Hamilton-Jacobi Equations for Controlled Magnetic Hamiltonian System with Nonholonomic Constraint

Hong Wang
School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, P.R.China
E-mail: hongwang@nankai.edu.cn

Dedicated to 110th anniversary of the birth of Professor Shiing Shen Chern
June 22, 2022

Abstract. In order to describe the impact of different geometric structures and constraints for the dynamics of a regular controlled Hamiltonian system, in this paper, we first define a kind of controlled magnetic Hamiltonian (CMH) system, and give a good expression of the dynamical vector field of the CMH system, such that we can describe the magnetic vanishing condition and the CMH-equivalence, and derive precisely the geometric constraint conditions of the magnetic symplectic form for the dynamical vector field of the CMH system, which are called the Type I and Type II of Hamilton-Jacobi equation. Secondly, we prove that the CMH-equivalence for the CMH systems leaves the solutions of corresponding to Hamilton-Jacobi equations invariant, if the associated magnetic Hamiltonian systems are equivalent. Thirdly, we consider the CMH system with nonholonomic constraint, and derive a distributional CMH system, which is determined by a non-degenerate distributional two-form induced from the magnetic symplectic form. Then we drive precisely two types of Hamilton-Jacobi equation for the distributional CMH system. Moreover, we generalize the above results for the nonholonomic reducible CMH system with symmetry, and prove two types of Hamilton-Jacobi theorems for the nonholonomic reduced distributional CMH system. These research works reveal the deeply internal relationships of the magnetic symplectic forms, the nonholonomic constraints, the dynamical vector fields and controls of the CMH systems.

Keywords: controlled magnetic Hamiltonian system, geometric constraint condition, Hamilton-Jacobi equation, CMH-equivalence, nonholonomic constraint.

AMS Classification: 53D20 70H20 70Q05.

Contents

1 Introduction 2
2 Controlled Magnetic Hamiltonian System 4
3 Two Types of Hamilton-Jacobi Equation for a CMH System 7
4 CMH-equivalence and the Solutions of Hamilton-Jacobi Equations 12
5 Nonholonomic CMH System and Distributional CMH System 16
6 Hamilton-Jacobi Equations for Distributional CMH System 19
1 Introduction

It is well-known that Hamilton-Jacobi theory is an important research subject in mathematics and analytical mechanics, see Abraham and Marsden [1], Arnold [2] and Marsden and Ratiu [15], and the Hamilton-Jacobi equation is also fundamental in the study of the quantum-classical relationship in quantization, and it also plays an important role in the study of stochastic dynamical systems, see Woodhouse [30], Ge and Marsden [7], and Lázaro-Camí and Ortega [9]. For these reasons it is described as a useful tool in the study of Hamiltonian system theory, and has been extensively developed in past many years and become one of the most active subjects in the study of modern applied mathematics and analytical mechanics.

Just as we have known that Hamilton-Jacobi theory from the variational point of view is originally developed by Jacobi in 1866, which state that the integral of Lagrangian of a mechanical system along the solution of its Euler-Lagrange equation satisfies the Hamilton-Jacobi equation. The classical description of this problem from the generating function and the geometrical point of view is given by Abraham and Marsden in [1] as follows: Let $Q$ be an $n$-dimensional smooth manifold and $TQ$ the tangent bundle, $T^*Q$ the cotangent bundle with a canonical symplectic form $\omega$ and the projection $\pi_Q: T^*Q \rightarrow Q$ induces the map $T\pi_Q: TT^*Q \rightarrow TQ$.

**Theorem 1.1** Assume that the triple $(T^*Q, \omega, H)$ is a Hamiltonian system with Hamiltonian vector field $X_H$, and $W: Q \rightarrow \mathbb{R}$ is a given generating function. Then the following two assertions are equivalent:

(i) For every curve $\sigma: \mathbb{R} \rightarrow Q$ satisfying $\dot{\sigma}(t) = T\pi_Q(X_H(dW(\sigma(t))))$, $\forall t \in \mathbb{R}$, then $dW \cdot \sigma$ is an integral curve of the Hamiltonian vector field $X_H$.

(ii) $W$ satisfies the Hamilton-Jacobi equation $H(q^i, \frac{\partial W}{\partial q^i}) = E$, where $E$ is a constant.

From the proof of the above theorem given in Abraham and Marsden [1], we know that the assertion (i) with equivalent to Hamilton-Jacobi equation (ii) by the generating function, gives a geometric constraint condition of the canonical symplectic form on the cotangent bundle $T^*Q$ for Hamiltonian vector field of the system. Thus, the Hamilton-Jacobi equation reveals the deeply internal relationships of the generating function, the canonical symplectic form and the dynamical vector field of a Hamiltonian system.

On the other hand, the authors in Marsden et al. [16] define a regular controlled Hamiltonian (RCH) system, which is a Hamiltonian system with external force and control. In general, an RCH system, under the actions of external force and control, is not Hamiltonian, however, it is a dynamical system closely related to a Hamiltonian system, and it can be explored and studied by extending the methods for external force and control in the study of Hamiltonian systems. Thus, one can emphasize explicitly the impact of external force and control in the study for the RCH systems. However, since an RCH system defined on the cotangent bundle $T^*Q$, may not be a Hamiltonian system, and it may have no generating function, we cannot give the Hamilton-Jacobi theorem for the RCH system just like same as the above Theorem 1.1. We have to look for a new way. It is worthy of noting that, in Wang [23] the author derives precisely the geometric constraint conditions of canonical symplectic form for the dynamical vector field of an RCH system. These conditions are called the Type I and Type II of Hamilton-Jacobi equation, which are the development of the Type I and Type II of Hamilton-Jacobi equation for a Hamiltonian system given in Wang [22]. Moreover, the author proves that the RCH-equivalence for the RCH systems leaves the solutions of corresponding to Hamilton-Jacobi equations invariant if the associated Hamiltonian systems are
It is well known that the different structures of geometry determine the different Hamiltonian systems and RCH systems, as well as their dynamics. In order to describe the impact of different structures of geometry for the dynamics of RCH system and Hamilton-Jacobi equations, we consider the magnetic symplectic form \( \omega^B = \omega - \pi_B^*B \), where \( \omega \) is the usual canonical symplectic form on \( T^*Q \), and \( B \) is the closed two-form on \( Q \). A magnetic Hamiltonian system is a Hamiltonian system defined by the magnetic symplectic form, which is a canonical Hamiltonian system coupling the action of a magnetic field \( B \). A controlled magnetic Hamiltonian (CMH) system on \( T^*Q \) is a magnetic Hamiltonian system \((T^*Q, \omega^B, H)\) with external force \( F \) and control \( W \), where \( F: T^*Q \to T^*Q \) is the fiber-preserving map, and \( W \subset T^*Q \) is a fiber submanifold of \( T^*Q \). see Wang [21]. Thus, it is a natural problem how to derive precisely the geometric constraint conditions of the magnetic symplectic form for the dynamical vector field of a CMH system, and how to describe explicitly the relationship between the CMH-equivalence and the solutions of corresponding Hamilton-Jacobi equations. These research are one of our goals in this paper.

We know that, in mechanics, it is very often that many systems have constraints. A nonholonomic CMH system is a CMH system with nonholonomic constraint. Usually, under the restriction given by nonholonomic constraint, in general, the dynamical vector field of a nonholonomic CMH system may not be (magnetic) Hamiltonian. Thus, we can not describe the Hamilton-Jacobi equations for nonholonomic CMH system from the viewpoint of generating function as in the classical Hamiltonian case. In consequence, when a CMH system with nonholonomic constrain, how to describe the impact of the constrain for the dynamics of CMH system and its reduced CMH system, and how to describe Hamilton-Jacobi equations for the nonholonomic CMH system and the nonholonomic reducible CMH system. These research are another of our goals in this paper.

A brief of outline of this paper is as follows. In the second section, we first define a kind of controlled magnetic Hamiltonian (CMH) system by using magnetic symplectic form, and then give a good expression of the dynamical vector field of the CMH system, which is the synthetic of magnetic Hamiltonian vector field and its changes under the actions of the external force and control law, such that we can describe the magnetic vanishing condition. In the third section, we first prove a key lemma, which is an important tool for the proofs of two types of Hamilton-Jacobi theorems of the CMH system. Then we derive precisely the geometric constraint conditions of the magnetic symplectic form for the dynamical vector field of a CMH system on the cotangent bundle of a configuration manifold, that is, the Type I and Type II of Hamilton-Jacobi equation for a CMH system. Moreover, in the fourth section, we describe the CMH-equivalence for the CMH system, and prove that the CMH-equivalence leaves invariant for the solutions of corresponding to Hamilton-Jacobi equations, if the associated magnetic Hamiltonian systems are equivalent. In the fifth section, we consider the CMH system with nonholonomic constraint, we define a distributional CMH system by analyzing carefully for the structure of the nonholonomic dynamical vector field, and this system is determined by a non-degenerate distributional two-form induced from the magnetic symplectic form. Moreover, in the sixth section, we derive precisely two types of Hamilton-Jacobi equation for the distributional CMH system. In the seventh section, we generalize the above results for the nonholonomic reducible RCH system with symmetry, and prove two types of Hamilton-Jacobi theorems for the nonholonomic reduced distributional RCH system. These research works reveal the deeply internal relationships of the geometrical structures of phase spaces, the dynamical vector fields and controls of the CMH systems, and make us have much deeper understanding and recognition for the structures of Hamiltonian system, RCH system and CMH system.
2 Controlled Magnetic Hamiltonian System

In this section, we first define a kind of controlled magnetic Hamiltonian (CMH) system by using magnetic symplectic form, and then give a good expression of the dynamical vector field of the CMH system, by using the vertical lift map of a vector along a fiber. This expression is the synthetic of magnetic Hamiltonian vector field and its changes under the actions of the external force and control law, such that we can describe the magnetic vanishing condition and CMH-equivalence. We shall follow some of the notations and conventions introduced in Abraham and Marsden [1], Arnold [2], Marsden et al. [13,16], Marsden and Ratiu [15], Ortega and Ratiu [18], Wang [21] and Wang [22]. For convenience, in this paper, we assume that all manifolds are real, smooth and finite dimensional and all actions are smooth left actions. and all controls appearing in this paper are the admissible controls.

In the reduction theory and application of Hamiltonian systems, the Marsden-Weinstein reduction for a Hamiltonian system with symmetry and momentum map is very important and foundational, see Marsden and Weinstein [17], and Libermann and Marle [11], Marsden [12]. But, from the classification of symplectic reduced space of the cotangent bundle $T^*Q$, see Marsden et al. [13] and Marsden and Perlmutter [14], we know that the set of Hamiltonian systems with symmetries and momentum maps on the cotangent bundle $T^*Q$ is not complete under the Marsden-Weinstein reduction, that is, the symplectic reduced system of a Hamiltonian system with symmetry and momentum map defined on the cotangent bundle $T^*Q$ may not be a Hamiltonian system on a cotangent bundle. In consequence, if we define directly a controlled Hamiltonian system with symmetries on the cotangent bundle $T^*Q$, then the symplectic reduced controlled Hamiltonian system may not have definition.

In order to describe uniformly RCH systems defined on a cotangent bundle and on the regular reduced spaces, in this section we first define an RCH system on a symplectic fiber bundle, see Marsden et al. [16]. Then we can obtain the RCH system and the CMH system on the cotangent bundle of a configuration manifold as the special cases, and give a good expression of the dynamical vector field of the CMH system, which is the synthetic of magnetic Hamiltonian vector field $X_B^H$ and its changes under the actions of the external force $F$ and control law $u$, such that we can describe the magnetic vanishing condition, and discuss CMH-equivalence in fourth section. In consequence, we can regard the associated Hamiltonian system and the magnetic Hamiltonian system on the cotangent bundle $T^*Q$, then the symplectic reduced controlled Hamiltonian system may not have definition.

Let $(E, M, \pi)$ be a fiber bundle, and for each point $x \in M$, assume that the fiber $E_x = \pi^{-1}(x)$ is a smooth submanifold of $E$ and with a symplectic form $\omega_E(x)$, that is, $(E, \omega_E)$ is a symplectic fiber bundle. If for any function $H : E \to \mathbb{R}$, we have a Hamiltonian vector field $X_H^E$ and its changes under the actions of the external force $F$ and control law $u$, such that we can describe the magnetic vanishing condition, and discuss CMH-equivalence in fourth section. In consequence, we can regard the associated Hamiltonian system and the magnetic Hamiltonian system on the cotangent bundle $E$ as follows.

**Definition 2.1 (RCH System)** A regular controlled Hamiltonian (RCH) system on $E$ is a 5-tuple $(E, \omega_E, H, F, W)$, where $(E, \omega_E, H)$ is a Hamiltonian system, and the function $H : E \to \mathbb{R}$ is called the Hamiltonian, a fiber-preserving map $F : E \to E$ is called the (external) force map, and a fiber submanifold $W$ of $E$ is called the control subset.

Sometimes, $W$ is also denoted the set of fiber-preserving maps from $E$ to $W$. When a feedback control law $u : E \to W$ is chosen, the 5-tuple $(E, \omega_E, H, F, u)$ is a regular closed-loop dynamic
system. In particular, when \( Q \) is an \( n \)-dimensional smooth manifold, and \( T^*Q \) its cotangent bundle with a symplectic form \( \omega \) (not necessarily canonical symplectic form), then \((T^*Q, \omega)\) is a symplectic vector bundle. If we take that \( E = T^*Q \), from above definition we can obtain an RCH system on the cotangent bundle \( T^*Q \), that is, 5-tuple \((T^*Q, \omega, H, F, W)\). Where the fiber-preserving map \( F : T^*Q \to T^*Q \) is the (external) force map, which is the reason that the fiber-preserving map \( F : E \to E \) is called an (external) force map in above definition.

In order to describe the impact of different structures of geometry for the RCH systems, we shall consider the magnetic symplectic form on \( T^*Q \) as follows: Assume that \( T^*Q \) with the canonical symplectic form \( \omega \), and \( B \) is a closed two-form on \( Q \), then \( \omega^B = \omega - \pi^*_B B \) is a symplectic form on \( T^*Q \), where \( \pi^*_B : T^*Q \to T^*T^*Q \). The \( \omega^B \) is called a magnetic symplectic form, and \( \pi^*_B B \) is called a magnetic term on \( T^*Q \), see Marsden et al. [13].

A magnetic Hamiltonian system is a 3-tuple \((T^*Q, \omega^B, H)\), which is a Hamiltonian system defined by the magnetic symplectic form \( \omega^B \), that is, a canonical Hamiltonian system coupling the action of a magnetic field \( B \). For a given Hamiltonian \( H \), the dynamical vector field \( X^B_H \), which is called the magnetic Hamiltonian vector field, satisfies the magnetic Hamilton’s equation, that is, \( 1_X^B_B \omega^B = dH \). In canonical cotangent bundle coordinates, for any \( q \in Q \), \((q, p) \in T^*Q \), we have that

\[
\omega = \sum_{i=1}^{n} dq^i \wedge dp_i, \quad B = \sum_{i,j=1}^{n} B_{ij} dq^i \wedge dq^j, \quad dB = 0,
\]

\[
\omega^B = \omega - \pi^*_B B = \sum_{i=1}^{n} dq^i \wedge dp_i - \sum_{i,j=1}^{n} B_{ij} dq^i \wedge dq^j,
\]

and the magnetic Hamiltonian vector field \( X^B_H \) with respect to the magnetic symplectic form \( \omega^B \) can be expressed that

\[
X^B_H = \sum_{i=1}^{n} \left( \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \right) - \sum_{i,j=1}^{n} B_{ij} \frac{\partial H}{\partial p_j} \frac{\partial}{\partial p_i}.
\]

See Marsden et al. [13].

Moreover, if considering the external force and control, we can define a kind of controlled magnetic Hamiltonian (CMH) system on \( T^*Q \) as follows.

**Definition 2.2 (CMH System)** A controlled magnetic Hamiltonian (CMH) system on \( T^*Q \) is a 5-tuple \((T^*Q, \omega^B, H, F, W)\), which is a magnetic Hamiltonian system \((T^*Q, \omega^B, H)\) with external force \( F \) and control \( W \), where \( F : T^*Q \to T^*Q \) is the fiber-preserving map, and \( W \subset T^*Q \) is a fiber submanifold, which is called the control subset.

From the above Definition 2.1 and Definition 2.2 we know that a CMH system on \( T^*Q \) is also an RCH system on \( T^*Q \), but its symplectic structure is given by a magnetic symplectic form, and the set of the CMH systems on \( T^*Q \) is a subset of the set of the RCH systems on \( T^*Q \). When a feedback control law \( u : T^*Q \to W \) is chosen, the 5-tuple \((T^*Q, \omega^B, H, F, u)\) is a regular closed-loop dynamic system.

In order to describe the dynamics of the CMH system \((T^*Q, \omega^B, H, F, W)\) with a control law \( u \), we need to give a good expression of the dynamical vector field of the CMH system. At first, we introduce a notations of vertical lift maps of a vector along a fiber, also see Marsden et al. [16]. For a smooth manifold \( E \), its tangent bundle \( TE \) is a vector bundle, and for the fiber bundle \( \pi : E \to M \),
we consider the tangent mapping $Tπ : TE \to TM$ and its kernel $ker(Tπ) = \{ρ ∈ TE | Tπ(ρ) = 0\}$, which is a vector subbundle of $TE$. Denote by $VE := ker(Tπ)$, which is called a vertical bundle of $E$. Assume that there is a metric on $E$, and we take a Levi-Civita connection $A$ on $TE$, and denote by $HE := ker(A)$, which is called a horizontal bundle of $E$, such that $TE = HE ⊕ VE$. For any $x ∈ M$, $a_x, b_x ∈ E_x$, any tangent vector $ρ(b_x) ∈ T_{b_x}E$ can be split into horizontal and vertical parts, that is, $ρ(b_x) = ρ^h(b_x) ⊕ ρ^v(b_x)$, where $ρ^h(b_x) ∈ H_{b_x}E$ and $ρ^v(b_x) ∈ V_{b_x}E$. Let $\gamma$ be a geodesic in $E_x$ connecting $a_x$ and $b_x$, and denote by $ρ^\gamma_x(a_x)$ a tangent vector at $a_x$, which is a parallel displacement of the vertical vector $ρ^v(b_x)$ along the geodesic $γ$ from $b_x$ to $a_x$. Since the angle between two vectors is invariant under a parallel displacement along a geodesic, then $Tπ(ρ^\gamma_x(a_x)) = 0$, and hence $ρ^\gamma_x(a_x) ∈ V_{a_x}E$. Now, for $a_x, b_x ∈ E_x$ and tangent vector $ρ(b_x) ∈ T_{b_x}E$, we can define the vertical lift map of a vector along a fiber given by

$$vlift : TE_x × E_x \to TE_x; \quad vlift(ρ(b_x), a_x) = ρ^\gamma_x(a_x).$$

It is easy to check from the basic fact in differential geometry that this map does not depend on the choice of $γ$. If $F : E → E$ is a fiber-preserving map, for any $x ∈ M$, we have that $F_x : E_x → E_x$ and $TF_x : TE_x → TE_x$, then for any $a_x ∈ E_x$ and $ρ ∈ TE_x$, the vertical lift of $ρ$ under the action of $F$ along a fiber is defined by

$$(vlift(F)(F_xρ)(a_x), a_x) = (TF_xρ)_\gamma(a_x),$$

where $γ$ is a geodesic in $E_x$ connecting $F_x(a_x)$ and $a_x$.

In particular, when $π : E → M$ is a vector bundle, for any $x ∈ M$, the fiber $E_x = π^{-1}(x)$ is a vector space. In this case, we can choose the geodesic $γ$ to be a straight line, and the vertical vector is invariant under a parallel displacement along a straight line, that is, $ρ^\gamma_x(a_x) = ρ^v(b_x)$. Moreover, when $E = T^*Q, M = Q$, by using the local trivialization of $TT^*Q$, we have that $TT^*Q ≃ TQ × T^*Q$. Because of $π : T^*Q → Q$, and $Tπ : TT^*Q → TQ$, then in this case, for any $α_x, β_x ∈ T^*_xQ, x ∈ Q$, we know that $(0, β_x) ∈ V_{β_x}T^*_xQ$, and hence we can get that

$$vlift((0, β_x))(β_x), α_x) = (0, β_x)(α_x) = \frac{d}{ds}\bigg|_{s=0} (α_x + sβ_x),$$

which is consistent with the definition of vertical lift map along fiber in Marsden and Ratiu [15].

For a given CMH system $(T^*Q, ω^B, H, F, W)$, the dynamical vector field of the associated magnetic Hamiltonian system $(T^*Q, ω^B, H)$ is $X^B_H$, which satisfies the equation $i_{X^B_H}ω^B = dH$. If considering the external force $F : T^*Q → T^*Q$, by using the above notation of vertical lift map of a vector along a fiber, the change of $X^B_H$ under the action of $F$ is that

$$vlift(F)X^B_H(α_x) = vlift((TFX^B_H)(F(α_x)), α_x) = (TFX^B_H)_\gamma(α_x),$$

where $α_x ∈ T^*_xQ, x ∈ Q$ and $γ$ is a straight line in $T^*_xQ$ connecting $F_x(α_x)$ and $α_x$. In the same way, when a feedback control law $u : T^*Q → W$ is chosen, the change of $X^B_H$ under the action of $u$ is that

$$vlift(u)X^B_H(α_x) = vlift((TuX^B_H)(u(α_x)), α_x) = (TuX^B_H)_\gamma(α_x).$$

In consequence, we can give an expression of the dynamical vector field of the CMH system as follows.

**Theorem 2.3** The dynamical vector field of a CMH system $(T^*Q, ω^B, H, F, W)$ with a control law $u$ is the synthetic of magnetic Hamiltonian vector field $X^B_H$ and its changes under the actions of the external force $F$ and control law $u$, that is,

$$X_{(T^*Q, ω^B, H, F, u)}(α_x) = X^B_H(α_x) + vlift(F)X^B_H(α_x) + vlift(u)X^B_H(α_x),$$

(2.1)
for any $\alpha_x \in T_x^*Q$, $x \in Q$. For convenience, it is simply written as
\[
X_{(T^*Q, \omega^B, H,F,u)} = X^B_H + \text{vlift}(F)^B + \text{vlift}(u)^B.
\] (2.2)

Where $\text{vlift}(F)^B = \text{vlift}(F)X^B_H$, and $\text{vlift}(u)^B = \text{vlift}(u)X^B_H$, are the changes of $X^B_H$ under the actions of $F$ and $u$. We also denote that $\text{vlift}(W)^B = \bigcup \{ \text{vlift}(u)X^B_H \mid u \in W \}$. It is worthy of noting that, in order to deduce and calculate easily, we always use the simple expression of dynamical vector field $X_{(T^*Q, \omega^B, H,F,u)}$.

From the expression (2.2) of the dynamical vector field of a CMH system, we know that under the actions of the external force $F$ and control law $u$, in general, the dynamical vector field may not be magnetic Hamiltonian, and hence the CMH system may not be yet a magnetic Hamiltonian system. However, it is a dynamical system closed relative to a magnetic Hamiltonian system, and it can be explored and studied by extending the methods for external force and control in the study of magnetic Hamiltonian system.

For the magnetic Hamiltonian system $(T^*Q, \omega^B, H)$, its magnetic Hamiltonian vector field $X^B_H$ satisfies the equation $i_{X^B_H} \omega = dH$, and for the associated canonical Hamiltonian system $(T^*Q, \omega, H)$, its canonical Hamiltonian vector field $X_H$ satisfies the equation $i_{X_H} \omega = dH$. Denote by the vector field $X^0 = X^B_H - X_H$, and from the magnetic symplectic form $\omega^B = \omega - \pi_Q B$, we have that
\[
i_{X^0} \omega = i_{(X^B_H - X_H)} \omega = i_{X^B_H} \omega - i_{X_H} \omega = i_{X^B_H} (\omega^B + \pi_Q B) - i_{X_H} \omega = i_{X^B_H} (\pi_Q B).
\]
Thus, $X^0$ is called the magnetic vector field and $i_{X^0} \omega = i_{X^B_H} (\pi_Q B)$ is called the magnetic equation, which is determined by the magnetic term $\pi_Q B$ on $T^*Q$. When $B = 0$, then $X^0 = 0$, the magnetic equation holds trivially. For the CMH system $(T^*Q, \omega^B, H,F,W)$, from the expression (2.2) of its dynamical vector field, we have that
\[
X_{(T^*Q, \omega^B, H,F,u)} = X_H + X^0 + \text{vlift}(F)^B + \text{vlift}(u)^B.
\] (2.3)

If we choose the external force $F$ and control law $u$, such that
\[
X^0 + \text{vlift}(F)^B + \text{vlift}(u)^B = 0,
\] (2.4)
then from (2.3) we have that $X_{(T^*Q, \omega^B, H,F,u)} = X_H$, that is, in this case the dynamical vector field of the CMH system is just the canonical Hamiltonian vector field, and the motion of the CMH system is just same like the motion of canonical Hamiltonian system without the actions of magnetic, external force and control. Thus, the condition (2.4) is called the magnetic vanishing condition for the CMH system $(T^*Q, \omega^B, H,F,W)$.

To sum up the above discussion, we have the following theorem.

**Theorem 2.4** If the external force $F$ and the control law $u$ for a CMH system $(T^*Q, \omega^B, H,F,u)$ satisfy the magnetic vanishing condition (2.4), then its dynamical vector field $X_{(T^*Q, \omega^B, H,F,u)}$ is just the canonical Hamiltonian vector field $X_H$ for the associated canonical Hamiltonian system $(T^*Q, \omega, H)$.

### 3 Two Types of Hamilton–Jacobi Equation for a CMH System

In this section, we shall derive precisely the geometric constraint conditions of the magnetic symplectic form for the dynamical vector field of a CMH system, that is, Type I and Type II of
Hamilton-Jacobi equation for the CMH system. In order to do this, in the following we first give an important notion and prove a key lemma, which is an important tool for the proofs of two types of Hamilton-Jacobi theorem for the CMH system.

Denote by $\Omega^i(Q)$ the set of all $i$-forms on $Q$, $i = 1, 2$. For any $\gamma \in \Omega^1(Q)$, $q \in Q$, then $\gamma(q) \in T^*_q Q$, and we can define a map $\gamma : Q \rightarrow T^*Q$, $q \rightarrow (q, \gamma(q))$. Hence we say often that the map $\gamma : Q \rightarrow T^*Q$ is an one-form on $Q$. If the one-form $\gamma$ is closed, then $d\gamma(x, y) = 0$, $\forall x, y \in TQ$. Note that for any $v, w \in TT^*Q$, we have that $d\gamma(T\pi_Q(v), T\pi_Q(w)) = \pi^*(d\gamma)(v, w)$ is a two-form on the cotangent bundle $T^*Q$, where $\pi^* : T^*Q \rightarrow TTT^*Q$. Thus, in the following we can give a weaker notion.

**Definition 3.1** The one-form $\gamma$ is called to be closed with respect to $T\pi_Q : TTT^*Q \rightarrow TQ$, if for any $v, w \in TT^*Q$, we have that $d\gamma(T\pi_Q(v), T\pi_Q(w)) = 0$.

From the above definition we know that, if $\gamma$ is a closed one-form, then it must be closed with respect to $T\pi_Q : TTT^*Q \rightarrow TQ$. Conversely, if $\gamma$ is closed with respect to $T\pi_Q : TTT^*Q \rightarrow TQ$, then it may not be closed. We can prove a general result as follows, its proof given in Wang [22], which states that the notion that $\gamma$ is closed with respect to $T\pi_Q : TTT^*Q \rightarrow TQ$, is not equivalent to the notion that $\gamma$ is closed.

**Proposition 3.2** Assume that $\gamma : Q \rightarrow T^*Q$ is an one-form on $Q$ and it is not closed. we define the set $N$, which is a subset of $TT^*Q$, such that the one-form $\gamma$ on $N$ satisfies the condition that for any $x, y \in N$, $d\gamma(x, y) \neq 0$. Denote by $Ker(T\pi_Q) = \{ u \in TT^*Q | T\pi_Q(u) = 0 \}$, and $T\gamma : TQ \rightarrow TT^*Q$ is the tangent map of $\gamma : Q \rightarrow T^*Q$. If $T\gamma(N) \subset Ker(T\pi_Q)$, then $\gamma$ is closed with respect to $T\pi_Q : TTT^*Q \rightarrow TQ$.

For the one-form $\gamma : Q \rightarrow T^*Q$, $d\gamma$ is a two-form on $Q$. Assume that $B$ is a closed two-form on $Q$, we say that the $\gamma$ satisfies condition $d\gamma = -B$, if for any $x, y \in TQ$, we have that $(d\gamma + B)(x, y) = 0$. In the following we can give a new notion.

**Definition 3.3** Assume that $\gamma : Q \rightarrow T^*Q$ is an one-form on $Q$, we say that the $\gamma$ satisfies condition $d\gamma = -B$ with respect to $T\pi_Q : TTT^*Q \rightarrow TQ$, if for any $v, w \in TT^*Q$, we have that $(d\gamma + B)(T\pi_Q(v), T\pi_Q(w)) = 0$.

From the above Definition 3.1 and Definition 3.3, we know that, when $B = 0$, the condition that, $\gamma$ satisfies condition $d\gamma = -B$ with respect to $T\pi_Q : TTT^*Q \rightarrow TQ$, become that $\gamma$ is closed with respect to $T\pi_Q : TTT^*Q \rightarrow TQ$. Now, we prove the following lemma, which is a generalization of a corresponding to lemma given by Wang [22], and the lemma is a very important tool for our research.

**Lemma 3.4** Assume that $\gamma : Q \rightarrow T^*Q$ is an one-form on $Q$, and $\lambda = \gamma \cdot \pi_Q : T^*Q \rightarrow T^*Q$. For the magnetic symplectic form $\omega^B = \omega - \pi_Q^*B$ on $T^*Q$, where $\omega$ is the canonical symplectic form on $T^*Q$, then we have that the following two assertions hold.

(i) For any $v, w \in TT^*Q$, $\lambda^*\omega^B(v, w) = -(d\gamma + B)(T\pi_Q(v), T\pi_Q(w))$;

(ii) For any $v, w \in TT^*Q$, $\omega^B(T\lambda \cdot v, w) = \omega^B(v, w - T\lambda \cdot w) - (d\gamma + B)(T\pi_Q(v), T\pi_Q(w))$.

**Proof:** We first prove the assertion (i). Since $\omega$ is the canonical symplectic form on $T^*Q$, we know that there is an unique canonical one-form $\theta$, such that $\omega = -d\theta$. From the Proposition 3.2.11 in Abraham and Marsden [1], we have that for the one-form $\gamma : Q \rightarrow T^*Q$, $\gamma^*\theta = \gamma$. Then we can obtain that for any $x, y \in TQ$,

$$\gamma^*\omega(x, y) = \gamma^*(-d\theta)(x, y) = -d(\gamma^*\theta)(x, y) = -d\gamma(x, y).$$
Note that $\lambda = \gamma \cdot \pi_Q : T^*Q \to T^*Q$, and $\lambda^* = \pi_Q^* \cdot \gamma^* : T^*T^*Q \to T^*T^*Q$, then we have that for any $v, w \in TT^*Q$,

$$\lambda^*\omega(v, w) = \lambda^*(-d\theta)(v, w) = -d(\lambda^*\theta)(v, w) = -d(\pi_Q^* \cdot \gamma^*)(v, w)$$

Hence, we have that

$$\lambda^*\omega^B(v, w) = \lambda^*\omega(v, w) - \lambda^* \cdot \pi_Q^* B(v, w)$$

$$= -d\gamma(T\pi_Q(v), T\pi_Q(w)) - (\pi_Q \cdot \gamma \cdot \pi_Q)^* B(v, w)$$

$$= -d\gamma(T\pi_Q(v), T\pi_Q(w)) - \pi_Q^* B(v, w)$$

$$= -(d\gamma + B)(T\pi_Q(v), T\pi_Q(w)),$$

where we have used the relation $\pi_Q \cdot \gamma \cdot \pi_Q = \pi_Q$. It follows that the assertion (i) holds.

Next, we prove the assertion (ii). For any $v, w \in TT^*Q$, note that $v - T(\gamma \cdot \pi_Q) \cdot v$ is vertical, because

$$T\pi_Q(v - T(\gamma \cdot \pi_Q) \cdot v) = T\pi_Q(v) - T(\pi_Q \cdot \gamma \cdot \pi_Q) \cdot v = T\pi_Q(v) - T\pi_Q(v) = 0,$$

Thus, $\omega(v - T(\gamma \cdot \pi_Q) \cdot v, w - T(\gamma \cdot \pi_Q) \cdot w) = 0$, and hence,

$$\omega(T(\gamma \cdot \pi_Q) \cdot v, w) = \omega(v, w - T(\gamma \cdot \pi_Q) \cdot w) + \omega(T(\gamma \cdot \pi_Q) \cdot v, T(\gamma \cdot \pi_Q) \cdot w).$$

However, the second term on the right-hand side is given by

$$\omega(T(\gamma \cdot \pi_Q) \cdot v, T(\gamma \cdot \pi_Q) \cdot w) = \gamma^*\omega(T\pi_Q(v), T\pi_Q(w)) = -d\gamma(T\pi_Q(v), T\pi_Q(w)),$$

It follows that

$$\omega(T\lambda \cdot v, w) = \omega(T(\gamma \cdot \pi_Q) \cdot v, w)$$

$$= \omega(v, w - T(\gamma \cdot \pi_Q) \cdot w) - d\gamma(T\pi_Q(v), T\pi_Q(w))$$

$$= \omega(v, w - T\lambda \cdot w) - d\gamma(T\pi_Q(v), T\pi_Q(w)).$$

Hence, we have that

$$\omega^B(T\lambda \cdot v, w) = \omega(T\lambda \cdot v, w) - \pi_Q^* B(T\lambda \cdot v, w)$$

$$= \omega(v, w - T\lambda \cdot w) - d(\pi_Q^*)(T\pi_Q(v), T\pi_Q(w)) - B(T\pi_Q \cdot T\lambda \cdot v, T\pi_Q(w))$$

$$= \omega^B(v, w - T\lambda \cdot w) + \pi_Q^* B(v, w - T\lambda \cdot w)$$

$$- d\gamma(T\pi_Q(v), T\pi_Q(w)) - B(T(\pi_Q \cdot \lambda) \cdot v, T\pi_Q(w))$$

$$= \omega^B(v, w - T\lambda \cdot w) + \pi_Q^* B(v, w) - B(T\pi_Q(v), T\pi_Q(w))$$

$$- d\gamma(T\pi_Q(v), T\pi_Q(w)) - B(T\pi_Q \cdot T\lambda \cdot w, T\pi_Q(w))$$

$$= \omega^B(v, w - T\lambda \cdot w) + \pi_Q^* B(v, w) - B(T\pi_Q(v), T\pi_Q(w))$$

$$- d\gamma(T\pi_Q(v), T\pi_Q(w)) - (d\gamma + B)(T\pi_Q(v), T\pi_Q(w))$$

$$= \omega^B(v, w - T\lambda \cdot w) - (d\gamma + B)(T\pi_Q(v), T\pi_Q(w)).$$

Thus, the assertion (ii) holds. ■

For a given CMH system $(T^*Q, \omega^B, H, F, W)$ on $T^*Q$, by using the above Lemma 3.4, we can derive precisely the geometric constraint conditions of the magnetic symplectic form $\omega^B$ for the dynamical vector field $X_{(T^*Q, \omega^B, H, F, W)}$ of the CMH system with a control law $u$, that is, Type I and Type II of Hamilton-Jacobi equation for the CMH system.
Theorem 3.5 (Type I of Hamilton-Jacobi Theorem for a CMH System) For the CMH system \((T^*Q, \omega^B, H, F, W)\) with the magnetic symplectic form \(\omega^B = \omega - \pi_Q^*B\) on \(T^*Q\), where \(\omega\) is the canonical symplectic form on \(T^*Q\) and \(B\) is a closed two-form on \(Q\), assume that \(\gamma : Q \to T^*Q\) is an one-form on \(Q\), and \(\tilde{X}^\gamma = T_\pi Q \cdot \tilde{X} \cdot \gamma\), where \(\tilde{X} = X_{(T^*Q, \omega^B, H, F, W)}\) is the dynamical vector field of the CMH system \((T^*Q, \omega^B, H, F, W)\) with a control law \(u\). If the one-form \(\gamma : Q \to T^*Q\) satisfies the condition \(d\gamma = -B\) with respect to \(T_\pi Q : TT^*Q \to TQ\), then \(\gamma\) is a solution of the equation \(T\gamma \cdot \tilde{X}^\gamma = X^B_H \cdot \gamma\), where \(X^B_H\) is the magnetic Hamiltonian vector field of the associated magnetic Hamiltonian system \((T^*Q, \omega^B, H)\), and the equation is called the Type I of Hamilton-Jacobi equation for the CMH system \((T^*Q, \omega^B, H, F, W)\) with a control law \(u\). The maps involved in the theorem are shown in the following Diagram-1.

\[
\begin{array}{c}
T^*Q \xrightarrow{\pi_Q} Q \xrightarrow{\gamma} T^*Q \\
X^B_H \bigg| \xrightarrow{T^\gamma} \tilde{X}^\gamma \bigg| \xrightarrow{T_\pi Q} T(T^*Q)
\end{array}
\]

Diagram-1

Proof: Since \(\tilde{X} = \tilde{X}_{(T^*Q, \omega^B, H, F, W)} = X^B_H + \operatorname{vlift}(F)^B + \operatorname{vlift}(u)^B\), and \(T_\pi Q \cdot \operatorname{vlift}(F)^B = T\pi_Q \cdot \operatorname{vlift}(u)^B = 0\), then we have that \(T_\pi Q \cdot (\tilde{X} \cdot \gamma) = T\pi_Q \cdot X^B_H \cdot \gamma\). If we take that \(v = X^B_H \cdot \gamma \in TT^*Q\), and for any \(w \in TT^*Q\), \(T_\pi Q(w) \neq 0\), from Lemma 3.4(ii) and \(d\gamma = -B\) with respect to \(T_\pi Q : TT^*Q \to TQ\), that is, \((d\gamma + B)(T_\pi Q \cdot X^B_H \cdot \gamma, T_\pi Q \cdot w) = 0\), we have that

\[
\omega^B(T\gamma \cdot \tilde{X}^\gamma, w) = \omega^B(T\gamma \cdot T_\pi Q \cdot \tilde{X} \cdot \gamma, w) = \omega^B(TT_\pi Q \cdot \tilde{X} \cdot \gamma, w) = \omega^B(T_\pi Q \cdot X^B_H \cdot \gamma, w) = \omega^B(T_\pi Q \cdot X^B_H \cdot \gamma, w) = \omega^B(T_\pi Q \cdot X^B_H \cdot \gamma, T_\pi Q \cdot w).
\]

Hence, we have that

\[
\omega^B(T\gamma \cdot \tilde{X}^\gamma, w) - \omega^B(X^B_H \cdot \gamma, w) = -\omega^B(X^B_H \cdot \gamma, T_\pi Q \cdot w).
\]

If \(\gamma\) satisfies the equation \(T\gamma \cdot \tilde{X}^\gamma = X^B_H \cdot \gamma\), from Lemma 3.4(i) we can obtain that

\[
\omega^B(X^B_H \cdot \gamma, T_\pi Q \cdot w) = \omega^B(T_\pi Q \cdot \tilde{X}^\gamma, T_\pi Q \cdot w) = \omega^B(T_\pi Q \cdot \tilde{X}^\gamma, T_\pi Q \cdot w) = \omega^B(T_\pi Q \cdot X^B_H \cdot \gamma, T_\pi Q \cdot w) = \omega^B(T_\pi Q \cdot X^B_H \cdot \gamma, T_\pi Q \cdot w) = \omega^B(T_\pi Q \cdot X^B_H \cdot \gamma, T_\pi Q \cdot w) = \omega^B(T_\pi Q \cdot X^B_H \cdot \gamma, T_\pi Q \cdot w) = 0.
\]

since \(\gamma : Q \to T^*Q\) satisfies the condition \(d\gamma = -B\) with respect to \(T_\pi Q : TT^*Q \to TQ\). But, because the magnetic symplectic form \(\omega^B\) is non-degenerate, the left side of (3.1) equals zero, only when \(\gamma\) satisfies the equation \(T\gamma \cdot \tilde{X}^\gamma = X^B_H \cdot \gamma\). Thus, if the one-form \(\gamma : Q \to T^*Q\) satisfies the condition \(d\gamma = -B\) with respect to \(T_\pi Q : TT^*Q \to TQ\), then \(\gamma\) must be a solution of the Type I of Hamilton-Jacobi equation \(T\gamma \cdot \tilde{X}^\gamma = X^B_H \cdot \gamma\), for the CMH system \((T^*Q, \omega^B, H, F, W)\) with a control law \(u\).

When \(B = 0\), in this case the magnetic symplectic form \(\omega^B\) is just the canonical symplectic form \(\omega\) on \(T^*Q\), and the CMH system \((T^*Q, \omega^B, H, F, W)\) is just a canonical RCH system.
(\(T^*Q, \omega, H, F, W\)) and the condition that, the one-form \(\gamma : Q \to T^*Q\) satisfies the condition \(d\gamma = -B\) with respect to \(T\pi_Q : TT^*Q \to TQ\), becomes that \(\gamma\) is closed with respect to \(T\pi_Q : TT^*Q \to TQ\). Thus, from above Theorem 3.5, we can obtain Theorem 2.6 in Wang [23]. On the other hand, it is a natural problem what and how we could do, if an one-form \(\gamma : Q \to T^*Q\) is not closed on \(Q\) with respect to \(T\pi_Q : TT^*Q \to TQ\) in Theorem 2.6 in Wang [23], and hence \(\gamma\) is not a solution of the Type I of Hamilton-Jacobi equation for the canonical RCH system. In this case, our idea is that we hope to look for a new RCH system, such that \(\gamma\) is a solution of the Type I of Hamilton-Jacobi equation for the new RCH system. Note that, if \(\gamma : Q \to T^*Q\) is not closed on \(Q\) with respect to \(T\pi_Q : TT^*Q \to TQ\), that is, there exist \(v, w \in TT^*Q\), such that \(d\gamma(T\pi_Q(v), T\pi_Q(w)) \neq 0\), and hence \(\gamma\) is not yet closed on \(Q\). In this case, we note that \(d \cdot d\gamma = d^2\gamma = 0\), and hence the \(d\gamma\) is a closed two-form on \(Q\). Thus, we can construct a magnetic symplectic form on \(T^*Q\), that is, \(\omega^B = \omega - \pi^*_Q B = \omega + \pi^*_Q (d\gamma)\), where \(B = -d\gamma\), and \(\omega\) is the canonical symplectic form on \(T^*Q\), and \(\pi^*_Q : T^*Q \to T^*Q\). In this case, for any \(x, y \in TQ\), we have that \((d\gamma + B)(x, y) = 0\), and hence for any \(v, w \in TT^*Q\), we have that \((d\gamma + B)(T\pi_Q(v), T\pi_Q(w)) = 0\), that is, the one-form \(\gamma : Q \to T^*Q\) satisfies the condition \(d\gamma = -B\) with respect to \(T\pi_Q : TT^*Q \to TQ\). Thus, we can construct a CMH system \((T^*Q, \omega^B, H, F, W)\), its dynamical vector field with a control law \(u\) is given by \(X_{(\omega^B, H, F, u)} = X^B_H + \text{vlift}(F)^B + \text{vlift}(u)^B\), where \(X^B_H\) satisfies the magnetic Hamiltonian equation, that is, \(\gamma, v^B = dH\), and \(\text{vlift}(F)^B = \text{vlift}(F)X^B_H\). In this case, by using Lemma 3.4 and the dynamical vector field \(X_{(\omega^B, H, F, u)}\) from Theorem 3.5 we can obtain the following theorem.

**Theorem 3.6** For a given RCH system \((T^*Q, \omega, H, F, W)\) with the canonical symplectic form \(\omega\) on \(T^*Q\), assume that the one-form \(\gamma : Q \to T^*Q\) is not closed with respect to \(T\pi_Q : TT^*Q \to TQ\). Construct a magnetic symplectic form on \(T^*Q\), \(\omega^B = \omega - \pi^*_Q B\), where \(B = -d\gamma\), and a CMH system \((T^*Q, \omega^B, H, F, W)\). Denote \(\bar{X}^\gamma = T\pi_Q : \bar{X} \cdot \gamma\), where \(\bar{X} = X_{(T^*Q, \omega^B, H, F, u)}\) is the dynamical vector field of the CMH system \((T^*Q, \omega^B, H, F, W)\) with a control law \(u\). Then the one-form \(\gamma\) is just a solution of the Type I of Hamilton-Jacobi equation \(T\gamma \cdot \bar{X}^\gamma = X^B_H \cdot \gamma\), for the CMH system with a control law \(u\).

Next, for any symplectic map \(\varepsilon : T^*Q \to T^*Q\) with respect to the magnetic symplectic form \(\omega^B\), we can prove the following Type II of geometric Hamilton-Jacobi theorem for the CMH system \((T^*Q, \omega^B, H, F, W)\). For convenience, the maps involved in the following theorem and its proof are shown in Diagram-2.

\[
\begin{align*}
T^*Q & \xrightarrow{\varepsilon} T^*Q & \varepsilon & \xrightarrow{\pi_Q} & Q & \xrightarrow{\gamma} & T^*Q \\
X^B_H & \xrightarrow{T\gamma} & \bar{X}^\gamma & \xrightarrow{T\pi_Q} & TQ & \xrightarrow{\bar{X}^\gamma} & T(T^*Q)
\end{align*}
\]

**Diagram-2**

**Theorem 3.7** (Type II of Hamilton-Jacobi Theorem for a CMH System) For the CMH system \((T^*Q, \omega^B, H, F, W)\) with the magnetic symplectic form \(\omega^B = \omega - \pi^*_Q B\) on \(T^*Q\), where \(\omega\) is the canonical symplectic form on \(T^*Q\) and \(B\) is a closed two-form on \(Q\), assume that \(\gamma : Q \to T^*Q\) is an one-form on \(Q\), and \(\lambda = \gamma \cdot \pi_Q : T^*Q \to T^*Q\), and for any symplectic map \(\varepsilon : T^*Q \to T^*Q\) with respect to \(\omega^B\), denote by \(\bar{X}^\varepsilon = T\pi_Q : \bar{X} \cdot \varepsilon\), where \(\bar{X} = X_{(T^*Q, \omega^B, H, F, u)}\) is the dynamical vector field of the CMH system \((T^*Q, \omega^B, H, F, W)\) with a control law \(u\). Then \(\varepsilon\) is a solution of the equation \(T\gamma \cdot \bar{X}^\varepsilon = T\lambda \cdot \bar{X} \cdot \varepsilon\), if and only if it is a solution of the equation \(T\gamma \cdot \bar{X}^\varepsilon = X^B_H \cdot \varepsilon\), where \(X^B_H\) and \(X^B_H \in TT^*Q\) are the magnetic Hamiltonian vector fields of the functions \(H\) and \(H \cdot \varepsilon : T^*Q \to \mathbb{R}\), respectively. The equation \(T\gamma \cdot \bar{X}^\varepsilon = X^B_H \cdot \varepsilon\), is called the Type II of Hamilton-Jacobi equation for the CMH system \((T^*Q, \omega^B, H, F, W)\) with a control law \(u\).
Because the magnetic symplectic form \( CMH \) system is just a magnetic Hamiltonian system, it is the equation of the differential one-form \( \gamma \).

Remark 3.8 It is worthy of noting that, the Type I of Hamilton-Jacobi equation \( T\gamma \cdot \tilde{X} = X_H^B \cdot \varepsilon \), is the equation of the differential one-form \( \gamma \); and the Type II of Hamilton-Jacobi equation \( T\gamma \cdot \tilde{X} = X_H^B \cdot \varepsilon \), is the equation of the symplectic diffeomorphism map \( \varepsilon \). If the external force and control of a CMH system \( (T^*Q, \omega^B, H, F, W) \) are both zeros, that is, \( F = 0 \) and \( W = \emptyset \), in this case the CMH system is just a magnetic Hamiltonian system \( (T^*Q, \omega^B, H) \), and from the proofs of the above Theorem 3.5-3.7, we can obtain two types of Hamilton-Jacobi equation for the associated magnetic Hamiltonian system, that is, Theorem 3.5-3.7 in Wang [27]. Thus, Theorem 3.5-3.7 can be regarded as an extension of two types of Hamilton-Jacobi equations for a magnetic Hamiltonian system to that for the system with external force and control. Moreover, if \( B = 0 \), in this case the magnetic symplectic form \( \omega^B \) is just the canonical symplectic form \( \omega \) on \( T^*Q \), and the condition that the one-form \( \gamma : Q \to T^*Q \) satisfies the condition \( d\gamma = -B \) with respect to \( T\pi_Q : TT^*Q \to TQ \), becomes that \( \gamma \) is closed with respect to \( T\pi_Q : TT^*Q \to TQ \). Thus, from above Theorem 3.5 and Theorem 3.7, we can obtain Theorem 2.5 and Theorem 2.6 in Wang [22]. Thus, Theorem 3.5 and Theorem 3.7 can be regarded as an extension of two types of Hamilton-Jacobi equation for a canonical Hamiltonian system to that for the system with magnetic, external force and control.

4 CMH-equivalence and the Solutions of Hamilton-Jacobi Equations

In the following we first give the definition of CMH-equivalence for the CMH systems, then prove that the solutions of corresponding to Hamilton-Jacobi equations leave invariant under the conditions of CMH-equivalence, if the associated magnetic Hamiltonian systems are equivalent. This result describes the relationship between the CMH-equivalence for the CMH systems and the solutions of the associated Hamilton-Jacobi equations.

For two given Hamiltonian systems \( (T^*Q_1, \omega_i, H_i), i = 1, 2 \), we say them to be equivalent, if there exists a diffeomorphism \( \varphi : Q_1 \to Q_2 \), such that their Hamiltonian vector fields \( X_{H_i}, i = 1, 2 \),...
satisfy the condition \( X_{H_1} \cdot \varphi^* = T(\varphi^*)X_{H_2} \), where the map \( \varphi^* = T^* \varphi : T^* Q_2 \to T^* Q_1 \) is the cotangent lifted map of \( \varphi \), and the map \( T(\varphi^*) : TT^* Q_2 \to TT^* Q_1 \) is the tangent map of \( \varphi^* \). From Marsden and Ratiu [15], we know that the condition \( X_{H_1} \cdot \varphi^* = T(\varphi^*)X_{H_2} \) is equivalent the fact that the map \( \varphi^* : T^* Q_2 \to T^* Q_1 \) is symplectic with respect to their canonical symplectic forms, on \( T^* Q_i \), \( i = 1, 2 \). In the same way, for two given magnetic Hamiltonian systems \((T^* Q_i, \omega_i^B, H_i)\), \( i = 1, 2 \), we say them to be equivalent, if there exists a diffeomorphism \( \varphi : Q_1 \to Q_2 \), which is symplectic with respect to their magnetic symplectic forms, such that their magnetic Hamiltonian vector fields \( X_{H_i}^B \), \( i = 1, 2 \) satisfy the condition \( X_{H_1}^B \cdot \varphi^* = T(\varphi^*)X_{H_2}^B \).

For two given CMH systems \((T^* Q_i, \omega_i^B, H_i, F_i, W_i)\), \( i = 1, 2 \), we also want to define their equivalence, that is, to look for a diffeomorphism \( \varphi : Q_1 \to Q_2 \), such that \( X_{(T^* Q_i, \omega_i^B, H_i, F_i, W_i)} \cdot \varphi^* = T(\varphi^*)X_{(T^* Q_2, \omega_2^B, H_2, F_2, W_2)} \). But, it is worthy of noting that, when a CMH system is given, the force map \( F \) is determined, but the feedback control law \( u : T^* Q \to W \) could be chosen. In order to describe the feedback control law to modify the structure of the CMH system, the controlled magnetic Hamiltonian matching conditions and CMH-equivalence are induced as follows.

**Definition 4.1 (CMH-equivalence)** Suppose that we have two CMH systems \((T^* Q_i, \omega_i^B, H_i, F_i, W_i)\), \( i = 1, 2 \), we say them to be CMH-equivalent, or simply, \((T^* Q_1, \omega_1^B, H_1, F_1, W_1) \sim_{CMH} (T^* Q_2, \omega_2^B, H_2, F_2, W_2)\), if there exists a diffeomorphism \( \varphi : Q_1 \to Q_2 \), such that the two closed-loop dynamical systems produce the same dynamical vector fields, that is, \( X_{(T^* Q_1, \omega_1^B, H_1, F_1, u_1)} \cdot \varphi^* = T(\varphi^*)X_{(T^* Q_2, \omega_2^B, H_2, F_2, u_2)} \), where the map \( T(\varphi^*) : TT^* Q_2 \to TT^* Q_1 \) is the tangent map of \( \varphi^* \).

From the expression (2.1) of the dynamical vector field of the CMH system and the condition \( X_{(T^* Q_1, \omega_1^B, H_1, F_1, u_1)} \cdot \varphi^* = T(\varphi^*)X_{(T^* Q_2, \omega_2^B, H_2, F_2, u_2)} \), we have that

\[
(X_{H_1}^B + \text{vlift}(F_1))X_{H_1}^B + \text{vlift}(u_1)X_{H_1}^B) \cdot \varphi^* = T(\varphi^*)[X_{H_2}^B + \text{vlift}(F_2)X_{H_2}^B + \text{vlift}(u_2)X_{H_2}^B].
\]

By using the notation of vertical lift map of a vector along a fiber, for \( \alpha_x \in T_x^* Q_2 \), \( x \in Q_2 \), we have that

\[
T(\varphi^*)\text{vlift}(F_2)X_{H_2}^B(\alpha_x) = T(\varphi^*)\text{vlift}((TF_2X_{H_2}^B)(F_2(\alpha_x)), \alpha_x)
\]

\[
= \text{vlift}(T(\varphi^*) \cdot TF_2 \cdot T(\varphi^*)X_{H_2}^B)(\varphi^* F_2 \varphi_*(\varphi^* \alpha_x)), \varphi^* \alpha
\]

\[
= \text{vlift}(T(\varphi^* F_2 \varphi_*)X_{H_2}^B(\varphi^* F_2 \varphi_*(\varphi^* \alpha_x)), \varphi^* \alpha)
\]

\[
= \text{vlift}(\varphi^* F_2 \varphi_*)X_{H_2}^B(\varphi^* \alpha_x),
\]

where the map \( \varphi_* = (\varphi^{-1})^* : T^* Q_1 \to T^* Q_2 \). In the same way, we have that \( T(\varphi^*)\text{vlift}(u_2)X_{H_2}^B = \text{vlift}(\varphi^* u_2 \varphi_*)X_{H_2}^B \cdot \varphi^* \). Note that \( \text{vlift}(F)^B = \text{vlift}(F)X_{H_2}^B \), and \( \text{vlift}(u)^B = \text{vlift}(u)X_{H_2}^B \), and hence we have that the explicit relation between the two control laws \( u_i \in W_i \), \( i = 1, 2 \) in **RCH-2** is given by

\[
(\text{vlift}(u_1)^B - \text{vlift}(\varphi^* u_2 \varphi_*)^B) \cdot \varphi^*
\]

\[
= -X_{H_1}^B \cdot \varphi^* + T(\varphi^*)(X_{H_2}^B) + (\text{vlift}(F_1)^B + \text{vlift}(\varphi^* F_2 \varphi_*)^B) \cdot \varphi^*. \quad (4.1)
\]

From the above relation (4.1) we know that, when two CMH systems \((T^* Q_i, \omega_i^B, H_i, F_i, W_i)\), \( i = 1, 2 \), are CMH-equivalent with respect to \( \varphi^* \), the associated magnetic Hamiltonian systems
If the associated canonical Hamiltonian systems (\(T^*Q_i, \omega^B_i, H_i\), \(i = 1, 2\)) are CMH-equivalent with respect to \(\varphi^*\), that is, \(T(\varphi^*) \cdot X_{H_2} = X_{H_1} \cdot \varphi^*\). In this case, from (2.4) and (4.2), we have the following theorem.

**Theorem 4.3** Suppose that two CMH systems \((T^*Q_i, \omega^B_i, H_i, F_i, W_i), i = 1, 2\), are CMH-equivalent with respect to \(\varphi^*\), and the associated canonical Hamiltonian systems \((T^*Q_i, \omega^B_i, H_i), i = 1, 2\), are also equivalent with respect to \(\varphi^*\). Then we have the following fact that, if one system satisfies the magnetic vanishing condition, then another CMH-equivalent system must satisfy the associated magnetic vanishing condition.

Moreover, if considering the CMH-equivalence of the CMH systems, we can prove the following Theorem 4.3, which states that the solutions of two types of Hamilton-Jacobi equations for the CMH systems leave invariant under the conditions of CMH-equivalence, if the associated magnetic Hamiltonian systems are equivalent.

**Theorem 4.3** Suppose that two CMH systems \((T^*Q_i, \omega^B_i, H_i, F_i, W_i), i = 1, 2\), are CMH-equivalent with an equivalent map \(\varphi: Q_1 \rightarrow Q_2\), and the associated magnetic Hamiltonian systems \((T^*Q_i, \omega^B_i, H_i), i = 1, 2\), are also equivalent with respect to \(\varphi^*\), under the hypotheses and notations of Theorem 3.5, Theorem 3.7, we have that

(i) If the one-form \(\gamma_2: Q_2 \rightarrow T^*Q_2\) satisfies the condition that \(d\gamma_2 = -B_2\) with respect to \(T\pi_{Q_2}: TT^*Q_2 \rightarrow TQ_2\), then \(\gamma_2 = \varphi^* \cdot \gamma_1\): \(Q_1 \rightarrow T^*Q_1\) satisfies also the condition that \(d\gamma_1 = -B_1\) with respect to \(T\pi_{Q_1}: TT^*Q_1 \rightarrow TQ_1\), and hence it is a solution of the Type I of Hamilton-Jacobi equation for the CMH system \((T^*Q_1, \omega^B_1, H_1, F_1, W_1)\). Vice versa;

(ii) If the symplectic map \(\varepsilon_2: T^*Q_2 \rightarrow T^*Q_2\) with respect to \(\omega^B_2\) is a solution of the Type II of Hamilton-Jacobi equation for the CMH system \((T^*Q_2, \omega^B_2, H_2, F_2, W_2)\), then \(\varepsilon_1 = \varphi^* \cdot \varepsilon_2 \cdot \varphi^*\): \(T^*Q_1 \rightarrow T^*Q_1\) is a symplectic map with respect to \(\omega^B_1\), and it is a solution of the Type II of Hamilton-Jacobi equation for the CMH system \((T^*Q_1, \omega^B_1, H_1, F_1, W_1)\). Vice versa.

**Proof:** We first prove the assertion (i). If two given CMH systems \((T^*Q_i, \omega^B_i, H_i, F_i, W_i), i = 1, 2\), are CMH-equivalent with an equivalent map \(\varphi: Q_1 \rightarrow Q_2\), from the definition of CMH-equivalence, we know that for each control law \(u_1: T^*Q_1 \rightarrow W_1\), there exists the control law \(u_2: T^*Q_2 \rightarrow W_2\), such that the two closed-loop dynamical systems produce the same dynamical vector fields, that is, \(\tilde{X}_1 \cdot \varphi^* = T(\varphi^*) \cdot \tilde{X}_2\), where \(\tilde{X}_i = X_{(T^*Q_i, \omega^B_i, H_i, F_i, u_i)}, i = 1, 2\). From the following commutative Diagram-3:

\[
\begin{array}{c}
\frac{\gamma_1}{Q_1 \rightarrow T^*Q_1} \xrightarrow{\varphi} \frac{\bar{X}_1}{TT^*Q_1} \xrightarrow{T^{\pi_{Q_1}}} TQ_1 \\
\gamma_2 \downarrow \quad \varphi \uparrow \quad T\varphi^* \uparrow \quad T\varphi \downarrow \\
\frac{\gamma_2}{Q_2 \rightarrow T^*Q_2} \xrightarrow{\varphi} \frac{\bar{X}_2}{TT^*Q_2} \xrightarrow{T^{\pi_{Q_2}}} TQ_2
\end{array}
\]
we have that $\gamma_1 = \varphi^* \cdot \gamma_2 \cdot \varphi$. $d\gamma_1 = \varphi^* \cdot d\gamma_2 \cdot \varphi$, $B_1 = \varphi^* \cdot B_2 \cdot \varphi$, and $T\varphi \cdot T\pi Q_1 \cdot T\varphi^* = T\pi Q_2$. For $x \in Q_1$, and $v, w \in T\pi x Q_1$, then $\varphi(x) \in Q_2$ and $T\varphi(v)$, $T\varphi(w) \in T\pi Q_2$. Since the one-form $\gamma_2 : Q_2 \to T^* Q_2$ satisfies the condition that $d\gamma_2 = -B_2$ with respect to $T\pi Q_2 : T\pi Q_2 \to TQ_2$, then

$$d\gamma_2 + B_2)(T\pi Q_2 \cdot T\varphi(v), T\pi Q_2 \cdot T\varphi(w))(\varphi(x)) = 0.$$

Thus,

$$(d\gamma_1 + B_1)(T\pi Q_1(v), T\pi Q_1(w))(x) = \varphi^* \cdot (d\gamma_2 + B_2) \cdot \varphi(T\pi Q_1(v), T\pi Q_1(w))(x) = (d\gamma_2 + B_2)(T\varphi \cdot T\pi Q_1(v), T\pi Q_1(w))(\varphi(x)) = (d\gamma_2 + B_2)(T\varphi \cdot T\pi Q_1(v), T\pi Q_1(w))(\varphi(x)) = (d\gamma_2 + B_2)(T\pi Q_2 \cdot T\varphi(v), T\pi Q_2 \cdot T\varphi(w))(\varphi(x)) = 0,$$

that is, the one-form $\gamma_1 = \varphi^* \cdot \gamma_2 \cdot \varphi : Q_1 \to T^* Q_1$ satisfies the condition that $d\gamma_1 = -B_1$ with respect to $T\pi Q_1 : T\pi Q_1 \to TQ_1$. Moreover, from Theorem 3.5 we know that, the one-form $\gamma_2$ is a solution of the Type I of Hamilton-Jacobi equation for the CMH system $(T^* Q_2, \omega^2, H_2, F_2, W_2)$, that is, $T\gamma_2 \cdot \hat{X}_\gamma = X_{H_2} \cdot \gamma_2$, where $\hat{X}_\gamma = T\pi Q_1 \cdot \hat{X}_\gamma \cdot \gamma_i$, $i = 1, 2$. Hence,

$$T\gamma_1 \cdot \hat{X}_\gamma = T(\varphi^* \cdot \gamma_2 \cdot \varphi) \cdot T\pi Q_1 \cdot \hat{X}_\gamma \cdot \gamma_1$$

$$= T(\varphi)^* \cdot T\gamma_2 \cdot T\varphi \cdot T\pi Q_1 \cdot \hat{X}_\gamma \cdot \gamma_1 \cdot (\varphi^* \cdot \gamma_2 \cdot \varphi)$$

$$= T(\varphi)^* \cdot T\gamma_2 \cdot T\varphi \cdot T\pi Q_1 \cdot (T(\varphi)^* \cdot \hat{X}_\gamma) \cdot \gamma_2 \cdot \varphi$$

$$= T(\varphi)^* \cdot T\gamma_2 \cdot (T\pi Q_1 \cdot \hat{X}_\gamma) \cdot T\varphi \cdot T\pi Q_1 \cdot \gamma_2 \cdot \varphi$$

$$= T(\varphi)^* \cdot X_{H_2} \cdot \gamma_2 \cdot \varphi = X_{H_2} \cdot \varphi \cdot \varphi = X_{H_1} \cdot \gamma_1,$$

where we have used that $T(\varphi)^* \cdot X_{H_2} = X_{H_1} \cdot \varphi^*$, because the associated magnetic Hamiltonian systems $(T^* Q_i, \omega^1, H_1, i = 1, 2$, are equivalent with respect to $\varphi^*$. Thus, the one-form $\gamma_1 = \varphi^* \cdot \gamma_2 \cdot \varphi$ is a solution of the Type I of Hamilton-Jacobi equation for the CMH system $(T^* Q_1, \omega^1, H_1, F_1, W_1)$. Note that the map $\varphi : Q_1 \to Q_2$ is a diffeomorphism, and $\varphi^* : T^* Q_2 \to T^* Q_1$ is a symplectic isomorphisms, vice versa. It follows that the assertion (i) of Theorem 4.3 holds.

Next, we prove the assertion (ii). From the following commutative Diagram-4:

$${\begin{array}{c}
\begin{array}{c}
Q_1 \xrightarrow{\gamma_1} T^* Q_1 \xrightarrow{\varphi^*} T^* Q_2 \xrightarrow{\varphi} TQ_2
\end{array}
\end{array}}$$

Diagram-4

we have that $\varepsilon_1 = \varphi^* \cdot \varepsilon_2 \cdot \varphi : T^* Q_1 \to T^* Q_1$. Since $\varepsilon_2 : T^* Q_2 \to T^* Q_2$ is symplectic with respect to $\omega^2$, then for $x \in Q_1$, $v, w \in T\pi x Q_1$, and $\varphi(x) \in Q_2$, $T\varphi(v)$, $T\varphi(w) \in T\pi Q_2$. We have that $\varepsilon^*_2 \cdot \omega^2(T\varphi(v), T\varphi(w))(\varphi(x)) = \omega^2(T\varphi(v), T\varphi(w))(\varphi(x))$. Note that the associated magnetic Hamiltonian systems $(T^* Q_i, \omega^1, H_i, i = 1, 2$, are equivalent with respect to $\varphi^*$, and hence $\varphi^* : T^* Q_2 \to T^* Q_1$ is symplectic with respect to their magnetic symplectic forms, that is, $(\varphi^*)^* \omega^1(w, x) = \omega^2(T\varphi(v), T\varphi(v))(\varphi(x))$, then we have that

$$\varepsilon^*_1 \cdot \omega^1(v, w)(x) = (\varphi^* \cdot \varepsilon_2 \cdot \varphi) \cdot \omega^1(v, w)(x) = (\varphi^* \cdot \varepsilon^*_2 \cdot (\varphi^*)^* \omega^1(v, w)(x)$$

$$= (\varphi^* \cdot \varepsilon^*_2 \cdot \omega^2(T\varphi(v), T\varphi(w))(\varphi(x)) = ((\varphi^*)^* \cdot \omega^2(T\varphi(v), T\varphi(v))(\varphi(x))$$

$$= \omega^1(T(\varphi)^* \cdot T\varphi(v), T(\varphi)^* \cdot T\varphi(v))(\varphi(x)) = \omega^1(v, w)(x),$$


that is, the map \( \varepsilon_1 = \phi^* \cdot \varepsilon_2 \cdot \varphi_* : T^*Q_1 \rightarrow T^*Q_1 \) is symplectic map with respect to \( \omega_1^B \). Moreover, because the symplectic map \( \varepsilon_2 : T^*Q_2 \rightarrow T^*Q_2 \) is a solution of the Type II of Hamilton-Jacobi equation for the CMH system \((T^*Q_2, \omega^B_2, H_2, F_2, W_2)\), that is, \( T\gamma_2 \cdot \tilde{X}_2^\varepsilon_2 = X^B_{H_2} \cdot \varepsilon_2 \), where \( \tilde{X}_i^\varepsilon_i = T\pi_{Q_i} \cdot \tilde{X}_i \cdot \varepsilon_i \), \( i = 1, 2 \). Hence, we have that

\[
T\gamma_1 \cdot \tilde{X}_1^\varepsilon_1 = T(\phi^* \cdot \gamma_2 \cdot \varphi) \cdot T\pi_{Q_1} \cdot \tilde{X}_1 \cdot \varepsilon_1 \\
= T(\phi^*) \cdot T\gamma_2 \cdot T\varphi \cdot T\pi_{Q_1} \cdot \tilde{X}_1 \cdot (\phi^* \cdot \varepsilon_2 \cdot \varphi_*) \\
= T(\phi^*) \cdot T\gamma_2 \cdot T\varphi \cdot T\pi_{Q_1} \cdot (T(\phi^*) \cdot \tilde{X}_2) \cdot \varepsilon_2 \cdot \varphi_* \\
= T(\phi^*) \cdot T\gamma_2 \cdot (T\pi_{Q_2} \cdot \tilde{X}_2 \cdot \varepsilon_2) \cdot \varphi_* = T(\phi^*) \cdot T\gamma_2 \cdot \tilde{X}_2^\varepsilon_2 \cdot \varphi_* \\
= T(\phi^*) \cdot X^B_{H_2} \cdot \varepsilon_2 \cdot \varphi_* = X^B_{H_1} \cdot \varphi^* \cdot \varepsilon_2 \cdot \varphi_* = X^B_{H_1} \cdot \varepsilon_1,
\]

that is, the symplectic map \( \varepsilon_1 = \phi^* \cdot \varepsilon_2 \cdot \varphi_* \) is a solution of the Type II of Hamilton-Jacobi equation for the CMH system \((T^*Q_1, \omega^B_1, H_1, F_1, W_1)\). In the same way, because the map \( \varphi : Q_1 \rightarrow Q_2 \) is a diffeomorphism, and \( \varphi^* : T^*Q_2 \rightarrow T^*Q_1 \) is a symplectic isomorphisms, vice versa. Hence we prove the assertion (ii) of Theorem 4.3.

5 Nonholonomic CMH System and Distributional CMH System

In order to describe the impact of nonholonomic constraints for Hamilton-Jacobi theory of the dynamics of a CMH system, in this section we first give some definitions and basic facts about the nonholonomic constraint and the nonholonomic CMH system. Moreover, by analyzing carefully the structure for the nonholonomic dynamical vector field, we give a geometric formulation of distributional CMH system, which is determined by a non-degenerate distributional two-form induced from the magnetic symplectic form, and which will be used in subsequent section.

In order to describe the nonholonomic CMH system, in the following we first give the completeness and regularity conditions for nonholonomic constraints of a mechanical system, see León and Wang [10] and Wang [27,28]. In fact, in order to describe the dynamics of a nonholonomic mechanical system, we need some restriction conditions for nonholonomic constraints of the system. At first, we note that the set of Hamiltonian vector fields forms a Lie algebra with respect to the Lie bracket, since \( [X_i, X_j] = -\{X_i, X_j\} \). But, the Lie bracket operator, in general, may not be closed on the restriction of a nonholonomic constraint. Thus, we have to give the following completeness condition for nonholonomic constraints of a system.

**D-completeness** Let \( Q \) be a smooth manifold and \( TQ \) its tangent bundle. A distribution \( \mathcal{D} \subset TQ \) is said to be **completely nonholonomic** (or bracket-generating) if \( \mathcal{D} \) along with all of its iterated Lie brackets \([\mathcal{D}, \mathcal{D}], [\mathcal{D}, [\mathcal{D}, \mathcal{D}]], \cdots\) spans the tangent bundle \( TQ \). Moreover, we consider a nonholonomic mechanical system on \( Q \), which is given by a Lagrangian function \( L : TQ \rightarrow \mathbb{R} \) subject to constraints determined by a nonholonomic distribution \( \mathcal{D} \subset TQ \) on the configuration manifold \( Q \). Then the nonholonomic system is said to be **completely nonholonomic**, if the distribution \( \mathcal{D} \subset TQ \) determined by the nonholonomic constraints is completely nonholonomic.

**D-regularity** In the following we always assume that \( Q \) is an \( n \)-dimensional smooth manifold with coordinates \((q^i)\), and \( TQ \) its tangent bundle with coordinates \((q^i, \dot{q}^i)\), and \( T^*Q \) its cotangent bundle with coordinates \((q^i, p_j)\), which are the canonical cotangent coordinates of \( T^*Q \) and \( \omega = \sum_{i=1}^{n} dq^i \wedge dp_i \) is canonical symplectic form on \( T^*Q \). If the Lagrangian \( L : TQ \rightarrow \mathbb{R} \) is hyperregular, that is, the Hessian matrix \((\partial^2 L / \partial \dot{q}^i \partial \dot{q}^j)\) is nondegenerate everywhere, then the Legendre transformation \( FL : TQ \rightarrow T^*Q \) is a diffeomorphism. In this case the Hamiltonian
\( H : T^*Q \to \mathbb{R} \) is given by \( H(q, p) = \dot{q} \cdot p - L(q, \dot{q}) \) with Hamiltonian vector field \( X_H \), which is defined by the Hamilton’s equation \( i_{X_H} \omega = dH \), and \( \mathcal{M} = \mathcal{F}L(\mathcal{D}) \) is a constraint submanifold in \( T^*Q \). In particular, for the nonholonomic constraint \( \mathcal{D} \subset TQ \), the Lagrangian \( L \) is said to be \( \mathcal{D}\)-regular, if the restriction of Hessian matrix \((\partial^2 L / \partial \dot{q}^i \partial \dot{q}^j)\) on \( \mathcal{D} \) is nondegenerate everywhere. Moreover, a nonholonomic system is said to be \( \mathcal{D}\)-regular, if its Lagrangian \( L \) is \( \mathcal{D}\)-regular. Note that the restriction of a positive definite symmetric bilinear form to a subspace is also positive definite, and hence nondegenerate. Thus, for a simple nonholonomic mechanical system, that is, whose Lagrangian is the total kinetic energy minus potential energy, it is \( \mathcal{D}\)-regular automatically.

A nonholonomic magnetic Hamiltonian system is the 4-tuple \((T^*Q, \omega^B, \mathcal{D}, H)\), which is a magnetic Hamiltonian system with a \( \mathcal{D}\)-completely and \( \mathcal{D}\)-regularly nonholonomic constraint \( \mathcal{D} \subset TQ \). A nonholonomic CMH system is the 6-tuple \((T^*Q, \omega^B, \mathcal{D}, H, F, W)\), which is a nonholonomic magnetic Hamiltonian system with external force \( F \) and control \( W \), where \( F : T^*Q \to T^*Q \) is the fiber-preserving map, and \( W \subset T^*Q \) is a fiber submanifold of \( T^*Q \). Under the restriction given by constraint, in general, the dynamical vector field of a nonholonomic CMH system may not be magnetic Hamiltonian, however the system is a dynamical system closely related to a magnetic Hamiltonian system. In the following we shall derive a distributional CMH system of the nonholonomic CMH system \((T^*Q, \omega^B, \mathcal{D}, H, F, W)\), by analyzing carefully the structure for the nonholonomic dynamical vector field similar to the method used in León and Wang \[10\] and Wang \[27, 28\].

We consider that the constraint submanifold \( \mathcal{M} = \mathcal{F}L(\mathcal{D}) \subset T^*Q \) and \( i_{\mathcal{M}} : \mathcal{M} \to T^*Q \) is the inclusion, the symplectic form \( \omega^B_{\mathcal{M}} = i_{\mathcal{M}}^*\omega^B \), is induced from the magnetic symplectic form \( \omega^B \) on \( T^*Q \). We define the distribution \( \mathcal{F} \) as the pre-image of the nonholonomic constraints \( \mathcal{D} \) for the map \( T\pi_Q : TT^*Q \to TQ \), that is, \( \mathcal{F} = (T\pi_Q)^{-1}(\mathcal{D}) \subset TT^*Q \), which is a distribution along \( \mathcal{M} \), and \( \mathcal{F}^\circ := \{ \alpha \in T^*T^*Q | \alpha, v = 0, \forall v \in TT^*Q \} \) is the annihilator of \( \mathcal{F} \) in \( T^*T^*Q_{|\mathcal{M}} \). We consider the following nonholonomic constraints condition

\[
(i_X \omega^B - dH) \in \mathcal{F}^\circ, \quad X \in T\mathcal{M}, \tag{5.1}
\]

from Cantrijn et al. \[4\], we know that there exists an unique nonholonomic vector field \( X_n \) satisfying the above condition (5.1), if the admissibility condition \( \text{dim}\mathcal{M} = \text{rank}\mathcal{F} \) and the compatibility condition \( T\mathcal{M} \cap \mathcal{F}^\perp = \{0\} \) hold, where \( \mathcal{F}^\perp \) denotes the magnetic symplectic orthogonal of \( \mathcal{F} \) with respect to the magnetic symplectic form \( \omega^B \) on \( T^*Q \). In particular, when we consider the Whitney sum decomposition \( T(T^*Q)_{|\mathcal{M}} = T\mathcal{M} \oplus \mathcal{F}^\perp \) and the canonical projection \( P : T(T^*Q)_{|\mathcal{M}} \to T\mathcal{M} \), then we have that \( X_n = P(X^B_H) \).

From the condition (5.1) we know that the nonholonomic vector field, in general, may not be magnetic Hamiltonian, because of the restriction of nonholonomic constraints. But, we hope to study the dynamical vector field of nonholonomic CMH system by using the similar method of studying magnetic Hamiltonian vector field. From León and Wang \[10\] and Bates and Śniatycki \[3\], by using the similar method, we can define the distribution \( \mathcal{K} = \mathcal{F} \cap T\mathcal{M} \), and \( \mathcal{K}^\perp = \mathcal{F}^\perp \cap T\mathcal{M} \), where \( \mathcal{K}^\perp \) denotes the magnetic symplectic orthogonal of \( \mathcal{K} \) with respect to the magnetic symplectic form \( \omega^B \), and the admissibility condition \( \text{dim}\mathcal{M} = \text{rank}\mathcal{F} \) and the compatibility condition \( T\mathcal{M} \cap \mathcal{F}^\perp = \{0\} \) hold, then we know that the restriction of the symplectic form \( \omega^B_{\mathcal{M}} \) on \( T^*\mathcal{M} \) fibrewise to the distribution \( \mathcal{K} \), that is, \( \omega^B_{\mathcal{K}} = \tau_{\mathcal{K}} \cdot \omega^B_{\mathcal{M}} \) is non-degenerate, where \( \tau_{\mathcal{K}} \) is the restriction map to distribution \( \mathcal{K} \). It is worthy of noting that \( \omega^B_{\mathcal{K}} \) is not a true two-form on a manifold, so it does not make sense to speak about it being closed. We call \( \omega^B_{\mathcal{K}} \) as a distributional two-form to avoid any confusion. Because \( \omega^B_{\mathcal{K}} \) is non-degenerate as a bilinear form on each fibre of \( \mathcal{K} \), there exists a vector field \( X^B_{\mathcal{K}} \) on \( \mathcal{M} \) which takes values in the constraint distribution \( \mathcal{K} \), such that the distributional
magnetic Hamiltonian equation holds, that is,

\[ i_{X^B}^\kappa \omega^B_\kappa = dH_\kappa, \]

(5.2)

where \( dH_\kappa \) is the restriction of \( dH_M \) to \( M \), and the function \( H_\kappa \) satisfies \( dH_\kappa = \tau_\kappa \cdot dH_M \), and \( H_M = \tau_M \cdot H \) is the restriction of \( H \) to \( M \). Moreover, from the distributional magnetic Hamiltonian equation (5.2), we have that \( X^B_\kappa = \tau_\kappa \cdot X^B_H \).

Moreover, if considering the external force \( F \) and control subset \( W \), and define \( F^B_\kappa = \tau_\kappa \cdot \text{vlift}(F_M)X^B_H \), and for a control law \( u \in W \), \( u^B_\kappa = \tau_\kappa \cdot \text{vlift}(u_M)X^B_H \), where \( F_M = \tau_M \cdot F \) and \( u_M = \tau_M \cdot u \) are the restrictions of \( F \) and \( u \) to \( M \), that is, \( F^B_\kappa \) and \( u^B_\kappa \) are the restrictions of the changes of magnetic Hamiltonian vector field \( X^B_H \) under the actions of \( F_M \) and \( u_M \) to \( M \). Then the 5-tuple \( (K, \omega^B_\kappa, H_\kappa, F^B_\kappa, u^B_\kappa) \) is a distributional CMH system of the nonholonomic CMH system \((T^*Q, \omega, D, H, F, W)\) with a control law \( u \). Thus, the geometric formulation of the distributional CMH system may be summarized as follows.

**Definition 5.1 (Distributional CMH System)** Assume that the 6-tuple \((T^*Q, \omega^B, D, H, F, W)\) is a nonholonomic CMH system, where the magnetic symplectic form \( \omega^B = \omega - \tau_Q^B \) on \( T^*Q \), and \( \omega \) is the canonical symplectic form on \( T^*Q \) and \( B \) is a closed two-form on \( Q \), and \( D \subset T^*Q \) is a \( D \)-completely and \( D \)-regularly nonholonomic constraint of the system, and the external force \( F : T^*Q \to T^*Q \) is the fiber-preserving map, and the control subset \( W \subset T^*Q \) is a fiber submanifold of \( T^*Q \). For a control law \( u \in W \), if there exist a distribution \( K \), an associated non-degenerate distributional two-form \( \omega^B_\kappa \) induced by the magnetic symplectic form \( \omega^B \) and a vector field \( X^B_\kappa \) on the constraint submanifold \( M = FL(D) \subset T^*Q \), such that the distributional magnetic Hamiltonian equation \( i_{X^B_\kappa}^\kappa \omega^B_\kappa = dH_\kappa \) holds, where \( dH_\kappa \) is the restriction of \( dH_M \) to \( K \), and the function \( H_\kappa \) satisfies \( dH_\kappa = \tau_\kappa \cdot dH_M \), and \( F^B_\kappa = \tau_\kappa \cdot \text{vlift}(F_M)X^B_H \), and \( u^B_\kappa = \tau_\kappa \cdot \text{vlift}(u_M)X^B_H \) as defined above, then the 5-tuple \((K, \omega^B_\kappa, H_\kappa, F^B_\kappa, u^B_\kappa)\) is called a distributional CMH system of the nonholonomic CMH system \((T^*Q, \omega^B, D, H, F, u)\), and \( X^B_\kappa \) is called a nonholonomic dynamical vector field. Denote by

\[ \tilde{X} = X^B_{(K, \omega^B_\kappa, H_\kappa, F^B_\kappa, u^B_\kappa)} = X^B_\kappa + F^B_\kappa + u^B_\kappa \]

(5.3)

is the dynamical vector field of the distributional CMH system \((K, \omega^B_\kappa, H_\kappa, F^B_\kappa, u^B_\kappa)\), which is the synthetic of the nonholonomic dynamical vector field \( X^B_\kappa \) and the vector fields \( F^B_\kappa \) and \( u^B_\kappa \). Under the above circumstances, we refer to \((T^*Q, \omega^B, D, H, F, u)\) as a nonholonomic CMH system with an associated distributional CMH system \((K, \omega^B_\kappa, H_\kappa, F^B_\kappa, u^B_\kappa)\).

**Remark 5.2** It is worthy of noting that, when \( B = 0 \), in this case the magnetic symplectic form \( \omega^B \) is just the canonical symplectic form \( \omega \) on \( T^*Q \), and the distributional CMH system \((K, \omega^B_\kappa, H_\kappa, F^B_\kappa, u^B_\kappa)\) becomes the distributional RCH system \((K, \omega_\kappa, H_\kappa, F_\kappa, u_\kappa)\), which is given in Wang [28]. Moreover, if the external force and control of a distributional CMH system \((K, \omega^B_\kappa, H_\kappa, F^B_\kappa, u^B_\kappa)\) are both zeros, that is, \( B = 0, F^B_\kappa = 0 \) and \( u^B_\kappa = 0 \), in this case, the distributional CMH system is just a distributional Hamiltonian system \((K, \omega_\kappa, H_\kappa)\), which is given in León and Wang [10]. Thus, the distributional CMH system can be regarded as an extension of the distributional Hamiltonian system to the system with magnetic, external force and control.

For the nonholonomic CMH system \((T^*Q, \omega^B, D, H, F, u)\) with an associated distributional CMH system \((K, \omega^B_\kappa, H_\kappa, F^B_\kappa, u^B_\kappa)\), the magnetic vector field \( X^0 = X^B_H - X_H \), which is determined by the magnetic equation \( i_{X^0}^\kappa \omega = i_{X^B_H}^\kappa (\pi_Q^*B) \) on \( T^*Q \). Denote by \( X^0_\kappa = \tau_\kappa(X^0) = \tau_\kappa(X^B_H) - \tau_\kappa(X_H) = X^B_\kappa - X_K \), from the expression (5.3) of the dynamical vector field of the distributional CMH system \((K, \omega^B_\kappa, H_\kappa, F^B_\kappa, u^B_\kappa)\), we have that

\[ \tilde{X} = X^B_\kappa + F^B_\kappa + u^B_\kappa = X_K + X^0_\kappa + F^B_\kappa + u^B_\kappa. \]

(5.4)
If the vector fields $F^B_k$ and $u^B_k$ satisfy the following condition

$$X^0_k + F^B_k + u^B_k = 0,$$  \hfill (5.5)

then from (5.4) we have that $X^B_{(K,\omega^B_K,\mathbf{H}_K,F^B_K,u^B_K)} = X_K$, that is, in this case the dynamical vector field of the distributional CMH system is just the dynamical vector field of the canonical distributional Hamiltonian system without the actions of magnetic, external force and control. Thus, the condition (5.5) is called the magnetic vanishing condition for the distributional CMH system $(K,\omega^B_K,\mathbf{H}_K,F^B_K,u^B_K)$.

6 Hamilton-Jacobi Equations for Distributional CMH System

In this section we shall derive precisely the geometric constraint conditions of the induced distributional two-form for the dynamical vector field of distributional CMH system, that is, the two types of Hamilton-Jacobi equation for the distributional CMH system. In order to do this, in the following we first give some important notions and prove a key lemma, which is an important tool for the proofs of two types of Hamilton-Jacobi theorem for the distributional CMH system.

Assume that $\mathcal{D} \subset TQ$ is a $\mathcal{D}$-regularly nonholonomic constraint, and the constraint submanifold $\mathcal{M} = \mathcal{F}L(\mathcal{D}) \subset T^*Q$, the distribution $\mathcal{F} = (T\pi_Q)^{-1}(\mathcal{D}) \subset TT^*Q$, where the projection $\pi_Q : T^*Q \rightarrow Q$ induces the map $T\pi_Q : TT^*Q \rightarrow TQ$. Assume that $\gamma : Q \rightarrow T^*Q$ is an one-form on $Q$, and $B$ is a closed two-form on $Q$. But, note that $d\gamma$ is a two-form on $Q$, and for any $v,w \in TT^*Q$, we have that $d\gamma(T\pi_Q(v),T\pi_Q(w)) = \pi^*(d\gamma)(v,w)$ is a two-form on the cotangent bundle $T^*Q$, where $\pi^* : T^*Q \rightarrow T^*T^*Q$. Thus, in the following we first introduce two weaker notions.

**Definition 6.1** (i) The one-form $\gamma$ is called to be closed on $\mathcal{D}$ with respect to $T\pi_Q : TT^*Q \rightarrow TQ$, if for any $v,w \in TT^*Q$, and $T\pi_Q(v), T\pi_Q(w) \in \mathcal{D}$, we have that $d\gamma(T\pi_Q(v),T\pi_Q(w)) = 0$;

(ii) The one-form $\gamma : Q \rightarrow T^*Q$ is called that satisfies condition that $d\gamma = -B$ on $\mathcal{D}$ with respect to $T\pi_Q : TT^*Q \rightarrow TQ$, if for any $v,w \in TT^*Q$, and $T\pi_Q(v), T\pi_Q(w) \in \mathcal{D}$, we have that $(d\gamma + B)(T\pi_Q(v),T\pi_Q(w)) = 0$.

From the above Definition 6.1, we know that, when $B = 0$, the notion that, $\gamma$ satisfies condition that $d\gamma = -B$ on $\mathcal{D}$ with respect to $T\pi_Q : TT^*Q \rightarrow TQ$, become the notion that $\gamma$ is closed on $\mathcal{D}$ with respect to $T\pi_Q : TT^*Q \rightarrow TQ$. On the other hand, it is worthy of noting that the notion that $\gamma$ satisfies condition that $d\gamma = -B$ on $\mathcal{D}$ with respect to $T\pi_Q : TT^*Q \rightarrow TQ$, is weaker than the notion that $\gamma$ satisfies condition $d\gamma = -B$ on $\mathcal{D}$ with respect to $T\pi_Q : TT^*Q \rightarrow TQ$. Conversely, if $\gamma$ satisfies condition that $d\gamma = -B$ on $\mathcal{D}$ with respect to $T\pi_Q : TT^*Q \rightarrow TQ$, then it may not satisfy condition $d\gamma = -B$ on $\mathcal{D}$. We can prove a general result as follows, which states that the notion that, the $\gamma$ satisfies condition that $d\gamma = -B$ on $\mathcal{D}$ with respect to $T\pi_Q : TT^*Q \rightarrow TQ$, is not equivalent to the notion that $\gamma$ satisfies condition $d\gamma = -B$ on $\mathcal{D}$, also see Wang [27].

**Proposition 6.2** Assume that $\gamma : Q \rightarrow T^*Q$ is an one-form on $Q$ and it doesn't satisfy condition $d\gamma = -B$ on $\mathcal{D}$. We define the set $N$, which is a subset of $TQ$, such that the one-form $\gamma$ on $N$ satisfies the condition that for any $x,y \in N$, $(d\gamma + B)(x,y) \neq 0$. Denote $\text{Ker}(T\pi_Q) = \{u \in TT^*Q | T\pi_Q(u) = 0\}$, and $\gamma : TQ \rightarrow TT^*Q$. If $T\gamma(N) \subset \text{Ker}(T\pi_Q)$, then $\gamma$ satisfies condition $d\gamma = -B$ with respect to $T\pi_Q : TT^*Q \rightarrow TQ$, and hence $\gamma$ satisfies condition $d\gamma = -B$ on $\mathcal{D}$ with respect to $T\pi_Q : TT^*Q \rightarrow TQ$. 

19
Proof: If the $\gamma : Q \to T^*Q$ doesn’t satisfy condition $d\gamma = -B$ on $D$, then it doesn’t yet satisfy condition $d\gamma = -B$. For any $v, w \in TT^*Q$, if $T\pi_Q(v) \notin N$, or $T\pi_Q(w) \notin N$, then by the definition of $N$, we know that $(d\gamma + B)(T\pi_Q(v), T\pi_Q(w)) = 0$; If $T\pi_Q(v) \in N$, and $T\pi_Q(w) \in N$, from the condition $T\gamma(N) \subset Ker(T\pi_Q)$, we know that $T\pi_Q \cdot T\gamma \cdot T\pi_Q(v) = T\pi_Q(v) = 0$, and $T\pi_Q \cdot T\gamma \cdot T\pi_Q(w) = T\pi_Q(w) = 0$, where we have used the relation $\pi_Q \cdot \gamma \cdot \pi_Q = \pi_Q$, and hence $(d\gamma + B)(T\pi_Q(v), T\pi_Q(w)) = 0$. Thus, for any $v, w \in TT^*Q$, we have always that $(d\gamma + B)(T\pi_Q(v), T\pi_Q(w)) = 0$. In particular, for any $v, w \in TT^*Q$, and $T\pi_Q(v), T\pi_Q(w) \in D$, we have $(d\gamma + B)(T\pi_Q(v), T\pi_Q(w)) = 0$. Thus, $\gamma$ satisfies condition that $d\gamma = -B$ on $D$ with respect to $T\pi_Q : TT^*Q \to TQ$.

Now, we prove the following Lemma 6.3. It is worthy of noting that this lemma and Lemma 3.4 given in §3 are the important tool for the proofs of the two types of Hamilton-Jacobi theorems for the distributional CMH system and the nonholonomic reduced distributional CMH system.

**Lemma 6.3** Assume that $\gamma : Q \to T^*Q$ is an one-form on $Q$, and $\lambda = \gamma \cdot \pi_Q : T^*Q \to T^*Q$, and $\omega$ is the canonical symplectic form on $T^*Q$, and $\omega^B = \omega - \pi_Q^*B$ is the magnetic symplectic form on $T^*Q$. If the Lagrangian $L$ is $D$-regular, and $\text{Im}(\gamma) \subset M = \mathcal{F}(D)$, then we have that $X^B_H \cdot \gamma \in \mathcal{F}$ along $\gamma$, and $X^B_H \cdot \lambda \in \mathcal{F}$ along $\lambda$, that is, $T\pi_Q(X^B_H \cdot \gamma(q)) \in D_q$, $\forall q \in Q$, and $T\pi_Q(X^B_H \cdot \lambda(q, p)) \in D_q$, $\forall q \in Q, (q, p) \in T^*Q$. Moreover, if a symplectic map $\varepsilon : T^*Q \to T^*Q$ with respect to the magnetic symplectic form $\omega^B$ satisfies the condition $\varepsilon(M) \subset M$, then we have that $X^B_H \cdot \varepsilon \in \mathcal{F}$ along $\varepsilon$.

**Proof:** Under the canonical cotangent bundle coordinates, for any $q \in Q, (q, p) \in T^*Q$, we have that

$$X^B_H \cdot \gamma(q) = \left(\sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \sum_{i,j=1}^{n} B_{ij} \frac{\partial H}{\partial p_j} \frac{\partial}{\partial p_i}\right) \gamma(q).$$

and

$$X^B_H \cdot \lambda(q, p) = \left(\sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \sum_{i,j=1}^{n} B_{ij} \frac{\partial H}{\partial p_j} \frac{\partial}{\partial p_i}\right) \gamma \cdot \pi_Q(q, p).$$

Then,

$$T\pi_Q(X^B_H \cdot \gamma(q)) = T\pi_Q(X^B_H \cdot \lambda(q, p)) = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} \gamma(q) \in T_q Q.$$ 

Since $\text{Im}(\gamma) \subset M$, and $\gamma(q) \in M(q, p) = \mathcal{F}(D_q)$, from the Lagrangian $L$ is $D$-regular, and $\mathcal{F}$ is a diffeomorphism, then there exists a point $(q, v_q) \in D_q$, such that $\mathcal{F}(q, v_q) = \gamma(q)$. Thus,

$$T\pi_Q(X^B_H \cdot \gamma(q)) = T\pi_Q(X^B_H \cdot \lambda(q, p)) = \mathcal{F}(q, v_q) \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} \in D_q,$$

it follows that $X^B_H \cdot \gamma \in \mathcal{F}$ along $\gamma$, and $X^B_H \cdot \lambda \in \mathcal{F}$ along $\lambda$. Moreover, for the symplectic map $\varepsilon : T^*Q \to T^*Q$ with respect to the magnetic symplectic form $\omega^B$, we have that

$$X^B_H \cdot \varepsilon(q, p) = \left(\sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \sum_{i,j=1}^{n} B_{ij} \frac{\partial H}{\partial p_j} \frac{\partial}{\partial p_i}\right) \varepsilon(q, p).$$

If $\varepsilon$ satisfies the condition $\varepsilon(M) \subset M$, then for any $(q, p) \in M(q, p)$, we have that $\varepsilon(q, p) \in M(q, p)$, and there exists a point $(q, v_q) \in D_q$, such that $\mathcal{F}(q, v_q) = \varepsilon(q, p)$. Thus,

$$T\pi_Q(X^B_H \cdot \varepsilon(q, p)) = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} \varepsilon(q, p) = \mathcal{F}(q, v_q) \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} \in D_q.$$
it follows that \( X^B_H \cdot \varepsilon \in \mathcal{F} \) along \( \varepsilon \). ■

We note that for a nonholonomic CMH system, under the restriction given by nonholonomic constraint, in general, the dynamical vector field of a nonholonomic CMH system may not be Hamiltonian. On the other hand, since the distributional CMH system is determined by a non-degenerate distributional two-form induced from the magnetic symplectic form, but, the non-degenerate distributional two-form is not a "true two-form" on a manifold, and hence the leading distributional CMH system can not be Hamiltonian. Thus, we can not describe the Hamilton-Jacobi equations for the distributional CMH system from the viewpoint of generating function as in the classical Hamiltonian case, that is, we cannot prove the Hamilton-Jacobi theorem for the distributional CMH system, just like same as the above Theorem 1.1. In the following by using Lemma 3.4, Lemma 6.3, and the non-degenerate distributional two-form \( \omega^B_K \) and the dynamical vector field \( X^B_{(K,\omega^B_K, H_K, F^B_K, u^B_K)} \) given in \( \S 5 \) for the distributional CMH system, we can derive precisely the geometric constraint conditions of the non-degenerate distributional two-form \( \omega^B_K \) for the dynamical vector field \( X^B_{(K,\omega^B_K, H_K, F^B_K, u^B_K)}, \) that is, the two types of Hamilton-Jacobi equation for the distributional CMH system \((K,\omega^B_K, H_K, F^B_K, u^B_K)\). At first, we prove the following Type I of Hamilton-Jacobi theorem for the distributional CMH system.

**Theorem 6.4 (Type I of Hamilton-Jacobi Theorem for the Distributional CMH System)** For the nonholonomic CMH system \((T^*Q, \omega^B, \mathcal{D}, H, F, u)\) with an associated distributional CMH system \((K,\omega^B_K, H_K, F^B_K, u^B_K),\) assume that \( \gamma : Q \rightarrow T^*Q \) is an one-form on \( Q \), and \( \tilde{X}^\gamma = T\pi_Q \cdot \tilde{X} \cdot \gamma \), where \( \tilde{X} = X_{(K,\omega^B_K, H_K, F^B_K, u^B_K)} = X^B_K + F^B_K + u^B_K \) is the dynamical vector field of the distributional CMH system. Moreover, assume that \( \text{Im}(\gamma) \subset M = \mathcal{F}L(\mathcal{D}) \), and \( \text{Im}(T\gamma) \subset K \). If the one-form \( \gamma : Q \rightarrow T^*Q \) satisfies the condition, \( d\gamma = -B \) on \( \mathcal{D} \) with respect to \( T\pi_Q : TT^*Q \rightarrow TQ \), then \( \gamma \) is a solution of the equation \( T\gamma \cdot \tilde{X}^\gamma = X^B_K \cdot \gamma \). Here \( X^B_K \) is the nonholonomic dynamical vector field of the distributional CMH system \((K,\omega^B_K, H_K, F^B_K, u^B_K)\). The equation \( T\gamma \cdot \tilde{X}^\gamma = X^B_K \cdot \gamma \) is called the Type I of Hamilton-Jacobi equation for the distributional CMH system \((K,\omega^B_K, H_K, F^B_K, u^B_K)\). Here the maps involved in the theorem are shown in the following Diagram-5.

![Diagram-5](image)

**Proof:** From Definition 5.1 we have that \( \tilde{X} = X^B_{(K,\omega^B_K, H_K, F^B_K, u^B_K)} = X^B_K + F^B_K + u^B_K \), and \( F^B_K = \tau_K \cdot \text{vlift}(F_{\mathcal{M}})X^B_H \), and \( u^B_K = \tau_K \cdot \text{vlift}(u_{\mathcal{M}})X^B_H \), note that \( T\pi_Q \cdot \text{vlift}(F_{\mathcal{M}})X^B_H = T\pi_Q \cdot \text{vlift}(u_{\mathcal{M}})X^B_H = 0 \), then we have that \( T\pi_Q \cdot F^B_K = T\pi_Q \cdot u^B_K = 0 \), and hence \( T\pi_Q \cdot \tilde{X} \cdot \gamma = T\pi_Q \cdot X^B_K \cdot \gamma \). On the other hand, we note that \( \text{Im}(\gamma) \subset \mathcal{M} \), and \( \text{Im}(T\gamma) \subset K \), in this case, \( \omega^B_K \cdot \tau_K = \tau_K \cdot \omega^B_M = \tau_K \cdot i^M_{\mathcal{M}} \cdot \omega^B \), along \( \text{Im}(T\gamma) \). Moreover, from the distributional magnetic Hamiltonian equation (5.2), we have that \( X^B_K = \tau_K \cdot X^B_H \), and \( \tau_K \cdot X^B_H \cdot \gamma = X^B_K \cdot \gamma \). Thus, using the non-degenerate distributional two-form \( \omega^B_K \), from Lemma 3.4(ii) and Lemma 6.3, if we take that \( v = X^B_K \cdot \gamma \in K(\subset \mathcal{F}) \), and for
any $w \in F$, $T\lambda(w) \neq 0$, and $\tau_{K} \cdot w \neq 0$, then we have that

\[
\omega_{K}^{B}(T\gamma \cdot X^\gamma, \tau_{K} \cdot w) - \omega_{K}^{B}(\tau_{K} \cdot T\gamma \cdot T\tau_{Q} \cdot \tilde{X} \cdot \gamma, \tau_{K} \cdot w) = \omega_{K}^{B}((\tau_{K} \cdot (T\gamma \cdot T\tau_{Q} \cdot X_{K}^{B}) \cdot \gamma, w) = \tau_{K} \cdot i_{M}^{*} \cdot \omega_{K}^{B}(T(\gamma \cdot T\tau_{Q}) \cdot X_{K}^{B} \cdot \gamma, w)
\]

where we have used that $\tau_{K} \cdot T\gamma = T\gamma$, and $\tau_{K} \cdot X_{K}^{B} \cdot \gamma = X_{K}^{B} \cdot \gamma$, since $\text{Im}(T\gamma) \subset K$. Note that $X_{K}^{B} \cdot \gamma, w \in F$, and $T\tau_{Q}(X_{K}^{B} \cdot \gamma), T\tau_{Q}(w) \in D$. If the one-form $\gamma : Q \to T^*Q$ satisfies the condition, $d\gamma = -B$ on $D$ with respect to $T\tau_{Q} : TT^*Q \to TQ$, then

\[
\tau_{K} \cdot i_{M}^{*} \cdot (d\gamma + B)(T\tau_{Q}(X_{K}^{B} \cdot \gamma), T\tau_{Q}(w)) = 0,
\]

Thus, we have that

\[
\omega_{K}^{B}(T\gamma \cdot X^\gamma, \tau_{K} \cdot w) - \omega_{K}^{B}(X_{K}^{B} \cdot \gamma, \tau_{K} \cdot w) = -\omega_{K}^{B}(X_{K}^{B} \cdot \gamma, T\gamma \cdot T\tau_{Q}(w)). \quad (6.1)
\]

If $\gamma$ satisfies the equation $T\gamma \cdot \tilde{X}^\gamma = X_{K}^{B} \cdot \gamma$, from Lemma 3.4(i) we know that the right side of (6.1) becomes that

\[
-\omega_{K}^{B}(X_{K}^{B} \cdot \gamma, T\gamma \cdot T\tau_{Q}(w)) = -\omega_{K}^{B}(T\gamma \cdot X^\gamma, T\gamma \cdot T\tau_{Q}(w)) = -\omega_{K}^{B}(T\gamma \cdot \tilde{X}^\gamma, T\gamma \cdot T\tau_{Q}(w)) = -\omega_{K}^{B}(T\gamma \cdot \tilde{X}^\gamma, \tau_{K} \cdot T\tau_{Q}(w)) = -\tau_{K} \cdot i_{M}^{*} \cdot \omega_{K}^{B}(T\gamma \cdot T\tau_{Q}(X_{K}^{B} \cdot \gamma), T\gamma \cdot T\tau_{Q}(w)) = -\tau_{K} \cdot i_{M}^{*} \cdot \lambda^{*} \omega_{K}^{B}(X_{K}^{B} \cdot \gamma, w) = \tau_{K} \cdot i_{M}^{*} \cdot (d\gamma + B)(T\tau_{Q} \cdot X_{K}^{B} \cdot \gamma, T\tau_{Q} \cdot w) = 0.
\]

Because the distributional two-form $\omega_{K}^{B}$ is non-degenerate, the left side of (6.1) equals zero, only when $\gamma$ satisfies the equation $T\gamma \cdot \tilde{X}^\gamma = X_{K}^{B} \cdot \gamma$. Thus, if the one-form $\gamma : Q \to T^*Q$ satisfies the condition that $d\gamma = -B$ on $D$ with respect to $T\tau_{Q} : TT^*Q \to TQ$, then $\gamma$ must be a solution of the Type I of Hamilton-Jacobi equation when $T\gamma \cdot \tilde{X}^\gamma = X_{K}^{B} \cdot \gamma$, for the distributional CMH system $(K, \omega_{K}^{B}, H_{K}, F_{K}^{B}, u_{K}^{B})$.

It is worthy of noting that, when $B = 0$, in this case the magnetic symplectic form $\omega_{K}^{B}$ is just the canonical symplectic form $\omega$ on $T^*Q$, and the nonholonomic CMH system $(T^*Q, \omega_{K}^{B}, H_{K}, F, u)$ becomes the nonholonomic RCH system $(T^*Q, \omega, D, H, F, u)$ with the canonical symplectic form $\omega$, and the distributional CMH system $(K, \omega_{K}^{B}, H_{K}, F_{K}^{B}, u_{K}^{B})$ becomes the distributional RCH system $(K, \omega_{K}, H_{K}, F_{K}, u_{K})$, and the condition that the one-form $\gamma : Q \to T^*Q$ satisfies the condition that $d\gamma = -B$ on $D$ with respect to $T\tau_{Q} : TT^*Q \to TQ$, becomes that $\gamma$ is closed on $D$ with respect to $T\tau_{Q} : TT^*Q \to TQ$. Thus, from above Theorem 6.4, we can obtain Theorem 3.4 given in Wang [28], that is, the Type I of Hamilton-Jacobi theorem for the distributional RCH system. On the other hand, from the proofs of Theorem 3.4 given in Wang [28], we know that, if the one-form $\gamma : Q \to T^*Q$ is not closed on $D$ with respect to $T\tau_{Q} : TT^*Q \to TQ$, then $\gamma$ is not a solution of the Type I of Hamilton-Jacobi equation for the distributional RCH system. In this case, our idea
is that we hope to look for a nonholonomic CMH system, such that \( \gamma \) is a solution of the Type I of Hamilton-Jacobi equation for the associated distributional CMH system. Since \( \gamma : Q \to T^*Q \) is not closed on \( \mathcal{D} \) with respect to \( T\pi_Q : T^*Q \to TQ \), then the \( \gamma \) is not yet closed on \( \mathcal{D} \), that is, \( d\gamma(x,y) \neq 0, \forall x,y \in \mathcal{D} \), and hence \( \gamma \) is not yet closed on \( Q \). However, in this case, we note that \( d \cdot d\gamma = d^2\gamma = 0 \), and hence the \( d\gamma \) is a closed two-form on \( Q \). Thus, we can construct a magnetic symplectic form on \( T^*Q, \omega^B = \omega - \pi^*_Q B = \omega + \pi^*_Q (d\gamma) \), where \( B = -d\gamma \). Moreover, we can also construct a nonholonomic CMH system \((T^*Q, \omega^B, \mathcal{D}, H, F, u)\) with an associated distributional CMH system \((K, \omega^B_K, H_K, F^B_K, u^B_K)\), which satisfies the distributional magnetic Hamiltonian equation (5.2), \( i_{X^B_K} \omega^B_K = dH_K, \) and \( F^B_K = \tau_K \cdot \text{vlift}(F_M)X^B_H, \) and \( u^B_K = \tau_K \cdot \text{vlift}(u_M)X^B_H \). In this case, the one-form \( \gamma : Q \to T^*Q \) satisfies the condition that \( d\gamma = -B \) with respect to \( T\pi_Q : T^*Q \to TQ \), and hence it satisfies also the condition that \( d\gamma = -B \) on \( \mathcal{D} \) with respect to \( T\pi_Q : T^*Q \to TQ \). By using Lemma 3.4, Lemma 6.3, and the non-degenerate distributional two-form \( \omega^B_K \) and the dynamical vector field \( X^B_K \), from Theorem 6.4 we can obtain the following Theorem 6.5.

**Theorem 6.5** For a given nonholonomic RCH system \((T^*Q, \omega, \mathcal{D}, H, F, u)\) with the canonical symplectic form \( \omega \) on \( T^*Q \) and \( \mathcal{D} \)-completely and \( \mathcal{D} \)-regularly nonholonomic constraint \( \mathcal{D} \subset TQ \), and assume that the one-form \( \gamma : Q \to T^*Q \) is not closed on \( \mathcal{D} \) with respect to \( T\pi_Q : T^*Q \to TQ \). Then one can construct a magnetic CMH system on \( T^*Q, \omega^B = \omega + \pi^*_Q (d\gamma) \), where \( B = -d\gamma \), and a nonholonomic CMH system \((T^*Q, \omega^B, \mathcal{D}, H, F, u)\) with an associated distributional CMH system \((K, \omega^B_K, H_K, F^B_K, u^B_K)\). Denote \( \tilde{X}^\gamma = T\pi_Q \cdot \tilde{X} \cdot \gamma, \) where \( \tilde{X} = X_{(K, \omega^B_K, H_K, F^B_K, u^B_K)} = X^B_K + F^B_K + u^B_K \) is the dynamical vector field of the distributional CMH system. Moreover, assume that \( \text{Im}(\gamma) \subset \mathcal{M} = FL(\mathcal{D}), \) and \( \varepsilon(\mathcal{M}) \subset \mathcal{M} \), and \( \text{Im}(T\gamma) \subset K \). Then \( \gamma \) is a solution of the Type I of Hamilton-Jacobi equation \( T\gamma \cdot \tilde{X}^\gamma = X^B_K \cdot \gamma, \) for the distributional CMH system \((K, \omega^B_K, H_K, F^B_K, u^B_K)\).

Next, for any symplectic map \( \varepsilon : T^*Q \to T^*Q \) with respect to the magnetic symplectic form \( \omega^B \), we can prove the following Type II of Hamilton-Jacobi theorem for the distributional CMH system. For convenience, the maps involved in the following theorem and its proof are shown in Diagram-6.

![Diagram-6](image-url)

**Theorem 6.6** *(Type II of Hamilton-Jacobi Theorem for a Distributional CMH System)* For the nonholonomic CMH system \((T^*Q, \omega^B, \mathcal{D}, H, F, u)\) with an associated distributional CMH system \((K, \omega^B_K, H_K, F^B_K, u^B_K)\), assume that \( \gamma : Q \to T^*Q \) is an one-form on \( Q \), and \( \lambda = \gamma \cdot \pi_Q : T^*Q \to T^*Q \), and for any symplectic map \( \varepsilon : T^*Q \to T^*Q \) with respect to the magnetic symplectic form \( \omega^B \), denote by \( \tilde{X}^\varepsilon = T\pi_Q \cdot \tilde{X} \cdot \varepsilon, \) where \( \tilde{X} = X_{(K, \omega^B_K, H_K, F^B_K, u^B_K)} = X^B_K + F^B_K + u^B_K \) is the dynamical vector field of the distributional CMH system. Moreover, assume that \( \text{Im}(\gamma) \subset \mathcal{M} = FL(\mathcal{D}), \) and \( \varepsilon(\mathcal{M}) \subset \mathcal{M} \), and \( \text{Im}(T\gamma) \subset K \). Then \( \varepsilon \) is a solution of the equation \( \tau_K \cdot \varepsilon(T\pi_{H\varepsilon}) = T\lambda \cdot \tilde{X} \cdot \varepsilon, \) if and only if it is a solution of the equation \( T\gamma \cdot \tilde{X}^\varepsilon = X^B_K \cdot \varepsilon. \) Here \( X^B_{H\varepsilon} \) is the magnetic Hamiltonian vector field of the function \( H \cdot \varepsilon : T^*Q \to \mathbb{R}, \) and \( X^B_K \) is the nonholonomic dynamical vector field of the distributional CMH system \((K, \omega^B_K, H_K, F^B_K, u^B_K)\). The equation \( T\gamma \cdot \tilde{X}^\varepsilon = X^B_K \cdot \varepsilon, \) is called the Type II of Hamilton-Jacobi equation for the distributional CMH system \((K, \omega^B_K, H_K, F^B_K, u^B_K)\).

**Proof:** In the same way, from Definition 5.1 we have that \( \tilde{X} = X^B_{(K, \omega^B_K, H_K, F^B_K, u^B_K)} = X^B_K + F^B_K + u^B_K \), and \( F^B_K = \tau_K \cdot \text{vlift}(F_M)X^B_H, \) and \( u^B_K = \tau_K \cdot \text{vlift}(u_M)X^B_H, \) note that \( T\pi_Q \cdot \text{vlift}(F_M)X^B_H = T\pi_Q \cdot \)
vlift(\(\omega_M\))\(X^B_H = 0\), then we have that \(T\pi_Q \cdot F^B = T\pi_Q \cdot u^B = 0\), and hence \(T\pi_Q \cdot \tilde{X} \cdot \epsilon = T\pi_Q \cdot X^B_H \cdot \epsilon\). On the other hand, we note that \(\text{Im}(\gamma) \subset M\), and \(\text{Im}(T\gamma) \subset K\), in this case, \(\omega^B_K \cdot \tau_K = \tau_K \cdot \omega^B_M = \tau_K \cdot i^*_M \cdot \omega^B\), along \(\text{Im}(T\gamma)\). Moreover, from the distributional magnetic Hamiltonian equation (5.2), we have that \(X^B_K = \tau_K \cdot X^B_H\), and \(\tau_K \cdot X^B_H \cdot \epsilon = X^B_K \cdot \epsilon\). Note that \(\epsilon(M) \subset M\), and \(T\pi_Q(X^B_K \cdot \epsilon(q, p)) \in D_q\), \(\forall q \in Q\), \((q, p) \in M(\subset T^*Q)\), and hence \(X^B_K \cdot \epsilon \in F\) along \(\epsilon\), and \(X^B_K \cdot \epsilon = \tau_K \cdot X^B_H \cdot \epsilon \in K(\subset F)\). Thus, using the non-degenerate distributional two-form \(\omega^B_K\), from Lemma 3.4 and Lemma 6.3, if we take that \(v = X^B_K \cdot \epsilon \in K(\subset F)\), and for any \(w \in F\), \(T\lambda(w) \neq 0\), and \(\tau_K \cdot w \neq 0\), then we have that

\[
\omega^B_K(T\gamma \cdot \tilde{X}^\epsilon, \tau_K \cdot w) = \omega^B_K(\tau_K \cdot T\gamma \cdot \tilde{X}^\epsilon, \tau_K \cdot w)
\]

\[
= \tau_K \cdot i^*_M \cdot \omega^B(T\gamma \cdot T\pi_Q \cdot \tilde{X} \cdot \epsilon, w) = \tau_K \cdot i^*_M \cdot \omega^B(T\gamma \cdot \pi_Q \cdot X^B_K \cdot \epsilon, w)
\]

\[
= \tau_K \cdot i^*_M \cdot \omega^B(X^B_K \cdot \epsilon, w - T\gamma \cdot \pi_Q \cdot \epsilon - (d\gamma + B)(T\pi_Q(X^B_K \cdot \epsilon), T\pi_Q(w)))
\]

\[
= \tau_K \cdot i^*_M \cdot \omega^B(X^B_K \cdot \epsilon, w)
\]

we have used that \(\tau_K \cdot T\gamma = T\gamma\), \(\tau_K \cdot T\lambda = T\lambda\), and \(\tau_K \cdot X^B_K \cdot \epsilon = X^B_K \cdot \epsilon\), since \(\text{Im}(T\gamma) \subset K\). From the distributional magnetic Hamiltonian equation (5.2), \(i^\gamma \omega^K_B = dH_K\), we have that \(X^B_K = \tau_K \cdot X^B_H\). Note that \(\epsilon : T^*Q \rightarrow T^*Q\) is symplectic with respect to the magnetic symplectic form \(\omega^B\), and \(X^B_H \cdot \epsilon = T\epsilon \cdot X^B_H\), along \(\epsilon\), and hence \(X^B_K \cdot \epsilon = \tau_K \cdot X^B_H \cdot \epsilon = \tau_K \cdot T\epsilon \cdot X^B_H\), along \(\epsilon\). Note that \(T\lambda \cdot X^B_K \cdot \epsilon = T\gamma \cdot T\pi_Q \cdot X^B_K \cdot \epsilon = T\gamma \cdot T\pi_Q \cdot \tilde{X} \cdot \epsilon = T\lambda \cdot \tilde{X} \cdot \epsilon\). Then we have that

\[
\omega^B_K(T\gamma \cdot \tilde{X}^\epsilon, \tau_K \cdot w) - \omega^B_K(X^B_K \cdot \epsilon, \tau_K \cdot w)
\]

\[
= -\omega^B_K(X^B_K \cdot \epsilon, T\lambda \cdot w) + \omega^B_K(T\lambda \cdot X^B_K \cdot \epsilon, T\lambda \cdot w)
\]

\[
= \omega^B_K(T\lambda \cdot \tilde{X} \cdot \epsilon - \tau_K \cdot T\epsilon \cdot X^B_H \cdot \epsilon, T\lambda \cdot w)
\]

Because the induced distributional two-form \(\omega^B_K\) is non-degenerate, it follows that the equation \(T\gamma \cdot \tilde{X}^\epsilon = X^B_K \cdot \epsilon\), is equivalent to the equation \(\tau_K \cdot T\epsilon \cdot X^B_H \cdot \epsilon = T\lambda \cdot \tilde{X} \cdot \epsilon\). Thus, \(\epsilon\) is a solution of the equation \(\tau_K \cdot T\epsilon \cdot X^B_H \cdot \epsilon = T\lambda \cdot \tilde{X} \cdot \epsilon\), if and only if it is a solution of the Type II of Hamilton-Jacobi equation \(T\gamma \cdot \tilde{X}^\epsilon = X^B_K \cdot \epsilon\). ■

**Remark 6.7** It is worthy of noting that, the Type I of Hamilton-Jacobi equation \(T\gamma \cdot \tilde{X}^\gamma = X^B_K \cdot \gamma\), is the equation of the differential one-form \(\gamma\); and the Type II of Hamilton-Jacobi equation \(T\gamma \cdot \tilde{X}^\epsilon = X^B_K \cdot \epsilon\), is the equation of the symplectic diffeomorphism map \(\epsilon\). If the nonholonomic CMH system considered has not any constrains, in this case, the distributional CMH system is just the CMH system itself. From the above Type I and Type II of Hamilton-Jacobi theorems, that is, Theorem 6.4- 6.6, we can get the Theorem 3.5- 3.7. It shows that Theorem 6.4- 6.6 can be regarded as an extension of two types of Hamilton-Jacobi theorem for the CMH system to the system with nonholonomic context. On the other hand, if both the external force and control of a nonholonomic CMH system \((T^*Q, \omega^B, D, H, F, W)\) are zero, that is, \(F = 0\) and \(W = 0\), in this case the nonholonomic CMH system is just a nonholonomic magnetic Hamiltonian system \((T^*Q, \omega^B, D, H)\), and from the proofs of the above Theorem 6.4- 6.6, we can obtain two types of Hamilton-Jacobi equation for the associated distributional magnetic Hamiltonian system, that is, Theorem 4.4- 4.6 given in Wang [27]. Thus, Theorem 6.4- 6.6 can be regarded as an extension of two types of Hamilton-Jacobi equation for a nonholonomic magnetic Hamiltonian system to
that for the system with external force and control. In particular, in this case, if \( B = 0 \), then the magnetic symplectic form \( \omega^B \) is just the canonical symplectic form \( \omega \) on \( T^*Q \), and the distributional magnetic Hamiltonian system is just the distributional Hamiltonian system itself. From the above Type I and Type II of Hamilton-Jacobi theorems, that is, Theorem 6.4 and Theorem 6.6, we can get the Theorem 3.5 and Theorem 3.6 given in León and Wang in \([10]\). It shows that Theorem 6.4 and Theorem 6.6 can be regarded as an extension of two types of Hamilton-Jacobi theorem for the distribational Hamiltonian system to that for the system with magnetic, external force and control.

7 Hamilton-Jacobi Equations for a Nonholonomic Reduced Distributinal CMH System

It is well-known that the reduction of nonholonomically constrained mechanical systems is a very important subject in geometric mechanics, and it is regarded as a useful tool for simplifying and studying concrete nonholonomic systems, see Bates and Śniatycki \([3]\), Cendra et al. \([5]\), Cushman et al. \([6]\), Koiller \([8]\), and León and Wang \([10]\) and so on, for more details and development.

In this section, we shall consider the nonholonomic reduction and Hamilton-Jacobi theory of a nonholonomic CMH system with symmetry. We first give the definition of a nonholonomic CMH system with symmetry. We first consider the nonholonomic reduction of a nonholonomic CMH system given in section 6 under nonholonomic reduction. Reduced distributional CMH system, which are an extension of the above two types of Hamilton-Jacobi equation for the nonholonomic reduced distributional two-form for the nonholonomic reducible dynamical vector field, that is, the two types of Hamilton-Jacobi equation for the nonholonomic reduced distributional CMH system, which are an extension of the above two types of Hamilton-Jacobi equation for the distributional CMH system given in section 6 under nonholonomic reduction.

Assume that the Lie group \( G \) acts smoothly on the manifold \( Q \) by the left, and we also consider the natural lifted actions on \( TQ \) and \( T^*Q \), and assume that the cotangent lifted left action \( \Phi^T : G \times T^*Q \to T^*Q \) is free, proper and symplectic with respect to the magnetic symplectic form \( \omega^B \) on \( T^*Q \). The orbit space \( T^*Q/G \) is a smooth manifold and the canonical projection \( \pi_i : T^*Q \to T^*Q/G \) is a surjective submersion. For the cotangent lifted left action \( \Phi^T : G \times T^*Q \to T^*Q \), assume that \( H : T^*Q \to \mathbb{R} \) is a \( G \)-invariant Hamiltonian, and the fiber-preserving map \( F : T^*Q \to T^*Q \) and the control subset \( W \) of \( T^*Q \) are both \( G \)-invariant, and the \( D \)-completely and \( D \)-regularly nonholonomic constraint \( D \subset TQ \) is a \( G \)-invariant distribution for the tangent lifted left action \( \Phi^T : G \times TQ \to TQ \), that is, the tangent of group action maps \( D_q \) to \( D_{gq} \) for any \( q \in Q \). A nonholonomic CMH system with symmetry is 7-tuple \( (T^*Q, G, \omega^B, D, H, F, W) \), which is an CMH system with symmetry and \( G \)-invariant nonholonomic constraint \( D \).

In the following we first consider the nonholonomic reduction of a nonholonomic CMH system with symmetry \( (T^*Q, G, \omega^B, D, H, F, W) \). Note that the Legendre transformation \( FL : TQ \to T^*Q \) is a fiber-preserving map, and \( D \subset TQ \) is \( G \)-invariant for the tangent lifted left action \( \Phi^T : G \times TQ \to TQ \), then the constraint submanifold \( M = FL(D) \subset T^*Q \) is \( G \)-invariant for the cotangent lifted left action \( \Phi^T : G \times T^*Q \to T^*Q \). For the nonholonomic CMH system with symmetry \( (T^*Q, G, \omega^B, D, H, F, W) \), in the same way, we define the distribution \( \mathcal{F} \), which is the pre-image of the nonholonomic constraints \( D \) for the map \( T\pi_Q : TT^*Q \to TQ \), that is, \( \mathcal{F} = (T\pi_Q)^{-1}(D) \), and the distribution \( \mathcal{K} = \mathcal{F} \cap TM \). Moreover, we can also define the distributional two-form \( \omega^B_{\mathcal{K}} \), which is induced from the magnetic symplectic form \( \omega^B \) on \( T^*Q \), that is, \( \omega^B_{\mathcal{K}} = \pi_\mathcal{K} \cdot \omega^B_{\mathcal{M}} \), and \( \omega^B_{\mathcal{M}} = \mathbf{i}_{TM} \omega^B \).

If the admissibility condition \( \text{dim} \mathcal{M} = \text{rank} \mathcal{F} \) and the compatibility condition \( TM \cap \mathcal{F}^\perp = \{0\} \).
hold, then \( \omega_B^\mathcal{K} \) is non-degenerate as a bilinear form on each fibre of \( \mathcal{K} \), there exists a vector field \( X^K_B \) on \( \mathcal{M} \) which takes values in the constraint distribution \( \mathcal{K} \), such that for the function \( H^\mathcal{K} \), the following distributional magnetic Hamiltonian equation holds, that is,

\[
i_{X^K_B} \omega^\mathcal{K}_B = dh^\mathcal{K},
\]

where the function \( H^\mathcal{K} \) satisfies \( dh^\mathcal{K} = \tau^\mathcal{K} \cdot dh^\mathcal{M} \), and \( H^\mathcal{M} = \tau^\mathcal{M} \cdot H \) is the restriction of \( H \) to \( \mathcal{M} \), and from the equation (7.1), we have that \( X^K_B = \tau^\mathcal{K} \cdot X^K_H \).

In the following we define that the quotient space \( \bar{\mathcal{M}} = \mathcal{M}/G \) of the \( G \)-orbit in \( \mathcal{M} \) is a smooth manifold with projection \( \pi^\mathcal{K} : \mathcal{M} \rightarrow \mathcal{M} (\subset T^*Q/G) \), which is a surjective submersion. The reduced magnetic symplectic form \( \omega_M^B = \pi^\mathcal{K} \cdot \omega_M^B \) on \( \mathcal{M} \) is induced from the magnetic symplectic form \( \omega_M^B = i_M^\mathcal{K} \omega^B \) on \( \mathcal{M} \). Since \( G \) is the symmetry group of the system \((T^*Q,G,\omega^B,D,H,F,W)\), all intrinsically defined vector fields and distributions are pushed down to \( \mathcal{M} \). In particular, the vector field \( X_M^B \) on \( \mathcal{M} \) is pushed down to a vector field \( \bar{X}_M^B = T\pi^\mathcal{K} \cdot X_M^B \), and the distribution \( \mathcal{K} \) is pushed down to a distribution \( T\pi^\mathcal{K} \cdot \mathcal{K} \) on \( \mathcal{M} \), and the Hamiltonian \( \bar{H} \) is pushed down to \( h^\mathcal{M} \), such that \( h^\mathcal{M} \cdot \pi^\mathcal{K} = \tau^\mathcal{M} \cdot H \). However, \( \omega^K_B \) need not to be pushed down to a distributional two-form defined on \( T\pi^\mathcal{K} \cdot \mathcal{K} \), despite of the fact that \( \omega^K_B \) is \( G \)-invariant. This is because there may be infinitesimal symmetry \( \eta^\mathcal{K} \) that lies in \( \mathcal{M} \), such that \( i_{\eta^\mathcal{K}} \omega^\mathcal{K}_B \neq 0 \). From Bates and Śniatycki [3], we know that in order to eliminate this difficulty, \( \omega^K_B \) is restricted to a sub-distribution \( \mathcal{U} \) of \( \mathcal{K} \) defined by

\[
\mathcal{U} = \{ u \in \mathcal{K} \mid \omega^K_B(u,v) = 0, \quad \forall v \in \mathcal{V} \cap \mathcal{K} \},
\]

where \( \mathcal{V} \) is the distribution on \( \mathcal{M} \) tangent to the orbits of \( G \) in \( \mathcal{M} \) and it is spanned by the infinitesimal symmetries. Clearly, \( \mathcal{U} \) and \( \mathcal{V} \) are both \( G \)-invariant, project down to \( \mathcal{M} \) and \( T\pi^\mathcal{K} \cdot \mathcal{V} = 0 \), and define the distribution \( \bar{\mathcal{K}} \) by \( \bar{\mathcal{K}} = T\pi^\mathcal{K} \cdot \mathcal{U} \). Moreover, we take that \( \omega_U^B = \tau^\mathcal{U} \cdot \omega_M^B \) is the restriction of the induced magnetic symplectic form \( \omega_M^B \) on \( T^*\mathcal{M} \) fibrewise to the distribution \( \mathcal{U} \), where \( \tau^\mathcal{U} \) is the restriction map to distribution \( \mathcal{U} \), and the \( \omega_U^B \) is pushed down to a distributional two-form \( \omega^K_B \) on \( \bar{\mathcal{K}} \), such that \( \pi^\mathcal{K} \omega^K_B = \omega_U^B \). We know that distributional two-form \( \omega^K_B \) is not a "true two-form" on a manifold, which is called the nonholonomic reduced distributional two-form to avoid any confusion.

From the above construction we know that, if the admissibility condition \( \text{dim} \mathcal{M} = \text{rank} \mathcal{F} \) and the compatibility condition \( TM \cap \mathcal{F}^\perp = \{0\} \) hold, where \( \mathcal{F}^\perp \) denotes the symplectic orthogonal of \( \mathcal{F} \) with respect to the reduced magnetic symplectic form \( \omega_M^B \), then the nonholonomic reduced distributional two-form \( \omega^K_B \) is non-degenerate as a bilinear form on each fibre of \( \bar{\mathcal{K}} \), and hence there exists a vector field \( X^K_B \) on \( \bar{\mathcal{M}} \) which takes values in the constraint distribution \( \bar{\mathcal{K}} \), such that the reduced distributional magnetic Hamiltonian equation holds, that is,

\[
i_{X^K_B} \omega^K_B = dh^K,
\]

where \( dh^K \) is the restriction of \( dh^\mathcal{M} \) to \( \bar{\mathcal{K}} \) and the function \( h^K : \bar{\mathcal{M}} (\subset T^*Q/G) \rightarrow \mathbb{R} \) satisfies \( dh^K = \tau^K \cdot dh^\mathcal{M} \), and \( h^\mathcal{M} \cdot \pi^\mathcal{K} = H^\mathcal{M} \) and \( H^\mathcal{M} \) is the restriction of the Hamiltonian function \( H \) to \( \mathcal{M} \), and the function \( h^\mathcal{M} : \bar{\mathcal{M}} (\subset T^*Q/G) \rightarrow \mathbb{R} \). In addition, from the distributional magnetic Hamiltonian equation (7.1), \( i_{X^K_B} \omega^K_B = dh^K \), we have that \( X^K_B = \tau^K \cdot X^K_H \), and from the reduced distributional magnetic Hamiltonian equation (7.2), \( i_{X^K_B} \omega^K_B = dh^K \), we have that \( X^K_B = \tau^K \cdot h^K \), where \( X^K_B \) is the magnetic Hamiltonian vector field of the function \( h^K \) with respect to the reduced magnetic symplectic form \( \omega_M^B \), and the vector fields \( X^K_B \) and \( X^K_B \) are \( \pi^\mathcal{K} \)-related, that is, \( X^K_B \cdot \pi^\mathcal{K} = T\pi^\mathcal{K} \cdot X^K_B \).
Moreover, if considering the external force $F$ and control subset $W$, and we define the vector fields $F^K_B = \tau_K \cdot \text{vlift}(F_M) X^K_B$, and for a control law $u \in W$, $u^K_B = \tau_K \cdot \text{vlift}(u_M) X^K_B$, where $F_M = \tau_M \cdot F$ and $u_M = \tau_M \cdot u$ are the restrictions of $F$ and $u$ to $M$, that is, $F^K_B$ and $u^K_B$ are the restrictions of the changes of magnetic Hamiltonian vector field $X^K_B$ under the actions of $F_M$ and $u_M$ to $K$, then the 5-tuple $(K, \omega^K_B, H_K, F^K_B, u^K_B)$ is a distributional CMH system corresponding to the nonholonomic CMH system with symmetry $(T^*Q, G, \omega^B, \mathcal{D}, H, F, u)$, and the dynamical vector field of the distributional CMH system can be expressed by

$$\ddot{X} = X^K_B = X^K_B + F^K_B + u^K_B,$$

which is the synthetic of the nonholonomic dynamical vector field $X^K_B$ and the vector fields $F^K_B$ and $u^K_B$. Assume that the vector fields $F^K_B$ and $u^K_B$ on $M$ are pushed down to the vector fields $F^K_B = T\pi/G \cdot F^K_B$ and $u^K_B = T\pi/G \cdot u^K_B$ on $\hat{M}$. Then we define that $F^K_B = T\tau_{\hat{K}} \cdot f^K_B$ and $u^K_B = T\tau_{\hat{K}} \cdot u^K_B$, that is, $f^K_B$ and $u^K_B$ are the restrictions of $f^K_M$ and $u^K_M$ to $\hat{K}$, where $\tau_{\hat{K}}$ is the restriction map to distribution $\hat{K}$, and $T\tau_{\hat{K}}$ is the tangent map of $\tau_{\hat{K}}$. Then the 5-tuple $(\hat{K}, \omega^K_B, H_K, f^K_B, u^K_B)$ is a nonholonomic reduced distributional CMH system of the nonholonomic reducible CMH system with symmetry $(T^*Q, G, \omega^B, \mathcal{D}, H, F, W)$, as well as with a control law $u \in W$. Thus, the geometrical formulation of a nonholonomic reduced distributional CMH system may be summarized as follows.

**Definition 7.1 (Nonholonomic Reduced Distributional CMH System)** Assume that the 7-tuple $(T^*Q, G, \omega^B, \mathcal{D}, H, F, W)$ is a nonholonomic reducible CMH system with symmetry, where $\omega^B$ is the magnetic symplectic form on $T^*Q$, and $\mathcal{D} \subset T^*Q$ is a $\mathcal{D}$-completely and $\mathcal{D}$-regularly nonholonomic constraint of the system, and $\mathcal{D}, H, F$ and $W$ are all $G$-invariant. If there exists a nonholonomic reduced distribution $\hat{K}$, an associated non-degenerate and nonholonomic reduced distributional two-form $\omega^K_B$ and a vector field $X^K_B$ on the reduced constraint submanifold $\hat{M} = M/G$, where $M = FL(\mathcal{D}) \subset T^*Q$, such that the nonholonomic reduced distributional magnetic Hamiltonian equation

$$i_X^K_B \omega^K_B = dh^K_{\hat{K}},$$

holds, where $dh^K_{\hat{K}}$ is the restriction of $dh_M$ to $\hat{K}$ and the function $h^K_{\hat{K}}$ satisfies

$$dh^K_{\hat{K}} = \tau_{\hat{K}} \cdot dh_M \text{ and } h_{\hat{M}} \cdot \pi/G = H_M,$$

and the vector fields $f^K_B = T\tau_{\hat{K}} \cdot f^K_M$ and $u^K_B = T\tau_{\hat{K}} \cdot u^K_M$ as defined above. Then the 5-tuple $(\hat{K}, \omega^K_B, h^K_{\hat{K}}, f^K_B, u^K_B)$ is called a nonholonomic reduced distributional CMH system of the nonholonomic reducible CMH system $(T^*Q, G, \omega^B, \mathcal{D}, H, F, W)$ with a control law $u \in W$, and $X^K_B$ is called a nonholonomic reduced dynamical vector field. Denote by

$$\ddot{X} = X^K_B = X^K_B + f^K_B + u^K_B$$

is the dynamical vector field of the nonholonomic reduced distributional CMH system $(\hat{K}, \omega^K_B, h^K_{\hat{K}}, f^K_B, u^K_B)$, which is the synthetic of the nonholonomic reduced dynamical vector field $X^K_B$ and the vector fields $F^K_B$ and $u^K_B$. Under the above circumstances, we refer to $(T^*Q, G, \omega^B, \mathcal{D}, H, F, u)$ as a nonholonomic reducible CMH system with the associated distributional CMH system $(\hat{K}, \omega^K_B, H_K, F^K_B, u^K_B)$ and the nonholonomic reduced distributional CMH system $(\hat{K}, \omega^K_B, h^K_{\hat{K}}, f^K_B, u^K_B)$. The dynamical vector fields $\ddot{X} = X^K_B$ and $\ddot{\tilde{X}} = X^K_B$ are $\pi/G$-related, that is, $\ddot{X} \cdot \pi/G = T\pi/G \cdot \ddot{\tilde{X}}$.

For a given nonholonomic reducible CMH system $(T^*Q, G, \omega^B, \mathcal{D}, H, F, u)$ with the associated distributional CMH system $(\hat{K}, \omega^K_B, H_K, F^K_B, u^K_B)$ and the nonholonomic reduced distributional CMH system $(\hat{K}, \omega^K_B, h^K_{\hat{K}}, f^K_B, u^K_B)$, the magnetic vector field $\dot{X}^0 = X^K_B - X_H$, which is determined by the magnetic equation $i_X^{\omega^B} = i_X^{\omega^B}(\pi^*_Q B)$ on $T^*Q$. Note that vector fields $X^0, X^K_B, X_H$ and distribution $K$ are pushed down to $\hat{M}$, that is, $X^0_M = T\pi/G \cdot X^0_H, X^K_B = T\tau_{\hat{K}} \cdot X^K_B, X_H = T\tau_{\hat{K}} \cdot X_H$ and $X^K_B = T\tau_{\hat{K}} \cdot X_M$. where the vector field $X^0_M, X^K_B$ and $X_M$ are the restrictions of $X^0, X^K_B$ and $X_H$ on $M$ and the distribution $K$ is pushed down to a distribution $T\tau_{\hat{K}} \cdot K$ on $\hat{M}$, and define the distribution
Denote by $X^0_{\bar{K}} = \tau_{\bar{K}}(X^0_{\bar{M}}) = \tau_{\bar{K}}(X^B_{\bar{M}}) = X^B_{\bar{K}} - X_{\bar{K}}$, from the expression (7.4) of the dynamical vector field of the nonholonomic reduced distributional CMH system $(\bar{K}, \omega^B_{\bar{K}}, h_{\bar{K}}, f^B_{\bar{K}}, u^B_{\bar{K}})$, we have that

$$X = X^B_{\bar{K}} + F^B_{\bar{K}} + u^B_{\bar{K}} = X_{\bar{K}} + X^0_{\bar{K}} + F^B_{\bar{K}} + u^B_{\bar{K}}.$$  \hspace{1cm} (7.5)

If the vector fields $F^B_{\bar{K}}$ and $u^B_{\bar{K}}$ satisfy the following condition

$$X^0_{\bar{K}} + F^B_{\bar{K}} + u^B_{\bar{K}} = 0,$$  \hspace{1cm} (7.6)

then from (7.5) we have that $X^B_{\bar{K}, \omega^B_{\bar{K}}, h_{\bar{K}}, f^B_{\bar{K}}, u^B_{\bar{K}}} = X_{\bar{K}}$, that is, in this case the dynamical vector field of the nonholonomic reduced distributional CMH system is just the dynamical vector field of the nonholonomic reduced canonical distributional Hamiltonian system without the actions of magnetic, external force and control. Thus, the condition (7.6) is called the magnetic vanishing condition for the nonholonomic reduced distributional CMH system $(\bar{K}, \omega^B_{\bar{K}}, h_{\bar{K}}, f^B_{\bar{K}}, u^B_{\bar{K}})$.

Since the non-degenerate and nonholonomic reduced distributional two-form $\omega^B_{\bar{K}}$ is not a "true two-form" on a manifold, and it is not symplectic, and hence the nonholonomic reduced distributional CMH system $(\bar{K}, \omega^B_{\bar{K}}, h_{\bar{K}}, f^B_{\bar{K}}, u^B_{\bar{K}})$ is not a Hamiltonian system, and has not yet generating function, and hence we can not describe the Hamilton-Jacobi equation for the nonholonomic reduced distributional CMH system just like as in Theorem 1.1. But, for a given nonholonomic reducible CMH system $(T^*Q, G, \omega^B, \mathcal{D}, H, F, u)$ with the associated distributional CMH system $(\bar{K}, \omega^B_{\bar{K}}, H_{\bar{K}}, F^B_{\bar{K}}, u^B_{\bar{K}})$ and the nonholonomic reduced distributional CMH system $(\bar{K}, \omega^B_{\bar{K}}, h_{\bar{K}}, f^B_{\bar{K}}, u^B_{\bar{K}})$, by using Lemma 3.4 and Lemma 6.3, we can derive precisely the geometric constraint conditions of the nonholonomic reduced distributional two-form $\omega^B_{\bar{K}}$ for the nonholonomic reducible dynamical vector field $\hat{X} = X^B_{\bar{K} \omega^B_{\bar{K}}, h_{\bar{K}}, f^B_{\bar{K}}, u^B_{\bar{K}}}$, that is, the two types of Hamilton-Jacobi equation for the nonholonomic reduced distributional CMH system $(\bar{K}, \omega^B_{\bar{K}}, h_{\bar{K}}, f^B_{\bar{K}}, u^B_{\bar{K}})$. At first, using the fact that the one-form $\gamma : Q \to T^*Q$ satisfies the condition, $d\gamma = -B$ on $\mathcal{D}$ with respect to $T\pi_Q : TT^*Q \to TQ$, $\text{Im}(\gamma) \subset \mathcal{M}$, and it is $G$-invariant, as well as $\text{Im}(T\gamma) \subset \bar{K}$, we can prove the Type I of Hamilton-Jacobi theorem for the nonholonomic reduced distributional CMH system. For convenience, the maps involved in the following theorem and its proof are shown in Diagram-7.

**Diagram-7**

**Theorem 7.2 (Type I of Hamilton-Jacobi Theorem for a Nonholonomic Reduced Distributional CMH System)** For a given nonholonomic reducible CMH system $(T^*Q, G, \omega^B, \mathcal{D}, H, F, u)$ with the associated distributional CMH system $(\bar{K}, \omega^B_{\bar{K}}, H_{\bar{K}}, F^B_{\bar{K}}, u^B_{\bar{K}})$ and the nonholonomic reduced distributional CMH system $(\bar{K}, \omega^B_{\bar{K}}, h_{\bar{K}}, f^B_{\bar{K}}, u^B_{\bar{K}})$, assume that $\gamma : Q \to T^*Q$ is an one-form on $Q$, and $\hat{X} = T\pi_Q \cdot \bar{X} \cdot \gamma$, where $\bar{X} = X^B_{\bar{K}, \omega^B_{\bar{K}}, h_{\bar{K}}, f^B_{\bar{K}}, u^B_{\bar{K}}} = X^B_{\bar{K}} + F^B_{\bar{K}} + u^B_{\bar{K}}$ is the dynamical vector field of the distributional CMH system corresponding to the nonholonomic reducible CMH system with symmetry $(T^*Q, G, \omega^B, \mathcal{D}, H, F, u)$. Moreover, assume that $\text{Im}(\gamma) \subset \mathcal{M}$, and it is $G$-invariant, $\text{Im}(T\gamma) \subset \bar{K}$, and $\bar{\gamma} = \pi_{\mathcal{D}}(\gamma) : Q \to T^*Q/G$. If the one-form $\gamma : Q \to T^*Q$ satisfies the condition, $d\gamma = -B$ on $\mathcal{D}$ with respect to $T\pi_Q : TT^*Q \to TQ$, then $\bar{\gamma}$ is a solution of the equation $T\bar{\gamma} : \bar{X} = X^B_{\bar{K}} \cdot \bar{\gamma}$. Here $X^B_{\bar{K}}$ is the nonholonomic reduced dynamical vector field. The equation $T\bar{\gamma} : \bar{X} = X^B_{\bar{K}} \cdot \bar{\gamma}$, is called the Type I of Hamilton-Jacobi equation for the nonholonomic reduced distributional CMH system $(\bar{K}, \omega^B_{\bar{K}}, h_{\bar{K}}, f^B_{\bar{K}}, u^B_{\bar{K}})$.
Proof: At first, for the dynamical vector field of the distributional CMH system $(\mathcal{K}, \omega_B^B, H_K, F_B, u_B^B)$, \(\hat{X} = X_{k(\omega_B^B,k,H_K,F_B^B,u_B^B)} = X_K^B + F_B^B + u_B^B\), and \(F_B = \tau_K \cdot \text{vlt}(F_M)X_B^H\), and \(u_B^B = \tau_K \cdot \text{vlt}(u_M)X_B^H\), note that \(T\pi_Q \cdot \text{vlt}(F_M)X_B^H = T\pi_Q \cdot \text{vlt}(u_M)X_B^H = 0\), then we have that \(T\pi_Q \cdot F_B^B = T\pi_Q \cdot u_B^B = 0\), and hence \(T\pi_Q \cdot \hat{X} \cdot \gamma = T\pi_Q \cdot X_K^B \cdot \gamma\). Moreover, from Theorem 6.4, we know that \(\gamma\) is a solution of the Type I of Hamilton-Jacobi equation \(T\gamma \cdot \hat{X}\gamma = X_K^B \cdot \gamma\). Next, we note that \(\text{Im}(\gamma) \subset K\), and it is G-invariant, \(\text{Im}(T\gamma) \subset K\), and hence \(\text{Im}(T\gamma) \subset \hat{K}\), in this case, \(\pi_{\hat{K}}^G \cdot \omega_B^B \cdot \tau_K = \pi_U \cdot \omega_B^B = \pi_U \cdot i_M^* \cdot \omega_B^B\), along \(\text{Im}(T\gamma)\). From the distributional Hamiltonian equation (7.1), we have that \(X_K^B = \tau_K \cdot X_B^H\), and \(\tau_K \cdot X_B^H \cdot \gamma = X_K^B \cdot \gamma \in K\). Because the vector fields \(X_K^B\) and \(X_B^H\) are \(\pi/G\)-related, \(T\pi/G(X_K^B) = X_K^B \cdot \pi/G\), and hence \(\tau_K \cdot T\pi/G(X_K^B) = \tau_K \cdot (T\pi/G)(X_K^B)\) \(\cdot \gamma = \tau_K \cdot (X_K^B \cdot \pi/G)(\gamma) = \tau_K \cdot X_K^B \cdot \pi/G(\gamma) = X_K^B \cdot \gamma\). Thus, using the non-degenerate nonholonomic reduced distributional two-form \(\omega_B^B\), from Lemma 3.4(ii) and Lemma 6.3, if we take that \(v = X_K^B \cdot \gamma \in \mathcal{K}(\subset F)\), and for any \(w \in F\), \(T\lambda(w) \neq 0\), and \(\tau_K \cdot T\pi/G \cdot w \neq 0\), then we have that

\[
\omega_K^B(T\gamma \cdot \hat{X}\gamma, \tau_K \cdot T\pi/G \cdot w) = \omega_K^B(\tau_K \cdot T(\pi/G) \cdot \hat{X}\gamma, \tau_K \cdot T\pi/G \cdot w) = \pi_{\hat{K}}^G \cdot \omega_K^B \cdot \tau_K \cdot T\pi/G \cdot \hat{X}\gamma, \tau_K \cdot T\pi/G \cdot w) = \pi_U \cdot i_M^* \cdot \omega_K^B \cdot (T(\pi/G) \cdot \hat{X}\gamma, \tau_K \cdot T\pi/G \cdot w) = \pi_U \cdot i_M^* \cdot \omega_K^B \cdot (\tau_K \cdot T\pi/G \cdot \hat{X}\gamma, \tau_K \cdot T\pi/G \cdot w)
\]

where we have used that \(\tau_K \cdot T\pi/G(X_K^B \cdot \gamma) = X_K^B \cdot \gamma\), and \(\tau_K \cdot T\gamma = T\gamma\), since \(\text{Im}(T\gamma) \subset \hat{K}\). If the one-form \(\gamma : Q \to T^*Q\) satisfies the condition, \(d\gamma = -B\) on \(\mathcal{D}\) with respect to \(T\pi_Q : TT^*Q \to TQ\), then we have that \((d\gamma + B)(T\pi_Q(X_K^B \cdot \gamma), T\pi_Q(w)) = 0\), since \(X_K^B \cdot \gamma, w \in F\), and \(T\pi_Q(X_K^B \cdot \gamma), T\pi_Q(w) \in \mathcal{D}\), and hence

\[
\pi_U \cdot i_M^* \cdot (d\gamma + B)(T\pi_Q(X_K^B \cdot \gamma), T\pi_Q(w)) = 0,
\]

and

\[
\omega_K^B(T\gamma \cdot \hat{X}\gamma, \tau_K \cdot T\pi/G \cdot w) - \omega_K^B(X_K^B \cdot \gamma, \tau_K \cdot T\pi/G \cdot w) = -\omega_K^B(X_K^B \cdot \gamma, T\gamma \cdot T\pi_Q(w)). \tag{7.7}
\]

If \(\gamma\) satisfies the equation \(T\gamma \cdot \hat{X}\gamma = X_K^B \cdot \gamma\), from Lemma 3.4(i) we know that the right side of (7.7) becomes that

\[
-\omega_K^B(X_K^B \cdot \gamma, T\gamma \cdot T\pi_Q(w)) = -\omega_K^B(T\gamma \cdot \hat{X}\gamma, T\gamma \cdot T\pi_Q(w)) = -\gamma^* \omega_K^B \cdot \tau_K(T\pi_Q \cdot \hat{X}\gamma, T\gamma \cdot T\pi_Q(w)) = -\gamma^* \cdot \pi_G^* \cdot \omega_K^B \cdot \tau_K(T\pi_Q \cdot X_K^B \cdot \gamma, T\pi_Q(w)) = -\gamma^* \cdot \pi_U \cdot i_M^* \cdot \omega_K^B(T\pi_Q(X_K^B \cdot \gamma), T\pi_Q(w)) = -\pi_U \cdot i_M^* \cdot \gamma^* \omega_K^B(T\pi_Q(X_K^B \cdot \gamma), T\pi_Q(w)) = \pi_U \cdot i_M^* \cdot (d\gamma + B)(T\pi_Q(X_K^B \cdot \gamma), T\pi_Q(w)) = 0,
\]

29
where $\gamma^* \cdot \tau_M \cdot i_M \cdot \omega^B = \tau_M \cdot i_M \cdot \gamma^* \cdot \omega^B$, because $\text{Im}(\gamma) \subset M$. But, since the nonholonomic reduced distributional two-form $\omega^B_K$ is non-degenerate, the left side of (7.7) equals zero, only when $\gamma$ satisfies the equation $T\gamma \cdot \tilde{X} = X^B_K \cdot \gamma$. Thus, if the one-form $\gamma : Q \to T\gamma Q$ satisfies the condition, $d\gamma = -B$ on $\mathcal{D}$ with respect to $T\pi_Q : T\gamma Q \to TQ$, then $\gamma$ must be a solution of the Type I of Hamilton-Jacobi equation $T\gamma \cdot \tilde{X} = X^B_K \cdot \gamma$. ■

Next, for any $G$-invariant symplectic map $\varepsilon : T^*Q \to T^*Q$ with respect to $\omega^B$, we can prove the following Type II of Hamilton-Jacobi theorem for the nonholonomic reduced distributional CMH system. For convenience, the maps involved in the following theorem and its proof are shown in Diagram-8.

**Diagram-8**

\[
\begin{array}{cccccc}
\mathcal{M} & \xrightarrow{i_M} & T^*Q & \xrightarrow{\pi_Q} & Q & \xrightarrow{\gamma} & T^*Q & \xrightarrow{\pi_G/G} & T^*Q \xrightarrow{i_M} & \mathcal{M} \\
X^B_K & \xleftarrow{T} & X^B_K & \xrightarrow{T} & \tilde{X} & \xrightarrow{T} & \tilde{X} & \xrightarrow{T} & \tilde{X} & \xleftarrow{T} & X^B_K \\
\mathcal{K} & \xleftarrow{T(T^*)} & T^*Q & \xrightarrow{T\pi_Q} & T^*Q & \xrightarrow{T\pi_G/G} & T^*Q/G & \xrightarrow{T\pi_G/G} & T^*Q/G \xrightarrow{T} & \mathcal{K} \\
\end{array}
\]

**Theorem 7.3 (Type II of Hamilton-Jacobi Theorem for a Nonholonomic Reduced Distributional CMH System)** For a given nonholonomic reducible CMH system $(T^*Q, G, \omega^B, D, H, F, u)$ with the associated distributional CMH system $(K, \omega^B_K, H_K, F^B_K, u^B_K)$ and the nonholonomic reduced distributional CMH system $(\tilde{K}, \omega^B_K, h_K, f^B_K, u^B_K)$, assume that $\gamma : Q \to T^*Q$ is an one-form on $Q$, and $\lambda = \gamma \cdot \pi_Q : T^*Q \to T^*Q$, and for any $G$-invariant symplectic map $\varepsilon : T^*Q \to T^*Q$ with respect to $\omega^B$, denote by $\tilde{X} = T\pi_Q \cdot \tilde{X} \cdot \varepsilon$, where $\tilde{X} = X^B_{(K, \omega^B_K, H_K, F^B_K, u^B_K)} = X^B_K + F^B_K + u^B_K$ is the dynamical vector field of the distributional CMH system corresponding to the nonholonomic reducible CMH system with symmetry $(T^*Q, G, \omega^B, D, H, F, u)$. Moreover, assume that $\text{Im}(\gamma) \subset M$, and it is $G$-invariant, $\varepsilon(M) \subset M$, $\text{Im}(T\gamma) \subset K$, and $\tilde{\gamma} = \pi_G/\gamma(Q) : T^*Q/G \to T^*Q/G$, and $\bar{\varepsilon} = \pi_G/\varepsilon(Q) : T^*Q \to T^*Q/G$. Then $\varepsilon$ and $\bar{\varepsilon}$ satisfy the equation $\tau_K \cdot T\bar{\varepsilon} \cdot X^B_{h_K} = T\bar{\lambda} \cdot \tilde{X} \cdot \varepsilon$, and if only they satisfy the equation $T\bar{\gamma} \cdot \tilde{X} \cdot X^B_{h_K} = T\bar{\lambda} \cdot \tilde{X} \cdot \varepsilon$, then $\tau_K \cdot T\bar{\varepsilon} \cdot X^B_{h_K} = T\bar{\lambda} \cdot \tilde{X} \cdot \varepsilon$, and thus $\tau_K \cdot T\bar{\varepsilon} \cdot X^B_{h_K} = T\bar{\lambda} \cdot \tilde{X} \cdot \varepsilon$. Note that $\tau_K \cdot T\bar{\varepsilon} \cdot X^B_{h_K} = T\bar{\lambda} \cdot \tilde{X} \cdot \varepsilon$, and for any $X^B \in \mathcal{K}$, $\tau_K \cdot T\bar{\varepsilon} \cdot X^B_{h_K} = T\bar{\lambda} \cdot \tilde{X} \cdot \varepsilon$, and thus $\tau_K \cdot T\bar{\varepsilon} \cdot X^B_{h_K} = T\bar{\lambda} \cdot \tilde{X} \cdot \varepsilon$. Thus, using the non-degenerate and nonholonomic reduced distributional two-form $\omega^B_K$, from Lemma 3.4 and Lemma 6.3, if we take that $v = X^B_K \cdot \varepsilon \in \mathcal{K}(\subset \mathcal{F})$, and for any $w \in \mathcal{F}$, $T\lambda(w) \neq 0$, $\tau_K \cdot T\pi_G/\cdot w \neq 0$, and $\tau_K \cdot T\pi_G/\cdot T\lambda(w) \neq 0$,
then we have that
\[
\omega^B_K(T_{\tilde{\gamma}} \cdot \tilde{X}^\varepsilon, \tau_K \cdot T_{\pi/G} \cdot w) = \omega^B_K(\tau_K \cdot T_{(\pi/G) \cdot \gamma} \cdot \tilde{X}^\varepsilon, \tau_K \cdot T_{\pi/G} \cdot w)
\]
\[
= \pi^*_G \cdot \omega^B_K \cdot \tau_K(T_{\gamma} \cdot \tilde{X}^\varepsilon, w) = \tau_U \cdot \imath^*_M \cdot \omega^B(T_{\gamma} \cdot T_{\pi Q} \cdot \tilde{X}^\varepsilon, w)
\]
\[
= \tau_U \cdot \imath^*_M \cdot \omega^B(T(\gamma \cdot \pi Q) \cdot X^B_K \cdot \varepsilon, w)
\]
\[
= \tau_U \cdot \imath^*_M \cdot (\omega^B(X^B_K \cdot \varepsilon, w - T(\gamma \cdot \pi Q) \cdot w) - (d\gamma + B)(T_{\pi Q}(X^B_K \cdot \varepsilon), T_{\pi Q}(w)))
\]
\[
= \tau_U \cdot \imath^*_M \cdot \omega^B(X^B_K \cdot \varepsilon, w) - \tau_U \cdot \imath^*_M \cdot \omega^B(X^B_K \cdot \varepsilon, T\lambda \cdot w)
\]
\[
- \tau_U \cdot \imath^*_M \cdot (d\gamma + B)(T_{\pi Q}(X^B_K \cdot \varepsilon), T_{\pi Q}(w))
\]
\[
= \pi^*_G \cdot \omega^B_K \cdot \tau_K(X^B_K \cdot \varepsilon, w) - \pi^*_G \cdot \omega^B_K \cdot \tau_K(X^B_K \cdot \varepsilon, T\lambda \cdot w) + \tau_U \cdot \imath^*_M \cdot \lambda^* \omega^B(X^B_K \cdot \varepsilon, w)
\]
\[
= \omega^B_K(\tau_K \cdot T_{\pi/G}(X^B_K \cdot \varepsilon), \tau_K \cdot T_{\pi/G} \cdot w) - \omega^B_K(\tau_K \cdot T_{\pi/G}(X^B_K \cdot \varepsilon), \tau_K \cdot (T_{\pi/G} \cdot \lambda \cdot w) + \pi^*_G \cdot \omega^B_K \cdot \tau_K(T\lambda \cdot X^B_K \cdot \varepsilon, T\lambda \cdot w)
\]
\[
= \omega^B_K(X^B_K \cdot \varepsilon, \tau_K \cdot T_{\pi/G} \cdot w) - \omega^B_K(X^B_K \cdot \varepsilon, \tau_K \cdot T_{\pi/G} \cdot \varepsilon, \tau_K \cdot T_{\pi/G} \cdot T\lambda \cdot \varepsilon)
\]
where we have used that \(\tau_K \cdot T_{\pi/G}(X^B_K \cdot \varepsilon) = X^B_{\tau_K} \cdot \varepsilon\) and \(\tau_K \cdot T_{\gamma} = T_{\tilde{\gamma}}\) and \(\tau_K \cdot \pi_{\gamma} \cdot \lambda \cdot T_{\pi/G} = T_{\pi/G} \cdot \varepsilon\), \(T\lambda = T\lambda\), since \(Im(T\gamma) \subset \tilde{K}\). From the nonholonomic reduced distributional magnetic Hamiltonian equation (7.2), \(X^B_K \cdot \omega^B_K = d\lambda^h_K\), we have that \(X^B_{\tau_K} = \lambda^h_K \cdot \lambda^* \omega^B\), where \(X^B_{\tau_K}\) is the magnetic Hamiltonian vector field of the function \(h^h_K : M(\subset T^*Q/G) \to \mathbb{R}\). Note that \(\varepsilon : T^*Q \to T^*Q\) is symplectic with respect to \(\omega^B\), and \(\tilde{\varepsilon} = \pi^*_G(\varepsilon) : T^*Q \to T^*Q/G\) is also symplectic along \(\varepsilon\), and hence \(X^B_{\tau_K} = \tau_K \cdot X^B_{h^h_K} \cdot \varepsilon\), along \(\varepsilon\), and hence \(X^B_{\tau_K} = \tau_K \cdot X^B_{h^h_K} = X^B_{\tilde{h}_K} \cdot \tilde{\varepsilon} \cdot X^B_{\tilde{h}_K} \cdot \tilde{\varepsilon} = T\tilde{\varepsilon} \cdot X^B_{\tilde{h}_K} \cdot \tilde{\varepsilon}\). Note that \(T\lambda \cdot X^B_{\varepsilon} = T_{\pi/G} \cdot T\lambda \cdot X^B_{\varepsilon} = T_{\pi/G} \cdot T\gamma \cdot T_{\pi Q} \cdot X^B_{\varepsilon} = T_{\pi/G} \cdot T\gamma \cdot T_{\pi Q} \cdot \tilde{X}^\varepsilon = T_{\pi/G} \cdot T\lambda \cdot \tilde{X}^\varepsilon = T\tilde{\lambda} \cdot \tilde{X} \cdot \varepsilon\). Then we have that
\[
\omega^B_K(T_{\tilde{\gamma}} \cdot \tilde{X}^\varepsilon, \tau_K \cdot T_{\pi/G} \cdot w) = \omega^B_K(X^B_{\tau_K} \cdot \varepsilon, \tau_K \cdot T_{\pi/G} \cdot w)
\]
Because the nonholonomic reduced distributional two-form \(\omega^B_K\) is non-degenerate, it follows that the equation \(T_{\tilde{\gamma}} \cdot \tilde{X}^\varepsilon = X^B_{\tau_K} \cdot \varepsilon\), is equivalent to the equation \(T\tilde{\lambda} \cdot \tilde{X} \cdot \varepsilon = \tau_K \cdot T\tilde{\varepsilon} \cdot X^B_{\tilde{h}_K} \cdot \tilde{\varepsilon}\). Thus, \(\varepsilon\) and \(\tilde{\varepsilon}\) satisfy the equation \(T\tilde{\lambda} \cdot \tilde{X} \cdot \varepsilon = \tau_K \cdot T\tilde{\varepsilon} \cdot X^B_{\tilde{h}_K} \cdot \tilde{\varepsilon}\), if and only if they satisfy the Type II of Hamilton-Jacobi equation \(T_{\tilde{\gamma}} \cdot \tilde{X}^\varepsilon = X^B_{\tau_K} \cdot \tilde{\varepsilon}\). ■

For a given nonholonomic reducible CMH system \((T^*Q, G, \omega^B, D, H, F, u)\) with the associated the distributional CMH system \((K, \omega^B_K, H^h_K, F^B_{\tilde{h}_K}, u^B_{\tilde{h}_K})\) and the nonholonomic reduced distributional CMH system \((\tilde{K}, \omega^B_{\tilde{h}_K}, h^h_{\tilde{h}_K}, f^B_{\tilde{h}_K}, u^B_{\tilde{h}_K})\), we know that the nonholonomic dynamical vector field \(X^B_K\) and the nonholonomic reduced dynamical vector field \(X^B_{\tilde{h}_K}\) are \(\pi_{/G}\)-related, that is, \(X^B_{\tilde{h}_K} \cdot \pi_{/G} = T_{\pi/G} \cdot X^B_K\). Then we can prove the following Theorem 7.4, which states the relationship between the solutions of Type II of Hamilton-Jacobi equations and nonholonomic reduction.

**Theorem 7.4** For a given nonholonomic reducible CMH system \((T^*Q, G, \omega^B, D, H, F, u)\) with the associated the distributional CMH system \((K, \omega^B_K, H_K, F^B_K, u^B_K)\) and the nonholonomic reduced distributional CMH system \((\tilde{K}, \omega^B_{\tilde{h}_K}, h^h_{\tilde{h}_K}, f^B_{\tilde{h}_K}, u^B_{\tilde{h}_K})\), assume that \(\gamma : Q \to T^*Q\) is an one-form on \(Q\), and \(\varepsilon : T^*Q \to T^*Q\) is a \(G\)-invariant symplectic map with respect to \(\omega^B\), and \(\tilde{\gamma} = \pi_{/G}(\gamma) : Q \to T^*Q/G\), and \(\tilde{\varepsilon} = \pi_{/G}(\varepsilon) : T^*Q \to T^*Q/G\). Under the hypotheses and notations of Theorem 7.3, then
we have that $\varepsilon$ is a solution of the Type II of Hamilton-Jacobi equation, $T\gamma \cdot \hat{X}^\varepsilon = X^K_B \cdot \varepsilon$, for the distributional CMH system $(K, \omega^K_B, H_B, F^K_B, u^K_B)$, if and only if $\varepsilon$ and $\bar{\varepsilon}$ satisfy the Type II of Hamilton-Jacobi equation $T\bar{\gamma} \cdot \bar{X}^\varepsilon = X^K_B \cdot \bar{\varepsilon}$, for the nonholonomic reduced distributional CMH system $(K, \omega^K_B, h^K_B, f^K_B, u^K_B)$.

**Proof:** At first, for the dynamical vector field of the distributional CMH system $(K, \omega^K_B, H_B, F^K_B, u^K_B)$, $\hat{X} = X^K_B (\omega^K_B, H_B, F^K_B, u^K_B) = X^K_B + F^K_B + u^K_B$, and $F^K_B = \tau_K \cdot \text{vlift}(F_M)X^K_B$, and $u^K_B = \tau_K \cdot \text{vlift}(u_M)X^K_B$, note that $T\pi_Q \cdot \text{vlift}(F_M)X^K_B = T\pi_Q \cdot \text{vlift}(u_M)X^K_B = 0$, then we have that $T\pi_Q \cdot F^K_B = T\pi_Q \cdot u^K_B = 0$, and hence $T\pi_Q \cdot \hat{X} \cdot \varepsilon = T\pi_Q \cdot X^K_B \cdot \varepsilon$. Next, under the hypotheses and notations of Theorem 7.3, $\text{Im}(\gamma) \subset M$, and it is $G$-invariant, $\text{Im}(T\gamma) \subset K$, and hence $\text{Im}(T\bar{\gamma}) \subset \bar{K}$, in this case, $\pi^*_G \cdot \omega^K_B \cdot \tau_K = \pi_K \cdot \omega^K_B = \pi_K \cdot i^*_M \cdot \omega^B$, along $\text{Im}(T\bar{\gamma})$. In addition, from the distributional magnetic Hamiltonian equation (7.1), we have that $X^K_B = \tau_K \cdot X^K_B$, and from the nonholonomic reduced distributional magnetic Hamiltonian equation (7.2), we have that $X^K_B = \tau_K \cdot X^K_B$, and the nonholonomic dynamical vector field $X^K_B$ and the nonholonomic reduced dynamical vector field $X^K_B$ are $\pi^*_G$-related, that is, $X^K_B \cdot \pi^*_G = T\pi/G \cdot X^K_B$. Note that $\varepsilon(M) \subset M$, and hence $X^K_B \cdot \varepsilon \in F$ along $\varepsilon$, and $\tau_K \cdot X^K_B \cdot \varepsilon = X^K_B \cdot \varepsilon \in \bar{K}$. Then $\tau_K \cdot T\pi/G(X^K_B \cdot \varepsilon) = \tau_K \cdot (T\pi/G(X^K_B)) \cdot \varepsilon = \tau_K \cdot (X^K_B \cdot \pi^*_G) \cdot \varepsilon = \tau_K \cdot X^K_B \cdot \pi^*_G \cdot \varepsilon = X^K_B \cdot \varepsilon$. Thus, using the non-degenerate and nonholonomic reduced distributional two-form $\omega^K_B$, note that $\tau_K \cdot T\bar{\gamma} = T\bar{\gamma}$, for any $w \in F$, $\tau_K \cdot w \neq 0$, and $\tau_K \cdot T\pi/G \cdot w \neq 0$, then we have that

$$\omega^K_B(T\bar{\gamma} \cdot \bar{X}^\varepsilon - X^K_B \cdot \bar{\varepsilon}, \tau_K \cdot T\pi/G \cdot w) = \omega^K_B(T\bar{\gamma} \cdot \bar{X}^\varepsilon, \tau_K \cdot T\pi/G \cdot w) - \omega^K_B(X^K_B \cdot \bar{\varepsilon}, \tau_K \cdot T\pi/G \cdot w) = \omega^K_B(\tau_K \cdot T\bar{\gamma} \cdot \bar{X}^\varepsilon, \tau_K \cdot T\pi/G \cdot w) - \omega^K_B(\tau_K \cdot X^K_B \cdot \pi^*_G \cdot \varepsilon, \tau_K \cdot T\pi/G \cdot w)$$

$$= \omega^K_B(\tau_K \cdot \tau_K \cdot T\bar{\gamma} \cdot \bar{X}^\varepsilon, \tau_K \cdot T\pi/G \cdot w) - \omega^K_B(\tau_K \cdot X^K_B \cdot \pi^*_G \cdot \varepsilon, \tau_K \cdot \tau_K \cdot T\pi/G \cdot w)$$

$$= \omega^K_B(\tau_K \cdot T\pi/G \cdot T\bar{\gamma} \cdot \bar{X}^\varepsilon, \tau_K \cdot T\pi/G \cdot w) - \omega^K_B(\tau_K \cdot T\pi/G \cdot X^K_B \cdot \varepsilon, \tau_K \cdot \tau_K \cdot T\pi/G \cdot w)$$

$$= \pi^*_G \cdot \omega^K_B \cdot \tau_K(T\bar{\gamma} \cdot \bar{X}^\varepsilon, w) - \pi^*_G \cdot \omega^K_B \cdot \tau_K(X^K_B \cdot \varepsilon, w)$$

$$= \pi^*_G \cdot \omega^K_B \cdot \tau_K(T\bar{\gamma} \cdot \bar{X}^\varepsilon, w) - \pi^*_G \cdot \omega^K_B \cdot \tau_K(X^K_B \cdot \varepsilon, w).$$

In the case we note that $\pi_K \cdot i^*_M \cdot \omega^B = \tau_K \cdot i^*_M \cdot \omega^B = \omega^K_B \cdot \tau_K$, and $\tau_K \cdot T\gamma = T\bar{\gamma}$, $\tau_K \cdot X^K_B = X^K_B$, since $\text{Im}(\gamma) \subset M$, and $\text{Im}(T\bar{\gamma}) \subset \bar{K}$. Thus, we have that

$$\omega^K_B(T\bar{\gamma} \cdot \bar{X}^\varepsilon - X^K_B \cdot \bar{\varepsilon}, \tau_K \cdot T\pi/G \cdot w) = \omega^K_B(\tau_K \cdot T\bar{\gamma} \cdot \bar{X}^\varepsilon, w) - \omega^K_B(\tau_K \cdot X^K_B \cdot \varepsilon, w)$$

$$= \omega^K_B(\tau_K \cdot T\bar{\gamma} \cdot \bar{X}^\varepsilon, w) - \omega^K_B(\tau_K \cdot X^K_B \cdot \varepsilon, w)$$

$$= \omega^K_B(T\bar{\gamma} \cdot \bar{X}^\varepsilon - X^K_B \cdot \bar{\varepsilon}, \tau_K \cdot w).$$

Because the distributional two-form $\omega^K_B$ and the nonholonomic reduced distributional two-form $\omega^K_B$ are both non-degenerate, it follows that the equation $T\bar{\gamma} \cdot \bar{X}^\varepsilon = X^K_B \cdot \bar{\varepsilon}$, is equivalent to the equation $T\bar{\gamma} \cdot \bar{X}^\varepsilon = X^K_B \cdot \bar{\varepsilon}$. Thus, $\bar{\varepsilon}$ is a solution of the Type II of Hamilton-Jacobi equation $T\bar{\gamma} \cdot \bar{X}^\varepsilon = X^K_B \cdot \bar{\varepsilon}$, for the distributional CMH system $(K, \omega^K_B, H_B, F^K_B, u^K_B)$, if and only if $\varepsilon$ and $\bar{\varepsilon}$ satisfy the Type II of Hamilton-Jacobi equation $T\bar{\gamma} \cdot \bar{X}^\varepsilon = X^K_B \cdot \bar{\varepsilon}$, for the nonholonomic reduced distributional CMH system $(K, \omega^K_B, h^K_B, f^K_B, u^K_B)$.

**Remark 7.5** It is worthy of noting that, the Type I of Hamilton-Jacobi equation $T\gamma \cdot \hat{X}^\varepsilon = X^K_B \cdot \gamma$, is the equation of the reduced differential one-form $\gamma$; and the Type II of Hamilton-Jacobi equation $T\bar{\gamma} \cdot \bar{X}^\varepsilon = X^K_B \cdot \bar{\varepsilon}$, is the equation of the symplectic diffeomorphism map $\varepsilon$ and the reduced symplectic diffeomorphism map $\bar{\varepsilon}$. If the nonholonomic CMH system with symmetry we considered has not
any the external force and control, that is, \( F = 0 \) and \( W = \emptyset \), in this case, the nonholonomic CMH system with symmetry \((T^*Q, G, \omega^B, D, H, F, W)\) is just the nonholonomic magnetic Hamiltonian system with symmetry \((T^*Q, G, \omega^B, D, H)\), and with the magnetic symplectic form \( \omega^B \) on \( T^*Q \). From the above Type I and Type II of Hamilton-Jacobi theorems, that is, Theorem 7.2 and Theorem 7.3, we can get the Theorem 5.2 and Theorem 5.3 in Wang [27]. It shows that Theorem 7.2 and Theorem 7.3 can be regarded as an extension of two types of Hamilton-Jacobi theorem for the nonholonomic magnetic Hamiltonian system with symmetry given in Wang [27] to that for the system with the external force and control. In particular, in this case, if \( B = 0 \), then the magnetic symplectic form \( \omega^B \) is just the canonical symplectic form \( \omega \) on \( T^*Q \), from the proofs of Theorem 7.2 and Theorem 7.3, we can also get the Theorem 4.2 and Theorem 4.3 in León and Wang [10]. It shows that Theorem 7.2 and Theorem 7.3 can be regarded as an extension of two types of Hamilton-Jacobi theorem for the nonholonomic Hamiltonian system with symmetry given in León and Wang [10] to that for the system with the magnetic, external force and control.

The theory of controlled mechanical system is a very important subject, its research gathers together some separate areas of research such as mechanics, differential geometry and nonlinear control theory, etc., and the emphasis of this research on geometry is motivated by the aim of understanding the structure of equations of motion of the system in a way that helps both analysis and design. Thus, it is natural to study the controlled mechanical systems by combining with the analysis of dynamical systems and the geometric reduction theory of Hamiltonian and Lagrangian systems. Following the theoretical development of geometric mechanics, a lot of important problems about this subject are being explored and studied, see León and Wang [10], Marsden et al. [16], Ratiu and Wang [19], Wang [20–28], and Wang and Zhang [29]. These research works from the geometrical point of view reveal the internal relationships of the geometrical structures of phase spaces, symmetric reductions, constraints, dynamical vector fields and controls of a mechanical system and its regular reduced systems. In particular, it is the key thought of the researches of geometrical mechanics of the professor Jerrold E. Marsden to explore and reveal the deeply internal relationship between the geometrical structure of phase space and the dynamical vector field of a mechanical system. It is also our goal of pursuing and inheriting.

References

[1] R. Abraham, J.E. Marsden, Foundations of Mechanics, second ed., Addison-Wesley, Reading, MA, 1978.

[2] V.I. Arnold, Mathematical Methods of Classical Mechanics, second ed., In: Graduate Texts in Mathematics, **60**, Springer-Verlag, 1989.

[3] L. Bates and J. Śniatycki, Nonholonomic reduction, Rep. Math. Phys. 32, 99-115(1993).

[4] F. Cantrijn, M. de León, J.C. Marrero. and D. Martin de Diego, Reduction of constrained systems with symmetries, J. Math. Phys., 40(2), 795-820(1999).

[5] H. Cendra, J.E. Marsden. and T.S. Ratiu, Geometric mechanics, Lagrangian reduction and nonholonomic systems, In ”Mathematics Unlimited 2001 and Beyond” (eds. B. Engquist and W. Schmid), Springer-Verlag, New York, 221-273(2001).

[6] R. Cushman, H. Duistermaat. and J. Śniatycki, Geometry of Nonholonomic Constrained Systems, Advanced series in nonlinear dynamics, 26, (2010).

[7] Z. Ge, J.E. Marsden, Lie-Poisson integrators and Lie-Poisson Hamilton-Jacobi theory, Phys. Lett. A, **133**(1988), 134-139.
[8] J. Koiller, Reduction of some classical non-holonomic systems with symmetry, Arch. Rational Mech. Anal. 118, 113-148(1992).

[9] J-A Lázaro-Camí, J-P Ortega, The stochastic Hamilton-Jacobi equation, J. Geom. Mech. 1(2009), 295-315.

[10] M. de León, H. Wang, Hamilton-Jacobi equations for nonholonomic reducible Hamiltonian systems on a cotangent bundle, (arXiv: 1508.07548, a revised version).

[11] P. Libermann, C.M. Marle, Symplectic Geometry and Analytical Mechanics, Kluwer Academic Publishers, 1987.

[12] J.E. Marsden, Lectures on Mechanics, In: London Mathematical Society Lecture Notes Series, 174, Cambridge University Press, 1992.

[13] J.E. Marsden, G. Misiolek, J.P. Ortega, M. Perlmutter, T.S. Ratiu, Hamiltonian Reduction by Stages, In: Lecture Notes in Mathematics, 1913, Springer, 2007.

[14] J.E. Marsden, M. Perlmutter, The orbit bundle picture of cotangent bundle reduction, C. R. Math. Acad. Sci. Soc. R. Can., 22(2000), 33-54.

[15] J.E. Marsden, T.S. Ratiu, Introduction to Mechanics and Symmetry, second ed., In: Texts in Applied Mathematics, 17, Springer-Verlag, New York, 1999.

[16] J.E. Marsden, H. Wang, Z.X. Zhang, Regular reduction of controlled Hamiltonian system with symplectic structure and symmetry, Diff. Geom. Appl., 33(3)(2014), 13-45, (arXiv: 1202.3564, a revised version).

[17] J.E. Marsden, A. Weinstein, Reduction of symplectic manifolds with symmetry, Rep. Math. Phys., 5(1974), 121–130.

[18] J.P. Ortega, T.S. Ratiu, Momentum Maps and Hamiltonian Reduction, In: Progress in Mathematics, 222, Birkhäuser, 2004.

[19] T.S. Ratiu, H. Wang, Poisson reduction by controllability distribution for a controlled Hamiltonian system, (arXiv: 1312.7047).

[20] H. Wang, The geometrical structure of phase space of the controlled Hamiltonian system with symmetry, (arXiv: 1802.01988).

[21] H. Wang, Regular reduction of a controlled magnetic Hamiltonian system with symmetry of the Heisenberg group, (arXiv: 1506.03640, a revised version).

[22] H. Wang, Hamilton-Jacobi theorems for regular reducible Hamiltonian system on a cotangent bundle, Jour. Geom. Phys., 119 (2017), 82-102.

[23] H. Wang, Hamilton-Jacobi equations for regular controlled Hamiltonian system and its reduced systems, (arXiv: 1305.3457, a revised version), To appear in Acta Mathematica Scientia, English Series, 2022.

[24] H. Wang, Symmetric Reduction of Regular Controlled Lagrangian System with Momentum Map, (arXiv: 2103.06563).

[25] H. Wang, Dynamical equations of the controlled rigid spacecraft with a rotor, (arXiv: 2005.02221).
[26] H. Wang, Symmetric reduction and Hamilton-Jacobi equations for the controlled underwater vehicle-rotor system, (arXiv: 1310.3014, a revised version).

[27] H. Wang, Hamilton-Jacobi equations for nonholonomic magnetic Hamiltonian system, (arXiv: 2112.00961).

[28] H. Wang, Nonholonomic controlled Hamiltonian system: symmetric reduction and Hamilton-Jacobi equations, (arXiv: 2205.01998).

[29] H. Wang, Z.X. Zhang, Optimal reduction of controlled Hamiltonian system with Poisson structure and symmetry, Jour. Geom. Phys., 62 (5)(2012), 953-975.

[30] N.M.J. Woodhouse, Geometric Quantization, second ed., Clarendon Press, Oxford, 1992.