We consider explicit type II string constructions of backgrounds containing warped and squashed anti–de Sitter spaces. These are obtained via T–duality from brane intersections including dyonic black strings, plane waves and monopoles. We also study the supersymmetry of these solutions and discuss special values of the deformation parameters.
1 Introduction

The study of three–dimensional maximally symmetric spaces has attracted a lot of attention over the years. They arise as near horizon geometries of various D–brane configurations and in some cases they can be shown to be exact string backgrounds. Furthermore, anti–de Sitter spaces have been used in the context of the AdS/CFT correspondence [1] leading to a dramatic progress in the understanding of gauge theories, and have found applications spanning from black hole physics to condensed matter physics or strongly coupled plasmas.

Recently, the subject has met with renewed interest [2, 3] in the context of the study of Topologically Massive Gravity (TMG) with a negative cosmological constant [4, 5, 6]. The theory admits a family of asymptotically AdS solutions parameterized by the value of the Chern–Simons coupling. Moreover, it was found to contain two stable warped AdS vacuum solutions for every value of the Chern–Simons coupling [7], where the term warped stands for the fact that AdS$_3$ is viewed as a Hopf fibration over AdS$_2$, with the fiber being lengthened by a constant factor.

In this context a warped AdS$_3$ geometry is obtained by changing the radius of the $S^1$ fiber over AdS$_2$. In a suitable coordinate system, we are considering Minkowskian three–manifolds endowed with a one–parameter family of metrics

$$ds^2[\text{WAdS}_3] = R^2 \left[ d\omega^2 - \cosh^2 \omega d\tau^2 + \frac{1}{\cosh^2 \Omega_w} (d\beta + \sinh \omega d\tau)^2 \right],$$

(1.1)

where $\Omega_w$ is the deformation parameter that interpolates between AdS$_3$ for $\Omega_w = 0$ and AdS$_2 \times S^1$ for $\Omega_w \to \infty$. Similarly, one can consider a squashed sphere obtained by changing the radius of the $S^1$ fiber over $S^2$. This is described by the one–parameter family of Euclidean metrics

$$ds^2[\text{SqS}^3] = R^2 \left[ d\theta^2 + \sin^2 \theta d\phi^2 + \frac{1}{\cosh^2 \Theta_m} (d\alpha + \cos \alpha d\phi)^2 \right],$$

(1.2)

where $\Theta_m$ interpolates between $S^3$ (for $\Theta_m = 0$) and $S^2 \times S^1$ (for $\Theta_m \to \infty$).

Squashed spacetime geometries are not new and have been studied in the context of deformed CFTs which were partially motivated by the search for black holes that generalised BTZ–type backgrounds. A string theory realisation of metrics including three–spheres and warped AdS$_3$ spaces was presented in [8, 9, 10]. There, such configurations were obtained as exact marginal deformations of SU(2) and SL(2, $\mathbb{R}$) Wess–Zumino–Witten models, thus providing by construction a worldsheet theory. In particular, it has been shown how to compute the partition function in the compact case and the spectrum of the primary operators in the non–compact case. Such configurations, that rely on a non–vanishing NS–NS field, are not the subject of this note but we expect them to be related by S–duality to the ones that we will describe.

It is natural to ask if one can use the AdS/CFT correspondence to study string theory on these three–dimensional warped/squashed backgrounds. Initial studies where made in the

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1For clarifications on the use of terminology with respect to the use of the terms “squashed” and “warped” we direct the reader to [6].
context of TMG in \[11\], since the thermodynamic properties pointed at the existence of a two dimensional boundary CFT dual to the geometry. Further examples were analysed in \[12\] as to include warped solutions arising from theories of Einstein gravity (such as Topologically Massive Electrodynamics and gravity coupled to a perfect fluid) and string theory \[10\] \[13\]. Particular attention has been given to Gödel black holes, as their non–flat part can be interpreted as resulting from the squashing of AdS lightcones \[14\] and are known to represent exact string theory backgrounds \[15\].

In all of the examples above it was possible to define an asymptotic symmetry algebra leading to well–defined conserved charges. The value of the central charge was then compared to that obtained from the Bekenstein–Hawking entropy. These observations and the fact that black strings with near–horizon geometries $W\text{AdS}_3 \times S^3$ arise as Hopf–T dualizations \[16\] of dyonic black strings in six dimensions (which in turn has an $\text{AdS}_3 \times S^3$ horizon) lead to the following duality chain \[12\]:

\[
CFT_1 \leftrightarrow \text{AdS}_3 \times S^3 \xrightarrow{\text{Hopf}} W\text{AdS}_3 \times S^3 \leftrightarrow CFT_2.
\] (1.3)

However, it was pointed out in the same paper, that there was no D–brane interpretation of the Hopf T–dual black string with near horizon geometry $W\text{AdS}_3 \times S^3$. A step towards that direction was given in \[17\], where Gödel space emerged from an M–theory compactification of the form Gödel $\times S^2 \times CY_3$, which was interpreted as coming from the backreaction of M2–branes wrapping the $S^2$.

In this note, our aim is to improve our understanding of these vacua by their explicit realisation in string theory and to study their supersymmetry properties (at the level of supergravity). In order to do so, we will make use of the standard brane intersection rules for building supergravity solutions, and the application of a form of T–duality at the level of the ten–dimensional theories. Our starting point will be the IIB setup obtained by T–dualizing the $D = 4$ extremal dyonic black string, which has been widely studied in the past \[18\] \[19\] \[20\] \[21\]. We will then show that by adding a plane wave or a monopole and T–dualizing along a fiber coordinate, it is possible to obtain backgrounds in which a maximally symmetric subspace becomes warped or squashed.

The plan of this paper is as follows. In section 2 we state our main results; we explicitly build the squashed/warped solutions in IIA supergravity and provide an interpretation in terms of D–branes with monopoles and/or waves. We then discuss the supersymmetry properties of the solutions in section 3 by direct computation of the associated Killing spinors and comment on specific limits of the deformation parameter. Finally we present our conclusions and provide some possible directions into future research.


2 T–duality for D1/D5/monopole/plane wave backgrounds

Main result

In this section we derive our main result. By T–dualizing the $D = 4$ extremal dyonic black string solution we can construct a type II string setup with metric

$$ds^2 = M_3 + E^3 + T^4,$$  

where $M_3$ is either AdS$_3$ or a warped anti–de Sitter space with radius $R$ and warping parameter sinh $\Theta_w$, and $E_3$ is either a three–sphere $S^3(R)$ or a squashed three–sphere with radius $R$ and squashing parameter sinh $\Theta_m$.

The extremal dyonic black string

Consider the type IIB setup obtained as the superposition of a D1/D5 system with a magnetic monopole and a plane wave. This was already described in [22] as the T–dual to the $D = 4$ extremal dyonic black string\footnote{The intersection of an NS–NS 1–brane and a NS–NS 5-brane generates a dyonic black string in ten-dimensions that when reduced to six yields the dyonic black string geometry.}. The metric, dilaton and RR three–form field are

$$ds^2 = H_1^{1/2}H_5^{1/2} \left( H_1^{-1}H_5^{-1} (du dv + Ku^2) + H_5^{-1} (dy_1^2 + \cdots dy_4^2) + V^{-1} (d\psi + A_i dx^i)^2 + V (dx_1^2 + \cdots dx_3^2) \right),$$  

$$e^{2\phi} = H_1^{-1}H_5, \quad F[3] = H_1^{-1}dt \wedge du \wedge dv - B_i dx^i \wedge d\psi,$$  

where $H_1(x), H_5(x), K(x), V(x), A_i(x), B_i(x)$ are harmonic functions of the transverse coordinates $x_i, i = 1, 2, 3$ and

$$dB = -* dH_5, \quad dA = -* dV.$$  

Passing to spherical coordinates $(r, \theta, \phi)$ for $x^i$ and taking the $r \to 0$ limit, the harmonic functions take the form

$$H_1 = \frac{Q_1}{r}, \quad H_5 = \frac{Q_5}{r}, \quad K = \frac{Q_w}{r}, \quad V = \frac{Q_m}{r},$$  

and in a suitable coordinate system the field configuration becomes

$$ds^2 = Q_mQ_1^{1/2}Q_5^{1/2} \left( -d\tau^2 + d\omega^2 + Q_w d\sigma^2 + 2Q_w^{1/2} \sinh \omega d\sigma d\tau \right) +$$  

$$+ Q_mQ_1^{1/2}Q_5^{1/2} (d\theta^2 + d\phi^2 + d\psi^2 + 2 \cos \theta d\psi d\phi) + Q_1^{1/2}Q_5^{-1/2} (dy_1^2 + \cdots + dy_4^2),$$
\[ e^{2\phi} = Q_1^{-1} Q_5, \quad F_{[3]} = Q_m Q_1^{1/2} Q_5^{1/2} (\cosh \omega d\tau \wedge d\omega \wedge d\sigma + \sin \theta d\phi \wedge d\psi \wedge d\theta). \quad (2.7) \]

Note that the metric is still AdS \(3 \times S^3 \times T^4\), just like in the near-horizon limit of the more standard \(D1/D5\) system\(^3\). Some of the variables are periodic by construction. Moreover, one can impose a discrete identification in the anti-de Sitter part, leading to a BTZ black hole \([23]\). In detail we have the following periodicities:

\[
\begin{cases}
\psi = \psi + 4\pi, \\
\sigma = \sigma + 4\pi, \\
y_i = y_i + 2\pi.
\end{cases}
\]

(2.8)

This is not unlike the BTZ identification, which strictly speaking differs from the black string geometry, but which is nevertheless well understood. Keeping this into account we can introduce a new pair of \(4\pi\)-periodic variables \(\alpha\) and \(\beta\):

\[
\psi = \alpha + 2y_1 \quad \quad \quad \quad \sigma = \beta + 2y_2,
\]

(2.9)

Notice that the \(y_i\) coordinates stop describing an external (to \(AdS_3 \times S^3\)) torus when we introduce the coordinates \(\alpha\) and \(\beta\), which are linear combinations of the \(y_i\) and angular coordinates in AdS and the sphere respectively. We now rewrite the metric as:

\[
ds^2 = R^2 \left[ d\omega^2 - \cosh^2 \omega d\tau^2 + \frac{1}{\cosh^2 \Theta_m} (d\beta + \sinh \omega d\tau)^2 \right] \\
+ R^2 \left[ d\theta^2 + \sin^2 \theta d\phi^2 + \frac{1}{\cosh^2 \Theta_m} (d\alpha + \cos \theta d\phi)^2 \right] + \frac{R^2}{\sinh^2 \Theta_m} (dy_3^2 + dy_4^2) \\
+ (dz_w + R \tanh \Theta_w (d\beta + \sinh \omega d\tau))^2 + (dz_m + R \tanh \Theta_m (d\alpha + \cos \theta d\phi))^2,\]

(2.10)

where the parameters \(R, \Theta_m\) and \(\Theta_w\) are related to the charges by

\[
R^2 = Q_m \sqrt{Q_1 Q_5}, \quad \sinh^2 \Theta_m = 4Q_m Q_5, \quad \sinh^2 \Theta_w = 4Q_w Q_m Q_5,
\]

(2.11)

and

\[
z_m = \frac{2R}{\tanh \Theta_m} \tilde{y}_1, \quad \quad \quad z_w = \frac{2R}{\tanh \Theta_w} \tilde{y}_2.
\]

(2.12)

where \(\tilde{y}_2 = \sqrt{Q_w} y_2\).

**Type IIA backgrounds**

Up to this point we have only obtained a rewriting of the background fields. Now comes the main point in our construction. Both the \(AdS_3\) and \(S^3\) geometries can be understood as Hopf fibrations (respectively of \(AdS_2\) and \(S^2\)), and performing a T-duality in the direction of the

\(^3\)Here the limit \(Q_m \to 0\) is singular, since in this configuration we only have 3 transverse coordinates.
fiber can undo the structure. The only technical problem that arises in this situation is related to the presence of the Ramond–Ramond fields that are not considered in the usual Buscher transformations [24]. This situation has already been studied in the literature (see e.g. [25, 26, 16, 27, 28, 29]). Nevertheless, it is instructive to consider a possible approach in detail. Starting with the type IIB background in Eq. (2.10) we obtain a type IIA T–dual background by the following set of transformations (described in more detail in Appendix B):

1. Reduce the type IIB background to nine dimensions in the direction $z_i$
2. Rewrite the nine–dimensional IIB fields in terms of nine–dimensional IIA fields (in nine dimensions type IIA and IIB are the same)
3. Oxidate to ten dimensional type IIA introducing the variables $\zeta_i$, T–dual of $z_i$.

**The Squashed Sphere.** As an example we will now apply this construction by performing a T–duality in the $z_m$ direction. If we single out the coordinates in the sphere component, the fields read

\[
ds_{10}^2 = \text{AdS}_3[R] + T^3 + R^2 \left[ d\theta^2 + \sin^2 \theta d\phi^2 + \frac{1}{\cosh^2 \Theta_m} (d\alpha + \cos \theta d\phi)^2 \right] + [dz_m + R \tanh \Theta_m (d\alpha + \cos \theta d\phi)]^2
\]

\[
F_3 = \omega_{\text{AdS}} + R^2 \sin \theta d\theta \wedge d\phi \wedge d\alpha + R \tanh \Theta_m \sin \theta d\theta \wedge d\phi \wedge dz,
\]

and we can perform a Kaluza-Klein reduction on $z_m$ and go to nine dimensions. The metric reads:

\[
ds_9^2 = \text{AdS}_3[R] + T^3 + R^2 \left[ d\theta^2 + \sin^2 \theta d\phi^2 + \frac{1}{\cosh^2 \Theta_m} (d\alpha + \cos \theta d\phi)^2 \right],
\]

and the gauge fields are obtained from:

\[
F_3 = F_3^{(3)} + F_2^{(3)} \wedge (dz_m + A),
\]

where $F_m^{(n)}$ is the $m$-form obtained from the reduction of a $n$-form and $A$ is the one-form

\[
A = R \tanh \Theta_m (d\alpha + \cos \theta d\phi).
\]

Explicitly, adding the extra Kaluza-Klein two-form:

\[
F_3^{(3)} = \omega_{\text{AdS}} + \frac{R^2}{\cosh^2 \Theta_m} \sin \theta d\theta \wedge d\phi \wedge d\alpha
\]

\[
F_2^{(3)} = R \tanh \Theta_m \sin \theta d\theta \wedge d\phi
\]

\[
F_2^{(g)} = dA = R \tanh \Theta_m \sin \theta d\theta \wedge d\phi.
\]

Now, let us perform a T-duality to go to type IIA. Given that there is only one supergravity theory in nine dimensions, the fields keep their expressions but the interpretation changes
according to Tab. 1, $F_3^{(3)}$ now comes from the reduction of a four-form in ten dimensions, $F_2^{(2)}$ from a two-form and $F_2^{(g)}$ is now obtained as the result of the reduction of the Kalb-Ramond field:

$$F_3^{(4)} = F_3^{(3)} \quad F_2^{(2)} = F_2^{(3)} \quad F_2^{(B)} = F_3^{(g)}.$$  \hspace{1cm} (2.20)

We can oxidise back to ten dimensions and get a IIA background:

$$d_{S^3_{10}} = AdS_3[R] + T^3 + R^2 \left[ d\theta^2 + \sin^2 \theta d\phi^2 + \frac{1}{\cosh^2 \Theta_m} (d\alpha + \cos \theta d\phi)^2 \right] + d\xi_m^2 \hspace{1cm} (2.21a)$$

$$F_4 = F_3^{(4)} \wedge d\xi_m = \left[ \omega_{AdS} + \frac{R^2}{\cosh^2 \Theta_m} \sin \theta d\theta \wedge d\phi \wedge d\alpha \right] \wedge d\xi_m \hspace{1cm} (2.21b)$$

$$F_2 = F_2^{(2)} = R \tanh \Theta_m \sin \theta d\theta \wedge d\phi \hspace{1cm} (2.21c)$$

$$H_3 = F_2^B \wedge d\xi_m = R \tanh \Theta_m \sin \theta d\theta \wedge d\phi \wedge d\xi_m. \hspace{1cm} (2.21d)$$

Even though the squashed sphere is not a group manifold, we can still use techniques borrowed from group theory. In particular, we can derive that the isometry group for this part of the metric is $SU(2) \times U(1)$ and that the spectrum of the scalar Laplacian is

$$\Delta_{\Theta_m} Y_{lj} = \frac{1}{R^2} \left\{ l (l + 1) + \sinh^2 \Theta_m j^2 \right\} Y_{lj} \quad l = 0, 1/2, 1, \ldots; \quad j = -l, \ldots, l, \hspace{1cm} (2.22)$$

where $Y_{lj}$ are the usual three–dimensional spherical harmonics. More details can be found in Appendix A.

It is worthwhile to emphasize that by construction $\alpha$ is $4\pi$-periodic and the geometry is the one of a respectable squashed three-sphere.

A similar construction was considered in [16]. In that case, though, the authors started with an $AdS_3 \times S^3$ geometry supported by both RR and NS–NS fields that was then reduced on one of the sphere isometries, yielding a Lens space $S^3 / \mathbb{Z}_p$ or a squashed version, where $p$ and the squashing depend on the values of the charges. This is clearly an orbifold of the solutions above. Our construction starts with a different background (which also includes a monopole and a plane wave) and RR fluxes and we consider T-duality along an extra-dimension which is a linear combination of the Hopf fiber coordinate on the $S^3$ (or $AdS_3$) and a torus direction. To be more specific, consider the $S^3 \times S^1$ part. The geometry can be understood as the fibration

$$S^1 \times S^1 \longrightarrow S^3 \times S^1 \quad \downarrow \quad S^2$$

where one of the directions in the torus fibration is the Hopf fiber in $S^3$. A $S_1$ sub-bundle $A$ of the torus, obtained as a rational linear combination of the two directions, describes a
where the squashing parameter depends on the coefficients in the linear combination. Note that just like in the pure $S^3$ case described in [16], since we only have RR fields, performing T–duality in the $A$ direction will "unwind" the fiber and lead to a geometry which is the direct product $\text{SqS}^3 \times S^1$. The same can be applied to the $\text{AdS}^3 \times S^1$ part.

Finally, given that our starting geometry is supported by RR fluxes only, our construction is not suited to describe the exact CFT backgrounds of [9, 10, 30]. Since these latter are based on Wess–Zumino–Witten models, they have NS–NS fields instead of R–R fields a T–duality in the $z_m$ direction would not trivialize the Hopf fibration.

Warped AdS. In principle these constructions can be extended to other group manifold geometries (e.g. the obvious choice leading to a squashed AdS$_3$) but in any case one should start from a configuration with RR fields, since the absence of NS–NS antisymmetric fields is the key ingredient for the trivialization of the fiber bundle. More general geometries can be obtained by starting with a mixed RR–NS–NS configuration. Starting with the background in Eq. (2.10), the construction can be performed in two more ways:

1. Performing the T–duality on the coordinate $z_w$. This leads to a type IIA background whose metric is $W\text{AdS}_3 \times S^3 \times T^4$, sustained by a four–form flux, a two–form flux and a Kalb–Ramond field. Explicitly:

$$\begin{align*}
\text{ds}^2 &= R^2 \left[ d\omega^2 - \cosh^2 \omega d\tau^2 + \frac{1}{\cosh^2 \Theta_w} (d\beta + \sinh \omega d\tau)^2 \right] + d\zeta_w^2 + S^3[R] + T^3 \, , \\
F_4 &= \left[ R^2 \omega_S + \frac{R^2}{\cosh^2 \Theta_w} \cosh \omega d\omega \wedge d\tau \wedge d\beta \right] \wedge d\zeta_w \, , \\
F_2 &= R \tanh \Theta_w \cosh \omega d\omega \wedge d\tau \, , \\
H_3 &= R \tanh \Theta_w \cosh \omega d\omega \wedge d\tau \wedge d\zeta_w \, .
\end{align*}$$

(2.25a)

2. Performing two T–dualities in both $z_m$ and $z_w$, which leads to a type IIB background with metric $W\text{AdS}_3 \times \text{SqS}^3 \times T^4$. In this case, following Appendix B, we find that the metric is sustained by a five–form flux, a three–form flux and a Kalb–Ramond field. Explicitly:

$$\begin{align*}
\text{ds}^2 &= R^2 \left[ d\theta^2 + \sin^2 \phi^2 + \frac{1}{\cosh^2 \Theta_m} (d\alpha + \cos \theta d\phi)^2 \right] \\
&\quad + R^2 \left[ - \cosh^2 \omega d\tau^2 + d\omega^2 + \frac{1}{\cosh^2 \Theta_w} (d\beta + \sinh \omega d\tau)^2 \right] + d\zeta_w^2 + d\zeta_m^2 + T^2 \, ,
\end{align*}$$

(2.26a)
\[ F_5 = R^2 \left[ \frac{1}{\cosh^2 \Theta_m} \sin \theta d\theta \wedge d\phi \wedge d\alpha + \frac{1}{\cosh^2 \Theta_w} \cosh \omega d\tau \wedge d\omega \wedge d\beta \right] \wedge d\zeta_w \wedge d\zeta_m, \]

(2.26b)

\[ F_3 = -R \tanh \Theta_w \cosh \omega d\tau \wedge d\omega \wedge d\zeta_m + R \tanh \Theta_m \sin \theta d\theta \wedge d\phi \wedge d\zeta_w, \]

(2.26c)

\[ H_3 = -R \tanh \Theta_w \cosh \omega d\tau \wedge d\omega \wedge d\zeta_w + R \tanh \Theta_m \sin \theta d\theta \wedge d\phi \wedge d\zeta_m. \]

(2.26d)

3 Supersymmetry Properties

T–duality transformations can break supersymmetries preserved by D–brane solutions. This has already been observed in the context of Hopf T-dualities on six-dimensional supergravity backgrounds of the form \( \text{AdS}_3 \times S^3 \) \cite{16}. The phenomenon is akin to the breaking of supersymmetry by compactification on a circle, so the resulting supersymmetries will be those that survive the circle compactification.

In the following we compute the explicit expressions of the Killing spinors for the geometries obtained above, and verify that under T–duality, the IIA backgrounds with squashed/warped spacetimes preserve \( 1/4 \) of the supersymmetries of the original D1/D5 background. We also show that for specific values of the deformation parameter some supersymmetries are restored and that generically, IIB backgrounds containing both warped AdS and squashed spheres preserve no supersymmetry.

3.1 \( \text{AdS}_3 \times S^3 \times T^4 \)

We start by calculating the Killing spinors in the \( \text{AdS}_3 \times S^3 \times T^4 \) background\footnote{Killing spinors for backgrounds of the form \( \text{AdS}_p \times S^q \) have been computed in horospherical coordinates in \cite{31}.}. The details can be found in appendix\[C\] here we will just quote the main results. The Killing spinors read:

\[ \epsilon = e^{-\frac{\omega}{2} \gamma^{12} \circ \mathcal{P}_+} e^{\frac{\theta}{2} \gamma^{31} \circ \mathcal{P}_+} e^{\frac{\theta}{2} \gamma^{34} \circ \mathcal{P}_+} e^{\frac{\phi}{2} \gamma^{24} \circ \mathcal{P}_+} e^{\frac{\phi}{2} \gamma^{23} \circ \mathcal{P}_-} \epsilon_0 \]

(3.1)

where \( \mathcal{P}_\pm = \frac{1 \pm \sigma^1}{2} \) are projection operators acting on the doublet \( \epsilon = (\epsilon_1 \epsilon_2) \) and the constant spinors \( \epsilon_0 \) are also complex Weyl spinors arising from the integration constants. These spinors are constrained as to satisfy the standard \( T^4 \) projection

\[ \left( \gamma^{012345} + 1 \right) \epsilon_0 = 0. \]

(3.2)

which implies the preservation of \( 1/2 \) of the supersymmetries (i.e. 16 supersymmetries remain unbroken).

Further insights can be obtained by looking at the explicit expressions of the Killing spinors. Half of them depend on the fiber coordinates \( \sigma, \phi \): we conclude \textit{a priori} that under T–duality, the spinors associated to those supersymmetries are broken and the IIA background will preserve \( 1/4 \) of the original supersymmetry.
3.2 Warped and Squashed Backgrounds

We now compute the Killing spinors for the T-dual backgrounds explicitly. For the technical details we refer the reader to appendix D. We take \( \epsilon \) to be Majorana but not Weyl.

If the T-duality is performed along the sphere fiber \( \alpha \), one obtains

\[
\epsilon = \epsilon_1(\omega, \tau, \sigma) \epsilon_2(\theta, \phi, \alpha) \epsilon_0,
\]

where \( \epsilon_0 \) is a constant Majorana spinor and

\[
\epsilon_1(\omega, \tau, \sigma) = \exp \left\{ -\frac{\omega}{2} \gamma^{02} Q_+ \right\} \exp \left\{ \frac{\tau}{2} \gamma^{12} Q_+ \right\} \exp \left\{ \frac{\sigma}{2} \gamma^{01} Q_- \right\}
\]

\[
\epsilon_2(\theta, \phi, \alpha) = \exp \left\{ \frac{\theta}{2} \gamma^{0123} Q_+ \right\} \left[ \exp \left\{ \tanh^2 \Theta_m \cos \left( \frac{\phi}{2} \right) \gamma^{34} \right\} Q_- - \exp \left\{ \frac{\phi}{2} \gamma^{34} \right\} Q_+ \right]
\]

(3.3)

\[
\epsilon_0 = \exp \left\{ \frac{\theta}{2} \gamma^{45} Q_+ \right\} \exp \left\{ \frac{\psi}{2} \gamma^{34} Q_+ \right\} \exp \left\{ \frac{-\phi}{2} \gamma^{34} Q_- \right\}
\]

(3.4)

and \( Q_\pm = \frac{1 + \gamma^9}{2} \). The Killing spinor is also subject to the constraint:

\[
\gamma^{34} Q_+ \Gamma^{11} \epsilon = 0
\]

(3.5)

which turns into a constraint on \( \epsilon_0 \). Writing \( \epsilon = \epsilon_+ + \epsilon_- \) with \( Q_\pm \epsilon_\pm = \pm \epsilon_\pm \) then one can verify that only the supersymmetries associated to the \( \epsilon_- \) spinors are preserved after T–duality. The solution is 1/4-BPS.

When the deformation parameter is such that \( \text{sech} \Theta_m = 0 \), there are no spinors depending on the T–dual coordinate and supersymmetry is restored to 1/2-BPS. This corresponds to the case in which the \( S^3 \) becomes \( S^2 \times S^1 \) [30,32].

In the other case, when the T–duality is done along the AdS fiber \( \beta \), one expresses the Killing spinor as \( \epsilon = \epsilon_1(\theta, \psi, \phi) \epsilon_2(\omega, \tau, \beta) \epsilon_0 \), with

\[
\epsilon_1(\theta, \psi, \phi) = \exp \left\{ \frac{\theta}{2} \gamma^{45} Q_+ \right\} \exp \left\{ \frac{\psi}{2} \gamma^{34} Q_+ \right\} \exp \left\{ \frac{-\phi}{2} \gamma^{34} Q_- \right\}
\]

(3.6)

\[
\epsilon_2(\omega, \tau, \beta) = \exp \left\{ -\frac{\omega}{2} \gamma^{1345} Q_- \right\} \left[ \exp \left\{ -\tanh \Theta_w^2 \sinh \omega \left( \frac{\tau}{2} \right) \gamma^{01} \right\} Q_+ \right]
\]

\[
+ \exp \left\{ \frac{\tau}{2} \gamma^{0545} \right\} Q_- \exp \left\{ -\text{sech}^2 \Theta_w^2 \frac{\beta}{2} \gamma^{01} Q_+ \right\}
\]

(3.7)

with the spinor satisfying the constraint

\[
\gamma^{01} Q_+ \Gamma^{11} \epsilon = 0
\]

(3.8)

which once more, turns into a constraint on \( \epsilon_0 \). For \( \epsilon = \epsilon_+ + \epsilon_- \), only the supersymmetries associated to the \( \epsilon_+ \) spinors will be preserved after T–duality and the solution is again 1/4-BPS.

Just as before, in the special case when the deformation parameter is such that \( \text{sech} \Theta_w = 0 \), AdS\(_3 \) becomes AdS\(_2 \times S^1 \), there are no spinors depending on the T–dual coordinate, and supersymmetry is restored to 1/2-BPS.
3.3 WAdS$_3 \times$ SqS$^3 \times T^4$

Instead of determining the expressions for the Killing spinors of the IIB background obtained after two T–dualities in which both the sphere and the anti-de Sitter space are squashed/warped, we can directly argue that all supersymmetries must be in general broken.

Let $\epsilon$ be a Weyl complex spinor in 10 dimensions and let $\mathcal{T}_\pm = \frac{1 \pm \gamma^{67}}{2}$ be a projection operators. Decomposing $\epsilon = \epsilon_+ + \epsilon_-$ with $\mathcal{T}_\pm \epsilon_\pm = \pm \epsilon_\pm$, one can check that for this background the following conditions need to be held.

- From the vanishing of the dilatino variation

$$\tanh \Theta_w \gamma^{01} \epsilon_- = 0 \quad \tanh \Theta_m \gamma^{34} \epsilon_+ = 0.$$  \hfill (3.9)

- The Killing spinor satisfies $\Gamma^{11} \epsilon = \epsilon$.

Imposing these projections necessarily yields the condition

$$\tanh \Theta_m = -i \tanh \Theta_w$$  \hfill (3.10)

for supersymmetry to be preserved. Hence, for generic values of the deformations all the associated supersymmetries must be broken. However, it is possible to see that there is restauration of 1/2-BPS in the special case of AdS$_2 \times S^2 \times T^6$ background.

4 Conclusions

In this note we have explicitly constructed string theory backgrounds which include three–dimensional squashed/warped spheres/anti-de Sitter spaces. They can be seen as T–dual brane intersections of D1 and D5 systems with monopole and/or plane waves. The deformation parameters of the squashed/warped spaces are interpreted in terms of the charges of the original background.

We also studied the supersymmetry properties of these backgrounds by explicitly calculating the Killing spinors. Under T–duality, IIA backgrounds containing SqS$^3$ or WAdS$_3$ preserve eight supersymmetries for generic values of the deformation parameter. In the special case of infinite deformation parameter $\cosh \Theta_{m,w} \rightarrow \infty$, some supersymmetries are restored and the backgrounds preserve sixteen supersymmetries. The same arguments can be applied to the IIB background in which both the sphere and the AdS space are squashed/warped, but for generic values of the deformation parameter, no supersymmetry is preserved.

It should be remarked that the previous results were obtained in the context of supergravity. It is well-known that spacetime supersymmetries that are manifest in some string background might very well be hidden in their T–dual [33, 34]. There are examples in the literature in which supersymmetry that seemed to be destroyed by duality could actually be restored by a non–local realization [35, 36]. To clarify these issues, it would be necessary to study the precise T–duality here addressed at the level of the GS action for a IIB AdS$_3 \times S^3 \times T^4$ background [37]. An analysis from the perspective of the dual CFT would
also be enlightening as all worldsheet and spacetime supersymmetries should remain symmetries of the underlying CFT. We leave these and related questions as future work.

The same construction works for any metric that possesses an $S^1$ fibration and formally also for $S^3$ fibers, even though in this case, some modifications would need to be included (in particular in order to generalize the T–duality to non–Abelian fields). Another possibility would be to follow the procedure along the time-like $S^1$ fiber in $\text{AdS}_3$, which would lead to a hyperbolic plane ($H^2$) geometry in type $\text{II}^*$ backgrounds with negative kinetic terms.

It would be interesting to use these solutions to compute holographic data, thus formulating in a more precise way the duality chain in \[1.3\]. For the case in which the $\text{AdS}$ space becomes warped after T–dualizing, an appropriate embedding in an asymptotically $\text{AdS}$ background would be required, following the procedure in \[17\]). We believe the geometries here presented constitute a first step towards attempting to uncover the microscopic duals of these backgrounds. This issue is currently under investigation.

Another possibility that we have not fully explored in the present note consists in T–dualizing the dyonic black string background along a direction that mixes the two initial coordinates $\psi$ and $\sigma$. The resulting geometry (in type IIA) is described by the metric

\[
d s^2 = R^2 \left[ d \omega^2 - \cosh^2 \omega d \tau^2 + \frac{1}{\cosh^2 \Theta_w} (d \beta + \sinh \omega d \tau)^2 \right] + \\
+ R^2 \left[ d \theta^2 + \sin^2 \theta d \phi^2 + \frac{1}{\cosh^2 \Theta_m} (d \alpha + \cos \theta d \phi)^2 \right] + \\
+ 2R^2 \tanh \Theta_w \tanh \Theta_m (d \beta + \sinh \omega d \tau) (d \alpha + \cos \theta d \phi) + T^4, \tag{4.1}
\]

which can be reduced to any of the previously discussed configurations, $\text{WAdS}_3 \times S^3$ and $\text{AdS}_3 \times \text{Sqs}_3$, by setting the relevant charges to zero.

Acknowledgements

We are indebted to José Figueroa-O’Farrill, Shigeki Sugimoto, Kostas Skenderis for enlightening discussions and especially to Arkady Tseytlin for comments and his careful revision of the manuscript. We would furthermore like to thank the participants of the IPMU string theory group meetings for directing our attention to the question investigated in this note. L.I.U. would like to thank IPMU for hospitality during the initial stages of this project. The research of D.O. was supported by the World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan. L.I.U. is supported by a STFC Postdoctoral Research Fellowship.

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A Group theoretical description of the geometry

Isometries. In order to understand the isometries of the squashed and warped spaces it is convenient to describe their geometry in algebraic terms. This is possible because both the three–sphere and AdS$_3$ are group manifolds, respectively for SU(2) and SL(2, R). It follows that their line elements can be written as

$$ds^2 = R^2 \text{Tr}[g^{-1}dg^{-1}dg] = R^2 \sum_a J^a \otimes J^a,$$  \hspace{1cm} (A.1)

where $g \in G$ is a general element of the group, $J^a = \text{Tr}[t^a g^{-1}dg]$ and $t^a$ are the generators of the Lie algebra (su(2) or sl$_2$). The effect of the T–duality amounts to adding an extra term to the metric, proportional to $(J^3)^2$ (in the sl$_2$ case, $J^3$ is the hyperbolic generator). The squashed/warped metric is written as

$$ds^2[\Theta] = R^2 \left( J^1 \otimes J^1 + J^2 \otimes J^2 + \frac{1}{\cosh^2 \Theta} J^3 \otimes J^3 \right).$$  \hspace{1cm} (A.2)

The initial group manifold has $G \times G$ isometry, generated by $J^a = \text{Tr}[t^a g^{-1}dg]$ and $J^a = \text{Tr}[t^a dgg^{-1}]$, but only part of this symmetry remains after the T–duality. To be precise, while the $J^a$ generators are preserved, since they commute with the current $J^3$, both $J^1$ and $J^2$ are
not Killing vectors anymore, as one can verify with a direct calculation of the Lie derivative of the metric:

\[ \mathcal{L}_J^1 [ds^2[\Theta]] = 2R^2 \tanh \Theta J^2 \otimes J^3, \quad (A.3) \]
\[ \mathcal{L}_J^2 [ds^2[\Theta]] = 2R^2 \tanh \Theta J^3 \otimes J^1. \quad (A.4) \]

The resulting isometry group is \( G \times U(1) \).

**Scalar Laplacian.** As an application of this description, let us derive the spectrum of the scalar Laplacian on the squashed/warped geometries. Consider in particular the squashed three–sphere \( \text{SqS}^3(R, \sinh \Theta_m) \). Let \( E^a \) be the basis vectors in \( S^3 \), such that \( \langle J^a, E_b \rangle = \delta^a_b \). The drei–bein \( \{ \theta^a \} \) and the dual basis vectors \( \{ e_a \} \) on \( \text{SqS}^3(R, \sinh \Theta_m) \), defined by

\[ ds^2[\text{SqS}^3(R, \sinh \Theta_m)] = \sum_{a=1}^{3} \theta^a \otimes \theta^a, \quad \langle \theta^a, e_b \rangle = \delta^a_b, \quad (A.5) \]

can be expressed as

\[ \theta^1 = RJ^1, \quad \theta^2 = RJ^2, \quad \theta^3 = \frac{R}{\cosh \Theta_m} J^3, \quad (A.6) \]
\[ e_1 = \frac{1}{R} E_1, \quad e_2 = \frac{1}{R} E_2, \quad e_3 = \frac{\cosh \Theta_m}{R} E_3. \quad (A.7) \]

By definition, the connection one–form \( \omega \) is given by

\[ d\theta^a = -\omega^a_b \wedge \theta^b, \quad (A.8) \]

and since

\[ dJ^a = \frac{1}{2} \epsilon_{abc} J^b \wedge J^c, \quad (A.9) \]

it is clear that \( \omega^a_b \) is proportional to \( \epsilon_{abc} \theta^c \) and hence \( \langle \omega^a_b, e_a \rangle = 0 \). We have now all the ingredients to express the scalar Laplacian operator on \( \text{SqS} \) in terms of operators on the three–sphere:

\[ \Delta_{\Theta_m} f = -e_a e_a f + \langle \omega^a_b, e_a \rangle e_b f = \frac{1}{R^2} \left\{ (E_1^2 + E_2^2 + E_3^2) + \sinh^2 \Theta_m E_3^2 \right\} = \frac{1}{R^2} \left( \Delta_0 + \sinh^2 \Theta_m E_3^2 \right). \quad (A.10) \]

Now observe the following:

1. \( E_1^2 + E_2^2 + E_3^2 = \Delta_0 \) is the Laplacian on the three–sphere. This has eigenvalues \( l (l + 1) \), \( l \in 0, 1/2, 1, \ldots \)
2. \( E_3 \) is the \( z \) component of the angular momentum on \( S^3 \). It has eigenvalue \( j, -l \leq j \leq l \).
3. Since the Laplacian on \( S^3 \) is also the Casimir of \( SU(2) \), \( [\Delta_0, E_3] = 0 \).
The operators $\Delta_{\Theta_m}$, $\Delta_0$ and $E_3$ are commuting and admit the same eigenvectors. Moreover, the spectrum of $\Delta_{\Theta_m}$ is given by the sum of the two contributions. Explicitly, the eigenvalue equation reads

$$\Delta_{\Theta_m} Y_{ij} = \frac{1}{R^2} \left( l (l + 1) + \sinh^2 \Theta_m j^2 \right) Y_{ij}, \quad l = 0, 1/2, 1, \ldots; \quad j = -l, \ldots, l,$$

(A.11)

where $Y_{ij}$ are the three-dimensional spherical harmonics.

In a similar way, one finds that the Laplacian spectrum on WAdS$_3(R, \sinh^2 \Theta_w)$ is given by the sum of the Laplacian of AdS$_3$ and an extra component $\sinh^2 \Theta_w / R^2 f^2$.

## B T duality with R–R fields

In the II solutions we consider, the rôle of sustaining the geometry is taken by R–R field strengths. In particular this means that the usual Buscher rules [24] prove insufficient and we are forced to follow a slightly more involved path to write T-duals: derive two low-energy effective actions and explicitly write down the transformations relating them (in this we will follow the same procedure as in [38][16]).

In ten dimensions, type IIA and IIB are related by a T-duality transformation, stating that the former theory compactified on a circle of radius $R$ is equivalent to the latter on a circle of radius $1/R$. This means in particular that there is only one possible nine-dimensional $N = 2$ SUGRA action. The rules of T-duality are easily obtained by explicitly writing the two low-energy actions and identifying the corresponding terms.

For sake of clarity let us just consider the bosonic sector of both theories. In [39][40] it was found that the IIA action in nine dimensions is given by

$$e^{-1} L_{IIA} = R - \frac{1}{2} (\partial \Phi)^2 - \frac{1}{2} (\partial \Phi)^2 - \frac{1}{2} (F_{[1]}^{(12)})^2 e^{3\Phi + \frac{2}{\sqrt{7}} \Phi} + \frac{1}{48} (F_{[4]}^{(12)})^2 e^{3\Phi + \frac{3}{2\sqrt{7}} \Phi} - \frac{1}{12} (F_{[3]}^{(1)})^2 e^{-\Phi + \frac{1}{2\sqrt{7}} \Phi} - \frac{1}{12} (F_{[3]}^{(2)})^2 e^{\frac{1}{2\sqrt{7}} \Phi - \frac{5}{2\sqrt{7}} \Phi} + \frac{1}{4} (F_{[2]}^{(12)})^2 e^{-\Phi - \frac{3}{2\sqrt{7}} \Phi} - \frac{1}{4} (F_{[2]}^{(1)})^2 e^{3\Phi + \frac{1}{2\sqrt{7}} \Phi} + \frac{1}{4} (F_{[2]}^{(2)})^2 e^{\frac{1}{2\sqrt{7}} \Phi} + \frac{1}{2e} \tilde{F}_{[3]}^{(1)} \wedge \tilde{A}_{[1]}^{(12)} - \frac{1}{e} \tilde{F}_{[3]}^{(1)} \wedge \tilde{A}_{[3]}^{(1)} - \frac{1}{e} \tilde{F}_{[3]}^{(1)} \wedge \tilde{A}_{[3]}^{(2)}$,

(B.1)

where $\Phi$ is the original dilaton, $\Phi$ is a scalar measuring the compact circle, defined by the reduction (in string frame)

$$ds^2 = e^{\Phi/2} ds_{10}^2 = e^{\Phi/2} \left( e^{-\Phi/(2\sqrt{7})} dz^2 + e^{\sqrt{7}\Phi/2} (dz + A_{[1]})^2 \right)^2$$

(B.2)

and $F_{[n]}$ are $n$-form field strengths defined as

$$F_{[n]} = \tilde{F}_{[n]} - \tilde{F}_{[3]}^{(1)} \wedge A_{[1]}^{(1)} - \tilde{F}_{[3]}^{(2)} \wedge A_{[1]}^{(2)} - \frac{1}{2} \tilde{F}_{[3]}^{(12)} \wedge A_{[1]}^{(1)} \wedge A_{[1]}^{(2)}$$

(B.3a)
This completes the T-duality relations generalizing the usual ones \[24\] valid in the NS sector.

In the same way, starting from the IIB action one obtains the following nine-dimensional IIB Lagrangian:

\[
e^{-1}L_{IIB} = R - \frac{1}{2} (\partial \Phi)^2 - \frac{1}{2} (\partial \Phi)^2 - \frac{1}{2} e^{2\Phi} (\partial \chi)^2 + \]
\[
- \frac{1}{48} e^{-\frac{3}{4} \Phi} F_{[4]}^2 - \frac{1}{12} e^{-\Phi_+ \frac{1}{2} \Phi (F_{[3]}^{(NS)})^2} - \frac{1}{2} e^{\Phi_+ \frac{1}{2} \Phi (F_{[3]}^{(R)})^2} + \]
\[
- \frac{1}{4} e^{\frac{1}{2} \Phi} (F_{[2]}^2) - \frac{1}{4} e^{\Phi_+ \frac{1}{2} \Phi (F_{[2]}^{(R)})^2} - \frac{1}{4} e^{-\Phi_+ \frac{1}{2} \Phi (F_{[2]}^{(NS)})^2} + \]
\[
- \frac{1}{2} e^{-\frac{1}{2} \Phi} F_{[4]} \wedge F_{[4]} \wedge A_{[1]} - \frac{1}{2} e^{-\frac{1}{2} \Phi} F_{[3]}^{(NS)} \wedge F_{[3]}^{(R)} \wedge A_{[3]} . \quad (B.4)
\]

Knowing that both describe the same theory we easily obtain the conversion table in Tab.1 which acts as a dictionary between IIA and IIB in ten dimensions plus the following relation between the scalar fields

\[
\left( \Phi \right)_{IIA} = \left( \begin{array}{cc}
3/4 & -\sqrt{7}/4 \\
-\sqrt{7}/4 & -3/4
\end{array} \right) \left( \Phi \right)_{IIB}.
\]

This completes the T-duality relations generalizing the usual ones \[24\] valid in the NS–NS sector.
C Killing spinor equations in Type IIB SUGRA

Type IIB supergravity contains a dilatino $\lambda$ and gravitino $\psi_M$ which can be expressed in terms of complex Weyl spinors. At tree level, unbroken supersymmetries are manifested by the invariance of the fermionic fields under the possible supersymmetry transformations. Supersymmetric configurations satisfy the following equations

\[
\delta \lambda = i \Gamma^M e^* P_M - \frac{i}{24} \Gamma^{KLN} G_{KLN} \epsilon = 0, \quad (C.1a)
\]

\[
\delta \psi_M = D_M \epsilon + \frac{1}{96} \left( \Gamma^M_{KLN} G_{KLN} - 9 \Gamma^L_{LN} G_{MLN} \right) \epsilon^* + \frac{i}{480} \Gamma^{M_1 \cdots M_5} F_{M_1 \cdots M_5} \Gamma_M \epsilon = 0, \quad (C.1b)
\]

where we have chosen the complex Weyl spinors to satisfy $\Gamma^{(11)} \psi_M = \psi_M$, $\Gamma^{(11)} \epsilon = \epsilon$, and $\Gamma^{(11)} \lambda = -\lambda$. The equations above are expressed in Einstein frame [41].

Let us now build the Killing spinors in the $\text{AdS}_3 \times S^3 \times T^4$ background. The vielbein for the Einstein metric reads

\[
e^{(0)} = R \cosh(\omega) d\tau \quad e^{(1)} = R d\omega \quad e^{(2)} = R \sinh \omega d\tau + R d\sigma \quad e^{(3)} = R d\theta \quad e^{(4)} = R \sin(\theta) d\psi \quad e^{(5)} = R \cos \theta d\theta + R d\phi \quad (C.2)
\]

(Note that here we have rescaled the torus coordinates with respect to the metric in (2.10)). The covariant derivatives are

\[
D_\tau = \partial_\tau - \frac{1}{4} \left[ \sinh \omega \gamma^0 + \cosh \omega \gamma^12 \right], \quad D_\omega = \partial_\omega + \frac{1}{4} \gamma^02, \quad D_\sigma = \partial_\sigma + \frac{1}{4} \gamma^01, \quad (C.3a)
\]

\[
D_\psi = \partial_\psi - \frac{1}{4} \left[ \cos \theta \gamma^34 - \sin \theta \gamma^35 \right], \quad D_\theta = \partial_\theta - \frac{1}{4} \gamma^45, \quad D_\phi = \partial_\phi + \frac{1}{4} \gamma^34 \quad (C.3b)
\]

\[
D_i = \partial_i, \quad (i = 6, \cdots, 9). \quad (C.3c)
\]

where we indicate with lower case greek letters the Dirac matrices in the orthonormal frame (tangent space). The gamma matrices in the coordinate frame $\Gamma_M$ can be expressed in terms of the gamma matrices $\gamma_a$ in the orthonormal frame as

\[
\Gamma_\tau = R \left[ \cosh \omega \gamma_0 + \sinh \omega \gamma_2 \right] \quad \Gamma_\omega = R \gamma_1 \quad \Gamma_\sigma = R \gamma_2 \quad (C.4a)
\]

\[
\Gamma_\theta = R \gamma_3 \quad \Gamma_\psi = R \left[ \sin \theta \gamma_4 + \cos \theta \gamma_5 \right] \quad \Gamma_\phi = R \gamma_5 \quad (C.4b)
\]

\[
\Gamma_i = \gamma_i, \quad (i = 6, \cdots, 9), \quad (C.4c)
\]

and

\[
\Gamma^\tau = \frac{1}{R} \text{sech} \omega \gamma^0 \quad \Gamma^\omega = \frac{1}{R} \gamma^1 \quad \Gamma^\sigma = \frac{1}{R} \left[ - \tanh \omega \gamma^0 + \gamma^2 \right] \quad (C.5a)
\]

\[
\Gamma^\theta = \frac{1}{R} \gamma^3 \quad \Gamma^\psi = \frac{1}{R} \csc \theta \gamma^4 \quad \Gamma^\phi = - \frac{1}{R} \left[ \cot \theta \gamma^4 - \gamma^5 \right] \quad (C.5b)
\]

\[
\Gamma^i = \gamma^i, \quad (i = 6, \cdots, 9). \quad (C.5c)
\]
The SU(1, 1)-invariant three-form field $G$ is given by

$$G[3] = \frac{i}{\sqrt{\tau^2}} \left( dC[2] - \tau dB[2] \right)$$  \hspace{1cm} (C.6)

where $\tau = \tau_1 + i\tau_2 = C_0 + i e^{-\Phi_{IIB}}$ is the complex scalar field which contains the R–R zero-form $C_0$ and the dilaton $\Phi_{IIB}$, $C[2]$ is the R–R two-form and $B[2]$ is the NS two-form field. For our field configuration:

$$G[3] = iR^2 \left[ \sin \theta d\theta \wedge d\psi \wedge d\phi + \cosh \omega d\tau \wedge d\omega \wedge d\sigma \right],$$  \hspace{1cm} (C.7)

and we can set the dilaton to zero. The dilatino variation reads

$$\delta \lambda = -\frac{i}{24} \Gamma^{KLM} G_{KLM} = 0$$  \hspace{1cm} (C.8a)

$$\left( \Gamma^{\phi \rho \psi} G_{\rho \phi \psi} + \Gamma^{\tau \omega \sigma} G_{\tau \omega \sigma} \right) \epsilon = i \frac{1}{R} \left( \gamma^{354} + \gamma^{012} \right) \epsilon = 0$$  \hspace{1cm} (C.8b)

so that

$$\left( \gamma^{012345} + 1 \right) \epsilon = 0$$  \hspace{1cm} (C.9)

The gravitino conditions are

$$\delta \psi_t = \partial_t \epsilon = \frac{1}{4} \left[ \sinh \omega \gamma^{01} + \cosh \omega \gamma^{12} \right] \epsilon$$

$$+ \frac{i}{16} \left[ - \cosh \omega \left( \gamma^{0345} + 3 \gamma^{12} \right) + \sinh \omega \left( \gamma^{2345} - 3 \gamma^{01} \right) \right] e^* = 0,$$

$$\delta \psi_\omega = \partial_\omega \epsilon + \frac{1}{4} \gamma^{02} \epsilon + \frac{i}{16} \left[ 3 \gamma^{02} + \gamma^{1345} \right] e^* = 0,$$

$$\delta \psi_\rho = \partial_\rho \epsilon + \frac{1}{4} \gamma^{01} \epsilon + \frac{i}{16} \left[ -3 \gamma^{01} + \gamma^{2345} \right] e^* = 0,$$

$$\delta \psi_\phi = \partial_\phi \epsilon + \frac{1}{4} \gamma^{45} \epsilon - \frac{i}{16} \left[ 3 \gamma^{45} + \gamma^{0123} \right] e^* = 0,$$

$$\delta \psi_\psi = \partial_\psi \epsilon + \frac{1}{4} \gamma^{35} \epsilon - \frac{i}{16} \left[ \sin \theta \gamma^{35} - \cos \theta \gamma^{34} \right] e$$

$$+ \frac{i}{16} \left[ \sin \theta \left( - \gamma^{0124} + 3 \gamma^{35} \right) - \cos \theta \left( \gamma^{0125} + 3 \gamma^{34} \right) \right] e^* = 0,$$

$$\delta \psi_\sigma = \partial_\sigma \epsilon = 0, \quad (i = 6, \cdots 9).$$  \hspace{1cm} (C.10)

The last equation tells us that the Killing spinors are constant around $T^4$. Imposing the dilatino condition (C.8), the previous equations simplify to

$$\partial_\mu \left( \begin{array}{c} \epsilon_1 \\ \epsilon_2 \end{array} \right) = \frac{1}{2} \Omega_\mu \left( \begin{array}{c} \epsilon_1 \\ \epsilon_2 \end{array} \right)$$  \hspace{1cm} (C.11)

\footnote{Notice that $\left( \gamma^{012345} + 1 \right) e^* = 0$}
with
\[ \Omega_t = \left( \sinh \omega \gamma^{01} + \cosh \omega \gamma^{12} \right) \otimes P_+ \quad \Omega_\omega = -\gamma^{02} \otimes P_+ \quad \Omega_\sigma = -\gamma^{01} \otimes P_- \quad (C.12a) \]
\[ \Omega_\psi = \left( -\sin \theta \gamma^{35} + \cos \theta \gamma^{34} \right) \otimes P_+ \quad \Omega_\theta = \gamma^{45} \otimes P_+ \quad \Omega_\phi = -\gamma^{34} \otimes P_- \quad (C.12b) \]

where \( \epsilon^* = -i \sigma^1 \epsilon \) and we have defined the projectors \( P_{\pm} = \frac{1 \pm \sigma^1}{2} \). Note that \( \Omega_{\mu} = A_{(32)} \otimes P_{\pm} \) so that \( A^{32}_{\mu} = \pm 1 \) for fixed value of \( \mu \). Hence schematically \( \Omega^{32}_{\mu} = \pm 1 \otimes P \). Making use of these facts and the following identity
\[ \exp \Omega x = (1 \otimes \bar{P}) + (\cosh x + A \sinh x) \otimes P \quad (C.13) \]

where \( \bar{P} = 1 - P \), one can integrate (C.11) and solve for the Killing spinor \( \epsilon \). We write
\[ \epsilon = \epsilon_\omega \epsilon_t \epsilon_\psi \epsilon_\phi \epsilon_\theta \epsilon_\sigma \epsilon_\beta \epsilon_\alpha \]

The constant spinors \( \epsilon_0 \) arising from the integration constants, is constrained by the projection
\[ \left( \gamma^{012345} + 1 \right) \epsilon_0 = 0. \quad (C.15) \]

so that 16 supersymmetries remain unbroken\(^6\)

### C.1 Squashed Sphere and Warped AdS

The field configuration for this type IIB background that arises after two Hopf T-dualities is determined by eqs. (2.26). The covariant derivatives are:
\[ \mathcal{D}_\tau = \partial_\tau + \frac{\text{sech}^2 \Theta w - 2}{4} \sinh \omega \gamma^{01} - \frac{\text{sech} \Theta w}{4} \cosh \omega \gamma^{12} \quad \mathcal{D}_\omega = \partial_\omega + \frac{\text{sech} \Theta w}{4} \gamma^{02} \]
\[ \mathcal{D}_\beta = \partial_\beta + \frac{\text{sech}^2 \Theta w}{4} \gamma^{01} \quad \mathcal{D}_\theta = \partial_\theta - \frac{\text{sech} \Theta m}{4} \gamma^{45} \quad \mathcal{D}_i = \partial_i, \quad i = \zeta_m, \zeta_w, 8, 9 \]
\[ \mathcal{D}_\phi = \partial_\phi + \frac{\text{sech}^2 \Theta m - 2}{4} \cos \theta \gamma^{34} + \frac{\text{sech} \Theta m}{4} \sin \theta \gamma^{35} \quad \mathcal{D}_\alpha = \partial_\alpha + \frac{\text{sech}^2 \Theta m}{4} \gamma^{34} \quad (C.16) \]

and the gamma matrices in the coordinate frame \( \Gamma_M \) read
\[ \Gamma_\tau = R(- \cosh \omega \gamma^0 + \text{sech} \Theta w \sinh \omega \gamma^2) \quad \Gamma_\phi = R(\sin \theta \gamma^4 + \text{sech} \Theta m \cos \theta \gamma^5) \]
\[ \Gamma_\omega = R \gamma^1 \quad \Gamma_\theta = R \gamma^3 \quad \Gamma_\beta = R \text{sech} \Theta w \gamma^2 \quad \Gamma_\alpha = \text{sech} \Theta m \gamma^5 \]
\[ \Gamma_{\zeta_m} = \gamma^6 \quad \Gamma_{\zeta_w} = \gamma^7 \quad \Gamma_i = \gamma^i \quad i = 8, 9 \quad (C.17) \]

\(^6\)This is just the usual \( T^4 \) projection, \( \Gamma^{0789} \epsilon_0 = \epsilon_0 \).
and

\[
\Gamma = \frac{1}{R} \text{sech} \omega \gamma^0 \quad \Gamma = \frac{1}{R} \csc \theta \gamma^A \quad \Gamma = \frac{1}{R} \gamma^1 \quad \Gamma = \frac{1}{R} \gamma^3
\]

\[
\Gamma_\beta = \frac{1}{R} \left( -\tan \omega \gamma^0 + \cosh \Theta_m \gamma^2 \right) \quad \Gamma_\theta = \frac{1}{R} \left( -\cot \theta \gamma^4 + \cosh \Theta_m \gamma^5 \right)
\]

\[
\Gamma_{\omega m} = \gamma^6 \quad \Gamma_{\bar{\omega} m} = \gamma^7 \quad \Gamma_i = \gamma^i \quad i = 8, 9
\]

(C.18)

The three-form field \( G_3 \) reads in this case

\[
G_3 = -R \tanh \Theta_m \cosh \omega \tau \wedge d\omega \wedge (id\zeta_m + d\xi_m) + R \tanh \Theta_m \sin \theta d\theta \wedge d\phi \wedge (id\zeta_m + d\xi_m)
\]

(C.19)

\[\text{D Killing spinor equations in Type IIA SUGRA}\]

The supersymmetry variations for the dilatino and the gravitino for type IIA supergravity read

\[
\sqrt{2} \delta \lambda = -\frac{1}{2} D_M \Phi_{IIA} \Gamma^M \Gamma^{11} \epsilon + \frac{1}{24} e^{-\frac{\Phi_{IIA}}{2}} H^{(3)}_{M_1 M_2 M_3} \Gamma_{M_1 M_2 M_3} \epsilon + \frac{1}{8 \cdot 4!} e^{-\frac{\Phi_{IIA}}{4}} F^{(4)}_{M_1 M_2 M_3 M_4} \Gamma_{M_1 M_2 M_3 M_4} \Gamma^{11} \epsilon = 0
\]

(D.1)

\[
\delta \psi = D_M \epsilon + \frac{1}{96} e^{-\frac{\Phi_{IIA}}{2}} H^{(3)}_{M_1 M_2 M_3} \left( \Gamma_{M_1 M_2 M_3} - 9 \delta_M \Gamma_{M_1 M_2 M_3} \right) \Gamma^{11} \epsilon
\]

\[
- \frac{1}{64} e^{-\frac{3\Phi_{IIA}}{4}} F^{(2)}_{M_1 M_2} \left( \Gamma_{M_1 M_2} - 14 \delta_M \Gamma_{M_1 M_2} \right) \Gamma^{11} \epsilon
\]

\[
+ \frac{1}{256} e^{-\frac{3\Phi_{IIA}}{4}} F^{(4)}_{M_1 M_2 M_3 M_4} \left( \Gamma_{M_1 M_2 M_3 M_4} - \frac{20}{3} \delta_M \Gamma_{M_1 M_2 M_3 M_4} \right) \epsilon = 0,
\]

(D.2)

where the spinor \( \epsilon \) is taken to be Majorana but not Weyl. Here \( \Phi_{IIA} \) is the dilaton field of type IIA supergravity and the three-form field, \( H^{(3)} = dB^{(2)} \) is the NS field strength.

We will consider two cases depending on how the circle chosen to perform T–duality. We first look at the case in which the sphere is squashed, followed by that in which the anti-de Sitter space becomes warped.

\[\text{D.1 Squashed Sphere}\]

Let the coordinates along the squashed sphere \( \text{SqS}^3 \) be denoted by \( x = \{ \theta, \phi, \alpha \} \) (the coordinates on AdS are \( \{ \tau, \omega, \sigma \} \)). The covariant derivatives along these directions are given by

\[
D_\phi = \partial_\phi + \frac{(\text{sech}^2 \Theta_m - 2)}{4} \cos \theta \gamma^{34} + \frac{\text{sech} \Theta_m}{4} \sin \theta \gamma^{35},
\]

(D.3a)

\[
D_\theta = \partial_\theta - \frac{\text{sech} \Theta_m}{4} \gamma^{35}, \quad D_\alpha = \partial_\alpha + \frac{\text{sech}^2 \Theta_m}{4} \gamma^{34}
\]

(D.3b)
The antisymmetric fields on this basis read

\[ H_3 = R \tanh \Theta_m \sin \theta d\theta \wedge d\phi \wedge d\xi_m \]  \hspace{1cm} (D.4)
\[ F_2 = R \tanh \Theta_m \sin \theta d\theta \wedge d\phi \]  \hspace{1cm} (D.5)
\[ F_4 = \left[ \frac{R^2}{\cosh \Theta_m^2} \sin \theta d\theta \wedge d\phi \wedge d\alpha + R^2 \cosh \omega dt \wedge d\omega \wedge d\sigma \right] \wedge d\xi_m \]  \hspace{1cm} (D.6)

The \( \Gamma \)-matrices in the background are related to those in the orthonormal frame as:

\[ \Gamma_\theta = R \gamma^3 \] \hspace{1cm} \[ \Gamma_\alpha = R \sech \Theta_m \gamma^5 \] \hspace{1cm} \[ \Gamma_\phi = R \left[ \sin \theta \gamma^4 + \sech \Theta_m \cos \theta \gamma^5 \right] \]  \hspace{1cm} (D.7)

and

\[ \Gamma^\theta = \frac{1}{R} \gamma^3 \] \hspace{1cm} \[ \Gamma^\phi = \frac{1}{R} \csc \theta \gamma^4 \] \hspace{1cm} \[ \Gamma^\alpha = \frac{1}{R} \left[ -\cot \theta \gamma^4 + \sech \Theta_m \gamma^5 \right] \]  \hspace{1cm} (D.8)

The variations of the gravitino, \( \delta \psi_M = 0 \), give rise to the equations:

\[
\delta \psi_\phi = \left[ \partial_\rho + \frac{\text{sech}^2 \Theta_m - 2}{4} \cos \theta \gamma^{34} + \frac{\text{sech} \Theta_m}{4} \sin \theta \gamma^{35} \right] \epsilon
+ \frac{1}{32} \tan \Theta_m \left( \sin \theta (6 \gamma^{39} - 7 \gamma^3) + \text{sech} \Theta_m \cos \theta (2 \gamma^{3459} - \gamma^{345}) \right) \Gamma^{11} \epsilon
+ \frac{1}{32} \left( \sin \theta (5 \text{sech} \Theta_m \gamma^{359} - 3 \gamma^{101249}) - \text{sech} \Theta_m \cos \theta (5 \text{sech} \Theta_m \gamma^{349} + 3 \gamma^{10259}) \right) \epsilon = 0
\]

\[
\delta \psi_\alpha = \left[ \partial_\alpha + \frac{\text{sech}^2 \Theta_m \gamma^{34}}{4} \right] \epsilon
+ \frac{1}{32} \tan \Theta_m \left( 2 \text{sech} \Theta_m \gamma^{3459} - \gamma^{345} \right) \Gamma^{11} \epsilon
- \frac{1}{32} \left( 5 \text{sech}^2 \Theta_m \gamma^{349} + 3 \text{sech} \Theta_m \gamma^{10259} \right) \epsilon = 0
\]

\[
\delta \psi_\theta = \left[ \partial_\theta - \frac{\text{sech} \Theta_m \gamma^{45}}{4} \right] \epsilon
+ \frac{1}{32} \tan \Theta_m \left( 2 \gamma^{2349} - \gamma^{234} \right) \Gamma^{11} \epsilon
- \frac{1}{32} \left( 5 \text{sech} \Theta_m \gamma^{459} + 3 \gamma^{10239} \right) \epsilon = 0
\]

\[
\delta \psi_\tau = \left[ \partial_\tau - \frac{1}{4} (\sinh \omega \gamma^{12} + \cosh \omega \gamma^{12}) \right] \epsilon
+ \frac{1}{32} \tan \Theta_m \left( \sinh \omega (2 \gamma^{2349} - \gamma^{234}) - \cosh \omega (2 \gamma^{2349} - \gamma^{234}) \right) \Gamma^{11} \epsilon
+ \frac{1}{32} \left( \sinh \omega (3 \text{sech} \Theta_m \gamma^{23459} - 5 \gamma^{1019}) - \cosh \omega (3 \text{sech} \Theta_m \gamma^{23459} + 5 \gamma^{1029}) \right) \epsilon = 0
\]

\[
\delta \psi_\nu = \left[ \partial_\nu + \frac{1}{4} \gamma^{01} \right] \epsilon
+ \frac{1}{32} \tan \Theta_m \left( 2 \gamma^{2349} - \gamma^{234} \right) \Gamma^{11} \epsilon
+ \frac{1}{32} \left( 3 \text{sech} \Theta_m \gamma^{23459} - 5 \gamma^{1019} \right) \epsilon = 0
\]

\[
\delta \psi_\omega = \left[ \partial_\omega + \frac{1}{4} \gamma^{02} \right] \epsilon
+ \frac{1}{32} \tan \Theta_m \left( 2 \gamma^{1349} - \gamma^{134} \right) \Gamma^{11} \epsilon
+ \frac{1}{32} \left( 3 \text{sech} \Theta_m \gamma^{13459} + 5 \gamma^{1029} \right) \epsilon = 0
\]

\[
\delta \psi_i = \partial_i \epsilon
+ \frac{1}{32} \frac{\tan \Theta_m}{R} \left( 2 \gamma^{349} - \gamma^{34} \right) \Gamma^{11} \epsilon
+ \frac{3}{32} \frac{1}{R} \left( \text{sech} \Theta_m \gamma^{3459} + \gamma^{10129} \right) \epsilon = 0
\]

\[
\delta \psi_{\xi_m} = \partial_{\xi_m} \epsilon
+ \frac{1}{32} \frac{\tan \Theta_m}{R} \left( -6 \gamma^{34} - \gamma^{349} \right) \Gamma^{11} \epsilon
+ \frac{5}{32} \frac{1}{R} \left( \text{sech} \Theta_m \gamma^{345} + \gamma^{1012} \right) \epsilon = 0
\]  \hspace{1cm} (D.9)
and the dilatino variation gives:

\[ \tanh \Theta_m \left( 2\gamma^{349} - 3\gamma^{34} \right) \epsilon + \left( \text{sech} \Theta_m \gamma^{3459} + \gamma^{0129} \right) \Gamma^{11} \epsilon = 0 \]  

(D.10)

Combining this equation with eqs. (D.9) one obtains:

\[ \partial_i \epsilon = \frac{1}{2R} \tanh \Theta_m \gamma^{34} Q^- \gamma \Gamma^{11} \epsilon \]
\[ \partial_{\gamma} \epsilon = \frac{1}{R} \tanh \Theta_m \gamma^{34} Q^+ \Gamma^{11} \epsilon \]
\[ \partial_\sigma \epsilon = -\frac{1}{2} \left[ \gamma^{01} Q^- + \text{tanh} \Theta_m \gamma^{234} Q^+ \Gamma^{11} \right] \epsilon \]
\[ \partial_\tau \epsilon = -\frac{1}{2} \left[ \gamma^{02} Q^+ + \text{tanh} \Theta_m \gamma^{134} Q^+ \Gamma^{11} \right] \epsilon \]
\[ \partial_\theta \epsilon = \frac{1}{2} \left[ \sinh \omega \gamma^{01} + \cosh \omega \gamma^{12} \right] Q^+ - \text{tanh} \Theta_m (\sinh \omega \gamma^{234} - \cosh \omega \gamma^{034}) Q^+ \Gamma^{11} \]
\[ \partial_\phi \epsilon = \frac{1}{2} \left[ \gamma^{0123} Q^- - \text{tanh} \Theta_m \gamma^4 Q^+ \Gamma^{11} \right] \epsilon \]
\[ \partial_\alpha \epsilon = -\frac{1}{2} \left[ \text{sech}^2 \Theta_m \gamma^{34} Q^- + \text{sech} \Theta_m \text{tanh} \Theta_m \gamma^{345} Q^+ \Gamma^{11} \right] \epsilon \]
\[ \partial_\chi \epsilon = \frac{1}{2} \left[ \sin \theta \gamma^{0124} Q^- - \text{sech}^2 \Theta_m Q^- - 1 \cos \theta \gamma^{34} \right] \]
\[ + \text{tanh} \Theta_m (\sin \theta \gamma^3 Q^+ - \text{sech} \Theta_m \cos \theta \gamma^{345} Q^+) \Gamma^{11} \epsilon \]  

(D.11)

Here \( Q_\pm = \frac{1+\gamma^9}{2} \). These equations all have solutions of the form \( \epsilon = \exp \{ x \epsilon_0 \} \). One needs to be careful with the boundary conditions when compactifying. Therefore, from the equation for the \( S^1 \), denoted by the coordinate \( z \), one obtains the following constraint on the Killing spinors

\[ \gamma^{34} Q^+ \Gamma^{11} \epsilon = 0 \]  

(D.12)

Hence the Killing spinors do not depend on the torus coordinates as it should be. Let us write \( \epsilon = \epsilon_1(\omega, \tau, \sigma) \epsilon_2(\theta, \phi, \alpha) \epsilon_0 \), with \( \epsilon_0 \) a constant Majorana spinor. The set of equations for the AdS coordinates becomes

\[ \partial_\omega \epsilon_1 = -\frac{1}{2} \gamma^{02} Q^+ \epsilon_1 \]  

(D.13)
\[ \partial_\tau \epsilon_1 = -\frac{1}{2} \gamma^{01} Q^- \epsilon_1 \]  

(D.14)
\[ \partial_\sigma \epsilon_1 = \frac{1}{2} \left( \sinh \omega \gamma^{01} + \cosh \omega \gamma^{12} \right) Q^+ \epsilon_1 \]  

(D.15)

Which can be immediately solved as we did on the previous section, by writing \( \epsilon_1(\omega, \tau, \sigma) = \epsilon_1(\omega) \epsilon_1(\tau) \epsilon_1(\sigma) \) (they are just the Killing spinor equations for \( AdS_3 \)). One obtains

\[ \epsilon_1(\omega, \tau, \sigma) = \exp \left\{ -\frac{\omega}{2} \gamma^{02} Q^+ \right\} \exp \left\{ \frac{\tau}{2} \gamma^{12} Q^+ \right\} \exp \left\{ \frac{\sigma}{2} \gamma^{01} Q^- \right\} \]  

(D.16)
The remaining equations can be solved in analogous fashion. Let $\epsilon_2(\theta, \phi, \alpha) = \epsilon_{2\theta}(\theta)\epsilon_{2\phi}(\theta, \phi)\epsilon_{2\alpha}(\theta, \psi, \alpha)$. The solution is:

$$
\epsilon_2(\theta, \phi, \alpha) = \exp\left\{\frac{\theta}{2} \gamma^{0123} Q_+ \right\} \left[ \exp \left\{ \tanh^2 \Theta_m \cos \theta \left( \frac{\phi}{2} \right) \gamma^{34} \right\} Q_- + \exp \left\{ \frac{\phi}{2} \gamma^{34} \right\} Q_+ \right]\exp \left\{ - \sech^2 \Theta_m \frac{\alpha}{2} \gamma^{34} Q_- \right\} \quad (D.17)
$$

If one writes $\epsilon = \epsilon_+ + \epsilon_-$ with $Q_\pm \epsilon_\pm = \pm \epsilon_\pm$ one can verify that only the supersymmetries associated to the $\epsilon_-$ spinors are preserved after T-duality. Hence the solution is 1/4-BPS. When the deformation parameter is such that $\sech \Theta_m = 0$, there are no spinors depending on the T-dual coordinate and supersymmetry is restored to 1/2-BPS. This corresponds to the case in which the geometry becomes $AdS_3 \times S^2 \times S^1 \times T^4$.

### D.2 Warped AdS

Let the coordinates along the warped anti-de Sitter space $WAdS_3$ be denoted by $x = \{t, \omega, \alpha\}$ (the coordinates on the sphere read $\{\theta, \psi, \phi\}$). The covariant derivatives along these directions are given by

$$
D_\tau = \partial_\tau + \left( \frac{\sech^2 \Theta_w - 2}{4} \right) \sinh \omega \gamma^{01} - \frac{\sech \Theta_w}{4} \cosh \omega \gamma^{12}, \quad (D.18a)
$$
$$
D_\omega = \partial_\omega + \frac{\sech \Theta_w}{4} \gamma^{02}, \quad D_\beta = \partial_\beta + \frac{\sech^2 \Theta_w}{4} \gamma^{01}. \quad (D.18b)
$$

The antisymmetric fields on this basis read

$$
H_3 = -R \tanh \Theta_w \cosh \omega d\tau \wedge d\omega \wedge d\zeta_w \quad (D.19)
$$
$$
F_2 = -R \tanh \Theta_w \cosh \omega d\tau \wedge d\omega \quad (D.20)
$$
$$
F_4 = \left[ -R^2 \sech^2 \Theta_w \cosh \omega d\tau \wedge d\omega \wedge d\beta + R^2 \sin \theta d\theta \wedge d\psi \wedge d\phi \right] \wedge d\zeta_w \quad (D.21)
$$

The $\Gamma$-matrices in the background are related to those in the orthonormal frame as:

$$
\Gamma_\omega = R \gamma^1 \quad \Gamma_\beta = R \sech \Theta_w \gamma^2 \quad \Gamma_\tau = R \left[ - \cosh \omega \gamma^0 + \sech \Theta_w \sinh \omega \gamma^2 \right] \quad (D.22)
$$

and

$$
\Gamma^\omega = \frac{1}{R} \gamma^1 \quad \Gamma^\tau = \frac{1}{R} \sech \omega \gamma^0 \quad \Gamma^\beta = \frac{1}{R} \left[ - \tanh \omega \gamma^0 + \cosh \Theta_w \gamma^2 \right]. \quad (D.23)
$$
The variations of the gravitino, $\delta \psi_M = 0$, give rise to the equations:

$$
\delta \psi_T = \left[ \partial_T + \left( \frac{\text{sech}^2 \Theta_w - 2}{4} \right) \sinh \omega \gamma^{01} - \frac{\text{sech} \Theta_w}{4} \cosh \omega \gamma^{12} \right] \epsilon 
+ \frac{1}{32} \tanh \Theta_w \left( \cosh \omega (6 \gamma^{19} - 7 \gamma^{01}) + \text{sech} \Theta_w \sinh \omega (-2 \gamma^{012} + \gamma^{021}) \right) \Gamma^{11} \epsilon 
+ \frac{1}{32} \left( \cosh \omega (5 \text{sech} \Theta_w \gamma^{129} - 3 \gamma^{03459}) + \text{sech} \Theta_w \sinh \omega (5 \text{sech} \Theta_w \gamma^{019} + 3 \gamma^{13459}) \right) \epsilon = 0
$$

$$
\delta \psi_\omega = \left[ \partial_\omega + \frac{\text{sech} \Theta_w}{4} \gamma^{02} \right] \epsilon 
+ \frac{1}{32} \tanh \Theta_w \left( -6 \gamma^{09} + 7 \gamma^0 \right) \Gamma^{11} \epsilon 
+ \frac{1}{32} \left( -5 \text{sech} \Theta_w \gamma^{029} + 3 \gamma^{13459} \right) \epsilon = 0
$$

$$
\delta \psi_\beta = \left[ \partial_\beta + \frac{\text{sech}^2 \Theta_w \gamma^{01}}{4} \right] \epsilon 
+ \frac{1}{32} \tanh \Theta_w \left( -2 \text{sech} \Theta_w \gamma^{012} + \gamma^{012} \right) \Gamma^{11} \epsilon 
+ \frac{1}{32} \left( 5 \text{sech}^2 \Theta_w \gamma^{109} + 3 \text{sech} \Theta_w \gamma^{23459} \right) \epsilon = 0
$$

$$
\delta \psi_\theta = \left[ \partial_\theta - \frac{1}{4} \gamma^{45} \right] \epsilon 
+ \frac{1}{32} \tanh \Theta_w \left( -2 \gamma^{0139} + \gamma^{013} \right) \Gamma^{11} \epsilon 
+ \frac{1}{32} \left( 3 \text{sech} \Theta_w \gamma^{0129} - 5 \gamma^{459} \right) \epsilon = 0
$$

$$
\delta \psi_\phi = \left[ \partial_\phi - \frac{1}{4} \left( \cos \theta \gamma^{34} + \sin \theta \gamma^{35} \right) \right] \epsilon 
+ \frac{1}{32} \tanh \Theta_w \left( \sin \theta (-2 \gamma^{0149} + \gamma^{014}) + \cos \theta (-2 \gamma^{0159} + \gamma^{015}) \right) \Gamma^{11} \epsilon 
+ \frac{1}{32} \left( \sin \theta (3 \text{sech} \Theta_w \gamma^{01249} + 5 \gamma^{359}) + \cos \theta (3 \text{sech} \Theta_w \gamma^{01259} - 5 \gamma^{349}) \right) \epsilon = 0
$$

$$
\delta \psi_i = \left[ \partial_i + \frac{1}{4} \gamma^{34} \right] \epsilon 
+ \frac{1}{32} \tanh \Theta_w \left( -2 \gamma^{015} + \gamma^{015} \right) \Gamma^{11} \epsilon 
+ \frac{1}{32} \left( 3 \text{sech} \Theta_w \gamma^{0125} - 5 \gamma^{349} \right) \epsilon = 0
$$

$$
\delta \psi_s = \left[ \partial_s + \frac{1}{32} Q_+ \right] \epsilon 
+ \frac{1}{32} \tanh \Theta_w \left( 6 \gamma^{01} + \gamma^{019} \right) \Gamma^{11} \epsilon 
+ \frac{5}{32 R} \left( - \text{sech} \Theta_w \gamma^{012} + \gamma^{345} \right) \epsilon = 0
$$

and the variation of the dilatino

$$
\tanh \Theta_w (-2 \gamma^{019} + 3 \gamma^{01}) \epsilon + (- \text{sech} \Theta_w \gamma^{012} + \gamma^{3459}) \Gamma^{11} \epsilon = 0
$$

Using (D.25) into the previous equations, yields

$$
\partial_i \epsilon = \frac{1}{2R} \tanh \Theta_w \gamma^{01} Q_+ \Gamma^{11} \epsilon
$$

$$
\partial_s \epsilon = -\frac{1}{R} \tanh \Theta_w \gamma^{01} Q_+ \Gamma^{11} \epsilon
$$

$$
\partial_\theta \epsilon = \frac{1}{2} \left[ \gamma^{45} Q_+ + \tanh \Theta_w \gamma^{013} Q_+ \Gamma^{11} \right] \epsilon
$$

$$
\partial_\phi \epsilon = \frac{1}{2} \left[ - \sin \theta \gamma^{35} + \cos \theta \gamma^{34} \right] Q_+ + \tanh \Theta_w (\sin \theta \gamma^{014} + \cos \theta \gamma^{015}) Q_+ \Gamma^{11} \epsilon
$$

$$
\partial_i \epsilon = \frac{1}{2} \left[ \gamma^{34} Q_- - \tanh \Theta_w \gamma^{015} Q_- \Gamma^{11} \right] \epsilon
$$

$$
\partial_\beta \epsilon = -\frac{1}{2} \left[ \text{sech} \Theta_w \gamma^{01} Q_- + \text{sech} \Theta_w \tanh \Theta_w \gamma^{012} Q_- \Gamma^{11} \right] \epsilon
$$
\[\partial_\omega \epsilon = -\frac{1}{2} \left[-\gamma^{1345} Q_- + \tanh \Theta_w \omega^0 Q + \Gamma^{11}\right] \epsilon\]
\[\partial_\tau \epsilon = \frac{1}{2} \left[(-\cosh \omega \gamma^{0345} Q_- - (\sech^2 \Theta_w Q_+ - 1) \sinh \omega \gamma^{01})
+ \tanh \Theta_w (\sech \Theta_w \sinh \omega \gamma^{012} Q_- - \cosh \omega \gamma^1 Q_+ \Gamma^{11}\right] \epsilon\]

(D.26)

Again, these equations have solutions of the form \(\epsilon = \exp x \epsilon_0 \). As before, from the equation for the \(z\) coordinate, one obtains the constraint on the Killing spinors
\[
\gamma^{01} Q_+ \Gamma^{11} \epsilon = 0 \quad (D.27)
\]

We write \(\epsilon = \epsilon_1(\theta, \psi, \phi) \epsilon_2(\omega, \tau, \beta) \epsilon_0\), with \(\epsilon_0\) a constant Majorana spinor. The set of equations for the sphere coordinates becomes
\[
\partial_\theta \epsilon_1 = \frac{1}{2} \gamma^{45} Q_+ \epsilon_1 \quad (D.28)
\]
\[
\partial_\psi \epsilon_1 = -\frac{1}{2} \gamma^{34} Q_- \epsilon_1 \quad (D.29)
\]
\[
\partial_\phi \epsilon_1 = \frac{1}{2} \left(-\sin \theta \gamma^{35} + \cos \theta \gamma^{34}\right) Q_+ \epsilon_1 \quad (D.30)
\]

Which can be immediately solved as we did on the previous section, by writing \(\epsilon_1(\theta, \psi, \phi) = \epsilon_{10}(\theta) \epsilon_{1\psi}(\theta, \psi) \epsilon_{1\phi}(\theta, \psi, \phi)\) (they are just the Killing spinor equations for \(S^3\)). One obtains
\[
\epsilon_1(\theta, \psi, \phi) = \exp \left\{ \frac{\theta}{2} \gamma^{45} Q_+\right\} \exp \left\{ \frac{\psi}{2} \gamma^{34} Q_+\right\} \exp \left\{ -\frac{\phi}{2} \gamma^{34} Q_-\right\} \quad (D.31)
\]

The remaining equations can be solved in analogous fashion. Let
\[
\epsilon_2(\omega, \tau, \beta) = \epsilon_{2\omega}(\omega) \epsilon_{2\tau}(\omega, \tau) \epsilon_{2\beta}(\omega, \tau, \beta) \quad . \quad (D.32)
\]

The solution is:
\[
\epsilon_2(\omega, \tau, \beta) = \exp \left\{ -\frac{\omega}{2} \gamma^{1345} Q_- \right\} \left[ \exp \left\{-\tanh \Theta_w \omega^0 \left( \frac{\tau}{2} \right) \gamma^{01}\right\} Q_+ 
+ \exp \left\{ -\frac{\tau}{2} \gamma^{0345} \right\} Q_- \exp \left\{-\sech^2 \Theta_w \frac{\beta}{2} \gamma^{01} Q_+\right\} \right] \quad (D.33)
\]

The arguments given in the previous section also apply to this case. For \(\epsilon = \epsilon_+ + \epsilon_-\) with \(Q_+ \epsilon_+ = Q_- \epsilon_- = \pm \epsilon_\pm\) only the supersymmetries associated to the \(\epsilon_+\) spinors are preserved after T-duality and the solution is 1/4-BPS. When the deformation parameter is such that \(\sech \Theta_w = 0\), there are no spinors depending on the T-dual coordinate and supersymmetry is restored to 1/2-BPS. This corresponds to the case in which the geometry becomes \(\text{AdS}_2 \times S^1 \times S^3 \times T^4\).