Weighted Bergman spaces: shift-invariant subspaces and input/state/output linear systems

Vladimir Bolotnikov (joint work with J. A. Ball)

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Given Hilbert spaces $\mathcal{X}, \mathcal{U}, \mathcal{Y}$ and given bounded linear operators $A \in \mathcal{L}(\mathcal{X}), B \in \mathcal{L}(\mathcal{U}, \mathcal{X}), C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $D \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$, consider the associated discrete-time linear time-invariant system

\[
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k \\
y_k &= Cx_k + Du_k
\end{aligned}
\]

$u_k \in \mathcal{U}, x_k \in \mathcal{X}, y_k \in \mathcal{Y}$.

Fix the initial state $x_0$ and evolve the system in $\mathbb{Z}_+$:

\[
\begin{aligned}
x_k &= A^k x_0 + \sum_{j=0}^{k-1} A^{k-1-j} Bu_j, \\
y_k &= CA^k x_0 + \sum_{j=0}^{k-1} CA^{k-1-j} Bu_j + Du_k
\end{aligned}
\]

Introduce the $Z$-transform $\{f_k\}_{n \geq 0} \mapsto \hat{f}(z) = \sum_{k=0}^{\infty} f_k z^k$ and write the latter relations in terms of $\hat{u}(z), \hat{x}(z), \hat{y}(z)$. 

Vladimir Bolotnikov (joint work with J. A. Ball) Weighted Bergman spaces: shift-invariant subspaces and input/output
\( \hat{x}(z) = (I - zA)^{-1}x_0 + z(I - zA)^{-1}B\hat{u}(z), \)
\( \hat{y}(z) = C(I - zA)^{-1}x_0 + [D + zC(I - zA)^{-1}B]\hat{u}(z) \)
\( = \mathcal{O}_{C,A}x_0 + \Theta(z)\hat{u}(z) \)

where

\( \mathcal{O}_{C,A}: \ x \mapsto C(I - zA)^{-1}x, \quad \Theta(z) = D + zC(I - zA)^{-1}B. \)
\[
\hat{x}(z) = (I - zA)^{-1}x_0 + z(I - zA)^{-1}B\hat{u}(z),
\]
\[
\hat{y}(z) = C(I - zA)^{-1}x_0 + [D + zC(I - zA)^{-1}B]\hat{u}(z)
\]
\[
= \mathcal{O}_{C,A}x_0 + \Theta(z)\hat{u}(z)
\]

where
\[
\mathcal{O}_{C,A} : \ x \mapsto C(I - zA)^{-1}x, \quad \Theta(z) = D + zC(I - zA)^{-1}B.
\]

A pair \((A, C)\) is called output stable if \(\mathcal{O}_{C,A} : \mathcal{X} \rightarrow H^2(\mathcal{Y})\) is bounded. Recall

\[
H^2(\mathcal{Y}) = \{ f(z) = \sum_{k=0}^{\infty} f_k z^k : \| f \| : = \sum_{k=0}^{\infty} \| f_k \|^2_{\mathcal{Y}} < \infty \} = \mathcal{H} \left( \frac{1}{1 - z\zeta} \right),
\]

If \((C, A)\) is output-stable, then the observability gramian \(G_{C,A} := \mathcal{O}_{C,A}^* \mathcal{O}_{C,A}\) is bounded on \(\mathcal{X}\) and can be represented via the strongly converging series

\[
G_{C,A} = \sum_{k=0}^{\infty} A^*k C^*CA^k.
\]
Connections between output stability, observability gramians and Stein equations and inequalities

1. \((C, A)\) is output-stable if and only if the Stein inequality

\[ H - A^* HA \geq C^* C \]  \hspace{1cm} (1)

has a positive semidefinite solution \(H \in \mathcal{L}(X)\).

2. If \((C, A)\) is output-stable, then \(G_{C, A}\) satisfies the Stein equality

\[ H - A^* HA = C^* C \]  \hspace{1cm} (2)

and is the minimal positive semidefinite solution of (1).

3. There is a unique positive semidefinite solution of (2) if \(A\) is strongly stable. If \(\|A\| \leq 1\), then the positive semidefinite solution of (2) is unique if and only if \(A\) is strongly stable.
Associated with an output-stable pair \((C, A)\) is the range of the observability operator

\[
\mathcal{M} = \operatorname{Ran} O_{C,A} = \{ C(I - zA)^{-1}x : x \in \mathcal{X} \} \subset H^2_Y.
\]
Associated with an output-stable pair \((C, A)\) is the range of the observability operator
\[\mathcal{M} = \text{Ran } O_{C,A} = \{ C(I - zA)^{-1}x : x \in \mathcal{X} \} \subset H^2_{\mathcal{Y}}.\]

1. \(\mathcal{M}\) is invariant under the backward shift \(S^*_1 : f(z) \mapsto \frac{f(z) - f(0)}{z}\).

2. If \(\mathcal{M} := \text{Ran } O_{C,A}\) is given the norm \(\|O_{C,A}x\|_\mathcal{M}^2 = \langle Hx, x \rangle_\mathcal{X}\), where \(H > 0\) solves \(H - A^*HA \geq C^*C\), then \(\mathcal{M}\) is contractively included in \(H^2_{\mathcal{Y}}\) and
\[
\|S^*_1 f\|_\mathcal{M}^2 \leq \|f\|_\mathcal{M}^2 - \|f(0)\|_{\mathcal{Y}}^2, \quad \text{for all } f \in \mathcal{M}. \tag{3}
\]
Associated with an output-stable pair \((C, A)\) is the range of the observability operator 
\[ \mathcal{M} = \text{Ran } O_{C,A} = \{ C(l - zA)^{-1}x : x \in \mathcal{X} \} \subset H^2_Y. \]

1. \( \mathcal{M} \) is invariant under the backward shift \( S_1^* : f(z) \rightarrow f(z) - f(0) \).

2. If \( \mathcal{M} := \text{Ran } O_{C,A} \) is given the norm \( \|O_{C,A}x\|_M^2 = \langle Hx, x \rangle_{\mathcal{X}} \), where \( H > 0 \) solves \( H - A^*HA \geq C^*C \), then \( \mathcal{M} \) is contractively included in \( H^2_Y \) and

\[ \|S_1^*f\|_M^2 \leq \|f\|_M^2 - \|f(0)\|_Y^2 \quad \text{for all } f \in M. \] (3)

3. Conversely, if \( \mathcal{M} \) is a Hilbert space contractively included in \( H^2_Y \) which is \( S_1^* \)-invariant and for which (3) holds, then there is a pair \((C, A)\) such that \( l - A^*A \geq C^*C \) and such that 
\[ \mathcal{M} = \text{Ran } O_{C,A}. \] Moreover, \( \mathcal{M} \) is the RKHS with RK

\[ K_M(z, \zeta) = C(l - zA)^{-1}(l - \zeta A^*)^{-1}C^*. \]
Assuming that the pair \((C, A)\) is output stable, look again at the formula

\[
\hat{y}(z) = \mathcal{O}_{C,A}x_0 + \Theta(z)\hat{u}(z),
\]

(3)

\[
\mathcal{O}_{C,A}: \quad x \mapsto C(I - zA)^{-1}x, \quad \Theta(z) = D + zC(I - zA)^{-1}B.
\]

Try to control \(\|\hat{y}\|_{H^2}\) in terms of \(\|\hat{u}\|_{H^2}\) and \(\|x_0\|_X\). We have
Assuming that the pair \((C, A)\) is output stable, look again at the formula

\[
\hat{y}(z) = O_{C,A}x_0 + \Theta(z)\hat{u}(z),
\]

(3)

\[
O_{C,A}: \quad x \mapsto C(I - zA)^{-1}x, \quad \Theta(z) = D + zC(I - zA)^{-1}B.
\]

Try to control \(\|\hat{y}\|_{H^2}\) in terms of \(\|\hat{u}\|_{H^2}\) and \(\|x_0\|_{\mathcal{X}}\). We have

\[
\|\hat{y}\|^2 = \|x_0\|^2 + \|\hat{u}\|^2 \quad \text{for all } x_0 \in \mathcal{X} \text{ and } \hat{u} \in H^2_{\mathcal{U}} \iff \quad \iff
\]

\[
D^*C + B^*G_{C,A}A = 0, \quad G_{C,A} = I, \quad D^*D + B^*G_{C,A}B = I.
\]
Assuming that the pair \((C, A)\) is output stable, look again at the formula

\[
\hat{y}(z) = O_{C,A}x_0 + \Theta(z)\hat{u}(z),
\]

(3)

\[
O_{C,A}: \quad x \mapsto C(I - zA)^{-1}x, \quad \Theta(z) = D + zC(I - zA)^{-1}B.
\]

Try to control \(\|\hat{y}\|_{H^2}\) in terms of \(\|\hat{u}\|_{H^2}\) and \(\|x_0\|_{\mathcal{X}}\). We have

\[
\|\hat{y}\|^2 = \|x_0\|^2 + \|\hat{u}\|^2 \quad \text{for all } x_0 \in \mathcal{X} \text{ and } \hat{u} \in H_U^2 \quad \iff
\]

\[
D^*C + B^*G_{C,A}A = 0, \quad G_{C,A} = I, \quad D^*D + B^*G_{C,A}B = I.
\]

We also have \(C^*C + A^*G_{C,A}A = G_{C,A}\). We conclude:

\[
\begin{bmatrix}
A^* & C^* \\
B^* & D^*
\end{bmatrix}
\begin{bmatrix}
G_{C,A} & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
= \begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}, \quad G_{C,A} = I.
Let $\mathcal{M} \subset H^2_Y$ be $S_1$-invariant. Then its RK equals

$$K_M(z, \zeta) = \frac{I_Y}{1 - z\zeta} - C(I - zA)^{-1}(I - \bar{\zeta}A^*)^{-1}C^*$$

for some pair $(C, A)$ such that $A^*A + C^*C = I$ and $G_{C,A} = I$. 
Let $\mathcal{M} \subset H^2_\mathcal{Y}$ be $S_1$-invariant. Then its RK equals

$$K_{\mathcal{M}}(z, \zeta) = \frac{I_{\mathcal{Y}}}{1 - z\zeta} - C(I - zA)^{-1}(I - \overline{\zeta}A^*)^{-1}C^*$$

for some pair $(C, A)$ such that $A^*A + C^*C = I$ and $G_{C,A} = I$.

Extend $\begin{bmatrix} A^* \\ C^* \end{bmatrix}$ to a coisometry $\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}$, let $\Theta(z) = D + zC(I - zA)^{-1}B$ and get

$$K_{\mathcal{M}}(z, \zeta) = \frac{\Theta(z)\Theta(\zeta)^*}{1 - z\zeta}$$
Let $\mathcal{M} \subset H^2_\mathcal{Y}$ be $S_1$-invariant. Then its RK equals

$$K_\mathcal{M}(z, \zeta) = \frac{I_\mathcal{Y}}{1 - z\zeta} - C(1 - zA)^{-1}(1 - \bar{\zeta}A^*)^{-1}C^*$$

for some pair $(C, A)$ such that $A^*A + C^*C = I$ and $G_{C, A} = I$.

Extend $\begin{bmatrix} A^* \\ C^* \end{bmatrix}$ to a coisometry $\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}$, let

$$\Theta(z) = D + zC(1 - zA)^{-1}B$$

and get

$$K_\mathcal{M}(z, \zeta) = \frac{\Theta(z)\Theta(\zeta)^*}{1 - z\zeta}$$

so that $\mathcal{M} = \Theta \cdot H^2_\mathcal{Y}$ and $\Theta$ is inner. Besides, $S^k_1 \Theta \mathcal{U} \perp S^m_1 \Theta \mathcal{U}$ for $m \neq k$ and $\mathcal{M} = \bigvee_{k \geq 0} S^k_1 \Theta \mathcal{U} = \bigvee_{k \geq 0} \left( S^k_1 \Theta \mathcal{U} \ominus S^{k+1}_1 \Theta \mathcal{U} \right)$. 

Vladimir Bolotnikov (joint work with J. A. Ball)
Given a positive sequence $\beta = \{\beta_j\}_{j \geq 0}$, define $H^2_\beta$ as the set of all functions analytic on $D$ and with finite norm given by

$$\|f\|_{H^2_\beta}^2 = \sum_{j=0}^{\infty} \beta_j |f_j|^2$$

if $f(z) = \sum_{j=0}^{\infty} f_j z^j$,

or alternatively, as the RKHS with RK $K_\beta(z, \zeta) = \sum_{j=0}^{\infty} \beta_j^{-1} z^j \zeta^j$. 

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Vladimir Bolotnikov (joint work with J. A. Ball)

Weighted Bergman spaces: shift-invariant subspaces and input/output...
Given a positive sequence $\beta = \{\beta_j\}_{j \geq 0}$, define $H^2_\beta$ as the set of all functions analytic on $\mathbb{D}$ and with finite norm given by

$$\|f\|_{H^2_\beta}^2 = \sum_{j=0}^{\infty} \beta_j |f_j|^2 \quad \text{if} \quad f(z) = \sum_{j=0}^{\infty} f_j z^j,$$

or alternatively, as the RKHS with RK $K_\beta(z, \zeta) = \sum_{j=0}^{\infty} \beta_j^{-1} \cdot z^j \bar{\zeta}^j$.

If $\beta_j = \frac{j!(n-1)!}{(j + n - 1)!}$, we get the standard weighted Bergman space $A^2_n$ with RK $K_n(z, \zeta) = \frac{1}{(1 - z \bar{\zeta})^n}$. 
H. Hedenmalm, 93: Let $\alpha \in \mathbb{D} \setminus \{0\}$. Define

$$G_\alpha(z) = -\frac{\overline{\alpha} \cdot b_\alpha(z)}{\sqrt{1 - (1 - |\alpha|^2)^n}} \cdot \sum_{k=0}^{n-1} \left( \frac{1 - |\alpha|^2}{1 - z\overline{\alpha}} \right)^k, \quad b_\alpha(z) = \frac{z - \alpha}{1 - z\overline{\alpha}}.$$
H. Hedenmalm, 93: Let $\alpha \in \mathbb{D} \setminus \{0\}$. Define

$$G_\alpha(z) = -\frac{\overline{\alpha} \cdot b_\alpha(z)}{\sqrt{1 - (1 - |\alpha|^2)^n}} \cdot \sum_{k=0}^{n-1} \left(\frac{1 - |\alpha|^2}{1 - z\overline{\alpha}}\right)^k, \quad b_\alpha(z) = \frac{z - \alpha}{1 - z\overline{\alpha}}.$$ 

1. $G_\alpha(\alpha) = 0$, $G_\alpha \in A_n^2$, $G_\alpha = b_\alpha$ if $n = 1$. 
2. For any $f \in A_n^2$ with $f(\alpha) = 0$, there is $g \in A_n^2$: $f = G_\alpha \cdot g$. 
3. The operator $M_{G_\alpha}$ is isometric from $\mathcal{U}$ to $A_n^2$, contractive from $H^2$ to $A_n^2$ and expansive from to $A_n^2$ to $A_n^2$. 
4. $G_\alpha \mathcal{U} \perp S_n^k G_\alpha \mathcal{U}$ for every $k \geq 1$ ($\mathcal{U} = \mathbb{C}$).
H. Hedenmalm, 93: Let $\alpha \in \mathbb{D} \setminus \{0\}$. Define

$$G_\alpha(z) = -\frac{\overline{\alpha} \cdot b_\alpha(z)}{\sqrt{1 - (1 - |\alpha|^2)^n}} \sum_{k=0}^{n-1} \left(\frac{1 - |\alpha|^2}{1 - z\overline{\alpha}}\right)^k, \quad b_\alpha(z) = \frac{z - \alpha}{1 - z\overline{\alpha}}.$$  

1. $G_\alpha(\alpha) = 0, \; G_\alpha \in A^2_n, \; G_\alpha = b_\alpha$ if $n = 1$.
2. For any $f \in A^2_n$ with $f(\alpha) = 0$, there is $g \in A^2_n$: $f = G_\alpha \cdot g$.
3. The operator $M_{G_\alpha}$ is isometric from $\mathcal{U}$ to $A^2_n$, contractive from $H^2$ to $A^2_n$ and expansive from to $A^2_n$ to $A^2_n$.
4. $G_\alpha \mathcal{U} \perp S^k_n G_\alpha \mathcal{U}$ for every $k \geq 1$ ($\mathcal{U} = \mathbb{C}$).
5. $G_\alpha$ can be realized as $G_\alpha(z) = D + zC \sum_{k=1}^{n} (I - zA)^{-k}B$ where

$$A = \overline{\alpha}, \; B = -\frac{\overline{\alpha} \cdot (1 - |\alpha|^2)^{\frac{n}{2}}}{\sqrt{1 - (1 - |\alpha|^2)^n}}, \; C = (1 - |\alpha|^2)^{\frac{n}{2}}, \; D = \sqrt{1 - (1 - |\alpha|^2)^n}.$$
A ∈ ℳ is an $n$-hypercontraction if

$$I - A^* A \geq 0, \quad I - 2A^* A + A^* A^2 \geq 0, \ldots$$

$$I - \binom{n}{1} A^* A + \binom{n}{2} A^* A^2 - \ldots + (-1)^n A^n A^n \geq 0.$$
Bergman-inner operator-valued functions

A ∈ X is an $n$-hypercontraction if

$$I - A^* A \geq 0, \quad I - 2A^* A + A^* A^2 \geq 0, \ldots$$

$$I - \binom{n}{1} A^* A + \binom{n}{2} A^* A^2 - \ldots + (-1)^n A^* A^2 \geq 0.$$

A. Olofsson, 06: Take $A ∈ \mathcal{L}(X)$ (an $n$-hypercontraction), $B ∈ \mathcal{L}(U, X)$, $C ∈ \mathcal{L}(X, Y)$, $D ∈ \mathcal{L}(U, Y)$ (such that) and define

$$Θ(z) = D + zC \sum_{k=1}^{n} (I - zA)^{-k} B.$$
Bergman-inner operator-valued functions

\( A \in \mathcal{X} \) is an \( n \)-hypercontraction if

\[
\begin{align*}
I - A^* A & \geq 0, \\
I - 2A^* A + A^2 A^2 & \geq 0, \\
I - \binom{n}{1} A^* A + \binom{n}{2} A^2 A^2 - \ldots + (-1)^n A^n A^n & \geq 0.
\end{align*}
\]

A. Olofsson, 06: Take \( A \in \mathcal{L}(\mathcal{X}) \) (an \( n \)-hypercontraction), \( B \in \mathcal{L}(\mathcal{U}, \mathcal{X}), \ C \in \mathcal{L}(\mathcal{X}, \mathcal{Y}), \ D \in \mathcal{L}(\mathcal{U}, \mathcal{Y}) \) (such that) and define

\[
\Theta(z) = D + zC \sum_{k=1}^{n} (I - zA)^{-k} B.
\]

1. The operator \( M_\Theta \) is isometric from \( \mathcal{U} \) to \( A^2_{n,\mathcal{Y}}, \) contractive from \( H^2_\mathcal{U} \) to \( A^2_{n,\mathcal{Y}} \) and expansive from to \( A^2_{n,\mathcal{U}} \) to \( A^2_{n,\mathcal{Y}}. \)

2. \( \Theta \mathcal{U} \perp S^k_n \Theta \mathcal{U} \) for every \( k \geq 1. \)
A relevant linear system

\[
\begin{align*}
    x_{k+1} &= Ax_k + Bu_k + \sum_{j=0}^{k} \binom{k - j + n - 1}{k - j + 1} A^{k-j} Bu_j, \quad x_0 = 0 \\
y_k &= Cx_k + Du_k, \quad u_k \in U, \quad x_k \in X, \quad y_k \in Y.
\end{align*}
\]

Passing to $Z$-transforms $\hat{u}(z), \hat{x}(z), \hat{y}(z)$ gives $\hat{y}(z) = \Theta(z)\hat{u}(z)$. 
A relevant linear system

\[
\begin{align*}
    x_{k+1} &= Ax_k + Bu_k + \sum_{j=0}^{k} \binom{k-j+n-1}{k-j+1} A^{k-j} B u_j, \quad x_0 = 0 \\
    y_k &= Cx_k + Du_k, \quad u_k \in U, \ x_k \in \mathcal{X}, \ y_k \in \mathcal{Y}.
\end{align*}
\]

Passing to \( Z \)-transforms \( \hat{u}(z), \hat{x}(z), \hat{y}(z) \) gives \( \hat{y}(z) = \Theta(z)\hat{u}(z) \).

However, if \( x_0 \neq 0 \), one gets

\[
\hat{y}(z) = C(I - zA)^{-1} x_0 + \Theta(z)\hat{u}(z)
\]

while we would prefer

\[
\hat{y}(z) = C(I - zA)^{-n} x_0 + \Theta(z)\hat{u}(z).
\]
Another relevant linear system

\[
\begin{align*}
  x_{k+1} &= \sum_{j=0}^{n-1} (-1)^j \binom{n}{j+1} A^{j+1} x_{k-j} + \sum_{j=0}^{n-1} (-1)^j \binom{n}{j+1} A^j B u_{k-j}, \\
  x_k &= 0 \text{ for } k < 0, \\
  y_k &= C x_k + D u_k.
\end{align*}
\]
Another relevant linear system

\[
\begin{cases}
    x_{k+1} = \sum_{j=0}^{n-1} (-1)^j \binom{n}{j+1} A^{j+1} x_{k-j} + \sum_{j=0}^{n-1} (-1)^j \binom{n}{j+1} A^j B u_{k-j}, \\
    x_k = 0 \text{ for } k < 0, \\
    y_k = C x_k + D u_k.
\end{cases}
\]

We have

\[
\hat{x}(z) = (I - zA)^{-n} x_0 + z \sum_{j=1}^{n} (I - zA)^{-j} B \hat{u}(z),
\]

\[
\hat{y}(z) = C I - zA)^{-n} x_0 + \Theta(z) \hat{u}(z).
\]
The right (supposedly) system

Given Hilbert spaces $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{U}_k$ ($k \geq 0$) and given operators

$$A \in \mathcal{L}(\mathcal{X}), \quad C \in \mathcal{L}(\mathcal{X}, \mathcal{Y}), \quad B_k \in \mathcal{L}(\mathcal{U}_k, \mathcal{X}), \quad D_k \in \mathcal{L}(\mathcal{U}_k, \mathcal{Y})$$

consider the associated discrete-time linear time-variant system

$$\begin{align*}
    x_{j+1} &= \frac{j + n}{j + 1} \cdot Ax_j + \binom{j + n}{j+1} \cdot B_j u_j, \\
    y_j &= Cx_j + \binom{j + n - 1}{j} \cdot D_j u_j.
\end{align*}$$
The right (supposedly) system

Given Hilbert spaces $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{U}_k$ ($k \geq 0$) and given operators

$$A \in \mathcal{L}(\mathcal{X}), \quad C \in \mathcal{L}(\mathcal{X}, \mathcal{Y}), \quad B_k \in \mathcal{L}(\mathcal{U}_k, \mathcal{X}), \quad D_k \in \mathcal{L}(\mathcal{U}_k, \mathcal{Y})$$

consider the associated discrete-time linear time-variant system

$$\begin{cases} 
    x_{j+1} = \frac{j + n}{j + 1} \cdot Ax_j + \binom{j+n}{j+1} \cdot B_j u_j, \\
    y_j = Cx_j + \binom{j+n-1}{j} \cdot D_j u_j. 
\end{cases}$$

More generally, if $\beta = \{\beta_j\}_{j \geq 0}$ is a given weight sequence, consider

$$\begin{cases} 
    x_{j+1} = \frac{\beta_j}{\beta_{j+1}} \cdot Ax_j + \frac{1}{\beta_{j+1}} \cdot B_j u_j, \\
    y_j = Cx_j + \frac{1}{\beta_j} \cdot D_j u_j, 
\end{cases} \quad x_j \in \mathcal{X}, \; y_j \in \mathcal{Y}, \; u_j \in \mathcal{U}_j.$$
To solve the system in terms of $\{u_k\}_{k \geq 0}$, $\hat{x}(z)$, $\hat{y}(z)$, let

$$R_n(z) = (1 - z)^{-n} = \sum_{j=0}^{\infty} \binom{n+j-1}{j} z^j,$$

$$R_{n,k}(z) = \sum_{j=0}^{\infty} \binom{n+j+k-1}{j+k} z^j = \sum_{\ell=1}^{n} \binom{\ell+k-2}{\ell-1} (1 - z)^{-(n-\ell+1)}.$$
To solve the system in terms of \( \{u_k\}_{k \geq 0}, \hat{x}(z), \hat{y}(z) \), let

\[
R_n(z) = (1 - z)^{-n} = \sum_{j=0}^{\infty} \binom{n+j-1}{j} z^j,
\]

\[
R_{n,k}(z) = \sum_{j=0}^{\infty} \binom{n+j+k-1}{j+k} z^j = \sum_{\ell=1}^{n} \binom{\ell+k-2}{\ell-1} (1 - z)^{-(n-\ell+1)}.
\]

We have

\[
\hat{x}(z) = \left( \sum_{j=0}^{\infty} \binom{j+n-1}{j} A^i z^j \right) x_0 + \sum_{k=1}^{\infty} \left( \sum_{j=k}^{\infty} \binom{j+n-1}{j} A^{i-k} z^j \right) B_{k-1} u_{k-1}
\]

\[
= (I - zA)^{-n} x_0 + \sum_{k=1}^{\infty} z^k \left( \sum_{j=0}^{\infty} \binom{j+k+n-1}{j+k} A^i z^j \right) B_{k-1} u_{k-1}
\]

\[
= (I - zA)^{-n} x_0 + \sum_{k=1}^{\infty} z^k R_{n,k}(zA) B_{k-1} u_{k-1}
\]

\[
= (I - zA)^{-n} x_0 + \sum_{k=0}^{\infty} z^{k+1} R_{n,k+1}(zA) B_k u_k.
\]

(4)
Then we get

\[ \hat{y}(z) = C(l - zA)^{-n}x_0 \]

\[ + \sum_{k=0}^{\infty} z^k \left( \begin{pmatrix} k + n - 1 \\ k \end{pmatrix} D_k + zCR_{n,k+1}(zA)B_k \right) u_k \]

\[ = \mathcal{O}_{n,C,A}x_0 + \sum_{k=0}^{\infty} z^k \Theta_k(z) u_k, \]
Then we get

\[ \hat{y}(z) = C(1 - zA)^{-n}x_0 + \sum_{k=0}^{\infty} z^k \left( \binom{k + n - 1}{k} D_k + zCR_{n,k+1}(zA)B_k \right) u_k \]

\[ = O_{n,C,A}x_0 + \sum_{k=0}^{\infty} z^k \Theta_k(z) u_k, \]

where \[ O_{n,C,A}: x \mapsto C(1 - zA)^{-n}x \] is the \( n \)-observability operator and where

\[ \Theta_k(z) = \binom{k + n - 1}{k} D_k + zCR_{n,k+1}(zA)B_k \quad (k = 0, 1, \ldots) \]

is the family of transfer functions.
Let us say that \((C, A)\) is \(n\)-output stable if \(O_{n,C,A}: \mathcal{X} \to A^2_{n,Y}\) is bounded. If \((C, A)\) is \(n\)-output-stable, then the observability gramian \(G_{n,C,A} := O^*_{n,C,A} O_{n,C,A}\) is bounded on \(\mathcal{X}\) and can be represented via the strongly converging series

\[
G_{n,C,A} = \sum_{k=0}^{\infty} \binom{k + n - 1}{k} A^k C^* C A^k
\]
Let us say that \((C, A)\) is \(n\)-output stable if \(O_{n, C, A}: \mathcal{X} \to A^2_{n, \mathcal{Y}}\) is bounded. If \((C, A)\) is \(n\)-output-stable, then the observability gramian \(G_{n, C, A} := O^*_{n, C, A} O_{n, C, A}\) is bounded on \(\mathcal{X}\) and can be represented via the strongly converging series

\[
G_{n, C, A} = \sum_{k=0}^{\infty} \binom{k + n - 1}{k} A^k C^* C A^k
\]

Define \(B_A : H \to A^* H A\) and let

\[
\Gamma_{n, A} = (I - B_A)^n : H \mapsto \sum_{k=0}^{n} (-1)^k \binom{n}{k} A^k H A^k.
\]
Let us say that \((C, A)\) is \(n\)-output stable if \(O_{n,C,A} : \mathcal{X} \to A_{n,Y}^2\) is bounded. If \((C, A)\) is \(n\)-output-stable, then the observability gramian \(G_{n,C,A} := O_{n,C,A}^* O_{n,C,A}\) is bounded on \(\mathcal{X}\) and can be represented via the strongly converging series

\[
G_{n,C,A} = \sum_{k=0}^{\infty} \binom{k + n - 1}{k} A^k C^* C A^k
\]

Define \(B_A : H \to A^* HA\) and let

\[
\Gamma_{n,A} = (I - B_A)^n : H \mapsto \sum_{k=0}^{n} (-1)^k \binom{n}{k} A^k HA^k.
\]

Observe that \(G_{n,C,A} = (I - B_A)^{-n} [C^* C]\).
1. \((C, A)\) is \(n\)-output-stable \iff there is an \(H \in \mathcal{L}(\mathcal{X})\) so that
\[
H \succeq A^*HA \succeq 0 \quad \text{and} \quad \Gamma_{n,A}[H] \succeq C^*C. \tag{5}
\]

2. If \((C, A)\) is \(n\)-output-stable, then \(G_{n,C,A}\) satisfies
\[
H \succeq A^*HA \succeq 0 \quad \text{and} \quad \Gamma_{n,A}[H] = C^*C \tag{6}
\]
and is the minimal solution of (5).

3. There is a unique solution \(H\) of (6) with \(H = G_{n,C,A}\) if \(A\) is strongly stable. If \(A\) is a contraction, then the solution of (6) is unique if and only if \(A\) is strongly stable.
1. \((C, A)\) is \(n\)-output-stable \(\iff\) there is an \(H \in \mathcal{L}(X)\) so that
\[ H \geq A^*HA \geq 0 \quad \text{and} \quad \Gamma_{n,A}[H] \geq C^*C. \quad (5) \]

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3. There is a unique solution \(H\) of (6) with \(H = G_{n,C,A}\) if \(A\) is strongly stable. If \(A\) is a contraction, then the solution of (6) is unique if and only if \(A\) is strongly stable.

\[ W_N = \sum_{k=0}^{N} \binom{k+n-1}{k} A^k C^*CA^k + \sum_{j=1}^{n} \binom{N+n}{N+j} A^{N+j} \Gamma_{n-j,A}[H]A^{N+j} \]

If \(H\) satisfies (5), then \(H \geq W_N \geq W_{N+1} \geq 0\).
Let $S_n : f(z) \mapsto zf(z)$ be the forward shift on $A^2_{n,Y}$. Then

$$S^*_n f = \sum_{k=0}^{\infty} \frac{k + 1}{n + k} \cdot f_{k+1} z^k \quad \text{if} \quad f(z) = \sum_{k=0}^{\infty} f_k z^k.$$

Let $\mathcal{M} = \text{Ran} \ O_{n,C,A} = \{ C(I - zA)^{-n}x : x \in \mathcal{X} \} \subset A^2_{n,Y}$. 

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Let $S_n : f(z) \mapsto zf(z)$ be the forward shift on $A^2_{n,Y}$. Then

$$S^*_n f = \sum_{k=0}^{\infty} \frac{k + 1}{n + k} \cdot f_{k+1} z^k \quad \text{if} \quad f(z) = \sum_{k=0}^{\infty} f_k z^k.$$ 

Let $\mathcal{M} = \text{Ran } O_{n,C,A} = \{ C(I - zA)^{-n} x : x \in \mathcal{X} \} \subset A^2_{n,Y}$.

1. $\mathcal{M}$ is invariant under $S^*_n$ and $S^*_n|\mathcal{M}$ is a contraction.
2. If $\mathcal{M} := \text{Ran } O_{C,A}$ is given the norm $\| O_{C,A} x \|_\mathcal{M}^2 = \langle Hx, x \rangle_\mathcal{X}$, where $H > 0$ is subject to $H - A^*HA \geq 0$ and $\Gamma_{n,A}[H] \geq C^*C$, then $\mathcal{M}$ is contractively included in $A^2_{n,Y}$ and

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} \| S^*_n f \|_\mathcal{M}^2 \geq \| f(0) \|_\mathcal{Y}^2 \quad \text{for all } f \in \mathcal{M}. \quad (7)$$
Let $S_n : f(z) \mapsto zf(z)$ be the forward shift on $A^2_{n,Y}$. Then

$$S_n^* f = \sum_{k=0}^{\infty} \frac{k+1}{n+k} \cdot f_{k+1} z^k \quad \text{if} \quad f(z) = \sum_{k=0}^{\infty} f_k z^k.$$

Let $\mathcal{M} = \text{Ran} \mathcal{O}_{n,C,A} = \{ C(I - zA)^{-n} x : x \in \mathcal{X} \} \subset A^2_{n,Y}$.

1. $\mathcal{M}$ is invariant under $S_n^*$ and $S_n^*|_\mathcal{M}$ is a contraction.
2. If $\mathcal{M} := \text{Ran} \mathcal{O}_{C,A}$ is given the norm $\| \mathcal{O}_{C,A} x \|_\mathcal{M}^2 = \langle Hx, x \rangle_\mathcal{X}$, where $H > 0$ is subject to $H - A^*HA \succeq 0$ and $\Gamma_{n,A}[H] \succeq C^* C$, then $\mathcal{M}$ is contractively included in $A^2_{n,Y}$ and

$$\sum_{j=0}^{n} (-1)^j \binom{k}{j} \| S_n^j f \|_\mathcal{M}^2 \geq \| f(0) \|_Y^2 \quad \text{for all} \quad f \in \mathcal{M}. \quad (7)$$

3. Conversely, if $\mathcal{M}$ is (1) contractively included in $A^2_{n,Y}$, (2) $S_n^*$-invariant, (3) $\| S_n^*|_\mathcal{M} \| \leq 1$ and (4) (7) holds, then there is a pair $(C, A)$ such that $A^*A \leq I$, $\Gamma_{n,A}[I] \succeq C^* C$ and such that $\mathcal{M} = \text{Ran} \mathcal{O}_{n,C,A}$. Moreover, $\mathcal{M}$ is the RKHS with RK

$$K_M(z, \zeta) = C(I - zA)^{-n}(I - \overline{\zeta} A^*)^{-n} C^*.$$
Look again at  
\[ \hat{y}(z) = \mathcal{O}_{n,C,A}x_0 + \sum_{k=0}^{\infty} z^k \Theta_k(z) u_k, \]
where
\[ \Theta_k(z) = \binom{k+n-1}{k} D_k + zC R_{n,k+1}(zA) B_k \]
and try to control \( \|\hat{y}\|_{A_n^2} \) in terms of \( \|u_k\| \) and \( \|x_0\| \).
Look again at

\[ \hat{y}(z) = O_n, C, A x_0 + \sum_{k=0}^{\infty} z^k \Theta_k(z) u_k, \]

where

\[ \Theta_k(z) = \binom{k+n-1}{k} D_k + z C R_{n, k+1} (zA) B_k \]

and try to control \( \| \hat{y} \|_{A_n^2} \) in terms of \( \| u_k \| \) and \( \| x_0 \| \). Let

\[ D_{k, C, A} = C R_{n, k}(z) = \sum_{j=0}^{\infty} \binom{n+j+k-1}{j+k} (CA^j x) z^j = S_1^k O_{n, C, A} \]
Look again at \( \hat{y}(z) = O_{n,C,A}x_0 + \sum_{k=0}^{\infty} z^k \Theta_k(z) u_k, \) where

\[
\Theta_k(z) = \binom{k+n-1}{k} D_k + z CR_{n,k+1}(zA) B_k
\]

and try to control \( \|\hat{y}\|_{A^2_n} \) in terms of \( \|u_k\| \) and \( \|x_0\| \). Let

\[
\mathcal{O}_{k,C,A} = CR_{n,k}(z) = \sum_{j=0}^{\infty} \binom{n+j+k-1}{j+k} (CA^j x) z^j = S_n^{*k} O_{n,C,A}
\]

\[
\mathcal{G}_{k,C,A} = \sum_{j=0}^{\infty} \binom{n+j+k-1}{j+k} A^j C^* CA^j = (S_n \mathcal{O}_{k,C,A})^* S_n \mathcal{O}_{k,C,A}.
\]
Look again at \( \hat{y}(z) = \mathcal{O}_{n,C,A}x_0 + \sum_{k=0}^{\infty} z^k \Theta_k(z) u_k \), where

\[ \Theta_k(z) = \left( \begin{array}{c} k+n-1 \\ k \end{array} \right) D_k + zCR_{n,k+1}(zA)B_k \]

and try to control \( \|\hat{y}\|_{A_n^2} \) in terms of \( \|u_k\| \) and \( \|x_0\| \). Let

\[ \mathcal{D}_{k,C,A} = CR_{n,k}(z) = \sum_{j=0}^{\infty} \left( \begin{array}{c} n+j+k-1 \\ j+k \end{array} \right) (CA^j x)z^j = S_1^k \mathcal{O}_{n,C,A} \]

\[ \mathcal{G}_{k,C,A} = \sum_{j=0}^{\infty} \left( \begin{array}{c} n+j+k-1 \\ j+k \end{array} \right) A^{j*}C^*CA^j = (S_n \mathcal{D}_{k,C,A})^* S_n \mathcal{D}_{k,C,A}. \]

We have \( \|\hat{y}\|^2 = \|x_0\|^2 + \sum_{k=0}^{\infty} \|u_k\|^2 \) for all \( x_0 \in \mathcal{X}, u_k \in \mathcal{U}_k \) \( \iff \)
Look again at $\hat{y}(z) = O_{n,c,A}x_0 + \sum_{k=0}^{\infty} z^k \Theta_k(z) u_k$, where

$$\Theta_k(z) = \binom{k+n-1}{k} D_k + z C R_{n,k+1}(zA) B_k$$

and try to control $||\hat{y}||_{A_n^2}$ in terms of $||u_k||$ and $||x_0||$. Let

$$\mathcal{D}_{k,C,A} = C R_{n,k}(z) = \sum_{j=0}^{\infty} \binom{n+j+k-1}{j+k} (CA^j x) z^j = S_1^{*k} O_{n,c,A}$$

$$\mathcal{G}_{k,C,A} = \sum_{j=0}^{\infty} \binom{n+j+k-1}{j+k} A^{j*} C^* C A^j = (S_n \mathcal{D}_{k,C,A})^* S_n \mathcal{D}_{k,C,A}.$$ 

We have $||\hat{y}||^2 = ||x_0||^2 + \sum_{k=0}^{\infty} ||u_k||^2$ for all $x_0 \in \mathcal{X}$, $u_k \in \mathcal{U}_k$ \iff

$$G_{n,c,A} = I, \quad \binom{k+n-1}{k} D_k^* C + B_k^* \mathcal{G}_{k+1,c,A} A = 0,$$

$$\binom{k+n-1}{k} D_k^* D_k + B_k^* \mathcal{G}_{k+1,c,A} B_k = I.$$
Look again at $\hat{y}(z) = O_{n,C,A}x_0 + \sum_{k=0}^{\infty} z^k \Theta_k(z) u_k$, where

$\Theta_k(z) = \binom{k+n-1}{k} D_k + zC R_{n,k+1}(zA) B_k$ and try to control $\|\hat{y}\|_{A_n^2}$ in terms of $\|u_k\|$ and $\|x_0\|$. Let

$\mathcal{O}_{k,C,A} = CR_{n,k}(z) = \sum_{j=0}^{\infty} \binom{n+j+k-1}{j+k} (CA^j x) z^j = S_1^k O_{n,C,A}$

$\mathcal{G}_{k,C,A} = \sum_{j=0}^{\infty} \binom{n+j+k-1}{j+k} A^j C^* C A^j = (S_n \mathcal{O}_{k,C,A})^* S_n \mathcal{O}_{k,C,A}$.

We have $\|\hat{y}\|^2 = \|x_0\|^2 + \sum_{k=0}^{\infty} \|u_k\|^2$ for all $x_0 \in \mathcal{X}$, $u_k \in \mathcal{U}_k$ $\iff$

$\mathcal{G}_{n,C,A} = I$, $\binom{k+n-1}{k} D_k^* C + B_k^* \mathcal{G}_{k+1,C,A} A = 0$,

$\binom{k+n-1}{k} D_k^* D_k + B_k^* \mathcal{G}_{k+1,C,A} B_k = I$.

We also have $\binom{k+n-1}{k} C^* C + A^* \mathcal{G}_{k+1,C,A} A = \mathcal{G}_{k+1,C,A}$. 

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Thus, if $A, C, B_k, C_k$ are such that $G_{n,C,A} = I$ and

$$
\begin{bmatrix}
A^* & C^* \\
B_k^* & D_k^*
\end{bmatrix}
\begin{bmatrix}
G_{k+1,C,A} & 0 \\
0 & (k+n-1)^{-1} \cdot I
\end{bmatrix}
\begin{bmatrix}
A & B_k \\
C & D_k
\end{bmatrix} =
\begin{bmatrix}
G_{k,C,A} & 0 \\
0 & I
\end{bmatrix}
$$

for $k = 0, 1, \ldots$, then $\|\hat{y}\|^2 = \|x_0\|^2 + \sum_{k=0}^{\infty} \|u_k\|^2$. 
Thus,

If $A, C, B_k, C_k$ are such that $G_{n,C,A} = I$ and

$$\begin{bmatrix} A^* & C^* \\ B_k^* & D_k^* \end{bmatrix} \begin{bmatrix} G_{k+1,C,A} & 0 \\ 0 & (k+n-1) \cdot I \end{bmatrix} \begin{bmatrix} A & B_k \\ C & D_k \end{bmatrix} = \begin{bmatrix} G_{k,C,A} & 0 \\ 0 & I \end{bmatrix}$$

for $k = 0, 1, \ldots$, then $\|\hat{y}\|^2 = \|x_0\|^2 + \sum_{k=0}^{\infty} \|u_k\|^2$.

1. For every $k \geq 0$, the function $z^k \Theta_k(z)$ is a contractive multiplier from $H^2_{U_k}$ to $A^2_{n,Y}$.

2. $S^k_{n} \Theta_k U_k$ is orthogonal to $S^m_{n} \Theta_k U_k$ for all $m > n$. 
Let $\mathcal{M} \subset A^2_{n,\mathcal{Y}}$ be $S_n$-invariant. Then its RK equals

$$K_{\mathcal{M}}(z, \zeta) = \frac{I_{\mathcal{Y}}}{(1 - z\zeta)^n} - C(I - zA)^{-n}(I - \zeta A^*)^{-n} C^*$$

for some pair $(C, A)$ such that $\Gamma_{n,A}[I] \geq C^* C$ and $G_{C, A} = I$. 
Beurling-Lax theorem

Let $\mathcal{M} \subset A^2_{n,Y}$ be $S_n$-invariant. Then its RK equals

$$K_{\mathcal{M}}(z, \zeta) = \frac{I_Y}{(1 - z\zeta)^n} - C(I - zA)^{-n}(I - \overline{\zeta}A^*)^{-n}C^*$$

for some pair $(C, A)$ such that $\Gamma_{n,A}[I] \geq C^*C$ and $G_{C,A} = I$.

Extend $\begin{bmatrix} A^* \\ C^* \end{bmatrix}$ to $\begin{bmatrix} A^* & C^* \\ B_k^* & D_k^* \end{bmatrix}$ for $k = 0, 1, \ldots$ so that

$$\begin{bmatrix} A^* & C^* \\ B_k^* & D_k^* \end{bmatrix} \begin{bmatrix} \mathcal{G}_{k+1,C,A} & 0 \\ 0 & \binom{k+n-1}{k} \cdot I \end{bmatrix} \begin{bmatrix} A & B_k \\ C & D_k \end{bmatrix} = \begin{bmatrix} \mathcal{G}_{k,C,A} & 0 \\ 0 & I \end{bmatrix}$$

and let $\Theta_k(z) = \binom{n}{k} D_k + zCR_{n,k+1}(z)B_k$. 

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Beurling-Lax theorem

Let $\mathcal{M} \subset A^2_{n,\mathcal{Y}}$ be $S_n$-invariant. Then its RK equals

$$K_{\mathcal{M}}(z, \zeta) = \frac{I_{\mathcal{Y}}}{(1 - z\bar{\zeta})^n} - C(I - zA)^{-n}(I - \bar{\zeta}A^*)^{-n}C^*$$

for some pair $(C, A)$ such that $\Gamma_{n,A}[I] \geq C^*C$ and $G_{C,A} = I$.

Extend $\begin{bmatrix} A^* \\ C^* \end{bmatrix}$ to $\begin{bmatrix} A^* & C^* \\ B_k^* & D_k^* \end{bmatrix}$ for $k = 0, 1, \ldots$ so that

$$\begin{pmatrix} A^* & C^* \\ B_k^* & D_k^* \end{pmatrix} \begin{bmatrix} \mathcal{G}_{k+1,C,A} & 0 \\ 0 & \binom{k+n-1}{k} \cdot I \end{bmatrix} \begin{pmatrix} A & B_k \\ C & D_k \end{pmatrix} = \begin{bmatrix} \mathcal{G}_{k,C,A} & 0 \\ 0 & I \end{bmatrix}$$

and let $\Theta_k(z) = \binom{n}{k} D_k + zCR_{n,k+1}(z)B_k$.

Then the RK for the subspace $S_n^k\mathcal{M} \ominus S_n^{k+1}\mathcal{M}$ equals

$$K_{S_n^k\mathcal{M} \ominus S_n^{k+1}\mathcal{M}}(z, \zeta) = z^{k-\bar{k}} \Theta_k(z) \Theta_k(\zeta)^*.$$
Thus,

\[ K_M(z, \zeta) = \sum_{k=0}^{\infty} z^k \overline{\zeta}^k \Theta_k(z) \Theta_k(\zeta)^*. \]
Thus,

\[ K_M(z, \zeta) = \sum_{k=0}^{\infty} z^k \zeta^k \Theta_k(z) \Theta_k(\zeta)^*. \]

In case \( n = 1 \) we have \( \Theta_k = \Theta_0 \) for all \( k \) and therefore,

\[ K_M(z, \zeta) = \frac{\Theta_0(z) \Theta_0(\zeta)^*}{1 - z\bar{\zeta}}. \]
Thus,

\[ K_M(z, \zeta) = \sum_{k=0}^{\infty} z^k \bar{\zeta}^k \Theta_k(z) \Theta_k(\zeta)^*. \]

In case \( n = 1 \) we have \( \Theta_k = \Theta_0 \) for all \( k \) and therefore,

\[ K_M(z, \zeta) = \frac{\Theta_0(z) \Theta_0(\zeta)^*}{1 - z \bar{\zeta}}. \]

Alternatively (for standard weighted Bergman spaces only),

\[ K_M(z, \zeta) = \sum_{k=1}^{n} \frac{F_j(z) F_j(\zeta)^*}{(1 - z \bar{\zeta})^j} \]

for some \( F_j \) contractive multipliers from \( A_{j,Y}^2 \) to \( A_{n,Y}^2 \).
Thus,

\[ K_M(z, \zeta) = \sum_{k=0}^{\infty} z^k \zeta^k \Theta_k(z) \Theta_k(\zeta)^*. \]

In case \( n = 1 \) we have \( \Theta_k = \Theta_0 \) for all \( k \) and therefore,

\[ K_M(z, \zeta) = \frac{\Theta_0(z) \Theta_0(\zeta)^*}{1 - z\zeta}. \]

Alternatively (for standard weighted Bergman spaces only),

\[ K_M(z, \zeta) = \sum_{k=1}^{n} \frac{F_j(z)F_j(\zeta)^*}{(1 - z\zeta)^j} \]

for some \( F_j \) contractive multipliers from \( A_{2,j,Y} \) to \( A_{2,n,Y} \) so that

\[ M = [F_1 \ F_2 \ \ldots \ F_n] \bigoplus_{j=1}^{n} A_{2,n,U_k}. \]
Another appearance of $\Theta_k$

In parallel to the space $A^2_{n,Y} \sim A^2_n \otimes Y$ we may consider the space $A^2_n(U,Y) \sim A^2_n \otimes \mathcal{L}(U,Y)$ consisting of $\mathcal{L}(U,Y)$-valued functions $F(z) = \sum_{j=0}^{\infty} F_k z^k$ analytic on $\mathbb{D}$ and with the finite norm given by

$$\|F\|^2_{A^2_n(U,Y)} = \sum_{j=0}^{\infty} \beta_j \cdot \text{Trace}(F^*_k F_k).$$

For an $F \in A^2_n(U,Y)$, define $[F,F] = \sum_{j=0}^{\infty} \beta_j F^*_k F_k$
Another appearance of $\Theta_k$

In parallel to the space $A^2_{n,Y} \sim A^2_n \otimes Y$ we may consider the space $A^2_n(U,Y) \sim A^2_n \otimes \mathcal{L}(U,Y)$ consisting of $\mathcal{L}(U,Y)$-valued functions $F(z) = \sum_{j=0}^{\infty} F_k z^k$ analytic on $\mathbb{D}$ and with the finite norm given by

$$\|F\|_{A^2_n(U,Y)}^2 = \sum_{j=0}^{\infty} \beta_j \cdot \text{Trace}(F_k^* F_k).$$

For an $F \in A^2_n(U,Y)$, define $[F, F] = \sum_{j=0}^{\infty} \beta_j F_k^* F_k$ and the left-tangential operator-argument evaluation

$$(C^* F)^{\wedge L}(A^*) = \sum_{j=0}^{\infty} A^j C^* F_j.$$
Another appearance of $\Theta_k$

In parallel to the space $A^2_{n,\mathcal{Y}} \sim A^2_n \otimes \mathcal{Y}$ we may consider the space $A^2_n(\mathcal{U}, \mathcal{Y}) \sim A^2_n \otimes \mathcal{L}(\mathcal{U}, \mathcal{Y})$ consisting of $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued functions $F(z) = \sum_{j=0}^{\infty} F_k z^k$ analytic on $\mathbb{D}$ and with the finite norm given by

$$\|F\|_{A^2_n(\mathcal{U}, \mathcal{Y})}^2 = \sum_{j=0}^{\infty} \beta_j \cdot \text{Trace}(F_k^* F_k).$$

For an $F \in A^2_n(\mathcal{U}, \mathcal{Y})$, define $[F, F] = \sum_{j=0}^{\infty} \beta_j F_k^* F_k$ and the left-tangential operator-argument evaluation

$$(C^* F)^\wedge^L (A^*) = \sum_{j=0}^{\infty} A^* j C^* F_j.$$

The $n$-output-stability of $(C, A)$ is exactly what is needed for the infinite series above to converge.
Given an $n$-output stable pair $(E, T)$ with $E \in \mathcal{L}(X, \mathcal{Y})$ and given $X \in \mathcal{L}(X, U)$, find all functions $F \in A^2_n(U, Y)$ such that

$$(E^* F)^L(T^*) = X^*.$$
Given an $n$-output stable pair $(E, T)$ with $E \in \mathcal{L}(X, Y)$ and given $X \in \mathcal{L}(X, U)$, find all functions $F \in A^2_n(U, Y)$ such that

$$(E^* F)^{\wedge L}(T^*) = X^*.$$

The function $F(z) = E(I - zT)^{-n} G_{n, E, T}^{-1} X^*$ is a solution with the minimally possible norm $[F, F] = X G_{n, E, T}^{-1} X^*$. 

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Extremal interpolation in $A^2_n(U, Y)$

Let $(C, A)$ be an $n$-output stable pair with $C \in \mathcal{L}(U, Y)$ and $A^{-1} \in \mathcal{L}(X)$, let $D \in \mathcal{L}(U, Y)$. Find an $F \in A^2_n(U, Y)$ such that

\[(C^* F)^\wedge (A^*) = 0 \quad \text{and} \quad F(0) = D. \quad (8)\]
Extremal interpolation in $A^2_n(\mathcal{U}, \mathcal{Y})$

Let $(C, A)$ be an $n$-output stable pair with $C \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ and $A^{-1} \in \mathcal{L}(\mathcal{X})$, let $D \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$. Find an $F \in A^2_n(\mathcal{U}, \mathcal{Y})$ such that

$$(C^* F)^\wedge L(A^*) = 0 \quad \text{and} \quad F(0) = D.$$ (8)

For what $D$, the minimal norm solution of the problem satisfies $[F, F] = I$?
Let \((C, A)\) be an \(n\)-output stable pair with \(C \in \mathcal{L}(U, \mathcal{Y})\) and \(A^{-1} \in \mathcal{L}(X)\), let \(D \in \mathcal{L}(U, \mathcal{Y})\). Find an \(F \in A^2_n(U, \mathcal{Y})\) such that

\[
(C^* F)^\wedge L(A^*) = 0 \quad \text{and} \quad F(0) = D.
\] (8)

For what \(D\), the minimal norm solution of the problem satisfies \([F, F] = I\)?

Let \(T = \begin{bmatrix} 0_Y & 0 \\ 0 & A \end{bmatrix}\), \(E = \begin{bmatrix} I_Y & C \end{bmatrix}\), \(X = \begin{bmatrix} D^* & 0 \end{bmatrix}\). Then condition

\[
(E^* F)^\wedge L(T^*) = X^*
\]

is equivalent to (8).
Extremal interpolation in $A^2_n(U, \mathcal{V})$

Let $(C, A)$ be an $n$-output stable pair with $C \in \mathcal{L}(U, \mathcal{V})$ and $A^{-1} \in \mathcal{L}(\mathcal{X})$, let $D \in \mathcal{L}(U, \mathcal{V})$. Find an $F \in A^2_n(U, \mathcal{V})$ such that

$$(C^* F)^L(A^*) = 0 \quad \text{and} \quad F(0) = D.$$  \hspace{1cm} (8)

For what $D$, the minimal norm solution of the problem satisfies $[F, F] = I$?

Let $T = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}$, $E = \begin{bmatrix} I & C \end{bmatrix}$, $X = \begin{bmatrix} D^* & 0 \end{bmatrix}$. Then condition

$$(E^* F)^L(T^*) = X^*$$

is equivalent to (8). Observe that

$$G^{-1}_{n,E,T} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -C \\ I \end{bmatrix} (G_{n,C,A} - C^* C)^{-1} \begin{bmatrix} -C^* & I \end{bmatrix}.$$
The minimal norm solution is

\[ F(z) = \begin{bmatrix} I & C \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & (I - zA)^{-n} \end{bmatrix} \left( \begin{bmatrix} D \\ 0 \end{bmatrix} - \begin{bmatrix} -C \\ I \end{bmatrix} (G_{n,C,A} - C^* C)^{-1} C^* D \right) \]

\[ = D - C \left( (I - zA)^{-n} - I \right) (G_{n,C,A} - C^* C)^{-1} C^* D \]

\[ = D - z R_{n,1}(zA) S_{1,C,A}^{-1} A^{-*} C^* D. \]
The minimal norm solution is

\[ F(z) = \begin{bmatrix} I & C \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & (I - zA)^{-n} \end{bmatrix} \left( \begin{bmatrix} D \\ 0 \end{bmatrix} - \begin{bmatrix} -C \\ I \end{bmatrix} \left( G_{n,C,A} - C^*C \right)^{-1} C^*D \right) \]

\[ = D - C \left( (I - zA)^{-n} - I \right) \left( G_{n,C,A} - C^*C \right)^{-1} C^*D \]

\[ = D - zR_{n,1}(zA) G_{1,C,A}^{-1} A^{-*} C^*D. \]

Its norm equals

\[ [F, F] = X G_{n,E,T}^{-1} X^* = D^* \left( I + C \left( G_{n,C,A} - C^*C \right)^{-1} C^* \right) D \]

\[ = D^* \left( I + CA^{-1} G_{1,C,A}^{-1} A^{-*} C^* \right) D = I \]
The minimal norm solution is
\[
F(z) = [I \ C] \begin{bmatrix} I & 0 \\ 0 & (I - zA)^{-n} \end{bmatrix} \left( \begin{bmatrix} D \\ 0 \end{bmatrix} - \begin{bmatrix} -C \\ I \end{bmatrix} (G_{n,C,A} - C^* C)^{-1} C^* D \right)
\]
\[
= D - C \left( (I - zA)^{-n} - I \right) (G_{n,C,A} - C^* C)^{-1} C^* D
\]
\[
= D - zR_{n,1}(zA) \mathcal{G}^{-1}_{1,C,A} A^{-*} C^* D.
\]
Its norm equals
\[
[F, F] = X \mathcal{G}_{n,E,T}^{-1} X^* = D^* \left( I + C(G_{n,C,A} - C^* C)^{-1} C^* \right) D
\]
\[
= D^* (I + CA^{-1} \mathcal{G}^{-1}_{1,C,A} A^{-*} C^*) D = I
\]
Letting
\[
B = -\mathcal{G}^{-1}_{1,C,A} A^{-*} C^* D,
\]
we see that \( F \) is of the the form and the following identity holds:
\[
\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} \mathcal{G}_{1,C,A} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} G_{n,C,A} & 0 \\ 0 & I \end{bmatrix}.
\]
If \( A \) is strongly stable, \( F \) is the Bergman inner function.