Research Article

Finite Iterative Algorithm for Solving a Complex of Conjugate and Transpose Matrix Equation

Mohamed A. Ramadan, 1 Talaat S. El-Danaf, 1 and Ahmed M. E. Bayoumi 2

1 Department of Mathematics, Faculty of Science, Menoufia University, Shebeen El-Koom, Egypt
2 Department of Mathematics, Faculty of Education, Ain Shams University, Cairo, Egypt

Correspondence should be addressed to Mohamed A. Ramadan; mramadan@eun.eg

Received 4 August 2012; Accepted 4 November 2012

Academic Editor: Franck Petit

Copyright © 2013 Mohamed A. Ramadan et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We consider an iterative algorithm for solving a complex matrix equation with conjugate and transpose of two unknowns of the form:

\[ A_1 V B_1 + C_1 W D_1 + A_2 V B_2 + C_2 W D_2 + A_3 V^H B_3 + C_3 W^H D_3 + A_4 V^T B_4 + C_4 W^T D_4 = E, \]

where \( A_1, A_2, C_1, C_2, A_3, A_4, C_3, C_4, E \in \mathbb{C}^{m \times r}, B_1, B_2, D_1, D_2 \in \mathbb{C}^{r \times s}, B_3, B_4, D_3, D_4 \in \mathbb{C}^{s \times t} \) are given matrices, while \( V, W \in \mathbb{C}^{r \times s} \) are matrices to be determined.

1. Introduction

Consider the complex matrix equation:

\[ A_1 V B_1 + C_1 W D_1 + A_2 V B_2 + C_2 W D_2 + A_3 V^H B_3 + C_3 W^H D_3 + A_4 V^T B_4 + C_4 W^T D_4 = E, \]

where \( A_1, A_2, C_1, C_2, A_3, A_4, C_3, C_4, E \in \mathbb{C}^{m \times r}, B_1, B_2, D_1, D_2 \in \mathbb{C}^{r \times s}, B_3, B_4, D_3, D_4 \in \mathbb{C}^{s \times t} \) are given matrices, while \( V, W \in \mathbb{C}^{r \times s} \) are matrices to be determined. In the field of linear algebra, iterative algorithms for solving matrix equations have received much attention. Based on the iterative solutions of matrix equations, Ding and Chen presented the hierarchical gradient iterative algorithms for general matrix equations [1, 2] and hierarchical least squares iterative algorithms for generalized coupled Sylvester matrix equations and general coupled matrix equations [3, 4]. The hierarchical gradient iterative algorithms [1, 2] and hierarchical least squares iterative algorithms [1, 4, 5] for solving general (coupled) matrix equations are innovative and computationally efficient numerical ones and were proposed based on the hierarchical identification principle [3, 6] which regards the unknown matrix as the system parameter matrix to be identified. Iterative algorithms were proposed for continuous and discrete Lyapunov matrix equations by applying the hierarchical identification principle [7]. Recently, the idea of the hierarchical identification was also utilized to solve the so-called extended Sylvester-conjugate matrix equations in [8]. From an optimization point of view, a gradient-based iteration was constructed in [9] to solve the coupled general matrix equation. A significant feature of the method in [9] is that a necessary and sufficient condition guaranteeing the convergence of the algorithm can be explicitly obtained.

Some complex matrix equations have attracted attention from many researchers since it was shown in [10] that the consistency of the matrix equation \( AX - \overline{XB} = C \) can be characterized by the consimilarity [11–13] of two partitioned matrices related to the coefficient matrices \( A, B, \) and \( C \). By consimilarity Jordan decomposition, explicit solutions were obtained in [10, 14]. Some explicit expressions of the solution to the matrix equation \( AX - \overline{XB} = C \) were established in [15], and it was shown that this matrix equation has a unique solution if and only if \( A \overline{A} \) and \( B \overline{B} \) have no common eigenvalues. Research on solving linear matrix equations has been actively engaged in for many years. For example, Navarra et al.
studied a representation of the general common solution of the matrix equations $A_1X B_1 = C_1$ and $A_2X B_2 = C_2$ [16]; Van der Woude obtained the existence of a common solution $X$ for the matrix equations $A_1X B_1 = C_1$ [17]; Bhimasankaram considered the linear matrix equations $AX = B$, $CX = D$, and $EXF = G$ [18]. Mitra has provided conditions for the existence of a solution and a representation of the general common solution of the matrix equations $AX = C$ and $XB = D$ and the matrix equations $A_1X B_1 = C_1$ and $A_2X B_2 = C_2$ [19, 20]. Ramadan et al. [21] introduced a complete, general, and explicit solution to the Yakubovich matrix equation $V − AVF = BW$, and the matrix equation $(AXB, GXH) = (C, D)$ has some important results that have been developed. In [22], necessary and sufficient conditions for its solvability and the expression of the solution were derived by means of generalized inverse. Moreover, in [22] the least-squares solution was also obtained by using the generalized singular value decomposition. While in [23], when this matrix equation is consistent, the minimum-norm solution was given by the use of the canonical correlation decomposition. In [24], based on the projection theorem in Hilbert space, an analytical expression of the least-squares solution was given for the matrix equation $(AXB, GXH) = (C, D)$ by making use of the generalized singular value decomposition and the canonical correlation decomposition. In [25], by using the matrix rank method a necessary and sufficient condition was derived for the matrix equations $AX, B = C$ and $GX, H = D$ to have a common least square solution. In the aforementioned methods, the coefficient matrices of the considered equations are required to be firstly transformed into some canonical forms. Recently, an iterative algorithm has presented in [26] to solve the matrix equation $(AXB, GXH) = (C, D)$. Different from the above mentioned methods, this algorithm can be implemented by initial coefficient matrices and can provide a solution within finite iterative steps for any initial values.

Very recently, in [27] a new operator of conjugate product for complex polynomial matrices is proposed. It is shown that an arbitrary complex polynomial matrix can be converted into the so-called Smith normal form by elementary transformations in the framework of conjugate product. Meanwhile, the conjugate product and the Sylvester-conjgate sum are also proposed in [28]. Based on the important properties of the above new operators, a unified approach to solve a general class of Sylvester-polynomial-conjugate matrix equations is given. The complete solution of the Sylvester-polynomial-conjugate matrix equation is obtained. In [29] by using a real inner product in complex matrix spaces, a solution can be obtained within finite iterative steps for any initial values in the absence of round-off errors. In [30] iterative solutions to a class of complex matrix equations are given by applying the hierarchical identification principle.

This paper is organized as follows. First, in Section 2, we introduce some notations, a lemma, and a theorem that will be needed to develop this work. In Section 3, we propose iterative methods to obtain numerical solution to the complex matrix equation with conjugate and transpose of two unknowns.

2. Preliminaries

The following notations, definitions, lemmas, and theorems will be used to develop the proposed work. We use $A^T$, $A^H$, $\text{tr}(A)$, and $|A|$ to denote the transpose, conjugate transpose, the trace, and the Frobenius norm of a matrix $A$, respectively. We denote the set of all $m \times n$ complex matrices by $C^{mn}$, and $\text{Re}(a)$ denote the real part of number $a$.

Definition 1 (inner product [31]). A real inner product space is a vector space $V$ over the real field $\mathbb{R}$ together with an inner product. That is, with a map

$$\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$$

Satisfying the following three axioms for all vectors $x, y, z \in V$ and all scalars $a \in \mathbb{R}$:

1. symmetry: $\langle x, y \rangle = \langle y, x \rangle$
2. linearity in the first argument:

$$\langle ax + y, z \rangle = a \langle x, z \rangle + \langle y, z \rangle$$
3. positive definiteness: $\langle x, x \rangle > 0$ for all $x \neq 0$,

Two vectors $u, v \in V$ are said to be orthogonal if $\langle u, v \rangle = 0$.

The following theorem defines a real inner product on space $C^{mn}$ over the field $\mathbb{R}$.

Theorem 2 (see [32]). In the space $C^{mn}$ over the field $\mathbb{R}$, an inner product can be defined as

$$\langle A, B \rangle = \text{Re} \left[ \text{tr} \left( A^H B \right) \right]$$

3. The Main Result

In this section, we propose an iterative solution to a complex matrix equation with conjugate and transpose of two unknowns:

$$A_1 V B_1 + C_1 W D_1 + A_2 V B_2 + C_2 W D_2 + A_3 V^H B_3$$
$$+ C_3 W^H D_3 + A_4 V^T B_4 + C_4 W^T D_4 = E$$

Defined in (1) where $A_1, A_2, C_1, C_2 \in C^{m \times r}$, $B_1, B_2, D_1, D_2 \in C^{n \times s}$, $A_3, A_4, C_3, C_4 \in C^{n \times r}$, $E \in C^{m \times s}$, and $B_3, B_4, D_3, D_4 \in C^{s \times r}$ are given matrices, while $V, W \in C^{r \times s}$ are matrices to be determined.

The following finite iterative algorithm is presented to solve it.

Algorithm 3. (1) Input $A_1, A_2, C_1, C_2, B_1, B_2, D_1, D_2, A_3, A_4, C_3, C_4, B_3, B_4, D_3, D_4, E$;
(2) Chosen arbitrary matrices $V_1, W_1 \in C^{r \times s}$;
(3) Set

\[ R_1 = E - A_1 V_1 B_1 - C_1 W_1 D_1 - A_2 V_2 B_2 - C_2 W_2 D_2 - A_3 V_1^H B_3 - C_3 W_1^H D_3 - A_4 V_4^H B_4 - C_4 W_4^H D_4, \]

\[ P_1 = A_1^H R_1^1 B_1^H + \frac{A_2^H R_{k+1}^1}{2} B_2^H + B_3 R_{k+1}^H A_3 + \frac{B_4 R_{k+1}^1 T}{A_4}, \]

\[ Q_1 = C_1^H R_1^1 D_1^H + \frac{C_2^H R_{k+1}^1 D_2^H}{2} + D_3 R_{k+1}^H C_3 + \frac{D_4 R_{k+1}^H T}{C_4}, \]

\[ k = 1; \]

(4) If \( R_k = 0 \), then stop; else go to Step 5;

(5) Set

\[ V_{k+1} = V_k + \frac{\|R_k\|^2}{\|P_k\|^2 + \|Q_k\|^2} P_k, \]

\[ W_{k+1} = W_k + \frac{\|R_k\|^2}{\|P_k\|^2 + \|Q_k\|^2} Q_k, \]

\[ R_{k+1} = E - A_1 V_{k+1} B_1 - C_1 W_{k+1} D_1 - A_2 V_{k+1} B_2 - C_2 W_{k+1} D_2 - A_3 V_{k+1}^H B_3 - C_3 W_{k+1}^H D_3 - A_4 V_{k+1}^H B_4 - C_4 W_{k+1}^H D_4, \]

\[ P_{k+1} = A_1^H R_{k+1} B_1^H + \frac{A_2^H R_{k+1}^1}{2} B_2^H + B_3 R_{k+1}^H A_3 + \frac{A_4 R_{k+1}^H}{P_k}, \]

\[ Q_{k+1} = C_1^H R_{k+1}^1 D_1^H + \frac{C_2^H R_{k+1}^1 D_2^H}{2} + D_3 R_{k+1}^H C_3 + \frac{D_4 R_{k+1}^H}{Q_k}; \]

(6) If \( R_{k+1} = 0 \), then stop; else let \( k = k + 1 \) go to Step 5.

To prove the convergence property of Algorithm 3, we first establish the following basic properties.

**Lemma 4.** Suppose that the matrix equation (1) is consistent and \( V^* \), \( W^* \) are arbitrary solutions of (1). Then for any initial matrices \( V_1 \) and \( W_1 \), we have

\[ \text{Re} \{ \text{tr} \left[ P_i^H (V^* - V_i) + Q_i^H (W^* - W_i) \right] \} = \| R_i \|^2, \]

where the sequence \( \{ V_i \} \), \( \{ P_i \} \), \( \{ W_i \} \), \( \{ Q_i \} \), and \( \{ R_i \} \) are generated by Algorithm 3 for \( i = 1, 2, \ldots \).

**Proof.** We apply mathematical induction to prove the conclusion.

For \( i = 1 \), from Algorithm 3 we have

\[ \text{Re} \{ \text{tr} \left[ P_1^H (V^* - V_1) + Q_1^H (W^* - W_1) \right] \} \]

\[ = \text{Re} \{ \text{tr} \left[ \left( A_1^H R_1 + A_2^H R_{k+1} \right) (V^* - V_1) \right. \}

\[ + B_3 R_{k+1} A_3 + B_4 R_{k+1} T A_4 \}

\[ + (C_1^H R_1^1 D_1 + C_2^H R_{k+1}^1 D_2 + D_3 R_{k+1}^H C_3 + D_4 R_{k+1}^H T C_4) \}

\[ \times (W^* - W_1) \} \}

\[ = \text{Re} \{ \text{tr} \left[ R_1^H A_1 (V^* - V_1) B_1 + R_1^H A_2 (V^* - V_1) B_2 \right. \}

\[ + R_1^H B_3 (V^* - V_1) A_3 + R_1^H B_4 (V^* - V_1) A_4 \}

\[ + R_1^H C_1 (W^* - W_1) D_1 + R_1^H C_2 (W^* - W_1) D_2 \}

\[ + R_1^H C_3 (W^* - W_1) D_3 + R_1^H C_4 (W^* - W_1) D_4 \} \}.

(9)

From properties of trace and conjugate

\[ \text{Re} \{ \text{tr} \left[ P_i^H (V^* - V_1) + Q_i^H (W^* - W_1) \right] \} \]

\[ = \text{Re} \{ \text{tr} \left[ R_i^H A_1 (V^* - V_1) B_1 + R_i^H A_2 (V^* - V_1) B_2 \right. \}

\[ + A_3 (V^* - V_1) B_3 R_i^H + A_4 (V^* - V_1) B_4 R_i^H \}

\[ + R_i^H C_1 (W^* - W_1) D_1 \}

\[ + R_i^H C_2 (W^* - W_1) D_2 \}

\[ + R_i^H C_3 (W^* - W_1) D_3 R_i^H \}

\[ + C_4 (W^* - W_1) D_4 R_i^H \} \}

\[ = \text{Re} \{ \text{tr} \left[ R_i^H A_1 (V^* - V_1) B_1 + R_i^H A_2 (V^* - V_1) B_2 \right. \}

\[ + R_i^H A_3 (V^* - V_1) B_3 \}

\[ + R_i^H A_4 (V^* - V_1) B_4 \}

\[ + R_i^H C_1 (W^* - W_1) D_1 \}

\[ + R_i^H C_2 (W^* - W_1) D_2 \}

\[ + R_i^H C_3 (W^* - W_1) D_3 \}

\[ + R_i^H C_4 (W^* - W_1) D_4 \} \}.

(9)
\[
\begin{align*}
= \text{Re} \left\{ \text{tr} \left[ R_1^H \left( A_1 V^* B_1 + C_1 W^* D_1 + A_2 V^* B_2 \right) + C_2 W^* D_2 + A_3 V^* H B_3 + C_3 W^* H D_3 \right.ight. \\
+ A_4 V^* T B_4 + C_4 W^* T D_4 - A_1 V_1 B_1 \\
- C_1 W_1 D_1 - A_3 V_1 H B_3 - C_3 W_1 H D_3 \\
- A_4 V_1 T B_4 - C_4 W_1 T D_4 \left. \right] \right\}.
\end{align*}
\]

(10)

In view that \( V^* \) and \( W^* \) are solutions of matrix equation (1), with this relation we have

\[
\begin{align*}
\text{Re} \left\{ \text{tr} \left[ P_1^H (V^* - V_1) + Q_1^H (W^* - W_1) \right] \right\} \\
= \text{Re} \left\{ \text{tr} \left[ \text{tr} \left[ \left( E - A_1 V_1 B_1 - C_1 W_1 D_1 \right) \\
- A_2 V_1 B_2 - C_2 W_1 D_2 \\
- A_3 V_1 H B_3 - C_3 W_1 H D_3 \\
- A_4 V_1 T B_4 - C_4 W_1 T D_4 \right] \right] \right\},
\end{align*}
\]

(11)

\[
= \text{Re} \left\{ \text{tr} \left[ R_1^H R_1 \right] \right\} = \|R_1\|^2.
\]

This implies that (8) holds for \( i = 1 \).

Assume that (8) holds for \( k \). That is,

\[
\text{Re} \left\{ \text{tr} \left[ P_k^H (V^* - V_k) + Q_k^H (W^* - W_k) \right] \right\} = \|R_k\|^2.
\]

(12)

Then we have to prove that the conclusion holds for \( i = k + 1 \); it follows from Algorithm 3 that

\[
\begin{align*}
\text{Re} \left\{ \text{tr} \left[ P_{k+1}^H (V^* - V_{k+1}) + Q_{k+1}^H (W^* - W_{k+1}) \right] \right\} \\
= \text{Re} \left\{ \text{tr} \left[ A_1^H R_{k+1} B_1^H + A_2^H R_{k+1} B_2^H \\
+ B_3 R_{k+1}^H A_3 + B_4 R_{k+1}^H A_4 \\
+ \|R_{k+1}\|^2 \left( \frac{P_k}{\|R_k\|^2} \right)^H (V^* - V_{k+1}) \\
+ \left( C_1^H R_{k+1} D_1^H + C_2^H R_{k+1} D_2^H \right) \\
+ D_3 R_{k+1}^H C_3 + D_4 R_{k+1}^H C_4 \\
+ \|R_{k+1}\|^2 \left( \frac{Q_k}{\|R_k\|^2} \right)^H (W^* - W_{k+1}) \right] \right\}
\end{align*}
\]

= \text{Re} \left\{ \text{tr} \left[ R_{k+1}^H A_1 (V^* - V_{k+1}) B_1 \\
+ R_{k+1}^T A_2 (V^* - V_{k+1}) B_2 \\
+ R_{k+1}^T A_3 (V^* - V_{k+1}) A_3^H \\
+ R_{k+1}^T A_4 (V^* - V_{k+1}) A_4^H \\
+ R_{k+1}^T A_3 (W^* - W_{k+1}) C_3^H \\
+ R_{k+1}^T A_4 (W^* - W_{k+1}) C_4^H \\
+ R_{k+1}^T A_4 (W^* - W_{k+1}) C_4^H \\
+ \|R_{k+1}\|^2 \left( \frac{P_k}{\|R_k\|^2} \right)^H (V^* - V_{k+1}) \\
+ \|R_{k+1}\|^2 \left( \frac{Q_k}{\|R_k\|^2} \right)^H (W^* - W_{k+1}) \right] \right\}.
\]

(13)

From properties of trace and conjugate we get

\[
\begin{align*}
\text{Re} \left\{ \text{tr} \left[ P_{k+1}^H (V^* - V_{k+1}) + Q_{k+1}^H (W^* - W_{k+1}) \right] \right\} \\
= \text{Re} \left\{ \text{tr} \left[ R_{k+1}^H A_1 (V^* - V_{k+1}) B_1 \\
+ A_3 (V^* - V_{k+1})^H B_3 R_{k+1}^H A_3 \\
+ A_4 (V^* - V_{k+1})^T B_4 R_{k+1}^H A_4 \\
+ R_{k+1}^T C_3 (W^* - W_{k+1}) D_3 R_{k+1} \\
+ C_4 (W^* - W_{k+1})^T D_4 R_{k+1}^H + \|R_{k+1}\|^2 \left( \frac{P_k}{\|R_k\|^2} \right)^H (V^* - V_{k+1}) \\
+ C_4 (W^* - W_{k+1})^T D_4 R_{k+1}^H + \|R_{k+1}\|^2 \left( \frac{Q_k}{\|R_k\|^2} \right)^H (W^* - W_{k+1}) \right] \right\}.
\end{align*}
\]
\[ \text{Re} \left\{ \operatorname{tr} \left[ R_{k+1}^H A_1 (V^* - V_{k+1}) B_1 \right] \right. \\
+ \left. R_{k+1}^H A_2 (V^* - V_{k+1}) B_2 \right. \\
+ \left. R_{k+1}^H A_3 (V^* - V_{k+1})^T B_3 \right. \\
+ \left. R_{k+1}^H A_4 (V^* - V_{k+1})^T B_4 \right. \\
+ \left. R_{k+1}^H C_1 (W^* - W_{k+1}) D_1 \right. \\
+ \left. R_{k+1}^H C_2 (W^* - W_{k+1}) D_2 \right. \\
+ \left. R_{k+1}^H C_3 (W^* - W_{k+1})^T D_3 \right. \\
+ \left. R_{k+1}^H C_4 (W^* - W_{k+1})^T D_4 \right) \\
\times \text{Re} \left\{ \operatorname{tr} \left[ \left( V^* - V_k - \frac{\|R_k\|_2}{\|P_k\|_2 + \|Q_k\|_2} P_k \right) \right. \\
+ \left. Q_k^H \left( W^* - W_k \right. \right. \\
\left. \left. - \frac{\|R_k\|_2}{\|P_k\|_2 + \|Q_k\|_2} Q_k \right) \right] \right\} \\
= \text{Re} \left\{ \operatorname{tr} \left[ R_{k+1}^H \left( A_1 V^* B_1 + C_1 W^* D_1 \right) \right. \\
+ \left. A_2 V B_2 + C_2 W^T D_2 \right. \\
+ \left. A_3 V^H B_3 + C_3 W^{HT} D_3 \right. \\
+ \left. A_4 V^T B_4 + C_4 W^T D_4 \right. \\
- \left. A_1 V_{k+1} B_1 - C_1 W_{k+1} D_1 - A_2 V_{k+1} B_2 \right. \\
- \left. C_2 W_{k+1} D_2 - A_3 V_{k+1}^H B_3 \right. \\
- \left. C_3 W_{k+1}^H D_3 - A_4 V_{k+1}^T B_4 \right. \\
- \left. C_4 W_{k+1}^T D_4 \right) \right\} \\
+ \frac{\|R_{k+1}\|_2^2}{\|R_k\|_2^2} \text{Re} \left\{ \operatorname{tr} \left[ P_k^H (V^* - V_k) + Q_k^H (W^* - W_k) \right. \right. \\
\left. \left. - \frac{\|R_k\|_2^2}{\|P_k\|_2^2 + \|Q_k\|_2^2} (P_k^H P_k + Q_k^H Q_k) \right) \right\}. \quad (14) \]

In view that \(V^*\) and \(W^*\) are solutions of matrix equation (1), with relation (14) one has

\[ \text{Re} \left\{ \operatorname{tr} \left[ P_{k+1}^H (V^* - V_{k+1}) + Q_{k+1}^H (W^* - W_{k+1}) \right] \right\} = \text{Re} \left\{ \operatorname{tr} \left[ R_{k+1}^H \left( E - A_1 V_{k+1} B_1 - C_1 W_{k+1} D_1 \right. \right. \\
\left. \left. - A_2 V_{k+1} B_2 - C_2 W_{k+1} D_2 \right. \right. \\
\left. \left. - A_3 V_{k+1}^H B_3 - C_3 W_{k+1}^H D_3 \right. \right. \\
\left. \left. - A_4 V_{k+1}^T B_4 - C_4 W_{k+1}^T D_4 \right) \right) \right\} \\
+ \frac{\|R_{k+1}\|_2^2}{\|R_k\|_2^2} \left\{ \frac{\|R_k\|_2^2}{\|P_k\|_2^2 + \|Q_k\|_2^2} (P_k^H P_k + Q_k^H Q_k) \right\}. \quad (15) \]

Then relation (8) holds by mathematical induction. \(\square\)

Lemma 5. Suppose that the matrix equation (1) is consistent and the sequences \(\{R_i\}, \{P_i\}, \text{and} \{Q_i\}\) are generated by Algorithm 3 with any initial matrices \(V_1, W_1, \text{such that} \ R_i \neq 0 \, \text{for all} \ i, 1, 2, \ldots, k, \text{and then} \)

\[ \text{Re} \left\{ \operatorname{tr} \left( R_i^H R_j \right) \right\} = 0, \quad i, j = 1, 2, \ldots, k, \ i \neq j. \]

Proof. We apply mathematical induction.

Step 1. We prove that

\[ \text{Re} \left\{ \operatorname{tr} \left( P_i^H P_j + Q_i^H Q_j \right) \right\} = 0, \quad i, j = 1, 2, \ldots, k. \]

First from Algorithm 3 we have

\[ R_{k+1} = E - A_1 V_{k+1} B_1 - C_1 W_{k+1} D_1 - A_2 V_{k+1} B_2 \]
\[ - C_2 W_{k+1} D_2 - A_3 V_{k+1}^H B_3 \]
\[ - C_3 W_{k+1}^H D_3 - A_4 V_{k+1}^T B_4 - C_4 W_{k+1}^T D_4, \]
\[ R_{k+1} = E - A_1 \left( V_k + \frac{\|R_k\|_2^2}{\|P_k\|_2^2 + \|Q_k\|_2^2} P_k \right) B_1 \]
\[ - C_1 \left( W_k + \frac{\|R_k\|_2^2}{\|P_k\|_2^2 + \|Q_k\|_2^2} Q_k \right) D_1 \]
\[ - A_2 \left( V_k + \frac{\|R_k\|_2^2}{\|P_k\|_2^2 + \|Q_k\|_2^2} P_k \right) B_2 \]
\[ - C_2 \left( W_k + \frac{\|R_k\|_2^2}{\|P_k\|_2^2 + \|Q_k\|_2^2} Q_k \right) D_2 \]
\[ - A_3 \left( V_k + \frac{\|R_k\|_2^2}{\|P_k\|_2^2 + \|Q_k\|_2^2} P_k \right)^T B_3 \]
\[ - C_3 \left( W_k + \frac{\|R_k\|_2^2}{\|P_k\|_2^2 + \|Q_k\|_2^2} Q_k \right)^T D_3 \]
\[ - A_4 \left( V_k + \frac{\|R_k\|_2^2}{\|P_k\|_2^2 + \|Q_k\|_2^2} P_k \right)^T B_4 \]
\[-C_4(W_k + \frac{\|R_k\|^2}{\|P_k\|^2 + \|Q_k\|^2}Q_k) D_4\]

\[= E - A_1V_kB_1 - C_1W_kD_1 - A_2\overline{V_k}B_2 - C_2\overline{W_k}D_2 - A_3V_k^H B_3 - C_3W_k^H D_3 - A_4V_k^T B_4\]

\[\times (A_1P_kB_1 + C_1Q_kD_1 + A_2P_k^T B_2 + C_2Q_k^T D_2 + A_3P_k^H B_3 + C_3Q_k^H D_3 + A_4P_k^T B_4 + C_4Q_k^T D_4),\]

\[R_{k+1} = R_k - \frac{\|R_k\|^2}{\|P_k\|^2 + \|Q_k\|^2} \times (A_1P_kB_1 + C_1Q_kD_1 + A_2P_k^T B_2 + C_2Q_k^T D_2 + A_3P_k^H B_3 + C_3Q_k^H D_3 + A_4P_k^T B_4 + C_4Q_k^T D_4).\]

For \(i = 1\), it follows from (19) that

\[\text{Re}\{\text{tr}\left(R_2^HR_1^H\right)\}\]

\[= \text{Re}\left\{\text{tr}\left(R_1 - \frac{\|R_1\|^2}{\|P_1\|^2 + \|Q_1\|^2} \times (A_1P_1B_1 + C_1Q_1D_1 + A_2P_1^T B_2 + C_2Q_1^T D_2 + A_3P_1^H B_3 + C_3Q_1^H D_3 + A_4P_1^T B_4 + C_4Q_1^T D_4)\right)^H R_1\right\}\]

\[= \text{Re}\left\{\text{tr}\left(R_1^H R_1\right) - \frac{\|R_1\|^2}{\|P_1\|^2 + \|Q_1\|^2} \times \text{tr}(P_1^H A_1^HR_1^H B_1^H + Q_1^H C_1^HR_1^H D_1^H + A_2P_1^T B_2^H + Q_1^H C_2^HR_1^H D_2^H + A_3P_1^H B_3^H + Q_1^H C_3^HR_1^H D_3^H + A_4P_1^T B_4^H + Q_1^H C_4^HR_1^H D_4^H)\right\}\]

\[= \|R_1\|^2 - \frac{\|R_1\|^2}{\|P_1\|^2 + \|Q_1\|^2} \times \text{Re}\{\text{tr}\left(P_1^H P_1 + Q_1^H Q_1\right)\},\]

\[\text{Re}\{\text{tr}\left(R_1^HR_1^H\right)\}\]

\[= \|R_1\|^2 - \frac{\|R_1\|^2}{\|P_1\|^2 + \|Q_1\|^2} \times \text{Re}\{\text{tr}\left(P_1^H P_1 + Q_1^H Q_1\right)\} = 0.\]

This implies that (17) is satisfied for \(i = 1\). From Algorithm 3 we have

\[\text{Re}\{\text{tr}\left(P_2^H P_1 + Q_2^H Q_1\right)\}\]

\[= \text{Re}\left\{\text{tr}\left(A_1^H R_2 B_1^H + A_2^H R_2^T B_2^H + B_3^H R_2 C_3\right)\right\}\]

\[= \text{Re}\left\{\text{tr}\left(A_1^H R_2 B_1^H + A_2^H R_2^T B_2^H + B_3^H R_2 C_3 + \frac{\|R_1\|^2}{\|P_1\|^2} P_1\right)\right\}\]

\[= \text{Re}\left\{\text{tr}\left(A_1^H R_2 B_1^H + A_2^H R_2^T B_2^H + B_3^H R_2 C_3 + \frac{\|R_1\|^2}{\|P_1\|^2} P_1 + \overline{A_1^H R_2 B_1^H} + \overline{A_2^H R_2^T B_2^H} + \overline{B_3^H R_2 C_3} + \overline{\frac{\|R_1\|^2}{\|P_1\|^2} P_1}\right)\right\}\]

\[= \text{Re}\left\{\text{tr}\left(A_1^H R_2 B_1^H + A_2^H R_2^T B_2^H + B_3^H R_2 C_3 + \frac{\|R_1\|^2}{\|P_1\|^2} P_1\right)\right\}\]

\[= \text{Re}\left\{\text{tr}\left(A_1^H R_2 B_1^H + A_2^H R_2^T B_2^H + B_3^H R_2 C_3 + \frac{\|R_1\|^2}{\|P_1\|^2} P_1 + \overline{A_1^H R_2 B_1^H} + \overline{A_2^H R_2^T B_2^H} + \overline{B_3^H R_2 C_3} + \overline{\frac{\|R_1\|^2}{\|P_1\|^2} P_1}\right)\right\}\]
Assume that (17) and (18) hold for \( i = k - 1 \), from Algorithm 3 we have

\[
\text{Re} \left\{ \text{tr} (R_{k-1}^H R_k) \right\} \]

\[
= \text{Re} \left\{ \text{tr} \left[ \frac{R_k^2}{L_k^2} R_k A_1^H R_k A_2^H + R_k^2 A_1^H A_2^H A_2 R_k B_2 + R_k^2 B_2 R_k P_1 B_3 \right] \right\} 
+ \frac{\| R_k \|^2}{\| R_k \|^2} \left( \| P_1 \|^2 + \| Q_1 \|^2 \right)
+ \frac{\| R_k \|^2}{\| R_k \|^2} \left( \| P_1 \|^2 + \| Q_1 \|^2 \right)

\[
\text{Re} \left\{ \text{tr} \left[ (R_1 - R_2)^H \right] \right\} 
+ \frac{\| R_k \|^2}{\| R_k \|^2} \left( \| P_1 \|^2 + \| Q_1 \|^2 \right)
+ \frac{\| R_k \|^2}{\| R_k \|^2} \left( \| P_1 \|^2 + \| Q_1 \|^2 \right) = 0. 
\]

(21)

This implies that (18) is satisfied for \( i = 1 \).
Thus (17) holds for $i = k$.
Also, from Algorithm 3 we have
\[
\text{Re} \left\{ \text{tr} \left( p_{k+1}^H P_k + Q_{k+1}^H Q_k \right) \right\}
\]
\[
= \text{Re} \left\{ \text{tr} \left[ A_1^H R_{k+1} B_1^H + A_2^H R_{k+1} B_2^H \right.ight.
\]
\[
\left. + B_3 R_{k+1}^H A_3 + \bar{B}_4 R_{k+1}^H A_4 \right) P_k
\]
\[
\left. + \left( C_1^H R_{k+1} D_1^H + C_2^H R_{k+1} D_2^H \right. \right]
\]
\[
\left. + D_3 R_{k+1}^H C_3 + \bar{D}_4 R_{k+1}^H C_4 \right) Q_k \right) \right\}
\]
\[
= \text{Re} \left\{ \text{tr} \left[ R_{k+1}^H A_1 P_k B_1 + \bar{R}_{k+1}^H A_2 P_k B_2 \right.ight.
\]
\[
\left. + R_{k+1} B_3^H P_k A_3 + \bar{R}_{k+1} B_4^H P_k A_4 \right) \right\}
\]
\[
= \text{Re} \left\{ \text{tr} \left[ R_{k+1}^H \left( A_1 P_k B_1 + \bar{A}_2 P_k B_2 \right) \right. \right.
\]
\[
\left. + R_{k+1} B_3^H P_k A_3 + \bar{R}_{k+1} B_4^H P_k A_4 \right) \right\}
\]
This implies that (17) and (18) hold for $i = k$.
Then relations (17) and (18) holds by mathematical induction.
Step 2. We want to show that
\[
\text{Re} \left( \text{tr} \left( R_{l+1}^H R_l \right) \right) = 0,
\]
\[
\text{Re} \left( \text{tr} \left( p_{l+1}^H p_l + Q_{l+1}^H Q_l \right) \right) = 0
\]
holds for $l \geq 1$. We will prove this conclusion by induction.
The case of $l = 1$ has been proven in Step 1. Now we assume that (24) holds for $l \leq s, s \geq 1$. The aim is to show that
\[
\text{Re} \left( \text{tr} \left( R_{l+1}^H R_l \right) \right) = 0,
\]
\[
\text{Re} \left( \text{tr} \left( p_{l+1}^H p_l + Q_{l+1}^H Q_l \right) \right) = 0.
\]
First we prove the following:
\[
\text{Re} \left( \text{tr} \left( R_{s+1}^H R_s \right) \right) = 0,
\]
\[
\text{Re} \left( \text{tr} \left( p_{s+1}^H p_s + Q_{s+1}^H Q_s \right) \right) = 0.
\]
By using Algorithm 3, from (19) and induction we have
\[
\text{Re} \left\{ \text{tr} \left( R_{s+1}^H R_0 \right) \right\} = \text{Re} \left\{ \text{tr} \left( R_s^H R_0 \right) - \frac{\|R_s\|^2}{\|P_s\|^2 + \|Q_s\|^2} \right\} \times \text{tr} \left( P_s^H A_s^H R_0 B_1^H + Q_s^H C_s^H R_0 D_1^H \right)
\]
\[
+ \frac{\|R_s\|^2}{\|P_s\|^2 + \|Q_s\|^2} \left( \frac{\|R_s\|^2}{\|P_s\|^2 + \|Q_s\|^2} \right) \left( A_s P_s B_1 + C_s Q_s D_1 + A_s P_s B_2 
+ C_s Q_s D_2 + A_s P_s^T B_3 + C_s Q_s^T D_3 
+ A_s P_s^T B_4 + C_s Q_s^T D_4 \right)^H \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \r
From Algorithm 3 we have

\[
\text{Re} \left\{ \text{tr} \left( P_{i+1}^H R_{i+1} + Q_i^H Q_i \right) \right\} = \text{Re} \left\{ \text{tr} \left[ \left( A_i^H R_{i+1} B_i^H + A_2^H R_{i+1} B_2^H \right. \right. \right. \\
\left. \left. \left. + B_3 R_{i+1}^H A_3 + B_4 R_{i+1}^T A_4 \right. \right. \right. \\
\left. \left. \left. \left. + \frac{\| R_{i+1} \|^2}{\| R_{i+1} \|} P_{i+1} \right) \right. \right. \right. \\
\left. \left. \left. \left. + \left( C_1^H R_{i+1} D_1^H + C_2^H R_{i+1} D_2^H \right. \right. \right. \right. \\
\left. \left. \left. \left. \left. + D_3 R_{i+1}^H C_3 + D_4 R_{i+1}^T C_4 \right. \right. \right. \right. \\
\left. \left. \left. \left. \left. + \frac{\| R_{i+1} \|^2}{\| R_{i+1} \|} Q_i \right) \right) \right) \right) \right) \right] \right) \right) \right) ^H \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right)
\]

Also from (19) we have

\[
\text{Re} \left\{ \text{tr} \left( R_{i+1}^H R_i \right) \right\} = \text{Re} \left\{ \text{tr} \left[ \left( R_{i+1} - \frac{\| R_{i+1} \|^2}{\| P_{i+1} \|^2 + \| Q_{i+1} \|^2} \times \left( A_1 P_{i+1} B_1 + C_1 Q_{i+1} D_1 \right. \right. \right. \\
\left. \left. \left. \left. + A_2 P_{i+1} B_2 + C_2 Q_{i+1} D_2 \right. \right. \right. \right. \\
\left. \left. \left. \left. \left. + A_3 P_{i+1} B_3 + C_3 Q_{i+1} D_3 \right. \right. \right. \right. \right. \\
\left. \left. \left. \left. \left. + A_4 P_{i+1} B_4 + C_4 Q_{i+1} D_4 \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right)
\]

(28)
\[
= -\frac{\|R_{i+s}\|^2}{\|P_{i+s}\|^2 + \|Q_{i+s}\|^2} \times \text{Re} \left\{ \text{tr} \left[ P_{i+s}^H \left( A_i R_i B_i^H + A_i R_i^T A_i \right) + Q_{i+s}^H \left( C_i R_i D_i^H + C_i R_i^T D_i \right) + D_3 R_i^T C_3 + D_4 R_i^T C_4 \right] \right\}
\]

\[
= -\frac{\|R_{i+s}\|^2}{\|P_{i+s}\|^2 + \|Q_{i+s}\|^2} \times \text{Re} \left\{ \text{tr} \left( P_{i+s}^H P_{i-1} + Q_{i+s}^H Q_{i-1} \right) \right\}.
\]

(29)

Repeating (28) and (29), one can easily obtain for certain \(\alpha\) and \(\beta\)
\[
\text{tr} \left( P_{i+s}^H P_{i-1} + Q_{i+s}^H Q_{i-1} \right) = \alpha \left[ \text{tr} \left( P_{i+s}^H P_{i} + Q_{i+s}^H Q_{i} \right) \right],
\]
\[
\text{tr} \left( R_{i+s}^H R_{i-1} \right) = \beta \left[ \text{tr} \left( R_{i+s}^H R_{i} \right) \right].
\]

(30)

Combining these two relations with (26) implies that (24) holds for \(i = s + 1\). From Steps 1 and 2 the conclusion holds by the principle of induction. With the above two lemmas, we have the following theorem.

**Theorem 6** (see [32]). If the matrix equation (1) is consistent, then a solution can be obtained within finite iteration steps by using Algorithm 3 for any initial matrices \(V_i, W_i\).

### 4. Numerical Example

In this section, we present numerical example to illustrate the application of our proposed methods.

**Example 7.** In this example we illustrate our theoretical results of Algorithm 3 for solving the system of matrix equation:

\[
A_1 V B_1 + C_1 W D_1 + A_2 V B_2 + C_2 W D_2 + A_3 V H B_3 \\
+ C_3 W H D_3 + A_4 V T B_4 + C_4 W T D_4 = E.
\]

(31)

Because of the influence of the error of calculation, the residual \(R(k)\) is usually unequal to zero in this process of the iteration. We regard the matrix \(R(k)\) as a zero matrix if \(R(k) < 10^{-10}\).

Given
\[
A_1 = \begin{bmatrix} 2 + 3i & -i & 1 + i \\ 5 & 1 + 2i & -3 \end{bmatrix},
\]
\[
A_2 = \begin{bmatrix} 2 + 3i & -i & 1 + i \\ 5 & 1 + 2i & -3 \end{bmatrix},
\]
\[
A_3 = \begin{bmatrix} 0 & 2 - i & i \\ -1 + 3i & 2 & 0 \end{bmatrix},
\]
\[
A_4 = \begin{bmatrix} 0 & 1 - 3i & 1 + i \\ 0 & 4i & -3i \end{bmatrix},
\]
\[
C_1 = \begin{bmatrix} 1 + 2i & 3 - 4i \\ -i & 2i & -3 \end{bmatrix},
\]
\[
C_2 = \begin{bmatrix} 3 + 2i & 0 & 1 + i \\ 0 & 4i & 1 - 2i \end{bmatrix},
\]
\[
C_3 = \begin{bmatrix} 1 - 3i & 2i & -3i \\ 1 & 2 + 3i & 4i \end{bmatrix},
\]
\[
C_4 = \begin{bmatrix} 1 - 2i & 0 \\ 2 & 3 - i & 1 + i \end{bmatrix},
\]
\[
B_1 = \begin{bmatrix} 4 + i & -i \\ 0 & 1 - i \\ 4i & 2 + 2i \end{bmatrix},
\]
\[
B_2 = \begin{bmatrix} 0 & i \\ 1 + i & 0 \\ -1 - 3i & 0 \end{bmatrix},
\]
\[
B_3 = \begin{bmatrix} 0 & 1 \\ -3i & 4 + i \\ 5 & 1 + 2i \end{bmatrix},
\]
\[
B_4 = \begin{bmatrix} 3 + i & -1 - i \\ 0 & 2 - i \\ -1 + i & 2 \end{bmatrix},
\]
\[
D_1 = \begin{bmatrix} 0 & 0 \\ 1 & 1 - 3i \\ 2i & -3i \end{bmatrix},
\]
\[
D_2 = \begin{bmatrix} 0 & i \\ 1 + i & 0 \\ -1 - 3i & 0 \end{bmatrix},
\]
\[
D_3 = \begin{bmatrix} 0 & 1 \\ -3i & 4 + i \\ 5 & 1 + 2i \end{bmatrix},
\]
\[
D_4 = \begin{bmatrix} 3i & -2 + i \\ 0 & i \\ -2i & -4i \end{bmatrix},
\]
\[
E = \begin{bmatrix} 42 + 55i & 115 + 25i \\ -38 - i & 132 + 44i \end{bmatrix}.
\]

(32)

Taking \(V_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\) and \(W_1 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}\), we apply Algorithm 3 to compute \(V_k, W_k\). And iterating 42 steps we get

\[
V = \begin{bmatrix} 0.0126 + 1.8415i & 0.0827 + 0.6381i \\ -0.6903 + 1.0185i & 1.8818 + 1.0203i \\ 0.5344 - 0.4909i & 0.9280 + 0.7169i \\ 0.4872 - 0.2734i \\ 0.6273 - 0.1216i & -0.3902 + 0.4313i \\ -0.7011 - 0.3418i & 0.3695 + 1.7627i \\ -0.3032 + 0.7073i \end{bmatrix}
\]

\[
W = \begin{bmatrix} 0.4218 - 0.9710i & 0.1763 + 0.5183i \\ 0.6273 - 0.1216i & -0.3902 + 0.4313i \\ -0.7011 - 0.3418i & 0.3695 + 1.7627i \\ -0.3032 + 0.7073i \end{bmatrix}
\]

(33)

which satisfy the matrix equation:

\[
A_1 V B_1 + C_1 W D_1 + A_2 V B_2 + C_2 W D_2 + A_3 V H B_3 \\
+ C_3 W H D_3 + A_4 V T B_4 + C_4 W T D_4 = E.
\]

(34)
With the corresponding residual
\[
\|R_{42}\| = \left\| \begin{bmatrix} E - A_1 V_{42} B_1 - C_1 W_{42} D_1 - A_2 V_{42} B_2 - C_2 W_{42} D_2 \\ \vdots \\ A_3 V_{42} B_3 - C_3 W_{42} D_3 - A_4 V_{42} B_4 - C_4 W_{42} D_4 \end{bmatrix} \right\| = 6.6115 \times 10^{-11}.
\]

(35)

5. Conclusions

The above Figure 1 shows the convergence curve for the residual function \( R(k) \). In this paper, an iterative algorithm constructed to solve a complex matrix equation with conjugate and transpose of two unknowns of the form: \( A_1 V B_1 + C_1 W D_1 + A_2 V B_2 + C_2 W D_2 + A_3 V^H B_3 + C_3 W^H D_3 + A_4 V^T B_4 + C_4 W^T D_4 = E \) is presented. We proved that the iterative algorithms always converge to the solution for any initial matrices. We stated and proved some lemmas and theorems where the solutions are obtained. The proposed method is illustrated by numerical example where the obtained numerical results show that our technique is very neat and efficient.

References

[1] F. Ding, P. X. Liu, and J. Ding, "Iterative solutions of the generalized Sylvester matrix equations by using the hierarchical identification principle," *Applied Mathematics and Computation*, vol. 197, no. 1, pp. 41–50, 2008.
[2] F. Ding and T. Chen, "Gradient based iterative algorithms for solving a class of matrix equations," *IEEE Transactions on Automatic Control*, vol. 50, no. 8, pp. 1261–1270, 2005.
[3] F. Ding and T. Chen, "Hierarchical gradient-based identification of multivariable discrete-time systems," *Automatica*, vol. 41, no. 2, pp. 315–325, 2005.
[4] F. Ding and T. Chen, "Iterative least-squares solutions of coupled Sylvester matrix equations," *Systems and Control Letters*, vol. 54, no. 2, pp. 95–107, 2005.
[5] F. Ding and T. Chen, "On iterative solutions of general coupled matrix equations," *SIAM Journal on Control and Optimization*, vol. 44, no. 6, pp. 2269–2284, 2006.
[6] F. Ding and T. Chen, "Hierarchical least squares identification methods for multivariable systems," *IEEE Transactions on Automatic Control*, vol. 50, no. 3, pp. 397–402, 2005.
[7] F. Ding and T. Chen, "Performance analysis of multi-innovation gradient type identification methods," *Automatica*, vol. 43, no. 1, pp. 1–14, 2007.
[8] A. G. Wu, X. Zeng, G. R. Duan, and W. J. Wu, "Iterative solutions to the extended Sylvester-conjugate matrix equations," *Applied Mathematics and Computation*, vol. 217, no. 1, pp. 130–142, 2010.
[9] B. Zhou, G. R. Duan, and Z. Y. Li, "Gradient based iterative algorithm for solving coupled matrix equations," *Systems and Control Letters*, vol. 58, no. 5, pp. 327–333, 2009.
[10] J. H. Bevis, F. J. Hall, and R. E. Hartwig, "Consimilarity and the matrix equation \( AX - XB = C \)," in *Current Trends in Matrix Theory*, pp. 51–64, North-Holland, New York, NY, USA, 1987.
[11] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, UK, 1990.
[12] H. Liping, "Consimilarity of quaternion matrices and complex matrices," *Linear Algebra and Its Applications*, vol. 331, no. 1–3, pp. 21–30, 2001.
[13] T. Jiang, X. Cheng, and L. Chen, "An algebraic relation between consimilarity and similarity of complex matrices and its applications," *Journal of Physics A*, vol. 39, no. 29, pp. 9215–9222, 2006.
[14] J. H. Bevis, F. J. Hall, and R. E. Hartwig, "The matrix equation \( AX - XB = C \) and its special cases," *SIAM Journal on Matrix Analysis and Applications*, vol. 9, no. 3, pp. 348–359, 1988.
[15] A. G. Wu, G. R. Duan, and H. H. Yu, "On solutions of the matrix equations \( XF - AX = C \) and \( XF - AX = C \)," *Applied Mathematics and Computation*, vol. 183, no. 2, pp. 932–941, 2006.
[16] A. Navarra, P. L. Odell, and D. M. Young, "Representation of the general common solution to the matrix equations \( A_1 X B_1 = C_1, A_2 X B_2 = C_2 \) with applications," *Computers and Mathematics with Applications*, vol. 41, no. 7–8, pp. 929–935, 2001.
[17] J. W. Van der Woude, "On the existence of a common solution \( X \) to the matrix equations \( A_1 X B_1 = C_1 (i, j) \) E.I.\", *Linear Algebra and Its Applications*, vol. 375, no. 1–3, pp. 135–145, 2003.
[18] P. Bhimasankaram, "Common solutions to the linear matrix equations \( AX = XB = C \) \( D \) and \( EF = G \)," *Sankhya Series A*, vol. 38, pp. 404–409, 1976.
[19] S. Kumar Mitra, "The matrix equations \( AX = C, XB = D \)," *Linear Algebra and Its Applications*, vol. 59, pp. 171–181, 1984.
[20] S. K. Mitra, "A pair of simultaneous linear matrix equations \( A_1 X B_1 = C_1, A_2 X B_2 = C_2 \) and a matrix programming problem," *Linear Algebra and Its Applications*, vol. 131, pp. 107–123, 1990.
[21] M. A. Ramadan, M. A. Abdel Naby, and A. M. E. Bayoumi, "On the explicit solutions of forms of the Sylvester and the Yakubovich matrix equations," *Mathematical and Computer Modelling*, vol. 50, no. 9–10, pp. 1400–1408, 2009.
[22] K. W. E. Chu, "Singular value and generalized singular value decompositions and the solution of linear matrix equations," *Linear Algebra and Its Applications*, vol. 88–89, pp. 83–98, 1987.
[23] Y. X. Yuan, “The optimal solution of linear matrix equation by matrix decompositions, Math,” *Numerica Sinica*, vol. 24, pp. 165–176, 2002.

[24] A. P. Liao and Y. Lei, “Least-squares solution with the minimum-norm for the matrix equation $(AXB, GXH) = (C, D)$,” *Computers and Mathematics with Applications*, vol. 50, no. 3–4, pp. 539–549, 2005.

[25] Y. H. Liu, "Ranks of least squares solutions of the matrix equation $AXB = C$,” *Computers and Mathematics with Applications*, vol. 55, no. 6, pp. 1270–1278, 2008.

[26] X. Sheng and G. Chen, "A finite iterative method for solving a pair of linear matrix equations $(AXB, CXD) = (E, F)$," *Applied Mathematics and Computation*, vol. 189, no. 2, pp. 1350–1358, 2007.

[27] A. G. Wu, W. Liu, and G. R. Duan, “On the conjugate product of complex polynomial matrices,” *Mathematical and Computer Modelling*, vol. 53, no. 9-10, pp. 2031–2043, 2011.

[28] A. G. Wu, G. Feng, W. Liu, and G. R. Duan, “The complete solution to the Sylvester-polynomial-conjugate matrix equations,” *Mathematical and Computer Modelling*, vol. 53, no. 9-10, pp. 2044–2056, 2011.

[29] A. G. Wu, B. Li, Y. Zhang, and G. R. Duan, "Finite iterative solutions to coupled Sylvester-conjugate matrix equations,” *Applied Mathematical Modelling*, vol. 35, no. 3, pp. 1065–1080, 2011.

[30] A. G. Wu, L. Lv, and G. R. Duan, “Iterative algorithms for solving a class of complex conjugate and transpose matrix equations,” *Applied Mathematics and Computation*, vol. 217, no. 21, pp. 8343–8353, 2011.

[31] X. Zhang, *Matrix Analysis and Application*, Tsinghua University Press, Beijing, China, 2004.

[32] A. G. Wu, L. Lv, and M. Z. Hou, "Finite iterative algorithms for extended Sylvester-conjugate matrix equations,” *Mathematical and Computer Modelling*, vol. 54, pp. 2363–2384, 2011.
Submit your manuscripts at http://www.hindawi.com