Correlation/Communication complexity of generating bipartite states

Rahul Jain∗ Yaoyun Shi† Zhaohui Wei‡ Shengyu Zhang§

Abstract

We study the correlation complexity (or equivalently, the communication complexity) of generating a bipartite quantum state $\rho$. When $\rho$ is a pure state, we completely characterize the complexity for approximately generating $\rho$ by a corresponding approximate rank, closing a gap left in Ambainis, Schulman, Ta-Shma, Vazirani and Wigderson (SIAM Journal on Computing, 32(6):1570-1585, 2003). When $\rho$ is a classical distribution $P(x,y)$, we tightly characterize the complexity of generating $P$ by the psd-rank, a measure recently proposed by Fiorini, Massar, Pokutta, Tiwary and de Wolf (STOC 2012). We also present a characterization of the complexity of generating a general quantum state $\rho$. 

∗ Department of Computer Science and Centre for Quantum Technologies, National University of Singapore, 2 Science Drive 3, Singapore 117542. Email: rahul@comp.nus.edu.sg
† Department of Electrical Engineering and Computer Science, 2260 Hayward Street, University of Michigan, Ann Arbor, Michigan, 48109-2121, USA. Email: shiyy@eecs.umich.edu
‡ Centre for Quantum Technologies, National University of Singapore, 2 Science Drive 3, Singapore 117542. Email: cqtwz@nus.edu.sg
§ Department of Computer Science and Engineering and The Institute of Theoretical Computer Science and Communications, The Chinese University of Hong Kong. Email: syzhang@cse.cuhk.edu.hk
1 Introduction

In [5], the following basic model was studied: Two parties, called Alice and Bob, aim to generate a target bipartite state \( \rho \in \mathcal{H}_A \otimes \mathcal{H}_B \) (Hilbert space \( \mathcal{H}_A \) is in possession of Alice and Hilbert space \( \mathcal{H}_B \) is in possession of Bob) using local quantum operations on a shared seed state \( \sigma \in \mathcal{H}_A \otimes \mathcal{H}_B \). The minimum size of this seed state is the quantum correlation complexity of \( \rho \), denoted \( Q(\rho) \). Since Alice and Bob can always just share \( \rho \) itself, \( Q(\rho) \) is at most the number of qubits of \( \rho \), so the correlation complexity is a sublinear complexity measure. Let \( \{|x\} | x \in [\dim(\mathcal{H}_A)]\) be the computational bases for \( \mathcal{H}_A \) and let \( \{|y\} | y \in [\dim(\mathcal{H}_B)]\) be the computational bases for \( \mathcal{H}_B \). We call a state \( \rho \) classical if its eigenvectors are the computational basis states \( \{|x\} \otimes |y\} | x \in [\dim(\mathcal{H}_A)], y \in [\dim(\mathcal{H}_B)]\). Equivalently, it is just a classical probability distribution on the computational bases of \( \mathcal{H}_A \otimes \mathcal{H}_B \). For a classical state \( \rho \), the minimum size of a classical seed state is the randomized correlation complexity of \( \rho \), denoted \( R(\rho) \). The work [5] exhibited a classical state \( \rho \) of size \( n \) with \( R(\rho) \geq \log_2(n) \) and \( Q(\rho) = 1 \).

Above we considered the model in which Alice and Bob start with some shared state \( \sigma \) and produce target state \( \rho \) by doing only local operations and no communication. On the other hand, we also consider the model in which Alice and Bob start with some tensor state \( \sigma_A \otimes \sigma_B \) and do some local operations and communication and produce \( \rho \) at the end of their protocol. The quantum communication complexity of \( \rho \), denoted \( Q\text{Comm}(\rho) \), is defined as the minimum number of qubits exchanged between Alice and Bob, such that at the end of their protocol they output \( \rho \). Again, when \( \rho \) is classical, one can also define the randomized communication complexity of \( \rho \), denoted \( R\text{Comm}(\rho) \), as the minimum number of bits exchanged between Alice and Bob, such that at the end of their protocol they output \( \rho \). In [5] it is shown that for any classical state \( \rho \),

\[
R\text{Comm}(\rho) = R(\rho) = \lceil \log_2 \text{rank}_+(P) \rceil
\]

where \( \text{rank}_+(P) \) is the nonnegative rank\(^2\) of the \( \dim(\mathcal{H}_A) \times \dim(\mathcal{H}_B) \) matrix \( P \) with \( P(x, y) \stackrel{\text{def}}{=} \langle x| \otimes |y\rangle \rho (|x\rangle \otimes |y\rangle) \). It turns out that for a general quantum state \( \rho \) it holds that \( Q\text{Comm}(\rho) = Q(\rho) \) as well. This fact was attributed to Nayak (personal communication) in [5], and we shall see the reason in a later section.

We have considered above two extreme models. Instead we can also consider the intermediate model where Alice and Bob start with some shared state \( \sigma \) and communicate between them to finally produce the target state \( \rho \). In this case we count the size of \( \sigma \) plus the communication as the resource used towards the complexity. Let us denote \( \hat{Q}(\rho) \) to be the minimum resource used by any protocol which produces \( \rho \). It is clearly seen that \( Q\text{Comm}(\rho) \leq \hat{Q}(\rho) \leq Q(\rho) \), and hence \( Q\text{Comm}(\rho) = \hat{Q}(\rho) = Q(\rho) \) since \( Q\text{Comm}(\rho) = Q(\rho) \).

Our results

In this paper, we conduct more studies on the fundamental question of bipartite state generation. We consider approximate versions of \( Q(\rho) \) defined as follows. Below \( F(\rho, \rho') \) represents the fidelity between \( \rho \) and \( \rho' \).

\(^1\)The size of a quantum state \( \sigma \) is defined to be half of the number of qubits of \( \sigma \).

\(^2\)The nonnegative rank of a nonnegative matrix \( A \) is the smallest number \( r \) such that \( A = \sum_{i=1}^{r} A_i \) where each \( A_i \) is a nonnegative rank-1 matrix.
**Definition 1.** Let \( \epsilon > 0 \). Let \( \rho \) be a quantum state in \( \mathcal{H}_A \otimes \mathcal{H}_B \). Define

\[
Q_\epsilon(\rho) \overset{\text{def}}{=} \min \{ Q(\rho') \mid F(\rho, \rho') \geq 1 - \epsilon; \; \rho' \in \mathcal{H}_A \otimes \mathcal{H}_B \}.
\]

\[
Q_\epsilon^{\text{pure}}(\rho) \overset{\text{def}}{=} \min \{ Q(\langle \phi | \phi \rangle) \mid F(\rho, \langle \phi | \phi \rangle) \geq 1 - \epsilon; \; \langle \phi | \phi \rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \}.
\]

In \cite{1}, Ambainis, Schulman, Ta-Shma, Vazirani and Wigderson showed that for any pure state \( |\psi\rangle = \sum_{x,y} a_{x,y} |x \rangle \otimes |y \rangle \),

\[
\lceil \log_2 \text{rank}_2(A) \rceil \leq Q_\epsilon^{\text{pure}}(\langle \psi | \psi \rangle) \leq \lceil \log_2 \text{rank}_\epsilon(A) \rceil.
\]

Above \( A \) is the \( \dim(\mathcal{H}_A) \times \dim(\mathcal{H}_B) \) matrix with \( A(x,y) = a_{x,y} \) and

\[
\text{rank}_\epsilon(A) \overset{\text{def}}{=} \min \{ \text{rank}(B) \mid \| A - B \|_2^2 \leq \epsilon \}.
\]

Using Lemma 4 (as mentioned in the next section), one can easily construct a state \( |\psi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n \) such that \( \text{rank}_2(A) = 1 \) but \( \text{rank}(A) = n/2 \), making the above two bounds arbitrarily far from each other. In this paper we show the following tight characterization.

**Theorem 1.** Let \( \epsilon > 0 \). Let \( \{ |x\rangle \mid x \in [\dim(\mathcal{H}_A)] \} \) be the computational bases for \( \mathcal{H}_A \) and let \( \{ |y\rangle \mid y \in [\dim(\mathcal{H}_B)] \} \) be the computational bases for \( \mathcal{H}_B \). Let \( |\psi\rangle = \sum_{x,y} a_{x,y} |x \rangle \otimes |y \rangle \). Let \( A \) be defined as \( A(x,y) = a_{x,y} \). Then

\[
Q_\epsilon(|\psi\rangle\langle \psi|) = Q_\epsilon^{\text{pure}}(|\psi\rangle\langle \psi|) = \lceil \log_2 \text{rank}_2(\psi) \rceil.
\]

Our result not only improves the bounds in \cite{1} to optimal, but also shows that allowing a mixed state to approximate a pure state \( |\psi\rangle \) does not help, for any \( |\psi\rangle \) and any approximation ratio \( \epsilon \).

Our second result is for the case of a classical state \( \rho \). Previously \cite{5} gave upper and lower bounds:

\[
\frac{1}{4} \log_2 \text{rank}(P) \leq Q(\rho) \leq \min_{Q: Q \circ P = P} \log_2 \text{rank}(Q).
\]

Above \( P \) is given by \( P(x,y) = (\langle x | \otimes | y \rangle) \rho (| x \rangle \otimes | y \rangle) \) and \( \circ \) is the Hadamard (i.e., entry-wise) product of matrices. How tight these bounds are is not clear yet, and an open question asked in \cite{5} was a characterization of \( Q(\rho) \). In this paper, we answer this question by showing a tight characterization in terms of \textit{psd-rank} of \( P \), a concept recently proposed in \cite{3} by Fiorini, Massar, Pokutta, Tiwary and de Wolf. For a nonnegative matrix \( P \), its psd-rank, denoted \( \text{rank}_{\text{psd}}(P) \), is the minimum \( r \) such that there are \( r \times r \) positive semi-definite matrices \( C_x, D_y \), satisfying that \( P(x,y) = \text{tr}(C_x D_y) \). We show the following result.

**Theorem 2.** Let \( \{ |x\rangle \mid x \in [\dim(\mathcal{H}_A)] \} \) be the computational bases for \( \mathcal{H}_A \) and let \( \{ |y\rangle \mid y \in [\dim(\mathcal{H}_B)] \} \) be the computational bases for \( \mathcal{H}_B \). Let

\[
\rho = \sum_{x \in [\dim(\mathcal{H}_A)]} \sum_{y \in [\dim(\mathcal{H}_B)]} p_{x,y} \cdot |x \rangle \langle x| \otimes |y \rangle \langle y|.
\]

Let \( P \) be a \( [\dim(\mathcal{H}_A)] \times [\dim(\mathcal{H}_B)] \) matrix with \( P(x,y) = p_{x,y} \). Then \( Q(\rho) = \lceil \log_2 \text{rank}_{\text{psd}}(P) \rceil \).
Along with the characterization $R(\rho) = \lceil \log_2 \text{rank}_+(P) \rceil$ (shown in [5]), it is interesting to see that for classical states $\rho$, randomized correlation/communication complexity is all about non-negative rank, and the quantum correlation/communication complexity is all about the psd-rank of the corresponding matrix $P$.

For a general quantum state $\rho$ we show the following characterization of $Q(\rho)$.

**Theorem 3.** Let $\rho$ be a quantum state in $\mathcal{H}_A \otimes \mathcal{H}_B$. Let $\{|x\rangle \mid x \in [\dim(\mathcal{H}_A)]\}$ be the computational bases for $\mathcal{H}_A$ and let $\{|y\rangle \mid y \in [\dim(\mathcal{H}_B)]\}$ be the computational bases for $\mathcal{H}_B$. Then $Q(\rho) = \lceil \log_2 r \rceil$ where $r$ is the minimum number such that there exist matrices $\{A_x \mid x \in [\dim(\mathcal{H}_A)]\}$ and $\{B_y \mid y \in [\dim(\mathcal{H}_B)]\}$, each with $r$ columns, and

$$
\rho = \sum_{x,x' \in [\dim(\mathcal{H}_A)]} |x\rangle \langle x'| \otimes |y\rangle \langle y'| \cdot \text{tr} \left( (A_x^\dagger A_x)^T (B_y^\dagger B_y) \right).
$$

The rest of the paper is organized as follows. In the next section we discuss our notation and some information theoretic preliminaries. In section 3 we prove Theorem 1. In section 4 we prove Theorem 2 and Theorem 3.

# 2 Preliminaries

## Matrix theory

For a natural number $n$ we let $[n]$ represent the set $\{1, 2, \ldots, n\}$. For a matrix $A$, we let $A^T$ represent the transpose of $A$, $A^*$ represent the conjugate of $A$ and $A^\dagger$ represent the conjugate transpose of $A$. An operator $A$ is said to be *Hermitian* if $A^\dagger = A$. A Hermitian operator $A$ is said to be positive semi-definite if all its eigenvalues are non-negative. We will use the following fact.

**Fact 1.** Let $|v_1\rangle, \ldots, |v_r\rangle$ be vectors in $\mathbb{C}^n$ for some $n \geq 1$. Then the $r \times r$ matrix $A$ defined by $A(i,j) \overset{\text{def}}{=} \langle v_i|v_j \rangle$ is positive semi-definite.

If $A$ is positive semi-definite then so is $A^T = A^*$. We let $\sigma_1(A) \geq \cdots \geq \sigma_n(A)$ denote singular values of $A$. The rank of $A$, denoted $\text{rank}(A)$, is defined to be the number of the non-zero singular values of $A$. The *Frobenius norm* of $A$ is defined as $\|A\|_2 = \sqrt{\sum_i \sigma_i(A)^2}$ and its *trace norm* is defined as $\|A\|_1 = \sum_i \sigma_i$. For $\epsilon > 0$, define

$$
\text{rank}_\epsilon(A) = \min\{\text{rank}(B) \mid \|A - B\|_2^2 \leq \epsilon\}.
$$

The following well-known result says that the best way to approximate $A$ (under the Frobenius norm) with the least rank is by taking the large singular values part.

**Lemma 4** (Eckart-Young, [2]). Let $\|A\|_2 = 1$ and $\epsilon > 0$. Then,

$$
\text{rank}_\epsilon(A) = \text{the minimum } k \text{ such that } \sum_{i=1}^k \sigma_i(A)^2 \geq 1 - \epsilon.
$$

The following definition of psd-rank of a matrix was proposed in [3].

**Definition 2** ([3]). For a matrix $P \in \mathbb{R}^{n \times m}$, its psd-rank, denoted $\text{rank}_{\text{psd}}(P)$, is the minimum number $r$ such that there are positive semi-definite matrices $C_x, D_y \in \mathbb{C}^{r \times r}$ with $\text{tr}(C_x D_y) = P(x, y), \forall x \in [n], y \in [m]$. 

3
Quantum computing

A quantum state \( \rho \) in Hilbert space \( \mathcal{H} \), denoted \( \rho \in \mathcal{H} \), is a trace one positive semi-definite operator acting on \( \mathcal{H} \). The size of a state \( \rho \) is defined to be half the number of qubits of \( \rho \). Here we take the factor of half because we shall talk about a correlation as a *shared* resource. It is consistent with the convention that when the two parties shares a classical correlation \((X,Y)\), where \( Y = X = R \) for a \( r \)-bit random string \( R \), we say that they share a random variable \( R \) of size \( r \). A quantum state is called pure if it is rank one. We often also identify a pure state with its unique eigenvector with non-zero eigenvalue. For quantum states \( \rho \) and \( \sigma \), their fidelity is defined as \( F(\rho, \sigma) \overset{\text{def}}{=} \text{tr}(\sqrt{\sigma^{1/2} \rho \sigma^{1/2}}) \). For \( \rho, \ket{\psi} \in \mathcal{H} \), we have \( F(\rho, \ket{\psi}\bra{\psi}) = \sqrt{\bra{\psi}\rho\ket{\psi}} \). We define norm of \( \ket{\psi} \) as \( \| \ket{\psi} \| \overset{\text{def}}{=} \sqrt{\bra{\psi}\psi} \). For a quantum state \( \rho \in \mathcal{H}_A \otimes \mathcal{H}_B \), we let \( \text{tr}_{\mathcal{H}_B} \rho \) represent the partial trace of \( \rho \) in \( \mathcal{H}_A \) after tracing out \( \mathcal{H}_B \). Let \( \rho \in \mathcal{H}_A \) and \( \ket{\phi} \in \mathcal{H}_A \otimes \mathcal{H}_B \) be such that \( \text{tr}_{\mathcal{H}_B} \ket{\phi}\bra{\phi} = \rho \), then we call \( \ket{\phi} \) a purification of \( \rho \). For a pure state \( \ket{\psi} \in \mathcal{H}_A \otimes \mathcal{H}_B \), its Schmidt decomposition is defined as \( \ket{\psi} = \sum_{i=1}^{r} \sqrt{\lambda_i} \ket{v_i} \otimes \ket{w_i} \), where \( \{\ket{v_i} \in \mathcal{H}_A\} \) are orthonormal, \( \{\ket{w_i} \in \mathcal{H}_B\} \) are orthonormal, \( \forall i : \lambda_i \geq 0 \) and \( \sum_{i=1}^{r} \lambda_i = 1 \). It is easily seen that \( r \) is also equal to \( \text{rank}(\text{tr}_{\mathcal{H}_A} \ket{\psi}\bra{\psi}) = \text{rank}(\text{tr}_{\mathcal{H}_B} \ket{\psi}\bra{\psi}) \) and is therefore the same in all Schmidt decompositions of \( \rho \). This number is often referred to as the *Schmidt rank* of \( \ket{\psi} \) and denoted \( S\text{-rank}(\ket{\psi}) \). Sometimes we absorb the coefficients \( \sqrt{\lambda_i} \) in \( \ket{v_i} \otimes \ket{w_i} \), in which case \( \ket{v_i}, \ket{w_i} \) may not be unit vectors. The following is easily verified.

**Fact 2.** Let \( U_A \) be a unitary operator on \( \mathcal{H}_A \) and let \( U_B \) be a unitary operator on \( \mathcal{H}_B \). Let \( \ket{\psi} \in \mathcal{H}_A \otimes \mathcal{H}_B \). Then \( S\text{-rank}(\ket{\psi}) = S\text{-rank}((U_A \otimes U_B)\ket{\psi}) \).

The following fact follows by considering Schmidt decomposition of the pure states involved; see, for example, Ex(2.81) of [4].

**Fact 3.** Let \( \ket{\chi}, \ket{\phi} \in \mathcal{H}_A \otimes \mathcal{H}_B \) be such that \( \text{tr}_{\mathcal{H}_B} \ket{\phi}\bra{\phi} = \text{tr}_{\mathcal{H}_B} \ket{\psi}\bra{\psi} \). There exists a unitary operation \( U \) on \( \mathcal{H}_B \) such that \( (I_{\mathcal{H}_A} \otimes U)\ket{\psi} = \ket{\phi} \), where \( I_{\mathcal{H}_A} \) is the identity operator on \( \mathcal{H}_A \).

The following fundamental fact is shown by Uhlmann [4].

**Fact 4** (Uhlmann, [4]). Let \( \rho, \sigma \in \mathcal{H}_A \). Let \( \ket{\psi} \in \mathcal{H}_A \otimes \mathcal{H}_B \) be a purification of \( \rho \) and \( \dim(\mathcal{H}_A) \leq \dim(\mathcal{H}_B) \). There exists a purification \( \ket{\phi} \in \mathcal{H}_A \otimes \mathcal{H}_B \) of \( \sigma \) such that \( F(\rho, \sigma) = |\bra{\phi}\psi| | \).

We define the approximate *Schmidt rank* as follows.

**Definition 3.** Let \( \epsilon > 0 \). Let \( \ket{\psi} \) be a pure state in \( \mathcal{H}_A \otimes \mathcal{H}_B \). Define

\[
S\text{-rank}_\epsilon(\ket{\psi}) \overset{\text{def}}{=} \min\{S\text{-rank}(\ket{\phi}) \mid \ket{\phi} \in \mathcal{H}_A \otimes \mathcal{H}_B \text{ and } F(\ket{\psi}\bra{\psi}, \ket{\phi}\bra{\phi}) \geq 1 - \epsilon\}.
\]

Let \( \{\ket{x} \mid x \in [\dim(\mathcal{H}_A)]\} \) be the computational bases for \( \mathcal{H}_A \) and let \( \{\ket{y} \mid y \in [\dim(\mathcal{H}_B)]\} \) be the computational bases for \( \mathcal{H}_B \). We define linear map vecin which takes vectors in \( \mathcal{H}_A \otimes \mathcal{H}_B \) and maps them to operators from \( \mathcal{H}_B \) to \( \mathcal{H}_A \). For all \( x \in [\dim(\mathcal{H}_A)], y \in [\dim(\mathcal{H}_B)] \) define \( \text{vecin}(\ket{x} \otimes \ket{y}) \overset{\text{def}}{=} \ket{x}\bra{y} \) and extend to all vectors in \( \mathcal{H}_A \otimes \mathcal{H}_B \) by linearity. For \( \ket{\psi} \in \mathcal{H}_A \otimes \mathcal{H}_B \), it is easily seen that \( \|\ket{\phi}\| = \|\text{vecin}(\ket{\psi})\|_2 \).

In the following sections we assume Hilbert spaces \( \mathcal{H}_A, \mathcal{H}_{A_1}, \mathcal{H}_{A_2}, \mathcal{H}_B, \mathcal{H}_{B_1}, \mathcal{H}_{B_2} \) etc. are possessed by Alice and Hilbert spaces \( \mathcal{H}_B, \mathcal{H}_{B_1}, \mathcal{H}_{B_2} \) etc. are possessed by Bob.

We start by showing the following key lemma which we will use many times in the following sections.
Lemma 5. Let $\rho$ be a quantum state in $\mathcal{H}_A \otimes \mathcal{H}_B$. Then,

$$Q(\rho) = \min_{\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1}} \left\{ \left\lceil \log_2 \left( S\text{-rank}(\rho) \right) \right\rceil \mid \rho = \text{tr}_{\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1}} |\psi\rangle\langle \psi| \right\}.$$

Proof. Let $r \overset{\text{def}}{=} \min_{\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1}} \left\{ \left\lceil \log_2 \left( S\text{-rank}(\rho) \right) \right\rceil \mid \rho = \text{tr}_{\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1}} |\psi\rangle\langle \psi| \right\}$. We first show $Q(\rho) \leq r$. Let $|\psi\rangle$ be such that

$$r = \left\lceil \log_2 \left( S\text{-rank}(\rho) \right) \right\rceil \text{ and } \rho = \text{tr}_{\mathcal{H}_A \otimes \mathcal{H}_B} |\psi\rangle\langle \psi|.$$

Let $t \overset{\text{def}}{=} S\text{-rank}(|\psi\rangle)$. Let $|\psi\rangle$ have a Schmidt decomposition

$$|\psi\rangle = \sum_{i=1}^{t} \sqrt{p_i} \cdot |v_i\rangle \otimes |w_i\rangle,$$

Let Alice and Bob start with the state

$$|\phi\rangle = \sum_{i=1}^{t} \sqrt{p_i} \cdot |i\rangle \otimes |i\rangle,$$

and transform $|\phi\rangle$ to $|\psi\rangle$ using local unitary transformations. This shows that $Q(\rho) \leq \left\lceil \log_2 t \right\rceil = r$.

For the other direction let $s \overset{\text{def}}{=} Q(\rho)$. Let Alice and Bob start with the seed state $\sigma$ and apply local completely positive trace preserving maps $\Phi_A, \Phi_B$ respectively to produce $\rho$. Let us assume without loss of generality that the number of qubits of $\sigma \overset{\text{def}}{=} \text{tr}_{\mathcal{H}_B} \sigma$ is at most $s$. Let $\sigma_A = \sum_{i=1}^{2^s} a_i |v_i\rangle \langle v_i|$, where $a_i \geq 0$ is the $i$-th eigenvalue of $\sigma_A$ with eigenvector $|v_i\rangle$. Define

$$|\phi\rangle \overset{\text{def}}{=} \sum_{i=1}^{2^s} \sqrt{a_i} \cdot |v_i\rangle \otimes |v_i\rangle$$

and let

$$|\phi'\rangle = \sum_{i=1}^{2^s} \sqrt{a_i} \cdot |v_i\rangle \otimes |w_i\rangle$$

be a purification of $\sigma$, where $\forall i: |v_i\rangle \in \mathcal{H}_A$ and $|w_i\rangle \in \mathcal{H}_B \otimes \mathcal{H}_{B_2}$.

Now consider the following operations by Alice and Bob. They start with the shared state $|\phi\rangle$. Bob using local unitary (after attaching ancilla $|0\rangle$ if needed) transforms $|\phi\rangle$ to $|\phi'\rangle$ (Bob can do this follows from Fact 3). Alice and Bob then simulate their maps $\Phi_A, \Phi_B$ on $\sigma$ by local unitaries (each after attaching ancilla $|0\rangle$ if needed on their parts; such a simulation is a standard fact, please refer to [4]) and finally produce a purification $|\theta\rangle \in \mathcal{H}_A \otimes \mathcal{H}_{A_1} \otimes \mathcal{H}_B \otimes \mathcal{H}_{B_1}$ of $\rho$. Since Alice and Bob, using local unitary operations and attaching ancilla $|0\rangle$, transform $|\phi\rangle$ to $|\theta\rangle$, we have (using Fact 2) $2^s \geq S\text{-rank}(|\phi\rangle) = S\text{-rank}(|\theta\rangle))$. This shows that $r \leq s$. \hfill \Box

The following lemma is credited to Nayak (personal communication) in [5].

Lemma 6. For a quantum state $\rho \in \mathcal{H}_A \times \mathcal{H}_B$, $Q(\rho) = Q\text{Comm}(\rho)$.  

5
Hence from Lemma 4, $S$-rank $F$. Then unitaries, exchange $\sigma$. Multiplying appropriate phase to $\rho$ at the end output $|\langle \theta \rangle\rangle$. This protocol can be converted into another protocol where Alice and Bob start with a purification $|\phi\rangle \in H_A \otimes H_B$ of $\sigma_A \otimes \sigma_B$ (with $S$-rank($|\phi\rangle\rangle$) = 1), do local unitaries, exchange $r$ qubits and at the end output a purification $|\psi\rangle \in H_A \otimes H_A \otimes H_B \otimes H_B$, of $\rho$. Since local unitaries do not increase the Schmidt rank of the shared state and exchanging $r$ qubits increases the Schmidt rank by a factor at most $2^r$ (since the rank of the marginal state possessed by Alice increases by at most a factor 2 on receiving a qubit from Bob, and similarly for Bob on receiving a qubit from Alice), we have $S$-rank($|\psi\rangle\rangle$) $\leq 2^r$. Hence from Lemma 5, $Q(\rho) \leq r$.

3 Correlation complexity of approximating a pure state

In this section we prove Theorem 1. We start by first characterizing the approximate Schmidt rank.

**Lemma 7.** Let $\epsilon > 0$. Let $|\psi\rangle$ be a pure state in $H_A \otimes H_B$ with a Schmidt decomposition $|\psi\rangle = \sum_{i=1}^{r'} \sqrt{p_i} \cdot |v_i\rangle \otimes |w_i\rangle$ (with $p_1 \geq p_2 \geq \ldots \geq p_{r'} > 0$ and $\sum_{i=1}^{r'} p_i = 1$). Let $r'$ be the minimum number such that $\sum_{i=1}^{r'} p_i \geq (1 - \epsilon)^2$. Then $r' = S$-rank$_s(|\psi\rangle\rangle)$.

**Proof.** We will first show that $r' \geq S$-rank$_s(|\psi\rangle\rangle)$. Let $q = \sum_{i=1}^{r'} p_i$. Define $|\phi\rangle \defeq \frac{1}{\sqrt{q}} \sum_{i=1}^{r'} \sqrt{p_i} \cdot |v_i\rangle \otimes |w_i\rangle$. Then $F(|\psi\rangle\langle \psi|, |\phi\rangle\langle \phi|) = |\langle \psi |\phi\rangle| = \sqrt{q} \geq 1 - \epsilon$. Clearly $S$-rank$_s(|\phi\rangle\rangle) = r'$ and hence $r' \geq S$-rank$_s(|\psi\rangle\rangle)$.

Now we will show $r' \leq S$-rank$_s(|\psi\rangle\rangle)$. Let $s = S$-rank$_s(|\psi\rangle\rangle)$. Let $|\theta\rangle \in H_A \otimes H_B$ be a pure state such that $|\langle \theta |\psi\rangle| = F(|\psi\rangle\langle \psi|, |\theta\rangle\langle \theta|) \geq 1 - \epsilon$ and $S$-rank$_s(|\theta\rangle\rangle) = s$. Without loss of generality (by multiplying appropriate phase to $|\theta\rangle$) let us assume that $\beta \defeq \langle \phi |\theta\rangle$ is real. Let $|\theta\rangle = \sum_{j=1}^{s} \sqrt{q_j} \cdot |v'_j\rangle \otimes |w'_j\rangle$ be a Schmidt decomposition of $|\theta\rangle$. Define $A \defeq \sum_{i=1}^{r} \sqrt{p_i} \cdot |v_i\rangle\langle w_i|$ and $B \defeq \beta \cdot \sum_{i=1}^{s} \sqrt{q_i} \cdot |v'_i\rangle\langle w'_i|$. Note that $A = \text{vecinv}(|\psi\rangle\rangle)$ and $B = \text{vecinv}(|\theta\rangle\rangle)$. Since $|v_i\rangle$ and $|w_i\rangle$ are orthonormal, $\{\sqrt{p_i}\}$ form the singular values of $A$. Similarly $\{\beta \cdot \sqrt{q_i}\}$ form the singular values of $B$. Now,

$$1 - (1 - \epsilon)^2 \geq 1 - \beta^2 = |||\phi\rangle||^2 + |||\theta\rangle||^2 - 2\langle \theta' |\phi\rangle = |||\phi\rangle - |\theta\rangle||^2$$

$$= ||\text{vecinv}(|\phi\rangle) - |\theta\rangle||^2 = ||\text{vecinv}(|\phi\rangle) - \text{vecinv}(|\theta\rangle)||^2$$

$$= ||A - B||_2^2.$$ 

Hence from Lemma 5, $S$-rank$_s(|\theta\rangle\rangle) = \text{rank}(B) \geq r'$.

\[6\]
We can now get the desired characterization for $Q^{\text{pure}}_\epsilon(|\psi\rangle\langle\psi|)$.

**Theorem 8.** Let $\epsilon > 0$. Let $|\psi\rangle$ be a pure state in $\mathcal{H}_A \otimes \mathcal{H}_B$ with a Schmidt decomposition $|\psi\rangle = \sum_{i=1}^r \sqrt{p_i} \cdot |v_i\rangle \otimes |w_i\rangle$ (with $p_1 \geq p_2 \geq \ldots \geq p_r > 0$ and $\sum_{i=1}^r p_i = 1$). Let $A = \sum_{i=1}^r \sqrt{p_i} \cdot |v_i\rangle \langle w_i| = \text{vecinv}(|\psi\rangle)$. Then, $Q^{\text{pure}}_\epsilon(|\psi\rangle\langle\psi|) = \lfloor \log_2 \text{rank}_{2\epsilon^2}(A) \rfloor$.

**Proof.** From the definitions and Lemma 5 it is clear that $Q^{\text{pure}}_\epsilon(|\psi\rangle\langle\psi|) = \lfloor \log_2 S\text{-rank}_\epsilon(|\psi\rangle) \rfloor$. Also from Lemma 4 and Lemma 7 it follows that $S\text{-rank}_\epsilon(|\psi\rangle) = \text{rank}_{2\epsilon^2}(A)$ (by noting that $\{|\sqrt{p_i}\rangle\}$ form singular values of $A$). \qed

The following lemma shows a monotonicity property for the approximate Schmidt rank.

**Lemma 9.** Let $\epsilon > 0$. Let $|\psi\rangle$ be a pure state in $\mathcal{H}_A \otimes \mathcal{H}_B$. Let $|\theta\rangle$ be a pure state in $\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1}$. Then,

$$S\text{-rank}_\epsilon(|\psi\rangle \otimes |\theta\rangle) \geq S\text{-rank}_\epsilon(|\psi\rangle).$$

Hence from Lemma 2

$$Q^{\text{pure}}_\epsilon(|\psi\rangle\langle\psi| \otimes |\theta\rangle\langle\theta|) \geq Q^{\text{pure}}_\epsilon(|\psi\rangle\langle\psi|).$$

**Proof.** Let $|\psi\rangle = \sum_{i=1}^r \sqrt{p_i} \cdot |u_i\rangle \otimes |v_i\rangle$ (with $p_1 \geq \ldots \geq p_r > 0$) and $|\theta\rangle = \sum_{i=1}^s \sqrt{q_i} \cdot |u_i^2\rangle \otimes |v_i^2\rangle$ be some Schmidt decompositions of $|\psi\rangle$ and $|\theta\rangle$ respectively. Then

$$|\psi\rangle \otimes |\theta\rangle = \sum_{i,j} \sqrt{p_i q_j} \cdot |u_i^1\rangle \otimes |u_i^2\rangle \otimes |v_i^1\rangle \otimes |v_i^2\rangle.$$

Fix a minimal set $S \subseteq [r] \times [s]$ with $\sum_{(i,j) \in S} p_i q_j \geq 1 - \epsilon$. Let $r' \overset{\text{def}}{=} S\text{-rank}_\epsilon(|\psi\rangle)$. We will show $|S| \geq r'$. Assume for contradiction $|S| \leq r' - 1$. Let $S_1 = \{i \mid \exists j \text{ such that } (i, j) \in S \}$, then $|S_1| \leq |S| \leq r' - 1$. We have

$$\sum_{(i,j) \in S} p_i q_j \leq \sum_{i \in S_1} p_i \leq p_1 + \ldots + p_{|S_1|} < 1 - \epsilon,$$

where the first inequality is because $\sum_{j \in S} q_j \leq 1$ for all $i$, the second inequality is because $p_i$’s are in the non-increasing order, and the last one is by the definition of $S\text{-rank}_\epsilon(|\psi\rangle) = r'$, the smallest number such that $p_1 + \ldots + p_{r'} \geq 1 - \epsilon$ (from Lemma 7). This contradicts the way we picked $S$ and hence

$$S\text{-rank}_\epsilon(|\psi\rangle \otimes |\theta\rangle) = |S| \geq r' = S\text{-rank}_\epsilon(|\psi\rangle). \quad \square$$

**Theorem 10.** Let $\epsilon > 0$. Let $|\psi\rangle$ be a pure state in $\mathcal{H}_A \otimes \mathcal{H}_B$. Then, $Q_\epsilon(|\psi\rangle\langle\psi|) = Q^{\text{pure}}_\epsilon(|\psi\rangle\langle\psi|)$.

**Proof.** By definition, we have $Q_\epsilon(|\psi\rangle\langle\psi|) \leq Q^{\text{pure}}_\epsilon(|\psi\rangle\langle\psi|)$. Now consider the other direction. By the definition of $Q_\epsilon(|\psi\rangle\langle\psi|)$, there exists a $\rho \in \mathcal{H}_A \otimes \mathcal{H}_B$ such that

$$Q_\epsilon(|\psi\rangle\langle\psi|) = Q(\rho) \text{ and } F(\rho, |\psi\rangle\langle\psi|) \geq 1 - \epsilon. \quad (1)$$

By Lemma 5 there exists a purification $|\phi\rangle$ in $\mathcal{H}_A \otimes \mathcal{H}_{A_1} \otimes \mathcal{H}_B \otimes \mathcal{H}_{B_1}$ of $\rho$ with

$$Q(\rho) = \lfloor \log_2 S\text{-rank}_\epsilon(|\phi\rangle\langle\phi|) \rfloor = Q(|\phi\rangle\langle\phi|). \quad (2)$$

Without loss of generality, we can assume that $\dim(\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1}) \geq \dim(\mathcal{H}_A \otimes \mathcal{H}_B)$ (otherwise we can attach $|0\rangle$ to $|\phi\rangle$ appropriately). Now by Uhlmann’s Theorem, there exists a purification
We first show the ‘only if’ implication. Let $\rho$ be a quantum state in $H_A \otimes H_B$. Let $\{\hat{x}\} \in [\dim(H_A)]$ be the computational bases for $H_A$ and let $\{\hat{y}\} \in [\dim(H_B)]$ be the computational bases for $H_B$. There exists a purification $|\psi\rangle$ of $\rho$, with $S$-rank($|\psi\rangle$) = $r$, if and only if there exist matrices $\{A_x \mid x \in [\dim(H_A)]\}$ and $\{B_y \mid y \in [\dim(H_B)]\}$, each with $r$ columns, such that

$$
\rho = \sum_{x,y,x',y' \in [\dim(H_A)], \dim(H_B)} |x\rangle \langle x' \otimes |y\rangle \langle y'| \cdot \text{tr} \left((A_x^\dagger A_x)^T (B_y^\dagger B_y)\right).
$$

Proof. We first show the ‘only if’ implication. Let $|\psi\rangle$ be a purification of $\rho$ in $H_A \otimes H_A \otimes H_B \otimes H_B$. Let $S$-rank($|\psi\rangle$) = $r$. Consider a Schmidt decomposition of $|\psi\rangle$.

$$
|\psi\rangle = \sum_{i=1}^r |v_i\rangle \otimes |w_i\rangle = \sum_{i=1}^r \left(\sum_{x \in [\dim(H_A)]} |x\rangle \otimes |v_{ix}\rangle \right) \otimes \left(\sum_{y \in [\dim(H_B)]} |y\rangle \otimes |w_{iy}\rangle \right).
$$

Above for any $i, x, y$, the vectors $|v_i\rangle, |w_i\rangle, |v_{ix}\rangle, |w_{iy}\rangle$ are not necessarily unit vectors. Consider

$$
\rho = \text{tr}_{H_A \otimes H_B} \langle \psi | \psi \rangle
$$

$$
= \text{tr}_{H_A \otimes H_B} \left(\sum_{i=1}^r \left(\sum_{x \in [\dim(H_A)]} |x\rangle \otimes |v_{ix}\rangle \right) \otimes \left(\sum_{y \in [\dim(H_B)]} |y\rangle \otimes |w_{iy}\rangle \right)\right)
$$

$$
= \text{tr}_{H_A \otimes H_B} \sum_{i,j=1}^r \left(\sum_{x \in [\dim(H_A)]} |x\rangle \langle x'| \otimes |v_{ix}\rangle \langle v_{ix'}| \right) \otimes \left(\sum_{y \in [\dim(H_B)]} |y\rangle \langle y'| \otimes |w_{iy}\rangle \langle w_{iy'}| \right)
$$

$$
= \sum_{i,j=1}^r \left(\sum_{x \in [\dim(H_A)]} \sum_{x' \in [\dim(H_A)]} |x\rangle \langle x'| \cdot \text{tr}(|v_{ix}\rangle \langle v_{ix'}|)\right) \otimes \left(\sum_{y \in [\dim(H_B)]} \sum_{y' \in [\dim(H_B)]} |y\rangle \langle y'| \cdot \text{tr}(|w_{iy}\rangle \langle w_{iy'}|)\right)
$$

$$
= \sum_{x,x',y,y' \in [\dim(H_A)], \dim(H_B)} |x\rangle \langle x'| \otimes |y\rangle \langle y'| \left(\sum_{i,j=1}^r |v_{ix}\rangle |v_{ix'}\rangle \cdot |w_{iy}\rangle |w_{iy'}\rangle\right).
$$

Theorem 11 now follows immediately by combining Theorem 8 and Theorem 10 and noting that the matrix $A$ as defined in the statement of Theorem 11 is $\text{vecinv}(|\psi\rangle)$.

4 Correlation complexity of a quantum state

In this section we show characterizations of correlation complexities for general quantum states and also for classical states and prove Theorem 2 and Theorem 3. We start with the following lemma.

Lemma 11. Let $\rho$ be a quantum state in $H_A \otimes H_B$. Let $\{\hat{x}\} \in [\dim(H_A)]$ be the computational bases for $H_A$ and let $\{\hat{y}\} \in [\dim(H_B)]$ be the computational bases for $H_B$. There exists a purification $|\psi\rangle$ of $\rho$, with $S$-rank($|\psi\rangle$) = $r$, if and only if there exist matrices $\{A_x \mid x \in [\dim(H_A)]\}$ and $\{B_y \mid y \in [\dim(H_B)]\}$, each with $r$ columns, such that

$$
Q^\text{pure}(\rho) = Q^\text{pure}(|\psi\rangle \langle \psi|)
$$

(from Lemma 9)

$$
= Q(|\psi\rangle \langle \psi|)
$$

(from the definition of $Q^\text{pure}(|\psi\rangle \langle \psi|)$)

$$
= Q(|\psi\rangle \langle \psi|)
$$

(from Eqs. 2)

$$
= Q^\text{pure}(\rho)
$$

(from Eq. 11)
For each $x \in [\dim(H_A)]$, let us define matrices $A_x \overset{\text{def}}{=} (|v_x^1\rangle, |v_x^2\rangle, \ldots, |v_x^r\rangle)$. Similarly for each $y \in [\dim(H_B)]$, let us define matrices $B_y \overset{\text{def}}{=} (|w_y^1\rangle, |w_y^2\rangle, \ldots, |w_y^r\rangle)$. Then from above,

$$
\rho = \sum_{x,x' \in [\dim(H_A)]} |x\rangle \langle x'| \otimes |y\rangle \langle y'| \cdot \text{tr} \left( (A_x^\dagger A_x)^T (B_y^\dagger B_y) \right).
$$

Next we show the ‘if’ implication. Let there exist matrices $\{A_x \mid x \in [\dim(H_A)]\}$ and $\{B_y \mid y \in [\dim(H_B)]\}$, each with $r$ columns, such that

$$
\rho = \sum_{x,x' \in [\dim(H_A)]} |x\rangle \langle x'| \otimes |y\rangle \langle y'| \cdot \text{tr} \left( (A_x^\dagger A_x)^T (B_y^\dagger B_y) \right).
$$

For $i \in [r]$, let $|v_x^i\rangle$ be the $i$-th column of $A_x$ and let $|w_y^i\rangle$ be the $i$-th column of $B_y$. Define

$$
|\psi\rangle \overset{\text{def}}{=} \sum_{i=1}^r \left( \sum_x |x\rangle \otimes |v_x^i\rangle \right) \otimes \left( \sum_y |y\rangle \otimes |w_y^i\rangle \right).
$$

It is clear that $S\text{-rank}(|\psi\rangle) = r$. We can check, by analogous calculations as above, that

$$
\rho = \text{tr}_{H_A \otimes H_B} |\psi\rangle \langle \psi|.
$$

By combining Lemma 5 and Lemma 11 we immediately get Theorem 3. We now show Theorem 2 which we restate below for convenience.

**Theorem 12.** Let $\{|x\rangle \mid x \in [\dim(H_A)]\}$ be the computational bases for $H_A$ and let $\{|y\rangle \mid y \in [\dim(H_B)]\}$ be the computational bases for $H_B$. Let $P$ be a $[\dim(H_A)] \times [\dim(H_B)]$ matrix with $P(x,y) = p_{x,y}$. Then $Q(\rho) = \lceil \log_2 \text{rank}_{\text{psd}}(P) \rceil$.

**Proof.** We will first show $Q(\rho) \leq [\log_2 \text{rank}_{\text{psd}}(P)]$. Let $r = \text{rank}_{\text{psd}}(P)$. We will exhibit a purification $|\psi\rangle$ of $\rho$ with $S\text{-rank}(|\psi\rangle) = r$. This combined with Lemma 5 will show $Q(\rho) \leq \lceil \log_2 r \rceil$.

Let $C_x, D_y \in \mathbb{C}^{r \times r}$ be positive semi-definite matrices with $\text{tr}(C_x D_y) = P(x,y)$, $\forall x \in [\dim(H_A)], y \in [\dim(H_B)]$. For $i \in [r]$, let $|v_x^i\rangle$ be the $i$-th column of $\sqrt{C_x}$ and let $|w_y^i\rangle$ be the $i$-th column of $\sqrt{D_y}$. Define $|\psi\rangle$ in $H_A \otimes H_A \otimes H_A \otimes H_B \otimes H_B \otimes H_B$ as follows.

$$
|\psi\rangle \overset{\text{def}}{=} \sum_{i=1}^r \left( \sum_x |x\rangle \otimes |v_x^i\rangle \right) \otimes \left( \sum_y |y\rangle \otimes |w_y^i\rangle \right).
$$

It is clear that $S\text{-rank}(|\psi\rangle) = r$. Also,

$$
\text{tr}_{H_A \otimes H_A \otimes H_B \otimes H_B} |\psi\rangle \langle \psi| = \sum_{x \in [\dim H_A]} |x\rangle \langle x| \otimes |y\rangle \langle y| \cdot \text{tr}(C_x D_y) = \rho.
$$
Note that Alice and Bob after sharing $|\psi\rangle$ can either just output their first registers or measure their first registers in their respective computational bases to obtain $\rho$.

Now we will show $Q(\rho) \geq \lceil \log_2 \text{rank}_{\text{psd}}(P) \rceil$. Let $|\psi\rangle \in H_A \otimes H_A \otimes H_B \otimes H_B$, be a purification of $\rho$ with $\mathcal{S}$-$\text{rank}(|\psi\rangle) = r$ and $Q(\rho) = \lceil \log_2 r \rceil$, as guaranteed by Lemma 5. We will show $r \geq \text{rank}_{\text{psd}}(P)$ and this will show the desired. Let

$$|\psi\rangle = \sum_{i=1}^{r} \left( \sum_{x \in \text{dim}(H_A)} |x\rangle \otimes |v_{ix}^x\rangle \right) \otimes \left( \sum_{y \in \text{dim}(H_B)} |y\rangle \otimes |w_{iy}^y\rangle \right).$$

For all $x \in \text{dim}(H_A)$, define $r \times r$ matrices $C_x$ such that $C_x(j, i) = \langle v_{ix}^x | v_{ij}^x \rangle$ for all $i, j \in [r]$. Similarly for all $y \in \text{dim}(H_B)$, define $r \times r$ matrices $D_y$ such that $D_y(i, j) = \langle w_{iy}^y | w_{ij}^y \rangle$ for all $i, j \in [r]$. From Fact 1, $C_x, D_y$ are positive semi-definite for all $x \in \text{dim}(H_A)$ and for all $y \in \text{dim}(H_B)$. Consider

$$\rho = \text{tr}_{H_A \otimes H_B} |\psi\rangle \langle \psi|$$

$$= \sum_{x \in \text{dim}(H_A)} \sum_{y \in \text{dim}(H_B)} |x\rangle \langle x| \otimes |y\rangle \langle y| \left( \sum_{i, j=1}^{r} \langle v_{ix}^x | v_{ij}^x \rangle \cdot \langle w_{iy}^y | w_{ij}^y \rangle \right)$$

$$= \sum_{x \in \text{dim}(H_A)} \sum_{y \in \text{dim}(H_B)} |x\rangle \langle x| \otimes |y\rangle \langle y| \cdot \text{tr}(C_x D_y).$$

Therefore for all $x \in \text{dim}(H_A)$ and for all $y \in \text{dim}(H_B)$ we have $p_{x,y} = P(x, y) = \text{tr}(C_x D_y)$. Hence $\text{rank}_{\text{psd}}(P) \leq r$. 

**Acknowledgments**

R.J. is supported by the internal grants of Centre for Quantum Technologies. Y.S. and S.Z. were partially supported by China Basic Research Grant 2011CBA00300 (sub-project 2011CBA00301) and 2007CB807900 (sub-project 2007CB807901). Y.S was also supported in part by US NSF (1017335). Z.W. would like to acknowledge the WBS grant under contract no. R-710-000-007-271. S.Z. was also supported by Research Grants Council of the Hong Kong S.A.R. (Project no. CUHK419309, CUHK418710, CUHK419011).

**References**

[1] Andris Ambainis, Leonard Schulman, Amnon Ta-Shma, Umesh Vazirani, and Avi Wigderson. The quantum communication complexity of sampling. *SIAM Journal on Computing*, 32(6):1570–1585, 2003.

[2] Carl Eckart and Gale Young. The approximation of one matrix by another of lower rank. *Psychometrika*, 1(3):211–218, 1936.

[3] Samuel Fiorini, Serge Massar, Sebastian Pokutta, Hans Raj Tiwary, and Ronald de Wolf. Linear vs. semidefinite extended formulations: Exponential separation and strong lower bounds. In *Proceedings of the 44th ACM Symposium on Theory of Computing*, 2012.
[4] Michael Nielsen and Isaac Chuang. Quantum Computation and Quantum Information. Cambridge University Press, Cambridge, UK, 2000.

[5] Shengyu Zhang. Quantum strategic game theory. In Proceedings of the 3rd Innovations in Theoretical Computer Science, pages 39–59, 2012. Earlier at arXiv:1012.5141 and QIP’11.