SOLUTIONS FOR A NONLOCAL ELLIPTIC EQUATION INVOLVING CRITICAL GROWTH AND HARDY POTENTIAL

CHUNHUA WANG, JING YANG, JING ZHOU

Abstract. In this paper, by an approximating argument, we obtain infinitely many solutions for the following Hardy-Sobolev fractional equation with critical growth

\[
\begin{cases}
(-\Delta)^su - \frac{\mu u}{|x|^{2s}} = |u|^{2^*_s-2}u + au, & \text{in } \Omega, \\
u = 0, & \text{on } \partial\Omega,
\end{cases}
\]

provided \(N > 6s, \mu \geq 0, 0 < s < 1, 2^*_s = \frac{2N}{N-2s}, a > 0\) is a constant and \(\Omega\) is an open bounded domain in \(\mathbb{R}^N\) which contains the origin.

Keywords: Hardy-Sobolev fractional equation; Infinitely many solutions; Variational methods.

1. Introduction

Let \(0 < s < 1, N > 6s, 2^*_s = \frac{2N}{N-2s}, \bar{\mu}\) be defined later, and \(\Omega\) be an open bounded domain in \(\mathbb{R}^N\) which contains the origin. We study the following nonlinear fractional problem

\[
\begin{cases}
(-\Delta)^su - \frac{\mu u}{|x|^{2s}} = |u|^{2^*_s-2}u + au, & \text{in } \Omega, \\
u = 0, & \text{on } \partial\Omega,
\end{cases}
\]

where \(\mu \geq 0\) satisfies

\[
\frac{2^*_s \sqrt{\bar{\mu}}}{\sqrt{s} - \sqrt{n - \mu}} > \frac{2N}{N-6s}, a > 0
\]

is a constant, and \((-\Delta)^s\) stands for the fractional Laplacian operator in \(\Omega\) with zero Dirichlet boundary values on \(\partial\Omega\).

To define the fractional Laplacian operator \((-\Delta)^s\) in \(\Omega\), let \(\{\lambda_k, \varphi_k\}\) be the eigenvalues and corresponding eigenfunctions of the Laplacian operator \(-\Delta\) in \(\Omega\) with zero Dirichlet boundary values on \(\partial\Omega\),

\[
\begin{cases}
-\Delta \varphi_k = \lambda_k \varphi_k, & \text{in } \Omega, \\
\varphi_k = 0, & \text{on } \partial\Omega,
\end{cases}
\]

normalized by \(||\varphi_k||_{L^2(\Omega)} = 1\). Then one can define \((-\Delta)^s\) for \(s \in (0, 1)\) by

\[(-\Delta)^s u = \sum_{k=1}^{\infty} \lambda_k^s c_k \varphi_k,\]

which clearly maps

\(H^s_0(\Omega) := \left\{ u = \sum_{k=1}^{\infty} c_k \varphi_k \in L^2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^s c_k^2 < \infty \right\}\)

into \(L^2(\Omega)\). Moreover, \(H^s_0(\Omega)\) is a Hilbert space equipped with an inner product

\[
\left\langle \sum_{k=1}^{\infty} c_k \varphi_k, \sum_{k=1}^{\infty} d_k \varphi_k \right\rangle_{H^s_0(\Omega)} = \sum_{k=1}^{\infty} \lambda_k^s c_k d_k, \quad \text{if } \sum_{k=1}^{\infty} c_k \varphi_k, \sum_{k=1}^{\infty} d_k \varphi_k \in H^s_0(\Omega).
\]

Now we can write the functional corresponding to (1.1) as

\[
I(u) = \frac{1}{2} \int_\Omega \left(|(-\Delta)^s u|^2 - \mu \frac{u^2}{|x|^{2s}} - au^2\right) dx - \frac{1}{2^*_s} \int_\Omega |u|^{2^*_s} dx, \quad u \in H^s_0(\Omega).
\]

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A great deal of work has currently been devoted to the study of the fractional Laplacian operator as it appears in several applications to some models related to anomalous diffusions in plasmas, flames propagation and chemical reactions in liquids, population dynamics, geophysical fluid dynamics, and American options in finance, see, e.g., [11][21][28][29][31] and the references therein. We refer the reader to [8][11][18][24][36] for a nice exposition and [5][9][12][40] for the operator defined via the classical spectral theory and [13][19][38] for the operator defined via the Riesz potential.

In this paper, our interest in problem (1.1) is related to the following Hardy inequality which was proved by Herbst in [30] (see also [7][43]):

\[
\mu \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2s}} \, dx \leq \int_{\mathbb{R}^N} |\xi|^{2s} \hat{u}^2 \, d\xi, \quad \forall u \in C_0^\infty(\mathbb{R}^N),
\]

where \( \hat{u} \) is the Fourier transform of \( u \) and

\[
\mu = 2^{2s} \frac{\Gamma(\frac{N+2s}{4})}{\Gamma(\frac{N-2s}{4})}.
\]

Here \( \Gamma \) is the usual gamma function, the constant \( \mu \) is optimal and converges to the classical Hardy constant \( \frac{(N-2)^2}{4} \) when \( s \to 1 \). Indeed, for \( \alpha \in [0, \frac{N-2s}{2}) \), if we denote

\[
\Upsilon_\alpha = 2^{2s} \frac{\Gamma(\frac{N+2s+2\alpha}{4})\Gamma(\frac{N+2s-2\alpha}{4})}{\Gamma(\frac{N-2s-2\alpha}{4})\Gamma(\frac{N-2s+2\alpha}{4})},
\]

then \( \mu = \Upsilon_0, \Upsilon_\alpha \to 0 \) when \( \alpha \to \frac{N-2s}{2} \) and the mapping \( \alpha \mapsto \Upsilon_\alpha \) is monotone decreasing (see [23]).

Taking into account the behavior of the Fourier transform with respect to the homogeneity, one has (see [20][26][32][35])

\[
\left\| (-\Delta)^s u \right\|_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\xi|^{2s} \hat{u}^2 \, d\xi, \quad \forall u \in C_0^\infty(\mathbb{R}^N),
\]

and then the Hardy inequality (1.3) can be rewritten as

\[
\left\| (-\Delta)^s u \right\|_{L^2(\mathbb{R}^N)} \geq \tilde{\mu} \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2s}} \, dx, \quad \forall u \in C_0^\infty(\mathbb{R}^N).
\]

In a more general setting, beyond of the Hilbertian framework, we can refer the reader to [25][27] where an improved inequality is proved.

Recently, the semilinear fractional elliptic equations involving Hardy potential

\[
(-\Delta)^s u - \frac{\mu u}{|x|^{2s}} = f(u), \quad \text{in } \Omega,
\]

have been widely studied since the operator \((-\Delta)^s - \mu |x|^{-2s}\) appears in the problem of stability of relativistic matter in magnetic fields. In [23], Fall studied (1.5) with \( f(u) = u^p \) and \( \Omega = \mathbb{B} \), and showed that (1.5) possesses a nonnegative distributional solution if \( \mu > 0 \) and \( p > 1 \) satisfying some suitable conditions. Replacing \( f(u) = u^p + |u|^q \), Barrios, Medina and Peral [6] considered (1.5) and discussed the existence and multiplicity of solutions depending on the value \( p \) to (1.5). Particularly, they verified the existence of (1.5) if \( p = p(\mu, s) = \frac{N+2s-2\alpha}{4} \) is the threshold for \( \alpha_s \in (0, \frac{N-2s}{2}) \). Choi and Seok [13] considered problem (1.4) with \( \mu = 0 \). They obtained the existence of infinity many solutions of (1.4) for any \( a > 0 \).

In [16], Cao and Yan also considered problem (1.1) with \( s = 1 \). It was proved that (1.1) has infinitely many solutions if \( N \geq 7 \) and \( 0 \leq \mu < \frac{(N-4)^2}{4} \) with \( a > 0 \) and \( s = 1 \). So motivated by [16][18], the aim of this paper is to study the existence of infinity many solutions of the Hardy-Sobolev fractional equation (1.1). Now we state our main result as follows.
Theorem 1.1. Suppose that $a > 0$, $N > 6s$ and $0 \leq \mu < \Upsilon_s$ satisfying $\frac{2\sqrt{N}}{\sqrt{\mu} - \sqrt{n - \mu}} > \frac{2N}{N - 6s}$. Then (1.1) has infinitely many solutions.

Remark 1.1. Our result extends the results in [18, 42] for the particular case of $\mu = 0$. Since there is no Hardy term in [18], they only required that $N > 6s$.

Remark 1.2. When $s \to 1$, the assumptions that $0 \leq \mu < \Upsilon_s$ and $\frac{2\sqrt{N}}{\sqrt{\mu} - \sqrt{n - \mu}} > \frac{2N}{N - 6s}$ in Theorem 1.1 are equivalent to $0 \leq \mu < \frac{N(N - 4)}{4}$ and $\mu < \frac{(N - 2)^2}{4} - 4$ respectively, that is just $0 \leq \mu < \frac{(N - 2)^2}{4} - 4$. So our result is uniform with the result in [16] when $s \to 1$.

As in [16, 18], one of the main difficulties to prove Theorem 1.1 by using variational methods is that $I(u)$ does not satisfy the Palais-Smale condition for large energy level, since $2^*_s$ is the critical exponent for the Sobolev embedding from $H^s(\Omega)$ to $L^{2^*_s}(\Omega)$. Another difficulty is that, unlike [18], every nontrivial solution of (1.1) is singular at $\{x = 0\}$ if $\mu \neq 0$ (see [6]). So different techniques are needed to deal with the case $\mu \neq 0$. In order to overcome the first difficulty, we first look at the following perturbed problem:

$$
\begin{align*}
(-\Delta)^s u - \frac{\mu u}{|x|^{2s}} &= |u|^{2^*_s - 2} u + a u, & \text{in } \Omega, \\
u &= 0, & \text{on } \partial \Omega,
\end{align*}
$$

where $\epsilon > 0$ is small.

The functional corresponding to (1.6) becomes

$$
I_{\epsilon}(u) = \frac{1}{2} \int_{\Omega} \left(|(-\Delta)^{\frac{\mu}{2}} u|^2 - \mu \frac{u^2}{|x|^{2s}} - au^2\right) dx - \frac{1}{2s} - \epsilon \int_{\Omega} |u|^{2^*_s - \epsilon} dx, \ u \in H^s_0(\Omega).
$$

Now $I_{\epsilon}(u)$ is an even functional and satisfies the Palais-Smale condition in all energy levels. It follows from the symmetric mountain-pass lemma [3, 33]. (1.6) has infinitely many solutions. More precisely, there are positive numbers $c_{\epsilon, l}, l = 1, 2, \ldots$, with $c_{\epsilon, l} \to \infty$ as $l \to +\infty$, and a solution $u_{\epsilon, l}$ for (1.6) satisfying

$$
I_{\epsilon}(u_{\epsilon, l}) = c_{\epsilon, l}.
$$

Moreover, $c_{\epsilon, l} \to c_{l} < +\infty$ as $\epsilon \to 0$. Now we want to study the behavior of $u_{\epsilon, l}$ as $\epsilon \to 0$ for each fixed $l$. If we can prove that $u_{\epsilon, l}$ converges strongly in $H^s_0(\Omega)$ to $u_l$ as $\epsilon \to 0$, then $u_l$ is a solution of (1.1) with $I(u_l) = c_{l}$. This will imply that (1.1) has infinitely many solutions. Thus Theorem 1.1 is a direct consequence of the following result.

Theorem 1.2. Suppose that $a > 0$, $N > 6s$ and $0 \leq \mu < \Upsilon_s$ satisfying $\frac{2\sqrt{N}}{\sqrt{\mu} - \sqrt{n - \mu}} > \frac{2N}{N - 6s}$. Then for any sequence $u_n$, which is a solution of (1.6) with $\epsilon = \epsilon_n \to 0$, satisfying $\|u_n\| \leq C$ for some constant independent of $n$, $u_n$ has a subsequence, which converges strongly in $H^s_0(\Omega)$ as $n \to \infty$.

Following the ideas in [15, 16, 18], to prove Theorem 1.2 we shall prove the strong convergence of $u_{\epsilon, l}$ by using a local Pohozaev identity to exclude the possibility of concentration. We would like to point out that since an important feature of the fractional Laplacian is its nonlocal property, it turns out from several technical reasons that studying our nonlocal equation (1.1) directly is not suitable for establishing Theorem 1.2. So different from [15, 16], like [18, 42], we need to realize the nonlocal operator $(-\Delta)^s$ in $\Omega$ through a local problem in $\Omega \times (0, \infty)$.

To explain this, we have to introduce some more function spaces on $D = \Omega \times (0, \infty)$, where $\Omega$ is either a smooth bounded domain or $\mathbb{R}^N$. If $\Omega$ is bounded, then we define the function space $H^s_0(t^{1 - 2s}, D)$ as the completion of

$$
C^{\infty}_{0, L}(D) := \{ \bar{u} \in C^\infty(\bar{D}) : \bar{u} = 0 \text{ on } \partial_L D = \partial \Omega \times [0, \infty) \}.
$$
with respect to the norm

\[ \| \bar{u} \|_{H^{s}_1(t^{1-2s}, \mathcal{D})} = \left( \int_{\mathcal{D}} t^{1-2s} |\nabla \bar{u}|^2 dx \right)^{\frac{1}{2}}. \]

Then it is a Hilbert space endowed with the inner product

\[ (\bar{u}, \bar{v})_{H^{s}_1(t^{1-2s}, \mathcal{D})} = \int_{\mathcal{D}} t^{1-2s} \nabla \bar{u} \nabla \bar{v} dx dt. \]

In the same manner, we define the space $D^1(t^{1-2s}, \mathbb{R}^{N+1}_+)$ as the completion of $C_0^\infty(\mathbb{R}^{N+1}_+)$ with respect to the norm

\[ \| \bar{u} \|_{D^1(t^{1-2s}, \mathbb{R}^{N+1}_+)} = \left( \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla \bar{u}|^2 dx dt \right)^{\frac{1}{2}}. \]

Recall that if $\Omega$ is a smooth bounded domain, then it is verified that (see [13], Proposition 2.1; [17], Proposition 2.1; [31], Section 2)

\[ H^s_0(\Omega) = \{ u = \text{tr}|_{\Omega \times \{0\}} \bar{u} : \bar{u} \in H^s_1(t^{1-2s}, \mathcal{D}) \}, \]

and

\[ \| \bar{u}(\cdot, 0) \|_{H^s_0(\Omega)} \leq C \| \bar{u} \|_{H^s_1(t^{1-2s}, \mathcal{D})} \]

for some $C > 0$ independent of $\bar{u} \in H^s_1(t^{1-2s}, \mathcal{D})$. Similarly, it holds by taking trace that

\[ D^s(\mathbb{R}^N) = \{ u = \text{tr}|_{\mathbb{R}^N \times \{0\}} \bar{u} : \bar{u} \in D^1(t^{1-2s}, \mathbb{R}^{N+1}_+) \}, \]

and

\[ \| \bar{u}(\cdot, 0) \|_{D^s(\mathbb{R}^N)} \leq C \| \bar{u} \|_{D^1(t^{1-2s}, \mathbb{R}^{N+1}_+)} \]

for some $C > 0$ independent of $\bar{u} \in D^1(t^{1-2s}, \mathbb{R}^{N+1}_+)$. Now we are ready to consider the fractional harmonic extension of a function $u$ defined in $\Omega$, where $\Omega$ is either a smooth bounded domain or $\mathbb{R}^N$. By the celebrated results of Caffarelli and Silvestre [13] (for $\mathbb{R}^N$) and Cabré and Tan [12] (for bounded domains, see also [9, 39, 41]), if we set $\bar{u} \in H^s_0(t^{1-2s}, \mathcal{D})$ (or $D^1(t^{1-2s}, \mathbb{R}^{N+1}_+)$) as the unique solution of the equation

\[ \begin{cases} 
    \text{div}(t^{1-2s} \nabla \bar{u}) = 0, & \text{in } \mathcal{D}, \\
    \bar{u} = 0, & \text{on } \partial L \mathcal{D}, \\
    \bar{u}(x, 0) = u(x) & \text{for } x \in \Omega
\end{cases} \]

for some fixed function $u \in H^s_0(\Omega)$ (or $D^s(\mathbb{R}^N)$), then

\[ A_s \bar{u} := -d_s \lim_{t \to 0^+} t^{1-2s} \frac{\partial \bar{u}}{\partial t}(x, t) \text{ for } x \in \Omega \]

is well defined and one must have

\[ (-\Delta)^s u = A_s \bar{u}, \]

with

\[ d_s := \frac{2^{1-2s} \Gamma(1-s)}{\Gamma(s)}. \]

Without loss of generality, we may assume throughout this paper that $d_s = 1$, that is,

\[ (-\Delta)^s u = A_s \bar{u} = -\lim_{t \to 0^+} t^{1-2s} \frac{\partial \bar{u}}{\partial t}(x, t). \]
We call this $\bar{u}$ the $s$-harmonic extension of $u$ and we point out that by a density argument, \( (1.12) \) is satisfied in weak sense for $u \in H^s_0(\Omega)$ (or $D^s(\mathbb{R}^N)$). In other words, for any $u, \phi \in H^s_0(\Omega)$ (or $D^s(\mathbb{R}^N)$), there holds

$$
\langle u, \phi \rangle_{H^s_0(\Omega)} = \langle \bar{u}, \bar{\phi} \rangle_{H^s_0(t^{1-2s}, \mathcal{D})}.
$$

Thus the trace inequality \( (1.9) \) is improved as

$$
(1.13) \quad \| \bar{u}(\cdot, 0) \|_{H^s_0(\Omega)} = \| \bar{u} \|_{H^s_0(t^{1-2s}, \mathcal{D})}.
$$

Therefore from the above analysis, we can deduce that if a function $u$ is a weak solution to the nonlocal problem

$$
(1.14) \quad \begin{cases}
(-\Delta)^s u = g, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
$$

if and only if its $s$-harmonic extension $\bar{u}$ is a weak solution to the local problem

$$
(1.15) \quad \begin{cases}
div(t^{1-2s}\nabla \bar{u}) = 0, & \text{in } \mathcal{D}, \\
\bar{u} = 0, & \text{on } \partial \mathcal{D}, \\
\mathcal{A}_s \bar{u} = g(x), & \text{on } \Omega \times \{0\},
\end{cases}
$$

where $g \in L^{2s/N}(\Omega)$. Moreover, we say that a function $u \in H^s_0(\Omega)$ is a weak solution of \( (1.14) \) provided

$$
(1.16) \quad \int_{\Omega} (-\Delta)^s \bar{u}(-\Delta)^s \phi dx = \int_{\Omega} g \phi dx
$$

for all $\phi \in H^s_0(\Omega)$ and a function $\bar{u}$ is a weak solution of \( (1.15) \) if

$$
(1.17) \quad \int_{\mathcal{D}} t^{1-2s} \nabla \bar{u} \nabla \tilde{\phi} dx dt = \int_{\Omega} g\tilde{\phi}(x, 0) dx
$$

for all $\tilde{\phi} \in H^s_0(t^{1-2s}, \mathcal{D})$.

Hence, as mentioned before, rather than studying the nonlocal problem \( (1.1) \) directly, it is better to consider the so-called $s$-harmonic extension problem

$$
(1.18) \quad \begin{cases}
div(t^{1-2s}\nabla \bar{u}) = 0, & \text{in } \mathcal{D}, \\
\bar{u} = 0, & \text{on } \partial \mathcal{D}, \\
\mathcal{A}_s \bar{u} = \frac{\mu u}{|x|^{2s}} + |u|^{2^*_s - 2}u + au, & \text{on } \Omega \times \{0\}.
\end{cases}
$$

By virtue of considering \( (1.18) \), one can easily obtain the decomposition of approximating solutions and establish a local Pohozaev identity in small balls which may contain the origin. Then applying this identity, we can exclude the possibility of concentration and prove the strong convergence of approximating solutions.

Theorems \( 1.1 \) and \( 1.2 \) extend the results in \( 18, 42 \) to the fractional Laplacian problem with Hardy term. We want to stress that it is more difficult to obtain the estimates in order to prove these results for \( (1.18) \). Like \( 18, 42 \), the main difficulty in the study of \( (1.18) \) is that we need to carry out the boundary estimates. This is greatly different from the usual Laplacian equations studied in \( 15, 16, 18 \) which mainly involving the interior estimates.

Throughout this paper, we denote the norm of $H^s_0(\Omega)$ by $\|u\| = (\int_{\Omega} |(-\Delta)^s u|^2 dx)^{\frac{1}{2}}$; the norm of $L^q(\Omega)(1 \leq q < \infty)$ by $\|u\|_q = (\int_{\Omega} |u|^q dx)^{\frac{1}{q}}$; the norm of $L^q(t^{1-2s}, \Omega)(1 \leq q < \infty)$ by $\|u\|_{L^q(t^{1-2s}, \Omega)} = (\int_{\Omega} t^{1-2s}|u|^q dx dt)^{\frac{1}{q}}$; moreover, we denote $B^N_r(x) := \{ y \in \mathbb{R}^N : |y - x| \leq r \},$
\(B^N_r(x) := \{ z = (y, t) \in \mathbb{R}^{N+1}_r : |z - (x, 0)| \leq r \}\), and positive constants (possibly different) by \(C\). For simplicity, sometimes we also write \(B^N_r(x)\) as \(B_r(x)\).

The organization of our paper is as follows. In Section 2, we will give some integral estimates. In Section 3, we obtain some estimates on safe regions. We will prove our main result in Section 4. In order that we can give a clear line of our framework, we will list some estimates for linear problems with Hardy potential, an iteration result, a decay estimate, a local Pohozaev identity and the decomposition of approximating solutions in Appendices A, B, C and D.

2. SOME INTEGRAL ESTIMATES

For any \(\Lambda > 0\) and \(x \in \mathbb{R}^N\), we define
\[
\rho_{x,\Lambda}(u) = \Lambda^\frac{N}{p} u(\Lambda \cdot - x), \quad u \in H_0^s(\Omega).
\]

Let \(u_n\) be a solution of (1.6) with \(\varepsilon = \varepsilon_n \rightarrow 0\), satisfying \(||u_n|| \leq C\) for some constant \(C\) independent of \(n\), by Proposition C.1, \(u_n\) can be decomposed as
\[
\eta_n = u_0 + \sum_{j=1}^{m} \rho_{0,\Lambda_n,j}(U_j) + \sum_{j=m+1}^{b} \rho_{\Lambda_n,j,\Lambda_n,j}(U_j) + \omega_n.
\]

In this section, we will prove a Brezis-Kato type estimate (see [10]).

In order to prove the strong convergence of \(u_n\) in \(H^s_0(\Omega)\), we only need to show that the bubbles \(\rho_{x,\Lambda_n,j}(U_j)\) will not appear in the decomposition of \(u_n\).

Among all the bubbles \(\rho_{x,\Lambda_n,j}(U_j)\), we can choose a bubble, such that this bubble has the slowest concentration rate. That is, there is \(j_0\) such that the corresponding \(\Lambda_{n,j_0}\) is the lowest order infinity among all the \(\Lambda_{n,j}\) appearing in the bubbles. For simplicity, we denote \(\Lambda_n\) the slowest concentration rate and \(x_n\) the corresponding concentration point.

Remark 2.1. Since \(\frac{1}{|x|^{2s}} \in C^{2s}(\Omega \setminus B_\delta(0))\) for any \(\delta > 0\) small, we know that \(u_n, u_0, U_j \in C^{2s}(\Omega \setminus B_\delta(0))\). As a result, \(\omega_n \in C^{2s}(\Omega \setminus B_\delta(0))\) for any \(\delta > 0\) small.

For any \(p_2 < 2_s < p_1, \alpha > 0\) and \(\Lambda \geq 0\), we consider the following relation:
\[
\left\{ \begin{array}{l}
|u_1|_{p_1} \leq \alpha, \\
|u_2|_{p_2} \leq \alpha \Lambda^\frac{N}{p} - \frac{N}{p_2}.
\end{array} \right.
\]

Define
\[
\|u\|_{p_1,p_2,\Lambda} = \inf \{ \alpha > 0 : \text{there are } u_1 \text{ and } u_2, \text{ such that (2.1) holds and } |u| \leq u_1 + u_2 \}.
\]

To deal with the Hardy potential, we need another norm. Consider the following relation:
\[
\left\{ \begin{array}{l}
|u_1|_{s,p_1} \leq \alpha, \\
|u_2|_{s,p_2} \leq \alpha \Lambda^\frac{N}{p} - \frac{N}{p_2},
\end{array} \right.
\]

where
\[
|u|_{s,p} = \|u\|_p + \left( \mu \int_{\Omega} \frac{|u|^{2p}}{|x|^{2s}} \, dx \right)^{\frac{2_s}{2p}}.
\]

Define
\[
\|u\|_{s,p_1,p_2,\Lambda} = \inf \{ \alpha > 0 : \text{there are } u_1 \text{ and } u_2, \text{ such that (2.2) holds and } |u| \leq u_1 + u_2 \}.
\]

From the definitions, it is easy to see that \(\|u\|_{p_1,p_2,\Lambda} \leq \|u\|_{s,p_1,p_2,\Lambda}\).
Let \( w_n(x) = |u_n(x)| \) in \( \Omega \); \( w_n(x) = 0 \) in \( \mathbb{R}^N \setminus \Omega \). Then it is easy to check that \( w_n \) satisfies the following inequality
\[
(2.3) \quad \int_{\mathbb{R}^N} t^{1-2s} \nabla w_n \cdot \nabla \phi \, dt - \mu \int_{\mathbb{R}^N} \frac{w_n \phi}{|x|^{2s}} \, dx \leq \int_{\mathbb{R}^N} (2w_n^{2s-1} + A) \phi \, dx, \quad \forall \phi \in H^1(t^{1-2s}, \mathbb{R}^N+), \phi \geq 0,
\]
where \( A > 0 \) is a large constant.

The main result of this section is the following proposition.

**Proposition 2.2.** Let \( w_n \) be a solution of \( (2.3) \). For any \( p_1, p_2 \in \left( \frac{2s}{N}, \frac{2s + 2s}{\sqrt{p_1 - \sqrt{p_1}}}, x \right) \) satisfying \( p_2 < 2s < p_1 \), there is a constant \( C \), depending on \( p_1 \) and \( p_2 \), such that
\[
\|w_n\|_{p_1, p_2, \Lambda} \leq C.
\]

To prove Proposition 2.2, we should show the following three lemmas.

**Lemma 2.3.** Let \( w \) be the solution of
\[
\begin{align*}
(-\Delta)^s w - \frac{\mu w}{|x|^{2s}} &= a(x)v, \quad \text{in } \Omega, \\
 w &= 0, \quad \text{on } \partial \Omega,
\end{align*}
\]
where \( a(x) \geq 0 \) and \( v \geq 0 \) are functions satisfying \( a, v \in C^{2s}(\Omega \setminus B_\delta(0)) \) for any \( \delta > 0 \). Then for any \( \frac{N}{N + 2s} < p_2 < 2s < p_1 \) and \( 0 \leq \mu \leq \mu \) satisfying \( p_1 < \frac{2s + 2s}{\sqrt{p_1 - \sqrt{p_1}}} \), there is a constant \( C = C(p_1, p_2) \) such that for any \( \Lambda \geq 1 \),
\[
\|w\|_{p_1, p_2, \Lambda} \leq C\|a\|_{L^1} \|v\|_{p_1, p_2, \Lambda}.
\]

**Proof.** For any small \( \theta > 0 \), let \( v_1, v_2 \in C^{2s}(\Omega \setminus B_\delta(0)) \) and \( v_1, v_2 \geq 0 \) be the functions such that \( v \leq v_1 + v_2 \) and \( 2.1 \) holds with \( a = \|v\|_{p_1, p_2, \Lambda} + \theta \). Consider
\[
\begin{align*}
(-\Delta)^s w_i - \frac{\mu w_i}{|x|^{2s}} &= a(x)v_i, \quad \text{in } \Omega, \\
w_i &= 0, \quad \text{on } \partial \Omega.
\end{align*}
\]

It follows from Lemma 2.2 that
\[
(2.4) \quad \|w_i\|_{p_1, p_2, \Lambda} \leq C\|a\|_{L^1} \|v_i\|_{p_1}, \quad i = 1, 2.
\]

On the other hand, by the maximum principle, we deduce \( w \leq w_1 + w_2 \). So the result follows. \( \square \)

**Remark 2.4.** We will let \( a(x) = |u_0| \frac{\mu}{|x|^{2s}} \) or \( a(x) = |\rho_0, \Lambda, \nu_{0,j}| \frac{\mu}{|x|^{2s}}, j = 1, 2, \cdots, m \), in Lemma 2.3 to obtain some desired estimates for \( w_n \). Here \( a(x) \) may have singularity at \( \{x = 0\} \). So, in Lemma 2.3 we only assume that \( a(x) \) and \( v(x) \) belong to \( C^{2s}(\Omega \setminus B_\delta(0)) \).

**Lemma 2.5.** Let \( w \geq 0 \) be a weak solution of
\[
\begin{align*}
(-\Delta)^s w - \frac{\mu w}{|x|^{2s}} &= 2\varphi^{2s-1} + A, \quad \text{in } \Omega, \\
w &= 0, \quad \text{on } \partial \Omega.
\end{align*}
\]
For any \( p_1, p_2 \in \left( \frac{N+2s}{N-2s}, \frac{N+2s}{N-2s}, \frac{N+2s}{N-2s} \right) \) with \( p_2 < 2s < p_1 \), let \( q_i \) be given by
\[
\frac{1}{q_i} = \frac{N+2s}{(N-2s)p_i} - \frac{2s}{N}, i = 1, 2.
\]
If \( q_1, q_2 < \frac{2s + 2s}{\sqrt{p_1 - \sqrt{p_1}}} \), then there is a constant \( C = C(p_1, p_2) \) such that for any \( \Lambda \geq 1 \),
\[
\|w\|_{q_1, q_2, \Lambda} \leq C\|v\|_{p_1, p_2, \Lambda} + C.
\]
Proof. For any small $\theta > 0$, let $v_1 \geq 0$ and $v_2 \geq 0$ be the functions such that $v_1 + v_2$, and (2.1) holds with $\alpha = \|v\|_{p_1,p_2,\Lambda} + \theta$.

Consider
\[
\begin{aligned}
\left\{
\begin{array}{ll}
(-\Delta)^\alpha w_1 - \frac{\mu w_1}{|x|^{2s}} = C v_1^{2^* - 1} + A, & \text{in } \Omega, \\
\end{array}
\right.
\end{aligned}
\]
and
\[
\begin{aligned}
\left\{
\begin{array}{ll}
(-\Delta)^\alpha w_2 - \frac{\mu w_2}{|x|^{2s}} = C v_2^{2^* - 1}, & \text{in } \Omega, \\
\end{array}
\right.
\end{aligned}
\]
where $C > 0$ is a large constant. Then by the maximum principle, $w \leq w_1 + w_2$.

Let $p_i = p_i N - 2s \in (1, \frac{N}{2s})$, then $q_i = \frac{N p_i}{N - 2sp_i}$, $i = 1, 2$. By Lemma A.3, we have
\[
\|w_i\|_{*,q_i} \leq C \|v_1^{2^* - 1} + A\|_{p_i} \leq C(\|v_1\|_{p_i}^{2^* - 1} + 1) \leq C(\|v\|_{p_1,p_2,\Lambda} + \theta)^{2^* - 1} + 1],
\]
and
\[
\|w_2\|_{*,q_2} \leq C \|v_2\|_{p_2}^{2^* - 1} \leq C(\|v\|_{p_1,p_2,\Lambda} + \theta)\Lambda^{\frac{N}{2s}} \frac{N}{q_2} \Lambda^{\frac{N}{2s}} \frac{N}{q_2},
\]
since
\[
\left(\frac{N}{2s} - \frac{N}{p_2} / \frac{N}{2s} - 1\right) = \frac{N}{q_2}.
\]
As a consequence, the result follows. \qed

Lemma 2.6. Let $w_n(x) = |u_n(x)|$ in $\Omega$; $w_n(x) = 0$ in $\mathbb{R}^N \setminus \Omega$. Then there are constants $C > 0$ and $p_1, p_2 \in (\frac{2s}{N}, +\infty)$ with $p_2 < 2^*_s < p_1$ such that
\[
\|w_n\|_{*,p_1,p_2,\Lambda} \leq C.
\]

Proof. From Proposition C.1, we have $u_n = u_0 + u_{n,1} + u_{n,2}$, where
\[
u_{n,1} = \sum_{j=1}^n \rho_{0,\Lambda_{n,j}}(U_j) + \sum_{j=m+1}^n \rho_{\Lambda_{n,j}}(U_j),
\]
and $u_{n,2} = w_n$.

Let $a_i = C|u_{n,i}|^{2^*_s - 1}$, $i = 0, 1, 2$, where $C > 0$ is a large constant. Then, we have
\[
(-\Delta)^\alpha w_n - \frac{\mu w_n}{|x|^{2s}} \leq (a_0 + a_1 + a_2)w_n + A.
\]

Let $w = G(v)$ be the solution of
\[
\begin{aligned}
\left\{
\begin{array}{ll}
(-\Delta)^\alpha w - \frac{\mu w}{|x|^{2s}} = v, & \text{in } \Omega, \\
\end{array}
\right.
\end{aligned}
\]
and
\[
\begin{aligned}
\left\{
\begin{array}{ll}
w = 0, & \text{on } \partial \Omega.
\end{array}
\right.
\end{aligned}
\]
Then, we have
\[
\|w_n\|_{*,p_i} \leq G(a_0w_n + A) + G(a_1w_n) + G(a_2w_n).
\]

We first deal with the term $G(a_0w_n + A)$. Let $q > \frac{2N}{N + 2s}$ such that $q - \frac{2s}{N + 2s}$ is so small that $p_1 := \frac{Nq}{N-2sq} \in (\frac{2s}{N + 2s}, \frac{2^*_s - \sqrt{q}}{\sqrt{q} - \sqrt{q}})$. Then it follows from Lemma A.3 that
\[
\|G(a_0w_n + A)\|_{*,p_i} \leq C\|a_0w_n + A\|_q \leq C + C\|a_0\|_{\frac{2^*_s}{q}} \|w_n\|_2 \leq C + C\|a_0\|_{\frac{2^*_s}{q}}.
\]
By Proposition 3.3 \(|u_0(x)| \leq C|x|^{-\left(\frac{N-2s}{2s} - \beta\right)}, \forall x \in \Omega\). So we see that if \(q - \frac{2N}{N+2s} > 0\) is small enough, then
\[
\left(\frac{N - 2s}{2} - \beta\right) 4s \frac{2^* q}{N - 2s 2^* - q} < N.
\]
As a result,
\[
\int_{\Omega} |a_0|^{\frac{2^*}{2^* - q}} dx \leq C \int_{\Omega} |x|^{-\left(\frac{N-2s}{2s} - \beta\right)\frac{4s}{N - 2s 2^* - q}} dx \leq C.
\]
Hence, we have proved that there is a \(p_1 > 2^*_s\) such that
\[
\|G(a_0 w_n + A)\|_{*,p_1} \leq C.
\]

Next we treat the term \(G(a_1 w_n)\). Let \(p_2 \in \left(\frac{N}{N-2s}, 2^*_s\right)\) be a constant. By Lemma A.4, we get
\[
\|G(a_1 w_n)\|_{*,p_2} \leq C\|a_1\|_r \|w_n\|_{2^*_s} \leq C\|a_1\|_r,
\]
where \(r\) is determined by \(\frac{1}{p_2} = \frac{1}{r} + \frac{2s}{N-2s} \).

But
\[
\int_{\Omega} |\rho_{x,j}(x)\Lambda_{n,j}(U_j)|^\frac{4s}{2s-4r} dx = \Lambda_{n,j}^{2s-4r-N} \int_{\Omega_{x,j}} |U_j|^\frac{4s}{2s-4r} dx,
\]
where \(\Omega_{x,A} = \{\hat{x} : x_{n,j} + \Lambda^{-1} \hat{x} \in \Omega\}\).

For \(j = m + 1, \ldots, h\), there is a \(C > 0\) such that (see [18])
\[
|U_j(x)| \leq \frac{C}{1 + |x|^N 2s}.
\]
Therefore, for any \(r \in \left(\frac{N}{2}, \frac{N}{2s}\right)\), we have
\[
\int_{\Omega_{x,j}} |U_j|^\frac{4s}{2s-4r} dx \leq C, \quad j = m + 1, \ldots, h.
\]
For \(j = 1, 2, \ldots, m\), by Lemma A.4 we have
\[
U_j \in L^p_{loc}(\mathbb{R}^N), \forall p < \frac{2s\sqrt{\mu}}{\sqrt{\mu} - \sqrt{\mu - \mu}},
\]
and by Lemma B.1
\[
|U_j(x)| \leq \frac{C}{|x|^\frac{N-2s}{2s} + \beta}, \quad \forall|x| \geq 1.
\]
Note that \(r \to \frac{N}{2s}\) as \(p_2 \to 2^*_s\). Now we choose \(p_2\) close to \(2^*_s\) so that
\[
\frac{4sr}{N - 2s} \left(\frac{N - 2s}{2} + \beta\right) > N,
\]
and
\[
\frac{4sr}{N - 2s} < \frac{2^* \sqrt{\mu}}{\sqrt{\mu} - \sqrt{\mu - \mu}}.
\]
As a result,
\[
\int_{\Omega_{x,j}} |U_j|^\frac{4s}{2s-4r} dx \leq C, \quad j = 1, 2, \ldots, m.
\]
Thus we have proved that there is a \(p_2 < 2^*_s\) close to \(2^*_s\) such that
\[
\|G(a_1 w_n)\|_{*,p_2} \leq C\Lambda_n^{2s - \frac{2^*}{2^*_s}} = C\Lambda_n^{2s - \frac{2^*}{2^*_s}}.
\]

Finally, we treat the term \(G(a_2 w_n)\). It follows from Lemma 2.3 that
\[
\|G(a_2 w_n)\|_{*,p_1, p_2, A_n} \leq C\|a_2\|_{\frac{2s}{2^*_s}} \|w_n\|_{p_1, p_2, A_n} \leq \frac{1}{2}\|w_n\|_{*,p_1, p_2, A_n},
\]
since \(\|a_2\|_{\frac{2s}{2^*_s}} = \|\omega_n\|_{2^*_s}^\frac{4s}{2^*_s} \to 0\) as \(n \to \infty\).
From (2.6), (2.7) and (2.8), we obtain
\[
\|w_n\|_{L^p_t L^2_x} \leq \|G(a_0 w_n + A)\|_{L^p_t L^2_x} + \|G(a_1 w_n)\|_{L^p_t L^2_x} + \|G(a_2 w_n)\|_{L^p_t L^2_x} \leq C + \frac{1}{2}\|w_n\|_{L^p_t L^2_x},
\]
Hence the result follows.

**Proof of Proposition 2.2.** Since \( w_n \) satisfies (2.3), we can use Lemmas 2.3 and 2.0 to prove that
\[
\|w_n\|_{L^p_t L^2_x} \leq C
\]
holds for any \( p_1, p_2 \) with \( p_1, p_2 \in \left( \frac{2}{2 - \frac{2}{p - 2}}, \frac{2}{2 - \frac{2}{p - 2}} \right) \) satisfying \( p_2 < 2^* < p_1 \).

3. Estimates on safe regions

Since the number of the bubbles of \( u_n \) is finite, by Proposition C.1 we can always find a constant \( \tilde{C} > 0 \), independent of \( n \), such that the region
\[
\left( B^N \left( \frac{1}{(C+5)\Lambda_n} \right) \right) \cap \Omega,
\]
does not contain any concentration point of \( u_n \) for any \( n \). We call this region a safe region for \( u_n \).

For \( d = N, N + 1 \), let
\[
A_{n,1}^d := \left( B^d \left( \frac{1}{(C+5)\Lambda_n} \right) \right) \cap \Omega (\text{or } D),
\]
and
\[
A_{n,2}^d := \left( B^d \left( \frac{1}{(C+4)\Lambda_n} \right) \right) \cap \Omega (\text{or } D).
\]

For a measurable set \( E \subset \mathbb{R}^{N+1}_+ \), we define a weighted measure
\[
m_s(E) := \int_E t^{1-2s} dx dt,
\]
and
\[
\int_E t^{1-2s} f(x,t) dx dt := \left( \frac{\int_E t^{1-2s} f(x,t) dx dt}{m_s(E)} \right).
\]

Firstly, we introduce the following two known results given in [22] and [18] respectively.

**Lemma 3.1.** (Theorem 1.3, [22]) Let \( \mathcal{F} \) be an open bounded set in \( \mathbb{R}^{N+1} \). Then there exists a constant \( C(N, s, \mathcal{F}) > 0 \) such that
\[
\left( \int_{\mathcal{F}} t^{1-2s}|\tilde{u}(x,t)|^{\frac{2(N+1)}{N}} dx dt \right)^{\frac{N}{2(N+1)}} \leq C \left( \int_{\mathcal{F}} t^{1-2s}|\nabla \tilde{u}(x,t)|^2 dx dt \right)^{\frac{1}{2}}.
\]

**Lemma 3.2.** (Lemma 5.2, [18]) For \( f \geq 0 \), assume that \( \bar{u} \in H^1_0(t^{-2s}, D) \) satisfies
\[
\left\{ \begin{array}{ll}
\text{div}(t^{1-2s}\nabla \bar{u}) = 0, & \text{in } D, \\
A_s(\bar{u}) = f, & \text{on } \Omega \times \{0\}, \\
\bar{u} = 0, & \text{on } \partial_0 D.
\end{array} \right.
\]
Then, for \( \gamma \in (1, \frac{2N+2}{2N+1}) \), there exists a constant \( C > 0 \) such that
\[
\left( \int_{B^{N+1}_1(x)} t^{1-2s}|\bar{u}|^\gamma dx dt \right)^{\frac{1}{\gamma}} \leq C \int_{B^{N+1}_1(x)} t^{1-2s}|\bar{u}|^\gamma dx dt + C \int_r^1 \left( \frac{1}{\rho^{N-2s}} \int_{B^N_{\rho}(x)} f(y) dy \right) d\rho.
\]
holds for any \( x \in \Omega \) and \( r \in (0, r_0) \), where \( r_0 = \text{dist}(x, \partial \Omega) \).

Now we come to our main result in this section.

**Proposition 3.3.** Let \( w_n \) be a weak solution of (2.3). Then there is a constant \( C > 0 \) independent of \( n \), such that

\[
\int_{A_{n,2}'} |w_n|^p dx \leq CA_n^{\frac{Np - N}{N+1}},
\]

and

\[
\int_{A_{n,2}''} t^{1-2s} |w_n|^p dx dt \leq CA_n^{\frac{Np - N + 2s - 2}{N+1}},
\]

where \( p_1 > 2^*_s \) and \( p \geq 2 \) are any constants, satisfying

\[
p, p_1 < \frac{2^* \sqrt{\mu}}{\sqrt{\mu - \mu + \mu}}.
\]

In order to prove Proposition 3.3 we need the following lemma.

**Lemma 3.4.** Suppose that \( w_n \) satisfies (2.3) with \( \epsilon = \epsilon_n \to 0 \). Then there is a constant \( C > 0 \) independent of \( n \), such that

\[
\sup_n \int_{B_{1}^{\nu+1}(y)} t^{1-2s} |\tilde{w}_n|^{\gamma} dx dt \leq CA_n^{\frac{2s}{N+1}}, \quad \forall y \in \mathbb{R}^N,
\]

for all \( r \in \left[ \tilde{C} \Lambda_n^{-\frac{s}{2}}, (\tilde{C} + 5) \Lambda_n^{-\frac{s}{2}} \right] \), where \( \gamma \in (1, \frac{2N+2}{2N+1}) \) and \( p_1 > 2^*_s \) is any constant satisfying

\[
p_1 < \frac{2^* \sqrt{\mu}}{\sqrt{\mu - \mu + \mu}}.
\]

**Proof.** Firstly, using Hölder inequality and (3.1), we find

\[
\int_{B_{1}^{\nu+1}(y)} t^{1-2s} |\tilde{w}_n|^{\gamma} dx dt
\]

\[
\leq \left( \int_{B_{1}^{\nu+1}(y)} t^{1-2s} |\tilde{w}_n|^{2(N+1)} dx dt \right)^{\frac{N+1}{2N+1}} \left( \int_{B_{1}^{\nu+1}(y)} t^{1-2s} dx dt \right)^{1-\frac{N+1}{2N+1}}
\]

\[
\leq C \left( \int_{B_{1}^{\nu+1}(y)} t^{1-2s} |\nabla \tilde{w}_n(x, t)|^2 dx dt \right)^{\frac{1}{2}} \leq C.
\]

So it follows from (2.3) and Lemma 3.3 that

\[
\left( \int_{B_{1}^{\nu+1}(y)} t^{1-2s} |\tilde{w}_n|^{\gamma} dx dt \right)^{\frac{1}{2}}
\]

\[
\leq C + C \int_r^{1} \frac{1}{\rho^{N-2s}} \int_{B_{1}^{\nu}(y)} \left( \frac{\mu}{|x|^{2s}} w_n + 2w_n^{2^*_s-1} + A \right) dx \frac{dp}{\rho}
\]

\[
\leq C + C \int_r^{1} \frac{1}{\rho^{N-2s+1}} \int_{B_{1}^{\nu}(y)} \frac{kw_n}{|x|^{2s}} dx d\rho + C \int_r^{1} \frac{1}{\rho^{N-2s+1}} \int_{B_{1}^{\nu}(y)} w_n^{2^*_s-1} dx d\rho.
\]

By Proposition 2.2, we know that \( \|w_n\|_{*; p_1, p_2, \Lambda_n} \leq C \) for any \( p_1, p_2 \in \left( \frac{2^*_s}{2}, \frac{2^* \sqrt{\mu}}{\sqrt{\mu - \mu + \mu}} \right) \) satisfying \( p_2 < 2^*_s < p_1 \).

Let \( p_1 \) be a constant satisfying \( 2^*_s < p_1 < \frac{2^* \sqrt{\mu}}{\sqrt{\mu - \mu + \mu}} \) and \( p_2 = \frac{2^*_s}{2} + \theta \), where \( \theta > 0 \) is a small constant. Then we can choose \( v_{1,n} \) and \( v_{2,n} \) such that \( w_n \leq v_{1,n} + v_{2,n} \), \( \|v_{1,n}\|_{*; p_1} \leq C \) and
\[ \|v_{2,n}\|_{s,\tilde{p}_2} \leq CA_n^{\frac{N}{\tilde{p}_2}} - \frac{N}{\tilde{p}_2}. \]

So,
\[
\int_r^1 \frac{1}{\rho^{N-2s+1}} \int_{B_{\rho}^N(y)} \frac{\mu v_{1,n}}{|x|^{2s+1}} dx d\rho \\
\leq \int_r^1 \frac{1}{\rho^{N-2s+1}} \left( \int_{B_{\rho}^N(y)} \frac{\mu |v_{1,n}|^{2p_1}}{|x|^{2s}} dx \right)^{\frac{\tilde{p}_2}{2p_1}} \left( \int_{B_{\rho}^N(y)} \frac{\mu}{|x|^{2s}} dx \right)^{1-\frac{\tilde{p}_2}{2p_1}} \\
\leq C \int_r^1 \rho^{2s-N-1+(N-2s)(1-\frac{\tilde{p}_2}{2p_1})} d\rho \leq C \rho^{-(\frac{N-2s}{2p_1})} \leq CA_n^{\frac{N}{\tilde{p}_2}} = CA_n^{\frac{N}{\tilde{p}_2}},
\]

and
\[
\int_r^1 \frac{1}{\rho^{N-2s+1}} \int_{B_{\rho}^N(y)} \frac{\mu v_{2,n}}{|x|^{2s+1}} dx d\rho \\
\leq \int_r^1 \frac{1}{\rho^{N-2s+1}} \left( \int_{B_{\rho}^N(y)} \frac{\mu |v_{2,n}|^{2p_1}}{|x|^{2s}} dx \right)^{\frac{\tilde{p}_2}{2p_1}} \left( \int_{B_{\rho}^N(y)} \frac{\mu}{|x|^{2s}} dx \right)^{1-\frac{\tilde{p}_2}{2p_1}} \\
\leq CA_n^{\frac{N}{\tilde{p}_2}} \int_r^1 \rho^{2s-N-1+(N-2s)(1-\frac{\tilde{p}_2}{2p_1})} d\rho \leq CA_n^{\frac{N}{\tilde{p}_2}} = CA_n^{\theta_1},
\]

where \( \theta_1 > 0 \) is a small constant if we choose \( \theta > 0 \) small enough.

Thus, we obtain that
\[
\int_r^1 \frac{1}{\rho^{N-2s+1}} \int_{B_{\rho}^N(y)} \frac{\mu w_n}{|x|^{2s}} dx d\rho \\
\leq \int_r^1 \frac{1}{\rho^{N-2s+1}} \int_{B_{\rho}^N(y)} \frac{\mu v_{1,n}}{|x|^{2s}} dx d\rho + \int_r^1 \frac{1}{\rho^{N-2s+1}} \int_{B_{\rho}^N(y)} \frac{\mu v_{2,n}}{|x|^{2s}} dx d\rho \\
\leq CA_n^{\frac{N}{\tilde{p}_2}} + CA_n^{\theta_1} \leq CA_n^{\frac{N}{\tilde{p}_2}}.
\]

Let \( \tilde{p}_2 = 2^* - 1 = \frac{N+2s}{N-2s} \) and let \( p_1 > 2^* \) with \( p_1 < \frac{2s+1}{\sqrt{N-2s-1}} \). Then we can choose \( \tilde{v}_1,n \) and \( \tilde{v}_2,n \) such that \( |w_n| \leq \tilde{v}_1,n + \tilde{v}_2,n \) and \( ||v_{1,n}||_{s,p_1} \leq C \) and \( ||v_{2,n}||_{s,\tilde{p}_2} \leq CA_n^{\frac{N}{\tilde{p}_2}} \). Since \( p_1 > 2^* \), we know that \( \frac{N(N+2s)}{2p_1(N-2s)} - s < \frac{N}{2p_1} \). Therefore,
\[
\int_r^1 \frac{1}{\rho^{N-2s+1}} \int_{B_{\rho}^N(y)} \tilde{v}_{1,n}^{2s-1} dx d\rho \leq \int_r^1 \frac{1}{\rho^{N-2s+1}} \left( \int_{B_{\rho}^N(y)} |\tilde{v}_{1,n}| dx \right)^{\frac{N+2s}{(N-2s)p_1}} \rho^{N(1-\frac{N+2s}{p_1(N-2s)})} d\rho \\
\leq C \int_r^1 \rho^{3s-1-\frac{N(N+2s)}{p_1(N-2s)}} d\rho \leq \Lambda_n \frac{N(N+2s)}{2p_1(N-2s)} - s \leq CA_n^{\frac{N}{\tilde{p}_2}},
\]

and
\[
\int_r^1 \frac{1}{\rho^{N-2s+1}} \int_{B_{\rho}^N(y)} \tilde{v}_{2,n}^{2s-1} dx d\rho \leq \int_r^1 \frac{1}{\rho^{N-2s+1}} \left( \Lambda_n^{\theta_1} \tilde{v}_{2,n}^{\frac{N}{\tilde{p}_2}} \right)^{\frac{N+2s}{N}} d\rho \\
\leq CA_n^{\frac{2s-N}{\tilde{p}_2}} \int_r^1 \frac{1}{\rho^{N-2s+1}} d\rho \leq C.
\]
Hence, we get
\[
\int_r^1 \frac{1}{\rho^{N-2s+1}} \int_{B^N_r(y)} w_n^2 v_n^{2s-1} \, dx \, d\rho
\]
(3.4)
\[
\leq C \int_r^1 \frac{1}{\rho^{N-2s+1}} \int_{B^N_r(y)} \tilde{v}_n^{2s-1} \, dx \, d\rho + C \int_r^1 \frac{1}{\rho^{N-2s+1}} \int_{B^N_r(y)} \tilde{v}_n^{2s-1} \, dx \, d\rho
\]
\[
\leq C \Lambda_n^{\frac{N}{N+1}}.
\]
From (3.4), (3.2) and (3.3), we have
\[
\left( \int_{B^N_{r+1}(y)} t^{1-2s}|\tilde{w}_n|^\gamma \, dx \, dt \right)^\frac{1}{\gamma} \leq C \Lambda_n^{\frac{N}{N+1}}
\]
for any \( r \in [\tilde{C}\Lambda_n^{-\frac{1}{2}}, (\tilde{C} + 5)\Lambda_n^{-\frac{1}{2}}] \). \( \square \)

**Proof of Proposition 3.5.** It follows from Lemma 3.4 that for any \( y \in A^N_{n,2} \) and \( r \in [\tilde{C}\Lambda_n^{-\frac{1}{2}}, (\tilde{C} + 5)\Lambda_n^{-\frac{1}{2}}] \), we get that if \( \gamma \in (1, 2\frac{N+2}{N+1}) \),
\[
\int_{B^N_{r+1}(y)} t^{1-2s}|\tilde{w}_n|^\gamma \, dx \, dt \leq C \int_{B^N_{r+1}(y)} t^{1-2s} \, dx \, dt \leq C \Lambda_n^{\frac{N}{N+1}} \Lambda_n^{-\frac{1}{2}(N+2-2s)}.
\]
(3.5)
Let \( \tilde{v}_n(z) = \tilde{w}_n(\Lambda_n^{-\frac{1}{2}} z), z \in \mathcal{D}_n \), where \( \mathcal{D}_n = \{ z : \Lambda_n^{-\frac{1}{2}} z \in \mathcal{D} \} \).
Then \( \tilde{v}_n \) satisfies
\[
\begin{cases}
div(t^{1-2s}\tilde{v}_n) = 0, & \text{in } \mathcal{D}_n, \\
\Lambda_n^{-\frac{1}{2}} \tilde{v}_n(x,0) \leq \frac{|\nabla u_n|}{\sqrt{2s}} + \Lambda_n^{-\frac{1}{2}} (|v_n|^{2s-2} + a) v_n, & \text{on } \Omega,
\end{cases}
\]
where \( \Omega_n = \{ x : \Lambda_n^{-\frac{1}{2}} x \in \Omega \} \).

Let \( \bar{x} = \Lambda_n^{\frac{1}{2}} y \). Since \( B_{\Lambda_n^{-\frac{1}{2}}}(y), y \in A^N_{n,2} \) does not contain any concentration point of \( u_n \), we can deduce that
\[
\int_{B^N_{\Lambda_n^{-\frac{1}{2}}}(\bar{x})} |\Lambda_n^{-\frac{1}{2}}(v_n|^{2s-2} + a)|^{\frac{N}{2s}} \, dx \leq C \int_{B^N_{\Lambda_n^{-\frac{1}{2}}}(y)} |u_n|^{2s} \, dx + C \Lambda_n^{-\frac{N}{2s}} \to 0,
\]
as \( n \to \infty \). Thus by Lemma 3.5 and (3.5), we obtain
\[
||v_n||_{L^p(B^N_{\frac{1}{2}}(\bar{x}))} \leq C \left( \int_{B^N_{1/2}(\bar{x})} t^{1-2s}|\tilde{w}_n|^\gamma \, dx \, dt \right)^\frac{1}{\gamma} \leq C \left( \Lambda_n^{\frac{N+2-2s}{2s}} \int_{B^N_{1/2}(\bar{x})} t^{1-2s}|\tilde{w}_n|^\gamma \, dx \, dt \right)^\frac{1}{\gamma} \leq C \Lambda_n^{\frac{N}{N+1}},
\]
and
\[
\left( \int_{B^N_{1/2}(\bar{x})} t^{1-2s}|\tilde{v}_n|^p \, dx \, dt \right)^\frac{1}{p} \leq C \left( \int_{B^N_{1/2}(\bar{x})} t^{1-2s}|\tilde{v}_n|^\gamma \, dx \, dt \right)^\frac{1}{\gamma} \leq C \Lambda_n^{\frac{N}{N+1}},
\]
for any \( p > \max\{2^s, 2^f\} \) with \( p < \min\{\frac{2^s \sqrt{p}}{\sqrt{p} - \sqrt{2^s}}, \frac{2^f \sqrt{p}}{\sqrt{p} - \sqrt{2^f}}\} \) and \( 2^f = \frac{2(N+1)}{N} \).
As a result,
\[
\Lambda_n^{\frac{N}{N+1}} \int_{B^N_{\Lambda_n^{-\frac{1}{2}}}(y)} |v_n|^p \, dx \leq C \Lambda_n^{\frac{2N}{N+1}}, \quad \forall y \in A^N_{n,2},
\]
and
\[
\Lambda_n^{\frac{N+2-2s}{2s}} \int_{B^N_{\Lambda_n^{-\frac{1}{2}}}(y)} t^{1-2s}|\tilde{w}_n|^p \, dx \, dt \leq C \Lambda_n^{\frac{2N}{N+1}}, \quad \forall y \in A^N_{n,2}.
\]
Hence, for any \( p > \max\{2^*_s, 2^\sharp\} \) with \( p < \min\{\frac{2^*_s\sqrt{\mu}}{\sqrt{\mu}-\sqrt{\mu-\mu}}, \frac{2^\sharp\sqrt{\mu}}{\sqrt{\mu}-\sqrt{\mu-\mu}}\} \), there holds

\[
\int_{A_{n,2}^N} |w_n|^p \, dx \leq CA_n^{\frac{Np}{p-2} - \frac{N}{2}},
\]

and

\[
\int_{A_{n,2}^{N+1}} t^{-2s}|\bar{w}_n|^p \, dx dt \leq CA_n^{\frac{Np}{p-2} - \frac{N+2-2s}{2}}.
\]

On the other hand, for any \( 2 \leq p \leq \max\{2^*_s, 2^\sharp\} \), take \( \bar{p} > \max\{2^*_s, 2^\sharp\} \) and \( \bar{p} < \min\{\frac{2^*_s\sqrt{\mu}}{\sqrt{\mu}-\sqrt{\mu-\mu}}, \frac{2^\sharp\sqrt{\mu}}{\sqrt{\mu}-\sqrt{\mu-\mu}}\} \). Then

\[
\int_{A_{n,2}^N} |w_n|^p \, dx \leq C \left( \int_{A_{n,2}^N} |w_n|^\bar{p} \, dx \right)^{\frac{\bar{p}}{p}} A_n^{-\frac{N}{2}(1-\frac{\bar{p}}{p})} \leq CA_n^{\frac{Np}{p-2} - \frac{N}{2}};
\]

and

\[
\int_{A_{n,2}^{N+1}} t^{-2s}|\bar{w}_n|^p \, dx dt \leq C \left( \int_{A_{n,2}^{N+1}} t^{-2s}|\bar{w}_n|^\bar{p} \, dx dt \right)^{\frac{\bar{p}}{p}} A_n^{-\frac{N+2-2s}{2}(1-\frac{\bar{p}}{p})} \leq CA_n^{\frac{Np}{p-2} - \frac{N+2-2s}{2}}.
\]

Hence, for any \( p \geq 2 \),

\[
\left( \int_{A_{n,2}^N} |w_n|^p \, dx \right)^{\frac{1}{p}} \leq CA_n^{\frac{N}{2} - \frac{N}{p}},
\]

and

\[
\int_{A_{n,2}^{N+1}} t^{-2s}|\bar{w}_n|^p \, dx dt \leq CA_n^{\frac{Np}{p-2} - \frac{N}{2}(N+2-2s)}.
\]

Let \( A_{n,3}^d = \left( B^d_{(\bar{c}+\delta)A_n^\frac{1}{2}}(x_n) \setminus B^d_{(\bar{c}+\delta)A_n^\frac{1}{2}}(x_n) \right) \cap \Omega \) (or \( D \)), \( d = N, N + 1 \).

**Proposition 3.5.** We have

\[
\int_{A_{n,3}^{N+1}} t^{-2s}|\nabla \bar{u}_n(x, t)|^2 \, dx dt, \int_{A_{n,3}^N} \frac{\mu |u_n|^2}{|x|^{2s}} \, dx \leq C \int_{A_{n,2}^N} (|u_n|^{2s}+1) \, dx + CA_n \int_{A_{n,2}^{N+1}} t^{-2s}|\bar{u}_n(x, t)|^2 \, dx dt.
\]

In particular,

\[
\int_{A_{n,3}^{N+1}} t^{-2s}|\nabla \bar{u}_n(x, t)|^2 \, dx dt, \int_{A_{n,3}^N} \frac{\mu |u_n|^2}{|x|^{2s}} \, dx \leq CA_n^{\frac{2s-N}{2p_1} + \frac{N}{p_1}}.
\]

**Proof.** Let \( \bar{\phi}_n \in C_0^\infty (A_{n,2}^{N+1}) \) be a function with \( \bar{\phi}_n = 1 \) in \( A_{n,3}^{N+1} \), \( 0 \leq \bar{\phi}_n \leq 1 \) and \( |\nabla \bar{\phi}_n| \leq CA_n^{\frac{1}{2}} \). From

\[
\int_D t^{-2s} \nabla \bar{u}_n \nabla (\bar{\phi}_n^2 \bar{u}_n) \, dx dt - \int_{\Omega} \frac{\mu \phi_n^2 u_n^2}{|x|^{2s}} \, dx \leq \int_D (2|u_n|^{2s-1} + A) \phi_n^2 |u_n| \, dx,
\]

we can prove (3.6). Since \( p_1 > 2^*_s \), we see

\[
\frac{2^*_s N}{2p_1} - \frac{N}{2} < \frac{N}{p_1} - \frac{N-2s}{2}.
\]

Thus from (3.6) and Proposition 3.3 we have

\[
\int_{A_{n,3}^{N+1}} t^{-2s}|\nabla \bar{u}_n(x, t)|^2 \, dx dt, \int_{A_{n,3}^N} \frac{\mu |u_n|^2}{|x|^{2s}} \, dx \leq CA_n^{\frac{2sN}{p_1} - \frac{N}{2}} + CA_n^{\frac{N}{p_1} - \frac{N-2s}{2}} \leq CA_n^{\frac{2sN}{p_1} + \frac{N}{p_1}}.
\]

\( \square \)
4. Proof of the Main Result

Choose an $\ell_n \in [\bar{C} + 2, \bar{C} + 3]$ such that

$$
\int_{\partial B^N_{\ell_n \Lambda_n^{\frac{1}{2}}}(x_n)} \left( \Lambda_n^{-1} |u_n|^{2^* - \epsilon_n} + |u_n|^2 + \Lambda_n^{-1} \frac{\mu_n^2}{|x|^{2s}} \right) dS_x
$$

(4.1)

$$
\leq C\Lambda_n^{2s-1} \int_{A_{n,3}^N} \left( \Lambda_n^{-1} |u_n|^{2^* - \epsilon_n} + |u_n|^2 + \Lambda_n^{-1} \frac{\mu_n^2}{|x|^{2s}} \right) dx,
$$

and

$$
\int_{\partial B^{N+1}_{\ell_n \Lambda_n^{\frac{1}{2}}}(x_n)} \Lambda_n^{-1} t^{1-2s}(|\nabla \bar{u}_n|^2 + |\bar{u}_n|^2) dS_x \leq C\Lambda_n^{2s} \int_{A_{n,3}^{N+1}} \Lambda_n^{-1} t^{1-2s}(|\nabla \bar{u}_n|^2 + |\bar{u}_n|^2) dx dt.
$$

(4.2)

Applying Proposition 3.3, 4.1, 4.2 and 5.7, we get

$$
\int_{\partial B^N_{\ell_n \Lambda_n^{\frac{1}{2}}}(x_n)} \left( \Lambda_n^{-1} |u_n|^{2^* - \epsilon_n} + |u_n|^2 + \Lambda_n^{-1} \frac{\mu_n^2}{|x|^{2s}} \right) dS_x
$$

$$
+ \int_{\partial B^{N+1}_{\ell_n \Lambda_n^{\frac{1}{2}}}(x_n)} \Lambda_n^{-1} t^{1-2s}(|\nabla \bar{u}_n|^2 + |\bar{u}_n|^2) dS_x
$$

(4.3)

$$
\leq C\Lambda_n^{2s-1} \left( C\Lambda_n^{-1} \Lambda_n^{-1} \frac{N(2^* - \epsilon_n)}{p \mu} + C\Lambda_n^{\frac{N}{2}} + C\Lambda_n^{-1} \Lambda_n^{\frac{N}{2} + \frac{2s-N}{2}} \right)
$$

$$
\leq C\Lambda_n^{2s-1} \left( C\Lambda_n^{-1} \Lambda_n^{-1} \frac{N(2^* - \epsilon_n)}{p \mu} + C\Lambda_n^{\frac{N}{2}} + C\Lambda_n^{-1} \Lambda_n^{\frac{N}{2} + \frac{2s-N}{2}} \right)
$$

since $-1 + \frac{N(2^* - \epsilon_n)}{2p} - \frac{N}{2} < -\frac{N}{2} + \frac{N}{2}$.  

Proof of Theorem 1.2: We have three different cases: (i) $B^N_{\ell_n \Lambda_n^{\frac{1}{2}}}(x_n) \cap (\mathbb{R}^N \setminus \Omega) \neq \emptyset$; (ii) $B^N_{\ell_n \Lambda_n^{\frac{1}{2}}}(x_n) \subset \Omega$ and $0 \notin B^N_{\ell_n \Lambda_n^{\frac{1}{2}}}(x_n)$; (iii) $B^{N+1}_{\ell_n \Lambda_n^{\frac{1}{2}}}(x_n) \subset \Omega$ and $0 \in B^N_{\ell_n \Lambda_n^{\frac{1}{2}}}(x_n)$.

Firstly, define $\partial_+ \mathcal{F} = \{ z = (x, t) \in \mathbb{R}_+^{N+1} : (x, t) \in \partial \mathcal{F}, t > 0 \}$ and $\partial_0 \mathcal{F} = \{ x \in \mathbb{R}^N : (x, 0) \in \partial \mathcal{F} \cap \mathbb{R}^N \times \{0\} \}$. Let $B^N_{\ell_n} = B^N_{\ell_n \Lambda_n^{\frac{1}{2}}}(x_n) \cap \Omega, B^{N+1}_{\ell_n} = B^{N+1}_{\ell_n \Lambda_n^{\frac{1}{2}}}(x_n) \cap \mathcal{D}$, and $p_n = 2^* - \epsilon_n$. Then from Proposition 3.1, we have the following local Pohozaev identity for $u_n$ on $B^{N+1}_{\ell_n}$,

$$
\left( \frac{N}{p_n} - \frac{N - 2s}{2} \right) \int_{B^N_n} |u_n|^p dx + s \int_{B^N_n} |u_n|^2 dx + s \int_{B^N_n} \frac{x \cdot x_0 |u_n|^2}{|x|^{2s+2}} dx
$$

$$
= \frac{1}{2} \int_{\partial B^N_n} \left( a + \frac{\mu}{|x|^{2s+2}} \right) |u_n|^2 (x - x_0) \cdot \nu_x dS_x + \frac{1}{p_n} \int_{\partial B^N_n} |u_n|^{p_n} (x - x_0) \cdot \nu_x dS_x
$$

(4.4)

$$
+ \int_{\partial_+ B^{N+1}_{\ell_n}} t^{1-2s} \left( (z_0, \nabla \bar{u}_n) \nabla \bar{u}_n - \frac{|\nabla \bar{u}_n|^2}{2}, \nu_z \right) dS_z
$$

$$
+ \frac{N - 2s}{2} \int_{\partial_+ B^{N+1}_{\ell_n}} t^{1-2s} \bar{u}_n \frac{\partial \bar{u}_n}{\partial \nu_z} dS_z,
$$

where $z_0 = (x_0, 0), z = (x, t)$ and $x_0$ in (4.4) is chosen as follows. In case (i), we take $x_0 \in \mathbb{R}^N \setminus \Omega$ with $|x_0 - x_n| \leq 2\ell_n \Lambda_n^{\frac{1}{2}}$ and $\nu_x \cdot (x_0 - x_n) \leq 0$ in $\partial \mathcal{F} \cap B^N_n$. Then we see from the fact $\nu_z = (\nu_x, 0)$ that $\nu_z \cdot (z_0, 0) = \nu_x \cdot (x_0 - x_n) \leq 0$ and with this $x_0$, we can check that $x_0, x \geq 0$ in $B^N_n$. In case (ii), we take a point $x_0 = x_0$ Then $x_0 \cdot x \geq 0$ in $B^N_n$. In case (iii), we take $x_0 = 0$. Thus, in any case $x_0 \cdot x \geq 0$ in $B^N_n$. 


In fact, in case (i) and case (ii), $u_n \in C^{2s}(B_n^N)$. So, (4.4) is the usual local Pohozaev identity. Now we prove that (4.4) holds as well in case (iii).

To see this, since
\[ \int_{\Omega} |(-\Delta)^{2s} u_n|^2 dx = \int_{D} t^{1-2s} |\nabla \bar{u}_n|^2 dx dt \leq C, \]
we can choose $\theta_j \rightarrow 0$ as $j \rightarrow +\infty$ such that
\[
(4.5) \quad \theta_j \int_{\partial B_{n,\theta_j}^{N+1}(0)} t^{1-2s} |\nabla \bar{u}_n|^2 dS_z + \theta_j \int_{\partial B_{n,\theta_j}^N(0)} |u_n|^p x \cdot \nu_z dS_x + \theta_j \int_{\partial B_{n,\theta_j}^N(0)} \left( a + \frac{\mu}{|x|^{2s}} \right) |u_n|^2 dS_x \rightarrow 0.
\]

Let $B_{n,\theta_j}^{N+1} = B_{n,\theta_j}^{N+1} \setminus B_{\theta_j}^{N+1}(0)$. Then $u_n \in C^{2s}(B_{n,\theta_j}^N)$ and
\[
\left( \frac{N}{p_n} - \frac{N - 2s}{2} \right) \int_{B_{n,\theta_j}^N} |u_n|^p n, \theta dx + s\int_{B_{n,\theta_j}^N} |u_n|^2 dx + s\mu \int_{B_{n,\theta_j}^N} \frac{x \cdot x_0 |u_n|^2}{|x|^{2s+2}} dx
\]
\[
= \frac{1}{2} \int_{\partial B_{n,\theta_j}^N} \left( a + \frac{\mu}{|x|^{2s}} \right) |u_n|^2 x \cdot \nu_z dS_x + \frac{1}{p_n} \int_{\partial B_{n,\theta_j}^N} |u_n|^p x \cdot \nu_z dS_x
\]
\[
+ \int_{\partial B_{n,\theta_j}^{N+1}} t^{1-2s} \left( (z, \nabla \bar{u}_n) \nabla \bar{u}_n - z \frac{|\nabla \bar{u}_n|^2}{2}, \nu_z \right) dS_z
\]
\[
+ \frac{N - 2s}{2} \int_{\partial B_{n,\theta_j}^{N+1}} t^{1-2s} \frac{\partial \bar{u}_n}{\partial \nu_z} dS_z.
\]

From (4.5) and Proposition B.3, we have
\[
(4.7) \quad \left| \int_{\partial B_{n,\theta_j}^{N+1}(0)} t^{1-2s} \frac{\partial \bar{u}_n}{\partial \nu_z} dS_z \right| \leq \left( \int_{\partial B_{n,\theta_j}^{N+1}(0)} t^{1-2s} |\nabla \bar{u}_n|^2 dS_z \right)^{\frac{1}{2}} \left( \int_{\partial B_{n,\theta_j}^{N+1}(0)} t^{1-2s} |\bar{u}_n|^2 dS_z \right)^{\frac{1}{2}}
\]
\[
\leq C \theta_j^{-\frac{4}{2}} \theta_j^{1+\beta} = o(1),
\]
and
\[
(4.8) \quad \frac{1}{2} \int_{\partial B_{n,\theta_j}^{N+1}(0)} \left( a + \frac{\mu}{|x|^{2s}} \right) |u_n|^2 x \cdot \nu_z dS_x + \frac{1}{p_n} \int_{\partial B_{n,\theta_j}^{N+1}(0)} |u_n|^p x \cdot \nu_z dS_x
\]
\[
+ \int_{\partial B_{n,\theta_j}^{N+1}(0)} t^{1-2s} \left( (z, \nabla \bar{u}_n) \nabla \bar{u}_n - z \frac{|\nabla \bar{u}_n|^2}{2}, \nu_z \right) dS_z
\]
\[
= O \left( \theta_j \int_{\partial B_{n,\theta_j}^N} |u_n|^p x \cdot \nu_z dS_x + \theta_j \int_{\partial B_{n,\theta_j}^N} \left( a + \frac{\mu}{|x|^{2s}} \right) |u_n|^2 dS_x + \theta_j \int_{\partial B_{n,\theta_j}^N} t^{1-2s} |\nabla \bar{u}_n|^2 dS_z \right) = o(1).
\]

So, letting $j \rightarrow +\infty$ in (4.4), from (4.7) and (4.8), we can get (4.4)

Since $p_n < 2s$, the first term in the left hand side of (4.4) is nonnegative and by the choice of $x_0$, the third term in the left hand side of (4.4) is also nonnegative. Hence (4.4) can be rewritten as
\[
(4.9) \quad \left| \int_{B_n^N} |u_n|^2 dx \right| \leq \frac{1}{2} \int_{\partial B_n^N} \left( a + \frac{\mu}{|x|^{2s}} \right) |u_n|^2 (x - x_0) \cdot \nu_x dS_x + \frac{1}{p_n} \int_{\partial B_n^N} |u_n|^p (x - x_0) \cdot \nu_x dS_x
\]
\[
+ \int_{\partial B_n^{N+2}} t^{1-2s} \left( (z - z_0, \nabla \bar{u}_n) \nabla \bar{u}_n - (z - z_0) \frac{|\nabla \bar{u}_n|^2}{2}, \nu_z \right) dS_z
\]
\[
+ \frac{N - 2s}{2} \int_{\partial B_n^{N+1}} t^{1-2s} \frac{\partial \bar{u}_n}{\partial \nu_z} dS_z.
\]
Now we decompose $\partial B^N_n$ into $\partial B^N_n = \partial_1 B^N_n \cup \partial_t B^N_n$, where $\partial_1 B^N_n = \partial B^N_n \cap \Omega$ and $\partial_t B^N_n = \partial B^N_n \cap \partial \Omega$. Similarly, $\partial_1 B^N_{n+1} = \partial_1 B^N_{n+1} \cup \partial_t B^N_{n+1}$, where $\partial_1 B^N_{n+1} = \partial_1 B^N_{n+1} \cap \Omega$ and $\partial_t B^N_{n+1} = \partial_1 B^N_{n+1} \cap \partial \Omega$.

Observing that $u_n = 0$ on $\partial_1 B^N_n$ and $\bar{u}_n = 0$ on $\partial_t B^N_{n+1}$, we have

\[ \frac{1}{2} \int_{\partial_1 B^N_n} (a + \frac{\mu}{|x|^2}) |u_n|^2 (x - x_0) \cdot \nu_x dS_x + \frac{1}{p_n} \int_{\partial_t B^N_n} |u_n|^p (x - x_0) \cdot \nu_x dS_x \]

\[ + \frac{N - 2s}{2} \int_{\partial_t B^N_{n+1}} |\partial_{\nu_n} u_n|^2 dS_z = 0. \]

Also, noting that $\nabla \bar{u}_n = \pm |\nabla \bar{u}_n| \nu_z$ on $\partial_t B^N_{n+1}$, we find

\[ \int_{\partial_t B^N_{n+1}} |\partial_{\nu_n} u_n|^2 dS_z \leq 0. \]

Hence, we can rewrite (4.9) as

\[ \text{RHS of (4.9)} \leq C \Lambda_n^{\frac{2s}{p}} \int_{\partial_1 B^N_n} (u_n^2 + |u_n|^p + \mu \frac{u_n^2}{|x|^{2s}}) dS_x + C \int_{\partial_t B^N_{n+1}} t^{1-2s} |\nabla u_n||u_n| dS_z \]

\[ + C \Lambda_n^{\frac{2s}{p}} \int_{\partial_t B^N_{n+1}} t^{1-2s} |\nabla u_n|^2 dS_z \]

\[ \leq C \Lambda_n^{\frac{2s-1-N}{p} + \frac{N}{p}} + C \Lambda_n^{1+ \frac{2s-1-N}{p} + \frac{N}{p}} \]

\[ \leq C \Lambda_n^{\frac{2s-1-N}{p} + \frac{N}{p}}. \]

Recalling that in the proof of Lemma 2.6, we have the decomposition

\[ u_n = u_0 + \sum_{j=1}^{m} \rho_{0, \Lambda_n^{+}}(U_j) + \sum_{j=m+1}^{h} \rho_{x_{n,j}, \Lambda_n^{+}}(U_j) + \omega_n =: u_0 + u_{n,1} + u_{n,2}, \]

with $\|u_{n,2}\| \to 0$ as $n \to +\infty$. By Proposition B.3 and Lemma B.1, we can verify that if $N > 6s$,

\[ \int_{\mathbb{R}^N} |U_j|^2 dx < +\infty, \quad j = 1, 2, \ldots, h. \]

On the other hand, let $B^N_{L,n} = B^N_{L \Lambda^{-1}_n}(x_n)$, where $L > 0$ is so large that

\[ \int_{B^N_{L}(0)} |U_j|^2 dx > 0, \quad j = 1, 2, \ldots, h. \]
Since $u_n = 0$ in $\mathbb{R}^N \setminus \Omega$, we have
\[
\int_{B_n^N} |u_n|^2 \, dx = \int_{\mathbb{R}^N_{\mathcal{E} - \frac{1}{2} (x_n)}} |u_n|^2 \, dx \geq \int_{B_n^N} |u_n|^2 \, dx
\]  
(4.13)
\[
\geq \frac{1}{2} \int_{B_n^N} |u_n,1|^2 \, dx - C \int_{B_n^N} |u_0|^2 \, dx - C \int_{B_n^N} |u_{n,2}|^2 \, dx.
\]

Moreover, we have
\[
\int_{B_n^N} |u_0|^2 \, dx \leq \left( \int_{B_n^N} |u_0|^2 \, dx \right)^{\frac{1}{2}} |B_n^N|^{1 - \frac{2}{s}} \leq C \Lambda_n^{-2s} \| u_0 \|_{L_{s}^{2s}(B_n^N)}^2 = o(1) \Lambda_n^{-2s},
\]  
(4.14)
and
\[
\int_{B_n^N} |u_{n,2}|^2 \, dx \leq C \left( \int_{B_n^N} |u_{n,2}|^2 \, dx \right)^{\frac{1}{2}} \Lambda_n^{-2s} = o(1) \Lambda_n^{-2s},
\]  
(4.15)
since $\| u_{n,2} \| \to 0$ as $n \to \infty$.

On the other hand, we may assume that $\rho_{x_n,1,\Lambda_n,1}(U_1)$ is the bubble with slowest concentration rate. Then
\[
\int_{B_n^N} |u_{n,1}|^2 \, dx \geq \frac{1}{2} \int_{B_n^N} |\rho_{x_n,1,\Lambda_n,1}(U_1)|^2 \, dx + O \left( \sum_{j=2}^k \int_{B_n^N} |\rho_{x_n,j,\Lambda_n,j}(U_j)|^2 \, dx \right).
\]

By direct calculations, we can obtain
\[
\int_{B_n^N} |\rho_{x_n,1,\Lambda_n,1}(U_1)|^2 \, dx = \Lambda_n^{-2s} \int_{B_n^N(0)} |U_1|^2 \, dx \geq C' \Lambda_n^{-2s},
\]
for some constant $C' > 0$. Similarly, we have
\[
\int_{B_n^N} |\rho_{x_n,j,\Lambda_n,j}(U_j)|^2 \, dx = \Lambda_n^{-2s} \int_{(B_n^N(0))_{x_n,j,\Lambda_n,j}} |U_j|^2 \, dx,
\]
(4.16)
where we use the notation $E_{x,\Lambda} = \{ y : \Lambda^{-1}y + x \in E \}$ for any set $E$.

If $\frac{\Lambda_n}{\Lambda_n,1} \to +\infty$, then we obtain from (4.16)
\[
\int_{B_n^N} |\rho_{x_n,j,\Lambda_n,j}(U_j)|^2 \, dx = o(\Lambda_n^{-2s}).
\]

If $\frac{\Lambda_n}{\Lambda_n,1} \leq C < +\infty$, then
\[
(B_n^N)_{x_n,j,\Lambda_n,j} = \{ y : \Lambda_n^{-1}y + x_n,j \in B_n^N \} = \{ y : |\Lambda_n^{-1}y + x_n,j - x_n,1| \leq \Lambda_n L \Lambda_n^{-1} \} \subset \{ y : |y + \Lambda_n (x_n,j - x_n,1)| \leq C \}.
\]

Since $|\Lambda_n (x_n,j - x_n,1)| \to +\infty$ as $n \to +\infty$, we find that $(B_n^N)_{x_n,j,\Lambda_n,j}$ moves to infinity. Hence it follows from (4.12) and (4.16) that
\[
\int_{B_n^N} |\rho_{x_n,j,\Lambda_n,j}(U_j)|^2 \, dx = o(\Lambda_n^{-2s}).
\]

Therefore, we have proved that there exists a constant $C' > 0$, such that
\[
\int_{B_n^N} |u_{n,1}|^2 \, dx \geq C' \Lambda_n^{-2s}.
\]  
(4.17)
Hence, from (4.13) to (4.17), we get
\begin{equation}
\text{LHS of (4.10)} \geq \frac{C'}{4} \Lambda_n^{-2s}.
\end{equation}

Combing (4.11) and (4.18), we obtain
\begin{equation}
\Lambda_n^{-2s} \leq CA_n^{\frac{2s-N}{2s} + \frac{N}{p_1}},
\end{equation}
where $p_1 > 2^*_s$ is any constant, satisfying $p_1 < \frac{2^*_s \sqrt{\mu}}{\sqrt{\mu} - \sqrt{\mu - \mu}}$. Choose $p_1 = \frac{2N}{N-6s}$ with $p_1 + \delta < \frac{2^*_s \sqrt{\mu}}{\sqrt{\mu} - \sqrt{\mu - \mu}}$, where $\delta > 0$ is a small constant. Then from the assumption on $\mu$, we see $2s < \frac{N-6s - N}{p_1}$.

So, we obtain a contradiction to (4.19).

\textbf{Proof of Theorem 1.1} This is a direct consequence of Theorem 1.2. See for example [15, 16, 18, 12].

\textbf{Appendix A. Some basic estimates on linear problems}

In this section, we deduce some elementary estimates for solutions of linear elliptic problem involving Hardy potential. These estimates are of independent interest.

\textbf{Lemma A.1.} Let $u \in H^p_0(\Omega)$ be a solution of (1.1). Then one has

\begin{equation}
u \in L^p(\Omega), \forall p < \frac{2^*_s \sqrt{\mu}}{\sqrt{\mu} - \sqrt{\mu - \mu}}.
\end{equation}

\textbf{Proof.} Just by the same argument as that of Lemma 2.1 in [14], we can prove our result. So we omit it here.

\textbf{Lemma A.2.} Let $w$ be a solution of

\begin{equation}
\begin{cases}
(-\Delta)^s w - \frac{\mu w}{|x|^{2s}} = a(x)v, & \text{in } \Omega, \\
w = 0, & \text{on } \partial \Omega,
\end{cases}
\end{equation}

where $a(x) \geq 0, v \geq 0$ are functions and $a, v \in C^2(\Omega \setminus B_\delta(0))$ for any $\delta > 0$ small. Then for any $p > \frac{N}{N-2s}$ and $0 \leq \mu < \tilde{\mu}$ satisfying $p < \frac{2^*_s \sqrt{\mu}}{\sqrt{\mu} - \sqrt{\mu - \mu}}$, there is a constant $C = C(p)$ such that

\begin{equation}
||w||_{L^p} \leq C ||a||_{L^p} ||v||_{L^p}.
\end{equation}

\textbf{Proof.} Let $q = \frac{p}{N-2s}$. Then $q > \frac{1}{2}$.

First we assume $p \geq 2^*_s$. In this case $q \geq 1$. Let $\tilde{\varphi} = \tilde{w}L\tilde{w}_L^{2(q-1)}$, where $\tilde{w}_L = \min\{\tilde{w}, L\}$. Then we have

\begin{equation}
\nabla \tilde{\varphi} = 2(q-1)\tilde{w}_L^{2(q-1)} \nabla \tilde{w}_L + \tilde{w}_L^{2(q-1)} \nabla \tilde{w}.
\end{equation}

Since $q > 1$, it is easy to see that $\nabla \tilde{\varphi} \in L^2(p^{1-2s}, D)$. Thus $\tilde{\varphi} \in H^q_0(t^{1-2s}, D)$. So, we have

\begin{equation}
\int_D t^{1-2s} \nabla \tilde{w} \nabla (\tilde{w}_L^{2(q-1)}) dx dt = \int_\Omega \frac{\mu \tilde{w}^{2(q-1)}}{|x|^{2s}} dx + \int_\Omega a(x) \tilde{w}L\tilde{w}_L^{2(q-1)} dx
\end{equation}

Letting $\tilde{\eta} = \tilde{w}L\tilde{w}_L^{2(q-1)}$, from Hardy-Sobolev inequality we find

\begin{equation}
\int_\Omega \frac{\mu \tilde{w}^{2(q-1)}}{|x|^{2s}} dx \leq \frac{\mu}{\tilde{\mu}} \int_\Omega \left| (-\Delta)^s \tilde{\eta} (\tilde{w}_L^{2(q-1)}) \right|^2 dx = \frac{\mu}{\tilde{\mu}} \int_\Omega \left| (-\Delta)^s \tilde{\eta} \right|^2 dx.
\end{equation}
Moreover, it follows from $|\nabla \bar{w}_L| \leq |\nabla \bar{w}|$ that

(A.3) \[
\int_\Omega |(-\Delta)^{q/2} \eta|^2 dx = \int_\mathcal{D} t^{1-2s} (\bar{w}_L^2)^{q-1} |\nabla \bar{w}|^2 + (q^2 - 1) \bar{w}_L^{2(q-1)} |\nabla \bar{w}_L|^2 dx dt \\
\leq \int_\mathcal{D} t^{1-2s} \left[ (q^2 - q - 2) \bar{w}_L^{2(q-1)} |\nabla \bar{w}|^2 + \frac{q^2(2q-2)}{q-1} \bar{w}_L^{2(q-1)} |\nabla \bar{w}_L|^2 \right] dx dt \\
= \frac{q^2}{2q-1} \int_\mathcal{D} t^{1-2s} \nabla \bar{w} \nabla (\bar{w} \bar{w}_L^{2(q-1)}) dx dt.
\]

From (A.1) - (A.3), we get

(A.4) 
\[
\left( \frac{2q-1}{q^2} - \frac{\mu}{\mu} \right) \int_\Omega |(-\Delta)^{q/2} \eta|^2 dx \leq \int_\Omega a(x)v \phi dx.
\]

Noting that $q < \frac{\sqrt{q}}{\sqrt{\mu} - \sqrt{\mu}}$ implies $\frac{2q-1}{q^2} - \frac{\mu}{\mu} \geq c_0 > 0$, we obtain from (A.4) that there is a $c' > 0$ such that

(A.5) 
\[
\int_\Omega a(x)v \phi dx \geq c_0 \int_\Omega |(-\Delta)^{q/2} \eta|^2 dx \geq c' \left( \int_\Omega |\eta|^{2q} dx \right)^{\frac{q}{2}}.
\]

On the other hand, by Hölder inequality we have

(A.6) 
\[
\int_\Omega a(x)v \phi dx \leq \left( \int_\Omega |v|^p dx \right)^{\frac{1}{p}} \left( \int_\Omega |a(x)|^{\frac{2q}{2q-1}} dx \right)^{\frac{2q-1}{2q}} \left( \int_\Omega |\phi|^{\frac{2q}{2q-1}} dx \right)^{\frac{2q-1}{2q}} \\
\leq \|v\|_p \|a\|^{rac{1}{p}} \left( \int_\Omega |\phi|^{\frac{2q}{2q-1}} dx \right)^{\frac{2q-1}{2q}} \\
\leq \|v\|_p \|a\|^{rac{1}{p}} \left( \int_\Omega |\eta|^{2q} dx \right)^{\frac{2q-1}{2q}},
\]

since $\frac{q}{2q-1} \leq 1$.

Thus,

(A.7) 
\[
c_0 \left( \int_\Omega |\eta|^{2q} dx \right)^{\frac{1}{2q}} \leq \|v\|_p \|a\|^{rac{1}{p}},
\]

From (A.2), (A.4), (A.6) and (A.7), we obtain

(A.8) 
\[
\int_\Omega \frac{\mu}{|x|^{2s}} \bar{w}^2 \bar{w}_L^{2(q-1)} dx \leq C \left( \|v\|_p \|a\|^\frac{1}{p} \right)^{2q}.
\]

Letting $L \to \infty$ in (A.7) and (A.8), we obtain the result.

Now we consider the case $q \in (\frac{2}{q}, 1)$. In this case, $w \bar{w}_L^{2(q-1)}$ may not be in $H_0^q(\Omega)$. Hence we have to deal with it differently.

By the comparison principle, we know that $w \geq 0$ in $\Omega$. For any $\theta > 0$ being a small number, let $\bar{\eta} = (\bar{w} + \theta)^{2q-1} \xi^2$, where $\xi \geq 0$ is a function satisfying $\xi = 0$ on $\partial \Omega \times [0, \infty)$; $\xi^2$ is in $D_\theta := \Omega_\theta \times [0, 1) = \{ x : x \in \Omega, d(x, \partial \Omega) \geq \theta^2 \} \times [0, 1)$, $0 < \xi < 1$ on $\Omega \setminus \Omega_\theta \times [0, 1)$; $\xi = 0$ in $\Omega \times [1, \infty)$ and $|\nabla \xi| \leq \frac{1}{\theta^2}$. Then $\bar{\eta} \in H_0^q(1-2s, \mathcal{D})$ and

\[
\nabla \bar{\eta} = (\bar{w} + \theta)^{2q-1} \nabla \xi^2 + (2q - 1)(\bar{w} + \theta)^{2(q-1)} \xi^2 \nabla \bar{w}.
\]
Moreover, from the assumption on $\xi$, $\xi$ satisfies $\xi \geq 0$, $\xi = 0$ on $\partial \Omega$, $\xi > 0$ in $\Omega$ and $\xi = 1$ in $\Omega_\theta$ and $|\nabla \xi| \leq \frac{2}{\theta}$. So, we have

$$\begin{align*}
(A.9) \quad \int_D t^{1-2s} \nabla \hat{w} \nabla \hat{q} dx dt &= \int_\Omega \mu \frac{w(w + \theta)^{2q-1} \xi^2}{|x|^{2s}} dx + \int_\Omega a(x)v(w + \theta)^{2q-1}\xi^2 dx.
\end{align*}$$

On the other hand,

$$\begin{align*}
(A.10) \quad \int_D t^{1-2s} \nabla \hat{w} \nabla \hat{q} dx dt &= (2q - 1) \int_D t^{1-2s} \hat{\xi}^2 (\hat{\omega} + \theta)^{2(q-1)} |\nabla (\hat{\omega} + \theta)|^2 dx dt + \int_D t^{1-2s} (\hat{\omega} + \theta)^{2q-1} \nabla (\hat{\omega} + \theta) \nabla \hat{\xi}^2 dx dt
\end{align*}$$

$$\begin{align*}
&= \frac{2q - 1}{q^2} \int_D t^{1-2s} \nabla (\hat{\xi}(\hat{\omega} + \theta))^q dx dt + \int_D t^{1-2s} (\hat{\omega} + \theta)^{2q-1} \nabla (\hat{\omega} + \theta) \nabla \hat{\xi}^2 dx dt
\end{align*}$$

$$\begin{align*}
&= \frac{2q - 1}{q^2} \int_D t^{1-2s} (\hat{\omega} + \theta)^{2q-1} \nabla \hat{\xi}^2 dx dt + \int_D t^{1-2s} (\hat{\omega} + \theta)^{2q-1} \nabla (\hat{\omega} + \theta) \nabla \hat{\xi}^2 dx dt.
\end{align*}$$

From $a, v \in C^{2s}(\Omega \setminus B_\delta(0))$ for any $\delta > 0$ small, it follows from [34] that $w \in C^{\beta}(\Omega \setminus B_\delta(0))$ for any $\beta \in [s, 1 + 2s)$ and $w(x) \leq Cd^s(x, \partial \Omega) \leq C\theta^s, |\nabla w| \leq C, \forall x \in \Omega \setminus \Omega_\theta$.

As a consequence, (A.10) becomes

$$\begin{align*}
(A.11) \quad \int_D t^{1-2s} \nabla \hat{w} \nabla \hat{q} dx dt &= \frac{2q - 1}{q^2} \int_D t^{1-2s} |\nabla (\hat{\xi}(\hat{\omega} + \theta))^q dx dt + O\left(\int_{(\Omega \setminus \Omega_\delta) \times (0, 1)} t^{1-2s}(\theta^s)^{2q-1} dx dt\right)
\end{align*}$$

$$\begin{align*}
&= \frac{2q - 1}{q^2} \int_D \left( (-\Delta)^s (\xi(w + \theta))^q \right)^2 dx + O((\theta^s)^{2q-1}).
\end{align*}$$

By (A.9) and (A.11), we get

$$\begin{align*}
(A.12) \quad &2q - 1 \int_\Omega (\Delta)^s (\xi(w + \theta))^q |^2 dx + O((\theta^s)^{2q-1}) - \int_\Omega \mu \frac{w(w + \theta)^{2q-1} \xi^2}{|x|^{2s}} dx
\end{align*}$$

$$\begin{align*}
&= \int_\Omega a(x)v(w + \theta)^{2q-1}\xi^2 dx.
\end{align*}$$

But,

$$\begin{align*}
(A.13) \quad &\mu \int_\Omega \frac{w(w + \theta)^{2q-1} \xi^2}{|x|^{2s}} dx \leq \mu \int_\Omega \frac{(w + \theta)^{2q} \xi^2}{|x|^{2s}} dx \leq \frac{\mu}{\mu} \int_\Omega (\Delta)^s (\xi(w + \theta))^q |^2 dx.
\end{align*}$$

From the assumptions on $q$ and $\mu$, (A.12) and (A.13), we can deduce

$$\begin{align*}
(A.14) \quad &C' \left( \int_\Omega (\xi(w + \theta))^q dx \right)^{q} + O((\theta^s)^{2q-1}) \leq \int_\Omega a(x)v(w + \theta)^{2q-1}\xi^2 dx,
\end{align*}$$

and

$$\begin{align*}
(A.15) \quad &C' \int_\Omega \mu \frac{(w + \theta)^{2q} \xi^2}{|x|^{2s}} dx + O((\theta^s)^{2q-1}) \leq \int_\Omega a(x)v(w + \theta)^{2q-1}\xi^2 dx.
\end{align*}$$
Letting $\theta \to 0$ in (A.12) and (A.13), we find
\[
C' \left( \int_{\Omega} w^{q_2^p} dx \right)^{\frac{2}{q_2^p}} \leq \int_{\Omega} a(x)v w^{2q-1} dx \leq ||a||_{\infty} ||v||_{p} ||w||_{2q-1}^{2q},
\]
and
\[
C' \int_{\Omega} \mu \frac{w^{2q}}{|x|^{2s}} dx \leq \int_{\Omega} a(x)v w^{2q-1} dx \leq ||a||_{\infty} ||v||_{p} ||w||_{2q-1}^{2q}.
\]
Therefore, the result follows.

**Lemma A.3.** Let $w$ be a solution of
\[
\begin{cases}
(-\Delta)^{s} w - \frac{\mu w}{|x|^{2s}} = f(x), & \text{in } \Omega, \\
w = 0, & \text{on } \partial \Omega.
\end{cases}
\]
Suppose that $f \in C^{s}(\Omega \setminus B_{\delta}(0))$ for any small $\delta > 0$. Then for any $\frac{N}{2s} > p \geq 1$ and $\mu$ with $\frac{Np}{N-2sp} < \frac{2^{*} \sqrt{\mu}}{\sqrt{\mu} - \mu}$, there is a constant $C = C(p)$ such that
\[
||w||_{s, \frac{Np}{N-2sp}} \leq C ||f||_{p}.
\]
**Proof.** Similar to the proof of Lemma A.2, we can deduce if $q > \frac{1}{2}$ with $q < \frac{\sqrt{\mu}}{\sqrt{\mu} - \mu}$, then
\[
C' ||w||_{2q}^{2q} \leq \int_{\Omega} f(x)w^{2q-1} dx,
\]
and
\[
C' \int_{\Omega} \mu \frac{|w|^{2q}}{|x|^{2s}} dx \leq \int_{\Omega} f(x)w^{2q-1} dx.
\]
For any $\frac{N}{2s} > p \geq 1$, see $2p - 2^{*}p > 0$. Let $q = \frac{p}{p - 2^{*}p} > \frac{1}{2}$. Then
\[
2^{*}q = \frac{2^{*}p}{2p - 2^{*}p} = \frac{Np}{N - 2sp} \quad \text{and} \quad \frac{2q - 1}{p - 1} = 2^{*}q.
\]
So,
\[
\left| \int_{\Omega} f(x)w^{2q-1} dx \right| \leq ||f||_{p} \left( \int_{\Omega} ||w||_{2^{*}p}^{2^{*}p-1} dx \right)^{1 - \frac{1}{2}} \leq ||f||_{p} ||w||_{2^{*}q}^{2q-1}.
\]
As a result,
\[
||w||_{2^{*}q} \leq C ||f||_{p},
\]
and
\[
\mu \int_{\Omega} \frac{|w|^{2q}}{|x|^{2s}} dx \leq C ||f||_{p}^{2q}.
\]
So the result follows.

**Lemma A.4.** Let $w \geq 0$ be a solution of
\[
\begin{cases}
(-\Delta)^{s} w - \frac{\mu w}{|x|^{2s}} = a(x)v, & \text{in } \Omega, \\
w = 0, & \text{on } \partial \Omega,
\end{cases}
\]
where $a(x) \geq 0$ and $v \geq 0$ are functions satisfying $a, v \in C^{2s}(\Omega \setminus B_{\delta}(0))$ for any small $\delta > 0$. Then for any $2^{*}s > p_{2} > \frac{N}{N - 2s}$, there is a constant $C = C(p_{2})$ such that
\[
||w||_{s, p_{2}} \leq C ||a||_{r} ||v||_{2^{*}},
\]
where $r$ is determined by $\frac{1}{r} = \frac{1}{p_{2}} + \frac{2s}{N} - \frac{1}{2^{*}}$. \hfill \qed
Proof. Let $q = \frac{2p_2}{2s}$. Since $p_2 \in \left(\frac{N}{N+2s}, 2s\right)$, we see $\frac{q}{r} \in (\frac{1}{2}, 1)$. Let $t = \frac{2N}{N+2s}$. Similar to the proof of Lemma A.2, we obtain

\begin{equation}
C \|w\|_{p_2}^q = C \|w\|_{p_2}^q \leq \int_{\Omega} a(x)wv^{q-1}dx \leq \|v\|_{p} \|a\|_{r} \left( \int_{\Omega} |w|^{\frac{(q-1)r}{r-t}}dx \right)^{\frac{r-t}{q}}.
\end{equation}

and

\begin{equation}
C \int_{\Omega} \frac{\mu |w|^q}{|x|^{2s}}dx \leq \|v\|_{p} \|a\|_{r} \left( \int_{\Omega} |w|^{\frac{(q-1)r}{r-t}}dx \right)^{\frac{r-t}{q}}.
\end{equation}

By the definition, choose

\begin{equation}
\frac{(q-1)r}{r-t} = \frac{2p_2 - 1}{\frac{1}{r} - \frac{2}{r}} = p_2.
\end{equation}

So,

\begin{equation}
\|w\|_{p_2}^q \leq C \|v\|_{p} \|a\|_{r} \|w\|_{p_2}^{q-1}.
\end{equation}

Moreover, it is easy to check

\begin{equation}
\frac{1}{p_2} = \frac{1}{r} + \frac{1}{2s} - \frac{2s}{N}.
\end{equation}

Therefore, the result follows. \qed

Lemma A.5. Let $w \geq 0$ be a weak solution of

\begin{equation}
(-\Delta)^s w - \frac{\mu w}{|x|^{2s}} = a(x)w \quad \text{in} \quad \mathbb{R}^N,
\end{equation}

where $a(x) \geq 0$. Suppose that there is a small constant $\delta > 0$ such that $\int_{B(\tilde{x})} |a|^{\frac{N}{N-2s}}dx \leq \delta$, then, for any $p > \max\{2^*, 2^t\}$ and $0 \leq \mu \leq \mu$ satisfying $p \leq \min\left\{ \frac{2^*}{\sqrt{N-2s}}, \frac{2^t}{\sqrt{N-2s}} \right\}$ there is a constant $C = C(p)$ such that

\begin{equation}
\|	ilde{w}\|_{L^p(I^{1-2s}, B^{N+1}(\tilde{x}))} + \|w\|_{L^p(I^{1-2s}, B^{N+1}(\tilde{x}))} \leq C \|\tilde{w}\|_{L^p(I^{1-2s}, B^{N+1}(\tilde{x}))},
\end{equation}

where $\gamma < \min\{2^*, 2^t\}$ and $2^t = \frac{2(N+1)}{N}$. \qed

Proof. Let $1 \geq R > r > 0$. Take $\xi \in C_0^\infty(B^N_{r+1}(\tilde{x}))$, with $\tilde{\xi} = 1$ in $B^N_{r+1}(\tilde{x})$, $0 \leq \tilde{\xi} \leq 1$, and $|\nabla \tilde{\xi}| \leq \frac{C}{R-r}$. Let $q = \frac{p}{2^t}, \tilde{\eta} = \tilde{\xi}^2 \tilde{w}w_0^{2(q-1)}$. We have

\begin{equation}
\int_{R^N} t^{1-2s} \nabla \tilde{w} \nabla \tilde{\eta} \xi dt - \int_{R^N} \frac{\mu}{|x|^{2s}} \tilde{w} \tilde{\eta} \xi dx \leq \int_{R^N} a(x) \tilde{w} \tilde{\eta} \xi dx.
\end{equation}

Firstly, by Hölder inequality, we have

\begin{equation}
\int_{R^N} a(x) \tilde{w} \tilde{\eta} \xi dx \leq \left( \int_{B(\tilde{x})} |a(x)|^{\frac{N}{N-2s}}dx \right)^{\frac{N-2s}{N}} \left( \int_{B(\tilde{x})} (\xi \tilde{w}w_0^{q-1})^{2^t}dx \right)^{\frac{1}{2^t}}.
\end{equation}

On the other hand, similar to the proof of Lemma A.2 by Hardy-Sobolev inequality we can deduce that if $q < \frac{\sqrt{N}}{\sqrt{N-2s} - \mu}$, then

\begin{equation}
\int_{R^N} t^{1-2s} \nabla \tilde{w} \nabla \tilde{\eta} \xi dx - \int_{R^N} \frac{\mu}{|x|} \tilde{w} \tilde{\eta} \xi dx \geq C' \int_{R^N} t^{1-2s} |\nabla (\tilde{\xi} \tilde{w}w_0^{q-1})|^2 dx dt - C \int_{R^N} t^{1-2s} |\nabla \tilde{\xi}|^2 (\tilde{w}w_0^{q-1})^2 dx dt.
\end{equation}
If $\delta$ is small enough, it follows from (A.18) and (A.19) that there exists $C > 0$ such that
\begin{equation}
(A.20) \quad \int_{R^{N+1}} t^{1-2s} |\nabla (\xi \bar{w} \bar{w}^{q-1}_L)|^2 dx \leq C \int_{R^{N+1}} t^{1-2s} |\nabla \xi|^2 (\bar{w} \bar{w}^{q-1}_L)^2 dx dt.
\end{equation}

Using the Sobolev inequality and Lemma 3.1, we obtain from (A.20) that
\begin{equation}
(A.21) \quad \left( \int_{R^{N+1}} (\xi \bar{w} \bar{w}^{q-1}_L)^2 dx \right)^{\frac{\gamma}{2}} + \left( \int_{R^{N+1}} t^{1-2s} (\xi \bar{w} \bar{w}^{q-1}_L)^{\frac{2N+1}{2N}} dx \right)^{\frac{2\gamma}{N}} \leq C \int_{R^{N+1}} t^{1-2s} |\nabla \xi|^2 (\bar{w} \bar{w}^{q-1}_L)^2 dx dt,
\end{equation}
which yields
\begin{equation}
(A.22) \quad \left( \int_{B_{1+r}(\bar{x})} |w|^q dx \right)^{\frac{1}{q}} \leq \left( \frac{C}{R-r} \right)^{\frac{1}{q}} \left( \int_{B_{N+1}(\bar{x}) \setminus B_{N+1}(\bar{x})} t^{1-2s} |\bar{w}|^2 dx \right)^{\frac{1}{2}}
\end{equation}
and
\begin{equation}
(A.23) \quad \left( \int_{B_{N+1}(\bar{x})} t^{1-2s} |\bar{w}|^q dx \right)^{\frac{1}{q}} \leq \left( \frac{C}{R-r} \right)^{\frac{1}{q}} \left( \int_{B_{N+1}(\bar{x})} t^{1-2s} |\bar{w}|^2 dx \right)^{\frac{1}{2}},
\end{equation}
where $2^f = \frac{2(N+1)}{N}$.

Let $\chi = \frac{2^f}{N} > 1$. For any $0 < r^* < R^* < 1$, define $r_i = r^* + \frac{i}{p} (R^* - r^*)$, $i = 0, 1, 2, \cdots$. Then $r_i - r_{i+1} = \frac{1}{p} (R^* - r^*)$. Taking $R = r_i, r = r_{i+1}, q = \chi^i$ in (A.23), we get
\begin{equation}
(A.24) \quad \left( \int_{B_{N+1}(\bar{x})} t^{1-2s} |\bar{w}|^q dx \right)^{\frac{1}{q}} \leq \left( \frac{C2^{i+1}}{R^* - r^*} \right)^{\frac{1}{q}} \left( \int_{B_{N+1}(\bar{x})} t^{1-2s} |\bar{w}|^2 dx \right)^{\frac{1}{2}}.
\end{equation}

By iteration, for any $0 < r^* < R^* < 1$, we can obtain from (A.24)
\begin{equation}
(A.25) \quad \left( \int_{B_{N+1}(\bar{x})} t^{1-2s} |\bar{w}|^q dx \right)^{\frac{1}{q}} \leq \frac{C}{(R-r)^{\sum_{j=1}^{i} \frac{1}{q}}} \left( \int_{B_{N+1}(\bar{x})} t^{1-2s} |\bar{w}|^2 dx \right)^{\frac{1}{2}}.
\end{equation}

Note that $\sum_{j=1}^{i} \frac{1}{\chi^j} < \sum_{j=1}^{\infty} \frac{1}{\chi^j} = 1 - \frac{1}{\chi} = N$. Hence, we have proved that for any $p > 2^f$ satisfying $p < \frac{2^f \sqrt{q}}{\sqrt{\mu} - \sqrt{p} - \mu}$, there is a $\sigma > 0$ depending on $p$ such that
\begin{equation}
(A.26) \quad ||\bar{w}||_{L^p(t^{1-2s}, B_{N+1}(\bar{x}))} \leq \frac{C}{(R-r)^{\sigma}} ||\bar{w}||_{L^2(t^{1-2s}, B_{N+1}(\bar{x}))}, \quad 0 < r < R \leq 1.
\end{equation}

Applying Young’s inequality, we have
\begin{equation}
(A.27) \quad \frac{C}{(R-r)^{\sigma}} \left( \int_{B_{N+1}(\bar{x})} t^{1-2s} |\bar{w}|^2 dx \right)^{\frac{1}{2}} \leq \frac{C}{(R-r)^{\sigma}} \left( \int_{B_{N+1}(\bar{x})} t^{1-2s} |\gamma dx \right)^{\frac{1}{2}} \left( \int_{B_{N+1}(\bar{x})} t^{1-2s} |\bar{w}|^p dx \right)^{\frac{1}{2}}
\end{equation}
\begin{equation}
\leq \frac{1}{2 ||\bar{w}||_{L^p(t^{1-2s}, B_{N+1}(\bar{x}))}} + \frac{C}{(R-r)^{\sigma}} ||\bar{w}||_{L^\gamma(t^{1-2s}, B_{N+1}(\bar{x}))},
\end{equation}
where $0 < \kappa < 1, \gamma < 2^f$ and $p > 2^f$ with $p < \frac{2^f \sqrt{q}}{\sqrt{\mu} - \sqrt{p} - \mu}$.

So,
\begin{equation}
(A.28) \quad ||\bar{w}||_{L^p(t^{1-2s}, B_{N+1}(\bar{x}))} \leq \frac{1}{2 ||\bar{w}||_{L^p(t^{1-2s}, B_{N+1}(\bar{x}))}} + \frac{C}{(R-r)^{\sigma}} ||\bar{w}||_{L^\gamma(t^{1-2s}, B_{N+1}(\bar{x}))},
\end{equation}
where $\gamma < 2^f$ and $2^f < p < \frac{2^f \sqrt{q}}{\sqrt{\mu} - \sqrt{p} - \mu}$. 
By using iteration argument, we deduce from (A.28) that for any $p > 2^\delta$
\[ \|\bar{w}\|_{L^p(t^{1-2\epsilon}, B^{N+1}_{R}(\bar{x}))} \leq \frac{C}{(R-r)^2} \|\bar{w}\|_{L^\gamma(t^{1-2\epsilon}, B^{N+1}_{R}(\bar{x}))}, \]
where $p > 2^\delta$ satisfies $p < \frac{2^\delta \sqrt{r}}{\sqrt{N-\sqrt{r}-\mu}}$ and $\gamma < 2^\delta$.
Similarly, by applying (A.22) and iteration argument, we can get that
\[ \|w\|_{L^p(B^{N}_{\frac{r}{2}}(\bar{x}))} \leq C\|\bar{w}\|_{L^\gamma(t^{1-2\epsilon}, B^{N+1}_{R}(\bar{x}))}, \]
where $p > 2^\ast$ satisfies $p < \frac{2^\ast \sqrt{r}}{\sqrt{N-\sqrt{r}-\mu}}$ and $\gamma < 2^\ast$. This completes our proof. \qed

**Appendix B. A Decay Estimate**

Let $u$ be a solution of
\[ (\Delta)^s u - \frac{\mu u}{|x|^2} = |u|^{2^\ast-2} u \quad \text{in} \mathbb{R}^N, \]
\[ u \in H^s(\mathbb{R}^N). \]
In this section, we will estimate the decay of the solution of (B.1). We have the following result:

**Lemma B.1.** Let $u$ be a solution of (B.1). Then there exists a constant $\beta \in (s, \frac{N-2\delta}{2})$ such that
\[ |u(x)| \leq \frac{C}{|x|^N + \beta}, \quad \forall |x| \geq 1. \]

First, similar to Proposition B.1 in [15], we can prove

**Lemma B.2.** Let $u$ be a solution of (B.1). Then there exists a constant $\tilde{\beta} > \frac{N-2\delta}{2}$ such that
\[ |u(x)| \leq \frac{C}{|x|^N + \tilde{\beta}}, \quad \forall |x| \geq 1. \]

**Proof.** Choose $R_0 > 0$ large. For any $R > r > R_0$, take $\bar{\xi} \in C^\infty_0(\mathbb{R}^{N+1})$ with $\bar{\xi} = 0$ in $B^{N+1}_r(0)$, $\bar{\xi} = 1$ in $\mathbb{R}^{N+1}_+ \setminus B^{N+1}_{r+1}(0)$, $0 \leq \xi \leq 1$ and $|\nabla \xi| \leq \frac{C}{r-\delta}$. Let $\tilde{\eta} = \bar{\xi}^2 \bar{u}_+^{2^\ast(q-1)}$. Since for any small $\delta > 0$,
\[ \int_{\mathbb{R}^N \setminus B_{R_0}(0)} |u|^{2^\ast-2} \tilde{\eta}^\frac{2^\ast}{2} dx = \int_{\mathbb{R}^N \setminus B_{R_0}(0)} |\bar{u}_+|^2 \tilde{\eta} dx \leq \delta, \]
if $R_0 > 0$ large enough, we can prove in a similar way as in (A.20) that for any $2^\delta < p < \frac{2^\delta \sqrt{r}}{\sqrt{N-\sqrt{r}-\mu}}$, there is an $R > 0$ large depending on $p$ such that
\[ \|\bar{u}_+\|_{L^p(t^{1-2\epsilon}, \mathbb{R}^{N+1}_+ \setminus B^{N+1}_{2R}(0))} \leq \frac{C}{R^{N-\delta}} \|\bar{u}_+\|_{L^{2^\delta(t^{1-2\epsilon}, \mathbb{R}^{N+1}_+ \setminus B^{N+1}_{2R}(0))}}, \]
where $\delta > 0$ depending on $p$ and $2^\delta = \frac{2(N+1)}{N}$.

Next we estimate $\|\bar{u}_+\|_{L^{2^\delta}(\mathbb{R}^{N+1}_+ \setminus B^{N+1}_{2R}(0))}$. Let $\tilde{\eta} = \bar{\xi}^2 \bar{u}_+$, $\bar{\xi} = 0$ in $B^{N+1}_R(0)$, $\bar{\xi} = 1$ in $\mathbb{R}^{N+1}_+ \setminus B^{N+1}_{2R}(0)$, $0 \leq \xi \leq 1$ and $|\nabla \xi| \leq \frac{C}{R}$. Then similar to the proof of (A.21), by Hölder inequality we have
\[ \left( \int_{\mathbb{R}^{N+1}_+} t^{1-2\epsilon} |\xi \bar{u}_+|^2 \right)^{\frac{1}{2}} \leq \frac{C}{R} \left( \int_{B^{N+1}_R(0) \setminus B^{N+1}_{2R}(0)} t^{1-2\epsilon} |\bar{u}_+|^2 dx dt \right)^{\frac{1}{2}} \]
\[ \leq \frac{C}{R^{1-\frac{2\epsilon}{2N+1}} \left( \int_{B^{N+1}_R(0) \setminus B^{N+1}_{2R}(0)} t^{1-2\epsilon} |\bar{u}_+|^2 dx dt \right)^{\frac{1}{2}}}. \]
As a result,
\begin{align}
(B.4) \quad \int_{\mathbb{R}^{N+1}\setminus B_{R}^{N+1}(0)} t^{1-2\tau}|\bar{u}_{+}|^{2} \, dx \, dt \\
\leq \left( \frac{C}{R^{\tau}} \right)^{2} \int_{B_{R}^{N+1}(0)\setminus B_{R}^{N+1}(0)} t^{1-2\tau}|\bar{u}_{+}|^{2} \, dx \, dt \\
= \left( \frac{C}{R^{\tau}} \right)^{2} \int_{B_{R}^{N+1}(0)\setminus B_{R}^{N+1}(0)} t^{1-2\tau}|\bar{u}_{+}|^{2} \, dx \, dt - \left( \frac{C}{R^{\tau}} \right)^{2} \int_{B_{R}^{N+1}(0)\setminus B_{R}^{N+1}(0)} t^{1-2\tau}|\bar{u}_{+}|^{2} \, dx \, dt,
\end{align}
where \( A = 1 - \frac{N+2-2\tau}{2N} \). So,

\begin{align}
(B.5) \quad \int_{R_{+}^{N+1}\setminus B_{R}^{N+1}(0)} t^{1-2\tau}|\bar{u}_{+}|^{2} \, dx \, dt \leq \frac{C'}{1 + C' \tau} \int_{R_{+}^{N+1}\setminus B_{R}^{N+1}(0)} t^{1-2\tau}|\bar{u}_{+}|^{2} \, dx \, dt,
\end{align}
where

\begin{align}
(C') = \left( \frac{C}{R^{\tau}} \right)^{2}.
\end{align}

Let \( \Psi(R) = \int_{R_{+}^{N+1}\setminus B_{R}^{N+1}(0)} t^{1-2\tau}|u_{+}|^{2} \, dx \, dt \) and \( \tau = \frac{C'}{1 + C' \tau} \). Then from (B.5),

\[ \Psi(2R) \leq \tau \Psi(R), \quad \forall R \geq R_{0}, \]

which implies that

\[ \Psi(2^{i}R_{0}) \leq \tau^{i} \Psi(R_{0}). \]

For any \(|(x,t)| \geq R_{0}\), there is an \( i \) such that

\[ 2^{i}R_{0} \leq |(x,t)| \leq 2^{i+1}R_{0}. \]

Hence

\[ \Psi(|(x,t)|) \leq \Psi(2^{i}R_{0}) \leq \tau^{i} \Psi(R_{0}) \leq \tau^{i} \Psi(|(x,t)|), \]

\[ \forall (x,t) \geq R_{0}. \]

Since \( \tau \log_{2} |(x,t)| = 2 \log_{2} |(x,t)| \log_{2} \tau = |(x,t)| \log_{2} \tau \), we have

\[ \Psi(|(x,t)|) \leq C |(x,t)|^{\log_{2} \tau} = \frac{C}{|(x,t)|^{\log_{2} \frac{1}{\tau}}}, \]

So we have proved that there is a \( \sigma > 0 \) independent of \( p \) such that

\[ \Psi(|(x,t)|) \leq \frac{C}{|(x,t)|^{\sigma}}, \quad |(x,t)| \geq R_{0}, \]

where \( \sigma = \log_{2} \frac{1}{\tau} \). Fix \( p > 2^{\frac{3}{2}} \) and \( \rho < \frac{2^{\frac{3}{2}}}{\sqrt{N+1}} \). It follows from (B.2) that

\[ ||\bar{u}_{+}||_{L^{p}(1-2\tau, R_{N+1}^{N+1}\setminus B_{R}^{N+1}(0))} \leq \frac{C}{R^{N+\delta p + \frac{3}{2}}}, \quad |(x,t)| = R \geq R_{0}. \]

Using the definition of \( \tau \), we can choose \( R_{0} \) large enough such that \( 2^{\frac{3}{2}} - \delta p > \beta > \frac{N-2\tau}{2} \) and then

\begin{align}
(B.7) \quad ||\bar{u}_{+}||_{L^{p}(1-2\tau, R_{N+1}^{N+1}\setminus B_{R}^{N+1}(0))} \leq \frac{C}{R^{N+\beta}}, \quad |x| = R \geq R_{0}.
\end{align}

Similarly,

\begin{align}
(B.8) \quad ||(-\bar{u})_{+}||_{L^{p}(1-2\tau, R_{N+1}^{N+1}\setminus B_{R}^{N+1}(0))} \leq \frac{C}{R^{N+\beta}}, \quad |x| = R \geq R_{0}.
\end{align}

Thus, we obtain

\[ ||\bar{u}||_{L^{p}(1-2\tau, R_{N+1}^{N+1}\setminus B_{R}^{N+1}(0))} \leq \frac{C}{R^{N+\beta}}, \quad |x| = R \geq R_{0}. \]
Now for any \((x, t)\) with \(|(x, t)| = 4R > 2R_0\),
\[
|\tilde{u}(x, t)| \leq \max_{(y, \hat{t}) \in B_{1}^{N+1}(x, t)} |\tilde{u}(y, \hat{t})| \leq C\|\tilde{u}\|_{L^p(B_{1-2s}^{N+1}(x, t))} \leq C\|\tilde{u}\|_{L^p(B_{1-2s}^{N+1}\setminus B_{2R}^{N+1}(0))} \leq \frac{C}{R^{N+\beta}}.
\]
By the fact that \(u(x) = \tilde{u}(x, 0)\), the result follows. \(\square\)

**Proof of Lemma B.1**  For every \(\alpha \in (-\frac{N}{2} - s, \frac{N}{2} - s)\), let \(\vartheta_\alpha = |x|^{2s-N+\alpha}\). Then it follows from Lemma 3.1 in [23] that
\[
(-\Delta)^{s}\vartheta_\alpha = \Upsilon_\alpha |x|^{-2s}\vartheta_\alpha \text{ in } \mathbb{R}^N \setminus \{0\}.
\]
where
\[
\Upsilon_\alpha = 2^{2s} \frac{\Gamma(N+2s+2\alpha)}{\Gamma\left(\frac{N-2s-2\alpha}{4}\right)} \frac{\Gamma\left(\frac{N+2s-2\alpha}{4}\right)}{\Gamma\left(\frac{N+2s+2\alpha}{4}\right)}
\]
So by the assumption \(0 \leq \mu < \Upsilon_\alpha\) and \(\Upsilon_\alpha\) is even on \(\alpha\), we can find \(\bar{\alpha} \in (-\frac{N-2s}{2}, -s)\) such that for \(|x| \geq 1,
\[
(-\Delta)^{s}\vartheta_{\bar{\alpha}} - \mu |x|^{-2s}\vartheta_{\bar{\alpha}} = (\Upsilon_{\bar{\alpha}} - \mu)|x|^{-2s-N+\bar{\alpha}} \geq \frac{C}{|x|^{(N+\beta)(2s-1)}}.
\]
On the other hand, from Lemma B.2
\[
(-\Delta)^{s}u_+ - \mu |x|^{-2s}u_+ \leq u_+^{2s-1} \leq \frac{C}{|x|^{(N+\beta)(2s-1)}}, \quad |x| \geq 1.
\]
Letting \(\beta = -\bar{\alpha}\), by comparison, we have
\[
u_+(x) \leq \frac{C}{|x|^{\frac{N-2s}{2}+\beta}}, \quad |x| \geq 1,
\]
where \(\beta \in (s, \frac{N-2s}{2})\). Similarly,
\[
(-u)_+ \leq \frac{C}{|x|^{\frac{N-2s}{2}+\beta}}, \quad |x| \geq 1.
\]
\(\square\)

**Proposition B.3.** Let \(u\) be a solution of (B.1). Then, we have
\[
|u(x)| \leq \frac{C}{|x|^{\frac{N-2s}{2}+\beta}}, \quad \forall |x| \leq 1,
\]
where \(\beta\) is given in Lemma B.1

**Proof.** Applying the Kelvin transformation \(v(x) = |x|^{2s-N}u\left(\frac{x}{|x|}\right)\), we know that \(v\) satisfies
\[
(-\Delta)^{s}v - \frac{\mu v}{|x|^{2s}} = |v|^{2s-2}v, \quad \forall |x| \geq 1,
\]
where \(0 \leq \mu < \bar{\mu}\). By Lemma B.1 we have
\[
|v(x)| \leq \frac{C}{|x|^{\frac{N-2s}{2}+\beta}}, \quad \forall |x| \geq 1.
\]
As a result,
\[
|u(x)| \leq \frac{C}{|x|^{\frac{N-2s}{2}-\beta}}, \quad \forall |x| \leq 1.
\]
Thus, the result follows. \(\square\)
**Appendix C. A local Pohozaev identity**

In this section, we give a local Pohozaev identity. For $F \subset \mathbb{R}^{N+1}_+$, we recall that $\partial_s F = \{ z = (x,t) \in \mathbb{R}^{N+1}_+ : (x,t) \in \partial F$ and $t > 0 \}$ and $\partial_0 F = \partial F \cap (\mathbb{R}^N \times \{0\})$. We have the following result.

**Proposition C.1.** Let $E \subset \mathbb{R}^{N+1}_+$ and we assume that $\bar{u}$ is a solution of

\[
\begin{aligned}
\begin{cases}
\text{div}(t^{1-2s} \nabla \bar{u}) = 0, & \text{in } E, \\
A_s(\bar{u}) = \frac{\mu}{|x|^{2s}}u + |u|^{p-2}u + au, & \text{on } \partial_0 E.
\end{cases}
\end{aligned}
\]

Then for $F \subset E$, there holds

\[
\begin{aligned}
\left( \frac{N}{p} - \frac{N-2s}{2} \right) \int_{\partial_0 F} |u|^p dx + sa & \int \nabla u^2 dx + s\mu \int \frac{x \cdot x_0 |u|^2}{|x|^{2s+2}} dx \\
= & \frac{1}{2} \int_{\partial_0 F} \left( a + \frac{\mu}{|x|^{2s}} \right) |u|^2(x - x_0) \cdot \nu_x dS_x + \frac{1}{p} \int_{\partial_0 F} |u|^p(x - x_0) \nu_x dS_x \\
& + \int_{\partial_s F} t^{-2s} \left( (z - z_0, \nabla \bar{u}) \nabla \bar{u} - (z - z_0) \frac{|\nabla \bar{u}|^2}{2}, \nu_z \right) dS_z \\
& + \frac{N-2s}{2} \int_{\partial_s F} t^{-2s} \frac{\partial \bar{u}}{\partial \nu_z} dS_z.
\end{aligned}
\]

**Proof.** Note that

\[
\begin{aligned}
div(t^{1-2s} \nabla \bar{u})(z - z_0, \nabla \bar{u})
& = div[t^{1-2s} \nabla \bar{u}(z - z_0, \nabla \bar{u})] - t^{1-2s} \nabla \bar{u}, \nabla (z - z_0, \nabla \bar{u}) \\
& = div[t^{1-2s} \nabla \bar{u}(z - z_0, \nabla \bar{u})] - t^{1-2s} \left[ (z - z_0, \nabla \frac{|\nabla \bar{u}|^2}{2}) + |\nabla \bar{u}|^2 \right] \\
& = div \left[ t^{1-2s} \nabla \bar{u}(z - z_0, \nabla \bar{u}) - t^{1-2s}(z - z_0) \frac{|\nabla \bar{u}|^2}{2} \right] + \frac{N-2s}{2} t^{-2s} |\nabla \bar{u}|^2.
\end{aligned}
\]

Then

\[
\begin{aligned}
div \left\{ t^{1-2s}(z - z_0, \nabla \bar{u}) \nabla \bar{u} - t^{1-2s} \frac{|\nabla \bar{u}|^2}{2}(z - z_0) \right\} + \frac{N-2s}{2} t^{-2s} |\nabla \bar{u}|^2 = 0.
\end{aligned}
\]

So we find

\[
\begin{aligned}
\int_{\partial_s F} t^{-2s} \left( (z - z_0, \nabla \bar{u}) \nabla \bar{u} - (z - z_0) \frac{|\nabla \bar{u}|^2}{2}, \nu_z \right) dS_z \\
= - \int_{\partial_0 F} \nabla_x \bar{u}, A_{s} \bar{u} dx - \frac{N-2s}{2} \int_F t^{-2s} |\nabla \bar{u}|^2 dxdt.
\end{aligned}
\]

By using the fact that $A_s \bar{u} = \frac{\mu}{|x|^{2s}}u + |u|^{p-2}u + au$ on $\partial_0 F$ and performing integration by parts, one has

\[
\begin{aligned}
\int_{\partial_0 F} (x - x_0, \nabla_x \bar{u}), A_{s} \bar{u} dx & = - \frac{N+2s}{2} \mu \int_{\partial_0 F} \frac{u^2}{|x|^{2s}} dx - sa \int_{\partial_0 F} \frac{x \cdot x_0 |u|^2}{|x|^{2s+2}} dx \\
& + \frac{\mu}{2} \int_{\partial_0 F} \frac{|u|^2}{|x|^{2s}}(x - x_0) \cdot \nu_x dS_x \\
& - \frac{N}{p} \int_{\partial_0 F} |u|^p dx + \frac{1}{p} \int_{\partial_0 F} |u|^p(x - x_0) \cdot \nu_x dS_x \\
& - \frac{N a}{2} \int_{\partial_0 F} |u|^2 dx + \frac{a}{2} \int_{\partial_0 F} |u|^2(x - x_0) \cdot \nu_x dS_x.
\end{aligned}
\]
where we use the fact that
\[
\int_{\partial \mathcal{F}} \frac{u}{|x|^{2s}} \, dx = -\frac{N + 2s}{2} \int_{\partial \mathcal{F}} \frac{u^2}{|x|^{2s}} \, dx - \frac{1}{s} \int_{\partial \mathcal{F}} \frac{u}{|x|^{2s+2}} \, dx + \frac{1}{2} \int_{\partial \mathcal{F}} \frac{|u|^2}{|x|^{2s}} (x - x_0) \cdot \nu_x \, dS_x,
\]
and
\[
\int_{\partial \mathcal{F}} |x - x_0, \nabla \tilde{u}| u^{p-1} u \, dx = -\frac{N}{p} \int_{\partial \mathcal{F}} |u|^p \, dx + \frac{1}{p} \int_{\partial \mathcal{F}} |u|^p (x - x_0) \cdot \nu_x \, dS_x.
\]
On the other hand,
\[
(C.4) \quad \int_{\mathcal{F}} t^{1-2s} |\nabla \tilde{u}|^2 \, dx dt = \int_{\partial \mathcal{F}} \left( \frac{\mu}{|x|^{2s}} u^2 + |u|^p + au^2 \right) \, dx + \int_{\partial \mathcal{F}} t^{1-2s} \frac{\partial}{\partial u_z} dS_z.
\]
Thus, from (C.2) to (C.4), the desired result follows.

\section*{Appendix D. Decomposition of approximating solutions}

In this section, we give a result describing the composition of approximating solutions bounded in $H^s_0(\Omega)$ obtained as a solution of (1.10) with $\epsilon = \epsilon_n$.

**Proposition D.1.** Suppose that $N > 6s$. Let $u_n$ be a solution of (1.10) with $\epsilon = \epsilon_n \to 0$, satisfying $\|u_n\| \leq C$ for some constant $C$. Then

(i) $u_n$ can be decomposed as

\[
(D.1) \quad u_n = u_0 + \sum_{j=1}^{m} \rho_{\Lambda_n, j}(U_j) + \sum_{j=m+1}^{\infty} \rho_{x_{n,j}, \Lambda_n, j}(U_j) + \omega_n,
\]

where $\omega_n \to 0$ in $H^s(\Omega)$, $u_0$ is a solution for (1.1). For $j = 1, 2, \ldots, m$, $\Lambda_{n,j} \to \infty$ as $n \to \infty$, and $U_j$ is a solution of

\[
(D.2) \quad (-\Delta)^s u - \mu \frac{u}{|x|^{2s}} = b_j |u|^{2^*_s - 2} u, \quad u \in D^s(\mathbb{R}^N),
\]

for some $b_j \in (0, 1)$. For $j = m + 1, m + 2, \ldots, h, x_{n,j} \in \Omega, \Lambda_{n,j} d(x_{n,j}, \partial \Omega) \to \infty, \Lambda_{n,j} |x_{n,j}| \to \infty$ as $n \to \infty$, and $U_j$ is a solution of

\[
(D.3) \quad (-\Delta)^s u = b_j |u|^{2^*_s - 2} u, \quad u \in D^s(\mathbb{R}^N),
\]

for some $b_j \in (0, 1)$. 

(ii) Set $x_{n,i} = 0$ for $i = 1, 2, \ldots, m$. For $i, j = 1, 2, \ldots, h$, if $i \neq j$, then as $n \to \infty$,

\[
(D.4) \quad \frac{\Lambda_{n,i}}{\Lambda_{n,i} + \Lambda_{n,j} \Lambda_{n,i} |x_{n,i} - x_{n,j}|^2} \to \infty.
\]

**Proof.** The proof is similar to [2,15,16,18] and we omit the details.

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