AUTOMORPHISMS OF HIGHER DIMENSIONAL RIGHT-ANGLED
ARTIN GROUPS

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Abstract. We study the algebraic structure of the automorphism group of a general
right-angled Artin group. We show that this group is virtually torsion-free and has
finite virtual cohomological dimension. This generalizes results proved in [CCV] for two-
dimensional right-angled Artin groups.

1. Introduction

In [CCV] we began a study of the group Out$(A_\Gamma)$ of outer automorphisms of a right-
angled Artin group by analyzing the case when the defining graph $\Gamma$ is connected and has
no triangles. We used both algebraic and geometric methods to establish cohomological
finiteness results and the Tits alternative for Out$(A_\Gamma)$, and remarked that much of the
algebra, in particular, extends to the general case. In this paper we give the details of this
extension. The resulting algebraic description of Out$(A_\Gamma)$ allows us to give an inductive
proof that Out$(A_\Gamma)$ is virtually torsion-free, so that the virtual cohomological dimension is
defined. We then prove that the virtual cohomological dimension of Out$(A_\Gamma)$ is finite for
all $\Gamma$.

We remark that a different algebraic approach to studying automorphisms of right-
angled Artin groups has recently been initiated by Duncan, Kazachkov and Remeslennikov
in [DKR].

2. Join subgroups

We consider a right-angled Artin group $A_\Gamma$ associated to a connected, simplicial graph
$\Gamma$ with vertex set $V$. For two vertices $v, w \in V$, let $d(v, w)$ denote the distance from $v$ to $w$
in $\Gamma$. The link $lk(v)$ is the subgraph spanned by vertices at distance one from $v$, and the
star $st(v)$ is the subgraph spanned by vertices at distance at most one from $v$. Note that
$v \in lk(w)$ if and only if $w \in lk(v)$, and $v \in st(w)$ if and only if $w \in st(v)$.

2.1. A partial ordering. Suppose $v$ and $w$ are elements of $V$. Write $v \leq w$ if $lk(v) \subseteq
st(w)$. Note that this may occur in three ways:

1. $v = w$,
2. $d(v, w) = 1$ and $st(v) \subseteq st(w)$,
3. $d(v, w) = 2$ and $lk(v) \subseteq lk(w)$.

Lemma 2.1. If $u \leq v \leq w$ and $d(u, v) = 1$, then $d(v, w) \leq 1$. 
Proof. Since \( d(u, v) = 1 \), \( u \in \text{lk}(v) \subseteq \text{st}(w) \). Therefore \( w \in \text{st}(u) \), i.e. either \( w = u \) or \( w \in \text{lk}(u) \subseteq \text{st}(v) \). In either case \( d(v, w) \leq 1 \).

Lemma 2.2. The relation \( \leq \) is transitive on the vertices of \( \Gamma \).

Proof. Suppose \( u \leq v \leq w \). We need to check that \( u \leq w \). If \( d(u, v) = 0 \) or \( 2 \), then \( \text{lk}(u) \subseteq \text{lk}(v) \subseteq \text{st}(w) \), so this is immediate. If \( d(u, v) = 1 \), then by Lemma 2.1, \( d(v, w) \leq 1 \). Since we already know that \( \text{lk}(v) \subset \text{st}(w) \), this gives \( \text{st}(v) \subseteq \text{st}(w) \), so \( \text{lk}(u) \subseteq \text{st}(v) \subseteq \text{st}(w) \).

Now define a relation on vertices by \( v \sim w \) if \( v \leq w \) and \( w \leq v \), or equivalently, if one of the following holds,

1. \( v = w \),
2. \( d(v, w) = 1 \) and \( \text{st}(v) = \text{st}(w) \),
3. \( d(v, w) = 2 \) and \( \text{lk}(v) = \text{lk}(w) \).

It follows from Lemma 2.2 that this is an equivalence relation and that \( \leq \) induces a partial ordering on the set of equivalence classes. We denote the equivalence class of \( v \) by \( [v] \).

Lemma 2.3. All elements of \([v]\) are the same distance from each other.

Proof. If \( x, y \in [v] \) and \( d(x, y) = 1 \), then any other \( z \in [v] \) satisfies \( y \leq x \leq z \), so by Lemma 2.1 \( d(x, z) = 1 \).

Thus \([v]\) generates either a free abelian subgroup of \( A_\Gamma \) or a non-abelian free subgroup. Let \( V_{ab} \) denote the set of vertices \( v \) with \( A_{[v]} \) abelian, and \( V_{fr} \) the set of vertices with \( A_{[v]} \) non-abelian.

Lemma 2.4. Suppose \([v_1] < [v_2] < \cdots < [v_k]\). Then there exists \( j, 0 \leq j \leq k \), such that \( d(v_i, v_{i+1}) = 2 \) for all \( i < j \) and \( d(v_i, v_{i+1}) = 1 \) for all \( i \geq j \). Moreover, for \( i > j \), \( v_i \in V_{ab} \).

Proof. This is immediate from Lemma 2.1.

It is not necessarily the case that \( v_i \) is non-abelian for \( i \leq j \), since an abelian \([v]\) can be less than a non-abelian \([w]\) if \([v]\) consists of just a single vertex.

2.2. The join associated to a vertex class. For two simplicial graphs \( \Theta_1 \) and \( \Theta_2 \), let \( \Theta_1 \ast \Theta_2 \) denote their join, that is, the graph formed by joining every vertex of \( \Theta_1 \) to every vertex of \( \Theta_2 \) by an edge. Note that if \( \Theta = \Theta_1 \ast \Theta_2 \), then \( A_\Theta = A_{\Theta_1} \times A_{\Theta_2} \).

For a vertex \( v \) in \( \Gamma \), define

\[
L_{[v]} = \text{lk}(v) \setminus [v] \\
J_{[v]} = L_{[v]} \ast [v]
\]

\( J_{[v]} \) is called the join associated to \([v]\). Note that it is always the case that \( \text{st}(v) \subseteq J_{[v]} \) and equality holds if and only if \( v \in V_{ab} \). Note also, that the relation \( w \in L_{[v]} \) is symmetric; that is, \( w \) lies in \( L_{[v]} \) if and only if \( v \) lies in \( L_{[w]} \).

The special subgroup generated by \( J_{[v]} \) is a direct product

\[
A_{J_{[v]}} = A_{L_{[v]}} \times A_{[v]}
\]

where the second factor is either free or free abelian.
2.3. The graph of maximal vertex classes. Define a graph $\Gamma_0$ as follows. The vertices of $\Gamma_0$ are the equivalence classes $[v]$ which are maximal with respect to the partial order $\leq$. Two vertices $[v]$ and $[w]$ are joined by an edge in $\Gamma_0$ if and only if $v$ and $w$ are adjacent in $\Gamma$. It follows from the definition of the equivalence relation that adjacency is independent of choice of representative.

Lemma 2.5. $\Gamma_0$ is a connected graph and every $w \in V$ lies in the join $J_v$ associated to some $[v]$ in $\Gamma_0$.

Proof. To prove connectivity of $\Gamma_0$, let $[v]$ and $[v']$ be vertices on $\Gamma_0$. Let $v = v_0, v_1, \ldots, v_n = v'$ be an edgepath in $\Gamma$. We proceed by induction on $n$. If $n = 1$, then $[v], [v']$ are adjacent in $\Gamma_0$. Suppose $n > 1$. Choose $[w]$ such that $[w]$ is maximal and $[v_1] \leq [w]$. Then $v$ and $v_2$ lie in $lk(v_1) \subseteq st(w)$. So either $[v] = [w]$ or $[v]$ is adjacent to $[w]$ in $\Gamma_0$, and there is an edgepath $w, v_2, \ldots, v_n$ in $\Gamma$. By induction, $[w]$ is connected to $[v']$ by a path in $\Gamma_0$.

For the second statement of the lemma, note that if $w$ is in the link of $u$, then for any $v$ with $[u] \leq [v]$, $w \in lk(u) \subseteq st(v) \subseteq J_v$.

2.4. Normalizers and centralizers. For a subgraph $\Theta$ of $\Gamma$, denote the normalizer, centralizer, and center of $A_\Theta$, respectively, by $N(\Theta)$, $C(\Theta)$ and $Z(\Theta)$. It is shown in [CCV], Proposition 2.2, that

$$N(\Theta) = A_{\Theta \cup \Theta^\perp} \quad C(\Theta) = A_{\Theta^\perp} \quad Z(\Theta) = A_{\Theta \cap \Theta^\perp},$$

where $\Theta^\perp$ is the set of vertices commuting with all elements of $\Theta$. We will be particularly interested in the case of $\Theta = J_v$.

Lemma 2.6. If $[v]$ is maximal, then the centralizer, center, and normalizer of $A_{J_v}$ are given by

$$C(J_v) = Z(J_v) = \begin{cases} A_v & \text{if } v \in V_{ab} \\ \{1\} & \text{if } v \in V_{fr} \end{cases}$$

$$N(J_v) = A_v.$$

Proof. Suppose $[v]$ is maximal and some $u$ commutes with all of $J_v$. Then $lk(v) \subseteq J_v \subseteq st(u)$, so by maximality of $[v], [u] = [v]$. Thus $J_v^\perp$ is contained in $[v]$ and the lemma follows from the descriptions above.

Lemma 2.7. Suppose $[v]$ and $[w]$ are adjacent vertices in $\Gamma_0$. Let $J_{v,w}$ denote the intersection of $J_v$ and $J_w$. Then

$$C(J_{v,w}) = Z(J_{v,w}) = Z([v]) \times Z([w]) \times Z(L_v \cap L_w)$$

$$N(J_{v,w}) = A_{J_{v,w}}.$$

Proof. Since $[v]$ and $[w]$ are adjacent, $[v] \subseteq L_w$ and $[w] \subseteq L_v$. Thus, $J_{v,w}$ decomposes as a join, $J_{v,w} = [v] \ast [w] \ast (L_v \cap L_w)$. Any generator commuting with both $[v]$ and $[w]$ lies in $J_{v,w}$, so $J_{v,w}^\perp \subseteq J_{v,w}$. The lemma now follows from the formulas above.
3. Restriction and Projection Homomorphisms

In this section we define two homomorphisms defined on a finite-index subgroup of $Out(A_\Gamma)$, and study their kernels. We begin by reviewing the work of M. Laurence [Lau95]. Building on the work of H. Servatius [Ser89], Laurence described a finite set of generators for $Aut(A_\Gamma)$ as follows.

1. **Inner automorphisms** conjugate the entire group by some generator $v$.
2. **Symmetries** are induced by symmetries of $\Gamma$ and permute the generators.
3. **Inversions** send a standard generator of $A_\Gamma$ to its inverse.
4. **Transvections** occur whenever $[v] \leq [w]$. In this case the transvection sends $v \mapsto vw$.
5. **Partial conjugations** exist when removal of the (closed) star of some vertex $v$ disconnects the graph $\Gamma$. In this case one obtains an automorphism by conjugating all of the generators in one of the components by $v$.

**Definition 3.1.** The subgroup of $Aut(A_\Gamma)$ generated by inner automorphisms, inversions, partial conjugations and transvections is called the **pure automorphism group** of $A_\Gamma$ and is denoted $Aut^0(A_\Gamma)$. The image of $Aut^0(A_\Gamma)$ in $Out(A_\Gamma)$ is the group of **pure outer automorphisms** and is denoted $Out^0(A_\Gamma)$.

The subgroups $Aut^0(A_\Gamma)$ and $Out^0(A_\Gamma)$ are easily seen to be normal and of finite index in $Aut(A_\Gamma)$ and $Out(A_\Gamma)$ respectively. We remark that if $A_\Gamma$ is a free group or free abelian group, then $Aut^0(A_\Gamma) = Aut(A_\Gamma)$.

3.1. Definition of restriction and projection.

**Proposition 3.2.** Let $\phi \in Out^0(A_\Gamma)$ and suppose $[v]$ is maximal. Then there is a representative automorphism $\phi_v$ of $\phi$ which preserves both the subgroup of $A_\Gamma$ generated by $[v]$ and the subgroup generated by $J_{[v]}$.

**Proof.** It suffices to check that the proposition holds for each of the generators of $Out^0(A_\Gamma)$. It is clear for inversions.

**Partial conjugations:** Let $\phi$ be a partial conjugation by $w$. If $w \notin J_{[v]}$, then $d(v, w) \geq 2$, so $st(w) \cap J_{[v]} \subseteq L_{[v]}$. If $[v]$ is abelian, $J_{[v]} - st(w)$ is clearly connected. If $[v]$ is nonabelian, then maximality of $[v]$ implies that $st(w) \cap J_{[v]}$ is not all of $L_{[v]}$, so that in this case too $J_{[v]} - st(w)$ is connected. Thus $\phi$ is either trivial on $A_{J_{[v]}}$ or acts as conjugation by $w$ on all of $A_{J_{[v]}}$. If $w \in J_{[v]} = [v] * L_{[v]}$, then conjugation by $w$ preserves the two factors $A_{[v]}$ and $A_{L_{[v]}}$, hence the same is true for the partial conjugation $\phi$.

**Transvections:** Suppose $\phi$ is the transvection $u \mapsto uw$ where $u \in J_{[v]}$ and $u \leq w$. If $u \in [v]$, then maximality of $[v]$ implies $w \in [v]$. If $u \in L_{[v]}$, then $v \in lk(u) \subseteq st(w)$, so $w \in st(v) \subseteq J_{[v]}$. In either case $\phi$ preserves both $A_{[v]}$ and $A_{J_{[v]}}$. \qed

The representative $\phi_v$ is well defined up to conjugation by an element of the normalizer of $A_{J_{[v]}}$. In light of Lemma 2.6, the restriction of $\phi_v$ to $A_{J_{[v]}}$ is well defined up to an inner automorphism of $A_{J_{[v]}}$. Moreover, since $\phi_v$ preserves $A_{[v]}$, it projects to an automorphism of $A_{L_{[v]}} \cong A_{J_{[v]}}/A_{[v]}$. 
Corollary 3.3. For every maximal \([v]\) there is a restriction homomorphism
\[
R_{[v]} : \text{Out}^0(A_\Gamma) \to \text{Out}(A_{J_{[v]}})
\]
and a projection homomorphism
\[
P_{[v]} : \text{Out}^0(A_\Gamma) \to \text{Out}(A_{L_{[v]}})
\]

3.2. The restriction kernel. Consider the product of the restriction homomorphisms taken over all maximal \([v]\),
\[
R = \prod R_{[v]} : \text{Out}^0(A_\Gamma) \to \prod \text{Out}(A_{J_{[v]}}).
\]

Theorem 3.4. The kernel \(K\) of the homomorphism \(R\) is a finitely generated free abelian group.

Proof. If \(\Gamma_0\) consists of a single vertex \([v]\), then \(\Gamma = J_{[v]}\) so the kernel is trivial. So assume that there is more than one maximal equivalence class.

By definition, elements of \(K\) are outer automorphisms. We begin by choosing a canonical automorphism to represent each element of \(K\). First note that for any element \(\phi\) of \(K\) and any maximal \([v]\), we can choose a representative automorphism \(\phi_v\) such that the restriction of \(\phi_v\) to \(A_{J_{[v]}}\) is the identity map. This representative is unique up to conjugation by an element of the centralizer \(C(J_{[v]})\). In particular, if \([v]\) is non-abelian, then \(\phi_v\) is unique.

Suppose \(V_{fr} \neq \emptyset\). Choose a non-abelian class \([y]\). Then for any element \(\phi\) of \(K\), we take \(\phi_0 = \phi_y\) as our canonical representative. If \(V_{fr} = \emptyset\), choose a pair of adjacent maximal classes \([y]\) and \([z]\). Note that \(\phi_y = c(a)\phi_z\) where \(c(a)\) denotes conjugation by an element \(a \in C(J_{y,z})\). By Lemma 2.7, \(a = rs\) for some \(r \in A_{[y]}, s \in A_{L_{[w]}}\). Set
\[
\phi_0 = c(r^{-1})\phi_y = c(s)\phi_z.
\]

Then \(\phi_0\) has the property that (i) it restricts to the identity on vertices of \(J_{[y]}\) and (ii) it acts on \(J_{[z]}\) as conjugation by an element of \(A_{L_{[w]}}\). If \(\phi_1\) is any other representative of \(\phi\) with these two properties, then it differs from \(\phi_0\) by conjugation by an element \(A_{[y]} \cap A_{L_{[w]}} = \{1\}\). Thus, \(\phi_0\) is the unique such representative and we designate it as our canonical representative.

The properties which characterize canonical representatives are preserved under composition, thus the map \(\phi \mapsto \phi_0\) defines a homomorphism of \(K\) into \(Aut(A_\Gamma)\). For the remainder of the proof we view \(K\) as a subgroup of the automorphism group by identifying \(\phi\) with \(\phi_0\).

To prove the theorem we will define a homomorphism \(f\) of \(K\) into a free abelian group and prove that \(f\) is injective. Recall that \(V\) is the set of vertices of \(\Gamma\) so abelianizing gives a homomorphism \(A_\Gamma \to \mathbb{Z}^V\). Denote the abelianization of \(g\) by \(\bar{g}\). For each maximal equivalence class \([v]\), define a homomorphism \(f_v : K \to \mathbb{Z}^{V-[v]}\) as follows. An element \(\phi \in K\), acts on \(J_{[v]}\) as conjugation by some \(g \in A_\Gamma\). The element \(g\) is unique up to multiplication by an element of \(C(J_{[v]}) \subset A_{[w]}\), thus \(\bar{g}\) determines a well defined element of \(\mathbb{Z}^{V-[v]}\) (which by abuse of notation we will also denote \(\bar{g}\)).

We claim that \(f_v(\phi) = \bar{g}\) is a homomorphism. For suppose \(\rho\) is another element of \(K\) and suppose \(\rho\) acts on \(J_{[v]}\) as conjugation by \(h\). Then \(\phi \circ \rho\) act on \(J_{[v]}\) as conjugation by
φ(h)g. Since φ takes every generator to a conjugate of itself, it leaves the abelianization unchanged. That is, (φ(h)) = h, so \( f_v(φ \circ ρ) = φ(h)g = g + h = f_v(φ) + f_v(ρ) \).

Now consider the product homomorphism \( f = \prod f_v \) taken over all maximal equivalence classes \([v]\),
\[
f : K \to \prod \mathbb{Z}^{v-[v]}.
\]
To complete the proof, we will show that \( f \) is injective. Suppose \( φ \) lies in the kernel of \( f \) and suppose \([v]\) and \([w]\) are adjacent maximal classes. If \( φ \) acts on \( J_{[v]} \) as conjugation by \( g_v \) and on \( J_{[w]} \) as conjugation by \( g_w \), then \( g_w^{-1}g_v \) lies in the centralizer of \( J_{[v,w]} \), namely in the abelian group \( Z([v]) \times Z([w]) \times Z(L_{[v]} \cap L_{[w]}) \). Since \( f_v(φ) = g_v = 0 \) and \( f_w(φ) = g_w = 0 \), the exponent sum of any \( u \in L_{[v]} \cap L_{[w]} \) is zero in both \( g_v \) and \( g_w \), hence also in \( g_w^{-1}g_v \). It follows that \( g_w^{-1}g_v \) lies in \( Z([v]) \times Z([w]) \).

Now consider an edge path in \( Γ_0 \) from the base vertex \([y]\) to an arbitrary vertex \([v]\),
\[
[y] = [v_0],[v_1],\ldots,[v_n] = [v],
\]
and suppose \( φ \) acts on \( J_{[v_i]} \) as conjugation by \( g_i \in AΓ \). Our canonical representative \( φ \) was chosen so that \( g_0 = 1 \). By the discussion above, \( g_n = (g_0^{-1}g_1)(g_1^{-1}g_2)\cdots(g_{n-1}^{-1}g_n) \) is of the form \( g_n = a_0a_1a_2\cdots a_n \), where \( a_i \in Z([v_i]) \). Since \( φ \) lies in the kernel of \( f \), \( f_{[v]}(φ) = 0 \), so all of the \( a_i \)'s, except possibly the last one \( a_n \), are trivial. Thus, \( φ \) acts on \( J_{[w]} \) as conjugation by an element of its center, \( Z([v]) \), i.e., \( φ \) acts trivially on \( J_{[v]} \). Since \([v]\) was arbitrary, we conclude that \( φ \) is the identity automorphism. □

In the case of a connected, 2-dimensional graph \( Γ \), a stronger version of this theorem is proved in the authors' previous paper with J. Crisp [CCV]. In that case, we give the exact dimension of \( K \) determine an explicit set of partial conjugations which generate \( K \).

3.3. The projection kernel. Now consider the kernel \( K_P \) of the product homomorphism
\[
P = \prod P_{[v]} : \text{Out}^0(AΓ) \to \prod \text{Out}(A_{L_{[v]}}).
\]
Say a vertex \( v \) is leaf-like if
1. \( L_{[v]} \) contains a unique maximal class \([w]\), and
2. \([v] < [w]\).
Since \( d(v, w) = 1 \), it follows from Lemma 2.34 that \( w \) must belong to \( V_{ab} \). An easy exercise shows that if \( Γ \) is 2-dimensional, then a vertex is leaf-like if and only if it is a leaf (valence 1). Since \([v] < [w]\), there is a transvection \( t(v, w) \) taking \( v \mapsto vw \). We will call this a leaf transvection.

Theorem 3.5. Assume \( Γ_0 \) has at least two vertices. Then the kernel \( K_P \) is a free abelian group generated by \( K \) and the set of leaf transvections.

Proof. From the definition of “leaf-like”, it is easy to see that the leaf transvections \( t(v, w) \) generate a free abelian group contained in the kernel of \( K_P \).

Since every element of \( K \) sends each generator to a conjugate of itself, Theorem 2.2 of [Lau95] says that \( K \) lies in the subgroup generated by partial conjugations. We claim that \( t(v, w) \) commutes with all partial conjugations and hence with all of \( K \). The only case in
which this could fail is the case of a partial conjugation by $u$ where $v$ and $w$ lie in different components of $\Gamma \setminus st(u)$. But this is impossible since $v$ and $w$ are connected by an edge. If the edge lies in $st(u)$, then so do $v$ and $w$, so $u$ commutes with both of them.

It remains to show that $K$ and the leaf transvections generate all of $K_P$. Suppose $\phi \in K_P$. Let $[w]$ be maximal and $\phi_w$ be a representative automorphism preserving $[w]$ and $J_{[w]}$. Then for any $v \in L_{[w]}$, $\phi_w(v) = v g$ for some $g \in A_{[w]}$. If $[u]$ is another maximal equivalence class with $v \in L_{[u]}$, then we also have $\phi_u(v) = vh$ for some $h \in A_{[u]}$. However, $\phi_u$ and $\phi_v$ differ by an inner automorphism. Since no (non-trivial) element of $A_{[u]}$ is conjugate to an element of $A_{[w]}$, this is impossible unless $g = h = 1$.

So suppose that $g \neq 1$ and $[w]$ is the unique maximal equivalence class with $v \in L_{[w]}$ (or equivalently, $[w]$ is the unique maximal class in $L_v$). Since $\phi_w$ preserves centralizers, the centralizer of $v$ must also centralize $g$, that is, $st(v) \subseteq C(g)$. In particular, $[w] \subseteq C(g)$. This is possible only if $w \in V_{ab}$ in which case $C(g) = J_{[w]} = st(w)$. Thus we conclude that $st(v) \subseteq st(w)$, so $[v] < [w]$ and $v$ is leaf-like.

In light of the discussion above, we can compose $\phi$ with an element of the leaf transvection group to obtain an outer automorphism $\tilde{\phi}$ such that for every $[w] \in \Gamma_0$, there is a representative $\tilde{\phi}_w$ which preserves $A_{J_{[w]}}$ and acts as the identity on $L_{[w]}$. If $[v]$ is adjacent to $[w]$ in $\Gamma_0$, then $[w]$ is contained in $L_{[v]}$ so there is another representative $\tilde{\phi}_w$ which preserves $A_{J_{[v]}}$ and acts as the identity on $[w]$. (Here we are using the hypothesis that $\Gamma_0$ contains at least two vertices.) These two representatives differ by an inner automorphism which preserves $A_{J_{[v]}}$, so $\tilde{\phi}_w$ acts on $[w]$ as conjugation by an element $g \in N(J_{[v]}) \subset A_{J_{[w]}}$. Since $[w]$ commutes with $L_{[w]}$, we may assume that $g$ lies in $A_{[w]}$. We conclude that the restriction of $\tilde{\phi}_w$ to $A_{J_{[w]}}$ is conjugation by an element of $A_{[w]}$, an inner automorphism. This holds for all maximal $[w]$, hence $\tilde{\phi}$ lies in $K$.

It remains to consider the special case in which $\Gamma_0$ consists of a single vertex.

**Lemma 3.6.** The following are equivalent

1. $\Gamma_0$ consists of a single vertex $[v]$.
2. $\Gamma = J_{[v]}$ for some $v \in V_{ab}$.
3. The center of $A_{\Gamma}$ is non-trivial.

**Proof.** (2) $\implies$ (3) follows from Lemma 2.6. (3) $\implies$ (1) is clear since if $v \in \Gamma^\perp$, then $st(v) = \Gamma$ so $[v]$ is necessarily the unique maximal class. For (1) $\implies$ (2), note that for any $u \in L_{[v]}$, $[v] \subset lk(u)$. If $[v]$ is non-abelian, then $[v] \not\subseteq st(v)$, so $[u] \not\subseteq [v]$. Hence if $[v]$ is the unique vertex in $\Gamma_0$, it must be abelian, and it follows from Lemma 2.5 that $\Gamma = J_{[v]}$. 

**Proposition 3.7.** If $\Gamma_0$ consists of a single vertex $[v]$, then

$$Out(A_{\Gamma}) = Tr \rtimes (GL(A_{[v]}) \times Out(A_{L_{[v]}}))$$

where $Tr$ is the free abelian group generated by the leaf transvections $t(u,w)$ with $u \in L_{[v]}$ and $w \in [v]$. 

Proof. Any outer automorphism of $A_\Gamma$ preserves the center $A_{[v]}$ and projects via $P_{[v]}$ to $Out(A_{L_{[v]}})$. This gives a homomorphism

$$Out(A_\Gamma) \to GL(A_{[v]}) \times Out(A_{L_{[v]}})$$

which is clearly split surjective. It is easy to check that the kernel of this homomorphism is the group generated by leaf transvections. □

4. Virtual cohomological dimension

If $\Gamma$ is discrete or is a a complete graph, then $A_\Gamma$ is free or free abelian and $Out(A_\Gamma)$ is known to have torsion-free subgroups of finite index, so that the virtual cohomological dimension (vcd) of $Out(A_\Gamma)$ is defined. In this section we prove that the same is true for arbitrary $\Gamma$, and we show furthermore that the vcd of $Out(A_\Gamma)$ is always finite.

The proof proceeds by induction on the dimension of the defining graph $\Gamma$, where the dimension of $\Gamma$ (or of the group $A_\Gamma$) is the maximal rank of a complete subgraph of $\Gamma$. Thus, $\Gamma$ has dimension 1 if and only if $A_\Gamma$ is free. If $\Gamma$ has dimension $n > 1$, then links of vertices in $\Gamma$ have dimension at most $n - 1$. The idea is to use the homomorphism $P$ to do the inductive arguments. The image of $P$ lies in the product of $Out(A_{L_{[v]}})$, for maximal $v$, but the subgraphs $L_{[v]}$ are not, in general, connected, so we must consider the disconnected case. To deal with this we use the work of Guirardel and Levitt [GL07].

If $\Gamma$ has $j$ components consisting of a single point and $k$ components, $\Gamma_1, \ldots, \Gamma_k$ consisting of more than one point, then $A_\Gamma$ splits as a free product

$$A_\Gamma = F_j \ast A_1 \ast \cdots \ast A_k$$

where $F_j$ is a free group and $A_i$ is the right-angled Artin group associated to $\Gamma_i$.

**Theorem 4.1** (Guirardel-Levitt). Suppose $G$ is a group which decomposes as a free product $G = G_1 \ast \cdots \ast G_n$ with at least one factor non-free. Assume that $G_i$ and $G_i/Z(G_i)$ are torsion-free (respectively, have finite virtual cohomological dimension) for all $i$. If the outer automorphism groups $Out(G_i)$ are virtually torsion-free (respectively, have finite virtual cohomological dimension) for each factor $G_i$, then the same is true for $Out(G)$.

**Theorem 4.2.** For any finite simplicial graph $\Gamma$, the group $Out(A_\Gamma)$ is virtually torsion-free and has finite virtual cohomological dimension. Moreover, every torsion-free subgroup of $Out(A_\Gamma)$ has finite cohomological dimension.

**Proof.** We proceed by induction on the dimension of $\Gamma$. If $\Gamma$ has dimension 1, then $A_\Gamma$ is free and the theorem follows from [CV86].

Now suppose that $\Gamma$ is connected and has dimension $\geq 2$. First assume $\Gamma_0$ has more than one vertex and consider the homomorphism $P$ defined in Section 3. By induction, for every maximal $[v]$, $Out(A_{L_{[v]}})$ is virtually torsion-free and has finite vcd, so the same holds for the image of $P$. By Theorem 3.5, the kernel $K_P$ is a finitely generated free abelian group, so in particular, it is torsion-free and has finite cohomological dimension. It now follows immediately that $Out^0(A_\Gamma)$ is virtually torsion-free and by the Serre spectral sequence, it
has finite vcd. Since $Out^0(A_\Gamma)$ is finite index in $Out(A_\Gamma)$, the same holds for the larger group.

If $\Gamma_0$ consists of a unique vertex $[v]$, we use Proposition 3.7. By induction, $Out(A_{L[v]})$ is virtually torsion-free and has finite vcd, and the same is classically true for $Out(A_{[v]}) = GL(A_{[v]})$. Since the transvection group $Tr$ is free abelian, the theorem follows as above.

Finally, applying Theorem 4.1, these results extend to $n$-dimensional graphs $\Gamma$ with more than one component. This completes the induction. □

In [CCV], the authors and J. Crisp studied the case in which $\Gamma$ is connected and 2-dimensional. They obtained explicit upper and lower bounds on the vcd of $Out(A_\Gamma)$ and constructed a contractible “outer space” with a proper $Out(A_\Gamma)$ action.

It was also shown in [CCV] that for connected, 2-dimensional $\Gamma$, $Out(A_\Gamma)$ satisfies the Tits alternative. One would like to do an inductive argument as above to show that this holds for all $\Gamma$. However, in this case, we do not have the analogue of Theorem [GL07] to pass from the connected to the disconnected case.

References

[CCV] Ruth Charney, John Crisp and Karen Vogtmann, Automorphisms of 2-dimensional right-angled Artin groups, to appear in Geometry and Topology.

[CV86] Marc Culler and Karen Vogtmann, Moduli of graphs and automorphisms of free groups, Invent. Math. 84 (1986), no. 1, 91–119.

[DKR] Andrew J Duncan, Ilya V Kazachkov and Vladimir N Remeslennikov, Orthogonal Systems in Finite Graphs, arXiv:0707.0087v1.

[GL07] Vincent Guirardel and Gilbert Levitt, The Outer space of a free product, Proc. Lond. Math. Soc. (3) 94 (2007) 695–714.

[Lau95] Michael R. Laurence, A generating set for the automorphism group of a graph group, J. London Math. Soc. (2) 52 (1995), no. 2, 318–334.

[Ser89] Herman Servatius, Automorphisms of graph groups, J. Algebra 126 (1989), no. 1, 34–60.