UNIQUENESS THEOREMS FOR FOURIER QUASICRYSTALS AND TEMPERED DISTRIBUTIONS WITH DISCRETE SUPPORT

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Abstract. It is proved that if some points of the supports of two Fourier quasicrystals approach each other while tending to infinity, then these quasicrystals coincide. A similar statement is obtained for a certain class of discrete tempered distributions.

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1. INTRODUCTION

P.Kurasov and R.Shur [6] noted that if zeros of two holomorphic almost periodic functions in a strip get closer at infinity, then the zeros sets of these functions coincide. This result can be interpreted as the coincidence of two almost periodic discrete sets if they get closer at infinity. It is natural to expect the same effect for other almost periodic objects, in particular, for Fourier quasicrystals or, in general, for distributions with discrete support.

Denote by \( S(\mathbb{R}^d) \) the Schwartz space of test functions \( \varphi \in C^\infty(\mathbb{R}^d) \) with finite norms

\[
N_m(\varphi) = \sup_{x \in \mathbb{R}^d} (\max\{1, |x|\})^m \max_{\|k\| \leq m} |(D^k \varphi)(x)|, \quad m = 0, 1, 2, \ldots,
\]

\( k = (k_1, \ldots, k_d) \in (\mathbb{N} \cup \{0\})^d, \quad \|k\| = \|k\|_\infty = \max\{k_1, \ldots, k_d\}, \quad D^k = \partial_{x_1}^{k_1} \cdots \partial_{x_d}^{k_d} \). These norms generate a topology on \( S(\mathbb{R}^d) \), and elements of the space \( S^*(\mathbb{R}^d) \) of continuous linear functionals on \( S(\mathbb{R}^d) \) are called tempered distributions.

The Fourier transform of a tempered distribution \( f \) is defined by the equality

\[
\hat{f}(\varphi) = f(\hat{\varphi}) \quad \text{for all} \quad \varphi \in S(\mathbb{R}^d),
\]

where

\[
\hat{\varphi}(y) = \int_{\mathbb{R}^d} \varphi(x) \exp\{-2\pi i \langle x, y \rangle\} dx
\]

is the Fourier transform of the function \( \varphi \).

We will say that a distribution (or a measure) \( f \) on \( \mathbb{R}^d \) is discrete, if for each \( \lambda \in \text{supp} \, f \) there is \( \varepsilon = \varepsilon(\lambda) > 0 \) such that \( B(\lambda, \varepsilon) \cap \text{supp} \, f = \{\lambda\} \), and uniformly discrete, if there is \( \varepsilon > 0 \) such that \( B(\lambda, \varepsilon) \cap B(\lambda', \varepsilon) = \emptyset \) for all \( \lambda, \lambda' \in \text{supp} \, f, \, \lambda \neq \lambda' \); a measure \( \mu \) is atomic if \( \mu = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda \) with \( a_\lambda \in \mathbb{C} \) and countable \( \Lambda \), in this case we will write \( a_\lambda = \mu(\lambda) \).

Here \( B(x, r) = \{y \in \mathbb{R}^d : |y - x| < r\} \), and \( \delta_\lambda \) means the unit mass at the point \( \lambda \in \mathbb{R}^d \).

A complex measure \( \mu \in S^*(\mathbb{R}^d) \) is a Fourier quasicrystal if \( \mu \) and \( \hat{\mu} \) are discrete measures, and measures \( |\mu| \) and \( |\hat{\mu}| \) belong to \( S^*(\mathbb{R}^d) \).

Note that condition \( \mu \in S^*(\mathbb{R}^d) \) do not imply \( |\mu| \in S^*(\mathbb{R}^d) \) (see [3], [12]).

Such measures are the main object in the theory of the Fourier quasicrystals (see [7], [10]-[14], [4]). The corresponding notion was inspired by experimental discovery of non-periodic atomic structures with diffraction patterns consisting of spots, which was made in the mid '80s.
We will say that a complex measure $\mu$ is a sparse Fourier quasicrystal, when $\mu$ is discrete, $\mu \in S^*(\mathbb{R}^d)$, $\hat{\mu}$ is atomic, $|\hat{\mu}| \in S^*(\mathbb{R}^d)$, and numbers of elements $\#\{\text{supp} \mu \cap B(x,1)\}$ are uniformly bounded in $x \in \mathbb{R}^d$.

Note that, compared with the classical definition of Fourier quasicrystal, we have weakened the conditions on the measure $\hat{\mu}$ and removed the requirement $|\mu| \in S^*(\mathbb{R}^d)$.

Clearly, a Fourier quasicrystal with a uniformly discrete support is a sparse Fourier quasicrystal.

**Theorem 1.** If two sparse Fourier quasicrystals $\mu = \sum_{\lambda \in \Lambda} \mu(\lambda) \delta_\lambda$, $\nu = \sum_{\gamma \in \Gamma} \nu(\gamma) \delta_\gamma$ under appropriate numbering $\Lambda = \{\lambda_n\}_{n=1}^\infty$, $\Gamma = \{\gamma_n\}_{n=1}^\infty$ have the properties
\[
\lambda_n - \gamma_n \to 0 \quad \text{and} \quad \mu(\lambda_n) - \nu(\gamma_n) \to 0 \quad \text{as} \quad n \to \infty,
\]
then the measures $\mu, \nu$ coincide.

The conditions of the theorem can be significantly weakened. So, the sparseness of measures and conditions (2) can only be checked on the set $E = \bigcup_k B(x_k, r_k)$, where $\{B(x_k, r_k)\}_{k=1}^\infty$ is an arbitrary sequence of mutually disjoint balls with radii $r_k \to \infty$, and mass of measures can be calculated by groups of points.

**Theorem 2.** Let $\mu, \nu$ be discrete measures from $S^*(\mathbb{R}^d)$ with supports $\Lambda, \Gamma$ respectively. Suppose that there exist disjoint sets $\Lambda_n$, and disjoint sets $\Gamma_n$ such that
\[
\Lambda \cap E = \cup_n \Lambda_n, \quad \Gamma \cap E = \cup_n \Gamma_n, \tag{3}
\]
(we do not require $\Lambda_n \neq \emptyset$ or $\Gamma_n \neq \emptyset$, but always $\Lambda_n \cup \Gamma_n \neq \emptyset$). Also, for some $N < \infty$ and all $x \in E$,
\[
\#\{n : \Lambda_n \cap B(x, 1) \neq \emptyset\} \leq N, \quad \#\{n : \Gamma_n \cap B(x, 1) \neq \emptyset\} \leq N. \tag{4}
\]
and
\[
diam\{\Lambda_n \cup \Gamma_n\} \to 0 \quad \text{as} \quad n \to \infty. \tag{5}
\]
If
\[
\hat{\mu}, \hat{\nu} \quad \text{are atomic measures, and} \quad |\hat{\mu}|, |\hat{\nu}| \in S^*(\mathbb{R}^d), \tag{6}
\]
\[
\mu(\Lambda_n) - \nu(\Gamma_n) \to 0 \quad \text{as} \quad n \to \infty, \tag{7}
\]
then the measures $\mu, \nu$ coincide.

**Remark 1.** In the case of one-point sets $\Lambda_n = \{\lambda_n\}$, $\Gamma_n = \{\gamma_n\}$ for all $n$ conditions (5) and (7) take, respectively, the form
\[
\lambda_n - \gamma_n \to 0 \quad \text{as} \quad n \to \infty, \quad \mu(\lambda_n) - \nu(\gamma_n) \to 0 \quad \text{as} \quad n \to \infty.
\]
Condition (4) means that quantities $\#\{\text{supp} \mu \cap E \cap B(x, 1)\}$ and $\#\{\text{supp} \nu \cap E \cap B(x, 1)\}$ are uniformly bounded.

**Remark 2.** Conditions $\mu, \nu \in S^*(\mathbb{R}^d)$ and (6) can be replaced by the following: the functions
\[
\int \varphi(x-t)\mu(dx) \quad \text{and} \quad \int \varphi(x-t)\nu(dx)
\]
are almost periodic for every $\varphi \in C^\infty$ with compact support. In particular, instead of discrete measures, we can consider discrete almost periodic multisets ([2], [3]) that are discrete almost periodic measures with integer positive masses.
Under some additional conditions, the uniqueness theorem also holds for distributions with discrete supports. Note that by [5], Proposition 3.1, for every tempered distribution \( F \) with discrete support there is \( m < \infty \) such that

\[
F = \sum_{\lambda \in \Lambda} \sum_{j : \|j\| \leq m} p_{\lambda,j} D^j \delta_{\lambda}, \quad j \in (\mathbb{N} \cup \{0\})^d, \quad p_{\lambda,j} \in \mathbb{C}.
\]

**Theorem 3.** Let

\[
f = \sum_{\lambda \in \Lambda} \sum_{j : \|j\| \leq m} p_{\lambda,j} D^j \delta_{\lambda}, \quad g = \sum_{\gamma \in \Gamma} \sum_{j : \|j\| \leq m} q_{\gamma,j} D^j \delta_{\gamma}
\]

be tempered distributions with discrete supports \( \Lambda, \Gamma \). Suppose that there exist disjoint sets \( \Lambda_n, \Gamma_n \) with properties (3), (4), and (5). If

\[
\hat{f}, \hat{g} \quad \text{are atomic measures, and} \quad |\hat{f}|, |\hat{g}| \in S^*(\mathbb{R}^d),
\]

\[
\sum_{\lambda \in \Lambda_n} p_{\lambda,j} - \sum_{\gamma \in \Gamma_n} q_{\gamma,j} \rightarrow 0 \quad \forall j, \quad \text{as} \quad n \rightarrow \infty,
\]

\[
\max_j \sup_n \sum_{\lambda \in \Lambda_n} |p_{\lambda,j}| < \infty, \quad \max_j \sup_n \sum_{\gamma \in \Gamma_n} |q_{\gamma,j}| < \infty,
\]

then the distributions \( f, g \) coincide.

Measures and their Fourier transforms can be interchanged in Theorems 2 and 3. Recall that the support of the Fourier transform of a measure \( \mu \) is called a *spectrum* of \( \mu \).

**Theorem 4.** Let \( \mu, \nu \) be atomic measures with discrete spectra \( \tilde{\Lambda}, \tilde{\Gamma} \) respectively, \(|\mu|, |\nu| \in S^*(\mathbb{R}^d)\), and there be disjoint sets \( \Lambda_n \) and disjoint sets \( \Gamma_n \) such that conditions (3), (4), (5) are met, with \( \Lambda \) replaced by \( \tilde{\Lambda} \), \( \Gamma \) by \( \tilde{\Gamma} \).

If either \( \hat{\mu}, \hat{\nu} \) are measures, and \( \hat{\mu}(\Lambda_n) - \hat{\nu}(\Gamma_n) \rightarrow 0 \) as \( n \rightarrow \infty \), or

\[
\hat{\mu} = \sum_{\lambda \in \tilde{\Lambda}} \sum_{j : \|j\| \leq m} \tilde{p}_{\lambda,j} D^j \delta_{\lambda}, \quad \hat{\nu} = \sum_{\gamma \in \tilde{\Gamma}} \sum_{j : \|j\| \leq m} \tilde{q}_{\gamma,j} D^j \delta_{\gamma},
\]

and

\[
\sum_{\lambda \in \Lambda_n} \tilde{p}_{\lambda,j} - \sum_{\gamma \in \Gamma_n} \tilde{q}_{\gamma,j} \rightarrow 0 \quad \forall j, \quad \text{as} \quad n \rightarrow \infty,
\]

\[
\max_j \sup_n \sum_{\lambda \in \Lambda_n} |\tilde{p}_{\lambda,j}| < \infty, \quad \max_j \sup_n \sum_{\gamma \in \Gamma_n} |\tilde{q}_{\gamma,j}| < \infty,
\]

then the measures \( \mu, \nu \) coincide.

We also give an analogue of Theorem 1 for classical Fourier quasicrystals.

**Theorem 5.** If two Fourier quasicrystals \( \mu, \nu \) with discrete "sparse" spectra \( \tilde{\Lambda}, \tilde{\Gamma} \) (this means that

\[
\sup_{y \in \mathbb{R}^d} \#\{\tilde{\Lambda} \cap B(y,1)\} < \infty, \quad \sup_{y \in \mathbb{R}^d} \#\{\tilde{\Gamma} \cap B(y,1)\} < \infty
\]

under appropriate numbering \( \tilde{\Lambda} = \{\lambda_n\}_{n=1}^{\infty}, \tilde{\Gamma} = \{\gamma_n\}_{n=1}^{\infty} \) have the properties

\[
\lambda_n - \gamma_n \rightarrow 0 \quad \text{and} \quad \hat{\mu}(\lambda_n) - \hat{\nu}(\gamma_n) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,
\]

then the measures \( \mu, \nu \) coincide.
2. Auxiliary Results

Lemma 1. Let a positive measure $\mu$ belong to $S^*(\mathbb{R}^d)$. Then there is $N < \infty$ such that $\mu(B(0, r)) = O(R^N)$, and for any Borel function $H(x)$ such that $\sup_{x \in \mathbb{R}^d} |H(x)|(1+|x|^T) < \infty$ for all $T < \infty$ we get

$$\int_{\mathbb{R}^d} |H(x)|\mu(dx) < \infty.$$

Proof of the Lemma. Assume the converse. Then there is a sequence $R_n \to \infty$ such that $\mu(B(0, R_n)) > R_n^n$. We may suppose that $R_{n+1} > 2R_n$ for all $n$. Take $\varphi(t) \in C^\infty(\mathbb{R})$, $0 \leq \varphi(t) \leq 1$ such that $\varphi(t) = 1$ for $t \leq 1$ and $\varphi(t) = 0$ for $t \geq 2$. Set

$$\Psi(x) = \sum_n R_n^{-n}\varphi(|x|/R_n).$$

Clearly, $\Psi \in C^\infty$ and

$$\int_{\mathbb{R}^d} \Psi(x)\mu(dx) = \sum_n R_n^{-n}\mu(B(0, R_n)) = \infty.$$

On the other hand, take any $K < \infty$ and $x$ such that $2R_{p-1} < |x| \leq 2R_p$ with $p > K$. We have

$$|x|^K \Psi(x) = \sum_n |x|^K R_n^{-n}\varphi(|x|/R_n) < 2^K R_p^{K-p} \sum_{n \geq p} R_n^p/R_n^n.$$

Taking into account that $R_n > 2^{-p}R_p$ and $p \to \infty$ as $|x| \to \infty$, we obtain

$$|x|^K \Psi(x) < 2^{K+1} R_p^{K-p} \to 0 \quad \text{as} \quad |x| \to \infty.$$

Similarly, one can check that $|x|^K \Psi^{(k)}(x) \to 0$ for all $K$ and $k \in (\mathbb{N} \cup \{0\})^d$, therefore, $\Psi \in S(\mathbb{R}^d)$. Since $\mu \in S^*(\mathbb{R}^d)$, we get the contradiction with (11). Hence there exists $N$ such that $M(R) := \mu(B(0, R)) \leq C \max(1, R^N)$.

Furthermore, let $|H(x)| \leq C_1|x|^{-N-1}$ for $|x| \geq 1$. Passing to polar coordinates and integrating in parts, we obtain

$$\int_{\mathbb{R}^d} |H(x)|\mu(dx) \leq C_0 + C_1 \int_{|x|>1} |x|^{-N-1}\mu(dx) = C_0 + C_2 \int_1^\infty r^{-N-1}M(dr)$$

$$= C_0 + C_2 \left( \lim_{R \to \infty} \frac{M(R)}{R^{N+1}} - M(1) + (N+1) \int_1^\infty \frac{M(r)}{r^{N+2}} dr \right) < \infty.$$

Lemma is proved.

The proofs of our theorems are also based on the properties of almost periodic functions and distributions. Recall some definitions related to the notion of almost periodicity (a detailed exposition of the theory of almost periodic functions on $\mathbb{R}$ see, for example, in [1] and [9], most of the results can easily be generalized to functions on $\mathbb{R}^d$: almost periodic measures and distributions were introduced in [8] and [15], see also [12], [13], [2], [5]).

A set $A \subset \mathbb{R}^d$ is relatively dense, if there is $R < \infty$ such that every ball of radius $R$ intersects with $A$.

A continuous function $f$ on $\mathbb{R}^d$ is almost periodic, if for every $\varepsilon > 0$ the set of $\varepsilon$-almost periods of $f$

$$\{\tau \in \mathbb{R}^d : \sup_{t \in \mathbb{R}^d} |f(t + \tau) - f(t)| < \varepsilon\}$$

is a relatively dense in $\mathbb{R}^d$.

For example, for arbitrary $s_n \in \mathbb{R}^d$ the function

$$f(t) = \sum_n a_n e^{2\pi i(t, s_n)}$$

is almost periodic on $\mathbb{R}^d$. \hfill \blacksquare
is almost periodic under the condition $\sum_n |a_n| < \infty$.

It was proved in [1] that a finite family $\{f_j\}_{j=1}^M$ of almost periodic functions on $\mathbb{R}$ has a common relatively dense set of $\varepsilon$-almost periods for every $\varepsilon$. The same result for almost periodic functions on $\mathbb{R}^d$ follows immediately from Bochner's criterion: a function $f(x)$ is almost periodic if and only if for every sequence $x_n$ there is a subsequence $x_{n'}$ such that the functions $f(x + x_{n'})$ converges uniformly in $x \in \mathbb{R}$. Its proof in [9] practically without changes is transferred to functions on $\mathbb{R}^d$ and even to mappings from $\mathbb{R}^d$ to $\mathbb{R}^M$.

Now, if each $f_j$ satisfies Bochner's criterion, then the mapping $F = (f_1(x), \ldots, f_M(x))$ satisfies this criterion too. It remains to notice that every $\varepsilon$-almost period of $F$ is an $\varepsilon$-almost period of every $f_j$.

Next, put $\varphi_t = \varphi(x - t)$ for any function $\varphi$ on $\mathbb{R}^d$.

A measure $\mu$ is almost periodic, if the function $F(t) = \int \psi_t(x) \mu(dx)$ is almost periodic for any continuous function $\psi(x)$ with compact support.

A tempered distribution $f$ is almost periodic, if for every $\varphi \in S(\mathbb{R}^d)$ the function $F(t) = f(\varphi_t)$ is almost periodic.

It can be proved that every almost periodic measure is an almost periodic tempered distribution, but there are discrete measures that almost periodic tempered distributions and not almost periodic measures. Nevertheless, if $\mu$ is a positive measure, or satisfies the condition $\sup_{x \in \mathbb{R}^d} |\mu(B(x,1))| < \infty$, then almost periodicity of $\mu$ in the sense of distributions implies almost periodicity in the sense of measures ([8], [12], [3]).

**Lemma 2.** If $f \in S^*(\mathbb{R}^d)$, $\hat{f}$ is an atomic measure, and $|\hat{f}| \in S^*(\mathbb{R}^d)$, then $f$ is an almost periodic distribution.

In particular, every Fourier quasicrystal is an almost periodic distribution.

**Proof of the Lemma.** Let $\hat{f} = \sum_n b_n \delta_{s_n}$, then $|\hat{f}| = \sum_n |b_n| \delta_{s_n}$. By (1), we have for each $\varphi \in S(\mathbb{R}^d)$

$$f(\varphi_t) = \int \varphi(y) e^{2\pi i \langle t, y \rangle} \hat{f}(dy) = \sum_n b_n \varphi(s_n) e^{2\pi i \langle t, s_n \rangle},$$

where $\varphi(y) e^{2\pi i \langle t, y \rangle}$ is the inverse Fourier transform of the function $\varphi(x - t)$. Since $\varphi(y) \in S(\mathbb{R}^d)$, we can apply Lemma 1. Therefore, series in (12) absolutely converge, and the function $f(\varphi_t)$ is almost periodic. \qed

3. PROOFS OF THE THEOREMS

**Proof of Theorem 2.** Assume the contrary $\mu \neq \nu$. Then there is $a \in \mathbb{R}^d$ such that $\nu(a) \neq \mu(a)$. Take $\varphi \in S(\mathbb{R}^d)$ such that $0 \leq \varphi \leq 1$, supp $\varphi \subset B(2)$, and $\varphi(x) = 1$ for $|x| < 1$. Since $\Lambda$ and $\Gamma$ are discrete, we can find $\rho \leq 1/2$ such that there are no points of $\Lambda \cup \Gamma$ other than $a$ in the ball $B(a, 2\rho)$. Therefore for all $\rho' \leq \rho$

$$\int \varphi \left( \frac{x - a}{\rho'} \right) \mu(dx) = \mu(a) \neq \nu(a) = \int \varphi \left( \frac{x - a}{\rho'} \right) \nu(dx).$$

Let $N$ be a number from (4). Set for $j = 1, \ldots, 2N + 1$

$$f_j(t) = \int \varphi \left( \frac{x - t}{2 - j \rho} \right) \mu(dx), \quad g_j(t) = \int \varphi \left( \frac{x - t}{2 - j \rho} \right) \nu(dx), \quad H_j(t) = f_j(t) - g_j(t).$$

Using (6) and applying Lemma 2, we obtain that all the functions $f_j(t)$ and $g_j(t)$ are almost periodic, and the functions $H_j(t)$ too. Moreover, for all $j$

$$H_j(a) = \mu(a) - \nu(a) \neq 0.$$
Set $\varepsilon = |\mu(a) - \nu(a)|/2$. Denote by $\mathcal{F}$ the set of all common $\varepsilon$-almost periods of the functions $H_j(t)$. We get
\begin{equation}
|H_j(a + \tau)| > \varepsilon \quad \forall \tau \in \mathcal{F}, \quad j = 1, \ldots, 2N + 1.
\end{equation}

Since $\mathcal{F}$ is relatively dense and $r_k \to \infty$, it follows that $B(x_k, r_k) \supset B(a + \tau_k, \rho)$ for every $k > k_1$ and some $\tau_k \in \mathcal{F}$. By the conditions of the theorem, $\Lambda \cup \Gamma$ is discrete, hence, we have
\begin{equation}
\min\{n : (\Lambda_n \cup \Gamma_n) \subset B(x_k, r_k)\} \to \infty \quad \text{as } k \to \infty,
\end{equation}
therefore, by (5),
\begin{equation}
\text{diam}(\Lambda_n \cup \Gamma_n) \to 0 \quad \text{as } k \to \infty \quad \text{for } \Lambda_n \cup \Gamma_n \subset B(x_k, r_k).
\end{equation}
In particular, there is $k_2$ such that for $k > k_2$ and $\Lambda_n \cup \Gamma_n \subset B(x_k, r_k)$ we get
\begin{equation}
\text{diam}(\Lambda_n \cup \Gamma_n) < 2^{-2N-1}\rho,
\end{equation}
and the set $\Lambda_n \cup \Gamma_n$ can intersect with only one of the spherical shells
\begin{equation*}
B(a + \tau_k, 2^{-j+1}\rho) \setminus B(a + \tau_k, 2^{-j}\rho), \quad j = 1, \ldots, 2N + 1.
\end{equation*}

On the other hand, by (4),
\begin{equation*}
\#\{n : (\Lambda_n \cup \Gamma_n) \subset B(a + \tau_k, 2\rho)\} \leq 2N,
\end{equation*}

hence there is $m = m(k), \ 1 \leq m \leq 2N + 1$, such that
\begin{equation*}
(\Lambda_n \cup \Gamma_n) \cap [B(a + \tau_k, 2^{-m+1}\rho) \setminus B(a + \tau_k, 2^{-m}\rho)] = \emptyset \quad \text{if } \Lambda_n \cup \Gamma_n \subset B(a + \tau_k, 2\rho).
\end{equation*}

Consequently, the sets $\Lambda_n, \Gamma_n$ are either both simultaneously subsets of $B(a + \tau_k, 2^{-m}\rho)$ and
\begin{equation*}
\varphi\left(\frac{\lambda - a - \tau_k}{2^{-m}\rho}\right) = \varphi\left(\frac{\gamma - a - \tau_k}{2^{-m}\rho}\right) = 1, \quad \forall \lambda \in \Lambda_n, \ \forall \gamma \in \Gamma_n,
\end{equation*}
or are not subsets of $B(a + \tau_k, 2^{-m+1}\rho)$, and
\begin{equation*}
\varphi\left(\frac{\lambda - a - \tau_k}{2^{-m}\rho}\right) = \varphi\left(\frac{\gamma - a - \tau_k}{2^{-m}\rho}\right) = 0 \quad \forall \lambda \in \Lambda_n, \ \forall \gamma \in \Gamma_n.
\end{equation*}
Hence,
\begin{equation*}
H_m(a + \tau_k) = \sum_{\lambda \in \Lambda \cap B(a + \tau_k, 2^{-m}\rho)} \mu(\lambda) - \sum_{\gamma \in \Gamma \cap B(a + \tau_k, 2^{-m}\rho)} \nu(\gamma) = \sum_{n : \Lambda_n \subset B(a + \tau_k, 2^{-m}\rho)} [\mu(\Lambda_n) - \nu(\Gamma_n)].
\end{equation*}

By (7) and (14), we obtain $\mu(\Lambda_n) - \nu(\Gamma_n) \to 0$ as $k \to \infty$. Since a number of members in the last sum is at most $N$, we get that the sequence $H_{m(k)}(a + \tau_k)$ tends to zero as $k \to \infty$, which contradicts to (13).

\textbf{Proof of Theorem 3.} Assume the contrary $f \neq g$. Then there is $a \in \text{supp } f$ such that either $a \in \Gamma$ and $p_{a, j_0} \neq q_{a, j_0}$ for some $j_0 \in (\mathbb{N} \cup \{0\})^d$, or $a \notin \Gamma$ and $p_{a, j_0} \neq 0$ for some $j_0 \in (\mathbb{N} \cup \{0\})^d$. In the latter case put $q_{a, j_0} = 0$. Take $\varphi \in S(\mathbb{R}^d)$ such that $0 \leq \varphi \leq 1$, $\text{supp } \varphi \subset B(0, 2)$, and $\varphi(x) = 1$ for $x \in B(0, 1)$. Since $\Lambda$ and $\Gamma$ are discrete, it follows that there are no points of $\Lambda \cup \Gamma$ other than $a$ in the ball $B(a, 2\rho)$ for some $\rho \leq 1/2$. Put
\begin{equation*}
\psi(x) = \varphi(x/\rho)x_{j_0}/j_0!.
\end{equation*}
Clearly, $(D^{j_0}\psi)(0) = 1$ and $(D^j\psi)(0) = 0$ for $j \neq j_0$. Therefore, $f(\psi_a) = p_{a, j_0} \neq g(\psi_a)$. Using (8) and applying Lemma 2, we obtain that the functions $f(\psi_t)$, $g(\psi_t)$ are almost periodic, and the function $H(t) = f(\psi_t) - g(\psi_t)$ too. Moreover, $H(a) = p_{a, j_0} - q_{a, j_0} \neq 0$.

Set $\varepsilon = |H(a)|/2$. Denote by $\mathcal{F}$ the set of all $\varepsilon$-almost period of the functions $H(t)$. We get
\begin{equation}
|H(a + \tau)| > \varepsilon \quad \forall \tau \in \mathcal{F}.
\end{equation}
Since $\mathcal{T}$ is relatively dense and $r_k \to \infty$, we see that $B(x_k, r_k) \supset B(a + \tau_k, 2\rho)$ for every $k > k_1$ and some $\tau_k \in \mathcal{T}$. Set

$$M_k = \{ n : (\Lambda_n \cup \Gamma_n) \cap B(a + \tau_k, 2\rho) \neq \emptyset \}.$$  

Note that $\Lambda \cup \Gamma$ is discrete, hence,

$$\min M_k \to \infty \quad \text{as} \quad k \to \infty. \quad (16)$$

Furthermore, for sufficiently large $k$

$$f(\psi_{a+\tau_k}) - g(\psi_{a+\tau_k}) = \sum_{n \in M_k} \sum_j \left[ \sum_{\lambda \in \Lambda_n} p_{\lambda,j}(D^j \psi)(\lambda - a - \tau_k) - \sum_{\gamma \in \Gamma_n} q_{\gamma,j}(D^j \psi)(\gamma - a - \tau_k) \right]. \quad (17)$$

All derivatives of $\psi$ are uniformly continuous, hence it follows from (5) and (10) that for some fixed points $b_n \in \Lambda_n \cup \Gamma_n$ and each $j$

$$\sum_{\lambda \in \Lambda_n} p_{\lambda,j} [(D^j \psi)(\lambda - a - \tau_k) - (D^j \psi)(b_n - a - \tau_k)] \to 0 \quad \text{as} \quad n \to \infty,$$

and

$$\sum_{\gamma \in \Gamma_n} q_{\gamma,j} [(D^j \psi)(\gamma - a - \tau_k) - (D^j \psi)(b_n - a - \tau_k)] \to 0 \quad \text{as} \quad n \to \infty.$$  

All derivatives of $\psi$ are uniformly bounded, hence it follows (9) that for each $j$

$$\left[ \sum_{\lambda \in \Lambda_n} p_{\lambda,j} - \sum_{\gamma \in \Gamma_n} q_{\gamma,j} \right] (D^j \psi)(b_n - a - \tau_k) \to 0 \quad \text{as} \quad n \to \infty.$$  

Hence,

$$\sum_{\lambda \in \Lambda_n} p_{\lambda,j}(D^j \psi)(\lambda - a - \tau_k) - \sum_{\gamma \in \Gamma_n} q_{\gamma,j}(D^j \psi)(\gamma - a - \tau_k) \to 0 \quad \text{as} \quad n \to \infty.$$  

Note that the number of values of $j$ does not exceed $(m + 1)^d$ and $\#M_k \leq 2N$. Thus we obtain from (16) and (17)

$$|H(a + \tau_k)| = |f(\psi_{a+\tau_k}) - g(\psi_{a+\tau_k})| \to 0 \quad \text{as} \quad k \to \infty,$$

which contradicts to (15). \qed

Theorem 4 follows from Theorems 2 and 3, if only we change the Fourier transform to the inverse Fourier transform. Theorem 1 follows from Theorem 2, and Theorem 5 follows from Theorem 4.

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