Exact solutions in multidimensional gravity
with antisymmetric forms

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Abstract

This topical review deals with a multidimensional gravitational model containing dilatonic scalar
fields and antisymmetric forms. The manifold is chosen in the form \( M = M_0 \times M_1 \times \ldots \times M_n \), where
\( M_i \) are Einstein spaces (\( i \geq 1 \)). The sigma-model approach and exact solutions in the model are
reviewed and the solutions with p-branes (e.g. Majumdar-Papapetrou-type, cosmological, spherically
symmetric, black-brane and Freund-Rubin-type ones) are considered.

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1 Introduction

1.1 Multidimensional models

The motivation for studying multidimensional models of gravitation and cosmology \cite{2, 3} is quite apparent for several reasons. Indeed, the main trend of modern physics is the unification of all known fundamental physical interactions: electromagnetic, weak, strong and gravitational ones. During the recent decades there has been significant progress in unifying weak and electromagnetic interactions, and some more modest achievements in GUT, supersymmetric, string and superstring theories.

Now, theories of membranes, \textit{p}-branes and (more vague) M- and F-theories are being created and studied (see subsect 1.2 below). Since no self-consistent successful theory of unification is currently available, it is desirable to study the common features of these theories and their applications to solving basic problems of modern gravity and cosmology.

Multidimensional gravitational models, as well as scalar-tensor theories of gravity, are theoretical frameworks for describing possible temporal and range variations of fundamental physical constants \cite{1, 4, 5, 6}. These ideas originated from the earlier papers of Milne (1935) and Dirac (1937) \cite{7} on relations between the phenomena of micro- and macro-worlds, and up until now they have been thoroughly studied both theoretically and experimentally.

On applying multidimensional gravitational models to the basic problems of modern cosmology and black-hole physics, we hope to find answers to such long-standing problems as the cosmological constant, acceleration, isotropization and graceful exit problems, stability and nature of fundamental constants \cite{4}, the possible number of extra dimensions, their stable compactification, etc.

Bearing in mind that multidimensional gravitational models are certain generalizations of general relativity which is tested reliably for weak fields up to 0.001 and partially in strong fields (binary pulsars), it is quite natural to inquire about their possible observational or experimental windows. From what we already know, among these windows are:

\begin{itemize}
  \item possible deviations from the Newton and Coulomb laws, or new interactions,
  \item possible variations of the effective gravitational constant with a time rate smaller than the Hubble one,
  \item possible existence of monopole modes in gravitational waves,
  \item different behaviour of strong field objects, such as multidimensional black holes, wormholes and \textit{p}-branes,
  \item standard cosmological tests etc.
\end{itemize}

As no accepted unified model exists, in our approach we adopt a simple, but general (from the point of view of number of dimensions) models based on multidimensional Einstein equations with or without sources of a different nature: cosmological constant, perfect and viscous fluids, scalar and electromagnetic fields, fields of antisymmetric forms (related to \textit{p}-branes), etc. Our programme’s main objective was and is to obtain exact self-consistent solutions (integrable models) for these models and then to analyze them in cosmological, spherically and axially symmetric cases. In our view this is natural and most reliable way to study highly nonlinear systems. Here (in the bulk of the paper) we review only exact solutions with scalar fields and antisymmetric forms.

The history of the multidimensional approach begins with the well known papers of Kaluza and Klein on five-dimensional theories which initiated interest in investigations in multidimensional gravity. These ideas were continued by Jordan who suggested considering the more general case $g_{55} \neq \text{const}$, leading to a theory with an additional scalar field. They were in some sense a source of inspiration for Brans and Dicke in their well known work on a scalar-tensor gravitational theory. After their work many investigations were performed in models with material or fundamental scalar fields, both conformal and non-conformal (see details given in \cite{11}).

A revival of the ideas of many dimensions started in the 1970s and continues now, mainly due to the development of unified theories. In the 1970s interest in multidimensional gravitational models was stimulated mainly by (i) the ideas of gauge theories leading to a non-Abelian generalization of the Kaluza-Klein approach and (ii) by supergravitational theories. In the 1980s the supergravitational theories were “replaced” by superstring models. Now it is driven by expectations connected with the overall M-theory. In all these theories, four-dimensional gravitational models with extra fields were obtained from some
multidimensional model by a dimensional reduction based on the decomposition of the manifold

\[ M = M^4 \times M_{\text{int}}, \]

where \( M^4 \) is a four-dimensional manifold and \( M_{\text{int}} \) is some internal manifold (widely considered to be compact).

The earlier papers on multidimensional gravity and cosmology dealt with multidimensional Einstein equations and with a block-diagonal cosmological or spherically symmetric metric, defined on the manifold \( M = \mathbb{R} \times M_0 \times \ldots \times M_n \) of the form

\[ g = -dt \otimes dt + \sum_{r=0}^n a_r(t)^2 g^r \]

where \( (M_r, g^r) \) are Einstein spaces, \( r = 0, \ldots, n \). In some of them a cosmological constant and simple scalar fields were also used [22].

Such models are usually reduced to pseudo-Euclidean Toda-like systems with the Lagrangian [13]

\[ L = \frac{1}{2} G_{ij} \dot{x}^i \dot{x}^j - \sum_{k=1}^m A_k e^{\nu_k \cdot x^i} \]

and the zero-energy constraint \( E = 0 \).

It should be noted that pseudo-Euclidean Toda-like systems are not well studied yet. There exists a special class of equations of state that gives rise to Euclidean Toda models [14].

It is well known that cosmological solutions are closely related to the solutions exhibiting spherical symmetry, and relevant schemes to obtain these solutions are quite similar to those applied in the cosmological approach [2]. The first multidimensional generalization of such a type was considered by Kramer [10] and rediscovered by Legkii, Gross and Perry [11, 12] (and also by Davidson and Owen). In [15] the Schwarzschild solution was generalized to the case of \( n \) internal Ricci-flat spaces, showing that a black hole configuration takes place when the scale factors of internal spaces are constants. Another important feature is that a minimally coupled scalar field is incompatible with the existence of black holes. Additionally, in [16] an analogous generalization of the Tangherlini solution was obtained, and an investigation of singularities was performed in [20]. These solutions were also generalized to the electrovacuum case with and without a scalar field [17, 18, 19]. Here, it was also proved that BHs exist only when a scalar field is switched off. Deviations from the Newton and Coulomb laws were obtained depending on mass, charge and number of dimensions. A theorem was proved in [19] that “cuts” all non-black-hole configurations as being unstable under even monopole perturbations. In [21] the extremely charged dilatonic black hole solution was generalized to a multicoaster (Majumdar-Papapetrou) case when the cosmological constant is non-zero.

We note that for \( D = 4 \) the pioneering Majumdar-Papapetrou solutions with a conformal scalar field and an electromagnetic field were considered in [23].

“Brane-world” approach. Recently, interest in the so-called “brane world” models (see [24]-[32] and therein) was greatly increased after the papers of [28, 29, 30]. It is supposed that we are living on a (1+3)-dimensional thin (or thick) layer (“3-brane”) in multidimensional space and there exists a potential preventing us from escaping from this layer, i.e. gauge and matter fields are localized on branes whereas gravity “lives” in a multidimensional bulk. Randall and Sundrum [29] suggested a rather elegant construction for the confining potential (see also [31]) using two symmetric copies of a part of 5-dimensional bulk anti-deSitter space-time. In framework of “brane-world” approach the modifications of Friedmann equations and Newton’s law were obtained. Nevertheless, at present the status quo of this approach is unclear, since in this rapidly developing field there exist a huge stream of publications, where any new paper may change drastically the “state of arts”.

1.2 Solutions with \( p \)-branes

In this review we consider several classes of the exact solutions for the multidimensional gravitational model governed (up to some details) by the Lagrangian

\[ L = R[g] - 2\Lambda - h_{\alpha\beta} g^{MN} \partial_M \varphi^\alpha \partial_N \varphi^\beta - \sum_a \frac{1}{n_a!} \exp(2\lambda a \cdot \varphi^\alpha)(F^a)^2, \quad (1.1) \]
where $g$ is metric, $F^a = dA^a$ are forms of ranks $n_a$ and $\varphi^a$ are scalar fields and $\Lambda$ is a cosmological constant (the matrix $h_{\alpha\beta}$ is invertible).

The simplest $D$-dimensional theory with scalar field, 2-form and dilatonic coupling $\lambda^2 = (D-1)/(D-2)$ may be obtained by dimensionally reducing the $(D+1)$-dimensional Kaluza-Klein theory (in this case the scalar field $\varphi$ is associated with the size of $(D+1)$ dimension). We note that the cosmological constant term can be imitated also by the form of rank $F = D$.

**Supergravities.** For certain field contents with distinguished values of total dimension $D$, ranks $n_a$, dilatonic couplings $\lambda_a$ and $\Lambda = 0$ such Lagrangians appear as “truncated” bosonic sectors (i.e. without Chern-Simons terms) of certain supergravitational theories or low-energy limit of superstring models [33, 34, 35]. For $D = 11$ supergravity [33] (that is considered now as a low-energy limit of some conjectured $M$-theory [30, 37, 38, 40, 42, 43, 41]) we have a metric and 4-form in the bosonic sector (there are no scalar fields). For $D = 10$ one may consider type I supergravity with metric, scalar field and 3-form, type IIA supergravity with bosonic fields of type I supergravity called as Neveu-Schwarz-Neveu-Schwarz (NS-NS) sector and additionally 2-form and 4-form Ramond-Ramond (R-R) sector, type IIB supergravity with bosonic fields of type I supergravity (NS-NS sector) and additionally 1-form, 3-form and (self-dual) 5-form (R-R sector). It is now believed that all five string theories (I, IIA, IIB and two heterotic ones with gauge groups $G = E_8 \times E_8$ and $\text{Spin}(32)/\mathbb{Z}_2$) [35] as well as 11-dimensional supergravity [33] are limiting case of $M$-theory. All these theories are conjectured to be related by a set of duality transformations: $S-, T-, \text{and more general } U-$ dualities [35, 41].

The list of supergravitational theories is not restricted only by dimensions $D = 10, 11$ and signature $(-, +, +, \ldots)$. One may consider also supergravities in dimensions $D < 11$ (e.g. those obtained by dimensional reduction from $D = 11$ supergravity [42], or $D = 12$ supergravity with two time dimensions [43], or even Euclidean supergravity model [48]. It was proposed earlier that $IIB$ string may have its origin in a 12-dimensional theory, known as $F$-theory [39, 44]. In [49] a low energy effective (bosonic) Lagrangian for $F$-theory was suggested. The field content of this 12-dimensional field model is the following one: metric, one scalar field (with negative kinetic term), 4-form and 5-form. In [50] a chain of so-called $B_D$-models in dimensions $D = 11, 12, \ldots$ was suggested. $B_D$-model contains $l = D - 11$ scalar fields with negative kinetic terms (i.e. $h_{\alpha\beta}$ in is negative definite) coupled to $(l+1)$ different forms of ranks $4, \ldots, 4+l$. These models were constructed using p-brane intersection rules that will be discussed below. For $D = 11$ ($l = 0$) $B_D$-model coincides with the truncated bosonic sector of $D = 11$ supergravity. For $D = 12$ ($l = 1$) it coincides with truncated $D = 12$ model from [43]. It was conjectured in [50] that these $B_D$-models for $D > 12$ may correspond to low energy limits of some unknown $F_D$-theories (analogues of $M-$ and $F$-theories).

**Setup for fields.** Here we review certain classes of (non-localized) p-brane solutions to field equations corresponding to the Lagrangian [141]. These solutions have a block-diagonal metrics defined on $D$-dimensional product manifold, i.e.

$$g = e^{2\gamma}g^0 + \sum_{i=1}^{n} e^{2\phi^i}g^i, \quad M_0 \times M_1 \times \ldots \times M_n,$$

(1.2)

where $g^0$ is a metric on $M_0$ and $g^i$ are fixed Ricci-flat (or Einstein) metrics on $M_i$ ($i > 0$). The moduli $\gamma, \phi^i$ and scalar fields $\varphi^a$ are functions on $M_0$ and fields of forms are also governed by several scalar functions on $M_0$. Any $F^a$ is supposed to be a sum of (linear independent) monoms, corresponding to electric or magnetic p-branes ($p$-dimensional analogues of membranes), i.e. the so-called composite p-brane ansatz is considered. (In non-composite case we have no more than one monom for each $F^a$.) $p = 0$ corresponds to a particle, $p = 1$ to a string, $p = 2$ to a membrane etc. The p-brane worldvolume (worldline for $p = 0$, worldsurface for $p = 1$ etc) is isomorphic to some product of internal manifolds: $M_I = M_{i_1} \times \ldots \times M_{i_k}$ where $1 \leq i_1 < \ldots < i_k \leq n$ and has dimension $p + 1 = d_{i_1} + \ldots + d_{i_k} = d(I)$, where $I = \{i_1, \ldots, i_k\}$ is a multiindex describing the location of p-brane and $d_i = \text{dim} M_i$. Any p-brane is described by the triplet (p-brane index) $s = (a, v, I)$, where $a$ is the color index labeling the form $F^a$, $v = \text{electric, magnetic}$ and $I$ is the multiindex defined above. For the electric and magnetic branes corresponding to form $F^a$ the worldvolume dimensions are $d(I) = n_a - 1$ and $d(I) = D - n_a - 1$, respectively. The sum of this dimensions is $D - 2$. For $D = 11$ supergravity we get $d(I) = 3$ and $d(I) = 6$, corresponding to electric M2-brane [57] and magnetic M5-brane [59], respectively (see also [58]).
Sigma model representation. Constraints. In \[23\] the model under consideration was reduced to gravitating self-interacting sigma-model with certain constraints imposed. These constraints coincide with non-block-diagonal part of Hilbert-Einstein equations. For \(d_0 = \dim M_0 \neq 2\) there are two groups of constraints: \((ee + mm)\) electric-electric plus magnetic-magnetic and \((em)\) electro-magnetic. In the first case \((ee + mm)\) the number of constraints is \(n_1(n_1 - 1)/2\), where \(n_1\) is number of 1-dimensional manifolds among \(M_i\). “Electro-magnetic” \((em)\) constraints appear for \(\dim M_0 = 1, 3\) and the number of these constraints is \(n_1\). We note that in \(d_0 \neq 2\) case a generalized harmonic gauge is used \(\gamma = \sum_{j=1}^n d_j \phi^j/(2 - d_0)\). For \(d_0 = \dim M_0 = 2\) the sigma model representation with \((ee + mm)\) constraints is also valid when additional restrictions on brane intersections are imposed. Here (see Section 2) we consider the sigma-model representation in a simplified form: all constraints are satisfied identically due to additional restrictions on brane intersections. There are three groups of restrictions: \((ee)\) electric-electric, \((mm)\) magnetic-magnetic and \((em)\) electric-magnetic. The restrictions of \((ee)\)- and \((mm)\)-types forbid the following intersections of two electric or two magnetic branes with the same color index: \(d(I \cap J) = d - 1\), where \(d = d(I) = d(J)\). Electro-magnetic restrictions forbid the following intersections: \(d(I \cap J) = 0\) for \(d_0 = 1\), and \(d(I \cap J) = 1\) for \(d_0 = 3\). All restrictions are satisfied identically in non-composite case when there are no two branes with the same color index. The restrictions are satisfied also when \(n_1\) is small enough: \(n_1 \leq 1\) in \((ee)\)- and \((mm)\)-cases and \(n_1 = 0\) for \((em)\)-case. (Notice, that \((ee)\)- and \((mm)\)-restrictions were considered first in \[39\]). The derivation of all constraints and restrictions on intersections was considered in detail in \[93\]. (The sigma-model representation for non-composite electric case was obtained earlier in \[87\] \[88\], for electric composite case see also \[94\]). We note that, recently, sigma model representation for non-block-diagonal metrics and two (intersecting) branes was obtained in \[128\].

The \(\sigma\)-model Lagrangian has the form \[35\] (see Section 2)

\[
\mathcal{L}_{\sigma} = R[g^0] - \hat{G}_{AB}g^{0\nu} \partial_{\nu} \sigma^A \partial_{\sigma} \sigma^B - \sum_s \varepsilon_s \exp(-2U^s) g^{0\nu} \partial_{\nu} \Phi^s \Phi^s - 2V, \tag{1.3}
\]

where \((\sigma^A) = (\phi^i, \varphi^a)\), \(V\) is a potential, \((\hat{G}_{AB})\) are components of (truncated) target space metric, \(\varepsilon_s = \pm 1\),

\[
U^s = U^s_\alpha \sigma^A = \sum_{i \in I_s} d_i \phi^i - \chi_s \lambda_{a,\alpha} \varphi^a
\]

are linear functions, \(\Phi^s\) are scalar functions on \(M_0\) (corresponding to forms), and \(s = (a_s, v_s, I_s)\). Here parameter \(\chi_s = +1\) for the electric brane \((v_s = e)\) and \(\chi_s = -1\) for the magnetic one \((v_s = m)\).

A pure gravitational sector of the sigma-model was considered earlier in \[165\] \[166\] \[167\] (notice, that ref. \[165\] contains a typo in the potential). For \(p\)-brane applications \(g^0\) is Euclidean, \((\hat{G}_{AB})\) is positive definite (for \(d_0 > 2\)) and \(\varepsilon_s = -1\), if pseudo-Euclidean (electric and magnetic) \(p\)-branes in a pseudo-Euclidean space-time are considered. The sigma-model \[113\] may be also considered for the pseudo-Euclidean metric \(g^0\) of signature \((-+, +, \ldots, +)\) (e.g. in investigations of gravitational waves). In this case for a positive definite matrix \((\hat{G}_{AB})\) and \(\varepsilon = 1\) we get a non-negative kinetic energy terms.

Brane \(U\)-vectors. The co-vectors \(U^s\) play a key role in studying the integrability of the field equations \[93\] \[117\] \[140\] and possible existence of stochastic behaviour near the singularity \[159\]. An important mathematical characteristic here is the matrix of scalar products \((U^s, U^{s'}) = G^{AB}U^s_A U^{s'}_B\), where \((\hat{G}_{AB}) = (\hat{G}_{AB})^{-1}\). The scalar products for co-vectors \(U^s\) were calculated in \[23\] (for electric case see \[57\] \[58\] \[74\])

\[
(U^s, U^{s'}) = d(I_s \cap I_{s'}) + \frac{d(I_s) d(I_{s'})}{2 - D} + \chi_s \chi_{s'} \lambda_{a,\alpha} \lambda_{a',\beta} h^{\alpha\beta},
\]

where \((h^{\alpha\beta}) = (h_{\alpha\beta})^{-1}; s = (a_s, v_s, I_s), s' = (a_{s'}, v_{s'}, I_{s'})\). They depend upon brane intersections (first term), dimensions of brane worldvolumes and total dimension \(D\) (second term), scalar products of dilatonic coupling vectors and electro-magnetic types of branes (third term). As will be shown below the so-called “intersections rules” (i.e. relations for \(d(I_s \cap I_{s'})\)) are defined by scalar products of \(U^s\)-vectors.

Solutions with harmonic functions. In Section 2 we consider a family of Majumdar-Papapetrou (MP) type solutions \[169\] for gravitating sigma-model \[113\]. An important class of solutions appear when all \"internal spaces\" \((M_i, g^i)\) are Ricci-flat, \(\Lambda = 0\) (in this case the potential is trivial \(V = 0\)) and brane vectors are orthogonal

\[
(U^s, U^{s'}) = 0 \tag{1.4}
\]
D = 11 and references therein). The curvature (e.g. Lobachevsky space, part of anti-de Sitter space) and several solvable Lie group manifolds. Space manifold was decomposed in a product of a flat space, several 2-dimensional spaces of constant curvature (e.g. Lobachevsky space, part of anti-de Sitter space) and several solvable Lie group manifolds. The target space was decomposed in a product of a flat space, several 2-dimensional spaces of constant curvature (e.g. Lobachevsky space, part of anti-de Sitter space) and several solvable Lie group manifolds. In Section 2 we present exact solutions for D = 11 supergravity and related 12-dimensional theory (F-theory) [83].

In [83], the solutions with Ricci-flat spaces M₁ and M₀ were generalized to the case of non-Ricci-flat M₀, when some additional "internal" Einstein spaces of non-zero curvature were added to M (the p-branes "live" only on products of Ricci-flat spaces). In this case the solution is a superposition of pure gravitational solution and p-brane one.

**Special solutions.** When the space (M, g) is the pseudo-Euclidean one, and spaces (M_i, g'_i) are flat, the special solutions with intersection rules [144] were considered earlier in numerous papers, e.g. in [81] (for one form F and two branes: one electric and one magnetic), [55] (in non-composite case, for two forms and two branes) and [81] (in electric case for one form F) (see also [51], [80] (for one and two forms), [50] [62] (d₁ = ... = dₙ = 1). In these solutions all "branes" contain a common manifold, say M₁ and time-submanifold belongs to M₁.

We note that there exists a lot of p-brane solutions in supergravitational theories, governed by harmonic functions with intersection rules [144] with flat M_i, see [51] [52] [83] for reviews and also [55] [77] and references therein. Some of them were obtained using supersymmetry arguments, duality transformations, dimensional reductions etc. For example, the brane solutions of type IIA supergravity have an eleven dimensional interpretation [93]: the fundamental string (FS1) [55] and five-brane (NS5) [56] are double dimensional reduction of M2-brane [57] and the direct dimensional reduction of M5-brane [59], respectively; the Dirichlet D2- and D4-branes may be obtained from M2- and M5-branes using direct and double dimensional reduction, respectively. M-branes (for D = 11) and D-branes (corresponding to Ramond-Ramond sectors of D = 10 supergravities) play a rather important role in non-perturbative analysis of superstring theories [46]. We note that the p-brane solutions of IIA supergravity (e.g. D4- and NS5-branes) play a rather important role in the so-called MQCD [108]–[115].

General solutions may be also considered for theories in dimensions D ≥ 12. In [83] a general composite solution for truncated 12-dimensional model (with 4-form and 5-form and one scalar field) from [49] (corresponding to low-energy limit of F-theory) was obtained. This solution contains electrically charged 2- and 3-branes and magnetically charged 5- and 6-branes. In [50] a chain of B₅-models in dimensions D = 11, 12, ... was constructed using intersection rules [144]: the scalar products of dilatonic couplings vectors were obtained from the requirement of the existence of binary configurations for any two p-branes. (It looks rather promising that the intersection rules may be used for constructing new Lagrangians with fields of forms and scalar fields.)

**Symmetries of target space metric.** The target space of the model is (R^K, G) (K is integer), where

\[ G = \hat{G}_{AB} d\sigma^A \otimes d\sigma^B + \sum_s \varepsilon_s \exp(-2U_s^a \sigma^A) d\Phi^a \otimes d\Phi^a. \]

It was proved in [149] [148] that the target space T = (R^K, G) is a homogeneous (coset) space G/H (G is the isometry group of T, H is the isotropy subgroup of G). T is (locally) symmetric (i.e. the Riemann tensor is covariantly constant: \( \nabla_M R_{M_1 M_2 M_3 M_4}(G) = 0 \)) if and only if

\[ (U^{s_1} - U^{s_2})(U^{s_1}, U^{s_2}) = 0 \]

for all \( s_1, s_2 \in S \), i.e. when any two vectors U^{s_1} and U^{s_2}, \( s_1 \neq s_2 \), are either coinciding \( U^{s_1} = U^{s_2} \) or orthogonal \( (U^{s_1}, U^{s_2}) = 0 \). For nonzero noncoinciding U-vectors the Killing equations were solved. Using a block-orthogonal decomposition of the set of the U-vectors it was shown that under rather general assumptions the algebra of Killing vectors is a direct sum of several copies of sl(2, R) algebras (corresponding to 1-vector blocks), several solvable Lie algebras \( g_i \) (with \( g_i^{(2)} = [g_i^{(1)}, g_i^{(1)}] = 0 \), where \( g_i^{(1)} = [g_i, g_i] \)) corresponding to multivector blocks and the Killing algebra of a flat space. The target space manifold was decomposed in a product of a flat space, several 2-dimensional spaces of constant curvature (e.g. Lobachevsky space, part of anti-de Sitter space) and several solvable Lie group manifolds. We note that recently solvable Lie algebras were studied for supergravity models in numerous papers (see [151] [152] and references therein).

**Block-orthogonal solutions.** In [117] the “orthogonal” (in target-space sense) solutions from [83]...
were generalized to a more general “block-orthogonal” case (see subsect. 3.1)

\[(U^s, U^{s'}) = 0, \quad s \in S_i, \quad s' \in S_j, \quad i \neq j;\]

\[i, j = 1, \ldots, k,\]

where the index set \(S\) is a union of \(k\) non-intersecting (non-empty) subsets \(S_i: S = S_1 \cup \ldots \cup S_k\). This means that the set of brane vectors \((U^s, s \in S)\) has a block-orthogonal structure with respect to the scalar product: it splits into \(k\) mutually orthogonal blocks \((U^s, s \in S_i), i = 1, \ldots, k\). The number of independent harmonic functions is \(k\). The solutions do exist, when the matrix of scalar products \(((U^s, U^{s''})\) and parameters \(\varepsilon_s\) satisfy the relations

\[\sum_{s'}(U^s, U^{s'})\varepsilon_s\nu_s^2 = -1\]

for some set of real \(\nu_s\) (meanwhile, the elementary monoms in composite forms \(F^s\) are proportional to \(\nu_s\)).

We note that first block-orthogonal solutions appeared for black and wormhole branes in \[127\]. When all \((U^s, U^*) \neq 0\) one can introduce a quasi-Cartan matrix

\[A_{ss'} = \frac{2(U^s, U^{s'})}{(U^*, U^{s'})},\]

\[s, s' \in S,\]

which coincides with the Cartan matrix, when \(U^s\) are simple roots of some Lie algebra and \((..)\) is standard bilinear form on the root space. It should be stressed here, that the quasi-Cartan matrix is a rather convenient tool for classification of \(p\)-brane solutions. For example, in \[117\] (see subsection 3.1.2 below) we analyzed three possibilities, when quasi-Cartan matrix coincides with the Cartan matrix of a (simple): a) finite-dimensional Lie algebra, \(\det(A_{ss'}) > 0\); b) hyperbolic Kac-Moody (KM) algebra, \(\det(A_{ss'}) < 0\); c) affine KM algebra, \(\det(A_{ss'}) = 0\) \[171\] \[172\]. It was shown that all \(\varepsilon_s = -1\) in the case a) and all \(\varepsilon_s = +1\) in the case b). The last possibility c) does not appear in the solutions under consideration. For fixed \(A\)-matrix the intersection rules read

\[d(I_s \cap I_{s'}) = \frac{d(I_s)d(I_{s'})}{D - 2} - \chi_s\chi_{s'}\lambda_{a,\alpha}\lambda_{a,\beta}h'^{\alpha\beta} + \frac{1}{2}(U^{s'}, U^{s'})A_{ss'}, \quad s \neq s'.\]

In supergravity examples all \((U^s, U^*) = 2\). Intersecting \(p\)-brane solutions with “orthogonal” intersection rules correspond to the Lie algebras \(A_1 \oplus \ldots \oplus A_1\), where \(A_1 = sl(2, \mathbb{C})\). In supergravitational models these solutions correspond to the so-called BPS saturated states preserving fractional supersymmetries. The MP solution \[159\] in this classification corresponds to the algebra \(A_1\). For Lie algebra case \(A_{ss'} = 0, -1, \ldots, \quad s \neq s',\) and hence, dimensions of \(p\)-brane intersections in this case are not greater than in the “orthogonal” case (if all \((U^*, U^*)\) are positive). In \[117\] \[130\] \[131\] some examples of solutions corresponding to hyperbolic and finite dimensional Lie algebras were considered (see subsect. 3.1.2): among them \(A_2\)-dyon solutions in \(D = 11\) supergravity and in \(B_D\)-models. The dyon solution for \(D = 11\) supergravity contains electric 2-brane and magnetic 5-brane. This solution has \(A_2 = sl(3, \mathbb{C})\) intersection rule: \(3 \cap 6 = 1\) instead of the “orthogonal” one \(3 \cap 6 = 2\) (here 3 and 6 are worldvolume dimensions). An analogous dyon solution for \(D = 10\) \(IIA\) supergravity was considered in \[129\]. For all simple finite-dimensional Lie algebras the parameters \(\nu_s\) were calculated in \[129\]. These parameters are related to integers \(n_s\) coinciding with the components of twice the dual Weyl vector in the basis of simple coroots \[172\] (see Appendix 3).

The quasi-Cartan matrix should not obviously coincide with the Cartan one. For instance, one may consider the dyon solution for truncated bosonic sector (i.e. without Chern-Simons term) of \(D = 11\) supergravity with intersection \(3 \cap 6 = 3\).

In \[117\] (see subsect. 3.1.3) the behavior of the Riemann tensor squared (Kretschmann scalar) for multicenter solutions

\[H_s(x) = 1 + \sum_b \frac{q_{sb}}{|x - b|^{d_0 - 2}},\]

was investigated and criteria for the existence of horizon and finiteness of the Kretschmann scalar were established. For \(d_0 > 2\) and "multicenter" harmonic functions the so-called \(\eta\)- and \(\xi\)- indicators were introduced. These indicators describe, under certain assumptions, the existence of a curvature singularity and a horizon, respectively, for \(x \to b\).
One-block solutions from null-geodesics. In \[140\] (see subsec. 3.2) the null-geodesic method (for \(D = 4\) see \[161\] \[162\]) was applied to the sigma-model with zero potential (i.e. when all spaces \(M_i\) are Ricci-flat) and a general class of solutions was obtained. These solutions are governed by one harmonic function \(H\) on \(M_0\) and functions \(f_s(H) = \exp(-q^s(H))\), where \(q^s\) are solutions to Toda-type equations

\[
\dot{q}^s = -B_s \exp(\sum_{s'} A_{ss'} q^{s'}),
\]

\(B_s \neq 0\). When \((A_{ss'})\) is a Cartan matrix of some semisimple finite-dimensional Lie algebra we get Toda lattice equations that are integrable \[175\] \[176\] \[177\]. Moreover, according to Adler-van Moerbeke criterion \[174\] the condition of the integrability in quadratures for Toda-type systems singles out Cartan matrices \((A_{ss'})\) among quasi-Cartan ones. Thus intersection rules \[174\] are restricted by condition of integrability of Toda-like system. In \[140\] special solutions corresponding to \(A_m\) Toda lattices (in parametrization of \[178\]) were considered. These solutions contain a class of "cosmological" solutions with Ricci-flat internal spaces and the so-called "matryoshka" solution from \[128\].

Cosmological and spherically symmetric solutions. A family of general cosmological type \(p\)-brane solutions with \(n\) Ricci-flat internal spaces was considered in \[140\], where also a generalization to the case of \(n - 1\) Ricci-flat spaces and one Einstein space of non-zero curvature (say \(M_1\)) was obtained (see subsect. 4.2). These solutions are defined up to solutions to Toda-type equations \[116\] and may be obtained using the Lagrange dynamics following from the sigma-model approach \[50\] (see subsect. 4.1). The solutions from \[140\] contain a subclass a spherically symmetric solutions (for \(M_1 = S^0\)). Special solutions with orthogonal and block-orthogonal sets of \(U\)-vectors were considered earlier in \[90\] and \[134\] \[135\], respectively. (For non-composite case, see \[79\] \[103\] \[102\] \[104\] and references therein.)

Toda solutions. In \[50\] the reduction of \(p\)-brane cosmological type solutions to Toda-like systems was performed (see also \[140\]). General classes of \(p\)-brane solutions (cosmological and spherically symmetric ones) related to Euclidean Toda lattices associated with Lie algebras (mainly \(A_m\), \(C_m\) ones) were obtained in \[138\] \[137\] \[140\] \[144\] \[145\] \[138\]. Special \(p\)-brane configurations were considered earlier in \[79\] \[80\] \[106\] (see also refs. therein).

Static solutions. In \[113\] \[132\] (see subsect. 4.3) general exact solutions defined on product of Einstein internal spaces with intersecting composite \(p\)-branes and constant scale factors were obtained and the effective cosmological constant was generated via \(p\)-branes. These solutions generalize well-known Freund-Rubin-type solutions \[124\] \[125\] of \(D = 11\) supergravity (for special supergravity solutions see also \[78\] \[122\] and refs. therein). It is remarkable, that special solutions of such kind with anti-de-Sitter factor-spaces appear in the “near-horizon” limit of extremely charged \(p\)-brane configurations (certain \(p\)-branes interpolate between flat and AdS spaces \[128\]). The interest to static configurations appeared due to papers on AdS/CFT correspondence, i.e. the duality between certain limit of some superconformal theories in \(d\)-dimensional space and string or M-theory compactified on \(AdS_{d+1} \times W\), where \(W\) is a compact manifold \[155\] \[157\] \[158\].

Quantum cosmology. In \[50\] \[119\] (see subsect. 4.4) the Wheeler-DeWitt (WDW) equation for the quantum cosmology with composite electro-magnetic \(p\)-branes defined on product of Einstein spaces was obtained (for non-composite electric case see also \[102\]). As in the pure gravitational case \[179\] this equation has a covariant and conformally covariant form (see also \[180\] \[181\] ). Moreover, in \[50\] \[119\] the WDW equation was integrated for intersecting \(p\)-branes with orthogonal \(U\)-vectors, when \(n - 1\) internal spaces are Ricci-flat and one is the Einstein space of a non-zero curvature (for non-composite electric case see \[102\]). It should be mentioned also, that a slightly different approach with classical field of forms (and rather special brane setup) was suggested in \[106\]. In \[146\] the solutions from \[50\] were used for constructing quantum analogues of black brane solutions (e.g. for \(M2\) and \(M5\) extremal branes).

Black brane solutions. In \[143\] \[145\] (see Section 5 below) a family of spherically-symmetric solutions from \[140\] was investigated and a subclass of black-hole configurations related to Toda-type equations with certain asymptotical conditions imposed was singled out. These black hole solutions are governed by functions \(H_s(z) > 0\), defined on the interval \((0, (2\mu)^{-1})\), where \(\mu > 0\) is the extremality parameter, and obey a set of differential equations (equivalent to Toda-type ones)

\[
\frac{d}{dz} \left( \frac{1 - 2\mu z}{H_s} \frac{d}{dz} H_s \right) = \tilde{B}_s \prod_{s'} H_s^{-A_{ss'}},
\]
with the following boundary conditions imposed: (i) \( H_s((2\mu)^{-1} - 0) = H_{s,0} \in (0, +\infty) \); (ii) \( H_s(+0) = 1, s \in S \). Here \( B_s \neq 0 \) and \((A_{ss'})\) is a quasi-Cartan matrix. It was shown, that for the positive definite scalar field metric \((h_{\alpha\beta})\) all \(p\)-branes in this solution should contain a time manifold (see Proposition 1 in [144, 145] and Theorem 3 from [139]: for “orthogonal” case see also [101]). In refs. [143, 144, 145] the following hypothesis was suggested: the functions \( H_s \) are polynomials when intersection rules correspond to semisimple Lie algebras, i.e. when \((A_{ss'})\) is a Cartan matrix. This hypothesis was verified for Lie algebras: \( A_m, C_{m+1} \), \( m = 1, 2, \ldots \), in [144, 145]. It was also confirmed by special black-hole “block orthogonal” solutions considered earlier in [124, 134, 135]. An analogue of this conjecture for extremal black holes was considered earlier in [100]. In Sect. 5 explicit formulas for the solution corresponding to the algebra \( A_2 \) are presented. These formulas are illustrated by two examples of \( A_2 \)-dyon solutions: a dyon in \( D = 11 \) supergravity (with \( M2 \) and \( M5 \) branes intersecting at a point) and Kaluza-Klein dyon. In subsect. 5.4 we deal with extremal configurations (e.g. with multi-black-hole extension.) We note, that special black hole solutions with orthogonal \( U \)-vectors were considered in [98, 99] \((d_1 = \ldots = d_n = 1)\), [101] (for non-composite case) and [50] (for earlier supergravity solutions see [96, 97] and refs. therein). In [101, 127] some propositions related to i) interconnection between the Hawking temperature and the singularity behaviour, and ii) multitemporal configurations were proved.

Now we briefly overview several important topics that are not considered in this short review.

**PPN parameters.** In [143, 144] the (parametrized) post-Newtonian (Eddington) parameters \( \beta \) and \( \gamma \) for 4-dimensional section of the metric were calculated. It was shown that \( \beta \) does not depend upon the \( p \)-brane intersections, while \( \gamma \) does depend. These results agree with the earlier calculations for block-orthogonal case [134, 135] (see also [147]).

**Stability of spherically symmetric solutions.** It was shown in [139] that single-brane black hole solutions are stable under spherically symmetric perturbations, whereas similar solutions possessing naked singularities turn out to be catastrophically unstable (this conclusion may be also extended to some configurations with intersecting branes). For possible other kinds of instabilities in multidimensional models (e.g. caused by waves in extra dimensions) see [187].

**Billiard representation near the singularity.** It is well-known, that the cosmological models with \( p \)-branes may have a “never ending” oscillating behaviour near the cosmological singularity as it takes place in Bianchi-IX model [189]. Remarkably, this oscillating behaviour may be described using the so-called billiard representation near the singularity (for multidimensional case see [191, 192, 193] and refs. therein). In [191, 190] the billiard representation for a cosmological model with a set of electro-magnetic \( d \)-vectors play a key role in determination of possible oscillating behaviour near the singularity. In [189] the relations (1.7) were also interpreted in terms of illumination of a (Kasner) sphere by point-like sources. We note that recently in [194, 195, 196] the relations (1.7) were applied, in fact, to \( D = 10, 11 \) supergravities and the never ending oscillatory behaviour of the generic solution near the cosmological singularity was announced to be established.

**Supersymmetries.** For flat internal spaces \( M_i \) and \( M_0 \) the supersymmetric (SUSY) solutions in supergravitational models were considered in numerous publications (see [51, 52, 53, 68, 69, 74, 75] and refs. therein). As is well-known, the major part of these solutions preserve a fractional number of SUSY of the form \( N = 2^{-k} \), where \( k \) is the number of intersecting \( p \)-branes. The “\( N = 2^{-k} \)-rule” was explained in [210] using the so-called \( 2^{-k} \)-splitting theorem for a family of \( k \) commuting linear operators. (We note that this rule is not a general one: there exists certain counterexamples.) Recently, certain SUSY
solutions in $D = 10, 11$ supergravities with several internal Ricci-flat internal spaces were considered in [198]-[210]. However, some of them may be obtained by a simple replacing of flat metrics by Ricci-flat ones. The major part of these solutions are not new ones but are special cases of those obtained before (see Section. 2). For example, the magnetic 5-brane solution from [200] with $N = 1/4$ SUSY is a special case of solutions from [87, 88, 93] etc. Notice, that in [210] the fractional number of SUSY were obtained for (non-marginal) $M2$- and $M5$-brane solutions, defined on product of two Ricci-flat spaces.

Reviews. We note, that there exist several good reviews devoted to certain aspects of solutions with $p$-branes (see, for example, [51, 52, 100]). However these reviews deal mainly with more or less special classes of $p$-brane solutions and their applications in superstring, M-theories, etc. Moreover, these reviews do not consider later results dealing with more general classes of $p$-brane solutions (see for references below). It should be stressed that one of the aims of our review is to improve the situations with citations in this area. Here we try to overview more general families of solutions with composite non-localized electromagnetic $p$-branes, when the block-diagonal metrics on product manifolds are considered. The main part of the solutions under consideration deals with Ricci-flat internal spaces of arbitrary dimensions and signatures (though certain solutions with Einstein internal spaces are also considered).

From a mathematical point of view, here we concentrate mainly on the following key items: (a) sigma-model representation; (b) Toda-like Lagrangians and Toda chains; (c) general intersection rules related to Cartan matrices in integrable cases.

Our approach cover more or less uniformly such topics as solutions with harmonic functions, classical and quantum cosmological solutions, spherically symmetric and black hole configurations, etc.

Meanwhile, some topics are out of consideration. Among them, there are localized branes, branes at angles, stability, ”brane-world” models based on $p$-branes, global properties of solutions (see [52] and refs. therein), black holes and branes in string theory [100] and microscopic origin of the Bekenstein-Hawking entropy [70, 71, 54] etc. The inclusion of these topics may be a subject of a future more extensive review.
2 The model

2.1 The action and equations of motion

We consider the model governed by an action

\[ S = \int_\mathcal{M} d^Dz \sqrt{|g|} \{ R[g] - 2\lambda - h_{\alpha\beta}g^{MN}\nabla_M\varphi^\alpha\nabla_N\varphi^\beta \} + S_{GH}, \]

(2.1)

where \( g = g_{MN}dz^M \otimes dz^N \) is the metric on the manifold \( \mathcal{M}, \dim \mathcal{M} = D, \varphi = (\varphi^\alpha) \in \mathbb{R}^l \) is a vector from dilatonic scalar fields, \( (h_{\alpha\beta}) \) is a non-degenerate symmetric \( l \times l \) matrix \( (l \in \mathbb{N}), \theta_a \neq 0, \)

\[ F^a = dA^a = \frac{1}{n_d} F_{a,M_1...M_n} dz^{M_1} \wedge ... \wedge dz^{M_n} \]

is a \( n_a \)-form \( (n_a \geq 2) \) on a \( D \)-dimensional manifold \( \mathcal{M} \), \( \Lambda \) is a cosmological constant and \( \lambda_a = 1 \)-form on \( \mathbb{R}^1 : \lambda_a(\varphi) = \lambda_{\alpha a}\varphi^\alpha, a \in \Delta, \alpha = 1, \ldots, l \). In (2.1) we denote \( |g| = |\text{det}(g_{MN})|, (F^a)^2_g = F_{a,M_1...M_n} F^{a}_{M_1...M_n} g^M_{N_1} ... g^{M_n}_{N_n}, a \in \Delta, \) where \( \Delta \) is some finite set, and \( S_{GH} \) is the standard Gibbons-Hawking boundary term \( \int_{\partial \mathcal{M}} \). In the models with one time all \( \theta_a = 1 \) when the signature of the metric is \( (-1, +1, \ldots, +1) \).

The equations of motion corresponding to (2.1) have the following form

\[ R_{MN} - \frac{1}{2} g_{MN}R = T_{MN} - \lambda g_{MN}, \]

(2.2)

\[ \triangle|g|\varphi^\alpha - \sum_{a\in\Delta} \theta_a \frac{\lambda_a}{n_d} 2\lambda_a(\varphi)(F^a)^2_g = 0, \]

(2.3)

\[ \nabla_M[|g|(e^{2\lambda_a(\varphi)}F_{a,M_1...M_n})] = 0, \]

(2.4)

\( a \in \Delta; \alpha = 1, \ldots, l \). In (2.3) \( \lambda^\alpha_a = h^{\alpha\beta}\lambda_{\beta a} \), where \( (h^{\alpha\beta}) \) is matrix inverse to \( (h_{\alpha\beta}) \). In (2.2)

\[ T_{MN} = T_{MN}[\varphi, g] + \sum_{a\in\Delta} \theta_a e^{2\lambda_a(\varphi)} T_{MN}[F^a, g], \]

(2.5)

is the stress-energy tensor where

\[ T_{MN}[\varphi, g] = \frac{1}{n_d} \left[ -\frac{1}{2} g_{MN}(F^a)^2_g + n_d F_{a,M_1...M_n} F^{a}_{M_1...M_n} \right]. \]

(2.7)

In (2.3) and (2.4) operators \( \triangle|g| \) and \( \nabla|g| \) are Laplace-Beltrami and covariant derivative operators, respectively, corresponding to \( g \).

2.2 Ansatz for composite p-branes

Let us consider the manifold

\[ M = M_0 \times M_1 \times ... \times M_n, \]

(2.8)

with the metric

\[ g = e^{2\gamma(x)}g^0 + \sum_{i=1}^n e^{2\varphi^i(x)}g^i, \]

(2.9)

where \( g^0 = g^0_{\mu\nu}(x)dx^\mu \otimes dx^\nu \) is an arbitrary metric with any signature on the manifold \( M_0 \) and \( g^i = g^i_{m,n_i}(y_i)dy^m_i \otimes dy^{n_i}_i \) is a metric on \( M_i \) satisfying the equation

\[ R_{m,n_i}[g^i] = \xi_i g^i_{m,n_i}, \]

(2.10)
manifold \( M \) spaces, are correctly defined for all \( \lambda, \phi : M_0 \to \mathbb{R} \) are smooth. We denote \( d_v = \dim M_v; \nu = 0, \ldots, n; D = \sum_{\nu=0}^n d_\nu \). We put any manifold \( M_\nu, \nu = 0, \ldots, n \), to be oriented and connected. Then the volume \( d_i \)-form
\[
\tau_i \equiv \sqrt{|g^i(y_i)|} \, dy^1 \wedge \ldots \wedge dy^d_i,
\]
and signature parameter
\[
\varepsilon(i) \equiv \text{sign}(\det(g^i_{m,n})) = \pm 1
\]
are correctly defined for all \( i = 1, \ldots, n \).

Let \( \Omega = \Omega(n) \) be a set of all non-empty subsets of \( \{1, \ldots, n\} \). The number of elements in \( \Omega \) is \( |\Omega| = 2^n - 1 \). For any \( I = \{i_1, \ldots, i_k\} \in \Omega, i_1 < \ldots < i_k \), we denote
\[
\tau(I) \equiv \tau_{i_1} \wedge \ldots \wedge \tau_{i_k},
\]
\[
\varepsilon(I) \equiv \varepsilon(i_1) \ldots \varepsilon(i_k),
\]
\[
M_I \equiv M_{i_1} \times \ldots \times M_{i_k},
\]
\[
d(I) \equiv \sum_{i \in I} d_i.
\]

Here \( \hat{\tau}_i = p_i^* \tau_i \) is the pullback of the form \( \tau_i \) to the manifold \( M \) by the canonical projection: \( p_i : M \to M_i, i = 1, \ldots, n \). We also put \( \tau(\emptyset) = \varepsilon(\emptyset) = 1 \) and \( d(\emptyset) = 0 \).

In the Appendix 1 we outline for completeness all relations for Riemann and Ricci tensors corresponding to the metric \( g \).

For fields of forms we consider the following composite electromagnetic ansatz
\[
F^a = \sum_{I \in \Omega_a,e} F^{(a,e,I)} + \sum_{J \in \Omega_a,m} F^{(a,m,J)}
\]
where
\[
F^{(a,e,I)} = d\Phi^{(a,e,I)} \wedge \tau(I),
\]
\[
F^{(a,m,J)} = e^{-2\lambda_a(\varphi)} \ast (d\Phi^{(a,m,J)} \wedge \tau(J))
\]
are elementary forms of electric and magnetic types respectively, \( a \in \Delta, I \in \Omega_{a,e}, J \in \Omega_{a,m} \) and \( \Omega_{a,v} \subset \Omega_v \), \( v = e, m \). In \( \Omega_{a,e} \) \( \ast = \ast g \) is the Hodge operator on \( (M, g) \)
\[
(\ast \varphi)_{M_1 \ldots M_{D-k}} = \frac{|g|^{1/2}}{k!} \varepsilon_{M_1 \ldots M_{D-k}N_1 \ldots N_k} \varphi^{N_1 \ldots N_k},
\]
where \( \text{rank} \omega = k \). For scalar functions we put
\[
\varphi^a = \varphi^a(x), \quad \Phi^s = \Phi^s(x),
\]
\( s \in S \). Thus \( \varphi^a \) and \( \Phi^s \) are functions on \( M_0 \).

Here and below
\[
S = S_e \cup S_m, \quad S_v = \sqcup_{a \in \Delta} \{a\} \times \{v\} \times \Omega_{a,v},
\]
\( v = e, m \). Here and in what follows \( \sqcup \) means the union of non-intersecting sets. The set \( S \) consists of elements \( s = (a_k, v_s, I_{s}), \) where \( a_k \in \Delta \) is colour index, \( v_s = e, m \) is electro-magnetic index and set \( I_s \in \Omega_{a_s, v_s} \) describes the location of brane.

Due to \( (2.13) \) and \( (2.14) \)
\[
d(I) = n_a - 1, \quad d(J) = D - n_a - 1,
\]
for \( I \in \Omega_{a,e} \) and \( J \in \Omega_{a,m} \) (i.e. in electric and magnetic case, respectively). The sum of worldvolume dimensions for electric and magnetic branes corresponding to the same form is equal to \( D - 2 \), it does not depend upon the rank of the form.
2.3 The sigma model

Let \( d_0 \neq 2 \) and

\[
\gamma = \gamma_0(\phi) \equiv \frac{1}{2 - d_0} \sum_{j=1}^{n} d_j \phi^j,
\]

i.e. the generalized harmonic gauge (frame) is used. As we shall see below the equations of motions have a rather simple form in this gauge. Moreover harmonic gauge preserve the harmonicity of coordinates: harmonic coordinates on \((M_0, g^0)\) (i.e. obeying \( \Delta [g^0] x^\mu = 0 \)) have harmonic pullbacks \( \hat{x}^\mu \) on \((M, g)\) (i.e. \( \Delta [g] \hat{x}^\mu = 0 \)) and analogous statement is valid for harmonic coordinates \( y^m \) on \((M, g')\), \( i = 1, \ldots, n \). It was shown recently in \([116]\) that the choice of harmonic coordinates is the most convenient coordinate choice for studying \( p \)-branes.

2.3.1 Restrictions on \( p \)-brane configurations.

Here we present two restrictions on the sets of \( p \)-branes that guarantee the block-diagonal form of the energy-momentum tensor and the existence of the sigma-model representation (without additional constraints).

We denote \( w_1 \equiv \{ i| i \in \{1, \ldots, n \}, \ d_i = 1 \} \), and \( n_1 = |w_1| \) (i.e. \( n_1 \) is the number of 1-dimensional spaces among \( M, i = 1, \ldots, n \)).

**Restriction 1.** For any \( a \in \Delta \) and \( v = e, m \) there are no \( I, J \in \Omega_{a,v} \) such that \( I = \{ i \} \cap (I \cap J) \), and \( J = (I \cap J) \cup \{ j \} \) for some \( i, j \in w_1, i \neq j \).

Let us define \( \tilde{I} \) as follows

\[
\tilde{I} \equiv I_0 \setminus I, \quad I_0 = \{1, \ldots, n\}.
\]

**Restriction 2** (only for \( d_0 = 1, 3 \)). For any \( a \in \Delta \) there are no \( I \in \Omega_{a,e} \) and \( J \in \Omega_{a,m} \) such that \( \tilde{J} = \{ i \} \cup I \) for \( d_0 = 1 \), and \( I = \{ i \} \cup \tilde{J} \) for \( d_0 = 3 \), where \( i \in w_1 \).

Restriction 1 is satisfied for \( n_1 \leq 1 \) and also in the non-composite case: \( |\Omega_{a,e}| + |\Omega_{a,m}| = 1 \) for all \( a \in \Delta \). For \( n_1 \geq 2 \) it forbids the following pairs of two electric or two magnetic \( p \)-branes, corresponding to the same form \( F^a, a \in \Delta \):

\[
\begin{array}{c}
\text{I} \\
\hline
\text{J}
\end{array}
\]

**Figure 1.** A forbidden by Restriction 1 pair of two electric or two magnetic \( p \)-branes.

Here \( d_i = d_j = 1, i \neq j, i, j = 1, \ldots, n \). Restriction 1 may be also rewritten in terms of intersections

\[
(R_1) \quad d(I \cap J) \leq d(I) - 2,
\]

for any \( I, J \in \Omega_{a,v}, a \in \Delta, v = e, m \) (here \( d(I) = d(J) \)).

Restriction 2 is satisfied for \( n_1 = 0 \) or when \( d_0 \neq 1, 3 \). For \( n_1 \geq 1 \) it forbids the following electromagnetic pairs, corresponding to the same form \( F^a, a \in \Delta \):

\[
\begin{array}{c}
\text{I} \\
\hline
\text{J}
\end{array}
\quad
\begin{array}{c}
\text{J} \\
\hline
\text{I}
\end{array}
\]

**Figure 2.** Forbidden by Restriction 2 electromagnetic pairs of \( p \)-branes for \( d_0 = 1 \) and \( d_0 = 3 \), respectively.

\[
(R_2) \quad d(I \cap J) \neq 0 \text{ for } d_0 = 1, \quad d(I \cap J) \neq 1 \text{ for } d_0 = 3
\]

(see \([222]\)).

It should be noted that possible non-diagonality of stress-energy tensor \( T^M_N \) for composite \( p \)-branes usually is not discussed in publications devoted to \( p \)-brane solutions. To our knowledge this topic was considered first in \([238]\) and (in detailed) in \([239]\).
2.3.2 Sigma-model action for harmonic gauge

It was proved in [93] that equations of motion for the model (2.1) and the Bianchi identities:

\[ dF^s = 0, \]  

(2.27)

\[ s \in S_m, \]  

for fields from (2.10), (2.17)–(2.20), when Restrictions 1 and 2 are imposed, are equivalent to equations of motion for the \( \sigma \)-model governed by the action

\[
S_{\sigma 0} = \frac{1}{2\kappa_0^2} \int d^6 x \sqrt{|g^0|} \left\{ R[g^0] - \hat{G}_{AB} g^{0\mu\nu} \partial_\mu \sigma^A \partial_\nu \sigma^B - \sum_{s \in S} \varepsilon_s \exp \left( -2U_s^A \sigma^A \right) g^{0\mu\nu} \partial_\mu \Phi^s \partial_\nu \Phi^s - 2V \right\} ,
\]

(2.28)

where \((\sigma^A) = (\phi^i, \varphi^\alpha), \) \( k_0 \neq 0, \) the index set \( S \) is defined in (2.21),

\[ V = V(\phi) = \Lambda e^{2\gamma_0(\phi)} - \frac{1}{2} \sum_{i=1}^n \xi_i d_i e^{-2\phi^i + 2\gamma_0(\phi)} \]

is the potential,

\[ (\hat{G}_{AB}) = \begin{pmatrix} G_{ij} & 0 \\ 0 & h_{\alpha\beta} \end{pmatrix}, \]

(2.29)

is the target space metric with

\[ G_{ij} = d_i \delta_{ij} + \frac{d_i d_j}{\alpha_0 - 2}, \]

(2.30)

and co-vectors

\[ U_s^A = U_s^A \sigma^A = \sum_{i \in I_s} d_i \delta^i - \chi_s \lambda_{a_s}(\varphi), \quad (U_s^A) = (d_i \delta_{1i_s}, -\chi_s \lambda_{a_s}), \]

(2.31)

\[ s = (a_s, v_s, I_s). \]  

Here \( \chi_e = +1 \) and \( \chi_m = -1; \)

\[ \delta_{II} = \sum_{j \in I} \delta_{ij} \]

(2.32)

is an indicator of \( i \) belonging to \( I: \delta_{II} = 1 \) for \( i \in I \) and \( \delta_{II} = 0 \) otherwise; and

\[ \varepsilon_s = (-\varepsilon[g])^{(1-\chi_s)/2} \varepsilon(I_s) \theta_{a_s}, \]

(2.33)

\[ s \in S, \varepsilon[g] \equiv \text{sign det}(g_{MN}). \]  

More explicitly (2.34) reads

\[ \varepsilon_s = \varepsilon(I_s) \theta_{a_s}, \]  

for \( v_s = e; \)

\[ \varepsilon_s = -\varepsilon[g] \varepsilon(I_s) \theta_{a_s}, \]  

for \( v_s = m. \)

(2.36)

For finite internal space volumes \( V_i \) (e.g. compact \( M_i \)) and electric \( p \)-branes (i.e. all \( \Omega_{a,m} = \emptyset \)) the action (2.10) coincides with the action (2.11) when \( \kappa^2 = \kappa_0^2 \prod_{i=1}^n V_i. \) This may be readily verified using the relations from Appendices 1 and 2. In general electro-magnetic case relation (2.39) can not be obtained from the action (2.11) by a straightforward integration over internal compact spaces \( M_i, \) since such integration will give wrong signs for kinetic terms corresponding to magnetic scalars \( \Phi^s. \) A simple explanation of this fact was given in [93] (see Sect. 5 therein). Thus in general case one should be careful in writing the sigma-model representation (especially when dealing with magnetic and composite branes). To our knowledge this (subtle) point is not widely discussed in literature.
Equations of motion corresponding to the action (2.28) with the potential (2.29) have the following form

\[ R_{\mu\nu}[g^0] = \hat{G}_{AB} \partial_{\mu} \sigma^A \partial_{\nu} \sigma^B + \sum_{s \in S} \varepsilon_s \exp(-2U_s^A \sigma^A) \partial_{\mu} \Phi_s \partial_{\nu} \Phi_s + \frac{2V}{d_0 - 2} g^0_{\mu\nu}, \]  

(2.36)

\[ \hat{G}_{AB} \Delta[g^0] \sigma^B + \sum_{s \in S} \varepsilon_s U_s^A \exp(-2U_s^A \sigma^A) g^0_{\mu\nu} \partial_{\mu} \Phi_s \partial_{\nu} \Phi_s = \frac{\partial V}{\partial \sigma^A}. \]  

(2.37)

\[ \partial_{\mu} \left( \sqrt{|g^0|} g^0_{\mu\nu} \exp(-2U_s^A \sigma^A) \partial_{\nu} \Phi_s \right) = 0, \]  

(2.38)

\( s \in S \). Here \( \Delta[g^0] \) is the Laplace-Beltrami operator corresponding to \( g^0 \).

**Sigma-model with constraints.** In [93] a general proposition concerning the sigma-model representation when the Restrictions 1 and 2 are removed is presented. In this case the stress-energy tensor \( T_{MN} \) is not identically block-diagonal as it takes place for \( R_{MN} \) and due to equations of motion the off-block-diagonal components of \( T_{MN} \) should be zero, hence, several additional constraints (or restrictions) on the field configurations appear [93].

### 2.3.3 General conformal gauges and \( d_0 = 2 \) case.

We may also fix the gauge \( \gamma = \gamma(\phi) \) (where \( \gamma(\phi) \) is a smooth function) by arbitrary manner or do not fix it. In this case the action (2.28) is simply replaced by the action

\[ S_\sigma = \frac{1}{2\kappa_0^2} \int_{M_0} d^d x \sqrt{|g^0|} e^{f(\gamma, \phi)} \left\{ R[g^0] - \sum_{i=1}^n d_i (\partial \phi^i)^2 - (d_0 - 2)(\partial \gamma)^2 \right. \]

\[ + (\partial f) \partial (f + 2\gamma) - 2\Lambda e^{2\gamma} + \sum_{i=1}^n \xi_i e^{2\phi^i} + 2\gamma - \sum_{s \in S} \varepsilon_s \exp(-2U_s^A \sigma^A) (\partial \Phi_s)^2 \left\}, \right. \]

where

\[ f = f(\gamma, \phi) = (d_0 - 2)\gamma + \sum_{j=1}^n d_j \phi^j. \]  

(2.40)

Here \( \partial f_1 \partial f_2 = g^{0\mu\nu} \partial_{\mu} f_1 \partial_{\nu} f_2 \) and \( \partial f \partial f = (\partial f)^2 \).

Now let us consider the case \( d_0 = 2 \). In this case the sigma-model representation holds if the Restriction 2 is replaced by the following restriction [93].

**Restriction 2** \( d_0 = 2 \). For any \( a \in \Delta \) there are no \( I \in \Omega_{a,e}, J \in \Omega_{a,m} \) such that \( \bar{I} = J \) or \( \bar{J} = \{i\} \cup (J \cap I) \), and \( I = (J \cap I) \cup \{j\} \) for some \( i, j \in w_1, i \neq j \).
3 Solutions governed by harmonic functions

3.1 Solutions with orthogonal and block-orthogonal $U^s$ and Ricci-flat $(M_\nu, g^0)$. Here we consider a special class of solutions to equations of motion governed by several harmonic functions when all factor spaces are Ricci-flat and cosmological constant is zero, i.e. $\xi = \Lambda = 0$, $i = 1, \ldots, n$. In certain situations these solutions describe extremal $p$-brane black holes charged by fields of forms.

The solutions crucially depend upon scalar products of $U^s$-vectors $(U^s, U^{s'})$; $s, s' \in S$, where

$$ (U, U') = \hat{G}^{AB} U_A U'_B, $$

for $U = (U_A), U' = (U'_A) \in \mathbb{R}^N$, $N = n + l$ and

$$ (\hat{G}^{AB}) = \begin{pmatrix} G^{ij} & 0 \\ 0 & h^{\alpha\beta} \end{pmatrix} $$

is matrix inverse to the matrix (2.30). Here (as in 179)

$$ G^{ij} = \frac{\delta^{ij}}{d_i} + \frac{1}{2 - D}, $$

$i, j = 1, \ldots, n$.

The scalar products (3.1) for vectors $U^s$ were calculated in [93]

$$ (U^s, U^{s'}) = d(I_s \cap I_{s'}) + \frac{d(I_s)d(I_{s'})}{2 - D} + \chi_s \chi_{s'} \lambda_{a,\alpha} \lambda_{a',\beta} h^{\alpha\beta}, $$

where $(h^{\alpha\beta}) = (h_{a\beta})^{-1}$; and $s = (a_s, v_s, I_s)$, $s' = (a_{s'}, v_{s'}, I_{s'})$ belong to $S$. This relation is a very important one since it encodes $p$-brane data (e.g. intersections) in scalar products of $U$-vectors. This relation simplify calculations, clarify the algebraic structure of the equations of motion and make transparent a reduction to Toda-like systems.

Let

$$ S = S_1 \sqcup \ldots \sqcup S_k, $$

$$ S_i \neq \emptyset, i = 1, \ldots, k, $$

and

$$ (U^s, U^{s'}) = 0 $$

for all $s \in S_i$, $s' \in S_j$, $i \neq j$; $i, j = 1, \ldots, k$. Relation (3.5) means that the set $S$ is a union of $k$ non-intersecting (non-empty) subsets $S_1, \ldots, S_k$. According to (3.5) the set of vectors $(U^s, s \in S)$ has a block-orthogonal structure with respect to the scalar product (3.1), i.e. it splits into $k$ mutually orthogonal blocks $(U^s, s \in S_i)$, $i = 1, \ldots, k$.

Here we consider exact solutions in the model (2.1), when vectors $(U^s, s \in S)$ obey the block-orthogonal decomposition (3.5), (3.6) with scalar products defined in (3.1) [117]. These solutions may be obtained from the corresponding solutions to the $\sigma$-model equations (2.30) - (2.33).

**Proposition 1** [117]. Let $(M_\nu, g^0)$ be Ricci-flat: $R_{\mu\nu}[g^0] = 0$. Then the field configuration

$$ g^0, \quad \sigma^A = \sum_{s \in S} \varepsilon_s U^{sA} \nu_s^2 \ln H_s, \quad \Phi^s = \frac{\nu_s}{H_s}, $$

$s \in S$, satisfies to field equations (2.30) - (2.33) with $V = 0$ if (real) numbers $\nu_s$ obey the relations

$$ \sum_{s' \in S} (U^s, U^{s'}) \varepsilon_s \nu_{s'}^2 = -1 $$

$s \in S$, functions $H_s > 0$ are harmonic, i.e. $\Delta[g^0]H_s = 0$, $s \in S$ and $H_s$ are coinciding inside blocks: $H_s = H_{s'}$ for $s, s' \in S_i$, $i = 1, \ldots, k$.

The Proposition 1 can be readily verified by a straightforward substitution of (3.7) - (3.8) into equations of motion (2.30) - (2.33). In the special (orthogonal) case, when any block contains only one vector (i.e. all $|S_i| = 1$) the Proposition 1 coincides with Proposition 1 of [93]. In general case vectors inside each block...
$S_i$ are not orthogonal. The solution under consideration depends on $k$ independent harmonic functions. For a given set of vectors $(U^s, s \in S)$ the maximal number $k$ arises for the irreducible block-orthogonal decomposition \cite{5.3, 5.6}, when any block $(U^s, s \in S_i)$ does not split into two mutually-orthogonal subblocks.

Using the sigma-model solution from Proposition 1 and relations for contravariant components \cite{8.3}:

$$\begin{aligned}
U^{si} &= \delta_{il} a_s - \frac{d(I_s)}{D-2} U^{sa}, \\
U^{sa} &= -\lambda_s^a \lambda_a^s,
\end{aligned} \quad (3.9)$$

$s = (a_s, v_s, I_s)$, we get \cite{11.7}:

$$g = \left( \prod_{s \in S} H_s^{2d(I_s)\varepsilon_s v_s^2} \right)^{1/(2-D)} \left\{ g^0 + \sum_{i=1}^n \left( \prod_{s \in S} H_s^{2\varepsilon_s \nu^2_{si} d(I_s)} \right) g^i \right\}, \quad (3.10)$$

$$\varphi^a = -\sum_{s \in S} \lambda_s^a \lambda_s \varepsilon_s \nu_s^2 \ln H_s, \quad (3.11)$$

$$F^a = \sum_{s \in S} F^a \delta^a_{\lambda_s^a}, \quad (3.12)$$

where $i = 1, \ldots, n$, $\alpha = 1, \ldots, l$, $a \in \Delta$ and

$$F^e = \nu_s dH_s^{-1} \wedge (I_s), \quad \text{for } v_s = e, \quad (3.13)$$

$$F^m = \nu_s (\mu_d H_s) \wedge (I_s), \quad \text{for } v_s = m, \quad (3.14)$$

$H_s$ are harmonic functions on $(M_0, g^0)$ coinciding inside blocks (i.e. $H_s = H_{s'}$ for $s, s' \in S_i, i = 1, \ldots, k$) and relations \cite{8.3} on parameters $\nu_s$, $s \in S$, are defined in \cite{8.4} and \cite{2.34}, respectively; $\lambda_s^a = h^{ab} \lambda_{\beta_s a}$, $*0 = *[g^0]$ is the Hodge operator on $(M_0, g^0)$ and $I$ is defined \cite{2.24}. In \cite{8.4} we redefined the sign of $\nu_s$-parameter (comparing to \cite{2.19}) as following: $\nu_s \mapsto -\varepsilon(I)\mu(I)\nu_s$.

Relation \cite{3.11} may be obtained from \cite{2.19} by use of the following identity

$$\exp(-2U_s^A \sigma^A) = H_s^2, \quad (3.15)$$

$s \in S$, following from Proposition 1. For certain models the latter appears as so-called "no-force" condition of the vanishing of the static force on a $p$-brane probe in the background of another $p$-brane \cite{5.1, 5.5, 5.7, 6.0, 7.3}.

**Remark 1.** The solution \cite{3.10-3.14} is also valid for $d_0 = 2$, if Restriction 2 from previous section is replaced by Restriction 2'. It may be verified using the sigma-model representation \cite{2.34}.

### 3.1.1 Solutions with orthogonal $U^s$:

Let us consider the orthogonal case \cite{8.3}

$$(U^s, U^{s'}) = 0, \quad s \neq s', \quad (3.16)$$

$s, s' \in S$. Then relation \cite{8.3} reads as follows

$$(U^s, U^s) \varepsilon_s \nu_s^2 = -1, \quad (3.17)$$

$s \in S$. This implies $(U^s, U^s) \neq 0$ and

$$\varepsilon_s (U^s, U^s) < 0, \quad (3.18)$$

for all $s \in S$. For $d(I_s) < D-2$ and $\lambda_{a, \alpha} \lambda_{s, \beta} \eta^\alpha \eta^\beta \geq 0$ we get from \cite{8.3} $(U^s, U^s) > 0$, and, hence, $\varepsilon_s < 0$, $s \in S$. If $\theta_a > 0$ for all $a \in \Delta$, then

$$\varepsilon(I_s) = -1 \text{ for } v_s = e; \quad \varepsilon(I_s) = \varepsilon[g] \text{ for } v_s = m. \quad (3.19)$$

For pseudo-Euclidean metric $g$ all $\varepsilon(I_s) = -1$ and, hence, all $p$-branes should contain time manifold. For the metric $g$ with the Euclidean signature only magnetic $p$-branes can exist in this case.
From scalar products \( (3.4) \) and the orthogonality condition \( (3.16) \) we get the "orthogonal" intersection rules \( [89, 90, 93] \)

\[
d(I_s \cap I_{s'}) = \frac{d(I_s) d(I_{s'})}{D - 2} - \chi_s \chi_{s'} \lambda_{\alpha, \beta} h^{\alpha \beta} = \Delta(s, s'), \tag{3.20}
\]

for \( s = (a_s, v_s, I_s) \neq s' = (a_{s'}, v_{s'}, I_{s'}) \). (For pure electric case see also \( [87, 88, 94] \).

**Example 1: \( D = 11 \) supergravity \( [33] \).** The action for the bosonic sector of \( D = 11 \) supergravity with omitted Chern-Simons term has the following form

\[
\hat{S}_{11} = \int_M d^{11}z \sqrt{|g|} \{ R[g] - \frac{1}{4!} F^2 \}. \tag{3.21}
\]

Here \( \text{rank} F = 4 \) and the signature of \( g \) is \( (-, +, \ldots, +) \).

The dimensions of p-brane worldvolumes are (see \( (2.22) \))

\[
d(I_s) = \begin{cases} 
3, & \text{for } v_s = e, \\
6, & \text{for } v_s = m.
\end{cases} \tag{3.22}
\]

The model describes electrically charged 2-branes and magnetically charged 5-branes.

From \( (3.20) \) we obtain the intersection rules \( (3.28) \)

\[
d(I_s \cap I_{s'}) = \Delta(s, s') + \frac{1}{2} K_s A_{ss'}, \tag{3.28}
\]

\( s \neq s' \), where \( \Delta(s, s') \) is defined in \( (3.20) \). These rules is a generalization of "orthogonal" intersection rules to the case of \( \).
For $\det A \neq 0$ relation (3.20) may be rewritten in the equivalent form

$$-\varepsilon_s \nu_1^2(U^s, U^s) = 2 \sum_{s' \in S} A^{ss'} \equiv b_s,$$

(3.29)

$s \in S$, where $(A^{ss'}) = A^{-1}$. Thus, eq. (3.20) may be resolved in terms of $\nu_1$ for certain $\varepsilon_s = \pm 1$, $s \in S$. We note that due to (3.4) the matrix $A$ has a block-diagonal structure and, hence, for any $i$-th block the set of parameters $(\nu_s, s \in S_i)$ depend upon the matrix inverse to the matrix $(A_{ss'}; s, s' \in S_i)$.

Now we consider one-block case when the $p$-brane intersections are related to some Lie algebras.

**3.1.2.1. Finite dimensional Lie algebras [129].**

Let $A$ be a Cartan matrix of a simple finite-dimensional Lie algebra. In this case $A_{ss'} \in \{0, -1, -2, -3\}$, $s \neq s'$. The elements of inverse matrix $A^{-1}$ are positive (see Ch. 7 in [172]) and hence we get from (3.20) the same signature relation (3.18) as in orthogonal case. Moreover, all $b_s$ are natural numbers:

$$b_s = n_s \in \mathbb{N},$$

(3.30)

$s \in S$. Integers $n_s$ coincide with the components of twice the so-called dual Weyl vector in the basis of simple coroots (see Ch. 3.1.7 in [172]). Explicit formulas for $n_s$, corresponding to simple finite dimensional Lie algebras are outlined in Appendix 3.

Here we consider three examples of solutions in $D = 11$ and $D = 10$ (IIA) supergravities and so-called $B_D$-models (see Example 4 below) corresponding to $A_2$-algebra with the Cartan matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

(3.31)

and $n_1 = n_2 = 2$.

**Example 2: $A_2$-dyon in $D = 11$ supergravity.** For the $D = 11$ supergravity with the bosonic part of the action (3.21) we get for $A_2$-solutions with two branes the intersections

$$3 \cap 6 = 1, \quad 6 \cap 6 = 3.$$  

(3.32)

Here and in what follows $(d_1 \cap d_2 = d) \Leftrightarrow (d(I) = d_1, d(J) = d_2, d(I \cap J) = d)$. The electromagnetic dyon (i.e. the bound state of electric and magnetic branes) with the intersection $3 \cap 6 = 1$ reads

$$g = H^2 g^0 - H^{-2} dt \otimes dt + \hat{g}^1 + \hat{g}^2,$$

$$F = \nu_1 dH^{-1} \wedge dt \wedge \hat{\tau}_1 + \nu_2 (*_g H) \wedge \hat{\tau}_1,$$

(3.33)

(3.34)

where $H$ is a harmonic function on $(M_0, g^0)$, $d_0 = 3$, $d_1 = 2$, $d_2 = 5$, $\nu_1^2 = \nu_2^2 = 1$, and metrics $g^\nu$ have Euclidean signature.

For $g^0 = \sum_{\mu=1}^3 d\mu^\mu \otimes d\mu^\mu$ and

$$H = 1 + \sum_{j=1}^N \frac{q_j}{|x - x_j|},$$

(3.35)

the 4-dimensional section of the metric (3.23) coincides with the metric of Majumdar-Papapetrou solution [159] describing $N$ extremal charged black holes with horizons at points $x_i$, and charges $q_i > 0$, $i = 1, \ldots, N$. The solution (3.33)–(3.34) with $H$ from (3.35) describes $N$ extremal $p$-brane dyonic black holes. Any dyon contains one electric ”brane” and one magnetic ”brane” with equal charge densities.

**Example 3: $A_2$-dyon in IIA supergravity.** The bosonic part of action for $D = 10$ IIA supergravity reads

$$S = \int d^{10} z \sqrt{|g|} \left\{ R[g] - (\partial \varphi)^2 - \sum_{a=2}^4 \epsilon^{2\lambda_a \varphi} (F^a)^2 \right\} - \frac{1}{2} \int F^4 \wedge F^4 \wedge A^2,$$

(3.36)

where $F^a = \delta A^{a-1} + \delta_S A^3 \wedge F^3$ is an $a$-form, $a = 2, 3, 4$, and $\lambda_3 = -2 \lambda_4$, $\lambda_2 = 3 \lambda_4$, $\lambda_4^2 = 1/8$. The dimensions of $p$-brane worldvolumes are

$$d(I) = \begin{cases} 1, 2, 3 & \text{in electric case,} \\ 7, 6, 5 & \text{in magnetic case,} \end{cases}$$

(3.37)
for $a = 2, 3, 4$, respectively.

We consider here the sector corresponding to $a = 3, 4$ describing electric $p$-branes: fundamental string (FS), $D2$-brane and magnetic $p$-branes: $NS5$- and $D4$-branes. We get $(U^s, U^s) = 2$ for all $s$. The solutions with $A_2$ intersection rules corresponding to relations

$$
2 \cap 6 = 3 \cap 6 = 3 \cap 5 = 1, \ 6 \cap 5 = 6 \cap 6 = 3, \ 5 \cap 5 = 2
$$

(3.38)

are valid in the "truncated case" (without Chern-Simons term) and in a general case, as well. Let us consider the solution describing the electromagnetic dyon with one of intersections 2-brane and magnetic 5-brane: fundamental string

$$
2 \cap 6 = 3 \cap 5 = 1
$$

. The solution is given by relations (3.33) and (3.34) with

$$
\text{Example 4:} \ A_2\text{-dyon in } B_D\text{-models.} \text{ Now we consider examples of solutions for } B_D\text{-models with the action [50]}
$$

$$
S_D = \int d^Dz \sqrt{|g|} \left( R[g] + g^{MN} \partial_M \varphi \partial_N \varphi - \sum_{a=4}^{D-7} \frac{1}{a!} \exp \left[ 2 \tilde{\lambda}_a \varphi \right] (F^a)^2 \right),
$$

(3.39)

where $\varphi = (\varphi^1, \ldots, \varphi^l) \in \mathbb{R}^l$, $\tilde{\lambda}_a = (\lambda_{a1}, \ldots, \lambda_{al}) \in \mathbb{R}^l$, $l = D - 11$, rank $F^a = a$, $a = 4, \ldots, D - 7$. Here vectors $\tilde{\lambda}_a$ satisfy the relations

$$
\tilde{\lambda}_a \tilde{\lambda}_b = N(a, b) - \frac{(a-1)(b-1)}{D-2},
$$

(3.40)

$$
N(a, b) = \min(a, b) - 3,
$$

(3.41)

$a, b = 4, \ldots, D - 7$ and $\tilde{\lambda}_{D-7} = -2 \tilde{\lambda}_4$. For $D > 11$ vectors $\tilde{\lambda}_4, \ldots, \tilde{\lambda}_{D-8}$ are linearly independent.

The model (3.39) contains $l$ scalar fields with a negative kinetic term (i.e. $h_{\alpha \beta} = -\delta_{\alpha \beta}$ in (2.1)) coupled to $(l+1)$ forms. For $D = 11$ ($l = 0$) the model (3.39) coincides with the truncated bosonic sector of $D = 11$ supergravity. For $D = 12$ ($l = 1$) (3.39) coincides with truncated $D = 12$ model from [49] (see also [23]).

For $p$-brane worldvolumes we have the following dimensions (see (2.22))

$$
d(I) = 3, \ldots, D - 8, \quad I \in \Omega_{a,c}, \\
d(I) = D - 5, \ldots, 6, \quad I \in \Omega_{a,m}.
$$

(3.42)

(3.43)

Thus, there are $(l+1)$ electric and $(l+1)$ magnetic $p$-branes, $p = d(I) + 1$. In $B_D$-model all $K_s = 2$.

Let us consider $B_D$-model, $D \geq 11$. Let $a \in \{4, \ldots, D - 7\}$, $g^3 = -dt \otimes dt$, $d_1 = a - 2$, $d_2 = D - 2 - a$, $d_0 = 3$ and metrics $g^0, g^1, g^2$ are Ricci-flat. The $A_2$-solution describing a dyon configuration with electric $d_1$-brane and magnetic $d_2$-brane, corresponding to $F^a$-form and intersecting in 1-dimensional time manifold reads as given by relations (3.33), (3.34) and $\varphi = 0$, where $H$ is the harmonic function on $(M_0, g^0)$ and $v_1^2 = v_2^2 = 1$. The case $D = 11$ was considered in Example 2. For $D = 12$ we have two possibilities: a) $a = 4, d_1 = 2, d_2 = 6$; b) $a = 5, d_1 = 3, d_2 = 5$. The signature restrictions on $g^1$ and $g^2$ are the following: $\varepsilon_1 = +1, \varepsilon_2 = -\varepsilon[g]$. They are satisfied when $g^0$ and $g^1$ are metrics of Euclidean signature.

Remark 3. In Examples 2, 3 and 4 the $A_2$-dyon solutions do not satisfy the Restriction 2 (or, equivalently, (2.22)) that guarantees the vanishing of non-block-diagonal components of stress-energy tensor, i.e. $T_{1, \mu_0} = 0, (\mu = 1, 2, 3, d_1 = 1)$. Nevertheless this vanishing does take place (due to formulas for non-diagonal components of stress-energy tensor in [93], see also Appendix 2).

### 3.1.2.2. Hyperbolic algebras

Let $\det A < 0$ and

$$
A_{ss'} = 0, -1, -2, \ldots
$$

(3.44)
s \neq s'. Among quasi-Cartan matrices there exists a large subclass of Cartan matrices, corresponding to infinite-dimensional simple hyperbolic generalized Kac-Moody (KM) algebras of ranks \( r = 2, \ldots, 10 \) [171] [172].

For the hyperbolic algebras the following relations are satisfied

\[
\varepsilon_s(U^s, U^s) > 0,
\]

\( s \in S \). This relation is valid, since \( A^{s's'} \leq 0 \), \( s, s' \in S \), for any hyperbolic algebra [170].

For \( (U^s, U^s) > 0 \) we get \( \varepsilon_s > 0 \), \( s \in S \). If \( \theta_{a_s} > 0 \) for all \( s \in S \), then

\[
\varepsilon(I_s) = 1 \text{ for } v_s = e; \quad \varepsilon(I_s) = -\varepsilon[g] \text{ for } v_s = m.
\]

For pseudo-Euclidean metric \( g \) all \( \varepsilon(I_s) = 1 \) and, hence, all \( p \)-branes are Euclidean or should contain even number of time directions: 2, 4, \ldots. For \( \varepsilon[g] = 1 \) only magnetic \( p \)-branes may be pseudo-Euclidean.

**Example 5.** \( \mathcal{F}_3 \) algebra [130]. Now we consider an example of the solution corresponding to the hyperbolic KM algebra \( \mathcal{F}_3 \) with the Cartan matrix

\[
A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix},
\]

\( \mathcal{F}_3 \) is an infinite dimensional Lie algebra generated by the (Serre) relations [171] [172]

\[
[h_i, h_j] = 0, \quad [e_i, f_j] = \delta_{ij} h_j,
\]

\[
[h_i, e_j] = A_{ij} e_j, \quad [h_i, f_j] = -A_{ij} f_j,
\]

\[
(ade_i)^{1-A_{ij}}(e_j) = 0 \quad (i \neq j),
\]

\[
(ad f_i)^{1-A_{ij}}(f_j) = 0 \quad (i \neq j).
\]

\( \mathcal{F}_3 \) contains \( A_1^{(1)} \) affine Kac-Moody subalgebra (it corresponds to the Geroch group) and \( A_2 \) subalgebra.

There exists an example of the solution with the \( A \)-matrix [3.44] for 11-dimensional model governed by the action

\[
S = \int dt d^2z \sqrt{|g|} \left\{ R[g] - \frac{1}{4!}(F^4)^2 - \frac{1}{4!}(F^{4*})^2 \right\},
\]

where \( \text{rank}F^4 = \text{rank}F^{4*} = 4 \). Here \( \Delta = \{4, 4*\} \). We consider a configuration with two magnetic 5-branes corresponding to the form \( F^4 \) and one electric 2-brane corresponding to the form \( F^{4*} \). We denote \( S = \{s_1, s_2, s_3\} \), \( a_{s_1} = a_{s_3} = 4 \), \( a_{s_2} = 4* \) and \( v_{s_1} = v_{s_3} = m, v_{s_2} = e \), where \( d(I_{s_1}) = d(I_{s_3}) = 6 \) and \( d(I_{s_2}) = 3 \). The intersection rules [3.28] read

\[
d(I_{s_1} \cap I_{s_2}) = 0, \quad d(I_{s_2} \cap I_{s_3}) = 1, \quad d(I_{s_1} \cap I_{s_3}) = 4.
\]

For the manifold [2.8] we put \( n = 5 \), \( d_1 = 2 \), \( d_2 = 4 \), \( d_3 = d_4 = 1 \), \( d_5 = 2 \). The corresponding sets for \( p \)-branes are the following: \( I_{s_1} = \{1, 2\} \), \( I_{s_2} = \{4, 5\} \), \( I_{s_3} = \{2, 3, 4\} \).

The corresponding solution reads

\[
g = H^{-12} \left\{ -dt \otimes dt + H^9 g^1 + H^{13} g^2 + H^{14} g^3 + H^{15} g^4 + H^{10} g^5 \right\},
\]

\[
F^4 = \frac{dH}{dt} \left\{ \nu_{s_1} \hat{\tau}_3 \wedge \hat{\tau}_4 \wedge \hat{\tau}_5 + \nu_{s_3} \hat{\tau}_1 \wedge \hat{\tau}_5 \right\},
\]

\[
F^{4*} = \frac{dH}{dt} \frac{\nu_{s_2}}{H^2} dt \wedge \hat{\tau}_3 \wedge \hat{\tau}_5,
\]

where \( \nu_{s_1} = \frac{9}{2}, \nu_{s_2} = 5 \) and \( \nu_{s_3} = 2 \) (see [3.24]).

All metrics \( g^i \) are Ricci-flat \( (i = 1, \ldots, 5) \) with the Euclidean signature (this agrees with relations [3.43] and [2.34]), and \( H = H_1 + H_0 > 0 \), where \( H_1, H_0 \) are constants. The metric [3.53] may be also rewritten using the synchronous time variable \( t_s \)

\[
g = -dt_s \otimes dt_s + f^{3/5} g^1 + f^{-1/5} g^2 + f^{8/5} g^3 + f^{-2/5} g^4 + f^{2/5} g^5,
\]
where \( f = 5ht_s = H^{-5} > 0 \), \( h > 0 \) and \( t_s > 0 \). The metric describes the power-law "inflation" in \( D = 11 \). It is singular for \( t_s \to +0 \). The powers in scale-factors \( f^{2\nu_i} \) do not satisfy Kasner-like relations: \( \sum_{i=1}^{5} d_i\alpha_i = \sum_{i=1}^{5} d_i(\alpha_i)^2 = 1 \). For flat \( g^i \) the calculation of the Riemann tensor squared gives us (see Appendix 1)

\[
\mathcal{K}[g] = 2,1428 t_s^{-4},
\]

where

\[
\mathcal{K}[g] = R_{MNPQ}[g]R^{MNPQ}[g]
\]

is also called the Kretschmann scalar.

**Example 6:** \( H_2(q, q) \) algebra. Let

\[
A = \begin{pmatrix}
2 & -q_1 \\
-q_2 & 2
\end{pmatrix}, \quad q_1q_2 > 4,
\]

\( q_1, q_2 \in \mathbb{N} \). This is the Cartan matrix for the hyperbolic KM algebra \( H_2(q, q) \) \cite{171}. Let us consider \( B_2 \)-model An example of the solution for \( B_2 \)-model with two electric \( p \)-branes \((p = d_1, d_2)\), corresponding to \( F^a \) and \( F^b \) fields and intersecting in time manifold, is the following:

\[
g = H^{-2/(q-2)}g^{0} - H^{2/(q-2)} dt \otimes dt + \hat{g}^1 + \hat{g}^2,
\]

\[
F = \nu_1 dH^{-1} \wedge dt \wedge \hat{\tau}_1 + \nu_2 dH^{-1} \wedge dt \wedge \hat{\tau}_2,
\]

\[
\hat{\varphi} = -(\hat{\lambda}_a + \hat{\lambda}_b)(q - 2)^{-1} \ln H
\]

where \( d_0 = 3, \ d_1 = a - 2, \ a = q + 4, \ b \geq a, \ d_2 = b - 2, \ d_0 = 3, \ D = a + b \). Here \( F = F^a + F^b \) for \( a < b \) and \( F = F^a \) for \( a = b \). The signature restrictions are \( : \varepsilon_1 = \varepsilon_2 = -1 \). Thus, the space-time \((M, g)\) should contain at least three time directions. The minimal \( D \) is 14. For \( D = 14 \) we get \( a = b = 7, \ d_1 = d_2 = 5, \ q = 3 \). In this case \( 6 \cap 6 = 1 \).

**3.1.2.3. Affine Lie algebras.**

We note that affine KM algebras (with \( \det A = 0 \)) do not appear in the solutions \cite{58,10,14}. Indeed, any affine Cartan matrix satisfy the relations

\[
\sum_{s' \in S} a_s' A_{s's} = 0,
\]

with \( a_s > 0 \) called Coxeter labels \cite{172}, \( s \in S \). This relation make impossible the existence of the solution to eq. \cite{38}.

Thus, affine Cartan matrices do not arise in our solutions and hence some configurations are forbidden. Let us consider \( A_1^{(1)} \) affine KM algebra with the Cartan matrix

\[
A = \begin{pmatrix}
2 & -2 \\
-2 & 2
\end{pmatrix}.
\]

For \( D = 11 \) supergravity the intersections: \( 3 \cap 6 = 0, \ 6 \cap 6 = 2 \), corresponding to the \( A \)-matrix \cite{58,68}, are forbidden.

**Remark 6.** In \cite{120} new solutions in the affine case were obtained. These solutions contain as a special case a solution in \( D = 11 \) supergravity from \cite{69} with the intersection \( 6' \cap 6 = 2 \). The solutions from \cite{120} use some modified ansatz for fields of forms (the ansatz for localized branes) and do not belong to scheme under consideration. The solutions of this section in the special case of \( D = 10, 11 \) supergravities are also different from the so-called non-marginal bound state solutions, since the latter have non-trivial Chern-Simons terms (see, for example, \cite{121,122} and references therein), although the rules for binary intersections may look similar.

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3.1.3 Kretschmann scalar, horizon and generalized MP solutions

Let $M_0 = \mathbb{R}^{d_0}$, $d_0 > 2$ and $g^0 = \delta_{\mu\nu} dx^\mu \otimes dx^\nu$. For

$$H_s = 1 + \sum_{b \in X_s} \frac{q_{sb}}{|x-b|^{d_0-2}},$$

(3.66)

where $X_s$ is finite non-empty subset $X_s \subset M_0$, $s \in S$, all $q_{sb} > 0$, and $X_s = X_s'$, $q_{sb} = q_{s'b}$ for

$b \in X_s = X_s'$, $s, s' \in S$, $j = 1, \ldots, k$. The harmonic functions (3.66) are defined in domain $M_0 \setminus X$, $X = \bigcup_{s \in S} X_s$, and generate the solutions (3.59)–(3.64).

Denote $S(b) = \{s \in S : b \in X_s\}$. We also put $M_i = \mathbb{R}^{d_i}$, $R[g^i] = K[g^i] = 0$, $i = 1, \ldots, n$ (see definition (3.59)). Then for the metric (3.10) we obtain

$$K[g] = \frac{C' + o(1)}{U^2|x-b|^4} = \left[C + o(1)|x-b|^{4(d_0-2)}\right](b) \eta(b)$$

(3.67)

for $x \rightarrow b \in X$, where

$$\eta(b) = \sum_{s \in S(b)} (-\varepsilon_s)\nu_s^2 \frac{d(I_s)}{D-2} - \frac{1}{d_0 - 2},$$

(3.68)

and $C = C(b) > 0$ is given in Appendix 1 (see (3.67)). In what follows we consider non-exceptional $b \in X$ defined by relations $C = C(b) > 0$.

Remark 7. It follows from relation (3.67) of Appendix 1 that an exceptional point $b \in X$, defined by relation $C = C(b) = 0$, if and only if

$$U(x) \sim c|x-b|^{-2\alpha}, \quad U(x)U_t(x) \sim c_1,$$

(3.69)

for $x \rightarrow b$, where $\alpha = 0, 2$ and $c, c_i \neq 0$ are constants, $i = 1, \ldots, n$.

Due to (3.67) the metric (3.10) has no curvature singularity when $x \rightarrow b \in X$, $C(b) > 0$, if and only if

$$\eta(b) \geq 0.$$  

(3.70)

From (3.68), we see that the metric (3.10) is regular at a “point” $b \in X$ for $\varepsilon_s = -1$ and large enough values of $\nu_s^2$, $s \in S(b)$. For $\varepsilon_s = +1$, $s \in S(b)$, we have a curvature singularity at non-exceptional point $b \in X$.

Now we consider a special case: $d_1 = 1$, $g^1 = -dt \otimes dt$. In this case we have a horizon w.r.t. time $t$, when $x \rightarrow b \in X$, if and only if

$$\xi_1(b) = \sum_{s \in S(b)} (-\varepsilon_s)\nu_s^2 \delta_{1s} - \frac{1}{d_0 - 2} \geq 0.$$  

(3.71)

This relation follows from the requirement of infinite time propagation of light to $b \in X$. If $\varepsilon_s = -1$, 1 $\in I_s$ for all $s \in S(b)$, we get

$$\eta(b) < \xi_1(b).$$  

(3.72)

$b \in X$. This follows from the inequalities $d(I_s) < D - 2$ ($d_0 > 2$).

We note that $g_{tt} \rightarrow 0$ for $x \rightarrow b \in X$, if (3.72) is satisfied. This follows from the relation

$$g_{tt} \sim \text{const}|x - b|^{2(d_0-2)(\xi_1(b) - \eta(b))},$$

(3.73)

$x \rightarrow b$.

Remark 8. Due to relations (3.69) and (3.73) the point $b \in X$ is non-exceptional if $g^1 = -dt \otimes dt$ and 1 $\in I_s$, $\varepsilon_s = -1$ for all $s \in S(b)$.

Thus, for the metric (3.10) with $H_s$ from (3.66) there are two dimensionless atoms at the non-exceptional point $b \in X$: a) horizon indicator $\xi_1(b)$ (corresponding to time $t$) and b) curvature singularity.
indicator $\eta(b)$. These indicators define (for our assumptions) the existence of a horizon and the singularity of the Kretschmann scalar (when $(M_i, g_i')$ are flat, $i = 1, \ldots, n$) at non-exceptional $b \in X$.

**Generalized MP solutions.** Here we consider special black hole solutions for the model (2.11) with all $\theta_a = 1$, $a \in \Delta$, when the signature of the metric $g$ is $(-1, +1, \ldots, +1)$. We put $\varepsilon_s = \varepsilon(I_s) = -1$ and $1 \in I_s$ for all $s \in S$ and $M_0 = \mathbb{R}^{d_0}$, $g^0 = \sum_{\mu=1}^{d_0} dx^\mu \otimes dx^\mu$, $M_1 = \mathbb{R}$, $g^1 = -dt \otimes dt$. Then, the metric (3.10) reads [117]

$$g = \left( \prod_{s \in S} H_s^{2d(I_s)\nu_s^2} \right)^{1/(D-2)} \left\{ \sum_{\mu=1}^{d_0} dx^\mu \otimes dx^\mu \right.$$ \hspace{1cm} \hfill (3.74) \hspace{1cm} \\
$$\left. - \left( \prod_{s \in S} H_s^{-2\nu_s^2} \right) dt \otimes dt + \sum_i^n \left( \prod_{s \in S} H_s^{-2\nu_s^2\delta(is)} g^i \right) \right\},$$

where $(M_i, g_i')$ are flat Euclidean spaces, $i = 2, \ldots, n$. Here all branes have a common time submanifold $M_1 = \mathbb{R}$, for all $a \in \Delta$, and

$$\eta(b) = \sum_{s \in S(b)} \nu_s^2 \frac{d(I_s)}{D - 2} - \frac{1}{d_0 - 2} \geq 0,$$ \hspace{1cm} \hfill (3.75) \hspace{1cm} \\

$b \in X$. This solution describes a set of extreme $p$-brane black holes with horizons at $b \in X$. The Riemann tensor squared has a finite limit at any $b \in X$.

Calculation of the Hawking "temperature" corresponding to $b \in X$ using standard formula (see, for example, [107, 111]) gives us

$$T_H(b) = 0,$$ \hspace{1cm} \hfill (3.76) \hspace{1cm} \\

for any $b \in X$ satisfying $\xi_1(b) > 0$.

**Example 7: MP solution.** The standard 4-dimensional Majumdar-Papapetrou solution [159] in our notations reads

$$g = H^2 g^0 - H^{-2} dt \otimes dt,$$ \hspace{1cm} \hfill (3.77) \hspace{1cm} \\
$$F = \nu dH^{-1} \wedge dt,$$ \hspace{1cm} \hfill (3.78) \hspace{1cm} \\

where $\nu^2 = 2$, $g^0 = \sum_{i=1}^{3} dx^i \otimes dx^i$ and $H$ is a harmonic function. We have one electric 0-brane (point) "attached" to the time manifold; $d(I_s) = 1$, $\varepsilon_s = -1$ and $(U^*, U^*) = 1/2$. In this case (e.g. for extremal Reissner-Nordström black hole) we get

$$\eta(b) = 0, \quad \xi_1(b) = 1,$$ \hspace{1cm} \hfill (3.79) \hspace{1cm} \\

and $T_H(b) = 0$, $b \in X$.

**Example 8: $D = 11$ supergravity.** In this case there are a lot of $p$-brane MP-type solutions with orthogonal intersection rules e.g. (i) solution with one electric 2-brane $(d(I_s) = 3)$ and $d_0 = 8$; (ii) solution with one magnetic 5-brane $(d(I_s) = 6)$ and $d_0 = 5$; (iii) solution with one electric 2-brane and one magnetic 5-brane $(d(I_{s_1} \cap I_{s_2}) = 2)$ and $d_0 = 4$; (iv) solution with two electric 2-branes $(d(I_{s_1} \cap I_{s_2}) = 1)$ and $d_0 = 5$. In the examples (iii), (iv) the harmonic functions $H_{s_1}$ and $H_{s_2}$ from (3.60) should have the coinciding sets of poles, i.e. $X_{s_1} = X_{s_2}$, to maintain the relation (3.73). The Chern-Simons terms are zero for these solutions. In all these examples $\eta(b) = 0$, $b \in X_s$, and $\nu_s^2 = 1/2$, $s \in S$. In examples (i) and (ii) $M2$ and $M5$ solutions are non-marginal, i.e. they have no internal spaces $M_i$ (margins) that are not occupied by branes. For marginal analogs of these two solutions we get $\eta(b) < 0$ for all $b \in X$. Thus, marginal $M$ branes have singularities at "points" $b$ (that are horizons for $d_0 \geq 4$, since all $\xi_1(b) \geq 0$). Let us compare the situation with $IIA$ supergravity. Performing the calculations for single $FS^-$, $NS5^-$, $D2$- and $D4$-branes in $IIA$ supergravity one get $\eta(b) < 0$ for all $b \in X$ in all (marginal and non-marginal) cases, i.e. all $b$ are singular "points" (horizons for $d_0 \geq 4$).

**Fundamental matrix.** Let

$$N(a, b) \equiv \frac{(n_a - 1)(n_b - 1)}{D - 2} - \lambda_a \cdot \lambda_b,$$ \hspace{1cm} \hfill (3.80) \hspace{1cm} \\

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where $\lambda_a \cdot \lambda_b = \lambda_{ab} \lambda_{\alpha \beta} h^{\alpha \beta}$, $a, b \in \Delta$. The matrix \( \text{(3.30)} \) is called the fundamental matrix of the model \( \text{(2.1)} \). It depends only on basic parameters of the model \( \text{(2.1)} \), i.e. ranks of forms, total dimensions and dilatonic couplings. For $s_1, s_2 \in S$, $s_1 \neq s_2$, the symbol \( \text{(3.21)} \) of orthogonal intersection may be expressed by means of the fundamental matrix \( \text{(3.21)} \)

\[
\Delta(s_1, s_2) = \bar{D} \tilde{\chi}_s \tilde{\chi}_s + \bar{n}_{a,s} \chi_s \chi_s + n_{a,s} \chi_s \tilde{\chi}_s + N(a_s, a_s) \chi_s \chi_s, \tag{3.81}
\]

where $\bar{D} = D - 2$, $\bar{n}_a = n_a - 1$, $\tilde{\chi}_s = \frac{1}{2}(1 - \chi_s)$. More explicitly \( \text{(3.81)} \) reads

\[
\begin{align*}
\Delta(s_1, s_2) &= N(a_s, a_s), \quad v_{s_1} = v_{s_2} = c; \\
\Delta(s_1, s_2) &= \bar{n}_{a,s} - N(a_s, a_s), \quad v_{s_1} = c, \quad v_{s_2} = m; \\
\Delta(s_1, s_2) &= \bar{D} - \bar{n}_{a,s} + N(a_s, a_s), \quad v_{s_1} = v_{s_2} = m.
\end{align*}
\]

This follows from the relations $d(I_s) = \bar{D} \tilde{\chi}_s + \bar{n}_a \chi_s$, equivalent to \( \text{(2.22)} \). Relation \( \text{(3.82)} \) means that $N(a, b)$ defines the dimension of intersection of two electric $p$-branes case corresponding to forms $F^a$ and $F^b$. Let

\[
K(a) \equiv n_a - 1 - N(a, a) = \frac{(n_a - 1)(D - n_a - 1)}{D - 2} + \lambda_a \cdot \lambda_a, \tag{3.85}
\]

$a \in \Delta$. The parameters play a rather important role in supergravitational theories, since they are preserved under Kaluza-Klein reduction \( \text{[52]} \) and define the norms of $U^s$-vectors: $K_s = (U^s, U^s) = K(a_s)$, $s \in S$.

**Intersection rules in $B_D$-models and F-theory.** Let us apply the general relations \( \text{(3.82)}-\text{(3.84)} \) to the $B_D$-model from \( \text{(3.39)} \). $D \geq 12$. The orthogonal intersection rules \( \text{(3.21)} \) read

\[
\begin{align*}
(a - 1)_c \cap (b - 1)_c &= N(a, b), \\
(a - 1)_c \cap (D - 1 - b)_m &= a - 1 - N(a, b), \\
(D - 1 - a)_m \cap (D - 1 - b)_m &= D - a - b + N(a, b),
\end{align*}
\]

\[a, b = 4, \ldots, D - 7,\] where subscripts $e$ and $m$ mark electric and magnetic branes, respectively ($N(a, b)$ is defined in \( \text{(3.11)} \)). For $D = 12$ (in the F-theory case) we get the following intersections \( \text{[43]} \)

\[
d(I \cap J) = 1, \quad \{d(I), d(J)\} = \{3, 3\}, \{3, 4\}, \{3, 6\}, \{3, 7\}, \{4, 4\}, \{4, 6\}, \{4, 7\}, \{6, 6\}, \{6, 7\}, \{7, 7\}. \tag{3.89}
\]

In most models including $D = 11$ supergravity, $B_{12}$ theory, $D < 11$ supergravities \( \text{[52]} \), all $K(a) = 2$ and \( \text{(3.21)} \) has the following form

\[
d(I_s \cap I_s) = \Delta(s_1, s_2) + A_{s_1 s_2}, \tag{3.90}
\]

$s_1 \neq s_2$, and get $A_{s_1 s_2} = A_{s_2 s_1}$, i.e. the Cartan matrix is symmetric. In a finite dimensional case we are led to the so-called simply laced or $A - D - E$ Lie algebras. The intersection rules are totally defined by the corresponding Dynkin diagram: $d(I_s \cap I_s) = \Delta(s_1, s_2) - 1$, when the vertices corresponding to $s_1$ and $s_2$ are connected by a line and $d(I_s \cap I_s) = \Delta(s_1, s_2)$ otherwise (since in $A - D - E$ case $A_{s_1 s_2} = 0, -1, s_1 \neq s_2$).

**Generalization to non-Ricci-flat internal spaces.** In \( \text{[43]} \) we presented a generalization of the solutions \( \text{(3.10)-\text{(3.14)}} \) to the case of non-Ricci-flat space $(M_0, g^0)$, when some additional internal Einstein spaces of non-zero curvature are included.

### 3.2 General Toda-type solutions with harmonic functions

It is well known that geodesics of the target space equipped with some harmonic function on a three-dimensional space generate a solution to the $\sigma$-model equations \( \text{[161, 162]} \). (It was observed in \( \text{[163]} \)
that null geodesics of the target space of stationary five-dimensional Kaluza-Klein theory may be used to
generate multisoliton solutions similar to the Israel-Wilson-Perjès solutions of Einstein-Maxwell theory.
Here we apply this null-geodesic method to our sigma-model and obtain a new class of solutions in
multidimensional gravity with p-branes governed by one harmonic function \( H \). The solutions from this
class correspond to null-geodesics of the target-space metric and are defined by some functions \( f_s(H) = \exp(-q^s(H)) \) with \( q^s(u) \) being solutions to Toda-type equations.

### 3.2.1 Toda-like Lagrangian

Action (3.28) may be also written in the form

\[
S_{\sigma 0} = \frac{1}{2\kappa_5^2} \int d^{10}x \sqrt{|g^0|} \{ R[g^0] - G_{\hat{A}\hat{B}}(X)g^{0\mu\nu} \partial_\mu X^{\hat{A}} \partial_\nu X^{\hat{B}} - 2V \} \quad (3.91)
\]

where \( X = (X^{\hat{A}}) = (\phi^i, \varphi^\alpha, \Phi^s) \in \mathbb{R}^N \), and minisuperspace \( G = G_{\hat{A}\hat{B}}(X)dX^{\hat{A}} \otimes dX^{\hat{B}} \) on minisuperspace \( \mathcal{M} = \mathbb{R}^N \), \( N = n + l + |S| \) (\(|S|\) is the number of elements in \( S \)) is defined by the relation

\[
(G_{\hat{A}\hat{B}}(X)) = \begin{pmatrix}
G_{ij} & 0 & 0 \\
0 & h_{\alpha\beta} & 0 \\
0 & 0 & \varepsilon_s \exp(-2U^s(\sigma)) \delta_{ss'}
\end{pmatrix}. \quad (3.92)
\]

Here we consider exact solutions to field equations corresponding to the action (3.91)

\[
R_{\mu\nu}[g^0] = G_{\hat{A}\hat{B}}(X)\partial_\mu X^{\hat{A}} \partial_\nu X^{\hat{B}} + \frac{2V}{d_0 - 2g^{0\mu\nu}}, \quad (3.93)
\]

\[
\frac{1}{\sqrt{|g^0|}} \partial_\mu [\sqrt{|g^0|}G_{\hat{A}\hat{B}}(X)g^{0\mu\nu} \partial_\nu X^{\hat{B}}] - \frac{1}{2} G_{\hat{A}\hat{B}, \hat{C}}(X)g^{0\mu\nu} \partial_\mu X^{\hat{A}} \partial_\nu X^{\hat{B}} = V_{,\hat{C}}, \quad (3.94)
\]

\( s \in S \). Here \( V_{,\hat{C}} = \partial V/\partial X^{\hat{C}} \).

We put

\[
X^{\hat{A}}(x) = F^{\hat{A}}(H(x)), \quad (3.95)
\]

where \( F : (u_-, u_+) \to \mathbb{R}^N \) is a smooth function, \( H : M_0 \to \mathbb{R} \) is a harmonic function on \( M_0 \) (i.e. \( \Delta[g^0]H = 0 \)), satisfying \( u_- < H(x) < u_+ \) for all \( x \in M_0 \). Let all factor spaces are Ricci-flat and cosmological constant is zero, i.e. relation \( \xi_i = \Lambda = 0 \) is satisfied. In this case the potential is zero : \( V = 0 \). It may be verified that the field equations (3.93) and (3.94) are satisfied identically if \( F = F(u) \) obey the Lagrange equations for the Lagrangian

\[
L = \frac{1}{2} G_{\hat{A}\hat{B}}(F) \dot{F}^{\hat{A}} \dot{F}^{\hat{B}} \quad (3.96)
\]

with the zero-energy constraint

\[
E = \frac{1}{2} G_{\hat{A}\hat{B}}(F) \dot{F}^{\hat{A}} \dot{F}^{\hat{B}} = 0. \quad (3.97)
\]

This means that \( F : (u_-, u_+) \to \mathbb{R}^N \) is a null-geodesic map for the minisupersymmetric \( G \). Thus, we are led to the Lagrange system (3.39) with the minisupersymmetric \( G \) defined in (3.92).

The problem of integrability will be simplified if we integrate the Lagrange equations corresponding to \( \Phi^s \) (i.e. the Maxwell equations for \( s \in S_c \) and Bianchi identities for \( s \in S_m \)):

\[
\frac{d}{du} \left( \exp(-2U^s(\sigma)) \Phi^s \right) = 0 \iff \Phi^s = Q_s \exp(2U^s(\sigma)), \quad (3.98)
\]

where \( Q_s \) are constants, \( s \in S \). Here \( (F^{\hat{A}}) = (\sigma^A, \Phi^s) \). We put \( Q_s \neq 0 \) for all \( s \in S \).
For fixed $Q = (Q_s, s \in S)$ the Lagrange equations for the Lagrangian (3.99) corresponding to $(\sigma^A) = (\phi^i, \varphi^\alpha)$, when equations (3.98) are substituted, are equivalent to the Lagrange equations for the Lagrangian

$$L_Q = \frac{1}{2} \hat{G}_{AB} \dot{\sigma}^A \dot{\sigma}^B - V_Q,$$

where

$$V_Q = \frac{1}{2} \sum_{s \in S} \varepsilon_s Q^2_s \exp[2U^s(\sigma)],$$

the matrix $(\hat{G}_{AB})$ is defined in (3.98). The zero-energy constraint (3.97) reads

$$E_Q = \frac{1}{2} \hat{G}_{AB} \dot{\sigma}^A \dot{\sigma}^B + V_Q = 0.$$ (3.101)

### 3.2.2 Toda-type solutions

Here we are interested in exact solutions for a special case when the vectors $U^s$ have non-zero length, i.e. $K_s = (U^s, U^s) \neq 0$, for all $s \in S$, and the quasi-Cartan matrix (3.27) is a non-degenerate one. Here some ordering in $S$ is assumed. It follows from the non-degeneracy of the matrix (3.27) that the vectors $U^s, s \in S$, are linearly independent. Hence, the number of the vectors $U^s$ should not exceed the dimension of $\mathbb{R}^{n+l}$, i.e. $|S| \leq n + l$.

The exact solutions could be readily obtained using the relations from Appendix 4 [140],

$$g = \left( \prod_{s \in S} f_s^{2d(I_s) \varepsilon_s/(D-2)} \right) \left\{ \exp(2c^0 H + 2c^0) \hat{g}^0 + \sum_{i=1}^n \left( \prod_{s \in S} f_s^{-2h_i \delta_i(s)} \right) \exp(2c^i H + 2c^i) \hat{g}^i \right\},$$

$$\exp(\varphi^\alpha) = \left( \prod_{s \in S} f_s^{h_i \chi_i \lambda_s^\alpha} \right) \exp(c^\alpha H + \bar{c}^\alpha),$$

where $s = 1, \ldots, l$ and $F^s = \sum_{s \in S} F_s \delta^s_{a_s}$ with

$$F_s = Q_s \left( \prod_{s' \in S \setminus S} f_s^{-A_{ss'}} \right) dH \cap \tau(I_s), \quad s \in S_e,$$

$$F_s = Q_s (s_0 dH) \cap \tau(I_s), \quad s \in S_m,$$

where $s_0 = \ast[g^0]$ is the Hodge operator on $(M_0, g^0)$. Here

$$f_s = f_s(H) = \exp(-q^s(H)),$$

where $q^s(u)$ is a solution to Toda-like equations

$$\dot{q}^s = -B_s \exp(\sum_{s' \in S} A_{ss'} q^{s'}),$$

with $B_s = K_s \varepsilon_s Q^2_s$, $s \in S$, and $H = H(x)$ ($x \in M_0$) is a harmonic function on $(M_0, g^0)$. Vectors $c^1 = (c^A)$ and $\bar{c} = (\bar{c}^A)$ satisfy the linear constraints (see Appendix 4)

$$U^s(c) = \sum_{i \in I_s} d_i c_i - \chi_s \lambda_{a,\alpha} c^\alpha = 0, \quad U^s(\bar{c}) = 0,$$

$s \in S$, and

$$c^0 = \frac{1}{2 - d_0} \sum_{j=1}^n d_j c_j^j, \quad \bar{c}^0 = \frac{1}{2 - d_0} \sum_{j=1}^n d_j \bar{c}^j.$$ (3.109)
The zero-energy constraint reads (see Appendix 4)

$$2E_{TL} + h_{\alpha\beta}c^\alpha c^\beta + \frac{n}{d_0 - 2} \left( \sum_{i=1}^{n} d_i c^i \right)^2 = 0,$$  \hspace{1cm} (3.110)

where

$$E_{TL} = \frac{1}{4} \sum_{s,s' \in S} h_s A_{ss'} q^s q^{s'} + \sum_{s \in S} A_s \exp(\sum_{s' \in S} A_{ss'} q^{s'}),$$  \hspace{1cm} (3.111)

is an integration constant (energy) for the solutions from $\text{(3.107)}$ and $A_s = \frac{1}{2} \varepsilon_s Q_s^2$.

We note that the equations $\text{(3.107)}$ correspond to the Lagrangian

$$L_{TL} = \frac{1}{4} \sum_{s,s' \in S} h_s A_{ss'} q^s q^{s'} - \sum_{s \in S} A_s \exp(\sum_{s' \in S} A_{ss'} q^{s'}),$$  \hspace{1cm} (3.112)

where $h_s = K_s^{-1}$.

Thus the solution is presented by relations $\text{(3.102)}-\text{(3.106)}$ with the functions $q^s$ defined in $\text{(3.107)}$ and the relations on the parameters of solutions $c^\lambda$, $\bar{c}^\lambda$ ($A = i, \alpha, 0$), imposed in $\text{(3.108)}$, $\text{(3.109)}$, $\text{(3.110)}$.

### 3.2.3 Solutions corresponding to $A_m$ Toda chain

Here we consider exact solutions to Toda-chain equations $\text{(3.107)}$ corresponding to the Lie algebra $A_m = \text{sl}(m + 1, \mathbb{C})$ $\text{(173)}$ $\text{(178)}$, $(m \geq 1)$ where

$$(A_{ss'}) = \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & 0 \\ 0 & -1 & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2 & -1 \\ 0 & 0 & 0 & \ldots & -1 & 2 \end{pmatrix} \hspace{1cm} (3.113)$$

is the Cartan matrix of the Lie algebra $A_m$ and $B_s > 0$, $s, s' = 1, \ldots, m$. Here we put $S = \{1, \ldots, m\}$.

The equations of motion $\text{(3.107)}$ correspond to the Lagrangian

$$L_T = \frac{1}{2} \sum_{s,s' = 1}^{m} A_{ss'} q^s q^{s'} - \sum_{s = 1}^{m} B_s \exp(\sum_{s' = 1}^{m} A_{ss'} q^{s'}) \hspace{1cm} (3.114)$$

This Lagrangian may be obtained from the standard one $\text{(173)}$ by separating a coordinate describing the motion of the center of mass.

Using the result of A. Anderson $\text{(178)}$ we present the solution to eqs. $\text{(3.107)}$ in the following form

$$C_s \exp(-q^s(u)) = \sum_{r_1 < \ldots < r_s}^{m+1} v_{r_1} \ldots v_{r_s} \Delta^2(w_{r_1}, \ldots, w_{r_s}) \exp[(w_{r_1} + \ldots + w_{r_s})u] \hspace{1cm} (3.115)$$

$s = 1, \ldots, m$, where $\Delta(w_{r_1}, \ldots, w_{r_s}) = \prod_{i<j}^{s} (w_{r_i} - w_{r_j})$; $\Delta(w_{r_1}) \equiv 1$, denotes the Vandermonde determinant. The real constants $v_r$ and $w_r$, $r = 1, \ldots, m + 1$, obey the relations

$$\prod_{r=1}^{m+1} v_r = \Delta^{-2}(w_1, \ldots, w_{m+1}), \quad \sum_{r=1}^{m+1} w_r = 0. \hspace{1cm} (3.116)$$

In $\text{(3.115)}$ $C_s = \prod_{s' = 1}^{m} B_{s'}^{-A_{ss'}}$, where $(A_{ss'}) = (A_{ss'})^{-1}$ is presented in $\text{(7.50)}$ of Appendix 3. Here $v_r \neq 0$, $w_r \neq w_{r'}$, $r \neq r'$, $r, r' = 1, \ldots, m + 1$. We note that the solution with $B_s > 0$ may be obtained from the solution with $B_s = 1$ (see $\text{(178)}$) by a certain shift $q^s \mapsto q^s + \delta^s$. 


The energy reads [178]

\[ E_T = \frac{1}{2} \sum_{s,s'=1}^{m} A_{ss'} \dot{q}^s \dot{q}^{s'} + \sum_{s=1}^{m} B_s \exp \left( \sum_{s'=1}^{m} A_{ss'} q^{s'} \right) = \frac{1}{2} \sum_{r=1}^{m+1} w_r^2. \] (3.117)

If \( B_s > 0, s \in S \), then all \( w_r, v_r \) are real and, moreover, all \( v_r > 0, r = 1, \ldots, m + 1 \). In a general case \( B_s \neq 0, s \in S \), relations (3.115) also describe real solutions to eqs. (3.107) for suitably chosen complex parameters \( v_r \) and \( w_r \). The parameters \( w_i \) are either real or belong to pairs of complex conjugate (non-equal) numbers, i.e., for example, \( w_1 = \bar{w}_2 \). When some of \( B_s \) are negative, there are also some special (degenerate) solutions to eqs. (3.107) that are not described by relations (3.115) but may be obtained from the latter by certain limits of parameters \( w_i \).

For the energy (3.111) we get \( E_{TL} = \frac{1}{2K} E_T \). Here \( K_s = K \). In the \( A_m \) Toda chain case eqs. (3.115) should be substituted into relations (3.102)-(3.104).

To our knowledge \( p \)-brane solutions governed by open Toda lattices with \( A_n \) Lie algebras were studied first in [105, 79]. In [106] open Toda lattices in maximal supergravities in \( D \) dimensions coming from \( D = 11 \) supergravity were considered. The appearance of \( A - D - E \) algebras is rather typical for supergravitational models (since all \( K_s = 2 \) and hence the roots of the Lie algebra have equal lengths).
4 Classical and quantum cosmological-type solutions

Here we consider the case $d_0 = 1$, $M_0 = \mathbb{R}$, i.e. we are interested in applications to the sector with one-variable dependence. We consider the manifold

$$M = (u_-, u_+) \times M_1 \times \ldots \times M_n$$

with the metric

$$g = we^{2\gamma(u)}du \otimes du + \sum_{i=1}^{n} e^{2\phi_i(u)}g^i,$$

where $w = \pm 1$, $u$ is a distinguished coordinate which, by convention, will be called “time”; $(M_i, g^i)$ are oriented and connected Einstein spaces (see (2.10)), $i = 1, \ldots, n$. The functions $\gamma, \phi^i$: $(u_-, u_+) \to \mathbb{R}$ are smooth.

Here we adopt the p-brane ansatz from Sect. 2. putting $g^0 = wdu \otimes du$.

4.1 Lagrange dynamics

It follows from Subsect. 2.3 that the equations of motion and the Bianchi identities for the field configuration under consideration (with the restrictions from Sect. 2.3.1 imposed) are equivalent to equations of motion for 1-dimensional variational under consideration (with the restrictions from Sect. 2.3.1 imposed) are equivalent to equations of motion for 1-dimensional $\sigma$-model with the action

$$S_\sigma = \frac{\mu}{2} \int du N \left\{ G_{ij}\dot{\phi}^i \dot{\phi}^j + h_{\alpha\beta}\phi^\alpha \phi^\beta + \sum_{s \in S} \varepsilon_s \exp[-2U^s(\phi, \varphi)](\Phi^s)^2 - 2N^{-2}V_w(\phi) \right\},$$

where $\dot{x} \equiv dx/du$,

$$V_w = -wV = -w\Lambda e^{2\gamma_0(\phi)} + \frac{\mu}{2} \sum_{i=1}^{n} \xi_i d_i e^{-2\phi^i + 2\gamma_0(\phi)}$$

is the potential with $\gamma_0(\phi) \equiv \sum_{i=1}^{n} d_i \phi^i$, and $N = \exp(\gamma_0 - \gamma) > 0$ is the lapse function, $U^s = U^s(\phi, \varphi)$ are defined in (2.34), $\varepsilon_s$ are defined in (2.31) for $s = (a_s, v_s, I_s) \in S$, and $G_{ij} = d_i \delta_{ij} - d_j$ are components of the "pure cosmological" minisuperspace metric, $i, j = 1, \ldots, n$.

In the electric case ($F^{a,m,l} = 0$) for finite internal space volumes $V_i$ the action (4.4) coincides with the action (4.3) if $\mu = -w/\kappa^2$, $\kappa^2 = \kappa_0^2 V_1 \ldots V_n$.

Action (4.4) may be also written in the form

$$S_\sigma = \frac{\mu}{2} \int du N \left\{ G_{AB}(X)\dot{X}^A \dot{X}^B - 2N^{-2}V_w \right\},$$

where $X = (X^A) = (\phi^i, \varphi^a, \Phi^s) \in \mathbb{R}^N$, $N = n + l + |S|$, and minisuperspace metric $G$ is defined in (3.92).

Scalar products. The minisuperspace metric (3.92) may be also written in the form $G = \hat{G} + \sum_{s \in S} \varepsilon_s e^{-2U^s(\sigma)}d\Phi^s \otimes d\Phi^s$, where $\sigma = (\sigma^A) = (\phi^i, \varphi^a)$,

$$\hat{G} = \hat{G}_{AB}d\sigma^A \otimes d\sigma^B = G_{ij}d\phi^i \otimes d\phi^j + h_{\alpha\beta}d\phi^\alpha \otimes d\phi^\beta,$$

is truncated minisuperspace metric and $U^s(\sigma) = U^s_A(\sigma)$ is defined in (2.32). The potential (4.4) reads

$$V_w = -(w\Lambda)e^{2U^\Lambda(\sigma)} + \sum_{j=1}^{n} \frac{\mu}{2} \xi_j d_j e^{2U^j(\sigma)},$$

where

$$U^j(\sigma) = U^j_A(\sigma) = -\delta^j_i + \gamma_0(\phi), \quad (U^j_A) = (-\delta^j_i + d_i, 0),$$

$$U^\Lambda(\sigma) = U^\Lambda_A(\sigma) = \gamma_0(\phi), \quad (U^\Lambda_A) = (d_i, 0).$$
The integrability of the Lagrange system (4.5) crucially depends upon the scalar products of co-vectors $U^\Lambda, U^i, U^s$ (see (3.1)). These products are defined by (3.4) and the following relations

\[(U^i, U^j) = \delta_{ij}, \quad (U^\Lambda, U^\Lambda) = \frac{D - 1}{D - 2},\]
\[(U^s, U^i) = -\delta_{iisl}, \quad (U^s, U^\Lambda) = \frac{d(I_s)}{2 - D},\]

where $s = (a_s, v_s, I_s) \in S$; $i, j = 1, \ldots, n$.

**Toda-like representation.** We put $\gamma = \gamma_0(\phi)$, i.e. the harmonic time gauge is considered. Integrating the Lagrange equations corresponding to $\Phi^s$ (see (3.98)) we are led to the Lagrangian from (3.99) and the zero-energy constraint (3.101) with the modified potential

\[V_Q = V_w + \frac{1}{2} \sum_{s \in S} \varepsilon_s Q^2_s \exp[2U^s(\sigma)],\]

where $V_w$ is defined in (4.4).

### 4.2 Classical solutions with $\Lambda = 0$

Here we consider classical solutions with $\Lambda = 0$.

#### 4.2.1 Solutions with Ricci-flat spaces

Let all spaces be Ricci-flat, i.e. $\xi_1 = \ldots = \xi_n = 0$.

Since $H(u) = u$ is a harmonic function on $(M_0, g^0)$ with $g^0 = wdu \otimes du$ we get for the metric and scalar fields from (3.102), (3.103)

\[g = \left( \prod_{s \in S} f_s^2 d(I_s)^{h_s/(D-2)} \right) \left\{ \exp(2c^0 u + 2c^0) wdu \otimes du \right\} \]
\[+ \sum_{i=1}^n \left( \prod_{s \in S} f_s^{-2h_s \delta_{is}} \right) \exp(2c^i u + 2c^i) g^i, \]
\[\exp(\varphi^\alpha) = \left( \prod_{s \in S} f_s^{h_s \chi_s^\alpha I_s^2} \right) \exp(c^\alpha u + \bar{c}^\alpha),\]

where $f_s = f_s(u) = \exp(-q^s(u))$ and $q^s(u)$ obey Toda-like equations (3.107).

Relations (3.105) and (3.110) take the form

\[c^0 = \sum_{j=1}^n d_j c^i, \quad \bar{c}^0 = \sum_{j=1}^n d_j \bar{c}^i, \]
\[2E_{TL} + h_{\alpha\beta} c^\alpha c^\beta + \sum_{i=1}^n d_i (c^i)^2 - \left( \sum_{i=1}^n d_i c^i \right)^2 = 0,\]

with $E_{TL}$ from (3.111) and all other relations (e.g. constraints (3.108) and relations for forms (3.104) and (3.110) with $H = u$) are unchanged. In a special $A_m$ Toda chain case this solution was considered previously in [137].

#### 4.2.2 Solutions with one curved space

The cosmological solution with Ricci-flat spaces may be also modified to the following case: $\xi_1 \neq 0, \xi_2 = \ldots = \xi_n = 0$, i.e. one space is curved and others are Ricci-flat and $\xi \notin I_s, s \in S$, i.e. all “brane” submanifolds do not contain $M_1$.
The potential (3.100) is modified for \( \xi_1 \neq 0 \) as follows (see (4.13))

\[
V_Q = \frac{1}{2} \sum_{s \in S} \varepsilon_s Q^2_s \exp[2U^*(\sigma)] + \frac{1}{2} w_1 d_1 \exp[2U^1(\sigma)],
\]

(4.18)

where \( U^1(\sigma) \) is defined in (4.3) \((d_1 > 1)\).

For the scalar products we get from (4.10) and (4.12)

\[
(U^1, U^1) = \frac{1}{d_1} - 1 < 0, \quad (U^1, U^s) = 0
\]

(4.19)

for all \( s \in S \).

The solution in the case under consideration may be obtained by a little modification of the solution from the previous section (using (4.10), relations \( U^{11} = -\delta_i / d_1, U^{10} = 0 \) and Appendix 4). We get (4.10)

\[
g = \left( \prod_{s \in S} [f_s(u)]^{2d(I_s)h_s/(D-2)} \right) \left\{ [f_1(u)]^{2d_1/(1-d_1)} \exp(2c^1u + 2\bar{c}^1) \right\}
\]

\[\times [wdu \otimes du + f_1^2(u)\bar{g}^1] + \sum_{i=2}^n \left( \prod_{s \in S} [f_s(u)]^{-2h_i\delta_i s} \right) \exp(2c_i u + 2\bar{c}_i) \bar{g}^1 \right\},
\]

\[
\exp(\varphi^w) = \left( \prod_{s \in S} f_s^{h_s, \lambda_s, \lambda_s^*} \right) \exp(c^a u + \bar{c}^a),
\]

(4.21)

and \( F^a = \sum_{s \in S} s^a, F^s \) with forms

\[
F^s = Q_s \left( \prod_{s' \in S} f_{s'}^{-A_{s,s'}} \right) du \wedge \tau(I_s), \quad s \in S_c,
\]

(4.22)

\[
F^s = Q_s \tau(\bar{I}_s), \quad s \in S_m
\]

Q_s \neq 0, s \in S. Here \( f_s = f_s(u) = \exp(-q^s(u)) \) where \( q^s(u) \) obeys Toda-like equations (3.108) and

\[
f_1(u) = R \sinh(\sqrt{C_1}(u - u_1)), \quad C_1 > 0, \quad \xi_1 w > 0;
\]

(4.24)

\[
R \sin(\sqrt{C_1}(u - u_1)), \quad C_1 < 0, \quad \xi_1 w > 0;
\]

(4.25)

\[
R \sinh(\sqrt{C_1}(u - u_1)), \quad C_1 > 0, \quad \xi_1 w < 0;
\]

(4.26)

\[
\left| \xi_1 (d_1 - 1) \right|^{1/2}, \quad C_1 = 0, \quad \xi_1 w > 0,
\]

(4.27)

\[u_1, C_1 \text{ are constants and } R = |\xi_1 (d_1 - 1) / C_1|^{1/2}.
\]

Vectors \( c = (c^A) \) and \( \bar{c} = (\bar{c}^A) \) satisfy the linear constraints

\[
U^r(c) = U^r(\bar{c}) = 0, \quad r = s, 1,
\]

(4.28)

(for \( r = s \) see 3.108) and the zero-energy constraint

\[
C_1 \frac{d_1}{d_1 - 1} = 2E_{TL} + h_{a\beta} c^\alpha \bar{c}^\beta + \sum_{i=2}^n d_i (c^i)^2 + \frac{1}{d_1 - 1} \left( \sum_{i=2}^n d_i c^i \right)^2.
\]

(4.29)

Restriction 1 (see Subsect. 2.3.1) forbids certain intersections of two p-branes with the same color index for \( n_1 \geq 2 \). Restriction 2 is satisfied identically in this case.

This solution in a special case of \( A_{m} \) Toda chain was obtained earlier in [133] (see also [137]). Some special configurations were considered earlier in [105][79][106].
4.2.3 Block-orthogonal solutions

Let us consider block-orthogonal case: \( \Theta_{\alpha}^{\dagger}, \Theta_{\alpha} \). In this case due to Appendix 4 we get \( f_s = \tilde{f}_s^{b_s} \) where \( b_s = 2 \sum_{s' \in S} A^{s' s'} \), \( (A^{s' s'})^{-1} \) and

\[
\tilde{f}_s(u) = R_s \sinh(\sqrt{C_s}(u - u_s)), \quad C_s > 0, \quad \eta_s \varepsilon_s < 0; \\
R_s \sin(\sqrt{C_s}(u - u_s)), \quad C_s < 0, \quad \eta_s \varepsilon_s < 0; \\
R_s \cosh(\sqrt{C_s}(u - u_s)), \quad C_s > 0, \quad \eta_s \varepsilon_s > 0;
\]

where \( R_s = |Q_s|/(|\nu_s||C_s|^{1/2}), \eta_s \nu_s^2 = b_s \bar{h}_s, \eta_s = \pm 1, C_s, u_s \) are constants, \( s \in S \). The constants \( C_s, u_s \) are coinciding inside blocks: \( u_s = u_{s'}, C_s = C_{s'}, s, s' \in S_i, i = 1, \ldots, k \) (see Appendix 4). The ratios \( \varepsilon_s Q_s^2/(b_s h_s) \) are also coinciding inside blocks, or, equivalently,

\[
\varepsilon_s Q_s^2 = \frac{\varepsilon_s Q_s^2}{b_s h_s},
\]

\( s, s' \in S_i, i = 1, \ldots, k. \)

Here

\[
E_{TL} = \sum_{s \in S} C_s \eta_s \nu_s^2.
\]

The solution with block-orthogonal set of vectors was obtained in (for non-composite case see also ). In the special orthogonal case when: \( |S_1| = \ldots = |S_k| = 1 \), the solution was obtained in . In non-composite case “orthogonal” solutions were considered in (electric case) and (electromagnetic case). For \( n = 1 \) see also .

4.3 Classical solutions with \( \Lambda \neq 0 \) on product of Einstein spaces

Here we describe an important class of classical solutions appearing when all scale factors are constant (Freund-Rubin-type solutions). The solutions with anti-de-Sitter spaces appear in the “near-horizon” limit of extremely charged \( p \)-brane configurations from Sect. 3.

We note, that recently an interest to Freund-Rubin-type solutions in multidimensional models with \( p \)-branes living on product of Einstein spaces appeared (see, for example, ). This interest was inspired by papers devoted to duality between certain limit of some superconformal theory in \( d \)-dimensional space and string or M-theory compactified on the space \( AdS_{d+1} \times W \), where \( W \) is a compact manifold (e.g. sphere) (see also etc.). It was shown in that the solutions in \( D = 10, 11 \) supergravities representing \( D3, M2, M5 \) branes interpolate between flat-space vacuum and compactifications to AdS space.

Here we consider a rather general class of solutions with spontaneous compactification for the model defined on the manifold \( M_0 \times M_1 \times \ldots \times M_n \), with the metric

\[
g = g_0 + g_1 + \ldots + g_n,
\]

where \( g^i \) is an Einstein metric on \( M_i \) satisfying the equation \( \Box q^i \), \( i = 0, \ldots, n \).

We note that in the pure gravitational model with cosmological constant \( \Lambda \) the equations of motion \( \text{Ric}[g] = \frac{2\Lambda}{D-2}g \) have a rather simple solution

\[
\xi_i = \frac{2\Lambda}{D-2},
\]

\( i = 0, \ldots, n \).

The solution is given by the relation and (below). In the non-composite case the “cosmological derivation” of the solution was obtained in . The Freund-Rubin solutions
in $D = 11$ supergravity: $AdS_4 \times S^7$ and $AdS_7 \times S^4$, correspond to M2-brane "living" on $AdS_4$ and $S^4$ respectively. The "popular" $AdS_5 \times S^5$ solution in $11B$ ($D = 10$) supergravity model [78] corresponds to composite self-dual configuration with two "branes" living on $AdS_5$ and $S^5$ and corresponding to 5-form.

We consider the model governed by the action (2.1). The equations of motion were presented in (2.2) - (2.4). The Hilbert-Einstein equations (2.2) may be written in the equivalent form (4.43). Here we keep the multi-index notations with $\Omega = \Omega^a$ being a set of all non-empty subsets of $\{0, \ldots, n\}$.

"Forbidden" due to relation (4.38) may be rewritten in the equivalent form (4.43). Here we consider some examples of the obtained solutions when $\varepsilon = 1$ and all $\varepsilon_i = 1$, $i = 1, \ldots, n$, i.e. "our space" $(M_0, g^0)$ is pseudo-Euclidean space and the "internal spaces" $(M_i, g^i)$ are Euclidean ones. We also put $\theta_a = 1$ and $n_a < D - 1$ for all $a \in \Delta$.

Due to relations from the Appendix 2 this restriction guarantees the block-diagonal structure of the energy-momentum tensor. We note that due to (4.38) the self-consistency condition should be satisfied $d(I ) = n_a$, for all $I \in \Omega_a$, $a \in \Delta$.

A simple example: for $D = 11$ supergravity with $n_a = 4$ the $p$-brane intersection $d(I_1 \cap I_2) = 3$ is "forbidden" due to Restriction 3.

The solution mentioned above may be obtained by a straightforward substitution of the fields (4.36), (4.37), (4.39) into equations of motion (2.2)-(2.4) while formulas from the Appendix 2 are keeping in mind. Eq. (2.2) should be written in the form

$$R_{MN} = Z_{MN} + \frac{2\Lambda}{D - 2} g_{MN},$$

where $Z_{MN} \equiv T_{MN} - \frac{T}{D - 2} g_{MN}$, and $T = T^M_M$.

**Electro-magnetic representation.** Due to relation

$$\ast \tau(I ) = \varepsilon(I ) \delta(I ) \tau(I ),$$

where $\ast = *[g]$ is the Hodge operator on $(M, g)$, $I = \{0, \ldots, n\} \setminus I$ is "dual" set and $\delta(I ) = \pm 1$ is defined by relation $\tau(I ) \wedge \tau(\bar{I}) = \delta(I ) \tau(\{0, \ldots, n\})$, the "electric brane" living on $M_I$ (see (2.1)) may be interpreted also as a "magnetic brane" living on $M_{\bar{I}}$. The relation (4.38) may be rewritten in the "electromagnetic" form

$$F^a = \sum_{I \in \Omega_a} Q_{aI} \tau(I ) + \sum_{J \in \Omega_a^\ast} Q_{aJ} \ast \tau(J),$$

where $\Omega_a = \Omega^{ae} \cup \Omega_{am}^\ast$, $\Omega^{ae} \cap \Omega_{am}^\ast = \emptyset$, $\Omega^\ast \equiv \{J | J = \bar{I}, I \in \Omega\}$, and $Q_{aI} = Q_{aI}$ for $I \in \Omega^{ae}$ and $Q_{aJ} = Q_{aJ}$ for $J \in \Omega_{am}^\ast$.

Here we consider some examples of the obtained solutions when $\varepsilon_0 = -1$ and all $\varepsilon_i = 1$, $i = 1, \ldots, n$, i.e. "our space" $(M_0, g^0)$ is pseudo-Euclidean space and the "internal spaces" $(M_i, g^i)$ are Euclidean ones. We also put $\theta_a = 1$ and $n_a < D - 1$ for all $a \in \Delta$. 

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Solution with one $p$-brane. Let $\Omega_a = \{I\}$, $\lambda_a = 0$ for some $a \in \Delta$ and $\Omega_b$ are empty for all $b \neq a$, $b \in \Delta$.

Equations (4.10) are satisfied identically in this case and (4.11) read

$$
\xi_i = \frac{2\Lambda}{D-2} + \varepsilon(I)Q^2\left[\delta_i^r - \frac{n_a - 1}{D-2}\right],
$$

(4.46)

$i = 0, \ldots, n$, where $Q = Q_{aI}$.

$p$-brane does not “live” in $M_0$. For $I = \{1, \ldots, k\}$, $1 \leq k \leq n$, we get $\varepsilon(I) = 1$ and $\xi_0 = \xi_{k+1} = \ldots = \xi_n = \lambda - Q^2 r_a$, $\xi_1 = \ldots = \xi_k = \lambda + Q^2(1 - r_a)$, where $\lambda = 2\Lambda/(D-2)$, $r_a = (n_a - 1)/(D-2)$.

For $\Lambda = 0$, $Q \neq 0$ we get $\xi_0 = \xi_{k+1} = \ldots = \xi_n < 0$ and $\xi_1 = \ldots = \xi_k > 0$. These solutions contain the solutions with the manifold

$$
M = AdS_{d_0} \times S^{d_1} \times \ldots \times S^{d_k} \times H^{d_{k+1}} \times \ldots \times M_n.
$$

(4.47)

Here $H^d$ is $d$-dimensional Lobachevsky space; $M_n = H^d_{\alpha}$ for $k < n$ and $M_n = S^{d_\alpha}$ for $k = n$.

For $2\Lambda = Q^2(n_n - 1)$ we get a solution with a flat “our” space: $M = \mathbb{R}^{d_0} \times S^{d_1} \times \ldots \times S^{d_k} \times \mathbb{R}^{d_{k+1}} \times \ldots$. One may consider the fine-tuning of the cosmological constant, when $\Lambda$ and $Q^2$ are of the Planck order but $\xi_0$ is small enough in agreement with observational data.

$p$-brane “lives” in $M_0$. For $I = \{0, \ldots, k\}$, $0 \leq k \leq n$, we get $\varepsilon(I) = -1$ and $\xi_{k+1} = \ldots = \xi_n = \lambda + Q^2 r_a$, $\xi_0 = \ldots = \xi_k = \lambda - Q^2(1 - r_a)$.

For $\Lambda = 0$, $Q \neq 0$ we get $\xi_{k+1} = \ldots = \xi_n > 0$ and $\xi_0 = \ldots = \xi_k < 0$. The solutions contain the solutions with the manifold

$$
M = AdS_{d_0} \times H^{d_1} \times \ldots \times H^{d_k} \times S^{d_{k+1}} \times \ldots \times M_n.
$$

(4.48)

Here $M_n = S^{d_\alpha}$ for $k < n$ and $M_n = H^{d_{\alpha}}$ for $k = n$.

For $2\Lambda = Q^2(D - n_n - 1)$ we get a solution with a flat our space: $M = \mathbb{R}^{d_0} \times S^{d_1} \times \ldots \times S^{d_k} \times \mathbb{R}^{d_{k+1}} \times \ldots$. We may also consider the fine-tuning mechanism here.

Solution with two $p$-branes. Let $n = 1$, $d_0 = d_1 = n_0 = d$, $\Omega_a = \{I_0 = \{0\}, I_1 = \{1\}\}$, for some $a$ and other $\Omega_b$ are empty. Denote $Q_0 = Q_{aI_0}$ and $Q_1 = Q_{aI_1}$. For the field of form we get from (4.38)

$$
F^a = Q_0 \tilde{\tau}_0 + Q_1 \tilde{\tau}_1.
$$

(4.49)

When $\lambda_a \neq 0$ the equations (4.10) are satisfied if and only if $Q_0^2 = Q_1^2 = Q^2$. Relations (4.11) read $\xi_0 = \lambda - Q^2 e^{2\lambda_0(\phi)}$, $\xi_1 = \lambda + Q^2 e^{2\lambda_0(\phi)}$.

For $\Lambda = 0$ and $Q \neq 0$ we get $\xi_{0} - \xi_{k+1} = \ldots = \xi_n < 0$ and $\xi_1 = \ldots = \xi_k > 0$. The solutions contain the solutions with the manifold

$$
M = AdS_2 \times S^2 \times M_2 \times M_3.
$$

(4.52)

Here $g[AdS_2] = R^{-2}[dR \times dR - dt \times dt]$ is the metric on $AdS_2$, $(M_1, g^i)$ are Ricci-flat, $i = 2, 3$, $\varepsilon_2 = +1$, $d_2 = 2$, $d_3 = 5$ and $\nu^2_1 = \nu^2_2 = 1$.

The solutions (4.50)-(4.52) may be generalized to $B_2$-models in dimension $D \geq 12$ (see (3.31)) with rank $F^a \in \{4, \ldots, D - 7\}$, $d_2 = a - 2, d_3 = D - 2 - a$ and all scalar fields are zero. For $M_2 = \mathbb{R}^2$ and $M_3 = \mathbb{R}^2 \times M_4$, the metric (4.50) may be obtained also for the solution with two M2 branes and two M5 branes (3.31).
4.4 Quantum solutions.

4.4.1 Wheeler–De Witt equation.

Here we fix the gauge as follows

\[ \gamma_0 - \gamma = f(X), \quad N = e^f, \]  

(4.53)

where \( f: \mathcal{M} \rightarrow \mathbb{R} \) is a smooth function. Then we obtain the Lagrange system with the Lagrangian

\[ L_f = \frac{\mu}{2}e^f \mathcal{G}_{\dot{A}\dot{B}}(X) \dot{X}^\dot{A} \dot{X}^\dot{B} - \mu e^{-f} V_w \]  

(4.54)

and the energy constraint

\[ E_f = \frac{\mu}{2}e^f \mathcal{G}_{\dot{A}\dot{B}}(X) \dot{X}^\dot{A} \dot{X}^\dot{B} + \mu e^{-f} V_w = 0. \]  

(4.55)

Using the standard prescriptions of (covariant and conformally covariant) quantization (see, for example, [179] [180] [181]) we are led to the Wheeler-DeWitt (WDW) equation

\[ \hat{H}^f \Psi^f \equiv \left(-\frac{1}{2\mu} \Delta [e^f \mathcal{G}] + \frac{\alpha}{\mu} R [e^f \mathcal{G}] + e^{-f} \mu V_w \right) \Psi^f = 0, \]  

(4.56)

where \( \alpha = a_N = (N - 2)/(N - 1), \)

\( N = n + l + |S|. \)

Here \( \Psi^f = \Psi^f(X) \) is the so-called “wave function of the universe” corresponding to the \( f \)-gauge and satisfying the relation

\[ \Psi^f = e^{bf} \Psi^f = 0, \quad b = (2 - N)/2, \]  

(4.58)

(\( \Delta[\mathcal{G}] \) and \( R[\mathcal{G}] \) denote the Laplace-Beltrami operator and the scalar curvature corresponding to \( \mathcal{G} \), respectively).

Harmonic-time gauge The WDW equation (4.56) for \( f = 0 \)

\[ \hat{H} \Psi \equiv \left(-\frac{1}{2\mu} \Delta [\mathcal{G}] + \frac{\alpha}{\mu} R [\mathcal{G}] + \mu V_w \right) \Psi = 0, \]  

(4.59)

where

\[ R[\mathcal{G}] = - \sum_{s \in S} (U^s, U^s) - \sum_{s,s' \in S} (U^s, U^{s'}). \]  

(4.60)

and

\[ \Delta[\mathcal{G}] = e^{U(\sigma)} \frac{\partial}{\partial \sigma} \left( \hat{G}^{AB} e^{-U(\sigma)} \frac{\partial}{\partial \sigma} \right) + \sum_{s \in S} \varepsilon_s e^{2U(\sigma)} \left( \frac{\partial}{\partial \Phi} \right)^2, \]  

(4.61)

with \( U(\sigma) = \sum_{s \in S} U^s(\sigma). \)

4.4.2 Quantum solutions with one curved factor space and orthogonal \( U^s \)

Here as in subsect. 4.2.2 we put \( \Lambda = 0, \xi_1 \neq 0, \xi_2 = \ldots = \xi_n = 0, \) and \( 1 \notin I_s, s \in S, \) i.e. the space \( M_1 \) is curved and others are Ricci-flat and all “brane” submanifolds do not contain \( M_1. \) We also put orthogonality restriction on the vectors \( U^s: (U^s, U^{s'}) = 0 \) for \( s \neq s' \) and \( K_s = (U^s, U^s) \neq 0 \) for all \( s \in S. \) In this case the potential (4.7) reads \( V_w = \frac{1}{2} w \xi_1 d_1 e^{2U(\sigma)}. \) The truncated minisuperspace metric (4.6) may be diagonalized by the linear transformation \( z^A = S^A B \sigma^B; (z^A) = (z^1, z^a, z^s) \) as follows

\[ \hat{G} = -dz^1 \otimes dz^1 + \sum_{s \in S} \eta_{ab} dz^a \otimes dz^b + dz^a \otimes dz^b \eta_{ab}, \]  

(4.62)

where \( a, b = 2, \ldots, n + l - |S|; \eta_{ab} = \eta_{aa} \delta_{ab}; \eta_{aa} = \pm 1, \eta_s = \text{sign}(U^s, U^s) \) and \( q_1 z^1 = U^1(\sigma), q_1 = \sqrt{|(U^1, U^1)|} = \sqrt{1 - d_1^{-1}}, q_s z^s = U^s(\sigma), q_s = \nu_s^{-1} = \sqrt{|(U^s, U^s)|}, s = (a_s, s, I_s) \in S. \)
We are seeking the solution to WDW equation (4.69) by the method of the separation of variables, i.e. we put
\[ \Psi_s(z) = \Psi_1(z^1) \prod_{s \in S} \Psi_s(z^s) e^{ip_s \varphi_s} e^{ip_s z^s}. \] (4.63)
It follows from the relation for the Laplace operator
\[ \Delta [g] = - \left( \frac{\partial}{\partial z^1} \right)^2 + \eta^{ab} \frac{\partial}{\partial z^a} \frac{\partial}{\partial z^b} + \sum_{s \in S} \eta_s e^{q_s z^s} \frac{\partial}{\partial z^s} \left( e^{-q_s z^s} \frac{\partial}{\partial z^s} \right) \]
+ \sum_{s \in S} \varepsilon_s e^{2q_s z^s} \left( \frac{\partial}{\partial \Phi_s} \right)^2. \] (4.64)
that \( \Psi_s(z) \) satisfies WDW equation (4.69) if
\[ 2 \hat{H}_1 \Psi_1 = \left\{ \left( \frac{\partial}{\partial z^1} \right)^2 + \mu^2 w \xi_1 d_1 e^{2q_1 z^1} \right\} \Psi_1 = 2 \mathcal{E}_1 \Psi_1; \] (4.65)
\[ 2 \hat{H}_s \Psi_s = \left\{ -\eta_s e^{q_s z^s} \frac{\partial}{\partial z^s} \left( e^{-q_s z^s} \frac{\partial}{\partial z^s} \right) + \varepsilon_s p_s^2 e^{2q_s z^s} \right\} \Psi_s = 2 \mathcal{E}_s \Psi_s, \] (4.66)
s \( \in S \), and
\[ 2 \mathcal{E}_1 + \eta^{ab} p_a p_b + 2 \sum_{s \in S} \mathcal{E}_s + 2 \mu \eta [g] = 0, \] (4.67)
where \( \mu \) is from (4.57) and \( R[g] = -2 \sum_{s \in S} (U_s^*, U_s^*) = -2 \sum_{s \in S} \eta_s q_s^2 \).
Using the relations from Appendix 5 we obtain linearly independent solutions to (4.65) and (4.66) respectively
\[ \Psi_1(z^1) = B_{\omega_1} \sqrt{-w \xi_1 d_1} e^{q_1 z^1} \] (4.68)
\[ \Psi_s(z^s) = e^{q_s z^s} B_{\omega_s} \sqrt{\eta_s p_s^2} e^{q_s z^s} \] (4.69)
where \( \omega_1 = \sqrt{2 \mathcal{E}_1/\eta_1}, \omega_s = \sqrt{1 - 2 \eta_s \mathcal{E}_s p_s^2}, s \in S \), and \( B_{\omega_1}, B_{\omega_s} = I_{\omega_1}, K_{\omega_s} \) are the modified Bessel function.

The general solution of the WDW equation (4.69) is a superposition of the "separated" solutions (4.68):\( (4.69) \)
\[ \Psi(z) = \sum_{B} \int dp d\mathcal{E} C(p, \mathcal{E}, B) \Psi_s(z|p, \mathcal{E}, B), \] (4.70)
where \( p = (p_a), \mathcal{E} = (\mathcal{E}_s, \mathcal{E}_1), B = (B_1, B_s), B_1, B_s = I, K; \) and \( \Psi_s = \Psi_s(z|p, \mathcal{E}, B) \) is given by relation (4.68), (4.68) with \( \mathcal{E}_1 \) from (4.67). Here \( C(p, \mathcal{E}, B) \) are smooth enough functions. In non-composite electric case these solutions were considered in [102].

### 4.4.3 WDW equation with fixed charges

We may consider also another scheme based on zero-energy constraint relation (4.101). The corresponding WDW equation in the harmonic gauge reads
\[ \hat{H}_Q \Psi \equiv \left( -\frac{1}{2\mu} \hat{G}^{AB} \frac{\partial}{\partial x^A} \frac{\partial}{\partial x^B} + \mu V_Q \right) \Psi = 0, \] (4.71)
where potential \( V_Q \) is defined in (4.13). This equation describes quantum cosmology with classical fields of forms and quantum scale factors and dilatonic fields. Such approach is equivalent to the scheme of quantization of multidimensional perfect fluid considered in [164].
Eq. (4.71) is readily solved in the orthogonal case (see Appendix 5). The solutions are given by the following modifications in eqs. (4.67), (4.69), respectively,

\[ 2E_1 + \eta^{ab} p_a p_b + 2 \sum_{s \in S} E_s = 0, \quad (4.72) \]

\[ \Psi_s(z^s) = B_s^{\bar{\omega}_s} \left( \sqrt{\eta_s \varepsilon_s Q_s^2 \bar{\omega}_s z^s} \right), \quad (4.73) \]

where \( \bar{\omega}_s = \sqrt{-2\eta_s \varepsilon_s \nu_s^2} \).

This solution for the special case with one internal space \( (n = 1) \) and non-composite \( p \)-branes was considered in [106].
5 Black hole solutions

5.1 Solutions with a horizon

Here we consider the spherically symmetric case of the metric (4.20), i.e. we put $w = 1$, $M_1 = S^{d_1}$, $g^1 = d\Omega^2_{d_1}$, where $d\Omega^2_{d_1}$ is the canonical metric on a unit sphere $S^{d_1}$, $d_1 \geq 2$. In this case $\xi^1 = d_1 - 1$. We put $M_2 = \mathbb{R}$, $g^2 = -dt \otimes dt$, i.e. $M_2$ is a time manifold. We also assume that $(U^s, U^s') \neq 0$, $s \in S$, and

$$\det((U^s, U^s')) \neq 0.$$  \hspace{1cm} (5.1)

We put $C_1 \geq 0$. In this case relations (4.24)-(4.27) read

$$f_1(u) = dC_1^{-1/2} \text{sh}(C_1^{1/2}u),$$  \hspace{1cm} (5.2)

for $C_1 > 0$, and $f_1(u) = du$, for $C_1 = 0$. Here and in what follows $d = d_1 - 1$.

Let us consider the null-geodesic equations for the light “moving” in the radial direction (following from $ds^2 = 0$):

$$\pm \frac{dt}{du} = \Phi \equiv f_1^{d_1/(1-d_1)} e^{(c_1-c_2)u+c_2^2} \prod_{s \in S} f_s^{-h_s \delta_{2s}}.$$  \hspace{1cm} (5.3)

We consider solutions defined on some interval $[u_0, +\infty)$ with a horizon at $u = +\infty$ satisfying

$$\int_{u_0}^{+\infty} du \Phi(u) = +\infty.$$  \hspace{1cm} (5.4)

When the matrix $(h_{\alpha\beta})$ is positive definite and

$$2 \in I_s, \quad \forall s \in S,$$  \hspace{1cm} (5.5)

i.e. all p-branes have a common time direction $t$, the horizon condition (5.3) singles out the unique solution with $C_1 > 0$ and linear asymptotics at infinity

$$q^s = -\beta^s u + \bar{\beta}^s + o(1),$$  \hspace{1cm} (5.6)

$$u \to +\infty,$$ where $\beta^s, \bar{\beta}^s$ are constants, $s \in S$, [144, 145].

In this case

$$c^A/\bar{\mu} = -\delta^A_2 + h_1 U^{1A} + \sum_{s \in S} h_s b_s U^{sA},$$  \hspace{1cm} (5.7)

$$\beta^s/\bar{\mu} = 2 \sum_{s' \in S} A^s_{s'} \equiv b_s,$$

where $s \in S$, $A = (i, \alpha)$, $\bar{\mu} = \sqrt{C_1}$, the matrix $(A^s_{s'})$ is inverse to the quasi-Cartan matrix $(A_{s,s'})$ and

$$h_1 = (U^1, U^1)^{-1} = d_1/(1 - d_1).$$

According to Proposition 1 from [144, 145] the condition (5.4) is also a necessary condition for the existence of the horizon at $u \to +\infty$ under the assumptions assumed: if there exists at least one brane not containing the time submanifold $\{t\}$, then the horizon with respect to time $t$ at $u \to +\infty$ is absent.

Remark 9. According to Lemma 2 from [143], black hole solutions can only exist for $C_1 \geq 0$ and the horizon is then at $u = \infty$. For the extremal case $C_1 = 0$ see Subsect. 5.5. below.

Let us introduce a new radial variable $R = R(u)$ by relations

$$\exp(-2\bar{\mu}u) = 1 - \frac{2\mu}{R^d}, \quad \mu = \bar{\mu}/\bar{d} > 0,$$  \hspace{1cm} (5.8)

where $u > 0$, $R^d > 2\mu$ ($\bar{d} = d_1 - 1$). We put $c^A = 0$ and

$$q^s(0) = 0,$$  \hspace{1cm} (5.9)

$A = (i, \alpha), s \in S$. These relations guarantee the asymptotical flatness (for $R \to +\infty$) of the $(2 + d_1)$-dimensional section of the metric.
Let us denote

$$H_s = f_s e^{-\beta u},$$

(5.10)

$s \in S$. Then, solutions (4.20)-(4.23) may be written as follows [143, 144, 145]

$$g = \left( \prod_{s \in S} H_s^{2h_s d(I_s)/(D-2)} \right) \left\{ \left( 1 - \frac{2\mu}{R^d} \right)^{-1} dR \otimes dR + R^2 d\Omega^2 \right\} - \left( \prod_{s \in S} H_s^{-2h_s} \right) \left( 1 - \frac{2\mu}{R^d} \right) dt \otimes dt + \sum_{i=3}^n \left( \prod_{s \in S} H_s^{-2h_s' \delta i_s} \right) \hat{g}^i \right\}, \quad \exp(\varphi^\alpha) = \prod_{s \in S} H_s^{h_s x_s \lambda_s^\alpha},$$

(5.11)

where $F^a = \sum_{s \in S} \delta^a_{s_s} F_s$, and

$$F^s = -\frac{Q_s}{R^{d_1}} \left( \prod_{s' \in S} H_{s'}^{A_{s'r}} \right) dR \wedge \tau(I_s),$$

(5.13)

$s \in S_e$, $F^s = Q_s \tau(I_s)$,

(5.14)

$s \in S_m$. Here $Q_s \neq 0$, $h_s = K_s^{-1}$, $s \in S$, and the quasi-Cartan matrix $(A_{ss'})$ is non-degenerate.

Functions $H_s > 0$ obey the equations

$$R^{d_1} \frac{d}{dR} \left[ \left( 1 - \frac{2\mu}{R^d} \right) \frac{R^{d_1} dH_s}{H_s} dR \right] = B_s \prod_{s' \in S} H_{s'}^{-A_{ss'}},$$

(5.15)

$s \in S$, where $B_s = \varepsilon_s K_s Q_s^2 \neq 0$. These equations follow from Toda-type equations (3.107) and the definitions (5.8) and (5.10).

It follows from (5.9), (5.10) and (5.11) that there exist finite limits

$$H_s \rightarrow H_{s0} \neq 0,$$

(5.16)

for $R^d \rightarrow 2\mu$, $s \in S$. In this case the metric (5.11) has a regular horizon at $R^d = 2\mu$. From (5.9) we get

$$H_s(R = +\infty) = 1,$$

(5.17)

$s \in S$.

The Hawking "temperature" corresponding to the solution is (see also [99, 101] for orthogonal case) found to be

$$T_H = \frac{\hat{d}}{4\pi(2\mu)^{1/\hat{d}}} \prod_{s \in S} H_{s0}^{-h_s},$$

(5.18)

where $H_{s0}$ are defined in (5.10).

The boundary conditions (5.10) and (5.11) play a crucial role here, since they single out, generally speaking, only few solutions to eqs. (5.15). Moreover, for some values of parameters $\mu = \hat{\mu}/\hat{d}$, $\varepsilon_s$ and $Q_s^2$ the solutions to eqs. (5.15)-(5.17) do not exist [144].

Thus, we obtained a general family of black hole solutions defined up to solutions of radial equations (5.15) with the boundary conditions (5.10) and (5.11). In the next sections we consider several exact solutions to eqs. (5.15)-(5.17).

**Remark 10.** Let $M_i = \mathbb{R}$ and $q^i = -dt \otimes dt$ for some $i \geq 3$. Then the metric (5.11) has no a horizon with respect to the "second time" $\bar{t}$ for $R^d \rightarrow 2\mu$. Thus, we are led to a "single-time" theorem from [139]. Relation (5.4) coincides with the "no-hair" theorem from [139].
5.2 Polynomial structure of $H_s$ for Lie algebras

5.2.1 Conjecture on polynomial structure

Now we deal with solutions to second order non-linear differential equations (5.15) that may be rewritten as follows

$$\frac{d}{dz} \left( \frac{1-2\mu z}{H_s} \frac{d}{dz} H_s \right) = \bar{B}_s \prod_{s' \in S} H_{s'}^{-A_{s,s'}}^-, \quad (5.19)$$

where $H_s(z) > 0$, $\mu > 0$, $z = R^{-d} \in (0, (2\mu)^{-1})$ and $\bar{B}_s = B_s/d^2 \neq 0$. Eqs. (5.16) and (5.17) read

$$H_s((2\mu)^{-1} - 0) = H_{s0} \in (0, +\infty), \quad H_s(+0) = 1, \quad (5.20)$$

$s \in S$.

It seems rather difficult to find the solutions to a set of eqs. (5.19)-(5.21) for arbitrary values of parameters $\mu$, $\bar{B}_s$, $s \in S$, and quasi-Cartan matrices $A = (A_{s,s'})$. But we may expect a drastic simplification of the problem under consideration for certain class of parameters and/or $A$-matrices.

In general we may try to seek solutions of (5.19) in a class of functions analytical in a disc $|z| < L$ and continuous in semi-interval $0 < z \leq (2\mu)^{-1}$. For $|z| < L$ we get

$$H_s(z) = 1 + \sum_{k=1}^{\infty} P_s^{(k)} z^k, \quad (5.22)$$

where $P_s^{(k)}$ are constants, $s \in S$. Substitution of (5.22) into (5.19) gives us an infinite chain of relations on parameters $P_s^{(k)}$ and $\bar{B}_s$. In general case it seems to be impossible to solve this chain of equations.

Meanwhile there exist solutions to eqs. (5.19)-(5.21) of polynomial type. The simplest example occurs in orthogonal case (5.19), (5.21) gives us an infinite chain of relations on parameters $P_s^{(k)}$ and $\bar{B}_s$. In general case it seems to be impossible to solve this chain of equations.

In [127, 134, 136] this solution was generalized to a block-orthogonal case (3.5), (3.6). In this case (5.23) is modified as follows

$$H_s(z) = 1 + P_s z, \quad (5.23)$$

with $P_s \neq 0$, satisfying

$$P_s (P_s + 2\mu) = -\bar{B}_s, \quad (5.24)$$

$s \in S$.

In [127, 134, 136] this solution was generalized to a block-orthogonal case (3.5), (3.6). In this case (5.23) is modified as follows

$$H_s(z) = (1 + P_s z)^{b_s}, \quad (5.25)$$

where $b_s$ are defined in (5.23) and parameters $P_s$ and are coinciding inside blocks, i.e. $P_s = P_s'$ for $s, s' \in S_i$, $i = 1, \ldots, k$. Parameters $P_s \neq 0$ satisfy the relations

$$P_s (P_s + 2\mu) = -\bar{B}_s/b_s, \quad (5.26)$$

$b_s \neq 0$, and parameters $\bar{B}_s/b_s$ are also coinciding inside blocks, i.e. $\bar{B}_s/b_s = \bar{B}_s'/b_s'$ for $s, s' \in S_i$, $i = 1, \ldots, k$. In this case $H_s$ are analytical in the disc $|z| < L$, where $L = \min(|P_s|^{-1}, s \in S)$.

Let $(A_{s,s'})$ be a Cartan matrix for a finite-dimensional semisimple Lie algebra $\mathfrak{g}$. In this case all powers in (5.27) are natural numbers coinciding with the components of twice the dual Weyl vector in the basis of simple coroots (1.72) (see Appendix 3) and hence, all functions $H_s$ are polynomials, $s \in S$.

**Conjecture 1.** Let $(A_{s,s'})$ be a Cartan matrix for a semisimple finite-dimensional Lie algebra $\mathfrak{g}$. Then the solution to eqs. (5.19)-(5.21) (if exists) is a polynomial

$$H_s(z) = 1 + \sum_{k=1}^{n_s} P_s^{(k)} z^k, \quad (5.26)$$

where $P_s^{(k)}$ are constants, $k = 1, \ldots, n_s$; $n_s = b_s = 2 \sum s' \in S A_{s,s'} \in \mathbb{N}$ and $P_s^{(n_s)} \neq 0, s \in S$.

In extremal case ($\mu = +0$) an analogue of this conjecture was suggested previously in [106]. Conjecture 1 was verified for $A_m$ and $C_{m+1}$ series of Lie algebras in [144, 145].
### 5.3 Examples

#### 5.3.1 Solution for \( A_2 \)

Here we consider solutions related to the Lie algebra \( A_2 = sl(3) \). According to the results of previous section we seek the solutions to eqs. \( 5.19-5.21 \) in the following form (see \( 5.26 \); here \( n_1 = n_2 = 2 \)):

\[
H_s = 1 + P_s z + P_s^{(2)} z^2,
\]

(5.27)

where \( P_s = P_s^{(1)} \) and \( P_s^{(2)} \neq 0 \) are constants, \( s = 1, 2 \).

From \( 5.19 \) we get for \( P_1 + P_2 + 4 \mu \neq 0 \)

\[
P_s^{(2)} = \frac{P_s P_{s+1}(P_s + 2 \mu)}{2(P_1 + P_2 + 4 \mu)}, \quad \tilde{B}_s = -\frac{P_s(P_s + 2 \mu)(P_s + 4 \mu)}{P_1 + P_2 + 4 \mu},
\]

(5.28)

\( s = 1, 2 \). Here we denote \( s + 1 = 2, 1 \) for \( s = 1, 2 \), respectively. For \( P_1 + P_2 + 4 \mu = 0 \) we get a special (exceptional) solution with \( P_1 = P_2 = -2 \mu, 2P_s^{(2)} = \tilde{B}_s > 0 \) and \( \tilde{B}_1 + \tilde{B}_2 = 4 \mu^2 \).

Thus, in the \( A_2 \)-case the non-exceptional solution is described by relations \( 5.21-5.24 \) with \( S = \{s_1, s_2\} \) (identified with \( \{1, 2\} \)), intersection rules

\[
d(I_{s_1} \cap I_{s_2}) = \Delta(s_1, s_2) - K,
\]

(5.29)

where symbol \( \Delta(s_1, s_2) \) is defined in \( 5.21 \) and \( K = K_{s_i} = (U^{s_i}, U^{s_i}) \neq 0 \); functions \( H_{s_i} = H_i \) are defined by relations \( 5.24 \) and \( 5.28 \) with \( z = R^{-d}, i = 1, 2 \).

#### 5.3.2 \( A_2 \)-dyon in \( D = 11 \) supergravity

Consider the “truncated” bosonic sector of \( D = 11 \) supergravity with the action \( 6.21 \). Let us consider a dyonic black-hole solutions with electric 2-brane and magnetic 5-brane defined on the manifold

\[
M = (2\mu, +\infty) \times (M_1 = S^2) \times (M_2 = \mathbb{R}) \times M_3 \times M_4,
\]

(5.30)

where \( \dim M_3 = 2 \) and \( \dim M_4 = 5 \). The solution reads,

\[
g = H_1^{1/3} H_2^{2/3} \left\{ \frac{dR \otimes dR}{1 - 2 \mu/R} + R^2 d\Omega_2^2 \right\},
\]

\[
- H_1^{-1} H_2^{-1} \left( 1 - \frac{2 \mu}{R} \right) dt \otimes dt + H_1^{-1} \hat{g}^3 + H_2^{-1} \hat{g}^4, \quad F = -\frac{Q_1}{R^2} H_1^{-2} H_2 dR \wedge dt \wedge \hat{\tau}_3 + Q_2 \hat{\tau}_1 \wedge \hat{\tau}_3,
\]

(5.32)

where metrics \( g^2 \) and \( g^3 \) are Ricci-flat metrics of Euclidean signature, and \( H_s \) are defined by \( 5.27 \), where \( z = R^{-1} \) and parameters \( P_s, P_s^{(2)}, \tilde{B}_s = B_s = -2Q^2_s, s = 1, 2 \), satisfy \( 5.28 \).

The solution describes \( A_2 \)-dyon consisting of electric 2-brane with worldvolume isomorphic to \( (M_2 = \mathbb{R}) \times M_3 \) and magnetic 5-brane with worldvolume isomorphic to \( (M_2 = \mathbb{R}) \times M_4 \). The “branes” are intersecting on the time manifold \( M_2 = \mathbb{R} \). Here \( K_s = (U^s, U^s) = 2, \varepsilon_s = -1 \) for all \( s \in S \). The \( A_2 \) intersection rule reads: \( 3 \cap 6 = 1 \).

The field configurations \( 5.31, 5.32 \) also satisfies to equations of motion for \( D = 11 \) supergravity (see \( 5.24, 5.25 \); in this case \( F \wedge F = 0 \)).

This solution in a special case \( H_1 = H_2 = H^2 \ (P_1 = P_2, Q_1^2 = Q_2^2) \) was considered in \( 13.1 \). The 4-dimensional section of the metric \( 5.31 \) in this special case coincides with the Reissner-Nordström metric. For the extremal case, \( \mu \to +0 \), and multi-black-hole generalization see also \( 17 \).
5.4 Extremal case

5.4.1 "One-pole" solution

Here we consider the extremal case: \( \mu \to +0 \). The relation for the metric (5.11) reads in this case as follows \[0x1]

\[ g = \left( \prod_{s \in S} H_s^{2h_s d(l_s)/(D-2)} \right) \left\{ dR \otimes dR + R^2 d\Omega_d^2 \right\} - \left( \prod_{s \in S} H_s^{-2h_s} dt \otimes dt + \sum_{i=3}^n \left( \prod_{s \in S} H_s^{-2h_s s_i} \right) \delta^i \right\}, \]

and the relations for scalar fields and fields of forms (5.12)-(5.14) are unchanged. Here \( H_s = H_s(z) > 0, z = R^{-d} \in (0, +\infty) \) and the following relations are satisfied

\[ \frac{d^2}{dz^2} \ln H_s = \tilde{B}_s \prod_{s' \in S} H_{s'}^{-A_{s's'}}, \quad H_s(+0) = 1, \]

\[ E_{TL} = \frac{d^2}{dz^2} \ln H_s = \tilde{B}_s \prod_{s' \in S} H_{s'}^{-A_{s's'}}, \quad H_s(+0) = 1, \]

where \( \tilde{B}_s = B_s/d_s \neq 0 \), and \( B_s = 2K_s A_s, A_s = \frac{1}{2} \varepsilon_s Q_s, s \in S \). These solution may be obtained as a special case of solutions from Subsect. 4.2.2 with \[0x2\]

\[ C_1 = E_{TL} = c^A = 0, \]

\[ A = (i, \alpha) \text{ and } u = z/d_s \ (H_s = f_s, \text{ see } 5.11). \]

Conjecture 2. Let \( (A_{s'}) \) be a Cartan matrix for a simple finite-dimensional Lie algebra. Then there exists a polynomial solution

\[ H_s(z) = 1 + \sum_{k=1}^{n_s} P_s^{(k)} z^k, \]

to eqs. (5.32), (5.33) for \( \tilde{B}_s < 0, s \in S \), where \( P_s^{(k)} \) are constants, \( k = 1, \ldots, n_s \); numbers \( n_s = b_s \) are defined in (5.27) and \( P_s^{(n_s)} \neq 0, s \in S \).

This conjecture was considered (in fact) previously in 106. The coefficients \( P_s^{(n_s)} = C_s > 0 \) may be calculated by substitution of asymptotical relations

\[ H_s(z) \sim C_s z^{b_s}, \quad z \to +\infty, \]

into eqs. (5.34), (5.35), \( s \in S \). This results in the relations

\[ C_s = \prod_{s' \in S} (-b_s^s \tilde{B}_s')^{A_{s's'}}, \]

\( s \in S \). We note that the asymptotical relations (5.38) are satisfied in a more general case, when \( \tilde{B}_s b_s < 0, s \in S \).

Let us consider the metric (5.33) with \( H_s \) obeying asymptotical relations (5.38). We have a horizon for \( R \to +0 \), if

\[ \xi = \sum_{s \in S} h_s b_s - \frac{1}{d_0 - 2} \geq 0, \]

where \( d_0 = d_1 + 1 \). This relation follows from the requirement of infinite time propagation of light to \( R \to +0 \).

For flat internal spaces \( M_i = \mathbb{R}^{d_i}, i = 3, \ldots, n \), we get for the Riemann tensor squared (Kretschmann scalar) from Appendix 1

\[ \mathcal{K}[g] = [C + o(1)] R^{4(d_0 - 2)} \eta \]
The solution reads

$$
\eta = \sum_{s \in S} h_s b_s \frac{d(I_s)}{D-2} - \frac{1}{d_0-2},
$$

(5.42)

and $C \geq 0$ ($C = \text{const}$). Due to (5.41) the metric (5.33) with flat internal spaces has no curvature singularity when $R \to +0$, if

$$
\eta \geq 0.
$$

(5.43)

For $h_s b_s > 0$, $d(I_s) < D-2$, $s \in S$, we get $\eta < \xi$ and relation (5.43) single out extremal charged black $p$-branes and with flat internal spaces.

### 5.4.2 Multi-black-hole extension

The solutions under consideration have a Majumdar-Papapetrou-type extension defined on the manifold

$$
M = M_0 \times (M_2 = \mathbb{R}) \times \ldots \times M_n,
$$

(5.44)

The solution reads

$$
g = \left( \prod_{s \in S} H_s^{2h_s d(I_s)/(D-2)} \right) \left\{ g^0 - \left( \prod_{s \in S} H_s^{-2h_s} \right) dt \otimes dt + \sum_{i=3}^n \left( \prod_{s \in S} H_s^{2h_s \delta_{i_s}} \right) g^i \right\},
$$

(5.45)

$$
\exp(\varphi^0) = \prod_{s \in S} H_s^{b_s \alpha_0 \lambda_{s_0}},
$$

(5.46)

$$
F^a = \sum_{s \in S} \delta^a_s F^s,
$$

(5.47)

where

$$
F^s = Q_s \left( \prod_{s' \in S} H_{s'}^{-A_{s'}} \right) dH \wedge \tau(I_s), \quad s \in S_e,
$$

(5.48)

$$
F^s = Q_s (\ast_0 dH) \wedge \tau(I_s), \quad s \in S_m.
$$

(5.49)

Here $\bar{I} \equiv \{2, \ldots, n\} \setminus I$, $g^0 = g^0_{\mu'\nu'}(x)dx^\mu \otimes dx^{\nu'}$ is a Ricci-flat metric on $M_0$ and $\ast_0 = \ast[g^0]$ is the Hodge operator on $(M_0, g^0)$ and

$$
H_s = H_s(H(x)),
$$

(5.50)

where functions $H_s = H_s(z) > 0$, $z \in (0, +\infty)$, $s \in S$, satisfy the relations (5.34) and (5.35) and $H = H(x) > 0$ is a harmonic function on $(M_0, g^0)$, i.e. $\Delta[g^0]H = 0$. This solution is a special case of the solutions from Subsect. 3.2.2 corresponding to restrictions (5.36).

Let us consider as an example a flat space: $M_0 = \mathbb{R}^{d_0} \setminus X$, $d_0 > 2$, and $g^0 = \delta_{\mu\nu} dx^\mu \otimes dx^{\nu'}$ and

$$
H(x) = \sum_{b \in X} \frac{q_b}{|x-b|^{d_0-2}},
$$

(5.51)

where $X$ is finite non-empty subset $X \subset M_0$ and all $q_b > 0$ for $b \in X$. For flat internal spaces $M_i = \mathbb{R}^{d_i}$, $i = 3, \ldots, n$, and non-negative indices $\eta$ and $\xi$ (see (5.40) and (5.43)) the solution describes a set of $|X|$ extremal $p$-brane black holes. Here relations $H(x) \to 0$ for $|x| \to +\infty$ and $H_s(+0) = 1$ ($s \in S$) imply the asymptotical flatness of the $(1 + d_0)$-dimensional section of the metric. A black hole corresponding to a “point” (horizon) $b \in X$ carries brane charges $Q_b q_b$, $s \in S$. Since the solution is invariant under the replacement of parameters: $Q_s \mapsto \alpha Q_s$, $q_b \mapsto Q_s/\alpha$, $\alpha > 0$, $b \in X$, $s \in S$, we may normalize parameters $q_b$ by the restriction $\sum_{b \in X} q_b = 1$. 

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The open problem is to find the solutions of WDW equation for more general $p$-branes: namely, when stress-energy tensor $T_{\mu\nu}$ has a diagonal structure. Here we presented three types of restrictions: electric-electric, magnetic-magnetic and electric-magnetic, which give a sufficient condition for the diagonality of $T_{\mu\nu}$. In [93], we obtained general and sufficient condition in terms of a set of constraints on all moduli scalar fields.

In this review (as in other authors publications) we were dealing with the most general intersection rules. Here we showed that in general intersection rules correspond to the quasi-Cartan matrix related to certain "brane" vectors $U^s$ belonging to some vector space (dual to truncated minisuperspace). A major part of well-known $p$-brane solutions have intersection rules equivalent to the orthogonality condition: $(U^s, U^{s'}) = 0$, $s \neq s'$, where $\langle \cdot, \cdot \rangle$ is the bilinear form dual to truncated minisupermetric (first it was understood in [57]). The quasi-Cartan matrix in this case is nothing more than the Cartan matrix for the semisimple Lie algebra $A_1 \oplus \ldots \oplus A_1$ ($r$ terms). For Majumdar-Papapetrou type solutions we got $r$ independent harmonic functions on $M_0$. Solutions of such class contain a large variety of supersymmetric (BPS saturated) solutions in supergravitational models. Here we also considered a more general class of "block-orthonal" solutions governed by harmonic functions and also considered a generalization in a one-block case governed by several functions (of one harmonic function) obeying Toda-type equations (e.g. Euclidean Toda lattices). The $p$-brane solutions may be considered as a nice tool for utilization of Euclidean and hyperbolic Toda lattices related to finite dimensional Lie algebras and hyperbolic Kac-Moody (KM) algebras. Here we considered also the Wheeler-DeWitt (WDW) equation for $p$-brane cosmology in d’Alembertian (covariant) and conformally covariant form and integrated it for orthogonal $U$-vectors. The open problem is to find the solutions of WDW equation for more general $p$-brane configurations.

We also considered general classes of "cosmological" and spherically symmetric solutions governed by Toda-type equations, e.g. black brane configurations. An interesting point here is the appearance of polynomials for black hole solutions for brane intersections governed by Cartan matrices of finite-dimensional simple Lie algebras [143, 144, 145]. An open problem here is to find all polynomials related to classical series of Lie algebras (at present only $A_1, A_2, A_3$ and $C_2$ solutions are known [211]). In extremal case $\mu \to +0$ several examples were presented earlier in [106]. Another open problem is to verify the Verlinde-Cardy formula (or modifications) for the entropy of black branes related to Lie algebras.

Due to dimensional restrictions, only a small number of solutions related to simple Lie algebras of different ranks appear in $D \leq 11$ supergravities (among them $A_2$-dyon solutions). We considered not obviously supergravitational theories, but also a chain of $B_D$-models in dimensions $D \geq 12$ (we remind the reader that $B_{12}$ model appear in the field limit of $F$-theory [49]).

Another topic of interest was the investigation of Toda-type $p$-brane solutions related to hyperbolic Kac-Moody (KM) algebras. This looks promising not only because of existence of $p$-brane solutions related to hyperbolic KM algebras (see subsect. 3.1.2) but also due to the explanation of never ending oscillating behaviour in supergravitational cosmologies and pure gravitational in terms of hyperbolic algebras [196, 197] (for general construction of $p$-brane billiards see [189, 193]).

It should be noted that here we were also trying to present the most general classes of classical composite non-localized $p$-branes solutions up to solutions of Laplace equations and/or Toda-like equations. Thus, any future progress in investigations of Toda-like differential equations may be immediately "recorded" in terms of new $p$-brane solutions. In other words, all "kynematics" (intersections, power coefficients, etc) may be solved and presented here, the only progress may be expected in "dynamics."
7 Appendix

7.1 Appendix 1

7.1.1 Ricci-tensor components

The nonzero Ricci tensor components for the metric (2.40) are the following

\[ R_{\mu\nu}[g] = R_{\mu\nu}[g^0] + g^0_{\mu\nu} \left[ -\Delta_0 \gamma + (2 - d_0)(\partial \gamma)^2 - \partial \gamma \sum_{j=1}^n d_j \partial \phi^j \right], \]  

\( \text{(7.1)} \)

\[ + (2 - d_0)(\gamma_{\mu\nu} - \gamma_{\mu\gamma\nu}) - \sum_{i=1}^n d_i (\phi^i_{\mu\nu} - \phi^i_{\mu\gamma\nu} - \phi^i_{\nu\gamma\mu} + \phi^i_{\mu\phi^i_{\nu\mu}}), \]

\[ R_{m_i,n}[g] = R_{m_i,n}[g^0] - e^{2\phi^i - 2\phi^i} \left\{ \Delta_0 \phi^i + (\partial \phi^i) [(d_0 - 2)(\partial \gamma)^2 + \sum_{j=1}^n d_j \partial \phi^j] \right\}. \]  

\( \text{(7.2)} \)

Here \( \partial \beta \partial \gamma \equiv g^0_{\mu\nu} \beta_{\mu\gamma\nu} \) and \( \Delta_0 = \Delta[g^0] \) is the Laplace-Beltrami operator corresponding to \( g^0 \) and all covariant derivatives correspond to \( g^0 \). The scalar curvature for (2.9) is (7.9) is

\[ R[g] = \sum_{i=1}^n e^{-2\phi} R[g^i] + e^{-2\gamma} \left\{ R[g^0] - \sum_{i=1}^n d_i (\partial \phi^i)^2 \right\}, \]  

\( \text{(7.3)} \)

\[ -(d_0 - 2)(\partial \gamma)^2 - (\partial f)^2 - 2\Delta_0 (f + \gamma) \}, \]

where \( f \) is defined in (2.40). Relations for the Ricci tensor may be obtained using the relations for the Riemann tensor from the next subsection and the relations for the conformal transformations from the last subsection.

7.1.2 Riemann tensor.

Let us denote \( g^0 = e^{2\gamma} g^0 \). The non-zero components of the Riemann tensor corresponding to metric The set \( S \) consists of elements \( s = (a_s, v_s, I_s) \), where \( a_s \in \Delta, v_s = e, m \) and \( I_s \in \Omega_{a_s, v_s} \), have the following form

\[ R_{\mu\nu\rho\sigma}[g] = R_{\mu\nu\rho\sigma}[g^0], \]  

\( \text{(7.4)} \)

\[ R_{m_i,n,m}[g] = -R_{m_i,m n}[g] = -R_{m_i,n,\nu}[g] = \]

\[ R_{m_i,n,\nu}[g] = - \exp(2\phi^i) g^0_{m_i,n}[\nabla \mu [g^0] (\partial \phi^i) + (\partial \mu \phi^i)(\partial \nu \phi^i)], \]  

\( \text{(7.5)} \)

\[ R_{m_i,p,q}[g] = \exp(2\phi^i) \delta_{ij} \delta_{kl} R_\mu \nu_{i,j,k,l} [g^0] + \]

\[ \exp(2\phi^i + 2\phi^j) g^0_{\mu\nu} (\partial \mu \phi^i)(\partial \nu \phi^i) [\delta_{ij} \phi^m_\mu g^i_{m,n} g^j_{n,p} - \delta_{ik} \phi^m_\nu g^i_{m,p} g^j_{n,q}], \]  

\( \text{(7.6)} \)

where indices \( \mu, \nu, \rho, \sigma \) correspond to \( M_0 \) and \( m_i, n_i, p_i, q_i \) to \( M_i; i, j, k, l = 1, \ldots, n; \nabla[g^0] \) is covariant derivative with respect to \( g^0 \). Here we consider the chart \( C_0 \times \ldots \times C_n \) on the manifold \( M_0 \times M_1 \times \ldots \times M_n \), where \( C_0 \) is a chart on \( M_0, \nu = 0, \ldots, n. \)

The relations (2.9)-(2.10) may be obtained from the following relations for the non-zero components of the Christoffel-Schwarz symbols

\[ \Gamma^\mu_{\nu\rho}[g] = \Gamma^\mu_{\nu\rho}[g^0], \]  

\( \text{(7.7)} \)

\[ \Gamma^m_{n,\nu}[g] = \Gamma^m_{n,\nu}[g] = \delta^m_{n} \partial \nu \phi^i, \]  

\( \text{(7.8)} \)

\[ \Gamma^m_{n,i}[g] = -g^0_{\mu\nu} (\partial \mu \phi^i) \exp(2\phi^i) g^i_{m,n}, \]  

\( \text{(7.9)} \)

\[ \Gamma^m_{n,p}[g] = \Gamma^m_{n,p}[g], \]  

\( \text{(7.10)} \)

\( i = 1, \ldots, n. \)
7.1.3 Riemann tensor squared (Kretschmann scalar).

It follows from the relations (7.4)–(7.7) that the Riemann tensor squared for the metric \( g \) with \( \gamma = 0 \) has the following form \[18\]

\[
K[g] = I[g^0] + \sum_{i=1}^{n} \left( e^{-4\phi^i} I[g^i] - 4e^{-2\phi} U[g^0, \phi] R[g^i] \right) - 2d_i U^2[g^0, \phi^i] + 4d_i V[g^0, \phi^i]) + \sum_{i,j=1}^{n} 2d_i d_j [g^{0\mu\nu} (\partial_\mu \phi^i) \partial_\nu \phi^j]^2,
\]

(7.11)

where \( R[g^i] \) is scalar curvature of \( g^i \) and \( d_i = \text{dim}M_i \) is dimension of \( M_i, i = 1, \ldots, n. \) In (7.11)

\[
U[g, \phi] \equiv g^{MN} (\partial_M \phi) \partial_N \phi,
\]

(7.12)

\[
V[g, \phi] \equiv g^{M_1 N_1} g^{M_2 N_2} [\nabla_M (\partial_{M_2} \phi) + (\partial_{M_1} \phi) \partial_{M_2} \phi] \times [\nabla_{N_1} (\partial_{N_2} \phi) + (\partial_{N_1} \phi) \partial_{N_2} \phi],
\]

(7.13)

where \( \nabla = \nabla[g] \) is covariant derivative with respect to \( g. \)

7.1.4 The cosmological case.

Now we consider the special case of the metric \( g \) with \( M_0 = (t_1, t_2), t_1 < t_2. \) Thus, we consider the metric

\[
g_c = -B(t) dt \otimes dt + \sum_{i=1}^{n} A_i(t) g^i,
\]

(7.14)

defined on the manifold

\[
M = (t_1, t_2) \times M_{1} \times \ldots \times M_{n}.
\]

(7.15)

Here \( g^i \) is a metric on \( M_i \) and \( B(t), A_i(t) \neq 0 \) are smooth functions, \( i = 1, \ldots, n. \) From (7.12) we obtain the Riemann tensor squared for the metric \( g \).

\[
K[g_c] = \sum_{i=1}^{n} \left( A_i^{-2} K[g^i] + A_i^{-3} B^{-1} \dot{A}_i^2 R[g^i] - \frac{1}{8} d_i B^{-2} A_i^{-4} \dot{A}_i^4 \right.
\]

\[
+ \frac{1}{4} d_i B^{-2} (2A_i^{-1} \ddot{A}_i - B^{-1} BA_i^{-1} \dot{A}_i - A_i^{-2} \ddot{A}_i^2) \}
\]

\[
+ \frac{1}{8} B^{-2} \sum_{i=1}^{n} d_i (A_i^{-1} \dot{A}_i)^2] \right. \]

(7.16)

7.1.5 Parameter \( C = C(b). \)

Here we also present the relation for the parameter \( C = C(b), b \in X, \) from (5.67)

\[
C = C_0 + C_1 + C_2,
\]

(7.17)

\[
C_0 = 2(d_0 - 1)(d_0 - 2) \alpha^2 (\alpha - 2)^2,
\]

(7.18)

\[
C_1 = 4[(d_0 - 1) \alpha^2 + \alpha - 1)^2] \sum_{i=1}^{n} d_i \alpha_i^2,
\]

(7.19)

\[
C_2 = 2 \left( \sum_{i=1}^{n} d_i \alpha_i^2 \right)^2 - 2 \sum_{i=1}^{n} d_i \alpha_i^4,
\]

(7.20)

where

\[
\alpha = \alpha(b) \equiv (d_0 - 2) \sum_{s \in S(b)} (-\varepsilon_s) \nu_s^2 \frac{d(I_s)}{D - 2} = (d_0 - 2) \eta(b) + 1,
\]

(7.21)

\[
\alpha_i = \alpha_i(b) \equiv (d_0 - 2) \sum_{s \in S(b)} (-\varepsilon_s) \nu_s^2 \left[ \delta_{ii} \nu_s^2 - \frac{d(I_s)}{D - 2} \right],
\]

(7.22)
It follows from definitions (7.17)-(7.20) that \( C \geq 0 \) and
\[
C = 0 \iff (\alpha = 0, 2, \ \alpha_i = 0, \ i = 1, \ldots, n). \tag{7.23}
\]

Parameter \( C \) appears in the Kretschmann scalar \( \frac{8}{5} \) for the metric
\[
g_\ast = r^{-2\alpha}[dr \otimes dr + r^2d\Omega_{d_0-1}^2] + \sum_{i=1}^{n} r^{2\alpha_i}g_i^i, \tag{7.24}
\]
with \( R[g^i] = \mathcal{K}[g^i] = 0, \ i = 1, \ldots, n \). Using formula the 7.16, we obtain
\[
\mathcal{K}[g_\ast] = Cr^{-4+4\alpha}. \tag{7.25}
\]

### 7.1.6 Conformal transformation

Here we also present for a convenience the well-known relations \([162]\)
\[
e^{-2\gamma}R_{\mu\nu\rho\sigma}g^\nu g^\sigma = R_{\mu\nu\rho\sigma}[g^0] + \\
Y_{\mu\nu\rho\sigma}g^\nu g^\sigma - Y_{\mu\rho\nu\sigma}g^\nu g^\sigma + Y_{\mu\sigma\nu\rho}g^\nu g^\sigma, \tag{7.26}
\]
\[
R_{\mu\nu}[e^{2\gamma}g^0] = R_{\mu\nu}[g^0] + (2 - d_0)Y_{\mu\nu} - g_{\mu\nu}g_{\rho\sigma}(g^{\rho\sigma}Y_{\mu\nu}), \tag{7.27}
\]
\[
\Delta[e^{2\gamma}g^0] = e^{-2\gamma}\{\Delta[g^0] + (d_0 - 2)g^{\mu\nu}(\partial_\mu \gamma)(\partial_\nu \gamma)\} \tag{7.28}
\]

where the metric \( g^0 \) is defined on \( M_0, \dim M_0 = d_0 \), \( \Delta[g^0] \) is Laplace-Beltrami operator on \( M_0 \) and
\[
Y_{\mu\nu} = \gamma_{\mu\nu} - \gamma_{\mu\gamma}\gamma^\gamma + \frac{1}{2}g^0_{\mu\nu}\gamma^\gamma. \tag{7.29}
\]

### 7.2 Appendix 2. Product of forms

Let \( F_1 \) and \( F_2 \) be forms of rank \( r \) on \((M, g)\) (\( M \) is a manifold and \( g \) is a metric on it). We define
\[
(F_1 \cdot F_2)_{MN} \equiv (F_1)_{MM_2 \ldots M_r}(F_2)_{N}^{M_2 \ldots M_r};
\]
\[
F_1 F_2 \equiv (F_1 \cdot F_2)_M = (F_1)_{M_1 \ldots M_r}(F_2)_{M_1 \ldots M_r}. \tag{7.31}
\]

It is clearly, that
\[
(F_1 \cdot F_2)_{MN} = (F_1 \cdot F_2)_{NM}, \quad F_1 F_2 = F_2 F_1. \tag{7.32}
\]

For the volume forms \( \mu_{\ast} \) we get
\[
\tau(I) = d(I)! \varepsilon(I), \tag{7.33}
\]
\[
(\tau(I) \cdot \tau(I))_{m_i n_i} = (d(I) - 1)!\varepsilon(I)\delta_{II}, \tag{7.34}
\]

where indices \( m_i, n_i \) correspond to the manifold \( M_i, i = 1, \ldots, n \). The symbols \( \varepsilon(I) \) and \( \delta_{II} \) are defined in \([2.14]\) and \([2.24]\) respectively.

For the form \( F_{(a,e,l)} \) from \([2.18]\) and metric \( g \) from \([2.9]\) we obtain from \([7.33]-7.34]\)
\[
\frac{1}{n_a!}(F_{(a,e,l)} \cdot F_{(a,e,l)})_{\mu\nu} = \frac{A(I)}{n_a} \partial_\mu \Phi_{(a,e,l)} \partial_\nu \Phi_{(a,e,l)} \exp(2\gamma); \tag{7.35}
\]
\[
\frac{1}{n_a!}(F_{(a,e,l)} \cdot F_{(a,e,l)})_{m_i n_i} = \delta_{II} g_{m_i n_i} \frac{A(I)}{n_a} (\partial \Phi_{(a,e,l)})^2 \exp(2\Phi^2), \tag{7.36}
\]

where indices \( m_i, n_i \) correspond to the manifold \( M_i, i = 1, \ldots, n \), and
\[
A(I) = \varepsilon(I) \exp(-2\gamma - 2 \sum_{i \in I} d_i \phi^i), \tag{7.37}
\]
\( I \in \Omega_{\alpha e}. \) All other components of \((F^{(\alpha \varepsilon, I) \cdot F^{(\alpha \varepsilon, I)})_{MN}}\) are zero. For the scalar invariant we have

\[
\frac{1}{n_a} (\mathcal{F}^{(\alpha \varepsilon, I)})^2 = \frac{1}{n_a} \mathcal{F}^{(\alpha \varepsilon, I)} \mathcal{F}^{(\alpha \varepsilon, I)} = A(I)(\partial \Phi^{(\alpha \varepsilon, I)})^2,
\]

(7.38)

\( I \in \Omega_{\alpha e}. \) Here, as above, we use the notations: \( \partial \Phi_1 \partial \Phi_2 = g^{\mu \nu} \partial_\mu \Phi_1 \partial_\nu \Phi_2 \) and \((\partial \Phi_1)^2 = \partial \Phi_1 \partial \Phi_1\) for functions \( \Phi_1 = \Phi_1(x) \) and \( \Phi_2 = \Phi_2(x) \) on \( M_0. \)

Analogous relations for magnetic case may be obtained using the formulas

\[
\frac{1}{k_s^2} (\ast F_1) (\ast F_2) = \frac{\varepsilon [g]}{k!} F_1 F_2,
\]

(7.39)

\[
\frac{1}{(k_s - 1)!} (\ast F_1) (\ast F_2)_{MN} = \frac{\varepsilon [g]}{k!} \{ g_{MN} (F_1 F_2) - k (F_2 : F_1)_{MN} \},
\]

(7.40)

where \( k = \text{rank} F_i \) and \( k_s = \text{rank}(\ast F_i) = D - k, i = 1, 2. \)

Let \( I, J \in \Omega, I \neq J \) and \( d(I) = d(J). \) Then

\[
\tau(I) \tau(J) = 0.
\]

(7.41)

Due to Restriction 1 from Section 2 (or Restriction 3 from Sect. 4.2)

\[
(\tau(I) \cdot \tau(J))_{MN} = 0.
\]

(7.42)

It follows from (7.39) and (7.41) that for \( I \neq J \)

\[
\mathcal{F}^{(\alpha \varepsilon, I)} \mathcal{F}^{(\alpha \varepsilon, J)} = 0,
\]

(7.43)

\( I, J \in \Omega_{a v}, v = c, m. \) For composite field

\[
F^{a v} = \sum_{I \in \Omega_{a v}} \mathcal{F}^{(a v, I)},
\]

(7.44)

\( a \in \Delta, v = c, m \) we get (see (7.38))

\[
(F^{a v})^2 = \sum_{I \in \Omega_{a v}} (\mathcal{F}^{(a v, I)})^2,
\]

(7.45)

\[
(F^{a v} \cdot F^{a v})_{MN} = \sum_{I \in \Omega_{a v}} (\mathcal{F}^{(a v, I)} \cdot \mathcal{F}^{(a v, I)})_{MN} + \sum_{I, J \in \Omega_{a v}, I \neq J} (\mathcal{F}^{(a v, I)} \cdot \mathcal{F}^{(a v, J)})_{MN}.
\]

(7.46)

The last term in (7.46) gives rise to off-block-diagonal components of stress-energy tensor.

For \( a \in \Delta \ (d_0 \neq 2) \) we obtain

\[
\mathcal{F}^{(a \varepsilon, I)} \mathcal{F}^{(a m, J)} = \mathcal{F}^{(a \varepsilon, J)} \mathcal{F}^{(a e, I)} = 0,
\]

(7.47)

\( I \in \Omega_{\alpha e}, J \in \Omega_{a m}, \) and hence

\[
F^{a e} F^{a m} = F^{a m} F^{a e} = 0, \quad (F^a)^2 = (F^{a e})^2 + (F^{a m})^2,
\]

(7.48)

where \( F^a = F^{a e} + F^{a m}. \) We also get

\[
(F^a \cdot F^a)_{MN} = (F^{a e} \cdot F^{a e})_{MN} + (F^{a m} \cdot F^{a m})_{MN} + (F^{a e} \cdot F^{a m})_{MN} + (F^{a m} \cdot F^{a e})_{MN}
\]

(7.49)

for \( a \in \Delta. \) The last two terms in (7.49) give rise to off-block-diagonal components of stress-energy tensor.
7.3 Appendix 3. Simple finite dimensional Lie algebras

In summary \[172\], there are four infinite series of simple Lie algebras, which are denoted by

\[ A_r \ (r \geq 1), \quad B_r \ (r \geq 3), \quad C_r \ (r \geq 2), \quad D_r \ (r \geq 4), \]

and in addition five isolated cases, which are called

\[ E_6, \quad E_7, \quad E_8, \quad G_2, \quad F_4. \]

In all cases the subscript denotes the rank of the algebra. The algebras in the infinite series of simple Lie algebras are called the classical (Lie) algebras. They are isomorphic to the matrix algebras

\[ A_r \cong \text{sl}(r+1), \quad B_r \cong \text{so}(2r+1), \quad C_r \cong \text{sp}(r), \quad D_r \cong \text{so}(2r). \]

The five isolated cases are referred to as the exceptional Lie algebras.

**A_r series.** Let \( A_r \) be \( r \times r \) Cartan matrix for the Lie algebra \( A_r = \text{sl}(r+1), \) \( r \geq 1. \) This matrix is described graphically by the Dynkin diagram pictured on Fig. A.1.

\[ \begin{align*}
1 & \quad 2 & \quad 3 & \quad \ldots & \quad r-1 & \quad r \\
\end{align*} \]

Fig. A.1. Dynkin diagram for \( A_r \) Lie algebra

Using the relation for the inverse matrix \( A^{-1} = (A^{ss'}) \) (see Sect.7.5 in \[172\])

\[ A^{ss'} = \frac{1}{r+1} \min(s, s')[r + 1 - \max(s, s')] \] (7.50)

we get for \( n_s = 2 \sum_{s'=1}^r A^{ss'}: \)

\[ n_s = s(r+1-s), \] (7.51)

\( s = 1, \ldots, r. \)

**B_r and C_r series.** Dynkin diagrams for these cases are pictured on Fig. A.2.

\[ \begin{align*}
1 & \quad 2 & \quad 3 & \quad \ldots & \quad r-1 & \quad r \\
\end{align*} \]

Fig.A.2. Dynkin diagrams for \( B_r \) and \( C_r \) Lie algebras

In these cases we have the following formulas for inverse Cartan matrices

\[ A^{ss'} = \begin{cases} 
\min(s, s') & \text{for } s \neq r, \\
\frac{1}{2}s' & \text{for } s = r,
\end{cases} \]

\[ A^{ss'} = \begin{cases} 
\min(s, s') & \text{for } s' \neq r, \\
\frac{1}{2}s & \text{for } s' = r,
\end{cases} \] (7.52)

and

\[ n_s = \begin{cases} 
s(2r+1-s) & \text{for } s \neq r, \\
\frac{r}{2}(r+1) & \text{for } s = r; \end{cases} \quad n_s = s(2r-s), \] (7.53)

for \( B_r \) and \( C_r \) series respectively, \( s = 1, \ldots, r. \)

**D_r series.** We have the following Dynkin diagram for this case (Fig. A.3):

\[ \begin{align*}
1 & \quad 2 & \quad 3 & \quad \ldots & \quad r-2 & \quad r-1 & \quad r \\
\end{align*} \]

Fig.A.3. Dynkin diagram for \( D_r \) Lie algebra
and formula for the inverse matrix \[ 172\]:

\[
A^{s s'} = \begin{cases}
  \min(s, s') & \text{for } s, s' \notin \{r, r - 1\}, \\
  \frac{1}{2} s & \text{for } s \notin \{r, r - 1\}, s' \in \{r, r - 1\}, \\
  \frac{1}{2} s' & \text{for } s \in \{r, r - 1\}, s' \notin \{r, r - 1\}, \\
  \frac{1}{r} & \text{for } s = s' = r \text{ or } s = s' = r - 1, \\
  \frac{1}{r} (r - 2) & \text{for } s = r, s' = r - 1 \text{ or vice versa}.
\end{cases}
\quad (7.54)
\]

Then

\[
n_s = \begin{cases}
  s(2r - 1 - s) & \text{for } s \notin \{r, r - 1\}, \\
  \frac{r}{2} (r - 1) & \text{for } s \in \{r, r - 1\},
\end{cases}
\quad (7.55)
\]

\(s = 1, \ldots, r\).

Let us consider the exceptional Lie algebras. Dynkin diagrams of these algebras are pictured on Fig. 4.A.

Using relations for inverse Cartan matrices from \[172\] we get

\[
n_{s/2} = \begin{cases}
  8, 15, 21, 15, 8, 11 & \text{for } E_6, \ s = 1, \ldots, 6; \\
  17, 33, 48, \frac{49}{2}, 26, \frac{27}{2}, \frac{49}{2} & \text{for } E_7, \ s = 1, \ldots, 7; \\
  29, 57, 84, 110, 135, 91, 46, 68 & \text{for } E_8, \ s = 1, \ldots, 8; \\
  11, 21, 15, 8 & \text{for } F_4, \ s = 1, \ldots, 4; \\
  5, 3 & \text{for } G_2, \ s = 1, 2.
\end{cases}
\quad (7.56)
\]

### 7.4 Appendix 4: Solutions for Toda-like system

#### 7.4.1 General solutions

Let

\[
L = \frac{1}{2} \langle \dot{x}, \dot{x} \rangle - \sum_{s \in S} A_s \exp(2 \langle u_s, x \rangle)
\]

be a Lagrangian, defined on \(V \times V\), where \(V\) is \(n\)-dimensional vector space over \(\mathbb{R}\), \(A_s \neq 0, s \in S; S \neq \emptyset\), and \(\langle \cdot, \cdot \rangle\) is non-degenerate real-valued quadratic form on \(V\). Let \(K_s = \langle u_s, u_s \rangle \neq 0\), for all \(s \in S\).

Then, the Euler-Lagrange equations for the Lagrangian (7.57)

\[
\ddot{x} + \sum_{s \in S} 2A_s u_s \exp(2 \langle u_s, x \rangle) = 0,
\quad (7.58)
\]
have the following solutions
\[
x(t) = \sum_{s \in S} q^t(t) u_s < u_s, u_s > + \alpha t + \beta,
\]
(7.59)

where \(\alpha, \beta \in V\),
\[
< \alpha, u_s > = < \beta, u_s > = 0,
\]
(7.60)
s \(\in S\), and functions \(q^t(u)\) satisfy the Toda-like equations
\[
\ddot{q}^s = -2A_s K_s \exp(\sum_{s' \in S} A_{ss'} q^{s'}),
\]
with
\[
A_{ss'} = \frac{2}{< u_s, u_{s'} >} \frac{< u_{s'}, u_{s'} >}{< u_{s'}, u_s >}.
\]
(7.62)
s, \(s' \in S\). Let the matrix \((A_{ss'})\) be a non-degenerate one. In this case vectors \(u_s, s \in S\), are linearly independent. Then eqs. (7.61) are field equations corresponding to the Lagrangian
\[
L_{TL} = \frac{1}{4} \sum_{s,s' \in S} K_s^{-1} A_{ss'} \dot{q}^s \dot{q}^{s'} - \sum_{s \in S} A_s \exp(\sum_{s' \in S} A_{ss'} q^{s'}) .
\]
(7.63)

For the energy corresponding to the solution (7.59) we get
\[
E = \frac{1}{2} < \dot{x}, \dot{x} > + \sum_{s \in S} \exp(2 < u_s, x >) = E_{TL} + \frac{1}{2} < \alpha, \alpha >,
\]
(7.64)
where
\[
E_{TL} = \frac{1}{4} \sum_{s,s' \in S} K_s^{-1} A_{ss'} \dot{q}^s \dot{q}^{s'} + \sum_{s \in S} A_s \exp(\sum_{s' \in S} A_{ss'} q^{s'}) .
\]
(7.65)
is the energy function corresponding to the Lagrangian (7.61).

For dual vectors \(u^s \in V^*\) defined as \(u^s(x) = < u_s, x >\), \(\forall x \in V\), we have \(< u^s, u^s > = < u_s, u_s >\), where \(< \cdot, \cdot >\) is dual form on \(V^*\). The orthogonality conditions (7.60) read
\[
u^s(\alpha) = u^s(\beta) = 0,
\]
(7.66)
s \(\in S\).

### 7.4.2 Solutions with block-orthogonal set of vectors

Let us consider the Lagrangian (7.61) with the set \(S = S_1 \cup \ldots \cup S_k\), all \(S_i \neq \emptyset\), and
\[
< u_s, u_{s'} > = 0,
\]
(7.67)
for all \(s \in S_i, s' \in S_j, i \neq j; i, j = 1, \ldots, k\).

Let \(h_s = K_s^{-1}, (A^{ss'}) = (A_{ss'})^{-1}, \)
\[
b_s = 2 \sum_{s' \in S} A^{ss'},
\]
(7.68)
for all \(s \in S\), and
\[
A_s / (b_s h_s) = A_{s'} / (b_{s'} h_{s'}),
\]
(7.69)
s, \(s' \in S_i, i = 1, \ldots, k\), (the ratio \(A_s / (b_s h_s)\) is constant inside \(S_i\)).

Then, there exists a special solution to eqs. (7.61)
\[
q^s(t) = -b_s \ln|y_s(t)|2A_s / (b_s h_s)|
\]
(7.70)
where functions \(y_s(t) \neq 0\) satisfy to equations
\[
\frac{d}{dt} \left( y_s^{-1} \frac{dy_s}{dt} \right) = -\xi_s y_s^{-2},
\]
(7.71)
with

\[ \xi_s = \text{sign} \left( \frac{A_s}{b_s h_s} \right), \tag{7.72} \]

\( s \in S \), and coincide inside blocks:

\[ y_s(t) = y_{s'}(t), \tag{7.73} \]

\( s, s' \in S_i, i = 1, \ldots, k \). More explicitly

\[ y_s(t) = s(t - t_s, \xi_s, C_s), \tag{7.74} \]

where constants \( t_s, C_s \in \mathbb{R} \) coincide inside blocks

\[ t_s = t_{s'}, \quad C_s = C_{s'}, \tag{7.75} \]

\( s, s' \in S_i, i = 1, \ldots, k \), and

\[ s(t, \xi, C) \equiv \begin{cases} \frac{1}{\sqrt{C}} \sinh(t\sqrt{C}), & \xi = +1, \quad C > 0; \\ \frac{1}{\sqrt{-C}} \sin(t\sqrt{-C}), & \xi = +1, \quad C < 0; \\ t, & \xi = +1, \quad C = 0; \\ \frac{1}{\sqrt{C}} \cosh(t\sqrt{C}), & \xi = -1, \quad C > 0. \end{cases} \tag{7.76-7.79} \]

For "Toda" part of energy we get

\[ E_{TL} = \frac{1}{2} \sum_{s \in S} C_s b_s h_s. \tag{7.80} \]

### 7.5 Appendix 5: Solutions with Bessel functions

Let us consider two differential operators

\[ 2 \hat{H}_0 = -\frac{\partial^2}{\partial z^2} + 2A \exp^{2qz}, \tag{7.81} \]

\[ 2 \hat{H}_1 = -\exp^{qz} \frac{\partial}{\partial z} \left( \exp^{-qz} \frac{\partial}{\partial z} \right) + 2A \exp^{2qz}. \tag{7.82} \]

Equation

\[ H_k \Psi_k = E \Psi_k \tag{7.83} \]

has the following linearly independent solutions for \( q \neq 0 \)

\[ \Psi_k(z) = e^{kqz/2} B_{\omega_k(E)} \left( \sqrt{2A} \frac{e^{qz}}{q} \right), \quad \omega_k(E) = \sqrt{\frac{k}{4} - \frac{2E}{q^2}}, \tag{7.84} \]

where \( k = 0, 1 \) and \( B_\omega, D_\omega = I_\omega, K_\omega \) are modified Bessel function.

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