On the nearly freeness of conic-line arrangements with nodes, tacnodes, and ordinary triple points

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Abstract
In the present note, we provide a partial classification of nearly free conic-line arrangements in the complex plane having nodes, tacnodes, and ordinary triple points. In this setting, our theoretical bound tells us that the degree of such an arrangement is bounded from above by 12. We construct examples of nearly free conic-line arrangements having degree 3, 4, 5, 6, 7, and we prove that in degree 10, 11, and 12, there is no such arrangement.

Keywords Nearly free curves · Conic-line arrangements

Mathematics Subject Classification 14C20 · 32S22

1 Introduction
The theory of plane curve arrangements has recently gained a lot of attention among researchers. It is worth recalling some recent papers devoted to rational curve arrangements in the complex projective plane, for instance [10–12]. One of the most interesting open questions that appears in the literature is the so-called numerical Terao’s conjecture. There are some variants of this conjecture, but we focus here on the following. Let $C, C' \subset \mathbb{P}^2_C$ be two reduced curves such that they have the same weak combinatorics, i.e., they have the same number of irreducible components of the same degree, and the same number of singularities of a given topological type. Assume that $C'$ is free, then $C$ has to be free. For instance, Pokora and Dimca in [6] proved that the numerical Terao’s conjecture holds in the class of conic-line arrangements with nodes, tacnodes, and ordinary triple points. On the other hand, it is
known that numerical Terao’s conjecture does not hold in the class of (triangular) line arrangements [8]. From this perspective, it is natural to understand wider classes of curves and hopefully this effort will help us to understand numerical Terao’s conjecture better. In the light of the original Terao’s conjecture for line arrangements, which tells us that the freeness is determined by the intersection poset, Dimca and Sticlaru in [3] defined a new class of curves that is called nearly free. Here our aim is to understand nearly free complex conic-line arrangement with nodes, tacnodes, and ordinary triple points. Our motivation comes from aforementioned paper by Dimca and Pokora [6], where the authors classify all free conic-line arrangement with nodes, tacnodes, and ordinary triple points. Let us briefly present the main outcome of the note. First, we observe in Proposition 4.2 that if $C$ is a conic-line arrangement in the complex plane with nodes, tacnodes, and ordinary triple points, then its degree is bounded from above by 12. Then we start to analyse, case by case, in which degree we can find a nearly free conic-line arrangement—it turns out that we can find such arrangements in degree 3, 4, 5, 6, 7. However, for the degree 10, 11, and 12, we show that there does not exist any conic-line arrangement in the complex plane with the prescribed above singularities that is nearly free. Based on the above discussion, one needs to decide the existence of nearly free conic-line arrangements in degree 8, 9.

Through the paper, we work exclusively over the complex numbers. Our computations are performed with Singular [1].

2 Conic-line arrangements with nodes, tacnodes, and ordinary triple points

Let $C \subset \mathbb{P}^2_C$ be an arrangement consisting of $d \geq 1$ lines and $k \geq 1$ smooth conics. We assume that $C$ has only $n_2$ nodes, $t$ tacnodes, and $n_3$ ordinary triple points. Denote by $m$ the degree of $C$ which is equal to $m = 2k + d$. Then we have the following combinatorial count

$$4 \cdot \binom{k}{2} + 2kd + \binom{d}{2} = \binom{m}{2} - k = n_2 + 2t + 3n_3,$$

and this formula follows from Bézout theorem.

Focusing on simple singularities like nodes, tacnodes, and ordinary triple points is the first non-trivial case when non-ordinary singularities occurs. For the discussion, we will need also the notion of the global and local Tjurina numbers of singularities.

**Definition 2.1** Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be the germ of an isolated singularity. Then the (local) Tjurina number at $p = (0, 0)$ is defined as

$$\tau_p := \dim_{\mathbb{C}} \frac{\mathbb{C}\{x, y\}}{\langle f, \partial_x f, \partial_y f \rangle}.$$ 

The total Tjurina number of a reduced curve $C$ is defined as
\[ \tau(C) := \sum_{p \in \text{Sing}(C)} \tau_p, \]

where the sum goes over all singular points of \( C \).

**Remark 2.2** Let us recall that if \( p \) is a node, then \( \tau_p = 1 \), if \( q \) is a tacnode, then \( \tau_q = 3 \), and if \( r \) is an ordinary triple point, then \( \tau_r = 4 \). Based on it, if \( C \) is a conic-line arrangement with \( n_2 \) nodes, \( t \) tacnodes, and \( n_3 \) ordinary triple points, then

\[ \tau(C) = n_2 + 3t + 4n_3. \]

### 3 Nearly freeness of reduced curves

Let \( C \) be a reduced curve \( \mathbb{P}^2_\mathbb{C} \) of degree \( m \) given by \( f \in S := \mathbb{C}[x, y, z] \). We denote by \( J_f \) the Jacobian ideal generated by the partial derivatives \( \partial_x f, \partial_y f, \partial_z f \). Moreover, we denote by \( r := \text{mdr}(f) \) the minimal degree of a non-trivial relation among the partial derivatives, i.e., the minimal degree \( r \) of a triple \( (a, b, c) \in S^3 \) such that

\[ a \cdot \partial_x f + b \cdot \partial_y f + c \cdot \partial_z f = 0. \]

We denote by \( m = \langle x, y, z \rangle \) the irrelevant ideal. Consider the graded \( S \)-module \( N(f) = I_f / J_f \), where \( I_f \) is the saturation of \( J_f \) with respect to \( m \).

**Definition 3.1** A reduced plane curve \( C \) is nearly free if \( N(f) \neq 0 \) and for every \( k \) one has \( \dim N(f)_k \leq 1 \).

In order to study nearly freeness of conic-line arrangements, we will use [2, Theorem 1.3].

**Theorem 3.2** (Dimca) Let \( C \subset \mathbb{P}^2_\mathbb{C} \) be a conic-line arrangement of degree \( m \) and let \( f = 0 \) be its defining equation. Denote by \( r := \text{mdr}(f) \). Then \( C \) is nearly free if and only if

\[ r^2 - r(m - 1) + (m - 1)^2 = \tau(C) + 1, \quad (1) \]

where \( \tau(C) \) is the total Tjurina number of \( C \).

**Remark 3.3** In the original formulation of the above result there was the assumption that \( r \leq m/2 \). However, it turns out that it is not necessary, and this follows from [7, Theorem 3.2].

Now we are going to discuss the freeness from the homological viewpoint. We need the following result which comes from [5].

**Theorem 3.4** (DimcaSticlaru) If \( C \subset \mathbb{P}^2_\mathbb{C} \) is a nearly free reduced curve of degree \( m \) given by \( f \in S_m \), then the minimal resolution of the Milnor algebra \( M(f) \) has the following form:
0 \to S(-d_2 - m) \to S(-d_1 - (m - 1)) \oplus S(-d_2 - (m - 1)) \oplus S(-d_2 - (m - 1)) \\
\quad \to S^3(-m + 1) \to S \to M(f) \to 0

for some integers \(d_1 \leq d_2\) such that \(d_1 + d_2 = m\).

In the setting of the above theorem, the pair \((d_1, d_2)\) is called the set of exponents of the nearly free curve \(C\).

### 4 Nearly free arrangements of conic and lines with nodes, tacnodes, and triple points

To understand conic-line arrangements with nodes, tacnodes, and ordinary triple points that are nearly free, we provide an upper-bound on the degree of such arrangements. We need the following result [6, Proposition 4.7].

**Proposition 4.1** Let \(C : f = 0\) be a conic-line arrangement of degree \(m\) in \(\mathbb{P}^2\) such that it has only nodes, tacnodes, and ordinary triple intersection points. Then one has

\[
\operatorname{mdr}(f) \geq \frac{2}{3} m - 2.
\]

If \(C : f = 0\) is a nearly free conic-line arrangement of degree \(m\) with nodes, tacnodes, and ordinary triple intersection points with the exponents \((d_1, d_2)\), \(d_1 \leq d_2\), then \(\operatorname{mdr}(f) = d_1\), and since

\[
2d_1 \leq d_1 + d_2 = m
\]

we obtain that \(\operatorname{mdr}(f) \leq m/2\). Combining it with the above proposition, we arrive at

\[
\frac{2}{3} m - 2 \leq \operatorname{mdr}(f) \leq m/2.
\]

It gives us the following result.

**Proposition 4.2** If \(C \subset \mathbb{P}^2\) is a nearly free conic-line arrangement of of degree \(m\) with nodes, tacnodes, and ordinary triple intersection points, then \(m \leq 12\).

Based on the above proposition, we can formulate the following problem.

**Problem 4.3** Classify all weak combinatorics of conic-line arrangements with nodes, tacnodes, and ordinary triple points in \(\mathbb{P}^2\) that are nearly free.

Here by a weak combinatorics we mean the vector \((d, k; n_2, t, n_3)\), where \(d\) is the number of lines, \(k\) is the number of conics (and of course \(m = 2k + d\)).

Let us pass to some combinatorial constraints on the singular points of such conic-line arrangements that come from the data of the exponents \((d_1, d_2)\). If \(C\) is a nearly
free conic-line arrangement with $k$ conics and $d$ lines, $d_1 + d_2 = m$, then following equality holds

$$d_1^2 + d_2^2 + d_1d_2 - d_1 - 2d_2 = n_2 + 3t + 4n_3. \quad (2)$$

Since $\left(\frac{d_1 + d_2}{2}\right)^2 - k = n_2 + 2t + 3n_3$, we can obtain the following equality

$$t + n_3 = d_1^2 + d_2^2 + d_1d_2 - d_1 - 2d_2 - \left(\frac{d_1 + d_2}{2}\right)^2 + k.$$ 

This gives

$$2(t + n_3) = d_1^2 + d_2^2 - d_1 - 3d_2 + 2k. \quad (3)$$

The last ingredient that we will use in our classification problem is the following proposition which is a direct consequence of the previously known results from [6, 9].

**Proposition 4.4** Let $\mathcal{C}$ be a conic-line arrangement in $\mathbb{P}_C^2$ of degree $m = 2k + d \geq 6$ having only $n_2$ nodes, $t$ tacnodes, and $n_3$ ordinary triple points. Then we have the following inequality

$$8k + n_2 + \frac{3}{4}n_3 \geq d + \frac{5}{2}t.$$ 

**Proof** It follows from the discussions in the framework of [6, Theorem 2.1] and [9, Theorem B]. \hfill \Box

## 5 Partial classification of nearly free conic-line arrangements

Here we perform a step-by-step approach towards the classification problem of our nearly free conic-line arrangements in the complex projective plane having $k \geq 1$ conics and $d \geq 1$ lines. We start with constructing explicit examples of nearly free conic-line arrangements having degree up to 7. Then we present our argument standing behind the non-existence of nearly free conic-line arrangement with nodes, tacnodes, and ordinary triple points having degree $m \in \{10, 11, 12\}$. Unfortunately, our method does not allow us to decide on the non-existence of conic-line arrangements having degree $m \in \{8, 9\}$, but we hope to resolve that problem using a different approach.

### 5.1 Case $m = 3$

Let us consider the following arrangement $\mathcal{C}_3 = \{\ell, \mathcal{C}\} \subset \mathbb{P}_C^2$ defined by
\[
F(x,y,z) = (x^2 + y^2 - 16z^2) \cdot (y - x + 4z).
\]

It is easy to see that \(C_3\) has only \(n_2 = 2\) and \(t = n_3 = 0\), so its total Tjurina number \(\tau(C_3)\) is equal to 2. Using \texttt{Singular}, we can compute \(\text{mdr}(F)\) which is equal to 1.

By Theorem 3.2 we see that
\[
3 = 1 - 2 + 4 = r^2 - r(m - 1) + (m - 1)^2 = \tau(C_3) + 1 = 2 + 1,
\]
so \(C_3\) is nearly free. Observe that this is the only possible nearly free conic-line arrangement with nodes, tacnodes, ordinary triple points and \(m = 3\). Since \(d_1 + d_2 = 3\) and \(d_1 \leq d_2\), we can have only \((d_1, d_2) = (1, 2)\). Recall that \(\text{mdr}(f) = d_1 = 1\), and
\[
3 = 1^2 - 1 \cdot (3 - 1) + (3 - 1)^2 = \tau(C) + 1.
\]
It means that \(\tau(C) = 2\), and the only possibility is to have \(n_2 = 2\) and \(t = 0\), which completes our justification (Fig. 1).

### 5.2 Case \(m=4\)

Now we consider the following arrangement \(C_4 = \{\ell_1, \ell_2, C\} \subset \mathbb{P}_C^2\) given by
\[
F(x,y,z) = (x^2 + y^2 - 16z^2) \cdot (y - x - 4z) \cdot (y + x - 4z).
\]

It is easy to see that for \(C_4\) we have \(n_2 = 2\) and \(n_3 = 1\), so its total Tjurina number \(\tau(C_4)\) is equal to 6. Using \texttt{Singular}, we can compute \(\text{mdr}(F)\) that is equal to 2.

Using Theorem 3.2, we see that
\[
7 = 4 - 6 + 9 = r^2 - r(m - 1) + (m - 1)^2 = \tau(C_4) + 1 = 6 + 1,
\]
so \(C_4\) is nearly free (Fig. 2).

Now we show that for \(m = 4\), there is another possible nearly free conic-line arrangement. Observe that for \(m = 4\), we have two possible pairs of the exponents, namely \((d_1, d_2) = (1, 3)\) and \((d_1, d_2) = (2, 2)\), and since \(k \geq 1, d \geq 1\), our conic-line arrangement \(C\) consists of \(k = 1\) conics and \(d = 2\) lines. We see that for both possibilities of the exponents we have \(\tau(C) = 6\). The following weak combinatorics

\[\text{Fig. 1} \quad \text{A nearly free arrangement with one conic and one line}\]
Fig. 2 A nearly free arrangement of degree $m = 4$ with two nodes and one triple point

for conic line arrangements with nodes, tacnodes, and ordinary triple points and $m = 4$ are admissible:

| $n_2$ | $t$ | $n_3$ |
|-------|-----|-------|
| 5     | 0   | 0     |
| 3     | 1   | 0     |
| 1     | 2   | 0     |
| 2     | 0   | 1     |

Based on that, an arrangement with 3 double points and one tacnode can be also nearly free. Consider the arrangement $C_4' = \{\ell_1, \ell_2, C\} \subset \mathbb{P}_C^2$ given by

$$G(x, y, z) = (x^2 + y^2 - 16z^2) \cdot (y - 4z) \cdot (y - x).$$

Using Singular, we can compute that $\text{mdr}(G)$ is equal to 2 and

Fig. 3 A nearly free arrangement of degree $m = 4$ with three nodes and one tacnode
\[ 7 = 4 - 6 + 9 = r^2 - r(m - 1) + (m - 1)^2 = \tau(C'_4) + 1 = 3 + 3 \cdot 1 + 1, \]

so \(C'_4\) is nearly free (Fig. 3).

### 5.3 Case \(m = 5\)

We consider here the following arrangement \(C_5 = \{\ell_1, \ell_2, \ell_3, C\} \subset \mathbb{P}^2_C\) given by

\[
F(x,y,z) = (x^2 + y^2 - 16z^2) \cdot (y - x + 4z) \cdot (y + x - 4z) \cdot (y - x - 4z).
\]

It is easy to see that for \(C_5\), we have \(n_2 = 3\) and \(n_3 = 2\), so its total Tjurina number \(\tau(C_5)\) is equal to 11. Using \texttt{Singular}, we can compute \(\text{mdr}(F)\) which is equal to 2.

Using Theorem 3.2, we have

\[
12 = 4 - 8 + 16 = r^2 - r(m - 1) + (m - 1)^2 = \tau(C_5) + 1 = 11 + 1,
\]

so \(C_5\) is nearly free (Fig. 4).

### 5.4 Case \(m = 6\)

We consider here the following arrangement \(C_6 = \{\ell_1, \ell_2, \ell_3, \ell_4, C\} \subset \mathbb{P}^2_C\) given by

\[
F(x,y,z) = (x^2 + y^2 - 16z^2) \cdot (y - x + 4z) \cdot (y + x - 4z) \cdot (y - x - 4z) \cdot (y + x + 4z).
\]

It is easy to see that for \(C_3\) we have \(n_2 = 2\) and \(n_3 = 4\), so its total Tjurina number \(\tau(C_6)\) is equal to 18. Using \texttt{Singular}, we can compute \(\text{mdr}(F)\) which is equal to 2.

Using Theorem 3.2, we obtain

\[
19 = 4 - 10 + 25 = r^2 - r(m - 1) + (m - 1)^2 = \tau(C_6) + 1 = 18 + 1,
\]

so \(C_6\) is nearly free (Fig. 5).
5.5 Case $m = 7$

Let us consider here the following arrangement $C_7 = \{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, C\} \subset \mathbb{P}^2_C$ given by

$$F(x, y, z) = (x^2 + y^2 - z^2) \cdot (x^2 - z^2) \cdot (y^2 - z^2) \cdot (y + x).$$

It is easy to see that for $C_7$ we have $n_2 = 6$, $t = 4$, and $n_3 = 2$, so its total Tjurina number $\tau(C_7)$ is equal to 26. Using Singular, we can compute $\text{mdr}(F)$ that is equal to 3.

Using Theorem 3.2, we see

$$27 = 9 - 18 + 36 = r^2 - r(m - 1) + (m - 1)^2 = \tau(C_7) + 1 = 26 + 1,$$

so $C_7$ is nearly free (Fig. 6).

5.6 Case $m = 10$.

Assume that $C \subset \mathbb{P}^2_C$ is a nearly free conic-line arrangement with $m = 10$ having the exponents $(d_1, d_2)$, $d_1 \leq d_2$. By Proposition 4.1, we have

Fig. 6 A nearly free arrangement with one conic and five lines
\[ d_1 = \text{mdr}(f) \geq \frac{2}{3}m - 2 = \frac{20}{3} - 2 = 4 \frac{2}{3}, \]

which implies that \( d_1 \geq 5 \). However, it means that we need to consider the only one case, namely \( (d_1, d_2) = (5, 5) \). Using Eq. (3) we obtain

\[ t + n_3 = 15 + k. \]

and it means, in particular, that \( k \in \{1, 2, 3, 4\} \).

By the combinatorial count, we have

\[ \binom{10}{2} - k = n_2 + 2(t + n_3) + n_3, \]

so we arrive at

\[ 15 - 3k = n_2 + n_3, \]

and it means, in particular, that \( k \in \{1, 2, 3, 4\} \). Hence

\[ t = 15 + k - n_3 = 15 + k - (15 - 3k - n_2) = 4k + n_2. \]

Now we are going plug these data into inequality from Proposition 4.4.

We have

\[ 8k + n_2 + \frac{3}{4}(15 - 3k - n_2) \geq (10 - 2k) + \frac{5}{2}(4k + n_2). \]

After some simple manipulations we obtain

\[ 5 \geq 9k + 9n_2, \]

which is a contradiction since \( k \in \{1, 2, 3, 4\} \). This proves the following result.

**Theorem 5.1** There does not exists any nearly free conic-line arrangement in the complex projective plane with nodes, tacnodes, and ordinary triple points having degree \( m = 10 \).

**5.7 Case \( m = 11 \).**

Assume that \( C \subset \mathbb{P}_C^2 \) is a nearly free conic-line arrangement with \( m = 11 \) having the exponents \( (d_1, d_2), d_1 \leq d_2 \). By Proposition 4.1, we have that

\[ d_1 = \text{mdr}(f) \geq \frac{2}{3}m - 2 = \frac{22}{3} - 2 = 5 \frac{1}{3}, \]

so \( d_1 \geq 6 \). However, it means that such a nearly free curve cannot exists, and we have the following proposition.

**Proposition 5.2** There does not exists any nearly free conic-line arrangement in the complex projective plane with nodes, tacnodes, and ordinary triple points having degree \( m = 11 \).
5.8 Case $m = 12$.

Assume that $C \subset \mathbb{P}^2_C$ is a nearly free conic-line arrangement with $m = 12$ having the exponents $(d_1, d_2)$ with $d_1 \leq d_2$. Using Proposition 4.1, we see that

$$d_1 = \text{mdr}(f) \geq \frac{2}{3} m - 2 = \frac{2}{3} \cdot 12 - 2 = 6,$$

so the only one case to consider is $(d_1, d_2) = (6, 6)$. Using Eq. (3) we obtain

$$2(t + n_3) = 48 + 2k,$$

and plugging this into the combinatorial count we get

$$n_2 + n_3 = 18 - 3k.$$

In particular, $k \in \{1, 2, 3, 4, 5\}$.

Using Proposition 4.4, we have

$$5k + 18 \geq d + \frac{5}{2} t,$$

so finally

$$t \leq \frac{2}{5} \left(5k + 18 - d\right).$$

We have the following possibilities, depending on $k$, namely

| $k$ | $n_3$ | $t \leq$ |
|-----|------|---------|
| 1   | 15   | 5       |
| 2   | 12   | 8       |
| 3   | 9    | 11      |
| 4   | 6    | 14      |
| 5   | 3    | 16      |

Since in all the cases listed above we have $t + n_3 \leq 20$, we arrive at a contradiction with respect to the condition that $t + n_3 = 24 + k$. This allows us to conclude our discussion by the following result.

**Theorem 5.3** There does not exists any nearly free conic-line arrangement in the complex projective plane with nodes, tacnodes, and ordinary triple points having degree $m = 12$.

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