An obstruction to the smoothability of singular nonpositively curved metrics on 4-manifolds by patterns of incompressible tori

Stephan Stadler *

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Abstract

We give new examples of closed smooth 4-manifolds which support singular metrics of nonpositive curvature, but no smooth ones, thereby answering affirmatively a question of Gromov. The obstruction comes from patterns of incompressible 2-tori sufficiently complicated to force branching of geodesics for nonpositively curved metrics.

Contents

1 Introduction 1
2 Preliminaries 2
   2.1 Quasi-isometry invariance of flats 2
   2.2 Product rigidity 3
   2.3 Coarse intersection of flats and quasi-isometry invariance 3
3 Configurations of convex product subsets in dimension 4 5
   3.1 Flat half-strips in CAT(0) surfaces with symmetries 5
   3.2 Configurations not occuring in smooth spaces 6
   3.3 Not equivariantly smoothable configurations 6
4 An example 8

1 Introduction

The goal of this note is to exhibit new examples of closed smooth 4-manifolds which support singular metrics of nonpositive (sectional) curvature, but no smooth ones.

*E-mail address: sstadler@math.uni-koeln.de
Such manifolds had first been found by Davis, Januszkiewicz and Lafont [DJL12]. The approaches are different, but they both rely on a basic rigidity phenomenon in nonpositive curvature, namely that free abelian subgroups in the fundamental group of a closed nonpositively curved manifold are carried by totally-geodesically immersed flat tori. The fundamental groups $\Gamma$ of the singular locally CAT(0) 4-manifolds $M$ studied in [DJL12] contain few (“isolated”) copies of $\mathbb{Z}^2$. The obstruction to the existence of a smooth nonpositively curved metric on $M$ is that they, respectively, the corresponding invariant flats in the universal covering $\tilde{M}$, are knotted at infinity. This is impossible for 2-flats in smooth Hadamard 4-manifolds. We consider fundamental groups $\Gamma$ which contain plenty of copies of $\mathbb{Z}^2$ and exploit the fact that this rigidifies the geometry of nonpositively curved metrics on $M$, singular or smooth, since it enforces a complicated pattern of immersed flat 2-tori. Extreme cases occur in “higher rank”: When $\Gamma$ splits as a product $\Gamma_1 \times \Gamma_2$ of subgroups, then the universal cover splits as a metric product. Or (in dimensions $\geq 5$) when $\Gamma$ is the fundamental group of an irreducible higher rank locally symmetric space of noncompact type, then the geometry of nonpositively curved metrics is completely rigidified by Mostow rigidity, i.e. it is essentially unique up to rescaling. We consider “rank one” situations where there are still plenty of subgroups isomorphic to $\mathbb{Z}^2$ which however only partially rigidify the geometry. Heuristically, singular nonpositively curved metrics allow more complicated patterns of tori than smooth ones because, due to possible branching, the tori can be packed “more densely”. It is therefore conceivable that sufficiently complicated patterns which occur for singular nonpositively curved metrics cannot occur in the smooth case because they enforce the branching of geodesics. Indeed, natural candidates to which this line of reasoning could apply had been pointed out by Gromov in (the first exercise of) [BGS85], namely the fundamental groups of branched coverings $M \to \Sigma \times \Sigma$ of products of higher genus surfaces $\Sigma$ with themselves with branching locus the diagonal. The purpose of this note is to do this exercise.

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2 Preliminaries

For notations and basics on CAT(0) spaces we refer the reader to the first two chapters of [B95] and Section 2 of [KL97].

2.1 Quasi-isometry invariance of flats

A flat $F$ in a CAT(0) space $X$ is a convex subset isometric to a Euclidean space.

If $\Gamma \acts X$ is an isometric action, then a flat $F \subset X$ is called $\Gamma$-periodic if its stabilizer $\text{Stab}_\Gamma(F)$ acts cocompactly on it. Note that if $\Gamma$ acts discretely on $X$, then a finite index subgroup of $\text{Stab}_\Gamma(F)$ acts on $F$ by translations, and hence $\text{Stab}_\Gamma(F)$ is virtually free abelian of rank equal to the dimension of $F$. 

2
An isometric action $\Gamma \curvearrowright X$ of a discrete group $\Gamma$ on a locally compact CAT(0) space $X$ is called geometric if it is properly discontinuous and cocompact. Then every abelian subgroup $A \subset \Gamma$ preserves a flat in $X$ on which it acts cocompactly.

Suppose that $\Gamma \curvearrowright X$ and $\Gamma \curvearrowright X'$ are geometric actions of the same group on two locally compact CAT(0) spaces. Then there exist $\Gamma$-equivariant quasi-isometries $\Phi : X \to X'$.

If $F \subset X$ is a $\Gamma$-periodic flat, then its stabilizer $\text{Stab}_\Gamma(F)$ is virtually abelian and preserves a flat $F' \subset X'$. Hence, a $\Gamma$-equivariant quasi-isometry $\Phi : X \to X'$ carries $\Gamma$-periodic flats in $X$ Hausdorff close to $\Gamma$-periodic flats in $X'$.

One can also say something regarding the quasi-isometry invariance of non-periodic flats: Recall that, for locally compact CAT(0) spaces with cocompact isometry group, the maximal dimension of flats equals the maximal dimension of quasi-flats and is in particular a quasi-isometry invariant [K99, Thm. C]. By Theorem B in [L S97], quasi-flats of maximal dimension which are within finite Hausdorff distance from (maximal) flats are actually within uniformly bounded Hausdorff distance from these flats. Combining these results, one obtains the following useful

**Proposition 1.** There exists a constant $D = D(L, A, X, X')$ such that a $\Gamma$-equivariant $(L, A)$-quasi-isometry $\Phi : X \to X'$ maps $\Gamma$-periodic flats of maximal dimension in $X$ $D$-Hausdorff close to such flats in $X'$. As a consequence, also pointed Hausdorff limits of $\Gamma$-periodic flats of maximal dimension in $X$ are carried $D$-Hausdorff close to such flats in $X'$.

### 2.2 Product rigidity

We will need the following product splitting result which is a special case of Corollary 10 in [M06]. See also Proposition 2.2 in [L00] and Theorem 1 in [S85].

**Proposition 2.** Let $X$ be a locally compact CAT(0) space and let $\Gamma \cong \Gamma_1 \times \Gamma_2$ be a product of non-abelian free groups $\Gamma_i$. Suppose that $\Gamma$ acts on $X$ discretely by axial isometries. Then there exists a minimal non-empty $\Gamma$-invariant closed convex subset $C \subset X$ which splits metrically as a product, $C \cong C_1 \times C_2$, such that $\Gamma$ preserves the product splitting and $\Gamma_i$ acts trivially on $C_{3-i}$.

**Remark 1.** If $X$ is 4-dimensional, then the set $C$ is unique. Indeed, the factors $C_i$ would have to be 2-dimensional. Since two minimal non-empty $\Gamma$-invariant closed convex subsets are parallel and $X$ is 4-dimensional, $C$ is unique.

### 2.3 Coarse intersection of flats and quasi-isometry invariance

Let $F_1, F_2 \subset X$ be flats. We say that they diverge if $\partial_\infty F_1 \cap \partial_\infty F_2 = \emptyset$. Equivalently, the distance function $d(\cdot, F_2)|_{F_1}$ is proper and grows (at least) linearly.

**Definition 1.** Let $F_1, F_2 \subset X$ be diverging flats. We say that $F_1$ coarsely intersects $F_2$ if for every $r > 0$ holds: If $B_1 \subset F_1$ is a round ball such that $F_1 \cap N_r(F_2) \subset \text{int}(B_1)$, then its boundary sphere $\partial B_1$ is not contractible inside $X \setminus N_r(F_2)$. Moreover, we say that two convex subsets $C_1, C_2 \subset X$ intersect coarsely, if they contain flats which intersect coarsely.
Remark 2. (i) This is independent of the choice of the ball $B_1 \subset F_1$.

(ii) The notion is asymptotic in the sense that it only depends on the ideal boundaries of the flats, i.e. passing to parallel flats does not affect coarse intersection.

(iii) It is unclear whether coarse intersection is a symmetric relation.

(iv) In general, disjoint flats can coarsely intersect. However, this cannot occur in geodesically complete smooth spaces, i.e. in Hadamard manifolds.

We need a criterion to recognize whether flats coarsely intersect. In the smooth case “coarse intersection” simply becomes “nontrivial transversal intersection”, i.e. two flats in a Hadamard manifold intersect coarsely if and only if they intersect transversely in one point. This is clear, because for a flat $F$ in a Hadamard manifold $X$ there is a deformation retraction of $X \setminus F$ onto $X \setminus \overline{N_r(F)}$ using the gradient flow of $d(\cdot, F)$.

More generally, we have:

**Lemma 1.** Let $F_1$ and $F_2$ be flats in a CAT(0) space $X$. Suppose that $F_2$ is contained in an open convex subset $C \subset X$ which is Riemannian, i.e. the metric on $C$ is induced by a smooth Riemannian metric. If $F_1$ and $F_2$ intersect transversely in one point, then $F_1$ coarsely intersects $F_2$.

**Proof.** Otherwise spheres in $F_1 \setminus F_2$ around the intersection point $F_1 \cap F_2$ could be contracted in $X \setminus F_2$. But this would be absurd since $X \setminus F_2$ retracts to $C \setminus F_2$ along normal geodesics. \qed

It will be crucial for us that coarse intersection is quasi-isometry invariant under suitable assumptions on the CAT(0) spaces involved.

Consider a quasi-isometry $\Phi : X \to X'$ of CAT(0) spaces with a quasi-inverse $\Phi' : X' \to X$. Let $F_1, F_2 \subset X$ and $F_1', F_2' \subset X'$ be flats such that $\Phi(F_i)$ is Hausdorff close to $F_i'$. Then $F_1$ and $F_2$ diverge if and only if $F_1'$ and $F_2'$ do. If one of the CAT(0) spaces is smooth Riemannian, we have:

**Lemma 2.** Suppose that $X'$ is a smooth Hadamard manifold. If $F_1$ coarsely intersects $F_2$, then $F_1'$ coarsely intersects $F_2'$.

**Proof.** The quasi-flat $\Phi|_{F_i}$ may not be continuous, but since $X$ is CAT(0), it is uniformly (in terms of the quasi-isometry constants) Hausdorff close to a continuous quasi-flat $q : F_1 \to X'$. Suppose that $q(F_1)$ is $D$-Hausdorff close to $F_1'$. If $F_1'$ does not coarsely intersect $F_2'$, then $F_1'$ and $F_2'$ are disjoint since $X'$ is smooth. The $q$-image of a sphere in $F_1$ can be homotoped to $F_1'$ with a $D$-short homotopy and then contracted inside $F_1'$. Hence, the image of a large sphere in $F_1$ can be contracted far away from $F_2'$, using the gradient flow of $d(\cdot, F_2')$ to push the (contracting) homotopy away from $F_2'$. The $\Phi'$-image of the homotopy is again uniformly Hausdorff close to a continuous map. Since $\Phi' \circ \Phi$ is at finite distance from $id_X$, it follows that we can for every radius $r > 0$ contract sufficiently large spheres in $F_1$ in the complement of the tubular $r$-neighborhood of $F_2$. Consequently, $F_1$ does not coarsely intersect $F_2$. \qed

**Remark 3.** The above remains valid for $X'$ a locally compact CAT(0) space with extendible geodesics which does not contain isometrically embedded tripods [St13].
3 Configurations of convex product subsets in dimension 4

3.1 Flat half-strips in CAT(0) surfaces with symmetries

By a flat strip, respectively, half-strip of width \( w \geq 0 \) in a CAT(0) space we mean a convex subset isometric to \( \mathbb{R} \times [0, w] \), respectively, to \( [0, +\infty) \times [0, w] \).

The following observation restricts the possible positions of flat half-strips in a CAT(0) surface relative to the action of its isometry group.

**Lemma 3.** Let \( Y \) be a smooth CAT(0) surface, and let \( h \subset Y \) be a flat half-strip. Suppose that \( h \) is asymptotic to a periodic geodesic \( c \subset Y \), i.e. to an axis \( c \) of an axial isometry \( \gamma \) of \( Y \).

Then either \( w = 0 \), or \( h \) extends to a (periodic) flat strip in \( Y \) parallel to \( c \).

**Proof.** We may assume that \( \gamma \) translates towards the ideal endpoint of \( h \) and preserves the orientation transversal to \( c \). If \( w > 0 \) and \( r(t) \) is a ray in \( \text{int}(h) \), then the ray \( \gamma^{-1}r \) is strongly asymptotic to \( r \), i.e. \( d(\gamma^{-1}r(t), r) \to 0 \) as \( t \to +\infty \). Therefore, \( \gamma^{-1}r \) must enter \( \text{int}(h) \), because \( \text{int}(h) \) is open in \( Y \). Consequently, \( \gamma^{-1}r \) extends \( r \), and \( \gamma^{-1}h \) extends \( h \). It follows by induction that \( h \) is contained in a \( \gamma \)-invariant flat strip. \( \square \)

3.2 Configurations not occurring in smooth spaces

Let \( X \) be a CAT(0) space.

We describe a configuration of convex product subsets which can occur if \( X \) is singular, but not if it is smooth.

We assume that \( X \) contains two closed convex subsets, namely a product \( Y_1 \times Y_2 \) of smooth CAT(0) surfaces \( Y_1 \) and \( Y_2 \) with boundary such that \( \text{int}(Y_1) \times \text{int}(Y_2) \) is open in \( X \); and a product \( Z \times \mathbb{R} \) whose (not necessarily smooth) cross section \( Z \) contains an ideal triangle with three ideal vertices \( \eta, \eta_+, \eta_- \). We denote the sides asymptotic to \( \eta \) and \( \eta_\pm \) by \( l_\pm \) and the side asymptotic to \( \eta_+ \) and \( \eta_- \) by \( l_{+-} \).

We assume furthermore, that these product subsets interact as follows:

(i) The intersection of the flat \( F_\pm = l_\pm \times \mathbb{R} \subset Z \times \mathbb{R} \) with \( Y_1 \times Y_2 \) contains a quadrant \( r_1^\pm \times r_2^\pm \), where \( r_i^\pm \) are asymptotic rays in \( Y_i \). We denote their common ideal endpoint by \( \xi_i \in \partial_\infty Y_i \).

(ii) \( \eta \) is an interior point of the Tits arc \( \xi_1 \xi_2 \) of length \( \frac{\pi}{2} \) in \( \partial_\infty X \).

Then the intersection \( Y_1 \times Y_2 \cap Z \times \mathbb{R} \) is nonempty and, by condition (ii), the product structures (i.e. the directions of the factors) do not match on it. The latter implies that the convex subset \( Y_1 \times Y_2 \cap Z \times \mathbb{R} \) is flat. As a consequence, subrays of the rays \( r_i^\pm \) bound a flat half-strip \( h_i \subset Y_i \).

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\(^1\)This follows from the fact that a geodesic triangle in a product of CAT(0) spaces is rigid if and only if its projections to the factors are rigid.
In addition, we impose a **periodicity** condition:

(iii) The rays \( r_i^\pm \) are asymptotic to a periodic geodesic \( c_i \subset Y_i \).

Using Lemma 3 above, we conclude: Either subrays of the rays \( r_i^\pm \) coincide, or subrays extend to geodesics \( c_i^\pm \subset Y_i \) parallel to \( c_i \).

Claim 1. If conditions (i)-(iii) hold, then \( X \) cannot be smooth.

**Proof.** Suppose that \( X \) is smooth. Then our discussion implies that the flats \( F_\pm \) either have a quadrant in common and therefore coincide, or contain parallel half-planes and their intersection of ideal boundaries \( \partial_\infty F_+ \cap \partial_\infty F_- \) contains an arc of length \( \pi \) of the form \( \xi_1 \xi_2 \xi_3 \) or \( \xi_2 \xi_1 \xi_3 \) with an antipode \( \xi_i \in \partial_\infty Y_i \) for \( i = 1 \) or 2. It follows that \( \angle_{Tits}(\eta_\pm, \xi_i) < \frac{\pi}{2} \) and hence \( \angle_{Tits}(\eta_+, \eta_-) < \pi \), a contradiction. \( \square \)

### 3.3 Not equivariantly smoothable configurations

Now, we restrict to periodic situations and consider geometric actions

\[ \Gamma \acts X \]

by discrete groups on locally compact CAT(0) spaces, i.e. actions which are isometric, properly discontinuous and cocompact.

We will tie the configuration considered above sufficiently closely to the action so that it will carry over to other geometric actions \( \Gamma \acts X' \) on CAT(0) spaces. This will then be used to rule out such actions on smooth CAT(0) spaces, i.e. on Hadamard 4-manifolds.

In addition to the conditions (i)-(iii) above, we assume:

(iv) \( X \) contains no 3-flats.

(v) \( Y_1 \times Y_2 \) is preserved by a subgroup \( \Gamma_1 \times \Gamma_2 \subset \Gamma \) with non-abelian free factors \( \Gamma_i \), and the restricted action \( \Gamma_1 \times \Gamma_2 \acts Y_1 \times Y_2 \) is a product action (not necessarily cocompact).

(vi) The flats \( F_\pm \) and the flat \( F_{+-} = l_{+-} \times \mathbb{R} \) in \( Z \times \mathbb{R} \) are \( \Gamma \)-periodically approximable (i.e. pointed Hausdorff limits of \( \Gamma \)-periodic flats).

(vii) The geodesics \( c_i \subset Y_i \) are \( \Gamma_i \)-periodic. Moreover, there exist \( \Gamma_i \)-periodic geodesics \( d_i \subset \text{int}(Y_i) \) which intersect the rays \( r_i^\pm \subset Y_i \) transversally in points.

Under the assumptions (i)-(vii), we look for a corresponding configuration in \( X' \). Let \( \Phi : X \to X' \) denote a \( \Gamma \)-equivariant quasi-isometry.

By (v) and Proposition 2 there exists a \( \Gamma_1 \times \Gamma_2 \)-invariant closed convex product subset (in general singular)

\[ Y'_1 \times Y'_2 \subset X' \]

on which \( \Gamma_1 \times \Gamma_2 \) acts by a product action. The \( \Gamma_i \)-periodic image quasigeodesics \( \Phi(c_i) \) are Hausdorff close to \( \Gamma_i \)-periodic geodesics \( c'_i \subset Y'_i \).

By (iv+vi) and Proposition 1 the quasi-flats \( \Phi(F_\pm) \) and \( \Phi(F_{+-}) \) are Hausdorff close to flats \( F'_\pm \) and \( F'_{+-} \). We have that any one of these flats is contained in a tubular neighborhood of the union of the other two. Hence its ideal boundary circle is contained in the union of the ideal boundary circles of the other two. This leaves
only the possibility that the union of their ideal boundaries is a spherical suspension of three points. It follows that the flats are contained in a closed convex product subset

\[ Z' \times \mathbb{R} \subset X' \]

whose cross section \( Z' \) contains an ideal triangle with corresponding ideal vertices \( \eta'_1, \eta'_+, \eta'_- \) and sides \( l'_+, l'_-, l'_{+/-} \), such that \( F'_{\pm} = l'_+ \times \mathbb{R} \) and \( F'_{+/-} = l'_{+/-} \times \mathbb{R} \). Furthermore, if \( \rho \subset Z \) is a ray asymptotic to one of the ideal vertices \( \eta, \eta_+ \) or \( \eta_- \), then \( \Phi \) carries the vertical half-plane \( \rho \times \mathbb{R} \subset Z \times \mathbb{R} \) Hausdorff close to a vertical half-plane \( \rho' \times \mathbb{R} \subset Z' \times \mathbb{R} \) where \( \rho' \subset Z' \) is a ray with corresponding ideal endpoint \( \eta'_1, \eta'_+ \) or \( \eta'_- \). This follows from the fact that the half-plane \( \rho \times \mathbb{R} \) is Hausdorff close to the intersection of sufficiently large tubular neighborhoods of two of the flats \( F_{\pm} \) and \( F_{+-} \).

Since the \( c_i \) are periodic, \( \Phi \) carries the quadrants \( r_1^+ \times r_2^+ \) Hausdorff close to a quadrant \( r'_1 \times r'_2 \) for rays \( r_i \subset c_i \). The quadrants \( r_1^+ \times r_2^+ \) are contained in vertical half-planes with ideal boundary semicircle \( \partial_{\infty}F_+ \cap \partial_{\infty}F_- \) and, by condition (ii), their ideal boundary arc \( \xi_1 \xi_2 \) of length \( \frac{\pi}{2} \) is contained in the interior of this semicircle. Denoting the ideal endpoints of the rays \( r_i \) by \( \xi_i = \partial_{\infty}r_i \) it follows that the arc \( \xi_1 \xi_2 \) of length \( \frac{\pi}{2} \) is contained in the interior of the semicircle \( \partial_{\infty}F'_+ \cap \partial_{\infty}F'_- \), and \( \eta' \) is an interior point of the arc \( \xi_1 \xi_2 \) of length \( \frac{\pi}{2} \).

In summary, the interaction of the product subsets \( Y_1' \times Y_2' \) and \( Z' \times \mathbb{R} \) at infinity is as for the configuration in \( X \). However, without further assumptions, the intersection \( Y_1' \times Y_2' \cap Z' \times \mathbb{R} \) could be empty.

**Claim 2.** \( X' \) cannot be smooth Riemannian.

**Proof.** Suppose that \( X' \) is smooth. We show that then the intersection \( Y_1' \times Y_2' \cap Z' \times \mathbb{R} \) must be nonempty.

Note that there exist \( \Gamma \)-periodic geodesics \( d'_i \subset Y_i' \) with the same stabilizers as the geodesics \( d_i \). By (vii), the periodic flat \( d_1 \times d_2 \) transversally intersects the flats \( F_{\pm} \) in points inside the smooth region \( \text{int}(Y_1) \times \text{int}(Y_2) \). Hence, by Lemma 2, \( d_1 \times d_2 \) coarsely intersects \( F_{\pm} \). It follows from Lemma 2 that \( d'_1 \times d'_2 \) coarsely intersects \( F'_{\pm} \). Now we use that \( X' \) is smooth to deduce that \( d'_1 \times d'_2 \) intersects \( F'_{\pm} \) transversally in a point. In particular, \( Y_1' \times Y_2' \cap Z' \times \mathbb{R} \neq \emptyset \).

It follows that conditions (i)-(iii) are satisfied by the product subsets \( Y_1' \times Y_2' \) and \( Z' \times \mathbb{R} \) of \( X' \). By Claim 1, this is a contradiction.

We have proved:

**Theorem 1** (Obstruction to smooth action). If a discrete group \( \Gamma \) admits a geometric action \( \Gamma \curvearrowright X \) on a locally compact CAT(0) space satisfying conditions (i)-(vii), then \( \Gamma \) does not act geometrically on any smooth Hadamard 4-manifold.

**Remark 4.** The regularity assumptions can be relaxed. The argument works more generally and shows that \( \Gamma \) does not act geometrically on locally compact, geodesically complete CAT(0) spaces \( X' \) without branching geodesics, for instance \( C^2 \)-smooth Hadamard 4-manifolds [St13].
4 An example

In this section, we consider the geometric actions on 4-dimensional singular CAT(0) spaces suggested by Gromov in the first exercise of [BGS85] and verify that they contain configurations satisfying conditions (i-vii).

Let \( \Sigma \) be a closed surface of genus \( \geq 2 \), and let \( \beta : V \to \Sigma \times \Sigma \) be a non-trivial finite branched covering with branching locus the diagonal \( \Delta_{\Sigma} \subset \Sigma \times \Sigma \). Then the group \( \Gamma := \pi_1(V) \) admits geometric actions on 4-dimensional singular CAT(0) spaces: Let \( \pi_V : X \to V \) denote the universal covering, and \( \pi := \beta \circ \pi_V : X \to \Sigma \times \Sigma \). We equip \( \Sigma \) with a hyperbolic metric and pull back the corresponding product metric on \( \Sigma \times \Sigma \) to singular metrics on \( V \) and \( X \). In this way the 4-manifold \( X \) becomes a CAT(0) space, and the deck action

\[ \Gamma \acts X \]

becomes a geometric action.

Regarding the geometry of \( X \), note first that the singular locus \( \pi^{-1}(\Delta_{\Sigma}) \subset X \) is a disjoint union of isometrically embedded hyperbolic planes. The restriction of \( \pi \) to any of them is a universal covering of the branching locus \( \Delta_{\Sigma} \subset \Sigma \times \Sigma \).

We look for patterns of flats in \( X \) which obstruct the existence of geometric \( \Gamma \)-actions on Hadamard manifolds, as described in sections 3.2 and 3.3.

The space \( X \) contains no 3-dimensional flats, but plenty of 2-dimensional ones. There are two kinds of them: flats disjoint from \( \pi^{-1}(\Delta_{\Sigma}) \), and flats which intersect \( \pi^{-1}(\Delta_{\Sigma}) \) orthogonally in one or several parallel geodesics.

Let \( F_0 \) denote the set of flats disjoint from \( \pi^{-1}(\Delta_{\Sigma}) \). There are obvious subfamilies of \( F_0 \) which occur in convex product subsets of \( X \). Namely, let

\[ \Sigma = \Sigma^+ \cup \Sigma^- \tag{1} \]

be a decomposition of \( \Sigma \) into two subsurfaces \( \Sigma^\pm \) along a finite family of disjoint closed geodesics. Then the open product block \( \text{int}(\Sigma^+ \times \Sigma^-) \subset \Sigma \times \Sigma \) is disjoint from \( \Delta_{\Sigma} \), and hence the connected components of its inverse image \( \pi^{-1}(\text{int}(\Sigma^+ \times \Sigma^-)) \) in \( X \) are convex subsets isometric to \( \text{int}(\tilde{\Sigma}^+ \times \tilde{\Sigma}^-) \) on which \( \pi \) restricts to a universal covering of \( \text{int}(\Sigma^+ \times \Sigma^-) \).

The other flats in \( X \) important for our argument are, somewhat unexpectedly, the flats which intersect \( \pi^{-1}(\Delta_{\Sigma}) \) in precisely one geodesic; let us denote the set of these flats by \( F_1 \). Understanding them leads us to considering flat half-planes.

We define \( \mathcal{H} \) as the set of injectively immersed flat half-planes \( H \subset \Sigma \times \Sigma \) which intersect the branching locus precisely along their boundary line, \( H \cap \Delta_{\Sigma} = \partial H \), and are orthogonal to it, \( H \perp \Delta_{\Sigma} \). Furthermore, we define \( \tilde{\mathcal{H}} \) as the set of isometrically embedded flat half-planes \( \tilde{H} \subset X \) such that \( \tilde{H} \cap \pi^{-1}(\Delta_{\Sigma}) = \partial \tilde{H} \) and \( \tilde{H} \perp \pi^{-1}(\Delta_{\Sigma}) \). We say that a half-plane \( \tilde{H} \in \tilde{\mathcal{H}} \) covers or is a lift of a half-plane \( H \in \mathcal{H} \) if \( \pi|_{\tilde{H}} \) is a
local isometry onto $H$. A flat in $\mathcal{F}_1$ is the union of two half-planes in $\tilde{H}$ with common boundary line.

We collect some facts about $\mathcal{H}$ and $\tilde{\mathcal{H}}$ needed for our argument.

If $H \in \mathcal{H}$, then $\partial H$ is an injectively immersed line in $\Delta_\Sigma$ and therefore of the form $\partial H = \Delta_e$ for a nonperiodic simple geodesic $c \subset \Sigma$. It follows that $H \subset c \times c$ because $H$ is flat. We also see that half-planes in $\mathcal{H}$ occur in pairs of opposite half-planes with common boundary line.

A half-plane $H \in \mathcal{H}$ lifts to a half-plane $\tilde{H} \in \tilde{\mathcal{H}}$ because it is simply-connected and the branched covering $\beta$ is a true covering over $\Sigma \times \Sigma - \Delta_{\Sigma_\infty}$. More precisely, for a point $p \in H - \partial H$ and a lift $\tilde{p}$ of $p$ there exists a unique lift $\tilde{H}$ of $H$ with $\tilde{p} \in \tilde{H}$. A lift $\tilde{F} \subset \pi^{-1}(\Delta_\Sigma)$ of the boundary line $\partial H$ extends in several ways to a lift $\tilde{H}$ of $H$, because points close to $\partial H$ can be lifted in several ways to points close to $\tilde{F}$. The number of lifts is given by the local branching order of $\pi$ at $\tilde{F}$.

If $\tilde{H} \in \tilde{\mathcal{H}}$, then its boundary line $\partial \tilde{H}$ projects to an immersed line $\Delta_e$ in $\Delta_\Sigma$. The geodesic $c \subset \Sigma$ must be nonperiodic simple, because otherwise $(\tilde{H} - \partial \tilde{H}) \cap \pi^{-1}(\Delta_\Sigma) \neq \emptyset$. Thus, all half-planes in $\tilde{\mathcal{H}}$ are lifts of half-planes in $\mathcal{H}$.

If $\tilde{H}_1, \tilde{H}_2 \in \tilde{\mathcal{H}}$ are distinct half-planes with the same boundary line, $\partial \tilde{H}_1 = \partial \tilde{H}_2$, then their projections $H_1, H_2 \in \mathcal{H}$ either coincide or are a pair of opposite half-planes. The local geometry of branched coverings implies that $\tilde{H}_1, \tilde{H}_2$ have angle $\pi$ along their common boundary line and their union $\tilde{H}_1 \cup \tilde{H}_2$ is a flat in $\mathcal{F}_1$.

We will use the following consequence of this discussion: Let $c \times c \subset \Sigma \times \Sigma$ be an injectively immersed plane, and let $H_{\pm}$ be the half-planes into which it is divided by $\Delta_e$. Then for every lift $\tilde{H}_+ \subset H_+$ there exist at least two distinct lifts $\tilde{H}_1, \tilde{H}_2$ of $H_-$ with the same boundary line $\partial \tilde{H}_- = \partial \tilde{H}_+$, and the union of any two of the three half-planes $\tilde{H}_+, \tilde{H}_1, \tilde{H}_2$ is a flat in $\mathcal{F}_1$.

The flats in $\mathcal{F}_1$ are nonperiodic. Nevertheless, they are useful for investigating geometric $\Gamma$-actions on other CAT(0) spaces. This is due to the following fact:

**Lemma 4.** Let $F \in \mathcal{F}_1$. Suppose that the nonperiodic simple geodesic $\pi(F \cap \pi^{-1}(\Delta_\Sigma))$ in $\Delta_\Sigma$ is the pointed Hausdorff limit of periodic simple geodesics in $\Delta_\Sigma$. Then $F$ is the pointed Hausdorff limit of $\Gamma$-periodic flats in $X$.

**Proof.** We denote $\tilde{F} = F \cap \pi^{-1}(\Delta_\Sigma)$. Let $(c_n, p_n) \to (c, p)$ be a sequence of pointed periodic simple geodesics in $\Sigma$ converging to the nonperiodic simple geodesic $c \subset \Sigma$ with $\pi(\tilde{F}) = \Delta_e$. There exist geodesics $\tilde{l}_n \subset \pi^{-1}(\Delta_\Sigma)$ lifting the $c_n$ and lifts $\tilde{p}_n, \tilde{p}$ of the base points $p_n, p$ such that $(\tilde{l}_n, \tilde{p}_n) \to (\tilde{l}, \tilde{p})$. We choose embedded subsegments $s_n \subset c_n$ of increasing lengths centered at the base points $p_n$ such that also $(s_n, p_n) \to (c, p)$ and lifted segments $\tilde{s}_n \subset \tilde{l}_n$ centered at $\tilde{p}$ such that $(\tilde{s}_n, \tilde{p}_n) \to (\tilde{l}, \tilde{p})$.

The main step of the argument is to approximate $F$ by isometrically embedded flat squares $\tilde{Q}_n \subset \pi^{-1}(s_n \times s_n)$ with diagonals $\tilde{s}_n$, $(\tilde{Q}_n, \tilde{p}_n) \to (F, \tilde{p})$. This will imply the assertion because isometrically embedded flat squares in $\pi^{-1}(c_n \times c_n)$ are contained in $\Gamma$-periodic flats. Indeed, the subsets $\pi^{-1}(c_n \times c_n) \subset X$ have cocompact stabilizers in $\Gamma$, and their connected components are convex subsets which split as metric products of the line with discrete metric trees. All flats contained in them are limits of $\Gamma$-periodic ones.
To find the squares $\tilde{Q}^n$, we proceed as follows. The flat $F$ is divided by $\tilde{l}$ into two half-planes $\tilde{H}_\pm \in \tilde{\mathcal{H}}$. We will approximate these simultaneously by isometrically embedded right-angled isosceles triangles $\tilde{T}_\pm^m \subset \pi^{-1}(s_n \times s_n)$ with sides $\tilde{s}_n$.

Let $\tilde{q}_\pm \in \tilde{H}_\pm - \partial \tilde{H}_\pm$ be base points close to $\tilde{p}$, and let $\tilde{q}_\pm = \pi(\tilde{q}_\pm) \in c \times c - \Delta_c$ denote their projections. There exist sequences of points $\tilde{q}_\pm^n \in s_n \times s_n - \Delta_{s_n}$ approximating them, $\tilde{q}_\pm^m \rightarrow \tilde{q}_\pm$. More precisely, we choose them such that they are close to $\Delta_{p_n} \in \Delta_{s_n}$ intrinsically in $s_n \times s_n$, i.e. such that the segments $\Delta_{p_n}\tilde{q}_\pm^n \subset s_n \times s_n$. Furthermore, there exists a sequence of lifts $\tilde{q}_\pm^n \in \pi^{-1}(\tilde{q}_\pm^m)$ close to $\tilde{p}_n$ such that $\tilde{q}_\pm^m \rightarrow \tilde{q}_\pm^n$.

The injectively immersed square $s_n \times s_n \subset \Sigma \times \Sigma$ is divided by $\Delta_{s_n}$ into two triangles. Let $T_{\pm}^m$ be the subtriangle containing $\tilde{q}_\pm^n$. (Possibly $T_{\pm}^m = T_n^m$.) Since the injectively immersed flat triangles $T_{\pm}^m$ meet $\Delta_{\Sigma}$ only along their hypotenuses $\Delta_{s_n}$, we can lift them to isometrically embedded flat triangles $\tilde{T}_{\pm}^m$ in $X$ with hypotenuses $\tilde{s}_n$, as we could lift the half-planes in $\tilde{\mathcal{H}}$ to half-planes in $\mathcal{H}$. The lifts are again uniquely determined by the lift of one off-hypotenuse point. Thus we can choose them such that $\tilde{q}_\pm^n \in \tilde{T}_\pm^m \subset \pi^{-1}(c_n \times c_n)$. Then the pointed triangles $(\tilde{T}_{\pm}^m, \tilde{q}_\pm^n)$ Hausdorff converge to a flat half-plane in $\tilde{\mathcal{H}}$ with base point $\tilde{q}_\pm^n$ and boundary line $\tilde{l}$. The only such half-plane is $\tilde{H}_\pm$, i.e. $(\tilde{T}_{\pm}^m, \tilde{q}_\pm^n) \rightarrow (\tilde{H}_\pm, \tilde{q}_\pm^n)$.

The two triangles $T_{\pm}^m$ either coincide or have angle $\pi$ along their common side $\Delta_{s_n}$. The local geometry of branched coverings implies that the lifted triangles $\tilde{T}_{\pm}^m$ have angle $\pi$ along their common side $\tilde{s}_n$. (They are distinct for large $n$, $\tilde{T}_+^m \cap \tilde{T}_-^m = \tilde{s}_n$.) Hence their union $\tilde{Q}^n = \tilde{T}_+^m \cup \tilde{T}_-^m$ is an embedded flat square in $X$. These are the squares we were looking for. As desired, they satisfy $(\tilde{Q}^n, \tilde{p}_n) \rightarrow (F, \tilde{p})$. This finishes the proof.

Now we describe a configuration in $X$ which satisfies conditions (i-vii) formulated in sections 3.2 and 3.3.

We consider a decomposition (I) of $\Sigma$ and choose an injectively immersed geodesic line $c \subset \Sigma$ which intersects $\Sigma^+ \cap \Sigma^-$ transversally in precisely one point $p$. The geodesic $c$ is divided by $p$ into the injectively immersed rays $r^\pm = c \cap \Sigma^\pm$. We can arrange our choices (of $\Sigma$, $\Sigma^\pm$ and $c$) so that

(a) $r^\pm$ is asymptotic to a simple closed geodesic $c^\pm \subset \text{int}(\Sigma^\pm)$, and

(b) $c$ is a pointed Hausdorff limit of simple closed geodesics $c_n \subset \Sigma$.

Indeed, if $\Sigma^\pm$ and $c^\pm$ are chosen appropriately then there exists a simple closed curve $a$, which intersects $c^+$ and $c^-$ transversally in one point each and $\Sigma^+ \cap \Sigma^-$ transversally in two points. It is divided by its intersection points with $c^\pm$ into two arcs $a_{++}$ and $a_{--}$. The concatenations $a_{++} \ast nc^- \ast a_{--} \ast nc^+$ are freely homotopic to simple closed geodesics $c_n$ which, when equipped with suitable base points, Hausdorff converge to an injectively immersed line $c$ with the desired properties.

Let $H \in \mathcal{H}$ be the half-plane $H \subset c \times c$ with boundary line $\partial H = \Delta_c$ and containing the quadrant $r^+ \times r^-$. There exist two distinct flats $F_1, F_2 \in \mathcal{F}_1$ which contain the same lift $\tilde{H} \in \tilde{\mathcal{H}}$ of $H$ (and branch along its boundary line $\partial \tilde{H}$). Their union $F_1 \cup F_2$ splits metrically as $Z \times \mathbb{R}$, and the cross section $Z$ is a degenerate ideal triangle (a tripod). By Lemma 4, the three flats contained in $Z \times \mathbb{R}$, i.e. $F_1, F_2$ and $(F_1 \cup F_2) - \text{int}(\tilde{H})$, are $\Gamma$-periodically approximable.
Let \( \tilde{r}^+ \times \tilde{r}^- \subset \tilde{H} \) be the quadrant lifting \( r^+ \times r^- \). There exists a closed convex product subset \( P = Y^+ \times Y^- \subset X \) such that \( \pi|_P \) is a universal covering of \( \Sigma^+ \times \Sigma^- \) and \( F_j \cap P = \tilde{r}^+ \times \tilde{r}^- \) for \( j = 1, 2 \).

The product subsets \( Y^+ \times Y^- \) and \( Z \times \mathbb{R} \) satisfy conditions (i)-(vii). Applying Theorem [1], we therefore obtain:

**Theorem 2** (Exercise 1 in [BGS85]). Let \( V \) be a closed 4-dimensional manifold which admits a non-trivial finite branched covering \( \beta : V \to \Sigma \times \Sigma \) over the product of a hyperbolic surface \( \Sigma \) with itself such that the branching locus equals the diagonal \( \Delta_{\Sigma} \subset \Sigma \times \Sigma \). Then \( V \) admits no smooth Riemannian metric of nonpositive sectional curvature.

**Remark 5.** As in Theorem [1], one can relax the regularity assumptions and rule out the existence of \( C^2 \)-smooth Riemannian metrics on \( V \) [St13].

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