Nonlinear Correction to the Longitudinal Structure Function at Small $x$

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We computed the longitudinal proton structure function $F_L$, using the nonlinear Dokshitzer-Gribov-Lipatov-Altarelli-parisi (NLDGLAP) evolution equation approach at small $x$. For the gluon distribution, the nonlinear effects are related to the longitudinal structure function. As, the very small $x$ behavior of the gluon distribution is obtained by solving the Gribov, Levin, Ryskin, Mueller and Qiu (GLR-MQ) evolution equation with the nonlinear shadowing term incorporated. We show, the strong rise that is corresponding to the linear QCD evolution equations, can be tamed by screening effects. Consequently, the obtained longitudinal structure function shows a tamed growth at small $x$. We computed the predictions for all detail of the nonlinear longitudinal structure function in the kinematic range where it has been measured by H1 collaboration and compared with computation Moch, Vermaseren and Vogt at the second order with input data from MRST QCD fit.

1 Introduction

The measurement of the longitudinal structure function $F_L(x, Q^2)$ is of great theoretical importance, since it may allow us to distinguish between different models describing the QCD evolution at low-$x$. In deep-inelastic scattering (DIS), the structure function measurements remain incomplete until the longitudinal structure function $F_L$ is actually measured [1]. At small $x$ values, the dominant contribution to $F_L(x, Q^2)$ comes from the gluon operators. Hence a measurement of $F_L(x, Q^2)$ can be used to extract the gluon structure function and therefore the measurement of $F_L$ provides a sensitive test of perturbative QCD [2-3]. As, at small $x$, the longitudinal structure function can be related to the gluon and sea-quark distribution. The behavior of the longitudinal structure function at small $x$ is given by the gluon behavior. The gluon behavior is observed that governs the physics of high energy processes in QCD. HERA shows [4-9] that the gluon distribution function has a steep behavior in the small $x$ region ($10^{-2}>x>10^{-5}$). This steep behavior is well described in the framework of the DGLAP [10-12] evolution equations.

In DIS at moderate values of $x$, the linear QCD evolution equations lead to good description of this process. But at small $x$, the problem is more complicated since recombination processes between gluons in a dense system have to be taken into account. This strong rise can eventually violate unitarity and so it has to be tamed by screening effects. These screening effects are provided by multiple gluon interaction which lead to the nonlinear terms in the DGLAP equations. These nonlinear terms reduce the growth of the gluon distribution in this kinematic region where $a_s$ is still small but the density of partons becomes so large. Gribov, Levin, Ryskin, Mueller and Qiu (GLR-MQ) performed a detailed study of this region. They argued that the physical processes of interaction and recombination of partons become important in the parton cascade at a large value of the parton density, and that these shadowing corrections could be expressed in a new evolution equation (the GLR-MQ equation) [13-14]. The main characteristic of this equation is that it predicts a saturation of the gluon distribution at very small $x$ [15-16]. This equation was based on two processes in a parton cascade:

i) The emission induced by the QCD vertex $G\to G + G$ with the probability which is proportional to $a_s \rho$ where $\rho(=\frac{zg(x, Q^2)}{2\pi R^2})$ is the density of gluon in the transverse plane, $\pi R^2$ is the target area, and $R$ is the size of the target which the gluons populate;

ii) The annihilation of a gluon by the same vertex $G + G \to G$ with the probability which is proportional to $a_s^{2}\rho^2$, where $a_s$ is probability of the processes.

Therefore, to obtain a precise evidence of the shadowing correction in the HERA kinematic region, we consider the longitudinal structure function that directly dependence on the behavior of the gluon distribution. In this paper we estimate the shadowing correction to the longitudinal structure function behavior. We calculate this observable using the Altarelli- Marinelii equation [17-18]. The longitudinal structure function $F_L$, projected from the hadronic tensor by combination of the metric and the spacelike momentum transferred by the virtual photon ($q_{\mu} = q_{\nu} = q^2/2$). Indeed, the longitudinal structure function is proportional to hadronic tensor as follows: $F_L(x, Q^2)/x = \frac{8\pi^2}{3F}p_\mu p_\nu W_{\mu\nu}(x, Q^2)$, where $p_\mu(p_\nu)$ is the hadron momentum and $W^\mu\nu$ is the hadronic tensor. In this relation we neglecting the hadron mass. The basic
hypothesis is that the total cross section of a hadronic process can be written as the sum of the contributions of each parton type (quarks, antiquarks, and gluons) carrying a fraction of the hadronic total momentum. In the case of deep inelastic scattering it reads:

\[ d\sigma_H(p) = \sum_i \int dy d\sigma_i(y)p \Pi_i^0(y), \]  

(1)

where \( d\sigma_i \) is the cross section corresponding to the parton \( i \) and \( \Pi_i^0(y) \) is the probability of finding this parton in the hadron target with the momentum fraction \( y \). Now, taking into account the kinematical constraints one gets the relation between the hadronic and the partonic structure functions:

\[ f_j(x, Q^2) = \sum_i \int \frac{dy}{y} f_j\left(\frac{x}{y}, Q^2\right) \Pi_i^0\left(\frac{y}{x}\right) \]

\[ = \sum_i f_j \otimes \Pi_i^0(y), \quad j = 2, L \]  

(2)

where \( f_j(x, Q^2) = F_j(x, Q^2)/x \). Equation (3) expresses the hadronic structure functions as the convolution of the partonic structure function, which are calculable in perturbation theory, and the probability of finding a parton in the hadron which is a nonperturbative function. So, in correspondence with Eq.(3) one can write Eq.(1) as follows:

\[ F_L/x = \frac{\alpha_s}{4\pi} f_{L,q}^{(1)} \otimes (a^0_q + a^0_{NS}) + f_{L,G}^{(1)} \otimes g^0 \]  

(3)

where \( a^0_q \) and \( a^0_{NS} \) are the singlet and nonsinglet quark distribution. \( f_{L,q}^{(1)} \) and \( f_{L,G}^{(1)} \) are the LO partonic longitudinal structure function corresponding to quarks and gluons, respectively [19-20]. At small \( x \) the second term with the gluon density is the dominant one. Here the representation for the gluon distribution \( G(x, Q^2) = xg(x, Q^2) \) is used, where \( g(x, Q^2) \) is the gluon density. After full agreement has been achieved, in the form of the gluon kernel \( K^G \), the standard collinear factorization formula for the longitudinal structure function at low \( x \) reads:

\[ F_L(x, Q^2) = \int \frac{dy}{y} K^G\left(\frac{x}{y}, Q^2\right)G(y, Q^2). \]  

(4)

where kernel \( K^G \) is defined by:

\[ K^G\left(\frac{x}{y}, Q^2\right) = \frac{\alpha_s}{4\pi}[8(x/y)^2(1-x/y)]\prod_{i=1}^{N_f} e_i^2]. \]  

(5)

and \( e_i \) are the quark charges.

One of the striking discoveries at HERA is the steep rise of the gluon distribution function with decreasing \( x \) value [6]. Indeed, considering the HERA data, as is shown, \( G(x, Q^2) = A_g x^{-\lambda_g(Q^2)} \), where \( \lambda_g(Q^2) \) is the Pomeron intercepts one. As \( x \rightarrow 0 \) the value of the gluon density becomes so large that the annihilation of gluons becomes important. So, this singular behavior is tamed by the shadowing effects. The strategy in this paper is based on the Regge-like behavior of the gluon distribution function that tamed with the shadowing correction. We assume this behavior as:

\[ G^{sh}(x, Q^2) = A_g x^{-\lambda_g(Q^2)}. \]  

(6)

We note that at \( x < x_0 = 10^{-2} \), shadowing gluon distribution (sh.) and unshadowing gluon distribution (unsh.) behavior are equal. At \( Q_0^2 \) the small \( x \) behavior of the shadowing gluon distribution assumed to be[21-24]:

\[ G^{sh}(x, Q_0^2) = G^{unsh}(x, Q_0^2)[1 + \theta(x_0 - x)]G^{unsh}(x, Q_0^2) \]

\[ -G^{unsh}(x_0, Q_0^2)/x_{\text{sat}}(x, Q_0^2)^{-1}, \]  

(7)

where \( x_{\text{sat}}(x, Q^2) = \frac{16\pi^2 Q^2}{\beta_0 R^2} \) is the value of the gluon which would saturate the unitarity limit in the leading shadowing approximation. Based on this behavior, the shadowing exponent of the gluon distribution can be determined as:

\[ \lambda_g^{sh}(Q_0^2) = \lambda_g^{unsh}(Q_0^2) + \frac{1}{L\ln x} \ln[1 + \theta(x_0 - x)] \]

\[ [G^{unsh}(x, Q_0^2) - G^{unsh}(x_0, Q_0^2)] \]

\[ \times \frac{2\pi^2}{4\beta_0 R^2 \Lambda^2 \exp(t_0)} \]  

(8)

where \( \alpha_s^{LO}(Q^2) = \frac{4\pi}{\beta_0 \ln(\frac{t_0}{Q^2})} \), \( \beta_0 = \frac{1}{4}(33 - 2N_f) \) and \( N_f \) being the number of active quark flavors (\( N_f = 4 \)), also \( t_0 = \ln(\frac{Q_0^2}{t}) \) (that \( \Lambda \) is the QCD cut-off parameter, i.e., \( \Lambda \approx 0.2 \text{ GeV} \)). The value of \( R \) depends on how the gluon ladders couple to the proton, or on how the gluons are distributed within the proton. \( R \) will be of the order of the proton radius (\( R \approx 5 \text{ GeV}^{-1} \)) if the gluons are spread throughout the entire nucleon, or much smaller (\( R \approx 2 \text{ GeV}^{-1} \)) if gluons are concentrated in hot-spot [25] within the proton. This equation (Eq.9) gives the shadowing exponent of the shadowing gluon distribution function at the scale \( Q^2 = Q_0^2 \). In order to solve this equation [26] we take \( \Lambda^{unsh}(Q_0^2) \) with respect to \( G^{unsh}(x, Q_0^2) \) that is the input unshadowing gluon distribution that take from QCD parametrisation.

Applying the dominant shadowing gluon distribution (i.e.Eq.7), in order to calculate of the shadowing longitudinal structure function at small \( x \) to equation (5). After integration, we find that:

\[ F_L^{sh}(x, t) = \frac{20\alpha_s}{9\pi} G^{sh}(x, t) Y_1(\lambda_g^{sh}(t)), \]  

(9)

where

\[ Y_1(\lambda_g^{sh}(t)) = \frac{(2 + \lambda_g^{sh}(t))(x^{3+\lambda_g^{sh}(t)} - (3 + \lambda_g^{sh}(t)x^2+\lambda_g^{sh}(t)) + 1}{(2 + \lambda_g^{sh}(t))(3 + \lambda_g^{sh}(t))}. \]  

(10)
The shadowing gluon distribution function should be defined in this equation (GLR-MQ equation) as:

\[
\frac{dG^{sh}(x, Q^2)}{d\ln Q^2} = \frac{dG(x, Q^2)}{d\ln Q^2}_{|_{DGLAP}} - \frac{8\alpha_s}{16\pi^2} \int_x^1 \frac{dy}{y} G^2(y, Q^2),
\]

where we used the modified gluon evolution equation arise from fusion of two gluon ladders. In this equation the first term is the standard DGLAP result that is linear into the parton distribution functions. Since we are interested to evolution of the longitudinal structure function with respect to nonlinear corrections, we can easily perform this behavior using the following equation:

\[
\frac{dF_L^{sh}(x, t)}{dt} = \frac{20 \alpha_s}{9\pi} \int_x^1 \frac{dy}{y} \left( \frac{x}{y} \right)^2 (1 - \frac{x}{y}) G^{sh}(y, Q^2) - \frac{20\alpha_s}{9\pi} \int_x^1 \frac{dy}{y} \left( \frac{x}{y} \right)^2 (1 - \frac{x}{y}) \frac{dG^{sh}(y, Q^2)}{dt},
\]

where the derivative of the shadowing gluon distribution with respect to \( t \) is given by Eq.12. Since we have assumed that the gluon have the Regge behavior at low \( x \) as controlled by shadowing corrections, we can easily solve this equation with respect to this behavior. We find the derivative of shadowing structure function with respect to \( t \) as:

\[
\frac{dF_L^{sh}(x, t)}{dt} = \left( -\frac{20}{9\pi} Y_1 + \frac{20\alpha_s^2}{3\pi^2} Y_2 \right) G^{sh}(x, t) - \frac{450\alpha_s^3}{8\pi R^2 Q^2} Y_3 G^{2sh}(x, t)
\]

where

\[
Y_2 = \frac{(1 - x^2 + \lambda_g^{sh}(t)) - (1 - x^2 + 3\lambda_g^{sh}(t))}{\lambda_g^{sh}(t)(2 + 2\lambda_g^{sh}(t))} - \frac{x\lambda_g^{sh}(t)(1 - x^2)}{2\lambda_g^{sh}(t)} + \frac{x\lambda_g^{sh}(t)(1 - x^3)}{3\lambda_g^{sh}(t)}
\]

and

\[
Y_3 = \frac{(1 - x^2 + 2\lambda_g^{sh}(t)) - (1 - x^2 + 3\lambda_g^{sh}(t))}{\lambda_g^{sh}(t)(2 + 2\lambda_g^{sh}(t))} - \frac{x\lambda_g^{sh}(t)(1 - x^2)}{2\lambda_g^{sh}(t)} + \frac{x\lambda_g^{sh}(t)(1 - x^3)}{3\lambda_g^{sh}(t)}
\]

Therefore, the following equation is a formula to extracted the shadowing longitudinal structure function, using the shadowing gluon distribution exponent and the shadowing gluon distribution determined in [26] at small \( x \) as a function of \( t \) value with respect to the initial conditions at \( t = t_0 \), as

\[
F_L^{sh}(x, t) = \frac{e^{\int_{t'}^{t} -Y_4(t')/t'dt'}}{\int e^{\int_{t'}^{t} -Y_4(t')dt'} Y_5(t')/t'dt' + C}
\]

where

\[
Y_4 = 1 - \frac{12 Y_5}{\beta_0 Y_1}
\]

and

\[
Y_5 = \frac{3645\pi^2}{800R^2 Q^2\beta_0 Y_1^2}
\]

in this equation \( C \) is a constant and dependence to the initial conditions at \( t = t_0 \) and \( x = x_0 \). The results are shown in Fig.1 for \( Q^2 = 20 GeV^2 \) at hot- spot point with \( R = 2 GeV^{-1} \).

In conclusion, in this paper we have obtained the effects of adding the nonlinear GLR-MQ corrections to the DGLAP evolution equation and especially the shadowing effects to the longitudinal structure function at low \( x \). We saw that the gluon recombination effects are expected to play an increasingly important role. These effects that arise from fusion of the two gluon ladders, slow down the evolution of the gluons from the standard DGLAP behavior. We show that the obtained results for the shadowing longitudinal structure function at small-\( x \) have a power- like behavior. As this growth tamed by the shadowing effects. This implies that the \( x \) dependence of the shadowing longitudinal structure function at low \( x \) is consistent with a power law, \( F_L^{sh}(x, Q^2) = A_L x^{-\lambda_L^{sh}(Q^2)} \), for fixed \( Q^2 \). This behavior is associated with the exchange of an object known as the hard Pomeron and also exponent of the shadowing longitudinal structure function defined as a polynomial function with respect to \( \ln Q^2 \).
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![Figure 1](https://example.com/figure1.png)

Fig 1:The values of the shadowing longitudinal structure function at $Q^2 = 20 GeV^2$ with $R = 2 GeV^{-1}$ (square) that accompanied with model error by solving the GLR-MQ evolution equation that compared with H1 Collab. data (up and down triangle). The error on the H1 data is the total uncertainty of the determination of $F_L$ representing the statistical, the systematic and the model errors added in quadrature. Circle data are the MVV prediction [31-33]. The solid line is the NLO QCD fit to the H1 data for $y < 0.35$ and $Q^2 \geq 3.5 GeV^2$. 