Optimal No-Regret Learning in Strongly Monotone Games with Bandit Feedback

Tianyi Lin
Department of Electrical Engineering and Computer Science, UC Berkeley, darren_lin@berkeley.edu

Zhengyuan Zhou
Stern School of Business, New York University, zzhou@stern.nyu.edu

Wenjia Ba
Stanford Graduate School of Business, Stanford University, wenjiaba@stanford.edu

Jiawei Zhang
Stern School of Business, New York University, jzhang@stern.nyu.edu

We consider online no-regret learning in unknown games with bandit feedback, where each agent only observes its reward at each time – determined by all players’ current joint action – rather than its gradient. We focus on the class of smooth and strongly monotone games and study optimal no-regret learning therein. Leveraging self-concordant barrier functions, we first construct an online bandit convex optimization algorithm and show that it achieves the single-agent optimal regret of \( \tilde{\Theta}(\sqrt{T}) \) under smooth and strongly-concave payoff functions. We then show that if each agent applies this no-regret learning algorithm in strongly monotone games, the joint action converges in last iterate to the unique Nash equilibrium at a rate of \( \tilde{\Theta}(1/\sqrt{T}) \). Prior to our work, the best known convergence rate in the same class of games is \( O(1/T^{1/3}) \) (achieved by a different algorithm), thus leaving open the problem of optimal no-regret learning algorithms (since the known lower bound is \( \Omega(1/\sqrt{T}) \)). Our results thus settle this open problem and contribute to the broad landscape of bandit game-theoretical learning by identifying the first doubly optimal bandit learning algorithm, in that it achieves (up to log factors) both optimal regret in the single-agent learning and optimal last-iterate convergence rate in the multi-agent learning. We also present results on several simulation studies – Cournot competition, Kelly auctions and distributed regularized logistic regression – to demonstrate the efficacy of our algorithm.

Key words: no-regret learning; bandit feedback model; strongly monotone games; optimal regret; optimal last-iterate convergence rate; eager projection
1. Introduction

In multi-agent online learning (Cesa-Bianchi and Lugosi 2006b, Shoham and Leyton-Brown 2008, Busoniu et al. 2010), a set of agents are repeatedly making decisions and accumulating rewards over time, where each agent’s action impacts not only its own reward, but that of the others. However, the mechanism of this interaction – the underlying game that specifies how an agent’s reward depends on the joint action of all – is unknown to agents, and agents may not even be aware that there is such a game. As such, from each agent’s own perspective, it is simply engaged in an online decision making process, where the environment consists of all other agents who are simultaneously making such sequential decisions, which are of consequence to all agents.

In the past two decades, the above problem has actively engaged researchers from two fields: machine learning (and online learning in particular), which aims to develop single-agent online learning algorithms that are no-regret in an arbitrarily time-varying and/or adversarial environment (Blum 1998, Shalev-Shwartz 2007, Shalev-Shwartz et al. 2012, Arora et al. 2012, Hazan 2016); and game theory, which aims to develop (ideally distributed) algorithms (see Fudenberg and Levine (1998b) and references therein) that efficiently compute a Nash equilibrium (a joint optimal outcome where no one can do better by deviating unilaterally) for games with special structures\(^1\).

Although these two research threads initially developed separately, they have subsequently merged and formed the core of multi-agent/game-theoretical online learning, whose main research agenda can be phrased as follows: Will joint no-regret learning lead to a Nash equilibrium, thereby reaping both the transient benefits (conferred by low finite-time regret) and the long-run benefits (conferred by Nash equilibria)?

More specifically, through the online learning lens, the agent \(i\)’s reward function at \(t\) – viewed as a function solely of its own action – is \(u_i^t(\cdot)\), and it needs to select an action \(x_i^t \in \mathcal{X}_i \subseteq \mathbb{R}^{n_i}\) before \(u_i^t(\cdot)\) – or other feedback associated with it – is revealed. In this context, no-regret algorithms ensure that the difference between the cumulative performance of the best fixed action and that of the learning algorithm, a widely adopted metric known as regret (\(\text{Reg}_{\text{Alg}}^{\text{Alg}} = \max_{x \in \mathcal{X}_i} \sum_{t=1}^{T} (u_i^t(x) - u_i^t(x_i^t))\)),
grows sublinearly in $T$. This problem has been extensively studied; in particular, when gradient feedback is available – $\nabla_{x_i} u_i^t(x_i^t)$ can be observed after $x_i^t$ is selected – the minimax optimal regret is $\Theta(\sqrt{T})$ for convex $u_i^t(\cdot)$ and $\Theta(\log T)$ for strongly convex $u_i^t(\cdot)$. Further, several algorithms have been developed that achieve these optimal regret bounds, including “follow-the-regularized-leader” (Kalai and Vempala, 2005), online gradient descent (Zinkevich, 2003b), multiplicative/exponential weights (Arora et al., 2012) and online mirror descent (Shalev-Shwartz and Singer, 2007). As these algorithms provide optimal regret bounds and hence naturally raise high expectations in terms of performance guarantees, a recent line of work has investigated when all the agents apply a no-regret learning algorithm, what the evolution of the joint action would be, and in particular, whether the joint action would converge in last iterate to a Nash equilibrium (if one exists).

These questions turn out to be difficult and had remained open until a few years ago, mainly because the traditional Nash-seeking algorithms in the economics literature are mostly either not no-regret or only exhibit convergence in time-average (ergodic convergence), or both. Despite the challenging landscape, in the past five years, affirmative answers have emerged from a fruitful line of work and the new analysis tools developed therein, first on the qualitative last-iterate convergence (Krichene et al., 2015, Balandat et al., 2016, Zhou et al., 2017, 2018, Mertikopoulos et al., 2019, Mertikopoulos and Zhou, 2019, Golowich et al., 2020b,a) in different classes of continuous games (such as variationally stable games), and then on the quantitative last-iterate convergence rates in more specially structured games such as co-coercive games or strongly monotone games (Lin et al., 2020, Zhou et al., 2021). In particular, very recently, Zhou et al. (2021) has shown that if each agent applies a version of online gradient descent, then the joint action converges in last iterate to the unique Nash equilibrium in a strongly monotone game at an optimal rate of $O(1/T)$.

Despite this remarkably pioneering line of work, which has elegantly bridged no-regret learning with convergence to Nash in continuous games and thus renewed the excitement of game-theoretical learning, a significant impediment still exists and limits their practical impact. Specifically, in the multi-agent learning setting, an agent is rarely able to observe gradient feedback. Instead, in most
cases, only bandit feedback is available: each agent observes only its own reward after choosing an action each time (rather than the gradient at the chosen action). This consideration of practical feasible algorithms then brings us to a more challenging and less explored desideratum in multi-agent learning: if each agent applies a no-regret online bandit convex optimization algorithm, would the joint action still converge to a Nash equilibrium? At what rate and in what class of games?

### 1.1. Related work

To appreciate the difficulty and the broad scope of this research agenda, we start by describing the existing related literature. First of all, we note that single-agent bandit convex optimization algorithms – and their theoretical regret characterizations – are not as well-developed as their gradient counterparts. More specifically, Flaxman et al. (2005) and Kleinberg (2004) provided the first bandit convex optimization algorithm – known as FKM – that achieved the regret bound of $O(T^{3/4})$ for convex and Lipschitz loss functions. However, it was unclear whether $O(T^{3/4})$ is optimal. Subsequently, Saha and Tewari (2011) developed a barrier-based online convex bandit optimization algorithm and established a $\tilde{O}(T^{2/3})$ regret bound for convex and smooth cost functions, a result that has further been improved to $\tilde{O}(T^{5/8})$ (Dekel et al. 2015) via a variant on the algorithm and new analysis. More recently, progress has been made on developing algorithms that achieve minimax-optimal regret. In particular, Bubeck et al. (2015) and Bubeck and Eldan (2016) provided non-constructive arguments showing that the minimax regret bound of $\tilde{O}(\sqrt{T})$ in one and high dimension are achievable respectively, without providing any algorithm. Later, Bubeck et al. (2017) developed a kernel method based bandit convex optimization algorithm which attains the bound of $\tilde{O}(\sqrt{T})$. Independently, Hazan and Li (2016) considered an ellipsoid method for bandit convex optimization that also achieves $\tilde{O}(\sqrt{T})$.

When the cost function is strongly convex and smooth, Agarwal et al. (2010) showed that the FKM algorithm achieves an improved regret bound of $\tilde{O}(T^{2/3})$. Later, Hazan and Levy (2014) established that another variant of the barrier based online bandit algorithm given in Saha and Tewari (2011) achieves the minimax-optimal regret of $\tilde{O}(\sqrt{T})$. For an overview of the
relevant theory and applications, we refer to the recent survey (Bubeck and Cesa-Bianchi 2012, Lattimore and Szepesvári 2020) and references therein.

However, much remains unknown in understanding the convergence of these no-regret bandit convex optimization algorithms to Nash equilibria. Bervoets et al. (2020) developed a specialized distributed payoff based algorithm that asymptotically converges to the unique Nash equilibrium in the class of strictly monotone games. However, the algorithm is not known to be no-regret and no rate is given. Héliou et al. (2020) considered a variant of the FKM algorithm and showed that it is no-regret even under delays; further, provided the delays are not too large, joint FKM learning would converge to the unique Nash equilibrium in strictly monotone games (again without rates). At this writing, the most relevant and the-state-of-the-art result on this topic is presented in Bravo et al. (2018), which showed that if each agent applies the FKM algorithm in strongly monotone games, then last-iterate convergence to the unique Nash equilibrium is guaranteed at a rate of $O(1/T^{1/3})$. Per Bravo et al. (2018), the analysis itself is unlikely to be improved to yield any tighter rate. However, a sizable gap still exists between this bound and the best known lower bound given in Shamir (2013), which established that in optimization problems with strongly convex and smooth objectives (which is a one-player Nash-seeking problem), no algorithm that uses only bandit feedback (i.e. zeroth-order oracle) can compute the unique optimal solution at a rate faster than $\Omega(1/T^{1/2})$. Consequently, it remains unknown as to whether other algorithms can improve the rate of $O(1/T^{1/3})$ as well as what the true optimal convergence rate is. In particular, since the $\Omega(1/T^{1/2})$ lower bound is established for the special case of optimization problems, it is plausible that in the multi-agent setting – where a natural potential function such as the cost in optimization does not exist – the problem is inherently more difficult, and hence the convergence intrinsically slower. Further, note that the lower bound in Shamir (2013) is established against the class of all bandit optimization algorithms, not necessarily no-regret; essentially, a priori, that could mean a larger lower bound when the algorithms are further restricted to be no-regret. As such, it has been a challenging open problem to close the gap.
1.2. Our Contributions

We tackle the problem of no-regret learning in strongly monotone games with bandit feedback and settle the above open problem by establishing that the convergence rate of $\tilde{O}(1/T^{1/2})$ – and hence minimax optimal (up to log factors) – is achievable. More specifically, we start by studying (in Section 2) single-agent learning with bandit feedback – in particular bandit convex optimization – and develop an eager variant of the barrier based family of online convex bandit optimization algorithms (Saha and Tewari 2011, Hazan and Levy 2014, Dekel et al. 2015). We establish that the algorithm achieves the minimax optimal regret bound of $\tilde{O}(\sqrt{T})$. Next, extending to multi-agent learning settings, we show that if all agents employ this optimal no-regret learning algorithm (see Algorithm 2 in Section 4), then the joint action converges in last iterate to the unique Nash equilibrium at a rate of $\tilde{O}(1/T^{1/2})$. As such, we provide the first online convex bandit learning algorithm (with continuous action) that is doubly optimal (up to log factors): it achieves optimal regret in single-agent settings under strongly convex cost functions and optimal convergence rate to Nash in multi-agent settings under strongly monotone games. Finally, we conduct extensive experiments on Cournot competition, Kelly auction and distributed regularized logistic regression in Section 5. The numerical results demonstrate that our algorithm outperforms the-state-of-the-art multi-agent FKM algorithm.

2. Single-Agent Learning with Bandit Feedback

In this section, we provide a simple single-agent bandit learning algorithm that players could employ to increase their individual rewards in an online manner and prove that it achieves the near-optimal regret minimization property for bandit concave optimization (BCO)².

In BCO, an adversary first chooses a sequence of concave reward functions $f_1, f_2, \ldots, f_T : \mathcal{X} \mapsto \mathbb{R}$, where $\mathcal{X}$ is a closed convex subset of $\mathbb{R}^n$. At each round $t = 1, 2, \ldots, T$, a (randomized) decision maker has to choose a point $x^t \in \mathcal{X}$ and incurs a reward of $f_t(x^t)$ after committing to her decision. Her expected reward (where the expectation is taken with respect to her random choice) is $\mathbb{E}[\sum_{t=1}^{T} f_t(x^t)]$ and the corresponding regret is defined by $\text{Reg}_T = \max_{x \in \mathcal{X}} \{\sum_{t=1}^{T} f_t(x) - \sum_{t=1}^{T} f_t(x^t)\}$.
\[ \mathbb{E}[\sum_{t=1}^{T} f_t(x^t)]. \] In the bandit setting, the feedback is limited to the reward at the point that she has chosen, i.e., \( f_t(x^t). \)

Algorithm 1 is inspired by the algorithms developed in Saha and Tewari (2011), Hazan and Levy (2014): the main difference lies in eager projection (ours) vs. lazy projection for updating \( x^{t+1}. \) This modification is crucial to last-iterate convergence analysis when we extend Algorithm 1 to the multi-agent setting. In what follows, we present the individual algorithm components.

**Algorithm 1** Eager Self-Concordant Barrier Bandit Learning

1: **Input:** step size \( \eta_t > 0, \) module \( \beta > 0 \) and barrier \( R : \text{int}(X) \mapsto \mathbb{R}. \)

2: **Initialization:** \( x^1 = \arg \min_{x \in X} R(x). \)

3: **for** \( t = 1, 2, \ldots \) **do**

4: \( A^t \leftarrow (\nabla^2 R(x^t) + \eta_t \beta (t + 1) I_n)^{-1}/2. \) \# SCALING MATRIX

5: draw \( z^t \sim \mathbb{S}^n. \) \# PERTURBATION DIRECTION

6: play \( \hat{x}^t \leftarrow x^t + A^t z^t. \) \# CHOOSE ACTION

7: receive \( \hat{f}^t \leftarrow f_t(\hat{x}^t). \) \# GET PAYOFF

8: set \( \hat{v}^t \leftarrow n \hat{f}^t (A^t)^{-1} z^t. \) \# ESTIMATE GRADIENT

9: update \( x^{t+1} \leftarrow P_R(x^t, \hat{v}^t, \eta_t). \) \# UPDATE PIVOT

2.1. Self-Concordant Barrier

Existing bandit learning algorithms can be interpreted under the framework of mirror descent (Cesa-Bianchi and Lugosi 2006a) and the common choice of regularization is self-concordant barrier, which is a key ingredient in regret-optimal bandit algorithms when the loss function is linear (Abernethy et al. 2008) or smooth and strongly convex (Hazan and Levy 2014). Here we provide an overview and refer to Nesterov and Nemirovskii (1994) for details.

**Definition 1.** A function \( R : \text{int}(X) \mapsto \mathbb{R} \) is called a \( \nu \)-self concordant barrier for a closed convex set \( X \subseteq \mathbb{R}^n \) if (i) \( R \) is three times continuously differentiable, (ii) \( R(x) \to \infty \) if \( x \to \partial X, \) and (iii) for \( \forall x \in \text{int}(X) \) and \( \forall h \in \mathbb{R}^n, \) we have \( |\nabla^3 R(x)[h, h, h]| \leq 2(h^\top \nabla^2 R(x) h) \beta/2 \) and \( |\nabla R(x)^\top h| \leq \sqrt{\nu}(h^\top \nabla^2 R(x) h)^{1/2} \) where \( \nabla^3 R(x)[h_1, h_2, h_3] = \frac{\partial^3}{\partial t_1 \partial t_2 \partial t_3} R(x + t_1 h_1 + t_2 h_2 + t_3 h_3) \bigg|_{t_1=t_2=t_3=0}. \)
Similar to the existing online bandit learning algorithms (Abernethy et al. 2008, Saha and Tewari 2011, Hazan and Levy 2014, Dekel et al. 2015), our algorithm requires \( \nu_i \)-self-concordant barriers over \( \mathcal{X}_i \) for all \( i \in \mathcal{N} \); see Algorithm 1. However, this does not weaken its generality. Indeed, it is known that any convex and compact set in \( \mathbb{R}^n \) admits a non-degenerate \( \nu \)-self-concordant barrier (Nesterov and Nemirovskii 1994) with \( \nu = O(n) \), and such barrier can be efficiently represented and evaluated for various choices of \( \mathcal{X}_i \) in game-theoretical setting. For example, \( -\log(b - a^\top x) \) is 1-self-concordant barrier for linear constraints \( a^\top x \leq b \) and similar function is \( n \)-self-concordant barrier for \( n \)-dimensional simplex or a cube. For \( n \)-dimensional ball, \( -\log(1 - \|x\|^2) \) is 1-self-concordant barrier and \( \nu \) is even independent of the dimension.

The above definition is only given for the sake of completeness and our analysis relies on some useful facts about self-concordant barriers. In particular, the Hessian of a self-concordant barrier \( R \) induces a local norm at \( \forall x \in \text{int}(\mathcal{X}) \); that is, \( \|h\|_x = \sqrt{h^\top \nabla^2 R(x) h} \) and \( \|h\|_{x,\star} = \sqrt{h^\top (\nabla^2 R(x))^{-1} h} \) for all \( h \in \mathbb{R}^n \). The nondegeneracy of \( R \) implies that \( \| \cdot \|_x \) and \( \| \cdot \|_{x,\star} \) are both norms.

First, we define the Dikin ellipsoid at any \( x \in \text{int}(\mathcal{X}) \) as

\[
W(x) = \{ x' \in \mathbb{R}^n : \|x' - x\|_x \leq 1 \},
\]

and present some nontrivial facts (see Nesterov and Nemirovskii (1994, Theorem 2.1.1) for a proof):

**Lemma 1.** Let \( W(x) \) be the Dikin ellipsoid at any \( x \in \text{int}(\mathcal{X}) \), the following statements hold true:

1. \( W(x) \subseteq \mathcal{X} \) for every \( x \in \text{int}(\mathcal{X}) \);
2. For \( \forall x' \in W(x) \), we have \( (1 - \|x' - x\|_x) \nabla^2 R(x) \preceq \nabla^2 R(x') \preceq (1 - \|x' - x\|_x)^{-2} \nabla^2 R(x) \).

**Remark 1.** Based on Lemma 1, the update of \( \hat{f}^t \) in Algorithm 1 is well-posed since we have \( \|\hat{x}^t - x^t\|_{x^t} \leq \|z^t\| \leq 1 \), which implies that \( \hat{x}^t \in W(x^t) \subseteq \mathcal{X} \).

Second, we define the Minkowski function (Nesterov and Nemirovskii 1994, Page 34) on \( \mathcal{X} \), parametrized by a point \( x \) as

\[
\pi_x(y) = \inf \left\{ t \geq 0 : x + \frac{1}{t}(y - x) \in \mathcal{X} \right\}.
\]
and a scaled version of $\mathcal{X}$ by

$$
\mathcal{X}_\varepsilon = \left\{ x \in \mathbb{R}^n : \pi_{\bar{x}}(x) \leq \frac{1}{1+\varepsilon} \right\}, \quad \forall \varepsilon \in (0, 1],
$$

where $\bar{x}$ be a “center” of $\mathcal{X}$ satisfying that $\bar{x} = \arg \min_{x \in \mathcal{X}} R(x)$, and $R$ is a $\nu$-self-concordant barrier function for $\mathcal{X}$. The following lemma shows that $R$ is quite flat at points that are far from the boundary (see Nesterov and Nemirovskii (1994, Proposition 2.3.2 and 2.3.3) for a proof):

**Lemma 2.** Suppose that $\mathcal{X}$ is a closed convex set, $R$ is a $\nu$-self-concordant barrier function $R$ for $\mathcal{X}$, and $\bar{x} = \arg \min_{x \in \mathcal{X}} R(x)$ is a center of $\mathcal{X}$. Then, we have

$$
R(x) - R(\bar{x}) \leq \nu \log \left( \frac{1}{1 - \pi_{\bar{x}}(x)} \right).
$$

For any $\varepsilon \in (0, 1)$ and $x \in \mathcal{X}_\varepsilon$, we have $\pi_{\bar{x}}(x) \leq \frac{1}{1+\varepsilon}$ and $R(x) - R(\bar{x}) \leq \nu \log(1 + \frac{1}{\varepsilon})$.

Finally, we define the *Newton decrement* for a self-concordant function $g$ (not necessarily a barrier function) as: (recall that $\| \cdot \|_x$ and $\| \cdot \|_{x,*}$ are a local norm and its dual norm)

$$
\lambda(x, g) := \| \nabla g(x) \|_{x,*} = \| (\nabla^2 g(x))^{-1} \nabla g(x) \|_x,
$$

which can be used to measure roughly how far a point is from a global optimum of $g$. Formally, we summarize the results in the following lemma (see Nemirovski and Todd (2008) for a proof):

**Lemma 3.** For any self-concordant function $g$, whenever $\lambda(x, g) \leq \frac{1}{2}$, we have

$$
\| x - \arg \min_{x' \in \mathcal{X}} g(x') \|_x \leq 2 \lambda(x, g),
$$

where the local norm $\| \cdot \|_x$ is defined with respect to $g$, i.e., $\| h \|_x := \sqrt{h^\top \nabla^2 g(x) h}$.

### 2.2. Single-Shot Ellipsoidal Estimator

It was Flaxman et al. (2005) that introduced a single-shot spherical estimator in the BCO literature and cleverly combined it with online gradient descent learning (Zinkevich 2003a). In particular, let $f : \mathbb{R}^n \to \mathbb{R}$ be a function, $\delta > 0$ and $z \sim S^n$, a single-shot spherical estimator is defined by

$$
\hat{v} = \left( \frac{n}{\delta} \right) f(x + \delta z) z.
$$
This estimator is an unbiased prediction for the gradient of a smoothed version; that is, $\mathbb{E}[\hat{v}] = \nabla \hat{f}(x)$ where $\hat{f}(x) = \mathbb{E}_{w \sim \mathbb{B}^n}[f(x + \delta w)]$. As $\delta \to 0^+$, the bias caused by the difference between $f$ and $\hat{f}$ vanishes while the variability of $\hat{v}_t$ explodes. This manifestation of the bias-variance dilemma plays a key role in designing bandit learning algorithms and a single-shot spherical estimator is known to be suboptimal in terms of bias-variance trade-off and hence regret minimization. This gap is closed by using a more sophisticated single-shot ellipsoidal estimator based on the self-concordant barrier function of $\mathcal{X}$ (Saha and Tewari 2011, Hazan and Levy 2014). Comparing to the spherical estimator in Eq. (1) based on the uniform sampling, Saha and Tewari (2011) and Hazan and Levy (2014) proposed to sample $f$ nonuniformly over all directions and create an unbiased gradient estimate of the scaled smooth version. In particular, a single-shot ellipsoidal estimator with respect to an invertible matrix $A$ is defined by

$$\hat{v} = nf(x + Az)A^{-1}z.$$  (2)

The following lemma is a simple modification of (Hazan and Levy 2014, Corollary 6 and Lemma 7).

**Lemma 4.** Suppose that $f$ is a continuous and concave function, $A \in \mathbb{R}^{n \times n}$ is an invertible matrix, $z \in \mathbb{S}^n$ and $w \in \mathbb{B}^n$, we define the smoothed version of $f$ with respect to $A$ by $\hat{f}(x) = \mathbb{E}[f(x + Aw)]$. Then, the following holds:

1. $\nabla \hat{f}(x) = \mathbb{E}[nf(x + Az)A^{-1}z]$.
2. If $f$ is $\beta$-strongly concave, then so is $\hat{f}$.
3. If $\nabla f$ is Lipschitz continuous with parameter $\ell$ and $\sigma_{\text{max}}(A)$ is the largest eigenvalue of $A$, then we have

$$0 \leq f(x) - \hat{f}(x) \leq \frac{\ell(\|A\|_2)^2}{2} = \frac{\ell(\sigma_{\text{max}}(A))^2}{2}.$$  

**Remark 2.** Lemma 4 shows that $\mathbb{E}[\hat{v}] = \nabla \hat{f}(x)$ where $\hat{f}(x) = \mathbb{E}_{w \sim \mathbb{B}^n}[f(x + Aw)]$. In Algorithm 11 we set $A$ using self-concordant barrier function $R$ for $\mathcal{X}$ and perform shrinking sampling (Hazan and Levy 2014), which makes the setup of $\delta \to 0^+$ unnecessary in the update, leading to a better bias-variance trade-off and a near-optimal regret minimization.
2.3. Eager Mirror Descent

The idea of eager mirror descent\(^3\) (Nemirovskij and Yudin 1983) is to generate a new feasible point \(x^+\) by taking a “mirror step” from a starting point \(x\) along an “approximate gradient” direction \(\hat{v}\).

By abuse of notation, we let \(R : \text{int}(\mathcal{X}) \mapsto \mathbb{R}\) be a continuous and strictly convex distance-generating (or regularizer) function, i.e., \(R(tx + (1 - t)x') \leq tR(x) + (1 - t)R(x')\) with equality if and only if \(x = x'\) for all \(x, x' \in \mathcal{X}\) and all \(t \in [0, 1]\). We also assume that \(R\) is continuously differentiable, i.e., \(\nabla R : \text{int}(\mathcal{X}) \mapsto \mathbb{R}^n\) is continuous. Then, we get a Bregman divergence on \(\mathcal{X}\) via the relation

\[
D_R(x', x) = R(x') - R(x) - \langle \nabla R(x), x' - x \rangle,
\]

for all \(x' \in \mathcal{X}\) and \(x \in \text{int}(\mathcal{X})\), which can fail to be symmetric and/or satisfy the triangle inequality. Nevertheless, \(D_R(x', x) \geq 0\) with equality if and only if \(x' = x\), so the asymptotic convergence of a sequence \(x^t\) to \(p\) can be checked by showing that \(D_R(p, x^t) \to 0\). For technical reasons, it will be convenient to assume the converse, i.e., \(D_R(p, x^t) \to 0\) when \(x^t \to p\). This condition is known in the literature as “Bregman reciprocity” (Chen and Teboulle 1993). We continue with some basic relations connecting the Bregman divergence relative to a target point before and after a prox-map. The key ingredient is “three-point identity” (Chen and Teboulle 1993) which generalizes the law of cosines, and widely used in the literature (Beck and Teboulle 2003, Nemirovski 2004).

**Lemma 5.** Let \(R\) be a regularizer on \(\mathcal{X}\). Then, for all \(p \in \mathcal{X}\) and all \(x, x' \in \text{dom}(R)\), we have

\[
D_R(p, x') = D_R(p, x) + D_R(x, x') + \langle \nabla R(x'), x' - x \rangle.
\]

**Remark 3.** In Algorithm 1 we set \(R\) as a self-concordant barrier function for \(\mathcal{X}\) and it is easy to check that \(D_R\) satisfies the aforementioned Bregman reciprocity for various constraint sets, e.g., \(n\)-dimensional simplex, a cube or \(n\)-dimensional ball.

The key notion for Bregman divergence \(D_R(x', x)\) is the induced prox-mapping given by

\[
\hat{P}_R(x, \hat{v}, \eta) = \arg \min_{x' \in \mathcal{X}} \eta \langle \hat{v}, x - x' \rangle + D_R(x', x),
\]
for all \( x \in \text{int}(\mathcal{X}) \) and all \( \hat{v} \in \mathbb{R}^n \), which reflects the intuition behind eager mirror descent. Indeed, it starts with a point \( x \in \text{int}(\mathcal{X}) \) and steps along the dual (gradient-like) vector \( \hat{v} \in \mathbb{R}^n \) to generate a new feasible point \( x^+ = \mathcal{P}_t(x, \hat{v}, \eta) \). However, the prox-mapping in Eq. (4) considers general Bregman divergence which does not exploit the structure of strong concave payoff function or constraint sets. In response to the above issues, we propose to use the prox-mapping given by

\[
P_R(x, \hat{v}, \eta) = \arg \min_{x' \in \mathcal{X}} \eta \langle \hat{v}, x - x' \rangle + \frac{\eta \beta (t+1)}{2} \| x - x' \|^2 + D_R(x', x),
\]

for all \( x \in \text{int}(\mathcal{X}) \) and all \( \hat{v} \in \mathbb{R}^n \). We remark that the above prox-mapping explicitly incorporates the problem structure information by considering a mixed Bregman divergence: the first term \( \frac{\eta \beta (t+1)}{2} \| x - x' \|^2 \) is Euclidean distance with the coefficient proportional to strong concavity parameter \( \beta > 0 \) and the second term is Bregman divergence defined with the self-concordant barrier function \( R \) for the constraint set \( \mathcal{X} \). It is also worth noting that we consider eager projection in Eq. (5), making our prox-mapping different from the one used in Hazan and Levy (2014) that also exploits the structure of strong concave payoff function and constraint sets but is with lazy projection.

With all these in mind, the *eager self-concordant barrier bandit learning* algorithm is given by the recursion \( x^{t+1} \leftarrow P_R(x^t, \hat{v}^t, \eta_t) \) in which \( \eta_t \) is a step-size and \( \hat{v}^t \) is a feedback of estimated gradient. In Algorithm 1 we generate \( \hat{v}^t \) by a single-shot ellipsoidal estimator as mentioned before.

### 2.4. Regret Bound

We consider the single-agent setting where the adversary is limited to choosing smooth and strongly concave functions. The following theorem shows that Algorithm 1 achieves the \( \tilde{O}(\sqrt{T}) \) regret bound, matching the lower bound in Shamir (2013).

**Theorem 1.** Suppose that the adversary is limited to choosing smooth and \( \beta \)-strongly concave functions \( f_1, f_2, \ldots, f_T \). Each function \( f_t \) is Lipschitz continuous and satisfies that \( |f_t(x)| \leq L \) for all \( x \in \mathcal{X} \). If a horizon \( T \geq 1 \) is fixed and each player follows Algorithm 1 with parameters \( \eta_t = \frac{1}{2nL\sqrt{T}} \), we have

\[
\text{Reg}_T = \max_{x \in \mathcal{X}} \left\{ \sum_{t=1}^{T} f_t(x) - \mathbb{E} \left[ \sum_{t=1}^{T} f_t(x^t) \right] \right\} = \tilde{O}(\sqrt{T}).
\]
Remark 4. Theorem 1 shows that Algorithm 1 is a near-regret-optimal bandit learning algorithm when the adversary is limited to choosing smooth and strongly concave functions. The algorithmic scheme is based on the eager mirror descent and thus appears to be different from the existing one presented in [Hazan and Levy (2014)].

To prove Theorem 1, we present our main descent lemma for the iterates generated by Algorithm 1.

Lemma 6. Suppose that the iterate \( \{x^t\}_{t \geq 1} \) is generated by Algorithm 1 and each function \( f_t \) satisfies that \( |f_t(x)| \leq L \) for all \( x \in X \), we have
\[
D_R(p, x^{t+1}) + \frac{\eta_t \beta (t+1)}{2} \|x^{t+1} - p\|^2 \leq D_R(p, x^t) + \frac{\eta_t \beta (t+1)}{2} \|x^t - p\|^2 + 2\eta_t^2 \| A^\top \hat{v}^t \|^2 + \eta_t \langle \hat{v}^t, x^t - p \rangle.
\]
where \( p \in X \) and \( \{\eta_t\}_{t \geq 1} \) is a nonincreasing sequence satisfying that \( 0 < \eta_t \leq \frac{1}{2nL} \).

See the proofs of Lemma 6 and Theorem 1 in Appendix A and B.

3. Multi-Agent Learning with Bandit Feedback

In this section, we consider multi-agent learning with bandit feedback and characterize the behavior of the system when each agent applies the eager self-concordant barrier bandit learning algorithm.

We first present basic definitions, and discuss a few important classes of games that are strongly monotone and finally discuss the multi-agent version of the learning algorithm.

3.1. Basic Definition and Notation

We focus on games played by a finite set of players \( i \in \mathcal{N} = \{1, 2, \ldots, N\} \). At each iteration of the game, each player selects an action \( x_i \) from a convex subset \( X_i \) of a finite-dimensional vector space \( \mathbb{R}^{n_i} \) and their reward is determined by the profile \( x = (x_i; x_{-i}) = (x_1, x_2, \ldots, x_N) \) of the action of all players; subsequently, each player receives the reward, and repeats this process. We denote \( \| \cdot \| \) as the Euclidean norm (in the corresponding vector space); other norms can be easily accommodated in our framework (and different \( X_i \)'s can in general have different norms), although we will not bother with all of these since we do not play with (and benefit from) complicated geometries.
Definition 2. A smooth concave game is a tuple \( G = (\mathcal{N}, \mathcal{X} = \prod_{i=1}^{N} \mathcal{X}_i; \{u_i\}_{i=1}^{N}) \), where \( \mathcal{N} \) is the set of \( N \) players, \( \mathcal{X}_i \) is a convex and compact set of finite-dimensional vector space \( \mathbb{R}^{n_i} \) representing the action space of player \( i \), and \( u_i : \mathcal{X} \mapsto \mathbb{R} \) is the \( i \)-th player’s payoff function satisfying: (i) \( u_i(x_i; x_{-i}) \) is continuous in \( x \) and concave in \( x_i \) for all fixed \( x_{-i} \); (ii) \( u_i(x_i; x_{-i}) \) is continuously differentiable in \( x_i \) and the individual payoff gradient \( v_i(x) = \nabla_i u_i(x_i; x_{-i}) \) is Lipschitz continuous.

A commonly used solution concept for non-cooperative games is Nash equilibrium (NE). For the smooth concave games considered in this paper, we are interested in the pure-strategy Nash equilibria. Indeed, for finite games, the mixed strategy NE is a probability distribution over the pure strategy NE. Our setting assumes continuous and convex action sets, where each action already lives in a continuum, and pursuing pure-strategy Nash equilibria is sufficient.

Definition 3. An action profile \( x^* \in \mathcal{X} \) is called a (pure-strategy) Nash equilibrium of a game \( G \) if it is resilient to unilateral deviations; that is, \( u_i(x^*_i; x^*_{-i}) \geq u_i(x_i; x^*_{-i}) \) for all \( x_i \in \mathcal{X}_i \) and \( i \in \mathcal{N} \).

It is known that every smooth concave game admits at least one Nash equilibrium when all action sets are compact (Debreu 1952) and Nash equilibria admit a variational characterization. We summarize this result in the following proposition.

Proposition 1. In a smooth concave game \( G \), the action profile \( x^* \in \mathcal{X} \) is a Nash equilibrium if and only if \( (x_i - x^*_i)^\top v_i(x^*) \leq 0 \) for all \( x_i \in \mathcal{X}_i \) and \( i \in \mathcal{N} \).

Proposition 1 shows that Nash equilibria of a smooth concave game can be precisely characterized as the solution set of a variational inequality (VI), so the existence results also follow from the standard results in VI literature (Facchinei and Pang 2007). We omit the proof here and refer to Mertikopoulos and Zhou (2019) for the reference.

3.2. Strongly Monotone Games

The study of (strongly) monotone games dates to Rosen (1965), with many subsequent developments; see, e.g., Facchinei and Pang (2007). Specifically, Rosen (1965) considered a class of games...
that satisfy the diagonal strict concavity (DSC) condition and prove that they admit a unique Nash equilibrium. Further work in this vein appeared in Sandholm (2015) and Sorin and Wan (2016), where games that satisfy DSC are referred to as “contractive” and “dissipative”. These conditions are equivalent to (strict) monotonicity in convex analysis (Bauschke and Combettes 2011).

To avoid confusion, we provide the formal definition of strongly monotone games considered in this paper.

**Definition 4.** A smooth concave game $G$ is called $\beta$-strongly monotone if there exist positive constants $\lambda_i > 0$ such that
\[
\sum_{i \in N} \lambda_i \langle x'_i - x_i, v_i(x') - v_i(x) \rangle \leq -\beta \|x - x'\|^2
\]
for any $x, x' \in X$.

The notion of (strong) monotonicity, which will play a crucial role in the subsequent analysis, is not necessarily theoretically artificial but encompasses a very rich class of games. We present three typical examples which satisfy the conditions in Definition 4 (see the references (Bravo et al. 2018, Mertikopoulos and Zhou 2019) for more details).

**Example 1 (Cournot Competition).** In the Cournot oligopoly model, there is a finite set $N = \{1, 2, \ldots, N\}$ of firms, each supplying the market with a quantity $x_i \in [0, B_i]$ of some good (or service) up to the firm’s production capacity, given here by a positive scalar $B_i > 0$. This good is then priced as a decreasing function $P(x)$ of the total supply to the market, as determined by each firm’s production; for concreteness, we focus on the standard linear model $P(x) = a - b \sum_{i \in N} x_i$ where $a$ and $b$ are positive constants. In this model, the utility of firm $i$ (considered here as a player) is given by
\[
u_i(x) = x_i P(x) - c_i(x_i),
\]
where $c_i(\cdot)$ represents the marginal production cost function of firm $i$ and is assumed to be strongly convex, i.e., as the income obtained by producing $x_i$ units of the good in question minus the corresponding production cost. Letting $X_i = [0, B_i]$ denote the space of possible production values for each firm, the resulting game $G \equiv G(N, X, u)$ is strongly monotone.
Example 2 (Strongly Concave Potential Games). A game $G = (N, X = \prod_{i=1}^{N} X_i, \{u_i\}_{i=1}^{N})$ is called a potential game (Monderer and Shapley 1996, Sandholm 2001) if there exists a potential function $f : \mathcal{X} \mapsto \mathbb{R}$ such that

$$u_i(x_i; x_{-i}) - u_i(\tilde{x}_i; x_{-i}) = f(x_i; x_{-i}) - f(\tilde{x}_i; x_{-i}),$$

for all $i \in N$, all $x \in \mathcal{X}$ and all $\tilde{x}_i \in X_i$. A potential game is called a strongly concave potential game if the potential function $f$ is strongly concave. Since the gradient of a strongly concave function is strongly monotone (Bauschke and Combettes 2011), a strongly concave potential game is a strongly monotone game.

Example 3 (Kelly Auctions). Consider a service provider with a number of splittable resources $s \in S = \{1, 2, \ldots, S\}$ (representing, e.g., bandwidth, server time, ad space on a website, etc.). These resources can be leased to a set of $N$ bidders (players) who can place monetary bids $x_{is} \geq 0$ for the utilization of each resource $s \in S$ up to each player’s total budget $B_i$, i.e., $\sum_{s \in S} x_{is} \leq B_i$. A popular and widely used mechanism to allocate resources in this case is the so-called Kelly mechanism (Kelly et al. 1998) whereby resources are allocated proportionally to each player’s bid, i.e., player $i$ gets

$$\rho_{is} = \frac{q_{is} x_{is}}{d_s + \sum_{j \in N} x_{js}},$$

units of the $s$-th resource (in the above, $q_s$ denotes the available units of said resource and $d_s \geq 0$ is the “entry barrier” for bidding on it). A simple model for the utility of player $i$ is then given by

$$u_i(x_i; x_{-i}) = \sum_{s \in S} (g_i \rho_{is} - c_i(x_{is})), $$

where $g_i$ denotes the player’s marginal gain from acquiring a unit slice of resources and $c_i(x_{is})$ denotes the cost if player $i$ place one unit monetary bid for the utilization of resource $s$ (the function $c_i(\cdot)$ is assumed to be strongly convex). If we write $\mathcal{X}_i = \{ x_i \in \mathbb{R}^S_{+} : \sum_{s \in S} x_{is} \leq b_i \}$ for the space of possible bids of player $i$ on the set of resources $S$, we obtain a strongly monotone game.
There are many other application problems that can be cast into strongly monotone games (Orda et al. 1993, Cesa-Bianchi and Lugosi 2006a, Sandholm 2015, Sorin and Wan 2016, Mertikopoulos et al. 2017). Typical examples include strongly-convex-strongly-concave zero-sum games, congestion games (Mertikopoulos and Zhou 2019), wireless network games (Weeraddana et al. 2012, Tan 2014, Zhou et al. 2021) and a wide range of online decision-making problems (Cesa-Bianchi and Lugosi 2006a). From an economic point of view, one appealing feature of strongly monotone games is that the last-iterate convergence can be achieved by some standard learning algorithms (Zhou et al. 2021), which is more natural than time-average-iterate convergence (Fudenberg and Levine 1998a, Cesa-Bianchi and Lugosi 2006a, Lin et al. 2020). The finite-time convergence rate is derived in terms of the distance between $x^t$ and $x^*$, where $x^t$ is the realized action and $x^*$ is the unique Nash equilibrium (under convex and compact action sets). In view of all this (and unless explicitly stated otherwise), we will focus throughout on strongly monotone games.

3.3. Multi-Agent Bandit Learning

In multi-agent learning with bandit feedback, at each round $t = 1, 2, \ldots, T$, each (possibly randomized) decision maker $i \in \mathcal{N}$ selects an action $x^t_i \in \mathcal{X}_i$. The reward $u_i(x^t)$ is realized after after all decision makers have chosen their actions. In addition to regret minimization, the convergence to Nash equilibria is an important criterion for measuring the performance of learning algorithms. In multi-agent bandit learning, the feedback is limited to the reward at the point that she has chosen, i.e., $u_i(x^t)$. Here, we propose a simple multi-agent bandit learning algorithm (see Algorithm 2) in which each player chooses her action using Algorithm 1. Algorithm 2 is a straightforward extension of Algorithm 1 from single-agent setting to multi-agent setting. It differs from Bravo et al. (2018, Algorithm 1) in two aspects: the ellipsoidal SPSA estimator v.s. spherical SPSA estimator and self-concordant Bregman divergence v.s. general Bregman divergence. We discuss these two crucial components next.
Single-shot ellipsoidal SPSA estimator. Recently, Bravo et al. (2018) have extended the spherical estimator in Eq. (1) to multi-agent setting by positing instead that players rely on a simultaneous perturbation stochastic approximation (SPSA) approach (Spall 1997) that allows them to estimate their individual payoff gradients \( \hat{v}_i \) based off a single function evaluation. In particular, let \( u_i : \mathbb{R}^n \to \mathbb{R} \) be a payoff function, \( \delta > 0 \) and the query directions \( z_i \sim S^n_i \) be drawn independently across players, a single-shot spherical SPSA estimator is defined by

\[
\hat{v}_i = \left( n_i / \delta \right) u_i(x_i + \delta z_i; x_{-i} + \delta z_{-i}) z_i.
\]

This estimator is an unbiased prediction for the partial gradient of a smoothed version of \( u_i \); that is \( \mathbb{E}[\hat{v}_i] = \nabla_i \tilde{u}_i(x) \) where \( \tilde{u}_i(x) = \mathbb{E}_{w_i \sim \mathbb{R}^n, \mathbb{E}_{z_{-i} \sim \mathbb{R}^n \setminus \{i\}}}[u_i(x_i + \delta w_i; x_{-i} + \delta z_{-i})] \). We can easily see the bias-variance dilemma here: as \( \delta \to 0^+ \), \( \hat{v}_i \) becomes more accurate since \( \|\nabla_i \tilde{u}_i(x) - \nabla_i u_i(x)\| = O(\delta) \), while the variability of \( \hat{v}_i \) grows unbounded since the second moment of \( \hat{v}_i \) grows as \( O(1/\delta^2) \). By carefully choosing \( \delta > 0 \), Bravo et al. (2018) provided the best-known last-iterate convergence rate of \( O(T^{-1/3}) \) which almost matches the lower bound in Shamir (2013). However, a gap still remains and they believe it can be closed by using a more sophisticated single-shot estimator.

Algorithm 2 Multi-Agent Eager Self-Concordant Barrier Bandit Learning

1: **Input:** step size \( \eta_t > 0 \), weight \( \lambda_i > 0 \), module \( \beta > 0 \), and barrier \( R_i : \text{int}(\mathcal{X}_i) \to \mathbb{R} \).

2: **Initialization:** \( x_1^i = \arg\min_{x_i \in \mathcal{X}_i} R_i(x_i) \).

3: for \( t = 1, 2, \ldots \) do

4: for \( i \in \mathcal{N} \) do

5: set \( A_t^i \leftarrow (\nabla^2 R_i(x_t^i) + \frac{n \beta (t+1)}{\lambda_i} I_{n_i})^{-1/2} \). \hfill \# SCALING MATRIX

6: draw \( z_t^i \sim S^n_i \). \hfill \# PERTURBATION DIRECTION

7: play \( \hat{x}_t^i \leftarrow x_t^i + A_t^i z_t^i \). \hfill \# CHOOSE ACTION

8: receive \( \hat{u}_t^i \leftarrow u_i(\hat{x}_t^i) + \xi_{i,t} \) for all \( i \in \mathcal{N} \). \hfill \# GET PAYOFF

9: for \( i \in \mathcal{N} \) do

10: set \( \hat{v}_t^i \leftarrow n_i \hat{u}_t^i (A_t^i)^{-1} z_t^i \). \hfill \# ESTIMATE GRADIENT

11: update \( x_{t+1}^i \leftarrow \mathcal{P}_{R_i}(x_t^i, \hat{v}_t^i, \eta_t, \lambda_i, \beta) \). \hfill \# UPDATE PIVOT
We now provide a single-shot ellipsoidal SPSA estimator by extending the estimator in Eq. (2) to the multi-agent setting. Using the SPSA approach, we let \( \hat{x}_i = x_i + A_i z_i \) and have
\[
\hat{v}_i = n_i u_i(\hat{x}_i; \hat{x}_{-i})(A_i)^{-1} z_i.
\]
(7)

The following lemma provides some results for a single-shot ellipsoidal SPSA estimator and the proof is given in Appendix C.

**Lemma 7.** The single-shot ellipsoidal SPSA estimator \((\hat{v}_i)_{i \in \mathcal{N}}\) given by Eq. (7) satisfies
\[
\mathbb{E}[\hat{v}_i | x] = \nabla_i \hat{u}_i(x)
\]
with \( \hat{u}_i(x) = \mathbb{E}_{w_i \sim B_{n_i}} \mathbb{E}_{z_{-i} \sim \Pi_{j \neq i} \mathcal{S}_{n_j}} [u_i(x_i + A_i w_i; \hat{x}_{-i})] \) where \( \hat{x}_i = x_i + A_i z_i \) for all \( i \in \mathcal{N} \). Moreover, if \( v_i(x) \) is \( \ell_i \)-Lipschitz continuous and \( \sigma_{\text{max}}(A) \) is the largest eigenvalue of \( A \), we have
\[
\|\nabla_i \hat{u}_i(x) - \nabla_i u_i(x)\| \leq \ell_i \sqrt{\sum_{j \in \mathcal{N}} \|A_j\|^2} = \ell_i \sqrt{\sum_{j \in \mathcal{N}} (\sigma_{\text{max}}(A_j))^2}.
\]

**Remark 5.** Lemma 7 generalizes Bravo et al. (2018, Lemma C.1) which is proved for a single-shot spherical SPSA estimator. It shows that \( \mathbb{E}[\hat{v}_i] = \nabla_i \hat{u}_i(x) \) where \( \hat{u}_i(x) = \mathbb{E}_{w_i \sim B_{n_i}} \mathbb{E}_{z_{-i} \sim \Pi_{j \neq i} \mathcal{S}_{n_j}} [u_i(x_i + A_i w_i; \hat{x}_{-i})] \). In Algorithm 2, we use self-concordant barrier function \( R_i \) for \( \mathcal{X}_i \) and perform shrinking sampling, leading to an optimal last-iterate convergence rate up to a log factor.

**Eager mirror descent.** The prox-mapping in Eq. (4) has been used in Bravo et al. (2018) to construct their multi-agent bandit learning algorithm, which achieves the suboptimal regret minimization and last-iterate convergence. One possible reason for such suboptimal guarantee is that the prox-mapping defined in Eq. (4) considers general Bregman divergence and takes care of neither strong monotone payoff gradient nor constraint sets. In Algorithm 2, we let each player update \( x_i^{t+1} \) using the prox-mapping in Eq. (5) and the rule is given by
\[
P_{R_i}(x_i, \hat{v}_i, \eta_i, \lambda_i, \beta) = \arg \min_{x_i' \in \mathcal{X}_i} \eta_i \langle \hat{v}_i, x_i - x_i' \rangle + \frac{\eta_i^2}{2\lambda_i} \|x_i - x_i'\|^2 + D_{R_i}(x_i', x_i),
\]
(8)
for all \( x_i \in \text{int}(\mathcal{X}_i) \) and all \( \hat{v}_i \in \mathbb{R}^{n_i} \). With all these in mind, the main step of Algorithm 2 is given by the recursion \( x_i^{t+1} \leftarrow P_{R_i}(x_i^{t}, \hat{v}_i^{t}, \eta_t) \) in which \( \eta_t \) is a step-size and \( \hat{v}_i^{t} \) is a feedback of estimated gradients for player \( i \) (we generate \( \hat{v}_i^{t} \) by a single-shot ellipsoidal SPSA estimator).
4. Finite-Time Convergence Rate

In this section, we establish that Algorithm 2 achieves the near-optimal last-iterate convergence rate for smooth and strongly monotone games if the perfect bandit feedback is available, improving the best-known last-iterate convergence rate in Bravo et al. (2018). The following theorem shows that Algorithm 2 achieves the $\tilde{O}(1/\sqrt{T})$ rate of last-iterate convergence to a unique Nash equilibrium if the perfect bandit feedback is available ($\xi_{i,t} \equiv 0$), improving the best-known rate of $O(1/T^{1/3})$ (Bravo et al. 2018) and matching the lower bound (Shamir 2013).

**Theorem 2.** Suppose that $x^* \in \mathcal{X}$ is a unique Nash equilibrium of a smooth and $\beta$-strongly monotone game. Each payoff function $u_i$ satisfies that $|u_i(x)| \leq L$ for all $x \in \mathcal{X}$. If each player follows Algorithm 2 with parameters $\eta_t = \frac{1}{2nL^2}$, we have

$$
\mathbb{E} \left[ \sum_{i \in \mathcal{N}} \| \hat{x}_i^T - x_i^* \|^2 \right] = \tilde{O} \left( \frac{1}{\sqrt{T}} \right).
$$

**Remark 6.** Theorem 2 shows that Algorithm 2 attains a near-optimal rate of last-iterate convergence in smooth and strongly monotone games. It extends Algorithm 1 from the single-agent setting to multi-agent setting, providing the first doubly optimal bandit learning algorithm, in that it achieves (up to log factors) both optimal regret in the single-agent learning and optimal last-iterate convergence rate in the multi-agent learning. In contrast, it remains unclear whether the multi-agent extension of Hazan and Levy (2014, Algorithm 1) can achieve the near-optimal last-iterate convergence rate or not.

To prove Theorem 2, we present our main descent lemma for the iterates generated by Algorithm 2.

**Lemma 8.** Suppose that the iterate $\{x^t\}_{t \geq 1}$ is generated by Algorithm 2 and each function $u_i$ satisfies that $|u_i(x)| \leq L$ for all $x \in \mathcal{X}$, we have

$$
\sum_{i \in \mathcal{N}} \lambda_i D_{R_i}(p_i, x_i^{t+1}) + \frac{\eta_{t+1} \beta(t+1)}{2} \left( \sum_{i \in \mathcal{N}} \| x_i^{t+1} - p_i \|^2 \right) \\
\leq \sum_{i \in \mathcal{N}} \lambda_i D_{R_i}(p_i, x_i^t) + \frac{\eta_t \beta(t+1)}{2} \left( \sum_{i \in \mathcal{N}} \| x_i^t - p_i \|^2 \right) + 2\eta_t^2 \left( \sum_{i \in \mathcal{N}} \lambda_i \| A_i^t \hat{v}_i^t \|^2 \right) + \eta_t \left( \sum_{i \in \mathcal{N}} \lambda_i \langle \hat{v}_i^t, x_i^t - p_i \rangle \right)
$$

where $p_i \in \mathcal{X}_i$ and $\{\eta_t\}_{t \geq 1}$ is a nonincreasing sequence satisfying that $0 < \eta_t \leq \frac{1}{2nL}$. 
See the proofs of Lemma 8 and Theorem 2 in Appendix D and E. We also provide the convergence results under the imperfect bandit feedback setting in Appendix F.

5. Numerical Experiments

In this section, we conduct the experiments using three different tasks: Cournot competition, Kelly auction, and distributed regularized logistic regression. We compare Algorithm 2 with Bravo et al. (2018, Algorithm 1). The implementation is done with MATLAB R2020b on a MacBook Pro with an Intel Core i9 2.4GHz (8 cores and 16 threads) and 16GB memory.

5.1. Cournot Competition

We show that Cournot competition is strongly monotone in the sense of Definition 4. In particular, each player’s payoff function is given by

$$u_i(x) = x_i \left( a - b \left( \sum_{j \in \mathcal{N}} x_j \right) \right) - c_i x_i, \quad \text{for some } a, b, c_i > 0.$$  

Taking the derivative of $u_i(x)$ with respect to $x_i$, we have

$$v_i(x) = a - c_i - b \left( \sum_{j \in \mathcal{N}} x_j \right) - bx_i.$$  

This implies that

$$\sum_{i \in \mathcal{N}} (x'_i - x_i, v_i(x') - v_i(x)) = -b \left( \sum_{i \in \mathcal{N}} x_i \right)^2 - b \|x' - x\|^2 \leq -b \|x' - x\|^2, \quad \text{for all } x, x' \in \mathcal{X}.$$  

Therefore, we conclude that Cournot competition is strongly monotone with the module $b > 0$.

Experimental setup. For all $i \in \mathcal{N}$, the production capacity is fixed as $B_i = 1$ and each $c_i$ is drawn independently from a uniform distribution on the interval $[0, 1]$. We evaluate the algorithms with $N \in \{10, 20, 50, 100\}$, $a \in \{10, 20\}$ and $b \in \{0.05, 0.1\}$.

For Algorithm 2 we set the parameters $\lambda_i = 1$, $\beta = b$, $\eta_t = \frac{1}{20\sqrt{t}}$, and define the barrier function as $R_i = -\log(x_i) - \log(B_i - x_i)$. For Bravo et al. (2018, Algorithm 1), we use the default setting in Bravo et al. (2018, Theorem 5.2) in which we apply Euclidean projections with the parameters $r_i = p_i = B_i^2$, $\delta_t = \min\{\min_{i \in \mathcal{N}} r_i, 1/\sqrt{\beta}\}$, $\beta = b$ and $\gamma_t = \frac{1}{5\beta t}$. Moreover, it is known that Cournot
Table 1 Numerical results on Cournot competition problem.

| (N, a, b)      | Bravo et al. (2018, Algorithm 1) | Our Algorithm |
|---------------|----------------------------------|---------------|
| (10, 10, 0.05)| 1.9e-01 ± 4.1e-02                | 1.3e-03 ± 3.6e-04 |
| (10, 10, 0.10)| 9.1e-02 ± 3.9e-02                | 1.4e-03 ± 3.3e-04 |
| (10, 20, 0.05)| 3.0e-01 ± 4.9e-02                | 8.9e-04 ± 5.7e-04 |
| (10, 20, 0.10)| 1.5e-01 ± 5.0e-02                | 7.0e-04 ± 2.2e-04 |
| (20, 10, 0.05)| 2.1e-01 ± 4.1e-02                | 1.7e-03 ± 2.7e-04 |
| (20, 10, 0.10)| 9.6e-02 ± 1.7e-02                | 1.9e-03 ± 4.1e-04 |
| (20, 20, 0.05)| 3.4e-01 ± 7.4e-02                | 9.4e-04 ± 2.4e-04 |
| (20, 20, 0.10)| 1.9e-01 ± 5.1e-02                | 9.4e-04 ± 2.4e-04 |
| (50, 10, 0.05)| 2.1e-01 ± 2.7e-02                | 2.4e-03 ± 1.7e-04 |
| (50, 10, 0.10)| 7.7e-02 ± 1.7e-02                | 4.1e-03 ± 2.0e-04 |
| (50, 20, 0.05)| 3.5e-01 ± 4.1e-02                | 1.1e-03 ± 1.7e-04 |
| (50, 20, 0.10)| 2.0e-01 ± 1.8e-02                | 1.1e-03 ± 2.0e-04 |
| (100, 10, 0.05)| 1.9e-01 ± 2.8e-02               | 2.5e-02 ± 4.6e-04 |
| (100, 10, 0.10)| 1.2e-02 ± 1.1e-03               | 4.4e-03 ± 1.0e-04 |
| (100, 20, 0.05)| 3.8e-01 ± 3.3e-02               | 1.2e-03 ± 1.2e-04 |
| (100, 20, 0.10)| 2.0e-01 ± 1.5e-02               | 1.5e-03 ± 8.9e-05 |

The unique Nash equilibrium can be achieved by the interior-point method with high accuracy, and we compute it by deploying QUADPROG in Matlab. This point will be a benchmark for evaluating the quality of the solution obtained by our algorithm and Bravo et al. (2018, Algorithm 1).

Experimental results. Fixing \((a, b) = (10, 0.05)\), we study the convergence behavior of both algorithms with an increasing number of agents \(N \in \{10, 20, 50, 100\}\). As indicated in Figure 1, our algorithm consistently outperforms Bravo et al. (2018, Algorithm 1) as it returns iterates that are closer to an unique Nash equilibrium in less iteration counts. To further facilitate the readers, we present the averaged results from 10 independent trials in Table 1.
Figure 1 The average distance between the Nash equilibrium and the iterates generated by Algorithm \[2\] with Bravo et al. (2018, Algorithm 1) from 10 independent trials when \((a, b) = (10, 0.05)\) is fixed.

5.2. Kelly Auction

We show that Kelly auction is strongly monotone in the sense of Definition \[4\]. In particular, each players’ payoff function is given by

\[
    u_i(x) = \left( \sum_{s \in S} g_i q_s x_{is} \right) - \sum_{s \in S} x_{is}, \quad \text{for some } g_i, d_s, q_s > 0.
\]

Define \(v_i(x) = \nabla_i u_i(x)\) for all \(i \in \mathcal{N}\), it suffices to prove that there exists positive constants \(\lambda_i > 0\) such that \(\sum_{i \in \mathcal{N}} \lambda_i (x'_i - x_i, v_i(x') - v_i(x)) \leq -\beta \|x' - x\|^2\) for all \(x, x' \in \mathcal{X}\). The following proposition is a restatement of Rosen (1965, Theorem 6) and plays an important role in the subsequent analysis.

**Proposition 2.** Given a continuous game \(\mathcal{G} = (\mathcal{N}, \mathcal{X} = \prod_{i=1}^{N} \mathcal{X}_i, \{u_i\}_{i=1}^{N})\), where each \(u_i\) is twice continuously differentiable. For each \(x \in \mathcal{X}\), define the \(\lambda\)-weighted Hessian matrix \(H^\lambda(x)\) as follows:

\[
    H^\lambda_{ij}(x) = \frac{1}{2} \lambda_i \nabla_j v_i(x) + \frac{1}{2} \lambda_j \nabla_i v_j(x)^\top.
\]
If $H^λ(x)$ is negative-definite for every $x \in X$, we have
\[
\sum_{i \in N} \lambda_i (x'_i - x_i, v_i(x') - v_i(x)) \leq 0 \quad \text{for all } x, x' \in X,
\]
where the equality holds true if and only if $x = x'$. As a consequence of Proposition 2, we have
\[
\sum_{i \in N} \lambda_i (x'_i - x_i, v_i(x') - v_i(x)) \leq -\beta \|x' - x\|^2 \quad \text{for all } x, x' \in X \text{ if } H^λ(x) \preceq -\beta I_n \text{ for all } x \in X. \text{ To show this, we consider the following weighted social welfare function which is given by}
\[
U(x) = \sum_{i \in N} g_i^{-1} u_i(x) = \sum_{i \in N} \sum_{s \in S} \frac{q_s x_{is}}{d_s + \sum_{j \in N} x_{js}} - \sum_{i \in N} \sum_{s \in S} x_{is} = \sum_{s \in S} q_s \left( \frac{\sum_{i \in N} x_{is}}{d_s + \sum_{i \in N} x_{is}} \right) - \sum_{i \in N} x_{is}.
\]
Since the $f(x) = \frac{x}{c + x}$ is concave for all $c > 0$, it readily follows that
(a) Each payoff function $u_i(x_i; x_{-i})$ is concave in $x_i$ and convex in $x_{-i}$;
(b) The welfare function $U(x)$ is concave in $x$.

Moreover, by appeal to the special structure of Kelly auction problems, we prove that the payoff function $\nabla^2_{i\bar{i}} u_i(x_i; x_{-i}) \preceq -\beta g_i I_{n_i}$ for all $x \in X$ for some $\beta > 0$. Indeed, $u_i(x)$ is the sum of $S$ functions in which the $s$-th one only depends on $(x_{1s}, x_{2s}, \ldots, x_{N_s})$. This implies that
\[
\nabla^2_{i\bar{i}} u_i(x_i; x_{-i}) = \left( \begin{array}{c}
-\frac{2g_i q_1 (d_1 + \sum_{j \in N} x_{1j})}{(d_1 + \sum_{j \in N} x_{1j})^2} \\
\vdots \\
-\frac{2g_i q_s (d_s + \sum_{j \in N} x_{js})}{(d_s + \sum_{j \in N} x_{js})^3}
\end{array} \right)
\]
Since $0 < d_s + \sum_{j \neq i} x_{js} \leq d_s + \sum_{j \in N} x_{js} \leq \sum_{s \in S} d_s + \sum_{i \in N} B_i$, we have
\[
\nabla^2_{i\bar{i}} u_i(x_i; x_{-i}) \preceq -\frac{2g_i \min_{s \in S} \{q_i d_s\}}{(\sum_{s \in S} d_s + \sum_{i \in N} B_i)^3} I_{n_i}.
\]
With these properties in mind, let $\lambda_i = 1/g_i$ and note that
\[
\nabla^2_{i\bar{i}} U(x) = \sum_{k \in N} \lambda_k \nabla^2_{i\bar{i}} u_k(x) = 2H^λ_{n_i}(x) - g_i^{-1} \nabla^2_{i\bar{i}} u_i(x) + \sum_{k \neq i} \lambda_k \nabla^2_{i\bar{i}} u_k(x).
\]
and
\[
\nabla^2_{ij} U(x) = \sum_{k \in N} \lambda_k \nabla^2_{ij} u_k(x) = 2H^λ_{ij}(x) + \sum_{k \neq i,j} \lambda_k \nabla^2_{ij} u_k(x).
\]
Bravo et al. (2018, Algorithm 1)

4.6e-01
1.9e-01
1.4e-01
1.6e-01
2.1e-01
1.9e+00
3.3e+00
1.7e-01
8.4e-01
4.5e+00
9.5e-01
2.2e+00
2.3e-01
9.4e-01
5.0e-01
2.0e-01
2.6e-01
1.9e-01
4.2e+00
1.6e+00
2.7e-01
2.2e-01
2.1e+00
3.2e-01
2.2e-01
2.8e+00
1.3e+00
3.1e+00
3.5e-01
1.4e+00
3.2e+00
1.2e+00

Our Algorithm

Therefore, the Kelly auction is strongly monotone with the module $N$ algorithms with $U$ Since $d$ independently drawn from an uniform distribution on the interval $[0, 1]$, we get

\begin{align*}
\text{Experimental setup.} & \quad \text{For all } i \in \mathcal{N}, \text{the total budget is fixed as } B_i = 1 \text{ and each of } g_i \text{ is independently drawn from an uniform distribution on the rectangular } [0, 1]. \text{ For all } s \in \mathcal{S}, \text{ each of } q_s \text{ is independently drawn from an uniform distribution on the interval } [0, 1] \text{ and } d_s \text{ is independently drawn from an uniform distribution on the interval } [0, \bar{d}] \text{ for some constants } \bar{d} > 0. \text{ We evaluate the algorithms with } N \in \{10, 20, 50, 100\}, \mathcal{S} \in \{2, 5\} \text{ and } \bar{d} \in \{0.5, 1\}.
\end{align*}
For Algorithm 2, we set the parameters $\lambda_i = \frac{1}{g_i}$, $\eta_t = \frac{1}{2\sqrt{N_\mathcal{S}(1+\min_{s \in \mathcal{S}} d_s/N_\mathcal{S})}}$, and define the barrier function as $R_i = -\sum_{s \in \mathcal{S}} \log(x_{is}) - \log(B_i - \sum_{s \in \mathcal{S}} x_{is})$. For Bravo et al. (2018, Algorithm 1), we use the default setting of Bravo et al. (2018, Theorem 5.2) in which we apply Euclidean projection with the parameters $r_i = B_i/(S(S+1))$, $p_i = B_i S$, $\delta_t = \min\{\min_{i \in \mathcal{N}} r_i, 1/t^{1/3}\}$ and $\gamma_t = 1/5 \beta_t$ where $1_S$ refers to a $S$-dimensional vector whose entries are all 1. Moreover, it is known that Kelly auction in the above form has a variational inequality (VI) reformulation (Mertikopoulos and Zhou 2019, Proposition 2.1) and can be solved by finding a point $x^* \in \mathcal{X}$ such that

$$\langle v(x^*), x - x^* \rangle \leq 0 \quad \text{for all } x \in \mathcal{X}.$$ 

The unique Nash equilibrium can be achieved by a few linearly convergent VI algorithms with high accuracy, and we compute it by deploying an operator extrapolation algorithm (see Kotsalis et al. 2020, Algorithm 1) for the scheme. This point will be a benchmark for evaluating the quality of the solution obtained by our algorithm and Bravo et al. (2018, Algorithm 1).

**Experimental results.** Fixing $(S, \bar{d}) = (5, 0.5)$, we study the convergence behavior of both algorithms with an increasing number of agents $N \in \{10, 20, 50, 100\}$. As indicated in Figure 2, our algorithm still outperforms Bravo et al. (2018, Algorithm 1) as it returns iterates that are closer to an unique Nash equilibrium in less iteration counts. Compared to Figure 1, the quality of the solutions obtained by both algorithm is lower. Indeed, the simplex constraint set is more complicated than the box constraint set and we make the conservative choice for parameters, e.g., $r = B_i/S(S+1)$, such that the algorithm works from a theoretical viewpoint. Bravo et al. (2018, Algorithm 1) is very sensitive to the parameters and unfortunately gets stuck after a few iterations, while our algorithm approaches the Nash equilibrium slowly but steadily as the iteration count increases. To further facilitate the readers, we present the averaged results from 10 independent trials in Table 2.

### 5.3. Distributed Regularized Logistic Regression

We provide a game-theoretical formulation of distributed $\ell_2$-regularized logistic regression problem and show that the resulting game is strongly monotone in the sense of Definition 4.
The problem of $\ell_2$-regularized logistic regression aims at minimizing the sum of a logistic loss function and a squared $\ell_2$-norm-based regularization given by,

$$
\min_{x \in \mathbb{R}^n} f(x) = \frac{1}{m} \left( \sum_{j=1}^{m} \log \left( 1 + \exp(-b_j \cdot a_j^T x) \right) \right) + \mu \|x\|^2, 
$$

(11)

where $(a_i, b_i)_{i=1}^m$ is a set of data samples with the binary label $b_i \in \{-1, 1\}$ and $\mu > 0$ is the regularization parameter. The term $\| \cdot \|^2$ prevents overfitting and the parameter $\mu > 0$ balances goodness-of-fit and generalization. In real applications, the dimension $n > 0$ can be very large and the key challenge is its computation. Practitioners tend to resolve this issue by solving the problem in a distributed computing environment.

While there is a vast literature on this topic (Wei and Ozdaglar 2012, Gopal and Yang 2013, Zhang and Kwok 2014, Liu et al. 2015, Mahajan et al. 2017, Aybat et al. 2017), the idea is similar...
Table 3  Statistics of datasets.

| Dataset | Number of Samples (m) | Dimension (n) |
|---------|-----------------------|---------------|
| a9a     | 32561                 | 123           |
| mushrooms | 8124                 | 112           |
| news20  | 16242                 | 100           |
| splice  | 1000                  | 60            |
| svmguide3 | 1243                 | 21            |
| w8a     | 64700                 | 300           |

and can be summarized in two aspects: (i) the reformulation based on consensus optimization; (ii) the algorithm based on either block coordinate gradient descent (BCD) [Wright 2015] or alternating direction method of multipliers (ADMM) [Boyd et al. 2011]. In contrast, we take another perspective by reformulating Eq. (11) using the multi-agent learning framework. Thus, the unique Nash equilibrium produced by Algorithm 2 is equivalent to the optimal solution of Eq. (11).

More specifically, there is a finite set $\mathcal{N} = \{1, 2, \ldots, n\}$ of blocks (players) and each one stands for the $i$-th block $x_i \in \mathbb{R}$. Suppose we have a shared memory such that all players can access a set of data samples $(a_i, b_i)_{i=1}^m$. A simple model for the utility of player $i$ is then given by

$$u_i(x) = -f(x).$$

Let $X_i = \mathbb{R}$ be the space of possible values of player $i$, the resulting game $\mathcal{G} \equiv \mathcal{G}(\mathcal{N}, X, u)$ is strongly monotone the module $2\mu > 0$. Indeed, note that $\mathcal{G}$ is a potential game with a $2\mu$-strongly concave potential function $-f$, hence the desired argument follows.

**Experimental setup.** We conduct the experiments on 5 binary classification datasets from LIBSVM collection\(^5\) and 20 newsgroup dataset\(^6\): a9a, mushrooms, news20, splice, svmguide3, w8a, and set parameter $\mu = 0.001$ and $\ell = \frac{1}{4}(\max_{1 \leq j \leq m} \|a_j\|^2)$ for each dataset.

For Algorithm 2 we set the parameters $\lambda_i = 2\mu$, $\beta = 2\mu$, $\eta_t = \frac{1}{\ell+2\sqrt{\alpha t}}$ and define the barrier function $R_i = 0$. For Bravo et al. (2018, Algorithm 1), we use the default setting of Bravo et al. (2018, Theorem 5.2) in which we apply Euclidean projection with the parameters $r_i = +\infty$, $p_i = 0$, $\delta_t = \frac{1}{t^{1/3}}$ and $\gamma_t = \frac{1}{t^{1/10}}$. Note that the scheme of Algorithm 2 is simplified for unconstrained setting and $A_t^i = \frac{\sqrt{\ell+2\sqrt{\alpha t}}}{\sqrt{t^{1/2}}} \propto \frac{1}{t^{1/4}}$ is analogue to $\delta_t$ in Bravo et al. (2018, Algorithm 1). Moreover, the unique
Nash equilibrium is equivalent to the optimal solution of Eq. \ref{eq:equilibrium} and can be achieved by a few linearly convergent optimization algorithms with high accuracy, and we compute it by deploying an accelerated gradient descent algorithm (see \cite{nesterov2018lectures} Section 2.2) for example. This point will be a benchmark for evaluating the quality of the solution obtained by our algorithm and \cite{bravo2018optimistic} Algorithm 1).
Experimental results. As indicated in Figure 3, our algorithm outperforms Bravo et al. (2018, Algorithm 1) as it exhibits a faster convergence to Nash equilibrium. Compared to the performance on Cournot competition and Kelly auction with synthetic data, the quality of the solutions obtained by both algorithms are lower, which is possibly caused by the ill-conditioning of real data.

6. Concluding Remarks and Future Direction

In a multi-agent online environment with bandit feedback, the most sensible choice for an agent who is oblivious to the presence of others (or who are conservative), is to deploy an optimal no-regret learning algorithm. With this in mind, we investigate the long-run behavior of individual optimal regularized no-regret learning policies. We show that, in strongly monotone games, the joint actions of all players converge to a (necessarily) unique Nash equilibrium, and the rate of convergence matches the optimal rates of single-agent, stochastic convex optimization up to log factors. We conduct extensive experiments using Cournot competition, Kelly auction, and distributed regularized logistic regression problems, the results demonstrate the superiority of our algorithm in practice. Our work thus settles an open problem and contributes to the broad landscape of bandit game-theoretical learning by identifying the first doubly optimal bandit learning algorithm, in that it achieves (up to log factors) both optimal regret in the single-agent learning and optimal last-iterate convergence rate in the multi-agent learning. Future work includes the extension to a fully decentralized setting where the players’ updates need not be synchronous and applications of our algorithms to online decision-making problems in practice.

Endnotes

1. Computing a Nash equilibrium is in general computationally intractable: the problem is indeed PPAD-complete (Daskalakis et al. 2009).

2. This setting is the same as bandit convex optimization in the literature and we consider maximization and concave reward functions instead of minimization and convex loss functions.

3. In utility maximization, we shall use mirror ascent instead of mirror descent because players seek to maximize their rewards (as opposed to minimizing their losses). Nonetheless, we keep the
term “descent” throughout because, despite the role reversal, it is the standard name associated with the method.

4. Randomization plays an important role in the online game playing literature. For example, the classical Follow The Leader (FTL) algorithm does not attain any non-trivial regret guarantee for linear cost functions (in the worst case it can be $\Omega(T)$ if the cost functions are chosen adversarially). However, Hannan (1957) proposed a randomized variant of FTL, called perturbed-follow-the-leader, which could attain an optimal regret of $O(\sqrt{T})$ for linear functions over the simplex.

5. https://www.csie.ntu.edu.tw/~cjlin/libsvm/

6. https://www.cs.nyu.edu/roweis/data.html
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Then we bound the term \( \langle \hat{v}^t, x^{t+1} - x^t \rangle \) by Lemma 5. Indeed, we let the function \( g(x) = \eta_t \langle \hat{v}^t, x^t - x \rangle + \eta_t \beta(t+1) \|x^t - x\|^2 + D_R(x, x^t) \) be defined as

\[
g(x) = \eta_t \langle \hat{v}^t, x^t - x \rangle + \eta_t \beta(t+1) \|x^t - x\|^2 + D_R(x, x^t)
\]

Since \( g \) is the sum of a self-concordant barrier function \( R \) and a quadratic function, we can easily verify that \( g \) is also a self-concordant function and have

\[
\nabla g(x^t) = -\eta_t \hat{v}^t, \quad \nabla^2 g(x^t) = \eta_t \beta(t+1) I_n + \nabla^2 R(x^t).
\]

### Appendix. Auxiliary Results and Missing Proofs
#### A. Proof of Lemma 6
By the definition of \( \mathcal{P}_R(\cdot, \cdot, \cdot) \) in Eq. (5), the iterate \( x^{t+1} = \mathcal{P}_R(x^t, \hat{v}^t, \eta_t) \) satisfies that

\[
-\eta_t \hat{v}^t + \eta_t \beta(t+1)(x^{t+1} - x^t) + \nabla R(x^{t+1}) - \nabla R(x^t) = 0.
\]

Using Lemma 5 with \( x = x^{t+1} \) and \( x' = x^t \), we have

\[
D_R(p, x^t) = D_R(p, x^{t+1}) + D_R(x^{t+1}, x^t) + \langle \nabla R(x^t) - \nabla R(x^{t+1}), x^{t+1} - p \rangle
\]

\[
= D_R(p, x^{t+1}) + D_R(x^{t+1}, x^t) - \eta_t \langle \hat{v}^t, x^{t+1} - p \rangle + \eta_t \beta(t+1)(x^{t+1} - x^t, x^{t+1} - p)
\]

\[
= D_R(p, x^{t+1}) - \eta_t \langle \hat{v}^t, x^{t+1} - x^t \rangle - \eta_t \langle \hat{v}^t, x^t - p \rangle + \left( D_R(x^{t+1}, x^t) + \frac{\eta_t \beta(t+1)}{2} \|x^{t+1} - x^t\|^2 \right)
\]

\[
+ \frac{\eta_t \beta(t+1)}{2} \|x^{t+1} - p\|^2 - \frac{\eta_t \beta(t+1)}{2} \|x^t - p\|^2.
\]

By the definition of Bregman divergence, we have \( D_R(x^{t+1}, x^t) \geq 0 \). Putting these pieces together yields that

\[
D_R(p, x^{t+1}) + \frac{\eta_t \beta(t+1)}{2} \|x^{t+1} - p\|^2 \leq D_R(p, x^t) + \frac{\eta_t \beta(t+1)}{2} \|x^t - p\|^2 + \eta_t \langle \hat{v}^t, x^{t+1} - x^t \rangle + \eta_t \langle \hat{v}^t, x^t - p \rangle.
\]

Then we bound the term \( \langle \hat{v}^t, x^{t+1} - x^t \rangle \) by Lemma 5. Indeed, we let the function \( g \) be defined as

\[
\eta_t \langle \hat{v}^t, x^t - x \rangle + \frac{\eta_t \beta(t+1)}{2} \|x^t - x\|^2 + D_R(x, x^t)
\]

Since \( g \) is the sum of a self-concordant barrier function \( R \) and a quadratic function, we can easily verify that \( g \) is also a self-concordant function and have

\[
\nabla g(x^t) = -\eta_t \hat{v}^t, \quad \nabla^2 g(x^t) = \eta_t \beta(t+1) I_n + \nabla^2 R(x^t).
\]
By definition of $A^t$, we have $A^t = (\nabla^2 g(x^t))^{-1/2}$. Using Eq. (13), we have

$$\lambda(x^t, g) = \|(\nabla^2 g(x^t))^{-1} \nabla g(x^t)\|_{x^t} = \eta_t \|\hat{v}^t\|_{x_t^t}. \tag{14}$$

To apply Lemma 3, we need to guarantee that $\lambda(x^t, g) \leq \frac{1}{\beta}$. Indeed, by the definition of $\hat{v}^t$ and using the fact that $A^t = (\nabla^2 g(x^t))^{-1/2}$, we have

$$\lambda(x^t, g) = \eta_t \|\hat{v}^t\|_{x_t^t, *} = \eta_t \|A^t \hat{v}^t\| = n\eta_t |f_t(x^t + A^t z^t)| \|z^t\| \leq n\eta_t |f_t(x^t + A^t z^t)|. \tag{15}$$

Since $x^t + A^t z^t \in X$ and each function $f_t$ satisfies that $|f_t(x)| \leq L$ for all $x \in X$, we have $\lambda(x^t, g) \leq nL\eta_t$. Combining it with the fact that $0 < \eta_t \leq \frac{1}{2nL}$ yields that $\lambda(x^t, g) \leq \frac{1}{\beta}$. Therefore, by Lemma 3, we have

$$\|x^{t+1} - x^t\|_{x^t} = \|x^t - \arg \min_{x \in X} g(x')\| \leq 2\lambda(x^t, g) \leq 2\eta_t \|\hat{v}^t\|_{x_t^t, *}. \tag{16}$$

This together with the Hölder’s inequality yields that

$$\langle \hat{v}^t, x^{t+1} - x^t \rangle \leq \|\hat{v}^t\|_{x_t^t, *} \|x^{t+1} - x^t\|_{x^t} \leq 2\eta_t \|\hat{v}^t\|_{x_t^t, *}^2. \tag{17}$$

Using the fact that $A^t = (\nabla^2 g(x^t))^{-1/2}$ again, we have

$$\langle \hat{v}^t, x^{t+1} - x^t \rangle \leq 2\eta_t \|A^t \hat{v}^t\|^2. \tag{18}$$

Plugging Eq. (15) into Eq. (12) yields the desired inequality.

**B. Proof of Theorem 1**

We are in a position to prove Theorem 1 regarding the regret bound of Algorithm 1. For simplicity, we assume that $\max_{x,x' \in X} \|x - x'\| \leq B$ for some $B > 0$. Fixing a point $p \in X$, we have

$$\sum_{t=1}^{T} f_t(p) - \mathbb{E} \left[ \sum_{t=1}^{T} f_t(\hat{x}^t) \right] = \sum_{t=1}^{T} \left[ f_t(p) - \hat{f}_t(p) \right] \tag{19}$$

$$+ \sum_{t=1}^{T} \mathbb{E} \left[ \hat{f}_t(p) - \hat{f}_t(x^t) \right] + \sum_{t=1}^{T} \mathbb{E} \left[ \hat{f}_t(x^t) - f_t(x^t) \right] + \sum_{t=1}^{T} \mathbb{E} \left[ f_t(x^t) - f_t(\hat{x}^t) \right], \tag{20}$$

where $\hat{f}_t$ is a smoothed version given by $\hat{f}_t(x) = \mathbb{E}_{w \sim \mathbb{B}} [f_t(x + A^t w)]$. Since $\nabla^2 R(x)$ is positive definite and $\eta_t = \frac{1}{2nL\sqrt{T}}$, we have

$$(\sigma_{\max}(A^t))^2 \leq \frac{1}{\eta_t \beta(t + 1)} = \frac{2nL\sqrt{T}}{\beta(t + 1)}. \tag{21}$$

By Lemma 4 and Eq. (17), we have

$$I \leq \frac{\ell(\sigma_{\max}(A^t))^2}{2} \leq \frac{2nL\sqrt{T}}{\beta(t + 1)}, \tag{22}$$
and

\[ \mathbf{III} = \mathbb{E}[\mathbb{E}[\hat{f}_t(x^t) - f_t(x^t) \mid x^t]] \leq 0. \]  \tag{19} \]

By the definition of \( \hat{f} \), we have

\[ \mathbf{IV} = \sum_{t=1}^{T} \mathbb{E}[f_t(x^t) - f_t(x^t)] = \sum_{t=1}^{T} \mathbb{E}[\mathbb{E}[f_t(x^t) - f_t(x^t + A^t z^t) \mid x^t]]. \]

By Lemma 4 and Eq. (17) again, we have

\[ \mathbf{IV} \leq \frac{\ell(\sigma_{\max}(A^t))^2}{2} \leq \frac{n\ell L \sqrt{T}}{\beta(t + 1)}. \]  \tag{20} \]

It remains to bound the second term. Indeed, since \( \eta_t = \frac{1}{2n L \sqrt{T}} \leq \frac{1}{2n L} \), Lemma 5 implies that

\[ D_R(p, x^{t+1}) + \eta_t \frac{\beta(t + 1)}{2} \|x^{t+1} - p\|^2 \leq D_R(p, x^t) + \frac{\eta_t \beta(t + 1)}{2} \|x^t - p\|^2 + 2\eta_t^2 \|A^t \hat{v}^t\|^2 + \eta_t \langle \hat{v}^t, x^t - p \rangle. \]

Taking the expectation of both sides conditioned on \( \mathcal{F}_t \) and dividing both sides by \( \eta_t \), we have

\[
\mathbb{E}[D_R(p, x^{t+1}) \mid \mathcal{F}_t] \leq \frac{\eta_t}{\beta(t + 1)} \mathbb{E}[\|x^{t+1} - p\|^2 \mid \mathcal{F}_t] \leq \frac{D_R(p, x^t)}{\eta_t} + \frac{\beta(t + 1)}{2} \|x^t - p\|^2 + 2\eta_t \mathbb{E}[\|A^t \hat{v}^t\|^2 \mid \mathcal{F}_t] + \mathbb{E}[\langle \hat{v}^t, x^t - p \rangle \mid \mathcal{F}_t].
\]

By the definition of \( \hat{v}^t \), we have

\[ \|A^t \hat{v}^t\|^2 = n^2 \|f_t(x^t + A^t z^t)\|^2 \|z^t\|^2 \leq n^2 L^2, \]

and

\[ \mathbb{E}[\langle \hat{v}^t, x^t - p \rangle \mid x^t] = \langle \nabla \hat{f}_t(x^t), x^t - p \rangle. \]

By Lemma 4 and the fact that \( f \) is \( \beta \)-strongly concave, we have \( \hat{f} \) is also \( \beta \)-strongly concave. This implies that

\[ \mathbb{E}[\langle \hat{v}^t, x^t - p \rangle \mid \mathcal{F}_t] \leq \hat{f}_t(x^t) - \hat{f}_t(p) - \frac{\beta}{2} \|x^t - p\|^2. \]

Putting these pieces together yields that

\[
\mathbb{E}[D_R(p, x^{t+1}) \mid \mathcal{F}_t] \leq \frac{D_R(p, x^t)}{\eta_t} + \frac{\beta(t + 1)}{2} \mathbb{E}[\|x^{t+1} - p\|^2 \mid \mathcal{F}_t] \leq \frac{D_R(p, x^t)}{\eta_t} + \frac{\beta(t + 1)}{2} \|x^t - p\|^2 + 2\eta_t n^2 L^2 + (\hat{f}_t(x^t) - \hat{f}_t(p)).
\]

Rearranging the above inequality and take the expectation of both sides, we have

\[ \mathbb{E}[\hat{f}_t(p) - \hat{f}_t(x^t)] \leq \mathbb{E}[D_R(p, x^t)] - \mathbb{E}[D_R(p, x^{t+1})] + \left( \frac{\beta t}{2} \mathbb{E}[\|x^t - p\|^2] - \frac{\beta(t + 1)}{2} \mathbb{E}[\|x^{t+1} - p\|^2] \right) + 2\eta_t n^2 L^2. \]
Therefore, we have

\[ \sum_{t=1}^{T} \mathbb{E}[\tilde{f}_t(p) - \tilde{f}_t(x^t)] \leq \frac{\sum_{t=1}^{T} \mathbb{E}[D_R(p, x^t)] - \mathbb{E}[D_R(p, x^{t+1})]}{\eta_t} \]

\[ + \frac{\beta}{2} \sum_{t=1}^{T} \mathbb{E}[||x^t - p||^2] - \frac{\beta(t + 1)}{2} \mathbb{E}[||x^{t+1} - p||^2] \] + 2n^2 L^2 \left( \sum_{t=1}^{T} \eta_t \right) \]

\[ = 2nL \sqrt{T} D_R(p, x^1) + \frac{\beta}{2} \left( (x^1 - p)^2 + nL \sqrt{T} \right) \]

\[ \leq nL \sqrt{T} (1 + 2D_R(p, x^1)) + \frac{\beta B^2}{2}. \] \hfill (21)

By the initialization \( x^1 = \arg \min_{x \in \mathcal{X}} R(x) \), we have \( \nabla R(x^1) = 0 \) which implies that \( D_R(p, x^1) = R(p) - R(x^1) \). We consider the following two cases:

- A point \( p \in \mathcal{X} \) satisfies that \( \pi_{x^1}(p) \leq 1 - \frac{1}{\sqrt{T}} \). Then, by Lemma [2] we have

\[ D_R(p, x^1) = R(p) - R(x^1) \leq \nu \log(T). \]

This together with Eq. (21) implies that

\[ \sum_{t=1}^{T} \mathbb{E}[\tilde{f}_t(p) - \tilde{f}_t(x^t)] \leq nL \sqrt{T} (1 + 2\nu \log(T)) + \frac{\beta B^2}{2}. \] \hfill (22)

Plugging Eq. (18), Eq. (19), Eq. (20) and Eq. (22) into Eq. (16) yields that

\[ \sum_{t=1}^{T} f_t(p) - \mathbb{E} \left[ \sum_{t=1}^{T} f_t(\hat{x}^t) \right] = \tilde{O}(\sqrt{T}). \] \hfill (23)

- A point \( p \in \mathcal{X} \) satisfies that \( \pi_{x^1}(p) > 1 - \frac{1}{\sqrt{T}} \). Then, we can always find \( p' \in \mathcal{X} \) such that \( ||p' - p|| = O\left(\frac{1}{\sqrt{T}}\right) \) and \( \pi_{x^1}(p') \leq 1 - \frac{1}{\sqrt{T}} \). Since Eq. (23) holds true for any fixed \( p \in \mathcal{X} \) satisfying \( \pi_{x^1}(p) \leq 1 - \frac{1}{\sqrt{T}} \), we have

\[ \sum_{t=1}^{T} f_t(p') - \mathbb{E} \left[ \sum_{t=1}^{T} f_t(\hat{x}^t) \right] = \tilde{O}(\sqrt{T}). \]

Recall that each function \( f_t \) is Lipschitz continuous, we have

\[ \left| \sum_{t=1}^{T} f_t(p') - f_t(p) \right| = O(\sqrt{T}). \]

Therefore, Eq. (23) also holds true in this case.

Combining the above two cases with the fact that \( p \in \mathcal{X} \) is arbitrary, we have

\[ \text{Reg}_T = \max_{p \in \mathcal{X}} \left\{ \sum_{t=1}^{T} f_t(p) - \mathbb{E} \left[ \sum_{t=1}^{T} f_t(\hat{x}^t) \right] \right\} = \tilde{O}(\sqrt{T}). \]

This completes the proof.
C. Proof of Lemma 7

By the definition, we have

$$
\mathbb{E}[\hat{v}_i | x] = \prod_{i \in \mathcal{N}} \frac{n_i}{\text{vol}(S_i)} \int_{S_1} \cdots \int_{S_N} u_i(x_1 + A_1 z_1, \ldots, x_N + A_N z_N)(A_i)^{-1} z_i \, dz_1 \cdots dz_N.
$$

Since all of $A_i$ are invertible, we can define auxiliary functions $U_i(x)$ by

$$
U_i(x_1, x_2, \ldots, x_N) = u_i(A_1 x_1, A_2 x_2, \ldots, A_N x_N), \quad \forall i \in \mathcal{N},
$$

and have

$$
\mathbb{E}[\hat{v}_i | x] = (A_i)^{-1} \left( \prod_{i \in \mathcal{N}} \frac{n_i}{\text{vol}(S_i)} \int_{S_1} \cdots \int_{S_N} U_i((A_i)^{-1} x_1 + z_1, \ldots, (A_i)^{-1} x_N + z_N) z_i \, dz_1 \cdots dz_N \right).
$$

For the ease of presentation, we let $\tilde{x}_i = (A_i)^{-1} x_i$. Then, by using the same argument as in [Bravo et al. 2018, Lemma C.1] with the independence of the sampling directions $\{z_i, i \in \mathcal{N}\}$, we have

$$
\prod_{i \in \mathcal{N}} \frac{n_i}{\text{vol}(S_i)} \int_{S_1} \cdots \int_{S_N} U_i(\tilde{x}_1 + z_1, \ldots, \tilde{x}_N + z_N) z_i \, dz_1 \cdots dz_N
$$

$$
= \prod_{i \in \mathcal{N}} \frac{n_i}{\text{vol}(S_i)} \int_{S_1} \cdots \int_{S_N} U_i(\tilde{x}_1 + z_1, \ldots, \tilde{x}_N + z_N) \frac{z_i}{\|z_i\|} \, dz_1 \cdots dz_N
$$

$$
= \prod_{i \in \mathcal{N}} \frac{n_i}{\text{vol}(S_i)} \int_{S_1} \left( \int_{\prod_{j \neq i} S_j} U_i(\tilde{x}_i + z_i; \tilde{x}_{-i} + z_{-i}) \frac{z_i}{\|z_i\|} \, dz_i dz_{-i} \right)
$$

$$
= \prod_{i \in \mathcal{N}} \frac{n_i}{\text{vol}(S_i)} \int_{B_i} \left( \int_{\prod_{j \neq i} S_j} \nabla_i U_i(\tilde{x}_i + w_i; \tilde{x}_{-i} + z_{-i}) \frac{z_i}{\|z_i\|} \, dw_i dz_{-i} \right),
$$

where, in the last equality, we use the identity

$$
\nabla \int_{\mathbb{B}} F(x + w) \, dw = \int_{\mathbb{B}} F(x + z) \frac{z}{\|z\|} \, dz,
$$

which, in turn, follows from Stoke’s theorem. Since $\text{vol}(B_i) = (1/n_i) \text{vol}(S_i)$, the above argument implies that $\mathbb{E}[\hat{v}_i | x] = (A_i)^{-1} \nabla_i \hat{U}_i(\tilde{x})$ with $\hat{U}_i$ given by

$$
\hat{U}_i(x) = \mathbb{E}_{w_i \sim \mathbb{B}_n} \mathbb{E}_{z_{-i} \sim \Pi_{j \neq i} S^n_j} [U_i(x_i + w_i; x_{-i} + z_{-i})].
$$

By the definition of $U_i$ in Eq. (24) and $\tilde{x}_i = (A_i)^{-1} x_i$, we have

$$
\mathbb{E}[\hat{v}_i | x] = (A_i)^{-1} A_i \nabla_i \hat{u}_i(x_i) = \nabla_i \hat{u}_i(x_i),
$$

with $\hat{u}_i(x) = \mathbb{E}_{w_i \sim \mathbb{B}_n} \mathbb{E}_{z_{-i} \sim \Pi_{j \neq i} S^n_j} [u_i(x_i + A_i w_i; \hat{x}_{-i})]$ where $\hat{x}_i = x_i + A_i z_i$ for all $i \in \mathcal{N}$. 
For the second part of the lemma, since \( v_i(x) \) is Lipschitz continuous with parameter \( \ell_i \), we have
\[
\| \nabla_i u_i(x_i + A_i w_i; \hat{x}_i) - \nabla_i u_i(x) \| = \| v_i(x_i + A_i w_i; \hat{x}_i) - v_i(x) \|
\leq \ell_i \sqrt{\| A_i w_i \|^2 + \sum_{j \neq i} \| A_j z_j \|^2} \leq \ell_i \sqrt{\sum_{j \in \mathcal{N}} \| A_j \|^2} = \ell_i \sqrt{\sum_{j \in \mathcal{N}} \sigma_{\text{max}}(A_j)^2}.
\]

This completes the proof.

D. Proof of Lemma 8

Since Algorithm 2 is developed in which each player chooses her decision using Algorithm 1, Lemma 6 implies that
\[
D_{R_i}(p_i, x_i^{t+1}) + \frac{\eta_i \beta(t+1)}{2 \lambda_i} \| x_i^{t+1} - p_i \|^2 \leq D_{R_i}(p_i, x_i^t) + \frac{\eta_i \beta(t+1)}{2 \lambda_i} \| x_i^t - p_i \|^2 + \frac{2 \eta_i^2}{} \| A_i \hat{v}_i^t \|^2 + \eta_i \langle \hat{v}_i^t, x_i^t - p_i \rangle,
\]
where \( p_i \in \mathcal{X}_i \) and \( \{ \eta_i \}_{i \geq 1} \) is a nonincreasing sequence satisfying that \( 0 < \eta_i \leq \frac{1}{2n L \ell_i} \). Since \( n = \sum_{i \in \mathcal{N}} n_i \) in the multi-agent setting, Eq. (25) holds true for all \( i \in \mathcal{N} \) if \( 0 < \eta_i \leq \frac{1}{2n L \ell_i} \). Multiplying Eq. (26) by \( \lambda_i \) and summing up the resulting inequalities over \( i \in \mathcal{N} \), then using the fact that \( \eta_i \leq \eta_{i+1} \) yields the desired inequality.

E. Proof of Theorem 2

We are in a position to prove Theorem 2 regarding the last-iterate convergence rate of Algorithm 2 for the case when the perfect bandit feedback is available \( (\xi_{i,t} \equiv 0) \). For simplicity, we assume that
\[
\max_{x, x' \in \mathcal{X}_i} \| x - x' \| \leq B_i \text{ for some } B_i > 0. \quad \text{Since } \eta_i = \frac{1}{2n L \ell_i} \leq \frac{1}{2n L}, \text{ Lemma 8 implies that}
\]
\[
\sum_{i \in \mathcal{N}} \lambda_i D_{R_i}(p_i, x_i^{t+1}) + \frac{\eta_i+1 \beta(t+1)}{2} \left( \sum_{i \in \mathcal{N}} \| x_i^{t+1} - p_i \|^2 \right)
\leq \sum_{i \in \mathcal{N}} \lambda_i D_{R_i}(p_i, x_i^t) + \frac{\eta_i \beta(t+1)}{2} \left( \sum_{i \in \mathcal{N}} \| x_i^t - p_i \|^2 \right) + 2n^2 \left( \sum_{i \in \mathcal{N}} \lambda_i \| A_i \hat{v}_i^t \|^2 \right) + \eta_i \left( \sum_{i \in \mathcal{N}} \lambda_i \langle \hat{v}_i^t, x_i^t - p_i \rangle \right).
\]
Taking the expectation of both sides conditioned on \( F_t \), we have
\[
\mathbb{E} \left[ \sum_{i \in \mathcal{N}} \lambda_i D_{R_i}(p_i, x_i^{t+1}) \mid F_t \right] + \frac{\eta_i+1 \beta(t+1)}{2} \mathbb{E} \left[ \sum_{i \in \mathcal{N}} \| x_i^{t+1} - p_i \|^2 \mid F_t \right]
\leq \sum_{i \in \mathcal{N}} \lambda_i D_{R_i}(p_i, x_i^t) + \frac{\eta_i \beta(t+1)}{2} \mathbb{E} \left[ \sum_{i \in \mathcal{N}} \| x_i^t - p_i \|^2 \right] + 2n^2 \mathbb{E} \left[ \sum_{i \in \mathcal{N}} \lambda_i \| A_i \hat{v}_i^t \|^2 \right] + \eta_i \mathbb{E} \left[ \sum_{i \in \mathcal{N}} \lambda_i \langle \hat{v}_i^t, x_i^t - p_i \rangle \mid F_t \right].
\]
By the definition of \( \hat{v}_i^t \) and Lemma 7, we have
\[
\| A_i \hat{v}_i^t \|^2 = n_i^2 \| u_i(x_i + A_i z_1^t, \ldots, x_N + A_N z_N) \| \| z_i^t \|^2 \leq n_i^2 L^2,
\]
(27)
and
\[ E[\langle \hat{\nu}_i^t, x_i^t - p_i \rangle | x_i^t] = \langle \nabla_i \hat{u}_i(x^t), x_i^t - p_i \rangle. \]

By Lemma 7, again and the Young’s inequality, we have
\[ E[\langle \hat{\nu}_i^t, x_i^t - p_i \rangle | F_i] = \langle \nabla_i u_i(x^t), x_i^t - p_i \rangle + \langle \nabla_i \hat{u}_i(x^t) - \nabla_i u_i(x^t), x_i^t - p_i \rangle \]
\[ \leq \langle \nabla_i u_i(x^t), x_i^t - p_i \rangle + \frac{\lambda_i}{2\beta} \| \nabla_i \hat{u}_i(x^t) - \nabla_i u_i(x^t) \|^2 + \frac{\beta}{2\lambda_i} \| x_i^t - p_i \|^2 \]
\[ \leq \langle \nabla_i u_i(x^t), x_i^t - p_i \rangle + \frac{\lambda_i^2 \ell_i^2}{2\beta} \left( \sum_{i \in \mathcal{N}} (\sigma_{\max}(A_i^t))^2 \right) + \frac{\beta}{2\lambda_i} \| x_i^t - p_i \|^2. \]

Since \( \nabla^2 R_i(x) \) is positive definite and \( \eta_t = \frac{1}{2nL\sqrt{t}} \), we have
\[ (\sigma_{\max}(A_i^t))^2 \leq \frac{\lambda_i}{\eta_t \beta(t + 1)} = \frac{2nL\lambda_i}{\beta \sqrt{t}}. \]

Plugging Eq. (29) into Eq. (28) and summing up the resulting inequality over \( i \in \mathcal{N} \), we have
\[ E \left[ \sum_{i \in \mathcal{N}} \lambda_i \langle \hat{\nu}_i^t, x_i^t - p_i \rangle \mid F_i \right] \leq \sum_{i \in \mathcal{N}} \lambda_i \langle \nabla_i u_i(x^t), x_i^t - p_i \rangle + \frac{nL}{\beta^2 \sqrt{t}} \left( \sum_{i \in \mathcal{N}} \lambda_i \right) \left( \sum_{i \in \mathcal{N}} \lambda_i^2 \ell_i^2 \right) + \frac{\beta}{2} \left( \sum_{i \in \mathcal{N}} \| x_i^t - p_i \|^2 \right). \]

By \( \beta \)-strongly monotonicity of the game, we have
\[ \sum_{i \in \mathcal{N}} \lambda_i \langle \nabla_i u_i(x^t), x_i^t - p_i \rangle \leq -\beta \left( \sum_{i \in \mathcal{N}} \| x_i^t - p_i \|^2 \right) + \sum_{i \in \mathcal{N}} \lambda_i \langle \nabla_i u_i(p), x_i^t - p_i \rangle. \]

Therefore, we have
\[ E \left[ \sum_{i \in \mathcal{N}} \lambda_i \langle \hat{\nu}_i^t, x_i^t - p_i \rangle \mid F_i \right] \leq \frac{nL}{\beta^2 \sqrt{t}} \left( \sum_{i \in \mathcal{N}} \lambda_i \right) \left( \sum_{i \in \mathcal{N}} \lambda_i^2 \ell_i^2 \right) + \frac{\beta}{2} \left( \sum_{i \in \mathcal{N}} \| x_i^t - p_i \|^2 \right) + \sum_{i \in \mathcal{N}} \lambda_i \langle \nabla_i u_i(p), x_i^t - p_i \rangle. \]

Plugging Eq. (27) and Eq. (30) into Eq. (26) and taking the expectation of both sides, we have
\[ E \left[ \sum_{i \in \mathcal{N}} \lambda_i D_{R_i}(p_i, x_i^{t+1}) \right] + \frac{\eta_t \beta(t + 1)}{2} E \left[ \sum_{i \in \mathcal{N}} \| x_i^{t+1} - p_i \|^2 \right] \]
\[ \leq E \left[ \sum_{i \in \mathcal{N}} \lambda_i D_{R_i}(p_i, x_i^t) \right] + \frac{\eta_t \beta t}{2} E \left[ \sum_{i \in \mathcal{N}} \| x_i^t - p_i \|^2 \right] + 2\eta_t^2 L^2 \left( \sum_{i \in \mathcal{N}} n_i^2 \lambda_i \right)
+ \frac{\eta_t nL}{\beta^2 \sqrt{t}} \left( \sum_{i \in \mathcal{N}} \lambda_i \right) \left( \sum_{i \in \mathcal{N}} \lambda_i^2 \ell_i^2 \right) + \eta_t E \left[ \sum_{i \in \mathcal{N}} \lambda_i \langle \nabla_i u_i(p), x_i^t - p_i \rangle \right]. \]

Since \( \eta_t = \frac{1}{2nL\sqrt{t}} \), we have
\[ E \left[ \sum_{i \in \mathcal{N}} \lambda_i D_{R_i}(p_i, x_i^T) \right] + \frac{\beta \sqrt{T}}{4nL} E \left[ \sum_{i \in \mathcal{N}} \| x_i^T - p_i \|^2 \right] \leq \sum_{t=1}^{T-1} \eta_t E \left[ \sum_{i \in \mathcal{N}} \lambda_i \langle \nabla_i u_i(p), x_i^t - p_i \rangle \right]
+ \sum_{i \in \mathcal{N}} \lambda_i D_{R_i}(p_i, x_i^1) + \frac{\beta}{4nL} \left( \sum_{i \in \mathcal{N}} \| x_i^1 - p_i \|^2 \right) + \frac{1}{2n^2} \left( \sum_{i \in \mathcal{N}} n_i^2 \lambda_i \right) \left( \sum_{t=1}^{T-1} \frac{1}{t} \right) + \frac{1}{2\beta^2} \left( \sum_{i \in \mathcal{N}} \lambda_i \right) \left( \sum_{i \in \mathcal{N}} \lambda_i^2 \ell_i^2 \right) \left( \sum_{t=1}^{T-1} \frac{1}{t} \right). \]
Rearranging the above inequality in the compact form and noting that $D_{R_i}(x_i^*, x_i^T) \geq 0$, we have

\[
\frac{\beta \sqrt{T}}{4nL} \mathbb{E} \left[ \| x^T - p \|^2 \right] \leq \sum_{t=1}^{T-1} \eta_t \mathbb{E} \left[ \sum_{i \in N} \lambda_i \langle \nabla_i u_i(p), x_i^t - p_i \rangle \right] + \sum_{i \in N} \lambda_i D_{R_i}(p_i, x_i^1) + \frac{\beta}{4nL} \left( \sum_{i \in N} B_i^2 \right) + \frac{1}{2n^2} \left( \sum_{i \in N} n_i^2 \lambda_i \right) + \frac{1}{2\beta^2} \left( \sum_{i \in N} \lambda_i \right) \left( \sum_{i \in N} \lambda_i^2 \ell_i^2 \right) \log(T).
\]

(31)

By the initialization $x_i^1 = \arg \min_{x \in X_i} R_i(x)$, we have $\nabla R_i(x_i^1) = 0$ which implies that $D_{R_i}(p_i, x_i^1) = R_i(p_i) - R(x_i^1)$. Then, let us inspect each coordinate of a unique Nash equilibrium $x^*$ and set coordinate by coordinate in Eq. (31).

- A point $x_i^* \in X_i$ satisfies that $\pi_{x_i^1}(x_i^*) \leq 1 - \frac{1}{\sqrt{T}}$. Then, by Lemma 2, we have

$$D_{R_i}(x_i^*, x_i^1) = R_i(x_i^*) - R_i(x_i^1) \leq \nu_i \log(T).$$

In this case, we choose $p_i = x_i^*$.

- A point $x_i^* \in X_i$ satisfies that $\pi_{x_i^1}(x_i^*) > 1 - \frac{1}{\sqrt{T}}$. Then, we can always find $\bar{x}_i \in X_i$ such that $\| x_i - x_i^* \| = O\left(\frac{1}{\sqrt{T}}\right)$ and $\pi_{x_i^1}(\bar{x}_i) \leq 1 - \frac{1}{\sqrt{T}}$. Then, by Lemma 2 we have

$$D_{R_i}(\bar{x}_i, x_i^1) = R_i(\bar{x}_i) - R_i(x_i^1) \leq \nu_i \log(T).$$

In this case, we choose $p_i = \bar{x}_i$.

For simplicity, we denote the set of coordinates in the former case by $\mathcal{I}$ which implies that the set of coordinates in the latter case is $\mathcal{N} \setminus \mathcal{I}$. Then, Eq. (31) becomes

\[
\frac{\beta \sqrt{T}}{4nL} \mathbb{E} \left[ \sum_{i \in \mathcal{I}} \| x_i^T - x_i^* \|^2 + \sum_{i \in \mathcal{N} \setminus \mathcal{I}} \| x_i^T - \bar{x}_i \|^2 \right] \leq \sum_{t=1}^{T-1} \eta_t \mathbb{E} \left[ \sum_{i \in \mathcal{I}} \lambda_i \langle \nabla_i u_i(x_i^*, \bar{x}_i, x_i^1), x_i^t - x_i^1 \rangle \right] + \sum_{i \in \mathcal{N} \setminus \mathcal{I}} \lambda_i D_{R_i}(\bar{x}_i, x_i^1)
\]

\[
+ \frac{\beta}{4nL} \left( \sum_{i \in \mathcal{N} \setminus \mathcal{I}} B_i^2 \right) + \frac{1}{2n^2} \left( \sum_{i \in \mathcal{N} \setminus \mathcal{I}} n_i^2 \lambda_i \right) + \frac{1}{2\beta^2} \left( \sum_{i \in \mathcal{N} \setminus \mathcal{I}} \lambda_i \right) \left( \sum_{i \in \mathcal{N} \setminus \mathcal{I}} \lambda_i^2 \ell_i^2 \right) \log(T)
\]

\[
= \sum_{t=1}^{T-1} \eta_t \mathbb{E} \left[ \sum_{i \in \mathcal{N} \setminus \mathcal{I}} \lambda_i \langle \nabla_i u_i(x_i^*, \bar{x}_i, x_i^1), x_i^t - x_i^1 \rangle \right] + \sum_{t=1}^{T-1} \eta_t \mathbb{E} \left[ \sum_{i \in \mathcal{N} \setminus \mathcal{I}} \lambda_i \langle \nabla_i u_i(x_i^*, \bar{x}_i, x_i^1), x_i^t - \bar{x}_i \rangle \right] + O(\log(T)).
\]

Since $\| \bar{x}_i - x_i^* \| = O\left(\frac{1}{\sqrt{T}}\right)$ and $x_i^*, \bar{x}_i \in X_i$ where $X_i$ is convex and compact, we have

$$\mathbb{E} \left[ \sum_{i \in \mathcal{N} \setminus \mathcal{I}} \| x_i^T - x_i^* \|^2 \right] \leq \mathbb{E} \left[ \sum_{i \in \mathcal{N} \setminus \mathcal{I}} \| x_i^T - \bar{x}_i \|^2 \right] + O\left(\frac{1}{\sqrt{T}}\right).$$
We turn to the imperfect bandit feedback setting and prove that Algorithm 2 still achieves the which returns an estimate of their payoff functions at their current action profile. This information This concludes the proof.

By the definition of \( \hat{x}^t \), we have

\[
\sum_{i \in \mathcal{N}} (\nabla_i u_i(x^*_{-i}, \bar{x}_{-i}, x_i^t - \bar{x}_i) \leq \sum_{i \in \mathcal{N}} \lambda_i \|\nabla_i u_i(x^*_{-i}, \bar{x}_{-i})\| \|x_i^t - \bar{x}_i\| = O \left( \frac{1}{\sqrt{T}} \right).
\]

Putting these pieces together yields that

\[
\mathbb{E} \left[ \sum_{t \in \mathcal{N}} \|x_i^t - x_i^*\|^2 \right] \leq \frac{4nL}{\beta \sqrt{T}} \mathbb{E} \left[ \sum_{t=1}^{T-1} \eta_t \left( \sum_{i \in \mathcal{N}} \lambda_i \nabla_i u_i(x^*_{-i}, x_i^t - \bar{x}_i) \right) \right] + O \left( \frac{\log(T)}{\sqrt{T}} \right).
\]

Since \( x^* \) is a Nash equilibrium, we have \( \sum_{i \in \mathcal{N}} \nabla_i u_i(x^*_{-i}, x_i - x_i^*) \leq 0 \) for all \( x \in \mathcal{X} \). This together with the above inequality yields that

\[
\mathbb{E} \left[ \sum_{t \in \mathcal{N}} \|x_i^t - x_i^*\|^2 \right] = \tilde{O} \left( \frac{1}{\sqrt{T}} \right).
\]

By the definition of \( \hat{x}^t \), we have

\[
\mathbb{E} \left[ \sum_{t \in \mathcal{N}} \|\hat{x}_i^t - x_i^*\|^2 \right] \leq 2 \mathbb{E} \left[ \sum_{t \in \mathcal{N}} \|\hat{x}_i^t - x_i^t\|^2 \right] + 2 \mathbb{E} \left[ \sum_{t \in \mathcal{N}} \|x_i^t - x_i^*\|^2 \right] \leq 2 \mathbb{E} \left[ \sum_{t \in \mathcal{N}} \|A_i^t\|^2 \right] + \tilde{O} \left( \frac{1}{\sqrt{T}} \right).
\]

This concludes the proof.

F. Imperfect Bandit Feedback

We turn to the imperfect bandit feedback setting and prove that Algorithm 2 still achieves the \( \tilde{O}(1/\sqrt{T}) \) rate of last-iterate convergence to a (necessarily) unique Nash equilibrium. In particular, we assume that each player \( i \in \mathcal{N} \) has access to a “black box” feedback mechanism — an oracle — which returns an estimate of their payoff functions at their current action profile. This information can be imperfect for a multitude of reasons: for instance, (i) estimates may be susceptible to random
measurement errors; (ii) the transmission of this information could be subject to noise; and/or (iii) the game’s payoff functions may be stochastic expectations of the form

\[ u_i(x) = \mathbb{E}[\hat{u}_i(x, \omega)] \quad \text{for some random variable } \omega. \]

With all this in mind, we will explore the noisy bandit feedback setting: \( \hat{u}_i^t \leftarrow u_i(\hat{x}^t) + \xi_{i,t} \) where the noise process \( \xi_t = (\xi_{i,t})_{i \in \mathcal{N}} \) is a bounded martingale difference adapted to the history \((\mathcal{F}_t)_{t=1}^\infty \) of \( \hat{x}^t \) (i.e., \( \xi_{i,t} \) is \( \mathcal{F}_t \)-measurable but \( \xi_t \) is not). More specifically, this implies that \((\xi_{i,t})_{t \geq 1}\) satisfies the statistical hypotheses (almost surely): (i) Zero-mean: \( \mathbb{E}[\xi_{i,t} | \mathcal{F}_t] = 0 \); and (ii) Boundedness: \( \|\xi_{i,t}\| \leq \sigma \) for some constant \( \sigma \geq 0 \). Alternatively, the above conditions simply posit that the players’ individual payoff estimates are conditionally unbiased and the difference remains bounded, i.e.,

\[ \mathbb{E}[\hat{u}_i^t | \mathcal{F}_t] = u_i(\hat{x}^t), \quad \|\hat{u}_i^t - u_i(\hat{x}^t)\| \leq \sigma, \quad \text{for all } i \in \mathcal{N}. \]

The above conditions are satisfied by a broad range of error processes, including all compactly supported distributions; further, many applications in operations research, control, network theory, and machine learning can be used as motivations for the bounded noise framework.

**Theorem 3.** Suppose that \( x^* \in \mathcal{X} \) is a unique Nash equilibrium of a smooth and \( \lambda \)-strongly monotone game. Each payoff function \( u_i \) satisfies that \( |u_i(x)| \leq L \) for all \( x \in \mathcal{X} \). If each player follows Algorithm 2 with parameters \( \eta_t = \frac{1}{2n(L+\sigma)\sqrt{t}} \) and \( \xi_{i,t} > 0 \) satisfies the aforementioned statistical hypotheses (almost surely), we have

\[ \mathbb{E}\left[ \sum_{i \in \mathcal{N}} \|\hat{x}_i^T - x_i^*\|^2 \right] = \tilde{O}\left(\frac{1}{\sqrt{T}}\right). \]

**Proof.** The key is to establish a descent lemma for the iterates generated by Algorithm 2 with the fully adaptive stepsize when only the imperfect bandit feedback is available (\( \xi_{i,t} > 0 \)).

**Lemma 9.** Suppose that the iterate \( \{x_i^t\}_{t \geq 1} \) is generated by Algorithm 2 and each function \( u_i \) satisfies that \( |u_i(x)| \leq L \) for all \( x \in \mathcal{X} \), we have

\[ \sum_{i \in \mathcal{N}} \lambda_i D_{R_i}(p_i, x_i^{t+1}) + \eta_t \beta(t+1) \left( \sum_{i \in \mathcal{N}} \|x_i^{t+1} - p_i\|^2 \right) \leq \sum_{i \in \mathcal{N}} \lambda_i D_{R_i}(p_i, x_i^t) + \eta_t \beta(t+1) \left( \sum_{i \in \mathcal{N}} \|x_i^t - p_i\|^2 \right) + 2\eta_t^2 \left( \sum_{i \in \mathcal{N}} \lambda_i \|A_i \hat{v}_i^t\|^2 \right) + \eta_t \left( \sum_{i \in \mathcal{N}} \lambda_i \langle \hat{v}_i^t, x_i^t - p_i \rangle \right). \]

where \( p_i \in \mathcal{X}_i \) and \( \{\eta_t\}_{t \geq 1} \) is a nonincreasing sequence satisfying that \( 0 < \eta \leq \frac{1}{2n(L+\sigma)} \).

**Proof.** Using the same argument as in Lemma 8, we have

\[ \sum_{i \in \mathcal{N}} \lambda_i D_{R_i}(p_i, x_i^{t+1}) + \eta_t \beta(t+1) \left( \sum_{i \in \mathcal{N}} \|x_i^{t+1} - p_i\|^2 \right) \leq \sum_{i \in \mathcal{N}} \lambda_i D_{R_i}(p_i, x_i^t) + \eta_t \beta(t+1) \left( \sum_{i \in \mathcal{N}} \|x_i^t - p_i\|^2 \right) + \eta_t \left( \sum_{i \in \mathcal{N}} \lambda_i \langle \hat{v}_i^t, x_i^t - p_i \rangle \right) + \eta_t \left( \sum_{i \in \mathcal{N}} \lambda_i \langle \hat{v}_i^t, x_i^t - p_i \rangle \right). \]
Next, we bound the term $\langle \hat{v}_t^i, x_{t+1}^i - x_t^i \rangle$ using Lemma 3. Indeed, we define the function $g$ by
\[ g(x_i) = \eta_t \langle \hat{v}_t^i, x_t^i - x_i \rangle + \frac{m_t}{2\lambda_i} \|x_t^i - x_i\|^2 + D_{R_i}(x_i, x_t^i). \]
and apply the same argument as in Lemma 8. The only difference is that we have a new upper bound for the term $\|A^t_i \hat{v}_t^i\|$ here:
\[ \|\hat{v}_t^i\|_{x_t^i,\star} = \|A^t_i \hat{v}_t^i\| = n|u_i(x^t + A^t z^t) + \xi_{i,t}\| z_t^i \| \leq n|u_i(x^t + A^t z^t) + \xi_{i,t}|. \]
Combining the above inequality with the fact that $0 < \eta_t \leq \frac{1}{2\lambda_i(L+\sigma)}$ gives $\lambda(x_t^i, g) \leq \frac{1}{2}$. Therefore, by Lemma 3 we have $\|x_{t+1}^i - x_t^i\|_{x_t^i} \leq 2\eta_t \|\hat{v}_t^i\|_{x_t^i,\star}$. This together with the Hölder’s inequality yields that
\[ \langle \hat{v}_t^i, x_{t+1}^i - x_t^i \rangle \leq \|\hat{v}_t^i\|_{x_t^i,\star} \|x_{t+1}^i - x_t^i\|_x^i \leq 2\eta_t \|\hat{v}_t^i\|^2_{x_t^i,\star} = 2\eta_t \|A^t_i \hat{v}_t^i\|^2. \]
Plugging the above inequality into Eq. (32) yields the desired inequality. □

Since the inequality in Lemma 9 is the same as that derived in Lemma 8 we can prove Theorem 3 by applying the same argument used in Theorem 2. □

Remark 7. Theorem 3 shows that Algorithm 2 can still achieve the near-optimal rate of last-iterate convergence in more challenging environment in which the bandit feedback is imperfect with zero-mean and bounded noises. Can we generalize the proposed algorithm to handle more general noises, e.g., the one with bounded variance? We leave the answers to future work.