Isolation of cycles

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Abstract

For any graph $G$, let $\iota_c(G)$ denote the size of a smallest set $D$ of vertices of $G$ such that the graph obtained from $G$ by deleting the closed neighbourhood of $D$ contains no cycle. We prove that if $G$ is a connected $n$-vertex graph that is not a triangle, then $\iota_c(G) \leq n/4$. We also show that the bound is sharp. Consequently, we solve a problem of Caro and Hansberg.

1 Introduction

Unless stated otherwise, we use small letters such as $x$ to denote non-negative integers or elements of sets, and capital letters such as $X$ to denote sets or graphs. The set of positive integers is denoted by $\mathbb{N}$. For $n \geq 1$, $[n]$ denotes the set $\{1, \ldots, n\}$ (that is, $[n] = \{i \in \mathbb{N}: i \leq n\}$). We take $[0]$ to be the empty set $\emptyset$. Arbitrary sets are assumed to be finite. For a set $X$, $X^2$ denotes the set of 2-element subsets of $X$ (that is, $X^2 = \{\{x, y\}: x, y \in X, x \neq y\}$).

If $Y$ is a subset of $X$ and $G$ is the pair $(X, E(G) \cap (X^2))$, then we say that $G$ is a graph, $X$ is called the vertex set of $G$, and $Y$ is called the edge set of $G$. We may represent an edge $\{v, w\}$ by $vw$. If $vw \in E(G)$, then we say that $w$ is a neighbour of $v$ in $G$ (and vice-versa). For $v \in V(G)$, $N_G(v)$ denotes the set of neighbours of $v$ in $G$, $N_G[v]$ denotes $N_G(v) \cup \{v\}$, and $d_G(v)$ denotes $|N_G(v)|$ and is called the degree of $v$ in $G$. For $S \subseteq V(G)$, $N_G(S)$ denotes $\bigcup_{v \in S} N_G[v]$ (the closed neighbourhood of $S$), $G[S]$ denotes $(S, E(G) \cap \binom{S}{2})$ (the subgraph of $G$ induced by $S$), and $G - S$ denotes $G[V(G) \setminus S]$ (the graph obtained by deleting $S$ from $G$). Where no confusion arises, the subscript $G$ may be omitted from the notation above that uses it; for example, $N_G(v)$ may be abbreviated to $N(v)$.

If $G$ and $H$ are graphs, $f : V(H) \to V(G)$ is a bijection, and $E(G) = \{f(v)f(w): vw \in E(H)\}$, then we say that $G$ is a copy of $H$, and we write $G \simeq H$. Thus, a copy of $H$ is a graph obtained by relabeling the vertices of $H$. 

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For $n \geq 1$, the graphs $([n], \binom{[n]}{2})$ and $([n], \{\{i, i+1\} : i \in [n-1]\})$ are denoted by $K_n$ and $P_n$, respectively. For $n \geq 3$, $C_n$ denotes the graph $([n], \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}, \{n, 1\}\}) = ([n], E(P_n) \cup \{n, 1\})$. A copy of $K_n$ is called a complete graph. A copy of $P_n$ is called an $n$-path or simply a path. A copy of $C_n$ is called an $n$-cycle or simply a cycle. We call a 3-cycle a triangle. Note that $K_3$ is the triangle $C_3$.

If $G$ and $H$ are graphs such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then $H$ is called a subgraph of $G$, and we say that $G$ contains $H$.

If $\mathcal{F}$ is a set of graphs and $F$ is a copy of a graph in $\mathcal{F}$, then we call $F$ an $\mathcal{F}$-graph. If $G$ is a graph and $D \subseteq V(G)$ such that $G - N[D]$ contains no $\mathcal{F}$-graph, then $D$ is called an $\mathcal{F}$-isolating set of $G$. Let $\iota(G, \mathcal{F})$ denote the size of a smallest $\mathcal{F}$-isolating set of $G$. We abbreviate $\iota(G, \{F\})$ to $\iota(G, F)$. The study of isolating sets was introduced by Caro and Hansberg [1]. It is an appealing and natural generalization of the classical domination problem [2, 3, 4, 5, 6, 7]. Indeed, $D$ is a $\{K_1\}$-isolating set of $G$ if and only if $D$ is a dominating set of $G$ (that is, $N[D] = V(G)$), so the $\{K_1\}$-isolation number is the domination number (the size of a smallest dominating set). Let $\mathcal{C}$ denote $\{C_k : k \geq 3\}$. In this paper, we obtain a sharp upper bound for $\iota(G, \mathcal{C})$, and consequently we solve a problem of Caro and Hansberg [1].

We call a subset $D$ of $V(G)$ a cycle isolating set of $G$ if $G - N[D]$ contains no cycle (that is, $G - N[D]$ is a forest). We denote the size of a smallest cycle isolating set of $G$ by $\iota_c(G)$. Thus, $\iota_c(G) = \iota(G, \mathcal{C})$.

If $G_1, \ldots, G_t$ are graphs such that $V(G_i) \cap V(G_j) = \emptyset$ for every $i, j \in [t]$ with $i \neq j$, then $G_1, \ldots, G_t$ are vertex-disjoint. A graph $G$ is connected if, for every $v, w \in V(G)$, $G$ contains a path $P$ with $v, w \in V(P)$. A connected subgraph $H$ of $G$ is a component of $G$ if, for each connected subgraph $K$ of $G$ with $K \neq H$, $H$ is not a subgraph of $K$. Clearly, any two distinct components of $G$ are vertex-disjoint.

For $n, k \in \mathbb{N}$, let $a_{n,k} = \left\lfloor \frac{n}{k+1} \right\rfloor$ and $b_{n,k} = n - ka_{n,k}$. Thus, $a_{n,k} \leq b_{n,k} \leq a_{n,k} + k$. If $F$ is a $k$-vertex graph and $n \leq k$, then let $B_{n,F} = P_n$. If $F$ is a $k$-vertex graph and $n \geq k + 1$, then let $F_1, \ldots, F_{a_{n,k}}$ be copies of $F$ such that $P_{b_{n,k}}, F_1, \ldots, F_{a_{n,k}}$ are vertex-disjoint, let $P_{b_{n,k}}^* = ([b_{n,k}], E(P_{b_{n,k}}) \cup \{a_{n,k}i : i \in [b_{n,k}] \setminus [a_{n,k}]\})$, and let $B_{n,F}$ be the connected $n$-vertex graph given by

$$B_{n,F} = \left(V(P_{b_{n,k}}^*) \cup \bigcup_{i=1}^{a_{n,k}} V(F_i), E(P_{b_{n,k}}^*) \cup \{iv : i \in [a_{n,k}], v \in V(F_i)\} \cup \bigcup_{i=1}^{a_{n,k}} E(F_i)\right).$$

Thus, $B_{n,F}$ is the graph obtained by taking $P_{b_{n,k}}^*, F_1, \ldots, F_{a_{n,k}}$ and joining $i$ (a vertex of $P_{b_{n,k}}^*$) to each vertex of $F_i$ for each $i \in [a_{n,k}]$.

For any $n \in \mathbb{N}$, any family $\mathcal{F}$ of graphs, and any $F \in \mathcal{F}$, let

$$\iota(n, \mathcal{F}, F) = \max \{\iota(G, \mathcal{F}) : G \text{ is a connected graph}, V(G) = [n], G \not\subseteq F\}.$$ 

We abbreviate $\iota(n, \{F\}, F)$ to $\iota(n, F)$. Let $\iota_c(n) = \iota(n, \mathcal{C}, K_3)$. In Section 2 we prove the following result.

**Theorem 1.1** If $G$ is a connected $n$-vertex graph that is not a triangle, then

$$\iota_c(G) \leq \frac{n}{4}.$$
Consequently, for any \( n \geq 1 \),
\[
\iota_c(n) = \iota_c(B_{n,K_3}) = \left\lfloor \frac{n}{4} \right\rfloor.
\]
The equality \( \iota_c(B_{n,K_3}) = \left\lfloor \frac{n}{4} \right\rfloor \) is generalized in the following result.

**Lemma 1.2** Let \( n, k \in \mathbb{N} \) and let \( F \) be a \( k \)-vertex graph.

(i) If \( n \neq k \) or \( F \neq P_k \), then
\[
\iota(B_{n,F}) = \left\lfloor \frac{n}{k+1} \right\rfloor.
\]

(ii) If \( \mathcal{F} \) is a family of graphs, \( F \in \mathcal{F} \), and \( n \neq k = |V(F)| = \min \{|V(H)| : H \in \mathcal{F}\} \),
then
\[
\left\lfloor \frac{n}{k+1} \right\rfloor = \iota(B_{n,F}) \leq \iota(n,F) \leq \iota(n,\mathcal{F},F).
\]

**Proof.** Let \( B = B_{n,F} \). If either \( n < k \) or \( n = k \) and \( F \neq P_k \), then \( \iota(B,F) = 0 \).
Suppose \( n \geq k + 1 \). Then, \( \iota(B,F) \leq a_{n,k} \) as \( [a_{n,k}] \) is a dominating set of \( B \). Let \( D \) be an \( \{F\}\)-isolating set of \( B \) of size \( \iota(B,F) \). For each \( i \in [a_{n,k}] \), \( D \cap (V(F) \cup \{i\}) \neq \emptyset \) as \( B - N_B[D] \) does not contain the copy \( F_i \) of \( F \). Thus, \( |D| \geq a_{n,k} \). Hence, (i) is proved.

Let \( \mathcal{F} \) and \( n \) be as in (ii). Since \( n \neq |V(F)| \), \( B \neq F \). Since \( |V(B)| = n \),
we can choose a copy \( B' \) of \( B \) with \( V(B') = [n] \). Since \( B \) is connected, \( B' \) is connected.
Thus, \( \iota(B',F) \leq \iota(n,F) \). Since \( F \in \mathcal{F} \), the \( \mathcal{F} \)-isolating sets of a graph \( G \)
are \( \{F\}\)-isolating sets of \( G \), so \( \iota(G,F) \leq \iota(G,\mathcal{F}) \). Thus, \( \iota(n,F) \leq \iota(n,\mathcal{F},F) \).
Now \( \iota(n,B') = \iota(n,B) = \left\lfloor \frac{n}{k+1} \right\rfloor \) by (i). Hence, (ii) is proved.

By the results above, \( \left\lfloor \frac{n}{4} \right\rfloor \) is a sharp upper bound on \( \iota(G,K_3) \) for connected \( n \)-vertex graphs \( G \neq K_3 \).

**Theorem 1.3** For any \( n \geq 1 \),
\[
\iota(n,K_3) = \iota(B_{n,K_3},K_3) = \left\lfloor \frac{n}{4} \right\rfloor.
\]

**Proof.** Let \( G \) be a connected \( n \)-vertex graph that is not a copy of \( K_3 \). By Lemma 1.2
(ii), \( \left\lfloor \frac{n}{4} \right\rfloor = \iota(B_{n,K_3},K_3) \leq \iota(n,K_3) \leq \iota(n,C,K_3) \) as \( K_3 = C_3 \in \mathcal{C} \). By Theorem 1.1,
\( \iota(n,C,K_3) = \left\lfloor \frac{n}{4} \right\rfloor \). The result follows.

In [1], Caro and Hansberg showed that \( \frac{1}{4} \leq \limsup_{n \to \infty} \frac{\iota_c(n)}{n} \leq \frac{1}{3} \). In Problem 7.3 of
the same paper, they asked for the value of \( \limsup_{n \to \infty} \frac{\iota_c(n)}{n} \). The answer is immediately
given by Theorem 1.1.

**Corollary 1.4** \( \limsup_{n \to \infty} \frac{\iota_c(n)}{n} = \frac{1}{4} \).

**Proof.** By Theorem 1.1, for any \( n \in \mathbb{N} \), we have \( \frac{1}{4} - \frac{3}{4n} = \frac{1}{n} \left( \frac{n-3}{4} \right) \leq \frac{\iota_c(n)}{n} \leq \frac{1}{4} \),
and, if \( n \)
is a multiple of 4, then \( \frac{\iota_c(n)}{n} = \frac{1}{4} \). Thus, \( \limn \sup \left\{ \frac{\iota_c(k)}{k} : k \geq n \right\} = \limn \frac{1}{4} = \frac{1}{4} \).
2 Proof of Theorem [1.1]

In this section, we prove Theorem [1.1]. We start with two lemmas that will be used repeatedly.

Lemma 2.1 If \( G \) is a graph, \( \mathcal{F} \) is a set of graphs, \( X \subseteq V(G) \), and \( Y \subseteq N[X] \), then

\[
\iota(G, \mathcal{F}) \leq |X| + \iota(G - Y, \mathcal{F}).
\]

Proof. Let \( D \) be an \( \mathcal{F} \)-isolating set of \( G - Y \) of size \( \iota(G - Y, \mathcal{F}) \). Clearly, \( \emptyset \neq V(F) \cap Y \subseteq V(F) \cap N[X] \) for each \( \mathcal{F} \)-graph \( F \) that is a subgraph of \( G \) and not a subgraph of \( G - Y \). Thus, \( D \cup X \) is an \( \mathcal{F} \)-isolating set of \( G \). The result follows. \( \square \)

For a graph \( G \) and a set \( \mathcal{F} \) of graphs, let \( C(G) \) denote the set of components of \( G \), and let \( C(G, \mathcal{F}) = \{ H \in C(G) : H \text{ is an } \mathcal{F} \text{-graph} \} \). We abbreviate \( C(G, \{ K_3 \}) \) to \( C'(G) \). Thus, \( C'(G) = \{ H \in C(G) : H \text{ is a triangle} \} \).

Lemma 2.2 If \( G \) is a graph and \( \mathcal{F} \) is a set of graphs, then

\[
\iota(G, \mathcal{F}) = \sum_{H \in C(G)} \iota(H, \mathcal{F}).
\]

Proof. For each \( H \in C(G) \), let \( D_H \) be a smallest \( \mathcal{F} \)-isolating set of \( H \). Then, \( \bigcup_{H \in C(G)} D_H \) is an \( \mathcal{F} \)-isolating set of \( G \), so \( \iota(G, \mathcal{F}) \leq \sum_{H \in C(G)} |D_H| = \sum_{H \in C(G)} \iota(H, \mathcal{F}) \).

Let \( D \) be a smallest \( \mathcal{F} \)-isolating set of \( G \). For each \( H \in C(G) \), \( D \cap V(H) \) is an \( \mathcal{F} \)-isolating set of \( H \). We have \( \sum_{H \in C(G)} \iota(H, \mathcal{F}) \leq \sum_{H \in C(G)} |D \cap V(H)| = |D| = \iota(G, \mathcal{F}) \).

The result follows. \( \square \)

Proof of Theorem [1.1] Let us first assume the bound in the first part of the theorem. Then, \( \iota_c(n) \leq \frac{n}{4} \). Since \( \iota_c(n) \) is an integer, \( \iota_c(n) \leq \left\lfloor \frac{n}{4} \right\rfloor \). Together with Lemma [1.2](i), this gives us \( \iota_c(n) = \iota_c(B_n, K_3, K_3) = \left\lfloor \frac{n}{4} \right\rfloor \).

We now prove the first part of the theorem. We use induction on \( n \). Let \( G \) be a connected \( n \)-vertex graph that is not a triangle. If \( n \leq 3 \), then, since \( G \) is not a triangle, \( G \) contains no cycle, and hence \( \iota_c(G) = 0 \). Suppose \( n \geq 4 \). Let \( k = \max\{d(v) : v \in V(G)\} \). Since \( G \) is connected, \( k \geq 2 \). Let \( v \in V(G) \) such that \( d(v) = k \). If \( k = 2 \), then \( G \) is a path or a cycle, \( \{v\} \) is a cycle isolating set of \( G \), and hence \( \iota_c(G) \leq 1 \leq \frac{n}{4} \).

Suppose \( d(v) \geq 3 \). Then, \( |N[v]| \geq 4 \). If \( V(G) = N[v] \), then \( \{v\} \) is a cycle isolating set of \( G \), so \( \iota(G) \leq 1 \leq \frac{n}{4} \). Suppose \( V(G) \neq N[v] \). Let \( G' = G - N[v] \) and \( n' = |V(G')| \). Then, \( n \geq n' + 4 \) and \( V(G') \neq \emptyset \). Let \( \mathcal{H} = C(G') \) and \( \mathcal{H}' = C'(G') \). By the induction hypothesis,

\[
\iota_c(H) \leq \frac{|V(H)|}{4} \quad \text{for each} \quad H \in \mathcal{H}' \setminus \mathcal{H}'.
\]

If \( \mathcal{H}' = \emptyset \), then, by Lemma [2.1] (with \( X = \{v\} \) and \( Y = N[v] \)) and Lemma [2.2]

\[
\iota_c(G) \leq 1 + \iota_c(G') = 1 + \sum_{H \in \mathcal{H}} \iota_c(H) \leq 1 + \sum_{H \in \mathcal{H}} \frac{|V(H)|}{4} = \frac{4 + n'}{4} \leq \frac{n}{4}.
\]
Suppose $\mathcal{H}' \neq \emptyset$. For any $H \in \mathcal{H}$ and any $x \in N(v)$ such that $xy \in E(G)$ for some $y \in V(H)$, we say that $H$ is linked to $x$ and that $x$ is linked to $H$. Since $G$ is connected, each member of $\mathcal{H}$ is linked to at least one member of $N(u)$. Let $L = \{x \in N(v) : x$ is linked to some member of $\mathcal{H}'\}$. Since $\mathcal{H}' \neq \emptyset, L \neq \emptyset$. Let $x \in L$. Let $\mathcal{H}_x^+ = \{H \in \mathcal{H}' : H$ is linked to $x\}$ and $\mathcal{H}_x^- = \{H \in \mathcal{H}\setminus \mathcal{H}' : H$ is linked to $x$ only\}. Let $U = \bigcup_{H \in \mathcal{H}_x^+} V(H)$. Let $U_x = N(x) \cap U$ and $U_x^+ = \{x\} \cup U_x$. Note that if a component $A$ of $G - U_x^+$ is a triangle, then $V(A) = N[v]\{x\}$.

Suppose $|U_x| \geq 3$. If no component of $G - U_x^+$ is a triangle, then, by Lemma 2.1 (with $X = \{x\}$ and $Y = U_x^+$), Lemma 2.2 and the induction hypothesis, we have

$$\iota_c(G) \leq 1 + \iota_c(G - U_x^+) = 1 + \sum_{H \in C(G - U_x^+)} \iota_c(H) \leq \frac{|U_x^+|}{4} + \sum_{H \in C(G - U_x^+)} \frac{|V(H)|}{4} = \frac{n}{4}.$$  

Suppose that a component $A$ of $G - U_x^+$ is a triangle. Then, $V(A) = N[v]\{x\}$. Let $Y = U_x^+ \cup V(A)$. Since $G - Y$ contains no triangle, $\iota_c(G - Y) \leq \frac{n-|Y|}{4}$ by Lemma 2.2 and the induction hypothesis. Let $D_{G-Y}$ be a cycle isolating set of $G - Y$ of size $\iota_c(G - Y)$. Since $y \in N(x) \cap V(A)$ and $U_x^+ \subset N[x]$, $\{x\} \cup D_{G-Y}$ is a cycle isolating set of $G$. Thus, $\iota_c(G) \leq 1 + \frac{n-|Y|}{4}$. Since $|Y| \geq 7$, $\iota_c(G) < \frac{n}{4}$.

Now suppose $|U_x| \leq 2$. Then, $1 \leq |\mathcal{H}_x^+| \leq 2$.

Case 1: $|\mathcal{H}_x^+| = 1$. Let $T$ be the member of $\mathcal{H}_x^+$. We have $xy \in E(G)$ for some $y \in V(T)$. Let $Y = \{x\} \cup V(T)$. Then, $Y \subseteq N[y]$. Also, $G - Y$ has a component $T'$ with $N[y]\{x\} \subseteq V(T')$. Since $T$ is the only member of $\mathcal{H}'$ that is linked to $x$, the components of $G - Y$ are $T'$ and the members of $\mathcal{H}_x^+$. Recall that no member of $\mathcal{H}_x^+$ is a triangle.

If $T'$ is not a triangle, then, by Lemma 2.1 (with $X = \{y\}$), Lemma 2.2 and the induction hypothesis, we have $\iota_c(G) \leq 1 + \iota_c(G - Y) \leq 1 + \frac{n-|Y|}{4} = \frac{n}{4}$. 

Suppose that $T'$ is a triangle. Let $W = V(T) \cup V(T')$. We have $x \notin W$, $y \in N(x) \cap V(T)$, and $v \in N(x) \cap V(T')$. Also, the components of $G - \{x\}$ are the components of $G[W]$ and the members of $\mathcal{H}_x^+$. By the induction hypothesis, each member $H$ of $\mathcal{H}_x^+$ has a cycle isolating set $D_H$ with $|D_H| \leq \frac{|V(H)|}{4}$.

Suppose that $G[W] - (N(x) \cap W)$ contains no cycle. Then, $\{x\} \cup \bigcup_{H \in \mathcal{H}_x^+} D_H$ is a cycle isolating set of $G$. Thus, $\iota_c(G) \leq 1 + \sum_{H \in \mathcal{H}_x^+} \frac{|V(H)|}{4} = 1 + \frac{n-|x|\cup W}{4} = 1 + \frac{n-7}{4} < \frac{n}{4}$.

Now suppose that $G[W] - (N(x) \cap W)$ contains a cycle $A$. Since $v, y \in N(x) \cap W$ and $|W| = 6$, either $A \simeq C_3$ or $A \simeq C_4$.

Suppose $A \simeq C_4$. Then, $N(x) \cap W = \{v, y\}$, $V(A) = (V(T)\{y\}) \cup (V(T')\{v\})$, and hence $uw \in E(A) \subseteq E(G)$ for some $u \in V(T)\{y\}$ and some $w \in V(T')\{v\}$. Let $Z = \{w\} \cup V(T)$ and let $x'$ be the member of $V(T')\{v, w\}$. We have $V(G - Z) = \{v, x, x'\} \cup \bigcup_{H \in \mathcal{H}_x^+} V(H)$ and $x, x' \in N(v)$. Thus, since the members of $\mathcal{H}_x^+$ are linked to $x$, $G - Z$ is connected. Since $N(x) \cap W = \{v, y\}$, we have $xx' \notin E(G - Z)$, so $G - Z$ is not a triangle. By the induction hypothesis, $\iota_c(G - Z) \leq \frac{n-|Z|}{4} = \frac{n-4}{4}$. Since $Z \subseteq N[u]$, Lemma 2.1 (with $X = \{u\}$) gives $\iota_c(G) \leq 1 + \iota_c(G - Z) \leq \frac{n}{4}$.

Now suppose $A \simeq C_3$. Since $V(A) \subseteq W \setminus N(x) \subseteq (V(T') \cup V(T')) \setminus \{v, y\}$, $V(A)$ contains either the two vertices in $V(T)\{y\}$ and one of the two vertices in $V(T')\{v\}$ or the two vertices in $V(T')\{v\}$ and one of the two vertices in $V(T)\{y\}$. Suppose $|V(A) \cap (V(T')\{v\})| = 1$. Then, $V(T)\{y\} \subseteq V(A)$. Let $x'$ be the member of
$V(A) \cap (V(T') \backslash \{v\})$. Let $Z = (V(T) \backslash \{y\}) \cup V(T')$. Since $V(A) \cup V(T') \subseteq N[x']$, $Z \subseteq N[x']$. We have $V(G - Z) = \{x, y\} \cup \bigcup_{H \in \mathcal{H}_x^*} V(H)$. Since $xy \in E(G)$ and the members of $\mathcal{H}_x^*$ are linked to $x$, $G - Z$ is connected. Since $N(y) \cap \left( \bigcup_{H \in \mathcal{H}_x^*} V(H) \right) = \emptyset$, $G - Z$ is not a triangle. By the induction hypothesis, $\iota_c(G - Z) \leq \frac{n - |Z|}{4} = \frac{n - 5}{4}$. Since $Z \subseteq N[x']$, Lemma 2.1 (with $X = \{x'\}$) gives us $\iota_c(G) \leq 1 + \iota_c(G - Z) < \frac{n}{4}$. Similarly, $\iota_c(G) < \frac{n}{4}$ if $|V(A) \cap (V(T') \backslash \{y\})| = 1$.

Case 2: $|\mathcal{H}_x'| = 2$. Let $T_1$ and $T_2$ be the two members of $\mathcal{H}_x'$.

Suppose that $T_2$ is linked to a member of $L \backslash \{x\}$. Since $T_1$ is linked to $y \in E(G)$ for some $y \in V(T_1)$, let $Y = \{x\} \cup V(T_1)$. Then, no component of $G - Y$ is a triangle $(G - Y)$ has a component $A$ with $(N[v] \backslash \{x\}) \cup V(T_2) \cup \bigcup_{H \in \mathcal{H}_x \backslash \{T_1, T_2\}} V(H) \subseteq V(A)$, and the other components of $G - Y$ are the members of $\mathcal{H}_x^*$. Also, $Y \subseteq N[y]$. By Lemma 2.1 (with $X = \{y\}$), Lemma 2.2 and the induction hypothesis, we have

$$\iota_c(G) \leq 1 + \iota_c(G - Y) = \frac{|Y|}{4} + \sum_{H \in \mathcal{C}(G - Y)} \iota_c(H) \leq \frac{|Y|}{4} + \sum_{H \in \mathcal{C}(G - Y)} \frac{|V(H)|}{4} = \frac{n}{4}.$$

Similarly, $\iota_c(G) \leq \frac{n}{4}$ if $T_1$ is linked to a member of $L \backslash \{x\}$.

Now suppose that, for each $i \in \{1, 2\}$ and each $x' \in L \backslash \{x\}$, $T_i$ is not linked to $x'$. Then, $T_1$ and $T_2$ are components of $G - \{x\}$. Let $Y = \{x\} \cup V(T_1) \cup V(T_2)$. Suppose that no component of $G - \{x\}$ other than $T_1$ and $T_2$ is a triangle. Then, $\iota_c(G - Y) \leq \frac{n - |Y|}{4}$ by Lemma 2.2 and the induction hypothesis. Let $D_{G - Y}$ be a cycle isolating set of $G - Y$ of size $\iota_c(G - Y)$. Since $N(x) \cap V(T_1) \neq \emptyset \neq N(x) \cap V(T_2), \{x\} \cup D_{G - Y}$ is a cycle isolating set of $G$, so $\iota_c(G) \leq 1 + \frac{n - |Y|}{4} = 1 + \frac{n - 5}{4} < \frac{n}{4}.$

Now suppose that $G - \{x\}$ has a component $T_3$ such that $T_3 \notin \{T_1, T_2\}$ and $T_3$ is a triangle. Since $T_1$ and $T_2$ are the only members of $\mathcal{H}$ that are linked to $x$, it follows that $V(T_3) = N[v] \backslash \{x\}$. Let $Y' = Y \cup V(T_3)$. Since $G - Y'$ contains no triangle, $\iota_c(G - Y') \leq \frac{n - |Y'|}{4}$ by Lemma 2.2 and the induction hypothesis. Let $D_{G - Y'}$ be a cycle isolating set of $G - Y'$ of size $\iota_c(G - Y')$. Since $v \in N(x) \cap V(T_3)$ and $N(x) \cap V(T_3) \neq \emptyset \neq N(x) \cap V(T_2)$, $\{x\} \cup D_{G - Y'}$ is a cycle isolating set of $G$. Thus, $\iota_c(G) \leq 1 + \frac{n - |Y'|}{4} = 1 + \frac{n - 10}{4} < \frac{n}{4}.$

\[\square\]

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