GENERALIZED ALGEBRAIC BARGMANN–DARBOUX TRANSFORMATIONS

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Algebraic Bargmann and Darboux transformations for equations of a more general form than the Schrödinger ones with an additional functional dependence \( h(r) \) in the right-hand side of equations are constructed. The suggested generalized transformations turn into the Bargmann and Darboux transformations for both fixed and variable values of energy and an angular momentum.

Key words: Darboux transformations, Bargmann potentials, inverse scattering problem.

1. Introduction

Algebraic Darboux and Bargmann transformations essentially broaden a class of exactly solvable problems in quantum mechanics and, consequently, allows one to derive solutions in a closed analytical form. In conventional statements the potentials and pertinent solutions are restored either at a fixed angular momentum \( l \) and variable energy \( E \) values or at a fixed \( E \) and different \( l \). In the papers 1, 2, 3 we suggested the generalized Darboux and Bargmann transformations to construct a wide class of potentials and appropriate solutions to the Schrödinger equation for variable \( l \) and energy \( E \) along arbitrary straight lines in the \((\lambda^2, E)\)–plane \((\lambda = l+1/2)\). In particular cases, when \( l = \text{const} \) or \( E = \text{const} \), the obtained relations are transformed into familiar expressions for potentials and solutions of the Bargmann type. This method was developed in 4 for constructing exactly solvable three-body models with two-central spheroidal potentials.

The studies performed for the Schrödinger equations with some varying parameters allowed us to construct algebraic Bargmann and Darboux transformations for equations of a more general form

\[
-\frac{d^2\phi(\gamma, r)}{dr^2} + V(r)\phi(\gamma, r) = \gamma^2 h(r)\phi(\gamma, r).
\]

The quantity \( \gamma^2 \) represents energy with a coefficient \( h(r) \) dependent on the coordinate variable; the \( h(r) \) should satisfy the general requirements imposed on the potential function by the scattering theory. These equations are applied in atomic physics, the theory of propagation of electromagnetic waves, acoustics, geophysics,

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and so on. The generalized Bargmann transformations are related to the generalized Darboux ones and can be obtained as their superposition. This technique of algebraic transformations is favored because it does not use integral equations of the inverse problem and, consequently, does not explicitly use the completeness of the set of eigenfunctions required for its derivation; at the same time this approach remains a closed algebraic procedure.

2. Darboux Transformations

Let us introduce the generalized Darboux transformations suggested in [5]. The solution of Eq. (3) with an unknown potential $V(r)$ has been sought in terms of the known solutions to this equation with the known potential $V^\circ(r)$ in the same form as for the standard Schrödinger equation

\[ \phi(\gamma, r) = y(r)W\{y^\circ(r), \phi^\circ(\gamma, r)\}. \]  

Here $W\{y^\circ(r), \phi^\circ(\gamma, r)\} = y^\circ(r)(d\phi^\circ(r)/dr) - (dy^\circ(r)/dr)\phi^\circ(r)$ is the Wronskian of the functions $y^\circ$ and $\phi^\circ$. The functions $y(r)$ and $y^\circ(r)$ obey Eq. (3) with $V(r)$ and $V^\circ(r)$, respectively, at a certain fixed value of $\gamma^2 = \gamma'^2$ that may correspond to a bound state. Multiplying Eq. (3) for $y^\circ(r)$ by function $\phi^\circ(\gamma, r)$ at arbitrary $\gamma$; whereas Eq. (3) for $\phi^\circ(\gamma, r)$, by $y^\circ(r)$, and then subtracting the resulting expressions from one another we obtain

\[ dW(r)/dr = h(r)(\gamma'^2 - \gamma^2)y^\circ(r)\phi^\circ(\gamma, r). \]  

Further, let us determine the second-order derivative $d^2\phi/dr^2$, by using Eqs. (3) and (4). Then taking Eq. (3) for $y(r)$ and making appropriate transformations we get

\[ d^2\phi(\gamma, r)/dr^2 = [V(r) - \gamma'^2 h(r)]y(r)W\{y^\circ(r), \phi^\circ(\gamma, r)\} \]
\[ + 2\frac{dy^\circ(r)}{dr}h(r)\phi^\circ(\gamma, r) + \frac{dh(r)y^\circ(r)}{dr}\phi^\circ(\gamma, r) + h(r)y^\circ(r)W\{y^\circ(r), \phi^\circ(\gamma, r)\}. \]

Using the definition (3) we rewrite the latter expression in the form

\[ \left(\frac{d^2}{dr^2} - V(r) + h(r)\gamma^2\right)\phi(\gamma, r) = 2h(r)\frac{dy^\circ(r)}{dr}\phi^\circ(\gamma, r) + y(r)y^\circ(r)\frac{dh(r)}{dr}\phi^\circ(\gamma, r). \]

It is clear that the condition for the right-hand side of the identity being zero $d(\ln(y(r)y^\circ(r)))/dr = d(\ln(h(r)))/dr$ makes the function $\phi(\gamma, r)$ defined by the expression (3) obey Eq. (3). This condition is equivalent to

\[ y(r) = \frac{1}{(\sqrt{h(r)}y^\circ(r))}. \]

Then, the solution to Eq. (3) with the definition (2) and at arbitrary $\gamma$ is written as

\[ \phi(\gamma, r) = \frac{1}{\sqrt{h(r)y^\circ(r)}}W\{y^\circ(r), \phi^\circ(\gamma, r)\}. \]

Now we will explicitly determine the potential $V(r)$ in terms of the known functions $h(r), y^\circ(r)$ and $V^\circ(r)$. Using the relation (3) in Eq. (3) for the function $y(r)$. 

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V(r) = \frac{d^2 y(r)}{dr^2} + h(r)\gamma^2 = 2\left(\frac{dy^0(r)}{y^0(r)}\right)^2 - \frac{d^2 y^0(r)}{y^0(r)} + \frac{dy^0(r)}{y^0(r)} \frac{dh(r)}{dr} \frac{dy^0(r)}{y^0(r)} + h(r)\gamma^2 - \frac{1}{2} \frac{d^2 h(r)}{h(r)} + \frac{3}{4} \left(\frac{dh(r)}{h(r)}\right)^2

and transforming this expression on the basis of the equality \(d^2 y^0(r)/dr^2/y^0(r) = V^0(r) - h(r)\gamma^2\), we finally obtain the following expression for the potential:

\[
V(r) = V^0(r) - 2\sqrt{h(r)} \frac{d}{dr} \left[ \frac{1}{\sqrt{h(r)}} \frac{d}{dr} \ln y^0(r) \right] + \sqrt{h(r)} \frac{d^2}{dr^2} \frac{1}{\sqrt{h(r)}}.
\]

3. Bargmann Transformations

We will look for the solution to Eq. (1) in a more general form, as compared to (2),

\[
\phi(\gamma, r) = \phi^0(\gamma, r) - \sum_{\mu} y_\mu(r) W\{\phi^0(\gamma_\mu, r), \phi^0(\gamma, r)\}/(\gamma_\mu^2 - \gamma^2),
\]

where \(y_\mu(r) = y(\gamma_\mu, r) = C_\mu \phi(\gamma_\mu, r)\). Let us now determine the conditions under which the function \(\phi(\gamma, r)\) given by (7) obeys Eq. (1) following a procedure analogous to that expounded in (1). To this end we differentiate (7) twice with respect to \(r\) and then obtain

\[
\frac{d^2 \phi(\gamma, r)}{dr^2} = \frac{d^2 \phi^0(\gamma, r)}{dr^2} - \sum_{\mu} \left\{ \frac{d^2 y_\mu(r)}{dr^2} W\{\phi^0(\gamma_\mu, r), \phi^0(\gamma, r)\} / (\gamma_\mu^2 - \gamma^2) \right\}
\]

\[
+ y_\mu(r) \left[ \frac{d[h(r)\phi^0(\gamma_\mu, r)\phi^0(\gamma, r)]}{dr} + 2\frac{dy_\mu(r)}{dr} h(r)\phi^0(\gamma_\mu, r)\phi^0(\gamma, r) \right]
\]

and transform it with \(y_\mu(r)\) given by (1) and the definition (3)

\[
\left( \frac{d^2}{dr^2} - V(r) + h(r)\gamma^2 \right) \phi(\gamma, r) = \left[ -V(r) + V^0(r) \right] \phi^0(\gamma, r)
\]

\[
-2 \sum_{\mu} \left\{ h(r) \frac{dy_\mu(r)}{dr} \phi^0(\gamma_\mu, r) + y_\mu(r) \phi^0(\gamma_\mu, r) \frac{dh(r)}{dr} \right\} \phi^0(\gamma, r).
\]

The function \(\phi(\gamma, r)\) satisfies Eq. (1) provided the right-hand side of the latter relation vanishes, which is equivalent to

\[
V(r) = V^0(r) - \sum_{\mu} \left\{ 2h(r) \frac{dy_\mu(r)}{dr} \phi^0(\gamma_\mu, r) + y_\mu(r) \phi^0(\gamma_\mu, r) \frac{dh(r)}{dr} \right\}.
\]

Using the connection \(y_\mu(r) = C_\mu \phi(\gamma_\mu, r)\), we determine the solution \(y_\mu(r)\) with the potential (8) from (9) as follows:

\[
y_\mu(r) = \sum_{\nu} C_\nu \phi^0(\gamma_\nu, r) P_{\nu\mu}^{-1}(r),
\]

where \(P_{\mu\nu}(r) = \delta_{\mu\nu} + C_\nu W\{\phi^0(\gamma_\mu, r), \phi^0(\gamma_\nu, r)\}/(\gamma_\mu^2 - \gamma_\nu^2)\).
Substituting (10) into (8) and (9) we can express the potential and corresponding solutions in terms of the known function \( h(r) \) and solutions \( \phi^\circ(\gamma, r) \)

\[
V(r) = V^\circ(r) - 2\sqrt{h(r)} \frac{d}{dr} \left[ \frac{1}{\sqrt{h(r)}} \frac{d}{dr} \ln P(r) \right];
\]

\[
\phi(\gamma, r) = \phi^\circ(\gamma, r) - \sum_{\mu} \sum_{\nu} C_{\nu} \phi^\circ(\gamma_{\nu}, r) P_{\nu\mu}^{-1}(r) \frac{W\{\phi^\circ(\gamma_{\mu}, r), \phi^\circ(\gamma, r)\}}{(\gamma_{\mu}^2 - \gamma^2)}.
\]

After the integration of Eq.(3) the Wronskian is expressed as follows

\[
W\{\phi^\circ(\gamma_{\mu}, r), \phi^\circ(\gamma, r)\} = (\gamma_{\mu}^2 - \gamma^2) \int_{0(r)}^{r(b)} h(r') \phi^\circ(\gamma_{\mu}, r') \phi^\circ(\gamma, r') dr'.
\]

With allowance of this expression, the transformation (8) and (12) can be represented in the integral form. In particular for regular solutions, the integration limits from "0" to "\( \infty\)" are the same as those in the Gelfand–Levitan approach; whereas the Jost solutions, the integration limits are as those in the Marchenko approach, i.e., from "\( r_{\nu} \)" to "\( \infty\)". It is now clear that \( \phi \) in formulae (8), (10)–(12) stands for any solutions, which are, generally, arbitrary, until boundary conditions are fixed.

Relations (8) or (12) may be represented as superposition of solutions of the type (7). Let us take a solution \( \phi_1(\gamma, r) \) of Eq.(1) in the form (5) with \( V(r) \) determined as (10). As two linearly independent solutions of (1) we take (see Eqs.(8),(10),(13))

\[
y_1(r) = \frac{1}{\sqrt{h(r)} y^\circ(r)}, \quad X_1(r) = \frac{1}{\sqrt{h(r)} y^\circ(r)} \int_{0(r)}^{r(b)} h(r') |y^\circ(r')|^2 dr'.
\]

Now we construct a solution \( \eta_1(r) \) of Eq.(1) with \( V_1(r) \) as a linear combination of these two solutions

\[
\eta_1(r) = \frac{1}{\sqrt{h(r)} y^\circ(r)} \left[ 1 + C \int_{0(r)}^{r(b)} h(r') |y^\circ(r')|^2 dr' \right] = \frac{1}{\sqrt{h(r)} y^\circ(r)} P(r).
\]

Then, the solution to Eq.(1) with the definition (10) and at arbitrary \( \gamma^2 \) can be rewritten in terms of the solutions \( \eta_1(r) \) and \( \phi_1(r) \) as

\[
\phi(\gamma, r) = \frac{1}{\sqrt{h(r)} \eta_1(r)} W\{\eta_1(r), \phi_1(\gamma, r)\} = \phi^\circ(\gamma, r) - \frac{C y^\circ(r) W\{y^\circ(r), \phi^\circ(\gamma, r)\}}{P(r)(\gamma^2 - \gamma^2)}.
\]

This is the solution of Eq.(1), coinciding with (12) at \( \mu = 1 \), with the potential (11) in the form

\[
V(r) = V^\circ(r) - 2\sqrt{h(r)} \frac{d}{dr} \left[ \frac{1}{\sqrt{h(r)}} \frac{d}{dr} \ln y^\circ(r) \right] - 2\sqrt{h(r)} \frac{d}{dr} \left[ \frac{1}{\sqrt{h(r)}} \frac{d}{dr} \ln \eta_1(r) \right] + 2\sqrt{h(r)} \frac{d^2}{dr^2} \frac{1}{\sqrt{h(r)}} = V^\circ(r) - 2\sqrt{h(r)} \frac{d}{dr} \left[ \frac{1}{\sqrt{h(r)}} \frac{d}{dr} \ln P(r) \right].
\]
Then, it is easy to show that the relations \( (11) \) and \( (12) \) as well as \( (11) \) and \( (12) \) are reduced to the corresponding relations for the Darboux and Bargmann transformations for both fixed or variable values of energy and orbital momentum.
For particular cases of a different choice of \( h(r) \), we may consider versions of these transformations for the Coulomb forces and a Coulomb coupling constant.

4. Multichannel exactly solvable models.
Matrix generalization of the Darboux transformation has been given in Refs. for multichannel systems of Schrödinger equations. In the paper matrices of solutions and potentials in a closed analytical form have been constructed for variable values of energy and angular momentum. Above approach is generalized for the multichannel system of coupled equations

\[
-\frac{d^2\phi_{\alpha\beta}(r)}{dr^2} + \sum_{\beta'} V_{\alpha\beta'}(r) \phi_{\beta'\beta}(r) = \gamma_{\alpha}^2 h(r) \phi_{\alpha\beta}(r). \tag{18}
\]

Here \( \gamma = \text{diag}(\gamma_{\alpha}) \). Let the vectors of solutions \( \psi_{\alpha}(r) = \sum_{\beta} \phi_{\alpha\beta}(r) c_{\beta} \); \( \psi_{\alpha}^*(r) = \sum_{\beta} \phi_{\beta\alpha}^*(r) c_{\beta} \) satisfy the system

\[
-\frac{d^2\psi_{\alpha}(r)}{dr^2} + \sum_{\beta} V_{\alpha\beta}(r) \psi_{\alpha}(r) = \gamma_{\alpha}^2 h(r) \psi_{\alpha}(r) \tag{19}
\]

with potential matrices \( V(r) \) and \( V^o(r) \), respectively, within the interval \( a < r < b \), where \( V(r) \) and \( V^o(r) \) are real, symmetric and continuous. In particular cases \( V_{\alpha\beta}^o(r) = V_{\alpha\alpha}^o(r) \delta_{\alpha\beta} \) or \( V_{\alpha\beta}(r) = 0 \). The system \( (19) \) is obtained from Eqs.\( (18) \) multiplied by \( c_{\beta} \), summed over \( \beta \) at fixed values \( \gamma_{\alpha} = \gamma_{\alpha}^* \); \( c_{\beta} \) are some spectral characteristics. For instance, if \( \gamma_{\alpha}^2 = E^* \) is the energy of the bound state for \( ||V_{\alpha\beta}|| \), then elements \( c_{\alpha} \) form its normalization matrix \( ||C_{\alpha\beta}|| = ||c_{\alpha}c_{\beta}|| \). We shall search for solutions \( \phi_{\alpha\beta}(r) \) of Eq.\( (18) \) with initially unknown potential matrix \( V_{\alpha\beta}(r) \) in terms of solutions appropriate for the known \( V_{\alpha\beta}^o(r) \) in one of the equivalent forms

\[
\phi_{\alpha\beta}(\gamma, r) = \phi_{\alpha\beta}^o(\gamma, r) - \psi_{\alpha}(r) \sum_{\beta} W_{\beta} \psi_{\alpha\beta}^*(\gamma, r) \sum_{\beta} W_{\beta} \phi_{\beta\beta}^o(\gamma, r) / (\gamma_{\beta}^2 - \gamma_{\beta}^*)
\]

\[
\phi_{\alpha\beta}(\gamma, r) = \phi_{\alpha\beta}^o(\gamma, r) - \psi_{\alpha}(r) \sum_{\beta} W_{\beta} \psi_{\alpha\beta}^*(\gamma, r) \sum_{\beta} W_{\beta} \phi_{\beta\beta}^o(\gamma, r) h(r) \psi_{\alpha\beta}^o(\gamma, r) d\tau'. \tag{20}
\]

With \( \gamma_{\alpha}^2 = \gamma_{\alpha}^* \), the second expression \( (21) \) after multiplication by \( c_{\beta} \) and summation over \( \beta \) can be rewritten as

\[
\psi_{\alpha}(r) = \frac{\psi_{\alpha}^o(r)}{1 + \sum_{\beta} W_{\beta} \psi_{\alpha\beta}^o(\gamma, r) \sum_{\beta} W_{\beta} \phi_{\beta\beta}^o(\gamma, r) h(r) \psi_{\alpha\beta}^o(\gamma, r) d\tau'}.
\tag{21}
\]

Let us determine the conditions for potential matrix \( ||V_{\alpha\beta}|| \) at which functions \( \phi \) and \( \psi \) specified by formulas \( (20) \) and \( (21) \) will satisfy the system of Eqs.\( (18), (19) \).
Perform double differentiation of relation (21) and transform the result with regard to Eqs. (19) for $\psi_\alpha$, (18) for $\phi^{\circ \alpha \beta}$, and definition (20) for $\phi_{\alpha \beta}$ and carrying further simplifications we obtain

$$
- \frac{d^2 \phi_{\alpha \beta}(r)}{dr^2} + \sum_{\beta'}^N V_{\alpha \beta'}(r) \phi_{\beta' \beta}(r) - \gamma^2 \alpha h(r) \phi_{\alpha \beta}(r) = \sum_{\beta'}^N \left[ V_{\alpha \beta'}(r) - V^{\circ \alpha \beta}(r) \right] \phi^{\circ \beta' \beta} \\
+ 2 \frac{d[h(r) \psi_\alpha(r) \sum_{\beta'}^N \psi^{\circ \beta}(r)]}{dr} \phi^{\circ \beta' \beta} + \frac{dh(r)}{dr} \psi_\alpha(r) \sum_{\beta'}^N \psi^{\circ \beta}(r) \phi^{\circ \beta' \beta}.
$$

One can easily see that the matrix of functions $\phi_{\alpha \beta}$ satisfy the system (18) if the right-hand side of Eq. (22) vanishes. In virtue of linear independence of functions $\phi^{\circ \alpha \beta}$ this is equivalent to

$$
V_{\alpha \beta}(r) = V^{\circ \alpha \beta}(r) - 2 \frac{d}{dr}[h(r) \psi_\alpha(r) \psi^{\circ \beta}(r)] + (dh(r)/dr) \psi_\alpha(r) \psi^{\circ \beta}(r)
$$

Taking account of definition (21) for $\psi_\alpha(r)$ in Eqs. (23) and (22), we find the analytical relationships between different potential matrices and their pertinent solutions. Now it is easy to establish analytical relationships between the solutions for different potential matrices and the potentials themselves in a more general case being the matrix generalization of the single-channel problem [1] – [12].

5. Conclusions.

Generalization is given for the algebraic transformations for equations (1) with a functional dependence $h(r)$ in the right-hand side of the Schrödinger equations. Analytical relationships between different potential matrices and their pertinent solutions are constructed that appear to be generalization of the corresponding single-channel formulae. Under a certain choice of $h(r)$, the Bargmann and Darboux transformations for both fixed and varying $l$ and $E$ are particular cases of the generalized transformations.

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