Lie Symmetries of the Self-Dual Yang-Mills Equations

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Abstract
We investigate Lie symmetries of the self-dual Yang-Mills equations in four-dimensional Euclidean space (SDYM). The first prolongation of the symmetry generating vector fields is written down, and its action on SDYM computed. Determining equations are then obtained and solved completely. Lie symmetries of SDYM in Euclidean space are in exact correspondence with symmetries of the full Yang-Mills equations in Minkowski space.

1 Introduction

The self-dual Yang-Mills equations in Euclidean space (SDYM) are the following first-order nonlinear partial differential equations:

\[
\left(\delta_{\mu\rho}\delta_{\nu\sigma} - \frac{1}{2} \epsilon_{\mu\nu\rho\sigma}\right) \left(\partial_\rho A_{a\sigma} - \partial_\sigma A_{a\rho} + C_{abc} A_{b\rho} A_{c\sigma}\right) = 0.
\] (1)

Greek indices label independent variables \(x_\nu\) and run from 0 to 3. Latin indices are associated with generators of a compact semisimple Lie algebra, with structure constants \(C_{abc}\). \(\delta_{\mu\rho}\) and \(\epsilon_{\mu\nu\rho\sigma}\) are the Kronecker and Levi-Civita symbols \((\epsilon_{0123} = 1)\), respectively. The \(A_{a\sigma}\) are dependent variables, namely, gauge potentials.

We shall be interested in Lie symmetries of Eq. (1). Consider a vector field \(v\) of the form

\[
v = H_\kappa \partial_\kappa + \Phi_{d\kappa} \frac{\partial}{\partial A_{d\kappa}},
\] (2)

where \(H_\kappa\) and \(\Phi_{d\kappa}\) are functions of \(x_\nu\) and \(A_{a\sigma}\). The first prolongation of \(v\) is defined as

\[
\text{pr}^{(1)}v = H_\kappa \partial_\kappa + \Phi_{d\kappa} \frac{\partial}{\partial A_{d\kappa}} + \Phi_{d\kappa\lambda} \frac{\partial}{\partial (\partial_\lambda A_{d\kappa})},
\] (3)

where \(\Phi_{d\kappa\lambda}\) is given by

\[
\Phi_{d\kappa\lambda} = \partial_\lambda \Phi_{d\kappa} - (\partial_\kappa H_\beta) \partial_\beta A_{d\kappa} + (\partial_\lambda A_{n\alpha}) \frac{\partial \Phi_{d\kappa}}{\partial A_{n\alpha}} - (\partial_\kappa A_{n\alpha}) (\partial_\beta A_{d\kappa}) \frac{\partial H_\beta}{\partial A_{n\alpha}}.
\] (4)

To obtain Lie symmetries of SDYM, we have to substitute (4) in (3) and let it act on (1). We obtain

\[
(\partial_\lambda \Phi_{a\kappa} + C_{abc} A_{b\lambda} \Phi_{c\kappa}) Z_{\mu\lambda\nu\kappa} - (\partial_\lambda A_{n\alpha})(\partial_\beta A_{a\kappa}) \frac{\partial H_\beta}{\partial A_{n\alpha}} Z_{\mu\lambda\nu\kappa}
+ (\partial_\lambda A_{n\alpha}) \left[ \frac{\partial \Phi_{a\kappa}}{\partial A_{n\alpha}} Z_{\mu\lambda\nu\kappa} - (\partial_\kappa H_\lambda) Z_{\mu\kappa\nu\alpha} \delta_{\alpha n} \right] = 0,
\] (5)
where
\[ Z_{\mu \lambda \nu \kappa} = \delta_{\mu \lambda} \delta_{\nu \kappa} - \delta_{\mu \kappa} \delta_{\nu \lambda} - \epsilon_{\mu \nu \lambda \kappa}. \] (6)

The vector field \( v \) generates a symmetry of SDYM provided that Eqs. (5) hold whenever SDYM hold. In other words, once SDYM are substituted in (5), the coefficients of each independent combination of derivatives of \( A_{a \sigma} \) must vanish. Note that SDYM can be written more explicitly as
\[ \partial_{\lambda} A_{n \alpha} - \partial_{\alpha} A_{n \lambda} = \partial_{0} A_{n 1} - \partial_{1} A_{n 0} + C_{n \lambda \kappa} (A_{b 0} A_{c 1} + A_{b 1} A_{c 0}), \] (7)
\[ \partial_{\lambda} A_{n 1} - \partial_{1} A_{n \lambda} = \partial_{0} A_{n 2} - \partial_{2} A_{n 0} + C_{n \kappa \alpha} (A_{b 0} A_{c 2} + A_{b 2} A_{c 0}), \] (8)
\[ \partial_{\lambda} A_{n 2} - \partial_{2} A_{n \lambda} = \partial_{0} A_{n 3} - \partial_{3} A_{n 0} + C_{n \kappa \alpha} (A_{b 0} A_{c 3} + A_{b 3} A_{c 0}). \] (9)

2 Determining equations

SDYM only involve combinations of derivatives \( \partial_{\lambda} A_{n \alpha} \) that are antisymmetric in \( \lambda \) and \( \alpha \). Coefficients of symmetric combinations must therefore vanish. Accordingly, we set to zero the coefficient of \( (\partial_{\lambda} A_{n \alpha})(\partial_{\mu} A_{m \nu}) \), symmetrized in \( \lambda \leftrightarrow \alpha \), \( \beta \leftrightarrow \kappa \) and, furthermore, in \( (\lambda n \alpha) \leftrightarrow (\beta m \kappa) \). The result is, \( \forall \mu, \nu, \lambda, \kappa, \beta, \alpha, a, m \) and \( n \)
\[ \delta_{am} \left\{ \frac{\partial H_{\beta}}{\partial A_{n \alpha}} Z_{\mu \lambda \nu \kappa} + \frac{\partial H_{\beta}}{\partial A_{n \lambda}} Z_{\mu \alpha \nu \kappa} + \frac{\partial H_{\kappa}}{\partial A_{n \alpha}} Z_{\mu \lambda \nu \beta} + \frac{\partial H_{\kappa}}{\partial A_{n \lambda}} Z_{\mu \alpha \nu \beta} \right\} 
+ \delta_{an} \left\{ \frac{\partial H_{\lambda}}{\partial A_{m \beta}} Z_{\mu \beta \nu \alpha} + \frac{\partial H_{\lambda}}{\partial A_{m \alpha}} Z_{\mu \kappa \nu \alpha} + \frac{\partial H_{\alpha}}{\partial A_{m \beta}} Z_{\mu \beta \nu \lambda} + \frac{\partial H_{\alpha}}{\partial A_{m \alpha}} Z_{\mu \kappa \nu \lambda} \right\} = 0. \] (10)

Taking \( \mu, \nu, \lambda \) and \( \kappa \) all different, \( \alpha = \lambda \) and \( a = m \neq n \), we get
\[ \epsilon_{\mu \nu \lambda \kappa} \frac{\partial H_{\beta}}{\partial A_{n \lambda}} + \epsilon_{\mu \nu \lambda \beta} \frac{\partial H_{\kappa}}{\partial A_{n \lambda}} = 0. \] (11)

The hat means that the summation convention is not to be carried over \( \lambda \). The last equation holds identically if and only if, \( \forall \beta, \lambda \) and \( n \)
\[ \frac{\partial H_{\beta}}{\partial A_{n \lambda}} = 0. \] (12)

Thus in Eqs. (5), all terms quadratic in partial derivatives of \( A_{d \kappa} \) vanish.

Let us now switch to terms linear in partial derivatives. The coefficient of \( \partial_{\lambda} A_{n \alpha} \), symmetrized in \( \lambda \leftrightarrow \alpha \), must vanish. Writing down the coefficient, and considering in turn all independent values of indices, one finds that conditions for this are that \( \forall a, n \) and \( \forall \lambda, \alpha \)
\[ \frac{\partial \Phi_{a \lambda}}{\partial A_{n \alpha}} - \frac{\partial \Phi_{a \alpha}}{\partial A_{n \lambda}} + (\partial_{\lambda} H_{\alpha} - \partial_{\alpha} H_{\lambda}) \delta_{an} = 0 \] (13)
and that, \( \forall a, n \) and \( \forall \nu, \alpha \neq \lambda \)
\[ \frac{\partial \Phi_{a \nu}}{\partial A_{n \alpha}} + \partial_{\nu} H_{\lambda} \delta_{an} = 0. \] (14)
For given $n$, of the six antisymmetric combinations $\partial_\lambda A_{n\alpha} - \partial_\alpha A_{n\lambda}$, only three are independent, others being constrained by Eqs. (7)–(9). The choice of independent combinations is arbitrary. We pick $(\lambda, \alpha) = (0, 1), (0, 2)$ and $(0, 3)$, and substitute dependent combinations as given by (7)–(9) in (5). The coefficient of independent combinations must then vanish. After some calculation we find that, $\forall a, n$ and $\forall \hat{\lambda}, \hat{\alpha}$

$$\frac{\partial \Phi_{a\hat{\alpha}}}{\partial A_{n\hat{\alpha}}} - \frac{\partial \Phi_{a\lambda}}{\partial A_{n\lambda}} - (\partial_{\hat{\alpha}} H_{\hat{\alpha}} - \partial_\lambda H_\lambda) \delta_{an} = 0$$

and that, $\forall a, n$ and $\forall \lambda, \alpha \neq$

$$\frac{\partial \Phi_{a\alpha}}{\partial A_{n\lambda}} + \frac{\partial \Phi_{a\lambda}}{\partial A_{n\alpha}} - (\partial_\alpha H_\lambda + \partial_\lambda H_\alpha) \delta_{an} = 0.$$  

There remain terms with no derivatives in $A_{d\kappa}$. To terms in (5) we must add the ones coming from the substitution of SDYM just before Eq. (15). The sum must vanish, yielding (indices $i$ and $j$ are summed from 1 to 3)

$$\left(\partial_\lambda \Phi_{a\kappa} + C_{abc} A_{b\alpha} \Phi_{c\kappa}\right) Z_{\mu\lambda\nu\kappa} + \frac{1}{4} C_{abc} \left(A_{bi} A_{cj} - \frac{1}{2} \epsilon_{ij\rho\sigma} A_{b\rho} A_{c\sigma}\right)$$

$$\cdot \left\{ \frac{\partial \Phi_{a\kappa}}{\partial A_{ni}} - \partial_{\kappa} H_i \delta_{an} \right\} Z_{\mu ij\nu\kappa} - \left(\frac{\partial \Phi_{a\kappa}}{\partial A_{nj}} - \partial_{\kappa} H_j \delta_{an} \right) Z_{\mu i\nu\kappa} = 0.$$  

### 3 Solution of Determining Equations

We proceed to solve Eqs. (12), (13), (14), (15), (16) and (17).

Eqs. (12) imply that $H_\beta$ is independent of $A_{n\lambda}$, that is, $H_\beta = H_\beta(x_\nu)$. Combining (13) with (15), we see that $\partial_\alpha H_\alpha$ is independent of $\hat{\alpha}$. Combining (14) with (16), we find that $\forall \lambda, \alpha \neq \hat{\alpha}$, $\partial_\alpha H_\lambda = -\partial_\lambda H_\alpha$. This means that

$$\partial_\kappa H_\lambda = f_{\lambda\kappa} + \delta_{\lambda\kappa} G,$$  

where $G$ and $f_{\lambda\kappa} = -f_{\kappa\lambda}$ are arbitrary functions of $x_\nu$. Coefficients of $\delta_{an}$ in (13), (15) and (16) have by now all vanished.

From (14) we see that $\Phi_{a\kappa}$ is independent of $A_{n\alpha}$ for $a \neq n$ and $\kappa \neq \alpha$. Moreover, $\Phi_{a\kappa}$ is linear in $A_{a\alpha}$. With (18), we can thus write

$$\Phi_{a\kappa} = f_{\kappa\alpha}(x_\nu) A_{a\alpha} + F_{a\kappa}(A_{m\kappa}, x_\nu).$$  

From (15), we see that $\forall a$ and $n, \partial \Phi_{a\alpha}/\partial A_{n\hat{\alpha}}$ is independent of $\hat{\alpha}$. A little thought shows that $\Phi_{a\kappa}$ must be linear in $A_{n\kappa}$, so that

$$\Phi_{a\kappa} = f_{\kappa\alpha}(x_\nu) A_{a\alpha} + h_{an}(x_\nu) A_{n\kappa} + F_{a\kappa}(x_\nu).$$

Eqs. (18) and (20) are the most general solutions of Eqs. (12), (13), (14), (15) and (16).
There remains to substitute (18) and (20) in (17), which must hold as an identity. In other words, the coefficients of each combinations of $A_{ab}$ must vanish. After some manipulations, there result the following equations:

\[ \partial_{\mu} F_{a\nu} - \partial_{\nu} F_{a\mu} - \epsilon_{\mu\nu\lambda\kappa} \partial_{\lambda} F_{a\kappa} = 0 \quad \forall \mu, \nu, a; \]  
\[ \partial_{\mu} h_{\hat{a}\hat{b}} - \partial_{\hat{b}} f_{\mu\hat{a}} - \epsilon_{\mu\hat{a}\hat{b}\lambda} \partial_{\lambda} F_{\mu\hat{a}} = 0 \quad \forall \hat{a}, \forall \mu, \hat{a} \neq; \]  
\[ \partial_{\mu} h_{an} - C_{anc} F_{c\mu} = 0 \quad \forall \mu, \forall a, n \neq; \]  
\[ C_{abn} h_{nc} - C_{acn} h_{nb} - C_{nbc} h_{an} + C_{abc} G = 0 \quad \forall a, b, c. \]

It was shown in [2] that the most general solution of Eqs. (24) is

\[ h_{an} = -G \delta_{an} + C_{anc} \chi_c, \]

where $\chi_c$ is an arbitrary function of $x_{\nu}$. Substituting (25) in (23), we see that

\[ F_{c\mu} = \partial_{\mu} \chi_c. \]

Thus, (21) holds identically. Substituting (25) in (22) yields $\forall \mu, \hat{a} \neq$

\[ - \partial_{\mu} G = \partial_{\hat{a}} f_{\mu\hat{a}} + \epsilon_{\mu\hat{a}\hat{b}\lambda} \partial_{\lambda} f_{\mu\hat{a}}. \]

There only remains to solve Eqs. (27). Note that they do not involve group indices. The general solution of (27) can be effected as follows.

1. Eliminate $G$ from the 12 Eqs. (27) to obtain 8 equations for $p = f_{10} + f_{23}, q = f_{20} + f_{31}$ and $r = f_{30} + f_{12}$. Show that all third derivatives of $p, q$ and $r$ vanish and that (all constants being arbitrary)

\[ p = e + e_0 t + e_1 x + e_2 y + e_3 z + e_{00}(t^2 + x^2 - y^2 - z^2) + 2e_{12}(xy + tz) + 2e_{13}(xz - ty), \]
\[ q = e' + e_3 t - e_2 x + e_1 y - e_0 z + e_{12}(t^2 - x^2 + y^2 - z^2) + 2e_{13}(tx + yz) + 2e_{00}(xy - tz), \]
\[ r = e'' - e_2 t - e_3 x + e_0 y + e_1 z + e_{13}(t^2 - x^2 - y^2 + z^2) + 2e_{12}(yz - tx) + 2e_{00}(xz + ty). \]

2. With the help of Eq. (18), express the 4 remaining equations in terms of the functions $p, q$ and $r$ and of second derivatives of $H_{\lambda}$.

3. Show that the resulting system has no solution for $e_{00}, e_{12}$ or $e_{13}$ different from zero, and that its most general solution is (with $b_{\lambda\alpha}$ antisymmetric and all constants arbitrary)

\[ H_{\lambda} = -\frac{1}{2} c_{\lambda\alpha} x_{\alpha} + c_{\alpha} x_{\lambda} x_{\alpha} + b_{\lambda\alpha} x_{\alpha} + dx_{\lambda} + a_{\lambda}. \]
Putting everything together, we find that

$$
\Phi_{\alpha\kappa} = (-c_\kappa x_\alpha + c_\alpha x_\kappa + b_{\kappa\alpha}) A_{\alpha\alpha} - (d + c_\alpha x_\alpha) A_{\alpha\kappa} + C_{\alpha\beta\gamma\delta} \chi_\delta A_{\beta\kappa} + \partial_\kappa \chi_\alpha.
$$

(32)

Eqs. (31) and (32) are the most general solution of the determining equations. Therefore, the corresponding vector field (2) generates Lie symmetries of SDYM. One can see that the constants $a_\mu$ correspond to uniform translations; that the $b_{\mu\nu}$ correspond to rotations in Euclidean space; that the $c_\mu$ correspond to uniform accelerations; that $d$ corresponds to dilatations; and that the functions $\chi_\alpha(x_\nu)$ correspond to local gauge transformations. This agrees with results found in the special case where the gauge group is SU(2) [3, 4]. Therefore, Lie symmetries of SDYM in Euclidean space correspond to symmetries of the Yang-Mills equations in Minkowski space, which are conformal transformations and gauge transformations [3, 4].

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