Hidden SUSY features, and Dawson’s function and the wrong-sign Hermite functions

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Abstract. We propose that some problems in mathematical physics show non-typical supersymmetric (SUSY) features which may still need to be thoroughly explored. In particular, we first show that the Pöschl-Teller problem posses some inherent ladder peculiarities which are not considered in the usual SUSY approach. Then we show that Dawson’s function and its derivatives show a SUSY type relation with the wrong-sign Hermite equation, found in a general factorization of the simple harmonic oscillator, which includes Mielnik’s SUSY partners as particular cases.

1. Introduction
One of the basic problems when trying to find analytical solutions to dynamic problems, is the fact that only a very small number of exact solutions to second order ordinary differential equations are known, the number being smaller if one wishes to work with non-linear problems. In the case of one-dimensional quantum mechanics (QM), only about a dozen of exact solutions are known, and therefore the appearance of supersymmetric (SUSY) quantum mechanics (QM) techniques in the 1980’s [1] was immediately embraced to find at least new problems with known old spectra. Work in this direction developed so fast that nowadays it seems that everything has been solved, leaving SUSY applications to cosmological problems, where new solutions are still being found.[2] However, as is usual with fast growing fields in science, some details in the subject were not thoroughly explored, and one finds from time to time new characteristics forgotten or hidden that still need to be solved out. Here we show how some SUSY aspects were forgotten in the settled work, and that some surprises can still be found that may deserve our attention.

We begin our article with a discussion of the Pöschl-Teller QM problem, to motivate the argument that some forgotten information may still need consideration. Afterwards, we show how new SUSY type factorizations lead to surprises, as can be seen in the case of the wrong-sign Hermite equation and the differential equation for Dawson’s function’s derivatives.

2. Unsolved features in the case of the Pöschl-Teller problem
First, let us consider the Hamiltonian of a particle in a modified Pöschl-Teller potential, whose time-independent Schrödinger equation is [3, 4, 5, 6]

\[
H_{m+1} \psi = \left( \frac{\hbar^2}{2m} \frac{d^2}{dx^2} - \frac{\alpha^2 m(m+1)}{\cosh^2 \alpha x} \right) \psi = E \psi ,
\]

where \(\alpha\) is a parameter and \(E\) is the energy of the particle. The Pöschl-Teller potential is known to possess some inherent ladder peculiarities which are not considered in the usual SUSY approach. These peculiarities are related to the fact that the Pöschl-Teller potential is a particular case of the modified Pöschl-Teller potential, where \(\alpha = 1\).

To see how these peculiarities manifest themselves, let us consider the following factorization of the modified Pöschl-Teller potential:

\[
V(x) = \frac{\alpha^2 m(m+1)}{\cosh^2 \alpha x} = \frac{\alpha^2 (m+1)}{\cosh^2 \alpha x} + \frac{\alpha^2 m}{\cosh^2 \alpha x}.
\]

The first term on the right-hand side of the equation can be interpreted as the potential of a harmonic oscillator, while the second term is the potential of a simple harmonic oscillator. The factorization can be viewed as a generalization of the factorization of the simple harmonic oscillator.

The factorization leads to the following differential equation for the wave function:\n
\[
\frac{d^2}{dx^2} \psi + \frac{\alpha^2 (m+1)}{\cosh^2 \alpha x} \psi = \frac{\alpha^2 m}{\cosh^2 \alpha x} \psi = E \psi
\]

This equation can be solved using the method of separation of variables, and the solutions are given by the Hermite polynomials.

The factorization also leads to the following differential equation for the derivative of the wave function:

\[
\frac{d^2}{dx^2} \frac{d \psi}{dx} + \frac{\alpha^2 (m+1)}{\cosh^2 \alpha x} \frac{d \psi}{dx} = \frac{\alpha^2 m}{\cosh^2 \alpha x} \frac{d \psi}{dx} = E \frac{d \psi}{dx}
\]

This equation can also be solved using the method of separation of variables, and the solutions are given by the Dawson’s integral function.

These results show that the Pöschl-Teller problem possesses some inherent ladder peculiarities which are not considered in the usual SUSY approach. The factorization also shows that Dawson’s function and its derivatives show a SUSY type relation with the wrong-sign Hermite equation, found in a general factorization of the simple harmonic oscillator, which includes Mielnik’s SUSY partners as particular cases.

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where $\alpha > 0$, and the integer $m$ is greater than 0. To shorten the algebraic equations we shall set $\frac{\hbar^2}{2m} = 1$.

The eigenvalue problem may be solved using the Infeld & Hull’s (IH) factorizations,[7]

\begin{align}
A_{m+1}^+ A_{m+1}^- \psi^m_\ell &= (H_{m+1} + \epsilon_{m+1}) \psi^m_\ell, \quad (2a) \\
A_m^- A_m^+ \psi^m_\ell &= (H_{m+1} + \epsilon_m) \psi^m_\ell, \quad (2b)
\end{align}

where the raising/lowering operators are given by

\begin{equation}
A_\pm^m = \alpha m \tanh \alpha x \pm \frac{d}{dx},
\end{equation}

where $\epsilon_m = \alpha^2 m^2$, and $\ell = m - n$, where $n$ is the eigenvalue index,

\begin{equation}
E_n \equiv E_{m,n} = -\alpha^2 (m - n)^2, \quad n = 0, 1, 2, \ldots < m.
\end{equation}

The coefficient out of the parenthesis in eq.(3) is only used to normalize the eigenfunctions, and we shall drop it here on.

Note that we have changed the usual notation for the eigenvalues in this problem, since we shall show below that there are some hidden features masked by the shorter notation. To do so, let us remember that the IH operators change the upper index of the eigenfunctions; in fact, the eigenfunctions $\psi_n(x) \equiv \psi_{m,n}^m(x)$ have to be found by successive applications of the raising operator on the zeroth order eigenfunctions defined by

\begin{equation}
\psi_0 \equiv \psi^0_\ell(x) = \sqrt{\frac{\alpha \Gamma(\ell + \frac{1}{2})}{\sqrt{\pi \Gamma(\ell)}}} \cosh^{-\ell} \alpha x.
\end{equation}

leaving the lower index $\ell$ unchanged, given that, up to a constant factor,

\begin{equation}
A_{s+1}^- \psi^s_\ell = \psi^{s+1}_\ell, \quad A_s^+ \psi^s_\ell = \psi^{s-1}_\ell.
\end{equation}

We would like to stress that in the usual SUSY treatment of this problem,[8] it seems that the following fact was not taken under consideration: if we look at the factorizations (2), we notice that interchanging the order of the IH operators seem to factorize the same hamiltonian, but only if the index of the raising/lowering operators is changed! A simple change of order produces another physical problem, with a different potential function. This becomes evident if we do the following algebra:

\begin{align}
A_m^- A_m^+ \psi^m_{m-n} &= (H_{m+1} + \epsilon_m) \psi^m_{m-n} = (E_{m,n} + \epsilon_m) \psi^m_{m-n}, \\
\end{align}

but also, due to (6) and (2a),

\begin{align}
A_m^- A_m^+ (A_m^- \psi^{m-1}_{m-n}) &= A_m^- \left( A_m^+ A_m^- \psi^{m-1}_{m-n-(n-1)} \right) = \quad (7a) \\
A_m^- (H_{m+1} + \epsilon_m) \psi^{m-1}_{m-n-(n-1)} &= (E_{m,n-1} + \epsilon_m) A_m^- \psi^{m-1}_{m-n} = (E_{m,n} + \epsilon_m) \psi^m_{m-n} \quad (7b)
\end{align}

where we used our notation to emphasise that in eq.(7a) the problem in parenthesis refers to a different potential than the problem in the preceding equation, however rendering the same energy eigenvalue! [9] Therefore, we think that it may be possible that the usual SUSY approach to this problem has to be reevaluated.[10]
3. Dawson’s function

Let us now consider Dawson’s function, which may be defined in terms of the error function. Dawson’s function is directly related to the error function of imaginary argument [11]

\[ F(x) = e^{-x^2} \int_0^x e^{y^2} dy = -\frac{\sqrt{\pi i}}{2} e^{-x^2} \text{erf}(ix) \]  

(8)

and its properties are better determined in terms of its derivatives,

\[ F'(x) = -2x F(x) + 1 \]  
\[ F^{(k+1)}(x) + 2xF^{(k)}(x) + 2kF^{(k-1)}(x) = 0. \]  

(9a)

(9b)

Attempts have been made to find a simple representation of this function in terms of elementary functions, for example, that of Cody et al.,[12] who proposed a representation of \( F(x) \) in terms of rational approximations.

We shall show here that there exists a singular SUSY type relation between Dawson’s function and its derivatives, to the eigenfunctions of the wrong-sign Hermite differential equation, which is found in a new factorization of the Hamiltonian of the simple harmonic oscillator (SHO).[13]

3.1. Generalized factorization of the SHO Hamiltonian

In a previous article [13] we have developed two factorizations of the SHO Hamiltonian in terms of two non-selfadjoint operators

\[ B^- = \frac{1}{\sqrt{2}} \left( \alpha^{-1}(x) \frac{d}{dx} + \beta(x) \right), \]  
\[ B^+ = \frac{1}{\sqrt{2}} \left( -\alpha(x) \frac{d}{dx} + \beta(x) \right). \]  

(10a)

(10b)

In the first factorization, we required that \( B^- B^+ = H + \frac{1}{2} \). Upon inverting the product, \( B^+ B^- \), the two parameter solutions for \( \alpha(x) \) and \( \beta(x) \) defined a Sturm-Liouville equation which included the quantum mechanics SHO equation, its SUSY partners,[1] and Hermite’s equation, as particular cases for defined regions of the two-parameter space.[13]

In a second factorization, we proposed that the Hamiltonian be factorized as \( B^+ B^- = H - \frac{1}{2} \), which is possible if now the functions \( \alpha(x) \) and \( \beta(x) \) depend on a single parameter, \( \gamma_3 \), and are given by

\[ \alpha_{\gamma_3}(x) = \sqrt{1 + \gamma_3 e^{x^2}}, \quad \beta_{\gamma_3}(x) = \frac{x}{\sqrt{1 + \gamma_3 e^{x^2}}}. \]  

(11)

The inverse operator product \( B^- B^+ \) now defines a new eigenvalue equation

\[ \mathcal{L}_{\gamma_3} H_{\gamma_3}^n + \lambda_n \omega_{\gamma_3}(x) H_{\gamma_3}^n = 0, \]  

(12)

where

\[ \mathcal{L}_{\gamma_3} = \left( 1 + \gamma_3 e^{x^2} \right) \frac{d^2}{dx^2} + 2\gamma_3 x e^{x^2} \frac{d}{dx} + \frac{\gamma_3 e^{x^2} + \gamma_3^2 e^{-x^2} - x^2}{1 + \gamma_3 e^{x^2}} \]  

(13)

is a one-parameter self-adjoint operator with the weight function \( \omega_{\gamma_3}(x) = 2 \left( 1 + \gamma_3 e^{x^2} \right) \), and it is isospectral to the quantum SHO Hamiltonian, which is obtained in the limit \( \gamma_3 \to 0 \). The eigenfunctions in this case are

\[ H_{\gamma_3}^n(x) = B^- \psi_{n+1}(x). \]  

(14)

where \( \psi_n(x) \) are the SHO eigenfunctions.
For this work, it is very interesting to note that in the large limit $\gamma_3 \gg 1$, one can obtain the wrong-sign Hermite’s differential equation

$$\left[ \frac{d^2}{dx^2} + 2x \frac{d}{dx} + 2(n + 1) \right] \tilde{H}_n(x) = 0,$$  
(15)

which differs from the Hermite equation only in the sign in front of the first derivative. The corresponding eigenfunctions are of the quantum oscillator type, but vanishing faster due to a squared exponential factor, $\tilde{H}_n(x) = e^{-x^2} H_n(x)$.

From the Hermite polynomials’ recursion relations it is easy to find the raising and lowering operators for these functions. They turn out to be

$$\left( \frac{d}{dx} + 2x \right) \tilde{H}_n(x) = 2n \tilde{H}_{n-1}(x),$$  
(16a)

$$-\frac{d}{dx} \tilde{H}_n(x) = \tilde{H}_{n+1}(x).$$  
(16b)

Note that the reversed sign in the first derivative term of eq.(15) produces these “reversed” Hermite polynomials’ recursion relations.

3.2. A singular SUSY relation and Dawson’s eigenfunctions

As one can see, equation (15) is the same as (9b) when $k = 1$ and $n = 0$, i.e., the ground state equation of the wrong-sign Hermite eigenvalue problem, whose eigenfunction is just the Gaussian function

$$\tilde{H}_0(x) = e^{-x^2}$$  
(17)

and it is not Dawson’s function! Now, $\tilde{H}_0(x)$ and $F(x)$ are completely different, and their first derivatives are also different,

$$\tilde{H}_0'(x) = -2x \tilde{H}_0(x),$$  
(18a)

$$F'(x) = -2xF(x) + 1,$$  
(18b)

however, they share the same second order differential equation. The reason for this is that Dawson’s function is the second solution of equation (15) when $n = 0$, since starting with $\tilde{H}_0(x)$, that solution is $[14]$

$$f(x) = \tilde{H}_0(x) \int_0^x \frac{e^{-f^2} 2z dz}{\tilde{H}_0(y)^2} dy = F(x)$$  
(19)

It is even more interesting to see that, using eqs.(9), (15) and (16), we can find a SUSY like ladder relation between the wrong-sign Hermite eigenfunctions $\tilde{H}_n(x)$, and the derivatives of Dawson’s functions, which we shall hereafter call the Dawson’s eigenfunctions, $D_n(x) \equiv F^{(n-1)}(x)$, for $n = 1, 2, 3, \ldots$, and hence $D_1(x) \equiv F(x)$. This ladder relation is shown in Fig.1.

The fact that $F(x)$ has one zero at $x = 0$, while the Gaussian does not have any zero for all finite $x$, implies that in order that both sets of eigenfunctions cover the whole space of non-singular functions $f(x)$, there must exist an additional function $D_0(x)$, however enlarging the eigenvalue equations for Dawson’s eigenfunctions. This feature is the equivalent to the SUSY-QM procedure, where the zero-th order SUSY partner eigenfunction is missing. To find the missing $D_0(x)$, we can see that eq.(9), in the case $k = 0$, has two solutions, the constant solution, and the error function. However, if we assume that the recursion relation (16a) gives rise to eq.(9a), then, the (non-trivial) zero-th eigenfunction ought to be $D_0(x) = const.$
We say here that Dawson’s eigenfunctions and the wrong-sign Hermite eigenfunctions possess a singular SUSY relation, although (i) there is no QM or any type of physics problem associated, (ii) their relation did not arise from an operator procedure; moreover, they share the recurrence relations and the second order differential equation, (iii) there does not exist an associated SUSY parameter in this relation, and (iv) in order to cover the space of nonsingular functions of $x$, there must exist a zero-less eigenfunction $D_0(x)$, which in this case is the constant function.

![Image of Dawson's and wrong-sign Hermite eigenfunctions](image)

**Figure 1.** The SUSY type ladder relation between Dawson’s eigenfunctions $D_n(x)$ and the wrong-sign Hermite eigenfunctions $\tilde{H}_n(x)$. The function $D_0(x)$ is missing, but can be found using the second order differential equation (9b) and the lowering operator (9a).

4. Conclusions

We have tried to motivate here that some aspects of the SUSY algebraic method still need to be considered, and some are hidden in known algebra, awaiting for a deeper or broadened approach in the field. The case of the Pöschl-Teller problem shows that we must reconsider some work already done in the area, while the relation of Dawson’s function to the wrong-sign Hermite eigenfunctions shows that some hidden SUSY aspects are still awaiting to be found, which may extend the SUSY techniques to other areas in mathematical physics.

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