Nearly resolution V plans on blocks of small size.

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Abstract

In Bagchi (2010) main effect plans “orthogonal through the block factor” (POTB) have been constructed. The main advantages of a POTB are that (a) it may exist in a set up where an “usual” orthogonal main effect plan (OMEP) cannot exist and (b) the data analysis is nearly as simple as an OMEP.

In the present paper we extend this idea and define the concept of orthogonality between a pair of factorial effects (main effects or interactions) “through the block factor” in the context of a symmetrical experiment. We consider plans generated from an initial plan by adding runs. For such a plan we have derived necessary and sufficient conditions for a pair of effects to be orthogonal through the block factor in terms of the generators. We have also derived a sufficient condition on the generators so as to turn a pair of effects aliased in the initial plan separated in the final plan.

The theory developed is illustrated with plans for experiments with three-level factors in situations where interactions between three or more factors are absent. We have constructed plans with blocks of size four and fewer runs than a resolution V plan estimating all main effects and all but at most one two-factor interactions.

Key words: Symmetrical experiment, blocking.

1 Introduction

A situation in which a treatment factor is neither orthogonal nor confounded to a nuisance factor in a main effect plan (MEP) was first explored in Morgan and Uddin (1996). They considered a nested row-column set up and derived a sufficient condition for a treatment factor, possibly non-orthogonal to the nuisance factors, to be orthogonal to every other treatment factor. They also derived sufficient condition for optimality and constructed several series of orthogonal and optimal MEPs.

Mukherjee, Dey and Chatterjee (2001) Constructed main effect plans on blocks of a small size. Their plans also satisfy the property that treatment factors are
possibly non-orthogonal to the block factor and possess optimality property. However, their method relies on an existing orthogonal main effect plan (OMEP) as a starting point. Bose and Bagchi (2007) provided examples of OMEPs in blocks of size 2 satisfying the same property as the plans of Mukherjee, Dey and Chatterjee (2001) but requiring fewer blocks. Extending this idea to an arbitrary block size, Bagchi (2010) defined orthogonality between a pair of treatment factors “through the block factor” and derived a sufficient condition for that. This condition is weaker (thus satisfied by more plans) than that of Mukherjee, Dey and Chatterjee (2001). Method of construction of plans orthogonal through the block factor or “POTB” were also presented, where it was seen that a POTB may exist in a setup where an “usual OMEP” cannot exist.

In this paper we extend the concept of orthogonality through the block factor to any pair of factorial effects (main effect or interaction) in a symmetric experiment. We compare this concept with the usual orthogonality. We also obtain necessary and sufficient condition on a plan which makes the inference on a given factorial effect free from the involvement of all other factorial effects, in the presence or absence of a blocking factor. [See Theorem 3.1 and Remark 3.2]

Next we concentrate on construction. We consider plans obtained by “expanding an initial plan along a subspace” [see Definition 4.1]. We derive sufficient conditions on the subspace so that the relation between the effects is improved upon in the final plan. That is, a pair of effects aliased in the initial plan is not so in the final plan. Similarly, a pair aliased or non-orthogonal in the initial plan becomes orthogonal (through the block factor) in the final plan.

We have illustrated these ideas using plans for three-level factors assuming all interactions involving three or more factors to be absent. We have obtained plans on blocks of size four for $3^3$ experiment on 6 blocks, $3^4$, $3^5$ experiments, each on 18 blocks and $3^6$ experiment on 24 blocks. The first two plans estimate all effects assumed in the model, while the last two plans estimate all but one two-factor interaction. [See Theorems 5.1, 5.2, 5.3 and 5.4]. Each plan requires fewer runs than a resolution V plan for the corresponding experiment.

2 Preliminaries

Let $s$ be a prime power. Let us recall the terminology of the $m$-dimensional Euclidean geometry and a few other terms and notations required for a symmetric experiment. We shall mostly follow the notations of Bose (1947).

Notation 2.1 (a) $F$ will denote the Galois field of order $s$. 0 will denote the additive identity of $F$.

(b) $F^m$ will denote the vector space of dimension $m$ over $F$. We shall think of the vectors in $F^m$ as column vectors.
(c) The points of an \( m \)-dimensional Euclidean geometry (\( \text{EG}(m, s) \)) are the vectors of \( F^m \). A point is denoted by \( x = (x_1, \cdots, x_m)' \), \( x_i \in F, i = 1, \cdots, m \).

(b) A hyperplane is a coset of an \( (m-1) \)-dimensional subspace of \( F^m \). Specifically, for \( a \in F^m \), \( a \neq 0 \) and \( t \in F \), the hyperplane \( H_a(t) \) is the set of points of \( \text{EG}(m, s) \) satisfying \( a'x = t \).

(c) A pencil is a set of \( s \) parallel hyperplanes of \( \text{EG}(m, s) \). The pencil \( P_a \) denotes the set of parallel hyperplanes \( \{H_a(t), t \in F\} \). Thus, the pencils are the sets \( P_a = \{H_a(t), t \in F\}, a \in F^m \setminus \{0\} \).

**Remark 2.1**: The pencil \( P_a \) is the same as the pencil \( P_b \), if \( b = pa, p \in F, p \neq 0 \).

**Notation 2.2** Consider an \( s^m \) experiment on \( n \) runs. Let us name the factors as \( A_i, i = 1, 2, \cdots, m \).

(i) A point \( x = (x_1, x_2, \cdots, x_m) \) in \( \text{EG}(m, s) \) represents a level combination (run), in which \( A_i \) is at level \( x_i, i = 1, 2, \cdots, m \).

(ii) The combined effect of all the main effects and interactions present on the level combination \( x \) will be denoted by \( \tau_x \).

(ii) By a (factorial) effect we mean a main effect or an interaction and it will generally be denoted by \( D, E \) etc.

(iii) For a vector \( a = (a_1, \cdots, a_m)' \in F^m \), \( E_a \) will denote the factorial effect \( A_1^{a_1} A_2^{a_2} \cdots A_m^{a_m} \). Thus, the main effect of \( A_i \) will be denoted by \( E_a \), where \( a \) has 1 in only the \( i \)th position and 0 elsewhere. Similarly, for a \( 3^5 \) experiment the interaction \( A_2 A_4 \) will be denoted by \( E_b \), where \( b = (0, 1, 0, 2, 0)' \).

(iv) The pencil \( P_a \) will represent the factorial effect \( E_a \).

(v) We shall say that a run \( x \) is in the \( t \)th level of the factorial effect \( E_a \) if \( x \) satisfies \( a'x = t \).

**Remark 2.2**: Recall that a contrast belonging to the factorial effect \( E_a \) is of the form \( \sum_{t \in F} l_t \sum_{x \in H_a(t)} \tau_x \), where \( l_t \) is a real number for each \( t \in F \), such that \( \sum_{t \in F} l_t = 0 \). Thus, for \( b = pa, p \in F, p \neq 0 \), a contrast belonging to the effect \( E_a \) also belongs to \( E_b \). This is consistent with the property of pencils noted in Remark 2.1.

**Notation 2.3** Consider an \( s^m \) experiment on \( b \) blocks of size \( k \) each.

(i) \( E \) will denote the set of effects believed to be present.

(ii) Consider an effect \( E \in E \). The replication number \( r_{Ef}^t \) of the level \( t \) of \( E \)
is the number of runs in which \( E \) is at level \( t \). The replication vector \( r^E \) is the \( s \times 1 \) vector with \( r^E_t \) as the \( t \)th entry, \( t \in F_s \).

(iii) For two effects \( D \) and \( E \), the \( D \)-versus \( E \) incidence matrix is a \( s \times s \) matrix denoted by \( N^{DE} = ((n^{DE}_{pq}))_{p,q \in F} \), where \( n^{DE}_{pq} \) denotes the number of runs which are in the \( p \)th level of \( D \) as well as the \( q \)th level of \( E \).

Note that when \( D \) and \( E \) are main effects, say of factors \( A_i \) and \( A_j \), \( n^{DE}_{pq} \) is the number of runs in which the \( i \)th entry is \( p \) and the \( j \)th entry is \( q \).

(iv) For an effect \( E \), the \( E \)-versus block incidence matrix is a \( s \times b \) matrix denoted by \( N^{E,bl} = ((n^{E,bl}_{pj}))_{p \in F, 1 \leq j \leq b} \), where \( n^{E,bl}_{pj} \) denotes the number of runs which are in the \( p \)th level of \( E \) and in the \( j \)th block.

We now define the concept of **orthogonality through the block factor**.

**Definition 2.1** Two effects \( D \) and \( E \) are said to be **orthogonal through the block factor (OTB)** if

\[
k N^{DE} = N^{D,bl}(N^{E,bl})'.
\]

We denote this by \( D \perp E(\text{bl}) \).

**Remark 2.3:** Condition (2.1) is equivalent to equation (7) of Morgan and Uddin (1996) in the context of nested row-column designs.

The next result follows immediately from the definition of OTB.

**Theorem 2.1** A sufficient condition for effects \( D \) and \( E \) to be ‘OTB’ is that each of the three incidence matrices \( N^{D,bl}, N^{E,bl} \) and \( N^{DE} \) have all entries equal. Specifically, the condition is

\[
N^{D,bl} = (n/(bs))J,
\]

\[
N^{E,bl} = (n/(bs))J
\]

and

\[
N^{DE} = (n/s^2)J.
\]

Here \( J \) is the all-one matrix.

Before closing this section we present a few well-known results of block designs.

Consider an equireplicate block design \( d \) with \( v \) treatments and \( b \) blocks of size \( k \) each. Let the replication number be \( r \).

**Theorem 2.2** The incidence matrix \( N \) of \( d \) satisfies

\[
N1_b = r \text{ and } N'1_v = k1_b.
\]
Consider a pair of equireplicate block designs $d_1$ and $d_2$ on the same set of $v$ treatments on $b$ blocks of size $k$ each. Let their incidence matrices be denoted by $N_1$ and $N_2$ respectively. Let the common replication number be $r$. Then, clearly we have:

in view of Lemma 2.2 we can say the following.

**Theorem 2.3**

$$N_1^*N_2^*1_v = rk1_v.$$ 

### 3 Orthogonality through block versus usual orthogonality

We shall now compare OTB with the usual orthogonality. For the sake of simplicity we assume a main effect plan, but the results are valid for any factorial effect.

We first present a few notation and well-known results.

**Notation 3.1**

(a) Consider a blocked MEP for $m - 1$ factors. The block factor is named as $A_m$ and the general effect as $A_{m+1}$. The model is expressed as

$$Y = X\beta,$$

where

$$X = [ \begin{array}{ccc} X_1 & \cdots & X_{m+1} \end{array} ]$$

and $\beta = \begin{bmatrix} \alpha^1 \\ \vdots \\ \alpha^{m+1} \end{bmatrix}.$

Here, $X_i$, the design matrix for $A_i$ is a $0 - 1$ matrix - the $(u, t)$th entry of $X_i$ is 1 if in the $u$th run the factor $A_i$ is set at level $t$ and 0 otherwise, $i = 1, 2, \cdots m$ and $X_{m+1} = 1_n$.

Throughout this section $r^i$ will denote the vector of replications of $A_i$, $N_{ij}$ the $A_i$-versus $A_j$ incidence matrix and $R_i$ will replace $N_{ii}$.

(b) $a_i \times 1$ vector $\alpha^i$ will denote the vector of unknown effects of $A_i$, $1 \leq i \leq m+1$. Thus, $\alpha_m$ is the vector of block effects and $\alpha_{m+1}$ is the general effect.

(c) For any $m \times n$ matrix $A$, $C(A)$ will denote the column space of $A$. Further, $P_A$ will denote the projection operator on the column space of $A$. In other words, $P_A = A(A'\ A)^{-}A'$, where $B^{-}$ denote a g-inverse of $B$.

(d) Let $I = \{1, 2, \cdots m + 1\}$ and $S = \{i, j, \cdots \}$ be a subset of $I$. We shall use the following notation for the sake of compactness.

(i) $X_S$ will denote $[ X_i \quad X_j \quad \cdots ].$
(ii) $\alpha^S$ will denote \[
\begin{bmatrix}
\alpha^1 \\
\alpha^2 \\
\vdots 
\end{bmatrix}.
\]

(iii) $P_i$ will denote the projection operator onto the column space of $X_{i}i \in I$. Further, $P_S$ will denote the projection operator onto the column space of $X_S$.

(e) Sum of squares: Fix a set of factors $T$ of $I$. For $i$ not in $T$, we define $SS_{i:T}$, the sum of squares for $F_i$, adjusted for the factors $F_t$, $t \in T$. More generally, we define the combined sum of squares for the set factors \{$F_i, i \in S$\}, $SS_{S:T}$, adjusted for the factors $F_t$, $t \in T$ ($T$ disjoint from $S$) as follows.

\[
SS_{i:T} = Q_{i:T} (C_{ii:T})^{-1} Q_{i:T} \\
and SS_{S:T} = Q_{S:T} (C_{S:S:T})^{-1} Q_{S:T}
\]

For ready reference, we present the following well-known results.

Lemma 3.1 (a) For $i \neq j$, $i, j = 1, \cdots m + 1$, $SS_{i:j}$ is the quadratic form $Y'P_{ij}Y$, where $U = (I - P_j)X_i$.

(b) More generally, for two disjoint subsets $S$ and $T$ of $I$, $SS_{S:T}$ is the quadratic form $Y'P_{S}Y$, where $V = (I - P_T)X_S$.

(c) The so-called unadjusted sum of squares for $F_i$ is $SS_i = m + 1 \cdot T_i^t (R_i)^{-1} T_i - G^2/n$.

Lemma 3.2 Consider a matrix $W$ partitioned as $[ U \ V ]$. Let $Z = (I - P_U)U$. Then, $P_W - P_V = P_Z$.

Corollary 3.1 Let $T \subset I$ and $i \in I \setminus T$. Let $D = (I - P_T)X_i$. Then,

$P_D = P_{T^*} - P_T$, where $T^* = T \cup \{i\}$.

Notation 3.2 (a) The total sum of squares and the error sum of squares will be denoted by $SS_{tot}$ and $SS_E$ respectively.

(b) Fix a factor $A_i, 1 \leq i \leq m$. Let $T = \{i + 1, \cdots m + 1\}$ and $\bar{i} = I \setminus \{i\}$.

(i) Let $SS_{i:all} = SS_{i:T}$ and

(ii) $SS_{i:all} = SS_{i:}^{\bar{i}}$.

Thus, $SS_{i:all}$ is the sum of squares for $A_i$, adjusted for the factors $A_{i+1}, \cdots A_{m+1}$, while $SS_{i:all}$ denotes the sum of squares for $A_i$, adjusted for all other factors.

Now we seek the answer to the following questions. Consider a main effect plan for $m$ factors ($m \geq 3$). Fix a factor, say $A_i$. What conditions
the design matrices must satisfy so that the sum of squares for $A_i$ adjusted for all others is the same as

(a) the unadjusted sum of squares for $A_i$?

(b) the sum of squares for $A_i$ adjusted for only one factor, (say $F_m$) ?

[That is so far as $A_i$ is concerned, other factors are virtually absent.]

**Theorem 3.1** Fix a factor, say $A_i$.

(a) A necessary and sufficient condition for $SS_{i:all} = SS_{i:m+1}$ is that the incidence matrix $N_{ij}$ satisfies the proportional frequency condition of Addelman (1962). That is $N_{ij} = r_i r_j'/n$, for every $j \neq i$.

(b) A necessary and sufficient condition for $SS_{i:all} = SS_{i:m}$ is that

$$N_{ij} = N_{im}(R_m)^{-1}N_{jm}', j \neq i, 1 \leq i, j \leq m - 1. \quad (3.3)$$

[Recall Notation 3.2]

The proof relies on a lemma that we present now.

**Lemma 3.3** Consider matrices $A(m \times n), B((m \times p)$ such that $C((m \times q) be any matrix. Then the necessary and sufficient condition that $C(P_B C) = C(P_A C)$ is that $(P_A - P_B)C = 0$.

**Proof of theorem 3.1** Let $T = \{1, 2, \cdots i - 1, i + 1, \cdots m - 1\}$, $T^* = T \cup \{m\}$ and $T^{**} = T^* \cup \{m + 1\}$. From Lemma 3.3 and Notation 3.2 we see that

$$SS_{i:all} = Y'P_U Y, \quad SS_{i:m+1} = Y'P_V Y, \quad SS_{i:m} = Y'P_W Y,$$

where $U = (I - P_{T^{**}})X_i$, $V = (I - P_{m+1})X_i$ and $W = (I - P_m)X_i$.

**Proof of (a):** From the expressions above, a necessary and sufficient condition for $SS_{i:all} = SS_{i:m+1}$ is that $P_U = P_V$.

Therefore, in view of lemma 3.3, the necessary and sufficient condition for $SS_{i:all} = SS_{i:m+1}$ is that

$$(P_{T^{**}} - P_{m+1})X_i = 0. \quad (3.4)$$

Now by Lemma 3.2 $P_{T^{**}} - P_{m+1} = P_Z$, where $Z = (I - P_{m+1})X_T$. Thus, (3.4) is $\iff P_Z X_i = 0 \iff X'_i(I - P_{m+1})X_T = 0 \iff X'_i(I - P_{m+1})X_j = 0, j \neq i$, which is the same as the proportional frequency condition.
Proof of (b): Proceeding along the lines as in the proof of (a), we find that the necessary and sufficient condition for \( SS_i;_{all} = SS_i;_m \) is that
\[
P_Z X_i = 0, \quad \text{where} \quad Z = (I - P_m)X_T. \tag{3.5}
\]
But this condition \( \iff X'_i(I - P_m)X_T = 0 \iff X'_i(I - P_m)X_j = 0, \quad j \neq i, \quad 1 \leq i, j \leq m - 1. \) This condition can be expressed in the same form as in the statement by using the relation between the design matrices and the incidence matrices. \( \square \)

Remark 3.1: The sufficiency part of (a) of Theorem 3.1 is already known. We have now shown that the condition is also necessary.

Remark 3.2: The results of Theorem 3.1 can be easily generalized to a situation where interactions are present. Part (a) says that the sums of squares for testing the significance of effect \( E \) (say \( SS_{E;adj} \)) is the same as the so-called unadjusted sum of squares for \( E \) if and only if \( E \) satisfies PFC with every other effect present. Similarly, part (b) says that \( SS_{E;adj} \) is the same as that sums of squares for \( E \) adjusted for the block effect if and only if \( E \perp D(b) \) for every other effect \( D \).

4 Construction

In the rest of this paper by orthogonality we mean orthogonality through the block factor.

Suppose our aim is to construct a plan for an \( s^m \) experiment on blocks of size \( k \) each. We begin with an initial plan \( P \) and then develop or expand it with the help of a vector subspace \( V \) of \( F^m \) to our final plan. In this section we find out conditions on \( V \), so that the final plan satisfies certain desirable properties.

Notation 4.1 Consider a plan \( P \) for an \( s^m \) experiment on \( b \) blocks of size \( k \) each. Let \( n = bk \). Thus, there is a total of \( n \) runs in the plan, which in general may not be distinct.

(a) Let \( B \) denote the set of all blocks in \( P \), say \( B = \{B_j, \quad j = 1, \cdots b\} \).

(b) Let \( E = \{E_a, a \in I\} \), be the set of effects believed to be present. Thus, \( I \subseteq F^m \setminus \{0\} \).

(c) For \( a \in I \), the replication number \( r^a_t \) of the level \( t \) of effect \( E_a \) is the number of runs in \( P \) in which \( E_a \) is at level \( t \). Thus, \( r^a_t = \sum_{j=1}^{b} \sum_{x \in B_j, a'x = t} 1 \).

The replication vector \( r^a \) is the \( s \times 1 \) vector with \( r^a_t \) as the \( t \)th entry, \( t \in F_s \).

(d) For \( a, b \in I \), the \( E_a \) versus \( E_b \) incidence matrix is the \( s \times s \) matrix \( \mathcal{N}^{ab} \),
where \( N^{ab} = (n^{ab}_{\alpha,\beta})_{\alpha,\beta \in F} \), and \( n^{ab}_{\alpha,\beta} \) is the number of runs in \( \mathcal{P} \) in which \( E_a \) is at level \( \alpha \) and \( E_b \) is at level \( \beta \). That is

\[
n^{ab}_{\alpha,\beta} = \sum_{j=1}^{b} \sum_{\substack{x \in B_j: \ a'x = \alpha, \ b'x = \beta}} 1.
\]

(e) For \( a \in I \), the \( E_a \)-versus block incidence matrix \( L^a \) is the \( s \times b \) matrix \( L^a = (l^a_{\alpha,j})_{\alpha \in F, 1 \leq j \leq b} \), where \( l^a_{\alpha,j} \) is the number of runs in the \( j \)th block of \( \mathcal{P} \) in which \( E_a \) is at level \( \alpha \). That is

\[
l^a_{\alpha,j} = \sum_{x \in B_j, a'x = \alpha} 1.
\]

**Definition 4.1** Let \( V \) be a vector subspace of \( F^m \), say of dimension \( t \geq 1 \). For any block \( B \in \mathcal{B} \), and any vector \( v \in V \), \( v + B \) will denote the set \( \{v + x : x \in B\} \) of runs. By the **expansion** \( V(\mathcal{P}) \) of \( \mathcal{P} \) along \( V \) we shall mean the plan (for the same experiment, with \( b.s^t \) blocks of size \( k \) each) whose set of blocks is \( \{v + B : v \in V, B \in \mathcal{B}\} \).

**Notation 4.2** We shall use the following notation for the replication vectors and incidence matrices for an expansion \( V(\mathcal{P}) \) of a plan \( \mathcal{P} \) along a vector subspace \( V \).

(a) \( \tilde{r}^a \) will denote the replication vector of \( E_a \) in \( V(\mathcal{P}) \).

(b) \( \tilde{N}^{ab} = (\tilde{n}^{ab}_{\alpha,\beta})_{\alpha,\beta \in F} \) will denote the \( E_a \) versus \( E_b \) incidence matrix in \( V(\mathcal{P}) \).

(c) Let \( J = \{1, 2, \cdots b\} \). The \( s \times b|V| \) matrix \( \tilde{L}^a \) will denote the \( E_a \) versus block incidence matrix in \( V(\mathcal{P}) \). Its entries will be denoted by \( \tilde{l}^a_{\alpha,j} \), \( \alpha \in F \), \( j \in J \times V \).

We now find the replication vectors and incidence matrices of effects in \( V(\mathcal{P}) \) in terms of those in the original plan \( \mathcal{P} \).

**Theorem 4.1** Consider an expansion of a plan \( \mathcal{P} \) along a vector subspace \( V \) as in Definition 4.1. Consider a pair of effects \( E_a \) and \( E_b \). Then, the replication vectors of these effects and the incidence matrices involving \( E_a, E_b \) and block for \( V(\mathcal{P}) \) are given in terms of the corresponding quantities for \( \mathcal{P} \) as follows.
\[ \tilde{\alpha}^{ab} = \sum_{v \in V} p^{ab}_{\alpha - a', \beta - b' v}, \quad \alpha, \beta \in F. \quad (4.6) \]

\[ \tilde{\alpha}^{a} = \sum_{v \in V} r^{a}_{\alpha - a' v}, \quad \alpha \in F. \quad (4.7) \]

\[ \tilde{r}_{\alpha, j} = r^{a}_{\alpha - a' v, j}, \quad \alpha \in F, \quad j = (j, v), \quad 1 \leq j \leq b, \quad v \in V. \quad (4.8) \]

**Corollary 4.1** For a plan \( V(P) \) which is an expansion of \( P \) along \( V \), the following hold. For \( \alpha, \beta \in F \), the \((\alpha, \beta)\)th element of \( L^{a}(L^{b})' \) is given by

\[ \sum_{v \in V} (L^{a}(L^{b})')_{\alpha - a' v, \beta - b' v}. \]

We now find the condition for orthogonality between a pair of effects in the final plan.

**Notation 4.3** For \( V \) as in Definition 4.1 let \( V^{\perp} \) denote its orthocomplement. That is \( V^{\perp} \) is vector subspace of \( F_{m} \) of dimension \( m - t \), given by \( V^{\perp} = \{ w \in F_{m} : w'v = 0, \forall v \in V \} \).

**Definition 4.2** Given a vector subspace \( V \) of \( F_{m} \), let \( \sim_{V} \) denote the binary relation on the set of all effects given by

\[ E_{a} \sim_{V} E_{b} \text{ if } < a > + V^{\perp} = < b > + V^{\perp}. \]

Clearly \( \sim_{V} \) is an equivalence relation and so it partitions the set of all effects into the corresponding equivalence classes. We define “the effects classes relative to \( V \)” to be the \( \sim_{V} \)-equivalence classes.

We now present our main result.

**Theorem 4.2** Consider an expansion \( V(P) \) of \( P \) along \( V \). Fix a pair of effects \( E_{a} \) and \( E_{b} \).

(a) If \( E_{a} \) and \( E_{b} \) are from different effects classes relative to \( V \), then \( E_{a} \perp E_{b}(bl) \) in \( V(P) \).

(b) Suppose \( E_{a} \) and \( E_{b} \) are from the same effects classes relative to \( V \). Then, the following hold.

(i) If both \( a \) and \( b \) are in \( V^{\perp} \), then the relation between \( E_{a} \) and \( E_{b} \) in \( V(P) \) is the same as that in \( P \). That is \( E_{a} \) is confounded, non-orthogonal or orthogonal to \( E_{b} \) in \( V(P) \) if it is so in \( P \).
(ii) If neither \(a\) nor \(b\) is in \(V^\perp\) and \(E_a\) is aliased with \(E_b\) in \(\mathcal{P}\), then \(E_a\) and \(E_b\) are no more aliased in \(V(\mathcal{P})\), provided \(V\) is non-trivial.

The proof is based on two lemmas we present now.

**Lemma 4.1** Let \(V\) be a \(t\)-dimensional vector subspace of \(F^m\), \(a, b \in F^m\) and \(\alpha, \beta \in F\). Consider the subset \(S(a, b, \alpha, \beta)\) of \(V\), namely \(S = \{v \in V : a'v = \alpha, b'v = \beta\}\). Then, the following hold.

(a) \(S\) is empty in the following cases. (i) \(a \in V^\perp\), \(\alpha \neq 0\), (ii) \(b \in V^\perp\), \(\beta \neq 0\) and (iii) \(a - cb \in V^\perp\), for some \(c \in F\), \(c \neq 0\), but \(\alpha \neq c\beta\).

(b) \(|S| = s^t\) if \(a, b \in V^\perp\), \(\alpha = \beta = 0\).

(c) \(|S| = s^{t-1}\) if one of the following conditions is satisfied. (i) \(a \in V^\perp\), \(\alpha = 0\), \(b \notin V^\perp\), (ii) \(b \in V^\perp\), \(\beta = 0\), \(a \notin V^\perp\), (iii) \(a \notin V^\perp\), \(b \notin V^\perp\) and \(\exists c \in F\) such that \(a - cb \in V^\perp\), \(\alpha = c\beta\).

(d) \(|S| = s^{t-2}\) if \(a \notin V^\perp\), \(b \notin V^\perp\), \(< a > + V^\perp \neq < b > + V^\perp\).

**Proof:** This is trivial if \(a \in V^\perp\) or \(b \in V^\perp\). So, we assume \(a \notin V^\perp\), \(b \notin V^\perp\).

**Case 1:** \(< a > + V^\perp = < b > + V^\perp\). Then, \(\exists c \in F\) such that \(a - cb \in V^\perp\). If \(\alpha \neq c\beta\), then the set is empty, proving (a)(iii). If \(\alpha = c\beta\), then the set is nothing but \(\{v \in V, a'v = \alpha\}\), which has size \(s^{t-1}\). Thus, (c) (iii) is proved.

**Case 2:** \(< a > + V^\perp \neq < b > + V^\perp\). Then, \(v \mapsto a'v\) and \(v \mapsto b'v\) are non-zero linear functionals on on \(V\). Their kernels are \((t - 1)\)-dimensional vector subspaces of \(V\), namely \(a^\perp \cap V\) and \(b^\perp \cap V\). These subspaces are distinct, since \(< a > + V^\perp \neq < b > + V^\perp\). The sets \(\{v \in V, a'v = \alpha\}\) and \(\{v \in V, b'v = \beta\}\) are cosets of these subspaces and therefore are euclidean hyperplanes in the \(t\)-dimensional euclidean space over \(F\). Since the vector subspaces are distinct, these hyperplanes are distinct and non-parallel. But any two non-parallel hyperplanes in the \(t\)-dimensional euclidean space meet in an euclidean subspace of dimension \(t - 2\), having \(s^{t-2}\) common points. This proves (d) and hence completes the proof.\(\square\).

**Lemma 4.2** Let \(a, b\) and \(V\) be as in Lemma 4.1. For any \(s \times s\) matrix \(M\) with rows and columns indexed by \(F\), we define the \(s \times s\) matrix \(\tilde{M}\) as follows.

\[
\tilde{m}_{\alpha, \beta} = \sum_{v \in V} m_{\alpha - a'v, \beta - b'v}, \quad \alpha, \beta \in F.
\]

Then, we have :-

(a) If \(a, b \in V^\perp\), then, \(\tilde{M} = s^tM\).
(b) If \( a \in V^\perp, b \not\in V^\perp \), then,
\[
\tilde{m}_{\alpha,\beta} = \sum_{\gamma \in F} m_{\alpha,\gamma}, \quad \alpha, \beta \in F.
\]

(c) If \( a \not\in V^\perp, b \in V^\perp \), then,
\[
\tilde{m}_{\alpha,\beta} = \sum_{\gamma \in F} m_{\alpha,\gamma}, \quad \alpha, \beta \in F.
\]

(d) If \( a \not\in V^\perp, b \not\in V^\perp, < a > + V^\perp = < b > + V^\perp \), then,
\[
\tilde{m}_{\alpha,\beta} = \sum_{u \in F} m_{\alpha-u,\beta-cu},
\]
for some \( c \neq 0 \in F \).

(e) If \( a \not\in V^\perp, b \not\in V^\perp, < a > + V^\perp \neq < b > + V^\perp \), then,
\[
\tilde{M} = s^{t-2}J, \quad \text{where} \quad c \text{ is the sum of all entries of } M \text{ and } J \text{ is the all-one matrix}.
\]

**Proof:** From the definition of \( \tilde{M} \), its \((\alpha, \beta)\)th entry is nothing but
\[
\sum_{z,w \in F} m_{z,w} \|\{v \in V, a'v = \alpha - z, b'v = \beta - w\}\|.
\]

Therefore the result is immediate from Lemma 4.1.

**Proof of Theorem 4.2**: Put \( P = L_a L'_b, M = N_{ab} \). Then, by Theorem 4.1 and Corollary 4.1, we have (in the notation of Lemma 4.2)
\[
\tilde{L}_a \tilde{L}_b = \tilde{P} \quad \text{and} \quad \tilde{N}_{ab} = \tilde{M}.
\]

Proof of (a): Consider two cases.

**Case 1:** Exactly one of \( a \) and \( b \) is in \( V^\perp \).

**Case 2:** \( a \not\in V^\perp, b \not\in V^\perp, < a > + V^\perp \neq < b > + V^\perp \).

We first consider Case 1. W.l.g, let \( b \in V^\perp \). Then, \( a \) is not in \( V^\perp \). So, Lemma 4.2 implies
\[
\tilde{Q}_{\alpha,\beta} = s^{t-1} \sum_{\gamma \in F} Q_{\gamma,\beta}, \quad \alpha, \beta \in F, Q = P, M.
\]

Now, by Theorems 2.2 and 2.3 it follows that
\[
\tilde{v}_{\alpha,\beta} = s^{t-1} r_{\alpha,\beta} \quad \text{and} \quad \tilde{L}_a \tilde{L}_b = s^{t-1} k_{\alpha,\beta}.
\]

Now, the result follows from Definition 2.1.

In Case 2, applying (e) of Lemma 4.2 we get
\[
\tilde{Q} = s^{t-2} c_q J, Q = P, M, \quad \text{where} \quad c_q \text{ is the total sum of the entries of } Q.
\]

But in view of Theorems 2.2 and 2.3, \( c_q = k c_m \). Hence the condition of Definition 2.1 is satisfied.

Proof of Case (b): (i) The result is immediate from (a) of Lemma 4.2.

(ii) It is enough to show that when \( a \not\in V^\perp, b \not\in V^\perp, < a > + V^\perp \neq < b > + V^\perp \) and \( M \) is positive diagonal, then every entry of \( \tilde{M} \) is non-zero. But (d) of Lemma 4.2 says that for every \( \alpha, \beta \in F \), \( \tilde{m}_{\alpha,\beta} \) is \( s^{t-1} \) times the sum of entries of \( M \) on a transversal. Since \( M \) is positive diagonal, the result follows.
5 Plans for three-level factors

In this section we concentrate on three-level factors and assume that interactions involving three or more factors are not present. In the preceding section we have obtained the conditions on the developing procedure, so that the final plan satisfies desirable properties. However, in order that the size of the experiment is not too large, one needs also to have a good start. We have been able to find such a good start - a plan \( P \) for a \( 3^4 \) experiment on two blocks of size four each - as shown below.

\[
P = \begin{array}{l|cccc}
\text{Blocks} & B_1 & \text{ } & B_2 \\
A & 0 & 1 & 1 & 2 \\
B & 0 & 1 & 2 & 0 \\
C & 0 & 1 & 0 & 1 \\
D & 0 & 0 & 1 & 1 \\
\end{array}
\]

Properties of \( P \) : The effects satisfy the following defining relations.

Block \( \equiv ABC \equiv AC^2D^2 \equiv AB^2D \equiv BC^2D \).

This implies that the main effects and the two-factor interactions form the following alias classes.

\[
\begin{align}
A_1 &= \{A, B^2C^2, BD^2, CD\}, \\
A_2 &= \{B, A^2C^2, AD, CD^2\}, \\
A_3 &= \{C, AD^2, A^2B^2, BD\}, \\
A_4 &= \{D, AC^2, A^2B, B^2C\}.
\end{align}
\]

The relation among the classes are shown in the following graph, where adjacency represents orthogonality.

We now modify \( P \) according to our requirement and expand along suitable subspaces to obtain the final plan \( V(P) \). Before that we need a definition.

**Definition 5.1** Let us consider a plan \( \rho \). Suppose the set \( \mathcal{E} \) of all effects of interest can be divided into several classes in such a way that every effect is orthogonal to every other from a different class. Then \( \rho \) is called inter-class orthogonal and the classes will be referred to as “orthogonal classes”.

**Remark 4.1:** The concept of inter-class orthogonality has been introduced and studied in Bagchi (1916).
Case 1: A $3^3$ experiment.

**Theorem 5.1** There exists an inter-class orthogonal plan for a $3^3$ experiment on six blocks of size four each estimating all main effects and all two-factor interactions.

**Proof**: We delete factor $D$ from $\mathcal{P}$ to get a plan, say $\mathcal{P}^3$. Now the alias classes are

\[
\begin{align*}
\mathcal{A}_1 &= \{A, B^2C^2\}, \\
\mathcal{A}_2 &= \{B, A^2C^2\}, \\
\mathcal{A}_3 &= \{C, A^2B^2\}, \\
\mathcal{A}_4 &= \{AC^2, A^2B, B^2C\}.
\end{align*}
\]

Now we expand $\mathcal{P}^3$ along $V = \langle \{(1,0,0)\} \rangle$. Using Theorem 4.2 one can verify that in $V(\mathcal{P}^3)$ the set of all effects in the model satisfy inter-class orthogonality, the orthogonal classes being as follows.

\[
\begin{align*}
C_1 &= \{A, AC\}, & C_2 &= \{B, BC\} \\
C_3 &= \{C, B^2C\}, & C_4 &= \{AB, AC^2, A^2B\}.
\end{align*}
\]

Since no effect is aliased with any other, the result follows. □

Case 2: A $3^4$ experiment.

**Theorem 5.2** There exists an inter-class orthogonal plan for a $3^4$ experiment on eighteen blocks of size four each estimating all main effects and all two-factor interactions.

**Proof**: We start with $\mathcal{P}$ in which the alias classes are as given in (5.9) and the equations next to it. Now we expand $\mathcal{P}$ along $V = \langle \{(0,1,0,2), (1,0,1,0)\} \rangle$. Theorem 4.2 implies that the resultant plan $V(\mathcal{P})$ is inter-class orthogonal for the set of all main effects and two-factor interactions. The classes are:

\[
\begin{align*}
C_1 &= \{A, AC\}, & C_2 &= \{B, BD^2\} \\
C_3 &= \{C\}, & C_4 &= \{D\} \\
C_5 &= \{BC, CD^2\}, & C_6 &= \{AD, CD\} \\
C_7 &= \{AB, AD^2\}, & C_8 &= \{AB^2, BC^2\} \\
C_9 &= \{AC^2, BD\}\Box
\end{align*}
\]

Since no effect is aliased with any other, the result follows. □

Case 3: A $3^5$ experiment.
**Theorem 5.3** There exists an inter-class orthogonal plan for a $3^5$ experiment on eighteen blocks of size four each estimating all main effects and all two-factor interactions, except $DE^2$, which is confounded with the block factor.

**Proof:** We obtain the following plan $P^5$ from $P$ by adding a factor $E$.

| Blocks | $B_1$ | $B_2$ |
|--------|-------|-------|
| $A$    | 0 1 1 2 | 0 0 2 2 |
| $B$    | 0 1 2 0 | 2 1 1 2 |
| $C$    | 0 1 0 1 | 1 2 0 2 |
| $D$    | 0 0 1 1 | 2 1 2 0 |
| $E$    | 0 0 1 1 | 2 1 2 0 |

The effects satisfy the following defining relations:

$$\text{Block} \equiv DE^2 \equiv ABC \equiv AC^2D^2 \equiv AC^2E^2,$$

$$\equiv AB^2D \equiv AB^2E \equiv BC^2D \equiv BC^2E^2.$$

Thus, interaction $DE^2$ is confounded with the block factor and the alias classes are as given below.

$$A_1 = \{A, B^2C^2, BD^2, CD, CE, BE^2\},$$
$$A_2 = \{B, A^2C^2, AD, AE, CD^2, CE^2\},$$
$$A_3 = \{C, AD^2, A^2B^2, AE^2, BD, BE\},$$
$$A_4 = \{D, E, D^2E^2, AC^2, A^2B, B^2C\}.$$

We expand $P$ along $V = \langle (0, 1, 0, 2, 0), (1, 0, 1, 0, 2) \rangle$ to obtain the final plan $V(P^5)$. Using Theorem 4.2 we find that if we forget $DE^2$, then $V(P^5)$ can be viewed as inter-class orthogonal plan: the classes being as listed below.

$$C_1 = \{A, AC, CE^2\}, \quad C_2 = \{B, BD^2\}$$
$$C_3 = \{C, E, AE^2\}, \quad C_4 = \{D\}$$
$$C_5 = \{BC, BE^2, CD^2\}, \quad C_6 = \{CD, AD\}$$
$$C_7 = \{AD^2, AB, DE\}, \quad C_8 = \{BE, BC^2, AB^2\}$$
$$C_9 = \{CE, AE\}, \quad C_{10} = \{BD, AC^2\}$$

**Remark 4.2:** A resolution $V$ plan for a $3^4$ as well as a $3^5$ experiment requires 81 runs. Thus, apart from providing more flexibility due to small size of the blocks, we have saved 9 runs in both the situations. While all effects in the model are estimable in the former plan, only one two-factor interaction is lost in the later.

**Case 4:** A $3^6$ experiment.
Theorem 5.4 There exists an inter-class orthogonal plan for a $3^5$ experiment
on eighteen blocks of size four each estimating all main effects and all but nine
two-factor interactions.

Proof: We obtain the following plan $\mathcal{P}^6$ from $\mathcal{P}$ by adding factors $E$ and $F$.

| Blocks | $B_1$ | $B_2$ |
|--------|-------|-------|
| A      | 0 1 1 2 | 0 0 2 2 |
| B      | 0 1 2 0 | 2 1 1 2 |
| C      | 0 1 0 1 | 1 2 0 2 |
| D      | 0 0 1 1 | 2 1 2 0 |
| E      | 0 0 1 1 | 2 1 2 0 |
| F      | 0 1 1 2 | 0 0 2 2 |

The effects satisfy the following defining relations:

\[ \text{Block} \equiv DE^2 \equiv AF^2 \equiv ABC \equiv AB^2D \equiv AB^2E \equiv AC^2D^2 \equiv AC^2E^2 \equiv BC^2D \equiv BC^2E \equiv BCF \equiv B^2DF \equiv B^2EF \equiv C^2E^2F \equiv C^2D^2F \]

Thus, interactions $DE^2$ and $AF^2$ are confounded with the block factor and the
alias classes are as given below.

\[ A_1 = \{ A, A^2F^2, B^2C^2, BD^2, BE^2, CD, CE, F \}, \]
\[ A_2 = \{ B, A^2C^2, AD, AE, CD^2, CE^2, C^2F^2, DF, EF \}, \]
\[ A_3 = \{ C, A^2B^2, AD^2, AE^2, BD, BE, B^2F^2, D^2F, EF \}, \]
\[ A_4 = \{ D, E, A^2B, AC^2, B^2C, BF^2, C^2F, D^2E^2, \}. \]

We expand $\mathcal{P}^6$ along $V = \langle \{1,1,0,1,0,0\}(0,0,1,0,1,1) \rangle$ to obtain the final
plan $V(\mathcal{P}^6)$. From Theorem 4.2 it follows that all the effects in the assumed
model, except $DE^2$ and $AF^2$ are divided into the orthogonal classes listed below.

The classes do contain pairs of aliased two-factor interactions, which are
presented within ().

\[ C_1 = \{ A, B, AD, \}, \quad C_2 = \{ C, E \}, \quad C_5 = \{ D, AB^2, BD \}, \]
\[ C_4 = \{ F, CE, CF, EF \}, \quad C_5 = \{ BD^2, CF^2 \}, \quad C_6 = \{ CD^2, BE^2 \}, \]
\[ C_7 = \{ BE, BF^2, D^2E^2 \}, \quad C_8 = \{ (AD^2, EF^2), (AB^2, CF^2) \}, \]
\[ C_9 = \{ AE^2, BC^2, (AC^2, BF^2) \}, \quad C_{10} = \{ CD, AC, (AE, DF), (AF, BC) \}, \]

We see that there are five pairs of mutually aliased two-factor interactions.
These together with the interactions confounded with the blocks make the size
of the set on non-estimable two-factor interactions as seven. Again, the classes
$C_4$ and $C_{10}$ are of size four (counting each alias pair as one effect). But in the
present set up at most three non-orthogonal effects can be estimated, so that
two more two-factor interactions are lost. Hence the result. □
Remark 4.3: In a plan of 18 blocks of size 4 each, the available treatment degrees of freedom is 54 and so at most 27 effects can be estimated. In $V(P^6)$ these treatment degrees of freedom are utilized to the full extent as 6 main effects and 30-9 = 21 two-factor interactions are estimated.

We now present another plan, which may be viewed as a supplement of the plan $V(P^6)$ of Theorem 5.4 in the sense that these two plans together estimates all main effects and all two-factor interactions.

Theorem 5.5 There exists a plan $V(P^6_2)$ on 6 blocks of size 4 each, which estimates all but one of the two-factor interactions lost in $V(P^6_1)$, so that these two plans together estimates all effects in the model except one two-factor interaction.

Proof: The plan is as given below.

| Blocks | B₁  | B₂  |
|--------|-----|-----|
| A      | 0 1 1 2 | 0 0 2 2 |
| B      | 0 1 2 0 | 2 1 1 2 |
| C      | 0 1 1 2 | 0 0 2 2 |
| D      | 0 1 2 0 | 2 1 1 2 |
| E      | 0 0 1 1 | 2 1 2 0 |
| F      | 0 1 0 1 | 1 2 0 2 |

We can see that the effects $AC^2$ and $BD^2$ are confounded with the block factor. The remaining effects form the following alias classes.

- $A_1 = \{A, C, AC, BE^2, B^2F^2, DE^2, D^2F^2, EF\}$
- $A_2 = \{B, D, AE, A^2F^2, BD, CE, C^2F^2, E^2F\}$
- $A_3 = \{F, A^2B^2, A^2D^2, AE^2, B^2C^2, BE, C^2D^2, CE^2, DE\}$
- $A_4 = \{E, A^2B, A^2D, AF^2, BC^2, B^2F, C^2D, CF^2, D^2F, \}$

Now, we expand the above plan along $V = \langle \{(1,0,0,1,2,0)\}\rangle$ and get the required plan $V(P^6_2)$. We now present the orthogonal classes obtained in view of Theorem 4.4. The effects within () are aliased.

- $C_1 = \{(C, DE^2, BF), (B, AE, CF)\}$
- $C_2 = \{(A, AC, BE^2), (D, BD, EF^2), (EF, DF), (AF, CE)\}$
- $C_3 = \{(F, DE, BC), (AD^2, BC^2, BF^2, CF^2)\}$
- $C_4 = \{(E, AB^2, DF^2), (AD, CE^2), (CD^2, AF^2), (AB, AE^2, BE, CD)\}$

We shall now see how the lost effects can be estimated. When two effects are aliased in $V(P^6)$ we need to see that one of them can be estimated from $V(P^6_2)$. 

17
When there are four effects among which at most three can be estimated from \( V(P^6) \), we need to find one member which can be estimated from \( V(P^2_6) \). We present the details in the following table.

| Effects to be Estimated | Class of the New plan Confounded with |
|------------------------|--------------------------------------|
| \( DE^2 \)             | \( C_1 \)                             | Block \|
| \( AF^2 \)             | \( C_4 \)                             | Block \|
| \( EF^2 \)             | \( C_2 \)                             | \( AD^2 \) |
| \( DF \)               | \( C_2 \)                             | \( AE \)   |
| \( BF^2 \)             | \( C_3 \)                             | \( AC^2 \) |
| \( BC \)               | \( C_3 \)                             | \( AF \)   |
| \( AB^2 \)             | \( C_4 \)                             | \( CF^2 \) |
| \( CF \)               | \( C_1 \)                             | one of \( F, CE, EF \) |
| \( CD \)               | \( C_4 \)                             | one of \( AC, BC, DF \) |

We note that \( C_4 \) contains three and each of the other classes contains two of the effects to be estimated. Further, one may check from equation (5.13) and the following three equations that the effects in the table, which belongs to the same class of \( V(P^2_6) \) are not aliased. But since from every class at most two effects can be estimated, one effect in \( C_4 \) is lost. Thus, all but one of the lost nine effects can be estimated. Hence the result. □

**Remark 4.4** It is known that a resolution \( V \) plan for a \( 3^6 \) experiment requires \( 3^5 = 243 \) runs [see Theorem 13.1 of Hinkelman and Kempthorn (205), for instance]. The plans \( V(P^6) \) and \( V(P^2_6) \) together provides estimability almost to the same extent, but requires only 96 runs.

**Acknowledgement**:

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