Sliding mode learning control for uncertain mechanical system: A dynamic output feedback approach

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Abstract
In this article, a dynamic output feedback based sliding mode learning control is proposed for uncertain mechanical system. After giving the model of uncertain mechanical system, the uncertainty and disturbance of it are discussed and they are assumed to be mismatched. The velocity of the uncertain mechanical system is assumed to be unmeasurable, and then a dynamic output feedback control strategy is utilized here. A dynamic output feedback-based sliding surface is constructed. The parameters of the designed surface are solved by Lyapunov function approach. Then a sliding mode learning controller is proposed for uncertain mechanical system to overcome the chattering of traditional sliding mode control. Finally, a numerical simulation is given to show the effectiveness of the proposed controller.

Keywords
Uncertain mechanical system, dynamic output feedback, sliding mode learning control

Introduction
In actual production process, a lot of industrial system can be represented by a mechanical system, such as active suspension of vehicle,¹ serial robot arm,² vessel-riser system,³ planar three-link mechanical system⁴ and rotational mechanical system.⁵ For the successful application of industrial system, the controller design of mechanical system has drawn much attention in recent years.⁶,⁷ However, for an actual industrial system, a specific sensor is needed if we want to get a certain variable, for example, a velocity sensor is need if we want to measure the velocity of the industrial system timely. Unfortunately, adding a specific sensor to the industrial system will certainly increase the cost. Simultaneously, the velocity information measured by the sensor is often affected by noise, so it is inconvenient to be applied directly. If we can design a controller without the sensor of velocity, the cost of the industrial system will certainly decrease, and the control performance will be improved. In this case, output feedback based controller design method is an effective way for the control of mechanical system.⁸–¹⁰

Because of the change of working environment and the existence of various external disturbances, uncertainty and disturbance are inevitable in mechanical system; in this case, a robust controller is needed. Sliding mode control (SMC) is a kind of robust control since it has great robustness for system disturbance.¹¹ For the controller design of SMC, a reduced order sliding surface is constructed first in which the reduced dynamics are stable. Then a discontinuous control law, which can force the system dynamics to the sliding surface, is designed. Based on the designed sliding surface and controller, the controlled system is unaffected by the disturbance.¹² SMC has been applied to the robust control of uncertain system, such as adaptive control of hypersonic flight vehicle¹³ and active suspension vehicle systems,¹⁰ adaptive control of fuzzy system,¹⁴,¹⁵ and sliding mode control of master-slave time-delay systems.⁶ Output feedback based SMC has also been widely studied, such as static output feedback (SOF) based SMC,¹⁶,¹⁷ and has been successfully utilized in Markovian jump systems¹⁸ and affine nonlinear system.¹⁹ But the information of SOF based SMC is one-sided, and partial states of the original system is missed.
In this case, dynamic output feedback (DOF) based SMC is needed.

Meanwhile, when utilizing SMC, the upper bound of the interference is assumed to be known. While for a real system, the interference and uncertainty are difficult to be modeled or measured. An effective way for this question is choosing a big enough upper bound for the interference and uncertainty, but this will cause chattering in the control input. Chattering is an obvious shortcoming of SMC, and is harmful to the stability of practical system, so it must be avoided or at least reduced. For reducing chattering, a lot of results have been listed in literature, such as adaptive law based SMC, but in real application, chattering is still an open problem, and is the main difficulty for the application of SMC. In this case, a novel DOF based SMC, which can greatly reduce the chattering of traditional SMC, is needed for the control of mechanical system.

Recently, sliding mode learning control (SMLC) has been proposed and successfully applied on the controller design of uncertain system. Similar with traditional SMC, a sliding surface is first constructed, and its stability is guaranteed by correctly selecting the parameter of the sliding surface. Then, a learning controller is introduced. The learning controller can greatly reduce chattering of traditional SMC on the basis of guaranteeing SMC’s robustness, so it is more practical in real application. But for the output feedback based SMLC, there are really few results can be found in literatures, let alone DOF-based SMLC. So for the control of mechanical system, a novel DOF-SMLC should be proposed first.

Motivated by the above discussions, a DOF-based improved SMLC is proposed for the robust control of mechanical system in this paper. An uncertain model of mechanical system is proposed first, and then the model is transformed into a standard one, more specifically, a linear uncertain system with disturbances. A DOF controller is designed, then a DOF-based sliding surface is designed and the stability of it is guaranteed by selecting parameters appropriately. After getting the sliding surface, a learning controller is proposed for the uncertain mechanical system. Finally, a solving algorithm is given for the proposed DOF-SMLC. The proposed DOF based SMLC is confirmed by a numerical example.

The novelties and main contributions of the paper can be summarized and listed as:

1. For an uncertain mechanical system with unknown parameter uncertainty and disturbance, a learning controller is designed.
2. A DOF-based sliding surface is proposed for uncertain mechanical system;
3. A novel SMLC is proposed and the proposed controller can greatly reduce the chattering of traditional SMC.

This paper is organized as follows. The uncertain mechanical system model is listed in section “Problem Formulation,” and the main results are given in section “Main Results.” Numerical simulation results are given in section “Simulation results.” This paper is summarized in section “Conclusion.”

### Problem formulation

#### Model of mechanical system

The mechanical system considered in this paper is a 2 degree-of-freedom mechanical system, and a sketch of the mechanical system is given in Figure 1. The system equation described in physical coordinate is listed as following:

\[
\begin{align*}
M \ddot{r}(t) + (G + \Delta G)\dot{r}(t) + (K + \Delta K)r(t) &= B u(t) + E d(t) \\
M' \ddot{r}(t) + (G' + \Delta G')\dot{r}(t) + (K' + \Delta K')r(t) &= B' u(t) + E' d(t)
\end{align*}
\]

where \(r(t) = [r_1(t) \; r_2(t)]^T\) represents the position vector of the mechanical system, correspondingly, \(\dot{r}(t)\) represents the velocity vector, and \(\ddot{r}(t)\) represents the acceleration vector. \(M'\) is the mass of the system, and \(G'\) represents the gyroscopic/dissipation characteristics, \(K'\) represents the stiffness characteristics, respectively. \(\Delta G'\) and \(\Delta K'\) are unknown uncertainties in gyroscopic/dissipation characteristics and stiffness characteristics, respectively, and

\[
\begin{align*}
M &= \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, & G' &= \begin{bmatrix} c_s & -c_s \\ -c_s & c_s \end{bmatrix}, \\
K' &= \begin{bmatrix} k_s & -k_s \\ -k_s & k_s \end{bmatrix}, \\
\Delta G' &= \begin{bmatrix} \Delta c_s & -\Delta c_s \\ -\Delta c_s & \Delta c_s \end{bmatrix}, & \Delta K' &= \begin{bmatrix} \Delta k_s & -\Delta k_s \\ -\Delta k_s & \Delta k_s \end{bmatrix}, \\
B &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & E' &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{align*}
\]

In equation (1), \(M', G'\) and \(K'\) are all known. \(u(t)\) is the control input and \(d(t)\) is unknown disturbance. \(B\) is the input matrix, and \(E'\) is the disturbance matrix. Then equation (1) can be rewrote as

![Figure 1. Geometry of the mechanical system.](image)
\[ \ddot{r}(t) = -M^{-1}(G + \Delta G)\dot{r}(t) - M^{-1}(K + \Delta K)r(t) \\
+ M^{-1}\dot{B}u(t) + M^{-1}\dot{E}d(t) \\
= -M^{-1}G\dot{r}(t) - M^{-1}Kr(t) + M^{-1}\dot{B}u(t) \\
- M^{-1}\Delta G\dot{r}(t) - M^{-1}\Delta K\dot{r}(t) + M^{-1}\dot{E}d(t) \]

(2)

For mechanical system model (2), choosing the state as
\[
x(t) = \begin{bmatrix} r_1(t) \\ r_2(t) \\ \dot{r}_1(t) \\ \dot{r}_2(t) \end{bmatrix}
\]
then system (2) can be rewrote as
\[
\dot{x}(t) = (A + \Delta A)x(t) + Bu(t) + Ed(t)
\]
where
\[ A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_x & k_x & c_x & 0 \\ m_1 & m_1 & m_1 & m_1 \end{bmatrix}, \quad \Delta A = \begin{bmatrix} -\Delta k_x & \Delta k_x & \Delta c_x & \Delta c_x \\ m_1 & m_1 & m_1 & m_1 \\ -\Delta k_x & \Delta k_x & \Delta c_x & \Delta c_x \\ m_2 & m_2 & m_2 & m_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ m_1 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ m_1 & m_1 & 0 & 0 \end{bmatrix}
\]

**Control objective**

For the original system equation (1), assuming that, the measurable output of the system (3) is
\[ y(t) = Cx(t) \]
then the equation of mechanical system can be summarized as
\[
\dot{x}(t) = (A + \Delta A)x(t) + Bu(t) + Ed(t), \\
y(t) = Cx(t)
\]

(4)

For the uncertain matrix \( \Delta A \), since \( \Delta G \) and \( \Delta K \) are all unknown, \( \Delta A \) is also unknown. Similarly, the disturbance \( d(t) \) is unknown. In equation (3), \( \text{rank}(\Delta A) = 2 \), \( \text{rank}(B) = 1 \), \( \Delta A \) doesn’t satisfy the matched condition, so it is mismatched. Similarly, \( d(t) \) is also mismatched. For \( \Delta A \) and \( d(t) \), the following assumption are made for the controller design of this paper.

**Assumption 1.** \( \Delta A \) is unknown but bounded, which means that,
\[ \|\Delta A\| \leq \bar{A} \]
where \( \bar{A} \) is a known constant vector.

**Assumption 2.** \( d(t) \) is unknown but bounded, which means that, there exist an known scalar \( \bar{d} \),
\[ \|d(t)\| \leq \bar{d} \]

**Remark 1.** In Assumptions 1 and 2, the unbound of \( \Delta A \) and \( d(t) \) are known. This is reasonable since the uncertainties of gyroscopic/dissipation characteristics and stiffness characteristics can be modeled according to the limitation of physical system.

Considering the existence of \( \Delta A \) and \( d(t) \), a robust controller is needed for equation (3). In consideration of the advantage of SMC in robust control, SMC is utilized here for the robust controller design. Also taking into account that, only \( y(t) \) is measurable in equation (4), an output feedback controller is needed here. Then the control goal for mechanical system in this paper is: Designing an output feedback-based SMC, which can guarantee the stability of the closed system in the existence of mismatched \( \Delta A \) and \( d(t) \).

**Main results**

Since only the output of equation (4) is measurable, output feedback-based controller design strategy is utilized here. Also considering the robust requirement for uncertainty and disturbance, SMC strategy is utilized here. In this case, DOF-based SMC approach is utilized here for the robust controller design of mechanical system.

**DOF based sliding surface design**

For system equation (4), a full-order DOF controller is constructed as
\[
\begin{align*}
\dot{x}(t) &= A_Dx(t) + B_Dy(t), \\
u(t) &= C_D\dot{x}(t) + u_i(t)
\end{align*}
\]
where \( x(t) \) is the state of the new system, \( y(t) \) is the measured output of equation (4), \( A_D, B_D \) and \( C_D \) are matrices needed to be determined, \( u(t) \) is the designed controller, \( u_i(t) \) is the nonlinear part related to SMC designing strategy, and needed to be designed later. Applying the above controller to system (4), the overall system can be obtained as
\[
\begin{align*}
\dot{x}(t) &= A_c\dot{x}(t) + \Delta A_c\dot{x}(t) + B_cu_i(t) + E_c\dot{d}(t), \\
y(t) &= C_c\dot{x}(t)
\end{align*}
\]
where

\[ A_c = A_D + \Delta A_c, \quad B_c = B_D, \quad C_c = C_D \]
\[
\chi(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}, \quad A_e = \begin{bmatrix} A & BC_D \\ BD_C & A_D \end{bmatrix},
\]
\[
\Delta A_e = \begin{bmatrix} \Delta A & 0 \\ 0 & 0 \end{bmatrix},
\]
\[
B_e = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad E_e = \begin{bmatrix} E \\ 0 \end{bmatrix}, \quad C_e = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}
\]

Then, the control of equation (4) is converted into the closed-loop stability of equation (6). According to the SMC designing strategy, an output feedback based sliding surface for equation (6) is defined as

\[
S(t) = K_1 \chi(t) = KC_e \chi(t) = [K_1 \ K_2] \begin{bmatrix} 0 \\ I \end{bmatrix} x(t)
\]

For system (6), \(u_e(t)\) is the discontinuous part of SMC, so equation (6) is an autonomous system and the equivalent control is not in existence. In this case, what we should do is just discussing the stability of \(A_e\). Similarly, for the designed sliding surface, we should only analyze the stability of \(S(t)\) without other influence. Then, for \(S(t)\), we have

\[
\dot{S}(t) = K_1 \dot{\chi}(t) = KC_e \dot{\chi}(t)
\]

For analyzing the stability of \(S\), we define the Lyapunov function for system (8) as

\[
V_s(t) = \frac{S^T(t)S(t)}{2}
\]

Then

\[
\dot{V}_s(t) = \frac{S^T(t)\dot{S}(t) + \dot{S}^T(t)S(t)}{2}
\]

For the stability analysis of \(S(t), \Delta A_e, u_e(t),\) and \(d(t)\), we can all be viewed as disturbance, the stability of \(S(t)\) is decided on the value of \(\chi^T(t)C_e^T KC_e \chi(t) + \dot{\chi}^T(t)C_e^T KC_e \dot{\chi}(t)\). Then we can decisively make the following conclusion: If the following inequation holds, the reduced dynamic on sliding surface equation (7) is stable

\[
C^T_e K^T KC_e A_e + A_e^T C_e^T K^T KC_e < -Q
\]

where \(Q\) is a positive matrix. If equation (9) is hold, \(\dot{V}_s(t)\) is reduced to

\[
\dot{V}_s(t) < -\frac{e_1^T Q e_1}{2} + \frac{e_2^T Q e_2}{2} + \frac{\dot{e}_1^T Q \dot{e}_1}{2} + \frac{\dot{e}_2^T Q \dot{e}_2}{2} + \frac{\dot{e}_1^T K \dot{e}_2}{2} + \frac{\dot{e}_2^T K \dot{e}_1}{2} + \dot{e}_1^T d_1 + \dot{e}_2^T d_2
\]

Since \(u_e(t)\) is the discontinuous part of SMC, \(u_e(t)\) is bounded. From Assumption 1 and 2, \(\Delta A_e\) and \(d(t)\) are all bounded. If

\[
\|Q\| > \|B_e u_e(t)\| + \|D_e d(t)\| + \|\Delta A_e\|
\]

\(\dot{V}_s\) is reduced to

\[
\dot{V}_s(t) < 0
\]

in this case, the stability of the reduced dynamics on sliding surface equation (7) is guaranteed. Then the stability of equation (7) can be summered as the following question

1. Finding a matrix \(K\) and the output feedback gain \(A_D\), \(B_D\) and \(C_D\), which can guarantee the following inequation:

\[
C^T_e K^T KC_e A_e + A_e^T C_e^T K^T KC_e < -Q
\]

2. The matrix \(K\) should satisfy \(\|Q\| > \|B_e u_e(t)\| + \|D_e d(t)\| + \|\Delta A_e\|\).

**Traditional discontinuous controller design**

If equation (9) is held, the stability of the sliding surface \(S(t)\) is guaranteed. For the stability of (6), the reachabil- ity of the sliding surface should also be guaranteed. For the reachability of \(S(t)\), the following inequation should be satisfied

\[
S^T(t)S(t) < 0
\]

which means that

\[
S^T(t)K C_e A_e \chi(t) + S^T(t)K C_e \Delta A_e \chi(t) + S^T(t)K C_e B_e u_e(t) + S^T(t)K C_e d(t) < 0
\]

For equation (10), it is difficult to get a feasible \(u_e\) since \(K\) is unknown. For the designed matrix \(K\)

\[
K C_e B_e = [K_1 \ K_2] \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} B \\ 0 \end{bmatrix} = K_1 C B
\]

and \(K_1 \in R^{1x6}, C\) and \(B\) are all known, then for the convenience of control design, we choose the value of \(K_1\) to make that, \(K_1 C B = I\), then

\[
K C_e B_e = I
\]
the SMC controller is

\[ S^T K_c A_c \chi(t) + S^T K_c \Delta A_c \chi(t) + S^T u_r(t) + S^T K_c E_c d(t) < 0 \]

Since

\[ S^T K_c A_c \chi(t) < \lambda_{\text{max}}(A_c) S^T S \]

and

\[ S^T K_c E_c d(t) < \lambda_{\text{max}}(D_c) d_{\text{max}} S^T S \]

\[ S^T K_c \Delta A_c \chi(t) < A S^T S \]

the discontinuous controller can be chosen as

\[ u_r = \begin{cases} -\rho \frac{S}{\|S\|} & \text{If } S \neq 0, \\ 0 & \text{If } S = 0 \end{cases} \]

where

\[ \rho > \lambda_{\text{max}}(A_c) + \lambda_{\text{max}}(D_c) d_{\text{max}} \|S\| \]

**Remark 2.** For solving \( u_r \), we make the assumption that \( K_1 C B = I \). Then \( K_c B_c = I \). \( K \in R^{1 \times 5} \), and \( K_1 \in R^{1 \times 2} \). In this case, we lost 2 dimension degree in the design of \( K \), but we still have 4 dimension degree since \( K_2 \in R^{1 \times 4} \).

Then for the designed sliding surface equation (7), the SMC controller is

\[ u(t) = C_D \dot{\chi}(t) + u_r(t) \]

\[ u_r = \begin{cases} -\rho \frac{S}{\|S\|} & \text{If } S \neq 0, \\ 0 & \text{If } S = 0 \end{cases} \]

**SMLC design**

In subsection “Traditional discontinuous controller design,” a discontinuous controller \( u_r(t) \) is designed for mechanical system (3). But when utilized the discontinuous controller \( u_r(t) \), chattering is inevitable. Obviously, chattering is unexpected for a real system, then a novel controller, which can retain the advantage of SMC while reduced chattering, is need. In this subsection, a sliding mode learning controller is proposed. The given learning controller is

\[ u(t) = u_r(t - \tau) + \Delta u_r(t) \quad (12) \]

where \( \tau \) is a time delay, \( \Delta u_r(t) \) is an adaptation term needed to be designed

\[ \Delta u_r(t) = \begin{cases} -\left( \alpha \dot{V}_3(t - \tau) + \beta V_3(t) \right) & \text{for } S(t) \neq 0, \\ 0 & \text{for } S(t) = 0 \end{cases} \quad (13) \]

where \( V_3(t) \) is a Lyapunov function which will be given later, \( \alpha \) and \( \beta \) are designed parameters. \( \dot{V}_3(t - \tau) \) is the numerical approximating of \( \dot{V}_3(t - \tau) \).

**Assumption 3.** The numerical approximating error is reasonable small, which means that, the sign of \( \dot{V}_3(t - \tau) \) and \( \dot{V}_3(t - \tau) \) is the same, and

\[ |\dot{V}_3(t - \tau) - \dot{V}_3(t - \tau)| < \varepsilon |\dot{V}_3(t - \tau)| \]

where \( \varepsilon \) is a positive constant, and \( \varepsilon \ll 1 \).

**Theorem 1.** For system (6), if Assumption 3 is hold, and \( u(t) \) are chosen as equations (12) and (13), the parameters \( \alpha \) and \( \beta \) are

\[ \frac{1}{M} < \alpha < 1 - \frac{1}{M} - \gamma, \beta > 0 \quad (14) \]

then equation (6) is stable.

**Proof.** Constructing Lyapunov function for equation (6) as

\[ V_3(t) = \frac{1}{2} S^T(t) S(t) \]

For sliding surface (7)

\[ \dot{S}(t) = K_c \dot{\chi}(t) = K_c \chi(t) \]

\[ = K_c A_c \chi(t) + K_c \Delta A_c \chi(t) + K_c B_c u(t) \]

\[ + K_c E_c d(t) \]

Then for \( V_3(t) \)

\[ \dot{V}_3(t) = S^T(t) \dot{S}(t) \]

\[ = S^T(t) (F(t) + u_r(t)) \]

Using the designed controller (12) we have

\[ \dot{V}_3(t) = S^T(t) F(t) + S^T(t) u_r(t - \tau) + S^T(t) \Delta u_r(t) \]

with the expression of \( \Delta u_r(t) \) we have

\[ S^T(t) \Delta u_r(t) = -\alpha \dot{V}_3(t - \tau) - \beta V_3(t) \]

Then

\[ \dot{V}_3(t) = S^T(t) F(t) + S^T(t) u(t - \tau) - \alpha \dot{V}_3(t - \tau) - \beta V_3(t) \]

Since \( \tau \) is reasonable small, if \( \dot{V}_3(t, t - \tau) \neq 0, \dot{V}_3(t - \tau) \neq 0 \) and \( \dot{V}_3(t - \tau) \neq 0 \), then

\[ |\dot{V}_3(t, t - \tau) - \dot{V}_3(t - \tau)| < \frac{1}{M} \dot{V}_3(t - \tau) \]

where \( M \gg 1 \). Then
\[
\dot{V}_3(t) \leq \dot{V}_3(t, t - \tau) + \dot{V}_3(t - \tau) - \alpha \dot{V}_3(t - \tau) - \beta V_3(t)
\]
\[
< \frac{1}{M} \dot{V}_3(t - \tau) + \dot{V}_3(t - \tau) - \alpha \dot{V}_3(t - \tau) - \beta V_3(t)
\]

Form the proof of Theorem 2 in Hu et al.,\textsuperscript{25} whether \(\dot{V}_3(t - \tau) > 0\) or \(\dot{V}_3(t - \tau) < 0\), we can always get that, \(V_3(t)\) are always reducing and from (14),

\[
V_3(t) < -\frac{1}{M} + \gamma - 1 + \alpha \left| \dot{V}_3(t - \tau) \right| - \beta V_3(t)
\]

Since \(\beta > 0\), the closed loop system is stable. The proof is completed.

**Solving algorithm**

For the successfully application of the proposed DOF-SMLC, a solving algorithm is presented in this subsection. In subsection 3.2, the following equations should be

\[
C_T K^T K C_e A_e + A^T C_T K^T K C_e < -Q,
\]

\[
K C_e B_e = I
\]

Associated with the Lyapunov stability theory, the following inequation should also be hold

\[
C_T K^T K C_e > 0
\]

Then

\[
C_T K^T K C_e = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} K_1 & K_2 \end{bmatrix}^T \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}
\]

the solving algorithm can be listed as follows

**Step 1:** Associated with (11), the value of \(K_1\) can be determined and

\[
K_1 C B = I
\]

**Step 2:** Find the matrix \(K_2\) by solving the inequation

\[
\begin{bmatrix} C_T K_1^T K_1 C_e & C_T K_1^T K_2 \\ K_2^T K_1 C & K_2^T K_2 \end{bmatrix} > 0
\]

**Step 3:** Getting \(A_D\), \(B_D\) and \(C_D\) by solving the equation

\[
C_T K^T K C_e A_e + A^T C_T K^T K C_e < -Q
\]

For the solving details of equations (15), (16), and (17), corresponding supporting materials can be found in Hu et al.\textsuperscript{26} Then by solving equations (15), (16), and (17), the DOF-SMLC can be constructed.

**Simulation results**

For demonstrating the effectiveness of the proposed DOF-SMLC, a numerical simulation example is considered in this section. The parameters of equation (1) are chosen as following

\[
m_{s1} = 1 \, \text{kg}, \quad m_{s2} = 2 \, \text{kg}, \quad c_s = 3 \, \text{Ns/m}, \quad k_s = 5 \, \text{N/m}
\]

\[
\Delta c_s = 0.5 \cos(t), \Delta k_s = 0.5 \sin(t)
\]

\[
d(t) = \begin{bmatrix} 0.2 \exp(-0.01t) \sin(2t) \\ 0.2 \exp(-0.01t) \cos(2t) \end{bmatrix}
\]

then the matrices of (3) are

\[
A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & 5 & -3 & 3 \\ 2.5 & -2.5 & 1.5 & -1.5 \end{bmatrix}
\]

\[
\Delta A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.5 \sin(t) & 0.5 \sin(t) & -0.5 \cos(t) & 0.5 \cos(t) \\ 0.25 \sin(t) & -0.25 \sin(t) & 0.25 \cos(t) & -0.25 \cos(t) \end{bmatrix}
\]

\[
B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}
\]

Then according to the proposed DOF-SMLC, \(A_D\), \(B_D\) and \(C_D\) can be computed and

\[
A_D = \begin{bmatrix} 158 & -122 & -703 & -674 \\ 14.16 & -4.10 & -15.3 & -28.1 \\ 4.19 & 1.31 & -0.411 & 0.72 \\ -13.1 & 0.78 & 8.93 & 14.86 \end{bmatrix}
\]

\[
B_D = \begin{bmatrix} -10.86 \\ 3.45 \\ 3.89 \\ -6.77 \end{bmatrix}
\]

\[
C_D = \begin{bmatrix} -276 & -194 & -293 & -369 \end{bmatrix}
\]

Then the sliding surface is constructed, and \(K = [1 \ 0.94 \ 2.31 \ 13.64 \ 70.8]\).

In the leaning controller, we choose \(\alpha = 0.3, \beta = 2\). For testing the performance of the proposed SMLC, the improved SMLC is marked as \(u_{\text{defsmc}}\), and is implemented on the 2 degree-of-freedom mechanical system together with a traditional sliding model control \(u_{\text{smc}}\). The initial states of mechanical system is set to be \(x(0) = [1 \ -0.8 \ 0 \ 0]^T\), and the simulation results are listed in Figures 2 and 3, and the dashdot line represents the responses under controller \(u_{\text{defsmc}}\), and solid line is for traditional controller SMC \(u_{\text{smc}}\). Figures 4 and Figures 5 are the input of \(u_{\text{smc}}\) and \(u_{\text{defsmc}}\). From Figures 2 and 3, we can see that, the system controlled by the proposed SMLC and SMC are all stable. But from equations (4) and (5) we can see that, the input of
udofsmlc is smooth while the input of usmc is chattering. The control performance of udofsmlc is obviously better than traditional controller usmc.

**Conclusion**

In this paper, a DOF-SMLC has been proposed for mechanical system. A DOF-based sliding mode surface is designed and then the stability of the designed sliding surface is discussed. A learning controller is designed instead of the discontinuous controller to guarantee the stability of the closed loop system. Finally, a numerical simulation example is given to show the good performance of the proposed strategy.

The proposed control method can be utilized by industrial system without enough sensor, or practical system which only partly states can be measured. But the upper bounds of parameter uncertainty and disturbance are assumed to be known in this paper, while in most instance, these special bounds are hard to be got. Further work is to design a controller without any prior information of them.

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