ADAPTIVE TESTING FOR THE GRAPHICAL LASSO

BY MAX GRAZIER G’SELL*, JONATHAN TAYLOR† AND ROBERT TIBSHIRANI‡

Stanford University

We consider tests of significance in the setting of the graphical lasso for inverse covariance matrix estimation. We propose a simple test statistic based on a subsequence of the knots in the graphical lasso path. We show that this statistic has an exponential asymptotic null distribution, under the null hypothesis that the model contains the true connected components.

Though the null distribution is asymptotic, we show through simulation that it provides a close approximation to the true distribution at reasonable sample sizes. Thus the test provides a simple, tractable test for the significance of new edges as they are introduced into the model. Finally, we show connections between our results and other results for regularized regression, as well as extensions of our results to other correlation matrix based methods like single-linkage clustering.

1. Introduction. We consider the problem of hypothesis testing for the components of the inverse covariance matrix. We focus on the particular case of the graphical lasso (Friedman, Hastie and Tibshirani, 2008), and construct a set of interpretable hypotheses tests and corresponding test statistics.

In the graphical lasso, the inverse covariance matrix $\Sigma^{-1}$ is estimated by the matrix $\Theta$ that maximizes

$$\log \det \Theta - \text{Tr}(S\Theta) - \rho ||\Theta||_1,$$

where $S$ is the sample covariance matrix of the data, $\rho > 0$, and $||\Theta||_1 = \sum_{i \neq j} |\Theta_{ij}|$, the element-wise $L_1$-norm excluding the diagonal. We restrict ourselves to the particular case where the data have been centered and scaled so that $S$ is the sample correlation matrix.

As the regularization parameter $\rho$ decreases, the resulting estimate $\Theta$ is correspondingly denser. Solutions can be computed over a range of $\rho$ values using existing approaches (e.g. Friedman, Hastie and Tibshirani, 2008; Witten, Friedman and Simon, 2011; Mazumder and Hastie, 2012a,b). Some

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theoretical guarantees also exist (e.g. Ravikumar et al., 2008) about the recovery of the appropriate sparsity patterns in $\Theta$ under certain conditions. However, there has not been much work in inference and testing in the graphical lasso setting.

A similar problem of testing for the lasso in linear regression was recently addressed by Lockhart et al. (2013). That paper constructs a covariance test statistic, based on the change in covariance between fitted values of the lasso and the original observed values, at points along the lasso path where new variables enter. Intuitively, this statistic should be large if the variables entering carry information. In the case of orthogonal predictors, this statistic simplifies to a knot-based statistic, $n\lambda_k(\lambda_k - \lambda_{k+1})$, based on the spacing between points in the lasso path where the set of selected variables change. In their paper, they show that these statistics have asymptotic $\text{Exp}(1/k)$ distributions for the $k^{th}$ null variable to enter the lasso estimate.

Inspired by their work, this paper proposes a set of hypotheses corresponding to a particular sequence of $\rho$ along the graphical lasso solution path. These hypotheses test the statistical significance of structure added after each of these points in the graphical lasso solution set. For these hypotheses, we construct simple test statistics, and prove that they have an exponential asymptotic null distribution. We also demonstrate through simulation that these distributions are practically useful at reasonable sample sizes.

Finally, our test statistics and the corresponding theory have close ties to particular functions of the order statistics of the correlation matrix. As a result, we are able to extend our results to other correlation-based methods. In particular, for the case of single-linkage clustering based on absolute correlations, we are able to provide similar test statistics and asymptotic null distributions.

Section 2 defines the proposed test statistic and gives an overview of its behavior, including a demonstration in Section 2.3 of this test statistic on the cell-signaling data set from Friedman, Hastie and Tibshirani (2008). Section 3 discusses the behavior of the proposed test statistic and derives its exponential asymptotic null distributions. Section 4 demonstrates empirically that this null distribution holds in a variety of settings and demonstrates desirable behavior at later steps that is yet unproven. We conclude in Section 6 with a discussion of the behavior and interpretation of this statistic, along with its extension to correlation-based clustering.

2. Significance testing in the graphical lasso. Estimation based on the graphical lasso is interesting, because it allows estimation of sparse
dependence structures of data. As the regularization parameter $\rho$ varies, so does the level of sparsity. This gives rise to an inferential question: how much structure can be justified in the presence of noise?

In this section, we define a simple set of hypotheses at a set of parameter values where the sparsity pattern changes dramatically. We characterize the behavior of the estimator at these points, and its relationship to order statistics of the correlation matrix. The estimate at these points resembles the situation that arises in Lockhart et al. (2013) for the lasso estimator in linear regression.

In Lockhart et al. (2013), the authors propose a covariance-based test statistic, which is a function of the lasso fitted values at the points in the lasso path where variables first enter. They show that this statistic has a simple Exp(1) asymptotic null distribution, under the null that the true signal variables have already been selected.

In this section, we propose a similar test statistic for our setting and the hypotheses mentioned above. We also provide an example demonstrating this test statistic on biological data. In later parts of this paper, we prove that this proposed statistic has similar desirable properties to the original lasso statistic of Lockhart et al. (2013).

2.1. The graphical lasso path. We begin by summarizing some known properties of the graphical lasso estimator. We refer to the graphical lasso path to be the set of solutions $\Theta(\rho)$ over all values of $\rho$. Broadly speaking, $\Theta(\rho)$ will be sparse for large values of $\rho$ and dense for small values of $\rho$.

More explicitly, there exists a finite set of graphical lasso knots, $\rho_1 = \max_{i \neq j} |S_{ij}| \geq \rho_2 \geq \rho_3 \geq \cdots \geq \min_{i \neq j} |S_{ij}| \geq 0$, which are the points where the sparsity pattern in the estimate $\Theta(\rho)$ changes. Above $\rho_1$, $\Theta(\rho)$ is diagonal; below the smallest knot, $\Theta(\rho)$ is dense.

![Fig 1. Illustration of the two types of sparsity pattern changes in the graphical lasso. In case I, an edge is formed between two previously disconnected components. In case II, an edge is formed within a previously connected component. This paper addresses the knots at which the first type of change occurs.](image-url)

If we consider the estimate as we move from larger parameter values to
smaller ones, one of two changes to the sparsity pattern can take place at a knot \( \rho_k \). These are illustrated in Figure 1. First, two groups of variables that were disconnected for all \( \rho > \rho_k \) can become connected by a new edge. Second, the sparsity pattern within a previously connected group of variables can change, without altering the overall set of connected components of variables.

The first of these changes is particularly interesting. These points have been characterized by Witten, Friedman and Simon (2011) and Mazumder and Hastie (2012a), where they are used to decompose the graphical lasso problem into smaller problems on the connected components. We are interested in them because they give rise to natural hypotheses to test, and because their special behavior will lead those hypotheses to be naturally nested.

Let \( \tilde{\rho}_1 \geq \cdots \geq \tilde{\rho}_M \) correspond to the subset of knots where the connected components of \( \Theta(\rho) \) change. As discussed in detail in the papers referenced above, the \( \Theta_{ij}(\rho) \neq 0 \) for all \( \rho < |S_{ij}| \), and in particular, disconnected components can only become connected through edges that enter in such a fashion. As a result, the \( \tilde{\rho}_k \) correspond to points where \( \rho = |S_{ij}| \) for some \( i \neq j \). Furthermore, \( \tilde{\rho}_1, \tilde{\rho}_2, \ldots \) corresponds to the subsequence of order statistics of \( |S_{ij}| \) obtained by ordering the \( |S_{ij}| \) from largest to smallest, and then eliminating those \( |S_{ij}| \) where \( i \) and \( j \) would already be in the same connected component based on the preceding elements.

This allows the points \( \tilde{\rho}_1, \ldots, \tilde{\rho}_M \) of the graphical lasso path to be characterized in terms of the extreme values of the off-diagonal elements of \( S \). This provides inspiration for the test statistics in the next section. It also forms the basis for the theory used in Section 3 to prove the asymptotic behavior of those test statistics.

2.2. Testing along the graphical lasso path. With \( \tilde{\rho}_1 \geq \cdots \geq \tilde{\rho}_M \) defined to be the points where the connected components of the graphical lasso estimator \( \Theta(\rho) \) change, we can define a corresponding sequence of hypotheses \( H_1, \ldots, H_M \). The hypothesis \( H_k \) is that each connected component of the true \( \Sigma^{-1} \) is contained within a connected component of \( \Theta(\rho) \), \( \forall \rho < \tilde{\rho}_k \).

Because the nature of the graphical lasso path is such that connected components can never become disconnected, these hypotheses are nested by construction. That is, if \( H_k \) is true, then \( H_\ell \) is true for all \( \ell > k \). These hypotheses correspond to natural questions about the dependence graph. They describe whether we have found all the important connections between groups of variables, where groups of variables can include singletons.

To test these hypotheses, we note that the \( \tilde{\rho}_k \) resemble the regularization
parameters of the lasso estimator in the case of linear regression with orthogonal $X$, including the relationship between the knot location and the ordered absolute values of the underlying data. The recent paper (Lockhart et al., 2013) constructs a test statistic for similar hypotheses in that setting; their work provides inspiration for the test statistics we propose here.

We propose the test statistic

$$T_k = n \hat{p}_k (\hat{p}_k - \hat{p}_{k+1}).$$

Intuitively, this statistic will tend to be large when signal is present, as the $\hat{p}$ values converge to large correlations and the $n$ grows. In the absence of signal, we will show that the spacings $\hat{p}_k - \hat{p}_{k+1}$ have simple limiting distributions when scaled by $n \hat{p}_k$.

The statistic $T_k$ is similar to the knot-form of the statistic discussed by Lockhart et al. (2013) and summarized here in Section 1; we discuss the relationship between $T_k$ and the more general covariance statistic form of Lockhart et al. (2013) in Section 2.4.

We will show in Section 3 that the $T_k$ have a simple asymptotic null distribution, under the hypotheses $H_k$. In particular, let $k$ be the smallest value such that $H_k$ is true. Then $T_k \xrightarrow{d} \text{Exp}(1)$, and $T_{k'} \xrightarrow{d} \text{Exp}(1/(k' - k + 1))$ for $k' > k$. We also demonstrate empirically that these asymptotic distributions are accurate even for reasonable sample sizes and common signal structures. This, combined with the simplicity of the null distributions, makes the proposed statistics a reasonable candidate for directly testing the significance of elements in the graphical lasso solution.

2.3. Example. Before proceeding with theoretical results, we demonstrate the proposed statistic on the cell signaling data from Friedman, Hastie and Tibshirani (2008). This data consists of a set of 7466 observations of 11 proteins. We apply the graphical lasso to estimate a sparse dependence graph describing the relationships between these proteins.

This experimental data set contains only a few variables, and was designed to contain strong correlations. Because our proposed statistic is really interesting in the presence of a mix of informative and uninformative variables, we augment the real data set with 100 noise variables, drawn from a Gaussian distribution with zero correlation between the variables. We then fit the graphical lasso on this augmented data set. With this construction, we expect the true connected components to involve the true proteins, and none of the noise variables.

We apply the tests from Section 2.2 to obtain test statistics and p-values for the first fifteen edges as they enter. This is performed on repeated sub-
samples of $n = 500$ observations to capture variation in the resulting statistics and $p$-values. The $p$-values are computed in comparison to an Exp(1) distribution, which should be accurate for the first null step and conservative for later null steps. The results are shown in Figure 2.

![Diagram](image)

**Fig 2.** The proposed test statistics and their $p$-values are shown, using data consisting of 11 related biological proteins and 100 simulated noise variables on repeated subsamples of size $n = 500$. The first four to five steps, depending on the realization, correspond to the entrance of edges between true protein variables, with the next step corresponding to a connection to a noise variable. The left panel shows boxplots of the $p$-values, where each statistic is compared to the CDF of an Exp(1) distribution. Note that this will be conservative after the first null step. The middle panel shows Exp(1) QQ-plots for the first five null steps, so that a line with slope $1/k$ should correspond to an Exponential distribution with mean $1/k$. The right plot shows a histogram of the $p$-value at the first null step.

With this sample size, the first four to five edges to enter (depending on the particular realization) correspond to edges in the original 11 signal variables. The next edge to enter is a noise edge, connecting either two noise variables or a real variable and a noise variable. The fact that the location of this step is realization-dependent makes it difficult to accurately assess the distribution of the first null step in the left plot.

The center plot shows QQ-plots for the distributions of the first five noise edges accepted, wherever they occur in a particular realization. This shows that the distributions are close to Exp$(1/k)$ distributions. The right plot shows a histogram of the $p$-values for the first of these noise edges, where they appear close to uniform. We attribute the conservative skew that can be seen in both the center and right plots to the presence of signal edges that have not yet been selected because their strength is on the same scale as the noise edges. In the more idealized simulations of Section 4, we see that this conservative trend vanishes.
2.4. Covariance Formulation. In Lockhart et al. (2013), the knot-based statistic we discussed in Section 2.2 was a special case of a more general covariance statistic, which applied more broadly. The covariance test statistic corresponded to a difference in inner products between the observed and fitted values. The equivalent formulation in the context of the graphical lasso would be

\[ \hat{T}_k = \frac{n}{2} (\text{Tr}(S\Sigma(\hat{\rho}_{k+1})) - \text{Tr}(S\Sigma_0(\hat{\rho}_{k+1}))) \]

where \( \Sigma_0(\hat{\rho}_{k+1}) \) is the graphical lasso solution \( \Sigma(\hat{\rho}_{k+1}) \), but without the change in nonzero pattern that would have happened at \( \hat{\rho}_k \), so that any zero elements before \( \hat{\rho}_k \) remain zero.

Let \( i^*, j^* \) be the element that entered at \( \hat{\rho}_k \). Since \( \Sigma_0(\rho_{k+1}) \) is identical to \( \Sigma(\rho_{k+1}) \) except at \( i^*, j^* \), the expression simplifies to

\[ \hat{T}_k = nS_{i^*,j^*} (S_{i^*,j^*} \pm \hat{\rho}_{k+1}) = n\hat{\rho}_k (\hat{\rho}_k - \hat{\rho}_{k+1}) = T_k, \]

so the covariance form of the statistic and the knot form of the statistic are the same for our problem.

It is interesting to note that if used the original graphical lasso knots \( \rho_k \) instead of the restricted sequence \( \hat{\rho}_k \) corresponding to changes in connected components, this connection between \( T_k \) and \( \hat{T}_k \) fails to hold.

3. Null behavior. Here we give justification and theory for the behavior of the statistic proposed in Section 2.2 in the null case. We begin with the global null of no correlation structure in Section 3.1, and then extend these ideas in Section 3.3 to a weaker null case where correlation structure exists but has been selected.

3.1. Global null, first step. We consider the test statistic, \( T_1 = n\hat{\rho}_1(\hat{\rho}_1 - \hat{\rho}_2) \), for the first edge to enter on the path, under the global null hypothesis that the data has no correlation structure (\( H_1 \) from Section 2.2). Explicitly, this is the hypothesis that \( \Sigma = I_p \), where \( \Sigma \) is the true correlation matrix and \( I_p \) is the \( p \times p \) identity matrix. We discuss this case first, since the proof is simpler and yet contains all of the important pieces for the more general case.

The test statistic is most easily described in terms of the order statistics of the correlation matrix. Let \( V_1, V_2 \) be the first and second largest off-diagonal elements of \( S \) (in absolute value). Since the first two edges selected necessarily connect previously disconnected components, we have \( \hat{\rho}_1 = V_1 \) and \( \hat{\rho}_2 = V_2 \), so \( T_1 = nV_1(V_1 - V_2) \).
Under the null distribution, $V_1$ and $V_2$ are the largest and second largest correlations out of $p$ i.i.d. spherically distributed vectors in $\mathbb{R}^n$. Distributions of these correlations and their extrema are discussed in Cai and Jiang (2011). We show that as $n,p \to \infty$ with $n$ growing faster than $\log p$, $T_1$ converges in distribution to $\text{Exp}(1)$.

**Theorem 1.** Let $V_1 \geq V_2$ be the first two order statistics of the absolute values of the correlations of $Z_1, \ldots, Z_p \in \mathbb{R}^n$, where the $Z_i$ are i.i.d. with a spherical distribution around 0. Then $nV_1(V_1 - V_2) \xrightarrow{d} \text{Exp}(1)$ as $n,p \to \infty$ with $\frac{\log p}{n} \to 0$.

**Proof.** Begin by defining the following quantities:

1. $M_{ij} = \max_{(i',j') \neq (i,j)} |S_{i'j'}|$ and $M_{ij}^* = \max_{(i',j') \neq (i,j)} |S_{i'j'}|$. Note that $M_{ij}^* \leq M_{ij}$.
2. Events $A_{ij} = \{|S_{ij}| > M_{ij}\}$ and $A_{ij}^* = \{|S_{ij}| > M_{ij}^*\}$. Note that $\{|S_{ij}|(|S_{ij}| - M_{ij}) > t/n\} \subseteq A_{ij}$, $\{|S_{ij}|(|S_{ij}| - M_{ij}^*) > t/n\} \subseteq A_{ij}^*$, that the $A_{ij}$ are disjoint, and that $A_{ij} \subseteq A_{ij}^*$. The events $A_{ij}, i < j,$ form a partition, since exactly one $|S_{ij}|$ is larger than all the others. The events $A_{ij}^*$ form an approximate partition; Lemma 12 shows that the approximation is close.
3. $(i^*, j^*) = \arg\max_{i<j} |S_{ij}|$, the indices of the largest off-diagonal element of $S$.
4. $E = \{M_{i^*j^*} = M_{i^*j^*}^*\}$. This is the event that the second largest element of $S$ shares no indices with the largest element. In particular, this is a subset of the event in Lemma 6, so $\mathbb{P}(E^c) \to 0$ as $n,p \to \infty$ with $\frac{\log p}{n} \to 0$.

We can expand $\mathbb{P}(T_1 > t) = \mathbb{P}(nV_1(V_1 - V_2) > t)$ as

$$\mathbb{P}(nV_1(V_1 - V_2) > t) = \mathbb{P}(V_1(V_1 - V_2) > t/n) = \sum_{i<j} \mathbb{P}( |S_{ij}| (|S_{ij}| - M_{ij}) > t/n).$$

Here we are taking advantage of the fact that $\mathbb{P} ( |S_{ij}| (|S_{ij}| - M_{ij}) > t/n)$ is only positive for one pair $i < j$, so the events being considered are disjoint and cover the event that $V_1(V_1 - V_2) > t/n$.

This sum can be further expanded based on intersection with $E$, obtaining

$$\mathbb{P}(nV_1(V_1 - V_2) > t) = \sum_{i<j} \mathbb{P} (\{ |S_{ij}| (|S_{ij}| - M_{ij}) > t/n \} \cap E) + \sum_{i<j} \mathbb{P} (\{ |S_{ij}| (|S_{ij}| - M_{ij}) > t/n \} \cap E^c).$$
Note that \( \mathbb{P}(\{|S_{ij}| (|S_{ij}| - M_{ij}) > t/n\} \cap E^c) \leq \mathbb{P}(A_{ij} \cap E^c) \), so

\[
\mathbb{P}(nV_1(V_1 - V_2) > t) - \sum_{i<j}^p \mathbb{P}(\{|S_{ij}| (|S_{ij}| - M_{ij}) > t/n\} \cap E) = \sum_{i<j} \mathbb{P}(\{|S_{ij}| (|S_{ij}| - M_{ij}) > t/n\} \cap E^c) \leq \sum_{i<j} \mathbb{P}(A_{ij} \cap E^c) = \mathbb{P}(E^c)
\]

By a similar argument,

\[
\sum_{i<j}^p \mathbb{P}(\{|S_{ij}| (|S_{ij}| - M^{*}_{ij}) > t/n\}) - \sum_{i<j}^p \mathbb{P}(\{|S_{ij}| (|S_{ij}| - M^{*}_{ij}) > t/n\} \cap E) = \sum_{i<j} \mathbb{P}(A^{*}_{ij} \cap E^c) = \sum_{i<j} \mathbb{P}(A^{*}_{ij} \cap E^c) + \sum_{i<j} \mathbb{P}((A^{*}_{ij} \setminus A_{ij}) \cap E^c) \leq \mathbb{P}(E^c) + \sum_{i<j} \mathbb{P}(A^{*}_{ij} \setminus A_{ij})
\]

Noting that \( \mathbb{P}(\{|S_{ij}| (|S_{ij}| - M_{ij}) > t/n\} \cap E) = \mathbb{P}(\{|S_{ij}| (|S_{ij}| - M^{*}_{ij}) > t/n\} \cap E) \), we can combine these two statements to obtain

\[
\mathbb{P}(nV_1(V_1 - V_2) > t) - \sum_{i<j}^p \mathbb{P}(\{|S_{ij}| (|S_{ij}| - M^{*}_{ij}) > t/n\}) \leq 2\mathbb{P}(E^c) + \sum_{i<j} \mathbb{P}(A^{*}_{ij} \setminus A_{ij})
\]

We apply Lemma 5 to \( M^{*}_{ij} \) (which is now \((p-2) \times (p-2)\)) to approximate the second term, giving

\[
\mathbb{P}(nV_1(V_1 - V_2) > t) - e^{-t} \left( \sum_{i<j}^p \mathbb{P}(A^{*}_{ij}) \right) \left( 1 + \frac{1}{\sqrt{\log p}} + \frac{\log p}{n} \right) \leq 2\mathbb{P}(E^c) + \sum_{i<j} \mathbb{P}(A^{*}_{ij} \setminus A_{ij}) + O(p^2 e^{-(p-2)^{3/4}/4\sqrt{\log(p-2)}})
\]

By Lemma 12, \( \sum_{i<j}^p \mathbb{P}(A^{*}_{ij}) \rightarrow 1 \), and since \( \sum_{i<j} \mathbb{P}(A_{ij}) = 1 \), \( \sum_{i<j} \mathbb{P}(A^{*}_{ij} \setminus A_{ij}) \), this implies that \( \mathbb{P}(T_1 > t) = \mathbb{P}(nV_1(V_1 - V_2) > t) \rightarrow e^{-t} \) and therefore \( T_1 \xrightarrow{d} \text{Exp}(1) \). □
Here, and throughout this paper, we use the notation $f(x) = O(g(x))$ to denote the situation where $\exists M \in \mathbb{R}^+, x_0 \in \mathbb{R}$ such that $|f(x)| \leq M|g(x)|$ for all $x > x_0$.

This result establishes that under the global null hypothesis, $T_1 \overset{d}{\rightarrow} \text{Exp}(1)$. The proofs of the supporting Lemmas are left for the Appendix. Note that throughout this paper, we define $\text{Exp}(\mu)$ to refer to the exponential distribution with mean $\mu$ (not rate $\mu$).

### 3.2. Global null, later steps

The results of Lockhart et al. (2013) for the lasso suggest that it should also be true that $T_k = n\hat{\rho}_k (\hat{\rho}_k - \hat{\rho}_{k+1}) \overset{d}{\rightarrow} \text{Exp}(1/k)$ under the same global null. This is supported by simulation results like those shown in Figure 3 and 4, which empirically observe that these distributions are nearly exponential, and that the means agree with the $1/k$ prediction.

The proof that $n\hat{\rho}_k (\hat{\rho}_k - \hat{\rho}_{k+1}) \overset{d}{\rightarrow} \text{Exp}(1/k)$ given in this section relies on a conjecture, which we have so far been unable to prove. This conjecture allows the approximation from Cai and Jiang (2011) to be applied when bounding the probability that $k$ independent correlations exceed the maximum correlation of a large correlation matrix. The conjecture is as follows.

**Conjecture 1.** Let $f_n(x)$ be the density of $\sqrt{n}|S_{ij}|$, $\bar{F}_n(x) = \int_x^{\sqrt{n}} f_n(x)$, and $G_{n,p}(x)$ be the CDF of the maximum of an independent $p \times p$ correlation matrix (based on $n$ observed $p$ vectors) scaled by $\sqrt{n}$. Then for fixed $k$,

$$\int_0^{\sqrt{(4 - \frac{2}{k+2}) \log p}} G_{n,p}(x) \bar{F}_n^{k-1}(x) f_n(x) dx = o \left( \frac{1}{p^{2k}} \right).$$

A stronger, but sufficient condition for this conjecture to hold is that $p^{2k} \left( \sqrt{n}M_{n,p} < \sqrt{(4 - \frac{2}{k+2}) \log p} \right) \rightarrow 0$ for fixed $k$.

We believe that this conjecture holds. The conjecture, and the stronger sufficient condition, would hold if the elements of $\sqrt{n}S$ were independent Gaussians. As $n$ and $p$ grow large, the elements of $\sqrt{n}S$ are very nearly independent Gaussians; the rest of the proofs in this paper take advantage of that near-Gaussianity.

Assuming Conjecture 1 holds, the following theorem proves that $T_k = n\hat{\rho}_k (\hat{\rho}_k - \hat{\rho}_{k+1}) \overset{d}{\rightarrow} \text{Exp}(1/k)$.

As in the previous theorem, we find it convenient to work with the order statistics of the absolute correlation matrix, $V_1, V_2, \ldots$. On the event $E_d$ that the first $d + 1$ largest elements of $S$ share no indices, an event which
is defined below and shown to satisfy $\Pr(E_d) \to 1$, $\tilde{\rho}_k = V_k$ for $k \leq d$, so the results hold equally well for the $\tilde{\rho}_1, \tilde{\rho}_2, \ldots$ sequence of interest.

**Theorem 2.** Let $V_1 \geq V_2 \geq \cdots \geq V_{p(p-1)/2}$ be the order statistics of the absolute values of the correlations of $Z_1, \ldots, Z_p \in \mathbb{R}^n$, where the $Z_i$ are i.i.d. with spherical distribution around 0. Assume that Conjecture 1 holds. Then for fixed $d \geq 0$ and $1 \leq k \leq d$, $nV_k(V_k - V_{k+1}) \xrightarrow{d} \text{Exp}(1/k)$ as $n, p \to \infty$ with $\frac{\log p}{n} \to 0$.

**Proof.** This proof follows nearly identically to the proof of Theorem 1, using more general forms of the same results.

Consider $T_k = nV_k(V_k - V_{k+1})$. We can expand $\Pr(T_k > t)$ similarly to the previous proof, but with $k$ greater than 1. To do this, we need a more complicated index set. Define

$$I_k = \left\{ i_1, \ldots, i_k, j_1, \ldots, j_k \in \{1, \cdots, p\} : i_\ell < j_\ell \text{ and } (i_\ell, j_\ell) \notin \bigcup_{\ell' < \ell} \{ (i_{\ell'}, j_{\ell'}) \}, \forall \ell \right\}$$

$$I_k^* = \left\{ i_1, \ldots, i_k, j_1, \ldots, j_k \in \{1, \cdots, p\} : i_\ell < j_\ell \text{ and } i_\ell, j_\ell \notin \bigcup_{\ell' < \ell} \{ i_{\ell'}, j_{\ell'} \}, \forall \ell \right\},$$

so that $I_k$ is the set of all selections of $k$ off-diagonal elements, and $I_k^*$ is the set of all selections of $k$ off-diagonal elements such that none share an index.

Then

$$\Pr(T_k > t) = \sum_{I_k} \Pr \left( \{|S_{i_k,j_k}|(S_{i_k,j_k} - M_{i_k,j_k,\ldots,i_k,j_k}) > t/n\} \cap \{S_{i_1,j_1} > \cdots > S_{i_{k-1},j_{k-1}}\} \right)$$

where $M_{i_1,j_1,\ldots,i_k,j_k} = \max_{i < j; (i,j) \notin \{(i_1,j_1), \cdots, (i_k,j_k)\}} |S_{ij}|$. Note that the function $x(x - M_{i_1,j_1,\ldots,i_k,j_k})$ is increasing for $x > M_{i_1,j_1,\ldots,i_k,j_k}$, so the above expression can be rewritten as

$$\Pr(T_k > t) = \sum_{I_k} \Pr \left( |S_{i_1,j_1}|(S_{i_1,j_1} - M_{i_1,j_1,\ldots,i_k,j_k}) > \cdots > |S_{i_k,j_k}|(|S_{i_k,j_k} - M_{i_1,j_1,\ldots,i_k,j_k}| > t/n) \right)$$

$$= \frac{1}{k!} \sum_{I_k} \Pr \left( \bigcap_{\ell \leq k} \{|S_{i_\ell,j_\ell}|(|S_{i_\ell,j_\ell} - M_{i_1,j_1,\ldots,i_k,j_k}| > t/n) \right),$$

where the $k!$ is included because the last event does not distinguish between permutations of the labels.
As in the previous proof, we now define several events:

\[ E_d = \{ \text{The largest } d + 1 \text{ elements of } S \text{ share no indices} \} \]

\[ A_{i_j} = \bigcap_{\ell \leq k} \{ S_{ij} > M_{i_j} \} \]

\[ A^*_{i_j} = \bigcap_{\ell \leq k} \{ S_{ij} > M^*_{i_j} \} \]

where \( M^*_{i_j} = \max_{i \neq j} \bigcup_{\ell \leq k} \{ S_{ij} \} \), the maximum outside of any indices in \( i_1, i_2, \ldots, i_k \).

Then we can expand the above expression in terms of intersections with \( E_d \) and \( E^*_d \).

\[
\frac{1}{k!} \sum_{I_k} \mathbb{P} \left( \bigcap_{\ell \leq k} \{ |S_{ij}| - M_{i_j} > t/n \} \bigg| E^*_d \right) \leq \frac{1}{k!} \sum_{I_k} \mathbb{P} \left( A_{i_j} \bigg| E^*_d \right) = \mathbb{P}(E^*_d)
\]

since the \( A_{i_j} \) are disjoint (up to permutations of the indices, which is accounted for by the \( \frac{1}{k!} \)). Similarly,

\[
\sum_{I_k^*} \frac{1}{k!} \mathbb{P} \left( \bigcap_{\ell \leq k} \{ |S_{ij}| - M^*_{i_j} > t/n \} \bigg| E^*_d \right) \leq \mathbb{P}(E^*_d) + \frac{1}{k!} \sum_{I_k} \mathbb{P} \left( A^*_{i_j} \bigg| E^*_d \right)
\]

Combining these, we obtain

\[
\left| P(T_k > t) - \frac{1}{k!} \sum_{I_k} \mathbb{P} \left( \bigcap_{\ell \leq k} \{ |S_{ij}| - M^*_{i_j} > t/n \} \right) \right| \leq 2\mathbb{P}(E^*_d) + \frac{1}{k!} \sum_{I_k} \mathbb{P} \left( A^*_{i_j} \bigg| E^*_d \right)
\]

where we take advantage of the fact that on \( E_d, M_{i_1, i_2, \ldots, i_k} = M^*_{i_1, i_2, \ldots, i_k} \), and all the terms in the sum over \( I_k \) that are not in \( I_k^* \) vanish.

Then, by Lemma 5 we can replace \( \mathbb{P} \left( \bigcap_{\ell \leq k} \{ |S_{ij}| - M^*_{i_j} > t/n \} \right) \), obtaining

\[
\left| P(T_k > t) - e^{-kt} \left( 1 + O \left( \frac{1}{\sqrt{\log p}} + \frac{\log p}{n} \right) \right) \right| \leq 2\mathbb{P}(E^*_d) + \frac{1}{k!} \sum_{I_k} \mathbb{P} \left( A^*_{i_j} \bigg| E^*_d \right) + O \left( e^{-(p-2k)^{3/4}/\sqrt{\log(p-2k)}} \right)
\]
By Lemma 6, \( P(E_k) \to 0 \), and by Lemma 12, \( \frac{1}{k^2} \sum_{k} P(A_{i_1j_1,\ldots,i_kj_k}) \to 1 \). Since \( \frac{1}{k^2} \sum_{k} P(A_{i_1j_1,\ldots,i_kj_k}) \to 1 \) and \( A_{i_1j_1,\ldots,i_kj_k} \subset A_{i_1j_1,\ldots,i_kj_k} \), \( \frac{1}{k^2} \sum_{k} P(A_{i_1j_1,\ldots,i_kj_k} \setminus P_{i_1j_1,\ldots,i_kj_k}) \to 0 \). Combining these, we obtain \( P(T_k > t) \to e^{-kt} \), so \( T_k \sim \text{Exp}(1/k) \).

The result of the previous theorems is that for a finite number of steps, the corresponding test statistics have asymptotic distributions under the global null that no signal is present. While this is a comforting fact, some dependence structure is expected in many of the situations where the graphical lasso is applied. In the next section, we relax the global null hypothesis to a weaker one where some correlation structure is allowed.

3.3. **Null after signal selection.** Instead of assuming no dependence structure as in the global null, suppose that the true structure is restricted to a small subset of the nodes, \( A \). By this we mean that there exists a fixed-size subset, \( A \subset \{1,\ldots,p\} \), such that the only off-diagonal nonzeros in \( \Sigma^{-1} \) occur in the \( A \times A \) block.

Then the test statistic \( T_k = n \hat{\rho}_k (\hat{\rho}_k - \hat{\rho}_{k+1}) \) corresponds to a test of a weaker null hypothesis. If \( V_k \) is the set of variables involved in the estimate by step \( k \), then the null hypothesis at step \( k \) is that the signal variables have already appeared: \( H_k : A \subseteq V_{k-1} \).

For this theorem, we have to be careful here to make sure that the signal on \( A \) is strong enough that those variables are selected first. Then the first step outside of \( A \times A \) is the same as the first step after the variables in \( A \) have been selected. The theorem presented here assumes that the signal variables are strong enough to be selected first; it is followed by two simple conditions under which this occurs.

We show in the following theorem that under the above condition, and defining \( m \) to be the last step inside \( A \times A \), the first null test statistic \( T_{m+1} = \sqrt{n} \hat{\rho}_{m+1} (\hat{\rho}_{m+1} - \hat{\rho}_{m+2}) \overset{d}{\to} \text{Exp}(1) \) and subsequent steps \( T_{m+k} = \sqrt{n} \hat{\rho}_{m+k} (\hat{\rho}_{m+k} - \hat{\rho}_{m+k+1}) \overset{d}{\to} \text{Exp}(1/k) \), for \( 1 < k \leq d \) and \( d \) finite.

As this theorem relies on Theorems 1 and 2, the first step \( (k = 1) \) is proven without Conjecture 1, and the later steps \( (k > 1) \) rely on the validity of Conjecture 1. Again, we work with the order statistics \( V_1, V_2, \ldots \) of the correlation matrix, as \( V_{m+1}, \ldots, V_{m+d+1} \) are identical to \( \hat{\rho}_{m+1}, \ldots, \hat{\rho}_{m+d+1} \) on the event that the largest \( d + 1 \) elements of \( S \) outside of \( A \times A \) share no indices, which has probability limiting to 1.

**Theorem 3.** Let \( Z_1, \ldots, Z_p \) be independent, and define \( A \subset \{1,\ldots,p\} \) so that for \( j \notin A \), \( Z_j \) are i.i.d. with spherical distribution around 0. Let
\[ V_1 \geq V_2 \geq \cdots \geq V_{p(p-1)/2} \] be the order statistics of the absolute values of the correlations of \( Z_1, \ldots, Z_p \).

Let \( B \) be the event that there exists a \( \delta \) such that for all \( i \in A \), there exists \( j \in A \) such that \( S_{ij} > \delta \), and furthermore \( S_{ij} < \delta \) for all \( i \notin A \). Let \( m \) denote the number of steps taken before the first edge enters outside of \( A \times A \). If \( \mathbb{P}(B) \to 1 \), then as \( n, p \to \infty \) with \( \log p \),
\[ n V_m + k (V_m + k - V_m + k + 1) \xrightarrow{d} \text{Exp}(1/k), \] for \( 1 \leq k \leq d \) and fixed \( d \geq 0 \).

Note that sufficient conditions for \( \mathbb{P}(B) \to 1 \) are either the conditions for recovery in Ravikumar et al. (2008), or that each variable in \( A \) has at least one true correlation with limit strictly greater than \( \sqrt{4 \log p/n} \).

**Proof.** For simplicity of notation, rearrange the variables so that the variables in \( A \) are the first \( |A| \) variables. Let \( E_{A,d} \) be the event that the first \( d+1 \) largest off-diagonal elements of \( S \) outside of \( A \times A \) share no indices, and let \( D \) be the event that these \( d \) elements have no indices in \( A \). By Lemma 6, \( \mathbb{P}(E_{A,d}) \to 1 \), and by Lemma 7, \( \mathbb{P}(D) \to 1 \). Therefore \( \mathbb{P}(B \cap D \cap E_{A,d}) \to 1 \). On \( B \cap D \cap E_{A,d} \), \( V_{m+k}(V_{m+k} - V_{m+k+1}) = \tilde{V}_k(\tilde{V}_k - \tilde{V}_{k+1}) \), where \( \tilde{V}_1 \geq \cdots \geq \tilde{V}_{(p-|A|)(p-|A|-1)/2} \) are the order statistics of the correlations of variables not in \( A \). Therefore, in general, \( V_{m+k}(V_{m+k} - V_{m+k+1}) = \tilde{V}_k(\tilde{V}_k - \tilde{V}_{k+1}) + o_P(1) \).

Note that Theorems 1 and 2 apply to \( \tilde{V}_1, \ldots, \tilde{V}_k \), so the rest of the proof follows.

4. **Simulations.** We explore the null distribution of the covariance statistic through simulations. These are carried out under the global null, where there is no correlation between the variables, and the weaker null, where correlation is present only on a subset \( A \) of the variables.

4.1. **Simulations under the global null.** We generate vectors \( X_1, \ldots, X_n \in \mathbb{R}^p \) as \( X_i \overset{i.i.d.}{\sim} N(0, I_p) \). Here \( n = 500, p = 100 \). The correlation matrix then has entries \( S_{ij} = X_i^T X_j / \|X_i\|_2 \|X_j\|_2 \), after first centering the \( X_i \). The simulations are all repeated 1000 times.
Figure 3 considers the corresponding covariance statistics. Only $k = 1, \ldots, 5$ are shown, for simplicity. The first plot shows box plots of the $p$-values $1 - F_1(T_k)$, where each $T_k$ is compared to the CDF for an Exp(1) distribution. Note that this is conservative for all $k > 1$, leading the later $p$-values to quickly approach 1. The second plot compares the distributions of $T_1, \ldots, T_5$ to an Exp(1) distribution in a QQ plot. On this plot, a line with slope $1/k$ corresponds to an exponential distribution with mean $1/k$. The last plot shows a histogram of the $p$-value for the first step, showing it to be close to uniform.

To more easily see that the statistic means at step $k$ are matching the theoretical value of $1/k$, Figure 4 shows 95% confidence intervals for the means of the test statistics for the first five steps, based on 1000 realizations. We see that the empirical means are in agreement with the theoretical predictions.

4.2. Simulations with signal, $H_A$. In this simulation, we select a subset $A \subset \{1, \ldots, p\}$ of size $|A| = 6$. The covariance matrix $\Sigma$ is then designed so that the $A \times A$ block has some correlation structure, while the rest of the covariance matrix remains diagonal. We consider two different structures on $A$:

1. Disconnected Pairs: The variables in $|A|$ are paired, and $\Sigma^{-1}$ is large on those pairs.

2. Clique: All the entries in the $A \times A$ block of $\Sigma^{-1}$ are made large.
Fig 4. Confidence intervals for the means of the statistics at the first five null steps. These agree with the theoretical prediction that the $k^{th}$ step should have mean $1/k$.

Fig 5. Distribution of test statistics under the alternative of three highly correlated pairs of variables, with $n = 500$ and $p = 100$. The left panel shows boxplots of the $p$-values, where each statistic is compared to the CDF of an Exp(1) distribution. Note that this is conservative after the first null step. The middle panel shows Exp(1) QQ-plots for the first five null steps, demonstrating that the distribution is very nearly exponential and has mean $1/k$ for the $k^{th}$ null step. The right plot shows a histogram of the $p$-values at the first null step, demonstrating them to be reasonably uniform.

The purpose of this section is to investigate whether the statistics retain the desired null behavior as edges are selected outside of the signal. Figures 5 and 6 demonstrate the behavior of our test statistic under scenarios 1 and 2, respectively. These simulations are both run for the case $n = 500$ and $p = 100$. The panels of each plot describe the same quantities as in Figure 3.
Note that the statistics exhibit the expected behavior: The statistic for the $k^{th}$ null step appears to closely follow an exponential distribution with mean $1/k$, as suggested by the theory in Section 3.3.

5. Connection to clustering. The theoretical results from this paper apply directly to the particular sequence $\tilde{\rho}_1, \ldots, \tilde{\rho}_M$ of absolute correlations described in Section 2.1. This allows the results from this paper to be applied to other statistical methods which rely on the same sequence. Hierarchical single-linkage clustering based on correlations is one of these methods.

Consider single linkage clustering of variables, based on their absolute pairwise correlations. Starting with all the variables disconnected, variable groups are merged at the level of the tree equal to the largest correlation between the two groups. The subtrees that arise in such a fashion correspond exactly to the connected components in the graphical lasso path. The sequence of levels at which the merges occur is exactly the sequence $\tilde{\rho}_1, \ldots, \tilde{\rho}_M$, illustrated in Figure 7. As a result, all the theoretical results from this paper apply to this clustering setting as well.

This connection implies that the test statistic $T_k$ corresponds to the $k^{th}$ merge in the tree generation process, and it is a function of the height of the tree at that merge and at the following merge. The null hypotheses being tested by $T_k$ is the hypothesis that there are no correlated variables in separate subtrees at this level of the tree or above. The statistic $T_k$ will
Fig 7. Illustration of single linkage clustering of correlations and the correspondence of levels in the tree to $\tilde{\rho}_1, \tilde{\rho}_2, \ldots$ in the graphical lasso setting. Because of this correspondence, the theoretical results from this paper also apply to single linkage clustering based on correlations.

have the same asymptotic exponential null distribution that we proved in section 3.

6. Discussion. The graphical lasso provides an attractive way to estimate sparse inverse covariance matrices. However, approaches to integrating hypotheses testing and inference into this setting have not yet been developed. Taking inspiration from recent results from Lockhart et al. (2013) on inference in lasso estimation, this paper constructs a series of hypotheses and test statistics along the graphical lasso path.

For these test statistics, simple asymptotic null distributions are proven under the corresponding sequence of hypotheses. It is also demonstrated empirically that these asymptotic distributions provide a good approximation to the finite sample distributions of these statistics, in several signal settings. Finally, extensions of these results are made to other correlation-based methods; in particular, single-linkage clustering based on absolute correlations.

The approach presented in this paper tests the hypotheses that all the variables that should be connected have been connected, in the sense that they are in the same connected component. Because of the subset of graphical lasso knots that are used, no statements can be made about the internal structure of the estimate on each connected component.

Similarly, we have only made statements about the distribution of this statistic under the null distribution. For practical application of this test, it is important to also understand how it behaves under the alternative.
Based on our empirical demonstrations, the statistic tends to be large in the presence of strong signal, leading to very small $p$-values when compared to the $\text{Exp}(1)$ distribution. This makes sense intuitively, as $n\tilde{\rho}_k(\tilde{\rho}_k - \tilde{\rho}_{k+1})$ should grow large as $\tilde{\rho}_k, \tilde{\rho}_{k+1}$ limit to finite, nonzero true correlations.

An special case of this occurs if two strong correlations have the same true nonzero value. Then the difference $\tilde{\rho}_k - \tilde{\rho}_k$ becomes smaller as $n$ increases, slowing the growth of the statistic. The result is that the statistic grows at rate $\sqrt{n}$, rather than the usual rate $n$. In practice, this can be seen with finite sample sizes and close nonzero correlations. This is illustrated in Figure 8, where the simulation from Figure 5 is repeated with an alternative of three pairs with exactly the same strong correlation. There the third step and all the null steps behave as before. However, the statistics for the first two non-null steps are quite large. This same behavior appears in the equivalent regression test with the lasso, and research is underway in determining good ways to handle this.

Finally, our eventual hope is to use results like those presented here and in Lockhart et al. (2013) to construct stopping rules for regularized regression methods like the lasso and the graphical lasso. The very strong similarities between the forms of the test statistics and null distributions between these two settings suggest that an overarching approach can be constructed that will apply broadly across these methods.

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clustering, and Jacob Bien for helpful feedback.

APPENDIX A: SUPPORTING LEMMAS

This appendix contains the supporting lemmas for Theorems 1, 2 and 3 from the paper, along with their proofs.

Throughout this section, we use the following notation:

- \( f_n(x) = c_n \left(1 - \frac{x^2}{n}\right)^{(n-4)/2} \), supported on \([0, \sqrt{n}]\), where \( c_n = \frac{2}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} = \sqrt{\frac{2}{\pi}} \left(1 + O \left(\frac{1}{n}\right)\right) \). This is the marginal density of \( \sqrt{n}|S_{ij}| \).
- \( F_n(x) = \int_0^x f_n(u) du \) and \( \bar{F}_n(x) = \int_x^{\sqrt{n}} f_n(u) du \).
- \( G_{n,p}(x) = P(\sqrt{n}M_{n,p} < x) \), where \( M_{n,p} \) is the maximum of a \( p \times p \) correlation matrix of spherically distributed variables (as in Theorem 1). \( g_{n,p}(x) \) is the corresponding density.
- \( f(x) = O(g(x)) \) if and only if \( \exists M \in \mathbb{R}^+, x_0 \in \mathbb{R} \) such that \( |f(x)| \leq M|g(x)| \) for all \( x > x_0 \).

A.1. Bounds on marginal distributions. This section presents results bounding the tails of the marginal distributions for \( \sqrt{n}|S_{ij}| \) and \( \sqrt{n}M_{n,p} \). Lemma 1 gives a Mills ratio bound for the tail of \( \sqrt{n}|S_{ij}| \), and Lemma 2 gives an important consequence of it. Lemmas 3 and 4 give bounds on the distributions of large elements of the correlation matrix. Lemma 5 combines these elements to provide an important result for proving the theorems in the paper.

**Lemma 1.** Let \( f_n \) be the density \( f_n(x) = \frac{2}{\sqrt{\pi n}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \left(1 - \frac{x^2}{n}\right)^{(n-4)/2} \) supported on \([0, \sqrt{n}]\) and let \( \bar{F}_n(x) = \int_x^{\sqrt{n}} f_n(w) dw \). Then for \( n \geq 3 \),

\[
\begin{align*}
(1) \quad & \frac{\bar{F}_n(x)}{f_n(x)} \leq \frac{n}{n-2} \cdot \frac{1}{x} \left(1 - \frac{x^2}{n}\right) \quad \text{for} \quad x \in (0, \sqrt{n}) \\
(2) \quad & \frac{\bar{F}_n(x)}{f_n(x)} \geq \frac{n+1}{n-2} \cdot \frac{x}{x^2 + 1} \left(1 - \frac{x^2}{n}\right) \quad \text{for} \quad x \in (a_n, \sqrt{n})
\end{align*}
\]

where \( a_n \) are constants satisfying \( a_n \leq \sqrt{3/5} \).

**Proof.** This proof follows the approach of Baricz (2008). Define \( r_n(x) = \bar{F}_n(x)/f_n(x) \).

To prove (1), define \( h_1(x) = \frac{1}{x} \left(1 - \frac{x^2}{n}\right) \). Note that \( h_1(x)f_n(x) \propto \frac{1}{x} \left(1 - \frac{x^2}{n}\right)^{(n-2)/2} \) is strictly decreasing and nonzero on \((0, \sqrt{n})\). Also note that \( \lim_{x \to \sqrt{n}} h_1(x)f_n(x) = 0 \) and \( \lim_{x \to 0} h_1(x)f_n(x) = 0 \).
Thus, by L’Hôpital’s rule,
\[
\lim_{x \to \sqrt{n}} \frac{r_n(x)}{h_1(x)} = \lim_{x \to \sqrt{n}} \frac{\tilde{F}_n(x)}{h_1(x)f_n(x)} = \lim_{x \to \sqrt{n}} \frac{\tilde{F}_n'(x)}{h_1(x)f_n(x)}.
\]
Defining \( g_1(x) \equiv \frac{\tilde{F}_n'(x)}{h_1(x)f_n(x)} \), we have
\[
g_1(x) = \frac{-f_n(x)}{f_n'(x)h_1(x) + f_n(x)h_1'(x)} = \frac{1}{n - 2 + \frac{1}{x}},
\]
and \( \lim_{x \to \sqrt{n}} g_1(x) = \frac{n}{n - 2} \).
Therefore, \( \lim_{x \to \sqrt{n}} r_n(x)/h_1(x) = \frac{n}{n - 2} \). Furthermore, by Lemma 2.1 of Baricz (2008), since \( g_1(x) \) is strictly increasing on \((0, \sqrt{n})\), \( r_n(x)/h_1(x) \) is also strictly increasing on that interval. Combining these facts gives \( r_n(x)/h_1(x) \leq \frac{n}{n - 2} \) or equivalently
\[
r_n(x) = \frac{\tilde{F}_n(x)}{f_n(x)} \leq \frac{n}{n - 2} \cdot \frac{1}{x} \left( 1 - \frac{x^2}{n} \right)
\]
for \( x \in (0, \sqrt{n}) \).

To prove (2), we similarly define \( h_2(x) = \frac{x}{1 + x^2} \left( 1 - \frac{x^2}{n} \right) \). Now \( h_2(x)f_n(x) \) nonzero with constant derivative on an interval \((a_n, \sqrt{n})\) for \( n \geq 3 \), with \( a_3 = \sqrt{3/5} \) and \( a_n = \sqrt{\frac{8n^2 - 16n + 1 - 2n + 1}{2(n - 3)}} \) for \( n \geq 4 \). In particular, \( a_n \leq \sqrt{3/5} \) for all \( n \geq 3 \), \( a_n \) is decreasing in \( n \), and \( \lim_{n \to \infty} a_n = \sqrt{2} - 1 \). Furthermore, \( \lim_{x \to \sqrt{n}} h_2(x)f_n(x) = 0 \).

We now have
\[
g_2(x) = \frac{\tilde{F}_n'(x)}{[f_n(x)h_2(x)]'} = \frac{(1 + x^2)^2}{n - 3} \cdot \frac{(1 + x^2)^2}{x^4 + 2n - 1} \cdot \frac{1}{x^2 - 1},
\]
where \( g_2(x) \) is strictly decreasing on \((a_n, \sqrt{n})\) and \( \lim_{x \to \sqrt{n}} g_2(x) = \frac{n + 1}{n - 2} \).

Then, by L’Hôpital’s rule, \( \lim_{x \to \sqrt{n}} r_n(x)/h_2(x) = \lim_{x \to \sqrt{n}} g_2(x) = \frac{n + 1}{n - 2} \), and furthermore by Lemma 2.1 of Baricz (2008), \( r_n(x)/h_2(x) \) is strictly decreasing on \((a_n, \sqrt{n})\). Therefore \( r_n(x)/h_2(x) \geq \frac{n + 1}{n - 2} \) on that interval, so
\[
r_n(x) = \frac{\tilde{F}_n(x)}{f_n(x)} \geq \frac{n + 1}{n - 2} \cdot \frac{x}{x^2 + 1} \left( 1 - \frac{x^2}{n} \right)
\]
for \( x \in (a_n, \sqrt{n}) \). \(\square\)
Lemma 2. For $\sqrt{n}|S_{ij}| \sim f_n(x)$, $t > 0$, and $x \in (\sqrt{\log p}, \sqrt{n})$, as $n, p \to \infty$ with $\frac{\log p}{n} \to 0$ we have

$$
\frac{\mathbb{P}(n^{1/2}|S_{ij}|(n^{1/2}|S_{ij}| - x) \geq t)}{\mathbb{P}(n^{1/2}|S_{ij}| \geq x)} = e^{-t} \left(1 + O \left(\frac{1}{\sqrt{\log p}} + \frac{\log p}{n}\right)\right)
$$

and

$$
\frac{\mathbb{P}(n^{1/2}|S_{ij}| \geq x + \frac{t}{2})}{\mathbb{P}(n^{1/2}|S_{ij}| \geq x)} = e^{-t} \left(1 + O \left(\frac{1}{\sqrt{\log p}} + \frac{\log p}{n}\right)\right).
$$

Proof. First, note that $|S_{ij}|(|S_{ij}| - x) > t/n$ \iff $|S_{ij}| > x + \frac{t}{2} \left(\sqrt{1 - \frac{4t}{x^2}} - 1\right)$. Applying the Lagrange form of the Taylor series remainder, this is equivalent to $|S_{ij}| > x + \frac{t}{2} \left(\frac{2t}{x^2} - \frac{1}{(1+\xi)^{3/2}} \frac{4t^2}{x^4}\right)$ for $\xi \in [0, \frac{4t}{x^2}]$. With some rearrangement, this becomes $|S_{ij}| > \frac{t}{x} - \frac{2t^2}{x^3}$ for a particular $\gamma \in \left[\frac{1}{(1+4t/x^2)^{3/2}}, 1\right] \subset [0, 1]$.

Next, from Cai and Jiang (2011), we know that $\sqrt{n}|S_{ij}|$ has distribution

$$
f_n(x) = \frac{2}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \left(1 - \frac{x^2}{n}\right)^{(n-4)/2}
$$

with support on $(0, \sqrt{n})$, so that Lemma 1 applies. Let

$$
U(z) = \frac{n}{n - 2} \frac{1}{z} \left(1 - \frac{z^2}{n}\right), \quad L(z) = \frac{n + 1}{n - 2} \frac{z}{z^2 + 1} \left(1 - \frac{z^2}{n}\right),
$$

the quantities from Lemma 1, so that

$$
L(z)f(z) \leq \tilde{F}(z) = \mathbb{P}(\sqrt{n}S_{ij} \geq z) \leq U(z)f(z).
$$

Evaluating this expression at $z = x + \frac{t}{x} - \frac{2t^2}{x^3}$ and $z = x$ and dividing yields the bounds

$$
\frac{L(x + \frac{t}{x} - \frac{2t^2}{x^3})}{U(x)} \cdot \frac{f(x + \frac{t}{x} - \frac{2t^2}{x^3})}{f_n(x)} \leq \frac{\mathbb{P}(n^{1/2}|S_{ij}| \geq x + \frac{t}{x} - \frac{2t^2}{x^3})}{\mathbb{P}(n^{1/2}|S_{ij}| \geq x)} \leq \frac{f(x + \frac{t}{x} - \frac{2t^2}{x^3})}{f_n(x)} \cdot \frac{U(x + \frac{t}{x} - \frac{2t^2}{x^3})}{L(x)}.
$$
Since $x \geq \sqrt{\log p}$, as $(\log p)/n \to \infty$ these expressions behave like

$$f(x + \frac{t}{2} - \frac{2t^2}{x^3}) = \left(1 - \frac{1}{n} (x + \frac{t}{x} - \frac{2t^2}{x^3})^2\right)^{(n-4)/2} = e^{-t} \left(1 + O\left( \frac{1}{\log p} + \frac{\log p}{n} \right) \right)$$

$$U(x + \frac{t}{2} - \frac{2t^2}{x^3}) = \frac{n}{n-2} \frac{1}{1-x^2/n} \left(1 - \frac{(x + \frac{t}{x} - \frac{2t^2}{x^3})^2}{n+1} \frac{n+1}{n-2} (1-x^2/n) \right) = 1 + O\left( \frac{1}{\sqrt{\log p}} + \frac{\log p}{n} \right)$$

and so

$$\frac{\mathbb{P}\left( n^{1/2} |S_{ij}| \geq x \right)}{\mathbb{P}\left( n^{1/2} |S_{ij}| \geq x \right)} = e^{-t} \left(1 + O\left( \frac{1}{\sqrt{\log p}} + \frac{\log p}{n} \right) \right).$$

The same argument holds with $\gamma = 0$, yielding

$$\frac{\mathbb{P}\left( n^{1/2} |S_{ij}| \geq x + \frac{t}{2} \right)}{\mathbb{P}\left( n^{1/2} |S_{ij}| \geq x \right)} = e^{-t} \left(1 + O\left( \frac{1}{\sqrt{\log p}} + \frac{\log p}{n} \right) \right).$$

\[\square\]

**Lemma 3.** Let $S$ be a $p \times p$ correlation matrix, with underlying data $Z_1, \ldots, Z_p \in \mathbb{R}^n$, where the $Z_j$ are independent and identically spherically distributed around the origin. Then

$$\mathbb{P}(\max_{i<j} \sqrt{n} |S_{ij}| < \sqrt{\log p}) \leq \exp \left( -\frac{p^{3/5}}{4\sqrt{\log p}} \right).$$

**Proof.** Let $\tilde{S}_j = S_{2j-1,2j}, j = 1, \ldots, \lfloor \frac{p}{2} \rfloor$. Note that the $\tilde{S}_j$ are mutually independent, and that $\sqrt{n} \tilde{S}_j \sim f_n(x)$. Then

$$\mathbb{P}(\max_{i<j} \sqrt{n} |S_{ij}| < \sqrt{\log p}) \leq \left( \mathbb{P}(|S_{ij}| \leq \sqrt{\log p}) \right)^{\lfloor \frac{p}{2} \rfloor} = (1 - F_n(\sqrt{\log p}))^{\lfloor \frac{p}{2} \rfloor}$$

Using Lemma 1,

$$F_n(\sqrt{\log p}) \geq \frac{1}{\sqrt{\log p}} \left(1 - \frac{\log p}{n}\right)^{n/2-1} \left(1 + O\left( \frac{1}{\log p} \right) \right) \geq \frac{1}{\sqrt{\log p}} \left(1 + O\left( \frac{1}{\log p} \right) \right) \exp \left(-\frac{\log p}{2} - \frac{\log^2 p}{4n(1-\xi)^2} \right)$$
Here \( \xi \in [0, \frac{\log p}{n}] \) from applying the Taylor series remainder theorem to the expansion of \( \log(1 - x) \) around 0. Suppose that \( \log p/n < \frac{1}{4} \), which must happen eventually since \( \log p/n \to 0 \). Then

\[
\exp \left( -\frac{\log^2 p}{4n(1-\xi)^2} \right) \geq p^{-1/9},
\]

so

\[
\bar{F}_n(\sqrt{\log p}) \geq \frac{1}{\sqrt{\log p}} p^{-13/36} \left( 1 + O \left( \frac{1}{\log p} \right) \right) \geq \frac{1}{2\sqrt{\log p}} p^{-2/5}
\]

where the last inequality assumes \( p \geq 8 \) and uses \( 2/5 > 13/36 \) for a simpler expression. Plugging this back into the initial expression and assuming \( p \) even for notational convenience,

\[
\mathbb{P} \left( \max_{i<j} \sqrt{n}S_{ij} < \sqrt{\log p} \right) \leq \left( 1 - \frac{1}{2\sqrt{\log p}} p^{-2/5} \right)^{p/2} \leq \exp \left( -\frac{p^{3/5}}{4\sqrt{\log p}} \right)
\]

Lemma 4. Let \( S \) be a \( p \times p \) correlation matrix, with underlying data \( Z_1, \ldots, Z_p \in \mathbb{R}^n \), where the \( Z_j \) are independent and identically spherically distributed around the origin.

Let \( \bar{Z} = \sum_{i<j} \left( F_n(x) - I_{ij}^{(x)} \right) \), where \( I_{ij}^{(x)} = 1_{\{\sqrt{n}\left|S_{ij}\right| > x\}} \). The third moment of \( |\bar{Z}| \) can be computed for use in a third-moment Chebyshev bound, which results in the bound

\[
\mathbb{P} \left( \sum_{i<j} I_{ij}^{(x)} < k \right) \leq \frac{1}{(\frac{p}{2})^2 \bar{F}_n^2(x)} \cdot \frac{1 + 4(p - 3)\bar{F}_n(x)}{\left( 1 - \frac{k}{(\frac{p}{2}) \bar{F}_n(x)} \right)^3},
\]

which simplifies when \( x < \sqrt{3.5 \log p} \) to

\[
\mathbb{P} \left( \sum_{i<j} I_{ij}^{(x)} < k \right) \leq \frac{1}{(\frac{p}{2})^2 \bar{F}_n^2(x)} \cdot \left( 1 + O \left( \frac{\sqrt{\log p}}{p^{1/4}} \right) \right)
\]

Proof. The sums involved in \( \mathbb{E}\bar{Z}^3 \) are simplified by the pairwise independence of the \( I_{ij}^{(x)} \) (due to the pairwise independence of the correlations in \( S \)). The only terms that are nonzero are those where all the terms correspond to the same indices \( ij \) (of which there are \( \binom{p}{2} \)), and those where the terms correspond to a cycle involving three indices \( ij, jk, ki \) (of which there
are \( p(p-1)(p-2) \). The expectation of the first is easy to compute.

\[
\mathbb{E} \left| \tilde{F}_n(x) - I_{ij}^{(x)} \right|^3 = \tilde{F}_n(x) \left| \tilde{F}_n(x) - 1 \right|^3 + (1 - \tilde{F}_n(x)) \tilde{F}_n^3(x)
\]

\[
= \tilde{F}_n(x)(1 - \tilde{F}_n(x)) \left( \tilde{F}_n^2(x) + (1 - \tilde{F}_n(x))^2 \right)
\]

\[
\leq \tilde{F}_n(x)
\]

The second is harder to compute, but can be bounded. Note first that

\[
\mathbb{P}(I_{jk} + I_{ki} > 0 | I_{ij}) \leq 2 \mathbb{P}(I_{jk} = 1 | I_{ij}) = 2 \tilde{F}_n(x)
\]

by applying a union bound and pairwise independence. Furthermore, if at least one of \( I_{jk} \) and \( I_{ki} \) are nonzero, then

\[
\mathbb{E} \left( \left| \tilde{F}_n(x) - I_{jk}^{(x)} \right| \cdot \left| \tilde{F}_n(x) - I_{ki}^{(x)} \right| \right) \leq 1 - \tilde{F}_n(x)
\]

Knowing this, we can condition on \( I_{ij} \) and expand the original expectation to obtain a bound.

\[
\mathbb{E} \left( \left| \tilde{F}_n(x) - I_{jk}^{(x)} \right| \cdot \left| \tilde{F}_n(x) - I_{ki}^{(x)} \right| \right) \leq 2 \tilde{F}_n(x)
\]

Using all this, we can bound \( \mathbb{E} |\tilde{Z}|^3 \)

\[
\mathbb{E} |\tilde{Z}|^3 \leq \left( \frac{p}{2} \right) \tilde{F}_n(x) + 4 \left( \frac{p}{2} \right) (p-3) \tilde{F}_n^2(x).
\]

Then applying Chebyshev’s inequality to \( \mathbb{P}(\sum_{i<j} I_{ij}^{(x)} < k) = \mathbb{P} \left( \tilde{Z} > \left( \frac{p}{2} \right) \tilde{F}_n(x) - k \right) \), we obtain

\[
\mathbb{P} \left( \sum_{i<j} I_{ij}^{(x)} < k \right) \leq \frac{\left( \frac{p}{2} \right) \tilde{F}_n(x) + 4 \left( \frac{p}{2} \right) (p-3) \tilde{F}_n^2(x)}{\left( \frac{p}{2} \tilde{F}_n(x) - k \right)^3} = \frac{1}{\left( \frac{p}{2} \tilde{F}_n(x) \right)^3} \cdot \left( 1 + 4(p-3) \tilde{F}_n(x) \right)^3
\]

**Lemma 5.** Consider mutually independent random variables \( \tilde{S}_1, \ldots, \tilde{S}_k \) with distribution \( f_n \) and \( M_{n,p} \) which is distributed as \( \max_{i<j} \sqrt{n} |S_{ij}| \) for \( S_{ij} \) as in Theorem 1. Then

\[
\mathbb{P} \left( \bigcap_i \left\{ \tilde{S}_i(\tilde{S}_i - M_{n,p}) > t \right\} \right)
\]

\[
= e^{-kt} \mathbb{P} \left( \bigcap_i \left\{ \tilde{S}_i > M_{n,p} \right\} \right) \left( 1 + O \left( \frac{1}{\sqrt{\log p}} + \frac{\log p}{n} \right) \right) + O \left( e^{-\frac{p^{3/5}}{4\sqrt{\log p}}} \right)
\]
PROOF. Let \( g_{n,p}(x) \) be the density of \( M_{n,p} \). We can expand the probability above by conditioning on \( M_{n,p} \) and taking advantage of the mutual independence of the variables. We obtain
\[
P \left( \bigcap_i \{ \tilde{S}_i(\tilde{S}_i - M_{n,p}) > t \} \right) = \int_0^{\sqrt{n}} P \left( \bigcap_i \{ \tilde{S}_i(\tilde{S}_i - x) > t \} \middle| M_{n,p} = x \right) g_{n,p}(x) dx
\]
\[
= \int_0^{\sqrt{n}} P \left( \tilde{S}_i(\tilde{S}_i - x) > t \right)^k g_{n,p}(x) dx
\]

Splitting the integral at \( \sqrt{\log p} \), the first piece can be bounded by Lemma 3, and the integrand of the second can be simplified by Lemma 2, yielding
\[
P \left( \bigcap_i \{ \tilde{S}_i(\tilde{S}_i - M_{n,p}) > t \} \right) = O \left( e^{-\frac{p^{3/5}}{4\sqrt{\log p}}} \right) + \int_{\sqrt{\log p}}^{\sqrt{n}} e^{-kt} P \left( \tilde{S}_i > x \right)^k \left( 1 + O \left( \frac{1}{\sqrt{\log p}} + \frac{\log p}{n} \right) \right) g_{n,p}(x) dx
\]
\[
= e^{-kt} \int_{\sqrt{\log p}}^{\sqrt{n}} P \left( \bigcap_i \{ \tilde{S}_i > M_{n,p} \} \right) \left( 1 + O \left( \frac{1}{\sqrt{\log p}} + \frac{\log p}{n} \right) \right) + O \left( \exp^{-\frac{p^{3/5}}{4\sqrt{\log p}}} \right)
\]

A.2. Locations of large correlations. The results in this section address the locations of large elements within the correlation matrices we consider. The first shows that with high probability, the largest correlations will have no overlapping variables. The second lemma shows that, with high probability, the largest correlations will not involve the finite set of variables, \( \mathcal{A} \), which have signal.

LEMMA 6. Let \( S \) be a \( p \times p \) correlation matrix, with underlying data \( Z_1, \ldots, Z_p \in \mathbb{R}^n \), where the \( Z_j \) are independent and identically spherically distributed around the origin.

Let \( E \) be the event that the first \( k \) largest off-diagonal elements of \( S \) share no indices. Then as \( n, p \to \infty \) with \( k \) fixed and \( \frac{\log p}{n} \to 0 \), \( P(E) \to 1 \).
Proof. Note that we can bound this probability by

\[ P(E^c) \leq P\left( \text{The } k^{th} \text{ largest } |S_{ij}| < \sqrt{\frac{3.5 \log p}{n}} \right) + P\left( \exists i, j, k : |S_{ik}| > \sqrt{\frac{3.5 \log p}{n}} \text{ and } |S_{jk}| > \sqrt{\frac{3.5 \log p}{n}} \right). \]

The first of these can be bounded by Lemma 4, giving

\[ P\left( \text{The } k^{th} \text{ largest } |S_{ij}| < \sqrt{\frac{3.5 \log p}{n}} \right) \leq \frac{1}{(\frac{p}{2})^2 F_n^2(\sqrt{3.5 \log p})} \cdot \left( 1 + O\left( \frac{\sqrt{\log p}}{p^{1/4}} \right) \right). \]

By Lemma 1,

\[ \frac{\sqrt{3.5 \log p}}{n} \geq \frac{1}{\sqrt{3.5 \log p}} e^{-1.75 \log p(1 + O(\frac{1}{n} + \frac{\log p}{n}))} \left( 1 + O\left( \frac{1}{n} + \frac{1}{\log p} \right) \right) \]

\[ \geq \frac{1}{p^{3/2} \sqrt{\log p}} \]

where we consider in the last step \( n, p \) large enough that the \( O\left( \frac{1}{n} + \frac{1}{\log p} \right) < 1 \) and \( O\left( \frac{1}{n} + \frac{\log p}{n} \right) < \frac{1}{7} \). Then

\[ P\left( \text{The } k^{th} \text{ largest } |S_{ij}| < \sqrt{\frac{3.5 \log p}{n}} \right) \leq \frac{p^3 \log p}{(\frac{p}{2})^2} \cdot \left( 1 + O\left( \frac{\sqrt{\log p}}{p^{1/4}} \right) \right) = O\left( \frac{\log p}{p} \right). \]

For the other term, first note that conditional on \( S_{ij} \), the pairs \( \{S_{ik}, S_{jk}\} \) are i.i.d. over all \( k \notin \{i, j\} \). Using this, and defining \( C = \left\{ s : |s| > \sqrt{\frac{3.5 \log p}{n}} \right\} \)

\[ P\left( |S_{ij}| > \sqrt{\frac{3.5 \log p}{n}} \text{ and } \exists k : \max\{|S_{ik}|, |S_{jk}|\} > \sqrt{\frac{3.5 \log p}{n}} \right) \]

\[ = \int_C P\left( \exists k : \max\{|S_{ik}|, |S_{jk}|\} > \sqrt{\frac{3.5 \log p}{n}} \mid S_{ij} = s \right) f_n(s)ds \]

\[ = \int_C \left( 1 - \left( 1 - P\left( \max\{|S_{ik}|, |S_{jk}|\} > \sqrt{\frac{3.5 \log p}{n}} \mid S_{ij} = s \right) \right)^{p-2} \right) f_n(s)ds. \]
We can then apply a union bound to obtain
\[
P \left( \max \{ |S_{ik}|, |S_{jk}| \} > \sqrt{\frac{3.5 \log p}{n}} \right) \leq 2P \left( |S_{ik}| > \sqrt{\frac{3.5 \log p}{n}} \right). \]
Since the elements of \( S \) are pairwise independent, this simplifies to
\[
2P \left( |S_{ik}| > \sqrt{\frac{3.5 \log p}{n}} \right). \]
Notating \( \bar{F}_n(\sqrt{3.5 \log p}) = P \left( |S_{ik}| > \sqrt{\frac{3.5 \log p}{n}} \right) \) as before, the bound becomes
\[
P \left( |S_{ij}| > \sqrt{\frac{3.5 \log p}{n}} \right) \leq \int_C \left( 1 - \left( 1 - 2\bar{F}_n(\sqrt{3.5 \log p}) \right)^{p-2} \right) f_n(s) ds
\]
\[
= \left( 1 - \left( 1 - 2\bar{F}_n(\sqrt{3.5 \log p}) \right)^{p-2} \right) \bar{F}_n(\sqrt{3.5 \log p})
\]
\[
\leq 1 - \left( 1 - \frac{2\sqrt{2p^{1.75}(1-2/n)}}{\sqrt{3.5\pi \log p}} \left( 1 + O \left( \frac{1}{n} \right) \right) \right)^{p-2} \left( \frac{\sqrt{2p^{1.75}(1-2/n)}}{\sqrt{3.5\pi \log p}} \left( 1 + O \left( \frac{1}{n} \right) \right) \right)
\]
\[
= \frac{4}{3.5\pi p^{2.5(1-2/n)} \log p} \left( 1 + O \left( \frac{1}{n} + \frac{1}{p^{3/4} \log p} \right) \right)
\]
where Lemma 1 was used in the last step to bound \( \bar{F}_n(\sqrt{3.5 \log p}) \).

Using this bound, the second term from the beginning of this lemma can be bounded,
\[
P \left( \exists i, j, k : |S_{ik}| \text{ and } |S_{jk}| > \sqrt{\frac{3.5 \log p}{n}} \right)
\]
\[
\leq \sum_{i<j} P \left( |S_{ij}| > \sqrt{\frac{3.5 \log p}{n}} \right) \text{ and } \exists k : \max \{ |S_{ik}|, |S_{jk}| \} > \sqrt{\frac{3.5 \log p}{n}}
\]
\[
\leq \sum_{i<j} \frac{4}{3.5\pi p^{2.5(1-2/n)} \log p} \left( 1 + O \left( \frac{1}{n} + \frac{1}{p^{3/4} \log p} \right) \right)
\]
\[
\leq \frac{2}{3.5\pi p^{1/2-5/n} \log p} \left( 1 + O \left( \frac{1}{n} + \frac{1}{p^{3/4} \log p} \right) \right) \leq o \left( \frac{1}{p^{1/2-\varepsilon} \log p} \right)
\]
for any \( \varepsilon > 0 \).

Combining these two bounds, we have \( P(E) \geq 1 - o \left( \frac{1}{p^{1/2-\varepsilon} \log p} \right) \), so \( P(E) \to 1 \).
Lemma 7. Consider the setting of Theorem 3. For convenience, arrange the $p$ variables so that the first $a \equiv |A|$ of them are the variables in $A$. Let $D$ be the event that the (finite) $k$ largest off-diagonal elements of $S$ outside of $A \times A$ appear in the $(p-a) \times (p-a)$ block involving only variables $Z_{a+1}, \ldots, Z_p$.

Then as $n, p \to \infty$ with $\frac{\log p}{n} \to 0$, $\mathbb{P}(D) \to 1$.

Proof. This argument follows directly from the proof approach of Lemma 6. The probability that at least $k$ elements of the $(p-a) \times (p-a)$ block are above $\sqrt{3.5 \log p/n}$ is $1 - O\left(\frac{\log(p-a)}{p-a}\right)$ by the argument provided there. Furthermore, even with structure in the $a \times a$ block of the true correlation matrix, the entries $S_{ij}, i \in \{1, \ldots, a\}, j \in \{a+1, \ldots, p\}$ still have the same marginal distribution, and furthermore the rows $S_{i1}, \ldots, S_{ia}$ are i.i.d. for different values of $i \in \{a+1, \ldots, p\}$. Therefore the same union bounding approach as in Lemma 6 works, with the 2 in the union bound being replaced by $a$, yielding

$$\mathbb{P}\left(\max_{i \geq a+1, j \leq a} |S_{ij}| > \sqrt{3.5 \log p/n}\right) \leq a \frac{2p^{-0.75(1-2/n)}}{\sqrt{3.5 \pi \log p}} \left(1 + O\left(\frac{1}{n} + \frac{1}{p^{3/4} \log p}\right)\right).$$

Therefore

$$\mathbb{P}(D) = 1 - O\left(\frac{p^{-0.75(1-2/n)}}{\sqrt{\log p}} + \frac{\log(p-a)}{p-a}\right) = 1 - o\left(\frac{1}{\sqrt{p}}\right),$$

so $\mathbb{P}(D) \to 1$.

\[\Box\]

A.3. Bounds relating to the approximate partitions, $A^*_i$. The results in this section are all developed to support Lemma 12, which shows that the sums of the probabilities of the events $A^*_{ij}$ and $A^*_{i_1, i_1, \ldots, i_k, j_k}$ have the necessary limits for the theorems presented in the paper. Lemma 8 is a special case of Conjecture 1 for $k = 1$, where a simple proof holds. Lemma 10 uses this result for $k = 1$, and depends on Conjecture 1 for $k > 1$.

Lemma 8. For $G_{n,p}(x)$ and $f_n(x)$ as defined previously,

$$\int_0^{\sqrt{3.5 \log p}} G_{n,p}(x)f_n(x)dx \sim o\left(\frac{1}{p^2}\right).$$
Proof. Applying the Chebyshev bound from Lemma 4 yields
\[
\mathbb{P}(\sqrt{nM_{ij}} < x) \leq \frac{1}{(\frac{p}{2})^2 F_n^2(x)} \left( 1 + O \left( \frac{\log p}{p^{1/4}} \right) \right)
\]
for \( x \in [0, \sqrt{3.5 \log p}] \). Recall from Lemma 1 that
\[
\frac{x}{x^2 + 1} \left( 1 - \frac{x^2}{n} \right) f_n(x) \leq \bar{F}_n(x) = \mathbb{P}(\sqrt{nS_{ij}} > x)
\]
The original integrand can be bounded above by these inequalities, yielding
\[
\int_0^{\sqrt{3.5 \log p}} G_{n,p}(x)f_n(x)dx \leq \left( 1 + O \left( \frac{1}{p} \right) \right) \int_0^{\sqrt{3.5 \log p}} \frac{4/c_n}{p^2(p - 1)^2 \frac{x^2}{x^2 + 1} \left( 1 - \frac{x^2}{n} \right)^{n/2}} dx.
\]
This integrand can be shown to be bounded above by
\[
\frac{7\sqrt{2\pi} \log p}{p^{2.25 - 1.75 \log p}} \left( 1 + O \left( \frac{1}{n} + \frac{1}{\log p} \right) \right),
\]
and so the integral is bounded by
\[
\int_0^{\sqrt{3.5 \log p}} \frac{4/c_n}{p^2(p - 1)^2 \frac{x^2}{x^2 + 1} \left( 1 - \frac{x^2}{n} \right)^{n/2}} dx.
\]
This implies that the integral is \( o \left( \frac{1}{p^2} \right) \) (as long as \( \log p < 4/49 \), which must happen eventually since \( \log p \to 0 \)). Furthermore, if \( (\log p)^2 \to 0 \), then the integral is \( O \left( \frac{\log p}{p^{2.25}} \right) \).

Lemma 9. For \( f_n(x), \bar{F}_n(x) \) as previously defined,
\[
\int_{\sqrt{\frac{4}{(4 - \frac{2}{\kappa + 2}) \log p}}}^{\sqrt{\frac{7}{4}} \log p} 2p^3 \bar{F}_n^{k+1}(x)f_n(x)dx = o \left( \frac{1}{p^2k} \right)
\]
Proof. Using the Mills ratio result from 1, we have
\[
\int_{\sqrt{\frac{4}{(4 - \frac{2}{\kappa + 2}) \log p}}}^{\sqrt{\frac{7}{4}} \log p} 2p^3 \bar{F}_n^{k+1}(x)f_n(x)dx \leq \int_{\sqrt{\frac{4}{(4 - \frac{2}{\kappa + 2}) \log p}}}^{\sqrt{\frac{7}{4}} \log p} 2c_n^3 \frac{x^3}{x^{k+1}} \left( 1 - \frac{x^2}{n} \right)^{-(k+2)n - k - 3} dx.
\]
Bounding the logarithm of \((1 - \frac{x^2}{n})^{-(k+2)n - k - 3}\) by its tangent at \( x = 0 \), we obtain \( (1 - \frac{x^2}{n})^{-(k+2)n - k - 3} \leq e^{-\frac{(k+2)n}{2} x^2} e^{(k+3)x^2/n} \). This bound is largest at the left endpoint of the integral, \( x^2 = \left( 4 - \frac{2}{\kappa + 2} \right) \log p \) (as long as \( n \geq 3 \)).
Substituting this bound into the integral yields

\[
\int_{\sqrt{(1-\frac{2}{k+2})\log p}}^{\sqrt{n}} 2p^{3\frac{1}{2}}F^{k+1}_n f_n(x) dx \leq 2e^{k+2}p^{-2k} \left( 1 + O \left( \frac{\log p}{n} \right) \right) \int_{\sqrt{(1-\frac{2}{k+2})\log p}}^{\sqrt{n}} \frac{1}{x^2} dx
\]

\[
\leq \frac{2e^{k+3}}{\sqrt{3} p^{2k} (\log p)^{k/2}} \left( 1 + O \left( \frac{\log p}{n} \right) \right) = o \left( \frac{1}{p^{2k}} \right)
\]

\[\square\]

**Lemma 10.** For \( f_n(x), \tilde{F}_n(x), G_{n,p}(x) \) as defined previously and \( k \geq 1 \),

\[
\int_{0}^{\sqrt{n}} G_{n,p}(x) \tilde{F}_n^{k-1}(x) f_n(x) dx \leq \int_{0}^{\sqrt{n}} e^{(\frac{1}{2})\tilde{F}_n(x)} \tilde{F}_n(x) f_n(x) dx + o \left( \frac{1}{p^{2k}} \right),
\]

where the result depends on Conjecture 1 for \( k > 1 \).

**Proof.** For the case \( k = 1 \), Lemma 8 implies that \( \int_{0}^{\sqrt{3.5 \log p}} G_{n,p}(x) f_n(x) dx = o \left( \frac{1}{p} \right) \), and therefore

\[
\int_{0}^{\sqrt{n}} G_{n,p}(x) f_n(x) dx \leq \int_{\sqrt{3.5 \log p}}^{\sqrt{n}} G_{n,p}(z) f_n(x) dx + o \left( \frac{1}{p^2} \right).
\]

The pointwise absolute error of the Chen-Stein approximation to \( G_{n,p}(x) \) presented in Lemmas 6.3 and 6.4 of Cai and Jiang (2011) is bounded by \( 2p^{3} \tilde{F}_n^2(x) \). The total error from replacing \( G_{n,p}(x) \) in our integral by the approximation \( e^{(\frac{1}{2})\tilde{F}_n(x)} \) is then bounded by \( \int_{0}^{\sqrt{n}} 2p^{3} \tilde{F}_n^2(x) f_n(x) dx \), which is in turn bounded by the result of Lemma 9. Therefore

\[
\int_{0}^{\sqrt{n}} G_{n,p}(x) f_n(x) dx \leq \int_{\sqrt{3.5 \log p}}^{\sqrt{n}} e^{(\frac{1}{2})\tilde{F}_n(x)} f_n(x) dx + o \left( \frac{1}{p^2} \right)
\]

\[
\leq \int_{0}^{\sqrt{n}} e^{(\frac{1}{2})\tilde{F}_n(x)} f_n(x) dx + o \left( \frac{1}{p^2} \right),
\]

so the lemma holds for \( k = 1 \).

For \( k > 1 \), we rely on Conjecture 1. As in the \( k = 1 \) case, we split the integral, this time at \( \sqrt{(4-\frac{2}{k+2})\log p} \). Invoking Conjecture 1, we obtain

\[
\int_{0}^{\sqrt{n}} G_{n,p}(x) \tilde{F}_n^{k-1}(x) f_n(x) dx \leq \int_{\sqrt{(4-\frac{2}{k+2})\log p}}^{\sqrt{n}} G_{n,p}(z) \tilde{F}_n^{k-1}(x) f_n(x) dx + o \left( \frac{1}{p^{2k}} \right)
\]
Then the error from replacing $G_{n,p}(x)$ by $e^{-\binom{x}{2}}\bar{F}_n(x)$ is bounded by
\[
\int_0^{\sqrt{n}} G_{n,p}(x)\bar{F}_n^{k-1}(x)f_n(x)dx \leq \int_0^{\sqrt{n}} e^{-\binom{x}{2}}\bar{F}_n^{k-1}(x)f_n(x)dx + o\left(\frac{1}{p^{2k}}\right),
\]
so the lemma holds for $k > 1$, conditional on Conjecture 1.

**Lemma 11.** Defining $\bar{F}_n(x)$ and $f_n(x)$ as above,
\[
\int_0^{\sqrt{n}} e^{-\binom{x}{2}}\bar{F}_n(x) f_n(x)dx \leq \frac{1}{(\frac{p}{2})}.
\]
\[
\int_0^{\sqrt{n}} e^{-\binom{x}{2}}\bar{F}_n(x) \bar{F}_n^{k}(x)f_n(x)dx \leq \frac{k!}{(\frac{p}{2})^{k+1}}
\]

**Proof.** To begin, note that $\frac{d}{dx}\bar{F}_n(x) = -f(x)$. The first integral can then be evaluated by substituting $z = \bar{F}_n(x)$ to obtain
\[
\int_0^{\sqrt{n}} e^{-\binom{x}{2}}\bar{F}_n(x) f_n(x)dx = \int_0^1 e^{-\binom{z}{2}}dz = \frac{1}{(\frac{p}{2})} \left(1 - e^{-p(p-1)/2}\right) \leq \frac{1}{(\frac{p}{2})}.
\]
Similarly, the same substitution can be applied to the second integral, followed by integration by parts, to obtain
\[
\int_0^{\sqrt{n}} e^{-\binom{x}{2}}\bar{F}_n(x) \bar{F}_n^{k}(x)f_n(x)dx = \int_0^1 e^{-\binom{z}{2}}z^k dz = -\frac{1}{(\frac{p}{2})} e^{-\binom{z}{2}} + \frac{k}{(\frac{p}{2})} \int_0^1 e^{-\binom{z}{2}}z^{k-1} dz
\]
\[
\leq \frac{k}{(\frac{p}{2})} \int_0^1 e^{-\binom{z}{2}}z^{k-1} dz.
\]
Using the result for $k = 1$ in the first integral, induction then implies that
\[
\int_0^{\sqrt{n}} e^{-\binom{x}{2}}\bar{F}_n(x) \bar{F}_n^{k}(x)f_n(x)dx \leq \frac{k!}{(\frac{p}{2})^{k+1}}.
\]

**Lemma 12.** Consider mutually independent random variables $\tilde{S}_1, \ldots, \tilde{S}_k$ with distribution $f_n$ and $M$ which is distributed as $\max_{i<j} \sqrt{n}|S_{ij}|$ for $S_{ij}$ as in the set up of Theorem 1.
Then
\[ P(\tilde{S}_1, \ldots, \tilde{S}_k > M) \leq \frac{k!}{(\frac{p}{2})^k} + o\left( \frac{1}{p^2k} \right), \]
and as a consequence,
\[ \sum_{i < j} P(A_{ij}^*) \to 1 \quad \text{and} \quad \frac{1}{k!} \sum_{i_1, \ldots, i_k, j_k} P(A_{i_1, \ldots, i_k, j_k}^*) \to 1. \]

**Proof.** Note that
\[ P(\tilde{S}_1, \ldots, \tilde{S}_k > M) = \int_0^{\sqrt{n}} G_{n,p}(x)k\tilde{F}_n^{k-1}(x)f_n(x)dx, \]
since the independence of the \( \tilde{S}_j \) means that the density of their minimum is \( k\tilde{F}_n^{k-1}(x)f_n(x) \). We then apply Lemma 10 to obtain the bound
\[ P(\tilde{S}_1, \ldots, \tilde{S}_k > M) \leq \int_0^{\sqrt{n}} e^{-(\frac{p}{2})\tilde{F}_n(x)k\tilde{F}_n^{k-1}(x)f_n(x)}dx + o\left( \frac{1}{p^2k} \right) \leq \frac{k!}{(\frac{p}{2})^k} + o\left( \frac{1}{p^2k} \right) \]
where the second inequality follows from Lemma 11.
To obtain the limit \( \sum_{i < j} A_{ij}^* \to 1 \), note that \( A_{ij} \subseteq A_{ij}^* \) and that \( P(A_{ij}) = \frac{2}{p(p-1)} \). Therefore
\[ \frac{2}{p(p-1)} \leq P(A_{ij}^*) \leq \frac{2}{p(p-1)} + o\left( \frac{1}{p^2} \right) \]
and so summing yields,
\[ 1 \leq \sum_{i < j} P(A_{ij}^*) \leq 1 + o(1). \]
Similarly, \( A_{i_1, j_1, \ldots, i_k, j_k} \subseteq A_{i_1, j_1, \ldots, i_k, j_k}^* \), so
\[ \frac{1}{k!} \sum_{\mathcal{T}_k} P(A_{i_1, j_1, \ldots, i_k, j_k}) \leq \frac{1}{k!} \sum_{\mathcal{T}_k} P(A_{i_1, j_1, \ldots, i_k, j_k}^*) \leq \frac{1}{k!} \sum_{\mathcal{T}_k} \frac{k!}{(\frac{p}{2})^k} + o(1). \]
Since both \( \frac{1}{k!} \sum_{\mathcal{T}_k} P(A_{i_1, j_1, \ldots, i_k, j_k}) \to 1 \) and \( \frac{1}{k!} \sum_{\mathcal{T}_k} \frac{k!}{(\frac{p}{2})^k} + o(1) \to 1, \)
\[ \frac{1}{k!} \sum_{\mathcal{T}_k} P(A_{i_1, j_1, \ldots, i_k, j_k}^*) \to 1. \]

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Department of Statistics
Stanford University
Sequoia Hall
Stanford, California 94305-4065
E-mail: maxg@stanford.edu
jonathan.taylor@stanford.edu
tibs@stanford.edu