INTERLEAVED PRANGE: A NEW GENERIC DECODER FOR INTERLEAVED CODES
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Abstract. Due to the recent challenges in post-quantum cryptography, several new approaches for code-based cryptography have been proposed. For example, a variant of the McEliece cryptosystem based on interleaved codes was proposed. In order to deem such new settings secure, we first need to understand and analyze the complexity of the underlying problem, in this case the problem of decoding a random interleaved code. A simple approach to decode such codes, would be to randomly choose a vector in the row span of the received matrix and run a classical information set decoding algorithm on this erroneous codeword. In this paper, we propose a new generic decoder for interleaved codes, which is an adaption of the classical idea of information set decoding by Prange and perfectly fits the interleaved setting. We then analyze the cost of the new algorithm and a comparison to the simple approach described above shows the superiority of Interleaved Prange.

1. Introduction

Code-based cryptography is one of the most promising and prominent candidates for post-quantum cryptography, which is reflected in the NIST standardization process [7]. Although the third round of submissions has already been completed, and the classical McEliece system [3] has been chosen as a finalist, there are still many open challenges in the area. For example the lack of efficient and secure signature schemes [14], but also the compelling task of reducing the key sizes of the original McEliece system is still open. For this reason, researchers have proposed several alternatives to the classical scheme of McEliece, not only by changing the underlying code family, but also by considering different settings, for example by employing the rank metric [1, 2], the Lee metric [6, 12, 20] or by using interleaved codes. The latter approach has been proposed in [8, 11, 17]. The simple reasoning behind this proposal is that an interleaved code has a larger error-correction capability than a non-interleaved code.

A codeword of an $\ell$-interleaved code is an $\ell \times n$ matrix over $\mathbb{F}_q$, where each row is a codeword of a constituent linear code of blocklength $n$ over $\mathbb{F}_q$. In this work, we consider the decoding problem for homogeneous interleaved codes, where the same constituent code is used for all the rows.

An interleaved code $C_\ell$ is especially well-suited for channels that are prone to burst errors, where $t$ burst errors can be modeled as the addition of an $\ell \times n$ matrix $E$ with $t$ non-zero columns to a codeword in $C_\ell$. We say that $E$ has column weight $t$.

A generic decoder for any linear interleaved code was proposed in [9, 13]. When the interleaving order $\ell$ is at least the number of column errors $t$, this decoder guarantees to correct (efficiently) any full-rank error of weight up to $d - 2$, where $d$ is the minimum distance of the constituent code. This decoder was generalized in [10, 18] for the case $\ell < t$ and guarantees to decode any error $E$ of weight $t$ if $2t - \text{rk}(E) \leq d - 2$. However, there is no known efficient decoder for interleaved codes with an arbitrary constituent code when $\ell \ll t$. In fact, it can be shown that the corresponding decisional problem, called Interleaved Decoding (ID) problem, is at least as hard as the decisional Syndrome Decoding (SD) problem.

This fact implies that interleaved codes are a well-suited alternative for code-based cryptography. It is therefore of interest to understand and analyze the complexity of decoding...
a generic interleaved code not only from a coding-theoretic perspective, but also in order to
assess the security of code-based cryptosystems based on interleaved codes.

In this paper, we consider algorithms for the ID problem when $\ell \ll t < d$ for arbitrary
linear constituent codes. We can categorize the generic decoding algorithms for interleaved
codes into three types:

1. Algorithms that reduce the problem to the classical SD problem.
2. Algorithms that reduce the problem to a low-weight codeword finding (CF) problem.
3. Algorithms that do not reduce the problem to either CF or SD. We present one
   such novel algorithm inspired by Prange’s information set decoder [16].

We remark that we will be content with finding just a subset of the $t$ error positions since
then the problem reduces to a much easier problem as the complexity is exponential in $t$.
For the third family of algorithms, we propose Interleaved Prange. The classical Prange
algorithm [16] can be described as picking $k$ columns of the generator matrix $G$, where the
algorithm is successful if the corresponding positions are error-free, i.e., their complement of
$n - k$ positions contains the support of the error. Alternatively one can pick $k + 1$ columns
from $[G \, r]$ where $r$ is the received word and check whether the $(k + 1) \times (k + 1)$ submatrix
formed by these columns is rank-deficient. This can be generalized to interleaved codes,
which is the main idea of our algorithm Interleaved Prange: we pick $k + \ell$ columns of $[G \, r]$ where $R$ is an $\ell \times n$ matrix containing the $\ell$ received words as rows, and check if the rank
of the $(k + \ell) \times (k + \ell)$ submatrix formed by these columns is less than $k + \ell$. The main
contribution of this paper is the proposal and the analysis of the new decoding algorithm
Interleaved Prange.

This paper is structured as follows. In Section 2 we introduce the notation and results for
interleaved codes which are essential for the remainder of the paper. We present the three
types of interleaved decoding algorithms in Section 3 together with their corresponding
complexity analysis. A comparison of their asymptotic cost, given in Section 3.4, then shows
that the newly proposed algorithm, Interleaved Prange, outperforms the straight-forward
decoders. Finally, we conclude this paper in Section 4.

2. Preliminaries

Let us first introduce the notation that is used throughout this paper. For a prime power
$q$, let us denote by $\mathbb{F}_q$ be a finite field with $q$ elements. We denote matrices and vectors
by bold capital, respectively lower case letters. For $k \leq n$ positive integers and a matrix
$G \in \mathbb{F}_q^{k \times n}$ we denote by $\langle G \rangle$ its rowspan, by $G^\top$ the transposed matrix and by $\mathrm{rk}(G)$ its
rank. For a vector $x \in \mathbb{F}_q^n$, we will denote by $\mathrm{wt}(x)$ the Hamming weight of $x$, that is the
size of its support $\mathrm{supp}(x)$. For a matrix $X \in \mathbb{F}_q^{k \times n}$ we will denote by $\mathrm{wt}(X)$ the number
of non-zero columns of $X$. For a set $S$ we will denote by $|S|$ its cardinality. The set of all
integers between 1 and $n$ is denoted by $[1, n]$. Finally, for a set $I \subseteq [1, n]$ of size $r$ and
a matrix $G \in \mathbb{F}_q^{k \times n}$, we denote by $G_I \in \mathbb{F}_q^{k \times r}$ the matrix consisting of all columns of $G$
indexed by $I$. For a vector $x \in \mathbb{F}_q^n$, we denote by $\mathrm{supp}(x)$ its support, that is the indices
of the non-zero entries of $x$. Similarly for a matrix $X \in \mathbb{F}_q^{k \times n}$ we denote by $\mathrm{supp}(X)$ the
indices of the non-zero columns.

A linear subspace $C \subseteq \mathbb{F}_q^n$ of dimension $k$ is called a linear code of length $n$ and dimension
$k$. We call this an $[n, k]_q$ code of rate $R = \frac{k}{n}$. For a linear code $C \subseteq \mathbb{F}_q^n$ we can also define
its minimum distance to be

$$d(C) = \min \{ \mathrm{wt}(c) \mid c \in C, c \neq 0 \}.$$ 

An $[n, k]_q$ linear code can be represented either through a generator matrix $G \in \mathbb{F}_q^{k \times n}$,
which has the code as image, or through a parity-check matrix $H \in \mathbb{F}_q^{(n-k) \times n}$, which has
the code as right kernel. For any $x \in \mathbb{F}_q^n$, we call $s = xH^\top \in \mathbb{F}_q^{n-k}$ the syndrome of $x$.

It is well known that random codes of large blocklength over $\mathbb{F}_q$ achieve with high
probability the minimum distance given by the Gilbert-Varshamov bound, that is

$$\delta = \frac{d(n)}{n} = H_q^{-1}(1 - R),$$
where we denote by $H_q$ the $q$-ary entropy function.

**Definition 1.** Let $C \subseteq \mathbb{F}_q^n$ be a linear code of dimension $k$ with generator matrix $G \in \mathbb{F}_q^{k \times n}$. The homogeneous interleaved code of interleaving order $\ell$ of $C$ is defined as

$$C_\ell = \{ C \in \mathbb{F}_q^{\ell \times n} \mid C = MG, M \in \mathbb{F}_q^{\ell \times k} \}.$$ 

Thus, the codewords of an interleaved code are $\ell \times n$ matrices. Let $H$ be a parity-check matrix of $C$ and consider the interleaved code $C_\ell$. The syndrome of $X \in \mathbb{F}_q^{\ell \times n}$ is then given by

$$S = XH \in \mathbb{F}_q^{\ell \times (n-k)}.$$ 

Decoding an interleaved code with an arbitrary constituent code can be seen as the following problem.

**Problem 1 (Interleaved Syndrome Decoding (ISD) Problem).** Let $\ell \geq 2$ be a positive integer. Given $H \in \mathbb{F}_q^{(n-k) \times n}, S \in \mathbb{F}_q^{\ell \times (n-k)},$ and $t \in \mathbb{N}$, decide if there exists a matrix $E \in \mathbb{F}_q^{\ell \times n}$ of weight at most $t$, such that $HE^\top = S^\top$.

This problem is equivalent to the Interleaved Decoding (ID) problem.

**Problem 2 (Interleaved Decoding (ID) Problem).** Given $G \in \mathbb{F}_q^{k \times n}, R \in \mathbb{F}_q^{\ell \times n}$, and $t \in \mathbb{N}$, decide if there exists a matrix $E \in \mathbb{F}_q^{\ell \times n}$ of column weight at most $t$, such that each row of $R - E$ is in $\langle G \rangle$.

This problem can be shown to be NP-hard by a reduction from the Hamming-Metric SD problem, which has been proven to be NP-complete in [5, 4].

**Problem 3 (Hamming Syndrome Decoding (SD) Problem).** Given $H \in \mathbb{F}_q^{(n-k) \times n}, S \in \mathbb{F}_q^{\ell \times n-k},$ and $t \in \mathbb{N}$, decide if there exists a $e \in \mathbb{F}_q^{\ell \times n-k}$ of weight at most $t$, such that $s = eH^\top$.

**Theorem 2.** The Interleaved Syndrome Decoding Problem (Problem 1) is NP-complete.

**Proof.** We show the NP-hardness of Problem 1 by a reduction from the classical Hamming SD. For this, take a random instance $H \in \mathbb{F}_q^{(n-k) \times n}, S \in \mathbb{F}_q^{\ell \times n-k}$ and $t \in \mathbb{N}$ of the Hamming SD. Now define $S = \begin{pmatrix} e \ \vdots \ e \end{pmatrix} \in \mathbb{F}_q^{\ell \times (n-k)}$. Assume we have an oracle for Problem 1.

- If the answer is ‘yes’ on the input $H, S, t$, then this is also the correct answer to the Hamming SD. In fact, if there exists $E \in \mathbb{F}_q^{\ell \times n}$, such that $HE^\top = S^\top$ and at most $t$ columns of $E$ are non-zero, then any column, e.g., the first column $e$, of $E$ is a solution to the Hamming SD, as $He = s$ and $wt(e) \leq t$.
- If the oracle returns ‘no’ on the input $H, S, t$, then this is also the correct answer to the Hamming SD. In fact, if there was a solution $e$ to the Hamming SD then $E = \begin{pmatrix} e \ \vdots \ e \end{pmatrix}$ would have been a solution to the interleaved SD.

Finally, we remark that for any candidate $E$ we can check in polynomial time, whether $E$ is a solution to the interleaved SD. Thus, the problem is also in NP.

### 3. Decoding Algorithms

In this section we present three types of generic decoding algorithms for interleaved codes. That is, given $G \in \mathbb{F}_q^{k \times n}, R \in \mathbb{F}_q^{\ell \times n}$, and $t \in \mathbb{N}$, these algorithms find a matrix $E \in \mathbb{F}_q^{\ell \times n}$ of column weight at most $t$, such that each row of $R - E$ is in $\langle G \rangle$.

In the following, we assume that $G \in \mathbb{F}_q^{k \times n}$ and a set of error positions $T \subseteq \{1, \ldots, n\}$ of size $t$ is chosen uniformly at random. Then ones takes a $\ell \times n$ zero matrix $E$ and sets each column at these $t$ error positions equal to a random vector in $\mathbb{F}_q^{\ell}$. Thus $E_T$ is a random matrix in $\mathbb{F}_q^{\ell \times t}$, and $E$ is a random matrix in $\mathbb{F}_q^{\ell \times n}$ of column weight at most $t$. Finally, we choose $M \in \mathbb{F}_q^{\ell \times k}$ uniformly at random and compute the received matrix $R = C + E$. 

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where \( C = MG \). Thus, we assume that at least one solution to the ID problem exists. For interleaved cryptosystems, \( t \) is typically close to the minimum distance of \( G \) which we denote by \( d \).

### 3.1. SD-based Algorithms.

The most straightforward way to solve the ID problem is to simply pick a random non-zero vector \( r \) in the rowspan of \( R \) and solve the resulting SD problem with the parity-check matrix \( H \in \mathbb{F}_q^{(n-k)\times n} \) of the constituent code and the syndrome \( s = rH^\top \in \mathbb{F}_q^{n-k} \). Since information set decoding (ISD) attacks are the best known algorithms to solve the SD problem, we call this Random \( \langle \text{ISD} \rangle \) (where \( \langle \text{ISD} \rangle \) can be any ISD algorithm such as Prange, Stern [19], etc.).

We assume that it is enough to recover only a part of the non-zero columns of \( E \) since this knowledge will reduce the problem already to a much easier problem. If the success probability of the employed ISD algorithm of finding an error of weight \( v \) is denoted by \( P(v) \), then the success probability of the Random \( \langle \text{ISD} \rangle \) approach is given by

\[
\sum_{v=0}^t \binom{t}{v} \left( q - 1 \right)^v \cdot P(v),
\]

Note that \( P(v) \) is simply zero for all those error weights \( v \) which the chosen ISD algorithm is not designed to solve for. Here \( \left( \binom{t}{v} \frac{q-1}{q}^v \right) \) denotes the probability that the chosen \( r \) has an error \( e \) of weight \( v \). In fact, by choosing a random codeword \( r \in \langle R \rangle \), this results in an error vector \( e \) which is a random linear combination of the rows of \( E \in \mathbb{F}_q^{\ell \times n} \) and thus when \( e \) is restricted to the \( t \) error positions it looks like a vector drawn uniformly at random from \( \mathbb{F}_q^t \). Note that this approach comes with a failure probability as the errors generally have weight greater than the unique decoding radius of \( G \). However, this probability is negligible as the error weights are less than the minimum distance of \( G \).

For the complexity analysis, let us consider first that we employ the ISD algorithm of Prange [16]. This algorithm has a success probability of

\[
P(v) = \binom{n-k}{v} \binom{n}{v}^{-1}.
\]

Hence the success probability of Random Prange is given by

\[
\sum_{v=0}^t \binom{t}{v} \left( q - 1 \right)^v \binom{n-k}{v} \binom{n}{v}^{-1}.
\]

To get an upper bound on the asymptotic complexity of Random Prange, we can give a lower bound on the success probability, e.g., by considering just the term in the summation where \( v = t \frac{q-1}{q} \) (a reasonable choice since this is the most likely error weight in the chosen \( r \), i.e., this \( v \) maximizes \( \binom{t}{v} \frac{q-1}{q}^v \)).
In order to give an asymptotic complexity, we first consider the parameters \( k, t \) as functions in \( n \) and define
\[
R = \lim_{n \to \infty} \frac{k(n)}{n},
\]
\[
T = \lim_{n \to \infty} \frac{t(n)}{n} = H_q^{-1}(1 - R).
\]

To ease the notation, we also introduce the asymptotics of the binomial coefficient, denoted by
\[
H(F, G) := \lim_{n \to \infty} \frac{1}{n} \log_q \left( \binom{f(n)}{g(n)} \right)
\]
\[
= F \log_q(F) - G \log_q(G) - (F - G) \log_q(F - G),
\]
where \( f(n), g(n) \) are integer-valued functions such that \( \lim_{n \to \infty} \frac{f(n)}{n} = F \) and \( \lim_{n \to \infty} \frac{g(n)}{n} = G \).

Thus, we get the following upper bound.

**Proposition 3.** The asymptotic complexity of Random Prange on an \( \ell \)-interleaved random code over \( \mathbb{F}_q \) with length \( n \) and dimension \( k \) is given by at most \( q^{\omega(R,q)} \), where
\[
e(R, q) = H(1, T(q - 1)/q) - H(1 - R, T(q - 1)/q).
\]

If we employ Stern’s ISD algorithm [19], we get a slight improvement. However, note that Stern’s algorithm (at least in its conventional formulation) only solves the SD problem for a fixed error weight \( w \). If instead in each iteration we run Stern \( t \) times for all error weights \( w \in [1, t] \), this gives us a straightforward extension of the algorithm that works for all errors with weights in \( [1, t] \). While this of course increases the cost of one iteration, it turns out that asymptotically the cost remains the same and since this formulation can only improve the probability of success of Random Stern, we will consider this version.

The cost of Random Stern’s algorithm is in \( O(I \cdot C) \), where \( I \) is the expected number of iterations and \( C \) the cost of one iteration. This is given by
\[
I = \left( \sum_{v=0}^{t} \frac{(q - 1)^v}{q^v} \left( \frac{(k + \ell_v')/2}{w_v'/2} \right)^2 \left( \frac{n - k - \ell_v'}{v} \right) \left( \frac{n}{v} \right)^{-1} \right)^{-1},
\]
\[
C = \sum_{v=1}^{t} C_v \text{ where } C_v = \left( \frac{(k + \ell_v')/2}{w_v'/2} \right) q^{w_v'/2} + \left( \frac{(k + \ell_v')/2}{w_v'/2} \right)^2 q^{w_v' - \ell_v'},
\]
where \( 0 \leq w_v' \leq \min\{k + \ell_v', v\}, 0 \leq \ell_v' \leq n - k \) are the internal parameters of Stern’s algorithm that can be optimized individually for each of the \( t \) runs to give the lowest cost.

To get an upper bound on the asymptotic complexity of Random Stern, we again just consider the \( v_0 = t2^{-1} q \) term in the summation in \( I \)’s formula. For this let us consider additionally the parameters \( w_{v_0}' \) and \( \ell_{v_0}' \) as functions in \( n \) and define
\[
W' = \lim_{n \to \infty} \frac{w_{v_0}'(n)}{n},
\]
\[
L' = \lim_{n \to \infty} \frac{\ell_{v_0}'(n)}{n}.
\]

**Proposition 4.** The asymptotic complexity of Random Stern on an \( \ell \)-interleaved random code over \( \mathbb{F}_q \) with length \( n \) and dimension \( k \) is given by at most \( q^{\omega(R,q)} \), where
\[
e(R, q) = H(1, T(q - 1)/q) - 2H((R + L')/2, W'/2)
\]
\[- H(1 - R - L', T(q - 1)/q - W')
\]+ \max\{H((R + L')/2, W'/2) + W'/2, 2H((R + L')/2, W'/2) + W' - L'\}.
This can be split into two cases:

with minimum distance

However, this approach comes with a possibly large

error in this word, giving us a subset of the

positions and so by performing the re-encoding step of Prange on such a word

In the first case, we succeed as at least one non-zero word in

If we treat

as choosing

novel approach: Interleaved Prange.

3.3. Novel approach: Interleaved Prange. We propose a new algorithm (Algorithm 3)

inspired by the classical attack of Prange. Note that Prange’s algorithm can be described

as choosing \( k + 1 \) columns in \( \begin{bmatrix} G \\ R \end{bmatrix} \) where \( r \) is the received word and checking whether the

\((k + 1) \times (k + 1)\) submatrix formed at these positions is rank deficient. This formulation can

neatly be generalized to interleaved codes, where we pick \( k + \ell \) columns in \( \begin{bmatrix} G \\ R \end{bmatrix} \) and check

if the rank of the \((k + \ell) \times (k + \ell)\) submatrix formed at these positions is less than \( k + \ell \).

In more details, we choose a set \( J \subset [1, n] \) of size \( k + \ell \), which contains an information set

\( I \) for \( G \) (in other words, \( G_I \) and hence \( G_J \) has full rank). Let us denote again by \( G' := \begin{bmatrix} G \\ R \end{bmatrix} \)

and check if the square submatrix \( G'_{JJ} \) (the blue region in Fig. 1) is rank-deficient, that is

\[ \text{rk} \left( \begin{bmatrix} G' \\ R \end{bmatrix} \right) < k + \ell. \]

This can be split into two cases:

1. \( E_J \) has linearly dependent rows (which implies \( G'_{JJ} \) is rank deficient).
2. \( E_J \) has linearly independent rows but \( G'_{JJ} \) is still rank deficient.

In the first case, we succeed as at least one non-zero word in \( (R) \) is error-free at these \( k + \ell \)

positions and so by performing the re-encoding step of Prange on such a word \( r \), we can find

the error in this word, giving us a subset of the \( t \) error positions. A naive way to find such

an \( r \) would be to do the re-encoding on all \( q^\ell - 1 \) non-zero words in \( (R) \). However, this step
will fail if we are in the second case. As it turns out, the second case is far more likely than
the first, so this naive re-encoding approach will make the entire algorithm very inefficient.
Instead we do the re-encoding for only those \( r \in \langle \mathbf{R} \rangle \) that actually belong to some linearly
dependent set of rows in \( \mathbf{G}'_\mathcal{J} \) which can be easily found by computing its left null space,
\[ \{ x \in \mathbb{F}_q^{k+\ell} : x\mathbf{G}'_\mathcal{J} = 0 \}. \] With this modification, the algorithm becomes efficient
again, though perhaps at the expense of a more involved complexity analysis.

**Figure 1.** Illustration of Interleaved Prange Algorithm.

**Algorithm 3: Interleaved Prange**

**Input:** A generator matrix \( \mathbf{G} \in \mathbb{F}_q^{k \times n} \) and a received matrix \( \mathbf{R} = \mathbf{C} + \mathbf{E} \in \mathbb{F}_q^{\ell \times n} \)
where \( \mathbf{E} \) has at most \( t \) non-zero columns.

**Output:** A nonempty subset \( \mathcal{U} \subseteq \text{supp}(\mathbf{E}) \)

1. Choose \( \mathcal{J} \subseteq [1, n] \) of size \( k + \ell \) such that \( \text{rk}(\mathbf{G}_\mathcal{J}) = k \)
2. if \( \text{rk}(\mathbf{G}'_\mathcal{J}) < k + \ell \) then
   3. for each \( x \in \mathbb{F}_q^{k+\ell} \setminus \{0\} \) in the left null space of \( \mathbf{G}'_\mathcal{J} \) do
   4. if \( \text{wt}(x\mathbf{G}') \leq t \) then return \( \text{supp}(x\mathbf{G}') \).
5. end
6. else
7. Go back to step 1.

**Theorem 5.** The cost of Interleaved Prange on an \( \ell \)-interleaved random code over \( \mathbb{F}_q \) with
length \( n \) and dimension \( k \) is in
\[
\mathcal{O} \left( P^{-1} C \right),
\]
where
\[
P = \sum_{i=0}^{\min\{\ell, k+\ell\}} \binom{n-t}{k+\ell-i} \left( \prod_{j=0}^{\ell-1} (1 - q^{j-i}) \right) \cdot \left( 1 - \prod_{j=0}^{\ell-1} (1 - q^{j-i}) \right),
\]
denotes the success probability and
\[
C = (k + \ell)^3 + \prod_{j=0}^{k-1} (1 - q^{j-k}) 16 \sum_{p=1}^{\ell} q^{-p^2+p} (k + \ell)(n - k - \ell)
\]
denotes the cost of one iteration.

**Proof.** This algorithm succeeds whenever the chosen set \( \mathcal{J} \) is such that the rows of \( \mathbf{E}_{\mathcal{J} \cap \mathcal{T}} \) are
linearly dependent and that \( \mathbf{G}_\mathcal{J} \) has rank \( k \). Since the latter is true with high probability, we
will assume this probability is one. Let \( i \) denote \( \vert \mathcal{J} \cap \mathcal{T} \vert \), i.e., the number of error positions
in the set \( \mathcal{J} \). Since \( \mathbf{E}_{\mathcal{J}\cap\mathcal{T}} \) has the distribution of a random matrix in \( \mathbb{F}_q^{\ell \times i} \), the probability that \( \mathbf{E}_{\mathcal{J}\cap\mathcal{T}} \) has linearly dependent rows is given by,

\[
\left( 1 - \prod_{j=0}^{\ell-1} (1 - q^{j-i}) \right).
\]

Next, we weight this term with the probability that exactly \( i \) errors land in \( \mathcal{J} \) and form the summation over all possible \( i \), giving us

\[
P = \sum_{i=0}^{\min(\ell, k+p)} \binom{n-i}{k-i} \binom{k}{k+\ell} \cdot \left( 1 - \prod_{j=0}^{\ell-1} (1 - q^{j-i}) \right).
\]

Hence, we will need \( P^{-1} \) many iterations until we succeed. Among these non-successful iterations, we could either have that \( \mathbf{G}'_{\mathcal{J}} \) was not rank deficient, which costs \( \mathcal{O}((k+\ell)^3) \) due to the Gaussian elimination to check \( \mathbf{G}'_{\mathcal{J}} \)'s rank (step 2) or we have that \( \mathbf{G}'_{\mathcal{J}} \) was indeed rank deficient but \( \mathbf{E}_{\mathcal{J}} \) had linearly independent rows. In this second case we incur the additional cost of step 3 since only after that we will recognize that \( \mathbf{E}_{\mathcal{J}} \) did not have linearly dependent rows. Note that step 3 consists of computing the left null space of \( \mathbf{G}' \) and then performing \( q^p \) re-encoding steps where \( p \) is the dimension of this space. The left null space can be found using Gaussian elimination and thus has the same cost as the rank-check in step 2 (in fact, it is possible to find the null space with effectively no additional work from this step).

In order to compute the cost of doing the \( q^p \) re-encodings, assume \( \mathbf{E}_{\mathcal{J}} \) has linearly independent rows. Let \( P(p) \) denote the probability that \( \mathbf{G}'_{\mathcal{J}} \) has rank deficiency \( p \), i.e., \( \text{rk}(\mathbf{G}'_{\mathcal{J}}) = k + \ell - p \) where \( p \in [1, \ell] \) (since \( \mathcal{J} \) is chosen such that \( \text{rk}(\mathbf{G}_{\mathcal{J}}) = k \)). Thus \( p \) is the dimension of the left null space of \( \mathbf{G}' \). Then the workfactor of the re-encoding is given by

\[
C' = \sum_{p=1}^{\ell} P(p) q^p \alpha
\]

where \( \alpha \in \mathcal{O}((k+\ell)(n-k-\ell)) \) is the cost of a single re-encoding step.

To compute \( P(p) \), we make use of the following result: if \( V \) is an \( n \)-dimensional vector space over \( \mathbb{F}_q \) and \( U \) is an \( m \)-dimensional subspace in \( V \), then the number of \( k \)-dimensional subspaces \( W \) over \( \mathbb{F}_q \) with \( \text{dim}(W \cap U) = d \) is given by

\[
\left[ \frac{n-m}{k-d} \right]_q \left[ \frac{m}{q} \right]_q q^{(m-d)(k-d)},
\]

where \( \left[ \frac{a}{b} \right]_q \) denotes the Gaussian binomial coefficient.

Since the number of \( \mathbf{G}'_{\mathcal{J}} \) with rank deficiency \( p \) is given by the number of \( \mathbf{E}_{\mathcal{J}\cap\mathcal{T}} \) of rank \( \ell \) times the number of \( \mathbf{G}_{\mathcal{J}} \) of rank \( k \) such that \( \text{dim}(\langle \mathbf{E}_{\mathcal{J}\cap\mathcal{T}} \rangle \cap \langle \mathbf{G}_{\mathcal{J}} \rangle) = p \), we get

\[
\prod_{j=0}^{\ell-1} (q^j - q^i) \prod_{j=0}^{k-1} (q^k - q^j) \left[ \frac{\ell}{p} \right]_q \left[ \frac{k}{q} \right]_q q^{(\ell-p)(k-p)},
\]

where the first term counts the number of rank \( \ell \) matrices \( \mathbf{E}_{\mathcal{J}\cap\mathcal{T}} \), the second term is the number of ways of picking an ordered basis of a \( k \)-dimensional subspace and the third term counts the number of \( k \)-dimensional subspaces (i.e. \( \langle \mathbf{G}_{\mathcal{J}} \rangle \)) inside a \( k + \ell \) dimensional space whose intersection with a fixed \( \ell \)-dimensional subspace (i.e. \( \langle \mathbf{E}_{\mathcal{J}\cap\mathcal{T}} \rangle \)) has dimension \( p \).

Dividing this by the total number of possible \( \mathbf{G}'_{\mathcal{J}} \), i.e.,

\[
q^{(k+\ell)k} \prod_{j=0}^{\ell-1} (q^j - q^i)
\]

we get the probability

\[
P(p) = \prod_{j=0}^{k-1} \frac{q^k - q^j}{q^{k+\ell}} \left[ \frac{\ell}{p} \right]_q \left[ \frac{k}{q} \right]_q q^{(\ell-p)(k-p)}.
\]
Hence the workfactor of one iteration is given by 

\[ C' = \sum_{p=1}^{\ell} P(p)q^p \alpha \]

\[ = \prod_{j=0}^{k-1} (1 - q^{j-k}) q^{-\ell k} \sum_{p=0}^{\ell} \left[ \sum_{p=0}^{k} \left( \quad \alpha \right) \sum_{p=1}^{q} (q^{(\ell-p)(k-p)}q^p) \alpha \right] \]

\[ \leq \prod_{j=0}^{k-1} (1 - q^{j-k}) q^{-\ell k} \sum_{p=1}^{\ell} \sum_{p=1}^{q} q^{-p^2+p} \alpha \]

\[ = \prod_{j=0}^{k-1} (1 - q^{j-k}) 16 \sum_{p=1}^{\ell} q^{-p^2+p} \alpha , \]

where we used that 

\[ q^{(a-b)b} < \left[ \frac{a}{b} \right] q \leq 4q^{(a-b)b}. \]

Again, we will give an upper bound on the asymptotic cost. For this it is enough to consider a lower bound on the success probability \( P \), as

\[ \lim_{n \to \infty} \frac{1}{n} \log_q(C) = \lim_{n \to \infty} \frac{1}{n} \log_q \left( (k+\ell)^3 + \prod_{j=0}^{k-1} (1 - q^{j-k}) 16 \sum_{p=1}^{\ell} q^{-p^2+p} \alpha \right) \]

\[ \leq \lim_{n \to \infty} \frac{1}{n} \log_q \left( ((k+\ell)^3 + \ell 16 (k+\ell) (n-k-\ell)) \right) = 0. \]

Note that the success probability can be written as

\[ P = \sum_{i=0}^{\min\{\ell,k+\ell\}} Q_i, \]

for

\[ Q_i = \left( \frac{n-i}{k+\ell-i} \right)^{\ell-1} \cdot \left( 1 - \prod_{j=0}^{\ell-1} (1 - q^{j-i}) \right). \]

To get a lower bound, we use that \( \sum_{i=0}^{\min\{\ell,k+\ell\}} Q_i \geq Q_{\ell-1} \), that is we just consider

\[ Q_{\ell-1} = \left( \frac{n-\ell}{k+\ell-1} \right) \left( \frac{\ell}{\ell-1} \right) \left( \frac{n}{k+\ell} \right)^{-1}. \]

Since the interleaving order \( \ell \) is usually very small compared to \( n \), we set

\[ L = \lim_{n \to \infty} \frac{\ell(n)}{n} = \frac{T}{\gamma}, \]

for some positive integer \( 2 < \gamma \).

**Proposition 6.** The asymptotic complexity of Interleaved Prange on an \( \ell \)-interleaved random code over \( \mathbb{F}_q \) with length \( n \) and dimension \( k \) is given by at most \( q^{\alpha e(R,q)} \), where

\[ e(R,q) = H(1, R + L) - H(1 - T, R) - H(T, L) + \min\{H(R + L, R), L\}. \]

**3.4. Comparison.** In general, for small \( \ell \), it would appear that CF-based algorithms have a lower complexity than SD-based algorithms because SD-based algorithms generally solve the problem for a larger error weight than CF-based ones (but only in a slightly larger code).

However, this comparison does not take into account the large failure probabilities of these algorithms. For this reason we will not compare the cost of these algorithms with the SD-based and the Interleaved Prange algorithm. In order to compare the different algorithms, we fix \( q = 7 \) and \( \ell \) the interleaving order to be such that \( L = \lim_{n \to \infty} \frac{\ell(n)}{n} = T/20 \), i.e., \( \gamma = 20 \). In addition, we denote by \( R^* = \arg\max_{0 \leq R \leq 1} (e(R,q)) \). We have two different
approaches for the comparison. The first one is to take \( \frac{1}{n} \log_q(\cdot) \) of the actual cost of the algorithms computed for large \( n \), which seem to converge rather quickly, thus Figure 2 and Table 1 gives a very accurate plot of the complexities in this case.

![Graph showing simulated asymptotic cost of algorithms for q = 7.](image)

**Figure 2.** Simulated asymptotic cost of the algorithms for \( q = 7 \).

| Algorithm                     | \( e(R^*, q) \) | \( R^* \) |
|-------------------------------|-----------------|---------|
| Simulated Interleaved Prange  | 0.06832         | 0.475   |
| Simulated Random Prange       | 0.07848         | 0.437   |

**Table 1.** Comparison of simulated asymptotic cost of different algorithms for \( q = 7 \).

The second approach is using the presented upper bounds on the asymptotic complexity, which can be seen in Figure 3 and Table 2.

| Algorithm                     | \( e(R^*, q) \) | \( R^* \) |
|-------------------------------|-----------------|---------|
| Upper Bound Interleaved Prange| 0.07961         | 0.524   |
| Upper Bound Random Prange     | 0.08621         | 0.468   |
| Upper Bound Random Stern      | 0.08510         | 0.465   |

**Table 2.** Comparison of upper bounds of asymptotic cost of different algorithms for \( q = 7 \).

Both approaches show the same predicted behaviour, that is Interleaved Prange has a much lower complexity than the straightforward approach. Note that in the simulated asymptotics we did not compare also to Random Stern, as the improvement on Random Prange is only marginal.
4. Conclusion

In this paper we presented several algorithms that decode a random homogeneous $\ell$-interleaved code, which also work for the missing case $\ell \ll t$. Two of these algorithms come from a straight-forward reduction to known ISD and CF algorithms in the classical case. In addition to those algorithms, we also presented a new generic decoding algorithm for interleaved codes, namely Interleaved Prange, which is an adaption of Prange’s classical idea to the interleaved setting. We provided a complexity analysis and compared the asymptotic costs of the considered algorithms.

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