Application of spectral collocation method to a class of nonlinear PDEs

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Abstract
In this paper approximate solutions to a class of nonlinear partial differential equations by means of the Chebyshev spectral collocation method is considered. First, properties of the Chebyshev spectral collocation method required for our subsequent development are given and utilized to reduce the computation of Fisher’s, generalized Burger’s-Fisher, generalized Huxley, and generalized Burger’s-Huxley equations to some system of ordinary differential equations. Then, we use fourth-order Runge-Kutta formula for the numerical solution of the system of ordinary differential equations. The method is applied to a few test examples to illustrate the accuracy and the implementation of the method.

Keywords: Chebyshev; Spectral method; Fisher; Generalized Burger-Fisher; Generalized Huxley; Generalized Burger-Huxley.

1 Introduction
In this paper a Chebyshev spectral collocation method is used for the numerical solution of the following nonlinear partial differential equations:
I) Fisher’s equation [1, 2]

\[ u_t = u_{xx} + u(u - 1), \quad (x,t) \in D \times [0,T], \tag{1.1} \]

II) generalized Burger’s-Fisher equation [3, 4]

\[ u_t + \alpha u^\delta u_x - u_{xx} = \beta u(1 - u^\delta), \quad (x,t) \in D \times [0,T], \tag{1.2} \]

where \( \alpha, \beta, \) and \( \delta \) are parameters,

III) generalized Huxley equation [5, 6]

\[ u_t - u_{xx} = \beta u(1 - u^\delta)(u^\delta - \gamma), \quad (x,t) \in D \times [0,T], \tag{1.3} \]

where \( \delta, \beta \geq 0 \) and \( \gamma \in (0,1) \) are given parameters,

IV) generalized Burger’s-Huxley equation [7, 8]

\[ u_t + \alpha u^\delta u_x - u_{xx} = \beta u(1 - u^\delta)(u^\delta - \gamma), \quad (x,t) \in D \times [0,T], \tag{1.4} \]

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where $\alpha, \beta, \gamma$ and $\delta$ are parameters, $\beta \geq 0$, $\delta > 0$, $\gamma \in (0,1)$.

For the problems (1.1)-(1.4), we consider initial condition

$$u(x,0) = f(x), \quad x \in D,$$  (1.5)

and boundary conditions

$$u(x,t) = g(t), \quad (x,t) \in \partial D \times [0,T],$$  (1.6)

where $D = \{x: a < x < b\}$ and $\partial D$ is its boundary. Nonlinear phenomena are of fundamental importance in various fields of science and engineering. Non-linear partial differential equations (1.1)-(1.4) play important roles in non-linear physics [9]-[17]. The Fishers equation is first introduced by Fisher as a deterministic model of the wave propagation of favored gene in population [9]. Also, the Fisher’s equation arises in many physical, biological, chemical, and engineering problems that are described by the interaction of diffusion and reaction process. For example, it plays a significant role include flame propagation, neutron flux in a nuclear reactor and the dynamics of defects in nematic liquid crystal [10]. Burger’s equation arises in many physical problems including one dimensional turbulence, gas dynamics, number theory, heat conduction, elasticity, sound waves in viscous medium [11], shock waves in viscous medium, waves in a medium with fluid filled viscoelastic tubes and magnetohydrodynamic waves in a medium with finite electrical conductivity. The core mathematical framework for modern biophysically based neural modeling was developed half a century ago by Hodgkin and Huxley [12]. In a series of papers published in 1952, they presented the results of an elegant series of electrophysiological experiments in which they investigated the flow of electric current through the surface membrane of the giant nerve fiber of a squid. The Burger’s-Huxley equation describes the interaction between diffusion, convection and reaction.

The layout of the paper is as follows. First, in Section 2 we review some of the main properties of Chebyshev polynomials that are necessary for the formulation of the discrete system. In Section 3, we illustrate how the Chebyshev spectral collocation method may be used to replace Eqs. (1.1)-(1.6) by explicit system of ordinary differential equations, which is solved by fourth-order Runge-Kutta method. In Section 4, we report our numerical results and demonstrate the efficiency and accuracy of the proposed numerical scheme by considering some numerical examples.

2 Preliminaries

The goal of this section is to recall notations and definition of the Chebyshev polynomials, state some known results, and derive useful formulas that are important for this paper. These are discussed thoroughly in [18]. The well known Chebyshev polynomial $T_n(x)$ of the first kind is a polynomial in $x$ of degree $n$, defined by the relation

$$T_n(x) = \cos n\theta \quad \text{when} \quad x = \cos \theta.$$  (2.7)

If the range of the variable $x$ is the interval [-1,1], then the range of the corresponding variable $\theta$ can be taken as [0, $\pi$]. These ranges are traversed in opposite directions, since $x = -1$ corresponds to $\theta = \pi$ and $x = 1$ corresponds to $\theta = 0$.

It is well known that $\cos n\theta$ is a polynomial of degree $n$ in $\cos \theta$, and indeed we are familiar with the elementary formulae

$$\cos 0\theta = 1, \quad \cos 1\theta = \cos \theta, \quad \cos 2\theta = 2\cos^2 \theta - 1,$$

$$\cos 3\theta = 4\cos^3 \theta - 3\cos \theta, \quad \cos 4\theta = 8\cos^4 \theta - 8\cos^2 \theta + 1, \ldots .$$

We may immediately deduce from (2.7), that the first few Chebyshev polynomials are

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1,$$

$$T_3(x) = 4x^3 - 3x, \quad T_4(x) = 8x^4 - 8x^2 + 1, \ldots .$$

In practice it is neither convenient nor efficient to work out each $T_n(x)$ from first principles. Rather by combining the trigonometric identity

$$\cos n\theta + \cos(n-2)\theta = 2\cos \theta \cos(n-1)\theta,$$
with equation (2.7), we obtain the fundamental recurrence relation

\[ T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n = 2, 3, \ldots, \]  

(2.8)

which together with the initial conditions

\[ T_0(x) = 1, \quad T_1(x) = x, \]  

(2.9)

recursively generates all the polynomials \( \{T_n(x)\} \) very efficiently.

Clenshaw and Curtis [19] introduced the following approximation of the function \( u(x,t) \):

\[ u(x,t) = \sum_{j=0}^{N} \alpha_{i,j} T^*_j(x_n)u(x_n, t), \]  

(2.10)

where \( T^*_j(x_n) = T_j((2x - (b + a))/(b - a)) \) denotes the \( j \)th shifted Chebyshev polynomial of the first kind. Note the double prime indicating that the first and last terms of the sum are to be halved.

We can use the discrete orthogonality relation

\[ \sum_{n=0}^{N} \alpha_{i,j} T^*_i(x_n)T^*_j(x_n) = \alpha_{i,j}, \]  

(2.11)

where

\[ \alpha_{i,j} = \begin{cases} 0, & i \neq j (\leq N), \\ \frac{1}{N}, & 0 < i = j < N, \\ \frac{1}{2}, & i = j = 0, N, \end{cases} \]  

(2.12)

and also, the collocation points \( x_n \) are given by

\[ x_n = \frac{1}{2} \left( (a + b) - (b - a) \cos \left( \frac{\pi n}{N} \right) \right), \quad n = 0, 1, \ldots, N. \]  

(2.13)

We can invert the interpolating polynomial defined as (2.13) and find

\[ a_j = \frac{2}{N} \sum_{n=0}^{N} \alpha_{i,j} T^*_j(x_n)u(x_n, t). \]  

(2.14)

The relation between the Chebyshev functions and the first derivative is given by [20]:

\[ T^*_j(x) = 2j\lambda \sum_{n=0, n+j \text{ odd}}^{j-1} c_n T_n(x), \]  

(2.15)

where \( \lambda = \frac{2}{b-a} \) and

\[ c_n = \begin{cases} 1, & 1 \leq n \leq N - 1, \\ \frac{1}{2}, & n = 0, N. \end{cases} \]  

(2.16)

3 The Chebyshev spectral collocation method

In this section, we use the spectral collocation method for some different kinds of nonlinear partial differential equations of the form (1.1)-(1.4) with initial and boundary conditions (1.5) and (1.6) by using the Chebyshev polynomials.
3.1 Fisher’s equation

Let us consider the Fisher’s equation

\[ u_t = u_{xx} + u(u - 1), \quad (x, t) \in D \times [0, T], \]  

(3.17)

with the initial condition

\[ u(x, 0) = f(x), \quad x \in D, \]  

(3.18)

and boundary conditions

\[ u(x, t) = g(t), \quad (x, t) \in \partial D \times [0, T], \]  

(3.19)

where \( D = \{ x : a < x < b \} \) and \( \partial D \) is its boundary. We assume \( u(x, t) \) defined over the \( D \times [0, T] \) be the exact solution of the problem (3.17)-(3.19) that is approximated as follows:

\[ u(x, t) = \sum_{j=0}^{N} a_j T_j^*(x). \]  

(3.20)

By considering the Equation (3.19) and the Chebyshev coefficients \( a_j \) that is defined by (2.14), we can obtain the first derivative of \( u(x, t) \) at the collocation points (2.13) as follows:

\[
\frac{d}{dx} u(x_i, t) = u_s(x_i, t) = \sum_{j=0}^{N} n'' a_j T_j''(x_i) \\
= \sum_{j=0}^{N} n \left( \frac{2}{N} \sum_{n=0}^{N} n'' T_j^n(x_n) u(x_n, t) \right) T_j''(x_i) \\
= \sum_{n=0}^{N} n \left( \frac{2}{N} \sum_{j=0}^{N} n'' T_j^n(x_i) T_j^n(x_n) \right) u(x_n, t) \\
= \sum_{n=0}^{N} D_{i,n}^s u(x_n, t),
\]  

(3.21)

where

\[ D_{i,n}^s = \frac{2c_n}{N} \sum_{j=0}^{N} n'' T_j^n(x_i) T_j^n(x_n), \quad i, n = 0, 1, \ldots, N - 1, N, \]  

(3.22)

and also, \( T_j^n(x_i) \) and \( c_n \) are defined by (2.15) and (2.16) respectively.

Having used the boundary conditions (3.19), we rewrite the Eq. (3.21) as follows:

\[ u_s(x_i, t) = \sum_{n=0}^{N} D_{i,n}^s u(x_n, t) = D_{i,0}^s u(x_0, t) + \sum_{n=1}^{N-1} D_{i,n}^s u(x_n, t) + D_{i,N}^s u(x_N, t). \]  

(3.23)

For the sake of simplicity, consider:

\[ F_i(t) = D_{i,0}^s u(x_0, t) + D_{i,N}^s u(x_N, t), \]

thus we can write:

\[ u_s(x_i, t) = F_i(t) + \sum_{n=1}^{N-1} D_{i,n}^s u(x_n, t). \]  

(3.24)
Now for the second derivative of $u(x, t)$ by similarly manner and using Equation (3.21), we obtain:

$$\frac{d^2}{dx^2}u(x_i, t) = u_{xx}(x_i, t) = \sum_{n=0}^{N} D_{i,n}^x \left( \frac{d}{dx}u(x_n, t) \right)$$

$$= \sum_{n=0}^{N} D_{i,n}^x \left( \sum_{j=0}^{N} D_{j,n}^x u(x_j, t) \right)$$

$$= \sum_{j=0}^{N} \left( \sum_{n=0}^{N} D_{i,n}^x D_{j,n}^x \right) u(x_j, t).$$  (3.25)

By assumption

$$D_{i,j}^{xx} = \sum_{n=0}^{N} D_{i,n}^x D_{n,j}^x, \quad i, j = 0, 1, \ldots, N,$$  (3.26)

we have:

$$u_{xx}(x_i, t) = \sum_{j=0}^{N} D_{i,j}^{xx} u(x_j, t).$$  (3.27)

By using the boundary conditions (3.19), we obtain:

$$u_{xx}(x_i, t) = \sum_{j=0}^{N} D_{i,j}^{xx} u(x_j, t) = D_{i,0}^{xx} u(x_0, t) + \sum_{n=1}^{N-1} D_{i,n}^{xx} u(x_n, t) + D_{i,N}^{xx} u(x_N, t).$$  (3.28)

We consider the notation $F_i^x(t)$ as follows:

$$F_i^x(t) = D_{i,0}^{xx} u(x_0, t) + D_{i,N}^{xx} u(x_N, t),$$  (3.29)

then we can write:

$$u_{xx}(x_i, t) = F_i^x(t) + \sum_{n=1}^{N-1} D_{i,n}^{xx} u(x_n, t).$$  (3.30)

Having replaced the first term on the right-hand side of (3.17) with the Eq. (3.30) and setting collocation points $x = x_i, \quad i = 0, 1, \ldots, N$ that are defined by (2.13), we get the collocation result as

$$u(x_i, t) = F_i^x(t) + \sum_{n=1}^{N-1} D_{i,n}^{xx} u(x_n, t) + u(x_i, t) \left( u(x_i, t) - 1 \right),$$  (3.31)

$$u(x_i, 0) = f(x_i).$$

We denote

$$G_i(t, u(t)) = F_i^x(t) + \sum_{n=1}^{N-1} D_{i,n}^{xx} u(x_n, t) + u(x_i, t) \left( u(x_i, t) - 1 \right),$$

$$u(t) = [u(x_1, t), u(x_2, t), \ldots, u(x_{N-1}, t)]^T,$$

$$u_0 = [u(x_1, 0), u(x_2, 0), \ldots, u(x_{N-1}, 0)]^T,$$

$$du(t) = [du(x_1, t), du(x_2, t), \ldots, du(x_{N-1}, t)]^T,$$

then the system of (3.31) can be given in the matrix form as:

$$du(t) = G(t, u(t)), \quad u_0 = P,$$  (3.32)
where
\[ G(t, u(t)) = [G_1(t, u(t)), G_2(t, u(t)), \ldots, G_{N-1}(t, u(t))]^T, \]
\[ P = [f(x_1), f(x_2), \ldots, f(x_{N-1})]^T. \]

The above system is a system of ordinary differential equations. Solving this system by the fourth-order Runge-Kutta method, we can obtain an approximation to the solution of (3.17). The fourth-order Runge-Kutta method that is one of the well-known numerical methods for differential equations, can be presented as:

\[
\begin{align*}
 & u_1 = h_G(t_n, u(t_n)), \\
 & u_2 = h_G(t_n + h, u(t_n + \frac{u1}{2})), \\
 & u_3 = h_G(t_n + h, u(t_n + \frac{u2}{2})), \\
 & u_4 = h_G(t_n + h, u(t_n + u3)), \\
 & u(t_{n+1}) = u(t_n) + \frac{1}{6} (u1 + 2u2 + 2u3 + u4).
\end{align*}
\]

### 3.2 Generalized Burger’s-Fisher equation

In this subsection, we consider the generalized Burger’’s-Fisher equation

\[ u_t + \alpha u^\delta u_x - u_{xx} = \beta u(1 - u^\delta), \quad (x, t) \in \mathcal{D} \times [0, T], \quad (3.34) \]

with the initial and boundary conditions (3.18) and (3.19). In Eq. (3.34) \( \alpha, \beta, \) and \( \delta \) are parameters. By considering approximate solution \( u(x, t) \) as in (2.10) and then setting \( x = x_i \) we get:

\[ u_i(t_i, t) = -\alpha u^\delta(x_i, t) u_x(x_i, t) + u_{xx}(x_i, t) + \beta u(x_i, t) (1 - u^\delta(x_i, t)), \quad (3.35) \]

where \( u_x(x_i, t) \) and \( u_{xx}(x_i, t) \) are approximated by (3.24) and (3.30) respectively. Having replaced the \( u_x(x_i, t) \) and \( u_{xx}(x_i, t) \) on the right-hand sides of (3.35) with the Eqs. (3.24) and (3.30), we get:

\[
\begin{align*}
 u_i(t_i, t) &= -\alpha u^\delta(x_i, t) \left( F_i(t) + \sum_{n=1}^{N-1} D^i_{x\alpha} u(x_n, t) \right) \\
 & \quad + \left( F^*_i(t) + \sum_{n=1}^{N-1} D^*_i x_{x\alpha} u(x_n, t) \right) + \beta u(x_i, t) (1 - u^\delta(x_i, t)), \quad (3.36) \\
 u(x_i, 0) &= f(x_i).
\end{align*}
\]

Now by assumption

\[
\begin{align*}
 G_i(t, u(t)) &= \left( -\alpha u^\delta(x_i, t) F_i(t) + F^*_i(t) \right) + \beta u(x_i, t) (1 - u^\delta(x_i, t)) \\
 & \quad + \left( -\alpha u^\delta(x_i, t) \sum_{n=1}^{N-1} D^i_{x\alpha} u(x_n, t) + \sum_{n=1}^{N-1} D^*_i x_{x\alpha} u(x_n, t) \right)
\end{align*}
\]
and also
\[
\begin{align*}
    u(t) &= [u(x_1,t), u(x_2,t), \ldots, u(x_{N-1},t)]^T, \\
    u_0 &= [u(x_1,0), u(x_2,0), \ldots, u(x_{N-1},0)]^T, \\
    du(t) &= [u_t(x_1,t), u_t(x_2,t), \ldots, u_t(x_{N-1},t)]^T,
\end{align*}
\] (3.37)

we may rewrite the system (3.36) in the form
\[
\begin{align*}
    du(t) &= G(t,u(t)), \\
    u_0 &= P, \\
\end{align*}
\] (3.38)

where
\[
G(t,u(t)) = [G_1(t,u(t)), G_2(t,u(t)), \ldots, G_{N-1}(t,u(t))]^T, \\
P = [f(x_1), f(x_2), \ldots, f(x_{N-1})]^T.
\] (3.39)

Solving system of ordinary differential equations (3.38) by Runge-Kutta method (3.33), we can obtain an approximation to the solution of (3.34).

3.3 Generalized Huxley equation

In this subsection the chebyshev spectral collocation procedure is developed for the numerical solution of the generalized Huxley equation:
\[
\begin{align*}
    u_t - u_{xx} &= \beta u(1-u^\delta)(u^\delta - \gamma), \quad (x,t) \in D \times [0,T], \\
    u(x,0) &= f(x), \quad x \in D, \\
    u(x,t) &= g(t), \quad (x,t) \in \partial D \times [0,T],
\end{align*}
\] (3.40)

with initial condition
\[
    u(x,0) = f(x), \quad x \in D,
\] (3.41)

and boundary conditions
\[
    u(x,t) = g(t), \quad (x,t) \in \partial D \times [0,T],
\] (3.42)

where \( D = \{ x : a < x < b \} \), \( \delta, \beta \geq 0 \) and \( \gamma \in (0,1) \) are given parameters and \( \partial D \) is its boundary.

By replacing the second term on the left-hand side of (3.40) with the Eq. (3.30) and setting the collocation points
\[
x_i = \frac{1}{2} \left( (a+b) - (b-a) \cos \left( \frac{\pi n}{N} \right) \right), \quad i = 0,1,\ldots,N,
\]

we get the collocation result as
\[
\begin{align*}
    u_t(x_i,t) - \left( F_1^*(t) + \sum_{n=1}^{N-1} D_{i,n}^* u(x_n,t) \right) &= \beta u(x_i,t)(1-u^\delta(x_i,t))(u^\delta(x_i,t) - \gamma), \\
\end{align*}
\] (3.43)

where \( D_{i,n}^* \) and \( F_1^*(t) \) are defined by (3.26) and (3.29).

Considering the Eqs. (3.37) and (3.39), the system (3.43) can be written in the following form
\[
\begin{align*}
    du(t) &= G(t,u(t)), \\
    u_0 &= P,
\end{align*}
\] (3.44)
where

\[ G(t, u(t)) = [G_1(t, u(t)), G_2(t, u(t)), \ldots, G_{N-1}(t, u(t))]^T, \]
\[ G_i(t, u(t)) = \beta u(x_i, t)(1 - u^\delta(x_i, t)) \left(u^\delta(x_i, t) - \gamma\right) + F^*_i(t) + \sum_{n=1}^{N-1} D_{i,n}^x u(x_n, t). \]

Having solved the system (3.44) by the fourth-order Runge-Kutta method, we obtain \( u(x, t) \) as (2.10) which is the computed solution for the Huxley equation (3.40).

### 3.4 Generalized Burger’s-Huxley equation

Finally in this subsection, we illustrate how the spectral collocation method based on the chebyshev polynomial may be used to find approximate solution for the generalized Burger’s-Huxley equation

\[ u_t + \alpha u^\delta u_x - u_{xx} = \beta u(1 - u^\delta)(u^\delta - \gamma), \quad (x, t) \in D \times [0, T], \tag{3.45} \]

with initial and boundary conditions (1.5) and (1.6). \( \alpha, \beta, \gamma \) and \( \delta \) are parameters, \( \beta \geq 0, \delta > 0 \) and \( \gamma \in (0, 1) \).

We assume the solution

\[ u(x, t) = \sum_{j=0}^{N} a_j T_j^*(x). \tag{3.46} \]

be the approximate solution of the problem (3.45).

Similarly, by applying the chebyshev spectral collocation method, using the relations (3.24) and (3.30) and substituting \( x = x_i \) for \( i = 0, \ldots, N \), we obtain the following system

\[ u_t(x_i, t) = -\alpha u^\delta(x_i, t) \left(F_i(t) + \sum_{n=1}^{N-1} D_{i,n}^x u(x_n, t)\right) + \left(F^*_i(t) + \sum_{n=1}^{N-1} D_{i,n}^{xx} u(x_n, t)\right) + \beta u(x_i, t)(1 - u^\delta(x_i, t)) \left(u^\delta(x_i, t) - \gamma\right), \]
\[ u(x_i, 0) = f(x_i). \tag{3.47} \]

By using the notations (3.37) and (3.39), and also,

\[ G_i(t, u(t)) = \left(-\alpha u^\delta(x_i, t) F_i(t) + F^*_i(t)\right) + \beta u(x_i, t)(1 - u^\delta(x_i, t)) \left(u^\delta(x_i, t) - \gamma\right) + \sum_{n=1}^{N-1} D_{i,n}^x u(x_n, t) - \alpha u^\delta(x_i, t) \sum_{n=1}^{N-1} D_{i,n}^{xx} u(x_n, t), \]
\[ u(x_i, 0) = f(x_i), \]

we then rewrite the system (3.47) in the following form which is the system of ordinary differential equations.

\[ du(t) = G(t, u(t)), \]
\[ u_0 = P. \tag{3.48} \]

Solving the system (3.48) by Runge-Kutta method (3.33), we can obtain an approximation to the solution of (3.45).
4 Numerical examples

In order to illustrate the performance of the Chebyshev spectral collocation method in solving the problems (1.1)-(1.6) and efficiency of the presented method, the following examples are considered. We assume \( u_i \) and \( u_i^* \) be exact and approximate solutions and use the maximum of absolute error, defined as

\[
\|E\|_\infty = \max_{0 \leq i < N} |u_i - u_i^*|, \quad (4.49)
\]

The numerical results are tabulated in Tables 1-5.

Example 4.1. Consider the Fisher’s equation [21]

\[
u_t = u_{xx} + u(u - 1), \quad 0 \leq x \leq 1, \quad t > 0, \quad (4.50)
\]

subject to initial condition

\[
u(x, 0) = \frac{1}{4} \left( -1 + \tanh \left( \frac{x}{2\sqrt{6}} \right) \right)^2, \quad 0 \leq x \leq 1, \quad (4.51)
\]

and boundary conditions

\[
u(0, t) = \frac{1}{4} \left( 1 + \tanh \left( \frac{5t}{12} \right) \right)^2, \quad \nu(1, t) = \frac{1}{4} \left( -1 + \tanh \left( \frac{1}{12} (\sqrt{6} - 5t) \right) \right)^2, \quad t > 0. \quad (4.52)
\]

which has the exact solution given by

\[
u(x, t) = \frac{1}{4} \left( -1 + \tanh \left( \frac{1}{12} (-5t + \sqrt{6}x) \right) \right)^2. \quad (4.53)
\]

We solve (4.50) for different values of \( t, \ h_t = 10^{-4} \). The maximum of absolute errors are tabulated in Table 1 for \( N = 4, 6, 8 \).

| \( t \) | \( N = 4 \) | \( N = 6 \) | \( N = 8 \) |
|---|---|---|---|
| 0.1 | 1.9927E-8 | 1.1487E-11 | 4.0745E-14 |
| 0.15 | 2.9992E-8 | 1.9907E-11 | 4.2410E-14 |
| 0.18 | 3.5647E-8 | 2.6276E-11 | 4.9516E-14 |
| 0.2 | 3.9275E-8 | 3.0459E-11 | 5.4345E-14 |
| 0.25 | 4.7892E-8 | 4.0610E-11 | 6.6058E-14 |
| 0.3 | 5.5879E-8 | 5.0175E-11 | 7.7355E-14 |

Example 4.2. For the sake of comparison, we consider the following generalized Burger’s-Fisher equation discussed by Zhu et al. [22] and Golbabai et al. [3]. The authors used the cubic B-spline quasi-interpolation (BSQI) and spectral domain decomposition (SDD) methods to obtain their numerical solution.

\[
u_t + \alpha u^\delta u_x - u_{xx} = \beta u(1 - u^\delta),
\]

\[
u(x, 0) = \left( \frac{1}{2} - \frac{1}{2} \tanh \left( \frac{x\alpha \delta}{2(1 + \delta)} \right) \right)^{\frac{1}{2}},
\]

\[
u(0, t) = \left( \frac{1}{2} + \frac{1}{2} \tanh \left( \frac{t\delta(\alpha^2 + \beta(1 + \delta)^2)}{2(1 + \delta)^2} \right) \right)^{\frac{1}{2}}, \quad (4.54)
\]

\[
u(1, t) = \left( \frac{1}{2} + \frac{1}{2} \tanh \left( \frac{-\alpha \delta}{2(1 + \delta)}(1 - \left( \frac{\alpha}{1 + \delta} + \frac{\beta(1 + \delta)}{\alpha} \right)t) \right) \right)^{\frac{1}{2}}
\]
with the exact solution

\[ u(x,t) = \left( \frac{1}{2} + \frac{1}{2} \tanh \left( \frac{-\alpha \delta}{2(1+\delta)} \left( x - \frac{\alpha (1+\delta) t}{\alpha} \right) \right) \right)^{\frac{1}{2}}, \]  \hspace{1cm} (4.55)

where \( \alpha, \beta \) and \( \delta \) are arbitrary constants.

We compare the results with the BSQI [22] and SDD [3] applied to same equation. For this purpose, we consider the same parameter values for the generalized Burgers- Fisher equation (4.54) as considered in [3] and [22], namely: \( N = 8, N = 16, \alpha = 0.1, \beta = -0.25 \) and \( h_t = 10^{-5} \). Table 2 exhibits the compared results.

### Table 2: The results for Example 4.2.

| \( t \) | \( N = 8 \) | \( N = 16 \) |
|--------|-------------|-------------|
|        | Present method | BSQI | SDD | Present method | BSQI | SDD |
| 0.3    | 1.0547E-15 | 3.6959E-10 | 2.1143E-12 | \( 3.8858E-15 \) | 3.3124E-11 | 1.4118E-10 |
| 4      | 2.5535E-15 | 1.5630E-8 | 3.8276E-12 | \( 1.4766E-14 \) | 1.7788E-8 | 2.3587E-10 |
| 8      | 1.8874E-15 | 4.8731E-8 | 3.9059E-12 | \( 1.2324E-14 \) | 5.0674E-8 | 2.6737E-10 |
| 0.9    | 4.9960E-14 | 9.1896E-10 | 7.3929E-12 | \( 1.2434E-14 \) | 6.4819E-11 | 4.9790E-10 |
| 4      | 2.3315E-15 | 1.0155E-8 | 1.2857E-11 | \( 1.7430E-14 \) | 1.0150E-8 | 8.1773E-10 |
| 8      | 9.2149E-15 | 1.2017E-8 | 1.2907E-11 | \( 8.8818E-14 \) | 1.2212E-8 | 9.0725E-10 |

Example 4.3. Consider the generalized Huxley equation [5]

\[ u_t - u_{xx} = \beta u(1-u^2)(u^2 - \gamma), \quad 0 \leq x \leq 1, \quad t > 0, \]  \hspace{1cm} (4.56)

where \( \delta, \beta \geq 0 \) and \( \gamma \in (0,1) \) are given parameters. The exact solution to Eq. (4.56) is given by

\[ u(x,t) = \left[ \frac{\gamma}{2} + \frac{\gamma}{2} \tanh \left( \frac{\sigma \gamma (x + \frac{\rho (1+\delta)}{2(1+\delta)} t)}{2(1+\delta)} \right) \right]^{\frac{1}{2}}, \]  \hspace{1cm} (4.57)

where \( \sigma = \delta \rho / (1+\delta) \) and \( \rho = 2\sqrt{\beta (1+\delta)} \). The initial and boundary conditions are taken from the exact solutions.

We solve the Example 3 for \( N = 10 \) and \( h_t = 10^{-4} \). The maximum of absolute errors are tabulated in Table 3 for the parameters \( \beta = 1, \delta = 1, 2, 3 \) and \( \gamma = 0.001 \).

### Table 3: The results for Example 4.3.

| \( t \) | \( \delta = 1 \) | \( \delta = 2 \) | \( \delta = 3 \) |
|--------|-------------|-------------|-------------|
| 0.05   | 2.3138E-8  | 1.0348E-6  | 3.6729E-6  |
| 0.1    | 3.8440E-8  | 1.7191E-6  | 6.1019E-6  |
| 0.5    | 4.7799E-8  | 2.1377E-6  | 7.5874E-6  |
| 0.2    | 5.3513E-8  | 2.3932E-6  | 8.4942E-6  |
| 0.25   | 5.7001E-8  | 2.5491E-6  | 9.0476E-6  |
| 0.3    | 5.9431E-8  | 2.6443E-6  | 9.3863E-6  |

In Table 4, we illustrate the accuracy and efficiency of the Chebyshev spectral collocation method applied to generalized Huxley equation (4.56) compared to the Adomian decomposition and homotopy perturbation methods [5, 6] applied to same equation with the same parameters \( \beta = 1, \gamma = 0.001 \) and \( \delta = 1 \) and consider \( h_t = 10^{-4} \) and \( N = 10 \).
Table 4: The results for Example 4.3.

| x  | t    | Exact        | present method      | ADM [5]        | HPM [6]        |
|----|------|--------------|---------------------|----------------|----------------|
| 0.1| 0.05 | 5.00030E-4   | 5.00020E-4          | 5.0006E-4      | 5.0006E-4      |
|    | 0.1  | 5.0043E-4    | 5.0028E-4           | 4.9993E-4      | 4.9993E-4      |
|    | 1    | 5.00268E-4   | 5.00245E-4          | 4.99768E-4     | 4.99768E-4     |
| 0.5| 0.05 | 5.00101E-4   | 5.00078E-4          | 5.00076E-4     | 5.00076E-4     |
|    | 0.1  | 5.00113E-4   | 5.00075E-4          | 5.00063E-4     | 5.00063E-4     |
|    | 1    | 5.00338E-4   | 5.00276E-4          | 4.99839E-4     | 4.99839E-4     |
| 0.9| 0.05 | 5.00172E-4   | 5.00161E-4          | 5.00147E-4     | 5.00147E-4     |
|    | 0.1  | 5.00184E-4   | 5.00169E-4          | 5.00134E-4     | 5.00134E-4     |
|    | 1    | 5.00409E-4   | 5.00380E-4          | 4.99909E-4     | 4.99909E-4     |

Example 4.4. We consider generalized Burger’s-Huxley equation [7]

\[ u_t + au^\delta u_x - u_{xx} = \beta u(1-u^\delta)(u^\delta - \gamma), \quad 0 \leq x \leq 1, \quad t \geq 0, \quad (4.58) \]

with initial condition

\[ u(x,0) = \left( \frac{\gamma}{2} + \frac{\gamma}{2} \tanh(w_1 x) \right)^{\frac{1}{\delta}}, \quad 0 \leq x \leq 1, \]

and boundary conditions

\[ u(0,t) = \left( \frac{\gamma}{2} + \frac{\gamma}{2} \tanh(-w_1 w_2 t) \right)^{\frac{1}{\delta}}, \quad t > 0, \]
\[ u(1,t) = \left( \frac{\gamma}{2} + \frac{\gamma}{2} \tanh (w_1 (1 - w_2 t)) \right)^{\frac{1}{\delta}}, \quad t > 0. \]

The exact solution of Eq. (4.58) is taken from [23], given by

\[ u(x,t) = \left( \frac{\gamma}{2} + \frac{\gamma}{2} \tanh \left(w_1 (x - w_2 t)\right) \right)^{\frac{1}{\delta}}, \quad (4.59) \]

where

\[ w_1 = -\alpha \delta + \delta \sqrt{\frac{\alpha^2 + 4\beta(1+\delta)}{2(1+\delta)}}, \quad (4.60) \]

and

\[ w_2 = \frac{\alpha \gamma}{1+\delta} - \frac{(1+\delta - \gamma)(-\alpha + \sqrt{\alpha^2 + 4\beta(1+\delta)})}{2(1+\delta)}, \quad (4.61) \]

where \( \alpha, \beta, \gamma \) and \( \delta \) are constant such that \( \beta \geq 0, \delta > 0, \gamma \in (0,1) \).

We solved Example 4.4 for different values of \( t, N = 10, h_i = 10^{-3} \) and \( \delta = 3 \). The maximum of absolute errors on the Chebyshev collocation points are tabulated in Table 5 for \( \alpha = 0, \beta = 1, \gamma = 0.001 \) and \( \alpha = 0.001, \beta = 0.001, \gamma = 0.001 \).
Table 5: The results for Example 4.4.

| t  | \(\alpha = 0, \beta = 1, \gamma = 0.001\) | \(\alpha = 0.001, \beta = 0.001, \gamma = 0.001\) |
|----|------------------------------------------|--------------------------------------------------|
| 0.01 | 7.9325E-7 | 7.8722E-10 |
| 0.1 | 6.1019E-6 | 6.0556E-9 |
| 0.2 | 8.4942E-6 | 8.4301E-9 |
| 0.4 | 9.7168E-6 | 9.6449E-9 |
| 0.6 | 9.8850E-6 | 9.8137E-9 |
| 0.8 | 9.9066E-6 | 9.8371E-9 |
| 1   | 9.9079E-6 | 9.8404E-9 |

5 Conclusion

The Chebyshev spectral collocation method is used to solve the Fisher’s, generalized Burger’s-Fisher, Huxley and generalized Burger’s-Huxley equations with initial and boundary conditions. From the numerical results and Tables 1-5, we can say that errors are very small and they are very better than the results of another papers cited in this article.

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