NON-COMPACT WZW CONFORMAL FIELD THEORIES

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ABSTRACT

We discuss non-compact WZW sigma models, especially the ones with symmetric space $H^C/H$ as the target, for $H$ a compact Lie group. They offer examples of non-rational conformal field theories. We remind their relation to the compact WZW models but stress their distinctive features like the continuous spectrum of conformal weights, diverging partition functions and the presence of two types of operators analogous to the local and non-local insertions recently discussed in the Liouville theory. Gauging non-compact abelian subgroups of $H^C$ leads to non-rational coset theories. In particular, gauging one-parameter boosts in the $SL(2,C)/SU(2)$ model gives an alternative, explicitly stable construction of a conformal sigma model with the euclidean 2D black hole target. We compute the (regularized) toroidal partition function and discuss the spectrum of the theory. A comparison is made with more standard approach based on the $U(1)$ coset of the $SU(1,1)$ WZW theory where stability is not evident but where unitarity becomes more transparent.

1. INTRODUCTION

The four years which passed since the previous Cargèse Institute of the series have brought a marked progress in the understanding of rational Conformal Field Theories (CFT’s), a class of 2D massless quantum field models, see e.g. [1]. The simplest of those theories is the free field with values in a circle of rational radius, more complicated examples are provided by the Wess-Zumino-Witten (WZW) sigma models with a general compact Lie group $G$ as the target [2],[3],[1] or by the coset theories obtained by gauging a subgroup of $G$ in the WZW theory [3],[1]. The characteristic property of the rational CFT’s is

1. decomposition of the euclidean Green functions into a finite sum of products of

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1extended version of lectures read at the Summer Institute “New Symmetry Principles in Quantum Field Theory”, Cargèse, July 16-27, 1991

2based on joint work in progress with Antti Kupiainen; these notes cover also some preliminary material far from being completely understood, reflecting there the present author’s point of view which has been changing with time and has not yet reached the final form
holomorphic and antiholomorphic “conformal blocks”.

This is accompanied by other simplifying features, which are more or less special from the point of view of the general quantum field theory, like

2. discreteness of finite volume energy spectrum,
3. one-to-one correspondence between states and operators,
4. simple structure of the operator product expansions,
5. finiteness of the partition functions,
6. simple factorization properties.

Our knowledge about the rational WZW and coset theories seems rather satisfactory today (although one might argue that a subtle cleaning of some fine mathematical points remains to be done; also the basic problem of classification of the rational CFT’s has not been solved). It contains the exact solution for the spectrum and for the low genus Green functions (see e.g. [7]). It seems then reasonable to go beyond the study of relatively simple conformal theories where properties 1-6 hold, especially since examples of conformal models without those properties appear rather naturally. The best known instance is the Liouville theory describing the conformal mode of 2D gravity [8]. It is in this model, essential for the treatment of non-critical string theory, where the new features related to the failure of 1-6 where first discussed, see inspiring lectures [9].

Mathematically, the passage from rational to irrational CFT’s involves a shift from purely algebraic treatment to more analysis. The parallel might be the passage from representation theory of compact Lie groups to the non-compact case. Indeed, the canonical WZW example of rational CFT is related to the representation theory of loop groups of compact groups [10] and it is expected that (largely non-existent, see however [11], [12]) theory of representations of loop groups of non-compact type will underlie an interesting class of irrational CFT’s. The first candidates which come to mind are the WZW theories with non-compact groups as targets. These however, if quantized as in the compact case, have unbounded below energy and, consequently, no stability and no euclidean picture. A possible solution is to pass to their coset models where in some cases one may expect to recover stability, see Sec. 5 below. In the present course, we shall start instead from a different series of non-rational CFT’s which have bounded below energy and stable euclidean picture but are non-unitary. These are the WZW-type sigma models with non-compact target spaces $H^C/H$ where $H$ is a compact (simple, connected, simply connected) Lie group. We shall call them shortly $H^C/H$ WZW models. It should be stressed that, contrary to what the name might suggest, this is a different class of models than the coset $G/H$ theories in the CFT sense. The latter are obtained by gauging a subgroup $H \subset G$ (or more generally $H \subset G_{\text{left}} \times G_{\text{right}}$) in the group $G$ WZW model and correspond rather to conformal sigma models with orbit space of the left-right action of $H$ on $G$ as the target. To avoid the terminological confusion, we shall label them as $G \text{ mod } H$ coset theories. In fact, a coset theory $G \text{ mod } H$ factorizes into the group $G$ WZW theory times the $H^C/H$ one, decoupled in the planar topology, and, in general, coupled only via zero modes. This is how the

\footnote{we are fully aware that this arrogant attempt to change accepted terminology is bound to be futile}
general $H^C/H$ WZW theories manifested themselves for the first time $[13],[14]$. The $SL(2,\mathbb{C})/SU(2)$ model has been discussed earlier in $[13]$. The Green functions of the $H^C/H$ models which appear in this context have also a 3D interpretation: they compute the scalar product of Schrödinger picture states of the 3D Chern-Simons $[16],[17],[7]$ field theory with gauge group $H$. In more geometric terms, they give the hermitian structure which pairs conformal blocks of the (rational) group $H$ WZW model into its Green functions. In this guise, the $H^C/H$ theories may be thought of as models dual to the ones with the compact group $H$ as the target. All this is briefly recalled in Sec. 2.

In Sec. 3, we discuss free field representations of the $H^C/H$ WZW models on the simplest example with $H = SU(2)$. We compute explicitly (the finite part of) the partition function of the model and discuss its spectrum and the relation between the space of states and the operators of the theory.

The coset scenario for producing new CFT’s may also work in the case of the $H^C/H$ WZW models if one gauges out a non-compact abelian subgroup $N \in H^C$ (the result will be called an $H^C/H \mod N$ theory). In Sec. 4, using free field representations, we show that the $SL(2,\mathbb{C})/SU(2) \mod R$ model where $R$ is embedded into $SL(2,\mathbb{C})$ by $t \mapsto e^{i\sigma^3}$ gives a conformal sigma model with the recently found $[18],[19]$ 2D euclidean black hole as the target. We discuss the partition functions and the spectrum of this model. A comparison is made between the $SL(2,\mathbb{C})/SU(2) \mod R$ theory and the rational parafermionic $SU(2) \mod U(1)$ model.

Finally, in Sec. 5, we contrast our approach to the black hole conformal sigma model with Witten’s original proposal $[18]$ based on the $SU(1,1) \mod U(1)$ coset theory, see also $[20]-[29]$. The free field calculation of the partition functions of the black hole model may be also repeated within Witten’s scenario giving the same result but it requires complex shifts and rotations of the fields in the functional integral. One may reasonably expect that both models coincide, the two approaches being complementary: the $SL(2,\mathbb{C})/SU(2) \mod R$ picture provides an explicitly stable construction whereas the $SU(1,1) \mod U(1)$ approach should be more useful for demonstrating unitarity of the theory.

2. ORIGIN OF THE $H^C/H$ WZW MODELS

2.1. From the coset $G \mod H$ theories

Let us start by recalling the formulation of a coset $G \mod H$ theory as a partially gauged, group $G$ WZW model (with compact $G$). The basic fields on the closed Riemann surface $\Sigma$ are $G^C$-valued functions $g$ and gauge fields $A = A_z dz + A_{\bar{z}} d\bar{z}$ with values in the complexified Lie algebra $H^C$ of a group $H \subset G$. $H$ is supposed to be embedded into $G$ in two possibly different ways: $\iota_{l,r} : H \hookrightarrow G$. We shall denote by “tr” the invariant form on the Lie algebra $G$ ($H$) of $G$ ($H$) normalized to give 2 as the length squared of the longest roots. We assume that via embeddings $\iota_{l,r}, \text{tr}$ on $G$ induces a single invariant form on $H$ equal $\eta$ times tr on $H$. The euclidean action of the coset model takes the
form \[ (\ref{eq:1}) \]

\[
kS(g, A) = kS(g) + \frac{ik}{\Sigma} \int \text{tr} [A_z^r(g^{-1}\partial_z g) + (g\partial_z g^{-1})A_z^l] + i\text{Ad}_g(A_z^r)A_z^l - i\eta A_z d^2z
\]  

(1)

where the superscripts “l, r” refer to the embeddings of \( H \) into \( G \). \( kS(g) \) is the pure WZW action

\[
S(g) = -\frac{1}{2\pi} \int \text{tr} (g^{-1}\partial_z g)(g^{-1}\partial_z g) d^2z + \frac{1}{24\pi} \int d^{-1}\text{tr} (g^{-1}dg)^3
\]

(2)

where we have used a shorthand notation for the Wess-Zumino topological term \[ (\ref{eq:2}) \]. Coupling constant \( k \) (“level”) is a positive integer. Under the complex (\( H^C \)-valued) chiral gauge transformations

\[
g \mapsto h_1^lgh_2^{r\dagger} \]
\[
A_z \mapsto h_1^lA_z \equiv \text{Ad}_{h_1}(A_z) - ih_1\partial_z h_1^{-1}, \]
\[
A_z \mapsto h_2^lA_z \equiv \text{Ad}_{h_2^{-1}}(A_z) - ih_2^{-1}\partial_z h_2^{\dagger},
\]

action (1) transforms like follows:

\[
S(h_1^lgh_2^{r\dagger}, h_2^lA_z dz + h_2^lA_z dz) = S(g, A) + \eta S(h_1^lA_z dz + h_2^lA_z dz).
\]

(3)

In particular, it is invariant under the unitary gauge transformations with \( H \)-valued \( h_1 = h_2 = h \).

The Green functions of the coset model are formally given by the functional integral

\[
\int e^{-kS(g, A)} Dg DA
\]

(4)

over \( G \)-valued fields \( g \) and real (i.e. \( H \)-valued) gauge fields \( A \). As the insertion, we should take an expression invariant under the unitary gauge transformations. An example is provided by

\[
\prod_{\alpha} \text{tr}_{R_\alpha} g(\xi_\alpha)n_{\alpha}
\]

(5)

where “\( \text{tr}_R \)” stands for the trace in representation \( R \) of \( G \) (in vector space \( V_R \)) and \( n_{\alpha} \in G \) satisfy

\[
u^l_n_v u^{r\dagger} = n_v
\]

(6)

for \( u \in H \). For example, if \( u^l = u^r \), we may take \( n_{\alpha} = 1 \).

On the Riemann sphere, we may parametrize real gauge fields \( A \) by \( H^C \)-valued gauge transformations by putting \( A_z(h) = h^{-1}\partial_z h \). Action (1) becomes then

\[
kS(g, A(h)) = kS(h^lgh^{r\dagger}) - \eta kS(hh^{\dagger}).
\]

(7)
The Jacobian of the change of variables is (we ignore the zero modes for the moment)
\[
\frac{\partial(A(h))}{\partial(h)} = \text{det}(\tilde{\partial}_{h}^* \tilde{\partial}_{h}) = e^{2h^\vee S(hh^\dagger)}(\text{det}(-\Delta))^{\text{dim}H}
\]  
where $\tilde{\partial}_{h} = d\bar{z}(\partial_{z} + \text{ad}_{A_{\kappa}(h)})$, $h^\vee$ is the dual Coxeter number of $H$ and $\Delta$ is the scalar Laplacian. More exactly, the change of variables $A \mapsto h$ gives the following expression for the Green functions (4) with insertion (5):
\[
C \int \left( \prod_{\alpha} \text{tr}_{R_{\alpha}} g(\xi_{\alpha})(h^\dagger n_{\alpha} h^\dagger)^{-1}(\xi_{\alpha}) \right) e^{-kS(g)} e^{(\kappa + 2h^\vee)S(hh^\dagger)} Dg \delta(h(\xi_{0})) Dh
\]  
where $C = (\text{det}'(-\Delta)/\text{area})^{\text{dim}H}$ with the determinant without the zero mode contribution. Expression (9) combines the Green functions of the compact group $G$ WZW model
\[
\Gamma = \int (\bigotimes_{\alpha} g_{R_{\alpha}}(\xi_{\alpha})) e^{-kS(g)} Dg \in \bigotimes_{\alpha} \text{End} V_{R_{\alpha}}
\]  
(10)
(\text{where } g_{R} \text{ denotes the representation } R \text{ matrix of } g) \text{ with those of a field theory with fields } hh^\dagger
\[
\int (\langle \Gamma, \otimes(h^\dagger n_{\alpha} h^\dagger)^{-1}(\xi_{\alpha}) \rangle) > e^{\kappa S(hh^\dagger)} \delta(hh^\dagger(\xi_{0})) D(hh^\dagger)
\]  
In the last expression $\Gamma$ may be any tensor in $\bigotimes \text{End} V_{R_{\alpha}}$ such that
\[
(\otimes \gamma'_{R_{\alpha}}) \Gamma(\otimes \gamma^\dagger_{R_{\alpha}}) = \Gamma
\]  
for $\gamma \in H^{C}$. This condition guarantees that the integral is independent of point $\xi_{0}$ in the $\delta$-function in (11) fixing the global $H^{C}$ invariance. Green functions (10) certainly satisfy condition (12). $\langle \cdot , \cdot \rangle$ in (11) stands for the scalar product induced from that of spaces $V_{R}$. Fields $hh^\dagger$ may be viewed as taking values in the non-compact symmetric space $H^{C}/H$ and functional integral (11) as defining the (euclidean) Green functions of the $H^{C}/H$ WZW theory (also in a general world-sheet topology). The euclidean action $-\kappa S(hh^\dagger)$ of the model is unambiguously defined and real, non-negative \footnote{4$H^{C}/H$ is topologically trivial}. We shall see that it leads to functional integrals of type (11) which are stable for any real $\kappa > h^\vee$. On the other hand, the Minkowskian action is not real: the Wess-Zumino term is purely imaginary so that we should not expect the theory to be unitary. We shall return to these issues below.

On a higher genus Riemann surface a similar treatment of the coset theory Green functions produces again a combination of the $G$ and $H^{C}/H$ WZW Green functions but this time both twisted by coupling to an external flat gauge field $A_{\text{flat}}$ and the result contains an integral over the moduli of $A_{\text{flat}}$ \footnote{4$H^{C}/H$ is topologically trivial}, essentially coinciding with the moduli of complex $H^{C}$-bundles.
2.2. From the scalar product of the Chern-Simons theory states

The Schrödinger picture states of the 3D Chern-Simons theory with gauge group $H$ on manifold $\Sigma \times \mathbb{R}$ and in the presence of the Wilson lines $\{\xi_\alpha\} \times \mathbb{R}$ in representations $R_\alpha$ are functionals

$$\psi : \mathcal{A} \rightarrow \bigotimes_\alpha V_{R_\alpha}$$

on space $\mathcal{A}$ of real gauge fields $A$. $\mathcal{A}$ has a natural complex structure obtained by identifying it with the space of forms $A \bar{z}$. Functionals $\psi$ are required to be holomorphic and to transform covariantly under the complex gauge transformations:

$$\psi(hA) = e^{kS(h^{-1}) + \pi^{-1}k \int \text{tr} (h^{-1} \partial_z h) A \bar{z} d^2z} \bigotimes_\alpha h_{R_\alpha}(\xi_\alpha) \psi(A).$$

$k$ this time denotes the coupling constant of the Chern-Simons theory. The space of states defined as above is finite-dimensional. The scalar product of the states is formally given by the functional integral

$$\|\psi\|^2 = \int <\psi(A), \psi(A)> e^{-\pi^{-1}k \int \text{tr} A \bar{z} A d^2z} DA.$$  

On $\Sigma = \mathbb{C}P^1$, upon the change of variables $A \mapsto h$, eq. (15) becomes

$$\|\psi\|^2 = \left(\text{det}'(-\Delta)/\text{area}\right)^{\text{dim}H} \cdot \int <\psi(0) \otimes \overline{\psi}(0), \otimes hh^\dagger R_\alpha(\xi_\alpha)> e^{(k+2h^\dagger)S(hh^\dagger)} \delta(hh^\dagger(\xi_0)) D(hh^\dagger)$$

which is a Green function of type (11) (for $G = H$ and $n_\alpha \equiv 1$).

2.3. From the hermitian structure coupling conformal blocks of the group $H$ WZW theory

Green functions of the group $H$ WZW model in an external $\mathcal{H}$-valued field $A$

$$\int (\bigotimes_\alpha g(\xi_\alpha)_{R_\alpha}) e^{-kS(g.A)} Dg$$

can be expressed as

$$\sum_{a,b} \Omega^{ab} \psi_a(A) \otimes \overline{\psi}_b(A) e^{-\pi^{-1}k \int \text{tr} A \bar{z} A d^2z}$$

where $(\psi_a)$ is a basis of the Chern-Simons states considered above and the inverse matrix

$$(\Omega^{-1})_{ab} = (\psi_a, \psi_b)$$

in the scalar product of (15), see [7], [31]. In the planar or toroidal geometry, the dependence of the basis vectors $\psi_a$ on the insertion points and the complex structure may be chosen analytic and such that the scalar products $(\psi_a, \psi_b)$ remain constant. Expression (18) gives then the decomposition of the Green functions into sum of combinations.
of conformal blocks demonstrating the rational character of the WZW theories with compact targets. As we see, scalar product (15) given by the Green functions of the $H^C/H$ theory determines the way the conformal blocks of group $H$ WZW theory are put together to build the complete Green functions.

The WZW theories with targets $H$ and $H^C/H$ may be considered as dual to each other. An elegant way to express this duality is to consider the coset $H \mod H$ model. This is a topological theory in the sense of [32]: its Green functions

$$\int \left( \prod_\alpha \text{tr}_R g(\xi_\alpha) \right) e^{-kS(g,A)} \, Dg \, DA$$

are independent of the location of the insertions and of the complex structure of the surface [14]. Integrating representation (18) over gauge fields $A$, one infers [33] that they are in fact equal to the dimensions of the spaces of states $\psi$ known explicitly due to [34]. On the other hand, the coset Green functions factorize, as we have seen, into a combination of products of those of the group $H$ and of the symmetric space $H^C/H$ WZW models. This is the precise expression of the duality between both theories.

In the planar case, the $H^C/H$ theory with Green functions (11) may be also viewed as an analytic continuation of those of the $H$ theory to negative levels. This relation becomes more complicated on higher genera as, for example, a look into the respective partition functions shows. It is not excluded, however, that both models describe different aspects of the same structure analytic in $k$.

3. FREE FIELD REPRESENTATION OF THE $H^C/H$ WZW THEORY

Functional integral (2.11) defining the Green functions of the $H^C/H$ WZW theory may be computed by iterative Gaussian integration. This was noticed in [15] for the $H = SU(2)$ case and was implemented in the present context and for general $H$ in [13], [14] for the twisted toroidal partition function and in [1], [33] for the planar Green functions. One can also compute toroidal Green functions. Free field representation for the model on a surface of genus $> 2$ is still an open problem. Below, we shall stick to the $SU(2)$ case, for simplicity.

Symmetric space $SL(2,\mathbb{C})/SU(2)$ coincides with the upper sheet $H^+_3$ of 3D mass hyperboloid. Convenient global coordinate system on $H^+_3$ is provided by the parametrization

$$hh^\dagger = \left( e^{\phi(1 + |v|^2)^{1/2}} \frac{v}{\bar{v}}, e^{-\phi(1 + |v|^2)^{1/2}} \right)$$

with $\phi$ real and $v$ complex. The $SL(2,\mathbb{C})$-invariant measure on $H^+_3$, $d(hh^\dagger) = d\phi \, d^2v$. In coordinates (1),

$$S(hh^\dagger) = -\frac{1}{\pi} \int [ (\partial_z \tilde{\phi})(\partial_{\bar{z}} \tilde{\phi}) + (\partial_{\bar{z}} \tilde{\phi})(\partial_{\bar{z}} \tilde{\phi}) ] \, d^2z$$

where $\tilde{\phi} \equiv \phi - \frac{i}{2} \log(1 + |v|^2)$. We shall also need a gauged version of the action. If we gauge the $U(1)$ group embedded into $SU(2)$ asymmetrically by $\nu(e^{i\theta}) = e^{i\theta \sigma^3}$,
\( \tau(e^{i\theta}) = e^{-i\theta \sigma^3} \), then the transformation law (2.3) implies, for \( h_1 = e^{\lambda \sigma^3} \) and \( h_2 = e^{-\lambda \sigma^3} \), that
\[
S(e^{\lambda \sigma^3}ge^{\lambda \sigma^3}, A + i\lambda) = S(g, A)
\]
for any \( SL(2, \mathbb{C}) \)-valued \( g \) (in particular for \( g = hh^\dagger \)) and for any complex 1-form \( A \). Consequently, taking \( A \) purely imaginary may be interpreted as gauging of subgroup \( \mathcal{R} \hookrightarrow \{ e^{\lambda \sigma^3} | \lambda \text{ real} \} \) in \( SL(2, \mathbb{C}) \) (which is the global symmetry group of the \( H_3^+ \) WZW model). \( \mathcal{R} \) corresponds to the boosts in the third direction under the standard relation between \( SL(2, \mathbb{C}) \) and the Lorentz group. A direct computation gives
\[
S(hh^\dagger, \frac{1}{2i}A) = \frac{1}{\pi} \int \left[ (\partial_z \tilde{\phi} + A_z)(\partial_{\bar{z}} \tilde{\phi} + A_{\bar{z}}) + (\partial_z + \partial_{\bar{z}} \tilde{\phi} + A_z)\bar{\nu} \left( \partial_z + \partial_{\bar{z}} \tilde{\phi} + A_z \right) \nu \right] d^2z.
\]
Invariance (3) becomes obvious in (4) since transformation \( hh^\dagger \mapsto e^{\lambda \sigma^3}hh^\dagger e^{\lambda \sigma^3} \) translates in coordinates (1) into \( (\phi, \nu) \mapsto (\phi + 2\lambda, \nu) \).

3.1. Toroidal partition function

First, let us describe the calculation \cite{4} of the twisted partition function \( Z_{H_3^+}^H(\tau, U) \) of the \( H_3^+ \) WZW theory on torus \( T_\tau \equiv \mathbb{C}/(2\pi\mathbb{Z} + 2\pi\tau \mathbb{Z}) \), \( \tau = \tau_1 + i\tau_2, \tau_2 > 0 \). It is given by the functional integral:
\[
Z_{H_3^+}^H(\tau, U) = \int e^{\kappa S(\gamma_U hh^\dagger \gamma_U^\dagger)} D(hh^\dagger)
\]
where \( \gamma_U = \exp[-\frac{1}{4\tau_2}U(z - \bar{z})\sigma^3] \), \( U \equiv U_1 + iU_2 \), satisfies
\[
\gamma_U(z + 2\pi) = \gamma(z) \quad \text{and} \quad \gamma_U(z + 2\pi\tau) = e^{-\pi i U \sigma_3} \gamma_U(z)
\]
and the action is extended to twisted field configurations \cite{4} by putting
\[
S(\gamma_U hh^\dagger \gamma_U^\dagger) = S(hh^\dagger, \frac{1}{4i} (\tau_2^{-1}\bar{U} dz + \tau_2^{-1}U d\bar{z})) + \frac{\pi}{\tau_2} U_1^2.
\]
Using the explicit form (4) of the action, we obtain
\[
Z_{H_3^+}^H(\tau, U) = e^{\pi \kappa \tau_2^{-1} U_1^2} \int e^{-\pi^{-1}\kappa \int (\partial_z \phi + \tau_2^{-1}\bar{U} / 2)(\partial_{\bar{z}} \phi + \tau_2^{-1}U / 2) d^2z} \cdot e^{\int (\partial_z + \partial_{\bar{z}} \phi + \tau_2^{-1}\bar{U}/2)(\partial_z + \partial_{\bar{z}} \phi + \tau_2^{-1}U/2) \nu d^2z} D(hh^\dagger).
\]
where we have shifted \( \tilde{\phi} \mapsto \phi \). The \( \nu \)-integral is gaussian and produces
\[
\det \left( (\bar{\partial} + \bar{\partial} \phi + \frac{1}{2} \tau_2^{-1} U d\bar{z})^* (\bar{\partial} + \bar{\partial} \phi + \frac{1}{2} \tau_2^{-1} U d\bar{z}) \right)^{-1} = e^{2\pi^{-1} \int (\partial_z \phi)(\partial_{\bar{z}} \phi) d^2z + (2\pi i)^{-1} \int \phi \mathcal{R} \det \left( (\bar{\partial} + \frac{1}{2} \tau_2^{-1} U d\bar{z})^* (\bar{\partial} + \frac{1}{2} \tau_2^{-1} U d\bar{z}) \right)^{-1}}
\]
where \( \mathcal{R} \) denotes the metric curvature form. Rather surprisingly, the resulting effective \( \phi \) theory is the free field with the background charge so that we obtain again a calculable
On the infinitesimal level, this action may be described by generators of $L$ is recovered. The space of states is unlike in the 2D theory, in the “mini” case also the real time action is real and unitarity is obtained here is the geodesic motion on $H$ (under the catchy name of “mini superspace”). The quantum-mechanical system that we contribute of stringy oscillations, has been widely used in 2D gravity where it goes configurations independent of the space coordinate (this approximation, ignoring the do it in the simpler quantum-mechanical case obtained from field theory by taking field

The presence of the (generic) twist $U$ breaks the global $SL(2, \mathbb{C})$ symmetry of the theory to the diagonal $U(1)^C$. The remaining symmetry results, however, in the divergence of the $\phi$-integral (and, consequently, of the partition function) due to the zero mode contribution. This divergence may be extracted in the usual way as the infinite volume of $U(1)^C$ leading to the insertion of $\delta(\phi(0))$ fixing the $\phi$ zero mode under the integral. The total central charge of the theory is easily computable from the standard dependence of the resulting determinants on the conformal factor of the metric. It is equal $c^\phi \equiv 3\kappa/(\kappa - 2)$ or $\kappa \dim H/(\kappa - h^*)$ for general $H$. The determinants are well known \cite{36}. The final result is (in the flat metric; $q \equiv e^{2\pi i \tau}$):

$$Z_{H^+}^H(\tau, U) = C\tau^{-1/2} qq^{-1/8} \exp \left[ -\pi(\kappa - 2)U_2^2/\tau_2 \right] \sin(\pi U)^{-2} \cdot \prod_{n=1}^\infty \left( 1 - e^{2\pi i U q^n} \right)^{-1} \left( 1 - q^n \right)(1 - e^{-2\pi i U q^n})^{-2}. \tag{9}$$

3.2. Quantum-mechanical model

It will be useful to interpret expression (9) in the hamiltonian language. Let us first do it in the simpler quantum-mechanical case obtained from field theory by taking field configurations independent of the space coordinate (this approximation, ignoring the contributions of stringy oscillations, has been widely used in 2D gravity where it goes under the catchy name of “mini superspace”). The quantum-mechanical system that we obtain here is the geodesic motion on $H_3^+$ with the euclidean action

$$- S_{\text{min}}(hh^+) = \frac{\kappa}{4} \int \left( (hh^+)^{-1} \partial_t (hh^+) \right)^2 dt. \tag{10}$$

Unlike in the 2D theory, in the “mini” case also the real time action is real and unitarity is recovered. The space of states is $L^2(H_3^+, d(hh^+)) \cong L^2(\mathbb{R} \times \mathbb{C}, d\phi d^2v)$ and it carries the unitary representation of $SL(2, \mathbb{C})$ defined by

$$(gf)(hh^+) = f(g^{-1}hh^+g^{+1}). \tag{11}$$

On the infinitesimal level, this action may be described by generators of $sl(2, \mathbb{C}) \oplus sl(2, \mathbb{C}) \cong sl(2, \mathbb{C})^C$:

$$J^1 = \frac{i}{4}(1 + |v|^2)^{-1/2}(ve^\phi - \bar{v}e^{\bar{\phi}})\partial_\phi - \frac{i}{4}(1 + |v|^2)^{1/2}(e^\phi \partial_v + e^{-\phi} \partial_{\bar{v}}),$$
$$J^2 = \frac{i}{4}(1 + |v|^2)^{-1/2}(ve^{\phi} + \bar{v}e^{\bar{\phi}})\partial_{\bar{\phi}} - \frac{i}{4}(1 + |v|^2)^{1/2}(e^{\phi} \partial_{\bar{v}} + e^{-\phi} \partial_v),$$
$$J^3 = -\frac{i}{2}\partial_v - \frac{i}{2}v\partial_{\bar{\phi}} + \frac{i}{2}\bar{v}\partial_\phi,$$

satisfying $[J^a, J^b] = i\epsilon^{abc}J^c$ and by $\bar{J}^a$’s given by the complex-conjugate vector fields. $J^a = -\bar{J}^a$ so that $J^a - \bar{J}^a$ and $i(J^a + \bar{J}^a)$ are the hermitian generators of $sl(2, \mathbb{C})$. The
Hamiltonian may be taken as $-2\kappa^{-1}\Delta$ where $\Delta$ denotes the Laplace-Beltrami operator on $H^+_3$ with the $SL(2,\mathbb{C})$-invariant metric,

$$\Delta = \vec{J}^2 = \vec{J}^2 = \frac{1}{4} \partial^2_{\phi} - \frac{1}{4} |v|^2 (1 + |v|^2)^{-1} \partial^2_{\phi}$$

$$+ (1 + |v|^2) \partial_v \partial_{\bar{v}} + \frac{1}{4} (v \partial_v - \bar{v} \partial_{\bar{v}})^2 + \frac{1}{2} (v \partial_v + \bar{v} \partial_{\bar{v}}).$$

(12)

$-\Delta$ has continuous bounded below spectrum starting from $\frac{1}{4}$ and induces the decomposition

$$L^2(H^+_3) \cong \int_{\rho > 0} \mathcal{H}_\rho \rho^2 d\rho$$

(13)

into the direct integral of irreducible unitary representations of $SL(2, \mathbb{C})$ from the principal continuous series $\mathbb{H}$ on which $-\Delta$ acts as multiplication by $(1 + \rho^2)/4$. $\mathcal{H}_\rho$ may be realized as the space of homogeneous functions of degree $-1 + i\rho$ on non-negative matrices $h'h'^\dagger$ with determinant zero, i.e. on the upper light cone $V^+_3$. The parametrization by $(\phi, v)$ together with all the formulae concerning the action of $SL(2, \mathbb{C})$ pass to the case of $V^+_3$ provided that we replace everywhere the factor $1 + |v|^2$ by $|v|^2$. The scalar product in $\mathcal{H}_\rho$ is that of $L^2(\delta(2 - \mathrm{tr} h'h'^\dagger) d(h'h'^\dagger))$. Operators $J^3 - \bar{J}^3 = -v \partial_v + \bar{v} \partial_{\bar{v}}$ and $i(J^3 + \bar{J}^3) = -i \partial_{\phi}$ may be diagonalized at the same time as $\Delta$ and their joint spectrum is $\mathbb{Z} \times \mathbb{R}$ in each $\mathcal{H}_\rho$ which, consequently, is very different from the highest- or lowest-weight representation spaces of $sl(2, \mathbb{C}) \oplus sl(2, \mathbb{C})$: both $J^3$ and $\bar{J}^3$ have continuous unbounded spectrum here!

The heat kernel on $H^+_3$ is known explicitly and it has a simple form:

$$e^{t\Delta}(h_1h_1^\dagger, h_2h_2^\dagger) = (\pi t)^{-3/2} d_{\sinh d}^{d/2} e^{-t/4 - d^2/t}$$

(14)

where $d$ is the hyperbolic distance between $h_1h_1^\dagger$ and $h_2h_2^\dagger$ or between $hh^\dagger$ and 1 where $h = h_2^{-1}h_1$. In the more standard parametrization of $H^+_3$

$$hh^\dagger = (1 + \vec{x}^2)^{1/2} + \vec{x} \cdot \vec{\sigma},$$

(15)

$d = \sinh^{-1}(|\vec{x}|)$. Operator $e^{t\Delta}$ is certainly not of trace class since $-\Delta$ has continuous spectrum and moreover of infinite multiplicity. In the formal expression

$$\int e^{t\Delta}(e^{-\pi i U^3} h_1h_1^\dagger e^{\pi i U^3}, hh^\dagger) d(hh^\dagger)$$

(16)

for $\mathrm{tr} e^{t\Delta} e^{2\pi i(UJ^3 - U\bar{J}^3)}$, the integral diverges due to the $U(1)^C$ symmetry of the integrated kernel. That is the familiar problem which we have encountered already in the two-dimensional theory. We solve it again by fixing the $U(1)^C$ invariance in the standard fashion. This leads to the insertion of $\delta(\phi)$ under the integral of the right hand side of (16)

\footnote{this is why formula (12) was written in a clumsy way}
which renders it finite (for $U \not\in \mathbb{Z}$). The hyperbolic distance between $e^{-\pi i U \sigma^3} hh^\dagger e^{\pi i U \sigma^3}$ and $hh^\dagger$

$$d = \cosh^{-1}\left(1 + |v|^2 \cosh(2\pi U_2) - |v|^2 \cos(2\pi U_1)\right)$$

(17)

for $hh^\dagger = \left((1 + |v|^2)^{1/2} \frac{v}{\bar{v}} (1 + |v|^2)^{1/2}\right)$ and an easy calculation gives

$$\text{tr}_{\text{ren}} e^{4\pi\tau_2 \kappa^{-1} \Delta} e^{2\pi i (U J^3 - \bar{U} \bar{J}^3)} = \int e^{4\pi\tau_2 \kappa^{-1} \Delta} \left(e^{-\pi i U \sigma^3} hh^\dagger e^{\pi i U \sigma^3}, hh^\dagger\right) \delta(\phi) d(hh^\dagger) = \frac{\kappa^{1/2}}{8\pi\tau_2^2} e^{-\pi\tau_2 / \kappa - \pi\kappa U_2^2 / \tau_2} |\sin(\pi U)|^{-2}.$$  

(18)

On the other hand, the quantum-mechanical partition function

$$Z^{H^+_3}_{\text{min}}(\tau, U) = \int e^{\kappa S_{\text{min}}(hh^\dagger)} \delta(\phi(t_0)) D(hh^\dagger)$$

over twisted paths on $[0, 2\pi \tau_2]$ satisfying $hh^\dagger(2\pi \tau_2) = e^{-\pi i U \sigma^3} hh^\dagger(0) e^{\pi i U \sigma^3}$ may be again computed by iterative gaussian integration. Not too surprisingly, one finds

$$Z^{H^+_3}_{\text{min}}(\tau, U) = C \tau_2^{-1/2} e^{-\pi \kappa U_2^2 / \tau_2} |\sin(\pi U)|^{-2}.$$  

(19)

Comparing eqs. (18) and (19), we find that

$$Z^{H^+_3}_{\text{min}}(\tau, U) = C \text{tr}_{\text{ren}} e^{4\pi\tau_2 \kappa^{-1} (\Delta + 1/4)} e^{2\pi i (U J^3 - \bar{U} \bar{J}^3)}$$

(20)

which establishes a Feynman-Kac type formula for the hyperbolic space $H^+_3$. Similar formulae may be produced for other symmetric spaces $H^C/H$.

3.3. Space of states

Let us return now to the interpretation of expression (9) for the 2D partition function which becomes now straightforward. Using eq. (19) and (20), we obtain

$$Z^{H^+_3}(\tau, U) = C \frac{q^{\kappa} e^{-\kappa/24}}{\text{tr}_{\text{ren}} e^{4\pi\tau_2 (\kappa - 2)^{-1} \Delta} e^{2\pi i (U J^3 - \bar{U} \bar{J}^3)}} \left|\prod_{n=1}^{\infty} (1 - e^{2\pi i U^q n})(1 - q^n)(1 - e^{-2\pi i U^q n})\right|^{-2}.$$  

(21)

The first term on the right is the familiar prefactor with the central charge. Next comes essentially the mini-space contribution with $\kappa \mapsto \kappa - 2$ and then, multiplicatively, the contribution of the oscillatory degrees of freedom. By studying the canonical quantization of the $H^+_3$ WZW theory, one may infer that its space of states should carry a representation of the affine algebra $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ of level $-\kappa$, extending the mini-space representation of $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$. Let $\hat{b}_\pm (\hat{n}_\pm)$ denote the subalgebras of $\hat{\mathfrak{sl}}(2, \mathbb{C})$ generated by $J_n^a$ with $n \geq 0$ ($\pm n > 0$). The action of $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ in $L^2(H^+_3)$ may be extended to a representation of $\hat{b}_\pm \oplus \hat{b}_\pm$ by making $J_n^a$ and $J_{n}^a$ for $n > 0$ act trivially (the bar refers to the second copy). Let us choose a dense invariant subdomain in $L^2(H^+_3)$.
like the space $\mathcal{S}(H^+_3)$ of fast decreasing functions (in $\vec{x}$ of (15)). $\mathfrak{sl}(2, \mathbb{C}) \oplus \hat{\mathfrak{sl}}(2, \mathbb{C})$ acts then in the space

$$\hat{\mathcal{H}}^{H^+_3} = \left( \mathcal{U}(\mathfrak{sl}(2, \mathbb{C})) \otimes \mathcal{U}(\hat{\mathfrak{sl}}(2, \mathbb{C})) \right) \otimes_{\mathcal{U}(\hat{\mathfrak{b}}_+)} \mathcal{S}(H^+_3)$$

(22)

where $\mathcal{U}$ denotes the enveloping algebra. This gives the representation of $\hat{\mathfrak{sl}}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ induced from the action of $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ in $L^2(H^+_3)$. In plain English, space $\hat{\mathcal{H}}^{H^+_3}$ is spanned by $\mathcal{S}(H^+_3)$ and by the descendents obtained by repeated action of $J^a_n$ and $\bar{J}^b_n$ with $n < 0$ on the states in $\mathcal{S}(H^+_3)$. As a vector space,

$$\hat{\mathcal{H}}^{H^+_3} \cong \text{Sym}(\hat{n}_-) \otimes \text{Sym}(\hat{n}_-) \otimes \mathcal{S}(H^+_3)$$

where Sym denotes the symmetric algebra. As usually, the Sugawara construction allows to define the action in $\hat{\mathcal{H}}^{H^+_3}$ of two commuting Virasoro algebras (of central charge $c_{-\kappa}$):

$$L_n = -\frac{1}{\kappa-2} \sum_{m,a} :J^a_m J^a_{-n-m}:$$

(23)

and similarly for $\bar{L}_n$. It is then the standard result that the contribution of the descendant states to $\text{tr} q^{L_0} \bar{q}^{\bar{L}_0} e^{4\pi i (U J^a_0 - \bar{U} \bar{J}^b_0)}$ is the infinite product factor in (21). Since

$$q^{L_0} \bar{q}^{\bar{L}_0} |_{L^2(H^+_3)} = e^{4\pi \tau_2 (\kappa - 2)^{-1} \Delta},$$

(24)

also the (renormalized) zero-level states contribution is recovered in (21).

The hamiltonian interpretation of the field-theoretic partition function may be then summarized in the following (Feynman-Kac type) formula:

$$\mathcal{Z}^{H^+_3}(\tau, U) = q^{c_{-\kappa}/24} \text{tr}_{\text{ren}} q^{L_0} \bar{q}^{\bar{L}_0} e^{2\pi i (U J^a_0 - \bar{U} \bar{J}^b_0)}$$

(25)

where on the right hand side the (renormalized) trace is taken over the space $\hat{\mathcal{H}}^{H^+_3}$ carrying the representation of $\hat{\mathfrak{sl}}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ induced from $L^2(H^+_3)$. The structure of the partition function of (21) and of the space of states appears to be much simpler here than in the case of compact WZW models. The probable reason is that $\hat{\mathcal{H}}^{H^+_3}$ may be decomposed into a direct integral of representations induced from $\mathcal{H}_\rho$, which we expect to be irreducible, at least in a suitable sense and for almost all $\rho$. Similar decomposition in the compact case (into a finite direct sum) yields representations which should be further reduced. $\hat{\mathcal{H}}^{H^+_3}$ carries a natural hermitian form $(\ , \ )$ extending the scalar product of $L^2(H^+_3)$. It may be characterized by the conjugacy relation $J^a_n = -\bar{J}^a_{-n}$. It is certainly non-positive since for $\chi \in L^2(H^+_3)$

$$((J^1_{-1} - \bar{J}^1_{-1}) \chi, (J^1_{-1} - \bar{J}^1_{-1}) \chi) = \frac{\kappa}{2} (\chi, \chi).$$

(26)

We expect however that $(\ , \ )$ is non-degenerate.
3.4. Green functions

In Sec. 2.1 and 2.2, we have seen that the matrix elements $hh^\dagger(\xi)_j$ of spin $j = 0, \pm 1, \ldots$ representations appear as natural insertions in the $SL(2, \mathbb{C})/SU(2)$ WZW theory, provided that they are arranged into combinations invariant under the global $SL(2, \mathbb{C})$ symmetry $hh^\dagger \mapsto \gamma hh^\dagger \gamma^\dagger$ (this is like the neutrality condition in the 2D Coulomb gas correlations). The corresponding Green functions are calculable by the iterative gaussian integration in parametrization (1). Let us explain how this works on the simplest example of the planar spin $\frac{1}{2}$ two-point function $[7]

$$
\int \left( \text{tr}_{1/2} hh^\dagger(\xi_1) (hh^\dagger)^{-1}(\xi_2) \right) e^{\kappa \int S(hh^\dagger)} \delta(hh^\dagger(\xi_0)) D(hh^\dagger)
$$

$$
= \int \left( |(e^{\phi} v)(\xi_1) - (e^{\phi} v)(\xi_2)|^2 + e^{\phi(\xi_1) - \phi(\xi_2)} + e^{\phi(\xi_2) - \phi(\xi_1)} \right) \cdot e^{-\pi^{-1} \kappa \int \left( \partial_x \phi(\partial_x \phi) + (\partial_x + \partial_y \phi)^\dagger \cdot (\partial_x + \partial_y \phi) v \right) d^2 z} \cdot \delta(\phi(\xi_0)) \delta^2(v(\xi_0)) D\phi Dv
$$

(27)

where we have already shifted $\bar{\phi} \mapsto \phi$. The $v$-integral is gaussian. It produces the partition function

$$
e^{2\pi^{-1} \int (\partial_x \phi)(\partial_x \phi) d^2 z + (2\pi i)^{-1} \int \phi R \left( \frac{\det(\bar{\partial} \partial)}{\text{area}} \right)^{-1}
$$

(28)

(which changes the coupling constant of the effective $\phi$-integral from $\kappa$ to $\kappa - 2$, compare eq. (8)) and the normalized expectation

$$
< |(e^{\phi} v)(\xi_1) - (e^{\phi} v)(\xi_2)|^2 >
$$

$$
= (\pi \kappa)^{-1} |\xi_1 - \xi_2|^2 e^{-\phi(\xi_1) - \phi(\xi_2)} \int e^{2\phi(\zeta)} |\xi_1 - \zeta|^{-2} |\xi_2 - \zeta|^{-2} d^2 \zeta .
$$

(29)

Notice the appearance of the linear term $\sim \int \phi R$ in the effective $\phi$-action and of the $e^{2\phi(\zeta)}$ insertion corresponding, respectively, to the background and screening charges in the Coulomb gas interpretation of the resulting $\phi$-field theory. The integral over $\phi$ is again gaussian but requires a renormalization of the polynomial in $e^{\pm \phi(\xi_0)}$ and $e^{2\phi(\xi)}$ to render it finite. If we extract the most divergent factor multiplicatively, the terms with milder divergences will not survive the renormalization. In the case at hand, these are terms $e^{\phi(\xi_1) - \phi(\xi_2)} + e^{\phi(\xi_2) - \phi(\xi_1)}$ on the right hand side of (27). They drop out leaving us with the result

$$
\text{const.} |\xi_1 - \xi_2|^{2-1/(\kappa-2)} \int |(\xi_1 - \zeta)(\xi_2 - \zeta)|^{-2+2/(\kappa-2)} d^2 \zeta
$$

$$
= \text{const.} |\xi_1 - \xi_2|^{3/(\kappa-2)}
$$

(30)

(in the flat metric). Replacing $\text{tr}_{1/2}$ in (27) by $\text{tr}_j$ for higher spins, we obtain a $\phi$-integral with $2j$ screening charges and finally

$$
\text{const.} |\xi_1 - \xi_2|^{4j(j+1)/(\kappa-2)}
$$

(31)
provided that $2j+1 < \kappa - 2$. Otherwise, the integral over the positions of the screening charges diverges. Higher Green functions may be computed similarly [7], [39], also for the general $H^C/H$ theories [35].

From the form of the general Green functions (also with the current and energy-momentum insertions) one infers that fields $h h^\dagger (\xi)$ are primary, both for the $\hat{s}l(2, \mathbb{C}) \oplus \hat{sl}(2, \mathbb{C})$ and $\text{Vir} \oplus \text{Vir}$ algebras. Their conformal weights $\Delta_j = \bar{\Delta}_j$ are, as read from eq. (31), $-\frac{j(j+1)}{\kappa-2} < 0$. Occurrence of fields with negative dimensions, so with Green functions growing with the distance, might seem incompatible with the stability although not necessarily in a non-unitary theory as ours. The point, however, lies elsewhere. Such fields (for $\alpha$ real) are clearly present for the massless free (uncompactified) field $\phi$ which gives a stable unitary theory and are also expected in the Liouville theory [9], believed to be stable and unitary (there, they correspond to the local operators in terminology of [1]). These operators escape the standard relation between the spectrum of energy and of conformal weights since they correspond to eigenfunctions of the Hamiltonian outside the generalized eigenspaces. This may be seen already in the “mini-space” quantum-mechanical picture which is stable and unitary for the $H^C_3$ theory: although

$$-\frac{1}{\kappa-2} \Delta h h^\dagger_j = -\frac{j(j+1)}{\kappa-2} h h^\dagger_j,$$

the matrix elements of $h h^\dagger_j$ are not the generalized eigenfunctions of $-\Delta$ due to their too rapid growth at infinity. Appearance of operators with negative conformal dimensions may be typical for irrational theories with continuous spectrum of $L_0, \bar{L}_0$. Notice nevertheless that in the $H^C/H$ WZW model they come in a finite number whereas for the massless free field and for the Liouville theory, there is a continuous family of such fields.

Besides fields with negative dimensions which do not correspond neither to true nor to generalized states of the theory, it is natural to expect existence of fields with positive dimensions corresponding to the states in the spectrum of $L_0, \bar{L}_0$. The natural candidates for such fields are given by $f_{\rho,m_l,m_r} (h h^\dagger (\xi))$ where $f_{\rho,m_l,m_r}$ is a joint generalized eigenfunction of $-\Delta, J^3, \bar{J}^3$ corresponding to eigenvalues $\frac{1}{2} (1 + \rho^2), m_l = \frac{i}{2} (n + i \omega), m_r = \frac{1}{2} (-n + i \omega)$ where $\rho \geq 0, n \in \mathbb{Z}$ and $\omega \in \mathbb{R}$. In the space $\mathcal{H}_\rho$ (of homogenous functions on $V^+_3$), the corresponding eigenfunction is

$$e^{-i \omega \phi - i \arg(v)} |v|^{-1+i\rho}. \quad (32)$$

Eigenfunction $f_{\rho,m_l,m_r}$ on $H^C_3$ is obtained by applying to (32) the Gelfand-Graev integral transformation [37] realizing the isomorphism (13):

$$f_{\rho,m_l,m_r} (\phi, v) = e^{-i \omega \phi - i \arg(v)} (1 + |v|^2)^{\omega/2} \cdot 2\pi \int_0^\infty d\theta \int_0^\infty dr \, r^{i\rho+\omega} [1 + 2 |v||r| \cos \theta + (1 + |v|^2) r^2]^{-1-i\rho}.$$ \quad (33)
For example, for $\rho = m_l = m_r = 0$, we obtain the elliptic integral

$$f_{0,0,0}(v) = \pi \int_0^{2\pi} (1 + |v|^2 \sin^2 \theta)^{-1/2} d\theta.$$  \hspace{1cm} (34)

Unfortunately, we were not able to compute the Green functions of fields $f_{\rho,m_l,m_r}$ exactly. It remains then to be seen if they indeed give rise, upon multiplicative renormalization, to primary fields with conformal weights $\Delta_\rho = \bar{\Delta}_\rho = 1 + \frac{\rho^2}{4(\kappa-2)}$.

4. $SL(2, \mathbb{C})/SU(2)$ mod $\mathbb{R}$ COSET THEORY

4.1 2D black hole sigma model

In Sec. 3, we have coupled the $SL(2, \mathbb{C})/SU(2) \equiv H^+_3$ WZW model to an abelian gauge field $A$ in the way which rendered the action invariant under the non-compact gauge transformations:

$$S(e^{\lambda \sigma^3/2} hh^\dagger e^{\lambda \sigma^3/2}, \frac{1}{2\pi}(A - d\lambda)) = S(hh^\dagger, \frac{1}{2\pi} A),$$  \hspace{1cm} (1)

see (3.3). Following the scenario for producing coset theories from compact WZW models, let us consider the functional integral

$$\int - e^{\kappa S(hh^\dagger, (2i)^{-1} A)} D(hh^\dagger) DA$$

$$= \int - e^{-\pi^{-1} \kappa \int[(\partial_z \bar{\phi} + A_z)(\partial_{\bar{z}} \bar{\phi} + \bar{A}_z) + (\partial_z \phi + A_z)(\partial_{\bar{z}} \phi + A_{\bar{z}})]d^2z} D\phi D\bar{\phi} dA$$  \hspace{1cm} (2)

with gauge invariant insertions. First notice that, by the gauge invariance, the integral over $\phi$ factors as the (infinite) volume of the gauge group. Since $A$ enters quadratically into the action, it may be integrated out (for appropriate, e.g. $A$-independent, insertions) giving

$$C \int - e^{-\pi^{-1} \kappa \int(1+|v|^2)^{-1}(\partial_z \bar{v})(\partial_{\bar{z}} v) d^2z} \prod_{\xi} \frac{d^2v(\xi)}{1+|v(\xi)|^2}.$$  \hspace{1cm} (3)

The effective action for $v$:

$$S_{\text{eff}}(v) \equiv \frac{\kappa}{\pi} \int(1+|v|^2)^{-1}(\partial_z \bar{v})(\partial_{\bar{z}} v) d^2z$$

$$= \frac{\kappa}{\pi} \sum_{a=1,2} \int(1+|v|^2)^{-1}(\partial_z v^a)(\partial_{\bar{z}} v^a) d^2z$$

if we integrate by parts. $v = v^1 + iv^2$. It is the action of a sigma model with the complex plane with metric

$$(1 + |v|^2)^{-1}(dz \otimes d\bar{z} + d\bar{z} \otimes dz)$$  \hspace{1cm} (4)

as the target. It was noticed recently [18],[19] that this target metric (together with the dilaton field $\Phi = \log(1 + |v|^2)$) forms a euclidean black hole solution of equations
of 2D gravity (with unit mass). It describes an infinite cigar becoming asymptotically a cylinder (the scalar curvature goes down as $|v|^{-2}$ as $v \to \infty$). The Minkowskian counterpart of this solution is the metric

$$(1 - v^+v^-)^{-1}(dv^+dv^- + dv^-dv^+)$$

with the asymptotically flat region $\pm v^\pm > 0$ with future horizon $v^- = 0$, $v^+ > 0$ and past horizon $v^+ = 0$, $v^- < 0$, another such region for $v^+ \leftrightarrow v^-$, and future and past singularities at $v^+v^- = 1$.

4.2 Toroidal partition function

As it stands, functional integral (3) for the black hole target is difficult to compute directly. Instead, we may go back to expression (2) and integrate first over $hh^\perp$ and then over $A$. Let us illustrate this on the example of the twisted toroidal partition function

$$Z^{bh}(\tau, U) = \int e^{\kappa S(\gamma Uhh^\perp\gamma^\dagger U, (2i)^{-1}A)} D(hh^\perp) DA$$

where the action for the twisted field configurations is coupled to the gauge field by putting

$$S(\gamma Uhh^\perp\gamma^\dagger U, (2i)^{-1}A) = S(hh^\perp, (2i)^{-1}A) = S(Uhh^\perp, \frac{1}{2i} \epsilon^{-1}Ud\nu + \frac{1}{2} \epsilon^{-1}Ud\bar{\nu})$$

$$+ \frac{1}{2\pi \tau_2} U_1 \int (\zeta_1 + \zeta_2) d^2\zeta + \frac{\pi}{\tau_2} U_1^2.$$  

(7)

The parametrization of $A$ by the Hodge decomposition

$$A = d\mu + \ast d\nu + \tau_2^{-1}(\bar{u}d\nu + u\bar{d}\nu)/2$$

(8)

$(\mu, \nu$ real functions, $u = u_1 + i u_2)$ gives for the volumes

$$DA = C\tau_2^{-2} \det'(\overline{\partial}^* \overline{\partial}) \delta(\mu(\xi_0)) \delta(\nu(\xi_0)) d^2u \ D\mu \ D\nu.$$ 

Due to the gauge invariance of the action, the integral over $\mu$ factors out as the (infinite) volume of the gauge group. The $\nu$-integral also factors out after unitary rotation $\nu \mapsto e^{-iu}\nu$ so that the $\nu$- and $\phi$-integrals produce the twisted partition function $Z^{H^+_3}(\tau, u)$ of the $H^+_3$ WZW theory. As the result, we obtain

$$Z^{bh}(\tau, U) = C\tau_2^{-2} \det'(\overline{\partial}^* \overline{\partial}) \int e^{-\pi^{-1}\kappa \int (\partial_\mu\nu)(\partial_\nu\mu) d^2z \ - \pi \kappa \tau_2^{-1}(U_1-u_1)^2}$$

$$\cdot Z^{H^+_3}(\tau, u) d^2u \ \delta(\nu(\xi_0)) \ D\nu.$$ 

(9)

The $\nu$-integral is straightforward and for $Z^{H^+_3}(\tau, u)$ we have expression (3.9). Hence

$$Z^{bh}(\tau, U) = C\tau_2^{-1/2} \int e^{-\pi \kappa \tau_2^{-1}(U_1-u_1)^2} Z(\tau, u) |\eta(\tau)|^2 d^2u$$

$$= C\tau_2^{-1} \bar{q}^{-1/12} \int e^{-\pi \kappa \tau_2^{-1}(U_1-u_1)^2 - \pi(\kappa-2)\tau_2^{-1}u_2^2} |\sin(\pi u)|^{-2}$$

$$\cdot \prod_{n=1}^\infty (1 - e^{2\pi i u q^n})(1 - e^{-2\pi i u q^n})^{-2} d^2u$$

(10)
where \( \eta(\tau) \equiv q^{1/24} \prod_{n \geq 1} (1 - q^n) \) is the Dedekind function. The \( u \)-integral diverges logarithmically due to the singularity \( \sim |u|^{-2} \) at zero. This singularity is repeated on the lattice \( \mathbb{Z} + \tau \mathbb{Z} \), as follows immediately from the bi-periodicity of expression
\[
e^{2\pi u^2} |\sin(\pi u)|^{-2} \prod_{n=1}^\infty (1 - e^{2\pi i u}) (1 - e^{-2\pi i u})^{-1} \cdot
\]
Let us explain this divergence of a relatively simple nature.

### 4.3 Mini-space partition function

It is instructive to start with the mini-space case (we remind that this means taking fields \( h h^{\dagger} \) and \( A_x, A_x \) independent of the space variable). For
\[
Z_{\min}^b(h, U) = \int e^{\kappa S_{\min}(\gamma u h h^{\dagger} \gamma u^\dagger, (2i)^{-1}A) D(h h^{\dagger}) DA} ,
\]
we may also proceed as before integrating first over \( A \) to get the twisted partition function for the quantum-mechanical particle moving on the euclidean black hole:
\[
Z_{\min}^b(\tau, U) = C \int e^{-(\kappa/2) \int_0^{2\pi \tau_2} (1 + |u|^2)^{-1} |(\partial - i \tau_2^{-1} U_1)v|^2 dl} \prod_\xi d^2 v(\xi) \prod_\xi d^2 u .
\]
On the other hand, integrating first over \( h h^{\dagger} \) and then over \( A \), we obtain:
\[
Z_{\min}^b(\tau, U) = C \tau_2^{-1} \int e^{-\kappa \tau_2^{-1}(U_1 - u_1)^2 + u_2^2} \sin(\pi u)|^{-2} \prod_\xi d^2 u .
\]
The right hand side of eq. (13) may be rewritten, with the use of eqs. (3.18)-(3.20), as
\[
C \tau_2^{-1/2} \int \text{tr}_{\text{ren}} (e^{4\pi \tau_2 \kappa^{-1}(\Delta + 1/4)} e^{2\pi i (u J^3 - U)} e^{-\kappa \tau_2^{-1}(U_1 - u_1)^2}) \prod_\xi d^2 u \prod_\xi d^2 v.
\]
Notice that
\[
\int e^{i \Delta}(2\pi u_2, v; 0, v') du_2 = \frac{1}{2\pi} e^{i \Delta_{\omega = 0}}(v; v')
\]
where \( \Delta_{\omega = 0} \) is the restriction of Laplacian \( \Delta \) to the generalized eigensubspace of operator \( i(J^3 + J^3) = -i\partial_\phi \) corresponding to eigenvalue 0. From the expression (3.12) for \( \Delta \), we infer that
\[
\Delta_{\omega = 0} = (1 + |v|^2) \partial_v \partial_v + \frac{1}{4} (v \partial_v - \bar{v} \partial_v)^2 + \frac{1}{2} (v \partial_v + \bar{v} \partial_v)
\]
and is a selfadjoint operator in \( L^2(d^2 v) \). Moreover,
\[
(\kappa/\tau_2)^{1/2} \int e^{4\pi \tau_2 \kappa^{-1} \Delta_{\omega = 0}}(e^{-2\pi i u_1} v; v') e^{-\kappa \tau_2^{-1} (U_1 - u_1)^2} du_1
\]
\[
= (\kappa/\tau_2)^{1/2} \int e^{4\pi \tau_2 \kappa^{-1} \Delta_{\omega = 0} + 2\pi i u_1 (J^3 - J^3)}(v; v') e^{-\kappa \tau_2^{-1} (U_1 - u_1)^2} du_1
\]
\[
= e^{4\pi \tau_2 \kappa^{-1} \Delta_{\omega = 0} - \pi \tau_2 \kappa^{-1} (J^3 - J^3)^2 + 2\pi i U_1 (J^3 - J^3)}(v; v') = e^{4\pi \tau_2 \kappa^{-1} \Delta^b}(e^{-2\pi i U_1} v; v')
\]
where we have introduced
\[- \Delta_\omega=0 + (J_3^3)^2 = -\Delta_\omega=0 + (\bar{J}^3)^2 \]
\[-\frac{1}{2} \partial_v(1 + |v|^2)\partial_v - \frac{1}{2} \partial_v(1 + |v|^2)\partial_v \equiv -\Delta^\text{bh}. \quad (18)\]

It is a Laplacian quantizing the classical Hamiltonian \( p_v p_v (1 + |v|^2) \) of the particle on the (euclidean) black hole, with a specific choice of ordering prescription (different from the Laplace-Beltrami operator which would correspond to \((1 + |v|^2)^{1/2} \partial_v (1 + |v|^2)^{1/2}\)).

We may finally rewrite the mini-space partition function as
\[ Z^\text{bh}_{\text{mini}}(\tau, U) = C \int e^{4\pi\tau_2 \kappa^{-1}(\Delta^\text{bh} + 1/4)} (e^{-2\pi i U_1 v}; v) \, d^2 v. \quad (19)\]

The integral is divergent but the nature of this divergence is quite simple. For \( v \to \infty \), where the metric becomes cylindrical in variable \( \log v \), \( \exp[t \Delta^\text{bh}(e^{-2\pi i U_1 v}; v)|v|^2] \) approaches a constant (equal to the free heat kernel between the points on the cylinder of constant difference). Hence the divergence due to the infinite volume of the black hole cigar. It may be easily regularized by cutting integral over \( v \) to \( |v| \leq R \). Going back to integral \((3.8)\), it is easy to see that such cutoff results in the replacement
\[ e^{-\pi \kappa \tau_2^{-1} u_2^2} \longrightarrow e^{-\pi \kappa \tau_2^{-1} u_2^2} - e^{-(4\pi \tau_2)^{-1} \kappa d^2_R} \quad (20)\]
in the integrand of \((13)\). Here \( d_R = \cosh^{-1}(\cosh(2\pi u_2) + 2R^2 \sinh(\pi u_2)^2) \) stands for the hyperbolic distance between \( e^{-\pi i U \sigma^3} h h^t e^{\pi i U \sigma^3} \) and \( h h^t = \left( \begin{array}{cc} (1 + R^2)^{1/2} & R \\ R & (1 + R^2)^{1/2} \end{array} \right) \).

Such a replacement makes the integral in \((13)\) convergent but behaving as \( O(\log R) \) (or more generally as \( O(\log MR) \) where \( M \) is the black hole mass; we consider here only the case \( M = 1 \)). We could define the finite part of \( Z^\text{bh}_{\text{mini}} \) by subtracting this logarithmic divergence, i.e. by comparing it to half the partition function of a particle on the cylinder.

Let us go back to the interpretation of the result \((10)\). As compared to expression \((13)\) for the mini-space case, the main differences in \((10)\) are the partial shift \( \kappa \mapsto \kappa - 2 \) and the presence of the big product inherited from the oscillatory modes of the \( H^+_3 \) theory. The shift of \( \kappa \) is easy: if we drop the infinite product from the right hand side of \((10)\) to get the level zero (i.e. zero mode) contribution, we obtain, proceeding as for the mini-space case,
\[ Z^\text{bh}_{\text{level}}(\tau, U) = C q q^{-(c - \kappa - 1)/24} \quad \text{tr}_{\omega=0} e^{4\pi \tau_2 (\kappa - 2)^{-1} \Delta - 2\pi \tau_2 \kappa^{-1} (J_3^3)^2 + (\bar{J}^3)^2 + 2\pi i U_1 (J_3^3 - \bar{J}^3)} \]
\[ = C q q^{-(c - \kappa - 1)/24} \text{tr}_{\text{level } 0} \cdot q^{\bar{L}_0^\text{cs}} \bar{q}^{L_0^\text{cs}} e^{2\pi i (U | \bar{J}_0^3 - U J_0^3)} \quad (21)\]

with the coset Virasoro generators
\[ L_0^\text{cs} = L_0 + \frac{1}{\kappa} \sum_{n} : J_3^3 J_3^{-n} : , \quad \bar{L}_0^\text{cs} = \bar{L}_0 + \frac{1}{\kappa} \sum_{n} : \bar{J}_3^3 \bar{J}_3^{-n} : \]
\[ \quad \text{with the coset Virasoro generators} \]
\[ L_0^\text{cs} = L_0 + \frac{1}{\kappa} \sum_{n} : J_3^3 J_3^{-n} : , \quad \bar{L}_0^\text{cs} = \bar{L}_0 + \frac{1}{\kappa} \sum_{n} : \bar{J}_3^3 \bar{J}_3^{-n} : . \quad (22)\]

The contribution of the higher level oscillatory modes is, however, less transparent than one may naively think if we want to interpret it in terms of gauge invariant states.
4.4 Asymmetric parafermions

Let us compare the situation to a somewhat similar case of a variant of rational parafermionic theory which may be described as the $SU(2)$ WZW model with the axial gauging of the $U(1)$ subgroup, i.e. with the diagonal $U(1)$ gauged asymmetrically. The twisted toroidal partition function for such parafermions is \[40, 14\]

$$Z^{pf}(U, \tau) = \int e^{-kS(g_1, g_2^\dagger U_1, A)} Dg DA. \quad (23)$$

The integration is now over real $A$. Parametrizing $A$ as before by the Hodge decomposition, one arrives at the formula

$$Z^{pf}(U, \tau) = C_2^{-1/2} \int_{C/(\mathbb{Z} + \tau \mathbb{Z})} e^{i\pi k r_2^{-1}(U_1 - iu_2)^2} Z^{SU(2)}(\tau, u) |\eta(\tau)|^2 d^2u \quad (24)$$

where $Z^{SU(2)}(\tau, u)$ is the asymmetrically twisted partition function of the rational $SU(2)$ WZW model:

$$Z^{SU(2)}(\tau, u) = q^{-ck/24} \text{tr} q^{L_0} \bar{q}^{\bar{L}_0} e^{2\pi i (u_3 J_0^3 + \bar{u}_3 J_0^3)} \quad (25)$$

The trace is taken over the space of states

$$\hat{\mathcal{H}}^{SU(2)} = \bigoplus_{j \leq k/2} \hat{\mathcal{H}}_j \otimes \overline{\check{\mathcal{H}}}_j \quad (26)$$

where $\hat{\mathcal{H}}_j$ carries the irreducible spin $j$ level $k$ representation of the Kac-Moody algebra $\hat{sl}(2, \mathbb{C})$. Notice the sign in front of $\bar{u}_3 J_0^3$ in (25). The integrand on the right hand side of eq. (24) is a function on $C/(\mathbb{Z} + \tau \mathbb{Z})$ only if $U_1 \in k^{-1}\mathbb{Z}$ and only such twists should be allowed. For other twists there is a global gauge anomaly: the ungauged global $U(1)$ symmetry is broken in the parafermionic theory to $\mathbb{Z}_k$. The spaces $\mathcal{H}_j$ may be decomposed into the weight spaces according to the integral or half-integral eigenvalue $m$ of $J_0^3$ and at the same time with respect to the level $k$ representations of the $\hat{U}(1)$ affine algebra (similarly for the complex conjugates):

$$\mathcal{H}_j \cong \bigoplus_m \hat{\mathcal{H}}^{sing}_{j,m} \otimes \check{\mathcal{H}}'_m \quad (27)$$

where $\hat{\mathcal{H}}^{sing}_{j,m}$ is the subspace of $\hat{\mathcal{H}}_j$ where $J^3_0 = m$ and $J^3_n = 0$ for $n > 0$. $\check{\mathcal{H}}'_m$ is the space of the level $k$ $J^3_0 = m$ irreducible representation of the $\hat{U}(1)$ algebra. The Sugawara Virasoro generator $L_0$ decomposes into the sum of $L^c_0 \equiv L_0 - \frac{i}{2} \sum_n J^3_n J^{-3}_n$, acting on spaces $\hat{\mathcal{H}}^{sing}_{j,m}$ and $L'_0 \equiv \frac{i}{2} \sum_n J^3_n J^{-3}_n$, acting on $\check{\mathcal{H}}'_m$ (in fact on $\hat{\mathcal{H}}^{sing}_{j,m}$, $L^c_0 = L_0 - \frac{i}{2} m^2$).

Accordingly, we obtain for the partition function of the $SU(2)$ WZW theory:

$$Z^{SU(2)}(\tau, u) = (qq)^{-(ck-1)/24} \sum_{m_l, m_r} Z^{sing}_{m_l, m_r} q^{m_l^2/k} \bar{q}^{m_r^2/k} |\eta(\tau)|^{-2} e^{2\pi i (u m_l + \bar{u} m_r)} \quad (28)$$

where

$$Z^{sing}_{m_l, m_r} = \text{tr}|\hat{\mathcal{H}}^{sing}_{m_l, m_r} q^{L^c_0} \bar{q}^{\bar{L}^c_0} \quad (29)$$
with

\[ \hat{\mathcal{H}}_{m_l, m_r}^{\text{sing}} \equiv \bigoplus_j \hat{\mathcal{H}}_{j, m_l}^{\text{sing}} \otimes \hat{\mathcal{H}}_{j, m_r}^{\text{sing}}. \]  

(30)

\[ Z_{m_l, m_r}^{\text{sing}} \] depends only on \( m_l \) and \( m_r \) mod \( k/2 \) [40] (essentially due to the compact nature of the gauged symmetry). More exactly,

\[ \text{tr} |_{\hat{\mathcal{H}}_{j, m}^{\text{sing}}} q^{L_0^a} = \text{tr} |_{\hat{\mathcal{H}}_{j, m+k}^{\text{sing}}} q^{L_0^a} = \text{tr} |_{\hat{\mathcal{H}}_{j, -m}^{\text{sing}}} q^{L_0^a}, \]

see [40]. Upon the insertion of (28) into the right hand side of (24), the \( u_1 \)-integral will enforce equality \( m_l = -m_r \equiv m \). The sum over \( m \) may be reduced mod \( k/2 \), with the sum over the integral part of \( 2m/k \) used to extend the integration over \( u_2 \) to a gaussian one over the entire real line. Finally we get

\[ Z_{\text{pf}}(\tau, U) = C \bar{q} q^{- (c_k - 1)/24} \sum_{m=0, \frac{k}{2}, ..., \frac{k}{2}} Z_{m_l, m_r}^{\text{sing}} e^{-4\pi i m U_1}. \]

(31)

As we see, the parafermionic partition function is consistent (modulo multiplicity) with the space of states of the coset theory obtained by imposing the gauge conditions

\[ J^3_0 + \bar{J}^3_0 = 0, \quad J^3_n = \bar{J}^3_n = 0 \quad \text{for} \quad n > 0 \]

(32)
in the space of states of the ungauged WZW theory with the Virasoro algebra given by the coset construction. On the other hand, we could replace the first gauge condition by \( J^3_0 + J^3 = kn \) for \( n \in \mathbb{Z} \) or by \( J^3 = -\bar{J}^3 \) and obtain equivalent theory. The latter means that the asymmetric parafermions are indistinguishable from the symmetric ones.

4.5 Space of states

The level zero contribution (21) to the black hole partition function is fully consistent with the gauge conditions (32) imposed on states of the \( H^+_3 \) WZW theory (for zero modes, only the first condition of (32) restricts the states). The problem appears on the excited levels of the space of states \( \hat{\mathcal{H}}_{H^+_3} \) of the ungauged theory. Let us consider, as an example, the first excited level with states of the form

\[ \sum_{a=\pm, 3} (J^a_{-1} \psi_a + \bar{J}^a_{-1} \bar{\psi}_a) \]

(33)

where \( \psi_a, \bar{\psi}_a \) are level zero states, i.e. functions on \( H^+_3 \). The \( J^3_0 + \bar{J}^3_0 = 0 \) condition translates into

\[ (J^3_0 + \bar{J}^3_0 \pm 1) \psi_\pm = 0, \quad (J^3_0 + \bar{J}^3_0 \pm 1) \bar{\psi}_\pm = 0, \quad (J^3_0 + \bar{J}^3_0) \psi_3 = 0. \]

(34)

The other conditions of (32) give

\[ \psi_3 = \frac{2}{\kappa} (J^+_0 \psi_+ - J^-_0 \psi_-), \quad \bar{\psi}_3 = \frac{2}{\kappa} (\bar{J}^+_0 \bar{\psi}_+ - \bar{J}^-_0 \bar{\psi}_-). \]

(35)
Notice, however, that in $L^2(H^+_0)$, $J_0^3 + J_0^3$ is antihermitian so it has imaginary spectrum. Thus non-trivial solutions of (34) and (35) are not only out of $L^2(H^+_0)$ but do not belong to the generalized eigenspaces of $J_0^3$, $J_0^3$ (they have $e^\pm \phi$ dependence on $\phi$). At best, we have to change the Hilbert space. Notice how the situation here differs from the case of parafermions where no such problems arise. We may understand the above difficulty also by looking at the level one contribution to the partition function (10) which involves integrals

$$
\tau_2^{-1} \int e^{-\pi \kappa \tau_2^{-1}(U_1-u_1)^2-\pi (\kappa-2) \tau_2^{-1} u_2^2} \left| \sin(\pi u) \right|^{-2} e^{\pm 2 \pi i u} d^2 u
$$

$$
= C \tau_2^{-1/2} \int e^{4 \pi \tau_2 (\kappa-2)^{-1} (\Delta+1/4)} (2\pi u_2, e^{-2\pi i u_1} v; 0, v) \cdot e^{\pm 2 \pi (i u_1-u_2)} e^{-\pi \kappa \tau_2^{-1} (U_1-u_1)^2} d^2 u d^2 v.
$$

(36)

By spectral analysis, we may decompose operators $e^{t \Delta}$ into the heat kernels acting in the generalized eigenspaces of $J_0^3$, $J_0^3$:

$$
e^{t(\Delta+1/4)} (2\pi u_2, e^{-2\pi i u_1} v'; 0, v) = \sum_n \int K_{n,\omega}(t; |v'|, |v|) e^{2\pi i u_1 - 2\pi i u_2} d\omega.
$$

(37)

This allows to rewrite integrals (36) as

$$
C \sum_n K_{n,\tau_2}(4\pi \tau_2 \kappa^{-1}; |v|, |v|) e^{-\pi \tau_2 \kappa^{-1} (n+1)^2 + 2\pi i (n+1) U_1} d|v|^2
$$

involving the analytic continuation of heat kernels $K_{n,\omega}$ to imaginary values of $\omega$. The question is whether such an analytic continuation (which exists) corresponds to a heat kernel in a different Hilbert space.

Summarizing the gauge conditions (32) do not determine unambiguously the space of states. We have to supplement them with regularity conditions specifying domains of the operators that they involve (the same applies to the BRST definition of gauge invariant states). Ultimately, we should be able to build a Hilbert space of states at each level and to compute the contribution to the partition function as a trace of a heat kernel in such a space. We shall discuss a candidate solution of this problem in Sec. 5.

On top of the above difficulties with the interpretation of the partition function (but not unrelated to them) comes the fact that, as it stands, the integral on the right hand side of eq. (10) diverges. The source of this divergence is, as in the mini-space approximation, the infinite volume of the target space. This may be regularized for example by defining

$$
\tilde{Z}_{\text{reg}}^{\text{hh}}(\tau, U; R) = C \tau_2^{-1} \int e^{-\pi \kappa \tau_2^{-1} |U-u|^2} S(\tau, u) \left( 1 - e^{R^2 S(\tau, u)^{-1}} \right) d^2 u
$$

(39)

where

$$
S(\tau, u) \equiv q^{-1/12} e^{2\pi \tau_2^{-1} u_2^2} |\sin(\pi u)|^{-2} \prod_{n=1}^\infty (1 - e^{2\pi i u^n q^n}) (1 - e^{-2\pi i u^n q^n})^{-2}.
$$

(40)
The partition function $\tilde{Z}_{\text{bh}}(\tau, U)$ is finite and when $R \to \infty$ and for $U_2 = 0$, we recover the infinite integral (10) (we have put the twists along both homology cycles in $\tilde{Z}_{\text{bh}}(\tau, U)$ so that in the limit $R \to \infty$ it corresponds to the black hole functional integral with boundary conditions $v(z + 2\pi) = e^{-2\pi i \Phi} v(z)$, $v(z + 2\pi \tau) = e^{-2\pi i \Theta} v(z)$ where $U = \Theta - \tau \Phi$; for $\Phi = 0$, we recover $Z_{\text{bh}}(\tau, U)$ with twist only along one cycle). $S(\tau, U)$ is invariant under translations $U \mapsto U + n + \tau m$ for $n, m$ integers and is modular invariant. As a result, under $SL(2, \mathbb{Z})$ transformations, 

$$\tilde{Z}_{\text{bh}}(\frac{a\tau + b}{c\tau + d}; R) = \tilde{Z}_{\text{bh}}(\tau; R) ,$$  

i.e. the regularized partition function is modular covariant. Again the divergence is logarithmic in $R$ and we could subtract it to define the renormalized partition function measuring the difference between the theories with the black hole and (half-)cylinder targets.

4.6 Partition functions at higher genera

On a higher genus Riemann surface $\Sigma$ with the homology basis $(a_\alpha, b_\beta)$, $\alpha, \beta = 1, \ldots, \text{genus}$, and with the basic holomorphic forms $\omega^\alpha$, $\int_{a_\alpha} \omega^\beta = \delta^{\alpha\beta}$, $\int_{b_\alpha} \omega^\beta = \tau^{\alpha\beta}$ $(\equiv \tau_1^{\alpha\beta} + i \tau_2^{\alpha\beta})$, let us define the multivalued field

$$\tilde{\gamma}_U(P) = e^{\frac{\pi \sigma^3}{\rho_0} \int_{U \tau_2^{-1}}^P (U \tau_1^{-1} \bar{\omega} - U \tau_2^{-1} \omega)/2}$$  

(42)

with values in the Cartan subgroup of $SU(2)$. Along the basic cycles

$$\tilde{\gamma}_U(a_\alpha P) = e^{-\pi i \Phi \sigma^3} \tilde{\gamma}_U(P) ,$$  

$$\tilde{\gamma}_U(b_\alpha P) = e^{-\pi i \Theta \sigma^3} \tilde{\gamma}_U(P)$$

where $U = \Theta - \tau \Phi$. The twisted partition function on $\Sigma$ is given by

$$\tilde{Z}^{\text{bh}}(\tau, U) = \int e^{\kappa S(\tilde{\gamma}_U, (2i)^{-1}A)} D(hh^\dagger) \ D A$$  

(43)

with

$$S(\tilde{\gamma}_U, hh^\dagger \tilde{\gamma}_U^\dagger, (2i)^{-1}A) = S(hh^\dagger, \frac{1}{2i} (A + \pi U^t \tau_2^{-1} \omega + \pi U^t \tau_2^{-1} \bar{\omega}))$$  

$$+ \frac{1}{2i} \int A \wedge (U^t \tau_2^{-1} \omega - U^t \tau_2^{-1} \bar{\omega}) + \pi U^t \tau_2^{-1} U .$$  

(44)

It defines the higher genus partition function for the black hole with twists of the $v$-field by $e^{-2\pi i \Phi}$ along the $a_\alpha$ cycles and by $e^{-2\pi i \Theta}$ along the $b_\beta$ ones. We decompose again the gauge field according to Hodge:

$$A = d\mu + \ast d\nu + \pi \bar{u}^t \tau_2^{-1} \omega + \pi u^t \tau_2^{-1} \bar{\omega}$$  

(45)
and integrate over the $v$-field (of $hh^\dagger$), $\nu$ and $\mu$ (the latter integral gives the volume of the gauge group). What is left is the $\phi$ functional integral and the integral over twists $u$:

$$
\tilde{Z}^{hh}(\tau, U) = C \left( \frac{\text{det}'(-\delta \partial)}{\text{area}} \right)^{1/2} \int e^{-\pi\kappa(U-\bar{u})^2(\bar{v}-u)+(2\pi i)^{-1}\int (\partial\bar{\phi})(\partial\phi)} \cdot \det \left( \partial + \bar{\phi} + \pi u^i \tau_2^{-1} \omega \right)^{-1} \delta(\phi(\xi_0)) \, D\phi \, du^{2 \text{ genus}}. \quad (46)
$$

By the chiral anomaly (compare the genus one formula (3.8)),

$$
\det \left( \partial + \bar{\phi} + \pi u^i \tau_2^{-1} \omega \right)^{-1} = e^{i\pi \int (\partial\bar{\phi})(\partial\phi) + (2\pi i)^{-1} \int \phi R} \cdot \left( \text{det}_{\alpha,\beta}(\int e^{2\phi \eta_{\alpha\beta}} / \text{det}_{\alpha,\beta}(\int \eta_{\alpha\beta})) \right)^{-1} \det \left( \partial_u^* \partial_u \right)^{-1} \quad (47)
$$

where $\partial_u \equiv \partial + \pi u^i \tau_2^{-1} \omega$ and $\eta_{\alpha\beta}, \alpha = 1, ..., \text{genus} - 1$, form a basis of the 01-forms in the kernel of $\partial_u^*$. Using eq. (47), we may rewrite the partition function as

$$
\tilde{Z}^{hh}(\tau, U) = C \left( \frac{\text{det}'(-\delta \partial)}{\text{area}} \right)^{1/2} \int e^{-\pi\kappa(U-\bar{u})^2(\bar{v}-u)+(2\pi i)^{-1}(\kappa-2)\int (\partial\bar{\phi})(\partial\phi) + (2\pi i)^{-1} \int \phi R} \cdot e^{-\int \eta_u \exp(2\phi) \eta_u} \text{det} \left( \partial_u^* \partial_u \right)^{-1} \delta(\phi(\xi_0)) \, D\phi \, d\eta_u \, du \quad (48)
$$

where the gaussian integral over $\eta_u \in \ker \partial_u^*$ was used to express the $\eta_{\alpha\beta}$ determinants. The expression is obviously similar to the Liouville partition function although the real relation between two theories lies probably deeper. In any way, we expect the $\phi$ and $\eta$ integrals to be finite and to lead to an expression regular in $u$ except for the contribution of $\det \left( \partial_u^* \partial_u \right)^{-1}$ which around $u = 0$ behaves as $|u|^{-2}$ which is integrable for genus $> 1$ and diverges logarithmically for genus 1 ($\eta_{\alpha\beta}$ may be chosen regular in $u$ around $u = 0$). This singularity is repeated around other points of $\mathbf{Z} + \tau \mathbf{Z}$. Thus, similarly as for the Liouville theory coupled to free bosonic field, see [11], [12], we expect the partition functions at higher genera to be convergent reflecting the finite dimension of the region in the target space relevant for the stringy interaction.

### 4.7 Green functions

Since the coset theory is an instance of a gauge theory, its Green functions should be given by functional integral with gauge invariant insertions. Examples of gauge invariant fields are $f_{\rho, m_1, m_2}(v(\xi))$ of eq. (3.33) with $m_1 = -m_r \equiv m$ whose conformal weights are

$$
\Delta_{\rho, m} = \tilde{\Delta}_{\rho, m} = \frac{1+\rho^2}{4(\kappa-2)} + \frac{m^2}{\kappa}. \quad (49)
$$

If we instead used $f_{\rho, m_1, m_2}(\phi(\xi), v(\xi))$ with $m_1 + m_r \neq 0$ as local fields, we could still maintain local gauge invariance by adding compensating currents, i.e. by considering insertions

$$
I(hh^\dagger, \frac{1}{2i} A) = \prod_{\alpha} f_{\rho_\alpha, m_\alpha, m_\alpha}(\phi(\xi_\alpha), v(\xi_\alpha)) \, e^{-\int \frac{A}{c}}, \quad (50)
$$

$$
\prod_{\alpha} f_{\rho_\alpha, m_\alpha, m_\alpha}(\phi(\xi_\alpha), v(\xi_\alpha)) \quad (51)
$$
where $c$ is a chain such that $\delta c = \sum_{\alpha}(m_{l\alpha} + m_{r\alpha})\xi_{\alpha}$. In the planar case, the functional integral over the gauge field may be easily done upon parametrization $A = d\mu + *d\nu$. The integral over $\mu$ drops out because of gauge invariance and the integral over $\nu$ gives expectation value of chiral vertex operators

$$
\int e^{i \int \partial_{\nu} - i \int \partial_{\nu} - \pi^{-1} \kappa (\partial_{\nu})(\partial_{\nu}) d^2 z}
$$

(51)

where $\delta c' = \sum_{\alpha}(m_{l\alpha} - m_{r\alpha})\xi_{\alpha}$ (compare [14] where similar calculation was done for the parafermions). Altogether, we obtain

$$
\int I(hh^\dagger, \frac{1}{2i} A) e^{\kappa S(hh^\dagger, (2i)^{-1} A)} D(hh^\dagger) D\nu
= \text{const.} \prod_{\alpha \neq \alpha'} (\xi_{\alpha} - \xi_{\alpha'})^{m_{l\alpha}m_{l\alpha'}/\kappa} (\bar{\xi}_{\alpha} - \bar{\xi}_{\alpha'})^{m_{r\alpha}m_{r\alpha'}/\kappa} \int I(hh^\dagger, 0) e^{\kappa S(hh^\dagger)} D(hh^\dagger)
$$

(52)

where the factors come from the (properly renormalized) free field integral (51). They modify the conformal dimensions of fields $f_{\rho, m_{l}, m_{r}}$ of the $H_3^+$ WZW theory to

$$
\Delta_{\rho, m_{l}} = \frac{1 + \rho^2}{4(\kappa - 2)} + \frac{m_{l}^2}{\kappa}, \quad \bar{\Delta}_{\rho, m_{r}} = \frac{1 + \rho^2}{4(\kappa - 2)} + \frac{m_{r}^2}{\kappa}.
$$

(53)

producing operators with imaginary spin and hence never local. It is possible, however, that correlations of fields coming from common eigenfunctions on $H_3^+$ of $\Delta, J^3, \bar{J}^3$ which do not correspond to the spectrum, for example for $\omega$ imaginary, may be given sense. If in the left hand side of (52) we integrated out the $A$-field, we would obtain the black hole functional integral with insertions which for large values of $|v(\xi_{\alpha})|$ take form

$$
\prod_{\alpha} \left( |v(\xi_{\alpha})|^{m_{l\alpha} + m_{r\alpha}} f_{\rho_{\alpha}, m_{l\alpha}, m_{r\alpha}}(0, |v(\xi_{\alpha})|) \right) e^{-i \int \partial arg(v) + i \int \bar{\partial} arg(v)}
$$

(54)

We recover then the chiral vertex operators of field $arg(v)(\xi)$ which, for large $|v|$, becomes a compactified free field. If fields with real $m_{l} + m_{r}$ existed, they would be mutually local for $m_{l} + m_{r} \in \kappa\mathbb{Z}$, as are their asymptotic versions. We shall return to the discussion of this possibility in the next section.

5. $SU(1, 1) \mod U(1)$ COSET THEORY

5.1 Functional integral formulation

The original proposal [18] for the conformal sigma model with 2D black hole target was based on a coset construction starting with $SU(1, 1) \cong SL(2, \mathbb{R})$ WZW model. The parametrization

$$
g = \begin{pmatrix}
e^{i\psi}(1 + |v|^2)^{1/2} & v \\
v & e^{-i\psi}(1 + |v|^2)^{1/2}
\end{pmatrix}
$$

(1)
where \( \psi \) is in \( \mathbb{R}/(2\pi \mathbb{Z}) \) and \( v \) is complex gives global coordinates on \( SU(1, 1) \). Comparing to parametrization (3.1) of positive elements in \( SL(2, \mathbb{C}) \), we see that it passes to the present one by simple substitution \( \phi \to i\psi \). Consequently, for the WZW action with the \( U(1) \subset SU(1, 1) \) gauged asymmetrically (i.e. with the axial \( U(1) \) gauge), we obtain from eq. (3.4)

\[
S(g, \frac{1}{2i}A) = -\frac{1}{\pi} \int \left[ (i\partial_z \tilde{\psi} + A_z)(i\partial_{\bar{z}} \tilde{\psi} + A_{\bar{z}}) + (\partial_z + i\partial_{\bar{z}} \tilde{\psi} + A_z)\bar{v} (\partial_{\bar{z}} + i\partial_z \tilde{\psi} + A_{\bar{z}})v \right] d^2z.
\]

where \( \tilde{\psi} \equiv \psi + \frac{i}{2}\log(1 + |v|^2) \). The axial gauge invariance is

\[
S(e^{i\lambda \sigma^3} g e^{i\lambda \sigma^3}, \frac{1}{2i}(A - 2id\lambda)).
\]

The euclidean action \( \pm \kappa S(g) \) for the \( SU(1, 1) \) WZW theory is not bounded below. For the minus sign (and \( \kappa \) positive) this is due to the \( \tilde{\psi} \)-field contribution. As a result, the stable euclidean picture is missing for this theory. In the coset functional integral

\[
\int - e^{\kappa S(g,(2i)^{-1}A)} Dg DA,
\]

however, the \( \tilde{\psi} \)-field may be gauged out and absorbed by a translation of \( A \). If \( A \) is taken real then the \( A \) integral is stable and the translation of \( A \) is complex (the axial gauge invariance requires imaginary \( A \)). In this case, moreover, after the translation, we recover the same integral as before for the \( SU(2, \mathbb{C})/SU(2) \) mod \( \mathbb{R} \) coset theory. It seems that the two coset theories coincide\(^8\). On the quantum-mechanical level, the equivalence of both approaches may be seen clearly.

5.2 Particle on \( SU(1,1) \)

The classical mini-space system which corresponds to the 2D WZW theory with target \( SU(1,1) \) is the geodesic motion in the invariant metric on \( SU(1,1) \) of signature, say, \((-,-,+,+\)). We may quantize it taking \( L^2(SU(1,1)) \) with the Haar measure (equal \( d\psi d^2v \) in parametrization (1)) as the space of states in which \( SU(1,1) \) acts unitarily. Infinitesimally, we get the action of \( sl(2,\mathbb{C}) \oplus sl(2,\mathbb{C}) \) generated by \( J^a \)'s and \( J^a \)'s given by the same formulae as in the case of \( L^2(H^+_3) \) except for the substitution \( \phi \to i\psi \). The hermiticity relations change, however, and we obtain

\[
J^a \ast = -J^a, \quad J^a \ast = -J^a \quad \text{for} \quad a = 1,2,3.
\]

Also \( -\Delta \equiv -\bar{J}^2 = -\bar{J}^2 \) is no more bounded below. It is again given explicitly by eq. (3.12) with \( \partial_\phi^2 \) replaced by \(-\partial_\psi^2 \). In fact

\[
L^2(SU(1,1)) \cong \int \mathbb{R}^+ e \otimes \mathbb{R}^+ e d\nu(\epsilon,\sigma) \bigoplus_{j=-1,-3/2,...} D_j^+ \otimes D_j^-.
\]

\(^8\)this is the point on which the present author’s opinion has waivered most and might continue to do so with the progress in the understanding of both theories.
\( \mathcal{D}_{\sigma, \epsilon} \) carry unitary irreducible representations of \( SU(1,1) \) of the principal continuous series which may be realized in the space of sections of a spin bundle on the circle \( SU(1,1) \) acts naturally on \( S^1 \), \( \epsilon \) corresponds to two choices of the spin structure. The eigenvalue of \( \vec{J}^2 \) on \( \mathcal{D}_{\rho, \epsilon} \) is \(-\frac{1}{4}(1 + \rho^2) \). Spaces \( \mathcal{D}^\pm_{-j} \) carry the lowest- (highest)-weight representations of \( sl(2, \mathbb{C}) \) of spin \( j \). They give the discrete series of unitary, irreducible representations of \( SU(1,1) \) with eigenvalue of \( \vec{J}^2 \) equal to \( j(j+1) \) which is \( \geq 0 \). If, instead of \( SU(1,1) \), we considered its simply-connected covering \( SU(1,1) \) (where \( \psi \) takes values in the non-compactified real line), the direct sums in decomposition (5) over \( \epsilon \) and \( j \) would be replaced by direct integrals over \( 0 \leq \epsilon < 1 \) and \( j < -1/2 \). Since \( -\Delta \) plays the role of Hamiltonian, the energy is not bounded below (nor above). This problem with stability renders the above quantization physically not very satisfactory. Indeed, the way we proceeded here is not the one used for example to quantize a particle in Minkowski space where one recovers satisfactory solution of the stability problem passing to the second-quantized level. Finding a stable quantization of the particle on \( SU(1,1) \) or, more importantly, of the \( SU(1,1) \) WZW field theory remains an open and seemingly very interesting problem. Here, however, we shall be interested only in the coset \( SU(1,1) \) mod \( U(1) \) theory where coupling to the gauge field removes the unstable \( \psi \) field. On the quantum-mechanical level, the gauge condition \( J^3 + \bar{J}^3 = i\partial_v = 0 \), cuts out from \( L^2(SU(1,1)) \) the contribution of the discrete series (and more) making \( -\Delta \) positive. Besides,

\[
L^2(SU(1,1))|_{J^3, \bar{J}^3 = 0} \cong L^2(d^2v) \cong L^2(H^+_3)|_{J^3, \bar{J}^3 = 0}
\]

in a natural way and this isomorphism preserves (restrictions of) \( \Delta, J^3 \) and \( \bar{J}^3 \). This proves on the mini-space level the identity of the coset theories \( SU(1,1) \) mod \( U(1) \) and \( SL(2, \mathbb{C})/SU(2) \) mod \( \mathbb{R} \). The generalized eigenfunctions \( f_{\rho, m_l, m_r} \) of \( \Delta, J^3, \bar{J}^3 \) on \( SU(1,1) \), corresponding to eigenvalues \(-\frac{1}{4}(1 + \rho^2) \), \( m_l, m_r \) with \( m_l \pm m_r \in \mathbb{Z} \), are given by Jacobi functions. For \( m_l + m_r = 0 \) they are independent of \( \psi \) and, although given by different expressions, coincide with similar eigenfunctions on \( H^+_3 \). For example, from the harmonic analysis on \( SU(1,1) \), we obtain

\[
f_{0,0,0}(v) = \pi \int_0^{2\pi} (1 + 2|v|^2 + 2v(1 + |v|^2)^{1/2}\cos\theta)^{-1/2} d\theta \tag{6}
\]

which should be compared with eq. (3.34). For both \( m_l + m_r \) equal and different from zero, eigenfunctions \( f_{\rho, m_l, m_r} \) seem to generate primary fields of dimensions given by eq. (4.53) (for \( m_l + m_r \neq 0 \), they should be dressed with line integrals of the gauge field, like in (4.50)). If \( m_l + m_r \in \kappa \mathbb{Z} \), the corresponding fields are mutually local.

5.3 Space of states, unitarity, duality, problems

On the level of 2D field theories, neither \( SL(2, \mathbb{C})/SU(2) \) mod \( \mathbb{R} \) nor \( SU(1,1) \) mod \( U(1) \) theory has been shown to exist, least solved completely, so comparison is more

\[\text{we thank G. Gibbons for attracting our attention to it}\]
difficult. The computation of the partition function in the first case did not require complex rotations or shifts of the fields so it seems more trustable. Nevertheless, we have seen that the interpretation of the excited contributions to it required analytic continuation of the heat kernels on the eigensubspaces of $J^3, \bar{J}^3$ in $L^2(H^3_5)$ to imaginary eigenvalues $\omega$ of $(J^3 + \bar{J}^3) = i\partial_\phi$. But this should be given by the heat kernels in the eigenspaces of $J^3, \bar{J}^3$ in $L^2(SU(1,1))$, or more generally in $L^2(SU(\tilde{1},1))$, obtained by the substitution $\phi \mapsto i\psi$. It is then possible that the partition function becomes a trace over the gauge-invariant states of the $SU(1,1)$ WZW theory. Superficially, the $U(1)$ coset of the latter has the same problem as the parafermionic model discussed in Sec. 4.4: the ungauged (vector) $U(1)$ symmetry has global anomaly which seems to break $U(1)$ to $\mathbb{Z}_k$. Here, this is a spurious problem, however: if we start from the $SU(\tilde{1},1)$ WZW theory rather than from the $SU(1,1)$ one, the coset theory is the same but the complete $U(1)$ symmetry is present. The space of states of the $SU(\tilde{1},1)$ WZW theory should be a subspace of

$$
\bigoplus_{\rho > 0, \ 0 \leq \epsilon < 1} \hat{\mathcal{D}}_{\rho, \epsilon} \otimes \hat{\mathcal{D}}_{\rho, \epsilon} \ d\nu(\epsilon, \sigma) \bigoplus \bigoplus_{j < -1/2} \hat{\mathcal{D}}^\pm_j \otimes \hat{\mathcal{D}}^\pm_j
$$

where “$^\pm$” denotes the representation space of the Kac-Moody algebra $\hat{sl}(2, \mathbb{C})$ induced (in the sense of Sec. 3.3) from the representations of $SU(\tilde{1},1)$. What exactly should be the subspace taken does not seem to be clear yet. A possibility is the appearance of the fusion rule $-\frac{1}{2}(\kappa - 1) < j$ in the discrete series, analogous to the rule $j \leq k/2$ of the $SU(2)$ WZW theory. Spaces $\hat{\mathcal{D}}$ may be provided with the hermitian form for which $J_{n}^a = J_{-n}^{-a}$ for $a = 1, 2$ and $J_{n}^3 = J_{-n}^{-3}$ (this agrees at level zero with the scalar product induced from $L^2(SU(1,1))$). The encouraging sign is the important result of Dixon-Lykken-Peskin [12] (see also [13]) who proved that the gauge conditions $J_{n}^3 = 0, \ n > 0$, cut out, under the restriction $-\frac{1}{2}\kappa \leq j$ on the discrete series, the negative norm states from the induced representations. Notice, that the latter condition disposes of the representations with negative eigenvalues of $L_0^a$. Their absence should then be assured by stability if the coincidence with the explicitly stable $H^3_5 \mod \mathbb{R}$ model really takes place. In that case, the $SU(1,1) \mod U(1)$ approach should allow to show the unitarity of the euclidean black hole CFT. Moreover, we should be able to assemble the calculated partition functions from the characters of the induced representations $\hat{\mathcal{D}}$. This is not simple even on the quantum mechanical level where we know that it works.

The gauge condition $J_0^3 + \bar{J}_0^3 = 0$ leaves us with spin-less, $U(1)$-charge zero sector of the theory. The functional integral for the partition function, as in any gauge theory, should be given by the trace over this subspace of states, as is also clearly indicated by the $U$ dependence of the result (4.10). The primary fields $f_{\rho,m_l,m_r}$ with $m_l + m_r = 0$ correspond to vectors in this sector. On the other hand, gauge conditions $J_0^3 + \bar{J}_0^3 = \ell\kappa$ should give for $0 \neq l \in \mathbb{Z}$ sectors with the spin and the $U(1)$ charge different from zero. Fields $f_{\rho,m_l,m_r}$ with $m_l + m_r = \ell\kappa$ should correspond to states in these sectors. From the point of view of the asymptotic free field with the cylindrical part of the cigar as
the target, these are the winding sectors, see formula (4.54). The partition functions corresponding to the winding sectors can be also computed, essentially by inserting a Polyakov line with charge \( l \kappa \) into the functional integral. We plan to return to these issues elsewhere.

Another open problem in the black hole CFT is a relation between the coset models \( SU(1,1) \mod U(1) \) obtained by gauging the axial and the vector \( U(1) \) subgroup. The vector theory has a more serious stability problem than the axial one since the vector gauging does not seem to remove completely the unbounded below modes. On a rather formal level one can argue that both theories have the same spectrum of mutually local operators \([20]-[22],[44]\). It was expected that they give the same CFT. The vector coset results in a sigma model with singular metric on the target. In the asymptotic region, the target also looks like a half-cylinder and the identity of the models would become there that of free fields compactified on dual radia \([45]\). We have not been able, however, to stabilize the functional integral for the vector theory in a sensible way to show that it has the same partition function as the axial coset. The situation should be contrasted with the case of parafermions. There, as we have seen in Sec. 4.4, both gaugings give the same theory, in fact already on the mini-space level. In particular, both partition functions coincide. The duality between the two \( U(1) \) cosets of the \( SU(1,1) \) WZW theory requires, in our opinion, further study. It may be that the vector description may be maintained only in the asymptotically flat region. The issue is important for understanding whether the coupling to dynamical gravity washes out the singularity at \( v^+ v^- = 1 \) of the classical Minkowskian metric (4.5) interchanged by the duality with the non-singular horizon \( v^+ v^- = 0 \), see \([20]\). Even less clear is what sense we can make of the sigma model which Minkowskian 2D black hole as the target which formally comes from gauging non-compact subgroup in \( SU(1,1) \) theory \([18]\) and how all these theories fit together. We clearly touch here on the relation between the stability and unitarity of the CFT’s and the signature of the effective targets. If a progress can be made in understanding such issues fundamental for quantum gravity, the effort invested in studying relatively simple non-rational theories may pay back.

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\(^{10}\)one should not confuse these sectors with the infinite volume superselection sectors obtained by sending some of the charges “behind the moon”; they are rather descendents of the charge sectors corresponding to infinitely heavy external charges - we thank E. Seiler for correcting some of the author’s original misconceptions about this point
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