Braid groups and quiver mutation

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Abstract

We describe presentations of braid groups of type $ADE$ and show how these presentations are compatible with mutation of quivers, building on work of Barot and Marsh for Coxeter groups. In types $A$ and $D$ these presentations can be understood geometrically using triangulated surfaces. We then give a categorical interpretation of the presentations, with the new generators acting as spherical twists at simple modules on derived categories of Ginzburg dg-algebras of quivers with potential.

Keywords: mutation, braid groups, cluster algebras, Ginzburg dg algebra, spherical twist

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1 Introduction

Let $\Delta$ be a Dynkin diagram of simply-laced type. Then a special case of result in [BM] gives a presentation of the Weyl group of type $\Delta$ for each quiver $Q$ mutation equivalent (in the sense of cluster algebras) to an orientation of $\Delta$. Here we give a presentation of the corresponding Artin braid group for each such quiver. In type $A$, these presentations coincide with some of the presentations given in [Ser]. In types $A$ and $D$ we give a topological interpretation of the generators, as braids on a disk in the former case and braids on a disk with a single cone point of degree two (i.e. an orbifold), in the latter case. In the first case this is related to the approach in [Ser], and in the second to...
the braid pictures of Allcock \[\text{[A]l}\], as well as to the surface realisations of the corresponding cluster algebras \[\text{[FoSTh]}\].

The construction of the presentation is via mutation, and employs a conjugation formula giving the new generators in terms of the old ones which specialises to the formula used in \[\text{[BM]}\] when passing to the Weyl group. In addition, the presentations appearing here pass to presentations of the Weyl group, and we explain how they are related to the presentations in \[\text{[BM]}\].

It is well known that the Artin braid group acts on certain derived categories via spherical twists, with the usual generators corresponding to twists at simple or projective objects \[\text{[ST]} \text{[RZ]}\]. We interpret the new generators of the braid groups as spherical twists associated to simple modules for Ginzburg dg-algebras associated to quivers with potential \[\text{[Gin]}\].

A potential for a quiver is a linear combination of oriented cycles in the quiver. Derksen, Weyman, and Zelevinsky showed how quiver mutation extends to mutation of quivers with potential \[\text{[DWZ]}\], and this was interpreted in terms of triangulations of surfaces by Labardini-Fragoso \[\text{[LF09]}\]. By applying an appropriate right equivalence to a result from \[\text{[LF12]}\], we have that repeatedly mutating a type A or D Dynkin quiver gives, up to right equivalence, the obvious potential: a sum of one copy of each chordless oriented cycle.

Associated to each quiver with potential is a differential graded algebra, known as the Ginzburg dg-algebra \[\text{[Gin]}\], and Keller and Yang showed that mutation of quivers with potential induces equivalences between the derived categories of modules over the corresponding Ginzburg dg-algebras \[\text{[KY]}\]. These derived categories are Calabi-Yau \[\text{[KVdB]}\], and the simple modules for the dg-algebras are spherical, so we have spherical twists which are natural candidates for the action of the new braid group generators.

A large part of Section 4 consists of setting up notation, recalling results about differential graded algebras and derived categories from papers of Keller, explaining basic facts about homological algebra which we will use, and checking some facts about spherical twists hold in the differential graded setting. Then, using Keller and Yang’s description of the images of simple modules under mutation, and our results on mutation of braid groups, we show that the Artin braid groups do act on the derived category of the Ginzburg dg-algebra with the new generators acting as the spherical twists at the simple modules.

Since we started work on this project, we have become aware of independent work by other authors. A. King and Y. Qiu have a related project, and were aware of the new relations between spherical twists and a topological interpretation of the spherical twist group; see \[\text{[Qiu]}\], particularly Appendix A. In particular, an independent proof of a version of Theorem 2.10 in types A and D was announced in \[\text{[Qiu]}\]. A key difference in our approach is the use of an orbifold with cone point of degree two in type D. In \[\text{[Nag]} \text{[§2.2]}\], K. Nagao refers to an action of the mapping class group of a marked surface on the derived category of a Ginzburg dg-algebra associated to a triangulation.

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2 Presentations of braid groups

2.1 Braid groups

Let $\Delta$ be a graph of ADE Dynkin type, i.e., $\Delta$ is a graph of type $A_n$ for $n \geq 1$, $D_n$ for $n \geq 4$, $E_6$, $E_7$, or $E_8$.

**Type $A_n$:**

$\bullet$ $1$ $\bullet$ $2$ $\bullet$ $3$ $\cdots$ $n - 1$ $\bullet$ $n$

**Type $D_n$:**

$\bullet$ $1$ $\bullet$ $2$ $\rightarrow$ $\bullet$ $3$ $\bullet$ $4$ $\cdots$ $n - 1$ $\bullet$ $n$

In particular, $\Delta$ has no double edges or cycles. Let $I$ be the set of vertices of $\Delta$. We can associate a group $B_\Delta$ to $\Delta$, which we call the braid group of $\Delta$. It has a distinguished set of generators $S_\Delta = \{s_i\}_{i \in I}$, and the relations depend on whether or not two vertices are connected by an edge. They are as follows:

(i) $s_is_j = s_js_i$ if $\bullet$ $i$ $\bullet$ $j$ ;

(ii) $s_is_js_i = s_js_is_j$ if $\bullet$ $i$ $\rightarrow$ $\bullet$ $j$ ;

If $\Delta$ is of type $A_n$ then we recover the “usual” braid group, sometimes denoted $B_{n+1}$. Its generators can be visualized as follows:

$s_i = \begin{array}{ccccccc}
1 & 2 & \cdots & i & i + 1 & n & n + 1 \\
1 & 2 & \cdots & i & i + 1 & n & n + 1 \\
\end{array}$

and the relations of type (i) record the fact that crossings of far-apart adjacent pairs of strings commute, while relations of type (ii) record a Reidemeister 3 move.

If we also impose the relation that $s_i^2 = 1$ for all $i \in I$ then we recover the Coxeter group of type $\Delta$. More information on Coxeter groups and braid groups can be found in [Hum] and [RT].

2.2 Mutation of quivers

A quiver is just a directed graph. Throughout this article we will only work with quivers with finitely many vertices and finitely many arrows that have no loops or oriented 2-cycles. For a given quiver $Q$, we again denote its set of vertices by $I$.

There is a procedure to obtain one quiver from another, called quiver mutation, due to Fomin and Zelevinsky [FZ1] §4. Fix $Q$ and let $k \in I$. Then we obtain the mutated quiver $\mu_k(Q)$ as follows:
(i) for each pair of arrows \( i \to k \to j \) through \( k \), add a formal composite \( i \to j \);
(ii) reverse the orientation of all arrows incident with \( k \);
(iii) remove a maximal set of 2-cycles (we may have created 2-cycles in the previous two steps).

It is a basic but important observation that quiver mutation does not change the set of vertices. One can also check that mutation is an involution.

We call a cycle in an unoriented graph (or in the underlying unoriented graph of a quiver) chordless if the full subgraph on the vertices of the cycle contains no edges which are not part of the cycle. We call a quiver Dynkin if its underlying unoriented graph is a Dynkin graph of type \( ADE \), and mutation Dynkin if it can be obtained by mutating a Dynkin quiver finitely many times. By a theorem of Fomin and Zelevinsky [FZ2, Thm. 1.4], there are only finitely many quivers that can be obtained by mutating a given Dynkin quiver.

The following fact will be useful to us.

**Proposition 2.1** (Fomin-Zelevinsky). *In any mutation Dynkin quiver, there are no double arrows and all chordless cycles are oriented.*

**Proof:** By [FZ2, Theorem 1.8], the entries in the corresponding exchange matrix \( B \) satisfy \(|B_{xy}B_{yx}| \leq 3\) for all \( x, y \) (known as being 2-finite). Hence there cannot be any double arrows in the quiver.

Now let \( Q \) be a mutation Dynkin quiver and \( C \) a chordless cycle in \( Q \). Then, since \( Q \) is 2-finite, so is \( C \). By [FZ2, Proposition 9.7], \( C \) must be an oriented cycle. \(\square\)

### 2.3 Groups from quivers

Let \( Q \) be a mutation Dynkin quiver.

**Definition 2.2.** From the quiver \( Q \) with vertex set \( I \), we define the group \( B_Q \) as follows: it has a distinguished generating set \( S_Q = \{s_i\}_{i \in I} \) and the relations given by:

(i) \( s_is_j = s_js_i \) if \( i \xrightarrow{\cdot} j \);
(ii) \( s_is_js_i = s_js_is_j \) if \( i \xrightarrow{\cdot \cdot} j \) or \( i \xleftarrow{\cdot \cdot} j \);
(iii) if we have an oriented chordless \( n \)-cycle

\[
\begin{array}{c}
1 \\
\downarrow \\
2 \\
\downarrow \\
n \\
\cdots
\end{array}
\]

then

\[
s_1s_2 \cdots s_ns_1s_2 \cdots s_{n-2} = s_2s_3 \cdots s_ns_1 \cdots s_{n-1}
\]

\[
= \cdots
\]

\[
= s_n s_1 s_2 \cdots s_n s_1 s_2 \cdots s_{n-3}
\]
Multiplying out the Barot-Marsh relation, we see it is equal to

\[ s \]

Using that Lemma 2.5.

groups defined by Barot and Marsh [BM]:

Though the relations look different, by taking an appropriate quotient we can define the Coxeter and the result follows.

So, we assume that (1) holds. Then we have:

\[ s_1 s_2 \cdots s_n s_1 s_2 \cdots s_{n-2} = s_2 s_3 \cdots s_n s_1 s_2 \cdots s_{n-1} \quad (1) \]

holds then

\[ s_1 s_2 \cdots s_n s_1 s_2 \cdots s_{n-2} = s_2 s_3 \cdots s_n s_1 s_2 \cdots s_n. \]

So, we assume that (1) holds. Then we have:

\[ s_2^{-1} s_1 s_2 \cdots s_n s_1 s_2 \cdots s_{n-2} s_n = s_3 \cdots s_n s_1 s_2 \cdots s_{n-1} s_n. \]

The left hand side can be rewritten, using relations of type (i) and (ii), as:

\[
\begin{align*}
    s_2^{-1} s_1 s_2 \cdots s_n s_1 s_2 \cdots s_{n-2} s_n &= s_1 s_2 s_3^{-1} s_2 \cdots s_n s_1 s_2 \cdots s_{n-2} s_n \\
    &= s_1 s_2 s_3 \cdots s_n s_1 s_2 \cdots s_{n-2} s_n \\
    &= s_2 \cdots s_n s_1 s_2 \cdots s_{n-2} s_n \\
    &= s_2 \cdots s_n s_1 s_2 \cdots s_{n-2}.
\end{align*}
\]

and the result follows. \(\Box\)

Though the relations look different, by taking an appropriate quotient we can define the Coxeter groups defined by Barot and Marsh [BM]:

**Lemma 2.4.** For each single chordless \(n\)-cycle, in the presence of the relations of type [(i)] and [(ii)], any one of the relations of type [(iii)] implies all the others.

**Proof:** It is enough to show that if the relation

\[ s_1 s_2 \cdots s_n s_1 s_2 \cdots s_{n-2} = s_2 s_3 \cdots s_n s_1 s_2 \cdots s_{n-1} \quad (1) \]

holds then

\[ s_1 s_2 \cdots s_n s_1 s_2 \cdots s_{n-2} = s_2 s_3 \cdots s_n s_1 s_2 \cdots s_n. \]

We need to show that, in the presence of relations [(i)] [(ii)] and \( s_i^2 = 1 \) for all \( i \in I \), our extra relation [(iii)] holds if and only if the relation

\[ (s_1 s_2 \cdots s_{n-1} s_n s_{n-1} \cdots s_2)^2 = 1 \]

and its rotations hold for each \( n \)-cycle \( 1 \to 2 \to \cdots \to n \to 1 \). By symmetry, it is enough to check that the relation above is equivalent to \( s_1 s_2 \cdots s_n s_1 s_2 \cdots s_{n-2} = s_2 s_3 \cdots s_n s_1 s_2 \cdots s_{n-1} \).

Using that \( s_i = s_i^{-1} \), we see that our our relation is equivalent to

\[ s_1 s_2 \cdots s_n s_1 s_2 \cdots s_{n-2} s_{n-1} s_{n-2} \cdots s_1 s_n \cdots s_3 s_2 = 1. \]

Multiplying out the Barot-Marsh relation, we see is equal to

\[ s_1 s_2 \cdots s_{n-1} s_n s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1} s_n s_{n-1} \cdots s_2 = 1. \]
Cancelling out $n$ terms on the left and $n - 1$ terms on the right of these two expressions, it just remains to show that

$$s_1 s_2 \ldots s_{n-2} s_{n-1} s_n = s_{n-1} s_{n-2} \ldots s_2 s_1 s_2 \ldots s_{n-2} s_{n-1}.$$  

As there is an arrow $i \to i + 1$ for each $i$ and the cycle is chordless, the subgroup generated by $s_1, \ldots, s_{n-1}$ is isomorphic to the symmetric group on $n$ letters, with $s_i$ being sent to the transposition which swaps $i$ and $i + 1$. It is easy to see that in this group both sides of the equation represent the transposition which swaps 1 and $n$. 

We will justify our choice of relations in Remark 4.20.

2.4 Mutation of groups

Let $B_Q$ be the group associated to the mutation Dynkin quiver $Q$, as above, and let $k$ be a vertex of $Q$. Denote $\mu_k(Q)$ by $Q'$. Our aim in this section is to show that $B_Q$ is isomorphic to $B_{Q'}$. We will do this by using a group homomorphism $\varphi_k : B_Q \to B_{Q'}$ defined using a formula which lifts the formula used in [BM, §5].

The following lemma follows from results in [FZ2] (see [BM, §2]).

**Lemma 2.6.** Let $Q$ be a quiver of mutation Dynkin type, and fix a vertex $k$ of $Q$. Suppose that $k$ has two neighbouring vertices. Then the possibilities for the induced subquiver of $Q$ containing vertex $k$ and its neighbours are shown in Figure 1. The effect of mutation is shown in each case.

The following lemma follows from [BM] Lemma 2.5.

**Lemma 2.7.** Let $Q$ be a quiver of mutation Dynkin type, and fix a vertex $k$ of $Q$. Let $C$ be an oriented cycle in $Q$. Then $C$ is one of the following. In each case we indicate what happens locally...
under mutation at \( k \).

(d) An oriented cycle containing exactly one neighbour of \( k \). Mutation at \( k \) reverses the arrow between \( k \) and its neighbour in \( C \).

(e) An oriented cycle containing no neighbours of \( k \). Mutation at \( k \) does not affect \( C \).

Recall that \( B_Q \) is defined using generators \( s_i \) for \( i \in I \). We denote the corresponding generating set for \( B_Q' \) by \( t_i, i \in I \). Let \( F_Q \) be the free group on the generators \( s_i \) for \( i \in I \).

**Definition 2.8.** Let \( \varphi_k : F_Q \to B_Q' \) be the group homomorphism defined by

\[
\varphi_k(s_i) = \begin{cases} 
  t_k t_i t_k^{-1} & \text{if } i \to k \text{ in } Q; \\
  t_i & \text{otherwise.}
\end{cases}
\]

**Proposition 2.9.** The group homomorphism \( \varphi_k \) induces a group homomorphism (which we also denote by \( \varphi_k \)) from \( B_Q \) to \( B_Q' \).

**Proof:** Let us write \( \bar{s}_i = \varphi_k(s_i) \). We must show that the elements \( \bar{s}_i \) in \( B_Q' \) satisfy the defining relations of \( B_Q \). Note that the \( t_i \) satisfy the defining relations for \( B_Q' \).

Firstly, we check the relations (ii) for an arrow incident with \( k \). Suppose that there is an arrow \( i \to k \). We have the following, using the fact that \( t_i t_k t_i = t_k t_i t_k^{-1} \):

\[
\bar{s}_i \bar{s}_k \bar{s}_i = t_k t_i t_k^{-1} = t_k^2 t_i t_k^{-1} = t_k^2 t_i;
\]

Also,

\[
\bar{s}_k \bar{s}_i \bar{s}_k = t_k^2 t_i t_k^{-1} t_k = t_k^2 t_i.
\]

So

\[
\bar{s}_i \bar{s}_k \bar{s}_i = \bar{s}_k \bar{s}_i \bar{s}_k,
\]

as required.
If there is an arrow \( i \rightarrow k \), then we have
\[
\bar{s}_i \bar{s}_k \bar{s}_i = t_i t_k t_j = t_k t_i t_k = \bar{s}_k \bar{s}_j \bar{s}_k.
\]

Next, we consider relations (i) and (ii) for all other arrows in \( Q \). Relations of this kind involving pairs of vertices which are not neighbours of \( k \) follow immediately from the corresponding relations in \( B_Q \). If only one of the vertices in the relation is a neighbour of \( k \), the relation again follows immediately since \( t_k \) commutes with any generator corresponding to a vertex not incident with \( k \) in \( Q' \) (equivalently, in \( Q \)). So we only need to consider the case where both of the vertices in the pair is incident with \( k \) and we can use Lemma \( \ref{lemma:relations} \).

Going in either direction in part (a) of Lemma \( \ref{lemma:relations} \), the relation \( \bar{s}_i \bar{s}_j = \bar{s}_j \bar{s}_i \) follows from the relation \( t_i t_j = t_j t_i \) in \( B_Q \), so we consider part (b), firstly from left to right. The cycle in \( Q' \) gives the relation \( t_k t_i = t_j t_k t_j t_k^{-1} t_j^{-1} \). Also applying the relation \( t_k^{-1} t_j^{-1} t_k^{-1} = t_j^{-1} t_k^{-1} t_j^{-1} \), we obtain
\[
\bar{s}_i \bar{s}_j = t_i t_k t_j^{-1} t_i = t_j t_i t_k^{-1} t_j^{-1} t_k^{-1} t_j = t_j t_k^{-1} t_j.
\]

Going from right to left in part (b), we have, using \( t_i t_k t_j = t_k t_j t_i \), \( t_i t_j = t_j t_i \) and \( t_i t_k t_i = t_k t_i t_k \),
\[
\bar{s}_i \bar{s}_j = t_k t_j^{-1} t_k^{-1} t_i t_k t_j = t_j^{-1} t_k t_j t_i = t_j^{-1} t_k^{-1} t_j = t_j^{-1} t_k^{-1} t_j = \bar{s}_i \bar{s}_j \bar{s}_i.
\]

Next, we have to check that the \( \bar{s}_i \) satisfy the relations of type (iii) for \( Q \), so we need to consider each the types of cycle described in Lemma \( \ref{lemma:cycle} \). By Lemma \( \ref{lemma:cycle} \) it is enough to check that, for any given cycle in \( Q \), one of the relations in (iii) holds.

For part (a), we have
\[
\bar{s}_k \bar{s}_i \bar{s}_j \bar{s}_k = t_k t_i t_j t_k^{-1} t_k = t_k t_i t_k t_j,
\]
while
\[
\bar{s}_k \bar{s}_j \bar{s}_k \bar{s}_i = t_i t_k t_j t_k^{-1} t_k = t_i t_k t_j,
\]
which is equal to \( \bar{s}_k \bar{s}_i \bar{s}_j \bar{s}_k \) as required.
For part (b) we have, applying a relation for the cycle in $Q'$ in the fourth step:

$$s_1 \cdots s_i s_i s_{i-1} s_{i-2} = t_k t_i t_k^{-1} t_i \cdots t_i$$

$$= t_k t_i t_k^{-1} t_i \cdots t_{i-2}$$

$$= t_k^{-1} t_k t_i t_i \cdots t_{i-2} t_k$$

$$= t_i^{-1} t_i \cdots t_i t_k t_i \cdots t_{i-2} t_k^{-1}$$

$$= t_i \cdots t_i t_k t_i \cdots t_{i-2} t_i$$

$$= s_k \cdots s_i s_i \cdots s_{i-1}$$

For part (c), we have, applying a relation for the cycle in $Q'$ in the fifth step:

$$s_1 \cdots s_i s_i s_{i-1} s_{i-2} = t_i \cdots t_{i-1} t_k t_i t_k^{-1}$$

$$= t_k t_i \cdots t_{i-1} t_k$$

$$= t_i \cdots t_{i-1} t_k t_i \cdots t_{i-1}$$

$$= t_i t_k t_i \cdots t_i \cdots t_{i-1}$$

$$= t_i t_k t_i \cdots t_i \cdots t_{i-2}$$

$$= t_i t_k t_i \cdots t_i \cdots t_{i-1} t_k t_i \cdots t_{i-2}$$

$$= s_k s_i \cdots s_i s_i \cdots s_{i-2} s_{i-2}$$

and we are done. \(\square\)

**Theorem 2.10.** \(\varphi_k : B_Q \rightarrow B_{Q'}\) is a group isomorphism.

**Proof:** As mutation is an involution, we can consider the composition

\(\varphi_k : B_Q \xrightarrow{\varphi_k} B_{Q'} \xrightarrow{\varphi_k} B_Q\).

Fix some \(i \in I\). Note that mutation at \(k\) does not change whether \(i\) and \(k\) are connected in the quiver: it just swaps the direction of any arrow that may exist between \(i\) and \(k\). So if we have \(i \rightarrow k\) then \(s_i \mapsto t_i t_k t_i^{-1} \mapsto s_k s_i s_k^{-1}\). If we have \(i \leftarrow k\) then \(s_i \mapsto t_i^{-1} \mapsto s_k s_i s_k^{-1}\). And if there is no arrow between \(i\) and \(k\) then \(s_i \mapsto t_i \mapsto s_i\). But in this case \(s_i\) and \(s_k\) commute, so \(s_i = s_k s_i s_k^{-1}\). Hence in every case \(\varphi_k(s_i) = s_k s_i s_k^{-1}\), so \(\varphi_k\) is just a conjugation map and therefore \(\varphi_k : B_Q \rightarrow B_{Q'}\) is an isomorphism. \(\square\)

**Remark 2.11.** The inverse of \(\varphi_k\) is the group isomorphism \(\varphi_k^{-1} : B_{Q'} \rightarrow B_Q\) defined by

\(\varphi_k^{-1}(t_i) = \begin{cases} s_k^{-1} s_k & \text{if } i \rightarrow k \text{ in } Q'; \\ s_i & \text{otherwise.} \end{cases}\)

Noting Remark 2.3, we have the following:

**Corollary 2.12.** If \(Q\) is a mutation Dynkin quiver of type \(\Delta\) then \(B_Q \cong B_{\Delta}\).

## 3 Topological interpretation of the generators

### 3.1 Braid groups

In this section we consider quivers \(Q\) which are mutation equivalent to an orientation of the Dynkin diagram of type \(\Delta\), where \(\Delta = A_n\) or \(D_n\). By Corollary 2.12, \(B_Q\) is isomorphic to the Artin braid
group $B_{\Delta}$ of the same Dynkin type. In other words, $B_Q$ gives a presentation of $B_{\Delta}$. In this section we give a geometric interpretation of this presentation.

We associate an oriented Riemann surface $S$ (with boundary) together with marked points $M$ to $\Delta$, as follows. If $\Delta = A_n$, we take $S$ to be a disk with $n-3$ marked points on its boundary, as in [FZ1, FZ2]. If $\Delta = D_n$, we take $S$ to be a disk with one marked point in its interior and $n$ marked points on its boundary, as in [FoSTh, Sch]. In each case, it was shown that every quiver of the corresponding mutation type arises from a triangulation of $(S, M)$ (tagged, in the type $D_n$ case) in the following way. We follow [FoSTh], in a generality great enough to cover both cases (noting that there is at most one interior marked point).

A (simple) arc in $(S, M)$ is a curve in $S$ (considered up to isotopy) whose endpoints are marked points in $M$ and which does not have any self-crossing points, except possibly at its endpoints. Apart from these endpoints, it must be disjoint from $M$ and the boundary of $S$, and it must not cut out an unpunctured one- or two-sided polygon.

Two arcs are said to be compatible if they are non-crossing in the interior of $S$. A maximal set of compatible arcs is a triangulation.

A tagged arc in $(S, M)$ is an arc which does not cut out a once-punctured monogon; each of its ends is tagged, either plain or notched. Plain tags are omitted, while notched tags are displayed using the bow-tie symbol $\bowtie$. An end incident with a boundary marked point is always tagged plain. Two tagged arcs $\alpha, \beta$ are compatible if

(i) the untagged arcs underlying $\alpha$ and $\beta$ are compatible, and

(ii) if the untagged versions of $\alpha$ and $\beta$ are different but share an endpoint, then the corresponding ends of $\alpha$ and $\beta$ are tagged in the same way.

A tagged triangulation $T$ of $(S, M)$ is a maximal collection of tagged arcs in $(S, M)$. Note that if none of the marked points in $M$ lies in the interior of $S$, every end of an arc in a tagged triangulation must be tagged plain, and tagged triangulations of $S$ can be identified with triangulations of $S$.

The set $M$ of marked points divides the boundary components of $(S, M)$ into connected components, which we call boundary arcs. Note that the boundary arcs do not lie in a triangulation or tagged triangulation of $(S, M)$, by definition.

The tagged triangulation $T$ can be built up by gluing together puzzle pieces of the two types shown in Figure 2 (see [FoSTh, Rk. 4.2]) by gluing together along boundary arcs. Note that the puzzle piece of type II can only occur in the type $D_n$ case, and then it occurs exactly once.

If $\alpha$ is an arc in a tagged triangulation $T$, then the flip of $T$ at $\alpha$ is the unique tagged triangulation containing $T \setminus \{\alpha\}$ but not containing $\alpha$. By [FoSTh], the set of tagged triangulations of $(S, M)$ is connected under flips.

The quiver $Q_T$ of a tagged triangulation $T$ has vertices corresponding to the arcs in $T$. The quiver is built up by associating a quiver to each puzzle piece; see Figure 2. If a boundary arc in the puzzle piece is also a boundary arc of $(S, M)$, then the corresponding vertex in the quiver is omitted, together with all incident arrows. The quivers are then glued together by identifying vertices whenever the corresponding edges are glued together in the puzzle pieces.

In order to discuss braid groups, we need to consider more general curves in $(S, M)$. We define a path in $(S, M)$ to be a (possibly non-simple) curve whose endpoints lie in $S$ (not necessarily in $M$).
**Definition 3.1.** Let $T$ be a tagged triangulation of $(S, M)$. We associate a graph to $T$, which we call the braid graph $G_T$ of $T$, as follows. The vertices $V_T$ of $G_T$ are in bijection with the connected components of the complement of $T$ in $(S, M)$ and, whenever two such connected components have a common tagged arc on their boundaries, there is an edge in $G_T$ between the corresponding vertices. Thus the edges in $G_T$ are in bijection with the arcs in $T$.

We choose an embedding $\iota$ of $G_T$ into $(S, M)$, mapping each vertex to an interior point of the corresponding connected component of the complement of $T$ in $(S, M)$ and each edge to a path between the images of its endpoints transverse to the corresponding edge in $T$. We identify $G_T$ with its image under $\iota$.

Note that in the type $A_n$ case the braid graph is the tree from Section 3.1 of [CCS].

We associate an orbifold $X$ to $S$ as follows. In the type $A_n$ case, we just take $X = S$, and in the type $D_n$ case we take $X$ to be $S$ with the interior marked point of $S$ interpreted as a cone point of order two. In each case, the set $M$ of marked points induces a corresponding set of marked points in $X$, which we also denote by $M$. Each arc or tagged arc $\alpha$ in $(S, M)$ induces a corresponding arc or tagged arc in $(X, M)$ which we also denote by $\alpha$. Thus each (tagged) triangulation $T$ of $(S, M)$ induces a corresponding set $T$ of (tagged) arcs in $(X, M)$.

Note also that orbifolds have been used to model cluster algebras in [FeSTu]. In this approach, the model for $B_n$ is an orbifold with a cone point of order 2, regarded as a folding of $D_n$, where $D_n$ is modelled by a disk with a single interior marked point (see also Lecture 15 of [Thu], which was given by A. Felikson).

We denote by $X^\circ$ the orbifold $X$ with the cone point (if there is one) removed (so $X^\circ = X$ in type $A_n$). Given any set $V$ of vertices in $X^\circ$, we may consider the corresponding braid group, $\Gamma(X, V)$ as defined in [All]. Each element of $\Gamma(X, V)$ (or braid) can be regarded as a tuple $\gamma = (\gamma_v)_{v \in V}$ of paths $\gamma_v : [0, 1] \to X^\circ$ such that $\gamma_v(0) = v$ and $\gamma_v(1) = g(v)$ for some permutation $g$ of $V$. In addition, for each $t \in [0, 1]$, the points $\gamma_v(t)$ for $v \in V$ must all be distinct for all $v \in V$. Braids are...
Each path $\pi$ in $X^o$ with endpoints $v_1, v_2$ in $V$ determines a braid $\sigma_{\pi}$ in $\Gamma(X, V)$ as follows (see [FN §7]). We thicken the path $\pi$ along its length, closing it off at the end points to form a (topological) disk. We give the boundary of the disk the clockwise orientation. The vertices $v_1$ and $v_2$ divide the boundary of the disk into two paths, one from $v_1$ to $v_2$ and the other from $v_2$ to $v_1$. We define $\gamma_{v_1}$ to be the former and $\gamma_{v_2}$ to be the latter. See Figure 3. For $v \in V$ such that $v \neq v_1, v_2$, we define $\gamma_v(t)$ to be $v$ for all $t \in [0, 1]$. Then $\sigma_{\pi}$ is the braid $(\gamma_v)_{v \in V}$. Note that $\sigma_{\pi}$ only depends on the isotopy class of the image of $\pi$ in $(X, V)$. In particular, it is unchanged if $\pi$ is reversed.

An example of a braid $\sigma_{\pi}$ is displayed as a picture (in the same way as in [All]) in Figure 4 (in this figure only, we display $\pi$ as a dashed line to distinguish it from the braid $\sigma_{\pi}$).

### 3.2 Interpretation of the generators

Let $T$ be a triangulation of $(S, M)$. Let $Q_T$ be the quiver of $T$. Then $Q_T$ has vertices $I$ corresponding to the arcs in $T$. We denote the arc in $T$ associated to $i \in I$ by $\alpha_i$. The corresponding edge in $G_T$ is denoted $\pi_i$. Let $\sigma_i = \sigma_{\pi_i} \in \Gamma(X, V_T)$ be the corresponding braid. We define $H_T$ to be the subgroup of $\Gamma(X, V_T)$ generated by the braids $\sigma_i$ for $i \in I$.

Let $T_0$ be an initial triangulation of $(S, M)$ defined as follows. In the type $A_n$ case, we choose a marked point $P$ in $M$ and take noncrossing arcs between $P$ and each of the other marked points in $M$ not incident with a boundary arc incident with $P$. In the type $D_n$ case, we choose two marked points $P, Q$ on the boundary of $S$. We take two arcs between the interior marked point and $Q$, one tagged plain at the interior marked point and the other one tagged notched, and an arc between $P$ and $Q$ (not homotopic to a boundary arc). We then take (noncrossing) arcs between $P$ and every
other marked point in $M$ on the boundary of $S$ not incident with a boundary arc incident with $P$. See Figure 5. Then the quiver $Q_{T_0}$ associated to $Q_{T_0}$ is a Dynkin quiver of type $\Delta$. By Remark 2.3, $B_{Q_{T_0}}$ is isomorphic to the Artin braid group of type $\Delta$.

**Proposition 3.3.** Let $T_0$ be the triangulation of $(X, M)$ defined as above. Then there is an isomorphism from $H_{T_0}$ to $B_{Q_{T_0}}$ taking the braid $\sigma_i$ to the generator $s_i$ of $B_{Q_{T_0}}$. Furthermore, in type $A_n$, $H_{T_0}$ coincides with $\Gamma(X, V_{T_0})$, while in type $D_n$, $H_{T_0}$ is a subgroup of $\Gamma(X, V_{T_0})$ of index two.

**Proof:** For type $A_n$, see [FN] and the explanation in [All §4]. For type $D_n$, note that the elements $\sigma_i$ for $i \in I$ coincide with the generators $h_i$ defined in [All §1] (via an isomorphism of the kind in Remark 3.2). The result then follows from [All Thm. 1].

The following lemma appears in [Ser Théorème, part (iv)].

**Lemma 3.4.** Let $A, B, C$ be three distinct points in $X^o$ and suppose there is a topological disk in $X^o$, with $A, B$ and $C$ lying in order clockwise around its boundary. Let $AB$ denote the arc on this boundary between $A$ and $B$. We define $BC$ and $CA$ similarly. Then $\sigma_{AB}\sigma_{BC} = \sigma_{BC}\sigma_{CA}$.

**Theorem 3.5.** Let $T$ be an arbitrary tagged triangulation of $(X, M)$. Then there is an isomorphism from $H_T$ to $B_{Q_T}$ taking the braid $\sigma_i$ to the generator $s_i$ of $B_{Q_T}$. Furthermore, in type $A_n$, $H_T$ coincides with $\Gamma(X, V)$, while in type $D_n$, $H_T$ is a subgroup of $\Gamma(X, V)$ of index two.

**Proof:** The result holds for $T = T_0$ by Proposition 3.3. Note that any triangulation can be obtained from $T_0$ by applying a finite number of flips of tagged triangulations. We show that the theorem is true for an arbitrary tagged triangulation $T$ by induction on the number of flips required to obtain $T$ from $T_0$. To do this, it is enough to show that if the theorem holds for a tagged triangulation $T$ and $\alpha_i$ is a tagged arc in $T$ then the theorem also holds for the flip of $T$ at $\alpha_i$.

So we assume the result holds for a tagged triangulation $T$. Thus there is an isomorphism $\psi_T : H_T \to B_{Q_T}$ sending $\sigma_i$ to $s_i$. We denote the corresponding elements of $H_T$ by $t_i$ and $t_i$. The tagged...
Figure 6: Flip involving an arc \( (\alpha_1) \) where two puzzle pieces of type I are glued

arcs in \( T \) are denoted by \( \alpha_i \), for \( i \in I \), and we denote the corresponding tagged arcs in \( T' \) by \( \beta_i \), for \( i \in I \). The edges of \( G_T \) are denote \( \pi_i \), and we denote the edges of \( G_{T'} \) by \( \rho_i \).

We define:

\[
\tilde{\tau}_i = \begin{cases} 
\sigma_k^{-1} \sigma_i \sigma_k, & \text{if } i \to k \text{ in } Q; \\
\sigma_i, & \text{otherwise}.
\end{cases}
\]

Then it is easy to see that \( H_T \) is generated by the \( \tilde{\tau}_i \) for \( i \in I \).

We consider the possible types of flip that can occur, which are determined by the fact that \( T \) can be constructed out of puzzle pieces. Suppose first that \( T' \) is the flip of \( T \) at an arc \( \alpha \) where two puzzle pieces of type (I) are glued together. We label the corresponding vertices in \( I \) by 1, 2, 3, 4, 5, for simplicity, and suppose we are flipping at the edge in \( T \) dual to \( \alpha_1 \). The braid graph local to the flip is shown in the left hand diagram in Figure 6. Applying Lemma 3.4, we see that the middle figure shows paths \( \tilde{\pi}_i \) with the property that \( \tilde{\tau}_i = \sigma_i \tilde{\pi}_i \) for \( i = 1, 2, 3, 4, 5 \).

Rotating vertices \( A \) and \( B \) clockwise, to get the right hand diagram in Figure 6, we obtain, via Remark 3.2 an isomorphism from \( H_T \) to \( H_{T'} \), taking \( \tilde{\tau}_i \) to \( \tau_i \) for all \( i \in I \). The inverse is an isomorphism from \( H_{T'} \) to \( H_T \), taking \( \tau_i \) to \( \sigma_k^{-1} \sigma_i \sigma_k \) if there is an arrow \( i \to k \) in \( Q \) and to \( \sigma_i \) otherwise. Composing with the isomorphism \( \psi \circ \psi_{T'} \), where \( \psi_k \) is the isomorphism in Proposition 2.9 we obtain an isomorphism from \( H_{T'} \) to \( B_{Q,T'} \), taking \( \tau_i \) to \( t_i \) as required. This proves the required result in type A, so we are left with the type D case, where puzzle pieces of type II may occur.

We next consider a flip inside a puzzle piece of type II. We can apply essentially the same argument: see Figures 7 and 8. Here we draw the puzzle piece together with the two adjacent triangles, necessarily of type I (since there is only one cone point). We use the fact that in the right hand diagram of Figure 8 the resulting path \( \tilde{\pi}_1 \) is isotopic to the path \( \rho_1 \) in \( G_{T'} \), using the fact that the cone point has order two.

Note that the adjoining type I puzzle pieces (in Figures 7 and 8) may not occur, but the argument is easily modified to cover these cases. We also need to consider the flips from the right hand diagram...
Figure 7: Flip (at $\alpha_1$) inside a puzzle piece of type II, first case

Figure 8: Flip (at $\alpha_2$) inside a puzzle piece of type II, second case
Finally, we need to consider a flip involving an arc where a puzzle piece of type I and a puzzle piece of type II have been glued together. These cases are shown in Figures 9 and 10. Figure 9 illustrates the case where the puzzle piece of type I is on the left of the puzzle piece of type II (when it is drawn as shown), while Figure 10 illustrates the case where it is on the right. Again, a similar argument applies in the case of flips from the right hand diagram to the left hand one in these cases. \[\square\]

4 Actions on categories

4.1 Quivers with potential

Fix an algebraically closed field \( \mathbb{F} \). To any quiver \( Q \) we can associate the path algebra \( \mathbb{F}Q \), which, as an \( \mathbb{F} \)-vector space, has basis given by all paths in \( Q \) of length \( \geq 0 \), and the multiplication of two paths \( p_1 \) and \( p_2 \) is their concatenation \( p_1p_2 \) if \( p_1 \) ends and \( p_2 \) starts at the same vertex, and is zero otherwise.

Let \( \mathbb{F}Q_{\geq n} \) be the ideal of \( \mathbb{F}Q \) generated by the paths in \( Q \) of length at least \( n \). We can take the completion \( \hat{\mathbb{F}}Q \) of \( \mathbb{F}Q \) with respect to \( \mathbb{F}Q_{\geq 1} \), which is defined as follows:

\[
\hat{\mathbb{F}}Q = \lim_{\longrightarrow} \mathbb{F}Q_{\geq n} = \{(a_n + \mathbb{F}Q_{\geq n})_{n=1}^{\infty} \mid a_n \in \mathbb{F}Q, \varphi(a_n + \mathbb{F}Q_{\geq n}) = a_{n-1} + \mathbb{F}Q_{\geq n-1}\}
\]

where the limit is taken along the chain of epimorphisms

\[
\begin{align*}
\mathbb{F}Q &\overset{\varphi_1}{\longrightarrow} \mathbb{F}Q_{\geq 2} \\
\mathbb{F}Q_{\geq 2} &\overset{\varphi_2}{\longrightarrow} \mathbb{F}Q_{\geq 3} \\
\mathbb{F}Q_{\geq 3} &\overset{\varphi_3}{\longrightarrow} \mathbb{F}Q_{\geq 4} \\
&\vdots
\end{align*}
\]
Let $\hat{F}_Q^\text{cyc}$ denote the subspace of (possibly infinite) linear combinations of cycles in $Q$. Recall that a potential for a quiver $Q$ is an element $W$ of $\hat{F}_Q^\text{cyc}$, regarded up to cyclic equivalence (and for which no two cyclically equivalent paths in $Q$ occur in the decomposition of $W$). The pair $(Q, W)$ is called a quiver with potential [DWZ], which we occasionally abbreviate to QP. The following definition is [DWZ, Definition 4.2].

**Definition 4.1 (Derksen-Weyman-Zelevinsky).** Let $Q_1$ and $Q_2$ be two quivers with the same vertex set $I$ and $(Q_1, W_1)$ and $(Q_2, W_2)$ be two QPs. A right equivalence between $(Q_1, W_1)$ and $(Q_2, W_2)$ is an algebra isomorphism $\varphi : \hat{F}_Q^1 \to \hat{F}_Q^2$ such that $\varphi(W_1)$ is cyclically equivalent to $W_2$ and $\varphi$ is the identity when restricted to the semisimple subalgebra $F I$ of $\hat{F}_Q^1$.

A quiver with potential $(Q, W)$ with $W$ containing paths of length two or more is trivial if $Q$ is a disjoint union of 2-cycles and there is an algebra automorphism of $kQ$ preserving the span of the arrows of $Q$ (a change of arrows) which takes $W$ to the sum of the 2-cycles in $Q$. A quiver with potential $(Q, W)$ is said to be reduced if $W$ is a linear combination of cycles in $Q$ of length 3 or more.

The splitting theorem [DWZ, Thm. 4.6] states that every quiver with potential can be written as a direct sum of a reduced quiver with potential and a trivial quiver with potential which are unique up to right equivalence.

Let $(Q, W)$ be a quiver with potential, and let $k$ be a vertex of $Q$ not involved in any 2-cycles. By replacing $W$ with a cyclically equivalent potential on $Q$ if necessary, we can assume that none of the cycles in the decomposition of $W$ start or end at $k$. We denote by $\pi_k(Q, W)$ the non-reduced mutation of $(Q, W)$ at $k$ in $Q$, as defined in [DWZ, §5]. Then the right equivalence class of $\pi_k(Q, W)$ is determined by the right equivalence class of $(Q, W)$ by [DWZ, Thm. 5.2]. The mutation $\mu_k(Q, W)$ of $(Q, W)$ at $k$ is then defined to be the reduced component of $\pi_k(Q, W)$, and is uniquely determined up to right equivalence, given the right equivalence class of $(Q, W)$.

As before, we will say that a quiver with potential $(Q, W)$ is Dynkin if the underlying unoriented graph of $Q$ is an orientation of a Dynkin quiver (and hence $W = 0$). We shall say that a quiver with potential $(Q, W)$ is trivial if $Q$ is a disjoint union of 2-cycles and there is an algebra automorphism of $kQ$ preserving the span of the arrows of $Q$ (a change of arrows) which takes $W$ to the sum of the 2-cycles in $Q$. A quiver with potential $(Q, W)$ is said to be reduced if $W$ is a linear combination of cycles in $Q$ of length 3 or more.

The splitting theorem [DWZ, Thm. 4.6] states that every quiver with potential can be written as a direct sum of a reduced quiver with potential and a trivial quiver with potential which are unique up to right equivalence.
potential \((Q', W')\) is mutation Dynkin if it can be obtained by repeatedly mutating a Dynkin quiver with potential in the above sense. For the rest of this article, we will restrict to Dynkin types \(A\) and \(D\).

Let \((S, M)\) be the Riemann surface with marked points associated to \(\Delta\) as in Section 3. So, if \(\Delta = A_n\), we take \(S\) to be a disk with \(n - 3\) points on its boundary, and if \(\Delta = D_n\), we take \(S\) to be a disk with one marked point in its interior and \(n\) marked points on its boundary.

Let \(Q\) be a mutation Dynkin quiver. By [FoSTh], \(Q = Q_T\) for some tagged triangulation \(T\) of \((S, M)\).

Let \(W, W'\) be the sum of the terms coming from local configurations in \(T\) as shown in Figure 11 (where in (c) and (d) there are at least three arcs incident with the interior marked point).

Then \(W_T\) is the potential given by taking the sum of the induced cycles in \(Q_T\) (i.e. induced subgraphs of \(Q_T\) which are cycles), and \(W'_T\) is the potential associated to \(T\) in [LF12, §3], taking the parameter associated to the internal marked point (if there is one) to be equal to \(-1\). Then we have the following:

**Lemma 4.2.** The potentials \(W_T\) and \(W'_T\) are right equivalent.

**Proof:** We assume we are in case \(D_n\), since the two potentials coincide in case \(A_n\). If the interior marked point is as in case (c) of Figure 11 (with at least 3 arcs incident with it), then there is a unique triangle in \(T\) with sides 1 and 2. We label the arrows in the corresponding 3-cycle in \(W_T\) or \(W'_T\) by \(a, x, y\), in order around the cycle. Then the automorphism \(\varphi\) of \(\hat{kQ_T}\) negating \(a\) and \(x\) and taking each other arrow to itself gives a right equivalence between \(W_T\) and \(W'_T\), since \(a\) and \(x\) are not involved in any other terms in any of these potentials.

If the interior marked point is as in case (d), then \(W_T\) and \(W'_T\) coincide. \(\square\)

We recall the following special case of [LF12 Thm. 8.1].

**Theorem 4.3.** [LF12] Let \(T, T'\) be triangulations of \((S, M)\). If \(T'\) is obtained from \(T\) by flipping at an arc \(\alpha_k\) then \(\mu_k(Q_T, W_T)\) is right equivalent to \((Q_{T'}, W_{T'})\).

By [DWZ] Thm. 7.1], it follows from this that the quiver of \(\mu_k(Q_T, W_T)\) coincides with the quiver obtained from \(Q_T\) by Fomin-Zelevinsky quiver mutation at \(k\).

**Proposition 4.4.** If \((\tilde{Q}, \tilde{W})\) is obtained from a Dynkin quiver with potential by iterated mutation in the sense of [DWZ], then \((\tilde{Q}, W)\) is right equivalent to \((Q_T, W_T)\) for some triangulation \(T\) of \((S, M)\).

**Proof:** This follows from Lemma 4.2 and Theorem 4.3. \(\square\)

Hence we can effectively ignore potentials:

**Proposition 4.5.** Any mutation Dynkin quiver with potential \((\tilde{Q}, \tilde{W})\) of type \(A\) or \(D\) is right equivalent to \((Q, W_Q)\), where \(W_Q\) is the sum of all chordless cycles in \(\tilde{Q}\).

**Proof:** Since a Dynkin quiver with potential is of the form \((Q_T, W_T)\) for some triangulation \(T\) (see [FoSTh]), this follows from Theorem 4.3 and the remark following it, using Lemma 4.2. \(\square\)
Figure 11: Terms in the potential $W_T$
4.2 Differential graded algebras and modules

Let $\mathbb{F}$ be an algebraically closed field. We think of $\mathbb{F}$ as a graded $\mathbb{F}$-algebra concentrated in degree 0. If $V = \bigoplus V_i$ is a graded $\mathbb{F}$-module then let $V[j]$ be the graded $\mathbb{F}$-module with $(V[j])_i = V_{i+j}$.

If $f : V \rightarrow W$ is a map of graded vector spaces with homogeneous components $f_i : V_i \rightarrow W_i$, then let $f[j] : V[j] \rightarrow W[j]$ be the map of graded vector spaces with homogeneous components $f[j]_i : V[j]_i \rightarrow W[j]_i$ defined by $f[j]_i(v) = (-1)^j f_{i+j}(v)$ for $v \in V[j]_i = V_{i+j}$. Thus [1] is an endofunctor of the category of graded $\mathbb{F}$-modules, called the shift functor.

We say that a map $f : V \rightarrow W$ of graded vector spaces has degree $i$ to mean that $f$ is a map $V \rightarrow W[i]$. We use the Koszul sign rule for graded $\mathbb{F}$-algebras, so if $f : V \rightarrow V'$ and $g : W \rightarrow W'$ are maps of graded $\mathbb{F}$-modules of degree $m$ and $n$ then

$$(f \otimes g)(v \otimes w) = (-1)^n f(v) \otimes g(w)$$

for $v \in V_i$ and $w \in W_j$.

A unital differential graded algebra (or dg-algebra, or dga) over $\mathbb{F}$ is a graded $\mathbb{F}$-algebra $A = \bigoplus_{i \in \mathbb{Z}} A_i$ with multiplication $m : A \otimes_{\mathbb{F}} A \rightarrow A$ of degree 0 together with a unit $\iota : \mathbb{F} \hookrightarrow A$ and an $\mathbb{F}$-linear differential $d : A \rightarrow A$ of degree +1. These should satisfy the following relations:

- the associativity relation $m \circ (1 \otimes m) = m \circ (m \otimes 1)$;
- the boundary relation $d^2 = 0$;
- the Liebniz relation $d \circ m = m \circ (1 \otimes d + d \otimes 1)$;
- the unital relation $m \circ (\iota \otimes \iota) = m \circ (\iota \otimes \iota)$, which should agree with the $\mathbb{F}$-algebra structure of $A$.

We often denote our dga by $(A, d)$, or simply by $A$. Each dga $(A, d)$ has an underlying unital graded algebra, obtained by simply forgetting the differential, which we denote $u(A)$.

A left module $M$ for $A$ is a graded left $\mathbb{F}$-module $M$ with a left action $m_M : A \otimes M \rightarrow M$ of $u(A)$ together with a map $d_M : M \rightarrow M$ of degree +1, called a differential, such that

$$d_M \circ m_M = m_M \circ (1 \otimes d_M + d \otimes 1).$$

We always have the regular module $M = A$ with $d_M = d$ and $m_M = m$. Similarly, a right module $M$ for $A$ is a graded right $\mathbb{F}$-module $M$ with a right action $m_M : M \otimes A \rightarrow M$ of $u(A)$ together with a differential $d_M$ such that $d_M \circ m_M = m_M \circ (1 \otimes d + d_M \otimes 1)$. If $(M, d_M)$ is an $A$-module, then $(M[1], d_M[1])$ is also an $A$-module, which we sometimes just write as $M[1]$. Modules for $A$ are modules for $u(A)$, simply by forgetting the differential.

A map $f : M \rightarrow N$ of left $A$-modules is a degree 0 map of $u(A)$-modules such that $f$ commutes with the differentials: $d_N \circ f = f \circ d_M$. We thus obtain a category $A$-$\text{Mod}$ of left $A$-modules, and we write the morphism spaces in this category as $\text{Hom}_{A$-$\text{Mod}}(M, N)$. $A$-$\text{Mod}$ is an $\mathbb{F}$-category: each morphism space is an $\mathbb{F}$-module.

Given two differential algebras $(A, d_A)$ and $(B, d_B)$, an $A$-$B$-bimodule $(M, d_M)$ is a graded $\mathbb{F}$-module which is a left $(A, d_A)$-module with left action $m'$ and a right $(B, d_B)$-module with right action $m''$ where the two actions commute: $m'' \circ (m' \otimes 1) = m' \circ (1 \otimes m'')$. We will always assume that $\mathbb{F}$ acts centrally. Under this assumption we can, and will, identify left $A$-modules with $A$-$k$-bimodules.
and $A$-$B$-bimodules with left $A \otimes_B B^{op}$-modules, where $B^{op}$ denotes the algebra $B$ with the order of multiplication reversed. A map of bimodules should commute with the differential on both the left and the right, and we obtain an $F$-category $A$-$Mod$-$B$ of $A$-$B$-bimodules.

Given a map $f : M \to N$ of left $A$-modules, we can construct a new left $A$-module called the cone of $f$, denoted cone$(f)$. As a left module for $u(A)$, we have cone$(f) = N \oplus M[1]$. The differential is given by:

$$
\begin{pmatrix}
d_N & 0 \\
d_M[1] & d_M[1]
\end{pmatrix}
$$

If $L$ is isomorphic to cone$(f)$ for some map $f : M \to N$, we say that $L$ is an extension of $M$ by $L[-1]$.

We will use the following lemma, whose proof follows immediately from the definitions, repeatedly.

**Lemma 4.6.** Let $f : M \to N$ be a map in $A$-$Mod$.

(i) Let $F : A$-$Mod \to B$-$Mod$ be an additive functor which commutes with the shift functor. Then we have an isomorphism cone$(f) \cong F$ cone$(f)$ in $B$-$Mod$.

(ii) For any commutative diagram

$$
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\sim & \sim & \sim \\
M' & \xrightarrow{f'} & N'
\end{array}
$$

in $A$-$Mod$ where both $\varphi_M$ and $\varphi_N$ are isomorphisms, we have an isomorphism $\varphi_N \oplus \varphi_M[1] : cone(f) \to cone(f')$ of $A$-modules.

Let $(A, d_A)$, $(B, d_B)$, and $(C, d_C)$ be dgas. If $(M, d_M)$ is an $A$-$B$-bimodule and $(N, d_N)$ is an $A$-$C$-bimodule then let $\text{Hom}_A^i(M, N)$ be the space of all graded left $u(A)$-module maps $f : M \to N$ of degree $i$. Note that we do not require that these maps commute with the differential. We define $\text{Hom}_A(M, N) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_A^i(M, N)$, and this is a graded $u(B)$-$u(C)$-bimodule. We also have a version for right modules, which we write as $\text{Hom}_{A^{op}}(M, N)$.

Note the distinction between $\text{Hom}_A(M, N)$ and the hom spaces in the category $A$-$Mod$. With the differential $d(f) = d_N \circ f - (-1)^i f \circ d_M$ for $f \in \text{Hom}_A^i(M, N)$, $\text{Hom}_A(M, N)$ becomes a $B$-$C$-bimodule. Similarly, if $(M, d_M)$ is an $B$-$A$-bimodule and $(N, d_N)$ is a $C$-$A$-bimodule, $\text{Hom}_{A^{op}}(M, N)$ is a $C$-$B$-bimodule. $\text{Hom}_A(-, -)$ is the internal hom in the bimodule category, and we can recover the hom spaces in $A$-$Mod$ as the 0-cycles of $\text{Hom}_A(M, N)$.

If $(M, d_M)$ is an $A$-$B$-bimodule and $(N, d_N)$ is a $B$-$C$-bimodule then let $M \otimes_B N$ denote the space $M \otimes_{u(B)} N$. It is a graded $u(A)$-$u(C)$-bimodule: if $m \in M_i$ and $n \in N_j$ then $m \otimes n$ has degree $i + j$.

With the differential $d_M \otimes 1 + 1 \otimes d_N$, it becomes an $A$-$C$-bimodule.

For an $A$-$B$-bimodule $(M, d_M)$, we thus have functors $M \otimes_B - : B$-$Mod \to A$-$Mod$ and $\text{Hom}_A(M, -) : A$-$Mod \to B$-$Mod$. $M \otimes_B -$ is left adjoint to $\text{Hom}_A(M, -)$. For $(N, d_N)$ a left $A$-module, the counit $e_N : M \otimes_B \text{Hom}_A(M, N) \to N$ of the adjunction is the evaluation map, which acts as $x \otimes f \mapsto (-1)^i f(m)$ for $x \in M_i$ and $f \in \text{Hom}_A^i(M, N)$.

### 4.3 Derived categories

Our references are [Kel94] [Kel06].
If $A$ is a graded vector space and $d$ is a differential, i.e., a degree $+1$ endomorphism of $A$ which satisfies $d^2 = 0$, then the $i$th homology of $A$, denoted $H_i(A)$, is the subquotient $\ker d_i / \operatorname{im} d_{i-1}$, where $d_i : A_i \to A_{i+1}$ denotes the restriction of $d$ to $A_i$. If $(A, d)$ is a dga then the homology $H(A) = \bigoplus H_i(A)$ is a graded algebra, and if $M$ is a left $A$-module then $H(M) = \bigoplus H_i(M)$ is a left $H(A)$-module. In fact, taking homology is a functor from the category of $A$-modules to the category of graded $H(A)$-modules. We say that a left $A$-module $M$ is acyclic if $H(A) = 0$, and that a map $f : M \to N$ of $A$-modules is a quasi-isomorphism if $H(f)$ is an isomorphism.

The category up to homotopy of $A$-Mod, denoted $K(A)$, is the $\mathbb{F}$-category whose objects are all left $A$-modules and whose morphism spaces, for $M, N \in A$-Mod, are $\operatorname{Hom}_{K(A)}(M, N) = H_0 \operatorname{Hom}_A(M, N)$. The derived category of $A$, denoted $D(A)$, is the $\mathbb{F}$-category obtained by localizing $K(A)$ at the full subcategory of acyclic $A$-modules. As a map of modules is a quasi-isomorphism if and only if its cone is acyclic, this is equivalent to localizing $K(A)$ at the class of all quasi-isomorphisms. So we have a canonical functor $K(A) \to D(A)$, which we call the projection functor. The finite-dimensional derived category, denoted $D_{\text{fd}}(A)$, is the full subcategory of $D(A)$ on objects with finite-dimensional total homology, i.e., on $A$-modules $M$ such that $H(M)$ is a finite-dimensional $\mathbb{F}$-vector space.

Let $(A, d_A)$ be a dga. We say that:

- $P \in A$-Mod is indecomposable projective if it is an indecomposable direct summand of the regular module,
- $P \in A$-Mod is relatively projective if it is a direct summand of shifts of indecomposable projective modules, and
- $P \in A$-Mod is cofibrant if, for each surjective quasi-isomorphism $f : M \to N$, the map $\operatorname{Hom}_{A \text{-Mod}}(P, f) : \operatorname{Hom}_{A \text{-Mod}}(P, M) \to \operatorname{Hom}_{A \text{-Mod}}(P, N)$ is surjective.

The following result (see [Kel94 Section 3] and [KY] Proposition 2.13) characterizes cofibrant modules.

**Proposition 4.7** (Keller). An $A$-module $P$ is cofibrant if and only if it is an iterated extension of a relatively projective module by relatively projective modules, possibly infinitely many times.

Let $A$-cofib denote the full subcategory of $K(A)$ on the cofibrant objects. The projection functor $K(A) \to D(A)$ induces an equivalence $A$-cofib $\to D(A)$. Each $A$-module $M$ has a cofibrant replacement, defined up to quasi-isomorphism and denoted $pM$, which can be realized as the image of $M$ under the left adjoint $D(A) \to K(A)$ to the canonical projection functor [Kel94 Proposition 3.1].

Let $(B, d_B)$ be another dga and let $F : A$-Mod $\to B$-Mod be an additive functor. Then $F$ preserves chain homotopies, so induces a functor $F(K) : K(A) \to K(B)$. If $K(F)$ preserves quasi-isomorphisms then, by the universal property of localization, it induces a functor $D(F) : D(A) \to D(B)$. If $P \in A$-Mod-$B$ is cofibrant as a left $A$-module then, by [Kel94 Theorem 3.1(a)] and [KY] Proposition 2.13, $\operatorname{Hom}_A(P, -)$ preserves acyclic modules, and so preserves quasi-isomorphisms. By imitating the proof of [Kel94 Theorem 3.1(a)] we see that if $P \in A$-Mod-$B$ is cofibrant as a right $B$-module then $P \otimes_B -$ also preserves acyclic modules. We often write $P \otimes_B -$ and $\operatorname{Hom}_A(P, -)$, instead of $D(P \otimes_B -)$ and $D(\operatorname{Hom}_A(P, -))$, for the induced functors $D(B) \to D(A)$ and $D(A) \to D(B)$.

For an arbitrary $M \in A$-Mod-$B$, we can obtain a functor $M \otimes_B^L - : D(B) \to D(A)$, known as the left derived functor of $M \otimes_B -, \text{ by composing the cofibrant replacement functor } D(B) \to K(B)$, the tensor functor $K(M \otimes_B -) : K(B) \to K(A)$, and the projection functor $K(A) \to D(B)$. By [Kel94 Lemma 6.3(a)], we have an isomorphism $M \otimes_B^L N \cong pM \otimes_B^L N$ for all $N \in D(B)$. The following basic, but useful, lemma says that this isomorphism is natural.
Lemma 4.8. Let $M \in \text{Mod-} B$.

(i) We have a natural isomorphism of functors $p M \otimes_B - \cong M \otimes^l_B -$.

(ii) If $M$ is cofibrant then we have a natural isomorphism of functors $M \otimes_B - \cong M \otimes^l_B -$.

Proof:

(i) We need to show that for each $N \in B\text{-Mod}$ there is a quasi-isomorphism $\varphi_N : p M \otimes_B N \to M \otimes_B p N$ such that, for all maps $f : N \to N'$, the diagram

$$
\begin{array}{ccc}
p M \otimes_B N & \xrightarrow{\varphi_N} & M \otimes_B p N \\
p M \otimes f & \downarrow & M \otimes f \\
p M \otimes_B N' & \xrightarrow{\varphi_N'} & M \otimes_B p N'
\end{array}
$$

commutes. Consider the following diagram:

As $p M$ and $p N$ are cofibrant and $\pi_M$ and $\pi_N$ are quasi-isomorphisms, both $p M \otimes \pi_N$ and $\pi_M \otimes p N$ are quasi-isomorphisms, so we can define $\varphi_N = (\pi_M \otimes p N) \circ (p M \otimes \pi_N)^{-1}$ and it is a quasi-isomorphism.

By the bifunctoriality of the tensor product, the left hand side is equal to

$$(M \otimes p f) \circ (\pi_M \otimes p N) \circ (p M \otimes \pi_N)^{-1} = (\pi_M \otimes p N') \circ (p M \otimes \pi_N')^{-1} \circ p M \otimes f$$

so we just need to show that

$$p f \circ \pi_N^{-1} = \pi_N' \circ f$$

but this follows from the functoriality $f \circ \pi_N = \pi_N' \circ p f$ of the cofibrant replacement functor $p$.

(ii) We just need to show that, for $M$ cofibrant, there is a natural isomorphism $M \otimes_B - \cong p M \otimes_B -$, and then the result will follow by part (i) of the lemma. This follows because $\pi_M : p M \to M$ is a quasi-isomorphism and by the bifunctoriality of the tensor product.

\[\square\]
If the functor $M \otimes_B -$ is an equivalence $\mathcal{D}(B) \simto \mathcal{D}(A)$, we say that $M$ is a \textit{tilting module}.

We say that a module $M$ is of \textit{finite global dimension} if its cofibrant replacement is an iterated extension of finitely many shifted indecomposable projective modules.

The following basic lemma will also be useful. It can be found as \cite[Lemma 6.2(a)]{Kel94}. We include a proof for the convenience of the reader.

\textbf{Lemma 4.9.} Let $M$ be a left $A$-module of finite global dimension and let $P$ be its cofibrant replacement. Then we have a natural isomorphism of functors

$$\Hom_A(P, A) \otimes_A - \simto \Hom_A(P, -) : \mathcal{A}\text{-Mod} \to \mathcal{F}\text{-Mod}.$$  

\textbf{Proof:} First note that, for any $P \in \mathcal{A}\text{-Mod}$, we always have a natural transformation

$$\Hom_A(P, A) \otimes_A - \to \Hom_A(P, -)$$

obtained by staring with the map

$$\text{ev} \otimes 1 : (P \otimes_{\mathcal{F}} \Hom_A(P, A)) \otimes_A M \to A \otimes_A M,$$

using the associativity isomorphism to obtain a map

$$P \otimes_{\mathcal{F}} (\Hom_A(P, A) \otimes_A M) \to A \otimes_A M,$$

using the adjunction

$$\Hom_{\mathcal{F}}(\Hom_A(P, A) \otimes_A M, \Hom_A(P, A \otimes_A M)) \cong \Hom_A(P \otimes_{\mathcal{F}} (\Hom_A(P, A) \otimes_A M), A \otimes_A M),$$

and finally using the natural isomorphism $A \otimes_A M \cong M$.

To show that our natural transformation is an isomorphism, we use induction on the number of times we need to extend a summand of the regular module to obtain $P$. We handle the base case as follows: the natural transformation is certainly an isomorphism when $P$ is the regular module and so, as hom functors commute with finite direct sums, it is an isomorphism for all summands of the regular module. For our inductive step, suppose the lemma holds for $P_1$ and $P_2$, and let $P = \text{cone}(f)$ for some map $f : P_1 \to P_2$. Then, for $M \in \mathcal{A}\text{-Mod}$, one can check that the map $\Hom_A(P, A) \otimes_A M \to \Hom_A(P, M)$ comes from the commutative diagram

$$\begin{array}{ccc}
\Hom_A(P_1, A) \otimes_A M & \xrightarrow{\Hom(f, A) \otimes M} & \Hom_A(P_2, A) \otimes_A M \\
\downarrow & & \downarrow \\
\Hom_A(P_1, M) & \xrightarrow{\Hom(f, M)} & \Hom_A(P_2, M)
\end{array}$$

as in the construction from the second half of Lemma 4.6, where the vertical maps come from the natural transformation described above. Therefore, as both vertical maps are isomorphisms by induction, $\Hom_A(P, A) \otimes_A M \to \Hom_A(P, M)$ is an isomorphism. \hfill \Box

\section{4.4 Spherical twists}

Our references are \cite{ST, RZ, Gra1}.

Let $(A, d)$ be a dga and $M$ be a left $A$-module with finite dimensional total homology. Let $d \in \mathbb{Z}$. Following \cite{ST}, we say that $M$ is \textit{$d$-spherical} if:
• $M$ is a $d$-Calabi-Yau object, i.e., we have an isomorphism
\[ \text{Hom}_{D^b(A)}(M, N) \cong \text{Hom}_{D^b(A)}(N, M[d])^\ast \]
which is functorial in $N$, and

• $\bigoplus_{i \in \mathbb{Z}} \text{Hom}_{D^b(A)}(M, M[i])$ is isomorphic as a graded algebra to $\mathbb{F}[x]/(x^2)$, with $x$ in degree $d$.

Associated to any spherical object $M$, we have a spherical twist functor $F_M : D^b(A) \to D^b(A)$ which is defined as follows. First, let $P = p_M$ be a cofibrant replacement of $M$. Then let $X_M$ be the cone of the following map of $A$-$A$-bimodules:

\[ P \otimes \mathbb{F} \text{Hom}_A(P, A) \xrightarrow{\sim} A \]

where the nonzero map is the obvious evaluation map. As both $\text{Hom}_A(P, A)$ and $A$ are cofibrant, $X_M$ is cofibrant as a right $A$-module. Then we define the spherical twist at $M$ by

\[ F_M = X_M \otimes_A - : D^b(A) \to D^b(A). \]

The spherical twist is an autoequivalence of $D^b(A)$ (so $X_M$ is a tilting module).

Note that, by Lemmas 4.6 and 4.9 if $M$ has finite global dimension then

\[ F_M(N) \cong P \otimes \mathbb{F} \text{Hom}_A(P, N) \cong N. \]

We will need a simple result on the commutation relation of spherical twists with derived equivalences. It is a generalization of [ST Lemma 2.11].

**Proposition 4.10.** Let $A, B$ be dgas. Let $T \in B \text{-Mod}$-$A$ be a tilting module and $\Phi = T \otimes^L_A - : D^b(A) \to D^b(B)$ be the associated derived equivalence. Let $M \in A \text{-Mod}$ have finite dimensional total homology and suppose it is $d$-spherical, for some $d \in \mathbb{Z}$. Suppose that $\Phi(M) \in B \text{-Mod}$ has finite dimensional total homology. Then $\Phi(M)$ is also $d$-spherical and we have an isomorphism of functors

\[ \Phi \circ F_M \cong F_{\Phi(M)} \circ \Phi : D^b(A) \xrightarrow{\sim} D^b(B). \]

In particular, we have an isomorphism

\[ F_{\Phi(M)} \cong \Phi \circ F_M \circ \Phi^{-1} : D^b(B) \xrightarrow{\sim} D^b(B) \]

where $\Phi^{-1}$ is the quasi-inverse functor of $\Phi$.

**Proof:** As $\Phi : D^b(A) \to D^b(B)$ is a derived equivalence it has quasi-inverse $\Phi^{-1} : D^b(B) \to D^b(A)$, and so we have isomorphisms

\[ \text{Hom}_{D^b(B)}(\Phi(M), \Phi(M)[i]) \cong \text{Hom}_{D^b(A)}(M, M[i]) \]

and

\[ \text{Hom}_{D^b(B)}(\Phi(M), N) \cong \text{Hom}_{D^b(A)}(M, \Phi^{-1}(N)) \cong \text{Hom}_{D^b(A)}(\Phi^{-1}(N), M[d])^\ast \cong \text{Hom}_{D^b(B)}(N, \Phi(M)[d])^\ast, \]

the second natural in $N \in D^b(B)$, using the facts that $M$ is a $d$-Calabi-Yau object and the shift functor $[d]$ commutes with all triangulated functors. Thus $\Phi(M)$ is $d$-spherical.
By Lemma 4.8 we may assume that $T$ is cofibrant as a right $B$-module and that $\Phi = T \otimes_A -$. We want to show that $T \otimes_A X_M \otimes_A - \cong X_{\Phi(M)} \otimes_B T \otimes_A -$, so it is enough to check that we have an isomorphism $T \otimes_A X_M \cong X_{\Phi(M)} \otimes_B T$ in $D_{\text{fd}}(B \otimes_B A^{\text{op}})$. To construct this isomorphism, we use the following extension of Lemma 4.6, which follows from the triangulated 5-lemma: for any commutative diagram

$$
\begin{array}{c}
M \\
\downarrow \varphi_M \\
M'
\end{array} \xrightarrow{f} \begin{array}{c}
N \\
\downarrow \varphi_N \\
N'
\end{array}
$$

in $B\text{-Mod-}A$ with $\varphi_M$ and $\varphi_N$ both quasi-isomorphisms, we have a quasi-isomorphism $\varphi_N \oplus \varphi_M^{-1} : \text{cone}(f) \to \text{cone}(f')$.

As above, write $P = pM$. Then, by the first part of Lemma 4.6, we just need to find two vertical maps which are quasi-isomorphisms and make the following diagram commute:

$$
\begin{array}{c}
T \otimes A P \otimes_B \text{Hom}_B(T \otimes_A P, B) \otimes_B T \\
\downarrow \sim \\
T \otimes A P \otimes_B \text{Hom}_B(P, A)
\end{array} \xrightarrow{1_T \otimes \text{ev}} \begin{array}{c}
B \otimes_B T \\
\downarrow \sim \\
T \otimes A A
\end{array}
$$

Our plan is to do this in stages: we will show that the vertical maps in the following diagram exist, and are quasi-isomorphisms, and that the diagram commutes:

$$
\begin{array}{c}
T \otimes A P \otimes_B \text{Hom}_B(T \otimes_A P, B) \otimes_B T \\
\downarrow \sim \\
T \otimes A P \otimes_B \text{Hom}_B(T \otimes_A P, B \otimes_B T)
\end{array} \xrightarrow{\text{ev}} \begin{array}{c}
B \otimes_B T \\
\downarrow \sim \\
T \otimes A A
\end{array}
$$

Let us show that the first square commutes. We'll introduce some temporary notation for the rest of this proof. Let $F$ and $G$ denote the functors $F = T \otimes A P \otimes_B -$ and $G = \text{Hom}_B(T \otimes_A P, -)$, so $F$ is left adjoint to $G$, and let $H$ denote the functor $- \otimes_B T$. Then we have unit and counit natural transformations $\varepsilon : FG \to 1$ and $\eta : 1 \to GF$, and a natural isomorphism $\zeta : HF \cong FH$ coming from the associativity isomorphism for tensor products. We first need to define a map $HFGB = T \otimes A P \otimes_F \text{Hom}_B(T \otimes_A P, B) \otimes_B T \to T \otimes A P \otimes_F \text{Hom}_B(T \otimes_A P, B \otimes_B T) = FGHB$

We define this as the composite $HFGB \xrightarrow{\zeta GB} FHGB \xrightarrow{F\text{HGB}} FGFHGB \xrightarrow{F\text{G\zeta^{-1}GB}} FGHFGB \xrightarrow{F\text{GHcB}} FGHB$.

One checks that this is an isomorphism using the same argument as in Lemma 4.3. To see that the
diagram commutes, we break it up into smaller diagrams as follows:

Now we see that both squares commute by the naturality of \( \varepsilon \), the triangle commutes by the triangle identity \( \varepsilon_F \circ F\eta = 1_F \), and the pentagon commutes because the isomorphisms are defined by \( \zeta \) and its inverse.

To define the second square we use the obvious composite isomorphism \( B \otimes_B T \cong T \cong T \otimes_A A \) and the fact that the evaluation map is a counit, and therefore a natural transformation.

To show that the third square commutes, we introduce some more notation. Let \( F' \) and \( G' \) denote the functors \( F' = P \otimes_B - \) and \( G' = \text{Hom}_A(P, -) \), so \( F' \) is left adjoint to \( G' \), and let \( H' \) and \( I' \) denote the functors \( H' = T \otimes_A - \) and \( I' = \text{Hom}_B(T, -) \), so \( H' \) is left adjoint (in fact, quasi-inverse) to \( I' \).

We denote the counit and unit maps of the first adjunction by \( \varepsilon' : F' G' \to 1 \) and \( \eta' : 1 \to G' F' \), and of the second adjunction by \( \varepsilon'' : H' I' \to 1 \) and \( \eta'' : 1 \to I' H' \). Note that, because \( H' \) induces an equivalence of derived categories, \( \varepsilon'' \) and \( \eta'' \) give quasi-isomorphisms when applied to any object.

Using the associativity isomorphism for tensor products we have a natural isomorphism of functors \( F \cong H' F' \), and by the uniqueness of right adjoints (or by using the tensor-hom adjunctions directly) this gives us another natural isomorphism \( G \cong G' I' \).

We now redraw our final square, breaking it up into smaller diagrams:

Here, the top square commutes by definition of the isomorphisms \( F \cong H' F' \) and \( G \cong G' I' \), the triangle commutes by the triangle identity \( \varepsilon'' H' \circ H' \eta'' = 1_{H'} \), and the bottom square commutes by the naturality of \( \varepsilon' \).

We now describe the braid relations for spherical twists, as in Propositions 2.12 and 2.13 of [ST] (see also [RZ,Gra2]).

**Proposition 4.11.** Suppose that \( M \) and \( N \) are spherical objects of \( \text{D}_{\text{id}}(A) \) and let

\[
(M, N) = \dim \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\text{D}_{\text{id}}(A)}(M, N[n]).
\]

Let \( F_M, F_N : \text{D}_{\text{id}}(A) \cong \text{D}_{\text{id}}(A) \) be the associated spherical twists.
• If \((M, N) = 0\) then \(F_M \circ F_N \cong F_N \circ F_M\);

• if \((M, N) = 1\) then \(F_M \circ F_N \circ F_M \cong F_N \circ F_M \circ F_N\).

4.5 Ginzburg dg-algebras

There is a by now well-known method to associate a differential graded algebra to a quiver with potential ([Gin Section 5], [KY Section 2.6]).

Let \((Q, W)\) be a quiver with potential. Construct a new quiver \(\overline{Q}\) by adding arrows to \(Q\): for each arrow \(a : i \rightarrow j\) in \(Q\) we add a new arrow \(a^* : j \rightarrow i\), and for each vertex \(i\) in \(Q\) we add a new arrow \(t_i : i \rightarrow i\). We view \(\overline{Q}\) as a graded quiver with the arrows of \(Q\) in degree 0, the arrows \(a^*\) in degree \(-1\), and the arrows \(t_i\) in degree \(-2\). This induces a grading on the path algebra \(\mathbb{F}Q\) of \(Q\) such that the degree 0 part \(\mathbb{F}Q_0\) is just the path algebra of \(\mathbb{F}Q\) of \(Q\). Let \(\overline{J}\) denote the ideal of \(\mathbb{F}\overline{Q}\) generated by the arrows of \(\overline{Q}\), and let \(\mathbb{F}\overline{Q}\) denote the completion of the graded algebra \(\mathbb{F}Q\) with respect to \(\overline{J}\), as in Section 4.4.

We define a differential \(d\) on \(\mathbb{F}\overline{Q}\) by imposing that \(d\) is zero on each idempotent \(e_t\) associated to a vertex \(t\) of \(Q\), specifying how \(d\) acts on arrows of \(\overline{Q}\), and then extending to the rest of \(\mathbb{F}\overline{Q}\) using the Liebniz rule and continuity. For degree reasons, we must have \(d(a) = 0\) for each arrow \(a\) of \(Q\).

For arrows \(a^*\), we set \(d(a^*) = \partial_a W\), where \(\partial_a\) denotes the cyclic derivative. For arrows \(t_i\), we set \(d(t_i) = e_t(\sum aa^* - a^*a) e_t\), where we sum over all arrows \(a\) of \(Q\). Then \(\Gamma_{Q,W} = (\mathbb{F}\overline{Q}, d)\) is called the Ginzburg dga of \((Q, W)\).

Note that [KY Lemma 2.9] if \((Q_1, W_1)\) and \((Q_2, W_2)\) are right equivalent, then we have an isomorphism of dgas \(\Gamma_{Q_1,W_1} \cong \Gamma_{Q_2,W_2}\). Hence, in our restricted situation where we only deal with quivers with potential of mutation type \(A\) or \(D\), by Proposition 4.5 we only need to consider the Ginzburg dgas \(\Gamma_{Q,W}\), and so can denote them \(\Gamma_Q\).

Keller and Yang showed [KY Theorem 3.2] that QP-mutation lifts to equivalences of derived categories of Ginzburg dgas. Suppose that \((Q, W)\) is a QP and that \((Q', W') = \mu_k(Q, W)\) for some \(k \in \mathbb{Z}\).

**Theorem 4.12 (Keller-Yang).** There is a tilting complex \(T\) which gives an equivalence of triangulated categories

\[
\mu_k = \text{Hom}_{\mathbb{F}Q,W'}(T, -) : D(\Gamma_{Q',W'}) \rightarrow D(\Gamma_{Q,W})
\]

and it restricts to an equivalence of triangulated categories

\[
\mu_k = \text{Hom}_{\mathbb{F}Q,W}(T, -) : D_{\text{id}}(\Gamma_{Q',W'}) \rightarrow D_{\text{id}}(\Gamma_{Q,W}).
\]

Recall that, for a dga \(A\), the finite-dimensional derived category \(D_{\text{id}}(A)\) is \(d\)-Calabi-Yau if there exists a bifunctorial isomorphism

\[
\text{Hom}_{D_{\text{id}}(A)}(M, N) \cong \text{Hom}_{D_{\text{id}}(A)}(N, M[d])^*
\]

where \((-)^*\) denotes the \(k\)-linear dual. We will need the following important result [KVdB Theorem 6.3 and Theorem A.12]:

**Theorem 4.13 (Keller, Van den Bergh).** \(D_{\text{id}}(\Gamma_{Q,W})\) is \(3\)-Calabi-Yau.
Let \((Q, W)\) be a QP and \(\Gamma = \Gamma_{Q,W}\). Associated to each vertex \(i\) of \(Q\), we have the one-dimensional simple left \(\Gamma\)-module, which we denote \(S_i\). In [KY, Section 2.14], Keller and Yang explain how to construct the cofibrant replacement of \(S_i\): as long as we remember the differential, we can proceed as if the Ginzburg dga were an ordinary hereditary algebra, and the underlying \(u(\Gamma)\)-module of \(p S_i\) is the direct sum of one copy of the projective \(P_i\) and one copy of the shifted projective \(P_i[1]\) for each arrow \(j \to i\) in \(Q\). Using this, they show [KY, Lemma 2.15]:

**Lemma 4.14.** Let \(i, j \in I\) and \(n \in \mathbb{Z}\) and \(\Gamma = \Gamma_{Q,W}\). Then \(\text{Hom}_{D^b(\Gamma)}(S_i, S_j[n]) = 0\) if \(n \neq 0, 1, 2, 3\), and

\[
\dim \mathbb{F} \text{Hom}_{D^b(\Gamma)}(S_i, S_j[n]) = \begin{cases} 
\delta_{ij} & \text{if } n = 0; \\
\#(\text{arrows } i \to j \text{ in } Q) & \text{if } n = 1; \\
\#(\text{arrows } j \to i \text{ in } Q) & \text{if } n = 2; \\
\delta_{ij} & \text{if } n = 3.,
\end{cases}
\]

where \(\delta_{ij}\) is the Kronecker delta.

### 4.6 Relations between functors

By Theorem 4.13 every object of \(D_{id}(\Gamma_{Q,W})\) is a 3-Calabi-Yau object. By Lemma 4.14

\[
\bigoplus_{j \in \mathbb{Z}} \text{Hom}_{D^b(\Gamma_{Q,W})}(S_i, S_j[j]) \cong \mathbb{F}[x]/(x^2)
\]

with \(x\) in degree 3. Hence \(S_i\) is 3-spherical, and we have a spherical twist \(F_{S_i}\) associated to \(S_i\). We will sometimes write \(F_i\) instead of \(F_{S_i}\).

Let \(k\) be a vertex of \(Q\), and write \((Q', W') = \mu_k(Q, W)\). Then write \(\Gamma = \Gamma_{Q,W}\) and \(\Gamma' = \Gamma_{Q', W'}\) for the associated Ginzburg dgas. Write \(T_i\) for the left \(\Gamma'\)-module associated to the vertex \(i\) of \(Q'\) and \(G_i\) for the associated autoequivalence \(F_{T_i}\) of \(D_{id}(\Gamma')\). In this section we will study how the spherical twists \(F_i : D_{id}(\Gamma) \to D_{id}(\Gamma)\) interact with the mutation functors \(\mu_k : D_{id}(\Gamma') \to D_{id}(\Gamma)\). Our key tools will be Proposition 4.11 and the results on the images of the simple modules under the mutation functors [KY, Lemma 3.12(a)], which we will describe below.

If \(A\) is a dga and \(M, N \in D_{id}(A)\), we have a natural map

\[
M \otimes_{\mathbb{F}} \text{Hom}_{D_{id}(A)}(M, N) \to N
\]

in \(D_{id}(A)\) given by evaluation. For any graded vector space \(V\), we have biadjoint functors \(- \otimes_{\mathbb{F}} V\) and \(- \otimes_{F} V^*\), and these respect the left \(A\)-module structure, so we also obtain a natural map

\[
M \to N \otimes_{\mathbb{F}} \text{Hom}_{D_{id}(A)}(M, N)^*
\]

in \(D_{id}(A)\). Now let \(L, N \in A\text{-Mod}\). The *universal extension of \(N\) by \(L\)* is the cone of the natural map

\[
N[-1] \to L \otimes_{\mathbb{F}} \text{Hom}_{D_{id}(A)}(N[-1], L)^*
\]

and the *universal coextension of \(L\) by \(N\)* is the cone of the natural map

\[
N[-1] \otimes_{\mathbb{F}} \text{Hom}_{D_{id}(A)}(N[-1], L) \to L.
\]

The following result is contained in the proof of [KY, Lemma 3.12(a)]:

\[
\text{dim } \mathbb{F} \text{Hom}_{D^b(\Gamma_{Q,W})}(S_i, S_j[n]) = \begin{cases} 
\delta_{ij} & \text{if } n = 0; \\
\#(\text{arrows } i \to j \text{ in } Q) & \text{if } n = 1; \\
\#(\text{arrows } j \to i \text{ in } Q) & \text{if } n = 2; \\
\delta_{ij} & \text{if } n = 3.,
\end{cases}
\]
Lemma 4.15 (Keller-Yang). We have $\mu_k(T_k) \cong S_k[1]$ and, for $i \neq k$, $\mu_k(T_i)$ is the universal extension of $S_i$ by $S_k$.

The following result should be compared to Definition 2.8

Proposition 4.16. We have a natural isomorphism of functors

$$F_{\mu_i^{-1}(S)} \cong \begin{cases} G_k G_i G_k^{-1} & \text{if } i \rightarrow k \text{ in } Q; \\ G_i & \text{otherwise.} \end{cases}$$

Proof: We first use Lemma 4.15 to calculate the images of the simple $\Gamma'$-modules under the inverse mutation functor $\mu_i^{-1}$, where $\mu = \mu_k$. We know that $\mu(T_k) \cong S_k[1]$, so $\mu_i^{-1}(S_k) \cong T_k[-1]$. As our quiver $Q$ is mutation Dynkin, we know that there is at most one arrow between any two vertices in $Q$. If $i \neq k$ and there is no arrow $i \rightarrow k$ in $Q$ then, by Lemma 4.15, $\text{Hom}_{\text{D}(\Gamma')}^d(S_k[-1], S_k) = 0$ and so $\mu(T_i) \cong \text{cone}(S_i[-1] \rightarrow 0) \cong S_i$, thus $\mu_i^{-1}(S_i) \cong T_i$.

If $i \neq k$ and there is an arrow $i \rightarrow k$ in $Q$ then $\text{Hom}_{\text{D}(\Gamma')}^d(S_k[-1], S_k)$ is 1-dimensional and so $\mu(T_i) \cong \text{cone}(S_i[-1] \rightarrow S_k)$, with the nonzero map determined up to scalar. We can then use Lemma 4.16 to calculate $\mu(\text{cone}(S_k[-1] \rightarrow T_i))$: this is cone($\mu(T_k)[-1] \rightarrow \mu(T_i)$) where, as $\mu$ is an equivalence, the map must again be nonzero and determined up to scalar. We know that $\mu(T_k)[-1] \cong S_k$ and $\mu(T_i)$ is $S_i \oplus S_k$ with appropriate differential. One can check that the injection $S_k \hookrightarrow S_i \oplus S_k$ respects the differentials, and so this must be our nonzero map. This is quasi-isomorphic to the map $0 \rightarrow S_i$, and so $\mu(\text{cone}(S_k[-1] \rightarrow T_i)) \cong S_i$, and hence $\mu_i^{-1}(S_i) \cong \text{cone}(T_k[-1] \rightarrow T_i)$. Note that this is the universal coextension of $T_i$ by $T_k$.

Now we check that our formula holds. If $i = k$ then $F_{\mu_i^{-1}(S)} = F_{T_i[1]}$, and as the shift functor on $\text{D}(\Gamma')$ is naturally isomorphic to $[1] \otimes \Gamma'$ we see that, by Proposition 4.10, $F_{T_i[1]} \cong [1] \otimes G_i \otimes [-1] \cong G_i$. If $i \neq k$ and there is no arrow $i \rightarrow k$ in $Q$ then $\mu_i^{-1}(S_i) \cong T_i$ so $F_{\mu_i^{-1}(T_i)} = G_i$.

Finally, suppose $i \neq k$ and there is an arrow $i \rightarrow k$ in $Q$. As mutation at $k$ reverses all arrows incident with $k$, and can never change the number of arrows incident with $k$, there must be exactly one arrow $k \rightarrow i$ in $Q'$. We first calculate $G_k(T_i)$: this is

$$\text{cone}(p T_k \otimes \text{Hom}_\Gamma(p T_k, T_i) \rightarrow T_i).$$

As $\text{Hom}_\Gamma(p T_k, T_i)$ is a differential graded $\mathbb{P}$-module, it is quasi-isomorphic to its homology, which is the direct sum $\bigoplus \text{Hom}_\Gamma(T_k, T_i[n])$ with $\text{Hom}_\Gamma(T_k, T_i[n]) \cong \text{Hom}_{\text{D}(\Gamma')}^d(T_k, T_i[n])$ in degree $n$. So by Lemma 4.15 the homology is only nonzero in degree 1, where it is 1-dimensional, and thus $G_k(T_i) \cong \text{cone}(T_k \otimes \mathbb{P}[-1] \rightarrow T_i)$. So we see that $\mu_i^{-1}(S_i) \cong G_k(T_i)$ and thus, using Proposition 4.10 again, $F_{\mu_i^{-1}(S)} \cong G_k G_i G_k^{-1}$.

We are now able to show that our braid groups $B_Q$ act via spherical twists on the category $\text{D}(\Gamma)$.

Theorem 4.17. Let $Q$ be a mutation Dynkin quiver of type A or D and let $W$ be the sum of cycles potential from Section 4.1. Then we have a group homomorphism

$$B_Q \rightarrow \text{Aut}_{\text{D}(\Gamma)}(G_{Q,W})$$

$$s_i \mapsto F_i$$

sending the group generator associated to the vertex $i \in I$ to the spherical twist at the simple $G_{Q,W}$-module $S_i$. 

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Proof: As \((Q, W)\) is mutation Dynkin, it is obtained by mutating a quiver with potential \((Q'', 0)\) finitely many times, where \(Q''\) is a Dynkin quiver. Then we have a group homomorphism \(B_{Q''} \to \text{Aut} \, D_{\text{id}}(\Gamma_{Q'', a})\) by Remark 2.3, Proposition 4.11 and Lemma 4.14. This gives the base case of an inductive argument, so we need to show that if the spherical twists \(F_i\) on \(\Gamma = \Gamma_{Q, W}\) satisfy the relations of \(B\), then the spherical twists \(G_i\) on \(\Gamma = \Gamma_{Q', W'}\) satisfy the relations of \(B_{Q'}\).

Assume the functors \(F_i: D_{\text{id}}(\Gamma) \to D_{\text{id}}(\Gamma)\) satisfy the relations of \(B\) and let \(\mu = \mu_k: D_{\text{id}}(\Gamma') \to D_{\text{id}}(\Gamma)\) be the Keller-Yang derived equivalence. Then the functors \(\mu^{-1} \circ F_i \circ \mu: D_{\text{id}}(\Gamma') \to D_{\text{id}}(\Gamma')\) also satisfy the relations of \(B\). By Proposition 4.10, we have \(\mu^{-1} \circ F_i \circ \mu \cong F_{\mu^{-1}(S_i)}\), i.e., the following diagram commutes:

\[
\begin{array}{ccc}
D_{\text{id}}(\Gamma') & \xrightarrow{\mu} & D_{\text{id}}(\Gamma) \\
\downarrow F_{\mu^{-1}(S_i)} & & \downarrow F_i \\
D_{\text{id}}(\Gamma') & \xrightarrow{\mu} & D_{\text{id}}(\Gamma)
\end{array}
\]

So we have a group homomorphism \(\rho: B_Q \xrightarrow{\rho} \text{Aut} \, D_{\text{id}}(\Gamma')\) sending \(s_i\) to \(F_{\mu^{-1}(S_i)}\) which, using Proposition 4.10, we can write as

\[
B_Q \xrightarrow{\rho} \text{Aut} \, D_{\text{id}}(\Gamma') \\
s_i \mapsto \begin{cases} G_k G_i G_k^{-1} & \text{if } i \to k \text{ in } Q; \\ G_i & \text{otherwise.} \end{cases}
\]

Precomposing this with the group isomorphism \(\varphi_k^{-1}: B_Q \xrightarrow{\sim} B_Q\) of Remark 2.11, we obtain the group homomorphism

\[
B_{Q'} \xrightarrow{\varphi_k^{-1}} B_Q \xrightarrow{\rho} \text{Aut} \, D_{\text{id}}(\Gamma')
\]

\[
t_i \mapsto \begin{cases} s_k^{-1} s_i s_k & \text{if } i \to k \text{ in } Q; \\ s_i & \text{otherwise;} \end{cases} \quad \begin{cases} G_k^{-1} G_k G_i G_k^{-1} G_k & \text{if } i \to k \text{ in } Q; \\ G_i & \text{otherwise;} \end{cases} \cong G_i
\]

as required. \(\square\)

Remark 4.18. In fact, one can use earlier results on braid group actions to see that the actions of Theorem 4.17 are faithful. It was shown by Seidel and Thomas [ST Theorem 2.18], building on work of Khovanov and Seidel [KS], that given a collection of \(d\)-spherical objects, with \(d \geq 2\), in a type \(A_n\)-configuration the action of the braid group by spherical twists is faithful. This was extended to all collections of 2-spherical objects in type \(ADE\) configurations by Brav and Thomas [BT], using the Garside structure of the braid monoid, and it appears that their argument may generalize to \(d\)-spherical objects for \(d \geq 2\). If so, then using Lemma 4.14, we know that if \(Q\) is an orientation of a type \(A_n\) or \(D_n\) graph then the action \(B_Q \to \text{Aut} \, D_{\text{id}}(\Gamma_{Q, 0})\) is faithful. From the proof of Theorem 4.17, we see that our actions of \(B_Q\) where \(Q\) is of mutation type \(A\) or \(D\) are just built by precomposing group isomorphisms with the usual actions, and so they are also faithful.

Remark 4.19. Although we have shown that our braid groups of mutation Dynkin quivers can be realized categorically, this is not a categorification of our earlier results because we cannot decategorify (see, for example, [BD]): we cannot recover Theorem 2.10 from Theorem 4.17 because we use Theorem 2.10 to prove Theorem 4.17. The problem is that, for an arbitrary mutation Dynkin quiver with potential \((Q, W)\), we do not in advance know the relations satisfied by the spherical twist functors \(F_i\). This question will be addressed in a forthcoming paper [Gra3].

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Remark 4.20. The arguments of [Gra2] generalize to show that, if vertices $i$ and $j$ of $Q$ are joined by an arrow, then $F_i F_j F_i \cong F_j F_i F_j$ can be realized as a single periodic twist. Similarly, one can show that if $i \to j \to k \to i$ is a 3-cycle in $Q$ then $F_1 F_2 F_3 F_1 \cong F_2 F_3 F_1 F_2 \cong F_3 F_1 F_2 F_3$ can be realized as a single periodic twist. This exhausts the possibilities in type $A$; type $D$ will be studied further in [Gra3].

References

[All] D. Allcock, *Braid pictures for Artin groups*, Trans. Amer. Math. Soc. 354 (2002), no. 9, 3455–3474

[BD] J. Baez and J. Dolan, *Categorification*, Higher category theory (Evanston, IL, 1997), 1–36, Contemp. Math., 230, Amer. Math. Soc., Providence, RI, 1998

[BM] M. Barot and R. J. Marsh, *Reflection group presentations arising from cluster algebras*, Preprint arXiv:1112.2300v2 [math.GR], 2011 To appear in Trans. Amer. Math. Soc.

[BT] C. Brav and H. Thomas, *Braid groups and Kleinian singularities*, Math. Ann. 351 (2011), no. 4, 1005–1017

[CCS] P. Caldero, F. Chapoton, and R. Schiffler, *Quivers with relations arising from clusters (A_n case)*, Trans. Amer. Math. Soc. 358 (2006), no. 3, 1347–1364

[DWZ] H. Derksen, J. Weyman, and A. Zelevinsky, *Quivers with potentials and their representations. I. Mutations*, Selecta Math. (N.S.) 14 (2008), no. 1, 59-119

[FeSTu] A. Felikson, M. Shapiro, P. Tumarkin, *Cluster algebras and triangulated orbifolds*, Adv. Math. 231 (2012), no. 5, 2953–3002

[FoSTh] S. Fomin, M. Shapiro, and D. Thurston, *Cluster algebras and triangulated surfaces. Part I: Cluster complexes*, Acta Math. 201 (2008), no. 1, 83–146

[FZ1] S. Fomin and A. Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. 15 (2002), no. 2, 497–529.

[FZ2] S. Fomin and A. Zelevinsky, *Cluster algebras. II. Finite type classification*, Invent. Math. 154 (2003), no. 1, 63–121.

[FN] R. Fox and L. Neuwirth, *The braid groups*, Math. Scand. 10 (1962), 119–126

[Gin] V. Ginzburg, *Calabi-Yau algebras*, arXiv:math/0612139v3 [math.AG]

[Gra1] J. Grant, *Derived autoequivalences from periodic algebras*, Proc. Lond. Math. Soc. (3) 106 (2013), no. 2, 375-409

[Gra2] J. Grant, *Lifts of longest elements to braid groups acting on derived categories*, Preprint arXiv:1207.2758 [math.RT], 2012 To appear in Trans. Amer. Math. Soc.

[Gra3] J. Grant, *n-cycle relations for spherical twists*, in preparation

[Hum] J. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Studies in Advanced Mathematics, 29. Cambridge University Press, Cambridge, 1990

[KT] C. Kassel and V. Turaev, *Braid Groups*, Graduate Texts in Mathematics, 247. Springer, New York, 2008
B. Keller, *Deriving DG categories*, Ann. Sci. École Norm. Sup. (4) 27 (1994), no. 1, 63-102.

B. Keller, *On differential graded categories*, International Congress of Mathematicians. Vol. II, 151–190, Eur. Math. Soc., Zürich, 2006.

B. Keller, with an appendix by M. Van den Bergh, *Deformed Calabi-Yau completions*, Journal für die Reine und Angewandte Mathematik (Crelle’s Journal), 654 (2011), 125-180

B. Keller and D. Yang, *Derived equivalences from mutations of quivers with potential*, Adv. Math. 226 (2011), no. 3, 2118-2168

M. Khovanov and P. Seidel, *Quivers, Floer cohomology, and braid group actions*, J. Amer. Math. Soc. 15 (2002), no. 1, 203–271.

D. Labardini-Fragoso, *Quivers with potentials associated to triangulated surfaces*, Proc. London Math. Soc. (2009) 98 (3): 7970–839

D. Labardini-Fragoso, *Quivers with potentials associated to triangulated surfaces, part IV: Removing boundary assumptions* Preprint arXiv:1206.1798v4 [math.CO], 2012

K. Nagao, *Triangulated surface, mapping class group and Donaldson-Thomas theory*, available at http://www.math.sci.osaka-u.ac.jp/ kazushi/kinosaki2010/proceeding.html

Y. Qiu, *On the spherical twists on 3-Calabi-Yau categories from marked surfaces*, Preprint arXiv:1407.0806 [math.RT]

R. Rouquier and A. Zimmermann, *Picard groups for derived module categories*, Proc. London Math. Soc. (2003) 87 (1): 197–225.

R. Schiffler, *A geometric model for cluster categories of type D_n*, J. Algebraic Combin. 27 (2008), no. 1, 1–21

P. Seidel and R. Thomas, *Braid group actions on derived categories of coherent sheaves*, Duke Math. J. 108 (2001), no. 1, 37-108

V. Sergiescu, *Graphes planaires et présentations des groupes de tresses [Planar graphs and presentations of braid groups]*, Math. Z. 214 (1993), no. 3, 477–490

D. Thurston, *The Geometry and Algebra of Curves on Surfaces*, Course at UC Berkeley, Fall 2014; notes taken by Qiaochu Yuan, available from http://math.berkeley.edu/~qchu/Notes/274/

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