Compact special Legendrian surfaces in $S^5$

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Abstract

A surface $\Sigma \subset S^5 \subset \mathbb{C}^3$ is called special Legendrian if the cone $0 \times \Sigma \subset \mathbb{C}^3$ is special Lagrangian. The purpose of this paper is to propose a general method toward constructing compact special Legendrian surfaces of high genus. It is proved there exists a compact, orientable, Hamiltonian stationary Lagrangian surface of genus $1 + \frac{k(k-3)}{2}$ in $\mathbb{C}P^2$ for each integer $k \geq 3$, which is a smooth branched surface except at most finitely many conical singularities. If this surface is smooth, it is minimal and the Legendrian lift of the surface is the desired compact special Legendrian surface.

We first establish the existence of a minimizer of area among Lagrangian disks in a relative homotopy class of a Kähler-Einstein surface without Lagrangian homotopy classes with respect to a configuration $\Gamma$ that consists of the fixed point loci of Kähler involutions. $\Gamma$ in addition must satisfy certain null relative homotopy conditions and angle criteria. The fundamental domain thus obtained is smooth along the boundary, and has finitely many interior singular points. We then apply successive reflection of this
fundamental domain along its boundary to obtain a complete or compact Lagrangian surface.

**Key words:** special Legendrian surface, Hamiltonian stationary, reflection principle

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Introduction

Bryant showed every compact Riemann surface admits conformal minimal embeddings in $S^4$ with vanishing complex quartic form [Br1]. With regard to the fibration

$$Sp(2) \to \mathbb{C}P^3 \to S^4 = \mathbb{H}P^1,$$

such an embedding corresponds to a holomorphic Legendrian curve in $\mathbb{C}P^3$ under $Sp(2)$ invariant holomorphic contact structure. Similarly, every compact Riemann surface can
be immersed as a branched complex curve in $S^6$ with null torsion with respect to $G_2$ invariant almost complex structure via fibration

$$G_2/U(2) = Gr^+(2, \mathbb{R}^7) \to S^6.$$  

The lift of such a complex curve is integral to a rank 3 holomorphic Pfaffian system on $Gr^+(2, \mathbb{R}^7) \subset \mathbb{C}P^6$, for which Cartan gave a local normal form in terms of birational coordinates [Br2]. These two differential systems for minimal immersion of surfaces have the same local generality of solutions, i.e., one holomorphic function of one complex variable [BCG].

The twistor constructions above, as they stand, do not apply to reduce to holomorphic category the differential systems for minimal immersion of surfaces in odd dimensional spheres with the same local generality of solutions. For instance, minimal surfaces in $S^3$ correspond to complex curves in $SO(4)/SO(2)$ with respect to a nonintegrable $CR$-structure, Section 7.

A surface $\Sigma \subset S^5 \subset \mathbb{C}^3$ is called special Legendrian if the cone $0 \times \Sigma \subset \mathbb{C}^3$ is special Lagrangian. Up to $U(3)$ motion, a special Legendrian surface is a minimal Legendrian surface, and hence corresponds to a minimal Lagrangian surface in $\mathbb{C}P^2$ via Hopf map.

The purpose of this paper is to propose a general method toward constructing compact special Legendrian surfaces of high genus. It is proved there exists a compact, orientable, Hamiltonian stationary Lagrangian surface of genus $1 + \frac{k(k-3)}{2}$ in $\mathbb{C}P^2$ for each integer $k \geq 3$, which is a smooth branched surface except at most finitely many conical singularities. If this surface is smooth, it is minimal and the Legendrian lift of the surface, Proposition 1.6, is then the desired compact special Legendrian surface. As of this writing, we are not able to resolve the singularities.

Our method of construction is analogous to that of Lawson’s on compact minimal surfaces in $S^3$ [La1]. We first find a minimizer of certain free boundary problem, and apply successive reflection of this fundamental domain to obtain a complete Lagrangian
surface. If the associated reflection group is finite, the resulting surface is compact. The reflection principle, to be explained below, is motivated from the reflection principle for minimal surfaces in $S^3$, and the similarity between the elliptic Monge-Ampère systems describing special Legendrian surfaces and minimal surfaces in $S^3$.

Let $\phi$ be a calibration on a Riemannian manifold $X$. Consider a compact calibrated submanifold $M$ with boundary $\partial M \subset N$, where $N$ is a $\phi$-null submanifold, a submanifold on which $\phi = 0$. Then via Stokes theorem, $M$ is absolutely mass minimizing in its relative homology class with respect to $N$. Thus $\phi$-null submanifolds serve as natural constraints for free boundary problem for $\phi$-submanifolds. For example, finding a special Lagrangian submanifold with boundary in a given complex hypersurface in a Calabi-Yau manifold roughly corresponds to solving Neumann problem.

Since such $\phi$-calibrated submanifold intersects a $\phi$-null submanifold orthogonally along the boundary, the observation above leads to the following reflection principles;

1. Suppose $N$ is a $\phi$-null submanifold which is a fixed point locus of a $\pm \phi$ isometric involution $\sigma$ of $X$, i.e., $\sigma^* \phi = \pm \phi$. Then any $\phi$-submanifold with boundary in $N$ and $C^1$ up to boundary can be analytically continued across $N$ via $\sigma$.

2. Suppose $M$ is a $\phi$-calibrated submanifold which is a fixed point locus of $\phi$ isometric involution $\sigma$. Then any minimal $\phi$-null submanifold with boundary in $M$ and $C^1$ up to boundary can be analytically continued across $M$ via $\sigma$.

Basic facts and fundamental equations for special Lagrangian submanifolds and special Legendrian surfaces are recorded in Section 1 and 2. After introducing the reflection principle for special Lagrangian 3-folds with complex, or anti-special Lagrangian boundary, we describe a set of conditions for the constraining polygon that is necessary for our construction. Section 5 contains the main theorem that there exists a minimizer of area of the free boundary problem for Lagrangian surfaces in $\mathbb{C}P^2$, which is smooth along the boundary and with finitely many interior singular points, if the constraining polygon satisfy certain vanishing relative homotopy conditions. It is based upon the fundamen-
tal work of Schoen and Wolfson [ScW], and holds for general Kähler-Einstein surfaces without Lagrangian homotopy classes. The main theorem is then used in Section 6 to construct a family of compact Hamiltonian stationary Legendrian surfaces. The constraining variety in this case is a complex geodesic triangle derived from the generating set of a finite unitary reflection group in \( \mathbb{C}^3 \) [Cox]. Section 7 is devoted to the study of surfaces associated to a given special Legendrian surface. We show in particular that a compact minimal Lagrangian surface in \( \mathbb{C}P^2 \) cannot lie in an open geodesic ball of radius \( \frac{\pi}{2} - \arccos\left(\frac{1}{\sqrt{3}}\right) \). In Section 8, we give an intrinsic characterization of the induced Riemannian metrics on special Legendrian surfaces or minimal Lagrangian surfaces in complex space forms.

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1 Special Lagrangian cones in \( \mathbb{C}^{n+1} \)

Let \( \mathbb{C}^{n+1} \) be the complex vector space of dimension \( n + 1 \), which we always regard as the real vector space \( \mathbb{R}^{2n+2} \) with coordinates \((x, y)\) and the complex structure

\[
J(x, y) = (-y, x),
\]

where \( x = (x^1, \ldots, x^{n+1}) \), \( y = (y^1, \ldots, y^{n+1}) \). Let

\[
\varpi = \sum_k dx^k \wedge dy^k
\]

\[
\langle v, w \rangle = \varpi(v, Jw) \quad \text{for } v, w \in \mathbb{C}^{n+1}
\]

be the standard Kähler form and the metric. Set \( z^k = x^k + i y^k \), and we denote

\[
\Upsilon = dz^1 \wedge \ldots \wedge dz^{n+1}
\]

\[
\Upsilon_\theta = e^{i\theta} \Upsilon
\]

the holomorphic volume form, and holomorphic volume form of phase \( e^{i\theta} \).
Let \( \text{Gr}_R^+(k, \mathbb{C}^{n+1}) \) be the set of oriented real Grassmann \( k \)-planes in \( \mathbb{C}^{n+1} \). An element \( \sigma \in \text{Gr}_R^+(k, \mathbb{C}^{n+1}) \) is called isotropic if \( \varpi|_{\sigma} = 0 \), \( 1 \leq k \leq n+1 \) necessarily, and it is called Lagrangian when \( k = n+1 \). We denote

\[
\text{Isot}^+(k, \mathbb{C}^{n+1}) = \{ \sigma \in \text{Gr}_R^+(k, \mathbb{C}^{n+1}) \mid \varpi|_{\sigma} = 0 \} = \text{SU}(n+1)/(\text{SO}(k) \times \text{SU}(n+1-k))
\]

\[
\text{Slag}(n+1) = \{ L \in \text{Gr}_R^+(n+1, \mathbb{C}^{n+1}) \mid \text{dvol}_L = \text{Re}(\Upsilon)|_L \} = \text{SU}(n+1)/\text{SO}(n+1)
\]

\[
\text{Slag}(n+1) = \{ L' \in \text{Gr}_R^+(n+1, \mathbb{C}^{n+1}) \mid \text{dvol}_{L'} = -\text{Im}(\Upsilon)|_{L'} \} = \left\{ (-1)^{\frac{n+1}{2}} J(L) = L' \mid L \in \text{Slag}(n+1) \right\} \text{ if } n+1 \text{ is odd.}
\]

An element of \( \text{Slag}(n+1) \) is called a special Lagrangian \( (n+1) \)-plane, and it is characterized, up to choice of right orientation, as a Lagrangian \( (n+1) \)-plane on which \( \text{Im}(\Upsilon) = 0 \).

**Lemma 1.1** [HL1] Given \( \sigma \in \text{Isot}^+(n, \mathbb{C}^{n+1}) \), there exists a unique special Lagrangian \( (n+1) \)-plane \( L_{\sigma} \) such that \( \sigma \subset L_{\sigma} \).

Lemma 1.1 provides a double fibration

\[
\begin{array}{ccc}
\text{Isot}^+(n, \mathbb{C}^{n+1}) & \xleftarrow{\pi_1} & S^{2n+1} \\
\downarrow{\pi_2} & & \downarrow{\pi_2} \\
\text{Slag}(n+1) & & \text{Slag}(n+1).
\end{array}
\]

Each fiber of the projection \( \pi_1 \) is identified with \( \text{Slag}(n) \) and each fiber of \( \pi_2 \) with \( S^n \).

Let \( M \) be an oriented manifold of dimension \( n+1 \).

**Definition 1.1** An immersion \( u : M \rightarrow \mathbb{C}^{n+1} \) is special Lagrangian if

\[
du(T_pM) \in \text{Slag}(n+1)
\]

for each \( p \in M \). The image \( u(M) \) of such an immersion is a special Lagrangian submanifold.
Note that a special Lagrangian submanifold is orientable.

**Proposition 1.1** [HL1] An immersion \( u : M \to \mathbb{C}^{n+1} \) is minimal Lagrangian if and only if
\[
u^* \text{Re}(\Upsilon_\theta) = d\text{vol}_M
\]
for some constant \( \theta \).

Such \( \theta \) is called the *phase* of the minimal Lagrangian submanifold. Minimal Lagrangian submanifolds of phase \( \theta \) are thus integral manifolds of the differential system
\[
\mathcal{I}_\theta = \{ \varpi, \text{Im}(\Upsilon_\theta) \}.
\]

Since each \( \text{Re}(\Upsilon_\theta) \) gives a calibration in \( \mathbb{C}^{n+1} \), a minimal Lagrangian submanifold is absolutely mass minimizing in the relative homology class with respect to its boundary. In particular, \( C^1 \) special Lagrangian submanifolds are real analytic.

The following is a direct consequence of Lemma 1.1 and the fact the differential system (1) is involutive [BCG].

**Proposition 1.2** Let \( N^n \subset \mathbb{C}^{n+1} \) be a connected, analytic, isotropic submanifold of dimension \( n \). There exists a unique maximal connected special Lagrangian submanifold \( M^{n+1} \) that contains \( N^n \).

We mention that real analyticity assumption on \( N \) is necessary both for existence and uniqueness.

Let \( S^{2n+1} = \{ v \in \mathbb{C}^{n+1} \mid |v| = 1 \} \) with the \( U(n+1) \) invariant contact structure generated by the contact 1-form
\[
r \wedge \varpi |_{S^{2n+1}},
\]
where \( r = \sum_k x^k \frac{\partial}{\partial x^k} + y^k \frac{\partial}{\partial y^k} \). A \( k \)-dimensional submanifold of \( S^{2n+1} \) is called *isotropic* if it is tangent to the contact hyperplane distribution, \( 1 \leq k \leq n \) necessarily, and it is called *Legendrian* when \( k = n \).

Let \( \Sigma \) be an oriented manifold of dimension \( n \).
**Definition 1.2** An immersion $u : \Sigma \rightarrow S^{2n+1} \subset \mathbb{C}^{n+1}$ is special Legendrian if the cone $0 \times u(\Sigma)$ is special Lagrangian. The image $u(\Sigma)$ of such an immersion is a special Legendrian submanifold.

Here the cone is given the product orientation of $\Sigma \times \mathbb{R}^+$. A special Legendrian submanifold is necessarily orientable.

**Proposition 1.3** Let $u : \Sigma \rightarrow S^{2n+1}$ be a minimal Legendrian immersion.

1. The cone $0 \times u(\Sigma)$ is minimal Lagrangian in $\mathbb{C}^{n+1}$. In particular, $C^1$ special Legendrian submanifolds are real analytic.

2. When coupled with the Hopf map $\pi : S^{2n+1} \rightarrow \mathbb{C}P^n$, $\pi \circ u : \Sigma \rightarrow \mathbb{C}P^n$ is a minimal Lagrangian immersion with respect to the standard Kähler structure of $\mathbb{C}P^n$.

By definition, special Legendrian submanifolds are integral manifolds of the involutive differential system

$$\mathcal{I} = \{ \vec{r} \cdot \vec{w}, \vec{w}, \vec{r} \cdot \text{Im}(\Upsilon) \}.$$  \hspace{1cm} (3)

**Proposition 1.4** Let $\gamma^{n-1} \subset S^{2n+1}$ be a connected, analytic, isotropic submanifold of dimension $n - 1$. There exists a unique maximal connected special Legendrian submanifold $\Sigma^n$ that contains $\gamma^{n-1}$.

**Example 1.1**

1. Let $L \subset \mathbb{C}^{n+1}$ be a special Lagrangian $(n+1)$-plane. The geodesic sphere $L \cap S^{2n+1}$ is special Legendrian in $S^{2n+1}$. Its image under Hopf map is the standard

$$RP^n \subset \mathbb{C}P^n.$$  \hspace{1cm} (4)

2. Let $M(n, \mathbb{C}) = \mathbb{C}^{n^2}$ be the space of $n$-by-$n$ complex matrices. Then the cone $0 \times SU(n) \subset M(n, \mathbb{C})$ is minimal Lagrangian with the associated minimal Lagrangian
embedding

$$SU(n)/Z_n \to \mathbb{C}P^{n^2-1},$$

(5)

where $Z_n$ is the center of $SU(n)$.

3. Let $\text{Sym}(n, \mathbb{C}) = \mathbb{C}^{n(n+1)/2}$ be the space of symmetric complex $n$-by-$n$ matrices. Consider $0 \times \text{Slag}(n) \subset \text{Sym}(n, \mathbb{C})$, where

$$\text{Slag}(n) = \{ UU^t \in \text{Sym}(n, \mathbb{C}) \mid U \in SU(n) \}.$$ 

It is a minimal Lagrangian cone with the associated minimal Lagrangian immersion

$$\text{Slag}(n) \to \mathbb{C}P^{n(n+1)/2-1}.$$  

(6)

Note that this observation implies a natural map

$$\text{Slag}(n) \to \text{Slag}(\frac{n(n+1)}{2}),$$

which is also the restriction of the Veronese embedding $\mathbb{C}P^m \to \mathbb{C}P^{2m(m+1)/2-1}$, $m = \frac{n(n+1)}{2} - 1$.

4. Let $A(2n, \mathbb{C}) = \mathbb{C}^{n(2n-1)}$ be the space of skew symmetric complex $2n$-by-$2n$ matrices. $U \in SU(2n)$ acts on this space by $A \to UA^t$ for $A \in A(2n, \mathbb{C})$. The orbit of the matrix

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

is $SU(2n)/Sp(n)$, and the cone $0 \times SU(2n)/Sp(n) \subset A(2n, \mathbb{C})$ is minimal Lagrangian. The associated minimal Lagrangian immersion is

$$SU(2n)/Sp(n) \to \mathbb{C}P^{(2n+1)(n-1)}.$$  

(7)

5. Let $V_3^{27}$ be the real vector space of Octonianic Hermitian 3-by-3 matrices and $V_3$ its complexification. The compact simple Lie group $E_6$ has a faithful special unitary representation on $V_3$ with $F_4$ as the stabilizer of the identity matrix. The cone $0 \times E_6/F_4$ is minimal Lagrangian in $V_3$ with the associated minimal Lagrangian immersion

$$E_6/F_4 \to \mathbb{C}P^{26}.$$  

(8)

Examples (4), (5), (6), (7), and (8) are in fact the only irreducible minimal Lagrangian submanifolds of complex projective spaces with parallel second fundamental form [Na].
Example 1.2  
Consider the minimal Lagrangian cone

\[ \{ (z^1, \ldots, z^{n+1}) \in \mathbb{C}^{n+1} \mid |z_1| = \ldots |z_{n+1}|, \text{Im}(z_1z_2 \ldots z_{n+1}) = 0 \} \].

It has the hexagonal \( n \)-torus as its link, where a hexagonal \( n \)-torus is the quotient of Clifford \((n + 1)\)-torus by the diagonal action of \( S^1 \). Hopf map induces an \((n + 1)\)-fold cover onto its image, which is again called a hexagonal torus in \( \mathbb{C}P^n \).

Special Legendrian differential system on \( S^{2n+1} \) can be considered as a compact real form of the differential system describing complex affine spheres in \( \mathbb{C}^{n+1} \). In this perspective, the product of minimal Lagrangian submanifolds below is a compact real form of the product for affine spheres introduced by Calabi [Cal].

Proposition 1.5  
Let \( M_i^{n_i} \subset \mathbb{C}P^{n_i}, n_i \geq 0 \), be a minimal Lagrangian submanifold for \( i = 1, \ldots, k \). There exists a product minimal Lagrangian submanifold

\[ (M_1 \otimes \ldots \otimes M_k)^n \subset \mathbb{C}P^n \]

where \( n = \sum_i (n_i + 1) - 1 \).

Proof. Let \( \pi^{-1}(M_i) \subset S^{2n_i+1} \) be the inverse image of \( M_i \) under the Hopf map, which is a minimal submanifold. We scale each of \( \pi^{-1}(M_i) \) by a factor \( r_i = (\frac{n_i+1}{n+1})^{\frac{1}{2}} \), denote it \( \tilde{M}_i \), so that the product \( \tilde{M}_1 \times \ldots \times \tilde{M}_k \subset S^{n+1} \) is a minimal submanifold. Now \( \pi(\tilde{M}_1 \times \ldots \times \tilde{M}_k) \subset \mathbb{C}P^n \) is a minimal Lagrangian submanifold. \( \square \)

Note that

\[ M_1 \otimes \ldots \otimes M_k \to M_1 \times \ldots \times M_k \]

is a \((k - 1)\) dimensional torus bundle. In fact, minimal Lagrangian torus in \( \mathbb{C}P^n \) of Example 1.2 is a product of \( n + 1 \) minimal Lagrangian submanifolds \( \mathbb{R}P^0 \subset \mathbb{C}P^0 \). We also mention that the product of Hamiltonian stable compact minimal Lagrangian submanifolds is again Hamiltonian stable, Theorem 7.3, [Oh].
In view of Proposition 1.5 and [Na], we have a complete description of minimal Lagrangian submanifolds in complex projective spaces with parallel second fundamental form.

**Theorem 1.1** Let \( M^n \subset \mathbb{C}P^n \) be a minimal Lagrangian submanifold with parallel second fundamental form. Then it is a part of a product of minimal Lagrangian submanifolds (4), (5), (6), (7), and (8).

Theorem 1.1 can be applied to improve the existing extrinsic or intrinsic curvature pinching results for minimal Lagrangian submanifolds [Ma][MRU].

**Example 1.3** Haskins provides families of ruled special Legendrian tori in \( S^5 \), and Joyce considers various symmetry reductions [Ha][Jo].

**Example 1.4** Every compact Riemann surface can be embedded in \( \mathbb{C}P^3 \) as a holomorphic Legendrian curve with respect to the \( Sp(2) \) invariant holomorphic contact structure [Br1]. Consider the inverse image of a holomorphic Legendrian curve in \( S^7 \subset \mathbb{C}^4 \) via Hopf map. Its projection in \( \mathbb{C}P^3 \) with respect to a different orthogonal complex structure on \( \mathbb{C}^4 = \mathbb{H}^2 \) is minimal Lagrangian.

**Proposition 1.6** [Wa1] Let \( u : M \to \mathbb{C}P^n \) be a connected minimal Lagrangian submanifold, and let

\[
\pi_1(M) \to Hol(M, S^1) \subset O(2)
\]

denote the holonomy of the associated flat \( S^1 \) bundle \( u^*(S^{2n+1}) \to M \). Then

\[
Hol(M, S^1) \subset \mathbb{Z}_{n+1} \subset SO(2) \quad \text{if and only if } M \text{ is orientable}
\]

\[
Hol(M, S^1) \subset \mathbb{D}_{n+1} \subset O(2) \quad \text{if } M \text{ is nonorientable}
\]

where \( \mathbb{D}_{n+1} \) is the dihedral group of order \( 2(n + 1) \). In particular, if \( M \) is compact or embedded, there exists a connected minimal Legendrian lift \( \bar{M} \subset S^{2n+1} \) that is compact or embedded respectively.
Examples 1.1, 1.2, 1.3, 1.4 and Proposition 1.5 and 1.6 provide many compact special Legendrian (minimal Lagrangian respectively) submanifolds in $S^{2n+1}$ ($\mathbb{C}P^n$) with nontrivial topology.

**Corollary 1.1** Let $\Sigma^n \subset S^{2n+1}$ be a connected minimal Legendrian submanifold of even dimension. Then the image of $\Sigma$ in $\mathbb{C}P^n$ under Hopf map is nonorientable whenever $\Sigma \subset S^{2n+1}$ is invariant under antipodal involution.

**Proposition 1.7** Let $\Sigma^n \subset S^{2n+1}$ be a compact special Legendrian submanifold of positive mass. Let $c(n+1)$ be the isoperimetric constant for $(n+1)$-dimensional varieties in $\mathbb{R}^{2n+2}$. Then

$$||\Sigma|| \geq \frac{c(n+1)}{(n+1)^n}.$$ 

**Corollary 1.2** There exists a uniform lower bound for the area of compact minimal Lagrangian submanifolds in $\mathbb{C}P^n$ depending only on the dimension $n$.

**Proof of the Proposition.** Let $0 \times_1 \Sigma = (0 \times \Sigma) \cap B(1, \mathbb{R}^{2n+2})$, where $B(1, \mathbb{R}^{2n+2})$ is the unit ball in $\mathbb{R}^{2n+2}$. Since $0 \times_1 \Sigma$ is mass minimizing,

$$||\Sigma||^{n+1} \geq c(n+1)||0 \times_1 \Sigma||^n = c(n+1)\frac{||\Sigma||^n}{(n+1)^n}.$$ 

We record the following topological obstruction for Lagrangian embedding.

**Proposition 1.8** Let $M^n$ be a compact, orientable, embedded Lagrangian submanifold in $\mathbb{C}P^n$. Then the Euler Characteristic $\chi(M) = 0$.

**Proof.** Since $M$ is Lagrangian, the tangent bundle $TM$ is isomorphic to the normal bundle $NM$. $M$ is orientable and embedded, which implies the Euler class of $NM$ comes from the restriction of an element in $H^n(\mathbb{C}P^n, \mathbb{R})$ [Mil]. But $H^*(\mathbb{C}P^n, \mathbb{R})$ is generated by the Kähler form. □
2 Special Legendrian surfaces in $S^5$

Let $\Sigma^2 \subset S^5 \subset \mathbb{C}^3$ be a special Legendrian surface. $\Sigma$ is by definition integral to the differential system

$$\mathcal{I} = \{\vec{r} \wedge \omega, \omega, \vec{r} \wedge \text{Im}(\Upsilon)\}$$

with

$$\omega = dx^1 \wedge dy^1 + dx^2 \wedge dy^2 + dx^3 \wedge dy^3$$

$$-\text{Im}(\Upsilon) = dy^{123} - dy^1 \wedge dx^{23} - dy^2 \wedge dx^{31} - dy^3 \wedge dx^{12},$$

where $dx^{23} = dx^2 \wedge dx^3$, etc. Let $e_3 : \Sigma \rightarrow S^5 \subset \mathbb{R}^6 = \mathbb{C}^3$ be the position function. Take an orthonormal basis $\{e_1, e_2\}$ of $T_{e_3}\Sigma$, which we may regard as an $SO(2)$ equivariant $\mathbb{R}^6$ valued functions on the unit tangent bundle $S\Sigma$.

Set

$$n_i = J(e_i),$$

$$e = (e_1, e_2, e_3), \quad n = (n_1, n_2, n_3).$$

Since $\Sigma$ is special Legendrian, $e_1 \wedge e_2 \wedge e_3 \in \text{Slag}(3)$ and we get an $SO(2)$ equivariant map

$$(e, n) : S\Sigma \rightarrow SU(3) \subset SO(6). \quad (10)$$

Differentiating (10), we get

$$d(e, n) = (e, n) \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

where $\alpha$ is an $\mathfrak{so}(3)$ valued, and $\beta$ is a trace free $\text{Sym}(3, \mathbb{R})$ valued one form.

Let $\{\omega^1, \omega^2\}$ be the basis of one forms dual to $\{e_1, e_2\}$, and $\rho$ be the connection form on $S\Sigma$. From the definition,

$$de_3 = \omega^1 e_1 + \omega^2 e_2$$
and (11) becomes
\[
\alpha = \begin{pmatrix}
0 & \rho & \omega^1 \\
-\rho & 0 & \omega^2 \\
-\omega^1 & -\omega^2 & 0
\end{pmatrix}, \quad \beta = \begin{pmatrix}
\beta^1_1 & \beta^1_2 & 0 \\
\beta^2_1 & -\beta^1_1 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\] \hfill (12)

Taking the exterior derivative of (11), we obtain the following structure equations.
\[
d(\omega^1 + i\omega^2) = i\rho \wedge (\omega^1 + i\omega^2) \quad \text{(13)}
\]
\[
\beta^1_1 - i \beta^1_2 = (a - i b)(\omega^1 + i\omega^2)
\]
\[
\quad d\rho = K \omega^1 \wedge \omega^2
\]
\[
K = 1 - 2(a^2 + b^2).
\]

Put \( h = a - i b \), then
\[
dh = -3i h \rho + h_1(\omega^1 + i\omega^2)
\]
\[
dh_1 \equiv -4i h_1 \rho + \frac{3}{2}h K (\omega^1 - i\omega^2) \mod \omega^1 + i\omega^2.
\]

Set
\[
\Phi = (a - i b)(\omega^1 + i\omega^2)^3.
\]

Then \( \Phi \) is a well defined cubic differential on \( \Sigma \) holomorphic with respect to the complex structure induced from the metric.

Bonnet type fundamental theorem for special Legendrian surfaces can now be stated as follows. \[Gr\].

**Theorem 2.1** Consider a triple \((\Sigma, g, \Phi)\) of a Riemann surface, a conformal metric, and a holomorphic cubic differential. It is called admissible if
\[
K = 1 - 2\|\Phi\|^2,
\] \hfill (14)

where \( \|\Phi\| \) is the norm with respect to \( g \), and \( K \) is the Gaussian curvature of the metric.

Let
\[
\pi : \tilde{\Sigma} \rightarrow \Sigma
\]

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be the universal covering of $\Sigma$. Then the triple $(\hat{\Sigma}, \pi^*g, \pi^*\Phi)$ is also admissible and $\hat{\Sigma}$ admits an isometric special Legendrian immersion in $S^5$ with $\pi^*\Phi$ as the associated holomorphic cubic differential. The immersion is unique up to motion by $SU(3)$.

Note that given an admissible triple, there is an $S^1$ family of admissible triples by taking $\Phi \rightarrow e^{i\tau} \Phi$ where $e^{i\tau}$ is any complex number of unit length.

Take a point $p \in \Sigma$ at which $\Phi$ is not 0, and a local coordinate $z$ in a neighborhood of $p$ so that $\Phi = (dz)^3$. A conformal metric $g = e^{2u}dzd\bar{z}$ is compatible, (14), if

$$\Delta u + e^{2u} - 2e^{-4u} = 0.$$  \hspace{1cm} (15)

(15) is also known as the Monge-Ampere equation describing 2-dimensional affine spheres. It has the largest, 8-dimensional, group of symmetry among nondegenerate(nonlinearity) elliptic Monge-Ampere equations.

**Definition 2.1** A point $p \in \Sigma$ of a special Legendrian surface is umbilic if the associated holomorphic differential $\Phi$ vanishes at $p$.

Let $d_p$ denote the degree of zero of $\Phi$ at $p$. Let $S_p$ be the geodesic 2-sphere tangent to $\Sigma$ at $p$ and $L_p$ be the special Lagrangian 3-plane such that $S_p = L_p \cap S^5$. It is clear then that $S_p$ has contact of order $(d_p + 1)$ with $\Sigma$ at $p$, i.e., $(d_p + 1)$th jet of $\phi$ at $p$ is contained in $L_p$.

**Proposition 2.1** Let $u : \Sigma \rightarrow S^5$ be a special Legendrian immersion of a compact orientable surface of genus $g$.

1. If $g = 0$, $u(\Sigma)$ is the totally geodesic sphere $S^5 \cap L$, $L \in \text{Slag}(3)$.
2. If $g \geq 1$, $\sum_{p \in \Sigma} d_p = 6g - 6$.

**Theorem 2.2** [Ha] Let $\Sigma \subset S^5$ be a special Legendrian surface with Gaussian curvature $K$. 

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1. Suppose $K$ is constant. Then either $K = 1$ and $\Sigma$ is totally geodesic, or $K = 0$ and $\Sigma$ is part of the hexagonal torus in Example 1.2.

2. If $\Sigma$ is not linearly full in $S^5 \subset \mathbb{R}^6$, it is totally geodesic.

3. Suppose $\Sigma$ is compact and $K \geq 0$. Then $\Sigma$ is either totally geodesic or the hexagonal torus.

Proof. 1 and 3 follow from $||dK||^2 = 16||h||^2||h_1||^2$ and Simons’ type identity

$$-\Delta K = 12K||h||^2 + 8||h_1||^2.$$ (16)

2 is immediate from the structure equations (11), (12), and (13). □

Special Legendrian surfaces in $S^5$ thus bear resemblance to minimal surfaces in $S^3$. The extrinsic geometric property, however, can be significantly different, for special Legendrian surfaces are integral to the first order differential system (9).

Proposition 2.2 Let $\Sigma \subset S^5$ be a special Legendrian surface, and $\gamma \subset \Sigma$ a smooth connected curve ($\gamma$ may be a boundary component of $\Sigma$).

1. If $\gamma$ contains an analytic arc, $\Sigma$ inherits all the symmetry of $\gamma$.

2. The Gaussian curvature of $\Sigma$ along $\gamma$ can be determined from $\gamma$ alone.

Proof. It follows from Proposition 1.2 and the structure equations (11), (13). □

In terms of the inclusion $\mathbb{C}^3 \subset Im \mathbb{O}$, where $\mathbb{O}$ is the algebra of Octonians, special Lagrangian cones are associative cones and their links, special Legendrian surfaces, are complex curves in $S^6$ with respect to $G_2$ invariant almost complex structure. Bryant showed every compact Riemann surface admits a branched immersion in $S^6$ as a complex curve with null torsion [Br2]. The only special Legendrian surface with null torsion in this sense, even locally however, is totally geodesic.
3 Reflection principle

Let

\[ L^⊥ = \{ (x, y) \in \mathbb{C}^3 \mid x = 0 \} = \frac{\partial}{\partial y^1} \wedge \frac{\partial}{\partial y^2} \wedge \frac{\partial}{\partial y^3} \]

\[ \Pi = \{ (z^1, z^2, z^3) \in \mathbb{C}^3 \mid z^3 = 0 \}. \]

**Definition 3.1** By geodesic reflection across \( L^⊥ \) we mean the map \( r_{L^⊥} : S^5 \to S^5 \) where

\[ r_{L^⊥}(x, y) = (-x, y). \]

By geodesic reflection across \( \Pi \) we mean the map \( r_\Pi : S^5 \to S^5 \) where

\[ r_\Pi(z^1, z^2, z^3) = (z^1, z^2, -z^3). \]

Note \( r_{L^⊥} \) and \( r_\Pi \) are symmetry of the special Legendrian differential system (9).

Consider an oriented isotropic 2-plane \( \sigma \) in \( \mathbb{C}^3 \). Since \( SU(3) \) acts transitively on the set of isotropic 2-planes, as well as on the set of special Lagrangian 3-planes, we may take the 2-plane

\[ \sigma = \frac{\partial}{\partial y^1} \wedge \frac{\partial}{\partial y^2}. \]

Then it follows from (9) that

\[ L_\sigma = -\frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial y^1} \wedge \frac{\partial}{\partial y^2} \]

is the unique special Lagrangian 3-plane containing \( \sigma \), and the 3-plane

\[ L^⊥ = \frac{\partial}{\partial y^1} \wedge \frac{\partial}{\partial y^2} \wedge \frac{\partial}{\partial y^3} \]

is the unique element in \( Slag^⊥(3) \) that contains \( \sigma \). Let

\[ \Pi = \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial y^1} \wedge \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial y^2} \]

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be the complex 2-plane generated by \( \sigma \). Note that both \( L^\perp \) and \( \Pi \) are orthogonal to \( L_\sigma \) along \( \sigma \).

The following reflection principle for special Legendrian surfaces can be considered as an integral of these linear relations.

**Proposition 3.1** 1. Let \( \Sigma \subset S^5 \) be a special Legendrian surface \( C^1 \) up to its boundary \( \partial \Sigma \). If part of \( \partial \Sigma \) lies in \( L^\perp \cap S^5 \), then \( \Sigma \) is necessarily orthogonal to \( L^\perp \cap S^5 \) and can be analytically continued across each component of \( \partial \Sigma \cap L^\perp \) by the reflection \( r_{L^\perp} \). Analogous reflection principle is true for \( \partial \Sigma \) that lies in a complex 2-plane \( \Pi \).

2. Let \( M \subset \mathbb{C}P^2 \) be a Hamiltonian stationary minimizing Lagrangian surface smooth up to its boundary \( \partial M \). If part of \( \partial M \) lies in \( \Pi \cap \mathbb{C}P^2 \), then \( M \) is necessarily orthogonal to \( \Pi \cap \mathbb{C}P^2 \) and can be analytically continued across each component of \( \partial M \cap \mathbb{C}P^2 \) by the reflection \( r_{\Pi} \).

**Proof.** 1. From the orthogonal relation above, the union of \( \Sigma \) and its reflected image is \( C^1 \) along the corresponding boundary component. By Proposition 1.3, it is in fact real analytic.

2. \( C^1 \) Hamiltonian stationary minimizing Lagrangian surface is smooth [ScW]. □

**Definition 3.2** Let \( \Sigma \subset S^5 \) be a special Legendrian surface. A curve \( \gamma \subset \Sigma \) is anti-special Lagrangian if it lies in \( L^\perp \in \text{Slag}(3)^\perp \), and it is complex if it lies in a complex 2-plane.

An anti-special Lagrangian or complex curve is necessarily a geodesic in the analytically continued surface. In fact, a curve on a special Legendrian surface is anti-special Lagrangian(complex) if it is a geodesic on which the associated cubic differential \( \Phi \) is real(purely imaginary), Proposition 8.1.

**Example 3.1** Consider the hexagonal torus in Example 1.2, which we parameterize by \( z_j = \frac{1}{\sqrt{3}} e^{i \theta_j} \) with \( \theta_1 + \theta_2 + \theta_3 = 0 \). Each of the curves \( \theta_j + \theta_k = \text{const} \) is anti-special.
Lagrangian and each curve \( \theta_j - \theta_k = \text{const} \) is complex.

Reflection principle analogous to Proposition 3.1 can be formulated for minimal Lagrangian submanifolds in Kähler manifolds with (anti)Kähler involution, Section 8.

4 Rational geodesic polygon

Let \( \Gamma = \{P_1, \ldots, P_m = P_0\} \subset S^5, \ m \geq 1, \) be a finite union of anti-special Lagrangian spheres \( L^\perp \cap S^5 \) and complex spheres \( \Pi \cap S^5 \). \( \Gamma \) is called a geodesic m-gon if \( \Gamma \) is connected and its image in \( \mathbb{C}P^2 \) is a \( m \)-gon, i.e., it consists of \( m \) vertices and two dimensional faces. We define \( \{v_i = P_i \cap P_{i+1}\}_{i=0}^{m-1} \) the vertices of the polygon with vertex angle \( \theta_i \) at \( v_i \), where \( \cos^2 \theta_i \) is the Jacobian of projection from \( P_i \) to \( P_{i+1} \). Note in case the adjacent faces are both complex, the corresponding vertex is a circle in \( S^5 \). Let \( G_\Gamma \) be the group generated by reflections across each faces of \( \Gamma \).

**Lemma 4.1** Let \( v = P_1 \cap P_2 \subset S^5 \) be a vertex, and \( r_1, r_2 \) the reflections across \( P_1, P_2 \) respectively. The group generated by \( \{r_1, r_2\} \) is finite if and only if the vertex angle at \( v \) is a rational multiple of \( 2 \pi \).

**Proof.** If \( P_1, P_2 \) are both complex or both anti-special Lagrangian, it follows form the observation that such pair of planes is isoclinic \([\text{TeT}]\). The mixed case follows similarly, Lemma 4.3. □

Note the finite group in each case is a Dihedral group.

**Definition 4.1** Let \( \Gamma \subset S^5 \) be a geodesic polygon. It is a rational polygon if each vertex angle is a rational multiple of \( 2 \pi \). It is of finite type if \( G_\Gamma \) is finite.

**Remark.** Let \( X \) be a Kähler surface. The above definition can be extended to a geodesic polygon \( \Gamma \subset X \), where \( \Gamma \) is a finite union of the fixed point locus of Kähler or anti-Kähler involutions. Note in this case we must require in addition the two associated angles, \([\text{TeT}]\), at every non complex-complex vertex be rational multiples of \( 2 \pi \).
Example 4.1 Consider a finite group $G \subset U(3)$ generated by unitary reflections of period two [Cox]. Let $E$ be the union of the axes of reflections. Any finite ordered set $\{ P_1, \ldots, P_m = P_0 \} \subset E$ of distinct elements in $E$ is a rational polygon of finite type. A particular class of such polygons will be examined in detail in Section 6.

We record the following for later application.

Lemma 4.2 Let $\Pi_1, \Pi_2$ be complex lines in $\mathbb{C}^2$, and $L \subset \mathbb{C}^2$ be a Lagrangian plane such that $\dim \Pi \cap L = 1$ for $i = 1, 2$. Then the angle of the wedge in $L$ formed by the intersection with $\Pi_1 \cup \Pi_2$ is determined from the angle between $\Pi_1, \Pi_2$.

Proof. Let $\{ E_1, N_1, E_2, N_2 \}$ be an orthonormal basis of $\mathbb{C}^2$ with $J(E_i) = N_i$. Without loss of generality, we may assume $\Pi_1 = E_1 \wedge N_1$, and $\Pi_2 = (\cos(\varphi)E_1 + \sin(\varphi)E_2) \wedge (\cos(\varphi)N_1 + \sin(\varphi)N_2)$, where $\varphi$ is the angle between $\Pi_1, \Pi_2$. Suppose $E_1 \in L$. Then since $L$ is Lagrangian, $L = E_1 \wedge E_2$. □

Lemma 4.3 Let $\gamma_1, \gamma_2$ be curves on a special Legendrian surface $\Sigma$ that meet at a vertex $p \in \Sigma$ with an angle $\varphi$. Let $L_1^+, L_2^+ \in \text{Slag}(3)^\perp$ and $\Pi_1, \Pi_2 \in \text{Gr}_C(2, \mathbb{C}^3)$.

1. Suppose $\gamma_i \subset L_i^+ \cap S^5$ or $\gamma_i \subset \Pi_i \cap S^5$ for $i = 1, 2$. Then $\cos^2(\varphi)$ is determined from the angle between $L_1^+$ and $L_2^+$ or $\Pi_1$ and $\Pi_2$.

2. Suppose $\gamma_1 \subset L_1^+ \cap S^5$ and $\gamma_2 \subset \Pi_1 \cap S^5$. Then $\sin(2\varphi)$ is determined from the angle between $L_1^+$ and $\Pi_1$.

Proof. We consider the case $\gamma_i \subset \Pi_i \cap S^5$ for $i = 1, 2$. Set $E_i = \frac{\partial}{\partial x_i}$ and $N_i = \frac{\partial}{\partial y_i}$. Let $\Pi_2 = E_2 \wedge N_2 \wedge E_3 \wedge N_3$ and without loss of generality, let $E_3 = p$ and $E_2$ be the tangent vector of $\gamma_2$ at $p$. Then $E_1 \wedge E_2$ is the tangent space of $\Sigma$ at $p$, and tangent vector of $\gamma_1$ at $p$ is of the form $v = \sin(\varphi)E_1 + \cos(\varphi)E_2$. Since $\gamma_1 \subset \Pi_1$, $\Pi_1 = E_3 \wedge N_3 \wedge v \wedge J(v)$, and $\cos^2(\varphi)$ is the Jacobian of the orthogonal projection from $\Pi_2$ to $\Pi_1$. □
5 Construction procedure

In this section, we describe a general method toward constructing complete or compact (branched) minimal Lagrangian surfaces in compact Kähler-Einstein surfaces without Lagrangian homotopy classes. A minimal Lagrangian surface in $\mathbb{C}P^2$ then lifts to a special Legendrian surface in $S^5$, Proposition 1.6. We remark however that main existence theorem Theorem 5.1 provides only a Hamiltonian stationary Lagrangian disk with at most finitely many interior singular points, and the disk is not minimal in general.

Let $D$ denote the closed unit disk in the complex plane with the standard Lebesgue measure, and $D^\pm = \{(x, y) \in D | x \gtrless 0\}$. Let $X$ be a compact Kähler-Einstein surface without Lagrangian homotopy classes. Define

$$W^{1,2}_L(D, X) = \{ l \in W^{1,2}(D, X) | l'(x) \varphi = 0 \text{ for a.e. } x \in D \}$$

where $W^{1,2}(D, X)$ is the Sobolev space of maps of square integrable first derivatives, and $\varphi$ is the symplectic form on $X$. For $l \in W^{1,2}_L(D, X)$, let $E(l)$ denote the Dirichlet energy and $\partial l = l|_{\partial D} \in W^{\frac{1}{2},2}(\partial D, X)$.

Let $\Gamma \subset X$ be a rational geodesic polygon. Define

$$W^c_\Gamma = \{ l \in W^{1,2}_L(D, X) | E(l) \leq c \}$$

$$W^{1,2}_{\Gamma} = \{ l \in W^{1,2}_L(D, X) | \partial l(x) \in \Gamma \text{ for a.e. } x \in \partial D \}$$

$$W^c_{\Gamma} = \{ l \in W^{1,2}_{\Gamma} | E(l) \leq c \}.$$

Then $W^c_{\Gamma}$ is weakly closed in $W^{1,2}(D, X)$ [ScW]. For simplicity, $\partial l \subset \Gamma$ would mean $\partial l(x) \in \Gamma$ for a.e. $x \in \partial D$.

**Lemma 5.1** $W^c_{\Gamma}$ is weakly closed in $W^{1,2}_L(D, X)$.

**Proof.** Suppose $l_i \to l$ weakly. By lower semicontinuity of energy and the arguments in Prop 2.6 [ScW], it suffices to show $\partial l \subset \Gamma$. Since the trace map $W^{1,2}(D) \to W^{\frac{1}{2},2}(\partial D)$...
is continuous, $\partial l_i \to \partial l$ weakly in $W^{1,2}(\partial D)$, and a subsequence $\partial l_{i_j} \to \partial l$ in $L^2$, and hence a subsequence $\partial l_{i_{jk}} \to \partial l$ almost everywhere. □

Let $\{P_1, .. P_m = P_0\}$ be the set of faces of $\Gamma$. $l \in W_1 \Gamma$ is called monotone if there exists a partition of $\partial D = S^1$ by a set of $m$ distinct points $\{q_1, q_2, .. q_m = q_0\} \subset \partial D$ called vertices such that $\partial l(q_i, q_{i+1}) \subset P_{i+1}$ as an $L^2$ map for $i = 0, ... m - 1$. Each of the arc $\overline{q_i q_{i+1}}$ is called an edge.

We begin with a relative version of the Gromov-Allcock isoperimetric inequality. Denote $S^- = \{ (x, y) \in \partial D^- | x^2 + y^2 = 1 \}$, and $I_y = \{ (x, y) \in \partial D^- | x = 0 \}$.

**Proposition 5.1** [Wa2] Let $\Gamma \subset \mathbb{C}^2$ be either a Lagrangian plane, or a complex plane, or a union of any two of Lagrangian or complex planes that intersect transversally at the origin. There exists a constant $\mu(\Gamma)$ such that for any $W^{1,2}$ curve $C : S^- \to \mathbb{C}^2$ which bounds a $W^{1,2}$ Lagrangian half disk(or a wedge respectively in case $\Gamma$ consists of two planes) $l_0 : D^- \to \mathbb{C}^2$ with $l_0(I_y) \subset \Gamma$ and $l_0|_{S^-} = C$, there exists a $W^{1,2}$ Lagrangian half disk(wedge) $l : D^- \to \mathbb{C}^2$ spanning $\Gamma$ and $C$ such that

$$\text{Area}(l(D^-)) \leq \mu \text{Length}^2(C).$$

If $\Gamma$ contains a complex plane, any curve $C$ with boundary in $\Gamma$ always bounds a Lagrangian half disk or a wedge with boundary in $\Gamma$.

These relative isoperimetric inequalities then give the relative analogues of the Collar Lemma 4.8 in [ScW].

**Corollary 5.1** Let $\Gamma \subset X$ be a rational polygon, and suppose $l \in W_1 \Gamma$ is a weakly conformal, monotone, weakly Lagrangian map that is minimizing in its relative homotopy class. Then $l$ is Hölder continuous on $D$ up to boundary.

**Proof.** The interior regularity is proved in Theorem 2.8 in [ScW]. Suppose $q \in \partial D$ is a vertex, and consider a small extension $\hat{l}$ of $l$ in a neighborhood of $q$ by finite successive
reflections across the corresponding faces of $\Gamma$, which is defined since $l$ is monotone and $\Gamma$ is rational. The corollary follows from the relative isoperimetric inequality and the interior regularity applied to $\hat{l}$. Regularity along the edges follows similarly. □

A weakly Lagrangian map $l \in W_\Gamma$ is called stationary if for any Lagrangian deformation vector field $V$ along $l$ which is tangent to $\Gamma$ on $\partial D$, the variation of area

$$\delta_V \text{Area}(l(D)) = 0.$$ 

It is Hamiltonian stationary if the variation vector field $V$ is assumed to be Hamiltonian.

**Proposition 5.2** Let $\Gamma \subset X$ be a rational polygon, and suppose $l \in W_\Gamma$ is monotone and stationary. For any $q \in \partial D$ consider $\hat{l}$, which is a slight continuation of $l$ by successive reflections across the corresponding face(s) of $\Gamma$ at $l(q)$. Then $\hat{l}$ is stationary with respect to compactly supported Lagrangian variations.

**Proof.** Suppose $q$ is an edge point, and $l(q)$ lies on a face $P$ of $\Gamma$. Take a representative $u = u^- \cup u^+$ of $\hat{l}$ near $q$ defined on $D^- \cup D^+$ so that $q$ corresponds to the origin and $u^+ = r_P \circ u^-$. Given any compactly supported Lagrangian vector field $V$ along $u$ in a neighborhood of $q$, let $\tilde{V}$ be the average of $V$ with respect to $r_P$. Then from the symmetry of $u$,

$$\delta_V \text{Area}(u(D)) = 2 \delta_{\tilde{V}} \text{Area}(u^-(D^-)) = 0,$$

for $\tilde{V}$ is an admissible variation of $u^-$ along $y-$axis of $D^-$. The case $q$ is a vertex follows by a similar argument. □

We now describe a set of conditions for the rational polygon $\Gamma$ that is necessary for our construction. For any configuration $Y \subset X$, $\pi_2^L(X,Y) \subset \pi_2(X,Y)$ denotes the set of relative homotopy classes that can be represented by Lagrangian disks.

**Definition 5.1** Let $\Gamma \subset X$ be a rational polygon with faces $\{ P_1, \ldots, P_m = P_0 \}$. $\Gamma$ is admissible if the relative Lagrangian homotopy classes $\pi_2^L(X,P_i)$, $\pi_2^L(X,P_i \cup P_{i+1})$, $i = 0, \ldots, m - 1$, are trivial.
Example 5.1  Suppose each faces of a polygon $\Gamma \subset X$ is $S^2$, and every adjacent pair of faces intersects at a single point. Then $\Gamma$ is admissible if $X$ is without Lagrangian homotopy class.

Let $\beta \in \pi^L_2(X, \Gamma)$ be a nonzero class which can be represented by a monotone Lagrangian disk. Define

$$W_{\Gamma, \beta} = \{ l \in W_\Gamma \mid [l] = \beta, \ l \text{ is monotone} \},$$

where $[l]$ is the homotopy class represented by $l$. Note the maps $W_\Gamma \to \pi_2(X, \Gamma)$ and $W_\Gamma \to W^{1,2}(\partial D, X) \to \pi_1(\Gamma)$ are well defined and continuous [ScU][Ku].

As remarked in [Ku], there exists a lower bound $\delta > 0$ such that $E(l) \geq \delta$ for any $l \in W_{\Gamma, \beta}$, $\beta \neq 0$. Set

$$E(\Gamma, \beta) = \inf_{W_{\Gamma, \beta}} E(l)$$

$$A(\Gamma, \beta) = \inf_{W_{\Gamma, \beta}} \text{Area}(l).$$

Then $A(\Gamma, \beta) = \frac{1}{2}E(\Gamma, \beta)$, and if $E(l) = E(\Gamma, \beta)$, $l$ is conformal and $A(l) = A(\Gamma, \beta)$.

The following is the main existence theorem of this section.

**Theorem 5.1** Let $\Gamma \subset X$ be a compact admissible complex rational polygon in a compact Kähler-Einstein surface $X$ without Lagrangian homotopy classes. Let $\beta \in \pi^L_2(X, \Gamma)$ be a nonzero class which can be represented by a monotone Lagrangian disk. There exists an area minimizing $l \in W_{\Gamma, \beta}$ with $A(l) = A(\Gamma, \beta)$ which is a Hamiltonian stationary branched Lagrangian surface with at most finitely many interior conical singular points and smooth up to $\partial D$. $l(D)$ generates a complete Lagrangian surface by successive reflections across its boundary. If $l(D)$ is smooth, it is minimal.

**Proof.** Let $\{l_k\}_{k=1}^\infty \subset W_{\Gamma, \beta}$ be a minimizing sequence of monotone Lagrangian maps with vertices $\{ q_k^j \}_{j=1}^m$ that weakly converges to $l \in W_\Gamma$.

Case $m \leq 3$ or $\Gamma$ is at most a 3-gon : In this case, we may assume each $q_k^j = q^j$, $j = 1, \ldots, m$, is fixed for all $k$, and hence $l$ is also monotone, proof of Lemma 5.1.
Interior regularity. It follows from Theorem 4.10, Proposition 5.7 of [ScW].

Boundary regularity.

1. Since \( \pi^L_2(X, P_j) = 0 \) and \( \pi^L_2(X, P_j \cup P_{j+1}) = 0 \), we can apply the relative collar lemma and reflection to show there exist at most finitely many edge points and vertices at which the energy measure of the sequence \( \{ l_k \} \) blow up. Each blow up points can then be used to construct a sequence of minimizing Lagrangian disks(edge point) or wedges(vertex) by the relative isoperimetric inequality. Again since \( \Gamma \) is admissible, such sequence is trivial, and the blow up loci in \( \partial D \) is empty and \( l_k \to l \) strongly in \( W^{1,2} \).

2. Let \( q \in \partial D \) be an edge point(or a vertex respectively), and let \( \hat{l} \) be the continuation of \( l \) in a neighborhood of \( q \) via (successive) reflection(s) across the face(s) of \( \Gamma \) that contain \( l(q) \). The relative isoperimetric inequality and collar lemma gives a relative version of the strong \( W^{1,2}_{loc} \) compactness theorem Proposition 4.7 of [ScW], which then shows the existence of the tangent cone of \( \hat{l} \) at \( q \).

3. From 1 and 2 and reflectional symmetry, the link of each tangent cone of \( \hat{l} \) along the edge points of \( \partial D \) is \( C^1 \), and from the normal form of the Hamiltonian stationary Lagrangian cones in \( \mathbb{C}^2 \), [ScW], it is in fact smooth. One can easily check that the only Hamiltonian stationary Lagrangian cone with reflectional symmetry across a complex plane is flat, and hence each tangent map is proper holomorphic with respect to a suitable coordinates via Schwartz reflection principle. Now boundary regularity follows as in Theorem 4.10 of [ScW] using partial regularity Theorem 4.1 of [ScW]. Regularity at the vertices is verified similarly.

Case \( m \geq 4 \): From the arguments in Case \( m \leq 3 \), it suffices to show \( l \) is monotone.

Suppose \( m = 4 \), and \( q^j_k = q^j \) is fixed for \( j = 1, 2, 3 \). If the sequence \( q^4_k \to q^1 \), we claim that the energy measure of the sequence \( l_k \) blows up at \( q^1 \). For if not, \( l_k \to l \) strongly in \( W^{1,2} \) in a neighborhood of \( q^1 \), and there exists a neighborhood \( U \) of \( q^1 \) for any neighborhood \( V \) of \( l(q^1) \) in \( X \) such that \( l_k(U) \subset V \) for all sufficiently large \( k \). Since each \( l_k \) is monotone, this is a contradiction if we choose \( V \) so small that \( V \cap P_0 = \emptyset \).
We now apply arguments in 1. of Case $m \leq 3$ above to show there exists a minimizing sequence of Lagrangian wedges spanning $P_0 \cup P_1$, which is trivial for $\pi_2^L(X, P_0 \cup P_1)$ is trivial. Hence the sequence $\{q^4_k\}$ has a limit point in $\overline{q^3 q^1} \subset \partial D$ distinct from $q^1$ and $q^3$, and thus $l$ is monotone.

The case $m > 4$ can be proved by successive application of the arguments for the case $m = 4$.

If $l$ is an immersion up to boundary, the mean curvature deformation is admissible along the boundary due to reflectional symmetry, and $l$ is minimal. General case with branch points follows similarly by applying Lemma 8.2 in [ScW]. □

Remark. For general Kähler-Einstein surface $X$ and a compact rational complex polygon $\Gamma$, a relative Lagrangian homotopy or homology class with respect to $\Gamma$ can be represented by a finite union of Hamiltonian stationary Lagrangian surfaces, wedges and disks with boundary in $\Gamma$, and Lagrangian spheres.

Example 5.2 Let $\Gamma \subset \mathbb{C}^2$ be a rational complex polygon, and let $\alpha_\Gamma \in \pi_1(\Gamma)$ be the class represented by a monotone closed curve that connects the vertices of $\Gamma$. By considering a sufficiently large torus $\mathbb{C}^2/\mathbb{Z}$ for some lattice $\mathbb{Z}$, Proposition 5.1 and Theorem 5.1 imply there exists a monotone Hamiltonian stationary Lagrangian disk $l_\Gamma$ with $\partial l_\Gamma \subset \Gamma$ such that $[\partial l_\Gamma] = \alpha_\Gamma$.

Suppose $l_\Gamma$ is smooth, hence minimal. Let $dz^1 \wedge dz^2$ be the holomorphic volume form of $\mathbb{C}^2$, and, with a slight abuse of notation, consider

$$A(\Gamma) e^{-i\Theta(\Gamma)} = \frac{1}{2} \int_{\alpha_\Gamma} z^1 dz^2 - z^2 dz^1.$$ 

Since $dz^1 \wedge dz^2 = 0$ on $\Gamma$, the above integral is well defined. $A(\Gamma)$ and $\Theta(\Gamma)$ represent the area and the phase of the minimal Lagrangian surface $l_\Gamma$ respectively.

**Corollary 5.2** Let $X$ be a compact Kähler-Einstein surface. Let $\sigma$ be a Kähler involution, and $\Gamma_\sigma$ be a compact fixed point locus of $\sigma$, which is a complex curve. Suppose
\[ \pi_2^L(X, \Gamma, \sigma) \] is nontrivial. Then there exists a Hamiltonian stationary Lagrangian sphere in \( X \), which is smooth except at most finitely many conical singularities.

**Proof.** Remark below Theorem 5.1. □

**Corollary 5.3** Let \( \Gamma \subset \mathbb{C}P^2 \) be a rational complex polygon, and let \( \alpha \in \pi_1(\Gamma) \) be the class which is represented by a monotone curve that connects the vertices of \( \Gamma \). Then there exists a Hamiltonian stationary Lagrangian disk \( l \in W_\Gamma \) with \( [\partial l] = \alpha \in \pi_1(\Gamma) \) which is smooth up to boundary except at most finitely many interior conical singularities.

**Proof.** Let \( \gamma_0 \in \Gamma \) be a monotone curve with \( [\gamma_0] = \alpha \). By local accessibility theorem for contact structures, \([BCG]\), there exists a Legendrian lift \( \gamma \subset S^5 \) of \( \gamma_0 \). Let \( D_0 \subset S^5 \) be a disk with boundary \( \gamma \). Then by triangulization and local accessibility theorem for contact structures again, we can deform \( D_0 \) to a Legendrian disk while keeping the boundary fixed. Since \( \Gamma \) is admissible by Example 5.1, the corollary follows from Theorem 5.1 and the fact the trace map \( W^{1,2}(D, \Gamma) \to W^{1/2,2}(\partial D, \Gamma) \) is continuous. □

Let \( M^0_{\Gamma, \alpha} \) be the Hamiltonian stationary disk in the above corollary. Then by successive reflection across each edges, we obtain a complete Hamiltonian stationary Lagrangian surface \( M_{\Gamma, \alpha} \) in \( \mathbb{C}P^2 \), or a compact surface if \( G_\Gamma \) is finite. Let \( \Sigma_{\Gamma, \alpha} \subset S^5 \) be a connected Legendrian lift of \( M_{\Gamma, \alpha} \). \( \Sigma_{\Gamma, \alpha} \to M_{\Gamma, \alpha} \) is a nontrivial covering when \( G_\Gamma \cap Z(U(3)) \neq I_3 \), where \( Z(U(3)) \) is the center of \( U(3) \). For instance, \( M_{\Gamma, \beta} \) is nonorientable whenever \(-I_3 \in G_\Gamma \), Corollary 1.1.

### 6 Surfaces \( \Sigma_{k, 3} \)

Let \( k \geq 3 \) be an integer and set
\[
\epsilon = \exp \frac{2\pi i}{k}
\]
Consider a set of lines in $\mathbb{C}P^2$

$$P_1 = \{ [z_1, z_2, z_3] \in \mathbb{C}P^2 \mid z_1 = \epsilon z_2 \}$$

$$P_2 = \{ [z_1, z_2, z_3] \in \mathbb{C}P^2 \mid z_1 = z_2 \}$$

$$P_3 = \{ [z_1, z_2, z_3] \in \mathbb{C}P^2 \mid z_2 = z_3 \},$$

and let $\{r_i\}_{i=1}^3$ be the corresponding reflections. Let $\Gamma = \bigcup_{i=1}^3 P_i$ be the complex geodesic triangle with vertices

$$v_i = P_j \cap P_k \ (ijk) = (123),$$

and the set of vertex angles

$$\{ \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{k} \}.$$  

The group $G_{k,3}$ generated by reflections $\{r_i\}$ is finite of order $6k^2$ with defining relations

$$r_i^2 = (r_2 r_3)^3 = (r_3 r_1)^3 = (r_1 r_2)^k = (r_1 r_2 r_3 r_2)^3 = 1.$$  

The image of $\Gamma$ under successive reflections across edges gives rise to a tessellation of $\mathbb{C}P^2$ with $6k^2$ triangular faces and $3k + 2k^2$ vertices [Cox].

Let $M_\Gamma$ be a minimizer of area among all Lagrangian disks spanning $\Gamma$, and let $M_{k,3}$ be the compact Hamiltonian stationary Lagrangian surface obtained by successive reflection of $M_\Gamma$. If $M_\Gamma$ is an immersed disk, we get from Gauss-Bonnet formula (since $M_\Gamma$ spans the smaller angles at each vertex by the minimizing property) and Lemma 4.2,

$$\int_{M_{k,3}} K \, dA = 6k^2 \pi \left( \frac{1}{k} - \frac{1}{3} \right),$$

and hence

$$\chi(M_{k,3}) = k(3 - k).$$

Consider now the Legendrian lift $\Sigma_{k,3} \subset S^5$. Since the center of $G_{k,3}$ is cyclic of order $(3, k)$, the greatest common divisor of 3 and $k$,

$$\Sigma_{k,3} \to M_{k,3}$$

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is 3:1 when \( k \) is a multiple of 3, and otherwise 1:1.

**Theorem 6.1** For each integer \( k \geq 3 \), there exist a compact Hamiltonian stationary Legendrian surface \( \Sigma_{k,3} \subset S^5 \) of genus \( 3(1 + \frac{k(k-3)}{2}) \) if \( 3 | k \), and genus \( 1 + \frac{k(k-3)}{2} \) otherwise. \( \Sigma_{k,3} \) is smooth except at most finitely many conical singularities. The image of this surface under Hopf map is a compact orientable Hamiltonian stationary Lagrangian surface in \( \mathbb{C}P^2 \) of genus \( 1 + \frac{k(k-3)}{2} \).

**Remark.** Compact orientable surface of genus \( \geq 2 \) cannot be embedded in \( \mathbb{C}P^2 \) as a Lagrangian surface by Proposition 1.8.

\( G_{k,3} \) acts on the vertices of the tessellation mentioned above with three orbits \( V_+, V_- \), and \( V_0 \) of order \( k^2, k^2, \) and \( 3k \) respectively. Moreover, there exists a Lagrangian reflection of \( \mathbb{C}P^2 \) that interchanges \( V_+, V_- \), and leaves \( V_0 \) invariant as a set [Cox]. Note this corresponds to adding a single generator of order two to \( G_{k,3} \). The enlarged group is also finite, and hence we obtain a rational Lagrangian polyhedral variety that consists of equilateral triangles of vertex angle \( \frac{\pi}{k} \). From the symmetry, it is likely that \( \Sigma_{k,3} \) is invariant under these enlarged group and hence can be constructed by successive reflections of a fundamental domain spanning the equilateral Lagrangian triangle.

Suppose \( M_{k,3} \) is smooth, hence minimal, and consider the subgroup \( G_{k,3}^{\text{even}} < G_{k,3} \) of even elements, which is well defined for the defining relations are all even. From Hurwitz formula,

\[
M_{k,3} \to M_{k,3}/G_{k,3}^{\text{even}} = \mathbb{C}P^1
\]

is a \( 3k^2 \)-fold cover branched over \( V_+, V_- \), and \( V_0 \) with branching degrees 3, 3, and \( k \) respectively. This leads to an alternative description of \( M_{k,3} \) in terms of a singular solution to (15) on a hexagonal torus as follows. The original idea of quotient by discrete symmetry is due to Robert Bryant in his unpublished note on Lawson’s minimal surfaces.

We follow the notations adopted in Section 2.
Let $w$ be a coordinate of $M_{k,3}/G_{k,3}^{even}$, and set

\[ w(V_+) = \pm 1 \]
\[ w(V_0) = 0. \]

As $\{ V_+, V_-, V_0 \}$ is the umbilic locus of $M_{k,3}$, the holomorphic cubic differential $\Phi$ of $M_{k,3}$, Section 2, is then a constant multiple of

\[ \frac{(dw)^3}{w^2(w^2 - 1)^2}. \]

Since $G_{k,3}^{even}$ acts as isometry, holomorphic cubic differential $\Phi$ and the metric $g$ on $M_{k,3}$ can be pushed forward to $M_{k,3}/G_{k,3}^{even}$ as a meromorphic differential $\Phi_0$ and a singular metric $g_0$. Let $T$ be the hexagonal torus in Example 1.2, and $\pi_3 : T \to \mathbb{C}P^1$ be the threefold covering branched at three points $\{ \pi_3^{-1}(1), \pi_3^{-1}(0), \pi_3^{-1}(1) \}$. Let $z$ be the standard coordinate on the universal covering $\mathbb{C} \to T$. Then $\pi_3^*(\Phi_0)$ is a holomorphic differential on $T$. In fact, we may choose $z$ so that

\[ \pi_3^*(\Phi_0) = \lambda (dz)^3 \]

for some real $\lambda > 0$. Similar analysis for the metric $g_0$ shows if we write

\[ \pi_3^*(g_0) = e^{2u}dzd\bar{z}, \]

then $u(z, \bar{z})$ is a function on $T$ that satisfies the compatibility condition

\[ \triangle u + e^{2u} - 2\lambda^2 e^{-4u} = 0 \]

with appropriate logarithmic singularities at the branch points of $\pi_3$.

7 Gauss map, polar surface, and bipolar surface

We now turn our attention to surfaces associated to a given special Legendrian surface. It will be shown for example that generating functions of the contact transformations of
$S^5$ induced by $SU(3)$ action, when restricted to a special Legendrian surface, describe a minimal surface in the unit sphere of the Lie algebra $\mathfrak{su}(3)$. To draw geometric conclusion from this, we apply maximum principle to obtain the following half space theorem for compact minimal Lagrangian surfaces in $\mathbb{C}P^2$.

A compact minimal Lagrangian surface in $\mathbb{C}P^2$ does not lie in any open geodesic ball of radius $\frac{\pi}{2} - \arccos\left(\frac{1}{\sqrt{3}}\right)$.

We continue to use the notation adopted in Section 2.

### 7.1 Gauss map

Let $u : \Sigma \to S^5$ be a special Legendrian surface. Gauss map $u^* : \Sigma \to Isot^+(2, \mathbb{C}^3) = SU(3)/SO(2)$ is defined by

$$u^* = e_1 \wedge e_2.$$  \hfill (17)

$u^*$ is then a conformal and minimal immersion of $\Sigma$ with the induced metric

$$du^* \circ du^* = (2 - K)du \circ du.$$

$Isot^+(2, \mathbb{C}^3)$ admits an integrable CR-structure as a real hypersurface of the hyperquadric $Gr^+_\mathbb{R}(2, \mathbb{C}^3) \subset \mathbb{C}P^5$. Gauss map $u^*$ is never a complex curve with respect to this CR-structure unless it is totally geodesic. Instead, consider the fibration in Lemma 1.1

$$Isot^+(2, \mathbb{C}^3) \to S^5$$

with $\mathbb{C}P^1$ fibers. Upon reversing the orientation of each fiber, we obtain a new CR-structure on $Isot^+(2, \mathbb{C}^3)$, under which the Gauss map is complex. This CR-structure is however not integrable [Sal].

Let $\gamma : S^1 \to S^5$ be an isotropic curve. Since the differential system for special Legendrian surfaces (9) is invariant under $SU(3)$ action, Noether’s theorem implies not every such $\gamma$ bounds a special Legendrian surface(variety or current). Equivalently, the first order characteristic cohomology of the special Legendrian differential system is at
least of dimension 8. Now by Lemma 1.1, every isotropic curve $\gamma$ has a unique lift $\gamma^*: S^1 \to Isot^+(2, \mathbb{C}^3)$. Thus, a necessary condition for $\gamma$ to bound a special Legendrian surface is that its lift $\gamma^*$ must bound a complex variety in $Isot^+(2, \mathbb{C}^3)$. More specifically, $\gamma^*$ must satisfy

$$\int_{\gamma^*} \eta = 0,$$

where $\eta$ is a 1-form on $Isot^+(2, \mathbb{C}^3)$ such that $d\eta$ modulo contact form is a linear combination of forms of type $(2, 0)$ or $(0, 2)$ with respect to the aforementioned nonintegrable $CR$-structure.

In case of a curve in a complex manifold, a necessary and sufficient condition for a curve to bound a complex variety is known, and has been generalized to higher dimensions by Harvey and Lawson [HL2].

### 7.2 Polar surface

We now view (17) as a map $u^*: \Sigma \to \bigwedge^2 \mathbb{R}^6$. Consider the endomorphism of $\bigwedge^2 \mathbb{R}^6$ induced from the complex structure $J$ on $\mathbb{C}^3 = \mathbb{R}^6$, which we continue to denote by $J$. Then $J \circ J = 1_{\bigwedge^2 \mathbb{R}^6}$, and let $\bigwedge^2 \mathbb{R}^6 = \bigwedge_+^9 \oplus \bigwedge_-^6$ be the $\pm 1$ eigenspace decomposition. $\bigwedge_+^9$ further decomposes as $\bigwedge_+^9 = \mathbb{R} \oplus W^8$ where $W$ is the adjoint representation of $SU(3)$.

Let

$$u_+^* = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 + n_1 \wedge n_2) \subset S^7 \subset W^8$$

$$u_-^* = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 - n_1 \wedge n_2) \subset S^5 \subset \bigwedge_-^6$$

be the respective projections, and note that $u_+^*$ is identified with $u$ under the isomorphism $\mathbb{C}^3 = \bigwedge^2 \mathbb{C}^3$. From the structure equations in Section 2,

$$du_+^* \circ du_+^* = (3 - 2K) du \circ du$$

$$\Delta u_+^* = -2 (3 - 2K) u_+^*$$

$$\Delta u_-^* = -2 u_-^*.$$  

(18)
Thus \( u_+^* \) is a minimal immersion into \( S^7 \), and we call \( u_+^* \) polar immersion of the special Legendrian immersion \( u \). Note that \( u_+^* \) is well defined on the minimal Lagrangian image of \( \Sigma \) under Hopf map.

Let \( \xi \in \Lambda^2 \mathbb{R}^6 \) be a 2-vector, and set
\[
\begin{align*}
  u_+^\xi &= \sqrt{2} \langle u_+^*, \xi \rangle = \langle e_1 \wedge e_2, \xi \pm J(\xi) \rangle.
\end{align*}
\]

From (18) and (19), we get
\[
\begin{align*}
  \Delta u_+^\xi &= -2 (3 - 2K) u_+^\xi \\
  \Delta u_-^\xi &= -2 u_-^\xi.
\end{align*}
\]

**Proposition 7.1** Let \( \Sigma \subset S^5 \) be a compact special Legendrian surface, and let \( e_1 \wedge e_2 \) be its oriented unit tangent plane field. For any 2-vector \( \xi \in \Lambda^2 \mathbb{R}^6 \),
\[
\begin{align*}
  \int_{\Sigma} (3 - 2K) \langle e_1 \wedge e_2, \xi + J(\xi) \rangle \, dvol_\Sigma &= 0 \\
  \int_{\Sigma} \langle e_1 \wedge e_2, \xi - J(\xi) \rangle \, dvol_\Sigma &= 0.
\end{align*}
\]

In particular, for any 2-vector \( \xi \in W^8 \) or \( \xi \in \Lambda^6_- \), there exists a point \( p \in \Sigma \) such that
\[
\langle e_1 \wedge e_2|_p, \xi \rangle = 0.
\]

It is easily verified that if \( \Sigma \) is compact with genus \( \geq 2 \), then its polar surface is linearly full in \( W^8 \).

### 7.3 Bipolar surface

Let \( \varpi^z = e_1 \wedge n_1 + e_2 \wedge n_2 + e_3 \wedge n_3 \) be the 2-vector dual to the Kähler form \( \varpi \) of \( \mathbb{C}^3 \), which is constant along \( \Sigma \). We define the bipolar immersion associated to a special Legendrian immersion \( u : \Sigma \to S^5 \) by
\[
  u_* = \sqrt{\frac{3}{2}} (e_3 \wedge n_3 - \frac{1}{3} \varpi^z) \subset S^7 \subset W^8.
\]
It is a conformal immersion with
\[
du_* \circ du_* = 3 du \circ du,
\]
\[
\Delta u_* = -6 u_*.
\]
(21)

Thus the bipolar immersion associated to a special Legendrian immersion is also minimal. \(u_*\) is also well defined on the minimal Lagrangian image of \(\Sigma\) under the Hopf map.

Coordinate functions of bipolar immersion admit a simple geometrical interpretation in terms of generating functions. Let \(\xi \in \mathfrak{su}(3) = W^8\), and denote its induced vector field on \(S^5\) by \(V_\xi\). Define the generating functions \(G_\xi \in C^\infty(S^5)\) by
\[
G_\xi(p) = \langle p \wedge J(p), \xi \rangle
= \langle V_\xi(p), J(p) \rangle.
\]
Each \(G_\xi\) is again a well defined function on \(\mathbb{CP}^2\).

Lemma 7.1 \(G_\xi : \mathfrak{su}(3) \to C^\infty(S^5)\) is one to one.

Proof. Suppose \(\langle V_\xi(p), J(p) \rangle \equiv 0\). Let \(\theta\) be the contact form (2), and \(\mathfrak{L}_{V_\xi} \theta\) the Lie derivative of \(\theta\). The lemma follows from the equation
\[
\mathfrak{L}_{V_\xi} \theta = 0 = d(V_\xi \lhd \theta) + V_\xi \lhd d\theta = V_\xi \lhd d\theta,
\]
for \(d\theta\) is nondegenerate. □

Let \(\Sigma\) be a special Legendrian surface in \(S^5\). Put
\[
\begin{align*}
\xi = & \sqrt{\frac{2}{3}} \langle u_*, \xi \rangle \\
& = G_\xi \circ u
\end{align*}
\]
and denote \(G(\Sigma) = \{ u_\xi \mid \xi \in \mathfrak{su}(3) \}\). Then
\[
\Delta u_\xi = -6 u_\xi,
\]
Lemma 7.1 implies $\mathcal{G}(\Sigma)$ can be identified with the set of Killing - Jacobi fields on $\Sigma$, i.e., the Jacobi fields along $\Sigma$ obtained by orthogonal projection to the normal bundle $N\Sigma$ of Killing fields generated by $SU(3)$ action.

**Proposition 7.2** Let $\xi \in su(3)$. Then $u_\xi \equiv 0$ on $\Sigma$ if and only if $V_\xi$ is tangent to $\Sigma$.

**Proof.** From the structure equations of the homogeneous space $S^5 = SU(3)/SU(2)$, there exists a set of 1-forms $\omega^1, \omega^2, \theta_1, \theta_2$ in a neighborhood of a point $p \in \Sigma$ so that $-d\theta = \omega^1 \wedge \theta^1 + \omega^2 \wedge \theta^2$ and $\theta_1, \theta_2 = 0$ on $\Sigma$. The proposition follows from (22). □

**Theorem 7.1** Let $\Sigma$ be a special Legendrian surface in $S^5$. Then $\dim \mathcal{G}(\Sigma) \geq 5$, and either

- $\dim \mathcal{G}(\Sigma) = 5$ and $\Sigma$ is totally geodesic, or
- $\dim \mathcal{G}(\Sigma) = 6$ and $\Sigma$ is a part of the hexagonal torus, or
- $\dim \mathcal{G}(\Sigma) = 7$ or $8$.

If $\Sigma$ is compact and $\dim \mathcal{G}(\Sigma) = 7$, $\Sigma$ is a torus. If $\Sigma$ is compact with genus $\geq 2$, then $\dim \mathcal{G}(\Sigma) = 8$.

**Corollary 7.1** Let $\Sigma$ be a compact special Legendrian surface of genus $\geq 2$. Then the eigenspace of $-\Delta$ on $C^\infty(\Sigma)$ with eigenvalue 6 is of dimension at least 8.

$\mathcal{G}(M)$ can be defined for a compact minimal Lagrangian submanifold $M$ in $CP^n$. For instance, $\dim \mathcal{G}(M) \geq n(n + 3)/2$ with equality only if $M$ is totally geodesic. $\mathcal{G}(M)$ is a subspace of the eigenspace of $-\Delta$ with eigenvalue $2n + 2$.

Equation (21) also has the following geometric consequences when integrated over a compact special Legendrian surface. Note (21) can be rewritten as

$$\Delta (e_3 \wedge J(e_3)) = -6 e_3 \wedge J(e_3) + 2\omega^4.$$
Lemma 7.2 Let $v', v$ be vectors in $\mathbb{R}^{2n} = \mathbb{C}^n$ with complex structure $J$. Then

$$\langle v' \wedge J(v'), v \wedge J(v) \rangle = \langle v, v' \rangle^2 + \langle Jv, v' \rangle^2 = \langle v, v' \rangle^2 + \langle v, J(v') \rangle^2.$$ 

Proposition 7.3 Let $e_3 : \Sigma \to S^5 \subset \mathbb{C}^3$ be a special Legendrian immersion of a compact surface $\Sigma$. For any vector $v \in \mathbb{C}^3$ of unit length, set

$$u_v = \langle e_3 \wedge J(e_3), v \wedge J(v) \rangle = \langle e_3, v \rangle^2 + \langle J(e_3), v \rangle^2,$$

which represents the length square of the orthogonal projection of $v$ onto the 2-plane $e_3 \wedge J(e_3)$. Then

$$\Delta u_v = -6u_v + 2.$$ 

Upon integration,

$$\int_{\Sigma} (u_v - \frac{1}{3}) d\text{vol}_{\Sigma} = 0.$$ 

Theorem 7.2 Let $M \subset \mathbb{C}P^2$ be a compact minimal Lagrangian surface, and let $\text{dist}(q, M)$ denote the distance of a point $q \in \mathbb{C}P^2$ to $M$. Then

$$\text{dist}(q, M) \leq \arccos\left(\frac{1}{\sqrt{3}}\right)$$

for any $q \in \mathbb{C}P^2$.

Proof. Take a point $\tilde{q} \in S^5$ in the inverse image of $q \in \mathbb{C}P^2$ under Hopf map. By Proposition 7.3, there exists a point $p \in M$ such that $\text{dist}(\tilde{q}, \pi^{-1}(p)) = \arccos(\frac{1}{\sqrt{3}})$. The theorem follows for Hopf map is an isometric submersion, and hence distance non-increasing. □

Corollary 7.2 Let $M \subset \mathbb{C}P^2$ be a compact minimal Lagrangian surface. Then $M$ does not lie in any open geodesic ball of radius $\frac{\pi}{2} - \arccos\left(\frac{1}{\sqrt{3}}\right)$. 

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Proof. Let \( q, q' \in \mathbb{C}P^2 \) be such that \( \text{dist}(q, q') = \frac{\pi}{2} \). The open geodesic ball of radius \( \frac{\pi}{2} - \arccos \left( \frac{1}{\sqrt{3}} \right) \) centered at \( q \) is disjoint from the closed geodesic ball of radius \( \arccos \left( \frac{1}{\sqrt{3}} \right) \) centered at \( q' \). \( \square \)

Theorem 7.2 agrees with Corollary 1.2 in that the area of a compact minimal Lagrangian surface cannot be too small. We also mention Theorem 7.2 and Corollary 7.2 is true for compact minimal Lagrangian submanifolds in \( \mathbb{C}P^n \), \( n \geq 2 \), with the constant \( \arccos \left( \frac{1}{\sqrt{n+1}} \right) \).

For minimal Lagrangian submanifolds in Kähler-Einstein manifolds, there is a notion of Hamiltonian stability and index introduced by Oh, which is finer or weaker than the usual stability and index for minimal submanifolds [Oh]. Since submanifolds in Example 1.1 are all Hamiltonian stable, [AO], we have many Hamiltonian stable compact minimal Lagrangian submanifolds in \( \mathbb{C}P^n \) via product, Proposition 1.5, whereas a compact minimal submanifold in \( \mathbb{C}P^n \) other than complex varieties is unstable [LS].

A compact minimal Lagrangian submanifold \( M \) in \( \mathbb{C}P^n \) is stable under Hamiltonian deformations near identity if \( \lambda_1 \geq 2(n+1) \), where \( \lambda_1 \) is the first nonzero eigenvalue of \( -\Delta \) on \( C^\infty(M) \) [Oh]. The remark below Corollary 7.1 gives a slight strengthening of this result.

**Theorem 7.3** A compact minimal Lagrangian submanifold \( M \) in \( \mathbb{C}P^n \) is (locally) Hamiltonian stable if \( \lambda_1 = 2(n+1) \), where \( \lambda_1 \) is the first nonzero eigenvalue of \( -\Delta \) on \( C^\infty(M) \).

Remark. Are examples (4), (5), (6), (7), (8) and their product the only Hamiltonian stable compact minimal Lagrangian submanifolds in complex projective spaces?
In this section, we consider minimal Lagrangian surfaces in a complex space form $X(c)$ of constant holomorphic curvature $4c$, $c = 1, 0$ or $-1$ for unified treatments. Special Legendrian surfaces in $S^5$ are naturally identified with minimal Lagrangian surfaces in $X(1)$.

We first recall the fundamental theorem for minimal Lagrangian surfaces in complex spaces forms, of which Theorem 2.1 is a particular case.

**Theorem 8.1** Let $\tilde{M}$ be a simply connected Riemann surface with a conformal metric $g$ and a holomorphic cubic differential $\Phi_0$ such that

$$K = c - 2||\Phi_0||^2,$$

where $||\Phi_0||$ is the norm with respect to $g$ and $K$ is the Gaussian curvature of the metric. Then there exists an $S^1$ family of isometric minimal Lagrangian immersions $u_\tau : \tilde{M} \rightarrow X(c)$ with the associated holomorphic cubic differentials $\Phi_\tau = e^{i\tau} \Phi_0$, $e^{i\tau} \in S^1$. The immersion is unique up to ambient isometry of $X(c)$ for each holomorphic cubic differential $\Phi_\tau$. This family exhausts all isometric minimal Lagrangian immersion of $\tilde{M}$ into $X(c)$.

The theorem follows from an application of Frobenius theorem [Gr]. For the last part, suppose $\Phi$ and $\Phi'$ are holomorphic cubic differentials induced from two isometric minimal Lagrangian immersions of a given metric $g$. From (23), the 0-divisors of $\Phi$ and $\Phi'$ are determined from the Gaussian curvature. Thus $\Phi/\Phi'$ is a holomorphic function of unit length, hence a constant.

The metric of a minimal Lagrangian surface in $X(c)$ admits a simple intrinsic characterization.
Theorem 8.2 Suppose \( g \) is a metric of a minimal Lagrangian surface in a complex space form \( X(c) \). Then the associated metric

\[
\tilde{g} = \left( \frac{c - K}{2} \right)^{\frac{1}{3}} g
\]

is flat. Conversely, let \( g \) be a metric on a simply connected surface \( \tilde{M} \) such that the Gaussian curvature

\[ K < c, \]

and that the associated metric \( \tilde{g} \) is flat. Then there exists an \( S^1 \) family of isometric minimal Lagrangian immersions \( u_\tau : \tilde{M} \to X(c) \), \( e^{i\tau} \in S^1 \).

Proof. We present a proof for the case \( c = 1 \), and use the notations in Section 2. For the first part of the theorem, take a local coordinate \( z \) on \( \tilde{M} \), away from the zero locus of \( \Phi \), so that \( \Phi = h(\omega^1 + i \omega^2)^3 = (dz)^3 \). Then the metric

\[
(\Phi \circ \tilde{\Phi})^{\frac{1}{3}} = dz \circ d\bar{z} = (h \bar{h})^{\frac{1}{3}} g
\]

is flat. The converse follows by reverse of this argument and Theorem 8.1. □

Given a minimal Lagrangian surface \( u : M \to X(c) \), we now define its associate family as follows. Let \( \pi : \tilde{M} \to M \) be the universal covering of \( M \) with the pulled back metric and holomorphic differential \( \pi^*(\Phi) \). By Theorem 8.1, there exist isometric minimal Lagrangian immersions \( u_\tau : \tilde{M} \to X(c) \) with the associated cubic differential \( e^{i\tau} \pi^*(\Phi) \). A surface \( u_\tau(\tilde{M}) \) is called an associate surface of \( u(M) \). \( u_\pi(\tilde{M}) = M^* \) is in particular the conjugate surface of \( u(M) \). Conjugate pair of surfaces satisfy the following dual reflection principle.

Let \( \gamma \) be a curve on a minimal Lagrangian surface \( M \) in \( X(c) \). \( \gamma \) is Lagrangian if \( \gamma \subset M \cap \mathcal{L} \) where \( \mathcal{L} \) is a totally geodesic Lagrangian surface in \( X(c) \) that has orthogonal intersection with \( M \). \( \gamma \) is complex if \( \gamma \subset M \cap \Pi \) where \( \Pi \) is a totally geodesic complex curve in \( X(c) \). In the latter case \( \Pi \) is necessarily orthogonal to \( M \). \( \Pi (\mathcal{L} \text{ respectively}) \) is
the fixed point locus of an Kähler (anti- Kähler) involution of $X(c)$, and as remarked in Section 3, the reflection principles continue to hold in this setting.

**Proposition 8.1** A curve on a minimal Lagrangian surface in a complex space form is Lagrangian (complex respectively) if it is a geodesic on which the associated holomorphic differential is real(purely imaginary).

*Proof.* We present a proof for the case $c = 1$. From the structure equations (11), assume $e_2$ is tangent to the curve $\gamma$, or $\omega^1 = 0$ on $\gamma$. Then the unique anti-special Lagrangian 3-plane that contains $e_2 \wedge e_3$ is $-n_1 \wedge e_2 \wedge e_3$. A computation shows $d(n_1 \wedge e_2 \wedge e_3) = 0$ if and only if $\rho = 0$ and $b = 0$. Similarly, the unique complex 2-plane generated by $e_2 \wedge e_3$ is $e_2 \wedge n_2 \wedge e_3 \wedge n_3$, and it is constant when $\rho = 0$ and $a = 0$. □

**Corollary 8.1** A Lagrangian geodesic on a minimal Lagrangian surface in a complex space form corresponds to a complex geodesic on its conjugate minimal Lagrangian surface, and vice versa.

For instance, if $M$ is obtained by successive reflection of a fundamental domain across its Lagrangian or complex boundaries, so is its conjugate $M^*$. 

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