Abstract. We study the torsion free generalized crystallographic groups with the indecomposable holonomy group which is isomorphic to either a cyclic group of order $p^n$ or a direct product of two cyclic groups of order $p$.

Introduction. A classical crystallographic group is a discrete cocompact subgroup of $I(\mathbb{R}^m)$, the isometry group of $\mathbb{R}^m$. Torsion-free crystallographic groups are called Bieberbach groups. The present state of the theory of crystallographic groups and a historical overview as well as its connections to other branches of mathematics are described in [16,17].

In this paper we consider generalized torsion-free crystallographic groups with indecomposable holonomy groups isomorphic to either $C_{p^n}$ or $C_p \times C_p$.

It was shown in [6, 7, 13] that the description of the $n$-dimensional crystallographic groups for an arbitrary $n$ is of wild type, in the sense that it relates to the classical unsolvable problem of describing the canonical forms of pairs of linear operators acting on finite dimensional vector spaces.

Using Diederichsen’s classification of integral representations of the cyclic group of prime order (see [5]), L. Charlap [4] gave a full classification of Bieberbach groups with the cyclic holonomy group $G$ of prime order. G. Hiss and A. Szczepánski [12] proved that Bieberbach groups with nontrivial irreducible holonomy group $G$ do not exist. G. Kopcha and V. Rudko [13] studied torsion-free crystallographic groups with indecomposable cyclic holonomy group of order $p^n$ the classification of which for $n \geq 5$ also has wild type.

J. Rossetti and P. Tirao [18, 19, 20] described the torsion-free crystallographic groups whose holonomy group are direct sums of indecomposable subgroups of $GL(n, \mathbb{Z})$ ($n \leq 5$) and isomorphic to $C_2 \times C_2$.

Similarly, interesting results on this topic were obtained in the research of N. Gupta and S. Sidki [8, 9].
We need the following definitions and notation for announcing our results.

Let $K$ be a principal domain, $F$ be a field containing $K$ and let $G$ be a finite group. Let $M$ be a $KG$-module of a faithful matrix $K$-representation $\Gamma$ of $G$ and let $FM$ be a linear space over $F$ in which the $K$-module $M$ is a full lattice. Let $\hat{M} = FM^+/M^+$ be the quotient group of the additive group $FM^+$ of the linear space $FM$ by the additive group $M^+$ of the module $M$. Then $FM$ is an $FG$-module and $\hat{M}$ is a $KG$-module with operations:

$$g \cdot (\alpha m) = \alpha g(m); \quad g \cdot (x + M) = g(x) + M,$$

where $g \in G$, $\alpha \in F$, $m \in M$, $x \in FM$.

Let $T : G \to \hat{M}$ be a 1-cocycle of $G$ with values in $\hat{M}$; that is, $T(g)$ is regarded as the set $x_1 + M$, where $x_1$ is an element from the coset of $T(g)$ in $\hat{M}$. We define the group

$$\mathcal{Crys}(G; M; T) = \{ (g, x) \mid g \in G, \ x \in T(g) \}$$

with the operation

$$(g, x) \cdot (g', x') = (gg', g'x + x'),$$

where $g, g' \in G$, $x \in T(g)$, $x' \in T(g')$.

The purpose of this paper is to study the group $\mathcal{Crys}(G; M; T)$, and in particular to determine when $\mathcal{Crys}(G; M; T)$ is a torsion free group. We note that if $K = \mathbb{Z}$ and $F = \mathbb{R}$, then $\mathcal{Crys}(G; M; T)$ is isomorphic to an $n$-dimensional classical crystallographic group, where $n = \text{rank}_\mathbb{Z} M$.

We use the terminology of the theory of group representations. The group $\mathcal{Crys}(G; M; T)$ is called irreducible ( indecomposable ), if $M$ is an irreducible ( indecomposable ) $KG$-module and the cocycle $T$ is not cohomologous to zero.

A cocycle $T : G \to \hat{M}$ is called coboundary, if for every $g \in G$ there exists an $x \in FM$ such that $T(g) = (g - 1)x + M$. The cocycles $T_1 : G \to \hat{M}$ and $T_2 : G \to \hat{M}$ are called cohomologous if $T_1 - T_2$ is a coboundary.

Let $C^1(G, \hat{M})$, $B^1(G, \hat{M})$, $H^1(G, \hat{M}) = C^1(G, \hat{M})/B^1(G, \hat{M})$ be the group of the cocycles, the coboundaries and the cohomologies of $G$ with values in the group $\hat{M}$, respectively. The group $\mathcal{Crys}(G; M; T)$ is an extension of $M^+$ by the group $G$. This extension splits if and only if $T \in B^1(G, \hat{M})$. Therefore, the group $\mathcal{Crys}(G; M; T)$ splits for all $T$ if and only if $H^1(G, \hat{M})$ is trivial.

**Main Results.** Using results from [2, 3, 10, 11, 14, 15], we prove the following three theorems. We note that Lemma 12 can considered as a separate result.

**Theorem 1.** Let $K$ be either the ring of rational integers $\mathbb{Z}$, or $p$-adic integers $\mathbb{Z}_p$, or the localization $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at $p$ and let $G \cong C_p^s$ be a cyclic group of order $p^s$. If $s \geq 3$, then the set of $K$-dimensions of the indecomposable $KC_p^s$-modules $M$, for which there exist torsion-free groups $\mathcal{Crys}(C_p^s; M; T)$, is unbounded.
Theorem 2. Let $K$ be either the localization $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at $p$, or the ring of $p$-adic integers $\mathbb{Z}_p$ and let $G = \langle a \rangle \cong C_{p^2}$ be a cyclic group of order $p^2$. Up to isomorphism, all torsion-free indecomposable groups $\text{Crys}(C_{p^2}; M; T)$ can be described with the help of the following indecomposable $KC_{p^2}$-modules $M$ and cocycles $T$ of the group $C_{p^2}$ with values in the groups $\tilde{M} = FM^+/M^+$:

1) $M = X_i = \langle (a - 1)\Phi(a^p), \Phi(a) + (a - 1)^{i+1} \rangle$, $T = T_i$, where $\Phi(x) = x^{p-1} + \cdots + x + 1$, $T_i(a) = p^{-2}\Phi(a)\Phi(a^p) + X_i$, and $i = 0, 1, \ldots, p - 2$;

2) $p > 2$ and $M = U_j = \langle (a - 1)^{i+1} + \Phi(a), (a - 1)^j \rangle$, $\Phi(a^p)(a - 1, 1)$, a $KC_{p^2}$-submodule in $(KC_{p^2})^2 = \{ (x_1, x_2) \mid x_1, x_2 \in KC_{p^2} \}$, and $T = f_j$, where $f_j(a) = p^{-2}\Phi(a)\Phi(a^p)(1, 0) + U_j$ and $j = 1, \ldots, p - 2$.

The number of these groups $\text{Crys}(C_{p^2}; M; T)$ is equal to $2p - 3$.

Corollary 1. There exist at least $2p - 3$ Bieberbach (in the classical sense) groups having a cyclic indecomposable holonomy group of order $p^2$.

Theorem 3. Let $G = \langle a \rangle \times \langle b \rangle \cong C_2 \times C_2$ and let $K$ be either the ring of rational integers $\mathbb{Z}$, or the ring of 2-adic integers $\mathbb{Z}_2$, or the localization $\mathbb{Z}_{(2)}$ of $\mathbb{Z}$ at the prime 2. Furthermore, let $F$ be a field containing $K$, and $M$ be a $KG$-module of the indecomposable $K$-representation $\Gamma$ of $G$, and let $\text{Crys}(G; M; f)$ be the group defined by the cocycle $f : G \rightarrow \tilde{M} = FM^+/M^+$. The following table exhibits indecomposable $K$-representations $\Gamma$ of $G$ and cocycles $f$, which define, up to isomorphism, all torsion-free indecomposable groups $\text{Crys}(G; M; f)$:

| $N$ | $m$ | $\Gamma$ | $f(a) = (x_1, \ldots, x_m) + M$, $f(b) = (y_1, \ldots, y_m) + M$ | $t_m$ |
|-----|-----|----------|-------------------------------------------------|-----|
| 1   | $4n + 1$ ($n \geq 1$) | $\Delta_n$ | $x_{n+1} = \frac{1}{2}$, $x_i = 0$ ($i \neq n + 1$), $y_1 = \frac{1}{2}$, $2y_2 = \cdots = 2y_{n+1} = 0$, $y_2 + \cdots + y_{n+1} = \frac{1}{2}$, $y_{n+2} = \cdots = y_{4n+1} = 0$ | $2^{n-1}$ |
| 2   | $4n + 4$ ($n \geq 0$) | $W_n^*$ | $x_{2n+3} = \frac{1}{2}$, $x_i = 0$ ($i \neq 2n + 3$), $y_1 = 0$, $y_2 = \frac{1}{2}$, $y_3 = \cdots = y_{3n+3} = 0$, $2y_{3n+4} = \cdots = 2y_{4n+3} = 0$, $y_{4n+4} = \frac{1}{2}$ | $2^n$ |
| 3   | 5   | $\Delta_1^*$ | $f(a) = (0, 0, 0, 0, 0, 0)$, $f(b) = (\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{4}, 0)$ | 1 |
| 4   | 8   | $W_1$ | $f(a) = (0, 0, 0, 0, 0, 0, 0, 0)$, $f(b) = (\frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{4}, 0, \frac{1}{4})$ | 1 |

where $m$ is the degree of the representation $\Gamma$, $f$ is the cocycle, and $t_m$ is the number of the torsion free groups.

Preliminary results. Let $K = \mathbb{Z}$, $\mathbb{Z}_{(p)}$ or $\mathbb{Z}_p$ as above. We point out that in these cases the group $H^1(G, \tilde{M})$ is finite. Denote by $C_{p^n} = \langle a \mid a^{p^n} = 1 \rangle$ the cyclic group of order $p^n$. The following three Lemmas and Corollary 2 are well-known and they can be found for example in [1].
Lemma 1. Let \( K \) be either the ring of rational integers \( \mathbb{Z} \), or the ring of \( p \)-adic integers \( \mathbb{Z}_p \), or the localization \( \mathbb{Z}_{(p)} \) of \( \mathbb{Z} \) at \( p \), respectively. Let \( G_i \) \((i = 1, 2)\) be a finite group and \( \Gamma_i, M_i, T_i \) \((i = 1, 2)\) be the representation, the module and the cocycle associated with \( G_i \) as in the introduction. The groups \( \mathfrak{Erg}(G_1; M_1; T_1) \) and \( \mathfrak{Erg}(G_2; M_2; T_2) \) are isomorphic if and only if there exist a group isomorphism \( \varepsilon : G_1 \to G_2 \) and a \( K \)-module isomorphism \( \tau : M_1 \to M_2 \) which satisfy the following conditions:

1) \( \varepsilon(g)\tau = \tau g \) in \( M_1 \), for all \( g \in G_1 \);
2) the cocycles \( T_2 \) and \( T_1^\varepsilon \) are cohomologous (here, \( T_1^\varepsilon(g) = \tau' T_1(\varepsilon^{-1} g), g \in G_2 \), where \( \tau' : \hat{M}_1 \to \hat{M}_2 \) is the homomorphism induced by the homomorphism \( \tau \)).

\[ \square \]

Lemma 2. Assume that the character of the \( K \)-representation \( \Gamma \) of the group \( C_n \) does not contain the trivial character as a summand. Then \( H^1(C_n, \hat{M}) \) is trivial.

Proof. Since \( 1 \) is not an eigenvalue of the operator \( a \), which acts on \( FM \), the operator \( a - 1 \) is a unit. This means that \( T(a) = (a - 1)x + M \) for some \( x \in FM \), i.e. \( B^1(C_n, \hat{M}) = C^1(C_n, \hat{M}) \).

\[ \square \]

Lemma 3. Let \( G \cong C_{p^s} \) and \( M \) be a projective \( KG \)-module. Then \( H^1(C_{p^s}, \hat{M}) \) is trivial.

Proof. It is well known that multiple direct sums \( M \oplus \cdots \oplus M \) of the module \( M \) are free \( KC_{p^s} \)-modules. Therefore, it is sufficient to prove the lemma for \( M = KC_{p^s} \). Let \( T(a) = x + M \) for some

\[
x = \lambda(1 + a + \cdots + a^{p^s - 1}) + u_1(a - 1) \in FM,
\]

where \( \lambda \in F \) and \( u_1 \in FC_{p^s} \). From the condition \((1 + a + \cdots + a^{p^s - 1})T(a) \subset M \) it follows that \( \lambda p^s \in K \). Then \( x - \lambda p^s = u_2(a - 1) \), where \( u_2 \in FC_{p^s} \). Therefore, \( T \) is coboundary.

\[ \square \]

Corollary 2. Assume that a \( K \)-representation \( \Gamma \) of the group \( C_p \) does not contain the trivial \( K \)-representation as a summand. Then \( H^1(C_p, \hat{M}) \) is trivial.

Proof. A \( K \)-representation \( \Gamma \) of the group \( C_p \) is a direct sum \( \Gamma = \Gamma_1 \oplus \Gamma_2 \), where \( \Gamma_1 \) is a sum of the irreducible \( K \)-representation of degree \( p - 1 \), and \( \Gamma_2 \) is a \( K \)-representation corresponding to a projective \( KC_p \)-module. The proof follows by applying Lemma 1 to \( \Gamma_1 \) and Lemma 3 to \( \Gamma_2 \).

\[ \square \]

For the proof of Theorem 1 we will consider a \( K \)-representation of the group \( \langle a \rangle \cong C_{p^s} \). Let \( \xi_t \) be a primitive \( p^s \)th root of unity and \( \xi_{t-1} = \xi_t^p \) \((t \geq 1)\). Put

\[
B_0 = \{1\}, \quad B_1 = \{1, \xi_1, \ldots, \xi_1^{p-2}\}, \quad B_j = \cup_{i=0}^{p-1} \xi_j^i B_{j-1} \quad (j \geq 2).
\]
Thus $B_t$ is a $K$-basis of the ring $K_t = K[\xi_t]$, which is a $KC_{p^s}$-module: $a(\alpha) = \xi_t \alpha$, where $\alpha \in K_t$, $t \leq s$. The set $B_t$ is also an $F$-basis of the space $FK_t$ ($t \geq 0$).

Let $\delta_t$ be a $K$-representation of $C_{p^s}$ corresponding to the $K$-basis of the module $K_t$ of this representation. We note that $\delta_t$ is an irreducible $K$-representation of $C_{p^s}$ and $\delta_t^2(a) = E_p \otimes \delta_{t-1}(a)$, where $E_p$ is the identity matrix of degree $p$. Let

$$\Delta_1 = \delta_0^{(n)} + \delta_1^{(n)}; \quad \Delta_2 = \delta_2^{(n)} + \delta_s^{(n)}$$

be the sums of $2n$ irreducible $K$-representations of $C_{p^s}$, where $\delta_i^{(n)} = \delta_i + \ldots + \delta_i$.

Let us consider the following $K$-representation $\Delta$ of the group $C_{p^s}$:

$$\Delta(a) = \begin{pmatrix} \Delta_1(a) & U(a) \\ 0 & \Delta_2(a) \end{pmatrix},$$

where

$$U(a) = \begin{pmatrix} E_n \otimes (1) & J_n \otimes (1) \\ E_n \otimes (1) & J_n \otimes (1) \end{pmatrix};$$

$J_n$ is the Jordan block of order $n$ with ones on the main diagonal and the symbol $\langle \omega \rangle_t$ denotes the matrix with zero columns except the last one, which consists of the coordinates of the element $\omega \in K_t$ written in the basis $B_t$ ($t = 0, 1, 2, \ldots$).

**Lemma 4.** (see [2, 3]) The $K$-representation $\Delta$ of $C_{p^s}$ is indecomposable.

**Lemma 5.** Let $x \in FK_t$ ($t > 0$), such that $(a - 1)x \in K_t$. Then $px \in K_t$ and all coordinates of the vector $px$ are multiples of the last coordinate.

**Proof.** The $K$-basis $B_t$ in $K_t$ will be an $F$-basis in $FK_t$. Let us consider the coordinates of the column vectors in $FK_t$ and the matrix $\delta_t(a)$ of the operator $a$ in this basis. The lemma is easily checked successively for $t = 1, t = 2$ and so on.

**Lemma 6.** The function $T_\Delta$ is a $1$-cocycle of the group $C_{p^s}$ with values in the group $F_{\Delta}$. The cocycle $T_\Delta$ is not cohomologous to the zero cocycle at the element $b = ap^{s-1}$ of order $p$.

**Proof.** The first statement about the function of $T_\Delta$ follows from the definition of this function. For proving the second part of the lemma, let us consider the $p$th
power $\Delta^p(a)$ of the representation $\Delta$. We note that $\Delta^p(a) = \begin{pmatrix} \Delta_0^p(a) & U'(a) \\ 0 & \Delta_2^p(a) \end{pmatrix}$ and $\Delta_1^p(a) = E$. Clearly, the first row in $U'(a)$ has the form:

$$(\langle 1 \rangle_0, \ldots, \langle 1 \rangle_0, \langle 1 \rangle_0, \ldots, \langle 1 \rangle_0)$$

and the row of matrices corresponding to the first of the representations $\delta_1^p$ will take the form of the following matrix:

$$(\langle 1 \rangle_1, \langle \xi_1 \rangle_1, \ldots, \langle \xi_1^{p-1} \rangle_1, \langle 1 \rangle_1, \ldots, \langle \xi_1^{p-1} \rangle_1).$$

If we sum the rows of this matrix and subtract the result from the 1st row in $U'(a)$, then we will get a row in which all the elements will be multiples of $p$. So this transformation of rows in $U'(a)$ corresponds to the replacement of some base elements $u \in B$ ($u \neq v$) to $u' = u \pm v$. Let us carry out this replacement; and let $\Delta'$ be a $K$-representation of the group $C_{p^s}$ in the new $K$-basis of the module $M_\Delta$.

It is easy to see that the replacement of the basis did not influence the values of the function $T_\Delta$.

Let $H = \langle b | b = a^{p^{s-1}} \rangle$ and $\Delta'_H$ be a restriction of the representation $\Delta'$ to the subgroup $H$. Then

$$\Delta'_H(b) = \begin{pmatrix} \delta_0^{(m_1)}(a) & U''(b) \\ 0 & \delta_1^{(m_2)}(a) \end{pmatrix},$$

where, as shown above, all elements of the first row in $U''(b)$ are multiples of $p$. Let $M_\Delta = M_1 \oplus M_2$ be a decomposition of the $K$-module $M_\Delta$ into the direct sum of $M_1$ and $M_2$, corresponding to the representations $\delta_0^{(m_1)}$ and $\delta_1^{(m_2)}$.

Suppose that the cocycle $T_\Delta$ is cohomologous to the trivial cocycle at $H$. Then there exists a vector $x \in FM_\Delta$, such that

$$T_\Delta(b) = (b - 1)x + M_\Delta.$$

Let $x = x_1 + x_2$ ($x_i \in FM_i, i = 1, 2$). Since the projection of $T_\Delta(b)$ on $FM_2$ is equal to zero (modulo $M_\Delta$), the projection of $(b - 1)x = (b - 1)x_2$ on $FM_2$ also equals to zero. From Lemma 5 it follows that $px_2 \in M_2$. Let $\lambda$ be the coefficient of the basis vector $v$ in $(b - 1)x$.

It is easy to see that $\lambda$ is a sum of products of the elements of the first row in $U''(b)$ (these elements are multiple to $p$) on the column, which consist of coordinates of the vector $x_2$. From the condition $px_2 \in M_2$ it follows that $\lambda \in K$. Since $T_\Delta(b) = p^{-1}v + M_\Delta$, we have $\lambda = p^{-1}$. But $p^{-1}$ does not belong to $K$, and, therefore, $\lambda \notin K$. This contradiction proves that $T_\Delta$ is not cohomologous to zero at $H$. The lemma is proved.

$\square$
Proof of Theorem 1. Let us consider the group $\text{Crys}(C_{p^s}; M_{\Delta}; T_{\Delta})$. If there exists an element of prime order in this group, then this order can only be $p$ and, moreover, the cocycle $T_{\Delta}$ must be cohomologous to the zero cocycle in the unique element $b = a^{p^s - 1}$ in the group $C_{p^s}$ of prime order $p$. According to Lemma 6, this is impossible. Therefore, the group $\text{Crys}(C_{p^s}; M_{\Delta}; T_{\Delta})$ has no torsion elements. Moreover, this group is indecomposable (see Lemma 4).

For the cyclic group $\langle a \rangle \cong C_{p^2}$ we want to find all those groups $\text{Crys}(C_{p^2}; M; T)$ which are torsion free. Put

$$\Phi(x) = x^{p-1} + x^{p-2} + \cdots + x + 1.$$ 

There exists a unit $\theta$ in $KC_{p^2}$ such that

$$(a - 1)^p \Phi(a^p) = p(a - 1)\theta \Phi(a^p).$$

Let $X_i$ be a $KC_{p^2}$-submodule in $KC_{p^2}$, generated by the following elements:

$$u = \Phi(a)\Phi(a^p), \quad \omega = (a - 1)\Phi(a^p), \quad v = \Phi(a) + (a - 1)^{i+1},$$

where $0 \leq i \leq p - 2$. It is easy to see that

$$(a - 1)u = 0, \quad \Phi(a)\omega = 0, \quad \Phi(a^p)v = u + (a - 1)^i\omega.$$ 

From these equations it follows that the $K$-representation $\Gamma_i$ of the group $C_{p^2}$ in the $K$-basis

$$u; \quad \omega, \quad a\omega, \quad a^2\omega, \quad \ldots, \quad a^{p-2}\omega;$$

$$a^l v, \quad a^{l+p} v, \quad a^{l+2p} v, \quad \ldots, \quad a^{l+p(p-2)} v,$$

$(l = 0, 1, \ldots, p - 1)$ corresponding to the module $X_i$, has the following form:

$$\Gamma_i(a) = \begin{pmatrix} 1 & 0 & \langle 1 \rangle_0 \\ \delta_1(a) & \alpha_i \langle \alpha_i \rangle_1 \\ \delta_2(a) \end{pmatrix},$$

where $\alpha_i = (\xi_1 - 1)^i$ and $i = 0, 1, \ldots, p - 2$.

Lemma 7. Let $H = \langle b \mid b = a^p \rangle$. The $KH$-module $X_i \mid_H$ is a direct sum of two $KH$-submodules, one of which coincides with $Ku$.

Proof. Let us examine the $K$-submodule $X'_i$ in $X_i$ generated by the following system of $p^2 - 1$ elements from $X_i$:

$$V = \{v, bv, \ldots, b^{p-2}v\}, \quad (a - 1)V, \ldots, (a - 1)^{p-2}V, \quad v' = (a - 1)^{p-1}v + \theta u,$$
where \( \omega_j = (a - 1)^j \Phi(b) = (a - 1)^{j-1} \omega \) and \( j = 1, \ldots, p - 1 \).

Clear, \( X_i = Ku \oplus X'_i \) is a direct sum of \( K \)-modules \( Ku \) and \( X'_i \). To prove the lemma it is sufficient to show that \( X'_i \) is a \( KH \)-module. We have:

\[
\begin{align*}
\Phi(b)v &= u + \omega_{i+1} \in X'_i, \\
\Phi(b)(a - 1)v &= \omega_{i+2} \in X'_i, \\
\vdots & \quad \vdots \\
\Phi(b)(a - 1)^{p-i-2}v &= \omega_{p-1} \in X'_i, \\
\Phi(b)(a - 1)^{p-r-2+j}v &= p\theta\omega_j \in X'_i,
\end{align*}
\]

where \( 0 < i, \ 0 \leq r \leq p - 2, \ j = 1, \ldots, r \) and

\[
\begin{align*}
\Phi(b)v' &= (a - 1)^{p-1}\Phi(b)v + p\theta u = (a - 1)^{p+i}\Phi(b) + p\theta u \\
&= p\theta(a - 1)^{1+i}\Phi(b) + p\theta u = p\theta(\omega_{i+1} + u) \in X'_i.
\end{align*}
\]

These equations show that \( X'_i \) is the \( KH \)-submodule in \( X_i \).

Let us introduce the cocycle

\[
T_i : C_{p^2} \to \hat{X}_i = F X_i^+/X_i^+;
\]

where \( T_i(a) = p^{-2}u + X_i \) and \( i = 0, 1, \ldots, p - 2 \).

**Lemma 8.** The group \( \mathfrak{C} \mathfrak{m} \mathfrak{s}(C_{p^2}; X_i; T_i) \) is torsion free \( (i = 0, 1, \ldots, p - 2) \).

**Proof.** Since \( T_i(a^p) = p^{-1}u + X_i \neq X_i \), from Lemma 7 it follows that

\[
((a^p - 1)FX_i + X_i) \cap (Fu + X_i) = X_i.
\]

These conditions show that the cocycle \( T_i \) is not cohomologous to the zero cocycle at the element \( a^p \). This means that \( \mathfrak{C} \mathfrak{m} \mathfrak{s}(C_{p^2}; X_i; T_i) \) is torsion free.

Let \( Y_i = \{ \Phi(a), (a - 1)^i \} \) be a \( KC_{p^2} \)-submodule in \( KC_{p^2} \), where \( i = 0, 1, \ldots, p - 1 \). The \( K \)-representation \( \Gamma'_i \) corresponding to \( Y_i \) has the following form

\[
\Gamma'_i(a) = \begin{pmatrix} 1 & 0 \\ \delta_1(a) & \delta_2(a) \end{pmatrix},
\]

where \( \alpha_i = (\xi_1 - 1)^i \) and \( i = 0, 1, \ldots, p - 1 \).
Lemma 9. For an arbitrary cocycle $T : C_{p^2} \to \widetilde{Y}_i = FY_i^+/Y_i^+$ the group $\mathcal{Crys}(C_{p^2}; Y_i; T)$ contains an element of order $p$.

Proof. It is easy to see that an arbitrary cocycle of $C_{p^2}$ with a value in $\widetilde{Y}_i$ will be cohomologous to a cocycle $T$, such that $T(a) = \lambda p^{-2}u + Y_i$, where $\lambda \in K$, $u = \Phi(a)\Phi(b)$. Thus, $T(a^p) = pT(a) = \lambda p^{-1}u + Y_i$, so to prove the lemma it is sufficient to show that $p^{-1}u \in (a^p - 1)FY_i + Y_i$. It is easy to see that

$$(a^{p-1} + a^{p-2} + \cdots + a + 1) - (a - 1)^{p-1} = p\omega_1(a),$$

(2)

where $\omega_1(a) \in KC_{p^2}$.

Let $v_1 = (a - 1)^j$. Then from (2) it follows that

$$(\Phi(a^p) - p)(a - 1)^{p-i-1}v_1 = u - p\omega_1(a)\Phi(a^p) - p(a - 1)^{p-i-1}v = u + py,$$

where $y \in Y_i$. Since $\Phi(a^p) - p = (a^p - 1)z$, where $z \in KC_{p^2}$, we have that $p^{-1}u + Y_i = (a^p - 1)p^{-1}z + Y_i$ which completes the proof of the lemma.

Let $p \neq 2$. In the free $KC_{p^2}$-module $(KC_{p^2})^{(2)} = \{(x_1, x_2) \mid x_1, x_2 \in KC_{p^2}\}$ let us consider $KC_{p^2}$-submodule

$$U_j = \langle \begin{array}{c} (a - 1)^{j+1} + \Phi(a), (a - 1)^j; \\ \Phi(a^p)(a - 1, 1) \end{array} \rangle,$$

where $1 \leq j \leq p - 2$. The $K$-representation of $C_{p^2}$, corresponding to the module $U_j$, has the following form

$$\Gamma''_j : a \to \begin{pmatrix} 0 & 0 & (1)_0 \\ 1 & (1)_0 & 0 \\ \delta_1(a) & \delta_2(a) & (a_j)_{1} \\ \delta_2(a) \end{pmatrix},$$

(3)

where $\alpha_j = (\xi_1 - 1)^j$ and $j = 1, 2, \ldots, p - 2$. Let us define a cocycle

$$f_j : C_{p^2} \to \widetilde{U}_j = FU_j^+/U_j^+$$

by $f_j(a) = p^{-2}\Phi(a)\Phi(a^p)(1, 0) + U_j$.

Lemma 10. The group $\mathcal{Crys}(C_{p^2}; U_j; f_j)$ is torsion free ( $j = 1, \ldots, p - 2$ ).

Proof. Let $u_1 = \Phi(a)\Phi(a^p)(1, 0)$ and $u_2 = \Phi(a)\Phi(a^p)(0, 1)$. It is easy to see that the sequence of $KC_{p^2}$-modules

$$0 \to Ku_2 \to U_j \to X_j \to 0$$

(4)

is exact. The cocycle $f_j$ induces a cocycle $T_j : C_{p^2} \to \widetilde{X}_j$ (see (1) which is not equal to the zero cocycle in the group $H = \langle a^p \rangle$ according to Lemma 8). Therefore $f_j$ is
also non-cohomologous to the zero cocycle in $H$. This means that $\mathfrak{K} \mathfrak{n} \mathfrak{s}(C_{p^2}; U_j; f_j)$ has no elements of order $p$.

\[ \square \]

Let us consider one more $KC_{p^2}$-module: $U_0 = KC_{p^2}\Phi(a)$ generated by $\Phi(a)$ in $KC_{p^2}$. The $K$-representation of the group $C_{p^2}$ corresponding to this module has the following form:

\[ a \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & \delta_2(a) \end{pmatrix}. \]

**Lemma 11.** For any cocycle $T : C_{p^2} \rightarrow U_0$ the group $\mathfrak{K} \mathfrak{n} \mathfrak{s}(C_{p^2}; U_0; T)$ contains an element of order $p$.

**Proof.** It is easy to see that any 1-cocycle of the group $C_{p^2}$ with values in the group $\hat{U}_0 = FU_0^+ / U_0^+$ is cohomologous to the cocycle $T$, such that

\[ T(a) = \lambda p^{-2}\Phi(a)\Phi(a^p) + U_0, \]

where $\lambda \in K$. Replacing $a$ with $a^p$ in (2) we have

\[ \frac{1}{p}\Phi(a)\Phi(a^p) = \frac{1}{p}(a^p - 1)^{p-1}\Phi(a) + \omega_1(a^p)\Phi(a). \]

Then $T(a^p) = (a - 1)z + U_0$, where $z \in FU_0$, which is proves the lemma.

\[ \square \]

**Proof of Theorem 2.** From the description of the $K$-representations of the group $C_{p^2}$ [2], it follows that all indecomposable $KC_{p^2}$-modules of those faithful $K$-representations of the group $C_{p^2}$, whose characters contain the trivial character of $C_{p^2}$, are the following:

\[ X_i \quad (i = 0, 1, \ldots, p - 2); \quad Y_j \quad (j = 0, 1, \ldots, p - 1); \quad U_0; \quad U_k \quad (k = 1, \ldots, p - 2). \]

According to Lemmas 9 and 11 we are interested only in the modules $X_i$ and $U_j$. Let us consider the module $X_i$ ($0 \leq i \leq p - 2$). It is easy to see that Lemma 2 can be applied to the factor module $X_i/Kv$, where $v = \Phi(a)\Phi(a^p)$. Therefore, any cocycle of the group $C_{p^2}$ with the values in $\hat{X}_i$ will be cohomologous to such a cocycle $T$ that

\[ T(a) = \lambda p^{-2}v + X_i, \quad (5) \]

where $\lambda \in K$. We will show that if in this equation $\lambda \equiv 0$ (mod $p$) then the cocycle $T$ is cohomologous to the trivial cocycle. From (2) it follows that

\[ \frac{1}{p}\Phi(a)\Phi(a^p) + \frac{1}{p}\Phi(a^p)(a - 1)^{i+1} = \frac{1}{p}(a^p - 1)^{p-1}\theta_i + \omega_1(a^p)\theta_i, \quad (6) \]
where $\theta_i = \Phi(a) + (a - 1)^{i+1} \in X_i$. We will use the equation
\[
\Phi(a^p)(a - 1)^p = p(a - 1)\Phi(a^p)\omega_2,
\]
where $\omega_2$ is a unit in $KC_{p^2}$. From (6) it follows that
\[
\frac{1}{p}(a - 1)^{i+1}\Phi(a^p) = \frac{1}{p}\Phi(a^p)(a - 1)^{p+i}\omega_2^{-1} \in (a - 1)FX_i,
\]
for all $i = 0, 1, \ldots, p - 2$. Then from (6) one finds
\[
\frac{1}{p}\Phi(a)\Phi(a^p) \in (a - 1)FX_i + X_i
\]
for all $i = 0, 1, \ldots, p - 2$. This means that if in (5) $\lambda \equiv 0 \pmod{p}$, then the cocycle $T$ is cohomologous to the zero cocycle.

It follows that $H^1(C_{p^2}, \widehat{X}_i)$ is a cyclic group of order $p$ and all elements of this group can be represented by the cocycles $T$ of (5) with $\lambda = 0, 1, \ldots, p - 1$.

We will show that each nonzero cocycle $T$ defines up to isomorphism one group $\mathcal{C}_\mathfrak{X}(C_{p^2}; X_i; T)$.

Let $\varepsilon$ be an automorphism of the group $C_{p^2}$ and $X_i^\varepsilon$ be the $KC_{p^2}$-module $X_i$ twisted by this automorphism, i.e.
\[
X_i^\varepsilon = X_i, \quad a \cdot x = \varepsilon(a)x, \quad x \in X_i.
\]
It is not difficult to show the existence of an automorphism $\tau$ of the $K$-module $X_i$ such that $\varepsilon(a)\tau = \tau a$ in $X_i$ and $\tau(v) = v$, where $v = \Phi(a)\Phi(a^p)$.

Let $\varepsilon^{-1}(a) = a^s$, with $(s, p) = 1$. Since $aT_i(a) = T_i(a)$ and $\tau'(\overline{v}) = \overline{v}$, where $\overline{v} = v + X_i$, we have that
\[
T_i^\varepsilon(a) = \tau' T_i(\varepsilon^{-1}(a)) = sT_i(a) = sp^{-2}v + X_i.
\]
From Lemma 1 it follows that $\mathcal{C}_\mathfrak{X}(C_{p^2}; X_i; T_i)$ is isomorphic to $\mathcal{C}_\mathfrak{X}(C_{p^2}; X_i; T)$, where $T(a) = sp^{-2}v + X_i$. We have shown that all groups $\mathcal{C}_\mathfrak{X}(C_{p^2}; X_i; T)$, where $T \neq 0$, are isomorphic to $\mathcal{C}_\mathfrak{X}(C_{p^2}; X_i; T_i)$.

Now let us consider the group $\mathcal{C}_\mathfrak{X}(C_{p^2}; U_j; T)$. First of all we remark that the group $H^1(C_{p^2}, \widehat{Y}_j)$ is a cyclic group of order $p$, where $j = 1, \ldots, p - 1$. The proof of this fact is similar to the proof for the group $H^1(C_{p^2}, \widehat{X}_i)$. Since $Y_0 = KC_{p^2}$, we have that $H^1(C_{p^2}, Y_0) = 0$ (see Lemma 3).

Let $u_1, u_2, \ldots, u_{p^2+1}$ be a $K$-basis in $U_j$, such that
\[
u_1 = \Phi(a)\Phi(a^p)(1, 0) \quad \text{and} \quad u_2 = \Phi(a)\Phi(a^p)(0, 1).
\]
We will use the exact sequence (4) and the exact sequence
\[
0 \rightarrow Ku_1 \rightarrow U_j \rightarrow Y_j \rightarrow 0.
\]
This will give us an opportunity to show that any cocycle \( T : C_{p^2} \rightarrow U_j \) is cohomologous to a cocycle \( T_{\alpha, \beta} \), such that

\[
T_{\alpha, \beta}(a) = p^{-2}(\alpha u_1 + \beta u_2) + U_j,
\]

where \( 0 \leq \alpha, \beta \leq p - 1 \).

If \( \alpha = 0 \) then according to Lemma 9 and (7) it follows that the cocycle \( T_{0, \beta} \) is cohomologous to the zero cocycle at the element \( a^p \) of the group \( C_{p^2} \) and, therefore, the group \( \mathcal{Crys}(C_{p^2}; U_j; T_{0, \beta}) \) will contain an element of order \( p \).

Now let \( \alpha \neq 0 \). Then \( \alpha \) is a unit in \( K \) and \( \tau(x) = ax \) (\( x \in U_j \)) is an automorphism of \( KC_{p^2} \)-module \( U_j \). From this it follows that the cocycle \( T_{\alpha, \beta} \) (\( \alpha \neq 0 \)) can be replaced by \( T_{1, \alpha-1, \beta} \). So it is enough to consider the cocycles \( T_{1, \beta} \), where \( \beta = 0, 1, \ldots, p - 1 \). We will show that \( \mathcal{Crys}(C_{p^2}; U_j; T_{1, \beta}) \) is isomorphic to \( \mathcal{Crys}(C_{p^2}; U_j; f_j) \) (note that \( f_j = T_{1,0} \)).

Let us replace the basis element \( u_1 \) by \( u_1' = u_1 + \beta u_2 \) in \( U_j \). Then

\[
T_{1, \beta}(a) = p^{-2}u_1' + U_j.
\]

Let \( Y_j' = U_j/Ku_1' \). Then the \( K \)-representation \( \Gamma_{j''} \) corresponding to the \( KC_{p^2} \)-module \( Y_j' \) is

\[
\Gamma_{j''} : a \mapsto \begin{pmatrix}
1 & (1)_0 & (-\beta)_0 \\
\delta_1(a) & (\alpha)_1 & (\beta)_1 \\
\delta_2(a) & (\alpha)_2 & (\beta)_2
\end{pmatrix}.
\]

This representation is equivalent to \( \Gamma_j \). According to this equivalence we will replace the basis elements \( u_2, \ldots, u_{p^2+1} \) with \( u_2', \ldots, u_{p^2+1}' \). Then in the \( K \)-basis \( u_1', u_2', \ldots, u_{p^2+1}' \) the operator \( a \) have the same matrix (3) as in the basis \( u_1, u_2, \ldots, u_{p^2+1} \). Let us define an automorphism \( \tau : U_j \rightarrow U_j \) of the \( K \)-module \( U_j \) by \( \tau(u_i') = u_i \), where \( i = 1, \ldots, p^2 + 1 \). As it was shown, \( \tau a = a \tau \). Moreover,

\[
\tau' T_{1, \beta}(a) = \tau'(p^{-2}u_1' + U_j) = p^{-2}u_1 + U_j = f_j(a).
\]

The isomorphism of the groups \( \mathcal{Crys}(C_{p^2}; U_j; T_{1, \beta}) \) and \( \mathcal{Crys}(C_{p^2}; U_j; f_j) \) follows from Lemma 1. So we have shown that among the groups \( \mathcal{Crys}(C_{p^2}; M; T) \) only those can be indecomposable and torsion free, for whose modules \( M \) and the cocycles \( T \) were characterized in this theorem.

Then Lemmas 8 and 10 complete the proof. \( \square \)

Let \( G \cong C_p \times C_p \) with generators \( a, b \) and let \( K \) be either the ring of rational integers \( \mathbb{Z} \), or the ring of \( p \)-adic integers \( \mathbb{Z}_p \), or the localization \( \mathbb{Z}(p) \) of \( \mathbb{Z} \) at \( p \), respectively. In case \( p = 2 \) we will give full description of the indecomposable torsion free groups \( \mathcal{Crys}(C_2 \times C_2; M; T) \).

In this case we will use the classification of the indecomposable \( K \)-representations of the group \( C_2 \times C_2 \), given by L. Nazarova in [14, 15].
Lemma 12. Let $M$ be a $K[C_p \times C_p]$-submodule in the free $K[C_p \times C_p]$-module $(K[C_p \times C_p])^{(2)}$ with the following system of generators:

$$M = \langle (\Phi(a), 0); (p, 0); (0, \Phi(b)); (0, p); (b - 1, 1 - a) \rangle.$$

Then the following conditions are satisfied:

1) $M$ is an indecomposable $K[C_p \times C_p]$-module and $\dim_K(M) = 2p^2$;
2) there exists a cocycle $T : C_p \times C_p \rightarrow \hat{M} = FM^+ / M^+$, such that:

$$T(a) = (1, 0) + M, \quad T(b) = (0, 1) + M;$$

3) the group $\text{Crys}(C_p \times C_p; M; T)$ is torsion free.

Proof. Let $\overline{Z_p} = K/pK$. Obviously, $\overline{Z_p}$ is a $K[C_p \times C_p]$-module with trivial action of $C_p \times C_p$. Let us consider the projective resolution of $K[C_p \times C_p]$-module $\overline{Z_p}$:

$$\cdots \longrightarrow (K[C_p \times C_p])^{(3)} \xrightarrow{\tau_1} (K[C_p \times C_p]) \xrightarrow{\tau_0} \overline{Z_p} \rightarrow 0. \quad (8)$$

It is easy to see that $\ker(\tau_0) = \langle a - 1, b - 1, p \rangle$, and

$$\ker(\tau_1) = \langle (\Phi(a), 0, 0); (0, \Phi(b), 0); (b - 1, 1 - a, 0); (p, 0, 1 - a); (0, p, 1 - b) \rangle.$$

The $K[C_p \times C_p]$-modules $\ker(\tau_0)$ and $\ker(\tau_1)$ are indecomposables of the $K$-representations of $C_p \times C_p$. Each $x \in \ker(\tau_1)$ has the following form:

$$x = (u_1 \Phi(a) + u_3 (b - 1) + pu_4; u_2 \Phi(b) + u_3 (1 - a) + pu_5; u_4 (1 - a) + u_5 (1 - b)), \quad (9)$$

where $u_i \in K[C_p \times C_p]$ and $i = 1, \ldots, 5$.

We will assign to each $x \in \ker(\tau_1)$ (see (9)) the element

$$(u_1 \Phi(a) + u_3 (b - 1) + pu_4; u_2 \Phi(b) + u_3 (1 - a) + pu_5)$$

of $M$. It is easy to check that this map generates an isomorphism of the $K[C_p \times C_p]$-modules $\ker(\tau_1)$ and $M$. Thus, we have shown that $M$ is an indecomposable $K[C_p \times C_p]$-module. Since $\dim_K(T_0) = p^2$, we have

$$\dim_K(M) = \dim_K(\ker(\tau_1)) = \dim_K(K[C_p \times C_p])^{(3)} - \dim_K(\ker(\tau_0)) = 2p^2.$$

2) Let us define a function $T : C_p \times C_p \rightarrow \hat{M}$ by:

$$T(a^i) = (1 + a + \cdots + a^{i-1}, 0) + M;$$
$$T(b^j) = (0, 1 + b + \cdots + b^{j-1}) + M;$$
\( T(a^i b^j) = a^i T(b^j) + T(a^i) + M; \quad T(1) = M, \)

where \( 0 < i, j \leq p - 2 \). It is easy to see that \( \Phi(a)T(a) \subset M, \Phi(b)T(b) \subset M \) and \( (a - 1)T(b) - (b - 1)T(b) \subset M \). It follows that \( T \) is a 1-cocycle of \( C_p \times C_p \) with values in \( \hat{M} = FM^+/M^+ \).

3) It is sufficient to show that the cocycle \( T \) is not cohomologous to the zero cocycle at every nontrivial element \( g \) of \( C_p \times C_p \). Let \( g = a^i b^j \), where \( 0 < i, j \leq p - 1 \). Suppose that there exists \( z \in FM \), such that

\[
T(g) = (g - 1)z + M. \tag{10}
\]

Then from the definition of \( T \) and from (10) it follows that

\[
( 1 + a + \cdots + a^{i-1}, \ a^i(1 + b + \cdots + b^{j-1}) ) = (g - 1)z + x,
\]

where \( x \in M \). Multiplying the last equality by \( \Phi(a)\Phi(b) \) taking into account that

\[
\Phi(a)\Phi(b)M = p\Phi(a)\Phi(b)(K, K),
\]

\[
\Phi(a)\Phi(b)(g - 1) = 0
\]

can conclude that \( (i, j) \in (pK, pK) \), which is impossible for \( 0 < i, j \leq p - 1 \). This contradicts to the assumption that the cocycle \( T \) is cohomologous to the zero cocycle at \( g \) (see (10)).

Similarly, we show that the cocycle \( T \) is not cohomologous to the zero cocycle at the rest of the nontrivial elements of the group \( G \cong C_p \times C_p \). Thus, the group \( \text{Crys}(G \times C_p; M; f) \) is torsion-free.

\[\square\]

Let \( p = 2 \), \( G = \langle a \rangle \times \langle b \rangle \cong C_2 \times C_2 \) and let \( K \) be either the ring of rational integers \( \mathbb{Z} \), or the ring of 2-adic integers \( \mathbb{Z}_2 \), or the localization \( \mathbb{Z}(2) \) of \( \mathbb{Z} \) at 2, respectively. We will study those groups \( \text{Crys}(G; M; T) \) which are torsion free.

The group \( G \) has the following irreducible \( K \)-representations:

\[
\begin{align*}
\chi_0 : & \quad a \rightarrow 1, \quad b \rightarrow 1; \quad \chi_1 : \quad a \rightarrow -1, \quad b \rightarrow 1; \\
\chi_2 : & \quad a \rightarrow -1, \quad b \rightarrow -1; \quad \chi_3 : \quad a \rightarrow 1, \quad b \rightarrow -1.
\end{align*}
\]

Let \( H = \langle h \rangle \) be a subgroup of \( G \) of order 2. The indecomposable \( K \)-representations of \( H \), up to equivalence, are one of the following:

\[
\begin{align*}
\gamma_0 : & \quad h \rightarrow 1; \quad \gamma_1 : h \rightarrow -1; \quad \gamma_2 : h \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.
\end{align*} \tag{11}
\]

Let \( \Gamma \) be a \( K \)-representation of \( G \), \( \Gamma|_H \) be the restriction of \( \Gamma \) to the subgroup \( H \), \( M \) be a \( KG \)-module of the \( K \)-representation \( \Gamma \) and \( T : G \rightarrow \hat{M} \) be an arbitrary cocycle of \( G \) with values in the group \( \hat{M} = FM^+/M^+ \) (\( F \) is a field containing \( K \)). The following Lemma gives the necessary conditions for \( \text{Crys}(G; M; T) \) to be torsion free.
Lemma 13. If $\mathcal{Crys}(G; M; T)$ is torsion free then for any one of the three nontrivial elements non-trivial subgroups $H$ of order 2, the restriction $\Gamma|_{H}$ of the representation $\Gamma$ contains the trivial representation $\gamma_0$ in the decomposition of $\Gamma|_{H}$ into the direct sum of indecomposable $K$-representations of $H = \langle h \rangle$.

Indirect proof. Assume that there is a nontrivial element $h$ in $G$, such that the $K$ representation $\Gamma|_{h}$ of $H$ is the direct sum of $K$-representation $\gamma_1$ and $\gamma_2$ (with the multiples) but does not include $\gamma_0$ in this sum (see (11)). Then from Lemma 2 and 3 it follows that any cocycle $T : G \to \hat{M}$ in the subgroup $H = \langle h \rangle$ will be cohomologous to the zero cocycle, which implies the existence of elements of order 2 in $\mathcal{Crys}(G; M; T).

\[\square\]

We make some remarks about the $K$-representations of $G \cong C_2 \times C_2$. Let $K$ be a $KG$-module of the trivial representation $\chi_0$ of $G$. Let us consider the projective resolution of the $KG$-module $K$:

$$\cdots \to (KG)^{(n)} \overset{\nu_n}{\to} (KG)^{(n-1)} \overset{\cdots}{\to} \cdots$$

$$\cdots \overset{\nu_2}{\to} (KG)^{(2)} \overset{\nu_2}{\to} (KG) \overset{\nu_1}{\to} K \to 0,$$

where $\nu_n$ is a homomorphism of the $KG$-modules ($n = 1, 2, \ldots$).

Then $\ker(\nu_n)$ is an indecomposable $KG$-module, and

$$\dim_K(\ker(\nu_n)) = 2n + 1,$$

where $n = 1, 2, \ldots$.

Let $\Gamma_n$ be the $K$-representation of $G$ corresponding to some $K$-basis in $\ker(\nu_n)$, and let $\Gamma_n^*$ be the contragradient $K$-representation of $\Gamma_n$, that is $\Gamma_n^*(g) = \Gamma^T(g^{-1})$, for all $g \in G$.

Lemma 14. (see [15, 21]) All indecomposable, pairwise nonequivalent $K$-representations of $G \cong C_2 \times C_2$ of odd degree are either $\chi_i$ or the tensor product $\Gamma_n \otimes_K \chi_i$ or $\Gamma_n^* \otimes_K \chi_i$, where $i = 0, 1, 2, 3$ and $n = 1, 2, \ldots$.

\[\square\]

Let $p = 2$ in (8) and let us consider the projective resolution for $\ker(\tau_0) = \langle a - 1, b - 1, 2 \rangle$:

$$\cdots \to (KG)^{(t_n)} \overset{\tau_n}{\to} (KG)^{(t_{n-1})} \overset{\cdots}{\to} \cdots$$

$$\cdots \overset{\tau_3}{\to} (KG)^{(t_2)} \overset{\tau_2}{\to} (KG)^{(t_1)} \overset{\tau_1}{\to} \ker(\tau_0) \to 0.$$ 

It is easy to show that in (13) $t_n = 2n + 1$ and

$$\dim_K(\ker(\tau_n)) = 4n + 4,$$
where \( n \geq 0 \). Moreover, all \( KG \)-modules \( \ker(\nu_n) \) are indecomposable. If we take the tensor product of the exact sequence (12) over the ring \( K \) by the \( KG \)-module \( \ker(\tau_0) \) and compare the result with the sequence (13), then it is easy to get the isomorphism:

\[
\ker(\tau_0) \otimes_K \ker(\nu_n) \cong \ker(\tau_n) \oplus P_n,
\]

where \( P_n \) is a projective \( KG \)-module.

**Lemma 15.** Let \( W_n \) be a \( K \)-representation of \( G = \langle a \rangle \times \langle b \rangle \cong C_2 \times C_2 \) corresponding to the module \( \ker(\tau_n) \) (\( n \geq 0 \)). This representation has the following form:

\[
W_0: \quad a \mapsto \begin{pmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix};
\]

\[
W_n: \quad a \mapsto \begin{pmatrix} D & 0 & 0 & 0 \\ E_n & 0 & V_n & 0 \\ -E_n & V_n & 0 & 0 \\ E_{n+1} & 0 & 0 & -E_{n+1} \end{pmatrix}, \quad b \mapsto \begin{pmatrix} D & 0 & S & 0 \\ E_n & 0 & V_n' & 0 \\ -E_n & 0 & V_n' & 0 \\ E_{n+1} & 0 & 0 & -E_{n+1} \end{pmatrix},
\]

where \( D = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \), \( S = \begin{pmatrix} 0 & \ldots & 0 & 1 & 1 \\ 0 & \ldots & 0 & 0 & 0 \end{pmatrix} \), and \( V_n = (0E_n) \), \( V_n' = (E_n0) \) are matrices with \( n \) rows and \( n + 1 \) columns (\( n \geq 1 \)).

**Proof.** The proof of the Lemma reduces to the determination of a \( K \)-bases of \( \ker(\tau_n) \), which is not difficult to construct inductively with respect to \( n \).

\( \square \)

**Lemma 16.** A faithful indecomposable \( K \)-representation of \( G = \langle a \rangle \times \langle b \rangle \cong C_2 \times C_2 \) which satisfies the necessary condition for the existence of the torsion-free group \( \text{Crys}(G; M; f) \) is one of the following:

\[
\Delta_n: \quad a \mapsto \Delta_n(a), \quad b \mapsto \Delta_n(b), \quad (n \geq 1);
\]

\[
\Delta_n^*: \quad a \mapsto \Delta_n(a)^T, \quad b \mapsto \Delta_n(b)^T, \quad (n \geq 1);
\]

\[
W_n: \quad a \mapsto W_n(a), \quad b \mapsto W_n(b), \quad (n \geq 0);
\]

\[
W_n^*: \quad a \mapsto W_n(a)^T, \quad b \mapsto W_n(b)^T, \quad (n \geq 0),
\]

where \( A^T \) is the transpose of \( A \);

\[
\Delta_n(a) = \begin{pmatrix} E_n & 0 & 0 & 0 \\ 0 & E_n & 0 & 0 \\ -E_n & 0 & E_n & 0 \\ -E_n & 0 & 0 & E_n \end{pmatrix}, \quad \Delta_n(b) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & E_n & 0 & 0 & 0 \\ -E_n & 0 & E_n & 0 & 0 \\ 0 & E_n & 0 & 0 & -E_n \end{pmatrix}.
\]
$\Delta^*_n$ and $W^*_n$ are $K$-representations of $G$, contragradient to the $K$-representations $\Delta_n$ and $W_n$, respectively.

Proof. All $K$-representations listed above satisfy the necessary condition for the existence of a torsion free group $\text{Crys}(G; M; f)$. The analysis of all representations of odd degree (see Lemma 14) shows that among the representations $\Gamma_n \otimes \chi_i$ the necessary condition is satisfied only by $\Delta^*$ which is equivalent to $\Gamma_{2n}$ ($n = 1, 2, \ldots$). Besides, the representations $W_n$ and $W^*_n$ the group $G$ has a parameterized series of representations, whose degrees are divisible by 4. In this series the following pairs of matrices correspond to the pairs of generator elements of $G$:

\[
\begin{pmatrix}
E_n & 0 & 0 & E_n \\
-E_n & E_n & 0 \\
E_n & 0 & -E_n
\end{pmatrix},
\begin{pmatrix}
E_n & 0 & \tilde{F} & 0 \\
-E_n & 0 & E_n
\end{pmatrix},
\]

where the $K$-matrix $\tilde{F}$ has indecomposable modulo 2K the Frobenius (i.e. rational) canonical normal form. Clearly, the representation of this series does not satisfy the necessary condition. Let us consider the following pair of matrices:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & E_n & 0 & 0 & 0 \\
0 & E_n & 0 & 0 \\
E_n & 0 & 0 & -E_n \\
0 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
E_{n+1} & 0 & 0 & E_{n+1} & 0 \\
0 & -E_n & 0 \\
0 & -E_{n+1} & 0 \\
E_n & 0 & -E_n
\end{pmatrix}.
\]

These matrices define the indecomposable $K$-representation of $G$ of degree congruent to 2 modulo 4. Obviously, this representation does not satisfy the necessary condition. We can obtain the remaining representations of degree $4n - 2$ either by the presented process of tensor multiplication by irreducible $K$-representation or by taking the contragradient representation.

As a result we get the representations which do not satisfy the necessary condition for the existence of the torsion free group $\text{Crys}(G; M; f)$. Thus, we have considered all indecomposable $K$-representations of $G$. The lemma is proved.

Proof of Theorem 3. Note that, the module $M$ of a $K$-representation $\Gamma$ of $G$ is a module with $m$-dimensional columns, whose entries belong to the ring $K$ ( $m$ is the degree of the representation $\Gamma$). Then $FM$ is a linear space of $m$-dimensional vector columns over the field $F$, $\hat{M} = FM^+/M^+$ is a group of $m$-dimensional columns, whose coordinates belong to the group $\hat{F} = F^+/K^+$. Let $f : G \to \hat{M}$ be a cocycle. The value $f(g)$ of $f$ at $g \in G$ is an $m$-dimensional column over $\hat{F}$. We note that if $g, h \in G$ then the multiplication $g \cdot f(h)$ is the multiplication of the matrices $\Gamma(g)$ and $f(h)$.

If we consider the coordinates of the vector $f(g)$ as elements of the field $F$, then the elements of the ring $K$ will be replaced by 0.
Let $\Gamma$ be any one of the representations of $G$ listed in Lemma 16, and let $M$ be the module of this representation, and $H = \langle h \rangle$ a non-trivial subgroup of $G$. There exists only one basis vector $v$ in $M$ such that the $KH$-module $M$ is a direct sum $M = Kv \oplus M'$ of the $KH$-module $Kv$ and the $KH$-module $M'$, generated by the rest of the basis vectors of $M$. In addition $hv = v$ and a $K$-representation $\Gamma'$ of $H$ corresponding to the module $M'$ is a sum of representations $\gamma_1$ and $\gamma_2$ (see (11)). This allows us the possibility to replace the cocycle $f$ by the cohomologous cocycle $f_1$ in such a way that the projection $f_1|_{M'}$ will be equal to zero in the element $h$ (see Lemmas 2–3).

The coordinate $x_v$ of the vector $f(h)$, corresponding to the basis vector $v$, will be called the special component of the vector $f(h)$. From $(1+h)f(h) = 0$ it follows that $2x_v = 0$ (in the group $\hat{F}$). Besides, for any vector $z \in \hat{M}$ the special component of the vector $(h-1)z + f(h)$ is always equal to $x_v$. If $x_v = \frac{1}{2}$, then the cocycle $f$ is not cohomologous to the zero cocycle at $h$.

These remarks justify the following plan for the construction of cocycles of the representations $\Gamma$ from Lemma 16. The form of the representation $\Gamma$ defines the special components of the vectors $f(a)$ and $f(b)$ ( $a$ and $b$ are generators of $G$). We choose the vector $f(a)$ such that we deduce that the special component is $\frac{1}{2}$ and all the rest are zero. The possible forms of the components of the vector $f(b)$ follow from the conditions:

\begin{align}
(1+b)f(b) &= 0, \\
(a-1)f(b) &= (b-1)f(a).
\end{align}

We will carry out the following operations on the vector $f(b)$: replace $f(b)$ by the vector

\begin{equation}
(b-1)z + f(b),
\end{equation}

where $z \in \hat{M}$ and $(a-1)z = 0$.

We discard all those forms of $f(b)$, with a zero special component. For the vector $f(b)$ whose special component equals to $\frac{1}{2}$, we find:

\begin{equation}
f(ab) = af(b) + f(a)
\end{equation}

and we examine the solvability of the following equation

\begin{equation}
(ab-1)z + f(ab) = 0
\end{equation}

with the variable $z \in \hat{M}$.

The group $\text{Crys}(G; M; f)$ is torsion free if and only if the equation (17) is unsolvable.

We consider the following seven cases:

Case 1. Let $\Gamma = \Delta_n$. The special components are the $(n+1)$th entry in $f(a)$ and the first one in $f(b)$. Set the $(n+1)$th coordinate of $f(a)$ to $\frac{1}{2}$ and let all the rest be 0. Let

\begin{equation}
f^T(b) = (y, Y_1, Y_2, Y_3, Y_4),
\end{equation}

\begin{equation}
\text{Crys}(G; M; f)
\end{equation}
where \( y \in \widehat{F} \), \( Y_i \in \widehat{F}^{(n)} \) and \( i = 1, 2, 3, 4 \).

The operation (16) can replace \( Y_2 \) by the zero vector. From (15) follows that \( Y_3 = Y_4 = 0 \), and from (14), it follows that \( 2y = 0 \) and \( 2Y_1 = 0 \). Let \( y = \frac{1}{2} \), \( Y_1 = (v_1, v_2, \ldots, v_n) \). Using (17), it is easy to transform (18) to a linear system of equations (over \( \widehat{F} \)) with the \((n + 1) \times n\)-matrix

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 & 0 \\
-1 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -1 & 1 \\
0 & 0 & \cdots & 0 & -1
\end{pmatrix}
\]

and free coefficients \( \frac{1}{2}, v_1, \ldots, v_{n-1}, v_n + \frac{1}{2} \). This system is solvable if and only if \( v_1 + \cdots + v_{n-1} + v_n = 0 \).

Case 2. Let \( \Gamma = \Delta_n^* \). The matrices of the \( K \)-representation were transposed to the matrices of \( \Delta_n \). The special components of \( f(a) \) and \( f(b) \) are the same as in Case 1. Let us assume that \( f(a) \) and \( f(b) \) are chosen at first in the same way as in the case of \( \Gamma = \Delta_n \) (see (19)). Condition (14) and operation (16) transform the vector \( f(b) \) to the following form:

\[
f^T(b) = (y, 0, -2Y_3, Y_3, 0).
\]

Let \( Y_3 = (v_1, \ldots, v_{n-1}, v_n) \). It follows from (15) that if \( n \geq 2 \) then

\[
y - 2v_1 = 0; \quad 2v_2 = \cdots = 2v_n = 0; \\
2v_1 = 0; \cdots; 2v_{n-1} = 0; \quad 2v_n = \frac{1}{2},
\]

and, if \( n = 1 \), then \( y - 2v_1 = 0, 2v_1 = \frac{1}{2} \).

If \( n > 1 \) and \( y = \frac{1}{2} \), then (19) leads to a contradiction. If \( n = 1 \) and \( y = \frac{1}{2} \), then \( v_1 = \frac{1}{4} \).

Thus, if \( n > 1 \) and \( f \) is a cocycle then the special component of the vector \( f(b) \) is equal to zero. Then the cocycle \( f \) is cohomologous to the zero cocycle in the element \( b \in G \). This means that the torsion free group \( \text{Crys}(G; M; f) \) does not exist for the representation \( \Gamma = \Delta_n^* \), if \( n > 1 \).

Let \( n = 1 \). Then

\[
f(a) = (0, \frac{1}{2}, 0, 0, 0); \quad f(b) = (\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{4}, 0); \quad f(ab) = (\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}).
\]

It is easy to check that (18) is unsolvable.

Case 3. Let \( \Gamma = W_n^* \) \( (n > 0) \). The special components are \((2n + 3)\)th of \( f(a) \) and the last of \( f(b) \). Let the special component of \( f(a) \) be equal to \( \frac{1}{2} \), and let all the rest be zero.

Let \( f^T(b) = (Y_0, Y_1, Y_2, Y_3, Y_4) \), where \( Y_0 \in \widehat{F}^{(2)}, Y_1, Y_2 \in \widehat{F}^{(n)}, Y_3, Y_4 \in \widehat{F}^{(n+1)} \).

Operation (16) allows us to replace \( Y_3 \) by the zero vector. It follows from (14) that
Y_1 = 0. Condition (15) shows that Y_2 = 0, Y_0 = (0, y) (y \in \hat{F}, 2y = 0) and 2Y_4 = 0. Consequently,

\[ f^T(b) = (0, y, 0, \ldots, 0, v_1, \ldots, v_n, \frac{1}{2}) \]

The special component of f(ab) is the second coordinate which, according to (17), equals to y. Therefore \( y = \frac{1}{2} \) and for any \( v_1, \ldots, v_n \) \((2v_1 = 2v_2 = \cdots = 2v_n = 0)\) the group \( \mathfrak{Crys}(G; M; f) \) is torsion free.

Case 4. Let \( \Gamma = W_0^* \). In this case it is easy to see that the cocycle \( f \) with

\[ f(a) = (0, 0, 0, \frac{1}{2}, 0); \quad f(b) = (0, \frac{1}{2}, 0, \frac{1}{2}) \]

determines a torsion free group \( \mathfrak{Crys}(G; M; f) \).

Case 5. Let \( \Gamma = W_n \ (n > 1) \). We take the vectors \( f(a) \) and \( f(b) \) in the same fashion as in the Case 3. Condition (14) shows that all components of the vector \( Y_4 \), except the last, are zero. Then condition (15) leads to contradiction. We obtain the contradiction by setting the special component in \( f(a) \) equal to \( \frac{1}{2} \).

Consequently, for \( \Gamma = W_n \ (n > 1) \) any cocycle \( f \) is cohomologous to the zero cocycle at the generator \( a \) of \( G \), this means that the torsion free group \( \mathfrak{Crys}(G; M; T) \) does not exist in this case.

Case 6. Let \( \Gamma = W_1 \). In a cocycle \( f \) with

\[ f(a) = (0, 0, 0, 0, \frac{1}{2}, 0, 0, 0); \quad f(b) = (0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}) \]

the special components of the vector \( f(a) \) (the fifth one) and \( f(b) \) (the last one), and \( f(ab) \) (the second one) are all equal to \( \frac{1}{2} \). The cocycle \( f \) determines the torsion free group \( \mathfrak{Crys}(G; M; f) \) (see also the Lemma 12).

Case 7. Let \( \Gamma = W_0 \). The special components are the 3\(^{rd}\) for \( f(a) \) and the 4\(^{th}\) for \( f(b) \). Let us assume that they are equal to \( \frac{1}{2} \). Then there exists only one cocycle

\[ f(a) = (0, 0, \frac{1}{2}, 0); \quad f(b) = (0, 0, 0, \frac{1}{2}) \]

Hence, \( f(ab) = (\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \) and the special component (the second one) for \( f(ab) \) is equal to zero. The cocycle \( f \) is cohomologous to zero on the element \( ab \). The group \( \mathfrak{Crys}(G; M; f) \) contains elements of order 2.

It follows from Lemma 16 that all the \( K \)-representations \( \Gamma \) of \( G \) have been enumerated, for the torsion-free group \( \mathfrak{Crys}(G; M; f) \) to exist. Consequently, the Theorem is proved.

\[ \square \]

References

1. Benson, D. J., Representations and cohomology. I, II. Cohomology of groups and modules, Cambridge Studies in Advanced Math., 31. Cambridge University Press, Cambridge, 1998.
2. Berman, S.D.; Gudivok, P.M., Integral representation of finite groups (Russian), Soviet Math. Dokladi 3 (1962), 1172–1174.
3. Berman, S.D.; Gudivok, P.M., *Indecomposable representation of finite group over the ring p-adic integers (Russian)*, Izvestia AN USSR **28**(4) (1964), 875-910.
4. Charlap, L.S., *Bieberbach groups and flat manifolds*, Springer-Verlag, New York (1986).
5. Curtis, C.W.; Reiner, I., *Methods of representation theory. Vol. I. With applications to finite groups and orders*, Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1990, pp. xxiv+819.
6. Gudivok, P.M., *Representations of finite groups over a complete discrete valuation ring (Russian) Algebra, number theory and their applications.*, Trudy Mat. Inst. Steklov. **148** (1978), 96-105.
7. Gudivok, P.M.; Shapochka, I.V., *On the wildness of the problem of description of some classes of groups (Russian)*, Uzhgorod State University Scientific Herald. Mathematical Series. **3** (1998), 69–77.
8. Gupta, N.; Sidki, S., *The group transfer theorem*, Arch. Math. (Basel) **64**(1) (1995), 5–7.
9. Gupta, N.; Sidki, S., *On torsion-free metabelian groups with commutator quotients of prime exponent*, Internat. J. Algebra Comput **9** (5) (1999), 493–520.
10. Heller, A.; Reiner, I., *Representations of cyclic groups in rings of integers I*, Ann. of Math. **76**(2) (1962), 73–92.
11. Heller, A.; Reiner, I., *Representations of cyclic groups in rings of integers II*, Ann. of Math. **77** (1963), 318–328.
12. Hiss, G.; Szczechonak, A., *On torsion free crystallographic groups*, J. Pure and Appl. Algebra **1**(74) (1991), 39–56.
13. Kopcha, G.M.; Rudko, V.P., *About torsion free crystallographic group with indecomposable point cyclic p-group (Ukrainian)*, Uzhgorod State University Scientific Herald. Mathematical Series. **3** (1998), 117–123.
14. Nazarova, L. A., *Unimodular representations of the four group (Russian)*, Dokl. Akad. Nauk SSSR **140**(5) (1961), 1011–1014.
15. Nazarova, L. A., *Representations of a tetrad. (Russian)*, Izv. Akad. Nauk SSSR Ser. Mat **31** (1967), 1361–1378.
16. Plesken, W., *Kristallographische Gruppen*, Group theory, algebra, and number theory (Saarbrücken, 1993) de Gruyter, Berlin (1996), 75–96.
17. Plesken, W., *Some applications of representation theory*, Prog. in Math. **95** (1991), 477-496.
18. Rossetti, J. P.; Tirao, P. A., *Compact flat manifolds with holonomy group \(\mathbb{Z}_2 \oplus \mathbb{Z}_2\)*, Proc. Am. Math. Soc. **124**(8) (1996), 2491-2499.
19. Rossetti, J. P.; Tirao, P. A., *Five-dimensional Bieberbach groups with holonomy group \(\mathbb{Z}_2 \oplus \mathbb{Z}_2\)*, Geometriae Dedicata **77** (1999), 149-172.
20. Rossetti, J. P.; Tirao, P. A., *Compact flat manifolds with holonomy group \(\mathbb{Z}_2 \oplus \mathbb{Z}_2\), II*, Rend. Sem. Mat. Univ. Padova **101** (1999), 99-136.
21. Rudko, V.P., *Algebras of integral p-adic representations of finite groups (Russian)*, Dokl. Akad. Nauk Ukrain. SSR Ser. A **11** (1979), 904–906.

V.A. Bovdi

Institute of Mathematics and Informatics
University of Debrecen
H-4010 Debrecen, P.O. Box 12
Hungary
vbovdi@mail.klte.hu

P.M. Gudivok, V.P. Rudko

Department of Algebra
Uzhgorod University
88 000, Uzhgorod
