DIOPHANTINE EQUATIONS DEFINED BY BINARY QUADRATIC FORMS
OVER RATIONAL FUNCTION FIELDS

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Abstract. We study the “imaginary” binary quadratic form equations $ax^2 + bxy + cy^2 + g = 0$
over $k[t]$ in rational function fields, showing that a condition with respect to the Artin reciprocity
map, is the only obstruction to the local-global principle for integral solutions of the equation.

1. Introduction

Consider the integral solvability of the generalized equation

\begin{equation}
ax^2 + bxy + cy^2 + g = 0
\end{equation}

over global fields, which amounts to the integral representability of $-g$ as the binary quadratic form
$ax^2 + bxy + cy^2$. Here “integral” means that we shall restrict the equation over rings of integers
or more generally $S$-integers. This problem is an old one which dates back to the theory of Gauss’
quadratic forms. Modern approaches involve the splitting of ideals in quadratic extensions and
class field theory. For example, the main theorem of Cox [4] gives a criterion of the solvability of
the diophantine equation

\begin{equation}
p = x^2 + ny^2
\end{equation}

for any positive integer $n$ and prime number $p$. Using a similar argument, the author and Deng [9]
generalized the base field of (1.2) to a class of imaginary quadratic fields. Maciak [11] treated the
same problem over rational function fields, and gave a similar criterion of the integral solvability of
(1.2).

Another approach is to use the point of view of arithmetic algebraic geometry. The integral
solvability of an equation amounts to the existence of integral points on the affine scheme defined
by it. Colliot-Thélène and Xu [3] studied the integral points on homogeneous spaces of semi-
simple and simply connected linear algebraic groups of non-compact type by using the strong
approximation theorem and the Brauer-Manin obstruction. They also applied the results to the
integral representation problem of quadratic forms. Harari [7] showed that the Brauer-Manin
obstruction is the only obstruction for the existence of integral points of a scheme over the ring of
integers of a number field, whose generic fiber is a principal homogeneous space (torsor) of a torus.
Although these results are applicable to (1.1), it can not yield an explicit criterion for the integral
solvability.

After then Wei and Xu [17, 18] showed that there exist idele groups which are the so-called
X-admissible subgroups for determining the integral points for multi-norm tori (more generally,
groups of multiplicative types), and interpreted the $X$-admissible subgroup in terms of finite Brauer-Manin obstruction. In [17, Section 3] Wei and Xu also showed how to apply this method to binary quadratic diophantine equations over rings of integers of number fields. As applications, they gave some explicit criteria of the solvability of equations of the form $x^2 \pm dy^2 = a$ over $\mathbb{Z}$ in [17] Sections 4 and 5, by constructing explicit admissible subgroups. Later Wei [15] applied the method in [17] to give some additional criteria of the solvability of the diophantine equation $x^2 - dy^2 = a$ over $\mathbb{Z}$ for some $d$. He also determined which integers can be written as a sum of two integral squares for some of the quadratic fields $\mathbb{Q}(\sqrt{\pm p})$ (in [14]), $\mathbb{Q}(\sqrt{-2p})$ (in [16]) and so on. The author et al. [10] also applied the method in [17] to (1.1) over $\mathbb{Z}$ and gave a criterion of the solvability with some additional assumptions, by constructing explicit admissible subgroups for (1.1).

In this text, we treat the equation (1.1) over $k[t]$, as an function field analogue of [10]. We generalize the method in [10] to construct explicit admissible subgroups for the equation (1.1). See Lemma 2.14. Specifically, the main result is:

**Theorem.** Let $k = \mathbb{F}_q$ be a finite field of odd characteristic, $F = k(t)$ a rational function field, $\mathfrak{o}_F = k[t]$. Suppose $a, b, c$ and $d$ are elements of $\mathfrak{o}_F$ such that $E = F(\sqrt{(b/2)^2 - ac})/F$ is an imaginary quadratic extension. Let $K_{\mathfrak{p}_\infty}^+$ be the class field corresponding to $E^\times \Xi_{\mathfrak{p}_\infty}^+$ and $X = \text{Spec}(\mathfrak{o}_F[x, y]/(a(x^2 + bxy + cy^2 + g)))$.

Then the equation (1.1) is solvable over $\mathfrak{o}_F$ if and only if there exists a local solution

$$\prod_{p \in \Omega_F} (x_p, y_p) \in \prod_{p \in \Omega_F} X(\mathfrak{o}_{F_p})$$

such that

$$\psi_{K_{\mathfrak{p}_\infty}^+/E}(\hat{f}_E(\prod_p (x_p, y_p))) = 1.$$

In the above theorem, “imaginary” means there is a unique place lying over $1/t$, $E^\times \Xi_{\mathfrak{p}_\infty}^+$ (depending on the sign function) is an open subgroup of finite index of the idele group $\mathbb{I}_E$ of $E$, $\hat{f}_E$ is a map from $\prod_p X(\mathfrak{o}_{F_p})$ to $\mathbb{I}_E$ which is constructed by using the fact that the generic fiber of $X$ admits the structure of a torsor of a torus, and $\psi_{K_{\mathfrak{p}_\infty}^+/E} : \mathbb{I}_E \to \text{Gal}(K_{\mathfrak{p}_\infty}^+/E)$ is the Artin reciprocity map. The condition $\psi_{K_{\mathfrak{p}_\infty}^+/E}(\hat{f}_E(\prod_p (x_p, y_p))) = 1$ is called the Artin condition. See Sections 2.1 and 3 for details.

In Section 2, we introduce from [17] notations and the general result we mainly use in this text, but in a modified way which focus on our goal. Then we give our result on the equation (1.1) over $k[t]$ in Section 3. The results state that the integral local solvability and the Artin condition (see Remark 2.11) completely describe the global integral solvability. In view of the result of Maciak, adding an assumption (see 3.13), we recover the main theorems in [11] by our result. At last, we ended this text by concrete examples showing the explicit criteria of the solvability.

### 2. Solvability by the Artin Condition

#### 2.1. Notations

Let $k = \mathbb{F}_q$ and $F/k(t)$ a algebraic function field with characteristic not 2, $\mathfrak{o}_F$ integral closure of $k[t]$ in $F$, $\Omega_F$ the set of all places in $F$. Thus $2 \in \mathfrak{o}_F^x$. Let $F_p$ be the completion of $F$ at $p$ and $\mathfrak{o}_{F_p}$ the valuation ring of $F_p$ for each $p \in \Omega_F$. Denote by $\infty_F \subset \Omega_F$ the set of infinite places of $F$, i.e., places lying over $1/t$. We also write $\mathfrak{o}_{F_p} = F_p$ for $p \in \infty_F$. The adele ring (resp. idele group) of $F$ is denoted by $\mathbb{A}_F$ (resp. $\mathbb{I}_F$).
Let \( a, b, c \) and \( g \) be elements in \( \mathfrak{o}_F \) such that \(-d = (b/2)^2 - ac\) is not a square in \( F \). Let 
\[ E = F(\sqrt{-d}), \] 
a quadratic extension of \( F \). Since the characteristic of \( F \) is not 2, the extension \( E/F \) is separable. Let
\[
X = \text{Spec}(\mathfrak{o}_F[x, y]/(a(ax^2 + bxy + cy^2 + g)))
\]
be the affine scheme defined by the equation
\[
a(ax^2 + bxy + cy^2 + g) = 0
\]
over \( \mathfrak{o}_F \). Since \(-d\) is not a square in \( F \), we have \( a \neq 0 \). Then the equation
\[
ax^2 + bxy + cy^2 + g = 0
\]
is solvable over \( \mathfrak{o}_F \) if and only if \( X(\mathfrak{o}_F) \neq \emptyset \).

Now we denote
\[
\tilde{x} = ax + \frac{b}{2}y, \quad \tilde{y} = y, \quad n = -ag.
\]
Then we can write (2.2) as
\[
\tilde{x}^2 + d\tilde{y}^2 = n.
\]
Denote by \( R_{E/F}(\mathbb{G}_m) \) the Weil restriction of \( \mathbb{G}_m,E \) to \( F \). Let
\[
\varphi : R_{E/F}(\mathbb{G}_m) \longrightarrow \mathbb{G}_m
\]
be the homomorphism of algebraic groups which represents
\[
x \longmapsto N_{E/F}(x) : (E \otimes_F A)^\times \longrightarrow A^\times
\]
for any \( F \)-algebra \( A \). Define the torus \( T = \ker \varphi \). Let \( X_F \) be the generic fiber of \( X \). We may write an element in \( T(A) \) (resp. \( X_F(A) \)) as \( u + \sqrt{-d}v \in E \otimes_F A \), with \( u, v \in A \), \( u^2 + dv^2 = 1 \) (resp. \( \tilde{x} + \sqrt{-d}\tilde{y} \in E \otimes_F A \) with \( x, y \in A \), \( \tilde{x}^2 + d\tilde{y}^2 = n \)). Then \( X_F \) is naturally a \( T \)-torsor by the action:
\[
T(A) \times X_F(A) \longrightarrow X_F(A), \quad (u + \sqrt{-d}v, x + \sqrt{-d}y) \longmapsto (u + \sqrt{-d}v)(\tilde{x} + \sqrt{-d}\tilde{y}).
\]
Obviously, \( T \) has an integral model \( T = \text{Spec}(\mathfrak{o}_F[x, y]/(x^2 + dy^2 - 1)) \) and since \( T \) is separated over \( \mathfrak{o}_F \), we can view \( T(\mathfrak{o}_{F_p}) \) as a subgroup of \( T(F_p) \). Note that for \( p \in \infty_F \) we also write \( T(\mathfrak{o}_{F_p}) \) (resp. \( X(\mathfrak{o}_{F_p}) \)) for \( T(F_p) \) (resp. \( X_F(F_p) \)).

Denote by \( \lambda \) the embedding of \( T \) into \( R_{E/F}(\mathbb{G}_m) \). Clearly \( \lambda \) induces a natural injective group homomorphism
\[
\lambda_E : T(\mathbb{A}_F) \longrightarrow I_E.
\]

Now we assume that
\[
X_F(F) \neq \emptyset,
\]
i.e. \( X_F \) is a trivial \( T \)-torsor. Fixing a rational point \( P \in X_F(F) \), for any \( F \)-algebra \( A \), we have an isomorphism
\[
\phi_P : X_F(A) \cong T(A), \quad x \longmapsto P^{-1}x.
\]
induced by $P$. That is, $P^{-1}x$ is the unique element in $T(A)$ sending $P$ to $x$. Since $X$ is separated over $\mathfrak{o}_F$, we can view $\prod_{p \in \Omega_F} X(\mathfrak{o}_{F_p})$ as a subset of $X_F(\mathbb{A}_F)$, and the composition
\[ f_E = \lambda_E \circ \phi_P : \prod_{p} X(\mathfrak{o}_{F_p}) \rightarrow \mathbb{I}_E \]
makes sense. Note that under the previous description for $X_F(A)$ where $A = F$ here, we have $P \in E^\times \subset \mathbb{I}_E$ since it is a rational point. It follows that we can define the map $\tilde{f}_E$ to be the composition
\[ \prod_{p} X(\mathfrak{o}_{F_p}) \xrightarrow{f_E} \mathbb{I}_E \xrightarrow{\times P} \mathbb{I}_E. \]

It can be seen that the restriction to $X(\mathfrak{o}_{F_p})$ of $\tilde{f}_E$ is defined by
\begin{equation}
\tilde{f}_E([x_p, y_p]) = \begin{cases} 
(x_p + \sqrt{-d}y_p, \bar{x}_p - \sqrt{-d}\bar{y}_p) \in E\mathfrak{p} \times E\mathfrak{p} & \text{if } p = \mathfrak{p}\mathfrak{q} \text{ splits in } E/F, \\
(x_p + \sqrt{-d}y_p, \bar{x}_p + \sqrt{-d}\bar{y}_p) \in E\mathfrak{q} & \text{otherwise,}
\end{cases}
\end{equation}
where $\mathfrak{p}$ and $\mathfrak{q}$ (resp. $\mathfrak{q}$) are places of $E$ above $p$ and $\bar{x}_p = ax_p + \frac{b}{2}y_p$, $\bar{y}_p = y_p$.

Let $\Xi$ be an open subgroup of $\mathbb{I}_E$ such that $E^\times \Xi$ is of finite index. Let $K_{\Xi}$ be the class field corresponding to $E^\times \Xi$ under class field theory, such that the Artin map gives the isomorphism
\[ \psi_{K_{\Xi}/E} : \mathbb{I}_E/E^\times \Xi \xrightarrow{\sim} \text{Gal}(K_{\Xi}/E). \]

For any $\prod_{p} (x_p, y_p) \in \prod_{p} X(\mathfrak{o}_{F_p})$, noting that $P$ is in $E^\times$, we have
\begin{equation}
\psi_{K_{\Xi}/E}(\tilde{f}_E(\prod_{p} (x_p, y_p))) = 1 \quad \text{if and only if} \quad \psi_{K_{\Xi}/E}(\tilde{f}_E(\prod_{p} (x_p, y_p))) = 1.
\end{equation}

**Remark 2.7.** The assumption (2.4) is easy to check by the Hasse-Minkowski theorem on quadratic equations. In particular, it holds if $\prod_{p} X(\mathfrak{o}_{F_p}) \neq \emptyset$, in which case, we can pick an $F$-point $P$ of $X_F$ and obtain $\phi_P$. Note that the map $\tilde{f}_E$ is independent of $P$.

2.2. A general result. For the integral points of the scheme $X$ over $\mathfrak{o}_F$ defined in (2.1), we observe that $T(\mathfrak{o}_{F_p})$ acts stably on $X(\mathfrak{o}_{F_p})$ for all $p \in \Omega_F$. To verify this, it suffice to show that for any $u, v, x, y \in \mathfrak{o}_{F_p}$,
\[ (u + v\sqrt{d})(ax + b\frac{y}{2}) + y\sqrt{d} = ((ax' + b\frac{y'}{2}) + y'\sqrt{d}) \]
for some $x', y' \in \mathfrak{o}_{F_p}$. Indeed, this is the case if we take
\[ x' = (u - \frac{b}{2}v)x + cvy, \]
\[ y' = uy + v(ax + \frac{b}{2}y). \]

Now we have the following general result, which is a function field analogue of [17] Corollary 1.6.

**Proposition 2.8.** Let $\Xi$ be an open subgroup of $\mathbb{I}_E$ described as before and suppose that
\begin{equation}
\lambda_{E}^{-1}(E^\times \Xi) \subseteq T(F)\prod_{p} T(\mathfrak{o}_{F_p}).
\end{equation}

Then $X(\mathfrak{o}_F) \neq \emptyset$ if and only if there exists a local solution
\[ \prod_{p \in \Omega_F} (x_p, y_p) \in \prod_{p \in \Omega_F} X(\mathfrak{o}_{F_p}) \]
such that

\[
\psi_{K_{\Xi}/E}(\tilde{f}_E(\prod_p (x_p, y_p))) = 1.
\]

Proof. The proof is similar to the number field case. See [17, Corollary 1.6]. To imitate the proof, the only nontrivial thing is that $T(o_{F_p})$ acts stably on $X(o_{F_p})$ for all $p \in \Omega_F$, as showed above. □

Remark 2.11. The equation (2.10) is called the Artin condition in Wei [15, 14, 16]. If the assumption in the proposition holds, the integral local solvability and the Artin condition completely describe the global integral solvability. As a result, in cases where $K_{\Xi}$ is known it is possible to calculate the Artin condition, and give explicit criteria for the solvability. Actually, the idele group $\Xi$ satisfying the assumption (2.9) is a variant of the definition of $X$-admissible subgroup in [17].

Remark 2.12. In the case that $K_{\Xi}/F$ is abelian, the Artin condition is trivially true for any local solution. Actually, since $K_{\Xi}/F$ is abelian, the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{I}_E & \xrightarrow{\psi_{K_{\Xi}/E}} & \text{Gal}(K_{\Xi}/E) \\
N_{E/F} \downarrow & & \downarrow \\
\mathbb{I}_F & \xrightarrow{\psi_{K_{\Xi}/F}} & \text{Gal}(K_{\Xi}/F)
\end{array}
\]

It follows that for any local solution $\prod_p (x_p, y_p)$,

\[
\psi_{K_{\Xi}/E}(\tilde{f}_E(\prod_p (x_p, y_p))) = \psi_{K_{\Xi}/F}(N_{E/F}(\tilde{f}_E(\prod_p (x_p, y_p)))) = \psi_{K_{\Xi}/F}(n) = 1,
\]

where the second equation comes from the definition (2.5) of $\tilde{f}_E$ and

\[
(\tilde{x}_p + \sqrt{-d} \tilde{y}_p)(\tilde{x}_p - \sqrt{-d} \tilde{y}_p) = n \text{ in } E_{\mathfrak{P}} \text{ with } \mathfrak{P} \mid p,
\]
and the last equality is obtained by the assumption that $n = -ag \in F$.

Let $L = o_F + o_F \sqrt{-d}$ in $E$ and $L_p = L \otimes_{o_F} o_{F_p}$ in $E_p = E \otimes_F F_p$. We also write $L_p = E_{\mathfrak{P}}$ for $p \in \infty_F$. Then $\prod_p L_{\mathfrak{P}}$ is an open subgroup of $\mathbb{I}_E$. Let $S \subseteq \Omega_E$ be a finite set of places of $E$. $U_{\mathfrak{P}} \subseteq o_{E_{\mathfrak{P}}}^\times$ be an open subgroup of $o_{E_{\mathfrak{P}}}^\times$ for $\mathfrak{P} \in S$ and

\[
W_{\mathfrak{P}} = \begin{cases} U_{\mathfrak{P}} & \text{for } \mathfrak{P} \in S, \\ o_{E_{\mathfrak{P}}}^\times & \text{for } \mathfrak{P} \notin S. \end{cases}
\]

We define the open subgroup of $\mathbb{I}_E$

\[
\Xi_W = \left( \prod_{p \in \Omega_F} L_p \right) \cap \left( \prod_{\mathfrak{P} \in \Omega_E} W_{\mathfrak{P}} \right) = \prod_p \left( L_p^\times \cap \prod_{\mathfrak{P}|p} W_{\mathfrak{P}} \right),
\]

and assume $E^\times \Xi_W$ is also of finite index in $\mathbb{I}_E$.

By some additional assumptions, we prove that $\Xi = \Xi_W$ satisfies the assumption (2.9) in Proposition 2.8, that is,

\[
\lambda_{E}^{-1}(E^\times \Xi_W) \subseteq T(F) \prod_p T(o_{F_p}).
\]
Lemma 2.14. Let $S$ and $W_\mathfrak{p}$ be as before. Suppose for every $u \in \mathfrak{o}_F^\times$, the equation

$$N_{E/F}(\alpha) = u, \quad \alpha \in L^\times$$

is solvable or the equation

$$N_{E_p/F_p}(\alpha) = u, \quad \alpha \in L^\times \cap \prod_{\mathfrak{p}|p} W_\mathfrak{p}$$

is not solvable for some place $p$. Then the assumption (2.9) in Proposition 2.8 is true.

Proof. The proof is similar to [10, Lemma 1] but a little different. Recall that $T = \ker(R_{E/F}(\mathbf{G}_m) \to \mathbf{G}_m)$ and $\mathbf{T}$ is the group scheme defined by the equation $x^2 + dy^2 = 1$ over $\mathfrak{o}_F$. Therefore we have

$$T(F) = \{ \beta \in E^\times \mid N_{E/F}(\beta) = 1 \}$$

and

$$\mathbf{T}(\mathfrak{o}_F) = \{ \beta \in L^\times_p \mid N_{E_p/F_p}(\beta) = 1 \}.$$

Suppose $t \in T(A_F)$ such that $\lambda_E(t) \in E^\times_\Xi_W$. Write $t = \beta i$ with $\beta \in E^\times$ and $i \in \Xi_W$. Since $t \in T(A_F)$ we have

$$N_{E/F}(\beta)N_{E/F}(i) = N_{E/F}(\beta i) = 1.$$

It follows that

$$N_{E/F}(i) = N_{E/F}(\beta^{-1}) \in F^\times \cap \prod_p \mathfrak{o}_F^\times_p = \mathfrak{o}_E^\times.$$

So we have $N_{E/F}(i) = u$ for some $u \in \mathfrak{o}_E^\times$. Note that $i \in \Xi_W$, and thus at each $p$ we have

$$N_{E_p/F_p}(i_p) = u, \quad i_p = (i_p)_W \in L^\times_p \cap \prod_{\mathfrak{p}|p} W_\mathfrak{p}.$$

Thus the assumption tells us that the equation

$$N_{E/F}(\alpha) = u, \quad \alpha \in L^\times$$

is solvable. Let $\alpha_0$ be such a solution and let

$$\gamma = \beta \alpha_0$$

and $j = i \alpha_0^{-1}$.

Then $N_{E/F}(\gamma) = N_{E/F}(j) = 1$. Note that $\alpha_0 \in L^\times$, and we have $\gamma \in T(F)$ and $j \in \prod_p \mathbf{T}(\mathfrak{o}_F)$. It follows that $t = \beta i = \gamma j \in T(F) \prod_p \mathbf{T}(\mathfrak{o}_F)$. This finishes the proof. \qed

Remark 2.15. In [10], the admissible subgroup $\Xi$ for the equation (1.1) is simply chosen to be $\prod_p L^\times_p$, which is generalized by the above lemma in the function field case, where we intersect $\prod_p L^\times_p$ with the open subgroup $\prod_\mathfrak{p} W_\mathfrak{p}$ (see (2.13)). This allows us to deal more difficult base global fields and parameters for (1.1). Previous method to do this [17, 15, 16] is to construct an Kummer extension $\Theta/E$ with low degree and choose the class group to be $E^\times \prod_p L^\times_p \cap E^\times N_{\Theta/E}^\times$. Using this lemma, we obtain the following corollary to Proposition 2.8.

Corollary 2.16. Let $\Xi_W$ be defined by (2.13) and suppose that $E^\times \Xi_W$ is of finite index in $\mathbb{I}_E$. Let $S$ and $W_\mathfrak{p}$ satisfy the assumption in Lemma 2.14 that is, for every $u \in \mathfrak{o}_F^\times$, the equation

$$N_{E/F}(\alpha) = u, \quad \alpha \in L^\times$$
is solvable or the equation
\[ N_{E_p/F_p}(\alpha) = u, \quad \alpha \in L^p_\chi \cap \prod_{\mathfrak{p} || \mathfrak{p}} W_{\mathfrak{p}} \]
is not solvable for some place \( \mathfrak{p} \). Then \( X(\mathfrak{o}_F) \neq \emptyset \) if and only if there exists a local solution
\[ \prod_{\mathfrak{p} \in \Omega_F} (x_\mathfrak{p}, y_\mathfrak{p}) \in \prod_{\mathfrak{p} \in \Omega_F} X(\mathfrak{o}_F_p) \]
such that
\[ \psi_E/F(\tilde{f}(\prod_{\mathfrak{p}} (x_\mathfrak{p}, y_\mathfrak{p}))) = 1, \]
where \( K_W \) is the class field corresponding to \( E \times \Xi_W \).

### 3. The integral representation of binary quadratic forms over \( k[t] \)

Now we consider our focus, the case where \( F = k(t) \) and \( k = \mathbb{F}_q \) is a finite field of characteristic \( p \neq 2 \). Hence we are interested in the diophantine equation
\[ ax^2 + bxy + cy^2 + g = 0 \]
over \( \mathfrak{o}_F = k[t] \). Suppose that \( -d = (b/2)^2 - ac \) is not a square in \( F \). Set \( E = F(\sqrt{-d}) \) and \( L = \mathfrak{o}_F + \mathfrak{o}_F \sqrt{-d} \) as previous sections. Let \( \mathfrak{p}_\infty = 1/t \) be the place of \( k(t) \) at infinity and suppose further that \( E/F \) is “imaginary”, that is,

\[ (3.1) \quad \text{there is a unique place } \mathfrak{p}_\infty \text{ in } E \text{ lying over } \mathfrak{p}_\infty. \]

We briefly introduce sign function here. In a completion \( K_p \) of a global function field \( K \), A sign function with respect to a uniformizer \( \pi \) is defined as
\[ \text{sgn} : K_p \rightarrow \mathfrak{o}_K/p \]
\[ x \mapsto c_{r_0}, \]
where \( c_{r_0} \neq 0 \) is the leading coefficient of the Laurent series \( x = \sum_{r=r_0}^{\infty} c_r \pi^r \) of \( x \) with coefficient in \( \mathfrak{o}_{K_p}/p \). An element \( x \in K_p \) is positive if \( \text{sgn}(x) = 1 \). Fix a sign function of \( \mathfrak{p}_\infty \), denoted by \( \text{sgn}(\cdot) \) and define the open subgroup
\[ (3.2) \quad E^+_{\mathfrak{p}_\infty} = \{ \alpha \in E^\times_{\mathfrak{p}_\infty} \mid \text{sgn}(\alpha) = 1 \} \subseteq E^\times_{\mathfrak{p}_\infty} \]
consisting all positive elements, \( S = \{ \mathfrak{p}_\infty \} \), and \( U_{\mathfrak{p}_\infty} = E^+_{\mathfrak{p}_\infty} \). Let \( \Xi^+_{\mathfrak{p}_\infty} \) be the subgroup \( \Xi_W \) defined in (2.13) for the chosen \( S \) and \( U_{\mathfrak{p}_\infty} \).

**Theorem 3.3.** With the above notations, we have:

(a) The open subgroup \( E^\times \Xi^+_{\mathfrak{p}_\infty} \) is of finite index in \( \mathbb{I}_E \).

(b) Let \( K^+_\mathfrak{p}_\infty \) be the class field corresponding to \( E^\times \Xi^+_{\mathfrak{p}_\infty} \) and
\[ X = \text{Spec}(\mathfrak{o}_F[x,y]/(a(ax^2 + bxy + cy^2 + g))). \]

Then \( X(\mathfrak{o}_F) \neq \emptyset \) if and only if there exists a local solution
\[ \prod_{\mathfrak{p} \in \Omega_F} (x_\mathfrak{p}, y_\mathfrak{p}) \in \prod_{\mathfrak{p} \in \Omega_F} X(\mathfrak{o}_F_p) \]
such that
\[ \psi_{K^+_\mathfrak{p}_\infty}/E(\tilde{f}(E(\prod_{\mathfrak{p}} (x_\mathfrak{p}, y_\mathfrak{p})))) = 1. \]
Proof. We first show that $\mathbb{I}_E / E^\times \Xi_{\mathfrak{p}_\infty}$ is finite. By the choice of $S$ and $U_{\mathfrak{p}_\infty}$ we know that

$$\Xi_{\mathfrak{p}_\infty}^+ = E_{\mathfrak{p}_\infty}^+ \times \prod_{p \neq \mathfrak{p}_\infty} L_p^\times$$

Define $(\sigma_E)_p = \sigma_E \otimes_{\sigma_p} \sigma_{F_p}$ for $p \neq \mathfrak{p}_\infty$ in $E_p = E \otimes_F F_p$ and

$$\Xi_{\mathfrak{p}_\infty}^+ = E_{\mathfrak{p}_\infty}^+ \times \prod_{p \neq \mathfrak{p}_\infty} (\sigma_E)_p = E_{\mathfrak{p}_\infty}^+ \times \prod_{p \neq \mathfrak{p}_\infty} \sigma_{E_p}^\times.$$

Since we have a surjection $\Xi_{\mathfrak{p}_\infty}^+ / \Xi_{\mathfrak{p}_\infty} = E^\times \Xi_{\mathfrak{p}_\infty}^+ / E^\times \Xi_{\mathfrak{p}_\infty}^+$ and

$$\Xi_{\mathfrak{p}_\infty}^+ / \Xi_{\mathfrak{p}_\infty} = \prod_{p \mid [\sigma_E : L]} (\sigma_E)_p / L_p^\times$$

is finite, we know that $E^\times \Xi_{\mathfrak{p}_\infty}^+ / E^\times \Xi_{\mathfrak{p}_\infty}$ is finite. Therefore we only need to show that $\mathbb{I}_E / E^\times \Xi_{\mathfrak{p}_\infty}$ is finite.

Define $\mathbb{I}_E^+ = \mathbb{I}_E \cap E_{\mathfrak{p}_\infty}^+$, the subgroup of $\mathbb{I}_E$ consisting elements whose projection to $E_{\mathfrak{p}_\infty}^+$ is in $E_{\mathfrak{p}_\infty}^+$. Let $E^+ = \mathbb{I}_E^+ \cap E^\times$. Then naturally we have an isomorphism $\mathbb{I}_E^+ / E^\times \Xi_{\mathfrak{p}_\infty} \cong Cl^+(\sigma_E)$, where $Cl^+(\sigma_E)$ is the narrow class group with respect to $(\mathfrak{p}_\infty, \text{sgn})$, which is finite (c.f. [6, p. 200]). Since $\mathbb{I}_E = E^\times \mathbb{I}_E^+$ by weak approximation theorem (c.f. [2, Chapter II.6]), we have $\mathbb{I}_E^+ / E^\times \Xi_{\mathfrak{p}_\infty} \cong \mathbb{I}_E / E^\times \Xi_{\mathfrak{p}_\infty}$ and thus $\mathbb{I}_E / E^\times \Xi_{\mathfrak{p}_\infty} \cong Cl^+(\sigma_E)$ is finite. This completes the proof for $\Xi_{\mathfrak{p}_\infty}$.

For the assertion (1) we apply Corollary 2.16. For any $u \neq 1$ in $\sigma_F^\times = K^\times$. Let $p = \mathfrak{p}_\infty$ so $L_p^\times \cap \prod_{p \mid [\sigma_E : L]} W_p = E_{\mathfrak{p}_\infty}^+$, since $\mathfrak{p}_\infty$ is the only place above $\mathfrak{p}_\infty$. Assume that there is $\alpha \in E_{\mathfrak{p}_\infty}^+$ such that $u = N_{E_p/F_p}(\alpha) = \alpha \bar{\alpha}$. By the definition of $E_{\mathfrak{p}_\infty}^+$ (3.2) we know that $\text{sgn}(\alpha) = \text{sgn}(\bar{\alpha}) = 1$. It follows that $u = \text{sgn}(u) = \text{sgn}(\alpha) \text{sgn}(\bar{\alpha}) = 1$, which is a contradiction and shows that the equation

$$N_{E_p/F_p}(\alpha) = u, \quad \alpha \in L_p^\times \cap \prod_{p \mid [\sigma_E : L]} W_p$$

is not solvable for $p = \mathfrak{p}_\infty$. Thus (1) follows form Corollary 2.16.

In contrast to the class field $K_{\mathfrak{p}_\infty}$ corresponding to $\Xi_{\mathfrak{p}_\infty}$, we denote $K_{\mathfrak{p}_\infty}$ the Hilbert class field of $E$ with respect to $\mathfrak{p}_\infty$, i.e. $K_{\mathfrak{p}_\infty}$ is the class field corresponding to the open subgroup $E^\times \Xi_{\mathfrak{p}_\infty}$ of finite index in $\mathbb{I}_E$, where

$$\Xi_{\mathfrak{p}_\infty} = E_{\mathfrak{p}_\infty}^\times \times \prod_{p \neq \mathfrak{p}_\infty} \sigma_{E_p}^\times,$$

which we will use later and basically we have $K_{\mathfrak{p}_\infty} \supseteq K_{\mathfrak{p}_\infty}$ since $E^\times \Xi_{\mathfrak{p}_\infty} \subseteq E^\times \Xi_{\mathfrak{p}_\infty}$.

We use the above theorem to derive a result similar to the main theorems of Maciak [11] considering the equation $l = x^2 + Dy^2$. First we need some notations and facts in [11]. Recall that we set $F = k(t)$, $k = \mathbb{F}_q$ and hence $\sigma_F = k[t]$. Let $D \in k[t]$ be square free with positive degree and $l \mid D$ an irreducible element of $k[t]$ and we consider the equation $l = x^2 + Dy^2$ over $k[t]$. For this equation, we have $a = 1$, $b = 0$, $c = D$, $g = -l$, $-d = (b/2)^2 - ac = -D$ and $E = \mathbb{Q}(\sqrt{-d})$. Thus $\bar{x} = x$, $\bar{y} = y$ and $n = 1$. Remember that $D$ is square free with positive degree, we know that $\sigma_F \circ \sigma_F \sqrt{-D} = \sigma_E$. Recall that [13] Proposition 14.6 $p_\infty$ ramifies, splits, or is inert in $E/F$ if $\deg D$ is odd, $\deg D$ is even and $\text{lc}(-D) \in k^{x^2}$, or $\deg D$ is even and $\text{lc}(-D) \not\in k^{x^2}$, respectively, where $\text{lc}$ means the leading coefficient. Suppose that $\deg D$ is odd or $\text{lc}(-D) \not\in k^{x^2}$, which is to say the assumption (3.1) holds. Then a necessary condition for $l = x^2 + Dy^2$ being solvable over $k[t]$ is

$$\deg l \text{ is even if deg D is}.$$
We will always assume this. Let $d_\infty$ be the relative degree of $\mathbb{P}_\infty | p_\infty$ and define $\deg^* l = \frac{\deg l}{d_\infty}$ as in [11 Section 4], which is an positive integer by the above assumption. Let $g$ be the genus of $E$. Following [11] we fix $\text{sgn}$ with respect to the uniformizer $t^g/\sqrt{-D}$ (i.e., $\text{sgn}(t^g/\sqrt{-D}) = 1$) in the following

**Theorem 3.5.** Let $k$, $D$ and $l$ in $k[t]$ be as before such that (3.5) holds. Then we have

(a) if $\text{sgn}(l)(-1)^{\deg^* l} E/F$ then $l = x^2 + Dy^2$ is solvable over $k[t]$ if and only if (3.3) is equivalent to

(b) if $\text{sgn}(l)(-1)^{\deg^* l} \not\in E$, then $l = x^2 + Dy^2$ is solvable over $k[t]$ if and only if (3.3) is equivalent to

Proof. In line with Theorem 3.3, recall that $F = k(t), E = F(\sqrt{-D}), L = o_F + o_F\sqrt{-D} = o_E$ and $K_\mathbb{P}_\infty$ (resp. $K_\mathbb{P}_\infty^+$) is the class field corresponding to $\mathbb{P}_\infty$ (resp. $\mathbb{P}_\infty^+$). We know by (2.5) that

\begin{equation}
\tilde{f}_E([x_p, y_p]) = \begin{cases} (x_p + \sqrt{-D}y_p, x_p - \sqrt{-D}y_p) & \text{if } p \text{ splits in } E/F, \\ (x_p + \sqrt{-D}y_p) & \text{otherwise.} \end{cases}
\end{equation}

Then by Theorem 3.3 the equation $l = x^2 + Dy^2$ is solvable over $k[t]$ if and only if there exists a local solution

\[ \prod_{p \in \Pi_F} (x_p, y_p) \subseteq \prod_{p \in \Pi_F} X(o_{F_p}) \]

such that

\[ \psi_{K_\mathbb{P}_\infty^+/E}(\tilde{f}_E(\prod_{p} (x_p, y_p))) = 1. \]

Next we verify these conditions in details. By a simple calculation we know the local condition

\[ \prod_{p} X(o_{F_p}) \neq \emptyset \]

is equivalent to

(I) \hspace{1cm} \text{l.c.}(l) \cdot \text{l.c.}(D)^{\deg l} \in k^\times \text{ if } D \text{ is odd,} 

(3.7) \hspace{1cm} l \text{ splits completely in } E,

(3.8) \hspace{1cm} \left(\frac{l}{r}\right) = 1, \text{ for each monic irreducible factor } r \mid D.

For the Artin condition, let $\prod_{p} (x_p, y_p) \subseteq \prod_{p} X(o_{F_p})$ be a local solution. Then

\begin{equation}
(x_p + \sqrt{-D}y_p)(x_p - \sqrt{-D}y_p) = l \text{ in } E_p \text{ with } \mathbb{P} | p.
\end{equation}

Let $l = t \sigma_F$. Thus for all $p \nmid p_\infty$, $\tilde{f}_E([x_p, y_p]) \in L^\times_p$ by (3.6) and (3.9). It follows that

\[ \psi_{K_\mathbb{P}_\infty^+/E}(\tilde{f}_E([x_p, y_p])) = 1 \text{ for all } p \nmid p_\infty, \]

where $\tilde{f}_E([x_p, y_p])$ is regarded as an element in $I_E$ such that the component above $p$ is given by the value of $\tilde{f}_E([x_p, y_p])$ and 1 otherwise. For $p = 1$, by the local condition we already know that $l$ splits completely in $E/F$. Hence (3.9) tells us that one of $v_1(x_1 \pm \sqrt{-D}y_1)$ is 1 and the other 0. Suppose
$v_1(x_t + \sqrt{-D}y_t) = 1$ and let $l = \mathfrak{L}\mathfrak{L}$ in $E$. Note that $L = \mathfrak{L}$ and $L^\times = (\mathfrak{L})^\times = \mathfrak{L}_E^\times \times \mathfrak{L}_E^\times$, so both $\mathfrak{L}$ and $\mathfrak{L}$ are unramified in $K_{\mathfrak{p}_\infty}^+ / E$. It follows that

$$\sigma_{\mathfrak{L}} = \psi_{K_{\mathfrak{p}_\infty}^+ / E}(f_E([x_t, y_t])) = \psi_{K_{\mathfrak{p}_\infty}^+ / E}(l_\mathfrak{L}) \in \text{Gal}(K_{\mathfrak{p}_\infty}^+ / E)$$

where $l_\mathfrak{L}$ is in $\mathfrak{L}$ such that its $\mathfrak{L}$ component is $l$ and the others 1, and $\sigma_{\mathfrak{L}}$ denotes the Frobenius automorphism of $\mathfrak{L}$ in $K_{\mathfrak{p}_\infty}^+ / E$.

For $p = p_\infty$, we have $l = \alpha \sigma$ where

$$\alpha = f_E([x_{p_\infty}, y_{p_\infty}]) = x_{p_\infty} + \sqrt{-D}y_{p_\infty}.$$ 

Then $\text{sgn}(\alpha) = (-1)^{\deg^* l}$ by (11 Proposition 4.3). Thus

$$\text{sgn}(\alpha) = \pm \sqrt{\text{sgn}(l)\text{sgn}(t)}.$$ 

Let

$$\sigma_{p_\infty} = \psi_{K_{\mathfrak{p}_\infty}^+ / E}(f_E([x_{p_\infty}, y_{p_\infty}])) \in \text{Gal}(K_{\mathfrak{p}_\infty}^+ / E).$$

So we obtain that the Artin condition $\psi_{K_{\mathfrak{p}_\infty}^+ / E}(f_E(\prod_p(x_p, y_p))) = 1$ is equivalent to

$$\sigma_{\mathfrak{L}} \sigma_{p_\infty} = 1.$$ 

Note that since $L = \mathfrak{L}$, we have

$$\Sigma_{p_\infty} = E_{p_\infty}^+ \times \prod_{p \neq p_\infty} I_p^\times = E_{p_\infty}^+ \times \prod_{p \neq p_\infty} \mathfrak{L}_p^\times.$$

It follows that for any $\beta \in E_{p_\infty}^+$, $\beta \in E_{p_\infty}^+ \Sigma_{p_\infty}$ if and only if $\text{sgn}(\beta) \in \mathfrak{L}_{p_\infty}^\times = k^\times$. We also note that $\text{sgn}(l)(-1)^{\deg^* l} \in k^\times$ since $\text{sgn}(l) = \text{lc}(l)\text{lc}(-D)^{-\deg^* l}$ by (1) of [11] p. 230. Using these facts, we distinguish two cases:

(i) $\text{sgn}(l)(-1)^{\deg^* l} \in k_{x^2}^\times$. Then $\text{sgn}(\alpha) = \pm \sqrt{\text{sgn}(l)(-1)^{\deg^* l}} \in k^\times$ and thus $\alpha_{p_\infty} \in E_{p_\infty}^+ \Sigma_{p_\infty}$. It follows that $\sigma_{p_\infty} = 1$. 

(ii) $\text{sgn}(l)(-1)^{\deg^* l} \notin k_{x^2}^\times$. But we already know that $\text{sgn}(l)(-1)^{\deg^* l} \in k^\times$. It follows that $\text{sgn}(\alpha) \notin k^\times$ and $\text{sgn}(\alpha^2) \in k_{x^2}^\times$, which is to say that $\alpha_{p_\infty}$ is of order 2 in $1_{E_{p_\infty} / E_{p_\infty}^+} \cong \text{Gal}(K_{\mathfrak{p}_\infty}^+ / E)$ and hence $\sigma_{p_\infty}$ is an element of order 2.

At this time we know that $l = x^2 + Dy^2$ is solvable over $k[t]$ if and only if $\mathfrak{L}$, $\mathfrak{L}_L$, $\mathfrak{L}_\mathfrak{L}$ hold and there is a local solution

$$\prod_{p \in \Omega_F} (x_p, y_p) \in \prod_{p \in \Omega_F} X(\mathfrak{L}_p)$$

such that the corresponding (3.10) holds. We will use this equivalence in the sequel.

We first consider $\mathfrak{L}$. Suppose that $\text{sgn}(l)(-1)^{\deg^* l} \in k_{x^2}^\times$. Then $\sigma_{p_\infty} = 1$. If $l = x^2 + Dy^2$ is solvable over $k[t]$, then by (3.7) $l$ splits completely in $E/F$. Moreover, (3.10) implies $\sigma_D = \sigma_\Sigma \sigma_{p_\infty} = 1$, i.e. $\Sigma$ splits completely in $K_{\mathfrak{p}_\infty}^+ / E$. Thus $l$ splits completely in $K_{\mathfrak{p}_\infty}^+ / E$ and $\left(\frac{\mathfrak{L}}{l}\right) = 1$ for each monic irreducible factor $l | D$ (which is $\mathfrak{L}$, and the same for the sequel). Conversely, if $l$ splits completely in $K_{\mathfrak{p}_\infty}^+ / E$ and $\left(\frac{\mathfrak{L}}{l}\right) = 1$ for each monic irreducible factor $r | D$, then $l$ also splits in $E$, i.e. (3.7) holds. Also we have $\sigma_D \sigma_{p_\infty} = \sigma_\Sigma = 1$ so (3.10) holds. At last note that $\text{sgn}(l)(-1)^{\deg^* l}$ is $k_{x^2}^\times$. Hence if $d$ odd, $\deg^* l = \deg^* D$ and

$$\text{sgn}(l)(-1)^{\deg^* l} = \text{lc}(l)\text{lc}(D)^{-\deg^* l} = \text{lc}(l)\text{lc}(D)^{-\deg l} \in k^\times.$$ 

Thus (11) holds. This completes the proof for $\mathfrak{L}$. 


To show (b), suppose that \( \text{sgn}(l)(-1)^{\deg l} \notin k^{\times 2} \). Then \( \sigma_{\mathfrak{p}_\infty} \) is of order 2. However, since \( \tilde{f}(x_{\mathfrak{p}_\infty}, y_{\mathfrak{p}_\infty}) \in \Xi_{\mathfrak{p}_\infty} \)

\[
\sigma_{\mathfrak{p}_\infty} \cap \mathfrak{K}_{\mathfrak{p}_\infty} = \psi_{\mathfrak{K}_{\mathfrak{p}_\infty}} / E(\tilde{f}(x_{\mathfrak{p}_\infty}, y_{\mathfrak{p}_\infty})) = 1.
\]

If \( l = x^2 + Dy^2 \) is solvable over \( k[l] \), then by (3.11) \( l \) splits in \( E/F \). By (3.10) we know that \( \sigma_E = \sigma_{\mathfrak{p}_\infty}^{-1} \) is of order 2. Also, by (3.11), \( \sigma_{E}|_{\mathfrak{K}_{\mathfrak{p}_\infty}} = (\sigma_{\mathfrak{p}_\infty}|_{\mathfrak{K}_{\mathfrak{p}_\infty}})^{-1} = 1 \). So we have \( l \) splits completely in \( \mathfrak{K}_{\mathfrak{p}_\infty} \) and the relative degree of \( l \) in \( \mathfrak{K}_{\mathfrak{p}_\infty} \) is 2. Conversely, if \( l \) splits completely in \( \mathfrak{K}_{\mathfrak{p}_\infty} \) and the relative degree of \( l \) in \( \mathfrak{K}_{\mathfrak{p}_\infty} \) is 2, then (3.11) holds for the same reason as in the prove for (a). In addition, if \( \deg D \) is odd,

\[
\text{sgn}(l)(-1)^{\deg l} = \text{lc}(l) \text{lc}(D)^{-\deg l} \text{lc}(D)^{-\deg l} \notin k^{\times 2},
\]

which is impossible. Thus (b) trivially holds. At last it suffices to show (3.10) for a local solution. Actually, since \( l \) splits completely in \( \mathfrak{K}_{\mathfrak{p}_\infty} \), we have \( \sigma_{E}|_{\mathfrak{K}_{\mathfrak{p}_\infty}} = 1 \). It follows that \( \sigma_E \in \text{Gal}(\mathfrak{K}_{\mathfrak{p}_\infty}/E) \). Also \( \sigma_E \) is of order 2 since the relative degree of \( l \) in \( \mathfrak{K}_{\mathfrak{p}_\infty} \) is 2. On the other hand, recall that \( \sigma_{\mathfrak{p}_\infty} \) is of order \( 2 \) in this case and we have \( \sigma_{\mathfrak{p}_\infty} \in \text{Gal}(\mathfrak{K}_{\mathfrak{p}_\infty}/E) \) by (3.11). Note that \( \text{Gal}(\mathfrak{K}_{\mathfrak{p}_\infty}/E) \) is cyclic (c.f. [16 Proposition 7.4.10]). Therefore \( \text{Gal}(\mathfrak{K}_{\mathfrak{p}_\infty}/E) \) has a unique subgroup \( \{\pm 1\} \) of order 2 and \( \sigma_E = \sigma_{\mathfrak{p}_\infty} = -1 \in \{\pm 1\} \). It follow that \( \sigma_E \sigma_{\mathfrak{p}_\infty} = 1 \), i.e. that (3.10) holds. The proof for (b) is finished.

**Remark 3.12.** If we assume that

\[
\deg D \text{ is odd or } D \text{ contains no odd degree irreducible factor,}
\]

then (3.13) is a special form of the above theorem in the case \( \text{sgn}(l) \in k^{\times 2} \). To see this it suffice to show that under the assumptions (3.13) and \( \text{sgn}(l) \in k^{\times 2} \), the local condition (3.8) is redundant. We will use a similar argument as in \( F = \mathbb{Q} \) case [17 Corollary 4.2]. Suppose that the solvable conditions in Theorem 3.5 hold. For monic irreducible \( r \mid D \), assuming (3.13) and \( \text{sgn}(l) \in k^{\times 2} \), quadratic reciprocity law implies that there exists \( u \in \mathfrak{O}_E^\times \) such that \( \left( \frac{r}{u} \right) = \left( \frac{u}{r} \right) \), and that one of \( ur \) and \( -D/(ur) \) have even degree and leading coefficient in \( k^{\times 2} \). Let \( r^* = ur \). We see that \( \sqrt{r^*} \in \mathfrak{K}_{\mathfrak{p}_\infty} \) if and only if

\[
\psi_{\mathfrak{p}}(i) \left( \sqrt{r^*} \right) = \sqrt{r^*} \text{ for all } i \in \Xi_{\mathfrak{p}_\infty}
\]

under the Artin map \( \psi_E \) of \( E \), which is equivalent to the product of quadratic Hilbert symbols

\[
\prod_{\mathfrak{p}} \left( \frac{r^*, i_\mathfrak{p}}{\mathfrak{p}} \right) = 1 \text{ for all } i = (i_\mathfrak{p})_\mathfrak{p} \in \Xi_{\mathfrak{p}_\infty}.
\]

Clearly \( E(\sqrt{r^*})/E \) is unramified at \( \mathfrak{p} \mid r \mathfrak{p}_\infty \), where \( r = r^* \mathfrak{O}_E \). Since one of \( r^* \) and \( -D/r^* \) has even degree and leading coefficient in \( k^{\times 2} \), \( \mathfrak{p}_\infty \) splits in one of \( E(\sqrt{r^*}) \) and \( E(\sqrt{-D/r^*}) \). It follows that \( \mathfrak{p}_\infty \) splits in \( E(\sqrt{r^*}) \) and then \( \left( \frac{r^*}{\mathfrak{p}_\infty} \right) = 1 \) for all \( \mathfrak{p} \mid r \). Since \( i \in \Xi_{\mathfrak{p}_\infty} = E(\sqrt{r^*}) \times \prod_{\mathfrak{p} \neq \mathfrak{p}_\infty} (\mathfrak{O}_E)^\times \), and \( \left( \sqrt{r^*} \right)_\mathfrak{p} = \left( \sqrt{r^*} \right) \mathfrak{O}_E \times \mathfrak{O}_E \), there exist \( a_p, b_p \in \mathfrak{O}_E \) for each \( \mathfrak{p} \neq \mathfrak{p}_\infty \) such that

\[
\begin{cases}
(i_\mathfrak{p}, i_\mathfrak{p}) = (a_p + \sqrt{-D}b_p, a_p - \sqrt{-D}b_p) & \text{if } \mathfrak{p} = \mathfrak{p}_\infty \text{ splits in } E/F, \\
i_\mathfrak{p} = a_p + \sqrt{-D}b_p & \text{otherwise}.
\end{cases}
\]

It follows that

\[
\prod_{\mathfrak{p} \mid r} \left( \frac{r^*, i_\mathfrak{p}}{\mathfrak{p}} \right) = \prod_{\mathfrak{p} \mid r} \left( \frac{r^*, a_p^2 + Db_p^2}{\mathfrak{p}} \right) = 1,
\]
where the last equality comes from [12] Ch.V (3.4) Proposition. Thus we have \( \sqrt{r} \in K_{\psi_{\infty}} \) and then \( l \) splits in \( F(\sqrt{r}) \) since it does in \( K_{\psi_{\infty}} \), which is to say \( \left( \frac{r}{l} \right) = (\frac{l}{r}) = 1 \) for each monic irreducible factor \( r \mid D \). This ensures the local condition (3.8).

We now give two examples where the explicit criteria are obtained using Theorem 3.3.

**Example 3.14.** Let \( k = \mathbb{F}_3 \) and \( g \in k[t] \), write
\[
g = u \times (t - 1)^{s_1} \times (t^2 - t - 1)^{s_2} \times \prod_{j=1}^{r} p_j^{m_j},
\]
where \( u \in k^\times, s_1, s_2, r \geq 0, m_j \geq 1, p_1, p_2, \ldots, p_r \neq t - 1, t^2 - t - 1 \) are distinct monic irreducible polynomial in \( k[t] \). Then the diophantine equation
\[
-x^2 + txy - (t^3 - t^2 + 1)y^2 + g = 0
\]
is solvable over \( k[t] \) if and only if
1. \( (\frac{2x + \tau(p)}{p}) = (-1)^{\nu_p(\theta)}, \) for \( p = t - 1 \) or \( t^2 - t - 1 \),
2. \( \left( \frac{(t - 1)(t^2 - t - 1)}{p} \right) = 1, \) for \( p \notdivides (t - 1)(t^2 - t - 1) \) with odd \( \nu_p(g) \).

**Proof.** In this example, we have \( a = -1, b = t, c = -(t^3 - t + 1) \) and \( -d = (b/2)^2 - ac = -(t - 1)(t^3 - t + 1) \). Since \( \deg d = 3 \) odd, the assumption (3.1) holds. In order to apply Theorem 3.3 let \( F = k(t), E = F(\sqrt{r^2 - t - 1}), L = \sigma_F + \sigma_F \sqrt{-d} \) and \( K_{\psi_{\infty}}^+ \) is the class field corresponding to \( \psi_{\infty} \), with respect to a fixed sign function. Since \( d \) is square free and \( \deg d \) is odd, we know that \( L = \sigma_E \) and \( K_{\psi_{\infty}}^+ = K_{\psi_{\infty}} \) is the Hilbert class field corresponding to
\[
E_{\psi_{\infty}}^\times \times \prod_{\psi \neq \psi_{\infty}} \sigma_{E_{\psi}}
\]
(c.f. [6] Proposition 7.4.10). Moreover, if the \( j \)-invariant of the Drinfeld \( \sigma_E \)-module corresponding to the lattice \( \sigma_E \), the Hilbert class field \( K_{\psi_{\infty}} \) is generated by \( j \) over \( E \). For proofs, see [5]. A concise introduction to the theory of Drinfeld modules can be found in Hayes [8]. Using Magma calculator [1] we have \( K_{\psi_{\infty}}^+ = K_{\psi_{\infty}} = E(\sqrt{t^2 - t - 1}) \). Thus \( K_{\psi_{\infty}}^+/F \) is abelian, and the Artin condition is trivially true by Remark 2.12 Then by Theorem 3.3 the equation (3.15) is solvable over \( k[t] \) if and only if it is locally solvable. By a simple calculation we know the local condition is equivalent to (1) and (2). The proof is complete. \( \square \)

**Example 3.16.** Let \( k = \mathbb{F}_3 \) and \( g \in k[t] \), write
\[
g = u \times q_1^{s_1} \times q_2^{s_2} \times \prod_{j=1}^{r} p_j^{m_j},
\]
where \( u \in k^\times, s_1, s_2, r \geq 0, m_j \geq 1, q_1 = t - 1, q_2 = t^2 + t - 1, \) and \( p_1, p_2, \ldots, p_r \neq q_1, q_2 \) are distinct monic irreducible polynomial in \( k[t] \). Let \( -d = -q_1q_2, \theta(X) = X^4 - (t^2 - t)X^2 - t^3 + 1 \in k[t][X] \) and
\[
D_1 = \{ p = p_1, \ldots, p_r \mid \left( \frac{-d}{p} \right) = 1 \text{ and } \theta(X) \text{ mod } p \text{ factors into two irreducible polynomials} \},
\]
\[
D_2 = \{ p = p_1, \ldots, p_r \mid \left( \frac{-d}{p} \right) = 1 \text{ and } \theta(X) \text{ mod } p \text{ is irreducible} \}.
\]
Then the diophantine equation
\[(3.17) \quad (t-1)x^2 + (t^2 + t-1)y^2 + g = 0\]
is solvable over \(k[t]\) if and only if
\[(1) \quad (\frac{a-x_p-v_p(g)}{p}) = (-1)^{\deg(p)} \text{, for } q_1 \text{ or } q_2,\]
\[(2) \quad (\frac{-\sqrt{d}}{p}) = 1, \text{ for } p \nmid d \text{ with odd } v_p(g),\]
and
\[(3) \quad D_2 = \emptyset \text{ and } \sum_{p \in (q_1,q_2) \cup D_1} v_p(g) \equiv 1 \pmod{2},\]
or
\[(4) \quad D_2 \neq \emptyset \text{ and } \sum_{p \in D_2} v_p(g) \equiv 0 \pmod{2}.\]

Proof. We have \(a = q_1, b = 0, c = q_2\) and \(-d = -q_1q_2\). Since \(-d\) is square free and \(\deg d\) is odd, we have \(F = k(t), E = F(\sqrt{-d}), L = \sigma_F + \sigma_F\sqrt{-d} \text{ and } K^+_{\wp,\infty} = K_{\wp,\infty}\) is the Hilbert class field as in the previous example. Using Magma calculator \([1]\) we have \(K^+_{\wp,\infty} = K_{\wp,\infty} = E[X]/\theta(X)\). Then by Theorem \([3.3]\) the equation \((3.17)\) is solvable over \(k[t]\) if and only if it is locally solvable and the Artin condition holds. It is easy to see that the local condition is equivalent to \((1)\) and \((2)\).

For the Artin condition, first we know that the discriminant of \(\theta\) is \(-q_1^4q_2\). Note that
\[
\text{Gal}(K^+_{\wp,\infty}/E) \cong \langle \sqrt{-1} \rangle
\]
is cyclic of order 4. This can be shown as follows. Let \(p_0 = t^2 + 1\). It is irreducible over \(k[x]\) and we have \((\frac{a}{p_0}) = 1\). Thus \(p_0 \sigma E = \wp_0 \wp_0\) splits. One can check that \(\theta(X)\) is irreducible over \(\wp_0/E\), \(\wp_0 \cong \wp/F_0\), it follows that the Frobenius automorphism \((\frac{a}{\wp_0})\) has order 4. Hence \(\text{Gal}(K^+_{\wp,\infty}/E)\) is cyclic of order 4. Next we only give a sketch of the calculation of Artin condition since it is very similar to \([10, \text{Example 1}]\) over \(F = \mathbb{Q}\). In the sequel we identify each finite place \(p\) of \(F\) as the unique monic irreducible polynomial in \(k[x]\) that generates it. We also write \(\infty\) for \(\wp\).

Let \(X = \text{Spec}(\sigma_F[x,y]/((t-1)((t-1)x^2 + (t^2 + t-1)y^2 + g))))\), and \((x_p,y_p) \in \prod_p X(\sigma_F)\). Note that \(\tilde{x} = q_1x, \tilde{y} = y \text{ and } n = -q_1g\).

(i) If \(p = q_1 \sigma F\), then \(\psi_{K^+_{\wp,\infty}/E} (\tilde{f}_E[(x_p,y_p)]) = (-1)^{v_p(g)+1}\).

(ii) If \(p = q_2 \sigma F\), then \(\psi_{K^+_{\wp,\infty}/E} (\tilde{f}_E[(x_p,y_p)]) = (-1)^{v_p(g)}\).

(iii) If \((\frac{-\sqrt{d}}{p}) = 1 \text{ and } \theta(X) \mod p \text{ splits into linear factors, then } \psi_{K^+_{\wp,\infty}/E} (\tilde{f}_E[(x_p,y_p)]) = 1\).

(iv) If \((\frac{-\sqrt{d}}{p}) = 1 \text{ and } \theta(X) \mod p \text{ splits into two irreducible factors, then } \psi_{K^+_{\wp,\infty}/E} (\tilde{f}_E[(x_p,y_p)]) = (-1)^{v_p(g)}\).

(v) If \((\frac{-\sqrt{d}}{p}) = 1 \text{ and } \theta(X) \mod p \text{ is irreducible, then } \psi_{K^+_{\wp,\infty}/E} (\tilde{f}_E[(x_p,y_p)]) = (\pm \sqrt{-1})^{v_p(g)}\)

with the sign chosen freely.

(vi) If \((\frac{-\sqrt{d}}{p}) = -1, \text{ then } \psi_{K^+_{\wp,\infty}/E} (\tilde{f}_E[(x_p,y_p)]) = 1\).

(vii) If \(p = \infty, \text{ then } \psi_{K^+_{\wp,\infty}/E} (\tilde{f}_E[(x_p,y_p)]) = 1 \text{ since } \infty \text{ splits completely in } K^+_{\wp,\infty}/E\).

Thus the Artin condition \(\psi_{K^+_{\wp,\infty}/E} (\tilde{f}_E(\prod_p (x_p,y_p))) = 1\) is exactly \((3)\). The proof is complete. \(\square\)
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