A Framework for Bounding Nonlocality of State Discrimination

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Abstract: We consider the class of protocols that can be implemented by local quantum operations and classical communication (LOCC) between two parties. In particular, we focus on the task of discriminating a known set of quantum states by LOCC. Building on the work in the paper Quantum nonlocality without entanglement (Bennett et al., Phys Rev A 59:1070–1091, 1999), we provide a framework for bounding the amount of nonlocality in a given set of bipartite quantum states in terms of a lower bound on the probability of error in any LOCC discrimination protocol. We apply our framework to an orthonormal product basis known as the domino states and obtain an alternative and simplified proof that quantifies its nonlocality. We generalize this result for similar bases in larger dimensions, as well as the “rotated” domino states, resolving a long-standing open question (Bennett et al., Phys Rev A 59:1070–1091, 1999).

Contents

1. Introduction .................................. 1122
2. Background ................................. 1123
   2.1 Notation .................................. 1123
   2.2 Separable and LOCC measurements .. 1124
      2.2.1 Separable measurements .......... 1124
      2.2.2 LOCC measurements ............ 1124
      2.2.3 Finite and asymptotic LOCC . 1125
      2.2.4 LOCC protocol as a tree ....... 1126
   2.3 Bipartite state discrimination problem . 1126
   2.4 Previous results ......................... 1127
3. Framework .................................. 1129
   3.1 Interpolated LOCC protocol .......... 1129
   3.2 Stopping condition .................... 1131
   3.3 Measure of disturbance .............. 1132
1. Introduction

The 1999 paper *Quantum nonlocality without entanglement* [BDF+99] exhibits an orthonormal basis $S \subseteq \mathbb{C}^3 \otimes \mathbb{C}^3$ of product states, known as domino states, shared between two separated parties. When the parties are restricted to perform only local quantum operations and classical communication (LOCC), it is impossible to discriminate the domino states arbitrarily well [BDF+99]. In such cases we say that perfect discrimination cannot be achieved with asymptotic LOCC. Moreover, [BDF+99] also quantifies the extent to which any LOCC protocol falls short of perfect discrimination of the domino states.

This result spurred interest in state discrimination with LOCC. Several alternative proofs [WH02,GV01,Coh07] of the impossibility of perfect LOCC discrimination of the domino states were given along with many other results concerning perfect state discrimination (e.g., [BDM+99,WSHV00,GKR+01,GV01,CY01,CY02,WH02,DMS+03,CL03,HSSH03,Ham03,Fan04,GKRS04,Che04,CL04,JCY05,Wat05,Nat05,NC06,DFJY07,FS09,DFXY09,DXY10]). However, the problem of asymptotic LOCC state discrimination has not received much attention since the initial study of nonlocality without entanglement [BDF+99].

The main motivation for our work is to better understand the phenomenon of quantum nonlocality without entanglement. More concretely, our goals are to

- simplify the original proof,
- render the technique applicable to a wider class of sets of bipartite states,
- exhibit new classes of product bases that cannot be asymptotically (as opposed to just perfectly) discriminated with LOCC,
- pin down where exactly the difference between LOCC and separable operations lies, and
- investigate the possibility of larger gaps between the sets of LOCC and separable operations.
In particular, we seek to exhibit quantitative gaps between the classes of LOCC and separable operations. Separable operations often serve as a relaxation of LOCC operations and such gaps show how imprecise this relaxation can be. The rationale behind this relaxation is that separable operations have a clean mathematical description whereas LOCC operations can be much harder to understand.

There is also an operational motivation to quantify the difference between separable measurements and those implemented by asymptotic LOCC: the former are precisely the measurements that cannot generate entangled states, while the latter are those that do not require entanglement to implement \([\text{BDF}^*99, \text{KTYI}07, \text{Koa}09]\). Thus, a separable measurement that cannot be implemented by asymptotic LOCC uses entanglement irreversibly.

**Our contributions.** In this paper, we develop a framework for obtaining quantitative results on the hardness of quantum state discrimination by LOCC. More precisely, we provide a method for proving a lower bound on the error probability of any LOCC measurement for discriminating states from a given set \(S\). Any strictly positive lower bound implies that the states from \(S\) cannot be even asymptotically discriminated with LOCC.

Our first main contribution (Theorem 10) is that any LOCC measurement for discriminating states from a set \(S\) errs with probability \(p_{\text{error}} \geq \frac{2}{27}\frac{\eta^2}{|S|^5}\), where \(\eta\) is a constant that depends on \(S\) (see Definition 7). Intuitively, \(\eta\) measures the nonlocality of \(S\).

Our second main contribution is a systematic method for bounding the nonlocality constant \(\eta\) for a large class of product bases. Together with the above theorem, this lets us quantify the hardness of LOCC discrimination for the following bases of product states:

1. **domino states**, the original set of nine states in \(3 \times 3\) dimensions first considered in \([\text{BDF}^*99]\), have \(p_{\text{error}} \geq 1.9 \times 10^{-8}\);
2. **domino-type states**, a generalization of domino states to higher dimensions corresponding to tilings of a rectangular \(d_A \times d_B\) grid by tiles of size at most two, have \(p_{\text{error}} \geq 1/((216D^2d_A^5d_B^5))\), where \(D\) is a property of the tiling that we call “diameter”;  
3. **\(\theta\)-rotated domino states**, a 1-parameter family that includes the domino states and the standard basis as extreme cases, have \(p_{\text{error}} \geq 2.4 \times 10^{-11} \sin^2 2\theta\) (determining whether these states can be discriminated perfectly by LOCC and finding a lower bound on the probability of error were left as open problems in \([\text{BDF}^*99]\)).

The rest of the paper is organized as follows. In Sect. 2 we introduce notation, give background on LOCC measurements and state discrimination, and summarize related prior work. In Sect. 3 we introduce our general framework for lower bounding the error probability of LOCC measurements, and in Sect. 3.5 we prove Theorem 10. In Sect. 4 we consider the case where \(S\) is a product basis and propose a method for bounding the nonlocality constant \(\eta\) by another quantity that we call “rigidity.” Our approach is based on a description of sets of bipartite states in terms of tilings. In Sect. 5 we define the three classes of states mentioned above and prove a bound on the rigidity of the domino states; bounds on the rigidity of the domino-type states and the rotated domino states appear in Appendices A and B, respectively. Finally, we discuss limitations of our framework in Sect. 6 and conclude with a discussion of open problems in Sect. 7.

2. **Background**

2.1. **Notation.** The following notation is used in this paper. Let \(L(\mathbb{C}^n, \mathbb{C}^m)\) be the set of all linear operators from \(\mathbb{C}^n\) to \(\mathbb{C}^m\) and let \(L(\mathbb{C}^n) := L(\mathbb{C}^n, \mathbb{C}^n)\). Next, let \(\text{Pos}(\mathbb{C}^n) \subset \)
\( L(\mathbb{C}^n) \) be the set of all positive semidefinite operators on \( \mathbb{C}^n \). Let \( \|M\|_{\max} := \max_{ij} |M_{ij}| \) denote the largest entry of \( M \in L(\mathbb{C}^n) \) in absolute value. Finally, for any natural number \( n \), let \([n] := \{1, \ldots, n\}\) and let \( I_n \) be the \( n \times n \) identity matrix.

### 2.2. Separable and LOCC measurements

A \( k \)-outcome (quantum) measurement \( \mathcal{M} \) on an \( n \)-dimensional vector space can be specified by a set of operators \( \{M_1, \ldots, M_k\} \subset L(\mathbb{C}^n, \mathbb{C}^m) \), where \( \sum_{i=1}^k M_i^\dagger M_i = I_n \) and \( m \) is finite. We refer to the operators \( M_i \) as the measurement operators of \( \mathcal{M} \). The probability of obtaining outcome \( i \) upon measuring state \( \rho \) is \( \text{Tr}(M_i^\dagger M_i \rho) \); the corresponding \( m \)-dimensional post-measurement state is \( M_i \rho M_i^\dagger / \text{Tr}(M_i^\dagger M_i \rho) \).

If we are only interested in the inner products between the post-measurement states, we can specify the measurement \( \mathcal{M} \) using its POVM elements \( \{E_1, \ldots, E_k\} \), where \( E_i := M_i^\dagger M_i \) for all \( i \in [k] \). Such a POVM only specifies the post-measurement states up to an isometry.

#### 2.2.1. Separable measurements

**Definition 1.** A measurement \( \mathcal{E} = \{E_1, \ldots, E_k\} \) on a bipartite state space \( \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \) is separable if all POVM elements \( E_i \) are separable, i.e.,

\[
E_i = \sum_j E_j^A \otimes E_j^B
\]

for some \( E_j^A \in \text{Pos}(\mathbb{C}^{d_A}) \) and \( E_j^B \in \text{Pos}(\mathbb{C}^{d_B}) \).

Note that the above definition is equivalent to saying that \( \mathcal{E} \) is obtained from a measurement with product POVM elements, followed by classical post-processing (coarse graining).

#### 2.2.2. LOCC measurements

We say that a POVM \( \mathcal{E} = \{E_i\}_{i \in [n]} \) is a 2-party LOCC measurement if it can be implemented by some LOCC protocol. Informally, such an LOCC protocol consists of the two parties taking finitely many turns (called rounds) of applying adaptive measurements to their state spaces and exchanging classical messages. This is followed by coarse graining all measurement records into \( n \) bins, each corresponding to one of the \( n \) outcomes of \( \mathcal{E} \). See [CLM+12] for a definition and discussion of more general LOCC instruments for which the LOCC protocol also has to produce correct post-measurement states.

Let us describe an LOCC protocol for implementing a measurement more formally, adopting notation similar to that of [BDF+99]. Let \( \Lambda \) denote the empty string, corresponding to no message being sent. The protocol begins when one of the parties, say Alice, applies a measurement

\[
\mathcal{A}(\Lambda) = \{A_1(\Lambda), \ldots, A_{k(\Lambda)}(\Lambda)\}
\]

to her state space and communicates the round 1 measurement outcome \( m_1 \in [k(\Lambda)] \) to Bob. Then, depending on the value of \( m_1 \) received, Bob applies a measurement

\[
\mathcal{B}(m_1) = \{B_1(m_1), \ldots, B_{k(m_1)}(m_1)\}
\]
to his state space and communicates the round 2 measurement outcome \( m_2 \in [k(m_1)] \) to Alice. The protocol proceeds with the two parties taking finitely many alternating turns of a similar form, where the measurement applied at round \( t \) depends on the measurement record \( m = (m_1, \ldots, m_{t-1}) \) accumulated during the previous rounds.

Let \( m \) be the measurement record after the execution of the first \( t \) rounds of the protocol. Then the measurement operator that Alice and Bob have effectively implemented is a product operator \( A_m \otimes B_m \), where
\[
A_m := A_{m_{t-1}}(m_1, \ldots, m_{t-2}) \cdots A_{m_3}(m_1, m_2) A_{m_1}(\Lambda),
B_m := B_{m_t}(m_1, \ldots, m_{t-1}) \cdots B_{m_4}(m_1, m_2, m_3) B_{m_2}(m_1).
\]

Certain measurement records will cause Alice and Bob to terminate the protocol. Prior to coarse-graining, the quantum operation implemented by the LOCC protocol acts on any state \( \rho \) as
\[
\bigoplus_m (A_m \otimes B_m) \rho (A_m \otimes B_m)^\dagger,
\]
where \( m \) ranges over all the terminating measurement records. When the output dimensions of \( A_m \) and \( B_m \) are equal for all \( m \), the above expression can be written as
\[
\sum_m |m\rangle \langle m| \otimes (A_m \otimes B_m) \rho (A_m \otimes B_m)^\dagger.
\]
Here the first register stores the classical measurement record and is shared between the two parties; the last two registers belong to Alice and Bob, respectively. At the end of the protocol Alice and Bob must output one of the \( n \) classical outcomes of \( \mathcal{E} \). Let \( L(k) \) be the set of all terminating measurement records corresponding to outcome \( k \in [n] \). Then coarse graining according to the partition induced by \( L(k) \) corresponds to measuring the classical register in Eq. (6) according to this partition. The \( k \)th POVM element of the resulting measurement is
\[
\sum_{m \in L(k)} A_m^\dagger A_m \otimes B_m^\dagger B_m,
\]
which must equal \( E_k \) if the LOCC protocol indeed implements measurement \( \mathcal{E} \). Since any operator of the form (8) is separable, any LOCC measurement is separable.

2.2.3. Finite and asymptotic LOCC. We consider two scenarios: when a measurement can be performed in a finite number of rounds or asymptotically.

**Definition 2.** We say that a measurement \( \mathcal{E} \) can be implemented by (finite) LOCC if there exists a finite-round LOCC protocol that, for any input state, produces the same distribution of measurement outcomes as \( \mathcal{E} \).

**Definition 3.** We say that a measurement \( \mathcal{E} \) can be implemented by asymptotic LOCC if there exists a sequence \( \mathcal{P}_1, \mathcal{P}_2, \ldots \) of finite-round LOCC protocols whose output distributions converge to that of \( \mathcal{E} \).

The exact implementation scenario is not practical since any real-world device is susceptible to errors due to imperfections in implementation. However, proving that a certain task cannot be performed asymptotically is considerably harder than showing that it cannot be done (exactly) by any finite LOCC protocol.

---

1 Here we assume for simplicity that \( t \) is even; in the odd case the operators \( A_m \) and \( B_m \) can be defined similarly.
Fig. 1. An example showing the tree structure of a specific LOCC measurement. In round one Alice performs a three-outcome measurement \( A(\Lambda) \); in round two, upon receiving message “1”, Bob performs a two-outcome measurement \( B(1) \) and upon receiving message “2” or “3” he terminates the protocol; in round three, upon receiving message “1”, Alice terminates the protocol and, upon receiving message “2”, she performs a two-outcome measurement \( A(1, 2) \). All nodes are labeled by the accumulated measurement record. The corresponding measurement operator is given below each leaf.

2.2.4. LOCC protocol as a tree. We represent an LOCC measurement protocol as a tree (see Fig. 1). The protocol begins at the root and proceeds downward along the edges. Each edge represents a certain measurement outcome obtained at its parent node, and leaves are the nodes where the protocol terminates. The set of all leaves is partitioned into subsets, each corresponding to an outcome of the LOCC measurement being implemented.

A path from the root to a leaf is called a branch. There is a one-to-one correspondence between the branches and the possible courses of execution of the LOCC protocol. Likewise, there is a one-to-one correspondence between the nodes of the tree and the accumulated measurement records.

The measurement at node \( u \) is the measurement performed by the acting party once the protocol has reached node \( u \). In contrast, the measurement operator corresponding to node \( u \) is the measurement operator that has been implemented upon reaching node \( u \). For example, consider the node \((1, 2)\). The measurement at node \((1, 2)\) is given by the POVM \( \{A_1(1, 2), A_2(1, 2)\} \), whereas the measurement operator corresponding to the node \((1, 2)\) is given by \( A(1) \otimes B_2(1) \). As another example, the measurement operators corresponding to the leaves are exactly the measurement operators of the LOCC protocol prior to coarse graining.

2.3. Bipartite state discrimination problem. The goal of this paper is to investigate the limitations of two-party LOCC protocols for the task of bipartite quantum state discrimination, which is as follows:

Let \( S = \{\ket{\psi_1}, \ldots, \ket{\psi_n}\} \subset \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \) be a known set of quantum states. Suppose that \( k \in [n] \) is selected uniformly at random and Alice and Bob are given the
corresponding parts of state $|\psi_k\rangle \in S$. Their task is to determine the index $k$ by performing a measurement on this state.

A case of special interest is when $S$ is an orthonormal product basis, i.e., each $|\psi_i\rangle = |\alpha_i\rangle|\beta_i\rangle$ for some orthonormal bases $|\alpha_i\rangle \in \mathbb{C}^{d_A}$ and $|\beta_i\rangle \in \mathbb{C}^{d_B}$. Such states can be perfectly discriminated by a separable measurement $E$ with POVM elements

$$E_i := |\alpha_i\rangle\langle\alpha_i| \otimes |\beta_i\rangle\langle\beta_i|.$$  \hspace{1cm} (9)

However, this measurement cannot always be implemented by finite [WH02, GV01] or even asymptotic LOCC [BDF+99]. In such cases we say that $S$ possesses nonlocality (without entanglement).

2.4. Previous results. The first example of an orthonormal product basis of bipartite quantum states that cannot be perfectly discriminated by (even asymptotic) LOCC was given in [BDF+99]. This is a striking illustration of the difference between the power of LOCC and separable operations. Furthermore, [BDF+99] quantifies the information deficit of any LOCC protocol for discriminating these states. This result has been a starting point for many other studies on state discrimination by LOCC, with the ultimate goal of understanding LOCC operations and how they differ from separable ones. We briefly describe some of the directions that have been explored. Unless otherwise stated, these results refer to the discrimination of pure states with finite LOCC.

First consider the problem of discriminating two states without any restrictions on their dimension. Surprisingly, any two orthogonal (possibly entangled) pure states can be perfectly discriminated by LOCC, even when they are held by more than two parties [WSHV00]. Furthermore, optimal discrimination of any two multipartite pure states can be achieved with LOCC both in the sense of minimum error probability [VSPM01] and unambiguous discrimination [CY01, CY02, JCY05]. Recently this has been generalized to implementing an arbitrary POVM by LOCC in any 2-dimensional subspace [Cro12].

Many authors have considered the problem of perfect state discrimination by finite LOCC. In particular, the case where one party holds a small-dimensional system is well understood. Reference [WH02] characterizes when a set of orthogonal (possibly entangled) states in $\mathbb{C}^2 \otimes \mathbb{C}^2$ can be perfectly discriminated by LOCC. A similar characterization for sets of orthogonal product states in $\mathbb{C}^3 \otimes \mathbb{C}^3$ has been given by [FS09]. In addition, [WH02] characterizes when a set of orthogonal states in $\mathbb{C}^2 \otimes \mathbb{C}^n$ can be perfectly discriminated by LOCC when Alice performs the first nontrivial measurement. It is also known that $\theta$-rotated domino states cannot be perfectly discriminated by LOCC (unless $\theta = 0$) [GV01]. Furthermore, the original domino states have inspired a construction of $n$-partite $d$-dimensional product bases that cannot be perfectly discriminated with LOCC [NC06].

The role of entanglement in perfect state discrimination by finite LOCC has also been considered. It is not possible to perfectly discriminate more than two Bell states by LOCC [GKR+01]. In fact, the same is true for any set of more than $n$ maximally entangled states in $\mathbb{C}^n \otimes \mathbb{C}^n$ [Nat05]. Multipartite states from an orthonormal basis can be perfectly discriminated by LOCC only if it is a product basis [HSSH03]. Also, no basis of the subspace orthogonal to a state with orthogonal Schmidt number 3 or greater can be perfectly discriminated by LOCC [DXY09]. On the other hand, any three orthogonal maximally entangled states in $\mathbb{C}^3 \otimes \mathbb{C}^3$ can be perfectly discriminated by LOCC [Nat05]. In fact, if the number of dimensions is not restricted, one can find
arbitrarily large sets of orthogonal maximally entangled states that can be perfectly discriminated by LOCC [Fan04]. Contrary to intuition, states with more entanglement can sometimes be discriminated perfectly with LOCC while their less entangled counterparts cannot [HSSH03]. Generally, however, a set of orthogonal multipartite states $S \subset \mathbb{C}^D$ can be perfectly discriminated with LOCC only if $|S| \leq \frac{D}{d(S)}$, where $d(S)$ measures the average entanglement of the states in $S$ [HMM+06].

It is known that local projective measurements are sufficient to discriminate states from an orthonormal product basis with LOCC [DR04,CL04]. Moreover, there is a polynomial-time (cubic in $\max\{d_A, d_B\}$) algorithm for deciding if states from a given orthonormal product basis of $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ can be perfectly discriminated with LOCC [DR04]. The state discrimination problem for incomplete orthonormal sets (i.e., orthonormal sets of states that do not span the entire space) seems to be harder to analyze. However, unextendible product bases might be an exception (although commonly referred to as “bases” these are in fact incomplete orthonormal sets). It is known that states from an unextendible product basis cannot be perfectly discriminated by finite LOCC [BDM+99]. In fact, the same holds for any basis of a subspace spanned by an unextendible product basis in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ [DXY10]. Curiously, there are only two families of unextendible product bases in $\mathbb{C}^3 \otimes \mathbb{C}^3$, one of which is closely related to the domino states [DMS+03].

The problem of state discrimination with asymptotic LOCC has been studied less. It is known that states from an unextendible orthonormal product set cannot be perfectly discriminated with LOCC even asymptotically [DR04]. Reference [KKB11] gives a necessary condition for perfect asymptotic LOCC discrimination, and also shows that for perfectly discriminating states from an orthonormal product basis, asymptotic LOCC gives no advantage over finite LOCC. The latter result implies that the algorithm from [DR04] also covers the asymptotic case. On the other hand, even in some very basic instances of state discrimination it remains unclear whether asymptotic LOCC is superior to finite LOCC (see [DFXY09,KKB11] for specific sets of states).

Another line of study originating from [BDF+99] aims at understanding the difference between the classes of separable and LOCC operations. To this end, [Coh11] constructs an $r$-round LOCC protocol implementing an arbitrary separable measurement whenever such a protocol exists. A different approach is to exhibit quantitative gaps between the two classes. To the best of our knowledge, only two quantitative gaps other than that of [BDF+99] are known. References [KTY10,Koa09] demonstrate a gap between the success probabilities achievable by bipartite separable and LOCC operations for unambiguously discriminating $|00\rangle$ from a fixed rank-2 mixed state. The largest known difference between the two classes is a gap of 0.125 between the achievable success probabilities for tripartite EPR pair distillation [CCL11]. Moreover, as the number of parties grows, the gap approaches 0.37 [CCL11].

At a first glance one might think that the nonlocality without entanglement phenomenon is related to quantum discord. However, the quantum discord value cannot be used to determine whether states from a given ensemble can be discriminated with LOCC [BT10].

Finally, if a set of orthogonal (product or entangled) states cannot be perfectly discriminated by LOCC, one can measure their nonlocality by considering how much entanglement is needed to achieve perfect discrimination [Coh08,BBKW09].
3. Framework

In this section we introduce a framework for proving lower bounds on the error probability of any LOCC measurement for discriminating bipartite states from a given set

\[ S := \{|\psi_1\rangle, \ldots, |\psi_n\rangle \} \subset \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}. \]  

We make no assumptions about the states \(|\psi_i\rangle\). In particular, they need not be product states or be mutually orthogonal.

From now on, \( \mathcal{P} \) denotes an arbitrary LOCC protocol for discriminating states from \( S \). In rough outline our argument proceeds as follows:

1. We modify \( \mathcal{P} \) so that it can be stopped when a specific amount of information \( \varepsilon \) has been obtained (see Sect. 3.1). This is done by terminating the protocol prematurely and possibly making the last measurement less informative (see Sect. 3.2).
2. When the information gain is \( \varepsilon \), we lower bound a measure of disturbance (defined in Sect. 3.3) by \( \eta \varepsilon \) for some constant \( \eta \) (see Sect. 3.4).
3. We show that at least two of the possible initial states have become nonorthogonal at this stage of the protocol, and we infer a lower bound on the error probability of \( \mathcal{P} \) (see Sect. 3.5).

Our framework reuses some ideas of the original approach [BDF+99]. However, instead of mutual information, we quantify how much an LOCC protocol has learned about the state using error probability. This allows us to replace the long mutual information analysis in the original paper with a simple application of Helstrom’s bound. The idea of relating information gain and disturbance also comes from [BDF+99]. Here, we analyze this tradeoff using the nonlocality constant (see Definition 7) which can be applied to any set of states. In Sect. 4 we give a method for lower bounding the nonlocality constant that applies specifically when \( S \) is an orthonormal basis of \( \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \). In Sect. 5 we apply this method for the domino states and some other related bases.

3.1. Interpolated LOCC protocol. Consider an arbitrary node in the tree representing the protocol \( \mathcal{P} \). Let \( m \) be the corresponding measurement record and let \( A \otimes B \) denote the Kraus operator that is applied to the initial state when this node is reached. Note that the output dimensions of operators \( A \) and \( B \) could be arbitrary.

The initial state \(|\psi_k\rangle\) yields measurement record \( m \) with probability

\[ p(m|\psi_k) := \text{Tr} [(A \otimes B) (A \otimes B)|\psi_k\rangle\langle\psi_k|] = \langle\psi_k|(a \otimes b)|\psi_k\rangle, \]  

where \( a := A^\dagger A \in \text{Pos}(\mathbb{C}^{d_A}) \) and \( b := B^\dagger B \in \text{Pos}(\mathbb{C}^{d_B}) \). Note that we need not concern ourselves with the arbitrary output dimensions of \( A \) and \( B \) from this point onward. We use Bayes’s rule and the uniformity of the probabilities \( p(\psi_k) \) to obtain the probability that the initial state was \(|\psi_k\rangle\) conditioned on the measurement record being \( m \):

\[ p(\psi_k|m) = \frac{p(\psi_k)p(m|\psi_k)}{\sum_{j=1}^n p(\psi_j)p(m|\psi_j)} = \frac{\langle\psi_k|(a \otimes b)|\psi_k\rangle}{\sum_{j=1}^n \langle\psi_j|(a \otimes b)|\psi_j\rangle}. \]  

At the root, the measurement record \( m \) is the empty string and \( p(\psi_k|m) = \frac{1}{n} \) for all \( k \). As we proceed toward the leaves, these probabilities fluctuate away from \( \frac{1}{n} \). For example, if \( \mathcal{P} \) discriminates the states perfectly, the distribution reaches a Kronecker delta function.
For a given node \( m \) let us define
\[
p_{\text{max}}(m) := \max_{k \in [n]} p(\psi_k|m).
\] (13)

Let \( \epsilon := p_{\text{max}}(m) - \frac{1}{n} \). Then \( \epsilon \) characterizes the non-uniformity of the distribution \( p(\psi_k|m) \) and thus the amount of information learned about the input state. The next theorem shows that we can modify the protocol \( \mathcal{P} \) so that it can be stopped when some but not too much information has been learned. While this idea originates from [BDF+99], we use a specific result from [KKB11].

**Theorem 4** (Kleinmann, Kampermann, Bruß [KKB11]). Let \( \mathcal{P} \) be an LOCC protocol for discriminating states from a set \( S \) of size \( n \). For any \( \epsilon > 0 \) there exists an LOCC protocol \( \mathcal{P}_\epsilon \) that has the same success probability as \( \mathcal{P} \), but each branch of \( \mathcal{P}_\epsilon \) has a node \( m \) such that either
\[
p_{\text{max}}(m) = \frac{1}{n} + \epsilon \quad \text{or} \quad p_{\text{max}}(m) < \frac{1}{n} + \epsilon \quad \text{and \( m \) is a leaf of \( \mathcal{P} \).}
\] (14)

**Proof idea.** Fix \( \epsilon > 0 \). Then in each branch of \( \mathcal{P} \) either \( p_{\text{max}} \leq \frac{1}{n} + \epsilon \) for all nodes in that branch or there exists a node at which \( \frac{1}{n} + \epsilon < p_{\text{max}} \). To obtain the interpolated protocol \( \mathcal{P}_\epsilon \), for each branch of the latter kind we identify the closest node to the root, say \( v_1 \), at which \( \frac{1}{n} + \epsilon < p_{\text{max}} \) and modify the measurement leading to it. Note that \( v_1 \) cannot be the root, since \( p_{\text{max}}(\text{root}) = \frac{1}{n} \). We now outline the modification procedure.

Let \( u \) be the parent of \( v_1 \) and \( v_1, \ldots, v_m \) be all the children of \( u \). Then
\[
p_{\text{max}}(u) < \frac{1}{n} + \epsilon < p_{\text{max}}(v_1),
\] (15)

which means that the measurement outcome corresponding to the edge \((u, v_1)\) is too informative. To rectify this, we break up the measurement at node \( u \) into two steps so that each individual measurement is less informative. We represent the outcomes of the first measurement by new nodes \( \tilde{v}_1, \ldots, \tilde{v}_m \) while the outcomes of the second measurement lead to the original nodes \( v_1, \ldots, v_m \) (see Fig. 2). A precise prescription of how to obtain the measurement operators for the newly introduced measurements can be found in [KKB11].

The first measurement interpolates between a completely uninformative trivial measurement and the original measurement at \( u \). The interpolation parameters are chosen so that \( p_{\text{max}}(\tilde{v}_i) = \frac{1}{n} + \epsilon \) for all \( i \) for which \( \frac{1}{n} + \epsilon < p_{\text{max}} \) in the original protocol \( \mathcal{P} \). The second measurement depends on the outcome of the first measurement. It produces the same set of post-measurement states as the original measurement at \( u \). Moreover, the total probability of obtaining each state is the same as in the case of the original measurement. After this we proceed according to the original protocol. \( \square \)

In the context of state discrimination, the possibility of interpolating a protocol to obtain some but not too much information is what distinguishes LOCC measurements from separable ones. In particular, a separable measurement for a set of states that cannot be distinguished by asymptotic LOCC cannot be divided into two steps, with the first yielding information precisely \( \epsilon \) and the second completing the measurement [CLM+13].
Protocol $\mathcal{P}$: $u$

$T_1$ $T_2$ $T_3$

Fig. 2. The protocol tree before (left) and after (right) splitting the measurement at node $u$ into two steps. (The graph on the right has been condensed for clarity, but it can be expanded into a tree by making a new copy of subtree $T_i$ for each incoming arc in $v_i$.) The amount of information learned in the first step is controlled by diluting the measurement operators, and the purpose of the second step is to complete the original measurement. The dotted line corresponds to the end of Stage I (see Definition 5).

Protocol $\mathcal{P}_\varepsilon$: $u$

$\tilde{v}_1$ $\tilde{v}_2$ $\tilde{v}_3$

Fig. 3. Probability distribution $p(\psi_k|m)$ at the end of Stage I. For all $k$ we have $\frac{1}{n} + \varepsilon \geq p(\psi_k|m) \geq \frac{1}{n} - (n-1)\varepsilon > 0$ where the first inequality is tight for some $k$.

3.2. Stopping condition. To control how much information the protocol has learned, we fix some $\varepsilon > 0$ and stop the execution of $\mathcal{P}_\varepsilon$ when we reach a node $m$ that satisfies the conditions in Eq. (14).

**Definition 5.** In any given execution of $\mathcal{P}_\varepsilon$, we say that **Stage I** is complete at the earliest point when Eq. (14) is satisfied.

We choose $\varepsilon < \frac{1}{n(n-1)}$ in our analysis. Operationally, this means that none of the $n$ states has been eliminated at the end of Stage I, since

$$\min_{k \in [n]} p(\psi_k|m) \geq 1 - (n - 1)p_{\max}(m) \geq \frac{1}{n} - (n-1)\varepsilon > 0. \quad (16)$$

This allows us to use Helstrom’s bound to lower bound the probability of error (see Sect. 3.5). It also ensures that the disturbance measure $\delta_S(a \otimes b)$ introduced in Sect. 3.3 is well defined at $m$. All constraints imposed on the distribution $p(\psi_k|m)$ are summarized in Fig. 3.

Since the error probability of the protocol $\mathcal{P}_\varepsilon$ is a weighted average of error probabilities of individual branches, it suffices to lower bound these individual error probabilities.
For any branch that terminates without a node satisfying
\[ p_{\text{max}}(m) = \frac{1}{n} + \epsilon. \]  
we can put a large lower bound on the error probability. In particular, for the optimal choice \( \epsilon = \frac{2}{3(n-1)} \) of Theorem 10 with \( n \geq 2 \),
\[ p_{\text{error}}(m) \geq 1 - p_{\text{max}}(m) > 1 - \left( \frac{1}{n} + \epsilon \right) = 1 - \frac{1}{n} - \frac{2}{3n(n-1)} \geq \frac{1}{6}, \]
which is much higher than the lower bound we obtain for other branches. We now consider the remaining case where Stage I ends with a node satisfying Eq. (17).

3.3. Measure of disturbance. Now we show that at least two possible post-measurement states \((A \otimes B)|\psi_i\rangle\) and \((A \otimes B)|\psi_j\rangle\) are nonorthogonal at the end of Stage I, and lower bound their overlap quantitatively. Assuming that the initial state was \(|\psi_i\rangle \in S\), the normalized post-measurement state at a node with corresponding measurement operator \(A \otimes B\) is
\[ |\phi_i\rangle := \frac{(A \otimes B)|\psi_i\rangle}{\sqrt{\langle \psi_i | (A \otimes b)|\psi_i\rangle}}. \]
where \(a := A^\dagger A\) and \(b := B^\dagger B\). We are interested in the overlaps \(\langle \phi_i | \phi_j \rangle\) rather than the actual post-measurement states. Hence, from now on we use the POVM elements \(a\) and \(b\) instead of the measurement operators \(A\) and \(B\).

Definition 6. The disturbance caused by the operator \(a \otimes b\) on the set of states \(S\) is defined as
\[ \delta_S(a \otimes b) := \max_{i \neq j} |\langle \phi_i | \phi_j \rangle| = \max_{i \neq j} \frac{|\langle \psi_i | (a \otimes b)|\psi_j\rangle|}{\sqrt{\langle \psi_i | (a \otimes b)|\psi_i\rangle \langle \psi_j | (a \otimes b)|\psi_j\rangle}}. \]  
We use \(\delta_S(a \otimes b)\) only in the context where \(a \otimes b\) is the operator corresponding to an end node of Stage I. In this case, from Eqs. (16) and (12) we get that \(0 < \min_{k \in [n]} p(|\psi_k\rangle|m) = \min_{k \in [n]} \frac{\langle \psi_k | (a \otimes b)|\psi_k\rangle}{\sum_{j=1}^{n} \langle \psi_j | (a \otimes b)|\psi_j\rangle}. \) Hence \(\langle \psi_i | (a \otimes b)|\psi_i\rangle > 0\) for all \(i \in [n]\), so \(\delta_S\) is well-defined.

Note that \(\delta_S(a \otimes b)\) measures the nonorthogonality of the post-measurement states \(|\phi_i\rangle\). If the initial states \(|\psi_i\rangle\) are orthogonal, \(\delta_S(a \otimes b)\) characterizes the disturbance imparted to the states at the end of Stage I in the branch corresponding to \(a \otimes b\).

3.4. Disturbance/information gain trade-off. Now we define the nonlocality constant \(\eta\). It relates the disturbance caused at the end of Stage I, minimized over all branches (see Definition 6), to the amount of information learned, \(\epsilon\).

Definition 7. The nonlocality constant \(\eta\) of \(S\) is the supremum over all \(\eta'\) such that
\[ \eta': \left( \frac{\max_{k \in [n]} \langle \psi_k | (a \otimes b)|\psi_k\rangle}{\sum_{j \in [n]} \langle \psi_j | (a \otimes b)|\psi_j\rangle} - \frac{1}{n} \right) \leq \delta_S(a \otimes b) \]
for all \(a \in \text{Pos}(\mathbb{C}^d_A), b \in \text{Pos}(\mathbb{C}^d_B)\) for which \(\langle \psi_i | (a \otimes b)|\psi_i\rangle \neq 0\) for all \(i \in [n]\).
Equivalently, if \( G_{ij} := \langle \psi_i | (a \otimes b) | \psi_j \rangle \) for \( i, j \in [n] \) then

\[
\eta := \inf_{a, b} \left\{ \max_{i \neq j \atop k} \frac{|G_{ij}|}{\sqrt{G_{ii} G_{jj}}} \right. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. 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As \(\varepsilon\) increases, the error probability for a specific branch changes in two opposite ways. On one hand, two post-measurement states \(|\psi_i\rangle\) and \(|\psi_j\rangle\) have overlap \(\delta \geq \eta \varepsilon\), and this lower bound increases with \(\varepsilon\). On the other hand, the probabilities of these two states, \(p(\psi_i|m)\) and \(p(\psi_j|m)\), are lower bounded by a function that decreases with \(\varepsilon\). Balancing these two effects, the choice \(\varepsilon = \frac{2}{3 n(n-1)}\) gives a lower bound on the error probability as follows.

**Lemma 9.** Let \(S\) be a set of quantum states in \(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}\) of size \(n \geq 2\). For any LOCC measurement discriminating states drawn uniformly from \(S\), each branch errs with probability

\[
P_{\text{error}} \geq \frac{2}{27} \frac{\eta^2}{n^5},
\]

where \(\eta\) is the nonlocality constant of \(S\) (see Definition 7).

**Proof.** At the end of Stage I, for each branch, there are two post-measurement states \(|\Phi_0\rangle\) and \(|\Phi_1\rangle\) with overlap \(\delta\). Let \(p_0\) and \(p_1\) be the posterior probabilities of these states. To lower bound the error probability of \(P_\varepsilon\) (and thus that of \(P\)), we give Alice and Bob extra power at this point:

- if the actual input state does not lead to \(|\Phi_0\rangle\) or \(|\Phi_1\rangle\), we assume that Alice and Bob succeed with certainty;
- otherwise Alice and Bob are allowed to perform the best joint measurement to discriminate the states \(|\Phi_0\rangle\) and \(|\Phi_1\rangle\).

For fixed \(\varepsilon\) and probabilities \(p_0\) and \(p_1\), we can lower bound the error probability by the following expression:

\[
P(p_0, p_1, \varepsilon) := (p_0 + p_1) \cdot Q\left(\frac{p_0}{p_0 + p_1}, \frac{p_1}{p_0 + p_1}, \delta\right).
\]

Using Eq. (26) and the inequality \(\delta \geq \eta \varepsilon\) from Lemma 8, we get that

\[
P(p_0, p_1, \varepsilon) \geq \frac{p_0 p_1}{p_0 + p_1} (\eta \varepsilon)^2.
\]

Recall that we stop the protocol at a point where we are guaranteed that \(0 < \varepsilon < \frac{1}{n(n-1)}\) and, by Eqs. (16) and (17),

\[
\frac{1}{n} - (n-1)\varepsilon \leq p_i \leq \frac{1}{n} + \varepsilon
\]

for all \(i\). Given these constraints on \(p_0\) and \(p_1\), we can choose the \(\varepsilon\) that maximizes \(P(p_0, p_1, \varepsilon)\) and guarantee that the error probability in the branch of the LOCC protocol being considered satisfies

\[
P_{\text{error}} \geq \max_{\varepsilon \in \left(0, \frac{1}{n(n-1)}\right)} \min_{p_0, p_1 \in \left[\frac{1}{n} - (n-1)\varepsilon, \frac{1}{n} + \varepsilon\right]} P(p_0, p_1, \varepsilon).
\]

From Eq. (29) we get

\[
P_{\text{error}} \geq \max_{\varepsilon \in \left(0, \frac{1}{n(n-1)}\right)} \frac{\min_{p_0, p_1 \in \left[\frac{1}{n} - (n-1)\varepsilon, \frac{1}{n} + \varepsilon\right]} P(p_0, p_1, \varepsilon)}{p_0 + p_1} (\eta \varepsilon)^2.
\]
The minimum is attained when $p_0 = p_1 = \frac{1}{n} - (n - 1)\varepsilon$ (i.e., the probabilities are equal and as small as possible), so the problem simplifies to

$$p_{\text{error}} \geq \max_{\varepsilon \in \left(0, \frac{1}{n(n-1)}\right)} \frac{1}{2} \left(\frac{1}{n} - (n - 1)\varepsilon\right)(\eta \varepsilon)^2 \geq \frac{2}{27} \frac{\eta^2}{n^3(n-1)^2} \geq \frac{2}{27} \frac{\eta^2}{n^5},$$

where the value

$$\varepsilon = \frac{1}{3} \frac{1}{n(n-1)}$$

achieves the maximum. □

The lower bound for the error probability for each branch implies the same lower bound for the LOCC measurement for discriminating states from the set $S$:

**Theorem 10.** Let $S$ be a set of quantum states in $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ of size $n \geq 2$. Any LOCC measurement for discriminating states drawn uniformly from $S$ errs with probability

$$p_{\text{error}} \geq \frac{2}{27} \frac{\eta^2}{n^5},$$

where $\eta$ is the nonlocality constant of $S$ (see Definition 7).

Theorem 10 shows that any LOCC protocol for discriminating states from $S$ errs with probability proportional to $\eta^2$, justifying the name “nonlocality constant.”

### 4. Bounding the Nonlocality Constant

The framework described in Sect. 3 reduces the problem of bounding the error probability for discriminating bipartite states by LOCC to the one of bounding the nonlocality constant $\eta$ (see Theorem 10). This reduction holds for any set of pure states $S$. In this section we assume that $S$ is an orthonormal basis of $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ and provide tools for bounding the nonlocality constant. In particular, we bound $\eta$ in terms of another quantity that we call “rigidity”.

For the remainder of the paper we represent pure states from $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ using “tiles” in a $d_A \times d_B$ grid. We first introduce some notations related to tilings in Sect. 4.1. Then we define rigidity and relate it to the nonlocality constant $\eta$ in Sect. 4.2. Sect. 4.3 provides a tool, the “pair of tiles” lemma, that we use to bound rigidity for specific sets of states in Sect. 5.

#### 4.1. Definitions.

Given a fixed orthonormal basis $\{|i\rangle: i \in [d]\}$, define the **support** of a pure state $|\psi\rangle \in \mathbb{C}^d$ as

$$\text{supp} |\psi\rangle := \{i \in [d]: \langle i | \psi \rangle \neq 0\}. \quad (36)$$

If $|\psi\rangle \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ then $\text{supp} |\psi\rangle \subseteq [d_A] \times [d_B]$. Consider $[d_A] \times [d_B]$ as a rectangular grid of size $d_A \times d_B$. Any region that corresponds to a submatrix of this grid is called a tile. More formally, a **tile** is a subset $T \subseteq [d_A] \times [d_B]$ such that $T = R \times C$ for some $R \subseteq [d_A]$ and $C \subseteq [d_B]$. (Note that a tile is not necessarily a contiguous region of the grid.) We use rows$(T) = R$ and cols$(T) = C$ to denote the **rows** and **columns** of this
Fig. 4. A domino-type tiling and the corresponding row and column graphs. This tiling is irreducible and has diameter two.

tile, respectively, and we use $|T|$ to denote the size or the area of $T$. If $|\psi\rangle = |\alpha\rangle|\beta\rangle$ is a product state, then $\text{supp}|\psi\rangle = \text{supp}|\alpha\rangle \times \text{supp}|\beta\rangle$ and thus $\text{supp}|\psi\rangle$ is a tile, which we call the tile induced by $|\psi\rangle$.

We say that an orthonormal set of product states $S \subset \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ induces a tiling of a $d_A \times d_B$ grid if the tiles induced by the states in $S$ are either disjoint or identical. Note that if $S$ is an orthonormal basis of $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$, then a tile of area $L$ is induced by $L$ states that form a basis of that tile. In a domino-type tiling, every tile has area 1 or 2 and the states inducing the tiles of size 2 are of the form $|i \pm j\rangle|k\rangle$ or $|k\rangle|i \pm j\rangle$, where $|i \pm j\rangle := (|i\rangle \pm |j\rangle)/\sqrt{2}$.

For a given tiling $T$ of a $d_A \times d_B$ grid let us define the corresponding row graph as follows: its vertex set is $[d_A]$ with two vertices $i$ and $j$ adjacent if and only if there exists a column $c$ such that $(i, c)$ and $(j, c)$ belong to the same tile. The column graph of a tiling is defined similarly. We say that a tiling is irreducible if its row graph and its column graph are both connected. The diameter of the tiling $T$ is the maximum of the diameters of its row and column graphs. See Fig. 4 for an example.

Without loss of generality we consider only irreducible tilings. Reducible tilings can be broken down into several smaller components without disturbing the underlying states. To do this, both parties simply perform a projective measurement with respect to the subspaces corresponding to the different components of the row and column graphs.

Note that in general, a tiling is not invariant under local unitaries. In particular, the irreducibility of the tiling induced by a given set of states is a basis-dependent property. The most extreme example of this phenomenon is the case of the standard basis. It induces a completely reducible tiling that consists only of $1 \times 1$ tiles. However, if both parties apply a generic local unitary transformation, the resulting tiling consists only of a single tile of maximal size.

4.2. Lower bounding the nonlocality constant using rigidity. In this section we assume that $S$ is an orthonormal basis of $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ (so in particular, $n = d_A d_B$) and discuss a particular strategy for lower bounding $\eta$ for such $S$. We apply this strategy to several sets of orthonormal product bases in Sect. 5.

We bound $\eta$ (quantifying a disturbance/strength tradeoff) by considering a quantitative property of the set $S$ called rigidity. Intuitively, we call a measurement operator strong if it is far from being proportional to the identity matrix; a set of states $S$ is rigid if there exists a strong measurement that leaves the set undisturbed. We formalize this as follows (recall that $\|\cdot\|_{\text{max}}$ denotes the largest entry of a matrix in absolute value):
Definition 11. We say that an orthonormal basis $S$ is $c$-rigid, or $c$ is an upper bound on the rigidity of $S$, if $c \in \mathbb{R}$ is such that

$$
\left\| \frac{a \otimes b}{\text{Tr}(a \otimes b)} - \frac{I}{n} \right\|_{\max} \leq c \cdot \delta_S(a \otimes b),
$$

(37)

for all $a \in \text{Pos}(\mathbb{C}^d_A)$, $b \in \text{Pos}(\mathbb{C}^d_B)$ for which $\langle \psi_i | (a \otimes b) | \psi_i \rangle \neq 0$ for all $i$.

When $S$ is rigid, the states can remain unchanged despite application of a strong measurement. For example, a tensor product basis is not $c$-rigid for any finite $c$ (i.e., such a basis is arbitrarily rigid). In contrast, if $c$ is small, then any strong measurement disturbs the set $S$, and Eq. (37) quantifies how weak a measurement operator $a \otimes b$ must be for the disturbance $\delta_S(a \otimes b)$ to be small.

We now relate upper bounds on the rigidity of $S$ to lower bounds on its nonlocality constant:

Lemma 12. Let $S$ be an orthonormal basis of $\mathbb{C}^d_A \otimes \mathbb{C}^d_B$. If $S$ is $c$-rigid then

$$
\eta \geq \frac{1}{cL},
$$

(38)

where $L$ is the size of the largest tile corresponding to states in $S$.

Proof. If $S$ is $c$-rigid, then for any $a \in \text{Pos}(\mathbb{C}^d_A)$ and $b \in \text{Pos}(\mathbb{C}^d_B)$ (such that $\langle \psi_k | (a \otimes b) | \psi_k \rangle \neq 0$ for all $k \in [n]$), we have

$$
\frac{a \otimes b}{\text{Tr}(a \otimes b)} - \frac{I}{n} = cM \cdot \delta_S(a \otimes b)
$$

(39)

for some Hermitian matrix $M \in \mathbb{L}(\mathbb{C}^d_A \otimes \mathbb{C}^d_B)$ with $\|M\|_{\max} \leq 1$. From this we get

$$
\max_{k \in [n]} \langle \psi_k | \frac{a \otimes b}{\text{Tr}(a \otimes b)} | \psi_k \rangle - \frac{1}{n} = c \max_{k \in [n]} \langle \psi_k | M | \psi_k \rangle \cdot \delta_S(a \otimes b)
$$

(40)

$$
\leq cL \cdot \delta_S(a \otimes b).
$$

(41)

By the definition of $\eta$ (Eq. (21)) and the fact that $\text{Tr}(a \otimes b) = \sum_{j \in [n]} \langle \psi_j | (a \otimes b) | \psi_j \rangle$ for any orthonormal basis $S$, we get the desired inequality.

Putting Lemma 12 and Theorem 10 together gives the following:

Theorem 13. Let $S$ be an orthonormal basis of $\mathbb{C}^d_A \otimes \mathbb{C}^d_B$. If $S$ is $c$-rigid then any LOCC measurement for discriminating states from $S$ errs with probability

$$
\frac{1}{n^5} \leq \frac{1}{(cL)^2n^5},
$$

(42)

$$
p_{\text{error}} \geq \frac{1}{27 \cdot (cL)^2n^5},
$$

where $L$ is the size of the largest tile of $S$. 
4.3. The “pair of tiles” lemma. In this section we present a lemma that serves as our main tool for bounding rigidity.

**Lemma 14.** Let \( U \in U(m), V \in U(n), \) and define \( |\varphi_i\rangle := U|i\rangle \) for \( i \in [m] \) and \( |\psi_j\rangle := V|j\rangle \) for \( j \in [n] \). Then for any \( M \in L(\mathbb{C}^n, \mathbb{C}^m) \) we have

\[
\sqrt{mn} \cdot \max_{i,j} |\langle \varphi_i|M|\psi_j\rangle| \geq \max_{k,l} |M_{kl}|. \tag{43}
\]

The main idea of the proof is that a unitary change of basis can only increase the largest entry of a vector by a multiplicative factor depending on the dimension of the vector.

**Proof.** Let us define a mapping \( \text{vec} : L(\mathbb{C}^n, \mathbb{C}^m) \to \mathbb{C}^n \otimes \mathbb{C}^m \) as

\[
\text{vec} : |i\rangle \langle j| \mapsto |i\rangle|j\rangle \tag{44}
\]

for \( i \in [m] \) and \( j \in [n] \) and extend it by linearity over \( \mathbb{C} \). One can check that \( \text{vec}(AXB) = (A \otimes B^T) \text{vec}(X) \). Using this and basic inequalities between the 2-norm and the \( \infty \)-norm, we get

\[
\max_{i,j} |\langle \varphi_i|M|\psi_j\rangle| = \left\| \text{vec} \left( \sum_{i,j} \langle \varphi_i|M|\psi_j\rangle |i\rangle\langle j| \right) \right\|_{\infty} \tag{45}
\]

\[
= \left\| \text{vec} \left( \sum_{i,j} \langle i|U^M V|j\rangle |i\rangle\langle j| \right) \right\|_{\infty} \tag{46}
\]

\[
= \left\| \text{vec}(U^M V) \right\|_{\infty} \tag{47}
\]

\[
= \left\| (U^\dagger \otimes V^T) \text{vec}(M) \right\|_{\infty} \tag{48}
\]

\[
\geq \frac{1}{\sqrt{mn}} \left\| (U^\dagger \otimes V^T) \text{vec}(M) \right\|_2 \tag{49}
\]

\[
= \frac{1}{\sqrt{mn}} \left\| \text{vec}(M) \right\|_2 \tag{50}
\]

\[
\geq \frac{1}{\sqrt{mn}} \left\| \text{vec}(M) \right\|_{\infty} \tag{51}
\]

\[
= \frac{1}{\sqrt{mn}} \max_{k,l} |M_{kl}|, \tag{52}
\]

as desired. \( \square \)

Let us restate Lemma 14 using the language of tilings:

**Lemma 15.** Let \( R_1, R_2 \subseteq [d_A] \times [d_B] \) be two arbitrary regions of a \( d_A \times d_B \) grid, and \( \{|\varphi_i\rangle\}_{i=1}^{\text{\#}R_1} \) and \( \{|\psi_j\rangle\}_{j=1}^{\text{\#}R_2} \subseteq C^{d_A} \otimes C^{d_B} \) be their bases (here \( |\varphi_i\rangle \) and \( |\psi_j\rangle \) need not be product states). Then for any matrices \( a \in L(\mathbb{C}^{d_A}) \) and \( b \in L(\mathbb{C}^{d_B}) \) we have

\[
\sqrt{|R_1| \cdot |R_2|} \max_{i,j} |\langle \varphi_i|(a \otimes b)|\psi_j\rangle| \geq \max_{(r_1,c_1) \in R_1 \atop (r_2,c_2) \in R_2} |a_{r_1c_1}| \cdot |b_{c_1c_2}|. \tag{53}
\]

This follows from Lemma 14 by restricting \( |\varphi_i\rangle \) and \( |\psi_j\rangle \) to regions \( R_1 \) and \( R_2 \), respectively, and choosing \( M \) to be a submatrix of \( a \otimes b \) with rows determined by \( R_1 \) and columns by \( R_2 \).
Proof. For $t \in \{1, 2\}$ let us enumerate the cells of region $R_t$ by integers from $\{1, \ldots, |R_t|\}$ arbitrarily, and let $(r_t(i), c_t(i))$ be the coordinates of the $i^{th}$ cell of region $R_t$. Let

$$\Pi_t := \sum_{i=1}^{|R_t|} |i\rangle \langle r_t(i), c_t(i)|$$

be a linear operator that restricts the space $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ to region $R_t$. Then $|\psi'_i\rangle := \Pi_1 |\psi_i\rangle$ is the restriction of $|\psi_i\rangle$ to region $R_1$ and $|\psi'_j\rangle := \Pi_2 |\psi_j\rangle$ is the restriction of $|\psi_j\rangle$ to $R_2$. Also, let $M := \Pi_1 (a \otimes b) \Pi_2^\dagger$.

Note that for all $i \in \{1, \ldots, |R_1|\}$ we have $\Pi_1^\dagger \Pi_1 |\psi_i\rangle = |\psi_i\rangle$ since the support of $|\psi_i\rangle$ lies entirely within region $R_1$ and $\Pi_1^\dagger \Pi_1$ is the projection onto $R_1$. Similarly, $\Pi_2^\dagger \Pi_2 |\psi_j\rangle = |\psi_j\rangle$ for all $j \in \{1, \ldots, |R_2|\}$. Hence

$$\langle \psi_i | (a \otimes b) | \psi_j \rangle = \langle \psi_i | \Pi_1^\dagger \Pi_1 (a \otimes b) \Pi_2^\dagger \Pi_2 | \psi_j \rangle = \langle \psi'_i | M | \psi'_j \rangle$$

(55)

during all $i$ and $j$. Finally, we apply Lemma 14 to $\{|\psi'_i\rangle\}_{i=1}^{|R_1|}$, $\{|\psi'_j\rangle\}_{j=1}^{|R_2|}$, and $M$:

$$\sqrt{|R_1| \cdot |R_2|} \max_{i,j} |\langle \psi_i | (a \otimes b) | \psi_j \rangle| = \sqrt{|R_1| \cdot |R_2|} \max_{i,j} |\langle \psi'_i | M | \psi'_j \rangle|$$

$$\geq \max_{k,l} |M_{kl}|$$

$$= \max_{k,l} |\langle k | \Pi_1 (a \otimes b) \Pi_2^\dagger | l \rangle|$$

$$= \max_{k,l} |\langle r_1(k) | a | r_2(l) \rangle| \cdot |\langle c_1(k) | b | c_2(l) \rangle|$$

$$= \max_{(r_1, c_1) \in R_1, (r_2, c_2) \in R_2} |a_{r_1r_2}| \cdot |b_{c_1c_2}|,$$

and the result follows. □

When regions $R_1$ and $R_2$ are two distinct tiles from the tiling induced by $S$, we can use Lemma 15 to get the following result:

Lemma 16 (“Pair of tiles” Lemma). Let $T_1$ and $T_2$ be two distinct tiles in the tiling induced by $S$, and let $a \in \text{Pos}(\mathbb{C}^{d_A})$ and $b \in \text{Pos}(\mathbb{C}^{d_B})$. Then

$$\sqrt{|T_1| \cdot |T_2|} \delta_S(a \otimes b) \text{ Tr}(a \otimes b) \geq |a_{r_1r_2}| \cdot |b_{c_1c_2}|$$

(56)

for any $r_t \in \text{rows}(T_t)$ and $c_t \in \text{cols}(T_t)$, where $t \in \{1, 2\}$.

Proof. We relax the inequality in Lemma 15 by observing that

$$\delta_S(a \otimes b) \geq \max_{i \neq j} \frac{|\langle \psi_i | (a \otimes b) | \psi_j \rangle|}{\|a \otimes b\|_\infty} \geq \frac{\max_{i \neq j} |\langle \psi_i | (a \otimes b) | \psi_j \rangle|}{\text{ Tr}(a \otimes b)},$$

(57)

which easily follows from the definition of $\delta_S(a \otimes b)$ in Eq. (20). □

Note that the tiles $T_1$ and $T_2$ in Lemma 16 have to be distinct since the maximization in the definition of $\delta_S(a \otimes b)$ is performed only over pairs of distinct states. This lemma will be used later to bound the off-diagonal entries of $a \otimes b$ (see Fig. 5).
5. Domino States

In this section we use the framework introduced earlier to give a lower bound on the error probability of any LOCC measurement for discriminating states from certain bipartite orthonormal product bases known as domino states. This provides an alternative proof of the quantitative separation between LOCC and separable measurements first given in [BDF+99] as well as generalizations to states corresponding to other domino-type tilings and a rotated version of the original domino states (Fig. 6).

5.1. Definition. The following orthonormal product basis is known as the domino states:

\[
|\psi_1\rangle = |1\rangle|1\rangle,
|\psi_2\rangle = |0\rangle|0+1\rangle,
|\psi_3\rangle = |0\rangle|0-1\rangle,
|\psi_4\rangle = |2\rangle|1+2\rangle,
|\psi_5\rangle = |2\rangle|1-2\rangle,
|\psi_6\rangle = |1+2\rangle|0\rangle,
|\psi_7\rangle = |1-2\rangle|0\rangle,
|\psi_8\rangle = |0+1\rangle|2\rangle,
|\psi_9\rangle = |0-1\rangle|2\rangle,
\]

where \(|i \pm j\rangle := (|i\rangle \pm |j\rangle)/\sqrt{2}\). In [BDF+99] it was shown that any LOCC protocol for discriminating these states has information deficit at least \(5.31 \times 10^{-6}\) (out of \(\log_2 9 \approx 3.17\)) bits.

In [BDF+99] the authors also consider a family of orthonormal product bases, the so-called rotated domino states, which are parametrized by four angles \(0 \leq \theta_1, \theta_2, \theta_3, \theta_4 \leq \pi/4\) and are defined as follows:
\[ |\psi_1\rangle = |1\rangle|1\rangle, \] (63)
\[ |\psi_2\rangle = |0\rangle(\cos \theta_1|0\rangle + \sin \theta_1|1\rangle), \] (64)
\[ |\psi_3\rangle = |0\rangle(-\sin \theta_1|0\rangle + \cos \theta_1|1\rangle), \] (65)
\[ |\psi_4\rangle = |2\rangle(\cos \theta_2|1\rangle + \sin \theta_2|2\rangle), \] (66)
\[ |\psi_5\rangle = |2\rangle(-\sin \theta_2|1\rangle + \cos \theta_2|2\rangle), \] (67)
\[ |\psi_6\rangle = (\cos \theta_3|1\rangle + \sin \theta_3|2\rangle)|0\rangle, \] (68)
\[ |\psi_7\rangle = (-\sin \theta_3|1\rangle + \cos \theta_3|2\rangle)|0\rangle, \] (69)
\[ |\psi_8\rangle = (\cos \theta_4|0\rangle + \sin \theta_4|1\rangle)|2\rangle, \] (70)
\[ |\psi_9\rangle = (-\sin \theta_4|0\rangle + \cos \theta_4|1\rangle)|2\rangle. \] (71)

Let \( S_3(\theta_1, \theta_2, \theta_3, \theta_4) \) denote the rotated domino basis parametrized as above. Then the original domino basis is \( S_3 := S_3(\pi/4, \pi/4, \pi/4, \pi/4) \).

Reference [BDF+99] shows that states from the domino basis \( S_3 \) cannot be perfectly discriminated by asymptotic LOCC and conjectures that the same holds for the rotated domino basis \( S_3(\theta_1, \theta_2, \theta_3, \theta_4) \) for any \( 0 < \theta_1, \theta_2, \theta_3, \theta_4 \leq \pi/4 \). In the next section we give an alternative proof that quantifies the nonlocality of the original domino states \( S_3 \) and then adapt the argument to the rotated domino states, thus resolving the conjecture.

### 5.2. Nonlocality of the domino states.

To lower bound the nonlocality constant of the domino states \( S_3 \), we put an upper bound on their rigidity. In other words, we show that measurement operators that only slightly disturb these states are weak (approximately proportional to the identity operator). The key ingredient of the proof is Lemma 16 from Sect. 4.

**Lemma 17.** The domino state basis \( S_3 \) is 4-rigid.

**Proof.** The claimed result can be restated as follows (see Definition 11):

\[
\left| a_{ij}b_{kl} - \frac{1}{9} \text{Tr}(a \otimes b) \right| \leq 4\delta \text{Tr}(a \otimes b),
\] (72)
\[
|a_{ij}b_{kl}| \leq 4\delta \text{Tr}(a \otimes b),
\] (73)

where \( i, j, k, l \in \{0, 1, 2\} \) and \( i \neq j \) or \( k \neq l \) in the second equation. First we prove the bound for the diagonal elements and then we proceed to bound the off-diagonal ones.

**Bounding the diagonal elements.** We start by bounding the differences of the diagonal elements of matrices \( a \) and \( b \) separately. Let us rewrite the definition of \( \delta \) from Eq. (20) in the case of product states \( |\psi_i\rangle = |\alpha_i\rangle|\beta_i\rangle \):

\[
\delta = \max_{i \neq j} \frac{|\langle \alpha_i | a | \alpha_j \rangle|}{\sqrt{\langle \alpha_i | a | \alpha_i \rangle \langle \alpha_j | a | \alpha_j \rangle}} \cdot \frac{|\langle \beta_i | b | \beta_j \rangle|}{\sqrt{\langle \beta_i | b | \beta_i \rangle \langle \beta_j | b | \beta_j \rangle}}. \] (74)

If we consider the pair of states \( |\psi_{2,3}\rangle = |0\rangle|0\rangle \pm 1\rangle \), we get

\[
\delta \geq \frac{|a_{00}|}{|a_{00}|} \cdot \frac{|b_{00} - b_{01} + b_{10} - b_{11}|}{\sqrt{(b_{00} + b_{01} + b_{10} + b_{11})(b_{00} - b_{01} - b_{10} + b_{11})}} \]
(75)
\[
= \frac{|b_{00} - b_{11}|}{|b_{00} + b_{11}|} \cdot \frac{|b_{00} - b_{11}|}{\sqrt{(b_{00} + b_{11})^2 - (b_{01} + b_{10})^2}} \]
(76)
\[
\geq \frac{|b_{00} - b_{11}|}{|b_{00} + b_{11}|} \cdot \frac{|b_{00} - b_{11}|}{\text{Tr}(b)} \].
(77)
Note that the cancellation of $|a_{00}|$ is valid since $a_{00} \neq 0$ by the definition of Stage 1. Applying a similar argument to the pairs of states from the other three tiles of size 2, we get that for any $i \in \{0, 2\}$,

$$\delta \text{Tr}(a) \geq |a_{11} - a_{ii}| \quad \text{and} \quad \delta \text{Tr}(b) \geq |b_{11} - b_{ii}|.$$  

(75)

Using these bounds and the triangle inequality, we can bound the difference between the first and last diagonal elements:

$$|a_{00} - a_{22}| \leq |a_{00} - a_{11}| + |a_{11} - a_{22}| \leq 2\delta \text{Tr}(a),$$

(76)

and similarly $|b_{00} - b_{22}| \leq 2\delta \text{Tr}(b)$.

Next, we use the bounds on the differences of the diagonal elements of $a$ and $b$ to bound the differences of the diagonal elements of $a \otimes b$. For all $i, j, k, t \in \{0, 1, 2\}$ we have

$$|a_{ii}b_{jj} - a_{kk}b_{tt}| \leq |a_{ii}b_{jj} - a_{kk}b_{jj}| + |a_{kk}b_{jj} - a_{kk}b_{tt}|$$

$$= |b_{jj}| \cdot |a_{ii} - a_{kk}| + |a_{kk}| \cdot |b_{jj} - b_{tt}|$$

$$\leq |b_{jj}| \cdot 2\delta \text{Tr}(a) + |a_{kk}| \cdot 2\delta \text{Tr}(b)$$

$$\leq 4\delta \text{Tr}(a \otimes b).$$

(80)

Using this inequality we can obtain the desired bound (68) for the diagonal elements: for all $i, j \in \{0, 1, 2\}$ we have

$$\left| a_{ii}b_{jj} - \frac{1}{9}\text{Tr}(a \otimes b) \right| = \left| a_{ii}b_{jj} - \frac{1}{9}\sum_{k,t \in \{0,1,2\}} a_{kk}b_{tt} \right|$$

$$\leq \frac{1}{9}\sum_{k,t \in \{0,1,2\}} |a_{ii}b_{jj} - a_{kk}b_{tt}|$$

$$\leq 4\delta \text{Tr}(a \otimes b).$$

(83)

Bounding the off-diagonal elements. From Lemma 16 we know that $\sqrt{|T_1|} \cdot |T_2| \cdot \delta \text{Tr}(a \otimes b) \geq |a_{r_1r_2}| \cdot |b_{c_1c_2}|$, where $T_1$ and $T_2$ are two distinct tiles and $|T_i|$ is the area of the tile containing $(r_i, c_i)$. For $(r_1, c_1) = (1, 1)$ and any $(r_2, c_2) \neq (1, 1)$ we get

$$\sqrt{2}\delta \text{Tr}(a \otimes b) \geq |a_{r_1r_2}| \cdot |b_{c_1c_2}|.$$  

(84)

Similarly, for any $(r_1, c_1)$ and $(r_2, c_2)$ that belong to distinct tiles of size two we get

$$2\delta \text{Tr}(a \otimes b) \geq |a_{r_1r_2}| \cdot |b_{c_1c_2}|.$$  

(85)

Now it only remains to bound the following four off-diagonal elements (each of which corresponds to one of the four tiles of size 2):

$$|a_{00}| \cdot |b_{01}|, \quad |a_{01}| \cdot |b_{22}|, \quad |a_{22}| \cdot |b_{12}|, \quad |a_{12}| \cdot |b_{00}|.$$  

(86)

To bound $|a_{00}| \cdot |b_{01}|$, first choose $(r_2, c_2) = (1, 0)$ and use Eq. (84):

$$\sqrt{2}\delta \text{Tr}(a \otimes b) \geq |a_{11}| \cdot |b_{10}| = |a_{11}| \cdot |b_{01}|.$$  

(87)
Now it only remains to replace \( a_{11} \) by \( a_{00} \). Notice from Eq. (75) that 
\[ \delta \text{Tr}(a) \geq |a_{11} - a_{00}| \geq |a_{00}| - |a_{11}|, \]
so
\[ \sqrt{2} \delta \text{Tr}(a \otimes b) \geq |a_{11}| \cdot |b_{01}| \geq (|a_{00}| - \delta \text{Tr}(a)) \cdot |b_{01}| \]  \( (88) \)
\[ \geq |a_{00}| \cdot |b_{01}| - \delta \text{Tr}(a \otimes b), \]  \( (89) \)
where the last inequality holds since \(|b_{01}| \leq \max \{b_{00}, b_{11}\} \leq \text{Tr}(b)\) as \( b \) is positive semidefinite. After rearranging the previous expression we obtain
\[ (1 + \sqrt{2}) \delta \text{Tr}(a \otimes b) \geq |a_{00}| \cdot |b_{01}|. \]  \( (90) \)
By appropriately choosing the value of \((r_2, c_2)\) and using a similar argument, we get the same upper bound for the remaining three off-diagonal elements listed in Eq. (86). Since the constants obtained in bounds (84), (85), and (90) satisfy 
\[ \max \left\{ \sqrt{2}, 2, 1 + \sqrt{2} \right\} \leq 4, \]
we have shown that Eq. (69) holds for all off-diagonal elements of \( a \otimes b \). \( \Box \)

Together with Eq. (38) this implies that the nonlocality constant for the domino states is \( \eta \geq 1/8 \). To get an explicit value for the lower bound on the error probability, we use Theorem 13 with \( n = 9, L = 2, \) and \( c = 4 \).

**Corollary 18.** Any LOCC measurement for discriminating the domino states \( S_3 \) errs with probability
\[ p_{\text{error}} \geq 1.9 \times 10^{-8}. \]  \( (91) \)

**5.3. Nonlocality of irreducible domino-type tilings.** Lemma 17 can be easily generalized to product bases that are similar to domino states on larger quantum systems. The main ideas of the proof are essentially the same as in Lemma 17, but the argument must be adapted to accommodate tiles in arbitrary positions. A complete proof can be found in Appendix A.

**Lemma 19.** Let \( d_A, d_B \geq 3 \) and let \( S \) be an orthonormal product basis of \( \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \). If \( S \) induces an irreducible domino-type tiling of diameter \( D \) then \( S \) is \( 2D \)-rigid (see Sect. 4.1 for terminology).

To bound the error probability, we use Theorem 13 with \( n = d_A d_B, L = 2, \) and \( c = 2D \).

**Corollary 20.** Any LOCC measurement for discriminating states from an orthonormal product basis of \( \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \) that induces an irreducible domino-type tiling of diameter \( D \) errs with probability
\[ p_{\text{error}} \geq \frac{1}{216D^2(d_A d_B)^5}. \]  \( (92) \)
5.4. Nonlocality of the rotated domino states. The following is an analog of Lemma 17 for rotated domino states.

**Lemma 21.** The rotated domino basis \( S_3(\theta_1, \theta_2, \theta_3, \theta_4) \) is \( \frac{C}{\sin 2\theta} \)-rigid, where
\[
C := 6 \left( 1 + 6\sqrt{2} + 2\sqrt{3(6 + \sqrt{2})} \right) \leq 114
\] (93)
and \( \theta := \min \{\theta_1, \theta_2, \theta_3, \theta_4\} \).

The proof appears in Appendix B.

Again, we use Theorem 13 to lower bound the error probability. Here the parameters are \( n = 9, L = 2, \) and \( c = 114/\sin(2\theta) \).

**Corollary 22.** Any LOCC measurement for discriminating \( S_3(\theta_1, \theta_2, \theta_3, \theta_4) \), the set of rotated domino states, errs with probability
\[
p_{\text{error}} \geq 2.4 \times 10^{-11} \sin^2(2\theta),
\] (94)
where \( \theta := \min \{\theta_1, \theta_2, \theta_3, \theta_4\} \).

Note that as \( \theta \) approaches zero, the rigidity bound tends to infinity and the bound on the error probability goes to zero. As the original domino basis is transformed continuously to the standard basis, the nonlocality decreases to zero. Moreover, since any orthonormal product basis of \( \mathbb{C}^3 \otimes \mathbb{C}^3 \) is equivalent to \( S_3(\theta_1, \theta_2, \theta_3, \theta_4) \) (up to local unitary transformations) for some angles \( \theta_i \) [FS09], Corollary 22 effectively covers all product bases of \( \mathbb{C}^3 \otimes \mathbb{C}^3 \).

6. Limitations of the Framework

6.1. Dependence of the nonlocality constant on \( n \). Recall that in Theorem 10 we established the lower bound \( p_{\text{error}} \geq \frac{\eta^2}{27 n^2} \) on the error probability, where \( \eta \) is the nonlocality constant and \( n \) is the number of states. Intuitively it seems that it should be possible to prove a stronger lower bound on \( p_{\text{error}} \) as \( n \) increases. However, to lower bound \( p_{\text{error}} \) by a fixed constant in any dimension using our framework, one would have to prove a lower bound on \( \eta \) that increases with \( n \).

Let us consider the problem of discriminating orthonormal product states. In the next lemma we show that it is not possible to obtain such strong error bounds using our framework in its present form. We do this by proving a fixed upper bound on the nonlocality constant in any dimension.

**Lemma 23.** Let \( S \) be a set of orthonormal product states in \( \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \). The nonlocality constant of \( S \) satisfies \( \eta \leq 2 \).

**Proof.** Let \( n = |S| \) and \( |\psi_i\rangle = |\alpha_i\rangle |\beta_i\rangle \). Fix some small \( \epsilon > 0 \), choose any \( i \in [n] \), and define
\[
a = |\alpha_i\rangle \langle \alpha_i| + \epsilon I_{d_A}, \quad b = |\beta_i\rangle \langle \beta_i| + \epsilon I_{d_B}.
\] (95)
Note that \( a \) and \( b \) have full rank and are positive semidefinite. We can easily check that
\[
\text{Tr}(a) = 1 + \epsilon d_A, \quad \text{Tr}(b) = 1 + \epsilon d_B, \quad \max_{k \in [n]} \langle \psi_k | (a \otimes b) | \psi_k \rangle = (1 + \epsilon)^2.
\] (96)
Using these observations together with the definition of $\eta$ in Eq. (21), we get

$$
\eta \left( \frac{(1 + \epsilon)^2}{(1 + \epsilon \text{d}_A)(1 + \epsilon \text{d}_B)} - \frac{1}{n} \right) \leq \eta \left( \frac{\max_{k \in [n]} \langle \psi_k | (a \otimes b) | \psi_k \rangle - \frac{1}{n}}{\sum_{j \in [n]} \langle \psi_j | (a \otimes b) | \psi_j \rangle} \right) \tag{97}
$$

$$
= \delta_S(a \otimes b) \tag{98}
$$

$$
\leq 1, \tag{99}
$$

where the last inequality follows directly from Definition 6. As $\epsilon \to 0$, the left-hand side goes to $\eta(1 - \frac{1}{n})$. We can choose $\epsilon$ arbitrarily small, so $\eta(1 - \frac{1}{n}) \leq 1$ and thus $\eta \leq \frac{n}{n-1} = 1 + \frac{1}{n-1} \leq 2$ since $n \geq 2$. $\Box$

### 6.2. Comparison to the result of Kleinmann, Kampermann, and Bruß.

The main application of the framework introduced in this paper is to show the impossibility of asymptotically discriminating a set of states $S$ with LOCC. We do this by showing that the nonlocality constant of $S$ is strictly positive. In other words, the nonlocality constant being zero is a necessary condition for the states in $S$ to be asymptotically distinguishable with LOCC. Another necessary condition is presented in recent work of Kleinmann, Kampermann, and Bruß:

**Theorem** ([KKB11]). Let $S = \{\rho_1, \ldots, \rho_n\}$ be a set of states such that $\bigcap_i \ker \rho_i$ does not contain any nonzero product vector. Then $S$ can be asymptotically discriminated with LOCC only if for all $\chi$ with $1/n \leq \chi \leq 1$ there exists a positive semidefinite product operator $E$ satisfying all of the following:

1. $\sum_i \text{Tr}(E\rho_i) = 1$,
2. $\max_i \text{Tr}(E\rho_i) = \chi$,
3. $\text{Tr}(E\rho_i E\rho_j) = 0$ for any $i \neq j$.

One should note however that in contrast to the above qualitative result, our framework can be applied to any set of orthogonal pure states (with no restriction on $\bigcap_i \ker \rho_i$) and can be used to obtain explicit lower bounds on the error probability. It is an open question whether our necessary condition (“the nonlocality constant of $S$ is zero”) or that of the above theorem is also sufficient. The lemma below shows that if our necessary condition is also sufficient then so is that of [KKB11].

**Lemma 24.** Let $S = \{\ket{\psi_i}\}_{i \in [n]}$ be a set of orthogonal pure states such that $\bigcap_i \ker(\ket{\psi_i}\bra{\psi_i})$ does not contain any nonzero product vector. If for all $\chi$ with $1/n \leq \chi \leq 1$ there exists a positive semidefinite product operator $E$ satisfying Conditions 1–3 from the above theorem, then the nonlocality constant $\eta$ of $S$ is zero.

**Proof.** Consider $\chi \in \left(\frac{1}{n}, \frac{1}{n-1}\right)$ and a positive semidefinite product operator $E_\chi$ satisfying Conditions 1–3. Conditions 1 and 2 imply that $\langle \psi_i | E_\chi | \psi_i \rangle > 0$, thus making $\delta_S(E_\chi)$ well defined (see Definition 6). Moreover, by Condition 3 we have that $|\langle \psi_i | E_\chi | \psi_j \rangle|^2 = 0$ for all $i \neq j$. Hence $\delta_S(E_\chi) = 0$ according to Definition 6. Finally, from Conditions 1 and 2, we get that

$$
\frac{\max_i \langle \psi_i | E_\chi | \psi_i \rangle \sum_j \langle \psi_j | E_\chi | \psi_j \rangle}{\sum_j \langle \psi_j | E_\chi | \psi_j \rangle} = \frac{\max_i \text{Tr}(E\rho_i)}{\sum_j \text{Tr}(E\rho_j)} = \chi. \tag{100}
$$
Using these observations we can rewrite Eq. (21) in the definition of $\eta$ as

$$\eta \left( \chi - \frac{1}{n} \right) \leq 0.$$  \hspace{1cm} (101)

Since $\chi > \frac{1}{n}$ it follows from the above inequality that $\eta = 0$. $\square$

7. Discussion and Open Problems

We have developed a framework for quantifying the hardness of distinguishing sets of bipartite pure states with LOCC. Using this framework, we proved lower bounds on the error probability of distinguishing several sets of states, as summarized in Table 1.

This work raises several open problems. While we were able to lower bound the non-locality constant $\eta$ in many cases, it could be useful to develop more generic approaches to computing or lower bounding this quantity. We are also interested in applying our method to other sets of states. For example, we would like to apply the method when $S$ is an incomplete orthonormal set (e.g., the domino basis with the middle tile omitted) or a product basis with tiles of size larger than two (see Fig. 7 for concrete examples of such tilings where no bounds on $p_{\text{error}}$ are known) or both (e.g., the states called GenTiles in [DMS+03]). It is unknown whether there exists a set $S$ of 2-qubit states that can be perfectly discriminated with separable operations, but for which any LOCC protocol has $p_{\text{error}}(S) > 0$ (see [DFXY09] for all possible candidate sets). Finally, it would be interesting to consider random product bases, since this would tell us how generic the phenomenon of nonlocality without entanglement is.

We discussed some limitations of our framework in Sect. 6, but we would like to better understand how broadly the framework can be applied. In particular, can it always

| Set of states          | $c$     | $\eta$  | $p_{\text{error}}$ |
|------------------------|---------|---------|---------------------|
| Domino states          | 4       | $\frac{1}{8}$ | $1.96 \times 10^{-8}$ |
| Domino-type states     | $2D$    | $\frac{1}{4D}$ | $\frac{1}{216 D^2 (d_A d_B)^5}$ |
| $\theta$-rotated domino states | $\frac{114}{\sin 2\theta}$ | $\frac{\sin 2\theta}{227}$ | $2.41 \times 10^{-11} \sin^2(2\theta)$ |

Fig. 7. Tilings corresponding to an incomplete orthonormal set in $\mathbb{C}^3 \otimes \mathbb{C}^4$ (left) and a product basis of $\mathbb{C}^5 \otimes \mathbb{C}^5$ with larger tiles (right). On the right, the tiles of size four are induced by states of the form $|\pm\rangle |\pm\rangle$ and one of the tiles corresponds to the four corners of the grid.
be used to obtain a lower bound on $p_{\text{error}}$ whenever such a bound exists? For example, from Sect. 5.4 we know that the answer to this question is “yes” for orthonormal product bases on two qudits.

Finally, the gaps between the classes of separable and LOCC operations exhibited by our framework are rather small (see Table 1). One cannot hope to do significantly better within our framework, as shown in Sect. 6.1. Is this due to limitations of our framework or because orthonormal product states in general can be discriminated well by LOCC?

Along these lines, a major open question raised by our work is the following: does there exist a sequence $S_1, S_2, S_3, \ldots$ of sets of orthonormal product states such that

$$\lim_{l \to \infty} p^{\text{LOCC}}(S_l) = 1?$$

Existence of such a sequence would give a strong separation between the classes of separable and LOCC measurements. Note that the local standard basis measurement followed by guessing gives the correct answer with probability at least $1/\mathcal{L}_l$, where $\mathcal{L}_l$ is the maximum number of states within a tile in the tiling induced by $S_l$. Thus for any such sequence, the value of $\mathcal{L}_l$ must grow with $l$. In particular, the number of states (and hence the local dimensions) must also grow with $l$.

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A. Rigidity of Domino-Type States (Lemma 19)

**Lemma 19.** Let $d_A, d_B \geq 3$ and let $S$ be an orthonormal product basis of $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$. If $S$ induces an irreducible domino-type tiling of diameter $D$ then $S$ is $2D$-rigid (see Sect. 4.1 for terminology).

**Proof.** We mimic the proof of Lemma 17 and make the appropriate generalizations when necessary. We want to show that

$$|a_{ii}b_{jj} - \frac{1}{d_Ad_B} \text{Tr}(a \otimes b)| \leq 2D\delta \text{Tr}(a \otimes b),$$

(102)

$$|a_{ij}b_{kt}| \leq 2D\delta \text{Tr}(a \otimes b),$$

(103)

where $i \neq j$ or $k \neq t$ in the second inequality.

**Bounding the diagonal elements.** Using the calculation in Eqs. (71–74) we can bound the difference of diagonal entries of $a$ and $b$. Whenever there is a $2 \times 1$ tile that connects rows $i$ and $j$, we get that

$$|a_{ii} - a_{jj}| \leq \delta \text{Tr}(a).$$

(104)

A similar equation holds for $b$ whenever there is a $1 \times 2$ tile that connects columns $i$ and $j$.

Since $T$ is irreducible, the row graph of $T$ is connected. Moreover, any two vertices of this graph are connected by a path of length at most $D$. We apply the triangle inequality along this path in the same way as in Eq. (76). After at most $D - 1$ repetitions we get that for any $i$ and $j$,

$$|a_{ii} - a_{jj}| \leq D\delta \text{Tr}(a).$$

(105)
A similar equation holds for $b$. When we repeat the calculation in Eqs. (77–80), we get that for any $i, j, k, t$,

$$|a_{ii}b_{jj} - a_{kk}b_{tt}| \leq 2D\delta Tr(a \otimes b). \quad (106)$$

Finally, we repeat the calculation in Eqs. (81–83) and get the desired bound stated in Eq. (102).

Bounding the off-diagonal elements. From Lemma 16 we get that

$$2\delta Tr(a \otimes b) \geq |a_{r_1r_2}| \cdot |b_{c_1c_2}| \quad (107)$$

for all $(r_1, c_1) \neq (r_2, c_2)$, except when $(r_1, c_1)$ and $(r_2, c_2)$ belong to the same tile of size two.

Suppose that we want to bound $|a_{rr}| \cdot |b_{c_1c_2}|$, where $(r, c_1)$ and $(r, c_2)$ belong to the same $1 \times 2$ tile. Since $T$ is irreducible, we can find a row $r'$ such that $(r', c_1)$ and $(r', c_2)$ belong to different tiles (if $\{r'\} \times \{c_1, c_2\}$ is a tile for each $r'$ then $\{c_1, c_2\}$ is a connected component of the column graph of $T$, contradicting the irreducibility of $T$). From Eq. (85) we get

$$2\delta Tr(a \otimes b) \geq |a_{r'r'}| \cdot |b_{c_1c_2}|. \quad (108)$$

According to Eq. (105), $D\delta Tr(a) \geq |a_{rr} - a_{r'r'}| \geq |a_{rr}| - |a_{r'r'}|$. Using this observation we repeat the calculation in Eqs. (88–89) and obtain

$$2\delta Tr(a \otimes b) \geq |a_{rr}| \cdot |b_{c_1c_2}| - D\delta Tr(a \otimes b). \quad (109)$$

After rearranging terms we get

$$(D + 2)\delta Tr(a \otimes b) \geq |a_{rr}| \cdot |b_{c_1c_2}|. \quad (110)$$

The same bound also holds for entries corresponding to $2 \times 1$ tiles. Together with Eq. (107) this establishes the desired bound in Eq. (103). \qed

Note that the above proof uses the assumption that the size of any tile is at most two in an essential way. Without this assumption, it might not be possible to find a row $r'$ such that $(r', c_1)$ and $(r', c_2)$ belong to different tiles when bounding the off-diagonal element $|a_{rr}| \cdot |b_{c_1c_2}|$.

B. Rigidity of Rotated Domino States (Lemma 21)

In this section we prove an analog of Lemma 17 for rotated domino states $S_3(\theta_1, \theta_2, \theta_3, \theta_4)$. For simplicity we consider only the set $S_3(\theta) := S_3(\theta, \theta, \theta, \theta)$ and obtain a bound as a function of $\theta$. In the more general case one can choose $\theta := \min \{\theta_1, \theta_2, \theta_3, \theta_4\}$ and use the same bound.

**Lemma 25.** For $j \in \{0, 2\}$ we have

$$|b_{11} - b_{jj}| \leq \frac{2}{\sin 2\theta} \left( \delta \|b\|_\infty + |\text{Re} b_{j1}| \right). \quad (111)$$

The same inequality holds for $a$. 

Proof. We show how to get the bound on $b$ for $j = 0$. The remaining three cases are similar.

We use the states $|\psi_2\rangle$ and $|\psi_3\rangle$ from Eq. (64) in the definition of $\delta$ in Eq. (70):

\[
\delta \|b\|_\infty \geq |\langle \beta_2 | b | \beta_3 \rangle| \quad (112)
\]

\[
= \left| (\cos \theta \sin \theta) \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} \right| \quad (113)
\]

\[
= \left| (b_{11} - b_{00}) \sin \theta \cos \theta - b_{10} \sin^2 \theta + b_{01} \cos^2 \theta \right| \quad (114)
\]

\[
= \left| (b_{11} - b_{00}) \sin \theta \cos \theta + \text{Re} b_{01} (\cos^2 \theta - \sin^2 \theta) + i \text{Im} b_{01} \right| \quad (115)
\]

\[
\geq \frac{|b_{11} - b_{00}|}{2} \sin 2\theta + \text{Re} b_{01} \cos 2\theta \quad (116)
\]

\[
\geq \frac{|b_{11} - b_{00}|}{2} \sin 2\theta - |\text{Re} b_{01}|. \quad (117)
\]

By rearranging terms we get the desired bound. □

Lemma 26. If $a_{11} \geq \frac{1}{s} \|a\|_\infty$ for some $s > 0$ then for $j \in \{0, 2\}$ we have

\[
|b_{j1}| \leq \sqrt{2s} \delta \|b\|_\infty, \quad |b_{11} - b_{jj}| \leq 2(1 + \sqrt{2s}) \frac{\delta}{\sin 2\theta} \|b\|_\infty. \quad (118)
\]

The same statement holds when the roles of $a$ and $b$ are exchanged.

Proof. We show how to get bounds on $b$ for $j = 0$. The remaining three cases are identical, except one has to use states from different tiles.

We use Lemma 16 with tiles corresponding to states $|\psi_{6,7}\rangle$ and $|\psi_1\rangle$:

\[
\sqrt{2s} \delta \|a \otimes b\|_\infty \geq |a_{11} b_{01}| \geq \frac{1}{s} \|a\|_\infty |b_{01}|, \quad (119)
\]

where the second inequality follows from our assumption $|a_{11}| \geq \frac{1}{s} \|a\|_\infty$. By rewriting this we get the first bound:

\[
|b_{01}| \leq \sqrt{2s} \delta \|b\|_\infty. \quad (120)
\]

Since $|\text{Re} b_{01}| \leq |b_{01}| \leq \sqrt{2s} \delta \|b\|_\infty$, we get the second bound from Lemma 25. □

Lemma 27. If $a_{11} \geq \frac{1}{s} \|a\|_\infty$ and $b_{11} \geq \frac{1}{s} \|b\|_\infty$ for some $s > 0$ then

\[
\left\| \frac{a \otimes b}{\text{Tr}(a \otimes b)} - \frac{I}{9} \right\|_{\text{max}} \leq 8(1 + \sqrt{2s}) \frac{\delta}{\sin 2\theta}. \quad (121)
\]

Proof. We follow the proof of Lemma 17 and show the following generalizations of Eqs. (68) and (69):

\[
|a_{ij} b_{jj} - \frac{1}{9} \text{Tr}(a \otimes b)| \leq 8(1 + \sqrt{2s}) \frac{\delta}{\sin 2\theta} \|a \otimes b\|_\infty, \quad (122)
\]

\[
|a_{ij} b_{kl}| \leq \max \left\{ \sqrt{2}, 2, \sqrt{2s} \right\} \delta \|a \otimes b\|_\infty. \quad (123)
\]

Note that the second inequality is stronger than we need, since $1/\sin 2\theta \geq 1$.
First, we use Lemma 26 to upper bound the difference of diagonal entries of \( a \) and \( b \). We use these bounds in the same way as in Lemma 17 to upper bound the differences of diagonal entries of \( a \otimes b \) and to get Eq. (122). Finally, we use Lemma 16 to upper bound most of the off-diagonal entries of \( a \otimes b \) and Lemma 26 to upper bound the remaining ones. This gives us Eq. (123).

**Bounding the diagonal elements.** From Lemma 26 we get bounds on
\[
|b_{11} - b_{ii}| \quad \text{and} \quad |a_{11} - a_{ii}| \quad \text{for} \quad i \in \{0, 2\}.
\]
Using the triangle inequality, we get
\[
|a_{ii} - a_{jj}| \leq 4\left(1 + \sqrt{2}s\right)\frac{\delta}{\sin 2\theta} \|a\|_{\infty}
\] (124)
for any \( i, j \in \{0, 1, 2\} \) (and the same for \( b \)). Using the triangle inequality once more we can bound the difference of any two diagonal entries of \( a \otimes b \):
\[
|a_{ii}b_{jj} - a_{kk}b_{tt}| \leq 8\left(1 + \sqrt{2}s\right)\frac{\delta}{\sin 2\theta} \|a \otimes b\|_{\infty}.
\] (125)
From this we obtain Eq. (122) in the same way as in Lemma 17.

**Bounding the off-diagonal elements.** Eq. (123) can be obtained from Lemma 16. For most of the entries the constant is either \( \sqrt{2} \) or 2, depending on the sizes of the tiles. For the remaining four entries, listed in Eq. (86), we proceed in a slightly different way. For example, for \( a_{00}b_{01} \) we use Eq. (120) to see that
\[
|a_{00}| \cdot |b_{01}| \leq \|a\|_{\infty} \cdot \sqrt{2}s \delta \|b\|_{\infty}.
\] (126)
A similar strategy works for the remaining three entries. \( \square \)

**Lemma 28.** Fix any \( s \geq 3 \) and let
\[
\frac{1}{r(s)} := \min \left\{ \frac{1}{14} \left( \frac{1}{3} - \frac{1}{s} \right), \frac{1}{2(1 + \sqrt{2}s)} \left( \frac{1}{3} - \frac{1}{s} \right) \right\}.
\] (127)
If \( \frac{\delta}{\sin 2\theta} \leq \frac{1}{r(s)} \) then \( a_{11} \geq \frac{1}{s} \|a\|_{\infty} \) and \( b_{11} \geq \frac{1}{s} \|b\|_{\infty} \).

**Proof.** We get one of the two lower bounds almost for free. We combine this with Lemma 16 and the triangle inequality to get the other lower bound.

If \( \max_i a_{ii} = a_{11} \) then \( a_{11} \geq \frac{1}{s} \|a\|_{\infty} \) and we are done with \( a \). Similarly, if \( \max_i b_{ii} = b_{11} \) then \( b_{11} \geq \frac{1}{s} \|b\|_{\infty} \). Thus it only remains to consider the cases when \( \max_i a_{ii} \in \{a_{00}, a_{22}\} \) and \( \max_i b_{ii} \in \{b_{00}, b_{22}\} \). By symmetry, it suffices to consider the case where \( \max_i a_{ii} = a_{22} \) and \( \max_i b_{ii} = b_{00} \). The remaining three cases are similar.

Using the tiles that correspond to states \( |\psi_{6,7}\rangle \) and \( |\psi_{4,5}\rangle \), we get
\[
2\delta \|a \otimes b\|_{\infty} \geq |a_{22}b_{01}| \geq \frac{1}{3} \|a\|_{\infty} |b_{01}|.
\] (128)
Thus \( |\Re b_{01}| \leq |b_{01}| \leq 6\delta \|b\|_{\infty} \) and using Lemma 25, we get
\[
b_{00} - b_{11} \leq |b_{11} - b_{00}|
\leq \frac{2}{\sin 2\theta} \left( \delta \|b\|_{\infty} + |\Re b_{01}| \right)
\leq 14 \frac{\delta}{\sin 2\theta} \|b\|_{\infty}.
\] (131)
We assumed that \( \max_i b_{ii} = b_{00} \), so
\[
\frac{1}{3} \| b \|_\infty \leq b_{00} \leq b_{11} + 14 \frac{\delta}{\sin 2\theta} \| b \|_\infty .
\] (132)

By assumption, \( \frac{\delta}{\sin 2\theta} \leq \frac{1}{r(s)} \leq \frac{1}{14} \left( \frac{1}{3} - \frac{1}{s} \right) \), so we get the desired bound \( b_{11} \geq \frac{1}{s} \| b \|_\infty \). As we have a lower bound on \( b_{11} \), we can use Lemma 26 and get
\[
| a_{11} - a_{22} | \leq 2 \left( 1 + \sqrt{2} s \right) \frac{\delta}{\sin 2\theta} \| a \|_\infty .
\] (133)

We assumed that \( \max_i a_{ii} = a_{22} \), so we can rewrite this as
\[
\frac{1}{3} \| a \|_\infty \leq a_{22} \leq a_{11} + 2 \left( 1 + \sqrt{2} s \right) \frac{\delta}{\sin 2\theta} \| a \|_\infty .
\] (134)

By assumption, \( \frac{\delta}{\sin 2\theta} \leq \frac{1}{r(s)} \left( \frac{1}{3} - \frac{1}{s} \right) \), so we get the desired bound \( a_{11} \geq \frac{1}{s} \| a \|_\infty \). \( \Box \)

**Lemma 29.** For any fixed \( s \geq 3 \) we have the following:

- if \( \frac{\delta}{\sin 2\theta} \leq \frac{1}{r(s)} \) then \( \left\| \frac{a \otimes b}{\text{Tr}(a \otimes b)} - \frac{I}{9} \right\|_{\max} \leq 8 \left( 1 + \sqrt{2} s \right) \frac{\delta}{\sin 2\theta} \),
- if \( \frac{\delta}{\sin 2\theta} \geq \frac{1}{r(s)} \) then \( \left\| \frac{a \otimes b}{\text{Tr}(a \otimes b)} - \frac{I}{9} \right\|_{\max} \leq r(s) \frac{\delta}{\sin 2\theta} \),

where \( r(s) \) is defined in Eq. (127).

**Proof.** The first part follows by combining Lemmas 27 and 28. To obtain the second part, notice that all diagonal entries of \( \frac{a \otimes b}{\text{Tr}(a \otimes b)} \) are at most 1. Since this matrix is positive semidefinite, the off-diagonal entries are also at most 1, so the bound follows. \( \Box \)

**Lemma 21.** The rotated domino basis \( S_3(\theta_1, \theta_2, \theta_3, \theta_4) \) is \( \frac{C}{\sin 2\theta} \)-rigid where
\[
C := 6 \left( 1 + 6 \sqrt{2} + 2 \sqrt{3(6 + \sqrt{2})} \right) \leq 114
\] (135)
and \( \theta := \min \{ \theta_1, \theta_2, \theta_3, \theta_4 \} \).

**Proof.** Let us denote the largest of the two constants in Lemma 29 by
\[
C(s) := \max \left\{ 8 \left( 1 + \sqrt{2} s \right), r(s) \right\}
\] (136)
\[
= \max \left\{ 8 \left( 1 + \sqrt{2} s \right), 14 \frac{3s}{s - 3}, 2 \left( 1 + \sqrt{2} s \right) \frac{3s}{s - 3} \right\},
\] (137)
where we substituted \( r(s) \) from Eq. (127). We want to make this constant as small as possible, so the best possible value is
\[
C = \min_{s \geq 3} C(s)
\] (138)
\[
= \min_{s \geq 3} \left( 2 \left( 1 + \sqrt{2} s \right) \frac{3s}{s - 3} \right)
\] (139)
\[
= 6 \left( 1 + 6 \sqrt{2} + 2 \sqrt{3(6 + \sqrt{2})} \right) \approx 6.33. \quad \Box
\] (140)
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