A Simple Invariance Theorem

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This is an old article (from May 2004), that will probably not be published, because a much improved paper with new results is in preparation. Still, I decided to put it in the archive because there are some things of interest here (in particular, the section on the S-K model) which will not appear in the new paper.

Abstract

We present a simple extension of Lindeberg’s argument for the Central Limit Theorem to get a general invariance result. We apply the technique to prove results from random matrix theory, spin glasses, and maxima of random fields.

1 Introduction and results

J. W. Lindeberg’s elegant proof of the Central Limit Theorem [15, 16], despite being in the shadow of Fourier analytic methods for a long time, is now well known. It was revived by Trotter [25] and has since been used successfully to derive CLTs in infinite dimensional spaces, where the Fourier analytic methods are not so useful. For more information on this topic, see the survey paper [3] and the monograph [19]. (Another possible source is Bergström’s books [4, 5]. It is also worth mentioning that LeCam [14] had a similar idea for Poisson approximation.) The ideas were carefully examined and generalized by Zolotarev [28] through the introduction of the so-called ζ metrics, which we shall not discuss here.

However, it seems that the basic method of replacing non-Gaussian random variables by Gaussians one by one and using Taylor expansion to get approximation bounds has been applied only for proving central limit theorems for sums of independent random elements, and its potential for proving more general invariance results has been overlooked in the literature. (After the preparation of the initial draft of this article, it came to our notice that indeed, there is an old article of Rotar [21] which examines the Lindeberg method polynomial maps in a limiting case. Also, earlier this year, Mossel, O’Donnell, and Oleszkiewicz [18] made some striking applications to problems from computer science and discrete mathematics using the Lindeberg method on polynomials.)

We shall derive a very simple extension of Lindeberg’s argument to obtain a result for general smooth functions. Basically, we shall show that if $f : \mathbb{R}^n \to \mathbb{R}$ is a function such that reasonable fluctuations in any single coordinate (keeping others fixed) do not affect the value of the function in...
a “big” way, then the distribution of \( f(X_1, \ldots, X_n) \), where \( X_i \)'s are independent random variables, depends mainly on the first two moments of the \( X_i \)'s.

To make things precise, we first need a suitable measure of the largest possible influence of any single coordinate on the outcome.

**Definition 1.1.** For any open interval \( I \) containing 0, any positive integer \( n \), any function \( f : I^n \to \mathbb{C} \) which is thrice differentiable in each coordinate, and \( 1 \leq r \leq 3 \), let

\[
\lambda_r(f) := \sup\{|\partial_i^p f(x)|^{r/p} : 1 \leq i \leq n, \ 1 \leq p \leq r, \ x \in I^n\}
\]

where \( \partial_i^p \) denotes \( p \)-fold differentiation with respect to the \( i^\text{th} \) coordinate. For a collection \( \mathcal{F} \) of such functions, define \( \lambda_r(\mathcal{F}) := \sup_{f \in \mathcal{F}} \lambda_r(f) \).

Note that the interval \( I \) can be bounded or unbounded. The numbers \( \lambda_r(f) \) jointly constitute a measure of the maximum possible influence of the fluctuation in a single coordinate on the value of \( f \) at any point in the set \( I^n \). We shall show that \( f \) will have the aforementioned invariance property when \( \lambda_2(f) \) and \( \lambda_3(f) \) are sufficiently small.

In this paper, we shall generally denote vectors by \( x, y \) etc. The \( i^\text{th} \) component of \( x \) will be denoted by \( x_i \), of \( y \) by \( y_i \) and so on.

In what follows, \( X = (X_1, \ldots, X_n) \) and \( Y = (Y_1, \ldots, Y_n) \) are two vectors of independent random variables with finite second moments, taking values in some open interval \( I \) and satisfying, for each \( i \), \( EX_i = EY_i \) and \( EX_i^2 = EY_i^2 \). We shall also assume that \( X \) and \( Y \) are defined on the same probability space and are independent. Finally, let \( \gamma = \max\{E|X_i|^3, E|Y_i|^3, 1 \leq i \leq n\} \). Note that \( \gamma \) may be \( \infty \).

Here is our main result:

**Theorem 1.1.** Let \( f : I^n \to \mathbb{R} \) be thrice differentiable in each argument. If we set \( U = f(X) \) and \( V = f(Y) \), then for any thrice differentiable \( g : \mathbb{R} \to \mathbb{R} \) and any \( K > 0 \),

\[
|\mathbb{E}g(U) - \mathbb{E}g(V)| \leq C_1(g)\lambda_2(f) \sum_{i=1}^n \left[ \mathbb{E}(|X_i|^2; |X_i| > K) + \mathbb{E}(|Y_i|^2; |Y_i| > K) \right] \\
+ C_2(g)\lambda_3(f) \sum_{i=1}^n \left[ \mathbb{E}(|X_i|^3; |X_i| \leq K) + \mathbb{E}(|Y_i|^3; |Y_i| \leq K) \right]
\]

where \( C_1(g) = \|g'\|_\infty + \|g''\|_\infty \) and \( C_2(g) = \frac{1}{6}\|g'\|_\infty + \frac{1}{2}\|g''\|_\infty + \frac{1}{6}\|g'''\|_\infty \).

The last term in the above bound is usually dealt with as follows: having chosen a suitable \( K \), we use \( \mathbb{E}(|X_i|^3; |X_i| \leq K) \leq KE(X_i^2) \). When \( \gamma < \infty \), we can do better:

**Corollary 1.2.** In the setting of the above Theorem, if we further have \( \gamma < \infty \), then \( |\mathbb{E}g(U) - \mathbb{E}g(V)| \leq 2C_2(g)\gamma n\lambda_3(f) \).

For a quick example to see how Theorem 1.1 can be applied, consider the function \( f(x) = n^{-1/2} \sum_{i=1}^n x_i \). It is very easy to compute \( \lambda_2(f) = n^{-1} \) and \( \lambda_3(f) = n^{-3/2} \). Now suppose \( X_i \)'s are...
i.i.d. and $Y_i$’s are also i.i.d. Further, assume $\mathbb{E}X_i = \mathbb{E}Y_i = 0$ and $\mathbb{E}X_i^2 = \mathbb{E}Y_i^2 = 1$ for all $i$. Then taking $K = \epsilon \sqrt{n}$ and using Theorem 1.1 we can easily get

$$|\mathbb{E}g\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i\right) - \mathbb{E}g\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i\right)| \leq C_1(g)\mathbb{E}(X^2_1; |X_1| > \epsilon \sqrt{n})$$

$$+ \mathbb{E}(Y^2_1; |Y_1| > \epsilon \sqrt{n}) + 2C_2(g)\epsilon.$$

Taking $n \to \infty$, this proves the classical CLT since $\epsilon$ is arbitrary. Furthermore, if we assume that $\mathbb{E}|X_1|^3 < \infty$ and $\mathbb{E}|Y_1|^3 < \infty$, then we also get an explicit error bound:

$$|\mathbb{E}g\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i\right) - \mathbb{E}g\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i\right)| \leq C_2(g)\mathbb{E}|X_1|^3 + \mathbb{E}|Y_1|^3 \sqrt{n}.$$

For a more complicated example, consider the Stieltjes transform of a Wigner matrix. For a given $z \in \mathbb{C} \setminus \mathbb{R}$, define a function $f$ as

$$f((x_{ij})_{1 \leq i \leq j \leq N}) = \frac{1}{N} \text{Tr}((A((x_{ij})) - zI)^{-1})$$

where $A((x_{ij}))$ is the $N$ by $N$ matrix whose $(i, j)^{th}$ element is $N^{-1/2}x_{ij}$ if $i \leq j$ and $N^{-1/2}x_{ji}$ otherwise, and $I$ is the $N$ by $N$ identity matrix, and “tr” stands for the trace of a matrix. In section 2 we shall use Theorem 1.1 to obtain invariance results about this function, which will in turn yield the weakest known condition for convergence of spectral measures to Wigner’s semicircle law.

Another nontrivial example that we shall consider (in section 3) is the free energy of the Sherrington-Kirkpatrick model of spin glass theory. Here the function $f$ is given by

$$f((x_{ij})_{1 \leq i < j \leq N}) = \frac{1}{N} \log \left[ \sum_{\sigma} \exp \left\{ \frac{\beta}{\sqrt{N}} \sum_{i<j} x_{ij}\sigma_i\sigma_j + \beta h \sum_i \sigma_i \right\} \right]$$

where the sum is taken over all $\sigma = (\sigma_1, \ldots, \sigma_N) \in \{-1, 1\}^N$, and $\beta, h$ are parameters. To deal with functions of this form, which commonly occur as free energy functions of various physical models, we have the following general Theorem:

**Theorem 1.3.** Suppose $\mathcal{F}$ is a finite collection of coordinatewise thrice differentiable functions from $I^n$ into $\mathbb{R}$, and $\alpha \geq 1$. If $F : I^n \to \mathbb{R}$ is defined as $F(x) := \alpha^{-1} \log \left( \sum_{f \in \mathcal{F}} e^{\alpha f(x)} \right)$, then $\lambda_2(F) \leq 3\alpha \lambda_2(\mathcal{F})$ and $\lambda_3(F) \leq 13\alpha^2 \lambda_3(\mathcal{F})$.

In section 3, we shall derive a condition under which the asymptotic behaviour of the free energy in the Sherrington-Kirkpatrick model is not dependent on the exact distributions of the entries. Our condition is weaker than the weakest known condition. In particular, it includes the “i.i.d. mean zero unit variance” case.

Besides the possible applications to free energy functions as mentioned before, Theorem 1.3 can have other important uses, as well. For example, the following result is an easy application of Theorems 1.1 and 1.3.
Theorem 1.4. Let $\mathcal{F}$ be as in Theorem 1.3. Let $U = \max_{f \in \mathcal{F}} f(X)$ and $V = \max_{f \in \mathcal{F}} f(Y)$. Then for any thrice differentiable $g : \mathbb{R} \to \mathbb{R}$, any $K > 0$, and any $\alpha \geq 1$, we have

$$|\mathbb{E}g(U) - \mathbb{E}g(V)| \leq 2\|g'\|_\infty \alpha^{-1} \log |\mathcal{F}| + 3\alpha C_1(g)\lambda_2(\mathcal{F})T_1(K) + 13\alpha^2 C_2(g)\lambda_3(\mathcal{F})T_2(K)$$

where $T_1(K) = \sum_{i=1}^n [\mathbb{E}(X_i^2; |X_i| > K) + \mathbb{E}(Y_i^2; |Y_i| > K)]$ and $T_2(K) = \sum_{i=1}^n [\mathbb{E}(|X_i|^3; |X_i| \leq K) + \mathbb{E}(|Y_i|^3; |Y_i| \leq K)].$

Again, we shall usually deal with $T_2(K)$ using $\mathbb{E}(|X|^3; |X| \leq K) \leq K^2 X^2$. If $\gamma < 0$, we have a more explicit bound:

Corollary 1.5. In the setting of the above Theorem, if we further have $\gamma < 0$, then

$$|\mathbb{E}g(U) - \mathbb{E}g(V)| \leq K(g)[(\gamma n \lambda_3(\mathcal{F}))^{1/3} (\log |\mathcal{F}|)^{2/3} + \gamma n \lambda_3(\mathcal{F})]$$

where $K(g) = 19^3 \|g'\|_\infty + 13\|g''\|_\infty + 15\|g'''\|_\infty$.

In section 4, we shall demonstrate an application of Theorem 1.4 involving the energy of the ground state in the Sherrington-Kirkpatrick model of spin glasses. Essentially, we shall show that under the same conditions on the $x_{ij}$'s as in section 3, the asymptotic behaviour of

$$N^{-3/2} \max_{\sigma} \sum_{1 \leq i < j \leq N} x_{ij} \sigma_i \sigma_j,$$

where the maximum is taken over all $\sigma \in \{-1, 1\}^N$, is not dependent on the exact distributions of the $x_{ij}$'s.

For an immediate application, consider the (very old) question raised by Erdős and Kac [8]: what is the limiting distribution of $\max_{1 \leq j \leq n} \frac{1}{\sqrt{n}} \sum_{i=1}^j X_i$ where $X_i$'s are i.i.d. with mean zero and unit variance? It is now well known that the limiting distribution is the same as that of $|Z|$, where $Z \sim N(0, 1)$. Erdős and Kac proved the result for the case of the simple random walk; the general result could be proved only after Donsker established the weak invariance principle. Using Corollary 1.5, we can easily establish concrete error bounds under finite third moments assumption for this problem.

To work things out, let $\mathcal{F} = \{f_i : 1 \leq i \leq n\}$, where $f_i(x) := n^{-1/2} \sum_{j=1}^i x_j$. Clearly, $\lambda_3(\mathcal{F}) = \max_{1 \leq i \leq n} \lambda_3(f_i) = n^{-3/2}$. Corollary 1.5 now gives the bound

$$|\mathbb{E}g(U) - \mathbb{E}g(V)| \leq K(g)[\gamma^{1/3} n^{-1/6} (\log n)^{2/3} + \gamma n^{-1/2}]$$

where $U = \max_{1 \leq i \leq n} \frac{1}{\sqrt{n}} \sum_{j=1}^i X_j$ and $V = \max_{1 \leq i \leq n} \frac{1}{\sqrt{n}} \sum_{j=1}^i Y_j$.

The three Theorems presented in this section are very general in applicability, and present a unifying approach to solving examples of the kind mentioned above, rather than applying different techniques for different problems. However, the method has its deficiencies, the greatest being that functions have to be smooth. This is a rather severe restriction, and eliminates a lot of interesting examples. For example, the method will not allow us to deal with non-smooth functionals like
stopping times (in the case of random walks) and empirical distribution functions (for random matrices). Smoothing approximations may sometimes give crude bounds. Furthermore, the restriction about the boundedness of derivatives hampers the applicability to many interesting functions like spectral radii of random matrices. Again, truncation techniques might work.

The next three sections will be devoted to working out in detail the examples mentioned before. Proofs of the Theorems and Corollaries will be presented in the last section.

2 Convergence of spectral distributions

In this section, we shall illustrate the application of our method to proving invariance results about random matrices. Specifically, we shall derive the weakest known condition under which the spectral measures of a sequence of Wigner matrices converge to the semicircle law. We begin with a very short introduction to some material from the spectral theory of large dimensional random matrices.

2.1 Spectral measures

The Empirical Spectral Distribution (ESD) of a square matrix is the probability distribution on the complex plane which puts equal mass on each eigenvalue of the matrix (repeated by multiplicities). The limit of a sequence of ESDs is called the Limiting Spectral Distribution (LSD) of the corresponding sequence of matrices. The existence and identification of LSDs for various kinds of random matrices is one of the main goals of random matrix theory.

For an excellent review of mathematical results known about limiting spectral behaviour and further references, see Bai [2]. For relevance in physics, see the book by Mehta [17].

2.2 Stieltjes transforms

A standard tool for identifying the LSD of a sequence of random matrices is the Stieltjes transform. To cut a long story short, we can say that the ESDs of a sequence \( \{A_N\}_{N=1}^{\infty} \) of random real symmetric matrices converge in probability (w.r.t. the Prokhorov metric, for example) to a probability distribution \( G \) if and only if

\[
\forall z \in \mathbb{C} \setminus \mathbb{R}, \quad \frac{1}{N} \text{Tr}\left( (A_N - zI_N)^{-1} \right) \xrightarrow{P} \int_{-\infty}^{\infty} \frac{1}{x - z} dG(x)
\]

where \( I_N \) is the identity matrix of order \( N \). The expression on the right is the Stieltjes transform of \( G \) evaluated at \( z \). Similarly, the expression on the left is the Stieltjes transform of the ESD of \( A_N \), evaluated at \( z \). Stieltjes transforms will be particularly useful for applying our technique, since they are infinitely differentiable as functions of the matrix entries.

2.3 Wigner matrices

A random Wigner matrix of order \( N \) is an \( N \) by \( N \) real symmetric matrix with independent entries on and above the diagonal.

More specifically, consider the map \( A \) which “constructs” Wigner matrices of order \( N \). Let \( n = N(N + 1)/2 \) and write elements of \( \mathbb{R}^n \) as \( x = (x_{ij})_{1 \leq i \leq j \leq N} \). For any \( x \in \mathbb{R}^n \), let \( A(x) \) be
the matrix whose \((i, j)\)th entry is \(N^{-1/2}x_{ij}\) if \(i \leq j\) and \(N^{-1/2}x_{ji}\) if \(i > j\). If \(X\) is a vector of \(n\) independent standard Gaussian random variables, then \(A(X)\) is a standard Gaussian Wigner matrix. Wigner [26] showed that the LSD for a sequence of standard Gaussian Wigner matrices is the semicircle law, which has density \((2\pi)^{-1/2} \sqrt{4-x^2}\) in \([-2, 2]\).

It was later shown that the distribution of the entries do not play a significant role: convergence to the semicircle law would hold under more general conditions (Cf. Arnold [1], Grenander [10] and Bai [2]). The weakest known condition under which the convergence to semicircle law holds was given by Pastur [20]. It is claimed that the condition was shown to be necessary by Girko [9]. For a detailed exposition, see Bai [2] or Khorunzhy, Khoruzhenko and Pastur [13].

The method of this paper will give an easy way to show the sufficiency of Pastur’s condition. Incidentally, somewhat similar ideas involving derivatives of empirical characteristic functions (instead of Stieltjes transforms) to get concentration bounds for ESDs have been explored in Chatterjee and Bose [7].

2.4 Derivation of Pastur’s condition

To get started, fix \(z = u + iv \in \mathbb{C}\), with \(v \neq 0\). Define \(f : \mathbb{R}^n \to \mathbb{R}\) as

\[ f(x) := \frac{1}{N} \text{Tr}((A(x) - zI)^{-1}). \]

Also, define \(G : \mathbb{R}^n \to \mathbb{C}^{N \times N}\) as \(G(x) := (A(x) - zI)^{-1}\). Now note that from matrix theory we know that inverting a matrix involves computing the classical adjoint and dividing by the determinant, which implies that the elements of the inverse are all rational functions of the elements of the original matrix. Also note that since all eigenvalues of \(A(x)\) are real, therefore \(\det(A(x) - zI) \neq 0\). Thus, \(G\) is infinitely differentiable along each coordinate. Also note that \((A(x) - zI)G(x) = I\) for each \(x\). Thus for \(1 \leq i \leq j \leq N\), \(\frac{\partial}{\partial x_{ij}}[(A - zI)G] \equiv 0\), which gives

\[ \frac{\partial G}{\partial x_{ij}} = -G \frac{\partial A}{\partial x_{ij}} G. \]

Also, note that higher order derivatives of \(A\) vanish identically. Combining everything we easily get

\[ \frac{\partial f}{\partial x_{ij}} = -\frac{1}{N} \text{Tr}(\frac{\partial A}{\partial x_{ij}} G^2), \quad (1) \]

\[ \frac{\partial^2 f}{\partial x_{ij}^2} = \frac{2}{N} \text{Tr}(\frac{\partial A}{\partial x_{ij}} \frac{\partial A}{\partial x_{ij}} G^2), \quad (2) \]

\[ \frac{\partial^3 f}{\partial x_{ij}^3} = -\frac{6}{N} \text{Tr}(\frac{\partial A}{\partial x_{ij}} G \frac{\partial A}{\partial x_{ij}} G \frac{\partial A}{\partial x_{ij}} G^2). \quad (3) \]

Now we need to find good bounds for the above quantities. For that, we need some preparation.

For an \(N \times N\) complex matrix \(B = ((b_{ij}))\), the Hilbert-Schmidt norm (or Schur norm, or Euclidean norm) of \(B\) is defined as \(\|B\| := (\sum_{i,j} |b_{ij}|^2)^{1/2}\). Besides the usual properties of a matrix norm, it also satisfies the following:
1. \(|\text{Tr}(BC)| \leq \|B\|\|C\|\).

2. If \(U\) is a unitary matrix, then for any \(C\) of the same order, \(\|CU\| = \|UC\| = \|C\|\).

3. For a normal matrix \(B\) (i.e. \(B^*B = BB^*\), \(B^*\) being the conjugate transpose of \(B\)) with eigenvalues \(\lambda_1, \ldots, \lambda_N\), and any \(C\), \(\max(\|BC\|, \|CB\|) \leq \max_{1 \leq i \leq N} |\lambda_i| \cdot \|C\|\).

The first property follows from the Cauchy-Schwarz inequality. The second is true because \(\|Uy\|_2 = \|y\|_2\) for any unitary matrix \(U\) and any vector \(y \in \mathbb{R}^N\), where \(\cdot\|\cdot\|_2\) denotes the Euclidean norm on \(\mathbb{R}^N\). For the last one, note that any normal matrix \(B\) can be written as \(B = U\Delta U^*\) where \(U\) is unitary and \(\Delta\) is diagonal, with the diagonal elements being the eigenvalues of \(B\), and then apply the second property.

The above facts are standard, and may be looked up in any standard text on matrix analysis. See Wilkinson [27] pp. 55-58, for example.

Now, it is easy to see that \(G\) and the derivatives of \(A\) are all normal matrices. Moreover, the eigenvalues of \(G\) are bounded by \(|v|^{-1}\) (where \(v = \text{Im} z\)) and the eigenvalues of \(\partial A/\partial x_{ij}\) are bounded by \(N^{-1/2}\). (Note that \(\partial A/\partial x_{ij}\) is the matrix which has \(N^{-1/2}\) at the \((i,j)^{th}\) and \((j,i)^{th}\) positions, and 0 elsewhere.)

Thus, from the spectral representation of \(G^2\) it follows that the elements of \(G^2\) are bounded by \(|v|^{-2}\). This fact, and the identity (11) imply that
\[
\left\| \frac{\partial f}{\partial x_{ij}} \right\|_\infty \leq 2|v|^{-2}N^{-3/2}.
\]

(4)

Next, using the expression (2) and the three properties of the Hilbert-Schmidt norm discussed above, we get
\[
\left\| \frac{\partial^2 f}{\partial x_{ij}^2} \right\|_\infty \leq \frac{2}{N} \left\| \frac{\partial A}{\partial x_{ij}} \right\| \left\| G \frac{\partial A}{\partial x_{ij}} G^2 \right\| \leq 4|v|^{-3}N^{-2}.
\]

(5)

Similarly, (3) gives
\[
\left\| \frac{\partial^3 f}{\partial x_{ij}^3} \right\|_\infty \leq 12|v|^{-4}N^{-5/2}.
\]

(6)

From (4), (5) and (6) it follows that
\[
\lambda_2(f) \leq 4 \max\{|v|^{-4}, |v|^{-3}\}N^{-2},
\]
\[
\lambda_3(f) \leq 12 \max\{|v|^{-6}, |v|^{-4}\}N^{-5/2}.
\]

Let \(X = (X_{ij})_{1 \leq i \leq j \leq N}\) and \(Y = (Y_{ij})_{1 \leq i \leq j \leq N}\) be collections of independent random variables with zero mean and unit variance. Let \(U = \text{Re} f(X)\) and \(V = \text{Re} f(Y)\), and let \(g : \mathbb{R} \to \mathbb{R}\) be any thrice differentiable function. Note that \(\text{Re} f\) is a smooth function and \(\lambda_r(\text{Re} f) \leq \lambda_r(f)\) for each \(r\). With \(K = \epsilon \sqrt{N}\), Theorem 1.1 immediately tells us that \(|\mathbb{E}g(U) - \mathbb{E}g(V)|\) can be bounded by a multiple (depending only on \(g\) and \(v\)) of
\[
N^{-2} \sum_{1 \leq i \leq j \leq N} \left[ \mathbb{E}(X_{ij}^2; |X_{ij}| > \epsilon \sqrt{N}) + \mathbb{E}(Y_{ij}^2; |Y_{ij}| > \epsilon \sqrt{N}) \right] + \epsilon.
\]
The same bound also works for functions of the imaginary parts. Using this result and Wigner’s Theorem for Gaussian matrices, we see that convergence to the semicircle law holds whenever $X_{ij}$’s are independent with zero mean and unit variance, and satisfy

$$\forall \epsilon > 0, \lim_{N \to \infty} N^{-2} \sum_{1 \leq i \leq j \leq N} \mathbb{E}(X_{ij}^2; |X_{ij}| > \epsilon \sqrt{N}) = 0.$$  \hspace{1cm} (7)

This is exactly Pastur’s condition, as mentioned before. The condition is satisfied, for example, if $X_{ij}$’s are i.i.d. with zero mean and unit variance. Also note that though this looks like Lindeberg’s condition for the central limit theorem, it is not exactly that.

3 Universality of a spin glass model

In this section, we obtain a condition for invariance (or, as physicists say, universality) of the limiting free energy of the Sherrington-Kirkpatrick model of spin glasses. We begin with a short introduction.

3.1 Spin glasses

Let $\Sigma_N = \{-1, 1\}^N$. This is the space of all possible spins of $N$ particles in statistical mechanics. The spins are random, but not independent — the spin of one particle exerts influence on the spin of another. The joint law of the $N$ spins is a matter of great interest and intrigue. Various models have been suggested over the years for various situations. Some of these models, like the famous Ising model, are deterministic in the sense that none of the model parameters are random, while some others, like the Sherrington-Kirkpatrick model which we shall discuss here, involve random variables as model parameters.

All models assign a probability proportional to $\exp(-\beta H_N(\sigma))$ to the configuration $\sigma$, where $H_N$ is the Hamiltonian, and $\beta = 1/T$, $T$ being the temperature. The partition function is $Z_N = \sum_{\sigma} \exp(-\beta H_N(\sigma))$, and the free energy is the log of the partition function divided by $N$. The asymptotic behaviour of the free energy is of great consequence and interest to physicists, and nowadays, to people in neural networks also.

For a detailed discussion of mathematical results about spin glass models and further references, see Talagrand [23], for instance.

3.2 The Sherrington-Kirkpatrick model

The Sherrington-Kirkpatrick (S-K) model, introduced in [22], can be briefly described as follows: For each $N \geq 1$ let $\{z_{ij}^N, 1 \leq i, j \leq N\}$ be a collection of i.i.d. $N(0, 1)$ random variables. The S-K model assigns a random probability distribution (the Gibbs measure) on $\Sigma_N$ as follows: For any configuration $\sigma \in \Sigma_N$, the probability of the system being in the state $\sigma = (\sigma_1, \ldots, \sigma_N)$ is given by

$$p_{N, \beta}(\sigma) = Z_{N, \beta}^{-1} \exp(-\beta H_{N, \beta}(\sigma))$$
where $H_{N,\beta}(\sigma) = -\frac{1}{\sqrt{N}} \sum_{i<j} J_{ij}^N \sigma_i \sigma_j - h \sum_{i \leq N} \sigma_i$, $\beta$ and $h$ are fixed parameters, and $Z_{N,\beta}$ is the normalising constant. Ideally, the subscripts should include $\beta$ and $h$, but we are considering them to be fixed. It has been shown by Guerra and Toninelli [12] that the limit
\[
\lim_{N \to \infty} \frac{1}{N} \mathbb{E}(\log Z_{N,\beta})
\]
exists for all $\beta$ and $h$. See Talagrand [23] Theorem 2.10.1, p. 140 for a proof. A formula for the limit was conjectured by Parisi and proved by Talagrand [24]. Talagrand ([23] Corollary 2.2.5, p. 32) also proves (in particular) that
\[
\frac{1}{N} (\log Z_{N,\beta} - \mathbb{E} \log Z_{N,\beta}) \xrightarrow{P} 0
\]
for any $\beta$ and $h$. Both the above facts were proved under the condition that $J_{ij}$ are i.i.d. $N(0,1)$. In fact, the rigorous proofs involve the use of intricate properties of Gaussian random variables.

Recently, in a paper which was archived at a time when this article was being written, Carmona and Hu [6] have proved that the limit will exist and be the same when $J_{ij}$ are i.i.d. with zero mean, unit variance and finite third moment. Their technique may be extended to the case of independent variables with uniformly bounded third absolute moments.

We shall derive a sufficient condition for invariance of the limiting free energy, which is weaker than the condition given by Carmona and Hu, and includes the case where $J_{ij}$’s are i.i.d. with zero mean and unit variance, with no assumption about the third moment.

### 3.3 Our condition

Let $\mathcal{F} = \{f_\sigma : \sigma \in \{-1,1\}^N\}$, where

\[
f_\sigma((x_{ij})) = \beta N^{-3/2} \sum_{i<j} x_{ij} \sigma_i \sigma_j + \beta h N^{-1} \sum_i \sigma_i.
\]

Then clearly, $\lambda_2(\mathcal{F}) = 2N^{-3}$, $\lambda_3(\mathcal{F}) = 3\beta^2 N^{-2}$ and $|\mathcal{F}| = 2^N$. Now, if we define $F(x) = N^{-1} \log(\sum_\sigma e^{N f_\sigma(x)})$, then by Theorem [1.3] $\lambda_2(F) \leq 3\beta^2 N^{-2}$ and $\lambda_3(F) \leq 13\beta^3 N^{-5/2}$.

Suppose $\mathcal{F}$ and $\mathcal{F}'$ are collections of independent random variables with zero mean and unit variance. If we let $U_N = F(\mathcal{F})$ and $V_N = F(\mathcal{F}')$, then by Theorem [1.4] for any thrice differentiable $g : \mathbb{R} \to \mathbb{R}$ and any fixed $\epsilon > 0$, $|\mathbb{E}g(U_N) - \mathbb{E}g(V_N)|$ is bounded by a constant multiple (depending only on $g$ and $\beta$) of

\[
N^{-2} \sum_{1 \leq i < j \leq N} \mathbb{E}(|\mathcal{F}_{ij}^2| \mathbb{1}_{|\mathcal{F}_{ij}| > \epsilon \sqrt{N}}) + \mathbb{E}(|\mathcal{F}_{ij}^2| \mathbb{1}_{|\mathcal{F}_{ij}'| > \epsilon \sqrt{N}}) + \epsilon.
\]

This shows that the limit of the free energy is the same as that in the i.i.d. standard Gaussian case whenever $\mathcal{F}_{ij}$’s are independent with zero mean and unit variance, and satisfy

\[
\forall \epsilon > 0, \lim_{N \to \infty} N^{-2} \sum_{1 \leq i < j \leq N} \mathbb{E}(|\mathcal{F}_{ij}^2| \mathbb{1}_{|\mathcal{F}_{ij}| > \epsilon \sqrt{N}}) = 0.
\]
Note that this is almost exactly condition (7), the only difference being that here we do not have terms corresponding to \(i = j\). In particular, it is satisfied when \(\beta_{ij}\)’s are i.i.d. with zero mean and unit variance.

Under the assumption of uniformly bounded third absolute moments, Corollary 1.2 can be applied to get an explicit error bound of order \(N^{-1/2}\), which is the same as that obtained by Carmona and Hu [6].

4 Ground state of the S-K model

The ground state in a spin glass model is the configuration which minimizes the Hamiltonian. With \(\beta = 1\) and \(h = 0\) for simplicity, the energy of the ground state is given by

\[
S_N(\beta) = \max_{\sigma \in \Sigma_N} \sum_{1 \leq i < j \leq N} \beta_{ij} \sigma_i \sigma_j.
\]

Guerra and Toninelli [11, 12] proved that \(N^{-3/2}S_N(\beta)\) converges almost surely and in average to a deterministic limit if \(\beta\) is a collection of standard Gaussian random variables. It was extended to the case of i.i.d. entries with zero mean, unit variance and finite third moment by Carmona and Hu [6]. We shall show that convergence in probability and in average to the same limit would hold if \(\beta_{ij}\)’s were independent and satisfied the same condition as in the previous section.

Let \(\mathcal{F}\), \(\mathcal{J}\) and \(\mathcal{J}'\) be as in the previous section, with \(\beta = 1\) and \(h = 0\). If we let \(U_N = \max_{\sigma} f_{\sigma}(\mathcal{J})\) and \(V_N = \max_{\sigma} f_{\sigma}(\mathcal{J}')\), then by Theorem 1.4 for any thrice differentiable \(g : \mathbb{R} \to \mathbb{R}\) and any fixed \(K > 0\) and \(\alpha \geq 1\), \(|E g(U_N) - E g(V_N)|\) is bounded by a constant multiple (depending only on \(g\)) of

\[
\alpha^{-1} N + \alpha N^{-3} \sum_{i < j} |E(\beta_{ij}^2; |\beta_{ij}| > K) + E(\beta_{ij}'^2; |\beta_{ij}'| > K)| + \alpha^2 N^{-5/2} K.
\]

Now choose any \(A \geq 1\) and \(\epsilon > 0\), and put \(\alpha = AN\) and \(K = \epsilon \sqrt{N}\). Substituting these values in the above expression, we get

\[
A^{-1} + AN^{-2} \sum_{i < j} |E(\beta_{ij}^2; |\beta_{ij}| > \epsilon \sqrt{N}) + E(\beta_{ij}'^2; |\beta_{ij}'| > \epsilon \sqrt{N})| + A^2 \epsilon.
\]

Thus, under condition (5) of the previous section, \(\limsup_{N \to \infty} |E g(U_N) - E g(V_N)| \leq A^{-1} + A^2 \epsilon\). This proves the claim, since \(A\) and \(\epsilon\) are arbitrary.

Again, Corollary 1.5 can be applied to obtain an error bound of order \(N^{-1/6}\) under the assumption of uniformly bounded third absolute moments.

5 Proofs

Proof of Theorem 1.1 As mentioned before, the proof is just an easy extension of Lindeberg’s argument for the classical central limit theorem. Fix \(f\) and \(g\) as in the statement of the Theorem. Let \(h = g \circ f\). Then observe that

\[
\partial_i^2 h(x) = g'(f(x)) \partial_i^2 f(x) + g''(f(x))(\partial_i f(x))^2,
\]

\[
\partial_i^3 h(x) = g'(f(x)) \partial_i^3 f(x) + 3g''(f(x))\partial_i f(x) \partial_i^2 f(x) + g'''(f(x))(\partial_i f(x))^3.
\]
It follows that for any $i$ and $x$, $|\partial_i^2 h(x)| \leq C_1 \lambda_2(f)$ and $|\partial_i^3 h(x)| \leq 6C_2 \lambda_3(f)$, where $C_1 = \|g''\|_{\infty} + \|g'''\|_{\infty}$ and $C_2 = \frac{1}{2}\|g''\|_{\infty} + \frac{1}{3}\|g'''\|_{\infty}$.

Next, for $0 \leq i \leq n$, define $Z_i := (X_1, \ldots, X_{i-1}, X_i, Y_{i+1}, \ldots, Y_n)$ and $W_i := (X_1, \ldots, X_{i-1}, 0, Y_{i+1}, \ldots, Y_n)$, with obvious meanings for $i = 0$ and $n$. For $1 \leq i \leq n$, define

\[ R_i := h(Z_i) - X_i \partial_i h(W_i) - \frac{1}{2} X_i^2 \partial_i^2 h(W_i), \]
\[ T_i := h(Z_{i-1}) - Y_i \partial_i h(W_i) - \frac{1}{2} Y_i^2 \partial_i^2 h(W_i). \]

By third order Taylor expansion and the bounds on the third partials of $h$ obtained above, we immediately see that $|R_i| \leq C_2 \lambda_3(f)|X_i|^3$ and $|T_i| \leq C_2 \lambda_3(f)|Y_i|^3$. Second order bounds, on the other hand, imply that $|R_i| \leq C_1 \lambda_2(f)|X_i|^2$ and $|T_i| \leq C_1 \lambda_2(f)|Y_i|^2$. Now for each $i$, $X_i$, $Y_i$ and $W_i$ are independent. Hence

\[
\mathbb{E}(X_i \partial_i f(W_i)) - \mathbb{E}(Y_i \partial_i f(W_i)) = \mathbb{E}(X_i - Y_i)\mathbb{E}(\partial_i f(W_i)) = 0.
\]

Similarly, $\mathbb{E}(X_i^2 \partial_i^2 f(W_i)) - \mathbb{E}(Y_i^2 \partial_i^2 f(W_i)) = 0$. Combining all these observations we have, for any $K > 0$,

\[
|\mathbb{E}_g(U) - \mathbb{E}_g(V)| = \left| \sum_{i=1}^{n} \mathbb{E}(h(Z_i) - h(Z_{i-1})) \right| \\
= \left| \sum_{i=1}^{n} \mathbb{E}(X_i \partial_i h(W_i) + \frac{1}{2} X_i^2 \partial_i^2 h(W_i) + R_i) - \sum_{i=1}^{n} \mathbb{E}(Y_i \partial_i h(W_i) + \frac{1}{2} Y_i^2 \partial_i^2 h(W_i) + T_i) \right| \\
\leq C_1 \lambda_2(f) \sum_{i=1}^{n} [\mathbb{E}(X_i^2; |X_i| > K) + \mathbb{E}(Y_i^2; |Y_i| > K)] \\
+ C_2 \lambda_3(f) \sum_{i=1}^{n} [\mathbb{E}(|X_i|^3; |X_i| \leq K) + \mathbb{E}(|Y_i|^3; |Y_i| \leq K)].
\]

The corollary follows by taking $K \rightarrow \infty$. \hfill \Box

**Proof of Theorem 1.3** We begin by defining a bunch of functions. The domains will be clear from the definitions. Let

\[
\psi(x, f) := e^{\alpha f(x)}, \\
Z(x) := \sum_{f \in \mathcal{F}} \psi(x, f), \\
p(x, f) := Z(x)^{-1} \psi(x, f), \\
a_i(x, f) := \alpha \delta_i f(x), \\
e_i(x) := \sum_{f \in \mathcal{F}} a_i(x, f)p(x, f).
\]
Note that for any $x$, $p(x, \cdot)$ is a probability on $\mathcal{F}$. This will be widely used without mention in obtaining the bounds below. Also, note that $F(x) = \alpha^{-1} \log Z(x)$.

We shall now find bounds on the partial derivatives of several orders for these functions. Function arguments will be suppressed for clarity. First, note that clearly from the given expressions,

\[
\partial_i \psi = a_i \psi, \quad (9)
\]

\[
\partial_i Z = \sum_{f \in \mathcal{F}} \partial_i \psi = \sum_{f \in \mathcal{F}} a_i \psi = Ze_i. \quad (10)
\]

Using (9) and (10) and the expression for $p$ we get

\[
\partial_i p = \frac{Z a_i \psi - Ze_i \psi}{Z^2} = (a_i - e_i)p. \quad (11)
\]

Now, directly from the expression for $e_i$ we get

\[
\partial_i e_i = \sum_{f \in \mathcal{F}} (p \partial_i a_i + a_i \partial_i p), \quad (12)
\]

\[
\partial^2_i e_i = \sum_{f \in \mathcal{F}} (p \partial_i^2 a_i + 2(\partial_i a_i)(\partial_i p) + a_i \partial_i^2 p). \quad (13)
\]

Using (11) and (12) we get

\[
\partial^2_i p = (\partial_i a_i - \partial_i e_i)p + (a_i - e_i)^2 p. \quad (14)
\]

Now for $1 \leq r \leq 3$, let $C_r = \sup \{|\partial^r_i f(x)| : 1 \leq i \leq n, f \in \mathcal{F}, x \in I^n\}$. Then note that for any $i$ we have the uniform bounds

\[
|a_i| \leq \alpha C_1, \quad |\partial_i a_i| \leq \alpha C_2, \quad |\partial_i^2 a_i| \leq \alpha C_3 \quad (15)
\]

In the following, we shall freely use the assumption that $\alpha \geq 1$. The first inequality above immediately gives

\[
|e_i| \leq \alpha C_1. \quad (16)
\]

From (11), (15) and (16), we get

\[
|\partial_i p| \leq 2\alpha C_1 p. \quad (17)
\]

Using (12), (15) and (16) we get

\[
|\partial_i e_i| \leq \alpha^2 (C_2 + 2C_1^2). \quad (18)
\]

Using (14), (16), (16) and (18) we get

\[
|\partial_i^2 p| \leq \alpha^2 (2C_2 + 6C_1^2)p. \quad (19)
\]

Using (13), (15), (17) and (18) we have

\[
|\partial_i^2 e_i| \leq \alpha^3 (C_3 + 6C_1 C_2 + 6C_1^3). \quad (20)
\]
The proof is completed by observing that $\partial_i F = \alpha^{-1} \partial_i \log Z = \alpha^{-1} e_i$ and using the bounds (16), (18) and (20) in Definition 1.1. □

**Proof of Theorem 1.4**

For each $\alpha \geq 1$, let $F_\alpha(x) = \alpha^{-1} \log [\sum_{f \in \mathcal{F}} e^{\alpha f(x)}]$. Also, let $F(x) = \max_{f \in \mathcal{F}} f(x)$. Then we have

$$F(x) = \alpha^{-1} \log [e^{\alpha \max_{f \in \mathcal{F}} f(x)}]$$
$$\leq \alpha^{-1} \log [\sum_{f \in \mathcal{F}} e^{\alpha f(x)}]$$
$$\leq \alpha^{-1} \log [\mathcal{F} e^{\alpha \max_{f \in \mathcal{F}} f(x)}]$$

which gives the uniform bound

$$|F(x) - F_\alpha(x)| \leq \alpha^{-1} \log |\mathcal{F}|.$$

Thus, by Theorem 1.3, for any $K > 0$,

$$|\mathbb{E} g(F(X)) - \mathbb{E} g(F(Y))| \leq 2 \|g\|_\infty \alpha^{-1} \log |\mathcal{F}| + 3\alpha C_1(g) \lambda_2(\mathcal{F}) T_1(K) + 13\alpha^2 C_2(g) \lambda_3(\mathcal{F}) T_2(K)$$

where $T_1(K) = \sum_{i=1}^n [\mathbb{E}(X_i^2; |X_i| > K) + \mathbb{E}(Y_i^2; |Y_i| > K)]$ and $T_2(K) = \sum_{i=1}^n [\mathbb{E}(|X_i|^3; |X_i| \leq K) + \mathbb{E}(|Y_i|^3; |Y_i| \leq K)]$. If $\gamma < \infty$, then we can let $K \to \infty$ and get

$$|\mathbb{E} g(F(X)) - \mathbb{E} g(F(Y))| \leq 2 \|g\|_\infty \alpha^{-1} \log |\mathcal{F}| + 26\alpha^2 C_2(g) \lambda_3(\mathcal{F}) \gamma n.$$  

Now choose $\alpha = [(\gamma n \lambda_3(\mathcal{F}))^{-2/3} (\log |\mathcal{F}|)^{2/3} + 1]^{1/2}$. Note that $\alpha \geq 1$ and $\alpha^{-1} \leq (\gamma n \lambda_3(\mathcal{F}))^{1/3} (\log |\mathcal{F}|)^{-1/3}$. The Corollary follows from this. □

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