Quantum Stability of the Phase Transition in Rigid QED

M. Awada*
D. Zoller, and J. F. Clark
Physics Department
University of Cincinnati, Cincinnati, OH-45221

Abstract

Rigid QED is a renormalizable generalization of Feynman’s space-time action characterized by the addition of the curvature of the world line (rigidity). We have recently shown that a phase transition occurs in the leading approximation of the large $N$ limit. The disordered phase essentially coincides with ordinary QED, while the ordered phase is a new theory. We have further shown that both phases of the quantum theory are free of ghosts and tachyons. In this letter, we study the first sub-leading quantum corrections leading to the renormalized mass gap equation. Our main result is that the phase transition does indeed survive these quantum fluctuations.

* E-Mail address: (moustafa@physunc. phy.uc.edu)
I- Phase Transition in Rigid Model Of QED

Recently we proposed a renormalizable generalization of the Feynman space-time picture of QED [1], [2]. In this picture the dynamical variables are the space-time position \( x^\mu, \mu = 1, 2, \ldots D \) of the point particle and the photon field \( A_\mu \). The usual Feynman action consists of the arc-length of the world line, the Maxwell action, and the usual point particle-Maxwell coupling. The renormalizable generalization is characterized by the addition of the scale invariant curvature of the world line. The origin of the term rigid refers to the Boltzmann suppression of curved trajectories by the curvature.

In a subsequent article [3] we proved a conjecture in [1] and [2] that there is phase transition in rigid QED. We found a critical line in the plane of the Coulomb coupling verses the curvature coupling below which there is a disordered phase and above which is a new ordered strongly coupled phase. The higher derivative nature of rigid QED should cause a serious pause as any higher derivative theory is typically pathological. Indeed, the arc-length plus the curvature term theory has classical runaway solutions which are tachyonic. Whether a higher derivative regulated quantum theory has such pathological behaviour is more subtle and depends on details of the continuum limit. A free scalar field theory on the lattice with spacing \( \frac{1}{\Lambda} \), has higher derivatives and ghosts. However these ghosts have mass of order \( \Lambda \) and decouple in the continuum limit as \( \Lambda \to \infty \). In the less trivial case of rigid QED we have shown that the ghosts have mass of order \( \Lambda \) and similarly decouple from the continuum limit. The necessity of the decoupling mechanism is associated with the absence of fine tuning of the curvature and the Coulomb coupling constants. Having phase transition would be of utmost importance because this would imply that the couplings of the theory are fixed by dimensional transmutation in both the disordered and ordered phases [4].

Our proof in [3] of the phase transition was based on the leading order approximation in large \( N \), where \( N \) is the space-time dimensions. Even though, the large \( N \) limit is a successful approximation for non-linear sigma models, and some spin systems it can sometimes lead to an incorrect conclusion. The leading order of the large \( N \) approximation is mean field theory which can give incorrect predictions in lower dimensions. For example mean field theory incorrectly gives a phase transition in the one dimensional Ising model. This discrepancy is resolved by carefully examining the sub-leading quantum corrections (loops) where one shows that such quantum corrections in fact destroy the phase transition. Therefore it is crucial to examine the quantum loop corrections to the mass gap, and the critical line of our model.

In this letter we will prove that the phase transition in our model of rigid QED survives quantum fluctuations and that the quantum loop corrections to the sub-leading order lead to mass and wave function renormalizations. As in non-linear sigma model [5], mass renormalization is equivalent to charge renormalization. Thus we obtain the renormalization group equation.
The effective action obtained after the Guassian integration of the gauge field sector is \([1], [2]\):

\[
\begin{align*}
\mu_0 \int_0^1 d\lambda \sqrt{\dot{x}^2} &+ \frac{1}{t_0} \int_0^1 d\lambda \frac{\dot{x}^2 - (\vec{x} \cdot \vec{\dot{x}})^2}{\dot{x}^2} + \frac{1}{2t_0} \int_0^1 d\lambda d\lambda' \dot{x}(\lambda) \dot{x}(\lambda') V(|\vec{x} - \vec{x}'|) \quad (1a) \\
\end{align*}
\]

where the first term is the arc-length \( ds = d\lambda \sqrt{\dot{x}^2} \) of a point particle of bare mass \( \mu_0 \), the second term is the curvature \( k(s) = \frac{|d^2 x(s)|}{ds^2} \) of the world line defined to be the length of the acceleration, \( t_0 \) is a dimensionless bare coupling constant (scale invariance of the curvature term) and \( V \) is the long range Coulomb potential:

\[
V(|\vec{x} - \vec{x}'|, a) = \frac{2g_0}{\pi} \frac{1}{|x(\lambda) - x(\lambda')|^2 + a^2} \quad (1b)
\]

We have introduced the cut-off "a" to avoid the singularity at \( \lambda = \lambda' \) and define \( g_0 = t_0 \alpha_{\text{Coulomb}} \approx t_0 \frac{e^2}{4\pi} \). In the arc-length gauge \( \dot{x}^2 = 1 \) we can gauge fix the action \((1a)\) and obtain:

\[
\tilde{S} = \frac{1}{2} \mu_0 L + \frac{1}{2t_0} \int_0^L d\lambda (e^{-1} \dot{x}^2 + e + \omega (\dot{x}^2 - 1)) + \frac{1}{2t_0} \int_0^L d\lambda d\lambda' \dot{x}(\lambda) \dot{x}(\lambda') V(|\vec{x} - \vec{x}'|, a) \quad (2)
\]

where \( e \) is an einbein to remove the square root of the acceleration, \( \omega \) is a Lagrange multiplier that enforces the constraint \( \dot{x}^2 = 1 \), and \( L \) is the length of the path. The partition function is:

\[
Z = \int D\omega De D\vec{x} \exp(-\tilde{S}) \quad (3)
\]

The long range Coulomb interactions are non-local and impossible to integrate. Therefore we consider

\[
x'^\nu(\lambda) = x'^\nu_0(\lambda) + x'^\nu_1(\lambda)
\]

and expand the action \((2)\) to quadratic order in \( x'^\mu_1(\lambda), \mu = 1, \ldots, D \) about the background straight line \( x_0 \). The x-integration is now Gaussian and to the leading \( D \)
approximation we obtain the following full effective action $S_{\text{eff}}$:

$$S_{\text{eff}} = \frac{1}{2t_0} \int d\lambda e(\lambda) - \omega(\lambda) + t_0 \text{tr} \ln A$$  \hspace{1cm} (4)

where $A$ is the operator

$$A = \partial^2 e^{-1} \partial^2 - \partial \omega \partial + V(\lambda, \lambda') .$$  \hspace{1cm} (5)

In the large D limit the stationary point equations resulting from varying $\omega$ and $e$ respectively are:

$$1 = t_0 \text{tr} G$$  \hspace{1cm} (6a)

$$1 = t_0 \text{tr} (e^{-2}(-\partial^2 G))$$  \hspace{1cm} (6b)

where the world line Green’s function is defined by:

$$G(\lambda, \lambda') = \langle \lambda|(-\partial^2)A^{-1}|\lambda' \rangle$$  \hspace{1cm} (7)

The stationary points are:

$$\omega(\lambda) = \omega_0, \quad \langle \lambda|e^{-1}|\lambda' \rangle = \int \frac{dp}{2\pi} \frac{e^{i(\lambda-\lambda')}}{|p|}$$  \hspace{1cm} (8)

where $\omega_0$ is the bare mass which is a positive constant. Using the stationary solutions (8) the operator $A$ is given by:

$$\text{tr} \ln A = \int \frac{dp}{2\pi} \ln |p|^3 + p^2 \omega_0 + p^2 V_0(|p|) + V_1(|p|)$$  \hspace{1cm} (9)

where

$$V_0(|p|) = \frac{2g_0}{a} e^{-a|p|}, \quad V_1(|p|) = \frac{2g_0}{a^2} [e^{-a|p|}(|p| + \frac{1}{a}) - \frac{1}{a}] .$$  \hspace{1cm} (10)

Thus Eq.(6) becomes the single mass gap equation [3]:

$$1 = D t_0 \int \frac{dp}{2\pi} \frac{p^2}{p^2(|p| + \omega_0) + p^2 V_0(|p|) + V_1(|p|)}$$  \hspace{1cm} (11)
where $\omega_0$ is now the mass associated with the propagator:

$$< \dot{x}^\mu(p)\dot{x}^\nu(-p) > = Dt_0 \frac{\delta^{\mu\nu}}{|p| + \omega_0 + V(|p|)}$$

(12)

where:

$$V(|p|) = V_0(|p|) + \frac{V_1(|p|)}{p^2}$$

(13)

which is regular at $p = 0$. To obtain a non-zero phase transition temperature the mass gap equation must be infra-red finite for $\omega_0 = 0$. Therefore without Coulomb long range interactions ($g_0 = 0$) the theory exists only in the disordered phase $t_0 > t_c$ and the U.V stable fixed point is $t_c = 0$. In this case the beta function of the pure curvature theory is asymptotically free, indicating the absence of the curvature term at large distance scales. In contrast to the naive classical limit the theory is therefore well behaved and free of ghosts. In [3] we calculated the poles of the Green’s function in the presence of Coulomb interactions ($g_0 \neq 0$) using the large D limit, and showed the absence of ghosts and tachyons in both the ordered and disordered phases which are separated by the critical line defined by eq.(11) at $\omega_0 = 0$:

$$1 = \frac{D t_0}{\pi} \int_0^\rho dy \frac{y^2}{y^3 + 2g_0(y^2e^{-y} + ye^{-y} + e^{-y} - 1)}$$

(14)

where we have made the change of variable $y = \alpha p$ and introduced the U.V. cut-off $\Lambda$, and $\rho = \frac{\Lambda}{\Lambda_0}$ where $\frac{1}{\alpha} := \Lambda_0$. It is straightforward to prove that there exist a $g^*_0$ (c.f. Fig.1) for which any choice of $\rho$ leads to phase transition as long as $g_0 < g^*_0$. We will set $\rho = 1$. Notice that eq.(14) is finite except at $g_0 = 0$ (absence of Coulomb interactions). The critical curve distinguishing the two phases in the $(t,g)$ plane is shown in Fig.1. The order parameter of the theory is the mass gap equation (14) where $\omega_0$ is the parameter that distinguishes the two phases. In the disordered phase $\omega_0 > 0$, while in the ordered phase it is straightforward to show that $\omega_0 = 0$. In the disordered phase the coupling constants $t_0$ and $g_0$ are completely fixed by dimensional transmutation in terms of the cut-off $\Lambda$ and $\omega_0$. Thus, they cannot be fine tuned. This is an important property that is vital in proving the absence of ghosts in our model [3]. From (14) we can immediately examine whether there is a phase transition in the pure Coulomb theory i.e ordinary QED. The curvature term would then be absent. This corresponds to the absence of the $y^3$ term in (14). If we choose the cut-off of the theory $\Lambda$ to be at the Compton wavelength i.e $\Lambda_{Compton} = \frac{2\pi}{a}$ one finds in this particular case that the integral (14) diverges implying an absence of a phase transition.
II - The Loop Corrected Gap Equation

In mean field theory i.e leading order in $\frac{1}{D}$, the relevant propagator is equation (12). In the sub-leading correction to mean field theory, the quantum fluctuation imply a new term corresponding to the self-energy of the $\dot{x}^{\mu}$-field

$$<\dot{x}^\mu(p)\dot{x}^\nu(-p)> = Dt_0 \frac{\delta^{\mu\nu}}{(|p| + \omega_0 + V(|p|) + \frac{1}{D}\Sigma(|p|))}. \quad (15)$$

The new contribution $\Sigma(|p|)$ arises from fluctuations of the Lagrange multiplier $\omega$ where the fluctuations $\eta$ are defined by:

$$\omega = \omega_0 + i \frac{1}{\sqrt(D/2)} \eta. \quad (16)$$

Expanding the effective action in powers of $\eta$, it is straightforward to extract the $\eta$ propagator (Fig (2)):

$$\Pi(|p|) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{1}{(|k| + \omega_0 + V(|k|))} \frac{1}{(|p + k| + \omega_0 + V(|p + k|))}. \quad (17)$$

The self energy $\Sigma$ can be computed from the diagrams of Fig(3). These diagrams are of order $\frac{1}{D}$ and represent the quantum fluctuations:

$$\Sigma(|p|) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{\Pi^{-1}(|k|)}{(|p + k| + \omega_0 + V(|p + k|))}$$

$$- \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \int_{-\infty}^{+\infty} \frac{dq}{2\pi} \frac{\Pi^{-1}(0)}{(|q| + \omega_0 + V(|q|))^2} \frac{\Pi^{-1}(|k|)}{(|q + k| + \omega_0 + V(|q + k|))}. \quad (18)$$

A Taylor expansion of the self energy about zero momentum leads to mass and wave function renormalizations and a remaining piece $\tilde{\Sigma}$. The propagator now
reads:

\[
Z = \frac{Z}{(|p| + \omega + \mathcal{V}(|p|) + \frac{1}{D} \hat{\Sigma}(|p|))}.
\]  

(19)

where

\[
Z = 1 - \frac{1}{D} \Sigma'(0)
\]  

(20a)

is the wave function renormalization,

\[
\omega = \omega_0 + \frac{1}{D} (\Sigma(0) - \omega_0 \Sigma'(0))
\]  

(20b)

is the renormalized mass and,

\[
g = Zg_0
\]  

(20c)

is the renormalized Coulomb coupling constant. The renormalized mass gap equation is

\[
1 = D t \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \frac{1}{(|p| + \omega + \mathcal{V}(|p|) + \frac{1}{D} \hat{\Sigma}(|p|))}.
\]  

(21)

To examine whether there is still a phase transition i.e the infra-red finiteness of the renormalized mass gap equation eq.(21) at \(\omega = 0\), we must examine the renormalizability of the theory. We have seen that the typically divergent terms \(\Sigma(0)\) and \(\Sigma'(0)\) can be absorbed in mass and wave function renormalizations. This will be true if we can regularize (18) so as to respect (19) and (20) with \(\hat{\Sigma}(|p|)\) being finite. Equivalently, \(\Sigma''(|p|)\) must be ultra-violet (U.V.) finite. To study this question we only need the asymptotic behaviour of \(\Pi\):

\[
\Pi_{as}(|p|) = \frac{2}{\pi} \frac{\log \frac{|p|}{\omega_0}}{(|p| + \mathcal{V}(|p|))}.
\]  

(22)

From (18) it is straightforward to find that

\[
\Sigma''(|p|) = \frac{1}{\Pi^{-1}(k)} \frac{\Pi^{-1}(|k|)}{(|p + k| + \omega_0 + \mathcal{V}(|p + k|))^2} \frac{d^2}{dk^2} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \left(1 + \mathcal{V}(|p + k|)\right)^2
\]

\[
- \frac{\Pi^{-1}(|p|)}{(\omega_0 + \frac{4g_0}{a})^2} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \mathcal{V}''(|p + k|) \frac{\Pi^{-1}(|k|)}{(|p + k| + \omega_0 + \mathcal{V}(|p + k|))^2}.
\]  

(23)

For a non zero cut-off "a" it is clear from (23) that the second term is a finite
function since $\Pi^{-1}(|p|)$ is a finite function of $|p|$. Inserting (22) into (23) one can easily prove that $\Sigma''(0)$ is ultra-violet (u.v) finite and thus $\Sigma''(|p|)$ is a finite function of $|p|$. Furthermore, although $V(a, |p|)$ diverges as $-\frac{2m^2}{a^2}$ as $a \to 0$, one can check that $\Sigma''(|p|)$ is still U.V. finite in this limit. Due to the complexity of the long range potential $V(a, |p|)$ we cannot calculate $\Sigma''(|p|)$ exactly. Therefore, to further confirm whether $\Sigma''(|p|)$ is indeed U.V. finite, Eq.(23) is calculated numerically for arbitrary fixed values of the cut-off $a$ and mass $\omega$.

To facilitate numerical calculations, $\Sigma''(|p|)$ is plotted in the range $0 < g < g^*$. In this range there are no real poles in the propagator (12) since $(|p| + V(|p|)) > 0$ for arbitrary $p$. Different choices of $a$ and $\omega$ reveal similar smooth surfaces, and, as $g \to g^*$ $\Sigma''(|p|)$ becomes more negative but always remains finite for arbitrary $p$.

Due to the asymptotic form of $\Pi_{as}(|k|)$ defined in Eq.(22), care must be taken in the numerical integration over $k$ in (23) to avoid artificial singularities in the small $k$ regime. Alternatively we can work out the regularized and finite self energy using the SM regularization scheme [5]:

\[
\tilde{\Sigma}_{finite}(|p|) = \Sigma(|p|) - \frac{\omega_0}{2}(I_0 - \frac{1}{\omega_0}I_1) + \frac{1}{2}(|p| + \omega_0)I_0
\]  

(24a)

\[
I_0(\frac{\Lambda}{\omega_0}, g) = \int_{\Lambda}^{\tilde{\Lambda}} dk \frac{1}{\log \frac{k}{\omega_0} (|k| + V(|k|))}
\]  

(24b)

\[
I_1(\frac{\Lambda}{\omega_0}, g) = (\Pi(0))^{-1} \int_{\Lambda}^{\tilde{\Lambda}} \frac{dk}{\pi} \frac{1}{(|k| + \omega_0 + V(|k|))} = \frac{2\Pi(0)^{-1}}{Dt_0}.
\]  

(24c)

To the first sub-leading order we have therefore shown that the theory is renormalizable. The renormalization group flow for the renormalized curvature coupling $t$ and the Coulomb coupling $g$ can be derived from (20c) and the mass renormalization by holding $\omega(\Lambda, t, g)$ fixed. From (11), (18) and (20) and (24) we obtain:

\[
\omega = \tilde{Z}\omega_0 = (Z - \frac{I_1(\frac{\Lambda}{\omega_0}, g)}{D\omega_0})\omega_0
\]  

(25a)

where $Z$ is given by:

\[
Z = 1 + \frac{1}{2D}I_0(\frac{\Lambda}{\omega_0}, g).
\]  

(25b)

Having shown that $\tilde{\Sigma}$ is indeed finite then the loop corrected critical line of the
theory defined by (21) at $\omega = 0$ is:

$$
1 = Dt \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \frac{1}{(|p| + \mathcal{V}(|p|)) + \frac{1}{D} \tilde{\Sigma}_{finite}(|p|)}.
$$

which is Infra-red finite since by its definition $\tilde{\Sigma}(0) = 0$ and (26) then has the exact behaviour as (14) at $y=0$. Furthermore, as long as $g < g^*$ we have shown that the critical line (14) has no real poles. i.e $(|p| + \mathcal{V}(|p|)) > 0$. The presence of $\tilde{\Sigma}$ will not affect the above conclusion to any sub-leading finite order in perturbation theory. In the sub-leading $\frac{1}{D}$ order the critical line is:

$$
1 = Dt \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \left( \frac{1}{(|p| + \mathcal{V}(|p|))} - \frac{1}{(|p| + \mathcal{V}(|p|))} \frac{1}{D} \tilde{\Sigma}_{finite}(|p|) \frac{1}{(|p| + \mathcal{V}(|p|))} \right).
$$

As in any quantum field theory the location of the poles in the presence of $\tilde{\Sigma}_{finite}(|p|)$ is a non-perturbative issue and requires the form of $\tilde{\Sigma}_{finite}(|p|)$ to all orders in perturbation theory.

**Acknowledgement**

We are very grateful to Prof. A. Polyakov for his constant encouragement and extensive support over the last year and a half and for suggesting that we address the quantum stability of the phase transition both in the model of rigid QED and that of rigid strings coupled to long range interactions [6]. In addition, one of us M.A is thankful for his long and patient conversations explaining the decoupling mechanism of ghosts and its connection with critical phenomena. We are also grateful to Prof. Y. Nambu for his extensive support, encouragement and long discussions over the last two years without which we could not have gone far in our investigations. M.A would like to thank P.Ramond, C. Thorn, and Z. Qui for constructive discussions and suggestions.

**References**

[1] M. Awada and D. Zoller, Phys.Lett B299 (1993) 151, M.Awada, M.Ma, and D.Zoller Mod.phys. Lett A8,(1993), 2585

[2] M.Awada and D.Zoller, Int. J. Mod. Phys. A9(1994) pp.4077-4099.

[3] M. Awada and D. Zoller, Phys.Lett B325 (1994) 119

[4] A. Polyakov, private communications.

[5] A. Polyakov, Gauge fields, and Strings, Vol.3, harwood academic publishers, J.Orloff and R.Brout, Nucl. Phys. B270 [FS16],273 (1986), M. Camponistrini and P.Rossi, Phys. Rev.D 45, 618 (1992) ; 46, 2741 (1992), H. Flyvberg, Nucl. Phys. B 348, 714, (1991).

[6] M. Awada and D. Zoller, Phys.Lett B325 (1994) 115
Short-Range Disordered Phase, Order Parameter: $w > 0$

Long-Range Ordered Phase, Order Parameter: $w = 0$
\[ \Pi(p) = \begin{align*} \end{align*} \]

Fig(2)

\[ \Sigma(p) = \begin{align*} \end{align*} \]

Fig(3)
Fig. 4