REPEATED DIFFERENTIATION AND FREE UNITARY POISSON PROCESS

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Abstract. We investigate the hydrodynamic behavior of zeroes of trigonometric polynomials under repeated differentiation. We show that if the zeroes of a real-rooted, degree $d$ trigonometric polynomial are distributed according to some probability measure $\nu$ in the large $d$ limit, then the zeroes of its $[2td]$-th derivative, where $t > 0$ is fixed, are distributed according to the free multiplicative convolution of $\nu$ and the free unitary Poisson distribution with parameter $t$. In the simplest special case, our result states that the zeroes of the $[2td]$-th derivative of the trigonometric polynomial $(\sin \frac{\theta}{2})^d$ (which can be thought of as the trigonometric analogue of the Laguerre polynomials) are distributed according to the free unitary Poisson distribution with parameter $t$, in the large $d$ limit. The latter distribution is defined in terms of the function $\zeta = \zeta_t(\theta)$ which solves the implicit equation $\zeta - t \tan \zeta = \theta$ and satisfies $\zeta_t(\theta) = \theta + t \tan(\theta + t \tan(\theta + \ldots))$, $\Im \theta > 0$, $t > 0$.

1. Main results

1.1. Introduction. Let $Q \in \mathbb{C}[z]$ be a polynomial with complex coefficients and degree $\deg Q \geq 1$. The empirical distribution of roots of $Q$ is the probability measure on the complex plane $\mathbb{C}$ given by

$$\mu[Q] := \frac{1}{\deg Q} \sum_{z \in \mathbb{C} : Q(z) = 0} m_Q(z) \delta_z,$$

where $m_Q(z)$ denotes the multiplicity of the root at $z$, and $\delta_z$ is the unit delta-measure at $z$. Let $(P_n(z))_{n \in \mathbb{N}}$ be a sequence of polynomials such that $\deg P_n = n$ and suppose that $\mu[P_n]$ converges weakly to some probability measure $\mu$ on $\mathbb{C}$, as $n \to \infty$. We are interested in the asymptotic distribution of complex roots of the repeated derivatives $P'_n, P''_n, \ldots$, in the large $n$ limit. For concreteness, let us consider random polynomials of the form

$$P_n(z) = (z - \xi_1) \ldots (z - \xi_n), \quad n \in \mathbb{N},$$

where $\xi_1, \xi_2, \ldots$ are independent complex-valued random variables with distribution $\mu$. By the law of large numbers, for almost every realization of the sequence $\xi_1, \xi_2, \ldots$ we have $\mu[P_n] \to \mu$ weakly on $\mathbb{C}$.

Regarding the first derivative, a result of [40], proving a conjecture of Pemantle and Rivin [60], states that $\mu[P'_n] \to \mu$ in probability on $\mathcal{M}_\mathbb{C}$, the space of probability measures on $\mathbb{C}$ endowed with the Lévy-Prokhorov metric (which generates the topology of weak convergence). For more results in this direction, we refer to [75, 76, 56, 59, 58, 67, 66, 14, 42]. In fact, Byun et al. [14] showed that the same conclusion continues to hold if $P'_n$ is replaced by the $k$-th derivative $P_n^{(k)}$, where $k \in \mathbb{N}$.

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is fixed. Recently, Angst et al. [4] (for $k = 1$) and Michelen and Vu [53] (for $k \in \mathbb{N}$) showed that convergence in probability can be replaced by a.s. convergence.

Knowing that under a single differentiation the empirical distribution of zeroes changes only infinitesimally, it is natural to ask about the distribution of zeroes of the $k$-th derivative $P_n^{(k)}$, where $k = k(n) \to \infty$ as $n \to \infty$. In fact, it is known that under one differentiation, the zeroes move by a quantity of order $1/n$. There exist several results [65, 83, 31, 42, 59, 58] stating the existence of a local “pairing” between the roots of $P_n$ and the roots of $P_n'$. More precisely, for every zero of $P_n$, say $\xi_1$, with high probability there exists a zero of $P_n'$ at $\xi_1 - 1/(nG(\xi_1)) + o(1/n)$, where

$$G(x) = \int_{\mathbb{C}} \frac{\mu(\mathrm{d}y)}{x - y}$$

is the Cauchy-Stieltjes transform of $\mu$. To derive this formula heuristically, let us write the equation for the zeroes of $P_n'(z)/P_n(z)$ in the form

$$\frac{1}{z - \xi_1} = -\sum_{j=2}^{n} \frac{1}{z - \xi_j} \approx -nG(z).$$

If we are looking for a solution $z$ close to $\xi_1$, then the right-hand side can be approximated by $-nG(\xi_1)$, and we obtain $z \approx \xi_1 - 1/(nG(\xi_1))$. Following [72, 57, 12] it is therefore natural to conjecture the emergence of a non-trivial hydrodynamic behavior of the roots in the regime when the number of differentiations satisfies $k(n)/n \to t$ as $n \to \infty$, where the parameter $t \in [0, 1)$ can be interpreted as the “time” of the system.

**Conjecture 1.1** (cf. [72, 57, 12]). Fix some $0 \leq t < 1$ and let $k = k(n)$ be such that $k/n \to t$ as $n \to \infty$. Suppose, for simplicity, that $\mu$ is compactly supported. Then, the random probability measure $\mu[P_n^{(k)}]$ converges\(^\dagger\) to a certain deterministic probability measure $\mu_t$, as $n \to \infty$.

In particular, in the regime when $k = o(n)$, it is conjectured that $\mu[P_n^{(k)}]$ converges to $\mu_0 = \mu$, the initial distribution of roots (see the paper of Michelen and Vu [53] for a result in this direction), while for $t \neq 0$ the measures $\mu_t$ and $\mu$ should be in general different. It follows from the Gauss-Lucas theorem that the convex hull of the support of $\mu_t$ is nonincreasing in $t$. Moreover, by [80, Theorem 3] (which is based on the work of Ravichandran [65]), the diameter of the support of $\mu_t$ converges to $0$ as $t \uparrow 1$ with an explicit bound on the rate of decrease. By a theorem of Malamud-Pereira, see [80, § 3], the function $t \mapsto \int_{\mathbb{C}} \varphi(x)\mu_t(\mathrm{d}x)$ is non-increasing for every convex function $\varphi : \mathbb{C} \to \mathbb{R}$. Although Conjecture 1.1 remains unproven in full generality, several approaches have been suggested to explicitly describe the measures $\mu_t$ under additional assumptions on the initial distribution $\mu$. Let us briefly review the existing results.

**Partial differential equations.** If at time $t = 0$ the zeroes have a probability density $p_0(x) = \mathrm{d}\mu(\mathrm{d}x)$ w.r.t. the two-dimensional Lebesgue measure, then one may conjecture that the density $p_t(x) = (1 - t)\mu_t(\mathrm{d}x)$ of $(1 - t)\mu_t$ satisfies the following non-local PDE:

$$\frac{\partial p_t(x)}{\partial t} = -\nabla \left( \tilde{v}_t(x)p_t(x) \right), \quad \tilde{v}_t(x) = -\left( \operatorname{Re} \frac{1}{G_t(x)}, \operatorname{Im} \frac{1}{G_t(x)} \right), \quad G_t(x) = \int_{\mathbb{C}} p_t(y)/x - y, \quad (1.1)$$

where $0 \leq t < 1$. Note that this is the convection equation in which $\tilde{v}_t(x)$ is the velocity of the flow at position $x$. There seem to be no results on this general PDE, but in the case when the initial distribution $\mu$ is rotationally invariant, the PDE simplifies considerably. The PDE corresponding to this isotropic special case has been first non-rigorously derived by O’Rourke and Steinerberger [57].

\(^\dagger\)In the sense of stochastic (or even a.s.) convergence of random $\mathcal{M}_\mathbb{C}$-valued elements.
and can be solved explicitly, as has been shown in \cite{37}; see also \cite{26} for a particular case when \( \mu \) is the uniform distribution on the unit circle.

Assume now that all zeroes of \( P_n \) are real and have some probability density \( u_0(x) \) w.r.t. the one-dimensional Lebesgue measure. By Rolle’s theorem, all zeroes of \( P_n, P_{n}', \ldots \) are real, too. Let \( u_t(x) \) be the density of zeroes at time \( t \in [0, 1) \), with the normalization \( \int_0^1 u_t(x) \, dx = 1 - t \). The following non-local PDE for \( u_t(x) \) has been non-rigorously derived by Steinerberger \cite{72}:

\[
\frac{\partial u_t(x)}{\partial t} + \frac{1}{\pi} \frac{\partial}{\partial x} \arctan \left( \frac{H u_t(x)}{u_t(x)} \right) = 0,
\]

where \( (Hu_t)(x) = \frac{1}{\pi} \text{P.V.} \int_{\mathbb{R}} \frac{u_t(y)}{x-y} \, dy \) is the Hilbert transform of \( u_t \) and the integral is understood in the sense of principal value. Local and global regularity results for this PDE with periodic initial condition have been obtained by Granero-Belinchón \cite{32}, Kiselev and Tan \cite{44} and Alazard et al. \cite{1}. As Kiselev and Tan \cite{44} rigorously showed, the periodic version of Steinerberger’s PDE \cite{12} describes the hydrodynamic limit of the roots of trigonometric polynomials, which is the setting we shall concentrate on in the present paper. A key ingredient used in \cite{72} and \cite{44} to derive the PDE is the fact that under repeated differentiation, the real roots crystallize \cite{61}, i.e. they approach locally an arithmetic progression.

**Free probability.** In the case when all roots are real, there is an intriguing connection between repeated differentiation and free probability \cite{74,73,38,70,37,5}. Steinerberger \cite{74} non-rigorously argued that \( \mu_t \), up to rescaling, coincides with the free additive self-convolution power \( \mu_t^{1/(1-t)} \). In fact, he showed that both objects satisfy the same PDE. Similar calculations appeared earlier in the paper of Shlyakhtenko and Tao \cite{70}. Steinerberger’s claim has been rigorously proven in \cite{37} without using PDE’s. A more natural proof, given by Arizmendi, Garza-Vargas and Perales \cite{5} Theorem 3.7], uses finite free probability, a subject developed recently in the papers of Marcus \cite{49}, Marcus, Spielman, Srivastava \cite{50} and Gorin and Marcus \cite{31}. In the approach of \cite{5}, the \( k \)-th derivative of \( P_n \) is essentially identified with the finite free multiplicative convolution of \( P_n \) and the polynomial \( x^k(1-x)^{d-k} \). Since the empirical distribution of the latter converges to \( t\delta_0 + (1-t)\delta_1 \) and since finite free probability converges to the usual free probability in the sense of \cite{5} Theorem 1.4], \( \mu_t \) can essentially be identified with the free multiplicative convolution of \( \mu \) and the probability measure \( t\delta_0 + (1-t)\delta_1 \). By accident, the latter convolution can be expressed through \( \mu_t^{1/(1-t)} \).

**Saddle point method.** Motivated by the Rodrigues formula for Legendre polynomials, Bøgvad, Hågg and Shapiro \cite{12} considered polynomials of the form

\[
R_{n,[nt],Q}(z) := \left( \frac{d}{dz} \right)^{[nt]} (Q(z))^n,
\]

where \( Q(z) \) is a fixed polynomial and \( 0 < t < \deg Q \). The main results of \cite{12} describe the asymptotic distribution of the zeroes of these so-called Rodrigues’ descendants, as \( n \to \infty \), and essentially prove Conjecture \cite{1} in the case when \( \mu \) is a finite convex combination of delta-measures. The approach of \cite{12} is to represent \( R_{n,[nt],Q}(z) \) as a contour integral using the Cauchy formula, and then to apply the saddle-point method to evaluate the large \( n \) limit of \( \frac{1}{n} \log |R_{n,[nt],Q}(z)| \), thus identifying the logarithmic potential of the limit distribution of zeroes of \( R_{n,[nt],Q}(z) \).

### 1.2. Main result

The aim of the present work is to study the hydrodynamic limit of roots of trigonometric polynomials under repeated differentiation and to relate it to the free multiplicative Poisson process on the unit circle. The same problem has been studied from the point of view of PDE’s in \cite{32,44,1}. Our approach is different and, in some sense, provides an explicit solution to
Denoting these zeroes by \( \theta \) is (The “fundamental solution”) when the trigonometric polynomial \( T \) this and other well-known facts about trigonometric polynomials. We are interested in the setting otherwise stated) in the interval \([\pi, \pi]\) converges weakly on \( T \) with real coefficients \( c \) with 1 flow instead of repeated differentiation. The dynamics of zeroes is very similar to that in (1.1), but this special case by the saddle-point method (which will be done in Section 3).

In this special case, Theorem 1.2 states that the empirical distribution of zeroes of the \( k \)-derivative of \( T \), where \( (k(d))_{d \in \mathbb{N}} \) is a sequence of natural numbers such that
\[
t := \lim_{d \to \infty} \frac{k(d)}{2d} \in [0, \infty).
\]

For example, we may take \( k(d) = [2td] \). Note that the property of real-rootedness is preserved under repeated differentiation by Rolle’s theorem. Our main result reads as follows.

**Theorem 1.2.** Let \( (T_{2d}(\theta))_{d \in \mathbb{N}} \) be a sequence of real-rooted trigonometric polynomials such that \( \nu[T_{2d}] \) converges weakly to some probability measure \( \nu \) on the unit circle \( \mathbb{T} \), as \( d \to \infty \). If \( k(d) \) satisfies (1.4), then the empirical distribution of zeroes of the \( k(d) \)-th derivative of \( T_{2d}(\theta) \) in \( \theta \) converges weakly on \( \mathbb{T} \) to the free multiplicative convolution of \( \nu \) and the free unitary Poisson distribution \( \Pi_t \) with parameter \( t \) (to be defined in Section 1.3 below). That is,
\[
\nu[T_{2d}^{(k(d))}] \xrightarrow{w} \nu \boxtimes \Pi_t.
\]

**Example 1.3** (The “fundamental solution”). The “simplest” real-rooted trigonometric polynomial is \( h_{2d}(\theta) := (\sin \frac{\theta}{2})^{2d} \). It has a zero of multiplicity \( 2d \) at \( \theta = 0 \). To see that \( h_{2d}(\theta) \) can be written in the form (1.3), use the formula \((\sin \frac{\theta}{2})^2 = (1 - \cos \theta)/2 = (2 - e^{i\theta} - e^{-i\theta})/4\), and expand its \( d \)-th power. In this special case, Theorem 1.2 states that the empirical distribution of zeroes of the \([2td]\)-th derivative of \( h_{2d}(\theta) \) converges weakly to \( \Pi_t \). Our strategy to prove Theorem 1.2 is to reduce it to this special case using finite free probability (which will be done in Section 2) and then to prove the special case by the saddle-point method (which will be done in Section 3).

In his blog, Tao [78, 79] discusses the evolution of zeroes of a polynomial which undergoes a heat flow instead of repeated differentiation. The dynamics of zeroes is very similar to that in (1.1), but with \( 1/G_t(x) \) replaced by \( G_t(x) \) in the velocity term. Stochastic differential equations of the same type arise in connection with random matrices (Dyson’s Brownian motion) and Bessel processes in Weyl chambers; see [2, Section 4.3]. Their hydrodynamic limits are related to free additive \( \text{Lévy} \)
processes generated by the Wigner or Marchenko-Pastur distributions, see [2] Section 4.3] and [80], while the limits when the variance of the driving white noise goes to zero are related to zeroes of Hermite and Laguerre polynomials [3 22 30 85].

1.3. Free probability on the unit circle. Let us recall the notions from free probability appearing in Theorem 1.2. Free multiplicative convolution of probability measures on the unit circle, introduced by Voiculescu in [82] and [83], can be defined as follows; see [84] § 3.6] and [83]. Let \( \mathcal{M}_T \) be the set of probability measures on \( \mathbb{T} \). Given \( \mu_1 \in \mathcal{M}_T \) and \( \mu_2 \in \mathcal{M}_T \) it is possible to construct a \( C^* \)-probability space and two mutually free unitaries \( u_1 \) (with spectral distribution \( \mu_1 \)) and \( u_2 \) (with spectral distribution \( \mu_2 \)). Then, \( \mu_1 \boxplus \mu_2 \in \mathcal{M}_T \) is the spectral distribution of \( u_1 u_2 \).

For our purposes, the following characterization of \( \mathbb{R} \) suffices. The \( \psi \)-transform of a probability measure \( \mu \in \mathcal{M}_T \) is the function \( \psi_\mu(z) \), defined on the open unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) by

\[
\psi_\mu(z) = \int_{\mathbb{T}} \frac{uz}{1-uz} \mu(du) = \sum_{n=0}^{\infty} z^n \int_{\mathbb{T}} u^n \mu(du), \quad z \in \mathbb{D}.
\]

Let \( \mathcal{M}_T^\bullet \) be the set of all \( \mu \in \mathcal{M}_T \) with \( \psi'_\mu(0) = \int_{\mathbb{T}} u \mu(du) \neq 0 \). If \( \mu \in \mathcal{M}_T^\bullet \), then the function \( \psi_\mu \) has an inverse on some sufficiently small disk around the origin. The \( S \)-transform and the \( \Sigma \)-transform of \( \mu \) are defined by

\[
S_\mu(z) = \frac{1+z}{z} \psi_\mu^{-1}(z), \quad \Sigma_\mu(z) = S_\mu \left( \frac{z}{1-z} \right) = \frac{1}{z} \psi_\mu^{-1} \left( \frac{z}{1-z} \right),
\]

respectively, if \( |z| \) is sufficiently small. Then, it is known [83] that

\[
S_{\mu \boxplus \nu}(z) = S_\mu(z) S_\nu(z), \quad \Sigma_{\mu \boxplus \nu}(z) = \Sigma_\mu(z) \Sigma_\nu(z), \quad \mu, \nu \in \mathcal{M}_T^\bullet.
\]

Let us now recall the analogue of the Poisson limit theorem for \( \mathbb{R} \). Fix some \( t \geq 0 \) and consider the probability measures \( \mu_n := (1 - \frac{t}{n}) \delta_1 + \frac{t}{n} \delta_{t-n} \), \( n \in \mathbb{N} \), on \( \mathbb{T} \), where \( \delta_z \) is the Dirac unit mass at \( z \). The Poisson limit theorem [8] Lemma 6.4] for \( \mathbb{R} \) states that, as \( n \to \infty \), the sequence of probability measures \( \mu_n \boxplus \ldots \boxplus \mu_n \) (\( n \) times) converges weakly to the free unitary Poisson distribution \( \Pi_t \) which is characterized by the following \( S \)- and \( \Sigma \)-transforms:

\[
S_{\Pi_t}(z) = \exp \left\{ \frac{t}{z + \frac{t}{z}} \right\}, \quad \Sigma_{\Pi_t}(z) = \exp \left\{ 2t \frac{1 - z}{1 + z} \right\}.
\]

In Section 3, we will derive some properties of \( \Pi_t \) including a characterization of its support and a series expansion for its absolutely continuous part. In fact, defining \( \mu_\zeta := (1 - \frac{t}{n}) \delta_1 + \frac{t}{n} \delta_{t-n} \) with arbitrary \( \zeta \in \mathbb{T} \) leads to a more general two-parameter family of free Poisson distributions [8] Lemma 6.4], but we shall need only the case \( \zeta = -1 \) here. For further information on \( \mathbb{R} \) including a characterization of \( \mathbb{R} \)-infinitely divisible distributions and more general limit theorems we refer to [7 9 10 16 17 88 89 87 15].

2. Proof of Theorem 1.2 Reduction to the fundamental solution

2.1. Trigonometric and algebraic polynomials. For \( d \in \mathbb{N} \) let \( \text{Trig}_{2d}[\theta] \) be the set of all trigonometric polynomials in the variable \( \theta \) of the form

\[
T_{2d}(\theta) = \sum_{\ell=-d}^{d} c_\ell e^{i\ell \theta},
\]

where \( c_{-d}, \ldots, c_d \in \mathbb{C} \) are complex coefficients. Clearly, \( \text{Trig}_{2d}[\theta] \) is a \( (2d + 1) \)-dimensional vector space over \( \mathbb{C} \) with basis \( 1, e^{i\theta}, \ldots, e^{i\ell \theta} \). Furthermore, for \( n \in \mathbb{N} \) let \( \mathbb{C}_n[z] \) be the set of all algebraic polynomials, with complex coefficients, of degree at most \( n \) in the variable \( z \). Clearly, \( \mathbb{C}_n[z] \) is
an \((n + 1)\)-dimensional vector space over \(\mathbb{C}\) with basis 1, \(z, \ldots, z^n\). We define an isomorphism 
\(\Psi : \operatorname{Trig}_{2d}[\theta] \to \mathbb{C}_{2d}[z]\) between these \((2d + 1)\)-dimensional vector spaces by 
\[
\Psi(e^{i\theta}) := z^{\ell + d}, \quad \ell \in \{-d, \ldots, d\},
\]
and then extending this map by linearity. Note that for every trigonometric polynomial \(T_{2d} \in \operatorname{Trig}_{2d}[\theta]\) the corresponding algebraic polynomial \(\Psi(T_{2d})\) satisfies 
\[
T_{2d}(\theta) = \frac{\Psi(T_{2d})(e^{i\theta})}{e^{i\theta}}. \tag{2.2}
\]
For arbitrary \(n \in \mathbb{N}\), consider the differential operator \(\mathcal{D}_n : \mathbb{C}_n[z] \to \mathbb{C}_n[z]\) defined by 
\[
\mathcal{D}_nP(z) = i \left( \frac{dP}{dz} - \frac{n}{2} P(z) \right), \quad P \in \mathbb{C}_n[z]. \tag{2.3}
\]

The action of \(\mathcal{D}_{2d}\) on the space \(\mathbb{C}_{2d}[z]\) of algebraic polynomials corresponds to the action of \(\frac{d}{d\theta}\) on the space \(\operatorname{Trig}_{2d}[\theta]\) of trigonometric polynomials in the sense that the following diagram is commutative:
\[
\begin{array}{ccc}
\operatorname{Trig}_{2d}[\theta] & \xrightarrow{\Psi} & \mathbb{C}_{2d}[z] \\
\downarrow{\frac{d}{d\theta}} & & \downarrow{\mathcal{D}_{2d}} \\
\operatorname{Trig}_{2d}[\theta] & \xrightarrow{\Psi} & \mathbb{C}_{2d}[z]
\end{array} \tag{2.4}
\]
Indeed, it follows from (2.3) that for all \(\ell \in \{-d, \ldots, d\}\) we have 
\[
\mathcal{D}_{2d}(\Psi(e^{i\theta})) = \mathcal{D}_{2d}(z^{\ell + d}) = i \left( (d + \ell) z^{\ell + d} - d z^{\ell + d} \right) = i\ell z^{\ell + d} = \Psi \left( \frac{d}{d\theta} e^{i\theta} \right). \tag{2.5}
\]

Knowing this, the identity \(\mathcal{D}_{2d}(\Psi(T_{2d})) = \Psi(\frac{d}{d\theta} T_{2d})\) for every \(T_{2d} \in \operatorname{Trig}_{2d}[\theta]\) follows by linearity. Thus, instead of studying the effect of repeated differentiation on trigonometric polynomials we may study the repeated action of the operator \(\mathcal{D}_{2d}\) on algebraic polynomials of degree \(2d\).

It follows from (2.2) that real-rooted trigonometric polynomials correspond to algebraic polynomials with roots on the unit circle. We are now going to state a result on such algebraic polynomials which is equivalent to Theorem 1.2. Consider an algebraic polynomial \(P_n(z)\) of degree \(n \in \mathbb{N}\) all of whose zeroes are located on the unit circle. We can represent it in the form 
\[
P_n(z) = a_n \cdot \prod_{j=1}^{n} (z - e^{i\theta_n}; j) \quad \text{for some } \theta_1; n, \ldots, \theta_n; n \in [-\pi, \pi) \text{ and } a_n \in \mathbb{C}\setminus\{0\}. \tag{2.6}
\]
Given arbitrary \(\theta_1; n, \ldots, \theta_n; n \in [-\pi, \pi)\) it is convenient to take \(a_n := (2i)^{-n} \prod_{j=1}^{n} e^{-i\theta_n; j}/2\) in (2.6).

With this choice, the following functional equation is satisfied:
\[
\bar{z}^n P_n(1/z) = P_n(z), \quad z \in \mathbb{C}\setminus\{0\}. \tag{2.7}
\]

By comparison of coefficients one shows that for a polynomial represented in the form \(P_n(z) = \sum_{j=0}^{n} a_j z^j\) the functional equation (2.7) is satisfied if and only if 
\[
a_j = a_{n-j} \quad \text{for all } j \in \{0, \ldots, n\}. \tag{2.8}
\]

If \(n = 2d\) is even, then \(P_n(z)\) corresponds to a trigonometric polynomial \(T_{2d}(\theta)\) which can be written in the form 
\[
T_n(\theta) := \frac{P_n(e^{i\theta})}{e^{i\theta}/2} = \prod_{j=1}^{n} \frac{e^{i\theta} - e^{i\theta_n; j}}{2i} = \prod_{j=1}^{n} \sin\left(\frac{\theta - \theta_n; j}{2}\right). \tag{2.9}
\]
If $n$ is odd, the above equation remains valid but does not define a trigonometric polynomial. Factorization of the form \((2.9)\) is well known; see, e.g., [23, Corollary 1.3]. The next lemma is an analogue of Rolle’s theorem for polynomials with roots on the unit circle.

**Lemma 2.1.** If all roots of a degree $n$ algebraic polynomial $P_n \in \mathbb{C}_n[z]$ are located on the unit circle, then all roots of $D_nP_n$ also lie on the unit circle.

**Proof.** Multiplying $P_n$ by a suitable $c \in \mathbb{C}\setminus\{0\}$ we may assume that \((2.9)\) holds implying that the function $T_n(\theta) = P_n(e^{i\theta})/e^{i\theta/2}$ takes real values for $\theta \in \mathbb{R}$. One checks that

$$\frac{d}{d\theta} T_n(\theta) = \frac{(D_nP_n)(e^{i\theta})}{e^{i\theta/2}}.$$  \hspace{1cm} (2.10)

Let $n = 2d$ be even. The real-valued function $T_n(\theta)$ is $2\pi$-periodic and has $n$ zeroes in $[-\pi, \pi)$, counting multiplicities. Rolle’s theorem implies that $D_nP_n$ has $n$ zeroes on $[-\pi, \pi)$ with multiplicities. It follows from \((2.10)\) that $D_nP_n$ has $n$ zeroes on the unit circle. Let now $n$ be odd. Then, the function $T_n(\theta)$ is periodic with period $4\pi$ and $T_n(\theta+2\pi) = -T_n(\theta)$. Since $P_n$ has $n$ zeroes on the unit circle, it follows from the definition of $T_n$ that it has $2n$ zeroes in $[-2\pi, 2\pi)$, counting multiplicities. Rolle’s theorem, applied to the $4\pi$-periodic function $T_n$, yields that $\frac{d}{d\theta} T_n(\theta)$ has $2n$ zeroes on $[-2\pi, 2\pi)$. Finally, \((2.10)\) implies that $D_nP_n$ has $n$ zeroes on the unit circle. \hfill $\Box$

The above discussion shows that the following proposition is essentially equivalent to Theorem 1.2. (Actually, the proposition is slightly stronger since for Theorem 1.2 we need only even $n = 2d$).

**Proposition 2.2.** For every $n \in \mathbb{N}$ let $P_n(z)$ be an algebraic polynomial with complex coefficients written in the form \((2.6)\). Suppose that the empirical distribution of roots of $P_n$ converges weakly to some probability measure $\nu$ on the unit circle, as $n \to \infty$, that is

$$\mu[P_n] = \frac{1}{n} \sum_{j=1}^{n} \delta_{\omega_{j,n}} \xrightarrow{n \to \infty} \nu.$$ Fix $t > 0$ and let $k = k(n)$ be a sequence of positive integers such that $k(n)/n \to t$ as $n \to \infty$. Then, the empirical distribution of roots of the polynomial $D_n^{k(n)}P_n$ converges to $\nu \boxtimes \Pi_t$ weakly on the unit circle $\mathbb{T}$, as $n \to \infty$.

2.2. **Finite free multiplicative convolution and circular Laguerre polynomials.** Let us recall some notions from finite free probability which was developed in [50] and [49]. The finite free multiplicative convolution $\boxtimes_n$ is a bilinear operation on the space $\mathbb{C}_n[z]$ of polynomials of degree at most $n$ which is defined as follows:

$$\left( \sum_{j=0}^{n} \alpha_j z^j \right) \boxtimes_n \left( \sum_{j=0}^{n} \beta_j z^j \right) = \sum_{j=0}^{n} (-1)^{n-j} \frac{\alpha_j \beta_j}{n} z^j.$$ The polynomial $(z-1)^n$ plays the role of the unit element under $\boxtimes_n$, namely

$$(z-1)^n \boxtimes_n p(z) = p(z) \quad \text{for all } p(z) \in \mathbb{C}_n[z].$$ Let $p(z) = \det(zI_{n \times n} - A)$ and $q(z) = \det(zI_{n \times n} - B)$ be characteristic polynomials of arbitrary normal $n \times n$-matrices $A$ and $B$, where $I_{n \times n}$ is the $n \times n$ identity matrix. Then it is known from [50, Theorem 1.5] that

$$p(z) \boxtimes_n q(z) = \mathbb{E} \det(zI_{n \times n} - AQBQ^*),$$ \hspace{1cm} (2.11)

where $Q$ is a random orthogonal (or unitary) $n \times n$-matrix with Haar distribution.
Figure 1. Zeros of $L_{n,k}(z)$ with $n = 400$ and $k = [tn]$, where $t \in \left\{ \frac{1}{7}, \frac{3}{7}, \frac{5}{7}, 1 \right\}$. The multiplicity of the zero at $z = 1$ is $n - k$.

The next lemma provides an interpretation of the repeated action of the differential operator $D_n$ on algebraic polynomials in terms of $\mathbb{E}_n$. A similar interpretation of the repeated action of $\frac{d}{dz}$ on algebraic polynomials can be found in [5, Lemma 3.5].

**Lemma 2.3.** For all $n \in \mathbb{N}$, $k \in \mathbb{N}_0$, and all $P \in \mathbb{C}_n[z]$ we have

$$D_n^k P(z) = P(z) \mathbb{E}_n L_{n,k}(z),$$

where

$$L_{n,k}(z) := i^k \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \left(j - \frac{n}{2}\right)^k z^j = D_n^k(z - 1)^n. \tag{2.13}$$

**Proof.** If we choose $1, z, \ldots, z^n$ as a basis of $\mathbb{C}_n[z]$, the linear operator $D_n$ becomes diagonal, namely

$$D_n z^j = i \left(j - \frac{n}{2}\right) z^j, \quad j \in \{0,\ldots,n\}. \tag{2.14}$$

Using this identity, (2.12) can be easily easily checked for $P(z) = z^j$, $j \in \{0,\ldots,n\}$. The general case follows by linearity. \hfill \square

The polynomials $L_{n,k}(z)$ will be referred to as the circular Laguerre polynomials\footnote{Circular Hermite polynomials appeared very recently in [55] and were studied in [41].} since they appear in the finite free multiplicative analogue of the Poisson limit theorem in the same way as the classical Laguerre polynomials appear in the finite free additive analogue [49, Section 6.2.3] of the Poisson limit theorem; see Section 2.3 for an explanation. The following semigroup property is a consequence of the definition of $\mathbb{E}_n$:

$$L_{n,k_1}(z) \mathbb{E}_n L_{n,k_2}(z) = L_{n,k_1+k_2}(z), \quad k_1, k_2 \in \mathbb{N}_0, \quad L_{n,0}(z) = (z-1)^n.$$

**Lemma 2.4.** For all $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ all roots of the algebraic polynomial $L_{n,k}(z)$ (whose degree is $n$) are located on the unit circle.

**Proof.** The claim follows from (2.13) and Lemma 2.1. \hfill \square

Conditions ensuring the real-rootedness of the classical Laguerre polynomials are discussed in [5, Example 2.8].

**Lemma 2.5.** For all $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ we have

$$T_{n,k}(\theta) := \frac{L_{n,k}(e^{i\theta})}{e^{i\theta/2}} = (2i)^n \frac{d^k}{d\theta^k} \left(\sin \left(\frac{\theta}{2}\right)\right)^n. \tag{2.15}$$

If $n = 2d$ is even, then $T_{2d,k}(\theta)$ is the trigonometric polynomial corresponding to $L_{2d,k}(z)$ via (2.2).
Proof. By the binomial theorem, we have
\[
(2i)^n \left( \sin \frac{\theta}{2} \right)^n = (e^{i\theta/2} - e^{-i\theta/2})^n = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} e^{i\theta(j-\frac{3}{2})}.
\]
The claim follows by differentiating this formula \( k \) times in \( \theta \). \( \square \)

Lemma 2.6. For \( n \in \mathbb{N} \) and \( 0 \leq k \leq n \), the polynomial \( L_{n,k}(z) \) is divisible by \( (z-1)^{n-k} \).

Proof. The function \( \theta \mapsto \frac{d^k}{d\theta^k} \left( \sin \frac{\theta}{2} \right)^n \) has a zero of multiplicity \( n-k \) at \( \theta = 0 \). By (2.15), it corresponds to a zero of \( L_{n,k}(z) \) of multiplicity \( n-k \) at \( z = 1 \). \( \square \)

2.3. Zeroes and asymptotics of the circular Laguerre polynomials. The next theorem, whose proof is postponed to Section 3, describes the asymptotic distribution of zeroes of \( L_{n,k} \) in the regime when \( n \to \infty \) and \( k = k(n) \sim tn \) for some constant \( t > 0 \); see Figure 1. Recall that the empirical distribution of zeroes of an algebraic polynomial \( P \) is the probability measure on \( \mathbb{C} \) defined by
\[
\mu[P] := \frac{1}{\deg P} \sum_{z \in \mathbb{C} : P(z) = 0} m_P(z) \delta_z,
\]
where \( m_P(z) \) is the multiplicity of the zero at \( z \).

Theorem 2.7. Fix \( t > 0 \) and let \( k = k(n) \) be a sequence of positive integers with \( k(n)/n \to t \) as \( n \to \infty \). Then the empirical distribution of zeroes of the algebraic polynomial \( L_{n,k(n)}(z) \) converges weakly on \( T \) to the free unitary Poisson distribution with parameter \( t \), that is
\[
\mu[L_{n,k(n)}] \xrightarrow{w} \Pi_t.
\]

Let us explain why the appearance of the free unitary Poisson distribution in Theorem 2.7 is natural. The empirical distribution of zeroes of the polynomial \( L_{n,1}(z) = D_n(z-1)^n = i(n/2)(z-1)^{n-1}(z+1) \) is given by
\[
\mu[L_{n,1}] = \left( 1 - \frac{1}{n} \right) \delta_1 + \frac{1}{n} \delta_{-1}.
\]
The classical Poisson limit theorem states that the \([tn]\)-th convolution of this distribution converges to the Poisson distribution with parameter \( t > 0 \). Its finite free multiplicative analogue states that for every \( k \in \mathbb{N} \),
\[
L_{n,k}(z) = \underbrace{L_{n,1}(z) \otimes_n \ldots \otimes_n L_{n,1}(z)}_{k \text{ times}} = (L_{n,1}(z))^\otimes_n (k \text{ times})
\]is the \( k \)-th \( \otimes \)-convolution of \( L_{n,1} \); see [19, Theorem 6.9] for the finite free additive analogue in which the classical Laguerre polynomials appear. Now we let \( n \to \infty \). Then, Theorem 2.7 states that for \( k \sim tn \) the empirical distribution of zeroes of these polynomials converges weakly to \( \Pi_t \). It has to be compared to the Poisson limit theorem for \( \otimes \) (not \( \otimes_n \)) on the unit circle, see [8, Lemma 6.4] or Section 1.3 which states that
\[
\left( \left( 1 - \frac{1}{n} \right) \delta_1 + \frac{1}{n} \delta_{-1} \right)^\otimes (k(n) \text{ times}) \xrightarrow{w} \Pi_t.
\]
To deduce Theorem 2.7 from this statement, one essentially needs to justify that replacing \( \otimes \) by \( \otimes_n \) (and measures by polynomials) does not change the large \( n \) limit. This task is less trivial than it looks at a first sight. We shall prove Theorem 2.7 in Section 3 by the saddle-point method.\(^3\) As a

\(^3\) After this paper appeared on the arXiv, a combinatorial proof has been found by Arizmendi et al. [6].
by-product, we shall identify the rate of growth of \( L_{n,k} \) and its logarithmic derivative on the open unit disk.

**Theorem 2.8.** Fix \( t > 0 \) and let \( k = k(n) \) be a sequence of positive integers with \( k(n)/n \to t \) as \( n \to \infty \). For every \( \theta \) in the open upper half-plane \( \mathbb{H} := \{ \theta \in \mathbb{C} : \text{Im} \theta > 0 \} \) we have

\[
\lim_{n \to \infty} \frac{1}{n} \log \left( \frac{L_{n,k}(e^{2i\theta})}{L_{n,k}(0)} \right) = -i(\zeta(\theta) - \theta - it) + \log(1 - e^{2i\zeta(\theta)}) - t \log \left( \frac{\zeta(\theta) - \theta}{it} \right), \tag{2.18}
\]

\[
\lim_{n \to \infty} \frac{1}{n} L'_{n,k}(e^{2i\theta}) = 1 - \frac{i \cot \zeta(\theta)}{2e^{2i\theta}}, \tag{2.19}
\]

where \( \zeta = \zeta(\theta) \) is the unique solution of the equation \( \zeta - t \tan \zeta = \theta \) in \( \mathbb{H} \). The function \( \zeta : \mathbb{H} \to \mathbb{H} \) is analytic and satisfies \( \zeta(\theta) - \theta \to it \) as \( \text{Im} \theta \to +\infty \) (uniformly in \( \text{Re} \theta \in \mathbb{R} \)); see Section 2 for more properties. The logarithms in (2.18) are chosen such that \( \log 1 = 0 \) and all functions of the form \( \log(\ldots) \) are continuous (and analytic).

It should be possible to prove more refined asymptotic properties of the zeroes of \( L_{n,k} \). For example, the zeroes should exhibit crystallization phenomenon in the bulk, that is they should approach locally an arithmetic progression; c.f. \[61\]. For \( t \in (0, 1) \), the minimal argument of the zeroes (excluding the zero at 1) is conjectured to converge to \( 2 \arccos \sqrt{t - 2\sqrt{t(1-t)}} \); see \( (5.2) \) for a corresponding formula on the support of \( \Pi_t \).

Asymptotic rate of growth, as in Theorem 2.8 and limiting distribution of zeroes, as in Theorem 2.7, are known for many sequences of polynomials; see, e.g., \[46\] and \[51\] for reviews and \[20, 21, 25, 28, 29, 47\] for specific examples.

**2.4. Products of random reflections.** Let \( U_n \) be a random vector distributed uniformly on the unit sphere in \( \mathbb{R}^n \) and let \( T_n : \mathbb{R}^n \to \mathbb{R}^n \) be a random reflection w.r.t. the linear hyperplane orthogonal to \( U_n \), that is \( T_n v = v - 2(v, U_n)U_n \) for \( v \in \mathbb{R}^n \). The characteristic polynomial of \( T_n \) is

\[
\det(zI_{n \times n} - T_n) = (z + 1)(z - 1)^{n-1} = -\frac{2i}{n} L_{n,1}(z).
\]

If \( T_{n,1}, \ldots, T_{n,k} \) are \( k = k(n) \) stochastically independent copies of \( T_n \), then by \[50\] Theorem 1.5, see \( (2.11) \), the expected characteristic polynomial of their product is given by

\[
\mathbb{E} \det(zI_{n \times n} - T_{n,1} \ldots T_{n,k}) = \left( (z + 1)(z - 1)^{n-1} \right)^{\otimes_n (k \text{ times})} = \left( -\frac{2i}{n} L_{n,1}(z) \right)^{\otimes_n (k \text{ times})}.
\]

Theorem 2.7 suggests that in the regime when \( k(n) \sim tn \) as \( n \to \infty \), the empirical eigenvalue distribution of the product \( T_{n,1} \ldots T_{n,k} \) converges to \( \Pi_t \) weakly on \( \mathcal{M}_T \), while in the regime when \( k(n)/n \to \infty \), the empirical eigenvalue distribution converges weakly to the uniform distribution on \( T \). The former claim is a special case of Theorem 3 in the paper of Marchenko and Pastur \[48\]. Later products of random reflections were studied by Diaconis and Shahshahani \[21\] and Porod \[62, 63, 64\] (see also \[68\] and \[51\] for related results) who showed that there is a phase transition for the total variation distance between the distribution of \( T_{n,1} \ldots T_{n,k} \) and the Haar measure on the orthogonal group in the regime when \( k(n) = \frac{1}{2}n \log n + O(n) \).

**2.5. Proof of Theorem 1.2 given Theorem 2.7.** As explained in Section 2.1, it suffices to prove Proposition 2.2. Recall from (2.12) that \( \mathcal{D}_n(k(n)) = P_n \otimes_n L_{n,k(n)} \). By Theorem 2.7, \( \mu[L_{n,k(n)}] \) converges weakly to \( \Pi_t \), while \( \mu[P_n] \) converges weakly to \( \nu \) by assumption. The subsequent proposition implies that \( \mu[\mathcal{D}_n(k(n))P_n] \) converges weakly to \( \nu \otimes \Pi_t \) and completes the proof of Proposition 2.2.
Proposition 2.9. Let \((p_n(z))_{n∈N}\) and \((q_n(z))_{n∈N}\) be sequences of polynomials in \(\mathbb{C}[z]\) with \(\deg p_n = \deg q_n = n\). Suppose that all roots of \(p_n\) and \(q_n\) are located on the unit circle and that \(\mu[p_n]\) and \(\mu[q_n]\) converge weakly to two probability measures \(ν\) and \(ρ\) on \(\mathbb{T}\), as \(n → ∞\). Then, all roots of the polynomial \(p_n ⊠ q_n\) are also located on the unit circle and \(\mu[p_n ⊠ q_n]\) converges weakly to \(ν ⊠ ρ\).

Proof. All roots of \(p_n ⊠ q_n\) are located on \(\mathbb{T}\) by [7] Satz 3, p. 36]. Since the empirical measures \(\mu[p_n]\) and \(\mu[q_n]\) are concentrated on \(\mathbb{T}\), the corresponding moments (or Fourier coefficients) also converge to the moments of \(ν\) and \(ρ\). By [5 Proposition 3.4], the moments of \(\mu[p_n ⊠ q_n]\) converge to the moments of \(ν ⊠ ρ\). Actually, this applies to the moments of positive order \(ℓ \in \mathbb{N}\), but the same argument applied to the polynomials \(p_n(z)\) and \(q_n(z)\) yields the same conclusion for moments of arbitrary integer order \(ℓ \in \mathbb{Z}\). To complete the proof, use the well-known fact [11 p. 50] that, for probability measures on the unit circle, the convergence of Fourier coefficients implies the weak convergence. □

3. Asymptotics of the circular Laguerre polynomials

3.1. Statement of results and method of proof. In this section we study the asymptotics of the circular Laguerre polynomials and their zeroes in the regime when \(k\) grows linearly with \(n\). Let \(k = k(n)\) be a sequence of natural numbers satisfying \(k(n) \sim tn\) for some fixed \(t > 0\). The zeroes of the circular Laguerre polynomial \(L_{n,k}(z)\) defined in (2.13) lie on the unit circle by Lemma 2.4, and we denote them by \(e^{iθ_1:n}, \ldots, e^{iθ_{n,n}}\) with some \(θ_1:n, \ldots, θ_{n,n} ∈ [−π, π)\). The next theorem is a restatement of Theorem 2.7.

Theorem 3.1. The following weak convergence of probability measures on the unit circle \(\mathbb{T}\) holds:

\[
\mu_{n,k} := \mu[L_{n,k}] = \frac{1}{n} \sum_{j=1}^{n} δ_{e^{iθ_j:n}} \xrightarrow{n \to ∞} Π_ℓ.
\]

Let us describe our strategy to prove Theorem 3.1. Recall that the \(ψ\)-transform of a probability measure \(μ\) on \(\mathbb{T}\) is given by

\[
ψ_μ(z) = \int_\mathbb{T} \frac{u}{1 - uz} μ(du) = \sum_{ℓ=1}^{∞} z^ℓ \int_\mathbb{T} u^ℓ μ(du), \quad |z| < 1.
\]

To prove (3.1), it suffices to show that for some \(0 < r_1 < r_2 < 1\) we have

\[
\lim_{n → ∞} ψ_{μ_{n,k}}(z) = ψ_{Π_ℓ}(z) \quad \text{uniformly on } r_1 ≤ |z| ≤ r_2.
\]

Indeed, by Cauchy’s formula, uniform convergence of analytic functions on the annulus \(\{r_1 ≤ |z| ≤ r_2\}\) implies convergence of their \(ℓ\)-th derivatives at 0, for all \(ℓ ∈ \mathbb{N}_0\). Applying (3.2) combined with this observation yields

\[
\int_\mathbb{T} u^ℓ ψ_{μ_{n,k}}(du) = \frac{ψ_{μ_{n,k}}^{(ℓ)}(0)}{ℓ!} \xrightarrow{n → ∞} \frac{ψ_{Π_ℓ}^{(ℓ)}(0)}{ℓ!} = \int_\mathbb{T} u^ℓ Π_ℓ(du),
\]

for all \(ℓ ∈ \mathbb{N}_0\). Since we are dealing with measures invariant under conjugation \(z → \bar{z}\), the same claim holds in fact for all \(ℓ ∈ \mathbb{Z}\). By a well-known criterion [11 p. 50], this implies the weak convergence \(μ_{n,k} → Π_ℓ\) stated in Theorem 3.1.

The \(ψ\)-transform of \(μ_{n,k}\). So, our aim is to prove (3.3). To express the \(ψ\)-transform of \(μ_{n,k}\), it will be convenient to introduce the following functions that are closely related to \(L_{n,k}(z)\):

\[
W_{n,k}(θ) := \frac{1}{k!} \left(\frac{d}{dθ}\right)^k \left(\frac{\sin θ}{2}\right)^n = \frac{1}{(2i)^n k!} \frac{L_{n,k}(e^{iθ})}{e^{iθn/2}}.
\]
Note that the second equality follows from Lemma 2.5.

**Lemma 3.2.** For all \( \theta \in \mathbb{C} \) with \( \text{Im} \theta > 0 \) and all \( n \in \mathbb{N} \), \( k \in \mathbb{N}_0 \) we have

\[
\psi_{\mu_{n,k}}(e^{i\theta}) = \frac{i}{n} \frac{W'_{n,k}(\theta)}{W_{n,k}(\theta)} - \frac{1}{2}.
\]

**Proof.** Recalling (2.13) we can write

\[
L_{n,k}(z) = i^k \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \left( j - \frac{n}{2} \right)^k z^j = \left( \frac{in}{2} \right)^k \prod_{j=1}^{n} (z - e^{i\theta_j,n}).
\]

Taking \( z = 0 \), we obtain \( \prod_{j=1}^{n} e^{i\theta_j,n} = (-1)^k \) and hence \( \prod_{j=1}^{n} e^{i\theta_j,n/2} = \pm i^k \). Then, by (3.4) we have

\[
W_{n,k}(\theta) = \frac{1}{(2i)^{nk}} L_{n,k}(e^{i\theta}) = \pm \frac{n^k}{2^{nk} k!} \prod_{j=1}^{n} \sin \frac{\theta - \theta_{jn}}{2}.
\]

Taking the logarithmic derivative and dividing by \( n \) we obtain

\[
\frac{1}{n} \frac{W'_{n,k}(\theta)}{W_{n,k}(\theta)} = \frac{1}{2n} \sum_{j=1}^{n} \cot \frac{\theta - \theta_{jn}}{2} = -\frac{i}{2} - i \sum_{\ell=1}^{\infty} e^{i\ell\theta} \frac{1}{n} \sum_{j=1}^{n} e^{-i\ell\theta_{jn}}, \tag{3.5}
\]

where we used the formula

\[
\cot \frac{\theta}{2} = \frac{e^{i\theta/2} + e^{-i\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}} = -i - \frac{1 + e^{i\theta}}{1 - e^{i\theta}} = -i \left( 1 + 2 \sum_{\ell=1}^{\infty} e^{i\ell\theta} \right), \quad \text{Im} \, \Theta > 0. \tag{3.6}
\]

By (3.4), the function \( W_{n,k}(\theta) \) is even (respectively, odd) if \( n - k \) is even (respectively, odd). In either case, its zeroes are symmetric w.r.t. \( 0 \) and we can replace \( -i\ell\theta_{jn} \) by \( i\ell\theta_{jn} \) in (3.5). Recall from (3.2) that the \( \psi \)-transform of \( \mu_{n,k} \) can be written as

\[
\psi_{\mu_{n,k}}(z) = \sum_{\ell=1}^{\infty} z^{-\ell} \sum_{j=1}^{n} e^{i\ell\theta_{jn}}, \quad |z| < 1.
\]

Together with (3.5), this yields the statement of the lemma if we put \( z := e^{i\theta} \in \mathbb{D} \). \qed

**Asymptotic results on** \( W_{n,k}(\theta) \). **In the following, we shall investigate the asymptotic behaviour of** \( \log W_{n,k}(\theta) \) **and** \( W'_{n,k}(\theta)/W_{n,k}(\theta) \) **as** \( n \to \infty \) **using the saddle-point method. Our results will be stated in terms of a family of analytic functions** \( \zeta_t(\theta) \) **(indexed by a parameter** \( t > 0 \) **that, as it will turn out, solve the saddle-point equation.**

**Theorem 3.3.** **Fix** \( t > 0 \). **Let** \( \mathbb{H} := \{ \theta \in \mathbb{C} : \text{Im} \theta > 0 \} \) **be the upper half-plane. For every** \( \theta \in \mathbb{H} \), the equation \( \zeta - t \tan \zeta = \theta \) **has a unique solution** \( \zeta = \zeta_t(\theta) \) **in** \( \mathbb{H} \), **and this solution is simple.** \( \overline{\mathbb{H}} \)** **admits a continuous extension to the closed upper half-plane** \( \mathbb{H} \), **and satisfies** \( \zeta_t(\theta + \pi) = \zeta_t(\theta) + \pi \) **as well as** \( \text{Im} \zeta_t(\theta) > \text{Im} \theta \) **for all** \( \theta \in \mathbb{H} \). **Finally, we have** \( \zeta_t(\theta) - \theta \to 0 \) **as** \( \text{Im} \theta \to +\infty \) **(uniformly in** \( \text{Re} \theta \in \mathbb{R} \).)

We shall prove Theorem 3.3 in Section 4, where more properties of the function \( \zeta_t(\theta) \) will be established. The next result on the asymptotics of \( W_{n,k} \) will be proved in Section 3.2.

---

4Simplicity means that the derivative of \( \zeta \mapsto \zeta - t \tan \zeta \) does not vanish at \( \zeta_t(\theta) \).
**Theorem 3.4.** Let $t > 0$ be fixed and recall that $k/n \to t$ as $n \to \infty$. Then, uniformly on compact subsets of the upper half-plane $\mathbb{H} = \{ \theta \in \mathbb{C} : \text{Im} \theta > 0 \}$ it holds that

$$
\lim_{n \to \infty} \frac{1}{n} \log |W_{n,k}(\theta)| = \log |\sin \zeta_t(\theta/2)| - t \log |2\zeta_t(\theta/2) - \theta|, 
$$

(3.7)

$$
\lim_{n \to \infty} \frac{1}{n} W_{n,k}'(\theta) = \frac{1}{2} \cot \zeta_t(\theta/2) = \frac{t}{2\zeta_t(\theta/2) - \theta}. 
$$

(3.8)

The next lemma expresses the $\psi$-transform of the free unitary Poisson distribution $\Pi_t$ in terms of the function $\zeta_t$.

**Lemma 3.5.** Fix some $t > 0$. Then, for all $\theta \in \mathbb{H}$ we have

$$
\psi_{\Pi_t}(e^{i\theta}) = \frac{i}{2} \cot \zeta_t(\theta/2) - \frac{1}{2} = \frac{it}{2\zeta_t(\theta/2) - \theta} - \frac{1}{2}. 
$$

(3.9)

*Proof.* By the definition of the free unitary Poisson distribution $\Pi_t$ given in (1.6) and (1.8), its $\psi$-transform satisfies

$$
w = \psi_{\Pi_t} \left( \frac{w}{1 + w} \exp \left\{ \frac{t}{w + \frac{1}{2}} \right\} \right) 
$$

(3.10)

provided $w \in \mathbb{C}$ and $|w|$ is sufficiently small. Let us put

$$
w := \frac{1}{2} \cot \zeta_t(\theta/2) - \frac{1}{2} = \frac{it}{2\zeta_t(\theta/2) - \theta} - \frac{1}{2}. 
$$

If $\text{Im} \theta$ is sufficiently large, then $|w|$ is sufficiently small (since $2\zeta_t(\theta/2) - \theta \to 2it$ as $\text{Im} \theta \to +\infty$) and (3.10) holds. Elementary transformations yield

$$
\frac{w}{1 + w} \exp \left\{ \frac{t}{w + \frac{1}{2}} \right\} = \frac{\frac{i}{2} \cot \zeta_t(\theta/2) - \frac{1}{2}}{\frac{i}{2} \cot \zeta_t(\theta/2) + \frac{1}{2}} e^{-i(2\zeta_t(\theta/2) - \theta)} = e^{2i\zeta_t(\theta/2)} e^{-i(2\zeta_t(\theta/2) - \theta)} = e^{i\theta}. 
$$

(3.11)

It follows that (3.9) holds provided $\text{Im} \theta$ is sufficiently large. By the principle of analytic continuation, (3.9) continues to hold for all $\theta \in \mathbb{H}$. \hfill \Box

**Completing the proof of Theorem 3.1.** As already explained above, it suffices to prove (3.3), i.e. that $\psi_{\mu_{n,k}}(z) \to \psi_{\Pi_t}(z)$ uniformly on $\{ r_1 \leq |z| \leq r_2 \}$. Combining Lemma 3.2, Theorem 3.4 and Lemma 3.5 we obtain

$$
\psi_{\mu_{n,k}}(e^{i\theta}) = \frac{i}{n} W_{n,k}'(\theta) - \frac{1}{2} \to \frac{it}{2\zeta_t(\theta/2) - \theta} - \frac{1}{2} = \psi_{\Pi_t}(e^{i\theta}) 
$$

for all $\theta \in \mathbb{H}$, the convergence being uniform on compact subsets of $\mathbb{H}$. This implies (3.3) and proves Theorem 3.1. Observe that for this proof a weaker form of Theorem 3.4 suffices: we only need that (3.8) holds uniformly on some rectangle of the form $\{-\pi \leq \text{Re} \theta \leq \pi, \theta_1 \leq \text{Im} \theta \leq \theta_2\}$. \hfill \Box

3.2. **Proof of Theorem 3.4:** A saddle-point argument. The proof is subdivided into several steps. It will be convenient to write

$$
V_{n,k}(\theta) := \frac{1}{(k-1)!} \left( \frac{d}{d\theta} \right)^{k-1} (\sin \theta)^n = 2^{k-1} W_{n,k-1}(2\theta). 
$$

(3.12)

**Cauchy’s formula.** We start by representing $V_{n,k}(\theta)$ as an integral. For all $\theta \in \mathbb{C}$, Cauchy’s formula yields

$$
V_{n,k}(\theta) = \frac{1}{2\pi i} \oint \frac{(\sin \theta)^n}{(z - \theta)^k} \, dz = \frac{1}{2\pi i} \oint (f_{k/n}(z; \theta))^k \, dz, 
$$

(3.13)
where the integral is taken over some small counterclockwise loop around $\theta$, and for $s > 0$ we defined

$$f_s(z; \theta) := \frac{\exp \left\{ \frac{1}{z} \log \sin z \right\}}{z - \theta}, \quad \text{Im} \ z > 0, \ z + \theta.$$  \hfill (3.14)

Since the function $z \mapsto \sin z$ vanishes if and only if $z \in \pi \mathbb{Z}$, restricting our attention to the upper half-plane $\mathbb{H}$ we can construct a well-defined branch of the function $\log \sin z$. In fact, there exist infinitely many branches differing by $2\pi i m$, $m \in \mathbb{Z}$. We can specify the concrete branch we are interested in by exhibiting its Fourier series:

$$\log \sin z = \log \frac{e^{-iz}(1 - e^{2iz})}{-2i} := -iz - \log 2 + \frac{\pi i}{2} - \sum_{\ell=1}^{\infty} \frac{e^{2i\ell z}}{\ell}. \quad \hfill (3.15)$$

Note that the series converges uniformly as long as $\text{Im} \ z \geq c$ for some $c > 0$.

**Saddle point equation.** We are looking for critical (saddle) points of $f_s(z; \theta)$ in the upper half-plane $\mathbb{H}$. The partial derivative of $\log f_s(z; \theta)$ in $z$ is given by

$$\frac{\partial}{\partial z} \log f_s(z; \theta) = \frac{1}{s} \cot z - \frac{1}{z - \theta}.$$  \hfill (3.16)

Hence, the saddle points are the solutions $\zeta \in \mathbb{H}$ of the equation

$$\zeta - s \tan \zeta = \theta.$$  \hfill (3.17)

It will be shown in Section 4 that for every $\theta \in \mathbb{H}$ this equation has a unique, simple solution $\zeta = \zeta_s(\theta)$ in $\mathbb{H}$. Moreover, we shall show that $\zeta_s(\theta) - \theta \to 0$ as $\text{Im} \ \theta \to +\infty$. If not otherwise stated, this and similar claims hold uniformly in $\text{Re} \ \theta \in \mathbb{R}$.

**Existence of the saddle-point contour.** To apply the saddle-point method, we need to deform the contour of integration in (3.13) to some closed $C^\infty$-contour $\gamma_{s,\theta}$ that depends on the parameters $s$ and $\theta$, passes through the saddle point $\zeta_s(\theta)$ and has the property that

$$|f_s(\zeta_s(\theta); \theta)| > |f_s(z; \theta)| \quad \text{for all} \quad z \in \gamma_{s,\theta} \setminus \{\zeta_s(\theta)\}. \hfill (3.18)$$

In the following we shall argue that a saddle-point contour satisfying (3.17) exists if $\text{Im} \ \theta$ is sufficiently large. Consider the level lines of the function $z \mapsto |f_s(z; \theta)|$, defined on the half-plane $\{z \in \mathbb{C} : \text{Im} \ z \geq 1\}$. Near the pole at $z = \theta$, the level lines are small loops enclosing $\theta$. Let us now decrease the level. According to Morse theory, the topology of the level lines can change only if they pass through some critical point or if the level lines cross the boundary $\text{Im} \ z = 1$. We know that the unique, non-degenerate critical point in the upper half-plane $\mathbb{H}$ is $\zeta_s(\theta)$. We shall prove in a moment that for sufficiently large $\text{Im} \ \theta$,

$$|f_s(\zeta_s(\theta); \theta)| > \sup_{z \in \mathbb{C} : \text{Im} \ z = 1} |f_s(z; \theta)|. \hfill (3.19)$$

Assuming (3.18) it follows that $\zeta_s(\theta)$ is the first critical point which appears on the level lines if we decrease the level. Since the critical point $\zeta_s(\theta)$ is simple, the level line passing through it has a $\gamma$-shaped topology and contains a loop enclosing $\theta$; see Figure 2. It follows that a saddle-point contour satisfying (3.17) and passing through $\zeta_s(\theta)$ exists.

Let us prove (3.18). It follows from (3.15) that $\log \sin z$ differs from $-iz$ by a function that stays bounded on $\{\text{Im} \ z \geq 1\}$. Therefore, uniformly over all $z \in \mathbb{C}$ with $\text{Im} \ z = 1$ we have the estimate

$$|f_s(z; \theta)| = \frac{\exp \left\{ \frac{1}{z} \text{Re} \log \sin z \right\}}{|z - \theta|} \leq \frac{\exp \{O(1)\}}{\text{Im} \ \theta - 1} \to 0,$$  \hfill (3.19)
as \( \text{Im} \theta \to +\infty \). On the other hand, recalling that \( \zeta_s(\theta) - \theta \to \text{Im} \theta \to +\infty \) we see that

\[
|f_s(\zeta_s(\theta); \theta)| = \exp \left\{ \frac{1}{s} \text{Re} \log \sin \zeta_s(\theta) \right\} = \exp \left\{ \frac{1}{s} \text{Im} \zeta_s(\theta) + O(1) \right\} = \exp \left\{ \frac{1}{s} \text{Im} \theta + O(1) \right\} \to +\infty,
\]

as \( \text{Im} \theta \to +\infty \). Taking (3.19) and (3.20) together we conclude that (3.18) holds provided \( \text{Im} \theta \) is sufficiently large.

Remark 3.6. We strongly believe that a saddle-point contour satisfying (3.17) exists for all \( \theta \in \mathbb{H} \). One way to prove this would be to show that the value of \( |f_s(z; \theta)| \) at \( z = \zeta_s(\theta) \) is larger than all values for \( z \in \mathbb{R} \setminus \pi \mathbb{Z} \). However, numerical simulations suggest that the latter claim is false without the assumption that \( \text{Im} \theta \) is sufficiently large.

The saddle-point asymptotics. Let \( s > 0 \) and \( \text{Im} \theta > \tau_0 \) with sufficiently large \( \tau_0 > 0 \). Then, (3.17) holds and the standard saddle-point method [71, §45.4] yields

\[
\frac{1}{2\pi i} \oint (f_s(z; \theta))^k dz \sim \frac{(f_s(\zeta_s(\theta); \theta))^k}{\sqrt{2\pi k \cdot (\log f_s)''(\zeta_s(\theta); \theta)}} \quad \text{as } k \to \infty.
\]

The choice of the branch of the square root is explained in [71, §45.4] and will be irrelevant for our purposes. Recalling (3.14) we obtain

\[
\lim_{k \to \infty} \sqrt{2\pi k} \cdot \frac{1}{2\pi i} \oint (f_s(z; \theta))^k dz = \left( \frac{1 - s}{(\zeta_s(\theta) - \theta)^2} - \frac{1}{s} \right)^{-1/2}.
\]

Note that the terms appearing in this formula cannot become 0 or \( \infty \) since \( \text{Im} \zeta_s(\theta) > \text{Im} \theta \) for all \( \theta \in \mathbb{H} \) by Lemma 4.6 and since the critical point at \( \zeta_s(\theta) \) is non-degenerate. In fact, the arguments of [71, §45.4] show that (3.21) holds uniformly as long as \((\theta, s)\) stays in a compact subset of \( \{\text{Im} \theta > \tau_0\} \times (0, \infty) \). Applying the function \( w \mapsto \frac{1}{k} \log |w| \) to both sides of (3.21) and removing
terms going to 0 yields
\[ \lim_{k \to \infty} \frac{1}{k} \log \left| \frac{1}{2\pi i} \oint (f_s(z; \theta))^k \, dz \right| = \frac{1}{s} \log |\sin \zeta_s(\theta)| - \log |\zeta_s(\theta) - \theta| \]
locally uniformly in \((\theta, s) \in \{\Im \theta > \tau_0\} \times (0, \infty)\). Recalling the formula for \(V_{n,k}(\theta)\) stated in (3.13) and the relation \(k/n \to t\), we arrive at
\[ \lim_{n \to \infty} \frac{1}{n} \log |V_{n,k}(\theta)| = \log |\sin \zeta_t(\theta)| - t \log |\zeta_t(\theta) - \theta|. \quad (3.22) \]
The above results can be summarized in the following lemma which proves Theorem 3.4 for sufficiently large \(\Im \theta\).

**Lemma 3.7.** Let \(t > 0\) be fixed and recall that \(k/n \to t\) as \(n \to \infty\). If \(\tau_0 > 0\) is sufficiently large, then locally uniformly on \(\{\theta \in \mathbb{C} : \Im \theta > \tau_0\}\), we have
\[ \lim_{n \to \infty} \frac{1}{n} \log |W_{n,k}(\theta)| = \log |\sin \zeta_t(\theta/2)| - t \log |2\zeta_t(\theta/2) - \theta|, \quad (3.23) \]
\[ \lim_{n \to \infty} \frac{1}{n} \frac{W'_{n,k}(\theta)}{W_{n,k}(\theta)} = \frac{1}{2} \cot \zeta_t(\theta/2) = \frac{t}{2\zeta_t(\theta/2) - \theta}. \quad (3.24) \]

**Proof.** The first formula is just a restatement of (3.22) using (3.12). To prove the second formula, consider the sequence of analytic functions
\[ h_n(\theta) := \frac{1}{n} \log W_{n,k}(\theta) - \log |\sin \zeta_t(\theta/2)| + t \log (2\zeta_t(\theta/2) - \theta), \quad \theta \in \mathbb{H}, \quad (3.25) \]
where the logarithms are chosen such that all functions of the form \(\log(\ldots)\) are continuous (and analytic) on \(\mathbb{H}\). From (3.23) we know that \(\Re h_n(\theta) \to 0\) locally uniformly on \(\{\Im \theta > \tau_0\}\), and it follows from Lemma 3.8 below that \(h'_n(\theta) \to 0\) locally uniformly. To complete the proof of (3.24) note that, after some calculus,
\[ h'_n(\theta) = \frac{1}{n} \frac{W'_{n,k}(\theta)}{W_{n,k}(\theta)} - \frac{t}{2\zeta_t(\theta/2) - \theta}. \quad (3.26) \]
The terms involving \(\zeta_t(\theta/2)\) cancel because \(\zeta_t(\theta/2) - \theta/2 = t \tan \zeta_t(\theta/2)\). \qed

**Lemma 3.8.** Let \(h_1(z), h_2(z), \ldots\) be a sequence of analytic functions defined on some domain \(D \subset \mathbb{C}\). If \(\Re h_n(z) \to 0\) locally uniformly on \(D\), then \(h'_n(z) \to 0\) locally uniformly on \(D\).

**Proof.** The claim follows from the integral representation
\[ h'_n(z) = \frac{1}{\pi} \oint \frac{\Re h_n(w)}{(w-z)^2} \, dw, \]
where the integration is over a sufficiently small counter-clockwise circle centered at \(z\). To prove this formula, we may assume that \(z = 0\). It suffices to verify the representation for individual terms in the Taylor series of \(h_n(w)\). Thus, we need to check that \(\frac{1}{\pi} \oint \Re (aw^\ell)w^{-2}\, dw = a1_{\ell=1}\) for all \(\ell \in \mathbb{N}_0\), \(a \in \mathbb{C}\). Substituting \(w = re^{i\phi}\) with \(\phi \in [-\pi, +\pi]\), the formula becomes
\[ \frac{1}{\pi} \int_{-\pi}^{+\pi} \Re (ae^{i\phi})e^{-i\phi} \, d\phi = a1_{\ell=1}, \]
which is elementary to verify. \qed

**Completing the proof of Theorem 3.4.** Let us prove that (3.24) holds locally uniformly on the whole of \(\mathbb{H}\). To this end we observe that in the above proof of Theorem 3.1 we used the already established Lemma 3.7 (rather than the full Theorem 3.4). So, we know that \(\mu_{n,k} \to \Pi_t\) weakly on \(\mathbb{T}\), as \(n \to \infty\), meaning that \(\int_T F(u)\mu_{n,k}(du) \to \int_T F(u)\Pi_t(du)\) for every continuous function \(F\) on \(\mathbb{T}\). From the first equality in (3.2) it follows that \(\psi^{\mu_{n,k}}(z) \to \psi^{\Pi_t}(z)\) for all \(z \in D\), which extends the already established convergence (3.3). In fact, the convergence is locally uniform on \(D\), which can be verified.
using that the family \((g_z)_{z \in K}\) of functions \(g_z(u) = \frac{ue^z}{1-ue^z}\), \(u \in \mathbb{T}\), is compact in \(C(\mathbb{T})\) for every compact set \(K \subset \mathbb{D}\). Applying Lemma 3.2 and then Lemma 3.5, we conclude that
\[
\frac{1}{n} W'_{n,k}(\theta) = -i\psi_{\mu_{n,k}}'(e^{i\theta}) - \frac{i}{2} n \xrightarrow{n \to \infty} -i\psi_{\mu_{n,k}}'(e^{i\theta}) - \frac{i}{2} = 2\cot(\theta/2) - \theta
\]
for all \(\theta \in \mathbb{H}\), and the convergence is locally uniform. This establishes (3.24). To prove that (3.23) holds locally uniformly on \(\mathbb{H}\), we take some \(\theta_0 \in \mathbb{H}\) with sufficiently large \(\text{Im} \theta_0\) and write
\[
\frac{1}{n} \log |W_{n,k}(\theta)| - \frac{1}{n} \log |W_{n,k}(\theta_0)| = \frac{1}{n} \text{Re} \left(\log W_{n,k}(\theta) - \log W_{n,k}(\theta_0)\right) = \frac{1}{n} \int_{\theta_0}^{\theta} \frac{1}{n} W'_{n,k}(x) \, dx.
\]
Now we let \(n \to \infty\). We can apply (3.23) to \(\frac{1}{n} \log |W_{n,k}(\theta_0)|\). Regarding the integral on the right-hand side, we can apply the just established relation (3.24) to obtain
\[
\int_{\theta_0}^{\theta} \frac{1}{n} W'_{n,k}(x) \, dx \to \int_{\theta_0}^{\theta} \frac{t}{2\zeta_t(x/2) - x} \left(\log \zeta_t(x/2) - t \log(2\zeta_t(x/2) - x)\right)_{x=\theta_0},
\]
see (3.25) and (3.26) for the last equality. Taking the real part yields (3.23).

**Proof of Theorem 2.8.** Recall from (3.4) that \(L_{n,k}(e^{2i\theta}) = (2i)^{n,k}e^{i\theta} W_{n,k}(2\theta)\). Taking into account (3.8) we obtain
\[
\frac{1}{n} L'_{n,k}(e^{2i\theta}) = e^{2i\theta} \left(\frac{1}{2} - \frac{i}{n} W'_{n,k}(2\theta)\right) \to \frac{1}{2e^{2i\theta} - 1} - i \cot(\theta/2) - \theta.
\]
This proves (2.19). To prove (2.18) we consider
\[
H_n(z) := \frac{1}{n} \log \left(\frac{L_{n,k}(e^{2i\theta})}{L_{n,k}(0)}\right) + i \log(1 - e^{2i\theta}) + t \log \left(\frac{\zeta_t(\theta) - \theta}{it}\right), \quad \theta \in \mathbb{H},
\]
as an analytic function of the argument \(z = e^{2i\theta} \in \mathbb{D} \setminus \{1\}\). Observe that \(H_n(z)\) becomes analytic on \(\mathbb{D}\) if we put \(H_n(0) := 0\). After some calculus, one gets
\[
\frac{d}{dz} H_n(z) = \frac{1}{n} \frac{L'_{n,k}(e^{2i\theta})}{L_{n,k}(e^{2i\theta})} - \frac{1}{2e^{2i\theta} - 1} - i \cot(\theta/2) - \theta.
\]
Note that the terms involving \(\zeta'_t(\theta)\) cancel. We already know that, as \(n \to \infty\), the right-hand side converges to 0 locally uniformly in \(\theta \in \mathbb{H}\) or, which is the same, \(z \in \mathbb{D} \setminus \{0\}\). However, we are dealing with analytic functions and Cauchy’s formula implies that the convergence is locally uniform in \(z \in \mathbb{D}\). Integrating and taking into account that \(H_n(0) = 0\) proves that \(H_n(z) \to 0\) locally uniformly in \(z \in \mathbb{D}\), thus establishing (2.18).

4. **Analysis of the equation \(\zeta - t \tan \zeta = \theta\)**

4.1. **Construction of the principal branch.** Fix some \(t > 0\). In the following we shall study the multivalued function \(\zeta\) of the complex variable \(\theta\) given by the implicit equation
\[
\zeta - t \tan \zeta = \theta.
\]
We are interested in constructing the principal branch \(\zeta_t(\theta)\) of this function which is analytic on the upper half-plane \(\mathbb{H}\), continuous on its closure \(\overline{\mathbb{H}}\), and satisfies the following boundary condition:
\[
\zeta_t(\theta) - \theta \text{ is bounded on } \{\theta \in \mathbb{C} : \text{Im} \theta > \tau_0\} \text{ for some } \tau_0 > 0.
\]
Proof. For every $$y \in \mathbb{C}$$, we have
$$\lim_{r \to +\infty} y(r) := e^{2t} \frac{1-r}{1+r}.$$ (4.3)

It is analytic on some neighborhood of $$r = 1$$ and satisfies $$y(1) = 0$$ and $$y'(1) = -\frac{1}{2} e^{2t} \neq 0$$. By the inverse function theorem, there is a well-defined inverse function $$r = r_1(y)$$ which is analytic on a small neighborhood of $$y = 0$$. Let us put $$\zeta_t(\theta) := \theta + it \cdot r_1(e^{2i\theta})$$, which is well-defined and analytic provided $$\Im \theta$$ is sufficiently large. With this definition, we have
$$\tan \zeta_t(\theta) = -i \cdot \frac{1 - e^{-2i\zeta_t(\theta)}}{1 + e^{-2i\zeta_t(\theta)}} = -i \cdot \frac{1 - e^{-2i\bar{t}} e^{2tr_1(e^{2i\theta})}}{1 + e^{-2i\bar{t}} e^{2tr_1(e^{2i\theta})}} = -i \cdot \frac{1 - 1 + r_1(e^{2i\theta})}{1 + 1 - r_1(e^{2i\theta})} = \frac{\zeta_t(\theta) - \theta}{t},$$
meaning that $$\zeta_t(\theta)$$ solves (4.1). It follows from $$r_1(0) = 1$$ that (4.2) holds and $$\zeta_t(\theta) - \theta \to it$$ as $$\Im \theta \to +\infty$$ (uniformly in $$\Re \theta \in \mathbb{R}$$). (4.4)

Uniqueness. We prove that an analytic function $$\zeta_t(\theta)$$ satisfying (4.1) and (4.2) is unique. Write $$\zeta_t(\theta) = \theta + it \cdot s_t(\theta)$$ for some analytic function $$s_t(\theta)$$. By (4.2) we have $$|s_t(\theta)| \leq B$$ for some constant $$B > 0$$ and all $$\theta \in \mathbb{C}$$ with $$\Im \theta > \tau_0$$. After some transformations, Equation (4.1) takes the form
$$G_\theta(s_t(\theta)) = 0,$$ where $$G_\theta(s) := s + \frac{e^{2ist} - e^{2is}}{e^{2ist} + e^{2is}}.$$ Note that for sufficiently large $$\Im \theta$$ the function $$s \mapsto G_\theta(s)$$ is analytic on $$\{|s| \leq 2B\}$$ and that $$G_\theta(s) \to s - 1$$ uniformly on that disc as $$\Im \theta \to +\infty$$. By Rouche’s theorem, the solution of $$G_\theta(s) = 0$$ is unique if $$\Im \theta$$ is sufficiently large, proving that $$s_t(\theta)$$ (and hence $$\zeta_t(\theta)$$) is uniquely defined if $$\Im \theta$$ is sufficiently large. The rest follows by the principle of analytic continuation. Note in passing that the uniqueness just proved implies that $$\zeta_t(\theta + \pi) = \zeta_t(\theta) + \pi$$ and $$\zeta_t(-\theta) = -\zeta_t(\theta)$$.

Behavior on the imaginary half-axis. Let us show that the principal branch $$\zeta_t(\theta)$$ that we defined for sufficiently large $$\Im \theta$$ can be continued to some open set of $$\theta$$’s containing the imaginary half-axis $$\{i\tau : \tau > 0\}$$. Let $$\theta = it$$ with $$\tau > 0$$ and $$\zeta = iy$$ with $$y > 0$$. Equation (4.1) takes the form
$$y - t \tanh y = \tau.$$ (4.5)
Our claim is a consequence of the following

**Lemma 4.1.** Fix $$t > 0$$. Then, for every $$\tau > 0$$, Equation (4.5) has a unique, simple solution $$y = y_t(\tau) > \tau$$. This solution continuously depends on $$\tau > 0$$ and satisfies $$y_t(\tau) - \tau \to t$$ as $$\tau \to +\infty$$. Moreover, we have
$$\lim_{\tau \to 0} y_t(\tau) = \begin{cases} 0, & \text{if } 0 < t \leq 1, \\ y_t(0), & \text{if } t > 1, \end{cases}$$ (4.6)
where $$y_t(0) > 0$$ is the unique strictly positive zero of the convex function $$y \mapsto y - t \tanh y$$, $$y \geq 0$$.

**Proof.** For every $$t > 0$$, the function $$g(y) = g_t(y) := y - t \tanh y$$, $$y \geq 0$$, is strictly convex since its second derivative equals $$2t \sinh y / (\cosh y)^3 > 0$$. Moreover, $$g(0) = 0$$, $$g'(0) = 1 - t$$ and $$g(y) \to +\infty$$ as $$y \to +\infty$$. It follows that for $$0 < t \leq 1$$, the function $$g$$ strictly increases from 0 to $$+\infty$$ on the interval $$[0, +\infty)$$, while for $$t > 1$$ it has a unique positive zero $$y_t(0) > 0$$ and is strictly increasing from 0 to $$+\infty$$ on the interval $$[y_t(0), +\infty)$$; see the first two panels of Figure 3. These properties imply the lemma. □
Since the solution mentioned in Lemma 4.1 is simple, it can be analytically continued to some open neighborhood of \( \{i \tau \} \). By the arguments used in the proof of uniqueness, we have \( \zeta_i(i \tau) = iy_i(\tau) \) for all \( \tau > 0 \). More generally, on the half-lines \( \{i \tau + i \pi n : \tau > 0\} \), \( n \in \mathbb{Z} \), we have \( \zeta_i(i \tau + i \pi n) = iy_i(\tau) + i \pi n \). Arguing in a similar way, it is possible to analyze the behavior of \( \zeta_i(\theta) \) on the half-lines \( \{i \tau + \frac{\pi}{2} + i \pi n : \tau \geq 0\} \), \( n \in \mathbb{Z} \), where we have \( \zeta_i(i \tau + \frac{\pi}{2} + i \pi n) = iy_i(\tau + \frac{\pi}{2} + i \pi n) \) for all \( \tau \geq 0 \) with \( \tilde{y} = \tilde{y}_i(\tau) > 0 \) being the unique solution to the equation \( \tilde{y} = y - t \cotanh \tilde{y} = \tau \). This solution continuously depends on \( \tau \geq 0 \), satisfies \( \tilde{y}_i(\tau) - \tau \to t \) as \( \tau \to +\infty \), and we have \( \lim_{\tau \to 0} \tilde{y}_i(\tau) = \tilde{y}_i(0) > 0 \), where \( \tilde{y}_i(0) > 0 \) is the unique zero of the concave function \( \tilde{y} \mapsto \tilde{y} - t \cotanh \tilde{y}, \ y \geq 0 \); see the right panel of Figure 3.

**Analytic continuation aside the ramification points.** Consider the meromorphic function \( f_i(\zeta) := \zeta - t \tan \zeta \) defined for \( \zeta \in \mathbb{C} \setminus \{ \frac{\pi}{2} + i \pi n : n \in \mathbb{Z} \} \). The critical points of \( f_i \) are the zeroes of its derivative, i.e. the solutions of the equation \( \cos^2 \zeta = t \). The images of the critical points under \( f_i \) are called the critical values of \( f_i \). Let \( \mathcal{D} \subset \mathbb{C} \) be any simply-connected domain in the \( \theta \)-plane not containing any critical value of \( f_i \). Take some \( \theta_* \in \mathcal{D} \) and let \( \zeta_* \in \mathbb{C} \setminus \{ \frac{\pi}{2} + i \pi n : n \in \mathbb{Z} \} \) be such that \( f_i(\zeta_*) = \theta_* \). Standard arguments [69, Chapter 16] show that there is a unique analytic function \( \zeta_i(\theta) \), defined on the whole of \( \mathcal{D} \) and satisfying \( f_i(\zeta_i(\theta)) = \theta \) for all \( \theta \in \mathcal{D} \) as well as \( \zeta_i(\theta_*) = \zeta_* \). Indeed, the local invertibility of the function \( f_i \) near its non-critical values is known and we need only to check that when continuing the function \( \zeta_i \) along some path \( \gamma : [0, 1] \to \mathcal{D} \) we cannot run into a singularity of \( \zeta_i(\theta) \), that is the function cannot become unbounded. Assume, by contraposition, that \( \zeta_i(\gamma(s)) \to \infty \) as \( s \uparrow s_0 \). Then, it follows from (4.1) that \( \tan \zeta_i(\gamma(s)) \to \infty \) as \( s \uparrow s_0 \), hence the distance between \( \zeta_i(\gamma(s)) \) and the set \( \frac{\pi}{2} + i \pi \mathbb{Z} \) goes to 0 as \( s \uparrow s_0 \), which is a contradiction since \( \zeta_i(\gamma(s)) \to \infty \) as \( s \uparrow s_0 \) and at the same time \( s \mapsto \zeta_i(\gamma(s)) \) is continuous for \( s < s_0 \). Hence, we can analytically continue \( \zeta_i(\theta) \) along any path in \( \mathcal{D} \).

**Critical points and values.** Recall that the critical points of \( \zeta \mapsto \zeta - t \tan \zeta \) are the solutions of the equation \( \cos^2 \zeta = t \). We consider three cases.

**Case 1:** \( 0 < t < 1 \). The critical points \( \zeta_{n, \varepsilon} \) and the corresponding critical values \( \theta_{n, \varepsilon} \) are given by

\[
\zeta_{n, \varepsilon} = \varepsilon \arccos \sqrt{1 + \pi n}, \quad \theta_{n, \varepsilon} = \varepsilon x_{t, \varepsilon} + \pi n, \quad n \in \mathbb{Z}, \quad \varepsilon \in \{ \pm 1 \},
\]

where

\[
x_{t, \varepsilon} := \arccos \sqrt{1 - \sqrt{t(1 - t)}} \in \left(0, \frac{\pi}{2}\right), \quad 0 < t < 1.
\]
Let us look more carefully at the square-root singularity of \( \zeta(t) \) at the branch point \( x_t \) (the analysis of \(-x_t\) being similar). We have the expansion

\[
\zeta - t \tan \zeta = x_t - \sqrt{1 - t} (\zeta - \arccos \sqrt{t})^2 + o((\zeta - \arccos \sqrt{t})^2), \quad \text{as } \zeta \to \arccos \sqrt{t}.
\]

Note that all critical values are real and it is easy to check\footnote{If \( \cos \varphi = \sqrt{t} \) for \( \varphi \in (0, \frac{\pi}{2}) \), then \( 2 \arccos \sqrt{t} - 2\sqrt{t(1-t)} = 2\varphi - \sin(2\varphi) > 0 \).} that \( -\frac{\pi}{2} < \theta_{0,-1} < 0 < \theta_{0,+1} < \frac{\pi}{2} \). It follows that the principal branch \( \zeta = \zeta(t) \) which we already defined for sufficiently large \( \Im \theta \) can be analytically extended to the complex plane with infinitely many vertical slits at \( (\theta_{n,\varepsilon} - i\infty, \theta_{n,\varepsilon}) \), \( n \in \mathbb{Z}, \varepsilon \in \{\pm 1\} \). This simply connected domain contains the upper half-plane \( \mathbb{H} \). In the next lemma we analyze the behavior of the function \( \Im \zeta(t) \) for real \( \theta \). Since it is periodic with period \( \pi \), it suffices to consider the interval \([ -\frac{\pi}{2}, +\frac{\pi}{2} ] \).

**Lemma 4.2.** Let \( 0 < t < 1 \). Then, \( \Im \zeta(t) = 0 \) for \( \theta \in [-x_t, x_t] \) and \( \Im \zeta(t) > 0 \) for \( \theta \in [ -\frac{\pi}{2}, +\frac{\pi}{2} ] \) \([-x_t, x_t] \); see the left panel of Figure 4.

**Proof.** We have already seen in Lemma 4.1 that \( \zeta_t(0) = 0 \), while \( \Im \zeta_t(\pm \pi/2) = \tilde{y}_t(0) > 0 \). The function \( \zeta \mapsto \zeta - t \tan \zeta \) is an increasing diffeomorphism between the interval \([ -\arccos \sqrt{t}, +\arccos \sqrt{t} ] \) and its image \([ -x_t, +x_t ] \). Its derivative vanishes at \( \pm \arccos \sqrt{t} \); see the left panel of Figure 5. Given the known value \( \zeta_t(0) = 0 \), we conclude that the principal branch on the interval \([ -x_t, x_t ] \) coincides with the inverse of this diffeomorphism. In particular, \( \Im \zeta_t(\theta) = 0 \) for \( \theta \in [-x_t, x_t] \). Also, \( \zeta_t(\pm x_t) = \pm \arccos \sqrt{t} \). Let us see what happens if \( \theta \in (x_t, \frac{\pi}{2}] \). Although the equation \( \zeta - t \tan \zeta = \theta \) has infinitely many real solutions, see the left panel of Figure 5, these solutions belong to the branches which cannot attain the value \( \arccos \sqrt{t} \) as \( \theta \downarrow x_t \). Therefore, the principal branch cannot attain real values on \([ x_t, \frac{\pi}{2} ] \). From the known value \( \Im \zeta_t(\pm \pi/2) = \tilde{y}_t(0) > 0 \) we conclude that \( \Im \zeta_t(\theta) > 0 \) for all \( \theta \in (x_t, \frac{\pi}{2}] \). The analysis of the interval \([ -\frac{\pi}{2}, -x_t ] \) is similar. \( \square \)

**Remark 4.3.**
Consequently, the Puiseux expansion of the inverse function begins with
\[ \zeta_t(\theta) = \arccos \sqrt{t} - \sqrt{\frac{t}{1-t}}(x_t - \theta)^{1/2} + o((x_t - \theta)^{1/2}), \quad \text{as } \theta \to x_t. \]

The branch of the square root appearing here is defined on \(-\mathbb{H}\) and satisfies \(\sqrt{t} = +1\) (because \(\zeta_t(\theta)\) is real and increases for \(\theta \uparrow x_t\)) and, consequently, \(\sqrt{-1} = -i\).

**Case 2:** \(t > 1\). The critical points \(\zeta_{n, \varepsilon}\) and the corresponding critical values \(\theta_{n, \varepsilon}\) are given by
\[ \zeta_{n, \varepsilon} = -\varepsilon \arccos \sqrt{t} + \pi n, \quad \theta_{n, \varepsilon} = \varepsilon \left(\sqrt{t(t-1)} - \arccosh \sqrt{t}\right) + \pi n, \quad n \in \mathbb{Z}, \varepsilon \in \{\pm 1\}. \]

It is easy to check\(^6\) that \(\text{Im} \theta_{n+1} > 0\) while \(\text{Im} \theta_{n-1} < 0\). Although the critical values \(\theta_{n+1}, n \in \mathbb{Z}\), lie in the upper half-plane \(\mathbb{H}\), we can argue that none of the points \((\theta_{n+1}, \zeta_{n+1})\) belong to the principal sheet of the Riemann surface we are interested in. Let us to do it for \(n = 0\). We have already identified the values of \(\zeta_t(\theta)\) on the imaginary half-axis. In particular, we know that \(\zeta_t(\theta_{n+1}) = iy\) with some \(y > 0\). On the other hand, we have \(\text{Im} \zeta_{0+1} = -\arccos \sqrt{t} < 0\), meaning that \(\zeta_t(\theta_{n+1}) \neq \zeta_{0+1}\). Therefore, the principal sheet of the Riemann surface avoids the branching points \((\theta_{n+1}, \zeta_{n+1})\) and it follows that \(\zeta_t(\theta)\) admits an analytic continuation to the complex plane with infinitely many vertical slits at \((\theta_{n-1} - i\infty, \theta_{n-1}), n \in \mathbb{Z}\). Note that this domain contains \(\mathbb{H}\) and, in fact, it contains the larger half-plane
\[ \left\{ \theta \in \mathbb{C} : \text{Im} \theta > \arccosh \sqrt{t} - \sqrt{t(t-1)} \right\}. \]

**Lemma 4.4.** Let \(t > 1\). Then, \(\text{Im} \zeta_t(\theta) > 0\) for all \(\theta \in \mathbb{R}\); see the right panel of Figure 4.

**Proof.** We have already seen in Lemma 4.1 that \(\zeta_t(0) = iy_t(0)\) with \(y_t(0) > 0\). Taking the derivative we see that for \(t > 1\) the function \(\zeta \mapsto \zeta - t \tan \zeta\) is strictly decreasing on every interval \((-\frac{\pi}{2} + \pi n, \frac{\pi}{2} + \pi n), n \in \mathbb{Z}\); see the right panel of Figure 5. It follows that the equation \(\zeta - t \tan \zeta = \theta\) has infinitely many simple real solutions for all real \(\theta\). Every such solution determines a branch of the inverse function which is different from the principal branch because at \(\theta = 0\), these solutions are real, while at the same time \(\text{Im} \zeta_t(0) > 0\). It follows that the curve \(\zeta_t(\mathbb{R})\) cannot cross the real axis. Hence it stays in \(\mathbb{H}\). \(\square\)

**Case 3:** \(t = 1\). The critical points are at \(\pi n, n \in \mathbb{Z}\), and the corresponding critical values also equal \(\pi n\). The principal branch is defined on the complex plane with slits at \((\pi n - i\infty, \pi n), n \in \mathbb{Z}\). This domain contains the upper half-plane \(\mathbb{H}\). Defining \(\zeta_t(\pi n) = \pi n, n \in \mathbb{Z}\), yields a continuous extension of the principal branch to the closed upper half-plane \(\overline{\mathbb{H}}\).

**Lemma 4.5.** Let \(t = 1\). Then, \(\text{Im} \zeta_1(\theta) > 0\) for all \(\theta \in \mathbb{R}\setminus\pi \mathbb{Z}\) and \(\text{Im} \zeta_1(\pi n) = 0\) for all \(n \in \mathbb{Z}\); see the middle panel of Figure 4. We also have
\[ \text{Im} \zeta_1(\theta) \sim (\sqrt{3}/2) \cdot |3\theta|^{1/3}, \quad \text{as } \theta \to 0. \quad (4.7) \]

**Proof.** Let us analyze the behavior of the principal branch near 0. Since \(\zeta - \tan \zeta = -\frac{1}{3} \zeta^3 + O(\zeta^5)\) as \(\zeta \to 0\), the inverse function has a Puiseux expansion in powers of \(\theta^{1/3}\) as \(\theta \to 0\). In fact, we have
\[ \zeta_1(\theta) = (-3\theta)^{1/3} - \frac{2}{15} (-3\theta) + \frac{3}{175} (-3\theta)^{5/3} - \frac{2}{1575} (-3\theta)^{7/3} + \frac{16}{202125} (-3\theta)^3 + \ldots \quad (4.8) \]

Different choices of the cubic root correspond to three different branches of the inverse function. Let us identify the choice corresponding to the principal branch \(\zeta_1(\theta)\). Since \(\zeta \mapsto \zeta - \tan \zeta\) is decreasing

\(^6\)If \(\cosh \varphi = \sqrt{t}\) for \(\varphi > 0\), then \(2\sqrt{t(t-1)} - 2 \arccosh \sqrt{t} = \sinh(2\varphi) - 2\varphi > 0\).
from $+\infty$ to $-\infty$ on ($-\frac{\pi}{2}, +\frac{\pi}{2}$) and the derivative is strictly negative for $\zeta \neq 0$, see the middle panel
of Figure 5; there exist two real branches, the first one taking real values for $\theta > 0$, the second one
being real for $\theta < 0$. On the other hand, the principal branch $\zeta_1(\theta)$ is obtained by choosing
the cubic root $\sqrt[3]{\omega}$, $\omega \in \mathbb{R}$, such that $\sqrt[3]{-1} = i$ (to be conform with the fact that $\zeta_1(i\tau) = \sinh(\tau)$ with
$\sinh(\tau) > 0$ for $\tau > 0$; see Lemma 4.1). Consequently, $\sqrt[3]{-1} = e^{2\pi i/3}$ and $\sqrt[3]{1} = e^{\pi i/3}$ for the principal
branch and it follows from the above expansion that $\Im \zeta_1(\theta) > 0$ for real, sufficiently small $\theta$. Since
$\zeta_1(\theta)$ cannot become real for $\theta \in (-\pi, \pi)$ (because the real solution belongs to one of the two real
branches), we conclude that $\Im \zeta_1(\theta) > 0$ for all $\theta \in (-\pi, \pi)$.

Lemma 4.6. For every $t > 0$, $\zeta_t(\theta)$ is a Nevanlinna function, that is $\zeta_t(\mathbb{H}) \subset \mathbb{H}$, and, moreover,$\Im \zeta_t(\theta) > \Im \theta$ for all $\theta \in \mathbb{H}$.

Proof. From the above analysis it follows that, in all three cases, $\Im \zeta_t(\theta) \geq 0$ for $\theta \in \mathbb{R}$. The rest
can be derived from the maximum principle. Indeed, the function $\zeta_t(\theta) - \theta$ is periodic with period
$\pi$ and hence can be written as $\zeta_t(\theta) - \theta = p_t(e^{2\pi i t})$ for some function $p_t$ analytic on $\mathbb{D}$ and continuous
on $\overline{\mathbb{D}}$. Since $\Im p_t(z)$ is non-negative on $\mathbb{T}$, by the maximum principle we conclude that $\Im p_t(z) \geq 0$ for all $z \in \mathbb{D}$, which proves that $\Im \zeta_t(\theta) \geq \Im \theta > 0$ for all $\theta \in \mathbb{H}$. To prove the strict inequality,
observe that $\zeta_t(\theta) - \theta = t \tan \zeta_t(\theta)$ and the right hand-side has strictly positive imaginary part
for all $\theta \in \mathbb{H}$ since, as already shown, $\zeta_t(\theta) \in \mathbb{H}$ and the tangent function maps $\mathbb{H}$ to $\mathbb{H}$.

4.2. Principal branch as limit of iterated functions. The implicit equation we are interested
in can be written as $\zeta = \theta + t \tan \zeta$. Iterating it, we obtain

$$\zeta = \theta + t \tan \zeta = \theta + t \tan(\theta + t \tan \zeta) = \theta + t \tan(\theta + t \tan(\theta + t \tan \zeta)) = \ldots$$

In the next theorem we investigate whether iterations of this type converge.

Theorem 4.7. Fix $t > 0$. Take some $\theta \in \mathbb{C}$ with $\Im \theta \geq 0$ and define the function $f_\theta(x) := \theta + t \tan x$
for $x \in \mathbb{H}$. Then, the sequence $x, f_\theta(x), f_\theta(f_\theta(x)), \ldots, f_\theta^n(x), \ldots$ of iterations of $f_\theta$
converges to $\zeta_t(\theta)$ (which does not depend on $x$) locally uniformly in $x \in \mathbb{H}$.

Proof. The function $f_\theta$ is a holomorphic self-map of $\mathbb{H}$ (because so is $z \mapsto \tan z = i \frac{1 - e^{2\pi iz}}{1 + e^{2\pi iz}}$). At the
same time, $f_\theta$ is not an automorphism of $\mathbb{H}$ (and, in fact, not bijective) since $f_\theta(x + \pi) = f_\theta(x)$.
By the Denjoy-Wolff theorem [13] Theorems 5.3, 5.4, the iterations $f_\theta^n(x)$ converge, as $n \to \infty$, to some constant limit $c \in \mathbb{H}$ or $c \in \mathbb{R} \cup \{\infty\}$ locally uniformly in $x \in \mathbb{H}$. We need to show that $c = \zeta_t(\theta)$.

If $\theta \in \mathbb{H}$, then we already know from Section 4.1 that $\zeta_t(\theta) \in \mathbb{H}$ is a fixed point of $f_\theta$, which
implies that $c = \zeta_t(\theta)$. Note in passing that we have shown that, for $\theta \in \mathbb{H}$, $\zeta_t(\theta)$ is the unique
solution to $\zeta - t \tan \zeta = \theta$ satisfying $\zeta \in \mathbb{H}$.

Consider now the case$^7$ when $\theta \in \mathbb{R}$. Let first $c \in \mathbb{H}$. Then, $f_\theta(c) = c$ and, by the Schwarz
lemma, $|f'_\theta(c)| < 1$ (equality is not possible since $f_\theta$ is not an automorphism of $\mathbb{H}$). Hence, $c$ is a simple
solution of the equation $f_\theta(c) = c$. Let us look what happens to the solution if we perturb
$\theta$. By the inverse function theorem, there is an analytic function $c(y)$ defined on a small disc
$\{|y| < \varepsilon\}$ such that $c(0) = c$ and $f_{\theta + y}(c(y)) = c(y)$. If $y \in \mathbb{H}$, then $\theta + y \in \mathbb{H}$ and we know from Section 4.1 and
the above discussion that $\zeta_t(\theta + y)$ is the unique fixed point of $f_{\theta + y}$ in $\mathbb{H}$, implying that
$c(y) = \zeta_t(\theta + y)$. Letting $y \to 0$ (in $\mathbb{H}$), we obtain that $c = c(0) = \lim_{y \to 0, \Im y > 0} \zeta_t(\theta + y) = \zeta_t(\theta)$.

Let now $\theta \in \mathbb{R}$ and $c \in \mathbb{R} \cup \{\infty\}$. Let us first exclude the case $c = \infty$. For $c = \infty$, the Denjoy-Wolff
theorem states that $\Im f_\theta^n(x) \to +\infty$ as $n \to \infty$, but then $f_\theta(f_\theta^n(x)) \to \theta + i$ since $\tan x \to i$ as
$\Im x \to +\infty$, which is a contradiction. So, let $c \in \mathbb{R}$. By Wolff's theorem [13] Theorem 3.1, $f_\theta$

$^7$The real $\theta$ case is more difficult but interesting because when combined with Proposition 5.1 it yields a method
for numerical computation of the density of the free unitary Poisson distribution; see Figure 6.
leaves invariant disks that are contained in $\mathbb{H}$ and tangential to $\mathbb{R}$ at $c$. One consequence is that $c \in \frac{\pi}{2} + \pi \mathbb{Z}$ (otherwise $\tan c = +\infty$). Hence, $f_0$ is well-defined in a neighborhood of $c$ and $f_0(c) = c$. Another consequence is that $0 \leq f_0'(c) \leq 1$. Consider first the case when $0 \leq f_0'(c) < 1$. Then, $c$ is a simple solution of the equation $f_\theta(c) = c$. Let us look what happens if we perturb $\theta$. By the inverse function theorem, there is an analytic function $c(y)$ defined on a small disc $\{ |y| < \varepsilon \}$ such that $c(0) = c$ and $f_{\theta+i\delta}(c(y)) = c(y)$. Moreover, the Taylor expansion of $c(y)$ around $y = 0$ begins with $c(y) = c - y/(f_0'(c) - 1) + o(y)$. Since the coefficient of $y$ in this expansion is $> 0$, it follows that $\Im c(i\delta) > 0$ for sufficiently small $\delta > 0$. However, we already know that $f_{\theta+i\delta}$ has a unique fixed point $\zeta(\theta + i\delta)$ in $\mathbb{H}$. We conclude that $c(i\delta) = \zeta(\theta + i\delta)$ provided $\delta > 0$ is sufficiently small. It follows that $c = c(0) = \lim_{\delta \to 0} \zeta(\theta + i\delta) = \zeta(\theta)$.

Finally, let us consider the case when $\theta \in \mathbb{R}$, $c \in \mathbb{R}$, $f_\theta(c) = c$ and $f_\theta'(c) = 1$. The latter equation means that $\cos^2 c = t$. It follows that $t \leq 1$ and $c = \varepsilon \arccos \sqrt{t} + \pi n$ with some $n \in \mathbb{Z}$ and $\varepsilon \in \{ \pm 1 \}$. From the equation $f_\theta(c) = \theta + t \tan c = c$ we conclude that $\theta = \varepsilon (\arccos \sqrt{t} - \sqrt{t(1-t)}) + \pi n$. In the notation of Section 4.1 we have $c = \zeta_n, \varepsilon$ and $\theta = \theta_n, \varepsilon$. Hence, $c = \zeta(\theta)$ as we argued in Section 4.1. □

**Corollary 4.8.** Fix $t > 0$. For $\theta \in \mathbb{H}$, the equation $\zeta - t \tan \zeta = \theta$ has a unique solution $\zeta = \zeta(\theta)$ in $\mathbb{H}$ and infinitely many solutions in $\mathbb{C} \setminus \mathbb{H}$. Furthermore, the solution $\zeta = \zeta(\theta)$ is simple, i.e. its multiplicity is 1.

**Proof.** The uniqueness of the fixed point of the map $f_\theta : \mathbb{H} \to \mathbb{H}$ follows from the Denjoy-Wolff theorem [13, Theorem 5.3]. On the other hand, the fact that the entire function $\zeta \mapsto \zeta \cos \zeta - t \sin \zeta - \theta \cos \zeta$ has infinitely many complex zeroes follows from the Hadamard factorisation theorem because any entire function of order 1 with finitely many zeroes has a representation $e^{Q(\zeta)}$, where $Q$ is a polynomial, which can be ruled out by looking at the asymptotics when $\zeta \to \pm \infty$. The simplicity of the solution $\zeta = \zeta(\theta)$ follows from the description of critical points and critical values given in Section 4.1 above. In fact, we have seen that every solution of the system $\zeta - t \tan \zeta = \theta$, $\cos^2 \zeta = t$ satisfies $\zeta \notin \mathbb{H}$ or $\theta \notin \mathbb{H}$. □

If $\Im \theta > 0$, we can take $x := \theta$ in Theorem 4.7, implying that the sequence $\theta, \theta + t \tan \theta, \theta + t \tan(\theta + t \tan \theta), \ldots$ converges to $\zeta(\theta)$ for all $\theta \in \mathbb{H}$ and justifying the formula

$$
\zeta(\theta) = \theta + t \tan(\theta + t \tan(\theta + t \tan(\theta + \ldots))), \quad \theta \in \mathbb{H}.
$$

Note that similar representations of the Lambert $W$-function, for example the following one in terms of the infinite tetration

$$
z^{e^z} = \frac{W(-\log z)}{-\log z}, \quad e^{-e} \leq z \leq e^{1/e},
$$

are well-known and go back to Euler; see [52, p. 213] or [19, 45]. The next result states that the convergence in (4.9) holds locally uniformly on $\mathbb{H}$.

**Corollary 4.9.** Fix $t > 0$. Uniformly on compact subsets of $\mathbb{H}$, we have $f_\theta^{\circ n}(\theta) \to \zeta(\theta)$ as $n \to \infty$.

**Proof.** The pointwise convergence for all $\theta \in \mathbb{H}$ follows by taking $x = \theta$ in Theorem 4.7. To prove local uniformity, it suffices to check that the family of functions $\theta \mapsto f_\theta^{\circ n}(\theta)$, $n \in \mathbb{N}$, is precompact in $C(K)$ for every compact set $K \subset \mathbb{H}$. We have $\Im \theta \geq \varepsilon > 0$ for all $\theta \in K$. It follows that

$$
\Im f_\theta^{\circ (n+1)}(\theta) = \Im \theta + t \Im \tan f_\theta^{\circ n}(\theta) \geq \Im \theta \geq \varepsilon,
$$
for all $n \in \mathbb{N}$ and $\theta \in K$. Since $z \mapsto \tan z$ is bounded on $\{ z \in \mathbb{C} : \text{Im } z \geq \varepsilon \}$, it follows that $f_\theta^{(n+2)}(\theta)$ is bounded uniformly in $n \in \mathbb{N}$ and $\theta \in K$. By Montel’s theorem, the family of analytic functions $\theta \mapsto f_\theta^{(n)}(\theta)$, $n \geq 3$, is normal on $\mathbb{H}$ and the claim follows.

Remark 4.10. Let $\theta$ be real. Then, the equation $\zeta - t \tan \zeta = \theta$ has infinitely many real solutions; see Figure 3. In the upper half-plane $\mathbb{H}$, the number of solutions is 0 or 1 (since any such solution remains in $\mathbb{H}$ after a small perturbation of $\theta$ and we can apply Corollary 4.8). Similarly, the number of solutions in $-\mathbb{H}$ is 0 or 1. Knowing this and extending the analysis of Section 4.1, it should be possible to identify all branches of the multivalued function given by the implicit equation $\zeta - t \tan \zeta = \theta$ as it has been done for the Lambert $W$-function in [19, 52, pp. 39–70]. We refrain from doing this since we need only the principal branch here.

4.3. Main properties of the principal branch. In the next theorem we summarize the main properties of the principal branch $\zeta_t(\theta)$. It will be convenient to state the result in terms of the function $r_t(z)$ that already appeared in Section 4.1.

**Theorem 4.11.** Fix some $t > 0$. The principal branch $\zeta_t(\theta)$ constructed in Section 4.1 admits a representation $\zeta_t(\theta) = \theta + it \cdot r_t(e^{2i\theta})$, $\theta \in \mathbb{H}$, where $r_t(z)$ is a function which is analytic on the unit disk $\mathbb{D}$ and satisfies

$$z = e^{2t r_t(z)} \frac{1 - r_t(z)}{1 + r_t(z)}, \quad z \in \mathbb{D}. \quad (4.10)$$

The Taylor expansion of $r_t(z)$ is given by

$$r_t(z) = 1 + 2 \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{\ell} e^{-2\ell t} q_{\ell-1}(-4\ell t) z^\ell, \quad z \in \mathbb{D}, \quad (4.11)$$

where $q_m(x)$ is a degree $m$ polynomial in $x$ given by

$$q_m(x) = \sum_{j=0}^{m} \frac{x^j}{j!} \binom{m+1}{j+1}, \quad m \in \mathbb{N}_0. \quad (4.12)$$

For every $z \in \mathbb{D}$ we have $\text{Re } r_t(z) > 0$. On the interval $(-1,1)$, the function $r_t(z)$ takes real values, is strictly decreasing and satisfies $r_t(0) = 1$. Regarding its behavior near the boundary of $\mathbb{D}$, we have the following claims.

(i) If $t > 1$, then $r_t(z)$ admits an analytic continuation to the disk $\{|z| \leq e^{2\sqrt{t(1-t)} - 2\arccosh \sqrt{t}}\}$ whose radius is strictly greater than 1, and the Taylor series (4.11) converges locally uniformly there. In particular, it converges uniformly on $\mathbb{T}$. We have $\text{Re } r_t(z) > 0$ for all $z \in \mathbb{T}$.

(ii) Let $0 < t \leq 1$. Then, $r_t(z)$ can be extended to a continuous function on the closed unit disk $\overline{\mathbb{D}}$ and the series (4.11) converges to $r_t(z)$ uniformly on $\overline{\mathbb{D}}$, but not on a larger disk. Also, we have $r_t(1) = 0$.

(iii) Let $0 < t < 1$. Then, for every $z_0 \in \mathbb{T}\{e^{2ix}, e^{-2ix}\}$, where $x_0 := \arccos \sqrt{t} - \sqrt{t(1-t)} \in (0, \frac{\pi}{2})$, the function $r_t(z)$ can be extended analytically to a small disk centered at $z_0$. At $z_0 = e^{2ix}$ and $z_0 = e^{-2ix}$, the function $r_t(z)$ has square-root singularities and Puiseux expansions in powers of $(z - e^{2ix})^{1/2}$. For $\theta \in [-x_1, x_1]$ we have $\text{Re } r_t(e^{2i\theta}) = 0$, while for $\theta \in [-\frac{T}{2} + x_1, x_1]$ we have $\text{Re } r_t(e^{2i\theta}) > 0$. Also, we have $r_t'(1) = 1/(2t - 2) = -r_t''(1)$.

(iv) Let $t = 1$. Then, for every $z_0 \in \mathbb{T}\{1\}$ the function $r_t(z)$ can be extended analytically to a small disk centered at $z_0$. At $z_0 = 1$, the function $r_t(z)$ has a cubic-root singularity and a
Poisson expansion \( r_1(z) = (\frac{3}{2}(1-z))^{1/3} + \ldots \) in powers of \((1-z)^{1/3}\). We have \( \text{Re} r_1(z) > 0 \) for all \( z \in \mathbb{T}\{1\} \) and \( r_1(1) = 0 \).

Proof. The existence of \( r_t(z) \), its analyticity on \( \mathbb{D} \) and \((4.10)\) follow from the analysis of Section \[4.1\] see, in particular, \((4.3)\). To prove \((4.11)\), let us write \( r_t(z) = 1 + p_t(z) \), where \( p(z) = p_t(z) \) is a series in \( z \) of the form \( p(z) = a_1z + a_2z^2 + \ldots \) solving

\[
  z = e^{2t(1+p)} \frac{-p}{2 + p} = \frac{p}{\phi(p)} \quad \text{with} \quad \phi(p) := -(2 + p)e^{-2t(1+p)}.
\]

The Lagrange inversion formula \[18, \text{p. 148, Theorem A}\] gives

\[
  [z^\ell]p(z) = \frac{1}{\ell} [p^{\ell-1}](\phi(p)^\ell) = \frac{(-1)^\ell}{\ell} e^{-2\ell t} [p^{\ell-1}](2 + p)e^{-2\ell tp} = \frac{(-1)^\ell}{\ell} e^{-2\ell t} \sum_{j=0}^{\ell-1} (-2\ell t)^j j! (j + 1)^2 \sum_{j=0}^{\ell} \frac{(-1)^{\ell-j}}{\ell} e^{-2\ell t} q_{\ell-1}(-4\ell t)
\]

with \( q_{\ell-1} \) defined as in \((4.12)\). This identifies the Taylor series of \( p_t \) (which converges on \( \mathbb{D} \)) and proves \((4.11)\). Properties (i), (iii), (iv) follow from the analysis of Section \[4.1\]. To prove (ii) observe that the existence of the continuous continuation of \( r_t \) to \( \overline{\mathbb{T}} \) follows from Section \[4.1\]. Moreover, the description of non-differentiability points of \( r_t(z) \) on \( \mathbb{T} \) implies that the function \( r_t(z) \) is \( \alpha \)-Hölder continuous with some \( \alpha > 0 \) on \( \mathbb{T} \). Hence, its Fourier series converges to \( r_t(z) \) uniformly by the Dini-Lipschitz condition; see \[39, \text{p.22, Corollary II}\] for a result on the speed of convergence. The maximum principle implies uniform convergence of \((4.11)\) on \( \mathbb{D} \).

Lemma 4.12. For \( 0 < t \leq 1 \), the unique zero of the function \( r_t(z) \) on \( \mathbb{T} \) is at \( z = 1 \). For \( t > 1 \), this function has no zeroes on \( \mathbb{T} \).

Proof. For \( 0 < t \leq 1 \) we know from Lemma \[4.1\] that \( \zeta_t(0) = 0 \), hence \( r_t(1) = 0 \). For \( t > 1 \), we know that \( \text{Im} \zeta_t(0) = 0 \), hence \( r_t(1) \neq 0 \). On the other hand, for every \( t > 0 \), if \( r_t(e^{2i\theta}) = 0 \) for some \( \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \), then \( \zeta_t(\theta) = \theta \), which implies that \( \tan \zeta_t(\theta) = 0 \) and hence \( \zeta_t(\theta) = \theta = 0 \).

Remark 4.13. Another natural function appearing in connection with \( \zeta_t(\theta) \) is the function \( v_t : \mathbb{D} \rightarrow \mathbb{D} \) defined by \( v_t(e^{2i\theta}) = e^{2i\zeta_t(\theta)} \) for \( \theta \in \mathbb{H} \). For every \( z \in \mathbb{D} \), \( v = v_t(z) \) is the unique solution of the equation

\[
  ve^{2t(1+v)} = z
\]

in the unit disk \( \mathbb{D} \). This follows from Corollary \[4.8\] after mapping \( \mathbb{H} \) to \( \mathbb{D} \) via \( \theta \rightarrow e^{2i\theta} \). The functions \( r_t(z) \) and \( v_t(z) \) are related by

\[
  \frac{1 - r_t(z)}{1 + r_t(z)} = v_t(z) \quad \text{and} \quad \frac{1 - v_t(z)}{1 + v_t(z)} = r_t(z), \quad z \in \mathbb{D}.
\]

Remark 4.14. It is well-known \[19, 52, \text{Section 1.1.3}\] that the equation \( e^x = a + bx \) can be solved in terms of the Lambert \( W \)-function. On the other hand, the equation \( e^{w} = a + \frac{b}{w} \) can be reduced to the equation \( \zeta - t \tan \zeta = \theta \) studied above. Indeed, it is elementary to check that \( \zeta \) solves \( \zeta - t \tan \zeta = \theta \) if and only if \( w := 2i(\zeta - \theta + it) \) solves \( e^{w} = a(1 + \frac{it}{w}) \) with \( a := -e^{-2t - 2i\theta} \). Let us also mention that the \( r \)-Lambert function \( W_r \), extensively studied in \[52, \text{Chapter 6}\], is defined as a solution \( z = W_r(b) \) of the implicit equation \( ze^{z} + rz = b \). It follows that \( \zeta_t \) is one of the branches of \( W_{-a}(4ta) \). The branch structure of \( W_r \), for real \( r_t \), is known \[52, \text{Chapter 6}\], but note that we need \( r = -a = e^{-2t - 2i\theta} \).
5. Properties of the free unitary Poisson distribution

5.1. Formulas for the density. In this section we shall derive some properties of the free unitary Poisson distribution $\Pi_t$ with parameter $t > 0$. Recall from Lemma 3.5 that the $\psi$-transform of $\Pi_t$ is given by

$$\psi_{\Pi_t}(e^{i\theta}) := \sum_{\ell=1}^{\infty} e^{i\ell\theta} \int_{T} u^\ell \Pi_t(du) = \frac{it}{2 \zeta_t(\theta/2) - \theta} - \frac{1}{2} = \frac{1}{2} r_t(e^{i\theta}) - \frac{1}{2}, \quad \theta \in \mathbb{H},$$

where $r_t(z)$ is the function whose properties are listed in Theorem 4.11. The next result describes the atoms of $\Pi_t$ and provides a formula for the density of the absolutely continuous part; see Figure 6.

**Proposition 5.1.** For $t > 1$ the distribution $\Pi_t$ is absolutely continuous w.r.t. the length measure on $T$ and its density is real-analytic, strictly positive and given by

$$f_{\Pi_t}(z) = \frac{1}{2\pi} \text{Re} \left( \frac{1}{r_t(z)} \right) = -\frac{1}{2\pi} \text{Im} \cot \zeta_t(\theta/2), \quad z = e^{i\theta} \in T.$$

For $0 < t \leq 1$, the distribution $\Pi_t$ is a sum of an atom at 1 with weight $1 - t$ and an absolutely continuous part $\Pi^*_t := \Pi_t - (1-t)\delta_1$ whose density w.r.t. the length measure on $T$ is given by

$$f_{\Pi^*_t}(z) = \frac{1}{2\pi} \text{Re} \left( \frac{1}{r_t(z)} \right) = -\frac{1}{2\pi} \text{Im} \cot \zeta_t(\theta/2), \quad z = e^{i\theta} \in \mathbb{T}\backslash\{1\}, \quad f_{\Pi^*_t}(1) = \begin{cases} 0, & \text{if } 0 < t < 1, \\ +\infty, & \text{if } t = 1. \end{cases}$$

For $0 < t < 1$, the function $\theta \mapsto f_{\Pi^*_t}(e^{i\theta})$ is continuous on $[-\pi, \pi]$, vanishes on the interval $[-2x_t, 2x_t]$, is real analytic and strictly positive outside this interval, and has square-root singularities at the end-points of the interval, where

$$x_t = \arccos \sqrt{t - \sqrt{t(1-t)}} = \frac{\pi}{2} - \int_0^t \sqrt{\frac{1-u}{u}} \, du > 0.$$  

(5.2)
For $t = 1$, the function $\theta \mapsto f_{\Pi_1}(e^{i\theta})$ is real-analytic and strictly positive on $[-\pi, \pi] \backslash \{0\}$, while $\theta = 0$ is a singularity with

$$f_{\Pi_1}(e^{i\theta}) \sim \frac{1}{2\pi} \cdot \left(\frac{\sqrt{3}}{4|\theta|}\right)^{1/3}, \quad \text{as } \theta \to 0. \tag{5.3}$$

All claims follow from the classical properties of the Poisson integral [69 Chapter 11], [27 Chapter 1], [36 Chapter 3] combined with Theorem 4.11. Recall that the Poisson integral of a finite, complex-valued measure $\mu$ on $T$ is defined by

$$F_{\mu}(z) := \frac{1}{2\pi} \Re \int_{T} \frac{u + z}{u - z} \mu(du) = \frac{1}{2\pi} \Re \int_{T} \frac{1 + z\bar{u}}{1 - z\bar{u}} \mu(du), \quad z \in \mathbb{D}.$$  

It is well-known [36 p. 33] that $\mu$ can be recovered from its Poisson integral as the weak limit, as $r \uparrow 1$, of the measures with density $z \mapsto F(rz)$, $z \in T$, w.r.t. the length measure on $T$.

**Lemma 5.2.** If the Poisson integral of some finite complex-valued measure $\mu$ on $T$ can be extended to a continuous function on $\hat{T}$, then $\mu$ is absolutely continuous w.r.t. the length measure on $T$ and its density is given by the restriction of the Poisson integral to $T$.

**Proof.** Let $\mu_1$ be the complex-valued measure with density $F_{\mu}(z)$ on $T$. The Poisson integral of $\mu_1$ coincides with $F_{\mu}(z)$ on $\mathbb{D}$; see [69 Theorem 11.9]. Hence the Poisson integral of $\mu - \mu_1$ vanishes on $\mathbb{D}$ and it follows that $\mu = \mu_1$. \hfill $\square$

**Proof of Proposition 5.1.** In view of the invariance of $\Pi_t$ w.r.t. the complex conjugation, the Poisson integral of $\Pi_t$ takes the form

$$F_{\Pi_t}(z) = \frac{1}{2\pi} \Re \int_{T} \frac{1 + z\bar{u}}{1 - z\bar{u}} \Pi_t(du) = \frac{1}{2\pi} \Re (1 + 2\psi_{\Pi_t}(z)) = \frac{1}{2\pi} \Re \left(\frac{1}{r_t(z)}\right), \quad z \in \mathbb{D}.$$  

For $t > 1$ the function $r_t(z)$ is analytic on a disk of radius $1$ and satisfies $\Re r_t(z) > 0$ for $z \in T$ (in particular, it has no zeroes on $T$); see Theorem 4.11 and Lemma 4.12. Hence, $z \mapsto \frac{1}{2\pi} \Re (1/r_t(z)) > 0$ is a real-analytic function on $T$ and the claim follows from Lemma 5.2.

For $0 < t \leq 1$, the function $r_t(z)$ admits a continuous extension to $\mathbb{D}$ and its unique zero on $T$ is at $z = 1$; see Lemma 4.12. Let first $0 < t < 1$. Then, the function $r_t(z)$ can be analytically continued to a small disk around $1$ and $r_t(1) = -r_t^\prime(1) = 1/(2t - 2)$; see Theorem 4.11. It follows that

$$z \mapsto \frac{1}{2\pi} \Re \left(\frac{1}{r_t(z)} - (1 - t) \frac{1 + z}{1 - z}\right), \quad z \in \mathbb{D}\backslash\{1\}$$

becomes a continuous function on $\hat{T}$ if we define its value at $z = 1$ to be $0$. The Poisson integral of the measure $\Pi_t^* = \Pi_t - (1 - t)\delta_1$ is given by

$$F_{\Pi_t^*}(z) = \frac{1}{2\pi} \Re \left(\frac{1}{r_t(z)} - (1 - t) \frac{1 + z}{1 - z}\right), \quad z \in \mathbb{D}.$$  

By Lemma 5.2 the density of the measure $\Pi_t^*$ is given by the restriction of $F_{\Pi_t^*}(z)$ to $T$ which can be simplified by dropping the $\frac{1 + z}{1 - z}$-term because its real part vanishes for $z \in T$. The remaining claims follow from the properties of the function $r_t(z)$ listed in Theorem 4.11.

For $t = 1$, the function $r_1(z)$ admits a continuous extension to $\mathbb{D}$ with the unique zero on $T$ being at $z = 1$, but this time $z = 1$ is a branch point where we have $r_1(z) \sim (\frac{3}{2}(1 - z))^{1/3}$ as $z \to 1$, $z \in \mathbb{D}$. Although the function $z \mapsto \frac{1}{2\pi} \Re (1/r_1(z))$ is not continuous on $T$, it belongs to $L^p$ and is the a.e. limit of the functions $z \mapsto F_{\Pi_t}(rz)$ as $r \uparrow 1$ whose $L^p$-norm stays bounded as $r \uparrow 1$, for all $1 \leq p < 3$. A uniform integrability argument implies that the weak limit of measures with densities
For every $p$ where $p_1(t) = 1, p_2(t) = 1 + 4t, \ldots$ are certain polynomials that are related to the moments (or rather Fourier coefficients) of $\Pi_t$ by $e^{-2\ell t} p_\ell(t) = \int_T w^\ell \Pi_t(du)$; see (1.5). The next proposition provides an explicit formula for $p_\ell(t)$.

**Proposition 5.3.** For every $t > 0$ the moments of $\Pi_t$ are given by

$$\int_T u^\ell \Pi_t(du) = \int_T u^{-\ell} \Pi_t(du) = e^{-2\ell t} p_\ell(t), \quad \ell \in \mathbb{N}_0,$$

where

$$p_\ell(t) = 1 + \frac{1}{2} \sum_{a, b, c \in \mathbb{N}_0, a + b + 1 \in \mathbb{N}_0} \left( \frac{\ell - 1}{a, b, c} \right) \frac{(2\ell t)^{a+b+1}(-1)^b}{(a+b)(a+b+1)}.$$

**Proof.** Writing $w = w(y) := \psi_{\Pi_t}(y)$ it follows from (1.6) and (1.8) that

$$y = \frac{w}{1 + w} \exp \left( \frac{t}{w + \frac{1}{2}} \right) = \frac{w}{\phi(w)} \quad \text{with} \quad \phi(w) = (1 + w) \exp \left( -\frac{2t}{2w + 1} \right).$$

The Lagrange inversion formula [18, p. 148, Theorem A] yields

$$[y^\ell] w(y) = \frac{1}{\ell} [w^{\ell-1}]((\phi(w))^\ell) = \frac{1}{\ell} [w^{\ell-1}][1 + w)^\ell e^{-2\ell t/(2w + 1)}] = \frac{e^{-2\ell t}}{\ell} [w^{\ell-1}][1 + w)^\ell e^{-2\ell t/(2w + 1)}],$$

for all $\ell \in \mathbb{N}$. Now we can use the expansion

$$a_{r\ell} = 1 + \sum_{m=1}^\infty \sum_{j=1}^m (-1)^{j-m} m \frac{m!}{j!} L(m, j)$$

which follows from the generating function of the Lah numbers $L(n, k) := (n-k)! k!$:

$$\sum_{m=j}^\infty L(m, j) \frac{x^m}{m!} = \frac{1}{j!} \left( \frac{x}{1-x} \right)^j, \quad j \in \mathbb{N}.$$
After some lengthy but elementary transformations, we arrive at
\[
[y^\ell] w(y) = e^{-2\ell t} + \frac{1}{2} e^{-2\ell t} \sum_{a,b,c,s \in \mathbb{N}_0, a \neq 0} \binom{\ell-1}{a,b,c} \frac{(2\ell)^a 2^{a+b+1}(-1)^b}{(a-1)!(a+b)(a+b+1)},
\]
and the proof is complete. \( \square \)

**Proposition 5.4.** For \( t > 1 \), the density of \( \Pi_t \) w.r.t. the length measure on \( \mathbb{T} \) is given by the uniformly convergent series
\[
f_{\Pi_t}(e^{i\theta}) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{\ell = 1}^{\infty} e^{-2\ell t} p_{\ell}(t) \cos(\ell \theta), \quad \theta \in [-\pi, \pi].
\]
(5.6)
For \( 0 < t < 1 \), the density of \( \Pi_t^* = \Pi_t - (1 - t)\delta_1 \) is given by the uniformly convergent series
\[
f_{\Pi_t^*}(e^{i\theta}) = \frac{t}{2\pi} + \frac{1}{\pi} \sum_{\ell = 1}^{\infty} (e^{-2\ell t} p_{\ell}(t) - 1 + t) \cos(\ell \theta), \quad \theta \in [-\pi, \pi].
\]
(5.7)
Finally, for \( t = 1 \), the density of \( \Pi_1 = \Pi_1^* \) is given by the series (5.6) which converges pointwise for \( \theta \in [-\pi, \pi] \backslash \{0\} \) and in \( L^p \) for \( 1 \leq p < 3 \).

**Proof.** By Proposition 5.3, the Fourier coefficients of \( f_{\Pi_t}(e^{i\theta}) \), respectively \( f_{\Pi_t^*}(e^{i\theta}) \), are given by
\[
\int_{-\pi}^{\pi} e^{i\ell \theta} f_{\Pi_t}(e^{i\theta}) d\theta = e^{-2|\ell| t} p_{\ell}(t), \quad \int_{-\pi}^{\pi} e^{i\ell \theta} f_{\Pi_t^*}(e^{i\theta}) d\theta = e^{-2|\ell| t} p_{\ell}(t) - (1 - t), \quad \ell \in \mathbb{Z}.
\]
The series in (5.6) and (5.7) are the Fourier series of the respective densities and their claimed convergence modes follow from the regularity properties of these densities established in the proof of Proposition 5.1. In particular, for \( t = 1 \) the density belongs to \( L^p \) for \( 1 \leq p < 3 \), see (5.3), and the \( L^p \)-convergence of its Fourier series follows from [43, p. 59], while the pointwise convergence follows from the local differentiability of the density at any \( \theta \in [-\pi, \pi] \backslash \{0\} \). \( \square \)

### 5.3. Formal solution to Steinerberger’s PDE
Let us finally comment on the solutions to Steinerberger’s PDE (1.2) with periodic initial condition. Let \( (T_{2\ell}t(\theta))_{\ell \in \mathbb{N}} \) be a sequence of real-rooted trigonometric polynomials such that \( \nu[\mathbb{T}_{2\ell}] \) converges weakly to some probability measure \( \nu = \nu_0 \) on the unit circle \( \mathbb{T} \) and assume that the density of \( \nu_0 \) w.r.t. the length measure on \( \mathbb{T} \) is \( e^{ix} \mapsto u_0(x) \), where \( u : \mathbb{R} \to [0, \infty) \) is some \( 2\pi \)-periodic function. Theorem 1.2 states that the empirical distribution of zeroes of the \( [2\ell] \)-th derivative of \( T_{2\ell}t \) converges to \( \nu_t := \nu_0 \otimes \Pi_t \). On the other hand, Kiselev and Tan [14] proved that (under certain regularity assumptions), the density of \( \nu_t \), which we write as \( e^{ix} \mapsto u_t(x) \), solves the PDE (1.2). With the notation \( m_\ell := \int_{-\pi}^{\pi} e^{i\ell x} u_0(x) dx \), \( \ell \in \mathbb{Z} \), the \( \psi \)-transform of \( \nu_0 \) is \( \psi_0(z) = \sum_{\ell \in \mathbb{Z}} m_\ell z^\ell \). Assuming that \( m_1 \neq 0 \) we can invert this series to compute the \( S \)-transform of \( \nu_0 \) via (1.6). Then, we can compute the \( S \)-transform of \( \nu_t = \nu_0 \otimes \Pi_t \) using (1.7) and (1.8). Using inversion we can compute the \( \psi \)-transform \( \psi_t \) of \( \nu_t \). This results in the following formula:
\[
\psi_t(z) = e^{-2t} m_1 z + e^{-4t} (4 t m_1^2 + m_2) z^2 + e^{-6t} (-8 t m_1^3 + 24 t^2 m_1^3 + 12 t m_1 m_2 + m_3) z^3 + \ldots.
\]
The formal solution to the PDE (1.2) is thus
\[
u_t(x) = \frac{1}{2\pi} \text{Re} \left( 1 + 2 \psi_t(e^{-ix}) \right) = \frac{1}{2\pi} \left( 1 + e^{-2t} (m_1 e^{-ix} + m_2 e^{ix} + \ldots) \right), \quad t \geq 0, \ x \in \mathbb{R}.
\]
It is interesting that only integer powers of \( e^{-2t} \) appear in the above series. Kiselev and Tan [14] have shown that as \( t \to \infty \), \( u_t(x) \) converges to the steady-state solution \( 1/(2\pi) \) exponentially fast. The above formula yields the precise exponential rate of convergence.
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