A new kind of numbers, the Non-Dedekindian Numbers, and the extension to them of the notion of algorithmic randomness

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A new number system, the set of the non-Dedekindian numbers, is introduced and characterized axiomatically.

It is then proved that any hypercontinuous hyperreal number system is strictly included in the set of the Non-Dedekindian Numbers.

The notion of algorithmic-randomness is then extended to non-Dedekindian numbers.

As a particular case, the notion of algorithmic randomness for the particular hyperreal number system of Non-Standard Analysis is explicitly analyzed.

1 The reported date is the one of the first public (i.e. appeared on my homepage http://www.gavrielsegre.com) version of this paper that has, anyway, received later improvements.
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I. INTRODUCTION

The non-euclidean revolution \[1\] consisted in the discovery of many kind of non-euclidean geometries corresponding to different choices in the imposed cardinality of the set of straight lines parallel to a given straight line and passing for a point not belonging to it (euclidean geometry consisting in the assumption, stated by Euclides’ Fifth Axiom, that such a cardinal number is equal to one).

In a completely different framework, the axiomatic definition of the set \( \mathbb{R} \) of the real numbers \[2\], Dedekind’s Continuity Axiom resembles, with this respect, Euclides’ Fifth Axiom since again it imposes that the cardinality of a suitable set (the intersection of a sequence of nested halving intervals) is equal to one.

Such a similarity naturally induces to investigate which kind of different numbers’ systems we obtain by replacing Dedekind’s Axiom with different choices in the imposed cardinality of the intersection of nested halving intervals.

What one obtains in this way is a new number system, the set \( \mathbb{ND} \) of the Non-Dedekindian numbers, that here we explicitly define axiomatically.

We then show that any hypercontinuous hyperreal number system is strictly included in \( \mathbb{ND} \).

Finally we extend the notion of algorithmic-randomness to non-Dedekindian numbers.

As a particular case, the notion of algorithmic randomness for the particular hyperreal number system of Non-Standard Analysis is explicitly analyzed.
II. DIFFERENT WAYS TO DEFINE THE REAL NUMBERS

As it is well known there exist many different equivalent ways of defining the set \( \mathbb{R} \) of the real numbers.

Many of them has the same structure: they define \( \mathbb{R} \) as a chain-ordered field satisfying some supplementary condition of completeness (belonging to a family of equivalent completeness’ conditions).

The simpler approach was given by Cantor: introduced on \( \mathbb{Q} \) the metric \( d(r_1, r_2) := |r_1 - r_2| \) one defines \( \mathbb{R} \) as the metric completion of the metric space \((\mathbb{Q}, d)\).

In this way a real number is then defined as an equivalence class of Cauchy sequences over \( \mathbb{Q} \) with respect to the following equivalence relation:

\[
\{ r_n \}_{n \in \mathbb{N}} \sim \{ s_n \}_{n \in \mathbb{N}} := \lim_{n \to +\infty} d(r_n, s_n) = 0
\]  

(2.1)

In this paper we will concentrate, anyway, our attention on Dedekind’s way of formalizing the supplementary completeness condition since:

1. it shows in such an intuitive way the evidence that the supplementary added completeness condition is a condition assuring the "continuity" of \( \mathbb{R} \) that it is usually called the Continuity Axiom

2. it has a natural link with Algorithmic Information Theory

3. it has a natural generalization that will allow us to introduce a new number system: the set \( \mathbb{ND} \) of the non-Dedekindian numbers
III. DEDEKIND CONTINUITY AXIOM

Let us denote by $\Sigma := \{0, 1\}$ the binary alphabet, by $\Sigma^* := \cup_{n \in \mathbb{N}_+} \Sigma^n$ the set of all the binary strings and by $\Sigma^\infty := \{ \bar{x} : \mathbb{N}_+ \rightarrow \Sigma \}$ the set of all the binary sequences. Given $\bar{x} \in \Sigma^\infty$ and $a_0, b_0 \in \mathbb{R}$: $a_0 < b_0$ let us introduce the following definition by induction:

**Definition III.1**

- $a_1(\bar{x}) := \begin{cases} a_0 & \text{if } x_1 = 0 \\ \frac{a_0 + b_0}{2} & \text{if } x_1 = 1 \end{cases}$ (3.1)

- $b_1(\bar{x}) := \begin{cases} \frac{a_0 + b_0}{2} & \text{if } x_1 = 0 \\ b_0 & \text{if } x_1 = 1 \end{cases}$ (3.2)

- $a_n(\bar{x}) := \begin{cases} a_{n-1} & \text{if } x_n = 0 \\ \frac{a_{n-1} + b_{n-1}}{2} & \text{if } x_n = 1 \end{cases}$ (3.3)

- $b_n(\bar{x}) := \begin{cases} \frac{a_{n-1} + b_{n-1}}{2} & \text{if } x_n = 0 \\ b_{n-1} & \text{if } x_n = 1 \end{cases}$ (3.4)

We can at last introduce the following:

**Definition III.2**

- set $\mathbb{R}$ of the real numbers: a chain-ordered field satisfying the following axiom [III.1]

**AXIOM III.1**

*Dedekind Continuity Axiom:*

$$\exists! N_{\text{Dedekind}}(a_0, b_0; \bar{x}) \in \mathbb{R} : N_{\text{Dedekind}}(a_0, b_0; \bar{x}) \in \cap_{n \in \mathbb{N}} [a_n(\bar{x}), b_n(\bar{x})] \quad \forall a_0, b_0 \in \mathbb{R} : a_0 < b_0, \forall \bar{x} \in \Sigma^\infty$$ (3.5)

It may be proved that:

**Proposition III.1**

- HP:

  $$(F, \leq) \text{ chain-ordered field}$$

- TH:

  Axiom [III.1] is equivalent to Dedekind completeness

**Remark III.1**

Owing to Proposition [III.1] the definition [III.2] of $\mathbb{R}$ is equivalent to the more usual definition of $\mathbb{R}$ as a Dedekind complete chain-ordered field.

Axiom [III.1] has, anyway, a constructive nature that lacks to the condition of Dedekind Completeness: given $n \in \mathbb{N}$, $a_n(\bar{x})$ and $b_n(\bar{x})$ may be computed through the following Mathematica [3] expressions:
leftextreme[n_, a_, b_, binarystring_] :=
   If[n == 1, If[Part[binarystring, 1] == 0, a, (a + b)/2],
      If[Part[binarystring, n] == 0, leftextreme[n - 1, a, b, binarystring],
          (leftextreme[n - 1, a, b, binarystring] +
           rightextreme[n - 1, a, b, binarystring])/2]]

rightextreme[n_, a_, b_, binarystring_] :=
   If[n == 1, If[Part[binarystring, 1] == 0, (a + b)/2, b],
      If[Part[binarystring, n] == 0,
          (leftextreme[n - 1, a, b, binarystring] +
           rightextreme[n - 1, a, b, binarystring])/2,
          rightextreme[n - 1, a, b, binarystring]]]

Let us now introduce the following:

**Definition III.3**

*Dedekind operator with respect to* \([a_0, b_0]\):

\[\mathcal{D}_{a_0, b_0} : \Sigma^\infty \mapsto [a_0, b_0] : \]

\[\mathcal{D}_{a_0, b_0}(\bar{x}) := N_{\text{Dedekind}}(a_0, b_0; \bar{x})\]  \hfill (3.6)

The axiom [III.1] implies that:

**Corollary III.1**

\[\mathcal{D}_{a_0, b_0} \text{ is bijective } \forall a_0, b_0 \in \mathbb{R} : a_0 < b_0\]  \hfill (3.7)

**PROOF:**

1. Let us prove that \(\mathcal{D}_{a_0, b_0}\) is injective.
   Given \(\bar{x}, \bar{y} \in \Sigma^\infty\) : \(\bar{x} \neq \bar{y}\) this means that:

\[\exists n \in \mathbb{N} : x_n \neq y_n\]  \hfill (3.8)

But then:

\[\left[a_n(\bar{x}), b_n(\bar{x})\right] \neq \left[a_n(\bar{y}), b_n(\bar{y})\right]\]  \hfill (3.9)

and hence:

\[\mathcal{D}_{a_0, b_0}(\bar{x}) \neq \mathcal{D}_{a_0, b_0}(\bar{y})\]  \hfill (3.10)

2. Let us prove that \(\mathcal{D}_{a_0, b_0}\) is surjective.
   Given \(c \in [a_0, b_0]\) let us choose at each step the value of \(x_n \in \Sigma\) such that \(c \in [a_n, b_n]\).
   Then by construction:

\[\mathcal{D}_{a_0, b_0}(\bar{x}) = c\]  \hfill (3.11)

■
IV. Algorithmic Information Theoretic Analysis of the Dedekind Operator

Let us observe, first of all, that denoted with $v_2$ the 2-ary map introduced in the definition B.2 of section B:

**Proposition IV.1**

$$P_{0,1}(\bar{x}) = v_2(\bar{x}) \quad \forall \bar{x} \in \Sigma^\infty \quad (4.1)$$

**PROOF:**

By construction:

$$\lim_{n \to +\infty} a_n(\bar{x}) = \sum_{n=1}^{\infty} \frac{\bar{x}_n}{2^n} \quad \forall \bar{x} \in \Sigma^\infty \quad (4.2)$$

But axiom III.1 implies that:

$$\lim_{n \to +\infty} a_n(\bar{x}) = \lim_{n \to +\infty} b_n(\bar{x}) = N_{\text{Dedekind}}(0, 1; \bar{x}) \quad \forall \bar{x} \in \Sigma^\infty \quad (4.3)$$

\[\blacksquare\]

i.e. $N_{\text{Dedekind}}(0, 1; \bar{x})$ is the number having $(0.\bar{x})_2$ as base-two representation.

Hence [4]:

**Corollary IV.1**

1. 

$$P_{0,1}(\bar{x}) \in \mathbb{Q} \iff \exists \bar{y} \in \Sigma^* \cup \{\lambda\}, \bar{z} \in \Sigma^* : \bar{x} = \bar{y} \cdot \bar{z}^\infty \quad (4.4)$$

where \(\cdot\) denotes concatenation, $\lambda$ is the empty string and where $\bar{z}^\infty$ denotes the infinite repetition of the string $\bar{z}$.

2. 

$$P_{0,1}(\text{RANDOM}(\Sigma^\infty)) = \text{RANDOM}([0, 1]) \quad (4.5)$$

where $\text{RANDOM}(0, 1)$ is the set of the random reals belonging to the interval $[0, 1]$ while $\text{RANDOM}(\Sigma^\infty)$ is the set of the random binary sequences.

Let us now consider two arbitrary $a_0, b_0 \in \mathbb{R} : a_0 < b_0$ and let us introduce the following set:

**Definition IV.1**

\emph{real random numbers with respect to $[a_0, b_0]$}:

$$\text{RANDOM}([a_0, b_0]) := D_{a_0, b_0}(\text{RANDOM}(\Sigma^\infty)) \quad (4.6)$$

Let us introduce the following map:

**Definition IV.2**

$$T_{a_0, b_0} : [a_0, b_0] \mapsto [0, 1] :$$

$$T_{a_0, b_0}(x) := \frac{x - a_0}{b_0 - a_0} \quad (4.7)$$

and its inverse:

**Definition IV.3**

$$T_{a_0, b_0}^{-1} : [0, 1] \mapsto [a_0, b_0] :$$

$$T_{a_0, b_0}^{-1}(x) := (b_0 - a_0)x + a_0 \quad (4.8)$$

Then by construction:
Proposition IV.2

\[ T_{a_0,b_0}(D_{a_0,b_0}(\bar{x})) = v_2(\bar{x}) \quad \forall \bar{x} \in \Sigma^\infty \]  

(4.9)

Corollary IV.2

1. \[ T_{a_0,b_0}(D_{a_0,b_0}(\bar{x})) \in \mathbb{Q} \iff \exists \bar{y} \in \Sigma^* \cup \{\lambda\}, \bar{z} \in \Sigma^* : \bar{x} = \bar{y} \cdot \bar{z}^\infty \]  

(4.10)

2. \[ T_{a_0,b_0}(RANDOM[a_0,b_0]) = RANDOM(0,1) \]  

(4.11)

Remark IV.1

It is extremely important to remark, at this point, that while \( RANDOM(0,1) \), according to definition B.13, is an intrinsic notion characterizing the set of the random reals belonging to the interval \([0,1]\), \( RANDOM([a_0,b_0]) \) is not an intrinsic notion that characterizes the set of the random numbers belonging to \([a_0,b_0]\) but a relative notion that characterizes the random reals with respect to \([a_0,b_0]\), i.e. the set of the random reals in the interval \([0,1]\) when such an interval is seen dilatated and translated by \( T_{a_0,b_0}^{-1} \).

That this is the case may be appreciated considering the following:

Proposition IV.3

HP:

\[ a_1, a_2, b_1, b_2 \in \mathbb{R} : [a_2, b_2] \subset [a_1, b_1] \]  

(4.12)

TH:

\[ RANDOM([a_2, b_2]) \neq RANDOM([a_1, b_1]) \cap [a_2, b_2] \]  

(4.13)

PROOF:

Since:

\[ \begin{align*}
RANDOM(0,1) &= T_{a_1,b_1}^{-1}\{RANDOM([a_1, b_1])\} = T_{a_2,b_2}^{-1}\{RANDOM([a_2, b_2])\} 
\end{align*} \]  

(4.14)

if follows that:

\[ RANDOM([a_2, b_2]) = \frac{(b_1 - a_1)RANDOM([a_1, b_1]) + a_1 - a_2}{b_2 - a_2} \neq RANDOM([a_1, b_1]) \cap [a_2, b_2] \]  

(4.15)
V. A NEW NUMBER SYSTEM: THE NON-DEDEKINDIAN NUMBERS

Given \( n \in \mathbb{N} \cup \{\aleph_n, n \in \mathbb{N}\} \) let \( \mathbb{G}_n \) be a chain-ordered field \( [2] \).

Given \( \bar{x} \in \Sigma^\infty \) and \( a_0, b_0 \in \mathbb{G}_n : a_0 < b_0 \) let us introduce the following definition by induction:

Definition V.1

\[
\begin{align*}
    a_1(\bar{x}) &:= \begin{cases} 
        a_0 & \text{if } x_1 = 0 \\
        \frac{a_0 + b_0}{2} & \text{if } x_1 = 1
    \end{cases} \\
    b_1(\bar{x}) &:= \begin{cases} 
        b_0 & \text{if } x_1 = 0 \\
        \frac{a_0 + b_0}{2} & \text{if } x_1 = 1
    \end{cases}
\end{align*}
\]

\[ (5.3) \]

\[
\begin{align*}
    a_n(\bar{x}) &:= \begin{cases} 
        a_{n-1} & \text{if } x_n = 0 \\
        \frac{a_{n-1} + b_{n-1}}{2} & \text{if } x_n = 1
    \end{cases} \\
    b_n(\bar{x}) &:= \begin{cases} 
        b_{n-1} & \text{if } x_n = 0 \\
        \frac{a_{n-1} + b_{n-1}}{2} & \text{if } x_n = 1
    \end{cases}
\end{align*}
\]

\[ (5.5) \]

\[ (5.6) \]

Definition V.2

Dedekind set with respect to \( [a_0, b_0] \) and \( \bar{x} \):

\[ S_{\text{Dedekind}}(a_0, b_0; \bar{x}) := \{ d \in \mathbb{G}_n : d \in \cap_{n \in \mathbb{N}} [a_n(\bar{x}), b_n(\bar{x})] \} \]

\[ (5.7) \]

Let us now introduce the following:

Definition V.3

set of the generalized numbers of order \( n \):

the chain-ordered field \( \mathbb{G}_n \) satisfying the following axiom \[ V.2 \]

\[ \text{Axiom V.1} \]

Generalized Continuum Hypothesis:

\[ 2^{\aleph_n} = \aleph_{n+1} \quad \forall n \in \mathbb{N} \]  \[ (5.1) \]

it follows that:

\[ \aleph_n = |\mathcal{P}^n(\mathbb{N})| \quad \forall n \in \mathbb{N} \]  \[ (5.2) \]

(where \( |S| \) denotes the cardinality of a set \( S \) and where \( \mathcal{P}^n \) denotes the \( n^{\text{th}} \) iterate of the power-set operator) while cardinals \( \geq \aleph_\omega \) cannot be obtained in this way.

\[ ^2 \text{ We have assumed that } n < \aleph_\omega. \text{ This has been done since assumed the following:} \]

\[ \text{Axiom V.1} \]

Generalized Continuum Hypothesis:

\[ 2^{\aleph_n} = \aleph_{n+1} \quad \forall n \in \mathbb{N} \]  \[ (5.1) \]

it follows that:

\[ \aleph_n = |\mathcal{P}^n(\mathbb{N})| \quad \forall n \in \mathbb{N} \]  \[ (5.2) \]

(where \( |S| \) denotes the cardinality of a set \( S \) and where \( \mathcal{P}^n \) denotes the \( n^{\text{th}} \) iterate of the power-set operator) while cardinals \( \geq \aleph_\omega \) cannot be obtained in this way.
AXIOM V.2

Generalized Dedekind Axiom of order $n$:

$$|S_{\text{Dedekind}}(a_0, b_0; \bar{x})| = n \quad \forall a_0, b_0 \in \mathbb{G}_n : a_0 < b_0, \forall \bar{x} \in \Sigma^\infty$$  \hspace{1cm} (5.8)

where $|S|$ denotes the cardinality of a set $S$.

Clearly:

**Proposition V.1**

$$\mathbb{G}_1 = \mathbb{R}$$  \hspace{1cm} (5.9)

**PROOF:**

For $n = 1$ the axiom $\text{V.2}$ reduces to Dedekind’s Continuity Axiom. ■

Given $n \in \mathbb{N}_+ \cup \{\aleph_n, n \in \mathbb{N}\} : n > 1$ we will call $\mathbb{G}_n$ the set of the non-Dedekindian numbers of order $n$ and we will call $\mathbb{ND} := \bigcup\{n \in \mathbb{N}_+: n > 1\} \cup \{\aleph_n, n \in \mathbb{N}\} \mathbb{G}_n$ the set of the Non-Dedekindian numbers.

**Remark V.1**

Clearly the furnished formal axiomatic definition of the Non-Dedekindian numbers is not the whole story. One has:

1. to prove that the involved formal system is consistent
2. to prove that $\mathbb{ND} \neq \emptyset$

**Remark V.2**

As to the proof of the consistence of the given axiomatic definition of generalized numbers of any order, let us remark that we know that for $n = 1$ this is true since we know that Dedekind’s axiomatization of real numbers is consistent.

Hence, to obtain a proof by induction, it would be sufficient to prove that the consistence of the axioms for $\mathbb{G}_n$ implies the consistence for the axioms for $\mathbb{G}_{n+1}$.

We leave this task open for future investigation.

The proof that $\mathbb{ND} \neq \emptyset$ will be given in the next section where we will prove that any hypercontinuous hyperreal number system is strictly included in $\mathbb{ND}$. 
VI. HYPERCONTINUOUS HYPERREAL NUMBERS AS PARTICULAR NON DEDEKINDIAN NUMBERS

Let us start from the following:

**Definition VI.1**

*hyperreal number system:*

A chain-ordered non-Archimedean field containing \( \mathbb{R} \) as a subfield.

**Definition VI.2**

*non-standard part of \( \mathbb{H} \):*

\[
\mathbb{H}_{nonstandard} := \mathbb{H} - \mathbb{R}
\]

**Definition VI.3**

*infinitesimals of \( \mathbb{H} \):*

\[
\mathbb{H}_{infinitesimals} := \{ x \in \mathbb{H} : x \in (-r, r) \forall r \in \mathbb{R}_+ \}
\]

**Definition VI.4**

*unboundeds of \( \mathbb{H} \):*

\[
\mathbb{H}_{unboundeds} := \{ x \in \mathbb{H} : \exists r \in \mathbb{R}_+ : x \in (-r, r) \}
\]

It may be proved that [2]:

**Proposition VI.1**

1.  

\[
\mathbb{H}_{infinitesimals} \cap \mathbb{R} = \{0\}
\]

2.  

\[
x \in \mathbb{H}_{infinitesimals} - \{0\} \iff \frac{1}{x} \in \mathbb{H}_{unboundeds}
\]

3.  

\[
|\mathbb{H}_{infinitesimals}| \geq \aleph_0
\]

so that obviously:

**Corollary VI.1**

\[
|\mathbb{H}_{unboundeds}| = |\mathbb{H}_{infinitesimals}| \geq \aleph_0
\]

Let us introduce the following:

**Definition VI.5**

\( \mathbb{H} \) *is hypercontinuous:*
1. 
\[ |\mathbb{H}_{\text{infinitesimals}}| \geq \aleph_1 \] (6.8)

2. 
\[ |[\epsilon_1, \epsilon_2]| = |\mathbb{H}_{\text{infinitesimals}}| \ \forall \epsilon_1, \epsilon_2 \in \mathbb{H}_{\text{infinitesimals}} : \epsilon_1 < \epsilon_2 \] (6.9)

**Remark VI.1**

We would like to caution the reader that the terminology of definition VI.5 is new.

Given \( x_1, x_2 \in \mathbb{H} \) let us introduce the following:

**Definition VI.6**

\( x_1 \) is infinitesimally closed to \( x_2 \): 

\[ x_1 \sim \text{infinitesimally closed } x_2 := x_1 - x_2 \in \mathbb{H}_{\text{infinitesimals}} \] (6.10)

and let us recall that [2]:

**Proposition VI.2**

1. \( \sim \text{infinitesimally closed} \) is an equivalence relation over \( \mathbb{H} \).

2. 
\[ \forall x \in \mathbb{R} \cup \mathbb{H}_{\text{infinitesimals}} \ \exists ! \ std(x) \in \mathbb{R} : x \sim \text{infinitesimally closed } std(x) \] (6.11)

\( \text{std}(x) \) is called the **standard part** of \( x \in \mathbb{H} \).

Let us now repeat the construction of the previous section.

Given \( \bar{x} \in \Sigma^\infty \) and \( a_0, b_0 \in \mathbb{H} : a_0 < b_0 \) let us introduce the following definition by induction:

**Definition VI.7**

- \[ a_1(\bar{x}) := \begin{cases} a_0 & \text{if } x_1 = 0 \\ \frac{a_0 + b_0}{2} & \text{if } x_1 = 1 \end{cases} \] (6.12)

- \[ b_1(\bar{x}) := \begin{cases} \frac{a_0 + b_0}{2} & \text{if } x_1 = 0 \\ b_0 & \text{if } x_1 = 1 \end{cases} \] (6.13)

- \[ a_n(\bar{x}) := \begin{cases} a_{n-1} - b_{n-1} & \text{if } x_n = 0 \\ \frac{a_{n-1} + b_{n-1}}{2} & \text{if } x_n = 1 \end{cases} \] (6.14)

- \[ b_n(\bar{x}) := \begin{cases} \frac{a_{n-1} + b_{n-1}}{2} & \text{if } x_n = 0 \\ b_{n-1} & \text{if } x_n = 1 \end{cases} \] (6.15)

**Definition VI.8**

**Dedekind set with respect to** \([a_0, b_0]\) **and** \(\bar{x}^\prime\):

\[ S_{\text{Dedekind}}(a_0, b_0; \bar{x}) := \{ d \in \mathbb{H} : d \in \bigcap_{n \in \mathbb{N}} [a_n(\bar{x}), b_n(\bar{x})] \} \] (6.16)

Then:
Proposition VI.3

HP:

\[ a_0, b_0 \in \mathbb{R} \cup \mathbb{H}_{\text{infinitesimals}} \land \text{std}(a_0) \neq \text{std}(b_0) \]  

(6.17)

TH:

1. 

\[ |\{\text{std}(d), d \in S_{\text{Dedekind}}(a_0, b_0; \bar{x})\}| = 1 \quad \forall \bar{x} \in \Sigma^\infty \]  

(6.18)

2. 

\[ |S_{\text{Dedekind}}(a_0, b_0; \bar{x})| = |\mathbb{H}_{\text{infinitesimals}}| \quad \forall \bar{x} \in \Sigma^\infty \]  

(6.19)

PROOF:

1. Let us observe first of all that:

\[ \text{std}(a_n(\bar{x})) \neq \text{std}(b_n(\bar{x})) \quad \forall n \in \mathbb{N}, \forall \bar{x} \in \Sigma^\infty \]  

(6.20)

The thesis follows applying the axiom III.1 to the set \([\text{std}(a_0), \text{std}(b_0)] \cap \mathbb{R}\)

2. the thesis follows observing that:

\[ d + \epsilon \in [a_n(\bar{x}), b_n(\bar{x})] \quad \forall n \in \mathbb{N}, \forall \epsilon \in \mathbb{H}_{\text{infinitesimals}}, \forall \bar{x} \in \Sigma^\infty \]  

(6.21)

\[ \blacksquare \]

Proposition VI.4

HP:

1. \(\mathbb{H}\) is hypercontinuous

2. 

\[ a_0, b_0 \in \mathbb{R} \cup \mathbb{H}_{\text{infinitesimals}} \land \text{std}(a_0) = \text{std}(b_0) \]  

(6.22)

TH:

\[ |S_{\text{Dedekind}}(a_0, b_0; \bar{x})| = |\mathbb{H}_{\text{infinitesimals}}| \quad \forall \bar{x} \in \Sigma^\infty \]  

(6.23)

PROOF:

The thesis follows applying the definition VI.5 \[ \blacksquare \]

Let us now include unbounded hyperreals in the game.

Given \(x_1, x_2 \in \mathbb{H}\):
Definition VI.9

$x_1$ is finitely distant from $x_2$:

\[ x_1 \sim \text{finitely distant from } x_2 := x_1 - x_2 \in \mathbb{R} \cup H_{infinitesimals} \quad (6.24) \]

Then \[2\]:

**Proposition VI.5**

\[ \sim \text{finitely distant from } \] is an equivalence relation over $H$

Let us now consider the various cases as to the cardinality of the Dedekind set:

**Proposition VI.6**

**HP:**

1. $H$ is hypercontinuous
2. 

\[ a_0 \sim \text{finitely distant from } b_0 \quad (6.25) \]

**TH:**

\[ |S_{\text{Dedekind}}(a_0, b_0; \bar{x})| = |H_{infinitesimals}| \quad \forall \bar{x} \in \Sigma^\infty \quad (6.26) \]

**PROOF:**

Clearly:

\[ a_n(\bar{x}) \sim \text{finitely distant from } b_n(\bar{x}) \quad \forall n \in \mathbb{N}, \forall \bar{x} \in \Sigma^\infty \quad (6.27) \]

the thesis follow applying Proposition VI.3 and Proposition VI.4 \[\blacksquare\]

**Proposition VI.7**

**HP:**

1. $H$ is hypercontinuous
2.

\[ a_0 \sim \text{finitely distant from } b_0 \quad (6.28) \]

**TH:**

\[ |S_{\text{Dedekind}}(a_0, b_0; \bar{x})| = |H_{infinitesimals}| \quad \forall \bar{x} \in \Sigma^\infty \quad (6.29) \]
PROOF:

Clearly:

\[ a_n(x) \sim \text{finitely distant from } b_n(x) \quad \forall n \in \mathbb{N}, \forall x \in \Sigma^\infty \]  

(6.30)

Hence:

\[ r \in [a_n(x), b_n(x)] \quad \forall r \in \mathbb{R}, \forall n \in \mathbb{N}, \forall x \in \Sigma^\infty \]  

(6.31)

so that:

\[ |S_{\text{Dedekind}}(a_0, b_0; \bar{x})| = \max(|\mathbb{R}|, |\mathbb{H}_{\text{infinitesimals}}|) = |\mathbb{H}_{\text{infinitesimals}}| \]  

(6.32)

where we have used the definition [VI.5] and the fact that the axiom [VI.1] implies that:

\[ |\mathbb{R}| = \aleph_1 \]  

(6.33)

Considering together all the different cases it follows that:

**Proposition VI.8**

**HP:**

\[ \mathbb{H} \text{ is hypercontinuous} \]

**TH:**

\[ \mathbb{H} = \mathcal{G}|\mathbb{H}_{\text{infinitesimals}}| \]  

(6.34)

**PROOF:**

The thesis is an immediate consequence of Proposition [VI.3] Proposition [VI.4] and Proposition [VI.7] ■

Proposition [VI.8] implies that:

**Proposition VI.9**

**HP:**

\[ \mathbb{H} \text{ is hypercontinuous} \]

**TH:**

\[ \mathbb{H} \subset \text{ND} \]  

(6.35)

Let us now consider the particular hyperreal number system \( \mathbb{R} \) of Nonstandard Analysis introduced in the definition [C.1]

Let us observe first of all that:

**Proposition VI.10**

\[ *\mathbb{R} \text{ is hypercontinuous} \]

**PROOF:**
Demanding to [2], [3], and [4] for all the mathematical-logical details let us recall that:

1. there exists a map, called the $\star$-transform, associating to each sentence $\phi$ of the language $L_R$ formalizing $\mathbb{R}$ a sentence $\star \phi$ of the language $L_\star R$ formalizing $\star \mathbb{R}$

2. there exists a principle, called the Transfer Principle, stating that a $L_R$-sentence $\phi$ is true if and only if $\star \phi$ is true

Let us then consider the following $L_R$-sentence:

$$\phi := \forall x_1, x_2 \in \mathbb{R} \exists f : [x_1, x_2] \mapsto \mathbb{R} \text{ bijective} \tag{6.36}$$

Applying to $\phi$ the $\star$-transform we obtain the following $L_\star R$-sentence:

$$\star \phi := \forall x_1, x_2 \in \star \mathbb{R} \exists f : [x_1, x_2] \mapsto \star \mathbb{R} \text{ bijective} \tag{6.37}$$

Since $\phi$ is true, the application of the Transfer Principle allows to infer that $\star \phi$ is true.

Choosing in particular $x_1 := + \epsilon \in \star \text{infinitesimals}$ and $x_2 := - \epsilon \in \star \text{infinitesimals}$ it follows that:

$$|[-\epsilon, \epsilon]| = |\star \mathbb{R}| \geq \aleph_1 \forall \epsilon \in \star \text{infinitesimals} \tag{6.38}$$

from which it follows that:

$$|\star \text{infinitesimals}| = ||[-\epsilon, \epsilon]| \geq \aleph_1 \forall \epsilon \in \star \text{infinitesimals} \tag{6.39}$$

Since, given $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in \star \text{infinitesimals}$ : $\epsilon_1 < \epsilon_2 < \epsilon_3 < \epsilon_4$, there exists clearly a bijection $g : [\epsilon_1, \epsilon_2] \mapsto [\epsilon_3, \epsilon_4]$, it follows that all the infinitesimal intervals have the same cardinality; hence:

$$|[\epsilon_1, \epsilon_2]| = |\star \text{infinitesimals}| \geq \aleph_1 \forall \epsilon_1, \epsilon_2 \in \star \text{infinitesimals} : \epsilon_1 < \epsilon_2 \tag{6.40}$$

We can then infer that:

**Proposition VI.11**

$$\star \mathbb{R} \subset \mathbb{N} \mathbb{D} \tag{6.41}$$

**PROOF:**

The thesis follows applying the Proposition [VI.10] and the Proposition [VI.9] \[\blacksquare\]
VII. ALGORITHMICALLY RANDOM NON-DEDEKINDIAN NUMBERS

Given \( n \in \mathbb{N}_+ \) and \( a_0, b_0 \in \mathbb{G}_n \) such that \( a_0 < b_0 \):

**Definition VII.1**

set of the random generalized numbers of order \( n \) with respect to \([a_0, b_0]\):

\[
\text{RANDOM}(\mathbb{G}_n; [a_0, b_0]) := \{ d \in S_{\text{Dedekind}}(a_0, b_0; \bar{x}) : \bar{x} \in \text{RANDOM}(\Sigma) \}
\]  

(7.1)

The definition VII.1 is a generalization to non-Dedekindian numbers of the notion of Martin Löf-Solovay-Chaitin algorithmic randomness as it is shown by the following:

**Proposition VII.1**

\[
\text{RANDOM}(\mathbb{G}_1; [a_0, b_0]) = \text{RANDOM}([a_0, b_0])
\]  

(7.2)

PROOF:

The thesis follows applying the Proposition V.1, the definition VII.1 and the definition IV.1. \(\blacksquare\)

As a particular case of the definition VII.1 we have extended the notion of algorithmic-randomness to the particular hyperreal number system \(*\mathbb{R}\) of Nonstandard Analysis introduced in the definition C.11.

Actually:

**Proposition VII.2**

\[
\text{RANDOM}(\star\mathbb{R}; [0, 1]) = \{ x + \epsilon : x \in \text{RANDOM}(0, 1), \epsilon \in \star\mathbb{R}_{\text{infinitesimals}} \}
\]  

(7.3)

PROOF:

The thesis follows by the definition VII.1 and the Proposition VII.1. \(\blacksquare\)
APPENDIX A: CHAIN ORDERED FIELDS

Let us recall the following:

Definition A.1
field:
a triple $(F, +, \cdot)$ where:

• $F$ is a non-empty set

• $+: F \times F \mapsto F, \cdot: F \times F \mapsto F$ are maps satisfying the following conditions:

1. commutativity of the sum:

\[ a + b = b + a \quad \forall a, b \in F \quad (A1) \]

2. associativity of the sum

\[ (a + b) + c = a + (b + c) \quad \forall a, b, c \in F \quad (A2) \]

3. existence of the zero element with respect to the sum

\[ \exists 0 \in F : (a + 0 = a) \quad \forall a \in F \quad (A3) \]

4. existence of the opposites with respect to the sum

\[ \forall a \in F \exists -a \in F : a + (-a) = 0 \quad (A4) \]

5. commutativity of the product

\[ a \cdot b = b \cdot a \quad \forall a, b \in F \quad (A5) \]

6. associativity of the product

\[ (a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \forall a, b, c \in F \quad (A6) \]

7. existence of the unity with respect to the product

\[ \exists 1 \in F : 1 \neq 0 \land (1 \cdot a = a \cdot 1 = a) \quad \forall a \in F \quad (A7) \]

8. existence of the inverse with respect to the product

\[ \forall a \in F : a \neq 0 \exists a^{-1} \in F : a \cdot a^{-1} = a^{-1} \cdot a = 1 \quad (A8) \]

9. distributivity of the product with respect to the sum

\[ a \cdot (b + c) = a \cdot b + a \cdot c \quad \forall a, b, c \in F \quad (A9) \]

Let us recall that given a set $S$:

Definition A.2
partial ordering on $S$:
\[ \preceq \in \mathcal{P}(S \times S) \text{ such that:} \]

1. reflectivity

\[ a \preceq a \quad \forall a \in S \quad (A10) \]

2. transitivity

\[ [(a \preceq b \land b \preceq c) \Rightarrow a \preceq c] \quad \forall a, b, c \in S \quad (A11) \]
3. antisymmetry

\[ (a \leq b \land b \leq a) \Rightarrow a = b \quad \forall a, b \in S \quad (A12) \]

**Definition A.3**

*total ordering on S:*

\[ (a \leq b \lor b \leq a) \quad \forall a, b \in S \quad (A13) \]

Given a field F:

**Definition A.4**

*chain ordering on F:*

\[ (a \leq b \lor 0 \leq b) \quad \forall a, b \in F \quad (A14) \]

1. translation invariance of the ordering

2. positive elements are closed under product

\[ [(0 \leq a \land 0 \leq b) \Rightarrow 0 \leq a \cdot b] \quad \forall a, b \in F \quad (A15) \]

**Definition A.5**

*chain ordered field:*

a couple \((F, \preceq)\) such that:

1. F is a field
2. \(\preceq\) is a chain-ordering over F

Given a chain ordered field \((F, \preceq)\) and \(a, b \in F\):

**Definition A.6**

\[ a \prec b := a \preceq b \land a \neq b \quad (A16) \]

Given \(a, b \in F : a \prec b\)

**Definition A.7**

\[ [a, b] := \{x \in F : a \preceq x \preceq b\} \quad (A17) \]

Given \(S \subseteq F : S \neq \emptyset\):

**Definition A.8**

*upper bounds of S:*

\[ UB(S) := \{b \in F : a \preceq b \quad \forall a \in S\} \quad (A18) \]

**Definition A.9**

*(F, \preceq) is Dedekind complete:*

\[ \forall S \subseteq F : S \neq \emptyset \land UB(S) \neq \emptyset \exists \sup(S) \in UB(S) : (\sup(S) \preceq b \quad \forall b \in UB(S)) \quad (A19) \]
Definition A.10

natural action of \( \mathbb{Z} \) over \( F \):
\[ \circ : \mathbb{Z} \times F \mapsto F \]
\[ n \circ a := \sum_{i=1}^{n} a \quad n \in \mathbb{N}, a \in F \]  
(A20)

\[ (-n) \circ a := -(n \circ a) \quad n \in \mathbb{N}, a \in F \]  
(A21)

Definition A.11

\( (F, \preceq) \) is Archimedean:
\[ (\exists b \in F : n \circ a \preceq b \quad \forall n \in \mathbb{Z}) \Rightarrow a = 0 \]  
(A22)
In this section we will briefly review the definition of algorithmically-random binary sequences. Given a number \( n \in \mathbb{N} : n \geq 2 \) let us introduce, preliminarily, the following:

**Definition B.1**

\( n \)-ary alphabet:

\[
\Sigma_n := \{ k \in \mathbb{N} : k \leq n - 1 \}
\]

Obviously:

**Proposition B.1**

\[
\Sigma_2 = \Sigma
\]

Denoted by \( \Sigma_n^* := \bigcup_{k \in \mathbb{N}_+} \Sigma_n^k \) the set of all the \( n \)-ary strings and by \( \Sigma_n^\infty := \{ \bar{x} : \mathbb{N}_+ \mapsto \Sigma_n \} \) the set of all the \( n \)-ary sequences, let us introduce the following:

**Definition B.2**

\( n \)-ary value:

the map \( v_n : \Sigma_n^\infty \mapsto [0, 1] \):

\[
v_n(\bar{x}) := \sum_{i=1}^{\infty} \frac{x_i}{n^i}
\]

and the more usual notation:

\[
(0.x_1 \cdots x_m \cdots)_n := v_n(\bar{x}) \quad \bar{x} \in \Sigma_n^\infty
\]

Let us introduce furthermore the following:

**Definition B.3**

\( n \)-ary nonterminating natural positional representation:

the map \( r_n : [0, 1] \mapsto \Sigma_n^\infty \):

\[
r_n((0.x_1 \cdots x_i \cdots)_n) := \bar{x}
\]

with the nonterminating condition requiring that the numbers of the form \( (0.x_1 \cdots x_i(n-1))_n = (0.\cdots(x_i + 1)\bar{0})_n \) are mapped into the sequence \( x_1 \cdots x_i(n-1) \).

Given \( n_1, n_2 \in \mathbb{N} : \min(n_1, n_2) \geq 2 \):

**Definition B.4**

change of basis from \( n_1 \) to \( n_2 \):

the map \( cb_{n_1, n_2} : \Sigma_n^\infty \mapsto \Sigma_n^\infty \):

\[
cb_{n_1, n_2}(\bar{x}) := r_{n_2}(v_{n_1}(\bar{x}))
\]

Given \( X \in \Sigma_n^\infty \):

**Definition B.5**

\[
X \Sigma_n^\infty := \{ \bar{x} \in \Sigma_n^\infty : (\exists n \in \mathbb{N}_+ : \bar{x}(n) \in S) \}
\]

Endowed \( \Sigma_n \) with the discrete topology and \( \Sigma_n^\infty \) with the induced product topology \( \tau \):

**Definition B.6**
$G \subset \Sigma_n^\infty$ is a constructively-open set:

1. $G \in \tau$ \hspace{1cm} (B8)

2. \[ \exists X \subset \Sigma_n^* \text{ recursively enumerable : } G = X \Sigma_n^\infty \] \hspace{1cm} (B9)

where we demand to [7] for the definition of recursive enumerability.

**Definition B.7**

*constructive sequence of constructively open sets (c.s.c.o. sets):*

\[ \{ G_k, k \in \mathbb{N} : k \geq 1 \} \text{ sequence of constructively-open sets } G_k = X_k \Sigma_n^\infty \text{ such that:} \]

\[ \exists X \subset \Sigma_n^* \times \mathbb{N} \text{ recursively enumerable : } X_k = \{ \vec{x} \in \Sigma_n^* : (\vec{x}, k) \in X \} \] \hspace{1cm} (B10)

**Definition B.8**

*cylinder set with respect to $\vec{x} \in \Sigma_n^*$:

\[ \Gamma_{\vec{x}} := \{ \vec{y} \in \Sigma_n^\infty : \vec{g}(|\vec{x}|) = \vec{x} \} \] \hspace{1cm} (B11)

**Definition B.9**

*cylinder - $\sigma$ - algebra on $\Sigma_n^\infty$:*

\[ \mathcal{F}_{cylinder} := \sigma - \text{algebra generated by } \{ \Gamma_{\vec{x}} : \vec{x} \in \Sigma_n^* \} \] \hspace{1cm} (B12)

**Definition B.10**

*Lebesgue measure:*

the probability measure over the measurable space $(\Sigma_n^\infty, \mathcal{F}_{cylinder})$:

\[ \mu_{\text{Lebesgue}}(\Gamma_{\vec{x}}) := \frac{1}{n^{|\vec{x}|}} \forall \vec{x} \in \Sigma_n^* \] \hspace{1cm} (B13)

Given $S \subset \Sigma_n^\infty$:

**Definition B.11**

*S is a constructively null set:*

\[ \exists \{ G_k, k \in \mathbb{N} : k \geq 1 \} \text{ c.s.c.o. sets } : S \subset \bigcap_{k \geq 1} G_k \land \lim_{k \to +\infty} \mu_{\text{Lebesgue}}(G_k) = 0 \text{ constructively} \] \hspace{1cm} (B14)

We can finally introduce the following:

**Definition B.12**

*Martin L"of - Solovay - Chaitin random sequences over $\Sigma_n$:*

\[ \text{RANDOM}(\Sigma_n^\infty) := \Sigma_n^\infty - \{ S \subset \Sigma_n^\infty \text{ constructively null set } \} \] \hspace{1cm} (B15)

A key feature of the Martin L"of - Solovay - Chaitin notion of algorithmic-randomness is the following [4]:

**Proposition B.2**

*Basis-independence of randomness:*

\[ \text{RANDOM}(\Sigma_n^\infty) = \text{cb}_{n_1,n_2}(\text{RANDOM}(\Sigma_n^\infty)) \forall n_1,n_2 \in \mathbb{N} : \min(n_1,n_2) \geq 2 \] \hspace{1cm} (B16)

Proposition [B.2] allows to restrict the analysis to algorithmically random binary sequences without any lost of generality and to introduce the following:

**Definition B.13**

*set of the algorithmically random numbers in the interval $[0,1]$:*

\[ \text{RANDOM}(0,1) := \{ v_2(\vec{x}) : \vec{x} \in \text{RANDOM}(\Sigma^\infty) \} \] \hspace{1cm} (B17)
APPENDIX C: THE PARTICULAR HYPERREAL NUMBER SYSTEM OF NON-STANDARD ANALYSIS

In this paper we work within the formal system ZFC, i.e. the Zermelo-Fraenkel axiomatization of Set Theory augmented with the Axiom of Choice (axiom C.1).

Given a set \( S \neq \emptyset \):

**Definition C.1**

*filter on \( S \):*

\[
\mathcal{F} \subseteq \mathcal{P}(S) : (A \cap B \in \mathcal{F} \ \forall A, B \in \mathcal{F}) \land (A \in \mathcal{F} \land A \subseteq B \subseteq S \Rightarrow B \in \mathcal{F})
\]  

(C1)

**Definition C.2**

*ultrafilter on \( S \):*

a filter \( \mathcal{F} \) on \( S \) such that:

\[
\mathcal{F} \neq \mathcal{P}(S) \land (A \in \mathcal{F} \lor S - A \in \mathcal{F} \ \forall A \in \mathcal{P}(S))
\]  

(C2)

**Definition C.3**

*principal filter generated by \( B \):*

\[
\mathcal{F}^B := \{ A \in \mathcal{P}(S) : A \supseteq B \}
\]  

(C3)

In this paper we assume the following:

**AXIOM C.1**

*Axiom of Choice:*

\[
\exists f : \mathcal{P}(S) \mapsto \bigcup_{B \in \mathcal{P}(S)} B : f(A) \in A \ \forall A \in \mathcal{P}(S) : A \neq \emptyset
\]  

(C4)

A consequence of the axiom C.1 is the following [6]:

**Proposition C.1**

\[
\vert S \vert \geq \aleph_0 \Rightarrow \exists \mathcal{F} \text{ nonprincipal ultrafilter on } S
\]  

(C5)

Uniforming our notation to the one adopted for sequences over finite alphabets let us introduce the following:

**Definition C.4**

\[
\mathbb{R}^\infty := \{ \bar{r} : \mathbb{N} \mapsto \mathbb{R} \}
\]  

(C6)

Given \( \bar{r} = \{ r_n \}_{n \in \mathbb{N}} \), \( \bar{s} = \{ s_n \}_{n \in \mathbb{N}} \in \mathbb{R}^\infty \):

**Definition C.5**

\[
\bar{r} \oplus \bar{s} := \{ r_n + s_n \}_{n \in \mathbb{N}}
\]  

\[
\bar{r} \odot \bar{s} := \{ r_n \cdot s_n \}_{n \in \mathbb{N}}
\]  

(C7) (C8)
Given $x \in \mathbb{R}$:

**Definition C.6**

\[ x^\infty := \{ r_n \}_{n \in \mathbb{N}} \in \mathbb{R}^\infty : r_n = x \ \forall n \in \mathbb{N} \quad (C9) \]

Let us now introduce the following:

**Definition C.7**

\[ NPU(\mathbb{N}) := \{ \mathcal{F} \text{ nonprincipal ultrafilter on } \mathbb{N} \} \quad (C10) \]

By Proposition C.11 it follows that:

**Proposition C.2**

\[ NPU(\mathbb{N}) \neq \emptyset \quad (C11) \]

Given $\mathcal{F} \in NPU(\mathbb{N})$ and $\bar{r} = \{ r_n \}_{n \in \mathbb{N}}, \bar{s} = \{ s_n \}_{n \in \mathbb{N}} \in \mathbb{R}^\infty$:

**Definition C.8**

$r$ and $s$ are equal $\mathcal{F}$-almost everywhere:

\[ \bar{r} \sim_{\mathcal{F}} \bar{s} := \{ n \in \mathbb{N} : r_n = s_n \} \in \mathcal{F} \quad (C12) \]

It may be proved that $\bar{r}$:

**Proposition C.3**

$\sim_{\mathcal{F}}$ is an equivalence relation over $\mathbb{R}^\infty$.

Let us finally introduce the following:

**Definition C.9**

\[ \star \mathbb{R}_\mathcal{F} = \frac{\mathbb{R}^\infty}{\sim_{\mathcal{F}}} \quad (C13) \]

Given $\bar{r} = \{ r_n \}_{n \in \mathbb{N}}, \bar{s} = \{ s_n \}_{n \in \mathbb{N}} \in \mathbb{R}^\infty$:

**Definition C.10**

1. \[ [\bar{r}]_\mathcal{F} + [\bar{s}]_\mathcal{F} := [\bar{r} \oplus \bar{s}]_\mathcal{F} \quad (C14) \]

2. \[ [\bar{r}]_\mathcal{F} \cdot [\bar{s}]_\mathcal{F} := [\bar{r} \odot \bar{s}]_\mathcal{F} \quad (C15) \]

3. \[ [\bar{r}]_\mathcal{F} \leq [\bar{s}]_\mathcal{F} := \{ n \in \mathbb{N} : r_n \leq s_n \} \in \mathcal{F} \quad (C16) \]
It may be proved that \( \mathbb{R} \):

**Proposition C.4**

\((\star \mathbb{R}, +, \cdot, \leq)\) is a hyperreal number system \( \forall \mathcal{F} \in NPU(\mathbb{N}) \) with zero \( [0^\infty]_\mathcal{F} \) and unity \( [1^\infty]_\mathcal{F} \) and where \( x \in \mathbb{R} \) is identified with \( [x^\infty]_\mathcal{F} \)

where we have used the definition \([VI.1]\) of a hyperreal number system.

Furthermore the assumption of the axiom \([V.1]\) implies that:

**Proposition C.5**

\((\star \mathbb{R}, +, \cdot, \leq)\) is isomorphic to \((\star \mathbb{R}, +, \cdot, \leq) \forall \mathcal{F}_1, \mathcal{F}_2 \in NPU(\mathbb{N}) \)

Proposition \([C.5]\) allows to give the following:

**Definition C.11**

*hyperreal number system of Non-Standard Analysis:*

\[ (\star \mathbb{R}, +, \cdot, \leq) := (\star \mathbb{R}, +, \cdot, \leq) \mathcal{F} \in NPU(\mathbb{N}) \]  \( (C17) \)
## APPENDIX D: NOTATION

| Symbol | Meaning |
|--------|---------|
| i.e.   | *id est* |
| ∀      | for all (universal quantificator) |
| ∃      | exists (existential quantificator) |
| ∃!     | exists and is unique |
| x = y  | x is equal to y |
| x := y | x is defined as y |
| ∧      | and (logical conjunction) |
| ∨      | or (logical conjunction) |
| gcd(n, m) | greatest common divisor of n and m |
| Σ      | binary alphabet |
| Σ∗     | set of the binary strings |
| Σ∞     | set of the binary sequences |
| RANDOM(Σ∞) | set of the random binary sequences |
| ⃗x     | binary string |
| ¯x     | binary sequence |
| x_n    | nth digit of the string ⃗x or of the sequence ¯x |
| ⃗x(n)  | prefix of length n of the string ⃗x or of the sequence ¯x |
| |length of the string ⃗x |
| ⃗x∞    | sequence made of infinite repetitions of the string ⃗x |
| ·      | concatenation operator |
| λ      | empty string |
| |cardinality of the set S |
| P(S)   | power set of the set S |
| ℵ_n    | nth infinite cardinal number |
| N      | set of the natural numbers |
| ω      | ordinality of N |
| N⁺     | set of the strictly positive natural numbers |
| Z      | set of the integer numbers |
| Q      | set of the rational numbers |
| R      | set of the real numbers |
| H      | an hyperreal number system |
| *R     | the hyperreal number system of Non-standard Analysis |
| ℏ infinities | set of the infinitesimal elements of H |
| ℏ unbounded | set of the unbounded elements of H |
| std(x) | standard part of x |
| D_{a_0,b_0} | Dedekind operator with respect to [a_0, b_0] |
| RANDOM([a_0, b_0]) | real random numbers with respect to [a_0, b_0] |
| S_{Dedekind}(a_0, b_0; ⃗x) | Dedekind set with respect to [a_0, b_0] and ⃗x |
| G_n    | set of the generalized numbers of order n |
| NID    | set of the non-Dedekindian numbers |
| RANDOM(G_n; [a_0, b_0]) | set of the random generalized numbers of order n with respect to [a_0, b_0] |
APPENDIX E: ACKNOWLEDGEMENTS

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[1] R. Trudeau. *The non-Euclidean revolution*. Birkhauser, Boston, 1987.
[2] E. Schechter. *Handbook of Analysis and Its Foundations. CD-Rom Edition*. Academic Press, 1998.
[3] S. Wolfram. *The Mathematica Book*. Cambridge University Press, 1996.
[4] C. Calude. *Information and Randomness. An Algorithmic Perspective*. Springer Verlag, Berlin, 2002.
[5] A. Robinson. *Non-standard Analysis*. Princeton University Press, Princeton, 1996.
[6] R. Goldblatt. *Lectures on the Hyperreals. An Introduction to Nonstandard Analysis*. Springer, New York, 1998.
[7] P. Odifreddi. *Classical Recursion Theory: vol. 1*. Elsevier Science, Amsterdam, 1989.