Vacuum Spacetimes with Future Trapped Surfaces

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Abstract

In this article we show that one can construct initial data for the Einstein equations which satisfy the vacuum constraints. This initial data is defined on a manifold with topology $\mathbb{R}^3$ with a regular center and is asymptotically flat. Further, this initial data will contain an annular region which is foliated by two-surfaces of topology $S^2$. These two-surfaces are future trapped in the language of Penrose. The Penrose singularity theorem guarantees that the vacuum spacetime which evolves from this initial data is future null incomplete.

*) Supported by Fonds zur Förderung der wissenschaftlichen Forschung in Österreich, Project No. P9376–PHY.

**) Partially supported by Forbairt Grant SC/94/225.
1 Introduction

This paper is the third in a series dealing with the existence of globally regular, asymptotically flat, maximal initial data for General Relativity in vacuo which have trapped surfaces. Our approach consists of considering certain classes of sequences of initial data called critical sequences (CS’s) which, for large values of the parameter, develop regions closely resembling the black hole region of the extended Schwarzschild (SS–) spacetime. We first recall (see [1,2]) the properties of the SS–geometry which form the model for our more general results.

Consider the spacetime metric
\[ ds^2 = -\left(1 - \frac{2m}{\tilde{r}}\right) dt^2 + \left(1 - \frac{2m}{\tilde{r}}\right)^{-1} d\tilde{r}^2 + \tilde{r}^2 dO^2 \]  
(1.1)
where \( 0 < 2m < \tilde{r} < \infty, -\infty < t < \infty \) and \( dO^2 \) is the line element on the unit two–sphere. The spacetime (1.1) can be viewed as the right quadrant of the Kruskal manifold. In particular it can be smoothly extended to the black hole region (i.e. the upper quadrant of Kruskal) by replacing \( t \) by the Eddington–Finkelstein coordinate \( \tau = t + \tilde{r} + 2m \log|\tilde{r} - 2m| - 1| \) and allowing \( \tilde{r} \) to vary over \( 0 < \tilde{r} < \infty \) and \( -\infty < \tau < \infty \). For \( \tilde{r} < 2m \) we can, if we wish, again represent \( ds^2 \) by Equ. (1.1). Now fix a constant \( C > 0 \) with
\[ C^2 < \frac{27m^4}{16} \]  
(1.2)
and consider the function \( \varphi : \mathbb{R}^+ \to \mathbb{R} \) defined by
\[ \varphi(\tilde{r}) = 1 - \frac{2m}{\tilde{r}} + \frac{C^2}{\tilde{r}^4}. \]  
(1.3)
\( \varphi \) is negative for \( \tilde{r} = 3m/2 \), positive for \( \tilde{r} = 2m \), goes to one as \( \tilde{r} \to \infty \) and is monotonically increasing for \( \tilde{r} \geq 3m/2 \). Thus, for some unique \( \tilde{r}_c \) with \( 3m/2 < \tilde{r}_c < 2m, \varphi(\tilde{r}_c) = 0 \) and \( \varphi > 0 \) for \( \tilde{r} > \tilde{r}_c \). Moreover \( \tilde{r}_c \) is a simple zero of \( \varphi \). Now define a function \( f(\tilde{r} \geq \tilde{r}_c) \) by
\[ f(\tilde{r}) = -C \int_{\tilde{r}_c}^{\tilde{r}} \frac{ds}{s^2(1 - 2m/s)\sqrt{\varphi(s)}}. \]  
(1.4)
This will be the height-function of a spherically symmetric maximal slice through the Schwarzschild solution of the kind discussed in [1,2]. If (1.4) is taken in the sense of the
Cauchy mean value at \( s = 2m \), \( f(\tilde{r}) \) is defined for \( \tilde{r}_c \leq \tilde{r} < 2m \) and \( 2m < \tilde{r} < \infty \). One can see that \( f(\tilde{r}) \) is everywhere positive and is logarithmically divergent at \( \tilde{r} = 2m \). One can also see that the function \( g \) given by
\[
g(\tilde{r}) = f(\tilde{r}) + 2m \log \left| \frac{\tilde{r}}{2m} - 1 \right| + \tilde{r} \tag{1.5}
\]
is \( C^\infty \) for all \( \tilde{r} \in (\tilde{r}_c, \infty) \). Thus the height-function diverges logarithmically at the horizon in the standard Schwarzschild coordinates but is everywhere regular in the regular Eddington-Finkelstein coordinates. Now consider the subset (of the extended manifold) given by \( \tau = g(\tilde{r}) \). For each fixed \( C \) satisfying (1.2) this is a smooth submanifold along which, for \( \tilde{r} \neq 2m \), one has \( t = f(\tilde{r}) \). This will lie in the upper half of the right-hand quadrant since we know that \( f > 0 \) and continues through to the middle of the upper quadrant of the Kruskal diagram. We can express this as \( \tilde{r} = f^{-1}(t), \ t > 0 \). The latter function can be smoothly extended to negative values of \( t \) by writing \( \tilde{r} = f^{-1}(|t|) \). This results in an extended submanifold \( \tilde{M}_C \) which is a Cauchy slice of the Kruskal manifold and lies symmetrical with respect to the timelike cylinder given by \( t = 0 \). (Note that the set where \( t = 0 \) consists of a time symmetric Cauchy slice plus a timelike cylinder which intersect orthogonally along the bifurcation sphere of the horizon.) The two asymptotic ends of \( \tilde{M}_C \) are translates under \( \partial/\partial t \) of one another by the amount \( \Delta t = 2 \lim_{\tilde{r} \to \infty} f(\tilde{r}) \). This surface \( \tilde{M}_C \) lies is the upper half of the Kruskal diagram. The ‘mirror’ surface in the lower half plane corresponds to choosing \( C < 0 \).

One can now compute the first and second fundamental form of \( \tilde{M}_C \). One gets \((\tilde{r}_c \leq r < \infty)\)
\[
\tilde{g}_{ab} dx^a dx^b = \frac{1}{\varphi} d\tilde{r}^2 + \tilde{r}^2 dO^2 \tag{1.6}
\]
and
\[
\tilde{p}_{ab} dx^a dx^b = C \left( \frac{2}{\varphi \tilde{r}^3} d\tilde{r}^2 - \frac{1}{\tilde{r}} dO^2 \right), \tag{1.7}
\]
where \( a, b \) run from 1 to 3. Note that our choice of the definition of extrinsic curvature starts from \( K_{\mu\nu} = \frac{1}{2} \mathcal{L}_u \gamma_{\mu\nu} \) which agrees with Wald[3] but is the negative of the choice made in MTW[4]. It now follows that \( \tilde{p} = \tilde{p}_{ab} \tilde{g}^{ab} = 0 \) and that \( \tilde{p}^{ab}_{\,\,\,a} = 0 \). There is a unique (up to a constant factor) spherically symmetric 2-tensor which is both divergence-free and
trace-free (TT-) on flat space [5]. TT-tensors are conformally covariant. The metric \( \tilde{g}_{ab} \) above is conformally flat and the tensor \( \tilde{p}_{ab} \) is the conformal transform of the flat-space spherically symmetric TT-tensor. In flat space the TT-tensor blows up at the origin, but since the slice through Schwarzschild has topology \( S^2 \times \mathbb{R} \) it is everywhere regular.

The complete 3–manifold \( \tilde{M}_C \) consists of two copies of the asymptotically flat manifold (1.6,7), glued together along the minimal surface \( \tilde{r} = \tilde{r}_c \) where \( \varphi \) vanishes. (The subsets \( \tilde{r} > 2m \) of these two copies are contained in the right and left quadrant of Kruskal, respectively.) Thus the \( \tilde{M}_C \)’s are maximal spherically symmetric Cauchy surfaces. Since \( \tilde{r}_c > 3m/2 \), they stay away from the singularity \( r = 0 \) uniformly in \( C \). It is well known that the spheres of constant \( t = t_0, \tilde{r} = \tilde{r}_0 \) for \( \tilde{r}_0 < 2m \) are future trapped surfaces (FTS’s). For spheres lying in \( \tilde{M}_C \) for some \( C \), we can express this fact in terms of \( \tilde{g}_{ab} \) and \( \tilde{p}_{ab} \). The divergence of the null normals to a submanifold \( \Sigma \) of a spacelike slice \( \tilde{M} \) is given in general by (note that our choice of sign for \( \tilde{p}_{ab} \) plays a role here)

\[
\Theta^\pm = H^{\pm} \tilde{p}_{ab} \tilde{n}^a \tilde{n}^b \pm \tilde{p}, \tag{1.8}
\]

where \( + (-) \) refers to the future (past) outer null normal, \( \tilde{n}^a \) is the outer normal to \( \Sigma \) embedded in \( \tilde{M} \) and \( H \) the mean curvature of \( \Sigma \) w.r. to \( \tilde{n}^a \). “Outer”, in the present context, is just conventional, say “right”. The surface \( \Sigma \) is called future–trapped iff \( \Theta^+ < 0 \) and \( \Theta^- > 0 \). In such a situation the Penrose singularity theorem [6] implies that any vacuum spacetime Cauchy evolving from \( M \) is future null incomplete.

Suppose, contrary to the Schwarzschild situation, that \( \Sigma \) divides \( M \) into an outer, non–compact region (containing spatial infinity) and a compact inner region. In this case \( \sigma \) is called an outer trapped surface (OTS) when merely \( \Theta^+ < 0 \). (Note that the notion of OTS, as opposed to that of FTS, is not necessarily time–asymmetric. We could have surfaces \( \Sigma \) for which both \( \Theta^+ \) and \( \Theta^- \) are negative.) It is known (see [7]) that the OTS–condition is already sufficient for the Penrose theorem to hold.

Let us remark at this point that the time–symmetric slices of SS have neither FTS’s nor OTS’s (at least no spherically symmetric ones), so the Penrose singularity theorem does not apply. But, due to the existence of two infinities (since the topology is \( S^2 \times \mathbb{R} \)),
the singularity theorems of Gannon [8] and Lee [9] can be used, again with the result that any Cauchy evolution is geodesically incomplete.

Applying (1.8) to \( \bar{r} = \text{const} \) in \( \tilde{M}_C \) we find that

\[
\Theta_\pm = \frac{2}{\bar{r}} \left( \sqrt{\varphi} \mp \frac{C}{\bar{r}^2} \right)
\]

(1.9)

for the right copy and

\[
\Theta_\pm = \frac{2}{\bar{r}} \left( -\sqrt{\varphi} \mp \frac{C}{\bar{r}^2} \right)
\]

(1.10)

for the left copy. Since \( \varphi > C^2/\bar{r}^4 \) for \( \bar{r} > 2m \) and \( \varphi < C^2/\bar{r}^4 \) for \( \bar{r} < 2m \), \( \Theta_+ \) and \( \Theta_- \) are both positive for \( \bar{r} > 2m \) in the right copy. At \( \bar{r} = 2m \) the quantity \( \Theta_+ \) changes sign and \( \Theta_- \) remains positive. As we cross \( \bar{r} = 2m \) in the left copy, \( \Theta_+ \) remains negative and \( \Theta_- \) changes sign. In particular, all spheres of constant \( \bar{r}_0 \) in \( \tilde{M}_C \) with \( \bar{r}_0 < 2m \) are FTS’s.

All spherically symmetric, complete maximal spacelike slices of SS are either the \( \tilde{M}_C \)’s and their translates under \( \partial/\partial t \) or the time symmetric slices \( t = \text{const} \) which can be viewed as limits of the \( \tilde{M}_C \)’s as \( C \) goes to zero. On the latter slices one has \( \Theta_+ = \Theta_- \) positive on the right copy, zero on the bifurcation 2–sphere and negative on the left copy.

In previous work [10,11] we have constructed sequences of solutions \((\tilde{g}_{ab}, \tilde{p}_{ab})\) with \( \tilde{p} = 0 \) of the Einstein vacuum constraints on \( \mathbb{R}^3 \) which are asymptotically flat, complete and which, as \( n \to \infty \), develop annular regions \( A_n \) foliated by 2–spheres for which \( \Theta_+ \) is negative. Although these data are not necessarily time–symmetric their behaviour in \( A_n \) is dominated by \( \tilde{g} \) for large \( n \) and, in particular, \( \Theta_- \) will also be negative in \( A_n \). In fact, for large \( n \), the geometry of \( A_n \) is approximated by an annulus of the left (in keeping with the previous convention) copy of a \( t = 0 \) slice of SS. For SS of course, the left and right copy are completely equivalent, but in the CS’s we consider the interior of \( A_n \) is topologically a ball rather than a second asymptotic end. Thus the (extended) Penrose theorem on OTS’s applies to spacetimes evolving from these data, and one concludes that, for large \( n \), they are null geodesically incomplete both in the future and in the past. In the present work we perform a similar construction, but where we are able to control the sign of the second term in Equ. (1.8) in such a way that, for large \( n \), we obtain annular regions \( A_n \) in our initial data sets which are modelled on the manifolds \((\tilde{M}_C, \tilde{g}_{ab}, \tilde{p}_{ab})\) of
SS described above in an annular region close to the minimal surface $\tilde{r} = \tilde{r}_c$. In particular the $A_n$'s for large $n$ are foliated by future trapped surfaces. Whether, as is suggested by the Schwarzschild case, there is inside of $A_n$ a region which is both outer future– and past trapped, is at present beyond our control. Ultimately we are interested in initial data with FTS's which result from Cauchy evolution of ones leaving neither OTS’s nor FTS’s, or, more ambitiously, which are asymptotically flat at past null infinity so that they, provided weak cosmic censorship is true, describe gravitational radiation collapsing to a black hole. To construct such data, or perhaps to show that the ones constructed here have this property, is an open problem.

In our most recent work on this topic we considered initial data sets with non-zero extrinsic curvature. However, we assumed that the extrinsic curvature fell off rapidly at infinity and played no significant role in trapped surface formation. The key improvement in this current article is that we control the asymptotic behaviour of the extrinsic curvature. In particular, we assume that the extrinsic curvature near infinity is dominated by the spherically symmetric TT-tensor of Equ.(1.7). Thus we construct a family of initial data, in which, to leading order, the intrinsic geometry looks like (1.6) and the extrinsic curvature looks like (1.7) with some constant $C$. This means that, to leading order, the null expansions approximate Equ.(1.9). Further, as in our previous work, we can allow the ADM mass to become unboundedly large while controlling the error terms. This means that we can construct initial data for which the now approximate formulae Equ.(1.9) are accurate for $\tilde{r} \geq 2m$. This initial data set will contain an annular region around $\tilde{r} = 2m$ of future trapped surfaces with a minimal surface in the annulus.

2 Definitions and results from previous work

Let $M$ be a compact manifold diffeomorphic to $S^3$. For $g$ a smooth Riemannian metric on $M$, define the conformal Laplacian acting on functions by

$$L_g = -\Delta_g + \frac{1}{8} \mathcal{R}[g],$$  \hspace{1cm} (2.1)
where $\Delta_g = g^{ab}D_a D_b$, $D_a$ the covariant derivative and $\mathcal{R}$ the scalar curvature of $g$. The elliptic operator $L_g$, viewed as a densely defined operator on the Hilbert space $L^2(M, g)$ is essentially self–adjoint, with real eigenvalues bounded from below. Let $\lambda_1(g)$ be it’s lowest eigenvalue. Suppose $\lambda_1(g) > 0$. Let $\Lambda$ be some arbitrary point in $M$. Then $L_g$ has a unique, positive Green function with source point $\Lambda$. More precisely, there exists $G : M \setminus \Lambda \to \mathbb{R}$, $G > 0$, satisfying

$$\int_M (L_g f) G dV_g = 4\pi f|_\Lambda$$  \hspace{1cm} (2.2)

for all $f \in C^\infty(M)$, or equivalently

$$L_g G = 4\pi \delta|_\Lambda,$$  \hspace{1cm} (2.3)

where $dV$ is the Riemannian volume element and $\delta|_\Lambda$ the Dirac delta distribution with source point $\Lambda$. The singularity of $G$ near $\Lambda$ can be described as follows. Let $\Omega$ be an asymptotic distance function (ADF). This should mean that $\Omega > 0$ outside $\Lambda$ and

$$\Omega|_\Lambda = 0, \quad D_a \Omega|_\Lambda = 0, \quad (D_a D_b \Omega - 2 g_{ab})|_\Lambda = 0,$$  \hspace{1cm} (2.4)

$$D_a D_b D_c \Omega|_\Lambda = 0.$$  \hspace{1cm} (2.5)

This, in a local coordinate neighbourhood centered at $\Lambda$, is equivalent to

$$\Omega = \Omega_0 + \frac{1}{2} g_{ab,c}(0)x^a x^b x^c + O^\infty(\Omega_0^2),$$  \hspace{1cm} (2.6)

where $\Omega_0 = g_{ab}(0)x^a x^b$ and $f = O^{\infty}(\Omega_0^{k/2})$ mean $f = O(\Omega_0^{k/2})$, $Df = O(\Omega_0^{(k-1)/2})$, $DDf = O(\Omega_0^{(k-2)/2})$, a.s.o. In particular $\Omega = O^{\infty}(\Omega_0)$. A function of the form $\Omega = \Omega_0 + O^{\infty}(\Omega_0^{3/2})$ satisfies (2.4) but not necessarily (2.5). Given an ADF, $G$ has the property that

$$G = \Omega^{-1/2} + \frac{m}{2} + O(\Omega^{1/2})$$  \hspace{1cm} (2.7)

for a constant $m$ and

$$\partial(G - \Omega^{-1/2}) = O(1).$$  \hspace{1cm} (2.8)

The constant $m$ has the interpretation of the ADM mass of the metric $\bar{g}_{ab} = G^4 g_{ab}$ on $M \setminus \Lambda \cong \mathbb{R}^3$. Let $\omega$ be positive. Then

$$L_g \Phi = \omega^{-5} L_g \Phi$$  \hspace{1cm} (2.9)
for all $\Phi$, where
\[
\bar{g} = \omega^4 g, \quad \bar{\Phi} = \omega^{-1}\Phi.
\]  
(2.10)

Letting $\omega = \Phi$ and $\Phi = G$, so that $\bar{\Phi} \equiv 1$, it follows from Equ. (2.3) that
\[
R[\bar{g}] = 0 \quad \text{on } \bar{M} = M \setminus \Lambda
\]  
(2.11)

where $\bar{g}_{ab} = G^4 g_{ab}$. By virtue of (2.4,5) and (2.7) $\bar{g}_{ab}$ is asymptotically flat near $\Lambda$. Thus, by the positive–mass theorem [12,13], we have that $m \geq 0$ and $m = 0$ implies that $\bar{g}$ is the flat metric on $\bar{M} = \mathbb{R}^3$, or, equivalently, $g$ is conformal to the standard metric on $S^3$.

Next let $\rho$ be smooth in $M \setminus \Lambda$, the singularity at $\Lambda$ being restricted by
\[
\rho = O(\Omega_0^{-3}), \quad \partial \rho = O(\Omega_0^{-7/2})
\]  
(2.12)

and consider the equation
\[
L_g \Phi = 4\pi \delta|_\Lambda + \frac{1}{8} \Phi^{-7} \rho.
\]  
(2.13)

This is treated as follows. Write
\[
\Phi = G + h,
\]  
(2.14)

so that (2.13) gets replaced by
\[
L_g h = \frac{1}{8} (1 + G^{-1} h)^{-7} G^{-7} \rho.
\]  
(2.15)

In order to solve Equ. (2.15) we use the Green function $G(x, x')$ with source at an arbitrary point $x' \in M$. Define
\[
\Omega_0(x, x') = g_{ab}(x')(x - x')^a(x - x')^b
\]  
(2.16)
\[
\Omega(x, x') = \Omega_0(x, x') + \frac{1}{2} g_{ab,c}(x')(x - x')^a(x - x')^b(x - x')^c + O^\infty(\Omega_0^2(x, x'))
\]  
(2.17)

in coordinate neighbourhoods of $(x, x') \in M \times M$, and extended to all of $M \times M$ as a smooth and positive function. Clearly we have
\[
G(x, x') = \Omega^{-1/2}(x, x') + \frac{\mu(x')}{2} + O(\Omega_0^{1/2}(x, x'))
\]  
(2.18)
\[
\partial_x [G(x, x') - \Omega^{-1/2}(x, x')] = O(1),
\]  
(2.19)
which relations also imply that

\[ G(x, x') = \Omega_0^{-1/2}(x - x') + O(1) \]  

(2.20)

\[ \partial_x[G(x, x') - \Omega_0^{-1/2}(x, x')] = O(1). \]  

(2.21)

It follows from the Appendix of Ref. [11] that there exists a unique positive solution \( h \) of (2.15), smooth on \( M \setminus \Lambda \), bounded and with bounded first derivative on \( M \). Thus

\[ h = O(1), \quad \partial h = O(1) \quad \text{at } \Lambda. \]  

(2.22)

We now come to the construction of maximal initial–data sets \((\mathbb{R}^3, \bar{g}, \bar{p})\) with \( \mathbb{R}^3 \) arising as \( M \setminus \Lambda = S^3 \setminus \Lambda \). Let \( p_{ab} \) be a symmetric tensor, smooth on \( M \setminus \Lambda \), which is trace– and divergence–free (TT–) with respect to \( g_{ab} \), where \( \lambda_1(g) > 0 \). Assume that

\[ p_{ab} = O(\Omega_0^{-3/2}), \quad \partial p_{ab} = O(\Omega_0^{-2}) \quad \text{at } \Lambda. \]  

(2.23)

Next solve Equ. (2.13), resp. (2.15) by taking \( \rho = p_{ab}p^{ab} \). It then follows that, with \( \bar{g}_{ab} = \Phi^4 g_{ab} \),

\[ \mathcal{R}[\bar{g}] = \frac{1}{8} \bar{p}_{ab} \bar{p}^{ab} \quad \text{on } \mathbb{R}^3, \]  

(2.24)

where \( \bar{p}_{ab} = \Phi^{-2} p_{ab} \). Furthermore

\[ \bar{\mathcal{D}}_a \bar{p}^a_b = 0, \quad \bar{p}_a^a = 0. \]  

(2.25)

Thus \((\mathbb{R}^3, \bar{g}, \bar{p})\) is a maximal solution of the Einstein vacuum constraints. Setting

\[ m = \mu + 2h|_\Lambda \]  

(2.26)

we have that \((\bar{r} := \Omega_0^{-1/2})\)

\[ \bar{g}_{ab} = \left(1 + \frac{m}{2\bar{r}}\right)^4 \delta_{ab} + O \left(\frac{1}{\bar{r}^2}\right) \]  

(2.27)

\[ \bar{\partial} \left[ \bar{g}_{ab} - \left(1 + \frac{m}{2\bar{r}}\right)^4 \delta_{ab} \right] = O \left(\frac{1}{\bar{r}^3}\right) \]  

(2.28)

\[ \bar{p}_{ab} = O \left(\frac{1}{\bar{r}^3}\right), \quad \bar{\partial} \bar{p}_{ab} = O \left(\frac{1}{\bar{r}^4}\right) \]  

(2.29)
in coordinates \( \bar{x}^a = \Omega^{-1}(x^a + \frac{1}{2} \Gamma^a_{bc} x^b x^c) \). Thus \((\bar{g}, \bar{p})\) is asymptotically flat (in fact: asymptotically Schwarzschildian) with mass \( m \) and vanishing momentum.

We now recall how TT–tensors \( p_{ab} \) having the asymptotic behaviour (2.23) are constructed. This is done on \( \mathbb{R}^3 \) with metric \( g'_{ab} \) given by

\[
g'_{ab} = \Omega^{-2} g_{ab},
\]

(2.30)

which satisfies

\[
g'_{ab} = \delta_{ab} + \left( \frac{1}{\bar{r}^2} \right), \quad \bar{\nabla} g'_{ab} = O^\infty \left( \frac{1}{\bar{r}^3} \right)
\]

(2.31)
in the \( \bar{x}^a \)-coordinates.

Again, by conformal invariance of the TT–condition we have to solve

\[
D' \cdot p'_{ab} = 0, \quad p^b_a = 0
\]

(2.32)

with

\[
p'_{ab} = O^1 \left( \frac{1}{\bar{r}^3} \right).
\]

(2.33)

More specifically we want to find a TT–tensor \( p'_{ab} \) with

\[
p'_{ab} = \frac{C}{\bar{r}^3} (3 \bar{x}_a \bar{x}_b - \bar{r}^2 \delta_{ab}) + O^1 \left( \frac{1}{\bar{r}^{3+\varepsilon}} \right),
\]

(2.34)

where \( C \) is a non–zero constant and \( \bar{x}_a = \delta_{ab} \bar{x}^b \). Note that (2.34) agrees asymptotically with (1.7), obtained for the SS metric. Using [14], this is accomplished as follows. Define the conformal Killing operator \( L \), acting on covector fields \( \lambda_a \) by

\[
(L' \lambda)_a = D'_a \lambda_b + D'_b \lambda_a - \frac{2}{3} g'_{ab} (D'c \lambda_c)
\]

(2.35)

and

\[
(D' \cdot L' \lambda)_b := D'^a (L' \lambda)_a = \Delta' \lambda_b + \frac{1}{3} D'_b (D'^a \lambda_a) + R'^b_a \lambda_b.
\]

(2.36)

With our asymptotic conditions on \( g' \), the operator \( D' \cdot L' \) is an isomorphism from the weighted Hölder space \( C^{k+2,\alpha}_\varepsilon(\mathbb{R}^3) \) to \( C^{k,\alpha}_\varepsilon(\mathbb{R}^3) \) for all \( k \in \mathbb{N} \) and all \( 0 < \varepsilon < 1 \) (see [15] for the precise definitions). We now seek fields \( \lambda_a, \bar{\lambda}_a, \lambda_a, \lambda_a \) solving

\[
(D' \cdot L' \lambda)_a = 0
\]

(2.37)
with

\[
\begin{align*}
\lambda_a^1 &= \mu_a + O^\infty \left( \frac{1}{r^\infty} \right) \\
\lambda_a^2 &= \mu_{ab} x^b + O^\infty \left( \frac{1}{r^\infty} \right) \\
\lambda_a^3 &= \varepsilon_{abc} \tilde{x}^b \kappa^c + O^\infty \left( \frac{1}{r^\infty} \right) \\
\lambda_a^4 &= 6 \tilde{x}_a \nu + O^\infty \left( \frac{1}{r^3} \right)
\end{align*}
\]

(2.38–41)

where \( \varepsilon_{abc} \) is the Euclidean volume element, \( \mu_a, \mu_{ab}, \kappa^c \) and \( \nu \) are constants with \( \mu_{ab} \) symmetric and tracefree w.r. to \( \delta_{ab} \). This is done by observing that the leading terms in (2.38–41), say \( \sigma_a \), have \( D' \cdot L' \sigma \) of fast decay, setting \( \lambda_a = \sigma_a + \delta \lambda_a \) and solving

\[
D' \cdot L' \delta \lambda = -D' \cdot L' \sigma.
\]

(2.42)

Denote by \( \lambda_a^A \) a basis of the 12–dimensional vector space spanned by \( \lambda_a^1, \lambda_a^2, \lambda_a^3, \lambda_a^4 \) and define a tracefree tensor \( Q_{ab} = Q_{(ab)} \) by

\[
Q_{ab} = f \sum_{A=1}^{12} c_A \left( L' \chi^A_{(ab)} \right),
\]

(2.43)

where \( f \in C_0^\infty(\mathbb{R}^3) \), non–negative and not identically zero. Then we seek constants \( c_A \) solving the linear system

\[
\sum_B E_{AB} c_B = -d_A,
\]

(2.44)

where \( d_A \) is the vector \( (\mu_a = 0, \mu_{ab} = 0, \kappa^a = 0, \nu = 12\pi C) \) and

\[
E_{AB} = E_{(AB)} = \int_{\mathbb{R}^3} f(L' \chi^A_{(ab)} L' \chi^B_{(ab)}) dV(g').
\]

(2.45)

It is shown in [11] that, when \( g_{ab}' \) has no conformal isometry, \( E_{AB} \) has trivial null space, and (2.44) has a unique solution. If \( g_{ab}' \) is only conformally non–flat, \( E_{AB} \) has at most a 1–dimensional null space given by \( \tilde{c}_A = (\tilde{\mu}_a, \tilde{\mu}_{ab} = 0, \tilde{\kappa}^c, \tilde{\nu} = 0) \) where \( \lambda_a = \lambda_a^1 (\tilde{\mu}_b) + \lambda_a^2 (\tilde{\kappa}^c) \) is a conformal Killing vector. Clearly \( d_A \) is orthogonal to this null space so that (2.44) can still be solved, and solved uniquely if we require in addition that

\[
\sum_A \tilde{c}_A c_A = 0.
\]

(2.46)
Thus, given \( C \) in the definition of \( d_A \), we can find \( c_A \), whence \( Q_{ab} \). For this \( Q_{ab} \) we solve

\[
(D' \cdot L'W)_a = D'^b Q_{ab}. \tag{2.47}
\]

There is a unique solution \( W_a \) to any equation of the form

\[
(D' \cdot L'W)_a = j_a, \tag{2.48}
\]

with \( j \in C^{k+2,\alpha}_\varepsilon(\mathbb{R}^3) \), lying in \( C^{k,\alpha}_\varepsilon(\mathbb{R}^3) \). Writing the differential operator as its flat–space analogue \( \delta L' \) plus the rest, we infer that

\[
(\delta L' W)_a = j_a + \delta j_a = \rho_a. \tag{2.49}
\]

Since \( g' - \delta = O^\infty(1/r^2) \), the Euclidean components of \( \rho_a \) are again in \( C^{k+2,\alpha}_\varepsilon(\mathbb{R}^3) \). Now the arguments in [11] show that

\[
W_a = \frac{\bar{x}_a P_a \bar{x}_b}{\bar{r}^3} + 7 P_a \bar{r}^2 + \frac{\varepsilon_a^{bc} \bar{x}_b L_c}{\bar{r}^3} + \frac{3 \bar{x}_a M_{bc} \bar{x}_b \bar{x}_c}{\bar{r}^5} + 6 M_{ab} \bar{x}_b \bar{r}^2 - \frac{C \bar{x}_a}{2 \bar{r}^3} + \delta W_a, \tag{2.50}
\]

where \( \delta W = O^\infty(1/r^{2+\varepsilon}) \) with the constants involved in \( \delta W = O(1/r^{2+\varepsilon}) \), \( \bar{\delta} \delta W = O(1/r^{3+\varepsilon}) \), \( \ldots \bar{\delta}^k \delta W = O(1/r^{2+k+\varepsilon}) \) depending only on pointwise weighted bounds on \( j \) and its first \( k \) derivatives.

In the particular case where \( j_a = D'^b Q_{ab} \), and with the choices made by (2.45) and by solving (2.46,47), it was shown in [17] that \( P_a \), \( L_{ab} \) and \( M_{ab} \) are all zero and \( C \neq 0 \). Thus, by a computation, the tensor

\[
p'_{ab} = Q_{ab} - (L'W)_{ab}, \tag{2.51}
\]

in addition to being TT, satisfies (2.34).

### 3 Critical Sequences

We now consider critical sequences (CS's) \( g_n \) of background metrics on \( M \). We require that these metrics all have \( \lambda_1(g_n) > 0 \) and tend smoothly to a metric \( g_\infty \) satisfying \( \lambda_1(g_\infty) = 0 \). This requirement of smooth convergence could be weakened considerably, but we shall not
attempt this. It is known that there are plenty of such sequences. In fact (see [16]), every metric $g$ with $\lambda_1(g) > 0$ can be deformed into a metric $h$ with $\lambda_1(h) < 0$ by changing it in an arbitrarily small subset of $M$, and then one could define $g_t = (1 - t)g + th$, which is a continuous sequence of Riemannian metrics for $t \in [0, 1]$. Since, by standard perturbation theory [17], $\lambda_1(g_t)$ depends continuously on $t$, there is a $t_0 \in (0, 1)$ such that $\lambda_1(g_{t_0}) = 0$ and $\lambda_1(g_t) > 0$ for $t \in [0, t_0)$. Then define $g_n = (1 - t_n)g + t_ng_{t_0}$, where $t_n$ is any infinite sequence in $[0, t_0)$ with $\lim_{n \to \infty} t_n = t_0$. This is a CS.

It was shown in [10] that, along any CS $g_n$, the time symmetric mass $\mu_n$ in (2.7) goes to infinity. Furthermore there exists a constant $E$ independent of $n$ such that

$$G_n - \Omega^{-1/2} - \frac{\mu_n}{2} \leq \mu_n E \Omega^{1/2}. \quad (3.1)$$

(We shall henceforth always denote positive, $n$–independent constants by $E$ or $E'$ with the understanding that they are not necessarily identical.) Also

$$\partial(G_n - \Omega^{-1/2}) \leq \mu_n E. \quad (3.2)$$

Note that $\Omega$, which because of Equ.’s (2.4,5) depends on $n$, also has an upper bound which is independent of $n$. Equ.’s (3.1,2) imply that

$$G_n - \Omega^{-1/2} \leq \mu_n E \quad (3.3)$$
$$\partial[G_n - \Omega^{-1/2}] \leq \mu_n E. \quad (3.4)$$

It also follows from [10] that

$$G_n - \Omega^{-1/2} \geq \mu_n E, \quad (3.5)$$

whence

$$G_n - \Omega_0^{-1/2} \geq \mu_n E. \quad (3.6)$$

All constants $E$, $E'$ originate from the Schauder and Harnack inequality on open subsets of $M$ and thus can be expressed in terms of global pointwise bounds on $g_n$ and a finite number of it’s derivatives. It follows that these constants, in addition to being independent of $n$, can be taken to be independent of the choice of source point $\Lambda$. Thus, in the Green
functions $G_n(x, x')$, the same bounds will hold with $\Omega$ replaced by $\Omega(x, x')$ and in turn by $\Omega_0(x, x')$, but with $\mu_n$ now a function of $x'$: $\mu_n = \mu_n(x')$. It is then easily shown [11] that, in fact, $\mu_n(x')$ has a uniform bound from above and below in terms of $\mu_n$ at any other point of $M$, say $\Lambda$. Writing now $\mu_n$ for $\mu_n|\Lambda$, we get

\begin{equation}
E'[\Omega^{-1/2}_0(x, x') + \mu_n] \leq G_n(x, x') \leq E[\Omega^{-1/2}_0(x, x') + \mu_n] \tag{3.7}
\end{equation}

\begin{equation}
\partial_x[G_n(x, x') - \Omega^{-1/2}_0(x, x')] \leq E\mu_n. \tag{3.8}
\end{equation}

Let $N_\Lambda$ be a fixed coordinate neighbourhood of $\Lambda$ with chart $y^a$ centered at $\Lambda$. Let $g_\delta$ be a metric on $M$ which coincides with the flat metric $\delta_{ab}$ in $N_\Lambda$ and let $\Omega^{-1/2}_\delta(x, x')$ be the Euclidean $|y - y'|$, extended smoothly as a positive function to all of $M \times M$. Clearly we have

\begin{equation}
E'[\Omega^{-1/2}_\delta(x, x') + \mu_n] \leq G_n(x, x') \leq E[\Omega^{-1/2}_\delta(x, x') + \mu_n] \tag{3.9}
\end{equation}

Thus

\begin{equation}
E'[\Omega^{-1/2}_\delta(x, x') + \mu_n] \leq G_n(x, x') \leq E[\Omega^{-1/2}_\delta(x, x') + \mu_n] \tag{3.10}
\end{equation}

\begin{equation}
\partial_x G_n(x, x') \leq D[\Omega^{-1}_\delta(x, x') + \mu_n]. \tag{3.11}
\end{equation}

Given the CS $g_n$ we can decompactify as shown in the last section, thus obtaining a sequence $g'_n$ of asymptotically flat metrics with components converging to a metric $g'_\infty$ in the Hölder norms $C^{k+3, \alpha}(\mathbb{R}^3)$ for all $k \in \mathbb{N}$. For each $g'_n$ and for $g'_\infty$ there is the operator $D' \cdot L'$, written down in Equ. (2.36) which maps $C^{k+2, \alpha}(\mathbb{R}^3)$ isomorphically onto $C^{k, \alpha}(\mathbb{R}^3)$. Thus, if there is a sequence of sources $j_n$ converging to some $j_\infty$ in the $C^{k, \alpha}_\varepsilon$-norm, the solution $W_n$ of $(D'_n \cdot L'_n)W_n = j_n$ converges to $W_\infty$ in the $C^{k+2, \alpha}_\varepsilon$-norm. Hence the fields $\lambda_n$, defined in the last section, converge to $\lambda_\infty$ corresponding to the metric $g'_\infty$.

In particular the constants involved in the remainder terms in (2.38 – 2.41) can be taken to be independent of $n$.

Let us now, for ease of presentation, suppose that the sequence $g_n$ and it’s limit $g_\infty$ are "generic". By this we mean that no linear combination of the $\lambda$’s for these metrics is a conformal Killing vector. Then, by performing for each $n$ the procedure described between Equ. (2.43) and Equ. (2.51), we obtain fields $W_n$ with $P_a = L_a = M_{ab} = 0$ and
fixed $C \neq 0$ converging, in the $C^{k+2,\alpha}_{\varepsilon}$-norms, to a $W_\infty$ with the same properties. Splitting the operators $D'_n \cdot L'_n$ into a flat-space part and the rest, it is standard to see [18] that the bounds on $\delta W$ in (2.50) are uniform in $n$. When we now undo the decompactification we obtain, for each metric $g_n$ in the CS, a TT-tensor $p_{ab}$ on $M$ obeying

$$p_{ab} = C \frac{3D_a(\Omega_0^{1/2})D_b(\Omega_0^{1/2}) - g_{ab}|_A}{\Omega_0^{3/2}} + O^\infty(\Omega_0^{-(3+\varepsilon)/2}), \quad C > 0$$

(3.12)

with $O^\infty$ independent of $n$. We now have to solve, for each $n$, the equation (2.13) with $\rho = p_{apb}$. This, in turn, requires to solve the integral form of equation (2.15), namely

$$h(x) = \frac{1}{8} \int_M G(x, x') \frac{G^{-7}(x')\rho(x')}{[1 + G^{-1}(x')h(x')]} dV_g(x').$$

(3.13)

Suppose $x \in N_\Lambda$. Then the integral in (3.13) is dominated by the contribution coming from integration over $x' \in N_\Lambda$. Using (3.9, 10, 11, 12) and that $E' \Omega_{\frac{7}{2}} / \delta \leq G^{-7}(x')$, (3.14)

we see, that $h$ is bounded by positive constants times the sum of the following integrals

$$A(y) = \int_{|y'| \leq R} \frac{1}{|y - y'|} \left( \frac{1}{|y'|} + \mu_nD \right)^{-7} \frac{1}{|y'|^6} d^3y'$$

(3.15)

and

$$B(y) = \mu_n \int_{|y'| \leq R} \left( \frac{1}{|y'|} + \mu_nD \right)^{-7} \frac{1}{|y'|^6} d^3y',$$

(3.16)

where $R$ is a bound on the size of $N_\Lambda$. It is elementary to check that

$$A = O(\mu_n^{-3}), \quad B = O(\mu_n^{-3})$$

(3.17)

with $O$ independent of $n$.

Similarly we find that $\partial h$ is uniformly bounded by

$$C(y) = \int_{|y'| \leq R} \frac{1}{|y - y'|^2} \left( \frac{1}{|y'|} + \mu_nD \right)^{-7} \frac{1}{|y'|^6} d^3y'$$

(3.18)

and

$$D(y) = \mu_n \int_{|y'| \leq R} \left( \frac{1}{|y'|} + \mu_nD \right)^{-7} \frac{1}{|y'|^6} d^3y'.$$

(3.19)
We find that
\[ C(y) = O(\mu_n^{-3}), \quad D(y) = O(\mu_n^{-2}). \] (3.20)

Consequently
\[ h = O(\mu_n^{-3}), \quad \partial h = O(\mu_n^{-2}). \] (3.21)

The above estimates provide information on the quantities \( \Theta \pm \) associated with surfaces “\( \Omega = \) small constant” as \( n \) gets large. Recall that
\[ \Theta \pm = H - \bar{p}_{ab} \bar{n}^a \bar{n}^b. \] (3.22)

The quantity \( H \), in the present case, is given by
\[ H = \tilde{D}_a \bar{n}^a = -\frac{4 \Omega^{-1/2} \Phi^{-3}}{(\Omega_c \Omega^c)^{1/2}} \left[ \frac{1}{4} \Omega^{1/2} \Phi \left( g^{ab} - \frac{\Omega^a \Omega^b}{\Omega_a \Omega_b} \right) \Omega_{ab} + \Omega^{1/2} \Phi_a \Omega^a \right]. \] (3.23)

In \( N_\Lambda \) we can, for each \( n \), change the \( y \)–coordinates centered at \( \Lambda \) to \( \bar{y}^a \) for which \( g_{ab}|_\Lambda = \delta_{ab} \) and then change these, in \( N_\Lambda \setminus \Lambda \) to \((\theta, \varphi, r)\), where \((\theta, \varphi, r)\) is related to \( \bar{y}^a \) like standard spherical coordinates to standard Euclidean coordinates. The quantity
\[ b = \Delta \Omega - \frac{\Omega^a \Omega^b g_{ab}}{\Omega_c \Omega^c} \] appearing in (3.23) can be expanded as
\[ b = 4 + f_1 r + f_2 r^2 + f_3 r^3 + O(r^4) \] (3.24)

with \( f_1, f_2, f_3 \) smooth functions on \((\theta, \varphi) \in S^2\).

Recall also that \( \Phi = G + h \), \( G \) being the conformal factor giving the time–symmetric solutions conformal to the given background metrics \( g_n \). The quantity \( G \), in turn, \([19]\) can be written as a sum
\[ G = G_{\text{loc}} + F, \] (3.25)

where \( G_{\text{loc}} \) is the Hadamard fundamental solution of \( L_g \) in a neighbourhood of \( \Lambda \), whence \( F \) is smooth and satisfies \( L_g F = 0 \) in this region. By \([19]\) this region can be chosen independent of \( n \). It is known \([19]\) that \( G_{\text{loc}} \) can be written as follows
\[ G_{\text{loc}} = \Omega^{-1/2} V + W, \] (3.26)
where $V = 1 + O(\Omega)$, $W|_\Lambda = 0$ and each of $V$ and $W$ permit an expansion in terms of $s^2$, the (geodesic distance)² to the point $\Lambda$. Clearly $s^2$ can be expanded as $s^2 = r^2 + g_1 r^3 + \ldots$ with $g_1, \ldots$ smooth on $S^2$, and the same is true for $\Omega$. Similar expansions hold for $\Omega_a$ and $G^{loc}$. Making $N_\Lambda$ smaller, if necessary, inserting this into (3.23) and using (3.21) we find that the square bracket in (3.23) can be written as $X + Y$ with

$$X = 1 - \frac{\mu_n}{2} r + \mu_n a_1 r^2 + \mu_n a_2 r^3 + O(\mu_n r^4) + O(\mu_n^{-3} r) + O(\mu_n^{-2} r^2).$$

(3.27)

The coefficients $a_1$ and $a_2$ are smooth on $S^2$ and remain bounded as $n \to \infty$. We now set

$$r_n = \frac{2}{\mu_n} + \frac{8 a_1}{\mu_n^2} + \frac{16 a_2}{\mu_n^3} + \frac{64 a_1^2}{\mu_n^4},$$

(3.28)

and take $n$ large enough so that $(\theta, \varphi, r_n) \in N_\Lambda$. Then Equ. (3.28) defines an embedded 2–sphere $\Sigma_n \subset N_\Lambda$. Evaluating $X$ along $\Sigma_n$, a computation gives that

$$X|_{\Sigma_n} = (\mu_n^{-3}).$$

(3.29)

Thus the surface $\Sigma_n$ becomes more and more minimal as $n \to \infty$. The second term in Equ. (3.21) can be estimated from

$$\tilde{\eta}_{ab} \tilde{n}^a \tilde{n}^b = \frac{4 \Omega^{-1/2} \Phi^{-3}}{(\Omega_c \Omega^c)^{1/2}} \left[ \frac{1}{4} (\Omega^a \Omega^b) \Omega^{1/2} \Omega^{1/2} \Phi^{-3} \tilde{\eta}_{ab} \tilde{n}^a \tilde{n}^b \right].$$

(3.30)

Now insert (3.12) into (3.30), using that

$$r = \Omega_0^{1/2} + O(\Omega_0) = \Omega_0^{1/2} + O(\mu_n^{-2}), \quad D_a r = D_a \Omega_0 + O(\mu_n^{-1}), \quad D_a r D^a r = 1 + O(\mu_n^{-1}).$$

(3.31)

We find that the square bracket in Equ. (3.30) is equal to

$$Y = \frac{1}{4} \Phi^{-3} [3 \mu_n C + O(\mu_n^{1-\epsilon})].$$

(3.32)

But, from (3.1),

$$\Phi|_{\Sigma_n} = \frac{3 \mu_n}{2} + O(1),$$

(3.33)

so that

$$Y = \frac{2 C}{9} \mu_n^{-2} + O(\mu_n^{-2-\epsilon}).$$

(3.34)

Thus, as $n$ gets large,

$$\Theta_+ < 0, \quad \Theta_- > 0 \quad \text{on } \Sigma_n.$$  

(3.35)

We have thus proved the
Theorem: Let \( g_n \) be a generic initial sequence and \((\tilde{g}_n, \tilde{p}_n)\) the unique initial–data set constructed for each \( n \) from the “free data” \( g_n, p_n \) with \( p_n \) coming from the “same tensors” \( Q^a_{ab} \), so that \( p_n \) is of the form (3.12) with a fixed constant \( C > 0 \). Then the 2–spheres \( \Sigma_n \) given by (3.28) are future trapped for sufficiently large \( n \).

We call a CS special when, for each \( n \) and for \( g_\infty \), there is a linear combination of the \( \lambda \)–fields defined in Sect. 2, which is a conformal Killing vector. (This has to be unique for \( g_\infty \) and for \( g_n \) with \( n \) large.) Then the above reasoning goes through unchanged, except that Equ. (2.44) is now solved with the side condition (2.46). Hence the above Theorem remains true with “generic CS” changed into “special CS”.

The construction we have described for given \( g_n \) and given function \( f \) appearing in (2.43), gives rise to a unique sequence of initial–data sets. There is of course much more freedom in the choice of extrinsic curvature. It is known [20] that, on the compact manifold \( M \), there is an infinite–dimensional set of smooth TT–tensor \( \tau_{ab} \) for any background metric \( g \). Adding such a tensor to the one constructed from \( Q_{ab} \) only changes \( \tilde{p}_{ab} \) at higher multipole order and thus gives rise to a critical sequence for which the Theorem remains to be true.

Finally, we could generalize our result by including angular momentum, that is to say construct data for which the \( W \)–field of Equ. (2.50) has no linear–momentum term \( P_a \) and no quadrupole–contribution \( M_{ab} \), but both \( L_a \) and \( C \) non–zero. The reason is that the angular momentum term in \( \tilde{p}_{ab} \) gives, to leading order, a vanishing contribution to \( \tilde{p}_{ab} \tilde{n}^a \tilde{n}^b \), so that \( \tilde{p}_{ab} \tilde{n}^a \tilde{n}^b \) will asymptotically still have a sign for large \( n \).
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