Induced magnetic moments in three-dimensional gauge theories with external magnetic fields

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Abstract

We study the appearance of induced parity-violating magnetic moment, in the presence of external magnetic fields, for even-number of fermion species coupled to dynamical fields in three dimensions. Specifically, we use a $SU(2) \times U(1)$ gauge model for dynamical gauge symmetry breaking, which has also been proposed recently as a field theoretical model for high-$T_c$ superconductors. By decomposing the fermionic degrees of freedom in terms of Landau levels, we show that, in the effective theory with the lowest Landau levels, a parity-violating magnetic moment interaction is induced by the higher Landau levels when the fermions are massive. The possible relevance of this result for a recently observed phenomenon in high-$T_c$ superconductors is also discussed.

I. INTRODUCTION

The generation of particle masses via dynamical symmetry has been studied in particle physics over three decades since the pioneering work of Nambu and Jona-Lasinio [1]. Apart from the realm of particle physics, this mechanism is also responsible for generating the mass gaps in condensed-matter systems like the BCS-type superconductors. Recently, it was found [2–5] that an external magnetic field can enhance such symmetry breaking, and specifically it leads to dynamical chiral symmetry breaking in QED [3]. The dynamically generated fermion mass then depends on the value of the external field. One can establish this fact by examining the two point function fermions in the presence of the external magnetic field [6]. However, due to the presence of dynamical QED interactions (apart

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from the external magnetic field), these two point functions are required to satisfy the Schwinger-Dyson (SD) equations. For strong magnetic fields, the effective fermionic degrees of freedom are the ones in the lowest Landau levels (LLs), as the energy gap between the Landau levels (LLs) becomes very large. The condensate is then obtained by considering the coincidence limit of the two-point function for these LLL fermions, in the presence of the external magnetic field.

In the context of four-dimensional QED, Hong has computed the Wilsonian effective action involving these LLL fermions by integrating out the higher Landau levels. The resulting action contains four-fermi interactions apart from the usual minimal coupling terms. These extra interactions are responsible for the appearance of the non-vanishing condensate for the LLL fermions. Though these four-fermi operators appear irrelevant (in a renormalization-group sense) via naive power counting, they indeed have non-trivial anomalous-dimensions which depend on the magnitude of the external magnetic field. Thus, above a critical value of the field, they become relevant operators and generate the chiral symmetry breaking. The situation is quite similar to the case of three-dimensional multicolor four-fermi theories, where naive power counting arguments contradict the relevance of the interactions, which can be established after a large-$N$ analysis.

The above-mentioned mass generation (at zero-temperature) also occurs in (2+1) dimensional QED as well as in the non-Abelian theory. This result can be relevant for high-temperature superconductors as effective field theories, like QED$_3$ (and variants of it) and non-Abelian gauge theory based on the group $SU(2) \times U(1)$, can demonstrate superconductivity. Indeed, there is experimental evidence for the opening of a second (superconducting) gap at the nodes of the Fermi surfaces in certain d-wave superconductors in the presence of strong external magnetic fields. In a rudimentary finite-temperature analysis was performed and it was found that the (magnetically induced) condensate disappears at a critical temperature, $T_c$ which scaled with the external magnetic field $B$ as,

$$T_c \sim \alpha \ln \left| \frac{\sqrt{eB}}{\alpha} \right|,$$

with $\alpha$ being the ‘fine structure’ constant, with dimensions of ‘mass’, for the $U(1)$ gauge group responsible for the generation of a mass gap for the charge-carrying fermions, which is enhanced by the presence of $B$. Note that this $U(1)$ gauge theory is not necessarily to be identified with electromagnetism. In $e$ is the electric charge (kept four-dimensional), and $\frac{1}{\sqrt{eB}}$ is the ‘magnetic length’.

Though there is no conclusive evidence for parity and time-reversal violation in high-temperature superconductors (see, however for a recent observation for time-reversal violation), Laughlin has suggested that during the abovementioned phase transition, the order parameter develops a P and T-violating state component, induced by the magnetic field. Therefore, it is interesting to ask whether the effective field theories mentioned above

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1 The relativistic (Dirac) nature of the fermion fields is justified by the fact that they describe the excitations about the nodes of a d-wave superconducting gap.
also lead to parity-violation in the presence of the external magnetic field. At first sight, the Vafa-Witten theorem \[17\] prevents one from obtaining such results, given that parity, as being a vector-like symmetry, cannot be broken in a vector-like theory, such as QED. Indeed, we note that the fermion condensates obtained in \[3,13\] are all parity-conserving. This is due to the fact that the effective Lagrangian for the LLL’s obtained in \[7\] involve parity-invariant terms only. However, the presence of these four-fermi terms violates one of the assumptions of the Vafa-Witten theorem which requires the effective action to be quadratic in fermion fields. Therefore, a possibility of parity-violation cannot be ruled out \textit{a priori} due to the presence of the fermi interactions.

A naive dimensional reduction of the 4-dimensional effective Lagrangian of \[7\] leads to a (2+1) dimensional theory with even number of fermion flavors and it does not lead to a parity violating condensate in (2+1) dimensions as well. The flavor number needs to be even if the theory is to describe high-$T_c$ superconducting system as they have an antiferromagnetic structure. Accordingly, the system comprises of two sublattices and within a spin-charge separation framework \[21\], there will be two species of charged fermion excitations (called holons), one associated with each sublattice \[11,12\].

In this article, however, we shall show that in the presence of the external field, charged fermions in such systems also attain a parity-violating induced magnetic moment, in their massive (i.e. superconducting) phase. This result can be established by studying the three-point vertex corrections depicted in Fig. 1. The internal lines in these graphs include fermions in higher Landau levels. In this work, we compute these one-loop corrections in the $SU(2) \times U_S(1)$ model of \[18\] which is a toy-model for dynamical gauge symmetry breaking \[18\] and has a rich phase structure \[12\] which can be of relevance for superconductors.

Interestingly, graphs similar to Figure 1 lead to magnetic moment for massive neutrinos \[19\]. In the latter situation, the role of the massive gauge bosons is played by the massive $W^\pm$ electroweak bosons while the role of the holon is played by the “massive” neutrino.

Naively, it seems that the induction of a magnetic moment for a spinless fermion invalidates the initial assumption of the spin-charge separation in superconductors, given that these fermions do not carry any spin degrees of freedom. However, in (2+1) dimensions one can define magnetic moment without resorting to a spin degree of freedom \[20\]. This is because, in (2+1)-dimensions, the $2 \times 2$ dimensional $\gamma$ matrices satisfy a $SO(2,1)$ algebra (in Minkowskian space times):

$$[\gamma^\mu, \gamma^\nu] \equiv 2\sigma^{\mu\nu} = -2i\epsilon^{\mu\nu\lambda}\gamma_\lambda.$$ (1.2)

Accordingly, the Pauli magnetic moment term can be written in terms of a (non-minimal) coupling to a current, $J_\lambda = \overline{\Psi}\gamma_\lambda\Psi$, for the Dirac fermion field $\Psi$:

$$\overline{\Psi}\sigma^{\mu\nu}\Psi F_{\mu\nu} = \frac{1}{2}\epsilon^{\lambda\mu\nu}F_{\mu\nu}J_\lambda$$ (1.3)

This implies that an induced magnetic moment interaction for holons in planar superconductors could still be consistent with the idea of spin-charge separation.

The paper is organized as follows. In section 2, we give a brief review of the $SU(2) \times U_S(1)$ model of \[12\], as well as the Dirac algebra in three dimensional spacetime with an even number of fermion flavors. In section 3, we present the induced magnetic moment calculation.
for the massive phase of the model, in which the fermions and some of the non-abelian
gauge bosons acquire (dynamically) non-trivial masses. In the final section, we present
our conclusions and discuss the relevance of our results to condensed-matter systems, in
particular planar high-temperature superconductors.

II. THE $SU(2) \times U(1)$ MODEL

The $SU(2) \times U(1)$ model of [18] is a toy model for dynamical electroweak gauge symmetry
breaking in three dimensions, while in the context of condensed-matter systems, the $SU(2) \times
U(1)$ model of [12] is based on a gauged particle-hole symmetry, via an appropriate extension
of the spin-charge separation [21]. The holons transform as a doublet under the $SU(2)$ (particle-hole) symmetry. In this respect the model is different from other $SU(2) \times U(1)$ spin-
charge separated theories, which are based on either direct gauging of genuine spin rotation
$SU(2)$ symmetries [22], or non-Abelian bosonization techniques [23,24]. The phase diagram
of the model of [12], and the associated symmetry-breaking patterns, are quite different from
these other models, and will be important for our purposes here.

The three-dimensional continuum Lagrangian of the model is given (in Euclidean metric,
which we use hereafter) by [18,13],

$$
\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 - \frac{1}{4}(G_{\mu\nu})^2 + \overline{\Psi} D_\mu \gamma_\mu \Psi - m \overline{\Psi} \Psi \quad (2.1)
$$

where $D_\mu = \partial_\mu - ig_1 a_\mu^S - ig_2 \sigma^a B_{a,\mu}$, and $F_{\mu\nu}, G_{\mu\nu}$ are the corresponding field strengths for
an abelian (‘statistical’) $U(1)$ gauge field $a_\mu^S$ and a non-abelian (‘spin’) $SU(2)$ gauge field
$B_{a,\mu}$, respectively. Due to the antiferromagnetic nature of the condensed matter system the
fermions $\Psi$ are four-component spinors. The presence of the even number of fermion species
allows us to define chiral symmetry and parity in three dimensions - which we discuss below.
The bare mass $m$ term is parity conserving and has been added by hand in the Lagrangian (2.1). In the model of [12,13], this term is generated dynamically via the formation of the
fermion condensate $< \Psi \Psi >$ by the strong $U(1)$ coupling. However, for our purposes, the
details of the dynamical mass generation is not important and hence, it will be sufficient to
include a bare mass term for the holons representing the mass generated by the (strongly
coupled) $U(1)$ interactions in the superconducting phase.

A. Dirac Algebra in $(2+1)$ dimensions

The $\gamma_\mu, \mu = 0, 1, 2$, matrices span the reducible $4 \times 4$ representation of the Dirac algebra
in three dimensions in a fermionic theory with an even number of fermion flavours [25]:

$$
\gamma^0 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & -i\sigma_3 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix}
$$

$$
\gamma^2 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix} \quad (2.2)
$$
where \( \sigma \) are \( 2 \times 2 \) Pauli matrices and the (continuum) space-time is taken to have Euclidean signature.

As well known [25] there exists two \( 4 \times 4 \) matrices which anticommute with \( \gamma_\mu, \mu = 0, 1, 2 \):

\[
\gamma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5 = i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

(2.3)

where the substructures are \( 2 \times 2 \) matrices. These are the generators of the ‘chiral’ symmetry for the massless-fermion theory

\[
\Psi \rightarrow \exp(i\theta \gamma_3)\Psi \\
\Psi \rightarrow \exp(i\omega \gamma_5)\Psi
\]

(2.4)

Note that these transformations do not exist in the fundamental two-component representation of the three-dimensional Dirac algebra, and therefore the above symmetry is valid for theories with even fermion flavours only.

For later convenience we list several useful identities of the Dirac algebra, to be used in this article:

\[
\gamma^\mu \gamma^\nu = -\delta^\mu_\nu - \tau_3 \epsilon^{\mu\nu\lambda} \gamma^\lambda ; \quad \tau_3 \equiv i\gamma_3\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

(2.5)

where repeated indices denote summation, the Greek indices are space time indices, and the latin indices are only spatial indices.

Parity in this formalism is defined as the transformation:

\[
P : \quad \Psi(x^0, x^1, x^2) \rightarrow -i\gamma_3\gamma_1\Psi(x^0, -x^1, x^2)
\]

(2.6)

and it is easy to see that a parity-invariant mass term for \( \Psi \) amounts to masses with opposite signs between the two species [25], while a parity-violating one corresponds to masses of equal signs.

The set of generators

\[
\mathcal{G} = \{1, \gamma_3, \gamma_5, \Delta \equiv i\gamma_3\gamma_5\}
\]

(2.7)

form [13] a global \( U(2) \simeq SU(2) \times U_S(1) \) symmetry. The identity matrix 1 generates the \( U_S(1) \) subgroup, while the other three form the \( SU(2) \) part of the group. The currents corresponding to the above transformations are:

\[
J^\Gamma_\mu = \overline{\Psi}\gamma_\mu \Gamma \Psi \quad \Gamma = \gamma_3, \gamma_5, i\gamma_3\gamma_5
\]

(2.8)

and are conserved in the absence of a fermionic mass term. It can be readily verified that the corresponding charges \( Q_\Gamma \equiv \int d^2x \overline{\Psi}\Gamma \Psi \) lead to an \( SU(2) \) algebra [15].
\[ [Q_3, Q_5] = 2iQ_\Delta \quad [Q_5, Q_\Delta] = 2iQ_3 \]
\[ [Q_\Delta, Q_3] = 2iQ_5 \]  

(2.9)

In the presence of a mass term an anomaly is present

\[ \partial^\mu J^\Gamma_\mu = 2m \overline{\Psi} \Gamma \Psi \]  

(2.10)

while the current corresponding to the generator \( \mathbf{1} \) is \textit{always} conserved, even in the presence of a fermion mass.

The bilinears

\[ \mathcal{A}_1 \equiv \overline{\Psi} \gamma_3 \Psi, \quad \mathcal{A}_2 \equiv \overline{\Psi} \gamma_5 \Psi, \quad \mathcal{A}_3 \equiv \overline{\Psi} \Psi \]
\[ B_{1\mu} \equiv \overline{\Psi} \gamma_\mu \gamma_3 \Psi, \quad B_{2\mu} \equiv \overline{\Psi} \gamma_\mu \gamma_5 \Psi, \quad B_{3\mu} \equiv \overline{\Psi} \gamma_\mu \Delta \Psi, \quad \mu = 0, 1, 2 \]  

(2.11)

transform as \textit{triplets} under \( SU(2) \). The \( SU(2) \) singlets are

\[ \mathcal{A}_4 \equiv \overline{\Psi} \Delta \Psi, \quad B_{4\mu} \equiv \overline{\Psi} \gamma_\mu \Psi \]  

(2.12)

i.e. the singlets are the parity violating mass term, and the four-component fermion number.

We now notice that in the case where the fermion condensate \( \mathcal{A}_3 \) is generated dynamically, energetics prohibits the generation of a parity-violating gauge invariant \( SU(2) \) term \( [17] \), and so a parity-conserving mass term necessarily breaks \( [13] \) the \( SU(2) \) group down to a \( \tau_3 - U(1) \) sector \( [11] \), generated by the \( \sigma_3 \) Pauli matrix in two-component notation. Upon coupling the system to external electromagnetic potentials, this phase with massive-fermions shows \textit{superconductivity}. The superconductivity is strongly type II \( [11, 13] \) as the Meissner penetration depth of external magnetic fields turn out to be very large, \( \int \) and hence the study of the response of the system to the external electromagnetic fields is justified.

III. EXTERNAL MAGNETIC FIELDS AND RADIATIVELY-INDUCED MAGNETIC MOMENT FOR FERMIONS

The point of this section is to study the induced magnetic moment for holons \( \Psi \) in the massive phase, after coupling to an external magnetic field. The magnetic moment can be computed by considering the graphs in Figure 1. The relevant terms in the effective Lagrangian, induced by the vertex correction of figure 1, will have the general form :

\[ \mathcal{L}_{\text{moment}} = \int d^3 x \overline{\Psi} (a + b \tau_3) \sigma^{\mu\nu} \Psi F_{\mu\nu}, \]  

(3.1)

where the coefficients \( a \) and \( b \) are to be computed. Also \( \sigma^{\mu\nu} \equiv \frac{1}{2} [\gamma^\mu, \gamma^\nu] \) but \( \gamma^\mu \) now being in the \( 4 \times 4 \) dimensional (reducible) representation of the 3-dimensional Dirac algebra (due to the even number of fermion species). \( \frac{1}{2} \epsilon^{ij} F_{ij} = B \) represents the external magnetic field.

\[ \int \]

\[ \text{Real high-temperature superconducting oxides are strongly type II superconductors. Therefore, the above theoretical model is of relevance.} \]
applied perpendicular to the spatial plane. Due to the identities (2.5), the magnetic moment interaction (3.1) can be written as:

$$\mathcal{L}_{\text{moment}} = - \int d^3 x \int d^3 x B \overline{\Psi} \gamma^0 (a \tau_3 + b) \Psi$$ \hspace{1cm} (3.2)$$

It can readily be seen that the term $\overline{\Psi} \gamma^0 \tau_3 \Psi$ changes sign under the parity transformation (2.6), whereas the term $\overline{\Psi} \gamma^0 \Psi$ remains invariant under the transformation (2.6).

The computation for the coefficients $a$ and $b$ is lengthy though straightforward. Let us now describe the basic steps of the calculation which basically involves the vertex-function corrections depicted by the Feynman graphs in Figure 1.

In momentum space, the vertex function in figure 1 is:

$$\Gamma^\lambda(k, q) = -3(g_2)^2 \int \frac{d^3 p}{(2\pi)^3} \gamma^\mu \tilde{S}(p) \gamma^\lambda \tilde{S}(p + q) \gamma^\nu \mathcal{D}^{(W)}_{\mu \nu}(k - p),$$ \hspace{1cm} (3.3)$$

where $\mathcal{D}^{(W)}_{\mu \nu}(p)$ is the propagator for the three (hence the factor 3) $SU(2)$ ‘massive’ gauge bosons, appearing in the superconducting phase of the model proposed in [13]. For convenience, we take the form of the massive propagators be -

$$\mathcal{D}^{(W)}_{\mu \nu}(p) = -\frac{\delta_{\mu \nu}}{p^2 + M_W^2},$$ \hspace{1cm} (3.4)$$

with $M_W$ being the mass of the gauge bosons, which is generated dynamically in this model [12]. However, there is also a contribution from the $U_s(1)$ gauge boson, which remain massless in the broken-symmetry phase of the model of [12]. This latter contribution is also given by an expression similar to (3.4) but with the massless propagator for the $U_s(1)$ gauge bosons and with $g_1^2$ replacing $3(g_2)^2$ in (3.4). We will comment on this contribution later.

Using (3.4), the vertex function (3.3) reads:

$$\Gamma^\lambda(k, q) = -3(g_2)^2 \int \frac{d^3 p}{(2\pi)^3} \frac{1}{(k - p)^2 + M_W^2} \mathcal{N}^\lambda(p, p + q),$$ \hspace{1cm} (3.5)$$

where

$$\mathcal{N}^\lambda(p, p + q) \equiv \gamma^\mu \tilde{S}(p) \gamma^\lambda \tilde{S}(p + q) \gamma^\mu.$$ \hspace{1cm} (3.6)$$

The momentum space fermion propagator $\tilde{S}(p)$ in the presence of an external magnetic field [3] can be expanded in terms of Landau levels [26,9]:

$$\tilde{S}(k) = -ie^{-\frac{k^2}{2m}} \sum_{n=0}^{\infty} (-1)^n \frac{D_n(k_0, k)}{k_0^2 + m^2 + 2eBn},$$ \hspace{1cm} (3.7)$$

where

$$D_n(k_0, k) \equiv \left[ (m - k_0 \gamma_0) \left\{ (1 - i \gamma^1 \gamma^2) L_n \left( \frac{2k_0^2}{eB} \right) - (1 + i \gamma^1 \gamma^2) L_{n-1} \left( \frac{2k_0^2}{eB} \right) \right\} + 4 (k \cdot \gamma) L_{n-1}^{1} \left( \frac{2k_0^2}{eB} \right) \right].$$ \hspace{1cm} (3.8)$$
where \( L_n(x) \) are the Laguerre polynomials. For later convenience, we use the following abbreviated form of the fermion propagator

\[
\tilde{S}(k) = -ie^{-\frac{k^2}{2\hbar}}[(m - k_0\gamma_0) \{(1 - i\gamma_1\gamma_2)A_1(k) - (1 + i\gamma_1\gamma_2)A_2(k)\} + 4(k \cdot \gamma)B(k) \quad \text{(3.9)}
\]

where the functions \( A_i \) and \( B \) can be read off from the expressions (3.7,3.8), namely

\[
A_1(p) = \sum_{n=0}^{\infty} \frac{(-1)^n L_n\left(\frac{2p^2}{eB}\right)}{p_0^n + m^2 + 2neB} \quad \text{(3.10)}
\]

\[
A_2(p) = \sum_{n=0}^{\infty} \frac{(-1)^n L_{n-1}\left(\frac{2p^2}{eB}\right)}{p_0^n + m^2 + 2neB} \quad \text{(3.11)}
\]

\[
B(p) = \sum_{n=0}^{\infty} \frac{(-1)^n L_n^1\left(\frac{2p^2}{eB}\right)}{p_0^n + m^2 + 2neB} \quad \text{(3.12)}
\]

The vertex function, obtained after performing the Dirac algebra, is a rather cumbersome expression and among other terms, contains the induced-magnetic moment term. Let us now define the quantity

\[
K(p) \equiv m(A_1(p) - A_2(p)) + i\tau_3p_0(A_1(p) + A_2(p)) \equiv mF(p) + i\tau_3p_0G(p), \quad \text{(3.13)}
\]

where the quantities \( A_i(p) \), \( i = 1, 2 \), \( B(p) \) are as defined in (3.7).

The contribution to \( N^0(p, p') \) from the magnetic moment interaction is then given by

\[
-\tau_3\gamma^0 \epsilon^{ij} \{p'_jK(p')B(p') - p_jB(p)K(p')\} e^{-\frac{(p^2 + p'^2)}{eB}} \equiv -\tau_3\gamma^0 \epsilon^{ij} T_j(p, q) \quad \text{(3.14)}
\]

where \( p' \equiv p + q \). Note that \( T_j(p, q) \) *vanishes* identically for \( p = p' \), i.e. \( q = 0 \).

Therefore, the magnetic moment contribution to the vertex function is given by

\[
\Gamma_{\text{moment}}^i(0, q) = \frac{3(g_3)^2}{(2\pi)^3} \tau_3 \epsilon^{ij} \gamma^0 \int dp_0 \int d^2p \frac{T_j(p, q)}{p_0^2 + p^2 + M_W^2} \quad \text{(3.15)}
\]

For our purposes of dealing with a a low-energy effective action, and a constant magnetic field, it will be sufficient to evaluate \( T_j \) for small momentum transfers \( q_j = p'_j - p_j \to 0 \), and \( q_0 = 0 \). Thus, we retain only the leading order term in the Taylor expansion for \( T_j \) around \( q_j = 0 \) which is linear in \( q_j \). In such a case of soft external photons only the second graph of figure \( \square \) contributes. To this order, one gets:

\[
T_j(p, q) = -[gj p_j]\{\mathcal{B}\partial_0\mathcal{F} - \mathcal{F}\partial_0\mathcal{B}\} + i\tau_3p_0\{\mathcal{B}\partial_0\mathcal{G} - \mathcal{G}\partial_0\mathcal{B}\} + q_j\mathcal{B}\{m\mathcal{F} + i\tau_3p_0\mathcal{G}\} e^{-\frac{2p^2}{eB}}. \quad \text{(3.16)}
\]

In the above expression \( \mathcal{F}, \mathcal{G}, \mathcal{B} \) are functions of \( p \) only.

Now, the matrix \( \tau_3 \) appear as the combination \( \tau_3p_0 \) in the function \( T_j(q, p) \). As the functions \( \mathcal{F}, \mathcal{G}, \mathcal{B} \) are even under transformation \( p_0 \to -p_0 \), the terms proportional to \( \tau_3p_0 \) in (3.16) give zero contributions to the vertex function \( \Gamma^i(0, q) \) after doing the \( p_0 \) integral.
Thus, $b = 0$ in (3.2), i.e. there is no parity-conserving induced magnetic moment. The presence of such terms would violate the Time-Reversal symmetry.

Dropping the terms proportional to $\tau_3$ and using the definitions (3.12) in (3.16) we get,

$$T_j = -m \sum_{k,n=0}^{\infty} \frac{(-1)^{k+n}}{(p_0^2 + m^2 + 2keB)(p_0^2 + m^2 + 2neB)} \left[ - \frac{4p_j(p \cdot q)}{eB} \left\{ L_{k-1} \left( \frac{2p^2}{eB} \right) L_{n-1} \left( \frac{2p^2}{eB} \right) \right\} ight. $$

$$\left. - L_k^{-1} \left( \frac{2p^2}{eB} \right) L_{n-2} \left( \frac{2p^2}{eB} \right) \right] + q_j L_k^{-1} \left( \frac{2p^2}{eB} \right) L_{n-1} \left( \frac{2p^2}{eB} \right) e^{-2p^2/\tau} \right) (3.17)$$

Note that when $m$ is zero $T_j(q,p)$ vanishes identically - even when all Landau levels are considered. This shows that, in the normal phase these fermions (holons) do not carry any induced magnetic moment.

In the dynamical mass generation scenario for the fermions in the model of [12], the mass $M_W$ of the $SU(2)$ gauge bosons (in the broken $SU(2)$ phase) is proportional to the fermion condensate, $u$: $M_W \sim Ku$, where $u = m^2$, and $K$ is a hopping matrix element for the fermions, a dimensionful constant depending on the details of the microscopic lattice model (doping concentration, Heisenberg exchange energies etc). The magnitude of $K$ determines the precise relation between $M_W$ and $m$. In the context of the effective (continuum) gauge field theory, $K$ is proportional to (the square of) the gauge couplings of the $SU(2)$ and $U_S(1)$ sectors (which are of equal magnitude in the model of [12]), and hence its magnitude depends crucially on the region of the phase diagram of the model where the analysis is performed.

To simplify our analysis in this article we shall assume that $M_W >> m$, which may occur for strong enough gauge couplings. This approximation is qualitatively sufficient to demonstrate the induction of a small but non-trivial magnetic moment. A more complete analysis, including loop corrections for the gauge boson propagators, and an extension to the case $M_W \sim m$ or $M_W < m$, will be addressed in the future.

Thus, in the approximation where $M_W$ is very large we can replace $\frac{1}{p_0^2 + p^2 + M_W^2}$ by $\frac{1}{M_W}$ in the gauge propagator and then the expression (3.13) simplifies to

$$\Gamma^i_{\text{moment}}(0,q) \simeq \frac{3}{(2\pi)^3 M_W^2} \tau_3 \epsilon^{ij} \gamma^0 \int dp^0 \int d^2p T_j(p,q) \quad (3.18)$$

where $T_j(p,q)$ is given by (3.17).

The spatial momentum integrals can be performed easily by the use of the identity [27]

$$\int_0^\infty x^{\alpha+n}e^{-x}L^\alpha_m(x)L^\beta_n(x)dx = (-1)^{m+n} \frac{1}{(\alpha + m + 1)(\beta + n + 1)B(\alpha + m - n + 1, n + 1)B(m + 1, \beta + n - m + 1)} \quad (3.19)$$

where $B(p,q)$ are the Euler’s beta function.

Subsequently, only terms with $k = n$ in the double sum of (3.17) survive. Performing the final momentum integral one gets:

$$\Gamma^i_{\text{moment}}(0,q) \simeq \frac{9\tau_3 \epsilon^{ij}q_j \gamma_0 (g_2)^2 m eB}{16\pi^2 M_W^2} \sum_{n=1}^{\infty} \frac{1}{(m^2 + 2neB)^{3/2}} \quad (3.20)$$
Note that the lowest Landau level (i.e. \( n = 0 \)) does not appear in the sum. Thus, the magnetic moment is induced by the higher Landau levels only.

Comparing (3.20) with the contribution for a magnetic moment term in the vertex function

\[
\Gamma^{\text{moment}}_i(0, q) = \mu_B e^{i j q} \gamma_0 \tau_3
\]

we can see that the induced magnetic moment due to the massive SU(2) gauge boson phase, is

\[
\mu_B |W| \approx \frac{9(g_2)^2 m e B}{16\pi^2 M_W^2} \sum_{n=1}^{\infty} \frac{1}{(m^2 + 2neB)^2}
\]

(3.22)

The important result is to notice that in the limit \( B \to 0 \) the interaction (3.20) vanishes, which is a nice consistency check of our approach, while in the limit \( eB >> m^2 \), we can ignore the \( m^2 \) in the denominators of the series in (3.22). Thus, we get an induced magnetic moment,

\[
\mu_B |W| \approx \frac{9(g_2)^2 m}{32\pi^2 M_W^2 \sqrt{2eB}} \zeta \left( \frac{3}{2} \right)
\]

(3.23)

which decreases as \( \frac{1}{\sqrt{eB}} \) with increasing external magnetic field. Therefore it vanishes in the formal limit \( B \to \infty \), thereby making a smooth connection with the results of [13]. However, in practice the limit \( B \to \infty \) is never reached, as is the case of high-temperature superconductors. Indeed, superconductivity is destroyed in that case by strong magnetic fields higher than a critical value of order 20 Tesla. For fields smaller than this critical value, which is the case encountered in the experiments of [14], the contribution (3.23) can be important.

Let us now discuss the magnetic moment induced by the massless \( U_S(1) \) gauge boson interactions. Again, we shall make use of the tree level propagator for the massless gauge boson, which will be sufficient for our purposes. The vertex function is then given, as before by (3.15), but with \( 3(g_2)^2 \) replaced by \( g_1^2 \) and with \( M_W = 0 \):

\[
\tilde{\Gamma}^{\text{moment}}_i(0, q) = \frac{(g_1)^2}{(2\pi)^3} \tau_3 \gamma_0 \int dp^0 \int d^2 p \frac{T_j(p, q)}{p_0^2 + p^2}
\]

(3.24)

where \( T_j \) is given, as before, by (3.16). The momentum integrals in the above expression cannot be computed in a closed form. However, we can attempt to contrast this result with that discussed for the massive boson case. Therefore, assuming that the external magnetic field is strong we truncate the infinite sums over the Landau Level poles entering in the quantities appearing in (3.16) by considering only the lowest and the first excited Landau Level pole in the propagator (3.7). In this case, the function \( T_j(q, p) \) assumes the form

\[
T_j = \frac{m}{p_0^2 + m^2 + 2eB} \left[ \frac{4p_j(p \cdot q)}{eB(p_0^2 + m^2 + 2eB)} - q_j \left\{ \frac{1}{p_0^2 + m^2} - \frac{2p_j^2}{p_0^2 + m^2 + 2eB} \right\} \right],
\]

(3.25)

where, as before, we have dropped the terms proportional to \( \tau_3 \), which vanish after performing the \( p_0 \) integration. Using the expression (3.24) we get the integral
\[ \int_{-\infty}^{\infty} dp_0 \frac{T_j d^2 p}{p_0^2 + \mathbf{p}^2} \]

\[= -2\pi m q_j \int dp_0 \left[ \frac{1}{(p_0^2 + m^2 + 2eB)^2} - \Phi\left(\frac{2p_0^2}{eB}\right) \frac{2p_0^2}{p_0^2 + m^2 + 2eB} - \frac{1}{2(p_0^2 + m^2)} \right] \]

\[= -2\pi m q_j \left[ \frac{\pi}{2(m^2 + 2eB)^2} - \int_{-\infty}^{\infty} dp_0 \frac{\Phi\left(\frac{2p_0^2}{eB}\right)}{p_0^2 + m^2 + 2eB} \left\{ \frac{2p_0^2}{p_0^2 + m^2 + 2eB} - \frac{1}{2(p_0^2 + m^2)} \right\} \right] \quad (3.26) \]

where the function \( \Phi(z) \) is defined as

\[ \Phi(z) \equiv \int_0^\infty \frac{e^{-x}dx}{x + z} \quad (3.27) \]

and is plotted in the figure [2].

Notice that the function \( \Phi(z) \) diverges as \( \ln z \) in the limit \( z \to 0 \). In the context of (3.26) this limit occurs for either \( p_0 \to 0 \) or \( eB \to \infty \). The limit \( eB \to \infty \) yields a zero asymptotic result for the magnetic moment, like the case of the magnetic moment induced by the massive gauge bosons. However, the magnetic moment due to the massless gauge bosons falls faster with increasing \( eB \), vanishing in the limit \( B \to \infty \) at least as \( \frac{1}{eB} \). On the other hand, the limit \( p_0 \to 0 \), yields finite contributions after the \( p_0 \) momentum integration in (3.26).

An accurate estimate of the induced moment cannot be made at this stage due to the approximation used in above calculations, as we have ignored the quantum corrections in the gauge boson propagators due to fermion loops. Nevertheless, the above calculation is sufficient to demonstrate our main point - the inclusion of the higher Landau levels in the loops induces P-violating terms in the effective action as induced magnetic moments.

As we have argued previously in the massless (i.e. normal) phase, the fermions and the gauge bosons are all massless and therefore the induced moment is absent in that phase. In this regard, the above phenomena can be treated as an interesting and verifiable prediction for gauge models of high-temperature superconductors [11,13].

IV. CONCLUSIONS

In this work we have argued how in three-dimensional gauge theories massive fermions acquire a magnetic moment in the presence of (strong) external magnetic fields. The relevant Feynman graphs for the induction of the magnetic moment are shown in figure [4]. It is essential for the phenomenon that the fermion degrees of freedom pertaining to the higher Landau level poles are included as internal lines in the diagrams. Even though we have used the model of [18,12] to demonstrate this phenomenon, this phenomenon seems to be model independent, the only requirement being that the fermion is massive and coupled to gauge fields in the presence of the external magnetic field.

The induction of the magnetic moment seems paradoxical at first sight, given that if one restricts oneself to the LLL case from the beginning, there is no induced magnetic moment. This is similar to the coupling of the neutral pions to photons, which is absent in the minimal-coupling scheme, but is induced via the anomalous fermionic loops.
An important feature of the induced magnetic moment interactions is that they violate parity. However, this does not necessarily imply that the fermion condensate in the presence of this extra interaction violate parity, as suggested recently by Laughlin [16]. One way to check this would be to include this interaction into the Schwinger-Dyson equations and check for a self-consistent solution for the dynamically-generated fermion mass gap, which violates parity.

We have also shown that the induced magnetic moment due to the massive gauge bosons and massless gauge bosons scale differently. Therefore, a magnetic susceptibility measurement on the superconducting samples would establish the relevance of the models [11,12]. In fact, the presence of a non-vanishing magnetic moment at very strong fields will be a strong indication in favour of the existence of these massive gauge-boson excitations.

It is also interesting to note that anyons, which interact via Chern-Simons interactions, also acquire an induced magnetic moment [20,28]. However, the induced magnetic moment discussed here scales with the external magnetic field differently than the induced magnetic moments for anyons: for anyons they scale as the inverse of the applied magnetic field, while the magnetic moments discussed here do not. However, our result for the induced magnetic moment due to the massless gauge bosons has been obtained including only the first excited Landau level. More accurate analyses, including the higher Landau levels are clearly desirable. Given that closed analytic formulas for such a case are probably not possible to obtain, Lattice analyses might be necessary [29]. Also, it would be interesting to see whether any connection between the two theories (in the presence of external magnetic fields) exist. It is also known that such induced magnetic moments lead to short range interaction between charged particles in three spacetime dimensions - leading to exotic bound states [28]. This result might again be relevant for the formation of a possible parity violating fermion condensate, as suggested in [16].

Finally, though our calculations are set in three dimensional space time, nevertheless it is natural to expect the phenomenon of the induced magnetic moment to occur in (3+1) dimensions as well. This could be relevant for the physics of chiral symmetry breaking in the Early Universe. We hope to report on this issue in the near future.

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FIG. 1. Vertex Corrections leading to a magnetic moment interaction. The curly line represents gauge bosons and the wavy line indicate an external static photon (due to the magnetic field). The thin straight lines denote Lowest-Landau-Level fermions, while the thick straight lines represent fermions in higher Landau levels. In the case of soft external photons, $q \to 0$, only the second graph contributes.

FIG. 2. The function $\Phi(z)$, defined in the text, (3.27), plotted vs. $z$. The function diverges as $\ln z$ as $z \to 0$. 

FIG. 2. The function $\Phi(z)$, defined in the text, (3.27), plotted vs. $z$. The function diverges as $\ln z$ as $z \to 0$. 
