Mode-I stress intensity factors for Periodic Cracked Stiffened Panels under tension

H. Yuan a, b, Y. J. Xie a, b, W. Wang a

a Department of Mechanical Engineering, Liaoning Shihua University, Fushun, 113001, LN, P. R. China
b College of Pipeline and Civil Engineering, China University of Petroleum (East China), Qingdao, 266000, SD, P. R. China

Corresponding authors. Y.J. Xie, E-mail address: yjxiefs@qq.com

Abstract: Stress intensity factor is the nuclear physical parameters in fracture mechanics. Stiffened panels structure is heavily used in actual engineering field, such as aviation, marine and other bearing structures. The SIF for periotic cracked stiffened panels under tension is a representative three-dimensional crack problem. In the present article, the solution of SIFs for stiffened panels periodic cracks are derived on the basis of materials mechanics and the conservation law $J_2$-integral.

1. Introductions

Stress intensity factor is the critical parameter which can describe singular stress intensification around the crack tip. The stiffened panels, as an enhance case of panels, will improve the ability of resisting fracture, which are heavily used in actual engineering applications. The stiffened panels are actually representative three-dimensional finite boundary problem, which is hard to give analytical solution by using traditional method. In 1998, $J_2$-integral theory was proposed to calculate the stress intensity factors for cracked beams based on conservation law [1], which greatly facilitates the analysis of stress intensity factors for cracked beams [1-6]. The theoretical relationship between the crack mouth widening energy release rate and SIF had been deduced by a special integral circular arc around K-dominant region. Present method are within elementary strength of materials, which is quiet simple to be applied.

Fig.1. Cracked periodic stiffened panels. (a) Periodic stiffened panels; (b) Cracked stiffened panel (m=1); (b) Periodic cracked stiffened panel (m=2).
2. Three-dimensional $J_2$-integral

For a three-dimensional deformation field, the displacement vector $\mathbf{u}_{ij}$ depends on $x_1$, $x_2$, and $x_3$. Based on the conservation law, the three-dimensional $J_2$-integral is similar to the two-dimensional one, for a closed surface $\Omega$, without any defects in the solids closed by which, the following integral are zero, [7-9]

$$J_j = \iiint_{\Omega} (w_n - T_{ij}) d\Omega = 0, \quad j=1, 2, 3.$$  \hspace{1cm} (1)

where $w$ is strain energy density; $n_j$ is unit outward normal to $\Omega$; $T_i$ is stress vector acting outside of $\Omega$. When the axis $x_2$ perpendicular to the cracked panels cross-section (see Fig.1), the $J_2$-integral can be utilized to calculate the stress intensity factor.

![Crack tip](image)

**Fig. 2. Local integration path of crack tip K-dominant region.**

For a two-dimensional deformation field, the counterpart of Eq. (1), the $J_2$-integral is

$$J_2(s) = \int_s (w_n - T_{ij}) ds = 0,$$  \hspace{1cm} (2)

where $s$ is a curve in the $x_1$-$x_2$ plane. For Mode-I crack with unit thickness, shown in Fig. 2, let $s$ denote the $s_{dke}$. Note that $s_{dke}$ is a straight line and $s_{ke}$ is a quarter of a circle. Along the path $s_{dke}$ and $s_{ke}$, Eq. (2) yields

$$J_2(s_{dke}) = \int_{s_{dke}} (w_n - T_{ij}) ds = 0,$$  \hspace{1cm} (3)

and

$$J_2(s_{ke}) = \int_{s_{ke}} (w_n - T_{ij}) ds = \frac{(1-\mu^2)K_I^2}{2\pi E}.$$  \hspace{1cm} (4)

More details on Eqs.(3) and (4) can be found in [1]. Based to the conservation law, for the closed path $s=s_{dke}+s_{ed}$, it follows that

$$J_2(s) = \int_s (w_n - T_{ij}) ds = \int_{s_{dke}} (w_n - T_{ij}) ds + \int_{s_{ed}} (w_n - T_{ij}) ds = \frac{(1-\mu^2)K_I^2}{2\pi E} + \int_{s_{ed}} (w_n - T_{ij}) ds = 0.$$  \hspace{1cm} (5)

Then

$$J_2(s) = \int_{s_{ed}} (w_n - T_{ij}) ds = \frac{(1-\mu^2)K_I^2}{2\pi E}.$$  \hspace{1cm} (6)

3. Three-dimensional $J_2$-integral for periodic cracked stiffened panels

The periotic cracked stiffened panels under tension as shown in Fig.1, there are two identical crack tips, and each of which has typical features of plane strain state. $N$ is the tension. The $x_2$, $x_3$ plane in Fig.1.
The $x_2$-$x_3$ plane in Fig. 1 is a symmetry plane for cracked stiffened panels, and all loads act in this plane; hence the bending deflections will take place in this plane.

Chose a three-dimensional closed surface $\Omega = mA_{\text{surface}} + mA^+ + mA^- + mA_c$ (see Fig. 1), where the symbol ‘+’ notes the cracked cross-section and ‘-’ the remote uncracked cross-section; then $A^-$ is the ligament area and $A^+$ the remote cross-section; $A_c$ the surface of web crack, i.e., $A_c = A_{ee} + A_{de} + A_{de'}$; $A_{out}$ is the outer free surfaces of the tube. Obviously,

$$J_2(mA_{\text{surface}}) = \iint_{mA_{\text{surface}}} (wn_2^+ - Ti_{i,2}) d\Omega = 0.$$  \hspace{1cm} (7)

By using elementary mechanics, it is not difficult to get

$$J_2(mA^+) = \iint_{mA^+} (wn_2^+ - Ti_{i,2}) d\Omega = (\bar{w}^+ - N_m \bar{u}_{2,2}^+) = \frac{1}{2} N_m \bar{u}_{2,2}^+$$  \hspace{1cm} (8)

and

$$J_2(mA^-) = \iint_{mA^-} (wn_2^- - Ti_{i,2}) d\Omega = \bar{w}^- - N_m \bar{u}_{2,2}^- = -\frac{1}{2} N_m \bar{u}_{2,2}^-,$$  \hspace{1cm} (9)

where $\bar{u}_{2,2}$ is the strain of the axis of the shells and $\bar{w}$ is strain energy per unit axis length. In the remote transverse cross-section

$$\bar{u}_{2,2} = \frac{N_m}{EAm}$$  \hspace{1cm} (10)

where $A_m$ is the area for remote uncracked cross-section and $A_m = mb t_f (m+1) vt_h$; $E$ the Young’s modulus. In the cracked cross-section, the crack is supposed to be an elliptical hole with $f \to 0$ [2] as shown in Fig. 3, the mean curvature becomes

$$\bar{u}_{2,2}^+ = \lim_{f \to 0} \frac{1}{f} \int_0^f N_m \frac{dx_2}{y_{\text{ref}}(a/b)} \frac{dx_2}{\sqrt{1-(x_2/f)^2/(1+(m+1)^2)} h t_w m b t_h},$$

$$= \frac{N_m}{EAm} y_m(a/b),$$  \hspace{1cm} (11)

where introducing the variable $\xi = x_2/f$, $\beta_m = \frac{1}{2} + \frac{(m+1) t_w h}{2 m t_f b}$, then

$$y_m(a/b) = J_0^1 \frac{dx_2}{y_{\text{ref}}(a/b)} = \frac{2\beta_m}{(2\pi)^2} \text{arctan} \left( \frac{\beta_m}{2(1/2)} \right) \sqrt{\beta_m} \sqrt{\beta_m + \pi}$$  \hspace{1cm} (12)

Neglecting the integral over free surface [1-6] such as $A_{ee}$ (see Fig. 2), and take the three-dimensional $J_2$-integral over $A_{de}$ as an instance from Eqs. (1) and (6)

$$mf_j^2(A_c) = 2mf_j^2(A_{de}) = m \iint_{A_{de}} (wn_2^+ - Ti_{i,2}) d\Omega = m t_f \int_{A_{de}} (wn_2^+ - Ti_{i,2}) ds = -mt_f \int_{A_{de}} (1-f^2)^{1/2} K_2^2$$

$$\frac{d}{m}$$  \hspace{1cm} (13)

4. Normalized SIFs

Substituting the Eqs. (7)-(13) into $J_2$-integral in Eq. (1) over the closed surface

$\Omega = mA_{\text{surface}} + mA^+ + mA^- + mA_c$, we have
\[ K_{ij}^m = \sigma_0^m \sqrt{\pi b f_m(a/b)} \] (14)

where \( \sigma_0 = N_m / A_m \), \( f_m(a/b) \) is the normalized \( K_i^m \), which is given by

\[
f_m(a/b) = \left\{ \beta_m \left( \frac{2\beta_m^2}{m(1-\mu^2)} \right) \frac{\text{arctan} \left( \frac{\beta_m+(a/b)}{\beta_m-(a/b)} \right)}{\sqrt{\beta_m^2-2(a/b)^2}} \right\}^{1/2}.
\] (15)

Fig. 4. The one-half symmetry finite element model (m=2), \( t_w/t_f=1, t_f/b=0.05, h/b=0.2, \mu=0.3 \). (a) Meshing of model. (b) Stress distribution of cracks. (a/b=0.25)

Fig. 5. SIFs for the cracked stiffened panels under tension. (a) m=1, (b) m=2.

ANSYS18.2 had been used for FEM analysis of present structure (\( E=200 \text{GPa} \) and \( \mu=0.3 \)). The SHELL 181 element had been selected, and the crack tip was surrounded by singular finite element for a highly refined-mesh. More than 25,000 nodes and 9000 elements had been used for meshing a half of symmetrical structures. Fig. 4 shows a partial amplifying version of finite element division around the crack tip region. The stress intensity factor was obtained by displacement extrapolation method. Fig. 5 is the comparison of present work and finite element method.

5. Conclusions
The stiffened panels are widely used in actual engineering problems and stress intensity factors of cracks in panels are key parameters in security evaluation. A method for calculating stress intensity factors of cracked periodic stiffened panels under tension have been established. The method is based on three-dimensional conservation law and elementary beam theory, which is extremely simple and flexible.
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