1. INTRODUCTION

The BFKL equation determines the high energy behaviour of the virtual gluon–gluon scattering amplitude in Regge limit, in which the cms energy $\sqrt{s}$ is much larger than the transverse momenta, $k, k'$, of the gluons. It was derived in the fixed coupling constant case by resumming all the Feynman diagrams describing gluon–gluon scattering in the leading or next-to-leading order. The scattering amplitude displays a scale invariance such that it can be described solely by functions of ratios of transverse momenta $k$ and $k'$.

A common application [1–3] of the BFKL equation is to use it as an evolution equation in rapidity $y$ and transverse momentum $t$ for some large-rapidity amplitude, namely

$$\frac{\partial}{\partial y} A(y, t) = \int d^2k d\alpha_y(t, t') A(y, t') \tag{1.1}$$

where

$$t = \ln \left( \frac{k^2}{\Lambda_{QCD}^2} \right).$$

and the coupling $\alpha_y$ runs with transverse momentum. This application is often used as it lends itself relatively easily to an extension of the DGLAP formalism to very low values of Bjorken-$x$, where the pure DGLAP formalism is known to break down.

In order to solve this evolution equation, one requires as input the amplitude at some rapidity $y$ for all values of transverse momentum $t'$. One could naively expect that Eq. (1.1) could be directly solved for large $y$ and $t$ well in the perturbative region because it is known that the BFKL kernel $K(t, t')$ is quasilocal in $t$, i.e. it diminishes when $|t - t'|$ is large. However, as was carefully investigated e.g. in [1] the BFKL equation is not only an evolution in $y$ but also in virtualities $t$, which leads to a substantial diffusion into the low transverse momenta region were perturbative QCD cannot be valid. Therefore, the authors of [1] proposed a modification of the BFKL equation by imposing a low and high cutoff in $t$ on the BFKL integral, a procedure which is today widely accepted. However, such a cutoff implies that the amplitude actually vanishes below a certain transverse momentum, rather than becoming non-perturbative. In fact, due to the growth of the coupling constant at small $k_T$, the amplitude could grow in this region and the vanishing of it at the infrared cutoff looks unnatural.

In [4] we proposed to solve Eq. (1.1) by the Green function method which does not require any cutoff on the BFKL integral. Instead, we assumed that the non-perturbative infrared region of QCD imposes a certain phase on the oscillatory parts of the eigenfunctions at some small transverse momentum. This treatment of the infrared boundary leads to a discrete set of eigenvalues, $\omega_n$, of the BFKL kernel, since only certain values permit the construction of eigenfunctions which simultaneously obey these phase conditions at low transverse momentum and the large transverse momentum boundary conditions imposed by the asymptotic freedom. This is in contrast to the “usual” treatment [1–3], in which a lower transverse momentum cutoff is imposed on the amplitude, i.e. the ampli-
tude is assumed to vanish below the cutoff. In our approach, the amplitudes are particularly sensitive to the exact values of the discrete $\omega_n$ which are related to the non-perturbative phases $\eta_n$ at the cutoff. These phases are determined, in turn, by the gluon–gluon interactions of the non-perturbative QCD, which lead to rich structures below this cutoff.

A rigid cutoff (either UV or IR) destroys the scale invariance of the BFKL kernel—and hence the validity of scale covariant solutions. As pointed out in [1], the imposition of such cutoffs has no effect on the position of the leading singularity (the exponent of $x$ in structure functions) but do affect the pre-factor, which is controlled by the form of this singularity. We find that the subleading singularities are also essential in order to obtain a good fit to HERA data. In our application of BFKL dynamics, the scale and conformal invariance (which is central to the BFKL formalism for fixed coupling) is broken in a controlled way, namely only through the running of the coupling and we assume “quasi-conformal” solutions in which the exponent of the transverse momentum varies slowly in order to compensate for the change in the coupling in an accordance with a generalized DGLAP dynamics.

In our previous paper [4] we have shown that HERA $F_2$ data, at low $x$, can be described very well by the gluon density constructed from the discrete spectrum of eigenfunctions of the BFKL kernel. The spectrum contained many eigenfunctions, $\mathcal{C}(100)$, with eigenvalues $\omega_n$ varying from $\omega_0 \sim 0.25$ to $\omega_n \sim 0.5/n$ for large $n$. This first successful confrontation of the BFKL formalism [6] with data led to the unexpected question as to whether the HERA data are sensitive to the Beyond Standard Model (BSM) effects. These effects, although only present at scales that are much higher than the region of HERA data, can nevertheless affect the quality of the fits to data since BSM effects change the running of the coupling and consequently also substantially change the values of $\omega_n$.

This seems somewhat counter-intuitive. One may ask how it can be possible that a fit to data at relatively low energies can be sensitive to corrections due to loops of particles whose masses are far in excess of those energies. The crucial point is that the above-mentioned large transverse momentum boundary conditions, imposed by the running of the coupling, can occur at very high $k_T$ scales. For $\omega \lesssim 0.1$ this is already above the scale at which one might expect BSM physics to occur. The value of the discrete eigenvalues arise from an interplay between these ultraviolet boundary conditions and the infrared boundary condition arising from the imposition of an infrared phase. It is in this sense that in our treatment of the BFKL formalism there is communication between high and low energy scales. We make the assumption that the allowed eigenvalues can be obtained from the BFKL equation supplemented by an infrared phase condition in a process-independent way, i.e. without needing to impose any infrared or ultraviolet cuts on the integration over transverse momentum. Once these eigenvalues are calculated their corresponding eigenfunctions are convoluted with the necessary impact factors in order to obtain the required amplitudes. In this way, it turns out that even though these amplitudes, in accordance with kinematical constraints, never involve diffusion into transverse momenta above the threshold for BSM physics, the rapidity dependence of such amplitudes is affected by the substantial changes in the eigenvalues arising as a result of BSM physics.

To understand how the running of the coupling constant can have such far-reaching consequences we derive analytically (in Section 2) the main properties of the discrete pomeron solution using the LO BFKL equation. This derivation provides a qualitative physical explanation of the mechanism by which the BSM effects modify the discrete pomeron structures and lead to a genuine change of the eigenvalues and eigenfunctions. It also elucidates the role of the infrared phases which define the boundary condition and which can be indirectly determined from data. This explanation is then carried over into the NLO evaluation, which was used for data analysis and was performed numerically.

As a popular example of BSM effects we have chosen the $\mathcal{N} = 1$ supersymmetry and modified the $\beta$-function and the kernel of the BFKL equation to include the contributions from the superpartners. We then describe the full NLO evaluation of the Discrete BFKL Pomerons (DP) with collinear resummation [10]. This allows us to show that the eigenvalues, $\omega_n$, at larger $n$, have a genuine sensitivity to BSM physics because the support of the corresponding eigenfunctions extends to very high virtualities. The values of $\omega_n$ are determined (to large extent) by the running of $\alpha_s$ and the properties of the BFKL kernel in the high virtuality regions, where BSM effects dominate and QCD NLO correction are very small. This is also the reason why these eigenvalues are not sensitive to a particular choice of the infrared boundary. All these properties are discussed in details in Section 2.

In Section 3 we then show that it is possible to construct a physically self-consistent infrared boundary condition which determines the properties of the gluon density to be in agreement with data. Finally we confront the DP gluon density with the HERA $F_2$ data and show that within our model for the infrared boundary condition we obtain indirect evidence of a supersymmetric threshold in a multi TeV range.

In Section 4 we discuss our results with particular emphasis on the role of the universal Green function and the momentum conservation. We also discuss the dependence of our fits on the choice of the infrared boundary condition. Section 5 presents a summary.
2. THE DISCRETE BFKL POMERON

The forward amplitude for a diffractive process with rapidity (or rapidity gap) $y$ is determined by the QCD pomeron and may be written

\[ \mathcal{A}(y) = \int d\omega \int dt \Phi_\omega(t)\Phi_\omega(t')x^{-\omega}\tilde{g}_\omega(t, t'), \]

where $\tilde{g}_\omega(t, t')$ is the Mellin transform of a universal (i.e. process independent) Green function $\tilde{g}_\omega(t, t', y-y')$.

\[ \tilde{g}_\omega(t, t', y-y') = \int d\omega \tilde{g}_\omega(t, t')e^{\omega(y-y')} \quad \text{ (2.2)} \]

The process dependence enters only through the impact factors, $\Phi_\omega$, $\Phi_\omega'$ at the top and bottom of the gluon ladder which depend on the transverse momenta of the gluons and may also depend on other kinematic variables. The integral over $\omega$ in the inverse Mellin transform, (2.2), is performed over a contour parallel to the imaginary axis, to the right of all singularities of the Green function. This Green function obeys the equation

\[ \omega \tilde{g}_\omega(t, t') - \int d\omega' \mathcal{H}(\tilde{g}_\omega(t, t') \tilde{g}_\omega(t', t')) = \delta(t-t'), \quad \text{ (2.3)} \]

which is solved by determining the set of eigenfunctions of the BFKL kernel, subject to certain boundary conditions. If we allow the coupling $\tilde{\alpha}_s$ to run with $t$, $t'$, which in LO means replacing it by $\sqrt{\tilde{\alpha}_s(t)\tilde{\alpha}_s(t')}$ then, as we explain below, the UV boundary condition—namely that the eigenfunctions decay as $t \to \infty$ is automatically implemented. The infrared boundary condition is imposed by requiring that the eigenfunctions have some given non-perturbative phases at some low value of $t$.\(^3\) As was first shown in [12] and we explain again below, the combination of the UV boundary condition (which determines the oscillation phase at $t=\infty$, where the oscillatory behaviour changes to an exponentially decaying behaviour, compatible with a DGLAP analysis in the double logarithmic limit) and the infrared phase condition leads (for positive $\omega$) to a discrete set of allowed eigenvalues $\omega_n$ with their attendant eigenfunctions, so that the Green function may be written as

\[ \tilde{g}_\omega(t, t') = \sum_n \frac{\omega_n^\omega(t')f_\omega(t')}{\omega - \omega_n} + \frac{1}{2\pi i} \int d\omega \sum_n \frac{\omega_n^\omega(t')f_\omega(t')}{{\omega - \omega_n} + i\epsilon} \quad \text{ (2.4)} \]

No further cuts from kinematic constraints on $t$ are imposed in the determination of the eigenfunctions and eigenvalues. However, as we discuss in detail in Section 4, owing to the quasi-local nature of the BFKL kernel $H$, this Green function is rapidly attenuated for large $|t-t'|$, which means that when it is inserted into the expression (2.1) in order to obtain the unintegrated gluon density, the diffusion into values of $t$ substantially above the region of support of the impact factor, $p(t)$ is highly suppressed, thereby automatically limiting the gluon virtuality to be small compared with the incoming energy.

The infrared non-perturbative phases, $\eta_n$, which determine the values of the discrete eigenvalues, $\omega_n$, are in general $\omega$ dependent, but they must lie within a range of magnitude $\pi$, so that the quantum number $n$ represents the number of oscillations of the eigenfunction between the scale, $t_0$, at which the infrared phase condition is applied and the ultra-violet scale, $t_c$, at which the oscillatory behaviour becomes an exponentially decaying one. As pointed out in ref. [4], an $\omega$-dependent infrared phase condition is necessary in order to be able to express an impact factor with support only for small $t$ in terms of the discrete eigenfunctions, since the frequency of oscillation of these eigenfunctions at small $t$ is always below $\sim 0.7$. In fact, after imposing this non-perturbative phase, the eigenfunctions form an almost complete set of functions in the region of comparatively small $k \sim 1 - 10$ GeV.

The existence of this set of discrete eigenfunctions is consistent with the known fact that in the Regge regime, the amplitude is determined by a set of Regge poles. The imposition of infrared phases does not in any way violate the kinematical constraint, but the ensuing discrete spectrum of eigenvalues has a very significant effect on the fitting of the results of the modified BFKL formalism to HERA data on the structure functions at low-$x$.

The continuum contribution (for negative $\omega$) is not significant for sufficiently small values of $x$ at any given $t$. However, for a given $x$, as $t$ increases these contributions become more significant and are essential in order for this formalism to match the double logarithmic limit of the DGLAP approach for sufficiently large $t$.

The value of $t_c$, at which the oscillatory behaviour converts into an exponentially decaying one increases linearly with eigenvalue number $n$. For $n \gtrsim 3$ this occurs at values of $t_c$ above the scale at which one may expect to see physics beyond the standard model. The running of the coupling is therefore affected by the presence of thresholds for such new physics and this in turn affects the positions of the discrete eigenvalues $\omega_n$. For sufficiently low values of $x$ the contribution from all but the first two or three eigenfunctions is negligible. However, we find that for values of $x$ which are probed at HERA there is a sizable contribution from these higher eigenfunctions. This means that even

\(^3\) These phases should be universal (process independent).
though the transverse momenta do not diffuse into regions of \(t\) where the particles of new physics are actually produced, the shifts in the positions of the eigenvalues due to new physics affects the \(x\)-dependence of the unintegrated gluon density. Despite the fact that these effects are small, the high quality of the HERA data means that the quality of the fit is significantly affected by the possibility of new physics at high energies.

We now show how this works in detail.

### 2.1. LO Evaluation

We begin this section by reviewing the argument of \([12]\) which led to a modification of the BFKL formalism which gives rise to discrete poles rather than a cut in the \(\omega\)-plane of the \(t\) channel partial waves.

We consider the case of the leading order BFKL equation \([6]\) with running coupling also taken to leading order so that (for \(t > 0\))

\[
\bar{\alpha}_s = \frac{C_A}{\pi} \frac{\alpha_s}{\bar{\beta}_0 t} \quad (2.5)
\]

where

\[
\bar{\beta}_0 = \frac{\beta_0}{4C_A} = \frac{11}{12} - \frac{n_f}{18} \quad (2.6)
\]

The Hermitian BFKL kernel is given by

\[
\sqrt{\bar{\alpha}_s(t)} \bar{\alpha}_s(t') \mathcal{K}_0(t, t') = \int dt' \mathcal{K}_0(t, t') e^{i\nu t'} = \chi_0(v)e^{i\nu t} \quad (2.7)
\]

\[
\chi_0(v) = 2 \Psi(1) - \Psi\left(\frac{1}{2} + iv\right) - \Psi\left(\frac{1}{2} - iv\right) \quad (2.8)
\]

Note that the parameter \(v\) may be real or imaginary for real eigenvalues \(\chi_0(v)\).

The eigenfunctions, \(g_\omega(t)\) of this Hermitian kernel obey the eigenvalue equation

\[
\int dt' \sqrt{\bar{\alpha}_s(t)} \bar{\alpha}_s(t') \mathcal{K}_0(t, t')g_\omega(t') = \omega g_\omega(t) \quad (2.9)
\]

These eigenfunctions form a complete orthonormal set

\[
\int d\omega g_\omega(t)g_{\omega'}(t) = 2\pi \delta(\omega - \omega') \quad (2.10)
\]

assuming (for the moment) a continuous spectrum for the eigenvalues, \(\omega\).

They can be obtained by defining a set of functions \(f_\omega(t)\):

\[
f_\omega(t) = \frac{g_\omega(t)}{\sqrt{t}} \quad (2.11)
\]

which obey the eigenvalue equation

\[
\bar{\alpha}_s(t) \int dt' \mathcal{K}_0(t, t')f_\omega(t') = \omega f_\omega(t) \quad (2.12)
\]

Using Eq. (2.5) we have

\[
\int dt' \mathcal{K}_0(t, t')f_\omega(t') = \bar{\beta}_0 \omega f_\omega(t) \quad (2.13)
\]

Taking the Fourier transform

\[
f_\omega(t) = \int d\nu e^{i\nu t}f_\omega(\nu) \quad (2.14)
\]

and using Eq. (2.7) we have a first-order differential equation

\[
\frac{d}{d\nu} f_\omega(\nu) = -i \chi_0(\nu)f_\omega(\nu) \quad (2.15)
\]

which has a well known solution

\[
f_\omega(\nu) = \exp\left[-i \bar{\beta}_0 \omega \int \nu(\nu')d\nu'\right] \quad (2.16)
\]

In this way we obtain

\[
f_\omega(t) = \frac{1}{\sqrt{2\pi\bar{\beta}_0 \omega}} \int_{-\infty}^{+\infty} d\nu e^{i\nu t} \left[\frac{\Gamma(1/2 + iv')}{\Gamma(1/2 - iv')}\right]^{1/(2\bar{\beta}_0 \omega)} \quad (2.17)
\]

where the pre-factor is taken such that the normalization condition \((2.10)\) is obeyed.

The integral over \(\nu\) can be performed numerically over a suitable contour. A very good approximation to this integral (for small \(\omega\)) is obtained by the saddle-point approximation, (equivalent to solving Eq. (2.13) using the semi-classical approximation). The saddle point, which is a function of \(t, \nu_s(t)\), is obtained from the solution to

\[
\chi_0(\nu_s(t)) = \bar{\beta}_0 \omega t. \quad (2.18)
\]

We consider two regions depending on whether \(t\) is greater or less than a critical point, \(t_c\), given by

\[
\bar{\beta}_0 \omega t_c = \chi_0(0) = 4\ln 2. \quad (2.19)
\]

\(t > t_c\):

In this case there is a single saddle-point on the positive imaginary axis, shown in Fig. 1. If we define \(\gamma_s\) by

\[
\gamma_s = \frac{1}{2} + iv_s \quad (2.20)
\]

then at the saddle-point, \(\gamma_s\), is the solution to

\[
\chi_0(\gamma_s) = 2 \Psi(1) - \Psi(\gamma_s) - \Psi(1 - \gamma_s) = \bar{\beta}_0 \omega t \quad (2.21)
\]
Moreover, real axis but for very large $\nu$ near the saddle point the contour runs parallel to the imaginary axis. In the saddle-point approximation, we obtain (using Eq. (2.21))

$$f_\omega(t) = \frac{2}{\sqrt{\gamma_s}(v_s)} \sin \left( v_s t + \frac{\phi(v_s)}{\beta_0 \omega} + \frac{\pi}{4} \right),$$

(2.25)

where

$$\phi(v_s) = \text{Arg} \left( e^{-2i\psi(1)v_s \Gamma(1/2 + iv_s)} \right).$$

(2.26)

The inclusion of $\pi/4$ in the phase in Eq. (2.25) ensures a matching of the solutions at $t = t_c$. Near $t = t_c$ the solution is given by an Airy function.

For $t < t_c$, we have an oscillatory solution which does not lend itself to a match to the DGLAP formalism—in this regime a DGLAP analysis is not appropriate, since in this region the saddle-point $\gamma_s$ is complex and double-valued, i.e.

$$\gamma_s = \frac{1}{2} \pm i|v_s|,$$

and cannot be related to the (real) anomalous dimension of the DGLAP formalism. In this region the DGLAP equation is not valid and the BFKL equation can be considered as a generalized (quantized) version of the DGLAP equation.

Recall that the saddle-point, $v_s(t)$, is a function of $t$ and so we do not have constant frequency oscillations. As $t \to 0$, $v_s$ tends to a value $v_0 \approx 0.635$ and we have constant frequency oscillations in the infrared limit. As $t$ increases this frequency decreases, becoming zero at $t = t_c$. 

Fig. 1. Integration contour on the $v$ plane for $t > t_c$. The black dot shows the position of the saddle point, $v_s$.

Fig. 2. Integration contour on the $v$ plane for $t < t_c$. The black dots show the positions of the saddle points, $\pm|v_s|$.
The infrared phase at \( t = 0 \), calculated from perturbative QCD (with \( t > 0 \)), is then given by

\[
\eta_0 \pi = \frac{\phi(v_0)}{\beta_0 \omega} + \frac{\pi}{4}, \quad (2.27)
\]

(\( \phi(v_0) \approx 0.96 \)). This phase is only determined up to a multiple of \( \pi \). We now make a very general assumption that the infrared properties of QCD fixes this phase (in general as a function of \( \omega \)) to be \( \eta(\omega) \), where the function \( \eta(\omega) \) is determined from the non-perturbative regime of QCD \( t \leq 0 \). The matching of the two phases \( \eta_0 \) and \( \eta(\omega) \) in the semi-classical solution, Eq. (2.25), then restricts the allowed values of \( \omega \) to a discrete set \( \omega_n \) that satisfy the equation

\[
\frac{\phi(v_0)}{\pi \beta_0 \omega_n} = \eta(\omega_n) + \left( n - \frac{1}{4} \right) \quad (n = 1, 2, 3 \ldots). \quad (2.28)
\]

The function \( \eta(\omega) \) could be a constant (as originally proposed in [12]) but in general it can vary with \( \omega \). Because of periodicity it can take values in the interval between 0.25 and -0.75 only. Although \( \eta(\omega) \) cannot be determined from the perturbative analysis described here, its restricted range limits its effect on the determination of the eigenvalues (see Section 2.5). However, its variation with \( \omega \) is very important in the construction of the gluon density, (see Section 3).

The above analysis shows clearly that the solution of the BFKL equation has to be given by the set of discrete eigenfunctions, whose support in the virtual gluon transverse momentum is determined by the critical point, \( t_c \), Eq. (2.19), and whose phase, \( \eta(\omega) \), at some low transverse momentum is determined by the non-perturbative sector of QCD. Effectively the boundary conditions modify the BFKL kernel so that it may be written as

\[
\mathcal{H}(t, t') = \sum_n \omega_n f_n(t)f_n(t'). \quad (2.29)
\]

This coincides with the (LO) kernel

\[
\bar{\sigma}_v(t) K_0(t, t')
\]

provided it acts on a function \( f(t) \) which may be written as a superposition of the eigen-functions \( f_n(t) \):

\[
f(t) = \sum_n a_n f_n(t).
\]

In this way we have supplemented the kernel with both infrared and ultraviolet boundary conditions. The infrared boundary conditions arise from the non-perturbative sector of QCD, but the ultraviolet boundary conditions arise naturally from the asymptotic freedom of QCD. Importantly, the behaviour of the eigenfunctions in the ultraviolet is controlled by a critical value, \( t_c \), of transverse momentum, which grows almost linearly with \( n \) in accordance with the fact that the period of oscillations is practically independent on \( n \). Therefore, the \( n \)-dependent boundary condition leads to qualitatively different results from those obtained using a kernel in which boundary conditions are effected simply by a cutoff on a wave function.

Furthermore we note that the value of this critical transverse momentum depends almost entirely on the eigenvalue, \( \omega \), which decreases like \( 1/n \), for large \( n \), as the quantum number \( n \) increases, (see Eq. (2.28)). This means that in turn the value of the critical transverse momenta, \( k_c \), increases exponentially as \( \omega \) decreases, so that the \( n \)-th critical momentum is given by (inserting Eq. (2.28) into Eq. (2.19))

\[
k^{(n)}_c = \Lambda_{QCD} \exp \left\{ \frac{2\pi \ln 2}{\phi(v_0)} \left[ n - \frac{1}{4} + \eta(\omega_n) \right] \right\} \approx \Lambda_{QCD} \exp^{4.5n}. \quad (2.30)
\]

Finally let us note that our solution of the BFKL equation is similar to the WKB method for the boundstate solution of the Schrodinger equation in the semiclassical approximation; the critical point, \( t_c \), is analogous to the turning point \( x_c \) where the potential is equal to the energy. Inside a potential well the solutions are oscillatory and outside they decay exponentially. This shows that the solution of the BFKL equation (1.1) consists of the superposition of the bound state eigenfunctions of the two gluon system with pseudo-energies given by the eigenvalues \( \omega_n \). Knowledge of the eigenvalues gives important information about the interactions between gluons, both in the infrared and ultraviolet regions of \( k \). We note that in the solution of the BFKL equation, the oscillations of the eigenfunctions at large \( k \) should cancel each other in accordance with the kinematical constrains provided by the beam energy of the experiment. This imposes additional restrictions on the non-perturbative phases \( \eta(\omega) \), see below.

### 2.2. Threshold Effects

The above analysis assumes that \( \beta_0 \) is a constant so that the coupling \( \bar{\sigma}_v(t) \) is given simply by Eq. (2.5). However, we know that there are thresholds at \( t = t_i \), where heavy flavour quarks can be produced, and also there may be extra thresholds arising from BSM physics with a threshold (that according to Eq. (2.30) can be large) below \( t = t_c \). This means that Eq. (2.17) can only be used as a solution for \( f_n(t) \) between thresholds. As an example, suppose that there is only one threshold, at \( t = t_i \) below the critical point, \( t_c \), and that \( \beta_0 \)
takes the value $\beta_0^>$ above this threshold and $\beta_0^<$ below. At $t \geq t_c$, we have
\[
 f_s(t) = \frac{1}{\sqrt{2\pi\beta_0\omega}} \int \! dv \, e^{i\beta_0\omega v} e^{i\phi(v,t)} \omega v \, f_s(t),
\]
(2.31)
and for $t < t_c$ we have
\[
 f_s(t) = A \int \! dv \, e^{i\phi(v,t)} e^{i\beta_0\omega v} f_s(t_c)
\]
(2.32)
with the constant $A$ chosen to be
\[
 A^{-1} = \int \! dv \, e^{i\phi(v,t)} e^{i\beta_0\omega v} f_s(t_c),
\]
(2.33)
so that the solutions match at $t = t_c$.

In the saddle-point approximation, we can handle such thresholds by noting that Eq. (2.17) can be written as
\[
 f_s(t) = \frac{1}{\sqrt{2\pi\beta_0\omega}} \int \! dv \, e^{i\phi(v,t)} e^{i\beta_0\omega v} f_s(t_c)
\]
(2.34)
where the “action” $S(v, t)$ is given by
\[
 S(v, t) = \omega v t - \frac{1}{\beta_0} \int \! \chi_0 \omega \, dv'.
\]
(2.35)

At the saddle-point, $v = v_s(t_c)$, upon integrating by parts this may be rewritten as
\[
 S(t) = \omega \int \! v_s(t') dt',
\]
(2.36)
where the function $v_s(t)$ is given by Eq. (2.18). (We have used the notation $S(t)$ to denote $S(v_s(t), t)$—it is now a function of $t$ only). Here we see explicitly that the saddle-point approximation for integral (2.34) is equivalent to the semi-classical approximation. Replacing the integral over $v$ in Eq. (2.34) by the value of the integrand at the saddle-point we obtain a solution which obeys the differential equation
\[
 \frac{d}{dt} f_s(t) = \chi^{-1} (\beta_0\omega t) f_s(t).
\]
(2.37)

The semi-classical approximation consists of the assumption that the solution to Eq. (2.13) is the solution to Eq. (2.37) multiplied by a slowly varying factor, which turns out to be the same as that obtained in the Gaussian integral around the saddle-point in Eq. (2.34). In analogy with the WKB approximation in the Schrodinger equation there exist a critical point, $t_c$, at which the approximate solution changes from an oscillatory function to an exponentially decaying one. The assumption of a slowly varying pre-factor breaks down at this point, but the solutions either side of the critical point can be matched using a suitable Airy function. It is this matching, together with some property of the behaviour at $t = 0$ that determines the allowed eigenvalues.

Thresholds are handled in general by replacing Eq. (2.18) by the more general relation
\[
 \chi_s(v_s(t)) = \frac{\omega}{\alpha_s(t)}.
\]
(2.38)
$\alpha_s(t)$ may now be determined using the $\beta$-function with appropriate thresholds.

The eigenfunctions for $t \leq t_c$ now take the form
\[
 f_s(t) = \frac{C}{\sqrt{\lambda_s \nu_s(t)}} \sin \left( \frac{S(t)}{\omega} + \frac{\pi}{4} \right)
\]
(2.39)
and the semiclassical quantization condition on the allowed eigenvalues becomes
\[
 \frac{S(t)}{\omega_n} = \left( \eta(n) + n - \frac{1}{4} \right) \pi.
\]
(2.40)

Comparing this with Eq. (2.28) we see that $S(0)$ is independent of $\omega$. However, its value clearly depends on the positions of the thresholds in $\beta_0$, Eqs. (2.31) and (2.32). Already for $n \geq 3$ the value of $k_c$ given by Eq. (2.30) exceeds many tens of TeV and this means that the spectrum is sensitive to any BSM physics. The BSM effect changes the perturbative phase $\eta$ if the corresponding threshold, $k_c$, is below the critical point, $k_c^{(n)}$. The effect of such a threshold can be readily estimated; let us assume that for a given eigenvalue, $\omega$, the threshold $t_t = \ln(k_c^{(n)} / \Lambda_{QCD})$ is the largest threshold, below the critical point $t_c$, and that in the range
\[
 t_t < t < t_c,
\]
$\beta_0$ takes the value $\beta_0^\sim$. The exact solution for the LO BFKL with running coupling at $t = t_t$ is
\[
 f(t_t) \propto \int \! dv \, e^{i\phi(v, t_t)}.
\]
(2.41)

\footnote{For sufficiently large $n$ we can see this directly since $S(0) \sim C(\omega_n \theta_c^{(n)})$ and whereas $\theta_c^{(n)} \propto 1/n$.}
where

\[
\phi(v, t) = v\left(t - 2\frac{\Psi(1)}{\bar{\beta}_0^c\omega}\right) + \frac{2}{\bar{\beta}_0^c\omega} \arg\left\{\Gamma\left(\frac{1}{2} + iv\right)\right\}.
\]

(2.42)

We evaluate this integral using the saddle-point method and assuming that the threshold occurs sufficiently close to the critical point \(t_c\), that \(v_t\) is sufficiently small for the diffusion approximation to be valid. The phase difference between the case where there is a threshold at \(t = t_c\) and the case where there is no threshold is then

\[
\Delta \phi = \frac{2}{3\omega_0^c\sqrt{14\xi(3)}} \left[ \frac{1}{\bar{\beta}_0^c} (4 \ln 2 - \bar{\beta}_0^c \omega t_c)^{3/2} - \frac{1}{\bar{\beta}_0^c} (4 \ln 2 - \bar{\beta}_0^c \omega t_c)^{3/2} \right].
\]

(2.43)

where \(\bar{\beta}_0^c\) denotes the value of \(\bar{\beta}_0\) below the threshold.

We observe that the above phase difference is substantial for \(\omega \leq 0.1\), and that this difference is not suppressed by the scale of the BSM physics. In this sense our analysis differs fundamentally from the treatment of the Standard Model as a low-energy effective theory in a way below the thresholds of new physics. In the latter case, logarithmic corrections can always be absorbed into unphysical renormalization constants of the renormalizable operators of the effective theory, leaving only higher dimension operators whose coefficients are suppressed by powers of the new-physics mass scale.

### 2.3. NLO Evaluation

We have shown in [4] that the BFKL integral equation can be cast in the form of the pseudo-differential equation

\[
\bar{\alpha}_\omega(t) \int dt' \mathcal{K}_\omega(t, t') f_\omega(t') = \chi_t(-i \frac{d}{dt} \alpha_\omega(t)) f_\omega(t) = \omega f_\omega(t).
\]

(2.44)

Making the simplifying assumption that \(\bar{\alpha}_\omega\) is given by Eq. (2.5) and taking into account the collinear resummation [10] we can write the BFKL equation in the next-to-leading order as

\[
\bar{\beta}_0^c \omega f_\omega(t) = \left[\tilde{\chi}_\omega(\hat{v}, \omega) + \frac{1}{t \bar{\beta}_0^c} \tilde{\xi}(\hat{v})\right] f_\omega(t),
\]

(2.45)

where the operator \(\hat{v} = -i d/dt\),

\[
\tilde{\chi}_\omega(\hat{v}, \omega) = 2\Psi(1) - \Psi\left(\frac{1}{2} + iv + \frac{\omega}{2}\right) - \Psi\left(\frac{1}{2} - iv + \frac{\omega}{2}\right)
\]

and

\[
\tilde{\xi}(v) = \chi_t(v)
\]

\[+ \frac{1}{2} \left[ 2\Psi(1) - \Psi\left(\frac{1}{2} + iv\right) - \Psi\left(\frac{1}{2} - iv\right) \right] \times \left\{ \Psi\left(\frac{1}{2} + iv\right) + \Psi\left(\frac{1}{2} - iv\right) \right\},
\]

where \(\chi_t\) denotes the NLO characteristic function [9]. The function \(\tilde{\xi}(v)\) does not have the poles of the third order of the form \(\sim 1/(1/2 \pm iv)^3\), in agreement with the renormalization group analysis [9].

The Eq. (2.45) can be considered as a quadratic equation in \(t\)

\[
[t \bar{\beta}_0^c - \tilde{\chi}_\omega(\hat{v}, \omega) - \tilde{\xi}(\hat{v})] f_\omega(t) = 0.
\]

(2.46)

We can convert this into a second order differential equation for the Fourier transform \(\hat{f}_\omega(v)\), where \(t\) is replaced by the operator \(\hat{t} = i d/dv\). In the semi-classical approximation in which \(\ln(\chi_\omega)\) and \(\ln(\tilde{\xi})\) are treated as slowly varying functions of \(v\), so that

\[
\hat{t}^2 \hat{f}_\omega \approx (\hat{t} \ln(\hat{f}_\omega)) \hat{f}_\omega,
\]

this second-order differential equation may be written as

\[
[i \bar{\beta}_0^c \hat{t} - X^\pm(v, \omega)] \hat{f}_\omega(v) = 0,
\]

(2.47)

where

\[
X^\pm(v, \omega) = \frac{1}{2\omega} \tilde{\chi}_\omega(\hat{v}, \omega) \pm \sqrt{\frac{1}{2\omega}} \tilde{\xi}(\hat{v}) \chi_t(v, \omega) + \frac{1}{\omega} \tilde{\xi}(v),
\]

with solution for \(f_\omega(t)\) which is analogous to the LO expressions of Eqs. (2.14) and (2.16):

\[
f_\omega(t) = \int_{-\infty}^{\infty} dv e^{ivt} \exp\left(-\frac{i}{\bar{\beta}_0^c} \int X^\pm(v, \omega) dv\right).
\]

(2.48)
For small \( \omega \), Eq. (2.48) may be approximated by
\[
f_{\omega}(t) = \frac{1}{\sqrt{2\pi \omega}} \int_{-\infty}^{+\infty} d\nu e^{i\nu t} \left[ \frac{\Gamma(1/2 + i\nu)}{\Gamma(1/2 - i\nu)} e^{-2i\nu \nu(1)} \right]^{1/|\nu|}\rho_{|\nu|}(0)\] (2.49)
Note that the saddle-point of the integral over \( \nu \) in Eq. (2.49) occurs at \( \nu_s \), where the RHS of Eq. (2.45) vanishes as required for \( \tilde{f}_\omega(\nu_s) \) to be a turning-point.

In a more general NLO approach, the BFKL Eq. (2.44) can be simplified using the semi-classical approximation, i.e. assuming that the \( t \)-dependence of \( \ln f_{\omega}(t) \) is large so that
\[
\left( \frac{d}{dt} \right)^{\nu} f_{\omega}(t) \approx f_{\omega}(t) \left( \frac{d\ln f_{\omega}(t)}{dt} \right)^{\nu}. \] (2.50)
The Eq. (2.44) looks then like the non-linear differential equation
\[
\chi\left( -\frac{d\ln f_{\omega}(t)}{dt}, \alpha_s(t) \right) = \chi(\nu(t), \alpha_s(t)) = \omega. \] (2.51)
As a result the frequency \( \nu(t) \) is a function of \( t \) such that
\[
\omega = \left( \frac{\alpha_s(t)\lambda_s}{\pi} \right) \chi_{\beta}(\nu) + \left( \frac{\alpha_s(t)\lambda_s}{\pi} \right)^2 \chi_{\eta}(\nu) + \ldots \] (2.52)
The expression (2.52)—including collinear resummation [10], is the NLO analogue of the Eq. (2.38). The Eq. (2.51) has a solution
\[
f_{\omega}(t) = e^{S(t)/\omega}, \] (2.53)
where
\[
S(t) = \omega \int_{t_0}^{t} \nu(t') dt', \] (2.54)
The critical logarithmic transverse momentum, \( t_\omega \), is the value of \( t \) for which \( \nu(t) = 0 \). This condition is the NLO analog of Eq. (2.19). For \( t < t_\omega \), there are two real solutions for \( \nu(t) \) generating an oscillatory solution with a given phase, whereas for \( t > t_\omega \) the solution is on the positive imaginary axis, generating an exponentially decaying function as \( t \to \infty \).

Thus we see, that the solution for the eigenfunctions in semi-classical approximation is analogous to that in leading order, but the function \( \nu(t) \) takes into account the NLO characteristic function as well as the NLO running of the coupling and the threshold effects. A further feature of threshold effects beyond leading order is that it is not only the \( \beta \)-function that has steps at the thresholds but also the NLO contributions to the characteristic function, \( \delta_{\eta} \), —corre-
\[ \alpha_s(k) \]

**Fig. 3.** The running of \( \alpha_s \) across a threshold for \( N = 1 \) SUSY at 10 TeV.

\( t_0 \), which can be safely reached by the perturbative calculation (see also the discussion in Section 3.2).

### 2.4. \( N = 1 \) Supersymmetry at Various Thresholds

We have chosen as example of “new physics” the popular \( N = 1 \) supersymmetric extension of the Standard model above a given threshold in \( k_T \), which for simplicity we assume to be a common mass threshold for all superpartners. Below this threshold the running of the coupling is governed by the \( \beta \)-function to two-loop order

\[ (2.58) \]

\[ \beta_\alpha = -\frac{\alpha_s}{4\pi} \left( \frac{11C_A}{3} - \frac{2}{3}n_f \right) \]

where for the case of QCD, \( C_A = 3, \ C_F = 4/3 \) and \( n_f \) is the number of active flavours. Above the threshold, the beta function is given by

\[ (2.59) \]

\[ \beta_\alpha = -\frac{\alpha_s}{4\pi} \left( 3C_A - n_f \right) - \frac{\alpha_s^3}{(4\pi)^2} \left( 6C_A^2 + \left( \frac{2C_A}{3} + 2C_F \right)n_f \right) \]

This leads to a “kink” (discontinuity in the derivative) in the running of \( \alpha_s \) at the threshold for \( N = 1 \) SUSY, which can be seen in Fig. 3.

Furthermore, above the SUSY threshold, the NLO characteristic function, \( \chi_1(v) \) acquires an additional contribution [14] of

\[ (2.60) \]

\[ \delta \chi_1(v) = \frac{\pi^2}{32} \frac{\sinh(\pi v)}{v(1 + v^2)\cosh^2(\pi v)} \left( \frac{11}{4} + 3v^2 \right) \]

from the octet of Majorana fermions (gluinos), and

\[ (2.61) \]

\[ \delta \chi_1(v) = \frac{\pi^2 n_f}{32v C_A^3 v(1 + v^2)\cosh^2(\pi v)} \left( \frac{5}{4} + v^2 \right) \]

from the squarks (in the fundamental representation).

Typical graphs contributing to \( \delta \chi_1(v) \) are shown in Fig. 4. They contribute only at NLO level since the exchange of a fermion or scalar particle in the \( t \)-channel is suppressed in LLA [9], and therefore only contributes at subleading logarithm order.

### 2.5. The Discrete Pomeron with and without SUSY

In this sub-section we investigate the properties of the Discrete BFKL Pomeron with and without SUSY contributions. For this example, we have assumed that the SUSY threshold is at 10 TeV. Figure 5 shows the spectrum of the eigenvalues \( \omega_n \) computed in the NLO computation assuming that the non-perturbative phase \( \eta(\omega_n) \) of the eigenvalue condition, Eq. (2.40), is \( \eta = -0.25 \), for all eigenfunctions. The perturbative phase, \( S(0) \), of Eq. (2.40), at the infrared boundary \( t = \)
0, is replaced by \( S(k_0) \), with \( k_0 = \Lambda_{\text{QCD}} \exp(t_0/2) \) and \( k_0 = 0.6 \text{ GeV} \). The eigenvalues determined with and without SUSY effects differ substantially for \( n \geq 3 \) whereas for \( n < 3 \) they show no difference. This is understandable from the Appelquist–Carrazone theorem [7] because the assumed SUSY threshold, that we have chosen, lies between the critical momenta for the second eigenfunction \( (k_c \sim 1 \text{ TeV}) \) and the third eigenfunction \( (k_c \sim 100 \text{ TeV}) \). The \( k_c \) values computed at NLO, with and without the SUSY threshold, are shown in Fig. 6. They turn out to be very close to the leading order values calculated from Eq. (2.30)—the difference being due to the fact that \( \alpha_s \) runs more rapidly for NLO than for LO. Furthermore these critical momenta show only small dependence on the presence on SUSY threshold.

On the other hand, the eigenvalues, \( \omega_n \), which are important for the description of the HERA structure-function data at low-\( x \), are very sensitive to possible threshold effects, i.e. they differ substantially already in LO and the differences are much larger than any possible uncertainties due to the unknown phase \( \eta \). For example, in LO, Eq. (2.28), the ratio of \( \tilde{\beta}_0 \)'s below and above the SUSY threshold is 7/3, which means that already for \( n \geq 3 \) the effect of the change in \( \tilde{\beta}_0 \) on the eigenvalues, is much larger than the maximal possible effects due to the uncertainty in \( \eta \) (\( \eta \) can only vary between \( \eta = 0.25 \) and \( \eta = -0.75 \)). At NLO, we find that the phase independent discrepancy between the eigenvalues with and without the SUSY threshold starts at \( n \geq 5 \) (see Fig. 5). These substantial differences are related to the fact that for \( \omega < 0.1 \), the change of phase of an eigenvalue, \( \Delta \phi \), arising from the change in \( \tilde{\beta}_0 \) as one crosses the SUSY threshold is large (as can be seen from Eq. (2.43)) and indeed much larger than the maximal possible \( \eta \) change, \( \Delta \eta \leq 1 \).

Since the properties of the eigenvalues are determined by the behaviour at very high virtualities, (of the order of \( k_c \)), it should be expected that the eigenvalues computed in NLO should approach the LO ones at large \( n \). Figure 7 shows the comparison of the eigenvalues computed using the NLO and LO approximations and confirms this expectation. The LO computation was made using Eq. (2.28) with \( \tilde{\beta}_0 \) values computed with \( n_f = 6 \) below the SUSY threshold of 10 TeV. It is interesting to observe that LO and NLO results approach each other more slowly in the case of the

\footnote{For sufficiently small \( \omega \), the NLO effects both in \( \tilde{\beta}_0 \) and the characteristic function, \( \chi \), become negligible.}
SM + SUSY than in the SM alone. This slower approach is due to the fact that $\alpha_s$ runs more slowly above the SUSY threshold. This means that the eigenvalues, $\omega_n$, approach zero at a different pace, as can be seen from the figure. We note that for small $\omega$, the eigenvalues are very closely packed and so the effect of the discrete nature of the solutions becomes less important. To a good approximation we could replace the sum over the eigenfunctions for small $\omega$ by an integral over a range of small $\omega$.

However, it is important to note that the Jacobian for the transition from a discrete sum to an integral is proportional to the gradient of the $\omega - n$ distribution shown in Fig. 7 and this is different in the two cases—leading to different pomeron amplitudes.

In addition to the change of the running of $\alpha_s$, there are also effects due to the $\delta_{k_i}$ contributions to $\chi_1$ which set in above the SUSY threshold, Eq. (2.60) and Eq. (2.61). It is this discontinuity which is responsible for the discontinuities in the frequencies $\nu$ at threshold, see Fig. 8, and not the change in the rate of running of the coupling, which remains a continuous function. The change in frequency thus compensates for the change in the characteristic function in order to ensure that the eigenvalues, $\omega_n$, remain unchanged as one passes through the threshold. The contribution, $\chi_1$, of these additional terms is shown as a function of frequency in Fig. 9 where it can be seen that this is a rapidly decreasing function, which explains why the discontinuities in frequency at threshold are much larger for the lower eigenfunctions for which the frequency at threshold is lower.

For lower $n$ eigenfunctions, the change of the frequencies due to the SUSY threshold leads also to the change of its shape. In Fig. 10 we show a representative subset of eigenfunctions in the Standard Model and the SUSY model in the transverse momentum region relevant for a fit to HERA data. The eigenfunctions are shown with values of $\eta = -0.25$ and $\eta = +0.25$ (in order to give an indication of the sensitivity of the allowed eigenvalues to the unknown infrared phases), with and without SUSY at a threshold of 10 TeV. As expected, the first two eigenfunctions are identical since their values of $k_c$ are below the SUSY threshold. The third and higher eigenfunctions display significant differences for both displayed $\eta$ values. Remarkably,

$$\delta\chi_1^{(SUSY)}$$

![Fig. 8. Oscillation frequencies as a function of gluon transverse momentum for various eigenfunctions. The left-hand panel is the case of the Standard Model and the right-hand panel is the case of $N = 1$ SUSY above a threshold of 10 TeV. For the purpose of this comparison it has been assumed that the infrared phases are the same in both cases.](image)

![Fig. 9. Decrease in the NLO characteristic function, $\chi_1$ as a function of frequency $\nu$.](image)
these differences diminish for higher eigenfunctions and for $n > 50$ the two eigenfunctions almost overlap in the displayed $k_T$ region (relevant for a fit to HERA data). The reason for this can be seen from Fig. 8, which shows that for the relatively low transverse momenta the differences in the frequencies between the two models decreases with increasing eigenvalue number, so that if the infrared phases are equal, the functions will be almost identical in this region.

In summary we can state that the Discrete BFKL Pomeron shows a clear sensitivity to BSM physics effects and that these effects cannot be absorbed into its only free parameters, the infrared phase $\eta(\omega)$. This is clearly seen in the eigenvalue dependence on the SUSY thresholds, both in the LO analytical approach, Eqs. (2.28) and (2.43), and in the NLO numerical evaluation, Fig. 5. In addition, the asymptotic behaviour of the eigenvalues with increasing $n$ (or decreasing $\omega$) is very different for SM and SM + SUSY irrespective of the possible higher order QCD corrections, Fig. 7. This means that we have here a very different situation from the scenario described by the decoupling theorem [7] where the large logarithmic corrections can be absorbed into unphysical renormalization constants leaving only higher dimension operators whose coefficients are suppressed by powers of the new-physics mass scale. In the case of the DP the effects of SUSY thresholds produce large changes of frequencies, Fig. 8, which modify the infrared perturbative phases $\eta$. These alter, in turn, the spectrum $\omega_n$ and hence the properties of the gluon density. Since the gluon density is a measurable quantity the non-perturbative phases, $\eta$, can also be measured although only indirectly.

3. COMPARISON WITH HERA DATA

3.1. General Considerations

One of the most important results of the HERA experiments is the measurement of the gluon density.
This density encompasses the properties of the pomeron in the sense that the same gluon density determines the dynamics of the inclusive $\gamma^* p$ (or $F_2$) and diffractive processes, in particular the exclusive vector meson production. Several investigations performed in the context of the dipole models [18–25] have shown that the effective intercept of the gluon density measured by the rise of $F_2$ with diminishing $x$, called $\lambda$, is properly translated by the optical theorem, to the effective intercepts seen in the exclusive vector meson production.

The effective intercept $\lambda$ measured at HERA varies between $\lambda \approx 0.2$ at $Q^2 = 10$ GeV$^2$ to about $\lambda = 0.35$ at $Q^2 = 100$ GeV$^2$, see Fig. 9 of [4]. The $Q^2$ dependence of $\lambda$ in $F_2$ and in its diffractive counterparts can be well reproduced by the DGLAP evolution in which the values of $\lambda$ are almost entirely of perturbative origin. In the well-known Donnachie–Landshoff (DL) [27] picture of the Pomeron the variation of $\lambda$ with $Q^2$ is due to the existence of a hard ($\lambda = 0.4$) and a soft, non-perturbative, ($\lambda = 0.08$) Pomeron which ad-mixtures varies with $Q^2$.

The properties of the gluon density corresponding to the DP are determined by the Green function constructed from the discrete eigenfunctions of the BFKL kernel (convoluted with the proton impact factor). In contrast to the DL Pomeron the DP is composed of infinite many eigenfunctions with eigenvalues varying like $\omega_n \approx 0.5/n$. The eigenvalues $\omega_n$ are almost entirely of perturbative origin because its only non-perturbative ingredients are the infrared phases, $\eta_n$, which have a negligible importance for larger $n$, as was explained in the previous section.

The infrared phases have, however, a strong influence on the shape of the gluon density since they determine how the contributing eigenfunctions add together. Let us recall that the un-integrated gluon density from the DP is of the form

$$\bar{g}(x, k^2)_{DP} = \frac{1}{x} \lambda^2 \sum_{n=1}^{n_{\text{max}}} \left( \frac{k}{\lambda} \right)^{\omega_n} A_{\phi_n}(k, \eta_n), \quad (3.1)$$

where $\bar{g}$ means differentiation of the gluon density w.r.t. $\ln(k^2)$. Here, the eigenfunctions as a function of $k$ (rather than $t$) are normalized w.r.t. $k$ and are related to $f_n(t)$ by

$$f_n(k, \eta_n) = \frac{1}{k} f_n \left( \ln \left( \frac{k^2}{\Lambda^2} \right) \right). \quad (3.2)$$

$A_{\phi}$ is the overlap integral of these eigenfunctions with the proton impact factor, Eq. (3.3), with the eigenfunctions $f_n(k, \eta_n)$ computed with a specific $\eta - n$ relation, $\eta_n$ (after accounting for the non-zero overlaps of the eigenfunctions of the non-Hermitian kernel,---for details see [4]). The sum over eigenfunctions in Eq. (3.1) is limited for numerical reasons (see below) to $n_{\text{max}} = O(100)$. The oscillation frequencies of the eigenfunctions, at transverse momenta relevant to HERA, vary very little from one eigenfunction to the next, so in order to obtain a positive gluon density, which grows with $k$, it is necessary to generate a strong $\eta - n$ (or equivalently $\eta - \omega$) dependence; the eigenfunctions oscillate in $\ln k$ and the only way to cancel these oscillations is to introduce a shift of the phase between the different eigenfunctions.

The sum of Eq. (3.1) determines the evolution properties of the gluon density in agreement with the BFKL Eq. (1.1). The infrared phases are determined by the dynamics of non-perturbative QCD and it should be possible, in principle, to estimate them using lattice calculations. It should be also possible to determine them directly from data at comparatively small $Q^2$. However, in order to be able to extract these phases accurately, the required data set should have a much larger $x$ and $Q^2$ range than the presently available HERA measurements. In addition the data set has to achieve the precision of the present HERA $F_2$ data.

Therefore at present, to be able to confront the DP with data, we have no other choice but to construct a heuristic model for the infrared boundary condition based on the general understanding of the non-perturbative physics. We first postulate that the form of the infrared phase function, $\eta(\omega)$, dictated by the infrared behaviour of QCD, is not sensitive to BSM physics and is a smooth function of $\omega$ rather than an arbitrary number for each eigen-function. We then choose to describe it in terms of a suitable parameterization, motivated by a similarity of the BFKL dynamics with the Schrodinger equation, described in detail below. In addition to the phases, we have also to specify the proton impact factor.

We consider this heuristic approach as a first step towards the determination of the infrared phases, which are important quantities of the non-perturbative QCD. The main purpose of the investigation reported in [4] was to check whether a physically plausible boundary condition provides a good description of data, i.e whether the Discrete BFKL Pomeron can describe the dynamics of the measured gluon density. The main purpose of the present investigation is to find out whether the genuine sensitivity of the DP to the presence of BSM physics at high-energy can improve or worsen the quality of the fit to data, notwithstanding the uncertainties associated with the infrared phase conditions.

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8 We have shown in [4] that an adequate description of HERA $F_2$ data requires $O(100)$ eigenfunctions. Nevertheless, the limit of $n_{\text{max}} \sim 100$ represents a model assumption which is sufficient for HERA data but which could be too low for evaluation of LHC DY data. For LHC data it could be also necessary to include the contributions of negative $\omega$’s, Eq. (2.4).
3.2. The Infrared Boundary

Our heuristic model of the infrared boundary consists of a set of physically well motivated assumptions about the proton impact factor and about the $\eta-n$ (or equivalently $\eta-\omega$) relation.

The proton impact factor has to be positive everywhere and concentrated at the values of $k < 0.1 \text{ GeV}$, in order to avoid "cross-over" between adjacent eigenvalues at L.O. We see that for large $\eta$ the factor and/or the exponent but found that the fit to data has no sensitivity to such alternatives. This is due to the fact that all eigenfunctions have a similar, oscillatory, shape near the infrared boundary and that the period of oscillations of the eigenfunctions is much larger than any physically possible support of the proton impact factor. Note that the form (3.3) vanishes as $\eta \rightarrow 0$, (at the two loop level), without approaching the Landau pole too closely. On the other hand, the infrared boundary conditions are to be imposed at a transverse momentum of order $Q_{\text{CD}}$. Moreover, we need to know the eigenfunctions below $k_0$ in order to perform a convolution with the proton impact factor, which has support mainly below $k_0$. Therefore, guided by the behaviour of the eigenfunctions in the perturbative region, we continue them down to a lower momentum $\tilde{k}_0$, which should be of order $Q_{\text{CD}}$, using the extrapolation of the phase $\phi_n(k)$

$$\phi_n(k) = \phi_n(k_0) - 2\nu_n^0 \ln\left(\frac{k}{k_0}\right), \quad (3.5)$$

where for each eigenfunction, with index $n$, $\nu_n^0$ is the frequency of the oscillations near $k = k_0$ [4]. We have assumed that this frequency is constant below $k_0$, an assumption which is correct for sufficiently small $k_0$, at least for the leading order BFKL kernel (see [12]). Any deviation from constant frequency should have a negligible effect as we are only extrapolating over a small range in gluon transverse momentum. The numerical values of $\nu_n^0$ are obtained by inverting the eigenvalue Eq. (2.52), modified according to [10].

The overlap integrals between the proton impact factor and the eigenfunctions must also start at $\tilde{k}_0$ (the support of these impact factors being significantly attenuated at $k_0$). We therefore use this momentum at which we impose the infrared phases of the eigenfunctions. The relation between the phases at $k_0$ and $\tilde{k}_0$ is given by Eq. (3.5). We leave the exact value of $\tilde{k}_0$ as a free parameter with the restriction that it must be $\mathcal{O}(\Lambda_{\text{QCD}})$.
phase of the leading eigenfunction vanishes (as can be seen from Eq. (3.4)).

3.3. Results of the Fit

Before a comparison can be made with the measured structure function, \( F_2 \), it is necessary to convolute the gluon density with the impact factor for the virtual photon (for details of the procedure see Section 6 of [4]). The impact factor for the virtual photon is calculable in perturbative QCD and has support which is peaked at transverse momenta of the order of the photon virtuality, \( \sqrt{Q^2} \).

The fits were performed using the HERA data [5] in the low-\( x \) region, \( x < 0.01 \). To avoid any saturation effects we have limited the fit to the \( Q^2 > 8 \text{ GeV}^2 \) region. We recall that the saturation scale at HERA was determined to be \( Q_s^2 = 0.5 \text{ GeV}^2 \) at \( x \approx 10^{-3} \) [23, 24], therefore our choice of the \( Q^2 \) region is very conservative. This choice means that, in this paper, we concentrate on the one-pomeron exchange, without any multi-pomeron contributions, which could induce saturation effects. The saturation effects could also play a role without multi-pomeron effects through a modification of the boundary conditions, see [26], which in turn could modify our ansatz for the infrared boundary. In any case, our choice of the \( Q^2 \) region for fits assures that saturation effects can be ignored in this first evaluation. In future we plan to extend our analysis into the \( Q^2 \) regions which could be more sensitive to saturation.

In the region of \( Q^2 > 8 \text{ GeV}^2 \) we have a total of 108 data points and a total of 5 parameters—so the number of degrees of freedom is \( N_{df} = 103 \). We consider the \( Q^2 > 8 \text{ GeV}^2 \) region as our main investigation region and use the \( Q^2 > 4 \text{ GeV}^2 \) as a cross check.

As discussed in the previous paper [4], in order to obtain the most accurate estimate of the un-integrated gluon density we should include in the fit as many of the higher \( n \) eigenfunctions as possible. Indeed, we observe that the fit quality improves with increasing number of included eigenfunctions and the series converges in \( \chi^2 \). In principle this convergence should improve as \( n \to \infty \), however in practice, the number of eigenfunctions used in a fit is limited by the numerical precision of our calculation. We have indications that if we take significantly more than 100 eigenfunctions our fit could be polluted by numerical instabilities arising from an accumulation of computational rounding errors. Moreover, we find no improvement in the quality of our fits, either in the case of the Standard Model or for MSSM SUSY at any of the thresholds investigated, when the maximum number of eigenfunctions, \( n_{\text{max}} \) exceeds 100. We have therefore taken \( n_{\text{max}} = 100 \) throughout.

In table we show our fits for various SUSY thresholds as well as the Standard Model. Let us first note that the \( \tilde{k}_0 \) values obtained in the unbiased fit, \( \tilde{k}_0 \approx 275 \text{ MeV} \), are close to \( \Lambda_{\text{QCD}} \). At the same time the value of \( b \) implies that the proton impact factor peaks around \( \Lambda_{\text{QCD}} \), as expected for a self consistent description. This together with the relatively low \( \chi^2 \)'s of all fits confirms the success of our construction of the infrared boundary.

The quality of the fits shows a clear preference of the evaluation with SUSY effects; the fit for the Standard model is worse than the fits with SUSY thresholds larger than 3 TeV. A SUSY threshold of 3 TeV, which is close to the reach of LHC also gives a worse fit. On the other hand for a SUSY threshold in the region of 10–15 TeV, the quality of the fit is significantly improved, but that for significantly larger SUSY thresholds worsens again.

Let us note that the differences between the \( \chi^2 \) fits shown in table are very significant because the Maximum-Likelihood method, which assures that the minimum of \( \chi^2 \) provides the best estimate of the parameter values, also states that 1σ error in determination of parameter values is given by \( \Delta \chi^2 = 1 \), irrespective of the

| SUSY Scale, TeV | \( \chi^2 \) | \( \kappa \) | \( \tilde{k}_0 \) (GeV) | \( \eta_r \) | \( A \) | \( b \), GeV\(^{-2} \) |
|----------------|----------|---------|-----------------|-------|-----|--------|
| 3              | 125.7    | 0.555   | 0.288           | -0.87 | 201.2| 10.6   |
| 6              | 114.1    | 0.575   | 0.279           | -0.880| 464.8| 15.0   |
| 10             | 109.9    | 0.565   | 0.275           | -0.860| 693.1| 17.4   |
| 15             | 110.1    | 0.555   | 0.279           | -0.860| 882.2| 18.6   |
| 30             | 117.8    | 0.582   | 0.278           | -0.870| 561.6| 16.2   |
| 50             | 114.9    | 0.580   | 0.279           | -0.870| 627.4| 16.8   |
| 90             | 114.8    | 0.580   | 0.279           | -0.870| 700.2| 17.5   |
| \( \infty \)   | 122.5    | 0.600   | 0.294           | -0.795| 813.1| 18.2   |
number of degrees of freedom. Therefore, the differences between the fits of table are a multi-σ effect, within our model of the infrared boundary.

The Δχ² = 1 rule is valid as an estimate of the parameter error only for estimates within one theoretical framework, i.e. for one likelihood function. On the other hand, when χ² is used to quantify the agreement of different theories with data, it is expected that the observed χ² can deviate from the optimal value because it fluctuates with the probability density function, f(χ², Nd), which is approximately a Gaussian with the average value equal to Nd and the variance, σ² = 2Nd. In the case of fits, presented in the table, the expected χ² should be around 103 and σ = 14. Therefore the χ² values obtained for best fits with the SUSY mass C(10) GeV lie well within one standard deviation. The DGLAP fits have a χ²/Nd ≈ 0.95. This would lead for our sample to χ²/Nd ≈ 98, which is also within one standard deviation of the optimal value, so that one cannot conclude that either fit is better. The evaluation with the goodness of fit criterion called p-value, which is more appropriate for Nd > 100, gives a p-value ≈ 30% for SUSY masses of C(10) GeV, which is again an excellent result, see [16, 17].

As a check we also performed the fits with a lower Q² cut, Q² > 4 GeV². We find that in this Q² region there is a significant increase of χ²/Nd presumably due to various higher order effects, such as the NLO contribution to the photon impact factor or valence quarks effects and possibly also the proximity of the saturation region. Although the overall quality of the fit for all data with Q² > 10 GeV² is significantly worse than with Q² > 8 GeV² the preference for N = 1 SUSY with the threshold region of 10–15 TeV is also clearly seen. In the Q² > 4 GeV² region there are 128 points and the χ²s of the best fits are 184.3 (3 TeV), 164.5 (6 TeV), 155.6 (10 TeV), 152.6 (15 TeV), 169.7 (30 TeV), 164.7 (50 TeV), 164.3 (90 TeV) The best χ² for Standard Model is 169.7. The values of the fit parameters are similar to the values shown in table, for the Q² > 8 GeV² region.

4. DISCUSSION

The main result of this paper—namely a possible sensitivity of HERA data to BSM effects is especially astonishing as it was not seen in the usual solutions to the BFKL equation [1–3]. This sensitivity is due to the fact that we use the Green function method, in which the Green function is considered to be a universal property of the BFKL equation and that apart from supplementing this equation with a given set of infrared boundary conditions for the eigenfunctions, no cuts on transverse momentum are applied, so that the conformal invariance is broken in a very smooth way, solely by the running of the coupling constant.

Had we imposed kinematic limits on the gluon virtuality for the individual eigenfunctions, then for those eigenfunctions with ω ≲ 0.1, they would never reach the region in which they decay as required for compatibility with a DGLAP analysis in the DLL limit. Alternatively, one might attempt to fit HERA data only with the first eigenfuction, which does indeed start to decay within the HERA region, making the assumption that the higher eigenfunctions (with lower ω) only couple weakly to the proton. However, the properties of this first eigenfunction are in clear contradiction with data; for example, the rate of rise of F₁ with diminishing x would be independent of Q² with a value of λ ≈ 0.25 whereas the λ value in data varies between 0.15 to 0.35 in the observed Q² region. The results of [11] indicate that one needs at least four eigenfunctions to describe HERA data (making an artificial assumption that the overlap constants Aₙ of Eq. (3.1) are totally unconstrained), which extends the virtuality region to kₐ ≈ 1000 TeV.

In the solution which we have adopted the momentum conservation is incorporated by convoluting the Green function with the proton form factor as defined by the solution of the BFKL Eq. (1.1)

\[
W[y, k^2] = \int dω \int \frac{dk'^2}{k'^2} e^{iωy} G_0(k, k') Φ(k'),
\]

Fig. 11. Comparison of the DP fit with M_{SUSY} = 10 TeV with HERA data.
where in analogy to Eq. (3.1) the Green function, \( \hat{G}_ \alpha(k, k') \), is written in terms of the eigenfunctions \( f_\alpha(k, \eta_\alpha) \) and \( f_\alpha(k', \eta_\alpha) \). This expression describes the evolution of a wave packet, \( \mathcal{W}\mathcal{P} \), from rapidity \( y = 0 \) to the larger rapidity \( y \) values. The (approximate) momentum conservation emerges here because the quasi-local nature of the kernel \( \mathcal{H}(t, \tau) \) ensures that there is no evolution into the very large transverse momenta, apart from the usual BFKL diffusion, \( \Delta k_T \sim \sqrt{\alpha_s} \ln s \). Figure 12 shows the wave packet as it evolves from the smallest to the largest rapidity values of the HERA region, \( y = \ln(1/x) \), and for the SM and SM + SUSY cases.

The initial wave packet, the curve \( \mathcal{W}\mathcal{P}(y = 0, k^2) \) shown in Fig. 12, is not quite the same as the function \( \Phi_\rho(k) \) defined in Eq. (3.3) (divided by \( k^2 \)) but rather it is related to it by Eq. (4.1), i.e. it is built only out of the eigenfunctions which obey the imposed boundary conditions on the Green function. For large \( y \), it is sufficient to consider only the discrete eigenvalue part of the Green function in Eq. (4.1), and so we are actually using that part of the proton impact factor which is orthogonal to the (continuum) negative \( \omega \) eigenfunctions. The initial wave packet is somewhat broader than the distribution of Eq. (3.3) but it has the required feature that it is localized to the transverse momenta of \( \sim 1 \) GeV. With increasing rapidity it broadens due to the BFKL diffusion but this broadening remains moderate, of a size of just few GeV, as in the usual solutions of the BFKL equation. Although our kernel is constructed from the oscillating eigenfunctions with properties determined by the BFKL dynamics at very high transverse momenta, the oscillations cancel away due to the choice of the phases \( \eta_\alpha \) in the fit procedure and what remains is only a slight broadening of the gluon diffusion spectrum with increasing \( y \).

In the application to DIS it is more customary to consider the evolution of the unintegrated gluon density given by

\[
xg(x, k^2) = k^2 \int d\omega \int \frac{dk'}{k'^2} (\frac{kx}{k'})^{-\omega_0} \hat{G}_\alpha(k, k') \Phi_\rho(k'),
\]

which differs from the evolution of the wave packet, Eq. (4.1), by the kinematical factor \( k^2 \) and the factor \( (k/k')^{-\omega_0} \). The latter factor encodes the difference between Bjorken-x and rapidity, \( y \). It is close to one and of minor importance. On the other hand, the factor of \( k^2 \), is important as it amplifies the high virtuality contributions to the structure function (note that \( F_2 \) is obtained from a convolution of the unintegrated gluon density with the photon impact factor which selects \( k^2 \) values close to the \( Q^2 \) of the experiment).

In Fig. 13 we show the unintegrated gluon density at two values of \( x \)—one in the middle of HERA region, \( x = 10^{-3} \), and one just above HERA low \( x \) region, \( x = 10^{-4} \). The figure shows again that the contribution of BSM physics changes the shape of the gluon density but does not lead to an increase of the large transverse momentum tail (in fact this tail turns out to be smaller for SM + SUSY) contrary to the naive expectation that BSM effects should show up as an increased parton activity at large virtualities. The figure also shows that the discrepancy between the unintegrated gluon density for the SM alone and for SM + SUSY increases substantially with diminishing \( x \) at larger \( k^2 \), which suggests that Drell–Yan measurements at LHC, in the region of \( x \sim 10^{-4} \) and \( k^2 \sim 50 \text{ GeV}^2 \), could have high sensitivity to BSM physics.

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9 Note that the analogy to the QM wave packet is not complete because in the BFKL eq. rapidity is analogous to imaginary time.

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**Fig. 12.** Evolution of the initial wave packet in the DP solution of the BFKL equation as determined from the fit to HERA \( F_2 \) data.

**Fig. 13.** The unintegrated gluon density using the DP solution of the BFKL equation, as determined from the fit to HERA \( F_2 \) data.
In contrast to the diffusion effect, the important point that we are stressing is that it is the spectrum of allowed eigenvalues (and also the shape of the eigenfunctions) that enters into the construction of the Green function which is sensitive to physics at high scales. This sensitivity emerges from the interplay of two main features of the DP solution: the identity of the boundary condition in respect to addition of \( n \pi \) to the phase and the running of the coupling constant. For \( n \geq 3 \) this interplay is affected by SUSY effects, whereas for \( n \leq 2 \) the eigenvalues are unaffected since the critical transverse momentum, \( t_c \), is below the SUSY threshold. For \( n \geq 3 \), the change of the eigenvalues is substantial. As \( n \) increases, the ratio of eigenvalues with and without SUSY rapidly reaches its asymptotic value of \( 7/3 \). In Fig. 14 we compare the summed contribution of the eigenfunctions which are sensitive to BSM effects (\( n = 3 \ldots 100 \)) with the sum of contribution of the first two eigenfunctions which are not sensitive to SUSY. The contributions of the \( n = 3 \ldots 100 \) eigenfunctions changes both in size and shape when evaluated with and without SUSY whereas the contribution of the first two eigenfunction remains unaltered.

Figure 15 shows the \( \eta \) variation as a function of the eigenvalue \( \omega \) for the fit with the SUSY threshold of 10 TeV and for the SM fit only. Both relations show a substantial variation of the phase \( \eta \) with decreasing \( \omega \). The substantial difference between the two relations reflects a large difference between the eigenvalues and eigenfunctions in both cases.

The changes due to BSM effects lead to a substantial increase of \( \chi^2 \) when the corresponding gluon densities are confronted with data because the changes are of the order of 10% whereas the data precision is around 1–2%. Indeed, the evolution of the unintegrated gluon density performed with the SM alone—using the same parameter values for \( \eta \) boundary as in the SUSY case (dotted line in Fig. 14, dashed lines in Fig. 13) would give an increase of \( \chi^2 \) by \( \Delta \chi^2 \approx 160 \) when compared to data. On the other hand, had we used the \( \eta \) boundary parameters as determined with the SM alone and performed the evolution with the SM + SUSY eigenvalues and eigenfunction, the increase of \( \chi^2 \) would be even larger, \( \Delta \chi^2 \approx 300 \). The differences between the \( \chi^2 \)'s of the SM and SM + SUSY fits shown in table are substantially smaller, \( \Delta \chi^2 \approx 12 \), because the parameters of the \( \eta \) boundary conditions and the proton form factors were fitted to data what diminishes substantially the sensitivity to BSM effects.

It is quite clear that the DP could provide an exciting framework to study BSM effects if more can be learned about its infrared boundary. One way to improve our knowledge of this boundary is to apply our analysis to another, independent set of data, such as the Drell–Yan processes at the LHC. The LHC Drell–Yan data extend in the low-\( x \) region to much larger \( Q^2 \) scales than HERA data. This will allow the study of the evolution in \( \ln k^2 \) which is dominated in the DP scheme by the low \( \omega \) region. This region and the region of very low \( x \) is very sensitive to SUSY effects and at the same time is much less sensitive to NLO effects which are difficult to compute.

Future HEP experiments which are now under discussion can also substantially improve the knowledge of the infrared boundary and of the properties of the DP. The LHeC project [28] could provide important information about the region of very low \( x \) and not so high scales. This could lead to a better understanding

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\(^{10}\)Natural candidates would be also the diffractive HERA processes. Unfortunately, they are not measured with sufficient accuracy because the HERA detectors were not designed to measure diffractive processes.
of the properties of the large $\omega$ contributions. In addition, it will be also possible to measure precisely the exclusive diffractive processes. This will provide an independent evaluation of the infrared boundary since in the exclusive processes (e.g. in the exclusive $J/\psi$ or $\Upsilon$ production) the gluon density contributes almost quadratically [21, 23].

The properties of the DP can also be studied very well in the $\gamma^*\gamma^*$ process in a future linear $e^+e^-$ collider [29]. The $\gamma^*\gamma^*$ process is very interesting because it would permit the direct test of the universality of the BFKL pomeron and its boundary condition. Of particular interest could also be the newly proposed electron wake field accelerator [30], which could accelerate electrons into the several TeV energy region. This would allow to measurement of the $\gamma^*\gamma^*$ structure function at very low $x$ and high $Q^2$s, which could further increase the sensitivity to BSM effects.

5. SUMMARY

In this paper we have analyzed the properties of the Discrete BFKL Pomeron (DP) using analytical and numerical methods. We show, as an example $N = 1$ SUSY, that BSM physics substantially alters its eigenvalue spectrum and the shape of its eigenfunctions. This is a genuine sensitivity insofar as it cannot be entirely absorbed into any free parameters of the Discrete Pomeron solution of the BFKL equation.

The physical origin of this sensitivity can be traced back to the fact that in the low Bjorken-$x$ regime the behaviour of the structure functions is directly related to the positions of the Regge poles (pomerons). In the BFKL approach, due to the approximate conformal invariance, the corresponding eigenfunctions and eigenvalues of the BFKL kernel are determined from exponentially large transverse moments, where contributions from particles of any BSM physics play an essential role. On the other hand the locality property of the BFKL equation assures that in any physical process performed at energy scales which are much lower than the BSM ones, the BSM quanta cannot be produced and the transverse momenta of the virtual particles are very limited. The effects of large transverse momenta appears, however, through the substantial alteration of the eigenstate spectrum of the BFKL Hamiltonian, which is important in the low-$x$ region. In our view, this provides a new mechanism for the detection of BSM effects, which has not previously been considered.

The eigenvalue spectrum of the DP cannot be directly measured because it determines the gluon density through a complicated superposition of Pomeron states. The result of their interference can be compared with data only after the free parameters of the BFKL solution, determining the infrared boundary, are specified. Although the infrared boundary is a physical quantity of non-perturbative QCD origin, we could only determine it in this and the previous paper within a heuristic approach. Our description of this boundary provides a very good fit to the data and shows that, the BSM effects are sensed by the HERA $F_2$ data, notwithstanding the large freedom of the parameter choice.

The analysis of HERA data indicates an improved quality of fit for the case of $N = 1$ SUSY, with the SUSY scale as being around 10 TeV. Needless to say that this determination is only possible within our heuristic model approach. Our limited knowledge of the infrared boundary diminishes substantially (but not completely) the sensitivity of the fit to BSM effects.

This sensitivity can be substantially improved by a better determination of the universal boundary condition. We can gain a better understanding of the infrared boundary from the analysis of additional data sets, especially of the LHC Drell–Yan data. The data from future experimental facilities like LHeC, the $e^+e^-$ linear collider or even higher energy plasma wakefield accelerators could also become crucial. The $\gamma^*\gamma^*$ process which can be very well measured at the linear colliders is of particular interest since in this reaction the properties of the discrete Pomeron solution are simplified owing to the absence of the proton.

The method described in this paper opens a new possibility of using high precision experiments to search for new physics at energy scales considerably larger than the scales at which the experiments are performed. We consider the approach formulated here, which involves a heuristic model for the parameterization of the infrared phases of the BFKL eigenfunctions, as a first step which should be substantially improved by involving more data and more theoretical analysis.

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