Inclusion of non-idealities in the continuous photodetection model

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Some non-ideal effects as non-unit quantum efficiency, dark counts, dead time and cavity losses that occur in experiments are incorporated within the continuous photodetection model by using the analytical quantum trajectories approach. We show that in standard photocounting experiments the validity of the model can be verified, and the formal expression for the quantum jump superoperator can also be checked.

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I. INTRODUCTION

The continuous photodetection model (CPM) was proposed in the early 1980’s in order to treat quantum optics situations in which a weak electromagnetic field enclosed in a cavity is continuously measured through the photocounting approach [1]. The theory has received considerable attention in the following years due to its new microscopic interpretation of the photodetection process [2, 3, 4, 5, 6], relation to the quantum trajectories approach [7, 8, 9, 10, 11, 12] and several proposals for applications. Among them we find studies of photocounts statistics in diverse systems [13, 14, 15, 16, 17], quantum non-demolition measurements [18, 19, 20], implementation of measurement schemes [21, 22, 23], quantum state preparation [24, 25, 26, 27, 28], quantum control via photodetection [29, 30], and quantum computation [31].

CPM is extensively discussed in the literature [3, 18, 32, 33, 34], so we shall mention only its main properties. The model, also referred as a theory, describes the field state evolution during the photodetection process in a closed cavity and is formulated in terms of two fundamental operations, assumed to represent the only events taking place at each infinitesimal time interval.

1) The one-count operation, represented by the Quantum Jump Superoperator (QJS), describes the detector’s action on the field upon a single count, and the trace calculation over the QJS gives the probability per unit time for occurrence of a detection.

2) The no-count operation describes the field non-unitary evolution in absence of counts.

If one sets the formal expressions for these operations, all possible outcomes of a photocounting experiment can be predicted. For instance, the photocounts [1, 2, 2] and the waiting time [35, 36, 37, 38] statistics are among the most common quantities to be studied both theoretically and experimentally. Moreover, CPM conferred a new step in photodetection theories by allowing to determine the field state after an arbitrary sequence of measurements, thus creating the possibility of controlling the field properties in real time experiments [15, 17, 34].

Actually, the QJS is the main formal ingredient within the theory, since it also dictates the form of the no-count superoperator [1]. Two different models for the QJS were proposed ad hoc. The first one was proposed by Srinivas and Davies [1], the SD-model, as

$$\dot{\rho} = \lambda a \rho a^\dagger,$$

where $\rho$ is the field density matrix, $a$ and $a^\dagger$ are the usual bosonic ladder operators and $\lambda$ is roughly the detector’s ideal counting rate [1, 39]. From the very beginning the authors [1] denounced the presence of some inconsistencies when the QJS [1] is employed for describing a real photodetection process, this point was also appointed in [39]. Nevertheless, this QJS is widely used in the literature [3, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30].

The other proposal [33, 40] assumes for the QJS an expression written in terms of the ladder operators $\hat{E}_- = (\hat{a}^\dagger \hat{a} + 1)^{-1/2} \hat{a}$ and $\hat{E}_+ = \hat{E}_-^\dagger$ (also known as exponential phase operators [41, 42, 43, 44, 45])

$$\dot{\rho} = \lambda \hat{E}_- \rho \hat{E}_+.$$

In [33] we called E-model such a choice, to differentiate from the SD QJS [1]. Besides eliminating the inconsistencies within the SD-model, the use of the E-model leads to different qualitative and quantitative predictions for several observable quantities. By an analysis of a microscopic model for the detector, it was recently shown that the QJS’s [1] and [2] are particular cases of a general time-dependent transition superoperator, each one occurring in a particular regime of the detector experimental parameters [46, 47]. Moreover, it was pointed out that by manipulating certain detector’s accessible parameters one could engineer the form of the QJS, thus changing the dynamics of the photodetection, as well as the field state after a sequence of measurements.

A way to check the validity of the CPM and to decide which QJS better describes the phenomenon in practice

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can be accomplished through photocount experiments in a high finesse cavity by comparing the results to the theoretical predictions. However, real detectors and cavities are far from ideal. So our first goal is to include into the CPM the effects of non-ideality, such as quantum efficiency (QE), dark counts, detector’s dead-time and cavity damping. Our second goal is to call attention to the fact that standard photodetection measurements could verify which of the QJS models actually prevails experimentally.

The plan of the paper is as follows. In Sec. II we present a simple model, which enables us to include the effects of non-ideality – QE and dark counts – into the CPM using the quantum trajectories approach. Then we calculate the main quantities characterizing the photodetection process – the photocounting and waiting time distributions. In subsection IIIA we do this using the QJS, and in subsection IIIB we repeat the same procedure for E-model. In Sec. III we analyze the behavior of the lower moments of the above distributions in realistic situations and point out how one could decide about a QJS from experimental data. Sec. IV contains conclusions. In the appendix A we treat the effects of dead-time and cavity damping: we show that 1) cavity losses are not significant compared to QE effect, 2) the dead-time effect leads to mathematical inconsistencies in SD-model, yet it is free of them in E-model, being however quite small compared to QE effect. Appendix B contains some mathematical details concerning evaluation of quantities of interest for different quantum states.

II. MODELS OF NON-IDEAL PHOTODETECTORS

A. SD-model

We consider a free electromagnetic mono-modal field of frequency $\omega$, enclosed in an ideal cavity together with a photodetector (in the appendix A we show that the cavity damping is not crucial if the detector has non-unit QE). The unconditional time evolution (UTE) of the field in the presence of the detector, i.e. the evolution when the detector is turned on but the outcomes of the measurements are disregarded (not registered), is described by the master equation [18, 22, 34]

$$\dot{\rho} = -i\omega (\hat{n}\rho - \rho\hat{n}) - \frac{\lambda}{2} \left( \hat{n}\rho + \rho\hat{n} - 2\hat{A}\rho \right),$$

where $\hat{A}\rho \equiv \hat{a}\rho\hat{a}^\dagger$ is a superoperator and $\hat{n} = \hat{a}^\dagger\hat{a}$ is the number operator. The first term stands for the free field evolution while the second describes the effect of the detector on the field due to their mutual interaction. The parameter $\lambda$ is the field-detector coupling constant, roughly equal to the ideal counting rate [46, 47].

To describe photocounting with QE $\eta$ and finite dark counts rate $\lambda d$ ($d$ is the ratio between the dark counts rate and the ideal photon counting rate), we assume the following expression for the QJS (c.f. the expression resulting from the microscopic model in [47])

$$\dot{J}_\rho = \lambda \left( \eta \hat{A} + d \right) \rho.$$

It describes the action of the detector on the field upon a photodetection, and its trace gives the probability per unit time of the click. Actually, the microscopic model [46] suggests that [47] has a diagonal form in the Fock basis, but this will not be important here, since we shall be interested only in diagonal elements. The first term within the parenthesis describes the absorption of a photon from the field with probability per unit time $\Tr [\eta \lambda \hat{A}\rho] = \eta \lambda \bar{n}$, where $\bar{n}$ is the field mean photon number – this means that the detector ‘sees’ all the photons. The second term describes the occurrence of a dark count with field-independent probability density $\lambda d$, and this event by itself does not modify the field state (the field state after a single dark count is $\lambda d/\Tr (\lambda d) \rho$).

However, when both terms are present, the field state upon a detector’s click becomes a mixture of both outcomes.

From the quantum trajectories approach and CPM [1, 2, 5], all the quantities related to photodetection can be calculated provided the complementary no-count superoperator $\hat{S}_t$ is known ($\hat{S}_t$ describes the action of the detector on the field during the time interval $t$ without registered counts). Acting $\hat{S}_t$ on the initial field state $\rho_0$, the no-count state $\rho_S \equiv \hat{S}_t \rho_0$ obeys Eq. (8) when one subtracts the term [1] on the RHS (see [46, 47]). Moreover, as we are interested in calculating probabilities, we shall disregard phase factors $\exp(\pm i\omega nt)$, since they are canceled in any trace evaluation. So the evolution equation of $\rho_S$ is

$$\dot{\rho}_S = -\frac{\lambda}{2} (\hat{n}\rho_S + \rho_S\hat{n}) + \lambda q_0 \hat{A}\rho_S - \lambda d\rho_S, \quad q \equiv 1 - \eta.$$  (5)

Setting the transformation

$$\rho_S = e^{-d\lambda t} \hat{U}_t \rho_1, \quad \hat{U}_t \rho = e^{-\lambda \overline{n} t/2} \rho e^{-\lambda \overline{n} t/2}$$

in Eq. (5) we obtain a simple equation for $\rho_1$

$$\dot{\rho}_1 = \lambda q_0 e^{-\lambda t} \hat{A}\rho_1,$$

whose solution is

$$\rho_1 = \sum_{l=0}^{\infty} \frac{(q_0)^l}{l!} \hat{a}_l^\dagger \rho_0 (\hat{a}_l^\dagger)^l \equiv \exp (q_0 \hat{A}) \rho_0,$$  (8)

where

$$\phi_t = 1 - e^{-\lambda t}.$$  (9)

Thus the no-count superoperator is

$$\hat{S}_t \rho_0 = e^{-d\lambda t} \hat{U}_t (e^{q_0 \hat{A}} \rho_0).$$  (10)
The field UTE superoperator $\hat{T}_t$, defined as the solution to Eq. (4), is naturally given by setting $d = \eta = 0$ in Eqs. (3) and (11), i.e.
\[
\hat{T}_t = \hat{U}_t(e^{\phi_t \hat{A}} \rho_0) .
\]
(11)

We introduced in Eq. (8) a compact notation for the infinite sum in terms of the exponential superoperator. We can deal with such superoperators as they were common operators, provided we use the ‘commutation relations’
\[
\hat{A}\hat{U}_t = e^{-\lambda t}\hat{U}_t\hat{A} , \quad e^{\hat{Y}\hat{U}_t} = \hat{U}_t\exp(ye^{-\lambda t}) ,
\]
(12)

obtained by expanding the superoperators in series.

Now we can calculate the $m$-counts superoperator $\hat{N}_t(m)$, that describes the field state after $m$ registered counts (whatever real or dark ones) in the time interval $(0, t)$, and whose trace gives the probability for this event. It reads
\[
\hat{N}_t(m)\rho = \int \cdots \int \hat{h}\rho ,
\]
where the integrals are evaluated over all the time intervals between the counts
\[
\int \cdots \int \equiv \int_0^t dt_m \int_0^{f_m} dt_{m-1} \cdots \int_0^{f_2} dt_1
\]
(14)
and the conditioned density operator is
\[
\hat{h}\rho = \hat{S}_t \hat{S}_{t-m} \cdots \hat{J} \hat{S}_t \rho .
\]
(15)

Expanding the QS [4] in Eq. (15) in terms of $\eta\hat{A}$ and $d$, one obtains a finite sum whose first term, proportional to $d^0$, describes the detection of $m$ photons:
\[
\hat{h}^{(0)} = (\lambda\eta)^m \hat{S}_t \hat{S}_{t-m} \cdots \hat{A} \hat{S}_t
\]
(16)
\[
= (\lambda\eta)^m e^{-\lambda(t_1 + t_2 + \cdots + t_m)} \hat{S}_t \hat{A}^m .
\]

After integrating (10) we obtain the first term in (13), describing the field state after the loss by absorption of $m$ photons,
\[
\hat{n}_t(m) \equiv \int \cdots \int \hat{h}^{(0)} = \hat{S}_t \frac{(\eta\phi_t \hat{A})^m}{m!} .
\]
(17)
Calculating in a similar way the contribution of the terms with higher powers in $d$ we arrive at the formula
\[
\hat{N}_t(m) = \sum_{k=0}^{m} \frac{(d\lambda t)^k}{k!} \hat{n}_t(m - k) = \hat{S}_t \frac{(d\lambda t + \eta\phi_t \hat{A})^m}{m!} .
\]
(18)

One can easily verify that the $m$-counts superoperators (18) satisfy identically the fundamental relation (11) i.e.
\[
\sum_{m=0}^{\infty} \hat{N}_t(m) = \hat{T}_t .
\]
(19)

The factorial moments of the photocounts distribution are easily evaluated as
\[
m(m-l) = \sum_{m=0}^{\infty} m(m-l) \text{Tr}[\hat{N}_t(m)\rho] = \text{Tr}[\hat{S}_t (d\lambda t + \eta\phi_t \hat{A})^{l+1} \exp(d\lambda t + \eta\phi_t \hat{A})\rho] = \text{Tr}[\hat{U}_t (d\lambda t + \eta\phi_t \hat{A})^{l+1} e^{\phi_t \hat{A}} \rho] .
\]
(20)

Thus we need to calculate the expression
\[
\Phi_k(b, x) \equiv \text{Tr}[\hat{U}_t e^{x \hat{A}} \hat{A}^k\rho]
\]
(22)

(see Eq. (8) for the expression of $\Phi_k$) we obtain general expressions for the lower factorial moments
\[
\bar{m}_t = d\lambda t + \eta\bar{n}\phi_t
\]
(23)
\[
\bar{m}(m-1) = (d\lambda t)^2 + 2\eta\bar{n}d\lambda t\phi_t + (\eta\phi_t)^2 n(n-1),
\]
(24)
where $\bar{n}$ and $n(n-1)$ are the factorial moments of the initial density operator.

Another measurable quantity we consider here is the waiting time distribution. It describes the probability density for registering two consecutive clicks separated by the time interval $\tau$, under the condition that the first one occurred at time $t$. Its non-normalized form is
\[
W_t(\tau) = \text{Tr}[\hat{S}_\tau \hat{J}\hat{T}_t \rho] ,
\]
(25)
and the mean waiting time is
\[
\bar{\tau} = N^{-1} \int_0^T d\tau W_t(\tau) , \quad N = \int_0^T d\tau W_t(\tau) ,
\]
(26)
where $T$ is the time interval during which one evaluates the averaging in experiments. As will be shown in section T $T$ is an important parameter due to the presence of dark counts. After straightforward manipulations, using the ‘commutation relations’ (12), we obtain
\[
W_t(\tau) = e^{-d\lambda\tau} \left[ \eta^2 e^{-\lambda(2\tau + \tau)} \Phi^W_2 + \eta d e^{-\lambda(1 + e^{-\lambda\tau})} \Phi^W_1 + d^2 \Phi^W_0 \right] ,
\]
where
\[
\Phi^W_k = \Phi_k(\tau + 1, 1 - e^{-\lambda\tau}(\eta + (1 - \eta)e^{-\lambda\tau})) .
\]
(27)
In the appendix [A] we consider the dead-time effect and show that it cannot be consistently incorporated into SD-model, because the QJS [4] is an unbounded superoperator and the resulting counting probability is non-normalizable. This is just one more mathematical inconsistency [39] of the SD-model. In the appendix [B] we evaluate the expression [21] for three kinds of states: coherent, number and thermal.

B. E-model

We now repeat the same procedures for E-model in which the QJS is

\[ \hat{J}_\rho = \lambda (\eta \hat{e} + d) \rho, \]  

(28)

where \( \hat{e} \equiv \hat{E}_- \rho \hat{E}_+ \). The probability per unit time for detecting a photon is \( \eta \lambda (1 - p_0) \), where \( p_0 = \langle 0 | \rho | 0 \rangle \), so the detector ‘sees’ whether there is any photon in the cavity. In principle, the parameter \( \lambda \) is different from the one in SD-model, but in the context of this paper it will be always clear which one we are dealing with. The field UTE is described by an equation similar to Eq. [33], obtained by doing the substitution \( \{ \hat{a}, \hat{a}^\dagger \} \rightarrow \{ \hat{E}_-, \hat{E}_+ \} \) in the non-unitary evolution (second term on the RHS). So the no-count state \( \rho_S \) obeys the equation

\[ \dot{\rho}_S = -\frac{\lambda}{2} \left( \hat{A} \rho_S + \rho_S \hat{A}^\dagger \right) + \lambda q \hat{e} \rho_S - d \lambda \rho_S, \]  

(29)

[similar to Eq. [5] where \( \hat{A} \equiv \hat{E}_+ \hat{E}_- = 1 - \hat{A}_0, \hat{A}_0 \equiv |0\rangle \langle 0 | \). Setting the transformation

\[ \rho_S = e^{-\lambda t} e^{-\lambda t/2} \rho_1 e^{-\lambda t/2} \]  

(30)

in Eq. [29] and using the property \( \exp(a \hat{A}) = \hat{A}_0 + e^a \hat{A} \), we obtain the differential equation for \( \rho_1 \)

\[ \dot{\rho}_1 = \lambda q e^{-\lambda t} \left( \hat{A}_0 + e^{\lambda t/2} \hat{A} \right) \hat{E}_- \left( \hat{A}_0 + e^{\lambda t/2} \hat{A} \right) \]  

(31)

We solve this equation by projecting it onto orthogonal subspaces spanned by projectors \( \{ \hat{A}_0, \hat{A}_0 \} \). Moreover, since at the end we shall be interested only in calculating probabilities, we consider only the diagonal part in Fock basis for quantities of interest, thus disregarding the terms whose trace is null, such as \( \hat{A}_0 \rho \hat{A}_0 \).

Multiplying Eq. [31] by \( \hat{A} \) on both sides we obtain

\[ \frac{d}{dt} \left( \hat{A} \rho_1 \hat{A} \right) = \lambda q \hat{A} \hat{E}_- \left( \hat{A}_0 \rho_1 \hat{A} \right) \hat{E}_+ \hat{A}, \]  

(32)

whose solution is

\[ \hat{A} \rho_1 \hat{A} = \hat{A} \left( e^{2 \lambda t} \rho_0 \right) \hat{A} \]  

(33)

and we note again that all the composite superoperators, such as \( \exp(y \hat{e}) \), are understood as power expansions. Now, multiplying [31] by \( \hat{A}_0 \) on both sides and using the solution [33] we get an equation for \( \hat{A}_0 \rho_1 \hat{A}_0 \)

\[ \frac{d}{dt} \left( \hat{A}_0 \rho_1 \hat{A}_0 \right) = \lambda q \hat{A}_0 \left( e^{-\lambda (1 - q \hat{e})} \rho_0 \right) \hat{A}_0 \]  

(34)

with solution

\[ \hat{A}_0 \rho_1 \hat{A}_0 = \hat{A}_0 \left( \frac{1 - q \hat{E}_t}{1 - q \hat{e}} \rho_0 \right) \hat{A}_0, \]  

(35)

Thus the diagonal form of the no-count superoperator, which we write just in terms of the projector \( \hat{A}_0 \) and the unit operator, is

\[ \hat{S}_t \rho_0 = e^{-d \lambda t} \left[ \hat{R}_t + \hat{A}_0 \left( \frac{1 - \hat{R}_t}{1 - q \hat{e}} \right) \hat{A}_0 \right] \rho_0, \]  

(36)

where we use the notation \( (\hat{A}_0 \hat{Q} \hat{A}_0) \rho = \hat{A}_0 (\hat{Q} \rho) \hat{A}_0 \).

Repeating the steps [16] – [18], we obtain first the conditioned density operator

\[ \hat{\rho}_t (m) = e^{-d \lambda t} \left[ \hat{R}_t + \hat{A}_0 \hat{R}_{tm} - \hat{R}_t \right] \left( \hat{A}_0 \hat{Q}_m \hat{A}_0 \right) \rho \]  

(37)

After evaluating the time integrals as in [17] we get

\[ \hat{\rho}_t (m) = e^{-d \lambda t} \left[ \hat{R}_t + \hat{A}_0 \hat{R}_{tm} - \hat{R}_t \right] \left( \hat{A}_0 \hat{Q}_m \hat{A}_0 \right) \rho \]  

(38)

for \( m > 0 \) and \( \hat{\rho}_t (0) = \hat{S}_t \). Finally, analogously to the expression [18], we obtain the m-counts superoperator

\[ \hat{N}_t (m) = e^{-d \lambda t} \left[ \hat{R}_t + \hat{A}_0 \hat{R}_{tm} - \hat{R}_t \right] \left( \hat{A}_0 \hat{Q}_m \hat{A}_0 \right) \rho \]  

(39)

where the last term is zero for \( m = 0 \). One can easily verify that the superoperator \( \hat{N}_t (m) \), Eq. [38], satisfies relation [19].

After lengthy however straightforward calculations we obtain the following expressions for the initial factorial photocounts moments

\[ m_t = d \lambda t + \eta n (1 - \Xi_1), \]  

(39)

\[ m (m - 1)_t = (d \lambda t)^2 + 2 \eta m d \lambda t (1 - \Xi_1) \]  

(40)

\[ + \eta^2 \left[ n (n - 1) (1 - \Omega) - 2 n \lambda \Xi_1 \right], \]
where
\[
\Xi_k \equiv \frac{1}{n} \text{Tr} \left[ \frac{\hat{\varepsilon}^k}{1 - \hat{\varepsilon}} \hat{R}_t^0 \rho \right], \quad \hat{R}_t^0 \equiv \hat{R}_t(q = 1),
\]
\[
\Omega \equiv \frac{2}{n(n - 1)} \text{Tr} \left[ \left( \frac{\hat{\varepsilon}}{1 - \hat{\varepsilon}} \right)^2 \hat{R}_t^0 \rho \right].
\]

Using Eq. (40), the waiting time distribution density is found to be
\[
W_t(\tau) = e^{-d\lambda t} \{ (\lambda d)^2 [1 - \text{Tr}(\hat{R}_t^0 \rho)]
+ \text{Tr}[(\hat{J}\hat{R}_t + \lambda \hat{d}\hat{A}_0 \frac{1 - \hat{R}_t}{1 - q\hat{\varepsilon}} \hat{A}_0) \hat{J}\hat{R}_t^0 \rho] \}.
\]

In the appendix A we show that the dead-time effect can be incorporated into E-model, however its effect is quite small compared to the non-unit QE effect, so we disregard it in this paper. In the appendix B we obtain formulas for Eqs. (11), (12) and (13) in terms of \(\rho_n\) and evaluate them for the coherent, number and thermal states.

III. VERIFYING CPM

Basing ourselves on published experimental data we chose the following numerical values for the model parameters: \(\eta = 0.6\) for the QE and \(d = 5 \cdot 10^{-3}\) for the dark counts rate (normalized by the ideal counting rate). We do not attribute any fixed value to \(\lambda\) since our analysis will be given in terms of the dimensionless \(\lambda t\). Many photodetection quantities in different contexts were reported in, e.g., \([18, 17, 32, 31, 34, 33, 32]\), so here we shall consider few of them that could check the validity of either, the SD- or the E-model in photocounting experiments.

First we analyze the counting statistics. In figure 1 we plot \(\bar{n}_t\) as function of \(\lambda t\) for both models for two values of the initial mean photon number, \(\bar{n} = 50\) and 100. Initially, \(\bar{n}_t\) increases steeply due to photons absorption, and after some time the growths turns linear with much smaller slope due to the dark counts. We call the time interval during which the photons are absorbed (representing the duration of the steep increase in the number of counts) the effective counting time \(t_E\). In the E-model \(t_E\) is proportional to the initial average photon number, contrary to the SD-model [as seen from the figure and formulas (23) and (59)]. So the experimental analysis of the dependence of \(t_E\) on \(\bar{n}\) seems to us a feasible way for verifying which model could hold in practice, because, according to the SD-model, \(t_E\) does not depend on \(\bar{n}\). Moreover, one could also check the validity of each model by verifying whether \(\bar{n}_t\) depends on the initial field state: in the SD-model it is independent of the field state, while in the E-model \(\bar{n}_t\) is quite sensible to it: in figure 1 one sees a notable difference between thermal and coherent states, although not so much between number and coherent states. This can be explained by a great difference in the values of Mandel’s Q-factor characterizing the statistics of photons in the initial state: it equals \(-1\) and 0 for number and coherent states, respectively, whereas it is very big \((Q_{th} = \bar{n})\) for the thermal states with big mean numbers of photons.

Now we analyze the normalized second factorial moment
\[
K_t \equiv \frac{m(m - 1)}{\bar{m}_t^2},
\]
for the same initial states with mean photon number \(\bar{n} = 50\). For the number and thermal states \(K_t\) as function of \(\lambda t\) is shown in figure 2 and for the coherent state we get \(K_t = 1\), so it is not plotted. In the asymptotic time limit and for non-zero dark counts rate, the same value \(K_\infty \to 1\) holds for both models, however the transient is model dependent. In the SD-model without considering dark counts \(K_t\) is time-independent, writing as \(K = n(n - 1)/\bar{n}^2\) (\(\bar{n}\) and \(n(n - 1)\) correspond to the initial field state), nevertheless it depends on the initial field state: \(K = 2\) for the thermal state and \(K = 1 - 1/\bar{n}\) for the number state. By including the dark counts in the analysis this constant behavior is slowly modified as time goes on, see figure 2.

In the E-model in the absence of dark counts \(K_t\) starts at the value
\[
\lim_{t \to 0} K_t = \frac{\text{Tr}(\hat{\varepsilon}^2 \rho)}{[\text{Tr}(\hat{\varepsilon} \rho)]^2} = \frac{1 - \rho_0 - \rho_1}{(1 - \rho_0)^2},
\]

FIG. 1: Mean photocounts number \(\bar{n}_t\) in E-model for coherent, number and thermal states (indicated in the figure, the lower curves are labeled analogously) as function of time for two values of the initial photon number: the lower curves correspond to \(\bar{n} = 50\) and the upper – to \(\bar{n} = 100\). In the inset we plot \(\bar{n}_t\) for the SD-model, which is independent from field state.
which is exactly 1 for the number state and very close to 1 for the thermal state with the chosen values of $\bar{n}$. With the course of time, $K_t$ attains the same values as for the SD-model (for respective initial field states) when all the photons have been counted. By taking in account the dark counts effect such a behavior is slightly modified, yet it is quite different from the behavior in the SD-model, as shown in the figure 2. This is another possible manner for verifying the applicability of SD- or E- models.

We now turn our attention to the waiting time analysis. It is important to define the time interval in which we do the average: if one has non-zero dark counting rate, then by performing the average over a very large time interval, we shall always get for the mean waiting time the value $\bar{\tau} \sim (\lambda d)^{-1}$, which is nothing but the mean time interval between consecutive dark counts. Since experimentally the average is done over finite time intervals, we shall proceed in the same way: the mean waiting time for initial times, when the photon number is significant, is roughly $(\eta \lambda)^{-1}$ (because $\eta \lambda$ is the effective counting rate), so we shall take the average over a time interval $\nu = 10(\eta \lambda)^{-1}$. This means that if one does not detect consecutive counts within the time $\nu$, such a measurement will not contribute to the average. In an ideal case this procedure is not necessary because the probability for registering consecutive clicks separated by a large time interval is zero.

In figure 3 we plot the mean waiting time for the SD- and E- models, for the number and thermal initial states (for the coherent state we obtain a curve almost identical to the one for a number state) with $\bar{n} = 100$ as function of the mean photon number in the cavity at the moment of the first click,

$$N_{CAV} = \text{Tr} \left[ \hat{n} \hat{T}_t \rho_0 \right] = \begin{cases} \bar{n}e^{-\lambda t} & \text{for SD-model} \\ \bar{n} & \text{for E-model.} \end{cases}$$

(For completeness, in the inset of figure 3 we plot $N_{CAV}$ as function of $\lambda t$ for both models.) For the E-model, we see that when $N_{CAV}$ becomes less than 1, the waiting time starts to increase dramatically due to the dominance of dark counts, which are much more rare events than absorption of photons. This is a drastic difference from the ideal case, in which no counts occur after all the photons having been absorbed, so the mean waiting time saturates at the inverse value of the counting rate, as shown in [39]. Moreover, from figure 3 one verifies that as long as there are photons in the cavity the mean waiting time is nearly time-independent within the E-model (and truly independent in the ideal case [39]), and do increase substantially in time for SD-model. This is another notable qualitative difference we suppose one could verify experimentally.

**IV. SUMMARY AND CONCLUSIONS**

In this paper we have generalized the continuous photodetection model through a careful quantum treatment of non-ideal effects that are ubiquitous in experiments. We derived general expressions for the fundamental operations in the presence of non-unit quantum efficiency and dark counts, and calculated explicitly the photocounts and the waiting time probability distributions for initial
coherent, number and thermal field states. By calculating the first and second factorial moments of the photocounts and the mean waiting time, we showed that in standard photodetection experiments one could check the applicability of the QJS of SD- or E- models. Namely, we indicated three different ways for revealing the actual QJS: (1) quantitatively, one should study the time dependence of the normalized second factorial photocounts moment. Qualitatively, we showed that the models can be also distinguished by measuring: (2) whether the effective detection time depends on the initial average photon number in the cavity and (3) whether the mean waiting time is modified as time goes on. To that end we have considered three different kinds of field in the cavity: the number, coherent and thermal states; each one on its own permitted to do comparisons between the two studied QJS’s. Results with other kinds of fields could also be presented here, as for instance the binomial state or the so-called squeezed state, however, no new physics related to the goals of the paper appears. A last remark, if the experimental data would depart significantly from the theoretical prediction one should reconsider both models and try to look for alternative mechanisms to reproduce the outcomes.

In conclusion, we believe that our theoretical treatment could provide clues for an experimental verification of the CPM, contributing with valuable insights about the quantum nature of the photodetection in cavities, as well as giving rise to the possibility of field state manipulation through detector post-action on the field.

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APPENDIX A: CAVITY DAMPING AND DEAD-TIME

First we include the effect of cavity damping in our treatment. In quantum optics experiments the background photons number is negligible, so we can model the cavity as a thermal reservoir with zero mean excitations number, described by the standard master equation [1]. Then the UTE equation in SD-model should be

\[ \dot{\rho} = -i\omega (\hat{n}\rho - \rho\hat{n}) - \frac{\lambda c}{2} \left( \hat{n}\rho + \rho\hat{n} - 2\hat{A}\rho \right) \]

where \( \lambda c \) is the cavity damping rate. From it, following the steps of sec. [1] we obtain the no-count superoperator

\[ \hat{S}_t\rho_0 = e^{-d\lambda t}\hat{U}_t\left( e^{\phi_t\hat{A}}\rho_0 \right), \quad \hat{\phi}_t = \frac{1}{p} \left( 1 - e^{-\lambda pt} \right), \]

\[ p \equiv 1 + c, \quad \tilde{q} \equiv 1 - n + c = p - n. \]  

The value of \( c \) should be at the most of order of \( 10^{-1} \) in order to make viable the CPM. In this case we see that if one takes into account the QE drawback, the cavity damping does not modify substantially the resulting expressions. Therefore we disregard its effect in this paper.

The dead-time effect means that immediately after a click the detector is unable to register another count within a quite small time interval \( x, \lambda x \ll 1 \). In our framework we can describe this effect as the occurrence of the UTE during the time \( x \) immediately after the count, so the conditioned density operator \( \hat{S}_t \) becomes

\[ \hat{h}\rho \equiv \hat{S}_{t-x}\hat{J}_2\hat{S}_{t-m-1-x}\hat{J}_2\ldots\hat{J}_2\hat{S}_t\rho \]

\[ = \hat{S}_{t-x}\hat{J}\hat{S}_{t-m-1-x}\hat{J}\ldots\hat{J}\hat{S}_t\rho, \]  

where the dead-time superoperator, under condition \( \lambda x \ll 1 \), is found to be

\[ \hat{\Theta} \equiv \hat{S}_x\hat{T}_x = \exp(x\hat{J}) \]

for both SD- and E- models, with respective QJS’s.

In SD-model the resulting dead-time superoperator is unbounded, as well as \( \hat{J} \), so it can bring some mathematical inconsistences. For example, the \( m \)-counts superoperator with dead-time effect is found to be

\[ \hat{\chi}_m(t) = \hat{S}_t\left[ d\lambda t + e^{dx_n x}(\phi x) \right]^m \]

where

\[ \hat{\chi}_m(t) \equiv e^{\phi x_{n+1}} - e^{\phi x_n} e^{-\lambda pt}. \]

If one evaluates, for instance, \( \text{Tr} \left[ \sum_m m^k \hat{N}_m(t)\rho \right] \) one will find a divergent result because \( \hat{\chi} \) increases much faster than the decreasing terms.

In the E-model \( \hat{J} \) is a bounded superoperator, so the dead-time corrections will be of order \( \eta x \ll 1 \), much less relevant than the non-unit QE drawback.

APPENDIX B: EVALUATION OF TRACES

In this appendix we derive general expressions for both SD- and E- models and evaluate them for a general initial density operator \( \rho = \sum \rho_n|n\rangle\langle n| \) (non-diagonal elements do not contribute to the trace in the expressions below). We shall analyze three particular field states: coherent state,

\[ \rho_n = e^{-\tilde{n}n}/n!, \quad n(n-1) = \tilde{n}^2 \],

number state,

\[ \rho_n = \delta_{n,\bar{n}} \quad \text{with integer} \quad \bar{n}, \]

and thermal state,

\[ \rho_n = (1 - \alpha)\alpha^n, \quad \alpha = n/\bar{n} + 1, \quad n(n-1) = 2\tilde{n}^2. \]

In the SD-model, formula [27] results in:
coherent state
\[ \Phi_k^W = \bar{n}^k \exp \left[ \eta \bar{n} \phi \rho e^{-\lambda t} \right] \]  \hspace{1cm} (B1)

number state
\[ \Phi_k^W = \frac{\bar{n}^k}{(\bar{n}-k)!} \left( 1 - \eta \phi \rho e^{-\lambda t} \right)^{\bar{n}-k} \]  \hspace{1cm} (B2)

thermal state
\[ \Phi_k^W = \frac{k!(1-\alpha)\alpha^k}{[1-\alpha(1-\eta \phi \rho e^{-\lambda t})]^{k+1}}. \]  \hspace{1cm} (B3)

The formula (46) yields \( N_{CAV} = \bar{n} e^{-\lambda t} \) for all the states.

In the E-model we need to expand the superoperators as series of \( \hat{\xi} \) and evaluate the sums. For Eqs. (11) and (42) we obtain
\[ \Xi_k = \frac{e^{-\lambda t}}{\bar{n}} \sum_{n, l, m=0}^{\infty} \frac{(\lambda t)^m}{m!} \rho_{n+l+m+k} \]
\[ = \frac{2 e^{-\lambda t}}{n(n-1)} \sum_{n, l, m=0}^{\infty} \frac{n(\lambda t)^m}{m!} \rho_{n+l+m+1} \]
\[ = \frac{e^{-\lambda t}}{n(n-1)} \sum_{n, m=0}^{\infty} n(n-1)(\lambda t)^m m! \rho_{n+m}. \]  \hspace{1cm} (B4)

For the coherent state we use the formula
\[ \sum_{k=0}^{\infty} \frac{x^k}{k!(k+n)!} = I_n(2\sqrt{x}), \]
where \( I_k(x) \) is the modified Bessel function \(^{50}\) to obtain
\[ \Xi_k = \frac{e^{-\lambda t}/\bar{n}}{n(n-1)} \sum_{n=0}^{\infty} (n+1) \left( \frac{\bar{n}}{\lambda t} \right)^{(n+k)/2} I_{n+k}(2\sqrt{\bar{n} \lambda t}), \]  \hspace{1cm} (B9)
\[ \Omega = \frac{e^{-\lambda t}/\bar{n}}{n(n-1)} \sum_{n=2}^{\infty} n(n-1) \left( \frac{\bar{n}}{\lambda t} \right)^{n/2} I_{n+k}(2\sqrt{\bar{n} \lambda t}), \]  \hspace{1cm} (B10)
\[ \Psi_k(q, \beta) = e^{-\bar{n}/(\lambda t)} \sum_{n=0}^{\infty} \left( \frac{\bar{n}q^2}{(\lambda t)^{2n}} \right)^{n/2} I_{n+k}(2\sqrt{\bar{n} \lambda t}), \]  \hspace{1cm} (B11)

The above series can be transformed in a finite integral using
\[ \sum_{k=0}^{\infty} k^k I_{k+\nu}(z) = \frac{e^{t z^2}}{z^n} \int_0^z \tau^n e^{-\tau^2/(2n)} I_{\nu-1}(\tau) d\tau, \]
valid for \( \text{Re}(\nu) > 0. \)

For the number state, using \( \sum_{k=0}^{\infty} x^k/k! = e^x \Gamma(n+1, x)/n! \), where \( \Gamma(\alpha, x) = \int_x^{\infty} t^{\alpha-1} e^{-t} dt \) is the incomplete complementary Gamma function \(^{50}\) we obtain
\[ \Xi_k = \frac{\Gamma(\bar{n} - k + 2, \lambda t) - \lambda t \Gamma(\bar{n} - k + 1, \lambda t)}{\bar{n}(\bar{n} - k)!}, \]  \hspace{1cm} (B12)
\[ \Omega = \frac{\Gamma(\bar{n} + 1, \lambda t) - 2\lambda \Gamma(\bar{n}, \lambda t) + (\lambda t^2) \Gamma(\bar{n} - 1, \lambda t)}{n(n-1)(\bar{n} - 2)!}, \]  \hspace{1cm} (B13)
\[ \Psi_k(q, \beta) = q^{\bar{n}-k} e^{\lambda \beta/q} \frac{\Gamma(\bar{n} - k + 1, \lambda \beta/q)}{(\bar{n} - k)!}. \]  \hspace{1cm} (B14)

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