}$ in one of the registers and zero in the other ([14] used $3^2n$ to encode the output but $2^n$ is enough, see [3]). Another result in [14] shows that the addition of yet another, third register would allow such a machine to compute $f(n)$ directly, starting with $n$ (rather than $2^n$).

A number of authors had proved that the addition of a third register is indeed necessary, and that the exponential encoding cannot be bypassed if one wishes to retain the full computational power of a TCM (see [17], [10], [3], and [8]). The authors of [10] prove that even a characteristic function of the set of exact squares, for example, is not computable by a TCM. Neither are elementary functions such as $2^n$, $n^2$, or $\lfloor \log n \rfloor$ (see [3] and [17]).

A TCM can be thought of as a program in a simple language \{ $x \leftarrow x + 1$, $x \leftarrow x - 1$, if $x = 0$ then goto $l$, goto $l$, halt \} where $x$ can be one of the two variables. A TCMI extends this language to include a number of ‘if $I = d$ goto $l$’ commands, one for each possible value $d$ of the special input variable $I$. It is convenient to think of a TCMI as a collection of several TCM’s sharing the two counters, each TCM dedicated to the processing of a specific input symbol. If such a TCM that processes the stop marker, $\diamond$ never changes the counters, we say that the TCMI processes its input online by analogy with the DPDA case. After each TCM is done with the processing it arrives at a line (which we will think of as a state, $w$) with one of the input commands. We will refer to such a line as a wait state.

We can show that a TCMI is capable of performing the task of radix con-
versions, and present a brief sketch of the proof of the following proposition.

**Proposition 1.** For every combination of radices $d$, $D$, and $b$, and any precision $n$ Clinger’s problem of the floating point arithmetic can be solved by a TCMI online.

**Proof.** The methods of [14] show that a TCM can simultaneously compute the values of several recursive functions $f_1, f_2, \ldots$ by keeping its input (and output) encoded in the form $2^{n_1}3^{n_2}5^{n_3} \ldots$. It should be noted that the original arguments of [14] used a more sophisticated encoding, namely $3^{2^n}$ for the output but several authors (see e.g. [3]) had observed that $p^n$ used for the input as well as $p^{f(n)}$ for the output where $p$ is some prime would suffice.

Thus if $n_1$ holds the value of the number input ‘so far’, $n_2$ holds the number of digits seen (to take into account any leading zeros in the case of a least significant digit first input), and $n_3$ represents the significand of the best $n$-digit approximation using some encoding, the computation can proceed by decoding the value of $5^{n_3}$ using successive divisions (of which there could only be a bounded number as there are only finitely many values a significand can assume).

Now, upon seeing the next digit the TCMI simply increments $n_2$ (multiplying the counter by 3), and computes the new value of $n_1$ using a straightforward computation followed by the computation of $n_3$ which is obviously a recursive function of $n_1$. \hfill \Box

## 5 Counting and the asymmetry of input

It is still unknown (at least to the authors) whether a TCM is capable of decoding its own output, i.e., for example to halt with the value of $k$ after having started with $2^k$ in one of the registers. It is known that a TCM is incapable of computing $\lfloor \log_2 k \rfloor$ (see [17]) but as R. Schroeppe points out in [17] these problems are not equivalent since a TCM that computes $k$ from $2^k$ is not even assumed to halt on any input other than $2^k$.

It is thus natural to ask whether a TCMI exists that decodes its input, i.e. terminates with the value of $n$ after being input the digits of $n$ followed by $\diamond$. This problem can be thought of as a conversion to radix 1 if desired. If the nature of the input is restricted, such decoders are certainly possible (see Proposition 2 below).
Let a TCMI $M$ compute the value of $n$ after being input the digits of $n$ in some radix $b$ followed by $\diamond$. If the digits are assumed to be presented starting with the least significant one, we say that $M$ *counts in radix* $b$ or simply *counts in* $b$. If the digits are presented starting with the most significant one we say that $M$ *counts in* $b$ *in reverse*. We have the following simple statement. Recall that computing *online* means that upon the input of $\diamond$ the values of the registers do not change.

**Proposition 2.** For any radix $b$ there exists a TCMI that counts in $b$ online in reverse.

**Proof.** When it is input $d$ the TCMI multiplies the value it has computed so far by $b$ and adds $d$ to the result. ☐

The authors of [10] introduce a number of tools that simplify dealing with TCM’s one of which is the idea of a *normalized* TCM (called nTCM below). By requiring that each individual TCM in a TCMI be normalized we arrive at the definition of a normalized TCM or nTCMI. While this concept is not required to carry out the proofs below, the handling of some border cases is simplified when a TCMI is actually an nTCMI so we tacitly assume that each TCMI is normalized (we only need this concept in Lemma 8 below). Since one of the results in [10] is the fact that each TCM can be replaced by an equivalent nTCM, this is not a significant restriction, and we thus omit the definition of nTCM and nTCMI here.

To simplify the notation, we will often record a configuration of a TCM $M$ as $(q^k, n^k_1, n^k_2)$ where $q^k$ is the state, $n^k_1$ and $n^k_2$ are the values of the counters, and $k$ is an integer index. Using this notation define $n^-_k = \min\{n^k_1, n^k_2\}$ and $n^+_k = \max\{n^k_1, n^k_2\}$. It is often important to know which counter holds the smaller (larger) value. We denote the index of the corresponding counter as $n^-_k = \min\{i \mid n^-_i = n^k_i\}$ ($n^+_k = \max\{i \mid n^+_i = n^k_i\}$ respectively). Sometimes we know only that one of the values is the value of the first counter, therefore the other value is the contents of the other, such as $n_u$ and $n_{3-u}$ for $u \in \{1, 2\}$. In such cases we write $(q, n_u n_{3-u})$, omitting the comma. Thus $(q, n_u n_{3-u}) = (q, n_1, n_2)$ regardless of what $u$ is.

To facilitate the handling of arguments involving computations performed by $M$ we introduce the notation $(w, n_1, n_2) \xrightarrow{d}(w', m_1, m_2)$ meaning $M$ will reach the configuration (although not necessarily halt) $(w', m_1, m_2)$ if started in $(w, n_1, n_2)$ and being input $d$. If $w = q_0$ and $n_1 = n_2 = 0$ we write $M \xrightarrow{d} \ldots$ instead.
Finally, each computation is analyzed in terms of *stages* or *phases* such that during each phase the values of both counters are nonempty. To simplify the terminology we introduce the notation \((w, n_1, n_2) \xrightarrow{I} (w', m_1, m_2)\) to mean that the first time \(M\) enters a configuration \((w'', m'_1, m'_2)\) such that \(m'_- = 0\) after having started in \((w, n_1, n_2)\) and while being input \(I\) (\(I\) can be a string so such a configuration can be encountered before \(I\) is fully consumed), this configuration is \((w'', m'_1, m'_2) = (w'', m_1, m_2)\). If having started in \((w, n_1, n_2)\) and being input a string \(I\) the machine never reaches a configuration with one of the counters empty, we write \((w, n_1, n_2) \xrightarrow{I} \emptyset\).

The analysis of a TCMI differs from that of a TCM in one important aspect: a TCM is usually started with at least one counter empty and terminates in a similar state. Simple arguments show that a TCM performing any useful computation can be restricted to operating in this manner (see Lemmas 9 and 10 below for one reason this is so). A TCMI is a different matter though. Imagine a TCMI that counts the number of 0’s in its input in one counter and the number of 1’s in the other, then multiplies the first value by the last digit seen (to make the example nontrivial assume that the radix is > 2) in the input and adds the two results. Even though the final output of the computation is a single value, it is unclear if the same result can be achieved by a TCMI that keeps only one of the counters nonempty before the next digit is input.

If in all such cases one of the values is bounded by a single constant this value can be kept in the final control ‘buffer’ instead, while emptying the corresponding counter. The advantage of this would be the availability of various reduction results for TCMs such as the one in [17] (see the proof of Theorem 3 below) for the analysis of the TCMI.

Our first goal is to show that such a bound exists for every TCMI that counts in some radix \(b\) thus showing that the anomaly described above does not occur in such TCMIs.

The following lemma is a corollary of Lemma 2.2 of [10] (see also [10], Lemma 2.3 and the definition of an MP1RM in [17] and below).

**Lemma 8.** Let \(s\) be the number of states of some \(nTCM\) \(M\). Let \((q^0, n^0_1, n^0_2), \ldots, (q^k, n^k_1, n^k_2)\) list all the stages of some computation performed by \(M\). Suppose \(n^i_- = 0\) and \(n^i_+ > s\) for all \(i \leq k\). Then there exist integer \(P, Q, R,\) and \(D\) such that for any \(m \geq 0\) and any computation \((q^0, m^0_1, m^0_2), \ldots, (p^k, m^k_1, m^k_2)\) such that \(m^0_0 = 0, -m^0 = -n^0,\) and \(m^0_+ = n^0_+ + mD\)

\[m^k_+ = P \lfloor m^k_+ / Q \rfloor + R\]

and \(p^i = q^i\) for \(i \leq k\).
The next statement describes what happens when both counters hold large enough values. As for most results of this kind, its proof is based on cycle analysis of the machine.

**Lemma 9.** Let \( s \) be the number of states of some TCM \( M \). Let \((q^0_0, n^0_1, n^0_2)\), \( n^0_2 > s+1 \) be a configuration of \( M \) with the following properties. \( M \) eventually reaches a waiting state when starting in \((q^0, n^1_0, n^2_0)\). There exists a \( k \geq 0 \) such that \((q^0, n^0_0, n^0_2), \ldots, (q^k, n^k_1, n^k_2)\) lists an initial stage of the computation performed by \( M \), where \( n^j_2 > 0 \) for \( j < k \), and either \( n^k_+ > n^k_- = 0 \) or \( M \) halts in \( q^k \) with \( n^k_- > 0 \). Then every configuration \((q^0, m^0_1, m^0_2)\) with \( m^i_0 \geq n^0_0 \), \( i = 1, 2 \) has the same property and one of the following two cases holds.

1. There exist \( r_1, r_2 \), with the property that \( |r_1| \leq s \) and \( m^k_1 = m^0_1 + r_1 \) for any computation \((q^0, m^0_1, m^0_2), \ldots, (p^k, m^k_1, m^k_2)\) such that \( m^0_1 \geq n^0_0 \), \( i = 1, 2 \). Moreover \( p^i = q^i \), \( k \leq s \), \( m^j_- > 0 \) for \( j < k \), and \( M \) halts in \( q^k \).

2. There exist integers \( \omega_1, \omega_2 \) with the following properties. Suppose \( m^0_1 \geq n^0_0 \), \( i = 1, 2 \). Then \((q^0, m^0_1, m^0_2) \Rightarrow (p^i, m^i_1, m^i_2)\) and the following additional properties hold.

   a. If \( \omega_1 \omega_2 \leq 0 \) then \( \omega_u \geq 0 \) and \( \omega_{3-u} < 0 \) for some \( u = 1, 2 \). Moreover there exist an \( r \) and a state \( q' \) such that \(|r| < 3s\), and whenever \( m^0_u > s+1 \), \( m^0_{3-u} = n^0_{3-u} + m\omega_{3-u} \), the state \( p' = q' \) and \( m^_+ = m^0_u + \omega_u m^0_{3-u}/|\omega_{3-u}| + r \).

   b. If \( \omega_1 \omega_2 > 0 \) then both \( \omega_1, \omega_2 < 0 \) and there exist \( r_1, r_2, |r_1|, |r_2| < 3s \), and states \( q \) and \( q' \) with the following properties. Suppose \( m^0_1 = n^0_1 + l_i |\omega_1| \) for some \( l_i > 0 \), \( i = 1, 2 \). Put \( u = \pm m^r \). Then \( m^0_+ = m^0_u + \omega_u m^0_{3-u}/|\omega_{3-u}| + r_u \) and \( p' = u q \). Moreover if \( m^0_u \) is replaced by a larger value, the length of the computation and the final state will not change and the expression above will remain valid. There also exists a constant \( D \) such that \( l_i > l_{3-i} + D \) implies \( i = u \). Finally, if \( l_1 = l_2 \) then \( p' = q^k \), \( u \) is independent of \( l_i \), and \( m^0_+ = n^0_u + \omega_u m^0_{3-u}/|\omega_{3-u}| + r_u \).

**Proof.** Suppose (1) does not hold. Then \( M \) does not reach a halting state before one of the counters is empty. Since \( n^0_2 > s+1 \), \( k > s+1 \). Therefore \( M \) enters a loop and passes through the following sequence of states (we are assuming \( M \) must halt eventually): \( q^0 \ldots q^{i_0} \sigma^0 \ldots \sigma^{i_{w-1}} q^{i_{w+1}} \ldots q^k \), where
\(\omega_0 < s, k - t\omega - \omega_0 < \omega \leq s\), and \((\sigma^{t+t'\omega}, n_1^{t+t'\omega}, n_2^{t+t'\omega}) = (\sigma^t, n_1^t + t'\omega_1, n_2^t + t'\omega_2)\) for some \(\omega_1\) and \(\omega_2\) such that \(|\omega_1|, |\omega_2| \leq \omega \leq s\). Thus \(M\) will reach the configuration \((q^k, n_1^k, n_2^k)\) with \(n_k^r = 0\) and \(n_k^s > 0\) for \(j < k\) for some \(k > 0\).

Suppose \(\omega_1, \omega_2 > 0\). Note then both \(\omega_1, \omega_2 < 0\). Otherwise the counters will be incremented indefinitely and the loop will not terminate. Thus for any configuration \((q^0, n_1^0, n_2^0)\) with \(m_i^0 \geq n_i^0, i = 1, 2\) there is a state \(p'\) such that \((q^0, m_1^0, m_2^0) \Rightarrow (p', m_1^0, m_2^0)\) with \(m_i^0 = 0\).

To prove (b) assume \(m_i^0 = n_i^0 + l_i|\omega_i|\) for some \(l_i > 0, i = 1, 2\), and \(u = \pm m_i\). Let \(u\) be \(p'\). As long as counter \(u\) stays nonempty throughout the whole computation, the value of \(l_u\) does not affect the final state. Therefore, \(p'\) depends only on the value of counter \(3 - u\) at the beginning of the last (possibly incomplete) iteration of the loop. This value is easily seen to be the same for any \(l_{3-u} > 0\) under the conditions listed above. It remains to pick \(r_u\). Let \(u_i, i = 1, 2\) be the amount counter \(i\) is incremented by before \(M\) enters the loop and let \(r_{3-u}'\) be the value of counter \(3 - u\) at the beginning of the last (possibly incomplete) iteration of the loop. Note that \(|u_i| < s\) and \(r_{3-u}' < s\) do not depend on the values of \(l_{3-u}\) and \(l_u\) as long as the conditions above hold (i.e. counter \(3 - u\) is emptied first). Now \(m_u^v = m_u^0 + u + \omega_u (m_{3-u}^0 + u_{3-u} - r_{3-u}')/|\omega_{3-u}| = m_u^0 + \omega_u [m_{3-u}^0/|\omega_{3-u}|] + r_u\) for some \(r_u\) independent of \(l_i\) and such that \(|r_u| < 3s\). Note that making the value of \(m_{3-u}^0\) larger at the beginning of the computation does not affect the argument above.

To find \(D\) simply note that it is sufficient to pick \(D\) large enough to ensure that counter \(i\) is nonempty after the loop has run \([(m_{3-i}^0 + s)/|\omega_{3-i}|] + 1\) times.

If \(l_1 = l_2\) then \(m_{r+}^v = m_{r+}^0 + l_u|\omega_u| - |\omega_u|[(n_{3-u}^0 + l_{3-u}|\omega_{3-u}|) + r_u = n_{r+}^0 + l_u|\omega_u| - |\omega_u|n_{3-u}^0/|\omega_{3-u}| + r_u = n_{r+}^0 + \omega_u [n_{3-u}^0/|\omega_{3-u}|] + r_u\). That \(p' = q^k\) follows from a simple observation that the values of both counters before the final iteration of the loop are the same whether \(M\) started in \((q^0, n_1^0, n_2^0)\) or \((q^0, m_1^0, m_2^0)\). Thus (b) holds.

Now suppose \(\omega_1, \omega_2 < 0\). Then, say \(\omega_1 < 0\) and \(\omega_2 > 0\). Thus counter 1 will be emptied first as long as \(m_{1}^0 > s + 1\) (to ensure that the value of counter 2 cannot become 0 during the first iteration of the loop). Now the argument similar to that for case (b) above finishes the proof of case (a).

If \(\omega_1 = 0\) then \(\omega_2 > 0\) would result in an infinite loop. Therefore \(\omega_2 < 0\) and counter 2 is emptied first. The rest of the argument is similar to the one for case (b) using the remark in the previous paragraph.

If a TCM goes through a configuration with an empty counter, the final
smaller value of the counter will be bounded. This is the statement of the following lemma.

**Lemma 10.** Let \( s \) be the number of states of some TCM \( M \). Let \((q_0, n_0^1, n_0^2), \ldots, (q_k, n_k^1, n_k^2)\) list a nontrivial last stage of some computation performed by \( M \) (i.e. \( n_0^i = 0, n_k^i > 0 \) for all \( 0 < i \leq k \) and \( M \) halts in \( q_k \)). Then \( n_k^i < s + 1 \) and any computation \((q_0, m_0^1, m_0^2), \ldots, (p_k, m_k^1, m_k^2)\) such that \( m_0^i \geq 2s + 1, -m_0^0 = -n_0^0, \) and \( m_0^0 = 0 \) satisfies \( m_k^i = n_k^i \) and \( p_i = q_i \) for \( i \leq k \).

**Proof.** First observe that the length of the last stage must be less than \( s + 1 \) in order for \( M \) to halt in \( q_k \) (otherwise \( M \) would have entered a loop which either does not terminate or empties one of the counters). Thus \( n_k^i < s + 1 \) and both \( n_k^+ \) and \( m_k^- \) are the contents of the counter that was nonzero at the beginning of the appropriate computation (i.e. the \(+m_0^0(= -n_0^0)\) th counter, we also use \( m_0^+ > 2s+1 \) here). Since the sequence of states \( M \) passes through must be identical in both cases, the other counter gets incremented exactly the same number of times in either computation.

**Corollary 2.** Let \( s \) be the number of states of some TCM \( M \) and \( d \) be an input symbol. Suppose \((w, n_0^1, n_0^2) \xrightarrow{d}(w', n_1^0, n_2^0)\) and \((w, n_1^0, n_2^0) \xrightarrow{d}(w'', m_1^0, m_2^0)\). Then \( m_+ < s + 1 \).

To ensure the applicability of Lemma 8 the following statement is used.

**Lemma 11.** For any \( K \) there exists a constant \( N_K > 0 \) such that any configuration \((q, n_1^0, n_2^0)\) that \( M \) passes through after inputting a string of digits longer than \( N_K \) satisfies \( n_0 > K \).

**Proof.** Otherwise \( M \) would output the same value for some strings of different lengths, a contradiction.

We can now turn our attention to the existence of the bound mentioned above.

**Lemma 12.** Suppose there exists a TCM \( M \) that counts in some radix \( b \). Let \( \{ w_i \mid i < \xi \} \) list all the waiting states of \( M \). Then there is a constant \( K > 0 \) such that for any input \((w_0, 0, 0) \xrightarrow{d_0}(w_{i(1)}, n_1^0, n_1^0) \xrightarrow{d_1} \ldots \xrightarrow{d_{\nu-1}}(w_{i(\nu)}, n_1^\nu, n_2^\nu)\) the bound \( n_1^i < K \) holds for \( i = 1, 2 \) and \( j \leq \nu \).
Before proceeding with the proof of Lemma 12 some preliminary properties need to be established.

**Lemma 13.** Suppose, using the notation of Lemma 12 there is no \( K \) with the claimed property. Then there exist a digit sequence \( \Omega \), a state \( w \), a constant \( s + 1 < v \leq (s + 1)^2 \), an index \( u \in \{1, 2\} \), and \( R_u > 0 \) and \( R_{3-u} < 0 \), \(|R_1|, |R_2| < (s + 1)^2\), with the following properties.

\( 1) \) for any \( K \) there exists a digit sequence \( I \) such that \( (w_0, 0, 0) \xrightarrow{I}(w, n_u n_{3-u}) \)
where \( n_u = v, n_{3-u} > K; \)

\( 2) \) \((w, m_1, m_2) \xrightarrow{\Omega}(w, m_u + R_u m_{3-u} + R_{3-u}) \) provided \( m_u \geq v, m_{3-u} > (s + 1)^2. \)

**Proof.** Let \( J \) be an input such that \( M \xrightarrow{J}(w', n'_1, n'_2) \) and \( n'_2 > \max\{K, (s + 1)^2\} \). If follows from Lemma 2 that there exists a string \( I' \) such that \( I' \leq J \), \( M \xrightarrow{I'}(w'', n''_1, n''_2) \) so that \( n''_2 < s + 1 \) and \( (w'', n''_1, n''_2) \xrightarrow{I''} \emptyset \) for any \( I'' \) such that \( I'I'' \leq J \). Let \( J = Id_1 d_2 \ldots d_k \) and \( (w'', n''_1 = n''_2 = n''_2) \xrightarrow{d_1}(w_1, n'_1, n'_2) \xrightarrow{d_2} \ldots \xrightarrow{d_k}(w^k = w', n^k_1 = n'_1, n^k_2 = n'_2) \) for some \( k \).

Since \((w^j, n^j_1, n^j_2) \xrightarrow{d_j} \emptyset \) one can apply Lemma 9(1) to find \( r^j_1 \) and \( r^j_2 \) such that \(|r^j_1| \leq s \) and \((w^j, m^j_1, m^j_2) \xrightarrow{d_j}(w^{j+1}, m^j_1 + r^j_1, m^j_2 + r^j_2) \) whenever \( m^j_1 > n^j \).

An easy inductive argument shows that \( w^0 \ldots w^k = \Lambda_1 \Lambda_2 \ldots \Lambda_l \), where \( l \leq s \) and \( \Lambda_j = w(j) \ldots w(j) \) or \( \Lambda_j = w(j) \). Let \( u \in \{1, 2\} \) be such that \( n''_u = n''_u \).

For \( \Lambda_j = w^{j'} \ldots w^{j''} \), \( j < l \) put \( R^j_u = \sum_{m=0}^{j''-j'} r^j_u + m \) and \( R^j_u = \sum_{m=0}^{j''-j'-1} r^j_u + m \). Then \( \sum_{m=1}^l R^m_u > (s + 1)^2 - (s + 1) = s^2 + s \), therefore there exists an \( m \) such that \( R^m_u \geq s + 1 \). Since \( |r^j_u| \leq s \) the sequence of states \( \Lambda_m = \Lambda w^{j''} \) for some \( \Lambda = w^{j''} \ldots \) such that \( w^{j''} = w^{j''} = w \). Let \( j' \) be the smallest such that \( m \) has the desired properties. Put \( \Omega = d^{j'+1} \ldots d^{j''} \), \( R^m = \sum_{j=j'}^{j''} r^j_1 = R^m_1 - r^{j''}_1 \) if \( m < l \) and \( R^m = R^m_1 \) if \( m = l \), where \( i = 1, 2 \). It follows from \(|r^j_1| < s + 1 \) that \( R_u \geq 0 \). Put \( v = n''_u \).

One can assume that the length of \( \Lambda \) is less than \( s + 1 \) using the following construction. Suppose \( \Lambda \) is longer than \( s + 1 \) and assume for the moment that \( K \) is large enough as explained below. Among the first \( s + 1 \) states in \( \Lambda \) find one, say \( q' \) that repeats at least twice. Let \( R'_u \) be the ‘contribution’ to the value of counter \( u \) by the subsequence of \( \Lambda \) that starts with \( q' \) and ends just before \( (q', \ldots) \) is reached again. If \( R'_u \geq 0 \) pick the corresponding input digits as \( \Omega \) and redefine \( R_u = R'_u \). Otherwise shorten \( \Lambda \) by removing
the subsequence. This will only increase $R_u$, and, if $n''_{u_3}$ is large enough will not cause counter 3 + $u$ to become 0 (counter u will only get larger than $n''_u$ where $w''^n$ is the first state after the subsequence above). Now $|R_v| < (s + 1)^2$. An argument similar to that of the proof of $R_u^n \geq s + 1$ shows that $s + 1 \leq v < (s + 1)^2$.

That $(w, m_1, m_2) \Omega \rightarrow (w, m_u + R_u, m_{3_u} + R_{3_u})$ for any $m_u \geq v, m_{3_u} > (s + 1)^2$ now follows from $|r_i| \leq s$. Since $\Omega$ is shorter than $s + 1$, counter 3 + $u$ will not become empty while the digits of $\Omega$ are input, whereas counter $u$ values will only increase if $\Lambda$ is trimmed as discussed above. Hence the sequence of states $M$ follows in $(w, m_1, m_2) \Omega \rightarrow \ldots$ is the same as long as $m_1$ and $m_2$ are large enough.

We have shown that for any $K$ there are $\Omega_K$ of length less than $s + 1$, $w_K$, $s + 1 < v_K \leq (s + 1)^2$, $u(K) \in \{1, 2\}$, $R_{u(K)} \geq 0$ and $R_{3_u(K)}$, $|R^K_1|, |R^K_2| < (s + 1)^2$ that satisfy property (2) in the statement of the lemma, as well as an $I_K$ such that $(w_0, 0, 0) I_K (w, n_u, n_{3_u})$ where $n_u = v$, $n_{3_u} > K$. Pick $\Omega, w, v, u, R_u, R_{3_u}$ such that $(\Omega, w, v, u, R_u, R_{3_u}) = (\Omega_K, w_K, v_K, u(K), R_{u(K)}, R_{3_u(K)})$ for arbitrarily large $K$. For any $K$ let $I(K)$ be an input such that $M I(K) (w, n_u, n_{3_u})$ where $n_u = v$, $n_{3_u} > K$.

To show that $R_{3_u} < 0$ suppose $R_{3_u} \geq 0$. Let $I = I((s + 1)^2)$. Then $M I(K) (w, n_u, n_{3_u})$ where $n_u = v$, $n_{3_u} > (s + 1)^2$. Consider what happens after the following input: $I \Omega \{l\} \Sigma_0$. In the trivial case of $R_u = R_{3_u} = 0$ inputting multiple copies of $\Omega$ does not change the configuration $M$ enters after inputting $I$. Thus $M$ produces the same value regardless of $l$, a contradiction. Hence either $R_u > 0$ or $R_{3_u} > 0$. If $(w, n_1, n_2) \Rightarrow \emptyset$, since $n_+ > s + 1$, Lemma [2] (1) shows that $(w, m_1, m_2) \Rightarrow (w', m_1 + C_1, m_2 + C_2)$ for some small $C_1$ and $C_2$ whose value does not depend on $m_1$ and $m_2$ as long as $m_+ > s + 1$.

Since $(w, n_1, n_2) \Rightarrow (q_1', n_1', n_2')$ for some $q_1', n_1' = 0$ the proof below will remain essentially the same if we assume that $(w, n_1, n_2) \Rightarrow (q_1', n_1', n_2')$. In this case Lemma [2] (2) applies since otherwise $n_+ > s + 1$ would imply $n_1' > 0$. Pick $\omega_1$ and $\omega_2$ as in Lemma [2], note that $(w, n_1, n_2) \Theta (\alpha(z)) \rightarrow (w, n_1 + zR_1l, n_2 + zR_2l)$ for arbitrarily large $l$ and pick and integer $z$ such that $|\omega_i|$ divides $zR_i$. Let $\Theta = \Omega \{z\}$, $Z_i = zR_i, i = 1, 2$, set $M \rightarrow (w, n_1, n_1) \Theta (l) (w, n_1 + Z_1l, n_2 + Z_2l) \Rightarrow (q_1', n_1', n_2')$ and consider the following cases, depending on $w$.

First suppose $\omega_1 \omega_2 \leq 0$. Suppose $\omega_1 \geq 0$ and use Lemma [2] property (a) to conclude that $n_{3_i} = 0, n_1' = n_i + lZ_i + \omega_i[(n_{3_i} + lZ_{3_i})|\omega_{3_i}|] + r =
In this case $M$ and $Al$ given by Lemma 9(1)(b). Thus $\omega_l \geq m$ does not depend on $l$. Second suppose $\omega_l \geq m$ does not depend on $l$. Then Lemma 9 implies that $n_u + lZ_u = n_u + lZ_u - \omega_u + (n_3-u + lZ_{3-u})/l(Z_{3-u}/\omega_{3-u}) + r = 0l + B$ where $B does not depend on $l$. Since $n_u + lZ_u = n_u + lZ_u + l(Z_{3-u}/\omega_{3-u}) = n_3-u + m|\omega_u|$ and $n_3-u + lZ_{3-u} = n_3-u + l(Z_{3-u}/\omega_{3-u}) = n_3-u + m|\omega_{3-u}|$ the state $q'$ or $n_u$ does not depend on $l$, $n'_u$ stays the same for all $l$ larger than some constant. $M$ is allowed to continue then following the input of $\phi$ it will produce the value of $n_1 + lZ_i + \omega_i l(Z_{3-i}/\omega_{3-i}) + r = l(Z_i + \omega_i (Z_{3-i}/\omega_{3-i})) + B = Al + B$ where $A$ and $B$ depend only on $n_1$, $n_2$, and $w$. Thus in all cases $M$ arrives at the configuration $(q', n'_1, n'_2) = (q', (Al + B)0)$ where $A$ and $B$ are independent of $l$ and $n'_u = i_w$ stays the same for all $l$ larger than some constant. If $M$ is allowed to continue then following the input of $\phi$ it will produce the value of $n_1 + lZ_i + \omega_i l(Z_{3-i}/\omega_{3-i}) + r = l(Z_i + \omega_i (Z_{3-i}/\omega_{3-i})) + B = Al + B$ where $A$ and $B$ depend only on $n_1$, $n_2$, and $w$.

Finally suppose $Z_i/\omega_l > Z_3-i/\omega_{3-i}$ for some $i \in \{1, 2\}$. If $l$ is large enough then $n_i + lZ_i = n_i + lZ_i - \omega_u l(Z_i/\omega_l) = n_i + al|\omega_l|$ and $n_3-i + lZ_{3-i} = n_3-i + lZ_{3-i} - \omega_u l(Z_3-i/\omega_{3-i}) = n_3-i + bl|\omega_{3-i}|$ with $(a - b)l > D$ where $D$ is given by Lemma 9(1)(b). Thus $n'_u = i$ for all such $l$ and $n'_3 = n_i + lZ_i + \omega_u l(Z_3-i/\omega_{3-i}) + r = l(Z_i + \omega_i (Z_{3-i}/\omega_{3-i})) + B = Al + B$ where $A$ and $B$ depend only on $n_1$, $n_2$, and $w$. We can now proceed with the proof of Lemma 12.

Proof of Lemma 12: Suppose the contrary and use Lemma 13 to find $\Omega$, $w$,
s + 1 < v ≤ (s + 1)^2, \ u \in \{1, 2\}, \ R_u > 0 \ and \ R_{3-u} < 0, \ with \ the \ appropriate \ properties. \ Let \ K \ be \ large \ enough. \ Consider \ what \ happens \ when \ the \ input \ \Omega\{m\} \ for \ small \ enough \ m \geq 0 \ is \ followed \ by \ 1v \ (the \ conditions \ on \ K \ and \ m \ are \ discussed \ below). \ Just \ as \ in \ the \ proof \ of \ Lemma \ 13 \ let \ M \xrightarrow{I}(w, n_1, n_1) \xrightarrow{\Theta}\ (w, n_1 + Z_1 l, n_2 + Z_2 l) \xrightarrow{(q', n'_1, n'_2)} \ for \ appropriately \ chosen \ \Theta, \ Z_1 \ and \ Z_2, \ let \ and \ \omega_1 \ and \ \omega_2 \ be \ the \ constants \ provided \ by \ Lemma 9(2) \ for \ the \ last \ stage \ of \ this \ computation. \ Note \ that \ these \ constants \ are \ the \ same \ for \ all \ configurations \ (w, n_1 + Z_1 l, n_2 + Z_2 l) \ such \ that \ n_1 + Z_1 l, n_2 + Z_2 l > s + 1.

We assume, as in Lemma 13 that both \ \omega_i, \ i \in \{1, 2\} \ (if \ \omega_i \ is \ 0, \ the \ appropriate \ requirement \ is \ considered \ satisfied \ automatically). \ Now \ consider \ the \ following \ three \ cases.

(1) \ \omega_{3-u} \geq 0 \ and \ \omega_u < 0. \ Then \ (w', n'_1, n'_2) = (w', (n'_u = 0) (n_{3-u} + \omega_{3-u}[v/|\omega_u|] + r)) = (w', (n'_u = 0) (n_{3-u} + B)) \ where \ both \ w' \ and \ B \ depend \ only \ on \ the \ value \ of \ v < (s + 1)^2. \ Moreover, \ (w, n_u + lZ_u n_{3-u} + lZ_{3-u}) \xrightarrow{1}(w', (n'_u = 0) (n_{3-u} + lZ_{3-u} + \omega_{3-u}(Z_u/|\omega_u|) + \omega_{3-u}[n_u/|\omega_u|] + r)) = (w', (n'_u = 0) (n_{3-u} + IA + B)) \ where \ the \ state \ w', \ and \ the \ constants \ A \ and \ B \ depend \ only \ on \ the \ value \ of \ v < (s + 1)^2, \ provided \ [(v + Z_u l)/|\omega_u|] < [(K + Z_{3-u} l - s - 1)/|\omega_{3-u}|] \ (if \ \omega_{3-u} = 0 \ this \ inequality \ is \ automatically \ satisfied).

(2) \ \omega_{3-u} < 0 \ and \ \omega_u \geq 0. \ Then \ (w, n_u + lZ_u n_{3-u} + lZ_{3-u}) \xrightarrow{1}(w', (n'_u = 0) (n_{3-u} + lZ_{3-u} + \omega_{3-u}[(v + Z_u l)/|\omega_u|] + r) (n'_{3-u} = 0)) = (w', (v + lZ_u + \omega_u lZ_u n_{3-u} + \omega_u n_{3-u} + |\omega_u| + r) (n'_{3-u} = 0)) \ where \ A \ does \ not \ depend \ on \ n_1 \ or \ n_2, \ m = |n_{3-u}/|\omega_2|, \ and \ r \ depends \ only \ on \ (n_{3-u} \ mod \ |\omega_2|) \ which \ is \ bounded \ by \ (s + 1).

(3) \ \omega_u < 0 \ and \ \omega_{3-u} < 0. \ Using \ (b) \ of \ Lemma 9 \ choosing \ l \ and \ k \ so \ that \ [(v + Z_u l)/|\omega_u|] < [(K + Z_{3-u} l - s - 1)/|\omega_{3-u}|] \ one \ can \ show \ that \ (w, n_u + lZ_u n_{3-u} + lZ_{3-u}) \xrightarrow{1}(w', (n'_u = 0) n_{3-u} + lZ_{3-u} + \omega_{3-u}[(v + lZ_u)/|\omega_u|] + r) = (w', (n'_u = 0) (n_{3-u} + lZ_{3-u} + \omega_{3-u}[Z_u/|\omega_u|] + \omega_{3-u}[v/|\omega_u|] + r) = (w', v + lA + B) \ where \ A \ does \ not \ depend \ on \ n_1 \ or \ n_2, \ B \ depends \ only \ on \ v.

Construct \ an \ infinite \ sequence \ of \ strings \ \{I_n \mid n \in \mathbb{N}\} \ satisfying \ the \ following \ properties. \ There \ are \ b < s + 1, \ and \ c \ such \ that \ for \ each \ j \ M \xrightarrow{I}(w, v n_{3-u}(j)) \ where \ n_{3-u}(j) > j, \ either \ \omega_{3-u} = 0 \ or \ the \ remainder \ b = n_{3-u} \ mod \ |\omega_{3-u}|, \ and \ c = n_{3-u} \ mod \ |A\omega_{3-u}| \ provided \ |A\omega_{3-u}| \ is \ not \ 0. \ Note \ that \ all \ (w, v n_{3-u}(j)) \ satisfy \ the \ same \ property \ (1)-(3) \ mentioned \ above.
Note that the conditions above and the cases (1)–(3) discussed before the previous paragraph imply for some state \(w'(w, v_{3-u}(j)) \Rightarrow (w', m_1, m_2)\).

Using Lemma 8 to ensure that all the configurations \(M\) enters after \(M \rightarrow (w, v_{3-u}(j))\) have at least one counter larger than \(s + 1\) and taking a large enough \(j\) consider the output of \(M\) given the input \(I_{j1\diamond}\). Using Lemma 8 find the constants \(P, Q, R,\) and \(D\) with the property that the output of \(M\) is \(P[m_+(j)/Q] + R\) where \((w', m_1(j), m_2(j))\) is the first configuration \(M\) enters after \(I_{j1}\) has been input such that \(m_-(j) = 0\).

By taking a multiple of \(D\) we can assume that every appropriate constant mentioned below divides \(D\). Next revisit the three cases above.

(1) Find \(k\) large enough so that one can find \(l\) and \(l + l'\) as described below, where \(l' > 2\) that are both solutions of \(n_{3-u}(k) + lA + B = n_{3-u}(j) + B + mD\) for some \(m\). To see that this is possible note that \(\omega_{3-u} \neq 0\) so either \(A = 0\) (which is impossible) or \(c = n_{3-u}(k) \mod |A\omega_{3-u}| = n_{3-u}(j) \mod |A\omega_{3-u}|\).

Let \(D = D'[\omega_{3-u}A]\) and let \(n_{3-u}(k) = n_{3-u}(j) + m'[\omega_{3-u}A] + m''D\) where \(m' < D' < D\). Now put \(l = (D' - (\text{sgn}A)m')/\omega_{3-u}\), \(l + l' = (2D' - (\text{sgn}A)m')/\omega_{3-u}\).

It is easy to see that \(l, l + l' \leq 3D[\omega_{3-u}]\) so the existence of a large enough \(k\) so that the conditions in case (1) above are satisfied is immediate. Then, on the one hand, \(M\) should output \(P[(n_{3-u}(k) + lA + B)/Q] + R\) using the choice of \(P, Q, R,\) and \(D\). One the other hand, \(M\) will output \(\gamma I_k\Omega\{l\} \mathbin{\uparrow} = UE^l + V\) where \(U = \frac{b^e(\gamma\Omega\uparrow + b^l - 1)}{b^l - 1}, V = \gamma I_k\uparrow - \frac{\gamma\Omega\uparrow b^e}{b^l - 1}, E = b^l,\) and \(e\) is the length of \(I_k\) and \(f\) is the length of \(\Omega\). Thus \(P[(n_{3-u}(k) + lA + B)/Q] + R = UE^l + V\) and \(P[(n_k(k) + (l + l')A + B)/Q] + R = UE^{l + l'} + V\) but \((P[(n_{3-u}(k) + (l + l')A + B)/Q] + R) - (P[(n_{3-u}(k) + lA + B)/Q] + R) \leq t^lPAQ + 1\) while \((UE^{l + l'} + V) - (UE^l + V) \geq U'\).

Since \(P, Q,\) and \(A\) are fixed and \(U\) increases indefinitely as the length of \(I_k\) increases choosing a large enough \(k\) results in a contradiction.

(2) Find \(k\) large enough so that one can find small enough \(l\) and \(l + l'\), where \(l' > 2\) that are both solutions of \(a + lA + \omega_u[n_{3-u}(k)/|\omega_{3-u}|] + r = a + \omega_u[n_{3-u}(j)/|\omega_{3-u}|] + r + mD\) for some \(m\). That such solutions exist follows from \(\omega_u[n_{3-u}(k)/|\omega_{3-u}|] = \omega_u(n_{3-u}(k) - n_{3-u}(k) \mod |A\omega_{3-u}|)/|\omega_{3-u}| + \omega_u[n_{3-u}(k) \mod |A\omega_{3-u}|/|\omega_{3-u}|] = Am' + \omega_u[c/|\omega_{3-u}|]\) and \(\omega_u[n_{3-u}(j)/|\omega_{3-u}|] = Am'' + \omega_u[c/|\omega_{3-u}|]\) for some integer \(m'\) and \(m''\). The rest is similar to case (1).

(3) Similar to case (1) with \(n_u(\cdot)\) replacing \(n_{3-u}(\cdot)\). \(\square\)

The following theorem is a corollary of Lemma 12.
Theorem 2. If there exists a TCMI that counts in some radix $b \geq 2$ then there exists a TCMI $M$ that counts in $b$ such that $M \xrightarrow{I}(w_I, n'_1, n'_2)$ where $n'_2 = 0$ for every input $I$.

Proof. Use Lemma [12] to find $K$ and change the finite control of the original TCMI so that all counter values less than $K$ are kept in the buffers in the finite control.

The utility of the theorem above lies in the idea that each computation performed by a TCMI between inputting two successive digits is a computation of a new value based on the intermediate result of the previous computation, i.e. only one value is needed to represent the intermediate result. Hence, if the TCM ‘responsible’ for each digit can be simulated by some other machine under the assumption that it starts with one empty register, the TCMI can now also be simulated by a family of such machines.

Paper [17] introduced the concept of a so-called more powerful one register machine or MP1RM. This machine with a somewhat tongue-in-cheek name has a single register $r$ and allows the following operations: $r \leftarrow r + K$, $r \leftarrow r \times K$, if $r - 1 \geq 0$ then $r \leftarrow r - 1$ else goto $l$, \{ $r \leftarrow r \div K$; goto $S(r \text{mod } K)$ \}; goto $l$, halt. We can now extend the definition of MP1RM by adding ‘if $I = d$ then goto $l$’ commands, analogous to the way a TCMI is derived from a TCM. Naturally there are only finitely many constants $K$ and $S(K)$ allowed.

Thus extended, an MP1RM becomes an MP1RM with input or MP1RMI. Other definitions (such as counting in some radix) can be naturally extended to MP1RMIs as well. R. Schroeppel shows in [17] that every TCM can be emulated with an MP1RM.

Theorem 3. There is no MP1RMI (therefore no TCMI) that counts online in some radix $b \geq 2$.

Proof. Suppose such a machine $M$ exists. Let $B$ be the product of all the constants mentioned in the instruction set of $M$ (see the description of an MP1RM above) as well as the input radix, $b$. First show that there exists a strictly increasing finite sequence $n_1, n_2, \ldots, n_k$, and a waiting state $q$ of $M$ such that $M$ enters $q$ after counting to $\sum_{i=1}^k B^{n_i}$ (which can be picked arbitrarily large) as well as after counting to $\sum_{i=1}^k B^{n_i} + \sum_{i=k+1}^K B^{n_i}$ for arbitrarily large $n_{k+1}$ and some strictly increasing $n_{k+1}, \ldots, n_K$. If there is no such $q$, one can construct, by induction, a finite increasing sequence.
such that for any \( n > n_T \) and any waiting state \( q \) of \( M \), the waiting state \( M \) enters after counting to \( \sum_{i=1}^{T} B^{n_i} + B^n \) is different from \( q \). A contradiction.

Let \( q \) be the waiting state mentioned above, let \( n_1, n_2, \ldots, n_k \) be the appropriate sequence and let \( L \) be the number of steps it takes \( M \) to process the next input digit, 1, after it has counted to \( \sum_{i=1}^{k} B^{n_i} \). Thus, after starting in \( q \) and \( \sum_{i=1}^{k} B^{n_i} \) in its only register, after inputting 1, \( M \) will output (in the register) \( \sum_{i=1}^{k} B^{n_i} + b^S \) where \( S \) is the smallest power such that \( b^S > \sum_{i=1}^{k} B^{n_i} \).

Pick \( n_{k+1} > \max\{L, n_k, R\} \), where \( R \), to be specified later, depends only on the computation \( M \) performs to input the last digit of \( \sum_{i=1}^{k} B^{n_i} + b^S \), such that \( M \) enters \( q \) after counting to \( \sum_{i=1}^{k} B^{n_i} + \sum_{i=k+1}^{K} B^{n_i} \) for some \( n_{k+2}, \ldots, n_K \). Note that the only (meaningful) branching can be performed by the various \( \ldots \div K \) instructions. By arranging \( \sum_{i=1}^{k} B^{n_i} \) to be large enough we can assume that the sole register of \( M \) is never equal 0 during the processing of the last input digit of \( \sum_{i=1}^{k} B^{n_i} + b^S \). It is easy to see that after counting to \( \sum_{i=1}^{k} B^{n_i} + \sum_{i=k+1}^{K} B^{n_i} \) and inputting 1, \( M \) must stop after the same \( L \) steps as it did after counting to \( \sum_{i=1}^{k} B^{n_i} \) and inputting 1. Indeed, \( M \) is in the same waiting state \( q \) before inputting 1 in both cases. Now, our choice of \( B \) and \( n_{k+1} > L \) imply that the remainders of the \( \ldots \div K \) instructions are unaffected by the addition of \( \sum_{i=k+1}^{K} B^{n_i} \).

Therefore, after counting to \( \sum_{i=1}^{k} B^{n_i} + \sum_{i=k+1}^{K} B^{n_i} \) and inputting 1, \( M \) will count to \( \sum_{i=1}^{k} B^{n_i} + b^S + (P/Q) \sum_{i=k+1}^{K} B^{n_i} \) for some integer \( P \) and \( Q \) such that \( Q \) divides \( \sum_{i=k+1}^{K} B^{n_i} \). Note that both \( P \) and \( Q \) are determined by the computation \( M \) performs after counting to \( \sum_{i=1}^{k} B^{n_i} \) and inputting 1. Thus choosing \( R \) large enough, it can be arranged that \( \sum_{i=1}^{k} B^{n_i} + b^S + (P/Q) \sum_{i=k+1}^{K} B^{n_i} \neq \sum_{i=1}^{k} B^{n_i} + \sum_{i=k+1}^{K} B^{n_i} + b^{S'} \), where \( S' \) is the smallest such that \( b^{S'} > \sum_{i=1}^{k} B^{n_i} + \sum_{i=k+1}^{K} B^{n_i} \). One way to see this is by noting that if \( n_{k+1} \) is large enough so that the representation of \( (1/Q)B^{n_{k+1}} \) in radix \( b \) has more than \( S \) initial zeroes then the required property holds.

The theorem above can be extended in a number of ways. With a slightly more complicated technique one can show that if the intermediate values differ from the online values by a bounded constant, the same result holds, i.e. such a machine cannot count in \( b \), or if the TCM stage that processes \( \diamond \) has a bounded number of reversals (or stages as they are called above), the result
still holds. If, on the other hand, there are no other restrictions on this post-processing stage, new ideas are needed to establish the nonexistence of such machines. Note that computations with MP1RMs can be very (potentially infinitely) long. In fact, the history of the famous Collatz conjecture or the $3n+1$ problem (see [12] for a discussion and a bibliography) shows that even analyzing the length of the computation performed by the simplest MP1RMs is very hard.

The last simple result shows that in order for TCMI s to be able to perform computations of the same level of sophistication as TCMs when the input is presented starting with the least significant digit, they need to be able to count in the appropriate radix. This can be viewed as a ‘factorization’ theorem where a computation on the input is split into a decoding stage followed by the actual computation.

**Theorem 4.** Suppose, for a total function $f$ with an unbounded range there exists a TCM $M_f$ that computes $f$. If $b$ is some radix and there is a TCMI $M$ such that $M \xrightarrow{I}(w, f(⌜I⌝), 0)$ for every string of $b$-digits then there exists a TCMI $M'$ that counts in $b$.

**Proof.** Using Theorem 3.4 of [10] find $a$, $b$, and $c$ such that $f(a + bn) = f(a) + cn$ for all $n$. Note that it follows from the proof of [10], Theorem 3.4 that $c > 0$. It is not very difficult to create a finite control that for any input $I$ holds $2m$ most significant digits of the expression $a + b⌜I⌝$ for any initial substring $I'$ of $I$ and correctly updates them upon the input of the next digit $d$ of $I$. The constant $m$ is chosen large enough so that all the relevant constants ($a$, $b$, $c$, $b$ and some of their products and sums) can be represented in under $m$ $b$-digits. Note that the $m$ least significant digits held by the finite control are the actual digits of the value of $a + b⌜I⌝$. Now the TCMI $M'$ will utilize the finite control just built to feed the digits of $a + b⌜I⌝$ to the input of $M$ with some delay and wait for $M$ to compute $f(a + b⌜I⌝) = f(a) + c⌜I⌝$. $M'$ then subtracts $f(a)$ and divides by $c$ to compute $⌜I⌝$. \[\square\]

Using the $M_f$ in the statement of the previous theorem as a ‘back-end’ to $M$ to convert between $n$ and $f(n)$ (and bypassing $M_f$ in the case of a leading 0 which can be flagged by $M$’s finite control) one can show that $M$ cannot compute the value of $f(n)$ online.

**Theorem 5.** Suppose, for a total function $f$ with an unbounded range there exists a TCM $M_f$ that computes $f$. Then for any radix $b$ there is no TCMI $M$ such that $M \xrightarrow{I}(w, f(⌜I⌝), 0)$ for every string of $b$-digits.
Proof. Otherwise using the proof of the previous theorem and the remark that follows it one can construct a TCMI that counts online in $b$ contradicting Theorem 3.

\[\square\]

6 Open Questions

We conclude this paper with some questions we have been unable to answer despite our best efforts.

It is easy to see that the languages $P_{m,n}$ defined in Section 2 are context free if and only if they are regular. It would be surprising if the answer to the question below is affirmative, however, the authors do not know how to show that it is negative.

**Question 1.** Do there exist incommensurable radices $d, D, a$ radix $b$, and a precision $n > 0$ such that all $P_{m,n}$’s for $d^{m-1} \leq m < d^n$ are regular assuming $n \geq 2$ if $d = 2$?

The proof of Theorem 1 establishes that neither $L_1$ nor $M_1$ is context-free. It is not clear at the moment how to show that all nontrivial $L_d$’s and $M_d$’s fail to be context-free.

**Question 2.** Given a radix $d > 2$ is it true that every $L_d$ and $M_d$, $0 < d < d$ is not a CFL?

It would also be interesting to know more about the computational power of TCMI.

**Question 3.** Does there exist a TCMI that counts in some radix $b$?

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