Membrane world model building
Putting the standard model on a domain-wall brane

Damien George

Theoretical Particle Physics group
School of Physics
University of Melbourne
Australia

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Context: physics beyond the standard model – extra dimensions.

We will cover:

- Randall-Sundrum 2 model with delta-function brane.
- Domain-walls (kinks/solitons).
- Trapping scalar and fermion fields to a domain-wall.
- Using Dvali-Shifman mechanism to trap gauge fields.
- $SU(5)$ grand unified domain-wall model.
We are inspired by the Randall-Sundrum warped metric solution.

RS1 is a compact extra dimension: provides a solution to the hierarchy problem – lots of work on this model. Branes are string theory like objects. Warped throats, inflation, dark matter, ...

RS2 is an infinite extra dimension: solves the trapping of low-energy gravity. Not as much interest because it doesn’t solve any major problems, just introduces another dimension.

We will pursue RS2 because it seems a natural extension of 3+1 space.

Most work done in collaboration with:
   Ray Volkas (Melbourne U) and Rhys Davies (Oxford U).
(Mem-)Brane worlds
Brane worlds

Premise:
- take the standard model and general relativity
- add an *infinite* extra space dimension
- recover the standard model and general relativity at low energies
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\[
S = \int d^4x \int dy \sqrt{|g|} \left[ -M^3 R + \delta(y)\mathcal{L}_{\text{SM}} \right]
\]
RS2 model

Need brane and bulk sources:

\[ S = \int d^4x \int dy \left[ \sqrt{|g|}(-M^3R - \Lambda_{\text{bulk}}) + \sqrt{|g^{(4)}|}\delta(y)(\mathcal{L}_{\text{SM}} - \Lambda_{\text{brane}}) \right] \]

(1)
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(1)

Solve the theory:

- Randall-Sundrum metric ansatz: \( ds^2 = e^{-2k|y|} g^{(4)}_{\mu\nu} dx^\mu dx^\nu - dy^2 \)
- Solve Einstein’s equations (\( \mathcal{L}_{\text{SM}} = 0 \) and \( R^{(4)} = 0 \)):
  \[ \Lambda_{\text{bulk}} = -12k^2 M^3 \quad \quad \Lambda_{\text{brane}} = 12k M^3 \]
- Write \( R \) in terms of \( R^{(4)} \): \( R = e^{2k|y|} R^{(4)} - 16k \delta(y) + 20k^2 \)
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Substitute into (1) and integrate over \( y \):

\[ S = \int d^4x \sqrt{|g^{(4)}|} \left[ -\frac{M^3}{k} R^{(4)} + \mathcal{L}_{\text{SM}} \right] \]

A dimensionally reduced theory.
Newton’s law

Just need to check Newton’s law. Linear tensor fluctuations are:

\[ g_{\mu\nu} = e^{-2k|y|} \eta_{\mu\nu} + \sum_n h^{(4)}_{n\mu\nu}(x^\mu) \psi_n(y) \]

The zero mode \( h^{(4)}_{0\mu\nu} \) dominates the Kaluza-Klein tower.
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\[ V(r) = -G_N \frac{m_1 m_2}{r} \left( 1 + \frac{\epsilon^2}{r^2} \right) \quad \text{(where } \epsilon = 1/k) \]

Current experimental bounds are very weak:

\[ \epsilon < 12\mu m \implies k > 16 \times 10^{-3}\text{eV} \]
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- We have what we wanted: a 5D theory that at low energies looks like our 4D universe.
- But almost no new phenomenology.
- Next step: brane forms naturally.
We want to remove the $\delta(y)$ part of the action:

$$S = \int d^4x \int dy \left[ \sqrt{|g|}(-M^3 R - \Lambda_{\text{bulk}}) + \sqrt{|g^{(4)}|} \delta(y) (\mathcal{L}_{\text{SM}} - \Lambda_{\text{brane}}) \right]$$

Everything from now on is one way of doing that.
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Geometry, hence gravity, is 5D. So why not try to make all fields 5D?

- First we show how to make a dynamical brane.
- Then we show how to trap scalars, fermions and gauge fields to the brane.
- Finally we present a 4+1-d $SU(5)$ based extension to the standard model.
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Turn off warped gravity for now – just think about trapping 5D fields.
A domain-wall as a brane

Idea: imagine the Higgs VEV had one value here, another there.

The interface is a \textit{domain-wall}.

\[ V = \lambda (\phi^* \phi - v^2)^2 \]
A domain-wall as a brane

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The interface is a domain-wall.

\[ V = \lambda (\phi^* \phi - v^2)^2 \]

This is unstable – vacua can be continuously deformed to each other.

(key: real, imaginary)
Disconnected degenerate vacua

We need a potential with disconnected and degenerate vacua:

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with \( \phi \) now a scalar.
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Lagrangian for \( \phi(x^\mu, y) \):

\[ \mathcal{L} = \frac{1}{2} \partial_M \phi \partial^M \phi - V(\phi) \]

A solution is the kink:

\[ \phi(y) = v \tanh(\sqrt{2\lambda} vy) \]

It is stable!
More complicated domain-walls

These examples use two scalar fields to form the wall.

\[ V = \lambda_1 (\phi_1^2 + \phi_2^2 - \nu^2)^2 + \lambda_2 \phi_1^2 \phi_2^2 \]

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In a realistic model, symmetries dictate \( V \).

To determine stability, expand \( \phi \) in normal modes about the background:

\[
\phi(x, t) = \phi_{bg}(x) + \sum_n \xi_n(x) e^{i\omega_n t}
\]

Make sure \( \omega_n^2 \geq 0 \).
Trapping matter fields
Aim: to trap a 5D scalar field $\Xi(x^\mu, y)$ to the brane.

A simple quartic coupling works:

$$S = \int d^4x \int dy \left[ \frac{1}{2} \partial^M \phi \partial_M \phi - V(\phi) + \frac{1}{2} \partial^M \Xi \partial_M \Xi - W(\Xi) - g\phi^2 \Xi^2 \right]$$
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Expand $\Xi$ in extra dimensional (Kaluza-Klein) modes:

$$\Xi(x^\mu, y) = \sum_n \xi_n(x^\mu) k_n(y)$$

$\xi_n$ are the 4D fields, $k_n$ their extra-dimensional profile. The profiles satisfy a Schrödinger equation:

$$\left( -\frac{d^2}{dy^2} + 2g\phi_{bg}^2 \right) k_n(y) = E_n^2 k_n(y)$$

The energy eigenvalues $E_n$ are related to the mass of the 4D field $\xi_n$. 
Trapping via a potential well

The effective potential acts like a well.

\[ \left( -\frac{d^2}{dy^2} + 2g\phi_{bg}^2 \right) k_n = E_n^2 k_n \]
Trapping via a potential well

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\[ (-\frac{d^2}{dy^2} + 2g\phi_{bg}^2) \ k_n = E_n^2 k_n \]

To get 4D theory, substitute mode expansion into action and integrate \( y \):

\[ S = \int d^4x \left[ \sum_n \left( \frac{1}{2} \partial^\mu \xi_n \partial_\mu \xi_n - m_n^2 \xi_n^2 \right) + \text{(higher order terms)} \right] \]

- Orthonormal basis \( k_n \) \( \implies \) diagonal kinetic and mass terms.
- \( m_n \) can be tuned.
We can trap a fermion $\Psi(x^\mu, y)$ to the brane with a Yukawa coupling:

$$S = \int d^4x \int dy \left[ \frac{1}{2} \partial^M \phi \partial_M \phi - V(\phi) + \bar{\Psi} i \Gamma^M \partial_M \Psi - h\phi \bar{\Psi} \Psi \right]$$
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Decompose into left- and right-chiral fields and Kaluza-Klein modes:

$$\Psi(x^\mu, y) = \sum_n \left[ \psi_{Ln}(x^\mu) f_{Ln}(y) + \psi_{Rn}(x^\mu) f_{Rn}(y) \right]$$
Trapping fermions

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$$\Psi(x^\mu, y) = \sum_n [\psi_{Ln}(x^\mu) f_{Ln}(y) + \psi_{Rn}(x^\mu) f_{Rn}(y)]$$

Schrödinger equation (mode index $n$ suppressed):

$$\left( -\frac{d^2}{dy^2} + (h^2 \phi_{bg}^2 \mp h \phi'_{bg}) \right) f_{L,R}(y) = m^2 f_{L,R}(y)$$
Gravity and matter fields

Brane (domain-wall/kink), trapped scalar and fermion. Plus gravity:

\[ S = \int d^4 x \int dy \sqrt{|g|} \left[ - M^3 R - \Lambda_{\text{bulk}} + \frac{1}{2} \partial^M \phi \partial_M \phi - V(\phi) \right. \\
+ \left. \frac{1}{2} \partial^M \Xi \partial_M \Xi - W(\Xi) - g\phi^2 \Xi^2 \right. \\
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Dimensionally reduce by integrating over \( y \):

\[ S = \int d^4 x \sqrt{|g^{(4)}|} \left[ -M_{4D}^2 R^{(4)} + (\text{brane dynamics}) \right. \]
\[ + \frac{1}{2} \partial^\mu \xi^\mu \partial_\mu \xi_n - m_n^2 \xi_n^2 - \tau_{mnop} \xi_m \xi_n \xi_o \xi_p - (\text{brane interactions}) \]
\[ + \overline{\Psi}_{L0} i \gamma^\mu \partial_\mu \Psi_{L0} + \overline{\Psi}_n (i \gamma^\mu \partial_\mu - \mu_n) \psi_n - (\text{brane interactions}) \right] \]

4D parameters \((M_{4D}, m_n, \tau_{mnop}, \mu_n, \text{brane dynamics})\) determined by eigenvalue spectra and overlap integrals.
Warped metric $ds^2 = e^{-2\sigma(y)} \eta_{\mu\nu} dx^\mu dx^\nu - dy^2$ modifies profile equation:

$$\left( -\frac{d^2}{dy^2} + 5\sigma' \frac{d}{dy} + 2\sigma'' - 6\sigma'^2 + U(y) \right) f_{Ln}(y) = m_n^2 e^{2\sigma} f_{Ln}(y)$$
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Conformal coordinates $ds^2 = e^{-2\sigma(y(z))}(\eta_{\mu\nu}dx^\mu dx^\nu - dz^2)$.

Rescale $f_{Ln}(y) = e^{2\sigma} \tilde{f}_{Ln}(z)$:

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Matter trapping potentials are warped down.
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Conformal coordinates $ds^2 = e^{-2\sigma(y(z))} (\eta_{\mu\nu} dx^\mu dx^\nu - d\bar{z}^2)$. Rescale $f_{Ln}(y) = e^{2\sigma} \tilde{f}_{Ln}(\bar{z})$:

$$\left( - \frac{d^2}{d\bar{z}^2} + e^{-2\sigma(y(z))} U(y(z)) \right) \tilde{f}_{Ln}(\bar{z}) = m_n^2 \tilde{f}_{Ln}(\bar{z})$$

Matter trapping potentials are warped down.

- Finite bound state lifetimes.
- Resonances.
- Tiny probability of interaction with continuum.

(a $\sim 1/M^3 \sim 5$D Newton’s constant)
Trapping gauge fields
Confining gauge fields

Need to trap gauge fields or e.g. Coulomb potential would be $V_{\text{Coulomb}} \sim 1/r^2$.

Not as simple as a Kaluza-Klein mode expansion:

- Photon and gluons must remain massless.
- Need to preserve gauge universality at 3+1-d level.

We use the Dvali-Shifman mechanism, following an argument due to arXiv:0710.5051 (Dvali et al).
Abelian Higgs model

$U(1)$ gauge theory, charged Higgs $\chi$:

$$S = \int d^4x \int dy \left[ -\frac{1}{4g^2} F^{MN} F_{MN} + \frac{1}{2} (D^M \chi)^\dagger D_M \chi - (|\chi|^2 - M^2)^2 \frac{|\chi|^2}{M^2} \right]$$
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In the bulk:
- $U(1)$ is broken, massive photon $\sim M_\chi$.
- Higgs vacuum is a superconductor.
- Electric charges are screened.

On the brane:
- $U(1)$ is restored, massless photon.
- Electric field ends on Higgs vacuum.

Charge screening leaks onto the brane!
Using a dual superconductor

$SU(2)$ gauge theory, adjoint Higgs $\chi^a$ ($a = 1, 2, 3$):

$$S = \int d^4x \int dy \left[ -\frac{1}{4g^2} G^{aMN} G^a_{MN} + \frac{1}{2} (D^M \chi^a)^\dagger D_M \chi^a - (\chi^a \chi^a - M_{\chi}^2)^2 \frac{\chi^a \chi^a}{M_{\chi}^2} \right]$$

![Diagram of a potential well and electric field repulsion](image)
Using a dual superconductor

\[ SU(2) \text{ gauge theory, adjoint Higgs } \chi^a \ (a = 1, 2, 3): \]

\[
S = \int d^4x \int dy \left[ -\frac{1}{4g^2} G^a_{MN} G^a_{MN} + \frac{1}{2} (D^M \chi^a) \dagger D_M \chi^a - (\chi^a \chi^a - M_\chi^2)^2 \frac{\chi^a \chi^a}{M_\chi^2} \right]
\]

In the bulk:
- \( SU(2) \) is restored, in confining regime.
- Large mass gap \( \sim M_\chi \) to colourless state.
- QCD-like vacuum is dual superconductor.

On the brane:
- \( SU(2) \) broken to \( U(1) \), massless photon.
- Electric field repelled from dual superconductor.

For distances much larger than brane width, electric potential \( \sim 1/r \).
Dvali-Shifman model

Stabilise the domain-wall with an extra uncharged scalar field $\eta$:

$$
S = \int d^4x \int dy \left[ -\frac{1}{4g^2} G^{aMN} G^a_{MN} + \frac{1}{2} \partial^M \eta \partial_M \eta + \frac{1}{2} (D^M \chi^a)^\dagger D_M \chi^a \\
- \lambda (\eta^2 - v^2)^2 - \frac{\lambda'}{2} (\chi^a \chi^a + \kappa^2 - v^2 + \eta^2)^2 \right]
$$

- $\eta$ has a kink profile.
- If $\kappa^2 - v^2 < 0$, $\chi$ becomes tachyonic near domain-wall (where $\eta \sim 0$).
- True vacuum has $\chi \neq 0$ near domain-wall.
- $\chi$ breaks symmetry near wall and confines gauge fields.
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Can add gravity: self consistently solve $\sigma$ (warped metric profile), $\eta$, $\chi$. 
The Dvali-Shifman mechanism:

- Works with any non-Abelian $SU(N)$ theory.
- Assumes the $SU(N)$ theory is confining (not proven for 5D).
- Has gauge universality:
  - Charges in the bulk are connected to the brane by a flux tube.
  - Coupling to gauge fields is independent of extra dimensional profile.
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Obvious choice for $SU(N)$ group is $SU(5)$. 
Quantum numbers of the standard model

Representations under $SU(3) \times SU(2)_L \times U(1)_Y$:

$q_L \sim (3, 2, 1/3)$  $u_R \sim (3, 1, 4/3)$  $d_R \sim (3, 1, -2/3)$

$l_L \sim (1, 2, -1)$  $\nu_R \sim (1, 1, 0)$  $e_R \sim (1, 1, -2)$
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$SU(2)_L$

$\nu_L \quad \bar{d}_R \quad u_L \quad \bar{e}_R \quad u_R \quad \bar{d}_L \quad \nu_R \quad \bar{d}_L \quad \bar{e}_L \quad \bar{e}_L$

$U(1)_Y$

$5^* \supset \bar{d}_L^{r,w,b} \quad \nu_L \quad e_L$

$(5 \times 5)_A = 10 \supset \bar{u}_L^{r,w,b} \quad u_L^{r,w,b} \quad d_L^{r,w,b} \quad \bar{e}_L$
Putting it all together
The $SU(5)$ model

Want the standard model on the brane: $SU(3) \times SU(2)_L \times U(1)_Y$. Dvali-Shifman needs a larger gauge group in the bulk:

$SU(5)$ is a perfect fit!

Unify the fermions as usual: $5^*$, $10$. Higgs doublet goes in a $5^*$. 

Summary:

- 4 + 1-dimensional theory – all spatial dimensions the same.

- SU(5) local gauge symmetry, Z\(_2\) discrete symmetry.

Field content:

- Gauge fields: $G_{MN} \sim 24$.
- Scalars: $\eta \sim 1$, $\chi \sim 24$, $\Phi \sim 5^*$.
- Fermions: $\Psi_{5} \sim 5^*$, $\Psi_{10} \sim 10$.

The standard model emerges as a low energy approximation.

Ignore gravity for now.
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  - fermions: $\Psi_5 \sim 5^*$, $\Psi_{10} \sim 10$.

- The standard model emerges as a low energy approximation.

Ignore gravity for now.
The action (without gravity)

The theory is described by:

\[ S = \int d^4x \int dy \left[ \frac{-1}{4g^2} G^{aMN} G_{aMN} + \frac{1}{2} \partial^M \eta \partial_M \eta + \text{Tr} \left( (D^M \chi)^\dagger (D_M \chi) \right) \right. \]

\[ + (D^M \Phi)^\dagger (D_M \Phi) + \bar{\Psi}_5 i \Gamma^M D_M \Psi_5 + \bar{\Psi}_{10} i \Gamma^M D_M \Psi_{10} \]

\[ - h_5 \eta \bar{\Psi}_5 \Psi_5 \eta - h_5 \chi \bar{\Psi}_5 \chi^T \Psi_5 \]

\[ - h_{10} \eta \text{Tr}(\bar{\Psi}_{10} \Psi_{10}) \eta + 2h_{10} \chi \text{Tr}(\bar{\Psi}_{10} \chi \Psi_{10}) \]

\[ - h_-(\bar{\Psi}_5 c \Psi_{10} \Phi) - h_+(\epsilon (\bar{\Psi}_{10} c \Psi_{10} \Phi^*) \right) + h.c. \]

\[ - (c\eta^2 - \mu_\chi^2) \text{Tr}(\chi^2) - d\eta \text{Tr}(\chi^3) \]

\[ - \lambda_1 \left[ \text{Tr}(\chi^2) \right]^2 - \lambda_2 \text{Tr}(\chi^4) - l(\eta^2 - v^2)^2 \]

\[ - \mu_\Phi \Phi^\dagger \Phi - \lambda_3 (\Phi^\dagger \Phi)^2 - \lambda_4 \Phi^\dagger \Phi \eta^2 \]

\[ - 2\lambda_5 \Phi^\dagger \Phi \text{Tr}(\chi^2) - \lambda_6 \Phi^\dagger (\chi^T)^2 \Phi - \lambda_7 \Phi^\dagger \chi^T \Phi \eta \]

with kinetic, brane trapping, mass and Dvali-Shifman terms.
Let $\Psi_{nY}$ be the components of $\Psi_5$ and $\Psi_{10}$ ($n = 5, 10$, $Y =$ hypercharge of component), e.g. $\Psi_5 \supset \Psi_{5,-1} = l_L$. Dirac equation:

$$\begin{bmatrix} i\Gamma^M \partial_M - h_{n\eta} \eta(y) - \sqrt{\frac{3}{5}} \frac{Y}{2} h_{n\chi} \chi_1(y) \end{bmatrix} \Psi_{nY}(x^\mu, y) = 0$$

Each $\Psi_{nY}$ is a non-chiral 5D field: need to extract the confined left-chiral zero-mode (recall the mode expansion and Schrödinger equation approach):

$$\Psi_{nY}(x^\mu, y) = \psi_{nY,L}(x^\mu) f_{nY}(y) + \text{massive modes}$$
Split fermions

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The effective Schrödinger potential depends on $Y$.

Thus each component $\psi_{nY,L}$ has a different profile $f_{nY}$. 

![Graph showing different profiles for $f_{nY}$ with dimensionless coordinate $k_y$](image-url)
Split Higgs

Φ contains the Higgs doublet \( \Phi_w \) and a coloured triplet \( \Phi_c \). Mode expand \( \Phi_{w,c}(x^\mu, y) = \phi_{w,c}(x^\mu)p_{w,c}(y) \). Schrödinger equation for \( p_{w,c} \) is:

\[
\left( -\frac{d^2}{dy^2} + \frac{3Y^2}{20}\lambda_6\chi_1^2 + \sqrt{\frac{3}{5}}\frac{Y}{2}\lambda_7\eta\chi_1 + \ldots \right) p_{w,c}(y) = m_{w,c}^2 p_{w,c}(y)
\]

*pCritical* that ground states have:

- \( m_w^2 < 0 \) to break electroweak symmetry.
- \( m_c^2 > 0 \) to preserve QCD.
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- $m_c^2 > 0$ to preserve QCD.

Large enough parameter space to allow this.
Features

Standard model parameters are computed from overlap integrals.

With one generation of fermions, parameters are easy to fit.

The model overcomes the major $SU(5)$ obstacles:

- $m_e = m_d$ not obtained due to naturally split fermions.
- Coloured Higgs induced proton decay is suppressed.
- Gauge coupling constant running modified due to Kaluza-Klein modes appearing (not analysed yet).
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Adding gravity:

- Solve for warped metric, kink and Dvali-Shifman background.
- Continuum fermion and scalar modes are highly suppressed on the brane.
- Main features remain.
Future work and extensions

Future work:

- Understand confinement of $SU(N)$ in 5D.
- Three families with full parameter fitting.
- Neutrino masses and mixings.
- Brane cosmology.

One promising extension is to the $E_6$ group:

$$E_6 \rightarrow SO(10)$$

$$SO(10) \rightarrow SU(5)$$

due to clash-of-symmetries and Dvali-Shifman.

Can eliminate kink scalar field $\eta$.

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Large reduction of free parameters.
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One promising extension is to the $E_6$ group:
- $E_6 \rightarrow SO(10)$ in the bulk.
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- Can eliminate kink scalar field $\eta$.
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- Large reduction of free parameters.
Main references:

- L. Randall and R. Sundrum
  *An alternative to compactification*
  Phys. Rev. Lett. 83 (1999) 4690-4693, arXiv:hep-th/9906064

- V.A. Rubakov and M.E. Shaposhnikov
  *Do we live inside a domain wall?*
  Phys. Lett. B125 (1983) 136-138

- G.R. Dvali and M.A. Shifman
  *Domain walls in strongly coupled theories*
  Phys. Lett. B396 (1997) 64-69, arXiv:hep-th/9612128

- G. Dvali, H. B. Nielsen and N. Tetradis
  *Localization of Gauge Fields and Monopole Tunnelling*
  arXiv:0710.5051

Our work:

- D.P. George and R.R. Volkas
  *Stability of domain walls coupled to Abelian gauge fields*
  Phys. Rev. D 72 (2005) 105011, arXiv:hep-ph/0508206

- D.P. George and R.R. Volkas
  *Kink modes and effective four dimensional fermion and Higgs brane models*
  Phys. Rev. D 75 (2007) 105007, arXiv:hep-ph/0612270

- R. Davies and D.P. George
  *Fermions, scalars and Randall-Sundrum gravity on domain-wall branes*
  Phys. Rev. D 76 (2007) 104010, arXiv:0705.1391

- R. Davies, D.P. George and R.R. Volkas
  *The standard model on a domain-wall brane*
  arXiv:0705.1584

- A. Davidson, D.P. George, A. Kobakhidze, R.R. Volkas and K.C. Wali
  *$SU(5)$ grand unification on a domain-wall brane from an $E_6$-invariant action*
  arXiv:0710.3432