ON ALGEBRAIC CONDITION FOR NULL CONTROLLABILITY OF SOME COUPLED DEGENERATE SYSTEMS

Ait Ben Hassi El Mustapha, Fadili Mohamed* and Maniar Lahcen

Département de Mathématiques, Faculté des Sciences Semlalia
LMDP, UMMISCO (IRD-UPMC)
Université Cadi Ayyad, Marrakech, 40000, B.P 2390, Morocco

Dedicated to Professor H. Bouslous on the occasion of his 65th birthday.

(Communicated by Piermarco Cannarsa )

Abstract. In this paper we will generalize the Kalman rank condition for the null controllability to $n$-coupled linear degenerate parabolic systems with constant coefficients, diagonalizable diffusion matrix, and $m$-controls. For that we prove a global Carleman estimate for the solutions of a scalar $2n$-order parabolic equation then we infer from it an observability inequality for the corresponding adjoint system, and thus the null controllability.

1. Introduction and main result. In this work, we focus on the following problem :

$$\begin{cases}
\partial_t Y = (D_M + A)Y + Bv1_\omega & \text{in } Q, \\
CY = 0 & \text{on } \Sigma, \\
Y(0) = Y_0 & \text{in } (0,1),
\end{cases}$$

where $Q := (0,T) \times (0,1)$, $\Sigma := (0,T) \times \{0,1\}$, $\omega \subset (0,1)$ is a nonempty open control region, $1_\omega$ denotes the characteristic function of $\omega$, $T > 0$, $D_M$ is a $n \times n$ matrix, $B$ is a $n \times m$ matrix, $v = (v_1, \cdots, v_m)^*$ is the control and $Y = (y_1, \cdots, y_n)^*$ is the state. In the sequel we denote also $Q_\omega := (0,T) \times \omega$. The operator $M$ is defined by $My = (ay_x)_x$ for $y \in D(M) \subset L^2(0,1)$. For $Y = (y_1, \cdots, y_n)^*$, $MY$ denotes $(M_y_1, \cdots, M_y_n)^*$. The function $a$ is a diffusion coefficient which degenerates at 0 (i.e., $a(0) = 0$) and which can be either weak degenerate (WD), i.e.,

$$\begin{cases}
(i) \ a \in C([0,1]) \cup C^1((0,1)), \ a > 0 \text{ in } (0,1), \ a(0) = 0, \\
(ii) \ \exists K \in [0,1] \text{ such that } xa'(x) \leq K a(x), \ \forall x \in [0,1],
\end{cases}$$

2010 Mathematics Subject Classification. 35K20, 35K65, 47D06, 93B05, 93B07.
Key words and phrases. Parabolic degenerate systems, Carleman estimate, null controllability, observability inequality, Kalman condition.

* Corresponding author: fadilimed@live.fr.
or strong degenerate (SD), i.e.,

\[
\begin{cases}
(i) \ a \in \mathcal{C}^1([0,1]), \ a > 0 \ in \ (0,1), \ a(0) = 0, \\
(ii) \ \exists K \in [1,2) \ such \ that \ xa'(x) \leq K a(x) \forall x \in [0,1], \\
(iii) \ \exists \theta \in (1,K) x \mapsto \frac{a(x)}{x^{\theta}} \ is \ nondecreasing \ near \ 0, \ if \ K > 1, \\
\quad \ \exists \theta \in (0,1) x \mapsto \frac{a(x)}{x^{\theta}} \ is \ nondecreasing \ near \ 0, \ if \ K = 1.
\end{cases}
\]

The boundary condition \( CY = 0 \) is either \( Y(0) = Y(1) = 0 \) in the weak degenerate case \((WD)\) or \( Y(1) = (aY_x)(0) = 0 \) in the strong degenerate case \((SD)\). Null controllability of nondegenerate \((a > 0)\) parabolic systems has been widely studied over the last 40 years and there have been a great number of results. In the case of one equation \((n = 1)\), the result was obtained by A. V. Fursikov and O. Y. Imanuvilov \[14\] and G. Lebeau and L. Robbiano \[18\]. In the case of coupled systems \( n \geq 2 \), M. Gonzalez-Burgos, L. de Teresa \[15\] provided a null controllability result for a cascade parabolic system. Recently, F. Ammar-Khodja et al. \[3, 4\] obtained several results characterizing the null controllability of fully coupled systems with \( m \)-control forces by a generalized Kalman rank condition.

For degenerate systems \((e.g., a(0) = 0)\), null controllability of one equation was studied in \[6, 9, 10\]. The case of two coupled equations \((n = 2)\), cascade systems are considered in \[11, 12\] and in \[1, 2\] the authors have studied the null controllability of degenerate nonecascade parabolic systems. In the case \( n > 2 \), in a recent work \[13\], we have extended the null controllability results obtained by Ammar-Khodja et al. \[4\] to a class of parabolic degenerate systems \((1)\) in the two following cases:

1. the coupling matrix \( A \) is a cascade one and the diffusion matrix \( D = \text{diag}(d_1, \ldots, d_n) \) where \( d_i > 0, i = 1, \ldots, n \),
2. the coupling matrix \( A \) is a full matrix (noncascade) and the diffusion matrix \( D = dI_n, \ d > 0 \).

In the present paper, we study the case where the coupling matrix \( A \) is a full matrix and the diffusion matrix \( D \) is a diagonalizable \( n \times n \) matrix with positive real eigenvalues, i.e.,

\[
D = P^{-1} JP, \ P \in \mathcal{L}(\mathbb{R}^n), \ det(P) \neq 0,
\]

where \( J = \text{diag}(d_1, \ldots, d_n) \), \( d_i > 0, 1 \leq i \leq n \). The strategy used in this case is quite different from the one used in \[13\], and follows the one used in \[5\]. To establish an observability inequality for the adjoint system of \((1)\), we prove a global Carleman estimate for a degenerate scalar equation \((40)\) of \( 2n \) order in space. This will lead to several Carleman estimates, and thus to an observability inequality, for our adjoint system. Another difference with \[13\] is that the Carleman estimates established in the current paper involve the terms \( y_t \) and \( (a(x)y_x)_x \) in addition to the state \( y \) and its space derivative \( y_x \).

Let us introduce the following weighted spaces. In the \((WD)\) case:

\[
H^1_a = \{ u \in L^2(0,1)/u \ absolutely \ continuous \ in [0,1], \sqrt{a}u_x \in L^2(0,1) \ and \ u(1) = u(0) = 0 \}
\]

and

\[
H^2_a = \{ u \in H^1_a(0,1)/au_x \in H^1(0,1) \}.
\]

In the \((SD)\) case:

\[
H^1_a = \{ u \in L^2(0,1)/u \ absolutely \ continuous \ in [0,1], \sqrt{a}u_x \in L^2(0,1) \ and \ u(1) = 0 \}
\]
and
\[ H^2_u = \{ u \in H^1(0,1)/au \in H^1(0,1) \} \]
\[ = \{ u \in L^2(0,1)/u \text{ absolutely continuous in } (0,1], \]
\[ au \in H^1_0(0,1), au \in H^1(0,1) \text{ and } (au_x)(0) = 0 \} . \]

In both cases, the norms are defined as follow
\[ \| u \|_{H^2_u}^2 = \| u \|_{L^2(0,1)}^2 + \| \sqrt{a} u_x \|_{L^2(0,1)}^2, \quad \| u \|_{H^2_u}^2 = \| u \|_{H^1_u}^2 + \| (au_x)_x \|_{L^2(0,1)}^2. \]

Let us note that, by using the assumptions on the operator \( \mathcal{M} \) and the condition (4) on the diffusion matrix \( D \), for every \( Y_0 \in L^2(0,1)^n \) and \( v \in L^2((0,T) \times (0,1))^m \)

system (1) possesses a unique solution \( Y \in L^2(0,T; H^2_u(0,1)^n) \cap C^0([0,T]; L^2(0,1)^n) \).

Let us denote \( L := D \mathcal{M} + A \), with \( D(L) = D(\mathcal{M})^n = H^2_u(0,1)^n \). Then the Kalman operator associated with \((L,B)\) is the matrix operator
\[ \begin{cases}
   K := [L|B] : D(K) \subset L^2(0,1)^{nm} \rightarrow L^2(0,1)^n, \\
   D(K) := \{ u \in L^2(0,1)^{nm} : Ku \in L^2(0,1)^n \},
\end{cases} \]

where
\[ [L|B] := [L^{n-1}B|L^{n-2}B| \cdots |LB|B] . \]

The adjoint system associated to the system (1) is the following
\[ \begin{cases}
   -\partial_t \phi = D^* \mathcal{M} \phi + A^* \phi & \text{in } Q, \\
   C \phi = 0 & \text{on } \Sigma, \\
   \phi(T) = \phi_T & \text{in } (0,1).
\end{cases} \]

To study the null controllability of the system (1), we need to establish an observability inequality of the corresponding adjoint problem (6). Indeed, we must prove the existence of a positive constant \( C \) such that, for every \( \phi_0 \in L^2(0,1)^n \), the solution \( \phi \in C^0([0,T],L^2(0,1)^n) \) of system (6) satisfies the following estimate
\[ \| \phi(0,\cdot) \|_{L^2(0,1)^n}^2 \leq C \int_{(0,T) \times \omega} |B^* \phi(t,x)|^2 dx dt. \]

The inequality (7) will be deduced from a global Carleman estimate satisfied by the solution of the adjoint system (6) (Corollary 2). To prove this, we first show a Carleman estimate (Theorem 4.1) which bounds a weighted global integral of \( K^* \phi \) by means of a weighted local integral of \( B^* \phi \). This last Carleman estimate is obtained by showing several intermediate Carleman estimates, and by assuming the generalized Kalman condition \( Ker(K^*) = \{0\} \), we will be able to obtain the desired Carleman estimate for system (6). Thus, we conclude with the observability inequality (7) and the null-controllability of system (1). At the end, we show that the generalized Kalman condition \( Ker(K^*) = \{0\} \) is also necessary. Thus our main result is the following.

**Theorem 1.1.** Let us assume that \( D \) satisfies (4). Then, system (1) is null controllable at any time \( T > 0 \) if and only if the Kalman operator \( K \) satisfies
\[ Ker(K^*) = \{0\}. \]

The rest of the work is organized as follows: In section 2, we state some properties of the unbounded operator \( K \) and give a useful characterization of the Kalman condition \( Ker(K^*) = \{0\} \) by using the spectrum of operator \( \mathcal{M} \). Section 3 is devoted to show several intermediate Carleman estimates for scalar parabolic degenerate...
equations of order 2 and 2n in space. In Section 4, the proof of Theorem 1.1 is
given in the end of Section 4.

All along the article, we use generic constants for the estimates, whose values
may change from line to line.

2. Spectrum of operator \( \mathcal{M} \) and some algebraic tools. This section will be
devoted to prove two crucial properties of the Kalman operator \( \mathcal{K} \) and to give an
equivalent algebraic condition to the condition (8). Let us focus on the spectrum of
the unbounded operator \( \mathcal{M} \) defined by \( \mathcal{M} u = (a(x)u_x)_x \), for \( u \in D(\mathcal{M}) = H^2_0(0, 1) \).

It is known that the operator \( -\mathcal{M} \) is a definite positive operator. We will use
the fact that \( H^2_0(0, 1) \) is compactly embedded in \( L^2(0, 1) \), see [7, 8, 19]. Thus,

\[ -\mathcal{M} \Phi_n = \lambda_n \Phi_n, \quad n \in \mathbb{N}^* \]  

Remark 1. In the case \( a(x) = x^\alpha \) with \( 0 < \alpha \leq 1 \) as in [16], the eigenfunctions
and eigenvalues of \( \mathcal{M} \) can be explicitly given using Bessel’s functions.

Now, we give some algebraic tools. It is known that \( D := \cap_{p \geq 0} D(\mathcal{M}^p) \) is dense
in \( D(\mathcal{M}^p) \) for every \( p \geq 0 \) and \( D^{nm} \subset D(\mathcal{K}) \). Thus, \( D(\mathcal{K}) = L^2(0, 1)^n \) and \( \mathcal{K} \)
is well defined from \( D(\mathcal{K}) \subset L^2(0, 1)^n \) into \( L^2(0, 1)^{nm} \). The formal adjoint of \( \mathcal{K} \),
again denoted by \( \mathcal{K}^* \) is given by

\[ \mathcal{K}^* = \begin{bmatrix} B^* (L^*)^n & -1 \\ B^* L^* & 0 \end{bmatrix} \]

and it coincides with the adjoint operator of \( \mathcal{K} \) on \( \mathbb{D}^n \). Moreover, we note that when
\( a \in C^\infty([0, 1]) \), from [7, Proposition 3.8], \( \mathbb{D} = C^\infty([0, 1]) \). Thereafter, we recall some
properties of the Kalman operator \( \mathcal{K} \) as it is given in [5]. For any \( j, p \in \mathbb{N}^* \), we consider the projection operator

\[ P^j_p : \Psi = (\Psi_k)_{1 \leq k \leq j} \in L^2(0, 1)^j \rightarrow P^j_p(\Psi) = (\langle \Psi_k, \Phi_p \rangle)_{1 \leq k \leq j} \in \mathbb{R}^j \]

where \( \langle \cdot, \cdot \rangle \) stands for the scalar product in \( L^2(0, 1) \). All along this paper, we denote by \( |\cdot| \) the euclidian norm in \( \mathbb{R}^j \). Thus, if \( j \in \mathbb{N}^* \), we have the following characterisation of \( \mathbb{D}^j \)

\[ \mathbb{D}^j = \left\{ \Psi = \sum_{p \geq 1} \Psi_p \Phi_p : \Psi_p \in \mathbb{R}^j \text{ and } \sum_{p \geq 1} \lambda_p^2 m |\Psi_p|^2 < \infty, \forall m \geq 0 \right\} \]

For \( p \in \mathbb{N}^* \), \( L_p := -\lambda_p \mathbb{D} + A \in \mathcal{L}(\mathbb{R}^n) \) and

\[ \mathcal{K}_p = [L_p B] = [L_p^{n-1} B \cdots L_p B B] \in \mathcal{L}(\mathbb{R}^{nm}, \mathbb{R}^n) \]

We have the following equalities

\[ \left\{ \begin{array}{l}
L(b \Phi_p) = (L_p b) \Phi_p, \quad b \in \mathbb{R}^n, p \geq 1, \\
\mathcal{K}(b \Phi_p) = (\mathcal{K}_p b) \Phi_p, \quad b \in \mathbb{R}^{nm}, p \geq 1.
\end{array} \right. \]

Since \( L \) and \( \mathcal{K} \) are closed unbounded operators, one has

\[ \left\{ \begin{array}{l}
L y = \sum_{p \geq 1} L_p P^p_p(y) \Phi_p, \quad y \in D(L), \\
\mathcal{K} u = \sum_{p \geq 1} \mathcal{K}_p P^{nm}_p(u) \Phi_p, \quad u \in D(\mathcal{K}).
\end{array} \right. \]
We have the following properties

and then
\[ D(K) = \left\{ u \in L^2(0,1)^{nm} : \sum_{p \geq 1} |K_pP_n(u)|^2 < \infty \right\}. \]

In a similar way, we obtain
\[ \begin{cases} 
K^* = \sum_{p \geq 1} K_p^*P_n(^\cdot)\Phi_p, \\
D(K^*) = \left\{ \varphi \in L^2(0,1)^n : \sum_{p \geq 1} |K_p^*P_n(\varphi)|^2 < \infty \right\}.
\end{cases} \]

We define also the operator \( KK^* : D(KK^*) \subset L^2(0,1)^n \rightarrow L^2(0,1)^n \), with domain
\[ D(KK^*) = \left\{ \varphi \in L^2(0,1)^n : K^*\varphi \in D(K), KK^*\varphi \in L^2(0,1)^n \right\}. \]

The operator \( KK^* \) is closed, and a simple computation provides
\[ \begin{cases} 
KK^*\varphi = \sum_{p \geq 1} K_pK_p^*P_n(\varphi)\Phi_p, \\
D(KK^*) = \left\{ \varphi \in L^2(0,1)^n : \sum_{p \geq 1} |K_pK_p^*P_n(\varphi)|^2 \leq \sum_{p \geq 1} |K_p^*P_n(\varphi)|^2 < \infty \right\}.
\end{cases} \]

As in [5], we obtain the following result.

**Proposition 1.** The following conditions are equivalent

1. \( Ker(K^*) = \{0\} \).
2. \( Ker(KK^*) = \{0\} \).
3. \( det(K_pK_p^*) \neq 0 \) for every \( p \geq 1 \).

**Proof.** To show (1) \( \Rightarrow \) (2), assume that \( Ker(KK^*) \neq \{0\} \). Then, there exists a non-zero element \( v \in L^2(0,1)^n \) such that \( KK^*v = 0 \). Thus, \( ||K^*v||^2 = 0 \). Therefore, \( K^*v = 0 \) and this contradicts (1). The implication (2) \( \Rightarrow \) (1) follows from \( Ker(K^*) \subset Ker(KK^*) \). For (2) \( \Rightarrow \) (3), assume \( det(K_pK_p^*) = 0 \) for some \( p \geq 1 \). Then, there exists a non-zero vector \( v_{p_0} \in \mathbb{R}^n \) such that \( K_p^*K_p^*v_{p_0} = 0 \). Therefore, \( \varphi = v_{p_0} \Phi_{p_0} \) is a non zero element in \( D(KK^*) \) such that
\[ KK^*(\varphi) = \sum_{p \geq 1} K_pK_p^*P_n(\varphi)\Phi_p = K_p^*K_p^*v_{p_0}\Phi_{p_0} = 0. \]

This would contradict the result \( Ker(KK^*) = \{0\} \). Finally, to show (3) \( \Rightarrow \) (2), let \( y \in Ker(KK^*) \). Hence \( \sum_{p \geq 1} K_pK_p^*P_n(y)\Phi_p = 0 \), and then \( K_pK_p^*P_n(y) = 0 \) for all \( p \geq 1 \). Therefore, \( P_n(y) = 0 \), since \( det(K_pK_p^*) \neq 0 \) for all \( p \geq 1 \). Thus, \( y = \sum_{p \geq 1} P_n(y)\Phi_p = 0 \).

The previous proposition is of great interest, since it allows us to check the following results whose proof, in our degenerate case, is similar to [5, Theorem 2.1].

**Theorem 2.1.** We have the following properties

1. there exists a constant \( C > 0 \) such that for all \( u \in D(M^{n-1})^{nm} \), \( Ku \in L^2(0,1)^n \) and
\[ ||Ku||^2_{L^2(0,1)^n} \leq C ||M^{n-1}u||^2_{L^2(0,1)^{nm}}, \]

2. there exists a constant \( C > 0 \) such that for all \( u \in D(M^{n-1})^n \), \( K^*u \in L^2(0,1)^{nm} \) and
\[ ||K^*u||^2_{L^2(0,1)^{nm}} \leq C ||M^{n-1}u||^2_{L^2(0,1)^n}, \]
3. assume \( \text{Ker}(\mathcal{K}^*) = \{0\} \), and let \( k \geq (2n-1)(n-1) \). Then, for every \( \phi \in L^2(0,1)^n \) satisfying \( \mathcal{K}^* \phi \in D(\mathcal{M}^k)^{nm} \), one has \( \phi \in D(\mathcal{M}^{k(2n-1)(n-1)})^n \) and

\[
\|\mathcal{M}^{k(2n-1)(n-1)} \phi\|_{L^2(0,1)^n}^2 \leq C\|\mathcal{M}^k \mathcal{K}^* \phi\|_{L^2(0,1)^{nm}}^2.
\]

By adapting the proof of [5, Theorem 2.1] to our case and using the fact that the polynomial \( P(\lambda) := \det[-\lambda \mathbf{D} + \mathcal{A}] \mathcal{B}^{-1}[-\lambda \mathbf{D} + \mathcal{A}] \) is either identically 0 or far from 0 for \( \lambda \) sufficiently large, one can deduce the following corollary.

**Corollary 1.** Either there exists \( p_0 \in \mathbb{N}^* \) such that rank \( \mathcal{K}_p = n \) for every \( p > p_0 \) or rank \( \mathcal{K}_p < n \) for every \( p \in \mathbb{N}^* \).

3. **Carleman estimates.** In this section we give a new global Carleman estimate for the adjoint problem (6). For this, we need to show first some Carleman estimates in the case of a single parabolic degenerate equation.

3.1. **Carleman estimate for a parabolic degenerate equation of order 2.** In this subsection we shall establish a new Carleman estimate for the solution of the following parabolic degenerate equation of order 2 in space

\[
\begin{cases}
  u_t - (a(x) u_x)_x + c u = f, & (t,x) \in Q, \\
  u(t,0) = 0, & \text{in case (WD)} \\
  (a(x)u_x)(t,0) = 0, & \text{in case (SD)} \\
  u(0,x) = u_0(x), & x \in (0,1).
\end{cases}
\]

Let us consider the following time and space weight functions

\[
\begin{aligned}
  \theta(t) &= \frac{1}{t^h(T-t)^{2h}} \psi(x) = \lambda \left( \int_0^x \frac{y}{a(y)} dy - c \right) \text{ and } \varphi(t,x) = \theta(t) \psi(x), \\
  \Phi(t,x) &= \theta(t) \Psi(x) \text{ and } \Psi(x) = e^{\rho \sigma(x)} - e^{2\rho\|\sigma\|_{\infty}},
\end{aligned}
\]

where \( \sigma \) a \( C^2([0,1]) \) function such that \( \sigma(x) > 0 \) in \( (0,1) \), \( \sigma(0) = \sigma(1) = 0 \) and \( \sigma_x(x) \neq 0 \) in \( [0,1] \setminus \omega_0 \), \( \omega_0 \) is an open subset of \( \omega \), and the parameters \( c, \rho \) and \( \lambda \) are chosen as in [13] such that

\[
c > 4^n c_0, \quad \rho > \frac{\ln \left( 4^n \frac{(c-c_0)}{c-4^n c_0} \right)}{\|\sigma\|_{\infty}}, \quad e^{2\rho\|\sigma\|_{\infty}} < \lambda < \frac{4^n}{(4^n - 1)c} \left( e^{2\rho\|\sigma\|_{\infty}} - e^{\rho\|\sigma\|_{\infty}} \right).
\]

where \( c_0 = \int_0^1 \frac{x}{a(x)} dx \).

Let’s recall some useful properties of the weighted functions, the proofs are given in [13].

**Lemma 3.1.** Under assumptions (12)-(13), we have

\[
0 < \frac{4^n}{4^n - 1} \Phi < \varphi < \Phi,
\]

and

\[
0 < \frac{4}{3} < \frac{4^2}{4^2 - 1} < \cdots < \frac{4^{(n-1)}}{4^{(n-1)} - 1} < \frac{4^n}{4^n - 1} < \frac{4^n}{4^n - 1} \Phi.
\]
Lemma 3.2. Let the sequence $\Phi_k$ defined by
\[ \Phi_k = 4^n - k(\Phi - \varphi) + \varphi, \quad k = 1, \ldots, n. \]
Then, we have
1. $\varphi < \Phi_k < 0$, $k = 1, \ldots, n$,
2. $\Phi_n < \Phi_1 < \cdots < \Phi_1 = 4^n - 1\Phi - (4^n - 1)\varphi$.

The following Carleman estimate will be crucial for the aim of this subsection. Note that the Carleman estimate needed in this work is different from the one shown in [6] and used in [13], since it involves in addition to $u$ and $u_x$ the terms $u_t$ and $\mathcal{M}u$.

Theorem 3.3. For $T > 0$, there exist two positive constants $C$ and $s_0$ such that, for $u_0 \in H^1_0$, the solution $u$ of equation (10) satisfies
\[
\int_Q \left( 1 \right) (u_t^2 + (\mathcal{M}u)^2) + s\theta a(x)u_x^2 + s^3 \theta^3 \frac{x^2}{a(x)} u^2 ) e^{2\varphi} dx dt 
\leq C \left( \int_Q f^2 e^{2\varphi} dx dt + sa(1) \int_0^T \theta u_x^2(t,1) e^{2\varphi} dt \right)
\]
for all $s \geq s_0$.

Proof. Let $u$ be the solution of equation (10). For $s > 0$, the function $w = e^{s\varphi} u$ satisfies
\[
-(aw_x)_x - s\varphi w - s^2 a\varphi^2 w + w_t + 2sa\varphi_x w_x + s(a\varphi_x)_x w = fe^{s\varphi} - \tilde{c}w.
\]
Moreover, from [6, Lemma 3.4, 3.5 and 3.6], we can deduce the following estimate
\[
\|L^+_x w\| + \|L^-_x w\|^2 + \int_Q \left( s^3 \theta^3 \frac{x^2}{a(x)} w^2 + s\theta a(x)w_t^2 \right) dx dt 
\leq C \left( \int_Q (f e^{s\varphi} - \tilde{c}w)^2 dx dt + sa(1) \int_0^T \theta(t)w_x^2(t,1) dt \right)
\]
\[
\leq C \left( \int_Q f^2 e^{2s\varphi} + \int_Q \tilde{c}^2 w^2 dx dt + sa(1) \int_0^T \theta(t) w_x^2(t,1) dt \right).
\]
Using the same technique as in [2, 6], the term $\int_Q \tilde{c}^2 w^2 dx dt$ can be absorbed by the last two terms in the left side of inequality (17). Thus
\[
\|L^+_x w\|^2 + \|L^-_x w\|^2 + \int_Q \left( s^3 \theta^3 \frac{x^2}{a(x)} w^2 + s\theta a(x)w_t^2 \right) dx dt 
\leq C \left( \int_Q f^2 e^{2s\varphi} dx dt + sa(1) \int_0^T \theta(t)w_x^2(t,1) dt \right).
\]
Using the previous estimate, we will bound the integral $\int_Q \frac{1}{s\theta} u_t^2 e^{2s\varphi} dx dt$. In fact, we have
\[
\frac{1}{\sqrt{s\theta}} L^-_x w = \frac{1}{\sqrt{s\theta}} (w_t + 2sa\varphi_x w_x + s(a\varphi_x)_x w)
= \frac{1}{\sqrt{s\theta}} w_t + 2\sqrt{s\theta} x w_x + \sqrt{s\theta} w.
\]
Therefore,
\[
\int_{Q} \frac{1}{s} w_{x}^{2} dx dt \leq C \left( \| L_{-}^{-1} \|^2 + \int_{Q} s\varphi \varphi_{w} dx dt + \int_{Q} s\theta w_{x}^{2} dx dt \right). \quad (19)
\]
Since the function \( x \mapsto \frac{x^{2}}{a} \) is nondecreasing, then one has
\[
\int_{Q} s\varphi \varphi_{w} dx dt \leq \int_{Q} s\varphi \varphi_{w} dx dt \leq \frac{1}{a(1)} \int_{Q} s\varphi \varphi_{w} dx dt. \quad (20)
\]
To estimate \( \int_{Q} s\theta w_{x}^{2} dx dt \), we have
\[
\int_{Q} s\theta w_{x}^{2} dx dt = s \int_{Q} \left( \theta \frac{a^{1/3}}{x^{2/3}} w^{3/4} \right)^{3/4} \left( \frac{x^{2}}{a} w^{1/4} \right) dx dt
\leq s \frac{3}{2} \int_{Q} \theta \frac{a^{1/3}}{x^{2/3}} w^{2} dx dt + \frac{s}{2} \int_{Q} \frac{x^{2}}{a} w^{2} dx dt.
\]
The function \( p(x) := \frac{x^{2}}{a} \) satisfies \( p(x) \leq Ca(x) \), since the function \( x \mapsto \frac{x^{2}}{a} \) is non-decreasing on \((0, 1)\). Hence by Hardy-Poincaré inequality [6, Proposition 2.1]
\[
\int_{0}^{1} \frac{a^{1/3}}{x^{2}} w^{2} dx = \int_{0}^{1} p(x) \frac{a^{1/3}}{x} w^{2} dx \leq C \int_{0}^{1} \left( \frac{a(x)}{x} \right) w^{2} dx \leq C \int_{0}^{1} a(x) w_{x}^{2} dx,
\]
and so
\[
\int_{Q} s\theta w_{x}^{2} dx dt \leq C \int_{Q} (s\varphi \varphi_{w} + s\theta \frac{a^{3}}{a} w^{2}) dx dt. \quad (22)
\]
From (19)-(22), we obtain
\[
\int_{Q} \frac{1}{s} w_{x}^{2} dx dt \leq C \left( \| L_{-}^{-1} \|^2 + \int_{Q} s\varphi \varphi_{w} dx dt + \int_{Q} s\theta \frac{a^{3}}{a} w^{2} dx dt \right). \quad (23)
\]
In a similar way, to bound the integral \( \int_{Q} \frac{1}{s\theta} (\mu w)^{2} dx dt \), we have
\[
\frac{1}{\sqrt{s\theta}} L_{-}^{+} w = \frac{1}{\sqrt{s\theta}} (-a\varphi_{w} x - s\varphi_{w} w - s^{2} a\varphi_{w}^{2} w)
= -\frac{\mu w}{\sqrt{s\theta}} - \sqrt{s\theta} \frac{\dot{\psi}}{\sqrt{\theta}} w - s^{2} \theta^{1/4} x^{2} a w.
\]
As \( |\dot{\psi}^{-1}(t)| = \left| \frac{-4(T - 2t)}{t(T - t)} \right| \leq 4T \theta^{1/4} \) and \( |\theta(t)| \geq \left( \frac{2}{7} \right)^{8} \), then we deduce \( \left| \frac{\dot{\psi}}{\sqrt{\theta}} \right| \leq C \theta^{2} \). Since \( \psi \) is bounded on \((0, 1)\) then
\[
\int_{Q} (\sqrt{s} \frac{\dot{\psi}}{\sqrt{\theta}} w^{2}) dx dt \leq C \int_{Q} s\theta w_{x}^{2} dx dt.
\]
By (21), we infer
\[
\int_{Q} s\theta^{2} w_{x}^{2} dx dt = s \int_{Q} \left( \theta \frac{a^{1/3}}{x^{2/3}} w^{2} \right)^{3/4} \left( \theta^{2} \frac{a^{3}}{a} w^{3} \right)^{1/4} dx dt
\leq \frac{3}{2} \int_{Q} \theta \frac{a^{1/3}}{x^{2/3}} w^{2} dx dt + \frac{8}{5} \int_{Q} \theta^{3} \frac{a^{3}}{a} w^{2} dx dt
\]
\[ \leq C \frac{3}{2} \int_Q \theta w^2 dx + \frac{s}{2} \int_Q \theta^3 a^{x^2} w^2 dx. \]

Therefore, for \( s \) large enough
\[ \int_Q (\sqrt{s} w)^2 dx \leq C \left( \int_Q \theta w^2 dx + \int_Q \theta^3 a^{x^2} w^2 dx \right). \]

Thus
\[ \int_Q \frac{1}{s}(Mw)^2 dx \leq C \left( \| L^+_Θ w \|^2 + \int_Q \theta w^2 dx + \int_Q \theta^3 a^{x^2} w^2 dx \right). \]

(24)

From inequalities (18), (23) and (24), one obtains
\[ \int_Q \frac{1}{s} w^2 + \int_Q (Mw)^2 + s^3 \theta^3 a(x) w^2 + \theta a(x) w^2 \]
\[ \leq C \left( \int_Q f^2 e^{2\phi} dx dt + s a(1) \int_0^T \theta(t) w^2(t) dt \right). \]

(25)

Consequently, we obtain the estimate (16) which completes the proof.

From the boundary Carleman estimate (16), we deduce Carleman estimates for equation (10) on the subregion \( \omega' \). Set \( \omega'' := (x''_1, x''_2) \subset \subset \omega' \) and \( \xi \in C^\infty((0,1]) \) such that \( 0 \leq \xi(x) \leq 1 \) for \( x \in (0,1) \), \( \xi(x) = 1 \) for \( x \in (0, x'') \) and \( \xi(x) = 0 \) for \( x \in (x'', 1) \).

State first the following intermediate Carleman estimate.

**Proposition 2.** For every \( T > 0 \), there exist two positive constants \( C \) and \( s_0 \) such that, for every \( u_0 \in L^2(0,1) \), the solution \( u \) of equation (10) satisfies
\[ \int_Q \left( \frac{1}{s} \xi^2 u_t^2 + \frac{1}{s} \xi^2 (Mu)^2 + s^3 \theta^3 \frac{x^2}{a(x)} \xi^2 u^2 + \theta a \xi^2 u_x^2 \right) e^{2s\phi} dt dx \]
\[ \leq C \left( \int_Q \xi^2 f^2 e^{2s\phi} dt dx + \int_{Q_{\omega''}} s \Theta^2 u^2 e^{2s\phi} dt dx \right). \]

(26)

for all \( s \geq s_0 \).

**Proof.** First, let \( u_0 \in H^1_\omega \). The function \( z := \xi u \) satisfies the following equation
\[
\begin{aligned}
&z_t = -(a(x)z) + \xi z = \xi f - \xi a(x) u_x - (a(x) \xi_x u_x), \quad (t,x) \in Q, \\
&z(t,0) = 0, \quad \text{in case (WD)} \\
&z(t,1) = 0, \quad \text{in case (SD)} \\
&z(0,x) = \xi(x) u_0, \quad x \in (0,1).
\end{aligned}
\]

(27)

The Carleman estimate (16) applied to equation (27) yields
\[ \int_Q \left( s a(x) z_x^2 + s^3 \theta^3 \frac{x^2}{a(x)} z_x^2 + \frac{1}{s} \xi_t^2 z_x^2 + \frac{1}{s} (Mz)^2 \right) e^{2s\phi} dx dt \]
\[ \leq C \int_Q \left( \xi^2 f^2 + (\xi_x a(x) u_x + (a(x) \xi_x u) u_x^2) \right) e^{2s\phi} dx dt. \]

(28)

From the definition of \( \xi \) and Caccioppoli inequality [2, Lemma 6.1], we obtain
\[ \int_Q (\xi_x a(x) u_x + (a(x) \xi_x u) u_x^2) e^{2s\phi} dx dt \leq C \int_{Q_{\omega''}} (u^2 + u_x^2) e^{2s\phi} dx dt. \]
Proposition 4. We are able to give a Carleman estimate to equation (10) on the interval $(\theta s, \theta s + 1)$.

Moreover, since $\xi u_x = z_x - \xi_x u$ and $\xi Mu = Mz - (a\xi u)_x - \xi_x au_x$, we obtain

$$\iint_Q s\theta a\xi_x^2 u^2 e^{2s\varphi} dxdt \leq 2 \iint_Q s\theta a\xi_x^2 u^2 e^{2s\varphi} dxdt + 2 \iint_{Q_{\varphi}} s^2 \theta^2 u^2 e^{2s\varphi} dxdt$$

(30)

and

$$\iint_{Q_{\varphi}} \frac{1}{s\theta} \xi^2 Mu^2 e^{2s\varphi} dxdt \leq 2 \iint_{Q_{\varphi}} \frac{1}{s\theta} Mz^2 e^{2s\varphi} dxdt + C \iint_{Q_{\varphi}} (s^2 \theta^2 u^2 + f^2) e^{2s\varphi} dxdt.$$  

(31)

Thus, from (28)-(31) and the definition of $\xi$ we deduce the desired estimate for $u_0 \in H^1_\omega$. Finally, by density, we conclude for $u_0 \in L^2$.

Using the previous Carleman estimate (26), by the same argument of [13, Proposition 2.4], we obtain the following general version.

Proposition 3. Let $T > 0$ and $\tau \in \mathbb{R}$. Then there exists two positive constants $C$ and $s_0$ such that, for all $u_0 \in L^2(0, 1)$, the solution $u$ of equation (10) satisfies

$$\iint_Q \left( (s\theta)^{-1} \xi^2 u_t^2 + (s\theta)^{-1} \xi^2 (Mu)^2 + (s\theta)^{1+\tau} a\xi^2 u_x^2 + (s\theta)^{1+\tau} \frac{a^2}{\alpha} \xi^2 u^2 \right) e^{2s\varphi} dxdt \leq C \left( \iint_{Q_{\varphi}} \xi^2 s^2 \theta^2 f^2(t, x) e^{2s\varphi(t, x)} dxdt + \iint_{Q_{\varphi}} s^{2+\tau+\tau} u^2 e^{2s\varphi(t, x)} dxdt \right)$$

for all $s \geq s_0$.

Proposition 2 gave a Carleman estimate in $(0, x'_1)$. For the interval $(x'_1, 1)$, similarly as in [17], [14, Lemma 1.2] remains true when we replace $\theta = \frac{1}{\frac{1}{r} - t}$ by $\theta = \frac{1}{\frac{1}{r} - t'}$. Now, with the nondegenerate Carleman estimate of [14, Lemma 1.2], we are able to give a Carleman estimate to equation (10) on the interval $(x'_1, 1)$.

Proposition 4. There exist two positive constants $C$ and $s_0$ such that for every $u_0 \in L^2(0, 1)$, the solution $u$ of equation (10) satisfies

$$\iint_Q \left( \frac{1}{s\theta^2} \xi^2 u_t^2 + \frac{1}{s\theta^2} \xi^2 (Mu)^2 + s^3 \theta^3 \frac{a^2}{\alpha} \xi^2 u_x^2 + s\theta a\xi u_x^2 \right) e^{2s\varphi} dxdt \leq C \left( \iint_{Q_{\varphi}} \xi^2 f^2 e^{2s\varphi} dxdt + \iint_{Q_{\varphi}} s^3 \theta^3 u^2 e^{2s\varphi} dxdt \right)$$

for all $s \geq s_0$, where $\zeta := 1 - \xi$.

Proof. The function $Z := \xi u$ has its support in $[0, T] \times (x'_1, 1)$, and it is also a solution of the nondegenerate parabolic equation

$$\begin{cases}
Z_t - (a(x)Z_x)_x + \delta Z = \xi f - \xi_x a(x) u_x - (a(x)\xi u)_x, & (t, x) \in Q, \\
Z(t, 1) = 0 \text{ and } Z(t, 0) = 0 & \text{on } (0, T), \\
Z(0, x) = \xi(x) u_0, & x \in (0, 1).
\end{cases}$$

(33)
Hence, by [14, Lemma 1.2], we have
\[
\iint_Q \left( \frac{1}{s\theta} \left( Z_t + (Z_{xx})^2 \right) + s\theta a Z_x^2 + s^3 \theta^3 \frac{x^2}{a} Z^2 \right) e^{2s\Phi} \, dx \, dt \\
\leq C \left( \iint_Q \left( \zeta^2 f^2 + (\zeta_x a(x) u_x + (a(x) \zeta_x u_x))^2 \right) e^{2s\Phi} \, dx \, dt \\
+ \iint_{\omega \times (0,T)} s^3 \theta^3 Z^2 e^{2s\Phi} \, dx \, dt \right).
\]
Again, from the definition of \( \zeta \) and the Cacciopoli inequality [2, Lemma 6.1], we obtain
\[
\iint_Q \left( \zeta_x a(x) u_x + (a(x) \zeta_x u_x) \right)^2 e^{2s\Phi} \, dx \, dt \\
\leq C \iint_{Q_{\omega'}} (u^2 + u_x^2) e^{2s\Phi} \, dx \, dt \\
\leq C \iint_{Q_{\omega'}} (s^2 \theta^2 u^2 + f^2) e^{2s\Phi} \, dx \, dt.
\]
Thus
\[
\iint_Q \left( \frac{1}{s\theta} \left( Z_t^2 + Z_{xx}^2 \right) + s\theta a Z_x^2 + s^3 \theta^3 \frac{x^2}{a} Z^2 \right) e^{2s\Phi} \, dx \, dt \\
\leq C \left( \iint_Q \zeta^2 f^2 e^{2s\Phi} \, dx \, dt + \iint_{\omega \times (0,T)} s^3 \theta^3 u^2 e^{2s\Phi} \, dx \, dt \right).
\]
From \( \zeta u_x = Z_x - \zeta_x u \) and \( \text{supp} \zeta_x \subset \omega'' \), we deduce
\[
\iint_Q s\theta \zeta^2 u_x^2 e^{2s\Phi} \, dx \, dt \leq C \left( \iint_Q s\theta a Z_x^2 e^{2s\Phi} \, dx \, dt + \iint_{Q_{\omega''}} s\theta u^2 e^{2s\Phi} \, dx \, dt \right) \\
\leq C \left( \iint_Q s\theta a Z_x^2 e^{2s\Phi} \, dx \, dt + \iint_{Q_{\omega'}} s^3 \theta^3 u^2 e^{2s\Phi} \, dx \, dt \right),
\]
for \( s \) large enough. Similarly since \( \zeta u_{xx} = Z_{xx} - \zeta_{xx} u - 2\zeta_x u_x \) and thanks to Cacciopoli inequality, we get
\[
\iint_Q \frac{1}{s\theta} \zeta^2 u_{xx}^2 e^{2s\Phi} \, dx \, dt \leq C \left( \iint_Q \frac{1}{s\theta} Z_{xx}^2 e^{2s\Phi} \, dx \, dt + \iint_{Q_{\omega''}} (\zeta_{xx} u + 2\zeta_x u_x)^2 e^{2s\Phi} \, dx \, dt \right) \\
\leq C \left( \iint_Q \frac{1}{s\theta} Z_{xx}^2 e^{2s\Phi} \, dx \, dt + \iint_{Q_{\omega'}} (s^2 \theta^2 u^2 + f^2) e^{2s\Phi} \, dx \, dt \right).
\]
Estimates (34)-(36) lead to
\[
\iint_Q \left( \frac{1}{s\theta} \zeta^2 u_t^2 + \frac{1}{s\theta} \zeta^2 (u_{xx})^2 + s\theta a \zeta^2 u_x^2 + s^3 \theta^3 \frac{x^2}{a} \zeta^2 u^2 \right) e^{2s\Phi} \, dx \, dt \\
\leq C \left( \iint_Q \zeta^2 f^2 e^{2s\Phi} \, dx \, dt + \iint_{\omega \times (0,T)} s^3 \theta^3 u^2 e^{2s\Phi} \, dx \, dt \right).
\]
Since \( a \) is continuous on \((x_1', 1]\) and by using \( Mu = a' u_x + au_{xx} \), we obtain
\[
\iint_Q \frac{1}{s\theta} \zeta^2 (Mu)^2 e^{2s\Phi} \, dx \, dt \leq C \left( \iint_Q \frac{1}{s\theta} \zeta^2 u_x^2 \, dx \, dt + \iint_Q s\theta a \zeta^2 u_x^2 \, dx \, dt \right).
\]
Thus, combining (37) and (38), we deduce the desired estimate.

Proposition 4 can also be generalized as follows.
Proposition 5. Let $T > 0$ and $\tau \in \mathbb{R}$. Then, there exist two positive constants $C$ and $s_0$ such that for every $u_0 \in L^2(0,1)$, the solution $u$ of equation (10) satisfies

\[
\int_Q \left( (s\theta)^{-1} \zeta^2 u_t^2 + (s\theta)^{-1} \zeta^2 (M u)^2 + (s\theta)^{1+\tau} a \zeta^2 u_z^2 + (s\theta)^{3+\tau} \frac{x^2}{a} \zeta^2 u^2 \right) e^{2s\Phi} \, dx \, dt \\
\leq C \left( \int_Q \zeta^2 s^\theta f^2(t,x) e^{2s\Phi} \, dx \, dt + \int_Q s^{3+\tau} \zeta^3 \Phi^3 e^{2s\Phi} \, dx \, dt \right)
\]

(39)

for all $s \geq s_0$, with $\zeta = 1 - \xi$.

Now, we show some Carleman estimates for an intermediate degenerate parabolic equation of order $2n$ in space.

3.2. Carleman estimate for a scalar degenerate parabolic equation of order $2n$. In this section we consider $z$, with the monomial derivative $M^t \partial_z^i z \in L^2(0,T,H^2_0(0,1))$ for every $i, j \in \mathbb{N}$ where $M^t = M \circ M^{t-1}$ and $\partial_i^j = \partial_i \circ \partial_{j-1}^t$, a solution of the following scalar degenerate parabolic equation of order $2n$ in space

\[
\begin{cases}
P(\partial_t, M) z = 0 & \text{in } Q, \\
CM^k z = 0, & k \geq 0, \text{ on } \Sigma,
\end{cases}
\]

(40)

where $P(\partial_t, M)$ is the operator defined by $P(\partial_t, M) = det(\partial_t I_d + M^t M + A^t)$. Since the matrix $D$ is diagonalizable by (4), one has

\[
P(\partial_t, M) = det(\partial_t I_d + JM + P^{*-1}A^*P^*)
\]

(41)

where $P_1 = \partial_t + d, M$, 1 $\leq i \leq n$, $d_i > 1$ and $\alpha, \alpha_i, \alpha_j, \alpha_n \in \mathbb{R}$ depend only on the matrices $D$ and $A$. The main result in this subsection is the following.

Theorem 3.4. Let us fix $k_1, k_2 \in \mathbb{N}$ and $\tau_0 \in \mathbb{R}$. Then, there exist positive constants $C_0, s_0$ and $r = r(n) \in \mathbb{N}$ such the following inequality

\[
\sum_{i=0}^{k_1} \sum_{j=0}^{k_2} J(\tau_0 - 6(i + j), M^t \partial_z^i z) \leq C_0 \int_Q (s\theta)^{7\tau + r} e^{2s\Phi} \, |z|^2
\]

(42)

holds for all $s \geq s_0$ and for every solution $z$ of equation (40) such that $M^t \partial_z^i z \in L^2(0,T,H^2_0(0,1))$ for $i \leq k_1, j \leq k_2$. The term $J(\tau, z)$ is given by

\[
J(\tau, z) = I(\tau + 3(n - 1), z) + \sum_{i=2}^{n} I(\tau + 3(n - 2), P_i z)
\]

(43)

\[
+ \sum_{p=2}^{n} \sum_{1 \leq i_1 < \cdots < i_p \leq n} I(\tau + 3(n - p - 1), P_{i_1} \cdots P_{i_p} z),
\]

(44)

and $\Phi_1$ the function defined in Lemma 3.2.
Proof. We adapt the technique used by Ammar-Khodja et al. in [5] to our degenerate case. All along this proof \( C \) will be a generic constant that may depend on \( \omega, n, a, D, A, \tau_0, k_1 \) and \( k_2 \). Let \( F \) denote the function

\[
F(z) = -\left( \sum_{p=2}^{n-1} \sum_{1 \leq i_1 < \cdots < i_p \leq n} \alpha_{i_1, \ldots, i_p} P_{i_1} \cdots P_{i_p} + \sum_{i=1}^{n} \alpha_i P_i + \alpha \right) z, \tag{43}
\]

and consider the following change of variables

\[
\begin{cases}
\psi_1 = z, \\
\psi_i = P_{i-1} \psi_{i-1} = (\partial_i + d_{i-1} \mathcal{M}) \psi_{i-1}, & 2 \leq i \leq n.
\end{cases} \tag{44}
\]

Having in mind the regularity assumptions on \( z \), one gets \( \psi_i \), \( 1 \leq i \leq n \), \( F(z) \in L^2(Q) \) and \( \Psi = (\psi_1, \ldots, \psi_n)^* \) satisfies the following cascade system

\[
\begin{align*}
(\partial_i + d_1 \mathcal{M}) \psi_1 &= \psi_2 & \text{in } Q, \\
(\partial_i + d_2 \mathcal{M}) \psi_2 &= \psi_3 & \text{in } Q, \\
& \quad \vdots \\
(\partial_i + d_n \mathcal{M}) \psi_n &= F(z) & \text{in } Q, \\
C \psi_i = 0, & 1 \leq i \leq n & \text{on } \Sigma.
\end{align*} \tag{45}
\]

Let us denote by

\[
I_{\alpha, \mathbf{A}}(\tau, z) = \iint_Q \left( (s\theta)^{\tau - 1} \alpha^2 z_x^2 + (s\theta)^{\tau - 1} \alpha^2 (\mathcal{M} z)^2 + (s\theta)^{\tau + 1} a(x) \alpha^2 z_x^2 \right. \\
+ (s\theta)^{\tau + 3} \frac{x^2}{a(x)} \alpha^2 \zeta^2 e^{2s\Phi} dxdt
\]

and

\[
\mathcal{J}_{\alpha, \mathbf{A}}(\tau, z) = I_{\alpha, \mathbf{A}}(\tau + 3(n - 1), z) + \sum_{i=2}^{n} I_{\alpha, \mathbf{A}}(\tau + 3(n - 2), P_{i-1}z) \\
+ \sum_{p=2}^{n-1} \sum_{1 \leq i_1 < \cdots < i_p \leq n} I_{\alpha, \mathbf{A}}(\tau + 3(n - p - 1), P_{i_p} \cdots P_{i_1} z).
\]

Applying Proposition 5 we get

\[
I_{\zeta, \Phi}(\tau_0 + 3(n - i), \psi_i) \leq C \left( \iint_Q (s\theta)^{\tau_0 + 3(n - i)} e^{2s\Phi} \zeta^2 |\psi_{i+1}|^2 \\
+ \iint_{Q_{\omega'}} (s\theta)^{\tau_0 + 3(n - i + 1)} e^{2s\Phi} |\psi_i|^2 \right)
\]

for \( i = 1, \ldots, n - 1 \), and, for \( i = n \), we obtain

\[
I_{\zeta, \Phi}(\tau_0, \psi_n) \leq C \left( \iint_Q (s\theta)^{\tau_0} e^{2s\Phi} \zeta^2 |F(z)|^2 + \iint_{Q_{\omega'}} (s\theta)^{\tau_0 + 3} e^{2s\Phi} |\psi_n|^2 \right)
\]

for every \( s \geq s_0 \). Thus, a suitable combination of the above inequalities leads to

\[
\sum_{i=1}^{n} I_{\zeta, \Phi}(\tau_0 + 3(n - i), \psi_i)
\]
The same procedure as above leads to the similar estimate as (46)

\[ I_{\zeta, \Phi}(\tau_0 + 3(n-1), z) + \sum_{i=2}^{n} I_{\zeta, \Phi}(\tau_0 + 3(n-i), P_{i-1} \cdots P_1 z) \]

\[ \leq C \left( \sum_{i=1}^{n} \int_{Q_{\omega_i}} (s\theta)^{\tau_0+3(n-i+1)} e^{2\phi \| |\psi_i|^2 + \int_{Q} (s\theta)^{\tau_0} e^{2\phi \zeta^2} |F(z)|^2 \right). \quad (46) \]

In order to absorb the term \( F(z) \), let \( \Pi \) denote any permutation of the set \( \{1, 2, \cdots, n\} \) and consider, instead of (44), the new change of variable

\[
\begin{align*}
\psi_1 &= z, \\
\psi_i &= P_{\Pi(i-1)} \psi_{i-1} = (\partial_t + d_{\Pi(i-1)} M) \psi_{i-1}, & 2 \leq i \leq n.
\end{align*}
\]

(47)

Hence, system (45) becomes

\[
\begin{align*}
(\partial_t + d_{\Pi(1)} M) \psi_1 &= \psi_2 & \text{in } Q, \\
(\partial_t + d_{\Pi(2)} M) \psi_2 &= \psi_3 & \text{in } Q, \\
& \vdots & \\
(\partial_t + d_{\Pi(n)} M) \psi_n &= F(z) & \text{in } Q, \\
C \psi_i &= 0, 1 \leq i \leq n, & \text{on } \Sigma.
\end{align*}
\]

The same procedure as above leads to the similar estimate as (46)

\[ I_{\zeta, \Phi}(\tau_0 + 3(n-1), z) + \sum_{i=2}^{n} I_{\zeta, \Phi}(\tau_0 + 3(n-i), P_{\Pi(i-1)} \cdots P_{\Pi(1)} z) \]

\[ \leq C \left( \sum_{i=1}^{n} \int_{Q_{\omega_i}} (s\theta)^{\tau_0+3(n-i+1)} e^{2\phi \| |\psi_i|^2 + \int_{Q} (s\theta)^{\tau_0} e^{2\phi \zeta^2} |F(z)|^2 \right). \quad (48) \]

Now, considering all such possible permutations with associated change of variable, we finally obtain

\[
I_{\zeta, \Phi}(\tau_0 + 3(n-1), z) + \sum_{i=2}^{n} I_{\zeta, \Phi}(\tau_0 + 3(n-i), P_i z)
\]

\[ + \sum_{p=2}^{n-1} \sum_{1 \leq i_1 < \cdots < i_p \leq n} I_{\zeta, \Phi}(\tau_0 + 3(n-p-1), P_p \cdots P_{i_1} z) \]

\[ \leq C \left( \sum_{i=1}^{n} \int_{Q_{\omega_i}} (s\theta)^{\tau_0+3(n-i+1)} e^{2\phi \| |\psi_i|^2 + \int_{Q} (s\theta)^{\tau_0} e^{2\phi \zeta^2} |F(z)|^2 \right), \quad (49) \]

where \( l(i) = \max(5, 3(n-\tau_0 - 1)), i = 1, \cdots, n \). From the definition of \( F(z) \), we have

\[
\int_{Q} (s\theta)^{\tau_0} e^{2\phi \zeta^2} |F(z)|^2 \, dx \, dt \leq C \int_{Q} (s\theta)^{\tau_0} e^{2\phi \zeta^2} (|z|^2 + \sum_{i=1}^{n} |P_i z|^2)
\]

\[ + \sum_{p=2}^{n-1} \sum_{1 \leq i_1 < \cdots < i_p \leq n} |P_p \cdots P_{i_1} z|^2 \, dx \, dt. \quad (50) \]

By choosing \( s \) large enough such that

\[ C(s\theta)^{\tau_0} \leq \frac{1}{2} (s\theta)^{\tau_0+3(n-p)} \quad \text{for all } 0 \leq p \leq n-1, \quad (51) \]
from (49)-(51), we obtain
\[
I_{\zeta, \phi}(\tau_0 + 3(n - 1), z) + \sum_{i=2}^{n} I_{\zeta, \phi}(\tau_0 + 3(n - i), P_i z) \\
+ \sum_{p=2}^{n-1} \sum_{1 \leq i_1 < \cdots < i_p \leq n} I_{\zeta, \phi}(\tau_0 + 3(n - p - 1), P_p \cdots P_{i_1} z)
\leq C \left( \sum_{i=1}^{n} \int_{Q_{\omega_i}} (s\theta)^{\tau_0 + l(i)} e^{2s\Phi |\psi_i|^2} \right).
\]

Thus
\[
\mathcal{J}_{\zeta, \phi}(\tau_0, z) \leq C \left( \sum_{i=1}^{n} \int_{Q_{\omega_i}} (s\theta)^{\tau_0 + l(i)} e^{2s\Phi |\psi_i|^2} \right). 
\]

In a similar way, using Proposition 3, we infer
\[
\mathcal{J}_{\zeta, \phi}(\tau_0, z) \leq C \left( \sum_{i=1}^{n} \int_{Q_{\omega_i}} (s\theta)^{\tau_0 + l(i)} e^{2s\varphi |\psi_i|^2} \right). 
\]

The previous estimates lead to
\[
\mathcal{J}(\tau_0, z) \leq C \left( \sum_{i=1}^{n} \int_{Q_{\omega_i}} (s\theta)^{\tau_0 + K} e^{2s\Phi |\psi_i|^2} \right). 
\]

With this intermediate estimate, using the same computation of [13, Lemma 3.7], we can show the existence of a constant $C > 0$ and an integer $K = K(n)$ such that
\[
\mathcal{J}(\tau_0, z) \leq C \int_{(0,T) \times \omega} (s\theta)^{\tau_0 + K} e^{2s\Phi_1 |z|}.
\]

From the regularity assumptions imposed on $z$, if $1 \leq i \leq k_1$ and $1 \leq j \leq k_2$, $M^i \partial^j_z z$ also satisfies the equation (40). Therefore, by applying the above estimate to $M^i \partial^j_z z$, there exist two positive constants $C_\tau$ and $s_\tau$ such that
\[
\mathcal{J}(\tau_0, z) \leq C_\tau \int_{(0,T) \times \omega} (s\theta)^{\tau_0 + K} e^{2s\Phi_1 |M^i \partial^j_z z|^2} \, dx \, dt
\]
for every $s \geq s_\tau$. Now for $s$ large enough we have
\[
\int_{(0,T) \times \omega} (s\theta)^{\tau_0} e^{2s\Phi_1} \left( |Mz|^2 + |\partial_t z|^2 \right) \, dx \, dt \leq I(\tau + 4, z) \leq \mathcal{J}(\tau + 4, z).
\]

Thus by iterating this process, one obtains
\[
\begin{aligned}
\mathcal{J}(\tau, M^i \partial^j_z z) &\leq C \mathcal{J}(\tau + 4, M^{i-1} \partial^j_z z), \\
\mathcal{J}(\tau, M^i \partial^j_z z) &\leq C \mathcal{J}(\tau + 4, M^i \partial^{j-1}_z z).
\end{aligned}
\]

Therefore, by applying successively the last inequalities we deduce
\[
\mathcal{J}(\tau, M^i \partial^j_z z) \leq C \mathcal{J}(\tau + (4 + K)(i + j), z),
\]
and summing on $i, j$, we obtain
\[
\sum_{i=0}^{k_1} \sum_{j=0}^{k_2} \mathcal{J}(\tau_0 - 4(i + j), M^i \partial^j_z z) \leq \sum_{i=0}^{k_1} \sum_{j=0}^{k_2} C \mathcal{J}(\tau_0 + K(i + j), z)
\leq C \mathcal{J}(\tau_0 + K(k_1 + k_2), z).
\]
Assume Theorem 1.1. The mate for the adjoint problem (6). Recall that $D$ is dense in $L^2(0,1)$. Consequently, we have

$$\sum_{i=0}^{k_1} \sum_{j=0}^{k_2} \mathcal{J}(\tau_0 - 4(i + j), M^j \partial^i_x) \leq C_0 \int_0^T \int_{\omega_T} (s\theta)^{r+\epsilon} e^{2x\psi_1} |z|^2 \, dx \, dt,$$

where $r = K(k_1 + k_2 + 1)$.

### 4. Null controllability of problem (1)

Now, we will show the Carleman estimate for the adjoint problem (6). Recall that $D = \cap_{p=0}^{\infty} D(M^p)$ is dense in $L^2(0,1)$. Thus, we have the following result see [5, Proposition 3.3].

**Proposition 6.** Let $\phi_0 \in \mathbb{D}^n$ and let $\phi = (\phi_1, \ldots, \phi_n)^*$ be the corresponding solution of problem (6). Then, $\phi \in \mathcal{C}^k([0,T]; D(M^p)^n)$ for every $k, p \geq 0$, and for every $i$, $\phi_i$ solves equation (40).

Now, the intermediate result that we present will be of great use to establish the main Theorem 1.1.

**Theorem 4.1.** Assume $D$ satisfies the condition (4). Then, given $\tau \in \mathbb{R}$ and $k \geq (n - 1)(2n - 1)$, there exist $r = r(n) \in \mathbb{N}$ and two positive constants $C$ and $\sigma$ such that for every $\phi_0 \in L^2(0,1)^n$ the corresponding solution $\phi$ to the adjoint problem (6) satisfies

$$\int_0^T (s\theta)^{r} e^{-2xM_0\theta} \|M^k \Phi \|^2_{L^2(0,1)^n} \leq C \int_0^T (s\theta)^{r+\kappa}\epsilon e^{2x\psi_1} |B^* \phi|^2,$$

where $M_0 = \max_{x \in (0,1)} \psi(x)$ and $\kappa = 4k + n - 4$.

**Proof.** Assume $\phi_0 \in \mathbb{D}^n$ and let $\phi$ the solution of the adjoint problem (6) corresponding to $\phi_0$. By Proposition 6 we have $\phi \in \mathcal{C}^l([0,T]; D(M^p)^n)$ for every $l, p \geq 0$; and $\phi_i$ solves equation (40) for every $i$, $1 \leq i \leq n$. Likewise $(B^* \phi)_j$ is in $\mathcal{C}^l([0,T]; D(M^p)^n)$; and solves equation (40), for all $j$ (with $1 \leq j \leq m$). Since

$$K^* \phi(t, \cdot) = ((-1)^{n-1} \partial^{n-1}_t B^* \phi, (-1)^{n-2} \partial^{n-2}_t B^* \phi, \ldots, (-1)^n B^* \phi, B^* \phi)^* (t, \cdot)$$

then

$$\int_0^T (s\theta)^{r} e^{-2xM_0\theta} \|M^k K^* \phi\|^2_{L^2(0,1)^n} \leq \sum_{i=1}^{m} \int_Q \sum_{j=0}^{n-1} (s\theta)^{r} e^{2x\psi_1} \|M^k \partial^j_x (B^* \phi)_i\|^2 \, dx \, dt$$

$$\leq \sum_{i=1}^{m} \int_Q \sum_{j=0}^{n-1} (s\theta)^{r} e^{2x\psi_1} \|M^k \partial^j_x (B^* \phi)_i\|^2 \, dx \, dt$$

$$\leq \sum_{i=1}^{m} \sum_{j=0}^{n-1} I(\tau - 3, M^k \partial^j_x (B^* \phi)_i)$$

$$\leq \sum_{i=1}^{m} \sum_{j=0}^{n-1} J(\tau - 3 - 3(n - 1), M^k \partial^j_x (B^* \phi)_i)$$

$$\leq \sum_{i=1}^{m} \sum_{j=0}^{n-1} J(\tau - 3n, M^k \partial^j_x (B^* \phi)_i).$$
For the choice of $\tau_0 = \tau + 4k + n - 4$ one gets $(s\theta)^\tau \leq C(s\theta)^{\tau_0 - 4(l+j)+3n}$ for every $l, j$ with $0 \leq l \leq k$ and $0 \leq j \leq n - 1$. Thus, using Theorem 3.4, we deduce

\[
\int_0^T (s\theta)^\tau e^{-2sM_0 \theta} \|M^k K^* \phi\|^2_{L^2(0,1)^n} \leq \sum_{i=1}^{m} \sum_{j=0}^{n-1} \mathcal{J}(\tau - 3n, M^k \partial^j_i (B^* \phi)_i) \\
\leq \sum_{i=1}^{m} \sum_{j=0}^{n-1} \mathcal{J}(\tau_0 - 4(k+j), M^k \partial^j_i (B^* \phi)_i) \\
\leq C \sum_{i=1}^{m} \int_{\omega_T} (s\theta)^{\tau_0 + \kappa + r} e^{2s\Phi_1} |B^* \phi_i|^2 \\
\leq C \int_{\omega_T} (s\theta)^{\tau + \kappa + r} e^{2s\Phi_1} |B^* \phi|^2,
\]

with $\kappa = 4k + n - 4$. Now, when $\phi_0 \in L^2(0,1)^n$, there exists a Cauchy sequence $(\phi'_0)_{j \geq 1} \subset \mathbb{D}^n$ such that $\phi'_0 \rightarrow \phi_0 \in L^2(0,1)^n$. Let $\phi'$ and $\phi$ be, respectively, the solution of the adjoint problem (6) corresponding to $\phi'_0$ and $\phi_0$. We have $\phi' \rightarrow \phi$ in $L^2(0,1)^n$ and $M^k K^* \phi' \rightarrow M^k K^* \phi$ in $\mathcal{D}'(Q)^n$ for every $k \geq 0$. Since $\phi'$ satisfies (55), then we deduce that $(M^k K^* \phi')_{j \geq 1}$ is a Cauchy sequence in the weighted space $L^2((s\theta)^{\tau} e^{-sM_0 \theta}, Q)$. Passing to the limit in the Carleman inequality (55) satisfied by $\phi'$, we obtain the result in the general case. This ends the proof. 

At present, using the condition $\text{Ker}(K^*) = \{0\}$, we state the following global Carleman estimate for the solution of Problem (6).

**Corollary 2.** In addition to the assumptions in Theorem 4.1, we assume the condition $\text{Ker}(K^*) = \{0\}$. Then, given $\tau \in \mathbb{R}$ and $k \geq (n-1)(2n-1)$, there exist two positive constants $C$ and $\sigma$ such that for every $\phi_0 \in L^2(0,1)^n$ the corresponding solution $\phi$ to the adjoint problem (6) satisfies

\[
\int_Q (s\theta)^\tau e^{-2sM_0 \theta} |M^{k-(n-1)(2n-1)} \phi|^2 \leq C \int_{(0,T) \times \omega} (s\theta)^{\tau + \kappa + r} e^{2s\Phi_1} |B^* \phi|^2 \tag{57}
\]

for every $s \geq \sigma$. The constants $M_0$, $\kappa$ and $r = r(n)$ are as in Theorem 4.1.

**Proof.** Since $\text{Ker}(K^*) = \{0\}$ and $k \geq (2n-1)(n-1)$, then, we infer from Theorem 2.1

\[
\|M^{k-(n-1)(2n-1)} \phi\|^2_{L^2(0,1)^n} \leq C \|M^k K^* \phi\|^2_{L^2(0,1)^n}.
\]

Now, using the inequality (55), we have

\[
\int_Q (s\theta)^\tau e^{-2sM_0 \theta} |M^{k-(n-1)(2n-1)} \phi|^2 \leq C \int_Q (s\theta)^{\tau} e^{-2sM_0 \theta} |M^k K^* \phi|^2 \\
\leq C \int_{(0,T) \times \omega} (s\theta)^{\tau + \kappa + r} e^{2s\Phi_1} |B^* \phi|^2.
\]

At present, we are ready to prove the main result.

**Proof of Theorem (1.1).** For the necessary part, we don’t need to use Carleman estimate. Indeed ; suppose $\text{Ker}(K^*) \neq \{0\}$, from Proposition 1, there exists $p_0 \in \mathbb{N}^*$
such that $\text{rank } K_{p_0} = \text{rank } [-\lambda_{p_0}D + A^*B] < n$. From the Kalman’s rank condition applied to ordinary differential system

$$y' = (\lambda_{p_0}D + A)y + Bv$$

is not controllable. Thus, there exists a nonzero solution $z_{p_0}(t) \in \mathbb{R}^n$ to the associated adjoint system

$$-z' = (\lambda_{p_0}D^* + A^*)z \quad \text{in } (0, T),$$

satisfying $B^*z_{p_0}(t) = 0$ for all $t \in [0, T]$. Then, let $\phi_0 = z_{p_0}(t)\Phi_{p_0}$ where $\Phi_{p_0}$ is the normalized eigenfunction associated with $\lambda_{p_0}$. The function $\phi(t, x) = z_{p_0}(t)\Phi_{p_0}$ is the solution of the adjoint problem (6), corresponding to $\phi_0$, which is nonzero and satisfies $B^*\phi(t, x) = 0$ in $Q$. So this solution does not satisfy the observability inequality (7) and thus (1) is not controllable.

For the sufficient part, let $\phi \in L^2(0, T, H^1_0(0, 1)^n)$ be the solution of Problem (6) corresponding to $\phi_0$. Using Corollary 2 with $\tau = 0$ and $k = (n - 1)(2n - 1)$, there exist two positive constants $C$ and $\sigma$ such that

$$\int_{\mathbb{T}_\sigma}^{\mathbb{T}_T} \int_0^1 e^{-2sM_0\theta} |\phi|^2 dxdt \leq C \int_{(0, T) \times \omega} (s\theta)^l e^{2s\theta_1} |B^*\phi|^2 dxdt$$

for all $s \geq \sigma$, where $l = 4(n - 1)(2n - 1) + n + r$. For all $t \in [\frac{T}{4}, \frac{3T}{4}]$, we have $-2sM_0(\frac{4}{T})^8 \leq -2sM_0(\frac{1}{T})^8$, and then $e^{-2sM_0\theta} \geq e^{-2sM_0(\frac{4}{T})^8}$. On the other hand, since $\Phi_1 = \theta(t)(4^{n-1}\Psi(x) - (4^{n-1} - 1)\phi(x))$ for $m_0 := \min_{x \in \omega} |4^{n-1}\Psi(x) - (4^{n-1} - 1)\phi(x)|$, we have

$$\int_{(0, T) \times \omega} (s\theta)^l e^{2s\theta_1} |B^*\phi|^2 dxdt \leq \int_{(0, T) \times \omega} (s\theta)^l e^{-2s\theta_0} |B^*\phi|^2 dxdt$$

$$\leq \int_0^T (s\theta)^l e^{-2s\theta_0} \int_\omega |B^*\phi|^2 dxdt.$$

Since $\lim_{t \to 0^+} (s\theta)^l e^{-2s\theta_0} = \lim_{t \to T^-} (s\theta)^l e^{-2s\theta_0} = 0$, we deduce

$$\int_{\mathbb{T}_\sigma}^{\mathbb{T}_T} \int_0^1 |\phi|^2 dxdt \leq C \int_{(0, T) \times \omega} |B^*\phi|^2 dxdt.$$

As in [5], there exists a positive constant $C$ depending on $D$ and $A$ such that

$$\frac{d}{dt} \left(e^{Ct}||\phi(t, \cdot)||^2\right) \geq 0, \quad t \in (0, T).$$

From this last inequality, we also infer

$$||\phi(0, \cdot)||^2 \leq e^{CT} ||\phi(T, \cdot)||^2 \leq \frac{2}{T} e^{C(\frac{T}{4})} \int_{\mathbb{T}_T}^{\mathbb{T}_T} \int_0^1 |\phi|^2 dxdt.$$

Therefore, this together with (59) imply the observability inequality for the solutions of the adjoint problem (6)

$$||\phi(0, \cdot)||^2 \leq C \int_{(0, T) \times \omega} |B^*\phi|^2 dxdt.$$

This completes the proof.

**Acknowledgments.** We thank the anonymous reviewers for their careful reading of our paper.
REFERENCES

[1] E. M. Ait Benhassi, F. Ammar Khodja, A. Hajjaj and L. Maniar, Null controllability of degenerate parabolic cascade systems, Portugal. Math., 68 (2011), 345–367.

[2] E. M. Ait Benhassi, F. Ammar Khodja, A. Hajjaj and L. Maniar, Carleman estimates and null controllability of coupled degenerate systems, Evol. Equ. Control Theory, 2 (2013), 441–459.

[3] F. Ammar-Khodja, A. Benabdallah, M. González-Burgos and L. de Teresa, Recent results on the controllability of linear coupled parabolic problems: A survey, Mathematical Control and Related Fields, 1 (2011), 267–306.

[4] F. Ammar-Khodja, A. Benabdallah, C. Dupaix and M. González-Burgos, A generalization of the Kalman rank condition for time-dependent coupled linear parabolic systems, Diff. Equ. Appl., 1 (2009), 427–457.

[5] F. Ammar-Khodja, A. Benabdallah, C. Dupaix and M. González-Burgos, A Kalman rank condition for the localized distributed controllability of a class of linear parabolic systems, J. Evol. Equ., 9 (2009), 267–291.

[6] F. Alabau-Boussouira, P. Cannarsa and G. Fragnelli, Carleman estimates for degenerate parabolic operators with application to nullcontrollability, J. evol. equ., 6 (2006), 161–204.

[7] M. Campiti, G. Metafune and D. Pallara, Degenerate self-adjoint evolution equations on the unit interval, Semigroup Forum, 57 (1998), 1–36.

[8] P. Cannarsa and G. Fragnelli, Null controllability of semilinear degenerate parabolic equations in bounded domains, Electronic Journal of Differential Equations, 136 (2006), 1–20.

[9] P. Cannarsa, P. Martinez and J. Vancostenoble, Null controllability of degenerate heat equations, Adv. Differential Equations, 10 (2005), 153–190.

[10] P. Cannarsa, P. Martinez and J. Vancostenoble, Carleman estimates for a class of degenerate parabolic operators, SIAM J. Control Optim., 47 (2008), 1–19.

[11] P. Cannarsa, P. Martinez and J. Vancostenoble, Global Carleman estimates for degenerate parabolic operators with applications, Memoirs of the American Mathematical Society, 239 (2016), ix+209 pp.

[12] P. Cannarsa and L. de Teresa, Controllability of 1-d coupled degenerate parabolic equations, Electronic Journal of Differential Equations, 73 (2009), 1–21.

[13] M. Fadili and L. Maniar, Null controllability of n-coupled degenerate parabolic systems with m-controls, J. Evol. Equ., 17 (2017), 1311–1340.

[14] A. V. Fursikov and O. Y. Imanuvilov, Controllability of Evolution Equations, Lectures notes series 34, Seoul National University Research Center, Seoul, 1996.

[15] M. Gonzalez-Burgos and L. De Teresa, Controllability results for cascade systems of m-coupled parabolic PDEs by one control force, Port. Math., 67 (2010), 91–113.

[16] M. Gueye, Exact boundary controllability of 1-D parabolic and hyperbolic degenerate equations, SIAM J. Control Optim., 52 (2014), 2037–2054.

[17] A. Hajjaj, Estimations de Carleman et Applications à la contrôlabilité à Zéro D’une Classe De Systèmes Paraboliques Dégénérés, Thèse d’État, Marrakech, 2013.

[18] G. Lebeau and L. Robbiano, Contrôle exact de l’équation de la chaleur, Comm. in PDE, 20 (1995), 335–356.

[19] R. D. Meyer, Degenerate elliptic differential systems, J. Math. Anal. Appl., 29 (1970), 436–442.

[20] J. Zabczyk, Mathematical Control Theory, Birkhäuser, Boston, 1995.

Received September 2017; 1st revision January 2018; 2nd revision February 2018.

E-mail address: m.benhassi@uca.ma
E-mail address: fadilimed@live.fr
E-mail address: maniar@uca.ma