Time Varying Isotropic Vector Random Fields on Spheres

Chunsheng Ma*

Abstract For a vector random field that is isotropic and mean square continuous on a sphere and stationary on a temporal domain, this paper derives a general form of its covariance matrix function and provides a series representation for the random field, which involve the ultraspherical polynomials. The series representation is somehow an imitator of the covariance matrix function, but differs from the the spectral representation in terms of the ordinary spherical harmonics, and is useful for modeling and simulation. Some semiparametric models are also illustrated.

Keywords Covariance matrix function · Elliptically contoured random field · Gaussian random field · Isotropic · Stationary · Ultraspherical polynomials

Mathematics Subject Classification (2010) 60G60 · 62M10 · 62M30

*Chunsheng Ma  chunsheng.ma@wichita.edu  Department of Mathematics, Statistics, and Physics, Wichita State University, Wichita, Kansas 67260-0033, USA; School of Economics, Wuhan University of Technology, Hubei 430070, China; and School of Mathematics and Statistics, Hubei Engineering University, Xiaogan, Hubei 432000, China
1 Introduction

Consider an $m$-variate spatio-temporal random field \( \{Z(x; t), x \in S^d, t \in T\} \), where \( S^d \) is the spherical shell of radius 1 and center \( \mathbf{0} \) in \( \mathbb{R}^{d+1} \), i.e., \( S^d = \{\|x\| = 1, x \in \mathbb{R}^{d+1}\} \), \( \|x\| \) is the Euclidean norm of \( x \in \mathbb{R}^{d+1} \), and \( T \) is either \( \mathbb{R} \) or \( \mathbb{Z} \). It is called a time varying or time dependent random field on the sphere \([8],[16],[33]\). When \( \{Z(x; t), x \in S^d, t \in T\} \) has finite second-order moments, its mean function and covariance matrix function are given respectively by \( E(Z(x; t)) \) and

\[
\text{cov}(Z(x_1; t_1), Z(x_2; t_2)) = E((Z(x_1; t_1) - E(Z(x_1; t_1)))(Z(x_2; t_2) - E(Z(x_2; t_2)))', \quad x_1, x_2 \in S^d, \; t_1, t_2 \in T.
\]

The primary goal of this paper is to explore the covariance matrix structure and the series representation of an \( m \)-variate random field \( \{Z(x; t), x \in S^d, t \in T\} \) that is isotropic and mean square continuous over the sphere \( S^d \) and stationary over the time domain \( T \), and is mean square continuous on \( S^d \times T \) if \( T = \mathbb{R} \).

For two points \( x_1 \) and \( x_2 \) on \( S^d \), their spherical (angular, or geodesic) distance is the distance between \( x_1 \) and \( x_2 \) on the largest circle on \( S^d \) that passes through them; more precisely,

\[
\vartheta(x_1, x_2) = \arccos(x_1' x_2), \quad x_1, x_2 \in S^d,
\]

or

\[
\vartheta(x_1, x_2) = \arccos \left( 1 - \frac{1}{2}\|x_1 - x_2\|^2 \right), \quad x_1, x_2 \in S^d,
\]

where \( x_1' x_2 \) is the inner product between \( x_1 \) and \( x_2 \). Evidently, \( 0 \leq \vartheta(x_1, x_2) \leq \pi \), \( S^d \) is a metric space under the spherical distance, and the Euclidean and spherical distances are closely connected on \( S^d \), with

\[
\|x_1 - x_2\| = (2 - 2x_1' x_2)^{1/2} = (2 - 2\cos \vartheta(x_1, x_2))^{1/2} = 2 \sin \left( \frac{\vartheta(x_1, x_2)}{2} \right), \quad x_1, x_2 \in S^d.
\]

An \( m \)-variate random field \( \{Z(x; t), x \in S^d, t \in T\} \) is said to be (wide-sense) isotropic over the sphere \( S^d \) and (wide-sense) stationary over the time domain \( T \), if its mean function \( E(Z(x; t)) \) equals a constant vector, and its covariance matrix function \( \text{cov}(Z(x_1; t_1), Z(x_2; t_2)) \) depends
only on the spherical distance \( \vartheta (x_1, x_2) \) between \( x_1 \) and \( x_2 \) and time lag \( t_1 - t_2 \) between \( t_1 \) and \( t_2 \). In such a case, it covariance matrix function is denoted by \( C(\vartheta; t) \), or

\[
C(\vartheta(x_1, x_2); t_1 - t_2) = \mathbb{E}\{(Z(x_1; t_1) - \mathbb{E}Z(x_1; t_1))(Z(x_2; t_2) - \mathbb{E}Z(x_2; t_2))\}' , \quad x_1, x_2 \in S^d, \quad t_1, t_2 \in \mathbb{R}.
\]

It is an \( m \times m \) matrix function, \( C(\vartheta; -t) = (C(\vartheta; t))' \), and inequality

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} a_i^* C(\vartheta(x_i, x_j); t_i - t_j) a_j \geq 0
\]

holds for every \( n \in \mathbb{N} \), any \( x_i \in S^d, t_i \in \mathbb{T} \), and \( a_i \in \mathbb{R}^m (i = 1, 2, \ldots, n) \), where \( \mathbb{N} \) stands for the set of positive integers. On the other hand, given an \( m \times m \) matrix function with these properties, there exists an \( m \)-variate Gaussian or elliptically contoured random field \( \{Z(x; t), x \in S^d, t \in \mathbb{T}\} \) with \( C(\vartheta; t) \) as its covariance matrix function [21].

In the scalar case \( m = 1 \), a spectral analysis is developed in [32], [33] when \( \mathbb{T} = \mathbb{Z} \) and \( d = 1 \) or 2, and a Fourier series expansion of \( \{Z(x; t), x \in S^d, t \in \mathbb{Z}\} \) is derived with the coefficients being stochastic processes indexed by the time only, as well as a spectral representation of its covariance function. For \( \mathbb{T} = \mathbb{R} \) and \( d \geq 2 \), the spectral expansion of a scalar random field \( \{Z(x; t), x \in S^d, t \in \mathbb{R}\} \) is described by [30],

\[
Z(x; t) = \sum_{n=0}^{\infty} \sum_{k=1}^{h(n)} U_{nk}(t) S_{n,k}(x), \quad x \in S^d, \quad t \in \mathbb{R},
\]

where \( h(n) = (2n + d - 1)!/(d-1)!n! \), \( S_{n,k}(x) \) \( (k = 1, \ldots, h(n)) \) are the orthonormal spherical harmonics of degree \( n \) on \( S^d \) [1], [31], \( S_{0,0} = 1/\sqrt{2^{d+1} \pi^{d+1/2}} \), \( \{U_{nk}(t), t \in \mathbb{R}\} (n \in \mathbb{N}_0, k \in \mathbb{N}_0) \) are stationary stochastic processes with

\[
\text{cov}(U_{nk}(t_1), U_{nj}(t_2)) = \delta_{n_1,n_2} \delta_{k_1,k_2} b_n(t_1 - t_2), \quad t_1, t_2 \in \mathbb{T},
\]

\( \{b_n(t), t \in \mathbb{R}, n \in \mathbb{N}_0\} \) is a sequence of stationary covariance functions with \( \sum_{n=0}^{\infty} b_n(0)P_n(\frac{d+1}{2})(1) < \infty \), \( \delta_{kj} \) is the Kronecker symbol, and \( \mathbb{N}_0 \) denotes the set of nonnegative integers. The covariance function of (2) is

\[
C(\vartheta; t) = \sum_{n=0}^{\infty} b_n(t) P_n(\frac{d+1}{2})(\cos \vartheta), \quad \vartheta \in [0, \pi], \quad t \in \mathbb{R},
\]
where \( P_n^{\frac{d-1}{2}}(x) \) \((n \in \mathbb{N}_0)\) are ultraspherical or Gegenbauer’s polynomials \([1], [35]\). Theoretical investigations and practical applications of scalar and vector random fields on spheres may be found in \([2], [4], [6], [9]-[12], [14], [16]-[20], [22]-[29], [32], [33], [37]-[39]\).

For an \( m \)-variate random field \( \{Z(x; t), x \in S^d, t \in T\} \) isotropic and mean square continuous on \( S^d \) and stationary on \( T \), a general form of its covariance matrix function is given in Section 2, and a series representation is presented in Section 3. The established forms of covariance matrix function and of series representation mimic each other, and are useful for modeling and simulation. Some concluding remarks are made in Section 4, and the theorems are proved in Section 5.

2 Covariance Matrix Structures

For an \( m \)-variate random field \( \{Z(x; t), x \in S^d, t \in T\} \) that is isotropic and mean square continuous over \( S^d \) and stationary on \( T \), its covariance matrix function \( C(\vartheta; t) \) is a continuous function of \( \vartheta \in [0, \pi] \), and is also a continuous function of \( t \in \mathbb{R} \) if \( T = \mathbb{R} \). This section derives the general form of \( C(\vartheta; t) \), which involves ultraspherical polynomials.

We start with a brief review of ultraspherical polynomials, and refer to \([1]\) and \([35]\) for comprehensive treatments. For \( \lambda > 0 \), the ultraspherical or Gegenbauer’s polynomials, \( P_n^{(\lambda)}(x) \), \( n \in \mathbb{N}_0 \), are the coefficients of \( u^n \) in the power series expansion of the function \((1 - 2ux + u^2)^{-\lambda}\), i.e.,

\[
(1 - 2ux + u^2)^{-\lambda} = \sum_{n=0}^{\infty} u^n P_n^{(\lambda)}(x), \quad x \in \mathbb{R}, \ |u| < 1.
\]

They can be alternatively defined through the recurrence formula

\[
\begin{align*}
P_0^{(\lambda)}(x) &\equiv 1, \\
P_1^{(\lambda)}(x) &= 2\lambda x, \\
P_n^{(\lambda)}(x) &= \frac{2(\lambda+n-1)\lambda P_{n-1}^{(\lambda)}(x) - (2\lambda+n-2)P_{n-2}^{(\lambda)}(x)}{n}, \quad x \in \mathbb{R}, \ n \geq 2.
\end{align*}
\]

Some special cases and particular values are

\[
P_n^{(1)}(\cos \vartheta) = \frac{\sin((n + 1)\vartheta)}{\sin \vartheta}, \quad \vartheta \in [0, \pi],
\]

\[
P_n^{(\lambda)}(1) = \binom{2\lambda + n - 1}{n},
\]
\[ |P_n^{(\lambda)}(x)| \leq P_n^{(\lambda)}(1), \quad |x| \leq 1. \]

In the particular case \( \lambda = \frac{1}{2} \), \( P_n^{(\frac{1}{2})}(x) \ (n \in \mathbb{N}_0) \) are the Legendre polynomials.

The ultraspherical polynomials are polynomial solutions of the differential equation
\[
(1 - x^2) \frac{d^2 y}{dx^2} - (2\lambda + 1)x \frac{dy}{dx} + n(2\lambda + n)y = 0,
\]
and possess two types of orthogonal properties. First, they are orthogonal with respective to the weight function \((1 - x^2)^{\lambda - \frac{1}{2}}\) on \([-1, 1]\), in the sense that
\[
\int_{-1}^{1} P_i^{(\lambda)}(x) P_j^{(\lambda)}(x)(1 - x^2)^{\lambda - \frac{1}{2}} \, dx = \begin{cases} 
\frac{\pi}{2} \alpha_i^{\lambda} \frac{\Gamma(i + 2\lambda)}{\Gamma(i + \lambda + \lambda)} & \text{if } i = j, \\
0 & \text{if } i \neq j.
\end{cases}
\]

Second, they are orthogonal over \( S^d \) \((d \geq 2)\), as the following lemma describes, which is a special case of the Funk-Hecke formula ([1], [31]) that is useful in simplifying calculations of certain integrals over \( S^d \).

**Lemma 1** For \( i, j \in \mathbb{N}_0 \), if \( d \geq 2 \), then
\[
\int_{S^d} P_i^{(\frac{d-1}{2})}(x_1'z) P_j^{(\frac{d-1}{2})}(x_2'z) d\mathbf{z} = \begin{cases} 
\frac{\omega_d}{\alpha_i^{\lambda}} P_i^{\left(\frac{d-1}{2}\right)}(x_1'x_2'), & \text{if } i = j, \\
0, & \text{if } i \neq j,
\end{cases}
\]
where \( \omega_d = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)} \) is the surface area of \( S^d \), and
\[
\alpha_n = \left(\frac{2n + d - 1}{d - 1}\right)^{\frac{1}{2}}, \quad n \in \mathbb{N}_0.
\]

In terms of the orthonormal spherical harmonics, \( P_n^{(\frac{d-1}{2})} \left( \cos \vartheta \right) \) or \( P_n^{(\frac{d-1}{2})}(x'y) \) can be expressed as (see, e.g., Theorem 9.6.1 of [1])
\[
P_n^{(\frac{d-1}{2})}(x'y) = \frac{\omega_d}{\alpha_n^{\lambda}} \sum_{k=1}^{h(n)} S_{n,k}(x) S_{n,k}(y), \quad x, y \in S^d, \ n \in \mathbb{N},
\]
from which its positive definiteness follows directly, noticing that \( \frac{h(n)}{P_n^{(\frac{d+1}{2})}}(1) = \alpha_n^2 \). The elementary positive definite spherical functions on \( S^d \) are the positive scalar products [5] of \( P_n^{(\frac{d+1}{2})}(\cos \vartheta) \), \( n \in \mathbb{N}_0 \), which actually form a basis [34] of the set of isotropic, continuous, and positive definite functions on \( S^d \). A probability interpretation for these elementary positive definite functions on the sphere is provided in Lemma 2 below [24], which illustrates a basis of the set of isotropic and mean square random fields on \( S^d \). It will be employed in the proofs of Theorems 1, 2 and 4.

**Lemma 2** If \( U \) is a \((d+1)\)-dimensional random vector uniformly distributed on \( S^d \) \((d \geq 2)\), then, for a fixed \( n \in \mathbb{N} \),

\[
Z_n(x) = \alpha_n P_n^{(\frac{d+1}{2})}(x^T U), \quad x \in S^d,
\]

is an isotropic random field with mean 0 and covariance function

\[
cov(Z_n(x_1), Z_n(x_2)) = P_n^{(\frac{d+1}{2})}(\cos \vartheta(x_1, x_2)), \quad x_1, x_2 \in S^d,
\]

where \( \alpha_n \) is defined in (6). Moreover, for \( i \neq j \), \( \{Z_i(x), x \in S^d\} \) and \( \{Z_j(x), x \in S^d\} \) are uncorrelated; that is

\[
cov(Z_i(x_1), Z_j(x_2)) = 0, \quad x_1, x_2 \in S^d.
\]

Alternatively, assume that \( Z_1, \ldots, Z_{h(n)} \) are uncorrelated random variables with mean 0 and variance 1. Then

\[
Z_n(x) = \frac{\omega_d^{\frac{1}{2}}}{\alpha_n} \sum_{k=1}^{h(n)} Z_k S_{n,k}(x), \quad x \in S^d,
\]

is an isotropic random field with mean 0 and covariance function \( P_n^{(\frac{d+1}{2})}(\cos \vartheta(x_1, x_2)) \); see page 77 of [37]. More interestingly, (8) may be thought of as a special case of (10) by selecting

\[
Z_k = \omega_d^{\frac{1}{2}} S_{n,k}(U), \quad k = 1, \ldots, h(n),
\]

with the help of identity (7).
Theorem 1  If an $m$-variate random field $\{Z(x; t), x \in \mathbb{S}^d, t \in T\}$ is isotropic and mean square continuous over $\mathbb{S}^d$ and stationary on $T$, then $\frac{C(\vartheta; t) + C(\vartheta; -t)}{2}$ is of the form

$$C(\vartheta; t) + C(\vartheta; -t) = \begin{cases} \sum_{n=0}^{\infty} B_n(t) \cos(n\vartheta), & d = 1, \\ \sum_{n=0}^{\infty} B_n(t) P_n^{(d-1)}(\cos \vartheta), & d \geq 2, \end{cases} \quad \vartheta \in [0, \pi], \ t \in T,$$

(11)

where, for each fixed $t \in T$, $B_n(t)$ ($n \in \mathbb{N}_0$) are $m \times m$ symmetric matrices and $\sum_{n=0}^{\infty} B_n(t)$ ($d = 1$) or $\sum_{n=0}^{\infty} B_n(t) P_n^{(d-1)}(1)$ ($d \geq 2$) converges, and, for each fixed $n \in \mathbb{N}_0$, $B_n(t)$ is a stationary covariance matrix function on $T$.

In particular, when $C(\vartheta; t)$ is spatio-temporal symmetric in the sense that

$$C(\vartheta; -t) = C(\vartheta; t), \quad \vartheta \in [0, \pi], \ t \in T,$$

it takes the form

$$C(\vartheta; t) = \begin{cases} \sum_{n=0}^{\infty} B_n(t) \cos(n\vartheta), & d = 1, \\ \sum_{n=0}^{\infty} B_n(t) P_n^{(d-1)}(\cos \vartheta), & d \geq 2. \end{cases} \quad \vartheta \in [0, \pi], \ t \in T,$$

In the next theorem $m \times m$ matrices $B_n(t)$ ($n \in \mathbb{N}_0$) are not necessarily symmetric. One simple such example is

$$B(t) = \begin{cases} \Sigma + \Phi \Sigma \Phi', & t = 0, \\ \Phi \Sigma, & t = -1, \\ \Sigma \Phi', & t = 1, \\ 0, & t = \pm 2, \pm 3, \ldots, \end{cases}$$

which is the covariance matrix function of an $m$-variate first order moving average time series $Z(t) = \varepsilon(t) + \Phi \varepsilon(t - 1), t \in \mathbb{Z}$, where $\{\varepsilon(t), t \in \mathbb{Z}\}$ is $m$-variate white noise with $\mathbb{E}\varepsilon(t) = 0$ and $\text{var}(\varepsilon(t)) = \Sigma$, and $\Phi$ is an $m \times m$ matrix.
Theorem 2  (i) An $m \times m$ matrix function

$$C(\vartheta; t) = \sum_{n=0}^{\infty} B_n(t) \cos(n\vartheta), \quad \vartheta \in [0, \pi], \ t \in \mathbb{T},$$

is the covariance matrix function of an $m$-variate Gaussian or elliptically contoured random field $\{Z(\mathbf{x}; t), \mathbf{x} \in S^1, t \in \mathbb{T}\}$ if and only if $\sum_{n=0}^{\infty} B_n(0)$ converges and $B_n(t)$ is a stationary covariance matrix function on $\mathbb{T}$ for each fixed $n \in \mathbb{N}_0$.

(ii) Let $d \geq 2$. An $m \times m$ matrix function

$$C(\vartheta; t) = \sum_{n=0}^{\infty} B_n(t) P_n^{(d-1)/2}(\cos \vartheta), \quad \vartheta \in [0, \pi], \ t \in \mathbb{T},$$

is the covariance matrix function of an $m$-variate Gaussian or elliptically contoured random field on $S^d \times \mathbb{T}$ if and only if $\sum_{n=0}^{\infty} B_n(0) P_n^{(d-1)/2}(1)$ converges and $B_n(t)$ is a stationary covariance matrix function on $\mathbb{T}$ for each fixed $n \in \mathbb{N}_0$.

Gaussian and second-order elliptically contoured random fields form one of the largest sets, if not the largest set, which allow any possible correlation structure [21]. The covariance matrix functions developed in Theorem 2 can be adopted for a Gaussian or elliptically contoured vector random field. However, they may not be available for other non-Gaussian random fields, such as a log-Gaussian, $\chi^2$, K-distributed, or skew-Gaussian one, for which admissible correlation structure must be investigated on a case-by-case basis.

Example 1  Given an $m \times m$ matrix function $B(t), t \in \mathbb{T}$, with all entries $b_{ij}(t)$ less than 1 in absolute value, consider an $m \times m$ matrix function $C(\vartheta; t)$ with entries

$$C_{ij}(\vartheta; t) = -\ln \left\{ \frac{1}{2} \left[ 1 - b_{ij}(t) \cos \vartheta + \left( 1 - 2b_{ij}(t) \cos \vartheta + b_{ij}^2(t) \right)^{1/2} \right] \right\}, \quad \vartheta \in [0, \pi], \ t \in \mathbb{T},$$

$$i, j = 1, \ldots, m.$$

It is the covariance matrix function of an $m$-variate Gaussian or elliptically contoured random field $\{Z(\mathbf{x}; t), \mathbf{x} \in S^2, t \in \mathbb{T}\}$ if and only if $B(t)$ is a stationary covariance matrix function on $\mathbb{T}$. In fact, a version (13) of $C(\vartheta; t)$ can be established by taking $B_0(t) \equiv 0, B_n(t) = \frac{1}{n}(B(t))^n, n \in \mathbb{N},$
and using the identity (see, e.g., (5) on page 128 of [28])

\[
\sum_{n=1}^{\infty} \frac{u^n}{n} P_n \left( \frac{x}{2} \right) = -\ln \left\{ \frac{1}{2} \left[ 1 - ux + (1 - 2ux + u^2)^{\frac{1}{2}} \right] \right\}, \quad |x| \leq 1, \; |u| < 1,
\]

where \( B^{op} \) denotes the Hadamard \( p \) power of \( B = (b_{ij}) \), whose entries are \( b_{ij}^p \), the \( p \) power of \( b_{ij}, i, j = 1, \ldots, m \).

A covariance matrix function \( C(\vartheta; t) \) defined on \( S^d \times T \) is also a covariance matrix function on \( S^{d_0} \times T \), provided that \( 1 \leq d_0 < d \), just as a point \( x \in S^{d_0} \) can be thought of as a point \( (x, 0)' \) on \( S^d \). A covariance matrix function \( C(\vartheta; t) \) on all \( S^d \times T \) \((d \in \mathbb{N})\) is called a covariance matrix function on \( S^\infty \times T \), with \( S^\infty \) being an infinite dimensional sphere in Hilbert space. A general form of this type of covariance matrix structures is given next.

**Theorem 3**

(i) If an \( m \)-variate random field \( \{Z(\mathbf{x}; t), \mathbf{x} \in S^\infty, t \in T \} \) is isotropic and mean square continuous over \( S^\infty \) and stationary on \( T \), then

\[
\frac{C(\vartheta; t) + C(\vartheta; -t)}{2} = \sum_{n=0}^{\infty} B_n(t) \cos^n \vartheta, \quad \vartheta \in [0, \pi], \; t \in T,
\]

where, for each fixed \( t \in T \), \( B_n(t) \) \((n \in \mathbb{N}_0)\) are \( m \times m \) symmetric matrices and \( \sum_{n=0}^{\infty} B_n(t) \) converges, and, for each fixed \( n \in \mathbb{N}_0 \), \( B_n(t) \) is a stationary covariance matrix function on \( T \).

In particular, a spatio-temporal symmetric \( C(\vartheta; t) \) is of the form

\[
C(\vartheta; t) = \sum_{n=0}^{\infty} B_n(t) \cos^n \vartheta, \quad \vartheta \in [0, \pi], \; t \in T.
\]

(ii) An \( m \times m \) matrix function

\[
C(\vartheta; t) = \sum_{n=0}^{\infty} B_n(t) \cos^n \vartheta, \quad \vartheta \in [0, \pi], \; t \in T,
\]

is the covariance matrix function of an \( m \)-variate Gaussian or elliptically contoured random field on \( S^\infty \times T \) if and only if \( \sum_{n=0}^{\infty} B_n(0) \) converges and \( B_n(t) \) is a stationary covariance matrix function on \( T \) for each fixed \( n \in \mathbb{N}_0 \).
One may use Lemma 1 of [34] to deduce (14). Instead, a more efficient approach is based on Lemma 3 below, which expresses $x^n$ as a convex combination of ultraspherical polynomials, explains a close connection between $\cos^n \vartheta$ and $P_k^{(d-1/2)}(\cos \vartheta)$ $(k = 0, 1, \ldots, n)$, where the former is for the basis of the covariance matrix structure on $S^\infty$ and the latter on $S^d$, and provides an approach to generate an isotropic random field on $S^d$ with covariance function $\cos^n \vartheta$.

**Lemma 3** Let $n \in \mathbb{N}_0$.

(i) $x^n$ can be expressed as

$$x^n = \left\lfloor \frac{n}{2} \right\rfloor \sum_{k=0}^{\left\lfloor n/2 \right\rfloor} \beta_{k,n}^{(d-1/2)} P_{n-2k}^{(d-1/2)}(x), \quad |x| \leq 1, \quad (16)$$

where $\left\lfloor u \right\rfloor$ denotes the integer part of a real number $u$, and

$$\beta_{k,n}^{(d-1/2)} = \frac{n! (n - 2k + \frac{d-1}{2}) \Gamma \left( \frac{d-1}{2} \right)}{2^n k! \Gamma \left( n - k + \frac{d+1}{2} \right)}, \quad k = 0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor;$$

(ii) $\cos^n \vartheta$ can be expressed as

$$\cos^n \vartheta = \sum_{k=0}^{\left\lfloor n/2 \right\rfloor} \beta_{k,n}^{(d-1/2)} P_{n-2k}^{(d-1/2)}(\cos \vartheta), \quad \vartheta \in [0, \pi]; \quad (17)$$

(iii) If $U$ is a $(d+1)$-dimensional random vector uniformly distributed on $S^d$ $(d \geq 2)$, then

$$Z(x) = \sum_{k=0}^{\left\lfloor n/2 \right\rfloor} \alpha_k \left( \beta_{k,n}^{(d-1/2)} \right) \frac{1}{2} P_{n-2k}^{(d-1/2)}(x'U), \quad x \in S^d, \quad (18)$$

is an isotropic random field with mean 0 and covariance function $\cos^n \vartheta$.

Identity (16) is an alternative version of Lemma 1 of [4], and Part (iii) of Lemma 3 follows from Lemma 2 and (17). Another method generating an isotropic random field on $S^d$ with covariance function $\cos^n \vartheta$ is presented in Subsection 12.3 of [7].
Example 2  An $m \times m$ matrix function $C(\vartheta; t)$ whose entries are second order polynomials of $\vartheta$,

$$C(\vartheta; t) = B_0(t) + B_1(t)\vartheta + B_2(t)\vartheta^2, \quad \vartheta \in [0, \pi], \ t \in \mathbb{T}, \quad (19)$$

is a covariance matrix function on $S^\infty \times \mathbb{T}$ if and only if $B_2(t), -B_1(t) - \pi B_2(t)$, and $B_0(t) + \frac{\pi}{2} B_1(t) + \frac{\pi^2}{4} B_2(t)$ are stationary covariance matrix functions on $\mathbb{T}$. To apply Theorem 3 to the function (19), we employ the formula

$$\vartheta = \frac{\pi}{2} - \arcsin(\cos \vartheta), \quad \vartheta \in [0, \pi],$$

and the Taylor expansions of $\arcsin x$ and $(\arcsin x)^2$,

$$\arcsin x = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} x^{2n+1}, \quad |x| \leq 1,$$

$$(\arcsin x)^2 = \sum_{n=1}^{\infty} \frac{2^{2n-1}((n-1)!)^2}{(2n)!} x^{2n}, \quad |x| \leq 1,$$

and obtain a version (15) of $C(\vartheta; t)$,

$$C(\vartheta; t) = B_0(t) + B_1(t)\left(\frac{\pi}{2} - \arcsin(\cos \vartheta)\right) + B_2(t)\left(\frac{\pi}{2} - \arcsin(\cos \vartheta)\right)^2$$

$$= B_0(t) + \frac{\pi}{2} B_1(t) + \frac{\pi^2}{4} B_2(t) - (B_1(t) + \pi B_2(t)) \arcsin(\cos \vartheta) + B_2(t)(\arcsin(\cos \vartheta))^2$$

$$= B_0(t) + \frac{\pi}{2} B_1(t) + \frac{\pi^2}{4} B_2(t) - (B_1(t) + \pi B_2(t)) \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} \cos^{2n+1} \vartheta$$

$$+ B_2(t) \sum_{n=1}^{\infty} \frac{2^{2n-1}((n-1)!)^2}{(2n)!} \cos^{2n} \vartheta, \quad \vartheta \in [0, \pi], \ t \in \mathbb{T},$$

whose coefficients are stationary covariance matrix functions on $\mathbb{T}$ if and only if $B_2(t), -B_1(t) - \pi B_2(t)$, and $B_0(t) + \frac{\pi}{2} B_1(t) + \frac{\pi^2}{4} B_2(t)$ are so.

In particular, in (19) taking $B_2(t) \equiv 0$ yields that

$$C(\vartheta; t) = B_0(t) + B_1(t)\vartheta, \quad \vartheta \in [0, \pi], \ t \in \mathbb{T},$$

is a covariance matrix function on $S^\infty \times \mathbb{T}$ if and only if $B_0(t) + \frac{\pi}{2} B_1(t)$ and $-B_1(t)$ are stationary covariance matrix functions on $\mathbb{T}$. Moreover, under these conditions,

$$C(\vartheta; t) = (B_0(t) - B_1(t)\vartheta)^{op}, \quad \vartheta \in [0, \pi], \ t \in \mathbb{T},$$
is also a covariance matrix function by Theorem 6 of [21], where \( p \) is a natural number.

**Example 3** An \( m \times m \) matrix function \( C(\vartheta; t) \) with entries

\[
C_{ij}(\vartheta; t) = \exp \left( -\frac{\pi}{2} b_{ij}(t) - b_{ij}(t) \vartheta \right), \quad \vartheta \in [0, \pi], \ t \in T, \ i, j = 1, \ldots, m, \quad (20)
\]

is a covariance matrix function on \( S^\infty \times T \) if and only if \( B(t) \) is a stationary covariance matrix function on \( T \). Theorem 3 is applicable, after we use the Taylor series of \( \exp(\arcsin x) \) (see, for instance, formula 1.216 of [13]),

\[
\exp(\arcsin x) = \sum_{n=0}^{\infty} \frac{(n + 1)x^n}{n!}, \quad |x| \leq 1,
\]

to represent (20) as the form of (15),

\[
C(\vartheta; t) = C \left( \frac{\pi}{2} - \arcsin(\cos \vartheta); t \right) = \sum_{n=0}^{\infty} \frac{n + 1}{n!} B^n(t) \cos^n \vartheta, \quad \vartheta \in [0, \pi], \ t \in T,
\]

whose coefficients are stationary covariance matrix functions on \( T \) if and only if \( B(t) \) is so.

### 3 Series Representations

For an \( m \)-variate random field with covariance matrix function (12) or (13) this section provides a series representation, which is a mimic of (12) or (13) involving ultraspherical polynomials. A purely spherical version is given in [24]. Two cases \( d \geq 2 \) and \( d = 1 \) are treated in Theorems 4 and 5 separately, since the main tool for the construction, Lemma 2, applies to the case \( d \geq 2 \) only.

**Theorem 4** Assume that \( \{V_n(t), t \in T\} \) is an \( m \)-variate stationary stochastic process with \( EV_n = 0 \) and \( \text{cov}(V_n(t_1), V_n(t_2)) = \alpha_n^2 B_n(t_1 - t_2) \) for each fixed \( n \in \mathbb{N}_0 \), \( U \) is a \((d+1)\)-dimensional random vector uniformly distributed on \( S^d \) \((d \geq 2)\), and \( U \) and \( \{V_n(t), t \in T\}, n \in \mathbb{N}_0 \), are independent. If \( \sum_{n=0}^{\infty} B_n(0) P_n \left( \frac{d-1}{2} \right) \) converges, then an \( m \)-variate random field

\[
Z(x; t) = \sum_{n=0}^{\infty} V_n(t) P_n \left( \frac{d-1}{2} \right) (x'U), \quad x \in S^d, \ t \in T,
\]

(21)
is isotropic and mean square continuous on \( S^d \), stationary on \( T \), and possesses mean 0 and covariance matrix function (13).

The distinct terms of (21) are uncorrelated each other, according to Lemma 2 and the independent assumption among \( U, V_i(t), V_j(t) \),

\[
\text{cov} \left( V_i(t)P_i^{(d-1)/2}(x'U), V_j(t)P_j^{(d-1)/2}(x'U) \right) = 0, \quad x \in S^d, \ t \in T, \ i \neq j.
\]

To see how \( Z(x; t), V_n(t) \) and \( U \) are related to each other in (21), we multiply both sides of (21) by \( P_n^{(d-1)/2}(x'U) \), integrate over \( S^d \), and obtain, in view of Lemma 2,

\[
\int_{S^d} Z(x; t)P_n^{(d-1)/2}(x'U)dx = \sum_{k=0}^{\infty} V_n(t) \int_{S^d} P_k^{(d-1)/2}(x'U)P_n^{(d-1)/2}(x'U)dx = \frac{\omega_d}{\alpha_n^2} P_n^{(d-1)/2}(1)V_n(t),
\]

or

\[
V_n(t) = \frac{\alpha_n^2}{\omega_d P_n^{(d-1)/2}(1)} \int_{S^d} Z(x; t)P_n^{(d-1)/2}(x'U)dx, \quad t \in T, \ n \in \mathbb{N}_0.
\]

**Example 4** Suppose that \( B(t) \) is an \( m \times m \) stationary covariance matrix function on \( T \) with all entries less than 1 in absolute value, all entries of \( B_0(t) \) equal 1, and \( B_n(t) = (B(t))^n, n \in \mathbb{N} \). Then (21) defines an \( m \)-variate isotropic random field on \( S^d \times \mathbb{R} (d \geq 2) \), with mean 0 and direct/cross covariance functions

\[
C_{ij}(\vartheta; t) = \left( 1 - 2b_{ij}(t) \cos \vartheta + b_{ij}^2(t) \right)^{-d-1/2}, \quad \vartheta \in [0, \pi], \ t \in T, \ i, j = 1, \ldots, m,
\]

which follows from (13) and the expansion (4).

Theorem 4 does not apply to the unit circle case \( d = 1 \), just as Lemma 2 is limited to \( d \geq 2 \). To deal with the case \( d = 1 \), for two points \( x_1 \) and \( x_2 \) on the unit circle \( S^1 \), denote their Cartesian coordinates by \( x_k = (\cos \theta_k, \sin \theta_k)' \), respectively, where \( \theta_k \) is the angular coordinate.
of \( x_k \) in polar coordinates with \( 0 \leq \theta_k \leq 2\pi, k = 1, 2 \). In terms of \( \theta_1 \) and \( \theta_2 \), the angular distance \( \vartheta(x_1, x_2) \) between \( x_1 \) and \( x_2 \) can be expressed as

\[
\vartheta(x_1, x_2) = \min(|\theta_1 - \theta_2|, 2\pi - |\theta_1 - \theta_2|),
\]

since \( x_1'x_2 = \cos(\vartheta(x_1, x_2)) = \cos(\theta_1 - \theta_2) \). The following theorem provides a series representation for an \( m \)-variate random field that is isotropic and mean square continuous on the unit circle and stationary on \( T \).

**Theorem 5** Suppose that, for each \( n \in \mathbb{N}_0 \), \( \{V_{n1}(t), t \in T\} \) and \( \{V_{n2}(t), t \in T\} \) are \( m \)-variate stationary stochastic processes with \( \mathbb{E}V_{nk}(t) = 0 \) and \( \text{cov}(V_{nk}(t_1), V_{nk}(t_2)) = B_n(t_1 - t_2), k = 1, 2 \), and that \( \{V_{n1}(t), t \in T\} \) and \( \{V_{n2}(t), t \in T\} \) \( (n \in \mathbb{N}_0) \) are independent. If \( \sum_{n=0}^{\infty} B_n(0) \) converges, then

\[
Z(x; t) = \sum_{n=0}^{\infty} (V_{n1}(t) \cos(n\theta) + V_{n2}(t) \sin(n\theta)), \quad x = (\cos \theta, \sin \theta)' \in S^1, \quad t \in T,
\]

is an \( m \)-variate random field on \( S^1 \times T \), with mean 0 and covariance matrix function (12).

**Example 5** Let \( B(t) \) be an \( m \times m \) stationary covariance matrix function on \( T \), all entries of \( B_0(t) \) equal 1, and \( B_n(t) = \frac{1}{n!}(B(t))^{\circ n}, n \in \mathbb{N} \). Then (22) is an \( m \)-variate isotropic random field on \( S^1 \times \mathbb{R} \), with mean 0 and direct/cross covariance functions

\[
C_{ij}(\vartheta; t) = \exp(b_{ij}(t) \cos \vartheta) \cos(b_{ij}(t) \sin \vartheta), \quad \vartheta \in [0, \pi], \quad t \in T, \quad i, j = 1, \ldots, m,
\]

which follows from (12) and the identity (see, e.g., (5) on page 98 of [28])

\[
\sum_{n=0}^{\infty} \frac{b^n}{n!} \cos(n\theta) = \exp(b \cos \theta) \cos(b \sin \theta), \quad b \in \mathbb{R}, \theta \in [0, \pi].
\]

4 Concluding Remarks

Our focus is mostly on the spherical domain, although the vector random field in this paper has a spatio-temporal domain. While the temporal domain \( T \) is assumed to be either \( \mathbb{Z} \) or \( \mathbb{R} \), the results deduced here may be easily extended to other cases.
The spatial domain $S^d$ may be substituted by a $d$-dimensional compact two-point homogeneous Riemannian manifold $M^d$. It is a compact Riemannian symmetric space of rank one, and belongs to one of the following categories ([15], [36]): the unit spheres $S^d$ ($d = 1, 2, \ldots$), the real projective spaces $\mathbb{P}^d(\mathbb{R})$ ($d = 2, \ldots$), the complex projective spaces $\mathbb{P}^d(\mathbb{C})$ ($d = 4, 6, \ldots$), the quaternionic projective spaces $\mathbb{P}^d(\mathbb{H})$ ($d = 8, 12, \ldots$), and the Cayley elliptic plane $\mathbb{P}^{16}(\text{Cay})$. For the lowest dimensions, $\mathbb{P}^1(\mathbb{R}) = S^1$, $\mathbb{P}^2(\mathbb{C}) = S^2$, and $\mathbb{P}^4(\mathbb{H}) = S^4$. A series representation of a continuous and isotropic covariance function on $M^d$ may be found in [2], [10], [26], with the ultraspherical polynomials substituted by Jacobi polynomials, which include the ultraspherical polynomial as a special case. The general form like those in Theorem 1 may be deduced for the covariance matrix function of an $m$-variate random field $\{Z(x; t), x \in M^d, t \in \mathbb{T}\}$ that is isotropic on $M^d$ and stationary on $\mathbb{T}$, although the approach in the proof of Theorem 1 may not be adopted, where Lemma 1 plays a key role. For an associated random field, is it possible to establish a series representation like (21) with $S^d \times \mathbb{T}$ substituted by $M^d \times \mathbb{T}$? This would highly depend on whether an orthogonal property like that in Lemma 1 holds for Jacobi polynomials over $M^d$ [23].

Theorem 2 characterizes a covariance matrix function on $S^\infty \times \mathbb{T}$, whose entries are isotropic and continuous on $S^\infty$ and stationary on $\mathbb{T}$. For an associated random field, it would be of interest to derive a series representation like (21). In the scalar and purely spherical case, a series representation of an associated Gaussian random field is given by [3],

$$Z(x) = \sum_{n=0}^{\infty} b_n Y_n(x), \quad x \in S^\infty,$$

where $\{b_n, n \in \mathbb{N}_0\}$ is a summable sequence of nonnegative numbers, for each $n \in \mathbb{N}_0$, $\{Y_n(x), x \in S^\infty\}$ is a Gaussian random field with mean 0 and covariance

$$\text{cov}(Y_n(x_1), Y_n(x_2)) = \cos^n(\vartheta(x_1, x_2)), \quad x_1, x_2 \in S^\infty,$$

and $\{Y_n(x), x \in S^\infty\}, n \in \mathbb{N}_0$, are mutually independent. Theorem 4 on page 83 of [37] gives an approach to generate each $Y_n(x)$ on $S^\infty$, while two generating methods are available on $S^d$, one in Subsection 12.3 of [7] and the other in Lemma 3.

The ultraspherical functions $P_n^{(d-1)}(\cos \vartheta), n \in \mathbb{N}_0$, are the basic spherical harmonics on $S^d$ ($d \geq 2$), analogous to $\cos(n \vartheta)$ on $S^1$. For every spherical harmonic $S_n(x)$, it is possible to choose
\( h(n) \) points \( y_1, \ldots, y_h(n) \) on \( S^d \) such that \( S_n(x) \) is a linear combination of \( P_{d-1}^{(d-1)}(x'y_k), k = 1, \ldots, h(n) \), according to Theorem 9.6.4 of [1]. With such substitutions, (2) can be rewritten in terms of the ultraspherical polynomials, although it is more completed than (21) in the scalar case. At each level \( n \), only one term gets involved in (21), in contrast to \( h(n) \) terms in (2). Intuitively, employing finitely truncated expansions of (21) for approximation or simulation would be more efficient than that of (2). An examination of the convergent rate would be expected if finitely truncated expansions of (21) are used for approximation or simulation. A purely spatial case with \( d = 2 \) is studied in [18] with respect to the spectral representation (2).

5 Proofs

5.1 Proof of Theorem 1

For a fixed \( t \in T \), consider two purely spatial random fields \( \{Z(x;0) + Z(x;t), x \in S^d\} \) and \( \{Z(x;0) - Z(x;t), x \in S^d\} \). In terms of \( C(\vartheta;t) \), their covariance matrix functions are, respectively,

\[
\text{cov} (Z(x_1;0) + Z(x_1;t), Z(x_2;0) + Z(x_2;t)) = 2C(\vartheta(x_1,x_2);0) + C(\vartheta(x_1,x_2);t) + C(\vartheta(x_1,x_2);-t),
\]

and

\[
\text{cov} (Z(x_1;0) - Z(x_1;t), Z(x_2;0) - Z(x_2;t)) = 2C(\vartheta(x_1,x_2);0) - C(\vartheta(x_1,x_2);t) - C(\vartheta(x_1,x_2);-t), \quad x_1, x_2 \in S^d.
\]

We consider the case \( d \geq 2 \) only, while a similar argument applies to the case \( d = 1 \). By Theorem 1 of [22], these two covariance matrix functions must take the form

\[
2C(\vartheta(x_1,x_2);0) + C(\vartheta(x_1,x_2);t) + C(\vartheta(x_1,x_2);-t) = \sum_{n=0}^{\infty} B_{n+}(t) P_{n}^{(d-1)}(\cos \theta(x_1,x_2)), \quad (23)
\]

\[
2C(\vartheta(x_1,x_2);0) - C(\vartheta(x_1,x_2);t) - C(\vartheta(x_1,x_2);-t) = \sum_{n=0}^{\infty} B_{n-}(t) P_{n}^{(d-1)}(\cos \theta(x_1,x_2)), \quad (24)
\]

16
where \( B_{n+}(t) \) and \( B_{n-}(t) \) (\( n \in \mathbb{N}_0 \)) are \( m \times m \) positive definite matrices, and \( \sum_{n=0}^{\infty} B_{n+}(t) P_n^{\left( \frac{d-1}{2} \right)}(\vartheta) \) and \( \sum_{n=0}^{\infty} B_{n-}(t) P_n^{\left( \frac{d-1}{2} \right)}(\vartheta) \) converge. Taking the difference between (23) and (24) results in (11), with

\[
B_n(t) = \frac{1}{4} B_{n+}(t) - \frac{1}{4} B_{n-}(t), \quad n \in \mathbb{N}_0.
\]

Clearly, \( B_n(t) \) is symmetric, and \( \sum_{n=0}^{\infty} B_n(t) P_n^{\left( \frac{d-1}{2} \right)}(\vartheta) \) converges.

What remains is to verify that \( B_n(t), t \in \mathbb{T} \), is a stationary covariance matrix function, for each fixed \( n \in \mathbb{N}_0 \). To this end, consider an \( m \)-variate stochastic process

\[
W_n(t) = \int_{S^d} \frac{Z(x; t) + \tilde{Z}(x; -t)}{\sqrt{2}} P_n^{\left( \frac{d-1}{2} \right)}(x'U) dx,
\]

where \( \{\tilde{Z}(x; t), x \in S^d, t \in \mathbb{T}\} \) is an independent copy of \( \{Z(x; t), x \in S^d, t \in \mathbb{T}\} \), \( U \) is an \((d+1)\)-variate random vector uniformly distributed on \( S^d \), and \( U, \{Z(x; t), x \in S^d, t \in \mathbb{T}\} \) and \( \{\tilde{Z}(x; t), x \in S^d, t \in \mathbb{T}\} \) are independent.

The mean function of \( \{W_n(t), t \in \mathbb{T}\} \) is

\[
E W_n(t) = E \int_{S^d} \frac{Z(x; t) + \tilde{Z}(x; -t)}{\sqrt{2}} P_n^{\left( \frac{d-1}{2} \right)}(x'U) dx
\]

\[
= \frac{1}{\omega_d} \int_{S^d} \int_{S^d} E \left( \frac{Z(x; t) + \tilde{Z}(x; -t)}{\sqrt{2}} P_n^{\left( \frac{d-1}{2} \right)}(x'u) \right) dx du
\]

\[
= \frac{\sqrt{2} \omega_d E Z(x; t)}{\omega_d} \int_{S^d} \int_{S^d} P_n^{\left( \frac{d-1}{2} \right)}(x'u) dx du
\]

\[
= \begin{cases} 
\sqrt{2} \omega_d E Z(x; t), & n = 0, \\
0, & n \in \mathbb{N},
\end{cases}
\]

where the last equality follows from Lemma 1. As is shown above, the covariance matrix function of an \( m \)-variate random field \( \left\{ \frac{Z(x; t) + \tilde{Z}(x; -t)}{\sqrt{2}} : x \in S^d, t \in \mathbb{T} \right\} \) is of the form

\[
\text{cov} \left( \frac{Z(x_1; t_1) + \tilde{Z}(x_1; -t_1)}{\sqrt{2}}, \frac{Z(x_2; t_2) + \tilde{Z}(x_2; -t_2)}{\sqrt{2}} \right)
\]

\[
= \frac{C(\vartheta(x_1, x_2); t_1 - t_2) + C(\vartheta(x_1, x_2); t_2 - t_1)}{2}
\]

\[
= \sum_{k=0}^{\infty} B_k(t_1 - t_2) P_k^{\left( \frac{d-1}{2} \right)}(\cos \vartheta(x_1, x_2))
\]

\[
= \sum_{k=0}^{\infty} B_k(t_1 - t_2) P_k^{\left( \frac{d-1}{2} \right)}(\cos \vartheta(x_1, x_2))
\]
= \sum_{k=0}^{\infty} B_k(t_1 - t_2) P_n^{(d-1)^2} (x_1, x_2), \quad x_1, x_2 \in S^d, t_1, t_2 \in \mathbb{T}.

From this observation and Lemma 1 we obtain the covariance matrix function of \{W_n(t), t \in \mathbb{T}\},

\begin{align*}
\text{cov}(W_n(t_1), W_n(t_2)) &= \text{cov} \left( \int_{S^d} Z(x; t_1) + \tilde{Z}(x; t_1) P_n^{(d-1)^2} (x' U) dx, \int_{S^d} Z(y; t_2) + \tilde{Z}(y; t_2) P_n^{(d-1)^2} (y' U) dy \right) \\
&= \frac{1}{\omega_d} \int_{S^d} \int_{S^d} \text{cov} \left( \int_{S^d} \frac{Z(x; t_1) + \tilde{Z}(x; t_1)}{\sqrt{2}} P_n^{(d-1)^2} (x' u) dx, \int_{S^d} \frac{Z(y; t_2) + \tilde{Z}(y; t_2)}{\sqrt{2}} P_n^{(d-1)^2} (y' u) dy \right) du \\
&= \frac{1}{\omega_d} \int_{S^d} \int_{S^d} \int_{S^d} \text{cov} \left( \frac{Z(x; t_1) + \tilde{Z}(x; t_1)}{\sqrt{2}}, \frac{Z(y; t_2) + \tilde{Z}(y; t_2)}{\sqrt{2}} \right) P_n^{(d-1)^2} (x' u) P_n^{(d-1)^2} (y' u) dx dy du \\
&= \frac{1}{\omega_d} \int_{S^d} \int_{S^d} \int_{S^d} \sum_{k=0}^{\infty} B_k(t_1 - t_2) P_n^{(d-1)^2} (x' y) P_n^{(d-1)^2} (x' U) P_n^{(d-1)^2} (y' u) dx dy du \\
&= \frac{1}{\omega_d} \sum_{k=0}^{\infty} B_k(t_1 - t_2) \int_{S^d} \left( \int_{S^d} P_n^{(d-1)^2} (x' y) P_n^{(d-1)^2} (x' U) dx \right) P_n^{(d-1)^2} (y' u) dy du \\
&= \frac{1}{\omega_d} B_n(t_1 - t_2) \int_{S^d} \left( \frac{\omega_d}{\alpha_n^2} \right)^2 P_n^{(d-1)^2} (1) du \\
&= B_n(t_1 - t_2) \left( \frac{\omega_d}{\alpha_n^2} \right)^2 P_n^{(d-1)^2} (1), \quad t_1, t_2 \in \mathbb{T},
\end{align*}

which implies that \(B_n(t)\) is a stationary covariance matrix function on \(\mathbb{T}\).

### 5.2 Proof of Theorem 2

(i) Suppose that (12) is the covariance matrix function of an \(m\)-variate random field \(\{Z(x; t), x \in S^d, t \in \mathbb{T}\}\). The existence of \(C(0; 0)\) ensures the convergence of \(\sum_{n=0}^{\infty} B_n(0)\). To verify that \(B_n(t)\) is a stationary covariance matrix function on \(\mathbb{T}\) for each fixed \(n \in \mathbb{N}_0\), consider an \(m\)-variate stochastic process

\[W_n(t) = \frac{1}{\pi} \int_{0}^{2\pi} Z(x; t) \cos(n \theta) d\theta, \quad t \in \mathbb{T},\]
where \( \mathbf{x} = (\cos \theta, \sin \theta)' \in S^1, \) \( 0 \leq \theta \leq 2\pi. \) The covariance matrix function of \( \{ \mathbf{W}_n(t), t \in T \} \) is given by

\[
\text{cov}(\mathbf{W}_n(t_1), \mathbf{W}_n(t_2)) = \frac{1}{\pi^2} \text{cov} \left( \int_0^{2\pi} Z(x_1; t_1) \cos(n\theta_1) d\theta_1, \int_0^{2\pi} Z(x_2; t_2) \cos(n\theta_2) d\theta_2 \right)
\]

\[
= \frac{1}{\pi^2} \int_0^{2\pi} \int_0^{2\pi} \text{cov}(Z(x_1; t_1), Z(x_2; t_2)) \cos(n\theta_1) \cos(n\theta_2) d\theta_1 d\theta_2
\]

\[
= \frac{1}{\pi^2} \int_0^{2\pi} \int_0^{2\pi} \sum_{k=0}^{\infty} B_k(t_1 - t_2) \cos(k\theta_1, x_1, x_2) \cos(n\theta_1) \cos(n\theta_2) d\theta_1 d\theta_2
\]

\[
= \frac{1}{\pi^2} \sum_{k=0}^{\infty} B_k(t_1 - t_2) \int_0^{2\pi} \int_0^{2\pi} \cos(k\theta) \cos(n\theta_1) \cos(n\theta_2) d\theta_1 d\theta_2
\]

\[
= \begin{cases} 
2B_0(t_1 - t_2), & n = 0, \\
B_n(t_1 - t_2), & n \in \mathbb{N}, \ t_1, t_2 \in T,
\end{cases}
\]

which implies that \( B_n(t) \) is a stationary covariance matrix function on \( T. \)

Conversely, if \( B_n(t) (n \in \mathbb{N}_0) \) are stationary covariance matrix functions on \( T \) and \( \sum_{n=0}^{\infty} B_n(0) \) converges, then, as Theorem 5 shows, we can generate an \( m \)-variate random field with (12) as its covariance matrix function, so that (12) satisfies inequality (1). By Theorem 8 of [21], there exists an \( m \)-variate Gaussian or elliptically contoured random field with (12) as its covariance matrix function.

(ii) We give a proof of the “only if” part here, while the “if” part is analogous to that in the proof of Part (i). Suppose that (13) is the covariance matrix function of an \( m \)-variate random field \( \{ Z(x; t), x \in S^d, t \in T \}. \) Evidently, the existence of \( C(0; 0) \) implies the convergence of \( \sum_{n=0}^{\infty} B_n(0) P_n^{(d-1)}(1). \) For each fixed \( n \in \mathbb{N}_0, \) consider an \( m \)-variate stochastic process

\[
\mathbf{W}_n(t) = \int_{S^d} Z(x; t) P_n^{(d-1)}(x'U) d\mathbf{x}, \quad t \in T,
\]

where \( U \) is a \((d + 1)\)-dimensional random vector uniformly distributed on \( S^d \) and independent with \( \{ Z(x; t), x \in S^d, t \in T \}. \) In a way similar to the proof of Theorem 1, we apply Lemma 1 to obtain that the covariance matrix function of \( \{ \mathbf{W}_n(t), t \in T \} \) is positively propositional to
\( \mathbf{B}_n(t) \). More precisely,
\[
\text{cov}(\mathbf{W}_n(t_1), \mathbf{W}_n(t_2)) = \mathbf{B}_n(t_1 - t_2) \left( \frac{\omega_d}{\alpha_n^2} \right)^2 P_n^{(d-1)}(1), \quad t_1, t_2 \in \mathbb{T},
\]
so that \( \mathbf{B}_n(t) \) is a stationary covariance matrix function on \( \mathbb{T} \).

### 5.3 Proof of Theorem 3

For a fixed \( t \in \mathbb{T} \), in a way similar to that in the proof of Theorem 1 it can be verify that
\[
2C(\vartheta(x_1, x_2); 0) + C(\vartheta(x_1, x_2); t) + C(\vartheta(x_1, x_2); -t) \quad \text{and} \quad 2C(\vartheta(x_1, x_2); 0) - C(\vartheta(x_1, x_2); t) - C(\vartheta(x_1, x_2); -t)
\]
are isotropic covariance matrix functions on \( \mathbb{S}^\infty \). They necessarily take the form, by Theorem 4 of [23],
\[
2C(\vartheta(x_1, x_2); 0) + C(\vartheta(x_1, x_2); t) + C(\vartheta(x_1, x_2); -t) = \sum_{n=0}^{\infty} \mathbf{B}_{n+}(t) \cos^n(\vartheta(x_1, x_2)), \quad (25)
\]
\[
2C(\vartheta(x_1, x_2); 0) - C(\vartheta(x_1, x_2); t) - C(\vartheta(x_1, x_2); -t) = \sum_{n=0}^{\infty} \mathbf{B}_{n-}(t) \cos^n(\vartheta(x_1, x_2)), \quad (26)
\]
where \( \mathbf{B}_{n+}(t) \) and \( \mathbf{B}_{n-}(t) \) (\( n \in \mathbb{N}_0 \)) are \( m \times m \) positive definite matrices, and \( \sum_{n=0}^{\infty} \mathbf{B}_{n+}(t) \) and \( \sum_{n=0}^{\infty} \mathbf{B}_{n-}(t) \) converge. The representation (14) results from taking the difference between (25) and (26), and
\[
\mathbf{B}_n(t) = \frac{1}{4} \mathbf{B}_{n+}(t) - \frac{1}{4} \mathbf{B}_{n-}(t), \quad n \in \mathbb{N}_0.
\]
Clear, \( \mathbf{B}_n(t) \) is symmetric, and \( \sum_{n=0}^{\infty} \mathbf{B}_n(t) \) converges.

In particular, \( \mathbf{B}_0(t) = \mathbf{C}(0; t), t \in \mathbb{T}, \) is a stationary covariance matrix function. For each \( n \in \mathbb{N} \), we are going to confirm that \( \mathbf{B}_n(t), t \in \mathbb{T}, \) is a stationary covariance matrix function. For every \( d \geq 2 \), a version (13) of \( \mathbf{C}(\vartheta; t) \) is derived from (14) by using the formula (17),
\[
\mathbf{C}(\vartheta; t) = \sum_{n=0}^{\infty} \mathbf{B}_n(t) \cos^n(\vartheta(x_1, x_2))
\]
\[
= \sum_{n=0}^{\infty} \mathbf{B}_n(t) \sum_{k=0}^{\left[ \frac{d}{2} \right]} \beta_{k,n}^{(d-1)} P_{n-2k}^{(d-1)} \cos \vartheta
\]
\[
= \sum_{n=0}^{\infty} \mathbf{A}_n^{(d-1)}(t) P_n^{(d-1)}(\cos \vartheta), \quad \vartheta \in [0, \pi], \ t \in \mathbb{T},
\]

20
Hence, \( \lim_{n \to \infty} A_n^{(d-1)}(t) = \sum_{k=0}^{\infty} \beta_{k,2k+n}^{(d-1)} B_{2k+n}(t), \quad n \in \mathbb{N}_0. \)

Since a covariance matrix function \( C(\vartheta; t) \) on \( \mathbb{S}^d \times T \) is also a covariance matrix function on \( \mathbb{S}^d \times T \) for every \( d \in \mathbb{N} \), applying Theorem 1 to \( C(\vartheta; t) \) on \( \mathbb{S}^d \times T \) we obtain that, for each \( n \in \mathbb{N}_0 \), \( A_n^{(d-1)}(t) \) is a stationary covariance matrix function on \( T \). So is \( \frac{A_n^{(d-1)}}{\beta_{0,n}^{(d-1)}}(t), \quad t \in T, \) by Theorem 6 of [21].

For \( k \in \mathbb{N} \), it follows from the formula \( \Gamma(x+1) = x\Gamma(x) \) that

\[
\frac{\beta_{k,2k+n}^{(d-1)}}{\beta_{0,n}^{(d-1)}} = \frac{(n+2k)! n + k + d-1}{2k n!} \frac{\Gamma(n+k+d-1)}{\Gamma(n+k+d+1)}
\]

\[
= \frac{(n+2k)! n + k + d-1}{2k n!} \frac{1}{\Gamma\left(n+k+\frac{d+1}{2}\right)}
\]

\[
\to 0, \quad \text{as} \quad d \to \infty.
\]

Hence, \( \lim_{d \to \infty} \frac{A_n^{(d-1)}}{\beta_{0,n}^{(d-1)}}(t) = B_n(t) \) is a stationary covariance matrix function on \( T \).

(ii) The “only if” part follows from Part (i), and the “if” part from Theorem 8 of [21].

### 5.4 Proof of Theorem 4

The convergent assumption of \( \sum_{n=0}^{\infty} B_n(0) P_n^{(d-1)}(1) \) ensures the mean square convergence of the series at the right hand of (21). In fact, for \( n_1, n_2 \in \mathbb{N} \), we have

\[
E\left( \sum_{i=n_1}^{n_1+n_2} \mathbf{V}_i(t) P_i^{(d-1)}(\mathbf{x}' \mathbf{U}) \right) \left( \sum_{j=n_1}^{n_1+n_2} \mathbf{V}_j(t) P_j^{(d-1)}(\mathbf{x}' \mathbf{U}) \right)'
\]

\[
= E\left( \sum_{i=n_1}^{n_1+n_2} \sum_{j=n_1}^{n_1+n_2} \mathbf{V}_i(t) \mathbf{V}_j(t) P_i^{(d-1)}(\mathbf{x}' \mathbf{U}) P_j^{(d-1)}(\mathbf{x}' \mathbf{U}) \right)'
\]

\[
= \sum_{i=n_1}^{n_1+n_2} \sum_{j=n_1}^{n_1+n_2} E(\mathbf{V}_i(t) \mathbf{V}_j(t)) E\left( P_i^{(d-1)}(\mathbf{x}' \mathbf{U}) P_j^{(d-1)}(\mathbf{x}' \mathbf{U}) \right)
\]

\[
= \omega_d \sum_{i=n_1}^{n_1+n_2} B_i(0) P_i^{(d-1)}(1)
\]

\[
\to 0, \quad \text{as} \quad n_1, n_2 \to \infty,
\]
where the second equality follows from the independent assumption between \( U \) and \( \{V_n(t), t \in \mathbb{T}\} \), and the third one from Lemma 2.

Under the independent assumption among \( U \) and \( \{V_n(t), t \in \mathbb{T}\} \), \( n \in \mathbb{N}_0 \), we obtain the mean and covariance matrix functions of \( \{Z(x; t), x \in S^d, t \in \mathbb{T}\} \) from Lemma 2, with

\[
\mathbb{E}Z(x; t) = \sum_{n=0}^{\infty} \mathbb{E}V_n(t)E P_n^{(\frac{d-1}{2})}(x'U) = 0, \quad x \in S^d, t \in \mathbb{T},
\]

and

\[
cov(Z(x_1; t_1), Z(x_2; t_2)) = \sum_{n=0}^{\infty} B_n(t_1 - t_2) \cos(\theta(x_1, x_2)), \quad x_1, x_2 \in S^d, \ t_1, t_2 \in \mathbb{T}.
\]

The latter is obviously isotropic and continuous on \( S^d \) and stationary on \( \mathbb{T} \).

### 5.5 Proof of Theorem 5

The series at the right hand side of (22) is convergent in mean square, since \( \sum_{n=0}^{\infty} B_n(0) \) is convergent and, for \( n_1, n_2 \in \mathbb{N}_0 \),

\[
\sum_{n=1}^{n_1+n_2} \sum_{j=1}^{n_1+n_2} E \{V_{i1}(t)V_{j1}(t)\cos(i\theta)\cos(j\theta) + V_{i1}(t)V_{j2}(t)\cos(i\theta)\sin(j\theta)
\]

\[
\quad + V_{i2}(t)V_{j1}(t)\sin(i\theta)\cos(j\theta) + V_{i2}(t)V_{j2}(t)\sin(i\theta)\sin(j\theta)\}
\]

\[
= \sum_{n=1}^{n_1+n_2} B_n(0)
\]

\[
\to 0, \quad \text{as} \quad n_1 \to \infty, \ n_2 \to \infty.
\]
where the second equality is due to the assumptions on \( \{V_{n_1}(t), t \in \mathbb{T}\} \) and \( \{V_{n_2}(t), t \in \mathbb{T}\} \), \( n \in \mathbb{N}_0 \).

Clearly, the mean function of \( \{Z(x; t), x \in S^1, t \in \mathbb{T}\} \) is identical to 0, and its covariance matrix function is

\[
\text{cov}(Z(x_1; t_1), Z(x_2; t_2)) = \sum_{n=0}^{\infty} B_n(t_1 - t_2) \cos(n(\theta_1 - \theta_2))
\]

\[
= \sum_{n=0}^{\infty} B_n \cos(n\vartheta(x_1, x_2)), \quad x_k = (\cos \theta_k, \sin \theta_k)' \in S^1, \ t_k \in \mathbb{T}, \quad k = 1, 2.
\]

**Acknowledgment**

A reviewer’s valuable comments and helpful suggestions are gratefully acknowledged.

**References**

[1] Andrews, G. E., Askey, R., Roy, R.: Special Functions. Cambridge University Press, Cambridge (1999)

[2] Askey, R., Bingham, N. H.: Gaussian processes on compact symmetric spaces. Z. Wahrscheinlichkeitstheorie verw. Gebiete 37, 127-143 (1976)

[3] Berman, S. M.: Isotropic Gaussian processes on the Hilbert sphere. Ann. Prob. 8, 1093-1106 (1980)
[4] Bingham, N. H.: Positive definite functions on spheres. Proc. Cambridge Phil. Soc. 73, 145-156 (1973)

[5] Cartan, É.: Sur la détermination d’un systém orthogonal complet dans un espace de Riemann symétrique clos. *Circolo matematico di Palermo*. Rendiconti 53, 217-252 (1929)

[6] Cheng, D., Xiao, Y.: Excursion probability of Gaussian random fields on sphere. Bernoulli 22, 1113-1130 (2016)

[7] Cohen, S., Lifshits, M. A.: Stationary Gaussian random fields on hyperbolic spaces and on Euclidean spheres. ESAIM 16, 165-221 (2012)

[8] D'Ovidio, M.: Coordinates changed random fields on the sphere. J. Stat. Phys. 154, 1153-1176 (2014)

[9] Du, J., Ma, C., Li, Y.: Isotropic variogram matrix functions on spheres. Math. Geosci. 45, 341-357 (2013)

[10] Gangolli, R.: Positive definite kernels on homogeneous spaces and certain stochastic processes related to Lévy’s Brownian motion of several parameters. Ann Inst H PoincaréB 3, 121-226 (1967)

[11] Gaspari, G., Cohn, S. E.: Construction of correlations in two and three dimensions. Q. J. R. Meteorol. Soc. 125 723-757 (1999)

[12] Gaspari, G., Cohn, S. E., Guo, J., Pawson, S.: Construction and application of covariance functions with variable length-fields. Q. J. R. Meteorol. Soc. 132, 815-1838 (2006)

[13] Gradshteyn, I. S., Ryzhik, I. M.: Tables of Integrals, Series, and Products, 7th edition. Academic Press, Amsterdam (2007)

[14] Hannan, E. J.: Multiple Time Series. Wiley, New York (1970)

[15] Helgason, S.: Integral Geometry and Radon Transforms. Springer, New York (2011)

[16] Jones, R. H.: Stochastic processes on a sphere. Ann. Math. Statist. 34, 213-218 (1963)
[17] Lamberg, L., Muinonen, K., Ylönen, J., Lumme, K.: Spectral estimation of Gaussian random circles and spheres. J. Comput. Appl. Math. 136, 109-121 (2001)

[18] Lang, A. and Schwab, C.: Isotropic Gaussian random fields on the sphere: Regularity, fast simulation and stochastic partial differential equations. Ann. Appl. Prob. 25, 3047-3094 (2015)

[19] Leonenko, N., Sakhno, L.: On spectral representation of tensor random fields on the sphere. Stoch. Anal. Appl. 31, 167-182 (2012)

[20] Leonenko, N., Shieh, N.: Rényi function for multifractal random fields. Fractals, 21, 1350009, 13 pp (2013)

[21] Ma, C.: Vector random fields with second-order moments or second-order increments. Stoch. Anal. Appl. 29, 197-215 (2011)

[22] Ma, C.: Stationary and isotropic vector random fields on spheres. Math. Geosci. 44, 765-778 (2012)

[23] Ma, C.: Isotropic covariance matrix functions on all spheres. Math. Geosci. 47, 699-717 (2015)

[24] Ma, C.: Stochastic representations of isotropic vector random fields on spheres. Stoch. Anal. Appl. 34, 389-403 (2016)

[25] Ma, C.: Isotropic covariance matrix polynomials on spheres. Stoch. Anal. Appl., To appear.

[26] Malyarenko, A.: Invariant Random Fields on Spaces with a Group Action. Springer, New York (2013)

[27] Malyarenko, A., Olenko, A.: Multidimensional covariant random fields on commutative locally compact groups. Ukrainian Math. J. 44, 1384-1389 (1992)

[28] Mangulis, V.: Handbook of Series for Scientists and Engineers. Academic Press, Inc., New York (1965)
[29] McLeod, M. G.: Stochastic processes on a sphere. Phy. Earth Plan. Interior 43, 283-299 (1986)

[30] Mokljacuk, M. P., Jadrenko, M. I.: Linear statistical problems for stationary isotropic random fields on a sphere, I. Theor. Prob. Math. Statist. 18, 115-124 (1979)

[31] Müller, C.: Analysis of Spherical Symmetries in Euclidean Spaces. Springer, New York (1998)

[32] Roy, R.: Spectral analysis for random process on the circle. J. Appl. Prob. 9, 745-757 (1972)

[33] Roy, R.: Spectral analysis for a random process on the sphere. Ann. Inst. Statist. Math. 28, 91-97 (1976)

[34] Schoenberg, I.: Positive definite functions on spheres. Duke Math. J. 9, 96-108 (1942)

[35] Szegö, G. Orthogonal Polynomials, 4th edition. Amer. Math. Soc. Colloq. Publ., vol 23. Amer. Math. Soc., Providence (1975)

[36] Wang, H.-C.: Two-point homogenous spaces. Ann. Math. 55, 177-191 (1959)

[37] Yadrenko, A. M. Spectral Theory of Random Fields. Optimization Software, New York (1983)

[38] Yaglom, A. M.: Second-order homogeneous random fields. Proc. 4th Berkeley Symp. Math. Stat. Prob. 2, 593-622 (1961)

[39] Yaglom, A. M.: Correlation Theory of Stationary and Related Random Functions. vol. I. Springer, New York (1987)