Motion of a carrier with a mobile load along a rough inclined plane

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Abstract. The mechanical system consisting of a carrier and a load is considered. The load can move respectively the carrier according to the preset given motion law. The carrier motion from rest caused the load motion is investigated. The carrier can move translationally along rectilinear trajectory along rough inclined plane. The trajectory is the line of the greatest descent. The axis of rectilinear channel along which the load moves is situated in vertical plane containing the carrier trajectory. The Coulomb model is taken to describe the friction forces on sloped plane. Differential equations of motion of carrier with load are obtained. The sufficient condition of the carrier motion without detachment from inclined plane is given. For two special cases of the channel installation angle and the plane inclination angle combination the motion types are described. The computation experiments results are presented: the carrier motions in the special cases are illustrated, the phase portraits for some types of motions are constructed.

1. Introduction
The vibration-driven systems are a new type of movable dynamical systems whose motion is carried out by periodic motion of internal masses. These systems application in disaster rescue, pipeline inspection and cardiovascular surgery is preferable in comparison with ordinary legged and wheeled robots because of their simplicity, controllability and miniaturization potential.

Such systems have been studied from many points of view. The laws of motion of the load, providing the required carrier motion, were found in [1, 2, 3, 4, 5]. The inverse problem of finding the carrier motion for the initial conditions

\[ x(t_0) = 0 \quad \text{and} \quad \dot{x}(t_0) = 0 \]

at \( t_0 = 0 \) for a preassigned law of load motion \( x_2(t) = \ell \cdot \sin(\omega t) \), where \( \ell = \text{Const} \), \( \omega = \text{Const} \), when the carrier moves along the horizontal plane, is posed and solved in [6, 7, 8]. The problem of definition of load motion influence on carrier dynamics is considered in [9] for the other load motion law.

In this paper the motion of carrier with mobile load along inclined rough plane is investigated. The author has no information on any research on motion of vibration-driven systems on inclined rough plane.
2. The carrier motion differential equations

The carrier motion dynamics is described exactly, if the carrier motion initial conditions are defined correctly, that allow avoid the situations like the P. Painlevé paradoxes \[10\]. It means that the carrier must be in rest on inclined plane before the load starts to move.

Let’s consider the vertical plane \( Q \) containing the carrier trajectory and the load motion channel axis as the system material symmetry plane. Then all the forces acting on the system: \( \vec{P}_C \) – the carrier weight, \( \vec{P}_L \) – the load weight, \( \vec{N} \) – the sloped inclined plane normal reaction and \( \vec{F}_{fr} \) – the sliding friction force in rest are situated in the plane \( Q \). Since \( \vec{P}_C \uparrow \uparrow \vec{P}_L \) and, hence, \( (\vec{P}_C, \vec{P}_L) \sim \vec{P} \), where \( \vec{P} = \vec{P}_C + \vec{P}_L \), then the carrier will be in rest on the inclined plane if the system of forces acting on it is balanced, i.e. \( (\vec{P}, \vec{N}, \vec{F}_{fr}) \sim 0 \). As the considered system is a system of convergent forces on plane, then writing its balance conditions we obtain (see fig. 1)

\[
\sum_{s=1}^{3} F_{sx} = -(P_C + P_L) \cdot \sin \alpha + F_{fr} = 0, \tag{1}
\]

\[
\sum_{s=1}^{3} F_{sz} = -(P_C + P_L) \cdot \cos \alpha + N = 0. \tag{2}
\]

![Figure 1. The carrier position on inclined plane scheme](image)

Let’s add to (1), (2) the Coulomb-Amonton law

\[
F_{fr} \leq F_{fr_{max}}, \tag{3}
\]

where \( F_{fr_{max}} = f \cdot N \) and \( f \) is the coefficient of sliding friction in rest for the pair of materials “carrier-inclined plane”. The conditions (1), (2) are the balance equations. Solving them we can find \( F_{fr} \) and \( N \). The substitution of \( F_{fr} \) and \( N \) into (3) allows obtain the condition for the plane inclination angle

\[
\tan \alpha \leq f.
\]

Further on the condition is presented as follows

\[
|\alpha| \leq \arctg f. \tag{4}
\]
**Note.** On fig. 1 \( HH \) is the line of intersection of the horizontal plane and the vertical plane \( Q \); the plane inclination angle \( \alpha > 0 \).

According to [6] let’s introduce:

- \( Oxyz \) is the motionless coordinates system being an inertial counting out system, where \( Ox \) is the plane coinciding with the inclined plane; the axis \( Ox \) is directed along the line of the greatest descent (to ascent for \( \alpha > 0 \) or to descent for \( \alpha < 0 \) ), and axis \( Oz \) is directed normally to inclined plane;
- \( O_{1}x_{1}y_{1}z_{1} \) is the mobile coordinates system fastened together with the carrier, besides, \( O_{1}x_{1} \) coincides with the vertical plane, and \( O_{1}x_{1} \uparrow \uparrow Ox \);
- \( O_{2}x_{2}y_{2}z_{2} \) is the mobile coordinates system fastened together with the channel, besides, \( O_{2}x_{2} \) coincides with the vertical plane, and \( \varphi \) is the channel installation angle – the angle between axes \( O_{1}x_{1} \) and \( O_{2}x_{2} \) counting up from inclined plane for \( \varphi > 0 \).

If the load motion in channel law is given as \( x_{2}(t) = \ell \cdot \sin(\omega t) \), where \( \ell = \text{Const} \), \( \omega = \text{Const} \), and the carrier motion medium resistance forces are the Coulomb friction type forces, then the *carrier motion differential equations* (CMDE) for the case inclined plane can be obtained analogously [6] and written as follows:

\[
\ddot{x} = \beta \left( \cos \varphi + f \cdot \sin \varphi \right) \cdot \sin(\omega t) - g \left( \cos \alpha + \sin \alpha \right) \quad \text{for} \quad \dot{x} > 0, \quad (5)
\]

\[
\ddot{x} = \beta \left( \cos \varphi - f \cdot \sin \varphi \right) \cdot \sin(\omega t) + g \left( \cos \alpha - \sin \alpha \right) \quad \text{for} \quad \dot{x} < 0, \quad (6)
\]

\[
\ddot{x} = 0 \quad \text{for} \quad \dot{x} = 0, \quad (7)
\]

where \( x \) is the carrier coordinate in \( Oxyz \); \( \beta = \frac{m}{M + m} \cdot \ell \cdot \omega^{2} \); \( M \) is the carrier mass; \( m \) is the load mass; \( g \) is the free fall acceleration; \( f \) is the sliding friction in motion coefficient, which is equal to the sliding friction in rest coefficient for the pair “carrier-inclined plane”.

Analogously [6] the following inequality is obtained

\[
\beta \cdot |\sin \varphi| < g \cdot \cos \alpha. \quad (8)
\]

Its fulfillment guarantees the carrier motion without detachment from inclined plane.

**3. The carrier with mobile load dynamics**

Let’s the system “carrier-load” parameters values \( m, M, \ell, \omega, f, \varphi, \alpha \) be given, and the parameters are such, that the conditions (4) and (8) are fulfilled. Let’s determine the *carrier motion* (CM) from rest.

The *necessary condition* of carrier motion from rest in the axis \( Ox \) **positive direction** is

\[
\beta \left( \cos \varphi + f \cdot \sin \varphi \right) > g \left( \cos \alpha + \sin \alpha \right). \quad (9)
\]

Then from

\[
\beta \left( \cos \varphi + f \cdot \sin \varphi \right) \cdot \sin(\omega \tau_{+}) = g \left( \cos \alpha + \sin \alpha \right)
\]

we can find the carrier start moment for positive direction

\[
\tau_{+} = \frac{1}{\omega} \arcsin \left( \frac{g}{\beta} \cdot \frac{f \cdot \cos \alpha + \sin \alpha}{\cos \varphi + f \cdot \sin \varphi} \right). \quad (10)
\]

The *necessary condition* of carrier motion from rest in the axis \( Ox \) **negative direction** is

\[
\beta \left( \cos \varphi - f \cdot \sin \varphi \right) > g \left( \cos \alpha - \sin \alpha \right). \quad (11)
\]
Then from
\[ \beta (\cos \varphi - f \cdot \sin \varphi) \cdot \sin \left( \omega \left( \frac{T}{2} + \tau_- \right) \right) = - g \left( f \cdot \cos \alpha - \sin \alpha \right), \]
where \( T = \frac{2\pi}{\omega} \), we can find the carrier start moment for negative direction
\[ \tau_- = \frac{1}{\omega} \arcsin \left( \frac{g}{\beta} \cdot \frac{f \cdot \cos \alpha - \sin \alpha}{\cos \varphi - f \cdot \sin \varphi} \right). \] (12)

If the (9) and (11) take place, then the carrier motion from rest along inclined plane is possible both in the axis \( O x \) positive and negative directions. Let’s consider a special case of carrier motion when
\[ \tau_+ = \tau_- = \tau. \] (13)
Taking \( \alpha \) as preset given from ratio
\[ \frac{f \cdot \cos \alpha + \sin \alpha}{\cos \varphi_+ + f \cdot \sin \varphi_+} = \frac{f \cdot \cos \alpha - \sin \alpha}{\cos \varphi_- - f \cdot \sin \varphi_-} \]
we can find
\[ \tan \varphi_* = \frac{1}{f^2} \cdot \tan \alpha. \] (14)
Hence, from (14) we obtain that \( \alpha \) and \( \varphi_* \) must have the same sign.
Let’s \( \alpha > 0 \), then \( \varphi_+ > 0 \). Since \( \tan \alpha \leq f \), then
\[ \tan \varphi_* \leq \frac{1}{f}. \]
Hence, with respect to (4) in the considered case of carrier motion the channel installation angle along which the load moves is such, that
\[ \varphi_* < \arctg \left( \frac{1}{f} \right). \] (15)
Taking into account (14), let’s find
\[ \cos \alpha = \frac{1}{\sqrt{1 + f^4 \cdot \tan^2 \varphi_*}} \quad \text{and} \quad \sin \alpha = \frac{f^2 \cdot \tan \varphi_*}{\sqrt{1 + f^4 \cdot \tan^2 \varphi_*}}. \] (16)
Then the CMDE for the considered special case are the following
\[ \ddot{x} = (\cos \varphi_* + f \cdot \sin \varphi_*) \cdot (\beta \sin (\omega t) - \tilde{\gamma}) \quad \text{for} \quad \dot{x} > 0, \] (17)
\[ \ddot{x} = (\cos \varphi_* - f \cdot \sin \varphi_*) \cdot (\beta \sin (\omega t) + \tilde{\gamma}) \quad \text{for} \quad \dot{x} < 0, \] (18)
\[ \ddot{x} = 0 \quad \text{for} \quad \dot{x} = 0, \] (19)
where
\[ \tilde{\gamma} = \frac{\gamma}{\sqrt{\cos^2 \varphi_* + f^4 \cdot \sin^2 \varphi_*}}, \]
and
\[ \gamma = g \cdot f. \]
For the CMDE (17)-(19) there is

$$\tau_+ = \tau_- = \tau = \frac{1}{\omega} \arcsin \left( \frac{\gamma}{\beta} \sqrt{\frac{1}{\cos^2 \varphi_* + f^4 \cdot \sin^2 \varphi_*}} \right). \quad (20)$$

Let’s find $\beta_1 > \bar{\gamma}$ such, for that

$$\int_{\tau_1}^{T+\tau_1} (\beta_1 \cdot \sin (\omega t) - \bar{\gamma}) \, dt = 0, \quad (21)$$

where

$$\tau_1 = \frac{1}{\omega} \arcsin \left( \frac{\bar{\gamma}}{\beta_1} \right).$$

From (21) we can find that

$$\beta_1 = \bar{\gamma} \cdot \frac{\sqrt{\pi^2 + 4}}{2} = \frac{\gamma}{2} \frac{\sqrt{\pi^2 + 4}}{\sqrt{\cos^2 \varphi_* + f^4 \cdot \sin^2 \varphi_*}}. \quad (22)$$

Analogously we can find $\beta_2 > \bar{\gamma}$ such, for that

$$\int_{\frac{T}{2} + \tau_2}^{T + \tau_2} (\beta_2 \cdot \sin (\omega t) + \bar{\gamma}) \, dt = 0, \quad (23)$$

where

$$\tau_2 = \frac{1}{\omega} \arcsin \left( \frac{\bar{\gamma}}{\beta_2} \right).$$

From (23) we can obtain

$$\beta_2 = \bar{\gamma} \cdot \frac{\sqrt{\pi^2 + 4}}{2} = \frac{\gamma}{2} \frac{\sqrt{\pi^2 + 4}}{\sqrt{\cos^2 \varphi_* + f^4 \cdot \sin^2 \varphi_*}}. \quad (24)$$

So, in the considered special case of carrier motion

$$\beta_1 = \beta_2 = \frac{\gamma}{2} \frac{\sqrt{\pi^2 + 4}}{\sqrt{\cos^2 \varphi_* + f^4 \cdot \sin^2 \varphi_*}} = \frac{\beta_0}{\sqrt{\cos^2 \varphi_* + f^4 \cdot \sin^2 \varphi_*}}, \quad (25)$$

where $\beta_0 = \frac{\gamma}{2} \cdot \sqrt{\pi^2 + 4}.$

Therefore, if in the studying system “carrier-load” the parameter

$$\beta < \frac{\beta_0}{\sqrt{\cos^2 \varphi_* + f^4 \cdot \sin^2 \varphi_*}},$$

then the CM belongs to the R2 type [6] for which the following regular time consequence of the CMDE (17)-(19) alternations takes place:

\[
\begin{array}{ccccccccc}
& t_0 & \overset{19}{\rightarrow} & t_1 & \overset{17}{\rightarrow} & t_2 & \overset{19}{\rightarrow} & t_3 & \overset{19}{\rightarrow} & t_4 & \overset{17}{\rightarrow} & t_5 & \overset{17}{\rightarrow} & \ldots
\end{array}
\]
Here \( t_0 = 0; \ t_1 = \tau; \ \dot{x}(t_2) = 0; \ t'_i = \frac{T}{2} \cdot (i - 1) + \tau, \) where \( i = 2, 3, \ldots; \ t_k = \frac{T}{2} \cdot (k - 2) + t_2, \) where \( k = 3, 4, \ldots \)

If on a level with the system (17), (18) we consider a similar system for which \( \beta_3 > \tilde{\gamma} \) and

\[
\int_{\tau_3}^{\frac{T}{2} + \theta} (\beta_3 \cdot \sin (\omega t) - \tilde{\gamma}) \cdot dt = 0
\]

and

\[
\int_{\frac{T}{2} + \theta}^{T + \tau_3} (\beta_3 \cdot \sin (\omega t) + \tilde{\gamma}) \cdot dt = 0,
\]

where

\[
\tau_3 = \frac{1}{\omega} \arcsin \left( \frac{\tilde{\gamma}}{\beta_3} \right),
\]

then we obtain from (26)

\[
\beta_3 \cdot \cos (\omega \theta) + \sqrt{\beta_3^2 - \tilde{\gamma}^2} = \tilde{\gamma} \left[ \pi + \omega \theta - \omega \tau_3 \right],
\]

and find from (27)

\[
\beta_3 \cdot \cos (\omega \theta) + \sqrt{\beta_3^2 - \tilde{\gamma}^2} = \tilde{\gamma} \left[ \pi + \omega \tau_3 - \omega \theta \right].
\]

Comparing (28) and (29) we ascertain that \( \omega \theta = \omega \tau_3, \) and then \( 2\sqrt{\beta_3^2 - \tilde{\gamma}^2} = \tilde{\gamma} \pi, \) so

\[
\beta_3 = \tilde{\gamma} \cdot \frac{\sqrt{\pi^2 + 4}}{2},
\]

i.e. it is clarified, that

\[
\beta_3 = \beta_2 = \beta_1 = \frac{\beta_0}{\sqrt{\cos^2 \varphi_* + f^4 \cdot \sin^2 \varphi_*}}.
\]

Hence, if in the investigated system “carrier-load” for the considered case the parameter

\[
\beta = \beta_3 = \frac{\beta_0}{\sqrt{\cos^2 \varphi_* + f^4 \cdot \sin^2 \varphi_*}},
\]

then the CM belongs to the \( R5 \) type for which the following regular time consequence of the CMDE (17)-(19) alternations takes place:

\[
t_0 \rightarrow t_1 \rightarrow t_2 \rightarrow t_3 \rightarrow t_4 \rightarrow t_5 \rightarrow \ldots
\]

Here \( t_0 = 0; \ t_i = \frac{T}{2} \cdot (i - 1) + \tau, \) where \( i = 1, 2, \ldots \)

If the parameter \( \beta > \beta_3, \) then the CM realizes according to the \( NR \) type with diverse moments of switching in each half-period of the load motion.

**Note.** If \( \alpha < 0 \) and hence \( \varphi_* < 0, \) then in this case all the above established (for \( \alpha > 0 \)) statements on the carrier motion are correct.
4. The computation experiments

For two cases of CM of $R$ type ($R_2$ and $R_5$) and for one case of CM of $NR$ type the mathematical modeling results are presented on figures 2-10. The left halves of fig. 2-10 correspond to the case $\alpha > 0$ and the rights ones correspond to the case $\alpha < 0$. In all the computation experiments the following parameter values are used: $f = 0.204$, $\omega = 6 \, [1/s]$ and $\alpha \cong \pm 0.0285$. Then $\gamma = gf \cong 2 \, [m/s^2]$ and $\varphi_* = \pm 0.6$. Let’s find

$$
\tilde{\gamma} = \frac{\gamma}{\sqrt{\cos^2 \varphi_* + f^4 \cdot \sin^2 \varphi_*}} \cong 2.42 \, [m/s^2] \quad \text{and} \quad \tilde{\beta} = \frac{\gamma}{2 \sqrt{\cos^2 \varphi_* + f^4 \cdot \sin^2 \varphi_*}} \cong 4.51 \, [m/s^2].
$$

So,

- if $\beta = 4.0$, then $\beta < \tilde{\beta}$ and the CM realizes according the $R_2$ type (see fig. 2-4);
- if $\beta = 4.51$, then $\beta = \tilde{\beta}$ and the CM realizes according the $R_5$ type (see fig. 5-7);
- if $\beta = 5.0$, then $\beta > \tilde{\beta}$ and the CM realizes according the $NR$ type (see fig. 8-10).

![Figure 2](image1.png) **Figure 2.** The $x$ dependence on $t$ for CM $R_2$ type

![Figure 3](image2.png) **Figure 3.** The $\dot{x}$ dependence on $t$ for CM $R_2$ type
Figure 4. The CM $R2$ type phase portrait

Figure 5. The $x$ dependence on $t$ for CM $R5$ type

Figure 6. The $\dot{x}$ dependence on $t$ for CM $R5$ type
\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{figure7.png}
\caption{The CM $R5$ type phase portrait}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{figure8.png}
\caption{The $x$ dependence on $t$ for CM $NR$ type}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{figure9.png}
\caption{The $\dot{x}$ dependence on $t$ for CM $NR$ type}
\end{figure}
Figure 10. The CM NR type phase portrait

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