Quantum Statistical Field Theory and Combinatorics

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To Margarita.
Contents

1. Preface iv

Chapter 1. Combinatorics 1
   1. Combinatorics and Moments 1
   2. Fundamental Enumerations 4
   3. Prime Decompositions 9

Chapter 2. Boson Fock Space 15
   1. Boson Statistics 15
   2. States 16

Chapter 3. Field Theory 23
   1. Introduction 23
   2. Field Calculus 24
   3. Green Functions 28
   4. Wick Ordering 36

Chapter 4. Stochastic Integrals 39
   1. Multiple Stochastic Integrals 40
   2. Itô-Fock Isomorphism 43

Bibliography 45
1. Preface

The purpose of these notes is to gather together several facts concerning the combinatorial aspects of diagram notation in field theory, as well as to say something of the use of combinatorics in probability theory and stochastic processes. For simplicity we have restricted to Boson systems only. The motivation comes from the fact that, for the full potential of the diagram notation to be realized, it must be supported by appropriate mathematical techniques. In the general physics community, there are those who never use diagrams and those who never use anything else. In favour of diagrammatic notation it should be emphasized that it is a powerful and appealing way of picturing fundamental processes that would otherwise require unwieldy and uninspired mathematical notation. It is also the setting in which a great many problems are generated and settled. On the other, it should be noted the diagrams are artifacts of perturbation theory and typically describe unphysical situations when treated in isolation. The main problem is, however, one of mathematical intuition. The author, for one, has always been troubled by the idea that, in a series expansion of some Green function or other, some diagrams my have been omitted, or some have been wrongly included, or some are equivalent but not recognized as such, or the wrong combinatorial weights are attached, etc. Ones suspects that a great many physicists labour similar concerns and would like some independent mechanism to check up on this.

Now, it is a fair comment to say that few people get into theoretical physics so that they can do combinatorics! Nevertheless, it turns out that any attempt to study statistical fields through perturbation theory becomes an exercise in manipulating expressions (or diagrams representing these expressions) and sooner or later we find ourselves asking combinatorial questions. The intention here is to gather together several combinatorial features arising from statistical field theory, quantum field theory and quantum probability: all are related in one way or another to the description of either random variables or observables through their moments in a given state. The connotation of the word combinatorics for most physicists is likely to be the tedious duty of counting out all diagrams to a particular order in a given perturbation problem and attaching the equally tedious combinatorial weight, if any. On the contrary, there is more combinatorics than mindless enumeration and the subject develops into a sophisticated and elegant process which has strong crossover with field theoretic techniques. It is hoped that the notes delve just enough into combinatorics, however, so as to cover the essentials we need in field theory, yet give some appreciation for its power, relevance and subtly.

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Frequently used symbols

$B_n$ Bell numbers (number of partitions of a set of $n$ items).
$\varepsilon(f)$ exponential vector for test function $f$.
$\Gamma_+(\mathfrak{h})$ the Bose Fock space over a Hilbert space $\mathfrak{h}$.
$h_n$ number of hierarchies on a set of $n$ items.
$\mathcal{H}(X)$ set of all hierarchies of a set $X$.
$\mu(\cdot)$ the Möbius function
$\prod(X)$ set of all partitions of a set $X$.
$\prod_n$ set of all partitions of $\{1, 2, \cdots n\}$.
$\prod_{n,m}$ set of all partitions of $\prod_n$ consisting of $m$ parts
$\prod^f(X)$ set of all partitions of a set $X$ finer than $\{X\}$.
$\prod^c(X)$ set of all partitions of a set $X$ coarser than $\{\{x\} : x \in X\}$.
$s(n,m)$ Stirling numbers of the first kind.
$S(n,m)$ Stirling numbers of the second kind.
$\mathfrak{S}_n$ set of permutations on $\{1, \cdots, n\}$.
$\mathfrak{S}_{n,m}$ set of permutations on $\{1, \cdots, n\}$ having exactly $m$ cycles.
$W_t$ the Wiener process.

$x^\downarrow$ falling factorial power.
$x^\uparrow$ rising factorial power.
$\oplus$ direct sum.
$\otimes$ tensor product.
$\hat{\otimes}$ symmetrized tensor product.
CHAPTER 1

Combinatorics

One of the basic problems in combinatorics related to over-counting and under-counting which occurs when we try to count unlabeled objects as if they were labeled and labeled objects as if they were unlabeled, respectively. It is curious that the solution to an early question in probability theory - namely, if three dice were rolled, would a 11 be more likely than 12? - was answered incorrectly by under-counting. The wrong answer was to say that both were equally likely as there are six ways to get 11 (6+4+1=6+3+2=5+5+1=5+4+2=5+3+3=4+4+3) and six ways to get 12 (6+5+1=6+4+2=6+3+3=5+5+2=5+4+3=4+4+4). The correct solution was provided by Pascal and takes into account that the dice are distinguishable and so, for instance, 6+5+1 can occur 3!=6 ways, 5+5+1 can occur 3 ways while 4+4+4 can occur only one way: so there are 6+6+3+6+3+1=27 ways to get 11 and only 6+6+3+6+1=25 ways to get 12. The wrong solution for dice, however, turns out to be the correct solution for indistinguishable subatomic particles obeying Bose statistics. The same over-counting of microscopic configurations is at the heart of Gibbs’ entropy of mixing paradox in statistical mechanics.

In this section, we look at some basic enumerations and show how they are of use in understanding moments in probability theory.

1. Combinatorics and Moments

1.1. Ordinary Moments. Let \( X \) a random variable and let \( F(x) = \Pr \{ X < x \} \). The expected value of some function \( f \) of the variable is then given by the Stieltjes integral \( \mathbb{E}[f(X)] = \int_{-\infty}^{\infty} f(x) \, dF(x) \). We assume that the variable has moments to all orders: that is \( \mathbb{E}[X^n] \) exists for each integer \( n \geq 0 \). The moment generating function is then defined to be

\[
M(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathbb{E}[X^n] t^n = \mathbb{E}[e^{tX}]
\]

and so \( M(t) = \int_{-\infty}^{\infty} e^{tx} \, dF(x) \) is Laplace transform of the probability distribution. The moments are recovered from \( M(t) \) by differentiating to the appropriate order and evaluating at \( t = 0 \), viz.

\[
\mathbb{E}[X^n] = \left. \frac{d^n}{dt^n} M(t) \right|_{t=0}.
\]

1.2. Factorial Moments. Suppose we now work with the parameter \( z = e^t - 1 \), then \( M(t) = \mathbb{E} \left[ (1+z)^X \right] \) and we consider Taylor expansion about \( z = 0 \). Now note that \( (1+z)^x = \sum_{n=0}^{\infty} \binom{x}{n} z^n \) where \( \binom{x}{n} = \frac{x(x-1)(x-2)\cdots(x-n+1)}{n!} \).
It is convenient to introduce the notion of falling factorial power
\[ x↓n := x(x-1)(x-2)\cdots(x-n+1) \] as well as a rising factorial power
\[ x↑n := x(x+1)(x+2)\cdots(x+n-1). \]
(We also set \( x↓0 = x↑0 = 1 \).)

It then follows that
\[ \mathbb{E}[zX] = \sum_{n=0}^{\infty} \frac{1}{n!} \mathbb{E}[X↓n] z^n \]
and we refer to \( \mathbb{E}[X↓n] \) as the \( n \)-th falling factorial moment: clearly \( \mathbb{E}[zX] \) acts as their generating function.

1.3. Stirling’s Numbers. We now derive the connection between ordinary and falling factorial moments. We begin by remarking that the right-hand sides of (1.3) and (1.4) may be expanded to obtain a polynomial in \( x \) of degree \( n \) with integer coefficients. The Stirling numbers of the first and second kind are defined as the coefficients appearing in the relations
\[ x↑n \equiv \sum_m s(n,m) x^m; \quad x^n \equiv \sum_m S(n,m) x^m \]
Evidently, the Stirling numbers of the first kind are integers satisfying \( s(n,m) \geq 0 \). It turns out that this is also true of the second kind numbers. We also have \( s(n,m) = 0 = S(n,m) \) for \( m \neq 1, \ldots, n \). It is easy to see that \( x↓n \equiv \sum_m (-1)^{n+m} s(n,m) x^m \) and \( x^n \equiv \sum_m (-1)^{n+m} S(n,m) x^m \) and from this we see that the Stirling numbers are dual in the sense that
\[ \sum_k (-1)^{n+k} s(n,k) S(k,m) = \delta_{nm}. \]
The Stirling’s numbers satisfy the recurrence relations (Stirling’s Identities)
\[ s(n+1,m) = s(n,m-1) + ns(n,m); \quad S(n+1,m) = S(n,m-1) + mS(n,m) \]
with \( s(1,1) = 1 = S(1,1) \). [Evidently \( s(n,n) = S(n,n) = S(n,1) = 1 \). The relations \( x↑(n+1) = x↑n \times (x+n) \) and \( x×x↓m = (x-m+m) x↓m = x↓(m+1)+mx↓m \) lead to the identities when we equate coefficients.]

This means that the Stirling numbers may then be generated recursively using a construction similar to Pascal’s triangle, vis.

| \( n \backslash m \) | 1 | 2 | 3 | 4 | 5 | 6 |
|-----------------|---|---|---|---|---|---|
| 1               | 1 |
| 2               | 1 | 1 |
| 3               | 2 | 3 | 1 |
| 4               | 6 | 11| 6 | 1 |
| 5               | 24| 50| 35| 10| 1 |
| 6               | 120|274|225|85 |15|1 |

| \( n \backslash m \) | 1 | 2 | 3 | 4 | 5 | 6 |
|-----------------|---|---|---|---|---|---|
| 1               | 1 |
| 2               | 1 | 1 |
| 3               | 3 | 1 | 3 | 1 |
| 4               | 4 | 1 | 7 | 6 | 1 |
| 5               | 5 | 1 | 15| 25|10|1 |
| 6               | 6 | 1 | 31| 90|65|15|1 |
1. COMBINATORICS AND MOMENTS

From relation (1.6) we see that the ordinary and falling factorial moments are related by

\[ E[X^n] \equiv \sum_m S(n,m) E[X^{\downarrow m}], \]

(1.10)

\[ E[X^{\downarrow n}] \equiv \sum_m (-1)^{n+m} s(n,m) E[X^{m}]. \]

1.4. Cumulant Moments. Cumulant moments \( \kappa_n \) are defined through the relation

\[ \sum_{n=1}^{\infty} \frac{1}{n!} \kappa_n t^n = \ln M(t) \]

or

\[ \sum_{n=0}^{\infty} \frac{1}{n!} E[X^n] t^n = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n!} \kappa_n t^n \right) \]

and one sees from expanding and comparing coefficients of \( t \) that

\[ E[X] = \kappa_1, \]

\[ E[X^2] = \kappa_2 + \kappa_1^2, \]

\[ E[X^3] = \kappa_3 + 3\kappa_2\kappa_1 + \kappa_1^3, \]

\[ E[X^4] = \kappa_4 + 4\kappa_3\kappa_1 + 3\kappa_2^2 + 6\kappa_2\kappa_1^2 + \kappa_1^4, \]

etc.,

or inversely

\[ \kappa_1 = E[X], \]

\[ \kappa_2 = E[X^2] - E[X]^2 \]

\[ \kappa_3 = E[X^3] - 3E[X^2]E[X] + 2E[X] \]

\[ \kappa_4 = E[X^4] - 4E[X^3]E[X] - 3E[X^2]^2 + 12E[X^2]E[X]^2 - 6E[X]^4, \]

etc.

1.5. Examples.

EXAMPLE 1 (Standard Gaussian). We take \( \mathbb{P}(x) = (2\pi)^{1/2} \int_{-\infty}^{x} e^{-y^2/2} dy \) leading to the moment generating function

\[ M(t) = e^{t^2/2}. \]

We see that all cumulant moments vanish except \( \kappa_2 = 1 \). Expanding the moment generating function yields

\[ E[X^n] = \begin{cases} \frac{(2k)!}{2^k k!}, & n = 2k; \\ 0, & n = 2k + 1. \end{cases} \]

EXAMPLE 2 (Poisson). We take \( X \) to be discrete with \( \Pr \{ X = n \} = \frac{\lambda^n}{n!} e^{-\lambda} \) for \( n = 0, 1, 2, \ldots \). The parameter \( \lambda \) must be positive. The moment generating function is readily computed and we obtain

\[ M(t) = \exp \{ \lambda (e^t - 1) \}. \]

Taking the logarithm shows that all cumulant moments of the Poisson distribution are equal to \( \lambda \):

\[ \kappa_n = \lambda, \text{ for all } n = 0, 1, 2, \ldots. \]
Likewise, \( M(t) \equiv \exp \{ \lambda z \} \) where \( z = e^t - 1 \) and so we see that the falling factorial moments are
\[
E[X^{\downarrow n}] = \lambda^n.
\]
The ordinary moments are more involved. From (1.11) we see that they are polynomials of degree \( n \) in \( \lambda \):
\[
E[X^n] = \sum_m S(n, m) \lambda^m
\]
and in particular \( \lambda \) is the mean. We may expand the moment generating function through the following series of steps
\[
M(t) = \sum_{m=0}^{\infty} \frac{1}{m!} \lambda^m (e^t - 1)^m
\]
\[
= \sum_{m=0}^{\infty} \frac{1}{m!} \lambda^m \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} e^{kt}
\]
\[
= \sum_{m=0}^{\infty} \frac{1}{m!} \lambda^m \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \sum_{n=0}^{\infty} \frac{1}{n!} (kt)^n
\]
and comparison with (1.11) yields the Stirling identity
\[
S(n, m) = \frac{1}{m!} \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} k^n.
\]

**Example 3 (Gamma).** Let \( X \) be a positive continuous variable with \( F(x) = \Gamma(\lambda)^{-1} \int_0^x y^{\lambda-1} e^{-y} dy \), (the Gamma function is \( \Gamma(\lambda) = \int_0^\infty y^{\lambda-1} e^{-y} dy \)), the its moment generating function is
\[
M(t) = (1 - t)^{-\lambda}.
\]
Now \( (1 - t)^{-\lambda} = \sum_{n=0}^{\infty} \binom{-\lambda}{n} (-t)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n t^n \) and so its moments are
\[
E[X^n] = \lambda^n = \sum_m S(n, m) \lambda^m.
\]

2. Fundamental Enumerations

We now want to draw attention to the fact that the various families of numbers, appearing in the formulas for the moments of our main examples above, have an importance in combinatorics [15].

For instance, the factor \( \frac{(2k)!}{2^k k!} \) occurring in the moments of the Gaussian are well-known as the count of the number of ways to partition a set of \( 2k \) items into pairs. The Stirling numbers also arise as counts of classes of permutations and partitions, as we shall see below.

2.1. Pair Partitions. A pair partition of the set \( \{1, 2, \ldots, 2k\} \) consists of \( k \) pairs taken from the set so that every element of the set is paired with another. We shall denote the collection of all such pair partitions by \( P_{2k} \). Evidently, \( |P_{2k}| = \frac{(2k)!}{2^k k!} \); we have \( 2k \times (2k - 1) \) choices for the first pair, then \( (2k - 2) \times (2k - 3) \) for the second, etc. This gives a total of \((2k)!\) however we have over-counted by a
factor of $k!$, as we do not label the pairs, and by $2^k$, as we do not label the elements within each pair either. It is convenient to set $|P_{2k+1}| = 0$ since clearly we cannot partition up an odd number of elements into pairs.

### 2.2. Permutations

The set of permutations, $\mathfrak{S}_n$, over $\{1, \ldots, n\}$ forms a non-abelian group under composition. We shall use the notation $\sigma^0 = id, \sigma^1 = \sigma, \sigma^2 = \sigma \circ \sigma$, etc.

Given a permutation $\sigma \in \mathfrak{S}_n$, the orbit of a number $i \in \{1, \ldots, n\}$ under $\sigma$ is the sequence $\{i, \sigma(i), \sigma^2(i), \ldots\}$. As the orbit must lie within $\{1, \ldots, n\}$ it is clear that $\sigma^k(i) = i$ for some $k > 0$: the smallest such value is called the period of the orbit and clearly orbit repeats itself beyond this point ($\sigma^{n+k}(i) = \sigma^n(i)$). The ordered collection $[i, \sigma(i); \sigma^2(i); \ldots; \sigma^{k-1}(i)]$ is referred to as a cycle or more explicitly a $k$-cycle. Cycles will be considered to be equivalent under cyclic permutation in the sense that $[x_1; x_2; \ldots; x_k]$ is not distinguished from $[x_2; x_3; \ldots; x_k; x_1]$, etc. Thus each $k$-cycle is equivalent to $k$ sequences depending on where on the orbit we choose to start. Clearly orbits arising from the same permutation $\sigma$ either coincide or are completely disjoint; this simple observation leads to the cyclic factorization theorem for permutations: each permutation $\sigma$ can be uniquely written as a collection of disjoint cycles.

**Lemma 1.** Let $\mathfrak{S}_{n,m}$ be the set of permutations in $\mathfrak{S}_n$ having exactly $m$ cycles. Then the number of permutations in $\mathfrak{S}_{n,m}$ is given by the Stirling numbers of the first kind

$$|\mathfrak{S}_{n,m}| = s(n, m).$$

**Proof.** This is proved by showing that $|\mathfrak{S}_{n,m}|$ satisfies the same recurrence relation as the Stirling numbers of the first kind, that is $|\mathfrak{S}_{n+1,m}| = |\mathfrak{S}_{n,m-1}| + n|\mathfrak{S}_{n,m}|$. Now $|\mathfrak{S}_{n+1,m}|$ counts the number of permutations of $\{1, 2, \ldots, n+1\}$ having $m$ cycles. Of these, some will have $n+1$ as a fixed point which here means that $n+1$ is unit-cycle: as we have $m-1$ cycles left to be made up from the remaining labels $\{1, 2, \ldots, n\}$, we see that there are $|\mathfrak{S}_{n,m-1}|$ such permutations. Otherwise, the label $n+1$ lies within a cycle of period two or more: now if we take any permutation in $\mathfrak{S}_{n,m}$ then we could insert the label $n+1$ before any one of the labels $i \in \{1, \ldots, n\}$ in the cyclic decomposition - there are $n|\mathfrak{S}_{n,m}|$ such possibilities and the second situation. Clearly $|\mathfrak{S}_{1,1}| = 1 = |\mathfrak{S}_{n,n}|$ while $|\mathfrak{S}_{n,m}| = 0$ if $m > n$. Therefore $|\mathfrak{S}_{n,m}| \equiv s(n, m).$

### 2.3. Partitions

Let $\mathcal{X}$ be a set. We denote by $\mathfrak{P}(\mathcal{X})$ the collection of all partitions of $\mathcal{X}$, that is, $\mathcal{A} = \{A_1, \ldots, A_m\} \in \mathfrak{P}(\mathcal{X})$ if the $A_j$ are mutually-disjoint non-empty subsets of $\mathcal{X}$ having $\mathcal{X}$ as their union. The subsets $A_j$ making up a partition are called parts. The set $\mathcal{X}$ is trivially a partition - the partition of $\mathcal{X}$ consisting of just one part, namely $\mathcal{X}$ itself. All other partitions are called proper partitions.

If $\mathcal{X} = \{1, \ldots, n\}$ then the collection of partitions of $\mathcal{X}$ will be denoted as $\mathfrak{P}_n$, while the collection of partitions of $\mathcal{X}$ having exactly $m$ parts will be denoted as $\mathfrak{P}_{n,m}$. 


The first few Bell numbers are

$$B_n = \sum_{m=1}^{n} S(n, m)$$

**Proof.** To prove this, we first of all show that we have the formula $|\mathcal{P}_{n,m}| = |\mathcal{P}_{n,m-1}| + m |\mathcal{P}_{n,m}|$. This is relatively straightforward. We see that $|\mathcal{P}_{n,m+1}|$ counts the number of partitions of a set $X = \{1, \ldots, n, n+1\}$ having $m$ parts. Some of these will have the singleton $\{n+1\}$ as a part: there will be $|\mathcal{P}_{n,m-1}|$ of these as we have to partition the remaining elements $\{1, \ldots, n\}$ into $m-1$ parts. The others will have $n+1$ appearing with at least some other elements in a part: we have $|\mathcal{P}_{n,m}|$ partitions of $\{1, \ldots, n\}$ into $m$ parts and we then may place $n+1$ into any one of these $m$ parts yielding $m |\mathcal{P}_{n,m}|$ possibilities. Clearly $|\mathcal{P}_{1,1}| = 1$ and $|\mathcal{P}_{n,n}| = 1$ while $|\mathcal{P}_{n,m}| = 0$ if $m > n$. The numbers $|\mathcal{P}_{n,m}|$ therefore satisfy the same generating relations (1.9) as the $S(n,m)$ and so are one and the same.

As a corollary, we get the following result.

The total number of partitions that can be made from $n$ symbols, termed the $n$-th Bell number and denoted by $B_n$, is given by

$$B_n = |\mathcal{P}_n| = \sum_{m=1}^{n} S(n, m)$$

The first few Bell numbers are

| $n$ | $B_n$ |
|-----|------|
| 1   | 1    |
| 2   | 2    |
| 3   | 5    |
| 4   | 15   |
| 5   | 52   |
| 6   | 203  |
| 7   | 877  |
| 8   | 4140 |

For instance, the set $\{1, 2, 3, 4\}$ can be partitioned into 2 parts in $S(4, 2) = 7$ ways, vis.

$$\{\{1,2\}, \{3,4\}\}, \{\{1\}, \{2,3,4\}\},$$

$$\{\{1,3\}, \{2,4\}\}, \{\{2\}, \{1,3,4\}\},$$

$$\{\{1,4\}, \{2,3\}\}, \{\{3\}, \{1,2,4\}\},$$

$$\{\{4\}, \{1,2,3\}\},$$

and into 3 parts in $S(4, 3) = 6$, vis.

$$\{\{1\}, \{2\}, \{3,4\}\}, \{\{1\}, \{3\}, \{2,4\}\},$$

$$\{\{1\}, \{4\}, \{2,3\}\}, \{\{2\}, \{3\}, \{1,4\}\},$$

$$\{\{2\}, \{4\}, \{2,4\}\}, \{\{3\}, \{4\}, \{1,2\}\}.$$

**2.3.1. Occupation Numbers for Partitions.** Given $\mathcal{A} \in \mathfrak{P}(\mathcal{X})$, we let $n_{j}(\mathcal{A})$ denote the number of parts in $\mathcal{A}$ having size $j$. We shall refer to the $n_{j}$ as occupation numbers and we introduce the functions

$$N(\mathcal{A}) = \sum_{j \geq 1} n_{j}(\mathcal{A}), \quad E(\mathcal{A}) = \sum_{j \geq 1} j n_{j}(\mathcal{A}).$$

If $\mathcal{X} = \{1, \ldots, n\}$ then the collection of partitions of $\mathcal{X}$ will be denoted as $\mathcal{P}_n$, while the collection of partitions having exactly $m$ parts will be denoted as $\mathcal{P}_{n,m}$.

Note that if $\mathcal{A} \in \mathcal{P}_{n,m}$ then $N(\mathcal{A}) = m$ and $E(\mathcal{A}) = n$.

It is sometimes convenient to replace sums over partitions with sums over occupation numbers. Recall that a partition $\mathcal{A}$ will have occupation numbers...
\( n = (n_1, n_2, n_3, \ldots) \) and we have \( N(A) = n_1 + n_2 + n_3 + \cdots \) and \( E(A) = n_1 + 2n_2 + 3n_3 + \cdots \).

We will need to count the number of partitions with \( E(A) = n \) leading to the same set of occupation numbers \( n \); this is given by

\[
\rho(n) = \frac{1}{n_1!n_2!n_3! \cdots} n_1 + 2n_2 + 3n_3 + \cdots
\]

The argument is that there are \( n! \) ways to distribute the \( n \) objects however we do not distinguish the \( n_j \) parts of size \( j \) nor their contents. We remark that the multinomials defined by

\[
B_{n,m}(z_1, z_2, \cdots) = \sum_{n_1, n_2, \cdots} \rho(n) z_1^{n_1} z_2^{n_2} \cdots
\]

are known as the Bell polynomials.

2.3.2. Coarse Graining and Möbius Inversion. A partial ordering of \( \mathcal{P}(X) \) is given by saying that \( A \preceq B \) if every part of \( A \) is a union of one or more parts of \( B \). In such situations we say that \( A \) is coarser than \( B \), or equivalently that \( B \) is finer than \( A \). The partition consisting of just the set \( X \) itself is coarser than every other partition of \( X \). Likewise, the partition consisting of only singletons is the finest possible.

Whenever \( A \preceq B \) we denote by \( n_j(A|B) \) the count of the number of parts of \( A \) that occur as the union of exactly \( j \) parts of \( B \). We also introduce the factor

\[
\mu(A|B) = -\prod_{j \geq 1} \left\{ -(j - 1)! \right\}^{n_j(A|B)}
\]

**Theorem 1.** Let \( \Psi : \mathcal{P}(X) \to \mathbb{C} \) be given and let a function be defined by \( \Phi(B) = \sum_{A \preceq B} \Psi(A) \). The relation may be inverted to give

\[
\Psi(B) = \sum_{A \preceq B} \mu(A|B) \Phi(A)
\]

**Proof.** Essentially we must show that

\[
\sum_{A \preceq B} \mu(A|B) = \delta_{A,B}.
\]

Suppose that \( A_j \in A \), then we may write \( A_j \) as the union of \( k_j \), say, parts of \( B \) and \( r_j \), say, parts of \( C \). Evidently we will have \( 1 \leq r_j \leq k_j \). By considering all the possible partitions of these \( k_j \) parts of \( C \) (for each \( j = 1, \ldots, l \)) we end up with all the partitions \( B \) finer than \( C \) but coarser than \( A \). The sum above then becomes

\[
\prod_{j=1}^{N(A)} \left\{ \sum_{r_j=1}^{k_j} (-1)^{r_j} (r_j - 1)! S(k_j, r_j) \right\},
\]

however, observing that \( (r - 1)! = s(r, 1) \) and using the duality of the first and second kind Stirling numbers \( [1.7] \), we see that this is proportional to \( \prod_{j=1}^{l} \delta_{1,k_j} \).

This gives the result. \( \blacksquare \)
Note that if $\psi$ is a function of the subsets of $X$ and if $\phi(A) = \sum_{B \subseteq A} \psi(B)$ then we have the relation $\psi(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} \phi(B)$ which is the so-called Möbius inversion formula \[13\]. The above result is therefore the corresponding Möbius inversion formula for functions of partitions.

2.4. Hierarchies. Let $X$ be a set and let

\[
\Psi^f(X) : = \text{"set of all finer partitions of } X", \\
\Psi^c(X) : = \text{"set of all coarser partitions of } X"
\]

where we mean $\Psi(X)$ excluding the coarsest, finest partition respectively.

A hierarchy on $X$ is a directed tree having subsets of $X$ as nodes, where $A$ is further along a branch from $B$ if and only if $A \subseteq B$, and where $X$ is the root of the tree and $\{x\}, x \in X$, are the leaves (terminal nodes).

For instance, let $X = \{1, \ldots, 6\}$ then a hierarchy is given by taking three nodes $A = \{2, 3\}, B = \{1, 2, 3\}$ and $C = \{4, 5, 6\}$.

There are two equivalent ways to describes hierarchies, both of which are useful.

2.4.1. Bottom-up description. Let us consider the following sequence of partitions:

\[
A^{(1)} = \left\{ A_1^{(1)}, A_2^{(1)}, A_3^{(1)} \right\} \in \Psi^c(X) : A_1^{(1)} = \{1\}, A_2^{(1)} = \{2, 3\}, A_3^{(1)} = \{4, 5, 6\}; \\
A^{(2)} = \left\{ A_1^{(2)}, A_2^{(2)} \right\} \in \Psi^c \left( A^{(1)} \right) : A_1^{(2)} = \left\{ A_1^{(1)}, A_2^{(1)} \right\}, A_2^{(2)} = \left\{ A_3^{(1)} \right\}; \\
A^{(3)} = \left\{ A_1^{(3)} \right\} \in \Psi^c \left( A^{(2)} \right) : A_1^{(3)} = \left\{ A_1^{(2)}, A_2^{(2)} \right\}.
\]

This equivalently describes or example above.

In general, every hierarchy is equivalent to such a sequence

\[
A^{(1)} \in \Psi^c(X), A^{(2)} \in \Psi^c \left( A^{(1)} \right), A^{(3)} \in \Psi^c \left( A^{(2)} \right), \ldots
\]

and as the partitions are increasing in coarseness (by combining parts of predecessors) the sequence must terminate. That is, there will be an $m$, which we refer to as the order of the hierarchy, such that $A^{(m)} = \left\{ A^{(m-1)} \right\}$. Evidently, $m$ measures the number of edges along the longest branch of the tree.
3. Prime Decompositions

In our example, we can represent the hierarchy as

$$\mathcal{H} = A_1^{(3)}$$

$$= \left\{ A_1^{(2)}, A_2^{(2)} \right\}$$

$$= \left\{ \left\{ A_1^{(1)}, A_2^{(1)} \right\}, \left\{ A_3^{(1)} \right\} \right\}$$

$$= \left\{ \left\{ \{1\}, \{2,3\} \right\}, \left\{ \{4,5,6\} \right\} \right\}.$$ 

This is an order three partition - each of the original elements of $X$ sits inside three braces.

2.4.2. Top-down description. Alternatively, we obtain the same hierarchy by first partitioning $\{1, \cdots, 6\}$ as $B_1^{(1)} = \{B_1^{(1)}_1, B_1^{(1)}_2\}$ where $B_1^{(1)}_1 = \{1,2,3\}$ and $B_1^{(1)}_2 = \{4,5,6\}$, then partitioning $B_1^{(1)}_1$ as $\{1\}$ and $\{2,3\}$, and finally partitioning all the parts at this stage into singletons.

In general, every hierarchy can be viewed as a progression:

$$\mathfrak{B}^{(1)} \in \mathfrak{P}^f(X), \{ \mathfrak{B}_B^{(2)} \in \mathfrak{P}^f(B) : B \in \mathfrak{B}^{(1)} \}, \{ \mathfrak{B}_B^{(3)} \in \mathfrak{P}^f(B) : B \in \mathfrak{B}^{(2)} \}, \cdots.$$ 

Eventually, this progression must bottom out as we can only subdivide $X$ into finer partitions so many times. Again the maximal number of subdivisions is again given by the order of the hierarchy.

We shall now introduce some notation. Let $\mathfrak{H}(X)$ denote the collection of all hierarchies on a set $X$. We would like to know the values of $h_n$, the number of hierarchies on a set of $n$ elements. We may work out the lowest enumerations:

When $n = 2$ we have the one tree $\text{ perseverance}$ and so $h_2 = 1$. When $n = 3$ we have the topologically distinct trees

and, when we count the number of ways to attach the leaves, we have $h_3 = 1+3 = 4$ possibilities.

When $n = 4$ when have the topologically distinct trees

which implies that $h_4 = 1 + 4 + 6 + 12 + 3 = 26$. We find that

$$\begin{array}{c|cccccccc}
 n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
 h_n & 1 & 1 & 4 & 26 & 236 & 2752 & 39208 & 660032 \\
\end{array}$$

This sequence is well known in combinatorics and appears as sequence A000311 on the ATT classification. It is known that the exponential generating series $h(x) = \sum_n \frac{1}{n!} h_n x^n$ converges and satisfies $\exp h(x) = 2h(x) - x + 1$.

3. Prime Decompositions

There results of this section are really necessary for our discussion on field theory, but have been included for completeness. As is well known, the primes are the indivisible natural numbers and every natural number can be decomposed...
uniquely into a product of primes. Each natural number determines an unique
sequence of occupation numbers consisting of the number of time as particular
prime divides into that number. In our discussion below, we meet a calculation
of functions that can be defined by their action on primes. Here we encounter
an argument for replacing a sum (over occupation sequences) of products with a
product of sums (over individual occupation numbers). This argument will recur
elsewhere.

It turns out that the cumulant moments play a role similar to primes insofar
as they are indivisible, in a sense to be made explicit, and every ordinary moment
can be uniquely decomposed into cumulant moments.

3.1. The Prime Numbers. The natural numbers are positive integers \(N = \{1, 2, 3, \cdots\}\). If a natural number \(m\) goes into another natural number \(n\) with no
remainder, then we say that \(m\) divides \(n\) and write this as \(m|n\). Given a natural
number \(n\), we shall denote the number of divisors of \(n\) by \(d(n)\), that is
\[
d(n) := \# \{m : m | n\}.
\]
Likewise, the sum of the divisors of \(n\) is denoted as
\[
s(n) := \sum_{k|n} k.
\]
A natural number, \(p\), is prime if is has no divisors other than itself and 1. That
is, \(d(p) = 2\). The collection of primes will be denoted as \(P\) and we list them, in
increasing order, as
\[
p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, p_5 = 11, \text{ etc.}
\]

Theorem 2 (Prime Decomposition Theorem). Any natural number \(m\) can be
uniquely decomposed as a product of primes:
\[
m = \prod_{j=1}^{\infty} (p_j)^{n_j}.
\]
The numbers \(n_j = n_j(m)\) give the number of times the \(j^{\text{th}}\) prime, \(p_j\), divides
an integer \(m\). In this way, we see that there is a one-to-one correspondence between
the natural numbers and the collection of “occupation numbers” \(n = (n_j)_{j=1}^{\infty}\) where
have \(0 < \sum_{j=1}^{\infty} n_j < \infty\).

Theorem 3 (Euclid). There are infinitely many primes.

Proof. Suppose that we new the first \(N\) primes where \(N < \infty\), we then
construct the number \(q = \prod_{j=1}^{N} p_j + 1\). If we try to divide \(q\) by any of the known
primes \(p_1, \cdots p_N\), we get a remainder of one each time. Since any potential divisor
of \(q\) must be factorizable as a product of the known primes, we conclude that \(q\) has
no divisors other that itself and one and is therefore prime itself. Therefore the list
of prime numbers is endless.
If \( m \) and \( n \) have no common factors then we say that they are \textit{relatively prime} and write this as \( m \perp n \). The Euler phi function, \( \varphi(n) \), counts the number of natural numbers less than and relatively prime to \( n 
 
\varphi(n) := \# \{ m : m < n, m \perp n \} .

\[ \text{3.2. Dirichlet Generating Functions.} \] Let \( a = (a_n)_{n=1}^{\infty} \) be a sequence of real numbers. Its \textit{Dirichlet generating function} is defined to be

\[ \mathcal{D}_a(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}. \]

Let \( a \) and \( b \) be sequences, then their Dirichlet convolution is the sequence \( c = a \ast_d c \) defined by

\[ c_n = \sum_{m|n} a_m b_{n/m}. \]

\textbf{Lemma 3.} Let \( a \) and \( b \) be sequences, then \( \mathcal{D}_a \mathcal{D}_b = \mathcal{D}_c \) where the sequence \( c \) is the Dirichlet convolution of \( a \) and \( b \).

\textbf{Proof.}

\[ \mathcal{D}_a \mathcal{D}_b = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n}{n^s} \frac{b_m}{m^s} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n b_m}{(nm)^s} = \sum_{k=1}^{\infty} c_k k^s. \]

Let \( f \) be a function defined on the natural numbers. The function is said to be \textit{multiplicative} if

\[ f(nm) = f(n)f(m), \quad \text{whenever} \quad m \perp n. \]

If we furthermore have \( f(nm) = f(n)f(m) \), for every pair of natural numbers, then the function is said to be \textit{strongly multiplicative}. For instance, \( f(n) = n^s \) is strongly multiplicative. However, the Euler phi function is multiplicative, but not strongly so.

\textbf{Lemma 4.} Let \( f \) be a multiplicative function of the natural numbers. Then

\[ \mathcal{D}_f(s) = \prod_{p \in \mathbb{P}} \sum_{m=0}^{\infty} \frac{f(p^m)}{p^{ms}}. \]

\textbf{Proof.} This is a consequence of the unique prime decomposition \( \prod_{j=1}^{\infty} (p_j)^{n_j} \) for any natural number. If \( f \) is multiplicative and \( m = \prod_{j=1}^{\infty} (p_j)^{n_j} \), then \( f(m) = \)
The Dirichlet generating function for the sequence \( f(m) \) is then
\[
D_f(s) = \sum_{m \geq 1} \frac{f(m)}{m^s} = \sum_n \prod_{j=1}^{\infty} f(p_j^n) p_j^{-ns} = \prod_{p \in \mathbb{P}} \sum_{n=0}^{\infty} f(p^n) p^{-ns}.
\]

The replacement \( \sum_n \prod_{j=1}^{\infty} g(n_j) \leftrightarrow \prod_{j=1}^{\infty} \sum_{n=0}^{\infty} g(n) \) used above is an elementary trick which we shall call the \( \sum \prod \leftrightarrow \prod \sum \) trick. (It’s the one that is used to compute the grand canonical partition function for the free Bose gas!)

**Lemma 5.** Let \( f \) be a strongly multiplicative function of the natural numbers. Then
\[
D_f(s) = \prod_{p \in \mathbb{P}} \left( 1 - \frac{f(p)}{p^s} \right)^{-1}.
\]

**Proof.** If \( f \) is strongly multiplicative, then \( f(p^n) = f(p)^n \). We then encounter the geometric sequence
\[
\sum_{n \geq 0} \frac{f(p^n)}{p^{ns}} = \sum_{n \geq 0} \frac{f(p)^n}{p^{ns}} = \left( 1 - \frac{f(p)}{p^s} \right)^{-1}.
\]

**3.3. The Riemann Zeta Function.** The Riemann zeta function, \( \zeta(s) \), is the Dirichlet generating function for the constant sequence \( 1 = (1)_{n=1}^{\infty} \). That is,
\[
\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.
\]

We note that \( \zeta(0) \) and \( \zeta(1) \) are clearly divergent. However, \( \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \). It is clear that \( \zeta(s) \) is in fact an analytic function of \( s \) for \( \text{Re}(s) > 1 \).

**Lemma 6 (Euler).** For \( \text{Re}(s) > 1 \),
\[
\zeta(s) = \prod_{p \in \mathbb{P}} \left( 1 - \frac{1}{p^s} \right)^{-1}.
\]

**Proof.** This follows from the observation that \( \zeta(s) = D_1(s) \) and that the constant sequence 1 corresponds to the trivially strongly multiplicative function \( f(n) = 1 \).
3. PRIME DECOMPOSITIONS

An immediate corollary to the previous lemma is that \( \zeta^2(s) = D_d(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}. \)

3.4. The Möbius Function. The Möbius function is the multiplicative function \( \mu \) determined by

\[
\mu(p^n) := \begin{cases} 
1, & n = 0; \\
-1, & n = 1; \\
0, & n \geq 2.
\end{cases}
\]

for each prime \( p \).

Lemma 7. \( D_\mu = \frac{1}{\zeta} \).

Proof. \( D_\mu(s) = \sum_{n \geq 1} \frac{\mu(n)}{n^s} = \prod_{p \in \mathcal{P}} \sum_{n \geq 0} \frac{\mu(p^n)}{p^{ns}} = \prod_{p \in \mathcal{P}} (1 - p^{-s}) = \frac{1}{\zeta(s)}. \)

Lemma 8. Let \( (b_n)_{n=1}^{\infty} \) be a given sequence and set \( a_n = \sum_{k \mid n} b_k \), then

\[
b_n = \sum_k \mu\left(\frac{n}{k}\right) a_k.
\]

Proof. We evidently have that \( D_a = D_b \zeta \) and so \( D_b = \zeta^{-1} D_a = D_\mu D_a \).

As an application, we take \( b_n = n \). Then \( a_n = \sum_{k \mid n} k \) which is just the sum, \( s(n) \), of the divisors of \( n \). We deduce that \( n = \sum_{k \mid n} \mu\left(\frac{n}{k}\right) s(k) \).
Boson Fock Space

Why do the Gaussian and Poissonian distributions emerge from basic enumerations? We shall try and answer this by looking at Bosonic field and their quantum expectations.

1. Boson Statistics

Identical sub-atomic particles are indistinguishable to the extent that there states must be invariant under arbitrary exchange (or, more generally permutation) of their labels we might attach to them. Suppose we have a system of \( n \) identical particles, then it happens in Nature that one of two possibilities can arise: either the total wave function \( \psi(x_1, x_2, \cdots, x_n) \) is completely symmetric, or completely anti-symmetric under interchange of particle labels. The former type are called bosons and we have

\[
\psi(x_{\sigma(1)}, x_{\sigma(2)}, \cdots, x_{\sigma(n)}) = \psi(x_1, x_2, \cdots, x_n), \quad \text{for all } \sigma \in S_n,
\]

while the latter species are called fermions and we have

\[
\psi(x_{\sigma(1)}, x_{\sigma(2)}, \cdots, x_{\sigma(n)}) = (-1)^\sigma \psi(x_1, x_2, \cdots, x_n), \quad \text{for all } \sigma \in S_n,
\]

where \((-1)^\sigma\) denotes the sign of the permutation. (Here, the coordinates \( x_j \) give all necessary labels we might attach to a particle: position, spin, color, etc.)

1.1. Fock Space. Let \( \mathfrak{h} \) be a fixed Hilbert space and denote by \( \mathfrak{h}^\hat{}^n \) the closed linear span of symmetrized vectors of the type

\[
\psi_1 \hat{} \cdots \hat{} \psi_n := \frac{1}{n!} \sum_{\sigma \in S_n} \psi_{\sigma(1)} \hat{} \cdots \hat{} \psi_{\sigma(n)},
\]

where \( \psi_j \in \mathfrak{h} \). To understand better the form of \( \mathfrak{h}^\hat{}^n \), let \( \{e_j : j = 1, \cdots\} \) be an orthonormal basis of \( \mathfrak{h} \). Suppose that we have \( n \) particles with \( n_1 \) particles are in state \( e_1 \), \( n_2 \) in state \( e_2 \), etc., then there is accordingly only one total state describing this: setting \( n = \{n_j : j = 1, 2, \cdots\} \), the appropriately normalized state is described by the vector -known as a number vector - given by

\[
|n\rangle = \left( \begin{array}{c} n \\ n_1, n_2, \cdots \end{array} \right)^{1/2} \left( e_1 \hat{} \cdots \hat{} e_1 \right)_{n_1 \text{ factors}} \left( e_2 \hat{} \cdots \hat{} e_2 \right)_{n_2 \text{ factors}} \cdots .
\]

The span of such states, with the obvious restriction that \( \sum_j n_j = n \), yields \( \mathfrak{h}^\hat{}^n \).

The Fock space over \( \mathfrak{h} \) is then the space spanned by all vectors \(|n\rangle\) when only the restriction \( \sum_j n_j < \infty \) applies. We also include a no-particle state called the
Fock vacuum and which we denote by \( \Omega \). The Fock space is then the direct sum
\[
\Gamma_+ (\mathfrak{h}) = \bigoplus_{n=0}^{\infty} \mathfrak{h}^\otimes n.
\]
The vacuum space \( \mathfrak{h}^\otimes 0 \) is taken to be spanned by the single vector \( \Omega \).

Often, the number state vectors are not the most convenient to use and we now give an alternative class. The exponential vector map \( \varepsilon : \mathfrak{h} \mapsto \Gamma_+ (\mathfrak{h}) \) is defined by
\[
\varepsilon (f) = \bigoplus_{n=0}^{\infty} \left( \frac{1}{\sqrt{n!}} f^\otimes n \right)
\]
with \( f^\otimes n \) the \( n \)-fold tensor product of \( f \) with itself. The Fock vacuum is, in particular, given by \( \Omega = \varepsilon (0) \). We note that \( \langle \varepsilon (f) | \varepsilon (g) \rangle = \sum_{n \geq 0} \frac{1}{\sqrt{n!}} \langle f^\otimes n | g^\otimes n \rangle = \exp \langle f|g \rangle \), whence the name exponential vectors. (These vectors are called Bargmann vector states in the physics literature, while their normalized versions are known as coherent state vectors.) The set \( \varepsilon (\mathfrak{h}) \), consisting of all exponential vectors generated by the test functions in \( \mathfrak{h} \), is linearly independent in \( \Gamma_+ (\mathfrak{h}) \). Moreover, they have the property that \( \varepsilon (S) \) will be dense in the Boson Fock space whenever \( S \) is dense in \( \mathfrak{h} \).

1.2. Creation, Annihilation and Conservation. Boson creation and annihilation fields are then defined on \( \Gamma_+ (\mathfrak{h}) \) by the actions:
\[
B^+ (\phi) f_1 \hat{\otimes} \cdots \hat{\otimes} f_n := \sqrt{n+1} \phi \hat{\otimes} f_1 \hat{\otimes} \cdots \hat{\otimes} f_n;
\]
\[
B^- (\phi) f_1 \hat{\otimes} \cdots \hat{\otimes} f_n := \frac{1}{\sqrt{n!}} \sum_{j} \langle \phi | f_j \rangle f_1 \hat{\otimes} \cdots \hat{\otimes} \hat{f}_j \hat{\otimes} \cdots \hat{\otimes} f_n.
\]
They satisfy the canonical commutation relations
\[
[B^- (\phi), B^+ (\psi)] = \langle \phi | \psi \rangle,
\]
\[
[B^- (\phi), B^- (\psi)] = 0 = [B^+ (\phi), B^+ (\psi)].
\]

Likewise, let \( M \) be an operator acting on \( \mathfrak{h} \). Its differential second quantization is the operator \( d\Gamma_+ (M) \) acting on \( \Gamma_+ (\mathfrak{h}) \) and defined by
\[
d\Gamma_+ (M) (f_1 \hat{\otimes} \cdots \hat{\otimes} f_n) := (M f_1) \hat{\otimes} \cdots \hat{\otimes} f_n + \cdots + f_1 \hat{\otimes} \cdots \hat{\otimes} (M f_n).
\]

2. States

2.1. Gaussian Fields. Consider \( \langle \Omega | B^{(k)} (f_n) \cdots B^{(2)} (f_2) B^{(1)} (f_1) \Omega \rangle \), where the \( \varepsilon \)'s stand for \( + \) or \( - \), this is a vacuum expectation and we may compute by the following scheme: every time we encounter an expression \( B^- (f_i) B^+ (f_j) \) we replace it with \( B^+ (f_i) B^- (f_j) + \langle f_i | f_j \rangle \). The term \( \langle f_i | f_j \rangle \) is scalar and can be brought outside the expectation leaving a product of two less fields to average. Ultimately we must pair up every creator with an annihilator otherwise we get a zero. Therefore only the even moments are non-zero and we obtain
\[
\sum_{\varepsilon \in \{+,-\}^n} \langle \Omega | B^{(2n)} (f_{2n}) \cdots B^{(2)} (f_2) B^{(1)} (f_1) \Omega \rangle = \sum_{(p_j, q_j)_{j=1}^n} \prod_{j=1}^n \langle f_{p_j} | f_{q_j} \rangle.
\]
Here \( (p_j, q_j)_{j=1}^{n} \in \mathcal{P}_{2n} \) is a pair partition: the \( p_j \) correspond to annihilators and the \( q_j \) to creators so we must have \( p_j > q_j \) for each \( j \); the ordering of the pairs is unimportant so for definiteness we take \( q_n > \ldots > q_2 > q_1 \). We may picture this as follows: for each \( i \in \{1, 2, \ldots, 2n\} \) we have a vertex; with \( B^+ (f_i) \) we associate a creator vertex with weight \( f_i \) and with \( B^- (f_i) \) we associate an annihilator vertex with weight \( f_i \). A matched creation-annihilation pair \( (p_j, q_j) \) is called a contraction over creator vertex \( q_j \) and annihilator vertex \( p_j \) and corresponds to a multiplicative factor \( \langle f_p | f_q \rangle \) and is shown pictorially as a single line.

\[
\begin{align*}
\text{annihilator vertex at } p & \quad \text{contraction between vertices } p \text{ and } q \\
\text{creator vertex at } q
\end{align*}
\]

We then consider a sum over all possible diagrams describing.

Setting all the \( f_j \) equal to a fixed test function \( f \) and let \( Q (f) = B^+ (f) + B^- (f) \) then we obtain

\[
\langle \Omega | Q (f)^k \Omega \rangle = \left\langle \Omega | \left[ B^+ (f) + B^- (f) \right]^k \Omega \right\rangle = \| f \|^k |\mathcal{P}_k|.
\]

The observable \( Q (f) \) therefore has a mean-zero Gaussian distribution in the Fock vacuum state:

\[
\langle \Omega | e^{i \left[ B^+ (f) + B^- (f) \right]} \Omega \rangle = e^{-\frac{1}{2} \| f \|^2}.
\]

For instance, we have \(|\mathcal{P}_4| = \frac{4!}{2! 2!} = 3\) and the three pair partitions \( \{(4, 3) (2, 1)\}, \{(4, 2) (3, 1)\} \) and \( \{(4, 1) (3, 2)\} \) are pictured below.

We remark that (2.1) is the basic Wick’s theorem in quantum field theory \([5]\), and can be realized in terms of Hafnians \([2]\). However, the result as it applies to multinomial moments of Gaussian variables goes back to Isserlis \([11]\) in 1918.

**2.2. Poissonian Fields.** More generally, we shall consider fields \( B^\pm (\cdot) \) on some Fock space on \( \Gamma_+ (\mathfrak{h}) \). Consider the vacuum average of an expression of the type

\[
\langle \Omega | \sqrt{n} (f_n)_{\alpha(n)} B^- (g_n)_{\beta(n)} \cdots \sqrt{n} (f_1)_{\alpha(1)} B^- (g_1)_{\beta(1)} \Omega \rangle
\]

where the \( \alpha_j \)'s and \( \beta_j \)'s are powers taking the values 0 or 1. This time, in the diagrammatic description, we have \( n \) vertices with each vertex being one of four possible types:
The typical situation is depicted below:

Evidently we must again join up all creation and annihilation operators into pairs; we however get creation, multiple scattering and annihilation as the rule; otherwise we have a stand-alone constant term at a vertex. In the figure, we can think of a particle being created at vertex $i(1)$ then scattered at $i(2), i(3), i(4)$ successively before being annihilated at $i(5)$. (This component has been highlighted using thick lines.) Now the argument: each such component corresponds to a unique part, here \{i(5), i(4), i(3), i(2), i(1)\}, having two or more elements; singletons may also occur and these are just the constant term vertices. Therefore every such diagram corresponds uniquely to a partition of \{1, \ldots, n\}. Once this link is made, it is easy to see that

\[
\sum_{\alpha, \beta \in \{0, 1\}^n} \left\langle \Omega | B^+ (f_1)^{\alpha(n)} B^- (g_1)^{\beta(1)} \cdots B^+ (f_1)^{\alpha(1)} B^- (g_1)^{\beta(1)} \frac{\Omega}{\Omega} \right\rangle = \sum_{\alpha, \beta \in \{0, 1\}^n} \left\langle g_{i(k)} | f_{i(k)} \right\rangle \cdots \left\langle g_{i(3)} | f_{i(2)} \right\rangle \left\langle g_{i(2)} | f_{i(1)} \right\rangle.
\]

(2.2)

If we now take all the $f_j$ and $g_j$ equal to a fixed $f$ then we arrive at

\[
\left\langle \Omega | \left[ (B^+ (f) + 1) (B^- (f) + 1) \right]^n \frac{\Omega}{\Omega} \right\rangle = \sum_{m=0}^{n} \sum_{\gamma \in P_{n,m}} \|f\|^{n-m} = \sum_{m=0}^{n} S(n, m) \|f\|^{2(n-m)}.
\]

Note that a part of size $k$ contributes $\|f\|^{2(k-1)}$ so a partition in $P_{n,m}$ with parts of size $k_1, \ldots, k_m$ contributes $\|f\|^{2(k_1+\cdots+k_m-m)} = \|f\|^{2(n-m)}$.

It therefore follows that the observable

\[
N(f) := \left( B^+ \left( \frac{f}{\|f\|} \right) + \|f\| \right) \left( B^- \left( \frac{f}{\|f\|} \right) + \|f\| \right)
\]

has a Poisson distribution of intensity $\|f\|^2$ in the Fock vacuum state.
2.3. Exponentially Distributed Fields. Is there a similar interpretation for Stirling numbers of the first kind as well? Here we should be dealing with cycles within permutations rather than parts in a partition. Consider the representation of a cycle \((i(1), i(2), \ldots, i(6))\) below:

![Diagram of cycle](https://example.com/diagram.png)

To make the sense of the cycle clear, we are forced to use arrows and therefore we have two types of lines. In any such diagrammatic representation of a permutation we will encounter five types of vertex:

- An uncontracted (constant) vertex indicates a fixed point for the permutation.

Let us consider the one-dimensional case first. The above suggests that we should use two independent (that is, commuting) Bose variables, say \(b^\pm_1, b^\pm_2\). Let us set

\[
\begin{aligned}
b^\pm f &:= b^\pm_1 \otimes 1_2 + 1_1 \otimes b^\pm_2 \\
t &:= b^+ - b^-
\end{aligned}
\]

then we see that \(t\) will commute with \(t\). We note that

\[
\begin{aligned}
t &= b^+_1 b^-_1 \otimes 1_2 + 1_2 \otimes b^+_2 b^-_2 + b^+_1 \otimes b^-_2 + b^-_1 \otimes b^+_2 + 1
\end{aligned}
\]

and here we see the five vertex terms we need.

Let \(\Omega_1\) and \(\Omega_2\) be the vacuum state for \(b^\pm_1\) and \(b^\pm_2\) respectively, then let \(\Omega = \Omega_1 \otimes \Omega_2\) be the joint vacuum state. We wish to show that \(N\) has a Gamma distribution of unit power (an Exponential distribution!) in this state.

First of all, let \(Q = b^+ + b^-\) so that \(Q = Q_1 \otimes 1_2 + 1_1 \otimes Q_2\) where \(Q_j = b^+_j + b^-_j\). Now each \(Q_j\) has a standard Gaussian distribution in the corresponding vacuum state \(\langle \Omega_j | e^{t Q_j} \Omega_j \rangle = e^{t^2/2}\) and so

\[
\langle \Omega | e^{t Q} \Omega \rangle = \langle \Omega_1 | e^{t Q_1} \Omega_1 \rangle \langle \Omega_2 | e^{t Q_2} \Omega_2 \rangle = e^{t^2}.
\]

Therefore \(\langle \Omega | Q^{2n} \Omega \rangle = 2^n |P_{2n}| = \frac{(2n)!}{n!}\). However \(Q^{2n} = \sum_m (\binom{2n}{m}) (b^+)^m (b^-)^{2n-m}\) (remember that \(b^+\) and \(b^-\) commute!) and so \(\langle \Omega | Q^{2n} \Omega \rangle \equiv \binom{2n}{n} \langle \Omega | (b^+)^n (b^-)^n \Omega \rangle\) which should equal \(\binom{2n}{n} \langle \Omega | (b^+ b^-)^n \Omega \rangle\). Therefore

\[
\langle \Omega | (b^+ b^-)^n \Omega \rangle = n!
\]
and so, for $t < 1$, 
\[
\langle \Omega | \exp \{ t b^+ b^- \} \Omega \rangle = \frac{1}{1 - t}.
\]
Therefore $N = b^+ b^-$ has an exponential distribution in the joint vacuum state.

The generalization of this result to Bosonic fields over a Hilbert space $\mathfrak{h}$ is straightforward enough. First we need the notion of a conjugation map on Hilbert spaces: this is a conjugate-linear map $j : \mathfrak{h} \rightarrow \mathfrak{h}$. For instance, let $\{ e_n \}_n$ be a fixed complete basis for $\mathfrak{h}$ then an example of a conjugation is given by $j(\sum_n c_n e_n) = \sum_n c^*_n e_n$.

**Theorem 4.** Let $\mathfrak{h}$ be a separable Hilbert space and $j$ an conjugation on $\mathfrak{h}$. Let $B^\pm$ be operator fields define on $\Gamma_+(\mathfrak{h}) \otimes \Gamma_+(\mathfrak{h})$ by
\[
B^\pm(f) = B^\pm_1(f) \otimes 1_2 + 1_1 \otimes B^\pm_2(jf)
\]
where the $B^\pm_i$ are usual Bosonic fields on the factors $\Gamma_+ (\mathfrak{h})$, $i (= 1, 2)$. Then $[B^-(f), B^+(g)] = 0$ for all $f, g \in \mathfrak{h}$ and if $N(f, g) := B^+(f) B^-(g)$ then we have the following expectations in the joint Fock vacuum state $\Omega = \Omega_1 \otimes \Omega_2$:
\[
\langle \Omega | N(f_n, g_n) \cdots N(f_1, g_1) \Omega \rangle
\]

The proof should be obvious from our discussions above so we omit it. The sum is over all permutations $\sigma \in S_n$ and each permutation is decomposed into its cycles; the product is then over all cycles $(i(1), i(2), \ldots, i(k))$ making up a particular permutation. We note that the representation corresponds to a type of infinite dimensional limit to the double-Fock representation for thermal states [1].

**2.4. Thermal Fields.** There is a well known trick for representing thermal states of Bose systems. To begin with, let $\mathfrak{h} = L^2(\mathbb{R}^d)$ be the space of momentum-representation wave functions, then the thermal state $\langle \cdot \rangle_{\beta, \mu}$ is characterized as the Gaussian (sometimes referred to as quasi-free), mean zero state with
\[
\langle B^+(f) B^-(f) \rangle_{\beta, \mu} = \int |f(k)|^2 \varrho_{\beta, \mu}(k) d^dk
\]
where
\[
\varrho_{\beta, \mu}(k) = \left( e^{\beta[E(k)-\mu]} - 1 \right)^{-1}.
\]
(We have the physical interpretation of $\beta$ as inverse temperature, $\mu$ as chemical potential, and $E(k)$ the energy spectrum function.) Let us denote by $\varrho_{\beta, \mu}$ the operation of pointwise multiplication by the function $\varrho_{\beta, \mu}(\cdot)$ on $\mathfrak{h}$, that is, $(\varrho_{\beta, \mu}f)(k) = \varrho_{\beta, \mu}(k) f(k)$. We check that
\[
\langle [B^+(f) + B^-(f)]^2 \rangle_{\beta, \mu} = \langle f | C_{\beta, \mu} f \rangle
\]
with $C_{\beta, \mu} = 2 \varrho_{\beta, \mu} + 1 = \coth \beta |E - \mu|/2$.

**Theorem 5.** Let $\mathfrak{h}$ be a separable Hilbert space and $j$ an conjugation on $\mathfrak{h}$. Let $B^\pm$ be operator fields define on $\Gamma_+ (\mathfrak{h}) \otimes \Gamma_+(\mathfrak{h})$ by
\[
B^\pm(f) = B^\pm_1\left( \sqrt{\varrho + 1}f \right) \otimes 1_2 + 1_1 \otimes B^\pm_2(j\sqrt{\varrho}f)
\]
where the $B^\pm_i(\cdot)$ are usual Bosonic fields on the factors $\Gamma_i(\mathfrak{h})$, $i (= 1, 2)$, and $\varrho$ is a positive operator on $\mathfrak{h}$. Then the fields satisfy the canonical commutation relations \(^{(1.3)}\) and their moments in the joint Fock vacuum state $\Omega = \Omega_1 \otimes \Omega_2$ are precisely the same as for the thermal state when we take $\varrho = \varrho_{\beta, \mu}$.

To see this, note that
\[
\langle \Omega | e^{i[B^+(f)+B^-_i(f)]} \Omega \rangle = \langle \Omega_1 | e^{i[B^+_1(f)+B^-_1(f)]} \Omega_1 \rangle \langle \Omega_2 | e^{i[B^+_2(f)+B^-_2(f)]} \Omega_2 \rangle
\]
\[
= \exp \left\{ -\frac{1}{2} \langle f| (\varrho + 1) f \rangle \right\} \exp \left\{ -\frac{1}{2} \langle f| (\varrho + 1) f \rangle \right\}
\]
\[
= \exp \left\{ -\frac{1}{2} \langle f| Cf \rangle \right\}
\]

where $C = 2\varrho + 1$ is the covariance of the state. The first factor describes the so-called spontaneous emissions and absorptions while the second factor describes the so-called stimulated emissions and absorptions. In the zero temperature limit, $\beta \to \infty$, we find that $\varrho \to 0$ and so the second factor fields (stimulated emissions and absorptions) become negligible.
CHAPTER 3

Field Theory

1. Introduction

Let us suppose that we have a fixed space $\Phi$ of functions over a set $\Lambda$. A function $\varphi = \{\varphi_x : x \in \Lambda\}$ in $\Phi$ will be called a field realization. By a field $\phi$ we mean an observable taking values in $\Phi$ (that is to say, the field realization are “eigen-values” for the field) and we suppose that we are supplied with state which we shall denote as $\langle \cdot \rangle$. Formally, we may think of the field as a family of operators $\phi = \{\phi_x : x \in \Lambda\}$ and we suppose that $[\phi_x, \phi_y] = 0$ for all $x, y \in \Lambda$. Since we are assuming that the field is commutative, we may in fact revert to classical probability and think of the field as being a random variable taking values in $\Phi$. The expectation of a (typically nonlinear) functional $F = F[\phi]$ of the field is to be understood as an integral over $\Phi$:

\[
\langle F[\phi] \rangle = \int_{\Phi} F[\varphi] \, d\nu(\varphi),
\]

where $\nu$ is the probability measure over $\Phi$ corresponding to the state. We shall frequently use the notation $\langle \cdot \rangle = \langle \cdot \rangle_\nu$ when we want to emphasize the correspondence.

The label $x \in \Lambda$ is assumed to give all relevant information such as position, spin, etc. When $\Lambda$ is a finite set, then the mathematical treatment is straightforward. Otherwise, we find ourselves having to resort to infinite dimensional analysis. We also introduce a dual space $J$ of fields $J = \{J^x : x \in \Lambda\}$ which we call the source fields. Our convention will be that the source fields carry a ‘contravariant’ index while the field, and its realizations, carry a ‘covariant’ index. The duality between fields and sources will be written as

\[
\langle \varphi, J \rangle = \varphi_x J^x.
\]

In the case where $\Lambda$ is a finite set, say of cardinality $N$, a realization $\varphi \in \Phi$ can be viewed as a set of $N$ numbers and so we can identify $\Phi$, and likewise $J$, as $\mathbb{R}^N$. In this case $\varphi_x J^x$ means $\sum_{x \in \Lambda} \varphi_x J^x$ and so the notation just implies an Einstein summation convention over $\Lambda$. If $d\varphi_x$ denotes standard Lebesgue measure, and let $S[\cdot]$ be some functional, called the action, such that $\Xi = \int_{\Phi} \exp \{S[\varphi]\} \, d\varphi < \infty$, then, we have the finite-dimensional probability measure, $\nu$, on $\Phi$ determined by

\[
d\nu(\varphi) = \frac{1}{\Xi^N} e^{S[\varphi]} \prod_{x \in \Lambda} d\varphi_x.
\]

The general situation which we are really interested is where $\Lambda$ is continuous. If we want $\Lambda = \mathbb{R}^d$, then we should take $J$ to be the space of Schwartz functions on $\mathbb{R}^d$ and $\Phi$ to be the tempered distributions. (The field realizations being more singular than the sources!) Here, the duality is denoted is $\langle \varphi, J \rangle = \int_{\mathbb{R}^d} \varphi_x J^x \, dx$ and we again
shorten to just \( \varphi_x J^x \). We shall therefore adopt an Einstein summation/integration convention over all indices from now on. The appropriate way to consider randomizing the field in the infinite-dimensional case, will then be to consider probability measures on the Borel sets of \( \mathcal{J} \), that is, on the \( \sigma \)-algebra generated by the weak topology of the Schwartz functions\(^1\).

2. Field Calculus

In this section, I want to present a way of handling functional derivatives exploiting the commutativity of multiple point derivatives. The notation that results has similarities to one introduced by Guichardet [8] for investigating Bosonic Fock spaces. We shall denote by \( \frac{\delta}{\delta J^x} \) the functional derivative wrt. \( J^x \). For \( X \) a finite subset of \( \Lambda \) we shall adopt the notations

\[
J^X = \prod_{x \in X} J^x, \quad \frac{\delta^{\lvert X \rvert}}{\delta J^X} = \prod_{x \in X} \frac{\delta}{\delta J^x}.
\]

(In both cases, we need only the set \( X \) - the ordering of the elements is irrelevant!) We of course have \( \frac{\delta}{\delta J^x} (J^y) = \delta^y_x \). Similarly, we shall write \( \varphi^X \) for \( \prod_{x \in \Lambda} \varphi_x \), and

\[
\frac{\delta^{\lvert X \rvert}}{\delta \varphi_X} = \prod_{x \in X} \frac{\delta}{\delta \varphi_x}.
\]

2.1. Analytic Functionals. A multi-linear map \( T : \times^n \Phi \to \mathbb{C} \) is called a tensor of covariant rank \( n \) and it will be determined by the components \( T_{x_1 \cdots x_n} \) such that \( T(J^{(1)}, \cdots, J^{(n)}) = T_{x_1 \cdots x_n} J^{x_1} (1) \cdots J^{x_n} (n) \). Likewise, we refer to a multilinear map from \( \times^n \Phi \) to the complex numbers as a tensor of contravariant rank \( n \).

A functional \( F = F[J] \) is said to be analytic in \( J \) if it admits a series expansion of the form \( F[J] = \sum_{n \geq 0} \frac{1}{n!} f_{x_1 \cdots x_n} J^{x_1} \cdots J^{x_n} \) where \( f_{x_1 \cdots x_n} \) are the components of a completely symmetric covariant tensor and as usual the repeated dummy indices are summed/integrated over. A more compact notation is to write the series expansion as

\[
F[J] = \sum_X \frac{1}{\lvert X \rvert !} f_X J^X.
\]

It is easy to see that

\[
\frac{\delta}{\delta J^y_1} \cdots \frac{\delta}{\delta J^y_m} F[J] = \sum_{n \geq 0} \frac{1}{n!} f_{y_1 \cdots y_m x_1 \cdots x_n} J^{x_1} \cdots J^{x_n} \quad \text{and this now reads as}
\]

\[
\frac{\delta^{\lvert Y \rvert}}{\delta J^Y} \left( \sum_X \frac{1}{\lvert X \rvert !} f_X J^X \right) = \sum_X \frac{1}{\lvert X \rvert !} f_{Y \cup X} J^X.
\]

2.2. The Leibniz rule. Functional derivatives obey all the standard rules of calculus, including the Leibniz rule and we have the natural extension to multiple derivatives:

\[
(2.1) \quad \frac{\delta^{\lvert X \rvert}}{\delta J^X} (FG) = \sum_{Y \subseteq X} \frac{\delta^{\lvert Y \rvert}}{\delta J^Y} \frac{\delta^{\lvert X \setminus Y \rvert}}{\delta J^{X \setminus Y}} G.
\]

\(^1\)At this stage we shall stop with the functional analysis details and proceed as if every thing is well-defined. The mathematically inclined reader can fill in the details, while everyone else could well live without them.
The generalization of this for several factors is

\[ \frac{\delta^{|X|}}{\delta J^X} (F_1 \cdots F_m) = \sum'_{Y_1, \ldots, Y_m \subseteq X} \frac{\delta^{|Y_1|}}{\delta J^{Y_1}} F_1 \cdots \frac{\delta^{|Y_m|}}{\delta J^{Y_m}} F_m \]

where the sum is over all partitions of \( X \) into \( m \) labelled parts (as opposed to the unlabeled ones we have considered up to now).

### 2.3. Differential Formulas.

We now derive a useful result.

**Lemma 9.** Let \( W = W[J] \) then

\[ \frac{\delta^{|X|}}{\delta J^X} e^W = e^W \sum_{\mathcal{A} \in \mathfrak{P}(X)} \prod_{A \in \mathcal{A}} \frac{\delta^{|A|} W}{\delta J^A}. \]

**Proof.** This is easily seen by induction on \( n \). As \( \frac{\delta^{|X|}}{\delta J^X} e^W = e^W \frac{\delta W}{\delta J^X} \), the identity is true for \( n = 1 \). Now assume that it is true for \( n \), then

\[ \frac{\delta^{|X|+1}}{\delta J^{X \cup \{x\}}} e^W = e^W \frac{\delta W}{\delta J^X} \sum_{\mathcal{A} \in \mathfrak{P}(X \cup \{x\})} \prod_{A \in \mathcal{A}} \frac{\delta^{|A|} W}{\delta J^A} + e^W \frac{\delta W}{\delta J^X} \sum_{\mathcal{A} \in \mathfrak{P}(X \cup \{x\})} \frac{\delta^{|A|} W}{\delta J^A} \]

however, the first term on the right hand side is a sum over all parts of \( X \cup \{x\} \) having \( x \) occurring as a singleton, while the second term, when differentiated wrt. \( J^x \), will be a sum over all parts of \( X \cup \{x\} \) having \( x \) in some part containing at least one element of \( X \). Thus we may write the above as

\[ \frac{\delta^{|X|+1}}{\delta J^{X \cup \{x\}}} e^W = e^W \sum_{\mathcal{A} \in \mathfrak{P}(X \cup \{x\})} \prod_{A \in \mathcal{A}} \frac{\delta^{|A|} W}{\delta J^A}. \]

The identity the follows by induction.

### 2.4. Legendre Transforms.

Suppose that \( W = W[J] \) is a convex analytic function. That is, \( W \) is a real-valued analytic functional with the property that

\[ W\{tJ_1 + (1-t)J_2\} \leq tW[J_1] + (1-t)W[J_2] \]

for all \( 0 < t < 1 \) and \( J_1, J_2 \in \mathfrak{J} \). The Legendre-Fenchel transform of \( W \) is then defined by

\[ \Gamma[\varphi] = \inf_{J \in \mathfrak{J}} \{ W[J] - \langle \varphi, J \rangle \}. \]

\( \Gamma[\varphi] \) will then be a concave (i.e. \( -\Gamma \) is convex) analytic functional in \( \varphi \) and we may invert the formula as follows:

\[ W[J] = \sup_{\varphi \in \Phi} \{ \Gamma[\varphi] + \langle \varphi, J \rangle \}. \]

If the functional \( W \) is taken to be strictly convex, that is, if we have strict inequality in (2.3), then the infimum is attained at a unique source \( \tilde{J} = \tilde{J}[\varphi] \) for each fixed \( \varphi \), and so \( \Gamma[\varphi] = W[J[\varphi]] - \langle \varphi, J[\varphi] \rangle \). Moreover, we may invert \( \tilde{J} : \Phi \to \mathfrak{J} \) to get a mapping \( \tilde{\varphi} : \mathfrak{J} \to \Phi \), and for fixed \( J \) the supremum is given by \( \tilde{\varphi}[J] \) and so \( W[J] = \Gamma[\tilde{\varphi}[J]] + \langle \tilde{\varphi}[J], J \rangle \). The extremal conditions are then

\[ J^x \equiv -\frac{\delta \Gamma}{\delta \varphi_x}, \quad \tilde{\varphi}_x \equiv \frac{\delta W}{\delta J^x}. \]
Let $W''[J]$ be the symmetric tensor with entries $W''_{xy}[J] = \frac{\delta^2 W}{\delta J^x \delta J^y}$. This will be positive definite - it will be interpreted below as the covariance of the field in the presence of the source $J$. Likewise, if we let $\Gamma''[\phi]$ be the linear operator with entries $\delta^2 \Gamma \delta \phi \delta \phi$, then we have

$$W''_{xy}[J] = \frac{\delta \phi_x}{\delta J^y} \delta \phi_y^2, \quad \Gamma''_{xy}[\phi] = -\frac{\delta \phi_y}{\delta \phi_x^2} \delta \phi_x^2,$$

and so we conclude that $W''[J]$ and $-\Gamma''[\phi^2]$ will be inverses for each other. In other words,

$$\frac{\delta^2 W}{\delta J^x \delta J^y} \frac{\delta^2 \Gamma}{\delta \phi_y \delta \phi_x} \bigg|_{\phi = \phi^2[J]} = -\delta^2 \Gamma \bigg|_{\phi = \phi^2[J]} \frac{\delta^2 W}{\delta \phi_x \delta \phi_y} = -\delta^2 \Gamma \bigg|_{\phi = \phi^2[J]} \frac{\delta^2 W}{\delta \phi_x \delta \phi_y} = -\delta^2 \Gamma \bigg|_{\phi = \phi^2[J]} \frac{\delta^2 W}{\delta \phi_x \delta \phi_y}.$$

**Lemma 10.** Let $F : \Phi \mapsto \mathbb{R}$ be a functional and suppose $\bar{F} : \mathcal{J} \mapsto \mathbb{R}$ is then given by $\bar{F}[J] := F[\phi^2(J)]$ then

$$\frac{\delta \bar{F}}{\delta J^x} = \frac{\delta^2 W}{\delta J^x \delta J^y} \frac{\delta F}{\delta \phi_x} \bigg|_{\phi = \phi^2[J]}.$$

**Proof.** This is just the chain rule, as $\frac{\delta \bar{F}}{\delta J^x} = \frac{\delta^2 W}{\delta J^x \delta J^y}$. \qed

**Lemma 11.** Let us introduce the tensor coefficients

$$\bar{\Gamma}^X[J] := \frac{\delta^{|X|} \Gamma}{\delta \phi^X} \bigg|_{\phi = \phi^2[J]},$$

then they satisfy the differential equations

$$\frac{\delta^{|Y|} \bar{\Gamma}^X}{\delta J^Y} = \sum_{A \in \mathcal{P}(Y)} \bar{\Gamma}^{X \cup A} \left( \prod_{A \in A} \frac{\delta^{|A|+1} W}{\delta J^{A \cup (z_A)}} \right),$$

where each $z_A$ is a dummy variable associated with each component part $A$ and $Z_A = \{ z_A : A \in \mathcal{A} \}$. There is, as usual, an implied contraction over the repeated $z_A$'s in $\bar{\Gamma}^{X \cup Z_A}$ and the $\frac{\delta^{|A|+1} W}{\delta J^{A \cup (z_A)}}$.

**Proof.** We shall prove this inductively. First note that for $F = \Gamma^X$, we have $\frac{\delta}{\delta J^z} \bar{\Gamma}^X = \frac{\delta^2 W}{\delta J^z \delta J^x} \bar{\Gamma}^{X \cup (z)}$ by the previous lemma. Assuming that the relation (2.8)
is true for a set \( Y \), let \( y \) be a free index then

\[
\frac{\delta}{\delta J^y} \frac{\delta^{|Y|}}{\delta J^y} \bar{\Gamma}^X = \frac{\delta}{\delta J^y} \sum_{A \in \Psi(Y)} \bar{\Gamma}^{X \cup Z_A} \left( \prod_{A \in A} \frac{\delta^{|A|+1} W}{\delta J^A \cup \{z_A\}} \right)
\]

\[
= \sum_{A \in \Psi(Y)} \left\{ \bar{\Gamma}^{X \cup Z_A} \frac{\delta}{\delta J^y} \left( \prod_{A \in A} \frac{\delta^{|A|+1} W}{\delta J^A \cup \{z_A\}} \right) + \bar{\Gamma}^{X \cup Z_A \cup \{z\}} \left( \prod_{A \in A} \frac{\delta^{|A|+1} W}{\delta J^A \cup \{z_A\}} \right) \right\}
\]

\[
= \sum_{A \in \Psi(Y \cup \{y\})} \bar{\Gamma}^{X \cup Z_A} \left( \prod_{A \in A} \frac{\delta^{|A|+1} W}{\delta J^A \cup \{z_A\}} \right)
\]

and so the relation holds for \( Y \cup \{y\} \).

**Theorem 6.** We have the following recurrence relation

\[
\frac{\delta^{X+1} W}{\delta J^{X \cup \{y\}}} = \sum_{A \in \Psi(X)} \frac{\delta^2 W}{\delta J^y \delta J^p} \bar{\Gamma}^{(p) \cup Z_A} \left( \prod_{A \in A} \frac{\delta^{|A|+1} W}{\delta J^A \cup \{z_A\}} \right),
\]

where, again, each \( z_A \) is a dummy variable associated with each component part \( A \) and \( Z_A = \{z_A : A \in A\} \).

**Proof.** Taking \( \frac{\delta^{|X|}}{\delta J^X} \) of (2.9) and using the multi-derivative form of the Leibniz rule, we find

\[
0 = \frac{\delta^{|X|}}{\delta J^X} \left( \frac{\delta^2 W}{\delta J^y \delta J^p} \bar{\Gamma}^{y_z} \right)
\]

\[
= \sum_{Y \subseteq X} \frac{\delta^{|Y|+1} W}{\delta J^y \delta J^p} \delta^{X/Y} \bar{\Gamma}^{y_z}
\]

\[
= \sum_{Y \subseteq X} \frac{\delta^{|Y|+1} W}{\delta J^y \delta J^p \delta J^{X/Y}} \sum_{A \in \Psi(X/Y)} \bar{\Gamma}^{y_z \cup Z_A} \left( \prod_{A \in A} \frac{\delta^{|A|+1} W}{\delta J^A \cup \{z_A\}} \right).
\]

Now the \( Y = X \) term in the last summation yields \( \frac{\delta^{X+1} W}{\delta J^{X \cup \{y\}}} \bar{\Gamma}^{y_z} \), which will be the highest order derivative appearing in the expression, and we take it over to the left hand side, we then multiply both sides by \( -\frac{\delta^2 W}{\delta J^x \delta J^y} \) which is the inverse of \( \bar{\Gamma}^{y_z} \), finally we get the expression

\[
\frac{\delta^{X+1} W}{\delta J^{X \cup \{y\}}} = \sum_{Y \subseteq X} \sum_{A \in \Psi(X/Y)} \frac{\delta^2 W}{\delta J^y \delta J^p \delta J^{Y \cup \{x\}}} \left( \prod_{A \in A} \frac{\delta^{|A|+1} W}{\delta J^A \cup \{z_A\}} \right) \bar{\Gamma}^{(p) \cup Z_A}.
\]

We note that the sum over all \( Y \cup \{x\} \) (where \( Y \subseteq X \), but not \( Y = X \)) and partitions of \( X/Y \) can be reconsidered as a sum over all partitions of \( X \cup \{x\} \), excepting the coarsest one. Let us do this and set \( X' = X \cup \{x\} \) and \( Z'_A = Z_A \cup \{q\} \); dropping the primes then yields (2.9).
3. Green Functions

3.1. Field Moments. The Green functions are the field moments \( \langle \phi_{x_1} \cdots \phi_{x_n} \rangle \), where \( X = \{x_1, \cdots, x_n\} \) is a subset of \( \Lambda \), and they form the components of a symmetric tensor of covariant rank \( n = |X| \) under the conventions above. We shall assume that field moments to all orders exist. It is convenient to introduce the shorthand notation

\[
\langle \phi_{X} \rangle = \langle \prod_{x \in X} \phi_x \rangle.
\]

The moment generating function is given by the Laplace transform

\[
Z[J] = \int_\Phi e^{\langle \phi, J \rangle} d\nu(\phi) \equiv \langle e^{\phi J} \rangle.
\]

In particular \( Z[0] = 1 \) and we have the expansion

\[
\frac{\delta^n Z}{\delta J^{x_1} \cdots \delta J^{x_n}} \bigg|_{J=0} = \langle \phi_{x_1} \cdots \phi_{x_n} \rangle ,
\]

or, in shorthand, this reads as

\[
\langle \phi_{X} \rangle = \frac{\delta^{|X|} Z[J]}{\delta J^X} \bigg|_{J=0}.
\]

3.2. Cumulant Field Moments. The cumulative field moments \( \langle\langle \phi_{X} \rangle\rangle \) (also known as Ursell functions in statistical mechanics) are defined through

\[
\langle\langle \phi_{X} \rangle \rangle = \sum_{A \in \mathcal{P}(X)} \mu(A) \prod_{A \in A} \langle\phi_{A} \rangle
\]

and using Möbius inversion, we get

\[
\langle\langle \phi_{X} \rangle \rangle = \sum_{A \in \mathcal{P}(X)} \mu(A) \prod_{A \in A} \langle\phi_{A} \rangle
\]

where \( \mu(A) = -\prod_{j \geq 1} \{- (j - 1)! \}^{a_j(A)} \). The first couple of cumulative moments are

\[
\langle\langle \phi_{x} \rangle \rangle = \langle \phi_{x} \rangle ,
\]

\[
\langle\langle \phi_{x} \phi_{y} \rangle \rangle = \langle \phi_{x} \phi_{y} \rangle - \langle \phi_{x} \rangle \langle \phi_{y} \rangle ,
\]

\[
\langle\langle \phi_{x} \phi_{y} \phi_{z} \rangle \rangle = \langle \phi_{x} \phi_{y} \phi_{z} \rangle - \langle \phi_{x} \phi_{y} \rangle \langle \phi_{z} \rangle - \langle \phi_{x} \phi_{z} \rangle \langle \phi_{y} \rangle - \langle \phi_{y} \phi_{z} \rangle \langle \phi_{x} \rangle + 2 \langle \phi_{x} \rangle \langle \phi_{y} \rangle \langle \phi_{z} \rangle .
\]

Note that \( \langle\langle \phi_{x} \rangle \rangle \) is just the mean field while \( \langle\langle \phi_{x} \phi_{y} \rangle \rangle \) is the covariance of \( \phi_x \) and \( \phi_y \) in the state.

**Theorem 7.** The cumulative Green functions are generated through the

\[
W[J] = \sum_{X} \frac{1}{|X|!} \langle\langle \phi_{X} \rangle \rangle J^X \equiv \ln Z[J]
\]
3. GREEN FUNCTIONS

Proof. For convenience, we write \((CJ)_j\) for \(\left\langle \phi_{x_1} \cdots \phi_{x_j} \right\rangle\), then

\[
Z[J] = \sum_{n \geq 0} \frac{1}{n!} \sum_{A \in \Psi_n} \prod_{A \in A} (CJ)_{|A|}.
\]

Now suppose that we have a partition consisting of \(n_j\) parts of size \(j\) then the occupation sequence is \(n = (n_1, n_2, n_3, \cdots)\). The number of parts in the partition is therefore \(\sum_j n_j\) while the number of indices being partition is \(n = \sum_j j n_j\). Recall that the number of different partitions leading to the same occupation sequence \(n\) is given by \(\rho(n)\) in equation (2.4) and so

\[
Z[J] = \sum_n \frac{1}{n_1! n_2! \cdots (1!)^{n_1} (2!)^{n_2} (3!)^{n_3} \cdots (cJ)_1^{n_1} (cJ)_2^{n_2} \cdots}
\]

\[
= \sum_n \prod_{j \geq 1} \frac{1}{n_j!} \left[ \frac{(CJ)_j}{j!} \right]^{n_j}
\]

\[
= \prod_{j \geq 1} \sum_n \frac{1}{n!} \left[ \frac{(CJ)_j}{j!} \right]^{n}
\]

\[
= \prod_{j \geq 1} \exp \left\{ \frac{(CJ)_j}{j!} \right\}
\]

\[
= \exp \sum_{j \geq 1} \frac{(CJ)_j}{j!}
\]

and so \(Z = \exp W\).

Note that we have used the \(\sum \prod \leftrightarrow \prod \sum\) trick from our section on prime decompositions. We could have alternatively used the formula (2.2) to derive the same result, however, the above proof suggests that we should think of cumulant moments as somehow being the ‘primes’ from which the ordinary moments are calculated.

3.3. Presence of Sources. Let \(J\) be a source field. Given a probability measure \(\nu\), we may introduce a modified probability measure \(\nu^J\), absolutely continuous wrt. \(\nu\), and having Radon-Nikodym derivative

\[
\frac{d\nu^J}{d\nu}(\varphi) = \frac{1}{Z_{\nu}[J]} \exp \{ \langle \varphi, J \rangle_{\nu} \}.
\]

The corresponding state is referred to as the state modified by the presence of a source field \(J\).

Evidently, we just recover the reference measure \(\nu\) when we put \(J = 0\). The Laplace transform of the modified state will be \(Z_{\nu^J}[K] = \langle e^{\varphi_x} K^x \rangle_{\nu^J}\) and it is readily seen that this reduces to

\[
Z_{\nu^J}[K] = \frac{Z[J + K]}{Z[J]}.
\]

In particular, the cumulants are obtained through \(W_{\nu^J}[K] = W_{\nu}[J + K] - W_{\nu}[J]\) and we find \(\langle \phi_X \rangle_{\nu^J} = \frac{\partial^{|X|} W_{\nu}[K]}{\delta K^X} \bigg|_{K=0} \).
It is, however, considerably simpler to treat $J$ as a free parameter and just consider $\nu$ as being the family \{\(\nu^J : J \in \mathbb{J}\}\}. In these terms, we have

\[
\langle \phi_X \rangle_{\nu^J} = \frac{\delta^{[X]} W_{\nu}[J]}{\delta J^X} = \sum_{Y} \frac{1}{|Y|!} \langle \phi_{X \cup Y} \rangle_{\nu^K} J^Y,
\]

We point out that it is likewise more convenient to write

\[
\langle \phi_X \rangle_{\nu^J} = \frac{1}{Z_{\nu}[J]} \langle \phi_X e^{\langle \phi, J \rangle} \rangle_{\nu^K} = \frac{1}{Z_{\nu}[J]} \delta_{|X|} \sum_{Y} \langle \phi_X \rangle_{\nu^K} J^Y,
\]

and again we can drop the superscripts $J$.

The mean field in the presence of the source, $\bar{\phi} [J] \in \Phi$, is defined to be $\bar{\phi}_x [J] = \langle \phi_x \rangle_{\nu^K}$ and is given by the expression

\[
\bar{\phi}_x [J] = \sum_{n \geq 0} \frac{1}{n!} \langle \phi_x \phi_{x_1} \cdots \phi_{x_n} \rangle_{\nu^K} J^{x_1} \cdots J^{x_n},
\]

and, of course, reduces to $\langle \phi_x \rangle_{\nu^K}$ when $J = 0$.

### 3.4. States

Our basic example of a state is a Gaussian state. We also show how we might perturb one state to get another.

#### 3.4.1. Gaussian States

Let $L$ be a linear, symmetric operator on $\Phi$ with well-defined inverse $G$. We shall write $g^{xy}$ for the components of $L$ and $g_{xy}$ for the components of $G$. That is, the equation $L\varphi = J$, or $g^{xy}\varphi_y = J^x$ will have unique solution $\varphi = GJ$, or $\varphi_x = g_{xy} J^y$. As $G$ is positive definite, symmetric it can be used as a metric. It can also be used to construct a Gaussian state.

We construct a Gaussian state explicitly in the finite dimensional case where $|\Lambda| = N < \infty$ by setting

\[
d\gamma (\varphi) = \frac{1}{\sqrt{(2\pi)^{N} \det G}} \exp \left\{ -\frac{1}{2} g^{xy} \varphi_x \varphi_y \right\} \prod_{x \in \Lambda} d\varphi_x
\]

which we may say is determined from the a quadratic action given by $S_\gamma [\varphi] = -\frac{1}{2} g^{xy} \varphi_x \varphi_y$. The moment generating function is then given by

\[
Z_\gamma [J] = \exp \left\{ \frac{1}{2} g_{xy} J^x J^y \right\}.
\]

In the infinite dimensional case, we may use (3.8) as the definition of the measure.

The measure is completely characterized by the fact that the only non-vanishing cumulant is $\langle \phi_x \phi_y \rangle_{\gamma} = g_{xy}$ and if we now use (3.8) to construct the Green's functions we see that all odd moments vanish while

\[
\langle \phi_{x(1)} \cdots \phi_{x(2k)} \rangle_{\gamma} = \sum_{p_k} g_{x(p_1)x(q_1)} \cdots g_{x(p_k)x(q_k)}
\]

where the sum is over all pair partitions of $\{1, \cdots, 2k\}$. The right-hand side will of course consist of $\frac{(2k)!}{2^k k!}$ terms.
3.4.2. Perturbations of a State. Suppose we are given a probability measure $\mu_0$, and suppose that $S_I [\cdot]$ is some analytic functional on $\Phi$, say $S_I [\varphi] = \sum_{n \geq 0} \frac{1}{n!} v_{y_1} \cdots y_n \varphi_{y_1} \cdots \varphi_{y_n}$, or more compactly

$$S_I [\varphi] = \sum_X \frac{1}{|X|!} v_X \varphi_X. \tag{3.10}$$

A probability measure $\mu$, absolutely continuous wrt. $\mu_0$, is then prescribed by taking its Radon-Nikodym to be

$$\frac{d\nu}{d\nu_0} (\varphi) = \frac{1}{\Xi} \exp \{ S_I [\varphi] \}$$

provided, of course, that the normalization $\Xi \equiv \langle \exp \{ S_I [\varphi] \} \rangle_0 < \infty$.

The generating functional for $\mu$ will then be

$$Z_\mu [J] = \frac{1}{\Xi} \exp \left\{ S_I \left[ \frac{\delta}{\delta J} \right] \right\} Z_{\mu_0} [J]. \tag{3.11}$$

We remark that, for finite dimensional fields, we may think of $\mu$ being determined by the action $S [\varphi] = S_0 [\varphi] + S_I [\varphi]$ where $S_0$ is the action of $\mu_0$.

3.5. The Dyson-Schwinger Equation. We now derive a functional differential equation for the generating function.

**Lemma 12.** The Gaussian generating functional $Z_\gamma$ satisfies the differential equations

$$\left\{ F^x_\gamma \left[ \frac{\delta}{\delta J} \right] + J^x \right\} Z_\gamma [J] = 0$$

where $F^x_\gamma [\varphi] = \frac{\delta}{\delta \varphi_x} S_\gamma [\varphi]$.

**Proof.** Explicitly, we have $Z_\gamma = \exp \left\{ \frac{1}{2} J^x g_{xy} J^y \right\}$ so that $\frac{\delta}{\delta J^x} Z_\gamma = g_{xy} J^y Z_\gamma$ which can be rearranged as

$$\left( -g^{xy} \frac{\delta}{\delta J^y} + J^x \right) Z_\gamma = 0.$$

However, $S_\gamma [\varphi] = -\frac{1}{2} g^{xy} \varphi_x \varphi_y$ and so $F^x_\gamma [\varphi] = -g^{xy} \varphi_x$.

**Lemma 13.** Suppose that $Z_{\mu_0}$ satisfies a differential equation of the form

$$\left\{ F^x_{\mu_0} \left[ \frac{\delta}{\delta J} \right] + J^x \right\} Z_{\mu_0} [J] = 0$$

and that $\mu$, is absolutely continuous wrt. $\mu_0$, is given by $\frac{d\nu}{d\nu_0} (\varphi) = \frac{1}{\Xi} \exp \{ S_I [\varphi] \}$. Then $Z_\mu$ satisfies

$$\left\{ F^x_I \left[ \frac{\delta}{\delta J} \right] + F^x_{\mu_0} \left[ \frac{\delta}{\delta J} \right] + J^x \right\} Z_\mu [J] = 0$$
where \( F_I^x [\varphi] = \frac{\delta}{\delta \varphi_x} S_I [\varphi] \).

**Proof.** We observe that from (3.11) we have

\[
J^x Z_\mu [J] = \frac{1}{\Xi} J^x \exp \left\{ S_I \left[ \frac{\delta}{\delta J} \right] \right\} Z_{\mu_0} [J]
\]

and using the commutation identity

\[
\left[ J^x, \exp \left\{ S_I \left[ \frac{\delta}{\delta J} \right] \right\} \right] = -F_I^x \left[ \frac{\delta}{\delta J} \right] \exp \left\{ S_I \left[ \frac{\delta}{\delta J} \right] \right\}
\]

we find

\[
J^x Z_\mu [J] = \frac{1}{\Xi} \exp \left\{ S_I \left[ \frac{\delta}{\delta J} \right] \right\} \left( J^x Z_\mu [J] - \frac{1}{\Xi} F_I^x \left[ \frac{\delta}{\delta J} \right] Z_{\mu_0} [J] \right)
\]

\[
= -F_{\mu_0}^x \left[ \frac{\delta}{\delta J} \right] Z_\mu [J] - F_I^x \left[ \frac{\delta}{\delta J} \right] Z_\mu [J]
\]

which gives the result. \( \blacksquare \)

Putting these two lemmas together we obtain the following result:

**Theorem 8 (Dyson-Schwinger).** The generating functional \( Z_\mu \) for a probability measure absolutely continuous wrt. \( \gamma \) satisfies the differential equation

\[
(3.12) \quad \left\{ F^x \left[ \frac{\delta}{\delta J} \right] + J^x \right\} Z [J] = 0
\]

where \( F^x [\varphi] = \frac{\delta S [\varphi]}{\delta \varphi_x} = -\frac{1}{2} g^{xy} \varphi_y + F_I^x [\varphi] \).

**Corollary 1.** Under the conditions of the above theorem, if \( S_I [\varphi] = \sum_X \frac{1}{|X|!} \varphi^X \varphi_X \), then the ordinary moments of \( \mu \) satisfy the algebraic equations

\[
(3.13) \quad \langle \varphi_{\{x\} \cup X} \rangle = \sum_{x' \in X} g_{xx'} \langle \varphi_X/\{x'\} \rangle + g_{xy} \sum_Y \frac{1}{|Y|!} \varphi^{(x) \cup Y} \langle \varphi_{X \cup Y} \rangle.
\]

**Proof.** We now have \( F^x = -g^{xy} \varphi_y + \sum_X \frac{1}{|X|} \varphi^X \varphi_X \). The Dyson-Schwinger equation then becomes

\[
-g^{xy} \frac{\delta Z}{\delta J^y} + \sum_Y \frac{1}{|Y|!} \varphi^{(x) \cup Y} \frac{\delta |Y|}{\delta J^Y} Z + J^x Z = 0
\]

and we consider applying the further differentiation \( \frac{\delta |X|}{\delta J^X} \) to obtain

\[
-g^{xy} \langle \varphi_{\{y\} \cup X} \rangle_J + \sum_Y \frac{1}{|Y|!} \varphi^{(x) \cup Y} \langle \varphi_{X \cup Y} \rangle_J + \frac{\delta |X|}{\delta J^X} (J^x Z) = 0.
\]

The result follows from setting \( J = 0 \). \( \blacksquare \)
We may write (3.13) in index notation as
\[
\langle \phi_x \phi_{x_1} \cdots \phi_{x_m} \rangle = m \sum_{i=1}^{m} g_{x_i} \langle \phi_{x_1} \cdots \hat{\phi}_{x_i} \cdots \phi_{x_m} \rangle + g_{xy} \sum_{n \geq 0} \frac{1}{n!} \phi_{y_1} \cdots \phi_{y_n} \langle \phi_{x_1} \cdots \phi_{y_1} \cdots \phi_{x_m} \rangle
\]
where the hat indicates an omission. This hierarchy of equations for the Green’s functions is equivalent to the Dyson-Schwinger equation.

We remark that the first term on the right hand side of (3.13) contains the moments \(\langle \phi_{X/x} \rangle\) which are of order two smaller than the left hand side \(\langle \phi_{x} \rangle\).

The second term on the right hand side of (3.13) contains the moments of higher order and so we generally cannot use this equation recursively. In the Gaussian case, however, we have
\[
\langle \phi_{x_1} \cdots \phi_{x_m} \rangle = \sum_{x' \in X} g_{x_x} \langle \phi_{x_1} \cdots \phi_{x_m} \rangle
\]
and so we can deduce (3.9) by just knowing that \(\langle \phi \rangle = 0\) and \(\langle \phi_x \phi_y \rangle = g_{xy}\).

The Dyson-Schwinger equations may alternatively stated for \(W\), they are
\[
-g_{xy} \frac{\delta W}{\delta J^y} + \sum_{X} \frac{1}{|X|!} \Gamma_{x \cup X} \sum_{A \in \Psi(X)} \frac{\delta |A|}{\delta J^A} + J^x = 0.
\]

### 3.6. Field-Source Relations

Recall that \(\bar{\phi}_x[J]\) is the mean field value in the presence of a source \(J \in \mathcal{J}\) and is given by
\[
(3.14) \quad \bar{\phi}_x[J] = \frac{\delta W}{\delta J^x}.
\]
Let us assume again that the map \(J \mapsto \bar{\phi}\) from \(\mathcal{J}\) to \(\Phi\) is invertible and denote the inverse by \(\bar{J}\). That is, \(\bar{\phi}_x[J[\varphi]] = \varphi_x\). Moreover, we suppose that \(\bar{J}\) admits an analytic expansion of the type \(\bar{J}^x[\varphi] = -\sum_{n \geq 0} \frac{1}{n!} \Gamma_{x_1 \cdots x_n} \varphi_{y_1} \cdots \varphi_{y_n}\), or
\[
\bar{J}^x[\varphi] = -\sum_{X} \frac{1}{|X|!} \Gamma_{x \cup X} \varphi_X.
\]
We introduce the functional
\[
\Gamma[\varphi] = \sum_{n \geq 0} \frac{1}{n!} \Gamma_{y_1 \cdots y_n} \varphi_{y_1} \cdots \varphi_{y_n} = \sum_{X} \frac{1}{|X|!} \Gamma_{x \cup X} \varphi_X
\]
so that
\[
(3.15) \quad \bar{J}^x[\varphi] = -\frac{\delta \Gamma}{\delta \varphi_x}.
\]
We then recognize \(\Gamma[\varphi]\) as the Legendre-Frenchel transform of \(W[J]\), that is, 
\(W[J] = \sup_{\varphi \in \Phi} \{\Gamma[\varphi] + \langle \varphi, J \rangle\}\) with the supremum attained at the mean-field \(\varphi = \bar{\phi}[J]\).

For the Gaussian state we have \(W[\varphi] = W_{\gamma}[\varphi] = \frac{1}{2} g_{xy} J^x J^y\) and we see that \(\bar{\phi}_x = g_{xy} \varphi_y\). The inverse map is therefore \(\bar{J}^x[\varphi] = g_{xy} \varphi_y\) and so we obtain
\[
\Gamma_{\gamma}[\varphi] = -\frac{1}{2} g_{xy} \varphi_x \varphi_y.
\]
More generally, we have $\Gamma = \Gamma_\gamma + \Gamma_I$ where

$$\Gamma_I[\varphi] = \Gamma^{xy}_{\varphi_x} + \frac{1}{2} \pi^{xy}_{\varphi_x \varphi_y} + \sum_{n \geq 3} \frac{1}{n!} \Gamma^{xy_1\cdots x_n}_{\varphi_{x_1} \cdots \varphi_{x_n}}$$

where it is customary to set $\Gamma^{xy}_{\varphi_x} = -g^{xy} + \pi^{xy}_{\varphi_x}$. Without loss of generality we shall assume that the lead term $\Gamma_x$ is equal to zero: this means that the state is centred so that $\bar{\phi}[0] = 0$. It follows that

$$(3.16) \quad \bar{J}^{xy}[\varphi] = (g^{xy} - \pi^{xy}_{\varphi_y}) \varphi_y - \frac{1}{2} \Gamma^{xyz}_{\varphi_y \varphi_z} \varphi_z - \cdots$$

and (substituting $\varphi = \bar{\phi}$) we may rearrange this to get

$$(3.17) \quad \bar{\phi}_x = g^{xy} \bar{J}^y + \pi^{yz}_{\varphi_z} \bar{\phi}_z + \frac{1}{2} \Gamma^{yzw}_{\varphi_z \varphi_w} \bar{\phi}_w + \cdots$$

We note that $\bar{\phi}$ appears on the right-hand side of (3.17) in a generally nonlinear manner: let us rewrite this as $\bar{\phi} = GJ + f(\bar{\phi})$ where $f$ satisfies $f(0) = 0$. We may re-iterate (3.17) to get a so-called tree expansion

$$\bar{\phi} = GJ + f(GJ + f(GJ + f(GJ + \cdots)))$$

and we know that this expansion should be re-summed to give the series expansion in terms of $J$ as in (3.7).

Now $\Gamma''^{xy} [\varphi] = -g^{xy} + \Gamma''^{xy} [\varphi]$ so we conclude that

$$(3.18) \quad W''[J] = \frac{1}{L - \Gamma''_I[J]}$$

where $(\bar{\Gamma}''_I[J])^{xy} = \pi^{xy} + \sum_{|X| \geq 1} \frac{1}{|X|} \Gamma^{(x,y) \cup X}_{\bar{\phi}_X}[J]$. This relation may be alternatively written as the series

$$(3.19) \quad W''[J] = G \frac{1}{1 - \bar{\Gamma}''_I[J]} = G + G \bar{\Gamma}''_I[J] G + G \bar{\Gamma}''_I[J] G \bar{\Gamma}''_I[J] G + \cdots$$

In particular, when we set $J = 0$, $W''[0]$ is the covariance matrix while $\bar{\Gamma}''_I[0] \equiv \pi$, provided we assume that the state is centred ($\bar{\phi} = 0$ when $J = 0$), and we obtain the series expansion

$$(3.20) \quad W''[0] = G + G \pi G + G \pi G \pi G + \cdots$$

We now wish to determine a formula relating the cumulant moments in the presence of the source $J$ to the tensor coefficients $\bar{\Gamma}^X[J]$.

**Theorem 9.** We have the following recurrence relation

$$(3.21) \quad \left\langle \left\langle \phi_{X \cup \{y\}} \right\rangle \right\rangle_{\nu} = \sum_{A \in \mathcal{P}(X)} \left\langle \left\langle \phi_{\nu} \phi_{\{y\}} \right\rangle \right\rangle_{\nu} \bar{\Gamma}^{\pi \cup Z_A}[J] \left( \prod_{A \in \mathcal{A}} \left\langle \left\langle \phi_{A \cup \{z_A\}} \right\rangle \right\rangle_{\nu} \right),$$

where, again, each $z_A$ is a dummy variable associated with each component part $A$ and $Z_A = \{ z_A : A \in \mathcal{A} \}$. 

This, of course, just follows straight from \((\ref{eq:kPhiPhi})\). The crucial thing about \((\ref{eq:kPhiPhi})\) is that the right hand side contains lower order cumulants and so can be used recursively. Let us iterate once:

\[
\left\langle \left\langle \phi_{\{y\} \cup X} \right\rangle \right\rangle_{\nu,j} = \left\langle \left\langle \phi_{\{y,p\}} \right\rangle \right\rangle_{\nu,j} \left( \sum_{A \in \Psi'(X)} \bar{\Gamma}^{(p)\cup Z_{A}} [J] \left( \prod_{A \in \mathcal{A}} \left\langle \left\langle \phi_{A\cup \{z_{A}\}} \right\rangle \right\rangle \right) \right)
\]

\[
= \left\langle \left\langle \phi_{\{y,p\}} \right\rangle \right\rangle_{\nu,j} \left( \sum_{A \in \Psi'(X)} \bar{\Gamma}^{(p)\cup Z_{A}} [J] \prod_{A \in \mathcal{A}} \left\langle \left\langle \phi_{\{z_{A},q\}} \right\rangle \right\rangle \right) \times \prod_{B \in \Psi(A)} \bar{\Gamma}^{(q)\cup Z_{B}} [J] \left( \prod_{B \in \mathcal{B}} \left\langle \left\langle \phi_{B,B} \right\rangle \right\rangle \right). \]

What happens is that each part \(A\) of the first partition gets properly divided up into sub-parts and we continue until eventually break down \(X\) into its singleton parts. However, this is just a top-down description of a hierarchy on \(X\). Proceeding in this way we should obtain a sum over all hierarchies of \(X\). At the root of the tree, we have the factor \(\left\langle \left\langle \phi_{y} \phi_{y} \right\rangle \right\rangle_{\nu,j}\) and for each node/part \(A\), appearing anywhere in the sequence, labelled by dummy index \(z_{A}\) say, we will break it into a proper partition \(B \in \Psi(A)\) and obtain a multiplicative factor \(\left\langle \left\langle \phi_{\{z_{A},q\}} \right\rangle \right\rangle \bar{\Gamma}^{(q)\cup Z_{B}} [J]\) where \(Z_{B}\) will be the set of labels for each part \(B \in \mathcal{B}\).

If we set \(X = \{x_{1},x_{2}\}\) then we have only one hierarchy to sum over and we find

\[
\left\langle \left\langle \phi_{y} \phi_{x_{1}} \phi_{x_{2}} \right\rangle \right\rangle_{\nu,j} = \bar{\Gamma}^{x_{1},x_{2}} [J] \left\langle \left\langle \phi_{x_{1}} \phi_{x_{2}} \right\rangle \right\rangle_{\nu,j}. \]

It is useful to use the bottom-up description to give the general result. First we introduce some new coefficients defined by

\[
\Upsilon_{x_{1},\ldots,x_{n}}^{y}[J] := \left\langle \left\langle \phi_{x_{1}} \phi_{x_{n}} \right\rangle \right\rangle_{\nu,j} \bar{\Gamma}^{x_{1},\ldots,x_{n} \cup y}. \]

with the exceptional case \(\Upsilon_{xy} := \left\langle \left\langle \phi_{x} \phi_{y} \right\rangle \right\rangle_{\nu,j}\). Then we find that

\[
\left\langle \left\langle \phi_{\{y\} \cup X} \right\rangle \right\rangle_{\nu,j} = \sum_{\mathcal{H} = \{A^{(1)},\ldots,A^{(m)}\} \in \delta(X)} \Upsilon_{y}^{z_{1},\ldots,z_{n} \cup \mathcal{H}} \times \prod_{A^{(m)} \in \mathcal{A}(m)} \Upsilon_{A^{(m)}}^{z_{A^{(m)}},\ldots} \prod_{A^{(1)} \in \mathcal{A}(1)} \Upsilon_{A^{(1)}}. \tag{3.22} \]

For instance, we have the following expansions for the lowest cumulants:

\[
\left\langle \left\langle \phi_{y} \phi_{x_{1}} \phi_{x_{2}} \right\rangle \right\rangle_{\nu,j} = \Upsilon_{y,x_{1},x_{2}} \]

\[
\left\langle \left\langle \phi_{y} \phi_{x_{1}} \phi_{x_{3}} \phi_{x_{4}} \right\rangle \right\rangle_{\nu,j} = \Upsilon_{y,x_{1},x_{2}} + (\Upsilon_{x_{1}} + \Upsilon_{x_{2}} + \cdots) \]

\[
\left\langle \left\langle \phi_{y} \phi_{x_{1}} \phi_{x_{2}} \phi_{x_{3}} \phi_{x_{4}} \phi_{x_{5}} \right\rangle \right\rangle_{\nu,j} = \Upsilon_{y,x_{1},x_{2},x_{3},x_{4}} + (\Upsilon_{x_{1}} + \Upsilon_{x_{2}} + \cdots) + (\Upsilon_{y,x_{1},x_{2}} + \Upsilon_{y,x_{3},x_{4}} + \cdots)
\]

\[
+ (\Upsilon_{x_{1},x_{2}} + \Upsilon_{y,x_{1},x_{2}} + \cdots) + (\Upsilon_{x_{1},x_{2},x_{3},x_{4}} + \cdots) + (\Upsilon_{y,x_{1},x_{2},x_{3},x_{4}} + \cdots) \]

etc.

The terms in round brackets involve permutations of the \(x_{j}\) indices leading to distinct terms. Thus, there are \(1 + \frac{1}{4}(\binom{5}{4}) = 4\) terms making up the right-hand side for the fourth order cumulant: The first term in round brackets corresponds to the hierarchy \(\{\{x_{1}\}\},\{\{x_{2}\}\},\{\{x_{3}\}\}\) and there are 3 such second order hierarchies. There are \(1 + \binom{3}{2} + \frac{1}{2}(\binom{5}{4}) = 26\) terms making up the right-hand side for the fifth order cumulant.
4. Wick Ordering

4.1. Wick Cumulants. Up to now, we have been considering cumulant moments of the field. Let $F_1, F_2, \ldots, F_n$ be some functionals of our field, then we may define the object $\langle\langle [F_1] \cdots [F_n] \rangle\rangle$ through

$$\langle\langle [F_1] \cdots [F_n] \rangle\rangle = \sum_{A \in \mathcal{P}(X)} \prod_{A \in \mathcal{A}} \langle\langle \prod_{i \in A} [F_i] \rangle\rangle$$

where $\langle [F_1] \cdots [F_n] \rangle \equiv \langle F_1 \cdots F_n \rangle$. Thus $\langle\langle [F] [G] \rangle\rangle = \langle FG \rangle - \langle F \rangle \langle G \rangle$, etc.

As the notation hopefully suggests, we treat each component in a square bracket as an indivisible whole object. For instance, we have that $\langle\langle [\phi_x] [\phi_y] [\phi_z] \rangle\rangle$ is a second order cumulant (even though it has three field factors!) and it equates to $\langle \phi_x \phi_y \phi_z \rangle - \langle \phi_x \rangle \langle \phi_y \phi_z \rangle$.

4.2. Wick Monomials. The following definition of Wick monomial is due to Djah, Gottschalk and Ouerdiane [4]. It is more general than the traditional definition, which uses the vacuum state, and we review it in terms of our field calculus notation: the theorems are otherwise after [4] directly.

**Definition 1.** For each finite subset $X$, the Wick ordered monomial $\phi_X$ is the operator defined through the property that

$$\langle\langle \phi_X : F \rangle\rangle = \langle\langle \prod_{x \in X} [\phi_x] [F] \rangle\rangle$$

for all appropriate $F$.

Let us first remark that the appropriate functionals $F = F [\phi]$ are those such that $F = F [\phi]$ is square-integrable wrt. the measure $\nu$. Secondly, it should be emphasized that the definition of $\phi_X$ depends on the choice of state or, equivalently, on $\nu$.

**Lemma 14.** The ordinary field operator products $\phi_X$ can be put together from the Wick monomials according to the formula

$$\phi_X = \langle \phi_X \rangle + \sum_{A \in \mathcal{P}(X)} \sum_{A \in \mathcal{A}} \left( \prod_{B \neq A} \langle\langle \phi_B \rangle\rangle \right) : \phi_A :$$

**Proof.** First of all, we observe that we can use the short-hand form $\langle\langle \phi_{X \cup \{\alpha\}} \rangle\rangle$ for $\langle\langle \prod_{x \in X} [\phi_x] [F] \rangle\rangle$ by setting $\phi_\alpha = F$ where $\alpha$ is an exceptional label, and we augment our configuration space to $X \cup \{\alpha\}$. Now the ordinary moment $\langle \phi_{X \cup \{\alpha\}} \rangle$ can be expanded as $\sum_{A \in \mathcal{P}(X \cup \{\alpha\})} \prod_{A \in \mathcal{A}} \langle\langle \phi_A \rangle\rangle$ and the partitions of $X \cup \{\alpha\}$ can be set out into two types: those that contain $\{\alpha\}$ as a singleton part, and those
that don’t. This leads to
\[
\left\langle \phi_{X \cup \{a\}} \right\rangle = \left( \sum_{A \in \mathfrak{P}(X)} \prod_{A \in A} \left\langle \phi_A \right\rangle \right) \left\langle \phi_\alpha \right\rangle + \sum_{A \in \mathfrak{P}(X)} \sum_{A \in A} \left( \prod_{B \notin A} \left\langle \phi_B \right\rangle \right) \left\langle \phi_{A \cup \alpha} \right\rangle
\]
\[
= \left\langle \phi_X \right\rangle \left\langle \phi_\alpha \right\rangle + \sum_{A \in \mathfrak{P}(X)} \sum_{A \in A} \left( \prod_{B \notin A} \left\langle \phi_B \right\rangle \right) \left\langle \phi_A : \phi_\alpha \right\rangle.
\]
However, as \(\phi_\alpha\) was arbitrary, we obtain the required identity. □

**Corollary 2.** The Wick monomials satisfy the following recursion relation
\[
\phi_X := \phi_X - \left\langle \phi_X \right\rangle - \sum_{A \in \mathfrak{P}(X)} \sum_{A \in A} \left( \prod_{B \notin A} \left\langle \phi_B \right\rangle \right) \left\langle \phi_A \right\rangle :.
\]

**Theorem 10.** Let \(X, Y_1, \ldots, Y_m\) be disjoint finite subsets, then
\[
\left\langle : \phi_{Y_1} : \cdots : \phi_{Y_m} : \phi_X \right\rangle = \sum_{A \in \mathfrak{P}'(Y_1, \ldots, Y_m; X)} \prod_{A \in A} \left\langle \phi_A \right\rangle
\]
where \(\mathfrak{P}'(Y_1, \ldots, Y_m; X)\) is the set of all partitions of \(Y_1 \cup \cdots \cup Y_m \cup X\) having no subset of any of the \(Y_j\)'s as a part.

**Proof.** Let \(n = \sum_{j=1}^m |Y_j|\), the result will be established by strong induction. The case \(n = 0\), to begin with, is just the expansion of \(\left\langle \phi_X \right\rangle\) in terms of its connected Green’s function. If the results is assumed to hold up to value \(n\) then
\[
\left\langle : \phi_{Y_1} : \cdots : \phi_{Y_{m+1}} : \phi_X \right\rangle = \left\langle : \phi_{Y_1} : \cdots : \phi_{Y_m} : \phi_{Y_{m+1}} \cup X \right\rangle
\]
\[
- \left\langle : \phi_{Y_1} : \cdots : \phi_{Y_m} : \phi_X \right\rangle \left\langle \phi_{Y_{m+1}} \right\rangle
\]
\[
- \sum_{A \in \mathfrak{P}'(Y_{m+1})} \sum_{A \in A} \left( \prod_{B \notin A} \left\langle \phi_B \right\rangle \right) \left\langle : \phi_{Y_1} : \cdots : \phi_{Y_m} : \phi_X \right\rangle
\]
and by induction up to order \(n\) we may rearrange this as
\[
\sum_{A \in \mathfrak{P}'(Y_1, \ldots, Y_{m+1}; X)} \prod_{A \in A} \left\langle \phi_A \right\rangle - \sum_{A \in \mathfrak{P}'(Y_1, \ldots, Y_m; X)} \sum_{B \in \mathfrak{P}(Y_{m+1})} \prod_{A \in A} \left\langle \phi_A \right\rangle \prod_{B \in B} \left\langle \phi_B \right\rangle
\]
\[
= \sum_{A \in \mathfrak{P}'(Y_{m+1})} \sum_{A \in A} \sum_{B \in \mathfrak{P}'} \sum_{A \in \mathfrak{P}(Y_{m+1})} \prod_{A \in A} \left\langle \phi_A \right\rangle \prod_{B \in B} \left\langle \phi_B \right\rangle \prod_{C \in C} \left\langle \phi_C \right\rangle.
\]

The claim is that this equals \(\sum_{A \in \mathfrak{P}'(Y_1, \ldots, Y_{m+1}; X)} \prod_{A \in A} \left\langle \phi_A \right\rangle\). Now the first term appearing in (4.2) is the sum over \(\mathfrak{P}'(Y_1, \ldots, Y_{m+1}; X)\) which is strictly larger than \(\mathfrak{P}'(Y_1, \ldots, Y_{m+1}; X)\) by means of including unwanted partitions of \(Y_1 \cup \cdots \cup Y_{m+1} \cup X\) in which subsets of \(Y_{m+1}\) appear as parts. The second term of (4.2) subtracts all such contributions from partitions for which every part \(B\) that contains an element of \(Y_{m+1}\) only contains elements of \(Y_{m+1}\). The third term subtracts the contribution from the remaining unwanted partitions, namely those where the offending parts \(B\) make up only some subset \(Y_{m+1}/A\) of \(Y_{m+1}\).

This establishes the result by induction. □
Stochastic Integrals

Stochastic processes are intermediary between random variables and quantum fields. That’s certainly not the view that most probabilists would have of their subject but for our purposes it is true enough. Indeed, there exists a natural theory of quantum stochastic calculus extending the classical theory of Itô to the quantum domain.

We can treat stochastic processes as a special case of quantum fields over one time dimension. More exactly, we may view stochastic processes as regularized forms of white noise processes - these are stochastic processes only in some distributional sense, but can be realized as quantum fields theory. Instead, we shall work with the more traditional viewpoint used by probabilists. We first recall some basic ideas.

We say that a random variable is second order if it has finite first and second moment and that a second order variable $X$ is the mean-square limit of a sequence of second order variables $(X_n)_n$ if $\mathbb{E}[(X - X_n)^2] \to 0$. A stochastic process is a family of random variables $\{X_t : t \geq 0\}$ labeled by time parameter $t$. The process is second order if $\mathbb{E}[X_t], \mathbb{E}[X_tX_s] < \infty$, for all times $t, s > 0$.

A process $\{X_t : t \geq 0\}$ is said to have independent increments if $X_t - X_s$ and $X_{t'} - X_{s'}$ are independent whenever the time intervals $(s,t)$ and $(s',t')$ do not overlap. The increments are stationary if the probability distribution of each $X_t - X_s$ depends only on $t - s$, where we assume $t > s$. The most important examples of second order processes are the Wiener process $\{W_t : t \geq 0\}$ and the Poisson process $\{N_t : t \geq 0\}$. The Wiener process is characterized by having increments $W_t - W_s$, $(t > s)$, distributed according to a Gaussian law of mean zero and variance $t - s$. The Poisson process is characterized by having increments $N_t - N_s$, $(t > s)$, distributed according to a Poisson law of intensity $t - s$.

In general, given a pair of stochastic processes $\{X_t\}$ and $\{Y_t\}$, we can try and give meaning to their stochastic integral $\int_a^b X_t dY_t$. What we do is to divide the interval $[a,b]$ up as $a = t_0 < t_1 < \cdots < t_n = b$ and consider the finite sum $\sum_{j=0}^n X_{t_j} (Y_{t_{j+1}} - Y_{t_j})$ and consider the limit in which $\max \{t_{j+1} - t_j\} \to 0$. If the sequence of finite sums has a mean-square limit then we call it the Itô integral $\int_a^b X_t dY_t$.

There is a problem, however. The Wiener process has increments that do not behave in the way we expect infinitesimals to behave. Let $\Delta t > 0$ and set $\Delta W_t = W_{t+\Delta t} - W_t$, then, far from being negligible, $(\Delta W_t)^2$ is a random variable of mean value $\Delta t$. The consequence is that the usual rules of calculus will not apply to stochastic integrals wrt. the Wiener process. Remarkably, if we work with suitable process there is a self-consistent theory, the Itô calculus, which extends the usual notions. In particular, the integration by parts formula is replaced by the Itô
formula, which states that
\[ d (X_s Y_t) = (dX_t) Y_t + X_t (dY_t) + (dX_t) (dY_t) \]
and is to be understood under the integral sign. We now present some combinatorial results on multiple Itô integrals: more details can be found in the paper of Rota and Wallstrom [16].

1. Multiple Stochastic Integrals

Let \( X^{(j)}_t \) be stochastic processes for \( j = 1, \ldots, n \). We use the following natural (notational) conventions:
\[ X_t^{(n)} \cdots X_t^{(1)} = \int_{[0,t]} dX_t^{(n)} \cdots \int_{[0,t]} dX_t^{(1)} = \int_{[0,t]^n} dX_t^{(n)} \cdots dX_t^{(1)}. \]

In the following we denote by \( \Delta^n_\sigma (t) \) the \( n \)-simplex in \([0,t]^n\) determined by a permutation \( \sigma \in \mathfrak{S}_n \): that is,
\[ \Delta^n_\sigma (t) = \left\{ (t_1, \ldots, t_n) \in (0,t)^n : t_{\sigma(1)} > t_{\sigma(2)} > \cdots > t_{\sigma(n)} \right\}. \]

We denote by \( \Delta^n (t) \) the simplex determined by the identity permutation: that is \( t > t_n > t_{n-1} > \cdots > t_1 > 0 \). Clearly \( \bigcup_{\sigma \in \mathfrak{S}_n} \Delta^n_\sigma (t) = [0,t]^n \) with the absence of the hypersurfaces (diagonals) of dimension less than \( n \) corresponding to the \( t_j \)'s being equal. Moreover, the \( \Delta^n_\sigma (t) \) are distinct for different \( \sigma \).

We also define the off-diagonal integral “\( \mathcal{J} \)” to be the expression with all the diagonal terms subtracted out. Explicitly
\[ \int_{[0,t]^n} dX_t^{(n)} \cdots dX_t^{(1)} := \sum_{\sigma \in \mathfrak{S}_n} \int_{\Delta^n_\sigma (t)} dX_t^{(n)} \cdots dX_t^{(1)}. \]

Take \( s_1, s_2, \ldots \) to be real variables and let \( A = \{ A_1, \ldots, A_m \} \) be a partition of \( \{1, \ldots, n\} \) then, for each \( i \in \{1, \ldots, n\} \), define \( s_A (i) \) to be the variable \( s_j \) where \( i \) lies in the part \( A_j \).

\textbf{Lemma 15.} The multiple stochastic integral can be decomposed as
\[ \int_{[0,t]^n} dX_t^{(n)} \cdots dX_t^{(1)} = \sum_{A \in \mathcal{P}_n} \int_{[0,t]^{N(T)}} dX_t^{(n)}_{s_A (n)} \cdots dX_t^{(1)}_{s_A (1)}. \]

Note that there \( n = 1 \) case is immediate as the \( \int \) and \( \mathcal{J} \) integrals coincide. In the situation \( n = 2 \) we have by the Itô formula
\[
\begin{align*}
X_t^{(2)} X_t^{(1)} &= \int_0^t dX_t^{(2)} X_t^{(1)} + \int_0^t X_t^{(2)} dX_t^{(1)} + \int_0^t dX_t^{(2)} dX_t^{(1)} \\
&= \int_{t > t_2 > t_1 > 0} dX_t^{(2)} dX_t^{(1)} + \int_{t > t_2 > t_1 > 0} dX_t^{(2)} dX_t^{(1)} + \int_0^t dX_t^{(2)} dX_t^{(1)} \\
&= \int_{[0,t]^2} dX_t^{(2)} dX_t^{(1)} + \int_{[0,t]} dX_t^{(2)} dX_t^{(1)} \\
&\text{and this is the required relation.}
\end{align*}
\]
The higher order terms are computed through repeated applications of the Itô formula. An inductive proof is arrived at along the following lines. Let $X_t^{(n+1)}$ be another quantum stochastic integral, then the Itô formula is

$$X_t^{(n+1)}Y_t = \int_{[0,t]^2} dx_{n+1}^{(1)} \int_{[0,t]} dY_s^{(n+1)} Y_s^{(n+1)}$$

and we take $Y_t = X_t^{(n)} \cdots X_1^{(1)}$. Assume the formula is true form $n$. The first term will be the sum over all partitions of $\{n+1, n, \ldots, 1\}$ in which $\{n+1\}$ appears as a singleton, the second term will be the sum over all partitions of $\{n+1, n, \ldots, 1\}$ in which $n+1$ appears as an extra in some part of a partition of $\{n, \ldots, 1\}$. In this way we arrive at the appropriate sum over $\mathfrak{P}_{n+1}$.

**Corollary 3.** The inversion formula for off-diagonal integrals is

$$\int_{[0,t]^n} dx_{n}^{(n)} \cdots dx_{1}^{(1)} = \sum_{A \in \mathfrak{P}_n} \mu(A) \int_{[0,t]^{N(A)}} dx_{s(A)}^{(n)} \cdots dx_{s(A)}^{(1)}$$

where $\mu(A)$ is the Möbius function introduced earlier.

**1.1. Multiple Martingale Integrals.** We suppose that $X_t$ is a martingale so that, in the particular, $E \left[ \int_{[0,t]} \phi(s) dX_s \right] = 0$ where $\phi$ is any adapted integrable function. In general, multiple integrals with respect to $X$ will not have zero expectation, however, this will be the case for the off-diagonal integrals:

$$E \left[ \int_{[0,t]^n} dx_{n}^{(n)} \cdots dx_{1}^{(1)} \right] = 0.$$ 

This property is in fact the main reason for introducing off-diagonal integrals in the first place.

In the special case of classical processes we can employ the commutativity to write

$$\int_{\Delta^n(t)} dx_{n}^{(n)} \cdots dx_{1}^{(1)} = \frac{1}{n!} \int_{[0,t]^n} dx_{n}^{(n)} \cdots dx_{1}^{(1)}.$$ 

We define a family of random variables $E^{X_t}$ by

$$E^{X_t} := \sum_{n \geq 0} \frac{1}{n!} \int_{[0,t]^n} dx_{n}^{(n)} \cdots dx_{1}^{(1)}$$

for $t \geq 0$. We refer to process $t \mapsto E^{X_t}$ as an exponentiated martingale.

**1.2. Wiener Integrals.** Let us take $X_t$ to be the Wiener process $W_x$. We will have

$$\int_{[0,t]^n} dW_{n}^{(n)} \cdots dW_{1}^{(1)} = \sum_{A \in \mathfrak{P}_n} \mu(A) \int_{[0,t]^{N(A)}} dW_{s(A)}^{(n)} \cdots dW_{s(A)}^{(1)}$$

Here we have $dW_s dW_s = ds$: this means that if we sum over partitions $A$ in the inversion formula then we need only consider those consisting of singletons and pairs only. We will then have $E(A) = n_1 + 2n_2 = n$ and $N(A) = n_1 + n_2 = n - n_2$. The combinatorial factor is $F(A) = 1.$
Now there are \( \binom{n}{n_1} = \binom{n}{2n_2} \) ways to choose the singletons and \( \frac{(2n_2)!}{2^{n_2}n_2!} \) ways to choose the pairs. This yields
\[
\int_{[0,t]^n} dW_{t_n}^{(n)} \cdots dW_{t_1}^{(1)} = \sum_{n_2=0}^{[n/2]} \frac{(-1)^{n-n_2} n!}{2^{n_2}n_2!(n-2n_2)!} (W_{t})^{n-2n_2} t^{n_2} \\
\quad = t^{n/2} H_n \left( \frac{W_t}{\sqrt{t}} \right)
\]
where \( H_n(x) = \sum_{k=0}^{[n/2]} (-1)^{n-k} \frac{n!}{2^k k!(n-2k)!} x^{n-k} \) are the well-known Hermite polynomials. The implication that \( \int_{\Delta_n(t)} dW_{t_n}^{(n)} \cdots dW_{t_1}^{(1)} = \frac{1}{n!} t^{n/2} H_n \left( \frac{W_t}{\sqrt{t}} \right) \) is a result due originally to Itô.

Using the relation \( \sum_{n=0}^{\infty} \frac{e^n}{n!} H_n(x) = \exp(xt - t^2/2) \) for generating the Hermite polynomials, we see that the exponentiated random variable corresponding to the choice of the Wiener process is \( E^{W_t} = e^{sW_t - s^2t/2} \).

More generally, if we take \( W_t(f) = \int_{[0,t]} f(s) \, dW_s \) for \( f \) square-integrable then its exponentiated random variable is
\[
E^{W_t(f)} = \exp \left\{ W_t(f) - \frac{1}{2} \int_{[0,t]} f(s)^2 \, ds \right\}.
\]

### 1.3. Compensated Poisson Process Integrals.

Let \( N_t \) be the Poisson process and \( Y_t = N_t - t \) be the compensated process. We have the differential rule \( (dN_t)^p = dN_t \) for all positive integer powers \( p \). The process \( Y_t \) is a martingale and we have \( (dY_t)^p = dN_t = dY_t + dt \) for all positive integer \( p \) with the obvious exception of \( p = 1 \).

It is convenient to replace sums over partitions with sums over occupation numbers. This time we find
\[
\int_{[0,t]^n} dY_{t_n}^{(n)} \cdots dY_{t_1}^{(1)} = \sum_{n_1, n_2, \ldots, n_m} \frac{(-1)^{n_1+2n_2+3n_3+\cdots} n_1!}{n_1!n_2!n_3!\cdots} (Y_{t})^{n_1} (Y_{t} + t)^{n_2} (Y_{t} + t)^{n_3} \cdots 
\]
Again we use a \( \prod \left\{ \prod \sum \text{ trick!} \right\} \)
\[
E^{zY_t} = \sum_{n_1, n_2, \ldots, n_m} \frac{(-1)^{n_1+2n_2+3n_3+\cdots} z^{n_1+2n_2+3n_3+\cdots}}{n_1!n_2!n_3!\cdots} (Y_{t} + t)^{n_1} (Y_{t} + t)^{n_2} (Y_{t} + t)^{n_3} \cdots 
\]
\[
= e^{-zt} \sum_{n_1, n_2, \ldots, n_m} \frac{(-1)^{n_1+2n_2+3n_3+\cdots}}{n_1!n_2!n_3!\cdots} (Y_{t} + t)^{n_1} (Y_{t} + t)^{n_2} (Y_{t} + t)^{n_3} \cdots 
\]
\[
= e^{-zt} \prod_{k=1}^{\infty} \left( \sum_{n_k=0}^{\infty} \frac{(-1)^{k-1} z^{n_k}}{n_k!} \frac{1}{k} \right) \left( \begin{array}{c} z \frac{Y_{t} + t}{1} \end{array} \right)^{n_k} 
\]
\[
= e^{-zt} \prod_{k=1}^{\infty} \exp \left( -\frac{k-1}{k} z \frac{Y_{t} + t}{1} \right) 
\]
and using \( \sum_{k=1}^{\infty} (-1)^{k-1} \frac{k}{k} = \ln (1 + z) \) we end up with
\[
E^{zY_t} = e^{-zt} (1 + z)^{Y_{t} + t}.
\]

This means that, in terms of the Poisson process \( N_t \), the exponentiated random variable associated with the compensated Poisson process \( Y_t \) is \( E^{Y_t} (z) = e^{-zt} (1 + z)^{N_t} \).
and we have
\[ \int_{[0,t]^n} dY_{t_n}^{(n)} \cdots dY_{t_1}^{(1)} = C_n (N, t) \]
where \( C_n(x,t) \) are the Charlier polynomials determined by the generating relation
\[ \sum_{n \geq 0} \frac{z^n}{n!} C_n(x,t) = e^{-zt} (1 + z)^x. \]
Explicitly we have
\[ C_n(x,t) = \sum_k (-t)^{n-k} \binom{n}{k} x^k = \sum_{k,r} (-t)^{n-k} \binom{n}{k} s(k,r) x^k. \]

2. Itô-Fock Isomorphism

Here we examine stochastic processes and exploit the fact that they can be viewed as commuting one-dimensional quantum fields (with base space typically taken as time). It should be remarked that the Wiener and Poisson process of classical probability can be naturally viewed as combinations of creation, annihilation and conservation processes introduced by Hudson and Parthasarathy \([10]\) as the basis of a quantum stochastic calculus which extends the Itô calculus to operator-valued processes. In fact the Itô correction can be alternatively understood as the additional term that arises from Wick ordering of stochastic integrals with respect to white noise creation/annihilation operators, see \([6]\), for instance, and also \([7]\).

2.1. Itô-Fock Isomorphism. Let \( X_t \) be a classical martingale with canonical probability space \((\Omega_X, \mathcal{F}_X, \mathbb{P})\) and let \( \mathfrak{h}_X = L^2(\Omega_X, \mathcal{F}_X, \mathbb{P}) \). We consider the function \( F(t) := \mathbb{E} [X_t^2] \) and this defines a monotone increasing function. We shall understand \( dF(t) \) to be the Stieltjes integrator in the following.

It turns out that our considerations so far allow us to construct a natural isomorphism between \( \mathfrak{h}_X \) and the Fock space \( \Gamma_+ (L^2(\mathbb{R}^+, dF)) \), see e.g. \([14]\).

For \( f \in L^2(\mathbb{R}^+, dF) \), we define the random variable \( \tilde{X} (f) := \int_{[0,\infty]} f(s) dX_s \) and \( \tilde{X}_t (f) := \tilde{X} (1_{[0,t]} f) \).

**Lemma 16.** Let \( \tilde{Z}^{(n)}_t (f) = \int_{\Delta^m(t)} d\tilde{X}_{t_n} (f) \cdots \tilde{X}_{t_1} (f) \) then
\[ \mathbb{E} [\tilde{Z}^{(n)}_t (f) \tilde{Z}^{(m)}_s (g)] = \frac{1}{n!} \left( \int_0^{t \wedge s} f(u) g(u) dF(u) \right)^n \delta_{n,m}. \]

**Proof.** For simplicity, we ignore the intensities. Let \( Z^{(n)}_t = \int_{\Delta^m(t)} dX_{t_n} \cdots dX_{t_1} \) then we have \( \mathbb{E} [Z^{(n)}_t Z^{(m)}_s] = \mathbb{E} [Z^{(n)}_t] = 0 \) whenever \( n > 0 \). Next suppose that \( n \) and \( m \) are positive integers, then
\[ \mathbb{E} [Z^{(n)}_t Z^{(m)}_s] = \mathbb{E} \left[ \int_0^t dX_u Z^{(n-1)}_u \int_0^s dX_v Z^{(m-1)}_v \right] = \int_0^{t \wedge s} dF(u) \mathbb{E} [Z^{(n-1)}_u Z^{(m-1)}_v] \]
and we may re-iterate until we reduce at least one of the orders to zero. We then have
\[ \mathbb{E} [Z^{(n)}_t Z^{(m)}_s] = \delta_{n,m} \int_0^{t \wedge s} dF(u_n) \int_0^{u_n} dF(u_{n-1}) \cdots \int_0^{u_1} dF(u_1) = \frac{1}{n!} F(t \wedge s)^n \delta_{n,m}. \]
The proof with the intensities from $L^2(\mathbb{R}^+, dF)$ included is then a straightforward generalization.

**Theorem 11.** The Hilbert spaces $\mathfrak{h}_X = L^2(\Omega_X, \mathcal{F}_X, \mathbb{P})$ and $\Gamma_+ \left( L^2(\mathbb{R}^+, dF) \right)$ are naturally isomorphic.

**Proof.** Consider the map into the exponential vectors (1.4) given by

$$E^{\tilde{X}(f)} \rightarrow \varepsilon(f)$$

for each $f \in L^2(\mathbb{R}^+, dF)$. We know that the exponential vectors are dense in Fock space and in a similar way the exponential martingales $E^{\tilde{X}(f)}$ are dense in $\mathfrak{h}_X$. The map may then be extended to one between the two Hilbert spaces.

Unitarity follows from the observation that

$$E \left[ E^{\tilde{X}(f)} E^{\tilde{X}(g)} \right] = e^{\int_{[0, \infty)} fg \ dF}$$

which is an immediate consequence of the previous lemma.

The choice of the Wiener process is especially widely used. Here we have the identification

$$L^2(\Omega_W, \mathcal{F}_W, \mathbb{P}) \cong \Gamma_+ \left( L^2(\mathbb{R}^+, dt) \right)$$

which goes under the name of the Wiener-Itô-Segal isomorphism. This result is one of the corner stones of Hida’s theory \cite{Hida9} of white noise analysis. The same Fock space occurs when we consider the compensated Poisson process also \cite{Parthasarathy14}.
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