SYSTOLIC INEQUALITIES, GINZBURG DG ALGEBRAS AND MILNOR FIBERS

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Abstract. We prove categorical systolic inequalities for the derived categories of 2-Calabi–Yau Ginzburg dg algebras associated to ADE quivers and explore their symplecto-geometric aspects.

1. INTRODUCTION

The systole \( \text{sys}(X, g) \) of a Riemannian manifold \((X, g)\) is defined to be the smallest length of non-contractible loops in \((X, g)\). Loewner showed that, for the 2-torus \( T^2 \), the inequality

\[
\text{sys}(T^2, g)^2 \leq \frac{2}{\sqrt{3}} \text{vol}(T^2, g)
\]

holds for every metric \( g \) on \( T^2 \).

This inequality, known as the systolic inequality for \( T^2 \), can be reinterpreted as follows [3]. First, let us write \( T^2_\tau = \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z}) \) for an element \( \tau \) of the upper half plane and equip \( T^2_\tau \) with the symplectic form \( \omega = \frac{i}{2} dz \wedge d\overline{z} \) and the holomorphic volume form \( \Omega = dz \). With respect to these structures, the special Lagrangian submanifolds in \( T^2_\tau \) are those come from the straight lines in \( \mathbb{C} \) and they coincide with the shortest non-contractible loops in \( T^2_\tau \). Therefore, the inequality (1.1) can be rewritten as

\[
\inf \left\{ \left| \int_L \Omega \right| \mid L \text{ is a special Lagrangian submanifold in } T^2_\tau \right\}^2 \leq \frac{1}{\sqrt{3}} \left| \int_{T^2_\tau} \Omega \wedge \overline{\Omega} \right|
\]

for every \( \tau \).

There is also a categorical interpretation of this inequality due to Fan [3] motivated by a conjecture of Bridgeland [2] and Joyce [8]. For a Calabi–Yau manifold \((X, \omega, \Omega)\), the conjecture asserts that the holomorphic volume form \( \Omega \) should correspond to a stability condition \( \sigma = (Z, P) \) on the derived Fukaya category \( D^\pi \text{Fuk}(X, \omega) \) (more precisely, the conjecture says that the complex moduli of \( X \) can be embedded into a quotient of the space of stability conditions) and, under this correspondence, the \( \sigma \)-semistable objects (resp. the central charge \( Z \)) should correspond to the special Lagrangian submanifolds in \((X, \omega)\) (resp. the period integral \( \int_L \Omega \)).
In view of this conjecture, Fan [3] defined the categorical systole $\text{sys}(\sigma)$ of a stability condition $\sigma = (Z, \mathcal{P})$ of a triangulated category $\mathcal{D}$ to be

$$\text{sys}(\sigma) = \inf \{ |Z(E)| \mid E \text{ is a } \sigma\text{-stable object of } \mathcal{D} \}$$

(Definition 2.5). There is also a notion of the categorical volume $\text{vol}(\sigma)$ of a stability condition $\sigma$ defined by Fan–Kanazawa–Yau [4] (Definition 2.7). Fan [3, Theorem 3.1] then showed the following categorical analogue of the inequality (1.2):

$$\text{sys}(\sigma)^2 \leq \frac{1}{\sqrt{3}} \text{vol}(\sigma)$$

for every stability condition $\sigma$ of $D^sFuk(T^2, \omega)$.

In this paper, we will show categorical inequalities for the derived category $\mathcal{D}_Q$ of 2-Calabi–Yau Ginzburg dg algebras associated to ADE quivers $Q$.

**Theorem 1.1.** Let $Q$ be an ADE quiver with $n$ vertices. Then, for every $\sigma \in \text{Stab}^c(\mathcal{D}_Q)$,

$$\text{sys}(\sigma)^2 \leq \frac{h_Q n + 1}{n} \text{vol}(\sigma)$$

where $h_Q$ is the Coxeter number of the underlying graph of $Q$.

| $Q$ | $A_n$ | $D_n$ | $E_6$ | $E_7$ | $E_8$ |
|-----|-------|-------|-------|-------|-------|
| $h_Q$ | $n + 1$ | $2(n - 1)$ | 12 | 18 | 30 |

We will also see a symplecto-geometric interpretation of Theorem 1.1 for $Q = A_n$. For that case, $\mathcal{D}_Q$ is equivalent to the derived Fukaya category $D^sFuk(X_n)$ of the Milnor fiber $X_n$ of type $A_n$ [18]. Thus, by the Bridgeland–Joyce’s conjecture, the inequality (1.3) should be described in terms of symplectic and complex geometry.

Now let $\mathcal{P}_n$ be the space of polynomials $z^{n+1} + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n \in \mathbb{C}[z]$ with only simple zeros. A polynomial $p \in \mathcal{P}_n$ gives rise to a holomorphic volume form $\Omega_p$ on $X_n$. Then, using $\Omega_p$ and special Lagrangian submanifolds in $X_n$ with respect to it, we can define the systole $\text{sys}(\Omega_p)$ and the volume $\text{vol}(\Omega_p)$.

**Theorem 1.2.** For every $p \in \mathcal{P}_n$,

$$\text{sys}(\Omega_p)^2 \leq \frac{n + 1}{n} \text{vol}(\Omega_p)$$

In Section 2 we review definitions and basic properties of stability conditions, categorical systole and volume. In Section 2.3 we first recall the definition of Ginzburg dg algebras and then prove Theorem 1.1. The main step of the proof is the calculation of the categorical volume (Proposition 3.8). Finally, in Section 4 we prove Theorem 1.2 which is a geometric counterpart of Theorem 1.1 for $Q = A_n$ case.

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Preliminaries

2.1. Stability conditions. Let $\mathcal{D}$ be a $\mathbb{K}$-linear triangulated category. Throughout this section, we assume $\mathcal{D}$ is of finite type, i.e., $\dim \text{Hom}^*(E, F) < \infty$ for all $E, F \in \mathcal{D}$. The Grothendieck group $K(\mathcal{D})$ of $\mathcal{D}$ is the abelian group generated by the objects of $\mathcal{D}$ with the relation $E - F + G$ whenever there is an exact triangle $E \to F \to G \to E[1]$ in $\mathcal{D}$. For $E, F \in \mathcal{D}$, their Euler form is defined by

$$\chi(E, F) = \sum_{i \in \mathbb{Z}} (-1)^i \dim \text{Hom}(E, F[i]).$$

Note that it descends to the map $\chi : K(\mathcal{D}) \times K(\mathcal{D}) \to \mathbb{Z}$ for which we use the same notation. We then define the numerical Grothendieck group by

$$N(\mathcal{D}) = K(\mathcal{D})/\langle E \in K(\mathcal{D}) | \chi(E, -) = 0 \rangle.$$

In what follows, we will also assume that $\text{rk} N(\mathcal{D}) < \infty$.

Definition 2.1 ([1]). A (numerical) stability condition $\sigma = (Z, P)$ on a triangulated category $\mathcal{D}$ consists of a group homomorphism $Z : N(\mathcal{D}) \to \mathbb{C}$ and a full additive subcategory $P(\phi) \subset \mathcal{D}$ for each $\phi \in \mathbb{R}$ which satisfy the following conditions:

1. if $0 \neq E \in P(\phi)$, then $Z(E) \in \mathbb{R}_{> 0}e^{i\pi \phi}$;
2. $P(\phi + 1) = P(\phi)[1]$ for every $\phi \in \mathbb{R}$;
3. if $\phi_1 > \phi_2$ and $E_i \in P(\phi_i)$, then $\text{Hom}(E_1, E_2) = 0$;
4. for every $0 \neq E \in \mathcal{D}$, there exist real numbers $\phi_1 > \cdots > \phi_k$ and $E_i \in P(\phi_i)$ which fit into an iterated exact triangle of the form (which is necessarily unique)

$$\begin{array}{cccccc}
0 & \overset{+1}{\longrightarrow} & \ast & \overset{+1}{\longrightarrow} & \ast & \cdots & \overset{+1}{\longrightarrow} E ; \\
& \downarrow & & \downarrow & & \downarrow & \\
E_1 & & E_2 & & \cdots & & E_k
\end{array}$$

5. there exists a constant $C > 0$ and a norm $\|\cdot\|$ on $N(\mathcal{D}) \otimes_{\mathbb{Z}} \mathbb{R}$ such that

$$\|E\| \leq C |Z(E)|$$

for any $E \in P(\phi)$ and $\phi \in \mathbb{R}$.

From the conditions of the above definition, it follows that the full subcategories $P(\phi)$ are abelian. For a stability condition $\sigma = (Z, P) \in \text{Stab}(\mathcal{D})$, we call $Z$ the central charge and an object (resp. a simple object) in the abelian category $P(\phi)$ to be $(\sigma)$-semistable (resp. $(\sigma)$-stable) of phase $\phi$.

Let us denote by $\text{Stab}(\mathcal{D})$ the space of (numerical) stability conditions on $\mathcal{D}$. Bridgeland [1] introduced a nice topology on $\text{Stab}(\mathcal{D})$ with respect to which the projection

$$\text{Stab}(\mathcal{D}) \to \text{Hom}(N(\mathcal{D}), \mathbb{C}); (Z, P) \mapsto Z$$

becomes a local homeomorphism. Thus the standard complex structure on $\text{Hom}(N(\mathcal{D}), \mathbb{C}) \cong \mathbb{C}^{\text{rk} N(\mathcal{D})}$ induces the one on $\text{Stab}(\mathcal{D})$. 
There are two natural actions on \( \text{Stab}(\mathcal{D}) \). The first one is the action of the group \( \text{Auteq}(\mathcal{D}) \) of exact autoequivalences of \( \mathcal{D} \). For \( \Phi \in \text{Auteq}(\mathcal{D}) \) and \( (Z, \mathcal{P}) \in \text{Stab}(\mathcal{D}) \), it is given by

\[
\Phi \cdot (Z, \mathcal{P}) = (Z \circ \Phi^{-1}, \Phi(\mathcal{P}))
\]

where \( \Phi(\mathcal{P})(\phi) = \Phi(\mathcal{P}(\phi)) \). The second one is the action of \( \mathbb{C} \). For \( \zeta \in \mathbb{C} \) and \( (Z, \mathcal{P}) \in \text{Stab}(\mathcal{D}) \), it is given by

\[
(Z, \mathcal{P}) \cdot \zeta = (e^{-i\pi \zeta}Z, \mathcal{P}[\text{Re}(\zeta)])
\]

where \( \mathcal{P}[\text{Re}(\zeta)](\phi) = \mathcal{P}(\phi + \text{Re}(\zeta)) \).

\[\text{2.2. Hearts and simple tilting.}\]

For the sake of simplicity, let us assume that \( \mathcal{N}(\mathcal{D}) = K(\mathcal{D}) \), i.e., the Euler form is non-degenerate, throughout this subsection.

Let \( \sigma = (Z, \mathcal{P}) \in \text{Stab}(\mathcal{D}) \) be a stability condition. For an interval \( I \subset \mathbb{R} \), define \( \mathcal{P}(I) \subset \mathcal{D} \) to be the smallest full extension closed subcategory containing \( \mathcal{P}(\phi) \) for all \( \phi \in I \). Then one can show that \( \mathcal{D}_{\geq 0} = \mathcal{P}(0, \infty) \) is a bounded t-structure on \( \mathcal{D} \). We call the heart \( \mathcal{A}_\sigma = \mathcal{P}(0, 1) \) of this bounded t-structure the heart of the stability condition \( \sigma \).

Let \( \mathcal{A} \) be an abelian category. A stability function on \( \mathcal{A} \) is a group homomorphism \( Z : K(\mathcal{A}) \to \mathbb{C} \) such that for every \( 0 \neq E \in \mathcal{A} \), \( Z(E) \) lies in \( \mathbb{H} = \{re^{i\pi \phi} \in \mathbb{C} \mid r \in \mathbb{R}_{>0} \text{ and } \phi \in (0, 1]\} \).

For a stability function \( Z : K(\mathcal{A}) \to \mathbb{C} \) and \( 0 \neq E \in \mathcal{A} \), define the phase of \( E \) as

\[
\phi(E) = \frac{1}{\pi} \arg Z(E) \in (0, 1].
\]

Then we call an object \( 0 \neq E \in \mathcal{A} \) to be \( (Z-)\text{semistable} \) (resp. \( (Z-)\text{stable} \)) if for every \( 0 \neq F \subsetneq E \) one has \( \phi(F) \leq \phi(E) \) (resp. \( \phi(F) < \phi(E) \)). If \( \mathcal{A} \) has a property analogous to Definition \([2.1](4)\) with respect to \( Z\)-semistable objects, we say \( Z \) has the Harder–Narasimhan property.

A stability condition can be thought of as a refinement of the heart of a bounded t-structure in the following sense.

**Proposition 2.2** (\([1\text{ Proposition 5.3}])\). To give a stability condition on \( \mathcal{D} \) is equivalent to give a bounded t-structure on \( \mathcal{D} \) and a stability function \( Z \) on its heart with the Harder–Narasimhan property.

**Proof.** For a stability condition \( \sigma = (Z, \mathcal{P}) \in \text{Stab}(\mathcal{D}) \), the corresponding t-structure and stability function are given by \( \mathcal{D}_{\geq 0} = \mathcal{P}(0, \infty) \) and \( Z : K(\mathcal{A}_\sigma) \cong K(\mathcal{D}) \to \mathbb{C} \). For details, see \([1\text{ Proposition 5.3}])\].

We denote by \( \text{Stab}_\mathcal{A}(\mathcal{D}) \subset \text{Stab}(\mathcal{D}) \) the space of stability conditions on \( \mathcal{D} \) whose heart is \( \mathcal{A} \). If \( \mathcal{A} \) has a finiteness property, \( \text{Stab}_\mathcal{A}(\mathcal{D}) \) can be easily described.

**Lemma 2.3** (\([2\text{ Lemma 5.2}])\). Let \( \mathcal{A} \) be the heart of a bounded t-structure on \( \mathcal{D} \). Suppose \( \mathcal{A} \) is a finite length abelian category with \( n \) simple objects \( S_1, \ldots, S_n \). Then \( \text{Stab}_\mathcal{A}(\mathcal{D}) \) is isomorphic to \( \mathbb{H}^n \).
Proof. The isomorphism is given by $\text{Stab}_A(D) \ni \sigma = (Z, \mathcal{P}) \mapsto (Z(S_i))_{i=1}^n \in \mathbb{H}^n$. For details, see [2 Lemma 5.2].

Let $A$ be the heart of a bounded t-structure on $D$ and $(\mathcal{T}, \mathcal{F})$ be a torsion pair on it. Define a full additive subcategory $A^\sharp$ (resp. $A^\flat$) $\subset D$ whose objects are those $E \in D$ such that

$$H^{-1}(E) \in \mathcal{F}, H^0(E) \in \mathcal{T} \text{ and } H^i(E) = 0 \text{ for all } i \neq -1, 0$$

(resp. $H^0(E) \in \mathcal{F}, H^1(E) \in \mathcal{T} \text{ and } H^i(E) = 0 \text{ for all } i \neq 0, 1$)

where $H^i$ denotes the $i$th cohomology with respect to the bounded t-structure corresponding to $A$. Then $A^\sharp$ (resp. $A^\flat$) is again the heart of a bounded t-structure with the torsion pair $(\mathcal{F}[1], \mathcal{T})$ (resp. $(\mathcal{F}, \mathcal{T}[−1])$) [6]. We call $A^\sharp$ (resp. $A^\flat$) the forward tilt (resp. backward tilt) of $A$ with respect to $(\mathcal{T}, \mathcal{F})$. In the case that $\mathcal{F}$ (resp. $\mathcal{T}$) is generated by a single simple object $S \in A$, the corresponding forward tilt (resp. backward tilt) is called simple and denoted by $A^\sharp_S$ (resp. $A^\flat_S$).

Denote by $\text{Sim}(A)$ the set of simple objects in an abelian category $A$.

**Proposition 2.4** ([12 Proposition 5.4]). Let $A$ be the heart of a bounded t-structure on $D$ which is of finite length and $S \in A$ be a rigid simple object. Then,

$$\text{Sim}(A^\sharp_S) = \{S[1]\} \cup \{\Phi^\sharp_S(M) \mid S \neq M \in \text{Sim}(A)\},$$

$$\text{Sim}(A^\flat_S) = \{S[-1]\} \cup \{\Phi^\flat_S(M) \mid S \neq M \in \text{Sim}(A)\}$$

where

$$\Phi^\sharp_S(M) = \text{Cone}(M \to \text{Hom}(M, S[1])^\vee \otimes S[1])[-1],$$

$$\Phi^\flat_S(M) = \text{Cone}(\text{Hom}(S[-1], M) \otimes S[-1] \to M).$$

In particular, $A^\sharp_S, A^\flat_S$ are again of finite length with $|\text{Sim}(A^\sharp_S)| = |\text{Sim}(A^\flat_S)| = |\text{Sim}(A)|$.

2.3. **Categorical systole and volume.** Let $(X, \omega, \Omega)$ be a compact Calabi–Yau manifold with a symplectic form $\omega$ and a holomorphic volume form $\Omega$. The systole of $(X, \omega, \Omega)$ can be defined as

$$\text{sys}(X, \omega, \Omega) = \inf \left\{ \left\| \int_{L} \Omega \right\| \mid L \text{ is a special Lagrangian submanifold in } (X, \omega, \Omega) \right\}.$$

In view of this observation and the conjectural description of stability conditions on Fukaya categories [2] [8] mentioned in the introduction, Fan [3] introduced a categorical analogue of systole.

**Definition 2.5** ([3]). The categorical systole of $\sigma = (Z, \mathcal{P}) \in \text{Stab}(D)$ is defined by

$$\text{sys}(\sigma) = \inf \{||Z(E)|| \mid E \text{ is a } \sigma\text{-stable object of } D\}.$$

**Remark 2.6.** By the condition Definition [21] (5), $\text{sys}(\sigma) > 0$ for any $\sigma \in \text{Stab}(D)$. 
On the other hand, the volume of \((X, \omega, \Omega)\) is given by

\[
\text{vol}(X, \omega, \Omega) = \left| \int_X \Omega \wedge \overline{\Omega} \right|
\]

For a basis \(L_1, \ldots, L_k\) of \(H_d(X, \mathbb{Z})/\text{Torsion}\) (where \(d = \dim \mathbb{C}X\)), this can be rewritten as

\[
\text{vol}(X, \omega, \Omega) = \left| \sum_{i,j=1}^k \gamma_{ij} \int_{L_i} \Omega \int_{L_j} \overline{\Omega} \right|
\]

where \(\gamma_{ij}\) is the \((i, j)\)-component of the inverse matrix of the intersection matrix \((L_i \cdot L_j)_{i,j}\).

This leads to the following definition.

**Definition 2.7** ([4]). Fix a basis \(E_1, \ldots, E_k\) of \(N(D)\). The categorical volume of \(\sigma = (Z, P) \in \text{Stab}(D)\) is defined by

\[
\text{vol}(\sigma) = \left| \sum_{i,j=1}^k \chi_{ij} Z(E_i) \overline{Z(E_j)} \right|
\]

where \(\chi_{ij}\) is the \((i, j)\)-component of the inverse matrix of the matrix \((\chi(E_i, E_j))_{i,j}\).

The following can be easily checked.

**Lemma 2.8** ([3, Lemmas 2.7 and 2.12]). For \(\Phi \in \text{Auteq}(D)\), \(\zeta \in \mathbb{C}\) and \(\sigma \in \text{Stab}(D)\),

1. \(\text{sys}(\Phi \cdot \sigma) = \text{sys}(\sigma)\);
2. \(\text{sys}(\sigma \cdot \zeta) = e^{2\pi \text{Im}(\zeta)} \text{sys}(\sigma)\);
3. \(\text{vol}(\Phi \cdot \sigma) = \text{vol}(\sigma)\);
4. \(\text{vol}(\sigma \cdot \zeta) = e^{2\pi \text{Im}(\zeta)} \text{vol}(\sigma)\).

**Corollary 2.9.** The map \(\text{vol}/\text{sys}^2\) defines a map from \(\text{Auteq}(D)\setminus\text{Stab}(D)/\mathbb{C}\) to \([0, \infty)\).

### 3. Categorical systolic inequalities

#### 3.1. Ginzburg dg algebras.

Let \(Q\) be an ADE quiver, i.e., a quiver whose underlying graph is an ADE graph. Let \(Q_0 = \{1, \ldots, n\}\) be the set of vertices of \(Q\) and \(Q_1\) be the set of arrows of \(Q\). The \(2\text{-Calabi–Yau Ginzburg dg algebra} \Gamma_Q = (\mathbb{K}\hat{Q}, d)\) associated to \(Q\) is defined as follows [5]. First, \(\mathbb{K}\hat{Q}\) is the graded path algebra of the extended quiver \(\hat{Q}\) with vertices \(\hat{Q}_0 = Q_0\) with the following arrows:

- the original arrow \(a \in Q_1\) (degree 0);
- the opposite arrow \(a^* : j \to i\) for each \(a : i \to j \in Q_1\) (degree 0);
- a loop \(t_i : i \to i\) for each \(i \in Q_0\) (degree −1).

The differential \(d : \mathbb{K}\hat{Q} \to \mathbb{K}\hat{Q}\) is then defined by

- \(da = da^* = 0\) for every \(a \in Q_1\);
- \(dt_i = \sum_{a \in Q_1} e_i(aa^* - a^*a)e_i\) where \(e_i\) denotes the constant path at \(i \in Q_0\).
Let $D(\Gamma_Q)$ be the derived category of dg modules over $\Gamma_Q$. The finite-dimensional derived category $D_Q = D_{\text{fd}}(\Gamma_Q)$ is defined to be the full triangulated subcategory of $D(\Gamma_Q)$ whose objects consist of dg modules $M$ such that $\dim H^*(M) < \infty$.

**Theorem 3.1** ([9 Theorem 6.3]). The category $D_Q$ is 2-Calabi–Yau, i.e., there is an isomorphism

$$\text{Hom}(M, N) \cong \text{Hom}(N, M[2])^\vee$$

which is functorial in both $M, N \in D_Q$.

There are $n$ simple dg modules $S_1, \ldots, S_n$ corresponding to each of the $n$ vertices of $Q$. It turns out that they generate $D_Q$ and their configuration can be described as follows.

**Proposition 3.2** ([10 Lemma 2.15]). The category $D_Q$ is generated by $S_1, \ldots, S_n$ and

$$\text{Hom}^*(S_i, S_j) = \begin{cases} \mathbb{K} \oplus \mathbb{K}[-2] & (i = j) \\ \mathbb{K}[-1] & (i \sim j \text{ in } Q) \\ 0 & \text{(otherwise)} \end{cases}$$

where $i \sim j$ in $Q$ if and only if $i$ is adjacent to $j$ in $Q$.

Let $D_Q^{\leq 0} \subset D_Q$ be the full subcategory consisting of dg modules $M$ with $H^i(M) = 0$ for all $i > 0$. This determines a bounded t-structure and its heart $A_{\text{can}}$ is of finite length and $\text{Sim}(A_{\text{can}}) = \{S_1, \ldots, S_n\}$.

Let us denote by $\text{Stab}^0(D_Q) \subset \text{Stab}(D_Q)$ the connected component containing $\text{Stab}A_{\text{can}}(D_Q)$. The following can be proved using [17 Theorem 2.12].

**Proposition 3.3** ([14 Corollary 5.3]). For an ADE quiver $Q$,

$$\text{Stab}^0(D_Q) = \bigcup A \text{ Stab}_A(D_Q)$$

where the union is over all hearts $A$ obtained from $A_{\text{can}}$ by iterated simple forward/backward tilts.

By Proposition 3.2 $S_1, \ldots, S_n$ are spherical objects in the sense of [15]. Thus each $S_i$ defines an exact autoequivalence $\Phi_i$ called the spherical twist which acts on $M \in D_Q$ as

$$\Phi_i(M) = \text{Cone}(\text{Hom}^*(S_i, M) \otimes S_i \to M),$$

$$\Phi_i^{-1}(M) = \text{Cone}(M \to \text{Hom}^*(M, S_i)^\vee \otimes S_i)[-1].$$

Let $\text{Sph}(D_Q)$ be the subgroup of $\text{Auteq}(D_Q)$ generated by $\Phi_1, \ldots, \Phi_n$.

**Corollary 3.4.** For an ADE quiver $Q$,

$$\text{Stab}^0(D_Q) = \bigcup_{\Phi \in \text{Sph}(D_Q)} \Phi \cdot \text{Stab}_{A_{\text{can}}}(D_Q).$$

**Proof.** Let $A_i^\sharp$ (resp. $A_i^\flat$) be the simple forward tilt (resp. backward tilt) of $A_{\text{can}}$ with respect to $S_i$. By Proposition 3.2 it follows that $\Phi_i^\sharp (M) = \Phi_i^{-1} (M)$ and $\Phi_i^\flat (M) = \Phi_i (M)$.
for all $S_i \neq M \in \text{Sim}(\mathcal{A}_{\text{can}})$. Since $\Phi_i(S_i) = S_i[-1]$, this implies that $\text{Sim}(\mathcal{A}_i^j) = \Phi_i^{-1}(\text{Sim}(\mathcal{A}_{\text{can}}))$ and $\text{Sim}(\mathcal{A}_i^j) = \Phi_i(\text{Sim}(\mathcal{A}_{\text{can}}))$ by Proposition 3.4. Therefore $\mathcal{A}_i^j = \Phi_i^{-1}(\mathcal{A}_{\text{can}})$, $\mathcal{A}_i^j = \Phi_i(\mathcal{A}_{\text{can}})$ and

$$\text{Stab}_{\mathcal{A}_i^j}(\mathcal{D}_Q) = \text{Stab}_{\Phi_i^{-1}(\mathcal{A}_{\text{can}})}(\mathcal{D}_Q) = \Phi_i^{-1} \cdot \text{Stab}_{\mathcal{A}_{\text{can}}}(\mathcal{D}_Q),$$

$$\text{Stab}_{\mathcal{A}_i^j}(\mathcal{D}_Q) = \text{Stab}_{\Phi_i(\mathcal{A}_{\text{can}})}(\mathcal{D}_Q) = \Phi_i \cdot \text{Stab}_{\mathcal{A}_{\text{can}}}(\mathcal{D}_Q).$$

The assertion follows by iterating this process and applying Proposition 3.3. □

Remark 3.5. If we consider the $d$-Calabi–Yau Ginzburg dg algebra $\Gamma_Q$ for general $d \geq 2$, we have $\mathcal{A}_i^{(d-1)} = \Phi_i^{-1}(\mathcal{A}_{\text{can}})$ and $\mathcal{A}_i^{(d-1)} = \Phi_i(\mathcal{A}_{\text{can}})$ where $\mathcal{A}_i^{(k)}$ and $\mathcal{A}_i^{(k)}$ are defined inductively by $\mathcal{A}_i^{(k)} = (\mathcal{A}_i^{(k-1)})^z_{S_i S_i}$ and $\mathcal{A}_i^{(k)} = (\mathcal{A}_i^{(k-1)})^z_{S_i S_i-1}$ (see [12], Corollary 8.4]). Accordingly, the description of $\text{Stab}_{\mathcal{A}_i^j}(\mathcal{D}_Q)$ becomes more complicated when $d \geq 3$. In this paper, we restrict our attention to the case $d = 2$ to make computations manageable.

3.2. Proof of Theorem 1.1. By Corollaries 2.9 and 3.4, it is enough to prove Theorem 1.1 for the stability conditions in $\text{Stab}_{\mathcal{A}_{\text{can}}}(\mathcal{D}_Q) \subset \text{Stab}_{\mathcal{D}_Q}$.  

Note that $K(\mathcal{D}_Q) \cong K(\mathcal{A}_{\text{can}}) \cong \mathbb{Z}^n$ is generated by the classes of $S_1, \ldots, S_n$. Moreover, by Proposition 3.2, $(\chi(S_i, S_j))_{i,j}$ is the Cartan matrix of the underlying graph of $Q$. In particular, it is non-degenerate so $\mathcal{N}(\mathcal{D}_Q) = K(\mathcal{D}_Q)$.

Definition 3.6. Let $\Delta_Q^+$ be the subset of $K(\mathcal{D}_Q) \cong \mathbb{Z}^n$ consisting of those elements corresponding to the positive roots of the underlying graph of $Q$ so that $S_1, \ldots, S_n$ correspond to the simple roots.

Example 3.7. Let $Q$ be an $A_3$ quiver. Then,

$$\Delta_Q^+ = \{S_1, S_2, S_3, S_1 + S_2, S_2 + S_3, S_1 + S_2 + S_3\} \subset K(\mathcal{D}_Q).$$

Proposition 3.8. Let $Q$ be an ADE quiver with $n$ vertices. Then, for any $\sigma = (Z, P) \in \text{Stab}_{\mathcal{A}_{\text{can}}}(\mathcal{D}_Q)$,

$$\text{vol}(\sigma) = \sum_{i,j=1}^n \chi_{ij} Z(S_i)Z(S_j) = \frac{1}{h_Q} \sum_{M \in \Delta_Q^+} |Z(M)|^2$$

(3.1)

where $h_Q$ is the Coxeter number of the underlying graph of $Q$.

Proof. It suffices to prove the second equality of (3.1). Indeed, it implies the middle of (3.1) is non-negative so the first equality follows from the definition of the categorical volume.

We will show the second equality of (3.1) by comparing the coefficients of $Z(S_i)Z(S_j)$ in both sides for every $i \leq j$. More specifically, we shall show that

$$\chi_{ij} = \frac{1}{h_Q} \sum_{M \in \Delta_Q^+} c_i(M)c_j(M)$$

(3.2)

for all $1 \leq i \leq j \leq n$, where $c_i(M) \in \mathbb{Z}_{\geq 0}$ denotes the coefficient of $S_i$ in $M$. 

(1) Let $Q$ be an $A_n$ quiver. Label the vertices of $Q$ as follows (the orientations of the arrows are suppressed from the picture):

\[1 \rightarrow 2 \rightarrow 3 \rightarrow \ldots \rightarrow n-1 \rightarrow n\]

Then $\chi^{ij}$, which is the $(i, j)$-component of the inverse matrix of the Cartan matrix $(\chi(S_i, S_j))_{i,j}$, is given by

$$\chi^{ij} = \min\{i, j\} - \frac{ij}{n+1}.$$  

Moreover $\Delta^+_Q$ consists of $\frac{n(n+1)}{2}$ elements $R_{ij} = S_i + \cdots + S_j$ ($1 \leq i \leq j \leq n$).

Now fix $i \leq j$. Then $c_i(R_{kl})c_j(R_{kl}) \neq 0$ if and only if $c_i(R_{kl})c_j(R_{kl}) = 1$ if and only if $k \leq i \leq j \leq l$. This implies that

$$\frac{1}{h_Q} \sum_{M \in \Delta^+_Q} c_i(M)c_j(M) = \frac{1}{n+1} \cdot i(n+1-j) = i - \frac{ij}{n+1} = \chi^{ij}.$$

(2) Let $Q$ be a $D_n$ quiver. Label the vertices of $Q$ as follows:

\[1 \rightarrow 2 \rightarrow 3 \rightarrow \ldots \rightarrow n-2 \rightarrow n-1 \rightarrow n\]

Then $\chi^{ij}$ for $i \leq j$ is given by

$$\chi^{ij} = \begin{cases} 
  i & (1 \leq i \leq j \leq n-2) \\
  \frac{i}{2} & (1 \leq i \leq n-2, j = n-1, n) \\
  \frac{n-2}{4} & ((i, j) = (n-1, n)) \\
  \frac{n}{4} & ((i, j) = (n-1, n-1), (n, n))
\end{cases}$$

and those for $i > j$ can be obtained from this using $\chi^{ij} = \chi^{ji}$. Moreover $\Delta^+_Q$ consists of $n(n-1)$ elements. Concretely, $\Delta^+_Q$ consists of $\frac{(n-2)(n-1)}{2}$ elements of the form $R_{ij} = S_i + \cdots + S_j$ ($1 \leq i \leq j \leq n-2$), $n-1$ elements of the form $R^+_i = S_i + \cdots + S_{n-2} + S_{n-1}$ ($1 \leq i \leq n-1$), $n-1$ elements of the form $R^-_i = S_i + \cdots + S_{n-2} + S_n$ ($1 \leq i \leq n-1$) and $\frac{(n-2)(n-1)}{2}$ elements of the form $R^a_i = S_i + \cdots + S_{n-a-2} + 2S_{n-a-1} + \cdots + 2S_{n-2} + S_{n-1} + S_n$ ($1 \leq i \leq n-a-2$) where $0 \leq a \leq n-3$. 
• Let \((i, j) = (n - 1, n - 1)\) (the case \((i, j) = (n, n)\) can be treated in the same way). The elements of \(\Delta_Q^+\) that contribute to the right hand side of (3.2) are \(R_k^+, R_k^a\) and each of them contributes 1. Therefore
\[
\frac{1}{h_Q} \sum_{M \in \Delta_Q^+} c_i(M) c_j(M) = \frac{1}{2(n-1)} (n-1) + \frac{(n-2)(n-1)}{2} = \frac{n}{4} = \chi_{ij}.
\]

• Next, let \((i, j) = (n - 1, n)\). In this case, the elements of \(\Delta_Q^+\) that contribute to the right hand side of (3.2) are \(R_k^a\) and each of them contributes 1. Thus
\[
\frac{1}{h_Q} \sum_{M \in \Delta_Q^+} c_i(M) c_j(M) = \frac{1}{2(n-1)} \cdot \frac{(n-2)(n-1)}{2} = \frac{n-2}{4} = \chi_{ij}.
\]

• Now fix \(1 \leq i \leq n - 2, j = n - 1\) (again, the case \(j = n\) can be treated similarly). The elements of \(\Delta_Q^+\) that contribute to the right hand side of (3.2) are \(R_k^+, R_k^a\) \((k \leq i)\). There are \(i\) such \(R_k^+\) and each of them contributes 1. On the other hand, if \(0 \leq a \leq n - i - 2\), each \(R_k^a\) contributes 1 and the number of such \(R_k^a\) is \((n - i - 1)i\). Moreover, if \(n - i - 1 \leq a \leq n - 3\), each \(R_k^a\) contributes 2 and the number of such \(R_k^a\) is \((i-1)i\). This shows that
\[
\frac{1}{h_Q} \sum_{M \in \Delta_Q^+} c_i(M) c_j(M) = \frac{1}{2(n-1)} \left[ i + (n - i - 1)i + 2 \cdot \frac{(i-1)i}{2} \right] = \frac{i}{2} = \chi_{ij}.
\]

• Finally, fix \(1 \leq i \leq j \leq n - 2\). The elements of \(\Delta_Q^+\) that contribute to the right hand side of (3.2) are \(R_{kl}, R_k^+, R_k^a, R_k^+\) \((k \leq i \leq j \leq l)\). Among them, each \(R_{kl}\) (resp. \(R_k^+\)) contributes 1 and there are \((n - j - 1)i\) (resp. \(i\)) such elements. On the other hand, each \(R_k^a\) contributes 1 if \(0 \leq a \leq n - j - 2\) and there are \((n - j - 1)i\) such \(R_k^a\). Moreover, if \(n - j - 1 \leq a \leq n - i - 2\), each \(R_k^a\) contributes 2 and the number of such \(R_k^a\) is \((j-i)i\). For the remaining case \(n - i - 1 \leq a \leq n - 3\), each \(R_k^a\) contributes 4 and there are \((i-1)i\) such \(R_k^a\). Consequently, we get
\[
\frac{1}{h_Q} \sum_{M \in \Delta_Q^+} c_i(M) c_j(M) = \frac{1}{2(n-1)} \left[ (n - j - 1)i + i + (n - j - 1)i + 2 \cdot (j-i)i + 4 \cdot \frac{(i-1)i}{2} \right] = \chi_{ij}.
\]

(3) The exceptional cases \(E_6, E_7, E_8\) can be verified by direct computations. \(\Box\)

Proof of Theorem 3.4. Let \(\sigma = (Z, P) \in \text{Stab}_{A_{can}}(D_Q)\). Since \(S_1, \ldots, S_n\) are simple in \(A_{can}\), they are stable for any \(\sigma \in \text{Stab}_{A_{can}}(D_Q)\). Therefore
\[
\text{sys}(\sigma) = \inf \{|Z(M)| \mid M \text{ is a } \sigma\text{-stable object of } D_Q\} \leq \inf_{1 \leq i \leq n} |Z(S_i)|.
\]
Then, by Proposition 3.8,

$$\text{vol}(\sigma) = \frac{1}{h_Q} \sum_{M \in \Delta_0} |Z(M)|^2 \geq \frac{1}{h_Q} \sum_{i=1}^{n} |Z(S_i)|^2$$

$$\geq \frac{n}{h_Q} \inf_{1 \leq i \leq n} |Z(S_i)|^2 \geq \frac{n}{h_Q} \text{sys}(\sigma)^2$$

as desired. \(\square\)

4. Geometric viewpoint

4.1. Milnor fibers. Let \(P_n\) be the space of polynomials

$$z^{n+1} + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n \in \mathbb{C}[z]$$

with only simple zeros. This can be identified with the configuration space \(\text{Conf}^{n+1}_{n+1}(\mathbb{C})\) of \(n+1\) points in \(\mathbb{C}\) with the center of mass 0 by sending \((z - \zeta_1) \cdots (z - \zeta_{n+1}) \in P_n\) to \([\zeta_1, \ldots, \zeta_{n+1}] \in \text{Conf}^{n+1}_{n+1}(\mathbb{C})\).

For \(p \in P_n\), we consider the associated Milnor fiber of type \(A_n\)

$$X_p = \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^2 = p(z)\}$$

equipped with the symplectic form \(\omega_p\) restricted from the standard symplectic form \(\omega_{\text{std}} = \frac{i}{2}(dx \wedge d\overline{x} + dy \wedge d\overline{y} + dz \wedge d\overline{z})\) on \(\mathbb{C}^3\). It is known that the symplectomorphism type of \((X_p, \omega_p)\) does not depend on the choice of \(p \in P_n\). Indeed, all of them are symplectomorphic to the \(A_n\)-plumbing of the cotangent bundles of \(S^2\).

Now consider the projection \(\pi : X_p \to \mathbb{C}; (x, y, z) \mapsto z\). Let \(\Delta_p \subset \mathbb{C}\) be the set of zeroes of \(p\) and

$$\Sigma_{p, \zeta} = \{((\sqrt{p}(\zeta) \cos \theta, \sqrt{p}(\zeta) \sin \theta, \zeta) \in \mathbb{C}^3 \mid \theta \in S^1 = \mathbb{R}/2\pi \mathbb{Z} \subset \pi^{-1}(\zeta)\)$$

(where \(\sqrt{p}\) is a suitably chosen smooth square root of \(p\)). Note that, for \(\zeta \in \Delta_p\) (resp. \(\zeta \in \mathbb{C} \setminus \Delta_p\)), \(\Sigma_{p, \zeta}\) is a point (resp. a circle). A simple curve \(\gamma : [0, 1] \to \mathbb{C}\) such that \(\gamma^{-1}(\Delta_p) = \{0, 1\}\) and \(\gamma(0) \neq \gamma(1)\) will be called a matching path. For a matching path \(\gamma\), we define the matching cycle associated to \(\gamma\) by

$$L_\gamma = \bigcup_{t \in [0, 1]} \Sigma_{p, \gamma(t)}.$$

This is a Lagrangian sphere of \((X_p, \omega_p)\) and isotopic matching paths give Hamiltonian isotopic matching cycles [11, Lemma 6.12].

Theorem 4.1 ([18, Theorem 1]). Every exact Lagrangian submanifold in \((X_p, \omega_p)\) is Hamiltonian isotopic to a matching cycle.

Note that a matching cycle is invariant under the \(U(1)\)-action on \(X_p\) given by

$$e^{i\theta} \cdot (x, y, z) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta, z).$$

It turns out that the converse also holds.

Lemma 4.2. An exact Lagrangian submanifold in \((X_p, \omega_p)\) is \(U(1)\)-invariant if and only if it is a matching cycle.
Proof. Every $U(1)$-orbit is contained in $\pi^{-1}(\zeta)$ for some $\zeta \in \mathbb{C}$. For $\zeta \in \Delta_p$, a $U(1)$-orbit inside $\pi^{-1}(\zeta)$ is of the form $\{re^{i\theta}, ire^{i\theta}, \zeta\} \subset \mathbb{C}^3 \mid \theta \in S^1$ or $\{re^{i\theta}, -ire^{i\theta}, \zeta\} \subset \mathbb{C}^3 \mid \theta \in S^1$ for some $r \in \mathbb{R}_{>0}$. On the other hand, for $\zeta \in \mathbb{C} \setminus \Delta_p$, a $U(1)$-orbit inside $\pi^{-1}(\zeta)$ can be written as

$$\left\{ \left( \frac{\sqrt{p(\zeta)}}{2} (re^{i\theta} + \frac{1}{r}e^{-i\theta}), \frac{\sqrt{p(\zeta)}}{2i} (re^{i\theta} - \frac{1}{r}e^{-i\theta}), \zeta \right) \mid \theta \in S^1 \right\}$$

for some $r \in \mathbb{R}_{>0}$. This can be seen from the $U(1)$-equivariant diffeomorphism

$$\mathbb{C}^* \to \pi^{-1}(\zeta); \ u \mapsto \left( \frac{\sqrt{p(\zeta)}}{2} \left( u + \frac{1}{u} \right), \frac{\sqrt{p(\zeta)}}{2i} \left( u - \frac{1}{u} \right), \zeta \right)$$

where $U(1)$ acts on $\mathbb{C}^*$ by $e^{i\theta} \cdot u = u e^{i\theta}$.

Let $L$ be a $U(1)$-invariant exact Lagrangian submanifold in $(X_p, \omega_p)$ which is necessarily a 2-sphere by Theorem 4.1. Moreover, by the above description of the $U(1)$-orbits, the projected image $\pi(L)$ must be the image of a curve $\gamma : [0, 1] \to \mathbb{C}$ such that $\gamma(0), \gamma(1) \in \Delta_p$, $\gamma(0) \neq \gamma(1)$ and $(0, 0, \gamma(0)), (0, 0, \gamma(1)) \in L$. Take $0 < \varepsilon < 1$ so that $\gamma(t) \in \mathbb{C} \setminus \Delta_p$ for all $0 < t < \varepsilon$. Near $(0, 0, \gamma(0)) \in L$, let us parametrize $L$ by $\iota : (0, \varepsilon) \times S^1 \to X_p$ as follows:

$$\iota(t, \theta) = (x(t, \theta), y(t, \theta), \gamma(t))$$

$$= \left( \frac{\sqrt{p(\gamma(t))}}{2} (r(t)e^{i\theta} + \frac{1}{r(t)}e^{-i\theta}), \frac{\sqrt{p(\gamma(t))}}{2i} (r(t)e^{i\theta} - \frac{1}{r(t)}e^{-i\theta}), \gamma(t) \right)$$

for some smooth function $r : (0, \varepsilon) \to \mathbb{R}_{>0}$. Below, we will see the Lagrangian condition $\iota^*\omega_p = 0$ implies that $\varepsilon$ can be taken to be 1 and $r(t) = 1$ for all $0 < t < \varepsilon = 1$. Then it follows that $\gamma$ is simple, $\gamma^{-1}(\Delta_p) = \{0, 1\}$, $\iota(t, S^1) = \Sigma_{p, \gamma(t)}$ for all $t \in [0, 1]$ and therefore $L = L_\gamma$.

By a direct computation, we have

$$\iota^*\omega_p = \frac{d}{dt} \left[ \frac{1}{4} |p(\gamma(t))| \left( r(t)^2 - \frac{1}{r(t)^2} \right) \right] dt \wedge d\theta$$

and so $\iota^*\omega_p = 0$ if and only if

$$\frac{1}{4} |p(\gamma(t))| \left( r(t)^2 - \frac{1}{r(t)^2} \right) = c$$

is a constant. On the other hand,

$$\frac{1}{4} |p(\gamma(t))| \left( r(t)^2 - \frac{1}{r(t)^2} \right) = \text{Im}(x(t, \theta)\bar{y}(t, \theta)) \to 0 \quad (t \to 0).$$

This shows that $c = 0$ and therefore $r(t) = 1$ for all $0 < t < \varepsilon$. A similar argument shows that $\varepsilon$ can be taken to be 1 (otherwise $L$ becomes an $A_k$-chain of 2-spheres).
4.2. Special Lagrangian submanifolds. We call \((X, \omega, \Omega)\) an almost Calabi-Yau manifold if \((X, \omega)\) is a (not necessarily compact) Kähler manifold and \(\Omega\) is a holomorphic volume form. Recall that a special Lagrangian submanifold of \((X, \omega, \Omega)\) is a Lagrangian submanifold \(L\) of \((X, \omega)\) such that \(\text{Im}(e^{-i\phi}\Omega|_L) = 0\) for some \(\phi \in \mathbb{R}\) called the phase.

As before, we define the systole of \((X, \omega, \Omega)\) by

\[
\text{sys}(\Omega) = \text{sys}(X, \omega, \Omega) = \inf \left\{ \left| \int_L \Omega \right| \mid L \text{ is a special Lagrangian submanifold in } (X, \omega, \Omega) \right\}.
\]

Now, for simplicity, assume that the intersection product on \(H^d(X, \mathbb{Z})/\text{Torsion}\) (where \(d = \dim_{\mathbb{C}} X\)) is non-degenerate. Fixing a basis \(L_1, \ldots, L_k\) of \(H^d(X, \mathbb{Z})/\text{Torsion}\), we define the volume of \((X, \omega, \Omega)\) by

\[
\text{vol}(\Omega) = \text{vol}(X, \omega, \Omega) = \left| \sum_{i,j=1}^{k} \gamma^{ij} \int_{L_i} \Omega \int_{L_j} \overline{\Omega} \right|
\]

where \(\gamma^{ij}\) is the \((i, j)\)-component of the inverse matrix of the intersection matrix \((L_i \cdot L_j)_{i,j}\).

**Remark 4.3.** If \(X\) is compact, the volume \(\text{vol}(X, \omega, \Omega)\) coincides with the usual volume \(\left| \int_X \Omega \wedge \overline{\Omega} \right|\). In general, these two volumes \(\text{vol}(X, \omega, \Omega)\) and \(\left| \int_X \Omega \wedge \overline{\Omega} \right|\) do not need to coincide for non-compact \(X\). It seems to be an interesting problem to study the relationship between \(\text{vol}(X, \omega, \Omega)\) and \(\left| \int_X \Omega \wedge \overline{\Omega} \right|\).

From now on, let us specialize to the case of Milnor fibers. As mentioned before, the Milnor fiber \((X_p, \omega_p)\) does not depend on the choice of \(p \in \mathcal{P}_n\) as a symplectic manifold. However, its complex structure depends on the choice of \(p \in \mathcal{P}_n\). More precisely, each \(p \in \mathcal{P}_n\) determines a holomorphic volume form on \(X_p\) by

\[
\Omega_p = \text{Res} \left( \frac{dx \wedge dy \wedge dz}{x^2 + y^2 - p(z)} \right)
\]

which also can be written as

\[
\Omega_p = \frac{dx \wedge dy|_{X_p}}{p'(z)} = -\frac{dy \wedge dz|_{X_p}}{2x} = \frac{dx \wedge dz|_{X_p}}{2y}.
\]

**Lemma 4.4.** A special Lagrangian submanifold in \((X_p, \omega_p, \Omega_p)\) is \(U(1)\)-invariant.

**Proof.** Let \(L\) be a special Lagrangian submanifold in \((X_p, \omega_p, \Omega_p)\). For every \(t \in \mathbb{R}\), \(L_t = e^{it} \cdot L\) is again a special Lagrangian submanifold (of the same phase). On the other hand, the space of infinitesimal special Lagrangian deformations of \(L\) can be identified with \(H^1(L, \mathbb{R}) \cong H^1(S^2, \mathbb{R}) = 0\) [13, Theorem 3.6]. It implies that \(L_t = L\) for all \(t \in \mathbb{R}\). \(\Box\)

**Lemma 4.5.** A matching cycle \(L_\gamma\) is a special Lagrangian submanifold in \((X_p, \omega_p, \Omega_p)\) if and only if \(\gamma\) is a line segment.

**Proof.** Consider the parametrization \(\iota : (0, 1) \times S^1 \to X_p\) of \(L_\gamma\) given by

\[
\iota(t, \theta) = (\sqrt{p(\gamma(t))} \cos \theta, \sqrt{p(\gamma(t))} \sin \theta, \gamma(t)).
\]
Then a direct computation shows that
\[ \iota^* \Omega_p = \frac{\gamma'(t)}{2} dt \wedge d\theta. \]

Thus \( L_\gamma \) being a special Lagrangian submanifold means that \( \arg \gamma'(t) \) is a constant, or equivalently that \( \gamma \) is a line segment. \( \square \)

Combining Lemmas 4.2, 4.4 and 4.5, we obtain the following classification of special Lagrangian submanifolds in \((X_p, \omega_p, \Omega_p)\).

**Corollary 4.6.** Special Lagrangian submanifolds in \((X_p, \omega_p, \Omega_p)\) are precisely those of the forms \( L_\gamma \) for line segments \( \gamma \).

### 4.3. Proof of Theorem 1.2

In this subsection, we will view \( \mathcal{P}_n \) as the configuration space \( \text{Conf}^0_{n+1}(\mathbb{C}) \). Let \( \mathcal{P}_n^0 \subset \mathcal{P}_n \) be the configuration space of \( n + 1 \) points in \( \mathbb{C} \) in general position (with the center of mass 0) in the sense that no 3 points of them lie on a single line. For \( p = [\zeta_1, \ldots, \zeta_{n+1}] \in \mathcal{P}_n^0 \), let us fix an order of \( n + 1 \) points \( \zeta_1, \ldots, \zeta_{n+1} \). We then define \( l_{ij}(p) \) (\( 1 \leq i \leq j \leq n \)) to be the length of the line segment connecting \( \zeta_i \) and \( \zeta_{j+1} \).

Fix \( p = [\zeta_1, \ldots, \zeta_{n+1}] \in \mathcal{P}_n^0 \) and let \( L_{ij} \) (\( 1 \leq i \leq j \leq n \)) be the matching cycle associated to the line segment connecting \( \zeta_i \) and \( \zeta_{j+1} \). By Corollary 4.6 these exhaust all special Lagrangian submanifolds in \((X_p, \omega_p, \Omega_p)\). Then since
\[ \left| \int_{L_{ij}} \Omega_p \right| = \pi \cdot l_{ij}(p), \]
we can write the systole of \((X_p, \omega_p, \Omega_p)\) as
\[ \text{sys}(\Omega_p) = \pi \cdot \inf_{1 \leq i \leq j \leq n} l_{ij}(p). \]

Now take \( L_1 = L_{11}, \ldots, L_n = L_{nn} \) as a basis of \( H_2(X_p, \mathbb{Z}) \cong \mathbb{Z}^n \). Their intersection matrix \((L_i \cdot L_j)_{i,j}\) is the Cartan matrix of type \( A_n \) (under a suitable choice of orientations) \[11\]. Then, as in the proof of Proposition 3.8 we can show that
\[ \text{vol}(\Omega_p) = \frac{\pi^2}{n + 1} \sum_{1 \leq i \leq j \leq n} l_{ij}(p)^2. \]

On the other hand, it is known that
\[ \text{Sph}(\mathcal{D}_{A_n}) \setminus \text{Stab}^\circ(\mathcal{D}_{A_n}) \simeq \mathcal{P}_n \quad (4.1) \]
[16] Theorem 6.4 (also see [7] Theorem 1.1)). For \( p \in \mathcal{P}_n^0 \), we can take a representative \( \sigma_p = (Z, \mathcal{P}) \in \text{Stab}^\circ(\mathcal{D}_{A_n}) \) under this correspondence with the properties that there exists a \( \sigma_p \)-stable object \( S_{ij} \) (\( 1 \leq i \leq j \leq n \)) in the class \( S_i + \cdots + S_j \in K(\mathcal{D}_{A_n}) \) satisfying \( Z(S_{ij}) = l_{ij}(p) \) and the set of \( \sigma_p \)-stable objects coincides with the set of shifts of \( S_{ij} \) \[16\]. Note that, by Lemma 2.8 \( \text{sys}(\sigma) = \text{sys}(\sigma_p) \), \( \text{vol}(\sigma) = \text{vol}(\sigma_p) \) for any representative \( \sigma \in \text{Stab}^\circ(\mathcal{D}_{A_n}) \) of the element corresponding to \( p \in \mathcal{P}_n \) under the correspondence \( (4.1) \).
Proposition 4.7. For every $p \in \mathcal{P}_n$,

$$\text{sys}(\Omega_p) = \pi \cdot \text{sys}(\sigma_p),$$

$$\text{vol}(\Omega_p) = \pi^2 \cdot \text{vol}(\sigma_p).$$

Proof. The above description of the representative $\sigma_p \in \text{Stab}^0(\mathcal{D}_{A_n})$ shows that the assertion holds for every $p \in \mathcal{P}_n^0$. As $\mathcal{P}_n^0$ is dense in $\mathcal{P}_n$, it also shows that the assertion holds for every $p \in \mathcal{P}_n$.

Proof of Theorem 1.2. Follows from Theorem 1.1 and Proposition 4.7.

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