The function $h^0$ for a number field is analogous to the dimension of the Riemann-Roch spaces at divisors on an algebraic curve. We provide a method to compute this function for number fields with unit group of rank at most 2, even with large discriminant. This method is based on using LLL-reduced bases, the “jump algorithm” and Poisson summation formula.

**Keywords:** Arakelov; effectivity divisor; size function; Poisson summation formula; jump algorithm.

Mathematics Subject Classification 2010: 11Y16, 11Y40, 11H06, 11H55

1. Introduction

As an analogue of the dimension of the Riemann-Roch spaces of divisors on an algebraic curve, van der Geer and Schoof introduced the function $h^0$ for a number field $F$ (see [15]). This function is also called the “size function” for $F$ (see [3],[8],[14]). The properties and an upper bound for $h^0$ were provided in [7],[8],[15]. After that, in [10] Section 10.9, Schoof proposed a method to compute this function by using reduced Arakelov divisors. Essentially, we can approximate the value of $h^0$ at a given class of divisor $D$ by knowing the short vectors of the lattice $L$ associated to $D$ because the main contributions to $h^0(D)$ come from the shortest vectors of $L$ (see [6] and [15]). This can be efficiently computed if $L$ comes with a good, i.e., reasonably orthogonal, basis.

Here we present a method to approximate the value of $h^0$ by using the Poisson summation formula as well as some “good” divisors which can be obtained by doing the “jump algorithm” [10] Algorithm 10.8 and from an LLL-reduction on $D$ (see Section [4] and [12],[13]). These divisors may not be reduced in the usual sense (see [10]) but they can be used to compute $h^0$ efficiently.

Let $F$ be a number field of degree $n$ and discriminant $\Delta_F$ with unit group of
rank at most 2. We compute an approximate value of $h^0$ at any class of Arakelov divisors on $\text{Pic}_F$. This method does not require a basis of the unit group of the ring of integers $O_F$ of $F$ and runs in polynomial time in $\log |\Delta_F|$ (see Section 5.2).

In Section 2, we give a brief introduction to Arakelov divisors, the Arakelov class group and the group $\text{Pic}_F$, the function $h^0$ for a number field and the Poisson summation formula for lattices. Then we discuss some results on Arakelov divisors obtained from the LLL-algorithm in Section 3. Section 4 provides an algorithm to approximate the values of $h^0$ for number fields with large discriminant. Section 5 is devoted to bounding the error as well determining the running time of the algorithm. We also present some numerical examples applying this method to compute $h^0$ for real quadratic fields and real cubic fields in Section 6.

2. Preliminaries

This section briefly recalls some basic definitions that are used in the next sections. See [10,15] for full details.

Let $F$ be a number field of degree $n$ with the ring of integers $O_F$ and $r_1, r_2$ the number of real and complex infinite primes (or infinite places). Let $\text{Cl}_F$ be the class group of $F$. Denote by $F_{\mathbb{R}} := F \otimes_{\mathbb{Q}} \mathbb{R} \simeq \prod_{\sigma \text{ real}} \mathbb{R} \times \prod_{\sigma \text{ complex}} \mathbb{C}$ where $\sigma$ runs through the infinite primes of $F$. Then $F_{\mathbb{R}}$ is an étale $\mathbb{R}$-algebra with the canonical Euclidean structure given by the scalar product

$$\langle u, v \rangle := \text{Tr}(uv^*)$$

for $u = (u_\sigma), v = (v_\sigma) \in F_{\mathbb{R}}$.

In particular, in terms of coordinates, we have

$$\|v\|^2 = \text{Tr}(uv^*) = \sum_\sigma \deg \sigma |u_\sigma|^2, \text{ for any } u = (u_\sigma) \in F_{\mathbb{R}}.$$

Here the degree of an infinite prime $\sigma$ is equal to 1 or 2 depending on whether $\sigma$ is real or complex.

The norm of an element $u = (u_\sigma)_\sigma$ of $F_{\mathbb{R}}$ is defined by $N(u) := \prod_{\sigma \text{ real}} u_\sigma \cdot \prod_{\sigma \text{ complex}} |u_\sigma|^2$.

2.1. Arakelov divisors

Here and in the rest of the paper we often call fractional ideals simply ‘ideals’. If we want to emphasize that an ideal is integral, we call it an integral ideal.

Definition 2.1. An Arakelov divisor is a pair $D = (I, u)$ where $I$ is an ideal and $u$ is an arbitrary unit in $\prod_{\sigma} \mathbb{R}^+_\sigma \subset F_{\mathbb{R}}$. The Arakelov divisors of $F$ form an additive group denoted by $\text{Div}_F$.

Let $D = (I, u)$ be an Arakelov divisor. The degree of $D$ is defined by $\deg(D) := -\log(N(u)N(I))$. We associate to $D$ the lattice $uI := \{uf = (u_\sigma \cdot \sigma(f))_\sigma : f \in I\} \subset F_{\mathbb{R}}$ with the inherited metric from $F_{\mathbb{R}}$ (see [10]). For each $f \in I$, by putting $\|f\|_D := \|uf\|$, we obtain a scalar product on $I$ that makes $I$ become an ideal.
lattice as well [10, Section 4]. The covolume of the lattice $L = uI$ associated to $D$ is \( \text{covol}(L) = \sqrt{|\Delta_F|} \text{N}(I) \text{N}(u) = \sqrt{|\Delta_F|} e^{-\text{deg}(D)} \).

To each element $f \in F^*$ is attached a principal Arakelov divisor $(f) = (f^{-1}O_F, |f|)$ where $f^{-1}O_F$ is the principal ideal generated by $f^{-1}$ and $|f| = (|\sigma(f)|)_{\sigma} \in \prod_{\sigma} R^*_F \subset F_R$. It has degree 0 by the product formula.

### 2.2. The group $\text{Pic}_F$ and the Arakelov class group $\text{Pic}_0^F$

#### Definition 2.2.

The quotient of $\text{Div}_F$ by its subgroup of principal Arakelov divisors is denoted by $\text{Pic}_F$.

The set of all Arakelov divisors of degree 0 form a subgroup of $\text{Div}_F$, denoted by $\text{Div}_0^F$. The Arakelov class group of a number field is analogous to the Picard group of an algebraic curve defined as follows.

#### Definition 2.3.

The Arakelov class group $\text{Pic}_0^F$ is the quotient of $\text{Div}_0^F$ by its subgroup of principal divisors.

Each $v = (v_{\sigma}) \in \oplus_{\sigma} R$ can be embedded into $\text{Div}_F$ as the divisor $D_v = (O_F, u)$ with $u = (e^{-v_{\sigma}})_{\sigma}$. Put $(\oplus_{\sigma} R)^0 = \{(v_{\sigma}) \in \oplus_{\sigma} R : \text{deg}(D_v) = 0\}$ and $\Lambda = \{(\log |\sigma(\epsilon)|)_{\sigma} : \epsilon \in O^*_F\}$. Then $\Lambda$ is a lattice contained in the vector space $(\oplus_{\sigma} R)^0$. We define

\[
T^0 = (\oplus_{\sigma} R)^0 / \Lambda.
\]

By Dirichlet’s unit theorem, $T^0$ is a compact real torus of dimension $r_1 + r_2 - 1$ [2, Section 4.9]. The structure of $\text{Pic}_0^F$ is described by the following proposition.

#### Proposition 2.4.

The map that sends the class of a divisor $(I, u)$ to the class of the ideal $I$ is a homomorphism from $\text{Pic}_0^F$ to the class group $\text{Cl}_F$ of $F$. It induces the exact sequence

\[
0 \rightarrow T^0 \rightarrow \text{Pic}_0^F \rightarrow \text{Cl}_F \rightarrow 0.
\]

#### Proof.

See [10, Proposition 2.2].

### 2.3. Metric on the Arakelov class group

For $u \in \prod_{\sigma} R^*_+$, we let $\log u$ denote the element $\log u := (\log(u_{\sigma}))_{\sigma} \in \prod_{\sigma} R \subset F_R$. By using the scalar product from $F_R$, this vector has length $\|\log u\|^2 = \sum_{\sigma} \text{deg}(\sigma) \log^2(u_{\sigma})$. We define

\[
\|u\|_{\text{Pic}} := \min_{u' \in \prod_{\sigma} R^*_+} \|\log u'\| = \min_{\epsilon \in O^*_F} \|\log(|\epsilon|u)\|.
\]

Now let $[D]$ and $[D']$ be two classes containing divisor $D$ and $D'$ respectively lying on the same connected component of $\text{Pic}_0^F$. Then by Proposition 2.4 there is some unique $u \in T^0$ such that $D - D' = (O_F, u)$. We define the distance between
two divisor classes containing $D$ and $D'$ to be $\|u\|_{\text{Pic}}$. The function $\| \|$ gives rise to a distance function that induces the natural topology of $\text{Pic}_F^0$. See Section 6 in [10] for more details.

2.4. The function $h^0$ for a number field

Let $D = (I, u)$ be an Arakelov divisor of $F$. The effectivity $e(D)$ of $D$ is the number defined by

$$e(D) = \begin{cases} 0 & \text{if } O_F \not\subseteq I \\ e^{-\pi \|u\|^2} & \text{if } O_F \subseteq I. \end{cases}$$

This number is between 0 and 1. A divisor $D$ is called effective if $e(D) > 0$. Similar to a Riemann-Roch space of an algebraic curve, we denote by $H^0(D)$ the union of $\{0\}$ and the set of all elements $f$ of $F$ for which the divisor $(f) + D$ is effective, i.e.,

$$H^0(D) := \{f \in F^* : e((f) + D) > 0\} \cup \{0\}.$$

Since $e((f) + D) > 0$ if and only if $f \in I \setminus \{0\}$, the set $H^0(D)$ is equal to the infinite group $I$. In order to measure its size, we weight each element $f \in I$ with the effectivity of the Arakelov divisor $(f) + D$:

$$e((f) + D) = e^{-\pi \|f\|^2}.$$

The value of the function $h^0$ at $D$ is obtained by summing up these terms for all $f$ in $I$ including 0 and then taking its logarithm as follows.

$$h^0(D) = \log \left( \sum_{f \in I} e^{-\pi \|f\|^2} \right) = \log \left( \sum_{f \in I} e^{-\pi \|fu\|^2} \right).$$

Since two Arakelov divisors in the same class in $\text{Pic}_F$ have isometric associated lattices, the function $h^0(D)$ only depends on the class $[D]$ of $D$ in $\text{Pic}_F$ and we may write $h^0([D])$. In other words, $h^0$ is well defined on $\text{Pic}_F$. See [15] for more details.

2.5. The Poisson summation formula for lattices

Let $L$ be a lattice in $\mathbb{R}^n$ and $L = L_1 \oplus L_2$. Note that we do not require that $L_1$ and $L_2$ are orthogonal to one another.

Consider the sum

$$S = \sum_{z \in L} e^{-\pi \|z\|^2}.$$

Let $V$ be the subspace $L_1 \otimes \mathbb{R}$ of $\mathbb{R}^n$ and let $\pi(b)$ denote the orthogonal projection onto $V$ of a vector $b \in \mathbb{R}^n$. We have the following.

Lemma 2.5. Let $\gamma$ be the covolume of the lattice $L_1$ inside $V$. Then

$$S = \frac{1}{\gamma} \sum_{b \in L_2} e^{-\pi \|b - \pi(b)\|^2} \sum_{a \in L_1'} e^{-\pi \|a\|^2 - 2\pi i \langle a, \pi(b) \rangle}.$$
Computing dimensions of spaces of Arakelov divisors of number fields

Proof. Let \( z \in L \). Then \( z = a + b \) for some \( a \in L_1 \) and \( b \in L_2 \). The vectors \( b - \pi(b) \) and \( a + \pi(b) \) are orthogonal. The Pythagorean Theorem implies therefore

\[
\|a + b\|^2 = \|b - \pi(b)\|^2 + \|a + \pi(b)\|^2.
\]

Thus, we can write

\[
S = \sum_{b \in L_2} \sum_{a \in L_1} e^{-\pi \|a+b\|^2} = \sum_{b \in L_2} e^{-\pi \|b - \pi(b)\|^2} \sum_{a \in L_1} e^{-\pi \|a + \pi(b)\|^2}.
\]

Applying the Poisson summation formula to the second sum (see [3] and Chapter 7 in [11]), we obtain that

\[
S = \frac{1}{\gamma} \sum_{b \in L_2} e^{-\pi \|b - \pi(b)\|^2} \sum_{a \in L_1^\gamma} e^{-\pi \|a\|^2 - 2\pi i \langle a, \pi(b) \rangle}.
\]

3. Some Results

In this section, we discuss “nice” properties of Arakelov divisors obtained from the LLL-algorithm.

From now on, we put

\[
\left( \prod_{\sigma} \mathbb{R}^*_+ \right)^0 = \left\{ s \in \prod_{\sigma} \mathbb{R}^*_+ : N(s) = 1 \right\}
\]

and

\[
\partial_F = \left( \frac{2}{\pi} \right)^{r_2} \sqrt{|\Delta_F|} \text{ and } D_F = \left( 2^{(n-1)/2} \sqrt{n} \right)^n \partial_F.
\]

Definition 3.1. Let \( J \) be an ideal of \( F \). Then the Arakelov divisor associated to \( J \) is \( d(J) := (J, u) \) where \( u = (u_\sigma) \in \prod_{\sigma} \mathbb{R}^*_+ \) and \( u_\sigma = N(J)^{-1/n} \) for all \( \sigma \).

Let \( D = (I, u) \) be an Arakelov divisor and let \( L = uI \) be the lattice associated to \( D \). Assume that a basis of \( L \) is given. By using the LLL-algorithm, we can find an LLL-reduced basis \( \{b_1, ..., b_n\} \) of \( L \). Since \( b_1 \in L = uI \), there is some nonzero element \( f \in I \) such that \( b_1 = u \cdot f \). Denote by

\[
J = b_1^{-1}L = u^{-1}f^{-1}(uI) = f^{-1}I.
\]

Then \( J \) is an ideal of \( F \). Therefore, we can define as follows.

Definition 3.2. Let \( D = (I, u) \) be an Arakelov divisor and let \( L = uI \) be the lattice associated to \( D \) with a known basis. We call an LLL-reduction on \( D \) the process of finding an LLL-reduced basis \( \{b_1, ..., b_n\} \) of \( L \), then computing a new ideal lattice \( J = b_1^{-1}L \) and a new divisor \( D' = d(J) \).

We first recall the following lemma [10] Proposition 4.4.

Lemma 3.3. Let \( D = (I, u) \) be a divisor of degree 0. Then there is a nonzero element \( f \in I \) such that \( u_\sigma |\sigma(f)| \leq \partial_F^{1/n} \) for all \( \sigma \). In particular \( \|uf\| \leq \sqrt{n} \partial_F^{1/n} \).
Proof. See [10, Proposition 4.4].

We prove the proposition below.

**Proposition 3.4.** Let \( D = (I, u) \) be an Arakelov divisor of degree 0 and \( D' = d(J) \) obtained by an LLL-reduction on \( D \). Then we have the following.

1. The ideal \( J^{-1} \) is integral with \( N(J^{-1}) \leq 2^{n(n-1)/2} \partial_F \). Moreover, we obtain that \( 2^{-n(n-1)/2} (\frac{2}{\pi})^{-\tau_2} \leq \text{covol}(J) \leq \sqrt{|\Delta_F|} \).

2. There is some \( s \in (\prod \mathbb{R}_+)^0 \) and some \( f \in I \) such that

\[
D - D' + (f) = (O_F, s)
\]

and

\[
\|D - D'\|_{Pic} = \|s\|_{Pic} < \log D_F.
\]

**Proof.** i) Since \( D' \) is obtained from an LLL-reduction on \( D \), there is an LLL-reduced basis \( \{b_1, \ldots, b_n\} \) of the lattice \( uI \) associated to \( D \) such that \( b_1 = uf \) and \( J = f^{-1}I \) for some \( f \in I \). As \( f \in I \), the ideal \( J \) contains 1. Thus \( J^{-1} \) is integral.

By Lemma 3.3, there is a nonzero element \( g \in J \) such that

\[
N(J) \leq \frac{\text{covol}(J)}{\partial_1^n}.
\]

Hence

\[
N(J)^{-1/n} \max_{\sigma} |\sigma(g)| \leq \partial_F^{1/n} \quad \forall \sigma.
\]

We have

\[
\|uf(g)\| = \|(uf)g\| = \|b_1g\| \leq \max_{\sigma} |\sigma(g)| \|b_1\|.
\]

Furthermore, \( \|b_1\| \leq 2^{(n-1)/2} \|u(fg)\| \) since \( u(fg) \in uI \) and by the property of LLL-reduced bases \([9, Section 10]\). As a consequence,

\[
\max_{\sigma} |\sigma(g)| \geq \frac{\|uf(g)\|}{\|b_1\|} \geq 2^{-(n-1)/2}.
\]

The inequalities (3.1) and (3.2) imply that

\[
N(J^{-1}) \leq \frac{\partial_F}{\max_{\sigma} \|\sigma(g)\|^n} \leq 2^{n(n-1)/2} \partial_F.
\]

Therefore the first statement in i) is proved. Since \( \text{covol}(J) = \sqrt{|\Delta_F|}N(J) \), the second statement in i) follows.

ii) The divisor \( D = (I, u) \) is of degree 0, by Lemma 3.3, there is a nonzero element \( f' \) in \( I \) such that \( \|uf'\| \leq \sqrt{n} \partial_F^{1/n} \). Since \( b_1 \) is the first vector in an LLL-reduced basis of the lattice \( uI \), we again have \( \|b_1\| \leq 2^{(n-1)/2} \|uf'\| \quad [9, Section 10] \). It follows that

\[
\|b_1\| \leq \sqrt{n} \partial_F^{1/n} D_F^{1/n}.
\]
Let $s = u|f|N(J)^{1/n} = |b_1|N(J)^{1/n}$. Then with the notation in i), we have that $D - D' + (f) = (O_F, s)$. In particular, $D - D' = (O_F, s) \in \text{Pic}_F$. By Section 2.3 this leads to the following.

$$\|D - D'\|_{\text{Pic}} = \|s\|_{\text{Pic}}.$$  

Part i) shows that $J^{-1}$ is integral, so $N(J) \leq 1$. Therefore, the following inequality holds.

$$s_\sigma \leq \|s\| = \|b_1\|N(J)^{1/n} \leq D_f^{1/n}$$

This leads to $\log(s_\sigma) \leq \frac{1}{n} \log D_f$ for all $\sigma$. Since $\sum_\sigma \deg(\sigma) \log s_\sigma = 0$, we can easily prove the following \cite[Lemma 7.5]{10}.

$$\| \log s \|^2 = \sum_\sigma \deg(\sigma) |\log s_\sigma|^2 \leq n(n-1) \left( \frac{1}{n} \log D_f \right)^2 < \log^2 D_f.$$  

Since $\|D - D'\|_{\text{Pic}} = \|s\|_{\text{Pic}} \leq \| \log s \|$, part ii) is proved. \hfill \square

**Definition 3.5.** Let $W = (I,v)$ be an Arakelov divisor of degree $d$. We call $D = (I,u)$ with $u = e^{d/n}v$ the divisor translated from $W$.

Note that if $D$ is translated from a divisor $W$ then $\deg(D) = 0$. In other words, the class of the divisor $D$ is in $\text{Pic}_F^0$.

We prove the following corollary.

**Corollary 3.6.** Let $W = (I,v)$ be an Arakelov divisor of degree $d$ and let $D$ be the divisor translated from $W$. Assume that $D' = d(J)$ is a divisor obtained by an LLL-reduction on $D$. Then $L = e^{-d/n}N(J)^{-1/n}sJ$ is the lattice associated to $W$ for some $s \in \{ \prod_\sigma \mathbb{R}_+ \}^0$ and $\|s\|_{\text{Pic}} < \log D_f$.

**Proof.** By Proposition 3.4, there exists some $s \in \{ \prod_\sigma \mathbb{R}_+ \}^0$ for which $\|s\|_{\text{Pic}} < \log D_f$ and $f \in I$ such that $D + (f) = D' + (O_F, s) \in \text{Div}_F^0$. Then so $D = D' + (O_F,s) = (J, N(J)^{-1/n}s) \in \text{Pic}_F^0$.

Since $W = D + (O_F,e^{-d/n}) \in \text{Pic}_F$, we have the following.

$$W = D + (O_F,e^{-d/n}) = (J, e^{-d/n}N(J)^{-1/n}s) \in \text{Pic}_F.$$  

Thus, the lattice associated to $W$ is $L = e^{-d/n}N(J)^{-1/n}sJ$. \hfill \square

**4. Computing The Function $h^0$**

Let $W = (I,v)$ be an Arakelov divisor of degree $d$ and let $D$ be the divisor translated from $W$. Assume that a basis for the ideal lattice $I$ and the coordinates of the vector $v$ are known. We compute an approximate value of $h^0$ at the class of the Arakelov divisor $W \in \text{Pic}_F$ with some given error $\delta$. We consider the case in which $v$ is a long vector and the discriminant $\Delta_F$ is quite large since it is quite trivial to compute $h^0(W)$ in other cases.
We have that
\[ h^0(W) = \log \sum_{f \in F} e^{-\pi \|f v\|^2}. \]

We approximate the value of \( h^0(W) \) with some small error. This can be done by summing up only the large terms, i.e., the terms \( e^{-\pi \|f v\|^2} \) for which \( \|f v\|^2 \leq M \) with some given \( M > 0 \). In case \( v \) is a long vector, while collecting such short vectors \( f \), it is quite easy to miss many of them. Consequently, the obtained value of \( h^0(W) \) may be smaller than the true value. Therefore, we will find some “good” divisor \( D' \), that is obtained from an LLL-reduction on \( D \) and has nice properties described in Section 3, then use it for computing \( h^0(W) \).

Note that by Proposition 3.4 for any given divisor \( D \) of degree 0, there exists a good divisor \( D' \) close to \( D \) in \( \text{Pic}\,^0_F \) in the sense that \( \|D - D'\|_{\text{Pic}} < \log D_F \).

We first describe the following algorithm that is similar to (cf. [10, Algorithm 10.4]).

**ALGORITHM 4.1.** Given two Arakelov divisors \( D_1 = d(J_1) \) and \( D_2 = d(J_2) \) such that \( N(J_1^{-1}) \leq 2^{n(n-1)/2} \partial_F \) and \( N(J_2^{-1}) \leq 2^{n(n-1)/2} \partial_F \), compute a divisor \( d(J) \) obtained from the LLL reduction on \( D_1 + D_2 \in \text{Pic}\,^0_F \) in polynomial time in \( \log |\Delta_F| \). Description. Since \( N(J_1^{-1}) \leq 2^{n(n-1)/2} \partial_F \) and \( N(J_2^{-1}) \leq 2^{n(n-1)/2} \partial_F \), the result \( D_3 = D_1 + D_2 = (J_1 J_2, N(J_1 J_2)) \) can be computed in time polynomial in \( \log |\Delta_F| \). Then one performs the LLL reduction on the divisor \( D_3 \). The resulting divisor \( d(J) \) is then close to \( D_1 + D_2 \) by Proposition 3.4. Since \( N(J_1 J_2)^{-1} \leq 2^{n(n-1)/2} \partial_F \), the running time of this second step is also polynomial in \( \log |\Delta_F| \).

Next, we explain how to compute efficiently a divisor \( D' \) obtained from some LLL reduction close to a given divisor \( D = (O_F, u) \) in \( \text{Pic}^0_F \). This process can be seen as performing repeatedly doubling and LLL-reduction to go from the origin \((O_F, 1)\) to \( D \). We apply the “jump algorithm” [10, Algorithm 10.8] with a minor modification to adapt to our situation. Indeed, in the reduction step, instead of using a shortest vector, we use the first vector of an LLL-reduced basis of the lattices associated to Arakelov divisors.

**ALGORITHM 4.2.** Given a divisor \( D = (O_F, u) \) of degree 0, compute a reduced Arakelov divisor whose image in \( \text{Pic}^0_F \) has distance less than \( \log D_F \) from \( D \). Description. Assume that \( u = (e^{-w^*})_{\sigma} \). Let \( t \geq 0 \) be the smallest integer for which \( n \cdot 2^{-t} \cdot |w_{\sigma}| < \log \partial_F \) for all \( \sigma \). Then \( z_{\sigma} = 2^{-t} \cdot w_{\sigma} \) satisfies \( n \cdot |z_{\sigma}| < \log \partial_F \) for all \( \sigma \). Let \( \omega = (e^{-z^*})_{\sigma} \). Then \( \omega^2 = u' \). In other words, from the point \((O_F, \omega)\) we can reach to \( D \) after \( t \) times doubling. Denote by \( W_i = (O_F, \omega^{2^i}) \) for \( i = 1, 2, \ldots, t \). We inductively compute Arakelov divisors \( D'_i = d(J_i) \) obtained by LLL reduction for which
\[ \|W_i - D'_i\|_{\text{Pic}} \leq \log D_F. \] (4.1)

We compute \( D'_{i+1} \) from \( W_i \) by doubling and doing LLL reduction. More precisely, by induction, there exists some \( \omega_i \in (\prod_{\sigma} \mathbb{R}^*_+) \) such that \( W_i = D'_i + (O_F, \omega_i) \) in...
Note that here \( \omega_{i+1} = \frac{1}{b_i}N(J_{i+1})^{1/n} \). Thus, we can construct all divisor \( D_i' \) satisfying (4.1) for \( i = 1, 2, \ldots, t \).

Now let \( s = \omega_t \) and let \( D' = d(J_t) \). Then

\[
D = D_i + (O_F, \omega_i) = D' + (O_F, s) \in \text{Pic}^0_F
\]

for \( s = \omega_t \in \left( \prod_j \mathbb{R}^*_+ \right)^0 \) and \( \|D - D'\|_{\text{Pic}} = \|s\|_{\text{Pic}} = \|\omega_t\|_{\text{Pic}} < \log D_F \). This completes the description of the algorithm.

With the notation of Proposition 3.4 and Corollary 3.6 let \( L = e^{-d/n}N(J)^{-1/n} sJ \) be the lattice associated to \( W \). Then

\[
h^0(W) = \log \sum_{f \in J} e^{-\pi e^{-2d/n}N(J)^{-1/n} \|fs\|^2} = \log \sum_{g' \in L} e^{-\pi \|g'\|^2}.
\]

Every vector of the lattice \( L \) has the form \( e^{-d/n}N(J)^{-1/n} sf \) for some \( f \in J \). As \( s \) is short, \( N(J)^{-1/n} \) is a small scalar and \( J \) is a nice lattice (see Proposition 3.4), we can easily compute an LLL-reduced basis of \( L \) then find short vectors of the lattice \( L \) more efficiently. Therefore, the value of \( h^0(W) \) can be computed more exactly. Computing \( h^0(W) \) is done in 3 steps described in Section 4.1, 4.2 and 4.3 in succession.

### 4.1. Finding a good divisor \( D' \) close to \( D \)

Assume that a basis for the ideal lattice \( I \) and the coordinates of the vector \( v \) are known. We will find a divisor \( D' \) that is obtained from some LLL reduction and with the property that \( D = D' + (O_F, s) \in \text{Pic}^0_F \) for some \( s \in \left( \prod_j \mathbb{R}^*_+ \right)^0 \) and \( \|D - D'\|_{\text{Pic}} < 3 \log D_F \).

Let \( D_1 = (I, N(I)^{-1/n}) \) and to \( D_2 = (O_F, N(I)^{1/n} u) \). Then \( D_1, D_2 \) have degree 0 and \( D = D_1 + D_2 \). We compute divisors \( D'_1 \) (see 4.1.1) and \( D'_2 \) (see 4.1.2) obtained from some LLL reduction so that \( \|D_1 - D'_1\|_{\text{Pic}} < \log D_F \) and \( \|D_2 - D'_2\|_{\text{Pic}} < \log D_F \). Then we find a divisor \( D' = d(J) \) close \( D'_1 + D'_2 \) in \( \text{Pic}^0_F \) (see 4.1.3). This process is described as follows.
4.1.1. Computing $D'_1$

We compute a divisor $D'_1$ close to $D_1$ in $\text{Pic}^0_F$ in the sense that its distance to $D_1$ is at most $\log D_F$. This can be done easily by performing the LLL reduction on $D_1$.

By Proposition 4.4, there is some $s_1 \in (\prod_{\sigma} \mathbb{R}^+_*)^0$ so that $D_1 - D'_1 = (O_F, s_1)$ in $\text{Pic}^0_F$ and $\|D_1 - D'_1\|_{\text{Pic}} = \|s_1\|_{\text{Pic}} < \log D_F$.

4.1.2. Computing $D'_2$

Use Algorithm 4.2 to compute a reduced Arakelov divisor $D'_2$ whose image in $\text{Pic}^0_F$ has distance less than $\log D_F$ from $D_2 = (O_F, u')$ with $u' = N(I)^{1/n}u$. We obtain that

$$D_2 = D'_2 + (O_F, s_2) \text{ in } \text{Pic}^0_F$$

for $s_2 \in (\prod_{\sigma} \mathbb{R}^+_*)^0$ and $\|D_2 - D'_2\|_{\text{Pic}} = \|s_2\|_{\text{Pic}} < \log D_F$.

4.1.3. Computing $D'$

Adding divisors $D'_1$ and $D'_2$ as described in Algorithm 4.1 we then compute a divisor $D' = d(J)$ close to $D'_1 + D'_2$ in $\text{Pic}^0_F$. Indeed, by performing LLL reduction on the divisor $D'_1 + D'_2$ we obtain $D' = d(J)$ and $(D'_1 + D'_2) - D' = (O_F, s_3)$ for some $s_3 \in (\prod_{\sigma} \mathbb{R}^+_*)^0$ and $\|D'_1 + D'_2\| - \|D'\|_{\text{Pic}} = \|s_3\|_{\text{Pic}} < \log D_F$.

Let $s = s_1 \cdot s_2 \cdot s_3 \in (\prod_{\sigma} \mathbb{R}^+_*)^0$. Then

$$D - D' = (D_1 - D'_1) + (D_2 - D'_2) - ((D'_1 + D'_2) - D') = (O_F, s) \in \text{Pic}^0_F.$$ 

Thus, $\|D - D'\|_{\text{Pic}} = \|s\|_{\text{Pic}} \leq \|s_1\|_{\text{Pic}} + \|s_2\|_{\text{Pic}} + \|s_3\|_{\text{Pic}} \leq 3 \log D_F$.

4.2. Applying Poisson summation for the lattice $L$

Since $D - D' = (O_F, s) \in \text{Pic}^0_F$, it follows that $D = (J, N(J)^{-1/n}s)$. By translating $D$ to the divisor $W$, we obtain that $W = D + (O_F, e^{-d/n}) = (J, e^{-d/n}N(J)^{-1/n}s)$.

Let $L = e^{-d/n}N(J)^{-1/n}sJ$ be the lattice associated to $D$. Assume that $L$ has an LLL-reduced basis $\{b_1, ..., b_n\}$. Let $\{b_1^*, ..., b_n^*\}$ denote the Gram-Schmidt orthogonalization of this basis and

$$\mu_{i,j} = \frac{(b_i, b_j^*)}{\|b_j^*\|^2} \text{ for all } 2 \leq i \leq n \text{ and } 1 \leq j \leq i - 1.$$

Then any element $z \in L$ can be written uniquely as $z = \sum_{i=1}^n x_i b_i$ with the coefficients $x_i \in \mathbb{Z}$ for all $i = 1, 2, ..., n$. Similar to Lemma 4.5 we can write $\|z\|^2$ as below.

$$\|z\|^2 = \sum_{i=1}^n A_{i,i} \left(x_i + \sum_{j=1}^{i-1} A_{i,j} x_j\right)^2 = g(x_1, x_2, ..., x_n).$$  \hspace{1cm} (4.3)
We only catch the vectors $x$ for which $\|x\|^2 < M$. In other words, we only compute vectors $x = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n$ satisfying $g(x) \leq M$. The Fincke–Pohst algorithm [4, Algorithm 2.12] or an LLL reduced basis of $L$ [9, Section 12] can be used to find the list $L$ of these vectors $x$.

An approximate value of $h^0$ is obtained by summing up only the terms $e^{-\pi q(x)}$ for which $x \in L$ as below.

$$h^0(W) \approx \log \sum_{x \in L} e^{-\pi q(x)}.$$  (4.5)

The lattice $L$ has covolume $\text{covol}(L) = \prod_{i=1}^{n} \|b_i^*\| = \sqrt{\prod_{i=1}^{n} A_{ii}}$. The list $L$ can have at most $\alpha(n) \frac{M^n}{n^{n/2}}$ vectors. Here $\alpha(n)$ is a function depending only on $n$. See Algorithm 2.12 in [4] and Section 12 in [9] for the explanation. Therefore, in order to reduce the number of vectors in the list $L$, we can “make” $\text{covol}(L)$ larger by using the Poisson summation formula as follows.

Assume that $\|b_i^*\| = \|b_2\| < 1$. Let $k$ be the largest index such that $A_{i,i} < 1$ for all $i \leq k$. Denote by

$$L_1 = \bigoplus_{i=1}^{k} \mathbb{Z} \cdot b_i \quad \text{and} \quad L_2 = \bigoplus_{j=k+1}^{n} \mathbb{Z} \cdot b_j.$$

Then $L = L_1 \oplus L_2$. Since $\{b_1, \ldots, b_n\}$ is LLL-reduced, the vectors $b_1, \ldots, b_k$ form an LLL-reduced basis for $L_1$ (see [9]).

**Remark 4.4.** Let $B$ be the matrix of which columns are vectors $b_1, \ldots, b_k$ and let $G = B^t \cdot B$. Then the columns of $B \cdot G^{-1}$ form a basis for the dual lattice $L_1^\vee$ of $L_1$.

Now we apply the Poisson summation formula for $L$. See Lemma 2.5. Let $\gamma$ be the covolume of the lattice $L_1$ inside $V = L_1 \otimes \mathbb{R}$. Then $\gamma = \prod_{i=1}^{k} \|b_i^*\|$ and

$$h^0(W) = \log \left( \frac{1}{\gamma} \sum_{b \in L_2} e^{-\pi \|b-b_2\|^2} \sum_{a \in L_1^\vee} e^{-\pi \|a\|^2 - 2\pi i \langle a, \pi(b) \rangle} \right). \quad (4.6)$$

Assume that $L_1^\vee$ has a basis $c_1, \ldots, c_k$ that is computed by Remark 4.4. Denote by $\{c_1^*, \ldots, c_k^*\}$ the Gram-Schmidt orthogonalization of the basis $\{c_1, \ldots, c_k\}$ and

$$C_{i,i} = \|c_i^*\|^2, \quad C_{i,j} = \frac{\langle c_i, c_j^* \rangle}{\|c_j^*\|^2} \quad \text{for all } 1 \leq i \leq k \text{ and } 1 \leq j \leq i - 1.$$

Now let $a = \sum_{i=1}^{k} x_i c_i \in L_1^\vee$ where $x_i \in \mathbb{Z}$ for all $i = 1, 2, \ldots, k$ and $b = \sum_{j=k+1}^{n} x_j b_j \in L_2$ where $x_j \in \mathbb{Z}$ for all $j = k+1, \ldots, n$.

**Lemma 4.5.** We have

$$C_{i,i} = \|c_i^*\|^2 = \frac{1}{\|b_i^*\|^2} \quad \text{for all } 1 \leq i \leq k$$
and moreover
\[ \|a\|^2 = \sum_{i=1}^{k} C_{i,j} \left( x_{i} + \sum_{r=i+1}^{k} C_{r,i} x_{r} \right)^2. \] (4.7)

**Proof.** This is easily proved by using Remark 4.4 and properties of the Gram-Schmidt orthogonal basis \( \{ c_1^*, \cdots, c_k^* \} \).

Recall that \( A_{j,j} = \| b_j^* \|^2 \) and \( A_{t,j} = \mu_{t,j} \) for all \( j = k+1, \cdots, n \) and \( t > j \). We have the lemmas below.

**Lemma 4.6.** We have
\[ \langle a, \pi(b) \rangle = \sum_{l=1}^{k} \sum_{j=k+1}^{n} \sum_{i=1}^{k} A_{j,i} \langle c_l, b_i^* \rangle x_l x_j. \] (4.8)

**Proof.** Because \( \pi(b) \) is the orthogonal projection of \( b \) on \( V \) that has an orthogonal basis \( b_1^*, \cdots, b_k^* \), we obtain that
\[ \pi(b) = \sum_{i=1}^{k} \frac{\langle b, b_i^* \rangle}{\| b_i^* \|^2} b_i^* = \sum_{i=1}^{k} \sum_{j=k+1}^{n} x_j A_{j,i} b_i^*. \] (4.9)

Then the result is implied by taking scalar product of \( \pi(b) \) with \( a = \sum_{l=1}^{k} x_l c_l \).

**Lemma 4.7.** We have
\[ \| b - \pi(b) \|^2 = \sum_{j=k+1}^{n} A_{j,j} \left( x_{j} + \sum_{t=j+1}^{n} A_{t,j} x_{t} \right)^2. \]

**Proof.** Since \( b_j = b_j^* + \sum_{t=j+1}^{n} \mu_{j,t} b_i^* \) for all \( j \geq 2 \), the vector \( b \) therefore can be rewritten as
\[ b = \sum_{j=k+1}^{n} \left( x_{j} + \sum_{t=j+1}^{n} \mu_{t,j} x_{t} \right) b_j^* + \sum_{i=1}^{k} \sum_{j=k+1}^{n} x_j \mu_{j,i} b_i^*. \] (4.10)

By using equalities (4.7) and (4.10), the result is obtained since the vectors \( b_{k+1}^*, \cdots, b_{n}^* \) are pairwise orthogonal.

**Lemma 4.5, 4.6, 4.7 and (4.6) lead to**
\[ h^0(W) = \log \left( \frac{1}{\gamma} \sum_{x_i \in Z} e^{-\pi Q(x_1, x_2, \ldots, x_n)} \right). \] (4.11)
where \( Q(x_1, \ldots, x_n) = Q_1(x_1, \ldots, x_n) + 2Q_2(x_1, \ldots, x_n)i \) with
\[
Q_1(x_1, \ldots, x_n) = \|b - \pi(b)\|^2 + \|a\|^2
\]
\[
= \sum_{j=k+1}^{n} A_{j,j} \left( x_j + \sum_{t=j+1}^{n} A_{t,j} x_t \right)^2 + \sum_{i=1}^{k} C_{i,j} \left( x_i + \sum_{r=i+1}^{k} C_{r,j} x_r \right)^2
\]  \hspace{1cm} (4.12)

and
\[
Q_2(x_1, \ldots, x_n) = \langle a, \pi(b) \rangle = \sum_{l=1}^{k} \sum_{j=k+1}^{n} \sum_{i=1}^{k} A_{j,i} (c_{i,b}) x_j x_i.
\]

### 4.3. Finding the short vectors of the lattice associated to \( D \)

An approximation of \( h^0(W) \) is obtained by summing up the terms \( e^{-\pi Q(x)} \) such that
\( Q_1(x) \leq M \). By using the Fincke–Pohst algorithm, we can find the list \( L_1 \) containing all vectors \( x = (x_1, \ldots, x_n) \in \mathbb{Z}^n \) such that \( Q_1(x) \leq M \). See Algorithm 2.12 in \[4\]. Then an approximate value of \( h^0(W) \) is obtained as follows.
\[
h^0(W) \approx \log \left( \frac{1}{\gamma} \sum_{x \in L_1} e^{-\pi Q(x_1, \ldots, x_n)} \right). \tag{4.13}
\]

**Remark 4.8.** Let \( L_1 = \{ x \in \mathbb{Z}^n : Q_1(x) \leq M \} \) and \( L = \{ x \in \mathbb{Z}^n : q(x) \leq M \} \). Let \( L' \) be the lattice associated to the quadratic form \( Q_1(x) \). Then
\[
\text{covol}(L')^2 = \prod_{i=1}^{k} C_{i,i} \prod_{j=k+1}^{n} A_{j,j} = \frac{1}{\prod_{i=1}^{k} A_{i,i}} \prod_{j=k+1}^{n} A_{j,j} = \frac{1}{\left( \prod_{i=1}^{k} A_{i,i} \right)^2} \text{covol}(L)^2.
\]  \hspace{1cm} (4.14)

We have that \( \#L \leq \alpha(n) M^{n/2} \text{covol}(L)^n \) and \( \#L_1 \leq \alpha(n) M^{n/2} \text{covol}(L)^n \) (see Algorithm 2.12 in \[4\] and Section 12 in \[9\]). Since \( A_{i,i} < 1 \) for all \( i \leq k \), it follows that \( \text{covol}(L) \geq \sqrt{\prod_{i=1}^{k} A_{i,i}} > 1 \). At the result, the sum in \( (4.13) \) converges better than in \( (4.1) \). Hence, we can compute \( (4.13) \) by only summing a small number of terms.

Note that the function \texttt{qfminim} in \texttt{pari-gp} that uses the Fincke–Pohst algorithm, can be used to find all nonzero vectors (up to a sign) with length bounded by \( M \) of a given lattice. Another method uses an LLL reduced basis of the lattice \( L' \); see Section 12 in \[9\]. For a fixed lattice, the complexity of both methods is in polynomial time in \( M \) (see Section 5.2).

By the proofs of Lemma 5.1, 5.2 and Proposition 5.3 to approximate \( h^0(W) \) with an error \( \delta \), we can choose \( M \approx \frac{1}{\gamma} \left( \log(1/\delta) + (n + 1) \log 3 + (n(n + 1)/2 - 1) \log 2 \right) \).

The algorithm below computes an approximate value of \( h^0(W) \) with a given error \( \delta \).

**Input:**
A basis for the lattice \(I\).

• The coordinates of \(v\).

• An error \(\delta\).

Output: An approximate value of \(h^0(W)\) with error \(\delta\).

**ALGORITHM 4.9.**

1. Find a divisor \(D'\) that is close to \(D\) in \(\text{Pic}^0_F\) as described in Section 4.1.
2. Apply Poisson summation formula.
   a. Find an LLL-reduced basis \(\{b_1, \ldots, b_n\}\) of \(L'\).
   b. Compute \(\{b_1^*, \ldots, b_n^*\}\) and \(A_{i,j} = \|b_i^*\|^2\) and \(A_{i,j} = \frac{(b_i, b_j^*)}{\|b_j^*\|^2}\) for all \(2 \leq i \leq n\) and \(1 \leq j \leq i - 1\).
   c. If \(\|b_1\| \geq 1\), then put \(Q(x_1, \ldots, x_n) = Q_1(x_1, \ldots, x_n) = q(x_1, \ldots, x_n)\) (see (4.3)) and \(Q_2(x_1, \ldots, x_n) = 0\), \(L' = L\) and \(\gamma = 1\). If \(\|b_1\| < 1\), then let \(k\) be the largest index such that \(\|b_j^*\| < 1\) for all \(j \leq k\). Denote by \(B\) the matrix of which columns are vectors \(b_1, \ldots, b_k\).
   i. Compute \(G = B' \cdot B\) and \(C = B \cdot G^{-1}\).
   ii. Let \(c_1, \ldots, c_k\) be the columns of \(C\). Compute \(\{c_1^*, \ldots, c_k^*\}\) and \(C_{i,j} = \frac{(c_i, c_j^*)}{\|c_j^*\|^2}\) for all \(1 \leq i \leq k\) and \(1 \leq j \leq i - 1\).
   iii. Compute \(\langle c_l, b_i^* \rangle\) for all \(l = 1, \ldots, k\) and \(i = 1, \ldots, k\).
   iv. Denote \(Q(x_1, \ldots, x_n), Q_1(x_1, \ldots, x_n)\) and \(Q_2(x_1, \ldots, x_n)\) as in (4.12) and let \(L'\) be the lattice associated to \(Q_1(x)\) and \(\gamma = \sqrt[1]{\prod_{i=1}^{k} A_{i,i}}\).
3. Find the short vectors of the lattice \(L'\).
   a. Compute \(M = \frac{1}{\pi \delta} \left(\log(1/\delta) + (n + 1) \log 3 + (n(n + 1)/2 - 1) \log 2\right)\).
   b. Find the list \(L_1 = \{x = (x_1, \ldots, x_n) \in \mathbb{Z}^n : Q_1(x) \leq M\}\) and approximate \(h^0(W)\) as (4.13).

5. The Error and Running Time of The Algorithm

5.1. **Bound for the error in Algorithm 4.9**

To find a bound for the error in approximating the value of \(h^0(W)\) in Algorithm 4.9 we use the idea of [8, Section 4] as below.

**Lemma 5.1.** Let \(L'\) be a lattice of rank \(n\). Assume that the length of shortest vector of the lattice \(L'\) is \(\lambda\) and \(M \geq \max\{\lambda^2, \frac{n}{2} \log \frac{n}{\delta}\}\).

Let

\[ S' = \sum_{x \in L', \|x\|^2 > M} e^{-\pi \|x\|^2}. \]
Then \( S' = O(\lambda^{-n}e^{-(\pi-1)M}) \). In particular, the bound for \( S' \) goes to zero when \( M \) tends to infinity.

**Proof.** Let \( B_t = \{ a \in L' : M \leq \|a\|^2 \leq t \} \) for each \( t > M \). The balls with centers \( a \in B_t \) and radius \( \lambda/2 \) are disjoint. Their union is contained in the (hyper) annular disk

\[
\{ z \in \mathbb{C} : \sqrt{M} - \lambda/2 \leq \|z\| \leq \sqrt{t} + \lambda/2 \}.
\]

Consequently, the following is implied.

\[
\left( \frac{\lambda}{2} \right)^n \# B_t \leq \left( \sqrt{t} + \frac{\lambda}{2} \right)^n - \left( \sqrt{M} - \frac{\lambda}{2} \right)^n.
\]

This leads to

\[
\# B_t \leq \left( 1 + \frac{2\sqrt{t}}{\lambda} \right)^n - \left( \frac{2\sqrt{M}}{\lambda} - 1 \right)^n < \left( \frac{3\sqrt{t}}{\lambda} \right)^n - \left( \frac{2\sqrt{M}}{\lambda} - 1 \right)^n.
\]

The second inequality is since \( t > M \geq \lambda^2 \). Using this inequality, we get

\[
S' = \sum_{\|a\|^2 > M} \pi e^{-\pi t} dt \leq \pi \int_M^\infty \# B_t e^{-\pi t} dt \leq \pi \int_M^\infty \# B_t e^{-\pi t} dt - \pi \int_M^\infty \left( \frac{3\sqrt{t}}{\lambda} \right)^n e^{-\pi t} dt - \pi \int_M^\infty \left( \frac{2\sqrt{M}}{\lambda} - 1 \right)^n e^{-\pi t} dt.
\]

Since \( M \geq \frac{\lambda}{2} \log \frac{2}{\lambda} \), we have \( \left( \frac{3\sqrt{t}}{\lambda} \right)^n < \left( \frac{3}{\lambda} \right)^n \). This implies that the first integral is at most \( \frac{1}{\pi - 1} \left( \frac{3}{\lambda} \right)^n e^{-(\pi-1)M} \). The second one is equal to \( \frac{1}{\pi} \left( \frac{2\sqrt{M}}{\lambda} - 1 \right)^n e^{-\pi M} \).

Hence

\[
S' \leq \frac{\pi}{\pi - 1} \left( \frac{3}{\lambda} \right)^n e^{-(\pi-1)M} - \left( \frac{2\sqrt{M}}{\lambda} - 1 \right)^n e^{-\pi M}.
\]

Thus, the lemma is proved. \( \square \)

**Lemma 5.2.** Let \( L' \) be the lattice in \( \mathbb{R}^n \) associated to the definite positive quadratic form \( Q_1(x) \) in \([4.12]\) of Section 4.3. Then the shortest vector of the lattice \( L' \) has length \( \lambda \geq 2^{(-n+1)/2} \).

**Proof.**

If \( A_{1,1} = \|b_1\|^2 \geq 1 \), then by the property of LLL-reduced bases, the length of the shortest vector of \( L' \) is at least \( 2^{(-n+1)/2} \|b_1\| \) [6, Section 10]. In other words, \( \lambda \geq 2^{(-n+1)/2} \).

If \( A_{1,1} < 1 \), then \( \frac{1}{A_{i,i}} > 1 \) for all \( i \leq k \) and \( A_{k+1,k+1} \geq 1 \) since \( k \) is the largest index such that \( A_{i,i} < 1 \) for all \( i \leq k \). On the other hand, we have \( A_{i,i} = \|b_i^*\|^2 \) for
all $i = 1, 2, \ldots, n$. So, if $k + 2 \leq j \leq n$ then $A_{j,j} = \|b_j^*\|^2 \geq 2^{-(j-k-1)}\|b_{k+1}^*\|^2 = 2^{-(j-k-1)}A_{k+1,k+1} \geq 2^{-n+1}$ \cite[Section 10]{10}. Thus, all the coefficients $C_{i,i} = \frac{1}{\lambda_i}$ with $i \leq k$ and $A_{jj}$ with $j \geq k + 1$ are at least $2^{-n+1}$. As the result,

$$Q_1(x) = \min \{C_{i,i}, A_{jj} : 1 \leq i \leq k \text{ and } k + 1 \leq j \leq n\} \geq 2^{-n+1}.$$ 

The result now follows since $\lambda^2 = \min \{Q_1(x) : x \in \mathbb{Z}\}$. \hfill \qed

**Proposition 5.3.** Let $\delta$ be the error in approximating $h^0(W)$ described in Algorithm 4.9. Then for fix degree of the number field, and for $M \geq \max \{\lambda^2, \frac{n}{2} \log \frac{n}{2}\}$, we have $\delta = O(e^{-(\pi-1)M})$.

**Proof.** Now let

$$S = \sum_{x \in \mathbb{Z}^n, Q_1(x) \leq M} e^{-\pi Q(x)} \quad \text{and} \quad S_0 = \sum_{x \in \mathbb{Z}^n} e^{-\pi Q(x)}.$$ 

Then $h^0(W) = \log(\frac{1}{\gamma} S_0)$ and we approximate it by $\log(\frac{1}{\gamma} S)$. The error in approximating $h^0(W)$ is

$$\delta = \left| \log \left( \frac{1}{\gamma} S_0 \right) - \log \left( \frac{1}{\gamma} S \right) \right| = \left| \log \frac{S_0}{S} \right|.$$ 

Furthermore,

$$|S_0 - S| = \left| \sum_{x \in \mathbb{Z}^n, Q_1(x) > M} e^{-\pi Q(x)} \right| \leq \sum_{x \in \mathbb{Z}^n, Q_1(x) > M} |e^{-\pi Q(x)}| = \sum_{x \in \mathbb{Z}^n, Q_1(x) > M} e^{-\pi Q_1(x)} = \sum_{a \in L'| \|a\|^2 > M} e^{-\pi \|a\|^2}.$$ 

Since $M \geq \max \{\lambda^2, \frac{n}{2} \log \frac{n}{2}\} \geq 2^{1-n}$, Lemma 5.1 and 5.2 show that $|S_0 - S| \leq S' = O(e^{-(\pi-1)M})$.

Since $S_0 > S > 1$, it follows that

$$\delta = \left| \log \frac{S_0}{S} \right| \leq |S - S_0| < O(e^{-(\pi-1)M}). \hfill \qed$$

**5.2. Run time of Algorithm 4.9**

We prove the proposition below.

**Prop 5.4.** Let $0 < \delta < 1$. Assume that the given basis of the ideal lattice $I$ and the vector $v$ have size bounded by $|\Delta_F|^{O(1)}$. If the degree $n$ of the number field is fixed, then Algorithm 4.9 with an error $\delta$ runs in time in poly($\log(1/\delta) \cdot \log |\Delta_F|$).

**Proof.** The basis $\{b_1, \ldots, b_n\}$ of the ideal lattice $I$ and vector $v$ have size at most $|\Delta_F|^{O(1)}$. Therefore, Step 1—finding a good divisor $D'$ close to $D$ by using the "jump algorithm"—runs in time polynomial in $\log |\Delta_F|$ \cite[Algorithm 10.8]{10}. In addition, the entries of the matrix $B$ bounded by $|\Delta_F|^{O(1)}$ since they are coordinates of
{b_1, ..., b_n}. Thus, each Step 2a), 2b) and 2c) and hence Step 2 can be done in polynomial time in \log |\Delta_F|.

The list \( \mathcal{L}_1 = \{x = (x_1, \cdots, x_n) \in \mathbb{Z}^n : Q_1(x) \leq M \} \) can be computed by the Fincke–Pohst algorithm (see Algorithm 2.12 in [4]). If the degree \( n \) of the number field is fixed, then the complexity of this algorithm is at most \( \frac{1}{\text{covol}(L')} \cdot \text{poly}(M) \).

See Section 3 in [4] for more details. The covolume of the lattice \( L' \) is \( \text{covol}(L') = \prod_{i=1,k} \|c_i\| \cdot \prod_{j=k+1,n} \|b_j\| \). By a similar argument in the proof of Lemma 5.2, one can show that \( \|c_i\| = C_{i,i} > 1 \) and \( \|b_j\| = \sqrt{A_{j,j}} \geq 2^{-(j-k-1)/2} \) for all \( 1 \leq i \leq k \) and \( k + 1 \leq j \leq n \). Consequently, \( \text{covol}(L') \geq 2^{-n(n+2)/4} \). Therefore, the complexity of Step 3 is bounded by \( \text{poly}(M) \). Proposition 5.3 says that \( M \) is bounded by \( O(\log(1/\delta)) \). As the result, Step 3 can be done in time \( \text{poly}(\log(1/\delta)) \).

Overall, the algorithm runs in time in \( \text{poly}(\log(1/\delta) \cdot \log |\Delta_F|) \) for fixed degree \( n \) of the number field. \( \square \)

6. Some Numerical Examples

We compute the value of \( h^0 \) for real quadratic fields and number fields with unit group of rank 2. In the examples below, we pick an irreducible polynomial \( P \) of large discriminant and compute \( h^0 \) for the number field \( F \) defined by \( P \). The algorithm works well without requiring the units of \( F \). Here \textit{pari - gp} is used to compute approximate values of \( h^0 \) and \textit{Mathematica} is used to plot it.

Since the symmetry induced by Riemann-Roch (see Proposition 1 in [15]), the graphs of \( h^0 \) on the cosets of Pic\(_F\) are similar. See Example 1, 2 and 3 in [15]. In the following examples, we compute \( h^0 \) on the coset Pic\(_F^{(d)}\) of class of divisors of degree \( d = 1/2 \log |\Delta_F| \).

**Example 6.1.** Let \( \Delta_F = 10^{80} + 129 \) and \( P = X^2 - \Delta_F \) be the polynomial defining \( F = \mathbb{Q}(\sqrt{\Delta_F}) \). Then \( F \) is a real quadratic field with the discriminant \( \Delta_F \) and with two real infinite primes \( \sigma_1 : F \to \mathbb{R}^2 \) sends \( \sqrt{\Delta_F} \) to itself and \( \sigma_2 : F \to \mathbb{R}^2 \) sends \( \sqrt{-\Delta_F} \) to \( -\sqrt{\Delta_F} \).

The class number of \( F \) is 1 and the group Pic\(_F\) is isomorphic to a cylinder. For every \( d \in \mathbb{R} \), the coset Pic\(_F^{(d)}\) of classes of degree \( d \) is a circle whose circumference is equal to the regulator \( R_F \) of \( F \).

We have \( (\mathbb{R}^2)^0 = \{(x_1, x_2) : x_1, x_2 \in \mathbb{R} \text{ and } x_1 + x_2 = 0\} = \{-x_1, x_1) : x_1 \in \mathbb{R}\} \) is the bisector of the second quadrant of the axes. It is a 1-dimensional subspace of \( \mathbb{R}^2 \) with an orthonormal basis \( e = (-1/\sqrt{2}, 1/\sqrt{2}) \). The connected component of identity of Pic\(_F^{0}\) is a circle \( T^0 = (\mathbb{R}^2)^0 / \Lambda \) where \( \Lambda \) is the lattice \( \{\log |\sigma_1(f)|, \log |\sigma_2(f)|) : f \in O_F^*\} \) (that is unknown).

Denote by \( w = 10^{20} \cdot e = (\frac{-10^{20}}{\sqrt{2}}, \frac{10^{20}}{\sqrt{2}}) \). Let \( W = (O_F, v) \in \text{Pic}^{(d)}_{F} \) where \( v = |\Delta_F|^{-1/4} \cdot \exp(w) = |\Delta_F|^{-1/4} \cdot (e^{-w_1}, e^{-w_2}) \). The divisor \( D \) translated from \( W \) is \( (O_F, u) \in T^0 \) with \( u = e^{d/n} \cdot v = (e^{-w_1}, e^{-w_2}) \) and \( \|w\| = 10^{20} \) can be seen as the distance from \( (O_F, 1) \) to \( D \).

**Input:**
Apply Poisson summation formula.

\[ v = |\Delta_F|^{-1/4} \cdot (e^{-w_1}, e^{-w_2}) \text{ and } (w_1, w_2) = \left(-\frac{10^{20}}{\sqrt{2}}, \frac{10^{20}}{\sqrt{2}}\right). \]

\[ \delta \approx 10^{-10}. \]

We apply Algorithm 4.9 as follows.

1. **Find a divisor \(D'\) close to \(D\) in \(\text{Pic}^0_F\).**
   
   The finite part of \(W\) is \(I = O_F\). As the notations in Section 4.1, \(D_1\) is the zero divisor \((O_F, 1)\), \(D_2 = D\) and \(v' = u\). We can skip part 4.1.1 and part 4.1.3 in Section 4.1. To find a divisor \(D'\) obtained from some LLL reduction close to the \(D\), it is sufficient to do part 4.1.2, i.e., do Algorithm 4.2, as follows.

   The smallest integer \(t\) such that \(n \cdot 2^{-t} \cdot |w_i| < \log \partial_P\) for \(i = 1, 2\) is \(t = 61\). Let \((z_1, z_2) = 2^{-61} \cdot \omega \approx (-30.66587, 30.66587)\) and \(\omega = (e^{-z_1}, e^{-z_2})\).

   As described in Algorithm 4.2, denote by \(W_i = (O_F, \omega^2)\) and \(W_i = D'_i + (O_F, \omega_i)\) with \(D'_i = d(J_i)\) a good divisor obtained from an LLL-reduction on \(W_i\). By performing doubling and LLL-reduction 61 times, we can reach to \(D\). The result can be seen in Table 1.

   In Table 1, the second column contains the matrices \(N_i\) for which \(M_i = M_0 \cdot N_i\) where \(M_0\) and \(M_i\) are the matrices of which columns form a basis of \(O_F\) and \(J_i\) respectively for all \(i = 1, 2, \ldots, 61\).

   Let \(J = J_{61}\) and \(s = \omega_{61} \approx (e^{-0.80975}, e^{0.80975})\). Then by choosing \(D' = D'_{61} = d(J)\), we obtain that \(D = D' + (O_F, s) = (J, sN(J)^{-1/n}) \in \text{Pic}^0_F\).

2. **Apply Poisson summation formula.**

   Since \(D = (J, sN(J)^{-1/n}) \in \text{Pic}^0_F\) and \(e^{-d/n} \cdot N(J)^{-1/n} = (\text{covol}(J))^{-1/2} \approx 13.46966\), we obtain that \(W = D + (O_F, e^{-d/n}) = (J, s(\text{covol}(J))^{-1/2})\). The lattice associated to \(W\) is \(L = (\text{covol}(J))^{-1/n} sJ\).

   (a) \(L\) has an LLL-reduced basis \(\{b_1, b_2\}\) with \(b_1 \approx (0.12124, 0.61234), b_2 \approx (-1.57394, 0.29870)\).

   (b) \(b_1^* = b_1, b_2^* = (-1.57148, 0.31114), A_{1,1} = \|b_1^*\|^2 \approx 0.38966\) and \(A_{2,2} = \|b_2^*\|^2 \approx 2.56635, A_{2,1} = \langle b_2, b_1^* \rangle / \|b_1^*\|^2 \approx -0.02033\).

   (c) Since \(\|b_1\| < 1\) and \(\|b_2^*\| \geq 1\), we have \(k = 1\). Let \(L_1 = Z \cdot b_1\) and \(L_2 = Z \cdot b_2\) as the notations in Section 2.5. Then \(B = (b_1) = \left(\begin{array}{c} 0.12124 \\ 0.61234 \end{array}\right)\).

   (i) \(G = B^t \cdot B = (0.38966)\) and \(C = B \cdot G^{-1} = \left(\begin{array}{c} 0.31115 \\ 1.57147 \end{array}\right)\).

   (ii) \(c_1 = (0.31115, 1.57147)\) is the column of \(C\) and \(c_1^* = (0.31115, 1.57147), C_{1,1} = \|c_1^*\|^2 = 2.56635\).

   (iii) \(\langle c_1, b_1^* \rangle = 1\) and \(\langle c_1, b_2^* \rangle = 0\).

   (iv) \(Q_1(x_1, x_2) = 2.56635x_1^2 + 2.56635x_2^2, Q_2(x_1, x_2) = -0.02033x_1x_2\) and \(Q(x_1, x_2) = Q_1(x_1, x_2) + 2Q_2(x_1, x_2)i\).

   Let \(L'\) be the lattice associated to \(Q_1(x)\).

(3) **Find short vectors of the lattice \(L'\).**
(a) To approximate \( h^0(W) \) for quadratic fields with an error \( \delta \approx 10^{-5} \), it is sufficient to choose \( M = 8 \).

(b) The Fincke–Pohst method (Algorithm 2.12 in [4]) is used to find the list \( L_2 \) of all column vectors \( x = (x_1, x_2) \in \mathbb{Z}^2 \setminus \{0, 0\} \) such that \( Q_1(x) = 2.56635x_1^2 + 2.56635x_2^2 \leq M \). Note that the function \texttt{qfminim} in \texttt{pari-gp} can be used to find \( L_2 \). Here \( L_2 \) contains only 4 vectors (up to a sign).

\[
L_2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \end{pmatrix}.
\]

By symmetry and since \( Q(0, 1) = Q(1, 0) \), an approximate value of \( h^0(W) \) is obtained as follows.

\[
h^0(W) \approx \log \left( \frac{1}{\sqrt{0.38966}} \left( 1 + 4e^{-\pi Q(0,1)} + 2e^{-\pi Q(1,1)} + 2e^{-\pi Q(1,-1)} \right) \right)
= 0.47250.
\]

\textbf{Output:} \( h^0(W) \approx 0.47250 \) with an error \( \delta = 10^{-5} \).

Recall that the coset \( \text{Pic}_d(F) \) is a circle containing the point \( W \). Let \( X = W \) and let \( Y = W + (O_F, v') \) be the points on \( \text{Pic}_d(F) \) where \( \log(v') = 50 \cdot e \). To see what \( h^0 \) looks like, we compute \( h^0 \) at the points in the interval \([X, Y]\) on \( \text{Pic}_d(F) \) and plot it. The points in this interval have corresponding translated divisors in the interval \([A, B]\) on \( T_0 \) where \( A = (O_F, u) \) and \( B = A + (O_F, v') \).

First, the interval \([A, B]\) is divided into small intervals of length 1. After that, we do LLL-reduction on the middle points of the small intervals to obtain good divisors. Let \( S \) be the set of all good divisors obtained by this way. Then \( S \) has 18 divisors in total (see Figure 1).

![Fig. 1: Good divisors in the interval \([A, B]\) on \( T_0 \).](image-url)

Now, let \( W' \) be an arbitrary divisor in \([X, Y]\). Then its translated divisor \( D \) is a divisor in \([A, B]\). We search for a good divisor \( D' \) in \( S \) which is the closest to \( D \) and use \( D' \) to compute \( h^0(W') \). Then \( h^0 \) is plotted in the interval \([X, Y]\) as in Figures 2 below. In which the red points are divisors on \( \text{Pic}_d(F) \) whose translated divisors are divisors in \( S \).
Example 6.2. Let $P = X^3 - 88998X^2 - 1090173446X - 1000470997815$. Then $P$ is an irreducible polynomial with 3 real roots denoted by $\alpha_1 = \alpha, \alpha_2, \alpha_3$. Let $F = \mathbb{Q}(\alpha)$. Thus, $F$ is a real cubic field with 3 real infinite primes $\sigma_i : F \to \mathbb{R}$ sending $\alpha$ to $\alpha_i$ for $i = 1, 2, 3$. The discriminant of $F$ is $\Delta_F = 10000820940380105429207549453 > 10^{28}$.

We have $(\mathbb{R}^3)^0 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$ a 2-dimensional subspace of $\mathbb{R}^3$ with an orthonormal basis $\{e_1 = (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}), e_2 = (\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}})\}$.

The connected component of identity of $\operatorname{Pic}^0_F$ is the Dirichlet torus $T^0 = (\mathbb{R}^3)^0 / \Lambda$ where $\Lambda$ is the lattice $\{(\log |\sigma_1(f)|, \log |\sigma_2(f)|, \log |\sigma_3(f)|) : f \in \mathcal{O}_F^*\}$.

Denote by $w = 10^{10} \cdot e_1 + 10^{10} \cdot e_2 \approx (11153550716.50411, -8164965809.27726, -298854907.22685) = (w_1, w_2, w_3)$. Let $W \in \operatorname{Pic}^{(d)}_F$ where $v = |\Delta_F|^{-1/6} \cdot \exp(w) = |\Delta_F|^{-1/6} \cdot (e^{-w_1}, e^{-w_2}, e^{-w_3})$. The divisor $D$ translated from $W$ is $(\mathcal{O}_F, u) \in T^0$ with $u = e^{d/n} \cdot v = (e^{-w_1}, e^{-w_2}, e^{-w_3})$ and $\|w\| \approx 14142135623.73095$ can be considered as the distance from $(\mathcal{O}_F, 1)$ to $D$.

Input:

- A basis for the lattice $\mathcal{O}_F$: $a_1 = (1, 1, 1), a_2 \approx (-39667.47830, -30666.32509, 70333.80339), a_3 \approx (451263681.71217, -491488332.71468, 40224652.00251)$.
- $v = |\Delta_F|^{-1/6} \cdot (e^{-w_1}, e^{-w_2}, e^{-w_3})$ where $(w_1, w_2, w_3) \approx (11153550716.50411, -8164965809.27726, -298854907.22685)$.
- $\delta \approx 10^{-5}$.

We compute $h^0(W)$ by using Algorithm 4.9.
(1) **Find a divisor** $D'$ **close to** $D$ **in** $\text{Pic}_F^0$.

As the notations in Section 4.1, $D_1 = (O_F, 1)$ the zero divisor, $D_2 = D$ and $u' = u$. Similar to Example 6.1, we can skip part 4.1.1 and part 4.1.3. To find a divisor $D'$ obtained from some LLL reduction close to the $D_i$, it is sufficient to do part 4.1.2 i.e., do Algorithm 4.2 as follows.

We have that $t = 30$ is the smallest integer for which $n \cdot 2^{-t} \cdot |w_i| < \log \partial_F$ for $i = 1, 2, 3$. Let $(z_1, z_2, z_3) = 2^{-30} \cdot w \approx (10.38755, -7.60422, -2.78333)$ and $\omega = (e^{-z_1}, e^{-z_2}, e^{-z_3})$.

As described in Algorithm 4.2, denote by $W_i = (O_F, \omega^{2_k})$ and $W_i = D_i' + (O_F, \omega_i)$ with $D_i' = d(J_i)$, a good divisor obtained from an LLL-reduction on $D_i$.

In Table 2, the second column contains the matrices of which columns form an LLL-reduced basis for all $i = 1, 2, 3, 30$. The coordinates of these vectors are computed with respect to the basis of $O_F$.

Let $J = J_{30}$ and $s = \omega_{30} \approx (e^{1.09384}, e^{-0.61605}, e^{-0.47780})$. Then by choosing $D' = D_{30}' = d(J)$, we obtain that $D = D' + (O_F, s) = (J, sN(J)^{-1/2})$ in $\text{Pic}_F^0$.

(2) **Apply Poisson summation formula.**

Since $D = (J, sN(J)^{-1/2})$ in $\text{Pic}_F^0$ and $e^{-d/n} \cdot N(J)^{-1/2} = (\text{covol}(J))^{-1/2}$, it follows that $W = D + (O_F, e^{-d/n}) = (J, s(\text{covol}(J))^{-1/2})$. Here $N(J)^{-1} = 938139713086$ and $\text{covol}(J) \approx 106.59831$. The lattice associated to $W$ is $L = (\text{covol}(J))^{-1/2} s J$.

(a) $L$ has an LLL-reduced basis $\{b_1, b_2, b_3\}$ with $b_1 \approx (0.07064, 0.39051, 0.34009)$, $b_2 \approx (0.83795, 0.53263, -0.87506)$, $b_3 \approx (1.35900, -0.65069, 0.22025)$.

(b) $b_1^* = b_1$, $b_2^* = (0.84581, 0.57611, -0.83720)$ and $b_3^* = (1.09497, -0.72624, 0.60648)$.

$A_{1,1} = \|b_1^*\|^2 \approx 0.27314$, $A_{2,2} = \|b_2^*\|^2 \approx 1.74820$, $A_{3,3} = \|b_3^*\|^2 \approx 0.20940$

and $A_{1,2} = (b_2, b_3^*)/\|b_2^*\| \approx -0.11133$, $A_{3,1} = (b_3, b_1^*)/\|b_1^*\| \approx -0.30460$

and $A_{3,2} = (b_3, b_2^*)/\|b_2^*\| \approx 0.33761$.

Let $L_1 = \mathbb{Z} \cdot b_1$ and $L_2 = \mathbb{Z} \cdot b_2 \oplus \mathbb{Z} \cdot b_3$.

(c) Since $\|b_1\| < 1$ and $\|b_2\| \geq 1$, we have $k = 1$ and $B = \{b_1\} = \begin{pmatrix} 0.07064 \\ 0.39051 \\ 0.34009 \end{pmatrix}$.

(i) $G = B^* \cdot B = \begin{pmatrix} 0.27314 \end{pmatrix}$ and $C = B \cdot G^{-1} = \begin{pmatrix} 0.258609 \\ 1.42968 \\ 1.24508 \end{pmatrix}$.

(ii) $c_1 = (0.258609, 1.42968, 1.24508)$ is the column of $C$ and $c_1^* = (0.258609, 1.42968, 1.24508)$, $C_{1,1} = \|c_1^*\|^2 = 3.66108$.

(iii) $\langle c_1, b_1^* \rangle = 1$ and $\langle c_1, b_2^* \rangle = \langle c_1, b_3^* \rangle = 0$.

(iv) Denote by $Q(x_1, x_2, x_3) = 3.66108x_1^2 + 1.74820(x_2 + 0.33761x_3)^2 + 2.09402x_2^2$, $Q_2(x_1, x_2, x_3) = 0.11133x_1x_2 + 0.30460x_1x_3$ and $Q(x_1, x_2, x_3) = Q_1(x_1, x_2, x_3) + 2Q_2(x_1, x_2, x_3)$.
Let $L'$ be the lattice associated to $Q_1(x)$.

3) **Find short vectors of the lattice $L'$**.

(a) By Proposition 5.3, to approximate $h^0(W)$ for cubic fields with an error $\delta \approx 10^{-5}$, it is sufficient to choose $M = 9$.

(b) The Fincke–Pohst method (Algorithm 2.12 in [4]) is used to find the list $L_2$ of all column vectors $x = (x_1, x_2, x_3) \in \mathbb{Z}^3 \backslash \{0, 0, 0\}$ such that $Q_1(x) \leq M$. Here $L_2$ has 17 vectors (up to a sign) obtained by using the function `qfminim` in pari-gp.

$$L_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 0 & 1 & -1 & -2 & -1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \\
\end{pmatrix}.$$

By symmetry, an approximate value of $h^0(W)$ is obtained as follows.

$$h^0(W) \approx \log \left( \frac{1}{\sqrt{0.27314}} \left( 1 + 2 \sum_{x \in L_2} e^{-\pi Q(x)} \right) \right) = 0.65882.$$

**Output**: $h^0(W) \approx 0.65882$ with error $\delta = 10^{-5}$.

Putting $v_1 = 1$, $v_2 = \exp(6 \cdot e_1)$, $v_3 = \exp(6 \cdot e_1 + 8 \cdot e_2)$ and $v_4 = \exp(8 \cdot e_2)$.

Let $X_i = W + (OF, v_i)$ be the points on $\text{Pic}_{d}^{(d)}$ for $i = 1, 2, 3, 4$. Similar to Example 6.1 to see what $h^0$ looks like, we compute $h^0$ at the points in the box $X_1X_2X_3X_4$ on $\text{Pic}_{d}^{(d)}$ and plot it. The points in this box have corresponding translated divisors in the rectangle $ABCE$ on $T^0$. Here $A = (OF, u), B = A + (OF, v_2), C = A + (OF, v_3)$ and $E = A + (OF, v_4)$.

![Image](image_url)

**Fig. 3**: The set $S$ of good divisors in the rectangle $ABCE$.

The rectangle $ABCE$ is divided into small squares, each one has sides of length 1. After that, we perform LLL-reduction at the center of such squares to obtain good divisors. Let $S$ be the set of all good divisors obtained by this way. Then $S$ has 15 points in total (see Figure 3).
Now, let $W'$ be an arbitrary divisor in $X_1X_2X_3X_4$. Then its translated divisor $D$ is a divisor in $ABCE$. We search for a good divisor $D'$ in $S$ which is the closest to $D$ and use $D'$ to compute $h^0(W')$. Then $h^0$ is plotted as in Figure 4 and 5 below in which the red points are divisors on $\text{Pic}_F^{(d)}$ whose translated divisors are good divisors in $S$. In Figure 4 the dark color area is correspondent to the large values of $h^0$.

![Fig. 4: The contour plot of $h^0$ in the box $X_1X_2X_3X_4$.](image)

![Fig. 5: $h^0$ for the real cubic field $F$.](image)
| i  | \( J_i \) | \( \frac{1}{N(J_i^{-1})} \) | \( N(J_i^{-1}) \) | \( \log \omega_i \) |
|----|------------|-------------------|-----------------|------------------|
| 0  | \( O_F \)  | 1                 | \((-30.66587, 30.66587)\) |
| 1  | \( \frac{1}{N(J_1^{-1})} \) \( \begin{pmatrix} 129.5 \cdot 10^{39} + 64 \\ 0 \\ -1 \end{pmatrix} \) | 129              | \((29.03491, -29.03491)\) |
| 2  | \( \frac{1}{N(J_2^{-1})} \) \( \begin{pmatrix} 129.5 \cdot 10^{39} + 25 \\ 0 \\ 1 \end{pmatrix} \) | 129              | \((-32.29683, 32.29683)\) |
| 3  | \( \frac{1}{N(J_3^{-1})} \) \( \begin{pmatrix} 2146689.5 \cdot 10^{39} + 261160 \\ 0 \\ -1 \end{pmatrix} \) | 2146689          | \((25.77299, -25.77299)\) |
| ... | ...       | ...              | ...             | ...              |
| 60 | \( \begin{pmatrix} 1 - \frac{3535142278155418356810286962804039637657}{N(J_{60}^{-1})} \\ 0 \\ -\frac{1}{N(J_{60}^{-1})} \end{pmatrix} \) | 19902657321594605283368410 59638321226594 | \((0.64301, -0.64301)\) |
| 61 | \( \begin{pmatrix} 1 - \frac{46378519698505151024763669502575362226}{N(J_{61}^{-1})} \\ 0 \\ -\frac{1}{N(J_{61}^{-1})} \end{pmatrix} \) | 742409069897505634669660076059992618001 | \((0.80975, -0.80975)\) |
| $i$ | $J_i$ | $N(J_i^{-1})$ | $\log \omega_i$ |
|-----|-------|---------------|----------------|
| 0   | $O_F$ | 1             | $(10.38755, -7.60422, -2.78334)$ |
| 1   | $\frac{1}{N(J_1^{-1})}$ | $\left( \begin{array}{ccc} 691582920399 & 118502077239 & -227460374946 \\ 0 & 40389624 & 42667155 \\ 0 & 3277 & -13661 \end{array} \right)$ | $(1.05601, -1.5469, 0.49083)$ |
| 2   | $\frac{1}{N(J_2^{-1})}$ | $\left( \begin{array}{ccc} 222208932162 & 82987136358 & 68645452464 \\ 0 & 19963734 & 42669156 \\ 0 & 4774 & -927 \end{array} \right)$ | $(-0.42917, -0.37727, 0.80643)$ |
| 3   | $\frac{1}{N(J_3^{-1})}$ | $\left( \begin{array}{ccc} 4204248079595 & 1034415034603 & 307388467818 \\ 0 & 60087131 & 218999921 \\ 0 & 14593 & -16782 \end{array} \right)$ | $(0.03212, 0.32161, -0.35373)$ |
| 29  | $\frac{1}{N(J_{29}^{-1})}$ | $\left( \begin{array}{ccc} 3063324517380 & 662309451492 & -144083620104 \\ 0 & 132428244 & 106307652 \\ 0 & 10287 & -14874 \end{array} \right)$ | $(-0.99628, -0.39485, 1.3911)$ |
| 30  | $\frac{1}{N(J_{30}^{-1})}$ | $\left( \begin{array}{ccc} 938139713086 & -448140560830 & 27116000512 \\ 0 & 3374186 & -90561028 \\ 0 & 10388 & -773 \end{array} \right)$ | $(1.09384, -0.61605, -0.47780)$ |
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