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Braids, mapping class groups, and categorical delooping

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Abstract Dehn twists around simple closed curves in oriented surfaces satisfy the braid relations. This gives rise to a group theoretic map \( \phi : \beta_{2g} \to \Gamma_{g,1} \) from the braid group to the mapping class group. We prove here that this map is trivial in homology with any trivial coefficients in degrees less than \( g/2 \). In particular this proves an old conjecture of J. Harer. The main tool is categorical delooping in the spirit of [25]. By extending the homomorphism to a functor of monoidal 2-categories, \( \phi \) is seen to induce a map of double loop spaces on the plus construction of the classifying spaces. Any such map is null-homotopic. In an appendix we show that geometrically defined homomorphisms from the braid group to the mapping class group behave similarly in stable homology.

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1 Introduction

Let \( \beta_k \) be Artin’s braid group on \( k \) strings [1]. In its standard presentation \( \beta_k \) has generators \( \sigma_1, \ldots, \sigma_{k-1} \) subject to the following relations:

\[
\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i - j| \geq 2;
\]

\[
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } i = 1, \ldots, k - 2.
\]

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Let $\Gamma_{g,1}$ be the mapping class group of an oriented surface with one boundary component. Wajnryb \cite{Wajnryb1}, \cite{Wajnryb2} gave a presentation for $\Gamma_{g,1}$ with generators the Dehn twists $\alpha_i, i = 0, \ldots, 2g-1$, and $\delta$ around the simple closed curves $a_i, i = 1, \ldots, 2g-1$, and $d$ as depicted in Figure 1. Among others, $\alpha_1, \ldots, \alpha_{2g-1}$ satisfy the same relations as the braid group generators above.

Hence, the assignment $\sigma_i \mapsto \alpha_i$ defines a purely group-theoretic homomorphism

$$\phi : \beta_{2g} \longrightarrow \Gamma_{g,1}.$$ 

$\phi$ extends to a homomorphism of stable groups from $\beta_\infty := \lim_{g \to \infty} \beta_{2g}$ to $\Gamma_\infty := \lim_{g \to \infty} \Gamma_{g,1}$. We study the effect of this map in group (co)homology.

By work of Fuks \cite{Fuks} and F. Cohen \cite{Cohen} the (co)homology of the braid groups has been well known. Though the (co)homology of the finite genus mapping class groups remains difficult, through recent work on a homotopy theoretic refinement of the Mumford conjecture, the (co)homology of the stable mapping class group is well understood rationally and integrally, cf. \cite{CF}, \cite{KM}, \cite{K}. In particular, both $\beta_\infty$ and $\Gamma_\infty$ have ample integral (co)homology.

**Theorem 1.1** The image of the map $\phi_* : H_*(\beta_\infty; \mathbb{Z}/2\mathbb{Z}) \to H_*(\Gamma_\infty; \mathbb{Z}/2\mathbb{Z})$ is zero for $* > 0$.

This is our main result. We believe though that the method of proof via categorical delooping is of interest in its own right. In particular, we introduce and study the homotopy type of a new monoidal category (operad) $\mathcal{T}$, built out of generalised Artin groups on trees and forests.

Most of the difficulty in proving Theorem 1.1 stems from the fact that $\phi$ is defined algebraically. In an Appendix we prove an analogue of Theorem 1.1 for two different geometrically defined maps.

**1.2 Background:** In the early 1980s, in conversation with E. Miller and F. Cohen, John Harer made the following conjecture based on many explicit calculations he had done using the Thurston-Hatcher method of giving a presentation of the mapping class group.

**Conjecture (J. Harer)** The map $\phi_* : H_*(\beta_\infty; \mathbb{Z}/2\mathbb{Z}) \to H_*(\Gamma_\infty; \mathbb{Z}/2\mathbb{Z})$ is trivial.
The approach suggested in [6], as ours in this paper, is based on product structures. Juxtaposition of strings defines a monoidal product on the disjoint union of classifying spaces of the braid groups, \( \coprod_{k \geq 0} B\beta_k \). Its group completion is the space

\[ \mathbb{Z} \times B\beta^+_\infty \simeq \Omega^2 S^2; \]

here “+” denotes Quillen’s plus construction. The loop space structure on \( \Omega^2 S^2 \) is induced by this monoidal product. The additional, two-fold loop space structure corresponds to a wreath product map of braid groups,

\[ \omega_\beta : \beta_q \wr \beta_k \to \beta_{qk} \]

which maps the element \((\sigma; \mu_1, \ldots, \mu_q)\) to the braid on \(qk\) strings where the braids \(\mu_1, \ldots, \mu_q\) are weaved together (each considered a single strand) as prescribed by \(\sigma\). Similarly, a monoidal product can be defined on \( \coprod_{g \geq 0} B\Gamma_{g,1} \) by joining two surfaces by a pair of pants surfaces. This induces a loop space structure on the group completion, \( \mathbb{Z} \times B\Gamma^+ \). Miller [19] (see also [4] and [24]) noted that this extends to a double loop space structure which corresponds to the wreath product

\[ \omega_\Gamma : \Gamma_{g,1} \to \Gamma_{qg,1}. \]

To define \(\omega_\Gamma\) one identifies \(\beta_q\) as a subgroup of the mapping class group of a \(q\)-legged pants surface (i.e. a sphere with \(q + 1\) disks removed) in which the \(q\) legs can be permuted, cf. [8].

In homology the wreath products give rise to the first Araki-Kudo-Dyer-Lashof operation \(Q_1 : H_* \to H_{2_*+1}\), and F. Cohen in [7] has computed \(Q_1\) in the homology of the braid groups with \(\mathbb{Z}/q\mathbb{Z}\)-coefficients. In particular, he proves

\[ H_*(\beta_{\infty}; \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}[a_1, a_2, \ldots] \]

as polynomial algebras, where \(|a_i| = 2^i - 1\) and \(Q_1(a_i) = a_{i+1}\). As by [21] the first homology group of \(\Gamma_{g,1}\) is trivial for \(g \geq 3\), Harer’s conjecture would follow if \(\phi\) commutes with the wreath products (for \(q = 2\)). However, this is not the case. Moreover, Maginnis [17] shows that this is also not the case in homology: the difference between \(\phi_* \circ Q_1\) and \(Q_1 \circ \phi_*\), when applied to the non-trivial 1-dimensional class, is a non-zero element in \(H_3(\Gamma_{2,1}; \mathbb{Z}/2\mathbb{Z})\).

1.3 Strategy: The fact that \(\phi\) does not commute with the wreath products means that \(\phi : \coprod_{g \geq 0} B\beta_{2g} \to \coprod_{g \geq 0} B\Gamma_{g,1}\) is not a map of \(E_2\)-spaces in the sense of May [21]. Nevertheless, we will show that the induced map on group completion is.

**Theorem 5.2** \(\phi : B\beta^+_\infty \to B\Gamma^+_\infty\) is a map of double loop spaces.

Theorem 1.1 and Harer’s conjecture follow immediately; see Lemma 5.3. We note here that Maginnis’ obstruction lies in the unstable range of the homology of the mapping class group, cf. [12], [13], and therefore there is no contradiction.
The idea of the proof is to use categorical delooping. Recall, when a connected category \( C \) is monoidal (symmetric) then its classifying space \( BC \) is a loop space (infinite loop space, cf. [18], [23]). Furthermore, monoidal (symmetric) functors induce maps of loop spaces (infinite loop spaces). By [20], there is a symmetric surface category \( S \) with \( \Omega BS \simeq \mathbb{Z} \times B\Gamma_\infty \). Our aim is therefore to construct a monoidal category \( T \) out of braid groups and extend \( \phi \) to a monoidal functor \( \Phi : T \to S \); the induced map \( \Omega B\Phi : \Omega BT \to \Omega BS \) will automatically be a map of double loop spaces.

1.4 The category \( T \): The construction of the category \( T \) imitates the construction of \( S \). The place of surfaces in \( S \) is taken by trees in \( T \). We are lead to consider generalised Artin groups associated with any tree and forest. Recall, given a graph \( \Sigma \), its Artin group \( \beta(\Sigma) \) is the group generated by the edges in \( \Sigma \). Two generators \( e, f \) commute if they are disjoint, and satisfy the braid relation \( efe = fef \) if they have a vertex in common. The composition (operad structure) in \( T \) is given by grafting of trees. The homomorphism \( \phi \) can now be extended to a functor from \( T \) to \( S \) through a careful choice of images for the generators at branch points; see Figure 7. Some complexity in the proof arises from the fact that \( \Omega BT \) is not homotopy equivalent to \( \mathbb{Z} \times B\beta_\infty^+ \). For comparison, we recall that Fuks [10] proves a homotopy equivalence

\[
B\beta(D_\infty)^+ \simeq \Omega^2 S^3 \times S^3\{2\}
\]

where \( S^3\{2\} \) denotes the homotopy fiber of the degree 2 map from \( S^3 \) to \( S^3 \) and \( D_\infty \) the infinite Coxeter graph of type \( D \). To prove Theorem 5.3 it is however enough to show that \( \phi \) factors as a map of double loop spaces through the identity component \( \Omega_0 BT \). Indeed, in Section 4, we prove

**Theorem 4.1** \( \Omega_0 BT \simeq \Omega^2 S^3 \times \Omega^2 W \) as double loop spaces for some \( W \).

The homotopy type of \( W \) remains undetermined.

2 Definition of tile categories

The categories we will consider in this paper are most naturally seen as strict 2-categories. Recall, a 2-category is a category \( C \) enriched over \( CAT \), the category of small categories. This means that instead of morphism sets, \( C \) has morphism categories \( C(a, b) \) and composition \( C(a, b) \times C(b, c) \to C(a, c) \) is a functor for all objects \( a, b, c \in C \). A 2-category \( C \) is strict if composition is strictly associative. We may apply the nerve construction to the morphism categories. Since the nerve construction commutes with products, this gives a category \( BC \) enriched over \( \Delta \)-sets, the category of simplicial sets. In other words, \( BC \) is a simplicial category with constant objects. Applying the nerve construction to \( BC \) yields a bisimplicial set. The realization of this bisimplicial set is the classifying space \( BC \) of \( C \).
The 2-category $\mathcal{T}$ may be thought of as a special type of cobordism category. Its objects are the natural numbers $\mathbb{N}$ where $n$ represents $n$ ordered copies of the unit interval. Its 1-morphisms $\mathcal{T}(n,m)$ are generated by three atomic tiles $D : 0 \to 1, P : 2 \to 1$ and $F : 1 \to 1$, as illustrated in Figure 2.

![Figure 2.](image)

These are three discs thought of as cobordisms between intervals. In addition, the tiles $P$ and $F$ have one respectively two marked points. The 1-morphisms are generated by these three atomic tiles by gluing along incoming and outgoing intervals and disjoint union. Note that all incoming and outgoing intervals are ordered. A typical 1-morphism is illustrated in Figure 3. Identity morphisms are added and may be thought of as the objects themselves, i.e. zero length cobordisms. Composition of 1-morphisms is defined by gluing. Note: The 1-morphisms are freely generated by the atomic tiles as morphisms of a monoidal category. Given any two tiles $T_1, T_2$, we identify $(T_1 \sqcup 1) \circ (1 \sqcup T_2) = (T_1 \sqcup T_2) = (1 \sqcup T_2) \circ (T_1 \sqcup 1)$. However, homeomorphic tiles are not identified in general and disjoint union is not symmetric. Indeed, $T_1 \sqcup T_2 = T_2 \sqcup T_1$ if and only if $T_1 = T_2$.

The morphism categories $\mathcal{T}(n,m)$ are disjoint unions of groups. Thus there are no morphisms between two different tiles, i.e. $\mathcal{T}(T_1,T_2) = \emptyset$ if $T_1 \neq T_2$. We will now describe a functorial choice of generators for the 2-endomorphism groups. Given a tile $T \in \mathcal{T}(n,m)$, join the marked points to form a graph $\Sigma_T$, as indicated in Figure 3.
Figure 3.

$\Sigma_T$ is a union of $m$ disjoint trees and can be built functorially out of trees on atomic tiles, see Figure 2. Any half-edges in $T$ that are not completed through gluing are not considered to be part of $\Sigma_T$. Then by definition, the construction of the graphs is functorial with respect to gluing and disjoint union, i.e. for two tiles $T_1$ and $T_2$, $\Sigma_{T_1}$ and $\Sigma_{T_2}$ can canonically be identified as subgraphs of $\Sigma_{T_1 \circ T_2}$ and $\Sigma_{T_1 \cup T_2}$. Define $\mathcal{T}(T, T)$ to be the group generated by the edges $e, f \in \Sigma_T$ satisfying the following relations.

\[
\begin{align*}
&ef = fe \quad \text{if} \quad e \cap f = \emptyset; \\
&efe = fef \quad \text{if} \quad e \cap f \neq \emptyset.
\end{align*}
\]

Thus $\mathcal{T}(T, T) = \beta(\Sigma_T)$ is the Artin group associated with the graph $\Sigma_T$; for further discussion see for example [27]. In particular, the group associated with the tile $F^k = F \circ \cdots \circ F$ ($k$ times) is the usual braid group on $2k$ strings,

\[
\mathcal{T}(F^k, F^k) = \beta(\Sigma_{F^k}) = \beta_{2k}.
\]

This completes the definition of $\mathcal{T}$.

We will need some information about the homotopy type of $\mathcal{T}$. For this purpose, we introduce an auxiliary tile category $\mathcal{T}$ whose homotopy type we are able to determine, see Proposition 3.2 below. The objects and 1-morphisms of $\mathcal{T}$ are the same as in $\mathcal{T}$. The morphism categories are groupoids: For tiles $T_1$ and $T_2$ in $\mathcal{T}(n, m) = \mathcal{T}(n, m)$, the set of 2-morphisms $\mathcal{T}(T_1, T_2)$ is the set of connected components of the space of homeomorphisms from $T_1$ to $T_2$ that identify the ordered incoming and outgoing boundary intervals, and map marked points bijectively to marked points. (The edges of the associated trees may not be preserved.) Hence, if $T_1$ and $T_2$ are not homeomorphic or don’t have the same number of marked points, $\mathcal{T}(T_1, T_2) = \emptyset$. On the other
hand, the endomorphisms of a tile \( T \) with \( k \) marked points is the mapping class group of \( T \):

\[
\hat{T}(T, T) = \Gamma(T) = \beta_k.
\]

In order to be able to compare the homotopy types of the two categories we define a functor

\[
\Theta : T \rightarrow \hat{T}.
\]

On objects and 1-morphisms \( \Theta \) is the identity, and we are left to define a homomorphism of 2-morphism groups associated to a tile \( T \). For each generator \( e \in T(T, T) = \beta(\Sigma_T) \), let \( \sigma_e \) in \( \hat{T}(T, T) = \Gamma(T) \) be the isotopy class of a half Dehn twist, the homeomorphism of \( T \) that turns the edge \( e \) by a half rotation in a clockwise direction and is the identity outside a contractible neighborhood of \( e \subset T \). By definition this assignment is functorial under gluing. It is also not difficult to see that the assignment \( e \mapsto \sigma_e \) induces a well-defined surjective group homomorphism

\[
\Theta : \beta(\Sigma_T) \rightarrow \Gamma(T).
\]

We expect this to be well-known and only sketch an argument. All trees that occur have vertices that are at most trivalent. By an inductive argument, it is enough to consider the tree \( T \) in Figure 4 with five vertices, 1, 2, 3, 4, 5, and edges \((1, 2), (2, 4), (3, 4), (4, 5)\). The corresponding generators are mapped to four elements \( \sigma_1, \tilde{\sigma}, \sigma_3, \sigma_4 \) of \( \Gamma(T) \simeq \beta_5 \).

![Figure 4.](image)

\( \tilde{\sigma} \) corresponds in the standard representation of \( \beta_5 \) to \( \sigma_2\sigma_3^{-1} \). It remains to check that the three relations in \( \beta(T) \) are satisfied in \( \beta_5 \). For example:

\[
\tilde{\sigma}\sigma_3\tilde{\sigma} = (\sigma_3\sigma_2\sigma_3^{-1})\sigma_3(\sigma_3\sigma_2\sigma_3^{-1}) = \sigma_3(\sigma_2\sigma_3\sigma_2)\sigma_3^{-1} = \sigma_3^{-1}\sigma_2 = \sigma_3(\sigma_3\sigma_2\sigma_3^{-1})\sigma_3 = \sigma_3\tilde{\sigma}\sigma_3.
\]

**Remark 2.2** The homomorphism \( \Theta \) is in general not injective. To invert the maps for the tree in Figure 4, we would need that the element corresponding to \( \sigma_2 = \sigma_3^{-1}\tilde{\sigma}\sigma_3 \) commutes with the one corresponding to \( \sigma_4 \). This, however, cannot be deduced from the relations (2.1). Note that this is the only additional relation satisfied by these generators in \( \beta_5 \).
Both categories $T$ and $\hat{T}$ contain the subcategory $M$ with one object, 1, and 1-morphisms generated by the atomic tile $F$:

$$M \subset T \xrightarrow{\theta} \hat{T}.$$ 

Disjoint union of cobordisms makes $T$ and $\hat{T}$ into strict monoidal 2-categories. Furthermore, the functor $\Theta : T \rightarrow \hat{T}$ is a map of strict monoidal 2-categories. Note that these monoidal structures are not symmetric as there are no morphisms that permute the ordered intervals and therefore no symmetries exist. Also, the subcategory $M$ is not monoidal.

3 Classifying spaces of tile categories

We will now study the homotopy type of these categories. The realization of $BM$ is the monoid $\coprod_{k \geq 0} B\beta_{2k}$. Hence, as is well known, by the group completion theorem for monoids, we have

**Proposition 3.1** $\Omega BM \simeq 2\mathbb{Z} \times B\beta_{\infty}^+.$

In order to calculate the homotopy type of $\hat{T}$ a generalisation of the group completion theorem for categories will be used, cf. [25], [20]. The main idea goes back to McDuff and Segal’s approach to the group completion theorem via homology fibrations [16].

**Proposition 3.2** $\Omega B\hat{T} \simeq \mathbb{Z} \times B\beta_{\infty}^+.$

**Proof** $B\hat{T}(n, 1)$ is the nerve of a groupoid in which each connected component is determined by the number of marked points of the underlying tiles; note, all such tiles are connected. Hence, its homotopy type is given by $\coprod_{k \geq n-1} B\beta_k$. Gluing the atomic tile $F \in \hat{T}(1, 1)$ on the right defines a self map of $B\hat{T}(n, 1)$. Let

$$B\hat{T}_\infty(n) = \text{hocolim}_F B\hat{T}(n, 1) \simeq \mathbb{Z} \times B\beta_{\infty}$$

be the corresponding homotopy colimit. Both $B\hat{T}(\_ , 1)$ and $B\hat{T}_\infty(\_ )$ define contravariant functors from $B\hat{T}$ to $\Delta$-sets. The homotopy colimit of the first is the nerve of Quillen’s category of objects over 1 in $B\hat{T}$, and is therefore contractible as this category has a final element. Hence, 

$$\text{hocolim}_{B\hat{T}} B\hat{T}(\_ , 1) \simeq \ast.$$ 

But then also the homotopy colimit of the second functor is contractible as the two homotopy colimits commute and the nerve of the translation category is contractible:

$$\text{hocolim}_{B\hat{T}} B\hat{T}_\infty(\_ ) = \text{hocolim}_F (\text{hocolim}_{B\hat{T}} B\hat{T}(\_ , 1)) \simeq \ast.$$
Next consider the canonical forgetful map

\[ \text{hocolim}_{\mathcal{T}} \mathcal{T}_\infty(\omega) \rightarrow \mathcal{T}_\infty \]

to the nerve of \( \mathcal{T}_\infty \). Each 1-morphism in \( \mathcal{T}(n, m) \) acts as a homology equivalence on the fiber

\[ \mathcal{T}_\infty(m) \simeq \mathbb{Z} \times B\beta_\infty. \]

This is immediate as the homotopy type of the fiber does not depend on the object \( m \) and left translation is conjugate to right translation in the braid group. Thus, \( \pi \) is a homology fibration, i.e. the canonical map of the fiber \( \mathcal{T}_\infty(m) \) into the homotopy fiber \( \text{hofib}(\pi) \simeq \Omega B\mathcal{T} \) is a homology isomorphism. By the Whitehead theorem for simple spaces this gives the desired homotopy equivalence after plus construction.

\[ \square \]

**Corollary 3.3** The inclusion \( \mathcal{M} \subset \mathcal{T} \) induces the natural inclusion

\[ \Omega B\mathcal{M} \simeq 2\mathbb{Z} \times B\beta_\infty^+ \rightarrow \Omega B\mathcal{T} \simeq \mathbb{Z} \times B\beta_\infty^+. \]

**Proof** Define \( \mathcal{M}_\infty(\omega) \) in analogue to \( \mathcal{T}_\infty(\omega) \) in the proof above. The inclusion \( \mathcal{M} \subset \mathcal{T} \) then induces a map of homology fibrations

\[ \text{hocolim}_{\mathcal{M}} \mathcal{M}_\infty(\omega) \rightarrow \text{hocolim}_{\mathcal{T}} \mathcal{T}_\infty(\omega) \]

which is a homotopy equivalence of total spaces (as both are contractible) and is homotopic to the inclusion \( 2\mathbb{Z} \times B\beta_\infty \rightarrow \mathbb{Z} \times B\beta_\infty \) on the fiber over the object 1. The corollary follows as before by an application of Whitehead’s theorem for simple spaces.

\[ \square \]

**4 Double loop space structures**

The monoidal structures on \( \mathcal{T} \) and \( \mathcal{T}_\infty \) induce double loop space structures on \( \Omega B\mathcal{T} \) and \( \Omega B\mathcal{T}_\infty \). A priori this double loop space structure may however not be compatible with the standard double loop space structure on

\[ B\beta_\infty^+ \simeq \Omega_0^2 S^2 \simeq \Omega^2 S^3; \]

here the subscript 0 indicates that only the 0-component is considered. Below \( \Omega^2 S^3 \) will be considered with its standard double loop space structure.
**Theorem 4.1** There is a splitting of double loop spaces

\[ \Omega_0BT \simeq \Omega^2S^3 \times \Omega^2W \]

for some 2-connected space \( W \). Furthermore, the inclusion \( \mathcal{M} \subset T \) induces a map of double loop spaces

\[ \Omega^2S^3 \simeq \Omega_0BM \to \Omega BT \]

which is homotopic to the inclusion of the first factor in the above splitting.

**Proof** We first recall that

\[ \Omega_0BM \simeq \Omega^2S^3 \]

is a homotopy equivalence of loop spaces, cf. [22], [7]. Consider now the maps of 0-components

\[ \Omega_0BM \to \Omega_0BT \xrightarrow{\Omega B \Theta} \Omega_0BT. \]

The first is a map of loop spaces by definition, while the second is a map of double loop spaces as \( \Theta \) is monoidal. Thus, by Corollary 3.3, the composition is a homotopy equivalence of loop spaces. By Lemma 4.3 below, the loop space structure on \( \Omega S^3 \) is unique. Hence,

\[ \Omega_0BT \simeq \Omega^2S^3 \]

is a homotopy equivalence of double loop spaces.

As \( \Omega B \Theta \) is a map of double loop spaces, its homotopy fiber is a connected double loop space \( \Omega^2W \) for some 2-connected space \( W \). Because \( \Omega_0BT \) is an H-space, the retraction above gives rise to a splitting of spaces,

(4.2) \[ \Omega_0BT \simeq \Omega^2S^3 \times \Omega^2W, \]

here the projection onto the first factor and the inclusion of the second factor are maps of double loop spaces.

\( S^1 \) embeds canonically in \( \Omega^2S^3 \). Let \( f : S^1 \to \Omega_0BT \) be the restriction of the inclusion of the first factor in (4.2) to \( S^1 \) with respect to the splitting (4.2). \( f \) may be written as \( (f_1, f_2) \) where \( f_1 \) is the inclusion and \( f_2 \) is the trivial map. \( \Omega^2S^3 = \Omega^2 \Sigma^2(S^1) \) is the free double loop space on \( S^1 \). By the universal property for free double loop spaces, \( f \) extends (up to homotopy) uniquely to a map of double loop spaces \( \bar{f} : \Omega^2S^3 \to \Omega_0BT \). Let \( \bar{f} = (\bar{f}_1, \bar{f}_2) \) with respect to the splitting (4.2), \( \Omega^2W \) is a double loop subspace of \( \Omega_0BT \), and hence \( \bar{f}_2 \) is a map of double loop spaces extending \( f_2 \simeq * \). Thus by the uniqueness property, \( \bar{f}_2 \) is therefore trivial as well. Finally, \( \bar{f}_1 = \Omega B \Theta \circ \bar{f} \) is a map of double loop spaces of \( \Omega^2S^3 \) to itself. As it extends \( f_1 \), by the uniqueness property, \( \bar{f}_1 \) is homotopic to the identity. Hence, the splitting (4.2) is a splitting of double loop spaces. In particular the inclusion of the first factor, which was induced by the inclusion \( \mathcal{M} \subset T \), is a map of double loop spaces. \( \square \)
Lemma 4.3  For n odd, the loop space structure on \( \Omega S^n \) is unique up to homotopy.

Proof For n odd, the cohomology of \( \Omega S^n \) is a divided power algebra, i.e. there are generators \( z_k \in H^{k(n-1)}(\Omega S^n) = \mathbb{Z} \) such that
\[
k! z_k = z_1^k.
\]
Let \( Z = \Omega S^n \) with some (potentially exotic) loop space structure, and let \( f : S^{n-1} \to Z \) be a generator of \( \pi_{n-1} Z = \mathbb{Z} \). By the universal property of free loop spaces, \( f \) extends to a map \( \tilde{f} : \Omega \Sigma(S^{n-1}) = \Omega S^n \to Z \) of loop spaces which is unique up to homotopy. Clearly,
\[
\tilde{f}^*(z_1) = f^*(z_1) = z_1,
\]
and, as \( \tilde{f}^* \) is a map of cohomology rings,
\[
k! \tilde{f}^*(z_k) = \tilde{f}^*(z_1^k) = z_1^k = k! z_k.
\]
Hence \( \tilde{f}^* \) is a homology equivalence and by Whitehead’s theorem for simple spaces a homotopy equivalence. It follows that \( Z \) and \( \Omega S^n \) are homotopy equivalent as loop spaces. \( \square \)

Remark 4.4  The loop space structure on \( \Omega S^2 \), and hence the double loop space structure on \( \Omega^2 S^2 \), are not unique which can be seen as follows. The generator of the first cohomology group defines a map \( g : \Omega S^2 \to S^1 \) which is a splitting of the embedding \( f : S^1 \to \Omega S^2 \). This induces a splitting of spaces
\[
\Omega S^2 \simeq S^1 \times F
\]
where \( F \) is the homotopy fiber of \( g \). \( F \) has a natural loop space structure as \( g \) is a map of loop spaces. Indeed, \( F \simeq \Omega S^3 \) as \( g \) is induced by the Hopf map
\[
\Omega S^3 \to \Omega S^2 \xrightarrow{g} S^1 \to S^3 \to S^2.
\]
However, the loop space structure of \( \Omega S^2 \) is not the same as the product loop space structure of \( S^1 \times \Omega S^3 \), for \( S^2 \) is not homotopy equivalent to \( \mathbb{C}P^\infty \times S^3 \).

The same argument shows that more generally the \( \Omega^{n-1} \)-structure on \( \Omega^n S^n \) and the \( \Omega^n \)-structure of \( \Omega^n S^n \) are not unique as \( S^n \) does not have \( K(\mathbb{Z}, n) \) as a retract.

Remark 4.5  Let \( T_0 \) denote the subcategory of \( T \) with the same objects and 1-morphisms but only identity 2-morphisms. As there are only endomorphisms in \( T \), there is a natural retraction \( \pi_0 : T \to T_0 \) mapping each 2-morphism group to its identity. \( T_0 \) inherits a monoidal structure from \( T \) and both the inclusion and the projection are monoidal. Thus, for some space \( Z \), there is a splitting of double loop spaces
\[
\Omega B T \simeq \Omega B T_0 \times \Omega^2 Z.
\]
Furthermore, from the proof of Theorem 4.1, it follows that for some two connected space $\tilde{W}$,

$$\Omega^2Z \simeq \Omega^2S^3 \times \Omega^2\tilde{W}.$$ 

We expect that $\Omega BT_0$ is homotopic to a discrete, free abelian group on infinitely many generators. This would, in particular, imply that $W = \tilde{W}$.

5 Surface category and the functor $\Phi$

We define a surface 2-category $\mathcal{S}$ following [25] and [26] which gives rise to a convenient categorical delooping of $B\Gamma^+_\infty$. Its objects are the natural numbers $\mathbb{N}$ with each $n$ representing $n$ disjoint, ordered circles. The 1-morphisms are generated by three atomic surfaces, a disk $D' : 0 \rightarrow 1$, a torus with two incoming and one outgoing boundary component $P' : 2 \rightarrow 1$, and a torus with one incoming and outgoing boundary component $T' : 1 \rightarrow 1$.

![Figure 5.](image)

In analogy with the tile category $\mathcal{T}$, any 1-morphism is obtained from these atomic surfaces by gluing incoming boundary circles of one surface to outgoing boundary circles of another surface and by disjoint union. In addition reordering of the incoming or outgoing boundary circles gives rise to new 1-morphisms. Identity 1-morphisms are adjoined and may be thought of as zero length cylinders, i.e. the objects themselves. A typical 1-morphism is illustrated in Figure 6.
Figure 6.

The set of 2-morphisms $\mathcal{S}(S_1, S_2)$ between two 1-morphisms $S_1, S_2 \in \mathcal{S}(n, m)$ is the set of connected components of the space of homeomorphisms from $S_1$ to $S_2$ which identify the $n$ incoming and $m$ outgoing boundary circles. When $S_1 = S_2$ this is the mapping class group of the surface.

Let $\mathcal{M}_T$ denote the subcategory of $\mathcal{S}$ with only one object, 1, 1-morphisms generated by $F'$ and a full set of 2-morphisms. Recall from [25, 26] that the group completion argument in conjunction with Harer’s homology stability theorem [12] for the mapping class groups proves

**Theorem 5.1** $\Omega B\mathcal{M}_T \simeq \Omega B\mathcal{S} \simeq \mathbb{Z} \times B\Gamma_\infty^+.$

Disjoint union makes $\mathcal{S}$ into a strict symmetric monoidal 2-category. As $\mathcal{S}$ is connected, its classifying space $B\mathcal{S}$ and thus its loop space are therefore infinite loop spaces. Wahl [28] proved that the induced infinite loop space structure on $\mathbb{Z} \times B\Gamma_\infty^+$ is compatible with the double loop space structure defined by the pairs of pants product discussed in Subsection 1.1 of the introduction.

We will now define the monoidal 2-functor $\Phi : \mathcal{T} \to \mathcal{S}$. On objects, $\Phi$ maps $n$ intervals to $n$ circles, i.e. $\Phi(n) = n$. For the atomic tiles, define

$$\Phi(D) = D', \quad \Phi(P) = P', \quad \Phi(F) = F'.$$

$\Phi$ extends to a map of all 1-morphisms as by definition the 1-morphisms in $\mathcal{T}$ and $\mathcal{S}$ are built from the atomic 1-morphisms in the same fashion. Under this map Figure 6 is the image of Figure 3 apart from a different labelling of the incoming boundary components.

To define the functor $\Phi$ on 2-morphisms, recall from Section 2 that a tile $T$ has an associated tree $\Sigma_T$. $\Phi$ will map this tree to a system of curves on the surface $\Phi(T)$. For the atomic tiles $P$ and $F$ this is indicated in Figure 7.
Through gluing this assignment extends to more general tiles. Half edges and half circles gain importance when under gluing they are completed and give rise to additional generators. Otherwise, they should be ignored. Thus, each edge $e \in \Sigma_T$ is mapped to a simple closed curve $C_e$, and two such curves $C_e$ and $C_f$ intersect (precisely once) if and only if the edges $e$ and $f$ share a vertex. Dehn twists around curves that don’t intersect commute. It is also well-known that Dehn twists around two simple closed curves which intersect once satisfy the second braid relation (2.1), cf. [2]. Thus the map of generators $e \mapsto C_e$ extends to a well-defined group homomorphism

$$\Phi: \mathcal{T}(T, T) = \beta(\Sigma_T) \longrightarrow S(\Phi(T), \Phi(T)) = \Gamma(\Phi(T)).$$

By definition the assignment of curve systems to trees is functorial under gluing. Furthermore, $\Phi$ clearly commutes with disjoint union, and hence defines a (strict) monoidal 2-functor.

**Theorem 5.2** The map $\phi : B\beta^+_{\infty} \to B\Gamma^+_{\infty}$ is a map of double loop spaces.

**Proof** On the subcategory $\mathcal{M}$ of $\mathcal{T}$ the functor $\Phi$ is identical to the homomorphism $\phi : \beta_{2g} \to \Gamma_{g,1+1}$ of the introduction. Hence the following diagram commutes up to homotopy.

$$
\begin{array}{ccc}
2\mathbb{Z} \times B\beta^+_{\infty} \simeq \Omega BM & \longrightarrow & \Omega BT \\
\phi \downarrow & & \downarrow \Omega B\phi \\
\mathbb{Z} \times B\Gamma^+_{\infty} \simeq \Omega BM_T & \longrightarrow & \Omega BS.
\end{array}
$$
By Theorem 4.1, the restriction to the 0-component of the top horizontal map is a map of double loop spaces. As $\Phi$ is monoidal, $\Omega B\Phi$ is a double loop space map, and hence so is $\phi$ when restricted to the 0-component.

\[\square\]

**Lemma 5.3** Any map $\varphi : B\beta_\infty^+ \to B\Gamma_\infty^+$ of double loop spaces is homotopically trivial.

**Proof** As $B\beta_\infty^+ \simeq \Omega^2 S^1$ is the free double loop space on the circle $S^1$, and as $\varphi$ is a map of double loop spaces, up to homotopy it is determined completely by its restriction to $S^1$. However, by [21], the mapping class group is perfect for $g \geq 3$, and $B\Gamma_\infty^+$ is simply connected. Hence, the restriction of $\varphi$ to $S^1$ is homotopically trivial, and so is $\varphi$.

As the plus construction does not change the homology, this also implies the triviality of the map $\varphi$ on group homology. In particular, for $\varphi = \phi$ this proves Theorem 1.1 of the introduction and Harer’s conjecture as a special case. By [12] and [13], we can state an unstable version of this result.

**Corollary 5.4** The image of $\phi : H_*(\beta_{2g};\mathbb{Z}) \to H_*(\Gamma_{g,1};\mathbb{Z})$ is zero for $0 < * < g/2$, and the image of any element of degree $* \geq g/2$ is zero or unstable.

### 6 Appendix: Geometrically defined homomorphisms

In this appendix we discuss maps from the braid group to the mapping class group which are defined geometrically, i.e. by identifying the braid group as (a subgroup of) the mapping class group of a subsurface. There are many such maps but we will discuss only two in detail. Others can be analyzed in a similar fashion.

The basic idea is to identify the braid group as a subgroup of the mapping class group of a genus zero surface with boundary components as follows. Let $S_{0,k+1}$ be a sphere with $k + 1$ disks removed and parametrised boundary circles $\partial_0$, $\partial_1$, $\ldots$, $\partial_k$. Consider the orientation preserving diffeomorphisms that fix the first boundary component $\partial_0$ pointwise but may permute the other $k$ boundary components as long as they preserve the parametrisation of each. The associated mapping class group, denoted by $\Gamma_{0,(k),1}$, is the ribbon braid group $R\beta_k$ on $k$ ribbons. $R\beta_k$ is the wreath product $\beta_k \wr \mathbb{Z}$, and $\beta_k$ can naturally be identified as a subgroup.

$$\beta_k \subset \beta_k \wr \mathbb{Z} = R\beta_k \simeq \Gamma_{0,(k),1}.$$
is obtained by gluing \( k \) copies of the surface \( S_{g,1} \) onto \( S_{0,k+1} \) along \( \partial_1, \ldots, \partial_k \), and then extending the diffeomorphism by the identity and permutation the \( k \) copies of \( S_{g,1} \), cf. Figure 8.

![Figure 8.](image)

On mapping class groups this map factors through the wreath product map \( \omega_\Gamma \) described in the introduction

\[
\varphi_1 : \beta_k \hookrightarrow \beta_k \ltimes \Gamma_{g,1} \xrightarrow{\omega_\Gamma} \Gamma_{kg,1}.
\]

As these give rise to Miller’s double loop space structure \([19]\), the induced map on group completions is a map of double loop spaces:

\[
\varphi_1 : \Omega^2 S^2 \to \Omega^\infty S^\infty \to \mathbb{Z} \times B\Gamma^+_\infty.
\]

It follows from Lemma 5.3 that

**Proposition 6.1** The image of \( \varphi_1 : H_*(\beta_k;\mathbb{Z}) \to H_*(\Gamma_{kg,1};\mathbb{Z}) \) is zero for \( 0 < * < kg/2 \).

The second family of homomorphisms is constructed as follows. Consider two copies of the surface \( S_{0,k+1} \) glued along their boundary components \( \partial_1, \ldots, \partial_k \) to form a surface \( S_{k-1,2} \) which in turn is a subsurface of a larger surface \( S_{g+k,n} \). We will only consider the case when one of the two boundary components \( \partial_0 \) remains a boundary component in \( S_{g+k,n} \) with \( n = 2 \), cf. Figure 9, and leave the other cases as an exercise.
Any diffeomorphism of $S_{0,k}$ as described above can be extended to $S_{k-1,2}$ by “mirroring” the action on the second copy of $S_{0,k}$, and can then be extended to $S_{g+k,n}$ by the identity diffeomorphism. This gives rise to a group homomorphism

$$\varphi_2 : \beta_k \to \beta_k \times \Sigma_k \beta_k \to \Gamma_{g+k,2},$$

where the group in the middle is defined as the pull-back in the following diagram:

$$\begin{array}{ccc}
\beta_k \times \Sigma_k \beta_k & \to & \Sigma_k \\
\downarrow & & \downarrow \triangle \\
\beta_k \times \beta_k & \to & \Sigma_k \times \Sigma_k.
\end{array}$$

Here $\triangle$ denotes the diagonal map, and $\pi : \beta_k \to \Sigma_k$ is the canonical surjection.

**Proposition 6.2** The image of $\varphi_2 : H_*(\beta_k; \mathbb{Z}) \to H_*(\Gamma_{g+k,2}; \mathbb{Z})$ contains at most 2-torsion for $0 < s < (g+k)/2$.

**Proof** Consider the following commutative diagram of group homomorphisms

$$\begin{array}{ccc}
\beta_k & \to & \beta_k \times \Sigma_k \beta_k \\
\downarrow s & & \downarrow s \times s \\
\Gamma_{g+k,(k),1} & \to & \Gamma_{g+k,(k),1} \times \Sigma_k \Gamma_{g+k,(k),1} \to \Gamma_{2(g+k)+(k-1),2}.
\end{array}$$

Here $\Gamma_{g+k,(k),1}$ denotes the mapping class group of the surface $S_{g+k,k+1}$ where $k$ of the boundary components may be permuted as long as the parametrisation of each component is preserved while one of the boundary components is fixed pointwise; the group in the middle on the bottom is defined as a pull-back as above. The map $s$ is induced by gluing one of the boundary components of $S_{g+k,2}$ along $\partial_0$ to a copy of $S_{0,k+1}$. The left horizontal maps are defined by “mirroring”, while the bottom right horizontal map is defined by identifying the two copies of the boundary components $\partial_1, \ldots, \partial_k$. As a
consequence of Harer-Ivanov stability \([12],[13]\), it was proved in [5] that the map \(\beta_k \to \Gamma_{g+k}^{(k)},1\) factors in homology in degrees \(s < (g + k)/2\) through \(\Sigma_k\). The right most vertical map is a homology isomorphism in these degrees, and hence the claim follows as the image of \(\pi : \beta_k \to \Sigma_k\) in homology contains only 2-torsion, cf. [7].

We expect that the image is actually zero in the given range though a proof of this seems at the moment out of reach.

\[\square\]

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