Symmetries and Special States
in
Two Dimensional String Theory

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Abstract

We use the $W_\infty$ symmetry of $c = 1$ quantum gravity to compute matrix model special state correlation functions. The results are compared, and found to agree, with expectations from the Liouville model.

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1 Introduction

The last few years have seen tremendous developments in the understanding of two dimensional quantum gravity and therefore non critical string theory. The first success came from field theory, [1], where the case with matter of central charge $c \leq 1$ coupled to quantum gravity was solved. Later these theories were also solved by the powerful matrix models, both for $c < 1$ [2], and for $c = 1$ [3]. They allow for exact, non perturbative solutions where one sums over all genus. This is an important improvement over field theory where higher genus contributions are extremely difficult to calculate. Most results have therefore been limited to the sphere or in some cases the torus.

These models are important for two reasons. First they give solutions of two dimensional quantum gravity and can serve as toy models for higher dimensional theories. Indeed, the matrix model provides directly the exact Wheeler de Witt equation summed over all genus, [4]. Clearly an important object for further study. Second, and perhaps even more interesting, they describe non critical string theories. The most physical example is clearly the case $c = 1$. It is now commonplace to identify the Liouville mode as an extra dimension, [5], and thereby obtaining a theory of strings moving in a two dimensional target space. Naively one would expect this theory to be very simple, just a single massless scalar particle, usually referred to as the tachyon. This can be argued by choosing lightcone gauge. Fortunately, this is not the whole story. Instead there are remnants of the massive excited string modes present for certain discrete values of the momentum, [4, 6, 7, 8]. They are usually called special or discrete states. Clearly it is very interesting to study these extra states if one wants to learn about truly stringy phenomena.

The special states have been the object for several recent studies. Both using Liouville theory and matrix models. An important discovery has been a huge set of symmetries. These symmetries obey a $W_\infty$ algebra, which can be thought of as a generalization of a Virasoro algebra. This has been shown in the Liouville theory with two different methods. In [9] the $W_\infty$ was found by explicitly calculating the operator product expansion of the special states. An important tool for doing so is the usual $SU(2)$ symmetry known for a long time. In [10] the symmetry was instead found from the construction of the ground ring. This gives a representation of the current algebra with the currents acting on a set of ghost number zero, spin zero fields. These fields can be shown to be primary, [11], and correspond to special states in addition to the standard ones. The meaning of these and other special states of non standard ghost numbers has as yet not been fully clarified. From the matrix model point of view, the emergence of the symmetry has been more gradual [10, 11, 12, 13, 14]. In [10] it was however clearly realized that the $W_\infty$ simply is generated by the matrix eigenvalue and its conjugated momentum through the Poisson brackets.
In this work we will use the $W_\infty$ to study the special operator correlation functions. Notations and conventions will be much the same as in a previous paper, [6]. We will be able to obtain many of the results in [6] using the simplifications the $W_\infty$ symmetry provides.

In section 2 we make some initial comments on Ward identities and symmetries relating to the results obtained in [6]. We will also make some comments on possible generalizations to non harmonic matrix model potentials based on a generalized Wheeler de Witt equation. In section 3 we calculate some matrix model special state correlation functions using the $W_\infty$ symmetry. We also make some comments on how to identify the counterparts of the Liouville model special states. Section 4 gives some illustrations of the structure one encounters at higher genus. Finally, section 5 is devoted to a comparison with Liouville theory. Although the success of the matrix model and its agreement with Liouville theory hardly is in doubt, it is important to make the connection as explicit as possible. In particular, it is so far not clear how to explicitly extract the space time structure from the matrix model. One would like to be able to study nontrivial space times like the recently discovered black hole solution [15]. We will not be able to address this question here, but we will be able to make a comparison between our matrix model results and some Liouville theory expectations. In the cases which we will examine we will find perfect agreement.

2 Ward Identities and Symmetries

Let us now consider the matrix model and its special states and operators. The matrix model represents the string theory Riemann surfaces by Feynman diagrams of interacting matrix variables which triangulate the surfaces. In the uncompactified case, or at least for large enough radius, we can simply integrate out most of the degrees of freedom. The only remaining will be the matrix eigenvalues. They will then behave as non interacting fermions in the matrix model potential. Obvious candidates for special state correlation functions are then correlation functions of powers of the matrix model eigenvalues. Such objects were studied in [6] and the expected poles for discrete momenta were found. In this section we will review and extend some of the results of [6] for Ward identities. These Ward identities can be used to recursively obtain the correlation functions.

The recent developments revealing the $W_\infty$ symmetries indicate however that this is not the whole story. One should also consider correlation functions involving powers of the conjugate momentum. In this section we will show the existence of this $W_\infty$ symmetry which greatly will simplify the subsequent calculations.

Ward identities for correlation functions are in general obtained by changes of variables in the path integral. Examples of such Ward identities were obtained in [6] from simple...
coordinate changes in the matrix eigenvalue. They can be thought of as generated by commutators, or classically i.e. on the sphere, Poisson brackets with $p\lambda^m$. They obviously obey a Virasoro algebra which is part of a $W_{\infty}$ algebra generated by all monomials $p^n\lambda^m$. We may also introduce time dependence and consider generators with certain momenta $q$, i.e. $p^n\lambda^m e^{iqt}$. Let us as an example make a $p\lambda^ke^{iqt}$ variation of the one puncture function. The two puncture function is schematically given by

$$<PP> = \text{Im} \int_0^\infty dT \int d\lambda G(\lambda, \lambda; T)$$

where the calculation is done at the Fermi surface. We will shift its energy to zero, hence putting the Fermi energy as a constant term in the potential. $G$ is the path integral given by, in Euclidean time,

$$G(\lambda_1, \lambda_2) = \int_{\lambda_1}^{\lambda_2} [dpd\lambda] e^{-\beta \int dt (p\dot{\lambda} - \frac{1}{2}p^2 + U(\lambda))}$$

where $U(\lambda) = \sum_p t_p \lambda^p$ is the potential. The variation of the partition function would have involved a sum over all states in the Fermi sea up to the Fermi level. By inserting a puncture, i.e. taking a derivative with respect to the Fermi energy, we restrict ourselves to the Fermi surface. Next we perform the variation of $<P>$. The measure, as given by (2), is invariant under the change of variables. (Clearly we are not supposed to differentiate with respect to $t$ when changing variables in the measure.) The change in the action give rise to the following identity among two point functions:

$$< (\int dt (iq\lambda^k p + k\lambda^{k-1} p^2 + \frac{\partial U}{\partial \lambda} \lambda^k) e^{iqt}) T > = 0$$

The puncture is now a tachyon, $T$, carrying away the momentum. The piece with a single $p$ is evaluated by integrating over $p$ in the path integral obtaining a $\dot{\lambda}$ which then is partially integrated. We then switch to a Hamiltonian formulation, remembering that we should use Weyl ordering. We finally obtain:

$$-\frac{1}{2}k(k-1)(k-2) < O_{k-3}T >_{q,g-1} + \sum_p (2k+p)t_p < O_{p+k-1}T >_{q,g}$$

$$+ \frac{q^2}{1+k} < O_{k+1}T >_{q,g} = 0$$

where $q$ indicates momentum and $g$ genus. We have introduced the notation $O_k$ for $\lambda^k$. The different genus for the first term is due to a $p, \lambda$ commutator which arises when we want to
evaluate the $p^2$ against the wave function of the Fermi surface. This gives an $\hbar$, which is the same as the genus coupling constant. Following [4] we may define the loop operator given by

$$w(l) = e^{l\lambda}$$

(5)

It corresponds to cutting out a hole in the surface with a boundary of length $l$. The reason is as follows. If we insert a power $n$ of the original matrix eigenvalue $m$ on the surface this creates a little hole, the length of the boundary being proportional to $n$ (the number of legs) and the lattice spacing $a$. For fixed $n$ the length clearly shrinks to zero in the double scaling limit. To get a finite length we must also take $n$ to infinity. Introducing $\lambda$ as $m$ expanded around the top of the potential, we find in the double scaling limit

$$m^n \sim (1 + a\lambda)^{l/a} \to e^{l\lambda}$$

(6)

We may then Fourier transform to obtain a differential equation in the loop length. We get

$$\left[\sum_p t_p (l^2 \frac{\partial p}{\partial b} + \frac{p}{2} l \frac{\partial p^{-1}}{\partial b^{-1}}) + t_0 l^2 - l^4 + q^2\right] < w(l) T > = 0$$

(7)

where the third term are of order one higher in the string coupling and do not contribute on the sphere. $t_0$ is the Fermi energy $\mu$ with the appropriate number of $\beta$’s absorbed. In the case of the usual harmonic oscillator potential, where $t_0 = \beta \mu$, the resulting equation is in fact the Wheeler de Witt equation obtained in [4] with a more indirect matrix model method. On the sphere this is one of the most striking verifications of the equivalence of the Liouville and matrix models. At zero momentum the Wheeler de Witt equation is just the Fourier transform of the Gelfand-Dikii equation for the resolvent for the Schrödinger operator. This was the way in which the zero momentum version of (4) was derived in [6]. If we want to be careful, see section 5, we need to rescale $\lambda$ by $\sqrt{-t_2}$ to get a dimensionless $l$. This is needed for the exact correspondence between the above result and the mini superspace canonical quantization of Liouville theory. Recall that, [6], $t_2 = -\frac{1}{2\alpha'}$. From (7) one might try to draw some conclusions about the Liouville theory correspondence to the more general potentials above. Clearly the last term, which corresponds to the matter piece, does not change while we change potential. Instead it is the piece which would be expected to arise from a canonically quantized kinetical term for the Liouville mode which gets modified. Hence one is lead to the conclusion that these more general models (however with $p$ independent potentials) may correspond not to modifications of the matter theory but rather to different theories for the Liouville part. This is also consistent with the point of view for which this paper will argue, that the special states must be represented using both $\lambda$’s and $p$’s, not just the $\lambda$’s.
In principle all correlation functions may be calculated with the help of Ward identities derived in this way. However, it is more convenient to make use of the large set of symmetries in the theory. As shown by Witten in [10] we may change basis to \((p - \lambda)^n (p + \lambda)^m\) and, for certain time dependence, obtain transformations which leave the action invariant. These transformations are generated by, in Minkowsky time,

\[ W^{r,s} = (p + \lambda)^r (p - \lambda)^s e^{(r-s)t} \]  

Again we get the \(W_\infty\) algebra

\[ \{W^{r_1,s_1}, W^{r_2,s_2}\} = (r_1 s_2 - r_2 s_1) W^{r_1+r_2-1, s_1+s_2-1} \]  

generated, classically, by the Poisson bracketts. For a general momentum \(q\) in (8) we find when acting on the Minkowsky action \(S = \int (p\dot{\lambda} - \frac{1}{2}(p^2 - \lambda^2))\)

\[ \{W^{r,s}, S\} = (r - s - q) W^{r,s} \]  

Hence a symmetry for appropriate discrete values of imaginary momentum. This is equivalent to saying that \(W = W(p, \lambda, t)\) is a solution of

\[ \frac{dW}{dt} = \frac{\partial W}{\partial t} + \{H, W\} = 0 \]  

Expressed in terms of the initial conditions, \(p_0\) and \(\lambda_0\), we have \(W = W(p_0, \lambda_0)\), i.e. any time independent function of the initial conditions. The generators (8) are then simply obtained through evolution in time. Hence the transformations can be understood as time independent canonical transformations of the initial conditions.

A more convenient way of labeling the generators is through their \(SU(2)\) quantum numbers \(J = (r + s)/2\) and \(m = (r - s)/2\). With the definition

\[ W_{j,m} = (p + \lambda)^{J+m}(p - \lambda)^{J-m} e^{2mt} \]  

one gets

\[ \{W_{j_1,m_1}, W_{j_2,m_2}\} = 2(m_1 J_2 - m_2 J_1) W_{j_1+j_2-1, m_1+m_2} \]  

In Euclidean time, which is what we will be using, one should take \(p \to ip\) and \(t \to it\) in (12). There is also an extra \(i\) in the structure constant of (13) and the eigenvalue of (10).
3 Matrix Model Correlation Functions

In this section we will calculate correlation functions involving the operators $W$ as defined above. The $W_\infty$ symmetries will help us organize the Ward identities. As an example, let us start with the two point function. Using

$$<PP> = \frac{1}{\pi} \text{Im} \sum_{n=0}^\infty \frac{1}{E_n + t_0}$$

and simple perturbation theory we get:

$$< W_1W_2P > = \frac{1}{\pi} \text{Im} \sum_{\text{timeord}} \sum_{n,k} \frac{<n|W_1|k><k|W_2|n>}{E_n + t_0} \frac{1}{ip_1 + E_k - E_n}$$

(15)

Since the $W$'s are of the form (8), continued to Euclidean time, they are simply raising or lowering operators in the inverted harmonic oscillator. This means that only a few of the matrix elements are nonzero. Since $W$ raises by $2m = r - s$ we get

$$< W_1W_2P > = \frac{1}{\pi} \text{Im} \sum_n \frac{<n|[W_1,W_2]|n>}{E_n + t_0} \frac{1}{ip_1 - 2m_1}$$

(16)

We have reduced the two point function to a one point function using the commutation relations. If we restrict ourselves to the sphere, use the algebra given by (9) (with an extra $i$ in the structure constant for Euclidean time) and directly calculate the one point function we get

$$< W_1W_2 > = \frac{2(m_1J_2 - m_2J_1)}{2m_1 - p_1} \frac{1}{\pi} \frac{\mu^{J_1+J_2}}{J_1 + J_2} | \log \mu | 2^{J_1+J_2-1}$$

(17)

or equivalently

$$< W_1W_2 > = \frac{m_1}{2m_1 - p_1} \frac{1}{\pi} \frac{\mu^{J_1+J_2}}{J_1 + J_2} | \log \mu | 2^{J_1+J_2}$$

(18)

The simplest way to obtain the one point function on the sphere is to use the classical Fermi liquid picture introduced by Polchinski [17]. We simply need to do the phase space integral:

$$< W > = \int dpd\lambda W(p,\lambda)$$

(19)

This was also discussed in [16]. To make everything well defined we need however to introduce an extra puncture, i.e. take a derivative with respect to the cosmological constant. Doing that the integral over the whole Fermi sea becomes just an integral over the Fermi surface:

$$< WP > = \int W(p,\lambda)$$

(20)
The integral is to be performed along a hyperbola \( \frac{x^2 - \lambda^2}{2} = \mu \). Although the answer is really infinite, we know that the piece nonanalytic in \( \mu \) is \( \frac{1}{\pi} | \log \mu | \) for \( PP \), i.e. when we just integrate 1. The other zero momentum operators simply involve integrations of \( (\frac{x^2 - \lambda^2}{2})^n = \mu^n \), again constants along the Fermi surface hyperbola, so we get

\[
< W^{n,n} P > = \frac{1}{\pi} \mu^n | \log \mu | 2^n
\]  

(21)

and from this (17) follows. Our conventions are such that \( \alpha' = 1 \). Another approach, which is convenient when calculating correlation functions of nonzero momentum, is to continue to the upside down oscillator where the Fermi surface is a circle. In that case, however, we need to remember to put the Liouville volume \( | \log \mu | \) in by hand. Parenthetically we may note how a general correlation function may be obtained in this way. For instance the two point function is obtained by perturbing the hamiltonian and hence the Fermi surface by one of the operators. If we integrate the other operator against the change in Fermi surface we get the correlation.

Let us now consider the more complicated case of a three point function. Again perturbation theory gives us

\[
\frac{1}{\pi} \text{Im} \sum_{\text{time order } n,m,k} \frac{< n | W_1 | m > < m | W_2 | k > < k | W_3 | n >}{E_n + t_0} \frac{1}{i p_1 + E_m - E_n} \frac{1}{i p_3 + E_n - E_k} 
\]  

(22)

Let us make the sum over time orderings more explicit. We find

\[
\frac{1}{\pi} \text{Im} \sum_n \left\{ \frac{< n | (W_1 W_2 W_3 + W_3 W_2 W_1) | n >}{E_n + t_0} \frac{1}{i (p_1 - 2m_1) i (p_3 - 2m_3)} + \text{perm} \right\} 
\]  

(23)

This may after some straightforward manipulations be rewritten as

\[
< W_1 W_2 W_3 > = \frac{< [W_1, [W_2, W_3]] > (p_1 - 2m_1)(p_3 + 2m_1 + 2m_2)}{(p_2 - 2m_2)(p_3 + 2m_1 + 2m_2)} + \frac{< [W_2, [W_1, W_3]] >}{(p_1 - 2m_1)(p_3 + 2m_1 + 2m_2)} 
\]  

(24)

For the sphere we now use the algebra (13) and the explicitly calculated one point function to get

\[
< W_1 W_2 W_3 > = \frac{(2m_1 - p_1)m_2 m_3 J_1 + (2m_2 - p_2)m_1 m_3 J_2 + (2m_3 - p_3)m_1 m_2 J_3}{(2m_1 - p_1)(2m_2 - p_2)(2m_3 - p_3)} 
\]

\[
\times \frac{1}{\pi} \mu \sum_n J_{n-1} | \log \mu | 2 \sum_n J_n 
\]  

(25)
The general higher point function can be obtained recursively from the three point by use of (9) and (10). To get the \( N \) point function with an additional operator \( W_N \) we vary the \( N-1 \) point function with \( W_N \) knowing that the total variation is zero. The variation consists of two terms. One from varying the action as given by (10) and a sum of terms from varying the other operators as given by (13). Each of the terms in this sum is obtained by shifting \( p_i \rightarrow p_i + p_N, \ m_i \rightarrow m_i + m_N \) and \( J_i \rightarrow J_i + J_{N+1} - 1 \). We also need to multiply with the Clebsch-Gordan coefficient \( 2(J_N m_N - J_N m_i) \). It is an easy exercise to check that (25) is obtained by applying this procedure to (17) or (18). There is one subtlety in the variational procedure which should be noted. The insertion of an operator \( W \) in the pathintegral does not involve only the operator itself, but also a delta function for its position in eigenvalue space. In general the delta function will also contribute to the variation. Luckily its contribution will be zero by invariance properties for the \( W_\infty \) generators.

Using this method one can write down several different recursion relations. One simple example is:

\[
<T_J, J W_{J_1, m_1} \prod_{i=2}^{N} T_{J_i, J_i} > = \frac{2J(J - m_1)}{2J - p} < W_{J + J_1 - 1, J + m_1} \prod_{i=2}^{N} T_{J_i, J_i} >
\]

(26)

We will come back to this relation later, when we compare with the Liouville model results.

Rather than considering these general expressions, let us look at a couple of important examples where the form of the general \( N \) point function is particularly simple.

The first example is the \( N \) point function of special tachyons. It is given by

\[
<T_n> = \frac{2 \prod_{n=1}^{N} J_n}{ \prod_{n=2}^{N} (2J_n - p_n) } \frac{1}{\pi} \frac{d^{N-3}}{d\mu^{N-3}} \mu^{\sum_{n=1}^{N} J_n - 1} | \log \mu |^{2 \sum_{n=1}^{N} J_n}
\]

(27)

The quantum numbers have been chosen as \( m_n = J_n \) for \( n > 1 \) and \( m_1 = -J_1 \). This is just the pole part of the general tachyon correlation function as computed both in the matrix model \([18]\) and in the Liouville theory \([18, 19]\), up to a factorized normalization factor. The proof is by varying the three point. We can not just vary the three point tachyon correlation function, since some of the \( J \)'s are really \( m \)'s in disguise and \( J \) and \( m \) vary differently. Instead we start with the general three point and make an arbitrary number of tachyon variations. A simplification is that we at each step only have to vary the single negative chirality tachyon. It is only from there were we will get a nonzero Clebsch-Gordan coefficient. Following the prescription above, performing \( N - 3 \) variations we get
\[< \prod_{n=1}^{N} T_n > = [(2m_1 - p_1 + \sum_{i=4}^{N} (2m_i - p_i))m_2m_3(J_1 + \sum_{i=4}^{N} J_i - N + 3) \]
\[+ (2m_2 - p_2)(m_1 + \sum_{i=4}^{N} m_i)m_3J_2 + (2m_3 - p_3)(m_1 + \sum_{i=4}^{N} m_i)m_2J_3] \]
\[\times \frac{\prod_{i=4}^{N}(J_i(m_1 + \sum_{j=i+1}^{N} m_j) - (J_1 + \sum_{j=i+1}^{N} J_j - N + i)m_i)}{(2m_1 - p_1 + \sum_{i=4}^{N}(2m_i - p_i))\prod_{n=2}^{N}(2m_n - p_n)} \frac{1}{\pi} \mu \sum_{n=1}^{N} J_n - N + 2 \mid \log \mu \mid 2 \sum_{n=1}^{N} J_n \] 

The product in the denominator is the product of all the Clebsch-Gordan coefficients of the variations. Note that each get shifted by the successive variations. By the use of momentum conservation and evaluating the \(m\)'s as \(J\)'s, the formula (27) is proved. This derivation shows how the combinatorial factor from the \(\mu\) derivatives is a consequence of the \(W_\infty\) symmetry.

If we want to consider the zero momentum operators, we have to be careful. The Clebsch-Gordan coefficients are zero in this case but these zeroes cancel precisely the momentum poles and leave a finite result. Also, we need to consider both signs of the \(m\) quantum number when we take the \(m \to 0\) limit. This gives a necessary extra factor of two. We get

\[< \prod_{n=1}^{N} W_n > = \sum_{n=1}^{N} J_n \frac{1}{\pi} \frac{d^{N-3}}{d \mu^{N-3}} \frac{\sum_{n=1}^{N} J_n - N - 2}{\mu \sum_{n=1}^{N} J_n} \mid \log \mu \mid 2 \sum_{n=1}^{N} J_n \] 

We will use induction for the proof. We find

\[< \prod_{n=1}^{N+1} W_N > = \sum_{k} 2(J_{N+1}m_k - J_km_{N+1}) \frac{\sum_{n=1}^{N+1} J_n - 1}{2m_{N+1} - p_{N+1}} \]
\[\times \frac{1}{\pi} \frac{d^{N-3}}{d \mu^{N-3}} \frac{\sum_{n=1}^{N+1} J_n - 2}{\mu \sum_{n=1}^{N+1} J_n - 2} \mid \log \mu \mid 2 \sum_{n=1}^{N+1} J_n - 1 + (m_{N+1} \to -m_{N+1}) \] 

If we then put \(p_{N+1} = 0\) and use that the sum of all \(m\)'s must be zero the result follows. This can also be checked by an explicit phase space calculation.

Given these expressions we may check the correlation functions calculated in [6] and independently in [4]. These were correlation functions of pure powers of the matrix eigenvalue without any momentum powers. In terms of the \(W\)'s they are given by

\[O_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k W^{n,k-n} \frac{1}{2^n} \]
From this it follows that the two point function is given by

\[ < O_n O_m >_q = \frac{1}{2^{n+m}} \sum_{k=0}^{n} \sum_{l=0}^{m} \binom{n}{k} \binom{m}{l} (-1)^{k+l} < W_{q^{-k,k}} W_{q^{-m,l}} > \]

Using (18) and some simple algebra we find

\[ \frac{1}{2^{(n+m)}} \mu^{(n+m)} | \log \mu | \sum_{k=0}^{n} \binom{n}{k} \left( \frac{n+m}{2} - k \right) \frac{4(n-k)^2}{4(n-k)^2 - q^2} \]

In precise agreement with [6] recalling our convention \( \alpha' = 1 \). We can now understand why the \( O \) operators gave correlation functions with sets of poles and were, depending on momentum, capable of exciting several special states [6]. They were, in fact, linear combinations of all special operators of a given gravitational dimension i.e. spin \( J \). The above construction with the generators (8) of the \( W_{\infty} \) disentangles the correlation functions. This means that the matrix model operators to be identified with the Liouville model special states are those defined in (12).

Finally let us consider the meaning of the momentum poles. As emphasized in [23] we should not treat the poles in (25) and the \( | \log \mu | \) asymmetrically since the source of the \( | \log \mu | \) is also a momentum pole. In fact, all the poles should be thought of as cut off by \( | \log \mu | \). A general \( N \) point function (without zero momentum operators) would then have \( | \log \mu |^N \). This proliferation of logarithms was also noted in [19].

### 4 Higher Genus

We have so far basically just treated the sphere, which means, in the matrix model, that we have been working at the classical level. The \( W_{\infty} \) has been generated by Poisson bracketts. Nothing can however stop us from considering the full quantum theory, i.e. all genus. It is just a matter of algebra to compute for instance the two or three point functions using (16) and (24) respectively.

More interestingly, the algebra changes at the quantum level. There is a deformation with \( \hbar \), the genus coupling, as parameter when we use commutators instead of Poisson bracketts. If we define our \( W \)’s using Weyl ordering, which is natural from the path integral point of view, the algebra may be conveniently represented using the Moyal bracketts [20]

\[ \{ W_1, W_2 \}_M = \frac{2}{\hbar} \sin \frac{\hbar}{2} (\frac{\partial}{\partial p_1} \frac{\partial}{\partial \lambda_2} - \frac{\partial}{\partial p_2} \frac{\partial}{\partial \lambda_1}) W_1 W_2 \]

(34)
which is a deformation of the usual Poisson brackett. From this we might conclude that also
the Liouville theory operator product expansions should receive higher genus corrections.
Presumably from handles getting caught inside the contour integrals defining the operator
product expansions.

Another way to exhibit the quantum deformation is through a generalized loop operator
\[ w(k, l) = e^{kp + l\lambda} \]  
where \( p \) and \( \lambda \) are the conjugate variables. It is a very old result, [22], that these operators
obey the algebra
\[ [w(k_1, l_1), w(k_2, l_2)] = \frac{1}{\hbar} \sin \hbar (k_1 l_2 - l_1 k_1) w(k_1 + k_2, l_1 + l_2) \]  
with \( \hbar \to 0 \) giving back a \( W_\infty \). Interestingly it can be shown [21] that (36) is a representation
of \( SU(N) \) with \( \hbar = 1/N \). This is reminiscent of the original unitary symmetry of the matrix
model.

Let us give an explicit example of a two point function to all genus. We choose the
correlator between spin \( J = 3/2, m = 1/2 \) and \( J = 3/2, m = -1/2 \). To do that we need to
calculate \( < W_{2,0}P > \). This is easy. We have
\[ < W_{2,0}P > = \frac{1}{\pi} \text{Im} \sum_{n=0}^{\infty} \frac{(2E_n)^2 + 1}{E_n + t_0} \]  
The extra term +1 comes from Weyl ordering. If we keep only terms nonanalytic in \( t_0 \) this
reduces to
\[ < W_{2,0}P > = (4t_0^2 + 1) < PP > \]  
To evaluate our two point we use the Moyal brackett to calculate
\[ [W_{3/2,1/2}, W_{3/2,-1/2}] = 6W_{2,0} - 4\hbar^2 W_{0,0} \]  
(10) and (38) then finally give
\[ < W_{3/2,1/2}W_{3/2,-1/2}P > = (48t_0^2 + 4) < PP > \frac{1/2}{1 - p} \]  
The same procedure may be used to calculate arbitrary correlation functions.
5 Comparison with Liouville Theory

We would like to understand the $W_\infty$ structure from the Liouville theory point of view. As shown in [10] and by more direct methods in [9], we indeed have the same algebraic structure present. Therefore one would expect the comparison of the Liouville and the matrix model to be straightforward. As we will see, the situation is more subtle. Let us first consider a very special case which also give us the opportunity to clarify some important points.

There are some very simple examples of correlation functions easily computable just using Liouville notions and no matrix model techniques. These are correlation functions involving the dilaton. We will in fact be able to obtain some results to all genus simply from dimensional arguments. Consider the Liouville partition function (or space time free energy)

$$E(\Delta) = \lim_{R \to \infty} \frac{1}{R} \int DXD\phi e^{-\int (-t_2 \partial X \partial \bar{X} + \partial \phi \partial \bar{\phi} + QR \phi + \Delta e^{\alpha \phi})}$$

where $t_2 = -\frac{1}{2\alpha}$, $Q = 2\sqrt{2}$, $\alpha = -\sqrt{2}$ and $R$ is the radius of the target space for the matter field $X$. $\Delta$ is the world sheet cosmological constant, dimensionless from the point of view of space time. The only dimensionful quantities are $R$ and $\alpha'$. In the noncompact case we have infact only $\alpha'$ at our disposal. From dimensional grounds and KPZ scaling we must have

$$E(\Delta) \sim (-t_2)^{1/2} \Delta^{2(1-g)}$$

at genus $g$. $E(\Delta)$ is the generator of connected amplitudes (in space time). Let us do a Legendre transform to obtain a generating functional for 1PI amplitudes with respect to the puncture, i.e. the zero momentum tachyon. This means taking away any pinches. We have

$$E(\Delta) = \Delta \mu - \Gamma(\mu)$$

with

$$\mu = \frac{\partial E}{\partial \Delta} = (-t_2)^{1/2} (\Delta + ...)$$

$\mu$ has the dimensions of energy. Hence

$$\Gamma(\mu) \sim (-t_2)^{\frac{1}{2}(2g-1)} \mu^{2-2g}$$

In the 1PI generator $\Gamma$ we should of course regard $\mu$ as independent of $t_2$. Since $t_2$ derivatives should generate dilaton insertions we find the following 1PI amplitude relation

$$< O_2 ... O_2 >_g = \frac{1}{(2t_2)^n} \prod_{p=1}^{n} (2g + 1 - 2p) < >_g$$

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which is identical to what was obtained in [3] using the matrix model recursion relations (generalizations of the zero momentum Wheeler de Witt equation). As noted in [3] the dilaton one point function involves a factor $2g - 1$ rather than the expected $2g - 2$. From above it is clear that this discrepancy is simply due to including the overall $(-t_2)^{1/2}$ in (42).

There is a further subtlety in how the dilaton is defined. As we have seen a pure $\lambda^2$ is not what we would expect to identify with the dilaton. Instead we should have $p^2 - \lambda^2$ if we keep the algebraic structure in mind. These are in general different in correlation functions. We will return to this shortly.

It is important to realize the difference between connected and 1PI amplitudes. In the matrix model 1PI amplitudes are the natural objects, in Liouville theory it is more common to treat the connected ones. Often the distinction is not very clearly made. Indeed if we consider generic nonzero momentum the difference is very easy to deal with. It amounts to a renormalization of the cosmological constant. We can simply replace $\Delta$ by $\mu$, [23]. At zero momentum we must be much more careful. In this case we may have internal puncture propagators, i.e. pinches. Some examples of this were obtained in [6]. This will turn out to be important later on.

Already at this point we may find traces of the $W_\infty$ structure. In fact, the seemingly innocent representation of the puncture and the dilaton as $\mu$ and $t^2$ derivatives respectively is a reflection of the $W_\infty$. Let us give a formal argument for this. First the puncture. Write the $SU(2)$ quantum numbers of the puncture as $J$ and $m$ which both will be taken to zero. Choose one of the operators in (25) to be a puncture. We get

$$< W_1 W_2 P > = (J_1 + J_2) \frac{m_1}{2m_1 - p_1} \frac{1}{\pi} \mu^{J_1 + J_2 - 1} | \log \mu | 2^{J_1 + J_2}$$

(47)

which by comparing with the two point function (17) shows how the puncture is represented as a $\mu$ derivative. The case of the dilaton is equally simple. Proceeding as above we find

$$< W_1 W_2 D > = [(J_1 + J_2) \frac{m_1}{2m_1 - p_1} + \frac{2m_1^2}{(2m_1 - p_1)^2}] \frac{1}{\pi} \mu^{J_1 + J_2} | \log \mu | 2^{J_1 + J_2 + 1}$$

(48)

If we introduce explicit $t_2$’s in the two point function we can write it as

$$< W_1 W_2 > = \frac{m_1}{2m_1 - p_1/(-2t_2)^{1/2}} \frac{1}{\pi} \mu^{J_1 + J_2} | \log \mu | 2^{J_1 + J_2}$$

(49)

Taking a $t_2^{1/2}$ derivative we indeed reproduce (48). We must now return to the issue of how precisely the dilaton is defined. Recall the original matrix model action:

$$\beta \int dt [p \dot{\lambda} - \frac{1}{2} p^2 - t_2 \lambda^2]$$

(50)
with $\beta$ dimensionless, $t_2$ having the dimension of energy squared and $p^2$ and $\lambda^2$ the dimensions of energy and one over energy respectively. To obtain (49) as a generating functional for dilaton insertions with the above definition of the dilaton we should rescale $\lambda$ and $p$ to make them dimensionless. We find

$$\beta \int dt [p\dot{\lambda} - \frac{1}{\sqrt{2}}(-t_2)^{1/2}(p^2 - \lambda^2)]$$

Hence the matrix model dilaton should be represented by $(-t_2)^{1/2}$ derivatives. This is the rescaling eluded to in section 2 in the context of the Wheeler de Witt equation.

To obtain the general special operator correlation function in the Liouville theory one would like to use the group theoretic information provided by the $W_\infty$ or, for given spin $J$, the $SU(2)$ symmetry. The states in the Liouville theory are given by combinations $W(z)\bar{W}(\bar{z})$ of the Liouville theory version of the special states $W(z)$. Given this it is tempting to believe that we have a representation of a $W_\infty \times W_\infty$ symmetry (left times right). This is however in general not correct. In the uncompactified case the left and the right moving states must be the same. The symmetry group is broken down to just the diagonal subgroup. This is achieved in two steps. First the gravitational dressing must be the same for left and right, otherwise we would be unable to screen using the cosmological constant which treats left and right in the same way. This means that we always must have the same spin $J$ for left and right. We get a reduction to the diagonal of the piece transverse to $SU(2) \times SU(2)$. This is true even for the compactified case. If we in addition are considering the uncompactified case, the left and right moving momenta must be the same and hence the $m$ quantum numbers. Consequently we just have a representation of the diagonal $W_\infty$. This is in precise agreement with the matrix model, where we indeed only see one $W_\infty$. There is however an apparent paradox here. If we would use the free field contractions in computing the correlation functions the results would seem to disagree since from this point of view left and right are still independent. We will return to this important point further on, and discover that there in fact seems to be no contradiction.

The symmetry may then be used to determine all correlation functions given the special tachyon correlation functions which may be computed using other means. The reason is that all $J$ and $m$ dependence of any correlation function is given by some combination of Clebsch-Gordan coefficients. For given $J$’s we need the Clebsch-Gordan coefficients of $SU(2)$, the $3j$ symbols, to get the $m$ dependence. In fact we have already seen the agreement for the tachyon correlation functions and if we believe that the group theoretic structure is the same in the matrix model and in the Liouville theory, we know that the expressions obtained in the matrix model must agree with Liouville theory. To be more explicit let us however look at an example, the three point function, to see how the invariance properties
determine the correlation functions. The three point function is obtained by considering coupling \((J_1, m_1), (J_2, m_2)\) and \((J_3, m_3)\) (with \(m_1 + m_2 + m_3 = 0\)) to \((J_1 + J_2 + J_3 - 2, 0)\). A complication is that there are in general several different channels to sum over. This is true already for the three point function. The reason is that we really should think of the three point function as a four point function. The fourth leg carries the excess Liouville momentum, i.e. \(J\) quantum number, into the vacuum. This is a consequence of the non conservation of Liouville momentum. Let us use the tachyon three point function for normalization. It is given by

\[
<T_1 T_2 T_3> = \frac{J_1 J_2 J_3}{(2m_2 - p_2)(2m_3 - p_3)} \frac{1}{\pi} \mu^{J_1 + J_2 + J_3 - 1} \log \mu | 2^{J_1 + J_2 + J_3 + 1} \tag{52}
\]

where we have kept the normalization choice of (27). Tachyons 2 and 3 are of positive chirality while tachyon 1 has negative chirality. There are two possible channels corresponding to either \(p_2 = m_2\) or \(p_3 = m_3\), i.e. 1 and 2 coming together or 1 and 3 coming together. The group theoretic factor in each case is simply proportional to a product of \(3j\) symbols. One for each vertex. For the 1-2 channel:

\[
\left( \begin{array}{c} J_1 J_2 J \\ m_1 m_2 m_3 \end{array} \right) \left( \begin{array}{c} J J_3 J' \\ -m_3 m_3 0 \end{array} \right) \tag{53}
\]

where \(J = J_1 + J_2 - 1\) and \(J' = J_1 + J_2 + J_3 - 2\). Just retaining the \(m\) dependence and adding the two channels we find

\[
(2m_2 - p_2)(J_1 m_2 - J_2 m_1) m_3 + (2m_3 - p_3)(J_1 m_3 - J_3 m_1) m_2 =
-(2m_1 - p_1) J_1 m_2 m_3 - (2m_2 - p_2) J_2 m_1 m_3 - (2m_3 - p_3) J_3 m_1 m_2
\]

which agrees with (25) after using (52) to fix the normalization and \(J\) dependence. This should come as no surprise since the calculations are almost identical.

Another convenient way to obtain more general correlation functions is through factorization. This is already implicit in our previous calculations. In fact, if we look at (28) we see the complete factorization of the tachyon correlation function into a product of three point functions, each given by a \(3j\) symbol, times a single zero momentum one point function. This last piece represents the extra leg in any correlation function which absorbs excess Liouville momenta. One may note that these three point functions in fact involve states of the wrong dressing. This was also pointed out in [24]. Strictly speaking the expression in (28) is just for one channel, the one where 1 fuses with 2 then with 3 etc. All channels give however identical contributions and can not be distinguished. Clearly the tachyon correlation function is consistent with the single \(W_\infty\) factorization result.
As has been remarked, this seems to be in contradiction with what to expect from the naive free field calculations in Liouville theory. From such a calculation you would expect to get a different result, all Clebsch-Gordan coefficients squared, one from the left and one from the right. We will however show that the results in the end turn out to be consistent. Let us begin by considering the tachyon correlation function as computed in \[19\]. As we have seen the result is in complete agreement with the matrix model results. On the other hand we have seen how the matrix model organizes its correlation functions using a single $W_\infty$. Let us consider the Liouville calculation more carefully. The result of \[19\] is obtained through arguments of analyticity and symmetry. In particular the by now well known factorized product of gamma functions is found \[18\] with a certain unknown coefficient independent of the particular momenta. This coefficient is then determined by sending all the momenta, except three, to zero. This reduces the expression to a three point function with $N-3$ extra punctures. Since the three point function is possible to evaluate directly, the general result follows. The extra $N-3$ punctures is simply represented as $\mu$ derivatives. This is the Liouville derivation of the expression \[27\]. The important point is that the use of a $\mu$ derivative for inserting a puncture is a consequence of having just one $W_\infty$! This means that the calculation in \[19\] automatically incorporates this feature.

For the more general case with nontachyonic special states, we return to the recursion relation \(26\). Let us redefine the fields according to

\[
W_{J,m} = \frac{2J}{2m-p} \tilde{W}_{J,m}
\]

Then the recursion relation takes the form

\[
< \tilde{T}_{J,J} \tilde{W}_{J_1,m_1} \prod_{i=2}^{N} \tilde{T}_{J_i,J_i} > = \frac{(J_1 - m_1)(J + J_1 - 1)}{J_1} \frac{< \tilde{W}_{J+J_1-1,m_1} \prod_{i=2}^{N} \tilde{T}_{J_i,J_i} >}{< \tilde{W}_{J_1,m_1} \prod_{i=2}^{N} \tilde{T}_{J_i,J_i} >}
\]

In \[25\] these very same recursion relations were obtained in the case $J = m = 1/2$ using Liouville methods. The coefficient in front of the right hand side were shown to be of the form $(2J_1-1)C^2$, where $C$ stands for the appropriate Clebsch-Gordan coefficient. The first factor comes from comparing with the purely tachyonic case where the answer is obtained from a simple Veneziano like integral. To see the agreement one uses the Clebsch-Gordan coefficients of the special operator algebra as obtained in \[9\].

\[
C_{J_3,m_3,J_1,m_1,J_2,m_2} = \frac{A(J_3,m_3)}{A(J_1,m_1)A(J_2,m_2)} (J_1m_2 - J_2m_1)
\]

where

\[
A(J,m) = -\frac{1}{2}[(2J)!((J+m)!(J-m)!)]^{1/2}
\]
At \( J = m = 1/2 \) one finds \( C^2 = \frac{J_1 - m_1}{2J_1} \) which then leads to (56). This is an important check on the equivalence between the Liouville and matrix model approaches. An everywhere present difficulty in these comparisons is, however, the fact that we are really sitting right on the momentum poles. Clearly one needs to carefully regularize all expressions.

Let us give some further illustrations in the case of puncture and dilaton insertions. We begin with the puncture. Starting with a general correlation function and inserting a puncture does not change the Veneziano like integral which has to be calculated. When we insert a puncture we also must remove one of the screening insertions. The only thing which changes is the zero mode part of the calculation. We recall the result

\[
\int d\phi e^{m\phi - \Delta e^{-\phi}} \sim \Gamma(-m)\Delta^m
\]  

(59)

If we start with \( \Gamma(-m)\mu^m \) we end up with \( \Gamma(-m + 1)\mu^{m-1} = -m\Gamma(-m)\mu^{m-1} \) when we remove a screening insertion. Comparing with (47) this shows the origin of the \( W_\infty \) related factor in the tachyon correlation function. It is a consequence of the changing number of puncture screening operators needed. For the tachyons the issue of connected or 1PI amplitudes is trivial for our case with just one tachyon of differing chirality. There can’t be any internal punctures just from kinematics. This is no longer the case when we turn to the dilaton. The crucial point is that the dilaton can be represented as a \( t_2 \) derivative. Usually this is precisely equivalent to using the ordinary free field contractions giving Veneziano like correlation functions. Inserting a dilaton in some tachyon correlation function means taking derivatives with respect to \( t_2 \) (i.e. \( 1/\alpha' \)). For dimensional reasons all tachyon momenta are accompanied by an \( t_2 \). Without explicit \( t_2 \)'s one could write \( k \frac{\partial}{\partial k} \) for the dilaton. For a dilaton insertion in a nonzero momentum correlation function the dominating pole contribution comes from letting the \( t_2 \) derivative act directly on the poles. All other terms are clearly less singular. This gives the second term in (48). At zero momentum we must also consider the dependence from the \( t_2 \)'s which go together with the \( \mu \)'s. The latter is a consequence of dealing with 1PI rather than connected correlation functions. One could in fact obtain the result by considering the explicit combinations of 1PI amplitudes into connected ones. To be more precise, if we want to obtain the 1PI amplitude from the connected amplitude, we must amputate external puncture legs but also subtract of the diagrams with internal puncture propagators. In particular we need to subtract a diagram where a puncture goes off and converts into a dilaton. This diagram therefore involves a puncture insertion and gives a contribution corresponding to a \( \mu \) derivative. This is then simulated by an explicit \( t_2 \) accompanying the \( \mu \)'s to assure the proper subtraction. This corresponds to the first term in (48), which only becomes relevant compared to the second term at zero momentum. Otherwise we will get one power less of \( | \log \mu | \)'s. We need not restrict ourselves to a
dilaton among special tachyons, the same reasoning works for a dilaton inserted in a general correlation function. From these two examples we can conclude that the 1PI nature of the correlation functions is very important in the case of zero momentum.

6 Conclusions

We have investigated the structure of special operator correlation functions in $c=1$ quantum gravity. Due to the presence of a $W_\infty$ symmetry the calculations become very simple. We have also investigated the connection between the Liouville and the matrix model, indicating the agreement for the correlation functions.

An important point is the existence in the uncompactified Liouville as well as matrix model formulation of $c=1$ of just one $W_\infty$. From the Liouville point of view this is somewhat obscure since the operator product expansion and its Clebsch-Gordan coefficients seem to give a structure corresponding to two $W_\infty$’s. Fortunately the final outcome of the explicit calculations are identical. Further work is however needed to establish the full equivalence.

A part of the problem is that many of the calculations are so ill defined. The reason is that we are sitting right on the discrete momentum poles. Especially in the Liouville theory this is a big technical problem. Often we must rely on guesswork concerning ill defined analytical continuations. It is very doubtful if many of the results would have been obtained correctly without knowing the answers in advance, given by the much more powerful matrix model.

Important issues for future research are to investigate multicritical points of matrix models with generalized potentials. It is natural to consider even non quadratic dependence on the momentum. Such perturbations could arise from adding the special states as we have seen. Another important point is to identify the ‘wrongly’ dressed special states in the matrix model. Such states have negative gravitational dimensions and are important in the context of the two dimensional black hole.

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