Cosmological tensor perturbations in the Randall–Sundrum model: evolution in the near-brane limit

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We discuss the evolution of cosmological tensor perturbations in the RSII model. In Gaussian normal coordinates the wave equation is non-separable, so we use the near-brane limit to perform the separation and study the evolution of perturbations. Massive excitations, which may also mix, decay outside the horizon which could lead to some novel cosmological signatures.

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I. INTRODUCTION

The brane-world idea, according to which the observable universe is a hypersurface (brane) embedded in a higher-dimensional spacetime (bulk), has attracted the attention of particle physicists and cosmologists in recent years — for reviews see [1, 2, 3, 4]. The idea is motivated from developments in string theory and M–theory, although many of the models which have been developed are phenomenological. Two such models were constructed by Randall and Sundrum [5, 6], in which the bulk spacetime is a five-dimensional Anti-de Sitter (AdS) spacetime with a small length-scale (∼1 mm or less). In this article, we will be considering the second [6] of these models which has only one brane with a large, positive, bare tension to balance the curvature of the AdS bulk. The cosmological background solutions in the bulk spacetime were calculated [7, 8, 9] shortly after the model was proposed.

Cosmological perturbations around this background, however, have resisted a solution in the face of a considerable research effort [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]. (For a review on cosmological perturbations in brane-worlds see [31].) Although diverse formalisms for cosmological perturbations in brane models exist, the evolution of perturbations during the different cosmological epochs is not yet fully understood. The problem is that the bulk equations are partial differential equations which are subject to spacetime dependent boundary conditions on the brane. Therefore, the full problem can cannot be reduced to a four-dimensional effective problem. In addition, there is a large difference in scales: the AdS length-scale is of the order of 1 mm or less, whereas, the perturbations have to be evolved over many orders of magnitudes of conformal time, a ratio in scales of about $10^{26}$. Because of this, the wave equation for the evolution of the perturbations is difficult to solve numerically, at least by brute-force numerical integration: the spatial length scale of 1 mm means that we would have to use a corresponding time scale as the time-step, meaning the code would have to be run for the order of $10^{26}$ time steps!

In order to test the brane-world idea against cosmological data, it is extremely important to understand the evolution of perturbations. One of the most powerful discriminating tools between cosmological models is the power spectrum of the cosmic microwave background (CMB) anisotropies, which is, effectively, a snapshot of the perturbations in the universe when it was about one thousandth of its current size. At about this time in the history of the universe, two related events occur: atomic recombination and last scattering of photons. Before atomic recombination, the temperature of the universe is so high that protons and electrons form a plasma, but during recombination they come together to form atomic hydrogen. At approximately the same time, the mean free path of the photons due Thomson scattering in the plasma increases rapidly from being very small to very large (effectively, from zero to infinity) meaning that most of the photons arriving at the earth from the cosmological plasma will not have scattered subsequently. Once foreground effects due to our galaxy and external galaxy clusters have been removed, the electromagnetic radiation seen from earth is an image of this surface of last-scattering, or, more precisely, the intersection of it with our past light-cone. However, because of perturbations in the early universe, this will occur at slightly earlier or later times at different points of space, so the CMB radiation will be slightly brighter or dimmer at different points on the sky. The upshot of this is that, by measuring the temperature fluctuations in the CMB, we can deduce the perturbations in the metric on the surface of last scattering.

The details of the perturbations in the metric at the time of last-scattering and their subsequent evolution depend on the theory of gravity. In brane-world models, Einstein’s equation becomes modified, at least at high energies, and it is plausible that these effects can propagate into the regime relevant to the creation of CMB anisotropies. Clearly, the evolution of cosmological perturbations in this model is of significant interest.
The modifications in the Randall–Sundrum (RS) model are twofold: first, in Einstein’s equation a term quadratic in the energy-momentum tensor appears. In the context of cosmology, this implies that at high energies the Hubble parameter on the brane is proportional the energy density of matter $\rho$ and not proportional to $\sqrt{\rho}$, as in General Relativity. However, in the RS model one can show that this term is negligible at low energies, i.e., at energies much smaller than the brane tension. The other correction which appears in Einstein’s field equation is the five-dimensional Weyl tensor projected onto the brane. This term encodes the influence of the bulk gravitational field on the dynamics of the brane. In the case of a homogeneous and isotropic brane this term vanishes in the case of an AdS bulk, but is non-zero if the bulk contains a black hole (Schwarzschild–AdS). From the point of view of 4D cosmology, this term behaves like a smooth energy component with equation of state $P = \rho/3$ and was therefore dubbed “dark radiation” [7]. Its response to perturbations is somewhat different [17].

The purpose of this article is to study the evolution of tensor perturbations in the RS brane-world. Although these perturbations are easier to understand than the observationally interesting case of scalar perturbations, we will see that their evolution is already very complicated. There are several coordinates system which one might use to tackle the problem. In the Gaussian normal (GN) coordinate system the brane is at rest but the bulk metric components have a complicated time-dependence: the brane boundary condition are easy to impose, but the solution of the corresponding wave equation is much more subtle since it is non-separable. Alternatively, one can formulate the problem in a coordinate system where the brane moves, but the bulk spacetime is manifestly static: one can then solve the wave equation in the bulk, but the boundary condition on the brane is difficult to impose. In this paper we will discuss solutions in the near-brane limit of the GN coordinate system. These solutions are the most direct analogue of the well studied mode functions of Minkowski space. Some of the material presented here first appeared in [32] and [33]. The fact that the equation of motion is not separable is likely to lead to mode mixing and this will be discussed in future work.

The paper is organized as follows: In Section II we discuss the different scales involved and discuss different ways to attack the problem. In Section III we present the setup and derive the bulk wave equation and the brane boundary conditions. We also discuss the issue of initial conditions. In Section IV we present the near-brane solutions in the different cosmological epochs. In Section V we discuss the problem of initial conditions. We present our conclusions in Section VI.

II. OVERVIEW

For simplicity, we will only consider the second RS model [6] which has a single brane of constant positive bare tension in an Anti-de Sitter (AdS) bulk which has a reflection symmetry between the two sides of the brane. Furthermore, we will restrict our attention to the case of a spatially flat universe, which is observationally favoured. The bulk Einstein equation for this model is then

$$R_{\mu\nu} = \frac{2}{3} \Lambda g_{\mu\nu}.$$  \hspace{1cm} (1)

The Gauss relation gives the four-dimensional Ricci tensor as

$$\bar{R}_{\mu\nu} = \Lambda \bar{g}_{\mu\nu} + KK_{\mu\nu} - K_{\mu}{}^{\rho}K_{\nu\rho} - W_{\mu\nu},$$  \hspace{1cm} (2)

where the extrinsic curvature tensor of the brane is defined as

$$K_{\mu\nu} = -\bar{g}_{\mu}{}^{\rho}\nabla_\rho n_\nu,$$  \hspace{1cm} (3)

with $n_\nu$ being the unit covector normal to the brane, and

$$W_{\mu\nu} = C_{\mu\nu\sigma\tau}n^\sigma n^\tau,$$  \hspace{1cm} (4)

is the contribution from the Weyl tensor of the bulk spacetime [47]. The junction conditions [32] become

$$K_{\alpha\beta} = \frac{\kappa}{6} \left\{ 3T_{\alpha\beta} + (T - \lambda)\bar{g}_{\alpha\beta} \right\},$$  \hspace{1cm} (5)

whilst the definition of $K_{\alpha\beta}$, combined with the bulk Einstein equations [11], yields

$$\mathcal{L}_n K_{\alpha\beta} = \bar{W}_{\alpha\beta} + \frac{\Lambda}{6} - K_{\alpha\gamma}K_{\beta}{}^{\gamma},$$  \hspace{1cm} (6)

$$\mathcal{L}_n \bar{g}_{\alpha\beta} = -2K_{\alpha\beta}.$$  \hspace{1cm} (7)

We will study perturbations by linearizing these equations about the cosmological background solution. For simplicity, we will focus on the case of a spatially flat universe, i.e., where the surfaces of isometry are flat planes, and consider only the case of a pure AdS bulk where $\bar{W}_{\alpha\beta} = 0$ for the background.
A. The variety of coordinates and gauges

There are a variety of formalisms available for studying this problem. In this section, we explore some of these possibilities and discuss their various merits and drawbacks. There is no clear favourite, but some gauges are better for studying certain aspects of the problem.

There are two different natural choices of coordinates in which it is convenient to express the background, in addition to the possibility of a coordinate-independent formalism. One natural choice is the Gaussian normal (GN) coordinate system. In these coordinates the metric for the background (non-perturbed) spacetime has the form

$$ds^2 = -n(\tau, \zeta)^2 d\tau^2 + a(\tau, \zeta)^2 \delta_{ij} dx^i dx^j + d\zeta^2.$$  \hfill (8)

The main advantage of this coordinate system is that the brane remains at a fixed value of one of the coordinates, meaning that imposing the boundary condition due to the perturbed junction conditions is simple. The main disadvantage is that the metric components $a$ and $n$ have complicated functional forms and, as a consequence, the equations of motion are not separable.

The other natural coordinate systems are ones where the static nature of the background is manifest. Another way of writing the line element for a flat cosmology in an AdS bulk is

$$ds^2 = l^2 z^2 (\eta_{\alpha\beta} + h_{\alpha\beta}) dx^\alpha dx^\beta = l^2 \left( -dt^2 + \delta_{ij} dx^i dx^j + dz^2 \right),$$  \hfill (9)

where the spacetime is manifestly conformally flat. (Here, $l$ is the AdS length-scale.) This coordinate system makes the five-dimensional linearized Einstein equations simple. The price paid for this simplicity is that the brane is no longer at a fixed coordinate value, but has locus given by $z = l/a$, making the boundary condition much more difficult to impose.

In addition to the various coordinate systems for expressing the background, there are also various gauge choices for expressing the perturbations, as well as gauge-independent formalisms such as those used in [10, 11, 25, 26, 30]. The gauge we will use is the GN gauge, which arises by requiring that both the perturbed and unperturbed metrics have GN form, so that the perturbed line element has the form

$$ds^2 = -n^2(1 + \phi) d\tau^2 + 2anb_i d\tau dx^i + a^2 (\delta_{ij} + h_{ij}) dx^i dx^j + d\zeta^2.$$  \hfill (10)

When using a GN background, this is a very natural choice and has the advantage of making it trivial to extract the perturbation of the brane metric, $\bar{g}_{\mu\nu}$. Note that this does not completely fix the gauge since it is possible to impose additional requirement on the values taken by certain components of perturbation variable on the brane. For example, we could impose $\phi = b_i = 0$ on the brane, which would make the perturbation of the brane metric that of the synchronous gauge, a gauge often used in the treatment of perturbations in the standard, four-dimensional cosmology [35].

If we chose to work in the conformally Minkowski coordinates of [4], we would naturally write the perturbed metric as

$$ds^2 = \frac{l^2}{z^2} (\eta_{\alpha\beta} + h_{\alpha\beta}) dx^\alpha dx^\beta ,$$  \hfill (11)

which would allow us to make various gauge choices. For example, [20] used the transverse-traceless (TT) gauge where $h_{\alpha\beta}$ satisfies

$$\eta^{\alpha\beta} h_{\alpha\beta} = 0, \quad \partial_\alpha h^\alpha_{\beta} = 0.$$  \hfill (12)

Late in the history of the universe, the motion of the brane in this coordinate system is slow, so that the time coordinate, $t$, is almost the same as conformal time on the brane and so the TT gauge is approximately a synchronous gauge. This setup has the advantage that the resulting perturbed Einstein equations exactly soluble. However, in this gauge the position of the brane is not on the same locus as for the background but is displaced, an effect which has been dubbed “brane-bending” in the literature.

B. Physical scales in the problem

We will be considering the evolution of perturbations by taking the Fourier transform in the three spatial directions of the surfaces of isotropy and then evolving each Fourier mode. Thus, each mode has associated with it a length-scale
given by $k^{-1}$, where $k$ is the magnitude of the Fourier transform variable. Physically, we interpret $k$ as the comoving momentum of the mode and $k^{-1}$ as the comoving wavelength. There is another length-scale, $H^{-1}$, given by the time dependent expansion rate of the universe, equivalent to the Hubble radius of the Universe. In the standard cosmology, these are the only two length-scales, giving rise to a single dimensionless parameter, $kH^{-1}$, which depends on time.

In the language of cosmological perturbation theory, this parameter tells us whether the mode is outside or inside the Hubble horizon.

For perturbations in the RS model, there is another length-scale, namely the AdS length-scale, $l$. Thus, we have three dimensionless parameters: $kH^{-1}$, $lH$, and $kl$, two of which are independent. The first two of these depend on time: $kH^{-1}$ becomes larger at later times, whereas, $lH$ is smaller at later times. Both will, of course, remain approximately constant during a period of inflation. As in the four-dimensional case, $kH^{-1}$ gives the scale of the mode relative to the horizon size. Typically, the AdS length-scale, $l$ will be less than 1 mm whereas the length-scale of the perturbations $k^{-1}$ will be greater than 1 Mpc, so $kl$ will be an extremely small number. Except in the very early universe, $lH$ will also be small since it is the Hubble parameter measured in time units corresponding to 1 mm.

### III. SETUP

As already discussed, there are a variety of formalisms, each with its own merits. Here, we shall use a GN coordinate system and gauge, so that the perturbed line element takes the form

$$ds^2 = -n^2(1 + \phi) \, dt^2 + 2anb_i \, d\tau \, dx^i + a^2(\delta_{ij} + h_{ij}) \, dx^i dx^j + d\zeta^2.$$  

The brane is located at $\zeta = 0$. Note that our choice does not completely fix the gauge as, for example, we can set $\phi$ and $b_i$ to be zero on the brane, thereby having the brane metric in synchronous gauge. As a shorthand, we will denote derivatives with respect to $\tau$ and $\zeta$ by dots and dashes respectively. Note that $\phi$ and $b_i$ are irrelevant for the tensor perturbations studied here.

The observed matter, $T_{\alpha\beta}$, will comprise a background part into a background part

$$T^\alpha_\beta = \left( \begin{array}{cc} -\rho & 0 \\ 0 & \rho \delta^\alpha_j \end{array} \right),$$

and a perturbation

$$\delta T^\alpha_\beta = \left( \begin{array}{cc} -\delta \rho & a^{-1}n(\rho + p)v^i \\ -a^{-1}n(\rho + p)v^i & \delta \rho \delta^i_j + \Sigma^i_j \end{array} \right),$$

with $\Sigma^i_j = 0$. This is similar to the formalism used in [27]. It has the great advantage that the brane metric in synchronous gauge can easily be read-off and standard CMB computer code (such as CMBFAST [35]) can be applied to solve the Boltzmann equations and determine the power spectrum of matter and CMB anisotropies.

For the RS model, the bulk spacetime is pure AdS, which has a length-scale, $l$, related to the cosmological constant by

$$l^2 = \frac{6}{\Lambda}. \quad (16)$$

The form of the functions $a(\tau, \zeta)$ and $n(\tau, \zeta)$ was found in [31] to be

$$a(\tau, \zeta) = a(\tau, 0) \left[ \frac{1}{2} \left( 1 + \frac{U(\tau)^2}{6\Lambda} \right) + \frac{1}{2} \left( 1 - \frac{U(\tau)^2}{6\Lambda} \right) \cosh(2\zeta/l) - \sqrt{\frac{-U(\tau)^2}{6\Lambda} \sinh(2\zeta/l)} \right]^{1/2}, \quad (17)$$

$$n(\tau, \zeta) = n(\tau, 0) = \frac{\dot{a}(\tau, \zeta)/a(\tau, 0)}{a(\tau, 0)}, \quad (18)$$

where $U(\tau) = \lambda + \rho(\tau)$ is the total energy density supported on the brane, i.e., the bare tension, $\lambda$, plus energy density of observed matter, $\rho(\tau)$ [32]. The metric component $n(\tau, \zeta)$ is chosen so that $\tau$ corresponds to conformal time on the brane. We will also tune the bare tension against the bulk cosmological constant so as to make the effective brane cosmological constant zero, for which we need to take $\lambda = 3/(4\pi G l^2)$. With this we get

$$a(\tau, \zeta) = a(\tau, 0) \left[ e^{-\zeta/l} - \frac{\rho}{\lambda} \tanh(\zeta/l) \right]. \quad (19)$$
For convenience, we list the non-zero Christoffel symbols for this background, which are
\[
\Gamma^0_{00} = \frac{\dot{n}}{n}, \quad \Gamma^0_{0i} = \frac{\dot{a}}{a} \delta^i_j, \quad \Gamma^0_{ij} = -\frac{\dot{a}}{n^2} \delta_{ij},
\]
and the non-zero components of the Riemann tensor, which are
\[
\bar{R}^0_{0ij} = \frac{n\dot{a} - \dot{n} \dot{a}}{an^3} \delta^i_j, \quad \bar{R}^i_{kl} = 2 \frac{\dot{a}^2}{a^2 n^2} \delta^{[i}_{[k} \delta^{j]}_{l]}.
\]
The bulk Einstein equations give the following relations between the metric components
\[
-2\Lambda = \frac{n''}{n} - 3 \frac{\dot{a} \dot{n}}{an} + 3 \frac{\dot{a}' n'}{an},
\]
\[
-2\Lambda = \frac{n''}{n} + 3 \frac{a'}{a},
\]
\[
-2\Lambda = -\frac{\ddot{a}}{an} + \frac{\dot{a} \dot{n}}{an} + \frac{a''}{a} - 2 \frac{\dot{a}^2}{a^2 n^2} + 2 \frac{a'^2}{a^2},
\]
\[
0 = \frac{\dot{a}'}{a} - \frac{\dot{a} n'}{an}.
\]
The brane, at \(\zeta = 0\) has extrinsic curvature given in Gaussian normal coordinates by
\[
K_{\alpha\beta} = -\frac{1}{2} \frac{d}{d\bar{\zeta}} \bar{g}_{\alpha\beta},
\]
which has components
\[
K_{00} = \dot{n}' n, \quad K_{0i} = 0, \quad K_{ij} = -a a' \delta_{ij},
\]
from which we can deduce that
\[
K = -\left(3\frac{a'}{a} + \frac{n'}{n}\right).
\]
Note that some authors use the opposite sign convention for the extrinsic curvature.

**A. Equations of motion and brane boundary conditions**

Since we have chosen GN gauge, where the perturbed metric is in GN form, it is natural to consider the perturbation of the Gauss relation (2) which is
\[
\delta \bar{R}_{\mu\nu} = \frac{\Lambda}{2} \delta \bar{g}_{\mu\nu} + \delta K K_{\mu\nu} + K \delta K_{\mu\nu} + K^\rho_{\mu} K^\sigma_{\nu} \delta \bar{g}_{\rho\sigma} - 2 K^\rho_{(\mu} \delta K^\nu)_{\rho} - \delta \bar{W}_{\mu\nu},
\]
and perturbing (6) and (7) gives
\[
\frac{d}{d\bar{\zeta}} \delta K_{\alpha\beta} = 2 K^\rho_{(\alpha} \delta K^\beta)_{\rho} - K^\rho_{\alpha} K^{\sigma}_{\beta} \delta \bar{g}_{\rho\sigma} - \frac{\Lambda}{6} \delta \bar{g}_{\alpha\beta} - \delta \bar{W}_{\alpha\beta}
\]
\[
\frac{d}{d\bar{\zeta}} \delta \bar{g}_{\alpha\beta} = 2 \delta K_{\alpha\beta}
\]
substituting these into (29) and equating with the perturbations of the Ricci tensor calculated from the metric gives the evolution equations for the metric perturbations. We will evaluate these evolution equations in the special case of tensor perturbations. Perturbing the Israel conditions of (5) gives
\[
\delta K_{00} = -\frac{\kappa}{6} n^2 \left\{2 \delta \rho + 3 \delta \phi + (2\rho + 3p)\phi\right\},
\]
\[
\delta K_{0i} = \frac{\kappa}{6} a n \left\{3(\rho + p)v_i + (2\rho + 3p - \lambda)h_i\right\},
\]
\[
\delta K_{ij} = -\frac{\kappa}{6} a^2 \left\{\delta \phi \delta_{ij} + (\rho + \lambda) h_{ij} + 3 \Sigma_{ij}\right\},
\]
which will provide the boundary conditions we need at the brane.
B. Initial conditions and boundary conditions at the horizon

Since we are aiming to evolve the equations of motion over the semi-infinite interval $0 \leq \zeta \leq \infty$ it would appear that another boundary condition is not necessary. However, the coordinates used do not span the universal covering set of the AdS spacetime (see [36] for more details on the universal covering set) so we could expect that gravitational waves could cross the AdS horizon. The choice of boundary condition on this horizon is somewhat arbitrary since we can have no knowledge of the other side; this being a philosophical shortcoming of the Randall–Sundrum model. A reasonable, and popular, choice is to have no incoming radiation at the horizon, as advocated in [37]. However, it is not possible to impose this boundary condition in addition to arbitrary initial conditions on some time-slice. So we can impose boundary data in two different ways:

1. Brane boundary condition and a condition at horizon.
2. Brane boundary condition and initial conditions at some time.

The second of these is more in keeping with the orthodox approach to cosmological perturbation theory. However, it requires knowledge of what created in the first place. These are not as different as they might seem, we can gain a better understanding of how they are related, and of when it is necessary to specify boundary data at the horizon by considering Carter–Penrose conformal diagrams of the spacetime, see Fig. 1. It is apparent from the diagrams that specifying data on the horizon is a limiting case of specifying data on a timelike initial surface.

To illustrate this further, it is useful to consider the solution of the ordinary, wave equation in one spatial dimension,

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial t^2},$$

(35)

with a boundary condition $\partial \phi / \partial x = 0$ at $x = 0$. The spatial part of the mode solutions are then of the form $\cos(\omega x)$, multiplied by a time dependent part of the same frequency, $\omega$. If we then impose a no incoming radiation condition, the solution is completely determined, and is constant, so it is not possible to impose initial data in addition. This can be seen more clearly by writing the general solution in the form $f(x - t) + g(x + t)$. The no radiation condition would force us to take $g \equiv 0$ and the brane boundary condition necessitates that $f \equiv \text{constant}$.

For studying the initial perturbation spectrum created in the inflationary era, it is useful to use the no-radiation boundary condition, as was done in [37]. Once we have the perturbations at the end of inflation, we can evolve them forward through the radiation eras, for which we will have initial data on a time-slice, and then through the matter era.

C. Tensor modes

The perturbations in the four-dimensional part of the metric naturally split into scalar, vector and tensor parts with respect to the Fourier transform variable, $k^i$. To linear order in perturbation theory, they do not mix. Of these
the scalar perturbations are the most important since these contain the underdensities and overdensities giving rise to large-scale structure and so necessarily exist. As we have already suggested, it is much easier to solve the equations for the tensor perturbations. We have formulated the equations of motion for all three of these but will only provide solutions for the tensor modes. Some of the lessons learned should assist in tackling the more difficult problem of the scalar modes.

The variation of the purely spatial part of the Ricci tensor is given by $\delta \bar{R}_{ij} = \delta \bar{R}^{0}_{ij} + \delta \bar{R}^{k}_{ikj}$, so the tensor part is given by

$$\delta \bar{R}^{T}_{ij} = \frac{a^2}{2n^2} \bar{h}^{T}_{ij} + \frac{a^2}{2n^2} \left( 3 \frac{\dot{a}}{a} - \frac{\dot{n}}{n} \right) \bar{h}^{T}_{ij} + \frac{a^2}{n^2} \left( \frac{\ddot{a}}{a} - \frac{\dot{a} \dot{n}}{an} + 2 \frac{\dot{n}^2}{a^2} \right) \bar{h}^{T}_{ij} + \frac{k^2}{2} \bar{h}^{T}_{ij},$$

(36)

where the superscript $T$ represents either of the two tensor modes, which will evolve separately, to linear order, and will both obey the same equation of motion. Equations (30) and (31) give us

$$K^{T}_{ij} = \frac{1}{2} \frac{d}{d\zeta} (a^2 \bar{h}^{T}_{ij}),$$

(37)

$$E^{T}_{ij} = \frac{a'}{a} \frac{d}{d\zeta} (a^2 \bar{h}^{T}_{ij}) - a'' \bar{h}^{T}_{ij} - \frac{\Lambda}{6} a^2 \bar{h}^{T}_{ij} - \frac{1}{2} \frac{d}{d\zeta^2} (a^2 \bar{h}^{T}_{ij}).$$

(38)

Substituting these into the tensor part of (29) gives

$$\delta \bar{R}^{T}_{ij} = \left( \frac{2\Lambda}{3} + 2 \frac{a'^2}{a^2} \right) a^2 \bar{h}^{T}_{ij} + \frac{1}{2} \left( \frac{n'}{n} - \frac{a'}{a} \right) \frac{d}{d\zeta} (a^2 \bar{h}^{T}_{ij}) + \frac{1}{2} \frac{d^2}{d\zeta^2} (a^2 \bar{h}^{T}_{ij}).$$

(39)

The purely spatial part of the Einstein equation (11) evaluated for the background metric gives

$$\frac{2\Lambda}{3} = \frac{\ddot{a}}{a} + 2 \frac{a''}{a^2} - \frac{\dot{a} \dot{n}}{an} - \frac{a''}{a} - \frac{\dot{a} \dot{n}}{an} - 2 \frac{a'^2}{a^2},$$

(40)

using this and equating (30) and (31) gives

$$\delta \bar{h}^{T}_{ij} + \left( 3 \frac{\dot{a}}{a} - \frac{\dot{n}}{n} \right) \bar{h}^{T}_{ij} + k^2 \frac{n^2}{a^2} \bar{h}^{T}_{ij} - n^2 \left( \frac{\ddot{a}}{a} + \frac{n'}{n} \right) \frac{d}{d\zeta} \bar{h}^{T}_{ij} - n^2 \frac{d^2}{d\zeta^2} \bar{h}^{T}_{ij} = 0,$$

(41)

in agreement with the result in (18).

The boundary condition on $\bar{h}^{T}_{ij}$ at the brane is given by the perturbation of the purely spatial part of the junction condition (33) which becomes

$$\bar{h}^{T}_{ij} \bigg|_{z=0} = -\kappa \bar{\Sigma}^{T}_{ij}.$$

(42)

The tensor part can be written in terms of two polarization tensors $e^+_{ij}$ and $e^\times_{ij}$ as

$$\bar{h}^{T}_{ij} = h^+ e^+_{ij} + h^\times e^\times_{ij},$$

(43)

so, in subsequent equations, we will simply write $h$ to represent either $h^+$ or $h^\times$.

For most of the history of the universe, there is effectively no matter source for the tensor modes so this condition will reduce to $h' = 0$ on the brane. Only when the CMB photons develop a quadrupole moment in the late universe is this assumption no longer valid.

The wave equation (11) is not separable and is difficult to solve in general, due to the complicated form of the scale factors $a$ and $n$, see (19). We will first study two cases where the equations can be solved exactly: the case of a Minkowski brane and the case of a de Sitter brane. The first of these was the case considered by Randall and Sundrum in their original paper (18); the second is very useful in cosmology for modeling an inflationary era in the early universe (18, 37).

### IV. BASIS SOLUTIONS

#### A. Near-brane limit

Our main interest is in cases which correspond to different cosmological eras. In an inflationary model, the universe will undergo a phase of accelerated expansion where the brane metric will be approximately de Sitter. After a
transition period, there will be an era where the matter content of the universe is dominated by a radiation fluid component, having equation of state $p = \rho/3$. There will then be another transition to evolution dominated by pressureless matter. Shortly after this transition, photons decouple from the cosmological plasma, producing the CMB radiation. Since the eventual aim of studying perturbations is to determine theoretical CMB power spectra for RS models and compare these to the observed CMB spectrum, we should try to solve the problem in each of these eras.

Unfortunately, the equation of motion is not separable in the radiation and matter dominated eras so we will approximate the solution by one valid near the brane, ignoring the $\sinh(\zeta/l)$ term in (19). The range of $\zeta$ for which the near-brane limit is a good approximation depends on the density of the cosmological fluid. The value of $\zeta$ for which $a(\tau, \zeta) = 0$ is given by

$$
\zeta = \frac{l}{2} \log \left( 1 + \frac{2\lambda}{\rho} \right),
$$

(44)

which will give us a qualitative idea of the domain of validity of such a solution. Since $\rho$ will decrease with time in the radiation and matter eras, this domain of validity will get wider later in the history of the universe. The near-brane limit can also be thought of in some sense as a low energy limit, but the fact that it is an approximation suggests that discussion of a low-energy effective theory needs to be carefully thought out.

Let us see where the locus of $a = 0$ intersects the past light cone of a point on the brane today since this will correspond to the limit of causal validity of the near-brane limit. It is simpler to work in proper time on the brane, $t$, so that a null geodesic satisfies

$$
\frac{d\zeta}{dt} = \pm n(t, \zeta) \approx \pm e^{-\zeta/l},
$$

(45)

near the brane. This integrates to give

$$
e^{\zeta/l} = 1 + \frac{t_0 - t}{l},
$$

(46)

for the past light-cone, where $t_0$ is the time today. We want to compare this to (44) to see when we can argue that the approximation is good on causality grounds alone. During a matter dominated era of the universe, the critical energy density is given by

$$
\rho = \frac{1}{6\pi G l^2},
$$

(47)

and the bare tension on the brane is related to the AdS length-scale by $\lambda = 3/(4\pi G l^2)$. So, in the matter dominated era, (44) becomes

$$
e^{\zeta/l} = \left( 1 + \frac{9t^2}{l^2} \right)^{1/2} \approx 3\frac{t}{l},
$$

(48)

which intersects the light-cone when $t \approx t_0/4$. Of course, the approximation used in (45) is good near the brane but breaks down as the null geodesic approaches the line (44) so this is only an order of magnitude estimate. Nonetheless, this means that we cannot use a causality argument to justify the approximation throughout the whole history — we would like to have obtained a time before the end of inflation as the answer! It seems clear that some mode mixing will take place.

Finally, we should point out that the near-brane approximation is also equivalent to an expansion in powers of $(Hl)^2$ (or $\rho/\lambda$). Since it involves ignoring the second term in (19), one could do this either because $\sinh(\zeta/l)$ is small, or because $\rho/\lambda \propto (Hl)^2$ is small.

**B. Minkowski brane**

For the RS case where the projected metric on the brane is Minkowski, the functions $a$ and $n$ have the simple, separable form

$$
a(\tau, \zeta) = n(\tau, \zeta) = e^{-\zeta/l}.
$$

(49)
This simplifies (39), giving
\[ \frac{1}{2} \ddot{h} + \frac{k^2}{2} h = 2 \left( \frac{\Lambda}{3} + \frac{1}{l^2} \right) a^2 h + \frac{1}{2} d^2 \left( a^2 h \right). \]

Or, equivalently, from (41)
\[ \ddot{h} + k^2 h = e^{-2\zeta/l} \left( h'' - \frac{4}{l} h' \right). \]

This equation has a constant solution, corresponding to a metric perturbation proportional to \( e^{-2\zeta/l} \), which is the same as the solution found by Randall and Sundrum. They interpreted this as a graviton zero-mode localised on the brane, thus having the appearance of a four-dimensional gravitational perturbation.

To find a set of basis solutions, we perform a separation of variables, using the ansatz
\[ h(\tau, \zeta) = \psi(\tau) \phi(\zeta). \]

The equation of motion then separates into
\[ \ddot{\psi}_m + \left( k^2 + m^2 a_0^2 \right) \psi_m = 0, \]
\[ \phi_m'' - \frac{4}{l} \phi'_m + m^2 e^{2\zeta/l} \phi_m = 0, \]

where \( m^2 \) is a separation constant. Because \( m^2 \) has dimensions of mass, modes with \( m^2 \neq 0 \) are referred to as massive modes, but it should be noted that \( m^2 \) can be negative. The solutions are
\[ \psi_m(\tau) = c \cos \left( \sqrt{m^2 + k^2} \tau \right) + d \sin \left( \sqrt{m^2 + k^2} \tau \right), \]
\[ \phi_0(\zeta) = C + D e^{\zeta/l}, \]
\[ \phi_m(\zeta) = e^{2\zeta/l} Z_\nu \left( m \rho e^{\zeta/l} \right), \]

where \( c, d, C \) and \( D \) are constants of integration and \( Z_\nu \) represents a linear combination of Bessel functions of order \( \nu \). (The boundary conditions will determine which particular combination to choose.) Randall and Sundrum also showed that, in the Newtonian limit, the contribution from the massive modes is sub–dominant, and so gravity has approximately four-dimensional behaviour.

C. In a de Sitter era

During an era of inflation in the primordial universe, the density of matter is almost constant, and so the spacetime is approximately a de Sitter universe. For this reason it is useful to consider a brane-world where the brane metric takes de Sitter form. For a de Sitter brane, the matter density of the background solution is a constant so scale factors \( a \) and \( n \) are separable functions. We shall work in conformal time, where they have the functional form
\[ a(\tau, \zeta) = n(\tau, \zeta) = a_0(\tau) A(\zeta), \]

where \( a_0(\tau) = a(\tau, 0) \) and \( A(\zeta) \) is given by
\[ A(\zeta) = \left( 1 + \frac{\rho}{2\lambda} \right) e^{-\zeta/l} + \frac{\rho}{2\lambda} e^{\zeta/l}. \]

The wave equation becomes
\[ \ddot{h} + 2 \frac{\dot{a}_0}{a_0} \dot{h} + k^2 h = A^2 h'' + 4 A' Ah'. \]

We can find a set of basis solutions to this by performing the same separation of variables used in the RS case. Writing \( h = \psi(\tau) \phi(\zeta) \), the wave equation splits into two parts:
\[ \ddot{\psi}_m + 2 \frac{\dot{a}_0}{a_0} \dot{\psi}_m + \left( k^2 + m^2 a_0^2 \right) \psi_m = 0, \]
\[ \phi_m'' - \frac{4}{A} \phi'_m + \frac{m^2}{A^2} \phi_m = 0, \]

where \( \psi_m(\tau) = \cos \left( \sqrt{m^2 + k^2} \tau \right) + \sin \left( \sqrt{m^2 + k^2} \tau \right), \]
\[ \phi_0(\zeta) = C + D e^{\zeta/l}, \]
\[ \phi_m(\zeta) = e^{2\zeta/l} Z_\nu \left( m \rho e^{\zeta/l} \right), \]

where \( m^2 \) is a separation constant. Because \( m^2 \) has dimensions of mass, modes with \( m^2 \neq 0 \) are referred to as massive modes, but it should be noted that \( m^2 \) can be negative. The solutions are
\[ \psi_m(\tau) = c \cos \left( \sqrt{m^2 + k^2} \tau \right) + d \sin \left( \sqrt{m^2 + k^2} \tau \right), \]
\[ \phi_0(\zeta) = C + D e^{\zeta/l}, \]
\[ \phi_m(\zeta) = e^{2\zeta/l} Z_\nu \left( m \rho e^{\zeta/l} \right), \]

where \( c, d, C \) and \( D \) are constants of integration and \( Z_\nu \) represents a linear combination of Bessel functions of order \( \nu \). (The boundary conditions will determine which particular combination to choose.) Randall and Sundrum also showed that, in the Newtonian limit, the contribution from the massive modes is sub–dominant, and so gravity has approximately four-dimensional behaviour.
where \( m^2 \) is a separation constant as before.

In a de Sitter era \( \rho \) is a constant so, in conformal time, we have \( a(\tau, 0) = -(H_{\text{inf}} \tau)^{-1} \) where \( H_{\text{inf}} \) is the energy scale for inflation, giving the evolution of the tensors as

\[
\dddot{\psi}_m - \frac{2}{\tau} \dot{\psi}_m + \left( k^2 + \frac{m^2}{H_{\text{inf}}^2 \tau^2} \right) \psi_m = 0 ,
\]

where \( m^2 \) is a separation constant as before.

In a de Sitter era \( \rho \) is a constant so, in conformal time, we have

\[
a(\tau, 0) = -\left( H_{\text{inf}} \tau \right)^{-1} - 1
\]

where \( H_{\text{inf}} \) is the energy scale for inflation, giving the evolution of the tensors as

\[
\dddot{\psi}_m - \frac{2}{\tau} \dot{\psi}_m + \left( k^2 + \frac{m^2}{H_{\text{inf}}^2 \tau^2} \right) \psi_m = 0 ,
\]

The solutions for \( \psi \) are

\[
\psi = \tau^{3/2} \left( c\tau^\alpha + d\tau^{-\alpha} \right) ,
\]

for \( k = 0 \), and

\[
\psi = \tau^{3/2} \left( cJ_\alpha(k\tau) + dY_\alpha(k\tau) \right) ,
\]

for \( k \neq 0 \), where \( \alpha = \sqrt{9 - (4m^2/H_{\text{inf}}^2)/2} \) and \( c \) and \( d \) are constants of integration. The equation for \( \phi \) reduces to a hypergeometric equation, which we can be solved exactly, but the solution is very complicated and not particularly informative. Let us state briefly the \( m = 0 \) solution, which is

\[
\phi_0 = C + De^{-4\zeta/l}.
\]

This de Sitter case has been studied extensively in the literature. It was found that the Kaluza–Klein modes are not continuous, but there is a mass gap between the zero mode and the first massive mode. If one requires that \( \alpha \) is pure imaginary then \( m > \left( 3/2 \right) H_{\text{inf}} \). Above that mass the spectrum is continuous. In [18] it was argued that the tensor perturbation left after an inflationary era would mainly be comprised of the zero-mode solution, \( h = \psi_0(\tau)\phi_0(\zeta) \), and this result was made more quantitative in [37] where Bogoliubov coefficients were calculated. Thus, the perturbations produced from a period of inflation will be the zero-mode, \( h = \psi_0(\tau)\phi_0(\zeta) \), to good approximation. This has been extended to the case of any conformally flat brane metric in [38].

**D. In the radiation era**

Let us now consider the evolution of the tensor perturbations in the radiation-dominated epoch. As mentioned earlier, we cannot solve the equations exactly and so we use the near-brane approximation. As for the de Sitter era, we work in conformal time, where the the scale factors \( a \) and \( n \) have the form

\[
a(\tau, \zeta) = n(\tau, \zeta) = a_0(\tau)e^{-\zeta/l}.
\]

We are assuming that the density is small compared to the bare tension of the brane in order for the near-brane solution to be a good approximation, so we can approximate the Friedmann equation by the usual, four-dimensional one

\[
\left( \frac{da}{d\tau} \right)^2 = \frac{8\pi G}{3} \rho a^4,
\]

which, if we choose the origin of time appropriately, has solution \( a = A\tau \) for some constant \( A \).

Our near-brane approximation allows us to separate variables, so we try to find solutions of the form \( h = \psi(\tau)\phi(\zeta) \) as before. The wave equation then splits into two equations of motion

\[
\dddot{\psi}_m - \frac{2}{\tau} \dot{\psi}_m + \left[ k^2 + \frac{m^2}{A^2 \tau^2} \right] \psi_m = 0 ,
\]

\[
\dddot{\phi}_m + \frac{4}{\tau} \phi_m' + \frac{m^2}{A^2} e^{2\zeta/l} \phi_m = 0 .
\]

The solution of (70) is independent of \( k \) and is given by

\[
\phi_0(\zeta) = C + De^{4\zeta/l} ,
\]

\[
\phi_m(\zeta) = e^{2\zeta/l} Z_2 \left( ml e^{\zeta/l} \right) ,
\]

\[(66)\]
where $C$ and $D$ are integration constants, and, as before, $Z$ represents a linear combination of Bessel functions of order $\nu$ with the boundary condition determining the particular combination. Assuming we have no source for the tensor perturbations, the boundary condition is simply $\phi' = 0$, which gives us basis solutions

$$\phi_m(\zeta) = e^{2\zeta/l} \left\{ Y_1(ml)J_2\left(ml\zeta/l\right) - J_1(ml)Y_2\left(ml\zeta/l\right) \right\}, \quad (73)$$

with $J_\nu$ and $Y_\nu$ being, respectively, Bessel functions of the first and second kind.

When $k = 0$, the solutions to (69) are

$$\psi_0(\tau) = c + \frac{d}{\tau}, \quad (74)$$

$$\psi_m(\tau) = \tau^{-1/2} Z_{1/4} \left( \frac{1}{2} mA \tau^2 \right), \quad (75)$$

and when $k \neq 0$ the solution for $m = 0$ is

$$\psi_0(\tau) = c \frac{\sin k\tau}{\tau} + d \frac{\cos k\tau}{\tau}, \quad (76)$$

where $c$ and $d$ are constants of integration.

For $m \neq 0$, the solutions can be written in terms of parabolic cylinder functions as follows. Writing $\psi = y/\tau$ and changing the independent variable to $t = k\tau$, (69) becomes

$$\frac{d^2 y}{dt^2} + \left( 1 + \frac{m^2 A^2}{k^4} t^2 \right) y = 0 \quad (77)$$

which is Weber’s equation and has parabolic cylinder functions as solutions

$$D_\nu \left( \pm(1 + i)\sqrt{mA} k^{-1} t \right), \quad \text{where} \quad \nu_m = \frac{k^2}{2mA} - \frac{1}{2}. \quad (78)$$

We can use the asymptotic form

$$D_\nu(z) \sim e^{-z^2/4} z^\nu, \quad \text{for} \quad |z| \gg 1, \quad |z| \gg |\nu|, \quad |\arg z| < 3\pi/4, \quad (79)$$

to approximate the solutions by

$$y_m \sim \frac{e^{k^2\pi/(8mA)}}{(2mA)^{1/4}\sqrt{\tau}} \exp \left\{ -\frac{i}{2} mA \tau^2 - \frac{i k^2}{2mA} \log \tau \right\}, \quad \text{as} \quad \tau \to \infty \quad (80)$$

which is used in Appendix A to derive the orthogonality relation. If $k = 0$ this can seen to have the same form as (74) as $\tau \to \infty$. The general solutions can then be written as a superposition of these modes of the form

$$h = C_0^+ \frac{\sin k\tau}{\tau} + C_0^- \frac{\cos k\tau}{\tau} + \sum \int_{\pm}^\infty \frac{C_m^\pm}{\tau} D_\nu_m \left( \pm(1 + i)\sqrt{mA} \tau \right) \left\{ Y_1(ml)J_2\left(ml\zeta/l\right) - J_1(ml)Y_2\left(ml\zeta/l\right) \right\} \, dm. \quad (81)$$

The evolution of the zero-mode is exactly the same as for the usual, four-dimensional cosmology based on General Relativity. For all values of $k$, the very heavy modes decay outside the horizon.

Since special functions of this nature are not always easy to visualize, let us try to approximate to solution to (77) using the WKB method. First define the dimensionless parameter

$$\epsilon = \frac{k^2}{mA} \quad (82)$$

The WKB approximation for $\epsilon$ small recovers the asymptotic expansion given above. When $\epsilon$ is large, we rescale the time coordinate by $T = t/\epsilon = mA\tau/k$, so that (77) is rewritten as

$$\frac{1}{\epsilon^2} \frac{d^2 y}{dT^2} + (1 + T^2) y = 0 \quad (83)$$
We will use the formulation of the WKB approximation as described in [39], where a small parameter $\delta$ is introduced and an asymptotic solution of the form

$$\exp\left\{ \frac{1}{\epsilon^2} \sum_{n=0}^{\infty} \delta^n S_n(T) \right\},$$

(84)
is postulated and the $S_n$ are functions to be determined. The parameter $\delta$ will be determined in terms of $\epsilon$ by a distinguished limit, i.e., the choice for which a non-trivial answer is obtained. Substituting (84) into the differential equation gives

$$\frac{1}{\epsilon^2 \delta^2} \left( \sum \delta^n \dot{S}_n \right)^2 + \frac{1}{\epsilon^2 \delta} \sum \delta^n \ddot{S}_n = -1 - T^2$$

(85)
The term proportional to $(\epsilon \delta)^{-2}$ is the leading term on the left and so must match the right hand side. Hence, we see that $\delta = 1/\epsilon$ is the distinguished limit and

$$\dot{S}_0 = \pm i \sqrt{1 + T^2},$$

(86)
The coefficients of the other powers of $\epsilon$ give differential equations for the other $S_n$: for the next three, we have

$$\dot{S}_1 = -\frac{T}{2(1 + T^2)}, \quad \dot{S}_2 = \pm i \frac{3T^2 - 2}{8(1 + T^2)^{3/2}}, \quad \dot{S}_3 = \frac{3T(2T^2 - 3)}{8(1 + T^2)^{4}},$$

(87)
The solutions to these equations are

$$S_0 = \pm \frac{i}{2} \left( T \sqrt{1 + T^2} + \text{arcsinh}(T) \right),$$

(88)
$$S_1 = -\frac{1}{4} \log(1 + T^2),$$

(89)
$$S_2 = \pm \frac{i}{24} \frac{T(T^2 + 6)}{(1 + T^2)^{3/2}},$$

(90)
$$S_3 = \frac{2 - 3T^2}{16(1 + T^2)^3},$$

(91)
which will give an approximation to $y_m$. Note that constants of integration have been omitted because they merely correspond to multiplying the solution by a constant factor.

It is useful to consider the early and late time behaviour of this approximation. When $T$ is large, the asymptotic expansion (80) is recovered. When $T$ is small, the $S_n$ are approximately

$$S_0 \approx \pm iT, \quad S_1 \approx -\frac{1}{4} T^2, \quad S_2 \approx \pm \frac{i}{4} T, \quad S_3 \approx -\frac{9}{16} T^2,$$

(92)
so the solution is

$$y_m \approx \exp\left[ -\frac{T^2}{4} \left( 1 + 9 \frac{\epsilon^2}{4} \right) \right] \exp\left[ \pm iT \left( \epsilon - \frac{1}{4} \epsilon \right) \right],$$

(93)
which is a good approximation for $k\tau \ll \epsilon$. We see that the solution which is bounded at the origin is

$$\psi_m \approx \frac{1}{T} \exp\left[ -\frac{T^2}{4} \left( 1 + 9 \frac{\epsilon^2}{4} \right) \right] \sin\left[ T \left( \epsilon - \frac{1}{4} \epsilon \right) \right],$$

(94)
which is flat near the origin and then begins to decay.

E. In the matter era

We now move on to the final phase of the evolution of cosmological perturbations when the universe is dominated by a pressureless fluid. As with the radiation era, we approximate the solution near the brane, where the scale factors $a$ and $n$ have the form (87). Again, we are quite late in the history of the universe, so (80) is valid to good
approximation and we have \( a = A\tau^2 \) if the origin of time is chosen appropriately. Making the ansatz \( h = \psi(\tau)\phi(\zeta) \), the equations of motion separate once again and the equation for \( \psi \) is
\[
\ddot{\psi}_m + \frac{4}{\tau} \dot{\psi}_m + \left[ k^2 + m^2 A^2 \tau^4 \right] \psi_m = 0 . \tag{95}
\]
The equation for \( \phi \) and its solutions are the same as in the radiation era \( \tau \). For \( k = 0 \), we can solve (95) exactly, giving
\[
\psi_0(\tau) = c + \frac{d}{\tau^3} , \tag{96}
\]
\[
\psi_m(\tau) = \frac{c}{\tau^3} \cos \left( \frac{1}{3} mA \tau^3 \right) + \frac{d}{\tau^3} \sin \left( \frac{1}{3} mA \tau^3 \right) . \tag{97}
\]
For \( k \neq 0 \), the zero-mode can be found exactly
\[
\psi_0(\tau) = \frac{c}{\tau^3} \left( k \tau \sin(k\tau) + \cos(k\tau) \right) + \frac{d}{\tau^3} \left( k \tau \cos(k\tau) - \sin(k\tau) \right) . \tag{98}
\]
We will study the modes with \( k \neq 0 \) and \( m \neq 0 \) by constructing approximate solutions, which will be appropriate in various limits. First, we will define the dimensionless variable
\[
\varepsilon = \frac{k^3}{mA} , \tag{99}
\]
and two rescaled time variables \( t = k\tau \) and \( T = \varepsilon^{-1/2} t = (mA/k)^{1/2} \tau \) which we will use in the limits \( \varepsilon \ll 1 \) and \( \varepsilon \gg 1 \) respectively. The equations of motion for \( \psi \) are then
\[
\varepsilon^2 \left( \frac{d^2 \psi}{dt^2} + \frac{4}{t} \frac{d\psi}{dt} + \psi \right) + t^4 \psi = 0 , \tag{100}
\]
and
\[
\varepsilon^{-1} \left( \frac{d^2 \psi}{dT^2} + \frac{4}{T} \frac{d\psi}{dT} \right) + (1 + T^4) \psi = 0 . \tag{101}
\]

Case \( \varepsilon \ll 1 \):

Let us consider first the case where \( \varepsilon \) is small and use (100). We can then find a solution using the WKB approximation as discussed in the previous section. We try to find an asymptotic solution of the form (84), which, when substituted into (100) gives
\[
\frac{\varepsilon^2}{\delta^2} \left( \sum \delta^n \dot{S}_n \right)^2 + \frac{\varepsilon^2}{\delta} \sum \delta^n \ddot{S}_n + \frac{4}{t} \frac{\varepsilon^2}{\delta} \sum \delta^n \dot{S}_n + \varepsilon^2 = -t^4 . \tag{102}
\]
We see that the distinguished limit is when \( \varepsilon = \delta \) and that the \( S_n \) are given by the differential equations
\[
\dot{S}_0 = -t^4 , \tag{103}
\]
\[
2\dot{S}_0 \dot{S}_1 + \ddot{S}_0 + \frac{4}{t} \dot{S}_0 = 0 , \tag{104}
\]
\[
2\dot{S}_0 \dot{S}_2 + \ddot{S}_1 + \frac{4}{t} \dot{S}_1 + 1 = 0 , \tag{105}
\]
\[
2\dot{S}_0 \dot{S}_n + \sum_{j=1}^{n-1} \ddot{S}_j \ddot{S}_{n-j} + \dddot{S}_{n-1} + \frac{4}{t} \dot{S}_{n-1} = 0 , \tag{106}
\]
for the first four terms. Solving for these, we find
\[
S_0 = \pm \frac{1}{3} it^4 , \quad S_1 = \log(t^{-3}) , \quad S_2 = \pm \frac{i}{2t} , \quad S_3 = -\frac{5}{16} . \tag{107}
\]
Substituting the first three of these into [S31] along with the expressions for $\varepsilon$ and $t$, the solution with the first correction term is

$$
\psi = \frac{e}{\tau^3} e^{-5/(m^2 A^2 \tau^6)} \cos \left( \frac{1}{3} m A T^3 - \frac{k^2}{2 m A \tau} \right) + \frac{d}{\tau^3} e^{-5/(m^2 A^2 \tau^6)} \sin \left( \frac{1}{3} m A T^3 - \frac{k^2}{2 m A \tau} \right),
$$

(108)

This solution is good for small $k$ and is progressively better for larger values of $\tau$. We see that in the limit $k \to 0$ this solution reverts to the form [97] as one would expect.

Case $\varepsilon \gg 1$:

Now consider the opposite limit where $\varepsilon$ is large using [101] which is in a form suitable for using the WKB method as formulated above. We get $\delta = \varepsilon^{-1/2}$ and the following differential equations,

$$
\dot{S}_0 = \pm i \sqrt{1 + T^4},
$$

(109)

$$
2 \dot{S}_0 \dot{S}_1 + S_0 + 4 T^{-1} \ddot{S}_0 = 0,
$$

(110)

$$
2 \dot{S}_0 \dot{S}_2 + \ddot{S}_1 + 4 T^{-1} \ddot{S}_1 = 0,
$$

(111)

$$
2 \dot{S}_0 \dot{S}_3 + 2 \dot{S}_1 \ddot{S}_2 + \ddot{S}_2 + 4 T^{-1} \ddot{S}_2 = 0,
$$

(112)

for the first four terms; the last three equations simplify to

$$
\dot{S}_1 = - \frac{T^3}{1 + T^4} - \frac{2}{T}, \quad \pm \dot{S}_2 = \frac{7 T^4 + 2}{2 T^2 (1 + T^4)^{3/2}}, \quad \dot{S}_3 = - \frac{35 T^8 + 7 T^4 + 2}{T^3 (1 + T^4)^4},
$$

(113)

These have solutions

$$
\pm S_0 = i \frac{3 T^3}{1 + T^4} + \frac{2}{3} e^{i \pi/4} F \left( e^{i \pi/4} T, i \right),
$$

(114)

$$
S_1 = - \log \left( T^2 (1 + T^4)^{1/4} \right),
$$

(115)

$$
\pm S_2 = \frac{i T^3}{T} - \frac{5 i T^3}{8 \sqrt{1 + T^4}} - \frac{7}{8} e^{-i \pi/4} \left[ F \left( e^{i \pi/4} T, i \right) - E \left( e^{i \pi/4} T, i \right) \right],
$$

(116)

$$
S_3 = \frac{7 T^4 + 2}{2 T^2 (1 + T^4)^{3/2}},
$$

(117)

where

$$
F \left( e^{i \pi/4} T, i \right) = \int_0^T \frac{dx}{\sqrt{1 + x^2}}, \quad E \left( e^{i \pi/4} T, i \right) = \int_0^T \sqrt{\frac{1 + ix^2}{1 - ix^2}} dx
$$

(118)

are, respectively, elliptic integrals of the first and second kinds. These expressions are quite complicated, so let us consider the two limits where $T$ is small or large.

When $T$ is small, the differential equations [109 112] are

$$
\dot{S}_0 \approx \pm i, \quad \dot{S}_1 \approx - \frac{2}{T}, \quad \dot{S}_2 \approx \pm \frac{i}{T^2}, \quad \dot{S}_3 \approx - \frac{2}{T^3},
$$

(119)

so that the solution is approximately

$$
\psi(T) \approx \frac{e^{1/(2 T^2)}}{T^2} \exp \left[ \pm i \left( \sqrt{\varepsilon} T + \frac{1}{\sqrt{\varepsilon} T} \right) \right] \propto \frac{e^{1/(k \tau^2)}}{\tau^2} \exp \left[ \pm i \left( k \tau + \frac{1}{k \tau} \right) \right]
$$

(120)

which should be a good approximation if $1 \ll k \tau \ll \sqrt{\varepsilon}$. This approximate solution for small $T$ does not depend on $m$, which is a consequence of the fact that the term involving $m$ is the equation of motion has a $T^3$ dependence. When $T$ is large equations [109 112] become

$$
\dot{S}_0 \sim \pm i T^2, \quad \dot{S}_1 \sim - \frac{3}{T}, \quad \dot{S}_2 \sim \pm \frac{7 i}{2 T^8}, \quad \dot{S}_3 \sim - \frac{35}{T^{11}},
$$

(121)
giving us
\[
\psi(T) \approx \frac{e^{\gamma/(2\epsilon T^\alpha)}}{T^3} \exp \left[ \pm i \left( \frac{1}{3} \sqrt{\epsilon} T^3 + \frac{1}{2 \sqrt{\epsilon} T^7} \right) \right] \propto \frac{1}{T^3} \exp \left( \frac{7 \kappa^2}{2 \kappa^4 A^4 T^{11}} \right) \exp \left[ \pm i \left( \frac{1}{3} m A T^3 + \frac{k^2}{2 m^4 A^4 T^7} \right) \right]
\]
valid for \( \kappa T \gg \sqrt{\epsilon} \).

V. IMPOSING THE INITIAL CONDITIONS

We are considering tensor perturbations here and since the matter on the brane has no tensor perturbation early in the universe, there is no matter source for these perturbations in the metric. So the boundary condition at the brane is \( h' = 0 \). The \( \zeta \) dependence of the solution near the brane is the same in all eras, and its evolution is given by \((70)\), with solutions given in \((71)\) and \((72)\). The derivative of \( \phi_m(\zeta) \) is

\[
\phi'_m(\zeta) = m e^{3\zeta/l} Z_1(m e^{\zeta/l}),
\]
so, if the junction condition is imposed, the massive modes are

\[
\phi_m(\zeta) = e^{2\zeta/l} \left( Y_1(m l) J_2(m e^{\zeta/l}) - J_1(m l) Y_2(m e^{\zeta/l}) \right),
\]
for which

\[
\phi'_m(\zeta) = m e^{3\zeta/l} \left( Y_1(m l) J_1(m e^{\zeta/l}) - J_1(m l) Y_1(m e^{\zeta/l}) \right).
\]
The zero-mode is simple a constant

\[
\phi_0(\zeta) = C,
\]
when the boundary condition is imposed.

Let us now try to impose initial conditions at some initial time, \( \tau_i \), which will, in practice, be the transition between two eras. This initial data must satisfy the boundary condition and will be expressible as a superposition of the mode solutions \( \phi_m(\zeta) \). This superposition is most easily calculated by considering the \( \zeta \) derivative of the initial data. So let us write

\[
h'(\tau_i, \zeta) = \int_0^\infty A(m) \phi'_m(\zeta) \, dm = \int_0^\infty m A(m) e^{3\zeta/l} C_1(m; le^{\zeta/l}) \, dm,
\]
where \( C_\nu(m; z) \) is the cylinder function

\[
C_\nu(m; z) = Y_1(m l) J_\nu(m z) - J_1(m l) Y_\nu(m z).
\]
We can deduce the \( A(m) \) from the orthogonality relation given in the appendix.

The contribution from the zero-mode is just a constant, and this is, of course, the bit that is undetermined by considering \( h' \). So we have a prescription for imposing any given initial conditions. The prescription given is somewhat academic, at least for the application under consideration here, because the spatial parts of the mode functions, \( \phi_m(\zeta) \), are the same in all eras, so the spectrum of modes present at the beginning of an era will be the same as the spectrum at the end of the previous era. In particular if inflation produces a perturbation which is well approximated by the zero-mode, as indeed it does according to \([18, 37]\), then only the zero-mode is present during subsequent eras too. Thus the evolution of tensor perturbations is the same as in the standard, four-dimensional case, and hence the contribution towards CMB anisotropies is the same. The absence of sources is responsible for this similarity, tensor matter perturbations on the brane would source the massive modes and give an answer different from the standard cosmology. The analysis presented here should extend to the scalar perturbations, although there are some additional technical difficulties. The scalar perturbations are not only the dominant contributions to the CMB anisotropies but are sourced by scalar matter perturbations so there will very probably be some signature of the RS model detectable in the CMB power spectrum.
VI. DISCUSSION AND CONCLUSIONS

In this article we have discussed some aspects of tensor perturbations in the Randall–Sundrum brane-world. The problem can be formulated in several coordinate systems, each of which has its shortcomings. In GN coordinates, the brane is static and the boundary condition can easily be imposed. However, the equations of motion for tensor perturbations are not separable. By formulating the problem in static bulk coordinates in which the brane moves, the bulk equation can be solved but the boundary conditions are difficult to impose on the (moving) brane. Therefore, an approximation scheme is necessary to make progress in both systems. We have presented solutions in the de Sitter era, the radiation and matter dominated era using the near-brane approximation first discussed in [32, 33].

The near-brane approximation is akin to an adiabatic treatment of the motion of the brane in that the warp factor \(a(\tau, \zeta)\) has the same \(\zeta\) dependence as in the standard Randall–Sundrum picture for a Minkowski brane, but is multiplied by the cosmological scale factor. In this lowest-order approximation, independent modes persist throughout the radiation and matter eras. If this approximation is universally valid, which we have argued against on causality grounds, and the initial spectrum includes only the zero mode created during inflation then the observed tensor spectrum will be exactly that of the standard cosmological model.

The zero mode is equivalent to the standard case of General Relativity, but the massive modes have some novel features. In particular they can be seen to decay and oscillate outside the horizon, whereas standard massless gravitons only begin to oscillate at horizon crossing. One important consequence of this is that initial power spectrum of fluctuations in the massive modes cannot be scale invariant as \(k \to 0\). There must exist a cut-off in the spectrum at very small values of \(k\). If one thinks of the perturbations as being particles created quantum mechanically during inflation then one can create those with \(k \ll m\) just on energetic grounds. Moreover, if massive modes exist and they decay outside the horizon, the photon quadrupole will increase without the need for Thomson scattering since it will be sourced by the decay of the super-horizon gravitational potential. This could lead to a significant polarization signature in the angular power spectrum on large angular scales.

It is worth noting that the equations of motion for the massive modes which we have solved here are equivalent to those of ordinary massless gravity with a mass term added to the Lagrangian in an ad hoc way via an addition term of the form

\[
\mathcal{L}_{\text{mass}} = -m^2 \left( h_{\mu\nu} h^{\mu\nu} - h^2 \right),
\]

where \(m\) is the graviton mass and \(h_{\mu\nu}\) is the 4D metric perturbation. Although such a theory is believed to suffer from the van Dam-Veltman discontinuity [44] in the graviton propagator in going from \(m = 0\) to \(m > 0\), this theorem will not apply in the RSII case where there is likely to be a spectrum of modes including the zero mode.

It should be possible to incorporate non-adiabatic effects, by taking into account the non-separability of the equations. We anticipate that this will lead to mixing of the near-brane modes discussed in this paper, as considered in [45, 46]. This is likely to lead to non-trivial time evolution of the total perturbation amplitude and interesting cosmological signatures. The solutions found in this paper can be used to gain insights into the full problem. The non-linearity in the wave equation induced next-order correction to the expressions (17) and (18) for the scale factors \(a(\tau, \zeta)\) and \(n(\tau, \zeta)\) is the origin of the mode mixing phenomena in [46].

The inclusion of a tensor matter component on the brane will source massive modes. The main source of this will be the photon quadrupole which develops late in the history of the universe. This issue will even more important in the case of scalar perturbations since the perturbations in the density, pressure and velocity will contribute to this. Some of the methods discussed here should be of use in taking this effect into account.

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APPENDIX: ORTHOGONALITY RELATIONS

Sturm–Liouville theory tells us that our basis functions will be orthogonal, but we need to know the normalization. We will derive the orthogonality here explicitly for the Bessel and parabolic cylinder functions used and determine the coefficient of the delta function in the orthogonality relation.
There is a standard formula for Bessel functions which tells us that
\[
\int x D_\nu(mx) \hat{D}_\nu(nx) \, dx = \frac{x}{m^2 - n^2} \left( m D_{\nu+1}(mx) \hat{D}_\nu(nx) - n D_\nu(mx) \hat{D}_{\nu+1}(nx) \right), \tag{A.1}
\]
where \( D_\nu \) and \( \hat{D}_\nu \) are any two Bessel functions of order \( \nu \) and \( m \neq n \). This can be derived by an argument given in [40]. If we consider two functions \( g_1(x) \) and \( g_2(x) \) which satisfy the differential equations \( g''_n + P_n(x) g_n = 0 \) for some functions \( P_n \), then
\[
\int (P_1 - P_2) g_1 g_2 \, dx = g_1' g_2' - g_1 g_2'. \tag{A.2}
\]
Applying this to Bessel's equation, we see that
\[
\int_{x_0}^\infty x D_\nu(mx) \hat{D}_\nu(nx) \, dx \propto \delta(m - n), \tag{A.3}
\]
if \( D_\nu(nx) \) and \( \hat{D}_\nu(nx) \) are chosen to vanish, or have vanishing derivatives, on the boundary. We are interested in the specific case
\[
D_\nu(mx) = C_\mu(m;x) = Y_1(ml) J_\nu(mx) - J_1(ml) Y_\nu(mx), \tag{A.4}
\]
with \( \nu = 1 \), and
\[
C_\mu(m;x) \sim \sqrt{\frac{2}{\pi mx}} \left( Y_1(ml)^2 + J_1(ml)^2 \right) \sin(mx + \text{phase angle}), \tag{A.5}
\]
asymptotically, when \( x \) is large. So the constant of proportionality can be deduced [41] to be
\[
\int_{x_0}^\infty D_\nu(mx) \hat{D}_\nu(nx) \, dx = \frac{1}{m} \left( Y_1(ml)^2 + J_1(ml)^2 \right) \delta(m - n). \tag{A.6}
\]
For the time-dependent parts of the mode solutions, \( \psi_m(\tau) = \tau^{-1} y_m(\tau) \), we can construct a similar orthogonality relation. The \( y_m \) satisfy the differential equation
\[
\frac{d^2 y_m}{dt^2} + \left( 1 + \frac{m^2 A^2}{k^2} \right) y_m = 0, \tag{A.7}
\]
so applying the same argument as before gives
\[
\frac{A^2}{k^2} (m^2 - n^2) \int t^2 y_m y_n \, dt = y_m y_n - y_m' y_n', \tag{A.8}
\]
where the r.h.s. will be zero due to the boundary conditions. To evaluate \( \int t^2 y_m y_n \, dt \) when \( m = n \), we can use the asymptotic form [30]
\[
y_m \sim \frac{e^{k^2 \pi/8mA^2}}{(2mA)^{1/4} \sqrt{\tau}} \exp \left\{ -\frac{i}{2} m A \tau^2 - \frac{ik^2}{2mA} \log \tau + i \theta_m \right\}, \tag{A.9}
\]
to write the integrand as
\[
\frac{e^{k^2 \pi/8mA^2} e^{k^2 \pi/8nA}}{\sqrt{2A(mn)^{1/4}}} \, kt \exp \left\{ -\frac{i}{2} (m + n) A t^2 - \frac{ik^2}{2mA} \log(k^{-1}t) - \frac{ik^2}{2nA} \log(k^{-1}t) \right\}. \tag{A.10}
\]
Neglecting the log terms in the exponential and making the change of variables \( x = A t^2 \)
\[
\int t^2 y_m y_n \, dt \sim \frac{k e^{k^2 \pi/8mA^2} e^{k^2 \pi/8nA}}{(2A)^{3/2} (mn)^{1/4}} \int e^{-imx/2} e^{-inx/2} \, dx \tag{A.11}
\]
which is asymptotically the same as \( \int x J_\mu(mx/2) J_\mu(nx/2) \), allowing us to calculate the coefficient of the delta function.

[1] P Brax and C van de Bruck (2003) Class. Quant. Grav. 20 R201
We are using the conventions of Misner, Thorne and Wheeler throughout. In particular, many authors use the opposite sign convention for the extrinsic curvature tensor.

Note that the solutions given in \[ \text{Ref.} \] are for the, more general, Schwarzschild-AdS spacetime and so look more complicated.