Exact solutions for some $\mathcal{N} = 2$ supersymmetric $SO(N)$ gauge theories with vectors and spinors

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We find exact solutions for $\mathcal{N} = 2$ supersymmetric $SO(N)$, $N = 7, 9, 10, 11, 12$ gauge theories with matter in the fundamental and spinor representation. These theories, with specific numbers of vectors and spinors, arise naturally in the compactification of type IIA string theory on suitably chosen Calabi–Yau threefolds. Exact solutions are obtained by using mirror symmetry to find the corresponding type IIB compactification. We propose generalizations of these results to cases with arbitrary numbers of massive vectors and spinors.

I. INTRODUCTION

Over the last few years our understanding of the low energy behavior of supersymmetric field theories has advanced substantially. Using field theory arguments, Seiberg and Witten found the exact solution of $\mathcal{N} = 2$ supersymmetric $SU(2)$ gauge theories with matter in the fundamental representation [1]. Subsequently many other gauge groups with fundamental matter were analyzed in this way [2–5]. It turned out to be rather difficult to generalize these results to theories with matter in any other representation because in these cases the curves encoding the gauge coupling are usually not hyperelliptic. General Riemann surfaces have more parameters than can be fixed by studying various limits of the gauge theory.

String theory offers a way to construct supersymmetric gauge theories geometrically. In this approach, the curves encoding the gauge couplings are real physical objects, which can be determined by string theory arguments. This provides a way to find exact solutions of gauge theories for which the curves are not hyperelliptic.

There are two methods of constructing four–dimensional gauge theories from string theory. The first uses configurations of branes in type IIA string theory [6,7]. Lifting the type IIA construction to M-theory, one can obtain the exact solution to the gauge theory living on the common directions of the branes. This approach allows the elegant construction of theories with matter in the fundamental or two-index tensor representation. However, at present there are no brane constructions for theories with higher index tensors or spinors of $SO(N)$.

The second approach involves compactifying type IIA string theory on $K3$ fibered Calabi–Yau threefolds [8–13]. There is a clear physical picture of how a theory with a gauge group corresponding to the singularity type [14] of the $K3$ arises in four dimensions. This provides a way to construct a large class of $d = 4$, $\mathcal{N} = 2$ gauge theories. Type IIA string theory, compactified on such a Calabi–Yau threefold, is conjectured to be dual to heterotic strings on $K3 \times T^2$, where the $E_8 \times E_8$ gauge group is broken by instantons [8]. One can use the breaking patterns of the $E_8 \times E_8$ adjoint to determine the charged matter content of the gauge theory [15]. At tree level this gives a complete description of the gauge theory in four dimensions. However, there are quantum corrections to the tree level results due to world sheet instantons in the type IIA theory. These instanton corrections can be summed up using mirror symmetry [16] (for reviews see [7]). Mirror symmetry pairs up two different Calabi–Yau manifolds such that type IIA compactified on the first gives rise to the same string world sheet theory, and therefore to the same gauge theory as type IIB compactified on the other. We obtain exact field theory results by studying the properties of the type IIB Calabi–Yau, because there are no quantum corrections to the tree level results in the type IIB compactification.

Since the description of the gauge theory depends only on the local properties of the Calabi–Yau, it is sufficient to consider local approximations to the threefolds. Constructing these without first constructing the entire Calabi–Yau is called ‘Geometric Engineering’ [8,9]. We will not use this approach for the analysis in this paper, because global descriptions of the Calabi–Yau manifolds we will consider are available.

The approach of constructing field theories from string theory compactifications may be more indirect than the brane picture, but it has the advantage of providing descriptions of theories with matter in higher index tensor representations, and in the spinor representation for $SO(N)$ theories [12,13]. In this paper we analyze $SO(N)$, $N = 7, 9, 10, 11, 12$ theories with vectors and spinors. We find exact solutions on the Coulomb branch of these theories in the form of ALE fibrations over a sphere.

So far, no theories of this type have been analyzed in the brane picture, but the solutions we present here may turn out to be useful in constructing appropriate brane configurations. The fact that our curves agree with known field theory results, in the cases where these are available, lends further support to the conjectured duality between type IIA compactified on a Calabi–Yau and heterotic strings on $K3 \times T^2$. It also provides additional examples where the instanton series can be summed up correctly, using the mirror map.
The paper is organized as follows: In the first two subsections of Sec. II we review the construction of a class of Calabi–Yau threefolds which give rise to $d = 4$, $N = 2$ $SO(10)$ and $SO(12)$ gauge theories with specific numbers of fundamentals and spinors. We use the toric description of these manifolds to find explicit expressions for the mirror manifolds. A local approximation to the mirror manifolds in the form of ALE fibrations provides the exact solutions for these theories. We also propose generalizations to arbitrary numbers of massive vectors and spinors and perform several consistency checks on our results. The non–simply laced cases $SO(7)$, $SO(9)$ and $SO(11)$ are the subject of the next three subsections. In these cases we slightly modify the conventional method of finding the mirror to obtain the exact solutions in the most convenient form. These modifications are explained in Sec. II C. We summarize our results in Sec. II D.

II. EXACT SOLUTIONS FROM MIRROR SYMMETRY

Type IIA string theory, compactified on a Calabi–Yau threefold that is both an elliptic and a $K3$ fibration, gives rise to an $N = 2$ gauge theory in four dimensions. Such elliptically fibered manifolds are defined by

$$y^2 = x^3 + x f(z_1, z_2) + g(z_1, z_2),$$

(2.1)

where $f$ and $g$ are functions of the base coordinates $z_1, z_2$. For this equation to define a Calabi–Yau, the functions $f$ and $g$ must be of the form

$$f(z_1, z_2) = \sum_{i=0}^{I} z_1^{8-i} f_{8+n(4-i)}(z_2)$$

$$g(z_1, z_2) = \sum_{j=0}^{J} z_1^{12-j} f_{12+n(4-j)}(z_2),$$

(2.2)

where the subscript on the polynomials $f$ and $g$ in the sums indicates their degree in $z_2$. $I$ and $J$ are the maximum values of $i$ and $j$ such that the degree is not negative. We can view this threefold as an elliptic fibration over the Hirzebruch surface $F_n$ or as a $K3$ fibration over a sphere parameterized by $z_2$.

Type IIA string theory compactified on this Calabi–Yau is conjectured to be dual to heterotic strings compactified on $K3 \times T^2$ with $12 - n$ and $12 + n$ instantons embedded in the first and second $E_8$ of the $E_8 \times E_8$ gauge group and all Wilson lines switched off. The coefficients of the monomials in Eq. (2.3) that are proportional to $xz_1^4$ and $z_1^6$ correspond to the moduli of the $K3$ and the other terms specify the $E_8 \times E_8$ gauge bundle. The coefficients of terms with lower powers of $z_1$ define the embedding of $12 - n$ instantons in the first $E_8$ and the remaining terms do the same for the $12 + n$ instantons in the second $E_8$.

For generic choices of the polynomials $f$ and $g$, the instantons break the $E_8 \times E_8$ gauge group of heterotic strings as far as possible. The $E_8$ with $12 + n$ instantons is broken completely (for $n \geq 0$) while the other is broken to some terminal group without matter. This is the case that was studied in [20] for various instanton embeddings.

Here we consider more restrictive instanton embeddings, which result in larger unbroken subgroups of the $E_8$ with $12 - n$ instantons. On the type IIA side such instanton embeddings correspond to choosing Calabi–Yau threefolds that have a more severe singularity in their $K3$ fiber than one would get from the generic choice of polynomials. For example, we can consider the Calabi–Yau defined by setting

$$f_{8-2n} = h_{4-n}^2$$

$$g_{12-3n} = h_{4-n}^3$$

$$g_{12-2n} = q_{6-n} - f_{8-n} h_{4-n}$$

(2.3)

and choosing the coefficients of lower powers of $z_1$ to vanish. $h_{4-n}$ and $q_{6-n}$ are polynomials in $z_2$ of the degree indicated by the subscripts. One can use Kodaira’s classification to determine the singularity type of the $K3$ fiber. The definitions above ensure that the fiber has a split $D_5$ singularity [12]. We can make this manifold smooth by blowing up a collections of spheres in the base of the $K3$, i.e., by modifying its Kähler structure. The intersection forms of these spheres give the entries in the Cartan matrix of the corresponding gauge group ($SO(10)$ for $D_5$). Compactifying type IIA on a Calabi–Yau with this blown–up $K3$ as a fiber results in a $d = 4$ $SO(10)$ gauge theory, where the $SO(10)$ is broken to its Cartan subalgebra. This situation arises because the 2-branes of type IIA can wrap around the blow–up spheres with two different orientations, giving rise to a pair of $W^\pm$ bosons with a mass proportional to the area of the spheres. Shrinking a sphere to zero size makes the $W^\pm$ massless, which corresponds to
unhiggsing an $SU(2)$ factor. Since the blow–up spheres have intersection forms determined by the singularity type, the corresponding $SU(2)$ factors link up to make the gauge group indicated by the singularity type. Thus it is clear that the Kähler structure moduli are related to the coordinates on the Coulomb branch of the $d = 4$ gauge theories in the type IIA picture. On the heterotic side, these blow–ups correspond to switching on Wilson lines to break the gauge group.

The moduli space of a Calabi–Yau is the space of all possible choices of its Kähler and complex structure. Locally, it is a direct product of the complex and Kähler structure moduli spaces. In type IIA compactifications, the Kähler moduli are vector multiplets while the complex structure moduli are hypermultiplets of the space–time theory. Since the dilaton is also a hypermultiplet, the Kähler moduli space does not have perturbative string corrections. However, there are world sheet instanton corrections to the Kähler moduli space, which are related to gauge theory instantons via the duality to heterotic strings [18]. Mirror symmetry provides a way to sum up these corrections.

To find the mirror manifold of the type IIA Calabi–Yau it is convenient to encode its salient properties using toric geometry [10,13,21]. For the cases we are considering here this was worked out in [12,13], so we will only summarize the results. The natural starting point for the application of toric methods is the representation of the Calabi–Yau as a hypersurface in a weighted projective space. The toric data consist of two polyhedra, the Newton polyhedron $\triangle$, and the dual polyhedron $\triangledown$. The vertices of $\triangledown$ are the normal vectors on the facets of $\triangle$ and vice versa. The Newton polyhedron encodes the complex structure of the manifold. For every monomial that appears in the defining equation of the manifold, the vector of exponents gives a corresponding point in the polyhedron. Some of the vertices of the dual polyhedron define the ambient space in which the Calabi–Yau is embedded and the others encode the Kähler structure of the blow–up spheres in the base of the $K3$.

The role of the two polyhedra $\triangle$ and $\triangledown$ are exchanged under mirror symmetry. In order to construct the manifold for type IIB compactifications, we take the dual polyhedron, $\triangledown$, to encode the complex structure of the mirror Calabi–Yau and the Newton polyhedron, $\triangle$, to define the toric variety in which it is embedded. The vertices of the dual polyhedron that encode the Kähler structure of the blow–ups on the type IIA side determine the complex structure of the type IIB manifold.

On the type IIB side, the complex structure moduli are vector multiplets and the Kähler structure and the dilaton are hypermultiplets. As on the type IIA side, the vector moduli space is not corrected by perturbative string effects but on the type IIB side the world sheet instanton corrections are absent as well [21]. Thus the classical description of the complex structure moduli space of the type IIB Calabi–Yau is exact. Since these moduli encode the behavior of the gauge theory, we can read off the exact solutions from the IIB manifold.

Below, we discuss a series of Calabi–Yau manifolds that give rise to $SO(10)$, $SO(12)$ and $SO(7)$, $SO(9)$ and $SO(11)$ gauge groups with spinors and fundamentals. In the first two cases we find exact solutions using Batyrev’s construction of the mirror [21]. For the non–simply laced cases we slightly modify the construction to simplify the resulting curves. These modifications are explained in Sec. [1C]

A. The Calabi–Yau for $SO(10)$

The dual polyhedron for the Calabi–Yau that gives rise to an $SO(10)$ gauge theory with $4−n$ spinors and $6−n$ vectors was constructed in [21]. The derivation there uses Tate’s algorithm and a more general form of the defining equation, Eq. (2.1), that makes it easier to encode the split or nonsplit property of the singularity. The same polyhedron was also found in [13], using toric arguments only. Using the basis of [13], the dual polyhedron, $\triangledown$, is given by the vertices

\[
\begin{align*}
\tilde{v}_1 &= (-1, 0, 2, 3) \\
\tilde{v}_2 &= (1, -n, 2, 3) \\
\tilde{v}_3 &= (0, -1, 2, 3) \\
\tilde{v}_4 &= (0, 0, -1, 0) \\
\tilde{v}_5 &= (0, 0, 0, -1) \\
\tilde{v}_6 &= (0, 0, 0, 0) \\
\tilde{v}_7 &= (0, 0, 2, 3) \\
\tilde{v}_8 &= (0, 1, 2, 3) \\
\tilde{v}_9 &= (0, -2, 2, 3) \\
\tilde{v}_{10} &= (0, -2, 1, 2) \\
\tilde{v}_{11} &= (0, -1, 1, 1) \\
\tilde{v}_{12} &= (0, -1, 0, 1) \\
\tilde{v}_{13} &= (0, -1, 0, 0).
\end{align*}
\]

(2.4)

This list of vertices includes all points that do not lie on codimension one facets of the dual polyhedron, i.e., this polyhedron encodes a fully blown–up type IIA manifold. The vertices $\tilde{v}_1, \ldots, \tilde{v}_8$ define the toric variety in which the type IIA manifold is embedded and the remaining vertices correspond to the blow–up spheres needed to repair the $D_5$ singularity. The vertices of the corresponding Newton polyhedron are

\[1\text{Note that in Refs. [2,13] the unhiggsing of the } E_8 \text{ with } 12+n \text{ instantons was studied while we are unhiggsing the } E_8 \text{ with } 12-n \text{ instantons.}\]
\[ v_1 = (2, 1, -1, 1) \quad v_2 = (3, 1, 1, 0) \quad v_3 = (0, 0, 1, -1) \]
\[ v_4 = (6, 1, 1, 1) \quad v_5 = (0, 0, -2, 1) \quad v_6 = (6, -6, 1, 1) \]
\[ v_7 = (-6 - 6n, -6, 1, 1) \quad v_8 = (n - 6, 1, 1, 1) \quad v_9 = (n - 3, 1, 1, 0) \]
\[ v_{10} = (n - 2, 1, -1, 1). \]  

Note that for \( n = 4 \) the vertices \( v_1 \) and \( v_{10} \) become identical which allows us to drop one of them.

We can use the information encoded in the dual pair of polyhedra, \( \Delta, \nabla \), to construct the mirror manifold of our initial Calabi–Yau. Batyrev’s construction of the mirror [23] requires that we switch the roles of the two polyhedra. An embedding polynomial defining the mirror manifold is given by

\[
W = \sum_j a_j \prod_i x_i^{v_j + 1} = 0,  \tag{2.6}
\]

where the \( x_i \) are coordinates in a weighted projective space (or more generally in a toric variety). In the cases we consider here there are nine or ten vertices in the Newton polyhedron, corresponding to the same number of coordinates in the hypersurface constraint, Eq. (2.6). We can eliminate some of these coordinates using the \( \mathbf{C}^* \) actions that define the identifications of coordinates in the embedding space. Sets of weights for the \( \mathbf{C}^* \) actions can be found by looking for sets of five vertices in \( \Delta \) such that

\[
\sum_i v_i k_i = 0,  \tag{2.7}
\]

where the coefficients satisfy \( k_i \neq 0 \). One can use these \( \mathbf{C}^* \) actions to set all but five of the coordinates in Eq. (2.6) to one. This gives a description of the Calabi–Yau in some local coordinate patch with one remaining \( \mathbf{C}^* \) action. For our purposes it is most convenient to retain \( x_1, x_2, x_3, x_6, x_7 \) and set the remaining coordinates to one. This amounts to choosing a patch in which the relevant properties of the Calabi–Yau are described most easily. Using these coordinates we find the following defining equation for the mirror manifold

\[
W = x_7^{12+6n} + a_9 x_1^{4-n} x_2^{6-n} x_6^{12+6n} + a_1 x_1 x_2^2 (x_6 x_7)^{12} + a_2 x_7^2 + a_3 x_2 x_3^2 + a_4 x_1 x_2 x_3 (x_6 x_7)
+ a_5 x_1^2 x_3^2 (x_6 x_7)^6 + a_6 x_1^4 x_3^4 + a_7 x_2 (x_6 x_7)^{18} + a_8 (x_6 x_7)^{16} + a_9 x_2 x_3 (x_6 x_7)^9
+ a_{10} x_1 (x_6 x_7)^8 + a_{11} x_3 (x_6 x_7)^7.  \tag{2.8}
\]

This Calabi–Yau is a K3 fibration. We can make this explicit by defining \( x_0 = x_6 x_7 \) and \( \zeta = (x_7/x_6)^{6+3n} \). Using the freedom to rescale \( x_1, x_2 \) and \( x_3 \) to eliminate three of the coefficients \( a_i \) we obtain

\[
W = \left( \zeta + a_9 \frac{x_1^{4-n} x_2^{6-n}}{x_6} \right) x_0^{6+3n} - 2x_1 x_2^2 x_0^{12} - x_1^3 + x_2 x_3^2 + a_4 x_1 x_2 x_3 x_0
+ a_5 x_1^2 x_2^6 + a_6 x_1^4 x_2^4 + a_7 x_2 x_0^{18} + a_8 x_0^{16} + a_9 x_2 x_3 x_0^9 + a_{10} x_1 x_0^8 + a_{11} x_3 x_0.  \tag{2.9}
\]

The first term in this equation describes the base sphere and the remaining terms define a K3. Approximating the K3 locally as an ALE space, we can bring this expression into a form that is equivalent to a Seiberg-Witten curve. In order to do this, we set \( x_0 = 1 \) and observe that the first three terms in the K3 part give a three–coordinate form of a \( D_5 \) singularity located at the origin. The terms with coefficients \( a_5 \) and \( a_6 \) are irrelevant near the singularity and can be neglected for our present purposes. The remaining terms are the versal deformations of the \( D_5 \) singularity. The following chain of substitutions brings the singularity into the standard form:

\[
\begin{align*}
x_3 &= y - \frac{1}{2} (a_9 + a_4 x_1) \\
a_8 &= c_1 + \frac{1}{16} (8a_{11} a_9 + 4a_{11} a_1 a_4 - 4a_{10}^2 - a_{11}^2 a_3^2) \\
a_7 &= c_2 + \frac{a_9}{8} (2a_9 + 2a_1 a_4 - a_{11} a_3^2) \\
a_{10} &= -c_3 + \frac{a_4}{16} (8a_{11} + 4a_4 a_3^2) \\
a_9 &= \frac{2}{a_4} c_4 \\
a_{11} &= -2(-a_0)^{1/2}.
\end{align*}  \tag{2.10}
\]

Neglecting an irrelevant term proportional to \( x_1^2 x_2 \) we obtain the standard form of the \( D_5 \) singularity after shifting
and defining $z = x_2$:

$$W = \left( \zeta + a_0 \frac{x_1^{4-n} z^{6-n}}{\zeta} \right) - x^2 + z^4 + y^2 z - 2(-c_0)^{1/2} y + c_4 z^3 + c_3 z^2 + c_2 z + c_1 + \cdots,$$

where Eq. (2.11) should be substituted for $x_1$. The ellipsis denotes contributions from terms that are irrelevant close to the singularity. Neglecting these terms amounts to switching off gravity or conversely taking the field theory limit.

The general method for converting $D_n$ type ALE fibrations into Seiberg-Witten curves was first introduced in [4] to find the curves for $SO(2N)$ gauge groups without matter. Using the same approach we integrate out $y$ from Eq. (2.12) and multiply by $z$. Absorbing a factor of $z$ into $\zeta$ gives

$$W = \left( \zeta + \Lambda^{2\beta_0} \frac{x_1^{4-n} z^{6-n+2}}{\zeta} \right) - x^2 + 2P(z),$$

where $P(z)$ is given by

$$P(z) = \frac{1}{2} \left( z^5 + c_4 z^4 + c_3 z^3 + c_2 z^2 + c_1 z + c_0 \right).$$

For $n = 4$, $x$ appears only quadratically and can be integrated out trivially. The substitutions $\zeta = y - P(z)$ and $z \rightarrow z^2$ result in a double cover version of the curve for $SO(10)$ with two fundamentals

$$y^2 = P^2(z^2) - \Lambda^{12} z^8.$$

Note that for the asymptotically free cases, $n = 2, 3, 4$, both $x$ and $y$ appear at most quadratically and can be integrated out. In the cases with one or two spinors of $SO(10)$, $n = 2, 3$, we still obtain a curve but it is no longer hyperelliptic. The $U(1)$ gauge couplings on the Coulomb branch are encoded in the normalized period matrix of this curve. The Seiberg-Witten 1-form needed to evaluate the period matrix, can be derived from the unique holomorphic 3-form, $\Omega$, of the original Calabi–Yau.

It is very tempting to modify Eq. (2.13) to allow an arbitrary number of massive spinors and vectors. This can probably be achieved by replacing the fibration over the sphere in Eq. (2.12) according to

$$\zeta + a_0 \frac{1}{\zeta} x_1^{4-n} z^{6-n} \rightarrow \zeta + a_0 \frac{1}{\zeta} \prod_{i=1}^{N_s} (x_1 - m_i^4) \prod_{j=1}^{N_f} (z - m_j^2),$$

where the $m_i$ are the masses of the $N_s$ spinors and the $m_j$ are the masses of the $N_f$ vectors. Using Eq. (2.16) and substituting $\zeta = y - P(z)$, $z \rightarrow z^2$ in Eq. (2.13), we get

$$y^2 = x^2 \left( y - P(z^2) \right) + P^2(z^2) - \Lambda^{2\beta_0} z^4 \prod_{i=1}^{N_s} (x_1 - m_i^4) \prod_{j=1}^{N_f} (z^2 - m_j^2).$$

The normalized period matrix of this surface encodes the gauge couplings on the Coulomb branch. Here, there is no natural 2-form inherited from $\Omega$, because Eq. (2.11) is generally not a parametrization of a local approximation to a Calabi–Yau. To compute the gauge couplings from this surface, one needs to identify the 2-cycles and construct a suitable 2-form directly.

Our proposal, Eq. (2.10), ensures plausible behavior when either a spinor or a vector is integrated out. Integrating out a vector and a spinor at the same time, we can flow between the theories we obtained from mirror symmetry. To check our solution further, we consider breaking the $SO(10)$ gauge group to $SO(8) \times U(1)$ by giving a large VEV, $M$, to one component of the $SO(10)$ adjoint. Under this breaking the fundamentals decompose into fundamentals of $SO(8)$ and singlets with $U(1)$ charge. The spinors decompose as $16 \rightarrow 8^4 \oplus 8^4$, where the superscripts denote the $U(1)$ charge. Both the singlets and the two spinor representations of $SO(8)$ acquire a large mass and should drop out from our solution. Taking $M$ to infinity, the piece proportional to $c_2^2 \approx M^4$ will dominate Eq. (2.11). Replacing $x_1$ by $M^4$, rescaling Eq. (2.13) by appropriate powers of $M$ and integrating out $x$ reduces it to the $SO(8)$ curve with vector matter only.
B. The Calabi–Yau for $SO(12)$

The analysis of the previous subsection can be repeated for $SO(12)$ with $r$ half hypermultiplets in the $32$, $(4-n-r)$ half hypermultiplets in the $32'$ and $8-n$ fundamentals. The restrictions on the polynomials $f$ and $g$ in Eq. (2.2) are more complicated for $SO(12)$ than for $SO(10)$ [24], partly because one has the freedom to trade matter fields in the $32$ for fields in the $32'$ representation. However, the curve of the $SO(12)$ theories depends only on the total number of fields in the $32$ and $32'$, so we will drop this distinction here. Using the vertices of the dual polyhedron given in [3],

\[
\begin{align*}
\bar{v}_1 &= (-1, 0, 2, 3) & \bar{v}_2 &= (1, -n, 2, 3) & \bar{v}_3 &= (0, -1, 2, 3) \\
\bar{v}_4 &= (0, 0, -1, 0) & \bar{v}_5 &= (0, 0, 0, -1) & \bar{v}_6 &= (0, 0, 0, 0) \\
\bar{v}_7 &= (0, 0, 2, 3) & \bar{v}_8 &= (0, 1, 2, 3) & \bar{v}_9 &= (0, -2, 2, 3) \\
\bar{v}_{10} &= (0, -2, 1, 2) & \bar{v}_{11} &= (0, -2, 0, 1) & \bar{v}_{12} &= (0, -1, 1, 1) \\
\bar{v}_{13} &= (0, -1, 0, 0) & \bar{v}_{14} &= (0, -1, -1, 0),
\end{align*}
\]

we find for the Newton polyhedron

\[
\begin{align*}
v_1 &= (2, 1, -1, 1) & v_2 &= (4, 1, 0, 1) & v_3 &= (0, 0, 1, -1) \\
v_4 &= (0, 0, -2, 1) & v_5 &= (-6, 0, 1, 1) & v_6 &= (6, 0, 1, 1) \\
v_7 &= (-6, 6, 1, 1) & v_8 &= (-6 - 6a, -6, 1, 1) & v_9 &= (n - 2, 1, -1, 1) \\
v_{10} &= (n - 4, 1, 0, 1),
\end{align*}
\]

In terms of $x_1, x_2, x_3, x_7$ and $x_8$ the hypersurface defining the Calabi–Yau, Eq. (2.6), is given by

\[
W = \left( \frac{\zeta + a_0 x_1^{4-n} x_8^{8-n}}{\zeta} \right) x_0^{6+3n} - 2 x_1 x_2 x_3 x_7 x_8 + 2 x_1 x_2 x_3 x_8 + a_4 x_1 x_2 x_3 x_9 \\
+ a_5 x_1^2 x_2 x_7 x_8 + a_6 x_1 x_2^2 x_8 + a_7 x_1^2 x_2 x_8 + a_8 x_2 x_7 x_8 + a_9 x_7 x_8 + a_{10} x_1 x_2 x_3 x_9 \\
+ a_{11} x_3 x_7 + a_{12} x_1 x_7 x_8^2,
\]

where we defined $x_0 = x_7 x_8$ and $\zeta = (x_8/x_7)^{6+3n}$ and rescaled the coordinates to eliminate the coefficients of the first three terms defining the fiber. The terms with coefficients $a_5$ and $a_6$ are again irrelevant near the singularity. Making the substitutions

\[
x_3 = x - \frac{1}{2} (a_{11} + a_{10} x_2 + a_4 x_1 x_2) \\
x_1 = y - \frac{1}{4} (a_{11} a_4 + a_{10} a_4 x_2 + 4 x_2^2)
\]

and neglecting an irrelevant piece proportional to $x_1^2 x_2^2$ brings Eq. (2.20) into the form

\[
W = \left( \frac{\zeta + a_0 x_1^{4-n} x_8^{8-n}}{\zeta} \right) + x^2 + y^2 + z^2 + 2(\beta_0)^{1/2} y + c_5 z^4 + c_4 z^3 + c_3 z^2 + c_2 z + c_1 + \cdots.
\]

In this expression, $x_1$ is given by

\[
x_1 = y - \frac{1}{8} (4 c_4 - c_5^2 + 4 c_5 z + 8 z^2).
\]

We can identify $a_0$ with the strong coupling scale of the $SO(12)$ gauge theory: $a_0 = \Lambda^{2/\beta_0}$. The $\beta$-function for this theory is given by $\beta_0 = 10 - N_f - 2 N_s$, where $N_s$ counts the number of half hypermultiplets in the spinor representation of $SO(12)$. One can check that for $n = 4$ Eq. (2.22) reduces to the known curve for $SO(12)$ with four fundamentals [34]. In the asymptotically free cases, $n = 2, 3, 4$, this expression reduces to a curve, because both $x$ and $y$ appear at most quadratically.

Again we conjecture that Eq. (2.23) can be modified to accommodate $N_s$ spinors with masses $m_i$ and $N_f$ vectors with masses $m_j$ by the following substitution

\[
\zeta + a_0 \frac{1}{\zeta} x_1^{4-n} x_8^{8-n} \rightarrow \zeta + a_0 \frac{1}{\zeta} \prod_{i=1}^{N_s} (x_1 - m_i^2) \prod_{j=1}^{N_f} (z - m_j^2).
\]

As in the $SO(10)$ case, this results in an expression that shows the expected behavior under adjoint breaking of the $SO(12)$ to $SO(10)$. The substitution above also ensures that spinors and vectors can be integrated out consistently.
C. The Calabi–Yau for $SO(7)$

The $SO(7)$ theory with $3 - n$ fundamentals and $8 - 2n$ spinors differs from the theories we considered above in several respects. It is our first example of a non–simply laced group. Unlike in the previous cases, the $K3$ part of the Calabi–Yau cannot have a singularity of a type that corresponds to the gauge group, since a $K3$ can only have ADE type singularities. Thus we should expect some mixture of fiber and base coordinates even if there is no matter in the theory. The second difference is that the $SO(7)$ theory makes sense only for $n = 2, 3$. For $n = 4$, the fiber of the type IIA manifold cannot have a semisplit $D_4$ singularity [12], which would give rise to an $SO(7)$ gauge theory. Thus we cannot consider the case without spinors to compare to known results. Apart from that, it will turn out that the most convenient representation of the $SO(7)$ curve requires a slight modification of Batyrev’s construction of the mirror.

The polar polyhedron giving rise to the $SO(7)$ gauge theory is defined by the vertices

\[
\begin{align*}
\tilde{v}_1 &= (1, 0, 2, 3) \\
\tilde{v}_2 &= (1, -n, 2, 3) \\
\tilde{v}_3 &= (0, -1, 2, 3) \\
\tilde{v}_4 &= (0, 0, -1, 0) \\
\tilde{v}_5 &= (0, 0, 0, -1) \\
\tilde{v}_6 &= (0, 0, 0, 0) \\
\tilde{v}_7 &= (0, 0, 2, 3) \\
\tilde{v}_8 &= (0, 1, 2, 3) \\
\tilde{v}_9 &= (0, -2, 2, 3) \\
\tilde{v}_{10} &= (0, -1, 1, 1) \\
\tilde{v}_{11} &= (0, -1, 0, 1)
\end{align*}
\]

(2.25)

and the corresponding Newton polyhedron is given by

\[
\begin{align*}
v_1 &= (4, 2, 0, 1) \\
v_2 &= (0, 0, -2, 1) \\
v_3 &= (0, 0, 1, -1) \\
v_4 &= (6, 2, 1, 1) \\
v_5 &= (6, -6, 1, 1) \\
v_6 &= (-6 - 6n, -6, 1, 1) \\
v_7 &= (2n - 6, 2, 1, 1) \\
v_8 &= (2n - 4, 2, 0, 1).
\end{align*}
\]

(2.26)

Using Eq. (2.6) and setting $x_4 = x_7 = x_8 = 1$, we find the defining equation of the Calabi–Yau

\[
\begin{align*}
W &= \left(\zeta + a_0 \frac{x_1^{4-2n}}{\zeta}\right) x_0^{6+3n} + x_1^2 x_0^{12} + x_1 x_2^3 + x_2^2 + a_4 x_1 x_2 x_3 x_0 + a_5 x_1^2 x_0^4 \\
&\quad + a_6 x_0^6 + a_7 x_0^{18} + a_8 x_3 x_0^9 + a_9 x_2 x_0^8.
\end{align*}
\]

(2.27)

The $K3$ part of this expression can be transformed into the standard form of the classical piece of the $SO(7)$ curve using coordinate redefinitions as in the previous subsections. This results in an expression of the form

\[
W = \left(\zeta + a_0 \frac{x_1^{4-2n}}{\zeta}\right) + x_2^2 + y^2 + z^6 + c_3 z^4 + c_2 z^2 + c_1 + \cdots,
\]

(2.28)

where $x_1$ is some function of $x, z$ and the Casimirs $c_i$. In this format there is no obvious way to identify the powers of the fiber coordinates that multiply the coordinate of the lower sphere with the number of matter fields.

This problem can be circumvented by replacing the Calabi–Yau, Eq. (2.27), with another Calabi–Yau that encodes the same field theory information. Recall that on the IIB side, the field theory information is encoded in the complex structure moduli, which in turn determine the period integrals over the three cycles of the Calabi–Yau. The Kähler structure moduli determine the integrals over two cycles but do not affect the integrals over the three cycles. Thus we can modify the Kähler structure of our manifold without changing the information about the gauge theory.

One way of seeing that the information encoded in the complex structure is invariant under changes of the Kähler structure is provided by the $\nabla$-hypergeometric system of partial differential equations (see, e.g., [23] for details). The period integral over the three cycles of the Calabi–Yau is given by

\[
\Pi_k(a) = \int_{\gamma_k} \frac{1}{W(a, x)} \prod_p \frac{dx_p}{x_p},
\]

(2.29)

where $W(a, x)$ is a hypersurface constraint such as Eq. (2.27), $x_p$ are the coordinates of the embedding space and $a$ denotes the set of complex structure moduli. The period integrals satisfy a set of differential equations

\[
D_l \Pi_k = 0, \quad Z_\alpha \Pi_k = 0,
\]

(2.30)

where the differential operators are given by

\[
D_l = \prod_{i, l_i > 0} \left(\frac{\partial}{\partial a_i}\right)^{-l_i} - \prod_{i, l_i < 0} \left(\frac{\partial}{\partial a_i}\right)^{-l_i}, \quad Z_\alpha = \sum_i \tilde{v}_{i, \alpha} a_i \frac{\partial}{\partial a_i}, \quad Z_0 = \sum_i a_i \frac{\partial}{\partial a_i} + 1.
\]

(2.31)
Here, $\tilde{v}_{i,\alpha}$ denotes the $\alpha$ component of the $i$-th vector in the dual polyhedron and the vectors $l$ define relations between the vertices $\tilde{v}_i$

$$\sum_i \tilde{v}_i l_i = 0, \quad \sum_i l_i = 0. \tag{2.32}$$

One can check that the hypersurface constraints obtained by Batyrev’s construction satisfy these relations.

However, this does not exhaust the list of hypersurface constraints that satisfy Eq. (2.30). One can find many additional manifolds by solving these equations directly. In this approach, one does not need the information encoded in the Newton polyhedron. This reflects the fact that all of the information on the behavior of the gauge theory is contained in the dual polyhedron. Different solutions to Eqs. (2.30) will describe different Calabi–Yau manifolds but in the Newton polyhedron. This reflects the fact that all of the information on the behavior of the gauge theory is contained in the dual polyhedron. Different solutions to Eqs. (2.30) will describe different Calabi–Yau manifolds but they will all have the same period integrals over the three cycles and therefore they encode the same gauge theory.

We can easily find other hypersurface constraints which satisfy Eqs. (2.30) by adding points to the Newton polyhedron that lie in its convex hull. Using the coordinates corresponding to these points to parametrize the hypersurface constraint guarantees that the resulting Calabi–Yau has the same period integrals as Eq. (2.27). Adding the vector $v_9 = (n-2,1,-1,1)$ to the Newton polyhedron and using the coordinates associated to $v_8, v_9, v_3, v_5$ and $v_6$ we find the hypersurface constraint

$$W = \left( \zeta + a_0 \frac{x_9^{8-2n} z^{2-n}}{\zeta} \right) x_0^{6+3n} - x_8 x_9 x_0^{12} + 2 x_8 x_9 x_0^2 + x_3^2 + a_4 x_3 x_8 x_9 x_0 + a_5 x_8^{4} x_9^2 + a_6 x_8^6 x_9^3 + a_7 x_8^{18} + a_8 x_3 x_9^9 + a_9 x_9 x_8^8. \tag{2.33}$$

Setting $x_0 = 1$, neglecting the terms with coefficients $a_5$ and $a_6$, and substituting

\begin{align*}
x_3 &= x - \frac{1}{2} (a_8 + a_4 x_8 x_9) \\
x_8 &= y + x_9 - \frac{1}{4} a_4 a_8 \\
x_9 &= z \tag{2.34}
\end{align*}

we find after redefining the complex structure parameters

$$W = \left( \zeta + a_0 \frac{x_8^{8-2n} z^{4-n}}{\zeta} \right) + x^2 + y^2 z + c_2 z^2 + c_1 z + c_0 + \cdots, \tag{2.35}$$

where $x_8 = y + z + c_2/2$. We can identify $a_0$ with $\Lambda^{2\beta_0}$ and the $c_i$ with the Casimirs of $SO(7)$. Since we cannot choose $n$ to eliminate all spinors, we cannot compare this curve directly to known results. However, higgsing $SO(7)$ to $SO(5)$ as in Sec. III A, we obtain the expected curve for $SO(5)$ with $3 - n$ fundamentals. If we modify Eq. (2.35) to allow arbitrary numbers of spinors and vectors with arbitrary masses by replacing

$$\zeta + a_0 \frac{1}{\zeta} x_8^{8-2n} z^{4-n} \to \zeta + a_0 \frac{1}{\zeta} \prod_{i=1}^{N_s} (x_8 - m_i^2) \prod_{j=1}^{N_l} (z - m_j^2), \tag{2.36}$$

we can integrate out all spinors in Eq. (2.33). Then $x$ and $y$ can be integrated out trivially and substituting $z \to z^2$, we find the double cover version of the $SO(7)$ curve with $3 - n$ fundamentals \cite{3}. Unlike in the previous cases, we can write Eq. (2.35) as a curve only for $n = 3$.

**D. The Calabi–Yau for $SO(9)$**

In this section we repeat the analysis of the previous sections for a class for Calabi–Yau manifolds that lead to an $SO(9)$ gauge theory with $5 - n$ vectors and $4 - n$ spinors. The toric description of these manifolds is given by the vertices

\begin{align*}
\tilde{v}_1 &= (-1,0,2,3) & \tilde{v}_2 &= (1,-n,2,3) & \tilde{v}_3 &= (0,-1,2,3) \\
\tilde{v}_4 &= (0,0,-1,0) & \tilde{v}_5 &= (0,0,0,-1) & \tilde{v}_6 &= (0,0,0,0) \\
\tilde{v}_7 &= (0,0,2,3) & \tilde{v}_8 &= (0,1,2,3) & \tilde{v}_9 &= (0,-2,2,3) \\
\tilde{v}_{10} &= (0,-2,1,2) & \tilde{v}_{11} &= (0,-1,1,1) & \tilde{v}_{12} &= (0,-1,0,1) \tag{2.37}
\end{align*}
of the dual polyhedron. The Newton polyhedron consists of the vertices

\[
\begin{align*}
\mathbf{v}_1 &= (2, 1, -1, 1) & \mathbf{v}_2 &= (6, 2, 1, 1) & \mathbf{v}_3 &= (0, 0, 1, -1) \\
\mathbf{v}_4 &= (6, -6, 1, 1) & \mathbf{v}_5 &= (0, 0, -2, 1) & \mathbf{v}_6 &= (-6 - 6n, -6, 1, 1) \\
\mathbf{v}_7 &= (2n - 6, 2, 1, 1) & \mathbf{v}_8 &= (n - 2, 1, -1, 1).
\end{align*}
\]

(2.38)

Using these vectors and Eq. (2.6), we can write down the mirror. It is convenient to use the \( \mathbb{C}^* \) actions to set all coordinates except \( x_1, x_2, x_3, x_4 \) and \( x_6 \) to one. Defining \( x_0 = x_4x_6 \) and \( \zeta = (x_4/x_6)^{6+3n} \) we get

\[
W = \left( \zeta + a_0 \frac{x_1^{4-n}x_2^{12-2n}}{\zeta} \right)^{6+3n} + 2x_1x_2x_0^{12} - x_1^2 + x_2^2 + a_4x_1x_2x_3x_0
+ a_5x_1^2x_2^6x_0^6 + a_6x_1^3x_2^8 + a_7x_1^2x_2x_0^{18} + a_8x_0^{16} + a_9x_2x_3x_0^9 + a_{10}x_1x_0^8.
\]

(2.39)

For \( SO(9) \), Batyrev’s construction gives a description of the mirror in which the matter content of the theory is visible in the fibration over the lower sphere. The terms with coefficients \( a_5 \) and \( a_6 \) are irrelevant near the singularity. We can transform the fiber into the standard form for an \( SO(9) \) theory by making the following substitutions

\[
\begin{align*}
x_3 &= x - \frac{1}{2}(a_4x_1x_2 + a_9x_2) \\
x_1 &= y + \frac{1}{4}(2a_{10} - a_4a_9x_2^2 + 4x_2^4). \\
x_2 &= z.
\end{align*}
\]

(2.40)

Neglecting an irrelevant term of the form \( x_1^2x_2^2 \) and renaming the coefficients, we find

\[
W = \left( \zeta + a_0 \frac{x_1^{4-n}z^{12-2n}}{\zeta} \right) + x_1^2 - y^2 + c_0 + c_1z^2 + c_2z^4 + c_3z^6 + z^8 + \cdots,
\]

(2.41)

where

\[
x_1 = y + \frac{1}{8}(4c_2 - c_3^2 + 4c_3z^2 + 8z^4).
\]

(2.42)

It is straightforward to check that for \( n = 4 \) this curve agrees with the curves in \[\text{[33]}, \text{[34]}\], once one identifies the \( c_i \) with the gauge invariant polynomials that parametrize the Coulomb branch and sets \( a_0 = \Lambda^{2\beta_0} \).

Again, the substitution

\[
\left( \zeta + a_0 \frac{x_1^{4-n}z^{12-2n}}{\zeta} \right) \rightarrow \zeta + a_0 \frac{1}{\zeta}z^2 \prod_{i=1}^{N_f} (x_1 - m_i^4) \prod_{j=1}^{N_f} (z^2 - m_j^2)
\]

(2.43)

presumably results in a solution of the theory with arbitrary numbers of massive vectors and spinors. Repeating the checks as in Sec. \[\text{IIA}\], we find consistent behavior.

E. The Calabi–Yau for \( SO(11) \)

For \( SO(11) \) with \( 4 - n \) half hypermultiplets in the spinor representation and \( 7 - n \) vectors we can repeat the steps that provided the curve for \( SO(7) \). The polar polyhedron is given by the vertices

\[
\begin{align*}
\tilde{v}_1 &= (-1, 0, 2, 3) & \tilde{v}_2 &= (1, -n, 2, 3) & \tilde{v}_3 &= (0, -1, 2, 3) \\
\tilde{v}_4 &= (0, 0, -1, 0) & \tilde{v}_5 &= (0, 0, 0, -1) & \tilde{v}_6 &= (0, 0, 0, 0) \\
\tilde{v}_7 &= (0, 0, 2, 3) & \tilde{v}_8 &= (0, 1, 2, 3) & \tilde{v}_9 &= (0, -2, 2, 3) \\
\tilde{v}_{10} &= (0, -2, 1, 2) & \tilde{v}_{11} &= (0, -2, 0, 1) & \tilde{v}_{12} &= (0, -1, 1, 1) \\
\tilde{v}_{13} &= (0, -1, 0, 0)
\end{align*}
\]

(2.44)

and the corresponding Newton polyhedron is defined by

\[
\begin{align*}
\mathbf{v}_1 &= (2, 1, -1, 1) & \mathbf{v}_2 &= (6, 1, 1, 1) & \mathbf{v}_3 &= (0, 0, 1, -1) \\
\mathbf{v}_4 &= (0, 0, -2, 1) & \mathbf{v}_5 &= (6, -6, 1, 1) & \mathbf{v}_6 &= (-6 - 6n, -6, 1, 1) \\
\mathbf{v}_7 &= (n - 6, 1, 1, 1) & \mathbf{v}_8 &= (n - 2, 1, -1, 1).
\end{align*}
\]

(2.45)
Using these polyhedra, we can write down the mirror Calabi–Yau but as in the $SO(7)$ case there is no choice of coordinates in which the fibration over the lower sphere has a simple interpretation in terms of the number of fundamentals and spinors. However, we can add the vector $v_0 = (n - 4, 1, 0, 1)$ to the Newton polyhedron and use $x_8, x_9, x_3, x_5, x_6$ with $x_0 = x_5 x_6$ and $\zeta = (x_5/x_6)^{6 + 3n}$ to parametrize the Calabi–Yau

$$W = \left( \zeta + a_0 x_8^{4-n} x_9^{8-n} \right) x_0^{6 + 3n} + 2 x_8 x_9 x_0^{12} - x_8^2 x_9 + x_3^2 + a_4 x_3 x_8 x_9 x_0 + a_5 x_8^2 x_9 x_0^6 + a_6 x_8^3 x_9^3 + a_7 x_8^2 x_9^2 x_0 + a_8 x_8 x_9 x_0^6 + a_9 x_0^{14} + a_{10} x_3 x_9 x_0^9 + a_{11} x_3 x_9 x_0^7.$$  \hspace{1cm} (2.46)

Near the singularity we can neglect the terms with coefficients $a_{5, 6}$. Substituting

$$x_3 = x - \frac{1}{2} (a_{11} + a_{10} x_9 + a_4 x_8 x_9),$$

$$x_8 = y - \frac{1}{4} (a_{11} a_4 + a_{10} a_4 x_9 - 4 x_9^2)$$

into the defining equation of the Calabi–Yau gives

$$W = \left( \zeta + a_0 \frac{x_8^{4-n} x_9^{8-n}}{\zeta} \right) + x^2 + z^5 - z y^2 + c_5 z^4 + c_4 z^3 + c_3 z^2 + c_2 z + c_1 + \cdots,$$  \hspace{1cm} (2.48)

where

$$x_8 = y - \frac{1}{8} \left( e_5^2 - 4 c_4 - 4 c_5 z - 8 z^2 \right).$$  \hspace{1cm} (2.49)

For $n = 4$ we can integrate out $y$ trivially. Substituting $z \to z^2$, Eq. (2.48) reduces to the double cover version of the curve for $SO(11)$ with three fundamentals [33]. In the other two asymptotically free cases $n = 2, 3$, we also obtain a curve but it is not hyperelliptic. Presumably we can obtain an exact solution for any number of massive vectors and spinors by substituting

$$\left( \zeta + a_0 \frac{x_8^{4-n} x_9^{8-n}}{\zeta} \right) \to \zeta + a_0 \frac{1}{\zeta} \prod_{i=1}^{N_s} (x_8 - m_i^1) \prod_{j=1}^{N_f} (z - m_j^2).$$  \hspace{1cm} (2.50)

Again, our solution passes the tests given in Sec. [31]

III. CONCLUSIONS

We have obtained exact solutions to $N = 2$ supersymmetric $SO(N)$ gauge theories for $N = 10, 12$ and $N = 7, 9, 11$ with massless matter in the spinor and the fundamental representation. We gave a description of the Coulomb branch of these theories in terms of ALE spaces fibered over a sphere.

These solutions were obtained by compactifying type IIA string theory on Calabi–Yau threefolds with singular $K3$ fibers. The singularity type of the $K3$ determines the gauge group of the $d = 4$ gauge theory and the duality to heterotic strings compactified on $K3 \times T^2$ can be used to determine the charged matter content of the theory. Mirror symmetry relates the Calabi–Yau for type IIA compactification to a different Calabi–Yau that gives rise to the same field theory when type IIB string theory is compactified on it. The exact solutions can be extracted from this mirror Calabi–Yau.

This approach provides exact solutions for the gauge theories listed above with specific matter contents. We proposed some generalizations of these results to arbitrary numbers of massive spinors and vectors and verified that our solutions are consistent under adjoint breaking and integrating out matter fields. Unfortunately, the list of asymptotically free $SO(N)$ theories with spinors is not exhausted by the cases we have studied. For $SO(8)$ there is no toric description of the corresponding type IIA and IIB Calabi–Yau manifolds and the higher rank groups $SO(N)$, $N = 13, 14, 15, 16$ cannot be obtained from compactifying type IIA on a Calabi–Yau threefold or conversely from breaking the adjoint of $E_8$ on the heterotic side.

The results presented may ultimately provide some insights into how to construct matter representations other than fundamentals and two index tensors from branes. In principle it should be possible to find a brane configuration
corresponding to the theories we analyzed here by studying an M-theory 5-brane wrapped on $R^4 \times \Sigma$ where $\Sigma$ is the curve encoding the gauge couplings on the Coulomb branch.

Since our solutions agree with known field theory results, in the cases where these are available, one can view the results of this paper as further confirmation of mirror symmetry and the duality between type IIA and heterotic strings.

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