String Supported Wormhole Spacetimes and Causality Violations

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Abstract

We construct a static axisymmetric wormhole from the gravitational field of two Schwarzschild particles which are kept in equilibrium by strings (ropes) extending to infinity. The wormhole is obtained by matching two three-dimensional timelike surfaces surrounding each of the particles and thus spacetime becomes non-simply connected. Although the matching will not be exact in general it is possible to make the error arbitrarily small by assuming that the distance between the particles is much larger than the radius of the wormhole mouths. Whenever the masses of the two wormhole mouths are different, causality violating effects will occur.
1 Introduction

The field equations of Einstein's theory, being local, do not fix the global structure of spacetime. Kurt Gödel [1] noted first in 1949 that general relativity admits topologically non-trivial solutions exhibiting causal paradoxes, including closed timelike curves. Whatever one’s attitude to these paradoxes - whether one views them simply as embarrassments, as a spur to re-examining the foundation of physics, as a challenge to philosophers of physics, or as a clue to new physical principles which might outlaw or tame the pathologies - the issues they raise are non-ignorable and have recently received a good deal of attention [2-5].

In 1988, Morris and Thorne [2] conceived the idea that an advanced civilization might be able to construct a traversible wormhole which connects two distant regions of space. To prevent the throat from closing, exotic material (with negative energy-density) needs to be packed into the hole. Such a wormhole can be employed as a time-machine by setting the two mouths into high speed relative motion. Frolov and Novikov [3] pointed out in 1990 that even a static wormhole functions as a time-machine if the two mouths are at different gravitational potentials, e.g. if one of the mouths is held near a neutron star. These are probably the simplest models for illustrating and testing ideas about causality-violating spacetimes.

The two mouths appear as two separate masses in the asymptotically flat exterior space. Spacetime can therefore not be spherically symmetric. Moreover, the mouths will naturally gravitate towards each other, thus violating the assumed staticity. The nonstaticity can be made arbitrarily small for sufficient large initial separation of the mouths. Alternatively, it can be eliminated altogether by anchoring the mouths, e.g. by strings held at infinity. It is the aim of this paper to develop an explicit, strictly static, wormhole model of this type.

In Section II we start from the so-called Bach-Weyl solution which describes two static particles held in equilibrium by a strut. The strut may be replaced by strings that are attached to each of the particles and run off to infinity. In Sec.III we construct a wormhole from the Bach-Weyl solution by cutting out the interior of two timelike tubes each surrounding one of the particles. Matching these boundaries gives rise to a layer of (non-standard) matter. The form of the boundaries is described by two "deformation-functions" which are required to make the induced metric continuous. We show that solutions for the deformation-functions exist (Appendix A) and give their explicit form for the case that the two wormhole mouths are far apart. However, this yields only two out of the
three components of the induced metric continuous. To smooth out the remaining metric component we introduce additional matter. The result is that this does not significantly contribute to the stress-energy tensor as long as the mouths are sufficiently separated. In Sec.IV we calculate the stress-energy tensor for the surface layer. Finally, we point out that such wormholes may be used as a time-machine whenever the mass parameters of the two wormhole mouths are different.

2 The Bach-Weyl-Solution

In this section we introduce the so-called Bach-Weyl solution which describes two static Schwarzschild particles held in equilibrium by a strut between them. For this we follow the work of W. Israel and K.A. Khan [6] who calculated the gravitational field of arbitrarily many collinear particles. Any static axially symmetric spacetime can be written in a cylindrical coordinate system $\tilde{I}(T, r, z, \varphi)$ in the form

$$ds^2 = -e^{2\lambda(r,z)}dT^2 + e^{2(\nu(r,z)-\lambda(r,z))}(dr^2 + dz^2) + r^2 e^{-2\lambda(r,z)}d\varphi^2 \tag{1}$$

depending on the two functions $\nu(r,z)$ and $\lambda(r,z)$. The vacuum Einstein equations require $\lambda(r,z)$ to be an axisymmetric Newtonian potential function of $(r, z, \varphi)$, treated as cylindrical coordinates on a fictitious flat background. It is known that if the Schwarzschild line element of a particle with mass $m$ is transformed to the coordinate system $\tilde{I}$, $\lambda$ is formally the Newtonian potential of a uniform rod with length $2m$. Therefore, we may use the following picture: Consider two non-overlapping rods placed along the z-axis, their centers located at $z = a_i$ (i=1,2). We assume that the $i^{th}$ rod has mass $b_i$ and define the quantities $\rho_i^{\pm}(r,z)$ which measure the distance from its ends (see Fig.1):

$$(\rho_i^+)^2 = r^2 + (z_i^+)^2, \quad (\rho_i^-)^2 = r^2 + (z_i^-)^2$$

$$z_i^+ = z - (a_i + b_i), \quad z_i^- = z - (a_i - b_i) \tag{2}$$

Following [6] we introduce the notation

$$E(i^+, j^+) = \rho_i^+ \rho_j^+ + z_i^+ z_j^+ + r^2$$

$$E(i^-, j^+) = E(j^+, i^-) = \rho_i^- \rho_j^+ + z_i^- z_j^+ + r^2 \tag{3}$$

with an analogous expression defining $E(i^-, j^-)$. As a solution of the Einstein equations for our two body problem we choose $\lambda$ to be the Newtonian potential of these rods:

$$\lambda(r, z) = \lambda_1(r, z) + \lambda_2(r, z), \quad \lambda_i = \frac{1}{2} \ln \frac{\rho_i^+ + \rho_i^- - 2b_i}{\rho_i^+ - \rho_i^- + 2b_i} \tag{4}$$
The corresponding \( \nu(r, z) \) is given by the sum
\[
\nu(r, z) = \sum_{i=1}^{2} \sum_{j=1}^{2} \nu_{ij}(r, z), \quad \text{where} \quad \nu_{ij} = \frac{1}{4} \ln \frac{E(i-, j^+)}{E(i^+, j^+)} \frac{E(i^+, j^-)}{E(i^-, j^-)}.
\]
(5)

With this choice the constant of integration has been adjusted to make \( \nu \) vanish at spatial infinity. Thus the spacetime is asymptotically flat and satisfies the vacuum equations everywhere except on the segment of the \( z \)-axis between the two rods.

\[
\nu_0 = \nu(r = 0, z) = \frac{1}{2} \ln \left( \frac{(a_1 - a_2)^2 - (b_1 + b_2)^2}{(a_1 - a_2)^2 - (b_1 - b_2)^2} \right), \quad z \in [a_1 + b_1, a_2 - b_2]
\]
(6)

This phenomenon was interpreted by Bach and Weyl as a strut which holds the two bodies apart. In [7] one of the authors has considered how a string energy density and tension can be defined for such line singularities. For our case the non-vanishing components of the energy-momentum tensor \( L^\nu_\mu \) of the line source are given by:

\[
L^0_0 = L^2_2 = \frac{1}{4}(1 - e^{\nu_0}) \quad \geq 0 \quad \text{if} \quad b_1, b_2 \geq 0
\]
(7)

The strut therefore has a negative energy density and a pressure numerically equal to it. Another possibility for keeping the two particles apart which avoids a strut with negative energy density is offered by a different choice for the constant of integration in equation (5). If e.g. we choose \( \nu = -\nu_0 \), our solution would be regular on the segment of the \( z \)-axis between the two rods. On the other hand, there are now two singularities extending from the two rods to spatial infinity. Staticity requires that these "strings" have tension and positive energy density.

### 3 Matching of Wormhole Mouths

We cut out the interior of the surfaces \( S_1 \) and \( S_2 \) surrounding each of the two particles - the wormhole mouths - and match the surfaces to get a non-simply connected spacetime. Let us introduce two spherical polar coordinate systems \( I \) and \( II \), one centered at \( z = a_1 \) the other at \( z = a_2 \) (see Fig.1):

\[
I(t, r_1, \vartheta, \varphi) : \quad t = e^{\Lambda_1} T \\
\quad r = \sqrt{r_1(r_1 - 2b_1)} \sin \vartheta \\
\quad z = a_1 + (r_1 - b_1) \cos \vartheta
\]
(8)

\[
II(t, r_2, \vartheta, \varphi) : \quad t = e^{\Lambda_2} T \\
\quad r = \sqrt{r_2(r_2 - 2b_2)} \sin \vartheta \\
\quad z = a_2 + (r_2 - b_2) \cos (\pi - \vartheta)
\]
(9)
We do not distinguish time and angle coordinates of the different coordinate patches because we want to identify points on the wormhole mouths with equal values of \((t, \vartheta, \varphi)\). The constants \(\Lambda_i\) determine the transformation of the time coordinate of the metric (1) to the charts I and II. We will see that they cannot be equal in general which is the crucial fact leading to the occurrence of closed timelike curves (CTCs). Notice that the polar axis of the spherical coordinate system near mouth \(S_1\) points to the +\(z\) - direction but the polar axis of the spherical coordinate system near mouth \(S_2\) points to the −\(z\) - direction. With this choice we identify the inner poles \(A\) and \(B\) of the wormhole mouths (see Fig.1).

In these coordinates the Bach-Weyl-solution becomes

\[
(ds^2)_j = e^{2(\sigma_j - \lambda_j)} \left\{ \frac{dr_j^2}{1 - 2b_j r_j} + r_j^2 d\vartheta^2 \right\} + e^{-2\lambda_j r_j^2} \sin^2 \varphi d\varphi^2 \\
- e^{2(\lambda_i - \lambda_j)} \left\{ 1 - \frac{2b_i}{r_j} \right\} dt^2, \quad i, j = 1, 2 \quad i \neq j
\]

where \(\sigma_i\) is defined as the perturbation of the function \(\nu(r, z)\) induced by the other particle, i.e.

\[
\nu = \nu_{11} + \nu_{22} + 2\nu_{12} = \nu_{11} + \sigma_1 = \nu_{22} + \sigma_2
\]

Now let the parametric equations of the surfaces \(S_1\) and \(S_2\) in our charts I and II be \(x_i^\alpha = x_i^\mu(\xi^a)\) with three-dimensional intrinsic coordinates \(\xi^a = (t, \vartheta, \varphi)\) - notice that \(i, j = 1, 2\) but \(a, b = 0, 2, 3\):

\[
S_1: \quad r_1 = Re^\delta(\vartheta) \\
S_2: \quad r_2 = Re^\epsilon(\vartheta)
\]

The deformation functions \(\epsilon(\vartheta)\) and \(\delta(\vartheta)\) just depend on the angle \(\vartheta\) and will be determined in the following. First we calculate the induced metric on \(S_1\) and \(S_2\). The three holonomic basis vectors \(e_{(a)}|s_i = \frac{\partial}{\partial \xi^a}|s_i\) tangent to \(S_i\) have components \(e_\alpha^\mu|s_i = \frac{\partial x_\alpha^\mu}{\partial \xi^a}\) and their scalar products define the metric induced on \(S_i\):

\[
g_{ab}|s_i = e_{(a)}|s_i \cdot e_{(b)}|s_i \equiv g_{\alpha\beta}e_{(a)}^\alpha e_{(b)}^\beta|s_i
\]

which for \(S_1\) reads:

\[
(ds^2)_{s_1} = e^{2(\sigma_1 - \lambda_2 + \delta)} R^2 \left\{ \frac{\left(\frac{d\delta}{d\vartheta}\right)^2}{1 - 2b_1 \frac{R e^\delta}{R_{Re^\delta}}} \right\} + e^{-2\lambda_2 R^2} e^{2\delta(\vartheta) \sin^2 (\vartheta)} d\vartheta^2 - e^{2(\lambda_2 - \Lambda_1)} \left(1 - \frac{2b_1}{R e^\delta}\right) dt^2
\]

To get the induced metric on \(S_2\) one has to interchange the indexes 1 and 2 and to replace \(\delta\) by \(\epsilon\). Next, we match these surfaces by identifying points with equal values of \((t, \vartheta, \varphi)\),
i.e. \( S_1 \equiv S_2 \equiv S \). To obtain a continuous four-metric the induced metric has to be the same on \( S_1 \) and \( S_2 \). We may achieve continuity on \( S \) for the metric coefficients \( g_{00} \) and \( g_{33} \) by specifying the functions \( \delta(\vartheta) \) and \( \epsilon(\vartheta) \) and the relation between the asymptotic time scales,

\[
C := e^{\Lambda_1 - \Lambda_2}.
\]

(15)

In general, continuity for \( g_{22} \) can not be achieved exactly. However, we will see that the error in the \( g_{22} \) component can be made arbitrarily small by enlarging the distance \( D = a_2 - a_1 \) between the wormhole mouths.

Throughout this paper we assume that the mass parameters \( b_i \) are positive. Hence, to make sure that the surfaces \( S_i \) are timelike we impose the condition

\[
2b_{1,2} < R \ll D
\]

(16)

In Appendix A we show that under this condition also the radii \( Re^\delta \) and \( Re^\epsilon \) are greater than the Schwarzschild-radii \( 2b_1 \) and \( 2b_2 \) respectively.

To calculate the deformation functions \( \delta \) and \( \epsilon \) we equate the metric-coefficients \( g_{33}|_{S_1,2} \) and \( g_{00}|_{S_1,2} \) which yields a system of two equations:

\[
- \lambda_1(\epsilon, \vartheta) + \epsilon(\vartheta) = -\lambda_2(\delta, \vartheta) + \delta(\vartheta)
\]

(17)

\[
\epsilon^{2(\lambda_2 - \lambda_1)} \left( 1 - \frac{2b_1}{Re^\delta} \right) = C^2 \left( 1 - \frac{2b_2}{Re^\epsilon} \right)
\]

(18)

There are different possibilities for fixing the size of the wormhole mouths, e.g. by choosing \( \delta \) at an arbitrary point on the surface \( S_1 \) or by choosing the constant \( C \). Notice that if the masses \( b_1 \) and \( b_2 \) are equal, exact matching of the two wormhole mouths is possible. In this case it follows from equations (17) and (18) that \( \delta(\vartheta) = \epsilon(\vartheta) \) and \( C = 1 \). Such a spacetime will be symmetric with respect to the plane \( z = \frac{1}{2}(a_2 - a_1) \) and will not contain a Cauchy horizon. The identification of the mouths can be chosen in such a way that either CTCs exist throughout the whole spacetime - an eternal time machine - or that there do not exist CTCs at all. Under the condition (16) we can show that if \( b_1 \neq b_2 \) then \( C \neq 1 \) in order that a solution to the system of equations (17) and (18) exists. Hence, although our constructed spacetime is static, there does not exist a global timelike Killing vector field.

In the terminology of V.Frolov and I.Novikov [3] the gravitational field is nonpotential and according to their general proof CTCs have to occur. V.Frolov and I.Novikov studied explicitly a spherically symmetric wormhole model with two asymptotic regions. In order to be able to compare our results with theirs we choose \( C \) to have the value

\[
C = \sqrt{(1 - \frac{2b_1}{R}) \frac{1 - \frac{2b_2}{R}}{1 - \frac{2b_1}{R}}}.
\]

(19)
With this choice of the constant $C$ and under the assumptions (16) we are able to show that a solution $\delta$ and $\epsilon$ to the system (17,18) exists for every $\vartheta$. To calculate the deformations we eliminate $(\lambda_2 - \lambda_1)$ from eqn. (18) by using eqn. (17) and solve for $\delta$:

$$
e^\delta = \frac{b_1}{R} + \sqrt{\left(\frac{b_1}{R}\right)^2 + C^2 \left(e^{2\epsilon} - \frac{2b_2}{R} e^{\epsilon}\right)}$$  \hspace{1cm} (20)

Putting this result back into eqn. (17) leads to an implicit equation for $\epsilon(\vartheta)$ which cannot be solved exactly:

$$F_D(\vartheta, e^\epsilon) := e^\epsilon - e^{\lambda_1(\epsilon) - \lambda_2(\delta)} e^{\delta(\epsilon)} = 0$$ \hspace{1cm} (21)

However, if the wormhole mouths are infinitely apart i.e. $D$ to infinity eq.(21) reduces to

$$F_\infty(\vartheta, e^\epsilon) := e^\epsilon - e^{\delta(\epsilon)} = 0$$ \hspace{1cm} (22)

By virtue of equations (19) and (20) one checks that $\epsilon = 0$ is a solution. This is to be expected since deviations from spherical symmetry should tend to zero as the distance increases. In Appendix A we show that for sufficiently large $D$ a solution to eq.(21) always exists. Moreover it is shown that the solution is well approximated by Newton’s method starting with $\epsilon = 0$ as zeroth approximation. To first order in $R/D$ the solution is:

$$e^{\epsilon(\vartheta)} \sim 1 - \frac{R}{D} (1 - \frac{2b_2}{R})(1 - \frac{b_1}{R})$$ \hspace{1cm} (23)

An analogous result holds for the function $e^{\delta(\vartheta)}$ with the mass-parameters $b_1$ and $b_2$ interchanged. We see that the wormhole mouths are slightly deformed spheres of radius $R$ and therefore our spacetime may be directly compared to the wormhole spacetimes treated in [3].

Now we focus attention on the metric component $g_{22}$. We have to check the behaviour of the quantities $\frac{d\delta}{d\vartheta}$, $\frac{d\epsilon}{d\vartheta}$ and $\sigma_i$ under our assumption (16). Using the implicit function theorem and expanding in powers of $\frac{R}{D}$ (see Appendix A) we find

$$\frac{d\delta}{d\vartheta} \cdot \frac{d\epsilon}{d\vartheta} \sim O \left(\frac{R^2}{D^2}\right)$$ \hspace{1cm} (24)

The discontinuity of the induced metric component $g_{22}$ (see eqn. (14)) depends on the angle $\vartheta$ and is given by:

$$g_{22} \big|_{S_2} - g_{22} \big|_{S_1} = e^{2(\sigma_1(\vartheta) - \lambda_2(\vartheta))} R^2 e^{2\delta} \Delta g_{22}(\vartheta)$$

where  $\Delta g_{22}(\vartheta) := \left[e^{2(\sigma_2(\vartheta) - \sigma_1(\vartheta))} \left\{ \frac{\left(\frac{d\epsilon}{d\vartheta}\right)^2}{(1 - \frac{2b_2}{R} e^{\epsilon})} + 1 \right\} - \left\{ \frac{(\frac{d\delta}{d\vartheta})^2}{(1 - \frac{b_1}{R} e^{\delta})} + 1 \right\} \right]$ \hspace{1cm} (25)
Using the definition of the functions \( \sigma_i(r, z) \) given by (11), inserting the functions \( v_{ij} \) from equation (5) and expanding this in powers of \( R/D \) yields the approximation:

\[
\Delta g_{22}(\vartheta) \sim -4(b_2 - b_1)b_1b_2D^3 \sin^2 \vartheta + O\left(\frac{R^4}{D^4}\right)
\]  

(26)

Thus, for given \( b_1/R \) and \( b_2/R \) the discontinuity can be made arbitrarily small by choosing the distance \( D \) to be suitably large without changing the constant \( C \). Causality violating effects will be unaffected by the distance of the wormhole mouths.

The price for constructing a transversable wormhole is of course that non-classical matter must be present. In our case there is an infinitely thin shell concentrated at the surface \( S \). Because of the discontinuity of the metric on this surface the Einstein-tensor is not well defined there, and in principle we cannot apply the thin shell formalism to calculate the stress-energy tensor. By a small modification of the metric - which physically means that we introduce some additional matter on one side of the shell - we circumvent this problem without strongly influencing the results obtained by following the standard shell formalism. We add a perturbation term to the component \( g_{\vartheta \vartheta} \) of the Bach-Weyl metric (10) in chart \( I \) which should exactly cancel the discontinuity of the induced metric on the surface \( S \):

\[
g_{\vartheta \vartheta}(r_1, \vartheta) \rightarrow e^{2(\sigma_1(r_1, \vartheta) - \lambda_2(r_1, \vartheta))}r_1^2\left\{1 + \Delta g_{22}(\vartheta)f_\alpha(r_1 - Re^{\delta(\vartheta)})\right\}
\]

(27)

where we have introduced a profile function \( f_\alpha(x) \). This function should be at least twice differentiable and satisfy the properties \( f_\alpha(0) = 1 \) and \( f_\alpha(x) = 0 \) for \( x \geq \alpha \). For example we may take it to be of the form:

\[
f_\alpha(x) = e^{\frac{a^2}{(a^2 - x^2)} + 1} \quad \text{for } |x| < \alpha
\]

\[
= 0 \quad \text{for } |x| \geq \alpha
\]

(28)

The stresses and the energy associated with the additional matter surrounding the shell can be analysed by calculating the Einstein-tensor of the new perturbed metric. As expected, it turns out that the energy- momentum tensor is of the same order as the perturbation function, i.e \( O(R^3/D^3) \), but also depends on the first and second derivatives of the profile function which are proportional to \( 1/\alpha \) and \( 1/\alpha^2 \) respectively. Therefore the matter distribution has to be spread over a region thicker than \( R/D \) to keep its density low. Notice that in general this matter will also violate the energy conditions.
4 Stress - Energy of the Wormhole Shell

The surface stress-energy tensor $S_{ab}$ of the layer is linked to the jump $[K_{ab}] := K_{ab}^{(II)} - K_{ab}^{(I)}$ of normal extrinsic curvature across $S$ \[, \] $K_{ab}^{I,II}$ are the extrinsic curvatures corresponding to the different imbeddings of $S$ in charts I and II, each defined as

$$K_{ab} := -n \cdot \nabla e_i e_a$$  \hspace{1cm} (29)

The vector $n$ is the normal to the surface $S$ pointing from chart I to II and is normalized to one, $e_i$ are the three holonomic basis vectors of the surface $S$ defined above. For non-lightlike surfaces the following distributional equivalent of Einstein’s field equations holds:

$$-8\pi (S_{ab} - \frac{1}{2}g_{ab}S) = [K_{ab}] = K_{ab}^{(II)} - K_{ab}^{(I)}$$  \hspace{1cm} (30)

Notice that the stress-energy-tensor does not depend on the sign of the normal vector $n$ because if it would point to the opposite direction, the two terms on the right hand side of equation (30) would have to be interchanged. For the unperturbed metric (11) the components of $n$ with respect to the different charts are:

\begin{align*}
n_{\mu}^{(I)} &= (0, -1, \frac{1}{Re^\delta} \frac{d\delta}{d\vartheta}, 0) \frac{e^{(\sigma_1 - \lambda_2)}}{\sqrt{1 - \frac{2b_1}{R}} + \left(\frac{d\delta}{d\vartheta}\right)^2} \\
n_{\mu}^{(II)} &= (0, 1, -\frac{1}{Re^\epsilon} \frac{d\epsilon}{d\vartheta}, 0) \frac{e^{(\sigma_2 - \lambda_1)}}{\sqrt{1 - \frac{2b_2}{R}} + \left(\frac{d\epsilon}{d\vartheta}\right)^2}
\end{align*}  \hspace{1cm} (31, 32)

As expected the calculation yields that the energy density of the wormhole mouth measured by any observer will be negative, pressures will be positive:

\begin{align*}
S_0^0 &\simeq \frac{1}{4\pi R} \left( \sqrt{1 - \frac{2b_1}{R}} + \sqrt{1 - \frac{2b_2}{R}} \right) + O(\frac{R}{D}) \\
S_0^2 &\simeq S_3^3 \\
S_2^2 &= S_3^3 \simeq \frac{1}{8\pi R} \left( \sqrt{\frac{1 - \frac{b_1}{R}}{1 - \frac{2b_1}{R}}} + \sqrt{\frac{1 - \frac{b_2}{R}}{1 - \frac{2b_2}{R}}} \right) + O(\frac{R}{D})
\end{align*}  \hspace{1cm} (33)

If we apply the thin shell formalism to the perturbed spacetime (27) the result for the stress-energy tensor of the shell does not differ from the calculation using the original metric up to order $O(R^3/D^3)$. This results from the fact that the perturbation in the metric and its first derivative with respect to $\vartheta$ are of this order. (For the profile function (28) the first derivative with respect to the $r_1$ vanishes on $S$.)
5 Conclusion

V. Frolov and I. Novikov [3] considered the general situation of static wormhole space-times where the gravitational field is non potential. They proved that the time gap for clock synchronization in the external space with respect to synchronization through the wormhole handle grows with time. The wormhole becomes a time machine as soon as this time gap is larger than the time needed for light propagation between the two mouths in the external space. In our case the creation of a time machine can be seen explicitly by studying the coordinate transformations (8, 9). Notice that these transformations are chosen in such a way that the resulting spacetime is symmetric with respect to the spacelike hypersurface \( T = t = 0 \). We see that any other slice of constant external time \( T \) which enters mouth \( S_1 \) will reemerge at mouth \( S_2 \) at a different time \( T' = CT \). Hence, there exists a time gap \( \Delta T_{\text{gap}} \) between the wormhole mouths which is changing linearly in \( T \):

\[
\Delta T_{\text{gap}} = |T' - T| = |(C - 1)T|
\]

(34)

For the case that the wormhole mouths are kept in equilibrium by strings extending to infinity (independent of the choice of the original (10) or the perturbed metric (27)) the external time interval \( \Delta T_{NG} \) needed by a null geodesic propagating from the inner pole \( A \) of wormhole mouth \( S_1 \) to the inner pole \( B \) of mouth \( S_2 \) is given by:

\[
\begin{align*}
\Delta T_{NG} &= D - R(e^{\delta(0)} + e^{\epsilon(0)}) + b_2 + b_1 \\
+2b_1\left(\frac{D - b_1 + b_2}{D - b_1 - b_2}\right) \log \left(\frac{D - Re^{\delta(0)} + b_2 - b_1}{Re^{\delta(0)} - 2b_1}\right) \\
+2b_2\left(\frac{D - b_2 + b_1}{D - b_1 - b_2}\right) \log \left(\frac{D - Re^{\epsilon(0)} + b_1 - b_2}{Re^{\epsilon(0)} - 2b_2}\right)
\end{align*}
\]

(35)

Because this is the shortest path between the two wormhole mouths, CTCs exist if the time gap \( \Delta T_{\text{gap}} \) is larger than this time interval \( \Delta T_{NG} \). By symmetry with respect to the hypersurface \( T = t = 0 \) the constructed spacetime consists of two regions containing CTCs separated by a region without CTCs. If for example \( b_2 > b_1 \) and therefore \( C > 1 \), a causality horizon will appear as soon as the null geodesic running along the \( z - \text{axis} \) from point \( A \) to \( B \) is closed. This happens at the value of exterior time \( T_{CH} \), where \( T_{CH} \) is given by

\[
\Delta T_{NG} = \Delta T_{\text{gap}} =: (C - 1)T_{CH}.
\]

(36)
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Appendix A

We prove that for large \( D \) there exists a deformation function \( \epsilon(\vartheta) \) which satisfies equation (37):

\[
F_D(\vartheta, \epsilon^\gamma) := \epsilon^\gamma - e^{\lambda_1(\vartheta) - \lambda_2(\delta, \vartheta)} e^\delta = 0
\]

Notice that \( e^\gamma \) is a function of \( e^\epsilon \) given by eqn. (20) and that \( \lambda_1 \) is to be taken on \( S_2 \) while \( \lambda_2 \) on \( S_1 \):

\[
e^{\lambda_1(\epsilon, \vartheta) - \lambda_2(\delta, \vartheta)} = \sqrt{1 - \frac{4b_1}{\rho_1^2 + \rho_1^2 + 2b_1}} \left( 1 + \frac{4b_2}{\rho_2^2 + \rho_2^2 - 2b_2} \right)
\]

\[
\rho_1^\pm|_{S_2} = \{ Re^\epsilon (Re^\epsilon - 2b_2) \sin(\vartheta)^2 + (D - (Re^\epsilon - b_2) \cos(\vartheta) = b_1)^2 \}^{\frac{1}{2}}
\]

\[
\rho_2^\pm|_{S_1} = \{ Re^\delta (Re^\delta - 2b_1) \sin(\vartheta)^2 + (D - (Re^\delta - b_1) \cos(\vartheta) = b_2)^2 \}^{\frac{1}{2}}
\]

We show that \( e^{\epsilon(\vartheta)} \) can be approximated by using Newton's method and additionally that its derivative \( \frac{d}{d\vartheta} e^{\epsilon(\vartheta)} \) exists and can be calculated by implicit differentiation of \( F_D(\vartheta, \epsilon^\gamma) \).

Since for \( b_1 = b_2 \) the function \( F_D(\vartheta, \epsilon^\gamma) \) vanishes identically, we assume - without loss of generality - that \( b_2 > b_1 \geq 0 \). Under this condition \( F_D(\vartheta, \epsilon^\gamma) \) satisfies the following properties which tell us that for every fixed \( \vartheta \) there exists at least one zero on the interval \( e^\epsilon \in I = [\frac{2b_2}{R}, 1] : \)

\[
F_D(\vartheta, \frac{2b_2}{R}) = \frac{2b_2}{R} - e^{\lambda_1 - \lambda_2} \frac{2b_1}{R} > 0
\]

\[
F_D(\vartheta, 1) = 1 - e^{\lambda_1 - \lambda_2} < 0
\]

These two inequalities can easily be verified for sufficient large \( D \) by expanding \( e^{\lambda_1 - \lambda_2} \) in powers of \( \frac{R}{D} \),

\[
e^{\lambda_1 - \lambda_2}(\frac{R}{D}) = 1 + \frac{b_2 - b_1}{D} + \frac{1}{2} \frac{d^2}{(dR/D)^2} e^{\lambda_1 - \lambda_2}(\xi) \left( \frac{R^2}{D^2} \right), \text{ where } \xi \in [0, \frac{R}{D}]
\]

By estimating the remainder of the expansion we actually can show that properties (40) and (41) hold for at least \( D > 4R \).
For large $D$ the function $F_D(\vartheta, e^\epsilon)$ is strictly monotonically decreasing and concave with respect to $e^\epsilon$. This can be seen by considering $F_D(\vartheta, e^\epsilon)$ and its first and second derivative with respect to $e^\epsilon$ as a sequence of functions labeled by $D$. For every fixed $\vartheta$ this sequence converges uniformly on $I$ to a limit function which can easily be estimated:

$$
\lim_{D \to \infty} d_{e^\epsilon} F_D(\vartheta, e^\epsilon) = 1 - C^2 \frac{(e^\delta - b_2 R)}{(R^2 - b_1 R)} \leq \frac{b_2 - b_1}{R} \left( \frac{1}{1 - 2b_2 R} \right) \quad \text{on } I \tag{43}
$$

$$
\lim_{D \to \infty} \frac{d^2}{d(e^\epsilon)^2} F_D(\vartheta, e^\epsilon) < 0 \quad \text{on } I \tag{44}
$$

A general theorem about the convergence of Newton’s method (see for example Heuser I, p.408) says that if conditions (40,41,43 and 44) hold Newton’s sequence, defined by

$$
e^\epsilon_{n+1} := e^\epsilon_n - \frac{d}{d(e^\epsilon)} F_D(\vartheta, e^\epsilon_n), \tag{45}
$$

converges to a unique zero $\zeta$ in $I$ when started from $e^\epsilon_0 = 1$. We perform one iteration and estimate the error by

$$
|e^\epsilon_1 - \zeta| \leq \frac{|F_D(\vartheta, e^\epsilon_1)|}{\mu} = C_1 \left( \frac{R}{D} \right)^2 \quad \text{where } \mu := \min_{e^\epsilon \in I} \left| \frac{d}{d(e^\epsilon)} F_D(\vartheta, e^\epsilon) \right| \tag{46}
$$

The existence of such a constant $C_1$ for large $D$ follows from the facts that $\mu$ is bounded away from zero (see eqn.(43)) for $D$ tending to infinity and that the first two terms of the expansion of the function $F_D(\vartheta, e^\epsilon_1)$ in powers of $\frac{R}{D}$ vanish. Without exactly determining $C_1$ we may conclude that $e^\epsilon_1$ (eqn.(23)) gives the zero $\zeta$ correctly to order $\frac{R}{D}$.

It is easy to see that the partial derivative $\frac{\partial}{\partial \vartheta} F_D(\vartheta, e^\epsilon)$ exists and on the other hand we have shown that for sufficient large $D$ the partial derivative $\frac{d}{d(e^\epsilon)} F_D(\vartheta, e^\epsilon)$ exists and is nonzero (see eqn.(43)). Therefore, it follows from the implicit function theorem that $e^\epsilon(\vartheta)$ is differentiable and $\frac{d}{d\vartheta} e^\epsilon(\vartheta)$ is given by

$$
\frac{d}{d\vartheta} e^\epsilon(\vartheta) = -\frac{\partial}{\partial \vartheta} F_D(e^\epsilon) \frac{d}{d(e^\epsilon)} F_D(e^\epsilon). \tag{47}
$$

Expanding this quantity in powers of $\frac{R}{D}$ leads to the estimate given by equation (24).

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Figure

Figure 1: The relation between the cylindrical coordinate system $\tilde{I}(T, r, z, \varphi)$ and the two spherical coordinate systems $I(t, r_1, \vartheta, \varphi)$ and $II(t, r_2, \vartheta, \varphi)$ centered at the wormhole mouths $S_1$ and $S_2$ respectively is shown. The time and angle coordinates $(T, t, \varphi)$ are suppressed and we have not distinguished angle coordinates $(\vartheta, \varphi)$ of the different patches $\tilde{I}$, $I$ and $II$. The figure also makes clear the geometrical meaning of the quantities $\rho_i^\pm$ defined in Section 2.