Ballistic transport through chaotic cavities: Can parametric correlations and the weak localization peak be described by a Brownian motion model?

Jochen Rau
Max-Planck-Institut für Kernphysik, Postfach 103980, 69029 Heidelberg, Germany
(Submitted to Phys. Rev. B, August 23, 1994)

A Brownian motion model is devised on the manifold of $S$-matrices, and applied to the calculation of conductance-conductance correlations and of the weak localization peak. The model predicts that (i) the correlation function in $B$ has the same shape and width as the weak localization peak; (ii) the functions behave as $\propto 1 - O(B^2)$, thus excluding a linear line shape; and (iii) their width increases as the square root of the number of channels in the leads. Some of these predictions agree with experiment and with other calculations only in the limit of small $B$ and a large number of channels.

72.10.Bg, 72.20.My, 05.45.+b, 05.40.+j

I. INTRODUCTION

The experimental observation of quantum interference effects in submicron devices has triggered rapidly growing interest in the properties of mesoscopic systems. Many investigations have focused on electronic transport through quasi-one-dimensional disordered structures and (more recently) in ballistic transport through chaotic cavities. In this paper I focus on the ballistic regime; moreover, I assume the ballistic cavities to be classically chaotic. For such systems the ratios $C(\Delta X)/C(0)$ and $W(B)/W(0)$ have been calculated semiclassically. For example, semiclassical analysis yields a Lorentzian,

$$C(\Delta X) := \langle \delta G(X) \delta G(X + \Delta X) \rangle,$$

(1)

where $\delta G := G - \langle G \rangle$ denotes the deviation of the conductance from its mean; while width and shape of the weak localization peak are encoded in the function

$$W(B) := \langle G \rangle(B) - \langle G \rangle(B = \infty).$$

(2)

(The averages are taken over an ensemble of probes with $B$ in the case of disordered structures – different impurity configurations, or $B$ in the case of chaotic cavities – different shapes. By a widely accepted ergodicity hypothesis, ensemble averages are equivalent to the average over some parameter such as $k$ or $B$.) Both $C(0)$ and $W(0)$ are well known, and the objects to be investigated are the ratios $C(\Delta X)/C(0)$ and $W(B)/W(0)$.

In this paper I focus on the ballistic regime; moreover, I assume the ballistic cavities to be classically chaotic. For such systems the ratios $C(\Delta X)/C(0)$ and $W(B)/W(0)$ have been calculated semiclassically. For example, semiclassical analysis yields a Lorentzian,

$$C(\Delta k)/C(0) = [1 + (\Delta k/\gamma_{cl})^2]^{-1},$$

(3)

or a Lorentzian-squared,

$$C(\Delta B)/C(0) = [1 + (\Delta B/\alpha_{cl}\phi_0)^2]^{-2},$$

(4)

respectively, for the correlation function. (Here $\gamma_{cl}$ and $\alpha_{cl}$ are characteristic inverse length and characteristic inverse enclosed area, respectively, of the classical trajectories: $\phi_0 = hc/e$ is the elementary flux quantum.) The semiclassical result for $C(\Delta B)/C(0)$ has been checked experimentally and appears to be in excellent agreement with the data.

For the weak localization peak, semiclassical theory predicts a Lorentzian

$$W(B)/W(0) = [1 + 2(B/\alpha_{cl}\phi_0)^2]^{-1}.$$}

(5)

This prediction, too, agrees well with experiments and with numerical simulations. We observe that the correlation function in $B$ and the weak localization peak look almost identical. Indeed, for small $B$

$$C(B)/C(0) \approx W(B)/W(0) \approx 1 - 2(B/\alpha_{cl}\phi_0)^2;$$

(6)
only the tails at large $B$ differ.

The above semiclassical calculations involve various approximations. In addition to (i) the semiclassical limit proper, it is assumed that (ii) the magnetic field is sufficiently weak so as to affect only phases, but not the classical trajectories, and that (iii) in the case of weak localization, only symmetry-related paths contribute (‘diagonal approximation’). The first two assumptions are expected to be justified only in the limit of weak fields and a large number of channels; while the third assumption has in fact been shown to be too crude. Moreover, a recent study has revealed that important contributions to weak localization stem from diagrams which appear to have no semiclassical analogues. In view of these theoretical limitations it is quite remarkable that semiclassics is so successful in describing the shapes of $C$ and $W$. Nevertheless, it seems highly desirable to derive parametric correlation functions and the weak localization peak by some alternative method.

The above results (3), (4) and (5) are universal in the sense that the geometry of the probe enters only via the characteristic scale $\gamma_c$ or $\alpha_d$. Such universality suggests that the phenomena in question should lend themselves to a treatment in the framework of random matrix theory. Indeed, Pfluhar et al. showed that the Lorentzian shape of the weak localization peak can be derived with Hamiltonian random matrix theory. In addition, they succeeded in proving that the width of the weak localization peak increases essentially as the square root of the number of channels. (This is a result which in a semiclassical theory can be inferred only indirectly.) As regards parametric correlations, Altland employed Hamiltonian random matrix theory to show that in the diffusive regime and quasi-one-dimensional limit, $C(\Delta E)$ is a Lorentzian. ($E$ denotes the Fermi energy of the electrons.) Other macroscopic calculations of parametric correlations have so far focused on correlations of the level density, rather than on correlations of the conductance. Their success certainly encourages further applications of macroscopic random matrix models.

A particularly simple and (arguably) generic model for any kind of parametric dependence is furnished by Dyson’s Brownian motion model. It is based on the assumption that variation of the parameter $X$ amounts to a random walk on the manifold of Hamiltonians, a process governed by a diffusion (or Fokker-Planck) equation. Beenakker successfully employed this model to calculate density-density correlations, and suggested that it might also be applied to the response of transmission eigenvalues (and hence of the conductance) to an external perturbation. This is difficult, however, because the conductance – in contrast to the level density – cannot be easily expressed as a function on the manifold of Hamiltonians. This difficulty can be avoided if one considers random $S$-matrices rather than random Hamiltonians. Random $S$-matrix theory, while consistent with Hamiltonian calculations, in the limit of a large number of channels, allows a much more direct calculation of conductance properties. Therefore, I propose that the Brownian motion model be reincarnated as a random walk on the manifold of $S$-matrices. To devise such a Brownian motion model for $S$-matrices, and to study its implications for conductance-conductance correlations and the weak localization peak, is the purpose of this paper.

II. THEORETICAL PRELIMINARIES

I will first review some basic definitions. Electronic transport through a microstructure constitutes a quantum mechanical scattering problem. Scattering at a probe with two leads, each with $N$ channels, is described by the $S$-matrix

$$S = \begin{pmatrix} t & t' \\ t' & t \end{pmatrix}$$

(7)

where $t$, $t'$ and $r$, $r'$ are the $N \times N$ transmission and reflection matrices, respectively. Both $(t' t)$ and $(t' t')$ have the same set of eigenvalues $\{\tau_\alpha \in [0,1]\}$. In terms of these eigenvalues, the $S$-matrix may be parametrized

$$S = \begin{pmatrix} v^{(1)} & 0 \\ 0 & v^{(2)} \end{pmatrix} \begin{pmatrix} -\sqrt{1-\tau} & \sqrt{\tau} \\ \sqrt{\tau} & \sqrt{1-\tau} \end{pmatrix} \begin{pmatrix} v^{(3)} & 0 \\ 0 & v^{(4)} \end{pmatrix}$$

(8)

with $\tau = \text{diag}(\tau_\alpha)$ and the $\{v^{(i)}\}$ unitary $N \times N$ matrices. The $\{\tau_\alpha\}$ and $\{v^{(i)}\}$ will be referred to as ‘radial’ and ‘angular’ degrees of freedom, respectively. In the absence of time-reversal symmetry (‘unitary case,’ $\beta = 2$) the $\{v^{(i)}\}$ are arbitrary; whereas the presence of time-reversal symmetry (‘orthogonal case,’ $\beta = 1$) imposes the additional constraint $S^T = S$ and hence $v^{(3)} = v^{(1)} T$, $v^{(4)} = v^{(2)} T$. Depending upon the symmetry, the manifold of $S$-matrices has the dimension

$$d_\beta := \begin{cases} 2N^2 + N & : \beta = 1 \\ 4N^2 & : \beta = 2 \end{cases} .$$

(9)

The manifold is endowed with a metric

$$g(dS_1, dS_2) := \text{tr}(dS_1^T dS_2)$$

(10)

which is invariant under the group action $dS_1 \rightarrow U dS_1 V$ ($U, V$ unitary). With respect to this metric, radial variations ($\tau_\alpha \rightarrow \tau_\alpha + d\tau_\alpha$) and angular variations ($v^{(i)} \rightarrow v^{(i)} + dv^{(i)}$) of the $S$-matrix are orthogonal; the metric tensor $(g_{\mu\nu})$ is thus built of two blocks, one ‘radial’ and one ‘angular.’ The radial part of the metric tensor reads

$$g_{ab} := g(\partial S / \partial \tau_a, \partial S / \partial \tau_b) = \frac{1}{2\tau_a(1-\tau_a)} \delta_{ab} ,$$

(11)

with inverse

$$g^{ab} = 2\tau_a(1-\tau_a) \delta^{ab} .$$

(12)
The metric defines an invariant volume element (Haar measure)
\[
d\mu(S) = \sqrt{|\det g|} \prod_a dx^a = J_\beta(\{\tau\}) \prod_a d\tau_a \prod d[\text{angles}] \tag{13}
\]
where \(J_\beta\) is the Jacobian
\[
J_\beta = \left\{ \prod_{a<b} |\tau_a - \tau_b|, \prod_c 1/\sqrt{\tau_c} : \beta = 1 \right\} = \left\{ \prod_{a<b} |\tau_a - \tau_b|, \prod_c 1/\sqrt{\tau_c} : \beta = 2 \right\} \tag{14}
\]
(The exact form of the angular factor need not concern us.) This volume element in turn permits the definition of probability densities \(P(S)\), with normalization
\[
\int d\mu(S) P(S) = 1 \tag{15}
\]
In the above parametrization \(\{\tau_a, \nu^{(i)}\}\), the conductance can be easily expressed as a function of \(S\). According to Landauer’s formula it reads
\[
G = (2e^2/h) \sum \tau_a \tag{16}
\]
(with the factor 2 accounting for spin). The sum \(T := \sum_a \tau_a\) is called the dimensionless conductance. Assuming a uniform probability density on the manifold of \(S\)-matrices, \(P(S) = \text{const}\), one obtains the well-known result
\[
\langle T \rangle_\beta = \frac{\beta N^2}{d_\beta} N = \frac{N}{2} - \delta_{\beta 1} \frac{N}{4N+2} \tag{17}
\]
and
\[
\text{var}_\beta(T) = \left\{ \frac{N(N+1)^2}{(2N+1)^2(2N+3)}, \frac{N^2}{(4N^2-1)} : \beta = 1 \right\} \tag{18}
\]
For \(N \to \infty\), these imply the universal magnitude of weak localization, \(\langle T \rangle_1 \to \langle T \rangle_2 \to (-1/4)\), and of conductance fluctuations, \(\text{var}_\beta(T) \to (1/8\beta)\).

III. BROWNIAN MOTION MODEL

In the Brownian motion model the probability density is no longer uniform and acquires an explicit dependence on the parameter \(X\). As \(X\) is being varied, \(X \to X + \Delta X\), the system executes a random walk on the manifold of \(S\)-matrices and, consequently, the probability density spreads according to a diffusion equation. In order to find the form of this diffusion equation, let us consider – as a simple example – a random walk on a flat \(d\)-dimensional manifold with Cartesian coordinates \(\{x^\mu, \mu = 1 \ldots d\}\). At each step the velocity \(\vec{v}\) of the moving ‘particle’ is random, with an isotropic distribution so that \(\langle v^\mu \rangle = 0\) and \(\langle v^\mu v^\nu \rangle = (1/d) \delta^\mu_\nu \delta^{\mu\nu}\). Hence
\[
P(X + \Delta X) = P(X) + \frac{1}{2d} (\Delta X)^2 \langle \vec{v}^2 \rangle \Delta P(X) + \ldots \tag{19}
\]
Upon identifying a fictitious ‘time’ \(t := (\Delta X)^2\) and the diffusion constant \(D := (1/2d) \langle \vec{v}^2 \rangle\), we arrive at the diffusion equation \(\partial P/\partial t = D \Delta P\). This is easily generalized to the (curved) manifold of \(S\)-matrices. There, too, increments \(\Delta X\) are related to a fictitious ‘time’ \(t\) by
\[
t = (\Delta X)^2 \tag{20}
\]
(Strictly speaking, this relationship holds only for infinitesimal parameter values; the exact relationship between \(t\) and finite values of the physical parameter remains unknown.) The diffusion equation then reads
\[
\frac{\partial}{\partial t} P = D_\beta \hat{\Delta}_\beta P \tag{21}
\]
Here \(\Delta_\beta\) is the Laplace-Beltrami operator (the generalization of the Laplace operator on curved Riemannian manifolds), and \(D_\beta\) is the diffusion constant
\[
D_\beta = \frac{1}{2d_\beta} \left\langle g \left( \frac{dS}{dX}, \frac{dS}{dX} \right) \right\rangle \tag{22}
\]
Both are labelled by the symmetry index \(\beta\). The average \(\langle g(\ldots)\rangle\) is a measure for the typical step size of the random walk; it in turn defines a characteristic ‘time’ scale
\[
t_0 := \left\langle g \left( \frac{dS}{dX}, \frac{dS}{dX} \right) \right\rangle^{-1} \tag{23}
\]
In order to make use of this model, one needs to know (i) the explicit form of the Laplace-Beltrami operator and (ii) how diffusion affects the conductance. The Laplace-Beltrami operator on an arbitrary Riemannian manifold with coordinates \(\{x^\mu\}\) is given by
\[
\hat{\Delta} = \sum_{\mu\nu} \frac{1}{\sqrt{|\det g|}} \frac{\partial}{\partial x^\mu} g^{\mu\nu} \sqrt{|\det g|} \frac{\partial}{\partial x^\nu} \tag{24}
\]
Its explicit form on the manifold of \(S\)-matrices, with parametrization \(\{\tau_a, \nu^{(i)}\}\), follows directly from the properties of the metric. As the metric tensor is composed of a radial and an angular block, the Laplace-Beltrami operator can be split into two parts,
\[
\hat{\Delta}_\beta = \hat{\Delta}_{\beta, \tau} + \hat{\Delta}_{\beta, \nu} \tag{25}
\]
The first containing derivatives with respect to \(\tau\) only, and the latter containing derivatives with respect to angles only. The two parts are called radial and angular, respectively. When applied to the conductance, only the
radial part \( \hat{\Delta}_\beta \) contributes. This radial part is obtained by inserting (12) and (13) into (24): all \( \tau \)-independent factors cancel, leaving
\[
\hat{\Delta}_{\beta,\tau} = 2 \sum_a \frac{1}{J_{\beta}} \frac{\partial}{\partial \tau_a} \left[ \tau_a (1 - \tau_a) J_\beta \frac{\partial}{\partial \tau_a} \right]. 
\] (26)

The Laplace-Beltrami operator has two important properties. First, it is Hermitian in the sense
\[
\int d\mu(S) f \hat{\Delta}_\beta h = \int d\mu(S) h \hat{\Delta}_\beta f 
\] (27)
provided \( f \) and \( h \) are sufficiently smooth. Second, the deviation of the dimensionless conductance from its mean, \( \delta G := \langle \hat{G} \rangle - \langle G \rangle \), or in the ‘Heisenberg picture,’
\[
\int d\mu(S) f \hat{\Delta}_\beta h = \int d\mu(S) h \hat{\Delta}_\beta f 
\] (27)
provided \( f \) and \( h \) are sufficiently smooth. Second, the deviation of the dimensionless conductance from its mean, \( \delta G := \langle \hat{G} \rangle - \langle G \rangle \), or in the ‘Heisenberg picture,’
\[
\hat{\Delta}_\beta \delta G = \hat{\Delta}_{\beta,\tau} \delta G
\] (28)
Since the Laplace-Beltrami operator is Hermitian, we are free to describe the evolution of expectation values \( \langle f \rangle = \int d\mu(S) f \) either in the ‘Schrödinger picture,’ \( P \rightarrow \hat{P}(t) \), or in the ‘Heisenberg picture,’ \( f \rightarrow \hat{f}(t) \). Choosing the latter, we see that diffusion causes the conductance to approach its equilibrium value exponentially:
\[
\delta G(t) := \exp[\tau_D \hat{\Delta}_\beta] \delta G = \exp[-t/(2Nt_0)] \delta G. 
\] (29)
This immediately yields the conductance-conductance correlation function
\[
C(t) = \langle \delta G(t) \delta G(0) \rangle = C(0) \cdot \exp[-t/(2Nt_0)]. 
\] (30)
The shape of the weak localization peak follows from a similar consideration. At \( t = B = 0 \) the probability density is non-zero only on the submanifold of time-reversal invariant \( \beta \)-matrices: \( P(0) \propto \delta(S^T - S) \). As \( B \) and hence \( t \) increase, this distribution diffuses over the entire submanifold of \( \beta \)-matrices. We thus find
\[
W(t) = \int d\mu(S) P(t) \delta_2 G = \int d\mu(S) P(0) \delta_2 G(t) = W(0) \cdot \exp[-t/(2Nt_0)]. 
\] (31)
Equations (30) and (31) together imply the key result of this paper:
\[
C(t)/C(0) = W(t)/W(0) = \exp[-t/(2Nt_0)]. 
\] (32)

IV. DISCUSSION

We are led to the following three conclusions. (i) The conductance-conductance correlation function in \( B \) has the same shape and width as the weak localization peak:
\[
C(B)/C(0) = W(B)/W(0) \quad \forall B. 
\] (33)
This is a stronger version of the approximate equality (2). Which shape the functions have, depends on the relationship between \( t \) and \( B \). For finite values of \( B \), this relationship is not known. (ii) For small \( B \), however, we know from equation (24) that \( t \sim B^2 \), hence
\[
C(B)/C(0) = W(B)/W(0) \approx 1 - O(B^2). 
\] (34)
This is not a very strong result, but it at least excludes linear line shapes of the kind which are observed in nonchaotic cavities. (iii) Provided we are in the regime where \( t^2 \sim B^2 \), the width of both functions increases as \( \sqrt{N} \).

The first conclusion is consistent with semiclassics only in the limit \( B \rightarrow 0 \) or \( N \rightarrow \infty \) (which implies \( \alpha \rightarrow \infty \)); the second conclusion is trivially correct; and the third conclusion agrees with Hamiltonian random matrix theory, except for very small \( N \). It therefore seems that our Brownian motion model is exact in the limit of small parameter values and a large number of channels. Outside this limit, however, the model can describe the phenomena at hand only qualitatively, not quantitatively.

The partial failure of the Brownian motion model reveals that the ‘walk’ on the manifold of \( \beta \)-matrices, induced by varying an external parameter such as \( B \), is not really ‘random’: it cannot be characterized solely by its typical step size. In other words, the second moment \( \langle g(dS/dX, dS/dX) \rangle \) is either not appropriate or not sufficient, or both, to characterize parametric motion; there must be other constraints. To find the nature and physical origin of these constraints, should be the goal of future investigations.

ACKNOWLEDGMENTS

I thank G. Hackenbroich, J. Müller, F. von Oppen, T. Papenbrock, H. A. Weidenmüller and J. A. Zuk for critical reading of the manuscript and helpful suggestions. Financial support by the Heidelberger Akademie der Wissenschaften is gratefully acknowledged.

1 For reviews see: S. Washburn and R. A. Webb, Adv. Phys. 35, 375 (1986); C. W. J. Beenakker and H. van Houten
in *Solid State Physics*, Vol. 44, ed. by H. Ehrenreich and D. Turnbull (Academic Press, New York, 1991); B. L. Altshuler, P. A. Lee, and R. A. Webb, eds., *Mesoscopic Phenomena in Solids* (North-Holland, New York, 1991).

2 C. M. Marcus, A. J. Rimberg, R. M. Westervelt, P. F. Hopkins, and A. C. Gossard, Phys. Rev. Lett. 69, 506 (1992); M. W. Keller, O. Millo, A. Mittal, and D. E. Prober, Surf. Sci. 305, 501 (1994).

3 B. L. Al’tshuler and B. I. Shklovskii, Zh. Eksp. Teor. Fiz. 91, 220 (1986) [Sov. Phys. JETP 64, 127 (1986)]; P. A. Lee, A. D. Stone, and H. Fukuyama, Phys. Rev. B 35, 1039 (1987).

4 P. A. Mello, Phys. Rev. Lett. 60, 1089 (1988); C. W. J. Beenakker, Phys. Rev. B 47, 15763 (1993).

5 S. Iida, H. A. Weidenmüller, and J. A. Zuk, Phys. Rev. Lett. 64, 583 (1990); Annals of Phys. 200, 219 (1990).

6 R. A. Jalabert and J.-L. Pichard, preprint (1994); R. A. Jalabert, J.-L. Pichard, and C. W. J. Beenakker, Europhys. Lett. 27, 255 (1994); H. U. Baranger and P. A. Mello, Phys. Rev. Lett. 73, 142 (1994).

7 R. A. Jalabert, H. U. Baranger, and A. D. Stone, Phys. Rev. Lett. 65, 2442 (1990).

8 H. U. Baranger, R. A. Jalabert, and A. D. Stone, Phys. Rev. Lett. 70, 3876 (1993).

9 A. M. Chang, H. U. Baranger, L. N. Pfeiffer, and K. W. West, cond-mat/9405077.

10 M. B. Hastings, A. D. Stone, and H. U. Baranger, cond-mat/9405080.

11 Z. Pluhar, H. A. Weidenmüller, J. A. Zuk, and C. H. Lewenkopf, preprints (1994).

12 A. Altland, Ph.D. thesis, Universität Heidelberg (1990).

13 B. D. Simons, P. A. Lee, B. L. Altshuler, Phys. Rev. Lett. 70, 4122 (1993); E. Brézin and A. Zee, Phys. Rev. E 49, 2588 (1994).

14 F. J. Dyson, J. Math. Phys. 13, 90 (1972).

15 C. W. J. Beenakker, Phys. Rev. Lett. 70, 4126 (1993).

16 A random walk on the manifold of $S$-matrices is certainly not exactly equivalent to a random walk on the manifold of Hamiltonians. How the change of base manifolds affects the final results is therefore not known; however, one would expect equivalence in the limit of a large number of channels.

17 R. Landauer, Phil. Mag. 21, 863 (1970); M. Büttiker, Phys. Rev. Lett. 57, 1761 (1986).

18 P. A. Mello and J.-L. Pichard, J. Phys. I 1, 493 (1991); P. A. Mello and A. D. Stone, Phys. Rev. B 44, 3559 (1991).

19 The 'symplectic case,' $\beta = 4$, will not be considered.

20 This assumption is reasonable as long as (i) the cavity is chaotic; (ii) dephasing effects (Coulomb interaction, phonons) may be neglected; and (iii) there are no barriers between cavity and leads, i.e., their coupling is maximal.

21 A. O. Barut and R. Raczka, *Theory of Group Representations and Applications* (World Scientific, Singapore, 1986).

22 It is assumed that $t_0$ is independent of $N$. This is equivalent to assuming $|dS_{ij}/dX|^2 \simeq O(1/N^2)$ for all $i,j$; or, in the Hamiltonian model $H = H(0) + i\sqrt{t/N}H^{(A)}$, to assuming $|H^{(A)}|^2 \simeq O(1/N)$.

23 That some additional information about the perturbation is needed, could have been guessed already from the different functional dependences of $C(\Delta k)$ (Lorentzian; eq. [1]) versus $C(\Delta B)$ (Lorentzian-squared; eq. [4]) on the respective parameter.