F-term spontaneous breaking of 3D-SUSY: an algebro-geometric treatment

José J. Ramón Marí, a, Y.M.P. Gomes, b J. A. Helayël-Neto b

a Instituto Militar de Engenharia (IME), Praça Gen. Tibúrcio 80, Urca, Rio de Janeiro, RJ, Brazil, CEP 22291-270
b Centro Brasileiro de Pesquisas Físicas (CBPF), Rua Dr. Xavier Sigaud 150, Urca, Rio de Janeiro, RJ, Brazil, CEP 22290-180

E-mail: jjramon@ime.eb.br, ymuller@cbpf.br, helayel@cbpf.br

Abstract: We settle a result on generic exactness of SUSY in 3-D, and provide a mechanism of F-term spontaneous breaking of 3-D SUSY, with a different set of tools from those used by O’Raifeartaigh in his seminal work on 4-D SUSY. In our study, we use techniques of projective algebraic geometry so as to deal successfully with cubic hypersurfaces.
1 Introduction

Supersymmetry (SUSY) appears in relativistic field theories (both classical and quantum) as a space-time symmetry with the remarkable consequence of placing bosons and fermions in the same (linear) representation space, called supermultiplet. This is why it is usually referred to as a fermion-boson symmetry. Besides being an elegant solution to the naturalness and gauge-hierarchy problems of the Standard Model of Particle Physics, SUSY also implies that the running coupling constants of the electroweak and strong interactions can be unified at a high energy scale, \(10^{16} \text{ GeV}\), providing then a viable path towards unification. Supersymmetric partners of the particles of the Standard Model have however not yet been observed at available accelerator energies (LHC, in Run II, has been exploiting physics up to 13 TeV centre-of-mass energy). Therefore, to describe some new physics beyond the Standard Model, SUSY needs to be broken by some particular mechanism (be it explicit by soft terms, spontaneous or dynamical). Even if the breaking should occur at very high energies, it can be communicated to the low-energy sector of the spectrum \([1]\). In four space-time dimensions, as is well-known, SUSY exhibits a complex structure: the fundamental representation of simple four-dimensional (4D) SUSY is realised in terms of complex superfields and their associated complex component fields. That is a crucial point in connection to a particular mechanism of SUSY spontaneous breaking, namely, F-term SUSY breaking, thoroughly studied by O’Raifeartaigh in his classical 1975 papers \([2]\). In \([2]\), the author turns the analysis of F-term spontaneous simple SUSY breaking into the study of a system of \(N\) quadratic equations in \(N\) complex unknowns (these \(N\) unknowns being space-time constant configurations of \(N\) complex scalar fields accommodated in \(N\) matter superfields), and shows that spontaneous breaking is possible only if there are at least \(N = 3\) superfields.
that display these scalar fields. The results of [2] gave rise to the so-called O’Raifeartaigh models. (Just to avoid any possible misunderstanding: N stands for the number of fields present in the particular model under consideration. Here we shall be dealing only with simple SUSY, and so N never refers to the number of extended supersymmetries.)

SUSY has been carefully explored in diverse space-time dimensions (more specifically, from two to eleven). In particular, SUSY in three space-time dimensions received a remarkable boost in relation to Chern-Simons and topological field-theoretic models in three dimensions (3D) [3]. More recently, renewed interest in 3D SUSY has arisen in connection to topological materials in lower space dimensions, for instance (1+2)-dimensional topological superconductors [4], where SUSY appears as an emergent symmetry of the action which describes the dynamics of the excitations. So, motivated by the relevance that SUSY is nowadays acquiring as an emergent symmetry in low-dimensional Condensed Matter systems such as graphene, topological insulators and topological superconductors, we re-examine the issue of F-term SUSY spontaneous breaking in three space-time dimensions.

Contrary to the 4D case, (1+2)-D SUSY has a real structure: simple SUSY in three space-time dimensions is realised in terms of real superfields and their corresponding real component fields. Renormalisation requirements in 3D allow the matter superfields to have up to quartic interactions, whereas in 4D, the coupling amongst matter superfields is at most cubic. This drastically changes the structure underneath spontaneous SUSY breaking in comparison to its four-dimensional counterpart: instead of a system of N quadratic equations in N complex unknowns, the conditions for 3D F-term spontaneous SUSY breaking hinge on the existence of real solutions of a system of N cubic equations in N real variables, which renders reference to the seminal results in [2] no longer suitable, and calls for different techniques. We carry out the analysis of the corresponding system of cubic equations derived from F-term breaking using projective algebraic geometry. Today, algebraic geometry is no stranger to theoretical physics, see the work of Candelas et al [5] for a recent example; for a sample of applications to biology, see [6]. Here we use classical tools from this field to solve a physical question.

We wish to emphasize that we do not attempt a phenomenological discussion here, but stick to the mathematical problem underlying F-term spontaneous breaking of SUSY, although we keep in mind applications to emergent SUSY in planar condensed matter phenomena, which shall be addressed elsewhere.

The outline of our paper is as follows: after presenting some background on algebraic geometry in Section 2, we provide a result on generic exactness in Section 3; next, in Section 4, we introduce an explicit family of potentials with F-term SUSY breaking, and provide a different example, this time numerical, of an even potential with F-term breaking in Section 5. Finally, in Section 6 we derive a small hypersurface of potentials satisfying F-term SUSY breaking in the 3-superfield case (33 dimensions) and its N-superfield analogue, building upon Section 5, and cast our conclusions and final remarks in Section 7.

**Notation:** We denote by $E = E_3$ the linear space of quartic scalar potentials $\Phi = \Phi(x, y, z) \in E$ on $\mathbb{R}^3$ ($E_N$ will be its N-superfield analogue), and $(F, G, H)$ to be the gradient of $\Phi$ in the N=3 case (quartic polynomials in N variables). Note that $E$ has di-
mension $1 + 3 + 6 + 10 + 14 = 34$. We denote by $\mathbb{R}_d[x,y,z]$ the space of polynomials in the variables $x, y, z$ of degree at most $d$, and write as $\mathbb{R}[x,y,z]_d$ the space of homogeneous polynomials in $x, y, z$. Given a polynomial $F$ in $x, y, z$, its homogenized form is denoted by $\tilde{F}$, i.e. $\tilde{F}(X_0, X_1, X_2, X_3) = X_0^{deg(F)}F(X_1/X_0, X_2/X_0, X_3/X_0)$.

2 Intersection product in projective spaces

See [8][9]. We work over the complex projective $n$-space $\mathbb{P}^n_C = \mathbb{P}^n$, with projective homogeneous coordinates $X_0, \ldots, X_n$. The affine coordinates are by default $x_i = \frac{X_i}{X_0}$, and a form of degree $d$ is a homogeneous polynomial of degree $d$. We start with a simple situation in the complex plane: clearly, a complex projective line $\ell \subset \mathbb{P}^2$ (say $X_1 = 0$) and a curve $F = 0$ (where $F(X_0, X_1, X_2)$ is a form of degree $d$ in the $X_i$’s) intersect in $d$ complex points, counting multiplicities, unless $X_1$ divides $F$. Likewise, we see that two conics with no common factors $Q_1, Q_2$ intersect in 4 points counted with multiplicities, which we write in intersection product notation as follows: $\{Q_1 = 0\} \cdot \{Q_2 = 0\} = \{Q_1 = 0\} \cdot \{Q_3 = 0\}$, where $Q_3 = Q_2 + \lambda Q_1$, where $\lambda$ is a solution of $\text{det}(Q_2 + xQ_1) = 0$, so $Q_3$ decomposes into linear factors, i.e.

$$\{Q_3 = 0\} = \ell + \ell'.$$

Thus,

$$\{Q_1 = 0\} \cdot \{Q_2 = 0\} = \{Q_1 = 0\} \cdot (\ell + \ell') = \{Q_1 = 0\} \cdot \ell + \{Q_1 = 0\} \cdot \ell' = 2 + 2 = 4.$$  

Bézout’s Theorem in $\mathbb{P}^2$ generalises this to the following result: if $F,G$ are forms of respective degrees $d,e$ in the variables $X_0, X_1, X_2$, and have no common factors, their intersection number in $\mathbb{P}^2$, $(F = 0) \cdot (G = 0) = F \cdot G = de$ (we abuse notation identifying the locus with its equation).

In the case of $\mathbb{P}^n$, one has $D_1, D_2, \ldots, D_n$ hypersurfaces, where $D_i$ is the set of zeros of a form $F_i$ in $X_0, \ldots, X_n$ of degree $d_i$. Assume that they properly intersect, i.e., that their intersection $F_1 = 0, \ldots, F_n = 0$ is finite. One may define an intersection 0-cycle

$$D_1 \cdot \cdot \cdot D_n = \sum \mu_P P,$$

where $\mu_P$ is the multiplicity of $P$, and all but a finite number of $\mu_P$ are zero. If $\mathcal{O}_P$ is the ring of germs of holomorphic functions on $\mathbb{P}^n$ around $P$ and $f_i = 0$ are local equations defined by $D_i$ (i.e. by $F_i$) one has:

$$\mu_P = \mu_P(F_1, \ldots, F_n) = \dim_{\mathbb{C}} \left( \mathcal{O}_P \bigg|_{(f_1, \ldots, f_n)} \right),$$

where $\mathcal{O}$ is taken to be the ring of germs of holomorphic functions at $P$, or the ring of rational functions in $n$ variables that are regular at $P$.

This generalises to the following. In $\mathbb{P}^2$ we may factor every (homogeneous) polynomial into (homogeneous) irreducible factors, $F = \prod F_i^{e_i}$, and there is an intersection product defined on integer linear combinations of hypersurfaces; thus if $D$ is the divisor associated with $F$, and $E_i$ are the divisors associated with $F_i$, one has $E = \sum e_k E_k$, and the intersection
product is linear on each argument, if each summand is defined. The intersection number is always an integer, it is invariant under deformation, depends only on the ideal generated by \((F_i)\) and if they do not intersect properly (i.e. their intersection does not have the right dimension) one may tweak the polynomials \(F_i\) so that their perturbed cousins \(H_i\) intersect properly (i.e. the \(n\) hypersurfaces intersect only in a finite set) and the intersection number of the \(F_i\) equals that of the \(H_i\). Back to the proper intersection case, the intersection of \(F_1, \ldots, F_{n-1}\) is a curve, consisting of a finite number of irreducible components endowed with a multiplicity. If we take a hypersurface \(F_n\) which, say, cuts this curve transversally (hence away from intersections of components), then the result is a 0-cycle \(\sum n_i P_i\), where the \(P_i\) are the points of intersection and the \(n_i\) are its multiplicities, which sum up to

\[
\sum n_i = d_1 d_2 \cdots d_n.
\]

This offers an upper bound in the case of finite intersection of the \(F_i\).

**Example 2.1** An extreme example of plane cubics having one real intersection point and none in the affine plane is that of the equations \(y^2 = f(x)\) and \(y^2 = f(x) + 1\). In this case, after homogenisation they intersect only on \((0 : 0 : 1)\) with multiplicity 9. Thus Bezout’s credits are all spent on one point, which lies outside the affine part. This example is easily generalised to the case of three cubic surfaces in projective 3-space, see for instance Theorem 4.1.

**Proposition 2.2** Let \(F\) be a degree-\(d\) form in the variables \(Z_0, \ldots, Z_n\), and let \(p \in \mathbb{P}^n\) lie on \(F = 0\). Being a singular point of \(F\) is a projective property; in other words, if in one affine chart one has the dehomogenised polynomial \(f = 0\), and the image \(p_0\) of \(p\) in the affine chart is a critical point of \(f\), then this is so in any coordinate chart.

**Proof:** One should recall Euler’s formula, which works for homogeneous functions of degree \(d\):

\[
\sum_{i=0}^{n} Z_i \frac{\partial F}{\partial Z_i} = dF.
\]

Now, assume that the 0-th coordinate of \(p\) is nonzero; in the affine chart \(Z_0 \neq 0\), the partial derivatives of \(F(1, x_1, \ldots, x_n)\) correspond to \(\partial Z_i F(1, x_1, \ldots, x_n)\), and by Euler’s formula, \(\partial Z_0 F(p) = 0\) as well. This fact is easily seen to be preserved under projective linear transformations and by restriction to any other affine chart. ■

**Proposition 2.3** ("Liouville’s Theorem") Let \(Z \subset \mathbb{P}^N\) be the zero set of \(r\) forms \(F_1, \ldots, F_r\). If \(Z\) is not finite, then for any hyperplane \(H\) the intersection \(H \cap Z\) is non-empty.

**Proof:** It suffices to apply [8, Th.1.6.6, p.76]. ■

3 Generic exactness of 3D SUSY

The result holds for \(N \geq 2\) superfields.
Theorem 3.1 (Main Result on Exactness) Given \( N \geq 2 \), outside a real hypersurface \( H \) in \( E_N \), every potential \( \Phi \in E_N \) gives rise to an exact (simple) SUSY in 3D. Equivalently, if the homogenised components of the gradient of \( \Phi \) have no common zeros at infinity, then \( \Phi \) has a real critical point. In particular, 3D SUSY with \( N \) superfields is generically exact.

Proof: By [8, I.6. Exercise 10], \( N \) cubic forms (or \( N \) forms of prescribed degrees \( m_1, \ldots, m_N \)) in \( N \) variables have a common (complex) zero if and only if a certain real polynomial \( R \) in their coefficients vanishes (\( R \) has its coefficients in \( \mathbb{Q} \), by Galois theory [10, Ch.I]). Thus the components of the gradient of the homogenised \( \Phi \in E_N \) have a common zero at infinity if and only if \( R(G_1, \ldots, G_N) = 0 \), where \( G_i(X_1, \ldots, X_N) = \partial X_i \tilde{\Phi}(0, X_1, \ldots, X_N) \), and the number of complex critical points of \( \Phi \) is finite if \( R \neq 0 \), for then the locus of critical points has empty intersection with the hyperplane \( X_0 = 0 \), see Proposition 2.3. Outside this real hypersurface \( R = 0 \) in \( E_N \), a potential \( \Phi \in E_N \) satisfies that the real hypersurfaces \( \partial X_i \tilde{\Phi}(X_0, X_1, \ldots, X_N) = 0, 1 \leq i \leq N \) intersect in \( 3^N \) points in \( \mathbb{C}^N \) (an odd number) counting multiplicities, as no point of intersection lies in \( X_0 = 0 \). Now, imaginary solutions come in complex conjugate pairs (and here \( \mu_P = \mu_F \), the system being defined over \( \mathbb{R} \)), so a real common zero of \( \partial_1 \Phi, \ldots, \partial_N \Phi \) must exist. ■

4 Explicit family with F-term breaking

Theorem 4.1 Let \( a, D \neq 0 \) be real parameters, and let \( A(u, v) \) be a quartic homogeneous polynomial. Define the potential \( \Phi = \Phi_1 + \Phi_4 \), where

\[
\Phi_4(x, y, z) = A(x + ay, z)
\]

and \( \Phi_1(x, y, z) = Dx \). The gradient \( \nabla \Phi = (F, G, H) \) is nonvanishing in \( \mathbb{R}^3 \). The above forms a 6-parameter family of such potentials.

Proof: Consider the equations \( \partial_u \Phi = \partial_v \Phi = 0 \), which in their explicit form are:

\[
\partial_u A = 0, \quad D + a\partial_u A = 0,
\]

which is clearly impossible for \( D \neq 0 \). This produces a 6-parameter example, counting \( D \) and the coefficients of \( A \). The Theorem is thus settled. ■

Remark: The family obtained in Theorem 4.1 is indeed not very big. Non-existence of affine real solutions relies on the fact that every complex common zero \( P_i \) of the homogenised versions of \( F, G, H \) lies at infinity, and the (projective) intersection cycle equals \( 3 \sum P_i (\sum P_i \) being the intersection cycle of the plane curves \( \tilde{F}(0, X_1, X_2, X_3), \tilde{H}(0, X_1, X_2, X_3) \)) if the intersection be proper. Indeed, the monomial \( Z_0^3 \) is a linear combination of \( \tilde{F}, \tilde{G} \). The rest follows from Section 2.

5 The even potential case

Let \( \Phi \) be even. This means that the cubic polynomials \( F, G, H \) are of the form

\[
F = F_1 + F_3, G = G_1 + G_3, H = H_1 + H_3.
\]
We wish to study the cases where no solution other than the origin exists. The assumption of having an ordinary double point below is at this stage unnecessary, but will later be very fruitful.

**Theorem 5.1** Assume that there is an even quartic potential \( V \in E \), which has an ordinary double point at the origin (i.e. such that the Hessian at 0 is nondegenerate), and no other critical points in \( \mathbb{P}^3_{\mathbb{R}} \). There exists an even quartic potential on \( \mathbb{R}^3 \), \( \Phi' \), with no critical points in \( \mathbb{R}^3 \).

**Proof:** In order to study the system \( F = 0, G = 0, H = 0 \) outside the origin, substitute \( x = \lambda v \), where \( v \in S^2, \lambda > 0 \).

The resulting system of equations is as follows:

\[
\begin{pmatrix}
F_1(v) & F_3(v) \\
G_1(v) & G_3(v) \\
H_1(v) & H_3(v)
\end{pmatrix}
\begin{pmatrix}
1 \\
\lambda^2
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
\]

Since \( \lambda \) is real, this tells us that not only the rank of the matrix is less than 2, but also that the signs of \( F_1(v), G_1(v), H_1(v) \) must be opposite, just as those of \( G_3(v), H_3(v) \). The condition on the Hessian of \( V \) means that \( F_1, G_1, H_1 \) form a basis of the dual space \( (\mathbb{R}^3)^\vee \) (i.e. transversal cut of \( F = 0, G = 0, H = 0 \) at the origin).

What this reveals is that, if \( \tilde{F}, \tilde{G}, \tilde{H} \) are the respective (cubic) homogenised forms corresponding to \( F, G, H \), then the intersection cycle

\[ \tilde{F} \bullet \tilde{G} \bullet \tilde{H} = O + Z, \]

where \( Z \) consists of pairs of complex conjugate points, with multiplicities, and \( O \) is the origin.

Note that, should we homogenise our quartic potential, the resulting quartic form \( \tilde{V} \) in the variables \( Z_0, Z_1, Z_2, Z_3 \) has respective partial derivatives \( \tilde{K}, \tilde{F}, \tilde{G}, \tilde{H} \). Euler’s formula yields

\[ Z_0 \tilde{K} + Z_1 \tilde{F} + Z_2 \tilde{G} + Z_3 \tilde{H} = 4 \tilde{V}. \]

Being a singular point of a hypersurface is a projective matter, by Proposition 2.2, and therefore shows in any affine chart we choose. Likewise, the fact of \( \tilde{V} \) having precisely one critical point in \( \mathbb{P}^3 \), which is an ordinary double point (i.e. the Hessian of \( \tilde{V} \) has rank 2 at that point) is invariant by real projective linear transformations and, since the origin becomes the point \( O = (1 : 0 : 0 : 0) \) in \( \mathbb{P}^3 \), placing infinity at, say, the plane \( Z_2 = 0 \) will furnish a dehomogenised quartic potential \( \Phi' \) with no real critical points, for the only possible candidate is now stashed at infinity. Thus, the new set of real cubic equations, \( F' = G' = H' = 0 \), has no real solutions in affine real space, as desired. ■

### 5.1 Final Ansatz and numerical example

Let \( \Phi \) be an even potential on \( \mathbb{R}^3 \) with \( \Phi(0) = 0 \), and assume that \( F_1, G_1, H_1 \) are linearly independent linear forms. Assume further that, say, \( H_1 = z, H_3 = zQ(x, y, z) \), where \( Q \) is a positive definite quadratic form (that is an open condition on the 6 parameters that

---

- 6 -
form $Q$). The condition $z + \lambda^2 z Q(x, y, z) = 0$ for $(x, y, z) \neq (0, 0, 0)$, $\lambda \neq 0$ forces $z = 0$, and it remains to find $\Phi$ with these constraints so that the remaining equations restricted to $z = 0$ have no solution other than the origin. Denote by $Q_0(x, y) = Q(x, y, 0)$ the $z$-free part of $Q$, and write $Q(x, y, z) = Q_0(x, y) + zL(x, y) + f z^2$, where $L$ is a linear form and $f > 0$. Thus $\Phi_4 = \frac{z^2}{2}Q_0(x, y) + \frac{z^2}{3}L(x, y) + \frac{1}{4}z^4 + A(x, y)$, where $A(x, y) = \Phi_4(x, y, 0)$ is an arbitrary quartic form. Now, by construction, a common zero of $F, G, H$ over $\mathbb{R}$ must have $z = 0$, and clearly $F_3(x, y, 0) = \partial_z A(x, y), G_3(x, y, 0) = \partial_y A(x, y)$.

Take now $\Phi_2 = \frac{1}{2}(z^2 + ax^2 + 2bxy + dy^2)$ nondegenerate. Then (5.1) becomes, after imposing the necessary $z = 0$:

$$
\begin{pmatrix}
ax + by & \partial_x A(x, y) \\
bx + dy & \partial_y A(x, y)
\end{pmatrix}
\begin{pmatrix}
1 \\
0
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}.
$$

(5.2)

Here, $ad - b^2 \neq 0$, which is tantamount to saying that $F_1, G_1$ and $H_1 = z$ are linearly independent. If the determinant

$$(ax + by)\partial_y A(x, y) - (bx + dy)\partial_x A(x, y)$$

(5.3)

is a (quartic) polynomial with no real linear factors, then no zero of (5.2) will exist in $\mathbb{P}_R^1$. In that case, the homogenisations of $F, G, H$ have no common zeros in $\mathbb{P}_R^3$ either. Indeed, $H_3(x, y, z)$ has no real zeros outside $z = 0$, but since $F_3(x, y, 0) = \partial_z A(x, y), G_3(x, y, 0) = \partial_y A(x, y)$, these cannot have common zeros in $\mathbb{P}_R^1$ if the quartic form in (5.3) has no real zeros in $\mathbb{P}_R^1$. In fact, no zeros of $(F, G, H)$ exist in $\mathbb{R}^3$ outside the origin, for if there was one $(x_0, y_0, 0)$, it should be a zero of (5.3).

Thus, if (5.3) has no nontrivial real zeros, the homogenised potential $\tilde{V}$ has precisely one critical point in $\mathbb{P}_R^3$, which is the origin in $\mathbb{R}^3$, and is an ordinary double point (the Hessian of $V$ is nonsingular), since this condition forces too $ad - b^2 \neq 0$. That is how we show that lying in the three surfaces $\tilde{F} = 0, \tilde{G} = 0, \tilde{H} = 0$ is only one real point, of multiplicity one (i.e. a transversal intersection), which is the origin in the affine part $\mathbb{R}^3$.

Take, for instance, the polynomial

$$A(x, y) = 0.673447x^4 + 0.299177x^3y + 0.269692x^2y^2 + 0.818559xy^3 + 0.44846y^4,$$

(5.4)

and take any positive definite quadratic form $Q(x, y, z)$ on $\mathbb{R}^3$ (this sweeps an open set of an $\mathbb{R}^6$). Take now $F_1 = 0.834982x + 0.547667y, G_1 = 0.547667x + 0.13926y$ (this determines $\Phi_2$ in our case). Then (5.3) has the following form:

$$1.36194x^4 - 0.340464x^3y - 0.643023x^2y^2 + 0.768934xy^3 + 0.72017y^4,$$

and has no nontrivial real zeros. We are now ready to apply Theorem 5.1, and thus establish the following result.

**Theorem 5.2** There is an even quartic potential $\Phi = \Phi_2 + \Phi_4$ on $\mathbb{R}^3$ with no critical points.
6 Sharp result on SUSY breaking

The numerical example given in Section 5 brings about a 33-parameter family out of the 34 parameters defining the potential \( \Phi \) (in the case of 3 superfields).

**Theorem 6.1** Consider the potential \( \Phi^0 \) obtained in our computer calculations (see proof of Theorem 5.1). Consider the following four equations in \( p \in \mathbb{R}^3 \) and in the coefficients of \( \Phi \):

\[
\partial_x \Phi(p) = 0, \partial_y \Phi(p) = 0, \partial_z \Phi(p) = 0, p_2 = 0.
\]

The four equations are satisfied by our initial, even potential \( \Phi^0 \) from Section 5, and this yields a family of \( 34 + 3 - 4 = 33 \) parameters around the potential \( \Phi^0 \), satisfying the four equations. Performing the ‘projective trick’ of swapping infinity planes (see Theorem 5.1) on them does provide a family of quartic potentials \( V \) with no critical points. In other words, the 33-parameter family of scalar potentials \( V \) obtained out of the \( \Phi \) by homogenising, then dehomogenising on the second variable, provides abundant examples of SUSY breaking at the F-term in 3D SUSY, with \( N = 3 \) superfields of simple SUSY.

By analogous arguments, one obtains a codimension-1 family with F-term supersymmetry breaking with \( N = 2 \) superfields of simple SUSY.

**Proof:** The four equations imposed on pairs \( (\Phi, p) \in E \times \mathbb{R}^3 = E^{(1)} \) provide a 33-parameter family in \( E^{(1)} \), by the Implicit Function Theorem. Should we drop the fourth equation \( p_2 = 0 \), we would have a diffeomorphism around the point \( (\Phi^0, (0, 0, 0)) \) with an open neighbourhood \( U \) of \( \Phi^0 \) in \( E \), where every \( \Phi \in U \) has a unique critical point on the plane \( y = 0 \) in \( \mathbb{R}^3 \), and no critical points at infinity (this property is preserved by deforming the equations for the gradient within the realm of real polynomials). Thus, by adding the condition \( p_2 = 0 \), we have found a hypersurface \( H \) in the small open subset \( U \), which makes for the 33 parameters, where every member of \( H \) has exactly one (real) critical point, which will lie on the coordinate plane \( y = 0 \). Now we homogenise \( \Phi \) and dehomogenise it again (analytic diffeomorphism of \( E_N \)), so the plane \( y = 0 \) will turn into the plane at infinity, and this produces the desired family of \( V \)'s with no critical points (see Proposition 2.2). This settles the case \( N = 3 \).

For the 2-superfield case, one may use the starting example in Section 5:

\[
\Phi^0(x, y) = \frac{1}{2} \left[ 0.834982 x + 2 \times 0.547667 xy + 0.13926 y^2 \right] + A(x, y),
\]

with \( A(x, y) \) as in (5.4), which will produce the expected family, after homogenising and dehomogenising again (a ball around the transformed potential \( \Phi^1 \)).

Just a little more work gives us an N-superfield analogue.

**Theorem 6.2** Given \( N \geq 2 \), there is a quartic potential in \( N \) variables \( \Phi^1 \) with no critical points in \( \mathbb{R}^N \), which gives rise to a small ball of dimension \( \dim E_N - 1 \) around \( \Phi^1 \) in \( E_N \) satisfying this condition (spontaneous F-term breaking of 3D SUSY with maximal dimension).
Proof: It suffices to take the example in 3 variables, $\Phi^0(x, y, z)$, and to define $\Phi^1(x_1, \ldots, x_N) = \Phi^0(x_1, x_2, x_3) + \sum_{i=4}^{N} x_i^2 + x_i^4 \left( x_i = X_i/X_0 \right)$. The very same argument that settled Theorem 6.1 works here: Implicit Function Theorem and swapping coordinates, hence infinity hyperplanes, to move the unique critical point to the infinity hyperplane.

7 Conclusion

We have found a family of scalar potentials with maximal number of free parameters, featuring F-term spontaneous breaking of 3D SUSY in the case of $N \geq 2$ superfields. However, only small balls around explicit even potentials have been obtained, without showing the global shape of the F-term breaking locus. An explicit, simpler albeit ‘smaller’ family in the case of 3 superfields has been provided in Theorem 4.1. In the latter case, the real zeros lie at infinity, but are multiple, which prevents from perturbing to bigger families. In our case, the locus of F-term spontaneous breaking within the parameter space of space-time configurations of $N \geq 2$ superfields is contained in a real (algebraic) hypersurface $H$, and contains at least small ball in $H$, whereas in 4D SUSY [2] its corresponding F-term breaking locus (only for $N \geq 3$) is contained within a complex hypersurface (real codimension two), as our proof of Theorem 3.1 and the reference therein show. This phenomenon is explained by the fact that simple SUSY in 4D corresponds to double SUSY in 3D.

References

[1] Dimopoulos, S.; Raby, S. Nucl. Phys. B192 (1981)353; Ibáñez, R., Ross, G.G., Phys.Lett.105B (1981)439.

[2] O’Raifeartaigh, L. Phys. Let. B, v. 56, n. 1, p. 41-44, 1975; O’Raifeartaigh, L., Nucl. Phys. B, v. 96, n. 2, p. 331-352, 1975. O’Raifeartaigh, L. and Parravicini, G. Nucl. Phys. B111(1976) 516. Intriligator, K., Seiberg, N., Lectures on supersymmetry breaking, arXiv:hep-ph/0702069v3.

[3] Gates JR, S. James et al. Superspace, or one thousand and one lessons in supersymmetry. arXiv preprint hep-th/0108200, 2001; and: Dunne, G.V., Aspects of Chern-Simons theory. In: Aspects topologiques de la physique en basse dimension. Topological aspects of low dimensional systems. Springer Berlin Heidelberg, 1999. p. 177-263.

[4] Grover, T., Sheng, D. N. and Vishwanath, A. Sci., v. 344, n. 6181, p. 280-283, 2014. and Lee, SS. TASI lectures on emergence of supersymmetry, gauge theory and string in condensed matter systems. arXiv:hep-th/1009.5127, 2010.

[5] Candelas, P. et al, Fortsch.Phys. 64 (2016) no.6-7, 463-509. See also: Candelas, P. et al, , Fields Inst.Commun. 38 (2013) 121-157.

[6] Casanellas M., Fernández J., Adv. App. Math, 41 (2008), 265-292; M. Casanellas et al, EMS Newsletter 86, (2011).

[7] Jian, S.K. et al. Phys. Rev. Let., v. 118, n. 16, p. 166802, 2017.

[8] I.R. Shafarevich, Basic Algebraic Geometry (2 Volumes), 2nd Ed., Springer-Verlag, 1994.

[9] P.A. Griffiths, J. Harris, Principles of Algebraic Geometry, Wiley, 1978.
[10] J.H. Silverman, *The Arithmetic of Elliptic Curves*, 2nd Ed., GTM 106, Springer-Verlag 2009.