Groups that have the same holomorph as a finite perfect group

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Four questions
Let $G$ be a group.
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Consider the opposite group $(G, \circ)$, where

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The opposite group of $G$ is isomorphic to $G$ via

$$\text{inv} : G \rightarrow (G, \circ)$$

$$x \mapsto x^{-1}$$
Let \( G = H \times K \) be a direct product.
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Let $(G, \circ)$ be obtained from $G$ by replacing $H$ with its opposite.
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$$(x_1, x_2) \circ (y_1, y_2) = (y_1 x_1, x_2 y_2).$$
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$$(x_1, x_2) \circ (y_1, y_2) = (y_1 x_1, x_2 y_2).$$

Is $(G, \circ)$ isomorphic to $G$? Yes, via

$$H \times K \to (H \times K, \circ)$$

$$(x, y) \mapsto (x^{-1}, y)$$
Let $G = HK$ be a central product
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The answer is possibly not obvious.
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Characteristic subgroups
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Assume all $H_i$ are characteristic in $G$.

Let $(G, \circ)$ be obtained from $G$ by replacing $H_1$ with its opposite.

Are the $H_i$ still characteristic in $(G, \circ)$?
(Multiple) holomorphs
We will be discussing *regular subgroups of the holomorph of a group.*
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- Skew right braces, which are equivalent to these subgroups.
- Hopf Galois extensions, which are linked to these subgroups.
  - C. Greither, B. Pareigis. Hopf Galois theory for separable field extensions. *J. Algebra* 106 (1987), 239–258.
  - N. P. Byott. Uniqueness of Hopf Galois structure of separable field extensions. *Comm. Algebra* 24 (1996), 3217–3228.
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The Holomorph

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The holomorph of a group \( G \) is the natural semidirect product \( \text{Aut}(G)G \).

If \( S(G) \) is the group of permutations on the set \( G \), and

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\rho : G \rightarrow S(G) \\
g \mapsto (x \mapsto xg)
\]

is the right regular representation

is (isomorphic to) the holomorph of \( G \). More generally, if \( N \leq S(G) \) is a regular subgroup, then

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If $G$ and $N$ have the same holomorph, and $G$ and $N$ are isomorphic, then $\rho(G)$ and $N$ are conjugate under an element of

$$N_{S(G)}(\operatorname{Hol}(G)) = N_{S(G)}(N_{S(G)}(\rho(G))),$$

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$$N_{S(G)}(\text{Hol}(G)) = N_{S(G)}(N_{S(G)}(\rho(G))),$$

the multiple (double) holomorph of $G$, and the group

$$T(G) = N_{S(G)}(\text{Hol}(G))/\text{Hol}(G)$$

acts regularly on the set

$$\mathcal{H}(G) = \{ N \leq S(G) : N \text{ is regular, } N_{S(G)}(N) = \text{Hol}(G) \text{ and } N \cong G \}.$$
Three sets

In increasing order

\[ H(G) = \{ N \leq S(G) : N \text{ is regular}, N \cap S(G) = \text{Hol}(G) \} \subseteq I(G) = \{ N \leq S(G) : N \text{ is regular} \} \subseteq J(G) = \{ N \leq S(G) : N \text{ is regular, } N \triangleright Hol(G) \} \]
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The latter appears to be easier to compute.
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The latter appears to be easier to compute.
Describing the regular (normal) subgroups of the holomorph
An element of the regular subgroup $N \leq \text{Hol}(G) = \text{Aut}(G)\rho(G)$ can be written uniquely as $\nu(g)$.
Regular subgroups of the holomorph

An element of the regular subgroup $N \leq \text{Hol}(G) = \text{Aut}(G) \rho(G)$ can be written uniquely as $\nu(g)$, with $1^{\nu(g)} = g$. 

Here $\cdot \circ y = x \circ y y$ is a group operation on $G$, $\cdot \circ : (G; \circ) \to N$ is an isomorphism, $x \circ (y) = x \circ (y) = x \circ (y) y = x \circ y$; to be compared with $x \circ (y) = xy$. 

10/22
An element of the regular subgroup $N \leq \text{Hol}(G) = \text{Aut}(G) \rho(G)$ can be written uniquely as $\nu(g)$, with $1^{\nu(g)} = g$.

Now

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10/22
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  - to be compared with $x^{\rho(y)} = xy$. 
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(Aside) Groups and rings

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Here the condition $\text{Aut}(G) \leq \text{Aut}(G, \circ)$ translates into the study of the commutative rings $(G, +, \cdot)$ such that every automorphism of the group $(G, +)$ is also an automorphism of the ring $(G, +, \cdot)$. 
\[ \iota : G \rightarrow \text{Inn}(G) \leq \text{Aut}(G) \]

\[ g \mapsto (x \mapsto g^{-1}xg) \]
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Take \( N = \lambda(G) \trianglelefteq \text{Hol}(G) \), where \( \lambda \) is the left regular representation.
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Take \( N = \lambda(G) \trianglelefteq \text{Hol}(G) \), where \( \lambda \) is the left regular representation. (Actually, \( N_{S(G)}(\lambda(G)) = \text{Hol}(G) \).)
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Take \( N = \lambda(G) \trianglelefteq \text{Hol}(G) \), where \( \lambda \) is the left regular representation. (Actually, \( N_{S(G)}(\lambda(G)) = \text{Hol}(G) \).) Then
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- Here \( y \circ x = y^{\gamma(x)}x = y^{x^{-1}}x = xy \) yields the opposite group.
Commutators and perfect groups
If \( N \trianglelefteq \text{Hol}(G) \) is regular, then \( \gamma : G \rightarrow \text{Aut}(G) \) satisfies

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\gamma(x^\beta) = \gamma(x)^\beta \quad \text{for} \quad \beta \in \text{Aut}(G),
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Perfect groups

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### Perfect groups

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The regular normal subgroups of the holomorph of a perfect group
Two decompositions

Theorem

Let $G$ be a finite perfect group.

- If $N \trianglelefteq \text{Hol}(G)$ is regular, then $\text{Inn}(G) = (G) \times (\ker)$.

- If $G$ is centreless, then $G = (\text{Hol}(G)) \times \ker$ is a product of two characteristic subgroups.

In the general case, $(h) = (h - 1)$; for $h \in (\text{Hol}(G))$, follows from $(x \circ y) = (yx)$. 

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Two decompositions

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Two decompositions

**Theorem**

Let $G$ be a finite perfect group.

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$$\gamma(h) = \iota(h^{-1}), \quad \text{for } h \in \iota^{-1}(\gamma(G)),$$
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Centreless groups: how to obtain all regular normal subgroups $N$ of the holomorph

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Divide the $L_i$ in two groups, say,

$$H = L_1 \times \cdots \times L_m, \quad K = L_{m+1} \times \cdots \times L_n.$$  

• is trivial on $K = \ker(\ )$ and $(h) = (h^{-1})$ on $H$.

• $(G; \circ) \cong N$ is obtained from $G = H \times K$ by replacing $H$ with its opposite, as in Question 2:

$$(x_1; x_2) \circ (y_1; y_2) = (y_1 x_1; x_2 y_2).$$  

• All these $N$ are isomorphic to $G$ (see Question 2), so that $T(G)$ is elementary abelian of order $2^n$. 

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The general case

When $G$ is allowed to have a nontrivial centre, things become more complicated.
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$$G = L_1 \cdots L_n,$$

with the $L_i \geq Z(G)$ characteristic, centrally indecomposable. The regular subgroups $N \unlhd \text{Hol}(G)$ are still obtained by replacing some of the $L_i$ with their opposites. But this time

- the groups $(G; \circ) = N$ need not have the same automorphism group of $G$;
- even if they do, $(G; \circ) = N$ need not be isomorphic to $G$.

Question 4: in this case $N_S(G)(N) > \text{Hol}(G)$;
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Examples
For primes $p \equiv 1 \pmod{3}$, there is a family of groups $Q_p$ which are

- pairwise non-isomorphic,
- perfect,
- centrally indecomposable, and such that
- $|\mathbb{Z}(Q_p)| = 3$,
- $\text{Aut}(Q_p)$ acts trivially on $\mathbb{Z}(Q_p)$.

These are obtained from $\text{SL}(3; p)$ by killing the transpose inverse automorphism via the insertion of a non self-dual module underneath.
Killing automorphisms

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Example 1 (Negative answer to Question 3)

\[ G = HK, \text{ a central product of two non-isomorphic } Q_p, \text{ with } Z(H) \text{ amalgamated with } Z(K). \]
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\[ G = HK, \] a central product of two non-isomorphic \( Q_p \), with \( Z(H) \) amalgamated with \( Z(K) \). (Both have order 3.)

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An isomorphism of \( G \) to \((G, \circ)\) would induce

- an automorphism of \( K \)
  - thereby inducing the identity on \( Z(K) \), and
Example 1 (Negative answer to Question 3)

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  - thereby inducing inversion on $Z(H)$. 
Example 2 (Negative answer to Question 4)

\[ G = L_1 L_2 M, \] a central product of \( Q_p \)'s, with \( L_1 \cong L_2 \not\cong M \), with amalgamated centres.
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\[ G = L_1 L_2 M, \] a central product of \( Q_p \)'s, with \( L_1 \cong L_2 \not\approx M \), with amalgamated centres.

- Let \( z_1 \) generate \( Z(L_1) \).
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- Fix an isomorphism \( \zeta : L_1 \to L_2, \) and define \( z_2 = z_1^\zeta \in L_2. \)
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Then every isomorphism \( L_1 \rightarrow L_2 \) takes \( z_1 \) to \( z_2 \).
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- Choose the amalgamation so that \( z_2 = z_1^{-1} \).
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- Then \( M \) and the \( L_i \) are characteristic in \( G \).
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- Fix an isomorphism $\zeta : L_1 \to L_2$, and define $z_2 = z_1^\zeta \in L_2$. Then every isomorphism $L_1 \to L_2$ takes $z_1$ to $z_2$.
- Choose the amalgamation so that $z_2 = z_1^{-1}$.
- Then $M$ and the $L_i$ are characteristic in $G$: an automorphism of $G$
  - takes $M$ to $M$, and thus fixes the centre elementwise;
Example 2 (Negative answer to Question 4)

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- Fix an isomorphism \( \zeta : L_1 \to L_2 \), and define \( z_2 = z_1^\zeta \in L_2 \).
  
  Then every isomorphism \( L_1 \to L_2 \) takes \( z_1 \) to \( z_2 \).

- Choose the amalgamation so that \( z_2 = z_1^{-1} \).

- Then \( M \) and the \( L_i \) are characteristic in \( G \): an automorphism of \( G \)
  
  - takes \( M \) to \( M \), and thus fixes the centre elementwise;
  - if it takes \( L_1 \) to \( L_2 \), then it takes \( z_1 \) to \( z_2 \).
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- Then \( M \) and the \( L_i \) are characteristic in \( G \): an automorphism of \( G \)
  - takes \( M \) to \( M \), and thus fixes the centre elementwise;
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Example 2 (Negative answer to Question 4)

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- Then \( M \) and the \( L_i \) are characteristic in \( G \): an automorphism of \( G \)
  - takes \( M \) to \( M \), and thus fixes the centre elementwise;
  - if it takes \( L_1 \) to \( L_2 \), then it takes \( z_1 \) to \( z_2 = z_1^{-1} \), i.e. it inverts the centre.
Example 2 (continued)

$G = L_1 L_2 M$, a central product of $Q_p$'s, with $L_1 \cong L_2 \not\cong M$, amalgamating $z_2 = z_1^{-1}$. 
Example 2 (continued)

\( G = L_1 L_2 M \), a central product of \( Q_p \)'s, with \( L_1 \cong L_2 \not\cong M \), amalgamating \( z_2 = z_1^{-1} \).

Get \((G, \circ)\) by replacing \( L_1 \) with its opposite.
Example 2 (continued)

\[ G = L_1L_2M, \text{ a central product of } Q_p \text{'s, with } L_1 \cong L_2 \not\cong M, \]

amalgamating \( z_2 = z_1^{-1} \).

Get \( (G, \circ) \) by replacing \( L_1 \) with its opposite.

There is an automorphism of \( (G, \circ) \) which
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There is an automorphism of \((G, \circ)\) which

- is the identity on \( M \),
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There is an automorphism of \((G, \circ)\) which

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Example 2 (continued)

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In fact for \( x, y \in L_1 \)

\[
(x \circ y)^{\zeta \text{ inv}} = (yx)^{\zeta \text{ inv}} = (y^\zeta x^\zeta)^{\text{ inv}} = x^{\zeta \text{ inv}} y^{\zeta \text{ inv}},
\]
Example 2 (continued)

\[ G = L_1 L_2 M, \]  
a central product of \( Q_p \)'s, with \( L_1 \cong L_2 \neq M \), 
amalgamating \( z_2 = z_1^{-1} \).

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\[ (x \circ y)^{\zeta \text{ inv}} = (yx)^{\zeta \text{ inv}} = (y^\zeta x^\zeta)^{\text{ inv}} = x^{\zeta \text{ inv}} y^{\zeta \text{ inv}}, \]

and

\[ z_1^{\zeta \text{ inv}} \]
Example 2 (continued)

\[ G = L_1 L_2 M, \] a central product of \( Q_p \)'s, with \( L_1 \cong L_2 \not\cong M \),
amalgamating \( z_2 = z_1^{-1} \).

Get \((G, \circ)\) by replacing \(L_1\) with its opposite.

There is an automorphism of \((G, \circ)\) which

- is the identity on \(M\),
- takes \(L_1\) to \(L_2\), acting like \(\zeta\ inv\) on \(L_1\).

In fact for \(x, y \in L_1\)

\[(x \circ y)^{\zeta \ inv} = (yx)^{\zeta \ inv} = (y^\zeta x^\zeta)^{\zeta \ inv} = x^{\zeta \ inv} y^{\zeta \ inv},\]

and

\[ z_1^{\zeta \ inv} = z_2^{\zeta \ inv} \]
Example 2 (continued)

\[ G = L_1L_2M, \] a central product of \( Q_p \)'s, with \( L_1 \cong L_2 \not\cong M \), amalgamating \( z_2 = z_1^{-1} \).

Get \((G, \circ)\) by replacing \( L_1 \) with its opposite.

There is an automorphism of \((G, \circ)\) which

- is the identity on \( M \),
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In fact for \( x, y \in L_1 \)

\[(x \circ y)^{\zeta \text{ inv}} = (yx)^{\zeta \text{ inv}} = (y^{\zeta} x^{\zeta})^{\text{ inv}} = x^{\zeta \text{ inv}} y^{\zeta \text{ inv}},\]

and

\[ z_1^{\zeta \text{ inv}} = z_2^{\text{ inv}} = z_2^{-1}. \]
Example 2 (continued)

\(G = L_1 L_2 M\), a central product of \(Q_p\)'s, with \(L_1 \cong L_2 \not\cong M\), amalgamating \(z_2 = z_1^{-1}\).

Get \((G, \circ)\) by replacing \(L_1\) with its opposite.

There is an automorphism of \((G, \circ)\) which

- is the identity on \(M\),
- takes \(L_1\) to \(L_2\), acting like \(\zeta^{\text{inv}}\) on \(L_1\).

In fact for \(x, y \in L_1\)

\[(x \circ y)^{\zeta^{\text{inv}}} = (yx)^{\zeta^{\text{inv}}} = (y \zeta x^{\zeta})^{\text{inv}} = x^{\zeta^{\text{inv}}} y^{\zeta^{\text{inv}}},\]

and

\[z_1^{\zeta^{\text{inv}}} = z_2^{\text{inv}} = z_2^{-1} = z_1.\]
Example 2 (continued)

\[
G = L_1 L_2 M, \text{ a central product of } Q_p \text{'s, with } L_1 \cong L_2 \not\cong M, \text{ amalgamating } z_2 = z_1^{-1}.
\]

Get \((G, \circ)\) by replacing \(L_1\) with its opposite.

There is an automorphism of \((G, \circ)\) which

- is the identity on \(M\),
- takes \(L_1\) to \(L_2\), acting like \(\zeta \text{ inv}\) on \(L_1\).

In fact for \(x, y \in L_1\)

\[
(x \circ y)^{\zeta \text{ inv}} = (yx)^{\zeta \text{ inv}} = (y^\zeta x^\zeta)^{\text{ inv}} = x^{\zeta \text{ inv} }y^{\zeta \text{ inv}},
\]

and

\[
z_1^{\zeta \text{ inv}} = z_2^{\text{ inv}} = z_2^{-1} = z_1
\]

is compatible with the identity on \(M\).
Example 2 (continued)

\[ G = L_1 L_2 M, \text{ a central product of } Q_p \text{'s, with } L_1 \cong L_2 \not\cong M, \]
amalgamating \( z_2 = z_1^{-1} \).

Get \((G, \circ)\) by replacing \( L_1 \) with its opposite.

There is an automorphism of \((G, \circ)\) which

- is the identity on \( M \),
- takes \( L_1 \) to \( L_2 \), acting like \( \zeta \text{ inv} \) on \( L_1 \).

In fact for \( x, y \in L_1 \)

\[
(x \circ y)^{\zeta \text{ inv}} = (yx)^{\zeta \text{ inv}} = (y^\zeta x^\zeta)^{\text{ inv}} = x^{\zeta \text{ inv}} y^{\zeta \text{ inv}},
\]

and

\[
\zeta_1^{\text{ inv}} = \zeta_2^{\text{ inv}} = \zeta_2^{-1} = z_1
\]
is compatible with the identity on \( M \).

So the \( L_i \) are not characteristic in \((G, \circ)\)
Example 2 (continued)

\[ G = L_1 L_2 M, \] a central product of \( Q_p \)'s, with \( L_1 \cong L_2 \not\cong M \), amalgamating \( z_2 = z_1^{-1} \).

Get \((G, \circ)\) by replacing \( L_1 \) with its opposite.

There is an automorphism of \((G, \circ)\) which

- is the identity on \( M \),
- takes \( L_1 \) to \( L_2 \), acting like \( \zeta \text{ inv} \) on \( L_1 \).

In fact for \( x, y \in L_1 \)

\[ (x \circ y)^{\zeta \text{ inv}} = (yx)^{\zeta \text{ inv}} = (y^\zeta x^\zeta)^{\text{ inv}} = x^{\zeta \text{ inv}} y^{\zeta \text{ inv}}, \]

and

\[ z_1^{\zeta \text{ inv}} = z_2^{\text{ inv}} = z_2^{-1} = z_1 \]

is compatible with the identity on \( M \).

So the \( L_i \) are not characteristic in \((G, \circ)\), and \( \text{Aut}(G, \circ) \) is twice as big as \( \text{Aut}(G) \).
That’s All, Thanks!