A note on non-repetitive colourings of planar graphs

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Abstract

Alon et al. introduced the concept of non-repetitive colourings of graphs. Here we address some questions regarding non-repetitive colourings of planar graphs. Specifically, we show that the faces of any outerplanar map can be non-repetitively coloured using at most five colours. We also give some lower bounds for the number of colours required to non-repetitively colour the vertices of both outerplanar and planar graphs.

1 Introduction

A sequence $a$ is called non-repetitive if $a$ contains no identical adjacent blocks. A vertex (resp. edge) colouring of a graph $G$ is called non-repetitive if for any open path $P$ in $G$, the sequence of vertex (resp. edge) colours along $P$ is non-repetitive. If $G$ is a planar map, then a colouring of the faces of $G$ is called non-repetitive if for any sequence of distinct faces such that each consecutive pair of faces shares an edge, the sequence of corresponding colours is non-repetitive.

Grytczuk [2] asked the following question: is there a natural number $k$ such that the faces of any planar map can be non-repetitively coloured using at most $k$ colours? We answer this question in the affirmative for all outerplanar maps.

Alon et al. [1] asked a similar question: is there a natural number $k$ such that the vertices of any planar graph can be non-repetitively coloured using at most $k$ colours? Here we give lower bounds for $k$ for both outerplanar and planar graphs.
2 Main results

Theorem 1. If $G$ is an outerplanar map, then the faces of $G$ can be non-repetitively coloured using at most five colours.

Proof. We assume that we have an outerplanar embedding of $G$. It suffices to show that the vertices of the dual graph $G^\ast$ can be coloured non-repetitively using at most five colours. Consider the weak dual $G^w$ formed by deleting the vertex of $G^\ast$ corresponding to the outer face of $G$. It is well known [3] that if $G$ is outerplanar, then $G^w$ is a forest of trees. Alon et al. [1] mention that the vertices of any tree can be non-repetitively coloured using at most four colours. Hence, the weak dual $G^w$ can be non-repetitively coloured with at most four colours. Finally, add back the vertex initially deleted from $G^\ast$ and colour it using a fifth colour. The resulting 5-colouring of the dual graph $G^\ast$ induces a non-repetitive colouring of the faces of the outerplanar map $G$. \qed

Theorem 2. There exists an outerplanar graph $G$ such that the vertices of $G$ cannot be non-repetitively coloured using fewer than five colours.

Proof. We will construct such a graph $G$. We begin with the graph $P_4$, i.e. the graph consisting of a simple path on four vertices. Since there are no non-repetitive binary sequences of length four, we require at least three colours to non-repetitively colour $P_4$. Next we add a vertex $v$, connecting it with an edge to each of the vertices of $P_4$, thus forming the so-called fan graph $F_4$. Let us call the vertex $v$ the rivet of the fan. Since $v$ is connected to vertices of three different colours, it is evident that $v$ must be coloured a fourth colour. The graph $G$ then consists of five disjoint copies of $F_4$, with an additional vertex $r$ connected to the rivet of each fan (see Figure 1). Clearly $G$ is an outerplanar graph. If we assume that we only have four colours with which to work, then by the pigeonhole principle, two rivets, say $v$ and $v'$, must be coloured the same colour, say $x$. The vertex $r$ cannot be coloured $x$, so $r$ must be coloured with one of the three remaining colours, say $y$. However, the subgraph $P_4$ connected to the rivet $v$ contains vertices coloured with three distinct colours different from $x$. Hence, we can always find a vertex $w$ such that the path $wvrw'$ has colouring $yxyx$. This is clearly a repetition, and so we see that we need at least five colours to non-repetitively colour $G$. \qed

Theorem 3. There exists a planar graph $G$ such that the vertices of $G$ cannot be non-repetitively coloured using fewer than seven colours.

Proof. The construction of $G$ is readily apparent from Figure 2. Let us label the two vertices of $G$ with degree eight $r$ and $s$. Let us call each of the connected components of the subgraph formed by deleting $r$ and $s$ from $G$ a diamond. By reasoning similar to that used in the proof of Theorem 2 we may conclude that each diamond requires at least five colours for a non-repetitive colouring. Assume that we have a non-repetitive 6-colouring of $G$. Now consider the seven vertices of $G$ with degree seven. By the pigeonhole principle, two of these vertices
must have the same colour. Let us call these two vertices \( v \) and \( v' \), and let us assume that they are each coloured \( x \). Let us call each of the two diamonds containing \( v \) and \( v' \) \( D \) and \( D' \) respectively. Suppose that \( D \) and \( D' \) are each coloured using exactly five colours, but the five colours used are not the same for each diamond. In this case, between \( D \) and \( D' \) all six colours are used, and so for all choices \( y, y \neq x \), for the colour of \( r \) we can always find a vertex \( w \) in one of \( D \) or \( D' \) such that the path \( wvr' \) has colouring \( yxyx \). Hence, it must be the case that \( D \) and \( D' \) are each coloured using exactly the same colours. If \( D \) and \( D' \) are each coloured using all six colours, then again we can always find a vertex \( w \) in one of \( D \) or \( D' \) such that the path \( wvr' \) has colouring \( yxyx \). It is therefore the case that \( D \) and \( D' \) are each coloured using exactly the same five colours. Thus we may colour \( r \) using the colour that does not appear in \( D \) or \( D' \); any other choice \( y, y \neq x \), will allow us to find a vertex \( w \) in one of \( D \) or \( D' \) such that the path \( wvr' \) has colouring \( yxyx \). However, by this same argument we find that \( s \) must be coloured the same colour as \( r \). Since \( r \) and \( s \) are adjacent, this is a contradiction, and so we have that \( G \) cannot be non-repetitively coloured using fewer than seven colours.

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References

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[3] H.J. Fleischner, D.P. Geller, F. Harary, “Outerplanar graphs and weak duals”, *J. Indian Math. Soc.* 38 (1974), 215–219.
Figure 2: Graph $G$ from Theorem 3.