JACQUET FUNCTOR AND DE CONCINI-PROCESI
COMPACTIFICATION

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Abstract. We give a geometric realization of the Jacquet functor using a
deformation of De Concini-Procesi compactification.

1. Introduction

The symmetric variety is used from a long time ago, when representations of a
real reductive group are studied by the analytic way. On the other hand, when
they are studied by the algebraic way, due to the localization theorem of Beilinson-
Bernstein [BB81], the flag variety is often used. However, the geometry of the
symmetric space is richer than that of the flag variety. For example, the symmetric
space has a boundary and one can take a “limit” to this boundary (cf. [KKM+78]).

Fortunately, the localization theorem gives a way to realize representations as
geometric objects on the symmetric variety $G/K$, where $G$ (resp. $K$) is the com-
plexification of a real reductive group $G_\mathbb{R}$ (resp. a maximal compact subgroup $K_\mathbb{R}$
of $G_\mathbb{R}$). However, as far as the authors know, little is studied by such a way. In this
paper, we use the symmetric variety and try to take a “limit” of such a geometric
object. The limit should become the Jacquet module [Cas80] since the Jacquet
module describes the asymptotic behavior of matrix coefficients [HS83]. In the
$p$-adic case, similar results can be found in a work of Schneider-Stuhler [SS97]. They
realize the Jacquet module on the boundary of the Borel-Serre compactification
of the Bruhat-Tits building. The vertices of the Bruhat-Tits building for a $p$-adic
semisimple group $G$ are in bijection with a union of sets of the form $G/K$, where
$K$ is a maximal compact subgroup of $G$. So it can be regarded as an analogue of
the symmetric variety. In this paper, we realize the Jacquet module by taking a
limit on the symmetric space. Notice that, if you use the flag variety instead of
the symmetric space, a realization of the Jacquet module has already been given
by Emerton-Nadler-Vilonen [ENV04].

We state our main results. Assume that $G_\mathbb{R}$ is of adjoint type. Let $G_\mathbb{R} =
K_\mathbb{R}A_\mathbb{R}N_\mathbb{R}$ be an Iwasawa decomposition, and $M_\mathbb{R}$ the centralizer of $A_\mathbb{R}$ in $K_\mathbb{R}$. Then
$P_\mathbb{R} = M_\mathbb{R}A_\mathbb{R}N_\mathbb{R}$ is a Langlands decomposition of a minimal parabolic subgroup.
We use lower-case fraktur letters to denote the corresponding Lie algebras and omit
the subscripts “$\mathbb{R}$” to denote complexifications. Let $X$ be the De Concini-Procesi
compactification of $G/K$ [DCP83]. The $G$-orbit of $X$ is parameterized by a subset
of $\Pi$, where $\Pi \subset \operatorname{Hom}_\mathbb{R}(\mathfrak{a}_\mathbb{R},\mathbb{R})$ is the set of simple restricted roots. Consider the
closures of the codimension 1 orbits $\{Y_\alpha\}_{\alpha \in \Pi}$. Then one can construct the variety
$\mathcal{X}$ over $\mathbb{A}^\Pi$ by iterating the deformation to the normal cone (see Section 2). The
subvariety $Y_\alpha \subset X$ defines the subvariety $\mathcal{Y}_\alpha \subset \mathcal{X}$. Put $Z = \mathcal{X} \setminus \bigcup_{\alpha \in \Pi} \mathcal{Y}_\alpha$. This

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is the variety which we will use. An important property of this variety is given by
the following proposition. For \( \Theta \subset \Pi \), let \( P_{\Theta, \mathbb{R}} = M_{\Theta, \mathbb{R}} A_{\Theta, \mathbb{R}} N_{\Theta, \mathbb{R}} \) be a parabolic
subgroup of \( G_{\mathbb{R}} \) corresponding to \( \Theta \). Put \( K_{\Theta} = M_{\Theta} \cap K \). Let \( f_{Z} : \mathbb{A}^{1} \to \mathbb{A}^{n} \) be the canonical morphism.

**Proposition 1.1** (Lemma \[5.4\] 5.5). For \( \Theta \subset \Pi \), we have \( f_{Z}^{-1}((G_{m})^{\Theta} \times \{0\}^{\Pi \setminus \Theta}) \cong G/K_{\Theta} N_{\Theta} \times (G_{m})^{\Theta} \).

Since \( X \) has many orbits, it is natural to consider the “partial” Jacquet modules.
Let \( \mathcal{H}C_{\Theta, \rho} \) be the category of finitely generated \((g, K_{\Theta} N_{\Theta})\)modules with the same infinitesimal characters as that of the trivial representation. For \( \Theta_{2} \subset \Theta_{1} \subset \Pi \) and \( V \in \mathcal{H}C_{\Theta_{1}, \rho} \), put
\[
J_{\Theta_{2}, \Theta_{1}}(V) = \{ v \in \lim_{\to \mathbb{k}} V/(m_{\Theta_{1}} \cap \mathfrak{n}_{\Theta_{2}})^{k} V \mid n_{\Theta_{2}} v = 0 \text{ for some } l \}
\]
where \( \mathfrak{n}_{\Theta_{2}} \) is the nilradical of the parabolic subalgebra opposite to \( p_{\Theta_{2}} \).
Then we can prove that \( J_{\Theta_{2}, \Theta_{1}}(V) \in \mathcal{H}C_{\Theta_{2}, \rho} \) (Proposition \[3.9\]).

**Theorem 1.2.** We have the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{H}C_{\Theta_{1}, \rho} & \xrightarrow{J_{\Theta_{2}, \Theta_{1}}} & \mathcal{H}C_{\Theta_{2}, \rho} \\
\downarrow & & \downarrow \\
\text{Perv}_{B}(G/K_{\Theta_{1}} N_{\Theta_{1}}) & \xrightarrow{\text{Kat}_{\nu}} & \text{Perv}_{B}(G/K_{\Theta_{2}} N_{\Theta_{2}}).
\end{array}
\]

We summarize the contents of this paper. In Section 2, we give preliminaries
on the deformation to the normal cone. The definition and the properties of the
partial Jacquet functor \( J_{\Theta_{2}, \Theta_{1}} \) are given in Section 3. We review the theorem of
Emerton-Nadler-Vilonen in Section 4. We will use their result to prove our theorem.
We finish a proof of the main theorem in Section 5.

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2. Deformation to normal cone

Let \( X \) be a scheme of finite type over \( \mathbb{C} \) and \( Y \) its closed subscheme.
Then we can construct a family \( f : \mathcal{A} \to \mathbb{A}^{1} \), called the deformation to the normal cone,
that satisfies the following:
over $\mathbb{G}_m$, it is a constant family $X \times \mathbb{G}_m \to \mathbb{G}_m$, and

the fiber $f^{-1}(0)$ at 0 is isomorphic to the normal cone $C_Y(X)$.

Recall that, if we write $I$ for the defining ideal of $Y \subset X$, the normal cone $C_Y(X)$ is the scheme $\text{Spec} \bigoplus_{k=0}^{\infty} T^k/I^{k+1}$ over $Y$. If $X$ and $Y$ are smooth over $\mathbb{C}$, $C_Y(X)$ is isomorphic to the normal bundle $T_Y(X)$.

Let us recall briefly its construction. For more detail, see [Ful98, Chapter 5]. Let $\mathcal{X}$ be the blow-up of $X \times \mathbb{A}^1$ along $Y \times \{0\}$. Then, by the universal property of the blow-up, we have a natural morphism $X \times \{0\} \to \mathcal{X}$, which is a closed immersion. We define $\mathcal{X}$ as the complement of its image in $\mathcal{X}$, and $f : \mathcal{X} \to \mathbb{A}^1$ as the composite of $\mathcal{X} \to \mathcal{X} \to X \times \mathbb{A}^1 \to \mathbb{A}^1$. If $X$ is an affine scheme Spec $A$, we may describe $\mathcal{X}$ more explicitly as follows. Let $I$ be the defining ideal of $Y \subset X$. Then $\mathcal{X} = \text{Spec} \bigoplus_{n \in \mathbb{Z}} I^{-n}T^n$, where $T$ is an indeterminate such that $A^1 = \text{Spec} \mathbb{C}[T]$. Note that we set $I^n = A$ for a negative integer $n$, and regard $\bigoplus_{n \in \mathbb{Z}} I^{-n}T^n$ as a subring of the Laurent polynomial ring $A[T^\pm 1]$.

To any subscheme $Z$ of $X$, we can attach a subscheme $\mathcal{Z}$ of $\mathcal{X}$ such that $\mathcal{Z} \to \mathbb{A}^1$ is the deformation to the normal cone with respect to $Y \cap Z \subset Z$. If $Z$ is open in $X$, then $\mathcal{Z}$ is simply the inverse image of $Z \times \mathbb{A}^1$ under $\mathcal{X} \to X \times \mathbb{A}^1$. On the other hand, if $Z$ is closed in $X$, then $\mathcal{Z}$ is the strict transform of $Z \times \mathbb{A}^1 \subset X \times \mathbb{A}^1$ in $\mathcal{X}$; namely, $\mathcal{Z}$ is the closure of $f^{-1}(Z \times \mathbb{A}_m)$ in $\mathcal{X}$.

For our purpose, iteration of this construction is important. Now let $X$ be a scheme which is smooth of finite type over $\mathbb{C}$, $Y$ its effective divisor, and $\bigcup_{i=1}^l Y_i$ the irreducible decomposition of $Y$. Assume that $Y$ is a strict normal crossing divisor. Namely, for each subset $\Theta \subset \{1, \ldots, l\}$, we assume that $Y_\Theta = \bigcap_{i \in \Theta} Y_i$ is smooth over $\mathbb{C}$. It is equivalent to saying that $Y \subset X$ is étale locally isomorphic to $(T_1 \cdots T_l = 0) \subset \mathbb{A}^n = \text{Spec} \mathbb{C}[T_1, \ldots, T_n]$ and every irreducible component of $Y$ is smooth over $\mathbb{C}$. Under this setting, let $\mathcal{X}^{(1)} \to \mathbb{A}^1$ be the deformation to the normal cone with respect to $Y_1 \subset X$. For each $i$, the closed subscheme $Y_i$ of $X$ induces a closed subscheme $\mathcal{Y}_i^{(1)}$ of $\mathcal{X}^{(1)}$. Next, consider the deformation to the normal cone $\mathcal{X}^{(2)} \to \mathbb{A}^1$ with respect to $\mathcal{Y}_2^{(1)}$ of $\mathcal{X}^{(1)}$ and closed subschemes $\mathcal{Y}_2^{(2)}$. Inductively, we can define a family $\mathcal{X}^{(k)} \to \mathbb{A}^1$ and closed subschemes $\mathcal{Y}_i^{(k)}$ of $\mathcal{X}^{(k)}$. Recall that, by construction, $\mathcal{X}^{(k)}$ is equipped with a natural structure morphism $\mathcal{X}^{(k)} \to \mathcal{X}^{(k-1)} \times \mathbb{A}^1$, where $\mathcal{X}^{(k)} \to \mathbb{A}^1$ is the composite of it with the second projection (here we put $\mathcal{X}^{(0)} = X$). Therefore, we get a natural morphism $\pi_k : \mathcal{X}^{(k)} \to X \times \mathbb{A}^k$ and $f_k = pr_2 \circ \pi : \mathcal{X}^{(k)} \to \mathbb{A}^k$. If $k = l$, we simply write $\mathcal{X}$, $\mathcal{Y}_i$, $\pi$, $f$ for $\mathcal{X}^{(i)}$, $\mathcal{Y}_i^{(i)}$, $\pi_i$, $f_i$, respectively.

First let us consider étale locally. Assume that $X = \mathbb{A}^n = \text{Spec} \mathbb{C}[S_1, \ldots, S_n]$ and $Y_i$ is given by the equation $S_i = 0$ for each $i$. Then we have

$\mathcal{X}^{(1)} = \text{Spec} \mathbb{C}[S_{T_1}, S_2, \ldots, S_n, T_1], \quad \mathcal{Y}_i^{(1)} : \frac{S_i}{T_i} = 0$ $(i = 1), \quad S_i = 0$ $(i \geq 2),

\mathcal{X}^{(2)} = \text{Spec} \mathbb{C} \left[ \frac{S_1}{T_1}, \frac{S_2}{T_2}, S_3, \ldots, S_n, T_1, T_2 \right], \quad \mathcal{Y}_i^{(2)} : \frac{S_i}{T_i} = 0$ $(i \leq 2), \quad S_i = 0$ $(i \geq 3),

\vdots

\mathcal{X}^{(l)} = \text{Spec} \mathbb{C} \left[ \frac{S_1}{T_1}, \ldots, \frac{S_l}{T_l}, S_{l+1}, \ldots, S_n, T_1, \ldots, T_l \right], \quad \mathcal{Y}_i^{(l)} : \frac{S_i}{T_i} = 0.

This computation can be generalized to the case where $X$ is affine:
Lemma 2.1. Assume that $X$ is an affine scheme $\text{Spec} \ A$. Let $I_i$ be the defining ideal of $Y_i$. Then, we have

$$\mathcal{X} = \text{Spec} \bigoplus_{n \in \mathbb{Z}^l} I^{-n} T^\mathbb{Z},$$

where $I^{-n} = I_1^{-n_1} \cdots I_l^{-n_l}$ and $T^\mathbb{Z} = T_1^{n_1} \cdots T_l^{n_l}$ for $n = (n_1, \ldots, n_l) \in \mathbb{Z}^l$.

Proof. As explained above, we have $\mathcal{X}^{(1)} = \text{Spec} B^{(1)}$ for $B^{(1)} = \bigoplus_{n \in \mathbb{Z}^l} I_1^{-n_1} T_1^{n_1}$. The local calculation tells us that the defining ideal of $\mathcal{X}^{(1)}_2$ is $I_2 B^{(1)}$. Thus we have $\mathcal{X}^{(2)} = \text{Spec} B^{(2)}$ for $B^{(2)} = \bigoplus_{n \in \mathbb{Z}^l} I_2^{-n_2} B^{(1)} T_2^{n_2} = \bigoplus_{n \in \mathbb{Z}^l} I_1^{-n_1} I_2^{-n_2} T_1^{n_1} T_2^{n_2}$. We can proceed similarly to obtain the desired formula.

Remark 2.2. By the lemma above, we know that $\mathcal{X}$ is independent of the labeling of $Y_1, \ldots, Y_l$.

For each subset $\Theta$ of $\{1, \ldots, l\}$, set $\mathcal{X}_\Theta = f^{-1}((\mathbb{G}_m)^\Theta \times \emptyset^\emptyset)$, where $\emptyset^\emptyset = \{1, \ldots, l\} \setminus \Theta$. Recall that we put $Y_\Theta = \bigcap_{i \in \Theta} Y_i$.

Lemma 2.3. We have $\mathcal{X}_\Theta \cong C_{Y_{\emptyset \emptyset}}(X) \times (\mathbb{G}_m)^\Theta$.

Proof. We may assume that $X$ is an affine scheme $\text{Spec} \ A$. Let $I_i$ be the defining ideal of $Y_i$ and put $B = \bigoplus_{n \in \mathbb{Z}^l} I^{-n} T^\mathbb{Z}$. By Lemma 2.1, we have

$$\mathcal{X}_\Theta = \text{Spec} B[T_i^{-1} \mid i \in \Theta]/(T_i \mid i \in \Theta^\emptyset),$$

where $(T_i \mid i \in \Theta^\emptyset)$ is the ideal generated by $T_i$ for $i \in \Theta^\emptyset$. Note that $B[T_i^{-1} \mid i \in \Theta]$ is isomorphic to $(\bigoplus_{n \in \mathbb{Z}^l} I^{-n} T^\mathbb{Z}) \otimes \mathbb{C}[T_i^{-1} \mid i \in \Theta]$. Therefore, by replacing $\{Y_1, \ldots, Y_l\}$ with $\{Y_i \mid i \in \Theta^\emptyset\}$, we may assume that $\Theta = \emptyset$.

Put $J = I_1 + \cdots + I_l$. By the definition, we have

$$\mathcal{X}_\emptyset = \text{Spec} B/(T_1, \ldots, T_l) = \text{Spec} \bigoplus_{n \in \mathbb{Z}^l} (I^n J/I^n) T^{-n} = \text{Spec} \bigoplus_{n \in \mathbb{Z}^l} I^n J/I^n.$$

On the other hand, we have $C_{Y_{\emptyset \emptyset}}(X) = \text{Spec} \bigoplus_{k=0}^\infty J^k/J^{k+1}$. Therefore we have a natural morphism $C_{Y_{\emptyset \emptyset}}(X) \to \mathcal{X}_\emptyset$ by sending $I^n J/I^n$ to $J^k/J^{k+1}$ where $k = n_1 + \cdots + n_l$. By étale local calculation, it is easily seen that this morphism is an isomorphism.

Lemma 2.4. Assume that $X = X' \times \mathbb{A}^l$ and $Y_i = \{(x, (c_i)) \in X \mid c_i = 0\}$.

1. We have an isomorphism $\mathcal{X} \cong X' \times \mathbb{A}^l \times \mathbb{A}^l$ under which $\pi : \mathcal{X} \to X \times \mathbb{A}^l = X' \times \mathbb{A}^l \times \mathbb{A}^l$ is given by $(x, (d_i), (t_i)) \mapsto (x, (d_i t_i), (t_i))$.
2. The projection $\mathcal{X}_\emptyset \cong C_{Y_{\emptyset \emptyset}}(X) \times (\mathbb{G}_m)^\emptyset \to C_{Y_{\emptyset \emptyset}}(X)$ is given by $(x, (d_i), (t_i)) \mapsto (x, (d_i t_i))$.
3. Let $G$ be an algebraic group over $\mathbb{C}$. Assume that we are given an action of $G$ on $X'$ and characters $\chi_i : G \to \mathbb{G}_m$. These induce an action of $G$ on $X$ by $(x, (c_i)) \mapsto (gx, (\chi_i(g) c_i))$ which preserves $Y_i$. Then, the induced action on $\mathcal{X} \cong X' \times \mathbb{A}^l \times \mathbb{A}^l$ is given by $(x, (d_i), (t_i)) \mapsto (gx, (\chi_i(g) d_i), (t_i))$.

Proof. (1) Since our construction clearly commutes with a smooth base change, we may assume that $X' = \text{Spec} \mathbb{C}$. Then the local calculation above gives us the desired isomorphism (we have only to take $n = l$).

(2) In the same way as in (1), we may assume that $X' = \text{Spec} \mathbb{C}$ and use the local calculation.
(3) Since $\pi : X \to X \times H^i$ is $G$-equivariant, for $(x, (d_i), (t_i)) \in X$, we have $\pi(g(x, (d_i), (t_i))) = g(x, (d_i), (t_i)) = (g \cdot x, (\chi_i(g)d_i, (t_i))) = \pi(g \cdot x, (\chi_i(g)d_i), (t_i))$. On the other hand, $\pi$ is an isomorphism over the open subset $X \times (\mathbb{G}_m)^i \subset X \times H^i$. Therefore, on the dense open subset $\pi^{-1}(X \times (\mathbb{G}_m)^i)$ of $X$, the action of $G$ is given by $(x, (d_i), (t_i)) \mapsto (g \cdot x, (\chi_i(g)d_i), (t_i))$. Hence it is given by the same formula over the whole $X$. □

3. THE JACQUET FUNCTORS

In this section, we recall some preliminaries on the Jacquet modules, which are well-known. (For example, some of them are proved in [HS83], [Wal88].) However, we give proofs for the sake of completeness.

Let $G_R$ be a connected reductive linear algebraic group over $\mathbb{R}$, $G_R = K_R A_R N_R$ an Iwasawa decomposition, and $M_R$ the centralizer of $A_R$ in $K_R$. Then $P_R = M_R A_R N_R$ is a Langlands decomposition of a minimal parabolic subgroup. We use lower-case fraktur letters to denote the corresponding Lie algebras and omit the subscripts "lower-case fraktur letters to denote the corresponding Lie algebras and omit the

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Lemma 3.2. 

(1) It is easy to see that \( \Gamma_{\theta_1}(V) \in HC_{\Theta_1} \) for \( \mu_1 \in a_{\Theta_1} \).

(2) If \( V \in HC_{\Theta_1}^{G_\Theta} \) then \( V/\pi_\Theta \) \( \in HC_{\Theta_1}^{L_\Theta, \pi} \).

(3) If \( V \in HC_{\Theta_1}^{G_\Theta} \) then \( V/\pi_\Theta \) \( \in HC_{\Theta_1}^{L_\Theta, \pi} \).

(4) For \( V \in HC_{\Theta_1} \) and \( \mu_1 \in a_{\Theta_1}^* \), \( \Gamma_{\mu_1}(V/(m_{\Theta_1} \cap \pi_{\Theta_2})^k V) \) is a Harish-Chandra module of \( L_{\Theta_2, \pi} \).

Proof. (1) It is easy to see that \( \Gamma_{\mu_1}(V) \) is a \( (g, K_{\Theta_1}(M_{\Theta_1} \cap N_{\Theta})) \)-module. It is sufficient to prove that \( \Gamma_{\mu_1}(V) \) is a finitely generated \( U(I_{\Theta_1}) \)-module and \( Z(I_{\Theta_1}) \)-finite.

Since \( V \in HC_{\Theta_1} \), \( V \) is generated by a finite-dimensional subspace \( W \) of \( V \). By Lemma 5.3 (3), the action of \( a_{\Theta_1} \) on \( V \) is locally finite. By the definition of \( HC_{\Theta_1} \), the action of \( n_{\Theta_1} \) on \( V \) is locally finite. Hence we may assume that \( W \) is \( n_{\Theta_1} \)-finite.

In particular, \( W \) is \( n_{\Theta_1} \)-stable. Therefore, we have \( V = U(m_{\Theta_1})U(n_{\Theta_1})W \).

From this, we get

\[
\Gamma_{\mu_1}(V) = U(m_{\Theta_1}) \left( \sum_{\mu'_1 \in wt_{\lambda_{\Theta_1}}(W)} \Gamma_{\mu_1-\mu'_1}(U(\pi_{\Theta_1}))\Gamma_{\mu'_1}(W) \right).
\]

Since \( W \) is finite-dimensional, \( wt_{\lambda_{\Theta_1}}(W) \) is finite. For each \( \mu'_1 \), \( \Gamma_{\mu_1-\mu'_1}(U(\pi_{\Theta_1})) \) is finite-dimensional. Therefore, \( \sum_{\mu'_1 \in wt_{\lambda_{\Theta_1}}(W)} \Gamma_{\mu_1-\mu'_1}(U(\pi_{\Theta_1}))\Gamma_{\mu'_1}(W) \) is finite-dimensional. Hence \( \Gamma_{\mu_1}(V) \) is a finitely generated \( U(m_{\Theta_1}) \)-module.

Put \( V_k = \{ v \in V \mid n_{\Theta_1}^k v = 0 \} \). Since \( \Gamma_{\mu_1}(V) \) is finitely generated \( U(m_{\Theta_1}) \)-module, we can take \( n_{\Theta_1} \)-stable subspace \( W' \) such that \( \Gamma_{\mu_1}(V) \subseteq U(m_{\Theta_1})W' \). Since \( W' \) is finite-dimensional, \( W' \subseteq V_k \) for some \( k \in \mathbb{Z}_{\geq 0} \). By Lemma 5.3, \( V_k \) is \( Z(I_{\Theta_1}) \)-finite. Hence \( \Gamma_{\mu_1}(V) \) is \( Z(I_{\Theta_1}) \)-finite.

(2) As above, we can take a finite-dimensional \( n_{\Theta_1} \)-stable submodule \( W \) which generates \( V \) as a \( g \)-module. Then we have \( V = U(\pi_{\Theta_1})U(m_{\Theta_1})W \). Hence we
get a surjective homomorphism $U(m_\Theta)W \to V/\pi_\Theta V$. It is sufficient to prove that $U(m_\Theta)W \in \mathcal{HC}_{\Theta}^{L_{\Theta},a}$. Since $W$ is finite-dimensional, $W \subset \bigoplus_{\mu \in \Lambda} \Gamma_{\mu}(V)$ for a finite subset $\Lambda \subset a_\Theta^\circ$. Each $\Gamma_{\mu}(V)$ is $m_\Theta$-stable. Therefore, $U(m_\Theta)W \subset \bigoplus_{\mu \in \Lambda} \Gamma_{\mu}(V)$.

By (1), $\Gamma_{\mu}(V) \in \mathcal{HC}_{\Theta}^{L_{\Theta},a}$. Hence we have $U(m_\Theta)W \in \mathcal{HC}_{\Theta}^{L_{\Theta},a}$.

(3) Recall that an object of $\mathcal{HC}_{\Theta}^{G_\Theta}$ is a Harish-Chandra module of $G_\Theta$. Hence this is [HS83a Proposition 2.24].

(4) By Lemma 3.1 we have $V = \bigoplus_{\mu_1 \in a_\Theta^\circ} \Gamma_{\mu_1}(V)$. Since $m_\Theta$ has an $a_\Theta^\circ$-weight 0, the action of $m_\Theta$ preserves each $\Gamma_{\mu_1}(V)$. Therefore, we have $\Gamma_{\mu_1}(V/(m_\Theta \cap \pi_{\Theta_2})^k) = \Gamma_{\mu_1}(V)/(m_\Theta \cap \pi_{\Theta_2})^k \Gamma_{\mu_1}(V)$. By (1), $\Gamma_{\mu_1}(V) \in \mathcal{HC}_{\Theta_1}^{L_{\Theta_1},a}$. Hence we may assume that $\Theta_1 = \Pi$. So we have $m_\Theta \cap \pi_{\Theta_2} = \pi_{\Theta_2}$.

By (2), $V' = V/\pi_\Theta V \in \mathcal{HC}_{\Theta_2}^{L_{\Theta_2},a}$. By (3), $V'' = V'/(m_\Theta \cap \pi_{\Theta_2})V' \in \mathcal{HC}_{\Theta_2}^{L_{\Theta_2},a}$. Since $V'' = V/\pi_{\Theta_2} V$, we get the lemma for $k = 1$.

For a general $k$, we can prove the lemma by induction on $k$ using an exact sequence $\pi_{\Theta_2} \otimes (V/\pi_{\Theta_2} V) \to V/\pi_{\Theta_2}^{k+1} V \to V/\pi_{\Theta_2} V \to 0$. □

We now prove that the length of an object of $\mathcal{HC}_{\Theta}$ is finite.

Lemma 3.3. Assume that a $(g, N_\Theta)$-module $V$ satisfies the following conditions:

1. The module $V$ is $Z(g)$-finite.
2. For all $\mu \in a_\Theta$, $\Gamma_{\mu}(V)$ has a finite length as a $m_\Theta$-module.

Then $V$ has a finite length.

Proof. Let $\varphi: Z(g) \to Z(\mathfrak{a}_\Theta)$ be a homomorphism defined in the proof of Lemma 3.1. Set $J = \varphi(\text{Ann}(Z(g)/Z(\mathfrak{a}_\Theta)))$. Then $J$ has a finite codimension. In particular, there exists a finite subset $\Lambda \subset a_\Theta^\circ$ which gives all maximal ideals of $U(\mathfrak{a}_\Theta)/(J \cap U(\mathfrak{a}_\Theta))$. Let $V'$ be a subquotient of $V$. Then we have $J(V')^{a_\Theta} = 0$, hence $(V')^{a_\Theta} \subset \bigoplus_{\nu \in \Lambda} \Gamma_{\nu}(V')$. Since $V'$ is a $(g, N_\Theta)$-module, the space $(V')^{a_\Theta}$ is non-zero. Hence the length of the $g$-module $V$ is less than or equal to the sum of the length of $m_\Theta$-modules $\Gamma_{\mu}(V)$ for $\mu \in \Lambda$. It is finite by the assumption. □

Corollary 3.4. Each object in $\mathcal{HC}_{\Theta}$ has a finite length.

Proof. The product of (1) in the previous lemma is satisfied by the definition of $\mathcal{HC}_{\Theta}$. For $V \in \mathcal{HC}_{\Theta}$ and $\mu \in a_\Theta^\circ$, $\Gamma_{\mu}(V)$ is a Harish-Chandra module of $L_{\Theta,\mathbb{R}}$ by Lemma 3.2 (4). (Take $\Theta_1 = \Theta_2 = \Theta$.) Hence it has a finite length. By the previous lemma, we get the corollary. □

For a subset $\Lambda \subset a_\Theta^\circ$, put $\Lambda - Z_{\geq 0} \Pi|_{a_\Theta} = \{\mu - \sum_{\alpha \in \Pi} n_\alpha \alpha|_{a_\Theta} \mid \mu \in \Lambda, n_\alpha \in \mathbb{Z}_{\geq 0}\}$.

Lemma 3.5. For $V \in \mathcal{HC}_{\Theta}$, there exists a finite subset $A_2$ of $a_{\Theta_2}^\circ$ such that $\text{wt}_{a_{\Theta_2}}(J_{\Theta, \Theta_2}(V)) \subset A_2 - Z_{\geq 0} \Pi|_{a_{\Theta_2}}$.

Proof. Put $c = m_\Theta \cap \pi_{\Theta_2}$. As in the proof of Lemma 3.2, we can take a finite-dimensional $m_\Theta \oplus a_\Theta^\circ$-stable subspace $W$ such that $V = U(g)W$. Put $A_1 = \text{wt}_{a_{\Theta_1}}(W)$. Then this is finite and, since $V = U(m_\Theta)U(\pi_\Theta)W$, $\text{wt}_{a_{\Theta_1}}(V) \subset A_1 - Z_{\geq 0} \Pi|_{a_{\Theta_1}}$. Set $A_2 = \bigcup_{\mu \in \Lambda_1} \text{wt}_{a_{\Theta_2}}(\Gamma_{\mu_1}(V/c(V))) = \{\mu_2 \in \text{wt}_{a_{\Theta_2}}(V/c(V)) \mid \mu_2|_{a_{\Theta_1}} \in \Lambda_1\}$. By Lemma 3.2 (4), $\Gamma_{\mu_1}(V/c(V))$ is a Harish-Chandra module of $L_{\Theta_2,\mathbb{R}}$. In particular, it is $Z(\Theta_0)$-finite. Therefore, it is $U(\mathfrak{a}_\Theta)$-finite since $\mathfrak{a}_\Theta$ is a subalgebra of the center of $g$. Hence $\text{wt}_{a_{\Theta_2}}(\Gamma_{\mu_1}(V/c(V)))$ is finite. This implies that $A_2$ is finite. We also have $\text{wt}_{a_{\Theta_2}}(V/c(V)) \subset A_2 - Z_{\geq 0} \Pi|_{a_{\Theta_2}}$. □
We prove $\text{wt}(V/c^kV) = \Lambda_2 - Z_{\geq 0} \Pi|_{a_{\Theta_2}}$ by induction on $k$. Then we get the lemma. From an exact sequence $c \otimes (V/c^kV) \to V/c^{k+1}V \to V/c^kV \to 0$, we have $\text{wt}(V/c^{k+1}V) \subset \text{wt}(c \otimes (V/c^kV)) \cup \text{wt}(V/c^kV) \subset \Lambda_2 - Z_{\geq 0} \Pi|_{a_{\Theta_2}}$ by the inductive hypothesis. 

**Lemma 3.6.** For $V \in \mathcal{HC}_{\Theta}$, we have $J_{\Theta_2, \Theta_1}(V) = \bigoplus_{\mu_2 \in a^*_{\Theta_2}} \Gamma_{\mu_2}(J_{\Theta_2, \Theta_1}(V))$. Therefore, $\Gamma_{\mu_2}(J_{\Theta_2, \Theta_1}(V)) = \Gamma_{\mu_2}(J_{\Theta_2, \Theta_1}(V))$.

**Proof.** By Lemma 3.5, the right hand side is contained in the left hand side. On the other hand, by Lemma 3.1 (3), we have $J_{\Theta_2, \Theta_1}(V) = \bigoplus_{\mu_2 \in a^*_{\Theta_2}} \Gamma_{\mu_2}(J_{\Theta_2, \Theta_1}(V))$. This is a subspace of the right hand side. □

Notice that $V/(m_{\Theta_1} \cap m_{\Theta_2}) k V \simeq \tilde{J}_{\Theta_2, \Theta_1}(V)/(m_{\Theta_1} \cap m_{\Theta_2}) k \tilde{J}_{\Theta_2, \Theta_1}(V)$ by the definition.

**Lemma 3.7.** For $\mu_2 \in a^*_{\Theta_2}$, there exists $k \in \mathbb{Z}_{\geq 0}$ such that $\Gamma_{\mu_2}(\tilde{J}_{\Theta_2, \Theta_1}(V)) \to \Gamma_{\mu_2}(V/(m_{\Theta_1} \cap m_{\Theta_2}) k V)$ is an isomorphism.

**Proof.** Put $c = m_{\Theta_1} \cap m_{\Theta_2}$. Take $\Lambda_2$ as in Lemma 3.5. Let $k \in \mathbb{Z}_{\geq 0}$ such that for any $\alpha_1, \ldots, \alpha_k \in \Sigma^+$ we have $\mu_2 \notin (\Lambda_2 - Z_{\geq 0} \Pi|_{a_{\Theta_2}})$, we have $\Gamma_{\mu_2}(\tilde{J}_{\Theta_2, \Theta_1}(V)) = 0$. By the exact sequence $0 \to c^k \tilde{J}_{\Theta_2, \Theta_1}(V) \to \tilde{J}_{\Theta_2, \Theta_1}(V) \to V/c^kV \to 0$, we have $0 = \Gamma_{\mu_2}(c^k \tilde{J}_{\Theta_2, \Theta_1}(V)) \to \Gamma_{\mu_2}(J_{\Theta_2, \Theta_1}(V)) \to \Gamma_{\mu_2}(c^k V) \to 0$. Hence we get the lemma. □

**Lemma 3.8.** For $V \in \mathcal{HC}_{\Theta}$ and $\mu_2 \in a^*_{\Theta_2}$, $\Gamma_{\mu_2}(J_{\Theta_2, \Theta_1}(V))$ is a Harish-Chandra module for $L_{\Theta_2, \Theta}$. 

**Proof.** This follows from Lemma 3.2, Lemma 3.6, and Lemma 3.1. □

If $\Theta = \Theta_1 = \Pi$, the following proposition is [HS83b] (34) Lemma.

**Proposition 3.9.** If $V \in \mathcal{HC}_{\Theta}$, then $J_{\Theta_2, \Theta_1}(V) \in \mathcal{HC}_{\Theta_2}$.

**Proof.** It is sufficient to prove that $J_{\Theta_2, \Theta_1}(V)$ has a finite length. This follows from Lemma 3.3 and the previous lemma. □

Hence $J_{\Theta_2, \Theta_1}$ defines a functor $\mathcal{HC}_{\Theta} \to \mathcal{HC}_{\Theta_2}$.

**Proposition 3.10.** For $\Theta_3 \subset \Theta_2 \subset \Theta \subset \Theta_1 \subset \Pi$, we have $J_{\Theta_3, \Theta_2} \circ J_{\Theta_2, \Theta_1} \simeq J_{\Theta_3, \Theta_1} : \mathcal{HC}_{\Theta} \to \mathcal{HC}_{\Theta_3}$.

We use the following lemma.

**Lemma 3.11.** Let $c$ be a finite-dimensional nilpotent Lie algebra, $c_1, c_2 \subset c$ Lie subalgebras such that:

- $c_2$ is an ideal of $c$.
- $c = c_1 \oplus c_2$.

Then for all $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ there exists $n \in \mathbb{Z}_{\geq 0}$ such that $c^n \subset c_{1^{k_1}} U(c) + c_{2^{k_2}} U(c)$.

**Proof.** Set $V = U(c)/(c_{1^{k_1}} U(c) + c_{2^{k_2}} U(c))$, $v_0 = 1 \in V$. Then $V$ is a right $U(c)$-module, $V = v_0 U(c)$ and $v_0 c_{1^{k_1}} = v_0 c_{2^{k_2}} = 0$. We have $V = v_0 U(c_1) U(c_2)$. Since $v_0 c_{1^{k_1}} = 0$, $v_0 U(c_1)$ is finite-dimensional. By the assumption, $c_2$ is an ideal of $c$. Therefore, $v_0 U(c_1) c_{2^{k_2}} = v_0 c_{2^{k_2}} U(c) = 0$. Hence $V$ is finite-dimensional. Since
a finite-dimensional irreducible representation of $\mathfrak{c}$ is a character, $V$ is given by an extension of characters. As $v_0\mathfrak{c}^l = v_0\mathfrak{c}^2 = 0$ and $\mathfrak{c}$ is nilpotent, for every $v \in V$ there exist integers $l_1, l_2$ such that $v_0^{l_1} = v_0^{l_2} = 0$. This implies that each irreducible subquotient of $V$ is the trivial representation. Hence there exists $n$ such that $v_0\mathfrak{c}^n = 0$. This completes the proof. □

**Proof of Proposition 3.10.** We prove that for each $\mu \in \mathfrak{a}_{\Theta}^\ast$, the generalized $\mu$-weight spaces of both sides are isomorphic. Put $c_1 = \mathfrak{m}_{\Theta_2} \cap \mathfrak{m}_{\Theta_3} \cap \mathfrak{m}_{\Theta_1}$ and $c = c_1 \oplus c_2$. Then $c_1, c_2$ satisfies the assumption of the previous lemma and $\mathfrak{c} = \mathfrak{m}_{\Theta_1} \cap \mathfrak{m}_{\Theta_2}$. Put $\mu_2 = \mu_3|_{\mathfrak{m}_{\Theta_2}}$. By Lemma 3.6 and Lemma 3.7 for sufficiently large $k_1, k_2$, we have

$$\begin{align*}
\Gamma_{\mu_3}((J_{\Theta_3, \Theta_2}(J_{\Theta_2, \Theta_1}(V)))) &= \Gamma_{\mu_3}(J_{\Theta_2, \Theta_1}(V)/\mathfrak{c}^k J_{\Theta_2, \Theta_1}(V)) \\
&= \Gamma_{\mu_3}(J_{\Theta_2, \Theta_1}(V)/\mathfrak{c}^k J_{\Theta_2, \Theta_1}(V)) \\
&= \Gamma_{\mu_3}(\Gamma_{\mu_2}(J_{\Theta_2, \Theta_1}(V))/\mathfrak{c}^k \Gamma_{\mu_2}(J_{\Theta_2, \Theta_1}(V))) \\
&= \Gamma_{\mu_3}(\Gamma_{\mu_2}(V/\mathfrak{c}^k V)/\mathfrak{c}^k \Gamma_{\mu_2}(V/\mathfrak{c}^k V)) \\
&= \Gamma_{\mu_3}(\Gamma_{\mu_2}((V/\mathfrak{c}^k V)/\mathfrak{c}^k \Gamma_{\mu_2}(V/\mathfrak{c}^k V))) \\
&= \Gamma_{\mu_3}(\Gamma_{\mu_2}(V/(\mathfrak{c}^k V + \mathfrak{c}^k V))) \\
&= \Gamma_{\mu_3}(V/(\mathfrak{c}^k V + \mathfrak{c}^k V)).
\end{align*}$$

We also have

$$\Gamma_{\mu_3}(J_{\Theta_3, \Theta_1}(V)) \simeq \Gamma_{\mu_3}(V/\mathfrak{c}^k V)$$

for sufficiently large $k$. Fix $k_1, k_2$ and take $n$ as in the previous lemma. We may assume $k \leq k_1, k_2 \leq n$. Consider the following homomorphism:

$$\Gamma_{\mu_3}(V/\mathfrak{c}^n V) \rightarrow \Gamma_{\mu_3}(V/(\mathfrak{c}^k V + \mathfrak{c}^k V)) \rightarrow \Gamma_{\mu_3}(V/\mathfrak{c}^k V).$$

Since $V/\mathfrak{c}^k V$ is decomposed into the generalized $\mathfrak{a}_{\Theta}^\ast$-weight spaces (this follows from the fact that $V/\mathfrak{c}^k V$ is a Harish-Chandra module of $L_{\Theta, \mathfrak{R}}$), the first homomorphism is surjective. If $k_1, k_2, n, k$ is sufficiently large, the composition of this homomorphism is isomorphic by Lemma 3.7. Hence the first homomorphism is injective. We get the proposition. □

**Proposition 3.12.** The functor $J_{\Theta_2, \Theta_1} : \mathcal{H}_{\Theta} \rightarrow \mathcal{H}_{\Theta_2}$ is independent of $\Theta_1$.

**Proof.** By the previous proposition, $J_{\Theta_2, \Theta_1} = J_{\Theta_2, \Theta_3} \circ J_{\Theta_3, \Theta_1}$. Hence it is sufficient to prove that $J_{\Theta_3, \Theta_1}(V) \simeq V$ for $V \in \mathcal{H}_{\Theta}$. We compare the $\mu$-weight spaces for each $\mu \in \mathfrak{a}_{\Theta_1}^\ast$. Put $\mathfrak{c} = \mathfrak{m}_{\Theta_1} \cap \mathfrak{m}_{\Theta_2} \cap \mathfrak{m}_{\Theta_3}$. Take a finite subset $\Lambda \subset \mathfrak{a}_{\Theta_1}^\ast$ such that $\mathrm{wt}_{\mathfrak{a}_{\Theta_1}}(V) \subseteq \Lambda - \mathbb{Z}_{\geq 0}\Pi_{\mathfrak{a}_{\Theta_1}}$. Then for a sufficiently large $k$, for any $\alpha_1, \ldots, \alpha_k \in \Pi$ we have $\mu \notin \Lambda - \mathbb{Z}_{\geq 0}\Pi_{\mathfrak{a}_{\Theta_1}} - (\alpha_1 + \cdots + \alpha_k)$. Hence we have $\mu \notin \mathrm{wt}_{\mathfrak{a}_{\Theta_1}}(\mathfrak{c}^k V)$. This implies $\Gamma_{\mu}(V) \simeq \Gamma_{\mu}(V/\mathfrak{c}^k V)$. On the other hand, the right hand side is isomorphic to $\Gamma_{\mu}(J_{\Theta_3, \Theta_1}(V))$ for a sufficiently large $k$ by Lemma 3.7. We get the proposition. □

**Lemma 3.13.** Each $V \in \mathcal{H}_{\Theta}$ is finitely generated as a $U(\mathfrak{m})$-module.

**Proof.** Take a finite-dimensional $\mathfrak{m}_{\Theta} \oplus \mathfrak{a}_{\Theta}$-stable subspace $W$ of $V$ which generates $V$ as a $\mathfrak{g}$-module. Then $U(\mathfrak{m}_{\Theta})W \subseteq \bigoplus_{\mu \in \Lambda} \Gamma_{\mu}(V)$ for some finite subset $\Lambda \subset \mathfrak{a}_{\Theta}^\ast$. Hence $U(\mathfrak{m}_{\Theta})W$ is a Harish-Chandra module of $L_{\Theta, \mathfrak{R}}$. Therefore, by a theorem of Casselman-Osborne [CO78, 2.3 Theorem], $U(\mathfrak{m}_{\Theta})W$ is finitely generated as a
(\mathfrak{m}_\Theta \cap \Pi)\)-module. Since \( V = U(\mathfrak{m}_\Theta)U(\mathfrak{m}_\Theta)W, \) \( V \) is a finitely generated \( U(\mathfrak{m}_\Theta)U(\mathfrak{m}_\Theta \cap \Pi) = U(\mathfrak{m}_\Theta) \)-module.

**Proposition 3.14.** The functor \( J_{\Theta_2,\Theta_1} : \mathcal{H}\mathcal{C}_{\Theta_1} \to \mathcal{H}\mathcal{C}_{\Theta_2} \) is exact.

**Proof.** If \( \Theta_2 = \emptyset \) and \( \Theta_1 = \Pi, \) this proposition is well-known [Wal88, 4.1.5, Theorem]. The key point of the proof is the Artin-Rees property and that \( V \in \mathcal{H}\mathcal{C}_\Pi \) is finitely generated as a \( U(\mathfrak{m}_\Theta) \)-module. Hence the usual proof is applicable for our situation using the above lemma. \( \square \)

## 4. The geometric Jacquet functor

In this section, we recall an argument of Emerton-Nadler-Vilonen [ENV04]. For \( \Theta \subset \Pi, \) let \( \mathcal{H}\mathcal{C}_{\Theta,\rho} \) be the category of \( V \in \mathcal{H}\mathcal{C}_\Theta \) whose infinitesimal character is the same as that of the trivial representation. Fix a Borel subgroup \( B \) of \( G. \)

Then by the Beilinson-Bernstein correspondence and the Riemann-Hilbert correspondence, we have an equivalence of categories \( \Delta : \mathcal{H}\mathcal{C}_{\Theta,\rho} \cong \text{Perv}_{K_\Theta N_\Theta}(G/B) \) where \( \text{Perv}_{K_\Theta N_\Theta}(G/B) \) is the category of \( K_\Theta N_\Theta \)-equivariant perverse sheaves on \( G/B. \)

Fix a cocharacter \( \nu : \mathbb{G}_m \to A \) such that \( \langle \nu, \alpha \rangle \geq 0 \) for all \( \alpha \in \Pi. \) Define \( a_\nu : G/B \times \mathbb{G}_m \to G/B \) by \( (gB, t) \mapsto \nu(t)gB. \) Let \( R^\psi \) be the nearby cycle functor with respect to \( G/B \times \mathbb{A}_1 \to \mathbb{A}_1. \) For \( F \in \text{Perv}(G/B), \) put \( \Psi_\nu(F) = R^\psi a_\nu^* F. \)

Then the main theorem of [ENV04] is the following.

**Theorem 4.1** (Emerton-Nadler-Vilonen [ENV04, Theorem 1.1]). Assume that \( \nu \) is regular. We have \( \Delta \circ J_{\emptyset,\Pi} \cong \Psi_\nu \circ \Delta : \mathcal{H}\mathcal{C}_{\Pi,\rho} \to \text{Perv}(G/B). \)

Their argument can be applicable for a general \( \nu. \) Namely, we can prove the following theorem.

**Theorem 4.2.** Set \( \Theta = \{ \alpha \in \Pi \mid \langle \alpha, \nu \rangle = 0 \}. \) Then we have \( \Delta \circ J_{\Theta,\Pi} \cong \Psi_\nu \circ \Delta : \mathcal{H}\mathcal{C}_{\Theta,\rho} \to \text{Perv}(G/B) \) for all \( \Theta' \subset \Pi \) such that \( \Theta \subset \Theta'. \)

We review the proof. Let \( V \in \mathcal{H}\mathcal{C}_{\Theta,\rho}. \) First we construct a filtration on \( V. \) To construct it, we prove the following lemma.

**Lemma 4.3.** We have the following.

1. We have \( J_{\emptyset,\Pi}(V) / \mathfrak{m}_{\emptyset}^k J_{\emptyset,\Pi}(V) \cong J_{\emptyset,\Pi}(V) / \mathfrak{m}_{\emptyset}^k J_{\emptyset,\Pi}(V). \)
2. We have \( J_{\emptyset,\Pi}(V) \cong J_{\emptyset,\Pi}(J_{\emptyset,\Pi}(V)). \)
3. We have \( \hat{J}_{\emptyset,\Pi}(V) \cong \prod_{\mu \in \mathfrak{a}_\emptyset^*} \Gamma_{\mu}(J_{\emptyset,\Pi}(V)). \)
4. The homomorphism \( V \to \hat{J}_{\emptyset,\Pi}(V) \) is injective.

**Proof.** (1) Since both sides are decomposed into generalized \( \mathfrak{a}_\emptyset^* \)-weight spaces, it is sufficient to prove that the \( \mu \)-weight spaces of both sides are isomorphic for every \( \mu \in \mathfrak{a}_\emptyset. \) We have

\[
\Gamma_{\mu}(J_{\emptyset,\Pi}(V)) / \mathfrak{m}_{\emptyset}^k J_{\emptyset,\Pi}(V) \cong \Gamma_{\mu}(J_{\emptyset,\Pi}(V)) / \mathfrak{m}_{\emptyset}^k J_{\emptyset,\Pi}(V).
\]

By Lemma 3.6, we have \( \Gamma_{\mu}(J_{\emptyset,\Pi}(V)) = \Gamma_{\mu}(\hat{J}_{\emptyset,\Pi}(V)). \) We also have

\[
\Gamma_{\mu}(\mathfrak{m}_{\emptyset}^k J_{\emptyset,\Pi}(V)) = \sum_{\mu' + \mu'' = \mu} \Gamma_{\mu'}(\mathfrak{m}_{\emptyset}^k) \Gamma_{\mu''}(J_{\emptyset,\Pi}(V)) = \sum_{\mu' + \mu'' = \mu} \Gamma_{\mu'}(\mathfrak{m}_{\emptyset}^k) \Gamma_{\mu''}(\hat{J}_{\emptyset,\Pi}(V)) = \Gamma_{\mu}(\mathfrak{m}_{\emptyset}^k \hat{J}_{\emptyset,\Pi}(V)).
\]
We get (1).
(2) This follows from (1).
(3) By (2), it is sufficient to prove that for \( V \in \mathcal{H}C_\Theta, \tilde{J}_{\Theta, \Pi}(V) \simeq \prod_{\mu \in n_\Theta} \Gamma(\mu(V)). \) We can use the proof of \([GW80\text{ Lemma 2.2}]\).
(4) The kernel of \( V \to \tilde{J}_{\Theta, \Pi}(V) \) satisfies \( \operatorname{Ker}(m_\Theta \cap \bar{m}_\Theta) \operatorname{Ker} = 0. \) Therefore, it is sufficient to prove that for \( V \in \mathcal{H}C_\Theta \) with \( V \neq 0, \ V/(m_\Theta \cap \bar{m}_\Theta)V \neq 0. \) We prove \( V/\bar{m}V \neq 0. \) Put \( V' = V/n_\Theta V. \) Take a maximal \( a_\Theta \)-weight \( \mu' \) of \( V. \) Then \( \Gamma(\mu'(n_\Theta V)) = 0. \) Hence \( \Gamma(\mu'(V')) = \Gamma(\mu'(V)). \) In particular, \( V' \neq 0. \) By \([GW80\text{ Lemma 2.2}]\) \( V' \) is a Harish-Chandra module of \( L_{\Theta, \mathbb{R}}. \) By Casselman's subrepresentation theorem, we have \( V/\bar{m}V = V'/(m_\Theta \cap \bar{m})V' \neq 0. \)

Using (4), we regard \( V \) as a submodule of \( \tilde{J}_{\Theta, \Pi}(V). \) Let \( dv : \mathbb{C} \to \mathfrak{a} \) be the differential of \( \nu \) and put \( H = dv(1). \) This is an integral dominant element of \( \mathfrak{a}_\Theta. \) In general, for an \( \mathfrak{a}\)-module \( V, \) let \( \Gamma_H(V) \) be the generalized \( H \)-eigenspace of \( V \) with an eigenvalue \( a. \) For \( a, a' \in \mathbb{C}, \) we define \( a \geq a' \) by \( a - a' \in \mathbb{Z}_{\geq 0}. \) For \( V \in \mathcal{H}C_{\Theta, \rho} \) and \( a \in \mathbb{C}, \) define \( F_a(\tilde{J}_{\Theta, \Pi}(V)) \subset \tilde{J}_{\Theta, \Pi}(V) \) and \( F_aV \subset V \) by

\[
F_a(\tilde{J}_{\Theta, \Pi}(V)) = \prod_{a' \leq a} \Gamma_{H, a'}(\tilde{J}_{\Theta, \Pi}(V)), \quad F_aV = V \cap F_a(\tilde{J}_{\Theta, \Pi}(V)).
\]

Let \( \mathcal{D} \) be the ring of differential operators on \( G/B. \) Set \( \mathcal{V}' = \mathcal{D} \otimes_{U(\mathfrak{g})} V, \ \tilde{V} = a_\Theta^* \mathcal{V}' \) and \( \tilde{\mathcal{V}} = \Gamma(G/B \times \mathbb{G}_m, \tilde{\mathcal{V}}) = \mathbb{C}[t, t^{-1}] \otimes V. \) As in \([ENV04]\), we define the filtration \( \mathcal{V}^a(\tilde{\mathcal{V}}) \) on \( \tilde{\mathcal{V}} \) by

\[
\mathcal{V}^a(\tilde{\mathcal{V}}) = \bigoplus_{k \in \mathbb{Z}} t^k F_{a+k}(V).
\]

**Lemma 4.4.** We have

\[
F_{-a}(V)/F_{-a-1}(V) \xrightarrow{\sim} F_{-a}(\tilde{J}_{\Theta, \Pi}(V))/F_{-a-1}(\tilde{J}_{\Theta, \Pi}(V)) \simeq \bigoplus_{\mu(H) = a} \Gamma_\mu(J_{\Theta, \Pi}(V))
\]

**Proof.** We prove that the first homomorphism is isomorphic. By the definition of \( F_{-a-1}(V), \) the homomorphism is injective. This homomorphism is surjective by \( \text{Lemma 3.7} \) The second homomorphism is obviously an isomorphism.

From this lemma, if we prove that \( \mathcal{V}^a(\tilde{\mathcal{V}}) \) is a \( V \)-filtration, then by the description of the nearby cycle functor in terms of \( \mathcal{D}\)-modules \([Kas83\text{ (see }ENV04\text{ 3)}] \) and \( \text{Lemma 3.6} \) we have \( \Gamma(G/B, R\tilde{\mathcal{V}}) = J_{\Theta, \Pi}(V). \) Hence Theorem 4.2 is proved.

To prove that this gives a \( V \)-filtration, it is sufficient to prove the following lemma. (See \([ENV04\text{ 4]}]\.) Define a filtration \( F_a(U(\bar{m}_\Theta)) \) by

\[
F_a(U(\bar{m}_\Theta)) = \bigoplus_{a' \leq a} \Gamma_{H, a'}(U(\bar{m}_\Theta)).
\]

**Lemma 4.5.** For \( a \in \mathbb{C} \) and \( k, l \in \mathbb{Z}, \) the following hold.

1. For a sufficiently large \( k, F_{-a+k}(V) \) is stable.
2. The module \( F_{-a-k}(V)/(F_{-k}(U(\bar{m}_\Theta))F_{-a}(V)) \) is a finitely generated \( U(\bar{m}_\Theta) \)-module.
3. For \( l \geq 0, \) we have \( F_{-a-k-l}(V) = F_{-l}(U(\bar{m}_\Theta))F_{-a-k}(V) \) for a sufficiently large \( k. \)
Proof (1) Take a finite subset $\Lambda \subset \alpha_i$ such that $\text{wt}_{\alpha_i}(\tilde{J}_{\Theta, \Pi}(V)) \subset \Lambda - Z_{\geq 0}\Pi|_{\alpha_i}$. Since $H$ is dominant integral, for all $\mu \in \text{wt}_{\alpha_i}(\tilde{J}_{\Theta, \Pi}(V))$ we have $\mu'(H) - \mu(H) \in Z_{\geq 0}$ for some $\mu' \in \Lambda$. Take $k$ such that $-a + k \geq \max\{\mu'(H) \mid \mu' \in \Lambda\}$. For such $k$, $F_{-a+k}(V)$ is stable.

(2, 3) We can use the same proof as that of [ENV04, Lemma 2.5].

From this, $V^*(\tilde{V})$ is a $V$-filtration. Hence we get Theorem 4.2.

5. Symmetric space

Assume that $G$ is of adjoint type. Let $\omega_\alpha$ be the fundamental coweight for $\alpha \in \Pi$, namely, it is a cocharacter $\omega_\alpha: \mathbb{G}_m \to G$ which satisfies $\langle \omega_\alpha, \alpha \rangle = 1$ and $\langle \omega_\alpha, \beta \rangle = 0$ for $\beta \in \Pi \setminus \{\alpha\}$. Since $G$ is of adjoint type, it exists. Define $\omega: (\mathbb{G}_m)^\Pi \to A$ by $(t_\alpha)_\alpha \mapsto \prod_{\alpha \in \Pi} \omega_\alpha(t_\alpha)$. Then $\omega$ gives an isomorphism.

De Concini and Procesi [DCPS83] constructed the wonderful compactification $X$ of $G/K$. This compactification satisfies the following conditions. Set $x_0 = K \in G/K$.

(C1) The variety $X$ is irreducible and proper smooth over $\mathbb{C}$.
(C2) A $G$-orbit of $X$ is parameterized by a subset of $\Pi$. We denote the $G$-orbit corresponding to $\Theta \subset \Pi$ by $X_\Theta$.
(C3) The $G$-orbit $X_\Pi$ is the unique open $G$-orbit and it is isomorphic to $G/K$.
(C4) The closure of each orbit is smooth.
(C5) We have an $\mathbb{N}$-equivariant open embedding $\overline{\mathbb{N}} \times A^\Pi \to X$ such that for all $a = (a_\alpha) \in (\mathbb{G}_m)^\Pi$, an element $\omega(a)t_0$ is the image of $(1, (a_\alpha^{-2})) \in \overline{\mathbb{N}} \times A^\Pi$.

Moreover, the intersection of $X_\Theta$ and $\overline{\mathbb{N}} \times A^\Pi$ is given by $\overline{\mathbb{N}} \times (\mathbb{G}_m)^\Theta \times \{0\}^\Pi \setminus \Theta$.

(C6) By the above condition, $\overline{\mathbb{N}} \times A^\Pi$ is regarded as an open subvariety of $X$.

Then the stabilizer of $(1, (1^\Theta, 0^\Pi \setminus \Theta))$ in $G$ is $K_\Theta A_\Theta N_\Theta$.

Remark 5.1. The parameterization of $G$-orbits in [DCPS83] is different from ours. In [DCPS83], the open orbit corresponds to $\Theta$.

For each $\alpha \in \Pi$, put $Y_\alpha = \overline{X_{\Pi \setminus \{\alpha\}}}$. By the conditions (C4) and (C5), $\bigcup_{\alpha \in \Pi} Y_\alpha$ is a strict normal crossing divisor. Let $f: \mathcal{X} \to A^\Pi$ be the variety constructed in Section 2 with respect to $\bigcup_{\alpha \in \Pi} Y_\alpha$. Each $Y_\alpha$ defines the subvariety $Y_\alpha \subset \mathcal{X}$. Put $X' = \mathbb{N} \times A^\Pi$ and regard it as an open subvariety of $X$ by the condition (C5). Then $X'$ defines an open subvariety $\mathcal{X}'$ of $\mathcal{X}$. Since $Y_\alpha \cap X'$ is isomorphic to $\{((\pi, (e_\beta)_{\beta \in \Pi}) \mid e_\alpha = 0\}$, $X'$ is isomorphic to $\overline{\mathbb{N}} \times A^\Pi \times A^\Pi$ by Lemma 2.4 (1).

Put $Z = X \setminus \bigcup_{\alpha \in \Pi} Y_\alpha$ and $Z' = X' \cap Z$. Let $f_Z: Z \to A^\Pi$. Then we have $Z' \simeq \overline{\mathbb{N}} \times (\mathbb{G}_m)^\Pi \times A^\Pi$. Define a section $s: A^\Pi \to Z$ of $f_Z: Z \to A^\Pi$ by $s(t) = (1, 1^\Pi, t) \in \overline{\mathbb{N}} \times (\mathbb{G}_m)^\Pi \times A^\Pi \simeq Z' \subset Z$. For $\Theta \subset \Pi$, set $t_{\Theta} = (1^\Theta, 0^\Pi \setminus \Theta)$ and $x_{\Theta} = s(t_{\Theta})$.

Lemma 5.2. For $(a_\alpha)_\alpha \in (\mathbb{G}_m)^\Pi$, the action of $\omega(a_\alpha) \in A$ on $\mathcal{X}' \simeq \overline{\mathbb{N}} \times A^\Pi \times A^\Pi$ is given by $(\pi, d, t) \mapsto (\text{Ad}(\omega(a_\alpha))\pi, (a_\alpha^{-2})d, t)$.

Proof. This follows from (C5) and Lemma 2.4 (3).□

By Lemma 2.3, we have $X' \simeq T_{x_{\Theta}}(X) \times (\mathbb{G}_m)^\Theta$. For each subvariety $W \subset \mathcal{X}$, put $W_{\Theta} = W \cap X_{\Theta}$.

Lemma 5.3. The open subvariety $Z_{\Theta} \subset X_{\Theta}$ is contained in $T_{x_{\Theta}}(X) \times (\mathbb{G}_m)^\Theta$.□
Proof. By Lemma 2.3 we have \( X_\Theta \simeq T_{X_\Theta}(X) \times (\mathbb{G}_m)^\Theta \) and \( (Y_\alpha)_\Theta = T_{Y_\alpha}(Y_\alpha) \times (\mathbb{G}_m)^\Theta \). For \( \alpha \in \Theta \), we have an obvious identity \( T_{X_\Theta}(X)|_{Y_\alpha \cap X_\Theta} = T_{Y_\alpha}(Y_\alpha) \). Thus we have \( (X' \setminus \bigcup_{\alpha \in \Theta} Y_\alpha)_\Theta \simeq T_{X_\Theta}(X' \setminus \bigcup_{\alpha \in \Theta} Y_\alpha) \times (\mathbb{G}_m)^\Theta = T_{X_\Theta}(X) \times (\mathbb{G}_m)^\Theta \). Hence \( Z_\Theta \subset X_\Theta \times (\mathbb{G}_m)^\Theta \). \( \Box \)

Lemma 5.4. The stabilizer of \( x_\Theta \) in \( G \) is \( K_\Theta N_\Theta \).

Proof. Consider the morphism \( Z_\Theta \to T_{X_\Theta}(X) \times (\mathbb{G}_m)^\Theta \to T_{X_\Theta}(X) \to X_\Theta \). Let \( y_\Theta \) be the image of \( x_\Theta \). Then \( y_\Theta = (1, (1^\Theta, 0^\Theta)) \in X \times K_\Theta \) by Lemma 2.3. Hence \( Stab_G(x_\Theta) \subset Stab_G(y_\Theta) = K_\Theta A_\Theta N_\Theta \) by (C2).

We prove \( K_\Theta N_\Theta \subset Stab_G(x_\Theta) \). By Lemma 2.3 we have \( X_\Theta \simeq X \times (\mathbb{G}_m)^\Theta \). Then \( s(t^2) \) is given by \( (\omega(t)^{-1}x_\Theta, t^2) \in X \times (\mathbb{G}_m)^\Theta \) for \( t \in (\mathbb{G}_m)^\Theta \) (cf. Lemma 2.4 (1)). Hence \( Stab_G(s(t^2)) = Ad(\omega(t)^{-1})K_\Theta \). Its Lie algebra is spanned by \( m \) and \( \{ Ad(\omega(t)^{-1})(X + \beta(X)) \mid X \in g_\beta, \beta \in \Sigma^+ \} \). Here, \( g_\beta \) is the root space for \( \beta \). Since \( Ad(\omega(t)^{-1})(X + \beta(X)) = \beta(\omega(t)^{-1})(X + \beta(\omega(t)^{-1})\beta(X)) \), the Lie algebra of \( Stab_G(s(t^2)) \) is spanned by \( m \) and \( \{ X + \beta(\omega(t)^{-1})\beta(X) \mid X \in g_\beta, \beta \in \Sigma^+ \} \). If \( \beta = \sum_{\alpha \in \Pi} n_\alpha \alpha, \) then \( \beta(\omega(t)^{-1}) = \sum_{\alpha \in \Pi} t^{2n_\alpha} \) for \( t = (t_\alpha) \in (\mathbb{G}_m)^\Theta \). Hence the Lie algebra of \( Stab_G(s(t^2)) \) contains Lie \( (K_\Theta N_\Theta) \). Therefore, \( Stab_G(x_\Theta) \supset (K_\Theta N_\Theta) \). Since \( K = MK_\Theta N_\Theta \) and \( M \) stabilizes \( t \) for all \( t \in (\mathbb{G}_m)^\Theta \), we have \( Stab_G(x_\Theta) \supset M(K_\Theta N_\Theta) = K_\Theta N_\Theta \).

Finally, we prove that \( Stab_{A_\Theta}(x_\Theta) \subset M \). Since \( M \subset K_\Theta \), this implies the lemma. For \( (a_\alpha) \in (\mathbb{G}_m)^\Theta \), we have \( \omega(a_\alpha)x_\Theta = (1, (a_\alpha^2, t_\Theta)) \in X \times K_\Theta \times \mathbb{A}^\Pi \simeq X' \). Hence if \( \omega(a_\alpha) \in Stab_{A_\Theta}(x_\Theta) \), then \( a_\alpha^2 = 1 \) for all \( \alpha \in \Pi \). Therefore, \( \omega(a_\alpha) \in M \).

Lemma 5.5. We have \( Gx_\Theta \) is a subvariety of \( X \) by \( T_{X_\Theta}(X) = T_{X_\Theta}(X) \times \{ 1^\Theta \} \subset T_{X_\Theta}(X) \times (\mathbb{G}_m)^\Theta \). By Lemma 5.3 we have \( f^{-1}(t_\Theta) = T_{X_\Theta}(X) \cap Z \). Let \( p: T_{X_\Theta}(X) \to X_\Theta \) be the projection. Then \( p(x_\Theta) = y_\Theta, \) where \( y_\Theta \) is given in the proof of the previous lemma. Since \( X_\Theta \) is a \( G \)-orbit, we have \( G_{y_\Theta} = X_\Theta \). Hence \( X_\Theta \) is a subvariety of \( X \). By (C6) it is equivalent to proving that \( Stab_G(y_\Theta) \) is a subvariety of \( Z \). By (C6) and Lemma 5.4 it is equivalent to proving that \( A_\Theta \subset M \). Since \( M \subset K_\Theta \), this implies the lemma. Hence the lemma follows from Lemma 5.5.

In general, for an algebraic group \( H \) and an \( H \)-variety \( Y \), let \( \text{Perv}_H(Y) \) be the category of \( H \)-equivariant perverse sheaves. Then \( \mathcal{H}_{\Theta, \rho} \simeq \text{Perv}_{K_\Theta N_\Theta}(G/B) \simeq \text{Perv}_{K_\Theta N_\Theta} \circ \mathcal{H}_{\rho} \simeq \text{Perv}_{M(K_\Theta N_\Theta)} \circ \mathcal{H}_{\rho} \). Write \( \Delta_\Theta \) for this equivalence. Let \( \Theta_1 \subset \Theta_2 \subset \Theta_2 \subset \Theta_1 \subset \Pi \). Take \( n_\alpha \in \mathbb{Z}_{\geq 1} \) for each \( \alpha \in \Theta_1 \setminus \Theta_2 \) and define \( \nu: \mathbb{A}^1 \to \mathbb{A}^1 \) by \( \nu(t) = (t^{n_\alpha})_{\alpha \in \Theta_1 \setminus \Theta_2} \times (1^\Theta) \). Put \( Z_\nu = Z \times \mathbb{A}^1 \) and denote the canonical morphism \( Z_\nu \to \mathbb{A}^1 \). Then, by Lemma 5.4 and Lemma 5.5 we have \( f^{-1}_{\nu}(0) \cong G/K_\Theta N_\Theta \). Since \( f^{-1}_{\nu}(G) \congruent G/K_\Theta N_\Theta \times \mathbb{G}_m \). Let \( p_{\nu}: f^{-1}_{\nu}(G) \congruent G/K_\Theta N_\Theta \times \mathbb{G}_m \to G/K_\Theta N_\Theta \) be the first projection and \( R_\psi \) be the nearby cycle functor with respect to \( f_\nu \). Define \( \text{Kat}_{\nu}: \text{Perv}_B(G/K_\Theta N_\Theta) \to \text{Perv}_B(G/K_\Theta N_\Theta) \) by \( \text{Kat}_{\nu} = R_\psi \circ R_{\nu} \). Now we prove the main theorem of this paper.
Theorem 5.6. As functors $\mathcal{H}C_{\Theta_1, \rho} \to \text{Perv}_B(G/K\Theta_2 N_{\Theta_2})$, we have $\text{Kat}_\nu \circ \Delta'_{\Theta_1} \simeq \Delta'_{\Theta_2} \circ J_{\Theta_2, \Theta_1}$.

Proof. Let $s_\nu : \mathbb{A}^1 \to Z_\nu$ be the section of $f_\nu$ obtained by the base change of $s$ under $\nu$. Consider the following diagram:

$$
\begin{array}{ccccccccc}
G/B & \overset{\rho \mapsto (g,t) \mapsto (g, t_\nu)}{\longrightarrow} & G/B \times \mathbb{G}_m & \overset{\nabla \mapsto (g,t) \mapsto (g, t_\nu)}{\longrightarrow} & G/B \times \mathbb{A}^1 & \overset{\nabla \mapsto (g,t) \mapsto (g, t_\nu)}{\longrightarrow} & G/B \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
B\backslash G & \overset{(g,t) \mapsto (\omega_\nu(t))^{-1}(g,t)}{\longrightarrow} & B\backslash G \times \mathbb{G}_m & \overset{(g,t) \mapsto g_{\nu}(t_\nu)}{\longrightarrow} & B\backslash G \times \mathbb{A}^1 & \overset{(g,t) \mapsto g_{\nu}(t_\nu)}{\longrightarrow} & B\backslash G \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
G & \overset{(g,t) \mapsto \nu(t)}{\longrightarrow} & G \times \mathbb{G}_m & \overset{(g,t) \mapsto \nu(t)}{\longrightarrow} & G \times \mathbb{A}^1 & \overset{(g,t) \mapsto \nu(t)}{\longrightarrow} & G \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
G/K\Theta_1 N_{\Theta_1} & \overset{(g,t) \mapsto \nu(t)}{\longrightarrow} & G/K\Theta_1 N_{\Theta_1} \times \mathbb{G}_m & \overset{(g,t) \mapsto \nu(t)}{\longrightarrow} & Z_\nu & \overset{(g,t) \mapsto \nu(t)}{\longrightarrow} & G/K\Theta_2 N_{\Theta_2} \\
\end{array}
$$

Every rectangle in the diagram above is cartesian, and every vertical arrow is smooth. The functor $\Psi_\nu$ of Emerton-Nadler-Vilonen is defined as the nearby cycle functor with respect to the top row, and the functor $\text{Kat}_\nu$ is defined as the nearby cycle functor with respect to the bottom row. Therefore, we get our theorem by Theorem 4.2 and the smooth base change theorem. □

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