EFFICIENT LIKELIHOOD ESTIMATION IN STATE SPACE MODELS

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Motivated by studying asymptotic properties of the maximum likelihood estimator (MLE) in stochastic volatility (SV) models, in this paper we investigate likelihood estimation in state space models. We first prove, under some regularity conditions, there is a consistent sequence of roots of the likelihood equation that is asymptotically normal with the inverse of the Fisher information as its variance. With an extra assumption that the likelihood equation has a unique root for each $n$, then there is a consistent sequence of estimators of the unknown parameters. If, in addition, the supremum of the log likelihood function is integrable, the MLE exists and is strongly consistent. Edgeworth expansion of the approximate solution of likelihood equation is also established. Several examples, including Markov switching models, ARMA models, (G)ARCH models and stochastic volatility (SV) models, are given for illustration.

1. Introduction. Motivated by studying asymptotic properties of the maximum likelihood estimator (MLE) in stochastic volatility (SV) models, in this paper we investigate likelihood estimation in state space models. A state space model is, loosely speaking, a sequence $\{\xi_n\}_{n=0}^{\infty}$ of random variables obtained in the following way. First, a realization of a Markov chain $X = \{X_n, n \geq 0\}$ is created. This chain is sometimes called the regime and is not observed. Then, conditional on $X$, the $\xi$-variables are generated. Usually the dependence of $\xi_n$ on $X$ is more or less local, as when $\xi_n = g(X_n, \xi_{n-1}, \eta_n)$ for some function $g$ and random sequence $\{\eta_n\}$, independent of $X$. $\xi_n$ itself is generally not Markov and may, in fact, have a complicated dependence structure. When the state space of $\{X_n, n \geq 0\}$ is finite, it is the so-called hidden Markov model or Markov switching model.
The statistical modeling and computation for state space models have attracted a great deal of attention recently because of their importance in applications to speech recognition [49], signal processing [17], ion channels [1], molecular biology [40] and economics [8, 19, 51]. The reader is referred to [20, 34, 41] for a comprehensive summary. The main focus of these efforts has been state space modeling and estimation, algorithms for fitting these models and the implementation of likelihood based methods.

The state space model here is defined in a general sense, in which the observations are conditionally Markovian dependent, and the state space of the driving Markov chain need not be finite or compact. When the state space is finite and the observation is a deterministic function of the state space, Baum and Petrie [3] established the consistency and asymptotic normality of the MLE. When the observed random variables are conditionally independent, Leroux [44] proved strong consistency of the MLE, while Bickel, Ritov and Rydén [7] established asymptotic normality of the MLE under mild conditions. Jensen and Petersen [39], Douc and Matias [14] and Douc, Moulines and Rydén [15] studied asymptotic properties of the MLE for general “pseudo-compact” state space models. By extending the inference problem to time series analysis where the state space is finite and the observed random variables are conditionally Markovian dependent, Goldfeld and Quandt [30] and Hamilton [33] considered the implementation of the maximum likelihood estimator in switching autoregressions with Markov regimes. Francq and Roussignol [21] studied the consistency of the MLE, while Fuh [23] established the Bahadur efficiency of the MLE in Markov switching models. We now give two examples of state space models.

**Example 1 [GARCH(p,q) model].** For given \( p \geq 1 \) and \( q \geq 0 \), let

\[
Y_n = \sigma_n \varepsilon_n \quad \text{and} \quad \sigma_n^2 = \delta + \sum_{i=1}^{p} \alpha_i \sigma_{n-i}^2 + \sum_{j=1}^{q} \beta_j Y_{n-j}^2,
\]

where \( \delta > 0, \alpha_i \geq 0 \) and \( \beta_j \geq 0 \) are constants, \( \varepsilon_n \) is a sequence of independent and identically distributed (i.i.d.) random variables, and \( \varepsilon_n \) is independent of \( \{Y_{n-k}, k \geq 1\} \) for all \( n \). This is the celebrated GARCH\((p,q)\) model proposed by Bollerslev [8]. When \( q = 0 \) or \( \beta_j = 0, \) for \( j = 1, \ldots, q \), this is the ARCH\((p)\) model first considered by Engle [19]. The reader is referred to [9] and [20] for a comprehensive summary.

For convenience of notation, we assume that \( p, q \geq 2 \), and by adding some \( \alpha_i \) or \( \beta_j \) equal to zero if necessary. Denote \( \eta_n = \sigma_n^{-1} Y_n, \tau_n = (\alpha_1 + \beta_1 \eta_n, \alpha_2, \ldots, \alpha_{p-1}) \in \mathbb{R}^{p-1}, \zeta_n = (\eta_n^2, 0, \ldots, 0) \in \mathbb{R}^{p-1}, \beta = (\beta_2, \ldots, \beta_{q-1}) \in \mathbb{R}^{q-2}, \) and let \( I_{p-1} \) and \( I_{q-2} \) be identity matrices. Let \( A_n \) be a \((p + q - 1) \times ...
$(p + q - 1)$ matrix written in block form as

\begin{equation}
A_n = \begin{bmatrix}
\tau_n & \alpha_p & \beta & \beta_q \\
I_{p-1} & 0 & 0 & 0 \\
\zeta_n & 0 & 0 & 0 \\
0 & 0 & I_{q-2} & 0
\end{bmatrix}.
\end{equation}

Note that $\{A_n, n \geq 0\}$ are i.i.d. random matrices.

Let $Z = (\delta, 0, \ldots, 0)' \in \mathbb{R}^{p+q-1}$ and $X_n = (\sigma_{n+1,2}, \ldots, \sigma_{n-p+2,2}, Y_{n,2}, \ldots, Y_{n-q+2,2})'$, where "$'$" denotes transpose. Following the idea of Bougerol and Picard \[10\], we have the following state space representation of the GARCH$(p,q)$ model:

\begin{equation}
X_{n+1} = A_{n+1}X_n + Z,
\end{equation}

and $\xi_n := g(X_n) = (Y_{n,2}, \ldots, Y_{n-q+2,2})'$, the observed random quantity, is a noninvertible function of $X_n$.

**Example 2** (Stochastic volatility models). Let

\begin{equation}
Y_n = \sigma_n \varepsilon_n,
\end{equation}

where $\log \sigma_n^2$ follows an AR(1) process and $\varepsilon_n$ is a sequence of i.i.d. random variables with standard normal probability density function. This is the discrete time stochastic volatility model proposed by Taylor \[51\]. The reader is referred to \[29, 50, 52\] for a comprehensive summary. Note that Genon-Catalot, Jeantheau and Larédo \[27\] studied the ergodicity and mixing properties of stochastic volatility models from the hidden Markov model point of view.

Write $X_n := \log \sigma_n^2$ and $Y_n = \sigma_n \varepsilon_n \exp(X_n/2)$, where $\sigma$ is a scale parameter. Squaring the observations in the above equation and taking logarithms gives $\log Y_n^2 = \log \sigma^2 + X_n + \log \varepsilon_n^2$. Alternatively, we have

\begin{equation}
\log Y_n^2 = \omega + X_n + \zeta_n,
\end{equation}

where $\omega = \log \sigma^2 + E \log \varepsilon_n^2$, so that the disturbance $\zeta_n$ has mean zero by construction. The scale parameter $\sigma$ also removes the need for a constant term in the stationary first-order autoregressive process

\begin{equation}
X_n = \alpha X_{n-1} + \eta_n, \quad |\alpha| < 1,
\end{equation}

where $\eta_n$ is a sequence of i.i.d. random variables distributed as $N(0, \sigma_\eta^2)$. Moreover, we assume that $\zeta_n$ and $\eta_n$ are independent. Note that in (1.5) and (1.6) the observed random quantity is $\xi_n := \log Y_n^2$. $\{X_n, n \geq 0\}$ and forms a Markov chain with transition probability

\begin{equation}
p(x_{k-1}, x_k) = (2\pi \sigma_\eta^2)^{-1/2} \exp \left\{ -\frac{1}{2} \frac{(x_k - \alpha x_{k-1})^2}{\sigma_\eta^2} \right\}
\end{equation}
and stationary distribution \( \pi \sim N(0, \sigma^2/\eta (1-\alpha)) \).

For given observations \( y = (\log y_1^2, \ldots, \log y_n^2) \) from the state space model (1.5) and (1.6), the likelihood function of the parameter \( \theta = (\alpha, \sigma^2) \) is

\[
l(y; \theta) = \int_{x_0 \in \mathcal{X}} \cdots \int_{x_n \in \mathcal{X}} \pi(x_0) \prod_{k=1}^{n} p(x_{k-1}, x_k) \times f_\zeta(\log y_k^2 - \omega - x_k) \, dx_n \cdots dx_0,
\]

where \( f_\zeta(\cdot) \) is the probability density function of \( \zeta_1 \).

A major difficulty in analyzing the likelihood function in state space models is that it can be expressed only in integral form; see equation (1.8), for instance. In this paper we provide a device which represents the integral likelihood function as the \( L_1 \)-norm of a Markovian iterated random functions system. This new representation enables us to apply results of the strong law of large numbers, central limit theorem and Edgeworth expansion for the distributions of Markov random walks, and to verify strong consistency of the MLE and first-order efficiency and Edgeworth expansion on the solution of the likelihood equation. Note that third-order efficiency follows from Edgeworth expansion by a standard argument (cf. [28]). Another essential point worth being mentioned is that we introduce a weight function in a suitable way [see (4.1)–(4.3), Assumptions K2, K3 and Definition 2 in Section 4, and C1 in Section 5] to relax the condition of a compact state space for the underlying Markov chain, and to cover several interesting examples.

The remainder of this paper is organized as follows. In Section 2 we define the state space model as a general state Markov chain in a Markovian random environment, and represent the likelihood function as the \( L_1 \)-norm of a Markovian iterated random functions system. In Section 3 we give a brief summary of a Markovian iterated random functions system, and provide an ergodic theorem and the strong law of large numbers. The multivariate central limit theorem and Edgeworth expansion for a Markovian iterated random functions system are given in Section 4. Section 5 contains our main results, where we consider efficient likelihood estimation in state space models, and state the main results. First, we compute Fisher information and prove the existence of an efficient estimator in a “Cramér fashion.” Second, we characterize Kullback–Leibler information, and prove strong consistency of the MLE. Last, we establish Edgeworth expansion of the approximate solution of the likelihood equation. In Section 6 we consider a few examples, including Markov switching models, ARMA models, (G)ARCH models and SV models, which are commonly used in financial economics. The proofs of the lemmas in Section 5 are given in Section 7. Other technical proofs are deferred to the Appendix.
2. State space models. A state space model is defined as a parameterized Markov chain in a Markovian random environment with the underlying environmental Markov chain viewed as missing data. Specifically, let $X = \{X_n, n \geq 0\}$ be a Markov chain on a general state space $X$, with transition probability kernel $P^\theta(x, \cdot) = P^\theta\{X_1 \in \cdot | X_0 = x\}$ and stationary probability $\pi(\cdot)$, where $\theta \in \Theta \subseteq \mathbb{R}^d$ denotes the unknown parameter. Suppose that a random sequence $\{\xi_n\}_{n=0}^\infty$, taking values in $\mathbb{R}^d$, is adjoined to the chain such that $\{(X_n, \xi_n), n \geq 0\}$ is a Markov chain on $X \times \mathbb{R}^d$ satisfying $P^\theta\{X_1 \in A | X_0 = x, \xi_0 = s\} = P^\theta\{X_1 \in A | X_0 = x\}$ for $A \in B(X)$, the $\sigma$-algebra of $X$. And conditioning on the full $X$ sequence, $\xi_n$ is a Markov chain with probability

$$P^\theta\{\xi_{n+1} \in B | X_0, X_1, \ldots; \xi_0, \xi_1, \ldots, \xi_n\}$$

(2.1)

$$= P^\theta\{\xi_{n+1} \in B | X_{n+1}; \xi_n\} \quad \text{a.s.}$$

for each $n$ and $B \in B(\mathbb{R}^d)$, the Borel $\sigma$-algebra on $\mathbb{R}^d$. Note that in (2.1) the conditional probability of $\xi_{n+1}$ depends on $X_{n+1}$ and $\xi_n$ only. Furthermore, we assume the existence of a transition probability density $p_\theta(x, y)$ for the Markov chain $\{X_n, n \geq 0\}$ with respect to a $\sigma$-finite measure $m$ on $X$ such that

$$P^\theta\{X_1 \in A, \xi_1 \in B | X_0 = x, \xi_0 = s\}$$

(2.2)

$$= \int_{y \in A} \int_{s \in B} p_\theta(x, y) f(s; \theta, s_0) Q(ds)m(dy),$$

where $f(\xi_k; \theta | X_k, \xi_{k-1})$ is the conditional probability density of $\xi_k$ given $\xi_{k-1}$ and $X_k$, with respect to a $\sigma$-finite measure $Q$ on $\mathbb{R}^d$. We also assume that the Markov chain $\{(X_n, \xi_n), n \geq 0\}$ has a stationary probability with probability density function $\pi(x)f(\cdot; \theta | x)$ with respect to $m \times Q$. In this paper we consider $\theta = (\theta_1, \ldots, \theta_q) \in \Theta \subseteq \mathbb{R}^d$ as the unknown parameter, and the true parameter value is denoted by $\theta_0$. We will use $\pi(x)$ for $\pi_\theta(x)$, $p(x, y)$ for $p_\theta(x, y)$, $f(\xi_0 | X_0)$ for $f(\xi_0; \theta | X_0)$, and $f(\xi_k | X_k, \xi_{k-1})$ for $f(\xi_k; \theta | X_k, \xi_{k-1})$, here and in the sequel, depending on our convenience. Now we give a formal definition as follows.

**Definition 1.** $\{\xi_n, n \geq 0\}$ is called a state space model if there is a Markov chain $\{X_n, n \geq 0\}$ such that the process $\{(X_n, \xi_n), n \geq 0\}$ satisfies (2.1).

Note that this setting includes several interesting examples of Markov-switching Gaussian autoregression of Hamilton [33], (G)ARCH models of Engle [19] and Bollerslev [8], and SV models of Clark [12] and Taylor [51]. When the state space $X$ is finite or compact, this reduces to the hidden
Markov model considered by Francq and Roussignol [21], Fuh [22, 23, 25] and Douc, Moulines and Rydén [15]. Denote \( S_n = \sum_{t=1}^n \xi_t \). When \( \xi_n \) are conditionally independent given \( X \), the Markov chain \( \{(X_n, S_n), n \geq 0\} \) is called a Markov additive process and \( S_n \) is called a Markov random walk. Furthermore, if the state space \( X \) is finite, \( \{\xi_n, n \geq 0\} \) is the hidden Markov model studied by Leroux [44], Bickel and Ritov [6] and Bickel, Ritov and Rydén [7]. When the state space \( X \) is “pseudo-compact” and \( \xi_n \) are conditionally independent given \( X \), \( \{\xi_n, n \geq 0\} \) is the state space model considered in [39] and [14].

For given observations \( s_0, s_1, \ldots, s_n \) from a state space model \( \{\xi_n, n \geq 0\} \), the likelihood function is

\[
p_n(s_0, s_1, \ldots, s_n; \theta) = \int_{x_0 \in X} \cdots \int_{x_n \in X} \pi_\theta(x_0) f(s_0; \theta|x_0) \\
\times \prod_{j=1}^n p_\theta(x_{j-1}, x_j) \\
\times f(s_j; \theta|x_j, s_{j-1}) m(dx_n) \cdots m(dx_0).
\]

Recall that \( \pi_\theta(x_0) f(s_0; \theta|x_0) \) is the stationary probability density with respect to \( m \times Q \) of the Markov chain \( \{(X_n, \xi_n), n \geq 0\} \).

To represent the likelihood \( p_n(\xi_0, \xi_1, \ldots, \xi_n; \theta) \) as the \( L_1 \)-norm of a Markovian iterated random functions system, let

\[
(2.4) \quad M = \left\{ h : h : X \to \mathbb{R}^+ \text{ is } m\text{-measurable and } \int_{x \in X} h(x)m(dx) < \infty \right\}.
\]

For each \( j = 1, \ldots, n \), define the random functions \( P_\theta(\xi_0) \) and \( P_\theta(\xi_j) \) on \( (X \times \mathbb{R}^d) \times M \) as

\[
(2.5) \quad P_\theta(\xi_0)h(x) = \int_{x \in X} f(\xi_0; \theta|x)h(x)m(dx), \quad \text{a constant,}
\]

\[
(2.6) \quad P_\theta(\xi_j)h(x) = \int_{y \in X} p_\theta(x, y)f(\xi_j; \theta|y, \xi_{j-1})h(y)m(dy).
\]

Define the composition of two random functions as

\[
P_\theta(\xi_{j+1}) \circ P_\theta(\xi_j)h(x)
\]

\[
(2.7) \quad = \int_{z \in X} p_\theta(x, z)f(\xi_j; \theta|z, \xi_{j-1})
\]

\[
\times \left( \int_{y \in X} p_\theta(z, y)f(\xi_{j+1}; \theta|y, \xi_j)h(y)m(dy) \right)m(dz).
\]
For $h \in M$, denote $\|h\| := \int_{x \in \mathcal{X}} h(x) m(dx)$ as the $L^1$-norm on $M$ with respect to $m$. Then the likelihood $p_n(\xi_0, \xi_1, \ldots, \xi_n; \theta)$ can be represented as

$$p_n(\xi_0, \xi_1, \ldots, \xi_n; \theta) = \int_{x_0 \in \mathcal{X}} \cdots \int_{x_n \in \mathcal{X}} \pi_\theta(x_0)f(\xi_0; \theta|x_0)$$

$$\times \prod_{j=1}^{n} p_\theta(x_{j-1}, x_j)$$

$$\times f(\xi_j; \theta|x_j, \xi_{j-1}) m(dx_n) \cdots m(dx_0) = \|P_\theta(\xi_n) \circ \cdots \circ P_\theta(\xi_1) \circ P_\theta(\xi_0)\pi_\theta\|.$$  

Note that, for $j = 1, \ldots, n$, the integrand $p_\theta(x, y)f(\xi_j; \theta|x, \xi_{j-1})$ of $P_\theta(\xi_j)$ in (2.6) and (2.8) represents $X_{j-1} = x$ and $X_j \in dy$, and $\xi_j$ is a Markov chain with transition probability density $f(\xi_j; \theta|y, \xi_{j-1})$ for given $X$. By definition (2.1), $\{(X_n, \xi_n), n \geq 0\}$ is a Markov chain, and this implies that $P_\theta(\xi_j)$ is a sequence of Markovian iterated random functions systems (see Section 5 for a formal definition). Therefore, by representation (2.8), $p_n(\xi_0, \xi_1, \ldots, \xi_n; \theta)$ is the $L_1$-norm of a Markovian iterated random functions system.

3. Ergodic theorems for a Markovian iterated random functions system.

To analyze the asymptotic properties of efficient likelihood estimators in state space models, in this section we study the ergodic theorem and the strong law of large numbers for a Markovian iterated random functions system. The Markovian iterated random functions system is a generalization of an iterated random functions system, in which the random functions are driven by a Markov chain. For a general account of an iterated random functions system, the reader is referred to [13] for a recent survey.

For simplicity in our notation, let $\{Y_n, n \geq 0\}$ [instead of $\{(X_n, \xi_n), n \geq 0\}$ in Section 2] be a Markov chain on a general state space $\mathcal{Y}$ with $\sigma$-algebra $\mathcal{A}$, which irreducible with respect to a maximal irreducibility measure on $(\mathcal{Y}, \mathcal{A})$ and is aperiodic. The transition kernel is denoted by $P(y, A)$. Let $(M, d)$ be a complete separable metric space with Borel $\sigma$-algebra $\mathcal{B}(M)$. Denote by $M_0$ a random variable which is independent of $\{Y_n, n \geq 0\}$. A sequence of the form

$$M_n = F(Y_n, M_{n-1}), \quad n \geq 1,$$

taking values in $(M, d)$ is called a Markovian iterated random functions system (MIRFS) of Lipschitz functions providing the following:

1. $\{Y_n, n \geq 0\}$ is a Markov chain taking values in a second countable measurable space $(\mathcal{Y}, \mathcal{A})$, with transition probability kernel $P(\cdot, \cdot)$ and stationary probability $\pi$, and $M_0$ is a random element on a probability space $(\Omega, \mathcal{F}, P)$, which is independent of $\{Y_n, n \geq 0\}$;
F : (Y \times M, \mathcal{A} \otimes \mathcal{B}(M)) \to (M, \mathcal{B}(M)) is jointly measurable and Lipschitz continuous in the second argument.

Clearly, \{ (Y_n, M_n), n \geq 0 \} constitutes a Markov chain with state space \( Y \times M \) and transition probability kernel \( P \), given by

\[
P((y, u), A \times B) := \int_{z \in A} I_{B}(F(z, u))P(y, dz)
\]

for all \( y \in Y, u \in M, A \in \mathcal{A} \) and \( B \in \mathcal{B}(M) \), where \( I \) denotes the indicator function. The \( n \)-step transition kernel is denoted \( P^n \). For \( (y, u) \in Y \times M \), let \( P_{yu} \) be the probability measure on the underlying measurable space under which \( Y_0 = y, M_0 = u \) a.s. The associated expectation is denoted \( E_{yu} \), as usual. For an arbitrary distribution \( \nu \) on \( Y \times M \), we put \( P_{\nu}(\cdot) := \int P_{yu}(\cdot) \nu(dy \times du) \) with associated expectation \( E_{\nu} \). We use \( P \) and \( E \) for probabilities and expectations, respectively, that do not depend on the initial distribution.

Let \( M_0 \) be a dense subset of \( M \) and \( \mathcal{M}(M_0, M) \) the space of all mappings \( h : M_0 \to M \) endowed with the product topology and product \( \sigma \)-algebra. Then the space \( \mathcal{L}_{\text{Lip}}(M, M) \) of all Lipschitz continuous mappings \( h : M \to M \) properly embedded forms a Borel subset of \( \mathcal{M}(M_0, M) \), and the mappings

\[
\mathcal{L}_{\text{Lip}}(M, M) \times M \ni (h, u) \mapsto h(u) \in M,
\]

\[
\mathcal{L}_{\text{Lip}}(M, M) \ni h \mapsto l(h) := \sup_{u \neq v} \frac{d(h(u), h(v))}{d(u, v)}
\]

are Borel; see Lemma 5.1 in [13] for details. Hence,

\[
L_n := l(F(Y_n, \cdot)), \quad n \geq 0,
\]

are also measurable and form a sequence of Markovian dependent random variables.

An important point to characterize the limit in the ergodic theorem will be the right use of the idea of duality. For this purpose, we introduce a time-reversed (or dual) Markov chain \( \{ \tilde{Y}_n, n \geq 0 \} \) of \( \{ Y_n, n \geq 0 \} \) as follows. Assume that there exists a \( \sigma \)-finite measure \( m \) on \( (Y, \mathcal{A}) \) such that the probability measure \( P \) on \( (Y, \mathcal{A}) \) defined by \( P(A) = P(Y_1 \in A | Y_0 = y) \) is absolutely continuous with respect to \( m \), so that \( P(A) = \int_A p(y, z)m(dz) \) for all \( A \in \mathcal{A} \), where \( p(y, \cdot) = dP/dm \). The Markov chain \( \{ Y_n, n \geq 0 \} \) is assumed to have an invariant probability measure \( \pi \) which has a positive probability density function \( \pi \) (without any confusion, we still use the same notation) with respect to \( m \). We shall use \( \sim \) to refer to the time-reversed (or dual) process \( \{ \tilde{Y}_n, n \geq 0 \} \) with transition probability density

\[
\tilde{p}(z, y) = p(y, z)\pi(y)/\pi(z).
\]
Denote \( \tilde{P} \) as the corresponding probability. It is easy to see that both \( Y_n \) and \( \tilde{Y}_n \) have the same stationary distribution \( \pi \). In this section we will assume that the initial distribution of \( Y_0 \) is the stationary distribution \( \pi \).

In the following, we write \( F_n(u) \) for \( F(Y_n, u) \). For all \( 1 \leq k \leq n \), let \( F_{k:n} := F_k \circ \cdots \circ F_n, F_{n:k} := F_n \circ \cdots \circ F_k \), where \( \circ \) denotes the composition of functions. Denote \( F_{n:n-1} \) as the identity on \( \mathbb{M} \). Hence

\[
M_n = F_n(M_{n-1}) = F_{n:1}(M_0)
\]

for all \( n \geq 0 \). Closely related to these forward iterations, and in fact a key tool to the analysis of the ergodic property, is the sequence of backward iterations

\[
\tilde{M}_n := F_{1:n}(M_0), \quad n \geq 0.
\]

The connection is established by the identity

\[
\pi(y)P(M_n \in \cdot \mid Y_0 = y) = \pi(z)\tilde{P}(\tilde{M}_n \in \cdot \mid \tilde{Y}_0 = z)
\]

for all \( n \geq 0 \). Put also \( M_n^u := F_{n:1}(u) \) and \( \tilde{M}_n^u := F_{1:n}(u) \) for \( u \in \mathbb{M} \) and note that

\[
\int_{z \in \mathcal{Y}} \int_{y \in \mathcal{Y}} P((M_n^u, \tilde{M}_n^u)_{n \geq 0} \in \cdot \mid Y_0 = y, \tilde{Y}_0 = z) \pi(dy)\pi(dz)
\]

\[
= \int_{z \in \mathcal{Y}} \int_{y \in \mathcal{Y}} P((M_n, \tilde{M}_n)_{n \geq 0} \in \cdot \mid Y_0 = y, \tilde{Y}_0 = z) \pi(dy)\pi(dz).
\]

Note that in (3.8), the probability \( P \) denotes a joint probability.

\( \{Y_n, n \geq 0\} \) is called Harris recurrent if there exist a set \( A \in \mathcal{A} \), a probability measure \( \Gamma \) concentrated on \( A \) and an \( \epsilon \) with \( 0 < \epsilon < 1 \) such that \( P^\epsilon(Y_n \in A \text{ i.o.}) = 1 \) for all \( y \in \mathcal{Y} \) and, furthermore, there exists \( n \) such that \( P^n(y, A') \geq \epsilon \Gamma(A') \) for all \( y \in A \) and all \( A' \in \mathcal{A} \).

A central question for an MIRFS \( (M_n)_{n \geq 0} \) is under which conditions it stabilizes, that is, converges to a stationary distribution \( \Pi \). The next theorem summarizes the results regarding this question.

**Theorem 1.** Let \( \{Y_n, n \geq 0\} \) be an aperiodic, irreducible and Harris recurrent Markov chain, and let \( (M_n)_{n \geq 0} \) be an MIRFS of Lipschitz functions. Suppose the initial distribution of \( Y_0 \) is \( \pi \), and

\[
E \log l(F_1) < 0 \quad \text{and} \quad E \log^+ d(F_1(u_0), u_0) < \infty
\]

for some \( u_0 \in \mathbb{M} \). Then the following assertions hold:

(i) \( \tilde{M}_n \) converges a.s. to a random element \( \tilde{M}_\infty \) which does not depend on the initial distribution.

(ii) \( M_n \) converges in distribution to \( \tilde{M}_\infty \) under \( \tilde{P} \).
(iii) Define $\Pi$ as the stationary distribution of $(\hat{Y}_\infty, \hat{M}_\infty)$. Then $\Pi$ is the unique stationary probability of the Markov chain $\{(Y_n, M_n), n \geq 0\}$.

(iv) $(M_n)_{n \geq 0}$ is ergodic under $P_\Pi$, that is, for any $u \in M$,

$$\frac{1}{n} \sum_{k=1}^{n} g(M_k) \rightarrow E_\Pi(g(\hat{M}_\infty)), \quad P_\Pi \text{-a.s.}$$

for all bounded continuous real-valued functions $g$ on $M$.

We remark that Elton [18] showed in the situation of a stationary sequence $(F_n)_{n \geq 1}$ that Theorem 1 holds whenever $E \log^+ l(F_1)$ and $E \log^+ d(F_1(u_0), u_0)$ are both finite for some (and then all) $u_0 \in M$ and the Lyapunov exponent $\gamma := \lim_{n \to \infty} n^{-1} \log l(F_{n:1})$, which exists by Kingman’s subadditive ergodic theorem, is a.s. negative. Since the initial distribution of $Y_0$ is the stationary distribution $\pi$, the Markov chain $Y_n$ is a stationary sequence, and hence, $M_n$ is a sequence of iterated random functions generated by stationary sequences. Here, we impose the Harris recurrent condition so that the invariant measure $\pi$ exists, and we are able to characterize $\hat{M}_\infty$ in a Markovian setting. Since the proof is similar to that in [2], it is omitted.

4. Central limit theorem and Edgeworth expansion for distributions of a Markovian iterated random functions system. Consider the Markovian iterated random functions system $\{(Y_n, M_n), n \geq 0\}$ defined in (3.1). Abuse the notation a little bit and let $g$ be an $R^p$-valued function on $M$. In this section we study the central limit theorem and Edgeworth expansion of the sum $S_n = \sum_{k=1}^{n} g(M_k)$ and $g(n^{-1} S_n)$ for a smooth function $g: R^p \to R^q$. Let $w: Y \to [1, \infty)$ be a measurable function, and let $B$ be the Banach space of measurable functions $h: Y \to C$ ($= \text{the set of complex numbers}$) with $\|h\|_w := \sup_y |h(y)|/w(y) < \infty$. Assume further that $\{Y_n, n \geq 0\}$ has a stationary distribution $\pi$ with $\int w(y) \pi(dy) < \infty$, and

$$\lim_{n \to \infty} \sup_y \left\{ \left| E[h(Y_n)|Y_0 = y] - \int h(z) \pi(dz) \right|/w(y) : y \in Y, |h| \leq w \right\} = 0,$$

$$\sup_y \{E[w(Y_0)|Y_0 = y]/w(y)\} < \infty,$$

for some $p \geq 1$. Condition (4.1) says that the chain is $w$-uniformly ergodic, which implies that there exist $\gamma > 0$ and $0 < \rho < 1$ such that, for all $h \in B$ and $n \geq 1$,

$$\sup_y \left| E[h(Y_n)|Y_0 = y] - \int h(z) \pi(dz) \right|/w(y) \leq \gamma \rho^n \|h\|_w,$$

(cf. pages 382–383 and Theorem 16.0.1 of [46]). We remark that, for $w = 1$, condition (4.1) is the classical uniform ergodicity condition for $\{Y_n, n \geq 0\}$.

The following assumption will be assumed throughout this section.
Assumption K.

K1. Let \( \{Y_n, n \geq 0\} \) be an aperiodic, irreducible Markov chain satisfying conditions (4.1)–(4.2). Furthermore, we assume the initial distribution of \( Y_0 \) is \( \pi \).

K2. The MIRFS \( (M_n)_{n \geq 0} \) has the weighted mean contraction property, that is, there exists a \( p \geq 1 \) such that

\[
\sup_y \left\{ \mathbb{E} \left( \log \frac{L^p w(Y_p)}{w(y)} \right) \bigg| Y_0 = y \right\} < 0.
\]

K3. There exists \( u_0 \in M \) for which

\[
\mathbb{E} d^2(F_1(u_0), u_0) < \infty \quad \text{and} \quad \sup_y \mathbb{E} \left( \frac{L^1 w(Y_1)}{w(y)} \bigg| Y_0 = y \right) < \infty.
\]

Remark 1. (a) Assumption K1 is a condition for the underlying Markov chain \( \{Y_n, n \geq 0\} \) which is general enough to include several practical used models studied in Section 6. Assumption K2 is a weighted mean contraction condition which is different from the standard mean contraction condition \( \mathbb{E} \log L_1 < 0 \) used in Theorem 1. Assumption K3 is a weighted moment condition. Note that under Assumptions K1–K3, and the extra assumption that \( \{(Y_n, M_n), n \geq 0\} \) is an irreducible, aperiodic and Harris recurrent Markov chain, Theorems 13.0.1 and 17.0.1(i) of [46] imply that Theorem 1 still holds.

Furthermore, we will prove the central limit theorem and Edgeworth expansion for the distributions of a Markovian iterated random functions system in Theorem 2.

(b) To have better understanding of Assumption K, we consider a simple state space model. Given \( p \geq 1 \) as in Assumption K2, and \( |\alpha| < 1 \), let \( Y_n = \alpha Y_{n-1} + \varepsilon_n, \xi_n = \beta Y_{n-1} + \eta_n \), where \( \varepsilon_n \) and \( \xi_n \) are i.i.d. random variables with \( E[\varepsilon_1] = c < \infty \), and \( \eta_n \) are i.i.d. random variables with \( E[\eta_1] < \infty \). Further, we assume both \( \varepsilon_1 \) and \( \eta_1 \) have positive probability density function with respect to Lebesgue measure, and that they are mutually independent. Denote \( b = (1 - |\alpha|^p)/(1 - |\alpha|) \) and \( a = 1/(bc + 1) < 1 \), and assume \( |\beta| < a^{1/p} < 1 \) for all \( y \in Y \). It is known that \( w(y) = |y| + 1 \) (cf. pages 380 and 383 of [46]). Let \( d(u, v) = |u - v| \). It is easy to see that Assumption K1 and the first part of Assumption K3 hold. To check Assumption K2, we have

\[
\begin{align*}
\sup_y \mathbb{E} \left( \log \frac{L^p w(Y_p)}{w(y)} \bigg| Y_0 = y \right) \\
= \sup_y \mathbb{E} \left( \log \frac{\beta_2 \cdots \beta_1 (|\alpha^p y + \sum_{k=0}^{p-1} \alpha^k \varepsilon_{p-k}| + 1)}{|y| + 1} \bigg| Y_0 = y \right) \\
< \log \sup_y \left\{ a(|\alpha^p y| + E[\sum_{k=0}^{p-1} \alpha^k \varepsilon_{p-k}] + 1) \right\}
\end{align*}
\]

\[\text{(4.4)}\]
\[ C.-D. \text{ Fuh} = \log \sup_y \left\{ \frac{a(|\alpha^p y| + bc + 1)}{|y| + 1} \right\} = 0. \]

By using the same argument, we have the second part of Assumption K3. When \( \varepsilon_n \) are i.i.d. \( N(0, 1) \), \( \eta_n \) are i.i.d. \( N(0, 1) \), and they are mutually independent. Then \( a = \sqrt{2\pi}/(2b + \sqrt{2\pi}) < 1. \)

Recall that \( \Pi \) is defined in Theorem 1(iii) and denote \( \Phi(B) := \Pi(Y \times B) \) for all \( B \in \mathcal{B}(M) \). Let \( g \in L^2_0(Q) \) be a square integrable function taking values in \( \mathbb{R}^p \) with mean 0, that is, \( g = (g_1, \ldots, g_p) \) with each \( g_k \) a real-valued function on \( M \), and

\[ \int_M g_k(u)Q(du) = 0, \quad \|g_k\|_2^2 = \int_M g_k^2(u)Q(du) < \infty, \]

for \( k = 1, \ldots, p \). Consider the sequence

\[ S_n = S_n(g) = g(M_1) + \cdots + g(M_n), \quad n \geq 1, \]

which may be viewed as a Markov random walk on the Markov chain \( \{(Y_n, M_n), n \geq 0\} \).

Note that there are two special properties of the Markov chain induced by the Markovian iterated random functions system (2.4)–(2.7). First, the hypothesis that the transition probability possesses a density leads to a classical situation in the context of the so-called “Doeblin condition” for Markov chains. Second, a positivity hypothesis on \( M \) defined in (2.4) in the support of the Markov chain leads to contraction properties, on which basis we will develop the spectral theory. The reader is referred to [37] for a general account of the perturbation theory of Markovian operators. We need the following notation first.

**Definition 2.** Let \( w : \mathcal{Y} \to [1, \infty) \) be a weight function. For any measurable function \( \varphi : \mathcal{Y} \times M \to [1, \infty) \), given \( u_0 \in M \), define

\[ \|\varphi\|_w := \sup_{y \in \mathcal{Y}, u \in M} \frac{\varphi(y, u)}{w(y)} \]

and

\[ \|\varphi\|_h := \sup_{y \in \mathcal{Y}, u, v : 0 < d(u, v) \leq 1} \frac{|\varphi(y, u) - \varphi(y, v)|}{(w(y)d(u, v))^\delta}, \]

for \( 0 < \delta < 1 \). We define \( \mathcal{H} \) as the set of \( \varphi \) on \( \mathcal{Y} \times M \) for which \( \|\varphi\|_{wh} := \|\varphi\|_w + \|\varphi\|_h \) is finite, where \( wh \) represents a combination of the weighted variation norm and the bounded weighted Hölder norm.

Let \( \nu \) be an initial distribution of \((Y_0, M_0)\) and let \( E_\nu \) denote expectation under the initial distribution \( \nu \) on \((Y_0, M_0)\). For \( \varphi \in \mathcal{H}, g \in L^2(Q), y \in \mathcal{Y}, \)
$u \in \mathbb{M}$ and $p \times 1$ vectors $\alpha = (\alpha_1, \ldots, \alpha_p)' \in \mathbb{R}^p$, define linear operators $T_\alpha$, $T$, $\nu_\alpha$ and $Q$ on the space $\mathcal{H}$ as

\begin{align*}
(4.7) \quad & (T_\alpha \varphi)(y, u) = \mathbb{E}\{e^{i\alpha'g(M_1)} \varphi(Y_1, M_1) | Y_0 = y, M_0 = u\}, \\
(4.8) \quad & (T \varphi)(y, u) = \mathbb{E}\{\varphi(Y_1, M_1) | Y_0 = y, M_0 = u\}, \\
(4.9) \quad & \nu_\alpha \varphi = \mathbb{E}_\nu\{e^{i\alpha'\nu(u)} \varphi(Y_0, u)\}, \quad Q \varphi = \mathbb{E}_\Pi\{\varphi(Y_0, u)\}.
\end{align*}

In the case of a $w$-uniformly ergodic Markov chain, Fuh and Lai [26] have shown that there exists a sufficiently small $\delta > 0$ such that, for $|\alpha| \leq \delta$, $\mathcal{H} = \mathcal{H}_1(\alpha) \oplus \mathcal{H}_2(\alpha)$ and

\begin{equation}
(4.10) \quad T_\alpha Q_\alpha \varphi = \lambda(\alpha) Q_\alpha \varphi \quad \text{for all } \varphi \in \mathcal{H},
\end{equation}

where $\mathcal{H}_1(\alpha)$ is a one-dimensional subspace of $\mathcal{H}$, $\lambda(\alpha)$ is the eigenvalue of $T_\alpha$ with corresponding eigenspace $\mathcal{H}_1(\alpha)$ and $Q_\alpha$ is the parallel projection of $\mathcal{H}$ onto the subspace $\mathcal{H}_1(\alpha)$ in the direction of $\mathcal{H}_2(\alpha)$. Extension of their argument to the weight functions $w$ and $l$ defined in Definition 2 is given in the Appendix, which also proves the following lemmas.

**Lemma 1.** Let $\{(Y_n, M_n), n \geq 0\}$ be the MIRFS of Lipschitz functions defined in (2.1) and satisfying Assumption K. Assume $g \in \mathcal{L}^r(Q)$ for some $r > 2$. Then $T$ and $Q$ are bounded linear operators on the Banach space $\mathcal{H}$ with norm $\| \cdot \|_{wh}$, and satisfy

\begin{equation}
(4.11) \quad \| T^n - Q \|_{wh} = \sup_{\varphi \in \mathcal{H}, \| \varphi \|_{wh} \leq 1} \| T^n \varphi - Q \varphi \|_{wh} < \gamma_s \rho_s^n,
\end{equation}

for some $\gamma_s > 0$ and $0 < \rho_s < 1$.

By using an argument similar to Proposition 1 of [24], we have the following:

**Lemma 2.** Let $\{(Y_n, M_n), n \geq 0\}$ be the MIRFS defined in (2.1) satisfying Assumption K, such that the induced Markov chain $\{(Y_n, M_n), n \geq 0\}$ with transition probability kernel (3.2) is irreducible, aperiodic and Harris recurrent. Assume $g \in \mathcal{L}^r(Q)$ for some $r > 2$. Then there exists $\delta > 0$ such that, for $\alpha \in \mathbb{R}^p$ with $|\alpha| < \delta$, and for $\varphi \in \mathcal{H},$

\begin{equation}
(4.12) \quad \mathbb{E}_\nu\{e^{i\alpha'g(M_n)} \varphi(Y_n, M_n)\} = \nu_\alpha T^n_\alpha \varphi = \nu_\alpha T^n_\alpha \{Q_\alpha + (I - Q_\alpha)\} \varphi
\end{equation}

\begin{equation*}
= \lambda^n(\alpha) \nu_\alpha Q_\alpha \varphi + \nu_\alpha Q^n_\alpha (I - Q_\alpha) \varphi,
\end{equation*}

and:

(i) $\lambda(\alpha)$ is the unique eigenvalue of the maximal modulus of $T_\alpha$;
(ii) $Q_\alpha$ is a rank-one projection;
(iii) \( \text{the mappings } \lambda(\alpha), Q_\alpha \text{ and } I - Q_\alpha \text{ are analytic;} \)

(iv) \( |\lambda(\alpha)| > \frac{2 + p_*}{3} \) and for each \( k \in \mathbb{N} \), the set of positive integers, there exists \( c > 0 \) such that, for each \( n \in \mathbb{N} \) and \( j_1, \ldots, j_p \) with \( j_1 + \cdots + j_p = k \),

\[
\left\| \frac{\partial^k}{\partial \alpha_1^{j_1} \cdots \partial \alpha_p^{j_p}} (I - Q_\alpha)^n \right\|_{wh} \leq c \left( \frac{1 + 2p_*}{3} \right)^n ;
\]

(v) denote \( g = (g_1, \ldots, g_p) \), and let \( \gamma_j := \lim_{n \to \infty} (1/n) E_{y,u} \log \|g_j(M_n)\| \), the upper Lyapunov exponent; it follows that

\[
(4.13) \quad \gamma_j = \left. \frac{\partial \lambda(\alpha)}{\partial \alpha_j} \right|_{\alpha=0} = \int E_{y,u} g_j(M_1) \Pi(dy \times du).
\]

Note that in Lemma 2 we need the extra assumption that the induced Markov chain \( \{(Y_n, M_n), n \geq 0\} \) with transition probability kernel (3.2) is irreducible, aperiodic and Harris recurrent. In Section 5 we will show that this condition is satisfied for the Markov chain induced by the Markovian iterated random functions system (2.4)–(2.7).

For given \( S_n = \sum_{k=1}^{n} g(M_k) \) of the MIRFS \( \{(Y_n, M_n), n \geq 0\} \), in this section we will obtain Edgeworth expansions for the standardized distribution of \( S_n \) via the representation (4.12) of the characteristic function \( E(e^{iu^T g(M_n)}) | Y_0 = y, M_0 = 0 \).

Note that Lemma 1 implies that \( \{(Y_n, M_n), n \geq 0\} \) is geometrically mixing in the sense that there exist \( r_1 > 0 \) and \( 0 < \gamma_1 < 1 \) such that, for all \( y \in \mathcal{Y}, u \in \mathcal{M}, k \geq 0 \) and \( n \geq 1 \) and for all measurable functions \( \varphi_1, \varphi_2 \) with \( \| \varphi_1 \|_{wh} < \infty \) and \( \| \varphi_2 \|_{wh} < \infty \),

\[
\| E\{ \varphi_1(Y_k, M_k) \varphi_2(Y_{k+n}, M_{k+n}) | Y_0 = y, M_0 = u \} - \{ E\varphi_1(Y_k, M_k) | Y_0 = y, M_0 = u \} \times \{ E\varphi_2(Y_{k+n}, M_{k+n}) | Y_0 = y, M_0 = u \} \|_{wh} \leq r_1 \gamma_1^n .
\]

Let \( \bar{\varphi}_1, \bar{\varphi}_2 \) be real-valued measurable functions on \( (\mathcal{Y} \times \mathcal{M}) \times (\mathcal{Y} \times \mathcal{M}) \). Denote \( \varphi_1(z, v) = E\{ \bar{\varphi}_1((z, v), (Y_1, M_1)) | Y_0 = z, M_0 = v \} \), and note that

\[
E\{ \bar{\varphi}_1((Y_k, M_k), (Y_{k+1}, M_{k+1})) | Y_0 = y, M_0 = u \}
= E\{ \varphi_1(Y_k, M_k) | Y_0 = y, M_0 = u \}.
\]

The same proof as that of Theorem 16.1.5 of [46] can be used to show that there exist \( r_1 > 0 \) and \( 0 < \gamma_1 < 1 \) such that, for all \( y \in \mathcal{Y}, u \in \mathcal{M}, k \geq 0 \) and \( n \geq 1 \) and for all measurable \( \bar{\varphi}_1, \bar{\varphi}_2 \) with \( \| \sup_{z,v} \bar{\varphi}_1((z, v)) \|_{wh} < \infty \) and \( \| \sup_{z,v} \bar{\varphi}_2((y, u)) \|_{wh} < \infty \),

\[
\| E\{ \bar{\varphi}_1((Y_k, M_k), (Y_{k+1}, M_{k+1})) \	imes \bar{\varphi}_2((Y_{k+n}, M_{k+n}), (Y_{k+n+1}, M_{k+n+1})) | Y_0 = y, M_0 = u \}
- E\{ \varphi(Y_k, M_k) | Y_0 = y, M_0 = u \} E\{ \varphi_2(Y_{k+n}, M_{k+n}) | Y_0 = y, M_0 = u \} \|_{wh}
\leq r_1 \gamma_1^{n-1}.
\]
To establish Edgeworth expansion for a Markovian iterated random functions system, we shall make use of (4.15) in conjunction with the following extension of Cramér (strongly nonlattice) condition:

\[ \inf_{|v|>|\alpha|} |1 - E_{\pi}\{\exp(iv' S_1(g))\}| > 0 \quad \text{for all } \alpha > 0. \]  

In addition, we also assume the conditional Cramér (strongly nonlattice) condition ((2.5) on page 216 in [31]): There exists \( \delta > 0 \) such that, for all \( m, n = 1, 2, \ldots, \delta^{-1} < m < n \), and all \( \alpha \in \mathbb{R}^p \) with \( |\alpha| \geq \delta \),

\[
E_{\pi}|E\{\exp(i\alpha'(g(M_{n-m}) + \cdots + g(M_{n+m}))\)}
\]
\[
|\{ (Y_{n-m}, M_{n-m}), \ldots, (Y_{n-1}, M_{n-1}),
\]
\[
(Y_{n+1}, M_{n+1}), \ldots, (Y_{n+m}, M_{n+m}), (Y_{n+m+1}, M_{n+m+1})\}| \leq e^{-\delta}.
\]

Let

\[
\gamma = \int E_{\nu}(dy_1)\Pi(dy \times du)\quad \text{for } \lambda(0),
\]

and denote by \( V = (\partial^2 \lambda(\alpha)/\partial \alpha_i \partial \alpha_j)_{|\alpha=0}|_{1 \leq i, j \leq p} \) the Hessian matrix of \( \lambda \) at 0. By Lemma 2,

\[
\lim_{n \to \infty} n^{-1}E_{\nu}\{(g(M_n) - n\gamma)(g(M_n) - n\gamma)'\} = V.
\]

Let \( \psi_n(\alpha) = E_{\nu}(e^{i\alpha'g(M_n)}) \). Then by Lemma 2 and the fact that \( \nu, Q_\alpha, h_1 \) has continuous partial derivatives of order \( r - 2 \) in some neighborhood of \( \alpha = 0 \), we have the Taylor series expansion of \( \psi_n(\alpha/\sqrt{n}) \) for |\( \alpha/\sqrt{n} \)| \( \leq \varepsilon \) (some sufficiently small positive number):

\[
\psi_n(\alpha/\sqrt{n})\left\{1 + \sum_{j=1}^{r-2} n^{-j/2} \hat{\pi}_j(i\alpha)\right\} e^{-\alpha'V\alpha/2} + o(n^{-(r-2)/2}),
\]

where \( \hat{\pi}_j(i\alpha) \) is a polynomial in \( i\alpha \) of degree \( 3j \) whose coefficients are smooth functions of the partial derivatives of \( \lambda(\alpha) \) at \( \alpha = 0 \) up to the order \( j + 2 \) and those of \( \nu, Q_\alpha, h_1 \) at \( \alpha = 0 \) up to the order \( j \). Letting \( D \) denote the \( p \times 1 \) vector whose \( j \)th component is the partial differentiation operator \( D_j \) with respect to the \( j \)th coordinate, define the differential operator \( \hat{\pi}_j(-D) \). As in the case of sums of i.i.d. zero-mean random vectors (cf. [5]), we obtain an Edgeworth expansion for the “formal density” of the distribution of \( g(M_n) \) by replacing the \( \hat{\pi}_j(i\alpha) \) and \( e^{-\alpha'V\alpha/2} \) in (4.20) by \( \hat{\pi}_j(-D) \) and \( \phi_V(y) \), respectively, where \( \phi_V \) is the density function of the \( q \)-variate normal distribution with mean 0 and covariance matrix \( V \). Throughout the sequel we let \( P_\nu \) denote the probability measure under which \( (Y_0, M_0) \) has initial distribution \( \nu \).
Theorem 2. Let \( \{(Y_n, M_n), n \geq 0\} \) be the MIRFS defined in (2.1) satisfying Assumption K, such that the induced Markov chain \( \{(Y_n, M_n), n \geq 0\} \), with transition probability kernel (3.2), is irreducible, aperiodic and Harris recurrent. Assuming \( g \in L^r(Q) \) for some \( r > 2 \), (4.16) and (4.17) hold. Let \( \phi_{j,V} = \tilde{\pi}_j(-D)\phi_V \) for \( j = 1, \ldots, r - 2 \). For \( 0 < a \leq 1 \) and \( c > 0 \), let \( B_{a,c} \) be the class of all Borel subsets \( B \) of \( \mathbb{R}^p \) such that \( \int_{(\partial B)^\varepsilon} \phi_V(y) \, dy \leq c \varepsilon^a \) for every \( \varepsilon > 0 \), where \( \partial B \) denotes the boundary of \( B \) and \( (\partial B)^\varepsilon \) denotes its \( \varepsilon \)-neighborhood. Then

\[
\sup_{B \in B_{a,c}} \left| P_{\nu}\{(S_n - n\gamma)/\sqrt{n} \in B\} - \int_B \left\{ \phi_V(y) + \sum_{j=1}^{r-2} n^{-j/2}\phi_{j,V}(y) \right\} \, dy \right| = o(n^{-(r-2)/2}).
\] (4.21)

A proof of Theorem 2 is given in the Appendix.

Note that under weaker moment conditions, and an alternative condition of (4.16) and (4.17) (see Condition 1 of [42]) Lahiri [42] proved the asymptotic expansions for sums of weakly dependent random vectors.

Letting \( r = 2 \) in Theorem 2, we have the following:

Corollary 1. With the same notation and assumptions as in Theorem 2, then

\[
\frac{1}{\sqrt{n}}(S_n - n\gamma) \to N(0, \Sigma) \quad \text{in distribution},
\]

where the variance–covariance matrix

\[
\Sigma = \left( \frac{\partial^2 \lambda(\alpha)}{\partial \alpha_i \partial \alpha_j} \right)_{i,j=1,\ldots,p}.
\] (4.22)

In statistical applications one often works with \( g(n^{-1}S_n) \) instead of \( S_n = \sum_{k=1}^n g(M_k) \), where \( g: \mathbb{R}^p \to \mathbb{R}^q \) is sufficiently smooth in some neighborhood of the mean \( \gamma := (\gamma_1, \ldots, \gamma_p) \). Denote \( g = (g_1, \ldots, g_q) \) with each \( g_i, 1 \leq i \leq q \), a real-valued function on \( \mathbb{R}^p \). For the case of a sum of i.i.d. random variables, Bhattacharya and Ghosh [4] made use of the Edgeworth expansion of the distribution of \( (S_n - n\gamma)/\sqrt{n} \) to derive an Edgeworth expansion of the distribution of \( \sqrt{n}\{g(n^{-1}S_n) - g(\gamma)\} \). Making use of Theorem 2 and a straightforward extension of their argument, we can generalize their result to the case where \( S_n \) is the partial sum of a Markovian iterated random functions system.

Theorem 3. Under the same assumptions as in Theorem 2, suppose that \( g: \mathbb{R}^p \to \mathbb{R}^q \) has continuous partial derivatives of order \( r \) in some neighborhood of \( \gamma \). Let \( \mathbf{J}_g = (D\mathbf{g}_i(\gamma))_{1 \leq i \leq q, 1 \leq j \leq p} \) be the \( q \times p \) Jacobian matrix and
let $V(g) = J_gVJ_g'$. Then
\[
\sup_{B \in B_{a,c}} \left| P_\nu \{ \sqrt{n}(g^{-1}S_n - g(\gamma)) \in B \} - \int_B \left( \phi_{V(g)}(y) + \sum_{j=1}^{r-2} n^{-j/2} \phi_{j,V,g}(y) \right) dy \right| = o(n^{-(r-2)/2}),
\]
where $\phi_{j,V,g} = \tilde{\pi}_{j,g}(-D)\phi_V$ and $\tilde{\pi}_{j,g}(y)$ is a polynomial in $y(\in \mathbb{R}^p)$ whose coefficients are smooth functions of the partial derivatives of $\lambda(\alpha)$ at $\alpha = 0$ up to order $j+2$ and those of $\nu_0\mathcal{Q}_0h_1$ at $\alpha = 0$ up to order $j$ together with those of $g$ at $\mu$ up to order $j+1$.

In the next theorem we consider $p = 1$.

**Theorem 4.** Under the same assumptions as in Theorem 2, assume $g \in L^r(Q)$ for some $r > 2$. Then
\[
\frac{1 - P_\nu\{(S_n - n\gamma)/\sqrt{n} \leq t\}}{1 - \Phi(t)} = \exp(t^3/\sqrt{n})\varphi(t/\sqrt{n}) \left( 1 + O\left( \frac{t}{\sqrt{n}} \right) \right),
\]
and
\[
\frac{P_\nu\{(S_n - n\gamma)/\sqrt{n} \leq -t\}}{\Phi(-t)} = \exp(-t^3/\sqrt{n})\varphi(-t/\sqrt{n}) \left( 1 + O\left( \frac{t}{\sqrt{n}} \right) \right),
\]
where $\Phi(t)$ is the standard normal distribution, and $\varphi(t)$ is a power series which converges for $t$ sufficiently small in absolute value.

Theorem 4 states the moderate deviations results for the distribution of an MIRFS, which will be used to prove Edgeworth expansion for the MLE in Section 5. Since the proof is a straightforward generalization of Theorem 6 in [47], it will not be repeated here.

**5. Efficient likelihood estimation.** For a given state space model defined in (2.1) which involves several parameters $\theta = (\theta_1, \ldots, \theta_q)$, the estimation problem we consider in this section is the case of estimating one of the parameters at a time; the other parameters play the role of nuisance parameters. The true parameter is denoted by $\theta_0$. Recall $p_n = p_n(\xi_0, \xi_1, \ldots, \xi_n; \theta_0)$ defined as (2.3). When $\partial \log p_n/\partial \theta$ exists, one can seek solutions of the likelihood equations
\[
\frac{\partial \log p_n}{\partial \theta} = 0.
\]
In the following, we denote $E^θ_x$ as the expectation defined under $P^θ(\cdot,\cdot)$ in (2.1) with initial state $X_0 = x$, and $E^θ_{(x,s)}$ as the expectation defined under $P^θ(\cdot,\cdot)$ in (2.1) with initial state $X_0 = x, ξ_0 = s$. The following conditions will be used throughout the rest of this paper.

C1. For given $θ ∈ Θ$, the Markov chain $\{(X_n, ξ_n), n ≥ 0\}$ defined in (2.1) and (2.2) is aperiodic, irreducible, and satisfies (4.1) and (4.2) with weight function $w(\cdot)$. Assume $0 < p_θ(x,y) < ∞$ for all $x,y ∈ X$, and $0 < \sup_{x ∈ X} f(s_1; \theta|x,s_0) < ∞$, for all $s_0, s_1 ∈ \mathbb{R}^d$. Denote $g_θ(s_0, ξ_1) = \sup_{x ∈ X} \int p_θ(x_0, x_1) × f(ξ_1; \theta|x_1, s_0) m(dx_1)$. Furthermore, we assume that there exists $p ≥ 1$ as in Assumption K2 such that

$$\sup_{(x_0,s_0) ∈ X × \mathbb{R}^d} E^θ_{(x_0,s_0)} \left\{ \log \left( g_θ(s_0, ξ_1) \frac{w(x_p, ξ_p)}{w(x_0, s_0)} \right) \right\} < 0,$$

(5.2)

$$\sup_{(x_0,s_0) ∈ X × \mathbb{R}^d} E^θ_{(x_0,s_0)} \left\{ \frac{g_θ(s_0, ξ_1)}{w(x_0, s_0)} \right\} < ∞.$$

(5.3)

C2. The true parameter $θ_0$ is an interior point of $Θ$. For all $x ∈ X, s_0, s_1 ∈ \mathbb{R}^d, θ ∈ Θ ⊂ \mathbb{R}^q$, and for $i,j,k = 1,\ldots,q$, the partial derivatives

$$\frac{∂f(s_0; θ|x)}{∂θ_i}, \quad \frac{∂^2 f(s_0; θ|x)}{∂θ_i ∂θ_j}, \quad \frac{∂^3 f(s_0; θ|x)}{∂θ_i ∂θ_j ∂θ_k}$$

exist, as well as the partial derivatives

$$\frac{∂f(s_1; θ|x, s_0)}{∂θ_i}, \quad \frac{∂^2 f(s_1; θ|x, s_0)}{∂θ_i ∂θ_j}, \quad \frac{∂^3 f(s_1; θ|x, s_0)}{∂θ_i ∂θ_j ∂θ_k},$$

and for all $x, y ∈ X, \theta → p_θ(x,y)$ and $\theta → \pi_θ(x)$ have twice continuous derivatives in some neighborhood $N_δ(θ_0) := \{θ: |θ - θ_0| < δ\}$ of $θ_0$.

C3.

$$\int \sup_{X θ ∈ N_δ(θ_0)} \left| \frac{∂\pi_θ(x)}{∂θ_i} \right| m(dx) < ∞, \quad \int \sup_{X θ ∈ N_δ(θ_0)} \left| \frac{∂^2 \pi_θ(x)}{∂θ_i ∂θ_j} \right| m(dx) < ∞,$$

and for all $x ∈ X, i,j = 1,\ldots,q$,

$$\int \sup_{X θ ∈ N_δ(θ_0)} \left| \frac{∂p_θ(x,y)}{∂θ_i} \right| m(dy) < ∞, \quad \int \sup_{X θ ∈ N_δ(θ_0)} \left| \frac{∂^2 p_θ(x,y)}{∂θ_i ∂θ_j} \right| m(dy) < ∞.$$

C4. For all $x ∈ X, s_0 ∈ \mathbb{R}^d$ and $θ ∈ Θ$,

$$E^θ_x \left| \frac{∂f(ξ_0; θ|x)}{∂θ_i} \right| < ∞, \quad E^θ_x \left| \frac{∂^2 f(ξ_0; θ|x)}{∂θ_i ∂θ_j} \right| < ∞,$$

$$E^θ_{(x,s_0)} \left| \frac{∂f(ξ_1; θ|x, s_0)}{∂θ_i} \right| < ∞, \quad E^θ_{(x,s_0)} \left| \frac{∂^2 f(ξ_1; θ|x, s_0)}{∂θ_i ∂θ_j} \right| < ∞.$$
Furthermore, we assume that, for all \( x \in \mathcal{X} \), \( s_0 \in \mathbb{R}^d \) and uniformly for \( \theta \in N_0(\theta_0) \),
\[
\left| \frac{\partial^3 \log f(\xi_0; \theta|x)}{\partial \theta_i \partial \theta_j \partial \theta_k} \right| < H_{ijk}(x, \xi_0), \quad \left| \frac{\partial^3 \log f(\xi_1; \theta|x, s_0)}{\partial \theta_i \partial \theta_j \partial \theta_k} \right| < G_{ijk}((x, s_0), \xi_1),
\]
where \( H_{ijk} \) and \( G_{ijk} \) are such that \( E_{x}^{\theta_0} H_{ijk}(x, \xi_0) < \infty \) and \( E_{(x,s_0)}^{\theta_0} G_{ijk}((x, s_0), \xi_1) < \infty \), for all \( i, j, k = 1, \ldots, q \) and for all \( x \in \mathcal{X}, s_0 \in \mathbb{R}^d \).

**Remark 2.** (a) Condition C1 is the \( w \)-uniform ergodicity condition for the underlying Markov chain, which is considerably weaker than the uniformly recurrent condition A1 of [39], and that of [14]. Furthermore, we impose conditions (5.2) and (5.3) to guarantee that the induced Markovian iterated random functions system satisfies Assumptions K2 and K3 in Section 4.

(b) To have better understanding of these properties, we first consider a simple state space model \( X_n = \alpha X_{n-1} + \varepsilon_n, \xi_n = X_n + \eta_n \), where \( |\alpha| < 1, \varepsilon_n \) and \( \eta_n \) are i.i.d. standard normal random variables, and they are mutually independent. Since \( \xi_n \) are independent for given \( X_n \), the weight function \( w \)
depends on \( X_0 \) only and we have \( w(x) = |x| + 1 \). Note that \( \mathcal{X} = \mathbb{R} \). Denote \( b = (1 - |\alpha|^p)/(1 - |\alpha|) \). Observe that

\[
\sup_{x \in \mathbb{R}} \int_{-\infty}^{\infty} \frac{\exp\{- (y - ax)^2/2\}}{\sqrt{2\pi}} \frac{\exp\{- (s - y)^2/2\}}{\sqrt{2\pi}} dy = \sup_{x \in \mathbb{R}} \sqrt{\frac{1}{2\pi}} \exp\{- (\alpha x - s)^2/4\} \times \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}(1/2)} \exp\{- (y - (\alpha x + s)/2)^2/2(1/2)\} dy
\]

\[
= \frac{\sqrt{1/2}}{\sqrt{2\pi}} \sup_{x \in \mathbb{R}} \exp\{- (\alpha x - s)^2/4\} = \frac{1}{\sqrt{4\pi}}.
\]

A simple calculation leads to

\[
\sup_{(x_0, s_0) \in \mathbb{R} \times \mathbb{R}} E_{(x_0, s_0)}^\alpha \left\{ \log \left( \frac{g(s_0, \xi_1)^p w(X_p, \xi_p)}{w(x_0, s_0)} \right) \right\}
\]

\[
< \log \sup_{x_0 \in \mathbb{R}} E_{x_0}^\alpha \left\{ \frac{|\alpha^p x_0 + \sum_{k=0}^{p-1} \alpha^k \varepsilon_{p-k}| + 1}{(4\pi)^{p/2}(|x_0| + 1)} \right\}
\]

\[
\leq \log \sup_{x_0 \in \mathbb{R}} \left\{ \frac{|\alpha^p x_0| + E_{x_0}^\alpha \sum_{k=0}^{p-1} \alpha^k \varepsilon_{p-k} + 1}{(4\pi)^{p/2}(|x_0| + 1)} \right\}
\]

\[
= \log \sup_{x_0 \in \mathbb{R}} \left\{ \frac{|\alpha^p x_0| + 2b + \sqrt{2\pi} + 1}{(4\pi)^{p/2}(|x_0| + 1)} \right\} < 0.
\]

This implies that (5.2) holds. By using the same argument, we see (5.3) holds.

Next, we consider the case that \( \varepsilon_n \) and \( \eta_n \) are i.i.d. double exponential(1) random variables. Observe that

\[
\sup_{x \in \mathbb{R}} \int_{-\infty}^{\infty} \frac{\exp\{- |y - ax|\}}{\sqrt{2}} \frac{\exp\{- |s - y|\}}{\sqrt{2}} dy
\]

\[
= \frac{1}{4} \sup_{x \in \mathbb{R}} \{(1 + |\alpha x - s|) \exp\{- |\alpha x - s|\} \} = \frac{1}{4}.
\]

By making use of the same argument as in (5.4), we see that (5.2) and (5.3) hold. The extension to \( \xi_n = \beta x_n \xi_n^{-1} + \eta_n \), studied in Remark 1(b), is straightforward and will not be repeated here. Other practical used models of the Markov-switching model, ARMA models, (G)ARCH models and SV models will be given in Section 6.

(c) Note that the mean contraction property \( \mathbb{E} \log L_1 < 0 \) is not satisfied in the above examples. Instead of applying Theorem 1 directly, we will explore
the special structure of the likelihood function in Lemma 4 below, such that \{((X_n, \xi_n), M_n), n \geq 0\} is an irreducible, aperiodic and Harris recurrent Markov chain. Hence, we can apply Theorem 1 for the Markovian iterated functions system on \(M\) induced from (2.4)–(2.7).

(d) C2–C4 are standard smoothness conditions. C5 is the technical condition for the existence of the Fisher information to be defined in (5.9) below. C8 and C9 are integrability conditions that will be used to prove strong consistency of the MLE. Condition C6 is the identifiability condition for state space models. That is, the family of mixtures of \(\{f(\xi_1; \theta| x, \xi_0) : \theta \in \Theta\}\) is identifiable. This condition will be used to prove strong consistency of the MLE. Although it is difficult to check this condition in a general state space model, in many models of interest the parameter itself is identifiable only up to a permutation of states such as a finite state hidden Markov model with normal distributions. A sufficient condition for the identifiable issue can be found in Theorem 1 of [14]. See also the paper by Itô, Amari and Kobayashi [38] for necessary and sufficient conditions in the case that the state space is finite and \(\xi_t\) is a deterministic function of \(X_t\).

(e) When the state space of the Markov chain \(\{X_n, n \geq 0\}\) is finite, and the observations \(\xi_n\) are conditionally independent, this reduces to the so-called hidden Markov model. It is easy to see that condition C1 implies (A1) by choosing \(w(x) = 1\), and conditions C2–C4 reduce to (A2), (A3) and (A5) of [7]. Conditions C6–C9 reduce to conditions C1–C6 in [38]. See also the paper by Itô, Amari and Kobayashi [38] for necessary and sufficient conditions in the case that the state space is finite and \(\xi_t\) is a deterministic function of \(X_t\).

Let \(\{(X_n, \xi_n), n \geq 0\}\) be the Markov chain defined in (2.1) and (2.2). Recall from (2.8) that the log likelihood can be written as

\[
l(\theta) = \log p_n(\xi_1, \ldots, \xi_n; \theta) = \log \|P_\theta(\xi_n) \circ \cdots \circ P_\theta(\xi_1) \circ P_\theta(\xi_0)\pi\|
\]

\[
= \log \frac{\|P_\theta(\xi_n) \circ \cdots \circ P_\theta(\xi_1) \circ P_\theta(\xi_0)\pi\|}{\|P_\theta(\xi_{n-1}) \circ \cdots \circ P_\theta(\xi_1) \circ P_\theta(\xi_0)\pi\|} + \cdots + \log \frac{\|P_\theta(\xi_1) \circ P_\theta(\xi_0)\pi\|}{\|P_\theta(\xi_0)\pi\|}.
\]

For each \(n\), denote

\[
M_n := P_\theta(\xi_n) \circ \cdots \circ P_\theta(\xi_1) \circ P_\theta(\xi_0)
\]

as the Markovian iterated random functions system on \(M\) induced from (2.4)–(2.7). Then \(\{(X_n, \xi_n), M_n\), \(n \geq 0\}\) is a Markov chain on the state space \((X \times \mathbb{R}^d) \times M\), with transition probability kernel \(P_\theta\) defined as in (3.2). Let \(P_\theta\) be the stationary distribution of \(\{(X_n, \xi_n), M_n\), \(n \geq 0\}\) defined in Theorem 1(iii). Then the log-likelihood function \(l(\theta)\) can be written as

\[
S_n := \sum_{k=1}^n g(M_{k-1}, M_k)
\]

with

\[
g(M_{k-1}, M_k) := \log \frac{\|P_\theta(\xi_k) \circ \cdots \circ P_\theta(\xi_1) \circ P_\theta(\xi_0)\pi\|}{\|P_\theta(\xi_{k-1}) \circ \cdots \circ P_\theta(\xi_1) \circ P_\theta(\xi_0)\pi\|}.
\]
In order to apply Theorems 1–4, we need to check that the Markovian iterated random functions system satisfies Assumption K, and the induced Markov chain is aperiodic, irreducible and Harris recurrent. For this purpose, we need to define a suitable metric on the space $M$, which has been defined in (2.4). First, we add a further condition on $M$ to have:

$$M = \left\{ h : h : X \to \mathbb{R}^+ \text{ is } m \text{-measurable, } \int h(x) m(dx) < \infty \text{ and } \sup_{x \in X} h(x) < \infty \right\}.$$ 

For convenience of notation, we still use the notation $M$, and will use $h$ to represent an element in $M$, which is different from the notation $u$ used in Sections 3 and 4. We define the variation distance between any two elements $h_1, h_2$ in $M$ by

$$d(h_1, h_2) = \sup_{x \in X} |h_1(x) - h_2(x)|.$$  

(5.8)

Note that $(M, d)$ is a complete metric space with Borel $\sigma$-algebra $\mathcal{B}(M)$, but it is not separable. Thus, Theorems 1–4 do not apply. However, rather than deal with the measure-theoretic technicalities created by an inseparable space, we can apply the results developed in Section 7 of [13] for a direct argument of convergence. Therefore, Theorems 1–4 still hold under the regularity conditions.

In order to describe our main results, we need the following lemmas first. Their proofs are given in Section 7.

**Lemma 3.** Assume C1–C5 hold or C1, C6–C9 hold. Then for each $\theta \in \Theta$ and $j = 1, \ldots, n$, the random functions $P_\theta(\xi_0)$ and $P_\theta(\xi_j)$, defined in (2.5) and (2.6), from $(X \times \mathbb{R}^d) \times M$ to $M$ are Lipschitz continuous in the second argument, and the Markovian iterated random functions system (2.4)–(2.7) satisfies Assumption K. Furthermore, the function $g$ defined in (5.7) belongs to $L^r(Q)$ for any $r > 0$.

For each $\theta \in \Theta$, recall that $\{(X_n, \xi_n), M_n\}$ is a Markov chain induced by the Markovian iterated random functions system (2.4)–(2.7) on the state space $(X \times \mathbb{R}^d) \times M$.

**Lemma 4.** Assume C1–C5 hold or C1, C6–C9 hold. Then for each $\theta \in \Theta$, $\{(X_n, \xi_n), M_n\}$ is an aperiodic, $(m \times Q \times Q)$-irreducible and Harris recurrent Markov chain.

**Lemma 5.** Assume C1–C5 hold. Then the Fisher information matrix

$$I(\theta) = (I_{ij}(\theta))$$

(5.9)

$$= \left( \mathbb{E}_\Pi^\theta \left[ \left( \frac{\partial \log \|P_\theta(\xi_1) \circ P_\theta(\xi_0)\pi\|}{\partial \theta_i} \right) \right] \right)$$
\[ \log \| P_\theta(\xi_1) \circ P_\theta(\xi_0) \pi \| \Bigg| \partial \theta_j \grad \quad \times \left( \frac{\partial \log \| P_\theta(\xi_1) \circ \cdots \circ P_\theta(\xi_0) \pi \|}{\partial \theta_j} \right) \Bigg) \]

is positive definite for \( \theta \) in a neighborhood \( N_\delta(\theta_0) \) of \( \theta_0 \). Recall that \( E^\theta_{\Pi} := E_{\Pi} \) is defined as the expectation under \( P_{\Pi} \) in (3.2).

**Remark 3.** Note that the Fisher information (5.9) is defined as the expected value under the stationary distribution \( \Pi^\theta \) of the Markov chain \( \{(X_n, \xi_n), M_n\), \( n \geq 0 \)}). It is worth mentioning that only \( \xi_n \) appears in \( M_n \), in which it reflects the nature of state space models.

When the state space \( \mathcal{X} \) is finite, and the random variables \( \xi_n \) are conditionally independent for given \( X_n \), let \( H := H(\xi_1, \xi_0, \xi_{-1}, \ldots) = \sum_{m=-\infty}^{1} H_m(\xi_1, \xi_0, \ldots) \), where

\[ H_m(\xi_1, \xi_0, \ldots) := E^{\theta_0} \left\{ \frac{\partial \log f(\xi_m; \theta | X_m)}{\partial \theta} \bigg| \xi_1, \xi_0, \ldots \right\} \]

\[ - E^{\theta_0} \left\{ \frac{\partial \log f(\xi_m; \theta | X_m)}{\partial \theta} \bigg| \xi_0, \xi_{-1}, \ldots \right\} \]

\[ + E^{\theta_0} \left\{ \frac{\partial \log p_\theta(X_m, X_{m+1})}{\partial \theta} \bigg| \xi_1, \xi_0, \ldots \right\} \]

\[ - E^{\theta_0} \left\{ \frac{\partial \log p_\theta(X_m, X_{m+1})}{\partial \theta} \bigg| \xi_0, \xi_{-1}, \ldots \right\} \].

Under their Assumptions 1–4, Bickel and Ritov [6] showed that \( H \in \mathcal{L}^2(P^{\theta_0}) \) and defined \( I_H(\theta_0) := E^{\theta_0}\{HH^t\} \). They also showed that

\[ \lim_{n \to \infty} E^{\theta_0} \left( \left( \frac{\partial \log \| T_n \pi \|_{\| \cdot \|_{\theta=\theta_0}}}{\partial \theta} \right) \left( \frac{\partial \log \| T_n \pi \|_{\| \cdot \|_{\theta=\theta_0}}}{\partial \theta} \right)^t \right) = I_H(\theta_0). \]

In this paper we represent the log likelihood function of an additive functional of the Markov chain \( \{(X_n, \xi_n), M_n\), \( n \geq 0 \)} in (5.7), and then apply the strong law of large numbers for Markovian iterated random functions given in Theorem 1(iv) to have, with probability 1,

\[ \lim_{n \to \infty} \frac{1}{n} \frac{\partial^2}{\partial \theta_i \partial \theta_j} \log \| P_\theta(\xi_n) \circ \cdots \circ P_\theta(\xi_1) \circ P_\theta(\xi_0) \pi \| = -I_{ij}(\theta). \]

Hence, under Assumptions 1–4 of [6], \( I(\theta) \) is well defined and is equal to \( I_H(\theta) \). The moment condition in Assumption 4 of [6] can be relaxed to the following: there exists a \( \delta > 0 \) with \( \rho_0(\xi) := \sup_{\| \theta - \theta_0 \| < \delta} \max_{x,y \in \mathcal{X}} \frac{f(\xi; \theta | x)}{f(\xi; \theta | y)} \), such that \( \sup_{x \in \mathcal{X}} P^{\theta_0}\{\rho_0(\xi_1) = \infty | X_0 = x\} < 1 \); see [7].
Lemma 6. Assume C1–C5 hold. Let \( l_j'(\theta_0) = \partial l(\theta)/\partial \theta_j |_{\theta = \theta_0} \). Then, as \( n \to \infty \),

\[
\frac{1}{\sqrt{n}} (l_j'(\theta_0))_{j=1,...,q} \to N(0, I(\theta_0)) \quad \text{in distribution.}
\]

Theorem 5. Assume C1–C5 hold. Then there exists a sequence of solutions \( \hat{\theta}_n \) of (5.1) such that \( \hat{\theta}_n \to \theta_0 \) in probability. Furthermore, \( \sqrt{n}(\hat{\theta}_n - \theta_0) \) is asymptotically normally distributed with mean zero and variance-covariance matrix \( I^{-1}(\theta_0) \).

Since the proof of Theorem 5 follows a standard argument, we will not give it here.

Corollary 2. Under the assumptions of Theorem 5, if the likelihood equation has a unique root for each \( n \) and all \( \xi_1,...,\xi_n \), then there is a consistent sequence of estimators \( \hat{\theta}_n \) of the unknown parameters \( \theta_0 \).

Next, we prove strong consistency of the MLE when the log likelihood function is integrable. A crucial step is to give an appropriate definition of the Kullback–Leibler information for state space models, so that we can apply Theorem 1 to have a standard argument of strong consistency for the MLE. Here, we define the Kullback–Leibler information as

\[
K(\theta_0, \theta) = E^\pi \left( \log \left| \frac{P_{\theta_0}(\xi_1) \circ P_{\theta_0}(\xi_0) \pi_{\theta_0}}{P_{\theta}(\xi_1) \circ P_{\theta}(\xi_0) \pi_{\theta}} \right| \right)
\]

\[
:= \int \log \left| \frac{P_{\theta_0}(\xi_1) \circ P_{\theta_0}(\xi_0) \pi_{\theta_0}}{P_{\theta}(\xi_1) \circ P_{\theta}(\xi_0) \pi_{\theta}} \right| \Pi(d(x, \xi) \times d\pi_{\theta_0}).
\]

Theorem 6. Assume that C1, C6–C9 hold and let \( \hat{\theta}_n \) be the MLE based on \( n \) observations \( \xi_0, \xi_1,...,\xi_n \). Then \( \hat{\theta}_n \to \theta_0 \) \( P_{\theta_0} \)-a.s. as \( n \to \infty \).

Since the proof of Theorem 6 follows a standard argument, we will not give it here.

To derive the Edgeworth expansion for the MLE, we need to define the following notation and assumptions first. For nonnegative integral vectors \( \nu = (\nu^{(1)}, ..., \nu^{(q)}) \), write \( |\nu| = \nu^{(1)} + \cdots + \nu^{(q)}, \nu! = \nu^{(1)}! \cdots \nu^{(q)}! \), and let \( D^\nu = (D_{1}^{\nu^{(1)}}) \cdots (D_{q}^{\nu^{(q)}}) \) denote the \( \nu \)th derivative with respect to \( \theta \). Suppose assumptions C2, C3, C4 and C5 are strengthened so that there exists \( r \geq 3 \), as follows.

C2': The true parameter \( \theta_0 \) is an interior point of \( \Theta \). For all \( x \in \mathcal{X}, s_0, s_1 \in \mathbb{R}^d, \theta \in \Theta \subset \mathbb{R}^q \), the partial derivatives

\[
D^1 f(s_0; \theta|x), \quad D^2 f(s_0; \theta|x), ..., D^r f(s_0; \theta|x),
\]

are defined and continuous.
as well as the partial derivatives
\[ D^1 f(s_1; \theta|x, s_0), \quad D^2 f(s_1; \theta|x, s_0), \ldots, D^r f(s_1; \theta|x, s_0), \]
and for all \( x, y \in X, \theta \rightarrow p_\theta(x, y) \) and \( \theta \rightarrow \pi_\theta(x) \) have \( r - 1 \) continuous derivatives in some neighborhood \( N_\delta(\theta_0) := \{ \theta : |\theta - \theta_0| < \delta \} \) of \( \theta_0 \).

For all \( x \in X, \)
\[ \int_X \sup_{\Theta \in N_\delta(\theta_0)} |D^1 \pi_\theta(x)| m(dx) < \infty, \ldots, \int_X \sup_{\Theta \in N_\delta(\theta_0)} |D^{r-1} \pi_\theta(x)| m(dx) < \infty, \]
and for all \( x \in X, \)
\[ \int_X \sup_{\Theta \in N_\delta(\theta_0)} |D^1 p_\theta(x, y)| m(dy) < \infty, \ldots, \int_X \sup_{\Theta \in N_\delta(\theta_0)} |D^{r-1} p_\theta(x, y)| m(dy) < \infty. \]

For all \( x \in X, s_0 \in \mathbb{R}^d \) and \( \theta \in \Theta, \)
\[ E^\theta_{(x, s_0)} |D^\nu f(x; \theta, s_0)| < \infty, \quad E^\theta_{(x, s_0)} |D^\nu f(x; \theta, s_0)| < \infty, \]
for \( 1 \leq |\nu| \leq r, \) and
\[ E^\theta_{x} \left( \sup_{\Theta \in N_\delta(\theta_0)} |D^\nu f(x; \theta, s_0)| \right) < \infty, \]
\[ E^\theta_{(x, s_0)} \left( \sup_{\Theta \in N_\delta(\theta_0)} |D^\nu f(x; \theta, s_0)| \right) < \infty, \]
for \( |\nu| = r + 1. \)

We will assume conditions (4.16) and (4.17) hold for \( Z_j^{(\nu)} := D^\nu \log p_1, \xi_1; \theta_0), 1 \leq |\nu| \leq r. \) Let \( Z_j := \{Z_j^{(\nu)} : 1 \leq |\nu| \leq r \} \) be \( p \)-dimensional random vectors for \( j \geq 1, \) where \( p \) is the number of all distinct multi-indices \( \nu, 1 \leq |\nu| \leq r. \) In the following, denote \( Z = (1/n) \sum_{j=1}^{n} Z_k. \)

Use a standard argument involving the sign change of a continuous function, or a fixed point theorem in the multi-parameter case (cf. [4]), to prove that the likelihood equation has a solution which converges in probability to \( \theta_0. \) Note that the following notation is interpreted in the multi-dimensional sense. Applying the moderate deviation result on \( Z \) in Theorem 4, it is possible to ensure that, with \( P^{\theta_0} \)-probability \( 1 - o(n^{-1}), \) \( \hat{\theta}_n \) satisfies the likelihood equation and lies on \( (\theta_0 \pm \log n / \sqrt{n}) \). It is this solution we take as our \( \hat{\theta}_n. \) If the likelihood equation has multiple roots, assume we have a consistent estimator \( T_n \) such that \( T_n \) lies in \( (\theta_0 \pm \log n / \sqrt{n}) \) with \( P^{\theta_0} \)-probability \( 1 - o(n^{-1}). \) In this case, we may take the solution nearest to
$T_n$. By the preceding reasoning, this solution, which is identifiable from the sample, will lie in $(\theta_0 \pm \log n/\sqrt{n})$ with $P^{\hat{\theta}_0}$-probability $1 - o(n^{-1})$.

Clearly, with $\hat{\theta}_n$ as above, with probability $1 - o(n^{-1})$,

$$0 = \tilde{Z}(e_s) + \sum_{|\nu|=1}^{r-1} \frac{1}{\nu!} \tilde{Z}(e_s+\nu)(\hat{\theta}_n - \theta_0)^\nu + R_{n,s}(\hat{\theta}_n), \quad 1 \leq s \leq q,$$

where $e_s$ has 1 as the $s$th coordinate and zeros otherwise.

We rewrite equation (5.12) as

$$0 = A(\tilde{Z}, \hat{\theta}_n) + R_n.$$  

Note $0 = A(\gamma(\theta_0), \theta_0)$ and $\left. \frac{\partial A}{\partial \theta} \right|_{\gamma(\theta_0), \theta_0} = -(\text{Fisher information}) \neq 0$.

Hence, by the implicit function theorem, there are a neighborhood $N$ of $\gamma$ and $q$ uniquely defined real-valued infinitely differentiable functions $g_i$ ($1 \leq i \leq q$) on $N$ such that $\theta = g(z) = (g_1(z), \ldots, g_q(z))$ satisfies (5.13). This implies, with probability $1 - o(n^{-1})$, $|\hat{\theta}_n - \theta_0| \leq K (\log n/\sqrt{n})^4$.

To derive the asymptotic expansion of $P^{\hat{\theta}_0}\{\sqrt{n}(\hat{\theta}_n - \theta_0) \in B\}$, note that $\hat{\theta}_n = g(n^{-1}Z_n)$, where $g : \mathbb{R}^p \to \mathbb{R}^q$ is sufficiently smooth in some neighborhood of $\gamma$. For the case of i.i.d. $\xi_n$, Bhattacharya and Ghosh [4] made use of the Edgeworth expansion of the distribution of $(S_n - n\gamma)/\sqrt{n}$ to derive an Edgeworth expansion of the distribution of $\sqrt{n}\{g(n^{-1}S_n) - g(\gamma)\}$. Making use of Theorem 4 and a straightforward extension of their argument, we can generalize their result to have the following theorem.

**Theorem 7.** Assume $C1, C2'-C5'$ hold for some $r \geq 3$. Assume (4.16) and (4.17) hold. Let $J_g = (J_j g_i(\gamma))_{1 \leq i \leq q, 1 \leq j \leq p}$ be the $q \times p$ Jacobian matrix and let $V(g) = J_g V J_g'$. Then there exists a sequence of solutions $\hat{\theta}_n$ of (5.1), and there exist polynomials $p_j$ in $q$ variables ($1 \leq j \leq r - 2$) such that

$$\sup_{B \in \mathcal{B}_{n,c}} \left| P_{\nu}^{\hat{\theta}_0}\{\sqrt{n}(\hat{\theta}_n - \theta_0) \in B\} - \int_{B} \left\{ \phi_{V(g)}(y) + \sum_{j=1}^{r-2} n^{-j/2} \phi_{j,V,g}(y) \right\} dy \right| = o(n^{-(r-2)/2}),$$

where $\phi_{j,V,g} = \tilde{\pi}_{j,g}(-D)\phi_V$ and $\tilde{\pi}_{j,g}(y)$ is a polynomial in $y \in \mathbb{R}^p$ whose coefficients are smooth functions of the partial derivatives of $\lambda(\alpha)$ at $\alpha = 0$ up to order $j + 2$, and those of $\nu_{0,Q_0 h_1}$ at $\alpha = 0$ up to order $j$ together with those of $g$ at $\mu$ up to order $j + 1$.

The application of Theorem 7 to third-order efficiency for the MLE and third-order efficient approximate solution of the likelihood equation follows directly from [28].
6. Examples. From a theoretical point of view, Theorems 5–7 are adequate for state space model estimation problems in providing assurance of the existence of efficient estimators, characterizing them as solutions of likelihood equations and prescribing their asymptotic behavior. In practice, however, one must still contend with certain statistical and numerical difficulties, such as implementation of the maximum likelihood estimator. In this section we apply our results to study some examples which include Markov switching models ARMA models, (G)ARCH models and SV models. For simplicity, in these examples we consider only specific structure of normal error assumption in most cases. Although strong consistency and asymptotic normality of the MLE in ARMA and GARCH\((p,q)\) have been known in the literature, we provide alternative proofs in the framework of state space models. Furthermore, we can apply Theorem 7 to have Edgeworth expansion for the MLE. To the best of our knowledge, the asymptotic normality of the MLE in the AR\((1)\)/ARCH\((1)\) model, considered in Section 6.3, seems to be new. The results of asymptotic properties for the MLE in stochastic volatility models not only provide theoretical justification, but also give some insight into the structure of the likelihood function, which can be used for further study.

6.1. Markov switching models. We start with a simple real-valued fourth-order autoregression around one of two constants, \(\mu_1\) or \(\mu_2\):

\[
\xi_n - \mu_{X_n} = 4 \sum_{k=1}^{4} \varphi_k (\xi_{n-k} - \mu_{X_{n-k}}) + \varepsilon_n, \tag{6.1}
\]

where \(\varepsilon_n \sim N(0, \sigma^2)\), and \(\{X_n, n \geq 0\}\) is a two-state Markov chain. This model was studied by Hamilton \cite{33} in order to analyze the behavior of U.S. real GNP. To apply our theory in the form of (6.1), we consider a simple case of order 1 in (6.1). In this case, the likelihood function for given \(X_n = x_n, n \geq 0\), is

\[
f(\xi_n|x_n; \theta) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{[\xi_n - \mu_{x_n} - \varphi_1 (\xi_{n-1} - \mu_{x_{n-1}})]^2}{2\sigma^2}\right). \tag{6.2}
\]

Denote by \([p_{xy}]_{x,y=1,2}\) the transition probability of the underlying Markov chain \(\{X_n, n \geq 0\}\) and let \(\theta = (p_{11}, p_{21}, \varphi_1, \mu_1, \mu_2, \sigma^2)\) be the unknown parameter. Assume that \(|\varphi_1| < 1\), and that there exists a constant \(c > 0\) such that \(\sigma^2 > c\). Moreover, we assume that \(\mu_1 \neq \mu_2\) such that the identifiability condition C6 holds. Since the state space of \(X_n\) is finite, we consider \(0 < p_{xy} < 1\) for all \(x, y = 1, 2\), and let \(w(x) = |x| + 1\) such that the condition C1 holds. Under the normal distribution assumption, it is easy to see that
conditions C2–C4 and C7–C9 are satisfied in this model. To check that C5 holds note that condition C5 reduces to

\[
\sup_{x \in \mathcal{X}} E_x^\theta \left( \sup_{|\theta - \theta_0| < \delta} \max_{y, z \in \mathcal{X}} f(\xi_0; \theta | y) f(\xi_1; \theta | y, \xi_0) f(\xi_0; \theta | z) f(\xi_1; \theta | z, \xi_0) \right)^2 < \infty.
\]

(6.3)

Since the maximum over \(x, y, z\) is applied to a finite set \(\mathcal{X}\), and \(f\) defined in (6.1) is a normal density, it is easy to check that (6.3) is satisfied.

When \(\xi_n = X_n\) as in (6.1), that is, \(\mu_1 = \mu_2 = \mu\) are given, this reduces to the classical autoregressive model with unknown parameters \(\theta = (\varphi_1, \ldots, \varphi_4, \sigma^2)\). The Fisher information matrix is then given by

\[
I(\theta) = \begin{pmatrix} \sigma^{-2} \Gamma & 0 \\ 0 & 2(\sigma^4)^{-1} \end{pmatrix},
\]

(6.4)

where \(\Gamma = (\gamma_{i-j})_{4 \times 4}\) for \(1 \leq i, j \leq 4\) with \(\gamma_k = EX_nX_{n+k}\). A simple calculation shows that (5.9) reduces to (6.4) in this case. When \(\varphi_k = 0\) as in (6.1), this is the hidden Markov model with normal mixture distributions considered in Example 1 of [7].

6.2. ARMA models. We start with a univariate Gaussian causal ARMA\((p, q)\) model which can be written as a state space model by defining \(r = \max\{p, q + 1\}\),

\[
\begin{align*}
\xi_n - \mu &= \alpha_1(\xi_{n-1} - \mu) + \alpha_2(\xi_{n-2} - \mu) + \cdots + \alpha_r(\xi_{n-r} - \mu) \\
&\quad + \varepsilon_n + \beta_1\varepsilon_{n-1} + \beta_2\varepsilon_{n-2} + \cdots + \beta_{r-1}\varepsilon_{n-r+1},
\end{align*}
\]

(6.5)

where \(\alpha_j = 0\) for \(j > p\) and \(\beta_j = 0\) for \(j > q\). Furthermore, we assume \(\varepsilon_n\) are i.i.d. random variables with distribution \(N(0, \sigma^2)\). Asymptotic properties of the MLE in the ARMA model can be found in [35] and [53]. A general treatment of the MLE in the Gaussian ARMAX model can be found in Chapter 7 of [11].

By using the same idea as that in [34], we consider the following state space representation of (6.5):

\[
X_{n+1} = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{r-1} & \alpha_r \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} X_n + \begin{bmatrix} \varepsilon_{n+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix},
\]

(6.6)

and

\[
\xi_n = \mu + [1 \ \beta_1 \ \beta_2 \cdots \beta_{r-1}] X_n.
\]

(6.7)

Assume that the roots of \(1 - \alpha_1 z - \alpha_2 z^2 - \cdots - \alpha_p z^p = 0\) lie outside the unit circle. It is easy to see that \(\{X_n, n \geq 0\}\) forms a \(w\)-uniformly ergodic
Markov chain with \( w(x) = \|x\|^2 \) (cf. Theorem 16.5.1 in [46]). And \( \xi_n \) are conditionally independent given \( \{X_n, n \geq 0\} \). Since the verification of the weighted mean contraction property and the weighted moment assumption is the same as those in Remark 2(b), it will not be repeated here. This implies that condition C1 holds. The assumption \( \varepsilon_n \sim N(0, \sigma^2) \) also implies that conditions C2–C5, C2’–C5’ and C7–C9 are satisfied in model (6.5). Since the verification is straightforward, we do not report it here. Suppose the conditional distribution of \( \xi_n \) given \( X_0, \ldots, X_n \) is of the form \( F_{X_n-1, X_n} \) from (6.7). The Cramér conditions (4.16) and (4.17) hold for \( Z_j(\nu) = D_\nu \log p_1(\xi_0, \xi_1; \theta_0) \), since the conditional density of \( \xi_n \) given \( \{x_n, n \geq 0\} \) is \( N(0, \sigma^2) \) and \( \limsup_{|\theta| \to 0} \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} e^{i\theta \xi} dF_{X_n, nx+z}(\xi) \right\} \varphi(z) dz \pi(dx) \right| < 1 \), (6.8)

where \( \varphi(\cdot) \) is the normal density function of \( \varepsilon_1, \) and \( \pi \) is the stationary distribution of \( \{X_n\} \). The identification issue in C6 can be found in Chapter 9 of [11] or Chapter 13 of [34].

6.3. (G)ARCH models. In this subsection we study two specific (G)ARCH models. To start with, we consider the AR(1)/ARCH(1) model

\[
X_n = \beta_0 + \beta_1 X_{n-1} + \sqrt{\alpha_0 + \alpha_1 X_{n-1}^2} \varepsilon_n,
\]

where \( \alpha_i, \beta_i \) are unknown parameters for \( i = 0, 1 \) with \( \alpha_0 > 0, 0 < \alpha_1 < 1, 3\alpha_1^2 < 1 \) and \( 0 < \beta_1 < 1 \). Here \( \varepsilon_n \) are i.i.d. random variables with the standard normal distribution. Note that in (6.9) \( X = (X_n) \) is defined as the autoregressive scheme AR(1) with ARCH(1) noise (\( \sqrt{\alpha_0 + \alpha_1 X_{n-1}^2} \varepsilon_n \)) \( n \geq 1 \). When \( \beta_0 = \beta_1 = 0 \), this is the classical ARCH(1) model first considered by Engle [19].

Model (6.9) is conditionally Gaussian, and therefore the likelihood function of the parameter \( \theta = (\alpha_0, \alpha_1, \beta_0, \beta_1) \) for given observations \( x = (x_0 = 0, x_1, \ldots, x_n) \) from (6.9) is

\[
\ell(x; \theta) = (2\pi)^{-n/2} \prod_{k=1}^{n} (\alpha_0 + \alpha_1 x_{k-1}^2)^{-1/2} \times \exp \left\{ -\frac{1}{2} \sum_{k=1}^{n} \frac{(x_k - \beta_0 - \beta_1 x_{k-1})^2}{\alpha_0 + \alpha_1 x_{k-1}^2} \right\}.
\]

Assume \( \beta_0 = 0 \) and \( \alpha_0, \alpha_1 \) are given. The maximum likelihood estimator \( \hat{\beta}_1 \) of \( \beta_1 \) is the root of the equation \( \partial \ell(x; \theta)/\partial \beta_1 = 0 \). In view of (6.9) and
(6.10), we obtain
\[
\hat{\beta}_1 = \frac{\sum_{k=1}^{n}(x_k - \beta_0)x_{k-1}/(\alpha_0 + \alpha_1x_{k-1}^2)}{\sum_{k=1}^{n}x_{k-1}^2/(\alpha_0 + \alpha_1x_{k-1}^2)}
\]
(6.11)
\[
= \beta_1 + \frac{\sum_{k=1}^{n}x_{k-1}\varepsilon_k/\sqrt{\alpha_0 + \alpha_1x_{k-1}^2}}{\sum_{k=1}^{n}x_{k-1}^2/(\alpha_0 + \alpha_1x_{k-1}^2)}.
\]

Meyn and Tweedie [46], pages 380 and 383, establish \(w\)-uniform ergodicity [with \(w(x) = |x| + 1\)] of the AR(1) model \(X_n = \beta_0 + \beta_1X_{n-1} + \varepsilon_n\) by proving that a drift condition is satisfied, where \(|\beta_1| < 1\) and the \(\varepsilon_n\) are i.i.d. random variables, with \(E|\varepsilon_n| < \infty\), whose common density function \(q\) with respect to Lebesgue measure is positive everywhere. The strongly nonlattice condition holds as that in model (6.5). By using an argument similar to Theorem 1 of [45], we have the asymptotic identifiability of the likelihood function (6.10). Letting \(\xi_n = X_n\), and using an argument similar to that in Remark 2(b), condition C1 holds. The verification of conditions C2–C9 and C2′–C5′ is straightforward and tedious, and is thus omitted. By Theorems 5–7, we have the strong consistency, asymptotic normality and Edgeworth expansion of the MLE \(\hat{\beta}_1\). The asymptotic properties of the MLE of \(\beta_0\), \(\alpha_0\) and \(\alpha_1\) can be verified in a similar way.

Next, we consider the GARCH\((p,q)\) model of (1.1) in Example 1. It is known that the necessary and sufficient condition for (1.1) defining a unique strictly stationary process \(\{Y_n, n \geq 0\}\) with \(EY_n^2 < \infty\) is
\[
\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1.
\]
(6.12)
We assume (6.12) holds.

Similar to the estimation for ARMA models, the most frequently used estimators for GARCH models are those derived from a (conditional) Gaussian likelihood function (cf. [20]). Without the normal assumption of \(\varepsilon_n\) in (1.1), and imposing the moment condition \(E(\varepsilon_1^4) < \infty\), Hall and Yao [32] established the asymptotic normality of the conditional maximum likelihood estimator in GARCH\((p,q)\). They also established asymptotic results when the case of the error distribution is heavy-tailed. Earlier in the literature, when \(p = q = 1\), Lee and Hansen [43] and Lumsdaine [45] proved, under some regularity conditions, the consistency and asymptotic normality for the quasi-maximum likelihood estimator in the GARCH\((1,1)\) model.

By using the state space representation (1.2) and (1.3), it is known (cf. Theorem 3.2 of [1]) that the Markov chain \(\{X_n, n \geq 0\}\) defined in (1.3) is stationary if and only if the top Lyapunov exponent \(\gamma\) of \(A_n\) is strictly
negative. It is easy to see that \( \{ X_n, n \geq 0 \} \) is an aperiodic, irreducible and \( w \)-uniformly \([w(x) = \|x\|^2]\) ergodic Markov chain. Furthermore, we assume \( \varepsilon_n \) are i.i.d. random variables with distribution \( N(0, \sigma^2) \). An argument similar to that in Remark 2(b) leads to condition C1 holding. The normal error assumption also implies that conditions C2–C5, C2’–C5’ and C7–C9 are satisfied in model (1.3). When \( p = q = 1 \), Theorem 1 of [45] proves the asymptotic identifiability of the likelihood function.

6.4. Stochastic volatility models. Consider the stochastic volatility model (1.4)–(1.8). To check that condition C1 holds, we note that \( w(x) = |x| + 1 \) in the AR(1) model \( X_n = \alpha X_{n-1} + \eta_n \) by proving that a drift condition is satisfied, where \( |\alpha| < 1 \) and the \( \eta_n \) are i.i.d. random variables, with \( E|\eta_1| < \infty \), whose common density function \( q \) with respect to Lebesgue measure is positive everywhere. Since \( \varepsilon_n \sim N(0, 1) \), \( \zeta_n = \log \varepsilon_n^2 \), \( \eta_n \sim N(0, \sigma_{\eta}^2) \), and \( \zeta_n \) and \( \eta_n \) are mutually independent, an argument similar to that in Remark 2(b) leads to the result that the rest of condition C1 holds. Conditions C2–C5, C2’–C5’ and C7–C9 are also satisfied in model (1.5) and (1.6) (cf. pages 22–23 of [50]). Denote \( \xi_n := \log Y_n^2 \). Note that the conditional density of \( X_n \) exists, and this implies that the conditional distribution of \( \xi_n \) given \( X_0, \ldots, X_n \) is of the form \( F_{X_{n-1}, X_n} \) such that

\[
(6.13) \limsup_{|t| \to 0} \left| \int_X \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} e^{it s} dF_{X, \alpha x + z}(s) \right\} \varphi(z) \, dz \, \pi(dx) \right| < 1,
\]

where \( \varphi(\cdot) \) is the normal density function of \( \zeta_1 \) and \( \pi \) is the stationary distribution of \( \{X_n\} \). Let \( S_n = \sum_{i=1}^n \xi_i \), \( S_0 = 0 \). Then \( \{(X_n, S_n), n \geq 0\} \) is strongly nonlattice. To check the identification condition C6, the reader is referred to Chapter 13 of [34] and Section 2.4.3 of [29].

Next, we assume that \( \varepsilon_n \sim N(0, 1) \), \( \zeta_n = \log \varepsilon_n^2 \), and \( \eta_n \) is a sequence of i.i.d. double exponential(1) random variables. Furthermore, we assume \( \zeta_n \) and \( \eta_n \) are mutually independent. By using an argument similar to that in Remark 2(b), condition C1 holds. Simple calculations also lead conditions C2–C5, C2’–C5’ and C7–C9 to hold in this case. Under the assumption that the conditional distribution of \( \xi_n \) given \( X_0, \ldots, X_n \) is of the form \( F_{X_{n-1}, X_n} \) such that (6.13) holds, \( \{(X_n, S_n), n \geq 0\} \) is strongly nonlattice.

Without the normal assumption, quasi-maximum likelihood (QML) estimators of the parameters are obtained by treating \( \zeta_n \) and \( \eta_n \) as though they were normal and maximizing the prediction error decomposition form of the likelihood obtained via the Kalman filter or implied volatility. That is, we assume that \( \zeta_n \) is a sequence of independent and identically distributed \( N(0, \sigma_\zeta^2) \) random variables. For given observations \( y = (\log y_1^2, \ldots, \log y_n^2) \) from (1.5) and (1.6), the likelihood function of the parameter \( \theta = (\alpha, \sigma_{\eta}^2, \sigma_\zeta^2) \)
is

\[ l(y; \theta) = \int_{x_0 \in \mathcal{X}} \cdots \int_{x_n \in \mathcal{X}} \pi(x_0)(2\pi \sigma^2)^{-n/2} \]

\[ \times \prod_{k=1}^{n} p(x_{k-1}, x_k) \]

\[ \times \exp \left\{ \frac{1}{2} \sum_{k=1}^{n} \frac{(\log y_k^2 - \omega - x_k)^2}{\sigma^2} \right\} dx_0 dx_1 \cdots dx_n, \]

where \( p(x_{k-1}, x_k) \) is defined in (1.7). By using the results of [16], Harvey, Ruiz and Shephard [36] showed that the quasi-maximum likelihood estimators are asymptotically normal under some regularity conditions. Further study of the MLE in stochastic volatility models will be published in a separate paper.

7. Proofs of Lemmas 3–6. For convenience of notation, denote \{\( Z_n, n \geq 0 \) := \{(X_n, \xi_n), M_n\), \( n \geq 0 \} as the Markov chain induced by the Markovian iterated random functions system (2.4)–(2.7) on the state space \((\mathcal{X} \times \mathbb{R}^d) \times \mathbb{M}\). In the proof of Lemma 3, we omit \( \theta \) in \( P_{\theta}(\cdot) \) for simplicity.

**Proof of Lemma 3.** We consider only the cases of \( P(\xi_1) \), since the cases of \( P(\xi_0) \) and \( P(\xi_j), for j = 2, \ldots, n \), are a straightforward consequence. For any two elements \( h_1, h_2 \in \mathbb{M} \), and two fixed elements \( s_0, s_1 \in \mathbb{R}^d \), by (5.8) we have

\[
d(P(s_1)h_1, P(s_1)h_2) = \sup_{x_0 \in \mathcal{X}} \left| \int p_\theta(x_0, x_1) f(s_1; \theta|x_1, s_0) h_1(x_1) m(dx_1) \\
- \int p_\theta(x_0, x_1) f(s_1; \theta|x_1, s_0) h_2(x_1) m(dx_1) \right| \]

\[ \leq d(h_1, h_2) \sup_{x_0 \in \mathcal{X}} \int p_\theta(x_0, x_1) f(s_1; \theta|x_1, s_0) m(dx_1) \]

\[ \leq C \left( \sup_{x_0 \in \mathcal{X}} \int p_\theta(x_0, x_1) m(dx_1) \right) d(h_1, h_2), \]

where \( 0 < C = \sup_{x_0 \in \mathcal{X}} f(s_1; \theta|x_1, s_0) < \infty \) by assumption C1 is a constant. Note that \( \sup_{x_0 \in \mathcal{X}} \int p_\theta(x_0, x_1) m(dx_1) = 1 \). The equality holds only if \( h_1 = h_2 \) \( m \)-almost surely. This proves the Lipschitz continuous condition in the second argument.
Note that C1 implies Assumption K1 holds. Recall that $M_n = \mathbf{P}(\xi_n) \circ \cdots \circ \mathbf{P}(\xi_1) \circ \mathbf{P}(\xi_0)$ in (5.6). To prove the weighted mean contraction property K2, we observe that, for $p \geq 1$,

$$
\sup_{x_0, s_0} \mathbf{E}_{(x_0, s_0)} \left\{ \log \left( L_p \frac{w(X_p, \xi_p)}{w(x_0, s_0)} \right) \right\} 
= \sup_{x_0, s_0} \mathbf{E}_{(x_0, s_0)} \left\{ \log \left( \sup_{h_1 \neq h_2} \frac{d(M_p h_1, M_p h_2)}{d(h_1, h_2)} \frac{w(X_p, \xi_p)}{w(x_0, s_0)} \right) \right\} 
< \sup_{x_0, s_0} \mathbf{E}_{(x_0, s_0)} \left\{ \log \left( \sup_{x_0 \in \mathcal{X}} \int p_\theta(x_0, x_1) f(\xi_1; \theta|x_1, s_0)m(dx_1) \right)^p \right. 
\times \left. \frac{w(X_p, \xi_p)}{w(x_0, s_0)} \right\} 
< 0.
$$

The last inequality follows from (5.2) in condition C1.

To verify that Assumption K3 holds, as $m$ is $\sigma$-finite, we have $\mathcal{X} = \bigcup_{n=1}^\infty \mathcal{X}_n$, where the $\mathcal{X}_n$ are pairwise disjoint and $0 < m(\mathcal{X}_n) < \infty$. Set

$$
h(x) = \sum_{n=1}^\infty \frac{I_{\mathcal{X}_n}(x)}{2^n m(\mathcal{X}_n)}.
$$

It is easy to see that $\int_{x \in \mathcal{X}} h(x) m(dx) = 1$ and, hence, belongs to M. Observe that

$$
\mathbf{E} d^2(\mathbf{P}(\xi) h, h)
= \mathbf{E} \sup_{x_{j-1} \in \mathcal{X}_j} \left| \int p_\theta(x_{j-1}, x_j) \right.
\times f(\xi_j; \theta|x_j, \xi_{j-1})h(x_j)m(dx_j) - h(x_{j-1}) \right|.
$$

By definition of $h(x)$ in (7.2), it is piecewise constant, and $p_\theta(x_{j-1}, x_j)f(\xi_j; \varphi_{x_j}(\theta)|\xi_{j-1})$ is a probability density function integrable over the subset $\mathcal{X}_n$. These imply (7.3) is finite.

Finally, we observe

$$
\sup_{x_0, s_0} \mathbf{E}_{(x_0, s_0)} \left\{ L_1 \frac{w(X_1, \xi_1)}{w(x_0, s_0)} \right\}
= \sup_{x_0, s_0} \mathbf{E}_{(x_0, s_0)} \left\{ \sup_{h_1 \neq h_2} \frac{d(\mathbf{P}(\xi_1) h_1, \mathbf{P}(\xi_1) h_2)}{d(h_1, h_2)} \frac{w(X_1, \xi_1)}{w(x_0, s_0)} \right\}
< \sup_{x_0, s_0} \mathbf{E}_{(x_0, s_0)} \left\{ \sup_{x_0 \in \mathcal{X}} \int p_\theta(x_0, x_1) f(\xi_1; \theta|x_1, s_0)m(dx_1) \frac{w(X_1, \xi_1)}{w(x_0, s_0)} \right\} < \infty.
$$
The last inequality follows from (5.3) in condition C1.

Note that C5 implies the exponential moment condition of $g$. Hence, the proof is complete. □

In the proof of Lemma 4 we omit $\theta$ for simplicity.

**Proof of Lemma 4.** We first prove that $\{Z_n, n \geq 0\}$ is Harris recurrent. Note that the transition probability kernel of the Markov chain $\{(X_n, \xi_n), n \geq 0\}$, defined in (2.1) and (2.2), has a probability density with respect to $m \times Q$. And the iterated random functions system, defined in (2.4)–(2.7), also has a probability density with respect to $Q$. By making use of the definition (3.2), there exists a measurable function $g: (\mathcal{X} \times \mathbb{R}^d \times \mathcal{M}) \times (\mathcal{X} \times \mathbb{R}^d \times \mathcal{M}) \rightarrow [0, \infty)$ such that

\[(7.4) \quad P(z, dz') = g(z, z')(m \times Q)(dz'),\]

where $\int_{(\mathcal{X} \times \mathbb{R}^d \times \mathcal{M})} g(z, z')(m \times Q)(dz') = 1$ for all $z \in (\mathcal{X} \times \mathbb{R}^d) \times \mathcal{M}$. For simplicity of notation, we let $\Lambda(\cdot) := (m \times Q)(\cdot)$ in the proof. For given $n > 1$, let $P^n(z, \cdot) := P_z(Z_n \in \cdot)$ for $z \in (\mathcal{X} \times \mathbb{R}^d) \times \mathcal{M}$. For $A \in \mathcal{B}(\mathcal{X} \times \mathbb{R}^d)$ and $B \in \mathcal{B}(\mathcal{M})$, define

$$\Lambda^n(A \times B) := \int_{(\mathcal{X} \times \mathbb{R}^d) \times \mathcal{M}} P_z\{Z_n \in A \times B\} \Lambda(dz').$$

Then for all $A \in \mathcal{B}(\mathcal{X} \times \mathbb{R}^d)$ and $B \in \mathcal{B}(\mathcal{M}),$

$$P^{n+1}(z, A \times B) = \int_{(\mathcal{X} \times \mathbb{R}^d) \times \mathcal{M}} P^n(z', A \times B) g(z, z') \Lambda(dz')$$

\[= \int_{(\mathcal{X} \times \mathbb{R}^d) \times \mathcal{M}} P_z\{Z_n \in A \times B\} g(z, z') \Lambda(dz').\]

It is easy to see that, for given any $n > 1$, the family $(P^{n+1}(z, \cdot))_{z \in (\mathcal{X} \times \mathbb{R}^d) \times \mathcal{M}}$ is absolutely continuous with respect to $\Lambda^n$. Therefore, by the Radon–Nikodym theorem, $P^n$ has a probability density with respect to $\Lambda^n$ for all $n \geq 1$. Let $g_n$ be such that

\[(7.5) \quad P^{n+1}(z, dz') = g_n(z, z') \Lambda^n(dz'), \quad z \in (\mathcal{X} \times \mathbb{R}^d) \times \mathcal{M},\]

where $\int_{(\mathcal{X} \times \mathbb{R}^d) \times \mathcal{M}} g_n(z, z') \Lambda^n(dz') = 1$ for all $z \in (\mathcal{X} \times \mathbb{R}^d) \times \mathcal{M}$. Note that $g_1 = g$. It is easy to check that all $\Lambda^n$ are absolutely continuous with respect to $\Pi$.

Denote $B^c$ as the complement of $B$. Since $\Pi((\mathcal{X} \times \mathbb{R}^d) \times \mathcal{M}) = 0$, also $\Lambda((\mathcal{X} \times \mathbb{R}^d) \times \mathcal{M}) = 0$. Recall $g$ is defined in (7.4). It is obvious from the previous considerations that we can choose $\delta > 0$ sufficiently small such that

\[
\int_{(\mathcal{X} \times \mathbb{R}^d) \times \mathcal{M}} \int_{(\mathcal{X} \times \mathbb{R}^d) \times \mathcal{M}} \int_{(\mathcal{X} \times \mathbb{R}^d) \times \mathcal{M}} \int_{(\mathcal{X} \times \mathbb{R}^d) \times \mathcal{M}} 1_{\{g_2 \geq \delta\}}(z_1, z_2)
\times 1_{\{g \geq \delta\}}(z_2, z_3) \Lambda(dz_3) \Lambda^2(dz_2) \Pi(dz_1) > 0.
\]
Hence, by Lemma 4.3 of [48], there exist a Π-positive set $\Gamma_1 \subset (\mathcal{X} \times \mathbb{R}^d) \times \mathbf{M}$ and a $\Lambda$-positive set $\Gamma_2 \subset (\mathcal{X} \times \mathbb{R}^d) \times \mathbf{M}$ such that

$$\alpha := \inf_{z_1 \in \Gamma_1, z_2 \in \Gamma_2} \Lambda^2 \{ z_2 \in (\mathcal{X} \times \mathbb{R}^d) \times \mathbf{M} : g_2(z_1, z_2) \geq \delta, g(z_2, z_3) \geq \delta \} > 0.$$ 

A combination of the above result with (7.4) and (7.5) implies

$$P^3(z_1, A \times B) = \int_{(\mathcal{X} \times \mathbb{R}^d) \times \mathbf{M}} P(z_2, A \times B)P^2(z_1, dz_2)$$

(7.6)

$$\geq \int_{(\mathcal{X} \times \mathbb{R}^d) \times \mathbf{M}} g_2(z_1, z_2) \int_{(A \times B) \cap \Gamma_2} g(z_2, z_3)\Lambda(dz_3)A^2(dz_2)$$

$$\geq \alpha \delta^2 \Lambda((A \times B) \cap \Gamma_2)$$

for all $z_1 \in \Gamma_1$ and $A \times B \in B((\mathcal{X} \times \mathbb{R}^d) \times \mathbf{M})$. Therefore, we obtain an absorbing set such that $\Gamma_1$ is a regeneration set for $\{Z_n, n \geq 0\}$ on $(\mathcal{X} \times \mathbb{R}^d) \times \mathbf{M}$, that is, $\Gamma_1$ is recurrent and satisfies a minorization condition, namely, (7.6). This proves the Harris recurrence of $\{Z_n, n \geq 0\}$ on $(\mathcal{X} \times \mathbb{R}^d) \times \mathbf{M}$. Since $\{Z_n, n \geq 0\}$ possesses a stationary distribution, it is clearly positive Harris recurrent.

Next, we give the proof of aperiodicity. If $\{Z_n, n \geq 0\}$ were $q$-periodic with cyclic classes $\Gamma_1, \ldots, \Gamma_q$, say, then the $q$-skeleton $(Z_{qn})_{n \geq 0}$ would have stationary distributions $\frac{\Pi(\cap \Gamma_k)}{\Pi(\Gamma_k)}$ for $k = 1, \ldots, q$. On the other hand, $Z_{qn}$ is aperiodic by definition, and $M_{qn}$ is also a Markovian iterated random functions system of Lipschitz maps, satisfying condition C1, and thus possesses only one stationary distribution. Consequently, $q = 1$ and $\{Z_n, n \geq 0\}$ is aperiodic. Since the Markov chain $\{(X_n, \xi_n), M_n\}, n \geq 0\}$ has a probability density with respect to $\Lambda$, it is obviously $\Lambda$-irreducible. The proof is complete. \(\Box\)

**Proof of Lemma 5.** In order to define the Fisher information (5.9), we need to verify that there exists a $\delta > 0$, such that $\partial \log \|P_\theta(\xi_1) \circ P_\theta(\xi_0)\pi\|/\partial \theta \in L_2(P^\theta_{\Pi})$ for $\theta \in N_\delta(\theta_0)$, a $\delta$-neighborhood of $\theta_0$. That is, we need to show

$$E_\Pi^\theta \left( \frac{\partial \log \|P_\theta(\xi_1) \circ P_\theta(\xi_0)\pi\|}{\partial \theta} \right)^2 < \infty,$$  

(7.7)

for $\theta \in N_\delta(\theta_0)$.

It is easy to see that C5 implies that

$$\sup_{x \in \mathcal{X}} E_\Pi^\theta \left( \frac{\partial \log \int_{y \in \mathcal{X}} \pi(x)p(x, y)f(\xi_0; \theta|x)f(\xi_1; \theta|y, \xi_0)m(dy)}{\partial \theta} \right)^2 < \infty$$

for $\theta \in N_\delta(\theta_0)$. And this leads to

$$\sup_{x \in \mathcal{X}} E_\Pi^\theta \left( \frac{\partial \log \int_{y \in \mathcal{X}} \pi(x)p(x, y)f(\xi_0; \theta|x)f(\xi_1; \theta|y, \xi_0)m(dy)}{\partial \theta} \right)^2 < \infty,$$  

(7.8)
by using an argument similar to that of Lemma (7.12) then Corollary (7.10)

where the variance-covariance matrix

\[ \Sigma(\theta_0) = (\Sigma_{ij}(\theta_0))_{i,j=1,...,q} \rightarrow N(0, \Sigma(\theta_0)) \text{ in distribution,} \]

where the variance—covariance matrix

\[ \Sigma(\theta_0) = (\Sigma_{ij}(\theta_0))_{i,j=1,...,q} \rightarrow N(0, \Sigma(\theta_0)) \text{ in distribution,} \]
In the following, we will verify that the variance–covariance matrix $\Sigma(\theta_0)$ defined as (7.12) is the Fisher information matrix $I(\theta_0)$. By Lemma 2 and Corollary 1, we have

$$E^{\theta_0}_{II} \left( \frac{\partial}{\partial \theta_j} \log \| M_n \pi \| \bigg|_{\theta = \theta_0} \right) \left( \frac{\partial}{\partial \theta_k} \log \| M_n \pi \| \bigg|_{\theta = \theta_0} \right) - n \frac{\partial^2}{\partial \alpha_j \partial \alpha_k} \chi_{T_1}(\alpha) \bigg|_{\alpha = 0}$$

$$\to 0$$
as $n \to \infty$. Therefore,

$$\Sigma_{jk}(\theta_0) = \left. \frac{\partial^2}{\partial \alpha_j \partial \alpha_k} \chi_{T_1}(\alpha) \right|_{\alpha = 0}$$

$$= \lim_{n \to \infty} \frac{1}{n} E^{\theta_0}_{II} \left( \frac{\partial}{\partial \theta_j} \log \| M_n \pi \| \bigg|_{\theta = \theta_0} \right) \left( \frac{\partial}{\partial \theta_k} \log \| M_n \pi \| \bigg|_{\theta = \theta_0} \right)$$

$$= \lim_{n \to \infty} - \frac{1}{n} E^{\theta_0}_{II} \left( \frac{\partial^2}{\partial \theta_j \partial \theta_k} \log \| M_n \pi \| \bigg|_{\theta = \theta_0} \right)$$

$$= - E^{\theta_0}_{II} \left( \frac{\partial^2}{\partial \theta_j \partial \theta_k} \log \| P_{\theta}(\xi_1) \circ P_{\theta}(\xi_0) \pi \| \bigg|_{\theta = \theta_0} \right)$$

$$= E^{\theta_0}_{II} \left( \frac{\partial}{\partial \theta_j} \log \| P_{\theta}(\xi_1) \circ P_{\theta}(\xi_0) \pi \| \bigg|_{\theta = \theta_0} \right)$$

$$\times \left( \frac{\partial}{\partial \theta_k} \log \| P_{\theta}(\xi_1) \circ P_{\theta}(\xi_0) \pi \| \bigg|_{\theta = \theta_0} \right)$$

$$= I_{jk}(\theta_0).$$

\[\square\]

**APPENDIX**

**Proofs of Lemma 1 and Theorem 2.** In the following proofs we will use the same notation as in Sections 3 and 4 unless specified. Without loss of generality, in this section we consider the case $M_0 = \text{Id}$, the identity, and the transition probability $P$ of the Markov chain $\{(Y_n, M_n), n \geq 0\}$ depends on the initial state $Y_0 = y$ only. Denote it as $P_y$, and let $E_y$ be the corresponding expectation. To prove Lemma 1, we need the following lemma first.

**Lemma A.1.** Let $\{(Y_n, M_n), n \geq 0\}$ be the MIRFS of Lipschitz functions defined in (2.1) satisfying Assumption K. There exists $0 < \delta_0 < 1$ such that, for all $0 < \delta \leq \delta_0$, there exist $K > 0$, and $0 < \eta < 1$, so that

$$\sup_y E_y \left\{ \left( \frac{d(M_n^u, M_n^v)}{d(u, v)} \frac{w(Y_n)}{w(y)} \right)^\delta \right\} \leq K \eta^n, \quad \text{for } n \in \mathbb{N} \text{ and } u, v \in \mathbb{M}.$$
Proof. For given $0 < \delta < 1$, and $y \in \mathcal{Y}$, denote

$$c_n(y) = \sup \left\{ E_y \left[ \frac{d(M_n^u, M_n^v) w(Y_n)}{d(u, v) w(y)} \right] : u, v \in M \right\},$$

and let $\eta_n = \sup \{c_n(y), y \in \mathcal{Y}\}$. Denote $u_m = M_m^u$ and $v_m = M_m^v$. Let $\mathcal{F}_m$ be the $\sigma$-algebra generated by $\{(Y_k, M_k), 0 \leq k \leq m\}$. Then

$$E_y \left\{ \delta \left( \frac{d(M_{n+m}^u, M_{n+m}^v) w(Y_{n+m})}{d(u, v) w(y)} \right) \right\}$$

$$= E_y \left\{ \delta \left( \frac{d(F_n(m) M_n^u, F_n(m) M_n^v) w(Y_{n+m})}{d(u, v) w(y)} \right) \right\}$$

$$= \delta \left( \frac{d(M_m^u, M_m^v) w(Y_m)}{d(u, v) w(y)} \right) E_{Y_n} \left\{ \delta \left( \frac{d(M_{m+n}^u, M_{m+n}^v) w(Y_{n+m})}{d(u, v) w(y)} \right) \right\}$$

$$\leq \delta \left( \frac{d(M_m^u, M_m^v) w(Y_m)}{d(u, v) w(y)} \right) \eta_n \left( \frac{d(M_m^u, M_m^v) w(Y_m)}{d(u, v) w(y)} \right)^\delta.$$
Note that in the last inequality we use $d$.

For $u, v \in M$, we have

$$\eta_p \leq 1 - d\delta + d^2 \sup_{y \in \mathcal{Y}} E_y \left\{ \left( G_p + \log \frac{w(Y_p)}{w(y)} \right)^2 \exp \left( \delta G_p + \delta \log \frac{w(Y_p)}{w(y)} \right) \right\}.$$

Therefore, we can choose $\delta_0 > 0$ small enough so that $\eta_p < 1$. Along with (A.1), we obtain the proof. \(\square\)

**Proof of Lemma 1.** For given $\varphi \in \mathcal{H}$, $y \in \mathcal{Y}$, and $u, v \in M$, if $m \leq n$, we have, for $0 < \delta \leq \delta_0 < 1$,

$$|T^n \varphi(y, u) - E_y \varphi(Y_n, F_{n:m}(v))|/w(y)$$

$$= |E_y \varphi(Y_n, M^n_u) - E_y \varphi(Y_n, F_{n:m}(v))|/w(y)$$

$$\leq \|\varphi\|_h E_y \left\{ d(M^n_u, F_{n:m}(v))^{\delta} \frac{w(Y_n)}{w(Y_{n-m})^{\delta}} \mathcal{F}_{n-m} \frac{w(Y_{n-m})^{\delta}}{w(y)} \right\}$$

$$\leq \|\varphi\|_h E_y \left\{ \sup_{u, v \in M} E_{Y_{n-m}} \left[ \left( d(M^n_u, M^n_v) \frac{w(Y_n)}{w(Y_{n-m})} \right)^{\delta} \frac{w(Y_{n-m})^{\delta}}{w(y)} \right] \right\}$$

Note that in the last inequality we use $d(u, v) \leq 1$ and $w(y) \geq 1$ for all $y \in \mathcal{Y}$.

By making use of Lemma A.1, and $\sup_{y \in \mathcal{Y}} E_y [w(Y_1)/w(y)] < \infty$ in (4.2), there exist $K > 0$ and $0 < \eta < 1$ such that

(A.2) $|T^n \varphi(y, u) - E_y \varphi(Y_n, F_{n:m}(v))|/w(y) \leq \|\varphi\|_h K \eta^m \leq \|\varphi\|_{w} K \eta^m$.

Denote $h(y) = E_y \varphi(Y_m, F_m(v))$. Then by assumption (4.1), there exist $\gamma > 0$ and $0 < \rho < 1$ such that

$$|E_y \varphi(Y_n, F_{n:m}(v)) - E \varphi(Y_m, F_m(v))|/w(y)$$

$$\leq |E_y \{ E_{Y_{n-m}} \varphi(Y_m, F_m(v)) \} - E \varphi(Y_m, F_m(v))|/w(y)$$

(A.3) $$\leq \|\varphi\|_{w} \gamma \rho^{n-m}.$$
For given $m, k \in \mathbb{N}$, by using Lemma A.1 again we have

\[ |E \varphi(Y_m, F_m(v)) - E \varphi(Y_{m+k}, F_{m+k}(v))|/w(y) \]
\[ \leq E \{ |\varphi(Y_{m+k}, F_{m+k}; f(M^n_k))|/w(y) \} \]
\[ \leq \| \varphi \|_h E \left\{ d(F_{m+k}; f), F_{m+k}(M^n_k) \delta w(Y_{m+k}) \right\} \]
\[ \leq \| \varphi \|_h E \left\{ \sup_{u, v \in M} E^y_m \left[ \left( \frac{d(M^n_m, M^n_n)}{d(u, v)} w(Y_{m+k}) \right)^\delta \right] \right\} \]
\[ \leq \| \varphi \|_{wh} K \eta^m. \]

By making use of (A.2), (A.3) and the above inequality, we have that for any given $n \geq m, k \geq 0$, and for all $u, v \in M$,

\[ |T^n \varphi(y, u) - E \varphi(Y_{m+k}, F_{m+k}(v))|/w(y) \leq \| \varphi \|_{wh} (2K \eta^m + \gamma \rho^{n-m}). \]

By setting $m = n/2$, we have that there exist $A > 0$ and $0 < r < 1$ such that

(A.4) \[ \| T^n \varphi(y, u) - Q \varphi(y, u) \|_w \leq \| \varphi \|_{wh} A r^n. \]

On the other hand, for $u, v \in M$,

\[ \frac{|(T^n - Q) \varphi(y, u) - (T^n - Q) \varphi(y, v)|}{(w(y) d(u, v))^\delta} \]
\[ = \left| E_y \varphi(Y_n, M^n_u) - \int \varphi(y, u) \Pi(dy \times du) \right. \]
\[ - E_y \varphi(Y_n, M^n_v) + \int \varphi(y, v) \Pi(dy \times dv) \]
\[ \times [(w(y) d(u, v))^{-\delta}]^{-1} \]
\[ \leq \frac{E_y \{ |\varphi(Y_n, M^n_u) - \varphi(Y_n, M^n_v)| \}}{(w(y) d(u, v))^\delta} \]
\[ \leq \| \varphi \|_h \sup_y E_y \left\{ \left( \frac{d(M^n_m, M^n_n)}{d(u, v)} w(Y_n) \right)^\delta \right\} \]
\[ \leq \| \varphi \|_{wh} K \eta^n \quad \text{by Lemma A.1}. \]

Denote $\rho_* = \min \{ \eta, r \}$ and $\gamma_* = A + K$. Combine (A.4) and (A.5) to get

\[ \| T^n - Q \|_{wh} = \sup_{\varphi \in \mathcal{H}_l, \| \varphi \|_{wh} \leq 1} \| T^n \varphi - Q \varphi \|_{wh} \leq \sup_{\varphi \in \mathcal{H}_l, \| \varphi \|_{wh} \leq 1} \| \varphi \|_{wh} \gamma_* \rho_*^n \leq \gamma_* \rho_*^n. \]

Then we have (4.11) and this completes the proof. \(\square\)
Proof of Theorem 2. By using Lemma 2, standard arguments involving smoothing inequalities and Fourier inversion (cf. Chapter 4 of [5]) reduce the proof to that of showing for every $\delta > 0, a > 0$ and $b > 1$,

$$(A.6) \quad \sup_{|\alpha| \leq n^a} |E_\pi(e^{i\alpha' S_n})| = o(n^{-b}).$$

To prove (A.6), we follow the same idea as (3.43) of [31], letting $\zeta_t = S_t - S_{t-1}$ $(t = 1, 2, \ldots)$, $\zeta_0 = S_0$ and $\tilde{\varphi}((y, u), (y', v)) = E\{e^{i\alpha' \zeta_1} | (Y_0 = y, M_0 = u), (Y_1 = y', M_1 = v)\}$.

Let $J = \{1, \ldots, n\}$, and fix $m > 1$ to be determined later. Divide $J$ into blocks $A_1, B_1, \ldots, A_l, B_l$ as follows. Define $j_1, \ldots, j_l$ by $j_1 = 1$, and $j_{k+1} = \inf\{j \geq j_k + 7m : j \in J\}$, and let $l$ be the smallest integer for which the inf is undefined. Write

$$A_k = \prod\{e^{n^{-1/2}i\alpha' \zeta_1} : |j - j_k| \leq m\}, \quad k = 1, \ldots, l,$$

$$B_k = \prod\{e^{n^{-1/2}i\alpha' \zeta_1} : j_k + m + 1 \leq j \leq j_{k+1} - m - 1\}, \quad k = 1, \ldots, l - 1,$$

$$B_l = \prod\{e^{n^{-1/2}i\alpha' \zeta_1} : j > j_l + m + 1\}.$$

Then $e^{i\alpha' S_n} = \prod_{k=1}^l A_k B_k$. Given $y \in \mathcal{Y}$, we have

$$(A.7) \quad \left| E_y \prod_{k=1}^l A_k B_k - E_y \prod_{k=1}^l B_k E(A_k | \zeta_j : j \neq j_k) \right| \leq \sum_{q=1}^l \left| E_y \prod_{k=1}^{q-1} A_k B_k (A_q - E(A_k | \zeta_j : j \neq j_q)) \prod_{q+1}^l B_k E(A_k | \zeta_j : j \neq j_k) \right|.$$

By using Lemma 2(iv), there exists $\delta > 0$ such that $E|E(A_k | \zeta_j : j \neq j_q) - E(A_k | \zeta_j : 0 < |j - j_k| \leq 3m)| \leq e^{-\delta m}$. Therefore, (A.7) $\leq$

$$\sum_{q=1}^l \left| E_y \prod_{k=1}^{q-1} A_k B_k (A_q - E(A_k | \zeta_j : j \neq j_q)) \right| \times \left| \prod_{q+1}^{l} B_k E(A_k | \zeta_j : 0 < |j - j_k| \leq 3m) \right| + \sum_{q=1}^l e^{-\delta m}.$$

The first summation term in (A.8) vanishes since $\prod_{q=1}^{l} A_k B_k$ and $\prod_{q+1}^{l} B_k \times E(A_k | \zeta_j : 0 < |j - j_k| \leq 3m)$ are both measurable with respect to the $\sigma$-field generated by $\zeta_j : j \neq j_q$. 

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Recall that the functions \(E(A_k | \zeta_j : 0 < |j - j_k| \leq 3m)\), for \(k = 1, \ldots, l\), are weakly dependent since \(j_{k+1} - j_k \geq 7m, k = 1, \ldots, l - 1\). Using Assumption K1, (4.14) and (4.15), we obtain

\[
\left| E_y \prod_1^l B_k E(A_k | \zeta_j : 0 < |j - j_k| \leq 3m) \right|
\leq E_y \left| \prod_1^l E(A_k | \zeta_j : 0 < |j - j_k| \leq 3m) \right|
\leq \prod_1^l E_y |E(A_k | \zeta_j : 0 < |j - j_k| \leq 3m)| + le^{-\delta m}.
\]

With the strong nonlattice condition (4.16), and conditional strong nonlattice condition (4.17), we find an upper bound for \(E_y |E(A_k | \zeta_j : 0 < |j - j_k| \leq 3m)|\).

We have for \(|\alpha| \geq \delta\) the relation \(E_y |E(A_k | \zeta_j : j \neq j_q)| \leq e^{-\delta}\) and, hence, by (4.17) for all \(\alpha \in R^p, |\alpha| \leq \delta, E_y |E(A_k | \zeta_j : j \neq j_q)| \leq \exp(-\delta|\alpha|^2/n)\). Therefore, for all \(\alpha \in R^p,\)

\[
E_y |E(A_k | \zeta_j : 0 < |j - j_k| \leq 3m)|
\leq e^{-\delta m} + E_y |E(A_k | \zeta_j : j \neq j_q)| \leq e^{-\delta m} + \max(\exp(-\delta|\alpha|^2/n), e^{-\delta}).
\]

If we choose \(K\) appropriately and let \(m\) be the integral part of \(K \log n\), then the assertion of the lemma follows from \(\exp(-\delta|\alpha|^2/n)^{n/m} \leq \exp(-\delta|\alpha|^2 / (K \log n)) \leq \exp(-\delta n^\epsilon/2)\) for \(|\alpha| \geq cn^\epsilon\) and some \(\delta > 0. \)

**Acknowledgments.** The author is grateful to the Editor Professor Jianqing Fan, an Associate Editor and a referee for constructive comments, suggestions and correction of some errors in the earlier version.

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ON SOME PROBLEMS IN THE ARTICLE EFFICIENT LIKELIHOOD ESTIMATION IN STATE SPACE MODELS
BY CHENG-DER FUH
[ANN. STATIST. 34 (2006) 2026–2068]

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1. Introduction. Upon reading the paper Efficient Likelihood Estimation in State Space Models by Cheng-Der Fuh I found a number of problems in the formulations and a number of mathematical errors. Together, these findings cast doubt on the validity of the main results in their present formulation. A reformulation and new proofs seem quite involved.

The paper, Efficient Likelihood Estimation in State Space Models deals with asymptotic properties of the maximum likelihood estimate in hidden Markov models. The hidden Markov chain is \( X_n \), and the observed process is \( \xi_n \) where \( \xi_n \) conditioned on the past and the hidden process depends on \((X_n, \xi_{n-1})\) only. The approach used is to add an iterated function system \( M_n \), and to consider the Markov process \((X_n, \xi_n, M_n)\). This is very much akin to the method in Douc and Matias [1], and I will use this article as a background for my comments.

2. Problems.

2.1. Definition of iterated function system. The first basic definition in the paper is a function \( P_\theta(\xi_j) : M \to M \) that maps a function \( h \in M \) into a new function in \( M \) (page 2031),

\[
P_\theta(\xi_j)h(x) = \int_{y \in \mathcal{X}} p_\theta(x, y)f(\xi_j; \theta | y, \xi_{j-1})h(y)m(dy).
\]

[It is unclear why the author states that \( P_\theta(\xi_j) \) is a function on \((\mathcal{X} \times \mathbb{R}^d) \times M\) where \( \mathcal{X} \) is the state space of the Markov chain.] The paper next defines the
composition $P_\theta(\xi_{j+1}) \circ P_\theta(\xi_j) \circ h$ by first applying $P_\theta(\xi_{j+1})$ to $h$ and then applying $P_\theta(\xi_j)$ to the result. Using these two definitions we have

$$P_\theta(\xi_n) \circ \cdots \circ P_\theta(\xi_1) \circ P_\theta(\xi_0) \pi_\theta = \int \pi_\theta(x_n) \left\{ \prod_{j=n}^{1} p_\theta(x_{j-1}, x_j) f(\xi_j; \theta|x_j, \xi_{j-1}) m(dx_j) \right\} f(\xi_0; \theta|x_0) m(dx_0).$$

The argument presented in the paper then appears to assume that this expression depends on some $x$ and performs an integration before claiming that the result is the joint density $p_n(\xi_0, \ldots, \xi_n; \theta)$. This is clearly not correct since $\pi_\theta(x_n)$ appears in the expression instead of $\pi_\theta(x_0)$.

Following the work of Douc and Matias [1] one would instead use the definition

$$P_\theta(\xi_j) h(x) = \int_{y \in X} p_\theta(y, x) f(\xi_j; \theta|y, \xi_{j-1}) h(y) m(dy);$$

that is, the integration is with respect to the first variable in $p_\theta(y, x)$ instead of the second. Changing the definition of $P_\theta(\xi_0)$ correspondingly and using ordinary composition of functions, one finds that $p_n(\xi_0, \ldots, \xi_n; \theta)$ equals the integral of $P_\theta(\xi_n) \circ \cdots \circ P_\theta(\xi_1) \circ P_\theta(\xi_0) \pi_\theta$ with respect to $x_{n+1}$. However, making this change necessitates a new proof for the first part of Lemma 3 on page 2056. Comparing with Douc and Matias ([1], Proposition 1) we see that this is one of the places where the latter authors use the stronger assumptions of that paper on the Markov chain.

Turning to the iterated function system, Fuh’s paper defines this as

$$M_n = P_\theta(\xi_n) \circ \cdots \circ P_\theta(\xi_1) \circ P_\theta(\xi_0)$$

[formula (5.6), page 2045]. Taking this literally, and using the definitions in Fuh’s paper, this is actually a mapping that takes a function as input and turns it into a constant. Instead $M_n$ should be a function obtained by applying a mapping to $M_{n-1}$. This is achieved when using the definition suggested in (1) and adding $\pi_\theta$ to the right-hand side of $M_n$ above.

2.2. Harris recurrence of iterated function. Whether or not we make the changes suggested in the previous subsection, $M_n$, defined on page 2045, is related to the density of $(\xi_0, \ldots, \xi_n)$. Making the change suggested in (1) above we have precisely $M_n(x_{n+1}) = p(x_{n+1}, \xi_0, \ldots, \xi_n)$. Such an expression will typically tend to either zero or infinity. However, in Lemma 4 on page 2046 Fuh claims that $(X_n, \xi_n, M_n)$ is a Harris recurrent Markov chain. It is difficult to pinpoint the exact origin of this problem. The Harris recurrence is established in Lemma 4 which in its formulation uses a measure $Q$ from Theorem 1 (in the formulation there are two $Q$’s, but these are different).
So we need to establish Theorem 1 before proving Lemma 4. In Lemma 3 it is stated that the Markov iterated function system satisfies Assumption K. In Remark 1 (page 2035) Fuh says that Assumption K is different from the assumptions of Theorem 1. He then goes on to say that if Assumption K is supplemented with the extra assumption that \((Y_n, M_n)\) is a Harris recurrent Markov chain, then Theorem 1 still holds. This, therefore, seemingly looks like a circular argument.

Comparing again with Douc and Matias [1] they consider instead
\[ M_n(x_{n+1}) = p(x_{n+1} | \xi_0, \ldots, \xi_n). \]
However, if we make this change we have introduced a new iterated function system, and a revised version of Lemma 3 is needed which presumably will lead to a different set of assumptions.

2.3. Asymptotic properties of score function and observed information.

The asymptotic normality of the score function is stated in Lemma 6 (page 2048). In the proof of Lemma 6 (page 2060) the author appeals to Corollary 1. The latter gives a central limit theorem for a sum of the form \( \sum_{j=1}^{n} g(M_j) \). However, the paper wants to use this result on the sum \( \sum_{j=1}^{n} \frac{\partial}{\partial \theta} g(M_{j-1}, M_j) \). This looks innocent, but since \( \theta \) appears in the iteration of \( M_n \) this is not on the form \( \sum_{j=1}^{n} \tilde{g}(M_{j-1}, M_j) \). Instead one needs to consider a new iterated function system. This is what is done in Appendix D of Douc and Matias [1].

Similarly, it is stated that the proof of the main Theorem 5 follows a standard argument. However, comparing with Douc and Matias [1] (Appendix D.3) it seems that yet another iterated function system is needed to deal with the convergence of the observed information.

2.4. Generality of conditions.

Assumption C5 on page 2043 restricts the dependency of the observed process on the hidden process. For the example considered in (b) on page 2044 one needs to consider
\[
\sup_{y, z \in X} \frac{f(\xi_0; \theta | y) f(\xi_1; \theta | y, \xi_0)}{f(\xi_0; \theta | z) f(\xi_1; \theta | z, \xi_0)}
= \sup_{y, z \in X} \exp\{-1/2(\xi_0 - y)^2 - 1/2(\xi_1 - y)^2\}
= \sup_{y, z \in X} \exp\{\xi_0^2 - 2\xi_0 y + (\xi_0 + \xi_1)(y - z)\} = \infty.
\]
Thus C5 is not satisfied (this seems to be contrary to the claim on page 2054 line 8 from the bottom).

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REPLY TO “ON SOME PROBLEMS IN THE ARTICLE EFFICIENT LIKELIHOOD ESTIMATION IN STATE SPACE MODELS”

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[ANN. STATIST. 34 (2006) 2026–2068]

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The author is grateful for the comments by Dr. Jensen. This note is in reply to his comments.

Problem 2.1. Definition of iterated function system.

(2.6) \[ P_\theta(\xi_j)h(x) = \int_{y \in \mathcal{X}} p_\theta(y, x)f(\xi_j; \theta|x, \xi_{j-1})h(y)m(dy). \]

Define the composition of two random functions as

\[ P_\theta(\xi_{j+1}) \circ P_\theta(\xi_j)h(x) = \int_{z \in \mathcal{X}} p_\theta(z, x)f(\xi_{j+1}; \theta|x, \xi_j) \]

\[ \times \left( \int_{y \in \mathcal{X}} p_\theta(y, z)f(\xi_j; \theta|z, \xi_{j-1})h(y)m(dy) \right)m(dz). \] (2.7)

Page 2042. C1. . . .

for all \( s_0, s_1 \in \mathbb{R}^d \), and \( \sup_{x \in \mathcal{X}} \int p_\theta(y, x)m(dy) < \infty \). Since \( m \) is \( \sigma \)-finite, there exist pairwise disjoint \( \mathcal{X}_n \) such that \( \mathcal{X} = \bigcup_{n=1}^\infty \mathcal{X}_n \), and \( 0 < m(\mathcal{X}_n) < \infty \). Assume \( E[\sum_{n=1}^\infty \frac{1}{\mathcal{X}_n} \sup_{x \in \mathcal{X}_n} f(\xi; \theta|x, s_0)] < \infty \) for all \( s_0 \in \mathbb{R}^d \). Denote \( g_\theta(\xi_0, \xi_1) = \sup_{x \in \mathcal{X}} \int p_\theta(y, x)f(\xi_1; \theta|x, \xi_0)m(dy) \). Furthermore, we assume that there exists \( p \geq 1 \) as in K2 such that

(5.2) \[ \sup_{(x_0, s_0) \in \mathcal{X} \times \mathbb{R}^d} E_{(x_0, s_0)}^\theta \left\{ \log \left( g_\theta(s_0, \xi_1) \cdots g_\theta(\xi_{p-1}, \xi_p) \frac{w(X_p, \xi_p)}{w(x_0, s_0)} \right) \right\} < 0. \]
The example on Page 2044, L12, holds if \( \alpha \neq 0 \). The original (5.6) was wrong; it should be

\[
M_n := P_\theta(\xi_n) \circ \cdots \circ P_\theta(\xi_1) \circ P_\theta(\xi_0) \pi \quad \text{(page 2045)}.
\]

Page 2046. **Lemma 3.** ... Furthermore, under conditions C1, C6–C9, the function \( g \) defined in (5.7) belongs to \( \mathcal{L}(Q \times Q) \).

**Proof of Lemma 3.** We consider only the case of \( P(\xi_1) \), since the case of \( P(\xi_0) \) and \( P(\xi_j) \), for \( j = 2, \ldots, n \), is a straightforward consequence. For any two elements \( h_1, h_2 \in M \), and two fixed elements \( s_0, s_1 \in R^d \), by (5.8) we have

\[
d(P(s_1)h_1, P(s_1)h_2)
\]

\[
= \sup_{x \in X} \left| \int_p p_\theta(y,x)f(s_1; \theta|x, s_0)h_1(y)m(dy) - \int_p p_\theta(y,x)f(s_1; \theta|x, s_0)h_2(y)m(dy) \right|
\]

\[
\leq d(h_1, h_2) \sup_{x \in X} \int_p p_\theta(y,x)f(s_1; \theta|x, s_0)m(dy)
\]

\[
\leq C \left( \sup_{x \in X} \int_p p_\theta(y,x)m(dy) \right) d(h_1, h_2),
\]

where \( 0 < C = \sup_{x \in X} f(s_1; \theta|x, s_0) < \infty \), and by assumption C1, is a constant. Note that \( \sup_{x \in X} \int_p p_\theta(y,x)m(dy) < \infty \) by assumption C1. The equality holds only if \( h_1 = h_2 \ m\text{-almost surely. This proves the condition of Lipschitz continuity in the second argument.} \)

Note that C1 implies that K1 holds. Recall that \( M_n = P(\xi_n) \circ \cdots \circ P(\xi_1) \circ P(\xi_0) \pi \) for \( \pi \in M \) in (5.6). To prove the weighted mean contraction property K2, we observe that for \( p \geq 1 \),

\[
\sup_{x_0, s_0} \mathbb{E}_{(x_0, s_0)} \left\{ \log \left( \frac{w(X_p, \xi_p)}{w(x_0, s_0)} \right) \right\}
\]

\[
= \sup_{x_0, s_0} \mathbb{E}_{(x_0, s_0)} \left\{ \log \left( \sup_{h_1 \neq h_2} \frac{d(M_p h_1, M_p h_2)}{d(h_1, h_2)} \frac{w(X_p, \xi_p)}{w(x_0, s_0)} \right) \right\}
\]

\[
< \sup_{x_0, s_0} \mathbb{E}_{(x_0, s_0)} \left\{ \log \left( \prod_{j=1}^p \sup_{x_j \in X} \int_p p_\theta(x_{j-1}, x_j)
\times f(\xi_j; \theta|x_j, s_{j-1})
\times m(dx_{j-1}) \right) \frac{w(X_p, \xi_p)}{w(x_0, s_0)} \right\} < 0.
\]


The last inequality follows from (5.2) in condition C1.

To verify that assumption K3 holds, as \( m \) is \( \sigma \)-finite, we have \( \mathcal{X} = \bigcup_{n=1}^{\infty} \mathcal{X}_n \) where the \( \mathcal{X}_n \) are pairwise disjoint and \( 0 < m(\mathcal{X}_n) < \infty \). Set

\[
(7.2) \quad h(x) = \sum_{n=1}^{\infty} \frac{I_{\mathcal{X}_n}(x)}{2^n m(\mathcal{X}_n)}.
\]

It is easy to see that \( \int_{x \in \mathcal{X}} h(x) m(dx) = 1 \) and hence belongs to \( \mathbf{M} \). Observe that

\[
\mathbf{E} d^2(\mathbf{P}(\xi_1)h, h)
= \mathbf{E} \left[ \sup_{x_1 \in \mathcal{X}} \int p_{\theta}(x_0, x_1) f(\xi_1; \theta|x_1, s_0) h(x_0) m(dx_0) - h(x_1) \right]
\leq \mathbf{E} \left[ \sum_{n=1}^{\infty} \frac{1}{2^n} \sup_{x_1 \in \mathcal{X}_n} f(\xi_1; \theta|x_1, s_0) \right] \left[ \sup_{x_1 \in \mathcal{X}} \int p_{\theta}(x_0, x_1) m(dx_0) \right] + \sup_{x_1 \in \mathcal{X}} |h(x_1)|.
\]

Note that \( h(x) \) is piecewise constant by definition (7.2), \( E[\sum_{n=1}^{\infty} \frac{1}{2^n} \sup_{x \in \mathcal{X}_n} f(\xi_1; \theta|x, s_0)] < \infty \) for all \( s_0 \in \mathbb{R}^n \) by assumption C1 and \( p_{\theta}(x_0, x_1) \) is integrable of \( x_0 \) over the subset \( \mathcal{X}_n \) by assumption C1. These imply that (7.3) is finite.

Finally, we observe

\[
\sup_{x_0, s_0} \mathbf{E}(x_0, s_0) \left\{ \frac{L_1 w(X_1, \xi_1)}{w(x_0, s_0)} \right\}
= \sup_{x_0, s_0} \mathbf{E}(x_0, s_0) \left\{ \sup_{h_1 \neq h_2} \frac{d(\mathbf{P}(\xi_1)h_1, \mathbf{P}(\xi_1)h_2)}{d(h_1, h_2)} \frac{w(X_1, \xi_1)}{w(x_0, s_0)} \right\}
< \sup_{x_0, s_0} \mathbf{E}(x_0, s_0) \left\{ \sup_{x_1 \in \mathcal{X}} \int p_{\theta}(x_0, x_1) f(\xi_1; \theta|x_1, s_0) m(dx_0) \right\} \frac{w(X_1, \xi_1)}{w(x_0, s_0)}
< \infty.
\]

The last inequality follows from (5.3) in condition C1.

Note that C8 and C9 imply that \( g \in \mathcal{L}(Q \times Q) \). Hence, the proof is complete. \( \square \)

**Problem 2.2. Harris recurrence of iterated function.** This paper is an extension of Fuh (2003) for finite state space in which the likelihood function can be expressed as the \( L_1 \)-norm of products of Markovian random matrices. Note that \( M_n \) defined in (5.6) is an iterated random functions system governed by a Markov chain \( Y_n \). And \( Y_n = (X_n, \xi_n) \) in the state space models case. In Theorem 1 I only assume \( Y_n = (X_n, \xi_n) \) is Harris recurrent.
The purpose of the statement, "Note that under K1–K3, . . . a Markovian iterated random functions system in Theorem 2," is to relate Theorems 1 and 2, to which I can apply limiting theorems in Markov chains to the law of large numbers and central limit theorem (and Edgeworth expansion) for \((Y_n, M_n)\).

In Lemma 4 I want to prove \(Z_n = ((X_n, \xi_n), M_n)\) is Harris recurrent (\(Z_n\) is defined in lines 1 and 2 on page 2056). In the proof, I can use the results in Theorem 1 since only \(Y_n = (X_n, \xi_n)\) is assumed to be Harris recurrent in Theorem 1. It is known that C1 implies that \(Y_n = (X_n, \xi_n)\) is Harris recurrent. A new proof of Lemma 3 was given on pages 1 and 2.

**Problem 2.3. Asymptotic properties of score function and observed information.** Page 2060, L12. In the proof of Lemma 6, (7.9) defined a new iterated functions system; therefore Corollary 1 cannot be used directly. The same situation happens for Theorems 5 and 7. The rigorous proofs of these results will be given in a separate paper.

**Problem 2.4. Generality of conditions.** C5. For \(\theta \in N_\delta(\theta_0)\),

\[
E^\theta_x \left( \frac{\partial \log \int_{y \in X} \pi(x)p(x, y) f(s_0; \theta|x)f(\xi_1; \theta|y, s_0)m(dy)}{\partial \theta_i} \right)^2 < w(x, s_0)
\]

for all \(i = 1, \ldots, q\).

Change C5 accordingly. It is straightforward to check that C5 holds for the examples considered in Section 6. The proof of Lemma 5 can be done under C5.

**Other typos and mistakes.** Page 2032, L1. \(\cdots p_\theta(y, x)f(\xi_j; \theta|x, \xi_{j-1}) \cdots \)

\(X_{j-1} = y\) and \(X_j \in dx, \ldots \)

(3.7) \(\pi(y)P(Y_n \in dz, M_n \in \cdot|Y_0 = y) = \pi(z)\tilde{P}(\tilde{Y}_n \in dy, \tilde{M}_n \in \cdot|\tilde{Y}_0 = z)\).

Page 2028, L5. \((1 - \alpha^2)\). Page 2043, C7, \(\theta \rightarrow \varphi_x(\theta)\) was a typo; delete it. Page 2047, L1, then, "each component of" the Fisher information matrix. L5, replace "positive definite" by "finite." Page 2048, Theorem 5, assume \(I(\theta_0)\) is invertible. Page 2057, L3, the notation \(m \times Q \times Q\) may be confusing; change it to \(m \times Q \times \bar{Q}\).