Matrix balls, radial analysis of Berezin kernels, and hypergeometric determinants

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The subject of this paper is a natural one-parametric family of Hilbert spaces $V_\alpha$ and representations interpolating $L^2$ on a Riemannian noncompact symmetric space $G/K$ and $L^2$ on the dual Riemannian compact symmetric space $G_{\text{comp}}/K$. The spaces $V_\alpha$ consist of holomorphic functions on some symmetric space $\tilde{G}/\tilde{K} \supset G/K$, the limit of $V_\alpha$ as $\alpha \to +\infty$ is $L^2(G/K)$, and the limit of $V_\alpha$ as $\alpha$ tends to $-\infty$ taking integer values is $L^2(G_{\text{comp}}/K)$. Even the case in which $G/K$ is the Lobachevskii plane $U(1,1)/U(1) \times U(1)$ and $G_{\text{comp}}/K$ is the two-dimensional sphere $U(2)/U(1) \times U(1)$, is nontrivial and is still not satisfactory clear today (see [57]).

More formally, we consider the problem of the restriction of a unitary highest weight representation of a semisimple group $G$ to a symmetric subgroup (see 4.7 below). This formulation is short and simple but it does not explain why this restriction problem differs from a nondenumerable collection of other restriction problems.

The objects of this paper first appeared in the short note of Berezin [4] in 1978. Berezin perished two years later and a complete text of his work never was published. For the second time, these objects occurred in a joint work of G.I.Olshanskii and author, but it was published only partially in two short notes [48] and [62].

As a result, this problem was "lost" and it became visible only in 1994-95 in [72] and [59]. Some other references in the last 5 years are [11], [29], [31], [32], [51]–[57], [60], [64], [79] (these papers can also be the source for many other references).

The present paper has three purposes.

1) We intend to survey phenomena arising in the analysis of the Berezin kernels. We consider a special case $G = U(p,q)$

In this case, there exist some specific tools that allow to avoid main difficulties existing in a general case. For instance, the complete Plancherel formula\footnote{In the case of Hermitian symmetric spaces $G/K$ the Plancherel formula for large values of the parameter $\alpha$ (see below) was announced in the Berezin work [4], proof was published by Upmeier and Unterberger in 1994 [72]; for rank 1 case it was obtained by van Dijk and Hille [11], for general situation it was obtained in [55],[57].} can be proved in a very simple way\footnote{Supported by the grants RFBR 98-01-00303 and NWO 047-008-009}.

2) For the case $G = U(p,q)$, we can also obtain some results that today are not achieved in the general situation. The main new result is an explicit construction of a unitary intertwining operator from $L^2(\mathbb{U}(p,q)/\mathbb{U}(q) \times \mathbb{U}(p))$ to the Berezin deformation of $L^2$.

3) Many multivariate special functions appear in a natural way in the harmonic analysis of the Berezin kernels (in particular, multivariate continuous dual Hahn, dual Hahn, Meixner–Pollachek, Krawtchouk, Laguerre, Jacobi orthogonal polynomials, Jack polynomials, Jack zonal functions, matrix $\Gamma$-function and B-function, matrix Bessel functions, Heckman-–Opdam multivariate hypergeometric functions). Also, it seems to me that the analysis of the Berezin kernels leads to some "new" special functions (to the A-function, see Section 11, and to generalizations of Gross–Richards kernels, see section 10). The relatively simple picture for $U(p,q)$ allows to touch some of these objects in a simple way.
It seems to me that the subject of this paper is elementary, this is some topic in analysis of the matrix variable. For this reason, I try to follow a simple approach to noncommutative harmonic analysis in the spirit of [17], [32], [80] whenever I can.

I am very grateful to Grigory Olshanski, Bent Ørsted and Vladimir Molchanov for meaningful discussion. I thanks the administration of the Erwin Schrödinger Institute for Mathematical Physics, where this text was prepared, for their hospitality.

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1. Preliminaries: positive definite kernels

Positive definite kernels machinery is a usual tool for work upon Hilbert spaces (70), (83). This section contains simple general facts concerning this subject.

1.1. Positive definite kernels. Let $X$ be a set, let $H$ be a Hilbert space, and let $\langle \cdot, \cdot \rangle$ denote the scalar product in $H$. Consider a map $x \mapsto v_x$ from $X$ to $H$. We define the function $L(x, y)$ on $X \times X$ by

$$L(x, y) = \langle v_x, v_y \rangle$$

Obviously, $L(x, y) = \overline{L(y, x)}$ and for any $x_1, \ldots, x_n \in X$ the matrix

$$\begin{pmatrix}
L(x_1, x_1) & \cdots & L(x_1, x_n) \\
\vdots & \ddots & \vdots \\
L(x_n, x_1) & \cdots & L(x_n, x_n)
\end{pmatrix}$$

is positive semidefinite.\footnote{An $n \times n$ Hermitian matrix $Q = \{q_{ij}\}$ is called positive semidefinite if $\sum q_{ij}w_i\overline{w_j} \geq 0$ for any $w_1, \ldots, w_n \in \mathbb{C}$.}

A function $L(x, y)$ on the set $X \times X$ is called a positive definite kernel on $X$ if $L(x, y) = \overline{L(y, x)}$ and for all $x_1, \ldots, x_n \in X$ the matrix (1.1) is positive semidefinite.

Nontrivial examples appear in Sections 3–4,

Theorem 1.1. \footnote{Let $L(x, y)$ be a positive definite kernel on $X$. Then}

a) There exists a Hilbert space $H$ and a system of vectors $v_x \in H$ enumerated by $x \in X$ such that $L(x, y) = \langle v_x, v_y \rangle$ and the linear span of the vectors $v_x$ is dense in $H$.

b) A space $H$ is unique in the following sense. Let $H'$ be another Hilbert space and let $v'_x \in H'$ be another system of vectors satisfying the same conditions. Then there exists a unique unitary operator $U : H \to H'$ such that $Uv_x = v'_x$ for all $x$. 

The linear span of the vectors $v$, where $x \in X$, be formal symbols. Consider the linear space $\tilde{H}$ consisting of all the formal finite linear combinations $\sum_{k=1}^{n} c_k v_{x_k}$, where $c_k \in \mathbb{C}$. We define a scalar product in $\tilde{H}$ by

$$\langle \sum_{k=1}^{n} c_k v_{x_k}, \sum_{l=1}^{m} \tilde{c}_l v_{y_l} \rangle = \sum_{k=1, l=1}^{n, m} c_k \tilde{c}_l L(x_k, y_l)$$

The space $\tilde{H}$ is a pre-Hilbert space and $H$ is the Hilbert space associated with $\tilde{H}$.

We denote by $\mathfrak{B}[L] = \mathfrak{B}[L; X]$ the space $H$ equipped with the distinguished system of vectors $v_x$. The set of vectors $v_x \in \mathfrak{B}[L]$ is called a supercomplete system, or an overfilled basis, or a system of coherent states.

Remark. A supercomplete system is nothing but a system of vectors with explicitly known pairwise scalar products. Nevertheless the knowledge of these data can be an effective tool for working in a given Hilbert space. □

If $X$ is a separable metric space and $L(x, y)$ is continuous on $X \times X$, then the Hilbert space $\mathfrak{B}[L]$ is separable and the map $x \mapsto v_x$ is continuous. In all natural cases, these conditions are satisfied. For certain technical reasons, the general definition given above is more convenient.

1.2. Functional realization $\mathfrak{B}^\circ[L]$. Consider the space $\mathfrak{B}[L]$ defined by a positive definite kernel $L$. To each $h \in \mathfrak{B}[L]$ we assign a function $f_h(x)$ on $X$ by

$$f_h(x) = \langle h, v_x \rangle_{\mathfrak{B}[L]}$$

The linear span of the vectors $v_x$ is dense in $\mathfrak{B}[L]$ and hence the map $h \mapsto f_h$ is an embedding of $\mathfrak{B}[L]$ to the space of functions on $X$. We denote by $\mathfrak{B}^\circ[L]$ the image of this embedding. A scalar product in $\mathfrak{B}^\circ[L]$ is defined by

$$\langle f_h, f_q \rangle_{\mathfrak{B}^\circ[L]} := \langle h, q \rangle_{\mathfrak{B}[L]} \quad (1.2)$$

The function $\theta_a$ corresponding to an element $v_a$ of the supercomplete system is given by

$$\theta_a(x) = L(a, x)$$

Proposition 1.2. (Reproducing property) For each $f \in \mathfrak{B}^\circ[L]$

$$\langle f, \theta_a \rangle_{\mathfrak{B}^\circ[L]} = f(a) \quad (1.3)$$

Proof. $\langle f_h, \theta_a \rangle_{\mathfrak{B}^\circ[L]} = \langle h, v_a \rangle_{\mathfrak{B}[L]} = f_h(a)$

1.3. Reconstruction of the kernel $L$ from $\mathfrak{B}^\circ[L]$. Let $\mathfrak{B}^\circ$ be some Hilbert space consisting of functions on $X$, and let the linear functional $u_x(f) := f(x)$ be continuous on $\mathfrak{B}^\circ$ for each $x \in X$. Then (see [17], Theorem 2.4) for any $x \in X$ there exists a unique function $\theta_x \in \mathfrak{B}^\circ$ such that

$$\langle f, \theta_x \rangle = f(x)$$

We define a positive definite kernel $L(x, y)$ on $X$ by

$$L(x, y) := \langle \theta_x, \theta_y \rangle_{\mathfrak{B}^\circ} = \theta_x(y) = \overline{\theta_y(x)}$$

for all $f \in \mathfrak{B}^\circ$. Then the space $\mathfrak{B}^\circ$ coincides with the space $\mathfrak{B}^\circ[L]$ defined by the positive definite kernel $L$.

The kernel $L$ is called the reproducing kernel of the space $\mathfrak{B}^\circ$.

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5 in honor of S.Bergman, V.Bargmann and F.A.Berezin
6 The space $L^2$ does not consist of functions!
1.4. Convergence in $\mathcal{B}^\circ[L]$.

Lemma 1.3. (see, for instance, [57]) Assume $X$ is a complete metric space, and the kernel $L$ is continuous. If a sequence $f_j \in \mathcal{B}^\circ[L]$ converges to $f$ in $\mathcal{B}^\circ[L]$, then it converges uniformly on compact sets in $X$.

Corollary 1.4. Any element of $\mathcal{B}^\circ[L]$ can be approximated by finite sums $\sum c_k L(a_k, x)$ in the topology of uniform convergence on compact sets.

1.5. Hilbert spaces of holomorphic functions. Let $\Omega$ be a bounded open domain in $\mathbb{C}^n$. Assume a positive definite kernel $L(z, u)$ on $\Omega$ is antiholomorphic in $z$ and holomorphic in $u$. Then, by Corollary 1.4, elements of the Hilbert space $\mathcal{B}^\circ[L]$ are holomorphic functions.

Let $\zeta(z)$ be a continuous function on $\Omega$ and let $\zeta(z) > 0$ for any $z \in \Omega$. Denote by $\mathcal{B}(\Omega, \zeta)$ the space of holomorphic functions on $\Omega$ satisfying the condition

$$\int\Omega |f(z)|^2 \zeta(z) \{dz\} < \infty$$

where

$$\{dz\} := \prod_{j=1}^n d(\text{Re} z_j) \prod_{j=1}^n d(\text{Im} z_j)$$

denotes the Lebesgue measure on $\Omega$. Consider the $L^2$ scalar product

$$\langle f, g \rangle = \int\Omega f(z) \overline{g(z)} \zeta(z) \{dz\}$$

in the space $\mathcal{B}(\Omega, \zeta)$.

Theorem 1.5 a) The space $\mathcal{B}(\Omega, \zeta)$ is closed in $L^2(\Omega, \zeta)$.

b) For any $u \in \Omega$ the linear functional $f \mapsto f(u)$ is continuous on $\mathcal{B}(\Omega, \zeta)$.

Proof. b) Let $\Omega$ be a polydisk $|z_j| < r_j$, $\zeta(z) = 1$ and $u = 0$. Then $f(0)$ coincides (up to a factor) with $\langle f, 1 \rangle$. Thus in this case the statement is obvious. Consider an arbitrary point $w \in \Omega$ and a small polydisk $D \subset \Omega$ with center in $w$. Then

$$\int\Omega |f(z)|^2 \zeta(z) \{dz\} \geq \min_{z \in D} \zeta(z) \int D |f(z)|^2 \{dz\}$$

Hence the convergence $f_n \to f$ in $\mathcal{B}(\Omega, \zeta)$ implies the convergence $f_n \to f$ in $L^2(D)$. Thus $f_n(w)$ converges to $f(w)$. This implies (b).

The assertion of (a) is a consequence of (b) and Lemma 1.3.

The spaces $\mathcal{B}(\Omega, \tau)$ are called weight spaces of holomorphic functions.

1.6. Another functional realization $\mathcal{B}^*[L]$. Let $X$ be a smooth manifold. Let a positive definite kernel $L$ be $C^\infty$-smooth. Let $\mathcal{E}'(X)$ be the space of compactly supported distributions on $X$. We define a scalar product in $\mathcal{E}'(X)$ by

$$\langle \chi_1, \chi_2 \rangle = \{L, \chi_1 \otimes \overline{\chi_2}\}$$

where $\{\cdot, \cdot\}$ denotes the pairing of smooth functions and distributions. We define the space $\mathcal{B}^*[L]$ as the Hilbert space associated with the pre-Hilbert space $\mathcal{E}'(X)$.

The canonical unitary operator $J : \mathcal{B}[L] \to \mathcal{B}^*[L]$ is defined by the condition: $Jv_x$ is the $\delta$-function supported by $x$.

Remark. In general, distributions do not represent all the elements of $\mathcal{B}^*[L]$.

1.7. Some operations with positive definite kernels.

\[\text{Recall the Weierstrass theorem. Let } f_n \text{ be holomorphic and } f_n \text{ converge to } f \text{ uniformly on compact sets. Then partial derivatives } \frac{\partial}{\partial x_n} f_n \text{ converge to } \frac{\partial}{\partial x_n} f \text{ uniformly on compact sets. In particular, } f \text{ is holomorphic.} \]
PROPOSITION 1.6. Let $L_1$, $L_2$ be positive definite kernels on $X$. Then $L_1 L_2$ and $L_1 + L_2$ are positive definite kernels.

PROOF. Let $v_x$ be the supercomplete system in $\mathfrak{B}[L_1]$ and $w_x$ be the supercomplete system in $\mathfrak{B}[L_2]$. Consider the space $\mathfrak{B}[L_1] \otimes \mathfrak{B}[L_2]$ with the supercomplete system $v_x \otimes w_x$ and the space $\mathfrak{B}[L_1] \oplus \mathfrak{B}[L_2]$ with the supercomplete system $v_x \oplus w_x$. Both statements are now obvious. $\Box$

PROPOSITION 1.7. Let $L(x,y)$ be a positive definite kernel on $X$. Then for an arbitrary function $\gamma(x)$ on $X$

- the kernel $M(x,y) := L(x,y)\gamma(x)\gamma(y)$ is positive definite.
- the operator $\mathfrak{B}^o[L] \rightarrow \mathfrak{B}^o[M]$ given by $f(x) \mapsto f(x)\gamma(x)$ is unitary.

PROOF. Consider the vectors $\gamma(x)v_x \in \mathfrak{B}[L]$. $\Box$

2. Groups $U(p,q)$ and matrix balls

Assume that $p \leq q$.

2.1. Groups $U(p,q)$. Consider the space $\mathbb{C}^p \oplus \mathbb{C}^q$ equipped with the indefinite Hermitian form $Q$ defined by the $(p+q) \times (p+q)$ matrix $(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix})$. We denote by $e_1^+, \ldots, e_p^+, e_1^-, \ldots, e_q^-$ the standard basis in $\mathbb{C}^p \oplus \mathbb{C}^q$. The pseudounitary group $G = U(p,q)$ is the group of all linear operators in $\mathbb{C}^p \oplus \mathbb{C}^q$ preserving the form $Q$. In other words, the group $U(p,q)$ consists of all $(p+q) \times (p+q)$ matrices satisfying the condition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.1}$$

2.2. Another realization of $U(q,q)$. Assume $p = q$. Consider the new basis

$$w_1^+ := \frac{1}{\sqrt{2}}(e_1^+ + e_1^-), \ldots, w_q^+ := \frac{1}{\sqrt{2}}(e_q^+ + e_q^-), \quad (2.2)$$

$$w_1^- := \frac{1}{\sqrt{2}}(e_1^+ - e_1^-), \ldots, w_q^- := \frac{1}{\sqrt{2}}(e_q^+ - e_q^-) \quad (2.3)$$

in $\mathbb{C}^q \oplus \mathbb{C}^q$. The matrix of the Hermitian form $Q$ in this basis is $(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$. Thus the group $U(q,q)$ can be represented as the group of all $(q+q) \times (q+q)$ matrices $(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix})$ satisfying

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.4}$$

2.3. Another realization of $U(p,q)$. Let us realize the groups $U(p,q)$ and $U(q,q)$ as in 2.1.

Consider the natural embedding of $U(p,q)$ to $U(q,q)$ given by

$$h \mapsto \begin{pmatrix} 1_{q-p} & 0 \\ 0 & h \end{pmatrix}$$

where $1_{q-p}$ denotes the unit matrix of the size $q-p$. Consider the realization of $U(q,q)$ described in 2.2. Then $U(p,q)$ becomes the group of $((q-p) + (q-p)) \times ((q-p) + (q-p))$ block matrices $g$ satisfying the conditions

$$g \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.5)$$

for all $v \in \mathbb{C}^{q-p}$. Sometimes this model is useful.

2.4. The symmetric space $G/K = U(p,q)/U(p) \times U(q)$. We say that a $p$-dimensional subspace $R \subset \mathbb{C}^p \oplus \mathbb{C}^q$ is positive, if the Hermitian form $Q$ is positive definite on $R$. Denote by

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8We also denote by $U(n)$ the group of unitary $n \times n$ matrices.
Gr
\sub{p,q} the space of all positive p-dimensional subspaces in \( \mathbb{C}^p \oplus \mathbb{C}^q \). The group \( U(p, q) \) acts on \( \mathbb{C}^p \oplus \mathbb{C}^q \) and hence it acts on \( \text{Gr}_{p,q}^+ \). Obviously (by the Witt theorem), this action is transitive. The stabilizer of the subspace \( \mathbb{C}^p \oplus 0 \in \text{Gr}_{p,q}^+ \) consists of matrices having the form

\[
\begin{pmatrix}
a & 0 \\
0 & d
\end{pmatrix}; \quad a \in U(p), \ b \in U(q)
\]

Hence the space \( \text{Gr}_{p,q}^+ \) is the homogeneous space

\[ G/K = U(p, q)/U(p) \times U(q) \]

2.5. Cartan matrix balls. Assume \( p \leq q \). A matrix ball \( B_{p,q} \) is the space of all complex \( p \times q \) matrices with norm\(^9\) less than 1. Also, \( z \in B_{p,q} \) iff \( zz^* < 1 \).

For \( z \in B_{p,q} \) we define the subspace \( \text{Graph}(z) \subset \mathbb{C}^p \oplus \mathbb{C}^q \) consisting of all the vectors of the form \((v, vz)\), where \( v \in \mathbb{C}^p \). It can easily be checked that the map \( z \mapsto \text{Graph}(z) \) is a bijection \( B_{p,q} \to \text{Gr}_{p,q}^+ \). Thus the space \( \text{Gr}_{p,q}^+ \cong G/K \) is parametrized by points of the matrix ball \( B_{p,q} \). The action of the group \( G \) in these coordinates is given by the linear-fractional transformations

\[ z \mapsto z^{[g]} := (a + zc)^{-1}(b + zd) \] (2.7)

where \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \).

The complex Jacobian of the transformation (2.7) is

\[ \det(a + zc)^{-p-q} \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \] (2.8)

**Lemma 2.1.** The \( G \)-invariant measure on \( B_{p,q} \) is given by

\[ \det(1 - zz^*)^{-p-q}\{dz\} \] (2.9)

where \( \{dz\} \) is the Lebesgue measure on \( B_{p,q} \).

**Proof.** This follows from (2.8) and the simple identity

\[ 1 - z^{[g]}(u^{[g]})^* = (a + zc)^{-1}(1 - zu^*)(a^* + c^*u^*)^{-1} \] (2.10)

2.6. Structure of the boundary of the matrix ball. Denote by \( \overline{B}_{p,q} \) the closure of \( B_{p,q} \) in \( \mathbb{C}^{pq} \), i.e the set of matrices with norm \( \leq 1 \). Denote by \( M_k \) the set of all matrices \( z \in \overline{B}_{p,q} \) satisfying the condition

\[ \text{rk}(1 - zz^*) = k; \quad k = 0, \ldots, p - 1 \]

The following statement is trivial

**Lemma 2.2.** The sets \( M_k \) are exactly the orbits of \( U(p, q) \) on the boundary of \( B_{p,q} \).

**Remark.** The orbit \( M_0 \) is the Shilov boundary of \( B_{p,q} \), see [15], X.5.

2.7. Siegel realizations of \( G/K \): matrix wedges. The Cartan realization of \( G/K \) will be basic for us. Nevertheless in some places we shall need of the Siegel realizations\(^{10}\)

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\(^9\)The term norm in this paper means the norm of an operator from Euclidean space \( \mathbb{C}^p \) to Euclidean space \( \mathbb{C}^q \). The notation \( A > 0 \) means that the operator \( A \) is positive i.e., \( \langle Ax, x \rangle > 0 \) for all \( x \neq 0 \). The notation \( A > B \) means \( A - B > 0 \).

\(^{10}\)The action of the compact subgroup \( K \subset G \) has the simplest form in the Cartan realization; if we want to write formulas related to the parabolic subgroup (see Subsection 6.2 below), then the Siegel realizations are more convenient.
Assume \( p = q \). Denote by \( W_q \) the space of all \( q \times q \) complex matrices \( T \) satisfying the condition
\[
T + T^* > 0
\]
Consider the basis (2.2)–(2.3), denote by \( Z_+ \) the subspace in \( \mathbb{C}^q \oplus \mathbb{C}^q \) spanned by \( w_j^+ \) and denote by \( Z_- \) the subspace spanned by \( w_j^- \). For \( T \in W_q \) consider the operator \( T : Z_+ \rightarrow Z_- \). It can easily be checked that the map \( T \mapsto \text{Graph}(T) \) is a bijection \( W_q \rightarrow \text{Gr}^+_{q,q} \).

The spaces \( B_{q,q} \) and \( W_q \) are identified by the Cayley transform \( T = -1 + (1 + z)^{-1} \).

2.8. Siegel realization in the case \( q \neq p \): sections of wedges. We define the space \( SW_{p,q} \) as the subset in \( W_q \) consisting of all block \( ((q - p) + p) \times ((q - p) + p) \)-matrices of the form
\[
S = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_p & 0 & \cdots & 0 \end{pmatrix}
\]
where
\[
1 > \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \geq 0
\]
are the eigenvalues of \( \sqrt{zz^*} \). We denote the simplex (2.13) by \( \Lambda_p \).

Consider the map \( \pi : B_{p,q} \rightarrow \Lambda_p \) taking each \( z \in B_{p,q} \) to the collection of the eigenvalues of \( \sqrt{zz^*} \). The image of the Lebesgue measure on \( B_{p,q} \) with respect to the map \( \pi \) is the measure on \( \Lambda_p \) given by
\[
\text{const} \cdot \prod_{i,j} (\lambda_i^2 - \lambda_j^2)^2 \prod_j \lambda_j^{2(q-p)+1} \prod d\lambda_j
\]
(see [26], X.1), see also numerous calculations of this type in [22], Chapter 3).

Here and below the symbol ‘const’ denotes a constant depending only on \( p,q \).

Thus the image of the \( G \)-invariant measure (2.9) is
\[
\text{const} \cdot \prod_{1 \leq i < j \leq p} (\lambda_i^2 - \lambda_j^2)^2 \prod_{j=1}^{p} \lambda_j^{2(q-p)+1} (1 - \lambda_j^2)^{-p-q} \prod_{j=1}^{p} d\lambda_j
\]
(2.14)

It will be convenient for us to define new coordinates
\[
x_j = \lambda_j^2/(1 - \lambda_j^2)
\]
(2.15)
Then the simplex $\Lambda_p$ transforms to the simplicial cone $\mathcal{X}_p$

$$\mathcal{X}_p : x_1 \geq x_2 \geq \ldots \geq x_p \geq 0$$

(2.16)

The measure $\mathcal{M}$ in the coordinates $x_j$ is of the form

$$\prod_{j=1}^{p} x_j^{q-p} \prod_{1 \leq k < l \leq p} (x_k - x_l)^2 \prod_{j=1}^{p} dx_k$$

(2.17)

Below we consider $K$-invariant functions on $G/K$ as symmetric functions in the variables $x_j \geq 0$ or as functions on the simplicial cone $\mathcal{X}$.

**Theorem 2.3.** (Hua)

$$\int_{B_{p,q}} \det(1 - zz^*)^{\alpha - p - q} (dz) = \omega \prod_{k=1}^{p} \Gamma(\alpha - q - k + 1) \Gamma(\alpha - k + 1)$$

(2.18)

where $\omega$ is the volume of $B_{p,q}$.

The problem is reduced to integrating of the function $\prod (1 + x_k)^{-\alpha}$ over the measure $\mathcal{M}$. This is a special case of the Selberg integral, see [1], chapter 8. Hua’s calculations ([32], chapter 2) are interesting and important by themselves. For other ways of calculation see [15], [54].

**2.10. Comments.** A matrix ball (see [19]) $B$ is

— the set of all $p \times q$ matrices over $\mathbb{R}$, $\mathbb{C}$ or the quaternion algebra $\mathbb{H}$ such that $zz^* < 1$

— or the set of all $n \times n$ matrices over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ satisfying $zz^* < 1$ and the natural symmetry condition.

The natural conditions of symmetry are

- $z = z^t$ and $z = -z^t$ over $\mathbb{R}$;
- $z = z^t$, $z = -z^t$, and $z = z^*$ over $\mathbb{C}$;
- $z = z^*$ and $z = -z^*$ over $\mathbb{H}$.

We consider the group $G$ of linear-fractional transformations ([2.7]) preserving $B$. Denote by $K$ the stabilizer of the point $0 \in B$. All classical Riemannian noncompact symmetric spaces $G/K$ can be obtained in this way. The table is contained in [21], Appendix A, see also [22].

Tools that are used below cannot be applied to arbitrary matrix ball.

### 3. Spaces of holomorphic functions in matrix balls. Berezin scale

A subject of this section is standard. Usually we give sketches of proofs.

**3.1. Berezin scale: large values of parameter.** Let $\alpha > p + q - 1$. Consider the weight space

$$H_{\alpha} := B(B_{p,q}, \det(1 - zz^*)^{\alpha - p - q} (dz))$$

consisting of holomorphic functions on $B_{p,q}$ (see 1.5). The scalar product in $H_{\alpha}$ is defined by

$$\langle f, g \rangle_\alpha = C(\alpha)^{-1} \int_{B_{p,q}} f(z) \overline{g(z)} \det(1 - zz^*)^{\alpha - p - q} (dz)$$

(3.1)

We define the normalization constant $C(\alpha)$ by (2.18), then $\langle 1, 1 \rangle_\alpha = 1$

Let us define the operators $\tau_\alpha(g)$, where $g = \begin{pmatrix} a & c \\ c & d \end{pmatrix} \in G = U(p, q)$, in $H_{\alpha}$ by

$$\tau_\alpha(g) f(z) = f ((a + z c)^{-1} (b + z d)) \det(a + z c)^{-\alpha}$$

(3.2)

A simple calculation (with applying (2.8), (2.10)) shows that the operators $\tau_\alpha(g)$ are unitary in $H_{\alpha}$. Hence we obtain a unitary representation of $G = U(p, q)$ in $H_{\alpha}$.

**Remark.** If $\alpha$ is integer, then the expression $\det(a + z c)^{-\alpha}$ is well defined. Otherwise the function

$$\det(a + z c)^{-\alpha} = \det(a)^{\alpha} \det[(1 + zca^{-1})^{-\alpha}] = e^{\alpha \ln \det a + 2\pi ki} \det[(1 + zca^{-1})^{-\alpha}]$$
on \(B_{p,q}\) has an infinite number of holomorphic branches, which differ by a constant factor. Indeed, \((2.1)\) implies \(\|a\| > \|c\|\), hence \(\|ca^{-1}\| < 1\), hence the expression \([\ldots]\) is well defined. Thus, for a noninteger \(\alpha\), the operators \(\tau_\alpha(y)\) are defined up to a scalar factor \(e^{\pi i \alpha k}\), and the representation \(\tau_\alpha\) is projective (see [34], 14). We can also consider the representation \(\tau_\alpha\) as a representation of the universal covering group of the group \(U(p,q)\), see [34], Corollary to Theorem 14.3.1.

**Theorem 3.1.** (Berezin, [3]) The reproducing kernel of the space \(H_\alpha\) is

\[
K_\alpha(z,u) = \det(1 - uz^*)^{-\alpha}
\]

**Proof.** We must find functions \(\theta_\alpha \in H_\alpha\) such that \(\langle f, \theta_\alpha \rangle = f(a)\). Expanding \(f(z)\) into its Taylor series, we obtain

\[
\frac{1}{C(\alpha)} \int_{B_{p,q}} f(z) \det(1 - zz^*)^{\alpha - p - q} \{dz\} = \frac{1}{C(\alpha)} \int_{B_{p,q}} (f(0) + \ldots) \det(1 - zz^*)^{\alpha - p - q} \{dz\} = \frac{1}{C(\alpha)} \int_{B_{p,q}} f(0) \det(1 - zz^*)^{\alpha - p - q} \{dz\} = f(0)
\]

Thus \(\theta_0(z) = 1\). Let us evaluate \(\langle \tau_\alpha(g)f, \theta_0 \rangle_\alpha\) in two ways. First, it equals

\[
\langle f(z^\alpha) \rangle \det(a + z) = f(a^{-1}b) \det(a)^{-\alpha}
\]

The operators \(\tau_\alpha(g)\) are unitary, hence it is equal to

\[
\langle f, \tau_\alpha(g^{-1}) \theta_0 \rangle_\alpha = \langle f, (a - b^*z)^{-\alpha} \rangle_a
\]

and we obtain an explicit expression for \(\theta_{\alpha^{-1}b}(z)\).

Formula \((3.1)\) defining the scalar product in \(H_\alpha\) has sense for \(\alpha > p + q - 1\). Nevertheless we shall see (Theorem 3.8) that the reproducing kernel \(K_\alpha\) is positive definite for \(\alpha > p - 1\) and also for \(\alpha = 0, 1, \ldots, p - 1\). Subsections 3.2–3.3 contains the preparation to the statement and proof of Theorem 3.8.

**3.2. Preliminaries: K-harmonics in the space of polynomials.** Let

\[
\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n \geq 0
\]

be integers. The **Schur function** \(s_\mu\) is defined by

\[
s_\mu(y_1, \ldots, y_n) = s^n_\mu(y_1, \ldots, y_n) := \frac{\det[y_i^{\mu_j + n - j}]}{\det[y_i^{n-j}]} = \frac{\det[y_i^{\mu_j + n - j}]}{\prod_{k<i}(y_k - y_i)}
\]

where \(y_i \in \mathbb{C}\) (see [33], I.3; [80], §73). The numerator is zero on the hyperplanes \(y_j = y_k\) and hence \(s^n_\mu\) is a symmetric polynomial in \(y_1, \ldots, y_n\). It is easy to prove that

\[
s^{n+m}_{\mu_1,\ldots,\mu_n,0,\ldots,0}(y_1, \ldots, y_n, 0, \ldots, 0) = s^n_{\mu_1,\ldots,\mu_n}(y_1, \ldots, y_n)
\]

Let \(A\) be a \(n \times n\) matrix. Let \(y_1, \ldots, y_n\) be the eigenvalues of \(A\). We define the **Schur function** \(S_\mu(A)\) by

\[
S_\mu(A) = s_\mu(y_1, \ldots, y_n)
\]

By \(\rho_\mu = \rho_{\mu_1,\ldots,\mu_n}\) we denote the representation of \(\text{GL}(n, \mathbb{C})\) with the signature \(\mu\) (see [80], §49; [29]; these objects are used only in this Subsection). The Schur function \(S_\mu(A)\) is the character of \(\rho_\mu\), i.e., \(S_\mu(A) = \text{tr} \rho_\mu(A)\). Recall (see [80], §73; see also [32], Theorem 1.2.4 on eliminating of undeterminacy) that the dimension of the representation \(\rho_\mu\) is given by the Weyl formula

\[
\dim \rho^n_\mu = s_\mu(1, \ldots, 1) = \prod_{\mu_i > j \geq 1} \frac{\mu_1 - \mu_j + i - j}{i - j}
\]
Denote by $\text{Mat}_{p,q}$ the space of all complex $p \times q$ matrices. Denote by $\text{Pol}_{p,q}$ the space of all holomorphic polynomials on $\text{Mat}_{p,q}$. The linear-fractional transformations $[2,7]$ for $g = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in \mathbf{K}$ reduce to the form

$$\tau_\alpha(g)f(z) = f(a^{-1}zd)\det(a)^{-\alpha}$$

Let us omit the unessential scalar factor $\det(a)^{-\alpha}$ and consider the action of $U(p) \times U(q)$ given by

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : f(z) \mapsto f(a^{-1}zd) = f(a^*zd)$$

It is natural to extend this action to a $GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$-action given by the formula $f(z) \mapsto f(a^*zd)$.[23]

**THEOREM 3.2.** (Hua, [32], see also [80], §56)

$$\text{Pol}_{p,q} \simeq \bigoplus_{\mu_1 \geq \mu_2 \geq \ldots \geq \mu_p \geq 0} \rho_{\mu_1}^p \otimes \rho_{\mu_2}^q \otimes \ldots \otimes \rho_{\mu_p}^q$$

(3.5)

This theorem is a consequence of the following Lemma.

**LEMMA 3.3.** Denote by $\Delta_j$ the determinant of the left upper $j \times j$ block of $z$. All $GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$-highest vectors in $\text{Pol}_{p,q}$ have the form $\prod_{j=1}^{p} \Delta_j^{\mu_j-\mu_{j+1}}$.

**Proof of the Lemma.** Denote by $N_q$ the group of all $d \in GL(q, \mathbb{C})$ with units on the diagonal and zeros under the diagonal. Denote by $N_p'$ the group of all $a \in GL(p, \mathbb{C})$ with units on the diagonal and zeros over the diagonal. Denote by $A_p$, $A_q$ the diagonal subgroups in $GL(p, \mathbb{C})$, $GL(q, \mathbb{C})$. We want to find all eigenfunctions of the subgroup $A_pN_p' \times A_qN_q \subset GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$. Denote by $\mathcal{R}$ the set of all $w \in \text{Mat}_{p,q}$ such that $w_{ij} = 0$ for $i \neq j$.

A highest vector is a $N_p' \times N_q$-invariant function on $\text{Mat}_{p,q}$. For a matrix $z \in \text{Mat}_{p,q}$ in a general position there exists $H \in \mathcal{R}$ that can be represented in the form $H = TzS$, where $T \in N_p'$, $S \in N_q$ (it is sufficient to apply the Gauss elimination algorithm). Moreover, the diagonal matrix elements $h_{ij}$ of $H$ are given by $h_{jj} = \Delta_j/\Delta_{j-1}$. Hence a highest vector is a rational function depending on $\Delta_j/\Delta_{j-1}$, $j = 1, 2, \ldots, p$.

Now we call to mind that a highest vector is an $A_p \times A_q$-eigenvector.

Denote by $\text{Pol}_{p,q}^\mu$ the subspaces in $\text{Pol}_{p,q}$ corresponding to the summands in (3.5).

**LEMMA 3.4.** a) The function $S_p^\mu(uz^*)$ is a positive definite kernel on $\text{Pol}_{p,q}$.

b) $S_{\mu_1, \ldots, \mu_p}(uz^*) = S_{\mu_1}^q(\mu_2, \ldots, \mu_p, 0, \ldots, 0)(z^*u)$

c) The subspace $\text{Pol}_{p,q}^\mu \cap \mathbb{B}[\mathbf{S}_p^\mu; \text{Mat}_{p,q}]$ defined by the positive definite kernel $S_p^\mu(uz^*)$.

**Proof.** Observe that $S_p^\mu$, like $\rho_\mu(A^*)$ extends canonically to the representation of the semigroup $\Gamma$ of all operators $C^p \to C^q$, and $\rho_\mu(A^*) = \rho_\mu(A)^*$ (see, for instance [32], Chapter I, Appendix A, [70], 3.3).

Let us show that $\text{tr} \rho_\mu(A^*)$ is a positive definite kernel on $\Gamma$. Let $v_j$ be an orthonormal basis in $\rho_\mu^p$. Then

$$\text{tr} \rho_\mu(A^*) = \sum_j \langle \rho_\mu(A^*)v_j, v_j \rangle = \langle \rho_\mu(B^*)v_j, \rho_\mu(A^*)v_j \rangle$$

The summands are positive definite (since $v_j \mapsto \rho_\mu(A^*)v_j$ is a function from $\Gamma$ to Euclidean space, see 1.1). By Proposition 1.6, the sum also is positive definite. Now we embed $\text{Mat}_{p,q}$ to $\Gamma$ adding $(q-p)$ zero rows at the bottom of the matrix.

b) The nonzero eigenvalues of $uz^*$ and $u^*z$ coincide (let $h$ be a vector; then $zu^*h = \lambda h$ implies $(u^*z)u^*h = \lambda u^*h$). Thus the statement follows from [1,3].

11 The groups $U(p) \times U(q)$ and $GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$ have the same invariant subspaces in $\text{Pol}_{p,q}$. Indeed a subspace of homogeneous polynomials of a given degree is invariant with respect to both groups. It remains to apply the Weyl’s unitary trick, see [80], §42.
c) First, the kernel $S_\mu(uz^*)$ is $K$-invariant and hence it defines a $K$-invariant subspace.

The elements of $Pol_{p,q}$ are holomorphic functions and hence they are uniquely determined by their restrictions to the submanifold $M_0$ defined in 2.6. The elements of $M_0$ are isometric embeddings $\mathbb{C}^p \to \mathbb{C}^q$. Consider the natural map $U(q) \to M_0$: we take a unitary matrix and delete its last $q - p$ rows. Hence a functions on $M_0$ can be regarded as a function on $U(q)$.

The pullback of the kernel $S_\mu(z^*u)$ from $M_0$ to $U(q)$ is

$$K_\mu(g_1, g_2) := S_\mu\left( g_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} g_2^{-1} \right)$$

The kernel $L_\mu(g_1, g_2) := S_\mu(g_1, g_2^{-1})$ on $U(q)$ defines the space $\rho_\mu^{q} \otimes \rho_\mu^{q} \otimes \rho_\mu^{q} \otimes \cdots \otimes \rho_\mu^{q}$. By Corollary 1.4, the space $H^0[L_\mu]$ is the linear span of the functions $S_\mu(gA)$, where $A$ ranges over invertible matrices. This space is finite-dimensional, and thus it is closed with respect to pointwise convergence. Hence the functions $S_\mu(gA)$ for all noninvertible matrices $A$. Thus $H^0[L_\mu; U(q)] \subset H^0[L_\mu; U(q)]$. Hence the representation of $U(q)$ in $H^0[S_\mu; Mat_{p,q}]$ is the direct sum of several copies of $\rho_\mu^{p} \otimes \rho_\mu^{p} \otimes \cdots \otimes \rho_\mu^{p}$. It remains to apply Theorem 3.2.

**3.3. Expansion of the Berezin kernel $K_\alpha$.**

**Theorem 3.5** (Hua [22])

$$\prod_{j=1}^{p} (1 - y_j)^{-\alpha} = \sum_{\mu_1 \geq \cdots \geq \mu_p \geq 0} c(\mu_1, \ldots, \mu_p; \alpha) s_{\mu_1, \ldots, \mu_p}(y_1, \ldots, y_p)$$

where

$$c(\mu_1, \ldots, \mu_p; \alpha) = \dim_{\rho_\mu^{\mu_1}, \ldots, \rho_\mu^{\mu_p}} \prod_{j=1}^{p} \frac{\Gamma(\alpha + \mu_j - j + 1)(p-j)!}{\Gamma(\alpha - j + 1)(\mu_j + p-j)!}$$

(3.6)

Proof is contained in [22], Theorem 1.2.5, [15].

**Corollary 3.6.** (Berezin [3])

$$\det(1 - uz^*)^{-\alpha} = \sum_{\mu_1 \geq \cdots \geq \mu_p \geq 0} c(\mu_1, \ldots, \mu_p; \alpha) s_{\mu_1, \ldots, \mu_p}(uz^*)$$

(3.7)

where the constants $c(\mu; \alpha)$ are the same as above (3.6).

**Corollary 3.7.** The kernels $c(\mu; \alpha)S_\mu(uz^*)$ and $\det(1 - uz^*)^{-\alpha}$ define the same scalar product in the subspace $Pol_{p,q}^\mu$.

**Proof.** The $H_\alpha$-scalar product in $Pol_{p,q}^\mu$ is $U(p) \times U(q)$-invariant. The scalar product in $Pol_{p,q}^\mu$ defined by $S_\mu(uz^*)$ also is $U(p) \times U(q)$-invariant. The $U(p) \times U(q)$-module $Pol_{p,q}^\mu$ is irreducible. Hence these scalar products differ by a scalar factor. Denote by $\sigma_\mu S_\mu(uz^*)$ the kernel defining the $H_\alpha$-scalar product in $Pol_{p,q}^\mu$. The subspaces $Pol_{p,q}^\mu$ are pairwise orthogonal, and hence (see the proof of Proposition 1.6) we have $\det(1 - uz^*)^{-\alpha} = \sum \sigma_\mu S_\mu(uz^*)$.

**3.4. Berezin scale: general values of parameter $\alpha$.**

**Theorem 3.8.** (Berezin [3], and also [13], [24], [73], [76]) a) The function $K_\alpha(z, u) = \det(1 - uz^*)^{-\alpha}$ is a positive definite kernel on $B_{p,q}$ if and only if $\alpha$ belongs to the set

$$\alpha = 0, 1, 2, \ldots, p - 1, \quad \text{or} \quad \alpha > p - 1$$

(3.8)

b) Denote by $H_\alpha$ the Hilbert space of holomorphic functions on $B_{p,q}$ defined by the kernel $K_\alpha$. Then the operators $\tau_\alpha(g)$, $g \in U(p,q)$, given by (3.2) are unitary in $H_\alpha$. 


c) If $\alpha > p - 1$, then the space $H_\alpha$ contains all holomorphic polynomials. If $\alpha = k = 0, 1, \ldots, p - 1$, then $H_\alpha$ has the following $U(p) \times U(q)$-module structure

$$H_k \simeq \bigoplus_{\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k \geq 0} \rho_{\mu_1, \ldots, \mu_k, 0, \ldots, 0} \otimes \rho_{\mu_1, \ldots, \mu_k, 0, \ldots, 0}$$

**Proof.** A sum of positive definite kernels is positive definite (see Proposition 1.6). Hence it is sufficient to check positivity of the coefficients in the expansion (3.7). The last item is trivial. \(\Box\)

**Remark.** The space $H_0$ is one-dimensional, it contains only constants.

**Remark.** The space $H_q$ coincides with the Hardy space $H^2(B_{p,q})$. The scalar product in $H_q$ is given by

$$\langle f, g \rangle_q = \text{const} \int_{M_0} f(z)g(z) \{dz\}$$

where $M_0$ is the boundary orbit defined in 2.6 and $\{dz\}$ is the unique $K$-invariant measure on $M_0$.

3.5. **Berezin scale: small values of the parameter $\alpha$.** Consider the case $\alpha = k = 0, 1, \ldots, p - 1$. Let

$$\partial_{kl} = \frac{\partial}{\partial z_{kl}}$$

Consider the matrix

$$D = \begin{pmatrix} \partial_{11} & \cdots & \partial_{1q} \\ \vdots & \ddots & \vdots \\ \partial_{p1} & \cdots & \partial_{pq} \end{pmatrix}$$

(3.9)

We regard the minors of the matrix $D$ as differential operators.

**Theorem 3.9.** Let $f \in H_k$, $k = 0, 1, \ldots, p - 1$. Then each $(k + 1) \times (k + 1)$ minor of $D$ annihilates $f$, i.e., for all $i_1 < i_2 < \cdots < i_k$, $j_1 < j_2 < \cdots < j_k$ the following identity holds

$$\det \begin{pmatrix} \partial_{i_1 j_1} & \cdots & \partial_{i_1 j_k+1} \\ \vdots & \ddots & \vdots \\ \partial_{i_k+1 j_1} & \cdots & \partial_{i_k+1 j_k+1} \end{pmatrix} f(z) = 0$$

(3.10)

**Proof.** By Corollary 1.4 and holomorphness of $f$, it is sufficient to prove the statement for elements of the supercomplete system, i.e., for the functions

$$f(z) = \det(1 - za^*)^{-k}$$

Obviously, the system of partial differential equations (3.10) is invariant with respect to the transformations

$$z \mapsto AzD, \quad \text{where} \quad A \in \text{GL}(p, \mathbb{C}), D \in \text{GL}(q, \mathbb{C})$$

Hence it is sufficient to consider the case in which block $p \times (p + (q - p))$ matrix $a$ has the form

$$a = \begin{pmatrix} 1_p & 0 \end{pmatrix}$$

where $1_p$ is the unit $p \times p$ matrix. Now $f(z) = \det^{-k}(1 - za^*)$ depends only on left $p \times p$ block of $z$. Hence, without loss of generality, we can consider only the case $p = q$ and $f(z) = \det(1 - z)^{-k}$. But the system (3.10) is invariant with respect to translations and we can change our function $f(z)$ to $\det(z)^{-k}$.

A direct (but pleasant) calculation shows that

$$(\partial_{11} \partial_{22} - \partial_{21} \partial_{21}) \det(z)^{-1} = 0$$
This implies that $3 \times 3$ minors of $D$ annihilate $\det z^{-2} = \det z^{-1} \det z^{-1}$ etc. 

Remark. Let $f$ be a solution of the system (3.10) in $B_{p,q}$. Generally, $f \notin H_k$, since $f$ can have too rapid growth near boundary. Nevertheless $f$ can be approximated by finite sums $\sum_j \det(1 - za_j^*)^{-k}$ in the topology of uniform convergence on compact sets. 

3.6. Gindikin–Vergne–Rossi description for $H_\alpha$. We consider only the case $p = q$. First we observe that the Berezin kernel $K_\alpha$ in the wedge realization $W_q$ has the form

$$K_\alpha(T, S) = \det(T^* + S)^{-\alpha} \quad (3.11)$$

We denote by $H_\alpha(W_q)$ the space of holomorphic functions on $W_q$ defined by the kernel (3.11). The group $U(q,q)$ acts in $H_\alpha(W_q)$ by the transformations

$$f(T) \mapsto f(A + TC)^{-1}(B + TD)) \det(A + TC)^{-\alpha}$$

where the matrix $\left( \begin{array}{cc} A & B \\ C & D \end{array} \right)$ satisfies condition (2.4).

Remark. Let $Z \in W_q$. Let $\lambda_j$ be the eigenvalues of $Z$, and $v_j$ be the eigenvectors. Then

$$0 < \langle (Z + Z^*)v_j, v_j \rangle = \langle Zv_j, v_j \rangle + \langle v_j, Zv_j \rangle = \langle \lambda_j + \overline{\lambda_j}, v_j \rangle$$

Hence $\Re \lambda_j > 0$ and hence the function $\det Z^{-\alpha} := \prod_j \lambda_j^{-\alpha}$ is well defined on $W_q$. Thus expression (3.11) makes sense.

Consider the cone $Pos_q$ consisting of all positive semidefinite complex $q \times q$ Hermitian matrices $X$. The group $\operatorname{GL}(q, \mathbb{C})$ acts on $Pos_q$ by the transformations $h : X \mapsto h^*Xh$, $h \in \operatorname{GL}(q, \mathbb{C})$. Denote by $N_k$, where $k = 0, 1, \ldots, q$, the set of all matrices $X \in Pos_q$ such that $rk X = k$. Obviously, the sets $N_k$ are the $\operatorname{GL}(q, \mathbb{C})$-orbits on $Pos_q$.

Let $\chi(X)$ be a tempered distribution supported by $Pos_q$. Its Laplace transform is

$$\tilde{\chi}(T) = \int_{Pos_q} \exp(-\operatorname{tr} XT)\chi(X)dX \quad (3.12)$$

The function $\tilde{\chi}(T)$ is a holomorphic function of polynomial growth on the Siegel wedge $W_q$ (see (3.14)).

Theorem 3.10. (19, 23) Let $\alpha$ satisfy the positive definiteness conditions (3.8). Then $\det(T)^{-\alpha}$ is the Laplace transform of some positive measure $\nu_\alpha$ on $Pos_q$. If $\alpha > q - 1$, then $\nu_\alpha = \det X^{\alpha - q}dX$. For $\alpha = k = 0, 1, \ldots, q - 1$, the measure $\nu_k$ is the unique up to a scalar factor $\operatorname{SL}(q, \mathbb{C})$-invariant measure on the boundary orbit $N_k$.

Proof. a) Let $\alpha > q - 1$. We must check the equality

$$\int_{Pos_q} \det X^{\alpha - q}\exp(-\operatorname{tr} XT)\,dX = \text{const} \cdot \det(T)^{-\alpha} \quad (3.13)$$

Since the both parts are holomorphic in $T \in W_q$, it is sufficient to consider the case $T = T^*$. Then the integral converts to

$$\int_{Pos_q} \det X^{\alpha - q}\exp(-\operatorname{tr} T^{1/2}XT^{1/2})\,dX =$$

$$= \det T^{-\alpha} \int_{Pos_q} \det(T^{1/2}XT^{1/2})^{\alpha - q}\exp(-\operatorname{tr} T^{1/2}XT^{1/2})\,d(T^{1/2}XT^{1/2}) =$$

$$= \det T^{-\alpha} \int_{Pos_q} \det Y^{\alpha - q}\exp(-\operatorname{tr} Y)\,dY \quad (3.14)$$

as required.
In fact, we only use the $GL(q,\mathbb{C})$-homogeneity of the measure $\nu_\alpha$:

$$\nu_\alpha(h^*Ah) = |\det(h)|^{2\alpha-2q}\nu_\alpha(A)$$  \hspace{1cm} (3.15)

where $h \in GL(q,\mathbb{C})$, and $A$ is a subset in $Pos_q$.

\textbf{b)} Let $\alpha \leq q - 1$. First we must check the existence of a $SL(q,\mathbb{C})$-invariant volume form on $N_k$. We have $N_k = SL(q,\mathbb{C})/\mathcal{H}$, where $\mathcal{H}$ is the stabilizer $\mathcal{H}$ of the point $u_k := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in N_k$. The subgroup $\mathcal{H}$ consists of matrices

$$g = \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix}, \quad \text{where } R \in U(p), \ S \in GL(q-p,\mathbb{C}) \quad \text{and} \quad \det(R)\det(S) = 1$$

The group $\mathcal{H}$ has no homomorphisms to $\mathbb{R}$. Hence the (unique) volume form on the tangent space to $N_k$ at the point $u_k$ is $H$-invariant. Then we define a volume form on $N_k$ by $SL(n,\mathbb{C})$-invariance.

Secondly we must evaluate the Laplace transform of the measure $\nu_k$. It is not hard to check, that the measure $\nu_k$ satisfies the same homogeneity condition $(3.13)$, $\alpha = k$, and this gives the formula for the Laplace transform of $\nu_k$.

For the case $p = q$, this gives an independent proof of Theorem 3.8 about positive definiteness conditions, i.e., we obtain the following corollary.

\textbf{COROLLARY 3.11.} The kernel $K_\alpha(T, S) = \det(T^* + S)^{-\alpha}$ on $W_q$ is positive definite if $\alpha > q - 1$ or $\alpha = 0, 1, \ldots, q - 1$.

\textbf{PROOF.} We use directly the definition of positive definiteness. We must show positivity of the expression

$$\sum_{1 \leq k,l \leq N} \det(T_k^* + T_l)^{-\alpha} \xi_k \xi_l \int_{Pos_q} \exp(-\text{tr}(T_k + T_l)X) d\nu_\alpha(X) =$$

$$= \int_{Pos_q} |\sum_{k} \xi_k \exp(-\text{tr}T_k X)|^2 d\nu_\alpha(X) \geq 0$$

as required. Of course, we repeated the proof of the trivial part of the Bochner theorem (see [67], v.2, Theorem IX.9).

\textbf{THEOREM 3.12.} The Laplace transform\footnote{We slightly change the notation with respect to (3.14)}

$$\tilde{f}(T) = \int_{Pos_q} f(X) \exp(-\text{tr}XT)d\nu_\alpha(X)$$  \hspace{1cm} (3.16)

is a unitary (up to a scalar factor) operator from $L^2(Pos_q,\nu_\alpha)$ to the Berezin space $H_\alpha(W_q)$.

\textbf{PROOF.} Consider the family of functions $\tilde{\alpha}_A(X) = \exp(-\text{tr}A^*X)$, where $A \in W_q$, on $Pos_q$. It is sufficient to prove that

$$\langle \tilde{\alpha}_A, \tilde{\alpha}_B \rangle_{L^2(Pos_q,\nu_\alpha)} = \text{const} \cdot (\tilde{\alpha}_A, \tilde{\alpha}_B)_{H_\alpha(W_q)}$$  \hspace{1cm} (3.17)

By $(3.13)$, the left-hand side is $\text{const} \cdot \det(A^* + B)^{-\alpha}$. By the same formula $(3.13)$, $\tilde{\alpha}_A(T) = \det(A^* + T)^{-\alpha}$. These $\tilde{\alpha}_A$ are the elements of the supercomplete system of $H_\alpha(W_q)$ and we know their scalar products.

\textbf{REMARK.} Assign $\tilde{\alpha}_A$ be a supercomplete system in $L^2(Pos_q,\nu_\alpha)$. Then the Laplace transform $(3.14)$ coincides with the transform $\mathcal{B} \rightarrow \mathcal{B}$ described in 1.2.
3.7. Comments. 1) The construction described in this Section for the symmetric spaces \( U(p, q)/U(p) \times U(q) \) works for all Hermitian noncompact symmetric spaces, i.e., also for \( \text{Sp}(2n, \mathbb{R})/U(n), \text{SO}^*(2n)/U(n), \text{SO}(p, 2)/\text{SO}(p) \times \text{SO}(2) \) and two exceptional spaces, see [8], [13].

2) The constant in formula (3.13) is a special case of Gindikin’s matrix \( \Gamma \)-function [8], see also the exposition in [7], [15].

3) After Theorem 3.12 there arises a problem of transferring of the action of the group \( U(q, q) \) to the space \( L^2(\mathbb{R}_+, \nu_\alpha) \). The Lie algebra \( u(q, q) \) acts in \( L^2(\mathbb{R}_+, \nu_\alpha) \) by second order partial differential operators, which can easily be written explicitly. The exponents of these differential operators (i.e elements of the group \( U(q, q) \) itself) are integral operators with kernels involving matrix Bessel functions, see [28], [37], [43].

Consider the case \( G = U(1, 1) \). Then \( W_q = W_1 \) is the Lobachevskii plane \( \text{Re} T > 0 \). The transformation \( T \to T^{-1} \) of \( W_1 \) corresponds to the Hankel transform in \( L^2(\mathbb{R}_+, \nu_\alpha) \) (the Tricomi theorem).

Rotations of the Lobachevskii plane with center at \( T = 1 \) corresponds to the Kepinski and Myller-Lebedeff explicit solution of the Cauchy problem for the Schrödinger equation

\[
\frac{i \partial}{\partial t} F(x, t) = \left( -\frac{\partial^2}{\partial x^2} + x^2 + \frac{a}{x^2} \right) F(x, t)
\]

4. Kernel representations and spaces \( V_\alpha \)

4.1. Definition of kernel representations. Let

\[ \alpha = 0, 1, 2, \ldots, p - 1, \quad \text{or} \quad \alpha > p - 1 \]

Consider the kernel \( L_\alpha(z, u) \) on \( B_{p, q} \) given by

\[
L_\alpha(z, u) := |\det(1 - zu^*)|^{-2\alpha} = |\det(1 - uz^*)|^{-2\alpha} = |\det(1 - zu^*)|^{-\alpha} |\det(1 - uz^*)|^{-\alpha} (4.1)
\]

The kernel \( L_\alpha \) is the product of two positive definite kernels and, by Proposition 1.6, \( L_\alpha \) is positive definite. We denote by \( V_\alpha \) the Hilbert space \( \mathcal{B}^\alpha[L_\alpha] \) defined by the kernel \( L_\alpha \).

The group \( G = U(p, q) \) acts in the space \( V_\alpha \) by the unitary operators

\[
T_\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) = f((a + zc)^{-1}(b + zd)) |\det(a + zc)|^{-2\alpha} \]

We say that the representation \( T_\alpha \) is a kernel-representation of the group \( G = U(p, q) \).

4.2. Decomposition of the kernel representations into a tensor product. Consider the unitary representation \( \tau_\alpha \) of \( G \) defined by (4.2) and the complex conjugate representation \( \overline{\tau}_\alpha \). Consider the tensor product \( \tau_\alpha \otimes \overline{\tau}_\alpha \). The space \( V_\alpha := \mathcal{H}_\alpha \otimes H_\alpha \) of the tensor product is the space of holomorphic functions on \( B_{p, q} \times B_{p, q} \) defined by the positive definite kernel

\[
\tilde{L}(z_1, z_2; u_1, u_2) = \det(1 - u_1 z_1^*)^{-\alpha} \det(1 - u_2 z_2^*)^{-\alpha} \quad (4.3)
\]

and the group \( G \) acts in \( V_\alpha \) by

\[
F(z_1, z_2) \mapsto F[(\overline{\tau} + z_1 \overline{\tau})^{-1}(b + z_1 d), (a + z_2 c)^{-1}(b + z_2 d)] |\det(\overline{\tau} + z_1 \overline{\tau})^{-\alpha} \det(a + z_2 c)^{-\alpha} \]

Denote by \( \Delta \) the diagonal \( z_1 = \overline{\tau}_2 \) in \( B_{p, q} \times B_{p, q} \).

First a holomorphic function on \( B_{p, q} \times B_{p, q} \) is determined by its restriction to \( \Delta \). Hence we can consider the space \( H_\alpha \otimes \mathcal{H}_\alpha \) as a space of functions on \( \Delta \). The reproducing kernel of the latter space can be obtained by the substitution \( z_1 = \overline{\tau}_2, u_1 = \overline{\tau}_2 \) to (1.3), see 1.3.
Secondly the transformations in square brackets in formula (4.4) preserve $\Delta$. Let us rewrite the operators (4.4) as operators in the space of functions on $\Delta$. This gives (4.2). Thus we obtain the following proposition.

**Proposition 4.1.** The operator of restriction of functions $F(z_1, z_2)$ to $\Delta$ is a unitary $G$-intertwining operator from $\tau_\alpha \otimes \tau_\alpha$ to the kernel representation $T_\alpha$.

### 4.3. Representation in the space of Hilbert–Schmidt operators.

Let $W_1, W_2$ be Hilbert spaces. Let $A$ be an operator $A : W_1 \to W_2$. Let $a_{ij}$ be the matrix elements of $A$ in some orthonormal bases. Recall that $A$ is called a Hilbert-Schmidt operator (see [67], v.1, VI.6) if $\sum |a_{ij}|^2 < \infty$.

The space $\mathcal{HS}(W_1, W_2)$ of all Hilbert–Schmidt operators $W_1 \to W_2$ is a Hilbert space with respect to the scalar product

$$\langle A, B \rangle_{\mathcal{HS}} = \text{tr} \, AB^*$$

There is an obvious canonical identification

$$\mathcal{HS}(W_1, W_2) \simeq \mathcal{HS}(H_\alpha, H_\alpha)$$

Hence we can identify

$$V_\alpha \simeq H_\alpha \otimes H_\alpha \simeq \mathcal{HS}(H_\alpha, H_\alpha)$$

The group $G$ acts in $\mathcal{HS}(H_\alpha, H_\alpha)$ by the conjugations

$$g : A \mapsto \tau_\alpha(g)^{-1} A \tau_\alpha(g)$$

### 4.4. Another version of the kernel $L$.

Let us define the modified Berezin kernel

$$L_\alpha(z, u) := \frac{\det(1 - zz^*)^\alpha \det(1 - uu^*)^\alpha}{|\det(1 - zu^*)|^2^\alpha}$$

We denote by $V_\alpha$ the space $\mathcal{B}^\alpha[L_\alpha]$. The kernel $L_\alpha$ is $G$-invariant in the following sense

$$L_\alpha((a + zc)^{-1}(b + zd), (a + uc)^{-1}(b + ud)) = L_\alpha(z, u)$$

(this easily follows from (2.10)). Hence the operators

$$\mathcal{T}_\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) = f((a + zc)^{-1}(b + zd))$$

are unitary in $V_\alpha$.

**Remark.** The last formula does not depend on $\alpha$. Nevertheless we shall see (Section 7) that generally the unitary representations $\mathcal{T}_\alpha$ are nonequivalent.

We also define the distinguished vector $\Xi = \Xi_\alpha \in V_\alpha$ by

$$\Xi_\alpha(z) := \det(1 - zz^*)^{-\alpha} = L_\alpha(z, 0)$$

The vector $\Xi_\alpha$ is an element of the supercomplete basis and the $G$-orbit of $\Xi_\alpha$ consists of all elements of the supercomplete system. Hence the vector $\Xi_\alpha$ is cyclic.

By Proposition 1.7, the kernels $L$ and $\mathcal{L}$ differ unessentially, and the canonical operator $\mathcal{L} : V_\alpha \to V_\alpha$, described in Proposition 1.7, is given by

$$\mathcal{L} f(z) = f(z) \det(1 - zz^*)^\alpha$$

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13 Let $\rho$ be a representation of a group $G$ in a topological vector space $W$ (see [24], 7.2). A vector $w \in W$ is called cyclic if the linear span of the vectors $\rho(g)w$, $g \in G$, is dense in $W$. For a subset $B \subset W$ we define its cyclic span as the closure of the linear span of all vectors $\rho(g)w$, where $g \in G$ and $w \in B$, see [31], 4.4.
This operator intertwines (4.2) and (4.9). Thus we identify the unitary representation $T_\alpha$ and $T_\alpha$ of the group $G$.

The last description of the kernel representation (kernel $\mathcal{L}_\alpha$ and the representation $T_\alpha$) will be main for us below.

4.5. Limit of kernel representations as $\alpha \to \infty$. Consider the space $\mathcal{B}_\alpha^*\mathcal{L}_\alpha$ defined in 1.6. Consider elements of $\mathcal{B}_\alpha^*\mathcal{L}_\alpha$ of the form

$$\varphi(z) \det(1 - zz^*)^{-p-q}\{dz\}$$

where $\varphi(z)$ are compactly supported smooth functions on $B_{p,q}$. The $\mathcal{B}_\alpha^*$-scalar product of such distributions is given by

$$\langle \varphi, \psi \rangle_\alpha := C(\alpha)^{-1} \int_{B_{p,q} \times B_{p,q}} \mathcal{L}_\alpha(z,u)\varphi(z)\overline{\psi(u)} \det(1 - zz^*)^{-p-q}\det(1 - uu^*)^{-p-q}\{dz\}\{du\}$$

(4.11)

We define the normalization constant $C(\alpha)$ as the right-hand side of (2.18). By (4.8), this scalar product is $G$-invariant.

**Lemma 4.2.** The family of distributions

$$C(\alpha)^{-1} \mathcal{L}_\alpha(z,u) \det(1 - zz^*)^{-p-q}\det(1 - uu^*)^{-p-q}\{dz\}\{du\}$$

converges to $\det(1 - uu^*)^{-p-q}\delta(z - u)$ as $\alpha \to +\infty$.

**Proof.** By the invariance property (4.8), it is sufficient to follow only limit behavior of the family of distributions

$$\Omega_\alpha(z) := C(\alpha)^{-1} L(z,0) \det(1 - zz^*)^{-p-q}\{dz\} = C(\alpha)^{-1} \det(1 - zz^*)^{\alpha-p-q}\{dz\}$$

By the choice of the normalization constant, the integral of the function $\Omega_\alpha$ is 1. We also have $\det(1 - zz^*) = 1$ if $z = 0$ and $\det(1 - zz^*) < 1$ otherwise. The last two remarks imply the convergence $\Omega_\alpha(z) \to \delta(z)$ as $\alpha \to +\infty$.

Lemma 4.2 implies the following statement.

**Theorem 4.3.** The family of scalar products (4.11) tends to the following $L^2$-scalar product with respect to the $G$-invariant measure $\det(1 - zz^*)^{-p-q}\{dz\}$:

$$\langle \varphi, \psi \rangle = \int_{B_{p,q}} \varphi(z)\overline{\psi(z)} \det(1 - zz^*)^{-p-q}\{dz\}$$

as $\alpha \to +\infty$.

We observed that the natural limit of kernel representations $T_\alpha$ as $\alpha \to +\infty$ is the space $L^2$ on Riemannian symmetric space $U(p,q)/U(p) \times U(q)$.

4.6. A canonical basis in space of $K$-invariant functions. Denote by $\mathcal{V}_\alpha^K$ the space of all functions $f \in \mathcal{V}_\alpha$ that are invariant with respect to the group $K = U(p) \times U(q)$.

**Proposition 4.4.** Let $\mu$ be a collection of integers ranging the domain $\mu_1 \geq \ldots \geq \mu_p \geq 0$ if $\alpha > p - 1$, and $\mu_1 \geq \ldots \geq \mu_k \geq \mu_{k+1} = \cdots = \mu_p = 0$ if $\alpha = k \leq p - 1$. Then the system of functions

$$\Delta_\mu(z) = \Delta_{\mu_1,\ldots,\mu_p}(z) := S_\mu(zz^*) \det(1 - zz^*)^\alpha$$

(4.12)

where $S_\mu$ are the Schur functions, forms an orthogonal basis in the space $\mathcal{V}_\alpha^K$ and

$$\|\Delta_\mu\|_{\mathcal{V}_\alpha^K}^2 = \prod_{j=1}^p \Gamma^2(\alpha - j + 1) \prod_{j=0}^{\alpha-p} (\mu_j + p - j)!(\mu_j + q - j)! \prod_{j=1}^p \frac{\Gamma^2(\alpha + \mu_j - j + 1)}{\Gamma^2(\alpha - j + 1)}$$

(4.13)
defines scalar $\Delta$ and we apply the Weyl formula (3.4). The space $\mathcal{W}^K_\alpha$ corresponds to the space of $K$-intertwining operators $H_\alpha \to H_\alpha$. In 3.2 we constructed the decomposition

$$H_\alpha = \bigoplus_\mu \text{Pol}^{\mu}_{p, q}$$

into the sum of pairwise distinct representations of $K$. Let $c(\mu; \alpha)$ be given by (3.6). By the Schur Lemma (see [34], 8.2), any $K$-intertwining operator is a scalar operator in each summand. Our basic element $\Delta_\mu$ corresponds to the operator $J_\mu$, that is $c(\mu; \alpha)$ on $\text{Pol}^{\mu}_{p, q}$ and 0 on $\text{Pol}^{\nu}_{p, q}$ for $\nu \neq \mu$.

It remains to evaluate the Hilbert–Schmidt norm of the operator $J_\mu$. By (4.3), it equals

$$\dim \text{Pol}^{\mu}_{p, q} = \dim \rho_{\mu_1, \ldots, \mu_p} \cdot \dim \rho_{\mu_1, \ldots, \mu_p, 0, \ldots, 0}$$

and we apply the Weyl formula [3.4].

4.7. Comments. 1) Formula (4.3), (4.9) make sense for an arbitrary matrix ball (see 2.10), and this defines scalar kernel representations of any classical group $G$, see [51], [56].

2) General kernel representations (the definition was given in [51], see also [59]) are realized in spaces of vector-valued functions on matrix balls, and analogues of the Berezin kernels are invariant matrix-valued positive definite kernels on matrix balls.

We shall give the definition of the kernel representations more formally. Consider an Hermitian symmetric space $G/\tilde{K}$ (see 3.7). Consider a symmetric subgroup $T \subset G$, denote by $K$ the maximal compact subgroup in $G$. It turns out to be that there are two possibilities.

a) The first case. $G/K$ is a complex submanifold in $G/\tilde{K}$ and $\dim G/K = \frac{1}{2} \dim \tilde{G}/\tilde{K}$.

b) The second case. $G/K$ is a totally real submanifold in $G/\tilde{K}$ and $\dim G/K = \frac{1}{2} \dim \tilde{G}/\tilde{K}$.

Consider a unitary highest weight representation $\tau$ of $G$ and its restriction to $K$. In the first case the spectrum of the restriction is discrete and the problem of decomposition is reduced to a purely combinatorial problem (3.3).

By definition, the restriction in the second case gives a kernel representation.

For a discussion of a priori relations of this problems with other spectral problems (Howe dual pairs, $L^2$ on Stiefel manifolds, $L^2$ on pseudo-Riemannian symmetric spaces) see [51].

3) Only 4 real exceptional groups (from 23) have kernel representations.

4) The case $\tilde{G} = O(2, n)$ differs from the cases $\tilde{G} = Sp(2n, \mathbb{R}), U(p, q), SO^*(2n)$ ("matrix balls cases") in many details.

5) Canonical bases exist for all scalar kernel representations (in fact, for any irreducible representation of $\tilde{K}$, the space of $K$-fixed vectors has dimension $\leq 1$). For matrix ball cases they consists of Jack polynomials (for a definition of the Jack polynomial see [39]); in our case $G = U(p, q)$, the Jack polynomial reduces to the Schur functions. Existence of the canonical bases for the case in which $G/K$ is an Hermitian space was observed in [64].

5. Index hypergeometric transform: preliminaries

We use the standard notation for the hypergeometric functions

$$\binom{a}{b} = \sum_{j=0}^{\infty} \frac{(a)_n (b)_n x^n}{n! (c)_n}$$

$$\binom{a, b}{c} = \sum_{j=0}^{\infty} \frac{(a)_n (b)_n (c)_n x^n}{n! (d)_n (e)_n}$$

where $(r)_n = r(r + 1)\ldots(r + n - 1)$ is the Pochhammer symbol.
5.1. Index hypergeometric transform. Fix \( b, c > 0 \). Consider a sufficiently decreasing function \( f(x) \) on the half-line \( \mathbb{R}_+ : x \geq 0 \). We define the integral transform \( J_{b,c} f \) by

\[
g(s) = J_{b,c} f(s) = \frac{1}{\Gamma(b + c)} \int_0^{+\infty} f(x) \, _2\text{F}_1(b + is, b - is; b + c - x) x^{b+c-1}(1 + x)^{b-c} \, dx \tag{5.1}
\]

The inverse transform is given by

\[
J_{b,c}^{-1} g(x) = \frac{1}{\Gamma(b + c)} \int_0^{+\infty} g(s) \, _2\text{F}_1(b + is, b - is; b + c; x) \Gamma(b + is) \Gamma(c + is) \, ds \tag{5.2}
\]

The integral transform \( J_{b,c} \) is called the index hypergeometric transform, or the Olevsky transform, or the generalized Fourier transform or the Jacobi transform, or the Fourier–Jacobi transform, see \([36]\), see also \([57]\). For the first time, the inversion formula was obtained by Weyl in 1910 (\([77]\)).

**Remark.**

Theorem 5.2. \([57]\) Theorem 5.2. Denote by \( W_{b,c}^a \) the Hilbert space of holomorphic functions in the disc \(|z| < 1\) with the scalar product

\[
(f, g) = \frac{1}{2\pi i} \int_{|z|<1} f(z)g(z) (1 - |z|^2)^{2a - 2} 2\text{F}_1(a - b, a - c; 2a - 1; 1 - |z|^2) \, ds
\]

where \( \{dz\} \) is the Lebesgue measure on the circle. A simple calculation shows that the reproducing kernel of the space \( W_{b,c}^a \) is

\[
N_{b,c}^a(z, u) = \frac{\Gamma(a + b)\Gamma(a + c)}{\Gamma(b + c)} 2\text{F}_1 \left[ \begin{array}{c} a + b, a + c, \\ b + c \end{array} ; z \right]
\]

Theorem 5.2. \([57]\) The operator

\[
J_{b,c}^a g(s) = \frac{1}{\Gamma(a + is)\Gamma(b + c)} \int_0^{+\infty} (1 + x)^{-a-b} g \left( \frac{x}{x + 1} \right) \times
\]

\[
\times 2\text{F}_1(b + is, b - is; b + c; -x) x^{b+c-1}(1 + x)^{b-c} \, dx \tag{5.4}
\]

is a unitary operator

\[
W_{b,c}^a \rightarrow L^2 \left( \mathbb{R}_{+}, \frac{\Gamma(a + b)\Gamma(b + c)}{\Gamma(2is)} \right) \tag{5.5}
\]

**Remark.** Let \( g \in W_{b,c}^a \). Consider the new function \( f(x) := g(x/(x + 1)) \). The function \( f \) is defined on the half-plane \( \Re x > -1 \) and thus we obtain a space of holomorphic functions on this half-plane. The kernel (5.3) is replaced by

\[
N_{b,c}^a(x, y) = \frac{\Gamma(a + b)\Gamma(a + c)}{\Gamma(b + c)} 2\text{F}_1 \left[ \begin{array}{c} a + b, a + c, \\ b + c \end{array} ; \frac{xy}{(1 + x)(1 + y)} \right]
\]

In formula (5.4) the factor \( g(x) \) changes to \( f(x) \) (and the formula will almost coincide with (5.1)). The Plancherel measure on the half-line \( \mathbb{R}^* \) will be the same as in (5.3). \( \square \)
5.4. Index transform and Hahn polynomials. Let $a, b, c > 0$. The continuous dual Hahn polynomials (see, for instance [1], 6.10, [35]) are defined by the formula

$$ S_n(s^2; a, b, c) = (a + b)_n (a + c)_n \, {}_3F_2 \left[ \begin{array}{c} -n, a + i s, a - i s \\ a + b, a + c \end{array} ; 1 \right] $$

(5.6)

The family of functions $S_n$ is an orthogonal basis in the space

$$ L^2 \left( \mathbb{R}_+, \left| \frac{\Gamma(a + i s)\Gamma(b + i s)\Gamma(c + i s)}{\Gamma(2i s)} \right|^2 ds \right) $$

and moreover

$$ \int_0^\infty S_n(s^2; a, b, c) S_m(s^2; a, b, c) \left| \frac{\Gamma(a + i s)\Gamma(b + i s)\Gamma(c + i s)}{\Gamma(2i s)} \right|^2 ds = \Gamma(a + b + n)\Gamma(a + c + n)\Gamma(b + c + n) n! \delta_{m,n} $$

(5.7)

**Lemma 5.3.** The image of the function

$$ \left( \frac{x}{x + 1} \right)^n (1 + x)^{-a-b} $$

under the index hypergeometric transform $J_{b,c}$ is

$$ \frac{\Gamma(a + i s)\Gamma(a - i s)}{\Gamma(a + b + n)\Gamma(a + c + n)} S_n(s^2; a, b, c) $$

and the image of the function $z^n \in W_{a,b,c}^a$ under $J_{a,b,c}^a$ is

$$ \frac{S_n(s^2; a, b, c)}{\Gamma(a + b + n)\Gamma(a + c + n)} $$

The Lemma can be checked by a more or less direct calculation (see [57]). The second part of the Lemma implies Theorem 5.2, since we have an explicit correspondence of the orthogonal bases ($z^n$ and the Hahn polynomials).

6. Helgason transform and spherical transform

Here we discuss the spherical transform (it is also called the Harish-Chandra transform) and the Helgason transform. The latter is used only for an explanation of the former. In 6.9–6.10 we dogmatically define the spherical transform independently on the Helgason transform. Proofs of all facts on the spherical representations and the spherical transform formulated in 6.1–6.8 are contained in Helgason [27], chapter 4.

6.1. Definition of spherical representations. An irreducible representation in a complete separable locally convex space (see [34], 7.2) $\rho$ of the group $\mathbf{G}$ in a space $W$ is called a spherical representation if $W$ contains a $\mathbf{K}$-invariant vector (this vector is called a spherical vector).

Under some minor natural conditions on the space and representation, a spherical vector is unique up to a factor (see [27], IV.4 and references in this book).

Denote by $\xi$ the spherical vector of a spherical representation $\rho$. Consider the operator in the space $W$ given by

$$ \Pi := \int_\mathbf{K} \rho(k) dk $$

(6.1)

where $dk$ is the Haar measure on $\mathbf{K}$ such that the measure of the whole group is 1. Obviously (see [34], 9.2.1) $\Pi$ is the $\mathbf{K}$-intertwining projection to the vector $\xi$. 

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The vector $\Pi \rho(g)\xi$ has the form $\Phi(g)\xi$, where $\Phi(g) \in \mathbb{C}$ is a scalar. The function $\Phi(g)$ is called the spherical function of the representation $\rho$.

Let $k_1, k_2 \in K$. Then

$$\Phi(k_1 g k_2)\xi = \Pi \rho(k_1) \rho(g) \rho(k_2)\xi = \rho(k_1) \Pi \rho(g)\xi = \rho(k_1) \Phi(g)\xi = \Phi(g)\xi$$

This implies the $K \times K$-invariance of the spherical function:

$$\Phi(k_1 g k_2) = \Phi(g)$$

Hence we can consider a spherical function as

a) a function on double cosets $K \setminus G / K$

b) a $K$-invariant function on $B_{p,q} = G / K$

As in 2.9, we shall consider $K$-invariant functions on $B_{p,q} = G / K$ as functions depending on variables $x_j$. Recall that $x_j$ are the eigenvalues of $z^*(1 - zz^*)^{-1}z$.

6.2. Construction of spherical representations. As in 2.1, consider the space $\mathbb{C}^p \oplus \mathbb{C}^q$ equipped with the form $Q$ having a matrix $(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix})$. A subspace $R \subset \mathbb{C}^p \oplus \mathbb{C}^q$ is called isotropic if the form $Q$ is the identical zero on $R$. Denote by $\text{IGr}_m$ (the isotropic Grassmannian) the set of all isotropic $m$-dimensional subspaces in $\mathbb{C}^p \oplus \mathbb{C}^q$. The space $\text{IGr}_m$ is a $G$-homogeneous space. It is also $K$-homogeneous, hence there exists a unique $K$-invariant measure (or a volume form) on $\text{IGr}_m$. For $g \in G$ we denote by $j_m(g, R)$ the Jacobian of the transformation $R \mapsto gR$.

An isotropic flag in $\mathbb{C}^p \oplus \mathbb{C}^q$ is a family of isotropic subspaces

$$\mathcal{R} : R_1 \subset R_2 \subset \cdots \subset R_p$$

such that $\dim R_j = j$. Denote by $\text{Fl}_{p,q}$ the space of all isotropic flags in $\mathbb{C}^p \oplus \mathbb{C}^q$. The space $\text{Fl}_{p,q}$ is a $G$-homogeneous space.

The stabilizer of a flag in $G$ is a minimal parabolic subgroup in $U(p,q)$. Denote by $e_1, \ldots, e_{p+q}$ the standard basis in $\mathbb{C}^p \oplus \mathbb{C}^q$. Consider the isotropic subspace $S_j$ spanned by the vectors $e_i + e_{p+i}$ for $i \leq j$. Consider the flag

$$S : S_1 \subset S_2 \subset \cdots \subset S_p$$

We define the standard parabolic subgroup $P \subset G$ as the stabilizer of the flag $S$ in $G$.

The space $\text{Fl}_{p,q}$ is also $K$-homogeneous,

$$\text{Fl}_{p,q} = K / M,$$

where $M \simeq U(q-p) \times U(1) \times \cdots \times U(1)$ \hspace{1cm} (6.2)

hence there exists a unique (up to a factor) $K$-invariant measure on $\text{Fl}_{p,q}$. For $g \in G$ we denote by $J(g, \mathcal{R})$ the Jacobian of the transformation $\mathcal{R} \mapsto g\mathcal{R}$.

A multiplier $\omega(g, x)$ on a homogeneous space $G / H$ is a function on $G \times (G / H)$ satisfying the condition

$$\omega(g_1 g_2, x) = \omega(g_1, g_2 x) \omega(g_2, x)$$

EXAMPLE. The Jacobian is a multiplier. In particular, $J(g, \mathcal{R})$ and $j_m(g, R_m)$, $m = 1, \ldots, p$ are multipliers on $\text{Fl}_{p,q}$.

Let $\omega$ be a multiplier. Then the formula

$$\rho_\omega(g) f(x) = f(g(x)) \omega(g, x)$$

defines a representation of $G$ in the space of functions on $G / H$. For a description of all multipliers in general case see [13], 13.2, see also [13], 13.5 for geometric explanations.
Consider the action of $G$ on the homogeneous space $F_{1,p,q}$. Obviously, any function of the form

$$\omega(g, R) = \prod_{m=1}^{p} j_m(g, R_m)^{\beta_m}$$

is a multiplier on $F_{1,p,q}$. It is more convenient to define the "basic" multipliers by the formula

$$\omega_1(g, P) = j_1(g, P_1)^{2(1-p+1)}$$

$$\omega_m(g, P) = j_m(g, P_m)^{2(q-1)+m} j_{m-1}(g, P_m)^{-2(q-1)+m-1}; \quad \text{for } m = 2, \ldots, p$$

In this notation,

$$J(g, P) = \prod_m \omega_m(g, P_m)^{2(q-p+1-2m)}$$

Fix $s_1, \ldots, s_p \in \mathbb{C}$. We define the representation $\tilde{\rho}_s$ of the group $G$ in the space of $C^\infty$-smooth functions on $F_{1,p,q}$ by the formula

$$\tilde{\rho}_s(g)f(P) = f(gP)J(g, P)^{1/2} \prod \omega_j(g, P_j)^{s_j} \quad (6.3)$$

For $(s_1, \ldots, s_p)$ in a general position, the representation $\tilde{\rho}_s$ is irreducible. Otherwise there exists a unique irreducible spherical subquotient in $\tilde{\rho}_s$. Let us describe it.

Denote by $M$ the minimal closed $\tilde{\rho}_s$-invariant subspace of $C^\infty(F_{1,p,q})$ containing the function $f(P) = 1$. Denote by $N$ the maximal proper closed $\tilde{\rho}_s$-invariant subspace in $M$ and consider the quotient $M/N$. We denote by $\rho_s$ the representation of $G$ in the space $M/N$ (if $\rho_s$ is reducible, then $\rho_s \cong \tilde{\rho}_s$).

Consider the space $\mathbb{C}^p$ with coordinates $s_1, \ldots, s_p$. Consider the hyperoctahedral group $D_p$, i.e., the group generated by all permutations of coordinates and by the reflections $(s_1, \ldots, s_p) \to (\sigma s_1, \ldots, \sigma_s s_p)$, where $\sigma_j = \pm 1$. This group also coincides with the so-called restricted Weyl group of $U(p, q)$.

**Theorem 6.1. a)** The representations $\rho_s$ are exactly all spherical representations of $G$. More precisely, the representations $\rho_s$ are all spherical Harish-Chandra modules up to equivalence of Harish-Chandra modules.

b) The representations $\rho_s$ and $\rho_{s'}$ are equivalent iff there exists $\gamma \in D_p$ such that $\gamma s = s'$.

Denote by $G_{sph}$ the set of all unitary spherical representations.

Explicit description of the set $G_{sph}$ is unknown. We shall formulate two facts about $G_{sph}$ that are necessary for understanding of the subsequent text.

1. Obviously, if all the coordinates $s_j$ are pure imaginary, then the representation $\tilde{\rho}_s$ is unitary in $L^2(F_{1,p,q})$. These representations are called *representations of the spherical unitary principal nondegenerate series*.

2. If a representation $\rho_s$ is unitary, then $s_j \in \mathbb{R} \cup i\mathbb{R}$ (this a corollary of Theorem 6.1.b; indeed a unitary representation $\rho$ is equivalent to its contragredient (= dual) representation, and the representation dual to $\rho_s$ is $\rho_{-s}$).

**6.3. Another realization of spherical representations.** Let $h$ be an element of the standard parabolic subgroup $P$. Then $h$ induces a linear transformation in each 1-dimensional quotient $S_j/S_{j-1}$, clearly, it is the multiplication by some complex number $\chi_j(g)$.\footnote{The space $M$ is the cyclic span of the vector $1$. Consider a proper $G$-invariant subspace $L \subset M$. Then $1 \notin L$. Hence $L$ is contained in the kernel of the $K$-invariant projection $P$ (see (6.1)). Thus a sum of all proper subspaces in $M$ is contained in $\ker P$.}

Let $s_j \in \mathbb{C}$. Consider the space $L_s$ of all smooth functions on $G$ satisfying the condition

\[ \psi \in L_s \iff \psi(hg) = \chi_j(g)\psi(g) \quad \text{for all } h \in P, g \in G. \]

\[ \psi \in L_s \iff \psi(hg) = \chi_j(g)\psi(g) \quad \text{for all } h \in P, g \in G. \]
\[ F(rh^{-1}) = F(r) \prod_{m=1}^{p} |\chi_m(h)|^{(q-p+1-2m)+s_m} \quad r \in G, h \in P \] (6.4)

Denote by \( \hat{\rho} \) the representation of \( G \) in \( L_s \) given by
\[ \hat{\rho}(g)F(r) = F(gr) \] (6.5)

**Lemma 6.2.** Any element of \( G \) has the decomposition \( g = hk \), where \( h \in P, k \in K \).

**Proof.** This is equivalent to the transitivity of \( K \) on \( Fl_{p,q} = G/P \). \( \square \)

Thus any function \( F \in L_s \) is determined by its restriction to the submanifold \( K \subset G \). In (6.2) we defined the subgroup \( M := K \cap P \). For \( h \in M \), we have \( |\chi(h)| = 1 \). Thus \( F \) can be regarded as a function on \( K/M = Fl_{p,q} \).

Thus we obtain a canonical operator from \( L_s \) to the space of \( C^\infty(\text{Fl}_{p,q}) \); it is not hard to check, that this operator intertwines \( \hat{\rho} \) with \( \hat{\mu} \).

**Remark.** This construction also explains the appearance of the "canonical multipliers" \( \omega_m \), they corresponds to the 1-dimensional characters \( |\chi_m(h)| \) of \( P \). \( \square \)

**6.4. Preliminary remarks on Plancherel formula.**

By a general abstract theorem (see [34], 4.5, 8.4), any unitary representation of a locally compact group is a direct integral of irreducible representations. For some types of groups (including semisimple groups and, hence, \( U(p,q) = U(1) \times SU(p,q) \)) this decomposition is unique in a natural sense.

**Theorem 6.3.** a) Representation of \( G \) in \( L^2(G/K) \) is a direct integral over \( \hat{G}_{sph} \) with multiplicities \( \leq 1 \).

b) A kernel representation \( T_\alpha \) is a direct integral over \( \hat{G}_{sph} \) with multiplicities \( \leq 1 \).

**6.5. Proof of Theorem 6.3.** Denote by \( M(G) \) the algebra of compactly supported (complex-valued) measures on \( G \); the multiplication in \( M(G) \) is the usual convolution \( * \). We define by \( \mu^\square \) the pushforward of a measure \( \mu \) under the map \( g \mapsto g^{-1} \). Obviously, \( \mu \mapsto \mu^\square \) is an involution on \( M(G) \): \( (\mu * \nu)^\square = \mu^\square * \nu^\square \).

Denote by \( \mathcal{C}(G) \) the subalgebra of \( M(G) \) consisting of measures invariant with respect to the transformations \( g \mapsto k_1 g k_2 \), where \( k_1, k_2 \in K \).

**Theorem 6.4.** (Gelfand, see [27], 5.1)

a) \( \mu^\square = \mu \) for all \( \mu \in \mathcal{C}(G) \).

b) The algebra \( \mathcal{C}(G) \) is commutative.

**Proof.** a) It easily can be checked that \( g^{-1} \) is contained in the double coset \( K g K \). Hence \( \mu = \mu^\square \).

b) By a), the identity \( (\mu * \nu)^\square = \nu^\square * \mu^\square \) coincides with \( \mu * \nu = \nu * \mu \). \( \square \)

Let \( \varkappa \) be a unitary representation of \( G \) in a Hilbert space \( H \). For \( \mu \in M(G) \) we define the operator
\[ \varkappa(\mu) = \int_G \varkappa(g) d\mu(g) \]
It is easily shown that \( \varkappa(\mu * \nu) = \varkappa(\mu) \varkappa(\nu) \) (see [34], 10.2).

Denote by \( H^K \) the space of all \( K \)-invariant vectors in \( H \). The projection \( \Pi \) to \( H^K \) is given by
\[ \Pi = \int_K \varkappa(k)dk \]
Obviously, \( \Pi \) has the form \( \varkappa(\delta_K) \), where \( \delta_K \) is the Haar measure on \( K \) normalized by the condition: the measure of the whole group is 1. We consider this measure as a \( \delta \)-measure on \( G \) supported by \( K \).
LEMMA 6.5. a) The operators \( \pi(\mu) \), where \( \mu \in \mathcal{C}(G) \), are zero on the orthocomplement to the space \( H^K \).

b) The operators \( \pi(\mu) \), where \( \mu \in \mathcal{C}(G) \), are self-adjoint.

c) Let \( R \) be an \( \mathcal{C}(G) \)-invariant subspace in \( H^K \). Let \( Z \) be the \( G \)-cyclic span of \( R \). Then the projection of \( Z \) to \( H^K \) coincides with \( R \).

d) If \( \pi \) is irreducible, then \( \dim H^K \leq 1 \).

e) If there exists a cyclic \( K \)-fixed vector \( \Xi \) in \( H \), then the spectrum of \( \pi \) contains only spherical representations and their multiplicities are \( \leq 1 \).

PROOF. a) \( \pi(\mu) = \pi(\mu * \delta_K) = \pi(\mu)\Pi \) for \( \mu \in \mathcal{C}(G) \).

b) We have \( \pi(\mu)^* = \pi(\mu^2) \). By Theorem 6.4.a, this coincides with \( \pi(\mu) \).

c) Let \( h \in R \), then

\[
\Pi \pi(\xi)h = \Pi \pi(\xi)\Pi h = \Pi \pi(\xi)\Pi h = \pi(\delta_K)\pi(\xi)\pi(\delta_K)h - \pi(\delta_K * \delta_K)h \in R
\]

d) The algebra \( \mathcal{C}(G) \) is commutative. Any commutative family of self-adjoint bounded op-erators in a Hilbert space of dimension \( > 1 \) has a proper invariant subspace. Now apply c).

e) First let \( L \) be a \( G \)-invariant subspace without \( K \)-invariant vectors. Then \( \Xi \) is contained in its orthocomplement \( L^\perp \). But \( \Xi \) is cyclic and hence \( H = L^\perp \).

Second, a cyclic representation of a commutative \( * \)-algebra has multiplicities \( \leq 1 \) (see [34], 4.4, Problem 3).

PROOF OF THEOREM 6.3 b) The distinguished vector \( \Xi_\alpha \) is cyclic (see 4.4). It remains to apply Lemma 6.5.e.

PROOF OF THEOREM 6.3.a. There exists a unique \( G \)-invariant differential operator \( \Delta \) of order 2 on \( G/K \). Consider the heat equation

\[
\frac{d}{dt}f(t,z) = \Delta f(t,z); \quad t \geq 0
\]

Let \( N(t; z, u) \) be the corresponding heat kernel and let \( A_t \) be the corresponding evolution operator, i.e.,

\[
f(t,z) = A_t f(0,z) = \int_{B_{r,q}} N(t; z, u) f(u) \det(1 - uu^*)^{-p-\eta} \{du\}
\]

Fix \( t > 0 \). Obviously, the function \( N(t; z, 0) \) is a cyclic vector in the (closed) image of the evolution operator \( A_t \). Thus the representation of \( G \) in the subspace \( \text{Im} A_t \) has a multiplicity free spectrum.

We have \( A_t A_{t'} = A_{t + t'} \). Hence \( \text{Im} A_t \supset \text{Im} A_{t'} \) if \( t < t' \). Thus \( L^2(G/K) \) is the closure of the union of increasing family of multiplicity free subrepresentations

\[
\text{Im} A_1 \subset \text{Im} A_{1/2} \subset \text{Im} A_{1/3} \subset \ldots
\]

Thus \( L^2(G/K) \) itself is multiplicity free.

6.6. Normalization of Plancherel measure. Let \( \rho_s \in \hat{G}_{\text{sph}} \) be a spherical representation. Denote by \( W_s \) the space of the representation \( \rho_s \), denote by \( \langle \cdot, \cdot \rangle_s \) the scalar product in \( W_s \), denote by \( \xi_s \) a spherical vector in \( W_s \), such that \( \langle \xi_s, \xi_s \rangle_s = 1 \).

We shall consider Borel measurable functions \( f \) on \( \hat{G}_{\text{sph}} \) such that a value of \( f \) at a point \( s \) is an element of the space \( W_s \) (we omit a definition of measurability).

Consider a Borel measure \( \nu \) on the set \( \hat{G}_{\text{sph}} \) of spherical unitary representations of \( G \), let \( M \) be the support of \( \nu \). We shall consider the Hilbert space \( \int W_s d\nu(s) \) of all Borel measurable functions \( f : s \mapsto f(s) \in W_s \) satisfying the condition

\[
\int_M \langle f(s), f(s) \rangle_s d\nu(s) < \infty
\]
The scalar product in this space is defined by
\[ [f, g] = \int_M (f(s), g(s))_s d\nu(s) \]
The group \( G \) acts in our space pointwise. We denote this representation (direct integral) by \( \int \rho_s d\nu(s) \) (for details of the definition of direct integral see [4]).

In this Section we shall present the classical description of the measure \( \nu_\infty(s) \) on \( \tilde{G}_{sph} \) such that \( \int \rho_s(g) d\nu_\infty(s) \) is equivalent to \( L^2(G/K) \) and of a canonical unitary operator \( U \) from \( L^2(G/K) \) to the direct integral \( \int W_s d\nu(s) \).

In the next Section we solve the same problem for the kernel representations of \( G = U(p, q) \). These measures are called the Plancherel measures.

Remark. A Plancherel measure is not canonically defined (see [34], 4.5, 8.4). Let \( \beta(s) \) be a positive function on the support \( M \) of the measure \( \nu \). Clearly, the direct integrals \( \int \rho_s(g) d\nu(s) \) and \( \int \rho_s(g) [\beta(s) d\nu(s)] \) are equivalent.

For the case \( L^2(G/K) \), we define a canonical normalization of a Plancherel measure by the following rule: the element \( s \mapsto \xi_s \) of the direct integral corresponds to the \( \delta \)-function on \( G/K \) supported by \( z = 0 \).

Let us repeat this more formally. Consider an element \( s \mapsto a(s) \xi_s \) of the direct integral. Let \( F_a \) be the image of \( s \mapsto a(s) \xi_s \) under the operator \( U^{-1} \). Then we require \( F_a(0) = \int a(s) d\nu_\infty(s) \) for all \( a(s) \).

This normalization is not an arbitrary rule. If we shall change it, then formulas for the measure \( \nu_\infty \) and for the operator \( U \) will change. It is easy to obtain a more complicated formula in this way, but it is impossible to obtain a simpler formula.

Now we shall describe the map \( U \) (the Helgason transform) and the measure \( \nu_\infty \) (the Gindikin–Karpelevich measure).

6.7. Helgason transform and Gindikin–Karpelevich measure. Consider the "section of wedge" model \( SW_{p,q} \) of the space \( G/K = U(p,q)/U(p) \times U(q) \), see 2.8. Fix \( s := (s_1, \ldots, s_p) \in \mathbb{C}^p \), assume \( s_{p+1} = 0 \). Let \( Z = \begin{pmatrix} 1 & 0 \\ 2K & L \end{pmatrix} \in SW_{p,q} \). Denote by \( [Z]_k \) the left upper \( k \times k \) block of \( Z \). We define the functions \( \Psi_s(Z) \) by
\[ \Psi_s \begin{pmatrix} 1 & 0 \\ 2K & L \end{pmatrix} = \prod_{j=1}^p \det \left[ \begin{array}{cc} 1 & K^* \\ K^*(L + L^*) & \frac{1}{2} \end{array} \right]^{-\sigma_j + (s_j - s_{j+1})/2} \]
where
\[ \sigma_1 = \cdots = \sigma_{p-1} = 1; \quad \sigma_p = q - p + 1 \] (6.7)

Lemma 6.6. For any \( h \) in the standard parabolic subgroup \( P \),
\[ \Psi_s([Z]^h) = \prod_{m=1}^p \chi_m(h)^{q-p+1-2m+s_j} \]
(6.8)

Proof. A calculation, see [54].

Let \( f \) be a \( C^\infty \)-function on \( SW_{p,q} \) with a compact support. Its Helgason transform is the function
\[ F(r; s_1, \ldots, s_p); \quad r \in G, s_j \in \mathbb{C} \]
defined by
\[ F(r; s_1, \ldots, s_p) = \int_{SW_{p,q}} f(Z) \Psi_s(Z^r) \det \left[ \begin{array}{cc} 1 & K^* \\ K^*(L + L^*) & \frac{1}{2} \end{array} \right]^{-p-q} \{dZ\} \] (6.9)
By (6.8), the functions $F(r; s_1, \ldots, s_p)$ satisfy (6.4), i.e., for a fixed $s \in \mathbb{C}^n$ a function $F(r, s)$ is an element of the space $L_s$ of the representation $\rho_s$ (see (6.3)).

Obviously, the pushforward of the transformation $f(Z) \mapsto f(Z^{[g]})$ under the Helgason transform is given by (6.5).

Above we identified the representations $\tilde{\rho_s}$ and $\tilde{s}_s$. Thus we can consider a function $F$ as a function on $\mathbb{P}_{i, q} \times \mathbb{C}^p$.

Remarks. 1) Functions $F(\ldots)$ are holomorphic in $s_1, \ldots, s_p \in \mathbb{C}^n$. □

2) For each element $\gamma$ of the hyperoctahedral group $D_p$ there exists an integral operator $A_\gamma$ on $\mathbb{P}_{i, q}$ (which can be written explicitly) such that $F(\mathbb{P}, \gamma s) = A_\gamma F(\mathbb{P}, s)$. □

Denote by $\Sigma_p$ the simplicial cone

$$\Sigma_p : s_1 \geq s_2 \geq \ldots \geq s_p \geq 0$$

Denote by $i\Sigma_p \subset \mathbb{C}^p$ its image under the multiplication by $i$.

**Theorem 6.7.** The Helgason transform is a unitary operator

$$L^2((\mathbb{P}_{i, q} \times \mathbb{C}^p), \mathcal{R}(s) \{ds\} d\mathbb{P}) \to L^2(\mathbb{P}_{i, q} \times (i\Sigma_p), \mathcal{R}(s) \{ds\} d\mathbb{P})$$

where $\{dZ\}$ is the Lebesgue measure on $\mathbb{P}_{i, q} \times \mathbb{C}^p$, $d\mathbb{P}$ is the $\mathbb{K}$-invariant measure on $\mathbb{P}_{i, q}$, $\{ds\}$ is the Lebesgue measure on the set $i\Sigma_p$ and $\mathcal{R}(s)$ the Gindikin–Karpelevich density

$$\prod_{k=1}^p \frac{|\Gamma(\frac{1}{2}(q-p+1)+s_k)|^2}{|\Gamma(2s_k)|^2} \prod_{1 \leq k < l \leq p} (s_k^2 - s_l^2)^2 \quad (6.10)$$

The pushforward of the transformation $f(Z) \mapsto f(Z^{[g]})$ under the Helgason transform is given by

$$F(\mathbb{P}; s_1, \ldots, s_p) \mapsto F(g\mathbb{P}; s_1, \ldots, s_p)J(g, \mathbb{P})^{1/2} \prod \omega_j(g, P_j)^{s_j}$$

**6.8. Spherical transform: preliminary remarks.** Consider the space of $\mathbb{K}$-invariant functions $f$ on $G/\mathbb{K}$. As above, we can consider elements of this space as functions in the variables $x_j > 0$ (see 2.9) symmetric with respect to the permutations.

Consider the Helgason transform $F$ of a $\mathbb{K}$-invariant function $f$. Obviously, $F$ is a $\mathbb{K}$-invariant function. Hence $F$ depends only in the variables $s_1, \ldots, s_p$. Thus we obtain the transform from the space of symmetric functions in the variables $x_1, \ldots, x_p$ to the space of $D_p$-symmetric functions in the variables $s_1, \ldots, s_p$.

If a function $f$ in formula (6.3) is $\mathbb{K}$-invariant, then we can replace the factor $\Psi_s$ by its average over the group $\mathbb{K}$. But this average is the spherical function $\Phi_s$ of the representation $\rho_s$ (see [27], 4.4.3). Thus the Helgason transform on the space of $\mathbb{K}$-invariant functions is given by

$$F(s) = \text{const} \cdot \int_{\mathbb{R}^p_+} f(x)\Phi_s(x)w(x)dx \quad (6.11)$$

where

$$w(x) \, dx := \prod_{j=1}^p x_j^{q-p} \prod_{1 \leq k < l \leq p} (x_k - x_l)^2 \prod_{j=1}^p dx_j \quad (6.12)$$

The map $f \mapsto F$ given by (6.11) is called the spherical transform.

Remark. In some sense, the spherical transform is simpler than the Helgason transform.

1. We have only $p$ variables instead of $2pq$ variables.
2. The function \( s \mapsto \Phi_s(x) \) is holomorphic in \( s \) and \( D_p \)-symmetric with respect to \( s \) (by Theorem 6.1b). Hence for a compactly supported \( f \) its spherical transform \( F \) is a \( D_p \)-symmetric holomorphic function.

3. The index hypergeometric transform \( J_{b,c} \) for some special values of the parameters \( b,c \) gives the spherical transforms for all rank 1 groups (\( O(p,1) \), \( U(p,1) \), \( Sp(p,1) \)). Heckman and Opdam [24] constructed the family of integral transforms interpolating the spherical transforms for simple Lie groups of arbitrary rank.

6.9. Spherical functions and Berezin–Karpelevich formula. First we fix the notation

\[
\det_{k,l} c_{k,l} := \det \begin{pmatrix} c_{11} & \cdots & c_{1p} \\ \vdots & \ddots & \vdots \\ c_{p1} & \cdots & c_{pp} \end{pmatrix}
\]

Below all determinants are of the size \( p \times p \).

**Theorem 6.8.** (Berezin–Karpelevich [5], Hoogenboom[31]) The spherical functions of the group \( G = U(p,q) \) are given by

\[
\Phi_s(x) = \text{const} \cdot \frac{\det_{k,j} \binom{F_1}{\frac{1}{2}(q-p+1) + s_j, \frac{1}{2}(q-p+1) - s_j; -x_k} \prod_{1 \leq k < l \leq p} (s_k^2 - s_l^2) \prod_{1 \leq k < l \leq p} (x_k - x_l)}{\prod_{1 \leq k \leq p} (s_k)}
\] (6.13)

In order to simplify this expression, we introduce the notation

\[
r = (q-p+1)/2
\]

6.10. Spherical transform. Thus the spherical transform for the group \( G \) is given by

\[
F(s) = \tilde{f}(s) = \text{const} \int_{x_1 \geq 0, \ldots, x_p \geq 0} f(x) \Phi_s(x) w(x) \, dx
\] (6.14)

where \( w(x) \) is given by (6.12) and \( \Phi_s \) is given by the Berezin–Karpelevich formula.

**Theorem 6.9.** Spherical transform is a unitary (up to a factor) operator from the space of symmetric functions with the scalar product

\[
\langle f, g \rangle = \int_{\mathbb{R}^p} f(x) \overline{g(x)} w(x) \, dx
\]

to the space of \( D_p \)-symmetric functions with the scalar product

\[
\langle F, G \rangle = \int_{i\mathbb{R}^p} F(s) \overline{G(s)} \mathfrak{R}(s) \, \{ds\}
\]

where \( \mathfrak{R}(s) \) is the Gindikin–Karpelevich density [6.10] and \( \{ds\} \) is the Lebesgue measure on \( i\mathbb{R}^p \). The inversion formula is given by

\[
f(x) = \widehat{F}(s) = \text{const} \int_{i\mathbb{R}^p} F(s) \Phi_s(x) \mathfrak{R}(s) \, \{ds\}
\] (6.15)

**Remark.** Theorems 6.7 and 6.9 coincide. The implicator 6.7 \( \Rightarrow \) 6.9 is obvious. Let us explain \( \Leftarrow \). The Helgason transform is an operator from \( L^2(G/K) \) to the direct integral of the principal nondegenerate series over the Gindikin–Karpelevich measure. We must show that this operator is unitary. Assume that the spherical transform is unitary. Then the Helgason transform preserves the scalar products \( \langle gv, w \rangle \), where \( v, w \) range the space of \( K \)-fixed vectors and \( gv \) denotes an action
of the group on the Hilbert space. Thus the Helgason transform preserves the scalar products $\langle g_v, g_w \rangle = \langle g_1^{-1} g_v, w \rangle$ and hence the Helgason transform is unitary (since the vectors $gv$ span the both Hilbert spaces).

6.11. Some integrals with determinants.

**Lemma 6.10.** Let $\mu$ be a measure on $\mathbb{R}$. Then

$$\int_{\mathbb{R}^p} \det\{f_k(x_l)\} \det\{g_k(x_l)\} \mu(x_1) \ldots \mu(x_p) = n! \det\left\{ \int_{\mathbb{R}} f_k(x) g_m(x) \, d\mu(x) \right\}$$  \hspace{1cm} (6.16)

if the right-hand side of the equation has sense.

**Proof.** Obvious. \hspace{1cm} $\Box$

6.12. Some spherical transforms.

**Lemma 6.11.** For functions $\beta_1, \ldots, \beta_p$ on $\mathbb{R}^+$ we define the function

$$\Theta_\beta(x) := \frac{\det\{\beta_k(x_l)\}}{\prod_{k<l}(x_l-x_k)}$$  \hspace{1cm} (6.17)

Then its spherical transform is

$$\tilde{\Theta}_\beta(x) = \frac{1}{\prod_{k<l}(s^2_k-s^2_l)} \det\left\{ \int_0^\infty \beta_k(x) F_1(r+s_l, r-s_l; 2r; -x) x^{q-p} \, dx \right\}$$

We see that the integral in the curly brackets is the index hypergeometric transform $J_{r,r}(\beta(s))$.

**Proof.** We evaluate the integral (6.14) by Lemma 6.10. \hspace{1cm} $\Box$

**Corollary 6.12.** The spherical transform of the function $\prod_j b(x_j)$ is

$$\frac{1}{\prod_{k<l}(s^2_k-s^2_l)} \det\left\{ \int_0^\infty x^{k-1} b(x) F_1(r+s_l, r-s_l; 2r; -x) x^{q-p} \, dx \right\}$$

Our next Section is based on the following simple formula

**Theorem 6.13.** The spherical transform of the function

$$\det(1 - zz^*)^\alpha = \prod_{j=1}^p (1 + x_j)^{-\alpha}$$

is

$$\prod_{k=1}^p \Gamma(\alpha - \frac{1}{2}(q+p-1) + s_k) \Gamma(\alpha - \frac{1}{2}(q+p-1) - s_k) \prod_{j=0}^{p-1} \Gamma^2(\alpha - j) \hspace{1cm} (6.18)$$

Below we shall use the notation

$$h := (q - p + 1)/2$$

**Proof.** We must evaluate

$$\int_{\mathbb{R}_+^p} \prod_{j=1}^p (1 + x_j)^{-\alpha} \det\left\{ 2F_1(r+s_k, r-s_k; 2r; -x_j) \right\} \prod_{1 \leq k < l \leq p} (s^2_k - s^2_l) \prod_{1 \leq m < n \leq p} (x_m - x_n) \times$$

$$\times \prod_{1 \leq m < n \leq p} (x_m - x_n)^2 \prod_{k=1}^p x^{q-p} \, dx_1 \ldots \, dx_p$$
It is possible to apply directly Corollary 6.12, but it is more convenient to write
\[
\prod_{1 \leq k < l \leq p} (x_k - x_l) = \prod_k (1 + x_k)^{p-1} \prod_{1 \leq k < l \leq p} \left( \frac{x_k}{1 + x_k} - \frac{x_l}{1 + x_l} \right) = \prod_k (1 + x_k)^{p-1} \det_{k,m} \frac{(x_k)}{1 + x_k}^{m-1}
\]
By Lemma 6.11, we obtain
\[
\frac{1}{\prod (s_k^2 - s_l^2)} \det_{m,l} \int_0^\infty (1 + x)^{-\alpha+p-1} \left( \frac{x}{1 + x} \right)^{m-1} \text{P} \frac{(r+s_l,r-s_l;2r;-x)x^{r-p} dx}
\]
By Lemma 5.3, the integrals under the determinant are the continuous dual Hahn polynomials \( s_{m-1} \). We obtain
\[
B(\alpha) \cdot \frac{\det_{k,m} \{ S_{m-1}(s_k^2; \alpha-h,r,r) \}}{\prod (s_k^2 - s_l^2)}
\]
where \( B(\alpha) \) is given by (6.18). By (5.4), \( s_{m-1}(s^2) = (s^2)^{m-1} + \ldots \) and hence the determinant in the numerator is the Vandermonde determinant.

6.13. Image of the canonical basis under the spherical transform. The canonical orthogonal basis \( \Delta_\mu \) in the space \( V^K_\alpha \) (see 4.6) in our coordinates \( x_k \) is given by
\[
\Delta_\mu(x) = \prod_{k=1}^p (1 + x_k)^{-\alpha+p-1} \cdot \det_{k,j} \{ x_k^{\alpha+j-p} \} \prod_{1 \leq k < m \leq p} (x_k - x_m)
\]

Theorem 6.14. The image of the function \( \Delta_\mu \) under the spherical transform is given by
\[
\text{const} \cdot \prod_{k=1}^p \Gamma(\alpha - h + s_k) \prod_{j=0}^{p-1} \Gamma^2(\alpha - j) \cdot \frac{\det_{k,d} \{ S_{n+p-1}(s_k^2; \alpha-h,r,r) \}}{\prod_{1 \leq k < l \leq p} (s_k^2 - s_l^2)}
\]
where \( S_n \) are the continuous dual Hahn polynomials (5.6).

Proof. We apply Lemma 6.10 and Lemma 5.3.

6.14. Reduction of the Gindikin–Karpelevich inversion formula to the Berezin–Karpelevich formula. Let \( \Theta_f(x) \) be the same as above (6.17). Then
\[
\widehat{\Theta}_f(s) = \det_{k,m} \{ J_{r,r} f_m(s_k) \} / \prod (s_k^2 - s_l^2)
\]
We must check the equality
\[
\langle \Theta_f, \Theta_g \rangle = \langle \widehat{\Theta}_f, \widehat{\Theta}_g \rangle
\]
By Lemma 6.10, this reduces to
\[
\det_{k,m} \int_0^\infty f_m(x) g_k(x) x^{r-p} dx = \det_{k,m} \int_0^\infty (J_{r,r} f_m)(s) (J_{r,r} g_k)(s) |\Gamma(r + is)|^4 / |\Gamma(2is)|^2 ds
\]
By Theorem 5.1, the matrix elements of these two matrices coincide.

6.15. Comments. 1) The main facts concerning the Helgason transform, the spherical functions and the spherical transform are the same for all semisimple groups. But generally, the spherical functions of the real semisimple groups are certain multivariate special functions, they are one of the natural multivariate analogues of the hypergeometric functions (see 24). Simple determinant formulas for spherical functions exist only for the complex groups \([17]\) and \( U(p,q) \).
2) In general, the image of the canonical basis under the spherical transform consists of multivariate continuous dual Hahn polynomials or multivariate Meixner–Pollachek (for series $GL_n$) polynomials. In our case, the multivariate Hahn polynomials admit a simple expression.

Koornwinder [37] constructed a multivariate analogue of the Askey–Wilson polynomials. All classical and neoclassical orthogonal polynomials in one variable can be obtained by a degeneration of the Askey–Wilson polynomials (see the treatise [35]). Multivariate versions of (neo)classical orthogonal polynomials can be obtained by a degeneration of the Koornwinder construction (the basic steps were done in [8], [9]).

In particular, the multivariate continuous dual Hahn polynomials or Meixner–Pollachek polynomials can be constructed in this way.

7. Plancherel formula for kernel representations

We preserve the notation

$$r = (q - p + 1)/2; \quad h = (q + p - 1)/2$$

We preserve the notation $\rho_s$ for a spherical representation, $W_s$ for its space, $\langle \cdot, \cdot \rangle_s$ for the scalar product in $W_s$, $\Phi_s(g) = \Phi_s(x)$ for the spherical function of $\rho_s$.

7.1. Normalization of the Plancherel measure. By Theorem 6.3b, any kernel representation $\mathcal{T}_\alpha$ is equivalent to a multiplicity-free direct integral $\int \rho_s d\nu_\alpha(s)$ of spherical representations over some measure $\nu = \nu_\alpha$ on the space $\hat{G}_{sph}$ of all spherical representations (this measure is called the Plancherel measure).

As we have seen in 6.6, the measure $\nu_\alpha$ is defined up to a multiplication by a positive function. Now we shall define a natural normalization of $\nu_\alpha$ and of a unitary intertwining operator $U_\alpha$ from $\mathcal{T}_\alpha$ to $\int \rho_s d\nu_\alpha(s)$. We require the image of the distinguished vector $\Xi_\alpha \in \mathcal{V}_\alpha$ under $U_\alpha$ to be the function $s \mapsto \xi$.

We want to find the measure $\nu_\alpha$ normalized in this way. To do this, we calculate the matrix element $\langle \mathcal{T}_\alpha(g) \Xi_\alpha, \Xi_\alpha \rangle$ in two ways. We recall that this matrix element can be regarded as $K$-invariant function on $G/K$. A calculation in the kernel representation $\mathcal{T}_\alpha$ gives

$$\langle \mathcal{T}_\alpha(g) \Xi_\alpha, \Xi_\alpha \rangle = \det(1 - zz^*)^\alpha = \prod_{k=1}^p (1 + x_k)^{-\alpha}$$

A calculation in the direct integral gives

$$\langle \mathcal{T}_\alpha(g) \Xi_\alpha, \Xi_\alpha \rangle = \int \langle \rho_s(g) \xi_s, \xi_s \rangle_s d\nu_\alpha(s) = \int \Phi_s(g) d\nu_\alpha(s)$$

Thus we must find the measure $\nu_\alpha$ on $\hat{G}_{sph}$ such that

$$\int \Phi_s(x) d\nu_\alpha(s) = \prod_{k=1}^p (1 + x_k)^{-\alpha}$$

Conversely, if we have a measure $\nu_\alpha$ on $\hat{G}_{sph}$ satisfying (7.2), then we have equality (7.1) for matrix elements. Hence the direct integral is equivalent to $\mathcal{T}_\alpha$.

---

18 This normalization is consistent with the normalization of the Plancherel measure for $L^2$ defined in 6.6, since the limit of $\Xi_\alpha \in \mathcal{V}_\alpha$ as $\alpha \to +\infty$ is the $\delta$-function $\delta(z)$

19 Let $\xi$ be a cyclic vector of a unitary representation of a group $G$ in a Hilbert space $H$. Assume we know the matrix element $\gamma(g) = \langle \rho(g) \xi, \xi \rangle$. Let us explain why we know the representation $\rho$ itself. Then

$$\gamma(g_2^{-1} g_1) = \langle \rho(g_2)^{-1} \rho(g_1) \xi, \xi \rangle = \langle \rho(g_1) \xi, \rho(g_2) \xi \rangle$$

is a positive definite kernel (see 1.1) on $G$ and after this we can reconstruct the Hilbert space $H$ with the distinguished system of vectors $\rho(g) \xi$ in the usual way.
7.2. Plancherel formula for large values of $\alpha$.

**Theorem 7.1.** (Berezin [4]) Let $\alpha > q - p + 1$. Then the Plancherel measure is supported by the pure imaginary $s$ and its density with respect to the Lebesgue measure on $i\mathbb{R}^p$ is

$$
\frac{1}{\prod_{j=0}^{p-1} \Gamma^2(\alpha - j)} \prod_{k=1}^{p} \Gamma(\alpha - h + s_k) \Gamma(\alpha - h - s_k) \times 
\prod_{k=1}^{p} \frac{\Gamma^2(r + s_k) \Gamma^2(r - s_k)}{\Gamma(2s_k) \Gamma(-2s_k)} \prod_{1 \leq i < m \leq p} (s_i^2 - s_m^2)^2 (7.3)
$$

In fact, this is the product of (6.18) and the Gindikin–Karpelevich density (6.10).

**Proof.** We want to represent $f(x) := \prod(1 + x_j)^{-\alpha}$ as the inverse spherical transform (6.13) of some function $F(s)$. It is sufficient to evaluate the direct spherical transform (6.14) for $f(x)$. This was done in Theorem 6.13. This operation is correct (see [3]) if $f \in L^2 \cap L^1$ (i.e., $\alpha > q + p - 1$). Then we consider analytic continuation. For $\alpha < h$ we have a pole of integrand and for $\alpha < h$ our Theorem 7.1 becomes incorrect, see below.

7.3. Analytic continuation of the Plancherel formula. In fact, the Plancherel formula obtained in Theorem 7.1 is the following identity for the hypergeometric functions

$$
\prod_{k=1}^{p} (1 + x_k)^{-\alpha} \prod_{1 \leq k < l \leq p} (x_k - x_l) = 
= \text{const} \cdot \frac{1}{\prod_{j=0}^{p-1} \Gamma^2(\alpha - j)} \int_{i\mathbb{R}^p} \det \{2F_1(r + s_m, r - s_m; 2r; -x_k)\} \times 
\prod_{k=1}^{p} \Gamma(\alpha - h + s_k) \Gamma(\alpha - h - s_k) \prod_{k=1}^{p} \frac{\Gamma^2(r + s_k) \Gamma^2(r - s_k)}{\Gamma(2s_k) \Gamma(-2s_k)} \prod_{1 \leq i < m \leq p} (s_i^2 - s_m^2) ds_1 \ldots ds_p (7.4)
$$

where $r = (q - p + 1)/2$, $h = (q + p - 1)/2$ (a direct verification of this formula is a nice exercise).

The left-hand side of the equation is holomorphic in $\alpha \in \mathbb{C}$. Let us discuss the right-hand side. The integrand has singularities if

$$
\text{Re} \alpha = h - k; \quad k = 0, 1, 2, \ldots (7.5)
$$

The $\Gamma$-function exponentially decreases in the imaginary direction (see, for instance, [3], v.1, (1.18.6))

$$
|\Gamma(\alpha + is)| = (2\pi)^{\frac{1}{2}} |s|^{\frac{1}{2} - a} e^{-\frac{\pi}{2} |s|} (1 + o(1)), \quad |s| \to \infty
$$

and the spherical functions of unitary representations are bounded by 1. Hence the right-hand side of (7.4) is holomorphic in $\alpha$ except for the lines (7.5).

Let us construct the analytic continuation of the right-hand side from the domain $\text{Re} \alpha > \frac{1}{2}(q + p - 1)$ to the whole $\mathbb{C}$.

By Lemma 6.10, we can represent the identity (7.4) in the form

$$
\prod_{k=1}^{p} (1 + x_k)^{-\alpha} \prod_{1 \leq k < l \leq p} (x_k - x_l) = \text{const} \cdot \frac{1}{\prod_{j=0}^{p-1} \Gamma^2(\alpha - j)} \times 
\det \{ \int_{-\infty}^{+\infty} s^{2(m-1)} \Gamma(\alpha - h + s) \Gamma(\alpha - h - s) \Gamma^2(r + s, r - s; 2r; -x_k) \times 
\times \frac{\Gamma^2(r + s) \Gamma^2(r - s)}{\Gamma(2s) \Gamma(-2s)} ds \} (7.6)
$$
Lemma 7.2. Denote by \( \int_{-\infty}^{\infty} I(\alpha,s)ds \) the integral in the curly brackets in (7.6). Then the meromorphic continuation of \( \int_{-\infty}^{\infty} I(\alpha,s) \) to the whole \( \mathbb{C} \) is given by

\[
\int_{-\infty}^{\infty} I(\alpha,s)ds + 4\pi \sum_{0 \leq k < h - \alpha} c_k(\alpha)_{2}F_1\left[\begin{array}{c}
\alpha - p + 1 + k, -\alpha + q - k; \\
2r
\end{array}\right]_{-x} (\alpha - h + k)^{2(m-1)}
\]

(7.7)

where the coefficients \( c_k(\alpha) \) are given by

\[
c_k(\alpha) = \frac{\Gamma(2\alpha - 2h + k)\Gamma^2(-p + 1 + \alpha + k)\Gamma^2(q - \alpha - k)(-1)^k}{\Gamma(2\alpha - 2h + 2k)\Gamma(-2\alpha + 2h - 2k)k!}
\]

(7.8)

Remark. The expression (7.8) has poles at the points \( \alpha = p - 1 - k, p - 2 - k, \ldots \). Thus expression (7.7) has poles at the points \( \alpha = p - 1, p - 2, \ldots \).

\( \square \)

Proof. We shall obtain the first summand of the formula. Let \( \text{Re} \alpha_0 = h \), assume \( \text{Im} \alpha_0 > 0 \). We want to construct an analytic continuation of the integral \( \int I(\alpha,s)ds \) to a small neighborhood of the point \( \alpha_0 \). Our integrand has poles at the points \( s = \pm \text{Im} \alpha_0 \). Consider the contour \( L \) shown on the Picture. In a small neighborhood of \( \alpha_0 \), the expression \( \int_L I(\alpha,s)ds \) depends on \( \alpha \) holomorphically and

\[
\int_{-\infty}^{\infty} I(\alpha,s)ds - \int_L I(\alpha,s)ds
\]

is the sum of the residues.

Thus we obtain the analytic continuation of the right-hand side of (7.6):

\[
\text{const} \cdot \frac{1}{\prod_{j=0}^{p-1} \Gamma^2(\alpha - j)} \det\left\{ \int \Gamma^{2(m-1)}_{k,m} F_1(r + s, r - s; 2r; -x_k) d\mu_\alpha(s) \right\}
\]

(7.9)

where the measure \( \mu_\alpha \) on the complex plane \( \mathbb{C} \) is the sum of a continuous measure on the imaginary axis and \( \delta \)-measures supported by the points \( \pm (\alpha + h - k) \):

\[
d\mu_\alpha(s) = \Gamma(\alpha - h + s)\Gamma(\alpha - h - s)\frac{\Gamma^2(r + s)\Gamma^2(r - s)}{\Gamma(2s)\Gamma(-2s)}\{ds\} + \\
+ 2\pi \sum_{0 \leq k < h - \alpha} c_k(\alpha)\delta(s \pm (\alpha + h - k))
\]
where \( ds \) denotes the Lebesgue measure on the imaginary axis and \( c_k(\alpha) \) are the same as above (7.8).

Remark. The measures \( \nu_\alpha \) are complex-valued if \( \alpha \in \mathbb{C} \). They are real-valued for \( \alpha \in \mathbb{R} \). □

Further, we apply Lemma 6.10 (from the right-hand side to the left-hand side) and obtain the equality

\[
\prod_{k=1}^p (1 + x_k)^{-\alpha} = \int_{\mathbb{C}^n} \Phi_s(x) d\nu_\alpha(s) \tag{7.10}
\]

where

\[
d\nu_\alpha(s) = \frac{1}{\prod_{j=0}^{p-1} \Gamma^2(\alpha - j)} \prod_{1 \leq k < l \leq p} (s_k^2 - s_l^2)^2 d\mu_\alpha(s_1) \ldots d\mu_\alpha(s_p) \tag{7.11}
\]

and the measures \( \mu_\alpha \) are the same as above.

The identity (7.10) is the required expansion of the distinguished matrix element in the spherical functions.

In fact, in this formula we have integration over the family of planes having (up to the action of the group \( D_p \)) the form

\[
\Pi_{k_1, \ldots, k_p}^\sigma : \ s_1 = \alpha - h + k_1, \ldots, s_\sigma = \alpha - h + k_\sigma, \quad s_{\sigma+1}, \ldots, s_p \in i\mathbb{R} \tag{7.12}
\]

where \( \sigma = 0, 1, \ldots, p \) and \( k_j \) are nonnegative integers. If some \( k_j \) coincide, then (due to the factor \( \prod (s_k^2 - s_l^2)^2 \)) the density of measure on the plane (7.12) is 0. Hence only the case

\[
k_1 > k_2 > \cdots > k_\sigma \geq 0
\]

really exists.

By construction, our family of measures is meromorphic in \( \alpha \in \mathbb{C} \). Nevertheless we have a factor \( \prod \Gamma^{-2}(\alpha - j) \), which is a zero at the poles of (7.8). It is easy to show that our family of measures \( \nu_\alpha \) is holomorphic in \( \alpha \in \mathbb{C} \). The final formula without poles can be easily obtained and hence we omit them.

7.4. Positive definiteness.

**Theorem 7.3.** \( \vdash \) The formula (7.10)–(7.11) is the Plancherel formula.

Identity (7.10) has the form (7.2) but our considerations do not imply that our measure \( \nu_\alpha \) is supported by unitary spherical representations. Hence some proof is necessary. An a priori proof of positive definiteness of all representations at the support of the Plancherel measure is given in [56]. For our case \( G = U(p, q) \), also it is possible to apply Molev unitarizability results [41].

**Corollary 7.4.** The Helgason transform is a unitary operator from the space of functions with the scalar product (4.11) to \( \int \rho_s d\nu_\alpha(s) \).

7.5. Discrete spectrum. For the case \( \sigma = p \), the plane \( \Pi_{k_1, \ldots, k_p}^\sigma \) is a one-point set. Hence the representation \( \rho_{\alpha-h+k_1+\cdots+k_p} \) is a direct summand in \( T_\alpha \). These representations were subject of papers [62], [41], [59].

All other planes \( \Pi_{k_1, \ldots, k_\sigma}^\sigma \) correspond to direct integrals of some spherical series of unitary representations.

7.6. Comments. 1) Theorem 7.1 (large values of \( \alpha \)) was announced by Berezin [4] for the series \( G = U(p, q), \text{Sp}(2n, \mathbb{R}), \text{SO}^*(2n), \text{SO}(n, 2) \), a proof was published by Unterberger and Upmeier in [72]. For other series, the problem was solved in [13], the construction is based on the matrix B-function, which was constructed by Gindikin [18] for the groups \( \text{GL}(n, \mathbb{R}), \text{GL}(n, \mathbb{C}), \text{GL}(n, \mathbb{H}) \) and by the author for other series of groups. The general Plancherel formula (for arbitrary \( \alpha \)) was obtained in [57].

\(^{20}\text{A partial result was obtained in [26].} \)
2) The simple proof of the Plancherel formula given above works also for series \(GL(n, \mathbb{C}), O(n, \mathbb{C}), \text{Sp}(2n, \mathbb{C})\).

3) There exists a counterpart of our analytic continuation construction on the level of orthogonal polynomials. It goes at least to Wilson [78], the most general construction is contained in [9].

8. Boundary behavior of holomorphic functions and separation of spectra.

Consider the planes \(\Pi_{k_1,\ldots,k_\sigma}^\alpha\) defined by (7.12). We have a canonical decomposition

\[
\mathcal{T}_\alpha \simeq \int \rho_s d\nu_\alpha(s) = \bigoplus_{\sigma; k_1,\ldots,k_\sigma} \int_{s \in \Pi_{k_1,\ldots,k_\sigma}^\alpha} \rho_s d\nu_\alpha(s)
\]

\(\sigma = 0,\ldots,p; \ h - \alpha > k_1 > \cdots > k_\sigma \geq 0\)

Our purpose is to obtain a natural realization of the summands of this decomposition.

8.1. Restriction of holomorphic functions to submanifolds in boundary. Let \(\Omega \subset \mathbb{C}^N\) be an open domain, and let \(\partial \Omega\) be its boundary. Suppose \(\Omega\) satisfies the conditions

1) If \(z \in \Omega, \lambda \in \mathbb{C}, \text{and} \ |\lambda| \leq 1, \text{then} \ \lambda z \in \Omega, \text{i.e.,} \ \Omega \text{is a circle domain.}\)

2) If \(z \in \partial \Omega \text{ and } |\lambda| < 1, \text{then} \ \lambda z \in \Omega\)

Let \(K(z, u)\) be a positive definite kernel on \(\Omega\). Let \(K(z, u)\) be holomorphic in \(u\) and antiholomorphic in \(z\) (and hence the space \(B^\circ[K; \Omega]\) consists of holomorphic functions).

Assume that the kernel \(K(z, u)\) is invariant with respect to the rotations

\(K(e^{i\varphi}z, e^{-i\varphi}u) = K(z, u)\)

**Theorem 8.1.** ([59]) Let \(\mu\) be a positive measure supported by a subset \(M \subset \partial \Omega\). Suppose that

1) The limit

\(K^*(z, u) = \lim_{\varepsilon \to 0} K((1 - \varepsilon)z, (1 - \varepsilon)u)\) (8.1)

exists almost everywhere on \(M \times M\) with respect to \(\mu \times \mu\).

2) \(K^* \in L^1(M \times M, \mu \times \mu)\) and the limit (8.1) is dominated, i.e., there exists a positive function \(\gamma(z, u) \in L^1(M \times M, \mu \times \mu)\) such that

\(|K((1 - \varepsilon)z, (1 - \varepsilon)u)| \leq \gamma(z, u)\)

Then

a) For any \(f \in B^\circ[K]\) the limit

\(R_\mu f(z) := \lim_{\varepsilon \to 0} f((1 - \varepsilon)z)\) (8.2)

exists almost everywhere on \(M\) with respect to \(\mu\), and the restriction operator \(R_\mu\) is a bounded operator \(B^\circ[K] \to L^1(M, \mu)\).

b) Let \(\chi\) be a bounded measurable function on \(M\). Then the limit

\(l_\chi(f) := \lim_{\varepsilon \to 0} \int_M f((1 - \varepsilon)z) \chi(z) d\mu(z)\)

exists for all \(f \in B^\circ[L]\) and \(l_\chi\) is a bounded linear functional on \(B^\circ[L]\). Moreover, the map \(\chi \mapsto l_\chi\) is a bounded operator from \(L^\infty(M, \mu)\) to the space of linear functionals on \(B^\circ[L]\).

**Remark.** Generally, functions \(f \in B^\circ[K]\) are discontinuous on the boundary. Hence they have no values at an individual point \(z \in \partial \Omega\). Theorem 8.1 claims that under some conditions there exists the operator of restriction of function to a submanifold in the boundary.
Remark. Statement b) of the Theorem means that the space $\mathcal{B}^*[K]$ contains distributions supported by the subset $M \subset \partial \Omega$.

In fact, the Theorem defines two following spaces of functions on $M$.
The first space $\mathcal{E}^*(M, \mu)$ consists of all functions on $M$ that can be obtained by the restriction of $f \in \mathcal{B}^0[L]$. This space $\mathcal{E}^0(M, \mu)$ is a quotient space of $\mathcal{B}^0[L]$

The second space $\mathcal{E}^*(M, \mu)$ is the subspace in $\mathcal{B}^*[L]$ spanned by the (complex-valued) measures $\chi(z)\mu(z)$, where $\chi(z)$ is in $L^\infty(M, \mu)$. The space $\mathcal{E}^*(M, \mu)$ can be described more directly in the following way. We consider the scalar product

$$\langle \chi_1, \chi_2 \rangle := \iint_{M \times M} K^*(z, u) \chi_1(u) \overline{\chi_2(z)} \, d\mu(z) \, d\mu(u)$$

in $L^\infty(M, \mu)$. Then $\mathcal{E}^*(M, \mu)$ is a Hilbert space associated with the pre-Hilbert space $L^\infty(M, \mu)$.
The space $\mathcal{E}^*(M, \mu)$ is a subspace of $\mathcal{B}^*[K]$.

8.2. Restriction operators in the spaces $V_\alpha$. We apply Theorem 8.1 to the objects described in 4.2. The domain $\Omega$ is $B_{p,q} \times B_{p,q}$, the kernel $K$ is the kernel $L_\alpha$ given by (4.3). The set $M$ is a submanifold lying in the boundary of the diagonal $\Delta$: $z_1 = \overline{z}_2$.

We obtain the following statement.

Corollary 8.2. Consider the space $V_\alpha = \mathcal{B}^0[L_\alpha]$ of functions on $B_{p,q}$ described in 4.1. Let $M$ be a subset in the boundary of $B_{p,q}$, and let $\mu$ be a measure supported by $M$. Assume $\mu$ satisfies conditions 1–2 of Theorem 8.1. Then the limit (8.2) exists and the restriction operator $R_\mu$ is a well-defined operator $\mathcal{B}^0[L_\alpha] \to L^1(M, \mu)$.

Now let us consider the $G$-orbits $M_k$ in the boundary of $B_{p,q}$ (see 2.6).

Theorem 8.3. Let $\alpha < (q - p + 1)/2 + k$. Then the restriction operator is a well-defined operator from $V_\alpha = \mathcal{B}^0[L_\alpha]$ to the space $L^1_{loc}(M_k)$ of locally integrable functions on $M_k$.

The question is reduced to an estimation of convergence of the integral

$$\iint_{A \times A \subset M_k \times M_k} L_\alpha(z, u) \, dl(z) \, dl(u)$$

where $A$ is a compact subset in $M_k$ and $l(z)$ is the surface Lebesgue measure on $M_k$. I cannot simplify estimates of [55] for our case $G = U(p, q)$ and hence I omit a proof.

Thus, we obtain the family of $G$-invariant subspaces $\mathcal{E}^*(M_k) \subset \mathcal{B}^*[L_\alpha]$ associated with the orbits $M_k$.

8.3. Restrictions of derivatives.
Theorem 8.4. Let $\alpha < (q - p + 1)/2 + k - l$. Then the operator of restriction of partial derivatives of order $l$ to $M_k$ is a well-defined operator.

Remark. Discrete part of the spectrum corresponds to the compact $G$-orbit $M_0$.

For further discussion see [53], [55].

8.4. Comments. 1) The problem of separation of spectra in noncommutative harmonic analysis goes back to Gelfand and Gindikin [16], Olshanskii [5] proposed a way, which in some cases separates one of the pieces of spectra. Our way differs from [16], [5], see also [4].

2) The condition $K^*(z, u) \in L^1$ is not necessary for existence of the restriction operator, some phenomena related to the restriction problem are discussed in [22].

3) The discrete part of the spectrum of $\mathcal{T}_\alpha$ corresponds to the minimal boundary orbit $M_0$. This part of the spectrum was the subject of the work [5].

9. Interpolation between $L^2(U(p, q)/U(p) \times U(q))$ and $L^2(U(p + q)/U(p) \times U(q))$. Pickrell formula

9.1. Analytic continuation of the Plancherel formula to negative integer $\alpha$. Assume that $\alpha$ in the Plancherel formula (7.10)–(7.11) is a nonpositive integer, $\alpha = -N$. 

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Proof of Theorem 9.1. The factor \( \prod_{j=0}^{p} \Gamma^{-2}(\alpha-j) \) of (7.11) has zero of order 2p at \( \alpha = -N \). Hence a summand of the Plancherel formula can be nonzero only in the case than the product \( \prod_{j} c_{k_{j}}(\alpha) \) has a pole of order 2p at \( \alpha = -N \). But only the factors

\[
\Gamma^{2}(-p+1+\alpha+k_{j})
\]

of \( c_{k_{j}}(\alpha) \) (see (7.8)) can give poles (the poles of the first factor in the numerator in (7.8) are annihilated by the poles of the first factor of the denominator). Hence all p factors (9.3) have to be present, and hence the continuous components of the Plancherel measure \( \nu_{\alpha} \) are absent. Now it remains to write the formula.
Our next aim is to explain the group-theoretical meaning of Theorem 9.1.

9.2. Symmetric spaces $U(p+q)/U(p) \times U(q)$. Consider the space $\mathbb{C}^p \oplus \mathbb{C}^q$ equipped with the standard scalar product. We denote by $U(p+q)$ the unitary group of this space. We write elements of $U(p+q)$ as $(p+q) \times (p+q)$ matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

We denote by $\text{Gr}_{p,q}$ the space of all $p$-dimensional subspaces in $\mathbb{C}^p \oplus \mathbb{C}^q$. Obviously, $\text{Gr}_{p,q}$ is a $U(p+q)$ homogeneous space $U(p+q)/U(p) \times U(q)$.

The graph of a linear operator $\mathbb{C}^p \to \mathbb{C}^q$ is an element of $\text{Gr}_{p,q}$, and elements of $\text{Gr}_{p,q}$ in a general position have this form. Hence we obtain a parametrization of an open dense subset in the formula as above (2.7).

The action of the group $U(p+q)$ on the space $\text{Mat}_{p,q}$ is given by the same linear-fractional formula as above (2.7).

The $U(p+q)$-invariant measure on the space $\text{Mat}_{p,q}$ is given by

$$\det(1 + zz^*)^{-p-q}dz$$  \hfill (9.4)

9.3. Representations $\tau_{-N}$ of the group $U(p+q)$. We define these representations in three ways.

1) $\tau_{-N}$ is the irreducible representation of $U(p+q)$ with the signature $(N, \ldots, N, 0 \ldots, 0)$, where $N$ is on the first $p$ places (see [80], §49).

2) Denote by $\text{Det}$ the determinant line bundle on $\text{Gr}_{p,q}$. Denote by $\text{Det}^\otimes N$ the $N$-the power of $\text{Det}$. Then $\tau_{-N}$ is the representation of $U(p+q)$ in the space of holomorphic sections of $\text{Det}^\otimes N$.

3) Consider the space $\mathfrak{B}^\circ[L_{-N}]$ of functions on $\text{Mat}_{p,q}$ defined by the positive definite kernel

$$L_{-N}(z, u) = \det(1 + uz^*)^N$$

**Lemma 9.2.** The kernel $L_{-N}$ is positive definite.

**Proof.** By Proposition 1.6, it is sufficient to prove the statement for $N = 1$. Consider Euclidean space $\mathbb{C}^{p+q}$ with an orthonormal basis $e_1, \ldots, e_p, h_1, \ldots, h_q$. Consider the $p$-the exterior power $\Lambda^p \mathbb{C}^{p+q}$ and the system of vectors

$$u_z := (e_1 + \sum_1^p \tau_1 h_j) \wedge (e_2 + \sum_2^p \tau_2 h_j) \wedge \cdots \wedge (e_p + \sum_p^p \tau_p h_j) \in \Lambda^p \mathbb{C}^{p+q}$$

Then $\langle u_z, v_u \rangle = \det(1 + zu^*)$.

The functions $\theta_z(u) := L_{-N}(z, u)$ are holomorphic polynomials of degree $\leq pN$. By Corollary 1.4, all elements of the space $\mathfrak{B}^\circ[L_{-N}]$ are holomorphic polynomials of degree $\leq pN$.

The group $U(p+q)$ acts in $\mathfrak{B}^\circ[L_{-N}]$ by the operators

$$\tau_{-N}(g)f(z) = f((a + z c)^{-1}b + z d)) \det(a + z c)^N$$

**Remark.** The image of an element $\theta_u(z) := \det(1 + z u^*)^N$ of the supercomplete system under the transformation $\tau_{-N}(g)$ is an element of the supercomplete system (up to a factor). Hence the operators $\tau_{-N}(g)$ preserve the space $\mathfrak{B}^\circ(L_{-N})$. \hfill $\square$

9.4. Kernel representations of $U(p+q)$. Consider the positive definite kernel

$$\mathcal{L}_{-N}(z, u) = \frac{\det(1 + zu^*)^{2N}}{\det(1 + z z^*)^N \det(1 + u u^*)^N}$$

and the space $\mathcal{V}_{-N} := \mathfrak{B}^\circ[L_{-N}]$ defined by this kernel. We define the kernel representation $\mathcal{T}_{-N}$ of the group $U(p+q)$ in the space $\mathcal{B}^\circ[L_{-N}]$ by the formula

$$\mathcal{T}_{-N}(g)f(z) = f((a + z c)^{-1}b + z d))$$
We also define the distinguished vector $\Xi_{-N}$. It is the function $f(z) = \det(1 + zz^*)^{-N}$. The orbit of the vector $\Xi_{-N}$ consists of all elements of the supercomplete system $\mathcal{L}_{-N}(z, a)$ and hence the vector $\Xi_{-N}$ is cyclic.

The same arguments as in 4.3 show that the representation $\mathcal{T}_{-N}$ can be decomposed into a tensor product of $\tau_{-N}$ and the complex conjugate representation $\mathcal{T}_{-N}$. Also the space $\mathfrak{B}^\circ[\mathcal{L}_{-N}]$ can be canonically identified with the space of operators $\mathfrak{B}^\circ[L_{-N}] \to \mathfrak{B}^\circ[L_{-N}]$.

9.5. Why formulas in Sections 4 and 9 are similar? Representations $\tau_\alpha$ (see 3.1) make sense for arbitrary complex $\alpha$. For arbitrary real $\alpha$ the kernel $K_\alpha(z, u) = \det(1 - uz^*)^{-\alpha}$ defines some scalar product in a space of holomorphic functions, but this scalar product is not positive definite. The work in such spaces is possible but it is difficult from analytical point of view.

For a negative integer $\alpha$ our representations are finite-dimensional and hence they can be extended holomorphically to $GL(p + q, \mathbb{C})$. It is more natural to consider them as representations of the compact form $U(p + q)$ of the group $GL(p + q, \mathbb{C})$.

In fact, the formulas of Sections 4 and 9 for actions of groups and scalar products really coincide and the reason of small difference in signs is the jump to another real form of the group $GL(p + q, \mathbb{C})$.

9.6. Plancherel formula. We say that an irreducible representation of $U(p+q)$ is $U(p) \times U(q)$-spherical if it contains a nonzero $U(p) \times U(q)$-fixed vector. By the same Gelfand Theorem 6.4, this vector is unique.

The $U(p) \times U(q)$-spherical functions of $U(p + q)$ are given by the same Berezin–Karpelevich formula\textsuperscript{21}.

**Lemma 9.3.** The kernel representation $\mathcal{T}_{-N}$ is a multiplicity free sum of the spherical representations.

**Proof.** This is almost a special case of Theorem 6.3b. Nevertheless we shall give an independent proof.

First the vector $\Xi_{-N}$ is cyclic and $U(p) \times U(q)$-invariant. Hence the orthogonal projection of $\Xi_{-N}$ to any subrepresentation is $U(p) \times U(q)$-invariant and cyclic in the subrepresentation. Thus all irreducible subrepresentations of $\mathcal{T}_{-N}$ are spherical.

Secondly, assume that $\mathcal{T}_{-N}$ contains two copies of the same irreducible representation $\rho'$ of $U(p + q)$. Let $V, V'$ be the corresponding orthogonal subspaces and let $\xi \in V, \xi' \in V'$ be the spherical vectors. Let $J : V \to V'$ be the unique intertwining linear operator $V \to V'$ such that $J\xi = \xi'$. The projection of $\Xi_{-N}$ to $V \oplus V'$ must have the form $\lambda \xi \oplus \mu \xi'$, where $\lambda, \mu \in \mathbb{C}$. But this vector is not cyclic in $V \oplus V'$ (since it is contained in the graph of the operator $\mu \lambda^{-1}J : V \to V'$).


\textsuperscript{21}In this case, $-1 \leq x_j \leq 0$ are the eigenvalues of the matrix $-z^*(1 + zz^*)z$, where $z \in \text{Mat}_{p,q}$.

The definition of the Plancherel measure given in 7.1 is valid in our situation. But direct integrals here reduce to finite sums and we shall give the definition again.

Let $\rho^j$ be the spherical representations that occur in the spectrum of $\mathcal{T}_{-N}$, let $\Phi^j$ be the spherical function of $\rho^j$, and let $\xi^j$ be the projection of the distinguished vector $\Xi_{-N}$ to the subspace $V^j$. The Plancherel measure in our case is the collection of numbers

$$\nu^j = \langle \xi^j, \xi^j \rangle$$

The matrix element of the distinguished vector can be expanded into the sum of spherical functions:

$$\langle \mathcal{T}_{-N}(g) \Xi_{-N}, \Xi_{-N} \rangle = \sum \rho^j(g) \xi^j, \sum \xi^j = \sum \langle \rho^j(g) \xi^j, \xi^j \rangle = \sum \langle \xi^j, \xi^j \rangle \Phi^j(g)$$

As a result, we obtain

**Theorem 9.4.** The Plancherel coefficients $\nu^j$ are the coefficients in formula (9.2).

9.7. Limit as $N \to \infty$. Consider the space $\mathfrak{B}^\circ[\mathcal{L}_{-N}]$ (see 1.6). Consider measures on $\text{Mat}_{p,q}$ of the form $\varphi(z) \det(1 + zz^*)^{-p-q} \{dz\}$, where $\varphi$ are compactly supported smooth functions. Then
we obtain a scalar product in the space of smooth compactly supported functions given by

$$\langle \varphi, \psi \rangle_{-N} = C(-N)^{-1} \int_{\text{Mat}_{p,q} \times \text{Mat}_{p,q}} \mathcal{L}_{-N}(z, u) \varphi(z) \overline{\psi(u)} \frac{\{du\} \{dz\}}{\det(1 + z z^*)^{p+q} \det(1 + u u^*)^{p+q}}$$

(9.5)

It is natural to choose the normalization constant by

$$C(-N) = \int_{\text{Mat}_{p,q}} (1 + zz^*)^{-N-p-q} \{dz\}$$

**Proposition 9.5.** The limit of scalar products (9.5) as $N \to +\infty$ is the $L^2$-scalar product with respect to the $U(p + q)$-invariant measure (9.4).

**Proof** is similar to the proof of Theorem 4.3. □

**9.8. Pickrell formula.**

**Theorem 9.6.** Let us omit the condition $N + p - 1 \geq m_1$ in the summation in (9.2). Then formula (9.2) remains valid for all complex numbers $N$ such that $\Re N > -1$.

For $p = q$ this gives the Pickrell formula (Carlson, see [1], 2.8.1) (9.7) as $N \to +\infty$.

**Remark.** Let $N$ be a positive integer. Then omitting of the condition $N + p - 1 \geq m_1$ does not change formula (9.2). All new summands, which appear in the formula, are zeros, since we have the factor $\Gamma(N + p - m_1)$ in the denominator.

**Theorem 9.7.** (Carlson, see [1], 2.8.1) Let $f(z)$ be holomorphic function for $\Re z > 0$, let $f(z) = O(e^{(\pi - \varepsilon)|z|})$, and $f(n) = 0$ for all integers. Then $f(z)$ is identically zero.

**Proof of Theorem 9.6.** A simple calculation with Lemma 6.10 shows that the functions $\Phi_{p - h - m_1 - 1, \ldots, p - h - m_p - 1}(1, \ldots, x_p)$ constitute an orthogonal system in the space $L^2$ of symmetric functions on the cube $[-1, 0] \leq x_j \leq 0$ with respect to the measure

$$d\sigma(x) := \prod_k (-x_k)^{q-p} \prod (x_k - x_l)^2 dx_1 \ldots dx_p$$

Hence the coefficients of the expansion of $\mathcal{L}_{-N}(1 + x_j)$ in the spherical functions $\Phi_{-}$ are $L^2$-scalar products

$$\tilde{c}_m(N) = \text{const}(m) \int_{[-1, 0]^p} \prod(1 + x_k)^N \Phi_{p - h - m_1 - 1, \ldots, p - h - m_p - 1}(x) d\sigma(x)$$

Obviously, these coefficients are bounded for $\Re N \leq 0$. Since

$$\Gamma(N + a)/\Gamma(N + b) \sim N^{a-b}, \quad N \to +\infty$$

(see [34], v.1 (1.19.4)), it follows that the coefficients $c_m(N)$ of (9.2) also are bounded. Now we apply the Carlson theorem to $\tilde{c}_m(N) - c_m(N)$.

**9.9. Comments.** 1) The formula (9.2) itself is an extension of the Pickrell formula (Carlson, see [1], 2.8.1). Nevertheless, our method of analytic continuation (see [24]) is the same for all symmetric spaces and the decompositions of the type (9.2) follow automatically from the B-integrals evaluated in [34].

2) The limit of formula (9.2) as $N \to +\infty$ gives the Plancherel formula for $L^2$ on the compact symmetric space $L^2(U(p + q)/U(p) \times U(q))$.

3) In particular, this gives the Helgason theorem on the description of spherical representations of compact groups, see [23], [24], Theorem 5.4.1.

4) As in Subsection 4.6, we obtain a canonical basis in the subspace $\mathcal{V}^K_{-N}$ of $K$-invariant functions in $\mathcal{V}_{-N}$ (since $\mathcal{V}^K_{-N}$ is equivalent to the space of operators in $\mathfrak{B}^2(L^2_{-N})$). Its image under the spherical transform consists of determinants of dual Hahn polynomials (see [34] on the dual Hahn polynomials).
For other series this gives multivariate dual Hahn polynomials or (for series GL\(_n\)) the multivariate Krawtchouk polynomials.

5) There exists a natural inverse (!) limit of the symmetric spaces

\[
\lim_{\downarrow} U(p + k + p)/U(p) \times U(k + p) \quad \text{as } p \to +\infty
\]

It was constructed in the important work of Pickrell \cite{P} (for \(k = 0\)). Pickrell’s type construction exists for all classical compact symmetric spaces \(\{\mathbb{R}, \mathbb{S}, \mathbb{P}\}\). The are many similarities between harmonic analysis on the inverse limits of symmetric spaces and the analysis of Berezin kernels (see \cite{P}, \cite{S}, \cite{S2}).

10. Radial part of the spaces \(\mathcal{V}_\alpha\) and Gross–Richards kernels

10.1. An orbital integral. Consider the space \(\mathcal{V}_\alpha^K\) consisting of \(K\)-invariant functions \(f \in \mathcal{V}_\alpha\), see 4.6.

**Proposition 10.1.** The space \(\mathcal{V}_\alpha^K\) has the form \(\mathfrak{B}^\alpha[R_\alpha]\), where the reproducing kernel \(R_\alpha(z, u)\), \(z, u \in B_{p,q}\), is given by

\[
R_\alpha(z, u) = \det(1 - zz^*)^\alpha \det(1 - uu^*)^\alpha \int_{k_1 \in U(p), k_2 \in U(q)} |\det(1 - k_1 z k_2^{-1} u^*)|^{-2\alpha} dk_1 \, dk_2
\]

where \(dk_1, dk_2\) denote the Haar measures on the unitary groups \(U(p), U(q)\) such that the measure of the whole group is 1.

**Remark.** The kernel \(R_\alpha(z, u)\) is invariant with respect to the transformations \(z \mapsto k_1 z k_2^{-1}, u \mapsto l_1 u l_2^{-1}\), where \(k_1, l_1 \in U(p), k_2, l_2 \in U(q)\). Hence all elements of the space \(\mathfrak{B}^\alpha[R_\alpha]\) are \(U(p) \times U(q)\)-invariant functions.

**Proof.** Let \(a \in B_{p,q}\). We want to find a function \(\varphi_a \in \mathcal{V}_\alpha^K\) such that for any \(f \in \mathcal{V}_\alpha^K\),

\[
f(a) = \langle f, \varphi_a \rangle_{\mathcal{V}_\alpha^K}
\]

Let

\[
\theta_a(z) = L_\alpha(z, a) = \frac{\det(1 - aa^*)^\alpha \det(1 - zz^*)^\alpha}{|\det(1 - za^*)|^{2\alpha}}
\]

Then, by the reproducing property (1.3)

\[
f(a) = \langle f, \theta_a \rangle_{\mathcal{V}_\alpha}\]

This implies boundedness of the linear functional \(f \mapsto f(a)\). Thus we can apply 1.3. Thus the reproducing kernel exists.

The function \(f\) is \(K\)-invariant and hence we can replace \(\theta_a\) in equation (10.3) by its average over the group \(K\). This gives the integral expression (10.1) for the function \(\varphi_a\) and for the reproducing kernel \(R_\alpha(z, u)\) of our space. 

10.2. Evaluation of the reproducing kernel. Let us write functions \(f \in \mathcal{V}_\alpha^K\) as functions depending of the variables \(x_1, \ldots, x_p\), see (2.3).

**Theorem 10.2.**

\[
R_\alpha(x, y) = \frac{\Gamma^2 p(\alpha^2 q - 1)}{(q - p)!^p \prod_{j=1}^p \Gamma^2 (\alpha - j + 1)} \prod_{k=1}^p (1 + x_k)^{-\alpha + p - 1} (1 + y_k)^{-\alpha - p - 1} \times
\]

\[
\times \prod_{1 \leq k < l \leq p} (x_k - x_l) \prod_{1 \leq k < l \leq p} (y_k - y_l)
\]

(10.4)
Remark. The kernel (10.4) is a special case of the Gross-Richards kernels [2]. □

Theorem 10.3. (S. Bergman, 1947) Let $K$ be a positive definite kernel on a set $X$. Let \( \zeta_1(x), \zeta_2(x), \ldots \) be an orthonormal basis in the space \( \mathcal{B}^s[K] \). Then
\[
K(x,y) = \sum_j \zeta_j(x) \zeta_j(y)
\]
and the series in the right-hand side converges on \( X \times X \).

Proof of Theorem 10.3. Consider the expansion of the function \( \theta_a(x) := K(a,x) \) with respect to the basis \( \zeta_j \)
\[
\theta_a(x) = \sum c_j(a) \zeta_j(x) \quad (10.5)
\]
This series converges in \( \mathcal{B}^s[K] \) and hence it converges pointwise. Let us evaluate \( \langle \theta_a(x), \zeta_k(x) \rangle \) in two ways. By (10.5), it is equal \( c_k(a) \). By the reproducing property (1.3), it equals to \( \zeta_k(a) \). □

Proof of Theorem 10.2. We must evaluate
\[
\sum_{\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p} \frac{\Delta_{\mu_1,\ldots,\mu_p}(x) \Delta_{\mu_1,\ldots,\mu_p}(y)}{\Delta_{\mu_1,\ldots,\mu_p},\Delta_{\mu_1,\ldots,\mu_p}} \quad (10.6)
\]
where \( \Delta_{\mu} \) is the canonical orthogonal basis in \( \mathcal{V}_x^k \) defined by (4.12). This reduces to the evaluation of the series
\[
\sum_{\mu_1 \geq \cdots \geq \mu_p \geq 0} \prod_{j=1}^p \frac{\Gamma(\alpha + \mu_j - j + 1)}{(\mu_j + p - j)!(\mu_j + q - j)!} s_{\mu_1,\ldots,\mu_p}(X_1,\ldots,X_p)s_{\mu_1,\ldots,\mu_p}(Y_1,\ldots,Y_p)
\]
where
\[
X_k = x_k/(1 + x_k), \quad Y_k = y_k/(1 + y_k)
\]
and \( s_{\mu} \) are the Schur functions.

Theorem 10.4. (Hua, [3])
\[
\det \left\{ \sum_{n \geq 0} a_n X_k^n Y_k^n \right\} = \sum_{n_1 > n_2 > \cdots > n_p \geq 0} a_{n_1} \cdots a_{n_p} \det \{X_k^n\} \det \{Y_k^n\}
\]
This can be verified directly. Let us write this identity in the form
\[
\prod_{1 \leq k < l \leq p} (X_k - X_l) \prod_{1 \leq k < l \leq p} (Y_k - Y_l) = \sum_{\mu_1 \geq \cdots \geq \mu_p \geq 0} a_{\mu_1+p+1} a_{\mu_2+p+2} \cdots a_{\mu_p} s_{\mu}(X)s_{\mu}(Y)
\]
Now Theorem 10.2 becomes obvious. □

10.3. Spherical transform in the spaces \( \mathcal{V}_x^k \).

Proposition 10.5. Let \( \alpha > h = \frac{1}{2}(q + p - 1) \). Then the spherical transform is a unitary operator from \( \mathcal{V}_x^k \) to the space \( L^2 \) with respect to the Plancherel measure (7.1).

This is a rephrasing of Theorem 7.1. This is also a simple corollary of the exotic Plancherel formula (see 5.3) for the index hypergeometric transform.

10.4. Comments. 1) Analogues of the orbital integral (10.1) (the average of the Berezin kernel over the compact subgroup) exist for all matrix balls.

2) The calculation of 10.2 is valid also for the series \( \text{Sp}(2n, \mathbb{R}), \text{SO}^*(2n), \text{GL}(n, \mathbb{C}) \). The last case was considered by Gross and Richards [2].
3) For other matrix ball series, the Schur functions in (10.6) are replaced by Jack polynomials. I cannot identify these kernels with some known special functions. It seems, that they define "new" scalar products in the space of symmetric functions.

11. Ørsted problem. Identification of kernel representations and $L^2(G/K)$

11.1. The problem of unitary equivalence. As we have seen, for $\alpha > (q + p - 1)/2$ the Plancherel measure $\nu_\alpha$ for $\mathcal{F}_\alpha$ differs from the Plancherel measure $\nu_\infty$ for $L^2(G/K)$ by a functional nonvanishing factor (6.18). Hence for $\alpha > (q + p - 1)/2$ the representation $\mathcal{F}_\alpha$ is equivalent to the representation of $G$ in $L^2(G/K)$.

It is easy to construct an intertwining operator from $\mathcal{F}_\alpha$ to $L^2(G/K)$. The simplest possibility is to consider the identity map $f \mapsto f^{[2]}$.

There arises a problem of construction of an explicit unitary operator

$$J_\alpha : L^2(G/K) \to \mathcal{V}_\alpha$$

(11.1)

11.2. A-function. Our construction is based on one special function. Let $a, b, c > 0$. Following [7], we define the $A$-function by

$$A_{b,c}^\alpha(x) = \frac{1}{\Gamma(b + c)} \int_0^\infty \Gamma(a + is) \frac{\Gamma(b + is) \Gamma(b - is) \Gamma(c + is) \Gamma(c - is)}{\Gamma(2is) \Gamma(-2is)} \times$$

$$\times _2 F_1 (b + is, b - is; b + c; -x) \, ds$$

It seems that $A$-function cannot be expressed in terms of the standard special functions by algebraic operations (except some special values of the parameters $b, c$). I discussed this function in detail in [7].

11.3. Construction of the unitary intertwining operator. We preserve the notation $\Xi_\alpha$ for the distinguished vector in $\mathcal{V}_\alpha$ (see 4.4) and the notation $\mathcal{L}_\alpha (z, u)$ for the reproducing kernel of the space $\mathcal{V}_\alpha$.

Suppose we know the function $\mathfrak{F} = J_\alpha^{-1} \Xi_\alpha \in L^2(G/K)$. For $g = (a \ b \ c \ d)$, the image of the function

$$\Xi_\alpha (z[^g]) = \mathcal{L}_\alpha (z, a^{-1}b)$$

under $J_\alpha^{-1}$ is $\mathfrak{F}(z[^g])$. Hence if we know $J_\alpha^{-1} \Xi_\alpha$, then we know the $J_\alpha^{-1}$-image of all supercomplete system $\mathfrak{L}_\alpha(z) = \mathcal{L}_\alpha(z, u)$. We define the unitary $G$-intertwining operator $J_\alpha : L^2(G/K) \to \mathcal{V}_\alpha$ by the formula

$$J_\alpha f (g) = \langle f, \mathfrak{F}(z[^g]) \rangle_{L^2} = \int_{B_{p,q}} f(z) \mathfrak{F}(z[^g]) \det(1 - zz^*)^{-p-q} \, dz$$

(11.2)

The distinguished vector $\Xi_\alpha$ is $K$-invariant, hence the function $\mathfrak{F}(z)$ also is $K$-invariant. Thus the function $J_\alpha f$ is a left $K$-invariant on the group $G$ and hence $J_\alpha f$ is a function on $G/K$.

Now we want to find a $K$-invariant function $\mathfrak{F}$ from the equality

$$\langle \mathfrak{F}(z), \Xi(z[^g]) \rangle_{L^2(G/K)} = \langle \Xi(z), \Xi(z[^g]) \rangle_{\mathcal{V}_\alpha}$$

(11.3)

22Scalar products in $\mathcal{F}_\alpha$ and $L^2(G/K)$ are different and problem of boundedness of the identity map is nontrivial. This identical operator $Id$ was discussed in many papers (see [4], [44], [45], [60]). In [44] there was announced boundedness of $Id$ for a large $\alpha$. Clearly, boundedness follows from the explicit Plancherel formula (the expression (6.18) is bounded for a fixed $\alpha$; this gives also an explicit formula for the norm of $Id$. On the other hand, the Plancherel formula for vector-valued kernel representations is not known, and hence a priori proofs of boundedness of $Id$ preserve sense.
or in explicit form
\[
\int_{B_{p,q}} \Phi(z) \Phi((a + az)^{-1}(b + zd)) \det(1 - zz^*)^{-p - q} \, dz = \det(1 - uu^*)^\alpha = \prod (1 + x_k)^{-\alpha} \tag{11.3}
\]
where \(u = 0|a| = a^{-1}b\) and \(x_j\) are the same as above (2.9).

**Theorem 11.1.** The function \(\Phi(x)\) given by

\[
\Phi(x) = \Lambda_{G/K}^\alpha(x) := \text{const} \left\{ \prod_{j=0}^{p-1} \Gamma(\alpha - j) \right\} \frac{\det\left\{ x_k^{1 - 2r} \frac{d^{p-1}}{dx_k^{p-1}} + x^{2r + j - 2} \Lambda_{G/K}^{\alpha - h}(x_k) \right\}}{\prod_{j=0}^{p-1} \Gamma(\alpha - j)}
\]

is a solution of the problem (11.3) and hence the \(G\)-intertwining operator \(J_\alpha\) given by (11.2) is unitary.

**Remark.** This solution is not unique. I think that it is the 'best' of possible solutions (but this can be regarded as my own opinion, this subject is discussed in [57]).

Let \(B\) be a \(p \times p\) matrix, let \(x_j\) be its eigenvalues. We define the \(\Lambda\)-function of the symmetric space \(G/K = U(p, q)/U(p) \times U(q)\) by

\[
\Lambda_{G/K}^\alpha(B) := \Lambda_{G/K}^\alpha(x)
\]

Formula (11.2) written in an explicit form gives the following statement.

**Corollary 11.2.** The unitary \(G\)-intertwining operator \(J_\alpha : L^2(G/K) \rightarrow V_\alpha\) is given by

\[
J_\alpha f(u) = \int_{B_{p,q}} f(z) \Lambda_{G/K}^\alpha(Q(z, u)) \det(1 - zz^*)^{-p - q} \, dz
\]

where

\[
Q(z, u) := (1 - zz^*)^{-1}(z - u)(1 - u^*u)^{-1}(z^* - u^*)
\]

**Proof of Theorem 11.1.** Let \(\xi_s\) be the spherical vector of a spherical representation \(\rho_s\) acting in the space \(W_s\) (as above).

Consider the decomposition \(\int W_s \mathcal{R}(s) ds\) of \(L^2(G/K)\) into the direct integral. The function \(\Phi\) is \(K\)-invariant. Hence the image \(\Phi\) of \(\Phi\) in the direct integral is a function of the form \(s \mapsto \gamma(s)\xi_s\), where \(\gamma(s)\) is the spherical transform of \(\Phi\). Under the action of an element \(g \in G\) the function \(\Phi\) transforms to \(s \mapsto \gamma(s)\rho_s(g)\xi_s\). Hence the left-hand side of (11.3) coincides with

\[
\int |\gamma(s)|^2 \xi_s, \rho_s(g)\xi_s W_s \mathcal{R}(s) \{ ds \} = \int |\gamma(s)|^2 \Phi_s(g) \mathcal{R}(s) \{ ds \} \tag{11.4}
\]

But the right-hand side of (11.3) is the matrix element of the distinguished vector \(\Xi_\alpha\). Hence \(|\gamma(s)|^2\) is the spherical transform of \(\prod (1 + x_k)^{-\alpha}\), and hence \(|\gamma(s)|^2\) is given by (6.18).

Now we assume

\[
\gamma(s) = \frac{1}{\prod_{j=0}^{p-1} \Gamma(\alpha - j)} \prod_{k=1}^p \Gamma(\alpha - (p + q - 1)/2 + s_k)
\]

(recall that \(s\) is imaginary). It remains to evaluate the inverse spherical transform of \(\gamma(s)\). This means that we must evaluate the integral

\[
\int_{s_1 \geq s_2 \geq \ldots \geq s_p \geq 0} \prod_{k, m} \Gamma(\alpha - h + is_k) \det(2F_1(r + is_m, r - is_m; 2r; -x_k)) \times \prod_{k=1}^p \frac{\Gamma^2(r + is_k)\Gamma^2(r - is_k)}{\Gamma(2is_k)\Gamma(-2is_k)} \prod_{1 \leq l < m \leq p} (s_l^2 - s_m^2) \, ds_1 \ldots ds_p
\]
The function $\Gamma(\alpha - h + is)$ is not even, hence we cannot change domain of integration to $\mathbb{R}^p$. Nevertheless the integrand is symmetric with respect to $s_k$ and we can replace the integral by the integral

$$\frac{1}{n!} \int_{s_1 \geq 0, \ldots, s_p \geq 0}$$

(with the same integrand).

Let us convert the last factor of the integrand to the form

$$\prod (s_i^2 - s_m^2) = \det \{(b + is_k)_{j-1}(b - is_k)_{j-1}\}$$

where $(b + is)_{j-1}$ is the Pochhammer symbol. By Lemma 6.10, the integral reduces to

$$\det \left\{ \int_0^\infty \Gamma(\alpha - h + is)\ {}_2F_1(r + is, r - is; 2r; -x_k) \times \frac{\Gamma(r + j - 1 + is)\Gamma(r + j - 1 - is)\Gamma(r + is)\Gamma(r - is)}{\Gamma(2is)\Gamma(-2is)} ds \right\}$$

Now we apply the formula (see [23], v.1, 2.8(22))

$$(\gamma)_{k} y^{\gamma-1} {}_2F_1(\alpha, \beta; \gamma; y) = \frac{d^k}{dy^k} y^{\gamma+k-1} {}_2F_1(\alpha, \beta; \gamma+k; y)$$

and obtain

$$\text{const} \cdot \det \left\{ \int_0^\infty \Gamma(\alpha - h + is)\ {}_2F_1(r + is, r - is; 2r + j - 1; -x_k) \times \frac{\Gamma(r + j - 1 + is)\Gamma(r + j - 1 - is)\Gamma(r + is)\Gamma(r - is)}{\Gamma(2is)\Gamma(-2is)} ds \right\}$$

This completes the proof.

**11.4. Hidden overgroup.** The operator $J_\alpha$ identifies the spaces $V_\alpha$ and $L^2(G/K)$. We have seen in 4.2 that the action of $G$ in $V_\alpha$ extends to the action of the group $G = G \times G$ containing $G$ as the diagonal subgroup. If we believe that our operator $J$ is canonical, then we obtain the following strange statement. *There exists a natural one-parametric family (depending of $\alpha > h$) of actions of the group $G \times G$ in the space $L^2(G/K)$.*

**11.5. Comments.** 1. The problem of explicit unitary identification goes back to the work [30], in this paper there was called attention to the coincidence of spectra of $L^2(G/K)$ and restriction of a highest weight representation of $G$ to $G$.

2. We define a $\Lambda$-function of a classical Riemannian noncompact symmetric space $G/K$ by

$$\Lambda_{G/K}^\alpha(x) = \int \prod_k \Gamma(\alpha + is_k)\Phi_\alpha(x)\mathcal{R}(s)ds$$

where $\Phi_\alpha$ are spherical functions of $G$ and $\mathcal{R}$ is the Gindikin–Karpelevich density. For six series of the classical groups the function $\Lambda_{G/K}^\alpha$ can be evaluated explicitly.

3. *The case $G = GL(q, \mathbb{C})$, GL$(q, \mathbb{R})$, GL$(q, \mathbb{H})$ is trivial. For definiteness, consider the kernel representation of GL$(q, \mathbb{C})$. It is the restriction of the representation $\tau_\alpha$ (see (3.2)) of the group $U(q, q)$ to the subgroup GL$(n, \mathbb{C})$ (see lists in [31], [37]). The symmetric space GL$(q, \mathbb{C})/U(q)$ is the cone $\text{Pos}_q$, defined in 3.6. The Laplace transform is a unitary GL$(q, \mathbb{C})$-intertwining operator from $L^2(\text{Pos}_q)$ to the space $H_\alpha(W_q)$ of the kernel representation.*

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4. The calculation given in 11.3 is valid also for the series $G = O(n, \mathbb{C})$ and $Sp(2n, \mathbb{C})$.

**Addendum. Pseudoriemannian symmetric spaces, Berezin forms, and some problems of non $L^2$ harmonic analysis**

Molchanov (see [3], [8], [2], [3]) introduced some types of representations of certain groups $G$ ("canonical representations") related to expressions similar to Berezin kernels (4.7). I shall try to give a general scheme for pseudo-Riemannian symmetric spaces, including kernel representations together with Molchanov’s, van Dijk’s, and others examples of "canonical representations". I shall also try to discuss similarities and differences of these constructions and the kernel representations (A.5–A.8).

In A.9–A.10 another (essentially different) problem of harmonic analysis on pseudo-Riemannian symmetric spaces is considered.

We need uniform models of all the classical pseudo-Riemannian symmetric spaces obtained in [53]. First we consider some examples and after this we describe the general construction.

**A.1. Several examples.** In all examples below a point of a symmetric space $G/H$ is an ordered pair of linear subspaces $W, Y$ in a fixed linear space $V$ such that $V = W \oplus Y$.

**Example 1.** The space $GL(n, \mathbb{R})/GL(k, \mathbb{R}) \times GL(n - k, \mathbb{R})$ consists of pairs $(W, Y)$ of subspaces in $\mathbb{R}^n$ such that $\dim W = k$, $\dim Y = n - k$, $\mathbb{R}^n = W \oplus Y$.

**Example 2.** Consider the linear space $\mathbb{C}^{2n}$ equipped with the operator $J$ of the complex conjugation

$$J(z_1, \ldots, z_{2n}) = (\bar{z}_1, \ldots, \bar{z}_{2n})$$

The group $G$ of all linear operators commuting with $J$ is $GL(2n, \mathbb{R})$. A point of the symmetric space $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$ is a pair of subspaces $(W, Y)$ in $\mathbb{C}^{2n}$ such that $JW = Y$ (and hence $JY = W$) and $\mathbb{C}^{2n} = W \oplus Y$.

**Example 3.** Consider the linear space $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ equipped with the operator $J$ with the matrix $\left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$. The group of all operators in $\mathbb{R}^{2n}$ commuting with $J$ is $G = GL(n, \mathbb{R}) \times GL(n, \mathbb{R})$. Consider the set $\mathcal{S}$ of all pairs of subspaces $(W, Y)$ in $\mathbb{R}^{2n}$ such that $JW = Y$ (and hence $JY = W$) and $W \oplus Y = \mathbb{R}^{2n}$. Obviously,

$$\mathcal{S} = \mathbb{C}^{2n} / \mathbb{C}^n \oplus \mathbb{C}^n$$

where the subgroup $H = GL(n, \mathbb{R})$ is the diagonal subgroup in $G$.

**Example 4.** Consider the linear space $\mathbb{C}^r \oplus \mathbb{C}^q$ equipped with the Hermitian form $Q$ with the matrix $\left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$ as above (Section 2). Consider the set $\mathcal{S}$ of all linear subspaces $W \subset \mathbb{C}^r \oplus \mathbb{C}^q$ such that $\dim W = r$ and the form $Q$ is nondegenerate on $W$. Obviously,

$$\mathcal{S} = \bigcup_{s, t: s + t = r, s \leq p, t \leq q} U(p, q)/(U(s, t) \times U(p - s, q - t))$$

(this construction includes the construction 2.4 above). Let us also introduce subspaces $Y$ which are the orthocomplements $W^\perp$ to $W$.

**Example 5.** Consider the space $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ equipped with the skew-symmetric bilinear form with the matrix $\left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$. Consider the space $\mathcal{S}$ of pairs $(W, Y)$ of maximal isotropic subspaces in $\mathbb{R}^{2n}$ such that $\mathbb{R}^{2n} = W \oplus Y$. Obviously,

$$\mathcal{S} = Sp(2n, \mathbb{R})/GL(n, \mathbb{R})$$

**Example 6.** Consider the space $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ equipped with the skew-symmetric bilinear form $B$ with the matrix $\left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$ and the symmetric bilinear form $Q = \left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$. Consider the space $\mathcal{S}$ of pairs $(W, Y)$ of maximal $B$-isotropic subspaces in $\mathbb{R}^{2n}$ such that $\mathbb{R}^{2n} = W \oplus Y$ and $Y$ is the orthocomplement of $W$ with respect to $Q$. It can easily be checked that

$$\mathcal{S} = \bigcup_{r=0}^{n} GL(n, \mathbb{R})/O(r, n - r)$$

**A.2. Uniform construction of classical pseudo-Riemannian symmetric spaces.** In all examples above a point of a pseudo-Riemannian symmetric space $G/H$ is a pair $(W, Y)$ of transversal subspaces satisfying some simple conditions. There are 3 types of conditions:

a) $W, Y$ are maximal isotropic subspaces
b) \( Y \) is the orthocomplement of \( W \)

c) \( Y = JW, W = JY \) for some fixed operator \( J \).

For a uniform description of all 54 series of pseudo-Riemannian classical symmetric spaces (see Berger classification), we must fix some notations and definitions from linear algebra (this general construction is not necessary for understanding Subsection A3-A10 below).

The term linear space below means a right finite-dimensional module \( V \) over \( \mathbb{R}, \mathbb{C} \) or the quaternions \( \mathbb{H} \).

A semiinvolution \( J \) is a linear or antilinear operator in a linear space \( V \) satisfying the condition \( J^2 = \pm 1 \). A semiinvolution \( J \) in \( V \) is split if there exists a subspace \( W \) such that \( V = W \oplus JW \).

The term form below means a nondegenerate form on a linear space \( V \) over \( K = \mathbb{R}, \mathbb{C}, \mathbb{H} \) of one of the following types

a) a symmetric or skew symmetric form over \( \mathbb{R} \)

b) a symmetric, skew symmetric or Hermitian form over \( \mathbb{C} \)

c) an Hermitian or anti-Hermitian form over \( \mathbb{H} \)

A linear semiinvolution \( J \) is consistent with a form \( B(\cdot, \cdot) \) if \( B(Jv, Jw) = \pm B(v, w) \). An antilinear semiinvolution \( J \) is consistent with \( B \) if \( B(Jv, Jw) = \pm B(v, w) \).

A subspace \( W \subset V \) is called isotropic with respect to a form \( B(\cdot, \cdot) \) if \( B(w, w') = 0 \) for all \( w, w' \in W \). A form \( B \) on \( V \) is split if there exists an isotropic subspace \( W \subset V \) such that \( \dim W = \frac{1}{2} \dim V \).

A Grassmannian is a set of all subspaces of a given dimension in a linear space or a set of all isotropic subspaces of a given dimension.

A real classical group is a group of all linear operators in a linear space over \( K \) (i.e., \( \text{GL}(n, \mathbb{R}), \text{GL}(n, \mathbb{C}), \text{GL}(n, \mathbb{H}) \)) or the group preserving some form in a linear space over \( K \), i.e.,

- the groups \( O(p, q), \text{Sp}(2n, \mathbb{R}) \) over \( \mathbb{R} \)
- the groups \( O(n, \mathbb{C}), \text{Sp}(2n, \mathbb{C}), \text{U}(p, q) \) over \( \mathbb{C} \)
- the groups \( \text{Sp}(p, q), \text{SO}^*(2n) \) over \( \mathbb{H} \).

A classical pseudo-Riemannian symmetric space is a homogeneous space of the form \( G/H \), where \( G \) is a classical group, and a subgroup \( H \) is the set of fixed points of some automorphism \( \sigma \) of \( G \) satisfying \( \sigma^2 = 1 \).

All classical symmetric spaces (up to covering and center) can be obtain in the following way.

Consider the following types of data on a linear space \( V \) over the field \( K \).

STRUCTURE 1'. A basic form \( B \) is a split form on \( V \).

STRUCTURE 2'. A control semiinvolution \( J \) is a split semiinvolution on \( V \).

STRUCTURE 3'. A control form is arbitrary form on \( V \).

Consider ordered pairs \((W, Y)\) of subspaces of \( V \) such that

\[
V = W \oplus Y
\]  \hspace{1cm} (A.1)

We say that a pair \((W, Y)\) is consistent with the basic form \( B \) if the both subspaces \( W, Y \) are \( B \)-isotropic.

We say that a pair \((W, Y)\) is consistent with the semiinvolution \( J \) if \( JW = Y \).

We say that a pair \((W, Y)\) is consistent with the control form \( C \) if \( Y \) is the orthocomplement of \( W \) with respect to the form \( C \).

We consider a linear space \( V \) without any additional structure, or a linear space \( V \) equipped with one of the structures 1' - 3', or a linear space \( V \) equipped with all structures 1' - 3'. In the last case we assume \( J \) is consistent with \( B \) and

\[
C(v, w) = B(Jv, w)
\]

**Lemma A.1.** If a pair \((W, Y)\) is consistent with two of the structures 1' - 3', then it is consistent with the third structure.

Fix a linear space \( V \) with such structure. Denote by \( G(V) \) the groups of all linear operators in \( V \) preserving all structures on \( V \). Consider the set \( \mathcal{G} \) of all pairs \((W, Y)\) such that

1) \( W \oplus Y = V \)

2) \((W, Y)\) is consistent with the structure of \( V \)

---

23 A map \( A : V \rightarrow V \) is called an antilinear operator if \( A(v + w) = Av + Aw, A\mu v = \overline{\mu} Av \) for all \( v, w \in V, \mu \in K \).

24 This definition is adapted to our field \( K = \mathbb{R}, \mathbb{C}, \mathbb{H} \), for a general definition of semiinvolution see [4].

25 A sesquilinear form \( B(v, w) \) is called anti-Hermitian if \( B(w, v) = -B(v, w) \).
3) dim $Y$ is fixed.

**Observation A.2.** \[**a)** If the set $\mathcal{S}$ is nonempty, then it is a pseudo-Riemannian symmetric space $G(V)/H$ or atheunion of a finite collection of pseudo-Riemannian symmetric spaces $G(V)/H_j$.

**b)** Each pseudo-Riemannian symmetric space can be obtained in this way.

Tables are contained in \[53\]. The examples given above give a representative sample.

**A.3. Matrix atlas.** Consider a symmetric space $G/H$ obtained in this way. Fix a point $(\tilde{W}, \tilde{Y}) \in G/H$. Then for $(W, Y) \in G/H$ in general position, the subspace $W$ is a graph of an operator $A : \tilde{W} \to \tilde{Y}$ and $Y$ is a graph of an operator $B : \tilde{Y} \to \tilde{W}$. Thus for any point of $G/H$ we obtain a pair of operators

$$A : \tilde{W} \to \tilde{Y}; \quad B : \tilde{Y} \to \tilde{W}$$

Thus for any point $(\tilde{W}, \tilde{Y})$ we obtain a coordinate system $(A, B)$ on $G/H$.

**Example.** For the spaces $U(p, q)/U(p) \times U(q)$ we obtain the Cartan matrix ball realization.

**Remark.** The matrices $A, B$ are not arbitrary. For instance, in Example 1 of A.1 a pair $(A, B)$ satisfy the unique condition $\det(1 - AB) \neq 0$. In Example 5 we have the same condition and also $A = A^t, B = B^t$. Generally the pair $(A, B)$ ranges in an open subset in some linear subspace in space of pairs of matrices.

**A.4. Hua Loo Keng double ratio.** Fix two pairs of subspaces $(W_1, Y_1), (W_2, Y_2) \in G/H$ in general position (we also assume $\dim W_1 \leq \dim Y_1$). Then $W_2$ is the graph of some operator $R : W_1 \to Y_1$, and $Y_2$ is the graph of some operator $S : Y_1 \to W_1$. We define the Hua double ratio operator

$$\mathcal{D}(W_1, Y_1; W_2, Y_2) := SR : W_1 \to W_1$$

In matrix coordinates, this operator is given by the formula

$$\mathcal{D}(A_1, B_1; A_2, B_2) = (1 - B_2 A_1)^{-1}(B_1 - B_2)(1 - A_2 B_1)^{-1}(A_1 - A_2)$$

**Lemma A.3.** \[**a)** $1$ is not an eigenvalue of $\mathcal{D}(W_1, Y_1; W_2, Y_2)$.

**b)** $1 - \mathcal{D}(A_1, B_1; A_2, B_2) = (1 - B_2 A_1)^{-1}(1 - B_2 A_2)(1 - B_1 A_2)^{-1}(1 - B_1 A_1)$

**c)** $\mathcal{D}/(1 - \mathcal{D}) = (1 - B_2 A_1)^{-1}(B_2 - B_1)(1 - A_2 B_1)^{-1}(A_2 - A_1)$

**Proof.** Statement a) is equivalent to $W_2 \cap Y_2 = 0$, and b), c) can checked by a simple calculation.

**A.5. Berezin form.** Consider an arbitrary character $\chi$ of the multiplicative group of $\mathbb{K}$ (for instance, $\chi(z) = |z|^\alpha$). Consider the kernel

$$\mathcal{L}_\chi(W_1, Y_1; W_2, Y_2) = \chi(\det[1 - D(W_1, Y_1; W_2, Y_2)])$$

We define the Berezin form on the space of smooth compactly supported functions on $G/H$ by

$$\langle f, g \rangle := \iint_{G/H \times G/H} \mathcal{L}_\chi(W_1, Y_1; W_2, Y_2)f_1(W_1, Y_1)\overline{f_2(W_2, Y_2)}\, d\mu(W_1, Y_1)\, d\mu(W_2, Y_2) \quad (A.2)$$

where $(W_1, Y_1), (W_2, Y_2)$ are points of the symmetric space and $\mu$ is the $G$-invariant measure on $G/H$.

**A.6. Comparison of Riemannian and pseudo-Riemannian cases.** Formally the construction A.5 in the Riemannian case gives kernel representations. But a serious divergence between the Riemannian and pseudo-Riemannian cases appears immediately.

I emphasize that the kernel $\mathcal{L}_\chi$ is smooth on the diagonal of $G/H \times G/H$ but (for non-Riemannian case) it has singularities outside the diagonal. It seems (but not proved carefully) that our Hermitian form always (except for the Riemannian case) is indefinite. This leads to serious technical difficulties. In particular, an indefinite Hermitian form does not define a topology in a functional space. So even the formulation of the spectral problem is not obvious, for a discussion see A.7, A.8 below.

Many other phenomena existing for the kernel representations do not survive in the pseudo-Riemannian case. For instance, there is no realization in holomorphic functions, there is no overgroup $\bar{G}$ described in 4.7 and Section 11,\[53\] etc. It seems that theory of the kernel representations cannot be a special case of pseudo-Riemannian theory.

\[26\] $\bar{G}$ has no relation to the overgroup $G^\circ$ described below.
A.7. Indefinite harmonic analysis: approach related to Krein structures. Consider a linear space \( X \) equipped with an indefinite Hermitian form \( Q \). Fix subspaces \( X_+ \), \( X_- \) in \( X \) such that the form \( Q \) is positive definite on \( X_+ \), negative definite on \( X_- \) and \( X = X_+ \oplus X_- \). Let \( X_\pm \) be the completion of the pre-Hilbert space \( X_\pm \) with respect to the form \((-Q)\). Thus we obtain a topological vector space \( X := X_+ \oplus X_- \) equipped with the form \( Q \) and with the fixed decomposition \( X_+ \oplus X_- \) (these data are called the Krein structure, \cite{3}).

A Krein structure is not canonically determined by the space \( X \) and the form \( Q \).

It is not clear if this approach is useful in representation theory of semisimple groups or not. It seems that even a problem of existence of nontrivial examples for groups of rank \( r > 1 \) is open. It seems to me that the following question is way to understand this.

Let \( G \) be a semisimple group and \( K \) be its compact subgroup. Let an irreducible representation (a Harish-Chandra module) of \( G \) in the space \( X \) admit a \( G \)-invariant Hermitian form \( Q \) and let the restriction of \( \rho \) to \( K \) be multiplicity free. Then \( X \) admits a canonical Krein structure (since the restriction of \( Q \) to any irreducible \( K \)-subrepresentation is positive definite or negative definite). The simplest nontrivial examples of such a picture are:

1. the representations \( \tau_\alpha(g) \) of the group \( G = U(p,q) \) for any noninteger real \( \alpha \).
2. Molchanov’s degenerate representations of \( O(p,q) \) (see \cite{33}).

There arises the following problem.

**Question A.4.** a) Is it possible in these two cases to write the projections to the subspaces \( X_\pm \) explicitly?

b) Are the operators of the representation continuous in the topology of the Krein space?

A.8. Indefinite harmonic analysis. Molchanov’s approach. This approach is not well-formalized, but there is a nice collection of explicit nontrivial examples. They have the following form.

Consider a representation \( \zeta \) of a group \( G \) in a topological vector space \( X \). Let \( Q \) be a \( G \)-invariant Hermitian form on \( X \) (the basic example is described above in A.5).

Also consider some family \( \rho_\ell \) of irreducible representations of \( G \) in the spaces \( Y_\ell \), say, and assume that each representation of this family admits a \( G \)-invariant Hermitian forms \( R_\ell(\cdot,\cdot) \). Consider a space \( \mathcal{Y} \) of functions \( f \) that takes any \( t \) to a vector \( y_t \in Y_\ell \). The space of such functions is some kind of a direct integral of representations, but it is not a direct integral in the formal sense.

Consider a \( G \)-intertwining operator from \( X \) to \( \mathcal{Y} \). The operator \( J \) takes any vector \( x \in X \) to some function \( Jx(t) \). An indefinite Plancherel formula is the identity

\[
Q(x_1, x_2) = \int R_\ell(Jx_1(t), Jx_2(t)) \, d\mu(t)
\]

where \( d\mu(t) \) is some ("Plancherel") measure.

In particular, this measure is obtained for Berezin forms on certain rank 1 pseudo-Riemannian symmetric spaces \( \big( \text{ cf. } \cite{4}, \text{ cf. } \cite{12}, \text{ cf. } \cite{13} \big) \).

Fix arbitrary (for simplicity noninteger) \( x \in \mathbb{R} \). Consider the space of smooth functions on \( B_{p,q} = G/K \) equipped with the (generally indefinite) Hermitian form (4.11).

**Conjecture A.5.** Our Plancherel formula (see Section 7) is valid for arbitrary \( x \in \mathbb{R} \) in Molchanov’s sense.

A.9. Conformal group of the symmetric space. Each classical pseudo-Riemannian symmetric space \( G/H \) admits a canonical open embedding to some space \( G^0/P^0 \), where the conformal group \( G^0 \supset G \) is a classical group and \( P^0 \supset H \) is some maximal parabolic subgroup in \( G^0 \). In fact the space \( G^0/P^0 \) is a Grassmannian or a product of two Grassmannians.

---

27 Let us describe the simplest example. Consider the space \( X \) consisting of finite linear combinations of vectors \( e_1, e_2, \ldots ; f_1, f_2, \ldots \). Assume all these vectors are pairwise orthogonal and \( (e_j, e_i) = 1 \) and \( (f_j, f_i) = -1 \). Consider the subspace \( X_+ \) generated by \( e_j \) and the subspace \( X_- \) generated by \( f_j \). Consider also the subspace \( X'_+ \) generated by the vectors \( \sqrt{3} + T e_j + \sqrt{3} T f_j \) and the subspace \( X'_- \) generated by the vectors \( \sqrt{3} e_j + \sqrt{3} T f_j \). Then \( X_+ \oplus X_- \) and \( X'_+ \oplus X'_- \) are different linear topological spaces containing \( X \), i.e., the identical operator \( X \to X \) cannot be extended to a bounded bijection \( X_+ \oplus X_- \to X'_+ \oplus X'_- \).

28 First example of this kind were observed on "physical level" in \cite{44}, for mathematically rigorous way of formulation of such problems see \cite{14}, see also \cite{12}, \cite{13}, \cite{12}.

29 The term "conformal" was introduced by Goncharov and Gindikin.

30 This fact can be extracted from Makarevich’s tables \cite{41}, but it never was claimed before \cite{12}.
Example. In the situation described in 2.4 \((G/H = G/K = U(p,q)/U(p) \times U(q))\) the space \(G^o/P^o\) is the Grassmannian of \(p\)-dimensional subspaces and \(G^o = GL(p + q, \mathbb{C})\).

Example. The conformal group in Example 1 in A.1 is \(G^o = GL(n, \mathbb{R}) \times GL(n, \mathbb{R})\), in Example 2 we have \(GL(2n, \mathbb{C})\), in Example 3 we have \(G^o = GL(2n, \mathbb{R})\), in Example 4 we have \(G^o = GL(p + q, \mathbb{C})\), in Example 5 we have \(G^o = Sp(2n, \mathbb{R})\times Sp(2n, \mathbb{R})\), in Example 6 we have \(G^o = Sp(2n, \mathbb{R})\).

Remark. The general algorithm for obtaining the conformal group \(G^o\) is following. 
1. Let a control semiinvolution and a control form be absent. Then \(G^o = G \times G\). The space \(G^o/P^o\) is the product of \(B\)-isotropic Grassmannians if a basic form \(B\) is present, and a product of the usual Grassmannians otherwise.

2. Otherwise, we "forget" the control semiinvolution and the control form on \(V\) and consider the group of automorphisms of the basic form \(B\) if the basic form is present, and the complete linear group if \(B\) is absent.

Remark. For 44 series of symmetric spaces, the subset \(G/H\) is dense in \(G^o/P^o\). For 10 series \(G/H\) is not dense in \(G^o/P^o\) but the group \(G\) has a finite number of open orbits in \(G^o/P^o\), all these orbits are symmetric spaces of the form \(G/H_1\), and the union of \(G/H_1\) is dense in \(G^o/P^o\). These 10 series are distinguished by the condition: a control form is present and it is an Hermitian form.

For Riemannian symmetric spaces \(G/H\) (i.e., \(G/K\) in the notation of Section 11) the conformal group \(G^o\) and the hidden overgroup \(G\) discussed in Section 11 are different objects. Hidden overgroup acts by integral operators in \(L^2\) and it does not act (even locally) on the symmetric space \(G/K\) itself.

A.10. Deformation of \(L^2\) on pseudo-Riemannian symmetric space. A natural representation \(\rho_0\) of \(G^o\) in \(L^2(G^o/P^o)\) is a representation of a principal degenerate series.

Observation A.6. (a) \(\rho_0\) is the product of \(B\)-isotropic Grassmannians but the group \(G\) has a finite number of open orbits in \(G^o/P^o\), all these orbits are symmetric spaces of the form \(G/H_1\), and the union of \(G/H_1\) is dense in \(G^o/P^o\). These 10 series are distinguished by the condition: a control form is present and it is an Hermitian form.

Observe that \(\rho_0\) is not dense in \(G^o/P^o\) but the group \(G\) has a finite number of open orbits in \(G^o/P^o\), all these orbits are symmetric spaces of the form \(G/H_1\), and the union of \(G/H_1\) is dense in \(G^o/P^o\). These 10 series are distinguished by the condition: a control form is present and it is an Hermitian form.

In many cases the representation \(\rho_0\) can be included in a degenerated complementary series \(\rho_s\). Thus there arises a problem of Plancherel formula for the restriction of \(\rho_s\) to \(G\). This restriction is some kind of deformation of \(L^2(G/H)\).

This problem survives even in the case in which the complementary series is absent. Consider the \(G\)-invariant kernel on pseudo-Riemannian symmetric space \(G/H\) given by

\[ \mathcal{L}_\chi(W_1, W_2, Y_2) := \chi \left( \frac{\det \frac{\mathcal{D}(W_1, Y_1, W_2, Y_2)}{1 - \mathcal{D}(W_1, Y_1, W_2, Y_2)}}{1 - \mathcal{D}(W_1, Y_1, W_2, Y_2)} \right) \]

Then the expression (A.2) is an Hermitian form on the space of compactly supported smooth functions on \(G/H\).

Example. For \(G/H = U(p,q)/U(p) \times U(q)\) we obtain the \(U(p,q)\)-invariant kernel

\[ \left| \det(1 - z^* z)^{-1}(z^* - u^*)^{-1}(1 - uu^*)^{-1}(z - u) \right|^p = \left( \frac{\det(z-u)(z^*-u^*)}{\det(1-2zz^*) \det(1-uu^*)} \right)^p \]

This kernel has the main singularity on the diagonal \(z = u\). Evidently the representation of \(U(p,q)\) associated with this kernel is not a kernel representation.

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