Results on independent sets in categorical products of graphs, the ultimate categorical independence ratio and the ultimate categorical independent domination ratio

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Abstract. We show that there are polynomial-time algorithms to compute maximum independent sets in the categorical products of two cographs and two split graphs. The ultimate categorical independence ratio of a graph \(G\) is defined as \(\lim_{k \to \infty} \alpha(G^k)/n^k\). The ultimate categorical independence ratio is polynomial for cographs, permutation graphs, interval graphs, graphs of bounded treewidth and split graphs. When \(G\) is a planar graph of maximal degree three then \(\alpha(G \times K_4)\) is NP-complete. We present a PTAS for the ultimate categorical independence ratio of planar graphs. We present an \(O^*(n^{n/3})\) exact, exponential algorithm for general graphs. We prove that the ultimate categorical independent domination ratio for complete multipartite graphs is zero, except when the graph is complete bipartite with color classes of equal size (in which case it is 1/2).

1 Introduction

Let \(G\) and \(H\) be two graphs. The categorical product also travels under the guise of tensor product, or direct product, or Kronecker product, and even more names have been given to it. It is defined as follows. It is a graph, denoted as \(G \times H\). Its vertices are the ordered pairs \((g, h)\) where \(g \in V(G)\) and \(h \in V(H)\). Two of its vertices, say \((g_1, h_1)\) and \((g_2, h_2)\) are adjacent if
\[
\{ g_1, g_2 \} \in E(G) \text{ and } \{ h_1, h_2 \} \in E(H).
\]

One of the reasons for its popularity is Hedetniemi’s conjecture, which is now more than 40 years old [8,17,19,25].

Conjecture 1. For any two graphs \(G\) and \(H\)
\[
\chi(G \times H) = \min \{ \chi(G), \chi(H) \}.
\]
It is easy to see that the right-hand side is an upperbound. Namely, if $f$ is a vertex coloring of $G$ then one can color $G \times H$ by defining a coloring $f'$ as follows

$$f'((g, h)) = f(g), \quad \text{for all } g \in V(G) \text{ and } h \in V(H).$$

Recently, it was shown that the fractional version of Hedetniemi’s conjecture is true [26].

When $G$ and $H$ are perfect then Hedetniemi’s conjecture is true. Namely, let $K$ be a clique of cardinality at most

$$|K| \leq \min \{ \omega(G), \omega(H) \}.$$ 

It is easy to check that $G \times H$ has a clique of cardinality $|K|$. One obtains an ‘elegant’ proof via homomorphisms as follows. By assumption, there exist homomorphisms $K \to G$ and $K \to H$. This implies that there also is a homomorphism $K \to G \times H$ (see, eg, [7,9]). (Actually, if $W$, $P$ and $Q$ are any graphs, then there exist homomorphisms $W \to P$ and $W \to Q$ if and only if there exists a homomorphism $W \to P \times Q$.) In other words [7, Observation 5.1],

$$\omega(G \times H) \geq \min \{ \omega(G), \omega(H) \}.$$ 

Since $G$ and $H$ are perfect, $\omega(G) = \chi(G)$ and $\omega(H) = \chi(H)$. This proves the claim, since

$$\chi(G \times H) \geq \omega(G \times H) \geq \min \{ \omega(G), \omega(H) \}$$

$$= \min \{ \chi(G), \chi(H) \} \geq \chi(G \times H).$$

(1)

Much less is known about the independence number of $G \times H$. It is easy to see that

$$\alpha(G \times H) \geq \max \{ \alpha(G) \cdot |V(H)|, \alpha(H) \cdot |V(G)| \}.$$ 

But this lowerbound can be arbitrarily bad, even for threshold graphs [11]. For any graph $G$ and any natural number $k$ there exists a threshold graph $H$ such that

$$\alpha(G \times H) \geq k + L(G, H),$$

where $L(G, H)$ is the lowerbound expressed in (2). Zhang recently proved that, when $G$ and $H$ are vertex transitive then equality holds in (2) [24]. Notice that, when $G$ is vertex transitive then $G^k$ is also vertex transitive and so, by the “no-homomorphism” lemma of Albertson and Collins, $\alpha(G^k) = \alpha(G) \cdot n^{k-1}$.

Definition 1. A graph is a cograph if it has no induced $P_4$, ie, a path with four vertices.

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3 Here we write $G^k$ for the $k$-fold product $G \times \cdots \times G$. 

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Cographs are characterized by the property that every induced subgraph \( H \) satisfies one of

(a) \( H \) has only one vertex, or  
(b) \( H \) is disconnected, or  
(c) \( \bar{H} \) is disconnected.

It follows that cographs can be represented by a cotree. This is pair \((T, f)\) where 

- \( T \) is a rooted tree and 
- \( f \) is a 1-1 map from the vertices of \( G \) to the leaves of \( T \).

Each internal node of \( T \), including the root, is labeled as \( \otimes \) or \( \oplus \). When the label is \( \oplus \) then the subgraph \( H \), induced by the vertices in the leaves, is disconnected. Each child of the node represents one component. When the node is labeled as \( \otimes \) then the complement of the induced subgraph \( H \) is disconnected. In that case, each component of the complement is represented by one child of the node.

When \( G \) is a cograph then a cotree for \( G \) can be obtained in linear time.

Cographs are perfect, see, eg, [13, Section 3.3]. When \( G \) and \( H \) are cographs then \( G \times H \) is not necessarily perfect. For example, when \( G \) is the paw, ie, \( G \cong K_1 \otimes (K_2 \oplus K_1) \) then \( G \times K_3 \) contains an induced \( C_5 \) [16]. Ravindra and Parthasarathy characterize the pairs \( G \) and \( H \) for which \( G \times H \) is perfect [16, Theorem 3.2].

2 Independence in categorical products of cographs

It is well-known that \( G \times H \) is connected if and only if both \( G \) and \( H \) are connected and at least one of them is not bipartite [22]. When \( G \) and \( H \) are connected and bipartite, then \( G \times H \) consists of two components. In that case, two vertices \((g_1, h_1)\) and \((g_2, h_2)\) belong to the same component if the distances \( d_G(g_1, g_2) \) and \( d_H(h_1, h_2) \) have the same parity.

**Definition 2.** The rook’s graph \( R(m, n) \) is the linegraph of the complete bipartite graph \( K_{m,n} \).

The rook’s graph \( R(m, n) \) has as its vertices the vertices of the grid, \((i, j)\), with \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \). Two vertices are adjacent if they are in the same row or column of the grid. The rook’s graph is perfect, since all linegraphs of bipartite graphs are perfect (see, eg, [13]). By the perfect graph theorem, also the complement of rook’s graph is perfect.

**Proposition 1.** Let \( m, n \in \mathbb{N} \). Then

\[
K_m \times K_n \cong \bar{R},
\]

where \( \bar{R} \) is the complement of the rook’s graph \( R = R(m, n) \).

**Lemma 1.** Let \( G \) and \( H \) be complete multipartite. Then \( G \times H \) is perfect.
Proof. Ravindra and Parthasarathy prove that $G \times H$ is perfect if and only if either

(a) $G$ or $H$ is bipartite, or
(b) Neither $G$ nor $H$ contains an induced odd cycle of length at least 5 nor an induced paw.

Since $G$ and $H$ are perfect, they do not contain an odd hole. Furthermore, the complement of $G$ and $H$ is a union of cliques, and so the complements are $P_3$-free. The complement of a paw is $K_1 \oplus P_3$ and so it has an induced $P_3$. This proves the claim. \qed

Let $G$ and $H$ be complete multipartite. Let $G$ be the join of $m$ independent sets, say with $p_1, \ldots, p_m$ vertices, and let $H$ be the join of $n$ independent sets, say with $q_1, \ldots, q_n$ vertices. We shortly describe how $G \times H$ is obtained from the complement of the rook’s graph $R(m, n)$. We call the structure a generalized rook’s graph.

Each vertex $(i, j)$ in $R(m, n)$ is replaced by an independent set $I(i, j)$ of cardinality $p_i \cdot q_j$. Denote the vertices of this independent set as

$$(i_s, j_t) \quad \text{where } 1 \leq s \leq p_i \text{ and } 1 \leq t \leq q_j.$$ 

Two vertices $(i_s, j_t)$ and $(i'_s, j'_t)$ are adjacent and these types of row- and column-adjacencies are the only adjacencies in this generalized rook’s graph. The graph $G \times H$ is obtained from the partial complement of the generalized rook’s graph.

Lemma 2. Let $G$ and $H$ be complete multipartite graphs. Then

$$\alpha(G \times H) = \kappa(G \times H) = \max \{ \alpha(G) \cdot |V(H)|, \alpha(H) \cdot |V(G)| \}. \quad (3)$$

Proof. Two vertices $(g_1, h_1)$ and $(g_2, h_2)$ are adjacent if $g_1$ and $g_2$ are not in a common independent set in $G$ and $h_1$ and $h_2$ are not in a common independent set in $H$.

Let $\Omega$ be a maximum independent set of $G$. Then

$$\{ (g, h) \mid g \in \Omega \text{ and } h \in V(H) \}$$

is an independent set in $G \times H$. We show that all maximal independent sets are of this form or of the symmetric form with $G$ and $H$ interchanged.

Consider the complement of the rook’s graph. Any independent set must have all its vertices in one row or in one column. This shows that every maximal independent set in $G \times H$ is a generalized row or column in the rook’s graph. Since the graphs are perfect, the number of cliques in a clique cover of $G \times H$ equals $\alpha(G \times H)$.

\qed
Remark 1. Notice that complete multipartite graphs are not vertex transitive, unless all independent sets have the same cardinality.

Proposition 2. Let \( G \) and \( H \) be cographs and assume that \( G \) is disconnected. Say that \( G = G_1 \oplus G_2 \). Then
\[
\alpha(G \times H) = \alpha(G_1 \times H) + \alpha(G_2 \times H).
\]

Lemma 3. Let \( G \) and \( H \) be connected cographs. Say \( G = G_1 \otimes G_2 \) and \( H = H_1 \otimes H_2 \). Then
\[
\alpha(G \times H) = \min \{ \alpha(G_1 \times H), \alpha(G_2 \times H), \alpha(G \times H_1), \alpha(G \times H_2) \}.
\]

Proof. Every vertex of \( V(G_1) \times V(H_1) \) is adjacent to every vertex of \( V(G_2) \times V(H_2) \) and, likewise, every vertex of \( V(G_1) \times V(H_2) \) is adjacent to every vertex of \( V(G_2) \times V(H_1) \). This proves the claim. \( \square \)

Theorem 1. There exists an \( O(n^2) \) algorithm which computes \( \alpha(G \times H) \) when \( G \) and \( H \) are cographs.

Proof. The proof follows easily from Proposition 2 and Lemma 3. \( \square \)

Remark 2. It seems not easy to extend the result of Theorem 1 to higher dimensions. It would be interesting to know whether \( \alpha(G^k) \), for \( k \in \mathbb{N} \), is computable in polynomial time when \( G \) is a cograph. Even for \( k = 3 \) we have no answer.

3 Splitgraphs

Földes and Hammer introduced splitgraphs [4]. We refer to [5, Chapter 6] and [15] for some background information on this class of graphs.

Definition 3. A graph \( G \) is a splitgraph if there is a partition \( \{ S, C \} \) of its vertices such that \( G|C \) is a clique and \( G|S \) is an independent set.

Theorem 2. Let \( G \) and \( H \) be splitgraphs. There exists a polynomial-time algorithm to compute the independence number of \( G \times H \).

Proof. Let \( \{ S_1, C_1 \} \) and \( \{ S_2, C_2 \} \) be the partition of \( V(G) \) and \( V(H) \), respectively, into independent sets and cliques. Let \( c_i = |C_i| \) and \( s_i = |S_i| \) for \( i \in \{ 1, 2 \} \). The vertices of \( C_1 \times C_2 \) form a rook’s graph.

We consider three cases. First consider the maximum independent sets without any vertex of \( V(C_1) \times V(C_2) \). Notice that the subgraph of \( G \times H \) induced by the vertices of
\[
V(S_1) \times V(C_2) \cup V(C_1) \times V(S_2) \cup V(S_1) \times V(S_2)
\]
is bipartite. A maximum independent set in a bipartite graph can be computed in polynomial time.
Consider maximum independent sets that contain exactly one vertex \((c_1, c_2)\) of \(V(C_1) \times V(C_2)\). The maximum independent set of this type can be computed as follows. Consider the bipartite graph of the previous case and remove the neighbors of \((c_1, c_2)\) from this graph. The remaining graph is bipartite. Maximizing over all pairs \((c_1, c_2)\) gives the maximum independent set of this type.

Consider maximum independent sets that contain at least two vertices of the rook's graph \(V(C_1) \times V(C_2)\). Then the two vertices must be in one row or in one column of the grid, since otherwise they are adjacent. Let the vertices of the independent set be contained in row \(c_1 \in V(C_1)\). Then the vertices of \(V(S_1) \times V(C_2)\) of the independent set are contained in

\[
W = \{ (s_1, c_2) \mid s_1 \notin N_G(c_1) \text{ and } c_2 \in C_2 \}.
\]

Consider the bipartite graph with one color class defined as the following set of vertices

\[
\{ (c_1, s_2) \mid c_1 \in C_1 \text{ and } s_2 \in S_2 \} \cup \{ (s_1, s_2) \mid s_1 \in V(S_1) \text{ and } s_2 \in V(S_2) \},
\]

and the other color class defined as

\[
W \cup \{ (c_1, c_2) \mid c_2 \in C_2 \}.
\]

Since this graph is bipartite, the maximum independent set of this type can be computed in polynomial time by maximizing over the rows \(c_1 \in C_1\) and columns \(c_2 \in C_2\).

This proves the theorem. 

\[\square\]

4 Tensor capacity

In this section we consider the powers of a graph under the categorical product.

**Definition 4.** The independence ratio of a graph \(G\) is defined as

\[
r(G) = \frac{\alpha(G)}{|V(G)|}. \tag{4}
\]

For background information on the related Hall-ratio we refer to [18,21].

By (2) for any two graphs \(G\) and \(H\) we have

\[
r(G \times H) \geq \max \{ r(G), r(H) \}. \tag{5}
\]

It follows that \(r(G^k)\) is non-decreasing. Also, it is bounded from above by 1 and so the limit when \(k \to \infty\) exists. This limit was introduced in [2] as the ‘ultimate categorical independence ratio.’ See also [16,10,14]. For simplicity we call it the tensor capacity of a graph. Alon and Lubetzky, and also Tóth claim that computing the tensor capacity is NP-complete but, unfortunately neither provides a proof [11,14,20].
Definition 5. Let $G$ be a graph. The tensor capacity of $G$ is

$$\Theta^T(G) = \lim_{k \to \infty} r(G^k).$$

(6)

Hahn, Hell and Poljak prove that for the Cartesian product,

$$\frac{1}{\chi(G)} \leq \lim_{k \to \infty} r(\square^k G) \leq \frac{1}{\chi_f(G)},$$

where $\chi_f(G)$ is the fractional chromatic number of $G$ [6]. This shows that it is computable in polynomial time for graphs that satisfy $\omega(G) = \chi(G)$.

Brown et al. [2, Theorem 3.3] obtain the following lower bound for the tensor capacity.

$$\Theta^T(G) \geq a(G) \quad \text{where} \quad a(G) = \max_{I \text{ an independent set}} \frac{|I|}{|I| + |N(I)|}. \quad (7)$$

It is related to the binding number $b(G)$ of the graph $G$. Actually, the binding number is less than 1 if and only if $a(G) > \frac{1}{2}$. In that case, the binding number is realized by an independent set and it is equal to $b(G) = \frac{1 - a(G)}{a(G)}$ [1220]. The binding number is computable in polynomial time [31223]. See also Corollary 1 below.

The following proposition was proved in [2].

Proposition 3. If $r(G) > \frac{1}{2}$ then $\Theta^T(G) = 1$.

Therefore, a better lower bound for $\Theta^T(G)$ is provided by

$$\Theta^T(G) \geq a^*(G) = \begin{cases} a(G) & \text{if } a(G) \leq \frac{1}{2} \\ 1 & \text{if } a(G) > \frac{1}{2}. \end{cases} \quad (8)$$

Definition 6. Let $G = (V, E)$ be a graph. A fractional matching is a function $f : E \to \mathbb{R}^+$, which assigns a non-negative real number to each edge, such that for every vertex $x$

$$\sum_{e \ni x} f(e) \leq 1.$$

A fractional matching $f$ is perfect if it achieves the maximum

$$f(E) = \sum_{e \in E} f(e) = \frac{|V|}{2}.$$

Alon and Lubetzky proved the following theorem in [11] (see also [12]).
**Theorem 3.** For every graph \( G \)
\[
\Theta^T(G) = 1 \iff a^*(G) = 1 \iff G \text{ has no fractional perfect matching.} \tag{9}
\]

**Corollary 1.** There exists a polynomial-time algorithm to decide whether
\[
\Theta^T(G) = 1 \quad \text{or} \quad \Theta^T(G) \leq \frac{1}{2}.
\]

The following theorem was raised as a question by Alon and Lubetzky in [1,14]. The theorem was proved by Agnes Tóth [20].

**Theorem 4.** For every graph \( G \)
\[
\Theta^T(G) = a^*(G).
\]

Equivalently, every graph \( G \) satisfies
\[
a^*(G^2) = a^*(G). \tag{10}
\]

Tóth proves that
\[
\text{if } a(G) \leq \frac{1}{2} \text{ or } a(H) \leq \frac{1}{2} \text{ then } a(G \times H) \leq \max \{ a(G), a(H) \}. \tag{11}
\]

Actually, Tóth shows that, if \( I \) is an independent set in \( G \times H \) then
\[
|N_{G \times H}(I)| \geq |I| \cdot \min \{ b(G), b(H) \}.
\]

From this, Theorem 4 easily follows. As a corollary (see [1,14,20]) one obtains that, for any two graphs \( G \) and \( H \)
\[
\tau(G \times H) \leq \max \{ a^*(G), a^*(H) \}.
\]

Tóth also proves the following theorem in [20]. This was conjectured by Brown et al. [2].

**Theorem 5.** For any two graphs \( G \) and \( H \),
\[
\Theta^T(G \oplus H) = \max \{ \Theta^T(G), \Theta^T(H) \}. \tag{12}
\]

Notice that the analogue of this statement, with \( a^* \) instead of \( \Theta^T \), is straightforward. The first part of the following theorem was proved by Alon and Lubetzky in [1].

**Theorem 6.** For any two graphs \( G \) and \( H \),
\[
\Theta^T(G \oplus H) = \Theta^T(G \times H) = \max \{ \Theta^T(G), \Theta^T(H) \}. \tag{13}
\]
For cographs we obtain the following theorem.

**Theorem 7.** There exists an efficient algorithm to compute the tensor capacity for cographs.

**Proof.** By Theorem 4 it is sufficient to compute \( a(G) \), as defined in (7).

Consider a cotree for \( G \). For each node the algorithm computes a table. The table contains numbers \( \ell(k) \), for \( k \in \mathbb{N} \), where

\[
\ell(k) = \min \{ |N(I)| \mid I \text{ is an independent set with } |I| = k \}.
\]

Notice that \( a(G) \) can be obtained from the table at the root node via

\[
a(G) = \max_k \frac{k}{k + \ell(k)}.
\]

Assume \( G \) is the union of two cographs \( G_1 \oplus G_2 \). An independent set \( I \) is the union of two independent sets \( I_1 \) in \( G_1 \) and \( I_2 \) in \( G_2 \). Let the table entries for \( G_1 \) and \( G_2 \) be denoted by the functions \( \ell_1 \) and \( \ell_2 \). Then

\[
\ell(k) = \min \{ \ell_1(k_1) + \ell_2(k_2) \mid k_1 + k_2 = k \}.
\]

Assume that \( G \) is the join of two cographs, say \( G = G_1 \otimes G_2 \). An independent set in \( G \) can have vertices in at most one of \( G_1 \) and \( G_2 \). Therefore,

\[
\ell(k) = \min \{ \ell_1(k) + |V(G_2)|, \ell_2(k) + |V(G_1)| \}.
\]

This proves the theorem. \( \Box \)

**Remark 3.** The tensor capacity is computable in polynomial time for many other classes of graphs via similar methods. We describe algorithms for some classes of graphs in Appendix A.

5 An exact exponential algorithm for the tensor capacity

Let \( G \) be a splitgraph with a partition \( \langle S, C \rangle \) of its \( n \) vertices such that \( G[C] \) is a clique and \( G[S] \) is an independent set. For any independent set \( I \) of \( G \), \( I \) can contain at most one vertex from \( C \). Define, for \( i \in \{0, 1\},
\[
a_i(G) = \max \left\{ \frac{|I|}{|I| + |N(I)|} \mid I \text{ an independent set with } |C \cap I| = i \right\}
\]

Then the value \( a(G) \) is obtained by

\[
\max \{ a_0(G), a_1(G) \}.
\]
To compute $a_0(G)$, we shall make use of the following simple observation: If $S$ can be partitioned into two sets $S_1$ and $S_2$, such that their neighbor sets $N(S_1)$ and $N(S_2)$ are disjoint, then there exists an optimal $I^*$ for $a_0(G)$, such that $I^* \subseteq S_1$ or $I^* \subseteq S_2$. To see this, suppose that it is not the case. Then, by assumption we can partition $I^*$ into non-empty sets $I_1 = I^* \cap S_1$ and $I_2 = I^* \cap S_2$, and we have $|I^*| = |I_1| + |I_2|$ and $|N(I^*)| = |N(I_1)| + |N(I_2)|$. Then

$$a_0(G) = \frac{|I^*|}{|I^*| + |N(I^*)|} \leq \max \left\{ \frac{|I_1|}{|I_1| + |N(I_1)|}, \frac{|I_2|}{|I_1| + |N(I_2)|} \right\} \leq a_0(G).$$

This proves the claim.

Based on this observation, we modify a technique described by, eg, Cunningham [3], that transforms the problem into a max-flow (min-cut) problem. We construct a flow network $F$ with vertices corresponding to each vertex of $S$ and $C$, a source vertex $s$ and a sink vertex $t$. We make the source $s$ adjacent to each vertex in $S$, with capacity 1, and the sink $t$ adjacent to each vertex in $C$, with capacity 1 as well. In addition, if $u \in S$ and $v \in C$ are adjacent in the original graph $G$, the corresponding vertices are adjacent in $F$, with capacity set to $\infty$. Note that we omit the edges between vertices in $C$.

Consider a minimum $s$-$t$ cut in $F$. Let $S_1$ be the subset of $S$ whose vertices are in the same partition as $s$, and $S_2 = S - S_1$. The weight of such a cut must be finite, as the maximum $s$-$t$ flow is bounded by $\min \{|S|, |C|\}$. Thus, we have that $N(S_1)$ and $N(S_2)$ are disjoint. Moreover, the total weight of the edges in the cut-set is $|S| - |S_1| + |N(S_1)|$, which implies that

$$S_1 = \arg \min_S \{ |N(S')| - |S'| \mid S' \subseteq S \}.$$

So after running the flow algorithm to obtain $S_1$, there will be three cases:

**Case 1**: the optimal $I^*$ for $a_0(G)$ is exactly $S_1$;

**Case 2**: the optimal $I^*$ for $a_0(G)$ is a proper subset of $S_1$;

**Case 3**: the optimal $I^*$ for $a_0(G)$ is a subset of $S_2$;

Note that Case 2 is impossible, since for any such proper subset $S_1'$, we have

$$|N(S_1')| - |S_1'| \geq |N(S_1)| - |S_1|$$

(by min-cut)

which implies

$$\frac{|N(S_1')| - |S_1'|}{|S_1'|} \geq \frac{|N(S_1')| - |S_1'|}{|S_1|} \geq \frac{|N(S_1)| - |S_1|}{|S_1|},$$

so that

$$\frac{|N(S_1')|}{|S_1'|} \geq \frac{|N(S_1)|}{|S_1|}.$$

Consequently, $S_1'$ cannot be an optimal set that achieves $a_0(G)$.  

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Thus, we have either Case 1 or Case 3. To handle Case 3, we simply remove $S_1$ and $N(S_1)$ from the graph, and solve it recursively. In total, finding $a_0(G)$ requires $O(|S|)$ runs of the max-flow algorithm, and can be solved in polynomial time.

**Remark 4.** There exists a somewhat faster algorithm, also proposed by Cunningham [3], which requires $O(\log |S|)$ runs of max-flow in a slightly different flow network; we omit the details for brevity.

Finally, to compute $a_1(G)$, notice that, if an independent set $I$ contains some vertex $v \in C$ then $N(I)$ contains all vertices of $C$. When $|I|/|I| + |N(I)|$ is maximal, $I$ will contain all the vertices in $S$ that are nonadjacent to $v$. Hence

$$a_1(G) = \frac{n - d}{n},$$

where $d$ denotes the minimum degree of a vertex in $C$. It follows that $a_1(G)$ can be obtained in linear time.

This proves the following theorem.

**Theorem 8.** There exists a polynomial-time algorithm to compute the tensor capacity for splitgraphs.

We modify the approach to obtain an exact algorithm for the tensor capacity of a general graph $H$. Let $n$ be the number of vertices in $H$. Assume we are given a maximal independent set $I$ of $H$. We let $I$ play the role of $S$ and $N(I)$ play the role of $C$ in the above transformation. Then, by the analysis above, we obtain a subset $I_1$ of $I$ with

$$I_1 = \text{arg max} \left\{ \frac{|I'|}{|I'| + |N(I')|} \mid I' \subseteq I \right\}.$$

The algorithm generates all the maximal independent sets $I_1$s, and finds the corresponding subset $I_1$s for each of them. This yields the value $a(H)$. By Moon and Moser’s classic result, $H$ contains at most $3^{n/3}$ maximal independent sets. Furthermore, by eg, the algorithm of Tsukiyama et al., they can be generated in polynomial time per maximal independent set. Thus we obtain the following theorem.

**Theorem 9.** There exists an $O^*(3^{n/3})$ algorithm to compute the tensor capacity for a graph with $n$ vertices.

**Remark 5.** We moved the section on the ultimate categorical independent domination ratio to Appendix C. Appendix B contains the NP-completeness proof for $\alpha(G \times K_4)$ when $G$ is a planar graph of degree three.
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A The ultimate categorical independence ratio for some classes of graphs

In this section we show that the tensor capacity is polynomial for permutation graphs, interval graphs, and graphs of bounded treewidth. The last result also shows that there is a PTAS for the ultimate categorical independence ratio of planar graphs.

A.1 The tensor capacity for permutation graphs

A permutation diagram is obtained as follows. Consider two horizontal lines, $L_1$ and $L_2$, and label $n$ distinct points on each by $1, \ldots, n$. For each label $i$ take the straight line segment that connects the points on $L_1$ and $L_2$ with that label. Pnueli et al. defined permutation graphs as follows [12].

**Definition 7.** A graph is a permutation graph if it is the intersection graph of the straight line segments in a permutation diagram.

Baker et al. characterized permutation graphs as follows [3].

**Theorem 10.** A graph $G$ is a permutation graph if and only if both $G$ and $\overline{G}$ are comparability graphs.

**Theorem 11.** There exists an $O(n^3)$ algorithm to compute the tensor capacity for permutation graphs.

**Proof.** Consider a permutation diagram. Notice that an independent set consists of line segments that are parallel.

For each line segment $x$, and for each integer $k$, compute the smallest neighborhood of an independent set of cardinality $k$ that has $x$ as its right-most line segment.

To compute this for $x$, consider the line segments $y$ that lie to the left of $x$. Let $N_{k-1}(y)$ be the smallest number of neighbors of an independent set with $k-1$ vertices that has $y$ as its right-most line segment. Let $N(x, y)$ be the number of neighbors of $x$ that are not neighbors of $y$. The value $N_k(x)$, for $k \in \mathbb{N}$, is defined as follows.

\[
N_k(x) = \begin{cases} 
|N(x)| & \text{if } k = 1 \\
\min \{ N_{k-1}(y) + N(x, y) \mid y \text{ lies to the left of } x \} & \text{otherwise}
\end{cases}
\]

The value

\[
a(G) = \max_{\text{1 an independent set}} \frac{|I|}{|I| + |N(I)|}
\]
is obtained by
\[ a(G) = \max \left\{ \frac{k}{k + N(x)} \mid x \in V, k \in \mathbb{N} \right\}. \]

The tensor capacity \( \Theta^T(G) \) is obtained from Theorem 4 on page 8 via Formula (8) on Page 7.

This proves the theorem.

\( \square \)

A.2 The tensor capacity for interval graphs

Hajós defined interval graphs as follows [8].

**Definition 8.** An interval graph is an intersection graph of a collection of intervals on the real line.

In the following we identify vertices and the intervals that represent them.

Notice that an independent set consists of a collection of disjoint intervals. So there is a linear, left-to-right ordering of the vertices of an independent set.

**Definition 9.** Let \( x \) be a vertex and let \( k \in \mathbb{N} \). Let \( I(x, k) \) denote the collection of independent sets of cardinality \( k \) in which \( x \) is the rightmost interval. Define
\[ i(x, k) = \min \{ |N(I)| \mid I \in I(x, k) \}. \tag{14} \]

To compute \( a(G) \), We can compute the value of \( i(x, k) \) via the recurrence relation,
\[ i(x, k) = \min_y \{ i(y, k - 1) + |N(x) \setminus N(y)| \} \tag{15} \]
where \( y \) is one of the intervals whose right endpoint is to the left of the left endpoint of \( x \). To avoid overcounting, we only add the neighbors of \( x \) that are not neighbors of \( y \). The correctness follows from the observation that if there is any interval overlapping with \( x \) and another interval \( z \) in the independent set, then \( z \) must also overlap with \( y \).

The algorithm computes \( a(G) \) via the following formula.
\[ a(G) = \max \left\{ \frac{k}{k + i(x, k)} \mid x \in V, k \in \mathbb{N} \right\}. \tag{16} \]

It is easy to see that the time complexity is bounded by \( O(n^3) \). This proves the following theorem.

**Theorem 12.** There exists an \( O(n^3) \) algorithm to compute the tensor capacity for interval graphs.

**Remark 6.** We leave it as an open problem whether the time complexity for interval graphs, or for permutation graphs, can be reduced to \( O(n^2) \).
A.3 The tensor capacity for graphs of bounded treewidth

Graphs of bounded treewidth were popularized by Robertson and Seymour during their work on graph minors \[13\].

**Definition 10.** A graph has treewidth at most \( k \) if it is a subgraph of a chordal graph with clique number \( k + 1 \).

For each \( k \), the class of graphs of treewidth at most \( k \) is closed under minors. The class plays a major role in the graph minor theory because every class of graphs that is closed under taking minors, which does not contain all planar graphs, has treewidth bounded by some \( k \). The class of graphs with treewidth at most \( k \) is recognizable in linear time \[4,9\]. For some background information on this class of graphs we refer to, eg, \[5,9\].

**Theorem 13.** Let \( k \in \mathbb{N} \). There exists a polynomial-time algorithm that computes the tensor capacity for the class of graphs that have treewidth at most \( k \).

**Proof.** Consider a nice tree-decomposition of width \( k \) \[9\]. Each node of the decomposition tree is of four possible types. The algorithm computes a table which contains some information of the graphs induced by the vertices that appear in bags of the subtree. For these induced subgraphs the table contains, for each value \( k \), the minimal number of neighbors of an independent set of cardinality \( k \). Each table entry further specifies

(a) the vertices in the bag that are contained in the independent set, and

(b) the vertices in the bag that are neighbors of vertices in the independent set.

We describe next how this information is computed for each type of the nodes in the tree-decomposition.

**Start node.** A start node \( s \) is a leaf of the decomposition tree. In that case the induced subgraph is just the subgraph induced by the vertices that appear in the bag, say \( S \). In that case, the table contains all the independent sets and all the neighbors of those independent sets.

**Join node.** A join node \( s \) has exactly two children, say \( s_1 \) and \( s_2 \). The three bags are the same, say \( S = S_1 = S_2 \). To construct the table for \( S \), consider table entries at \( s_1 \) and \( s_2 \) that have identical independent set in \( S_1 \) and \( S_2 \). For the neighborhoods in \( S \) the algorithm takes the union of the neighbors indicated in \( S_1 \) and \( S_2 \). The total number of vertices in the independent set is the sum of the numbers at the nodes \( s_1 \) and \( s_2 \), avoiding double counting the number that are in \( S \). The number of neighbors is also the union of the neighbors in the subtree at \( s_1 \) and \( s_2 \), again avoiding double counting the neighbors that are in both \( S_1 \) and \( S_2 \).

**Introduce node.** An introduce node \( s \) has exactly one child \( s' \). The bag \( S \) of \( s \) has exactly one vertex more than the bag \( S' \) of \( s' \). Say \( S = S' \cup \{x\} \). All neighbors of \( x \) are in \( S \). To compute the table at the node \( s \) we consider the cases where \( x \) is in the independent set, in the neighborhood of the independent set, or unrelated to the independent set. Since all neighbors of \( x \) are in \( S \), the table entries at \( s' \) are easily extended to make up table entries for the node \( s \).
**Forget node.** A node \( s \) is a forget node if it has exactly one child, say \( s' \), and the bag of \( S \) has exactly one vertex \( x \) less than the bag \( S' \). Say \( S' = S \cup \{ x \} \).

The table at \( s \) is easily obtained from the table at \( s' \). The values for the independent sets and their numbers of neighbors don’t change. Simply the information whether the vertex \( x \) is a vertex of the independent set, or if it is a neighbor of the independent set, or if it is unrelated to the independent set, disappears. Of course, this may cause some table entries to coincide.

This describes the dynamic programming algorithm. The timebound is determined by the size of the tables. Each table entry is characterized by a 3-coloring of the vertices in the bag; namely as a vertex of the independent set, as a neighbor of the independent set, or as a vertex which is not related to the independent set. Since each bag contains at most \( k + 1 \) vertices, there are \( O(3^{k+1}) \) different types. For each type, the table entry contains two numbers, namely the total size of the independent set and the total number of neighbors. Thus the size of each table is bounded by \( O(3^{k+1} \cdot n^2) \).

The decomposition tree has \( O(n) \) different nodes. Each table is computed in constant time per table entry. Thus the total time is bounded by \( O(3^{k+1} \cdot n^3) \) time.

Via Baker’s method we easily obtain the following result [2]. For brevity we omit the (standard) details.

**Theorem 14.** There exists a PTAS to approximate the ultimate categorical independence ratio in planar graphs.

**B NP-Completeness of independence in categorical products of planar graphs**

**Theorem 15.** Let \( G \) be a planar graph of maximum vertex degree 3. It is NP-complete to compute the maximum independent set of \( G \times K_4 \).

**Proof.** Clearly, the problem is in NP. We show that, to decide whether there is an independent set of size \( 4k \) is NP-hard for \( G \times K_4 \) when \( G \) is a planar graph of maximal degree three.

We reduce the decision problem of deciding whether there is an independent set for \( G \) of size \( k \), which is known to be NP-complete [10], to this problem.

Let \( K_4 = \{ t_1, t_2, t_3, t_4 \} \). Now suppose that if \( G \) has an independent set \( S \) of size \( k \), then for each vertex \( s \) in \( S \), we select the four vertices \( \{ s, t_1 \}, \{ s, t_2 \}, \{ s, t_3 \}, \) and \( \{ s, t_4 \} \) in \( G \times K_4 \). Clearly, the selected 4s vertices \( S' \) is an independent set in \( G \times K_4 \).

On the other hand, suppose that \( G \times K_4 \) has an independent set \( S' \) of size \( 4s \) in \( G \times K_4 \). Unfortunately, the related vertices \( S \) in \( G \) corresponding to the vertices in \( S' \) are not necessarily independent. We transform \( S \) so that it becomes independent.
For any vertex \( s \) in \( S \), it has at most three neighbors in \( S \), say \( s_1, s_2 \) and \( s_3 \). Without loss of generality, we assume that \( (s, t_4) \) belongs to \( S' \). Then clearly, any of \( (s_i, t_j) \) where \( i = 1, 2, 3 \) and \( j = 1, 2, 3 \) does not belong to \( S' \) because otherwise, \( (s_i, t_j) \) would be adjacent to \( (s, t_4) \), which is impossible since \( S' \) is an independent set. Thus the three vertices \( (s_1, t_4), (s_2, t_4) \) and \( (s_3, t_4) \) must all belong to \( S' \).

We transform these three vertices in \( S' \) to become \( (s, t_1), (s, t_2) \) and \( (s, t_3) \). It is clear that the resulted \( S' \) is still independent. Consequently, we also remove all \( s_1, s_2, s_3 \) from \( S \). If the new \( S \) is not an independent set, then we apply the above transformation step on another vertex in \( S \) which has at least one more neighbor in \( S \). At the end, we obtain an independent set \( S \) such that \( |S| \geq 4k/4 = k \) since the size of \( K_4 \) is 4.

This completes our hardness proof. \( \square \)

C The ultimate categorical independent domination ratio for complete multipartite graphs

In this section we assume that the graphs have no isolated vertices.

**Definition 11.** Let \( G \) be a graph. The independent domination number \( i(G) \) is the smallest cardinality of an independent dominating set in \( G \). That is, \( i(G) \) is the cardinality of a smallest maximal independent set in \( G \).

In [6], Farber studies the following ‘independent domination capacity’ for the strong product \( G \boxtimes \cdots \boxtimes G \).

\[
i_s(G) = \lim_{k \to \infty} \sqrt[k]{i(G^k)}.
\]

For chordal graphs \( G \) the fractional independent domination number, \( i_f(G) \), equals the independent domination number. Farber shows that there are infinitely many trees \( T \) for which \( i_s(T) < i(T) \). It seems difficult to get a grip on the parameter. Farber conjectures that \( i_s(C_4) = \sqrt{4} \).

In the rest of this section we concentrate on the categorical product. To start with, the following conjecture appears in [11].

**Conjecture 2.** For all graphs \( G \) and \( H \)

\[
i(G \times H) \geq i(G) \cdot i(H). \tag{17}
\]

**Definition 12.** The independent domination ratio of a graph \( G \) is defined as

\[
\tau_i(G) = \frac{i(G)}{|V(G)|}. \tag{18}
\]
Lemma 4. Let $G$ and $H$ be graphs without isolated vertices. Then

$$i(G \times H) \leq i(G) \cdot |V(H)|.$$  \hfill (19)

Proof. Let $A$ be a minimum independent dominating set in $G$. Let

$$S = \{ (a, h) \mid a \in A \text{ and } h \in V(H) \}.$$  

Since $H$ has no isolated vertices, $S$ is a dominating set in $G \times H$ with cardinality

$$|S| = i(G) |V(H)|.$$

This proves the lemma. \qed

Lemma 5. The sequence $r_i(G^k), k \in \mathbb{N},$ is non-increasing. Thus the limit

$$I(G) = \lim_{k \to \infty} r_i(G^k)$$

exists.

Proof. Notice that, by Lemma 4,

$$i(G^k) \leq i(G^{k-1}) \cdot |V(G)| \Rightarrow \frac{i(G^k)}{|V(G)|} \leq i(G^{k-1}) \Rightarrow \frac{i(G^k)}{|V(G)|^k} \leq \frac{i(G^{k-1})}{|V(G)|^{k-1}}.$$

This proves the claim. \qed

Remark 7. Finbow shows in [7, Page 57] that for the complete bipartite graph

$$I(K_{m,n}) = \lim_{k \to \infty} r_i(K^k_{m,m}) = \lim_{k \to \infty} \frac{i(\times^k K_{m,m})}{(2m)^k} = \lim_{k \to \infty} \frac{2^{k-1} \cdot m^k}{2^k \cdot m^k} = \frac{1}{2}.$$  

Lemma 6. Let $G \simeq K(m,n)$ be the complete bipartite graph with $m$ and $n$ vertices in the two color classes. Then

$$r_i(G^k) = \frac{1}{(m+n)^k} \cdot \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} \cdot \min \{ m^{k-\ell} n^\ell, m^\ell n^{k-\ell} \}. \hfill (20)$$

This implies that $I(K_{m,n}) = 0$ when $0 < m < n.$
Proof. According to [7, Page 57],

\[ K(m, n) \times K(p, q) = K(mp, nq) \oplus K(mq, np). \]

Via induction, it follows that

\[ \times^k K(m, n) = \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} K(m^{k-\ell} n^{\ell}, m^{\ell} n^{k-\ell}), \]

where the sum denotes union. For a complete bipartite graph \( K(p, q) \) the independent domination number is \( \min\{p, q\} \).

Let \( m = \alpha \cdot n \) for some \( 0 < \alpha < 1 \). Then (20) yields

\[
\begin{align*}
I(G) &= \frac{1}{(m+n)^k} \cdot \left[ \sum_{0 \leq \ell \leq k/2} \binom{k-1}{\ell} m^{k-\ell} n^{\ell} + \sum_{1 \leq \ell < k/2} \binom{k-1}{\ell-1} m^{k-\ell} n^{\ell} \right] \\
&\leq \frac{1}{(m+n)^k} \cdot \sum_{0 \leq \ell \leq k/2} \binom{k}{\ell} m^{k-\ell} n^{\ell} \\
&\leq \sqrt{k/(2 \cdot \pi)} \cdot \left( \frac{2\sqrt{\alpha}}{1 + \alpha} \right)^k \rightarrow 0 \quad (k \rightarrow \infty).
\end{align*}
\]

This proves the lemma. \( \square \)

**Theorem 16.** Let \( G \) be a complete multipartite graph with \( t \) color classes of size

\[ n_1 \leq \cdots \leq n_t. \]

Then \( I(G) = 0 \) unless \( t = 2 \) and \( n_1 = n_2 \), in which case \( I(G) = \frac{1}{2} \).

Proof. For the case where \( G \simeq K(m, m) \), Finbow proved that \( I(G) = \frac{1}{2} \) [7]. When \( t = 2 \) and \( n_1 < n_2 \) then \( I(G) = 0 \), as is shown in Lemma 6.

Assume \( t \geq 3 \). Let \( G' \) be the subgraph of \( G \) obtained from \( G \) by removing all edges except those with one endpoint in the smallest color class. Then, obviously,

\[ i(G') \geq i(G) \quad \text{and} \quad i((G')^k) \geq i(G^k). \]

The graph \( G' \) is complete bipartite and the two color classes do not have the same size. Therefore,

\[
\lim_{k \rightarrow \infty} r_t((G')^k) = 0 \quad \Rightarrow \quad I(G) = \lim_{k \rightarrow \infty} r_t(G^k) = 0.
\]

This proves the theorem. \( \square \)
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