Quark mass and field anomalous dimensions to $\mathcal{O}(\alpha_s^5)$

P.A. Baikov,* K.G. Chetyrkin † and J.H. Kühn †

*Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University, 1(2), Leninskie gory, Moscow 119991, Russian Federation
†Institut für Theoretische Teilchenphysik, Karlsruhe Institute of Technology (KIT), Wolfgang-Gaede-Straße 1, 726128 Karlsruhe, Germany

E-mail: baikov@theory.sinp.msu.ru, Konstantin.Chetyrkin@kit.edu, johann.kuehn@kit.edu

ABSTRACT: We present the results of the first complete analytic calculation of the quark mass and field anomalous dimensions to $\mathcal{O}(\alpha_s^5)$ in QCD.

KEYWORDS: Renormalization Group, QCD, Quark Masses and SM Parameters

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1 Introduction

The quark masses depend on a renormalization scale. The dependence is usually referred to as “running” and is governed by the quark mass anomalous dimension, $\gamma_m$, defined as:

$$\mu^2 \frac{d}{d\mu^2} m |_{g_0, m_0} = m \gamma_m(a_s) \equiv -m \sum_{i \geq 0} \gamma_i a_{s}^{i+1},$$

(1.1)

where $a_s = \frac{\alpha_s}{\pi} = \frac{g^2}{4\pi^2}$, $g$ is the renormalized strong coupling constant and $\mu$ is the normalization scale in the customarily used $\overline{\text{MS}}$ renormalization scheme. Up to and including four loop level the anomalous dimension is known since long [1–5]. In this paper we will describe the results of calculation of $\gamma_m$ and a related quantity — the quark field anomalous dimension — in the five-loop order.

The evaluation of the quark mass anomalous dimension with five-loop accuracy has important implications. The Higgs boson decay rate into charm and bottom quarks is proportional to the square of the respective quark mass at the scale of $m_H$ and the uncertainty from the presently unknown 5-loop terms in the running of the quark mass is of order $10^{-3}$. This is comparable to the precision advocated for experiments e.g. at TLEP [6]. Similarly, the issue of Yukawa unification is affected by precise predictions for the anomalous quark mass dimension.

The paper is organized as follows. The next section deals with the overall set-up of the calculations. Then we present our results (section 3), and a brief discussion (section 4) as well as a couple of selected applications (section 5). Our short conclusions are given in section 6.
2 Technical preliminaries

To calculate $\gamma_m$ one needs to find the so-called quark mass renormalization constant, $Z_m$, which is defined as the ratio of the bare and renormalized quark masses, viz.

$$Z_m = \frac{m^0}{m} = 1 + \sum_{i,j}^{0<j\leq i} (Z_m)_{ij} \frac{a_s^{i}}{\epsilon}.$$  \hspace{1cm} (2.1)

Within the $\overline{\text{MS}}$ scheme [7, 8] the coefficients $(Z_m)_{ij}$ are just numbers [9]; $\epsilon \equiv 2 - D/2$ and $D$ stands for the space-time dimension. Combining eqs. (1.1), (2.1) and using the RG-invariance of $m^0$, one arrives at the following formula for $\gamma_m$:

$$\gamma_m = \sum_{i\geq 0} (Z_m)_{ii} i a_s^i.$$  \hspace{1cm} (2.2)

To find $Z_m$ one should compute the vector and scalar parts of the quark self-energy $\Sigma_V(p^2)$ and $\Sigma_S(p^2)$. In our convention, the bare quark propagator is proportional to \[\not{p} \left(1 + \Sigma_0^V(p^2)\right) - m^0 \not{p} \left(1 - \Sigma_0^S(p^2)\right)\]. Requiring the finiteness of the renormalized quark propagator and keeping only massless and terms linear in $m_q$, one arrives at the following recursive equations to find $Z_m$:

$$Z_m Z_2 = 1 + K_\epsilon \left\{ Z_m Z_2 \Sigma_0^S(p^2) \right\}, \quad Z_2 = 1 - K_\epsilon \left\{ Z_2 \Sigma_0^V(p^2) \right\},$$  \hspace{1cm} (2.3)

where $K_\epsilon \{ f(\epsilon) \}$ stands for the singular part of the Laurent expansion of $f(\epsilon)$ in $\epsilon$ near $\epsilon = 0$ and $Z_2$ is the quark wave function renormalization constant. Eqs. (2.3) express $Z_m$ through massless propagator-type (that is dependent on one external momentum only) Feynman integrals (FI), denoted as $p$-integrals below.

Eqs. (2.3) require the calculation of a large number\(^1\) of the five-loop p-integrals to find $Z_m$ and $Z_2$ to $\mathcal{O}(\alpha_s^5)$.

At present there exists no direct way to analytically evaluate five-loop p-integrals. However, according to (2.1) for a given five-loop p-integral we need to know only its pole part in $\epsilon$ in the limit of $\epsilon \to 0$. A proper use of this fact can significantly simplify our task. The corresponding method — so-called Infrared Rearrangement (IRR)—first suggested in [11] and elaborated further in [12–14] allows to effectively decrease number of loops to be computed by one.\(^2\) In its initial version IRR was not really universal; it was not applicable in some (though rather rare) cases of complicated FT’s. The problem was solved by elaborating a special technique of subtraction of IR divergences — the $R^*$-operation [15, 16]. This technique succeeds in expressing the UV counterterm of every L-loop Feynman integral in terms of divergent and finite parts of some (L-1)-loop massless propagators.

In our case $L = 5$ and, using IRR, one arrives at around 10\(^5\) four-loop p-integrals. These can, subsequently, be reduced to 28 four-loop master-integrals, which are known analytically, including their finite parts, from [17, 18] as well as numerically from [19].

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\(^1\)We have used QGRAF [10] to produce around 10\(^5\) FI’s contributing to the quark self-energy at $\mathcal{O}(\alpha_s^5)$.

\(^2\)With the price that resulting one-loop-less p-integrals should be evaluated up to and including their constant part in the small $\epsilon$-expansion.
We need, thus, to compute around $10^5$ p-integrals. Their singular parts, in turn, can be algebraically reduced to only 28 master 4-loop p-integrals. The reduction is based on evaluating sufficiently many terms of the $1/D$ expansion \cite{20} of the corresponding coefficient functions \cite{21}.

All our calculations have been performed on a SGI ALTIX 24-node IB-interconnected cluster of eight-cores Xeon computers using parallel MPI-based \cite{22} as well as thread-based \cite{23} versions of FORM \cite{24}.

### 3 Results

Our result for the anomalous dimension

$$\gamma_m = -\sum_{i=0}^{\infty} (\gamma_m)_i a_s^{i+1}$$

reads:

$$(\gamma_m)_0 = 1, \quad (\gamma_m)_1 = \frac{1}{16} \left\{ \frac{202}{3} + n_f \left[ -\frac{20}{9} \right] \right\},$$

$$(\gamma_m)_2 = \frac{1}{64} \left\{ 1249 + n_f \left[ -\frac{2216}{27} - \frac{160}{3} \zeta_3 + \frac{18400}{9} \zeta_5 \right] + n_f^2 \left[ -\frac{140}{81} \right] \right\},$$

$$(\gamma_m)_3 = \frac{1}{256} \left\{ \frac{4603055}{162} + \frac{135680}{27} \zeta_3 - 8800 \zeta_5 
+ n_f \left[ \frac{91723}{27} - \frac{34192}{9} \zeta_3 + 880 \zeta_4 + \frac{18400}{9} \zeta_5 \right] 
+ n_f^2 \left[ \frac{5242}{243} + \frac{800}{9} \zeta_3 - \frac{160}{3} \zeta_4 \right] + n_f^3 \left[ -\frac{332}{243} + \frac{64}{27} \zeta_3 \right] \right\},$$

$$(\gamma_m)_4 = \frac{1}{45} \left\{ \frac{9951327}{162} + \frac{46402466}{243} \zeta_3 + 96800 \zeta_5 \zeta_3 - \frac{698126}{9} \zeta_4 
- \frac{231757160}{243} \zeta_5 + 242000 \zeta_6 + 412720 \zeta_7 
+ n_f \left[ -\frac{150736283}{1458} - \frac{12538016}{81} \zeta_3 - \frac{75680}{9} \zeta_4^2 + \frac{2038742}{27} \zeta_4 \right] 
+ n_f^2 \left[ -\frac{49876180}{243} - \frac{638000}{9} \zeta_5 - \frac{1820000}{27} \zeta_7 \right] 
+ n_f^3 \left[ \frac{1320742}{729} + \frac{2010824}{243} \zeta_3 + \frac{46400}{27} \zeta_3^2 - \frac{166300}{27} \zeta_4 - \frac{264040}{81} \zeta_5 + \frac{92000}{27} \zeta_6 \right] 
+ n_f^4 \left[ -\frac{260}{243} - \frac{320}{243} \zeta_3 + \frac{64}{27} \zeta_4 \right] \right\}. \quad (3.4)$$

Here $\zeta$ is the Riemann zeta-function ($\zeta_3 = 1.202056903 \ldots$, $\zeta_4 = \pi^4/90$, $\zeta_5 = 1.036927755 \ldots$, $\zeta_6 = 1.017343062 \ldots$ and $\zeta_7 = 1.008349277 \ldots$). Note that in four-loop order we exactly\footnote{This agreement can be also considered as an important check of all our setup which is completely different from the ones utilized at the four-loop calculations.} reproduce well-known results obtained in \cite{4, 5}. The $n_f^3$ and $n_f^4$
terms in (3.4) are in full agreement with the results derived previously on the basis of the $1/n_f$ method in [25–27].

For completeness we present below the result for the quark field anomalous dimension $\gamma_2 = -\sum_{i=0}^{\infty} (\gamma_2)_i a_s^{i+1}$:

\[ (\gamma_2)_4 = \frac{1}{4^5} \left\{ \frac{2798900231}{7776} + \frac{17969627}{864} - \frac{13214911}{648} \zeta_3 + \frac{16730765}{864} \zeta_4 - \frac{832567417}{3888} \zeta_5 + \frac{40109575}{1296} \zeta_6 + \frac{124597529}{1728} \zeta_7 \\
+ n_f \left[ -\frac{861347053}{11664} - \frac{274621439}{11664} \zeta_3 + \frac{1960337}{972} \zeta_3^2 + \frac{465395}{1296} \zeta_4 + \frac{22169149}{5832} \zeta_5 + \frac{16743377}{1944} \zeta_6 + \frac{3443909}{216} \zeta_7 \right] \\
+ n_f^2 \left[ -\frac{37300355}{11664} + \frac{1349831}{486} \zeta_3 - \frac{128}{9} \zeta_3^2 - \frac{27415}{54} \zeta_4 - \frac{12079}{27} \zeta_5 - \frac{800}{9} \zeta_6 - \frac{1323}{2} \zeta_7 \right] \\
+ n_f^3 \left[ -\frac{114049}{8748} - \frac{1396}{81} \zeta_3 + \frac{208}{9} \zeta_4 \right] + n_f^4 \left[ \frac{332}{729} - \frac{64}{81} \zeta_4 \right]\right\} \] (3.5)

The above result is presented for the Feynman gauge; the coefficients $(\gamma_2)_i$ with $i \leq 3$ can be found in [28] (for the case of a general covariant gauge and SU(N) gauge group).

4 Discussion

In numerical form $\gamma_m$ reads

\[ \gamma_m = -a_s - a_s^2 (4.20833 - 0.138889n_f) - a_s^3 (19.5156 - 2.28412n_f - 0.0270062n_f^2) - a_s^4 (98.9434 - 19.1075n_f + 0.276163n_f^2 + 0.00579322n_f^3) - a_s^5 (559.7069 - 143.6864n_f + 7.4824n_f^2 + 0.1083n_f^3 - 0.000085359n_f^4) \] (4.1)

and

\[ \gamma_m \overset{n_f=3}{=} -a_s - 3.79167a_s^2 - 12.4202a_s^3 - 44.2629a_s^4 - 198.907a_s^5, \]

\[ \gamma_m \overset{n_f=4}{=} -a_s - 3.65278a_s^2 - 9.94704a_s^3 - 27.3029a_s^4 - 111.59a_s^5, \]

\[ \gamma_m \overset{n_f=5}{=} -a_s - 3.51389a_s^2 - 7.41986a_s^3 - 11.0343a_s^4 - 41.8205a_s^5, \]

\[ \gamma_m \overset{n_f=6}{=} -a_s - 3.37500a_s^2 - 4.83867a_s^3 + 4.50817a_s^4 + 9.76016a_s^5. \] (4.2)

Note that significant cancellations between $n_f^0$ and $n_f^1$ terms for the values of $n_f$ around 3 or so persist also at five-loop order. As a result we observe a moderate growth of the series in $a_s$ appearing in the quark mass anomalous dimension at various values of active quark flavours (recall that even for scales as small as 2 GeV $a_s \equiv \frac{\alpha_s}{\pi} \approx 0.1$).
\( n_f \) & 3 & 4 & 5 & 6 \\
\hline
\( (\gamma_m)_{\text{exact}} \) & 198.899 & 111.579 & 41.807 & -9.777 \\
\hline
\( (\gamma_m)_{\text{APAP}}^{[29]} \) & 162.0 & 67.1 & -13.7 & -80.0 \\
\( (\gamma_m)_{\text{APAP}}^{[30]} \) & 163.0 & 75.2 & 12.6 & 12.2 \\
\( (\gamma_m)_{\text{APAP}}^{[31]} \) & 164.0 & 71.6 & -4.8 & -64.6 \\
\hline

Table 1. The exact results for \( (\gamma_m) \) together with the predictions made with the help of the original APAP method and its two somewhat modified versions.

Similar behavior shows up for \( \gamma_2 \):

\[
\gamma_2 = -0.3333 a_s - a_s^2 (-1.9583 + 0.08333 n_f) \\
- a_s^3 (-10.3370 + 1.0877 n_f - 0.01157 n_f^2) \\
- a_s^4 (-53.0220 + 10.1090 n_f - 0.27703 n_f^2 - 0.0023 n_f^3) \\
- a_s^5 (-310.0700 + 76.3260 n_f - 4.6339 n_f^2 + 0.0085 n_f^3 + 0.00048 n_f^4) \\
\] (4.3)

and

\[
\gamma_2_{\text{APAP}} \begin{array}{c}
\text{for } n_f = 3 \\
\text{for } n_f = 4 \\
\text{for } n_f = 5 \\
\text{for } n_f = 6 \\
\end{array} \\
-0.3333 a_s - 1.7083 a_s^2 - 7.1779 a_s^3 - 25.2480 a_s^4 - 122.5300 a_s^5, \\
-0.3333 a_s - 1.6250 a_s^2 - 6.1712 a_s^3 - 17.1610 a_s^4 - 78.2430 a_s^5, \\
-0.3333 a_s - 1.5417 a_s^2 - 5.1877 a_s^3 - 9.6824 a_s^4 - 42.9240 a_s^5, \\
-0.3333 a_s - 1.4583 a_s^2 - 4.2274 a_s^3 - 2.8251 a_s^4 - 16.4710 a_s^5. \\
\] (4.4)

It is instructive to compare our numerical result for \( (\gamma_m) \)

\[
(\gamma_m)_{\text{APAP}} = 559.71 - 143.6 n_f + 7.4824 n_f^2 + 0.1083 n_f^3 - 0.00008535 n_f^4 \\
\] (4.5)

with a 15 years old prediction based on the “Asymptotic Pade Approximants” (APAP) method \([29]\) (the \( n_f^4 \) term below was used as the input)

\[
(\gamma_m)_{\text{APAP}} = 530 - 143 n_f + 6.67 n_f^2 + 0.037 n_f^3 - 0.00008535 n_f^4. \\
\] (4.6)

Unfortunately, this impressively good agreement does not survive for fixed values of \( n_f \) due to severe cancellations between different powers of \( n_f \) as one can see from the table 1.

The solution of eq. (1.1) reads:

\[
\frac{m(\mu)}{m(\mu_0)} = \frac{c(a_s(\mu))}{c(a_s(\mu_0))}, \quad c(x) = \exp \left\{ \int dx' \frac{\gamma_m(x')}{\beta(x')} \right\}. \\
\] (4.7)

\[
c(x) = (x)^{\gamma_0} \left\{ 1 + d_1 x + (d_1^2/2 + d_2) x^2 + (d_1^3/6 + d_1 d_2 + d_3) x^3 \\
+ (d_1^4/24 + d_1^2 d_2/2 + d_2^2/2 + d_1 d_3 + d_4) x^4 + O(x^5) \right\}, \\
\] (4.8)
\[ d_1 = -\bar{\beta}_1 \bar{\gamma}_0 + \bar{\gamma}_1, \]
\[ d_2 = \bar{\beta}_1^2 \bar{\gamma}_0/2 - \bar{\beta}_2 \bar{\gamma}_0/2 - \bar{\beta}_1 \bar{\gamma}_1/2 + \bar{\gamma}_2/2, \]
\[ d_3 = -\bar{\beta}_1^3 \bar{\gamma}_0/3 + 2 \bar{\beta}_1 \bar{\beta}_2 \bar{\gamma}_0/3 - \bar{\beta}_3 \bar{\gamma}_0/3 + \bar{\beta}_1^2 \bar{\gamma}_1/3 - \bar{\beta}_2 \bar{\gamma}_1/3 - \bar{\beta}_1 \bar{\gamma}_2/3 + \bar{\gamma}_3/3, \]
\[ d_4 = \bar{\beta}_1^4 \bar{\gamma}_0/4 - 3 \bar{\beta}_1^2 \bar{\beta}_2 \bar{\gamma}_0/4 + \bar{\beta}_2^2 \bar{\gamma}_0/4 + \bar{\beta}_1 \bar{\beta}_3 \bar{\gamma}_0/2 - \bar{\beta}_4 \bar{\gamma}_0/4 - \bar{\beta}_3 \bar{\gamma}_1/4 + \bar{\beta}_1 \bar{\beta}_2 \bar{\gamma}_1/2 - \bar{\beta}_3 \bar{\gamma}_1/4 + \bar{\beta}_2 \bar{\gamma}_2/4 - \bar{\beta}_1 \bar{\gamma}_3/4 + \bar{\gamma}_4/4. \]

Here \( \bar{\gamma}_i = (\gamma_m)_i / \beta_0 \), \( \bar{\beta}_i = \beta_i / \beta_0 \) and

\[ \beta(a_s) = - \sum_{i \geq 0} \beta_i a_s^{i+2} = - \beta_0 \left\{ \sum_{i \geq 0} \bar{\beta}_i a_s^{i+2} \right\} \]

is the QCD \( \beta \)-function. Unfortunately, the coefficient \( d_4 \) in eq. (4.12) does depend on the yet unknown five-loop coefficient \( \beta_4 \) (up to four loops the \( \beta \)-function is known from [14, 32–39]).

Numerically, the \( c \)-function reads:

\[ c(x) \equiv \frac{1}{n_f - 3} x^{4/9} c_s(x), \quad c(x) \equiv \frac{1}{n_f - 4} x^{12/25} c_c(x), \quad c(x) \equiv \frac{1}{n_f - 5} x^{12/23} c_b(x), \quad c(x) \equiv \frac{1}{n_f - 6} x^{4/7} c_l(x), \]

with

\[ c_s(x) = 1 + 0.8950 x + 1.3714 x^2 + 1.9517 x^3 + (15.6982 - 0.1111 \bar{\beta}_4) x^4, \]
\[ c_c(x) = 1 + 1.0141 x + 1.3892 x^2 + 1.0905 x^3 + (9.1104 - 0.1200 \bar{\beta}_4) x^4, \]
\[ c_b(x) = 1 + 1.1755 x + 1.5007 x^2 + 0.17248 x^3 + (2.69277 - 0.13046 \bar{\beta}_4) x^4, \]
\[ c_l(x) = 1 + 1.3980 x + 1.7935 x^2 - 0.68343 x^3 + (-3.5130 - 0.14286 \bar{\beta}_4) x^4. \]

5 Applications

5.1 RGI mass

Eq. (4.7) naturally leads to an important concept: the RGI mass

\[ m^{\text{RGI}} = m(\mu_0) / c(a_s(\mu_0)), \]

which is often used in the context of lattice calculations. The mass is \( \mu \) and scheme independent; in any (mass-independent) scheme

\[ \lim_{\mu \to \infty} a_s(\mu)^{-\gamma_0} m(\mu) = m^{\text{RGI}}. \]

The function \( c_s(x) \) is used, e.g., by the ALPHAv4 lattice collaboration to find the \( \overline{\text{MS}} \) mass of the strange quark at a lower scale, say, \( m_s(2 \text{ GeV}) \) from the \( m_s^{\text{RGI}} \) mass determined from lattice simulations (see, e.g., [40]). For example, setting \( a_s(\mu = 2 \text{ GeV}) = \frac{a_s(\mu)}{\bar{\beta}_4} = 0.1 \), we arrive at (\( h \) counts loops):

\[ m_s(2 \text{ GeV}) = m_s^{\text{RGI}} (a_s(2 \text{ GeV}))^{\frac{1}{2}} \left( 1 + 0.0895 h^2 + 0.0137 h^3 + 0.00195 h^4 + (0.00157 - 0.000011 \bar{\beta}_4) h^5 \right) \]
In order to have an idea of effects due to the five-loop term in (5.2) one should make a guess about \( \bar{\beta}_4 \). By inspecting lower orders in

\[
\beta(n_f = 3) = -\left( \frac{4}{9} \right) (a_s + 1.777 a_s^2 + 4.4711 a_s^3 + 20.990 a_s^4 + \bar{\beta}_4 a_s^5)
\]

one can assume a natural estimate of \( \bar{\beta}_4 \) as laying in the interval 50–100. With this choice we conclude that the (apparent) convergence of the above series is quite good even at a rather small energy scale of 2 GeV.

On the other hand, the authors of [30] estimate \( \bar{\beta}_4 \) in the \( n_f = 3 \) QCD as large as -850! With such a huge and negative value of \( \bar{\beta}_4 \) the five-loop term in (5.2) would amount to 0.01092 and, thus, would significantly exceed the four-loop contribution (0.00195).

### 5.2 Higgs decay into quarks

The decay width of the Higgs boson into a pair of quarks can be written in the form

\[
\Gamma(H \to \bar{f}f) = \frac{G_F M_H}{4\sqrt{2}\pi} m_f^2(\mu) R^S(s = M_H^2, \mu)
\]

where \( \mu \) is the normalization scale and \( R^S \) is the spectral density of the scalar correlator, known to \( \alpha_s^4 \) from [41]

\[
R^S(s = M_H^2, \mu = M_H) = 1 + 5.667 a_s + 29.147 a_s^2 + 41.758 a_s^3 - 825.7 a_s^4
= 1 + 0.2041 + 0.0379 + 0.0020 - 0.00140
\]

where we set \( a_s = \alpha_s / \pi = 0.0360 \) (for the Higgs mass value \( M_H = 125 \) GeV and \( \alpha_s(M_Z) = 0.118 \)).

Expression (5.3) depends on two phenomenological parameters, namely, \( \alpha_s(M_H) \) and the quark running mass \( m_q \). In what follows we consider, for definiteness, the dominant decay mode \( H \to \bar{b}b \). To avoid the appearance of large logarithms of the type \( \ln \mu^2 / M_H^2 \) the parameter \( \mu \) is customarily chosen to be around \( M_H \). However, the starting value of \( m_b \) is usually determined at a much smaller scale (typically around 5-10 GeV [42]). The evolution of \( m_b(\mu) \) from a lower scale to \( \mu = M_H \) is described by a corresponding RG equation which is completely fixed by the quark mass anomalous dimension \( \gamma(\alpha_s) \) and the QCD beta function \( \beta(\alpha_s) \) (for QCD with \( n_f = 5 \)). In order to match the \( \mathcal{O}(\alpha_s^4) \) accuracy of (5.4) one should know both RG functions \( \beta \) and \( \gamma_m \) in the five-loop approximation. Let us proceed, assuming conservatively that \( 0 \leq \bar{\beta}_4^{n_f=5} \leq 200 \).

The value of \( m_b(\mu = M_H) \) is to be obtained with RG running from \( m_b(\mu = 10 \) GeV) and, thus, depends on \( \beta \) and \( \gamma_m \). Using the Mathematica package RunDec\(^4\) [43] and eq. (4.13) we find for the shift from the five-loop term

\[
\frac{\delta m_b^2(M_H)}{m_b^2(M_H)} = -1.3 \cdot 10^{-4}(\bar{\beta}_4 = 0) - 4.3 \cdot 10^{-4}(\bar{\beta}_4 = 100) - 7.3 \cdot 10^{-4}(\bar{\beta}_4 = 200)
\]

\(^4\)We have extended the package by including the five-loop effects to the running of \( \alpha_s \) and quark masses.
If we set $\mu = M_H$, then the combined effect of $\mathcal{O}(\alpha_s^4)$ terms as coming from the five-loop running and four-loop contribution to $R^S$ on

$$\Gamma(H \to b\bar{b}) = \frac{G_F M_H}{4\sqrt{2}\pi} m_f^2(M_H) R^S(s = M_H^2, M_H)$$

(5.5)

is around -2‰ (for $\bar{\beta}_4 = 100$). This should be contrasted to the parametric uncertainties coming from the input parameters $\alpha_s(M_Z) = 0.1185(6)$ [44] and $m_b(m_b) = 4.169(8) \text{ GeV}$ [45] which correspond to $\pm 1\%$ and $\pm 4\%$ respectively. We conclude, that the $\mathcal{O}(\alpha_s^4)$ terms in (5.4), (5.5) are of no phenomenological relevancy at present. But, the situation could be different if the project of TLEP [6] is implemented. For instance, the uncertainty in $\alpha_s(M_Z)$ could be reduced to $\pm 2\%$ and Higgs boson branching ratios with precisions in the permille range are advertised.

6 Conclusions

We have analytically computed the anomalous dimensions of the quark mass $\gamma_m$ and field $\gamma_2$ in the five loop approximation. The self-consistent description of the quark mass evolution at five loop requires the knowledge of the QCD $\beta$-function to the same number of loops. The corresponding, significantly more complicated calculation is under consideration.

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References

[1] R. Tarrach, The Pole Mass in Perturbative QCD, Nucl. Phys. B 183 (1981) 384 [nSPIRE].
[2] O.V. Tarasov, Anomalous dimensions of quark masses in three loop approximation, JINR-P2-82-900.
[3] S.A. Larin, The Renormalization of the axial anomaly in dimensional regularization, Phys. Lett. B 303 (1993) 113 [hep-ph/9302240] [nSPIRE].
[4] K.G. Chetyrkin, Quark mass anomalous dimension to $O(\alpha_s^4)$, Phys. Lett. B 404 (1997) 161 [hep-ph/9703278] [nSPIRE].
[5] J.A.M. Vermaseren, S.A. Larin and T. van Ritbergen, The four loop quark mass anomalous dimension and the invariant quark mass, Phys. Lett. B 405 (1997) 327 [hep-ph/9703284] [nSPIRE].
[6] TLEP DESIGN STUDY WORKING GROUP collaboration, M. Bicer et al., First Look at the Physics Case of TLEP, JHEP 01 (2014) 164 [arXiv:1308.6176] [nSPIRE].
[7] G. ’t Hooft and M.J.G. Veltman, *Regularization and Renormalization of Gauge Fields*, Nucl. Phys. B 44 (1972) 189 [SPIRE].

[8] W.A. Bardeen, A.J. Buras, D.W. Duke and T. Muta, *Deep Inelastic Scattering Beyond the Leading Order in Asymptotically Free Gauge Theories*, Phys. Rev. D 18 (1978) 3998 [SPIRE].

[9] J.C. Collins, *Normal Products in Dimensional Regularization*, Nucl. Phys. B 92 (1975) 477 [SPIRE].

[10] P. Nogueira, *Automatic Feynman graph generation*, J. Comput. Phys. 105 (1993) 279 [SPIRE].

[11] A.A. Vladimirov, *Method for Computing Renormalization Group Functions in Dimensional Renormalization Scheme*, Theor. Math. Phys. 43 (1980) 417 [SPIRE].

[12] D.I. Kazakov, O.V. Tarasov and A.A. Vladimirov, *Calculation of Critical Exponents by Quantum Field Theory Methods*, Sov. Phys. JETP 50 (1980) 345 [SPIRE].

[13] K.G. Chetyrkin and V.A. Smirnov, *New Approach to Evaluation of Multiloop Feynman Integrals: The Gegenbauer Polynomial x Space Technique*, Nucl. Phys. B 174 (1980) 345 [SPIRE].

[14] O.V. Tarasov, A.A. Vladimirov and A.Y. Zharkov, *The Gell-Mann-Low Function of QCD in the Three Loop Approximation*, Phys. Lett. B 93 (1980) 429 [SPIRE].

[15] K.G. Chetyrkin and V.A. Smirnov, *R* operation corrected, Phys. Lett. B 144 (1984) 419 [SPIRE].

[16] K.G. Chetyrkin, *Corrections of order α³ to Rhad in pQCD with light gluinos*, Phys. Lett. B 391 (1997) 402 [hep-ph/9608480] [SPIRE].

[17] P.A. Baikov and K.G. Chetyrkin, *Four Loop Massless Propagators: An Algebraic Evaluation of All Master Integrals*, Nucl. Phys. B 837 (2010) 40 [arXiv:1004.1153] [SPIRE].

[18] R.N. Lee, A.V. Smirnov and V.A. Smirnov, *Master Integrals for Four-Loop Massless Propagators up to Transcendentality Weight Twelve*, Nucl. Phys. B 856 (2012) 95 [arXiv:1108.0732] [SPIRE].

[19] A.V. Smirnov and M. Tentyukov, *Four Loop Massless Propagators: a Numerical Evaluation of All Master Integrals*, Nucl. Phys. B 837 (2010) 40 [arXiv:1004.1149] [SPIRE].

[20] P.A. Baikov, *A Practical criterion of irreducibility of multi-loop Feynman integrals*, Phys. Lett. B 634 (2006) 325 [hep-ph/0507053] [SPIRE].

[21] P.A. Baikov, *Explicit solutions of the three loop vacuum integral recurrence relations*, Phys. Lett. B 385 (1996) 404 [hep-ph/9603267] [SPIRE].

[22] M. Tentyukov et al., *ParFORM: Parallel Version of the Symbolic Manipulation Program FORM*, arXiv:cs/0407066.

[23] M. Tentyukov and J.A.M. Vermaseren, *The Multithreaded version of FORM*, Comput. Phys. Commun. 181 (2010) 1419 [hep-ph/0702279] [SPIRE].

[24] J.A.M. Vermaseren, *New features of FORM*, math-ph/0010025 [SPIRE].

[25] A. Palanques-Mestre and P. Pascual, *The 1/Nf Expansion of the γ and β-functions in QED*, Commun. Math. Phys. 95 (1984) 277 [SPIRE].

[26] M. Ciuchini, S.E. Derkachov, J.A. Gracey and A.N. Manashov, *Computation of quark mass anomalous dimension at O(1/N_f^3) in quantum chromodynamics*, Nucl. Phys. B 579 (2000) 56 [hep-ph/9912221] [SPIRE].
[27] M. Ciuchini, S.E. Derkachov, J.A. Gracey and A.N. Manashov, Quark mass anomalous dimension at $O(1/N_c^2)$ in QCD, Phys. Lett. B 458 (1999) 117 [hep-ph/9903410] [inSPIRE].

[28] K.G. Chetyrkin and A. Retey, Renormalization and running of quark mass and field in the regularization invariant and MS-bar schemes at three loops and four loops, Nucl. Phys. B 583 (2000) 3 [hep-ph/9910332] [inSPIRE].

[29] J.R. Ellis, I. Jack, D.R.T. Jones, M. Karliner and M.A. Samuel, Asymptotic Pade approximant predictions: Up to five loops in QCD and SQCD, Phys. Rev. D 57 (1998) 2665 [hep-ph/9710302] [inSPIRE].

[30] V. Elias, T.G. Steele, F. Chishtie, R. Migneron and K.B. Sprague, Pade improvement of QCD running coupling constants, running masses, Higgs decay rates and scalar channel sum rules, Phys. Rev. D 58 (1998) 116007 [hep-ph/9806324] [inSPIRE].

[31] A.L. Kataev and V.T. Kim, Higgs boson decay into bottom quarks and uncertainties of perturbative QCD predictions, arXiv:0804.3992 [inSPIRE].

[32] D.J. Gross and F. Wilczek, Ultraviolet Behavior of Nonabelian Gauge Theories, Phys. Rev. Lett. 30 (1973) 1343 [inSPIRE].

[33] H.D. Politzer, Reliable Perturbative Results for Strong Interactions?, Phys. Rev. Lett. 30 (1973) 1346 [inSPIRE].

[34] W.E. Caswell, Asymptotic Behavior of Nonabelian Gauge Theories to Two Loop Order, Phys. Rev. Lett. 33 (1974) 244 [inSPIRE].

[35] D.R.T. Jones, Two Loop Diagrams in Yang-Mills Theory, Nucl. Phys. B 75 (1974) 531 [inSPIRE].

[36] E. Egorian and O.V. Tarasov, Two Loop Renormalization of the QCD in an Arbitrary Gauge, Teor. Mat. Fiz. 41 (1979) 26 [inSPIRE].

[37] S.A. Larin and J.A.M. Vermaseren, The Three loop QCD $\beta$-function and anomalous dimensions, Phys. Lett. B 303 (1993) 334 [hep-ph/9302208] [inSPIRE].

[38] T. van Ritbergen, J.A.M. Vermaseren and S.A. Larin, The Four loop $\beta$-function in quantum chromodynamics, Phys. Lett. B 400 (1997) 379 [hep-ph/9701390] [inSPIRE].

[39] M. Czakon, The Four-loop QCD $\beta$-function and anomalous dimensions, Nucl. Phys. B 710 (2005) 485 [hep-ph/0411261] [inSPIRE].

[40] ALPHA collaboration, M. Della Morte et al., Non-perturbative quark mass renormalization in two-flavor QCD, Nucl. Phys. B 729 (2005) 117 [hep-lat/0507035] [inSPIRE].

[41] P.A. Baikov, K.G. Chetyrkin and J.H. Kuhn, Scalar correlator at $O(\alpha_s^3)$, Higgs decay into $b$-quarks and bounds on the light quark masses, Phys. Rev. Lett. 96 (2006) 012003 [hep-ph/0511063] [inSPIRE].

[42] K.G. Chetyrkin, J.H. Kuhn, A. Maier, P. Maierhofer, P. Marquard et al., Charm and Bottom Quark Masses: An Update, Phys. Rev. D 80 (2009) 074010 [arXiv:0907.2110] [inSPIRE].

[43] K.G. Chetyrkin, J.H. Kuhn and M. Steinhauser, RunDec: A Mathematica package for running and decoupling of the strong coupling and quark masses, Comput. Phys. Commun. 133 (2000) 43 [hep-ph/0004189] [inSPIRE].

[44] Particle Data Group collaboration, J. Beringer et al., Review of Particle Physics (RPP), Phys. Rev. D 86 (2012) 010001 [inSPIRE].

[45] A.A. Penin and N. Zerf, Bottom Quark Mass from $\Upsilon$ Sum Rules to $O(\alpha_s^3)$, JHEP 04 (2014) 120 [arXiv:1401.7038] [inSPIRE].