THE METHOD OF FORCING

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Abstract. The purpose of this article is to give a presentation of the method of forcing aimed at someone with a minimal knowledge of set theory and logic. The emphasis will be on how the method can be used to prove theorems in ZFC.

1. Introduction

Let us begin with two thought experiments. First consider the following “paradox” in probability: if $Z$ is a continuous random variable, then for any possible outcome $z$ in $\mathbb{R}$, the event $Z \neq z$ occurs almost surely (i.e. with probability 1). How does one reconcile this with the fact that, in a truly random outcome, every event having probability 1 should occur? Recasting this in more formal language we have that, “for all $z \in \mathbb{R}$, almost surely $Z \neq z$”, while “almost surely there exists a $z \in \mathbb{R}$, $Z = z$.”

Next suppose that, for some index set $I$, $(Z_i : i \in I)$ is a family of independent continuous random variables. It is a trivial matter that for each pair $i \neq j$, the inequality $Z_i \neq Z_j$ holds with probability 1. For large index sets $I$, however,

$$|\{Z_i : i \in I\}| = |I|$$

holds with probability 0; in fact this event contains no outcomes if $I$ is larger in cardinality than $\mathbb{R}$. In terms of the formal logic, we have that, “for all $i \neq j$ in $I$, almost surely the event $Z_i \neq Z_j$ occurs”, while “almost surely it is false that for all $i \neq j \in I$, the event $Z_i \neq Z_j$ occurs”.

It is natural to ask whether it is possible to revise the notion of almost surely so that its meaning remains unchanged for simple logical assertions such as $Z_i \neq Z_j$ but such that it commutes with quantification. For instance one might reasonably ask that, in the second example, $|\{Z_i : i \in I\}| = |I|$ should occur almost surely regardless of the cardinality of the index set. Such a formalism would describe truth in a necessarily larger model of mathematics, one in which there are new outcomes to the random experiment which did not exist before the experiment was performed.

The method of forcing, which was introduced by Paul Cohen to establish the independence of the Continuum Hypothesis [5] and put into its current form by

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Scott [22] and Solovay [25], addresses issues precisely of this kind. From a modern perspective, forcing provides a formalism for examining what occurs almost surely not only in probability spaces but also in a much more general setting than what is provided by our conventional notion of randomness. Forcing has proved extremely useful in developing and understanding of models of set theory and in determining what can and cannot be proved within the standard axiomatization of mathematics (which we will take to be ZFC). In fact it is a heuristic of modern set theory that if a statement arises naturally in mathematics and is consistent, then its consistency can be established using forcing, possibly starting from a large cardinal hypothesis.

The focus of this article, however, is of a different nature: the aim is to demonstrate how the method of forcing can be used to prove theorems as opposed to establish consistency results. Forcing itself concerns the study of adding generic objects to a model of set theory, resulting in a larger model of set theory. One of the key aspects of forcing is that it provides a formalism for studying what happens almost surely as the result of introducing a generic object. An analysis of this formalism sometimes leads to new results concerning the original model itself — results which are in fact independent of the model entirely. This can be likened to how the probabilistic method is used in finite combinatorics in settings where more constructive methods fail (see, e.g., [1]).

In what follows, we will examine several examples of how forcing can be used to prove theorems. Admittedly there are relatively few examples of such applications thus far. It is my hope that by reaching out to a broader audience, this article will inspire more innovative uses of forcing in the future.

Even though the goals and examples are somewhat unconventional, the forcings themselves and the development of the theory are much the same as one would encounter in a more standard treatment of forcing. The article will only assume a minimal knowledge of set theory and logic, similar to what a graduate or advanced undergraduate student might encounter in their core curriculum. In particular, no prior exposure to forcing will be assumed.

The topics which will be covered include the following: genericity, names, the forcing relation, absoluteness, the countable chain condition, countable closure, and homogeneity arguments. These concepts will be put to use though several case studies:

1. partition theorems of Galvin, Nash-Williams, and Prikry for finite and infinite subsets of \(\omega\);
2. intersection properties of uncountable families of events in a probability space;
3. a partition theorem of Halpern and Läuchli for products of finitely branching trees;
4. a property of marker sequences for Bernoulli shifts;
5. Todorcevic’s analysis of Rosenthal compacta.

Sections marked with a ‘*’ will not be needed for later sections.

While we will generally avoid proving consistency results, the temptation to establish the consistency of the Continuum Hypothesis and its negation along the way is too great — this will be for free given what needs to be developed. For those interested in supplementing the material in this article with a more conventional approach to forcing, Kunen’s [16] is widely considered to be the standard treatment. It also provides a complete introduction to combinatorial set theory.
and independence results; the reader looking for further background on set theory is referred there. See also [12, 22, 24, 25, 31]. The last section contains a list of additional suggestions for further reading.

This exposition grew out of a set of lecture notes prepared for workshop 13w5026 on “Axiomatic approaches to forcing techniques in set theory” at the Banff International Research Station in November 2013. None of the results presented below are my own. I’ll finish by saying that this project was inspired by countless conversations with Stevo Todorcevic over the years, starting with my time as his student in the 1990s.

2. Preliminaries

Before beginning, we will fix some conventions and review some of the basic concepts from set theory which will be needed. Further reading and background can be found in [16]. A set \( x \) is transitive if whenever \( z \in y \) and \( y \in x \), then \( z \in x \).

An ordinal \( \alpha \) is a set which is transitive and wellordered by \( \in \). It is readily checked that every element of an ordinal is also an ordinal. Every wellorder is isomorphic to an ordinal; moreover this ordinal and the isomorphism are unique. If \( \alpha \) and \( \beta \) are two ordinals, then exactly one of the following is true: \( \alpha < \beta \), \( \beta < \alpha \), or \( \alpha = \beta \). We will often write \( \alpha < \beta \) to denote \( \alpha \in \beta \) if \( \alpha \) and \( \beta \) are ordinals.

Notice that an ordinal is the set of ordinals smaller than it. The least ordinal is the emptyset, which is denoted 0. If \( \alpha \) is an ordinal, then \( \alpha + 1 \) is the least ordinal greater than \( \alpha \); this coincides with the set \( \alpha \cup \{ \alpha \} \).

The finite ordinals therefore coincide with the nonnegative integers: \( n := \{0, \ldots, n-1\} \). The least infinite ordinal is \( \omega := \{0, 1, \ldots\} \), which coincides with the set of natural numbers. We will adopt the convention that the set \( \mathbb{N} \) of natural numbers does not include 0. Unless otherwise specified, \( i, j, k, l, m, n \) will be used to denote finite ordinals.

An ordinal \( \kappa \) is a cardinal if whenever \( \alpha < \kappa \), \( |\alpha| < |\kappa| \). If \( \alpha \) is an ordinal which is not a successor, then we say that \( \alpha \) is a limit ordinal. In this case, the cofinality of \( \alpha \) is the minimum cardinality of a cofinal subset of \( \alpha \). A cardinal \( \kappa \) is regular if its cofinality is \( \kappa \).

The regularity of a cardinal \( \kappa \) is equivalent to the assertion that if \( \kappa \) is partitioned into fewer than \( \kappa \) sets, then one of these sets has cardinality \( \kappa \). If \( \kappa \) is a cardinal, then \( \kappa^+ \) denotes the least cardinal greater than \( \kappa \). Cardinals of the form \( \kappa^+ \) are called successor cardinals and are always regular.

Since every set can be wellordered, every set has the same cardinality as some (unique) cardinal; we will adopt the convention that \( |x| \) is the cardinal \( \kappa \) such that \( |x| = |\kappa| \). If \( \alpha \) is an ordinal, then we define \( \omega_\alpha \) to be the \( \alpha \)-th infinite cardinal. Thus \( \omega_0 := \omega \) and \( \omega_\beta := \sup_{\alpha < \beta} (\omega_\alpha)^+ \) if \( \beta > 0 \). The Greek letters \( \alpha, \beta, \gamma, \xi, \) and \( \eta \) will be used to refer to ordinals; the letters \( \kappa, \lambda, \mu, \) and \( \theta \) will be reserved for cardinals.

If \( A \) and \( B \) are sets, then \( B^A \) will be used to denote the collection of all functions from \( A \) into \( B \). For us, a function is simply a set of ordered pairs. Thus if \( B \subseteq C \), then \( B^A \subseteq C^A \) and if \( f \) and \( g \) are functions, \( f \subseteq g \) means that \( f \) is a restriction of \( g \).

There is one exception to this notation worth noting. We will follow the custom of writing \( \aleph_\alpha \) for \( \omega_\alpha \) in situations where the underlying order structure is unimportant (formally \( \aleph_\alpha \) equals \( \omega_\alpha \)). Arithmetic expressions involving \( \aleph_\alpha \)'s will be used to refer to the cardinality of the resulting set. For instance \( 2^{\aleph_1} \) is a collection of functions whose cardinality is \( 2^{\aleph_1} \), which is a cardinal.
If \((X, d)\) is a metric space, then we define its \textit{completion} to be the set of equivalence classes of Cauchy sequences, where \((x_n : n < \infty)\) is equivalent to \((y_n : n < \infty)\) if \(d(x_n, y_n) \to 0\). In particular, we will regard the set of real numbers \(\mathbb{R}\) as being the completion of the rational numbers \(\mathbb{Q}\) with the usual metric \(d(p, q) := |p - q|\). Notice that even if \(X\) is complete, the completion is not literally equal to \(X\), even though it is canonically isometric to \(X\). This will serve as a minor annoyance when we define names for complete metric spaces in Section \[9\].

Finally, we will need some notation from first order logic. The \textit{language of set theory} is the first order language with a single binary relation \(\in\). If \(\phi\) is a formula in the language of set theory, then \(\phi(v_1, \ldots, v_n)\) will be used to indicate that every free variable in \(\phi\) is \(v_i\) for some \(i = 1, \ldots, n\). If \(x_1, \ldots, x_n\) are constants, then \(\phi(x_1, \ldots, x_n)\) is the result of simultaneously substituting \(x_i\) for \(v_i\) for each \(i\). If \(\phi\) is a formula and \(v\) is variable and \(x\) is a term, then \(\phi[x/v]\) is the result of substituting \(x\) for every free occurrence of \(v\) in \(\phi\) (of which there may be none). If \(\phi\) has no free variables, then we say that \(\phi\) is a \textit{sentence}. If every quantifier in \(\phi\) is of the form \(\exists x \in y\) or \(\forall x \in y\) for some variables \(x\) and \(y\), then we say that the quantification in \(\phi\) is \textit{bounded}. Many assertions can be expressed using only bounded quantification: for instance the assertions “\(A = \bigcup B\)” and “\((\mathbb{Q}, \leq)\) is a partially ordered set” are expressible by formulas with only bounded quantification.

We now recall some foundational results in set theory which justify our emphasis on transitive models of set theory below. A binary relation \(R\) is \textit{well founded} if there is no infinite sequence \((x_n : n < \infty)\) such that \(x_{n+1}Rx_n\). A binary relation \(R\) on a set \(X\) is \textit{extensional} if for all \(x\) and \(y\) in \(X\), \(\{z \in X : zRx\} = \{z \in X : zRy\}\) implies \(x = y\). Among the axioms of ZFC are the assertions that \(\in\) is well founded and extensional.

**Proposition 2.1.** Suppose that \((X, E)\) is a binary relation and \(E\) is well founded and extensional. Then \((X, E)\) is uniquely isomorphic to a transitive set equipped with \(\in\). In particular, if \((X, E)\) is a model of ZFC and \(E\) is well founded, then \((X, E) \simeq (M, \in)\) for some transitive set \(M\).

**Proposition 2.2.** If \(M\) is a transitive set, \(\phi(v_1, \ldots, v_n)\) is a first order formula with only bounded quantification and \(a_1, \ldots, a_n \in M\), then \((M, \in) \models \phi(a_1, \ldots, a_n)\) if and only if \(\phi(a_1, \ldots, a_n)\) is true.

Thus, for example, if \(M\) is a transitive set and \(\mathbb{Q}\) is a partial order in \(M\), then \((M, \in)\) satisfies that \(\mathbb{Q}\) is a partial order.

3. \textbf{What is forcing?}

Forcing is the procedure of adjoining to a model \(M\) of set theory a new \textit{generic object} \(G\) in order to create a larger model \(M[G]\). In this context, \(M\) is referred to as the \textit{ground model} and \(M[G]\) is a \textit{generic extension} of \(M\). For us, the generic object will always be a new subset of some partially ordered set \(\mathbb{Q}\) in \(M\), known as a \textit{forcing}. This procedure has the following desirable properties:

1. \(M[G]\) is also a model of set theory and is the minimal model of set theory which has as members all the elements of \(M\) and also the generic object \(G\).
2. The truth of mathematical statements in \(M[G]\) can be determined by a formalism within \(M\), known as the \textit{forcing relation}, which is completely specified by \(\mathbb{Q}\). The workings of this formalism are purely internal to \(M\).
While it will generally not concern us in this article, the key meta-mathematical feature of forcing is that it is often the case that it is easier to determine truth in the generic extension $M[G]$ than in the *ground model* $M$. For instance Cohen specified the description of a forcing $Q$ with the property that if $M[G]$ is any generic extension created by forcing with $Q$, then $M[G]$ necessarily satisfies that the Continuum Hypothesis is false [5] (see Section 8 below). In fact the second thought experiment presented at the beginning of the introduction is derived from a variation of this forcing. It is also not difficult to specify different forcings which always produce generic extensions satisfying the Continuum Hypothesis (see Section 14 below).

There are two perspectives one can have of forcing: one which is primarily semantic and one which is primarily syntactic. Each has its own advantages and disadvantages. The semantic approach makes certain properties of the forcing relation and the generic extension intuitive and transparent. On the other hand, it is fraught with metamathematical issues and philosophical hangups. The syntactic approach is less intuitive but more elementary and makes certain other features of forcing constructions more transparent. We will tend to favor the syntactic approach in what follows. We will now fix some terminology.

**Definition 3.1** (forcing). A *forcing* is a set $Q$ equipped with a transitive reflexive relation $\leq$ which contains a greatest element $1_Q$. If $Q$ is clear from the context, the subscripts are usually suppressed.

Our prototypical example of a forcing is $R$, the collection of all measurable subsets of $[0,1]$ having positive Lebesgue measure, ordered by containment. Elements of a forcing are often referred to as *conditions* and are regarded as being approximations to a desired *generic object*. In the analogy with randomness, the conditions correspond to the events of the probability space which have positive measure. If $q \leq p$, then we sometime say that $q$ is *stronger than* $p$ or that $q$ *extends* $p$. We think of $q$ as providing a better approximation to the generic object. It will be helpful to abstract the notion of an outcome in terms of a collection of mutually compatible events. A set $G \subseteq Q$ is a *filter* if $G$ is nonempty, upward closed, and downward directed in $Q$: 

(8) if $q$ is in $G$, $p$ is in $Q$ and $q \leq p$, then $p$ is in $G$;

(9) if $p$ and $q$ are in $G$, then there is an $r$ in $G$ such that $r \leq p,q$.

If $p$ and $q$ are in $Q$, then we say that $p$ and $q$ are *compatible* if there is a $r$ in $Q$ such that $r \leq p,q$. Otherwise we say that $p$ and $q$ are *incompatible*. Notice that two conditions are compatible exactly when there is a filter which contains both of them. Of course two events in $R$ are compatible exactly when they intersect in a set of positive measure.

A forcing $Q$ is *separative* if whenever $p \not\leq q$, there is an $r \leq p$ such that $r$ and $q$ are incompatible. Notice that if $Q$ is any forcing, we can define an equivalence relation $\equiv$ on $Q$ by $q \equiv p$ if

\[ \{ r \in Q : r \text{ is compatible with } p \} = \{ r \in Q : r \text{ is compatible with } q \} . \]

The quotient is ordered by $[q] \leq [p]$ if

\[ \{ r \in Q : r \text{ is compatible with } q \} \subseteq \{ r \in Q : r \text{ is compatible with } p \} . \]

This quotient ordering is separative and is known as the *separative quotient*. Notice that if $Q$ is separative, then $\equiv$ is just equality and the quotient ordering is just the
usual ordering. The forcing $\mathcal{R}$ is not separative; in this example $p \equiv q$ if $p$ and $q$
differ by a measure 0 set. It is often convenient to assume forcings are separative
and we will often pass to the separative quotient without further mention (just as
one often writes equality of functions in analysis when they really mean equality
modulo a measure 0 set).

The following definition will play a central role in all that follows.

**Definition 3.2** (generic). If $M$ is a collection of sets and $Q$ is a forcing, then we
say that a filter $G \subseteq Q$ is $M$-*generic* if whenever $E \subseteq Q$ is in $M$, there is a $p \in G$
which is either in $E$ or is incompatible with every element of $E$.

A family $E$ of conditions is said to be *exhaustive* if whenever $p$ is an element
of $Q$, there is an element $q$ of $E$ which is compatible with $p$. Notice that if $E$ is a
collection of exhaustive sets and $G \subseteq Q$ is an $E$-generic filter, then $G$ must intersect
every element of $E$. Also observe that if $\mathcal{S} := \{\{q\} : q \in Q\}$, then the $\mathcal{S}$-generic
filters are exactly the *ultrafilters* — those filters which are maximal.

In order to illustrate the parallel with randomness, take $Q = \mathcal{R}$. Observe that
if $E \subseteq \mathcal{R}$ is exhaustive, then its union has full measure. Conversely, if $E \subseteq \mathcal{R}$ is
countable and $\bigcup E$ has full measure, then $E$ is exhaustive. Thus in this setting,
genericity is an assertion that certain measure 1 events occur.

There are two other order-theoretic notions closely related to being exhaustive
which will be useful to define. A family of pairwise incompatible conditions is said
to be an *antichain*. Notice that this differs from the usual notion of an antichain
in a poset, where antichain would mean pairwise *incomparable*. Observe that any
maximal antichain is exhaustive but that in general exhaustive families need not
be pairwise incompatible. A family $D$ of conditions is *dense* if every element of $Q$
has an extension in $D$. For example, the collection of all elements of $\mathcal{R}$ which
are compact is dense in $\mathcal{R}$. Observe that, by Zorn’s Lemma, every dense set in a
partial order contains a maximal antichain. Two forcings are said to be *equivalent*
if they have dense suborders which are isomorphic. The reason for this is that such
forcings generate the same generic extensions.

4. **A Precursor to the Forcing Relation: A Partition Theorem of Galvin and Nash-Williams**

In this section, we will prove the following theorem of Galvin and Nash-Williams
which generalizes Ramsey’s theorem. The proof is elementary, but crucially employs
the forcing relation, albeit implicitly. We will also use this partition relation in
Section 13 The presentation in this section follows [30, §5]. If $A \subseteq \omega$, let $[A]^{\omega}$
denote all infinite subsets of $A$.

**Theorem 4.1** (see [8]). If $\mathcal{F}$ is a family of nonempty finite subsets of $\omega$, then there
is an infinite subset $H$ of $\omega$ such that either:

a. no element of $\mathcal{F}$ is a subset of $H$ or
b. every infinite subset of $H$ has an initial segment which is in $\mathcal{F}$.

Notice that Ramsey’s theorem is the special case of this theorem in which all
elements of $\mathcal{F}$ have the same cardinality. We will now introduce some terminology
which will be useful in organizing the proof of Theorem 4.1 Fix $\mathcal{F}$ as in the
statement of the theorem. If $a \subseteq \omega$, $A \subseteq \omega$ with a finite and $A$ infinite, then we
say that $A$ *accepts* $a$ if whenever $B \subseteq A$ is infinite with $\max(a) < \min(B)$, then
a ∪ B has an initial segment in \( F \). We say that \( A \) rejects \( a \) if no infinite subset of \( A \) accepts \( a \) and that \( A \) decides \( a \) if it either accepts or rejects \( A \).

We will prove the conclusion of the theorem through a series of lemmas.

**Lemma 4.2.** If \( A \) rejects \( a \), then \( \{ k ∈ A : A \) accepts \( a ∪ \{ k \} \} \) is finite.

**Proof.** If \( B := \{ k ∈ A : A \) accepts \( a ∪ \{ k \} \} \) is infinite, then \( B \) is an infinite subset of \( A \) which accepts \( a \). □

**Lemma 4.3.** There is an infinite set \( H ⊆ \omega \) which decides all of its finite subsets.

**Proof.** Recursively construct infinite sets \( \omega ⊇ H_0 ⊇ H_1 ⊇ \cdots \) such that if \( n_k := \min(H_k) \) then \( n_k−1 < n_k \) and \( H_k \) decides all subsets of \( \{ n_i : i < k \} \). It follows that \( H := \{ n_i : i < ∞ \} \) decides all finite subsets of \( \omega \). □

**Lemma 4.4.** If \( H ⊆ \omega \) is infinite and decides all of its finite subsets, then either \( H \) accepts \( ∅ \) or else there is an infinite \( A ⊆ H \) which rejects all of its finite subsets.

**Proof.** If \( H \) rejects the emptyset and decides all of its finite subsets, then recursively construct \( n_0 < n_1 < \cdots \) in \( H \) so that for each \( k \), \( H \) rejects all subsets of \( \{ n_i : i < k \} \). The choice of the next \( n_k \) is always possible since

\[
\{ n : ∃ a ⊆ \{ n_i : i < k \} ( H \) accepts \( a ∪ \{ n \} ) \}
\]

is finite. The set \( A := \{ n_i : i < ∞ \} \) now rejects all of its finite subsets. □

In order to finish the proof of Theorem 4.1 observe that if \( H \) accepts the emptyset, then every infinite subset of \( H \) contains an initial segment in \( F \). By the previous lemmas, it therefore suffices to show that if \( A \) is an infinite set which rejects all of its finite subsets, then no element of \( F \) is a subset of \( A \). If \( a ∈ F \) with \( a ⊆ A \), then \( B := A \setminus \{ 0, \ldots, \max(a) \} \) would accept \( a \), which is impossible. This finishes the proof of Theorem 4.1.

5. The formalism of the forcing relation

In this section we will develop the forcing relation and the forcing language axiomatically, treating the notion of a \( Q \)-name and the forcing relation \( ⊩ \) as undefined concepts; the definitions are postponed until Section 6. The advantage of this approach is that it emphasizes the aspects of the formalism which are actually used in practice.

Let \( Q \) be a forcing, fixed for the duration of the section. As we stated earlier, one can view \( Q \) as providing the collection of events of positive measure with respect to some abstract notion of randomness. In this analogy, a \( Q \)-name would correspond to a set-valued random variable. It is conventional to denote \( Q \)-names by letters with a “dot” over them.

There are two examples of \( Q \)-names which deserve special mention. The first is the “check name”: for each set \( x \), there is a \( Q \)-name \( \check{x} \). This corresponds to a random variable which is constant — it does not depend on the outcome. The other is the \( Q \)-name \( \check{G} \) for the generic filter; this corresponds to the random variable representing the outcome of the random experiment.

The forcing language associated to \( Q \) is the collection of all first order formulas in the language of set theory augmented by adding a constant symbol for each \( Q \)-name. If \( q \) is in \( Q \) and \( ϕ \) is a sentence in the forcing language, then informally the forcing relation \( q \models ϕ \) asserts that if the event corresponding to \( q \) occurs, then
almost surely \( \phi \) will be true. In the absence of the definitions of “\( Q \)-name” and “\( \vdash \),” the following properties can be regarded as axioms which govern the behavior of these primitive concepts. They can be proved from the definitions of \( Q \)-names and the forcing relation which will be given in Section 6.

**Property 1.** For any \( p \in Q \) and any sets \( x \) and \( y \):

- a. \( p \vdash \bar{x} \in \bar{y} \) if and only if \( x \in y \);
- b. \( p \vdash \bar{x} = \bar{y} \) if and only if \( x = y \);

**Property 2.** For \( p, q \in Q \), \( p \vdash \bar{q} \in \hat{G} \) if and only if whenever \( r \in Q \) is compatible with \( p \), \( r \) is compatible with \( q \).

If \( Q \) is separative, then this property takes a simpler form: \( p \vdash \bar{q} \in \hat{G} \) if and only if \( p \leq q \).

**Property 3.** For any \( \bar{x} \), any \( Q \)-name \( \dot{y} \), and \( p \in Q \), if \( p \vdash \dot{y} \in \bar{x} \), then there is a \( q \leq p \) which decides \( \dot{y} \).

**Property 4.** For any \( x \), any \( Q \)-name \( \dot{y} \), and \( p \in Q \), if \( p \vdash \dot{y} \in \bar{x} \), then there is a \( q \leq p \) which decides \( \dot{y} \).

**Property 5.** If \( \bar{x} \) is a \( Q \)-name and \( p \in Q \), then the collection of all \( Q \)-names \( \dot{y} \) such that \( p \vdash \dot{y} \in \bar{x} \) forms a set and the collection of all \( Q \)-names \( \dot{y} \) such that \( p \vdash \dot{y} = \bar{x} \) forms a set.

**Remark 5.1.** Unlike the other properties, this one is dependent on the definition of \( Q \)-name which we will give in the next section.

**Property 6.** If \( p \in Q \) and \( \phi \) is a formula in the forcing language, then \( p \vdash \neg \phi \) if and only if there is no \( q \leq p \) such that \( q \vdash \phi \).

Observe that this property implies that if \( p \vdash \phi \) and \( q \leq p \), then \( q \vdash \phi \). Property 6 can be seen as providing an organizational tool in the proof of Theorem 4.1. If \( Q := (\omega^{\omega}, \subseteq) \) then an \( A \in [\omega^{\omega}]^{\omega} \) accepts \( a \) if and only if \( A \) forces that every element of the generic filter contains an infinite set with an initial part in \( F \). An infinite \( A \) rejects \( a \) if it forces the negation of this assertion.

**Property 7.** If \( p \in Q \), then \( p \vdash \exists v \phi \) if and only if there is a \( Q \)-name \( \bar{x} \) such that \( p \vdash \phi[\bar{x}/v] \).

**Property 8.** For any \( q \in Q \), the collection of sentences in the forcing language which are forced by \( q \) contains the ZFC axioms, the axioms of first order logic, and is closed under modus ponens. Moreover, if the axioms of ZFC are consistent, then so are the sentences forced by \( q \).

If \( 1 \vdash Q \phi \), then we will sometimes say that “\( Q \) forces \( \phi \)” or, if \( Q \) is clear from the context, that “\( \phi \) is forced.” Similarly, we will write “\( x \) is a \( Q \)-name for...” to mean “\( x \) is a \( Q \)-name and \( Q \) forces that \( x \) is...”.

In order to demonstrate how these properties can be used, we will prove the following useful propositions.
strictly decreasing map from $T$ by every forcing involving only bounded quantification. By Proposition 5.3, this statement is forced by a decreasing infinite sequence of ordinals. Observe that the assertion that such a sequence $\sigma$ exists, then there is a function $\rho$ from $T$ into the ordinals such that if $s$ is a proper initial segment of $t$, then $\rho(t) \in \rho(s)$. Such a $\rho$ certifies the nonexistence of such a $\sigma$ since such a $\sigma$ would define a strictly decreasing infinite sequence of ordinals. Observe that the assertion that $\rho$ is a strictly decreasing map from $T$ into the ordinals is a statement about $\rho$ and $T$ involving only bounded quantification. By Proposition 5.3, this statement is forced by every forcing $Q$.

There is a special class of forcings for which there is a more conceptual picture of the forcing relation. We begin by stating a general fact about forcings.

**Theorem 5.5.** For every forcing $Q$, $Q$ is isomorphic to a dense suborder of the positive elements of a complete Boolean algebra.

Here we recall that a Boolean algebra is complete if every subset has a least upper bound. A typical example of a complete Boolean algebra is the algebra of measurable subsets of $[0,1]$ modulo the ideal of measure zero sets. The algebra of Borel subsets of $[0,1]$ modulo the ideal of first category sets is similarly a complete Boolean algebra. Random and Cohen forcing, respectively, are isomorphic to dense suborders of the positive elements of these complete Boolean algebras.

Suppose now that $Q$ is the positive elements of some complete Boolean algebra $B$. If $\phi$ is a formula in the forcing language, then define the truth value $\ truth value$ $\langle \phi \rangle$ of $\phi$ to be the least upper bound of all $b \in B$ such that $b \vdash \phi$. Observe that if $a \leq \langle \phi \rangle$, then $a \vdash \phi$. If $a$ does not force $\phi$, then $1 \not \vdash \phi$. By Proposition 5.2, it follows that $1 \vdash \phi(1)$. By Property 6, there is an assertion that $\phi$ is atomic, then this follows from Property 1. If $\phi$ is a formula in the forcing language, then define the truth value $\langle \phi \rangle$ of $\phi$ to be the least upper bound of all $b \in B$ such that $b \vdash \phi$. Observe that if $a \leq \langle \phi \rangle$, then $a \vdash \phi$. If $a$ does not force $\phi$, then $1 \not \vdash \phi$. By Proposition 5.2, it follows that $1 \vdash \phi(1)$.

**Proposition 5.3.** Suppose that $\phi(v_1,\ldots,v_n)$ is a formula in the language of set theory with only bounded quantification. If $x_1,\ldots,x_n$ are sets and $\phi(x_1,\ldots,x_n)$ is true, then $1 \vdash \phi(x_1,\ldots,x_n)$.

**Proof.** The proof is by induction on the length of $\phi$. If $\phi$ is atomic, then this follows from Property 1. If $\phi$ is a conjunct, disjunct, or a negation, then the proposition follows from Property 8 and the induction hypothesis. Finally, suppose $\phi(v_1,\ldots,v_n)$ is of the form $\forall v \in v_1,\psi(v_1,\ldots,v_n,w)$. If $\forall w\psi(x_1,\ldots,v_n,w)$ is true, then for each $w$, $\psi(x_1,\ldots,x_n,w)$ is true. By our induction hypothesis, $1 \vdash \phi(1,\ldots,x_n,w)$ for each $w$. By Proposition 5.2, it follows that $1 \vdash \forall w \in x_n\psi(x_1,\ldots,x_n,w)$.

**Proposition 5.4.** Suppose that $T$ is a set consisting of finite length sequences, closed under taking initial segments. If there is a forcing $Q$ and some $q \in Q$ forces “there is an infinite sequence $\sigma$, all of whose finite initial parts are in $T$,” then such a sequence $\sigma$ exists.

**Proof.** If no such sequence $\sigma$ exists, then there is a function $\rho$ from $T$ into the ordinals such that if $s$ is a proper initial segment of $t$, then $\rho(t) \in \rho(s)$. Such a $\rho$ certifies the nonexistence of such a $\sigma$ since such a $\sigma$ would define a strictly decreasing infinite sequence of ordinals. Observe that the assertion that $\rho$ is a strictly decreasing map from $T$ into the ordinals is a statement about $\rho$ and $T$ involving only bounded quantification. By Proposition 5.3, this statement is forced by every forcing $Q$.

There is a special class of forcings for which there is a more conceptual picture of the forcing relation. We begin by stating a general fact about forcings.
then $a$ cannot force $\neg \phi$. Hence $\llbracket \phi \rrbracket$ forces $\phi$. The rules which govern the logical connectives now take a particularly nice form:

\[
\begin{align*}
\llbracket \neg \phi \rrbracket &= \llbracket \phi \rrbracket^* \\
\llbracket \phi \land \psi \rrbracket &= \llbracket \phi \rrbracket \land \llbracket \psi \rrbracket \\
\llbracket \phi \lor \psi \rrbracket &= \llbracket \phi \rrbracket \lor \llbracket \psi \rrbracket \\
\llbracket \forall x \phi \rrbracket &= \bigwedge \llbracket \phi[\check{x}/v] \rrbracket \\
\llbracket \exists x \phi \rrbracket &= \bigvee \llbracket \phi[\check{x}/v] \rrbracket
\end{align*}
\]

Notice that while $\check{x}$ ranges over all $\mathcal{Q}$-names in the last equations — a proper class — the collection of all possible values of $\llbracket \phi[\check{x}/v] \rrbracket$ is a set and therefore the last items are meaningful.

In spite of the usefulness of complete Boolean algebras in understanding forcing and also in some of the development of the abstract theory of forcing, forcings of interest rarely present themselves as complete Boolean algebras (the notable exceptions being Cohen and random forcing). While Theorem 5.5 allows us to represent any forcing inside a complete Boolean algebra, defining forcing strictly in terms of complete Boolean algebras would prove cumbersome in practice.

6. Names, Interpretation, and Semantics

In this section we will turn to the task of giving a formal definition of what is meant by a $\mathcal{Q}$-name and $q \models \phi$. This will in turn be used to give a semantic perspective of forcing. The definitions in this section are not essential for understanding most forcing arguments and the reader may wish to skip this section on their first reading of the material. Others, however, may wish to have a tangible model of the axioms.

Before proceeding, we need to recall the notion of the rank of a set. If $x$ is a set, then the rank of $x$ is defined recursively: the rank of the emptyset is 0 and the rank of $x$ is the least ordinal which is strictly greater than the ranks of its elements. This is always a well defined quantity and it will sometimes be necessary to give definitions by recursion on rank. We recall that formally an ordered pair $(x, y)$ is defined to be \{x, \{x, y\}\}; this is only relevant in that the rank of $(x, y)$ is greater than the ranks of either $x$ or $y$.

Now let $\mathcal{Q}$ be a forcing, fixed for the duration of the section. If $q \in \mathcal{Q}$, let $\mathcal{Q}_q$ denote the forcing $(\{r \in \mathcal{Q} : r \leq q\}, \leq)$.

**Definition 6.1** (name). A set $\check{x}$ is a $\mathcal{Q}$-name if every element of $\check{x}$ is of the form $(\check{y}, q)$ where $\check{y}$ is a $\mathcal{Q}_q$-name and $q$ is in $\mathcal{Q}$.

(The requirement that $\check{y}$ be a $\mathcal{Q}_q$-name is to help ensure that Property 5 is satisfied.) Notice that this apparently implicit definition is actually a definition by recursion on rank, as discussed in Section 2. Furthermore, if $\mathcal{P} \subseteq \mathcal{Q}$ is a suborder, then any $\mathcal{P}$-name is also a $\mathcal{Q}$-name.

The following provide two important examples of $\mathcal{Q}$-names.

**Definition 6.2** (check names). If $x$ is a set, $\check{x}$ is defined recursively by

$$\{(\check{y}, 1) : y \in x\}.$$

**Definition 6.3** (name for the generic filter). $\check{G} := \{(\check{q}, q) : q \in \mathcal{Q}\}$.

As mentioned in the previous section, the notion of a $\mathcal{Q}$-name is intended to describe a procedure for building a new set from a given filter $G \subseteq \mathcal{Q}$. This procedure is formally described as follows.
Definition 6.4 (interpretation). If $G$ is any filter and $\dot{x}$ is any $Q$-name, define $\dot{x}(G)$ recursively by

$$\dot{x}(G) := \{ \dot{y}(G) : \exists p \in G \ ( (\dot{y}, p) \in \dot{x}) \}$$

Again, this is a definition by recursion on rank. In the analogy with randomness, $\dot{x}(G)$ corresponds to evaluating a random variable at a given outcome.

The following gives the motivation for the definitions of $\dot{x}$ and $\dot{G}$.

Proposition 6.5. If $H$ is any filter and $x$ is any set, then $\dot{x}(H) = x$.

Proposition 6.6. If $H$ is any filter, then $\dot{G}(H) = H$.

Remark 6.7. It is possible to define $Q$-name to just be a synonym for set. The definition of $\dot{x}(G)$ would be left unchanged so that only those elements of $\dot{x}$ which are ordered pairs with a second coordinate in $Q$ play any role in the interpretation. This alternative has the advantage of brevity and much of what is stated in the previous section remains true with this alteration. On the other hand, it is easily seen that Property 5 fails. For instance those sets which do not contain any ordered pairs forms a proper class and each member of this class is forced by the trivial condition to be equal to the emptyset.

We now turn to the formal definition of the forcing relation. The main complexity of the definition of the forcing relation is tied up in the formal definition of $p \models \dot{x} \in \dot{y}$.

Definition 6.8 (forcing relation: atomic formulae). If $Q$ is a forcing and $\dot{x}$ and $\dot{y}$ are $Q$-names, then we define the meaning of $p \models \dot{x} = \dot{y}$ and $p \models \dot{x} \in \dot{y}$ as follows (the definition is by simultaneous recursion on rank):

a. $p \models \dot{x} = \dot{y}$ if and only if for all $\dot{z}$ and $p' \leq p$,

$$(p' \models \dot{z} \in \dot{x}) \leftrightarrow (p' \models \dot{z} \in \dot{y}).$$

b. $p \models \dot{x} \in \dot{y}$ if and only if for every $p' \leq p$ there is a $p'' \leq p' \leq q$ and a $(\dot{z}, q)$ in $\dot{y}$ such that $p'' \leq q$ and $p'' \models \dot{x} = \dot{z}$.

Notice that the definition of $p \models \dot{x} = \dot{y}$ is precisely to ensure that the Axiom of Extensionality — which asserts that two sets are equal if they have the same set of elements — is forced by any condition. The definition of the forcing relation for nonatomic formulas is straightforward and is essentially determined by the properties of the forcing relation mentioned already in Section 5.

Definition 6.9 (forcing relation: logical connectives). Suppose that $p \in Q$ and $\phi$ and $\psi$ are formulas in the forcing language. The following are true:

a. $p \models \neg \phi$ if there does not exist a $q \leq p$ such that $q \models \phi$.

b. $p \models \phi \land \psi$ if and only if $p \models \phi$ and $p \models \psi$.

c. $p \models \phi \lor \psi$ if there does not exist a $q \leq p$ such that $q \models \neg \phi \land \neg \psi$.

d. $p \models \forall v \phi$ if and only if for all $\dot{x}$, $p \models \phi[\dot{x}/v]$.

e. $p \models \exists v \phi$ if and only if there is an $\dot{x}$ such that $p \models \phi[\dot{x}/v]$.

The interested reader may wish to stop and verify that the definitions of $\models Q$ and $Q$-name given in this section satisfy the properties stated in Section 5.

The following theorem is one of the fundamental results about forcing. It connects the syntactic properties of the forcing relation with truth in generic extensions.
of models of set theory. If $M$ is a countable transitive model of ZFC, $Q$ is a forcing in $M$, and $G \subseteq Q$ is an $M$-generic filter, define

$$M[G] := \{ \dot{x}(G) : \dot{x} \in M \text{ and } \dot{x} \text{ is a } Q\text{-name} \}.$$ 

In this context, $M[G]$ is the generic extension of $M$ by $G$ and $M$ is referred to as the ground model. Notice that $M = \{ \check{x} : x \in M \} \subseteq M[G]$ and $G = \check{G} \in M[G]$.

The following theorem relates the semantics of forcing (i.e. truth in the generic extension) with the syntax (i.e. the forcing relation).

**Theorem 6.10.** Suppose that $M$ is a countable transitive model of ZFC and that $Q$ is a forcing which is in $M$. If $q \in Q$, $\phi(v_1, \ldots, v_n)$ is a formula in the language of set theory, and $\dot{x}_1, \ldots, \dot{x}_n$ are in $M$, then the following are equivalent:

a. $q \Vdash \phi(\dot{x}_1, \ldots, \dot{x}_n)$.

b. $M[G] \models \phi(\dot{x}_1(G), \ldots, \dot{x}_n(G))$ whenever $G \subseteq Q$ is an $M$-generic filter and $q$ is in $G$.

**Remark 6.11.** This theorem can be modified to cover countable transitive models of sufficiently large finite fragments of ZFC. In fact this is crucial if one wishes to give a rigorous treatment of the semantics. By Gödel’s second incompleteness theorem, ZFC alone does not prove that there are any set models of ZFC (countable or otherwise). This is in fact our main reason for de-emphasizing the semantics: while it is formally necessary to work with models of finite fragments of ZFC, this only introduces technicalities which are inessential to understanding what can be achieved with forcing.

While we will generally not work with the semantics of forcing, let us note here that it is conventional to use $\dot{x}$ to denote a $Q$-name for an element $x$ of a generic extension $M[G]$. While such names are not unique, the choice generally does not matter and this informal convention affords a great deal of notational economy.

We will now finish this section with some further discussion and notational conventions concerning names. It is frequently the case in a forcing construction that one encounters a $Q$-name for a function $\dot{f}$ whose domain is forced by some condition to be a ground model set; that is, for some set $D$, $p \Vdash \text{dom}(\dot{f}) = D$. A particularly common occurrence is when $D = \omega$ or, more generally, some ordinal. Under these circumstances, it is common to abuse notation and regard $\dot{f}$ as a function defined on $D$, whose values are themselves names: $\dot{f}(\check{x})$ is a $Q$-name $\dot{y}$ such that it is forced that $\dot{f}(\check{x}) = \check{y}$. Notice that if, for some sets $A$ and $B$, $p \Vdash \dot{f} : A \rightarrow B$, it need not be the case that $\dot{f}(a)$ is of the form $\check{b}$ for some $b$ in $B$ — i.e. $p$ need not decide the value of $\dot{f}(a)$ for a given $a \in A$.

In most cases, names are not constructed explicitly. Rather a procedure is described for how to build the object to which the name is referring. Properties and $\exists$ are then implicitly invoked. For example, if $\dot{x}$ is a $Q$-name, $\bigcup \dot{x}$ is the $Q$-name for the unique set which is forced to be equal to the union of $\dot{x}$. Notice that there is an abuse of notation at work here: formally, $\dot{x}$ is a set which has a union $y$. It need not be the case that $y$ is even a $Q$-name and certainly one should not expect $1 \Vdash \bigcup \dot{x} = \check{y}$. This is one of the reasons for using “dot notation”: it emphasizes the role of the object as a name.
A more typical example of is $\omega_1$, the least uncountable ordinal. Since ZFC proves "there is a unique set $\omega_1$ such that $\omega_1$ is an ordinal, $\omega_1$ is uncountable, and every element of $\omega_1$ is countable," it follows that if $Q$ is any forcing, $1 \Vdash Q \exists x \phi(x)$, where $\phi(x)$ asserts $x$ is the least uncountable ordinal. In particular there is a $Q$-name $\dot{x}$ such that $1 \Vdash Q \phi(\dot{x})$. Unless readability dictates otherwise, such names are denoted by adding a “dot” above the usual notation (e.g. $\dot{\omega}_1$).

Another example is $R$. Recall that $R$ is the completion of $Q$ with respect to its metric — formally the collection of all equivalence classes of Cauchy sequences of rationals. We use this same formal definition of $R$ to define $\dot{R}$: if $Q$ is a forcing, $\dot{R}$ is the collection of all $Q$-names for equivalence classes of Cauchy sequences of rational numbers. Notice that $\dot{R}$ is not the same as $\check{R}$ and, more to the point, we need not even have that $1 \Vdash Q \dot{R} = \check{R}$ for a given forcing $Q$. This construction also readily generalizes to define $\dot{X}$ if $X$ is a complete metric space. The $Q$-name $\dot{X}$ is then the collection of all $Q$-names $\dot{x}$ such that $1$ forces that $\dot{x}$ is an equivalence class of Cauchy sequences of elements of $X$. That is, $\dot{X}$ is a $Q$-name for the completion of $X$.

Finally, there are some definable sets which are always interpreted as ground model sets and do not depend on the generic filter. Two typical examples are finite and countable ordinals such as $0$, $1$, $\omega$, and $\omega^2$ as well as sets such as $Q$. In such cases, checks are suppressed in writing the names for ease of readability — we will write $Q$ and not $\check{Q}$ or $\dot{Q}$ in formulae which occur in the forcing language.

7. The cast

We will now introduce the examples which we will put to work throughout the rest of the article. The first class of examples provides the justification for viewing forcings as abstract notions of randomness.

Example 7.1 (random forcing). Define $R$ to be the collection of all measurable subsets of $[0,1]$ which have positive measure. If $I$ is any index set, let $R_I$ denote the collection of all measurable subsets of $[0,1]^I$ which have positive measure. Here $[0,1]$ is equipped with Lebesgue measure and $[0,1]^I$ is given the product measure. Define $q \leq p$ to mean $q \subseteq p$. This order is not separative so formally here we define $R$ and $R_I$ to be the corresponding separative quotients. This amounts to identifying those measurable sets which differ by a measure zero set. Notice that every element of $R_I$ contains a compact set in $R_I$ — the compact elements of $R_I$ are dense. Furthermore, any two elements of $R_I$ are compatible if their intersection has positive measure.

When working with a forcing $Q$, one is rarely interested in the generic filter itself but rather in some generic object which can be derived in some natural way from the generic filter. For instance, in $R_I$ it is forced that

$$\bigcap \{cl(q) : q \in \dot{G}\}$$

contains a unique element. We will let $\dot{r}$ denote a fixed $R_I$-name for this element. For each $i \in I$, let $\dot{r}_i$ denote a fixed $R_I$-name for the $i$th coordinate of $\dot{r}$ and observe that for all $i \neq j$ in $I$,

$$D_{i,j} := \{q \in R_I : (x \in cl(q)) \rightarrow (x(i) \neq x(j))\}$$
is dense. Therefore $1 \Vdash_{\mathcal{R}_I} \forall i \neq j \in I \ (\dot{r}_i \neq \dot{r}_j)$. In particular, it is forced by $\mathcal{R}_I$ that $|\dot{R}| \geq |\dot{I}|$. (Notice however, that we have not established that if, e.g., $I = \aleph_2$, then $1 \Vdash_{\mathcal{R}_I} \aleph_2 = \aleph_2$. This will be established in Section 8.)

In the context of $\mathcal{R}$, we will use $\dot{r}$ to denote a $\mathcal{R}$-name for the unique element of $\bigcap\{\text{cl}(q) : q \in G\}$. If $M$ is a transitive model of ZFC, then $r \in [0, 1]$ is in every measure 1 Borel set coded in $M$ if and only if $\{q \in \mathcal{R} \cap M : r \in q\}$ is a $M$-generic filter. Such an $r$ is commonly referred to as a random real over $M$. The notion of a random real was first introduced by Solovay [25].

The next class of examples includes Cohen’s original forcing from [5]. Just as random forcing is rooted in measure theory, Cohen forcing is rooted in the notion of Baire category.

**Example 7.2** (Cohen forcing). Let $\mathcal{C}$ denote the collection of all finite partial functions from $\omega$ to 2: all functions $q$ such that the domain of $q$ is a finite subset of $\omega$ and the range of $q$ is contained in $2 = \{0, 1\}$. We order $\mathcal{C}$ by $q \leq p$ if $q$ extends $p$ as a function. If $I$ is a set, let $\mathcal{C}_I$ denote the collection of all finite partial functions from $I \times \omega$ to 2, similarly ordered by extension. It is not difficult to show that $\mathcal{C}_I$ is isomorphic to a dense suborder of the collection of all nonempty open subsets of $[0, 1]^I$, ordered by containment. This makes $\mathcal{C}_I$ analogous to $\mathcal{R}_I$ (in fact it is a suborder), although viewing $\mathcal{C}_I$ as a collection of finite partial functions will often be more convenient from the point of view of notation.

It is very often the case that forcings consist of a collection of partial functions ordered by extension. By this we mean that $q \leq p$ means that $p$ is the restriction of $q$ to the domain of $p$. A filter in the forcing is then a collection of functions which is directed under containment and whose union is therefore also a function. This union is the generic object derived from the generic filter.

In the case of $\mathcal{C}_I$, observe that for each $i \neq j$ in $I$ and $n < \omega$, both

$$\{q \in \mathcal{C}_I : (i, n) \in \text{dom}(q)\}$$

are dense. In particular, the generic object will be a function from $I \times \omega$ into 2. As in the case of $\mathcal{R}_I$, such a generic object naturally corresponds to an indexed family $(r_i : i \in I)$ of elements of $[0, 1]$ and genericity ensures that these elements are all distinct.

If $M$ is a transitive model of ZFC, then $r \in [0, 1]$ is in every dense open set coded in $M$ if and only if the set of finite restrictions of the binary expansion of $r$ is an $M$-generic filter for the forcing $\mathcal{C}$. Such an $r$ is commonly referred to as a Cohen real over $M$. Notice that $[0, 1]$ is a union of a measure 0 set and a set of first category: for every $n$, the rationals in $[0, 1]$ are contained in a relatively dense open set of measure less than $1/n$. Thus no element of $[0, 1]$ is both a Cohen real and a random real over a transitive model of ZFC. In fact there are qualitative difference between Cohen and random reals as well. For instance, in the case of random forcing it is forced that

$$\lim_{n \to \infty} \frac{1}{n} |\{i < n : r(i) = 1\}| = \frac{1}{2}$$

where as in the case of Cohen forcing, it is forced that the limit does not exist.

Therefore $1 \Vdash_{\mathcal{R}_I} \forall i \neq j \in I \ (\dot{r}_i \neq \dot{r}_j)$. (Notice however, that we have not established that if, e.g., $I = \aleph_2$, then $1 \Vdash_{\mathcal{R}_I} \aleph_2 = \aleph_2$. This will be established in Section 8.)

In the context of $\mathcal{R}$, we will use $\dot{r}$ to denote a $\mathcal{R}$-name for the unique element of $\bigcap\{\text{cl}(q) : q \in G\}$. If $M$ is a transitive model of ZFC, then $r \in [0, 1]$ is in every measure 1 Borel set coded in $M$ if and only if $\{q \in \mathcal{R} \cap M : r \in q\}$ is a $M$-generic filter. Such an $r$ is commonly referred to as a random real over $M$. The notion of a random real was first introduced by Solovay [25].

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In the case of $\mathcal{C}_I$, observe that for each $i \neq j$ in $I$ and $n < \omega$, both

$$\{q \in \mathcal{C}_I : (i, n) \in \text{dom}(q)\}$$

and

$$\{q \in \mathcal{C}_I : \exists m \ ((\{(i, m), (j, m)\} \subseteq \text{dom}(q)) \land (q(i, m) \neq q(j, m)))\}$$

are dense. In particular, the generic object will be a function from $I \times \omega$ into 2. As in the case of $\mathcal{R}_I$, such a generic object naturally corresponds to an indexed family $(r_i : i \in I)$ of elements of $[0, 1]$ and genericity ensures that these elements are all distinct.

If $M$ is a transitive model of ZFC, then $r \in [0, 1]$ is in every dense open set coded in $M$ if and only if the set of finite restrictions of the binary expansion of $r$ is an $M$-generic filter for the forcing $\mathcal{C}$. Such an $r$ is commonly referred to as a Cohen real over $M$. Notice that $[0, 1]$ is a union of a measure 0 set and a set of first category: for every $n$, the rationals in $[0, 1]$ are contained in a relatively dense open set of measure less than $1/n$. Thus no element of $[0, 1]$ is both a Cohen real and a random real over a transitive model of ZFC. In fact there are qualitative difference between Cohen and random reals as well. For instance, in the case of random forcing it is forced that

$$\lim_{n \to \infty} \frac{1}{n} |\{i < n : r(i) = 1\}| = \frac{1}{2}$$

where as in the case of Cohen forcing, it is forced that the limit does not exist.
The next example of a forcing appears similar at first to random forcing, but in fact it is quite different in nature.

**Example 7.3 (Amoeba forcing).** If \( 1 > \epsilon > 0 \), then define \( A_\epsilon \) to be the collection of all elements of \( \mathcal{R} \) of measure greater than \( \epsilon \). This is regarded as a forcing with the order induced from \( \mathcal{R} \).

Notice that the compatibility relation on \( A_\epsilon \) differs from that inherited from \( \mathcal{R} \): two conditions in \( A_\epsilon \) are compatible in \( A_\epsilon \) if and only if their intersection has measure greater than \( \epsilon \).

The previous forcings all introduce a new subset of \( \omega \) to the ground model. The next example adds a new ultrafilter on \( \omega \) but, as we will see in Section 14, it does not introduce a new subset of \( \omega \).

**Example 7.4.** Let \( [\omega]^\omega \) denote the collection of all infinite subsets of \( \omega \). The ordering of containment on \( [\omega]^\omega \) is not separative; its separative quotient is obtained by identifying those \( x \) and \( y \) which have finite symmetric difference. We will abuse notation and denote this quotient by \( [\omega]^\omega \) as well.

The next forcing was introduced by Mathias \[19\] to study infinite dimensional generalizations of Ramsey’s theorem.

**Example 7.5 (Mathias forcing).** Let \( \mathcal{M} \) denote the collection of all pairs \( p = (a_q, A_p) \) such that \( A_p \) is in \( [\omega]^\omega \) and \( a_p \) is a finite initial part of \( A_p \). Define \( q \leq p \) to mean \( a_p \subseteq a_q \) and \( A_q \subseteq A_p \). Note in particular that in this situation \( a_p \) is an initial segment of \( a_q \) and \( a_q \setminus a_p \) is contained in \( A_p \). This forcing is known as Mathias forcing.

The final example is an illustration of the potential raw power of forcing. Typically the phenomenon of collapsing cardinals to \( \aleph_0 \) is something one wishes to avoid.

**Example 7.6 (collapsing to \( \aleph_0 \)).** If \( X \) is a set, consider the collection \( X^{<\omega} \) of all finite sequences of elements of \( X \), ordered by extension. Observe that if \( x \) is in \( X \), then the collection of all elements of \( X^{<\omega} \) which contain \( x \) in their range is dense. Thus \( X^{<\omega} \) forces that \( |X| = \aleph_0 \). Notice that if \( X = \mathbb{R} \) in this example, then it is forced that \( |\mathbb{R}| = \aleph_0 < |\hat{\mathbb{R}}| \).

8. **THE COUNTABLE CHAIN CONDITION**

Something which is often an important consideration in the analysis of a forcing is whether uncountability is preserved. That is, does \( 1 \Vdash Q \check{\aleph}_1 = \check{\aleph}_1 \)? More generally, one can ask whether cardinals are preserved by forcing with \( Q \): if \( X \) and \( Y \) are sets such that \( |X| < |Y| \), then does \( 1 \Vdash Q |X| < |Y| \)?

In general, this can be a very subtle matter (and even can be influenced by forcing). One way to demonstrate that a forcing preserves cardinals is to verify that it satisfies the countable chain condition (c.c.c.). A forcing \( Q \) satisfies the c.c.c. if every family of pairwise incompatible elements of \( Q \) is at most countable. The following proposition dates to Cohen’s proof that the Continuum Hypothesis is independent of ZFC.

**Proposition 8.1.** Suppose that \( Q \) is a c.c.c. forcing. If \( \kappa \) is a regular cardinal, then \( 1 \Vdash \check{\kappa} \) is a regular cardinal. In particular, \( \kappa \) is a cardinal then \( 1 \Vdash \check{\kappa} \) is a cardinal and hence for every ordinal \( \alpha \), \( 1 \Vdash \check{\aleph}_\alpha = \check{\aleph}_\alpha \).
Proof. The second conclusion follows from the first since every cardinal is the supremum of a set of successor cardinals and every supremum of a set of successor cardinals is a cardinal. Let \( \kappa \) be a regular cardinal and \( Q \) be a given forcing. Suppose that \( \dot{f} \) and \( \dot{\lambda} \) are \( Q \)-names and that \( p \) is an element of \( Q \) such that
\[
p \forces (\dot{\lambda} \in \check{\kappa}) \wedge (\dot{f} : \dot{\lambda} \to \check{\kappa})
\]
By extending \( p \) if necessary, we may assume without loss of generality that \( \dot{\lambda} = \check{\lambda} \) for some \( \lambda < \kappa \). It is sufficient to show that \( p \) forces that \( \dot{f} \) is not a surjection. If \( \kappa \) is countable, then \( \dot{\lambda} \) is finite and it is possible to decide \( \dot{f} \) by deciding its values one at a time (this does not require that \( Q \) is c.c.c.). Thus we will assume that \( \kappa \) is uncountable.

For each \( \alpha < \lambda \), define
\[
F(\alpha) := \{ \beta < \kappa : \exists q \leq p(q \forces \dot{f}(\alpha) = \check{\beta}) \}.
\]
Notice that if \( \beta \neq \beta' \) are in \( F(\alpha) \) and \( q \) forces that \( \dot{f}(\alpha) = \check{\beta} \) and \( q' \) forces that \( \dot{f}(\alpha) = \check{\beta'} \), then \( q \) and \( q' \) are incompatible (otherwise any extension \( \bar{q} \) would force \( \dot{\beta} = \dot{f}(\alpha) = \check{\beta'} \)). Since \( Q \) is c.c.c., \( F(\alpha) \) is countable and has an upper bound \( g(\alpha) < \kappa \).

Since \( \kappa \) is regular, the range of \( g \) is bounded. We are therefore finished once we show that
\[
p \forces \forall \alpha \in \dot{\lambda} (\dot{f}(\alpha) \in \check{F}(\alpha)).
\]
By Proposition 5.2, this is equivalent to showing that for all \( \alpha \) in \( \lambda \), \( p \forces \dot{f}(\alpha) \in \check{F}(\alpha) \). Suppose for contradiction that this is not the case. Then there is a \( q \leq p \) and an \( \alpha \) in \( \lambda \) such that \( q \forces \dot{f}(\alpha) \notin \check{F}(\alpha) \). By Proposition 8.1 there is a \( q' \leq q \) and a \( \beta < \kappa \) such that \( q' \forces \dot{f}(\alpha) = \check{\beta} \). But now \( \beta \in F(\alpha) \), a contradiction. \( \Box \)

Proposition 8.2. Suppose that \( Q \) is a c.c.c. forcing and that \( (q_\xi : \xi < \omega_1) \) is a sequence of conditions in \( Q \). Then there is a \( p \) such that
\[
p \forces \{ \xi \in \check{\omega}_1 : q_\xi \in \check{G} \}
\]
is uncountable.

Remark 8.3. Notice that this characterizes the c.c.c.: if \( A \) is an uncountable antichain in a forcing \( Q \), then any condition forces that \( A \cap \check{G} \) contains at most one element.

Proof. Suppose that this is not the case. Then
\[
1 \models \exists \beta \in \check{\omega}_1 \forall \xi \in \check{\omega}_1 (q_\xi \in \check{G} \rightarrow \xi < \beta).
\]
By Property 7 of the forcing relation, there is a \( Q \)-name \( \dot{\beta} \) for an element of \( \omega_1 \) such that
\[
1 \models \forall \xi \in \check{\omega}_1 (q_\xi \in \check{G} \rightarrow \xi < \dot{\beta}).
\]
As in the proof of Proposition 8.1 the set of \( \alpha < \omega_1 \) such that, for some \( q \in Q \), \( q \forces \dot{\beta} = \check{\alpha} \) is countable and therefore bounded by some \( \gamma \). That is
\[
1 \models \forall \xi \in \check{\omega}_1 (q_\xi \in \check{G} \rightarrow \xi < \check{\gamma}).
\]
But now \( q_\gamma \) forces that \( \check{q}_\gamma \), is in \( \check{G} \), a contradiction. \( \Box \)

We will now return to some of the examples introduced in Section 7.

Proposition 8.4. For any index set \( I \), both \( R_I \) and \( C_I \) are c.c.c. forcings.
Proof. In the case of $R_I$, this is just a reformulation of the assertion that if $\mathcal{F}$ is an uncountable family of measurable subsets of $[0,1]^I$, each having positive measure, then there are two elements of $\mathcal{F}$ which intersect in a set of positive measure. The reason for this is that if $\mathcal{F}$ is uncountable, then for some $\epsilon > 0$ there are more than $1/\epsilon$ elements of $\mathcal{F}$ with measure at least $\epsilon$. At least two of these elements must intersect in a set of positive measure. The same argument applies to $C_I$, by observing that we may view $C_I$ as a dense suborder of the collection of all nonempty open subsets of $[0,1]^I$, ordered by containment. Since any nonempty open subset of $[0,1]^I$ has positive measure we may view $C_I$ as a suborder of $R_I$. Moreover, conditions $p,q \in C_I$ which are compatible in $R_I$ are compatible in $C_I$. □

Proposition 8.5. The forcing $A_\epsilon$ satisfies the c.c.c. for every $\epsilon > 0$.

Proof. Let $D$ denote the collection of all elements of $A_\epsilon$ which are finite unions of rational intervals. Notice that $D$ is countable. For each $p$ in $D$, let $F_p$ denote the collection of all elements $q$ of $A_\epsilon$ such that $q \subseteq p$ and

$$\lambda(p) - \lambda(q) < \frac{\lambda(p) - \epsilon}{2}.$$  

Notice that $\bigcup_{p \in D} F_p$ contains all of the compact sets in $A_\epsilon$ which are in turn dense in $A_\epsilon$. Moreover, any two elements of $F_p$ intersect in a set of measure greater than $\epsilon$ and hence have a common lower bound in $A_\epsilon$. If $X \subseteq A_\epsilon$ is uncountable, two distinct elements of $X$ must have extensions in the same $F_p$ for some $p$ and thus be compatible. Hence any antichain in $A_\epsilon$ is countable and $A_\epsilon$ is c.c.c.. □

Remark 8.6. The reader may wonder why we have not bothered to generalize $A_\epsilon$ to a larger index set, given that we did this for $R$. The reason is that, for uncountable index sets, the analog of $A_\epsilon$ is not c.c.c. and in fact collapses the cardinality of $I$ to become countable.

We finish this section by demonstrating that the Continuum Hypothesis isn’t provable within ZFC. By Theorem 8.1, $R_{\omega_2}$ forces that $\check{\aleph}_1 = \check{\aleph}_1$ and $\check{\aleph}_2 = \check{\aleph}_2$. On the other hand, we have already observed that for all $\alpha < \beta < \omega_2$,

$$1 \vDash R_{\omega_2} \check{i}_\alpha \neq \check{i}_\beta.$$  

Hence $R_{\omega_2}$ forces that $|\check{R}| \geq \check{\aleph}_2 = \check{\aleph}_2$. Since the set of formulas which are forced by 1 is a consistent theory extending ZFC and containing $|\check{R}| \geq \check{\aleph}_2$, this establishes that ZFC cannot prove the Continuum Hypothesis. The same argument shows that $\check{C}_{\omega_2}$ forces that CH is false; this was the essence of Cohen’s proof.

9. AN INTERSECTION PROPERTY OF FAMILIES OF SETS OF POSITIVE MEASURE*

The purpose of this section is to use the tools which we have developed in order to prove the following intersection property of sets of positive measure in $[0,1]$.

Proposition 9.1. If $X \subseteq \mathbb{R}$ is uncountable and $(B_x : x \in X)$ is an indexed collection of Borel subsets of $[0,1]$, each having positive measure, then there is a nonempty set $Y \subseteq X$ such that $Y$ has no isolated points and such that $\bigcap\{B_y : y \in Y\}$ has positive measure.

Proof. By replacing each $B_x$ with a subset if necessary, we may assume that each $B_x$ is compact. Similarly, by replacing $X$ with a subset if necessary, we may assume
that there is an \( \epsilon > 0 \) such that if \( x \) is in \( X \), then \( B_x \) has measure greater than \( \epsilon \).

Let \( T \) consist of all finite length sequences \( \sigma = (\sigma_i : i < n) \) such that:

\[
\begin{align*}
(10) & \quad \sigma \text{ is an increasing sequence of finite subsets of } X; \\
(11) & \quad \bigcap \{ B_x : x \in \sigma_i \} \text{ has measure greater than } \epsilon \text{ for all } i < n; \\
(12) & \quad \text{for each } i < n, \text{ if } x \text{ is in } \sigma_i, \text{ then there is a } y \text{ distinct from } x \text{ in } \sigma_i \text{ such that } |x - y| < 1/i.
\end{align*}
\]

Observe that if \( \sigma \) is an infinite sequence all of whose initial parts are in \( T \), then \( Y := \bigcup \{ \sigma_i : i < \infty \} \) has no isolated points and \( \bigcap \{ B_y : y \in Y \} \) has measure at least \( \epsilon \). Conversely, if there is a countable \( Y \subseteq X \) with no isolated points and \( \bigcap_{x \in F} B_x \) has measure greater than \( \epsilon \) whenever \( F \subseteq Y \) is finite, then \( T \) has an infinite path. Thus by Proposition 5.4 it is sufficient to show that the conclusion of the proposition is forced by some condition in some forcing.

Consider the Amoeba forcing \( \mathcal{A}_\epsilon \) and let \( \tilde{Z} \) be the \( \mathcal{Q} \)-name for the set \( \{ x \in \tilde{X} : \tilde{B}_x \in \tilde{G} \} \). Observe that every condition forces that the intersection of every finite subset of \( \{ \tilde{B}_x : x \in \tilde{Z} \} \subseteq \tilde{G} \) is in \( \tilde{G} \) and hence in \( \mathcal{A}_\epsilon \). By Proposition 5.2 there is a \( q \) in \( \mathcal{A}_\epsilon \) such that \( q \) forces that \( \tilde{Z} \) is uncountable. By Property 8 of the forcing relation, \( q \) forces that \( \tilde{Z} \) contains a countable subset \( \tilde{Y} \) with no isolated points.

This finishes the proof. \( \square \)

10. THE HALPERN-LÄUCHLI THEOREM*

The Halpern-Läuchli Theorem is a Ramsey-theoretic result concerning colorings of products of finitely branching trees. Before stating the theorem, we need to first define some terminology. Recall that a subset \( T \) of \( \omega^{<\omega} \) is a tree if it is closed under initial segments: whenever \( t \) is in \( T \) and \( s \) is an initial part of \( t \), it follows that \( s \) is in \( T \). A tree \( T \subseteq \omega^{<\omega} \) comes equipped with a natural partial order: \( s \leq t \) if and only if \( s \) is an initial part of \( t \). If \( T \subseteq \omega^{<\omega} \) is a tree and \( l < \omega \), the \( l \)th level of \( T \) consists of all elements of \( T \) of length \( l \) and is denoted \( (T)_l \).

All trees considered in this section will be assumed to be pruned without further mention: every element will have at least one immediate successor. A tree \( T \subseteq \omega^{<\omega} \) is finitely branching if every element of \( T \) has only finitely many immediate successors in \( T \). If \( S \subseteq T \subseteq \omega^{<\omega} \) are trees and \( J \subseteq \omega \) is infinite, then we say that \( S \) is a strong subtree of \( T \) based on \( J \) if whenever \( s \) is in \( S \) with length in \( J \), every immediate successor of \( s \) in \( T \) is in \( S \). The Halpern-Läuchli Theorem can now be stated as follows.

**Theorem 10.1.** [10] If \( (T_i : i < d) \) is a sequence of finitely branching subtrees of \( \omega^{<\omega} \) and

\[
f : \bigcup_{i=0}^\infty \prod_{i < d} (T_i)_l \rightarrow k
\]

then there exists an infinite set \( L \subseteq \omega \) and strong subtrees \( S_i \subseteq T_i \) based on \( L \) for each \( i < d \) such that \( f \) is constant when restricted to \( \bigcup_{l \in L} \prod_{i < d} (S_i)_l \).

Unlike essentially all other Ramsey-theoretic statements concerning the countably infinite, the full form of the Halpern-Läuchli Theorem — at least at present — cannot be derived from the machinery of semigroup dynamics of spaces of ultrafilters (see [11], [28]). The special case of the Halpern-Läuchli Theorem for \( n \)-ary trees is a consequence of a form of the Hales-Jewett Theorem, which can be proved using semigroup dynamics — see [28]. The proof which is presented in this section
is based on forcing and is an inessential modification of an argument due to Leo Harrington (see [31]).

In order to prove the Halpern-Läuchli Theorem, we will derive it from the so-called dense set form of the theorem. If \( T \subseteq \omega^{<\omega} \) is a tree and \( t \) is in \( S \), then a set \( D \subseteq T \) is \((m,n)\)-dense in \( T \) above \( t \) if \( D \subseteq (T)_n \) and whenever \( u \) is in \( (T)_m \) with \( t \subseteq u \), there is a \( v \) in \( D \) such that \( u \subseteq v \). If \( t \) is the null string, then we just say that \( D \) is \((m,n)\)-dense in \( T \).

**Theorem 10.2.** If \( (T_i : i < d) \) is a sequence of finitely branching subtrees of \( \omega^{<\omega} \) and

\[
 f : \bigcup_{i=0}^{\infty} \prod_{i < d}(T_i) \rightarrow k
\]

then there is an \( l \) and a \( \xi \) in \( \prod_{i < d}(T_i)_\xi \) such that for every \( m \geq l \) there is an \( n \geq m \) and sets \( (D_i : i < d) \) such that for each \( i < d \), \( D_i \) is \((m,n)\)-dense above \( t_i \) in \( T_i \) and such that \( f \) is constant on \( \prod_{i < d}D_i \).

The original form of the Halpern-Läuchli Theorem is an immediate consequence of the dense set version and the following observation.

**Observation 10.3.** Let \( T \subseteq \omega^{<\omega} \) be a tree and \( t \) be an element of \( T \). If \( (D_p : p < \infty) \) is a sequence of subsets of \( T \) such that for some increasing sequence \( (m_p : p < \infty) \), \( D_p \) is \((m_p,m_{p+1})\)-dense in \( T \) above \( t \), then the downward closure of \( \bigcup_{p=0}^{\infty} D_p \) contains a strong subtree of \( T \) which is based on \( \{m_p : p < \infty\} \).

It is also not difficult to see that, unlike the standard formulation of the Halpern-Läuchli Theorem, the special case of Theorem 10.2 in which each \( T_i \) is \( 2^{<\omega} \) is equivalent to the theorem in its full generality.

Harrington’s proof of the Halpern-Läuchli theorem uses the forcing relation to reduce the desired Ramsey-theoretic properties of trees to Ramsey-theoretic properties of cardinals. In the proof we will need some standard definitions and facts from combinatorial set theory (see, e.g., [16 Ch.III]). If \( \kappa \) is a regular cardinal, a subset \( S \) of \( \kappa \) is stationary if it intersects every closed and unbounded subset of \( \kappa \). Clearly every stationary subset of \( \kappa \) has cardinality \( \kappa \). Furthermore, if \( \mu \) is an infinite cardinal less than \( \kappa \), then the set of all ordinals in \( \kappa \) of cofinality \( \mu \) is stationary. We will need the following property of stationary sets.

**Lemma 10.4** (Pressing Down Lemma; see [16]). Suppose that \( \theta \) is a regular cardinal and \( S \subseteq \theta \) is a stationary set. If \( r : S \rightarrow \theta \) satisfies that \( r(\xi) < \xi \) for all \( \xi \in S \), then \( r \) is constant on a stationary subset of \( S \). In particular if a stationary subset of \( \theta \) is partitioned into fewer than \( \theta \) sets, then one of the pieces of the partition is stationary.

We will need the following variant of the \( \Delta \)-System Lemma.

**Lemma 10.5.** Suppose that \( X \) is a set, \( \theta \) is the successor of a regular cardinal, and \( \{p_\xi : \xi < \theta\} \) is a family of partial functions from \( X \) to \( 2 \) such that for every \( \xi < \theta \), \( 2^{\text{dom}(p_\xi)} < \theta \). Then there exists a cofinal \( H \subseteq \theta \) such that \( \bigcup_{\xi \in H} p_\xi \) is a function.

**Proof.** Set \( \theta := \kappa^+ \). Observe that by replacing \( X \) with the union of the domains of the \( p_\xi \)’s if necessary, we may assume that \( |X| < \theta \) and thus moreover that \( X \subseteq \theta \). For each \( \xi < \theta \), define \( a_\xi := \text{dom}(p_\xi) \cap \xi \). Observe that \( |\text{dom}(p_\xi)| \) must be less than \( \kappa \) for each \( \xi \) and thus \( a_\xi \) is a bounded subset of \( \xi \) whenever \( \text{cf}(\xi) = \kappa \). Let \( E \subseteq \theta \)
consist of all $\delta$ such that if $\xi < \delta$, then $\sup(\text{dom}(p_\xi)) < \delta$. It is easily checked that $E$ is a closed and unbounded set. By Lemma 10.4, there is a stationary $S \subseteq E$ consisting of ordinals of cofinality $\alpha$ and a $\zeta$ such that if $\xi$ is in $S$, $\sup a_\xi < \zeta$. By the pigeonhole principle, there is a stationary set $H \subseteq S$ and partial function $r$ from $\theta$ to $2$ such that if $\xi$ is in $H$, then $p_\xi \upharpoonright \xi = r$. Now if $\xi < \eta$ are in $H$, then $p_\xi \cup p_\eta$ is a function. To see this, suppose that $\alpha$ is in $\text{dom}(p_\xi) \cap \text{dom}(p_\eta)$. Since $\eta$ is in $E$, it must be that $\alpha < \eta$. Thus $\alpha$ is in $a_\xi = a_\eta = \text{dom}(r)$ and hence $p_\xi(\alpha) = r(\alpha) = p_\eta(\alpha)$.

Next, we will need two closely related Ramsey-theoretic statements which are relatives of the Erdős-Rado Theorem but which have simpler proofs.

**Lemma 10.6.** Suppose that $(\theta_i : i < d)$ is a sequence of uncountable regular cardinals satisfying $2^{\theta_i} < \theta_{i+1}$ if $i < d - 1$. If $f : \prod_{i<d} \theta_i \rightarrow \omega$, then there exist cofinal sets $H_i \subseteq \theta_i$ for each $i < d$ such that $f$ is constant on $\prod_{i<d} H_i$.

**Proof.** The proof is by induction on $d$. If $d$ is given and $\xi < \theta_{d-1}$, fix $H_i^\xi \subseteq \theta_i$ for each $i < d - 1$ such that $f$ takes the constant value $g(\xi)$ on

$$\left( \prod_{i<d-1} H_i \right) \times \{\xi\}$$

(if $d = 1$, then the product over the emptyset is the trivial map with domain $\emptyset$ and this is vacuously true). By applying the pigeonhole principle and our cardinal arithmetic assumption, there is an $H_{d-1} \subseteq \theta_{d-1}$ such that $g$ is constant on $H_{d-1}$ and $H_i^\xi$ does not depend on $\xi$ for $\xi \in H_{d-1}$. It follows that $f$ is constant when restricted to $\prod_{i<d} H_i$, where $H_i := H_i^\xi$ for some (equivalently any) $\xi$ in $H_{d-1}$. $\square$

**Lemma 10.7.** Suppose that $X$ is a set and $(\theta_i : i < d)$ is a sequence of successors of infinite regular cardinals such that $2^{\theta_i} < \theta_{i+1}$ if $i < d - 1$. If $\{p_\sigma : \sigma \in \prod_{i<d} \theta_i\}$ is a family of finite partial functions from $X$ into a countable set, then there are $H_i \subseteq \theta_i$ of cardinality $\theta_i$ such that

$$\bigcup\{p_\sigma : \sigma \in \prod_{i<d} H_i\}$$

is a function.

**Proof.** The proof is by induction on $d$. The case $d = 1$ follows from Lemma 10.5. Now suppose $(\theta_i : i \leq d)$ and $(p_\sigma : \sigma \in \prod_{i \leq d} \theta_i)$ are given. For each $\xi$ in $\theta_d$, find $(H_i^\xi : i < d)$ such that $H_i^\xi \subseteq \theta_i$ and such that

$$\bigcup\{p_{\sigma \upharpoonright \xi} : \sigma \in \prod_{i<d} H_i\}$$

is a function, which we will denote by $q_\xi$. Applying the pigeonhole principle, find a cofinal $\Gamma \subseteq \theta_d$ such that, for some $(H_i : i < d)$, $H_i^\xi = \Gamma$ if $\xi$ is in $\Gamma$. Now apply Lemma 10.5 to $(q_\xi : \xi \in \Gamma)$ to find $H_d \subseteq \Gamma$ of cardinality $\theta_d$ such that

$$\bigcup_{\xi \in H_d} q_\xi = \bigcup_{\xi \in H_d} \{p_\sigma : \sigma \in \prod_{i\leq d} H_i\}$$

is a function. $\square$

Finally we turn to the task of proving the dense set form of the Halpern-Läuchli Theorem.
Suppose that

$$f : \bigcup_{k=0}^{\infty} \prod_{i<d} 2^k \to 2$$

is given and define \((\theta_i : i < d)\) by \(\theta_0 = \aleph_1\) and \(\theta_{i+1} = (\theta_i)^{++}\). Set \(Q\) to be the collection of all finite partial functions from \(\theta_{d-1}\) into \(2^{<\omega}\). (This is an inessential modification of the forcing \(C_{\theta_{d-1}}\).) The order on \(Q\) is defined by \(q \leq p\) if the domain of \(q\) contains the domain of \(p\) and \(p(\alpha)\) is an initial part \(q(\alpha)\) whenever \(\alpha\) is in the domain of \(p\). Observe that \(\dot{r}_\xi := \bigcup_{q \in \dot{G}} q(\dot{\xi})\) describes an element of \(2^\omega\).

Applying Property 8 of the forcing relation, fix a \(Q\)-name \(\dot{U}\) for a nonprincipal ultrafilter on \(\omega\). Since \(\dot{U}\) is forced to be an ultrafilter, for each \(\sigma \in \prod_{i<d} \theta_i\) there is a \(\dot{e}_\sigma\) such that it is forced that there such that

$$\dot{U}_\sigma = \{ m \in \omega : f(\dot{r}_{\sigma(0)} \upharpoonright m, \ldots, \dot{r}_{\sigma(d-1)} \upharpoonright m) = \dot{e}_\sigma \}$$

is in \(\dot{U}\). By Property 4 of the forcing relation, there is a \(p_\sigma\) which decides \(\dot{e}_\sigma\) to be some \(e_\sigma \in \{0, 1\}\). By extending \(p_\sigma\) if necessary, we may assume that there is an \(l_\sigma\) such that if \(\alpha\) is in the domain of \(p_\sigma\), \(p_\sigma(\alpha)\) has length \(l_\sigma\). Define

$$g(\sigma) := (e_\sigma, l_\sigma, p_\sigma(\sigma(0)), \ldots, p_\sigma(\sigma(d-1))).$$

By Lemmas 10.6 and 10.7 there are cofinal sets \(H_i \subseteq \theta_i\) such that:

13. \(g\) is constantly \((e, l, t_0, \ldots, t_{d-1})\) on \(\prod_{i<d} H_i\) for some \((e, l)\) and \(t_0, \ldots, t_{d-1}\) in \(2^d\);

14. every finite subset of \(\{p_\sigma : \sigma \in \prod_{i<d} H_i\}\) has a common lower bound in \(Q\).

Now let \(m \geq l\) be given. For each \(i < d\), let \(A_i\) be a subset of \(H_i\) of cardinality \(2^{m-l}\) and fix a bijection between \(A_i\) and the set of binary sequences of length \(m-l\). Let \(q\) be a condition in \(Q\) which is a common lower bound for

$$\{p_\sigma : \sigma \in \prod_{i<d} A_i\}$$

and such that if \(\alpha\) is the element of \(A_i\) which corresponds to \(u \in 2^{m-l}\) under the bijection, then \(q(\alpha)\) has \(t_i \upharpoonright u\) as an initial part. That is, \(q\) forces that \(\dot{r}_n \upharpoonright m = t_i \upharpoonright u\).

Let \(\bar{q}\) be an extension of \(q\) such that for some \(n > m\), \(\bar{q}\) forces that

$$\bar{n} \in \bigcap_{i<d} \{ \dot{U}_\sigma : \sigma \in \prod_{i<d} A_i \}.$$ 

By extending \(\bar{q}\) if necessary, we may assume that for each \(i < d\) and \(\alpha\) in \(A_i\), \(\bar{q}(\alpha)\) has length at least \(n\). Finally, set \(D_i\) to be the set of all \(w \in 2^n\) such that for some \(\alpha\) in \(A_i\), \(\bar{q}(\alpha) \upharpoonright n = w\).

We will now show that \(D_i\) is \((m, n)\) dense above \(t_i\) for each \(i < d\) and that \(f \upharpoonright \prod_{i<d} D_i\) is constantly \(e\). To see the former, fix \(i < d\) and let \(u\) be in \(2^{m-l}\) and let \(\alpha\) be the corresponding element of \(A_i\). By our choice of \(q\), \(q(\alpha) \upharpoonright m = t_i \upharpoonright u\) and by our choice of \(\bar{q}\) and the definition of \(D_i\), there is a \(w\) in \(D_i\) such that \(\bar{q}(\alpha) \upharpoonright n = w\). To see the latter, suppose \((w_i : i < d) \in \prod_{i<d} D_i\) and let \(\sigma \in \prod_{i<d} A_i\) be such that \(\bar{q}(\alpha_i) \upharpoontright n = w_i\). Clearly \(\bar{q}\) forces that

$$(\dot{r}_{\sigma(i)} \upharpoonright n : i < d) = (w_i : i < d).$$

Furthermore, by the definition of \(U_\sigma\) and \(n\), we have that

$$f((\dot{r}_{\sigma(i)} \upharpoonright n : i < d)) = f((w_i : i < d)) = e.$$
11. UNIVERSALLY BAIRE SETS AND ABSOLUTENESS

In this section, we will introduce an abstract notion of regularity for subsets of complete metric spaces which is useful in proving absoluteness results. Let \((X,d)\) be a (not necessarily separable) complete metric space. Recall that the completion of a metric space is taken to be the collection of all equivalence classes of its Cauchy sequences. Recall also that if \(Q\) is a forcing, then \(\dot{X}\) represents a \(Q\)-name for the completion of \(X\).

In this section we will be interested in interpreting names by filters which are not fully generic. Notice, for instance, that it is possible that \(\dot{y}\) is forced to be equal to \(\dot{x}\), even though there are some (non generic) ultrafilters which interpret \(\dot{y}\) to be different than \(x\). For this reason it is necessary to work with names which have better properties with respect to arbitrary interpretations.

Definition 11.1. If \(Q\) is a forcing, then a nice \(Q\)-name for an element of \(\dot{X}\) is a \(Q\)-name \(\dot{x}\) such that, for some countable collection of dense subsets \(D\) of \(Q\), \(\dot{x}(G)\) is a Cauchy sequence in \((X,d)\) whenever \(G\) is \(D\)-generic.

Remark 11.2. For technical reasons we need to make nice \(Q\)-names for elements of a complete metric space \(\dot{X}\) to formally be a Cauchy sequence rather than an equivalence class of a Cauchy sequence, even though the intent is only to refer to the limit point corresponding to the equivalence class. Also, while the completion of a complete metric space is not literally equal to the original space, there is a canonical isometry between the two and usually there is no need to distinguish them. The point in the above definition is that the only meaningful way to define \(\dot{X}\) is as the name for the completion of \(\dot{X}\). Hence names for elements of \(\dot{X}\) are names for equivalence classes of Cauchy sequences. When they are interpreted by a sufficiently generic filter, they will typically result in elements of the completion of \(X\), not in elements of \(X\).

The next lemma shows nice \(Q\)-names can be used to represent any element of \(\dot{X}\) whenever \(Q\) is a forcing and \(X\) is a complete metric space.

Lemma 11.3. If \((\dot{x}_n : n < \infty)\) is a \(Q\)-name for a Cauchy sequence in \(\dot{X}\), then there exists a nice \(Q\)-name \((\dot{y}_n : n < \infty)\) for an element of \(\dot{X}\) such that it is forced that \((\dot{y}_n : n < \infty) = (\dot{x}_n : n < \infty)\).

Proof. Define
\[
\dot{y}_n := \{(\dot{y},q) \in X \times Q : q \vDash \dot{x}_n = \dot{y}\}
\]
and let \(D_n\) be the elements of \(Q\) which decide both \(\dot{x}_n\) and the least \(\bar{m}\) such that for all \(i, j > \bar{m}\), \(d(\dot{x}_i, \dot{x}_j) < 1/n\). It is readily verified that \(D := \{D_n : n < \infty\}\) witnesses that \((\dot{y}_n : n < \infty)\) is a nice \(Q\)-name for an element of \(\dot{X}\).

Definition 11.4. (see [6]) Let \((X,d)\) be a complete metric space. A subset \(A\) of \(X\) is universally Baire if whenever \(Q\) is a forcing there is a \(Q\)-name \(\dot{A}\) such that for every nice \(Q\)-name \(\dot{x}\) for an element of \(\dot{X}\), there is a countable collection of dense subsets \(D\) of \(Q\) such that:

a. \(\{q \in Q : q\) decides \(\dot{x} \in \dot{A}\}\) is in \(D\);

b. whenever \(G\) is a \(D\)-generic filter in \(Q\), \(\dot{x}(G)\) is in (the completion of) \(X\) and \(\dot{x}(G)\) is in \(A\) if and only if there is a \(q\) in \(G\) such that \(q \vDash \dot{x} \in \dot{A}\).
The following proposition, while easy to establish, is important in what follows.

**Proposition 11.5.** If \( \dot{A} \) and \( \dot{B} \) are \( Q \)-names which both witness that \( A \) is universally Baire with respect to \( Q \), then \( 1 \Vdash \dot{A} = \dot{B} \).

**Proof.** If this were not the case, then there would exist a nice \( Q \)-name \( \dot{x} \) for an element of \( \dot{X} \) and a \( p \) in \( Q \) such that
\[ p \Vdash \dot{x} \in (\dot{A} \triangle \dot{B}). \]
Suppose without loss of generality that \( p \Vdash \dot{x} \in (\dot{A} \setminus \dot{B}) \). If \( G \) is a sufficiently generic filter containing \( p \), then \( \dot{x}(G) \) will be in \( A \) since \( p \) is in \( G \) and \( p \) forces that \( \dot{x} \) is in \( A \). On the other hand, \( \dot{x}(G) \) can’t be in \( A \) since \( p \) is in \( G \) and \( p \) forces that \( \dot{x} \) is not in \( B \), a contradiction. \( \square \)

The following is also easy to establish. The proof is left to the interested reader.

**Proposition 11.6.** The universally Baire subsets of a complete metric space form a \( \sigma \)-algebra which includes the open subsets of \( X \). In particular, every Borel set in a complete metric space is universally Baire.

Putting this all together, we have the following proposition which will be used in establishing absoluteness results. If \( \phi(v_1, \ldots, v_n) \) is a logical formula and \( x_1, \ldots, x_n \) are sets, then we say that \( \phi(x_1, \ldots, x_n) \) is generically absolute if whenever \( Q \) is a forcing and \( q \) is in \( Q \), \( q \Vdash \phi(\dot{x}_1, \ldots, \dot{x}_n) \) if and only if \( \phi(x_1, \ldots, x_n) \) is true.

**Proposition 11.7.** The assertion that a given countable Boolean combination of universally Baire subsets of a complete metric space is nonempty is generically absolute.

12. A property of marker sequences for Bernoulli shifts*

In this section we will give an example of how homogeneity properties of a forcing can be put to use. The goal of the section is to prove a special case of a theorem of Gao, Jackson, and Seward concerning marker sequences in Bernoulli shift actions. Let \( \Gamma \) be a countable discrete group acting continuously on a Polish space \( X \). A decreasing sequence of Borel subsets \( (A_n : n < \infty) \) of \( X \) is a vanishing marker sequence for the action if each \( A_n \) intersects every orbit of the action and \( \bigcap_{n=0}^{\infty} A_n = \emptyset \). Certainly a necessary requirement for such a sequence to exist is that every orbit of \( \Gamma \) is infinite. In fact this is also a sufficient condition; this is the content of the so-called Marker Lemma (see, e.g., [14]). The following result of Gao, Jackson, and Seward grew out of their analysis of the Borel chromatic number of the free part of the shift graph on \( 2^{2^\mathbb{Z}} \).

**Theorem 12.1.** [9] Suppose that \( \Gamma \) is a countable group, \( k \) is a natural number, and \( (A_n : n < \infty) \) is a vanishing marker sequence for the free part of the shift action of \( \Gamma \) on \( k^\Gamma \). For every increasing sequence \( (F_n : n < \infty) \) of finite sets which covers \( \Gamma \), there is an \( x \in k^\Gamma \) such that:

a. the closure of the orbit of \( x \) is contained in the free part of the action;
b. the closure of the orbit of \( x \) is a minimal nonempty closed subset of \( k^\Gamma \) which is invariant under the action;
c. there are infinitely many \( n \) such that, for some \( g \) in \( F_n \), \( g \cdot x \) is in \( A_n \).
Our interest will be primarily in the last clause, although in this generality, \[9\] represents the first proof of the first two clauses in the theorem.

We will focus our attention on the special case \( \Gamma = \mathbb{Z} \) and \( k = 2 \) (the case \( \mathbb{Z}^d \) and \( k \) arbitrary is just notationally more complicated, but the full generality of the theorem requires a different argument). In this context, the above theorem can be rephrased as follows.

**Theorem 12.2.** \[9\] Suppose that \( (A_n : n < \infty) \) is a vanishing marker sequence for the free part of the action of \( \mathbb{Z} \) on \( 2^\mathbb{Z} \) by shift. For every \( f : \mathbb{N} \to \mathbb{N} \) such that \( \lim_n f(n) = \infty \), there exists an \( x \) in \( 2^\mathbb{Z} \) such that:

a. the closure of the orbit of \( x \) is contained in the free part of the action;

b. the closure of the orbit of \( x \) is a minimal nonempty closed subset of \( 2^\mathbb{Z} \) which is invariant under the action;

c. there are infinitely many \( n \) such that, for some \( i \in [-f(n), f(n)] \), \( x + i \) is in \( A_n \).

It will be useful to have a more combinatorial way of formulating the first two conclusions of the theorem. They are provided by the following lemmas.

**Lemma 12.3.** (see \[9\]) If \( x \) is in \( 2^\mathbb{Z} \), then the closure of the orbit of \( x \) is contained in the free part of the action exactly when the following condition holds: for all \( a \in \mathbb{Z} \setminus \{0\} \), there exists a finite interval \( B \subseteq \mathbb{Z} \) such that for all \( c \in \mathbb{Z} \) there is a \( b \in B \) with

\[
x(a + b + c) \neq x(b + c).
\]

**Lemma 12.4.** (see \[9\]) If \( x \) is in \( 2^\mathbb{Z} \), then the closure of the orbit of \( x \) is a minimal closed invariant subset of \( 2^\mathbb{Z} \) if and only if the following condition holds: for every finite interval \( A \subseteq \mathbb{Z} \), there is a finite interval \( B \subseteq \mathbb{Z} \) such that for all \( c \in \mathbb{Z} \) there is \( a \) in \( A \) such that for all \( b \) in \( B \) with

\[
x(a + c) = x(a + b).
\]

Define \( Q \) to consist of all finite partial functions \( q : \mathbb{Z} \to 2 \) such that the domain of \( q \) is an interval of integers, denoted \( I_q \). If \( n \) is in \( \mathbb{Z} \) and \( q \) is in \( Q \), then the translate of \( q \) by \( n \) is denoted \( q + n \) and is defined by \( (q + n)(i) = q(i - n) \) (with \( q + n \) having domain \( \{i \in \mathbb{Z} : i - n \in I_q\} \)). If \( q \) is in \( Q \), define \( \bar{q} \) to be the bitwise complement of \( q \): \( \bar{q}(i) := 1 - q(i) \). A disjoint union of conditions which is again a condition will be referred to as a *concatenation*.

The order on \( Q \) is defined by \( q \leq p \) if \( q = p \) or else \( q \) is a concatenation of a set \( S \) of conditions, each of which is a translate or \( p \) or \( \bar{p} \) and such that \( p \) and a translate of \( \bar{p} \) are in \( S \). If \( G \) is a filter in \( Q \) such that \( \bigcup G \) is a total function \( x \) from \( \mathbb{Z} \) to \( 2 \), then it is straightforward to check that \( x \) satisfies the conclusion of Lemma \[12.3\] and thus that the closure of the orbit of \( x \) is a minimal invariant closed subset of \( 2^\mathbb{Z} \).

Under a mild genericity assumption on \( G \), the closure of the orbit of \( x \) will be contained in the free part of the action. For each \( p \neq 1 \) in \( Q \) and \( i < \infty \), define \( D_{p,i} \) to be the set of all \( q \) in \( Q \) such that either \( q \) is incompatible with \( p \) or else \( q \leq p \), \( m + i \) is in the domain of \( q \), and \( q(m + i) \neq q(m) \), where \( m := \max(I_p) \).

**Lemma 12.5.** Each \( D_{p,i} \) is a dense subset of \( Q \) and if \( G \subseteq Q \) is a filter which intersects \( D_{p,i} \), for each \( p \in Q \setminus \{1\} \) and \( i < \infty \), then \( x := \bigcup G \) satisfies that the closure of the orbit of \( x \) is contained in the free part of the action.
Thus we have shown that 1 forces that \( \dot{x} = \bigcup G \) satisfies that the closure of the orbit of \( \dot{x} \) is minimal and contained in the free part of the Bernoulli shift. The following proposition, when combined with Proposition 11.7 implies Theorem 12.2.

**Proposition 12.6.** Suppose that \((A_n : n < \infty)\) is a vanishing sequence of markers for the free part of the Bernoulli shift \( \mathbb{Z} \overset{\cdot}{\rightarrow} 2^\mathbb{Z} \) and that \( f : \mathbb{N} \to \mathbb{N} \) is a function such that \( \lim_n f(n) = \infty \). Every condition in \( Q \) forces that for every \( m \) there is an \( n \geq m \) such that \( \dot{x} + k \in A_n \) for some \( k \) with \(-f(n) \leq k \leq f(n)\).

**Proof.** Suppose for contradiction that this is not the case. Then there is a \( p \in Q \) such that \( f(p) \) forces: there is an \( m \) such that for every \( n \geq m \), if \(-f(n) \leq k \leq f(n)\) then \( \dot{x} + k \notin A_m \). By replacing \( p \) with a stronger condition if necessary, we may assume that there is an \( m \) such that \( p \) forces that for every \( n \geq m \), if \(-f(n) \leq k \leq f(n)\) then \( \dot{x} + k \notin A_m \). Let \( q := p \cup (\bar{p} + l) \) where \( l \) is the length of \( I_p \). Observe that \( q \leq p \) and that if \( r \leq q \) and \( i \in \mathbb{Z} \), then there is a \( j \) with \( 0 \leq j < 2l \) such that \( r - i + j \) is compatible with \( p \); simply choose \( j \) such that \( i - j \) is a multiple of \( 2l \).

Let \( n \geq m \) be such that \( f(n) \) is greater than \( 2l \) and find \( r \leq q \) and an \( i \in \mathbb{Z} \) such that \( r \) forces that \( \dot{x} + i \) is in \( A_n \). This is possible since it is forced that \( A_n \) meets every orbit (strictly speaking, we are appealing to Proposition 11.7 here).

Now, let \( j < 2|p| \) be such that \( r \) is compatible with \( p + i + j \).

We now have that \( r - i + j \) forces that \( \dot{x} + j \) is in \( A_n \). This follows from the fact that

\[ 1 \models \dot{x} - i + j \text{ is generic over the ground model.} \]

(This follows from the observation that if \( E \subseteq Q \) is exhaustive, then so is any translate of \( E \).) Recall that \( r - i + j \) is compatible with \( p \) and let \( s \) be a common lower bound for \( r - i + j \) and \( p \). It follows that \( s \) forces that both \( \dot{x} + j \notin A_n \) and \( \dot{x} + j \in A_n \), a contradiction. \( \square \)

### 13. Todorcevic’s absoluteness theorem for Rosenthal compacta

We will now use the results of Section 11 to prove Todorcevic’s absoluteness theorem for Rosenthal compacta. Fix a Polish space \( X \). Recall that a real valued function defined on \( X \) is **Baire class 1** if it is the limit of a pointwise convergent sequence of continuous functions. Baire characterized functions \( f \) which are **not** Baire class 1 as those for which there exist rational numbers \( p < q \) and nonempty sets \( D_0, D_1 \subseteq X \) such that the closures of \( D_0 \) and \( D_1 \) coincide, have no isolated points, and

\[ \sup_{x \in D_0} f(x) \leq p < q \leq \inf_{x \in D_1} f(x) \]

(see [2]). The collection of all Baire class 1 functions on a Polish space \( X \) is denoted \( BC_1(X) \) and is equipped with the topology of pointwise convergence.

A compact topological space which is homeomorphic to a subspace of \( BC_1(X) \) is said to be a **Rosenthal compactum**. This class includes all compact metric spaces and is closed under taking closed subspaces and countable products. The following are typical nonmetrizable examples.

**Example 13.1** (Helly’s space; the double arrow). The collection of all nondecreasing functions from \([0, 1]\) to \([0, 1]\) is known as Helly’s space. It is convex as a subset of \( \mathbb{R}^{[0,1]} \). The extreme points of this set are the characteristic functions of the intervals \([r, 1]\) and \([r, 1]\). This subspace is homeomorphic to the so-called **double arrow**
space: the set \([0, 1] \times 2\) equipped with the order topology from the lexicographic order.

**Example 13.2** (one point compactification). The constant 0 function, together with the functions \(\delta_r : [0, 1] \to \mathbb{R}\) defined by

\[
\delta_r(t) := \begin{cases} 
1 & \text{if } t = r \\
0 & \text{otherwise.}
\end{cases}
\]

This is homeomorphic to the one point compactification of a discrete set of cardinality \(2^{\aleph_0}\).

Rosenthal compacta enjoy a number of strong properties similar to those of compact metric spaces. One which will play an important role below is **countable tightness**: a topological space \(Z\) is **countably tight** if whenever \(a\) is in the closure of \(A \subseteq Z\), there is a countable \(A_0 \subseteq A\) such that \(a\) is in the closure of \(A_0\).

**Theorem 13.3.** \([21]\) Rosenthal compacta are countably tight.

In \([27]\), Todorcevic derived a number of properties of Rosenthal compacta by showing that there is a natural way to reinterpret such spaces as Rosenthal compacta in generic extensions. This result is in fact a fairly routine consequence of the machinery which was developed in Section 11 above.

First, we must verify that elements of \(BC_1(X)\) extend to elements of \(BC_1(X)\) in the generic extension.

**Lemma 13.4.** \([27]\) Suppose that \((f_n : n < \infty)\) is a sequence of continuous functions on a Polish space \(X\). The assertion that \((f_n : n < \infty)\) converges pointwise is generically absolute. Furthermore, if \((f_n : n < \infty)\) and \((g_n : n < \infty)\) are sequences of continuous functions on \(X\), the assertion that \(f_n - g_n \to 0\) pointwise on \(X\) is generically absolute.

**Proof.** Let \((f_n : n < \infty)\) be sequence of continuous functions. Observe that

\[
\bigcup_{\epsilon > 0} \bigcap_{n=0}^{\infty} \bigcup_{i,j \geq n} \{ x \in X : |f_i(x) - f_j(x)| > \epsilon \}
\]

specifies a countable Boolean combination of open subsets of \(X\) which is empty if and only if \((f_n : n < \infty)\) converges pointwise. Thus the assertion that \((f_n : n < \infty)\) converges pointwise is generically absolute by Proposition \([11.7]\). The second conclusion is verified in a similar manner. \(\Box\)

Now suppose that \(Q\) is a forcing, \(X\) is a Polish space, and \(f\) is in \(BC_1(X)\). By Lemma \([13.4]\) \(Q\) forces that there is a unique element of \(BC_1(X)\) which extends \(\check{f}\); fix a \(Q\)-name \(\check{f}\) for this extension. If \(K \subseteq BC_1(X)\) is a Rosenthal compactum, then \(\check{K}\) is a \(Q\)-name for the closure of the set of extensions of elements of \(K\) to \(X\). (Specifically, it is a \(Q\)-name for the closure of \(\{(\check{f}, 1) : f \in K\}\) in \(R^X\).) Todorcevic’s absoluteness theorem can now be stated as follows.

**Theorem 13.5.** \([27]\) Suppose that \(X\) is a Polish space and \(F\) is a family of Baire class 1 functions. The assertion that every accumulation point of \(F\) is Baire class 1 is generically absolute.
Proof. Let $X$ and $\mathcal{F}$ be fixed and let $Q$ be a forcing. It is sufficient to show that the assertion that $\mathcal{F}$ has a pointwise accumulation point which is not in $BC_1(X)$ is equivalent to a certain countable Boolean combination of open sets in a completely metrizable space being nonempty. Let $Z$ be the set of all sequences

$$((f_{k,i} : i < \infty) : k < \infty)$$

such that, for each $k$, $(f_{k,i} : i < \infty)$ is a sequence of continuous functions which converges pointwise to an element of $\mathcal{F}$. We will regard $Z$ as being a product of discrete spaces, noting that with this topology, $Z$ is completely metrizable.

Observe that if $g$ is a limit point of $\mathcal{F}$ which is not in $BC_1(X)$, then by Baire’s characterization, there are rational numbers $p < q$, sets $A := \{a_k : k < \infty\}$, $B := \{b_k : k < \infty\}$, and $\{f_k : k < \infty\} \subseteq \mathcal{F}$ such that:

15. $A$ and $B$ are contained in $X$, have no isolated points, and have the same closures;

16. if $k < l$, then $f_l(a_k) < p < q < f_l(b_k)$.

Moreover, one can select sequences $(f_{k,i} : i < \infty)$ of continuous functions such that $f_{k,i} \to f_k$ pointwise for each $k$. Thus we have that for every $k < l$ there is an $n$ such that if $n < j$, then $f_{l,j}(a_k) < p < q < f_{l,j}(b_k)$. It follows that there exist

$$((a_k : k < \infty), (b_k : k < \infty), ((f_{k,i} : i < \infty) : k < \infty))$$

in $X^\omega \times X^\omega \times Z$ specifying objects with the above properties if and only if $\mathcal{F}$ has an accumulation point outside of $BC_1(X)$. Notice however, that these properties define a countable Boolean combination of open subsets of $X^\omega \times X^\omega \times Z$ and therefore the theorem follows from Proposition 11.7. \qed

14. $\sigma$-closed forcings

There are two basic aspects of a forcing which are of fundamental importance in understanding its properties: how large are its families of pairwise incompatible elements and how frequently do directed families have lower bounds. Properties of the former type are often referred to loosely as chain conditions; we have already seen the most important of these in Section 8. Properties of the latter type are known as closure properties of a forcing. In this section, we will discuss the simplest and most important example of a closure property.

Definition 14.1 ($\sigma$-closed). A forcing $Q$ is $\sigma$-closed if whenever $(q_n : n < \infty)$ is a $\leq$-decreasing sequence of elements of $Q$, there is a $\bar{q}$ in $Q$ such that $\bar{q} \leq q_n$ for all $n$.

It is perhaps worth remarking that any forcing which is $\sigma$-closed and atomless (i.e. every element has two incompatible extensions), necessarily has an antichain of cardinality of the continuum and so in particular is not c.c.c.. Like c.c.c. forcings, however, $\sigma$-closed forcings also preserve uncountability, although for a quite different reason.

Proposition 14.2. Suppose that $Q$ is a $\sigma$-closed forcing. If $\dot{f}$ is a $Q$-name and $p \in Q$ forces that $\dot{f}$ is a function with domain $\omega$, then there is a $q \leq p$ and a function $g$ such that $q \models \dot{f} = \check{g}$. In particular $1 \models \check{R}_1 = \check{R}_1$ and $1 \models \check{R} = \check{R}$. 

Proof. Let \( p \) and \( f \) be given as in the statement of the proposition. By repeatedly appealing to Property 4, recursively construct a sequence of conditions \( (p_n : n < \infty) \) and values \( g(n) \) of a function \( g \) defined on \( \omega \) such that for all \( n \), \( p_{n+1} \leq p_n \leq p \) and
\[
p_n \Vdash f(\check{n}) = \check{g}(\check{n}).
\]
Since \( Q \) is \( \sigma \)-closed, there is a \( q \) in \( Q \) such that \( q \leq p_n \) for all \( n \). Thus by Proposition 5.2 it follows that
\[
q \Vdash \forall n (f(n) = g(n)).
\]
\( \square \)

We will now consider some examples. The first forcing provides a means for forcing the Continuum Hypothesis over a given model of set theory, complementing the discussion at the end of Section 8.

Example 14.3. Let \( Q \) denote the collection of all countable partial functions from \( \omega_1 \) to \( \mathbb{R} \), ordered by extension. Let \( \check{g} \) be the \( Q \)-name for the union of the generic filter. It is easily verified that \( Q \) forces that \( \check{g} \) is defined on all of \( \check{\omega}_1 \) and maps \( \check{\omega}_1 \) onto \( \check{\mathbb{R}} \). Furthermore, if \( (q_n : n < \infty) \) is a descending sequence of conditions, then \( \bigcup_{n=0}^{\infty} q_n \) is a condition: it is a function and its domain is countable, being a countable union of countable sets. Thus \( Q \) is \( \sigma \)-closed and hence forces that \( \check{\mathbb{R}} = \check{\mathbb{R}} \) and \( \check{\aleph}_1 = \check{\aleph}_1 \). Hence \( Q \) forces that \( |\check{\mathbb{R}}| = \check{\aleph}_1 \) (i.e. that the Continuum Hypothesis is true).

Example 14.4. Consider the forcing \( ([\omega]^\omega, \subset) \). This forcing is neither separative nor \( \sigma \)-closed. The separative quotient is obtained by identifying sets \( a \) and \( b \) which have a finite symmetric difference. If we define \( a \subseteq^* b \) to mean that \( a \setminus b \) is finite, then \( \subseteq^* \) induces the order on the separative quotient. If \( (A_n : n < \infty) \) is a \( \subseteq^* \)-decreasing sequence of infinite subsets of \( \omega \), let \( n_k \) be the least element of \( \bigcap_{i \leq k} A_i \) which is greater than \( n_i \) for each \( i < k \). Notice that \( B := \{n_i : i < \infty\} \) is an infinite set and that \( \{n_i : i \geq k\} \) is a subset of \( A_k \). Thus \( B \subseteq^* A_k \) for all \( k \). This shows that the separative quotient is \( \sigma \)-closed. Notice that, by Ramsey’s theorem, if \( f : [\omega]^d \to 2 \), then
\[
\{q \in [\omega]^\omega : f \upharpoonright [q]^d \text{ is constant}\}
\]
is dense in \( [\omega]^\omega \) (here \( [A]^d \) denotes the \( d \)-element subsets of \( A \)). Since the separative quotient of \( [\omega]^\omega \) is \( \sigma \)-closed, forcing with it does not add new subsets of \( \omega \). Thus it forces that \( \check{G} \) is a Ramsey ultrafilter on \( \omega \): if \( f : [\omega]^d \to 2 \) is a coloring of the \( d \)-element subsets of \( \omega \), there is an \( H \) in the ultrafilter such that \( f \) is constant on the \( d \)-element subsets of \( H \). Kunen has shown, on the other hand, that whenever \( \theta > 2^{\aleph_0} \), \( \mathcal{R}_\theta \) forces that there does not exist a Ramsey ultrafilter on \( \omega \) [15]. (Kunen actually proved this in the special case in which the ground model satisfies the Continuum Hypothesis. The general case follows by an absoluteness argument — forcing with the poset \( Q \) of the previous example does not change the truth of \( \text{"} \mathcal{R}_\theta \text{ forces that there are no Ramsey ultrafilters on } \omega \text{."} \)

We are now in a position to derive another property of Rosenthal compacta. The proof below is a reproduction of Todorcevic’s proof in [27]; the result itself was originally proved by Bourgain [4] using classical methods.

Theorem 14.5. If \( K \) is a Rosenthal compactum, then \( K \) contains a dense set of points with a countable neighborhood base.
Proof. Observe that it is sufficient to show that every Rosenthal compactum contains a point with a countable base. Recall the following result of Čech and Pošpišil: if \( K \) is a compact topological space of cardinality at most \( \aleph_1 \), then \( K \) contains a point with a countable neighborhood base. Let \( Q \) be the forcing from the previous example. We have seen that \( Q \) forces that \( |\mathbb{R}| = \aleph_1 \) and hence that the collection of all real valued Borel functions on a given Polish space has cardinality \( \aleph_1 \). In particular, \( Q \) forces that any Rosenthal compactum has cardinality \( \aleph_1 \).

Now, let \( K \) be a Rosenthal compactum consisting of Baire class 1 functions on some Polish space \( X \). By Theorem 13.5, \( Q \) forces that the closure of \( K \) inside of \( \mathbb{R}^X \) still consists only of Baire class 1 functions. Since \( Q \) is \( \sigma \)-closed, it follows that \( 1 \) forces that \( K \) is closed and hence a compact space of cardinality \( \aleph_1 \). Therefore by the Čech-Pošpišil Theorem, there are \( Q \)-names \( \dot{g} \) and \( \dot{U}_n \) for each \( n \) such that \( 1 \) forces that \( \dot{g} \) is an element of \( \dot{K} \) and that \( \{ \dot{U}_n : n < \infty \} \) is a countable neighborhood base for \( \dot{g} \) consisting of basic open sets. Since \( Q \) is \( \sigma \)-closed, there is a \( q \) in \( Q \) which decides \( \dot{g} \) to be some \( f \) and \( \dot{U}_n \) to be some \( V_n \) for each \( n \). It follows that \( \{ V_n : n < \infty \} \) is a countable neighborhood base.

15. Mathias reals and a theorem of Galvin and Prikry

In this section we will give a forcing proof of the Galvin-Prikry Theorem, which is an infinite dimensional form of Ramsey’s Theorem:

**Theorem 15.1.** [8] If \( X \subseteq [\omega]^\omega \) is Borel, then there is an \( H \in [\omega]^\omega \) such that either \( [H]^\omega \subseteq X \) or else \( [H]^\omega \cap X = \emptyset \).

Recall that Mathias forcing \( M \) consists of all pairs \( p = (a_p, A_p) \) such that \( A_p \) is an infinite subset of \( \omega \) and \( a_p \) is a finite initial segment of \( A_p \). The order on \( M \) is such that \( q \) extends \( p \) if \( a_p \) is an initial part of \( a_q \) and \( A_q \subseteq A_p \). A Mathias real is a subset \( X \) of \( \omega \) such that

\[
G_X := \{ p \in M : a_p \subseteq X \subseteq A_p \}
\]

is a generic filter. If \( D \) is a collection of subsets of \( M \), then we say that \( X \) is \( D \)-generic if \( G_X \) is \( D \)-generic. If \( D \subseteq M \), we will say that \( X \) is \( D \)-generic if it is \( \{ D \} \)-generic. We say that \( D \subseteq M \) is dense above \( n \) if whenever \( p \in M \) and \( n \leq \min(a_p), p \) has an extension in \( D \).

**Lemma 15.2.** Suppose that \( D \subseteq M \) is dense above \( n \). There is a dense set of \( H \) in \( ([\omega]^\omega, \subseteq) \) such that any infinite subset of \( H \) is \( D \)-generic.

**Proof.** Let \( D \) and \( n \) be given as in the statement of the lemma and let \( A \subseteq \omega \) be arbitrary with \( n < \min(A) \). Construct a sequence of infinite subsets \( H_k \subseteq A \) for each \( k \) such that, setting \( n_k := \min(H_k) \):

1. \( H_0 := A \) and \( H_{k+1} \subseteq H_k \);
2. \( n_k < n_{k+1} \);
3. for each \( x \subseteq \{ i : i < k \} \) either there is a \( p \in D \) such that \( a_p = x \) and \( H_k \subseteq A_p \) or else whenever \( p \in D \) with \( a_p = x, A_p \cap H_k \) is finite.

Define \( B := \{ n_k : k < \infty \} \) and set

\[
F := \{ x \in [B]^\omega : \exists p \in D((a_p = x) \text{ and } (A_p \subseteq B)) \}.
\]

By Theorem 13.4, there is an infinite \( H \subseteq B \) such that either \( H \) has no subset in \( F \) or else every infinite subset of \( H \) has an initial segment in \( F \). Since \( (\emptyset, H) \) is in
Proposition 15.3. Suppose that \( D \) is a countable collection of dense subsets of \( M \). For every \( x \in [\omega]^\omega \) there is a dense set of \( H \) in \( [\omega]^\omega \) such that if \( X \subseteq H \) is infinite then \( x \cup X \) is \( D \)-generic.

Proof. Let \( D \) and \( x \) be given as in the statement of the proposition. Fix an enumeration \( \{D_k : k < \infty \} \) of \( D \) and let \( A \in [\omega]^\omega \) be arbitrary with \( \max(x) < \min(A) \).

If \( y \) is a finite set and \( k < \infty \), define

\[
D_{k,y} := \{ p \in M : (\max(y) < \min(a_p)) \text{ and } ((y \cup a_p, y \cup A_p) \in D_k) \}.
\]

Observe that \( D_{k,y} \) is dense above \( \max(y) + 1 \) and if \( X \subseteq \omega \) with \( \max(y) < \min(X) \), then \( y \cup X \) is \( D_k \)-generic if \( X \) is \( D_{k,y} \)-generic.

Using Lemma 15.2. construct infinite sets \( \{H_k : k < \infty \} \) so that:

1. \( H_0 := A \) and \( H_{k+1} \subseteq H_k \);
2. setting \( n_k := \min(H_k) \), we have \( n_k < n_{k+1} \);
3. any infinite subset of \( H_k \) is \( D_{j,y} \)-generic whenever \( j < k \) and \( x \subseteq y \subseteq x \cup \{n_i : i < k \} \).

Define \( H := \{n_k : k < \infty \} \) and suppose that \( X \) is an infinite subset of \( H \). Let \( k \) be given and set \( y := x \cup (X \cap \{n_i : i < k \}) \). Since \( X \cap H_k = X \setminus x \) is \( D_{k,y} \)-generic, \( x \cup X \) is \( D_k \)-generic. Thus \( H \) satisfies the conclusion of the proposition. \( \square \)

If \( p, q \in M \), then we say that \( q \) is a pure extension of \( p \) if \( q \leq p \) and \( a_p = a_q \). The following proposition is central to the analysis of \( M \) and related posets.

Proposition 15.4. If \( \phi \) is a formula in the forcing language and \( p \in M \), \( p \) has a pure extension which decides \( \phi \).

Proof. Let \( p \) and \( \phi \) be given as in the statement of the proposition. Define \( D \) to be the set of all conditions in \( M \) which decide \( \phi \), noting that \( D \) is dense. By Proposition 15.3 there is an infinite \( B \subseteq A_p \) such that if \( X \subseteq B \) is infinite, then \( a_p \cup X \) is \( D \)-generic. Set

\[
\mathcal{F} := \{ x \in [B]^\omega : \exists p \in M((p \models \phi) \text{ and } (a_p = x) \text{ and } (A_p \subseteq B)) \}.
\]

By Theorem 11.1 there is an infinite \( H \subseteq B \) such that either \( H \) has no subset in \( \mathcal{F} \) or else every infinite subset of \( H \) has an initial part in \( \mathcal{F} \). If the first conclusion is true, then \( (a_p, H) \) forces \( \neg\phi \). If the second conclusion is true, then \( (a_p, H) \) forces \( \phi \). \( \square \)

Since every Borel set is universally Baire by Proposition 11.6 the next theorem implies the Galvin-Prikry Theorem 5.

Theorem 15.5. If \( X \subseteq [\omega]^\omega \) is universally Baire, then there is an \( H \in [\omega]^\omega \) such that either \( [H]^\omega \subseteq X \) or else \( [H]^\omega \cap X = \emptyset \).
Proof. Since $\mathcal{X}$ is universally Baire, there is an $\mathcal{M}$-name $\check{\mathcal{X}}$, countably many dense sets $\mathcal{D}$, and a name $\check{X}$ for the Mathias real such that if $G$ is a $\mathcal{D}$-generic filter, then $X(G) \in \check{\mathcal{X}}$ if and only if there is a $p \in G$ such that $p \Vdash \check{X} \in \check{\mathcal{X}}$ if and only if there is no $p \in G$ such that $p \Vdash \check{X} \notin \check{\mathcal{X}}$. By Proposition 15.3 there is $\mathcal{H} \in [\mathcal{A}]^\omega$ such that every infinite subset of $\mathcal{H}$ is $\mathcal{D}$-generic. It follows that $\mathcal{H}$ satisfies the conclusion of the theorem. □

16. WHEN COMPACTA HAVE DENSE METRIZABLE SUBSPACES*

Suppose that $K$ is a compact Hausdorff space. In this section we will reformulate the question of when $K$ contains a dense metrizable subspace in terms of the language of forcing. Recall that every compact Hausdorff space is homeomorphic to a closed subspace of $[0, 1]^I$ for some index set $I$. In this section, when we reinterpret $K$ in a generic extension, we will take $\check{K}$ to be the name for the closure of $\check{\mathcal{H}}$ in $[0, 1]^I$.

Recall that a regular pair in $K$ is a pair $(F, G)$ such that $F$ and $G$ are disjoint closed $G_\delta$ subsets of $K$. If $\Xi$ is an ordered set and $((F_\xi, G_\xi) : \xi \in \Xi)$ is a sequence of regular pairs, then we say that $((F_\xi, G_\xi) : \xi \in \Xi)$ is a free sequence if whenever $A, B \subseteq \Xi$ are finite and satisfy $\max(A) < \min(B)$, it follows that

$$\bigcap_{\xi \in A} G_\xi \cap \bigcap_{\xi \in B} F_\xi \neq \emptyset.$$ 

Recall also that a collection $\mathcal{B}$ of nonempty open subsets of $K$ is a $\pi$-base if every nonempty open set in $K$ contains an element of $\mathcal{B}$.

We note the following result of Todorcevic.

Theorem 16.1. [26] If $K$ is any compact Hausdorff space, there is a sequence $((F_\xi, G_\xi) : \xi \in \Pi)$ of regular pairs in $K$ such that $\{\text{int}(G_\xi) : \xi \in \Pi\}$ forms a $\pi$-base for $K$ of minimum cardinality and such that whenever $\Xi \subseteq \Pi$ satisfies that $\{G_\xi : \xi \in \Xi\}$ has the finite intersection property, $((F_\xi, G_\xi) : \xi \in \Xi)$ is a free sequence.

The following result is implicit in [27] and is a key component in Todorcevic’s proof that every Rosenthal compactum contains a dense metrizable subspace. Let $\mathcal{Q}_K$ denote the forcing consisting of all nonempty open subsets of $K$ ordered so that $q < p$ means that the closure of $q$ is contained in $p$. We will let $\check{x}_G$ denote the $\mathcal{Q}_K$-name for the unique element of the intersection of $\check{G}$, when regarded as a collection of open sets.

Theorem 16.2. Suppose that $K$ is a compact Hausdorff space and $((F_\xi, G_\xi) : \xi \in \Pi)$ is a sequence satisfying the conclusion of Theorem 16.1. The following are equivalent:

a. $K$ has a $\sigma$-disjoint $\pi$-base.

b. $\mathcal{Q}_K$ forces that $\check{x}_G$ has a countable neighborhood base.

c. $\mathcal{Q}_K$ forces that $|\{\xi \in \Pi : G_\xi \in \check{\mathcal{G}}\}| \leq \aleph_0$.

Proof. To see that (a) implies (b), first observe that if $\mathcal{U}$ is a $\pi$-base for the topology on $K$, then $\mathcal{U}$ is dense as a subset of $\mathcal{Q}_K$. Hence $\mathcal{Q}_K$ forces that $\mathcal{U} \cap \check{\mathcal{G}}$ generates $G$. Also, if $\mathcal{O}$ is a pairwise disjoint family of open sets, then it is forced that
\(|\hat{O} \cap \hat{G}| \leq 1\). Hence if \(\mathcal{U}\) is a \(\sigma\)-disjoint \(\pi\)-base, then it is forced that \(\hat{U} \cap \hat{G}\) is a countable neighborhood base of \(\hat{x}_G\).

The equivalence between \(\text{(b)}\) and \(\text{(c)}\) follows from the fact that
\[
\{G_\xi : (\xi \in \Pi) \text{ and } (\hat{x}_G \in G_\xi)\}
\]
is forced to be a neighborhood base for \(\hat{x}_G\) and that
\[
\{(F_\xi, G_\xi) : (\xi \in \Pi) \text{ and } (\hat{x}_G \in G_\xi)\}
\]
is a free sequence and hence no smaller neighborhood base can suffice.

Finally, to see that \(\text{(c)}\) implies \(\text{(a)}\), suppose that every condition forces that
\[
|\{\xi \in \hat{\Pi} : \hat{G}_\xi \in \hat{G}\}| \leq \aleph_0.
\]
Let \((\hat{\xi}_n)_{n < \infty}\) be a sequence of \(Q_K\)-names such that every condition of \(Q_K\) forces that
\[
\hat{\xi}_n = \{\xi \in \hat{\Pi} : \hat{x}_G \in \hat{G}_\xi\}.
\]
Let \(O_n\) be a maximal antichain in \(Q_K\) such that elements of \(O_n\) decide \(\xi_n\) and set
\[
\mathcal{U} := \bigcup_{n=0}^{\infty} O_n.
\]
Clearly \(\mathcal{U}\) is \(\sigma\)-disjoint; it suffices to show that it is a \(\pi\)-base. To see this, suppose that \(V\) is a nonempty open subset of \(K\). Let \(p\) be a nonempty regular open subset of \(V\) and let \(\hat{n}\) be such that \(p\) forces that \(\hat{G}_{\xi_{\hat{n}}} \subseteq \hat{V}\). Now let \(U\) be an element of \(\mathcal{U}\) which decides \(\hat{\xi}_{\hat{n}}\) to be \(\xi\). Notice that we must have that \(U \subseteq \hat{G}_\xi \subseteq V\).

Recall that a topological space \(X\) is countably tight if whenever \(A \subseteq X\) and \(x \in \text{cl}(A)\), there is a countable \(A_0 \subseteq A\) such that \(x \in \text{cl}(A_0)\). It is easy to show that continuous images of countably tight spaces are countably tight. It is well known that in the class of compact Hausdorff spaces, countable tightness is equivalent to the nonexistence of uncountable free sequences of regular pairs. We now have the following corollary.

**Corollary 16.3.** Let \(\mathcal{P}\) be a forcing. If \(K\) is compact, contains a dense first countable subspace, and
\[1 \forces \hat{K} \text{ countably tight,}\]
then \(K\) contains a dense metrizable subspace.

**Proof.** By our assumption and Theorem 16.2, \(K\) has a \(\sigma\)-disjoint \(\pi\)-base and thus so does the dense first countable subspace. By a result of H.E. White [33], any first countable Hausdorff space with a \(\sigma\)-disjoint \(\pi\)-base has a dense metrizable subspace. \(\square\)

An immediate consequence of the results we have developed so far is the following result of Todorcevic. Previously it had not been known whether there were nonseparable Rosenthal compacta which had no uncountable family of pairwise disjoint open sets or whether certain specific Rosenthal compacta had dense metrizable subspaces (see the discussion in [27]).

**Theorem 16.4.** [27] Rosenthal compacta contain dense metrizable subspaces.

**Proof.** Let \(K\) be a Rosenthal compactum and let \(\hat{K}\) be the \(Q_K\)-name for the reinterpretation of \(K\) as a Rosenthal compactum in the generic extension by \(Q_K\). By Theorem 14.3, \(K\) contains a dense first countable subspace. By Theorem 16.3, \(Q_K\) forces that \(K\) is a Rosenthal compactum and hence is countably tight by Theorem
Since $\hat{K}$, as defined in the beginning of this section, is a continuous projection of $\hat{K}$, it follows that $\hat{K}$ is forced to be countably tight. By Corollary 16.3, $K$ contains a dense metrizable subspace.

17. Further reading

As was mentioned earlier, Kunen’s book [16] is a good next step if one is interested in further reading on forcing. It also contains a large number of exercises. Chapters VII and VIII provide a standard treatment of forcing, presented with a more semantic orientation, and Chapter II provides some useful background on combinatorial set theory.

Further reading on forcings which add a single real — such as $C$, $R$, $M$, $A_\alpha$ — can be found in [9]. Also, Laver’s work on the Borel Conjecture [18] is a significant early paper on the subject which already contains important techniques in the modern set theory such as countable support iteration. Zapletal’s [35] gives a different perspective on forcings related to set theory of the reals.

For those who can find a copy, [31] is also good further reading on forcing and provides a different perspective than [16]. Those readers who have studied the material on Martin’s Axiom in [16, II, VIII] and/or forcing axioms in [31] are referred to [20], [29], and [21] where this concept is further developed and the literature is surveyed.

Solovay’s analysis of the model $L(R)$ [25] is a landmark result in the study of forcing and large cardinals. (Solovay actually analyzed a larger model than $L(R)$, but $L(R)$ has since shown itself to be more fundamental and now bears the name Solovay model.) At the same time, [25] should be accessible to readers who have been through this article. A proof of Solovay’s theorem is reproduced in [13] which is also a standard encyclopedic reference on large cardinals. See also Mathias’s infinite dimensional generalization of Ramsey’s Theorem which holds in $L(R)$ after collapsing an appropriate large cardinal [19]. An explanation of the special role Solovay’s model plays in the foundations of mathematics can be found in [23] [34].

[17] provides a good introduction to the methods needed to establish absoluteness results about $L(R)$.

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