Schwarzschild-like exteriors for stars in Kaluza-Klein gravity

J. Ponce de Leon*
Laboratory of Theoretical Physics, Department of Physics
University of Puerto Rico, P.O. Box 23343, San Juan,
PR 00931, USA

March 2010

Abstract

In this work we examine the effective four-dimensional world that emanates from a general class of static spherical Ricci-flat solutions in Kaluza-Klein gravity in $D$-dimensions. By means of dimensional reduction we obtain a family of asymptotically flat Schwarzschild-like metrics for which all the components of the Ricci tensor, except for $R_{11}$, are zero. Although the reduced spacetime is not empty, it is similar to vacuum in the sense that the effective matter satisfies an equation of state which is the generalization of $(\rho + 3p) = 0$ for “nongravitating matter” in 4D. In Kaluza-Klein gravity these Schwarzschild-like metrics describe the exterior of a spherical star without rotation. In this framework, we generalize the well-known Buchdahl’s theorem for perfect fluid spheres whose mass density does not increase outward. Without any additional assumptions, we develop the most general expression for the compactness limit of a star. We provide some numerical values for it, which in principle are observationally testable and allow us to compare and contrast different theories and exteriors. We find that in Kaluza-Klein gravity the compactness limit of a star can be larger than 1/2, without being a black hole: the general-relativistic upper limit $M/R < 4/9$ is increased as we go away from the Schwarzschild vacuum exterior. We show how this limit depends on the number of dimensions of spacetime, and demonstrate that the effects of gravity are stronger in 4D than in any other number of dimensions.

PACS: 04.50.+h; 04.20.Cv

Keywords: Kaluza-Klein Theory; General Relativity; Stellar Models.

*E-mail: jpdel@ltp.upr.clu.edu; jpdel1@hotmail.com
1 Introduction

The concept of black holes in the context of general relativity theory arises with the well-known Schwarzschild solution, which according to Birkhoff’s theorem is the most general spherically symmetric, asymptotically flat, vacuum solution of the Einstein field equations. It describes the gravitational field outside an isolated spherical, non-rotating mass or black hole. For a perfect fluid sphere with a mass density which does not increase outwards, Buchdahl [1] showed that the ratio of its gravitational mass $M$ to the coordinate radius $R$ satisfies the inequality $GM/R \leq 4/9$, i.e. under these conditions no static spherically symmetric star is possible in general relativity with a radius less than 9/8 of the Schwarzschild radius $R_{Schw} = 2GM$. By definition, any non-rotating and non-charged mass that is smaller than the Schwarzschild radius (horizon) constitutes a black hole.

Nowadays there are a number of non-equivalent theories that suggest the existence of extra dimensions. Kaluza-Klein gravity [2]-[4] and braneworld theory [5]-[8] are well known examples. The study of stellar structure and stellar evolution might constitute an important approach to predict observable effects from extra dimensions. However, there is a fundamental limitation. Namely, that Birkhoff’s theorem is no longer valid in more than four dimensions, i.e. there is no an unique asymptotically flat vacuum solution with spatial spherical symmetry. As a consequence, the effective picture in four dimensions allows the existence of different possible non-Schwarzschild scenarios for the description of the spacetime outside of a spherical star [9]-[21].

Therefore, the study of stars in these theories leads to two main questions [22]:

1. How does the existence of stars restrict the class of admissible non-Schwarzschild exteriors?
2. How does a possible deviation from the Schwarzschild vacuum exterior can affect the star parameters?

In a recent series of papers we have investigated different aspects of these questions. In the framework of five-dimensional Kaluza-Klein gravity, without making any assumption about the stellar structure, we have shown that the condition that in the weak-field limit we recover the usual Newtonian physics singles out an *unique* effective exterior for a spherically symmetric star [22], [23]. On the other hand, in the framework of Randall-Sundrum II braneworld scenario, without making any assumption about the bulk, or the material medium inside the star, we have proved that for *any* nonstatic spherical star, without rotation, there are only two possible static exteriors; these are the Schwarzschild and the ‘Reissner-Nordström-like’ exteriors [24]. This is quite distinct from the case of stars in hydrostatic equilibrium which admit a much larger family of non-Schwarzschild static exteriors.

We have also studied the second question mentioned above in the context of static, perfect fluid, spherical stars of *uniform density* [22], [25]. We have shown that, in principle, the compactness limit of such a star can be larger than 1/2, without being a black hole. In this work we generalize this result to *any* static spherical perfect fluid star in Kaluza-Klein gravity.

Thus, here we get rid of the unrealistic assumption of uniform density, but keep the isotropy condition[1]. The generalization is attained by using an extension, to more than four dimensions, of an inequality originally discovered by Buchdahl [1]. This inequality and the matching conditions allow us to obtain the compactness limit of a star for *any* given exterior spacetime. This limit is important because it indicates how much perfect fluid matter can be packed in a given volume, without provoking gravitational collapse. Here we develop the most general expression for this limit and provide some numerical values for it, which in principle are observationally testable and allow us to compare and contrast different theories and exteriors.

This paper is organized as follows. In section 2, we derive and analyze a general class of static, spherical vacuum solutions in Kaluza-Klein gravity in $D$-dimensions, where the metric functions are independent of the extra coordinates. They generalize some well-known solutions in the literature as the Schwarzschild-Tangherlini spacetimes and other ones with spherical symmetry in the three usual space dimensions. In section 3, we carry out the dimensional reduction of the solutions. We obtain a family of asymptotically flat Schwarzschild-like metrics for which all the components of the Ricci tensor, except for $R_{11}$, are zero. In section 4, we use them to construct the spacetime outside of a spherical star. In this framework, we generalize the well-known Buchdahl’s theorem for perfect fluid spheres whose mass density does not increase outward. Without any additional assumptions, we show that the compactness

---

[1] Is important to emphasize the role of isotropic pressures, because for anisotropic pressures there is no upper bound on the gravitational potential of a star. See [26], [27] and references therein.
limit of a star in Kaluza-Klein gravity can be larger than $1/2$, without being a black hole: the general-relativistic upper limit $M/R < 4/9$ increases as we go away from the Schwarzschild vacuum exterior. Our results are consistent with our previous findings and show that, as in general relativity, all these inequalities can be saturated in the case of uniform proper density from the condition that the isotropic pressure does not become infinity at the center. In section 5 we summarize our results. Finally, in the Appendix following our previous work [28] we present some general inequalities for the metric functions in the interior of a static star, which extend to $D$-dimensions those obtained by Buchdahl.

2 Static spherical vacuum solutions in Kaluza-Klein gravity

In five-dimensional general relativity the most general time-independent vacuum solutions, with spherical symmetry in the three usual space dimensions and a Killing vector in the fifth direction, have been found and classified by Chodos and Detweiler [29]. The physical properties of these solutions with zero electric charge have been discussed by a number of authors [4], [30]-[39]. Although they are frequently described as black holes, many researchers have noticed that instead they represent naked singularities because the event horizon is reduced to a singular point at the center of ordinary space [40]-[43]. Truly higher dimensional extensions of the Schwarzschild black hole metric have been obtained by Tangherlini [44] and generalized by Myers and Perry [45], who considered spacetimes with spherical symmetry in $(D - 1)$, rather than three spatial dimensions.

In this work we consider static spacetimes with topology $R^1 \times S^{n+3} \times K^m$, where $R^1$ corresponds to the time dimension and $K^m$ is a $m$-dimensional manifold. The $D = (4 + n + m)$ metric is assumed to be independent of the extra coordinates and in isotropic coordinates can be written as:

$$ds^2 = A^2(r)dt^2 - B^2(r) \left[ dr^2 + r^2 d\Omega^2_{(n+2)} \right] - C^2(r)\delta_{ab}dy^ad^b.$$ (1)

Here $n$ and $m$ represent the number of “internal” and “external” dimensions, respectively; $d\Omega_{(2+n)}$ is the metric on a unit $(n + 2)$-sphere $(n = 0, 1, 2...)$: the coordinates along the external dimensions are denoted as $y^a$ with $a, b = 1, 2...m$. These spacetimes generalize the ones discussed by Kramer [17], Davidson and Owen [18], Chatterjee [22], Millward [19], Dereli [20], Tangherlini [23], as well as the non-rotating black holes of and Myers and Perry [45].

**Notation:** Before going on, it is useful to establish our notations and conventions: Indices labeled by Latin capital letters $A$, $B$ run over the full $D = (4 + n + m)$ space; indices labeled by $\mu, \nu$ run over the $(4 + n)$ subspace and the metric is called $\gamma_{\mu\nu}$; the indices in the conventional spacetime, with signature is $(+, -, -, -)$, are labeled $\mu, \nu = (0, 1, 2, 3)$; primes denote differentiation with respect to $r$; we follow the definitions of Landau and Lifshitz [50]; and the speed of light, as well as the gravitational constant are taken to be unity.

The metric functions in (1) are solutions of the vacuum Einstein field equations $R_{AB} = 0$. The non-vanishing equations are:

$$\frac{A''}{A} + (n + 1) \frac{B'}{B} + \frac{n + 2}{r} + m \frac{C'}{C} = 0,$$ (2)

$$\frac{A''}{A} + (n + 2) \frac{B''}{B} + m \frac{C''}{C} + \frac{B'}{B} \left[ \frac{n + 2}{r} - \frac{A'}{A} - (n + 2) \frac{B'}{B} - m \frac{C'}{C} \right] = 0,$$ (3)

$$\frac{B''}{B} + \frac{B'}{B} \left[ \frac{2n + 3}{r} + \frac{A'}{A} + m \frac{C'}{C} \right] + 1 \frac{A'}{A} + m \frac{C'}{C} = 0,$$ (4)

$$\frac{A'}{A} + (n + 1) \frac{B'}{B} + \frac{n + 2}{r} + (m - 1) \frac{C''}{C} + \frac{C'}{C} = 0,$$ (5)

$^2$A more general line element is studied in [10].
which correspond to $R^0_0 = 0$, $R^1_1 = 0$, $R^2_2 = \ldots = R^{n+2}_{n+2} = 0$ and $R^3_3 = 0$, respectively. Now combining (4) and (5) we get an equation that can be easily integrated, viz., $A'/A = -k C'/C$, where $k$ is a constant of integration. Thus, $A \propto C^{-k}$ and $B^{n+1} \propto C^{k+1-m}/r^{n+2}C'$. Substituting into (4) we obtain an equation for $C(r)$, whose solution is

$$C(r) = C_0 \left( \frac{ar^{n+1} + 1}{ar^{n+1} - 1} \right)^{\sigma},$$

(6)

where $a$, $C_0$ are constants of integration and $\sigma$ is subjected to the condition

$$\sigma^2 \left[ (n + 2)k^2 - 2km + m(m + n + 1) \right] = n + 2,$$

(7)

which is imposed by (3). The final form of the remaining metric functions is given by

$$A(r) = \left( \frac{ar^{n+1} - 1}{ar^{n+1} + 1} \right)^{\sigma k}, \quad B^{n+1}(r) = \frac{1}{a^{2(2n+1)}} \frac{(ar^{n+1} + 1)}{(ar^{n+1} - 1)^{\sigma(k-m)+1}}.$$

(8)

We will see below that the constant $a$ is related to the total gravitational mass. Therefore, in what follows we take $a > 0$. Also, without loss of generality we set $C_0 = 1$. In the case where $D = 5$ with $n = 0$ and $m = 1$, (7) reduces to $\sigma^2(k^2 - k + 1) = 1$ which is the consistency condition in Davidson-Owen solution [38].

In the above equations $k$ is an arbitrary real number, i.e.

$$-\infty < k < \infty,$$

(9)

and, as a consequence of (7), $\sigma$ is bounded from bellow and above. Namely,

$$|\sigma| \leq \sigma_{\text{max}} = \frac{n + 2}{\sqrt{m(n + 1)(n + m + 2)},}

(10)

where the maximum value is attained at $k = m/(n + 2)$. Thus, the range of $\sigma$ decreases, moving closer to zero, with the increase of $m$. Therefore, for any fixed $n$ the effects of the external extra dimensions (measured by the deviation of $\sigma$ from zero) become weaker with the increase of $m$. Also, for $m = 1$, $\sigma_{\text{max}}$ steadily decreases with the increase of $n$. However, for $m > 1$ this is not so. Indeed, in this case $\sigma_{\text{max}}$ first decreases and subsequently increases with the increase of $n$. As an illustration, let us take

$$m = (1, 2, 5, 7, 11),$$

then

$$n = 0, \quad |\sigma| \leq (1.15, 0.71, 0.34, 0.25, 0.17),$$

$$n = 1, \quad |\sigma| \leq (1.06, 0.67, 0.34, 0.25, 0.17),$$

$$n = 5, \quad |\sigma| \leq (1.01, 0.67, 0.37, 0.29, 0.20).$$

The behavior of the metric functions near $ar^{n+1} \sim 1$ depends on the choice of $\sigma$ and $k$. For example, when $\sigma > 0$ we find that $B \to 0$ as $ar^{n+1} \to 1^+$, in the whole range of $k$. However, when $\sigma < 0$ the same limit gives: (i) $B \to \infty$ for $k < (m - 1)/2$, (ii) $B \to 0$ for $k > (m - 1)/2$, and (iii) $B \to 2^{2/(n+1)}$ for $k = (m - 1)/2$. In addition, for $\sigma > 0$, $k \geq 0$ we find that $A \to 0$, $B \to 0$, $C \to \infty$ as $ar^{n+1} \to 1^+$ for any number of dimensions $n$ and $m$. Consequently, in the allowed range of $\sigma$ and $k$, there are several families of solutions with different geometrical and physical properties. For $n = 0$ and $m = 1$, these have been thoroughly discussed in the literature [38], [39].

A simple classification of the solutions can be obtained from the analysis of the physical radius $R(r)$ of a $(n + 2)$-sphere. In the present case it is given by

$$R(r) = rB(r) = \frac{1}{a^{2/(n+1)}} \frac{\left[(ar^{n+1} + 1)^{\sigma(k-m)+1} \right]^{1/(n+1)}}{\left[(ar^{n+1} - 1)^{\sigma(k-m)-1} \right]}.$$  

(11)
For $ar^{n+1} \gg 1$, $R \approx r$, regardless of the number of dimensions and the choice of $\sigma$ and $k$. However, near $ar^{n+1} \sim 1$ we find

$$\lim_{ar^{n+1} \to 1+} R(r) = \begin{cases} 
0, & \text{for } \sigma > 0, -\infty < k < \infty, \text{ and } \sigma < 0, k > (m-1)/2, \\
\infty, & \text{for } \sigma < 0, -\infty < k < (m-1)/2, \\
\frac{1}{a} \frac{1}{n+1}, & \text{for } \sigma = 0, \text{ and } \sigma < 0, k = (m-1)/2.
\end{cases} \quad (12)$$

For the parameters $\sigma$, $k$ in the first expression of (12) we cannot interpret (6)–(8) as a black hole solution because the “event horizon” $ar^{n+1} = 1$ occurs at $R = 0$, which by itself is a singular point, line, plane...for $m = 0, 1, 2$, etc. The second expression in (12) is a consequence of the fact that $dR/dr$ vanishes at some finite value of $r$, say $\bar{r}$. Since $R_{\min} = R(\bar{r}) > 0$, the solutions with $\sigma < 0, k < (m-1)/2$ can be used to generate the higher dimensional counterpart of the wormholes solutions discussed by Agnese et al.\textsuperscript{34}. The solutions with $\sigma = 0$ and $\sigma < 0, k = (m-1)/2$ are specially simple because now $k$ and $\sigma$ are fixed. Below we discuss them separately.

**Solutions with $\sigma = 0$:** In the limit $\sigma \to 0$, from (7) it follows that $\sigma^2 k^2 \to 1$. If we take $\sigma k = 1$, then

$$A(r) = \frac{a r^{n+1} - 1}{a r^{n+1} + 1}, \quad B^{n+1}(r) = \frac{(a r^{n+1} + 1)^2}{a^2 r^{2(n+1)}}. \quad (13)$$

Far away from a stationary source $g_{00} \sim (1 + 2\phi)$, where $\phi$ is the Newtonian gravitational potential which goes as $-\mathcal{M}/r^{n+1}$. Therefore, in the present case the total gravitational mass $\mathcal{M}$ is given by

$$\mathcal{M} = \frac{2}{a}. \quad (14)$$

(If we had taken $\sigma k = -1$, we would have obtained a negative mass, viz., $\mathcal{M} = -2/a$). With the transformation

$$R = \left[\frac{a r^{n+1} + 1}{a^2 r^{2(n+1)}}\right]^{2/(n+1)}, \quad (15)$$

the solution becomes

$$dS^2 = \left(1 - \frac{2\mathcal{M}}{R^2 + r^2}\right) dt^2 - \left(1 - \frac{2\mathcal{M}}{R^2 + r^2}\right)^{-1} dR^2 - R^2 d\Omega_{n+2}^2 - \delta_{ab} dy^a dy^b, \quad (16)$$

which, up to the innocuous $m$ flat extra dimensions, describes the so-called Schwarzschild-Tangherlini black holes with spherical symmetry in $(n + 3)$ rather than three spatial dimensions. The radius $R_h$ of the horizon of the black hole is given by $R_h = (4/a)^{1/(n+1)}$, which in isotropic coordinates corresponds to $ar_h^{n+1} = 1$, as expected. For $n = 0$ they reduce to the conventional Schwarzschild solution of general relativity.

**Solutions with $\sigma < 0, k = (m-1)/2$:** Substituting $k = (m-1)/2$ into (7) we find $\sigma = -2/(m+1)$. To illustrate the properties of this class of solutions let us momentarily take $n = 0, m = 2$. For this choice, using (6)–(8) we obtain

$$dS^2 = \left(\frac{ar + 1}{ar - 1}\right)^{2/3} dt^2 - \frac{(ar + 1)^4}{a^4 r^4} \left[dr^2 + r^2 d\Omega_{2}^2\right] + \left(\frac{ar - 1}{ar + 1}\right)^{4/3} \left[dy_1^2 + dy_2^2\right]. \quad (17)$$

The interpretation of this metric is difficult because far from the origin $g_{00} \sim [1 + (4/3ar)]$, which implies a negative mass parameter, viz., $\mathcal{M} = -(2/3a)$. However, using a double Wick rotation $t \to iy_1$, $y_1 \to it$ we generate a solution of the field equations with positive mass. Namely,

$$dS^2 = \left(\frac{ar - 1}{ar + 1}\right)^{4/3} dt^2 - \frac{(ar + 1)^4}{a^4 r^4} \left[dr^2 + r^2 d\Omega_{2}^2\right] - \left(\frac{ar + 1}{ar - 1}\right)^{2/3} dy_1^2 - \left(\frac{ar - 1}{ar + 1}\right)^{4/3} dy_2^2. \quad (18)$$

\textsuperscript{3}For black holes in general relativity, the event horizon is defined as the surface where the norm of the timelike Killing vector vanishes. In our case the Killing vector is just $(1, 0, 0, \ldots 0)$ so its norm vanishes where $g_{00}$ does.
for which $M = (4/3a)$. Using the transformation of coordinates $aR = [(ar + 1)^2/ar]$ this metric becomes

$$dS^2 = \left(1 - \frac{3M}{R}\right)^{2/3} dt^2 - \frac{dR^2}{(1 - 3M/R)^2} - R^2 d\Omega^2_{(2)} - \frac{dy_1^2}{(1 - 3M/R)^{1/3}} - \left(1 - \frac{3M}{R}\right)^{2/3} dy_2^2. \quad (19)$$

We see that $g_{TT} = 0$ and $g_{RR} = -\infty$ at $R = 3M$. However, this is not a horizon but a singularity. This may be verified by evaluating the invariant geometric scalars. For example, the Kretschmann curvature scalar $I = R_{ABCD}R^{ABCD}$ is

$$I = \frac{12M^2(9R^2 - 52RM + 76M^2)}{(R - 3M)^2R^6}, \quad (20)$$

which is manifestly divergent at $R = 3M$. The above analysis can be extended to any $n$ and $m$. The result is that the line element (19) is a member of the family of solutions

$$dS^2 = F^{2/(m+1)}dt^2 - F^{-1}dR^2 - R^2 d\Omega^2_{(n+2)} - F^{(1-m)/(1+m)}dy_1^2 - F^{2/(m+1)}[dy_2^2 + dy_3^2 + \ldots + dy_m^2], \quad (21)$$

with

$$F = 1 - \frac{(m + 1)M}{R^{n+1}}. \quad (22)$$

These are singular at $R = [(m + 1)M]^{1/(n+1)}$ for any $m \neq 1$. However, for $m = 1$ they yield Schwarzschild-Tangherlini spacetimes with one flat extra dimension. It is important to note that (19) and (21) are the only solutions for which $g_{TT} = 0$ and $g_{RR} = -\infty$ in the same region of the spacetime. In fact, from (8) and (11) it follows that

$$B^2(r)dr^2 = \frac{[a^2r^{2(n+1)} - 1]^2 dR^2}{[a^2r^{2(n+1)} + 2\alpha(m-k)ar^{(n+1)} + 1]} = -g_{RR}dR^2. \quad (23)$$

Thus, in general $g_{RR} \to 0$ as $ar^{(n+1)} \to 1^+$, except in the case where the denominator in (23) also vanishes in this limit, i.e. for

$$1 + \sigma(m-k) = 0. \quad (24)$$

This occurs in two cases only: (i) $\sigma = 0$, $(\sigma k = 1)$ which corresponds to Schwarzschild-Tangherlini’s spacetimes, and

(ii) $\sigma(k - m) = 1$, which by virtue of (7) implies $k = (m - 1)/2$ and thus generates the family (21) - (22).

To conclude this section, we notice that for $k = 0$ the solution (21) generalizes the well-known zero-dipole moment soliton of Gross and Perry [53] to any number of internal dimensions. As an illustration, let us take $m = 1$,

$$dS^2 = dt^2 - \left(1 - \frac{1}{ar^{n+1}}\right)^{4/(n+1)} [dr^2 + r^2 d\Omega^2_{(n+2)}] - \left(\frac{ar^{n+1} + 1}{ar^{n+1} - 1}\right)^2 dy_1^2. \quad (25)$$

We note that here $a$ does not have to be positive. If we assume $a = -\alpha^2 < 0$ and perform a double Wick rotation $t \to iy_1, y_1 \to it$, then we obtain the metric

$$dS^2 = \left(1 - \frac{\alpha^2r^{n+1}}{1 + \alpha^2r^{n+1}}\right)^2 dt^2 - \left(1 + \frac{1}{\alpha^2r^{n+1}}\right)^{4/(n+1)} [dr^2 + r^2 d\Omega^2_{(n+2)}] - dy_1^2, \quad (26)$$

which after a simple coordinate transformation reduces to (10) with mass parameter $M = 2/\alpha^2$. This suggests that, as in 5D, the above $(4 + n)$ metrics represent limiting configurations that are unstable to metric perturbations [39].

---

4A generalization of Schwarzschild-Tangherlini’s spacetimes to a chain of several Ricci-flat internal spaces has been obtained in [51]. In turn, those generalized solutions are extended to include a massless scalar field in [52].

5To avoid a misunderstanding we note that the Kaluza-Klein monopole of Gross, Perry and Sorkin [54] does not belong to the class of spacetimes investigated here.
3 Dimensional reduction

The class of Kaluza-Klein solutions discussed in the previous section clearly demonstrates the wealth of possible physical scenarios in higher dimensions. In order to study their observational implications, and test possible deviations from general relativity, we have to examine the effective four-dimensional world that emanates from them. Thus, we consider the case where \( n = 0 \), which physically corresponds to spherical symmetry in the three usual spatial dimensions. For the sake of generality, we also discuss the dimensional reduction for \( n \neq 0 \). Also, at the end of this section, we discuss some similarities and differences between spacetimes with different \( n \).

To this end, we recall that the dimensional reduction of \( R_{(D)} \), the curvature scalar associated with the metric

\[
dS^2 = \gamma_{\mu\nu}(x)dx^\mu dx^\nu - \sum_{i=1}^{m} N_i^2(x)dy_i^2,
\]

(27)
can be expressed as

\[
\sqrt{|g_{(D)}|} R_{(D)} \propto \sqrt{|g_{(4)}|} R_{(4)} + \text{other terms},
\]

(28)
where \( R_{(4)} \) is the four dimensional curvature scalar calculated from the effective 4D metric tensor. \(^6\)

\[g_{(D)} \text{ and } g_{(4)} \text{ denote the determinants of the D-dimensional metric (27) and effective 4D metric (29), respectively, and the “other terms” are proportional to the Lagrangian of an effective energy-momentum tensor (EMT) } T_{\mu\nu} \text{ in } 4D. \text{ If we use the fact that the metric functions satisfy the field equations } R_{AB} = 0, \text{ then we obtain a nice expression for (28), viz.,}
\]

\[
\sqrt{|g_D|}R_D = \sqrt{|g_{(4)}|} \left[ R_{(4)} - \sum_{a=1}^m \frac{\partial_a N_a \partial^\mu N_a}{N_a^2} - \frac{1}{2} \sum_{a=1}^m \sum_{b=1}^m \left( \frac{\partial_a N_a}{N_a} \right) \left( \frac{\partial_b^\mu N_b}{N_b} \right) \right].
\]

(30)
Here \( D = 4 + m \) and \( m \geq 1 \). It shows that the choice of the factor \( \prod_{i=1}^m N_i \) in (29) assures that the first term in (30) yields the conventional general relativity action. In the case under consideration \( N_i = C(x) \). Therefore, the physics in 4D can be extracted from the four-dimensional effective action

\[
S_{(4)} = -\frac{1}{k_4} \int d^4x \sqrt{|g_{(4)}|} \left[ R_{(4)} - m \left( 1 + \frac{m}{2} \right) \frac{\partial_a C \partial^a C}{C^2} \right],
\]

(31)
where \( k_4 \) is a positive constant. From the variational principle, \( \delta S_{(4)} = 0 \), we get the effective equations in 4D, viz.,

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{m(m+2)}{2C^2} \left( \partial_\mu C \partial_\nu C - \frac{1}{2} g_{\mu\nu} \partial_\alpha C \partial^\alpha C \right) \equiv 8\pi T_{\mu\nu}.
\]

(32)
Here \( g_{\mu\nu} \equiv g_{\mu\nu}^{(4)} \), and \( R_{\mu\nu} \) as well as \( R \) are calculated with \( g_{\mu\nu} \).

Thus, for the Kaluza-Klein metric (1), (6), (8), with \( n = 0 \), the physics in 4D is concentrated in the effective line element

\[
ds^2 = \left( \frac{ar - 1}{ar + 1} \right)^{2\varepsilon} dt^2 - \frac{1}{a^4 r^4 \left( ar - 1 \right)^{2(\varepsilon - 1)}} \left[ dr^2 + r^2 d\Omega^2 \right],
\]

(33)
with

\[
\varepsilon \equiv \frac{\sigma(2k - m)}{2}.
\]

(34)
\(^6\)For the effective action in 4D to contain the exact Einstein Lagrangian, i.e. to deal with a constant effective gravitational constant, \( g_{\mu\nu}^{(4)} \) should be identified with the physical metric in ordinary 4D spacetime [2], [25], [25].
The effective 4D energy-momentum tensor $T_{\mu\nu}$ can be calculated by substituting (33) into the l.h.s of (32). However, it is easier to use the r.h.s., viz.,

$$8\pi T^{\mu\nu}_0 = \frac{m(m+2)}{4B^2C^{m+2}} \left( \frac{dC}{dr} \right)^2, \quad T^1_1 = -T^0_0, \quad T^3_3 = T^2_2 = T^0_0. \quad (35)$$

Taking $B$ and $C$ from (6), (8) and using the condition (7), we find

$$8\pi T^{\mu\nu}_0 = \frac{4a^6r^4(1-\varepsilon^2)(ar-1)^2(\varepsilon-2)}{(ar+1)^2(\varepsilon+2)}. \quad (36)$$

It should be noted that for an observer in 4D, who is not aware of the extra dimensions, $\varepsilon$ is a free parameter in the solution and measures the deviation from Schwarzschild.

The line element (33) acquires a more familiar form in terms of the Schwarzschild-like coordinate $R$ defined by

$$R = r\left(1 + \frac{1}{ar}\right)^2, \quad (37)$$

and setting $a = (2/\varepsilon M)$. Indeed, (33) becomes

$$ds^2 = \left(1 - \frac{2M/\varepsilon}{R}\right)^{\varepsilon} dt^2 - \frac{dR^2}{\left(1 - \frac{2M/\varepsilon}{R}\right)^{\varepsilon}} - R^2 \left(1 - \frac{2M/\varepsilon}{R}\right)^{1-\varepsilon} (d\theta^2 + \sin^2 \theta d\phi^2). \quad (38)$$

This 4D metric has been obtained previously by the present author in the context of Davidson-Owen solutions [48] (these correspond to $m = 1$). This means that the effective metric in 4D is independent of the number of external extra coordinates. It "generalizes" the Schwarzschild vacuum metric ($\varepsilon = 1$), and has been widely discussed as a possible non-Schwarzschild exterior for stellar models within the context of Kaluza-Klein gravity [22]-[23]. We have shown that (38) is compatible with (i) Newtonian physics, in the weak-field limit; (ii) the general-relativistic Schwarzschild limit for $\varepsilon = 1$; (iii) the dominant energy condition, and (iv) the (weak) equivalence principle, even in the non-Schwarzschild case where $\varepsilon \neq 1$.

In terms of $R$ the EMT becomes

$$8\pi T^{\mu\nu}_0 = \frac{(1-\varepsilon^2)M^2}{\varepsilon^2R^4} \left(1 - \frac{2M/\varepsilon}{R}\right)^{(\varepsilon-2)}, \quad T^1_1 = -T^0_0, \quad T^3_3 = T^2_2 = T^0_0. \quad (39)$$

Since $(T^0_0 - T^1_1 - T^2_2 - T^3_3) = 0$, this spacetime is ‘similar’ to Schwarzschild vacuum in the sense that it has no effect on gravitational interactions. However, it is more realistic because instead of being absolutely empty $T^0_0 = T^1_1 = T^2_2 = T^3_3 = 0$, it is consistent with the existence of quantum zero-point fields [56]. It is not difficult to show that (38) and (39) are equivalent to the static, spherically symmetric solution of the coupled Einstein-massless scalar field equations originally discovered by Fisher [57] and rediscovered by Janis, Newman and Winicour [58].

### 3.0.1 Dimensional reduction for $n > 0$

Although from an observational point of view the models with $n > 0$ might not be of prime concern, it is of theoretical interest to study how the physics depends on the number of internal dimensions.

It is not difficult to verify that $g_{\mu\bar{\nu}}^{\text{eff}}$, the effective metric in $D = (4+n)$, is obtained from $\gamma_{\mu\bar{\nu}}$ as

$$g_{\mu\bar{\nu}}^{\text{eff}} = \gamma_{\mu\bar{\nu}} \prod_{i=1}^{m} N_i^{q(n)}, \quad q(n) = \frac{2}{n+2}. \quad (40)$$

Similar to the discussion leading to (30), with this choice the dimensional reduction of the $m$ external dimensions yields
\[ \sqrt{|g_{(D)}|} R_{(D)} \propto \sqrt{|g_{(4+n)}^{\text{eff}}|} \left[ R_{(4+n)} + \text{constant} \times \frac{\partial \mu C \partial \nu C}{C^2} \right], \]  
where the constants depend on the choice of \( n \) and \( m \). In this way, the gravitational action has the standard form

\[ S_{(D)} = -\frac{1}{k_D} \int d^D x \sqrt{|g_{(D)}|} R_{(D)}, \]

in any number of dimensions.

In the case under consideration \( g_{00}^{\text{eff}} = C^{2m/(n+2)} A^2, g_{11}^{\text{eff}} = -C^{2m/(n+2)} B^2, g_{22}^{\text{eff}} = -C^{2m/(n+2)} B^2 r^2 \), etc. Thus the effective gravity in \( 4 + n \) is governed by the line element

\[ ds^2 = \left( \frac{a r^{n+1} - 1}{a r^{n+1} + 1} \right)^{2\varepsilon} dt^2 - \frac{1}{(a r^{n+1})^{4/(n+1)}} \frac{(a r^{n+1} + 1)^{2(\varepsilon+1)/(n+1)} - (a r^{n+1} - 1)^{2(\varepsilon-1)/(n+1)}}{(a r^{n+1} - 1)^{2(\varepsilon-1)/(n+1)}} \left[ dr^2 + r^2 d\Omega^2_{(n+2)} \right], \]

where

\[ \varepsilon = \frac{\sigma (n + 2) k - m}{n + 2}. \]

From the asymptotic behavior of the metric we get the total mass

\[ M = \frac{2\varepsilon}{a}. \]

Now, introducing the Schwarzschild-like coordinate

\[ R = r \left( 1 + \frac{1}{a r^{n+1}} \right)^{2/(n+1)} \]

the metric becomes

\[ ds^2 = \left( 1 - \frac{2M/\varepsilon}{R_{n+1}} \right)^{\varepsilon} dt^2 - \left( 1 - \frac{2M/\varepsilon}{R_{n+1}} \right)^{-(n+\varepsilon)/(n+1)} dR^2 - R^2 \left( 1 - \frac{2M/\varepsilon}{R_{n+1}} \right)^{(1-\varepsilon)/(n+1)} d\Omega^2_{(n+2)}. \]

For \( n = 0 \) we recover \( (55) \). Also, for \( n \neq 0 \) and \( \varepsilon = 1 \) it reduces to the Schwarzschild-Tangherlini black hole solutions \( (10) \). For any other \( \varepsilon \) the effective \( (4 + n) \) spacetime is not empty. In fact, the EMT is given by

\[ 8\pi T_0^0 = \frac{(n + 1)(n + 2)(1 - \varepsilon^2)M^2}{2 \varepsilon^2 R^{2(n+2)}} \left( 1 - \frac{2M/\varepsilon}{R_{n+1}} \right)^{(-n-2)/(n+1)} \left( 1 - \frac{2M/\varepsilon}{R_{n+1}} \right)^{-n/2}, \quad T_1^1 = -T_0^0, \quad T_2^2 = T_3^3 = \cdots = T_{n+3}^{n+3} = T_0^0, \]

which can be interpreted as a massless scalar field in \( (4 + n) \) dimensions. For \( n = 0 \) it reduces to \( (39) \), as expected.

### 3.1 Properties of the effective spacetimes

It should be noted that the dimensional reduction eradicates the geometrical and physical differences between the three families of higher-dimensional solutions discussed in \( (12) \). In particular, the effective \( (4 + n) \) spacetime shows no evidence of the different nature of the singularity of \( g_{RR} \) near \( ar^{n+1} = 1 \), which is revealed by \( (23) \). Thus, regardless of their specific properties, all the solutions discussed in section 2 yield the same effective spacetime in \( (4 + n) \).

From \( (18) \) we find that the components of the EMT satisfy the equation

\[ T = (n + 2)T_0^0. \]

Substituting this into \( (A-1) \) we find that the effective spacetimes satisfy the equations

\[ R_0^0 = R_2^2 = R_3^3 = \cdots = R_{n+3}^{n+3} = 0, \quad R_1^1 = 16\pi T_1^1. \]
We now recall that in the case of a constant, asymptotically flat, gravitational field there is an expression for the total energy of matter plus field, which is an integral of $R^0_0$ over the volume $V$ occupied by the matter\footnote{Landau and Lifshitz \cite{50} the discussion is in 4D, however it can be extended to any number of dimensions}, viz.,

$$M = \alpha \int \sqrt{|g|} R^0_0 dV,$$

where the constant of proportionality $\alpha$ depends on the number of dimensions, e.g., $\alpha = 1/4\pi$ in 4D. In conventional general relativity this expression is known as the Tolman-Wittaker formula.

Since $R^0_0 = 0$, it follows that the gravitational mass of any spherical shell is just zero. This conclusion holds for any $n$ and $\varepsilon$, which include the Schwarzschild-Tangherlini black holes, as well as the familiar Schwarzschild solution of general relativity. Clearly, this is a consequence of the fact that the scalar field is massless.

We note that

$$R^0_0 = \rho + p_r + (n + 2)p_\perp = 0, \quad p_r = -T^1_1, \quad p_\perp = -T^2_2,$$  \hspace{1cm} (51)

generalizes to $n$ dimensions the well-known equation of state $(\rho + p_r + 2p_\perp) = 0$ for non-gravitating matter in 4D, which in turn generalizes to anisotropic matter the equation of state $(\rho + 3p) = 0$ for a perfect fluid that has no effect on gravitational interactions \footnote{Note that the $r$ in (47), as well as in the rest of this section, represents what we have previously called ‘Schwarzschild-like coordinate’, e.g., in (41) and (45). Please, do not confuse it with the radial coordinate in isotropic coordinates (1).}.

Finally, we note that at large distances from the origin, i.e. for $R \gg (2M)^{1/(n+1)}$, the line element \footnote{Note that the $r$ in (47), as well as in the rest of this section, represents what we have previously called ‘Schwarzschild-like coordinate’, e.g., in (41) and (45). Please, do not confuse it with the radial coordinate in isotropic coordinates (1).} becomes

$$ds^2 = ds_0^2 - \frac{2M}{R^{n+1}} \left[ d\ell^2 + \frac{(\varepsilon + n)}{\varepsilon(n + 1)} R^2 d\Omega_{n+2} \right] + O \left( \frac{M}{R^{n+1}} \right)^2. \hspace{1cm} (52)$$

The second term represents a small correction to the Minkowski metric $ds_0^2$ in $(4 + n)$. Since at large distances from the sources every field appears centrally symmetric, it follows that \footnote{Note that the $r$ in (47), as well as in the rest of this section, represents what we have previously called ‘Schwarzschild-like coordinate’, e.g., in (41) and (45). Please, do not confuse it with the radial coordinate in isotropic coordinates (1).} determines the metric at large distances from any system of bodies \footnote{Note that the $r$ in (47), as well as in the rest of this section, represents what we have previously called ‘Schwarzschild-like coordinate’, e.g., in (41) and (45). Please, do not confuse it with the radial coordinate in isotropic coordinates (1).}. For $\varepsilon = 1$ the correction is identical to the one in general relativity, for any $n$. However, this is not so for $\varepsilon \neq 1$. This could serve in astrophysical observations to detect possible deviations from general relativity.

### 4 Physical interpretation

The metric of the effective spacetime \footnote{Note that the $r$ in (47), as well as in the rest of this section, represents what we have previously called ‘Schwarzschild-like coordinate’, e.g., in (41) and (45). Please, do not confuse it with the radial coordinate in isotropic coordinates (1).} contains a singularity at $ar^{n+1} = 1$, which, in principle, can be visible to an external observer. Only in the Schwarzschild-Tangherlini limit ($\varepsilon \to 1$) it is covered by an event horizon; for any other value of $\varepsilon$ the horizon is reduced to a singular point.

However, the presence of naked singularities makes everybody uncomfortable and according to the cosmic censorship hypothesis they should not be realized in nature. In order to avoid them, we have to exclude the central region and require $ar^{n+1} > 1$. What this suggests is that the asymptotically flat metric \footnote{Note that the $r$ in (47), as well as in the rest of this section, represents what we have previously called ‘Schwarzschild-like coordinate’, e.g., in (41) and (45). Please, do not confuse it with the radial coordinate in isotropic coordinates (1).} should be used to describe the gravitational field outside of the core of a spherical matter distribution. The “interior” region has to be described by some solution of the field equations, which must be regular at the origin and not necessarily asymptotically flat.

In this interpretation, the effective exterior is not Ricci-flat \footnote{Note that the $r$ in (47), as well as in the rest of this section, represents what we have previously called ‘Schwarzschild-like coordinate’, e.g., in (41) and (45). Please, do not confuse it with the radial coordinate in isotropic coordinates (1).}, except for $\varepsilon = 1$. However, we have just seen that the exterior scalar field is gravitationally innocuous, in the sense that, as in conventional general relativity, any shell outside of the source carries zero gravitational mass.

We now proceed to investigate this interpretation. We use the standard boundary conditions to study the question of how a possible deviation from the Schwarzschild vacuum exterior can affect the star parameters. Instead of restricting our discussion to a particular equation of state for the stellar interior, here we only assume that (1) the matter inside the star is a perfect fluid, and that (2) the energy density is positive and does not increase outward. Under these conditions we will be able to extend the well-known Buchdahl’s theorem of general relativity to stellar models in Kaluza-Klein gravity.

Thus, we assume that the spacetime outside of a spherical body is described by the line element \footnote{Note that the $r$ in (47), as well as in the rest of this section, represents what we have previously called ‘Schwarzschild-like coordinate’, e.g., in (41) and (45). Please, do not confuse it with the radial coordinate in isotropic coordinates (1).}, which we now denote us:\footnote{Note that the $r$ in (47), as well as in the rest of this section, represents what we have previously called ‘Schwarzschild-like coordinate’, e.g., in (41) and (45). Please, do not confuse it with the radial coordinate in isotropic coordinates (1).}

$$ds^2 = e^{\alpha(r)} dt^2 - e^{\beta(r)} dr^2 - r^2 e^{\mu(r)} d\Omega_{n+2}, \hspace{1cm} (53)$$

Note that the $r$ in (47), as well as in the rest of this section, represents what we have previously called ‘Schwarzschild-like coordinate’, e.g., in (41) and (45). Please, do not confuse it with the radial coordinate in isotropic coordinates (1).
with
\[ e^{\alpha(r)} = \left(1 - \frac{2M/\varepsilon}{r^{n+1}}\right)^\varepsilon, \quad e^{\beta(r)} = \left(1 - \frac{2M/\varepsilon}{r^{n+1}}\right)^{-(n+\varepsilon)/(n+1)}, \quad e^{\mu(r)} = \left(1 - \frac{2M/\varepsilon}{r^{n+1}}\right)^{(1-\varepsilon)/(n+1)}. \] (54)

The interior of a spherical star is assumed to be described by
\[ ds^2 = e^{\nu(R)}dt^2 - e^{\lambda(R)}dR^2 - R^2d\Omega_{(2+n)}, \] (55)
which is the most general line element describing the interior of a spherical non-rotating star in \((4+n)\) dimensions. The functions \(e^{\nu(R)}\) and \(e^{\lambda(R)}\) are solutions of the Einstein field equations (A-1).

The exterior boundary of the star is a hypersurface \(\Sigma\) defined as \(R = R_b\), and \(r = r_b\) from inside and outside respectively. Standard matching conditions require continuity of the first and second fundamental forms at \(\Sigma\) [62]. For the metrics under consideration they demand
\[ e^{\nu(R_b)} = e^{\alpha(r_b)}, \quad R_b = r_b e^{\mu(r_b)/2}, \] (56)
and
\[ e^{-\lambda(R_b)/2} \left. \frac{d\alpha}{dr} \right|_{r=r_b} = e^{-\beta(r_b)/2} \left. \frac{d\mu}{dr} \right|_{r=r_b}, \]
\[ e^{-\lambda(R_b)/2} = e^{-\beta(r_b)/2} \left[ 1 + \frac{1}{2} \left( \frac{d\mu}{dr} \right) \right] |_{r=r_b} R_b e^{\mu(r_b)/2}. \] (57)

Setting
\[ e^{-\lambda(R)} = 1 - \frac{2m(R)}{R^{n+1}}, \] (58)
from the second equation in (57) we get
\[ m(R_b) = \frac{1}{2} \left\{ 1 - \left[ 1 - \left( \frac{\phi_g}{\varepsilon} \left( \frac{R_b}{r_b} \right)^{n+1} \right) \right]^2 \left[ 1 - \frac{2\phi_g}{\varepsilon} \left( \frac{R_b}{r_b} \right)^{n+1} \right]^{-1} \}, \] (59)
where \(\phi_g\) is the surface gravitational potential, viz.,
\[ \phi_g \equiv \frac{M}{R_b^{n+1}}. \] (60)

The above conditions require continuity of \(T^1_1\), i.e. the “radial” pressure, across \(\Sigma\). We find,
\[ 8\pi R_b^2 p(R_b) = \frac{(n+1)(n+2)(1-\varepsilon^2)\phi_g^2}{2\varepsilon^2} \left( \frac{R_b}{r_b} \right)^{2(n+2)} \left[ 1 - \frac{2\phi_g}{\varepsilon} \left( \frac{R_b}{r_b} \right)^{n+1} \right]^{(\varepsilon-2)/(n+1)}. \] (61)

**Buchdahl’s limit for \(\varepsilon = 1\):** We note that for \(\varepsilon = 1\), we recover the Schwarzschild case, i.e., \(R_b = r_b\) and
\[ m(R_b) = M, \quad p(R_b) = 0. \] (62)
Substituting this into (A-21), after a simple algebra we obtain
\[ \frac{M}{R_b^{n+1}} \leq \frac{2(n+2)}{(n+3)^2}. \] (63)
Thus,
\[ g_{00} \geq \left( \frac{n+1}{n+3} \right)^2. \] (64)
For \( n = (0, 1, 2, 3, 4, 5, 6, 7) \), i.e. \( D = (4, 5, 6, 7, 8, 9, 10, 11) \) we find

\[
\frac{M}{R_b^{n+1}} \leq (4/9, 0.375, 0.320, 0.278, 0.245, 0.219, 0.198, 0.180),
\]
\[
g_{00} \geq (1/9, 0.250, 0.360, 0.444, 0.510, 0.563, 0.605, 0.640),
\]

respectively. We note that for \( n = 0 \), which corresponds to a matter distribution that has spherical symmetry in three spatial dimensions (rather than in \( n + 3 \)), we recover the usual Schwarzschild values as expected.

An important observational parameter is the redshift \( Z(R) = 1/\sqrt{g_{00}(R)} - 1 \) of the light emitted from a point \( R \) inside the sphere to infinity. For Schwarzschild-Tangherlini’s exteriors \( Z_b \), the redshift of the light emitted from the boundary surface, is given by

\[
Z_b^{\text{max}} \leq \frac{2}{n+1}
\]

**Buchdahl’s limit for \( \varepsilon \neq 1 \):** In this case \( m(R_b) \neq M \) and \( p(R_b) \neq 0 \). From the second equation in (56) we get

\[
\phi_g = \frac{\varepsilon}{2 (R_b/r_b)^{n+1}} \left[ 1 - \left( \frac{R_b}{r_b} \right)^{2(n+1)/(1-\varepsilon)} \right].
\]

Consequently, at the surface of a body

\[
g_{00}(R_b) = e^\nu(R_b) = \left[ 1 - \frac{2\phi_g}{\varepsilon} \left( \frac{R_b}{r_b} \right)^{n+1} \right]^{\varepsilon} = \left( \frac{R_b}{r_b} \right)^{2\varepsilon(n+1)/(1-\varepsilon)}.
\]

Substituting (59), (61) and (67) into (A-21) we obtain an inequality for \( (R_b/r_b) \). Unfortunately, it is very cumbersome, so we omit it here. However, it can be solved numerically for any given value of \( n \) and \( 0 < \varepsilon < 1 \). Then, using the allowed values of \( (R_b/r_b) \) in (69) we obtain the range of \( \phi_g \). Here we present the solution for \( n = 0, 1, 2 \) and some selected values\(^9\) of \( \varepsilon \), viz.,

\[
\varepsilon = (0.90, 0.80, 0.75, 0.60).
\]

We also present the redshift of the light emitted from the boundary surface \( Z_b = Z(R_b) = 1/\sqrt{g_{00}(R_b)} - 1 \), which in the present case is given by

\[
Z_b = \left( \frac{R_b}{r_b} \right)^{\varepsilon(n+1)/(\varepsilon-1)} - 1.
\]

- For \( n = 0 \), which corresponds to spherical symmetry in ordinary three space, the boundary conditions are satisfied for

\[
(0.882, 0.746, 0.669, 0.383) \leq \frac{R_b}{r_b} < 1 \quad \Longrightarrow \quad 0 < \phi_g \leq (0.469, 0.508, 0.538, 0.777).
\]

It is important to note that the upper values of \( \phi_g \) give the Buchdahl’s limit for a star of uniform density (See equation (83) in [22]). Using (58) and (70) we find

\[
g_{00}(R_b) \geq (0.104, 0.096, 0.090, 0.056), \quad \Longrightarrow \quad Z_b \leq (2.096, 2.229, 2.340, 3.219).
\]

- For \( n = 1 \), and the same values selected in (69), we get

\[
(0.962, 0.915, 0.887, 0.778) \leq \frac{R_b}{r_b} < 1 \quad \Longrightarrow \quad 0 < \phi_g \leq (0.383, 0.397, 0.407, 0.455),
\]

and

\[\text{We emphasize that for } \varepsilon > 1 \text{ the boundary conditions do not admit solutions in the realm of positive real numbers.}\]
\[ g_{00}(R_b) \geq (0.248, 0.241, 0.237, 0.222) \quad \implies \quad Z_b \leq (1.008, 1.035, 1.053, 1.124). \]

- For \( n = 2 \), under the same conditions, we find

\[ (0.981, 0.958, 0.944, 0.888) \leq \frac{R_b}{r_b} < 1 \quad \implies \quad 0 < \phi_g \leq (0.326, 0.329, 0.334, 0.356), \quad (73) \]

and

\[ g_{00}(R_b) \geq (0.355, 0.357, 0.354, 0.343) \quad \implies \quad Z_b \leq (0.679, 0.673, 0.680, 0.707). \]

The above calculations show how Buchdahl’s limit depends on \( n \) and \( \varepsilon \): (i) For a fixed \( n \), the upper limit of \( \phi_g \) increases as we go away from the Schwarzschild vacuum exterior, i.e., with the increase of \((1 + \varepsilon)\); (ii) For a fixed \( \varepsilon \), the upper limit of \( \phi_g \) decreases with the increase of \( n \), which implies that the effects of gravity are stronger in 4D than in any other number of dimensions.

5 Summary and concluding remarks

The main question under investigation here has been how a possible deviation from the Schwarzschild vacuum exterior can affect the compactness of spherical stars in equilibrium. We have discussed this question within the context of Kaluza-Klein gravity, by using a general class of Ricci-flat metrics in \( D = (4 + n + m) \)-dimensions, namely (6)-8, which generalize a number of solutions in the literature.

Following a standard technique, based on the assumption that the gravitational action has the standard form (42) in any number of dimensions, we have reduced the \( m \) external dimensions. We have seen that the reduction procedure flattens out the rich diversity of higher-dimensional solutions. The effective metrics in \((4 + n)\) constitute a one-parameter family of asymptotically-flat metrics given by (47), which contain Schwarzschild-Tangherlini’s spacetimes in \((4 + n)\) dimensions for \( \varepsilon = 1 \) \((m = 0, \sigma = 0)\). For any other value of \( \varepsilon \) the effective spacetime is not Ricci-flat because \( R_{11} \neq 0 \), while all the other components of the Ricci tensor vanish identically (50). The fact that \( R^0_0 = 0 \) implies that the effective (or geometrical) matter in \((4 + n)\) satisfies the equation of state \((\rho + p_r + (n + 2)p_\perp = 0)\), which generalizes to \( n \) dimensions the well-known equation of state \((\rho + 3p) = 0\) for nongravitating matter in 4D (gravitational or Tolman-Wittaker mass is proportional to \( R^0_0 \)). Thus, for any value of \( \varepsilon \), the effective spacetime is similar to Schwarzschild vacuum in the sense that it has no effect on gravitational interactions. Consequently, we can interpret it as describing the gravitational field outside of a spherical star.

To put the discussion in perspective, let us notice that in the Randall & Sundrum braneworld scenario the concept of empty space requires only the vanishing of the Ricci scalar, while the components of the Ricci tensor are unknown without specifying the metric in the bulk (See, e.g., [24]). Here the situation is much more restricted because only \( R^i_1 \) is allowed to be different from zero. The crucial point is that Kaluza-Klein and Braneworld theories are alike in one important aspect: the effective vacuum spacetime outside of an isolated star does not have to be Ricci-flat, as in conventional 4D general relativity.

The line element (47) provides the Kaluza-Klein corrections to the Minkowski metric at large distances from any system of bodies (52), which could serve in astrophysical observations to detect possible deviations from general relativity. Also, it allowed us to study the question under consideration. Namely, using the standard matching conditions, i.e. the continuity of the first and second fundamental forms across the boundary, as well as the generalized Buchdahl’s inequality (A-21), we have obtained the compactness limit for various values of \( \varepsilon \) and \( n \), for any perfect fluid star with a mass density which does not increase outward.

Our analysis shows that in Kaluza-Klein gravity the compactness limit of a star can be larger than 1/2, without being a black hole: the general-relativistic upper limit \( M/R < 4/9 \) is increased as we go away from the Schwarzschild vacuum exterior. Our results are consistent with our previous findings in [22], [28]. They show that, as in general relativity, the compactness limit can be saturated in the case of stars with uniform proper density from the condition that the isotropic pressure does not become infinity at the center.
It should be noted that, for any \( n \), the boundary conditions require \( 0 < \varepsilon \leq 1 \), otherwise they have no real solutions. From (44) it follows that \( \varepsilon \to 0 \) as \( k \to m/(n+2) \), which according to (11) corresponds to the maximum value of \( \sigma \). In the other extreme, for \( \varepsilon = 1 \) we recover the Schwarzschild-Tangherlini spacetimes. From a physical point of view \( 0 < \varepsilon \leq 1 \) ensures the positivity of the effective energy density (48).

Our approach allowed us to determine some similarities and differences between spacetimes with different number of internal dimensions: (i) Regardless of \( \varepsilon \), they satisfy similar equations, viz., (49), (50); (ii) The corrections to Minkowski metric (52) manifestly depend on \( n \); (iii) the effects of gravity decrease with the increase of \( n \).

**Appendix A: Buchdahl’s inequalities in \( D \)-dimensions**

In this appendix, following our previous work [28] we show how Buchdahl’s inequalities can be extended to any number of internal dimensions. We start with the Einstein field equations in \( D \) dimensions,

\[
R_{AB} = 8\pi G \left[ T_{AB} - \frac{1}{D-2} g_{AB} T \right],
\]

where \( G, T_{AB}, \) and \( T \) represent: the gravitational constant; the energy momentum tensor in \( D \)-dimensions; and its trace respectively. In what follows we set \( G = 1 \).

We will consider the \( D \)-dimensional spherically symmetric metric, given by

\[
ds^2 = e^{\nu(R)} dt^2 - e^{\lambda(R)} dR^2 - R^2 d\Omega_{(2+n)},
\]

where \( d\Omega_{(2+n)} \) is the line element on a unit \((n+2)\) sphere; \( n = D - 4 \).

Now, let us assume that the \( D \)-dimensional energy-momentum tensor has the form

\[
T^A_B = \text{diag}(\rho, -p, -p, ..., -p),
\]

where \( \rho \) is the energy density and \( p \) is the isotropic pressure. With this choice the field equations (A-1) reduce to

\[
e^{-\lambda(R)} \left[ \frac{1}{R} \left( \frac{d\lambda}{dR} \right) - \frac{n + 1}{R^2} \right] + \frac{n + 1}{2R^2} = \frac{16\pi \rho}{n + 2}.
\]

\[
e^{-\lambda(R)} \left[ \frac{1}{R} \left( \frac{d\nu}{dR} \right) + \frac{n + 1}{2R^2} \right] \left[ \frac{n + 1}{R^2} \right] = \frac{16\pi p}{n + 2}.
\]

\[
\frac{1}{2} \frac{d^2 \nu}{dR^2} + \frac{1}{4} \left( \frac{d\nu}{dR} \right)^2 - \frac{1}{4} \left( \frac{d\lambda}{dR} \right) \left( \frac{d\nu}{dR} \right) - \frac{1}{2R} \left[ (n + 1) \left( \frac{d\lambda}{dR} \right) + \left( \frac{d\nu}{dR} \right) \right] + \frac{n + 1}{2R^2} \left[ e^{\lambda(R)} - 1 \right] = 0.
\]

We note that the density and pressure satisfy the relation

\[
\frac{dp}{dR} = -\frac{(\rho + p)}{2} \frac{d\nu}{dR},
\]

which is equivalent to the conservation equation \( T^B_{A:B} = 0 \).

The isotropy condition given by (A-6) takes a remarkable simple form with the introduction of the following notation [63]

\[
e^{-\lambda} = 1 - \frac{2m(R)}{R^{n+1}} = Z, \quad e^{\nu(R)} = Y^2, \quad R^2 = x.
\]

and

\[
u = \int_0^x \frac{dx'}{\sqrt{Z(x')}}.
\]
Indeed, (A-6) reduces to
\[ 2 \frac{d^2Y}{du^2} = (n+1) \frac{Y}{\frac{m}{R^{n+1}}} \] (A-10)

The function \( m(R) \) can be obtained from the integration of (A-4), viz.,
\[ m(R) = \frac{8\pi}{n+2} \int_0^R \rho(\bar{R}) \bar{R}^{n+2} d\bar{R}, \] (A-11)

where the constant of integration has been set equal to zero to remove singularities at the origin. Thus,
\[ \frac{d}{dR} \left( \frac{m}{R^{n+3}} \right) = \frac{8\pi \rho(R)}{(n+2)R} - \frac{(n+3)m(R)}{R^{n+4}} \] (A-12)

Now, following Buchdahl we assume that the energy density is positive and does not increase outward, i.e.,
\[ \frac{d\rho}{dR} \leq 0. \] (A-13)

This implies that \( \rho(\bar{R}) \geq \rho(R) \) in (A-11). Consequently,
\[ m(R) \geq \frac{8\pi R^{n+3} \rho(R)}{(n+2)(n+3)}. \]

Substituting into (A-12) we find
\[ \frac{d}{dR} \left( \frac{m}{R^{n+3}} \right) \leq 0. \] (A-14)

We note that the quantity \( m/R^{n+3} \) can be identified with the mean density of the fluid sphere in \( D \) dimensions.

Now Eq. (A-10) gives
\[ \frac{d^2Y}{du^2} \leq 0, \] (A-15)

which means that \( dY/du \) decreases monotonically. This in turn implies
\[ \frac{dY}{du} \leq \frac{Y(u) - Y(0)}{u}. \] (A-16)

Since both \( Y(0) \) and \( u \) are non-negative, it follows that
\[ Y^{-1} \frac{dY}{du} \leq \frac{1}{u}. \] (A-17)

In terms of the original variables this equation reads
\[ \left( 1 - \frac{2m(R)}{R^{n+1}} \right)^{1/2} \frac{dv}{dR} \leq 2R \left[ \int_0^R \bar{R} \left( 1 - \frac{2m(\bar{R})}{R^{n+1}} \right)^{-1/2} d\bar{R} \right]^{-1}. \] (A-18)

Using the fact that the average density decreases outward (A-14), we can evaluate the integral in (A-18) as follows
\[ \int_0^R \bar{R} \left( 1 - \frac{2m(\bar{R})}{R^{n+1}} \right)^{-1/2} d\bar{R} \geq \int_0^R \bar{R} \left( 1 - \frac{2m(R)}{R^{n+3}} \right)^{-1/2} d\bar{R} \]
\[ = \frac{R^{n+3}}{2m(R)} \left[ 1 - \left( 1 - \frac{2m(R)}{R^{n+1}} \right)^{1/2} \right]. \] (A-19)
On the other hand, using (A-5) and (A-8), \( \frac{dv}{dR} \) can be expressed as

\[
\frac{dv}{dR} = 2 \left[ \frac{(n + 2)(n + 1)m(R)/R^{n+1} + 8\pi p R^2}{(n + 2)(1 - 2m(R)/R^{n+1}) R} \right].
\] (A-20)

Now, substituting (A-19) and (A-20) into (A-18) we obtain

\[
\frac{(n + 2)(n + 1)m(R)/R^{n+1} + 8\pi p R^2}{(n + 2) \left(1 - \left(1 - \frac{2m(R)}{R^{n+1}}\right)^{1/2}\right)^{1/2}} \leq 2m(R) \left[1 - \left(1 - \frac{2m(R)}{R^{n+1}}\right)^{1/2}\right]^{-1}.
\] (A-21)

Evaluating this expression at the outer surface of a static spherical star, and using the standard matching conditions, i.e. the continuity of the first and second fundamental forms across the boundary, we obtain the compactness limit of a such star for any given exterior spacetime. In particular, for the Schwarzschild exterior, (A-21) leads to the well-known upper mass limit \( M/R_b \leq 4/9 \).

References

[1] H. A. Buchdahl, *Phys. Rev.* **116** (1959) 1027.

[2] G.C. Segré, “Physics in more than four-dimensions, another look at the Kaluza-Klein theory”, in Cosmology and Elementary particles. Proceedings of the first winter school of physics. World Scientific Publishing Co. Pte. Ltd. 1989.

[3] J. M. Overduin and P. S. Wesson, *Phys.Rept.* **283** (1997) 303, arXiv:gr-qc/9805018

[4] P.S. Wesson, *Space-Time-Matter* (World Scientific Publishing Co. Pte. Ltd. 1999).

[5] L. Randall and R. Sundrum, *Phys. Rev. Lett.* **83** (1999) 4690; arXiv:hep-th/9906064

[6] N. Arkani-Hamed, S. Dimopoulos, G. Dvali and N. Kaloper, *Phys.Rev.Lett.* **84** (2000) 586, arXiv:hep-th/9907209

[7] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, *Phys.Lett.* **B429** (1998) 263, arXiv:hep-ph/9803315

[8] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, *Phys.Rev.* **D59** (1999) 086004, arXiv:hep-ph/9807344.

[9] I. Antoniadis, *Phys. Lett.* **B246** (1990) 3171.

[10] R. Maartens, *Phys. Rev.* **D62** (2000) 084023, arXiv:hep-th/0004166

[11] Roy Maartens, Frames and Gravitomagnetism, ed. J Pascual-Sanchez et al. (World Sci., 2001) pp 93-119, arXiv:gr-qc/0101059.

[12] N. Dadhich and S.G. Gosh, *Phys. Lett.* **B518** (2001) 1, arXiv:hep-th/0101019.

[13] M. Govender and N. Dadhich, *Phys.Lett.* **B538** (2002) 233, arXiv:hep-th/0109086.

[14] C. Germani and Roy Maartens, *Phys. Rev.* **D64** (2001) 124010, arXiv:hep-th/0107011.

[15] M. Bruni, C. Germani and R. Maartens, *Phys. Rev. Lett.* **87** (2001) 231302, arXiv:gr-qc/0108013

[16] G. Kofinas and E. Papantonopoulos, *J. Cosmol. Astropart. Phys.* **12** (2004) 11, arXiv:gr-qc/0401047

[17] M. Visser and D. L. Wiltshire, *Phys.Rev.* **D67** (2003) 104004, arXiv:hep-th/0212333

[18] N. Dadhich, R. Maartens, P. Papadopoulos and V. Rezania, *Phys.Lett.* **B487** (2000) 1, arXiv:hep-th/0003061
[19] K.A. Bronnikov, H. Dehnen and V.N. Melnikov, Phys.Rev. D68 (2003) 024025, arXiv:gr-qc/0304068.
[20] K.A. Bronnikov and S-W Kim, Phys.Rev. D67 (2003) 064027, arXiv:gr-qc/0212112.
[21] R. Casadio, A. Fabbri and L. Mazzacurati, Phys.Rev. D65 (2002) 084040, arXiv:gr-qc/0111072.
[22] J. Ponce de Leon, Class.Quant.Grav. 24 (2007) 1755, arXiv:gr-qc/0701129.
[23] J. Ponce de Leon, Int. J. Mod. Phys. D18 (2009) 251, arXiv:gr-qc/0703094.
[24] J. Ponce de Leon, Class.Quant.Grav. 25 (2008) 075012, arXiv:gr-qc/0711.4415.
[25] J. Ponce de Leon, Grav.Cosmol. 14 (2008) 65, arXiv:0711.0998.
[26] L. Bowers and E.P.T. Liang, Astrophys. J. 188 (1974) 657.
[27] J. Ponce de Leon, Phys. Rev. D37 (1988) 309.
[28] J. Ponce de Leon and N. Cruz, Gen.Rel.Gravit. 32 (2000) 1207, arXiv:gr-qc/0207050.
[29] A. Chodos and S. Detweiler, Gen. Rel. Gravit. 14 (1992) 879.
[30] J.A. Casas, C.P. Martin and A.H. Vozmediano, Phys. Lett. B 186 (1987) 29.
[31] V.S. Gurin and A.P. Trofimenko, Phys. Lett. B 241 (1990) 328.
[32] P.S. Wesson and J. Ponce de Leon, Class. Quantum Grav. 11 (1994) 1341.
[33] P. Lim, J. Overduin and P.S. Wesson, J. Math. Phys. 36 (1995) 6907.
[34] A.G.Agnese, A.P. Billyard, H. Liu and P.S. Wesson, Gen. Rel. Gravit. 31 (1999) 527.
[35] W.N. Sajko and P.S. Wesson, Gen. Rel. Gravit. 32 (2000) 1381.
[36] H. Liu and J. Overduin, Astrophys. J. 538 (2000) 386, arXiv:gr-qc/0003034.
[37] J. Overduin, Phys. Rev. D62 (2000) 102001, arXiv:gr-qc/0007047.
[38] K. Lake, Class. Quant. Grav. 23 (2006) 5876, arXiv:gr-qc/0606005.
[39] J. Ponce de Leon, Int.J.Mod.Phys. D17 (2008) 237, arXiv:gr-qc/0611082.
[40] T. Dereli, Phys. Lett. B161 (1985) 307.
[41] L. Sokolowski and B. Carr, Phys. Lett. B 176 (1986) 334.
[42] S. Chatterjee, Astron. Astrophys. 230 (1990) 1.
[43] H. Liu, Gen. Rel. Gravit. 23 (1991) 759.
[44] F. R. Tangherlini, Nuovo Cimento 27 (1963) 636.
[45] R.C. Myers and M.J. Perry, Annals of Physics 172 (1986) 304.
[46] J. Ponce de Leon, Grav. Cosmol. 15 (2009) 345, arXiv:0905.2010.
[47] D. Kramer, Acta Phys. Polon. B2 (1970) 807.
[48] A. Davidson and D. Owen, Phys. Lett. B 155 (1985) 247.
[49] R.S. Millward, “A five-dimensional Schwarzschild-like solution”, arXiv:gr-qc/0603132.
[50] L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, Fourth Edition (Butterworth-Heinemann, 2002).

[51] S.B. Fadeev, V.D. Ivashchuk and V.N. Melnikov, *Phys. Lett.* A **161** (1991) 98.

[52] V.D. Ivashchuk and V.N. Melnikov, *Grav. Cosmol.* 1 (1995) 133, arXiv:hep-th/9503223

[53] D.J. Gross and M.J. Perry, *Nucl. Phys.* B**226** (1983) 29.

[54] D.J. Gross, M.J. Perry and R.D. Sorkin, *Phys. Rev. Lett.* 51 (1983) 87.

[55] L. Dolan and M.J. Duff, *Phys. Rev. Lett.* 52 (1984) 14.

[56] P.S. Wesson, *Phys. Essays, Orion* 5 (1992) 591.

[57] Z. Fisher, *Zh. Eksp. Teor. Fiz.* 18 (1948) 636 (in Russian), arXiv:gr-qc/9911008

[58] A.I. Janis, E.T. Newman and J. Winicour, *Phys. Rev. Lett.* 20 (1968) 878.

[59] J.R. Gott and M.J. Rees, *Mon. Not. R. Astron. Soc.* 227 (1987) 453.

[60] E. Kolb, *Astrophys J.* 344 (1989) 543.

[61] J. Ponce de Leon, *Gen. Rel. and Gravit.* 11 (1993) 1123.

[62] W. Israel, *Nuovo Cimento* B**44**, (1966) 1;[Erratum-ibid. B**48** (1967) 463].

[63] J. Ponce de Leon, *J. Math. Phys.* 29 (1988) 197.