Actions, angles and frequencies for numerically integrated orbits

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ABSTRACT

We present a method for extracting actions, angles and frequencies from an orbit’s time series. The method recovers the generating function that maps an analytic phase-space torus to the torus to which the orbit is confined by simultaneously solving the constraints provided by each time step. We test the method by recovering the actions and frequencies of a triaxial Stäckel potential, and use it to investigate the structure of orbits in a triaxial potential that has been fitted to our Galaxy’s Sagittarius stream. The method promises to be useful for analysing N-body simulations. It also takes a step towards constructing distribution functions for the triaxial components of our Galaxy, such as the bar and dark halo.

Key words: methods: numerical – Galaxy: kinematics and dynamics – galaxies: kinematics and dynamics

1 INTRODUCTION

Although no galaxy is ever in perfect dynamical equilibrium, equilibrium dynamical models are central to the interpretation of observations of both our Galaxy and external galaxies. A major reason for the importance of equilibrium models is that we can infer a galaxy’s gravitational potential, and thus its dark-matter distribution, only to the extent that it is in equilibrium. Moreover, equilibrium models are the simplest models and more complex configurations, involving spiral structure or an on-going minor merger for example, are best modelled as perturbations of an equilibrium model.

Globular clusters are the stellar systems that are most completely understood, and the theory of these systems illustrates the importance of equilibrium models: at each instant the cluster is assumed to be in dynamical equilibrium, so, by Jeans’ theorem, its distribution function (DF) is a function of the relevant isolating integrals, such as stellar energy $E$, total angular momentum $L$, or angular momentum about a symmetry axis, $L_z$. Over many dynamical times encounters between stars and stellar evolution cause the DF to change, but in such a way that the DF continues to satisfy Jeans’ theorem, so the cluster evolves through a series of dynamical equilibria.

N-body simulations of cosmological clustering likewise yield a picture in which dark-matter halos are far from dynamical equilibrium only during short-lived and quite rare major mergers. In general a dark-matter halo can be well approximated by a dynamical equilibrium that is mildly perturbed by accretion.

The natural way to model a dynamical equilibrium is via Jeans theorem, which assures us that the system’s DF can be assumed to be a non-negative function of isolating integrals. Since one expects a smooth time-independent gravitational potential to admit up to three functionally independent isolating integrals, Jeans theorem states that we should be able to represent an equilibrium stellar system by the density of stars in a three-dimensional space of integrals rather than in full six-dimensional phase space. This reduction in dimensionality makes the system very much easier to comprehend and model.

Since any function of integrals is itself an integral, infinitely many different integrals may be used as arguments of the DF. However, the action integrals $J_i$ stand out as uniquely suited to be used as arguments of the DF. What makes actions special is that they can be complemented by canonically conjugate variables, the angles $\theta_i$, to form a complete set of canonical phase-space coordinates. Obviously the equations of motion of the actions are trivial: $\dot{J}_i = 0$. More remarkably the equations of motion of the angles are almost as trivial: $\dot{\theta}_i = \Omega_i(J) = \text{constant}$. Thus the angle variables increase linearly in time and if we use angle-action coordinates, the unperturbed motion of stars becomes trivial. This fact makes angle-action coordinates uniquely suited to work involving perturbation theory, and indeed the angle-action coordinates of the Kepler problem were invented to explore the role played by planet-planet interactions in the dynamics of the Solar System.

McMillan & Binney (2008) have shown that angle-action coordinates make it possible to identify stars near the Sun that have been stripped from an object that was tidally disrupted gigayears ago, and even to determine the date of the disruption to good precision. Sellwood (2010) and McMillan (2013) have used angle-action coordinates to identify stars near the Sun that are resonantly trapped by spiral structure. Sanders & Binney (2013) have shown how angle-action coordinates for the stars of a stream enable one to constrain the gravitational potential in which the stream moves.

Cosmological simulations have shown that triaxial dark matter halos are to be expected, at least up to the point at which baryons become gravitationally dominant (Valluri et al. 2010). Moreover, Law & Majewski (2010) present evidence that the tidal tails of the...
Sagittarius dwarf galaxy can only be fitted using a triaxial dark matter halo. Hence we need to be able to determine angle-action coordinates for stars in triaxial potentials. In this paper we show how to evaluate the angles and actions of particles in a given triaxial potential. If the potential is axisymmetric, the actions can be evaluated using the algorithm given by Binney (2012a).

In Section 2 we derive the equations that yield values of angles, frequencies and actions. In Section 3 we test our solutions of these equations by comparing the resulting angles, frequencies and actions for two orbits in a Stäckel potential with analytic values. In Section 4 we use the equations to explore a constant-energy surface of the action space of the triaxial potential for our Galaxy that Law & Majewski (2010) fitted to the tidal stream of the Sagittarius dwarf. In Section 5 we relate our work to previous work in the field and discuss possible extensions. Section 6 sums up and looks to the future.

2 FORMALISM

Angles and actions can be assigned to orbits that are “regular” or quasiperiodic because such an orbit is confined to a torus labelled by the actions (Arnold 1978). We will work in three dimensions so will have three actions denoted as $J = (J_1, J_2, J_3)$. Each action quantifies the magnitude of the oscillation in a suitable coordinate. For a box orbit $J_3$ quantifies the oscillation in the $z$ direction, $J_2$ in the $y$ direction and $J_1$ in the $x$ direction. For a (short-axis) loop orbit, $J_1$ quantifies oscillation in a generalised radial coordinate, while $J_2$ quantifies the particle’s circulation around some axis and $J_3$ quantifies oscillation parallel to that axis.

The transformation from ordinary phase-space coordinates $(x, v)$ to angle-action coordinates $(\theta, J)$ is possible analytically in only a few cases. McGill & Binney (1990) used one of these cases as a starting point for the numerical construction of more general transformations by “torus mapping”. The key point about torus mapping is that it yields orbits with specified actions rather than orbits with specified initial conditions $(x, v)$. When analysing an N-body model, we require actions given an initial condition and not vice-versa. Here we adapt the approach of McGill & Binney (1990) into a procedure which finds the actions, angles, and frequencies given a series of phase-space coordinates $(x_i, v_i)$ sampled along an orbit at times $t_i$.

With this time series we seek a generating function that will map a “toy torus” of a simple “toy potential” into the “target torus” to which the orbit is confined. The toy potential must have analytically tractable angles and actions and permit orbits that have the correct geometry.

In the absence of figure rotation, a general triaxial potential admits two basic classes of non-resonant orbit: loop orbits and box orbits (Schwarzschild 1979; de Zeeuw 1985). Loop orbits have a definite sense of rotation either around the long- or short-axis of the potential, whilst a box orbit has no sense of rotation and can reach down to the centre of the potential. Hence the class of an orbit can be determined by inspection of components of the angular momentum along the orbit: if all components of the angular momentum change sign, the orbit has no sense of circulation and is a box orbit; when a component of the angular momentum retains its sign, the orbit is a loop orbit around the corresponding axis (Carpintero & Aguilar 1998). For each class of orbit we use a toy potential that provides tori with the same geometrical structure as the tori of the given orbit class.

2.1 Toy potentials

2.1.1 Triaxial harmonic oscillator

For box orbits we use the potential of the triaxial harmonic oscillator,

$$\Phi_{\text{ho}}(x) = \frac{1}{2} \sum_{i=1}^{3} \omega_i x_i^2,$$

which has three parameters, $\omega_i$. Here we have chosen the principal axes of the potential to lie along the Cartesian $x, y, z$ directions on the assumption that the time series has already been rotated into the coordinate system that is aligned with the principal axes of the true potential. The actions and angles in this potential are given by

$$J_i = \frac{p_i^2 + \omega_i x_i^2}{2\omega_i},$$
$$\theta_i = \arctan \left( \frac{p_i}{\omega_i x_i} \right).$$

2.1.2 Isochrone sphere

For loop orbits we use the isochrone potential,

$$\Phi_{\text{iso}}(x) = \frac{-GM}{b + \sqrt{b^2 + r^2}},$$

where $r$ is the spherical radius. This potential has two free parameters: the mass $M$ and the scale radius, $b$. The expressions for the actions and angles in this potential are more involved than for the harmonic oscillator so are not repeated here. Readers can consult Binney & Tremaine (2008) for the appropriate equations.

2.1.3 Offsets

We include an arbitrary potential offset term, $\Phi_0$ so the potential we use is

$$\Phi(x) = \Phi_{\text{iso}}(x) + \Phi_0 \quad \text{or} \quad \Phi(x) = \Phi_{\text{iso}}(x) + \Phi_0.$$

The offset $\Phi_0$ obviously does not affect the equations of motion or the determination of the actions and angles but it is essential for finding the best-fitting parameters of the toy potential. For instance, a logarithmic potential is always positive while $\Phi_{\text{iso}}$ is always negative, so the latter cannot be fitted to the former without an offset.

One might also offset the centre of the potential from the coordinate centre, but we shall not do so here, presuming instead that the time samples $x_i$ have already been adjusted to be relative to one’s best estimate of the centre of the true potential.

2.1.4 Parameter choice

Once a class of potential has been chosen, we set the parameters of the potential by minimizing (McGill & Binney 1990)

$$\chi^2 = \sum_i (H_i - H')^2,$$

where the sum is over the times, $H_i$ is the value of the toy Hamiltonian at $(x_i, v_i)$, and $H'$ is the value of the target Hamiltonian, which is the same for all times. The minimization of $\chi^2$ is done using the Levenberg–Marquardt algorithm (Press et al. 2002).

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2.2 Generating Function

With a toy potential chosen, we construct the generating function to transform between the angle-actions \((\theta, J)\) of the toy potential, and those \((\theta', J')\) of the target potential. The generating function for this transformation, \(S(\theta, J')\), can be written

\[
S(\theta, J') = \theta \cdot J' - i \sum_{n \neq 0} S_n(J') e^{i n \cdot \theta},
\]

(6)

where the vector \(n\) has integer components. The first term on the right generates the identity transformation, whilst the structure of the second part is required by the periodicity of the angle variables.

McGill & Binney (1990) show that if the Hamiltonian is time-reversible, the reality of the generating function requires the \(S_n\) to satisfy

\[
S_n = -S_{-n}.
\]

(7)

For this condition to be satisfied there must exist a point on the toy torus at which \(J = 0\) – in Appendix A we demonstrate that this is true for the toy potentials of the previous section. With this constraint, the generating function can be written

\[
S(J', \theta) = \theta \cdot J' + 2 \sum_{n \in \mathbb{N}} S_n(J') \cos n \cdot \theta,
\]

(8)

where the integer vectors \(n\) are now restricted to just half of a three-dimensional lattice. We take this half to be the set \(\mathbb{N} = \{ (i, j, k) \}\), where either \((k > 0), (k = 0, j > 0)\) or \((k = 0, j = 0, i > 0)\).

From the generating function (8) we find the toy actions are

\[
J = \frac{\partial S}{\partial \theta} = J' + 2 \sum_{n \in \mathbb{N}} n S_n(J') \cos n \cdot \theta,
\]

(9)

and the target angles are

\[
\theta' = \frac{\partial S}{\partial J'} = \theta + 2 \sum_{n \in \mathbb{N}} \frac{\partial S_n}{\partial J'} (J') \sin n \cdot \theta.
\]

(10)

Note that by the choice of our generating function, the target angle zero-point coincides with the toy angle zero-point.

Given the choice of a toy Hamiltonian, we may find the toy actions and angles \((J(t_i), \theta(t_i))\) at each time. Then each time produces a separate equation (9) with common unknowns: the target actions and the Fourier components of the generating function, \(S_n\).

We cannot solve these equations exactly because we are dealing with equations in an infinite number of unknowns. Because we can include only a finite number of terms on the right side of each equation, the right sides should not agree exactly with the left sides, and the correct procedure is to minimise the sum of the squares of the residuals of individual equations. This sum is

\[
E = \sum_i \sum_k \left( J_k(t_i) - J_k' - 2 \sum_{n \in \mathbb{N}} n S_n(J') \cos n \cdot \theta(t_i) \right)^2,
\]

(11)

where the inner sum is over the dimension of the action space and the set \(\mathbb{N}\) is limited to a finite number of vectors \(n\). We take this set to be the \(N\) vectors that satisfy the condition \(|n| \leq N_{\text{max}}\), where \(N_{\text{max}} \approx 8\).

We minimise \(E\) by setting to zero its derivatives with respect to the unknowns:

\[
0 = \frac{\partial E}{\partial J_k'}
\]

\[
= \sum_i \left( J_k(t_i) - J_k' - 2 \sum_{n \in \mathbb{N}} n S_n(J') \cos n \cdot \theta(t_i) \right)
\]

(12)

\[
0 = \frac{\partial E}{\partial S_n}
\]

\[
= - \sum_i \sum_k 2 m_k \cos m \cdot \theta(t_i)
\]

\[
\times \left( J_k(t_i) - J_k' - 2 \sum_{n \in \mathbb{N}} n S_n(J') \cos n \cdot \theta(t_i) \right).
\]

To solve these equations we define the \(N\)-by-3 matrix

\[
c_{nk}(t_i) \equiv 2 n_k \cos(n \cdot \theta(t_i)), \quad \text{(no sum over } n)\]

(13)

We further define two \((3 + N)\)-vectors

\[
x_J \equiv (J', S_n), \quad b_J \equiv \sum_i (J(t_i), c_{nk}(t_i) \cdot J(t_i)),
\]

(14)

and the symmetric matrix

\[
A_J \equiv \sum_i \begin{pmatrix} l_3 & c(t_i) & c(t_i) \cdot c(t_i) \\ c(t_i) & c(t_i) & c(t_i) \\ c(t_i) \cdot c(t_i) & c(t_i) \cdot c(t_i) & c(t_i) \cdot c(t_i) \end{pmatrix}.
\]

(15)

Here \(l_3\) is the 3-by-3 identity matrix. With these definitions, the equations (12) to be solved can be written

\[
A_J \cdot x_J = b_J.
\]

(16)

We solve these equations for \(x_J\) by LU decomposition (Press et al. 2002).

A similar procedure yields the target angles from equation (10). We note that at time \(t_i\), the orbit has \(\theta'(t_i) = \theta'(0) + \Omega t_i\), where \(\Omega\) is the target frequency, and \(\theta'(0)\) is the angle corresponding to the initial point in the orbit integration. The relevant sum of squared residuals is

\[
F = \sum_i \sum_k \left( \theta_k'(0) + \Omega t_i - \theta_k(t_i) - 2 \sum_{n \in \mathbb{N}} \frac{\partial S_n}{\partial J_k'} (J') \sin n \cdot \theta \right)^2.
\]

(17)

The unknowns now are \(\theta'(0), \Omega\) and the set of \(\partial S_n/\partial J'\). The requirement of vanishing partial derivatives of \(F\) with respect to the unknowns yields the matrix equation

\[
A_\theta \cdot x_\theta = b_\theta.
\]

(18)

These symbols are defined in Appendix B. The toy angles will be \(2\pi\)-periodic, and we require the same for the target angles \(\theta'(0) + \Omega t_i\). However, in order to solve the matrix equation we must first make the \(\theta(t_i)\) from the orbit integration continuously increase, and then we solve for the target angles and take the \(2\pi\)-modulus.

2.3 Choice of \(N_T\) and \(N_{\text{max}}\)

Given the scheme presented above, the only question that remains is how to select the \(N_T\) times and what value to use for \(N_{\text{max}}\), which determines the number \(N\) of Fourier components we solve for. Let’s first consider an idealised 1D case. If we were able to sample uniformly in the toy angle of a 1D system, we would select \(N_T\) points in a single period, \(T\), separated in toy angle by \(\Delta = 2\pi/N_T\). With this sampling rate we would be able to constrain all modes \(e^{in\theta}\) with \(n\Delta \leq \pi\). We can choose to constrain

\[\ldots\]
only the $N_{\text{max}}$ modes with $n < \pi/\Delta$ as then we would be super-sampling the highest considered modes. Here we are using a time series that is not uniformly spaced in toy angles – its distribution depends on the target Hamiltonian, the toy potential and the distribution of sampling times. The recovery of Fourier components from non-uniform samples is discussed in Marvasti (2001). If on average we sample at the Nyquist frequency, $N_{\text{max}}/\pi$, then we are able to recover all Fourier components up to $N_{\text{max}}$. Additionally it is shown that the Kadec condition,

$$\min_i |\theta_i - \frac{k\pi}{N_{\text{max}}} | < \frac{\pi}{4N_{\text{max}}} \quad \text{for all} \quad k < N_{\text{max}},$$  

(19)

is a sufficient condition for reconstruction.

In higher dimensions the situation is more complex. There exists an equivalent condition to equation (19) for 2D signals (Marvasti 2001), but we were unable to find any condition for 3D signals. It is clear in any dimension that we need a good coverage of the toy angle space in order to constrain all required modes. If we could choose where to place the samples in toy angles, we would select a uniform grid with spacing $\pi/N_{\text{max}}$ as this would satisfy the Nyquist criterion in each dimension for all required modes. For a non-uniform sampling we would like to satisfy the Nyquist criterion on average along each vector $\mathbf{n}$. Also, we would like to produce a series of points such that we have at least one sample within each Nyquist volume. We need to have points, $\theta_i$, such that

$$\min_i |\theta_i - \frac{k\pi}{2N_{\text{max}}} | < \frac{\pi}{2N_{\text{max}}} \quad \text{for all} \quad k$$  

(20)

where $k_i$ takes odd integer values in the interval $(1, 2N_{\text{max}} + 1)$ so we have at least one point in each Nyquist volume. This condition is used by Warnock (1991). We will see below that adequate recovery of the actions is achieved when this condition is not satisfied. However, when this condition is not satisfied the $S_n$ will not be found accurately.

For near-resonant orbits we require time samples which span a very large number of periods such that the orbit covers the toy angle-space and satisfies equation (20). This is an inevitable drawback of the approach taken here. We have very little control over the sampling in the toy angle space. However our approach is not so sensitive to the density of points in the toy angle space.

3 EXAMPLE

As a test of the above, let’s look at an example. The most general separable triaxial potential is the triaxial St"uckel potential (de Zeeuw 1985). We choose to work with the perfect ellipsoid, which has density profile

$$\rho(x, y, z) = \frac{\rho_0}{(1 + m^2)^2},$$  

(21)

where

$$m^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}, \quad a \geq b \geq c \geq 0.$$  

(22)

The associated coordinates are confocal ellipsoidal coordinates in which the actions can be expressed as one-dimensional integrals. These may be calculated numerically using Gauss-Legendre quadrature. Similarly the frequencies can also be determined from one-dimensional integrals. Equations for these quantities are given in de Zeeuw (1985). Here we work with the potential with parameters $\rho_0 = 7.2 \times 10^9 M_\odot \text{kpc}^{-3}$, $a = 5.5 \text{kpc}$, $b = 4.5 \text{kpc}$ and $c = 1 \text{kpc}$.

In this potential we examine two orbits – a short-axis loop orbit and a box orbit. Each orbit was integrated for approximately 10 periods of the lowest frequency. From the integration we extracted $N_T = 400$ equally-spaced time samples so we have at least 40 time-samples per period. From a loose application of Nyquist’s theorem we expect to be able to constrain modes with $|n| < 20$ from this sampling. We set $N_{\text{max}} = 8$ such that we are solving square matrices of size 131. However, our ability to extract the relevant frequency components also depends on the coverage of the toy angle space from the time-sampling. Note that the condition of equation (20) is not satisfied for this $N_{\text{max}}$. In Figures 1 and 2 we show the orbits in the $(x, y)$ and $(x, z)$ planes, the sampling of the toy angle space and the resultant actions. We also show, in faint red, the result of integrating in the best-fitting toy potential. This gives us an idea of the work that the generating function has to do to deform the toy torus into the target torus.

For the loop orbit the true actions are $J = (270.3, 1099.5, 666.2) \text{kpc km s}^{-1}$, and the recovered actions are $J' = (272.2, 1098.8, 667.0) \text{kpc km s}^{-1}$. The actions are recovered within $\sim 1$ per cent. In Figure 4 we show two cross-
sections of $n$-space showing the absolute value of the components of the generating function. We see that the two most significant modes are $n = (2, -2, 0)$, which causes a mixing between the radial motion and azimuthal motion, and $n = (0, 0, 2)$.

For the box orbit the true actions are $J = (336.38, 137.78, 237.96) \text{ kpc km s}^{-1}$, and the recovered actions are $J' = (336.32, 137.70, 238.11) \text{ kpc km s}^{-1}$. The error in all the actions is $\sim 0.05$ per cent. In Figure 5 we show two cross-sections of $n$-space showing the absolute value of the components of the generating function. The two most significant modes are $n = (2, -2, 0)$, which causes a mixing between the $x$ motion and the $y$ motion, and $n = (2, 0, -2)$, which mixes the $x$ and $z$ motions. These modes are required to distort the rectangular orbits of the triaxial harmonic oscillator into those bounded by surfaces of constant confocal ellipsoidal coordinate.

Fig. 3 shows errors in $J_3$ and $\Omega_3$ for the box orbit as a function of $N_{\text{max}}$ for various choices of the number of time samples $N_T$ sampled along the orbit. Increasing $N_{\text{max}}$ without increasing $N_T$ to ensure the Nyquist condition remains satisfied can increase the errors. This is particularly so in the case of $\Omega_3$ because the equations from which it is recovered contain terms that secularly increase with time. In fact, the most accurate values of $\Omega_3$ are recovered with small $N_{\text{max}}$. For the two test cases presented above we recover $\Omega$ within 0.005 per cent using $N_{\text{max}} = 4$.

Figure 3. Error in $J_3$ and $\Omega_3$ for the box orbit as a function of $N_{\text{max}}$ and the number of time samples $N_T$ taken over $\sim 10$ fundamental periods. In general, $N_{\text{max}}$ can be usefully increased only if $N_T$ is large enough. The adverse effect of using too large a value of $N_{\text{max}}$ is very much stronger in the case of the frequency than the action.

Figure 4. Cross-sections of the $S_n$ as a function of $n$ for the loop orbit. In the top panel we show the cross-section $n_3 = 0$. The most significant mode in this plane is $(2, -2, 0)$, which causes a mixing between the radial motion and azimuthal motion. In the lower plane we show the cross-section $n_2 = 0$, in which the mode $(0, 0, 2)$ is the most significant.
The error in the actions and frequencies can be estimated by \( J_1 = 106 \), \( J_2 = 1056 \), and \( J_3 = 23 \) kpc km s\(^{-1}\) and use different, but overlapping, 375 Myr long segments of the orbit with \( N_T = 1000 \) to calculate the actions, angles and frequencies. The error in the actions and frequencies can be estimated by

\[
\Delta J \approx (0.8, 0.2, 0.5) \text{ km s}^{-1}, \quad \Delta \Omega \approx (0.1, 0.01, 0.6) \text{ Gyr}^{-1}.
\]

For each orbit segment we find \( \theta_0 \), which we expect to all lie along straight lines with gradients given by the derived frequencies. In Fig. 6 we show that the condition is well satisfied.

Using different orbit segments is perhaps the only way to estimate the error in an action or frequency found using the present method. It is simplest to use consecutive orbit segments as we have here. However, a better method is to use orbit segments separated by a large time interval. This can be achieved most effectively by utilising the estimated generating function to find an initial condition for a second orbit integration. A simple choice is to increase one of the derived angle coordinates by \( \pi/2 \).

4 A typical constant energy surface

Now we turn to constructing the action diagram for the chosen potential. For a given energy we launched particles at a series of points along the potential’s intermediate axis (\( \lesssim 18 \) kpc) with the velocity vector perpendicular to the axis and inclined at varying angles to the z-axis. We then calculated the actions from the time series. Figure 7 shows each orbit as a point in 3D action-space. We see that the surface of constant energy is a triangle-shaped plane in action-space. The points are coloured based on their orbit classification. An equivalent figure for a Stöckel potential can be found in de Zeeuw (1985).

In a triaxial potential the loop orbits can be divided into two classes: the short-axis loops that loop around the short axis (in our case the z-axis) and the long-axis loops that loop around the long axis (the x-axis). Along with the box orbits these three classes of orbit occupy distinct regions on the action-space plane of constant energy. At each corner of the plane only one action is non-zero and the corresponding orbit is the parent orbit of each of the three classes: the \( J_2 = 0, J_3 = 0 \) orbit is a radial orbit along the long axis, the \( J_1 = 0, J_3 = 0 \) orbit is a closed orbit in the \((x, y)\) plane and the \( J_1 = 0, J_2 = 0 \) orbit is a closed orbit in the \((y, z)\) plane. We note that the interface between the different orbit classes consists of points, with some overlap between the different orbit classes in the action space. These features are due to the presence of resonant islands with surrounding chaotic orbits at the interface of the regular orbit regions (see Section 5.3).

5 DISCUSSION

5.1 Relation to previous work

The problem addressed here goes back to Binney & Spergel (1982, 1984), who Fourier transformed the time series \( x(t) \) of individual coordinates and assigned to each line in the resulting spectrum appropriate integers \( n_j \) so that \( \omega t \) could be identified with \( \sum_j n_j \Omega_j t. \)

1 Note that the intermediate axis of the halo model proposed by Law & Majewski is actually the z-axis. However, at small radii (\( \lesssim 18 \) kpc) the intermediate axis of the full potential is in the \((x, y)\) plane due to the disc contribution, and the z-axis is the short axis.

2 To produce a continuous plane in action-space we must scale the ‘radial’ actions of the loop orbits, \( J_1 \), by a factor of 2. \( J_1 \) for a loop orbit corresponds to a single oscillation from maximum to minimum coordinate and back, whilst for a box orbit a single oscillation covers the interval 0 to maximum coordinate four times.
Figure 6. An example orbit in the Law & Majewski (2010) potential. It is a short-axis loop orbit with actions \( \mathbf{J} \approx (63, 1056, 23) \) kpc km s\(^{-1}\). In the top panel we show a 750 Myr long orbit segment in the \((x, y)\) and \((x, z)\) planes. In the central two panels we show \( J_1 \) and \( \Omega_1 \) calculated using 1000 time-samples from a 375 Myr orbit segment labelled by its initial time sample. In the bottom panel we show the calculated angles at these times with black dots. We also show the toy angles in red, and the ‘true’ angles found using \( \theta_0 + \Omega_1 t \) with one of the calculated frequencies and initial angles in blue.

Once this identification had been successfully accomplished, \( \Omega_1 t \) could be replaced with \( \theta_1 \) to yield the orbit’s angle representation. This approach is inferior to that introduced here in several respects: (i) Whereas the generating function is a scalar, a star’s location is described by a vector, so it is wasteful to construct the angle representations of all three coordinates rather than the angle representation of the generating function: Binney & Spurgeon (1984) failed to take advantage of the strong restrictions on tori that arise from angle-action coordinates being canonical. (ii) It is not straightforward to measure correctly the complex amplitudes \( \mathcal{A} \) from the discrete Fourier transform of a time series such as \( x(t_i) \) because the required amplitude will in general not lie at one of the discrete frequencies sampled. (iii) When an orbit is near-resonant there is often dangerous ambiguity in the integers \( n_i \) that should be assigned to a particular line. With the present technique we work from the outset with periodic functions and their Fourier series so the issue of how frequencies fall on a discrete grid does not arise. Moreover, the assignment of integers \( n_i \) to Fourier terms is unambiguous.

The method described here has significant overlap with the work of Warnock (1991) on the construction of magnetic coordinates and the related method of Kaasalainen & Binney (1994) for the construction of angle coordinates. In both these studies angle-action variables were evaluated along numerically computed orbits. The coordinates evaluated were not those of a toy potential but of a trial torus that had been previously constructed: Warnock (1991) was refining the Fourier coefficients \( S_n \) while Kaasalainen & Binney (1994) were solving for the \( \partial_t S_n \) given the \( S_n \). In both these studies, several initial conditions for orbit integration were chosen on each torus to overcome the problem that with a single short integration a resonant orbit yields a highly non-uniform distribution of sample points on the torus. Since we do not have a good representation of the target torus until the equations have been set up and solved, we cannot take advantage of this possibility.

Warnock (1991) solved for the discrete Fourier transforms of the \( n S_n \) rather than for the \( S_n \) because the matrix that then has to be inverted is nearly diagonal when the toy and target tori are close to one another and the sample points provide a nearly regular grid in the space of toy angles. Since our toy and target tori can be quite different, and it is hard to achieve a uniform sampling of toy-angle space, we have not used Warnock’s technique.
5.2 Possibility of using Stäckel tori

We have used completely different toy potentials for each class of orbit, and it is natural to ask whether it would not be advantageous to use always a Stäckel potential since such a potential has tori of every type. We have not pursued this option for two reasons. First, the actions and angles of Stäckel potentials require the evaluation of integrals whereas the potentials we have used yield algebraic expressions for angles and actions. Second, and more fundamentally, when integrating an orbit that lies close to the box/loop interface, it would be non-trivial to ensure that the toy torus with the actions of the target orbit had the same geometry as the target torus. By using potentials that support only one type of torus, we are assured from the outset that this condition is satisfied.

5.3 Resonances and chaos

We have focused here on orbits that are non-resonant members of the major orbital families. In real galactic potentials significant numbers of orbits are either resonantly trapped or chaotic. Chaotic orbits can be thought of as sequences of sections of resonantly trapped orbits, so these two types of orbit raise similar issues.

In a generic integrable potential, the frequencies $\Omega$ depend on the actions, so on some tori a resonant condition $n \cdot \Omega = 0$ is satisfied. Consequently, individual orbits on these resonant tori do not cover the entire torus since the condition $n \cdot \theta = \text{constant}$ constrains the angle variables. This lack of coverage makes it impossible to determine some of the Fourier coefficients $S_n$. When the potential is strictly integrable, orbits on tori that are adjacent to a resonant torus completely cover their tori although they take a long time to do so. In a generic potential, however, such orbits move over a series of tori without covering any of them, as they librate around the strictly resonant orbit. Consequently, these orbits have some of the characteristics of a strictly resonant torus. When the present technique is used on a resonantly trapped orbit, the generating function will map the toy torus into a close approximation to the strictly resonant torus, so in an N-body model the density of stars on this torus will seem to be larger than it really is. Hence with the present technique, resonantly trapped orbits will give rise to apparent crowding in action space that is analogous to the signature of resonances when particles are mapped into frequency space by determining orbital frequencies by Fourier decomposition of coordinates (Dumas & Laskar 1993): when the ratios $\Omega_2/\Omega_1$ and $\Omega_3/\Omega_1$ are used to place orbits in frequency-ratio space, the existence of resonantly trapped orbits leads to a crowding of points along the straight lines associated with certain resonance conditions $n \cdot \theta = \text{constant}$ (Binney & Tremaine 2008, §3.7.3(b)).

6 CONCLUSIONS

We have presented a method for finding actions, frequencies and angles from numerically integrated orbits in a general potential. The method relies on estimating the Fourier components of the generating function that maps a toy torus into the torus on which the computed orbit lies by solving systems of linear algebraic equations. This method enables one to determine the angle-action coordinates $(\theta, J)$ of a given phase-space point $(x, v)$ and has numerous possible applications in astronomy.

Ours is the first method presented in the literature for finding the actions in a general triaxial potential. Triaxiality is an essential ingredient of dark-matter distributions, and a realistic Galactic model which should include non-axisymmetric features such as the bar, and the potentially triaxial halo. This method is a necessary first step towards constructing distribution functions, $f(J)$, for these more complex Galactic components.

An important application is to the analysis of N-body simulations. A single N-body snapshot consists of 3D positions and velocities for $10^9$ particles. Letting the simulation evolve for a few time steps produces another snapshot with a completely different set of $10^9$ positions and velocities. Thus the particles’ phase-space coordinates constitute a highly degenerate and non-compact representation of the simulation. Effective analysis of the simulation should start by condensing the coordinates into a smaller set of numbers. This can be done by replacing the $6N_T$ numbers $(x_i, v_i)$ with just three numbers $J_i$ and plotting each particle as a point in 3D action space. The simulation then becomes a density of particles in a 3D space. This representation will greatly facilitate the comparison of different N-body models. Also it may prove possible to find good fits to the star density in terms of analytic functions, as Pontzen & Governato (2013) have done for numerical dark-matter halos and appears to be possible for the Galactic discs (Binney 2012b; Binney et al. 2013). We hope to report on an application of this method to an N-body simulation soon.

Here we discussed time-reversible triaxial potentials. In this case we can determine a priori the phases of the terms in the generating function. Rotation of the figure of the potential destroys the time-reversibility of the Hamiltonian and we lose the ability to set the phases a priori. In the worst case, the $S_n$ in equation (6) become complex numbers that are only limited by the condition $S_{-n} = S_n$ required to make the generating function real. Extending the current framework to this case ~ doubles the dimensionality of the matrices we must solve for given $N_{\text{max}}$.

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APPENDIX A: SYMMETRIES

In Section 2.2 we asserted that for a time-reversible Hamiltonian the Fourier components of the generating function, $S_n$, are real. However, it must also be true that there is a point on the target torus where $J = 0$. McGill & Binney (1990) show that this is true if the potential is an isochrone and the target Hamiltonian is axisymmetric. Additionally they demonstrated that when the potential is symmetric about the plane $z = 0$, Fourier components of the generating function with odd $n$ vanish. Here we repeat these arguments extended to the 3D triaxial case. Let’s first consider the loop orbits. Suppose we have a target Hamiltonian of the form

$$H(r, \phi, \vartheta) = \frac{1}{2} p_\phi^2 + \frac{p_y^2}{2 r^2 \sin^2 \vartheta} + \frac{p_z^2}{2 r^2} + \Phi(r, \phi, \vartheta),$$

where $(r, \phi, \vartheta)$ are standard spherical polar coordinates. The equations of motion for the toy actions are

$$\dot{J}_i = -\frac{\partial H}{\partial \dot{\theta}_i} = \left(\frac{p_\phi^2}{r^2} + \frac{p_y^2}{r^2 \sin^2 \vartheta} - \frac{\partial \Phi}{\partial r} \right) \frac{\partial r}{\partial \theta_i} + \left(\frac{p_\phi^2 \cos \vartheta}{r^2 \sin^2 \vartheta} - \frac{\partial \Phi}{\partial \vartheta} \right) \frac{\partial \vartheta}{\partial \theta_i} - \frac{\partial \Phi}{\partial \phi} \frac{\partial \phi}{\partial \theta_i},$$

(A2)

Now let us consider the point $\theta = (0, 0, \pi/2)$: at this point the particle is at pericentre, at a maximum in its vertical oscillation and at $\phi = 0$. Therefore at this point we have that

$$\frac{\partial r}{\partial \theta_i} = \frac{\partial \vartheta}{\partial \theta_i} = p_r = p_\theta = \frac{\partial \phi}{\partial \theta_i} = 0,$$

(A3)

so

$$\dot{J}_i = -\frac{\partial \Phi}{\partial \phi} \frac{\partial \phi}{\partial \theta_i}.$$

(A4)

In a triaxial potential $x = 0$ is a symmetry plane of the potential so $\partial \Phi/\partial \phi|_{x=0} = 0$ and $\dot{J}_i = 0$. This is the requirement introduced in Section 2.2 for the Fourier generating function to be real. Now let’s consider the point $\theta = (0, 0, 0)$. Here the particle is at pericentre, crossing the $z = 0$ plane, and at $\phi = 0$. At this point we have

$$\frac{\partial r}{\partial \theta_i} = \cos \vartheta = p_r = \frac{\partial p_\phi}{\partial \theta_i} = \frac{\partial \phi}{\partial \theta_i} = 0,$$

(A5)

so

$$\dot{J}_i = -\frac{\partial \Phi}{\partial \phi} \frac{\partial \phi}{\partial \theta_i}.$$

(A6)

As we saw before the first term is zero as $x = 0$ is a symmetry plane of the potential. The second term is also zero as $z = 0$ is also a symmetry plane. By a similar argument at $\theta = (0, \pi/2, 0)$, $\partial \Phi/\partial \phi|_{x=y/2} = 0$ as $y = 0$ is a symmetry plane of the potential.

We calculate $\mathbf{J}$ from equation (9) as

$$\mathbf{J} = \sum_{n \in \mathbb{N}} 2 n \left(\mathbf{n} \cdot \hat{\theta}\right) S_n(\mathbf{J}) \sin \mathbf{n} \cdot \mathbf{\theta}$$

(A7)

At the point $\theta = (0, 0, \pi/2)$ we know $\mathbf{J} = 0$ so we require $\sin n \pi/2 = 0$ so $n \pi/2$ must be even. Similarly we know $\mathbf{J} = 0$ at $\theta = (0, \pi/2, 0)$ so $n_2$ is restricted to even values. However, $n_1$ can take any integer value.

Now let us consider the box orbits. We have a target Hamiltonian of the form

$$H = \frac{1}{2} \sum_i p_i^2 + \Phi(x, y, z)$$

(A8)

where $p_i = (p_x, p_y, p_z)$ and the equations of motion for the toy actions are

$$\dot{J}_i = -\sum_j \frac{\partial \Phi}{\partial x_j} \frac{\partial x_j}{\partial \theta_i} - p_i \frac{\partial p_i}{\partial \theta_i}.$$

(A9)

Consider the point $\theta = (0, 0, 0)$. Here the orbit is turning in all three coordinates so $p = 0$ and $\partial x/\partial \theta_i = 0$ so $J = 0$ as required in Section 2.2. Now let’s consider the point $\theta = (\pi/2, 0, 0)$. Here the orbit is turning in $y$ and $z$ and is passing through the $x = 0$ plane at which point $\partial p_x/\partial \theta_i = 0$ as $p_x$ is at a maximum. Therefore we have

$$\dot{J}_i = -\frac{\partial \Phi}{\partial x} \frac{\partial x}{\partial \theta_i}.$$ (A10)

For a triaxial potential aligned with our choice of Cartesian axes $x = 0$ is a symmetry plane so $\partial \Phi/\partial x|_{x=0} = 0$. Therefore $J = 0$ here and by similar arguments to the loop orbit case we are restricted to even $n_1$. We can employ the same arguments by considering the stationary points $\theta = (0, \pi/2, 0)$ and $\theta = (0, 0, \pi/2)$ to show that $n_2$ and $n_3$ must be even.

APPENDIX B: ANGLES AND FREQUENCIES

To find the angles and frequencies from an orbit timeseries we must minimize equation (17) with respect to the unknowns. The unknowns are $\theta'(0), \Omega'$ and the set of $\partial S_n/\partial J'$, which we denote as $(\partial_1 S_n, \partial_2 S_n, \partial_3 S_n)$. For each time we define the $N$-vector $s_n(t_i) = -2 \sin(\mathbf{n} \cdot \mathbf{\theta}(t_i))$.

(B1)

We also define the $3(2 + N)$-vectors

$$x_0 \equiv (\theta'(0), \Omega', \partial_1 S_n, \partial_2 S_n, \partial_3 S_n),$$

(B2)

and the symmetric matrix

$$A_0 \equiv \sum_i \left( \begin{array}{c} I_3 \ t_1 l_1 \\ t_1 l_1 \ t_2 l_2 \\ \vdots \ t_2 l_2 \ t_3 l_3 \ s^T \ s^T \ s^T \ s^T \ s^T \ s^T \end{array} \right)$$

(B3)

where each $s^m$ is an $N$-by-$3$ matrix with the $N$-vector $\mathbf{s}$ in the $m$th column, and each $s$ is evaluated at the $i$th time. Setting the partial derivatives of $F$ with respect to the unknowns to zero yields the matrix equation.

$$A_0 \cdot x_0 = b_0.$$ (B4)

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