ON UPPER TAIL LARGE DEVIATION RATE FUNCTION FOR CHEMICAL DISTANCE IN SUPERCRITICAL PERCOLATION

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Abstract. We consider the supercritical bond percolation on \( \mathbb{Z}^d \) and study the graph distance on the percolation graph called the chemical distance. It is well-known that there exists a deterministic constant \( \mu(x) \) such that the chemical distance \( D(0, nx) \) between two connected points 0 and \( nx \) grows like \( nx \mu(x) \). Garet and Marchand [21] proved that the probability of the upper tail large deviation event \( \{n\mu(x)(1 + \varepsilon) < D(0, nx) < \infty\} \) decays exponentially with respect to \( n \). In this paper, we prove the existence of the rate function for upper tail large deviation when \( d \geq 3 \) and \( \varepsilon > 0 \) is small enough. Moreover, we show that for any \( \varepsilon > 0 \), the upper tail large deviation event is created by space-time cut-points (points that all paths from 0 to \( nx \) must cross after a given time) that force the geodesics to consume more time by going in a non-optimal direction or by wiggling considerably. This enables us to express the rate function in regards to space-time cut-points.

1. Introduction and main results

The model of Bernoulli bond percolation was introduced by Broadbent and Hammersley in 1957 to model the circulation of water in a porous medium [7]. Since then, various models of percolation have been developed, such as level-sets of random fields, Boolean and Voronoi percolation, random interlacements, etc. A common feature among all of these models is that they involve a random subset of the underlying space, which is characterized by a parameter. This random subset grows in size as the parameter increases. The primary focus is on studying the connectivity of the random subset, specifically identifying the parameter values at which the random subset has an unbounded connected component. Of particular interest is the existence of a phase transition that occurs in the parameter when transitioning from a regime where there is no infinite connected component to a regime where one exists.

The study of these problems has uncovered numerous properties and techniques, giving rise to the so-called percolation theory. This theory holds significant importance in probability theory and statistical physics, especially regarding its relationship with other statistical physics models like lattice spin models, interacting particle systems, and random walks in random environments. These connections enable the successful application of powerful results and arguments from percolation theory to other fields, even in the absence of a direct link. Refer to [23] for comprehensive background information and known results on percolation theory.

In the model of Bernoulli bond percolation, each edge of \( (\mathbb{Z}^d, E^d) \) is independently removed with probability \( 1 - p \). When \( p \) increases the set of remaining edges increases. In particular, Aizenman–Kesten–Newman [11] proved the existence of a phase transition in the sense that there exists a special parameter \( p_\ast(d) \in (0, 1) \) such that below \( p_\ast(d) \), there is no infinite connected component; above \( p_\ast(d) \), there exists a unique one. In Bernoulli percolation, the subcritical regime \( p < p_\ast(d) \) and the supercritical regime \( p > p_\ast(d) \) are both well-studied and a lot of important results have been obtained rigorously. On the other hand, the study of critical percolation (i.e. the case \( p = p_\ast(d) \)) have appeared considerably more complicated. One of the major challenges in percolation is to exclude the existence of an infinite connected component in critical percolation. In \( d = 2 \), Kesten [20] proved that \( p_\ast(2) = 1/2 \) and

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there is no infinite connected component at \( p = p_c(2) \). Additionally, planar critical percolation was hypothesized to be associated with Schramm-Loewner evolution with \( \kappa = 6 \). As a result, this area has been an active research field for the past three decades. For \( d \geq 11 \), the absence of an infinite cluster in critical percolation has been conclusively proven through the use of lace expansion and mean-field approximation techniques \cite{25,20}. Nonetheless, for \( d \in \{3, \ldots, 10\} \), this problem remains unsolved and is regarded as one of the most significant conjectures in probability theory.

From the viewpoint of graph theory, it is also important to study the graph distance in a graph since the graph distance gives even more in-depth information on geometric structures of the graph compared to connectivity alone. In a percolation graph, the graph distance is referred to as the chemical distance, a name coined by a physicist S. Alexander. A central theme in this field is the asymptotic behavior of the chemical distance between two distant endpoints. This question has been explored for various models, including those with long-range interactions \cite{2,6,13,18,9}. The extensive study of the chemical distance started in the 1980s. Physicists were particularly interested in the growth exponent for the chemical distance in critical percolation, known as the chemical distance exponent. Despite numerous efforts and advancements in this area, there is still no consensus on the exact values of the chemical distance exponents even for the planar critical percolation. Recently, Damron–Hanson–Sosoe \cite{15} established an upper bound, showing that the chemical distance exponent is strictly smaller than the growth exponent for the lowest left/right crossing. This latter exponent is related to the three-arm exponent in percolation and its value is expected to be universal over the 2-dimensional lattices, which was, in particular, rigorously obtained for the triangular lattice. This suggests that the chemical distance exponent is not characterized by a well-known quantity in connectivity. For more on the background and known results on chemical distance, we refer the reader to \cite{14}.

In Bernoulli percolation on \( \mathbb{Z}^d \) lattice, the properties of the chemical distance in supercritical percolation are also of great interest both in mathematics and physics. Garet and Marchand proved a kind of law of large numbers for the chemical distance \cite{28}. So, the next step is to study the fluctuations and large deviations. However, there have been few results on the fluctuations of the chemical distance. Recently, Dembin \cite{16} proved super-concentration for the chemical distance, that is, the variance of the chemical distance increases at most sublinearly in the distance between the endpoints. However, as far as we know, there are no predictions in mathematical literature regarding the expected behavior of the variance of chemical distance. For the lower bound, the problem whether the variance diverges or not is important but still open for any dimension larger than 1.

Moving on to the large deviations, Antal–Pisztora and subsequently Garet–Marchand determined the correct speed of the upper tail large deviations \cite{2,24}. However, previous studies did not establish the existence of the rate function and a precise description for upper tail large deviations. The aim of this paper is to obtain the rate function by identifying the correct scenario responsible for upper tail large deviations. We believe that the methods and concepts introduced in this paper provide new insights on upper tail large deviations for other percolation models.

1.1. Chemical distance in Bernoulli percolation. The model of Bernoulli percolation is formally defined as follows. Let \( \mathbb{E}^d \) be the set of all pairs of nearest neighbours in \( \mathbb{Z}^d \). We consider i.i.d. Bernoulli random variables \( (B_e)_{e \in \mathbb{E}^d} \) of parameter \( p \in [0,1] \). If \( B_e = 1 \), then the edge \( e \) is called open; otherwise, the edge is called closed. Let \( G_p \) be the graph of the open edges:

\[
G_p := (\mathbb{Z}^d, \{e \in \mathbb{E}^d : B_e = 1\}).
\]

The graph \( G_p \) is called the percolation graph. A path is said to be open if the path consists only of open edges. We write \( x \leftrightarrow y \) if \( x \) and \( y \) are connected in \( G_p \). This model exhibits a phase transition. Indeed, when \( d \geq 2 \), there exists a critical parameter \( p_c(d) \in (0,1) \) such that for \( p > p_c(d) \) (supercritical regime), there almost surely exists a unique infinite open cluster \( \mathcal{C}_\infty \) in \( G_p \). In contrast, for \( p < p_c(d) \) (subcritical regime), there are no infinite open clusters. We refer to \cite{23} for general backgrounds and known results on Bernoulli percolation. Throughout the paper, we always
assume $p > p_c(d)$. We denote by $D_{\mathcal{G}_p}$ the graph distance on the graph $\mathcal{G}_p$, i.e. for $x, y \in \mathbb{Z}^d$,
\begin{align}
D_{\mathcal{G}_p}(x, y) := \inf \left\{ |r| : r \text{ is a path from } x \text{ to } y \text{ in } \mathcal{G}_p \right\},
\end{align}
where $|r|$ denotes the number of edges in the path $r$ and we use the convention $\inf \emptyset = +\infty$. In particular, if $x$ and $y$ are not connected in $\mathcal{G}_p$, then $D_{\mathcal{G}_p}(x, y) = \infty$. For later purposes, we extend the chemical distance to a function of real vectors by setting $D_{\mathcal{G}_p}(x, y) = D_{\mathcal{G}_p}([x], [y])$ for $x, y \in \mathbb{R}^d$, where $[\cdot]$ stands for the floor function. This $D_{\mathcal{G}_p}$ is the so-called chemical distance. We interpret the quantity $D_{\mathcal{G}_p}(x, y)$ as the time to go from $x$ to $y$ (see Section 1.4). Any path achieving the infimum in $D_{\mathcal{G}_p}(x, y)$ is called a geodesic. Note that any geodesic is self-avoiding.

**Time constant.** Garet and Marchand [28] obtained an asymptotic behavior of $D_{\mathcal{G}_p}(0, nx)$ as $n \to \infty$: for any $p > p_c(d)$, there exists a deterministic norm $\mu : \mathbb{R}^d \to [0, +\infty)$ such that
\begin{align}
\forall x \in \mathbb{R}^d \lim_{n \to \infty, \frac{n}{|x|} \to 0} \frac{D_{\mathcal{G}_p}(0, nx)}{n} = \mu(x) \quad \text{a.s.}
\end{align}
The function $\mu$ is the so-called time constant. We remark that they obtain (1.2) in a more general context of stationary integrable ergodic fields. It is well-known that $\mu$ is a norm, in particular convex and continuous.

**Upper tail large deviations.** Our main goal is to study the upper tail large deviation event for $x \in \mathbb{R}^d$:
\begin{align}
\{ \mu(x)(1 + \xi)n < D_{\mathcal{G}_p}(0, nx) < \infty \}, \quad \xi > 0.
\end{align}
We here consider the case $x = e_1 = (1, \ldots, 0)$. To better understand the decay rate of its probability, let us give an example configuration. Picture the configuration where all the horizontal edges along the segment from $-\xi ne_1$ to 0 are open, and all edges sharing exactly one endpoint with that segment are closed except for the horizontal edge $(-(\xi n + 1)e_1, -\xi ne_1)$ (see Figure 1). Moreover, we further assume that $-\xi ne_1$ is connected to $ne_1$. With this configuration, the geodesic is forced to go to $-\xi ne_1$, hence the upper tail large deviation event occurs with high probability. Note that the probability of such configurations decays exponentially with respect to $n$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{example_configuration.png}
\caption{Construction of a cut-point at $-\xi ne_1$. The red edges are closed; the black edges are open.}
\end{figure}

Garet and Marchand proved in [21] that the probability of the event (1.3) decays exponentially with respect to $n$. We aim to prove the existence of the so-called rate function $J_x$ such that
\[ P(\mu(x)(1 + \xi)n < D_{\mathcal{G}_p}(0, nx) < \infty) = e^{-nJ_x(\xi) + o(n)}, \]
and give an explicit description of the rate function. In particular, we prove that on the upper tail large deviation event, typical scenarios are similar to the example we gave above (a cut-point where all the geodesics are forced to pass). For the study of cut-point, we should not only record its position but also the time when the geodesics pass through the cut-point (that is the graph distance from 0 to this cut-point, see (1.4) for a formal definition). We call this a space-time cut-point. Space-time cut-points play a central role in studying upper tail large deviation for chemical distance as we will see.
later. In the example above, we created a cut-point in a non-optimal location so that the geodesics make a detour and even go in the opposite direction $-e_1$. However, there may exist other scenarios where cut-points are located in optimal directions, but the upper tail large deviation event still occurs. For instance, picture a space-time cut-point located around 0 whose chemical distance to 0 is larger than $\xi_n$ (see Figure 2). Hence, we need to compare the probabilities of all the space-time cut-points leading to the upper tail large deviations and determine the best scenario among them.

![Figure 2. Construction of a cut-point $w_n$ close to 0. The path corresponds to a geodesic from 0 to $ne_1$.](image)

1.2. Main results. We assume $p > p_c(d)$. Let us give a formal definition of space-time cut-point. For $t \geq 1$ and $z \in \mathbb{R}^d$, define

$$B_t(z) := \{x \in \mathbb{Z}^d : \mathcal{D}^{D_p}(z, x) \leq t\}.$$  

For short, we write $B_t$ instead of $B_t(0)$. Let $s > 0$ and $x \in \mathbb{R}^d$. We denote $\Lambda_s(x) := x + [-s, s]^d$. We define $\alpha_d := 1 - \frac{1}{\mathbb{E}} < 1$. The value of $\alpha_d$ is due to technical reasons, and its precise value is not important. We define $A_{s,x}(n)$ the event that there is a cut-point located around $nx$ whose chemical distance from 0 is larger than or equal to $sn$:

$$A_{s,x}(n) := \{\exists n \geq sn \ \exists x_n \in \Lambda_{n\alpha_d}(nx) ; B_{sn} \setminus B_{sn-1} = \{x_n\}\}.$$  

The most technical and innovative part of this paper is the following.

**Theorem 1.1** (Rate function for space-time cut-point). There exists a function $I : [0, \infty) \times \mathbb{R}^d \mapsto [0, +\infty)$ such that for all $(s, x) \in [0, +\infty) \times \mathbb{R}^d$,

$$\lim_{n \to \infty} -\frac{1}{n} \log \mathbb{P}(A_{s,x}(n)) = I(s, x),$$

where the convergence is locally uniform on $[0, +\infty) \times \mathbb{R}^d$. For any $s > 0$ and $x \in \mathbb{R}^d$, $I(s, x) > 0$.

Moreover, the function $I$ is convex and homogeneous (i.e. $I(\lambda s, \lambda x) = \lambda I(s, x)$ for $\lambda \geq 0$).

We define the rate function $J_x$ for $x \in \mathbb{Z}^d$ as

$$J_x(\xi) := \inf \left\{ I(s, y) : y \in \mathbb{R}^d, s \geq 0 ; s + \mu(y - x) \geq (1 + \xi)\mu(x) \right\}.$$  

**Proposition 1.2.** For any $\xi > 0$, $J_x(\xi) > 0$ and the function $\xi \mapsto J_x(\xi)$ is non-decreasing and continuous.

The following is our main result.

**Theorem 1.3.** For any $x \in \mathbb{R}^d \setminus \{0\}$, there exists $\xi_0 = \xi_0(x) > 0$ such that for any $\xi \in (0, \xi_0)$,

$$\lim_{n \to \infty} -\frac{1}{n} \log \mathbb{P}(\mu(x)(1 + \xi)n < \mathcal{D}^{D_p}(0, nx) < \infty) = -J_x(\xi).$$

An explicit expression of $\xi_0(x)$ is given in (4.2).

**Remark 1.4.** We shall discuss the expression as defined in (1.5). Let us examine the point $(s, y) \in [0, +\infty) \times \mathbb{R}^d$ such that

$$s + \mu(y - x) \geq (1 + \xi)\mu(x).$$

Upon the occurrence of event $A_{s,y}(n)$, a geodesic connecting the origin and $ne_1$ arrives later than time $sn$ at a cut-point in proximity to $ny$. According to (1.2), the chemical distance from this cut-point to $nx$ is typically about $n\mu(y - x)$. Consequently, it is not possible for the geodesic to arrive at $nx$ earlier than
(1 + \xi)\mu(x)n. Consequently, J_x can be interpreted as the minimum cost across all cut-point situations that result in the occurrence of a large deviation event in the upper tail.

**Remark 1.5.** In this remark, we provide a rationale for our exclusive focus on the regime \( \xi < \xi_0 \) in the main result. We will demonstrate that the large deviation event arises from the presence of two cut-points, each connected to one of the endpoints, as detailed in Proposition 5.3. When \( \xi \leq \xi_0 \), the balls associated with these cut-points are in close enough proximity to their corresponding endpoints, ensuring that they do not intersect. Consequently, the scenario can be simplified to a single cut-point. However, when \( \xi \) is large, the two cut-points may leverage shared closed edges to increase the probability relative to a single cut-point. Hence, this larger \( \xi \) regime might necessitate further studies of more complicated interacting cut-points.

1.3. **2D v.s. 3D and higher.** We expect that the upper tail large deviations behave differently in \( d = 2 \) and \( d \geq 3 \). For \( d \geq 3 \), only space-time cut-points can create an upper tail large deviation event. Whereas, for \( d = 2 \), we need to consider other scenarios unrelated to space-time cut-points, which create the event. One possible scenario is to add a close wall in the middle that forces the geodesic to deviate from the horizontal line (see figure 3). In \( d = 2 \), the cost of adding the wall is of order \( e^{\alpha}n \), so it is of the same order as the speed of large deviations. However, in dimensions \( d \geq 3 \), the cost becomes of order \( e^{\alpha d-1} \). According to this picture, we have the following theorem, which will be proved in Section 2.4.

**Theorem 1.6.** For any \( \varepsilon > 0 \) and \( \xi > 0 \), there exists \( c > 0 \) such that for \( n \) large enough,

\[
P(D_G^p(\Lambda_{\varepsilon n}(0), \Lambda_{\varepsilon n}(ne_1)) > (1 + \xi)\mu(e_1)n \leq e^{\alpha d-1},
\]

where \( D_G^p(A, B) := \inf_{x \in A, y \in B} D_G^p(x, y) \) for \( A, B \subset \mathbb{Z}^d \).

The aforementioned theorem emphasizes the distinction between dimension 2 and dimensions \( d \geq 3 \). In dimensions \( d \geq 3 \), the upper tail large deviation event for the chemical distance between two macroscopic boxes decays significantly more rapidly than the upper tail large deviation event for the chemical distance between two points. This implies that if the balls \( B_{\varepsilon n}(0) \) and \( B_{\varepsilon n}(ne_1) \) are both substantially large (as would be the case in a typical supercritical percolation), the probability of the upper tail large deviation event decays much faster than \( e^{-\alpha n} \). Consequently, during the upper tail large deviation event, the balls at the endpoints do not exhibit typical behavior and are noticeably small. In this context, we describe the upper tail large deviation as **local**, since it only pertains to edges in close proximity to the endpoints. However, in \( d = 2 \), the balls may maintain their usual volumetric properties, and both global scenarios (e.g. closed walls mentioned earlier) and local scenarios (e.g. space-time cut-points) may coexist, necessitating the identification of the most advantageous scenario among them.

![Figure 3. Construction of a closed wall. The edges in red are closed and force the geodesic to deviate a lot from the straight line.](image-url)
1.4. Link between First-passage percolation and chemical distance. The model of First-passage percolation (FPP) was first introduced by Hammersley and Welsh \[24\] as a model for the spread of a fluid in a porous medium. In the model, we assign a non-negative random variable $\tau_e$ to each edge $e$ such that the family $(\tau_e)_{e \in \mathbb{Z}^d}$ is independent and identically distributed with a distribution $F$. The random variable $\tau_e$ may be interpreted as the time needed for the fluid to cross the edge $e$. For any pair of vertices $x, y \in \mathbb{Z}^d$, the random variable $T(x, y)$, called the first passage time, is the shortest time to go from $x$ to $y$. We are interested in the asymptotic behavior of the quantity $T(0, x)$ when $\|x\|$ goes to infinity. Under some integrability conditions on $F$, it is proved that

\[ \lim_{n \to \infty} \frac{1}{n} T(0, nx) = \inf_{n \in \mathbb{N}} \frac{1}{n} E T(0, nx) := \mu_F(x) \quad \text{a.s.,} \]

where $\mu_F$ is a semi-norm associated to the distribution $F$ called the time constant for FPP. Indeed, Cox and Durrett \[13\] proved (1.6) under necessary and sufficient integrability conditions on the distribution $F$. Kesten \[27\] extended the result to dimensions $d \geq 2$, and proved that $\mu_F$ is exactly a norm if and only if $\mathbb{P}(\tau_e = 0) < p_c$. See \[3\] for more detailed backgrounds on FPP.

We here mention an important correspondence between the chemical distance in percolation and FPP. Consider the bond percolation $G_p$ introduced in Section 1.1. Let us couple the bond percolation with the time configuration $(\tau_e)_{e \in \mathbb{Z}^d}$ in the following way:

\[ \tau_e := \begin{cases} 1 & \text{if } e \text{ is open}, \\ \infty & \text{if } e \text{ is closed}. \end{cases} \]

In this setting, the infinite cluster made of the edges with passage time 1 corresponds to the infinite cluster $G_\infty$. For $x, y \in \mathbb{Z}^d$, we have

\[ T(x, y) := \inf_{\gamma: x \leftrightarrow y} \sum_{\gamma \in \gamma} \tau_e = \inf_{\gamma: x \leftrightarrow y \subset G_p} |\gamma| = D^{G_p}(x, y), \]

where the first infimum is taken over all paths from $x$ to $y$ in $\mathbb{Z}^d$; the second infimum is taken over all open paths from $x$ to $y$. Hence, the first passage time for the distribution $G_p = p\delta_1 + (1 - p)\delta_\infty$ is the same as the chemical distance in percolation.

1.5. Related work. In this section, we discuss related work on the large deviations for chemical distance.

1.5.1. Large deviations for the chemical distance in supercritical percolation. The first upper tail large deviation bound on the chemical distance for supercritical percolation was obtained by Antal and Pisztora \[2\]. They obtained the correct speed in the regime of large $\xi$: for every $p > p_c(d)$, there exists $C \geq 1$ such that

\[ \limsup_{\|x\| \to \infty, \|x\|_1} \frac{1}{\|x\|_1} \log \mathbb{P} \left( C \|x\|_1 \leq D^{G_p}(0, x) < \infty \right) < 0. \]

**Remark 1.7.** Antal and Pisztora actually proved a stronger claim \[2\] (4.9): there exists $C \geq 1$ and $c > 0$ such that

\[ \forall x \in \mathbb{Z}^d \forall l \geq C \|x\|_1 \quad \mathbb{P} \left( l \leq D^{G_p}(0, x) < \infty \right) \leq \exp(-cl). \]

We use this claim several times in this paper.

The result was later extended to the entire regime of $\xi$ by Garet and Marchand in \[21\]: for every $p > p_c(d)$ and $\xi > 0$,

\[ \limsup_{\|x\| \to \infty, \|x\|_1} \frac{1}{\|x\|_1} \log \mathbb{P} \left( (1 + \xi)\mu(x) \leq D^{G_p}(0, x) < \infty \right) < 0. \]
In the same paper, they also proved the existence of the rate function for lower tail large deviation: for every $p > p_c(d)$, $\xi > 0$, and $x \in \mathbb{R}^d \setminus \{0\}$, the following limit exists and is negative or $-\infty$:

\[
(1.11) \quad \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(D_{G^p}(0, nx) < \mu(nx)(1 - \xi)).
\]

### 1.5.2. Upper tail large deviations in FPP with compactly supported distributions.

Basu, Ganguly, and Sly [5] established the existence of a rate function for upper tail large deviation when the distribution $F$ is both compactly supported and possesses a continuous density. It is important to note the contrast between the behavior of the upper tail large deviation for the chemical distance and that for the FPP with a compactly supported distribution. In the first scenario, the speed of large deviation exhibits a linear order. Conversely, in the second scenario, increasing the first passage time necessitates the involvement of the entire environment, resulting in a volumetric speed, i.e. of order $n^d$. The foundation of the proof in [5] lies in the presence of an underlying limiting metric accountable for the upper tail large deviation. Although the authors did not formalize this limiting metric, it enabled them to derive an involved subadditivity for the upper tail large deviation event. Dembin and Théret [17] later formalized the concept of the limiting environment for maximal flow in FPP in order to demonstrate the upper tail large deviation principle.

### 1.5.3. Upper tail large deviations in FPP with distributions under tail estimates.

Our approach was initially inspired by the arguments in [11]. We briefly explain the main results and the sketch of the proof in [11] to discuss obstacles when adapting the strategies there to chemical distance. Cosco and Nakajima [11] considered the FPP on $\mathbb{Z}^d$ with distributions satisfying

\[
\mathbb{P}(\tau_{e} \geq t) \asymp e^{-\alpha t^r} \quad \text{as } t \to \infty \text{ with some constants } r \in (0, d], \alpha > 0.
\]

They derived a specific rate function for upper tail large deviations, known as the discrete p-capacity. In order to demonstrate this, they considered numerous slabs and found a good slab among them where the first passage time between any pair of points $x, y$ inside the slab is approximately equal to the time constant $\mu_F(y - x)$. This suggests that on the upper tail large deviation event, the first passage time to join one of the endpoints and this good slab is abnormally large. This enables us to replace the large deviation event $\{T(0, ne_1) > \mu_F(e_1)(1 + \xi)n\}$ with a local event where there are many high weights around the endpoints. To calculate the rate function, they explicitly estimated the probability of the local event, invoking the Laplace principle.

Nonetheless, these arguments are inapplicable to the current model. Contrary to FPP, there are closed edges that a geodesic is unable to traverse. In FPP, when a geodesic reaches a suitable slab, it is understood that the geodesic moves at a typical speed. In this study, however, arriving at an appropriate slab does not necessarily guarantee that a geodesic will maintain a typical speed within the slab. In fact, even when the geodesic crosses a slab that possesses beneficial connectivity within its infinite cluster, if the geodesic avoids this infinite cluster, the slab’s advantageous connectivity cannot be utilized.

Furthermore, the high weights around the endpoints in FPP are substituted by the presence of space-time cut-points in our situation. Specifically, we cannot employ the Laplace principle, so we resort to the rate function for space-time cut-point instead.

### 1.6. Sketch of the proofs.

#### 1.6.1. Sketch of the proof of Theorem 1.6

The proof is inspired by the arguments in [11]. Let $N$ be a fixed large integer. We consider disjoint $\mathbb{Z}^2$-slabs of order $n^{d-2}$ and of thickness $N$ that intersect $\Lambda_{en}(0), \Lambda_{en}(ne_1)$. Thanks to a renormalization argument, we can prove that for each slab the probability that the time in the slab between the two boxes $\Lambda_{en}(0), \Lambda_{en}(ne_1)$ is larger than $(1 + \xi)\mu(e_1)$ decays exponentially fast in $n$. Thus, if $D^{G_p}_{\tau}(\Lambda_{en}(0), \Lambda_{en}(ne_1)) > (1 + \xi)\mu(e_1)n$, then the time between $\Lambda_{en}(0), \Lambda_{en}(ne_1)$ on each slab is abnormally large. Since we consider disjoint slabs of order $n^{d-2}$, it follows that the decay of this event is of order $\exp(-cn^{d-1})$. 

1.6.2. Sketch of the proof of Theorem 1.1 Consider two configurations in \( A_{s,x}(n) \) and \( nx + A_{s,x}(m) \), where

\[
y + A_{s,x}(m) := \{ \exists t \geq sm \ \exists x_m \in \Lambda_{m+1}(y + mx) : B_t(y) \setminus B_{s-1}(y) = \{x_m\} \}.
\]

Let \( s_n \geq s, s_m \geq s, \) and \( w_n, z_m \in \mathbb{Z}^d \) such that \( B_{s_n} \setminus B_{s_n-1} = \{w_n\} \) and \( B_{s_m}(nx) \setminus B_{s_m-1}(nx) = \{z_m\} \). The two balls come from different time configurations. To build a configuration in \( A_{s,x}(n + m) \), we want to join \( w_n \) to \( xn \) without changing the metric structure so that \( z_m \) is a still cut-point and the chemical distance between 0 and \( z_m \) is larger than \( s(n + m) \). A first problem is when the two balls intersect. We can circumvent this problem by finding an appropriate translation of \( B_{s_m}(nx) \) not far from its original position. Such a translation can be found using the ball size control.

The main problem comes from the situations where \( w_n \) is not connected to infinity in \( \mathbb{Z}^d \setminus B_{s_n} \), or where it is connected to infinity but a path to join \( w_n \) and \( nx \) wiggles a lot. These scenarios may seem pathological, but we believe there are no easy ways to discard these scenarios. Indeed, when \( x = 0 \), we can consider the events where the ball \( B_{s_n} \) fills most of the box \( \Lambda_{n+1/4}(0) \) (e.g. a space-filling spiral) such that 0 is not connected to \( \infty \) in \( \mathbb{Z}^d \setminus B_{s_n} \) or 0 is connected to \( \partial \Lambda_{n+1/4}(0) \) but with a very large graph distance (of order \( n \)). In the first scenario, it is not possible to open edges to create a connection between the cut-point \( w_n \) of the first ball to the origin \( nx \) of the second ball without modifying the second ball structure. In the second scenario, the cost to join \( w_n \) and \( nx \) is at least of order exponential in \( n \) (in particular, it does not vanish in the limit).

To solve this issue, we modify the ball structure to create a free line, that is a line intersecting \( B_{s_n} \) only at \( w_n \) (see Lemma 3.7). The main difficulty of this step is to only slightly modify the ball structure so that the new space-time cut-point is still close to the original one. In particular, the metric structure of the ball has to be preserved as much as possible to ensure that the event \( A_{s,x}(n) \) still occurs for the modified environment. The technical part in this procedure is that the cost of modifying the structure of the ball has to vanish in the limit. Creating the free lines at \( w_n \) and \( nx \) gives more freedom to build a connection between these two vertices.

1.6.3. Sketch of the proof of Theorem 1.6 Consider a couple \((s, x) \in \mathbb{R}_+ \times \mathbb{R}^d\) such that

\[
s + \mu(x - e_1) > (1 + \xi)\mu(e_1).
\]

On the event \( A_{s,x}(n) \), any geodesic between 0 and \( ne_1 \) arrives at a cut-point located close to \( nx \) at time larger than \( sn \). Moreover, the time needed to go from \( nx \) to \( ne_1 \) is typically at least \( n(1 + \xi)\mu(e_1) - s \) thanks to (1.12) and (1.13). Hence, the event \( \{D^{\beta_p}(0, ne_1) > \mu(e_1)(1 + \xi)n\} \) is very likely to occur if the event \( A_{s,x}(n) \) occurs. Using Theorem 1.1, it follows that

\[
-\liminf_{n \to \infty} -\frac{1}{n} \log \mathbb{P}(\mu(e_1)(1 + \xi)n < D^{\beta_p}(0, ne_1) < \infty) \geq J(s, x).
\]

Taking the infimum over \((s, x)\) satisfying (1.13) on the right hand side, we get

\[
-\liminf_{n \to \infty} -\frac{1}{n} \log \mathbb{P}(\mu(e_1)(1 + \xi)n < D^{\beta_p}(0, ne_1) < \infty) \geq J(\xi).
\]

To get the converse inequality, we have to prove that if the upper tail large deviation event occurs, then there exists a space-time cut-point. The idea goes as follows. Consider first the balls \( B_s(0), B_t(ne_1) \) at both endpoints at the moments \( s \) and \( t \) when their respective size reaches \( n^{7/4} \) (See (5.8) for formal definitions). Since the balls are large enough, it is very likely to find \( x \in B_s(0) \) and \( y \in B_t(ne_1) \) such that \( D^{\beta_p}(x, y) \leq \mu(x - y) + o(n) \). Moreover, since the balls are of size negligible compared to \( n^2 \), we can make \( x \) and \( y \) cut-points (see Lemma 5.6). Besides, since we are on the upper tail large deviation event, these cut-points must satisfy

\[
D^{\beta_p}(0, x) + \mu(x - y) + D^{\beta_p}(y, ne_1) + o(n) \geq \mu(e_1)(1 + \xi)n.
\]
Hence, one expects from the Laplace principle and the continuity of \( I \) (Theorem 1.1) that
\[
\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\mu(e_1)(1 + \xi)n < D^{G_r}(0, ne_1) < \infty)
\]
\[
\leq \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \bigcup_{s,t \geq 0, x,y \in \mathbb{R}^d; \ s + t + \mu(x-y-e_1) \geq (1+\xi)\mu(e_1)} A_{s,x}(n) \cap (ne_1 + A_{t,y}(n)) \right)
\]
\[
\leq \sup_{s,t \geq 0, x,y \in \mathbb{R}^d; \ s + t + \mu(x-y-e_1) \geq (1+\xi)\mu(e_1)} \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( A_{s,x}(n) \cap (ne_1 + A_{t,y}(n)) \right).
\]

When \( \xi \leq \xi_0 \), the two space-time cut-points at 0 and \( ne_1 \) occur disjointly (see the proof of Proposition 5.6). Using the property of \( I \) (convexity and homogeneity), we argue that the cost of having a unique cut-point at one endpoint \( A_{s+t,x-y}(n) \) is smaller than the cost of having two cut-points one at each endpoint of \( A_{s,x}(n) \cap (ne_1 + A_{t,y}(n)) \). Therefore, (1.15) is further bounded from above by

\[
\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \bigcup_{s \geq 0, x \in \mathbb{R}^d; \ s + \mu(x-e_1) \geq (1+\xi)\mu(e_1)} A_{s,x}(n) \right) \leq -J(\xi).
\]

The result follows combining (1.14) and (1.15).

1.7. Notations and terminology. In this section, we collect useful notations, terminologies, and claims. For \( m \geq 1 \), set \([m] := \{1, \ldots, m\} \).

Distances. We denote by \( \| \cdot \|_1, \| \cdot \|_{\infty}, \| \cdot \|_2 \) respectively the \( \ell_1 \), \( \ell_\infty \) and \( \ell_2 \) norms. For \( A, B \subset \mathbb{R}^d \) and \( i \in \{1, 2, \infty\} \), we denote
\[
d_i(A, B) := \inf_{x \in A, y \in B} \| x - y \|_i.
\]

Diameter. For a set \( S \subset \mathbb{Z}^d \) and \( i \in [d] \), we define
\[
\text{Diam}_i(S) := \max_{x, y \in S} |x_i - y_i| \quad \text{and} \quad \text{Diam}(S) = \max_{1 \leq i \leq d} \text{Diam}_i(S).
\]

\( \mathbb{Z}^d \)-path. We say that a sequence \( \gamma = (v_0, \ldots, v_n) \in \mathbb{Z}^d \) is a \( \mathbb{Z}^d \)-path if for all \( i \in [n] \), \( \| v_i - v_{i-1} \|_1 = 1 \). Note that a \( \mathbb{Z}^d \)-path may also be seen as a set of edges \( \{v_i, v_{i+1}\}, 0 \leq i \leq n-1 \). In what follows, a path will implicitly mean a \( \mathbb{Z}^d \)-path.

Cluster. Given \( x, y \in \mathbb{Z}^d \) and \( A \subset \mathbb{Z}^d \), if there exists a \( \mathbb{Z}^d \)-path \( \gamma \subset A \) from \( x \) to \( y \) in \( G_p \), then we write \( x \leftrightarrow y \). When \( A = \mathbb{Z}^d \), we simply write \( x \leftrightarrow y \). Given \( x \in \mathbb{Z}^d \), we define the open cluster containing \( x \) as
\[
\mathcal{C}(x) := \{ y \in \mathbb{Z}^d : x \leftrightarrow y \}.
\]

Paths concatenation. Given two paths \( \gamma^1, \gamma^2 \) such that the end point of \( \gamma^1 \) is the same as the starting point of \( \gamma^2 \), we denote by \( \gamma^1 \oplus \gamma^2 \) the concatenation of \( \gamma^1 \) and \( \gamma^2 \). Inductively, we can define the concatenation of \( n \) paths \( \gamma^1 \oplus \cdots \oplus \gamma^n \).

Interior and exterior boundary. For a set \( \Gamma \subset \mathbb{Z}^d \), we denote by \( \text{Int}(\Gamma) \) the set of points in \( \mathbb{Z}^d \) enclosed by \( \Gamma \):
\[
\text{Int}(\Gamma) := \{ x \in \mathbb{Z}^d \setminus \Gamma : x \text{ is not connected to } \infty \text{ in } \mathbb{Z}^d \setminus \Gamma \}.
\]

We define the outer boundary of \( \Gamma \) as
\[
\partial^{\text{ext}} \Gamma := \{ x \in \mathbb{Z}^d \setminus \Gamma : x \text{ is connected to infinity in } \mathbb{Z}^d \setminus \Gamma \text{ and has a neighbour in } \Gamma \}.
\]
There exists a constant $\kappa_d > 1$ such that for every connected set $\Gamma \subset \mathbb{Z}^d$, we have the following discrete isoperimetric inequality (see, for instance, [12, Theorem 1])

\[
|\Gamma| \leq \kappa_d |\partial^\text{ext} \Gamma|^{d/(d-1)}.
\]

\textit{*-connected and lattice animals.} We say that $x$ and $y$ are *-neighbor if $|x-y|_\infty = 1$. We say that a sequence $\gamma = (v_0, \ldots, v_n) \subset \mathbb{Z}^d$ is a *-path from $x$ to $y$ in $\Gamma \subset \mathbb{Z}^d$ if $v_0 = x \in \Gamma$, $v_n = y$ and for all $i \in [n]$, $v_i \in \Gamma$ and $|v_i - v_{i-1}|_\infty = 1$. We say that $x$ and $y$ are *-connected in $\Gamma$ if such a path exists. We say that $\Gamma$ is *-connected if for any $x, y \in \Gamma$, $x$ and $y$ are *-connected in $\Gamma$. We denote by $\text{Animal}^*_k$ the set of *-connected components of $\mathbb{Z}^d$ of size $k$ containing $i \in \mathbb{Z}^d$. We have (see for instance Grimmett [23, p85])

\[
|\text{Animal}^*_k| \leq \tau^{dk}.
\]

\textit{Lines and hyperplanes.} Given $w \in \mathbb{Z}^d$, $n \geq 1$ and $i \in [d]$, we define

\[
L_n(w) := \{w + ke_i : k \in \mathbb{Z}\},
L^n_i(w) := \{w + ke_i : |k| \leq n\},
H_i(w) := \{w + \sum_{j \neq i} k_j e_j : k_j \in \mathbb{Z}\}.
\]

Let $P_i$ be the projection on the hyperplane $H_i(0)$.

1.8. \textbf{Organisation of the paper.} In Section 2 we introduce a renormalization procedure and prove Theorem 1.6. In Section 3 we prove the existence of the rate function for space-time cut-point (Theorem 1.3). Finally, in Section 5 we prove that on the upper tail large deviation event, there exist space-time cut-points and deduce the existence of the rate function for upper tail large deviation (Theorem 1.6). As before, we always assume $p > p_c(d)$ in the rest of the paper.

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2. \textbf{Renormalization}

For simplicity, we write $\Lambda_s := \Lambda_s(0) = [-s, s]^d$.

2.1. \textbf{Preliminary on percolation.} The following estimate controls the probability that the density of the infinite cluster is atypically small.

\textbf{Theorem 2.1.} [29, Theorem 2] \textit{Let $\varepsilon > 0$. There exists $c > 0$ such that for $n$ large enough}

\[
\mathbb{P}\left(\frac{|c_\infty \cap \Lambda_n(w)|}{|\Lambda_n(w)|} \leq \theta(p) - \varepsilon\right) \leq e^{-cn^{d-1}},
\]

\textit{where $\theta(p) := \mathbb{P}(0 \in c_\infty)$ corresponds to the density of the infinite cluster.}

The following theorem may be seen as a corollary of the previous theorem. It controls the probability that the infinite cluster does not intersect a given box.

\textbf{Theorem 2.2} (Holes). \textit{There exists $c > 0$ such that for $n \in \mathbb{N}$ large enough,}

\[
\mathbb{P}(c_\infty \cap \Lambda_n = \emptyset) \leq e^{-cn^{d-1}}.
\]

The following theorem enables us to control the probability of the existence of two large disjoint clusters (e.g. see [23, Lemma 7.104] for a reference).

\textbf{Theorem 2.3} (Distinct clusters). \textit{For any $\varepsilon > 0$, there exists $c > 0$ such that for $n \in \mathbb{N}$ large enough,}

\[
\mathbb{P}(\exists \text{two disjoint open clusters of diameter at least } \varepsilon n \text{ in } \Lambda_n) \leq e^{-cn}.
\]
2.2. Macroscopic lattice. We define the $N$-box and the enlarged box for $k \in \mathbb{Z}^d$:

$$\Lambda_N(k) := [-N, N]^d \cap \mathbb{Z}^d + 2kN, \quad \Lambda'_N(k) := \bigcup_{|i-k|\leq 1} \Lambda_N(i).$$

Then, we have the decomposition of $\mathbb{Z}^d$ as $\mathbb{Z}^d = \bigcup_{i \in \mathbb{Z}^d} \Lambda_N(i)$ where $\sqcup$ denotes the disjoint union. Thus, for any $x \in \mathbb{Z}^d$, there exists a unique $i \in \mathbb{Z}^d$ such that $x \in \Lambda_N(i)$. With abuse of notation, we use the notation $\Lambda_N(x)$ for the box $\Lambda_N(i)$ containing $x$. The sites corresponding to the boxes are the so-called macroscopic lattice of sidelength $N$; whereas the standard vertices in $\mathbb{Z}^d$ correspond to the microscopic lattice.

**Definition 2.4.** Let $\varepsilon \in (0, \frac{1}{2})$ and $N \in \mathbb{N}$. We say that a site $i \in \mathbb{Z}^d$ is $\varepsilon$-good if the following hold:

1. There exists a unique open cluster, denoted by $\mathcal{C}(i)$, in $\Lambda'_N(i)$ with diameter at least $\frac{N}{2}$;
2. This cluster $\mathcal{C}(i)$ intersects with every subbox in $\Lambda'_N(i)$ of sidelength $\varepsilon N$;
3. For all $x, y \in \Lambda'_N(i) \cap \mathcal{C}(i)$, we have $D^p(x, y) \leq \mu(x - y) + \varepsilon N$.

Otherwise, if at least one of the three conditions does not hold, we say that the site $i$ is $\varepsilon$-bad.

**Remark 2.5.** Thanks to our definition of $\varepsilon$-good, for any *-neighbours $i$ and $j$ that are $\varepsilon$-good, the clusters $\mathcal{C}(i)$ and $\mathcal{C}(j)$ are connected in $\Lambda'_N(j)$. Indeed, thanks to Property (2), $\mathcal{C}(i) \cap \Lambda'_N(j)$ has a connected component of diameter at least $N/2$ and the claim follows from Property (1) for the site $j$.

When there is no confusion, we say good instead of $\varepsilon$-good. The states of the boxes have a short-range dependence. To see this, set

$$\rho := \lfloor 10d\mu(e_1) \rfloor.$$ 

Thanks to the definition of $\varepsilon$-good, for $i, j \in \mathbb{Z}^d$ with $|i - j|_{\infty} \geq \rho$, the states of the sites $i$ and $j$ are independent. Finally, set

$$\Lambda^e_N(i) := \bigcup_{j:|i-j|_{\infty} \leq \rho} \Lambda_N(j).$$

**Proposition 2.6.** Let $\varepsilon > 0$. For $p > p_c(d)$, there exists $c > 0$ such that for $N \in \mathbb{N}$ large enough,

$$\mathbb{P}(\Lambda_N(0) \text{ is } \varepsilon\text{-bad}) \leq e^{-cN}.$$

**Proof.** Thanks to Theorems 2.2 and 2.3, $\mathbb{P}(\Lambda_{\infty} \cap \Lambda_N(0) = \emptyset) \leq e^{-cN}$. Hence, we suppose $\Lambda_{\infty} \cap \Lambda_N(0) \neq \emptyset$. Then, we can take a connected component $\mathcal{C}(0)$ in $\Lambda_{\infty} \cap \Lambda_N'(0)$ with diameter at least $N/2$. Thanks to Theorem 2.3, this is a unique connected component of diameter at least $N/2$ with probability at least $1 - e^{-cN^2}$. The result follows from the union bound with (1.9), (1.10), and Theorem 2.2. 

The following lemma controls the number of bad boxes.

**Lemma 2.7.** Let $\varepsilon > 0$ and $N = N_n := \log^2 n := \lfloor (\log n)^2 \rfloor$. For $n$ large enough,

$$\mathbb{P} \left( \# \{i \in \Lambda_{\infty} \cap \mathbb{Z}^d : i \text{ is } \varepsilon\text{-bad} \} > n \right) \leq e^{-n \log n}.$$ 

**Proof.** By the union bound, we have

$$\mathbb{P}(\# \{i \in \Lambda_{\infty} \cap \mathbb{Z}^d : i \text{ is } \varepsilon\text{-bad} \} > n) \leq \sum_{I \subseteq \Lambda_{\infty} \cap \mathbb{Z}^d: |I| > n} \mathbb{P}(\forall i \in I \ i \text{ is } \varepsilon\text{-bad}).$$

If $|i - j|_{\infty} \geq \rho$, then the states of $i$ and $j$ are independent. It is easy to see (e.g. [10 Lemma 4.3]) that there exists $J \subset I$ such that $|J| \geq |I|/(4\rho)^d$ and for all $i \neq j \in J$, $|i - j|_{\infty} \geq \rho$. It follows from
Lemma 2.9. Combined with (2.4), this yields the claim.

Proof. Let \( \varepsilon \in (0, 1/2) \) and \( N \in \mathbb{N} \). Consider the macroscopic lattice of side length \( N \). For \( w \in \mathbb{Z}^d \), denote by \( C(w) \) the \( \varepsilon \)-bad \( \mathbb{Z}^d \)-cluster of \( w \) in the macroscopic lattice, that is, the set of all \( \varepsilon \)-bad sites connected to \( w \) by a macroscopic \( \mathbb{Z}^d \)-path of bad sites. We define \( C(w) := \emptyset \) if \( w \) is \( \varepsilon \)-good.

Lemma 2.8. Let \( \delta, \varepsilon > 0 \). If \( N \) is large enough, then for any set of macroscopic sites \( \Gamma \),

\[
P \left( \left| \bigcup_{w \in \Gamma} C(w) \right| \geq \delta |\Gamma| \right) \leq \exp(-\delta |\Gamma|/2).
\]

Proof. Let \( (C_i)_{i \geq 1} \) be a family of independent cluster following the law of \( C(0) \). We have

\[
(2.4) \quad P \left( \left| \bigcup_{w \in \Gamma} C(w) \right| \geq \delta |\Gamma| \right) \leq P \left( \sum_{i=1}^{|\Gamma|} |C_i| \geq \delta |\Gamma| \right) \leq E(\exp(|C(0)| - \delta))^{\lceil |\Gamma| \rceil},
\]

where we reference the proof of [22, Lemma 3.6] for the first inequality, while the second inequality employs the exponential Markov inequality. For \( N \) large enough, using the same arguments as in (2.3), by [1.18],

\[
(2.5) \quad P(|C(0)| \geq k) \leq \sum_{\ell \geq k} \sum_{A \in \text{Animal}_k} P(\forall i \in A \ i \text{ is } \varepsilon \text{-bad}) \leq \sum_{\ell \geq k} \tau^\ell \delta \left(2\rho \right)^d \leq \exp(-ckN/(8\rho^d)).
\]

If \( N \) is large enough depending on \( \rho, c \) and \( \delta \), then we have

\[
E(\exp(|C(0)|)) \leq 1 + \sum_{k \geq 1} \exp(k(1 - cN/(8\rho^d))) \leq \exp(\delta/2).
\]

Combined with (2.4), this yields the claim. \( \square \)

We will also need the following lemma that is an easy adaptation of [11, Lemma 3.2].

Lemma 2.9. Let \( \Gamma \) be a \(*\)-connected set of \( \varepsilon \)-good sites. Let \( x \in C(j) \cap \Delta_N(j) \) and \( y \in C(k) \cap \Delta_N(k) \) with \( j, k \in \Gamma \). Then, we can find a microscopic, open path joining \( x \) and \( y \) of length at most \( 2d\mu(e_1)N|\Gamma| \) in \( \cup_{i \in \Gamma} \Delta_N(i) \).
Proposition 2.10. Let $\varepsilon > 0$. There exist $c > 0$ and $N \in \mathbb{N}$ such that for any macroscopic slab $S$ and $x, y \in S \cap \mathbb{Z}^d$ satisfying $x - y \in 2\varepsilon e_k$ with some $k \in [d]$,
\begin{equation}
\mathbb{P}(x, y \in \mathcal{C}_\infty(S), \exists x \in \mathcal{C}(x) \cap \Delta_N(x) \exists y \in \mathcal{C}(y) \cap \Delta_N(y); \mathcal{D}_{G}^\varepsilon(x, y) \geq (1+\varepsilon)^{-1}\mu(x-y)) \leq e^{-c\|x-y\|^2}.
\end{equation}

Proof. Without loss of generality, we assume $y = x + Ke_1$ with some $K \in \mathbb{N}$. Let $\varepsilon_0 := \varepsilon/(2\varepsilon + 4d\mu(e_1))^3 \in (0, 1/2)$. We take $N = N(\varepsilon_0) \in \mathbb{N}$ large enough as in Lemma 2.8 with $\varepsilon_0$ in place of $\varepsilon$ and $\delta$. We consider the macroscopic lattice of side length $N$ with $\varepsilon_0$-good. Since the probability in the claim is always less than 1, we can assume $\|x - y\|_\infty \geq 2N$ by taking $c = c(\varepsilon_0, N)$ small enough. Let $\Gamma \subset \mathbb{Z}^d$ be the set of macroscopic sites intersecting the line between $x$ and $y$, that is
$$\Gamma := \{x + i e_1 : i \in \{0, \ldots, K\}\}.$$ 

Our aim is to prove
\begin{equation}
\mathbb{P}(x, y \in \mathcal{C}_\infty(S), \exists x \in \mathcal{C}(x), \exists y \notin \mathcal{C}(y), \mathcal{D}_{G}^\varepsilon(x, y) \geq (1+\varepsilon)^{-1}\mu(x-y)) \leq \left\{ \left| \bigcup_{w \in \Gamma} \mathcal{C}(w) \right| \geq \varepsilon_0 |\Gamma| \right\}.
\end{equation}

Note that (2.7) implies the claim for sufficiently large $N = N(\varepsilon_0)$ and sufficiently small $c = c(\varepsilon_0, N)$ since by Lemma 2.8, the LHS of (2.6) is bounded from above by
$$\exp(-\varepsilon_0 |\Gamma|/2) \leq \exp(-c\|x - y\|^2).$$

To prove the contrapositive of (2.7), we suppose $\left| \bigcup_{w \in \Gamma} \mathcal{C}(w) \right| < \varepsilon_0 |\Gamma|$, $x, y \in \mathcal{C}_\infty(S)$, and $x \in \mathcal{C}(x), y \notin \mathcal{C}(y)$. Denote by $\mathcal{C}_S(w)$ the set of all $\varepsilon$-bad sites connected to $w$ by a macroscopic $\mathbb{Z}^d$-path of bad sites included in $S$. Since $\mathcal{C}_S(w) \subset \mathcal{C}(w)$, $\left| \bigcup_{w \in \Gamma} \mathcal{C}_S(w) \right| < \varepsilon_0 |\Gamma|$. Let $x \in \mathcal{C}(x)$ and $y \in \mathcal{C}(y)$ and denote by $L$ the line joining $x$ and $y$. Since $\|x - y\|_\infty = \|x - y\|_1 \geq 2N$, we can take $(x_k)_{k=0}^K \subset L$ such that $x_0 = x$, $x_K = y$, and $\|x_i - x_{i-1}\|_\infty \in [N, 2N]$ and $x_i \in \Delta_N(y, x_i)$ where $x_i := x + i e_1 \in \Gamma$. To each good site $x_k \in \Gamma$, choose the closest point $x'_k$ from $x_k$ in $\mathcal{C}(x_k)$ in a deterministic rule breaking ties. Thanks to Property (3) of $\varepsilon_0$-good, we have $\|x_k - x'_k\|_\infty \leq \varepsilon_0 N$. We decompose $\Gamma$ into portions of paths consisting of only good boxes as follows: We define
$$\tau_{in}(1) := 1, \quad \tau_{out}(1) := \max\{j \geq \tau_{in}(1) : \forall i \in \{\tau_{in}(1), \ldots, j\}, \quad x_i \text{ is good}\}.$$ 

Suppose that $\tau_{in}(1), \ldots, \tau_{in}(k)$ and $\tau_{out}(1), \ldots, \tau_{out}(k)$ have been defined. We define
$$\tau_{in}(k+1) := \min\{j \geq \tau_{out}(k) : \forall i \geq j, \quad x_i \notin \mathcal{C}_S(x_{\tau_{out}(k)+1})\},$$
$$\tau_{out}(k+1) := \max\{j \geq \tau_{in}(k+1) : \forall i \in \{\tau_{in}(k+1), \ldots, j\}, \quad x_i \text{ is good}\}.$$ 

We stop this procedure once $\tau_{out}(k+1) = K$. By construction, using $x, y \in \mathcal{C}_\infty(S)$, we have $x_{\tau_{out}(k)}, x'_{\tau_{out}(k)}, x_{\tau_{out}(k)+1} \in \mathcal{D}_G^\varepsilon \mathcal{C}(x_{\tau_{out}(k)+1})$, where
$$\partial_G^\varepsilon A := \{x \in S \setminus A : x \text{ is connected to infinity in } S \setminus A \text{ and has a neighbor in } A\}.$$ 

Since $\partial_G^\varepsilon \mathcal{C}_S(x_{\tau_{out}(k)+1})$ consists only of good boxes and is $\varepsilon$-connected in $S$ by Lemma 2.9, there exists an $\varepsilon$-good $\varepsilon$-path from $x_{\tau_{out}(k)}$ to $x_{\tau_{out}(k)+1}$ in $\partial_G^\varepsilon \mathcal{C}_S(x_{\tau_{out}(k)+1})$. By Lemma 2.9, we can therefore build a microscopic, open path between $x'_{\tau_{out}(k)}$ and $x'_{\tau_{out}(k)+1}$ of length at most
$$2dN \mu(e_1) |\partial_G^\varepsilon \mathcal{C}_S(x_{\tau_{out}(k)+1})| \leq (2d^2 N \mu(e_1)) |\mathcal{C}_S(x_{\tau_{out}(k)+1})|.$$ 

Between $x'_{\tau_{out}(k)}$ and $x'_{\tau_{out}(k)+1}$, thanks to Property (3) of $\varepsilon_0$-good, we can build a microscopic, open path of length at most
$$\sum_{j=\tau_{out}(k)}^{\tau_{out}(k)-1} [\mu(x'_{j+1} - x'_j) + \varepsilon_0 N] \leq \sum_{j=\tau_{in}(k)}^{\tau_{out}(k)-1} [\mu(x_{j+1} - x_j) + 4d \mu(e_1) \varepsilon_0 N]$$
$$\leq \mu(x_{\tau_{out}(k)} - x_{\tau_{in}(k)}) + 4d \mu(e_1) \varepsilon_0 N |\partial_G^\varepsilon \mathcal{C}_S(x_{\tau_{out}(k)-1}) - \tau_{in}(k)|,$$
where we have used $x_k \in L \cap \Delta_N(x_k)$ in the last inequality.
Finally, by the assumption \(|\bigcup_{w \in \Gamma} \mathcal{E}_S(w)| < \varepsilon_0|\Gamma|\),
\[
\sum_k |\mathcal{E}_S(x_{\tau_{\text{out}}(k)+1})| \leq \varepsilon_0|\Gamma|, \quad \sum_k (\tau_{\text{out}}(k) - \tau_{\text{in}}(k)) \leq |\Gamma|.
\]
Therefore, since \(|x - y|_\infty \geq 2N\) and \(|\Gamma| \leq 2d\mu(x - y)/N\), and \(\varepsilon_0 = \varepsilon/(2\varepsilon + 4d\mu(e_1))^3\), we can build a microscopic, open path from \(x\) to \(y\) of length at most
\[
\mu(x - y) + (4d)^2\mu(e_1)\varepsilon_0 N|\Gamma| \leq (1 + \varepsilon)\mu(x - y).
\]
Therefore, we have \([2.7]\).

\[\Box\]

2.4. Proof of Theorem 1.6
To prove Theorem 1.6, we need the following proposition.

Proposition 2.11. Suppose \(d \geq 3\). Let \(S = \mathbb{Z}^d \times \{0\}^{d-2}\). We denote \(\Lambda_{\varepsilon,\rho}(n; N) := [-\varepsilon n, \varepsilon n]^2 \times [-\rho N, \rho N]^{d-2}\).

For any \(\xi, \varepsilon > 0\), there exist \(N \in \mathbb{N}\) and \(c > 0\) such that for \(n\) large enough,
\[
\mathbb{P}\left(\mathcal{D}^{\rho}_{\xi}\left(\Lambda_{\varepsilon,\rho}(n; N), n\mathbf{e}_1 + \Lambda_{\varepsilon,\rho}(n; N)\right) > (\mu + \xi)n\right) \leq e^{-cn}.
\]

Proof. Let \(\xi, \varepsilon > 0\), and \(N \in \mathbb{N}\) large enough. Let \(F_n\) be the following event
\[
F_n := \{\mathcal{E}_{\infty}(S) \cap \{(k_1, 0, 0, \ldots, 0) : k_1 \in [-\varepsilon n/(2N), \varepsilon n/(2N)]\} = \emptyset\ \cup \{\mathcal{E}_{\infty}(S) \cap \{([n/N] + k_1, 0, 0, \ldots, 0) : k_1 \in [-\varepsilon n/(2N), \varepsilon n/(2N)]\} = \emptyset\}.
\]

If the event occurs, then there exists a \(\ast\)-connected component of bad sites in \(S\) such that it encloses \(\{(k_1, 0, 0, \ldots, 0) : k_1 \in [-\varepsilon n/(2N), \varepsilon n/(2N)]\} \cap \mathbb{Z}\). In particular, this \(\ast\)-connected component is of size at least \(\varepsilon n/N\). By a similar computation as in \([2.5]\), there exists \(c = c(\varepsilon, \rho, N, \kappa_2) > 0\) such that for \(n\) large enough, \(\mathbb{P}(F_n) \leq e^{-cn}\).

On the event \(F_n\), there exist \(x \in \mathcal{E}_{\infty}(S) \cap \{(k_1, 0, 0, \ldots, 0) : k_1 \in [-\varepsilon n/(2N), \varepsilon n/(2N)]\}\) and \(y \in \mathcal{E}_{\infty}(S) \cap \{([n/N] + k_1, 0, 0, \ldots, 0) : k_1 \in [-\varepsilon n/(2N), \varepsilon n/(2N)]\}\). The result follows by using Proposition 2.10.

\[\Box\]

Proof of Theorem 1.6. Let \(\xi, \varepsilon > 0\). Let \(N = N(\xi, \varepsilon) \in \mathbb{N}\) be as in Proposition 2.11. For \(z \in 2\rho \mathbb{Z}^{d-2}\), denote \(S(z) := \mathbb{Z}^d \times \{z\}\). Note that the slabs \((S(z) : z \in 2\rho \mathbb{Z}^{d-2})\) are all disjoint. Let \(\bar{z} := (0, 0, z)\).

We define \(\mathcal{E}(z) := \{\mathcal{D}^{\rho}_{\bar{z}}(\bar{x} + \Lambda_{\varepsilon,\rho}(n; N), \bar{z} + n\mathbf{e}_1 + \Lambda_{\varepsilon,\rho}(n; N)) > (\mu + \xi)n\}\).

Notice that
\[
\{\mathcal{D}^{\rho}_{\bar{z}}(\Lambda_{2\varepsilon n}(0), \Lambda_{2\varepsilon n}(n\mathbf{e}_1)) > (\mu + \xi)n\} \subset \bigcap_{z \in 2\rho \mathbb{Z}^{d-2} \cap [-\varepsilon n/(2N), \varepsilon n/(2N)]^{d-2}} \mathcal{E}(z).
\]

Since the events on the right hand side are independent, using Proposition 2.11 we get
\[
\mathbb{P}(\mathcal{D}^{\rho}_{\bar{z}}(\Lambda_{2\varepsilon n}(0), \Lambda_{2\varepsilon n}(n\mathbf{e}_1)) > (\mu + \xi)n) \leq \exp \left(-cn \left(\frac{\varepsilon N}{4N\rho}\right)^{d-2}\right).
\]

\[\Box\]
3. Space-time cut-points

3.1. Rate function for space-time cut-point. The proof of Theorem 1.1 relies on the following key lemma. We postpone its proof until Section 3.5.

Lemma 3.1. Let \( s_0 > 0 \). There exist \( c = c(s_0) > 0 \) and \( n_0 = n_0(s_0) \in \mathbb{N} \) such that the following holds. For any \( s, s' \in [0, s_0] \), \( x, x' \in [-s_0, s_0]^d \) and \( n \geq m \geq n_0 \),
\[
\mathbb{P}(A_{s,x}(n)) \mathbb{P}(A_{s',x'}(m)) \leq e^{cn^d} \mathbb{P}(A_{s+m+s', n+m} \cap A_{s+x+x'}(n+m)).
\]
Moreover, when \( n = m \), we have
\[
\mathbb{P}(A_{s,x}(n)) \mathbb{P}(A_{s',x'}(n)) \leq e^{cn^d} \mathbb{P}(A_{s+s', x+x'}(n)).
\]

Remark 3.2. Note that \( A_{s/2,x/2}(2n) \neq A_{s,x}(n) \) in general. Hence, the second inequality does not directly follow from the first inequality in Lemma 3.1 by setting \( n = m \).

We have the following corollary:

Corollary 3.3. Let \( s_0 > 0 \). There exist \( c = c(s_0) > 0 \) and \( n_0 = n_0(s_0) \in \mathbb{N} \) such that the following holds. For any \( \delta > 0 \), \( s, s' \in [0, s_0] \) and \( x, x' \in [-s_0, s_0]^d \) with \( |s - s'| \leq \delta \) and \( \|x - x'\|_1 \leq \delta \), and for any \( n \geq n_0 \),
\[
\mathbb{P}(A_{s,x}(n)) \leq e^{c(n+n\delta)} \mathbb{P}(A_{s',x'}(n)).
\]

Proof. We first suppose \( s \leq s' \). Thanks to Lemma 3.1, we have
\[
\mathbb{P}(A_{s,x}(n)) \mathbb{P}(A_{s'-x'-x}(n)) \leq \mathbb{P}(A_{s',x'}(n)) e^{c'n^d},
\]
with some constant \( c' > 0 \) independent of \( \delta, n \). Since \( |s' - s| + \|x' - x\|_1 \leq 2\delta \), one can check that (as in [3.5] below), we have
\[
\mathbb{P}(A_{s'-x'-x}(n)) \geq e^{c'n\delta}
\]
with some constant \( c'' > 0 \) independent of \( \delta, n \). Therefore, with \( c := c' + c'' \),
\[
\mathbb{P}(A_{s,x}(n)) \leq e^{c(n+n\delta)} \mathbb{P}(A_{s',x'}(n)).
\]
If \( s > s' \), by the result above, then
\[
\mathbb{P}(A_{s,x}(n)) \leq \mathbb{P}(A_{s',x'}(n)) \leq e^{c(n+n\delta)} \mathbb{P}(A_{s',x'}(n)).
\]

Let us see how this lemma and corollary imply Theorem 1.1.

Proof of Theorem 1.1. Let \( s \geq 0 \) and \( x \in \mathbb{R}^d \). Let \( n \geq m \geq 1 \). Thanks to Lemma 3.1,
\[
\mathbb{P}(A_{s,x}(n)) \mathbb{P}(A_{s,x}(m)) \leq e^{cn^d} \mathbb{P}(A_{s,x}(n+m)),
\]
with some constant \( c > 0 \) independent of \( n, m \). It follows that
\[
- \log \mathbb{P}(A_{s,x}(n+m)) \leq - \log \mathbb{P}(A_{s,x}(n)) - \log \mathbb{P}(A_{s,x}(m)) + cn^d.
\]
Using deBruijn and Erdős’s subadditive lemma [8], the following limit exists:
\[
I(s, x) := \lim_{n \to \infty} - \frac{1}{n} \log \mathbb{P}(A_{s,x}(n)) \in [0, +\infty).
\]
By Corollary 3.3, \( I \) is continuous. In particular, \( I \) is uniformly continuous on a compact set.

Let \( K \subset [0, \infty) \times \mathbb{R}^d \) be a compact set and \( s_0 := \text{diam}(K) + 1 \). Let \( \delta \in (0, 1/(4ds_0)) \) arbitrary. Since the cardinality of \( K_\delta := (\delta \mathbb{Z})^{d+1} \cap K \) is finite, the convergence (3.1) is uniform over \( K_\delta \). We take \( (s, x) \in K \) arbitrary. Let \( (s', x') \) be the closest point from \( (s, x) \) in \( K_\delta \) satisfying \( |s' - s| + \|x' - x\|_1 \leq 4d\delta \) with a deterministic rule breaking ties. By Corollary 3.3 and the uniform convergence over \( K_\delta \), since \( I \) is uniformly continuous on \( K \), we have
\[
\mathbb{P}(A_{s,x}(n)) \leq e^{o(n)} \mathbb{P}(A_{s',x'}(n)) \leq e^{o(n)} e^{-I(s', x') n} \leq e^{-I(s, x) n + o(n)},
\]
where $a_0(n)$ are some positive constants such that $a_0(n)/n$ converges to 0 uniformly over $(s, x) \in K$ when $n \to \infty$, and then $\delta \to 0$. Similarly, we have

$$\mathbb{P}(A_{s,x}(n)) \geq e^{-a_0(n)}\mathbb{P}(A_{s',x'}(n)) \geq e^{-a_0(n)}e^{-I(s',x')}n \geq e^{-I(s,x)n-a_0(n)}.$$ 

Thus, letting $\delta \to 0$, we find that the convergence (3.1) is uniform on $K$.

Let $s_0 > 0$. We take $c > 0$ and $n_0 \in \mathbb{N}$ as in Lemma 3.1. Let $s, s' \in [0, s_0], x, x' \in [-s_0, s_0]^d$, and let $\lambda \in (0, 1) \cap \mathbb{Q}$. Let $(u_n)_{n \geq 1}$ be a sequence of integers larger than $n_0/\min(\lambda, 1 - \lambda)$ such that $\lambda u_n \in \mathbb{N}$ and $u_n \to \infty$ as $n \to \infty$. Thanks to Lemma 3.1, we have

$$\mathbb{P}(A_{s,x}(\lambda u_n))\mathbb{P}(A_{s',x'}((1 - \lambda)u_n)) \leq e^{-a_0d}\mathbb{P}(A_{s,x+1}(\lambda s, x))\mathbb{P}(A_{s',x'}((1 - \lambda)u_n)).$$

By passing to the limit using (3.1) with $u_n$ in place of $n$, we get

$$I(\lambda s + (1 - \lambda)s', \lambda x + (1 - \lambda)x') \leq \lambda I(s, x) + (1 - \lambda)I(s', x').$$

This together with the continuity of $I$ implies (3.2) for general $\lambda$. Moreover, thanks to Lemma 3.1 with $n = m$, we have

$$\mathbb{P}(A_{s,x}(n))\mathbb{P}(A_{s',x'}(n)) \leq e^{-a_0d}\mathbb{P}(A_{s+s',x+x'}(n)).$$

By passing to the limit using (3.1), we get

$$I(s + s', x + x') \leq I(s, x) + I(s', x').$$

Next, we consider the homogeneity. Let $k, m \in \mathbb{N}, s \geq 0$, and $x \in \mathbb{R}^d$. By (3.3), $I(ks, kx) \leq kI(s, x)$. On the other hand, the opposite inequality is trivial because of $A_{ks,kx}(n) \subset A_{s,x}(kn)$. Therefore, we have $I(ks, kx) = I(s, x)$. This implies

$$I(ks/m, kx/m) = mI(ks/m, kx/m)/m = I(ks/kx)/m = kI(s, x)/m.$$ 

For general $\lambda \geq 0$, consider a sequence $\lambda_n \in [0, \infty) \cap \mathbb{Q}$ whose limit is $\lambda$. The homogeneity follows by the continuity of $I$ and (3.4) with $k/m = \lambda_n$.

Finally, we prove $I(s, x) > 0$ for any $s > 0$ and $x \in \mathbb{R}^d$. Note that by (3.3)

$$I(2s, 0) \leq I(s, x) + I(s, -x) = 2I(s, x).$$

Moreover, thanks to (1.9), $I(2s, 0) > 0$. Thus, we have $I(s, x) > 0$. \hfill \Box

3.2. Size of the ball corresponding to cut-point. For $K > 0$, define

$$A^K_{s,x}(n) := \left\{ \begin{array}{l} \exists s_n \geq s \cap \exists w_n \in \Lambda_{n_0d}(nx), \exists \Gamma \subset \mathbb{Z}^d; \\
B_{sn} \setminus B_{sn-1} = \{w_n\}, \{\Gamma\} \subseteq Kn, B_{sn} \subset \text{Int}(\Gamma) \end{array} \right\}.$$ 

Proposition 3.4. For any $s_0 > 0$, there exist $n_0 = n_0(s_0) \in \mathbb{N}$ and $K = K(s_0) > 0$ such that for any $n \geq n_0$,

$$\mathbb{P} \left( \bigcup_{s \in [0,s_0], x \in [-s_0,s_0]^d} (A_{s,x}(n) \setminus A^K_{s,x}(n)) \right) \leq e^{-n} \min_{s \in [0,s_0], x \in [-s_0,s_0]^d} \mathbb{P}(A_{s,x}(n)).$$

Proof. Let $s \in [0, s_0], x \in [-s_0, s_0]^d$, and $n \geq 1$. Let $w_n \in \Lambda_{n_0d}(nx) \setminus \{0\}$ and $\gamma$ be a deterministic self-avoiding path from 0 to $w_n$ such that $s_0 \gamma \leq |\gamma| \leq 2ds_0n$. Let us define

$$\mathcal{E} := \{ e \in \gamma, e \text{ is open}, \forall e \notin \gamma \text{ with } |e \cap \gamma| = 1, e \text{ is closed} \}.$$ 

By construction, we have $\mathcal{E} \subset A_{s,x}(n)$, and

$$\mathbb{P}(\mathcal{E}) \geq p^{\rho|\gamma|(1-p)^{2d|\gamma|}} \geq (p(1-p)^{2d})^{2ds_0n}.$$ 

Hence, there exists $C > 0$ depending on $s_0$, $p$, and $d$ such that for all $n \geq 1$,

$$\mathbb{P}(A_{s,x}(n)) \geq e^{-Cn}.$$
Let $s_n \geq s_n$ be such that $B_{s_n} \setminus B_{s_n-1} = \{w_n\}$. Set $\varepsilon = 1/2$. Let $N, K \in \mathbb{N}$ be chosen later. Let $\Gamma_N$ be the set of boxes intersecting $B_{s_n}$:

$$\Gamma_N := \{i \in \mathbb{Z}^d : \Lambda_N(i) \cap B_{s_n} \neq \emptyset\}.$$ 

Note that $\Gamma_N$ is connected in $\mathbb{Z}^d$. Let us consider the exterior boundary of $\Gamma_N$ as

$$\partial^{\text{ext}} \Gamma_N := \{i \in \mathbb{Z}^d \setminus \Gamma_N : \exists j \in \Gamma_N; \|i - j\|_1 = 1, i \text{ is connected to infinity in } \mathbb{Z}^d \setminus \Gamma_N\}.$$ 

Note that $\partial^{\text{ext}} \Gamma_N$ is $*$-connected by [30, Lemma 2]. Let $i \in \Gamma_N$ and $j \in \partial^{\text{ext}} \Gamma_N$ be such that $\|i - j\|_1 = 1$ and $w \notin \Lambda_N(i)$. Let us prove that $i$ or $j$ is $\varepsilon$-bad. By definition of $\Gamma_N$ and $\partial^{\text{ext}} \Gamma_N$, we have $\Lambda_N(j) \cap B_{s_n} = \emptyset$. Let us assume that $i$ and $j$ are both good. Let $x \in B_{s_n} \cap \Lambda_N(i)$. Note that $x$ is connected to $w \notin \Lambda_N(i)$ by a macroscopic, open path inside $B_{s_n}$. By definition of good box, this yields that $x \in \mathcal{C}(i) \cap B_{s_n} \cap \Lambda_N(i)$. Since $\Lambda_N(j) \cap B_{s_n} = \emptyset$, and $i$ and $j$ are good, there exists $y \in \mathcal{C}(j) \cap \mathcal{C}(i) \cap B_{s_n}$. Thus, $x$ and $y$ are connected by a macroscopic, open path in $\mathcal{C}(i) \subset \Lambda_N(i)$. Hence, by definition of cut-point, any macroscopic, open path from $x \in B_{s_n}$ to $y \notin B_{s_n}$ must contain $w$, which derives a contradiction since $w \notin \Lambda_N(i)$. It follows that any $j \in \partial^{\text{ext}} \Gamma_N$ is $\varepsilon$-bad, or has a $\varepsilon$-bad neighbor, or has a neighbor whose enlarged box contains $w_n$. Let $A^+ := \{i \in \mathbb{Z}^d : \exists j \in A; \|i - j\|_1 \leq 1\}$ for $A \subset \mathbb{Z}^d$. Therefore, $\partial^{\text{ext}} \Gamma_N$ is $*$-connected, its interior contains $\Gamma_N$, and

$$\{|i \in (\partial^{\text{ext}} \Gamma_N)^+ : i \text{ is bad}| \geq \frac{\|\partial^{\text{ext}} \Gamma_N\|}{(4d)^d} - (4d)^d.$$ 

Let $\Gamma := \bigcup_{i \in \partial^{\text{ext}} \Gamma_N} \Lambda_{N}(i)$. Since the interior of $\partial^{\text{ext}} \Gamma_N$ contains $\Gamma_N$, $B_{s_n} \subset \text{Int}(\Gamma)$.

Let $\kappa = 1/(10d)^d$ and $K \geq (10d)^{10d}$. Denote by $\mathcal{F}$ the event that there exists a macroscopic, $*$-connected set $A$ such that $|A| \geq Kn$, $\{|i \in A^+ : i \text{ is bad}| \geq \kappa|A|\}$, and its interior contains $0$. To compute its probability, we first fix such a $*$-connected set $A$. By pigeon-hole principle as in Proposition 2.6, there exists a subset of $A^+$ containing at least $\frac{\kappa}{4\rho^d}|A|$ bad sites at $\ell_\infty$ distance at least $\rho$ from each other (recall the notation $\rho$ from (2.1)). In particular, the states of these sites are independent. By Proposition 2.6, for $N$ large enough depending on $d, p$,

$$\mathbb{P}(\{|x \in A^+ : x \text{ is bad}| \geq \kappa|A|\} \leq 2|A^+|\exp\left(-cN\frac{K}{4\rho^d}|A|\right).$$

This yields

$$\mathbb{P}(\mathcal{F}) \leq \sum_{k \geq Kn} \sum_{y \in [-k,k]^{d} \cap \mathbb{Z}^d} \sum_{A \in \text{Animal}^d_{y}} \mathbb{P}(\{|x \in A^+ : x \text{ is bad}| \geq \kappa|A|\}) \leq \sum_{k \geq Kn} 7^{dk}(2k + 1)^d 2^{d^2k} \exp\left(-cN\frac{K}{4\rho^d}k\right) \leq \exp\left(-cN\frac{K}{(8p^d)K_n}\right),$$

where we have used (1.18) in the last line. Hence, thanks to (3.5), for $K$ large enough depending on $s_0, p, d$, we have for all $n \in \mathbb{N}$,

$$\mathbb{P}(\mathcal{F}) \leq e^{-(1+C)n} \leq e^{-n} \min_{s \in [0,s_0], x \in [-s_0,s_0]^d} \mathbb{P}(\mathcal{A}_{s,x}(n)).$$

On the event $\mathcal{F}^c$, $|\Gamma| \leq (2N)^d|\partial^{\text{ext}} \Gamma_N| \leq (2N)^dKn =: K'n$. Therefore, we obtain $\mathcal{A}_{s,x}(n) \setminus \mathcal{A}_{s,x}(n)^K(n) \subset \mathcal{F}$, and by using (3.6), it follows that

$$\mathbb{P}\left(\bigcup_{s \in [0,s_0], x \in [-s_0,s_0]^d} (\mathcal{A}_{s,x}(n) \setminus \mathcal{A}_{s,x}(n)^K(n))\right) \leq \mathbb{P}(\mathcal{F}) \leq e^{-n} \min_{s \in [0,s_0], x \in [-s_0,s_0]^d} \mathbb{P}(\mathcal{A}_{s,x}(n)).$$

□
3.3. Resampling arguments.

**Lemma 3.5 (Resampling lemma).** Let $A \in \mathbb{N}$. Recall $\tau = (\tau_e)_{e \in \mathbb{R}^d}$ from \[ \Box \]. Let $E_0 = E_0(\tau)$, $E_1 = E_1(\tau)$ be random sets of edges inside $[-n^2, n^2]^d$ depending on $\tau$ such that

$$E_0 \cap E_1 = \emptyset, \quad \max\{|E_0|, |E_1|\} \leq A \quad \text{a.s.}$$

Then, there exists another random configuration $(\tau^*_e)_{e \in \mathbb{R}^d} \in \{1, \infty\}^{\mathbb{R}^d}$ with the same law as $\tau$ such that for any event $\mathcal{E} \subset \{1, \infty\}^{\mathbb{R}^d}$ and $n \geq \max\{4d, (p(1-p))^{-1}\}$,

$$\mathbb{P}(\tau \notin \mathcal{E}; \forall e \in E_0(\tau), \tau^*_e = +\infty; \forall e \in E_1(\tau), \tau^*_e = 1; \forall e \notin E_0(\tau) \cup E_1(\tau), \tau^*_e = \tau_e) \geq e^{-8dA \log n} \mathbb{P}(\tau \in \mathcal{E}).$$

The configuration $\tau^* = (\tau^*_e)_{e \in \mathbb{R}^d}$ is called the resampled configuration.

**Proof.** Let $\tau^* = (\tau^*_e)_{e \in \mathbb{R}^d}$ be an independent copy of $\tau$. Let $\mathfrak{E}_0$ and $\mathfrak{E}_1$ be two independent random variables distributed uniformly on the set

$$\{ E \subset \mathbb{R}^d : \forall e \in E, e \subset [-n^2, n^2]^d, |E| \leq A \}.$$

In particular, we have

$$\mathbb{P}(\mathfrak{E}_0 = E_0, \mathfrak{E}_1 = E_1) \geq e^{-5dA \log n}.$$

We define the resampled configuration $\tau^r$ as

$$\tau^r_e = \begin{cases} \tau^*_e & \text{if } e \in \mathfrak{E}_0 \cup \mathfrak{E}_1, \\ \tau_e & \text{otherwise.} \end{cases}$$

Recall that $n \geq \max\{4d, (p(1-p))^{-1}\}$. We conclude

$$e^{-8dA \log n} \mathbb{P}(\tau \in \mathcal{E}) \leq \mathbb{P}(\tau \in \mathcal{E}, \forall e \in \mathfrak{E}_1, \tau^r_e = +\infty, \forall e \notin \mathfrak{E}_0, \tau^r_e = 1; \mathfrak{E}_0 = E_0, \mathfrak{E}_1 = E_1) \leq \mathbb{P}(\tau \in \mathcal{E}; \forall e \in E_0(\tau), \tau^r_e = +\infty; \forall e \in E_1(\tau), \tau^r_e = 1; \forall e \notin E_0(\tau) \cup E_1(\tau), \tau^r_e = \tau_e).$$

We say that an event $\mathcal{E}$ is decreasing if for any $\tau \in \mathcal{E}$ and $\tau^r$ satisfying for all $e \in \mathbb{R}^d \tau^r_e \geq \tau_e$, $\tau^r \in \mathcal{E}$. The terminology decreasing is here related to percolation. In other words, the less open edges there are, the more likely the event occurs.

**Lemma 3.6 (Making a cut-point).** Let $t_0 \in \mathbb{N}$. Consider a family of decreasing events $(\mathcal{E}(w), w \in \mathbb{Z}^d)$, $\mathcal{E}(w) \subset \{1, \infty\}^{\mathbb{R}^d}$. For any $k \geq (4d + (p(1-p))^{-1})^2$, $t_0 \geq \sqrt{k}$, and $\Lambda \subset \mathbb{Z}^d$,

$$e^{-(8d)2\sqrt{k} \log k} \mathbb{P}(\exists t \geq t_0 \exists x \in \Lambda : |B_t| \leq k, x \in B_{t-1} \cap \Lambda) \leq \mathbb{P}(\exists t \geq t_0 \exists x \in \Lambda : B_t \setminus B_{t-1} = \{x\}, \tau \in \mathcal{E}(x)).$$

**Proof.** Set

$$\mathcal{E} := \{ \tau \in \{1, \infty\}^{\mathbb{R}^d} : \exists t \geq t_0 \exists x \in \Lambda : |B_t| \leq k, x \in B_t \setminus B_{t-1}, \tau \in \mathcal{E}(x) \}.$$

Given $\tau \in \mathcal{E}$, let $t = t(\tau) \geq t_0$ and $x = x(\tau) \in \Lambda$ be such that $|B_t| \leq k, x \in B_t \setminus B_{t-1}, \tau \in \mathcal{E}(x)$. We take a geodesic $\gamma_x$ from 0 to $x$. In case there are several choices, we pick one of them according to a deterministic rule. Since the sets $(B_r \setminus B_{r-1}, r = 1 \ldots t)$ are disjoint, we have

$$\sum_{r=t-(\sqrt{k})}^{t} |B_r \setminus B_{r-1}| \leq |B_t| \leq k.$$

Hence, there exists $r \in [t] \setminus [t - (\sqrt{k})]$ such that $|B_r \setminus B_{r-1}| \leq \sqrt{k}$. Let $E_1(\tau) = \emptyset$ and

$$E_0(\tau) := \{ (v, w) \in \mathbb{R}^d : v \in \gamma_x \setminus B_{r-1} \} \cup \{ e \in \mathbb{R}^d : e \text{ connects } B_r \setminus B_{r-1} \text{ and } B_{r-1} \}.$$
Note that $|E_0(\tau)| \leq 4d\sqrt{k}$ and $E_0(\tau) \subset [-k,k]^d$. Hence, we can take a resampled configuration $\tau^f = (\tau^f_e)_{e \in \mathbb{E}^d}$ as in Lemma 3.5 with $\mathcal{E} \subset \{1,\infty\}^{\mathbb{Z}^d}$ so that $\tau^f_e \geq \tau_e$. Denote by $B^s_r$ the ball of radius $s$ centered at 0 for $\tau^f$. Since $\mathcal{E}(x)$ is decreasing, $\tau^f \in \mathcal{E}(x)$. Thus, we get
\[
e^{-\frac{8d}{2}(s\log k + 1)}P(\exists t \geq t_0 \ \exists x \in \Lambda \mid B^s_{k} \leq k, x \in B^s_{1}\setminus B^s_{t-1}, \tau \in \mathcal{E}(x))
\leq P(\exists t \geq t_0 \ \exists x \in \Lambda \mid B^s_{k} \leq k, x \in B^s_{1}\setminus B^s_{t-1}, \forall e \in E_0(\tau) \ \tau^t_e = \infty, \tau \in \mathcal{E}(x))
\leq P(\exists t \geq t_0 \ \exists x \in \Lambda \mid B^s_{k} \leq k, B^s_{t} \setminus B^s_{t-1} = \{x\}, \tau^t \in \mathcal{E}(x))
= P(\exists t \geq t_0 \ \exists x \in \Lambda \mid B^s_{k} \leq k, \tau \in \mathcal{E}(x)).
\]

3.4. Drilling free line. We define the number of lines in direction $e_i$ intersecting $A \subset \mathbb{Z}^d$ as
\[
N_i(A) := |\{z \in \mathbb{Z}^d \mid z_i = 0, L_i(z) \cap A \neq \emptyset\}|.
\]
Given $s > 0$ and $x \in \mathbb{R}^d$, let us define
\[
A_{s,x}^{\text{free}}(n) := \{\exists i \neq j \in [d], \exists s_n \geq sn - 3n^{2/3}, \exists w_n \in \Lambda_{4n^3}(nx); B_{sn} \setminus B_{sn-1} = \{w_n\}, L_i(w_n) \cap B_{sn} = \{w_n\}, \forall k \neq j \ N_k(H_j(w_n) \cap B_{sn}) \leq n^{\alpha_d}, \ |B_{sn}| \leq n^{7/4}\}.
\]
The following is a key result of this section, which is the most technical part of this paper.

Lemma 3.7 (Drilling free line at cut-point). Let $s_0 > 0$ and $K = K(s_0) > 0$ be as in Proposition 3.4. There exists $n_0 = n(s_0) \in \mathbb{N}$ such that the following holds. For any $s \in [0, s_0]$ and $x \in [-s_0, s_0]^d$, and $n \geq n_0$,
\[
e^{-n^{\alpha_d}}P(A_{s,x}^{K}(n)) \leq P(A_{s,x}^{\text{free}}(n)).
\]
We will need the following deterministic lemma. We postpone its proof until the appendix.

Lemma 3.8. Suppose $d \geq 2$. Let $S \subset \mathbb{Z}^d$. We define
\[
m(d, S) := \left(\frac{|S|}{2^{d-1}Diam(S)}\right)^{\frac{1}{d-1}}.
\]
There exist $i \neq j \in [d]$ and $S' \subset S$ with $|S'| \geq m(d, S)$ such that $z_i \neq z'_i$ and $z_j \neq z'_j$ for any $z \neq z'$ in $S'$.

From now on, if it is clear from context, we simply write $n^a$ instead of $|n^a|$ for $a > 0$. On the event $A_{s,x}^{K}(n)$, let $s_n \geq sn$ and $w_n \in \Lambda_{4n^3}(nx)$ such that $B_{sn} \setminus B_{sn-1} = \{w_n\}$. We take $\gamma_{0,w_n} = (w_1^n, \cdots, w_{n^n})$ to be a geodesic between 0 and $w_n$ with a deterministic rule breaking ties. Recall $\alpha_d = 1 - \frac{1}{6d}$. Set $\beta_d = \frac{5\alpha_d - 4}{5(d-1)}$. In particular, we have $1 - \beta_d < \alpha_d$.

If $sn < 3n^{\alpha_d}$ and $x \in \Lambda_{4n^3}(0)$, then $A_{s,x}^{\text{free}}(n)$ occurs with $s_n = 0$ and $w_n = 0$, in particular
\[
P(A_{s,x}^{K}(n)) \leq 1 = P(A_{s,x}^{\text{free}}(n)).
\]
Otherwise, then since $s_n \geq 3n^{\alpha_d}$ or $w_n \notin \Lambda_{3n^{\alpha_d}}(0)$, we eventually have $s_n \geq 3n^{\alpha_d}$ that we assume from now on. Let
\[
S := \{w^{s_n-2n^{\alpha_d}}, \cdots, w^{s_n-n^{\alpha_d}}\}.
\]
We consider the events:
\[
A_{n}^{[1]} := A_{s,x}^{K}(n) \cap \{\text{Diam}(S) \geq 2^{-(d-1)}n^{\frac{d}{2}}\},
A_{n}^{[2]} := A_{s,x}^{K}(n) \cap \{\exists i \neq j \in [d], \exists S' \subset S; |S'| = n^{\beta_d}, \forall z \neq z' \in S'z_i \neq z'_i, z_j \neq z'_j, \}.\]

By Lemma 3.8, $A_{s,x}^{K}(n) \subset A_{n}^{[1]} \cup A_{n}^{[2]}$. Hence, Lemma 3.7 follows from the following.
Lemma 3.9. For \( n \in \mathbb{N} \) large enough,

\[
\mathbb{P}(\mathcal{A}_n^{(1)}) \leq \frac{1}{2} e^{n^d} \mathbb{P}\left( A_{s,x}^{\text{free}}(n) \right),
\]

\[
\mathbb{P}(\mathcal{A}_n^{(2)}) \leq \frac{1}{2} e^{n^d} \mathbb{P}\left( A_{s,x}^{\text{free}}(n) \right).
\]

The proof of Lemma 3.9 involves modifying the initial environment and adjusting the time of the edges such that a new ball in the resampled environment possesses a free line at its cut-point, meaning a line that intersects the ball solely at the cut-point. To achieve this, we drill a line near the end of geodesic \( \gamma_{0,w_n} \), which is in close proximity to the cut-point \( w_n \). The primary challenge in this process is maintaining the overall structure of the ball, ensuring the new space-time cut-point remains close to the original one in both position and time. We need to be cautious in order to prevent the creation of a shortcut within the ball during the resampling process.

The first case (3.9) is simpler, as it pertains to situations where the geodesic \( \gamma_{0,w_n} \) does not wiggle too much (i.e. the diameter of \( S \) is sufficiently large). In this case, we can identify a hyperplane intersecting \( S \) where the size of the ball’s intersection with the hyperplane is negligible compared to \( n \). As a result, we only incur a minimal cost to maintain the connections after the shift and avoid a specific line. However, the second case (3.10) is considerably more difficult. This case applies to instances where the path has significant fluctuations around the endpoint, and the intersections of the ball with the hyperplane are not negligible. In this scenario, the shifting procedure differs; we employ two hyperplanes and shift each quadrant. It is not feasible to preserve all connections within the ball, as the expense may become exponentially high. Instead, we maintain the connections of the geodesic while blocking connections around bad boxes and the ball’s boundary. We will show that this resampling process does not generate a shortcut inside the ball.

Proof of (3.9). We suppose the event \( \mathcal{A}_n^{(1)} \) occurs. Without loss of generality, we assume \( \text{Diam}_1(S) \geq n^2/2^{d-1} \). Then, there exists \( S' \subset S = \{w^{s_n-2n^d}, \ldots, w^{s_n-n^d}\} \) such that \(|S'| = n^{4/3}/2^{d-1}\) and the hyperplanes \((H_1(z), z \in S')\) are disjoint. Let \( \Gamma \subset \mathbb{Z}^d \) be such that \(|\Gamma| \leq Kn\) and \( B_{sn} \subset \text{Int}(\Gamma) \). Then, there exists \( w^s = w^* \in S' \) such that \(|(H_1(w^s - e_1) \cup H_1(w^*)) \cap \Gamma| \leq 2^dK\). Hence, by the isoperimetry inequality [1.17], we have

\[
|(H_1(w^s - e_1) \cup H_1(w^*)) \cap B_{sn}| \leq Cn^{2/5}
\]

with some \( C = C(d,K) \in \mathbb{N} \). We write \( \overline{H}_1 := H_1(w^s - e_1) \cup H_1(w^*) \) and \( H_1^* := H_1(w^*) \).

Recall \((\tau_e)_{e \in E_0}\) from (1.7). Let \((\tau_e^s)_{e \in E_0}\) be an independent copy of \((\tau_e)\), and \( X = (X_i)_{i=1}^d \) a uniform random variable on \([-Kn, Kn]^d \cap \mathbb{Z}^d\) independent both from \((\tau_e), (\tau_e^s)\). Let \( \varphi_n = 2Cn^{2/5} \in 2\mathbb{N} \). We consider the following shift:

\[
\mathcal{S}(x) = \mathcal{S}(x) := \begin{cases} 
 x - (\varphi_n - 1)e_1 & \text{if } x_1 < X_1, \\
 x + \varphi_n e_1 & \text{if } x_1 \geq X_1.
\end{cases}
\]

Then, we define the resampled configuration \((\tau_e^{\mathcal{S}})\) as follows: for \( e = (x,y)\),

\[
\tau_e^{\mathcal{S}} := \begin{cases} 
 \tau_e^s & \text{if } x_1 \text{ or } y_1 \in (X_1 - \varphi_n, X_1 + \varphi_n), \\
 \tau_{(\mathcal{S}^{-1}(x), \mathcal{S}^{-1}(y))} & \text{otherwise}.
\end{cases}
\]

In the proof, we simply write \( D = D^{\bar{\mathcal{S}}} \) for the chemical distance for the configuration \( \tau \). Denote by \( D^\tau \) the chemical distance for the configuration \( \tau^\mathcal{S} \). We need the following claim.

Claim 3.10. Let \( n \in \mathbb{N} \) and \((x^i)_{i=1}^n, (y^i)_{i=1}^n \subset \mathbb{Z}^d \) such that

\[
2n = -x^i = y^i, \quad x^i = y^j \forall j \geq 2.
\]

Then, there exist disjoint paths \((\gamma^i)_{i=1}^n \) such that

\[
\gamma^i : x^i \to y^i, \quad \gamma^i \subset (-2n, 2n) \times \mathbb{Z}^{d-1} \cup \{x^i, y^i\}, \quad \gamma^i \cap L_3(0) = \emptyset, \quad |\gamma^i| \leq 8n.
\]
Proof. Without loss of generality, we can suppose $x_2^2 \leq x_2^2 \leq \cdots x_2^2$. We denote by $L(x, y)$ the straight line between $x$ and $y$. By abuse of notation, we can see $L(x, y)$ as a path by considering all the edges $e$ such that $e \subset L(x, y)$. Set

$\ell := \inf\{k \geq 1 : x_2^2 \geq 0\}$.

For $i < \ell$, we take $\gamma^i = L(x^i, y^i)$; for $i \geq \ell$, we take $\gamma^i$ to be

$L(x^i, x^i + 2(n - i)e_1) \oplus L(x^i + 2(n - i)e_1, x^i + 2(n - i)e_1 + e_2)$

$\oplus L(x^i + 2(n - i)e_1 + e_2, y^i - 2(n - i)e_1 + e_2) \oplus L(y^i - 2(n - i)e_1 + e_2, y^i)$.

From the definition, $(\gamma^i)_{i=1}^n$ are disjoint paths and do not intersect $L_3(0)$ (See Figure 4). \hfill $\square$

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{Construction of $(\gamma^i)$ colored in red.}
\end{figure}

From now on, we suppose $X = w^*$. Since $|H_1^i \cap B_n| \leq \varphi_n/2$, by Claim 10 with $\varphi_n/2$ in place of $n$, there exist disjoint paths $\tilde{\gamma}_{x,y} = \tilde{\gamma}_{x,y} : x - \varphi_n e_1 \rightarrow x + \varphi_n e_1$ for each pair $x, y \in H_1^i \cap B_n$ with $y = x - e_1$ such that

$\tilde{\gamma}_{x,y} \setminus \{x - \varphi_n e_1, x + \varphi_n e_1\} \subset \Z^d \setminus (\varphi_n \Z^d \cup L_3(w^*))$,

and $|\tilde{\gamma}_{x,y}| \leq 4\varphi_n$. We say that $e \in \Z^d$ crosses $H_1^i$ if there exists $z \in H_1^i$ such that $e = (z - e_1, z)$. Note that here we make no distinction on the orientation of the crossing. Let $(i_j)_{j=1}^k$ be such that $(w^{i_j}, w^{i_j+1})$ is the $j$-th crossing of $\gamma_{0,w_n}$ over $H_1^i$ until $s_*$. Let us define

$E_1 := \left\{ e \in \Z^d : e \in \tilde{\gamma}_{w^{i_j},w^{i_j+1}} \text{ with } j \leq k, \text{ or } e \in L(w^*, s(w^*)) \right\}$,

$E_2 := \left\{ e \in \Z^d \setminus E_1 : |e \cap e'| = 1 \text{ with some } e' \in E_1 \text{ or } e \in s(B_n \cap H_1^i) \neq \emptyset \right\}$.

Then, we consider the event

$A^i := \{ \text{X = w*; } \tau_+^e = 1 \text{ for } e \in E_1; \tau_+^e = \infty \text{ for } e \in E_2 \}$.

Since $|E_1 \cup E_2| \leq 4d\varphi_n^2 \leq 16C^2dn^{4/5}$ on $A^i_{[1]}$, for $n$ large enough depending on $p, d$, and $s_0$,

$\mathbb{P}(A^i \cap A^i_{[1]}) = \mathbb{E}[\mathbb{P}(X = w^*, \tau_+^e = 1 \text{ for } e \in E_1; \tau_+^e = \infty \text{ for } e \in E_2 \mid A^i_{[1]} \cap A^i_{[1]})]$

$\geq \mathbb{E}[(2KN_1 + 1)^{-d}p|E_1|(1 - p)|E_2|1_{A^i_{[1]}}] \geq e^{-c_0n^{4/5} \mathbb{P}(A^i_{[1]}),}$

where $c_0$ is a positive constant depending on $d, s_0$ and $p$. Set $\tilde{0} := s_X(0)$ and $t_* := D^s(\tilde{0}, w^*)$. We will prove that on $A^i \cap A^i_{[1]}$, the following occur:

(i) $t_* \in [sn - 3n^d, \infty)$ and $w^* \in A_4(n^d, nx + \tilde{0})$;
(ii) $L_3(w^*) \cap B_{t_*}^i(\tilde{0}) = \{w^*\}$;
(iii) $\forall k \neq 1 \ N_k(H_1^i \cap B_{t_*}^i(\tilde{0})) \leq C_0n^{4/5}$ where $C_0 := 16dC^2$.
(iv) $|B_{s_+}^r(0)| \leq n^{7/4}$.

Since $(\tau^r_{s+})_{s+\rightarrow 0} \overset{\text{law}}{=} (\tau_0)$, together with (3.12), this will imply

$$
P \left( \exists i \neq j \in [d], \exists t \geq sn - 3n^{\alpha_d}, \exists w_n \in \Lambda_{4n^{\alpha_d}}(nx); \right.
$$

$$
\left. \quad w_n \in B_1 \setminus B_{t-1}, L_i(w_n) \cap B_1 = \{ w_n \}, N_k(H_i(w_n) \cap B_1) \leq C_0 n^{1/5} \forall k \neq j, |B_t| \leq n^{7/4} \right) \geq \mathbb{P} \left( \exists i \neq j \in [d], \exists t \geq sn - 3n^{\alpha_d}, \exists w_n \in \Lambda_{4n^{\alpha_d}}(nx + \tilde{0}); w_n \in B^i_r(\tilde{0}) \setminus B^r_{t-1}(\tilde{0}) \right) \geq e^{-\text{const}^{3/5}} \mathbb{P}(A_r^i[1]).
$$

Together with Lemma 3.6, this yields the claim. Let us now prove the conditions (i)–(iv).

Since $\|\tilde{0} - 0\|_{\infty} \leq \varphi_n$ and $\|w^* - w_n\|_{\infty} \leq 2n^{\alpha_d}$, for $n$ large enough depending on $d, s_0$, we have $w^* \in \Lambda_{4n^{\alpha_d}}(nx + \tilde{0})$. We define a modified path:

$$
\tilde{\gamma} := (s(w^1), s(w^2), \ldots, s(w^i)) \oplus \tilde{\gamma}_{w^i_{1}, w^i_{1+1}} \oplus (s(w^i + 1), \ldots, s(w^j))
$$

$$
\oplus \tilde{\gamma}_{w^j_{1}, w^j_{1+1}} \ldots \oplus (s(w^k + 1), \ldots, s(w^*)) \oplus L(s(w^*), w^*).
$$

On the event $A^i \cap A_r^i[1]$, by (3.11), $\tilde{\gamma}$ is an open path from $\tilde{0} = s(w^1)$ to $w^*$ for $\tau^r$ such that

$$
|\tilde{\gamma}| \leq s_* + |\Pi^*_r \cap B_{s_n}| \times 8\varphi_n + \varphi_n \leq s_* + 5\varphi_n^2
$$

Hence, since $s_* \leq s - n^{\alpha_d}$, for $n$ large enough depending on $p, d, s_0$,

$$
(3.13) \quad \max \{ D^r(\tilde{0}, s(w^*)), D^r(\tilde{0}, w^*) \} \leq s_* + 20C_2 n^{4/5} < s_n.
$$

Next, we will prove that

$$
(3.14) \quad t_* := D^r(\tilde{0}, w^*) \geq s_* - \varphi_n.
$$

To this end, we take a geodesic $\gamma^r = (v^i)_{i=1}^{t_*}$ from $\tilde{0}$ to $s(w^*)$ for $\tau^r$. Let

$$
i_1 := \inf \{ k \geq 1 : v^k_1 \in (w^*_1 - \varphi_n, w^*_1 + \varphi_n) \} - 1,
$$

$$
\quad j_1 := \inf \{ k \geq i_1 : v^k_1 \in (w^*_1 - \varphi_n, w^*_1 + \varphi_n) \} - 1,
$$

$$
\quad i : = \inf \{ k \geq j_1 : v^k_1 \in (w^*_1 - \varphi_n, w^*_1 + \varphi_n) \} - 1,
$$

$$
\quad j := \inf \{ k \geq i : v^k_1 \in (w^*_1 - \varphi_n, w^*_1 + \varphi_n) \} - 1.
$$

Assuming that $i_1, j_1, \ldots, i_{t-1}, j_{t-1}$ have been defined, we define

$$
i_l := \inf \{ k > j_{l-1} : v^k_1 \in (w^*_1 - \varphi_n, w^*_1 + \varphi_n) \} - 1,
$$

$$
\quad j_l := \inf \{ k > i_l : v^k_1 \in (w^*_1 - \varphi_n, w^*_1 + \varphi_n) \} - 1.
$$

We continue this procedure until $i_l = \infty$ (using the convention $\inf \emptyset = \infty$). Let $m$ be the first number such that $i_{m+1} = \infty$. Note that $i_{l+1} > j_l$ for any $l < m$, since $\gamma^r$ is self-avoiding. Let us prove by induction that for all $l \leq m$,

$$
(3.15) \quad s^{-1}(v^i_l) \in B_{s_n}, s^{-1}(v^j_l) \in B_{s_n}, \tau_{s^{-1}(v^i_l), s^{-1}(v^j_l)} = 1, \text{ and } D^r(\tilde{0}, v^i_l) \geq D(0, s^{-1}(v^i_l)).
$$

We first note that the path $(s^{-1}(v^1), \ldots, s^{-1}(v^i_l))$ is open for $\tau$ by definition of the resampled environment and $(v^1, \ldots, v^i_l) \subset s(\mathbb{Z}^d)$. Hence, by (3.13),

$$
s_n > D^r(\tilde{0}, v^i_l) \geq D(0, s^{-1}(v^i_l)),
$$

which implies $s^{-1}(v^i_l) \in B_{s_n}$. Let us assume $s^{-1}(v^i_l) \in B_{s_n} \setminus \Pi^*_1$. Since $s^{-1}(v^i_l) \in \Pi^*_1$, $s^{-1}(v^i_l) \in B_{s_n} \cap \Pi^*_1$. On the event $A^i \cap A_r^i[1]$, we have necessarily that $(s^{-1}(v^i_l), s^{-1}(v^j_l))$ is an edge collinear to $e_1$ and corresponds to a crossing of $\gamma_{0,w_n}$ over $H^*_1$. In particular, the edge $(s^{-1}(v^i_l), s^{-1}(v^j_l))$
is open for $\tau$, and it follows that $s^{-1}(v_j) \in B_{s_n}$. The path $(s^{-1}(v_1), \ldots, s^{-1}(v_{l+1}))$ is open for $\tau$ by definition of the resampled environment. Recalling $0 = s^{-1}(0)$, by induction hypothesis and (3.13),

$$s_n > D^f(0, v_{l+1}) = D^f(0, v_l) + D^f(v_l, v_j) + D^f(v_j, v_{l+1})$$

$$\geq D(0, s^{-1}(v_l)) + 1 + D(s^{-1}(v_j), s^{−1}(v_{l+1}))$$

$$\geq D(0, s^{-1}(v_{l+1})).$$

It follows that $s^{-1}(v_{l+1}) \in B_{s_n-1}$. Since $(s^{-1}(v_{l+1}), s^{-1}(v_{l+1}))$ is an open edge for $\tau$ on $\mathcal{A}^r \cap A_n^{[1]}$ as before, $s^{-1}(v_{l+1}) \in B_{s_n}$. This concludes the induction.

Since $s(w^*) \in s(\mathbb{Z}^d)$, by maximality of $m$, the path $(v^{m}, \ldots, v^f)$ stays in $s(\mathbb{Z}^d)$. In particular, by definition of the resampling, the path $(s^{-1}(v^{m}), \ldots, s^{-1}(v^f))$ is open for $\tau$. Thanks to (3.15),

$$D^f(0, s(w^*)) = D^f(0, v^{m}) + D^f(v^{m}, v^{j}) + D^f(v^{j}, s(w^*))$$

$$\geq D(0, s^{-1}(v^{m})) + 1 + D(s^{-1}(v^{j}), s^{-1}(v^f)) \geq D(0, w^*) = s_\ast.$$

Thus, we obtain for $n$ large enough depending on $s_0, p, d$,

$$t_\ast = D^f(0, w^*) \geq D^f(0, s(w^*)) - D(w^*, s(w^*))$$

$$\geq s_\ast - \varphi_n \geq sn - 2n^{\alpha_d} - 2Cn^{d/5} \geq sn - 3n^{\alpha_d}.$$

This concludes the proof of (3).

Next, we will prove that

$$B^*_i(\bar{0}) \subset s(B_{s_n}) \cup \bigcup_{j=1}^{k} \bar{\gamma}_{u^{j}, w^{j+1}} \cup L(w^*, s(w^*)).$$

The proof is similar to that of (3.14). To end the proof, we take $v \in B^*_i(\bar{0})$ arbitrary, and take a geodesic $\gamma^* = (v^{\ell}_{\ell=0})$ from $\bar{0}$ to $v$. We define $(i_\ell, j_\ell)_{\ell=1}^{\infty}$ as in (3.14) with $v$ in place of $s(w^*)$. If $v \in s(\mathbb{Z}^d)$, then by (3.13), the same argument as in (3.16) shows

$$s_n > t_\ast \geq D^f(0, v) \geq D(0, s^{-1}(v)).$$

Hence, $v \in s(B_{s_n})$. Let us assume $v \notin s(\mathbb{Z}^d)$. Thanks to (3.15), we have $s^{-1}(v^{m}) \in B_{s_n} \cap H_1^*$. Thus, on the event $\mathcal{A}^r \cap A_n^{[1]}$, and $v \in \bar{\gamma}_{u^{j}, w^{j+1}}$ with some $j \leq k$ or $v \in L(w^*, s(w^*))$, and (3.17) follows. By the definition of $\bar{\gamma}_{u^{j}, w^{j+1}}$ and isoperimetry (1.16), since $s$ is injective, one can check that (iv) follows from (3.17).

Finally, by $|B_{s_n} \cap H_1^*| \leq \varphi_n/2$, (3.17), and $H_1^* \cap s(\mathbb{Z}^d) = \emptyset$, we have

$$|H_1^* \cap B_{s_n}^*(\bar{0})| \leq \left\lfloor k \sum_{j=1}^{k} \bar{\gamma}_{u^{j}, w^{j+1}} \cup L(w^*, s(w^*)) \right\rfloor \leq 4d\varphi_n^2 \leq 16dC^2n^{d/5}.$$

Thus, we have $N_k(H_1^* \cap B_{s_n}^*(\bar{0})) \leq 16dC^2n^{d/5}$ for $k \neq 1$. Therefore, we obtain (iii) for $n$ large enough depending on $s_0, p, d$. This concludes the proof.
Therefore, we have 
\[ P(\mathcal{A}_n^{[2]}) \leq P(\mathcal{A}_n^{[2]}, |V_n| \leq (2N)^d n, s_n \leq n \log n) + 2(5s_0n)^d e^{-cn \log n}. \]

We distinguish two cases, either \( P(\mathcal{A}_n^{[2]}) \leq e^{-\frac{1}{2}n \log n} \) or \( P(\mathcal{A}_n^{[2]}) > e^{-\frac{1}{2}n \log n} \). There is no need to examine the first case since it is negligible compared to \( P(\mathcal{A}_n^{[1]}) \) by (3.5). For the second case, we have for \( n \) large enough,

\[ P(\mathcal{A}_n^{[2]}) \leq 2P(\mathcal{A}_n^{[2]}, |V_n| \leq (2N)^d n, s_n \leq n \log n). \]

From now on, with abuse of notation, we write \( \mathcal{A}_n^{[2]} \) instead of \( \mathcal{A}_n^{[2]} \cap \{|V_n| \leq (2N)^d n\} \cap \{s_n \leq n \log n\} \).

Let \( S' \subset S \) and \( \Gamma \subset \mathbb{Z}^d \) be such that \( |\Gamma| \leq Kn \) and \( B_{s_n} \subset \text{Int}(\Gamma) \) as in the event \( \mathcal{A}_n^{[2]} \). Recall that \( \gamma_{0,w_n} \) is a geodesic from 0 to \( w_n \). Since the hyperplanes \( (H_1(z), z \in S') \) are disjoint by definition of \( S' \), there exist \( (z^k)_{k=1}^{n_d/2} \subset S' \) such that for any \( k \in \lceil n_d/2 \rceil \),

\[ |(H_1(z^k) \cup H_1(z^k - e_1)) \cap (\gamma_{0,w_n} \cup V_n \cup \Gamma)| \leq \frac{5|\gamma_{0,w_n} \cup V_n \cup \Gamma|}{n^{\beta_d}}. \]

Similarly, there exists \( w^* \in (z^k)_{k=1}^{n_d/2} \) such that

\[ |(H_2(w^*) \cup H_2(w^* - e_2)) \cap (\gamma_{0,w_n} \cup V_n \cup \Gamma)| \leq \frac{5|\gamma_{0,w_n} \cup V_n \cup \Gamma|}{n^{\beta_d}}. \]

Therefore,

\[ |(H_1(w^*) \cup H_1(w^* - e_1) \cup H_2(w^*) \cup H_2(w^* - e_2)) \cap (\gamma_{0,w_n} \cup V_n \cup \Gamma)| \]
\[ \leq |(H_1(w^*) \cup H_1(w^* - e_1)) \cap (\gamma_{0,w_n} \cup V_n \cup \Gamma)| + |(H_2(w^*) \cup H_2(w^* - e_2)) \cap (\gamma_{0,w_n} \cup V_n \cup \Gamma)| \]
\[ \leq \frac{10|\gamma_{0,w_n} \cup V_n \cup \Gamma|}{n^{\beta_d}}. \]

Recall that \( s_n \leq n \log n \) on \( \mathcal{A}_n^{[2]} \) and so \( |\gamma_{0,w_n}| \leq n \log n \). Thus, on the event \( \mathcal{A}_n^{[2]} \), we have \( |\gamma_{0,w_n} \cup V_n \cup \Gamma| \leq 4^d nN^d \). Let \( s_* \in \{s_n - 2n^{\alpha_d} + 1, \ldots, s_n - n^{\alpha_d}\} \) be such that \( w^* = w^{s_*} \). Set \( H^* := H_1(w^*) \cup H_2(w^*) \cup H_1(w^* - e_1) \cup H_2(w^* - e_2) \). It yields

\[ |H^* \cap (\gamma_{0,w_n} \cup V_n \cup \Gamma)| \leq 40^d n^{1-\beta_d} N^d. \]

Let \( (\tau_e^i) \) be an independent copy of \( (\tau_e) \), and \( X = (X_i)_{i=1}^d \) an independent uniform random variable on \([-Kn,Kn]^d \cap \mathbb{Z}^d \). We consider the following shift: for \( x = (x_i)_{i=1}^d \in \mathbb{Z}^d \),

\[ s(x) = s_X(x) := \begin{cases} 
  x + e_1 + e_2 & \text{if } x_1 \geq X_1 \text{ and } x_2 \geq X_2, \\
  x - e_1 + e_2 & \text{if } x_1 < X_1 \text{ and } x_2 \geq X_2, \\
  x - e_1 - e_2 & \text{if } x_1 < X_1 \text{ and } x_2 < X_2, \\
  x + e_1 - e_2 & \text{if } x_1 \geq X_1 \text{ and } x_2 < X_2.
\end{cases} \]

We note that for any \( x \in \mathbb{Z}^d \), \( s_X(x) \neq X_i \) for \( i = 1, 2 \), \( H^* = \mathbb{Z}^d \setminus s_X(\mathbb{Z}^d) \) and \( s_X \) is injective. We define the resampled configuration \( (\tau_e^r) \) by \( \tau_e^r = (\tau_{e^r}(x), y) \), for \( e = (x,y) \),

\[ \tau_e^r = \begin{cases} \tau_{(n-1)(x),y} & \text{if } x, y \in \mathbb{Z}^d, \\
 \tau_e^r & \text{otherwise.}
\end{cases} \]

From now on, we assume \( X = w^* \). We write \( H_1^* = H_1(w^*), H_2^* = H_2(w^*) \). For any \( x \in H_1^* \), we consider the straight path \( \gamma_{x-e_1,x} = \gamma_{x,x-e_1} \), between \( s(x) \) and \( s(x-e_1) \); for any \( x \in H_2^* \), we consider the straight path \( \gamma_{x-e_2,x} = \gamma_{x,x-e_2} \), between \( s(x) \) and \( s(x-e_2) \). In particular, these paths are made of three edges. We say that \( e \in \mathbb{Z}^d \) crosses \( H^* \) if there exist \( i \in \{1, 2\} \) and \( z \in H_i^* \) such that \( e = (z - e_i, z) \). Let \( (i_j)_{j=1}^k \) be such that \( \langle w^{i_j}, w^{i_{j+1}} \rangle \) is the \( j \)-th crossing of \( \gamma_{0,w_n} \) over \( H^* \) until \( s_* \). Since \( \gamma_{0,w_n} \) is self-avoiding and
$w^{s*} = w^*$, we have $w^{s^j} \neq w^{s*}$. Moreover, there exists a path $\tilde{\gamma}_{s(w^*),w^*}$ between $s(w^*)$ and $w^*$ with $|\tilde{\gamma}_{s(w^*),w^*}| = 2$ that do not have any edge in common with $\tilde{\gamma}_{w^{s^j},w^{s^j+1}}$, $j \leq k$; namely

$$\tilde{\gamma}_{s(w^*),w^*} := \begin{cases} (s_{w^*(w^*)}, s_{w^*(w^*)} - e_1, w^*), & \text{if } w^{s*-1} = w^{s*} - e_2, \\ (s_{w^*(w^*)}, s_{w^*(w^*)} - e_2, w^*), & \text{otherwise.} \end{cases}$$

Define

$$E_1 := \{e \in \mathbb{E}^d : e \in \tilde{\gamma}_{w^{s^j},w^{s^j+1}} \text{ with } j \leq k, \text{ or } e \in \tilde{\gamma}_{s(w^*),w^*}\},$$

$$E_2 := \{e \in \mathbb{E}^d \setminus E_1 : d_{\infty}(e, \gamma_{0,w_n} \cup \mathcal{V}_n \cup \Gamma \cap H^*) \leq 2N\},$$

$$E_3 := \{e \in \mathbb{E}^d \setminus E_1 : |e \cap L_3(w^*)| = 1, d_{\infty}(e, \text{Int} (\Gamma \cap H^*) \leq N\}.$$

Let $\mathcal{H}_1^* = H_1(w^*) \cup H_1(w^* - e_1)$ and $\mathcal{H}_2^* = H_2(w^*) \cup H_2(w^* - e_2)$. We now prove

$$|E_3| \leq 12dN(N_2(\text{Int}(\Gamma) \cap \mathcal{H}_1^*) + N_1(\text{Int}(\Gamma) \cap \mathcal{H}_2^*)).$$

Let us take $e = (u, v) \in E_3$ with $u = u^c \in L_3(w^*)$. Note that $u_3 = v_3$. Then, we select $z^c \in \text{Int}(\Gamma) \cap H^*$ such that $|z_3^c - u_3| \leq N$. If $z^c \in \text{Int}(\Gamma) \cap \mathcal{H}_1^*$, setting $g_3^c = 0$ and $g_{i}^c = z_i^c$ for $i \neq 2$, we have $z^c \in L_2(g^c) \cap \text{Int}(\Gamma) \cap \mathcal{H}_1^*$. In particular, $L_2(g^c) \cap \text{Int}(\Gamma) \cap \mathcal{H}_1^* \neq \emptyset$. Note that $u_3^c \in [g_3^c - N, g_3^c + N] \cap \mathcal{H}$. Similarly, if $z^c \in \text{Int}(\Gamma) \cap \mathcal{H}_2^*$, setting $r_3^c = 0$ and $r_{i}^c = z_i^c$ for $i \neq 2$, we have $L_1(r^c) \cap \text{Int}(\Gamma) \cap \mathcal{H}_2^* \neq \emptyset$. Note that $u_3^c \in [r_3^c - N, r_3^c + N] \cap \mathcal{H}$. Therefore, (3.20) follows from the fact that $|A| \leq |B| \max_{y \in B} |\mathcal{H}|^{-1}(y)$ for any map $f : A \rightarrow B$, and for any $g = (g_i) \in \mathcal{H}$ and $r = (r_i) \in \mathcal{H}$ with $g_2 = 0$ and $r_1 = 0$,

$$\max\{|e \in E_3 : g = g^c\}, |e \in E_3 : r = r^c\} \leq 12dN.$$

Next we estimate $N_k(\{z \in \mathcal{H} : d_{\infty}(z, \text{Int}(\Gamma) \cap \mathcal{H}_1^*) \leq N\})$ for $k \neq 1$. Note that we here consider a stronger estimate for a later purpose. Let $z \in \mathcal{H}$ be such that $z_k = 0$ and $L_k(z) \subset \mathcal{H} : d_{\infty}(z', \text{Int}(\Gamma) \cap \mathcal{H}_1^*) \leq N \neq \emptyset$. Then, there exists $z^0 = (z_i^0) \in \mathcal{H}$ such that $\|z - z^0\|_{\infty} \leq N, z_k^0 = 0$, and $L_k(z^0) \cap \text{Int}(\Gamma) \cap \mathcal{H}_1^* \neq \emptyset$. It follows that $L_k(z^0) \cap \Gamma \cap \mathcal{H}_1^* \neq \emptyset$. Recalling that $|\Gamma \cap H^*| \leq 4dN^{1-\beta_d} N^d$, this yields

$$N_k(\{z \in \mathcal{H} : d_{\infty}(z, \text{Int}(\Gamma) \cap \mathcal{H}_1^*) \leq N\}) \leq |\{z^0 \in \mathcal{H} : \|z^0\|_{\infty} \leq N\}| \times |\Gamma \cap \mathcal{H}_1^*| \leq 10dN^{1-\beta_d} N^{2d}.$$

By the same argument, we have for $k \neq 2$,

$$N_k(\{z \in \mathcal{H} : d_{\infty}(z, \text{Int}(\Gamma) \cap \mathcal{H}_2^*) \leq N\}) \leq 10dN^{1-\beta_d} N^{2d}.$$

Compared with (3.20) and (3.21), this gives

$$|E_3| \leq 10^3dN^{1-\beta_d} N^{2d+1}.$$
We consider the event
\[ A^r := \{ X = w^r; \tau^r_e = 1 \text{ for } e \in E_1; \tau^r_e = \infty \text{ for } e \in E_2 \cup E_3 \}. \]

Thanks to (3.19), \(|E_1 \cup E_2| \leq 10^d n^{1-\beta_d} N^{2d}\). Recall that \(1 - \beta_d < \alpha_d\). Combining this with (3.22), we have for \(n\) large enough depending on \(s_0, d, p\),
\[
\mathbb{P}(A^r \cap A_n^{[2]}[L]) = \mathbb{E}[\mathbb{P}(A^r \cap A_n^{[2]}[L]|A_n^{[2]}[L])]
\geq \mathbb{E}[(2Kn + 1)^{-d}p|E_1|(1-p)|E_2|\cup E_3|A_n^{[2]}[L]) \geq e^{-n^{\alpha_d}}\mathbb{P}(A_n^{[2]}[L]).
\]

Let \(\tilde{\tau} := s_X(0)\) and \(t_* := D^r(\tilde{\tau}, w^r)\). We will prove that on \(A^r \cap A_n^{[2]}\), the following occur:

(i) \(t_* \in [sn - 3n^{\alpha_d}, \infty)\) and \(w^r \in \Lambda_{4n^{\alpha_d}}(nx + \tilde{\tau})\);
(ii) \(L_\gamma(w^r) \cap B_{t_*}^0(\tilde{\tau}) = \{w^r\}\);
(iii) \(\forall k \neq 1\) \(N_k(H^r_i \cap B_{t_*}^0(\tilde{\tau})) \leq n^{\alpha_d}\);
(iv) \(|B_{t_*}^0(\tilde{\tau})| \leq n^{7/4}\).

Since \((\tau^r_{x+e})_{x \in \mathbb{Z}^d} \overset{law}{=} (\tau_e)\), this implies
\[
\mathbb{P} \left( \exists i \neq j \in [d], \exists t \geq sn - 3n^{\alpha_d}, \exists w_n \in \Lambda_{4n^{\alpha_d}}(nx); L_i(w_n) \cap B_t = \{w_n\}, N_k(H_j(w_n) \cap B_t) \leq n^{\alpha_d}\forall k \neq j, |B_t| \leq n^{7/4} \right)
\geq \mathbb{P} \left( \exists i \neq j \in [d], \exists t \geq sn - 3n^{\alpha_d}, \exists w_n \in \Lambda_{4n^{\alpha_d}}(nx + \tilde{\tau}); w_n \in B_{t_*}^0(\tilde{\tau}) \cap B_{t_*}^0(\tilde{\tau}); L_i(w_n) \cap B_t = \{w_n\}, N_k(H_j(w_n) \cap B_t) \leq n^{\alpha_d}\forall k \neq j, |B_t| \leq n^{7/4} \right)
\geq e^{-n^{\alpha_d}}\mathbb{P}(A_n^{[2]}[L]).

Combined with Lemma 3.6, this yields the claim.

We prepare some claims to prove (i) (iv). Given \(x \neq y \in H^r\), we define
\[ x \overset{H^r}{=} y \overset{def}{=} \exists \gamma : x \leftrightarrow y \subset H^r \setminus (\gamma_{0, w_n} \cup \Gamma \cup V_n), \]

with the convention \(x \overset{H^r}{=} x\). Let \(C_H(x)\) be the set of all points connected to \(x\) in \(H^r\) for the relation \(\overset{H^r}{=}\). Note that this relation only depends on the configuration \(\tau\). Note that
\[
(3.23) \quad x \in \gamma_{0, w_n} \Rightarrow C_H(x) = \{x\}, \quad (x, y) \in \gamma_{0, w_n} \Rightarrow y \notin C_H(x).
\]

**Claim 3.11.** Assume \(A^r \cap A_n^{[2]}\). Let \(a, b \in \mathfrak{s}(H^*)\) and a self-avoiding open path \(\gamma : a \leftrightarrow b\) for \(\tau^r\) such that \(\gamma \setminus \{a, b\} \subset H^r\). Then \(s^{-1}(a) \overset{H^r}{=} s^{-1}(b)\) or \(\langle s^{-1}(a), s^{-1}(b) \rangle \in \gamma_{0, w_n}\).

**Proof.** Note that \(s^{-1}(a) \oplus (\gamma \setminus \{a, b\}) \oplus s^{-1}(b)\) is a \(\mathbb{Z}^d\)-path inside \(H^r\). We extract a self-avoiding path \(\gamma'\) from this path. If \(\gamma' \cap (\gamma_{0, w_n} \cup \Gamma \cup V_n) = \emptyset\), then by definition we have \(s^{-1}(a) \overset{H^r}{=} s^{-1}(b)\). Otherwise, there exists \(e \in \gamma \setminus \{a, b\}\) at \(\ell_1\)-distance less than 1 from \(\gamma_{0, w_n} \cup \Gamma \cup V_n\). On the event \(A^r \cap A_n^{[2]}\), this implies that \(e \in E_1\) since otherwise the edge would be closed for \(\tau^r\). On the event \(A^r \cap A_n^{[2]}\), since the edges with one endpoint in common with the set \(E_1\) are closed, the path \(\gamma \setminus \{a, b\}\) has to be included in \(\gamma_{s(w^r), w^r}\) or \(\gamma_{s(w^r), w^r+1}\) for some \(j\). However, since \(\gamma\) is a path between \(a, b \in \mathfrak{s}(H^r)\), it cannot be included in \(\gamma_{s(w^r), w^r}\). Therefore, \(a \) and \(b\) are the extremities of \(\gamma_{s(w^r), w^r+1}\) for some \(j\) and \(\langle s^{-1}(a), s^{-1}(b) \rangle \in \gamma_{0, w_n}\).

Recall that \(C(a)\) denotes the largest open cluster of the \(\varepsilon\)-good box \(\Lambda^\varepsilon_N(a)\).

**Claim 3.12.** Let \(a, b \in \mathbb{Z}^d\) be good macroscopic sites. Let \(a \in H^* \cap C(a)\) and \(b \in C(b) \cap C_H(a)\). It holds that
\[ D(a, b) \leq 8d^2 \mu(e_1)N \{ e \in E^d : e \subset H^*, |e \cap C_H(a)| = 1 \}. \]
Proof. We claim that there exists a $\mathbb{Z}^d$-path $\gamma \subset C_H(a)$ from $a$ to $b$ in $H^*$ such that $|\gamma| \leq 4d|\{e \in \mathbb{E}^d : e \subset H^*, |e \cap C_H(a)| = 1\}|$.

Indeed, consider a $\mathbb{Z}^d$-path $\gamma'$ from $a$ to $b$ such that $|\gamma'| = \|a - b\|_1$; we arbitrarily take a path $\gamma$ from $a$ to $b$ in

$$(\gamma' \cap C_H(a)) \cup \{x \in C_H(a) : \exists y \in H^* \setminus C_H(a) \ | x - y \|_\infty = 1\}.$$ 

Note that the set $\{x \in C_H(a) : \exists y \in H^* \setminus C_H(a) \ | x - y \|_\infty = 1\}$ is $\mathbb{Z}^d$-connected (see for instance Lemma 2.23)). Moreover, we have

$$|\{x \in C_H(a) : \exists y \in H^* \setminus C_H(a) \ | x - y \|_\infty = 1\}| \leq 2d|\{e \in \mathbb{E}^d : e \subset H^*, |e \cap C_H(a)| = 1\}|.$$ 

Note that since $b \in C_H(a)$,

$$\|a - b\|_1 \leq |\{x \in C_H(a) : \exists y \in H^* \setminus C_H(a) \ | x - y \|_\infty = 1\}| \leq |\{e \in \mathbb{E}^d : e \subset H^*, |e \cap C_H(a)| = 1\}|.$$ 

This yields the existence of such a path $\gamma$. By construction, $\gamma$ only intersects good boxes since $C_H(a) \cap (\gamma_{0,\infty}) \cap \gamma \subset C_H(a)$. The path $\gamma$ intersects at most $|\gamma|$ boxes and all these boxes are good by construction and connected. By Lemma 2.9 there exists an open path from $a$ to $b$ of length at most $2d\mu(e_1)N|\gamma|$ and

$$D(a, b) \leq 2d\mu(e_1)N|\gamma| \leq 8d^2\mu(e_1)N|\{e \in \mathbb{E}^d : e \subset H^*, |e \cap C_H(a)| = 1\}|.$$

\[\square\]

Proof of (i). Note that $|0 - \tilde{0}| \leq 1$ and $\|w^* - w_n\|_\infty < 2n^{\alpha_d}$. It follows from $w_n \in \Lambda_{\alpha_d}(0 + nx)$ that $w^* \in \Lambda_{\alpha_d}(0 + nx)$. We define a modified path as

$$\tilde{\gamma} := (0, s(u_2^1) \cdots , s(u_i^1)) \oplus \gamma_{w^1,w^1+1} + \gamma_{w^1,w^1+1} \oplus \cdots \oplus (s(u_i^{k+1}), \cdots , s(u^k)) \oplus \gamma_{s(u^k),w^*}.$$ 

Recall that $s_s \leq s_n - n^{\alpha_d}$. On the event $A'$, $\tilde{\gamma}$ is still an open path from $\tilde{0}$ to $w^*$ for $\tau^f$. Hence, we have using (3.19) for $n$ large enough depending on $d, \alpha_0$, $p$,

$$(3.24) \quad D(0, w^*) \leq |\tilde{\gamma}| \leq |\gamma| + 3|H^* \cap \gamma_{0,\infty}| + |\gamma_{s(u^k),w^*}| \leq s_s + 10^{3d}n^{-\beta_d}N^d < s_n - \frac{1}{2}n^{\alpha_d}.$$ 

Let $v \in B^*_n - n^{\alpha_d}/2$. Let us take a geodesic $\gamma^f : \tilde{0} \rightarrow v$ with $\gamma^f = \{(v^f)^i\}_{i=1}^\tau$ for $\tau^f$. We define $i_1, i_2, \ldots , i_m, j_m$ inductively as follows: Let

$$i_1 := \inf \{k : v^k \in s(H^*) , v^{k+1} \in H^*\},$$

$$j_1 := \sup \{k \geq i_1 : v^k \in s(C_H(s^{-1}(v^i))) \text{ or } \langle s^{-1}(v^i), s^{-1}(v^k) \rangle \in \gamma_{0,\infty}\}.$$ 

Assuming that $i_1, i_2, \ldots , i_{j-1}, j_1-1$ have been defined, we define

$$i_j := \inf \{k \geq j_{i-1} : v^{k} \in s(H^*) \setminus s(C_H(s^{-1}(v^{j_{i-1}}))) , v^{k+1} \in H^*\},$$

$$j_i := \sup \{k \geq i_j : v^{k} \in s(C_H(s^{-1}(v^{j_{i-1}}))) \text{ or } \langle s^{-1}(v^i), s^{-1}(v^k) \rangle \in \gamma_{0,\infty}\}.$$ 

Let $m$ be the smallest integer such that $i_{m+1} = \infty$. For $k \leq m$, we set

$$x^k := s^{-1}(v^k) \text{ and } y^k := s^{-1}(y^k).$$ 

Note that if $i_{i+1} < \infty$, by Claim 3.11 then $v^{k} \in s(C_H(s^{-1}(v^{j_{i-1}}))) \text{ or } \langle s^{-1}(v^i), s^{-1}(v^k) \rangle \in \gamma_{0,\infty}$ with $k := \min\{j > i_i : v^j \in s(\mathbb{Z}^d)\}$, thus we have $i_j < j_i$. Therefore, $m$ is finite.

Claim 3.13. The paths $(v^{j_1}, \cdots , v^{k+1})$ for $k < m$ are contained in $s(\mathbb{Z}^d)$. In particular, the paths $\langle s^{-1}(v^{j_1}), \cdots , s^{-1}(v^{j})) \rangle$ for $k < m$ are open for $\tau$. Moreover, the path $(v^{j_1}, \cdots , v^{j})$ is either included in $s(\mathbb{Z}^d)$ or included in $H^*$. 
Proof. Let \( k < m \). Note that \( j_k > i_k \) mentioned above, and thus \( x^k \neq y^k \). If \( j_k = i_{k+1} \), then the claim is trivial. Hence, we assume \( j_k < i_{k+1} \). To prove the claim, we assume the contrary, i.e. there exists \( i \in (j_k, i_{k+1}) \) such that \( v^i \in H^* \). Consider the smallest such \( i \). If \( i > j_k + 1 \), then the edge \( \langle v^{i-1}, v^i \rangle \) crosses \( s(H^*) \) before \( i_{k+1} \) and \( v^{i-1} \in s(H^*) \backslash s(C_H(x^k)) \) due to \( i - 1 > j_k \), which contradicts the minimality of \( i_{k+1} \). Hence, we have \( i = j_k + 1 \), i.e. \( v^{j_k+1} \in H^* \). Note that, by (3.23), \( \langle x^k, y^k \rangle \in \gamma_{0, w_n} \) implies \( y^k \notin C_H(x^k) \) and \( j_k = i_{k+1} \), which contradicts the assumption \( j_k < i_{k+1} \). Thus, it holds \( x^k \not\preceq y^k \). Let \( j := \min\{i' \geq i : v^{i'} \in s(\mathbb{Z}^d)\} \). Since \( \langle y^k, s^{-1}(v^j) \rangle \in \gamma_{0, w_n} \) again implies \( j_k = i_{k+1} \) by (3.23), we have \( v^j \in s(C_H(y^k)) = s(C_H(x^k)) \) by Claim 3.11 which contradicts the maximality of \( j_k \). Therefore the path \( \langle v^{j_k}, \ldots, v^{j_{k+1}} \rangle \) is contained in \( s(\mathbb{Z}^d) \). Hence, by definition of \( \tau^i \), the path \( (s^{-1}(v^{j_k}), \ldots, s^{-1}(v^{j_{k+1}})) \) is open for \( \tau \).

We assume \( (v^{j_m+1}, \ldots, v^\ell) \) is not included in \( s(\mathbb{Z}^d) \), i.e. there exists \( i \in \{j_m + 1, \ldots, \ell\} \) such that \( v^i \notin H^* \). We take the smallest such \( i \). If \( i > j_{m+1} \), then by the same reasoning as above, it contradicts the maximality of \( m \). Hence, we have \( i = j_{m+1} \). We further suppose that \( (v^{j_m+1}, \ldots, v^\ell) \) is not included in \( H^* \). Define \( j = \min\{j > j_m : v^j \in s(\mathbb{Z}^d)\} \). If \( y^m \in C_H(x^m) \), then by (3.23) and Claim 3.11, \( \langle x^m, s^{-1}(v^j) \rangle = (y^m, s^{-1}(v^j)) \in \gamma_{0, w_n} \) or \( s^{-1}(v^j) \notin C_H(y^m) = C_H(x^m) \), which contradicts the maximality of \( j_m \). If \( y^m \notin C_H(x^m) \), then we have \( j_m = i_{m+1} \) by definition of \( i_k \), which contradicts the maximality of \( m \). □

Let us prove by induction that
\[
\forall k < m, \ x^k, y^k \in B_{s_{n-1}} \text{ and } x^m \in B_{s_{n-1}}.
\]
Let \( k \leq m \) and suppose that we have proved the claim for any \( i < k \), i.e. \( x^i, y^i \in B_{s_{n-1}} \). Thanks to Claim 3.13, we have
\[
(3.26) \quad \mathcal{D}(0, x^1) + \sum_{i=1}^{k-1} \mathcal{D}(y^i, x^{i+1}) \leq \mathcal{D}(0, s(x^1)) + \sum_{i=1}^{k-1} \mathcal{D}(s(y^i), s(x^{i+1})) \leq \mathcal{D}(0, v^{i_{k}}) \leq \ell.
\]
Let \( C := 10^{3d} \mu(e_1) \). Next, let us prove
\[
(3.27) \quad \sum_{i=1}^{k-1} \mathcal{D}(x^i, y^i) \leq CN^{d+1}n^{1-\beta_d}.
\]
For any \( i < k \), either (1) \( \langle x^i, y^i \rangle \in \gamma_{0, w_n} \), or (2) \( y^i \in C_H(x^i) \) holds. For case (1), we have \( \mathcal{D}(x^i, y^i) = 1 \). For case (2), by induction hypothesis \( x^i, y^i \in B_{s_{n-1}} \), they belong to the largest open cluster of their respective good boxes. Thanks to Claim 3.12, we have
\[
\mathcal{D}(x^i, y^i) \leq 8d^2 \mu(e_1) |\{e \in \mathbb{E}^d : e \subset H^*, |e \cap C_H(x^i)| = 1\}|.
\]
Finally, since \( \{i \leq k : |e \cap C_H(x^i)| = 1\} \leq 2 \) for any \( e \subset H^* \), using (3.19)
\[
(3.28) \quad \sum_{i=1}^{k-1} \mathcal{D}(x^i, y^i) \leq 8d^2 \mu(e_1) N \sum_{i=1}^{k-1} |\{e \in \mathbb{E}^d : e \subset H^*, |e \cap C_H(x^i)| = 1\}| + |H^* \cap \gamma_{0, w_n}| \leq 32d^2 \mu(e_1) N \mu(e_1) |H^* \cap (\mathbb{V}_n \cup \Gamma \cup \gamma_{0, w_n})| \leq CN^{d+1}n^{1-\beta_d}.
\]
Therefore, since \( \ell \leq s_n - \frac{1}{2} n^{\alpha_d} \) by \( v \in B_{\frac{s_n}{2}n^{\alpha_d}} \), (3.26) and (3.27) imply
\[
(3.29) \quad \mathcal{D}(0, x^k) \leq \mathcal{D}(0, x^1) + \sum_{i=1}^{k-1} \mathcal{D}(y^i, x^{i+1}) + \sum_{i=1}^{k-1} \mathcal{D}(x^i, y^i)
\]
\[
\leq \ell + \sum_{i=1}^{k-1} \mathcal{D}(x^i, y^i) \leq \ell + CN^{d+1}n^{1-\beta_d} \leq s_n - \frac{1}{4} n^{\alpha_d}.
\]
This, in particular, yields \( x^k \in B_{s_{n-2}} \).
It remains to prove \( y^k = s^{-1}(v^k) \in B_{s_{n-1}} \) for \( k < m \). Let us assume \( y^k \notin B_{s_{n-1}} \). Then, since \( D(x^k, y^k) \geq 2 \), \( (x^k, y^k) \notin \gamma_{0,w_n} \) and thus \( y^k \notin C_H(x^k) \setminus \{ x^k \} \). In particular, the boxes containing \( x^k \) and \( y^k \) are good, and the two points \( x^k \) and \( y^k \) must be connected by a path in \( H^* \setminus (\gamma_{0,w_n} \cup \Gamma \cup V_n) \). If \( y^k \) is in the largest open cluster, since \( x^k \in B_{s_n} \) is in the largest open cluster of its good box, by the same argument as in (3.28), we have \( D(x^k, y^k) \leq CN^{d+1}n^{1-\beta_d} \), which implies \( y^k \in B_{s_{n-1}} \) by (3.29). Otherwise, if \( y^k \) is not in the largest open cluster, since by Claim 3.13 the path \( (y^k, s^{-1}(v^k+1)), \ldots, s^{-1}(v^k+1), x^k+1) \) is open for \( \tau \). Since there are no two distinct large open clusters in \( \Lambda_N(y^k) \), \( \| y^k - x^{k+1} \| \leq \frac{N}{2} \). Moreover, since \( x^k+1 \in \gamma_{0,w_n} \) implies that \( y^k \) must be in the largest open cluster, \( x^{k+1} = s^{-1}(v^k+1) \notin \gamma_{0,w_n} \). Thus, the edge \( (v^k+1, v^k+1) \) does not belong to \( E_1 \). Moreover, since the edge \( (v^k+1, v^k+1) \) crosses \( s(H^*) \) and all the edges in \( E_2 \) are closed, \( d_\infty(x^{k+1}, H^* \cap (\gamma_{0,w_n} \cup \Gamma \cup V_n)) > 2N \). Thus, there exists a path from \( y^k \) to \( x^{k+1} \) inside \( H^* \) not intersecting with \( H^* \cap (\gamma_{0,w_n} \cup \Gamma \cup V_n) \), and \( x^{k+1} \in C_H(y^k) = C_H(x^k) \), which derives a contradiction. Therefore, we have (3.23) with \( k < m \).

We can also conclude from what is above that either \( y^m \in B_{s_{n-1}} \) or \( y^m \in C_H(x^m) \setminus \{ x^m \} \). In both cases, we have that \( y^m \in \text{Int}(\Gamma) \cap H^* \). When \( y^m \in B_{s_{n-1}} \), by the same argument as in (3.26) and (3.27), we have

\[
D(0, y^m) \leq D(0, s^{-1}(v^m)) + CN^{d+1}n^{1-\beta_d}.
\]

If \( y^m \notin B_{s_{n-1}} \), then \( y^m \) does not belong to the largest cluster of its good box, as we proved its contrapositive. If, in addition, \( v \in s(\mathbb{Z}^d) \), by Claim 3.13, the path \( (v^m, \ldots, v^f) \) is included in \( s(\mathbb{Z}^d) \). Since there are no two distinct large open clusters in \( \Lambda_N(y^m) \), \( (s^{-1}(v^m), \ldots, s^{-1}(v^f)) \) is open for \( \tau \) and inside \( \Lambda_{N/2}(y) \). In particular,

\[
D(y^m, s^{-1}(v)) \leq N_d.
\]

Let us now furthermore assume that \( v \in s(B_{s_{n-\alpha_d}}) \cap B_{s_n - \frac{3}{2} n^{\alpha_d}}(0) \). We will prove that

\[
D(0, s^{-1}(v)) \geq D(0, s^{-1}(v)) - \frac{1}{2} n^{\alpha_d}.
\]

If \( y^m \notin B_{s_{n-1}} \), then by (3.31), we have

\[
D(0, y^m) = D(0, s^{-1}(v)) + N \leq s_n - n^{\alpha_d} + N < s_n,
\]

which contradicts \( y^m \notin B_{s_{n-1}} \). Therefore, we have \( y^m \in B_{s_{n-1}} \). Moreover, by (3.30),

\[
D(0, s^{-1}(v)) \leq D(0, y^m) + D(y^m, s^{-1}(v)) \leq D(0, v) + CN^{d+1}n^{1-\beta_d} \leq D(0, v) + \frac{1}{2} n^{1-\alpha_d}.
\]

This yields (3.32). In particular, since \( D(0, w^*) \leq s_n - \frac{1}{2} n^{\alpha_d} \) by (3.24), applying the previous inequality for \( v = w^* \), we have

\[
D(0, w^*) \geq D(0, s(w^*)) - D(w^*, s(w^*)) \geq s_n - CN^{d+1}n^{1-\beta_d} - 2 > s_n - 3n^{\alpha_d}.
\]

**Proof of (iv)** Recall \( t_* := D(0, w^*) \leq s_n - \frac{1}{2} n^{\alpha_d} \). Let us prove that

\[
B_{t_*}^* \subseteq s(B_{s_n}) \cup \{ z \in \mathbb{Z}^d : d_\infty(z, \text{Int}(\Gamma) \cap H^*) \leq N \}.
\]

From now on, we assume that \( v \in B_{t_*}^* \). We keep the same notation \( \gamma^i \) to be a geodesic for \( \tau^i \) between \( 0 \) and \( v \), and \( j_k, k_j \) are defined in the same way as before. Thanks to (3.29), we have \( x^m \in B_{s_{n-2}} \). Recall that we have proved \( y^m \in \text{Int}(\Gamma) \cap H^* \) above (3.31). We divide the proof into several cases. Let us first assume that \( v \notin s(\mathbb{Z}^d) \). By Claim 3.13, the path \( (v^{j_{m+1}}, \ldots, v) \) is included in \( H^* \). On the event \( \mathcal{A}_1 \cap \mathcal{A}_2 \), since \( y^m \in \text{Int}(\Gamma) \cap H^* \) and the path \( (v^{j_{m+1}}, \ldots, v) \) is open for \( \tau^i \), we
have either that the path is included in \(E_1\) or it cannot go at distance less than \(2N\) from \(\Gamma\). In both cases, \(d_\infty(v, \text{Int}(\Gamma) \cap H^*) \leq N\).

Let us next assume that \(y^m \in B_{s_n-1}^\ast\) and \(v \in s(\mathbb{Z}^d)\). By (3.30), we have
\[
\mathcal{D}(0, y^m) \leq \mathcal{D}(0, s(y^m)) + CN^{d+1}n^{1-\beta_d}.
\]
By Claim 3.13, the path \((v, y^m, \ldots, v)\) is in \(s(\mathbb{Z}^d)\) and the path \((s^{-1}(v), s^{-1}(y^m), \ldots, s^{-1}(v))\) is open for \(\tau\). Hence, by \(s(y^m) = v^m \in \gamma^r\) and \(t_s \leq s_n - \frac{2}{7}n^\alpha_d\), those imply
\[
\mathcal{D}(0, s^{-1}(v)) \leq \mathcal{D}(0, y^m) + \mathcal{D}(y^m, s^{-1}(v)) \\
\leq \mathcal{D}(0, s(y^m)) + CN^{d+1}n^{1-\beta_d} + \mathcal{D}(s(y^m), v) \\
= \mathcal{D}(0, v) + CN^{d+1}n^{1-\beta_d} \leq t_s + CN^{d+1}n^{1-\beta_d} \leq s_n,
\]
which yields \(v \in s(B_{s_n}^\ast)\).

Finally, assume \(y^m \notin B_{s_n-1}^\ast\) and \(v \in s(\mathbb{Z}^d)\). Thanks to (3.31) and a claim thereover, we have \(y^m \in \text{Int}(\Gamma) \cap H^*\) and
\[
d_\infty(v, \text{Int}(\Gamma) \cap H^*) \leq N.
\]
Thus, we have (3.34). Condition (iv) follows from (1.17), \(B_{s_n} \subset \text{Int}(\Gamma)\), \(|\Gamma| \leq Kn\), and (3.34).

**Proof of (iii).** Let us prove that \(B_{1}^\ast(\tilde{0}) \cap L_3(w^*) = \{w^*\}\) on the event \(\mathcal{A}^r \cap \mathcal{A}^k_{n}\). To this end, let us take \(v \in B_{1}^\ast(\tilde{0}) \cap L_3(w^*)\) and a geodesic \(\gamma^r = \gamma^r_{\ell-1} \) from \(\tilde{0}\) to \(v\) for \(\tau^r\). Let
\[
i_\tau := \min\{i \leq \ell : v^r \in L_3(w^*)\}.
\]
If \(v^r = w^*\), then \(i_\tau = \ell\) and \(v = w^*\). Indeed, since any edge \(e \in E_2\) is closed and the path \(\gamma^r\) is self-avoiding, it must end at \(w^*\). Otherwise, if \(v^r \neq w^*\), then \(d_\infty(v^r, \text{Int}(\Gamma) \cap H^*) > N\) since any edge \(e \in E_3\) is closed. However, since \(s(\mathbb{Z}^d) \cap L_3(w^*) = \emptyset\), \(v^r \notin s(B_{s_n}^\ast)\). By (3.34), this implies \(d_\infty(v^r, \text{Int}(\Gamma) \cap H^*) \leq N\), which leads to a contradiction.

**Proof of (iii).** Finally, we prove \(N_k(H_1^\ast \cap B_{1}^\ast(\tilde{0})) \leq n^\alpha_d\) for any \(k \neq 1\). We fix \(k \neq 1\). Suppose \(L_k(z') \cap H_1^\ast \cap B_{1}^\ast(\tilde{0}) \neq \emptyset\) with \(z' \in \mathbb{Z}^d\) such that \(z_k' = 0\). By (3.34) and \(H_1^\ast \cap s(\mathbb{Z}^d) = \emptyset\),
\[
(3.35) \quad L_k(z') \cap \{z \in \mathbb{Z}^d : d_\infty(z, \text{Int}(\Gamma) \cap H_1^\ast) \leq N\} \neq \emptyset.
\]
Combined with (3.21), this implies that for \(n\) large enough depending on \(s_0, p, d\),
\[
N_k(H_1^\ast \cap B_{1}^\ast(\tilde{0})) \leq N_k(\{z \in \mathbb{Z}^d : d_\infty(z, \text{Int}(\Gamma) \cap H_1^\ast) \leq N\}) \leq n^\alpha_d.
\]
\[\square\]

### 3.5. Proof of Lemma 3.1

We will need the following lemma that gives a lower bound for the number of almost disjoint paths connecting two given subsets of two separate hyperplanes. We postpone its proof until the appendix.

**Lemma 3.14.** Let \(d \geq 3\). There exists \(C_d > 0\) such that the following holds. Let \(S_1, S_2 \subset \mathbb{Z}^d\) that satisfy one of the following.

- There exist \(i \in [d]\), \(K \geq \ell \geq 1\) such that \(S_1 \subset H_i(0), S_2 \subset H_i(te_i)\) and
\[
(3.36) \quad \max_{x \in S_1, y \in S_2} \|x - y\|_\infty = K.
\]

- There exist \(i \neq j \in [d]\), \(K \geq 1\) such that \(S_1 \subset H_i(0), S_2 \subset H_j(0)\) and \(S_1 \cup S_2 \subset [-K, K]^d\).

Then, we can find \(\mathbb{Z}^d\)-paths \((p_i)_{1 \leq i \leq m}\) from \(S_1\) to \(S_2\) with \(m := \min(|S_1|, |S_2|)/2\) such that the length of each path is less than \(2dK\) and there exists a constant \(\chi_d \in \mathbb{N}\) depending only on \(d\) such that
\[
\forall x \in \mathbb{Z}^d \quad |\{i \in [m] : x \in p_i\}| \leq \chi_d\left(\frac{K}{\ell}\right)^{d-1}
\].

Proof of Lemma 3.7. Let \( s_0 > 0 \). Let \( K \) be as in Proposition 3.4 with \( 2s_0 \) in place of \( s_0 \). Let \( n_0 \) be so that Proposition 3.4 and Lemma 3.7 hold. Let \( s, s' \in [0, s_0], t, x' \in [-2s_0, 2s_0]^d \) and \( n \geq m \geq n_0 \). Denote

\[
A_{s,x}^{\text{free},K}(n) := \left\{ B_{s_n} \setminus B_{s_{n-1}} = \{ w_n \}, L_i(w_n) \cap B_{s_n} = \{ w_n \}, \forall k \neq j, N_k(H_j(w_n) \cap B_{s_n}) \leq n^\alpha, \exists \Gamma \subset \mathbb{Z}^d; B_{s_n} \subset \text{Int}(\Gamma), |\Gamma| \leq Kn \right\}.
\]

By Proposition 3.4 we have

\[
\mathbb{P}(A_{s,x}^{\text{free}}(n) \setminus A_{s,x}^{\text{free},K}(n)) \leq \mathbb{P} \left( \bigcup_{t \in [0,2s_0], y \in [-2s_0,2s_0]^d} (A_{t,y}^{K}(n) \setminus A_{t,y}^{K}(n)) \right) \leq e^{-n} \mathbb{P}(A_{s,x}(n)).
\]

It follows that by Lemma 3.7 and Proposition 3.4

\[
\frac{1}{2} \mathbb{P}(A_{s,x}(n)) \leq \mathbb{P}(A_{s,x}^{K}(n)) \leq e^{n^\alpha} \mathbb{P}(A_{s,x}^{\text{free}}(n)) \leq e^{n^\alpha} (\mathbb{P}(A_{s,x}^{\text{free},K}(n)) + e^{-n} \mathbb{P}(A_{s,x}(n))).
\]

Finally, we have for \( n \) large enough depending on \( s_0 \),

\[
(3.37) \quad \mathbb{P}(A_{s,x}(n)) \leq 4e^{n^\alpha} \mathbb{P}(A_{s,x}^{\text{free},K}(n)).
\]

Define for \( t \geq 1, i \neq j \in [d], w \in \mathbb{Z}^d \),

\[
\mathcal{E}_i^{e,\omega,i,j} := \left\{ B_t \setminus B_{t-1} = \{ w \}, L_i(w) \cap B_t = \{ w \}, \exists \Gamma \subset \mathbb{Z}^d; B_t \subset \text{Int}(\Gamma), |\Gamma| \leq Kn, \forall k \neq j, N_k(H_j(w) \cap B_{s_n}) \leq n^\alpha \right\},
\]

\[
\mathcal{E}_i^{b,\omega,i,j} := \left\{ B_t \setminus B_{t-1} = \{ w \}, L_i(0) \cap B_t = \{ 0 \}, \exists \Gamma \subset \mathbb{Z}^d; B_t \subset \text{Int}(\Gamma), |\Gamma| \leq Kn, \forall k \neq j, N_k(H_j(w) \cap B_{s_n}) \leq n^\alpha \right\}.
\]

Note that on the event \( A_{s,x}^{\text{free},K}(n) \), since there exists \( \Gamma \subset \mathbb{Z}^d \) such that \( |\Gamma| \leq Kn \) and \( B_t \subset \text{Int}(\Gamma) \), it follows that \( t \leq |\text{Int}(\Gamma)| \leq (Kn)^d \). By pigeon-hole principle, using (3.37), there exist \( s_n \in [sn - 3n^\alpha, (Kn)^d], w_m \in \Lambda_{1m^\alpha}(nx_i), i_n \neq j_n \in [d] \) such that

\[
(3.38) \quad \mathbb{P}(\mathcal{E}_{s_n,w,m,i_n,j_n}^{e}) \geq \frac{1}{(Kn)^d} e^{-n^\alpha} \mathbb{P}(A_{s,x}(n)).
\]

In order to make \( \mathcal{E}_{s_m,w,m,i_m,j_m}^{b} \) occur from \( A_{s',x'}(m) \), we first apply Lemma 3.6 to create a cut point at 0, i.e. \( B_{s_n}(w) \setminus B_{s_{n-1}}(w) = \{ 0 \} \), and then we apply Lemma 3.7 to create a free line at 0. Next, we use Lemma 3.6 again to make \( w \) a cut-point. Finally, by the same argument as in (3.37), we can find a desired \( \Gamma \) as in \( \mathcal{E}_{s_m,w,m,i_m,j_m}^{b} \). Hence, there exist \( s_m \in [s'm - 3m^\alpha, (Kn)^d], w_m \in \Lambda_{im^\alpha}(xm), i_m \neq j_m \in [d] \) such that

\[
(3.39) \quad \mathbb{P}(\mathcal{E}_{s_m,w,m,i_m,j_m}^{b}) \geq \frac{1}{(Kn)^d} e^{-m^\alpha} \mathbb{P}(A_{s',x'}(m)).
\]

Denote by \( \mathcal{E}_{t,w,i,j}^{e} \) (respectively \( \mathcal{E}_{t,w,i,j}^{b} \)) the set of admissible connected graph \( C = B_t \) for the event \( \mathcal{E}_{t,w,i,j}^{e} \) (respectively \( \mathcal{E}_{t,w,i,j}^{b} \)). We have

\[
\mathbb{P}(\mathcal{E}_{s_n,w,m,i_n,j_n}^{e})\mathbb{P}(\mathcal{E}_{s_m,w,m,i_m,j_m}^{b}) = \sum_{C_1 \in \mathcal{E}_{s_n,w,m,i_n,j_n}} \sum_{C_2 \in \mathcal{E}_{s_m,w,m,i_m,j_m}} \mathbb{P}(B_{s_n} = C_1) \mathbb{P}(B_{s_m} = C_2).
\]

Let \( C_1 \in \mathcal{E}_{s_n,w,m,i_n,j_n}^{e} \) and \( C_2 \in \mathcal{E}_{s_m,w,m,i_m,j_m}^{b} \). Let us first prove that we can do a small translation of \( w_n + C_2 \) such that its intersection with \( C_1 \) is empty. We take \( \Gamma_1, \Gamma_2 \subset \mathbb{Z}^d \) such that \( \text{max}(|\Gamma_1|, |\Gamma_2|) \leq Kn \) and \( C_1 \subset \text{Int}(\Gamma_1), C_2 \subset \text{Int}(\Gamma_2) \). By isoperimetry of \( \mathbb{Z}^d \) (see (1.17)),

\[
(3.40) \quad \text{max}(|\text{Int}(\Gamma_1)|, |\text{Int}(\Gamma_2)|) \leq \kappa_d(Kn)^{d-1} \leq \kappa_d K^2 n^{d-1}.
\]
Since $\sum_{z \in \mathbb{Z}^d} 1_{x \in (x+A)} = |A|$ for any $x \in \mathbb{Z}^d$ and $A \subset \mathbb{Z}^d$, we have

$$\left\lfloor \{ z \in \Lambda_{9n^{\alpha_d}}(w_n) \cap \mathbb{Z}^d : (\text{Int}(\Gamma_1) \cup L_{\text{in}}^{K_n}(w_n)) \cap (z + (G_2 \cup L_{\text{im}}^{K_n}(0))) \neq \emptyset \} \right\rfloor$$

$$\leq \sum_{z \in \Lambda_{9n^{\alpha_d}}(w_n) \cap \mathbb{Z}^d} |(\text{Int}(\Gamma_1) \cup L_{\text{in}}^{K_n}(w_n)) \cap (z + (G_2 \cup L_{\text{im}}^{K_n}(0)))|$$

$$= \sum_{x \in \text{Int}(\Gamma_1) \cup L_{\text{in}}^{K_n}(w_n)} \sum_{z \in \Lambda_{9n^{\alpha_d}}(w_n) \cap \mathbb{Z}^d} 1_{x \in (z+G_2 \cup L_{\text{im}}^{K_n}(0))}$$

$$\leq (|G_2| + 4K_n) (|\text{Int}(\Gamma_1)| + 4K_n) \leq c_d n^{1+\frac{d}{d-1}}.$$  

for some constant $c_d > 0$ depending only on $d, K$. Besides, since $\|nx - w_n\|_\infty \leq 4n^{\alpha_d}$ and $\|mx' - w_m\|_\infty \leq 4m^{\alpha_d} \leq 4n^{\alpha_d}$, $z \notin \Lambda_{9n^{\alpha_d}}(w_n)$ (or equivalently $\|z - w_n\|_\infty > 9n^{\alpha_d}$) implies

$$\|z + w_m - nx - mx'\|_\infty \geq \|z - w_n\|_\infty - \|nx - w_n\|_\infty - \|mx' - w_m\|_\infty \geq n^{\alpha_d}.$$  

Therefore, we reach

$$\left\lfloor \{ z \in \Lambda_{9n^{\alpha_d}}(w_n) \cap \mathbb{Z}^d : \|z + w_m - nx - mx'\|_\infty \leq n^{\alpha_d} \} \right\rfloor$$

$$= \left\lfloor \{ z \in \mathbb{Z}^d : \|z + w_m - nx - mx'\|_\infty \leq n^{\alpha_d} \} \right\rfloor = |\Lambda_{n^{\alpha_d}}(nx + mx' - w_m) \cap \mathbb{Z}^d| \geq 2n^{d\alpha_d}.$$  

As a consequence, we have

$$\left\lfloor \{ z \in \Lambda_{9n^{\alpha_d}}(w_n) \cap \mathbb{Z}^d : (\text{Int}(\Gamma_1) \cup L_{\text{in}}^{K_n}(w_n)) \cap (z + (G_2 \cup L_{\text{im}}^{K_n}(0))) = \emptyset, \|z + w_m - nx - mx'\|_\infty \leq n^{\alpha_d} \} \right\rfloor \geq 2n^{d\alpha_d} - c_d n^{1+\frac{d}{d-1}} \geq n^{\alpha_d}.$$  

Thus, by pigeon-hole argument and max($|\Gamma_1|, |\Gamma_2|$) $\leq Kn$, there exists $z \in \Lambda_{9n^{\alpha_d}}(w_n)$;

- $(\text{Int}(\Gamma_1) \cup L_{\text{in}}^{K_n}(w_n)) \cap (z + (G_2 \cup L_{\text{im}}^{K_n}(0))) = \emptyset$;
- $\|(z + w_m) - (nx + mx')\|_\infty \leq n^{\alpha_d}$;
- $|\Gamma_1 \cap H_{j_m}(z)| \leq K n^{1-\alpha_d} \leq n^{\alpha_d}$;
- $|(z + G_2) \cap H_{j_n}(w_n)| \leq K n^{1-\alpha_d} \leq n^{\alpha_d}$;
- if $j_m = j_n$ then $|z - w_n| \cdot e_{j_n} > n^{\alpha_d}/4$.

Let us fix such a $z$. By the first condition together with $|\Gamma_2| \leq Kn$, $\text{Int}(\Gamma_1) \cap (z + \text{Int}(\Gamma_2)) = \emptyset$. We will establish a path $w_n$ with $z$ avoiding $C_1$ and $z + C_2$. Let $k_m \in [d] \setminus \{i_m, j_m\}$. Define

$$E_n := \{ \ell \in \{5n^{\alpha_d}, \ldots, 10n^{\alpha_d}\} : L_{k_m}(w_n + \ell e_{i_m}) \cap (C_1 \cup (z + G_2)) = \emptyset \}.$$  

Since the lines are disjoint, $|H_{j_m}(w_n) \cap (z + G_2)| \leq n^{\alpha_d}$, and $N_{k_m}(H_{j_m}(w_n) \cap C_1) \leq n^{\alpha_d}$, we have $|E_n| \geq n^{\alpha_d}$. It follows that all these lines do not intersect with either $C_1$ or $z + C_2$. Similarly, let $k_m \in [d] \setminus \{i_m, j_m\}$ and

$$E_m := \{ \ell \in \{5n^{\alpha_d}, \ldots, 10n^{\alpha_d}\} : L_{k_m}(z + \ell e_{i_m}) \cap (C_1 \cup (z + G_2)) = \emptyset \}.$$  

It holds that $|E_m| \geq n^{\alpha_d}$. Denote

$$E_n := \{ x \in L_{k_m}^{\alpha_d}(w_n + \ell e_{i_m}) : \ell \in E_n \}, \quad E_m := \{ x \in L_{\text{in}}^{\alpha_d}(z + \ell e_{i_m}) : \ell \in E_m \}.$$  

It is easy to check that the sets $E_n, E_m$ are contained in $\Lambda_{20n^{\alpha_d}}(w_n)$. In the case $j_n = j_m$, recall that we chose $z$ so that the hyperplanes $H_{j_n}(w_n)$ and $H_{j_m}(z)$ are separated at distance at least $n^{\alpha_d}/4$.

Thanks to Lemma 3.14 with $K = 20n^{\alpha_d} = 80\ell$, there exist $Z^d$-paths $(p_i)_{1 \leq i \leq 2n^{\alpha_d}}$ from $E_n$ to $E_m$ such that each path has a length less than $100dn^{\alpha_d}$ and

$$\forall x \in Z^d \quad |\{ i \in [n^{2\alpha_d}] : x \in p_i \}| \leq 80^{d-1} \chi_d =: \chi_d'.$$  

Let us assume all these paths intersect $C_1 \cup (z + C_2)$. Since each vertex cannot be contained in more than $\chi_d'$ paths, it follows that $|C_1 \cup (z + C_2)| \geq n^{2\alpha_d}/\chi_d'$, which contradicts 3.40. It follows that, there
exists at least one path \( p_t \) that does not intersect \( C_1 \cup (z + C_2) \) between some \( x_1 \in L_n^{\alpha_d} (w_n + \ell_1 e_i) \) and \( x_2 \in L_n^{\alpha_d} (z + \ell_2 e_i) \) with min \( \{ |\ell_1|, |\ell_2| \} \geq 5n^{\alpha_d} \). Consider the path:

\[
p := L(w_n, w_n + \ell_1 e_i) \oplus L(w_n + \ell_1 e_i, x_1) \oplus p_t \oplus L(x_2, z + \ell_2 e_i) \oplus L(z + \ell_2 e_i, z).
\]

It is easy to check that \( p \setminus \{ w_n, z \} \subset \mathbb{Z}^d \setminus (C_1 \cup (z + C_2)) \) and \( 10n^{\alpha_d} \leq |p| \leq 20dn^{\alpha_d} \). Moreover, by construction, \( B_{s_n + s_m + |p|} = C_1 \cup p \cup (z + C_2) \) implies \( B_{s_n + s_m + |p| - 1} = \{ z + w_m \} \).

Let us now build a resampled configuration. We denote by \( E(C_1) \) the set of edges whose values are involved with the event \( \{ B_{s_n} = C_1 \} \). We define \( E(C_2) \) similarly. Given a set \( E \subset E^d \) and \( z \in \mathbb{Z}^d \), we define \( E + z := \{ (x + z, y + z) : (x, y) \in E \} \). Let \( \tau^* \) be a configuration independent of \( \tau \). Consider the following resampled configuration:

\[
\forall e \in E^d \quad \tau^*_e := \begin{cases} 
\tau_e & \text{if } e \in E(C_1) \cup (E(C_2) + z), \\
\tau^*_e & \text{otherwise}.
\end{cases}
\]

We denote by \( B^e \) the ball for the configuration \( \tau^* \). It follows that there exists \( c' > 0 \) depending on \( p \) and \( d \) such that

\[
e^{-c' n^{\alpha_d}} \mathbb{P}(B_{s_n} = C_1) \mathbb{P}(B_{s_m} = C_2)
\]

\[
\leq \mathbb{P}(B_{s_n} = C_1, B_{s_m}(z) = C_2, \forall e \in p \quad \tau^*_e = 1, \forall |e \cap p| = 1 \quad \tau_e = \infty)
\]

\[
\leq \mathbb{P}(B_{s_n + s_m + |p|} = C_1 \cup p \cup (z + C_2))
\]

\[
= \mathbb{P}(B_{s_n + s_m + |p|} = C_1 \cup p \cup (z + C_2)) =: \mathbb{P}(p_{s_n + s_m + |p|} = \pi(C_1, C_2)),
\]

where we denote by \( \pi \) the map associating \( (C_1, C_2) \in \mathcal{E}^e_{s_n, w_n, i_n, j_n} \times \mathcal{E}^b_{s_m, w_m, i_m, j_m} \) to the graph \( C_1 \cup p \cup (z + C_2) \). One can check that \( \pi \) is injective. Indeed, if \( \pi(C_1, C_2) \) is given, then \( C_1 \) is recovered by considering all the points at distance at most \( s_n \) from 0. To recover \( C_2 \), we find \( z + w_m \) as the furthest point from 0 in \( \pi(C_1, C_2) \), this enables us to recover \( z \) and then \( C_2 \), by considering all the points that are at distance at most \( s_m \) from \( z \) not connected to \( C_1 \) by a path in \( \pi(C_1, C_2) \setminus \{ z \} \). Note that thanks to our choice of \( z \), we have

\[
\| (z + w_m) - (nx + mx') \|_\infty \leq n^{\alpha_d} \leq (n + m)^{\alpha_d}.
\]

Finally, since

\[
s_n + s_m + |p| \geq s_n + s'm - 6n^{\alpha_d} + 10n^{\alpha_d} \geq s_n + s'm,
\]

we have

\[
\mathbb{P}(\mathcal{E}^e_{s_n, w_n, i_n, j_n}) \mathbb{P}(\mathcal{E}^b_{s_m, w_m, i_m, j_m})
\]

\[
= \sum_{C_1 \in \mathcal{E}^e_{s_n, w_n, i_n, j_n}} \sum_{C_2 \in \mathcal{E}^b_{s_m, w_m, i_m, j_m}} \mathbb{P}(B_{s_n} = C_1) \mathbb{P}(B_{s_m} = C_2)
\]

\[
\leq \sum_{C_1 \in \mathcal{E}^e_{s_n, w_n, i_n, j_n}} \sum_{C_2 \in \mathcal{E}^b_{s_m, w_m, i_m, j_m}} e^{-c' n^{\alpha_d}} \mathbb{P}(B_{s_n + s_m + |p|} = \pi(C_1, C_2))
\]

\[
\leq e^{-c' n^{\alpha_d}} \mathbb{P}(A_{\frac{n}{n+m} x + \frac{m}{n+m} x'}(n + m))
\]

Moreover, when \( n = m \),

\[
\mathbb{P}(\mathcal{E}^e_{s_n, w_n, i_n, j_n}) \mathbb{P}(\mathcal{E}^b_{s_m, w_m, i_m, j_m}) \leq e^{-c' n^{\alpha_d}} \mathbb{P}(A_{s' x x'}(n)).
\]

The result follows by combining the last three inequalities with (3.38) and (3.39). \( \square \)
4. On properties of $J_x$

The following lemma shows that we can constrain a range of the infimum in (1.5) to a compact set.

**Lemma 4.1.** Let $x \in \mathbb{R}^d \setminus \{0\}$. We define $R = R_{x,\xi} := J_x(\xi)/I(1,0) \geq 0$. It holds:

$$J_x(\xi) = \inf_{y \in [-R,R]^d, s \in [0,R]} I(s,y). \quad (4.1)$$

**Proof.** Recall that $I(s,y) = I(s \vee \|y\|_1, y)$. If $s \wedge \|y\|_\infty \geq R$, by Theorem 1.1 then

$$I(s,y) \geq I(R,y) = \frac{1}{2}(I(R,y) + I(R,-y)) \geq \frac{1}{2}I(2R,0) = RI(1,0) = J_x(\xi).$$

Therefore, the further restriction in the infimum does not change the value. \qed

**Proof of Proposition 1.2.** By definition, it is trivial to see that $J_x$ is non-decreasing. Moreover, by Lemma 4.1, since $I$ and $\mu$ are continuous (see a remark below (1.2) and Theorem 1.1), there is a minimizer in (4.1), say $(s_*,y_*)$, such that $(s_*,y_*) \neq (0,0)$ because of the condition $s + \mu(y - x) \geq (1 + \xi)\mu(x)$. By $I(s_*,y_*) = I(s_* \vee \|y_*\|_1, y_*) > 0$, we have $J_x(\xi) > 0$ for any $\xi > 0$.

Next, we will prove the continuity. Let $\xi, \varepsilon > 0$. We take $y \in \mathbb{R}^d, s \geq 0$ to be such that

$$s + \mu(y - x) \geq (1 + \xi)\mu(x), \quad J_x(\xi) \geq I(s,y) - \varepsilon.$$

By continuity of $I$ (Theorem 1.1), there exists $\delta > 0$ small enough such that

$$I(s + \delta\mu(x), y) \leq I(s,y) + \varepsilon.$$

We have

$$s + \delta\mu(x) + \mu(y - x) \geq (1 + \xi + \delta)\mu(x).$$

Hence, it yields

$$J_x(\xi) \leq J_x(\xi + \delta) \leq I(s + \delta\mu(x), y) \leq I(s,y) + \varepsilon \leq J_x(\xi) + 2\varepsilon.$$

Therefore, $J_x(\xi) = \lim_{a \to \xi} J_x(a)$.

Finally, we consider the left-limit. Given $n \in \mathbb{N}$, by Lemma 4.1, since the function $\xi \to R_{x,\xi}$ is non-decreasing, for any $\varepsilon > 0$, there exists $(s_n,y_n) \in [0,R] \times [R,R]^d$ such that

$$s_n + \mu(y_n - x) \geq (1 + \xi - (1/n))\mu(x), \quad J_x(\xi - (1/n)) \geq I(s_n,y_n) - \varepsilon.$$

By the Bolzano–Weierstrass theorem, we can find a subsequence $(n_k)_{k \in \mathbb{N}}$ and $(s_*,y_*) \in [0,\infty) \times \mathbb{R}^d$ such that $s_{n_k} \to s_*$ and $y_{n_k} \to y_*$. By continuity of the time constant $\mu$ and $I$, since $J$ is non-decreasing, $(s_*,y_*)$ satisfies

$$s_* + \mu(y_* - x) \geq (1 + \xi)\mu(x), \quad \lim_{k \to \infty} J_x(\xi - (1/n_k)) \geq I(s_*,y_*) - \varepsilon.$$

Therefore, since $J_x$ is non-decreasing, we have

$$\lim_{a \to \xi^-} J_x(a) = \lim_{k \to \infty} J_x(\xi - (1/n_k)) \geq I(s_*,y_*) - \varepsilon \geq J_x(\xi) - \varepsilon \geq \lim_{a \to \xi^-} J_x(a) - \varepsilon.$$

We conclude the proof by letting $\varepsilon$ go to 0. \qed

We introduce $\xi_0$ appearing in Theorem 1.3

**Lemma 4.2.** Let $x \in \mathbb{R}^d \setminus \{0\}$. The following quantity is positive:

$$\xi_0(x) := \sup \left\{ \xi > 0 : J_x(\xi) < \inf_{y \in \mathbb{R}^d, s \geq 1/2} I(s,y) \right\}. \quad (4.2)$$
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Proof. By (3.3),
\[ I(2s, 0) \leq I(s, x) + I(s, -x) = 2I(s, x). \]
Thus,
\[ \inf_{y \in \mathbb{R}^d, s \geq 1/2} 2I(s, y) \geq I(1, 0) > 0. \]
Since \( J_x \) is continuous and \( J_x(0) = I(0, 0) = 0, \) \( \xi_0(x) > 0. \)

5. Rate function for upper tail large deviations (Theorem 1.3)

The aim of this section is to prove that the rate function for upper tail large deviations coincides with our function \( J_x \). Hereafter, we focus only on the direction \( e_1 \) and we write \( J(x) := J_{e_1}(x) \) and \( \xi_0 := \xi_0(e_1) \). However, all of our results can be extended to any direction without difficulty. In particular, the symmetries in regards to the direction \( e_1 \) are never used in our proofs. We split the proof of the theorem into two subsections; namely the lower bound and the upper bound.

5.1. Lower bound. We will need the following lemma for the lower bound. Let \( \beta \in (\alpha_d, 1) \).

Lemma 5.1. There exists \( n_0 \in \mathbb{N} \) such that for any \( s \in [0, 1/2), \) \( \|x\|_1 \leq 1/2 \) with \( I(s, x) \leq \inf_{y \in \mathbb{R}^d} I(1/2, y) \), and \( n \geq n_0 \),
\[ \mathbb{P}(\exists s_n \geq sn \ w \in \Lambda_n \alpha^d \ni B_{s_n} \backslash B_{s_n-1} = \{ w \}, ne_1 \notin B_{s_n}, w \leftrightarrow ne_1 \) denotes \( \mathbb{P}(A_{s,x}(n)) \).

Let \( n_0 \) be as in the statement of Lemma 3.7 and \( K \) be as in the statement of Proposition 3.4.

Thanks to Lemma 3.7 and Proposition 3.4 for \( n \geq n_0 \)
\[ \frac{1}{2} e^{-n^d} \mathbb{P}(A_{s,x}(n)) \leq e^{-n^d} \mathbb{P}(A_{s,x}^K(n)) \leq \mathbb{P}(A_{s,x}^{\text{free}}(n)). \]

Let \( \beta \in (\alpha_d, 1) \). We define the event \( F \) as
\[ F := \{ \exists w \in \Lambda_n \alpha^d \ni |C_\infty \cap \Lambda_n \alpha^d | / |\Lambda_n \alpha^d | \leq \frac{3}{4} \theta(p) \} \cup \{ C_\infty \cap \Lambda_n \alpha^d \ni ne_1 = \emptyset \}. \]

Thanks to Theorem 2.2 and Theorem 2.1 we have for \( n \) large enough depending on \( p, d \),
\[ \mathbb{P}(F) \leq 2e^{-cn^2d}. \]

Besides, note that
\[ I(\frac{2}{3}, x) = \frac{4}{3} I(\frac{1}{2}, \frac{3}{4} x) \geq \frac{4}{3} \inf_{y \in \mathbb{R}^d} I(\frac{1}{2}, y) > I(s, x). \]

Denote
\[ A_{2/3}(n) := \{ \exists s_n \geq \frac{2}{3} n, \exists w_n \in \Lambda_n \ni B_{s_n} \backslash B_{s_n-1} = \{ w_n \} \}. \]

In particular, this yields for \( n \) large enough depending on \( p, d \),
\[ \mathbb{P}(A_{s,x}^{\text{free}}(n)) \leq \mathbb{P}(A_{s,x}^{\text{free}}(n) \backslash A_{2/3}(n)) + \mathbb{P}(A_{2/3}(n)) \leq 2\mathbb{P}(A_{s,x}^{\text{free}}(n) \backslash A_{2/3}(n)). \]

Therefore, we have for \( n \) large enough depending on \( p, d \),
\[ \mathbb{P}(A_{s,x}^{\text{free}}(n)) \leq \mathbb{P}(A_{s,x}^{\text{free}}(n) \backslash F^c \backslash A_{2/3}(n)) + \mathbb{P}(F) \leq 3\mathbb{P}(A_{s,x}^{\text{free}}(n) \backslash F^c \backslash A_{2/3}(n)). \]

On the event \( A_{s,x}^{\text{free}}(n) \backslash F^c \backslash A_{2/3}(n) \), denote by \( w_n \) the cut-point and \( s_n \) the associated time, where we have \( s_n \leq 2n/3 \) and \( \Lambda_n \alpha^d \ni ne_1 = \emptyset \). Denote by \( P \) the set of points in \( \Lambda_n \alpha^d \ni ne_1 = \emptyset \) connected to \( w_n \) in \( \mathbb{Z}^d \backslash B_{s_n-1} \) with less than \( dn^2 \) edges. On the event \( A_{s,x}^{\text{free}}(n) \backslash F^c \backslash A_{2/3}(n) \), we claim that
\[ \frac{|P|}{|\Lambda_n \alpha^d|} \geq 1 - \frac{2}{3} \theta(p). \]

Since \( |\Lambda_n \alpha^d| > 3\theta(p)/4 \), this implies \( P \cap C_\infty \neq \emptyset \).
We move on to (5.4). Let $i, j$ the indices in the event $A_{s,n}^{\text{free}}(n)$. Without loss of generality, we assume that $i = 2$ and $j = 1$. In particular, we have $L_2(w_n) \cap B_{s_n} = \{w\}$ and $\forall k \neq 1 \ N_k(H_1(w_n) \cap B_{s_n}) \leq n^{\alpha_d}$. We define $E_1, E_2, \ldots, E_n$ inductively as follows: Set

$$E_1 := \{w_n\}.$$ 

Assume $E_1, E_2, \ldots, E_{k-1}$ have already been defined. We define

$$E_k := \bigcup_{\gamma \in E_{k-1}} L_k^\gamma(y).$$

Finally, define

$$T := \bigcup_{\gamma \in E_d} L_1^\gamma(x).$$

By construction, all the points in $E_{k+1}$ are connected to a point in $E_k$ by a path in $Z^d \setminus B_{s_n-1}$ of length at most $n^{\beta}$. It yields that all the points in $T$ are connected to $w_n$ by a path in $Z^d \setminus B_{s_n-1}$ of length at most $dn^{\alpha_d}$. Since $N_{k+1}(H_1(w_n) \cap B_{s_n}) \leq n^{\alpha_d}$, we have $|E_{k+1}| \geq 2n^{\beta}(|E_k| - n^{\alpha_d})$. Moreover, since $|T| \leq Kn$, we have $|T| \geq 2n^{\beta}(|E_k| - Kn)$. Finally it is easy to check that for $n$ large enough the density of $T$ is larger than $1 - 2\theta(p)/3$ and that $T \subset \mathcal{P}$.

As we mentioned above (5.4), on the event $A_{s,n}^{\text{free}}(n) \cap \mathcal{F}^c$, the set $\mathcal{P}$ must intersect $\mathcal{C}_\infty$. On the event $A_{s,n}^{\text{free}}(n) \cap \mathcal{F}^c \setminus A_{2,1/3}(n)$, there exists a path $p$ in $Z^d \setminus B_{s_n-1}$ starting at $w_n$ and ending in $\mathcal{P} \cap \mathcal{C}_\infty$ of length at most $dn^{\alpha_d}$. Moreover, on the event $A_{s,n}^{\text{free}}(n) \cap \mathcal{F}^c \setminus A_{2,1/3}(n)$, since $s_n \leq 2n/3$, there exists a path $p'$ between $ne_1$ and $\mathcal{C}_\infty \setminus B_{s_n}$ of length at most $dn^{\alpha_d}$. Applying Lemma 3.5 with $E_1 = p \cup p'$ and $E_0 = \emptyset$, we have for $n$ large enough depending on $d, p,$

$$10e^{-(log n)^2n^\beta}P(A_{s,n}^{\text{free}}(n) \cap \mathcal{F}^c \setminus A_{2,1/3}(n)) \leq P(3t \geq n \exists w_n \in \Lambda_{n^{\alpha_d}}(nx); B_t \setminus B_{t-1} = \{w_n\}, w_n \leftrightarrow ne_1, ne_1 \notin B_{s_n}).$$

Combining this with (5.1) and (5.3), The result follows. □

**Proposition 5.2.** Let $\xi \in (0, \xi_0)$. it holds that

$$\liminf_{n \to \infty} -\frac{1}{n} \log P(\mu(\mathbf{e}_1)(1 + \xi)n < D^{\theta}_p(0, ne_1) < \infty) \geq -J(\xi).$$

**Proof.** Let $s > 0$ and $x \in \mathbb{R}^d$ be such that

$$s + \mu(x - \mathbf{e}_1) > (1 + \xi)\mu(\mathbf{e}_1).$$

Set

$$\Lambda_{s,n}(x) := \{x \in \Lambda_{n^{\alpha_d}}(nx) : B_{s_n} \setminus B_{s_n-1} = \{x_n\}, ne_1 \notin B_{s_n}\}. $$

Thanks to the lower tail large deviation ([1.1]), there exists $c > 0$ such that for $n$ large enough,

$$\mathbb{P}(D^{\theta}_p(\Lambda_{n^{\alpha_d}}(nx), ne_1) \leq \mu(\mathbf{e}_1)(1 + \xi)n - sn) < e^{-cn}.$$ 

On the event $\Lambda_{s,n}(x)$, let $s_n$ be the smallest integer at least $sn$ such that $|B_{s_n} \setminus B_{s_n-1}| = 1$ and $B_{s_n} \setminus B_{s_n-1} \subset \Lambda_{n^{\alpha_d}}(nx)$. Let $\mathcal{C}_n$ be the set of admissible $B_{s_n}$ where $s_n$ is the smallest integer at least $sn$ such that $|B_{s_n} \setminus B_{s_n-1}| = 1$, $ne_1 \notin B_{s_n}$ and $B_{s_n} \setminus B_{s_n-1} \subset \Lambda_{n^{\alpha_d}}(nx)$. For $C \in \mathcal{C}_n$, denote by $E(C)$ the edges that determine $B_{s_n} = C$:

$$E(C) := \{(x, y) \in E^d : x \in C\} \setminus \{(w_n, y) \in E^d : y \notin C\}.$$

where $\{w_n\} := B_{s_n} \setminus B_{s_n-1}$. We have

$$\mathbb{P}(\Lambda_{s,n}(x), D^{\theta}_p(0, ne_1) \leq \mu(\mathbf{e}_1)(1 + \xi)n) \leq \mathbb{P}(\exists s_n \geq sn, \exists w_n \in \Lambda_{n^{\alpha_d}}(nx); B_{s_n} \setminus B_{s_n-1} = \{w_n\}, D^{\theta}_p(\Lambda(\Lambda_{n^{\alpha_d}}(nx), ne_1) \leq \mu(\mathbf{e}_1)(1 + \xi)n - sn.$$ 

$$\leq \mathbb{P}(\exists s_n \geq sn, \exists w_n \in \Lambda_{n^{\alpha_d}}(nx); B_{s_n} \setminus B_{s_n-1} = \{w_n\}, D^{\theta}_p(\Lambda_{n^{\alpha_d}}(nx), ne_1) \leq \mu(\mathbf{e}_1)(1 + \xi)n - sn).$$
This is further bounded from above by
\[
\sum_{C \in \mathcal{C}_n} \mathbb{P}(B_{s_n} = C) \mathbb{P}(\mathcal{D}^\mathcal{G}_n \setminus E(C)(\Lambda^{n,x}(nx), ne_1) \leq \mu(e_1)(1 + \xi) n - sn) \\
\leq \mathbb{P}(\mathcal{A}'_{s,x}(n)) \mathbb{P}(\mathcal{D}^\mathcal{G}_p(\Lambda^{n,x}(nx), ne_1) < \mu(e_1)(1 + \xi) n - sn) \leq \mathbb{P}(\mathcal{A}'_{s,x}(n)) e^{-cn} \leq \mathbb{P}(\mathcal{A}_{s,x}(n)) e^{-cn}.
\]
where in the last inequality we have used (5.6). Therefore, we have
\[
\mathbb{P}(\exists s_n \geq sn, w_n \in \Lambda^{n,x}(nx), B_{s_n} \setminus B_{s_n-1} = \{w_n\}, ne_1 \notin B_{s_n}, w_n \leftrightarrow ne_1) \\
\leq \mathbb{P}(\mathcal{A}'_{s,x}(n), D^\mathcal{G}_p(0, ne_1) < \infty) \\
\leq \mathbb{P}(\mathcal{A}'_{s,x}(n), \mu(e_1)(1 + \xi) n < D^\mathcal{G}_p(0, ne_1) < \infty) + \mathbb{P}(\mathcal{A}_{s,x}(n), D^\mathcal{G}_p(0, ne_1) \leq \mu(e_1)(1 + \xi) n) \\
\leq \mathbb{P}(\mu(e_1)(1 + \xi) n < D^\mathcal{G}_p(0, ne_1) < \infty) + \mathbb{P}(\mathcal{A}_{s,x}(n)) e^{-cn}.
\]
Besides, thanks to Lemma 5.1, we have
\[
e^{o(n)} \mathbb{P}((\mathcal{A}_{s,x}(n)) \leq \mathbb{P}(\mu(e_1)(1 + \xi) n < D^\mathcal{G}_p(0, ne_1) < \infty) + \mathbb{P}(\mathcal{A}_{s,x}(n)) e^{-cn}.
\]
Using Theorem 1.1 we get by taking the liminf in the previous inequality
\[
\liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\mu(e_1)(1 + \xi) n < D^\mathcal{G}_p(0, ne_1) < \infty) \geq -I(s, x).
\]
Taking the supremum over \((s, x)\) with (5.5) and the continuity of \(I\) (Theorem 1.1), the claim follows. \(\square\)

5.2. Upper bound. We first prove that on the upper tail large deviation event, there is a space-time cut-point with high probability.

**Proposition 5.3** (Creation of cut-points on the large deviation event). For any \(\xi > 0\), there exists \(s_0 > 0\) such that for any \(\varepsilon > 0\),
\[
\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\mu(e_1)(1 + \xi) n < D^\mathcal{G}_p(0, ne_1) < \infty) \\
\leq \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left(\bigcup_{s,t \in \mathbb{Z}[\varepsilon] \cap [0,s_0], x,y \in \mathbb{Z}^d /n \cap [-s_0,s_0]^d: s+\ell+(1+\varepsilon)\mu(x-y-e_1)+x \geq (1+\xi)\mu(e_1)} A_{s,x}(n) \cap (ne_1 + A_{t,y}(n))\right).
\]

We need the following lemmas. We postpone their proofs to appendix.

**Lemma 5.4.** For any finite \(S \subset \mathbb{Z}^d\), there exists \(i \in [d]\) such that
\[
|P_i(S)| \geq \frac{1}{2}|S|^\frac{1}{2}.
\]

For \(x, y \in \mathbb{R}^d\), we denote by \([x,y]\) the segment joining \(x\) and \(y\).

**Lemma 5.5.** Let \(d \geq 3, K \geq \ell \geq 1\) and \(m \geq 1\). Consider \(S_1 \subset H_1(0) \cap \mathbb{Z}^d\) and \(S_2 \subset H_1(e_1) \cap \mathbb{Z}^d\) such that \(|S_1| = |S_2| = m\) and
\[
(5.7) \max_{x \in S_1, y \in S_2} \|x - y\|_\infty \leq K.
\]
Then, there exists a bijection \(\sigma\) from \(S_1\) to \(S_2\) such that
\[
\forall x \neq x' \in S_1 \quad d_2([x, \sigma(x)], [x', \sigma(x')]) \geq \frac{1}{\sqrt{2} K},
\]
where \(d_2\) is the Euclidean distance. In particular, the segments do not intersect.
Proof of Proposition 3.3. Let $\xi, \varepsilon > 0$. Thanks to (1.9), there exists $K > 0$ depending on $\xi$ such that for $n$ large enough,

$$
\mathbb{P}(\mu(e_1)(1 + \xi) n < D_{\mathbb{R}^d}(0, ne_1) < \infty) \leq 2\mathbb{P}(\mu(e_1)(1 + \xi) n < D_{\mathbb{R}^d}(0, ne_1) < K n).
$$

From now on, we will work on the event $\{\mu(e_1)(1 + \xi) n < D_{\mathbb{R}^d}(0, ne_1) < K n\}$. Set

$$
(5.8) \quad s^*_n := \inf\{i \geq 1 : |B_i| \geq n^{7/4}\} \quad \text{and} \quad t^*_n := \inf\{i \geq 1 : |B_i(ne_1)| \geq n^{7/4}\},
$$

with the convention $\inf\emptyset = +\infty$. Let $s_0 := 3K + 1$. We consider two cases.

Case 1. Suppose that $s^*_n + t^*_n \geq D_{\mathbb{R}^d}(0, ne_1)$. Then, $B_{s^*_n} \cap B_{t^*_n}(ne_1) \neq \emptyset$ and there exists $x_n \in \mathbb{Z}^d$ such that

$$
D_{\mathbb{R}^d}(0, x_n) \leq s^*_n, \quad D_{\mathbb{R}^d}(x_n, ne_1) \leq t^*_n, \quad \text{and} \quad D_{\mathbb{R}^d}(0, ne_1) = D_{\mathbb{R}^d}(0, x_n) + D_{\mathbb{R}^d}(x_n, ne_1).
$$

Denote by $s_n = D_{\mathbb{R}^d}(0, x_n)$ and $t_n = D_{\mathbb{R}^d}(x_n, ne_1)$. We claim that if

$$
(5.9) \quad B_{s_n} \setminus B_{s_n-1} = \{x_n\} \quad \text{and} \quad B_{t_n}(x_n) \setminus (B_{t_n-1}(x_n) \cup B_{s_n}) = \{ne_1\},
$$

then $A_{\mu(e_1)(1 + \xi), e_1}(n)$ occurs. Indeed let $y \in B_{s_n+t_n} \setminus B_{s_n+t_n-1}$ and $\gamma$ be a geodesic from $0$ to $y$. By hypothesis, $\gamma$ passes through $x_n$ at time $s_n$ and then passes through $ne_1$ at time $s_n + t_n = D_{\mathbb{R}^d}(0, ne_1) > \mu(e_1)(1 + \xi) n$, which implies $y = ne_1$ and the claim follows.

Let us resample the configuration to create a cut-point at $ne_1$ via Lemma 3.5. We take a geodesic $\gamma$ from $0$ to $ne_1$ with a deterministic rule breaking ties. Since $|B_{s_n-1}| \leq n^{7/4}$ and $|B_n(x_n) \cap B_n(ne_1)| \leq n^{7/4}$, if $s_n \geq 2n^{7/8}$ and $t_n \geq 2n^{7/8}$, then there exist $r_0 \in [s_n - 1] \setminus [s_n - 2n^{7/8}]$ and $r_1 \in [t_n] \setminus [t_n - 2n^{7/8}]$ such that

$$
|B_{r_0} \setminus B_{r_0-1}| \leq n^{7/8} \quad \text{and} \quad |(B_{r_1}(x_n) \setminus B_{r_1-1}(x_n)) \cap B_{t_n-1}(ne_1)| \leq n^{7/8}.
$$

If $s_n < 2n^{7/8}$, then we can simply set $r_0 := 0$; if $t_n < 2n^{7/8}$, we set $r_1 := 0$. Set

$$
E_0 := \left\{\{v, w\} \in \mathbb{E}^d \setminus \gamma : v \in \gamma \cap \left(B_{s_n} \setminus B_{r_0} \cup (B_{t_n}(x_n) \setminus B_{t_n-1}(ne_1)) \right) \right\}
$$

$$
\cup \left\{e \in \mathbb{E}^d \setminus \gamma : e \text{ connects } B_{r_0} \setminus B_{r_0-1} \right\}
$$

$$
\cup \left\{e \in \mathbb{E}^d \setminus \gamma : e \text{ connects } (B_{r_1}(x_n) \setminus B_{r_1-1}(x_n)) \cap B_{t_n-1}(ne_1) \right\},
$$

Consider a resampled configuration $(\tau^e_{\mathbb{E}^d}, e \in \mathbb{E}^d)$ as in Lemma 3.5 with $E_0$ defined above and $E_1 = \emptyset$. We now prove that if all the edges of $\tau^e$ in $E_0$ are closed, then $|B_{r_1}(x_n) \setminus B_{r_1-1}(x_n)| \leq n^{7/8}$, which is the desired result. The first part of (5.9) follows easily. For the second part, by the choice of $E_0$, $B_{r_1}(x_n) \subset B_{s_n} \cup B_{t_n-1}(x_n) \cup \gamma$. Therefore, the only vertex in $B_{r_1}(x_n) \setminus B_{s_n}$ at distance $t_n$ from $x_n$ is $ne_1$, and thus (5.9) follows. Since $|E_0| \leq (2d)^{2d} n^{7/8}$ and $E_0 \subset [-n^2, n^2]^d$ for $n$ large enough, by Lemma 3.5

$$
e^{-8d^2(\log n)^{n/8}} \mathbb{P}(\mu(e_1)(1 + \xi) n < D_{\mathbb{R}^d}(0, ne_1) < \infty, s^*_n + t^*_n \geq D_{\mathbb{R}^d}(0, ne_1))
$$

$$
(5.10) \quad \leq \mathbb{P} \left( A_{\mu(e_1)(1 + \xi), e_1}(n) \right) \leq \mathbb{P} \left( \bigcup_{s, t \in \mathbb{Z}^d \cap [0, n_\delta], x_n, y_n \in \mathbb{Z}^d \cap [-n_\delta, n_\delta]^d, \varepsilon > 0} \left( A_{s, x}(n) \cap (ne_1 + A_{t, y}(n)) \right) \right),
$$

where $A_{0,0} = \{B_0 = \emptyset\}$ always occurs.

Case 2. Suppose that $s^*_n + t^*_n < D_{\mathbb{R}^d}(0, ne_1)$. In particular, we have $s^*_n + t^*_n \leq Kn$ and $B_{s^*_n} \cap B_{t^*_n}(ne_1) = \emptyset$. Let $\delta := 1/(18d)$, $N = \log^2 n$ and $\varepsilon > 0$ be chosen later. We consider the macroscopic lattice of sidelength $N$ for the parameter $\varepsilon$. Set

$$
\mathcal{E}_n := \left\{ i \in \Lambda_\varepsilon \cap \mathbb{Z}^d : i \text{ is } \varepsilon \text{-bad} \right\}.
$$
By Lemma \[2.7\] we have \(P(\mathcal{E}_n) \leq e^{-n \log n}\). Hence, we suppose \(\mathcal{E}_n^c\) from now on. Let \(C_1\) be the set of boxes intersected by \(B_{\ast, 1}\) and \(C_2\) the set of boxes intersected by \(B_{\ast, 2}(ne_1)\). Note that \(|C_1| \geq n^{7/4}/(2^{d}N^d)\).

Thanks to Lemma \[5.3\] there exists \(i \in [d]\) such that for \(n\) large enough,

\[
|P_i(C_1)| \geq \frac{n^{7/6}}{2^{d+1} \log^2 2 n} \geq 2n^{10/9}.
\]

Denote by \(C'_i\) the set of macroscopic sites that are connected in the macroscopic lattice by a macroscopic, good path of length at most \(n^{1-\delta}\) to a site in \(C_1\). The number of disjoint macroscopic lines \(L_{i_1}(w)\) with \(w \in C_1\) is at least \(2n^{10/9}\). We say that a line \(L_{i_1}(w)\) is good if all sites in \(L_{i_1}(w)\) are good. On the event \(\mathcal{E}_n^c\), there are at least \(2n^{10/9} - n \geq n^{10/9}\) good lines of these disjoint macroscopic lines. It follows that \(|C'_1| \geq n^{19/9 - \delta}\). We can define similarly \(C'_2\). Since \(s_n < D_{\mathcal{H}_2}(0, ne_1) \leq Kn\), we have \(C'_1, C'_2 \in \bigcup_{k = -K}^K \mathcal{H}_1(ke_1)\).

Besides, since \((H_1(ke_1), -Kn \leq k \leq 2Kn)\) are disjoint, by pigeon-hole principle, there exists \(w_1 \in \{ke_1 : -Kn \leq k \leq 2Kn\}\) such that

\[
|H_1(w_1) \cap C'_1| \geq \frac{1}{4K} n^{10/9 - \delta}.
\]

Similarly, there exists \(w_2 \in \{ke_1 : -Kn \leq k \leq 2Kn\}\) such that

\[
|H_1(w_2) \cap C'_2| \geq \frac{1}{4K} n^{10/9 - \delta}.
\]

Let us take \(S_i \subset H_1(w_i) \cap C'_i\) for \(i = 1, 2\) such that \(|S_1| = |S_2| =: m \geq \frac{1}{4K} n^{10/9 - \delta}\). We write \(S_1 =: \{x^1, \ldots, x^m\}\) and \(S_2 =: \{y^1, \ldots, y^m\}\).

We take an integer \(|k| \leq n^{1-\delta}\) such that

\[
|(w_1 - w_2) \cdot e_1 - k| \geq n^{1-\delta}.
\]

Consider now the set \(S'_2 := S_2 + ke_1\). By using Lemma \[5.5\] we obtain a bijection \(\sigma : S_1 \to S'_2\) and straight lines \(L'_1, L'_2\) joining points \(x_i\) and \(\sigma(x_i)\), where \(d_2(L'_1, L'_2) \geq n^{\delta}/4K\). We denote by \(L_i\) the concatenation of \(L'_i\) with the straight line joining \(\sigma(x_i)\) and \(\sigma(x_i) - ke_1\) in \(S_2\). This implies that for each macroscopic site \(x \in \mathbb{Z}^d\), we have

\[
\#\{i : d_\infty(x, L_i) \leq 1\} \leq O(n^{\delta(d-1)}) \leq n^{\delta d}.
\]

Note that \(1 + \delta d < \frac{10}{9} - \delta\) with \(\delta = 1/(18d)\). On the event \(\mathcal{E}_n^c\), the number of \(L_i\) crossing at least one bad box is at most \(n^{1+\delta d}\), hence there exists at least one path \(L_i\) between some \(x_1 \in S_1\) and \(x_2 \in S_2\) such that all the macroscopic sites \(x\) with \(d_\infty(L_i, x) \leq 1\) are good. In particular, there exists a macroscopic path of good sites joining \(x_1 \in S_1\) and \(x_2 \in S_2\). Let \(x_n, y_n, z_n\) be macroscopic points in the largest open cluster of the boxes corresponding to \(x_1, x_2, x_2 + ke_1\), respectively. By definition of \(C'_1\) and \(C'_2\), there exist good, macroscopic paths of length at most \(n^{1-\delta}\) to boxes in \(C_1\) and \(C_2\), respectively. We conclude using Lemma \[2.9\] that

\[
D_{\mathcal{H}_2}(0, x_n) < s^*_n + O(N^d n^{1-\delta}), \quad D_{\mathcal{H}_2}(y_n, ne_1) < t^*_n + O(N^d n^{1-\delta}).
\]

Moreover, since the path \(L'_i\) joining \(x_1\) and \(x_2 + ke_1\) is a straight line, and all the macroscopic sites \(x\) with \(d_\infty(L_i, x) \leq 1\) are good, using a similar argument as in the proof of Proposition \[2.10\] we have

\[
D_{\mathcal{H}_2}(x_n, y_n) \leq D_{\mathcal{H}_2}(x_n, z_n) + D_{\mathcal{H}_2}(z_n, y_n) \\
\leq (1 + 2d\varepsilon)\mu(x_n - z_n) + O(N^d n^{1-\delta}) \leq (1 + 2d\varepsilon)\mu(x_n - y_n) + O(N^d n^{1-\delta})
\]

Next, we create two cut-points at \(x_n\) and \(y_n\). Let \(\gamma_{v,v}\) be a geodesic from \(v\) to \(w\) with a deterministic rule breaking ties. Set \(s_n := D(0, x_n) \leq s^*_n + O(N^d n^{1-\delta})\) and \(t_n := D(ne_1, y_n) \leq t^*_n + O(N^d n^{1-\delta})\). We denote by \(\gamma_{v,v}^\prime\) the concatenation of \(\gamma_{v,v}\) and \(\gamma_{x_n,y_n}\), and \(\gamma_{y_n,ne_1}\). i.e. \(\gamma_{v,v}^\prime := \gamma_{v,v} \oplus \gamma_{x_n,y_n} \oplus \gamma_{y_n,ne_1}\). First, since we assumed \(D_{\mathcal{H}_2}(0, ne_1) \geq \mu(e_1)(1 + \xi)n\), we have

\[
s_n + t_n + (1 + 2d\varepsilon)\mu(x - y) + 2d\varepsilon n \geq D_{\mathcal{H}_2}(0, x_n) + D_{\mathcal{H}_2}(x_n, y_n) + D_{\mathcal{H}_2}(y_n, ne_1) \geq \mu(e_1)(1 + \xi)n.
\]
Since $|B_{\text{min}(s_n^* - 1, s_n)}| \leq n^{7/4}$ and $|B_{\text{min}(t_n^* - 1, t_n)}(ne_1)| \leq n^{7/4}$, there exist $r_0 \in [\min(s_n^* - 1, s_n)] \setminus [\min(s_n^* - 1, s_n) - n^{7/8}]$ and $r_1 \in [\min(t_n^* - 1, t_n)] \setminus [\min(t_n^* - 1, t_n) - n^{7/8}]$ such that
$$|B_{r_0} \setminus B_{r_0 - 1}| \leq n^{7/8}, \quad |B_{r_1}(ne_1) \setminus B_{r_1 - 1}(ne_1)| \leq n^{7/8},$$
where we can simply set $r_0 := 0$ if $\min(s_n^* - 1, s_n) < n^{7/8}$; $r_1 := 0$ if $\min(t_n^* - 1, t_n) < n^{7/8}$. Set
$$E_0 := \{ (v, w) \in \mathbb{E}^d \setminus \gamma' : v \in (\gamma_{0, x_n} \setminus B_{r_0}) \cup (\gamma_{y_n, ne_1} \setminus B_{r_1}(ne_1)) \}$$
and $E_1$ defined as in Proposition 5.6. Using the proposition, we prove the upper bound.

$$(5.11) \quad \leq \exp(O(n^{d-1})) \mathbb{P} \left( \bigcup_{s, t \in \{\mathbb{E}^d \setminus \gamma' : e \text{ connects } B_{r_0} \setminus B_{r_0 - 1} \text{ and } B_{r_1}(ne_1) \}} A_{s, x}(n) \cap (ne_1 + A_{e, y}(n)) \right).$$

Together with (5.10), the result follows by taking the limit sup and changing $2\varepsilon$ by $\varepsilon$. \hfill \Box

Using the proposition, we prove the upper bound.

**Proposition 5.6.** Let $\xi \in (0, \xi_0)$. We have
$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\mu(e_1)(1 + \xi)n < \mathcal{D}^p(0, ne_1)) < \infty, \quad s_n^* + t_n^* < \mathcal{D}^p(0, ne_1)$$

Proof. Let $\xi \in (0, \xi_0)$ and $\varepsilon > 0$. By continuity of $J$ (Proposition 1.2), $J(\xi_0) = \inf_{x \in \mathbb{R}^d, s \geq 1/2} I(s, x)$. Hence, by definition of $\xi_0$, $J(\xi_0) < J(\xi_0)$ for any $\xi < \xi_0$. Thanks to Proposition 5.2 we have

$$(5.12) \quad \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\mu(e_1)(1 + \xi)n < \mathcal{D}^p(0, ne_1)) < \infty \geq -J(\xi) > -J(\xi_0).$$

Thanks to Proposition 5.3, we have
$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\mu(e_1)(1 + \xi)n < \mathcal{D}^p(0, ne_1)) < \infty$$

$$(5.13) \quad \leq \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \bigcup_{s, t \in \{\mathbb{E}^d \setminus \gamma' : e \text{ connects } B_{r_0} \setminus B_{r_0 - 1} \text{ and } B_{r_1}(ne_1) \}} A_{s, x}(n) \cap (ne_1 + A_{e, y}(n)) \right).$$

Let $n_0$ be as in the statement of Lemma 3.1 (depending on $s_0, d, p$). Let $n \geq n_0$. By pigeon-hole principle, there exist $x^*, y^* \in (\mathbb{Z}^d/n) \cup [-s_0, s_0]^d$ and $s^*, t^* \in (\mathbb{Z}/n) \cap [0, s_0]$ (that may depend on $\varepsilon, n$) such that
$$s^* + t^* + (1 + \varepsilon) \mu(x^* - y^* - e_1) + \varepsilon \geq (1 + \xi) \mu(e_1).$$
and
\begin{align*}
\frac{1}{(4s_0n)^{2(d+1)}} \mathbb{P} \left( \bigcup_{s,t \in (Z/n) \cap [0,s_0], x,y \in (Z^d/n) \cap [-s_0,s_0]^d} \mathcal{A}_{s,x}(n) \cap (n \mathbf{e}_1 + \mathcal{A}_{t,y}(n)) \right) \\
\leq \mathbb{P}(\bar{\mathcal{A}}_{s^*,x^*}(n) \cap (n \mathbf{e}_1 + \bar{\mathcal{A}}_{t^*,y^*}(n)),
\end{align*}
(5.13)
where $\bar{\mathcal{A}}_{s^*,x^*}(n) := \mathcal{A}_{s^*,x^*}(n) \setminus \mathcal{A}_{s^*+(1/n),x^*}(n)$.

Let us first assume that these two events $\bar{\mathcal{A}}_{s^*,x^*}(n)$ and $(n \mathbf{e}_1 + \bar{\mathcal{A}}_{t^*,y^*}(n))$ do not occur disjointly, that is, there exists an edge used to achieve both two events. Note that on the event $\mathcal{A}_{s,x}(n)$, the cut-point is exactly located at time $sn$ for $s \in Z/n$. Since we assumed $\mathcal{B}_{s^*n}(0) \cap \mathcal{B}_{t^*n}(n \mathbf{e}_1) \neq \emptyset$, this implies that $s^* + t^* \geq 1$. Without loss of generality, let us assume $s^* \geq 1/2$. By the uniform convergence on a compact set in Theorem 1.1 together with (5.13), we have

\begin{align*}
\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \bigcup_{s,t \in (Z/n) \cap [0,s_0], x,y \in (Z^d/n) \cap [-s_0,s_0]^d} \mathcal{A}_{s,x}(n) \cap (n \mathbf{e}_1 + \mathcal{A}_{t,y}(n)) \right) \\
\leq - \inf_{x \in R^d, s \geq 1/2} J(s, x) = -J(\xi_0),
\end{align*}
(5.14)
which contradicts (5.12). Therefore, the occurrences $\bar{\mathcal{A}}_{s^*,x^*}(n)$ and $(n \mathbf{e}_1 + \bar{\mathcal{A}}_{t^*,y^*}(n))$ are disjoint.

Since it is a disjoint occurrence, by BK inequality, we have
\begin{align*}
\mathbb{P}(\bar{\mathcal{A}}_{s^*,x^*}(n) \cap (n \mathbf{e}_1 + \bar{\mathcal{A}}_{t^*,y^*}(n))) \leq \mathbb{P}(\mathcal{A}_{s^*,x^*}(n))\mathbb{P}(n \mathbf{e}_1 + \bar{\mathcal{A}}_{t^*,y^*}(n)) = \mathbb{P}(\mathcal{A}_{s^*,x^*}(n))\mathbb{P}(\mathcal{A}_{t^*,y^*}(n)).
\end{align*}

It follows that for $n \in N$ large enough, the right-hand side of (5.13) is bounded from above by
\begin{align*}
\max_{s,t \in [0,s_0], x,y \in [-s_0,s_0]^d} \mathbb{P}(\mathcal{A}_{s,x}(n))\mathbb{P}(\mathcal{A}_{t,y}(n)).
\end{align*}

By the uniform convergence on a compact set in Theorem 1.1
\begin{align*}
\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \bigcup_{s,t \in (Z/n) \cap [0,s_0], x,y \in (Z^d/n) \cap [-s_0,s_0]^d} \mathcal{A}_{s,x}(n) \cap (n \mathbf{e}_1 + \mathcal{A}_{t,y}(n)) \right) \\
\leq - \inf_{s,t \in [0,s_0], x,y \in [-s_0,s_0]^d} (I(s, x) + I(t, y)).
\end{align*}

Putting things together with letting $\varepsilon \to 0$, we have
\begin{align*}
\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \bigcup_{s,t \in Z/n, x,y \in Z^d/n} \mathcal{A}_{s,x}(n) \cap (n \mathbf{e}_1 + \mathcal{A}_{t,y}(n)) \right) \\
\leq - \inf_{s,t \in [0,s_0], x,y \in [-s_0,s_0]^d} (I(s, x) + I(t, y)).
\end{align*}
Moreover, by (3.3) and $I(t, y) = I(t, -y)$,
\[
\inf_{x,y \in \mathbb{R}^d, t \geq 0: s + t + \mu(x-y-e_1) \geq (1+\xi)\mu(e_1)} I(s, x) + I(t, y) \geq \inf_{x,y \in \mathbb{R}^d, s \geq 0: s + \mu(x-y-e_1) \geq (1+\xi)\mu(e_1)} I(s + t, x - y) = \inf_{x \in \mathbb{R}^d, s \geq 0: s + \mu(x-e_1) \geq (1+\xi)\mu(e_1)} I(s, x) = J(\xi).
\]
This yields the result. \qed

**Appendix A. Combinatorial Lemmas**

Let us denote by $\mathcal{S}_m$ the set of all the permutations of $[m]$. Recall that $[x, y]$ stands for the segment between $x$ and $y$.

**Proof of Lemma 5.4**. Let us prove the result in the case $\ell = K$. The result for general $\ell$ follows by dilating the space by a factor $\ell/K$ in the $e_1$ direction.

Let $S_1 = \{x^1, \ldots, x^m\}$ and $S_2 = \{y^1, \ldots, y^m\}$ be such that $\max_{x \in S_1, y \in S_2} \|x - y\|_\infty \leq K$. We will find a permutation $\sigma \in \mathcal{S}_m$ such that
\[
d_2([x^i, y^{\sigma(i)}], [x^j, y^{\sigma(j)}]) \geq \frac{1}{\sqrt{2}}, \quad \forall i \neq j \in [m].
\]

For $k, l \in [d]$, denote by $\pi_{k,l}$ the projection on the plane spanned by $e_k$ and $e_l$. Let $\sigma \in \mathcal{S}_m$ be such that $\sigma$ minimizes
\[
\sum_{i=1}^m \sum_{k=2}^d \|\pi_{1,k}(x^i - y^{\sigma(i)})\|_2.
\]

Denote by $L_i$ the segment joining $x^i$ and $y^{\sigma(i)}$. And denote by $v^i$ the unit vector associated to the direction of the segment $L_i$. We claim $L_i \cap L_j = \emptyset$ for all $i \neq j$. To this end, we first suppose that $L_i \cap L_j \neq \emptyset$ for some $i \neq j \in [m]$, and we shall derive a contradiction. We note that for all $k \neq 1$, we have $\pi_{1,k}(L_i) \cap \pi_{1,k}(L_j) \neq \emptyset$, so we can take $z \in \pi_{1,k}(L_i) \cap \pi_{1,k}(L_j)$. By the triangular inequality,
\[
\|\pi_{1,k}(x^i - y^{\sigma(i)})\|_2 + \|\pi_{1,k}(x^j - y^{\sigma(i)})\|_2 \leq \|\pi_{1,k}(x^i - z)\|_2 + \|z - \pi_{1,k}(y^{\sigma(i)})\|_2 + \|\pi_{1,k}(x^j) - z\|_2 + \|z - \pi_{1,k}(y^{\sigma(i)})\|_2
\]
\[
= \|\pi_{1,k}(x^i - y^{\sigma(i)})\|_2 + \|\pi_{1,k}(x^j - y^{\sigma(i)})\|_2.
\]

Moreover, since $v^i, v^j$ are not colinear due to $L_i \cap L_j \neq \emptyset$, there exists $k \in \{2, \cdots, d\}$ such that $\pi_{1,k}(v^i)$ is not colinear to $\pi_{1,k}(v^j)$, where $\pi_{1,k}(L_i)$ and $\pi_{1,k}(L_j)$ intersect only at one point. Then, the inequality in (A.1) becomes strict, i.e.
\[
\|\pi_{1,k}(x^i - y^{\sigma(i)})\|_2 + \|\pi_{1,k}(x^j - y^{\sigma(i)})\|_2 < \|\pi_{1,k}(x^i - y^{\sigma(i)})\|_2 + \|\pi_{1,k}(x^j - y^{\sigma(i)})\|_2.
\]

Therefore, we arrive at
\[
\sum_{k=2}^d \|\pi_{1,k}(x^i - y^{\sigma(i)})\|_2 + \|\pi_{1,k}(x^j - y^{\sigma(i)})\|_2 < \sum_{k=2}^d \|\pi_{1,k}(x^i - y^{\sigma(i)})\|_2 + \sum_{k=2}^d \|\pi_{1,k}(x^j - y^{\sigma(i)})\|_2,
\]
which contradicts that $\sigma$ is a minimizer. Therefore, we have $L_i \cap L_j = \emptyset$ for all $i \neq j$.

Let us next assume that for all $k \neq 1$ the projection of the segments $\pi_{1,k}(L_i)$ and $\pi_{1,k}(L_j)$ intersect. By similar reasoning as above, $\pi_{1,k}(v^i)$ and $\pi_{1,k}(v^j)$ are colinear for all $k \neq 1$, since otherwise it contradicts the minimality of $\sigma$. Hence $v^i$ and $v^j$ are colinear. However, since $x^i \neq x^j$ and $x^i_1 = x^j_1$, there exists $k \in \{2, \cdots, d\}$ such that the $k$-th coordinates of $x^i$ and $x^j$ do not coincide, i.e. $x^i_k \neq x^j_k$. Thus, $\pi_{1,k}(L_i)$ and $\pi_{1,k}(L_j)$ do not intersect, which is a contradiction. Therefore, there exists $k \neq 1$ such that $\pi_{1,k}(L_i)$ and $\pi_{1,k}(L_j)$ do not intersect.
Let us now compute the distance between \( L_i \) and \( L_j \). Since the minimal distance of non-intersecting segments in \( \mathbb{R}^2 \) is attained at one of the endpoints, without loss of generality, we can assume that
\[
d_2(\pi_{1,k}(L_i), \pi_{1,k}(L_j)) = d_2(\pi_{1,k}(x^i), \pi_{1,k}(L_j)).
\]
Denote by \( \mathbf{u} \) a unit vector in \( \mathbb{R}^2 \) orthogonal to \( \pi_{1,k}(L_j) \). By \( \max_{x \in S_1, y \in S_2} \|x - y\|_\infty \leq K \), we have
\[
|\mathbf{u} \cdot \mathbf{e}_i| \leq \frac{1}{\sqrt{2}} \leq |\mathbf{u} \cdot \mathbf{e}_k|.
\]
Besides, since \( \pi_{1,k}(L_i) \) and \( \pi_{1,k}(L_j) \) do not intersect, \( x^i, x^j \in \mathbb{Z}^d \), and \( x^i = x^j \), \( |x_k^i - x_k^j| \geq 1 \). Thus, we have
\[
d_2(\pi_{1,k}(x^i), \pi_{1,k}(L_j)) = |\pi_{1,k}(x^i - x^j) \cdot \mathbf{u}|
\geq |\pi_{1,k}(x^i - x^j) \cdot \mathbf{e}_k| |\mathbf{u} \cdot \mathbf{e}_k|
= |x_k^i - x_k^j| |\mathbf{u} \cdot \mathbf{e}_k| \geq \frac{1}{\sqrt{2}}.
\]
Therefore, we conclude
\[
d_2(L_i, L_j) \geq d_2(\pi_{1,k}(L_i), \pi_{1,k}(L_j)) \geq \frac{1}{\sqrt{2}}.
\]

Proof of Lemma 3.14: By taking a subset, without loss of generality, we assume \( m := |S_1| = |S_2| \).

Case 1. Let us first study the case where \( i = j \). We write
\[
S_1 := \{x^1, \ldots, x^m\} \quad \text{and} \quad S_2 := \{y^1, \ldots, y^m\}.
\]
We can assume without loss of generality that \( i = 1 \). Consider \( \sigma \in \mathfrak{S}_m \) as in Lemma 5.5 applied to the sets \( S_1 \) and \( S_2 \). In particular, for \( i \neq j \), we have
\[
d_2([x^i, y^{\sigma(i)}], [x^j, y^{\sigma(j)}]) \geq \frac{\ell}{K \sqrt{2}}.
\]

It is easy to check that there exists a \( \mathbb{Z}^d \)-path \( p_t \) from \( x^i \) and \( y^{\sigma(i)} \) of length at most \( 2dK \) included in
\[
\{x \in \mathbb{Z}^d : [x^i, y^{\sigma(i)}] \cap (x + [-1, 1]^d) \neq \emptyset\}.
\]

We claim that
\[
\forall x \in \mathbb{Z}^d \quad \#\{i \in [m] : x \in p_t\} \leq (2d)^{2d} \left( \frac{K}{\ell} \right)^{d-1}.
\]
To this end, fix \( x \in \mathbb{Z}^d \). We have
\[
\#\{i \in [m] : x \in p_t\} \leq \#\{i \in [m] : [x^i, y^{\sigma(i)}] \cap (x + [-1, 1]^d) \neq \emptyset\}.
\]

Since the minimal distance between two lines is at least \( \frac{\ell}{K \sqrt{2}} \), the number of lines intersecting the cube \( x + [-1, 1]^d \) is at most \( (2d)^{2d} \left( \frac{K}{\ell} \right)^{d-1} \).

Case 2. Let us now assume that \( i \neq j \). Let \( K \geq 1 \) such that \( S_1 \cup S_2 \subset [-K, K]^d \). Let \( m' := \lceil m/2 \rceil \) where we assumed \( m := |S_1| = |S_2| \). By reflection symmetry, without loss of generality, we can assume that
\[
\#\{x \in S_1 : x \cdot \mathbf{e}_j \geq 0\} \geq \frac{|S_1|}{2}, \quad \text{and} \quad \#\{x \in S_2 : x \cdot \mathbf{e}_j \geq 0\} \geq \frac{|S_2|}{2}.
\]
Hence, we can take \( S_1^+ := \{x \in S_1 : x \cdot \mathbf{e}_i \geq 0\} \) and \( S_2^+ := \{x \in S_2 : x \cdot \mathbf{e}_i \geq 0\} \) such that
\[
S_1^+ = \{x^1, \ldots, x^{m'}\}, \quad \text{and} \quad S_2^+ = \{z^1, \ldots, z^{m'}\}.
\]

Set \( y^i \) to be the intersection between \( H_i(Ke_j) \) and the line passing through \( y^i \in S_2^+ \) and of direction \( \mathbf{e}_i + \mathbf{e}_j \). Denote \( S_2 := \{y^1, \ldots, y^{m'}\} \). One can check that
\[
\max_{x \in S_1^+, y \in S_2} \|x - y\|_\infty \leq 4K.
\]
We can apply Lemma 5.5 to find a matching \( \sigma \in \mathcal{G}_m \) such that the corresponding segments are at distance at least \( 1/(4\sqrt{2}) \) from each other. We find \( \mathbb{Z}^d \)-paths \( (p_{i}^{1})_{i=1}^{m} \) joining \( z^{i} \in S_{2}^{+} \) to \( y^{i} \) such that \( p_{i}^{1} \subset \{ x^{i}+t(e_{i}+e_{j}) : t \in \mathbb{R} \} \), which implies that each vertex is crossed by at most \( 4d \) paths. By a similar argument as in Case 1, we find \( \mathbb{Z}^d \)-paths \( (p_{i}^{2})_{i=1}^{m} \) joining \( x^{i} \) to \( y^{\sigma(i)} \) such that each vertex is crossed by at least \( (2d)^{2d} \) paths. We can obtain a path going from \( x^{i} \) to \( y^{\sigma(i)} \) by considering concatenation of \( p_{i}^{1} \) and \( p_{\sigma(i)}^{1} \). This concludes the proof.

\[ \square \]

**Proof of Lemma 5.4.** Our goal is to prove that there exists \( i \in [d] \) such that \( P_{i}(S) \geq |S|^{2/3}/2 \).

Let \( n = |S| \). If \( |P_{i}(S)| \geq n^{2/3}/2 \), then the claim holds with \( i = 1 \). Hence, we assume \( |P_{i}(S)| < n^{2/3}/2 \). Define \( S_{1} := P_{i}(S) \). Since

\[
\sum_{z \in S_{1}} |P_{i}^{-1}(z) \cap S| \geq n^{1/3} \leq |S_{1}|n^{1/3} < \frac{n}{2},
\]

we have

\[
\sum_{z \in S_{1}} |P_{i}^{-1}(z) \cap S| \geq |P_{i}^{-1}(z) \cap S| \geq \frac{n}{2}.
\]

Set

\[ S_{1}^{'}, S_{2} := \{ z \in S_{1} : |P_{i}^{-1}(z) \cap S| \geq n^{1/3} \} \]

Let us first assume that \( |S_{2}| \geq n^{1/3} \). Then, there exist \( z^{1}, \ldots, z^{m} \in S_{1}^{'}, m \geq n^{1/3} \), with \( m \geq n^{1/3} \) such that \( (P_{i}(z^{j}))_{j=1}^{m} \) are all distinct. Hence, \( (P_{i}(x), x \in \cup_{i=1}^{m} P_{i}^{-1}(z^{j}) \cap S) \) are also all distinct. It follows that

\[
|P_{i}(S)| \geq \sum_{i=1}^{m} |P_{i}^{-1}(z^{j}) \cap S| \geq \sum_{i=1}^{m} |P_{i}^{-1}(z^{j}) \cap S| \geq \frac{n}{2}.
\]

Otherwise, if \( |S_{2}| < n^{1/3} \), then

\[
\sum_{z^{j} \in S_{2}} \sum_{z \in S_{1}} |P_{i}^{-1}(z) \cap S| \geq n^{1/3} \leq |S_{2}|n^{1/3} \geq \frac{n}{2}.
\]

Hence, by pigeon-hole principle, there exists \( z \in S_{2} \) such that

\[
\sum_{w \in S} |P_{i}(w) \cap S| \geq \frac{n}{2|S_{2}|} \geq \frac{n^{2/3}}{2},
\]

which implies \( |P_{i}(S)| \geq n^{2/3} \). This concludes the proof.

\[ \square \]

**Proof of Lemma 3.8.** Recall \( m(d, S) = \left( \frac{|S|}{2d-1 \text{Diam}(S)} \right)^{\frac{1}{d-1}} \). We will find, by induction on dimension, \( i \neq j \in [d], S' \subset S \) with \( |S'| \geq m(d, S) \) such that \( z_{i} \neq z'_{i} \) and \( z_{j} \neq z'_{j} \) for any \( z \neq z' \in S' \).

Let us start by proving the result for \( d = 2 \). Given \( u \in \mathbb{Z}^2 \), define \( L'(u) := L_{1}(u) \cup L_{2}(u) \).

Now, we construct \( u^{k} \) as follows: Set \( u^{1} = u \) for some arbitrary \( u \in S \). Suppose that \( u^{1}, \ldots, u^{l} \) have been defined. If there exists \( u \in S \setminus \cup_{i=1}^{l} L'(u^{i}) \), then we set \( u^{l+1} = u \); otherwise, we stop this procedure and set \( N = \ell \). Since the set \( S \) is finite, this procedure will eventually stop. Since \( |S \cap L'(u^{k})| \leq 2 \text{Diam}(S) \),

\[
2 \text{Diam}(S)N \geq |S|,
\]

so

\[
N \geq \frac{|S|}{2 \text{Diam}(S)} = m(2, S).
\]
The family $S' = (u^k)_{k=1}^N$ satisfies the requirements. The proof is completed.

Let us now assume that the claim holds for $d-1 \geq 2$. Let $S \subset \mathbb{Z}^d$. Given $z \in S$, set
$$L(z) := \{w \in \mathbb{Z}^d : w_{d-1} = z_{d-1}\} \cup \{w \in \mathbb{Z}^d : w_d = z_d\}.$$We define $v^k$ recursively as follows: Set $v^1 = z$ for some arbitrary $z \in S$. Suppose that $v^1, \ldots, v^\ell$ have been defined. If there exists $z \in S \setminus \bigcup_{i=1}^\ell L(v^i)$, then we set $v^{\ell+1} = z$; otherwise, we stop this procedure and set $N = \ell$. Since the set $S$ is finite, this procedure will eventually stop. If $N \geq m(d, S)$, then the proof is completed with $i = d-1, j = d$ and $S' = (v_k)_{k=1}^N$. Otherwise, if $N < m(d, S)$, then by pigeon-hole principle, there exists $k \in [N]$ such that
$$|\{x \in S : x \in L(v^k)\}| \geq \frac{|S|}{m(d, S)}.$$By pigeon-hole principle, there exists $r \in \{d-1, d\}$ such that $\#\{x \in S : x_r = v^k_r\} \geq \frac{|S|}{2m(d, S)}$. Without loss of generality, we suppose $r = d$, i.e.
$$\#\{x \in S : x_d = v^k_d\} \geq \frac{|S|}{2m(d, S)}.$$It follows that the set $S := \{(x_1, \ldots, x_{d-1}) : x \in S\}$ is of size at least $|S|/(2m(d, S))$. Applying the induction hypothesis to the set $S \subset \mathbb{Z}^{d-1}$, we find $i \neq j \in [d-1]$ and $S' \subset S$ such that for all $y \neq y' \in S'$, $y_i \neq y'_i$ and $y_j \neq y'_j$, and
$$|S'| \geq \frac{|S|}{2^{d-2} \text{Diam}(S)}^{1/(d-2)} \geq \frac{|S|}{2^{d-1} \text{Diam}(S)} \frac{|S|}{m(d, S)}^{1/(d-2)} \geq \frac{|S|}{2^{d-1} \text{Diam}(S)}^{\pi^{-1}(1-\frac{1}{\pi-1})} = m(d, S).$$We take $u^y \in S$ for $y \in S'$ such that $(u^y_1, \ldots, u^y_{d-1}) = y$. The result follows with the family $S' := (u^y)_{y \in S'}$ and $i, j$ chosen above. This completes the induction. 

\[\square\]

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