Purification and correlated measurements of bipartite mixed states

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We prove that all purifications of a non-factorable state (i.e., the state which cannot be expressed in a form \( \rho_{AB} = \rho_A \otimes \rho_B \)) are entangled. We also show that for any bipartite state there exists a pair of measurements which are correlated on this state if and only if the state is non-factorable.

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**I. INTRODUCTION**

Quantum entanglement is one of the most important ingredients of the paradigm of the quantum theory \[1,2\]. It plays the central role in quantum teleportation \[3\], quantum dense coding \[4\], quantum secret sharing \[5\], and other quantum information processes \[6\]. Quantum entanglement can be manipulated using the entanglement swapping \[7,8\] and it can be concentrated via quantum distillation techniques \[6\].

It is well known that pure entangled states violate the so-called Bell inequalities \[9\], which implies that these states have nonlocal properties. This means that pure entangled states cannot be created locally. Moreover, for each bipartite pure state there exists a pair of correlated measurements \[10\] if and only if the state is entangled.

In the case of mixed states the situation is more complex. In 1989 Werner \[10\] have introduced the following definition of entanglement for mixed states: the bipartite mixed state is entangled if and only if it is inseparable. In addition Werner has shown that any separable state can be created exclusively via local operations and classical communication (and hence it doesn’t have nonlocal properties).

In this Brief Report we will concentrate our attention on correlations in measurements performed on mixed entangled states. The problem of correlations in measurements of two qubits has been studied by Englert \[11,12\]. Specifically, we will derive the necessary and sufficient condition for existence of correlated measurements on bipartite mixed states.

In what follows we will utilize the purification anzats as proposed by Uhlmann \[13\] via which an impure state of a given quantum system can be purified with the help of ancillas. Our main motivation to study purification of mixed states is to determine the relation between the entanglement present in purified states and the existence of correlations in measurements performed on original bipartite mixed states. We will also study whether these correlations are related to non-locality of purified states.

In Section \[1\] we introduce the notion of factorability and we derive the relation between the factorability of bipartite density matrix \( \rho \) and the entanglement of any purification of \( \rho \). In Section \[II\] we prove that for any bipartite density matrix \( \rho \) there exists a pair of measurements which are correlated on \( \rho \) if and only if \( \rho \) is not factorable.

In order to unify the notation and terminology we define correlations in measurements and present two examples which clarify the problem we address.

Let \( \rho_{AB} \) is a bipartite density matrix while \( \rho_A = \text{Tr}_B(\rho_{AB}) \) and \( \rho_B = \text{Tr}_A(\rho_{AB}) \) are reduced density matrices of the subsystems \( A, B \), respectively. Let \( E, F \) are measurements on the subsystems \( A, B \), respectively, and \( \hat{E}, \hat{F} \) are the corresponding observables. Then the measurements \( E \) and \( F \) are correlated on \( \rho_{AB} \) if and only if

\[
\text{Tr}(\rho_{AB} \hat{E} \otimes \hat{F}) \neq \text{Tr}(\rho_A \hat{E})\text{Tr}(\rho_B \hat{F}).
\]

**Example 1:** Let us consider two correlated sources \( A, B \) emitting spin-\( \frac{1}{2} \) particles (qubits) such that with the probability \( \frac{1}{2} \) both sources simultaneously produce particles in the state \( |0\rangle \) and with the probability \( \frac{1}{2} \) both particles are simultaneously in the state \( |1\rangle \). Hence, the sources produce states \( |00\rangle_{AB} \) or \( |11\rangle_{AB} \) and the density matrix describing this source is

\[
\rho_{AB} = \frac{1}{2} (|00\rangle_{AB} \langle 00| + |11\rangle_{AB} \langle 11|).
\]

If we subject such pair of particles to orthogonal (projective) measurements in the bases \( \{|0\rangle_A, |1\rangle_A\} \) and \( \{|0\rangle_B, |1\rangle_B\} \), then the results of measurements of the state of particles \( A \) and \( B \) are the same. The reason is, that the pairs were produced in such a way that they are both in the same state. In this case we can apply the formalism of a micro-canonical ensemble since we have a set of pairs (of particles) denoted \( p_1, p_2, \ldots \) in pure states \( |\phi_1\rangle, |\phi_2\rangle, \ldots \), where each of the states \( |\phi_i\rangle \) is either \( |00\rangle \) or \( |11\rangle \). Results of the measurements are in this case two sequences of random variables \( a_1, a_2, \ldots \) and \( b_1, b_2, \ldots \). Each pair of random variables \( a_i, b_i \) is not correlated. But the ensemble of the pairs (which is described as a statistical mixture) exhibits correlations. As pointed out by Werner \[10\] these correlations have nothing to do with quantum non-locality and they are caused by classical correlations of the sources.

**Example 2:** Now let us consider a source which repeatedly produces three spin-\( \frac{1}{2} \) particles \( A, B, C \) in the Greenberger-Horn-Zeilinger (GHZ) state \[1\]

\[
|\phi\rangle_{ABC} = \frac{1}{\sqrt{2}} (|000\rangle_{ABC} + |111\rangle_{ABC}).
\]
Obviously, the reduced density matrix $\rho_{AB}$ of particles $A, B$ is the same as in the previous example described by Eq. (1.2). This means that measurements in the bases $\{|0\rangle_A, |1\rangle_A\}$ and $\{|0\rangle_B, |1\rangle_B\}$ yield the same results as in the previous example. Nevertheless, in order to describe the present situation we have to employ the macro-canonical formalism. Results of the measurements are two random variables, which are correlated, and this correlation is caused by a quantum non-locality, which follows from the fact that measurements $\{|0\rangle_A, |1\rangle_A\}$ and $\{|0\rangle_B, |1\rangle_B\}$ performed on $|\phi\rangle_{ABC}$ are correlated due to the present quantum entanglement.

In order to appreciate the relevance of these two examples we note that in spite of the fact that the measurements performed on the two systems generate the same experimental results their interpretation might be totally different. The in-spite of this effect is that although the properties of separable states can be explained locally (i.e. without employing entanglement), the actual physical reason behind these “classical” correlations can be related to the quantum non-locality in preparation of the system.

II. PURIFICATIONS AND FACTORABILITY

We start this section with the definitions of factorability, separability and purification of density matrices:

The density matrix $\rho_{AB}$ is factorable if it can be written in the form

$$\rho_{AB} = \rho_A \otimes \rho_B. \quad (2.1)$$

The density matrix $\rho_{AB}$ is separable if it can be written in the form

$$\rho_{AB} = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)}. \quad (2.2)$$

Let us consider a bipartite system $AB$ in the state described by a density matrix $\rho_{AB}$. Let $AB$ is a subsystem of some larger system $ABC_1C_2$, which is in a pure state $|\psi\rangle$. Obviously, there is a whole class of states $|\psi\rangle$, which represent purifications of the density matrix $\rho_{AB}$, i.e. which fulfill the condition

$$\text{Tr}_{C_1C_2} (|\psi\rangle \langle \psi|) = \rho_{AB}. \quad (2.3)$$

It is important to note that the purification of a given state $\rho_{AB}$ is not unique. Firstly, from the Schmidt decomposition it follows that we can choose auxiliary systems $C_1$ and $C_2$ of an arbitrary dimension such that $\dim(C_1C_2) \geq \dim(AB)$. Secondly, the purification is not unique even when we fix dimensions of Hilbert spaces $C_1$ and $C_2$ because if $|\psi\rangle_{ABC_1C_2}$ is a purification of $\rho_{AB}$, then $U_{C_1C_2}|\psi\rangle_{ABC_1C_2}$ is also a purification of $\rho_{AB}$ for any unitary operator $U_{C_1C_2}$ acting on $C_1C_2$.

**Theorem 1.** Let $\rho_{AB}$ is a non-factorable density matrix. Then any purification $|\psi\rangle_{ABC_1C_2}$ of $\rho_{AB}$ is entangled in a sense that it cannot be written in the factorized form

$$|\psi\rangle_{ABC_1C_2} = |\psi_1\rangle_{AC_1} \otimes |\psi_2\rangle_{BC_2}. \quad (2.4)$$

Conversely, if all purifications of $\rho_{AB}$ are entangled, then $\rho_{AB}$ is non-factorable.

In order to prove this theorem let us suppose that there is a purification $|\psi\rangle_{ABC_1C_2} = |\psi_1\rangle_{AC_1} \otimes |\psi_2\rangle_{BC_2}$ of $\rho_{AB}$. From the definition of purification it holds that

$$\text{Tr}_{C_1C_2} (|\psi\rangle_{ABC_1C_2} \langle \psi|) = \rho_{AB}. \quad (2.5)$$

However, from the definition of the partial trace we have

$$\text{Tr}_{C_1C_2} (|\psi_1\rangle_{AC_1} \langle \psi_2\rangle_{BC_2} \text{ Tr}_{C_1C_2} (|\psi_1\rangle_{AC_1} \langle \psi_1|) \otimes \text{ Tr}_{C_1C_2} (|\psi_2\rangle_{BC_2} \langle \psi_2|)) = \rho_A' \otimes \rho_B', \quad (2.6)$$

which is in a contradiction with the fact, that $\rho_{AB}$ is a non-factorable density matrix.

In order to prove the second implication we will prove the following: If $\rho_{AB}$ is factorable, then there exists a purification of $\rho_{AB}$ which is not entangled. In fact, we will prove a stronger statement by restricting the dimension of the purification. Let $\dim(H_{AB}) = n$. Then there exists a purification of $\rho_{AB}$ of dimension $n^2$, which is not entangled. It is well known, that there exist purifications $|\phi_1\rangle_{AC_1}$ of $\rho_A$ and $|\phi_2\rangle_{BC_2}$ of $\rho_B$ such that $\dim(H_A) = \dim(H_{C_1})$ and $\dim(H_B) = \dim(H_{C_2})$. Then $|\psi\rangle_{ABC_1C_2} = |\phi_1\rangle_{AC_1} \otimes |\phi_2\rangle_{BC_2}$ is a purification of $\rho_{AB}$ of the desired dimension, which is not entangled.

Theorem 1 can be easily generalized for $n$–partite systems in the following way: Let $\rho_{A_1...A_n}$ is a density matrix, which is not factorable in the sense that it cannot be written as $\rho_{A_1...A_n} = \rho_{A_1} \otimes ... \otimes \rho_{A_n}$. Then any purification $|\psi\rangle_{A_1...A_nC_1...C_n}$ of $\rho_{A_1...A_n}$ is entangled in the sense that it cannot be written in the form

$$|\psi\rangle_{A_1...A_nC_1...C_n} = |\psi_1\rangle_{A_1C_1} \otimes ... \otimes |\psi_n\rangle_{A_nC_n}. \quad (2.7)$$

Conversely, if each purification of $\rho_{A_1...A_n}$ is entangled, then $\rho_{A_1...A_n}$ is not factorable.

From above it follows that if $\rho_{AB}$ is a non-factorable state and $\rho_{ABC_1C_2}$ an arbitrary mixed state such that

$$\text{Tr}_{C_1C_2} (\rho_{ABC_1C_2}) = \rho_{AB}, \quad \text{then} \quad \rho_{ABC_1C_2} \text{ is not factorable in the sense that it cannot be written as} \quad \rho_{ABC_1C_2} = \rho_{AC_1} \otimes \rho_{BC_2}. \quad \text{This follows from Theorem 1, because each purification of } \rho_{ABC_1C_2} \text{ is also a purification of } \rho_{AB} \text{ and thus it is entangled.}$$

It is also straightforward to show that a factorable density matrix $\rho_{AB}$ has both entangled and unentangled purifications. Specifically, from the factorability we have $\rho_{AB} = \rho_A \otimes \rho_B$. Let $|\psi\rangle_{AC_1}$ is a purification of $\rho_A$ and $|\phi\rangle_{BC_2}$ is a purification of $\rho_B$. Then $|\psi\rangle_{AC_1} \otimes |\phi\rangle_{BC_2}$ is a purification of $\rho_{AB}$. Let $|\omega\rangle = U_{C_1C_2} (|\psi\rangle_{AC_1} \otimes |\phi\rangle_{BC_2})$, where $U_{C_1C_2}$ is a unitary operator acting on $C_1C_2$. Clearly $|\omega\rangle$ is a purification of $\rho_{AB}$ for any $U_{C_1C_2}$ and moreover there is a $U_{C_1C_2}$ such that $|\omega\rangle$ is entangled.
We conclude the present section by the following observation: If a system \( AB \), which is a part of a larger system \( ABC_1C_2 \), is in a non-factorable state \( \rho_{AB} \), then it must be a part of a larger system which is entangled. In other words, when we have a non-factorable system, then any larger system (in a pure state) containing this system is entangled. Moreover, for each purification \( |\psi\rangle \) of \( \rho_{AB} \) no unitary \( U_{C_1C_2} \) operation can be found such that \( U_{C_1C_2} |\psi\rangle \) is unentangled. Hence, the non-factorability of \( |\psi\rangle \) is not caused by the correlation between \( C_1 \) and \( C_2 \). The most interesting fact is that all previous statements hold regardless if \( \rho_{AB} \) is separable or not.

### III. CORRELATIONS IN MEASUREMENTS

**Theorem 2.** Let \( \rho_{AB} \) is a non-factorable density matrix. Then there exists a pair of orthogonal measurements represented by observables \( E \) and \( F \) (measured on \( \mathcal{H}_A \) and \( \mathcal{H}_B \), respectively), which are correlated on \( \rho_{AB} \).

**Proof.** In order to prove the theorem we will use the negated implication. That is, let \( \rho_{AB} \) is a density matrix such that any two orthogonal measurements \( E \) and \( F \) performed on \( \rho_{AB} \) are uncorrelated. Then \( \rho_{AB} = \rho_A \otimes \rho_B \) is factorable.

A result of a measurement can be represented as a random variable. Therefore the results of the measurement are uncorrelated iff the corresponding random variables are uncorrelated, i.e., the covariance \( C(E,F) \) fulfills the condition \( C(E,F) = 0 \). Let \( \rho_A = \text{Tr}_B(\rho_{AB}) \) and \( \rho_B = \text{Tr}_A(\rho_{AB}) \), then the covariance of uncorrelated measurements fulfills the condition

\[
C(E,F) = \langle E \otimes F \rangle_{\rho_{AB}} - \langle E \rangle_{\rho_A} \langle F \rangle_{\rho_B} = 0, \quad (3.1)
\]

from which it follows that

\[
\text{Tr}(E \otimes F \rho_{AB}) = \text{Tr}(E \rho_A) \text{Tr}(F \rho_B) = \text{Tr}(E \otimes F \rho_A \otimes \rho_B). \quad (3.2)
\]

We want to show, that this identity implies \( \rho_{AB} = \rho_A \otimes \rho_B \) and hence that \( \rho_{AB} \) is factorable.

The condition \((3.2)\) holds for any two Hermitian operators \( E \) and \( F \). Let us choose some fixed basis \( \{|\phi_i\rangle_{AB}\}_i \) on \( \mathcal{H}_A \otimes \mathcal{H}_B \). We will show that

\[
\forall i,j : (\rho_{AB})_{ij} = (\rho_A \otimes \rho_B)_{ij}. \quad (3.3)
\]

Because Eq. \((3.2)\) holds for any two Hermitian operators \( E_i \) and \( F_i \) it also holds that

\[
\sum_i \alpha_i \text{Tr}(E_i \otimes F_i \rho_{AB}) = \sum_i \alpha_i \text{Tr}(E_i \otimes F_i \rho_A \otimes \rho_B) \quad (3.4)
\]

for any \( \alpha_i \in C \) and hence

\[
\text{Tr} \left( \sum_i \alpha_i E_i \otimes F_i \rho_{AB} \right) = \text{Tr} \left( \sum_i \alpha_i E_i \otimes F_i \rho_A \otimes \rho_B \right). \quad (3.5)
\]

To prove Eq. \((3.3)\) it is enough to show that

\[
\text{Tr}(A \rho_{AB}) = \text{Tr}(A \rho_A \otimes \rho_B) \quad (3.6)
\]

for any matrix \( A \) such that

\[
A_{ij} = 1 \text{ for fixed } i, j \text{ and } A_{xy} = 0 \text{ otherwise}. \quad (3.7)
\]

However, an arbitrary matrix on \( \mathcal{H}_A \otimes \mathcal{H}_B \) can be expressed as

\[
\sum_i \alpha_i E_i \otimes F_i, \quad (3.8)
\]

where \( E_i \) and \( F_i \) are Hermitian matrices and \( \alpha_i \) is an arbitrary complex number. This completes the proof.

The remaining part of this problem is trivial. When \( \rho_{AB} = \rho_A \otimes \rho_B \) (\( \rho_{AB} \) is factorable), then the systems \( A \) and \( B \) are not correlated which follows from Theorem 2.

### IV. CONCLUSION

We proved that any purification of a non-factorable state is always entangled. This means that any system which contains a non-factorable subsystem is also non-factorable. Moreover, we described purifications of factorable states and we proved that for any bipartite density matrix \( \rho \) there exists a pair of measurements which are correlated on \( \rho \) if and only if \( \rho \) is non-factorable. Taking into account the fact that any purification of a non-factorable state is entangled we conclude that these correlations have their origin in quantum non-locality.

This can be interpreted as an alternative approach to Werner’s explanation of the origin of correlations in measurements on separable (but non-factorable) states. Our approach supplements the original work of Werner [10]. Specifically, we showed that correlations on bipartite mixed state exists if and only if the state is non-factorable. These correlations can be explained locally (see Werner [11]) when the state is separable, or they can be explained via quantum entanglement of purified states (see Section II).

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