Simulations of quantum double models

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Abstract. We demonstrate how to build a simulation of two-dimensional (2D) physical theories describing topologically ordered systems whose excitations are in one-to-one correspondence with irreducible representations of a Hopf algebra, $D(G)$, the quantum double of a finite group $G$. Our simulation uses a digital sequence of operations in a spin lattice model originally due to Kitaev to prepare a ground ‘vacuum’ state and to create, braid and fuse anyonic excitations. The simulation works with or without the presence of a background Hamiltonian though only in the latter case is the system topologically protected. We describe a physical realization of a simulation of the simplest non-Abelian model, $D(S_3)$, using trapped neutral atoms in a 2D optical lattice and provide a sequence of steps to perform universal quantum computation with anyons. The use of ancillary spin degrees of freedom figures prominently in our construction and provides a novel technique to prepare and probe these systems.
1. Introduction

Broadly speaking, topology deals with the properties of spaces invariant under continuous transformations. Topology appears in a physical setting, typically, when a quantity $X$ can be shown to take values in a discrete set $\{x_1, x_2, \ldots\}$ when the parameters of a configuration space $\mathcal{F}$ vary over a continuous range, in which case $\mathcal{F}$ splits into sectors labelled by the discrete values of $X$,

$$\mathcal{F} = \bigcup_{x_i \in X} \mathcal{F}_i,$$

these values providing a topological classification of configurations. Two configurations in different sectors cannot be transformed into one another continuously; in other words, quantity $X$ is topologically stable and is, for instance, conserved during (smooth) time evolutions both in the classical and quantum worlds. Examples range from soliton theory in hydrodynamics, where soliton charges determine the stability of solitary waves, to the classification of instantons, the key to chiral symmetry breakdown in four-dimensional (4D) Yang–Mills theory.

Topological phases of matter and lattice systems [1] are highly correlated phases whose order cannot be described in the local group symmetry and order parameter paradigm. Such systems exhibit a topological order usually associated with a gapped ground level with degeneracy dependent on the topological properties (typically, a Betti number) of the underlying spatial manifold—and, in 2D, with particle-like excitations with anyonic statistics, and gapless edge modes when boundaries are present. Fractional quantum Hall states exhibit such an order and provide an experimental setting where Abelian anyonic statistics have been shown to exist, and non-Abelian statistics are widely expected to appear for certain filling fractions. Topologically ordered lattice systems have also been theoretically constructed, and their experimental simulation complements their analytical and numerical study.

Topological orders have recently attracted considerable interest in the field of quantum information, due to proposals to use topological phases of matter and lattice systems as quantum
memories, whereby information is stored in topologically stable quantities and quantum computers. In the most widely explored scenario, gates are performed on the stored information by creation, braiding and fusion of anyonic excitations. While it is not clear whether usual topological codes in 2D remain useful at finite temperatures or in the presence of noise [2], models in 2D have been proposed featuring string tensions between anyons [3], therefore confining anyonic defects; on the other hand, a kind of topological memories in 4D is also expected to show resilience to thermal effects. The anyon braiding paradigm of topological quantum computation hence deserves to be further studied and realized in the laboratory.

In this respect, it is instructive to consider, following DiVincenzo’s example, the requirements an experimental setting must fulfil in order to implement anyonic topological quantum computation based on anyon braiding (see also [4]):

- **Existence:**
  
  **E0** Existence of anyons.
  
  Anyonic statistics is used to act on code quantum states by manipulating anyonic objects. Quantum information may also be encoded in static configurations of anyons.

  **E1** Controlled initialization of anyons.
  
  Anyons may be present from the beginning in the system, or may need to be created; manipulation of anyons by transport or fusion may be required to bring the system to the desired initial configuration (e.g. a product state).

  **E2** Implementation of gates by anyon braiding.
  
  Braiding non-Abelian anyons results in the application of quantum gates. The statistics should be rich enough to provide a universal set of braiding quantum gates, in which case we speak of universal anyons.

  **E3** Measurement of topological charge.
  
  This is the read-out part of a computation, and can be achieved in a number of ways, e.g. by using interferometry. Such methods may involve the controlled fusion of anyons.

- **Experimental feasibility:** These are not independent of the existence requirements, since, e.g., the ability to perform gates imply that the life of an anyon is long enough to carry out the task, but their importance as criteria for practical implementations warrants a separate listing.

  **S0** Scalability of the implementation.
  
  A practical implementation should maximize the encoded information while ensuring control of the anyonic population. This leads to spatial efficiency as a design goal.

  **S1** Robustness of the implementation.
  
  For instance, the anyon lifetime should be larger than the gate application time, or than computation time for information-encoding anyons.

The basic tools to perform anyonic TQC can be seen to be the controlled creation, transport and fusion of anyons. The status of anyonic computation in different experimental settings is as follows:

- **Quantum Hall effect.** At fractional filling of the particle orbitals in 2D electron gasses in the quantum Hall effect, the low lying excitations are particle-like and at particular filling fractions predicted to behave like anyons [5, 6]. Some experiments have reported interference measurements consistent with the existence of Abelian anyons in [7, 8].
although there are some caveats to the interpretation of the data [9, 10]. Non-Abelian excitations have been argued to appear for filling fraction $5/2$ and $12/5$ [5, 11], and in the latter case, braiding would be universal for quantum computation. See recent experimental progress in [12].

- **Josephson junction arrays.** The theory was put forward in [13], leading to the experiments reported in [14].
- **Photonic systems.** See [15, 16] for recent experiments using entangled states of photons.
- **Optical lattices:** satisfy all of E0–E3 as argued in this article but no experiments have been conducted as yet.

Simulation of lattice systems using cold atoms and molecules in optical lattices offers an attractive setting where a high degree of control over the parameters of the system can be achieved. Minimal building blocks of topologically ordered systems have been considered in [17]. The implementation of Kitaev’s honeycomb lattice model was discussed in [18, 19]; braiding and interferometry experiments mediated by optical cavity modes were considered in [20]. Braiding in the Abelian phase was also considered in [21]; however, the degraded visibility of the anyonic statistics with this approach due to the perturbative treatment of the honeycomb Hamiltonian was pointed out in [22], and a method to construct Kitaev’s toric code from the cluster state was outlined in [23].

In this work we develop a practical, universal method, introduced in [24], to perform anyonic interferometry and arbitrary quantum computation tasks based on anyon braiding in spin lattices, using controlled operators applied by manipulating an extra species of ancillary particles to construct all the needed primitives: anyon creation, transport and fusion. We illustrate the method using the D($S_3$) quantum double model, whose excitations allow for universal quantum computation by braiding [25]. Our construction is based on the spin lattice model introduced by Kitaev [26] with the new feature that we give explicit protocols for manipulating anyonic excitations and read-out of fusion products. We also describe a method to construct the relevant topologically ordered states efficiently in the absence of a simulated topological Hamiltonian. While this construction is of interest in itself, we want to emphasize that the anyon manipulation procedure is independent and can be used whenever the topological phase is achieved, e.g. by simulation of the Hamiltonian.

### 2. Spin lattice models for the quantum double of a finite group

Denote by $D(G)$ the quantum double of a finite group $G = \{g_j\}$, which is a quasi-triangular Hopf algebra. An algebraic construction of these models was first given by Bais et al [27, 28] and for a brief introduction see appendix A. We simulate a spin lattice Hamiltonian $H_{TO}$, which is a sum of quasi-local operators and has localized particle-like excitations that are in one-to-one correspondence with the irreducible representations (irreps) of $D(G)$. Consider a two-complex $\Gamma$, which is a cellulation of a 2D surface with vertex set $\mathcal{V} = \{v_i\}$, edge set $\mathcal{E} = \{e_j\}$ and face set $\mathcal{F} = \{f_j\}$. Particles with $d = |G|$ internal levels (qudits) are placed on the edges and physical states live in a Hilbert space $\mathcal{H} = \mathcal{H}(d)^{\mathcal{E} / \mathcal{F}}$, where $\mathcal{H}(d) = \mathbb{C}|0\rangle + \cdots + \mathbb{C}|d - 1\rangle$. Particles on edges that meet at a vertex $v$ all interact via a vertex operator $A(v)$. Similarly, all particles on edges that are on the boundary of a face $f$ interact via a face operator $B(f)$. We pick an orientation for each edge with $e = [v_j, v_k]$ denoting an edge with arrow pointing from vertex $v_j$ to $v_k$. We assume an orientable complex $\Gamma$ and each face $f$ is given an orientation (say, counterclockwise) independent of the orientation of the edges. The Hamiltonian is a
sum of constraints chosen such that the ground states of $H_{TO}$ are invariant under local gauge transformations

$$T_g(v) = \prod_{e_j \in [s,v]} L_g(e_j) \prod_{e_j \in [s,v]} R_{g^{-1}}(e_j).$$

(2)

Here $L_g(e_j), R_g(e_j) \in U(d)$ the $d$ dimensional unitary group, are the permutation representations of the left and right action of multiplication by the group element $g \in G$ on the system particle located at edge $e_j$. For the particle states we make the identification $|j\rangle \equiv |g_j\rangle$, where by convention $|0\rangle \equiv |g_0\rangle \equiv |e\rangle$, with $e$ being the identity element. The action of left and right group multiplication on the basis states is then $L_h|j\rangle = |hj\rangle$, $R_h|j\rangle = |g_j h\rangle$.

A suitable spin lattice model was provided by Kitaev [26]:

$$H_{TO} = -\sum_v A(v) - \sum_f B(f),$$

(3)

where

$$A(v) = \frac{1}{|G|} \sum_{g \in G} T_g(v),$$

$$B(f) = \sum_{\{\Pi_{e_j \in f} h_k^{-o_j(\alpha)}\}} \otimes_{\ell \in f} |h_k^{o_j(\alpha)}\rangle \langle h_k^{-o_j(\alpha)}|.$$  

(4)

In the definition of $B(f)$, the sum is taken over all products of group elements $h_k$ acting on a counterclockwise cycle of edges on the boundary of $f$ such that the accumulated left action is the identity element $e \in G$ (i.e. $h_f h_{f-1} \ldots h_1 h_f = e$ for the counterclockwise cycle starting at edge $e_1$ and ending at edge $e_f$). The function $o_f(e_j) = \pm 1$ according to whether the orientation of the edge is the same as (opposite to) the face orientation. By construction $[A(v), A(v')] = [B(f), B(f')] = [A(v), B(f)] = 0$. Furthermore, it is straightforward to verify that since $A(v)$ is a symmetrized gauge transformation it is a projection as is $B(f)$. The ground states of $H_{TO}$ are then manifestly gauge invariant states. Excited states are described by violations of the local constraints $A(v)$, $B(f)$ and are particle-like corresponding to the irrep of $D(G)$ labelled by $\Pi_{\chi \in R(\alpha)}$, where $\alpha$ denotes a conjugacy class of $G$ that labels the magnetic charge, and $R(N_{h \gamma})$ denotes a unitary irrep $R$ of the centralizer of an element in the conjugacy class $\alpha$ that labels the electric charge. Note that there is an arbitrariness in how one picks the fiducial element of the conjugacy class. However, $N_{gh^{-1}} = gN_hg^{-1}$ so that the centralizers for the elements in a given conjugacy class are isomorphic (they are equal up to a gauge transformation), and we can index them just by the conjugacy class.

We focus on a specific two-complex $\Gamma$, which is a square lattice with boundary (see figure 1). For any two complexes with boundary and without holes, there exists a ground state $|GS\rangle$ such that $H_{TO}|GS\rangle = −(|V| + |F|)|GS\rangle$ and it is unique (for the argument see e.g. [29]).

The convention for edge and face orientations is shown in figure 1. We slightly abuse notation by labelling the particles according to the location relative to a face index $f_i$ and vertex index $v_j$ (see figure 1(a)). For instance, a vertex ancillary particle at vertex $v_{i,j}$ will be labelled $v_{i,j}$ and a face ancillary particle at face $f_{i,j}$ will be labelled $f_{i,j}$. The system particle on edge $e = [v_{i,j}, v_{k,l}]$ will be labelled $e_{i,j,k,l}$. When we are referring to the actual spatial locations $f$ and $v$ it will be made clear.

In table 1 we give an algorithmic procedure to prepare the ground state of the Hamiltonian $H_{TO}$ for an arbitrary finite group $G$. We begin with all system particles and face ancillae in

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Figure 1. A spin lattice model for the quantum double of a finite group. The system particles (blue) reside on the edges of a two-complex $\Gamma$. In a Hamiltonian formulation, all particles on edges that meet at a vertex $v$ interact as do all edges that surround a face $f$. The orientation of edges and faces is indicated. An ancillary set of particles (red) is placed on the vertices of the complex $\Gamma$ and the faces of $\Gamma$ (equivalently, the vertices of the dual complex $\tilde{\Gamma}$). The ancillary particles on the vertices (faces) afford a handle on operations with system particles to create and guide electric (magnetic) charges, which are indicated by diamonds (squares). The braiding of one member of a chargeless magnetic flux $[\ell]$ pair around one member of a fluxless electric charge $R$ pair is shown: $R^2|0_{[\ell]}; (v_{3,1}, f_{3,1}), (v_{1,1}, f_{1,1})\rangle|1^R; (v_{4,4}, v_{1,5})\rangle \rightarrow \frac{1}{|G|} \sum_{\ell \in [\ell]} |(P^R_{\ell}, \ell); (v_{3,1}, f_{3,1})\rangle \langle (P^R_{\ell}, \ell^{-1}); (v_{1,1}, f_{1,1})\rangle |R(\ell); (v_{4,4}, v_{1,5})\rangle$.

The system particles (blue) reside on the edges of a two-complex $\Gamma$. In a Hamiltonian formulation, all particles on edges that meet at a vertex $v$ interact as do all edges that surround a face $f$. The orientation of edges and faces is indicated. An ancillary set of particles (red) is placed on the vertices of the complex $\Gamma$ and the faces of $\Gamma$ (equivalently, the vertices of the dual complex $\tilde{\Gamma}$). The ancillary particles on the vertices (faces) afford a handle on operations with system particles to create and guide electric (magnetic) charges, which are indicated by diamonds (squares). The braiding of one member of a chargeless magnetic flux $[\ell]$ pair around one member of a fluxless electric charge $R$ pair is shown: $R^2|0_{[\ell]}; (v_{3,1}, f_{3,1}), (v_{1,1}, f_{1,1})\rangle|1^R; (v_{4,4}, v_{1,5})\rangle \rightarrow \frac{1}{|G|} \sum_{\ell \in [\ell]} |(P^R_{\ell}, \ell); (v_{3,1}, f_{3,1})\rangle \langle (P^R_{\ell}, \ell^{-1}); (v_{1,1}, f_{1,1})\rangle |R(\ell); (v_{4,4}, v_{1,5})\rangle$.

state $|e\rangle = |0\rangle$. This guarantees that the initial system state satisfies the zero flux condition, i.e. $B(f)|\psi\rangle_S = |\psi\rangle_S \forall f$. All vertex ancillae are prepared in the state $|0\rangle$ where $|j\rangle = \frac{1}{|G|} \sum_{k=0}^{[G]-1} e^{2\pi i j k / |G|} |k\rangle$. In the case $G = \mathbb{Z}_2$, this algorithm produces the ground state of the planar version of Kitaev’s toric code [26]. For that model all operations can be done with qubits, and the permutation rep of the group is $L_e = R_e = I_2$ and $L_g = R_g = \sigma^x$. Controlled operations involve only CNOT gates and the correction gates $Z^j = (\sigma^z)^j$. Another scheme for constructing the ground state of the toric code is proposed in [23] using single qubit measurements and feedforward on a prepared cluster state.

Our algorithm has a computational depth of $O(|G|(m+n))$ but one might wonder if a faster ground state preparation procedure is possible. The answer is no if the initial state is uncorrelated and the available set of operations is quasi-local. The reason is that the final state has global correlations that are created by quasi-local operations. In our algorithm these operations are measurements but they could also be an adiabatic turn on of the summands

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Table 1. Algorithm: Ground state synthesis. An algorithm for preparation of the ground state $|GS\rangle$ of $H_{TO}$ over a finite group $G$ on an $(n+1) \times (m+1)$ square lattice with boundary satisfying $B(f)|GS\rangle = A(v)|GS\rangle = |GS\rangle \forall v \in \mathcal{V}, f \in \mathcal{F}$. The algorithm works by beginning in a $+1$ co-eigenstate of all face operators $B(f)$ and applying vertex ancilla-assisted projections $A(v)$. Using single particle operations conditioned on measurement outcomes of vertex ancilla, this algorithm outputs the state $|\Psi_{\text{out}}\rangle = \prod_{v \in \mathcal{V} \setminus v_{n,m}} \frac{1}{\sqrt{|G|}} \sum_{h \in G} T_h(v) |\Psi_m\rangle$. No projection $A_{v_{n,m}}$ is needed because for a two-complex group $\Gamma$ with boundary, $\prod_{v \in \mathcal{V}} A(v)|e\rangle^{\otimes |\mathcal{E}|} = \prod_{v \in \mathcal{V} \setminus v_{n,m}} A(v)|e\rangle^{\otimes |\mathcal{E}|}$. Each controlled gate operation requires $O(|G|)$ elementary single and two qudit operations and the algorithm has complexity $O(|G|nm)$. Since $[L_h, R_{hv}] = 0$, the operators performing local gauge transformations commute: $[W(v), W(v')] = 0$. Up to the last column then, all columnwise unitary operations $W(v)$ and subsequent correction gates $Z^j(e)$ and $Z^j(e)$ can be done in parallel. For the last column the correction gates do not commute with the operators $W(v)$; hence the operations are done serially so that the computational depth of the algorithm is $O((|G|(m+n)))$.

By single particle measurement, prepare the initial state $|\Psi_m\rangle = \otimes_{v \in \mathcal{E}} |0\rangle_v \otimes_{e \in \mathcal{V}} |0\rangle_v \otimes_{f \in \mathcal{F}} |0\rangle_f$.

for $k = 0:n - 1$
  for $j = 0:n$
    Apply the unitary $W(v_{j,k})$ where $W(v) = \sum_{h \in G} |h\rangle_v \langle h| \otimes T_h(v)$.
    Measure ancilla $v_{j,k}$ in the basis $|\{j\}\rangle$ (probability for outcome $j$ is $1/|G|$).
    For the outcome $|j\rangle$ apply the single qudit operation $Z^j(e_{j,k;j,k+1})$ where $Z^j(e) = \sum_{k=0}^{G-1} e^{i2\pi jk/|G|} |k\rangle_e \langle k|.$
    $j++$
  end
  $k++$
end

$j = 0$

for $j = 0:n - 1$
  Apply the unitary $W(v_{j,k})$.
  Measure ancilla $v_{j,k}$ in the basis $|\{j\}\rangle$ (probability for outcome $j$ is $1/|G|$).
  For the outcome $|j\rangle$ apply $Z^j(e_{j,m;j+1,m})$.
  $j++$
end

of $H_{TO}$. The time scale to perform the quasi-local operations (here the measurement time of $A(v)$) establishes a light cone for the flow of correlations. In [30] it was shown by an application of the Lieb–Robinson bound that the minimal time to prepare a topologically ordered state beginning in a completely uncorrelated state is of the order of the length of the correlations. For topologically ordered states there exist global correlation functions, which here correspond to ribbons of operators that extend across the linear dimension of the lattice [26]. Since our algorithm has time complexity scaling as the linear dimension of the lattice it is essentially optimal.
3. Simulation of D(S₃)

The physical operations to simulate this quantum double model depend on the properties of the symmetric group S₃, which are given in appendix B.

3.1. Ground state preparation

The algorithm *Ground state synthesis* can be built using the tools for single- and two-qudit gates described above. The controlled gauge transformations W(v) are built up by applying Lₜ or Rₜ to the left, right, top or bottom neighbors of the vertex v controlled on v beginning in state |h⟩. Notice that unitaries Lₜ and Rₜ are maximally sparse in the logical basis; hence using the QR decomposition for simulating unitaries, each such operation can be built using a small number of two-qudit diagonal phase phase gates U = e^{iθ|g⟩⟩⟨g|} conjugated by single qudit Givens rotations. Givens rotations are unitaries in SU(2) which act as a generic SU(2) rotation on a 2D subspace and as the identity on the complementary vector space [31]. Since any unitary in SU(d) can be constructed out of O(d²) such rotations by a QR decomposition, the Givens rotations count is a good measure of the complexity of simulating unitary operations. Measurement of vertex ancilla is performed in the Fourier basis |w⟩; hence a one-qudit Fourier transform Fᵥ is needed before measuring in the logical basis. That operator is not sparse but can be constructed with fewer than 13 steps using parallel Givens rotations [32].

3.2. The particle spectrum

The eight irreps for D(S₃) are listed below with their corresponding quantum dimensions:

\[
\begin{align*}
\Pi^{[e]}_{R^e_1}, d &= 1 \quad \text{(vacuum)} \\
\Pi^{[e]}_{R^e_0}, \Pi^{[e]}_{J^e_0}, d &= 2, 3 \quad \text{(pure magnetic charges)} \\
\Pi^{[e]}_{R^e_1}, \Pi^{[e]}_{J^e_2}, d &= 1, 2 \quad \text{(pure electric charges)} \\
\Pi^{[e]}_{J^e_1}, \Pi^{[e]}_{J^e_2}, \Pi^{[e]}_{J^e_3}, d &= 2, 2, 3 \quad \text{(dyonic combinations)}.
\end{align*}
\]

A complete derivation of the fusion rules for this model is given in [34].

Magnetic charges are labelled by conjugacy classes of the group G. Recall that the conjugacy class Cₜ is defined by Cₜ = {ghg⁻¹ | g ∈ G}. For S₃ we label [e] = Cₑ, [c] = Cₑ⁺, [t] = Cₑ⁻. The magnetic flux across a face is given by the ordered product of group elements represented by the basis states of the edges surrounding the face. The order is taken along a closed counterclockwise cycle, beginning at an origin v. Except for the trivial flux e case, the origin from which the product is taken is also important as not labelling the origin is equivalent to only specifying the conjugacy class or magnetic charge. The projector onto states with magnetic flux ℓ at face f as computed taking a connected counterclockwise cycle around f with the origin at vertex v is

\[
B_ℓ(v, f) = \sum_{[\Pi_{e, f} | h = ℓ]} \otimes_{e_1 \in [f]} |h^{-α_{j}(e_1)}⟩_{e_1} ⟨h^{-α_{j}(e_1)}|.
\]

Using the software package available at [33] we find a construction for the operator \(\sum_{h ∈ S₃} |h⟩ ⟨h| \otimes Lₜ\) or \(\sum_{h ∈ S₃} |h⟩ ⟨h| \otimes Rₜ\) in 37 controlled phase gates. For some of the operators, the count may be reduced to eight controlled phase gates using a qutrit–qubit encoding for the group elements (see appendix B).
The projector onto states with magnetic charge \([\ell]\) at face \(f\) is then the sum over all fluxes in the same conjugacy class: \(\sum_{t \in [\ell]} B_t(v, f)\), in which case the specification of origin is unnecessary. The identity element \(e\) is its own conjugacy class; hence we use the convention \(B_e(v, f) \equiv B(f)\).

Electric charges are labelled by irreps of the centralizer of a conjugacy class of \(G\). The centralizer is defined by 
\[ N_h = \{ g \in G | gh = hg \}. \] 
For \(S_3\), we have \(N_{[e]} = S_3, N_{[t]} \cong \{ e, t_0 \} \cong \mathbb{Z}_2, N_{[c_\pm]} \cong \{ e, c_+, c_- \} \cong \mathbb{Z}_3\).

For a finite group \(G\) every representation is equivalent to a unitary representation. Consider projection operators onto subspaces belonging to the unitary irreducible representation, or unirep, \(R\) with matrix representation \(R\) and dimension \(|R|\):
\[
P^R_{\mu,\nu} = \frac{|R|}{|G|} \sum_{g \in G} [R(g)^\ast]_{\mu,\nu} g.
\]
This projection operator satisfies
\[
P^R_{\mu,\nu} P^R_{\kappa,\lambda} = \delta_{R,R'} \delta_{\nu,\kappa} P^R_{\mu,\lambda},
\]
as is verified by taking the inner product over \(G\),
\[
P^R_{\mu,\nu} P^R_{\kappa,\lambda} = \frac{|R||R'|}{|G|^2} \sum_{h_1, h_2 \in G} [R(h_1)^\ast]_{\mu,\nu} [R'(h_2)^\ast]_{\kappa,\lambda} h_1 h_2
\]
\[
= \frac{|R||R'|}{|G|^2} \sum_{h_1, h_2 \in G} [R(h_1)^\ast]_{\mu,\nu} \sum_{\gamma} [R'(h_1^{-1})^\ast]_{\kappa,\gamma} [R'(h_1 h_2)^\ast]_{\gamma,\lambda} h_1 h_2
\]
\[
= \frac{|R||R'|}{|G|^2} \sum_{h' \in G} \sum_{\gamma} \left| \sum_{h_1 \in G} [R(h_1)^\ast]_{\mu,\nu} [R'(h_1)\gamma] \right|^2 [R'(h')^\ast]_{\gamma,\lambda} h'
\]
\[
= \frac{|R||R'|}{|G|^2} \sum_{h' \in G} \sum_{\gamma} \frac{|G|}{|R|} \delta_{R,R'} \delta_{\nu,\kappa} [R(h')^\ast]_{\gamma,\lambda} h'
\]
\[
= \delta_{R,R'} \delta_{\nu,\kappa} P^R_{\mu,\lambda}.
\]
We can obtain a set of \(|R|\) orthonormal basis states \(|R_\lambda\rangle\) that satisfy
\[
P^R_{\mu,\nu} |R_\lambda\rangle = \delta_{R,R'} \delta_{\nu,\lambda} |R_\mu\rangle
\]
by applying the set \(P^R_{\mu,\nu}\) over all \(\mu\) for \(v\) fixed onto some vector \(|\chi\rangle\) with \(P^R_{\nu,\kappa} |\chi\rangle \neq 0\), and normalizing.

For pure electric charges in \(D(S_3)\), we are interested in unireps of \(N_{[e]} = S_3\). There are three unireps: the 1D identity rep \(R^+_1\), the 1D signed rep \(R^-_1\) and the 2D rep \(R_2\). The projection operators are
\[
P^{R_1}_t = \frac{1}{3} (e + t_0 + t_1 + t_2 + c_+ + c_-)\]
\[
P^{R_2}_t = \frac{1}{2} (e + t_0) + \frac{1}{2} (t_1 + t_2 + c_+ + c_-)\]
\[
P^{R_2}_0 = \frac{1}{2} (e + t_0) - \frac{1}{2} (t_1 + t_2 + c_+ + c_-)\]
for the identity irrep, and

\[ P^{R_1}_i = \frac{1}{6} (e - t_0 - t_1 - t_2 + c_+ + c_-) \]  \hspace{1cm} (9)

for the signed irrep, and

\[ P^{R_2}_i = \frac{1}{3} \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} t_0 + \begin{pmatrix} 0 & \xi^* \\ \xi & 0 \end{pmatrix} t_1 + \begin{pmatrix} \xi^* & 0 \\ 0 & \xi \end{pmatrix} t_2 + \begin{pmatrix} \xi & 0 \\ 0 & \xi^* \end{pmatrix} c_+ + \begin{pmatrix} \xi & 0 \\ 0 & \xi^* \end{pmatrix} c_- \right] \]  \hspace{1cm} (10)

for the 2D irrep \( R_2 \) with \( \xi = e^{i\pi/3} \). In the context of the lattice spin model, charge at a vertex \( v \) will correspond to applying the components of these projection operators, with group action being local gauge transformations at vertex \( v \), to some system state \( |\Psi\rangle \)

\[ P^{R_1}_i(v)|\Psi\rangle = \frac{1}{6} [T_c(v) + T_{b_0}(v) + T_{t_1}(v) + T_{c_+}(v) + T_{c_-}(v)]|\Psi\rangle, \]

\[ P^{R_2}_i(v)|\Psi\rangle = \frac{1}{3} [T_c(v) - T_{b_0}(v) - T_{t_1}(v) - T_{c_+}(v) + T_{c_-}(v)]|\Psi\rangle, \]

\[ P^{R_3}_i(v)|\Psi\rangle = \frac{1}{4} [T_c(v) + \xi T_{c_+}(v) + \xi T_{c_-}(v)]|\Psi\rangle, \]

\[ P^{R_4}_i(v)|\Psi\rangle = \frac{1}{4} [T_{b_0}(v) + \xi T_{t_1}(v) + \xi T_{c_+}(v)]|\Psi\rangle, \]

\[ P^{R_5}_i(v)|\Psi\rangle = \frac{1}{4} [T_{b_0}(v) + \xi T_{t_1}(v) + \xi T_{c_-}(v)]|\Psi\rangle. \]  \hspace{1cm} (11)

We obtain two copies of a basis for the 2D irrep \( R_2 \), namely \( \{P^{R_2}_{00}, P^{R_2}_{10}\} \) and the conjugate basis \( \{P^{R_2}_{01}, P^{R_2}_{11}\} \). The system state \( |\Psi\rangle \) here can be interpreted as the state at some intermediate stage of the state synthesis algorithm before a gauge symmetrization at vertex \( v \) has been performed.

For the dyonic combination with flux \([c]\) the charges are labelled by unireps of the \( N_{[c]} \cong \mathbb{Z}_3 \). There are three unireps of \( \mathbb{Z}_3 \): \( R^0_1 \), \( R^1_1 \) and \( R^2_1 \), and since it is an Abelian group, they are all 1D. The projection operators for \( N_{[c]} \) are

\[ P^{R^0_1}_i = \frac{1}{3} (e + c_+, c_-), \]

\[ P^{R^1_1}_i = \frac{1}{3} (e + \xi c_+, \xi^* c_-), \]

\[ P^{R^2_1}_i = \frac{1}{3} (e + \xi^* c_+, \xi c_-). \]  \hspace{1cm} (12)

For the dyonic combination with flux \([t]\) the charges are labelled by unireps of \( N_{[t]} \cong \mathbb{Z}_2 \). There are two 1D unireps of \( \mathbb{Z}_2 \): \( R^3_1 \) and \( R^4_1 \). The projection operators for \( N_{[t]} \) are

\[ P^{R^3_1}_i = \frac{1}{2} (e + t_1), \]

\[ P^{R^4_1}_i = \frac{1}{2} (e - t_1). \]  \hspace{1cm} (13)

In the spin lattice model electric (magnetic) charges correspond to violations of the local vertex (face) constraints and the state of a dyonic particle at vertex and face location \((v, f)\) will be denoted by

\[ |(P^R_{\mu,v}, g); (v, f)\rangle, \]
where $g$ is the flux at face $f$ evaluated taking a counterclockwise cycle with base point at $v$, and $P_{\mu,\nu}^R$ labels an irrep of the centralizer of an element of the conjugacy class $[g]$. For example, the state of magnetic flux $\ell$ particle located at face $f$ and its anti-particle located at face $f'$ (with magnetic flux evaluated with respect to the origin $v$) is

$$|(P_{\mu,\nu}^R, \ell); (v, f)) (P_{\mu,\nu}^R, \ell^{-1}); (v, f')\rangle.$$

For an earlier, formal construction of projectors onto the irreps of quasi-Hopf algebras associated with finite groups see [35].

For our model excitations created in the bulk of the lattice appear as particle anti-particle pairs (such that the total charge of the pair is zero). Although single particle excitations can be made by creating them at the boundary. We will describe digital simulations of braiding of pure charges and pure fluxes.

### 3.3. Anyonic dynamics

Before deriving a sequence of operations to create and move anyons in the spin lattice let us review the rules for braiding charges and fluxes in anyonic models. We will write $|a\rangle$ to represent a magnetic flux of value $a$ and $|a, a^{-1}\rangle$ for a flux anti-flux pair. For electric charge pairs we have one charge that transforms under the irrep $R$ and the anti-charge which transforms under the complex conjugate representation $R^\ast$. We introduce the bases $\{|\mu\rangle_R\}_{\mu=0}^{\lvert R\rvert-1}$, $\{|v\rangle_{R^\ast}\}_{j=0}^{\lvert R\rvert-1}$ on which the representations act and write a generic state of an electric charge anti-charge pair as a $|R| \times |R|$ matrix:

$$|M_R\rangle = \frac{1}{\sqrt{|R|}} \sum_{\mu,\nu} M_{\mu,\nu}^R |\mu\rangle_R \otimes |\nu\rangle_{R^\ast},$$

with the normalization chosen such that $\sum_{\mu,\nu} |M_{\mu,\nu}^R|^2 = |R|$. Interchanging two fluxes, a left flux $a$ and a right flux $b$, in a counterclockwise sense is described by the action of the monodromy operator $R$:

$$R|a\rangle|b\rangle = \sigma|a\rangle|aba^{-1}\rangle = |aba^{-1}\rangle|a\rangle,$$

where $\sigma$ is the particle interchange operator. Squaring the monodromy operator gives the action of braiding two fluxes

$$R^2|a\rangle|b\rangle = |(ab)a(ab^{-1})|aba^{-1}\rangle.$$

(14)

Braiding a flux $b$ around a flux anti-flux pair $(a, a^{-1})$ is equivalent to braiding first around one and then around the other (we can order the particles left to right $(1, 2, 3)$)

$$R_{1,2}^2 \otimes R_{1,3}^2 |b\rangle|a, a^{-1}\rangle = |b\rangle|bab^{-1}, ba^{-1}b^{-1}\rangle.$$

(15)

If $|bab^{-1}, ba^{-1}b^{-1}\rangle \neq |a, a^{-1}\rangle$ then we say that $|a, a^{-1}\rangle$ has magnetic charge. For each conjugacy class $[\ell]$ there is a unique state of a pair of pure flux $[\ell]$ and its conjugate with overall trivial topological charge; we will refer to this as the chargeless state for conjugacy class $[\ell]$:

$$|0_{[\ell]}\rangle = \frac{1}{\sqrt{|[\ell]|}} \sum_{\ell \in [\ell]} |\ell, \ell^{-1}\rangle.$$

(16)
For each irrep $|\mu\rangle$ for the conjugacy class $[\ell]$, charges has an effect. Specifically, if we braid a flux $|h\rangle$ around one electric charge in the pair $|M^R\rangle$, we obtain
\[ R_{1,2}^2|h\rangle|M^R\rangle = |h\rangle|R(h)M^R\rangle \] (17)
and if we braid around the anti-charge we obtain
\[ R_{1,3}^2|h\rangle|M^R\rangle = |h\rangle|M^R R(h^{-1})\rangle, \] (18)
where $R(h)$ is the matrix representation $R$ of the group element $h$. Braiding around both charges is a conjugation
\[ R_{1,2}^2 \otimes R_{1,3}^2|h\rangle|M^R\rangle = |h\rangle|R(h)M^R R(h^{-1})\rangle. \]
For each irrep $R$ there is one unique fluxless state that is invariant under conjugation:
\[ |1_{[R]}\rangle = \frac{1}{\sqrt{|R|}} \sum_{\mu} |\mu\rangle_R \otimes |\mu\rangle_{R^*}. \] (19)

### 3.3.1. Magnetic charges in $D(S_3)$. Consider the creation of the chargeless magnetic flux pair for the conjugacy class $[\ell]$
\[ |0_{[\ell]}; (v_{i+1,j+1}, f_{i,j}), (v_{i+1,j+1}, f_{i,j})\rangle = \frac{1}{\sqrt{|[\ell]|}} \sum_{\ell \in [\ell]} |(P_R^{R^*}, \ell^{-1}); (v_{i+1,j+1}, f_{i,j})\rangle |(P_R^{R^*}, \ell); \]
\[ \times (v_{i+1,j+1}, f_{i,j+1})\rangle. \] (20)
This state can be created by starting in the ground state $|GS\rangle$ and acting on one edge which is a shared boundary of the two faces:
\[ |0_{[\ell]}; (v_{i+1,j+1}, f_{i,j}), (v_{i+1,j+1}, f_{i,j+1})\rangle = \frac{1}{\sqrt{|[\ell]|}} \sum_{\ell \in [\ell]} R_{\ell}(e_{i+1,j+1;i+1,j+1})|GS\rangle. \]
Note that the right multiplication operator $R_{\ell}(e_{i,j+1;i+1,j+1})$ commutes with all vertex operators except $A(v_{i+1,j+1})$. However, the sum of $R_{\ell}$ over all elements of the conjugacy class does commute with it. To reduce clutter here, we write $v = v_{i+1,j+1}$ and $e = e_{i+1,j+1;i+1,j+1}$, then:
\[
\sum_{\ell \in [\ell]} A(v) R_{\ell}(e) A(v) = \frac{1}{|G|^2} \sum_{g,g'} \tilde{T}_g(v) \tilde{T}_{g'}(v) \otimes \left[ \sum_{\ell \in [\ell]} R_{g^{-1}}(e) R_{\ell}(e) R_{g^{-1}}(e) \right]
\[
= \frac{1}{|G|^2} \sum_{g,g'} \tilde{T}_g(v) \tilde{T}_{g'}(v) \otimes \left[ \sum_{\ell \in [\ell]} R_{g^{-1}}(e) R_{g^{-1}}(e) R_{g^{-1}}(e) \right]
\[
= \frac{1}{|G|^2} \sum_{g,g'} \tilde{T}_g(v) \tilde{T}_{g'}(v) \otimes \left[ \sum_{\ell \in [\ell]} R_{\ell}(e) R_{g^{-1}}(e) R_{g^{-1}}(e) \right]
\]

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where \( \tilde{T}_g(v) R_{g^{-1}}(e) \equiv T_g(v) \), since \( A(v) |GS \rangle = |GS \rangle \). Therefore, only the face constraints \( B(f_{i,j}), B(f_{i,j+1}) \) are violated. Starting from the vacuum (ground) state \( |GS \rangle \), this state is created by preparing the ancilla \( f_{i,j} \) in the state \( |0_{(\ell)}\rangle_{f_{i,j}} \), where \( |0_{(\ell)}\rangle = \frac{1}{\sqrt{\ell}} \sum_{\ell \in \{\ell\}} |\ell\rangle \), applying the two-qudit unitary

\[
F_{(\ell)}(f_{i,j}) = 1_{|G| - |(\ell)|} \otimes 1_{|G|} + \sum_{\ell \in \{\ell\}} |\ell\rangle_{f_{i,j}} \langle \ell| \otimes R_{(\ell)}(e_{i,j+1;i,j+1})
\]

and measuring the face ancilla in the basis \( \{ |k_{(\ell)}\rangle = Z_{(|\ell|)} |0_{(\ell)}\rangle \}_{k=0}^{(|\ell|) - 1} \), where \( Z_{(|\ell|)} = \sum_{\ell \in \{\ell\}} e^{2\pi n k m / |(\ell)|} |\ell\rangle \langle \ell|_m \) (where we have labelled the group elements in \( \{\ell\} = \{0, \ldots, |\ell| - 1\} \)). For the outcome \( 0_{(\ell)} \) the target magnetic charge state is created. Otherwise for outcome \( k_{(\ell)} \), we need a correction step. To do this prepare the ancilla \( f_{i,j} \) in the state \( |e\rangle_{f_{i,j}} \). Apply the controlled operation \( \Lambda(v_{i+1;j+1}, f_{i,j}) \) where

\[
\Lambda(v, f) = \sum_{g \in G} B_g(v, f) \otimes L_g(f),
\]

which maps the ancilla \( f \) to state \( |g\rangle_f \) when the flux at \( f \) (with respect to the base point \( v \)) is \( g \). Such a controlled operation can be decomposed into elementary two-qudit controlled rotation operators with each edge \( e_k \) surrounding \( f \) as a control and the ancilla as the target, viz.

\[
\begin{align*}
\Lambda(v_{i+1;j+1}, f_{i,j}) & = \Lambda_{\text{right}}(f_{i,j}) \Lambda_{\text{bottom}}(f_{i,j}) \Lambda_{\text{left}}(f_{i,j}) \Lambda_{\text{top}}(f_{i,j}), \\
\Lambda(v_{i;j+1}, f_{i,j}) & = \Lambda_{\text{top}}(f_{i,j}) \Lambda_{\text{right}}(f_{i,j}) \Lambda_{\text{bottom}}(f_{i,j}) \Lambda_{\text{left}}(f_{i,j}), \\
\Lambda(v_{i,j}, f_{i,j}) & = \Lambda_{\text{left}}(f_{i,j}) \Lambda_{\text{top}}(f_{i,j}) \Lambda_{\text{right}}(f_{i,j}) \Lambda_{\text{bottom}}(f_{i,j}), \\
\Lambda(v_{i,j+1}, f_{i,j}) & = \Lambda_{\text{bottom}}(f_{i,j}) \Lambda_{\text{left}}(f_{i,j}) \Lambda_{\text{top}}(f_{i,j}) \Lambda_{\text{right}}(f_{i,j}),
\end{align*}
\]

where

\[
\begin{align*}
\Lambda_{\text{right}}(f_{i,j}) & = \sum_{g \in G} |g\rangle_{e_{i,j+1;i,j+1}} |g\rangle \otimes L_{g^{-1}}(f_{i,j}), \\
\Lambda_{\text{bottom}}(f_{i,j}) & = \sum_{g \in G} |g\rangle_{e_{i,j;i+1,j}} |g\rangle \otimes L_{g^{-1}}(f_{i,j}), \\
\Lambda_{\text{left}}(f_{i,j}) & = \sum_{g \in G} |g\rangle_{e_{i;i+1,j}} |g\rangle \otimes L_g(f_{i,j}).
\end{align*}
\]
\[ \Lambda_{\text{top}}(f_{i,j}) = \sum_{g \in G} |g\rangle_{e_{i1,j+1},j+1} \langle g| \otimes L_{g^{-1}}(f_{i,j}) \].

The convention here is that one takes the product of controlled left multiplication operations on the face ancilla along a counterclockwise path around \( f \) beginning at vertex \( v \) where the operator applied to \( f \) is \( L_{g^{+1}} \) depending on the orientation of the edge relative to the face. Next apply the single qudit phase gate \( |1\rangle_{[G_{v}] \otimes [\ell_{f}]} \otimes Z_{[\ell_{e}]}(f_{i,j}) \) and finally apply \( \Lambda(f_{i,j})^{-1} \) to disentangle the ancilla from the system.

We now describe in detail how to move magnetic charges from one face \( f \) to an adjacent face \( f' \). Essentially it involves coherently mapping the value of flux at \( f \) to the face ancilla \( f \) and applying a controlled operation on an edge \( e \in \partial f \cup \partial f' \) (the shared boundary of the faces \( f, f' \)). After this controlled operation the face ancilla \( f \) is disentangled from the system by mapping the flux at face \( f' \) to ancilla \( f' \) and performing a controlled operation between ancillae \( f, f' \) and finally reversing the mapping on \( f' \). In this protocol we are careful to demand only single qudit and nearest neighbor two-qudit interactions.

Here is the protocol to move a magnetic flux one face unit to the right:

\[
|(P^{R}_{i}, x); (v_{i+1,j+1}, f_{i,j})\rangle \rightarrow \Lambda_{\text{top}}(f_{i,j}) |(P^{R}_{i}, x); (v_{i+1,j+1}, f_{i,j+1})\rangle.
\]

- Prepare the face ancillae \( f_{i,j}, f_{i,j+1} \) in state \( |e\rangle \)
- Coherently map the state of the flux at face \( f_{i,j} \) to the ancilla \( f_{i,j} \) via \( \Lambda(v_{i+1,j+1}, f_{i,j}) \)
- Apply the controlled unitary \( Y(f_{i,j}, e_{i+1,j+1}, f_{i,j}) \) where we define

\[
Y(f, e) = \begin{cases} 
    \sum_{h \in G} |h\rangle_f \langle h| \otimes R_h(e) & \text{if } o_f(e) = +1 \\
    \sum_{h \in G} |h\rangle_f \langle h| \otimes L_{h^{-1}}(e) & \text{if } o_f(e) = -1 
\end{cases}
\]

The remaining steps disentangle the ancilla from the system.

- Map the flux value at the face \( f_{i,j+1} \) to the face ancilla \( f_{i,j+1} \) by applying \( \Lambda(v_{i+1,j+1}, f_{i,j+1}) \)
- Swap qudits \( e_{i+1,j+1}, f_{i,j} \) and \( f_{i,j+1} \)
- Apply the unitary \( u = \sum_{g \in G} |g\rangle_{f_{i,j+1}} \langle g| \otimes L_{g^{-1}}(f_{i,j}) \)
- Swap qudits \( e_{i+1,j+1}, f_{i,j} \) and \( f_{i,j+1} \)
- Apply \( \Lambda(v_{i+1,j+1}, f_{i,j+1})^{-1} \).

This entire process respects superpositions over flux states and can therefore be used to propagate magnetic charges around the lattice. A simple adaptation allows a magnetic flux to move one face unit in the other directions.

Fusion of a magnetic charge particle–anti-particle pair can be measured by using controlled operations to bring the constituent charges in conjugacy class \( [\ell] \) adjacent to one another at faces \( (f, f') \) with shared edge \( e \) and applying first the operator \( \Lambda(v, f) = \sum_{g \in G} B_g(v, f) \otimes L_g(f) \) followed by the operator \( \sum_{g \in G} g \langle g| \otimes L_{g^{-1}}(e) \) and then measurement of the ancilla \( f \) in the basis \( |g_{[\ell]}\rangle_f \). The probability of obtaining the outcome \( 0_{[\ell]} \) equals the probability for the pair to fuse into the vacuum.

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3.3.2. Electric charges in $D(S_3)$. Consider the following electric charge $R(N(|e|))$ particle at $v_{i,j}$ and its anti-particle pair at $v_{i,j+1}$

$$|(P^R_{\mu,\beta}, e); (v_{i,j}, -)|| (P^{R*}_{v,\beta}, e); (v_{i,j+1}, -)\rangle,$$

which is a labelling of the state by basis states of the irrep $R$ and its conjugate $R^*$. The index $\beta$ labels different copies of higher dimensional irreps and for physical states we will sum over the copies. A generic state of an electric charge–anti-charge pair at vertices $(v, v')$ will then be represented as a matrix

$$|M^R; (v, v')\rangle = \frac{1}{|R|} \sum_{\mu,\nu,v=0}^{[R]-1} M^R_{\mu,\nu} |(P^R_{\mu,\beta}, e); (v, -)|| (P^{R*}_{\nu,\beta}, e); (v', -)\rangle$$

with the normalization chosen so that $\sum_{\mu,\nu,v=0}^{[R]-1} |M^R_{\mu,\nu}|^2 = |R|$. Let us see how such a state arises in this spin lattice model. Recall that charge states are obtained by acting on a state $|\Psi\rangle$ at an intermediate stage of ground state synthesis where the gauge symmetrization has been performed at all vertices except at $v_{i,j}, v_{i,j+1}$. Applying the following joint charge projection operators onto $|\Psi\rangle$ gives

$$P^R_{\mu,\beta}(v_{i,j}) P^{R*}_{v,\beta}(v_{i,j+1}) |\Psi\rangle/\sqrt{|| \cdot ||}$$

$$= \frac{|R|^2}{|G|^2} \sum_{h,k \in G} |\mathcal{R}(h^*)_{\mu,\beta} [\mathcal{R}(h')]_{v,\beta} T_h(v_{i,j}) T_k(v_{i,j+1}) |\Psi\rangle/\sqrt{|| \cdot ||}$$

$$= \frac{|R|^2}{|G|^2} \sum_{h \in G} \sum_{\gamma} \sum_{h' \in G} |\mathcal{R}(hh'^{-1})^*_{\mu,\gamma} [\mathcal{R}(h')^*_{\gamma,\beta} [\mathcal{R}(h')]_{v,\beta} T_{hh'^{-1}}(v_{i,j}, v_{i,j+1}) |\Psi\rangle/\sqrt{|| \cdot ||}$$

$$= \frac{|R|}{|G|} \sum_{h \in G} \sum_{\gamma} |\mathcal{R}(hh'^{-1})^*_{\mu,\gamma} T_{hh'^{-1}}(v_{i,j}, v_{i,j+1}) |\Psi\rangle/\sqrt{|| \cdot ||}$$

$$= \frac{|R|}{|G|} \sum_{h \in G} |\mathcal{R}(hh'^{-1})^*_{\mu,\gamma} T_{hh'^{-1}}(v_{i,j}, v_{i,j+1}) |\Psi\rangle/\sqrt{|| \cdot ||}.$$  

(22)

The operator $T_{hh'^{-1}}(v_{i,j}, v_{i,j+1}) = T_h(v_{i,j}) T_k(v_{i,j+1})$ acts as $L_{h-R_{hh'^{-1}}} e_{i,j,i,j+1}$ on the connecting edge. Before the projections, the system particle $e_{i,j,i,j+1}$ on the edge connecting the vertices is in state $|e\rangle$, while after the local gauge transformations on the boundaries it is in the state $|hh'^{-1}\rangle$. Hence the charge state equation (21) can equivalently be obtained by beginning in the fully gauge invariant ground state $|GS\rangle$ and applying a projection operator that depends on the local state of the connecting edge $e_{i,j,i,j+1}$:

$$|(P^R_{\mu,\beta}, e); (v_{i,j}, -)|| (P^{R*}_{v,\beta}, e); (v_{i,j+1}, -)\rangle = \sum_{g \in G} |g\rangle e_{i,j,i,j+1} \langle g| [\mathcal{R}(g)^*]_{\mu,\gamma} |GS\rangle.$$

(23)

This argument in fact extends to electric charge–anti-charge pairs separated by longer chains. Consider the chain $[v_1, v_2, \ldots, v_k]$ and an initial state $|\Psi\rangle$ which is gauge symmetrized over all
vertices but those in the chain. Applying the projections onto the charge pair at the boundaries and onto charge zero for the vertices \(\{v_2, v_3, \ldots, v_{k-1}\}\), we have

\[
\frac{P_{\mu, \beta}^R(v_1) P_{\mu, \beta}^R(v_k)}{\sqrt{|| \cdot ||}} = \frac{|R|}{|G|} \sum_{h_1, h_k^{-1} \in G} [R(h_1 h_k^{-1})]^*_{\mu, \nu} T_{h_1 h_k^{-1}}(v_1, v_k) \sum_{h_2, \ldots, h_{k-1}} \prod_{j=2}^{k-1} T_{h_j}(v_j) |\Psi\rangle.
\]

The gauge transformations act on a state for the chain as

\[
T_{h_1 h_k^{-1}}(v_1, v_k) \sum_{h_2, \ldots, h_{k-1}} \prod_{j=2}^{k-1} T_{h_j}(v_j) |\ell_1\rangle_{[v_1, v_2]} |\ell_2\rangle_{[v_2, v_3]} \cdots |\ell_{k-1}\rangle_{[v_{k-1}, v_k]} = \sum_{h_2, \ldots, h_{k-1}} [h_1 \ell_1 h_2^{-1}]_{[v_1, v_2]} [h_2 \ell_2 h_3^{-1}]_{[v_2, v_3]} \cdots [h_{k-1} \ell_{k-1} h_k^{-1}]_{[v_{k-1}, v_k]}.
\]

It is then the product of the group elements \(\ell_1 \ell_2 \cdots \ell_{k-1}\) along the edges of the chain that is an invariant under local gauge transformations on the vertices (excluding the boundaries). Notice that if one of the edges had opposite orientation to the chain, say \(e = [v_j, v_{j-1}]\), then the invariant would be \(\ell_1 \cdots \ell_{j-1} \ell_{j}^{-1} \ell_{j+1} \cdots \ell_{k-1}\). Hence we find that an arbitrary state of an electric charge pair along the chain \([v_1, \ldots, v_k]\) spanned by the edges \(\{e_j\}\) is

\[
|(P_{\mu, \beta}^R, e); (v_1, -)\rangle |(P_{\mu, \beta}^R, e); (v_k, -)\rangle = \left[ \sum_{h_1, \ldots, h_{k-1} \in G} \otimes_{j=1}^{k-1} |h_j\rangle e_j \langle h_j| [R\left(\prod_{j=1}^{k-1} h_j^{o(e_j)}\right)]^*_{\mu, \nu} \right] |GS\rangle,
\]

where \(o(e_j) = 1\) if \(e_j = [v_j, v_{j+1}]\) and \(o(e_j) = -1\) if \(e_j = [v_{j+1}, v_j]\).

We now describe how to construct the charge state of equation (21). This state does not violate any face constraints (since the local gauge transformations do not change magnetic flux), but it does violate the vertex constraints \(A(v_{i,j}), A(v_{i,j+1})\). To create this state, first prepare the vertex ancilla \(v_{i,j}\) in state \(|e\rangle_{v_{i,j}}\). Apply the conditional unitary \(K(v_{i,j}, e_{i,j}; i, j+1)\) defined by

\[
K(v, e) = \begin{cases} 
\sum_{g \in G} |g\rangle e \langle g| \otimes R_g(v) & e = [v, *] \\
\sum_{g \in G} |g\rangle e \langle g| \otimes R_{g^{-1}}(v) & e = [* , v]
\end{cases}
\]

and measure ancilla \(v_{i,j}\) in the basis

\[
|R_{\mu, \nu}\rangle = \sqrt{|R|} \sum_{g \in G} [R(g)^*]_{\mu, \nu} |g\rangle.
\]

The orientation of the edge dictates the left or right action taken by the operator \(K\) on the vertex ancilla.

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The above method only works probabilistically with probability $1/|G|$. It is possible to prepare the vacuum electric charge states with unit probability as we now describe. For equation (19), the vacuum electric charge states at the boundaries of the edge $e_{i,j;i,j+1}$ are

$$|1_{[R]}(v_{i,j} ,v_{i,j+1}) \rangle = \frac{1}{|R|} \sum_{\mu, \beta} |(P^R_{\mu, \beta}, e) ; (v_{i,j} , -) \rangle |(P^R_{\mu, \beta}, e) ; (v_{i,j+1} , -) \rangle .$$

(25)

Using equation (23) this can be rewritten as the effect of a single qudit edge operator acting on the ground state,

$$|1_{[R]}(v_{i,j} ,v_{i,j+1}) \rangle = \sum_{\mu \in G} \sum_{g \in G} |g \rangle \left( e_{i,j;i,j+1} \langle g | \sum_{j} (R(g)^+)^{\mu, \mu} |GS \rangle \right) = W_R(e_{i,j;i,j+1}) |GS \rangle ,$$

(26)

where

$$W_R(e) = \sum_{g \in G} |g \rangle \langle g | \chi^i_R(g) .$$

These operators obey the sum rule, $\sum_R |R| W_R = |G| |e \rangle \langle e |$. For $S_3$ we have explicitly

$$W_R^i = \mathbf{1}_6 ,$$

$$W_R^{-i} = |e \rangle \langle e | + |c_+ \rangle \langle c_+ | + |c_- \rangle \langle c_- | - |t_0 \rangle \langle t_0 | - |t_1 \rangle \langle t_1 | - |t_2 \rangle \langle t_2 | ,$$

$$W_R^+ = 2 |e \rangle \langle e | - |c_+ \rangle \langle c_+ | - |c_- \rangle \langle c_- | .$$

(27)

Notice, that for the 1D irreps, the electric charge creation operator is a diagonal unitary acting on the edge. The operator $W_{R_i}$ is not unitary but we can construct it using adaptive measurements. First prepare the vertex ancilla $v_{i,j}$ in the state $|+ \rangle_{v_{i,j}}$ where $|\pm \rangle = (|0 \rangle \pm |1 \rangle) \sqrt{2}$ and apply the controlled unitary operation $|0 \rangle_{v_{i,j}} |0 \rangle \otimes U_0(e_{i,j;i,j+1}) + |1 \rangle_{v_{i,j}} |1 \rangle \otimes U_1(e_{i,j;i,j+1})$, where $U_0 = \text{diag}(1, 1, \xi, \xi^*, 1, 1) \otimes U_1 = \text{diag}(1, -1, \xi^{-1/2}, \xi^{1/2}, \xi^*, -\xi)$ in the basis $|e \rangle, |t_0 \rangle, |t_1 \rangle, |t_2 \rangle, |c_+ \rangle, |c_- \rangle$. Next measure the ancilla in the basis $|\pm \rangle_{v_{i,j}}$. The outcomes $\pm 1$ occur with probability $p_\pm = \frac{1}{4} \mp \frac{1}{4}$. For the outcome $+1$, the operator $W_{R_2}$ is successfully applied and the vacuum charge state is created. Otherwise we must correct. One strategy is to first apply the operator $U_2(e_{i,j;i,j+1})$ where $U_2 = \text{diag}(1, 1, \xi, \xi, \xi, -i)$. This corresponds to a combined action on the system $\text{diag}(0, 1, 1, 1, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) |GS \rangle$. Now, re-prepare the ancilla in the state $|e \rangle_{v_{i,j}}$ and entangle the edge with it by the operation $X = \sum_{j=0}^2 |t_j \rangle \otimes R_j(v_{i,j}) + \mathbf{1}_3 \otimes \mathbf{1}_6$. Next, apply a controlled gauge transformation at vertex $v_{i,j}$ that depends on the state of the ancilla via the operator $Y = \sum_{j=0}^2 |t_j \rangle \otimes T_{j,1}^{-1}(v_{i,j}) + \mathbf{1}_3 \otimes \mathbf{1}_6$. This acts to map the three components of the code wavefunction with edge states in $|t_j \rangle$ to the state $|e \rangle$. Since all these operations preserve the zero flux condition on neighboring faces, the states of the remaining lattice spins are disentangled from the $|t_j \rangle$ components of the ancilla. Finally, we need to disentangle the ancilla from the lattice spins. This can be realized by first applying the single qudit operation on the ancilla $U_3 = (|e \rangle \langle t_0 | + |t_1 | + |t_2 |) \sqrt{3} + |t_1 | + \xi |t_1 | + \xi^* |t_2 |) \sqrt{3} + |t_2 | + \xi^* |t_2 | + |t_0 | + |e \rangle | \otimes \mathbf{1}_2$ and then measuring the ancilla in the Fourier basis $|\tilde{k} \rangle_{v_{i,j}}$. For outcome $k$ apply the correction gate $Z^k(e_{i,j;i,j+1})$. This then realizes $W_{R_3}(e_{i,j;i,j+1}) |GS \rangle$.

One can prepare states with electric charge particle–anti-particle pairs that are separated further than one edge, e.g. $|(P^R_{\mu, 0}, e) ; (v_{i,j} , -) \rangle |(P^R_{v, 0}, e) ; (v_{i+2,j}, -) \rangle$ as follows. Prepare vertex

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ancilla \( v_{i,j} \) in state \( |e\rangle_{v_{i,j}} \) and apply the conditional unitary \( K(v_{i,j}, e_{i,j;i,j+1}) \). Swap the qudits at locations \( v_{i,j} \) and \( e_{i,j;i,j+1} \) and then swap qudits at locations \( e_{i,j;i,j+1} \) and \( v_{i,j+1} \). Apply the conditional unitary \( K(v_{i,j+1}, e_{i,j+1;i,j+2}) \) and measure \( v_{i,j} \) in the basis \( \{|R^{\mu}_{i,j}\rangle\} \). To deterministically prepare the vacuum state \( |I|_{R^{\mu}}; (v_{i,j}, v_{i,j+2}) \), apply the same sequence of swaps but instead of measuring the ancilla, apply the gate \( W_R \) to the ancilla and then invert the sequence of conditional unitaries and swaps. This protocol easily extends to preparing arbitrarily long separated electric charge pairs provided care is taken during the controlled flip operations on the ancilla to keep track of the orientation of the edge for the system control. In the case of the non-unitary gate \( W_{R_{2}} \), another ancilla should be used to perform the operations on the ancilla that carries the information on the accumulated product of group elements along the path between the boundary vertices where the charges will be created.

To move an electric charge from a vertex \( v \) to a neighboring vertex \( v' \) consider that the edges are oriented \([*, v], [v, v']\). Then we need to coherently map the state of the edge \([*, v]\) to the product of the states of \([*, v]\) and \([v, v']\). This is achieved by the following operator:

\[
A(v) \sum_{g, g' \in G} T_g(v') T_g(v) |g\rangle_{[*, v]} \langle g| \otimes |g'\rangle_{[v, v']} \langle g'|.
\]

The last projector \( A(v) \) reinforces the gauge symmetry at \( v \) so that total charge is conserved. More concretely, say, we wish to move an electric charge one unit to the right:

\[
|(P_{\mu, \beta}^{R}, e); (v_{i,j}, -)\rangle \rightarrow |(P_{\mu, \beta}^{R}, e); (v_{i,j+1}, -)\rangle,
\]

then the following protocol will work:

- Prepare the vertex ancillae \( v_{i,j}, v_{i,j+1} \) in state \( |e\rangle \).
- Coherently map the state at edge \( e_{i,j-1;i,j} \) to vertex ancilla \( v_{i,j} \) using the operator \( K^{-1}(v_{i,j}, e_{i,j-1;i,j}) \) and similarly for the other edge \( e_{i,j;i,j+1} \) using \( K^{-1}(v_{i,j+1}, e_{i,j;i,j+1}) \).
- Apply the controlled gauge transformations \( W(v_{i,j}) \) and \( W(v_{i,j+1}) \).
- Disentangle the ancilla \( v_{i,j} \) by applying \( K(v_{i,j}, e_{i,j;i,j+1}) \).
- Prepare the vertex ancilla \( v_{i,j} \) in state \( |0\rangle_{v_{i,j}} \).
- Apply the controlled gauge transformation \( W(v_{i,j}) \).
- Measure vertex ancilla \( v_{i,j} \) in basis \( \{|\tilde{j}\rangle\} \).
- For the outcome \( |\tilde{j}\rangle \) apply the single qudit operation \( Z^j(e_{i,j-1;i,j}) \).
- Disentangle the ancilla \( v_{i,j+1} \) by measuring in the basis \( |R^{\mu}_{i,j}\rangle \).

Moving charges in other directions is straightforward keeping in mind that if the edge orientations are reversed, then the inverse gauge transformations should be applied.

Finally, fusion of an electric charge pair \( |M^{R}; (v, v')\rangle \) is realized by moving the charges until they overlap at one vertex (say, \( v' \)). The outcome of measurement on the ancilla \( v' \) in the transport steps above determines the residual charge. For outcome \( |R^{\mu}_{1}\rangle = |0\rangle \) the charges are perfectly annihilated into the vacuum, otherwise there is some residual charge.

3.3.3. Dyons in \( D(S_3) \). We briefly mention how to create dyonic excitations without discussing how to move them. Recall from equation (5) that there are three dyonic particles in \( D(S_3) \).
The explicit representations of the dyonic particle–anti-particle pairs located at \((v_{i,j}, f_{i,j})\) and \((v_{i,j+1}, f_{i-1,j})\) are

\[
\frac{1}{\sqrt{2}} \sum_{\rho = \pm} |(P_{Ri}^1, c_\rho); (v_{i,j}, f_{i,j})\rangle (P_{Ri}^0, c_\rho^{-1}); (v_{i,j+1}, f_{i-1,j})\rangle,
\]

\[
\frac{1}{\sqrt{3}} \sum_{\rho = \pm} |(P_{Ri}^2, c_\rho); (v_{i,j}, f_{i,j})\rangle (P_{Ri}^2, c_\rho^{-1}); (v_{i,j+1}, f_{i-1,j})\rangle,
\]

\[
\frac{1}{\sqrt{3}} \sum_{j = 0}^2 |(P_{Ri}^3, t_j); (v_{i,j}, f_{i,j})\rangle (P_{Ri}^3, t_j); (v_{i,j+1}, f_{i-1,j})\rangle.
\]

To create these states, first create the vacuum magnetic charge states \([0_{\rho}; (v_{i,j}, f_{i,j}), (v_{i,j+1}, f_{i-1,j})]\) with \([c]\) for the first two and \([t]\) for the third state. Next apply one of the following projection operators:

\[
W_{Ri}^1 = \frac{1}{2} (|e\rangle \langle e| + \xi |c_+\rangle \langle c_+| + \xi^* |c_-\rangle \langle c_-|),
\]

\[
W_{Ri}^2 = W_{Ri}^a,
\]

\[
W_{Ri}^3 = \frac{1}{2} (|e\rangle \langle e| - |t_0\rangle \langle t_0|).
\]

These projections cannot be done unitarily with unit probability but similar to the implementations of \(W_{Ri}\) one can use an adaptive protocol to realize them with unit probability.

3.4. Anyonic interferometry

Let us see how our spin lattice model reproduces these braiding relations given in section 3.3. First consider braiding of fluxes. In section 3.3.1, we described how to create a total magnetic charge zero state like equation (16). One process we could attempt is to create two pairs of total charge zero states, braid a member of one pair around a member of the other, and annihilate.

Say, we begin in the state

\[
|\Psi_{in}\rangle = \frac{1}{\sqrt{2}} \sum_{\rho = \pm} |(P_{Ri}^1, c_\rho); (v_{2,4}, f_{1,3})\rangle (P_{Ri}^0, c_\rho^{-1}); (v_{2,4}, f_{1,4})\rangle
\]

\[
\times \frac{1}{\sqrt{3}} \sum_{j = 0, 1, 2} |(P_{Ri}^1, t_j); (v_{3,3}, f_{2,2})\rangle (P_{Ri}^1, t_j); (v_{3,3}, f_{2,4})\rangle.
\]

If we braid the flux \([c]\) at \(f_{1,3}\) around the flux \([t]\) at \(f_{2,2}\), we obtain the output state

\[
|\Psi_{out}\rangle = \frac{1}{\sqrt{6}} \sum_{j = 0, 1, 2} \sum_{\rho = \pm} |(P_{Ri}^1, c_\rho^{-1}); (v_{2,4}, f_{1,3})\rangle (P_{Ri}^0, c_\rho^{-1}); (v_{2,4}, f_{1,4})\rangle
\]

\[
\times |(P_{Ri}^1, t_{j+\rho}); (v_{3,3}, f_{2,2})\rangle (P_{Ri}^1, t_{j}); (v_{3,3}, f_{2,4})\rangle
\]

using the relations: \(c_{\rho j} c_{-\rho} = t_{j+\rho}\) and \(c_{\rho j} c_{\rho j} c_{-\rho} = c_{\rho j} t_{j+\rho} = c_{\rho} c_{\epsilon_{j+\rho}} = c_{-\rho}\). Now we can attempt to fuse the \([c]\) flux–anti-flux pair by propagating the flux at \((v_{1,3}, f_{1,3})\) one unit to the right. This is done by preparing the ancilla \(f_{1,3}\) in state \(|e\rangle\), mapping the flux to \(f_{1,3}\) by \(\Lambda(v_{2,4}, f_{1,3})\) and applying the controlled unitary \(Y(f_{1,3}, e_{1,4,2,4})\). If no braiding had been done by the \([t]\) flux, then the ancilla would be left in the state \(|\phi^+\rangle_{f_{1,3}}\) where \(|\phi^\pm\rangle = (|c_+\rangle \pm |c_-\rangle)/\sqrt{2}\).

What we find is that the ancilla will not be disentangled from the system because the fusion is
consider that all the edges have the same orientation as the chain, i.e. one simply takes the inverse group element in the projection. The action is then trivial in the spin lattice quantum double model here because all the operations for creating and moving electric charges are diagonal in the logical basis and hence must commute. To obtain this incomplete fusion, we measure $p = (\phi^- | \rho(f_{1,3}) | \phi^-)$. Before braiding, $p = 0$, whereas afterwards $p = 1/2$.

Next we examine the braiding of charges. Braiding of charges around each other acts trivially in the spin lattice quantum double model here because all the operations for creating and moving electric charges are diagonal in the logical basis and hence must commute. To obtain the other relations first consider the effect of applying a local gauge transformation to an electric charge–anti-charge pair separated by a chain of vertices $[v_1, v_2, \ldots, v_k]$. For simplicity we consider that all the edges have the same orientation as the chain, i.e. $e_1 = [v_1, v_2], e_2 = [v_2, v_3], \ldots$, etc, though this is not necessary as, in the case of an edge with opposite orientation to the chain, one simply takes the inverse group element in the projection. The action is

\[
T_h(v_1)(P_{\mu,0}^R, e); (v_1, -))(P_{v,0}^{R*}, e); (v_k, -))
\]

\[
= \frac{|R|}{|G|} T_h(v_1) \left[ \sum_{h_1, \ldots, h_k \in G} |h_1\rangle_{e_1} \langle h_1| \otimes \cdots \otimes |h_k\rangle_{e_k} \langle h_k| [R(h_1 \ldots h_k)^*]_{\mu, v} \right] T_{h^{-1}}(v_1)|GS\rangle
\]

\[
= \frac{|R|}{|G|} \left[ \sum_{h_1, \ldots, h_k \in G} |h_1h_1\rangle_{e_1} \langle h_1h_1| \otimes \cdots \otimes |h_k\rangle_{e_k} \langle h_k| [R(h_1 \ldots h_k)^*]_{\mu, v} \right] |GS\rangle
\]

\[
= \frac{|R|}{|G|} \left[ \sum_{h_1, \ldots, h_k \in G} |h_1\rangle_{e_1} \langle h_1| \otimes \cdots \otimes |h_k\rangle_{e_k} \langle h_k| [R(h^{-1}h_1 \ldots h_k)^*]_{\mu, v} \right] |GS\rangle
\]

\[
= \sum_{h'} \sum_{h_1, \ldots, h_k \in G} [R(h^{-1})^*]_{\mu, h'} (P_{\mu,0}^R, e); (v_1, -))(P_{v,0}^{R*}, e); (v_k, -))
\]

\[
= \sum_{h'} \sum_{h_1, \ldots, h_k \in G} [R(h)_{\mu, h}] (P_{\mu,0}^R, e); (v_1, -))(P_{v,0}^{R*}, e); (v_k, -)),
\]

where we used the gauge invariance of the ground state. Similarly, we can compute the action of a gauge transformation on the anti-charge, and we find for a general electric charge vacuum state:

\[
T_h(v)|M^R; (v, v')\rangle = |R(h)M^R; (v, v')\rangle, \quad T_h(v')|M^R; (v, v')\rangle = |M^R R(h^{-1}); (v, v')\rangle.
\]

But this local gauge transformation $T_h(v)$ can also be viewed as creating a flux–anti-flux pair $h, h^{-1}$ and braiding it in a counterclockwise sense around the electric charge at $v$ followed by annihilation of the flux pair. The resultant action on the charge state reproduces the correct braiding relations for a flux around an electric charge. In particular, given that we can create the vacuum state for the representation $R$: $| \mathbf{1}_R; (v_{i,j}, v_{p,q}) \rangle$, we can create the state $|R(h); (v_{i,j}, v_{p,q})\rangle$, by applying $T_h(v_{i,j})$, which is a product of four single qudit unitaries. How
can we measure the effect of this braiding? One way is to invert the vertex ancilla steps that prepared the electric charge pair. For the state $|1_R; (v, v')\rangle$, the inverted sequence will leave the ancilla disentangled from the system, whereas if the initial state is $|R(h); (v, v')\rangle$ then the ancilla will be left in a mixed state.

To actually extract the value of amplitudes for fusion of charges into the vacuum we can use the vertex ancilla prepared in $|1/\rangle$ where $|\psi_{1/\rangle} = (|e\rangle_v \pm |h\rangle_v)/\sqrt{2}$, and apply the controlled operation $W(v)$ followed by measurement of the ancilla in the basis $|\psi_{1/\rangle}$ with outcome $m = \pm 1$. The outcome distribution satisfies

$$P(m = 1) - P(m = -1) = \Re[\langle 1_R; (v, v')|R(h); (v, v')\rangle] = \frac{\Re[\langle\tr{R(h)}\rangle]}{|R|},$$

which is the real part of the fusion amplitude for $R(h) \rightarrow$ vacuum. Similarly, measuring the ancilla in the basis $|\psi_{/\rangle} = (|e\rangle_v \pm i|h\rangle_v)/\sqrt{2}$ yields the imaginary part of the fusion amplitude for $R(h) \rightarrow$ vacuum.

### 3.5. Elementary operations for quantum computation with anyons

We now show explicitly how to create anyonic states and perform braiding and fusion operations, which are universal for computation. This section follows closely the work of Mochon [25] who proved two important facts: firstly, that by working with magnetic charge anyons alone from non-solvable, non-nilpotent groups, universal quantum computation is possible, and secondly, that for some groups that are solvable but not nilpotent, in particular $S_3$, universal quantum computation is also possible if one includes some operations using electric charges.

The first step is to identify the logical basis. We will encode a qutrit in the three charge $|t\rangle$ magnetic fluxes: $t_0, t_1, t_2$. We will work with pairs of flux with total trivial flux and will adopt the simplified notation for pure magnetic charges located at positions $r$ and $s$ to $|t_j, t_j^{-1}; (r, s)\rangle \equiv |(P^{R_k^+}, t_j); (v, f_j))\rangle$ and for an electric charge pair at positions $r, s$, $|M^{R_k^+}; (r, s)\rangle = |M^{R_k^+}; (v_r, v_s)\rangle$. Since we will be working with magnetic fluxes that may not have zero charge, the base point vertex $v$ is important, but for convenience we suppress it keeping in mind that all fluxes should be valued with respect to some fiducial base point. The $Z$ basis is $\{|t_j, t_j^{-1}\}\rangle$ and the $X$ basis is $\{\tilde{t}_j, \tilde{t}_j^{-1}\}\rangle$ where

$$|\tilde{t_j}, \tilde{t}_j^{-1}; (r, s)\rangle = \frac{1}{\sqrt{3}} \sum_{k=0}^{2} \xi^{-j} |t_k, t_k^{-1}; (r, s)\rangle.$$

The state $|\tilde{0}, \tilde{0}^{-1}\rangle$ is just the vacuum magnetic charge state $|0_{10}\rangle$. The generators of the Pauli group operations on this basis are

$$X(r, s) = \sum_{j=0}^{2} |t_{j+1}, t_{j+1}^{-1}; (r, s)\rangle\langle t_j, t_j^{-1}; (r, s)|, \quad Z(r, s) = \sum_{j=0}^{2} \xi^{j} |t_j, t_j^{-1}; (r, s)\rangle\langle t_j, t_j^{-1}; (r, s)|.$$

Encoded information is always stored in flux–anti-flux pairs, and the braiding operations we employ will always braid the pair together, thus perserving the total zero flux condition (see equation (15)) though there may be a residual electric charge. First we describe how to initialize single qudits in the $X$ or $Z$ basis. The initialization of the state $|\tilde{t}_j, \tilde{t}_j^{-1}\rangle$ is equivalent to

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preparation of the vacuum state described in section 3.3.1 except where we alter the correction step via the face ancilla $f$ accordingly, i.e. we apply $[1_{[G]-[L]} \oplus Z_{[G]}^{k_f^i}](f)$ given a measurement outcome $|k_f^i\rangle_f$. To initialize the state $|t_j, t_j^{-1}; (f_1, f_2)\rangle$, we require a means to perform a projection. The main ingredient necessary is the projection of the qutrits onto the subspace $K^\perp = \text{span}_C \{|t_k, t_k^{-1}, k \neq j\}$. This can be accomplished with the assistance of an ancilla pair in the vacuum 2D electric charge state $|1_{R_2}; (r_1, r_2)\rangle$. Say, the initial state of the qutrit is

$$|\Psi; (1, 2)\rangle = \sum_{j=0}^{2} c_j |t_j, t_j^{-1}; (1, 2)\rangle$$

and we want to project out the $t_0$ component, i.e. we want to project onto $K^\perp$. To do so we braid flux 1 around the ancillary electric charge at $r_2$ creating the correlated state

$$\mathcal{R}_{1, r_2}^2 |\Psi\rangle |1_{R_2}; (r_1, r_2)\rangle = \sum_{j=0}^{2} c_j |t_j, t_j^{-1}; (1, 2)\rangle |R_2(t_j); (r_1, r_2)\rangle.$$

Next we apply the ancilla-assisted local gauge transformation $T_{t_0^{-1}}(r_2)$ so that the joint state is

$$c_0 |t_0, t_0^{-1}; (1, 2)\rangle |1_{R_2}; (r_1, r_2)\rangle + c_1 |t_1, t_1^{-1}; (1, 2)\rangle |R_2(c_+); (r_1, r_2)\rangle + c_2 |t_2, t_2^{-1}; (1, 2)\rangle |R_2(c_-); (r_1, r_2)\rangle.$$

Then we annihilate the electric charge pair and observe whether it is fused into the vacuum (using the vertex ancilla protocol described in section 3.4). The states $|R_2(c_{\pm})\rangle$ always fuse into 1D representations (either the vacuum state or the sign representation each with probability $\frac{1}{2}$). If the charges fuse into the vacuum then we throw the qutrit out, whereas if the outcome is imperfect annihilation then the projection onto the subspace $K^\perp$ is successful. Similarly, to project onto $K^\perp$, we use the same protocol but with the local gauge transformation $T_{t_0^{-1}}(r_2)$. By composition of projections we can prepare any basis state beginning in $|0_{t_0}\rangle$, e.g. $|t_1, t_1^{-1}; (1, 2)\rangle = K^\perp K^\perp |0_{t_1}; (1, 2)/\sqrt{|\perp|}\rangle$.

Let us see how to generate the controlled sum operation $\Sigma((1, 2); (3, 4))$, where

$$\Sigma((r, s); (p, q)) = \sum_{j=0}^{2} |t_j, t_j^{-1}; (r, s)\rangle \langle t_j, t_j^{-1}; (r, s)| \otimes X^j(p, q).$$

To realize this, we need two primitive braiding operations. Firstly, consider the ancilla-assisted operation with the ancilla prepared in $|t_0, t_0^{-1}; (5, 6)\rangle$,

$$\sigma((3, 4); (5, 6)) [\mathcal{R}_{3, 5}^2 \mathcal{R}_{3, 6}^2 |\mathcal{R}_{3, 5}\mathcal{R}_{3, 6}^2 |t_k, t_k^{-1}; (3, 4)|t_0, t_0^{-1}; (5, 6)\rangle$$

$$= \sigma((3, 4); (5, 6)) [\mathcal{R}_{3, 5}^2 \mathcal{R}_{3, 6}^2 |t_k, t_k^{-1}; (3, 4)|X^{2k}|t_0, t_0^{-1}; (5, 6)\rangle$$

$$= \sigma((3, 4); (5, 6)) [t_{2k} |t_{2k}, t_{2k}^{-1}; t_{2k}, t_{2k}^{-1}; (3, 4)|t_{2k}, t_{2k}^{-1}; (5, 6)\rangle$$

$$= |t_{2k}, t_{2k}^{-1}; (3, 4)|t_0, t_0^{-1}; (5, 6)\rangle.$$

The final swap $\sigma$ on the two pairs is just a classical operation since each pair has total charge zero. Secondly, consider the action of braiding one control flux around the target pair

$$\mathcal{R}_{1, 3}^2 \mathcal{R}_{1, 4}^2 |t_j, t_j^{-1}; (1, 2)|t_k, t_k^{-1}; (3, 4)\rangle = |t_j, t_j^{-1}; (1, 2)|t_{2j+2k}, t_{2j+2k}^{-1}; (3, 4)\rangle.$$
The composition of these two operations realizes the inverse controlled-sum gate:

$$|t_j, t_j^{-1}; (1, 2)⟩|t_k, t_k^{-1}; (3, 4)⟩ \rightarrow |t_j, t_j^{-1}; (1, 2)⟩|t_{2j+k}, t_{2j+k}^{-1}; (3, 4)⟩$$

$$= |t_j, t_j^{-1}; (1, 2)⟩|t_{2j+k}, t_{2j+k}^{-1}; (3, 4)⟩$$

$$= [\Sigma((1, 2); (3, 4))]^{-1}|t_j, t_j^{-1}; (1, 2)⟩|t_k, t_k^{-1}; (3, 4)⟩. \quad (36)$$

Now $[\Sigma((1, 2); (3, 4))]^{-1}I^2 = \Sigma((1, 2); (3, 4))$, hence by two applications we obtain the controlled sum gate. Using a controlled-sum gate with the control prepared in state $|t_j, t_j^{-1}; (1, 2)⟩$ allows the implementation of an $X^j$ gate on the target. Imagine that we had an ancillary source prepared in $|\tilde{t}_j, \tilde{t}_j^{-1}; (3, 4)⟩$. Using a controlled-sum gate with the target being this ancilla we can perform the one-qudit operation $Z^j$ on the control, which is verified by considering the action on a complete basis for the control:

$$\Sigma((1, 2); (3, 4))|\tilde{t}_j, \tilde{t}_j^{-1}; (1, 2)⟩|\tilde{t}_j, \tilde{t}_j^{-1}; (3, 4)⟩ = [Z^j|\tilde{t}_j, \tilde{t}_j^{-1}; (1, 2)⟩|\tilde{t}_j, \tilde{t}_j^{-1}; (3, 4)⟩.$$

How do we prepare such an ancilla? We can do this by preparing two ancillary pairs in the state $|\Psi⟩ = |0t⟩; (1, 2)⟩|0, t_0^{-1}; (3, 4)⟩$ and applying the controlled sum gate $\Sigma((3, 4); (1, 2))$. This prepares a maximally correlated state and if we trace over the pair $(1, 2)$, we obtain the maximally mixed state $\rho_{3,4} = \frac{1}{3} \sum_{j=0}^{2} |t_j, t_j^{-1}; (3, 4)⟩⟨t_j, t_j^{-1}; (3, 4)|$. Now using the methods of [25] one can measure $\rho_{1,2}$ in the $X$ basis, which will project the state onto $|\tilde{t}_j, \tilde{t}_j^{-1}; (3, 4)⟩$ with probability $\frac{1}{3}$. This process can be repeated until the desired outcome is obtained. Such a measurement requires only the preparation of many ancilla pairs prepared in $|0t⟩$. Similarly, one can measure in the $Z$ basis with the assistance of many ancilla pairs prepared in $|0, t_0^{-1};$.

Up to now we have shown how to implement the Clifford group operations: controlled-sum, and $X^aZ^b$ as well as preparation in an $X$ or $Z$ eigenstate and measurement in the $X$ and the $Z$ basis. In order to realize universal quantum computation we require the ability to perform a non-Clifford operation such as the three-qutrit Toffoli gate defined by the action:

$$T((1, 2); (3, 4); (5, 6))|t_j, t_j^{-1}; (1, 2)⟩|t_k, t_k^{-1}; (3, 4)⟩|t_\ell, t_\ell^{-1}; (5, 6)⟩$$

$$= |t_j, t_j^{-1}; (1, 2)⟩|t_k, t_k^{-1}; (3, 4)⟩|t_{jk+\ell}, t_{jk+\ell}^{-1}; (5, 6)⟩.$$

Because the group $S_3$ is solvable we cannot implement the Toffoli gate by braiding magnetic charges alone [36]. However, as shown by Mochon [25], we can use allowed braiding operations with electric charges to aid in preparing magic states that facilitate the Toffoli gate. We require two types:

$$|\phi_{M1}; (r_1, r_2); (r_3, r_4); (r_5, r_6)⟩ = \frac{1}{2} \sum_{j,k=0}^{2} |t_j, t_j^{-1}; (r_1, r_2)⟩|t_k, t_k^{-1}; (r_3, r_4)⟩|t_{jk}, t_{jk}^{-1}; (r_5, r_6)⟩,$$

$$|\phi_{M2}; (r_1, r_2); (r_3, r_4)⟩ = \frac{1}{2} \sum_{j,k=0}^{2} \xi^{k_0, \beta_0} |t_j, t_j^{-1}; (r_1, r_2)⟩|t_k, t_k^{-1}; (r_3, r_4)⟩.$$
To realize the Toffoli gate on the initial state of three charge pairs

\[ |\Psi; (1, 2; 3, 4; 5, 6)\rangle = \sum_{a,b,c=0}^2 \beta_{a,b,c} |t_a, t_a^{-1}; (1, 2)\rangle |t_b, t_b^{-1}; (3, 4)\rangle |t_c, t_c^{-1}; (5, 6)\rangle, \]

we append the magic state \( |\phi_{M1}; (r_1, r_2); (r_3, r_4); (r_5, r_6)\rangle \) and apply

\[ [\Sigma((5, 6); (r_5, r_6))]^{-1}\Sigma((r_3, r_4); (3, 4))^{-1}[\Sigma((r_1, r_2); (1, 2))]^{-1} \]

followed by measurement of 1 and 3 in the \( Z \) basis with outcomes \( m_1, m_3 \) and 5 in the \( X \) basis with outcome \( m_5 \) and subsequent correction gates \( X(r_1, r_2)^{m_1}X(r_3, r_4)^{m_3}X(r_5, r_6)^{-m_1m_3} \).

The outcome on the ancilla states is

\[ \sum_{a,b,c=0}^2 \beta_{a,b,c} \xi^{m_{ab}} |t_a, t_a^{-1}; (1, 2)\rangle |t_b, t_b^{-1}; (3, 4)\rangle |t_{ab}^{-1}; (r_5, r_6)\rangle. \]

Finally, we apply \([Z(r_5, r_6)]^{-m_5}[\Sigma((r_3, r_4); (r_5, r_6))]^{m_1}[\Sigma((r_1, r_2); (r_5, r_6))]^{m_3}\) to realize

\[ |\chi\rangle = \sum_{a,b,c=0}^2 \beta_{a,b,c} \xi^{m_{ab}} |t_a, t_a^{-1}; (1, 2)\rangle |t_b, t_b^{-1}; (3, 4)\rangle |t_{ab}^{-1}; (r_5, r_6)\rangle. \]

This set of operations has the action of the Toffoli gate acting on \( |\Psi\rangle \) up to a phase \( \xi^{-m_{ab}} \) on the first two qudits.

The correction gate \( C = \sum_{a,b} \xi^{ab} |t_a, t_a^{-1}; t_b, t_b^{-1}\rangle \langle t_a, t_a^{-1}; t_b, t_b^{-1}| \) is in the Clifford group and one might hope that it could be generated from our set of available operations, e.g. using the techniques in [37]. However, to do so requires the single qudit discrete Fourier transform gate, which is not accessible using braiding operations. Hence we use the second magic state \( |\phi_{M2}; (r_7, r_8; r_{10})\rangle \) and apply the gates \([\Sigma((r_3, r_4); (r_9, r_{10}))]^{m_1}[\Sigma((r_1, r_2); (r_7, r_8))]^{m_3}\) followed by measurement of \( r_7 \) and \( r_9 \) in the \( Z \) basis with outcomes \( m_7, m_9 \). The outcome is

\[ |\chi\rangle \rightarrow \sum_{a,b,c=0}^2 \beta_{a,b,c} \xi^{-m_{ab}+m_{7a}+m_{9b}} |t_a, t_a^{-1}; (1, 2)\rangle |t_b, t_b^{-1}; (3, 4)\rangle |t_{ab}^{-1}; (r_5, r_6)\rangle. \]

We can append another magic state \( |\phi_{M2}; (r_{11}, r_{12}; r_{13}, r_{14})\rangle \) and measure again with outcomes \( m_{11}, m_{13} \). Repeating \( k \) times the total phase accumulated is \( \delta(k, a, b) = \sum_{l=1}^k \delta_{m_{la}, a} \delta_{m_{lb}, b} \). In order to realize the Toffoli gate we demand that for some \( k, \delta(k, a, b) = m_{ab} \mod d^2a, b \), where \( d \) is the qudit dimension (here \( d = 3 \)). Each step of appending a magic state, applying controlled operations and measuring applies a phase on the state. This is akin to picking \( k \) balls with \( d^2 \) ‘colors’, which are labelled by the pair \( (a, b) \), from an independent and identically distributed probability distribution and placing them into \( d^2 \) bins each labelled by \( (a, b) \). We find a satisfying distribution if the number of balls in each bin \( n_{a,b} = m_{ab} \mod d^2a, b \). The probability that after \( k \) trials the condition is not satisfied falls off exponentially with \( k \) (although the expected number of operations can be quite large). The preparation of the magic states follows identically the procedure in [25] with frequent use of the projection onto \( \mathcal{K}^{2^d} \).
4. Physical implementation in atomic spin lattices

An experimental candidate for realization of the ideas presented herein is cold trapped atoms in a 2D optical lattice (with motion quenched in the third dimension). Single lattice site occupancy of atoms prepared in motional ground states can be prepared in a region of an optical lattice using a variety of loading techniques [38]. In our construction, we use an antiferromagnetic array of two types of particles: system particles or A species, and ancillary particles or B species, which will assist in state preparation and generation and dynamical propagation of anyonic excitations. This bipartite division could simply be a spatial labelling of the same type of physical particle or it could correspond to physically different atomic species or subspaces of the same species. The advantage of using different species for system and ancilla particles is that it could assist in addressability of the controlled unitary operations and measurements in our simulation. In figure 1(b), we suggest a suitable 2D lattice architecture for trapping and manipulating the atoms. By tuning the intensities and phase of the trapping fields, particles can be brought together pairwise with left, right, top and bottom neighbors to facilitate controlled gate operations. Single particle coherent control and pairwise entanglement generated by controlled collisional exchange interactions have been experimentally demonstrated with this architecture [39].

The six group elements of $S_3$ can be encoded into the ground electronic hyperfine states of a trapped alkali atom with enough available levels. Candidates include $^{87}$Rb or $^{23}$Na each having eight available ground hyperfine states or $^{133}$Cs having 16 levels. Arbitrary single qudit operations can be done using Raman laser pulses or microwave pulses that connect pairs of hyperfine states and measurement can be done using a cycling transition (see figure 2). The atoms are prepared in an optical lattice with spacing $\lambda/2$ where $\lambda$ is an optical wavelength. At this spacing and for tight lattices, the atoms are well localized and do not interact. In order to perform controlled unitary operations, one strategy is to bring atoms closer together pairwise and turn on a magnetic field that would generate a trap-induced shape resonance on a pair of internal states. Ideally one would like this collision to be stable in the sense that it would be diagonal: $H_{\text{int}} = U |g_j\rangle_A \langle g_j| \otimes |g_j\rangle_B \langle g_j|$. This could be provided for by choosing the collision on maximal angular momentum projection states of the ground hyperfine level, choosing e.g. $|g_j\rangle = |g_j\rangle = |F_{\downarrow}, m_F = -F_{\downarrow}\rangle$. Quantum gates using such diagonal collisional interactions generated via trap-induced state resonances have been proposed in references [40, 41]. There it was shown that a robust controlled phase gate between a pair (A, B) of $^{133}$Cs atoms is generated by collisions between the maximally projected angular momentum pair $|F_{\downarrow}, m_F = F_{\uparrow}\rangle_A |F_{\downarrow}, m_F = F_{\uparrow}\rangle_B$ at a particular lattice well spacing. In order to have high fidelity gates, it is important to work in a parameter regime where there are negligible interactions between other states for two reasons. Firstly, non-maximally projected angular momentum states have elastic collisions that are non-diagonal interactions, making a target two-qudit unitary difficult to achieve, and secondly, inelastic collisions between states in the upper hyperfine manifold $F_{\uparrow}$ can destroy internal state coherences and convert the internal energy into kinetic energy of the dimer.

Another possibility is to use the collision gate suggested in [42]. This would work by first using Raman pulses to map the states $|g_j\rangle_{A,B}$ to the first excited vibrational state of lattice wells A and B. Then adjusting the intensity of the trapping fields such that the well minima are tilted with respect to each other and bringing the wells together (but not overlapping) pairwise and finally removing the bias in the wells. At an appropriate well minima separation, there will be a
Figure 2. The proposed implementation used atoms trapped in an optical lattice. (a) In the 2D retroreflected optical lattice of [46], when the trapping laser field polarization is in plane, the lattice has periodicity $\lambda/2$, whereas if the polarization is out of plane, the periodicity is $\lambda$ with a relative shift of $\lambda/2$ resulting in a tunable array of double well potentials. The upper panel shows the optical lattice with the field having purely in plane polarization. In the lower panel, nearest neighbors along 1D are brought together pairwise by increasing the intensity of the field with out of plane polarization. Left neighbors or right neighbors are joined depending on the sign of phase of the out of plane polarization light. Top and bottom neighboring pairs can be joined by increasing the out of plane polarization intensity and changing the phase of the field component with polarization in plane. We note that this architecture allows for global addressing of the A or B species by virtue of distinguishability in the motional states of the two types of lattice wells. (b) Encoding a $d = 6$ qudit into the ground hyperfine levels of an $^{87}$Rb atom. The logical states are labelled by group elements of $S_3$. In the figure, the states $|c_+\rangle$ and $|t_2\rangle$ are shown coupled by a Raman laser pulse with Rabi frequencies $\Omega_{1,2}$ by the Hamiltonian $H_{AL} = \Omega_{\text{eff}} (e^{i\phi} |c_+\rangle\langle t_2| + e^{-i\phi} |t_2\rangle\langle c_+|)$, where $\Omega_{\text{eff}} = |\Omega_1\Omega_2|/\Delta$ and $\phi = \arg(\Omega_1\Omega_2^*)$. Arbitrary unitaries in the 2D subspace can be generated from this family of Hamiltonians, and given a connected coupling graph between basis states, which is provided for here using polarization and frequency selectivity of Raman pulses, any $U \in SU(d)$ can be generated [31]. Projective measurements can be made by mapping population in each sublevel to $|F_\uparrow = 2, m_F = 2\rangle \rightarrow |F' = 3, m_F = 3\rangle$ transition. The state $|c_-\rangle$ can be used for entangling gates via ground state collisions, described by the interaction $H_{\text{int}} = U |c_-\rangle_A \langle c_-| \otimes |c_-\rangle_B \langle c_-|$ between neighboring atoms A and B. This interaction is diagonal due to total angular momentum and energy conservation. The state $|r\rangle$ is used as an ancillary state to shelve amplitude in the state $|c_-\rangle$ of atoms that we do not want to participate in a controlled interaction.
large tunnelling and large collisional shift acquired on atoms in the first vibrational states, which are nearly degenerate, and near zero tunnelling shift for ground vibrational state atoms, which are shifted in energy by an amount large compared to the collisional energy. Then the wells can be moved apart again and the population in $|g_j\rangle_{A,B}$ mapped back to the vibrational ground state. Whatever entangling gate is used, given the ability to generate a phase on one basis state pair, arbitrary two-qudit gates can be generated [31].

In order to perform the quantum simulation here, some amount of addressability of the atoms will be necessary. This is slightly non-trivial as the atoms are separated by $\lambda/2$ and therefore cannot be directly addressed by optical fields. One solution is to use gradient field spectroscopy wherein a strong local electric field gradient (created e.g. using optical tweezers) is applied in the vicinity of the target atom. This field will also have some crosstalk with neighbors of the target atom but will shift the energy levels strong enough at the target location such that chirped microwave or Raman operations coupling hyperfine states act trivially at any location other than the target [43]. To implement two-qudit gates on a target pair of atoms A and B, we suggest the following strategy: firstly, for all atoms except A or except B, map population in state $|g_j\rangle$ to the spectator internal state $|r\rangle$; secondly, tune the lattice trapping fields to bring A and B together and apply the unitary $U = e^{i\phi|x_g\rangle\langle x_g|\otimes|x_{g'}\rangle\langle x_{g'}|}$ in parallel on all nearest neighbor pairs; thirdly, invert the mapping to return population to $|g_j\rangle$. The invertible mapping on non-target atoms can be done using stimulated Raman rapid adiabatic passage (STIRAP) pulses with shaped intensity profiles as proposed in [44, 45]. For example, using a pair of Raman fields that couple states $|c_-\rangle \rightarrow |r\rangle$ (in the encoding of figure 2) one could make the Raman pair out of standing waves of light with a node at the target atom A or B. For any atom not at a node, there will be an invertible adiabatic mapping $|c_-\rangle \rightarrow |r\rangle$ which works with fidelity approaching unity. This technique is particularly well suited to addressing a line of atoms using a 1D standing wave for the STIRAP pulse. One could also use gradient field spectroscopy to perform the mapping although this may not be of high fidelity due to the crosstalk with neighboring atoms.

5. Conclusions

We have shown how universal quantum computation by anyonic braiding can be demonstrated in an optical lattice using the method introduced in [24], based on the creation and manipulation of anyons via controlled interactions of a code lattice with an ancillary species. In addition, we show how to construct the relevant topological state using the same method in the absence of the background Hamiltonian $H_{TO}$. The system considered is the $D(S_3)$ quantum double lattice model introduced by Kitaev [26]. In all cases, protocols are thoroughly described and the experimental techniques required are discussed in detail.

The experimental realization of this scheme relies on methods already in the literature. For the procedures described, which exhibit optimal scaling, the auxiliary species is distributed in a lattice with the same spacing as the code lattice, so addressability is an issue, but we propose ways to overcome this problem. We believe that these techniques are of sufficient interest to warrant experimental work. In particular, non-Abelian anyonic statistics can be demonstrated with our interferometry protocols.
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Appendix A. The quantum double of a finite group

Given a finite group \( G = \{g_i\} \), the \textit{group algebra} \( \mathbb{C}[G] \) is a complex vector space spanned by basis vectors \( g_i \) (which can be chosen orthonormal to define an inner product). Indeed, the multiplication \( g_i \otimes g_j \mapsto g_i g_j \) inherited from \( G \) makes \( \mathbb{C}[G] \) an algebra. The unit element \( e \) can be viewed as embedding the complex numbers in \( \mathbb{C} \). This allows us to use Drinfel’d’s quantum double construction to define a quasitriangular \textit{multiplication} \( \otimes \) of \( \mathbb{C}[G] \) satisfying

\[
\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad (A.1)
\]

while for \( \mathcal{F}(G) \),

\[
\Delta(P_h) = \sum_{h_1 h_2 = h} P_{h_1} \otimes P_{h_2}, \quad \varepsilon(P_h) = \delta_{h,e}. \quad (A.2)
\]

For a general element,

\[
\Delta(P_h g) = \sum_{h_1 h_2 = h} P_{h_1} g \otimes P_{h_2} g, \quad \varepsilon(P_h g) = \delta_{h,e} 1. \quad (A.3)
\]

(ii) There exists an antilinear mapping \( S : A \to A \), the antipode, satisfying \( m(S \otimes \text{id}) \Delta = m(\text{id} \otimes S) \Delta = \eta \varepsilon \). The antipodes of \( \mathbb{C}[G] \) and \( \mathcal{F}(G) \) are determined, respectively, by \( S(g) = g^{-1} \) and \( S(P_h) = P_{h^{-1}} \).

Moreover, the structures of algebra and coalgebra of \( \mathbb{C}[G] \) and \( \mathcal{F}(G) \) are dual to each other. This allows us to use Drinfel’d’s quantum double construction to define a quasitriangular Hopf algebra structure on \( \mathcal{F}(G) \times \mathbb{C}[G] \) with multiplication \( P_{h g} \otimes P_{h' g'} \mapsto \delta_{h,g h' / g^{-1}} P_h g g' \), unit \( 1 e \), comultiplication \( P_{h g} \mapsto \sum_{h_1 h_2 = h} P_{h_1} g \otimes P_{h_2} g \), counit \( P_h g \mapsto \delta_{h,e} \) and antipode \( P_h g \mapsto P_{g^{-1} h^{-1} g} g^{-1} \). This is precisely the quantum double \( D(G) \) of the finite group \( G \). It is generated as an algebra by the elements \( \{1g | g \in G\} \) that form the electric gauge group \( G \), and \( \{P_g e | g \in G\} \), which are projections onto the set of states with flux \( g \) in the theory.

The representation theory of \( D(G) \) is well known. Each irreducible representation is determined by a conjugacy class \( [\ell] = \{u \ell u^{-1} | u \in G\} \) in \( G \), and an irreducible representation \( R \) of the centralizer of an element \( \ell \in [\ell] \), \( N_{[\ell]} = \{u \in G | u \ell = \ell u\} \). Let us label the different elements of the conjugacy class \([\alpha]\) as

\[
[\alpha] = \{h_1^{[\alpha]}, h_2^{[\alpha]}, \ldots, h_k^{[\alpha]}\}.
\]
The centralizer $N_{[\alpha]} \subseteq G$, will be defined as the centralizer for the first element $h_1^{[\alpha]}$. We can relate the different elements of $[\alpha]$ in terms of the first element $h_1^{[\alpha]}$, via a set of representatives of the equivalence classes of $G/N_{[\alpha]}$

$$X^{[\alpha]} = \{x_{h_1^{[\alpha]}}^{[\alpha]}, x_{h_1^{[\alpha]}}^{[\alpha]}, \ldots, x_{h_2^{[\alpha]}}^{[\alpha]} \mid h_1^{[\alpha]} = x_{h_1^{[\alpha]}}^{[\alpha]} h_1^{[\alpha]} (x_{h_1^{[\alpha]}}^{[\alpha]})^{-1} \},$$

and adopt the convention that $x_{h_1^{[\alpha]}}^{[\alpha]} = e$. The carrier space for the irrep $\Pi^{[\alpha]}_{R(N_{[\alpha]})}$ is

$$V_{R(N_{[\alpha]})}^{[\alpha]} = \{|h_1^{[\alpha]}_i, v_j^R \mid h_1^{[\alpha]}_i \in [\alpha], 0 \leq j \leq |R| - 1 \}.$$ 

The action of the irrep of an element of the Hopf algebra on this space is

$$\Pi^{[\alpha]}_{R(N_{[\alpha]})}(P_h g)|h_1^{[\alpha]}_i, v_j^R \rangle = \delta_{h_1^{[\alpha]}_i h_1^{[\alpha]} g^{-1}} \sum_m |g h_1^{[\alpha]}_i g^{-1} h_1^{[\alpha]}_m, R(g)_{m,j} v_m^R \rangle.$$ 

The element $\tilde{g} = (x_{h_1^{[\alpha]}}^{[\alpha]} g^{-1})^{-1} g x_{h_1^{[\alpha]}}^{[\alpha]} g$ is constructed from the gauge transformation $g$ and the flux $h_1^{[\alpha]}$ to satisfy $[\tilde{g}, h_1^{[\alpha]}] = 0$ (verified using the fact that $(x_{h_1^{[\alpha]}}^{[\alpha]} g^{-1})^{-1} g h_1^{[\alpha]} g^{-1} x_{h_1^{[\alpha]}}^{[\alpha]} g^{-1} = h_1^{[\alpha]}$) implying that $\tilde{g} \in N_{[\alpha]}$. In this way, we see that the action of the irrep $\Pi^{[\alpha]}_{R(N_{[\alpha]})}(P_h \otimes g)$ on the carrier space is to perform a gauge transformation on the centralizer charge followed by a projection onto the flux conjugated by $g$. The dimension of the carrier space, also known as the quantum dimension of the particle $\Pi^{[\alpha]}_{R(N_{[\alpha]})}$, is

$$d_{R(N_{[\alpha]})}^{[\alpha]} = ||[\alpha]|| R(N_{[\alpha]}).$$

The quantum dimension satisfies the sum rule

$$\sum_{[\alpha]} (d_{R(N_{[\alpha]})}^{[\alpha]})^2 = \sum_{[\alpha]} ||[\alpha]||^2 \sum_R R(N_{[\alpha]})^2 = \sum_{[\alpha]} ||[\alpha]||^2 |N_{[\alpha]}| = \sum_{[\alpha]} ||[\alpha]|| |G| = |G|^2.$$ 

There is some arbitrariness in the choice of representative for the equivalence classes and the ordering of the elements therein; however, different choices lead to unitarily equivalent representations of the quantum double.

The operation of braiding two particles is generated by the monodromy $\mathcal{R}$, which effects a counterclockwise interchange of two particles:

$$\mathcal{R} = \sigma \circ \Pi^{[\alpha]}_{R(N_{[\alpha]})} \otimes \Pi^{[\alpha]'}_{R(N_{[\alpha]'})} \left( \sum_{h, g \in G} P_g \otimes P_h g \right).$$

Here $\sigma$ is the particle interchange operator and the operator in parentheses is the universal $R$-matrix, an element of $D(G) \times D(G)$ describing a gauge transformation on the second particle by the flux of the first. The explicit action on the two-particle state space $V_{R(N_{[\alpha]})}^{[\alpha]} \otimes V_{R(N_{[\alpha]})}^{[\alpha]'}$ is

$$\mathcal{R}|h_1^{[\alpha]}_i, v_j^R \rangle|h_1^{[\alpha]'}_m, v_j^R \rangle = \sum_{\ell} |h_1^{[\alpha]}_i h_1^{[\alpha]'}_m (\tilde{h}_1^{[\alpha]}_i)^{-1}, R' (\tilde{h}_1^{[\alpha]}_i)_{\ell,n} v^R_{\ell} \rangle|h_1^{[\alpha]'}_m, v_j^R \rangle \text{,}$$

where $\tilde{h}_1^{[\alpha]} = (x_{h_1^{[\alpha]}}^{[\alpha]} h_1^{[\alpha]'} h_1^{[\alpha]} h_1^{[\alpha]'} h_1^{[\alpha]})^{-1} h_1^{[\alpha]} x_{h_1^{[\alpha]}}^{[\alpha]}$ is defined as above. Frequently, we are interested in the action of a full counterclockwise braid, $\mathcal{R}^2$, of one particle around another. For the case of a pure flux braiding another pure flux:

$$\mathcal{R}^2|h_1^{[\alpha]}_i, 0 \rangle|h_1^{[\alpha]'}_m, 0 \rangle = (h_1^{[\alpha]}_i h_1^{[\alpha]'} h_1^{[\alpha]} (h_1^{[\alpha]} h_1^{[\alpha]'})^{-1}, 0 \rangle |h_1^{[\alpha]'}_m (h_1^{[\alpha]} h_1^{[\alpha]'} h_1^{[\alpha]} (h_1^{[\alpha]} h_1^{[\alpha]'} h_1^{[\alpha]'} h_1^{[\alpha]})^{-1}, 0 \rangle \text{.}$$
and for a pure flux braiding a pure charge:

\[ R^2 |h_i^\alpha, 0\rangle |e, u_R^0\rangle = \sum_\ell |h_i^\alpha, 0\rangle |e, R(h_i^\alpha)_{\ell,n} u_R^\ell\rangle. \]

**Appendix B. Some properties of the group \( S_3 \)**

\( S_3 \) is the group of permutations of three objects, which we label \{0, 1, 2\}, and is the smallest non-Abelian group. Elements of \( S_3 \) are organized in three conjugacy classes, namely,

- identity \( e \)
- transpositions (reflections) \( t_0 = (01), t_1 = (12), t_2 = (20) \)
- three-cycles (rotations) \( c_+ = (012), c_- = (021) \).

The multiplication rules for \( S_3 \) are as follows:

\[
\begin{align*}
t_i t_i &= e, & t_i t_k &= c_{\varepsilon_{i,k}} & \text{for } j \neq k, \\
t_i c_\rho &= t_{i+\rho}, & c_\rho t_i &= t_{i-\rho}, \\
c_{\rho} c_{\rho} &= c_{-\rho}, & c_\sigma c_\tau &= e & \text{for } \sigma \neq \tau
\end{align*}
\]

Together with the trivial operations involving \( e \). Here \( \varepsilon_{i,k} = \pm 1 \) is such that \( k = j + \varepsilon_{i,k} \) (modulo 3, as indicated).

In particular \( t_i^{-1} = t_i, c_+^{-1} = c_- \), and conjugation relations are

\[
\begin{align*}
t_i t_i t_i &= t_i, \\
t_j t_k t_j &= t_j, & \text{where all of } i, j, k \text{ are different,} \\
c_\rho t_i c_{-\rho} &= t_{i+\rho}, \\
t_i c_\rho t_i &= c_{-\rho}, \\
c_\rho c_\sigma c_{-\rho} &= c_\sigma.
\end{align*}
\]

The group has three irreducible representations (irreps): the two 1D irreps are the trivial one \( R_1^+ (g) = 1 \), the signature representation

\[ R_1^- (e) = +1 = R_1^- (c_\rho), \quad R_1^- (t_i) = -1, \tag{B.1} \]

and the 2D irrep

\[ R_2 (e) = 1_2, \quad R_2 (t_k) = \sigma^x \exp \left( i \frac{2\pi}{3} k \sigma^z \right), \quad R_2 (c_\rho) = \exp \left( i \rho \frac{2\pi}{3} \sigma^z \right). \]

Explicitly,

\[
\begin{align*}
R_2 (e) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & R_2 (c_+) &= \begin{pmatrix} \xi & 0 \\ 0 & \xi^* \end{pmatrix}, & R_2 (c_-) &= \begin{pmatrix} \xi^* & 0 \\ 0 & \xi \end{pmatrix}, \\
R_2 (t_0) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & R_2 (t_1) &= \begin{pmatrix} 0 & \xi^* \\ \xi & 0 \end{pmatrix}, & R_2 (t_2) &= \begin{pmatrix} 0 & \xi \\ \xi^* & 0 \end{pmatrix},
\end{align*}
\]

where \( \xi = e^{i2\pi/3}, \xi^* = e^{i4\pi/3} \).
The characters $\chi_R(g) = tr[R(g)]$ are all equal to $\pm 1$ for the 1D reps and
\[
\chi_{R_2}(e) = 2, \quad \chi_{R_2}(t_j) = 0, \quad \chi_{R_2}(c_\rho) = -1.
\] (B.2)

The permutation representation for $S_3$ is a set of $6 \times 6$ matrices that faithfully represent group left action on the basis $\{|e\rangle, |t_0\rangle, |t_1\rangle, |t_2\rangle, |c_+\rangle, |c_-\rangle\}$, i.e. $L_h|g\rangle = |hg\rangle$. Similarly, we have unitaries for right multiplication $R_h|g\rangle = |gh\rangle$. The unitary matrices satisfy $[L_h, R_h] = 0$ and are given by

\[
\begin{align*}
L_e &= \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}, &
L_{t_0} &= \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}, &
L_{t_1} &= \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}, \\
L_{t_2} &= \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}, &
L_{c_+} &= \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}, &
L_{c_-} &= \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}, \\
R_e &= \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}, &
R_{t_0} &= \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}, &
R_{t_1} &= \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}, \\
R_{t_2} &= \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}, &
R_{c_+} &= \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}, &
R_{c_-} &= \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}.
\end{align*}
\]

Actually, $g \mapsto L_g$ and $g \mapsto R_{g^{-1}}$ determine the left and right regular representations of $G$, respectively. This is because right multiplication inverts the order of group multiplication $R_g R_{g'} = R_{g'g}$, whereas $\bar{R}_g = R_{g^{-1}}$ defines a representation.

Finally, we point out that the group $S_3$ has a semi-direct product structure, which may be exploited to simplify physical realizations. Recall the definition of the semi-direct product. Suppose that we are given a group $G$ with a normal subgroup $N$, a subgroup $H$, and the property that any $g \in G$ can be written $g = nh$ for $n \in N$ and $h \in H$. Let $\phi$ be the homomorphism $\phi : H \to Aut(N)$, where $\phi(h)(n) = hnh^{-1}$. Then $G$ is isomorphic to the semi-direct product $N \rtimes_{\phi} H$ and the isomorphism identifies the product $nh \in G$ with the pair $(n, h) \in N \times H$. We have $S_3 \cong \mathbb{Z}_3 \rtimes_{\phi} \mathbb{Z}_2 = \langle a, b | a^3 = e, b^2 = e, b^{-1}ab = a^{-2} \rangle$. Here the homomorphism is specified by

\[
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\]
\[ \phi_h(a) = bab^{-1} = a^2. \] Using the notation above we can choose \( \mathbb{Z}_3 = \{ e, c_+, c_- \} \) and \( \mathbb{Z}_2 = \{ e, t_0 \} \), and any element \( g \in S_3 \) can be written \( g = c_i^r t_0^s \) for \( r \in \{ 0, 1, 2 \}, s \in \{ 0, 1 \} \). Introducing the basis for group elements \( \{|r\rangle |s\rangle = |c_i^r t_0^s\rangle \} \), i.e. a product basis for a qutrit and a qubit, we have compact representation of the left and right action operators:

\[
\begin{align*}
L_e &= 1_3 \otimes 1_2, & L_{t_0} &= F(1, 2) \otimes \sigma^x, & L_{t_1} &= F(0, 2) \otimes \sigma^x, & L_{t_2} &= F(0, 1) \otimes \sigma^x, \\
L_{c_+} &= X^{-1} \otimes 1_2, & L_{c_-} &= X \otimes 1_2, & R_e &= 1_3 \otimes 1_2, & R_{t_0} &= 1_3 \otimes \sigma^x, \\
R_{t_1} &= X^{-1} \otimes \sigma^- + X \otimes \sigma^+, & R_{t_2} &= X^{-1} \otimes \sigma^+ + X \otimes \sigma^-, \\
R_{c_+} &= X \otimes |0\rangle \langle 0| + X^{-1} \otimes |1\rangle \langle 1|, & R_{c_-} &= X^{-1} \otimes |0\rangle \langle 0| + X \otimes |1\rangle \langle 1|,
\end{align*}
\]

where \( F(i, j) = (|i\rangle \langle j| + |j\rangle \langle i|) \oplus 1 \) flips two basis states of the qutrit. The electric charge creation operators of equation (27) assume a particularly simple form:

\[
W_{K_i} = 1_3 \otimes \sigma^z \quad \text{and} \quad W_{K_i} = \text{diag}(2, -1, -1) \otimes |0\rangle \langle 0|, \quad \text{as do the dyonic projectors:} \quad W_{K_i}^+ = \text{diag}(1, \frac{\xi}{3}, \frac{\xi^*}{3}) \otimes |0\rangle \langle 0|, \quad W_{K_i}^- = W_{K_i}^*, \quad W_{K_i}^a = |0\rangle \langle 0| \otimes \sigma^z.
\]

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