A NEW FORM OF THE HAHN - BANACH THEOREM

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Abstract. In this paper, we present a new form of the Hahn - Banach Theorem in terms of the sub-additive convex functions.

1. Introduction and Preliminaries

The Hahn-Banach theorem is one of the central results in functional analysis. This theorem originated from Hahn [5] and Banach [1] is of basic importance in the analysis of problems concerning the existence of continuous linear functionals. Its principal formulations are as a dominated extension theorem (analytic form) and as a separation theorem (geometric form), c.f. [12, Theorem II, 3.2] and [12, Theorem II, 3.1]. It is known that [12, Theorem II, 3.2] and [12, Theorem II, 3.1] imply each other.

Throughout this paper, $L$ denotes a vector space over $\Phi$. The scalar field $\Phi$ is the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. We use the terminologies in the book [12].

A function $p : L \rightarrow \mathbb{R}$ is said to be a sub-linear function if for every $x, y \in L$ and for every $\lambda \geq 0$, we have

1. $p(x + y) \leq p(x) + p(y),$
2. $p(\lambda x) = \lambda p(x)$.

The function $p$ is said to be a semi-norm if $p$ is a sub-linear function such that for every $x \in L$ and for every $\lambda \in \Phi$, we have

$$p(\lambda x) = |\lambda| p(x).$$

A function $\varphi : L \rightarrow \mathbb{R}$ is said to be a convex function if for every $x, y \in L$ and for every $\lambda \in \mathbb{R}$, we have

$$0 < \lambda < 1 \Rightarrow \varphi(\lambda x + (1-\lambda)y) \leq \lambda \varphi(x) + (1-\lambda)\varphi(y).$$

We say that $\varphi$ is a sub-additive function if

$$\varphi(x + y) \leq \varphi(x) + \varphi(y),$$

for all $x, y \in L$, and $\varphi(\theta) = 0$, where $\theta$ is the zero vector. The function $\varphi$ is called a sub-additive convex function if $\varphi$ is a sub-additive and convex function.

Let $L$ be a topological vector space with 0-neighborhood base $\mathcal{B}$ and let $\varphi : L \rightarrow \mathbb{R}$ be a convex function. The function $\varphi$ is said to be upper semi-continuous at $x_0 \in L$ if given $\varepsilon > 0$ there exists a neighborhood $U_\varepsilon \in \mathcal{B}$ such that

$$x \in x_0 + U_\varepsilon \Rightarrow \varphi(x) < \varphi(x_0) + \varepsilon;$$

$\varphi$ is said to be upper semi-continuous on a subset $G \subseteq L$ if for each $x \in G$, $\varphi$ is upper semi-continuous at $x$.

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The following version of the analytic Hahn-Banach theorem first appears in Banach [1] (see also [3, Theorem II.3.10]).

**Theorem 1.1.** Let \( Y \) be a vector subspace of the real vector space \( X \) and let \( p : X \to \mathbb{R} \) be a sub-linear function. Let \( f : Y \to \mathbb{R} \) be a linear function with \( f(x) \leq p(x), \ x \in Y \).

Then there exists a linear function \( F : X \to \mathbb{R} \) for which \( F(x) = f(x), \ x \in Y; \ F(x) \leq p(x), \ x \in X \).

In this paper, we present a new form of the analytic Hahn-Banach theorem in terms of the sub-additive convex functions, see Theorem 2.3. Since each sub-linear function is sub-additive convex, Theorem 2.3 implies Theorem 1.1. It is also shown that Theorem 2.3 implies [12, Theorem II, 3.2], see Corollary 2.4.

In the papers [8], [9] and [14] are presented generalizations of the Hahn-Banach theorem in terms of the convex functionals. There are also generalizations of the Hahn-Banach theorem in the papers [6], [2], [10], [11] and [13]. Kakutani [7] gave a proof of the analytic Hahn-Banach theorem by using the Markov-Kakutani fixed-point theorem.

2. The Main Result

The main result is Theorem 2.3. Let us start with two auxiliary lemmas.

**Lemma 2.1.** Let \( L \) be a topological vector space over \( \mathbb{R} \), let \( B \) be the family of all circled \( 0 \)-neighborhood and let \( \varphi : L \to \mathbb{R} \) be a convex function. If \( G \) is a non-empty, convex, open subset of \( L \) on which \( \varphi \) is bounded above, then \( \varphi \) is upper semi-continuous on \( G \).

**Proof.** Let \( x_0 \in G \) be arbitrary vector and let \( 0 < \varepsilon < 1 \). Then by hypothesis there exists \( M > 0 \) such that

\[ \varphi(x) \leq M, \text{ for all } x \in G \]

and there exists \( B \in B \) such that \( x_0 + B \subset G \). Thus, \( \varphi(x) \leq M \) for all \( x \in x_0 + B \).

If we choose

\[ \eta = \frac{\varepsilon}{3(1 + M)(1 + |\varphi(x_0)|)} \]

then \( \eta B \subset B \), since \( B \) is circled. For every \( u \in B \) the equality

\[ x_0 + \eta u = \eta(x_0 + u) + (1 - \eta)x_0 \]

yields

\[ \varphi(x_0 + \eta u) \leq \eta \varphi(x_0 + u) + (1 - \eta)\varphi(x_0) \]

\[ = \varphi(x_0) + \eta \varphi(x_0 + u) - \eta \varphi(x_0) \leq \varphi(x_0) + \eta M - \eta \varphi(x_0) \]

\[ \leq \varphi(x_0) + \frac{\varepsilon M}{3(1 + M)(1 + |\varphi(x_0)|)} + \frac{\varepsilon(-\varphi(x_0))}{3(1 + M)(1 + |\varphi(x_0)|)} \]

\[ \leq \varphi(x_0) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varphi(x_0) + \varepsilon. \]

Thus, for every \( x \in x_0 + \eta B \), we have

\[ \varphi(x) < \varphi(x_0) + \varepsilon. \]

This means that \( \varphi \) is upper semi-continuous at \( x_0 \in G \), and since \( x_0 \) was arbitrary it follows that \( \varphi \) is upper semi-continuous on \( G \) and the proof is finished. \( \square \)
Lemma 2.2. Let $L$ be a topological vector space over $\mathbb{R}$, let $\mathcal{B}$ be the family of all circled $0$-neighborhood and let $\varphi : L \to \mathbb{R}$ be a convex function. Then the following statements are equivalent:

(i) $\varphi$ is continuous on $L$.
(ii) $\varphi$ is upper semi-continuous on $L$.
(iii) There exists a non-empty, convex, open subset of $L$ on which $\varphi$ is bounded above.

Proof. (i) $\Leftrightarrow$ (ii) Clearly, if $\varphi$ is continuous on $L$, then $\varphi$ is upper semi-continuous on $L$. Conversely, assume that $\varphi$ is upper semi-continuous at $x_0 \in L$. Then, given $\varepsilon > 0$ there exists $U_\varepsilon \in \mathcal{B}$ such that

$$u \in U_\varepsilon = -U_\varepsilon \Rightarrow \max\{\varphi(x_0 + u), \varphi(x_0 - u)\} < \varphi(x_0) + \varepsilon$$

and since

$$2\varphi(x_0) \leq \varphi(x_0 + u) + \varphi(x_0 - u)$$

it follows that

$$u \in U_\varepsilon \Rightarrow 2\varphi(x_0) \leq \varphi(x_0 + u) + \varphi(x_0 - u) < 2\varphi(x_0) + 2\varepsilon.$$

The last result together with (2.1) yields

$$u \in U_\varepsilon \Rightarrow \varphi(x_0) - \varepsilon < \varphi(x_0 + u) < \varphi(x_0) + \varepsilon,$$

since $\varphi(x_0 + u) \leq \varphi(x_0) - \varepsilon$ together with $\varphi(x_0 - u) \leq \varphi(x_0) + \varepsilon$ yields

$$\varphi(x_0 + u) + \varphi(x_0 - u) < 2\varphi(x_0)$$

contradicting (2.2). This means that $\varphi$ is continuous at $x_0$.

(ii) $\Leftrightarrow$ (iii) Assume that $\varphi$ is upper semi-continuous on $L$ and let $G = \{x \in L : \varphi(x) < 1\}$. It is easy to see that $G$ is non-empty convex subset of $L$. Let $x_0 \in G$ be an arbitrary vector and let $\varepsilon = \frac{1 - \varphi(x_0)}{2}$. Then there exists $U_\varepsilon \in \mathcal{B}$ such that

$$x \in x_0 + U_\varepsilon \Rightarrow \varphi(x) < \varphi(x_0) + \varepsilon < 1 \Rightarrow x \in G.$$ 

Thus, $x_0 + U_\varepsilon \subset G$, and since $x_0$ was arbitrary it follows that $G$ is an open subset of $L$.

Conversely, assume that $G$ is a non-empty, convex, open subset of $L$ on which $\varphi$ is bounded above and let $x_0 \in L$ be an arbitrary vector. Then, there exists $M > 0$ such that $\varphi(x) \leq M$ for all $x \in G$. If $x_0 \in G$, then by Lemma 2.1 $\varphi$ is upper semi-continuous at $x_0$.

It remains to consider $x_0 \notin G$. Fix a vector $y_0 \in G$ and a real positive number $\rho > 1$ and define $w = y_0 + \rho(x_0 - y_0)$. The function $h : L \to L$ defined by

$$h(y) = \frac{\rho - 1}{\rho} y + \frac{1}{\rho} w, \quad \text{for all } y \in L$$

is a homeomorphism, c.f. [12] 1.1, p.13. The function $h$ transforms $y_0$ into $x_0$ and $G$ into an open and convex set $h(G)$ containing $x_0$. For every $x \in h(G)$, we have

$$x = h(h^{-1}(x)) = \frac{\rho - 1}{\rho} h^{-1}(x) + \frac{1}{\rho} w$$

and

$$\varphi(x) = \varphi \left( \frac{\rho - 1}{\rho} h^{-1}(x) + \frac{1}{\rho} w \right) \leq \frac{\rho - 1}{\rho} \varphi(h^{-1}(x)) + \frac{1}{\rho} \varphi(w)$$

$$\leq \frac{\rho - 1}{\rho} M + \frac{1}{\rho} \varphi(w) = M' < +\infty.$$
Thus, \( \varphi \) is bounded over \( h(G) \). Therefore, by Lemma 2.2 \( \varphi \) is upper semi-continuous at \( x_0 \in h(G) \). Since \( x_0 \) was arbitrary \( \varphi \) is upper semi-continuous at every \( x \in G \) and at every \( x \notin G \). Thus, \( \varphi \) is upper semi-continuous on \( L \), and this ends the proof. \( \square \)

We are now ready to present the main result.

**Theorem 2.3.** Let \( M \) be a subspace of a vector space \( L \) over \( \mathbb{R} \) and let \( \varphi : L \to \mathbb{R} \) be a sub-additive convex function. If \( g : M \to \mathbb{R} \) is a linear function such that \( g \) "dominated" by \( \varphi \), i.e.,

\[
(\forall x \in M)[g(x) \leq \varphi(x)],
\]

then there exists a linear function \( f : L \to \mathbb{R} \) such that

(i) \( f \) extends \( g \) to \( L \), i.e.,

\[
f|_M = g,
\]

(ii) \( f \) "dominated" by \( \varphi \), i.e.,

\[
(\forall x \in L)[f(x) \leq \varphi(x)].
\]

**Proof.** Define

\[
U_n = \left\{ x \in L : \varphi(x) < \frac{1}{n} \right\}, n \in \mathbb{N}
\]

and

\[
\mathcal{B} = \{V_n \in 2^L : V_n = U_n \cap (-U_n), n \in \mathbb{N}\},
\]

where

\[
-U_n = \left\{ -x \in L : \varphi(x) < \frac{1}{n} \right\} = \left\{ x \in L : \varphi(-x) < \frac{1}{n} \right\}.
\]

The family \( \mathcal{B} \) is a filter base in \( L \), since

\[
V_n \cap V_m \supset V_m, \quad m > n
\]

and \( \theta \in V_n \) for all \( n \in \mathbb{N} \).

Note that \( (-1)V_n = V_n \) and \( 0V_n = \{ \theta \} \subset V_n \); given \( 0 < \lambda < 1 \) or \( -1 < \lambda < 0 \), we have also

\[
\lambda V_n \subset V_n,
\]

since

\[
\varphi(x) = \varphi\left(\frac{x}{\lambda}\right) = \varphi\left(\frac{1}{\lambda}x + (1 - \lambda)\theta\right) \leq \lambda \varphi\left(\frac{1}{\lambda}x\right).
\]

Thus, \( V_n \) is circled. Let us show next that \( V_n \) is radial. Since \( V_n \) is circled, it is enough to show that given \( x_0 \in L \) there exists \( \mu \in \mathbb{R} \) such that \( x_0 \in \mu V_n \). If \( x_0 \in V_n \) then \( \mu = 1 \). Otherwise \( x_0 \notin U_n \) or \( x_0 \notin -U_n \); \( \varphi(x_0) \geq \frac{1}{n} \) or \( \varphi(-x_0) \geq \frac{1}{n} \). There exists \( \mu > 1 \) such that \( \frac{1}{\mu}\varphi(x_0) < \frac{1}{n} \) and \( \frac{1}{\mu}\varphi(-x_0) < \frac{1}{n} \). Since

\[
\varphi\left(\frac{1}{\mu}x_0\right) = \varphi\left(\frac{1}{\mu}x_0 + \left(1 - \frac{1}{\mu}\right)\theta\right) \leq \frac{1}{\mu}\varphi(x_0) < \frac{1}{n},
\]

\[
\varphi\left(-\frac{1}{\mu}x_0\right) = \varphi\left(\frac{1}{\mu}(-x_0) + \left(1 - \frac{1}{\mu}\right)\theta\right) \leq \frac{1}{\mu}\varphi(-x_0) < \frac{1}{n},
\]

it follows that \( x_0 \in \mu U_n \cap \mu(-U_n) = \mu V_n \). Thus, \( \mathcal{B} \) is a filter base in \( L \) such that for each \( n \in \mathbb{N} \), we have
\begin{itemize}
  \item[(a)] \( V_{2n} + V_{2n} \subset V_n \), since \( \varphi \) is sub-additive function,
  \item[(b)] \( V_n \) is radial and circled.
\end{itemize}
Therefore, by [12, I, 2.1, p.15] the family $\mathcal{B}$ is a 0-neighborhood base for a unique topology $\mathcal{T}$ under which $L$ is a topological vector space.

Given any vector $x_0 \in U_1$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < 1 - \varphi(x_0)$. Then, $x_0 + V_n \subset U_1$. This means that $U_1$ is an open set, and since $U_1$ is also convex, by Lemma 2.2 it follows that $\varphi$ is continuous on $L$.

We now consider $L \times \mathbb{R}$ as the product of two topological vector spaces. Note that if $T : L \times \mathbb{R} \rightarrow \mathbb{R}$ is a linear function, then

$$T(x, t) = h(x) + \alpha t, \text{ for all } (x, t) \in L \times \mathbb{R},$$

where $h(x) = T(x, 0)$ for all $x \in L$ and, $\alpha = T(0, 1)$. Hence,

$$H = \{(x, t) \in M \times \mathbb{R} : g(x) - t = 1\}$$

is a linear manifold in the subspace $M \times \mathbb{R}$. Since $(x, t) \rightarrow S(x, t) = g(x) - t$ is a non-zero linear functional, by [12, I, 4.1] $H$ is a hyperplane in $M \times \mathbb{R}$ and a linear manifold in $L \times \mathbb{R}$.

Since $\varphi$ is a continuous function on $L$ it follows that

$$G = \{(x, t) \in L \times \mathbb{R} : \varphi(x) - t < 1\}$$

is an open set, and since $g(x) \leq \varphi(x)$ for $x \in M$ it follows that $G \cap H = \emptyset$; $G$ is also a non-empty convex set. Therefore, by [12, Theorem II, 3.1] there exists a closed hyperplane $H_1$ in $L \times \mathbb{R}$ such that $H_1 \supset H$ and $H_1 \cap G = \emptyset$. Since $H_1$ is a hyperplane, $H_1 = (x_0, t_0) + H_0$ where $H_0$ is a maximal subspace of $L \times \mathbb{R}$. If we choose $(x_0, t_0) \in H \subset H_1 \cap M \times \mathbb{R}$, then

$$H_1 \cap (M \times \mathbb{R}) = (x_0, t_0) + (H_0 \cap (M \times \mathbb{R})),$$

and since

$$H_1 \cap (M \times \mathbb{R}) \neq (M \times \mathbb{R}), \quad ((0, 0) \notin H_1 \text{ and } (0, 0) \in (M \times \mathbb{R}))$$

it follows that $H_1 \cap (M \times \mathbb{R})$ is a hyperplane in $M \times \mathbb{R}$. Further, $H_1 \cap (M \times \mathbb{R}) \supset H$ implies

$$H_1 \cap (M \times \mathbb{R}) = H = (x_0, t_0) + (H_0 \cap (M \times \mathbb{R}));$$

($M \times \mathbb{R}$ is the algebraic sum of subspaces $H_0 \cap (M \times \mathbb{R})$ and $\{\lambda(x_0, t_0) : \lambda \in \mathbb{R}\}$).

By [12, I,4.1], there exits a non-zero linear functional $F : L \times \mathbb{R} \rightarrow \mathbb{R}$ such that $H_1 = \{(x, t) \in L \times \mathbb{R} : F(x, t) = 1\}$. The least result together with (2.3) yields

$$F(x, 0) = S(x, 0) = g(x), \text{ for all } x \in M;$$

that is, the linear function $f(\cdot) = F(\cdot, 0)$ is an extension of $g$ to $L$.

It remains to prove that $f$ "dominated" by $\varphi$. Note that

$$F(x, t) = f(x) - t, \text{ for all } (x, t) \in L \times \mathbb{R},$$

since $(0, -1) \in H \subset H_1$. By $H_1 \cap G = \emptyset$ it follows that

$$f(x) = 1 + t \leq \varphi(x), \text{ for all } (x, t) \in H_1.$$  \hspace{1cm} (2.4)

While $(x, t) \notin H_1$ implies that there exists a real number $r \neq 1$ such that

$$f(x) - t = r.$$  \hspace{1cm} (2.3)

Then, $(x, t + r - 1) \in H_1$ and therefore (2.4) yields

$$f(x) = 1 + (t + r - 1) \leq \varphi(x).$$

Thus, $f$ "dominated" by $\varphi$ and this ends the proof. \hspace{1cm} $\Box$
Corollary 2.4. Let $M$ be a subspace of a vector space $L$ over $\mathbb{C}$ and let $p : L \to [0, +\infty]$ be a semi-norm. If $g : M \to \mathbb{C}$ is a linear functional such that $g$ "dominated" by $p$, i.e.,
$$\forall x \in M \exists \|g(x)\| \leq p(x),$$
then there exists a linear functional $f : L \to \mathbb{C}$ such that
- $f$ extends $g$ to $L$, i.e., $f|_M = g$,
- $f$ "dominated" by $p$, i.e.,
$$\forall x \in L \exists \|f(x)\| \leq p(x).$$

Proof. There exists a linear functional $g_0 : M \to \mathbb{R}$ such that
$$g(x) = g_0(x) - ig_0(ix), \text{ for all } x \in M.$$ 
We have also that $p$ is a convex functional such that $p(\lambda x) = \lambda p(x)$ for every $\lambda \geq 0$ and $x \in L$. Thus, regarding $L$ as a real linear space, and applying Theorem 2.3 a real linear function $f_0 : L \to \mathbb{R}$ is obtained for which $f_0|_M = g_0$ and $f_0(x) \leq p(x)$ for all $x \in L$. Let the function $f : L \to \mathbb{C}$ be defined by the equation
$$f(x) = f_0(x) - if_0(ix).$$
Clearly, $f$ is a linear functional and
$$f(x) = f_0(x) - if_0(ix) = g_0(x) - ig_0(ix) = g(x), \text{ for all } x \in M.$$ 
Thus $f$ is an extension of $g$. Finally, let $f(x) = |f(x)|e^{i\theta(x)}$, where $\theta(x)$ is an argument of the complex number $f(x)$; then
$$|f(x)| = f_0 \left( \frac{x}{e^{i\theta(x)}} \right) \leq p \left( \frac{x}{e^{i\theta(x)}} \right) = p(x),$$
which proves that $|f(\cdot)| \leq p(\cdot)$ and this ends the proof.

Since [12], Theorem II, 3.1 implies Theorem 2.4 and Theorem 2.3 implies [12], Theorem II, 3.2, we conclude that Theorem 2.3 [12], Theorem II, 3.2 and [12], Theorem II, 3.1 imply each other.

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