The natural metric in the Horrocks-Mumford bundle is not Hermitian-Einstein.

O.F.B. van Koert, M. Lübke

Abstract. The Horrocks-Mumford bundle $E$ is a famous stable complex vector bundle of rank 2 on 4-dimensional complex projective space. By construction, $E$ has a natural Hermitian metric $h_1$. On the other hand, stability implies the existence of a Hermitian-Einstein metric in $E$ which is unique up to a positive scalar. Now the obvious question is if $h_1$ is in fact the Hermitian-Einstein metric. In this note we indicate how to show by computation that this is not the case.

1 Introduction and main result

Let $N$ be a null correlation bundle on complex projective 3-space $\mathbb{P}_3$, i.e. a quotient of $\Omega^1_{\mathbb{P}_3}$ by $\mathcal{O}_{\mathbb{P}_3}(-1)$ (see e.g. [OSS]). Then $N$ is stable in the sense of [OSS], or equivalently, $g$-stable in the sense of [LT], where $g$ is the Fubini-Study metric in $\mathbb{P}_3$, so the Kobayashi-Hitchin correspondence tells us that there exists a $g$-Hermitian-Einstein metric $h_0$ in $N$, which is unique up to a constant positive factor. On the other hand, the standard metric in $\mathbb{C}^4$ not only induces the Fubini-Study metric in $\mathbb{P}_3$, but also natural metrics in $\Omega^1_{\mathbb{P}_3}$ and $\mathcal{O}_{\mathbb{P}_3}(-1)$, and hence a metric $h_1$ in the quotient $N$, too. Now the obvious question arises:

(Q) Does it hold $h_1 = c \cdot h_0$ with a positive constant $c$, or equivalently, does $h_1$ satisfy the $g$-Hermitian-Einstein equation

\[(HE) \quad K_{h_1} = \lambda \cdot \text{id}_E\]

where $K_{h_1}$ is the mean curvature of $h_1$ and $\lambda$ a real constant?

This question was answered in the affirmative in [L] by manual computations with respect to local coordinates and a local holomorphic frame field.

1
In this note we consider the following similar situation. It is well known that on the 4-dimensional complex projective space \(\mathbb{P}^4 = \mathbb{P}(\mathbb{C}^5)\) there exists a stable holomorphic 2-bundle \(E\) with Chern numbers \(c_1(E) = 5\) and \(c_2(E) = 10\), the Horrocks-Mumford bundle [HM], [OSS]. Again, stability of \(E\) in the sense of [OSS] is the same as \(g\)-stability in the sense of [LT], where \(g\) is the Fubini-Study metric in \(\mathbb{P}^4\), so there exists a \(g\)-Hermitian-Einstein metric \(h_0\) in \(E\), which is unique up to a constant positive factor. On the other hand, using the construction of \(E\) given in [OSS] one gets in a natural way an explicit metric \(h_1\) in \(E\) induced by the standard metric in \(\mathbb{C}^5\), hence question (Q) arises for \(E\), too. Again we used explicit calculations in local coordinates to tackle this problem, and the result is

**Theorem 1** The metric natural metric \(h_1\) in the Horrocks-Mumford bundle is NOT \(g\)-Hermitian-Einstein.

In section 2 we sketch our approach to the problem, and in section 3 we give some details and explicit formulae which should be sufficient to make our calculations reproducible.

### 2 Our approach

The construction in [OSS] we use does not produce the bundle \(E\) directly, but the bundle \(E(-2) = E \otimes \mathcal{O}_{\mathbb{P}^4}(-2)\) and a metric \(h\) in it. Let \(h_2\) be the standard metric in \(\mathcal{O}_{\mathbb{P}^4}(2) = \mathcal{O}_{\mathbb{P}^4}(1)^{\otimes 2}\), induced by the canonical inclusion \(\mathcal{O}_{\mathbb{P}^4}(1)^* = \mathcal{O}_{\mathbb{P}^4}(-1) \hookrightarrow \mathbb{P}^4 \times \mathbb{C}^5\) and the standard metric in \(\mathbb{C}^5\), then the natural metric \(h_1\) in \(E = E(-2) \otimes \mathcal{O}_{\mathbb{P}^4}(2)\) is the metric induced by \(h\) and \(h_2\).

Our initial guess was that \(h_1\) would be indeed \(g\)-Hermitian-Einstein. Since \(h_2\) is known to be \(g\)-Hermitian-Einstein, this is equivalent to \(h\) being \(g\)-Hermitian-Einstein. Hence we attempted to show that the equation \((HE)\) holds for \(h\); by continuity, it suffices to do that on some open dense subset \(U_0^*\) of \(\mathbb{P}^4\). So in a suitable chart for \(\mathbb{P}^4\) we explicitly determined a matrix representation \(H\) of the metric \(h\) with respect to a holomorphic frame field; this already involved algebraic calculations which where impossible to do by hand (\(H\) contains rational expressions in the 8 real variables \(x_1, \bar{x}_1, \ldots, x_4, \bar{x}_4\), with numerators of degree up to 16), so we used the computer package MAPLE. The next step would have been the calculation of the mean curvature, i.e. essentially the matrix \(K = (K_{ij})_{i,j=1}^4\) where

\[
K_{ij} = -\sum_{\alpha,\beta=1}^4 g^{\alpha\beta} \left( \sum_{k=1}^2 \frac{\partial^2 H_{ik}}{\partial x_\alpha \partial \bar{x}_\beta} H^{kj} - \sum_{k,l,m=1}^2 \frac{\partial H_{ik}}{\partial x_\alpha} \frac{\partial H_{lm}}{\partial \bar{x}_\beta} H^{mj} \right).
\]
Here the $g_{\alpha\beta}$ are the coefficients of the Fubini-Study metric with respect to the local holomorphic coordinates $x_\alpha$, and upper indices mean coefficients of the inverse matrix. Now if the metric $h$ was $g$-Hermitian, then since $c_1(E(-2)) = 1$ this would be equivalent to $K(x) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ for all $x \in U_0^\ast$. (Notice that the constant $\lambda$ in $(HE)$ is determined by the topology of $E$ (see e.g. [LT]) and can therefore be determined a priori.)

Unfortunately, MAPLE was not able (at least on our computer) to calculate $K$ in a general point $x$ (the main problem being the inverse of $H$), so we decided to do some testing.

For this, we first let MAPLE determine the derivatives involved in the formula for $K_{ij}$ in a general point. Then we took the particular point $x_0 = (x_0^1, \ldots, x_0^4) = (1, 1, 1, 1)$, and could calculate $K(x^0)$ (inversion of the scalar matrix $H(x^0)$ is easy). The result was (as we had hoped) indeed

$$K(x^0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$ 

We repeated the procedure with the point $x^1 = (2, 1, 1, 1)$, and again we got

$$K(x^1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$ 

This seemed to indicate that we had in fact some chance to be right with our first guess.

In the meantime, we had started a different test by looking at the determinant line bundle $L := \det E(-2)$. The induced metric $\det h$ in $L$ is given over $U_0^\ast$ by the function $\det H$, and if $h$ was $g$-Hermitian-Einstein, then $\det h$ would be $g$-Hermitian-Einstein, too; more precisely, the mean curvature $K_{\det H}$ of $\det h$ would be the constant function 4. Again we got a problem: MAPLE could calculate $\det H$, but was not able to simplify the resulting rational function to a form from which it could determine $K_{\det H}$. But we made the motivated guess that

$$\det H = \frac{|x_1|^2|x_2|^2|x_3|^2|x_4|^2}{1 + \|x\|^2},$$

and where able (using MAPLE) to verify the correctness of this formula. Now it was easy to check (even by hand) that indeed $K_{\det H} \equiv 4$, i.e. that the induced metric in $\det E(-2)$, and hence that in $\det E$, is $g$-Hermitian-Einstein.

So far everything seemed to be okay, but testing of a third point $x^2 = (1 + i, 1, 1, 1)$ gave the disappointing result

$$K(x^2) = \begin{pmatrix} 2 & -217 \\ -\frac{217}{25992} & \frac{217}{25992} \end{pmatrix}.$$
This of course meant precisely what we did not want to show, namely that the metric in $E(-2)$, and hence the metric in $E$, is not $g$-Hermitian-Einstein.

3 Some details and formulae

The bundle $E(-2)$ is the cohomology of a monad

$$(M) \quad 0 \to \mathbb{C}^5 \otimes \mathcal{O}_{P_4} \xrightarrow{a} (\Lambda^2 Q)^{\oplus 2} \oplus (\mathbb{C}^5)^* \otimes \Lambda^4 Q \to 0,$$

i.e. $a$ is injective, $b$ is surjective, it holds $\text{im}(a) \subset \ker(b)$, and $E(-2) = \frac{\ker(b)}{\text{im}(a)}$; here $Q = T_{P_4}(-1)$ (see [OSS]). Let $\pi : \mathbb{C}^5 \otimes \mathcal{O}_{P_4} \to Q$ be the natural projection in the Euler sequence. The standard Hermitian inner product in $\mathbb{C}^5$ defines the standard flat Hermitian metric in the trivial bundle $\mathbb{C}^5 \otimes \mathcal{O}_{P_4}$, and hence a quotient metric $h_Q$ in $Q$. This induces a metric $\Lambda^2 h_Q$ in $\Lambda^2 Q$, and hence a metric $h_3$ in $(\Lambda^2 Q)^{\oplus 2}$ by taking the two summands as orthogonal. Next we get a metric $h_b$ in $\ker(b)$ by restricting $h_3$, and finally a quotient metric $h$ in $E(-2)$.

Let $x = (x_0 : x_1 : \ldots : x_4)$ be the homogeneous coordinates in $P_4$ with respect to the standard basis $e_0, \ldots, e_4$ of $\mathbb{C}^5$. The holomorphic section in $\mathbb{C}^5 \otimes \mathcal{O}_{P_4}$ defined by $e_i$ is denoted $\overline{e}_i$, and we define $v_i := \pi(\overline{e}_i) \in H^0(P_4, Q), i = 0, \ldots, 4$.

Over $U_0 := \{ x \in P_4 | x_0 \neq 0 \}$, $u := (v_1, \ldots, v_4)$ is a holomorphic frame field for $Q$. For $x = (1 : x_1 : \ldots : x_4) \in U_0$ the quotient metric in $Q(x)$ is given by

$$h_Q(v_i, v_j)(x) = \delta_{ij} - \frac{\overline{x}_i x_j}{n}, \quad 1 \leq i, j \leq 4,$$

where $n = 1 + \sum_{i=1}^4 |x_i|^2$. A holomorphic frame field $u = (u_1, \ldots, u_6)$ for $\Lambda^2 Q$ over $U_0$ is given by $u_1 := v_1 \wedge v_2$, $u_2 := v_1 \wedge v_3$, $u_3 := v_1 \wedge v_4$, $u_4 := v_2 \wedge v_3$, $u_5 := v_2 \wedge v_4$, $u_6 := v_3 \wedge v_4$. Since $\Lambda^2 h_Q(v_i \wedge v_j, v_k \wedge v_l) = h_Q(v_i, v_k)h_Q(v_j, v_l) - h_Q(v_i, v_l)h_Q(v_j, v_k)$, it is easy to determine the matrix representation of $\Lambda^2 h_Q(x)$ with respect to $u(x)$. The holomorphic frame field $b := (b_1, \ldots, b_{12})$ for $(\Lambda^2 Q)^{\oplus 2}$ is defined by $b_i := (u_i, 0)$ for $1 \leq i \leq 6$ and $b_i := (0, u_{i-6})$ for $7 \leq i \leq 12$; then the matrix representation of $h_3(x)$ with respect to $b(x)$ is $h_3(x) = \begin{pmatrix} \Lambda^2 h_Q(x) & 0 \\ 0 & \Lambda^2 h_Q(x) \end{pmatrix}$.

Define $a_+ : \mathbb{C}^5 \to \Lambda^2 \mathbb{C}^5$ by $a_+(e_i) := e_{i+2} \wedge e_{i+3}$, $a_-(e_i) := e_{i+1} \wedge e_{i+4}, 0 \leq i \leq 4$ (indices mod 5). Then the map $a$ in $(M)$ is defined as the composition

$$a(x) : \mathbb{C}^5 \xrightarrow{(a_+, a_-)} (\Lambda^2 \mathbb{C}^5)^{\oplus 2} \xrightarrow{(\Lambda^2 \pi(x))^{\oplus 2}} (\Lambda^2 Q(x))^{\oplus 2}.$$
3 SOME DETAILS AND FORMULAE

Using \( \pi(x)(e_0) = -\sum_{i=1}^{4} x_i v_i \), it follows that the basis \( (a_3, \ldots, a_7) \), \( a_i := a(e_i) \), of \( \text{im}(a) \) is given in coordinates with respect to \( b \) as

\[
\begin{align*}
a_3(x) &= (0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0) , \\
a_4(x) &= (0, 0, 0, 0, 1, x_1, 0, 0, -x_3, -x_4, 0) , \\
a_5(x) &= (0, 0, x_1, 0, x_2, x_3, 0, -1, 0, 0, 0) , \\
a_6(x) &= (x_2, x_3, x_4, 0, 0, 0, 0, 0, -1, 0) , \\
a_7(x) &= (1, 0, 0, 0, 0, 0, 0, -x_1, 0, -x_2, -x_3) .
\end{align*}
\]

The map \( b : (\Lambda^2Q)^{\otimes 2} \rightarrow (\mathbb{C}^5)^* \otimes \Lambda^4Q = \text{Hom}(\mathbb{C}^5 \otimes \mathcal{O}_{\mathbb{P}^5}, \Lambda^4Q) \) in \( (M) \) is defined by

\[
b(x)(\xi, \eta)(v) := -\eta \wedge (\Lambda^2\pi(x))(a_+(v)) + \xi \wedge (\Lambda^2\pi(x))(a_-(v))
\]

for \( v \in \mathbb{C}^5 \) and \( \xi, \eta \in \Lambda^2Q(x) \). It is easily checked that the vectors

\[
\begin{align*}
a_1(x) &= (x_1 x_2, 0, x_1 x_4, 0, 0, x_3 x_4, 0, 0, 0, 0, 0) , \\
a_2(x) &= (0, 0, 0, 0, 0, 0, x_1 x_3, 0, x_2 x_3, x_2 x_4, 0)
\end{align*}
\]

are in \( \ker(b(x)) \), and that \( \underline{a} := (a_1, \ldots, a_7) \) is a holomorphic frame field for \( \ker(b) \) over \( U_0^* := \{ x \in U_0 \mid x_i \neq 0 \, , \, 1 \leq i \leq 4 \} \). Since \( (a_3, \ldots, a_7) \) is a basis of \( \text{im}(a) \), the projection \( \ker(b) \rightarrow \ker(b)/\text{im}(a) = \mathbb{P}^1 \) maps \( a_1, a_2 \) to a holomorphic frame field \( \tilde{\underline{a}} := (\tilde{a}_1, \tilde{a}_2) \) of \( \mathbb{P}^1 \) over \( U_0^* \). We write the matrix representation of \( h_3(x)|_{\ker(b)} \) with respect to \( \underline{a} \) as block matrix

\[
h_3(x)|_{\ker(b)} = \frac{1}{n} \begin{pmatrix} C & \tilde{B}^t \\ B & A \end{pmatrix},
\]

where \( A \) is the \( 5 \times 5 \)-matrix representing \( h_3(x)|_{\text{im}(a)} \). This can be calculated explicitly, using the matrix for \( h_3(x) \); the result is

\[
C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}
\]

where

\[
\begin{align*}
c_1 &= |x_1|^2 |x_2|^2 (1 + |x_3|^2) + |x_1|^2 |x_4|^2 + |x_3|^2 |x_4|^2 (1 + |x_2|^2) , \\
c_2 &= |x_1|^2 |x_3|^2 (1 + |x_4|^2) + |x_2|^2 |x_3|^2 + |x_2|^2 |x_4|^2 (1 + |x_1|^2) ,
\end{align*}
\]

\[
B = \begin{pmatrix}
|\bar{x}_1|^2 + |\bar{x}_3|^2 & -|\bar{x}_2|^2 + |\bar{x}_3|^2 & (|x_1|^2 + |x_2|^2) \bar{x}_2 \bar{x}_3 \\
(1 + |x_2|^2) \bar{x}_3 \bar{x}_4 & -|\bar{x}_3|^2 + |\bar{x}_4|^2 & -|x_2|^2 + |x_3|^2 \\
(|x_1|^2 + |x_3|^2) \bar{x}_4 & -(1 + |x_4|^2) \bar{x}_1 \bar{x}_3 & (|x_1|^2 + |x_2|^2) \bar{x}_1 \\
|\bar{x}_2|^2 + |\bar{x}_4|^2 & -|x_2|^2 + |x_3|^2 & -(1 + |x_1|^2) \bar{x}_2 \bar{x}_4 \\
(1 + |x_3|^2) \bar{x}_1 \bar{x}_2 & -(|x_1|^2 + |x_2|^2) \bar{x}_3 & 0
\end{pmatrix},
\]
and

\[
A = \begin{pmatrix}
  n+1 & \bar{x}_2x_4 & \bar{x}_4x_3 & \bar{x}_1x_2 & \bar{x}_3x_1 \\
  \bar{x}_4x_2 & n + |x_1|^2 & \bar{x}_3 & x_4 & \bar{x}_2x_3 \\
  \bar{x}_3x_4 & x_3 & n + |x_2|^2 & \bar{x}_4 & x_1 \\
  \bar{x}_2x_1 & \bar{x}_4 & \bar{x}_1x_4 & n + |x_3|^2 & x_2 \\
  \bar{x}_1x_3 & \bar{x}_3x_2 & x_1 & \bar{x}_2 & n + |x_4|^2 \\
\end{pmatrix}.
\]

The matrix representation of the metric \( h(x) \) in \( E(x) \) with respect to \( \tilde{a}(x) \) is now given by

\[
h(x) = \frac{1}{n} (C - B^t \cdot A^{-1} \cdot B).
\]

We used MAPLE to explicitly calculate \( h(x) \), but the resulting expression is too large to write down here.

We view \( x = (x_1, \ldots, x_4) \) as holomorphic coordinates in \( U_0 \) via the standard chart \( (x_1, \ldots, x_4) \mapsto (1 : x_1 : \ldots : x_4) \). With respect to these coordinates, the Kähler form of the Fubini-Study metric \( g \) is \( \omega_g = \frac{i}{2} \sum_{\alpha, \beta=1}^4 g_{\alpha\beta} dx_\alpha \wedge d\bar{x}_\beta \), where \( g_{\alpha\beta} = \frac{\delta_{\alpha\beta}}{n} - \frac{\bar{x}_\alpha x_\beta}{n^2} \) with \( n = 1 + \sum_{i=1}^4 |x_i|^2 \) as above.

Let \( D \) be the Chern connection in \( (E(-2), h) \), i.e. the unique \( h \)-unitary connection compatible with the holomorphic structure in \( E(-2) \) (compare [K],[LT]), and \( F = D \circ D \) its curvature. With respect to the holomorphic frame field \( \tilde{a} \) for \( E(-2) \) over \( U_0^* \), we write \( F = (F_{ij})_{i,j=1,2} \) and \( F_{ij} = \sum_{\alpha, \beta=1}^4 F_{ij\alpha\beta} dx_\alpha \wedge d\bar{x}_\beta \), \( 1 \leq i,j \leq 2 \). Let be \( h = (H_{ij})_{i,j=1,2} \) with respect to \( \tilde{a} \), and \( (H^{ij})_{i,j=1,2} := h^{-1} \). Then it holds

\[
F_{ij\alpha\beta} = - \sum_{k=1}^2 \frac{\partial^2 H_{ik}}{\partial x_\alpha \partial \bar{x}_\beta} H^{kj} + \sum_{k,l,m=1}^2 \frac{\partial H_{ik}}{\partial x_\alpha} H^{kl} \frac{\partial H_{lm}}{\partial \bar{x}_\beta} H^{mj}.
\]

The mean curvature \( K \) of \( h \) (with respect to \( g \)) is defined by the relation

\[
F \wedge \omega_g^3 = -\frac{i}{2} K \omega_g^4.
\]

With respect to \( \tilde{a} \) we write \( K = (K_{ij})_{i,j=1,2} \), then it holds

\[
(*) \quad K_{ij} = \sum_{\alpha, \beta=1}^4 g^{\alpha\beta} F_{ij\alpha\beta} = - \sum_{\alpha, \beta=1}^4 g^{\alpha\beta} \left( \sum_{k=1}^2 \frac{\partial^2 H_{ik}}{\partial x_\alpha \partial \bar{x}_\beta} H^{kj} - \sum_{k,l,m=1}^2 \frac{\partial H_{ik}}{\partial x_\alpha} H^{kl} \frac{\partial H_{lm}}{\partial \bar{x}_\beta} H^{mj} \right).
\]

where \( (g^{\alpha\beta})_{\alpha, \beta=1, \ldots, 4} := ((g_{\alpha\beta})_{\alpha, \beta=1, \ldots, 4})^{-1} \), i.e. \( g^{\alpha\beta} = n(\delta_{\alpha\beta} + \bar{x}_\alpha x_\beta) \).
Since the $H_{ij}$ and $g^{\alpha\beta}$ are explicitly given, the calculation of $K(x^0)$ for a given point $x^0$ can now be done as follows (using MAPLE where necessary):

- determine $\frac{\partial^2 H_{ik}}{\partial x^\alpha \partial \bar{x}^\beta}(x), \frac{\partial H_{ik}}{\partial x^\alpha}(x), \frac{\partial H_{ik}}{\partial \bar{x}^\beta}(x), 1 \leq i, k \leq 2, 1 \leq \alpha, \beta \leq 4$, for a general point $x$;
- substitute $x^0$ into $h$, and invert the scalar matrix $h(x^0)$ to get the $H^{ij}(x^0)$’s;
- substitute $x^0$ into $\frac{\partial^2 H_{ik}}{\partial x^\alpha \partial \bar{x}^\beta}, \frac{\partial H_{ik}}{\partial x^\alpha}, \frac{\partial H_{ik}}{\partial \bar{x}^\beta}, g^{\alpha\beta}, 1 \leq i, k \leq 2, 1 \leq \alpha, \beta \leq 4$;
- substitute the resulting scalars into the right hand side of equation (*), and evaluate.

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O.F.B. van Koert
M. Lübke

Mathematical Institute
Leiden University
PO Box 9512
NL 2300 RA Leiden

okoert@math.leidenuniv.nl
lubke@math.leidenuniv.nl