OPTIMAL GEVREY STABILITY OF HYDROSTATIC APPROXIMATION FOR THE NAVIER-STOKES EQUATIONS IN A THIN DOMAIN

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Abstract. In this paper, we study the hydrostatic approximation for the Navier-Stokes system in a thin domain. When the convex initial data with Gevrey regularity of optimal index \( \frac{3}{2} \) in \( x \) variable and Sobolev regularity in \( y \) variable, we justify the limit from the anisotropic Navier-Stokes system to the hydrostatic Navier-Stokes/Prandtl system. Due to our method in the paper is independent of \( \varepsilon \), by the same argument, we also get the hydrostatic Navier-Stokes/Prandtl system is well-posedness in the optimal Gevrey space. Our results improve the Gevrey index in [15, 35] whose Gevrey index is \( \frac{9}{8} \).

1. INTRODUCTION

1.1. Presentation the problem and related results. In this article, we study 2-D incompressible Navier-Stokes equations in a thin domain where the aspect ratio and the Reynolds number have certain constraints:

\[
\begin{aligned}
\partial_t U + U \cdot \nabla U - \varepsilon^2 (\partial_x^2 + \eta \partial_y^2) U + \nabla P &= 0, \\
\text{div } U &= 0, \\
U|_{y=0} &= U|_{y=\varepsilon} = 0,
\end{aligned}
\]

where \( t \geq 0, (x, y) \in S^\varepsilon = \{(x, y) \in \mathbb{T} \times \mathbb{R} : 0 < y < \varepsilon\} \). Here, \( U(t, x, y), P(t, x, y) \) stand for the velocity and pressure function respectively and \( \eta \) is a positive constant independent of \( \varepsilon \). The width of domain \( S^\varepsilon \) is \( \varepsilon \), and the boundary condition in (1.1) corresponds to non-slip condition at the walls \( y = 0, \varepsilon \). In addition, the system is prescribed with the initial data of the form

\[
U|_{t=0} = \left( u_0 \left( x, \frac{y}{\varepsilon} \right), \varepsilon v_0 \left( x, \frac{y}{\varepsilon} \right) \right) = U_0^\varepsilon \quad \text{in} \quad S^\varepsilon.
\]

This is a classical model with applications to oceanography, meteorology and geophysical flows, where the vertical dimension of the domain is very small compared with the horizontal dimension of the domain.

To study the process \( \varepsilon \to 0 \), we firstly fix the domain independent of \( \varepsilon \). Here, we rescale the system (1.1) as follows:

\[
U(t, x, y) = \left( u^\varepsilon \left( t, x, \frac{y}{\varepsilon} \right), \varepsilon v^\varepsilon \left( t, x, \frac{y}{\varepsilon} \right) \right) \quad \text{and} \quad P(t, x, y) = p^\varepsilon \left( t, x, \frac{y}{\varepsilon} \right).
\]
We put above relations into (1.1), then (1.1) is reduced to a scaled anisotropic Navier-Stokes system:

\[
\begin{align*}
\partial_t u^\varepsilon + u^\varepsilon \partial_x u^\varepsilon + v^\varepsilon \partial_y u^\varepsilon - \varepsilon^2 \partial_x^2 u^\varepsilon - \eta \partial_y^2 u^\varepsilon + \partial_x p^\varepsilon &= 0, \\
\varepsilon^2 (\partial_t v^\varepsilon + u^\varepsilon \partial_x v^\varepsilon + v^\varepsilon \partial_y v^\varepsilon - \varepsilon^2 \partial_x^2 v^\varepsilon - \eta \partial_y^2 v^\varepsilon) + \partial_y p^\varepsilon &= 0, \\
\partial_x u^\varepsilon + \partial_y v^\varepsilon &= 0, \\
(u^\varepsilon, v^\varepsilon)|_{y=0,1} &= 0, \\
(u^\varepsilon, v^\varepsilon)|_{t=0} &= (u_0, v_0),
\end{align*}
\]

(1.3)

where \((x, y) \in S = \{(x, y) \in \mathbb{T} \times (0, 1)\} \). To simplify the notations, we take \(\eta = 1\) in this paper and denote \(\Delta_x = \varepsilon^2 \partial_x^2 + \partial_y^2\).

Formally, taking \(\varepsilon \to 0\) in (1.3), we derive the hydrostatic Navier-Stokes/Prandtl system (see [22, 31]):

\[
\begin{align*}
\partial_t u_p + u_p \partial_x u_p + v_p \partial_y u_p - \eta \partial_y^2 u_p + \partial_x p_p &= 0 \quad \text{in} \ S \times (0, \infty), \\
\partial_y p_p &= 0 \quad \text{in} \ S \times (0, \infty), \\
\partial_x u_p + \partial_y v_p &= 0 \quad \text{in} \ S \times (0, \infty), \\
(u_p, v_p)|_{y=0,1} &= 0, \\
u_p|_{t=0} &= u_0 \quad \text{in} \ S.
\end{align*}
\]

(1.4)

The goal in this paper is to justify the limit from the scaled anisotropic Navier-Stokes system (1.3) to the hydrostatic Navier-Stokes/Prandtl system (1.4), for a class of convex data in the optimal Gevrey class with index \(\gamma = \frac{3}{2}\).

Before presenting the precise statement of the main result in this paper, we recall some results on system (1.4). If \(\eta = 0\) in the system (1.4), we get the hydrostatic Euler system:

\[
\begin{align*}
\partial_t u + u \partial_x u + v \partial_y u + \partial_x p &= 0 \quad \text{in} \ S \times (0, \infty), \\
\partial_y p &= 0 \quad \text{in} \ S \times (0, \infty), \\
\partial_x u + \partial_y v &= 0 \quad \text{in} \ S \times (0, \infty), \\
(u, v)|_{y=0,1} &= 0, \\
u|_{t=0} &= u_0 \quad \text{in} \ S.
\end{align*}
\]

(1.5)

There are a lot of research on the system (1.5), and readers can refer to [3, 4, 5, 16, 20, 19, 26, 31, 38]. Renardy [31] proved the linearization of (1.5) has a growth like \(e^{k|t|}\) if the initial data is not uniform convexity (or concavity) in variable \(y\). Local well-posedness in the analytic setting was established in [20]. Under the convexity condition, Masmoudi and Vicol [26] got the well-posedness of (1.5) in the Sobolev space.

Next, we recall some results on the well-posedness of the hydrostatic Navier-Stokes/Prandtl system (1.4). Similar to the classical Prandtl equation, (1.4) lose one derivative because of term \(v_p \partial_y u_p\). Paicu, Zhang and the Zhang [30] obtained the global well-posedness of system (1.4) when the initial data is small in the analytical space. Meantime, Renardy [31] also proved that the linearization of the hydrostatic Navier-Stokes equations at certain parallel shear flows is ill-posed, and may have a growth \(e^{k|t|}\) which is the same as (1.5) when the initial data is not convex. Thus, in order to obtain well-posedness results that break through the analytic space, one may need the convexity condition on the velocity. For that, under the convexity condition, Gérard-Varet, Masmoudi and Vicol proved the (1.4) is local well-posedness in the Gevrey class with index 9/8 in [15]. In [15], they firstly derive the vorticity...
equations \( \omega = \partial_y u \):
\[
\partial_t (\partial_x \omega) + \partial_x v \partial_y \omega + \cdots = 0,
\]
where the worst term is \( \partial_x v \) leading to one derivative loss. Then, they use the "hydrostatic trick" which means that they take inner product with \( \partial_x \omega / \partial_y \omega \) instead of \( \partial_x \omega \) to take advantage of the cancellation:
\[
\hat{\partial}_x v \partial_y \omega \cdot \partial_x \omega \partial_y \omega = \hat{\partial}_x v \partial_x \omega = -\hat{\partial}_x \partial_y v \partial_x u = 0.
\]
Such an idea was used previously in [26]. To close the energy estimates, "hydrostatic trick" is not enough due to the "bad" boundary condition of \( \omega \)
\[
\partial_y \omega |_{y=0} = -\partial_x \int_0^1 u^2 dy + \cdots.
\]
which lose one derivative too. To overcome that, [15] introduce the following decomposition
\[
\omega = \omega^{bl} + \omega^{in},
\]
where \( \omega^{bl} \) is the boundary corrector which satisfies that
\[
\partial_t \omega^{bl} - \partial_y^2 \omega^{bl} = 0, \quad \partial_y \omega^{bl} |_{y=0} = -\partial_x \int_0^1 u^2 dy.
\]
Following the above decomposition, [15] obtain the well-posedness results of (1.4) in the Gevrey class with index \( \gamma = \frac{9}{8} \).

To search the best functional space for the system (1.4), based on the Tollmien-Schlichting instabilities for Navier-Stokes [17], Gérard-Varet, Masmoudi and Vicol also give the following conjecture: "Our conjecture - based on a formal parallel with Tollmien-Schlichting instabilities for Navier-Stokes [18] - is that the best exponent possible should be \( \gamma = \frac{3}{2} \), but such result is for the time being out of reach."

During studying the anisotropic Navier-Stokes system (1.3) and the hydrostatic Navier-Stokes/Prandtl system (1.4), another important problem is to justify the inviscid limit. Under analytical setting, Paicu, Zhang and Zhang [30] justified the limit from (1.3) to (1.4). Based on the work [15], we [35] justified the limit in the Gevrey class with index \( \gamma = \frac{9}{8} \).

In this paper, we aim to prove the conjecture of Gérard-Varet, Masmoudi and Vicol. To do that, we use some idea from the classical inviscid limit theory. Next, we recall the recent development on the classical Prandtl equation and the inviscid limit theory.

There are a lot of papers studying the well-posedness of Prandtl equation in some special functional space. For monotonic initial data, [29, 1, 27] used different method to get local existence and uniqueness of classical solutions to the Prandtl equation in Sobolev space.

Without monotonic condition, [24, 32] proved that Prandtl equation are well-posedness in the analytic class; [14, 23, 6] proved well-posedness of the Prandtl equations in Gevrey class for a class of concave initial data. Without any structure assumption, Dietert and Gérard-Varet [8] proved well-posedness in Gevrey space with index \( \gamma = 2 \). According to [10], \( \gamma = 2 \) may be the optimal index for the well-posedness theory. For more results on the Prandtl equation, see [18, 37, 36, 39, 40].

On the inviscid limit problem, we refer to [33, 34, 21, 28, 25, 9] for the analytical class. Note that to go from analytic to Gevrey data is a challenging problem. The first results in Gevrey class is given by [12]. Gérard-Varet, Masmoudi and Maekawa [12] proved stability of the
Prandtl expansion for the perturbations in the Gevrey class when $U^{BL}(t, Y)$ is a monotone and concave function where the boundary layer is the shear type like

$$u_\varepsilon^\nu = (U^\varepsilon(t, y), 0) + (U^{BL}(t, \frac{y}{\sqrt{\nu}}), 0),$$

where $\nu$ is the viscosity coefficient. Later, Chen, Wu and Zhang [7] improved the results in [12] to get the $L^2 \cap L^\infty$ stability. Very recently, Gérard-Varet, Masmoudi and Mekawa [13] used a very clever decomposition to get the optimal Prandtl expansion around concave boundary layer. Their results generalized the one obtained in [12, 7] which restricted to expansions of shear flow type. In their paper, they decompose the stream function $\phi$ as follows:

$$\phi = \phi_{\text{slip}} + \phi_{bc}$$

where $\phi_{\text{slip}}$ enjoys a "good" boundary condition and $\phi_{bc}$ is a corrector which recover the boundary condition. This kind of decomposition is also used in [7]. To estimate $\phi_{bc}$, they also need the following decomposition

$$\phi_{bc} = \phi_{bc,S} + \phi_{bc,T} + \phi_{bc,R},$$

where $\phi_{bc,S}$ satisfies the Stokes equation, $\phi_{bc,T}$ is to correct the stretching term with "good" boundary condition and $\phi_{bc,R}$ solves formally the same system as $\phi_{\text{slip}}$. In this paper, we apply the decomposition in [13] to justify the limit from (1.3) to (1.4).

1.2. Statement of the main results. Before stating the main results, we give some assumption on initial data. Assume that initial data belong into the following Gevrey class:

$$(1.6) \| e^{D_x} \partial_y u_0 \|_{H^{1.4,0}} + \| e^{D_x} \partial_y^3 u_0 \|_{H^{10,0}} := M < +\infty,$$

where $H^{r,s}$ is the anisotropic Sobolev space defined by

$$\|f\|_{H^{r,s}} = \|\|f\|_{H^r(T)}\|_{H^s(0,1)}. $$

More precisely, we consider initial data of the form

$$(1.7) \quad \partial_x u_0 + \partial_y v_0(t, x, y) = 0, \quad u_0(t, x, 0) = u_0(t, x, 1) = v_0(t, x, 0) = v_0(t, x, 1) = 0,$$

$$(1.8) \quad \int_0^1 \partial_x u_0 dy = 0, \quad \partial_y^2 u_0|_{y=0,1} = \int_0^1 (-\partial_x u_0^2 + \partial_y^2 u_0) - \int_S \partial_y^2 u_0 dx dy.$$  

Moreover, we assume the initial velocity satisfies the convex condition

$$(1.9) \quad \inf_S \partial_y^2 u_0 \geq 2c_0 > 0.$$  

Now, we are in the position to state the main results in our paper.

**Theorem 1.1.** Let initial data $u_0$ satisfy (1.6)-(1.9). Then there exist $T > 0$ and $C > 0$ independent of $\varepsilon$ such that there exists a unique solution of the scaled anisotropic Navier–Stokes equations (1.3) in $[0, T]$, which satisfies that for any $t \in [0, T]$, it holds that

$$\|(u^\varepsilon - u_p, \varepsilon v^\varepsilon - \varepsilon v_p)\|_{L^2_x \cap L^\infty} \leq C\varepsilon^2,$$

where $(u_p, v_p)$ is the solution to (1.4).
Remark 1.2. Although we do not give the proof that the system (1.3) is well-posedness in Gevrey class $\frac{3}{2}$, one can follow the proof of Theorem [1.4] to obtain the well-posedness. To avoid repetability in the proof, we omit the details. Actually, the main difference between $\varepsilon = 0$ and $\varepsilon \neq 0$ is on the construction boundary corrector $\phi_{bc,S}$, and readers can find more details in Remark [8.1].

Remark 1.3. In the recent work [11] by Gérard-Varet, Iyer and Maekawa, they establish well-posedness of Hydrostatic Navier-Stokes system in Gevrey class $\frac{3}{2}$. In our present work, we focus on the inviscid limit problem.

1.3. Sketch of the proof. In this subsection, we sketch the main ingredients in our proof.

1) Introduce the error equations. In Section 3, we deduce the error equations. We introduce the error
\[ u^R = u^\varepsilon - u^p, \quad v^R = v^\varepsilon - v^p, \quad p^R = p^\varepsilon - p^p, \]
which satisfies
\[ \begin{cases} 
\partial_t u^R - \Delta \varepsilon u^R + v^R \partial_y p^R + \partial_x p^R = \cdots, \\
\varepsilon^2 (\partial_t v^R - \Delta \varepsilon v^R) + \partial_y p^R = \cdots.
\end{cases} \tag{1.10} \]
Here $(u^p, v^p, p^p)$ is approximate solution given in (3.1). The key point in this paper is to obtain the uniform estimate (in $\varepsilon$) of $(u^R, \varepsilon v^R)$ in the Gevrey class with index $\gamma = \frac{3}{2}$. In view of (1.10), since $v^R$ is controlled via the relation $v^R = -\int_0^y \partial_x u^R dy'$, the main difficulty comes from the term $v^R \partial_y p^R$, which loses one tangential derivative.

2) Introduce the vorticity formulation. In order to eliminate $p^R$, we introduce vorticity $\omega^R = -\varepsilon^2 \partial_x v^R + \partial_y u^R$ and rewrite the equation of $\omega^R$ by stream function $\phi$ which satisfies
\[ v^R = -\partial_x \phi, \quad u^R = \partial_y \phi + C(t), \quad C(t) = \frac{1}{2\pi} \int_S u^R dxdy. \]
Thus, we get
\[ \begin{cases} 
(\partial_t - \Delta \varepsilon) \Delta \varepsilon \phi - \partial_x \phi \partial_y \omega^p = \cdots, & (x, y) \in S, \\
\phi|_{y=0,1} = 0, \quad \partial_y \phi|_{y=0,1} = C(t), & x \in \mathbb{T}.
\end{cases} \tag{1.11} \]
We notice the term $\partial_x \phi \partial_y \omega^p$ also lose one tangential derivative. But under the convexity condition $\partial_y \omega^p \geq c_0 > 0$, one can use "hydrostatic trick" to deal with this term. Testing $\frac{\omega^R}{\partial_y \omega^p}$ to the (1.11) instead of $\omega^R$, we have the following cancellation:
\[ -\int_S \partial_x \phi \partial_y \omega^p \cdot \omega^R \partial_y \omega^p dxdy = -\int_S \partial_x \phi \Delta \varepsilon \phi dxdy = \int_S \partial_x |\nabla \varepsilon \phi|^2 dxdy = 0, \]
where we use $\phi|_{y=0,1} = 0$. However, the boundary condition of $\phi$ is $\partial_y \phi|_{y=0,1} = C(t)$ which brings an essential difficulty.

By the energy estimates, taking inner product in $X^r$ (the definition is given in section 2) with $-\partial_t \phi$, we get
\[ \sup_{s \in [0, T]} \left( \lambda \|\nabla \varepsilon \phi(s)\|^2_{X^r} + \|\Delta \varepsilon \phi(s)\|^2_{X^2} \right) \]
\[ C \int_0^t \left( \varepsilon^{-2} \| \varphi \Delta_x \phi \|_{X^2}^2 + \varepsilon^{-2} \| \nabla_x \phi \|_{X^2}^2 + \cdots \right) ds, \]

where \( \varphi(y) = y(1 - y) \). All we need to do is to control \( \varepsilon^{-1} \| \varphi \Delta_x \phi \|_{X^2} \) and \( \varepsilon^{-1} \| \nabla_x \phi \|_{X^2} \) by the left hand side of (1.12).

Motivated by [13], we expect to achieve that by a decomposition of stream function \( \hat{\phi} = \phi_{\text{slip}} + \phi_{bc} \) in Gevrey \( \frac{3}{2} \) regularity. Here \( \phi_{\text{slip}} \) enjoys a "good" boundary condition and \( \phi_{bc} \) is a corrector which recover the boundary condition. In the following, we present the decomposition precisely.

**(3) Gevrey estimate under artificial boundary conditions.** \( \phi_{\text{slip}} \) enjoying a good boundary condition is defined by

\[
\begin{align*}
& (\partial_t - \Delta_x) \omega_{\text{slip}} - \partial_x \phi_{\text{slip}} \partial_y \omega^p = \cdots, \quad (x, y) \in S \\
& \phi_{\text{slip}}|_{y=0,1} = 0, \quad \omega_{\text{slip}}|_{y=0,1} = 0, \quad x \in \mathbb{T},
\end{align*}
\]

where \( \omega_{\text{slip}} = \Delta_x \phi_{\text{slip}} \). By "hydrostatic trick" and Navier-slip boundary conditions to obtain

\[
\lambda \int_0^t \left( \| \omega_{\text{slip}} \|_{X^\frac{3}{2}}^2 + \| \nabla_x \omega_{\text{slip}} \|_{X^\frac{3}{2}}^2 + \| \nabla_x \phi_{\text{slip}} \|_{y=0,1}^2 \right) ds \leq \frac{C}{\lambda} \int_0^t \| \Delta_x \phi \|_{X^2}^2 ds + \cdots.
\]

The full study of the Orr-Sommerfeld formulation (1.13) with Navier-slip boundary conditions is given in Section 7.

**(4) Recovery the non-slip boundary condition.** In Step (3), we use the slip boundary condition, not the real boundary condition \( \partial_y \phi|_{y=0,1} = C(t) \). To recover the boundary condition, we introduce the following system:

\[
\begin{align*}
& (\partial_t - \Delta_x) \phi_{bc} - \partial_x \phi_{bc} \partial_y \omega^p = 0, \quad (x, y) \in S \\
& \phi_{bc}|_{y=0,1} = 0, \quad \partial_y \phi_{bc}|_{y=i} = h^i, \quad x \in \mathbb{T},
\end{align*}
\]

where \( \omega_{bc} = \Delta_x \phi_{bc} \) and \( i = 0, 1 \). And we need to choose a suitable \( h^i \) such that

\[ \partial_y \phi_{bc}|_{y=0,1} = -\partial_y \phi_{\text{slip}}|_{y=0,1} + C(t). \]

Next, we give the main idea for proving the existence of \( h^i \):

1. We define \( \phi_{bc,S} = \phi_{bc,S}^0 + \phi_{bc,S}^1 \), where \( \phi_{bc,S}^0 \) solve

\[
\begin{align*}
& (\partial_t - \Delta_x) \Delta_x \phi_{bc,S}^i = 0, \\
& \phi_{bc,S}^i|_{y=0} = 0, \quad \partial_y \phi_{bc,S}^i|_{y=i} = h^i, \\
& \phi_{bc,S}^i|_{t=0} = 0,
\end{align*}
\]

where \( x \in \mathbb{T}, \ y \in (0, +\infty) \) for \( i = 0 \) and \( y \in (-\infty, 1) \) for \( i = 1 \). Taking Fourier transformation on \( t \) and \( x \), we can write the precise expression of the solution to obtain the Gevrey estimate for \( \phi_{bc,S}^i \):

\[
\int_0^t \left( \| \nabla_x \phi_{bc,S}^i \|_{X^\frac{3}{2}}^2 + \| \varphi^i \Delta_x \phi_{bc,S}^i \|_{X^\frac{3}{2}}^2 + \| \partial_x \phi_{bc,S}^i \|_{X^\frac{3}{2}}^2 \right) ds \leq \frac{C}{\lambda^\frac{3}{2}} \int_0^t |h^i|^2 \| \varphi^i \|_{X^\frac{3}{2}}^2 ds,
\]

where \( \varphi^0(y) = y, \ \varphi^1(y) = 1 - y \). Compared with the decomposition in [35], we get more regularity of \( \partial_x \phi_{bc,S}^i \) which is a key point to get the optimal Gevrey regularity. The details for this step is given in Section 8.1.
2. We correct the nonlocal term constructed in the above step, by consider the following equations:

\[
\begin{aligned}
\begin{cases}
(\partial_t - \Delta \varepsilon) \Delta \phi_{bc,R}^i - \partial_x \phi_{bc,R}^i \partial_y \omega^p = \partial_x \phi_{bc,S}^i \partial_y \omega^p, & (x, y) \in \mathcal{S} \\
\phi_{bc,R}^i|_{t=0} = 0, & (x, y) \in \mathcal{S}
\end{cases}
\end{aligned}
\]

with Navier-slip conditions. By the same process as Step (3) and combining with the sharp estimate (1.17) to get estimate for \( \phi_{bc,R}^i \):

\[
\lambda \int_0^t \| \omega_{bc,R}^i \|^2_{\mathcal{X}^\mathcal{Z}} \, ds + \int_0^t \left( \| \nabla \phi_{bc,R}^i \|^2_{\mathcal{X}^\mathcal{Z}} + |\partial_y \phi_{bc,R}^i|_{y=0,1}^2 \right) ds \leq \frac{C}{\lambda \mathcal{Z}} \int_0^t |h^i|^2_{\mathcal{X}^\mathcal{Z}} ds + \cdots, \quad t \in [0, T],
\]

More details are given in Section 8.3.

3. We define \( \phi_{bc} = \phi_{bc,S} + \phi_{bc,R} \), where \( \phi_{bc,S} = \sum_{i=0,1} \phi_{bc,S}^i \) and \( \phi_{bc,R} = \sum_{i=0,1} \phi_{bc,R}^i \), which solve system (1.15). To match the boundary condition on the derivative of \( \partial_y \phi|_{y=0,1} = C(t) \), we need

\[
\partial_y \phi_{bc,S}|_{y=0,1} + \partial_y \phi_{bc,R}|_{y=0,1} = \partial_y \phi_{bc}|_{y=0,1} = -\partial_y \phi_{slip}|_{y=0,1} + C(t).
\]

On one hand, \( \phi_{bc,S} \) and \( \phi_{bc,R} \) are defined by \( h^i \). We define a 0-order operator \( R_{bc} \) given in (8.71) such that

\[
(1 + R_{bc}) h^i = -\partial_y \phi_{slip}|_{y=0,1} + C(t).
\]

Moreover, by the estimate in Step 1 and Step 2, we can get

\[
\int_0^t |R_{bc}[h^0, h^1]|_{\mathcal{X}^\mathcal{Z}}^2 \, ds \leq \frac{C}{\lambda \mathcal{Z}} \int_0^t |(h^0, h^1)|_{\mathcal{X}^\mathcal{Z}}^2 \, ds.
\]

which means that \( 1 + R_{bc} \) is an invertible operator when \( \lambda \) is large. That means that \( \phi_{bc,S} \) and \( \phi_{bc,R} \) are well-defined and (1.15) is well-posedness. Details are given in Section 8.4.

Due to the transport terms, we need to introduce a new auxiliary function \( \phi_{bc,T} \) between Step 1 and Step 2. For more details, see Section 8.2.

(5) **Close the energy estimates** (1.12). Summing estimates (1.17) and (1.19) in Step (4), we get estimate for \( \phi_{bc} \):

\[
\int_0^t \| \nabla \phi_{bc} \|^2_{\mathcal{X}^\mathcal{Z}} + |\varphi \Delta \phi_{bc}|_{\mathcal{X}^\mathcal{Z}}^2 \, ds \leq \frac{C}{\lambda \mathcal{Z}} \int_0^t |(h^0, h^1)|_{\mathcal{X}^\mathcal{Z}}^2 \, ds \leq C \int_0^t |\nabla \phi_{slip}|_{y=0,1}^2_{\mathcal{X}^\mathcal{Z}} \, ds + \cdots,
\]

which along with (1.14) to have

\[
\int_0^t (\| \varphi \omega \|^2_{\mathcal{X}^\mathcal{Z}} + \| \nabla \phi \|^2_{\mathcal{X}^\mathcal{Z}}) ds \leq \frac{C}{\lambda} \int_0^t \| \Delta \phi \|^2_{\mathcal{X}^\mathcal{Z}} ds + \cdots,
\]

then putting above estimate into (1.12) to close the estimate for system (1.11).
1.4. Notations. - $S^\varepsilon = \{(x,y) \in \mathbb{T} \times \mathbb{R} : 0 < y < \varepsilon\}$ and $\mathcal{S} = \{(x,y) \in \mathbb{T} \times \mathbb{R} : 0 < y < 1\}$. 
- $\nabla \varepsilon = (\varepsilon \partial_x, \partial_y)$ and $\Delta \varepsilon = \varepsilon^2 \partial_x^2 + \partial_y^2$.
- Vorticity of Prandtl part $\omega^p$ is defined by $\omega^p = \partial_y u^p$.
- Vorticity of reminder part $\omega^R = \Delta \varepsilon \phi$ is defined by $\omega^R = \varepsilon^2 \partial_x v^R - \partial_y u^R$. In this paper, we also define $\omega^p_{bc,j}$ where $i = 0, 1$ and $j \in \{R, T\}$.
- Cut-off functions $\varphi(y) = y(1-y)$ and $\varphi^i(y) = i + (-1)^i y$.
- $C(t) = \frac{1}{2\pi} \int_{\mathbb{S}} u^R dxdy$.
- The Fourier transform of $f_\Phi$ is defined by $e^{(1-\lambda)(k)^2} \hat{f}(k)$.

2. Gevrey norms and preliminary lemmas

At the beginning of this section, we give the definition of the functional space $X^r$ and the Gevrey class. First, we define

\begin{equation}
(2.1) 
\begin{align*}
\Phi(t,k) & = e^{\Phi(t,D_x) f}, \\
\Phi(t,k) & \overset{def}{=} \tau(t)(k)^2,
\end{align*}
\end{equation}

where $\tau(t) \geq 0$. Moreover, it is easy to get that $\Phi(t,k)$ satisfies the subadditive inequality

\begin{equation}
(2.2) 
\Phi(t,k) \leq \Phi(t,k-\ell) + \Phi(t,\ell).
\end{equation}

Now, we are in the position to define $X^r_T$ which is defined by

$$
||f||_{X^r_T} = ||f_\Phi||_{H^r(\mathbb{T})}.
$$

We say that a function $f$ belongs to Gevrey class $\frac{3}{2}$ if $||f||_{X^r_T} < +\infty$.

Moreover, we need to deal with some Gevrey class functions defined on the boundary. Thus, we introduce the following functional space:

$$
|f|_{X^r_T} = ||f_\Phi||_{H^r(\mathbb{T})}.
$$

where $f$ depends on variable $x$.

By the definition of $X^r_T$, it is easy to see that if $r' \geq r$, then $|| \cdot ||_{X^{r'}_T} \geq || \cdot ||_{X^r_T}$. For simplicity, we drop subscript $\tau$ in the notations $||f||_{X^r_T}, ||f||_{X^{r'}_T}$ etc. In the sequel, we always take

$$
\tau(t) = 1 - \lambda t,
$$

with $\lambda \geq 1$ determined later. Thus, if we take $t$ small enough, we have $\tau > 0$.

In the following, we present some lemmas on product estimates in Gevrey class and the readers can refer to Lemma 2.1-2.3 in \[34\] for details. The first lemma is the commutator estimate in Sobolev space:

**Lemma 2.1.** Let $r \geq 0$, $s_1 > \frac{3}{2}$, $s > \frac{1}{2}$ and $0 \leq \delta \leq 1$. Then it holds that

$$
\| \langle (D)^r \rangle f |\partial_x g \rangle \|_{L^2} \leq C \| f \|_{H^{s_1}} \| g \|_{H^s} + C \| f \|_{H^{s+1-\delta}} \| g \|_{H^{s+\delta}}.
$$

In Gevrey class, we have

**Lemma 2.2.** Let $r \geq 0$ and $s > \frac{1}{2}$. Then it holds that

$$
\| f g \|_{X^r} \leq C \| f \|_{X^{s}} \| g \|_{X^r} + C \| f \|_{X^{s}} \| g \|_{X^r}.
$$

For the commutator in Gevrey class, we have

**Lemma 2.3.** Let $r \geq 0$, $s_1 > \frac{3}{2}$, $s > \frac{1}{2}$ and $0 \leq \delta \leq 1$. Then it holds that

$$
\| (f \partial_x g)_\Phi - f \partial_x g \|_{H^s} \leq C \| f \|_{X^{s_1}} \| g \|_{X^{r+\delta}} + C \| f \|_{X^{r+1-\delta}} \| g \|_{X^{s+\delta}}.
$$
3. Approximate equations and Error equations

3.1. Approximate equations. By Hilbert asymptotic method, we can obtain the approximate solutions. We define approximate solutions as following

\[
\begin{align*}
&u^p(t, x, y) = u_p^0(t, x, y) + \varepsilon^2 u_p^2(t, x, y), \\
v^p(t, x, y) = v_p^0(t, x, y) + \varepsilon^2 v_p^2(t, x, y), \\
p^p(t, x, y) = p_p^0(t, x, y) + \varepsilon^2 p_p^2(t, x, y),
\end{align*}
\]

(3.1)

where \((u_p^0, v_p^0, p_p^0)\) satisfies equation (1.4) and \((u_p^2, v_p^2, p_p^2)\) satisfies equation

\[
\begin{align*}
\partial_t u_p^2 + u_p^0 \partial_x u_p^2 + v_p^0 \partial_y u_p^2 + \varepsilon^2 \partial_y u_p^0 + \partial_x v_p^0 - \partial_y^2 u_p^2 = 0, \\
\partial_y p_p^0 = -(\partial_t v_p^0 + u_p^0 \partial_x v_p^0 + v_p^0 \partial_y v_p^0 - \partial_y^2 v_p^0), \\
\partial_x u_p^2 + \partial_y v_p^2 = 0, \\
(u_p^2, v_p^2)|_{y=0,1} = 0, \\
u_p^2|_{t=0} = 0.
\end{align*}
\]

(3.2)

Based on the equation of \((u_p^0, v_p^0, p_p^0)\) and \((u_p^2, v_p^2, p_p^2)\), we deduce approximate solution \((u^p, v^p, p^p)\) which satisfies the following equation:

\[
\begin{align*}
&\partial_t u^p + u^0 \partial_x u^p + v^0 \partial_y u^p + \partial_x p^p - \Delta_v u^p = -R_1, \\
&\varepsilon^2 (\partial_t v^p + u^0 \partial_x v^p + v^0 \partial_y v^p - \Delta_v v^p) + \partial_y p^p = -R_2, \\
&\partial_x u^p + \partial_y v^p = 0, \\
&(u^p, v^p)|_{y=0,1} = 0, \\
&(u^p, v^p)|_{t=0} = (u_0, v_0),
\end{align*}
\]

(3.3)

where reminder \((R_1, R_2)\) is given by

\[
\begin{align*}
R_1 &= \varepsilon^4 (u_p^0 \partial_x u_p^2 + v_p^0 \partial_y v_p^2 - \partial_x^2 u_p^2), \\
R_2 &= \varepsilon^4 \left( \partial_t v_p^2 + u_p^0 \partial_x v_p^2 + v_p^0 \partial_y v_p^2 + \varepsilon^2 u_p^2 \partial_x v_p^2 + v_p^0 \partial_y v_p^0 + v_p^2 \partial_y v_p^0 \\
&\quad\quad\quad\quad\quad\quad\quad\quad+ \varepsilon^2 \partial_y v_p^2 - \partial_x^2 v_p^0 + \varepsilon^2 v_p^2 - \partial_y^2 v_p^2 \right).
\end{align*}
\]

(3.4)

(3.5)

By the definition of \(R_1\) and \(R_2\), it is easy to get that

\[(R_1, R_2) \sim O(\varepsilon^4).\]

3.2. Equations of error functions. We define error functions \((u^R, v^R, p^R)\):

\[
\begin{align*}
u^R &= u^\varepsilon - u^p, \\
v^R &= v^\varepsilon - v^p, \\
p^R &= p^\varepsilon - p^p.
\end{align*}
\]

It is easy to deduce the system of error functions:

\[
\begin{align*}
&\partial_t u^R - \Delta_x u^R + \partial_x p^R + u^\varepsilon \partial_x u^R + u^R \partial_x u^p + v^\varepsilon \partial_y u^R + v^R \partial_y u^p = R_1, \\
&\varepsilon^2 (\partial_t v^R - \Delta_x v^R + u^\varepsilon \partial_x v^R + u^R \partial_x v^p + v^\varepsilon \partial_y v^R + v^R \partial_y v^p) + \partial_y p^R = R_2, \\
&\partial_x u^R + \partial_y v^R = 0, \\
&(u^R, v^R)|_{y=0} = (u^R, v^R)|_{y=1} = 0, \\
&(u^R, v^R)|_{t=0} = 0.
\end{align*}
\]

(3.6)
For convenience, we rewrite (3.6) as

\[
\begin{align*}
\left\{ \begin{array}{l}
\partial_t u^R - \Delta_x u^R + u^p \partial_x u^R + u^R \partial_x u^p + v^R \partial_y u^p + v^p \partial_y u^R + \partial_x p^R = N_u + R_1, \\
\varepsilon^2 (\partial_t v^R - \Delta_x v^R + u^p \partial_x v^R + u^R \partial_x v^p + v^p \partial_y v^p + v^p \partial_y v^R) + \partial_y p^R = \varepsilon^2 N_v + R_2,
\end{array} \right.
\end{align*}
\]

(3.7)

where \((u^R, v^R)|_{y=0} = (u^R, v^R)|_{y=1} = 0,

\((u^R, v^R)|_{t=0} = 0\).

Here \((N_u, N_v)\) is nonlinear term given by

\[
N_u = -(u^R \partial_x u^R + v^R \partial_y u^R), \quad N_v = -(u^R \partial_x v^R + v^R \partial_y v^R).
\]

Based on the above system, we get the equations of the vorticity \(\omega^R = \partial_y u^R - \varepsilon^2 \partial_x v^R\).

\[
\begin{align*}
\partial_t \omega^R - \Delta_x \omega^R + u^p \partial_x \omega^R + u^R \partial_x \omega^p + v^p \partial_y \omega^p + v^p \partial_y \omega^R + \partial_y p^R = \partial_y N_u - \varepsilon^2 \partial_x N_v + \varepsilon^2 f_1 + f_2,
\end{align*}
\]

where \(f_1, f_2\) are defined by

\[
\begin{align*}
f_1 &= -(u^R \partial_x^2 v^p + v^R \partial_x \partial_y v^p), \\
f_2 &= \partial_y R_1 - \varepsilon^2 \partial_x R_2, \\
\omega^p &= \partial_y u^p.
\end{align*}
\]

Moreover, following the calculations in [34], we can obtain the boundary conditions of \(\omega^R\):

\[
\begin{align*}
(\partial_y + \varepsilon |D|)\omega^R|_{y=0} &= \partial_y (\Delta_x D)^{-1} (f - N)|_{y=0} + \frac{1}{2\pi} \int_S \partial_t u^R dx dy, \\
(\partial_y - \varepsilon |D|)\omega^R|_{y=1} &= \partial_y (\Delta_x D)^{-1} (f - N)|_{y=1} + \frac{1}{2\pi} \int_S \partial_t u^R dx dy,
\end{align*}
\]

where

\[
N = \partial_y N_u - \varepsilon^2 \partial_x N_v = -u^R \partial_x \omega^R - v^R \partial_y \omega^R.
\]

\[
\begin{align*}
f &= f_3 - \varepsilon^2 f_1 - f_2, \\
f_3 &= u^p \partial_x \omega^R + u^R \partial_x \omega^p + v^p \partial_y \omega^R + v^R \partial_y \omega^p, \quad \omega^p = \partial_y u^p.
\end{align*}
\]

3.3. Equations of stream function. Thanks to \(\partial_x u^R + \partial_y v^R = 0\) and \(v^R|_{y=0,1} = 0\), there exists a stream function \(\phi\) satisfying the following system:

\[
-\partial_x \phi = v^R, \quad \partial_y \phi = u^R - \frac{1}{2\pi} \int_S u^R dx dy,
\]

Since \(\int_S v^R dx = 0\), the function \(\phi\) is periodic in \(x\). Thanks to \(\partial_x \phi|_{y=0,1} = 0\) and \(\phi(1, x) - \phi(0, x) = 0\), we may assume that \(\phi|_{y=0,1} = 0\). Thus, there holds that

\[
\Delta_x \phi = \omega^R \text{ in } S, \quad \phi|_{y=0,1} = 0.
\]

Taking (3.18) and (3.19) into (3.9) and using the boundary condition \((u^R, v^R)|_{y=0,1} = 0\), we obtain

\[
\begin{align*}
\left\{ \begin{array}{l}
(\partial_t - \Delta_x) \Delta_x \phi + u^p \partial_x \Delta_x \phi + v^p \partial_y \Delta_x \phi + \partial_y \phi \partial_x \omega^p - \partial_x \phi \partial_y \omega^p \\
\quad = \partial_y N_u - \varepsilon^2 \partial_x N_v + \varepsilon^2 f_1 + f_2 - C(t) \partial_x \omega^p,
\end{array} \right.
\end{align*}
\]

(3.20)

\[
\begin{align*}
\phi|_{y=0,1} = 0, \quad \partial_y \phi|_{y=0,1} = C(t),
\end{align*}
\]
where \( C(t) = \frac{1}{2\pi} \int_S u^r dxdy \) and \( (N_u, N_v), f_1, f_2 \) are given in (3.8), (3.10) and (3.11).

In the end of subsection, we state some elliptic estimates which can be got by classical theory. First, by elliptic estimate and Hardy inequality, we have

\[
\|\nabla \phi\|_{L^2} \leq C \|\varphi\omega\|_{L^2},
\]

where \( \varphi(y) = y(1 - y) \) and \( \nabla \phi = (\partial_y, \varepsilon \partial_x) \).

Since \((u^r, \varepsilon v^r)\) satisfies the following elliptic equations

\[
\begin{cases}
\Delta u^R = \partial_y \omega^R, \\
u^R|_{y=0,1} = 0,
\end{cases}
\]

we arrive at

\[
\|(u^R, \varepsilon v^R, \partial_y u^R, \varepsilon \partial_x u^R, \varepsilon \partial_y v^R, \varepsilon^2 \partial_x v^R)\|_{X^r} \leq C \|\omega^R\|_{X^r},
\]

for any \( r \geq 0 \).

4. Estimate of \( \nabla \phi \) and \( \Delta \phi \) in Gevrey space

Before giving the estimate of \( \nabla \phi \) and \( \Delta \phi \), we need the estimates of the reminder terms \( R_1 \) and \( R_2 \) which are defined by the approximate solution \( u^p \) and \( v^p \). For \((u^p, v^p)\), we have the following bound:

**Lemma 4.1.** Let initial data \( u_0 \) of (1.4) satisfy (1.0) - (1.9). There exists a time \( T_p \) such that \((u^p_i, v^p_i), i = 0, 2\) defined in (1.4) and (3.2) have the following estimates

\[
\begin{align*}
\|v^0_p\|_{X^{11}} + \|u^0_p\|_{X^{12}} + \|\partial_y u^0_p\|_{X^{12}} + \|\partial_y^2 u^0_p\|_{X^8} & \leq C, \\
\|v^2_p\|_{X^9} + \|u^2_p\|_{X^{10}} + \|\partial_y u^2_p\|_{X^{10}} + \|\partial_y^2 u^2_p\|_{X^6} & \leq C,
\end{align*}
\]

for \( t \in [0, T_p] \).

Moreover, according to (3.1), it holds that

\[
\begin{align*}
\|v^p\|_{X^9} + \|(u^p, \varepsilon v^p)\|_{X^{10}} + \|\partial_y u^p\|_{X^{10}} + \|\partial_y^2 u^p\|_{X^6} & \leq C, & t & \in [0, T_p],
\end{align*}
\]

and

\[
\partial_y \omega^p \geq c_0, \quad t \in [0, T_p].
\]

**Proof.** Here, the key of this lemma is to prove the (1.4) is well-posedness in the Gevrey class \( \frac{3}{2} \), which is the conjecture in [15]. If we set \( \varepsilon = 0 \) and follow the step by step in this paper, we can get the conjecture proved. Here, to avoid the repeatability, we leave the proof to the readers.

Then, by the definition of \((R_1, R_2)\) in (3.4) to (3.5), using Lemma 2.2 to get that

**Lemma 4.2.** It holds that

\[
\|(R_1, R_2)\|_{X^3} \leq C \varepsilon^4, \quad \|\nabla (R_1, R_2)\|_{X^2} \leq C \varepsilon^4, \quad t \in [0, T_p].
\]

Now, we state our main result in this section:
Proposition 4.3. There exist $0 < T < \min\{T_p, \frac{1}{\Delta t}\}$ and $\lambda_0 \geq 1$, such that for any $t \in [0, T]$ and $\lambda \geq \lambda_0$, it holds that

\[
\sup_{s \in [0, t]} \left( \lambda \|\nabla_x \phi(s)\|^2_{L^2} + \|\Delta_x \phi(s)\|^2_{L^2} \right) + \int_0^t \left( \|\partial_t \nabla_x \phi\|_{H^2,0}^2 + \|\nabla_{x} R\|_{L^2}^2 \right) ds
\leq C \int_0^t \left( \epsilon^{-2} \|\phi \Delta_x \phi\|^2_{L^2} + \epsilon^{-2} \|\nabla_x \phi\|^2_{L^2} + \|(N_u, \epsilon \eta)\|^2_{L^2} + \|\eta\|^2_{L^2} + \epsilon^8 \right) ds,
\]

where $\Delta_x \phi = \omega^R$, $\phi(y) = y(1 - y)$ and $C$ is a constant independent of $\epsilon$.

Proof. Acting $e^{\Phi(t/D_x)}$ on the both sides of the first equation of (13.20), we get

\[
(\partial_t + \lambda(D_x)^2 - \Delta_x)\Delta_x \phi + (u^p \partial_x \Delta_x \phi + v^p \partial_y \Delta_x \phi) \phi + (\partial_y \phi \partial_x \omega^p - \partial_x \phi \partial_y \omega^p) \phi = \partial_y(N_u) \phi - \epsilon^2 \partial_x(N_\eta) \phi + (\epsilon^2 f_1 + f_2) \phi - C(t) \partial_x \omega^p
\]

Taking $H^{2,0}$ inner product with $-\partial_t \phi$ and using boundary conditions $\phi|_{y=0} = 0$, $\partial_y \phi|_{y=0} = C(t)$, we integrate by parts to arrive at

\[
(4.1) \quad \frac{1}{2} \frac{d}{dt} \left( \lambda \|\nabla_x \phi\|^2_{L^2} + \|\Delta_x \phi\|^2_{L^2} \right) + \|\partial_t \nabla_x \phi\|_{H^2,0}^2 - \left( \Delta_x \phi, \partial_t \phi \right)_{H^2,0} = I_1 + \cdots + I_5.
\]

Firstly, let’s estimate of $I_i$, $i = 1, \cdots, 5$ term by term.

**Estimate of $I_1$.** Since divergence free condition $\partial_x u^p + \partial_y v^p = 0$, we get $(u^p \partial_x \Delta_x \phi + v^p \partial_y \Delta_x \phi) \phi = \partial_x(u^p \Delta_x \phi) + \partial_y(v^p \Delta_x \phi) \phi$.

According to $(u^p, v^p)|_{y=0} = 0$, we use integration by parts and Lemma 2.2 to have

\[
I_1 = - \left( \left( u^p \Delta_x \phi, \partial_t \Delta_x \phi \right)_{H^2,0} - \left( v^p \Delta_x \phi, \partial_t \Delta_x \phi \right)_{H^2,0} \right)
\leq C \left( \frac{u^p}{\varphi} |\phi\|^2_{L^2} |\Delta_x \phi|_{L^2} \right)_{H^2,0} + C \left( \frac{v^p}{\varphi} |\phi\|^2_{L^2} |\Delta_x \phi|_{L^2} \right)_{H^2,0}
\leq C \epsilon^{-1} |\Delta_x \phi|_{L^2} |\partial_t \Delta_x \phi|_{H^2,0}.
\]

**Estimate of $I_2$.** Similarly, we write

\[
(\partial_y \phi \partial_x \omega^p - \partial_x \phi \partial_y \omega^p) \phi = \left( \partial_x(\partial_y \phi \omega^p) - \partial_y(\partial_x \phi \omega^p) \right) \phi,
\]

then along with $\phi|_{y=0} = 0$, we use integration by parts and Lemma 2.2 to deduce

\[
I_2 = \left( \partial_x(u^p \phi \omega^p) + \partial_y(u^p \phi \omega^p) \right)_{H^2,0} + \left( \partial_y(v^p \phi \omega^p) + \partial_x(v^p \phi \omega^p) \right)_{H^2,0}
\]

we integrate by parts to arrive at

\[
(4.1) \quad \frac{1}{2} \frac{d}{dt} \left( \lambda \|\nabla_x \phi\|^2_{L^2} + \|\Delta_x \phi\|^2_{L^2} \right) + \|\partial_t \nabla_x \phi\|_{H^2,0}^2 - \left( \Delta_x \phi, \partial_t \phi \right)_{H^2,0} = I_1 + \cdots + I_5.
\]
Recall where we used (4.3) in the last step, which gives
\[ C \varepsilon^{-1} \| \nabla_x \phi \|_{L^2} \| \partial_t \nabla_x \phi \|_{H^{2,0}}. \]

Estimate of $I_3$. Due to $\phi|_{y=0,1} = 0$, taking integration by parts, it yields that
\[ I_3 \leq C \| (N_u, \varepsilon N_v) \|_{X^2} \| \partial_t \nabla_x \phi \|_{H^{2,0}}. \]

Estimate of $I_4$. Recall $f_1$ and $f_2$ in (3.10)-(3.11). According to (3.22) and Lemma 4.2 we have
\[
I_4 \leq C(\varepsilon^2 \| u^R \|_{X^2} + \varepsilon \| u^R \|_{X^2} + \varepsilon^4) \| \partial_t \partial_y \phi \|_{H^{2,0}}.
\]

Estimate of $I_5$. Poincaré inequality implies
\[ \| \partial_t \phi \|_{H^{2,0}} \leq C \| \partial_t \partial_y \phi \|_{H^{2,0}}, \]
for $\phi|_{y=0,1} = 0$. So that
\[
|C(t)| = \frac{1}{2\pi} \int_S u^R dxdy \leq C \| u^R \|_{L^2} \leq C \| \omega^R \|_{L^2} \leq C \| \Delta \phi \|_{L^2},
\]
we get
\[ I_5 \leq C \| \Delta \phi \|_{L^2} \| \partial_t \partial_y \phi \|_{H^{2,0}}. \]

Collecting $I_1 - I_5$ together, it holds that
\[
I_1 + \cdots + I_5 \leq C \varepsilon^{-1} \| \Delta \phi \|_{X^2} \| \partial_t \nabla_x \phi \|_{H^{2,0}} + C \varepsilon^{-1} \| \nabla_x \phi \|_{X^2} \| \partial_t \nabla_x \phi \|_{H^{2,0}}
+ C \| (N_u, \varepsilon N_v) \|_{X^2} \| \partial_t \nabla_x \phi \|_{H^{2,0}} + C \| \Delta \phi \|_{L^2} \| \partial_t \partial_y \phi \|_{H^{2,0}}
+ C(\varepsilon \| \Delta \phi \|_{X^2} + \varepsilon^4) \| \partial_t \partial_y \phi \|_{H^{2,0}}
\leq \frac{1}{10} \| \nabla_x \phi \|_{H^{2,0}}^2 + C \left( \varepsilon^{-2} \| \Delta \phi \|_{X^2}^2 + \varepsilon^{-2} \| \nabla_x \phi \|_{X^2}^2 + \| \Delta \phi \|_{X^2}^2 \right)
+ C \left( \| (N_u, \varepsilon N_v) \|_{X^2} + \varepsilon^8 \right).
\]

Next, we focus on the boundary term $\left( \Delta \phi, \partial_t \partial_y \phi \right)_{H^2_y \mid y=0}$. First, we give the estimate of $C'(t)$.
\[
\int_S \partial_t u^R dxdy = \int_S \partial_y^2 u^R dxdy = \int_S \partial_y \omega^R dxdy,
\]
which gives
\[
\left| \int_S \partial_t u^R dxdy \right| \leq \| \partial_y \omega^R \|_{L^1}.
\]

Owing to
\[ \partial_t \partial_y \phi|_{y=0,1} = C'(t) \delta(k), \]
where $\delta(k)$ is a dirac function and $k \in \mathbb{Z}$, we have
\[
\left( \Delta \phi, \partial_t \partial_y \phi \right)_{H^2_y \mid y=0} = \left( \Delta \phi |_{y=0}, C'(t) \right)_{L^2_y} = \left( \int_0^1 \partial_y \Delta \phi dy, C'(t) \right)_{L^2_y}
\leq CC'(t) \| \partial_y \omega^R \|_{L^2} \leq C \| \partial_y \omega^R \|_{L^2}^2,
\]
where we used (4.3) in the last step.
Putting above estimate and (4.2) into (4.1), we get
\[
\frac{1}{2} \frac{d}{dt} (\lambda \|\nabla \phi\|^2 + \|\Delta \phi\|^2) + \|\nabla \phi\|^2 + \|\Delta \phi\|^2 + \|\partial_y \omega^R\|^2 \leq C \left( \varepsilon^2 \|\Delta \phi\|^2 + \|\nabla \phi\|^2 + \|\Delta \phi\|^2 + \|\partial_y \omega^R\|^2 + \varepsilon^8 \right)
\]  
(4.4)

Next, we give the estimates of \(\|\partial_y \omega^R\|^2\). Firstly, recalling the equation of \(\omega^R\):
\[
\partial_t \omega^R - \Delta_x \omega^R + u^p \partial_x \omega^R + u^R \partial_y \omega^p + v^p \partial_y \omega^R + v^R \partial_y \omega^p = \partial_y N_u - \varepsilon^2 \partial_y N_v + \varepsilon^2 f_1 + f_2,
\]
with boundary conditions
\[
(\partial_y + \varepsilon |D|) \omega^R|_{y=0} = \partial_y (\Delta, D)^{-1}(f - \mathcal{N})|_{y=0} + \frac{1}{2\pi} \int_\mathcal{S} \partial_t u^R dxdy,
\]
(4.6)
\[
(\partial_y - \varepsilon |D|) \omega^R|_{y=1} = \partial_y (\Delta, D)^{-1}(f - \mathcal{N})|_{y=1} + \frac{1}{2\pi} \int_\mathcal{S} \partial_t u^R dxdy,
\]
(4.7)
where \(f_1, f_2, f\) and \(\mathcal{N}\) are given in (3.10)-(3.16).

Taking \(L^2\) inner product with \(\omega^R\) on (4.5) and integration by parts, it follows from \((N_u, \varepsilon N_v)|_{y=0,1} = 0\) and \((u^p, \varepsilon v^p)|_{y=0,1} = 0\) to obtain
\[
\frac{1}{2} \frac{d}{dt} \|\omega^R\|^2 + \|\nabla \omega^R\|^2 - \int_\mathcal{T} \partial_y \omega^R \omega^R dx = \int_\mathcal{T} (\varepsilon |D| \omega^R|_{y=1} + \partial_y (\Delta, D)^{-1}(f - \mathcal{N})|_{y=1} + C(t)) \omega^R|_{y=1} dx
\]  
(4.8)
\[
= \int_\mathcal{T} (\varepsilon |D| \omega^R|_{y=0} + \partial_y (\Delta, D)^{-1}(f - \mathcal{N})|_{y=0} + C(t)) \omega^R|_{y=0} dx + \int_\mathcal{S} \partial_y \left( \partial_y (\Delta, D)^{-1}(f - \mathcal{N}) \omega^R \right) dxdy = B_1 + B_2 + B_3.
\]

For the boundary term, we use (4.6)-(4.7) to write
\[
\int_\mathcal{T} \partial_y \omega^R \omega^R dx = \int_\mathcal{T} (\varepsilon |D| \omega^R|_{y=1} + \partial_y (\Delta, D)^{-1}(f - \mathcal{N})|_{y=1} + C(t)) \omega^R|_{y=1} dx
\]
\[
= \int_\mathcal{T} (\varepsilon |D| \omega^R|_{y=0} + \partial_y (\Delta, D)^{-1}(f - \mathcal{N})|_{y=0} + C(t)) \omega^R|_{y=0} dx + \int_\mathcal{S} \partial_y \left( \partial_y (\Delta, D)^{-1}(f - \mathcal{N}) \omega^R \right) dxdy = B_1 + B_2 + B_3.
\]

Let \(y_0 \in [0,1]\) so that
\[
\|\varepsilon |D| \omega^R(y_0)\|_{L^2} \leq \|\varepsilon |D| \omega^R\|_{L^2},
\]
then along with Gagliardo-Nirenberg inequality
\[
\|g\|_{L^\infty} \leq C \|g\|_2 \left( \|g\|_2^2 + \|\partial_y g\|_2^2 \right),
\]
(4.9)
\[
\text{it infers that}
\]
\[
B_1 = \int_{y_0}^1 \partial_y (\varepsilon |D| \omega^R) dxdy + \int_0^{y_0} \partial_y (\varepsilon |D| \omega^R) dxdy + 2 \int_\mathcal{T} (\varepsilon |D| \omega^R) |y=y_0| dx \leq C \|\varepsilon |D| \omega^R\|_{L^2} \|\partial_y \omega^R\|_{L^2} + C \|\varepsilon |D| \omega^R\|_{L^2} \|\omega^R\|_{L^\infty}.
\]
\[ \leq C \varepsilon \| \omega^R \|^2_{L^2_{1.0}} + C \varepsilon \| \partial_y \omega^R \|^2_{L^2} \]

Similarly, we use (4.9) and \( |C(t)| \leq C \|u^R\|_{L^2} \leq C \|\omega^R\|_{L^2} \) to have
\[
B_2 \leq C |C(t)| \| \omega^R \|_{L^\infty_y(L^2)} \leq C \| \omega^R \|^3_{L^2} \left( \| \omega^R \|^2_{L^2} + \| \partial_y \omega^R \|^2_{L^2} \right) \\
\leq \frac{1}{10} \| \partial_y \omega^R \|^2_{L^2} + C \| \omega^R \|^2_{L^2}.
\]

All we left is to do \( B_3 \). With the fact: operator \( \partial_y (\Delta, \varepsilon, D)^{-1} \), \( \partial_y (\Delta, \varepsilon, D)^{-1} (\partial_y, \varepsilon \partial_x) \) and \( \partial_y (\Delta, \varepsilon, D)^{-1} \) are bounded from \( L^2 \rightarrow L^2 \), we have
\[
B_3 = \int_S \partial_y (\Delta, \varepsilon, D)^{-1} (f - N) \omega^R dx dy + \int_S \partial_y (\Delta, \varepsilon, D)^{-1} \partial_y (v^p \omega^R) \partial_y \omega^R dx dy \\
+ \int_S (f - \partial_y (v^p \omega^R) - \partial_y N_u - \varepsilon^2 \partial_x N_v) \partial_y \omega^R dx dy \\
\leq C \| f - N \|_{L^2} \| \omega^R \|_{L^2} + C \| v^p \omega^R \|_{L^2} \| \partial_y \omega^R \|_{L^2} \\
+ C \| f - \partial_y (v^p \omega^R) \|_{L^2} + \| (N_u, \varepsilon N_v) \|_{L^2} \| \partial_y \omega^R \|_{L^2}.
\]

According to the definition of (3.16) and (3.15), we have
\[
\| f \|_{L^2} \leq C (\| \partial_x \omega^R \|_{L^2} + \| \partial_y \omega^R \|_{L^2} + \| (u^R, v^R) \|_{L^2} + \varepsilon^4) \\
\leq C (\| \omega^R \|_{H_{1.0}} + \| \partial_y \omega^R \|_{L^2} + \varepsilon^4),
\]

and
\[
\| f - \partial_y (v^p \omega^R) \|_{L^2} \leq C (\| \partial_x \omega^R \|_{L^2} + \| (u^R, v^R) \|_{L^2} + \varepsilon^4) \\
\leq C (\| \omega^R \|_{H_{1.0}} + \varepsilon^4),
\]

which give that
\[
B_3 \leq C (\| \omega^R \|_{H_{1.0}} + \| \partial_y \omega^R \|_{L^2} + \| N \|_{L^2} + \varepsilon^4) \| \omega^R \|_{L^2} \\
+ C (\| \omega^R \|_{H_{1.0}} + \| (N_u, \varepsilon N_v) \|_{L^2} + \varepsilon^4) \| \partial_y \omega^R \|_{L^2} \\
\leq \frac{1}{10} \| \partial_y \omega^R \|^2_{L^2} + C (\| \omega^R \|^2_{H_{1.0}} + \| (N_u, \varepsilon N_v) \|^2_{L^2} + \| N \|^2_{L^2} + \varepsilon^8).
\]

Summarizing \( B_1 - B_3 \) together, we obtain
\[
(4.10) \quad \left| \int_T \partial_y \omega^R \omega^R dx \right|_{y=0}^{y=1} \leq \left( \frac{1}{5} + C \varepsilon \right) \| \partial_y \omega^R \|^2_{L^2} + C (\| \omega^R \|^2_{H_{1.0}} + \| (N_u, \varepsilon N_v) \|^2_{L^2} + \| N \|^2_{L^2} + \varepsilon^8).
\]

Substituting (4.10) into (4.8), we take \( \varepsilon \) small enough to arrive at
\[
\frac{1}{2} \frac{d}{dt} \| \omega^R \|^2_{L^2} + \frac{1}{2} \| \nabla \omega^R \|^2_{L^2} \\
\leq C (\| (N_u, \varepsilon N_v) \|^2_{L^2} + \| \omega^R \|^2_{H_{1.0}} + \varepsilon^8) \\
+ C (\| \omega^R \|^2_{H_{1.0}} + \| (N_u, \varepsilon N_v) \|^2_{L^2} + \| N \|^2_{L^2} + \varepsilon^8) \\
\leq C (\| \Delta \varepsilon^2 \|^2_{H_{1.0}} + \| (N_u, \varepsilon N_v) \|^2_{L^2} + \| N \|^2_{L^2} + \varepsilon^8).
\]

Bring the above estimate into (4.3) and integrate time from 0 to \( t \) to get the desired results. \( \square \)
5. Sketch the proof to Theorem 1.1

In this section, we shall sketch the proof of Theorem 1.1. In the paper, we use the continue argument. Here, we define

\[ T^* \overset{\text{def}}{=} \sup\{ t > 0 \mid \sup_{s \in [0,t]} \| \omega^R \|_{X^2} \leq C \varepsilon^3 \}. \]

5.1. The key a priori estimates. In this subsection, we shall present the key a priori estimates used in the proof of Theorem 1.1.

By Proposition 4.3, we need the estimates of \( \int_0^t (\| \nabla \phi \|_{X^2}^2 + \| \varphi \Delta \phi \|_{X^2}^2) ds \) to close the energy.

**Proposition 5.1.** Let \( \phi \) be the solution of (3.20). Then there exists \( \lambda_0 \geq 1 \) and \( 0 < T \leq \min\{ T_p, \frac{1}{T^*} \} \) such that for \( \lambda \geq \lambda_0 \) and \( t \in [0, T] \), it holds that

\[ \int_0^t (\| \nabla \phi \|_{X^2}^2 + \| \varphi \Delta \phi \|_{X^2}^2) ds \leq C T \int_0^t (\| (\lambda u, \varepsilon N_v) \|_{X^2}^2 + \| \varepsilon \Delta \phi \|_{X^2}^2 + \varepsilon^6) ds, \]

with \( t \in [0, T] \).

The proof of the above proposition is the main part in this paper and we prove it in the section 6.

5.2. Proof of Theorem 1.1. Before we prove the Theorem 1.1, we firstly give the estimates for the nonlinear terms:

**Proposition 5.2.** Under the assumption (5.1), there holds that

\[ \int_0^t \| (\lambda u_v, \varepsilon N_v) \|_{X^2}^2 ds \leq C \varepsilon^4 \int_0^t \| \omega^R \|_{X^2}^2 ds, \]

\[ \int_0^t \| N \|_{L^2}^2 ds \leq C \varepsilon^4 \int_0^t \| \nabla \omega^R \|_{L^2}^2 ds, \]

where \( t \in [0, T^*] \).

**Proof.** By the definition of \( N_u \), we have

\[ \int_0^t \| N_u \|_{X^2}^2 ds \leq \int_0^t \| u^R \partial_x u^R \|_{X^2}^2 ds + \int_0^t \| v^R \partial_y u^R \|_{X^2}^2 ds = I_1 + I_2. \]

It follows from Lemma 2.2 and (3.22) that

\[ I_1 \leq C \int_0^t \| \frac{u^R}{\varepsilon} \|_{L^{\infty}(H^2)}^2 \| \varepsilon \partial_x u^R \|_{X^2}^2 ds \]

\[ \leq C \varepsilon^{-2} \int_0^t \| u^R \|_{X^2} (\| u^R \|_{X^2} + \| \partial_y u^R \|_{X^2}) \| \varepsilon \partial_x u^R \|_{X^2}^2 ds \]

\[ \leq C \varepsilon^{-2} \int_0^t \| \omega^R \|_{X^2}^4 ds, \]

where we use Gagliardo-Nirenberg inequality (4.9) in the second step.

Similarly, we use Lemma 2.2 and (3.22) to deduce

\[ I_2 \leq C \int_0^t \| \frac{v^R}{\varepsilon} \|_{L^{\infty}(H^2)}^2 \| \partial_y u^R \|_{X^2}^2 ds \leq C \varepsilon^{-2} \int_0^t \| \varepsilon \partial_x u^R \|_{X^2}^2 \| \partial_y u^R \|_{X^2}^2 ds \]

\[ \leq C \varepsilon^{-2} \int_0^t \| \omega^R \|_{X^2}^4 ds. \]
\[ \leq C^2 \varepsilon^{-2} \int_0^t \| \omega^R \|_{X^2}^2 ds, \]

where we use \( \nu^R = -\int_0^y \partial_s u^R dy' \) in the second step.

Collecting \( I_1 \) and \( I_2 \) together and using (5.1), it holds that

\[ \int_0^t \| N \|_{L^2}^2 ds \leq C \varepsilon^{-2} \int_0^t \| \omega^R \|_{X^2}^2 ds \leq C \varepsilon^4 \int_0^t \| \omega^R \|_{X^2}^2 ds. \]

The estimate for \( \varepsilon N_v \) is obtained by changing \( u^R \) into \( \varepsilon v^R \) in the above argument and we omit details. Thus we obtain (5.3).

For (5.4), we use the definition of \( N \) to have

\[ \leq \varepsilon^{-2} \int_0^t \| u^R \|_{L^2(H^1_x)}^2 \| \varepsilon \partial_x \omega^R \|_{L^2}^2 ds + \int_0^t \| v^R \|_{L^2(H^1_x)}^2 \| \partial_y \omega^R \|_{L^2}^2 ds \]

\[ \leq C \varepsilon^{-2} \int_0^t \| \omega^R \|_{H^{1,0}}^2 \| \varepsilon \partial_x \omega^R \|_{L^2}^2 ds + C \int_0^t \| \omega^R \|_{H^{2,0}}^2 \| \partial_y \omega^R \|_{L^2}^2 ds \]

\[ \leq C \varepsilon^{-2} \sup_{s \in [0, t]} \| \omega^R \|_{X^2}^2 \int_0^t \| \nabla \omega^R \|_{L^2}^2 ds \]

\[ \leq C \varepsilon^{-4} \int_0^t \| \nabla \omega^R \|_{L^2}^2 ds, \]

by (5.1) and we obtain (5.3).

With Proposition 5.1 and Proposition 5.2 in hand, we are in the position to prove the Theorem 1.1. By Proposition 4.3, Proposition 5.1 and Proposition 5.2, we get

\[ \sup_{s \in [0, t]} (\lambda \| \nabla \phi(s) \|_{X^2}^2 + \| \Delta \phi(s) \|_{X^2}^2) + \int_0^t \| \partial_t \nabla \phi \|_{H^{2,0}}^2 + C t \varepsilon^6 + C \int_0^t \| \Delta \phi(s) \|_{X^2}^2 ds, \]

for \( t \in [0, T] \). By Gronwall inequality and choosing a small \( T < \min \{ T_p, \frac{1}{2 \lambda} \} \), we get that

\[ \sup_{s \in [0, t]} (\lambda \| \nabla \phi(s) \|_{X^2}^2 + \| \omega^R \|_{X^2}^2) + \int_0^t \| \partial_t \nabla \phi \|_{H^{2,0}}^2 + \leq \frac{c}{2} \varepsilon^6. \]

By Sobolev embedding theorem and Lemma 4.1, we get the Theorem 1.1 proved.

6. The proof of Proposition 5.1

All we left is the Proposition 5.1. To prove that, we firstly give the following decomposition of \( \phi \):

\[ \phi = \phi_{\text{slip}} + \phi_{\text{bc}}, \]

\[ (6.1) \]
where $\phi_{\text{slip}}$ satisfies that
\[
\begin{cases}
(\partial_t - \Delta)\Delta \phi_{\text{slip}} + u^p \partial_x \Delta \phi_{\text{slip}} + v^p \partial_y \Delta \phi_{\text{slip}} + \partial_y \phi_{\text{slip}} \partial_x \omega^p - \partial_x \phi_{\text{slip}} \partial_y \omega^p = \partial_y N_u - \varepsilon^2 \partial_x N_v + \varepsilon^2 f_1 + f_2 - C(t) \partial_x \omega^p, \\
\phi_{\text{slip}}|_{y=0,1} = 0, \\
\phi_{\text{slip}}|_{t=0} = 0,
\end{cases}
\]
(6.2)
and $\phi_{bc}$ satisfies that
\[
\begin{cases}
(\partial_t - \Delta)\Delta \phi_{bc} + u^p \partial_x \Delta \phi_{bc} + v^p \partial_y \Delta \phi_{bc} + \partial_y \phi_{bc} \partial_x \omega^p - \partial_x \phi_{bc} \partial_y \omega^p = 0, \\
\phi_{bc}|_{y=0,1} = 0, \\
\phi_{bc}|_{t=0} = 0.
\end{cases}
\]
(6.3)

To prove Proposition 5.1, we need the estimates of $\phi_{\text{slip}}$ and $\phi_{bc}$. First, we notice that $\phi_{\text{slip}}$ has a good boundary condition. We use "hydrostatic trick" method to get its estimates. The proof of following proposition is given in section 7.

**Proposition 6.1.** There exists $\lambda_0 > 1$ and $0 < T < \min\{T_p, \frac{1}{2\lambda}\}$ such that for all $t \in [0, T]$, $\lambda \geq \lambda_0$, there holds that
\[
\|\Delta \phi_{\text{slip}}\|_{L^2}^2 + \lambda \int_0^t \left(\|\Delta \phi_{\text{slip}}\|_{X^2}^2 + \|\nabla \varepsilon \phi_{\text{slip}}\|_{X^2}^2 + |\nabla \varepsilon \phi_{\text{slip}}|_{y=0,1}\|_{X^2}^2\right) ds + \int_0^t \|\nabla \varepsilon \phi_{\text{slip}}\|_{X^2}^2 ds 
\leq C \int_0^t \|\nabla \varepsilon \phi_{bc}\|_{X^2}^2 ds + \frac{C}{\lambda^2} \int_0^t \|\nabla \varepsilon \phi_{bc}\|_{X^2}^2 ds.
\]

The estimates of $\phi_{bc}$ is much more difficult. Here, we state the main results on it:

**Proposition 6.2.** There exists $\lambda_0 > 1$ and $0 < T < \min\{T_p, \frac{1}{2\lambda}\}$ such that for all $t \in [0, T]$, $\lambda \geq \lambda_0$, there holds that
\[
\int_0^t \|\nabla \varepsilon \phi_{bc}\|_{X^2}^2 ds + \int_0^t \|\varphi \Delta \phi_{bc}\|_{X^2}^2 ds 
\leq C \int_0^t \left(\|\nabla \varepsilon \phi_{\text{slip}}|_{y=0,1}\|_{X^2}^2 + |C(s)|^2\right) ds,
\]
where $C$ is a universal constant.

The proof of Proposition 6.2 is given in section 8.

Based on the above two propositions, we are in the position to prove Proposition 5.1. Firstly, we give the estimates of $\|u^R\|_{L^2}$ which is used to control the $C(t)$.

**Lemma 6.3.** There exist $0 < T < \min\{T_p, \frac{1}{2\lambda}\}$ and $\lambda_0 \geq 1$ such that for $t \in [0, T]$ and $\lambda \geq \lambda_0$, it holds that
\[
\int_0^t \left(\|\nabla \varepsilon \phi_{bc}\|_{X^2}^2 + \|\varphi \Delta \phi_{bc}\|_{X^2}^2\right) ds \leq C \frac{1}{\lambda^2} \int_0^t \left(\|\nabla \varepsilon \phi_{\text{slip}}|_{y=0,1}\|_{X^2}^2 + |C(s)|^2\right) ds,
\]
(6.4)
where $C$ is a universal constant.

**Remark 6.4.** We use weighted quantity $\|\varepsilon(1-\lambda)^{\frac{1}{2}}(u^R, \varepsilon v^R)\|_{L^2}$ instead of $\|u^R, \varepsilon v^R)\|_{L^2}$ to obtain small constant factor in front of $\int_0^t \|\nabla \varepsilon \phi\|_{X^2}^2 ds$ in (6.5).

**Proof.** Taking $L^2$ inner product with $\varepsilon(1-\lambda)^{\frac{1}{2}}u^R$ in the first equation of (3.7) and with $\varepsilon(1-\lambda)^{\frac{1}{2}}v^R$ in the second equation of (3.7), we use the fact
\[
\partial_t(\varepsilon(1-\lambda)^{\frac{1}{2}} f) = \varepsilon(1-\lambda)^{\frac{1}{2}} \partial_t f + 2\lambda \varepsilon(1-\lambda)^{\frac{1}{2}} f
\]
and integrate by parts by boundary condition \((u^R, v^R)[y=0,1] = 0\) to yield
\[
\frac{1}{2} \frac{d}{dt} \|e^{(1-\lambda t)}(u^R, v^R)\|^2_{L^2} + \lambda \|e^{(1-\lambda t)}(u^R, v^R)\|^2_{L^2} + \|e^{(1-\lambda t)}\nabla \epsilon(u^R, v^R)\|^2_{L^2} \\
\leq C(\|e^{(1-\lambda t)}(u^R, v^R)\|^2_{L^2} + \|e^{(1-\lambda t)}(N_u, \epsilon N_v)\|^2_{L^2} + \|\langle R_1, R_2 \rangle\|^2_{L^2}) \|e^{(1-\lambda t)}(u^R, v^R)\|^2_{L^2} \\
\leq \frac{\lambda}{2} \|e^{(1-\lambda t)}(u^R, v^R)\|^2_{L^2} + C(\|e^{(1-\lambda t)}(N_u, \epsilon N_v)\|^2_{L^2} + \epsilon^8) + \frac{C}{\lambda} \|e^{(1-\lambda t)}\partial_x u^R\|^2_{L^2},
\]
where we write \(v^R = -\int_0^y \partial_x u^R dy'\) and use the fact \(\partial_x u^p + \partial_y v^p = 0\) to eliminate transport term and \(\partial_x u^R + \partial_y v^R = 0\) to eliminate pressure term respectively.

Afterwards, integrating time from 0 to \(t\) and using \(\partial_x u^R = -\partial_x \partial_y \phi\), we obtain
\[
\|e^{(1-\lambda t)}(u^R, v^R)(t)\|^2_{L^2} + \lambda \int_0^t \|e^{(1-\lambda s)}(u^R, v^R)\|^2_{L^2} ds + \int_0^t \|e^{(1-\lambda s)}\nabla \epsilon(u^R, v^R)\|^2_{L^2} ds \\
\leq C \int_0^t (\|e^{(1-\lambda s)}(N_u, \epsilon N_v)\|^2_{L^2} + \epsilon^8) ds + \frac{C}{\lambda} \int_0^t \|e^{(1-\lambda s)}\partial_x \partial_y \phi\|^2_{L^2} ds.
\]
Finally, we use \(\|e^{(1-\lambda t)}\partial_x \partial_y \phi\|^2_{L^2} \leq C\|\nabla \epsilon \phi\|^2_{X^2}\) to complete the proof. 

\[\square\]

**Proof of Proposition [5.1]**: Now, we give the proof Proposition [5.1]. We divide this proof into two parts.

**Estimates of \(\int_0^t \|\nabla \phi\|^2_{X^2}\)**. Since
\[
|C(t)| = \left| \frac{1}{2\pi} \int \int_S u^R dx dy \right| \leq C \|u^R\|^2_{L^2} \leq C \|e^{(1-\lambda t)}u^R\|^2_{L^2},
\]
by Lemma [5.3] to ensure
\[
\|C(t)\|^2 \leq C \int_0^t (\|e^{(1-\lambda s)}(N_u, \epsilon N_v)\|^2_{L^2} + \epsilon^8) ds + \frac{C}{\lambda} \int_0^t \|\nabla \epsilon \phi\|^2_{X^2} ds.
\]
By the definition of \(f_1\) and \(f_2\), we obtain that
\[
\int_0^t \|f_1\|^2_{X^\frac{3}{2}} + \|f_2\|^2_{X^\frac{3}{2}} ds \leq C \int_0^t (\|\epsilon \Delta \phi\|^2_{X^\frac{3}{2}} + \epsilon^8) ds,
\]
we get
\[
\lambda \int_0^t (\|\Delta \phi_{slip}\|^2_{X^\frac{3}{2}} + \|\nabla \phi_{slip}\|^2_{X^\frac{3}{2}} + |\nabla \phi_{slip}|_{y=0,1}|^2_{X^\frac{3}{2}}) ds \\
\leq C \int_0^t (\|N_u, \epsilon N_v\|^2_{X^2} ds + \frac{C}{\lambda^2} (|C(t)|^2 + \int_0^t (\|\epsilon \Delta \phi\|^2_{X^\frac{3}{2}} + \epsilon^8) ds).
\]
Then, it follows \(\phi = \phi_{slip} + \phi_{bc}\) and \([0.4]\) to deduce
\[
\int_0^t \|\nabla \phi\|^2_{X^2} ds \leq \int_0^t \|\nabla \phi_{slip}\|^2_{X^2} ds + \int_0^t \|\nabla \phi_{bc}\|^2_{X^2} ds \\
\leq \frac{C}{\lambda^2} \int_0^t (\|N_u, \epsilon N_v\|^2_{X^2} ds + \frac{C}{\lambda^2} (\int_0^t \|\epsilon \Delta \phi\|^2_{X^\frac{3}{2}} + \epsilon^8 ds) \\
+ \frac{C}{\lambda^2} \int_0^t |\nabla \phi_{slip}|_{y=0,1}|^2_{X^\frac{3}{2}} ds + \frac{C}{\lambda^2} |C(t)|^2, \\
\leq \frac{C}{\lambda^2} |C(t)|^2 + \frac{C}{\lambda} \int_0^t (\|N_u, \epsilon N_v\|^2_{X^2} ds + \frac{C}{\lambda^2} \int_0^t (\|\epsilon \Delta \phi\|^2_{X^\frac{3}{2}} + \epsilon^8) ds.
\]
Hence, we use the "hydrostatic trick" to get the desired results. Firstly, acting operator \(\geq\) (6.9) and above estimates together and taking \(\lambda\) large enough to get
\[
(6.9) \quad |C(t)|^2 + \int_0^t \|\nabla \phi\|^2_{X^2} ds \leq C \int_0^t \|\langle N_u, \varepsilon N_v \rangle\|^2_{X^2} ds + C \int_0^t (\|\varepsilon \Delta \phi\|^2_{X^{3/2}} + \varepsilon^8) ds.
\]

Estimates of \(\int_0^t \|\varphi \Delta \phi\|^2_{X^2} ds\).
It follows from (6.7) and (6.9) that
\[
\int_0^t \|\Delta \phi\|_{X^2}^2 ds \leq \frac{C}{\lambda} \int_0^t \|\langle N_u, \varepsilon N_v \rangle\|^2_{X^2} ds + \frac{C}{\lambda} \left( (C(t))^2 + \int_0^t (\|\varepsilon \Delta \phi\|^2_{X^{3/2}} + \varepsilon^8) ds \right)
\]
\[
\leq \frac{C}{\lambda} \int_0^t \|\langle N_u, \varepsilon N_v \rangle\|^2_{X^2} + \|\varepsilon \Delta \phi\|^2_{X^{3/2}} + \varepsilon^8) ds.
\]

Applying Proposition 6.2 again, we get
\[
\int_0^t \|\varphi \Delta \phi\|_{X^2}^2 ds \leq C \int_0^t \|\langle N_u, \varepsilon N_v \rangle\|^2_{X^2} + \|\varepsilon \Delta \phi\|^2_{X^{3/2}} + \varepsilon^8) ds.
\]

Combining above two estimates, we get
\[
\int_0^t \|\varphi \Delta \phi\|^2_{X^2} ds \leq C \int_0^t \|\langle N_u, \varepsilon N_v \rangle\|^2_{X^2} + \|\varepsilon \Delta \phi\|^2_{X^{3/2}} + \varepsilon^8) ds.
\]

By now, we get the desired results.

7. Vorticity estimates under artificial boundary condition: Proof of Proposition 6.1

In the section, we give the proof of Proposition 6.1. To simplify the notation, we drop the subscript in the system (6.2):
\[
(\partial_t - \Delta \varepsilon) \Delta \varepsilon \phi + u^p \partial_x \Delta \varepsilon \phi + v^p \partial_y \Delta \varepsilon \phi + \partial_y \phi \partial_x \omega^p - \partial_x \phi \partial_y \omega^p = \partial_y N_u - \varepsilon^2 \partial_x N_v + \varepsilon^2 f_1 + f_2 - C(t) \partial_x \omega^p,
\]
\[
\phi|_{y=0,1} = 0, \quad \Delta \varepsilon \phi|_{y=0,1} = 0, \quad \phi|_{t=0} = 0.
\]

The goal in this section is to establish uniform (in \(\varepsilon\)) estimate of vorticity \(\omega = \Delta \varepsilon \phi\).

**Proposition 7.1.** There exists \(\lambda_0 > 0\) and \(0 < T < \min\{T_p, \frac{1}{2\lambda}\}\) such that for all \(t \in [0, T]\), \(\lambda \geq \lambda_0\), the following holds that
\[
\|\omega(t)\|^2_{X^2} + \lambda \int_0^t (\|\omega\|^2_{X^{3/2}} + \|\Delta \phi\|^2_{X^{3/2}} + \|\nabla \phi\|^2_{X^{3/2}}) ds + \int_0^t \|\nabla \omega\|^2_{X^2} ds \leq C \int_0^t \|\langle N_u, \varepsilon N_v \rangle\|^2_{X^2} ds + \frac{C}{\lambda} \int_0^t (\|f_1 + f_2 + C(t)\|^2_{X^{3/2}}) ds.
\]

**Proof.** By Lemma 4.1, we have \(\partial_y \omega^p \geq c_0 > 0\).

Hence, we use the "hydrostatic trick" to get the desired results. Firstly, acting operator \(e^{\Phi(t, D_x)}\) on the first equation of (6.2) to get
\[
(\partial_t + \lambda (D_x)^{\frac{3}{2}} - \Delta \varepsilon) \omega \phi^p + u^p \partial_x \omega \phi^p + v^p \partial_y \omega \phi^p - \partial_x \phi \partial_y \omega^p = - (\partial_y \phi \partial_x \omega^p) \phi - [e^{\Phi(t, D_x)}, u^p \partial_x] \omega - [e^{\Phi(t, D_x)}, v^p \partial_y] \omega,
\]
By using Lemma 2.1, we get it is easy to see product with \( \langle \nabla_x \rangle \),

In view of (7.2), the terrible term comes from \( \partial_x \phi \partial_y \omega^p \), which lose one tangential derivative. In order to overcome the derivative loss, we take \( \langle D_x \rangle^2 \) on the (7.2) and then take \( L^2 \) inner product with \( \langle D_x \rangle^2 \omega^p \) to obtain that

\[
\frac{1}{2} \frac{d}{dt} \| \langle D_x \rangle^2 \omega^p \|_{L^2}^2 + \lambda \| \langle D_x \rangle^2 \omega^p \|_{L^2}^2 + \| \nabla_x \langle D_x \rangle^2 \omega^p \|_{L^2}^2 = -\int_S \langle D_x \rangle^2 \omega^p \cdot (\varepsilon \partial_x, \partial_y) \frac{1}{\partial_y \omega^p} \cdot (\varepsilon \partial_x, \partial_y) \langle D_x \rangle^2 \omega^p dxdy
\]

\[
+ \int_S \| \langle D_x \rangle^2 \omega^p \|^2 (\partial_x (\frac{u^p}{\partial_y \omega^p}) + \partial_y (\frac{v^p}{\partial_y \omega^p})) dxdy - \int_S \| \langle D_x \rangle^2, u^p \partial_x + v^p \partial_y \| \omega^p \langle D_x \rangle^2 \omega^p dxdy
\]

\[
- \int_S \langle D_x \rangle^2 (\partial_y \phi \partial_x \omega^p) \omega^p dxdy + \int_S \langle D_x \rangle^2, \partial_y \omega^p \partial_x \phi \omega^p dxdy
\]

\[
+ \int_S \langle D_x \rangle^2 (\partial_y (\nabla_x \phi) - \varepsilon^2 \partial_x (\nabla_y \phi) \omega^p dxdy
\]

\[
+ \int_S \langle D_x \rangle^2 (\varepsilon^2 f_1 + f_2 - C(t) \partial_x \omega^p) \omega^p dxdy
\]

\[= T^0 + \ldots + T^0 \]

The boundary term is zero due to artificial boundary condition \( \omega \big|_{y=0,1} = \Delta_x \phi \big|_{y=0,1} = 0 \).

Integrating on \( [0, t] \) with \( t \leq T \) and using \( \partial_y \omega^p \geq c_0 \), we obtain

\[
\| \omega(t) \|^2_{X^2} + 2\lambda \int_0^t \| \omega \|^2_{X^2} ds + 2 \int_0^t \| \nabla \varepsilon \omega \|^2_{X^2} ds \leq C \int_0^t |T^0| + \ldots + |T^1| ds.
\]

Now, we estimate \( T^i, i = 0, \ldots, 10 \) one by one.

**Estimate of \( T^0 \) and \( T^1 \).** Since \( \partial_y \omega^p \geq c_0 > 0 \) and Lemma 4.1 imply

\[
\| \varepsilon \partial_x, \partial_y \| \geq c_0 \geq c_1 \| \varepsilon \partial_x, \partial_y \|,
\]

\[
| \partial_x (\frac{u^p}{\partial_y \omega^p}) | + | \partial_y (\frac{v^p}{\partial_y \omega^p}) | \leq C,
\]

it is easy to see

\[
|T^0| + |T^1| \leq C \| \omega \|_{X^2} (\| \nabla \varepsilon \omega \|_{X^2} + \| \omega \|_{X^2}).
\]

**Estimate of \( T^2 \) and \( T^4 \).** By using Lemma 2.1, we get

\[
\| \langle D_x \rangle^2, u^p \partial_x + v^p \partial_y \|_{L^2} \leq C \| \omega \|_{X^2} + \| \partial_y \omega \|_{X^2},
\]

\[
\| \langle D_x \rangle^2, \partial_y \omega^p \|_{L^2} \leq C \| \omega \|_{X^2} \leq C \| \partial_y \omega \|_{X^2},
\]

where we used Poincaré inequality and \( \phi \big|_{y=0,1} = 0 \) to ensure

\[
(7.3) \quad \| \phi \|_{X^r} \leq C \| \partial_y \phi \|_{X^r}, \quad r \geq 0.
\]
in the last step.

According to
\[ \Delta_\varepsilon \phi = \omega, \quad \phi|_{y=0,1} = 0, \]
classical elliptic estimate and (7.3) imply
\[ \| \nabla \varepsilon \phi \|_{L^2}^2 \leq \| \omega \|_{L^2} \| \phi \|_{L^2} \leq C \| \omega \|_{L^2} \| \partial_y \phi \|_{L^2}, \]
which gives
\[ \| \nabla \varepsilon \phi \|_{X^r} \leq \| \omega \|_{X^r}, \quad r \geq 0. \]
Therefore, it follows from \( \partial_y \omega^p \geq c_0 > 0 \) to get
\[ |T^2| + |T^4| \leq C(\| \partial_y \omega \|_{X^2} + \| \omega \|_{X^2}) \| \omega \|_{X^2}. \]

**Estimate of \( T_3 \).** Using Lemma 2.2 and (7.5), it shows
\[ |T^3| \leq C \| \partial_y \phi \|_{X^2} \| \omega \|_{X^2} \leq C \| \omega \|_{X^2}^2. \]

**Estimate of \( T^5 \).** This term is the trouble term because it loses one tangential derivative. However, hydrostatic trick implies
\[
T^5 = \int_S \langle D_\varepsilon \rangle^2 \partial_x \phi \langle D_\varepsilon \rangle^2 \Delta_\varepsilon \phi \omega dx dy = -\int_S \langle D_\varepsilon \rangle^2 \partial_x \nabla_\varepsilon \phi \langle D_\varepsilon \rangle^2 \nabla_\varepsilon \phi \omega dx dy
\]
\[ = -\frac{1}{2} \int_S \partial_x |\langle D_\varepsilon \rangle^2 \nabla_\varepsilon \phi|^2 dx dy = 0, \]
by using \( \phi|_{y=0,1} = 0 \).

**Estimate of \( T^6, T^7 \) and \( T^8 \).** Let’s estimate commutators by Lemma 2.3. Since \( \partial_y \omega^p \geq c_0 > 0 \), we use Lemma 2.3 to ensure that
\[ |T^6| \leq C(\| u^p \partial_x \omega \|_{X^2} + \| u^p \partial_x \omega \|_{H^{z,0}}) \| \omega \|_{X^2} \leq C \| \omega \|_{X^2} \| \omega \|_{X^2} = C \| \omega \|_{X^2}^2, \]
\[ |T^7| \leq C(\| \partial_y \omega \|_{X^2} \| \omega \|_{X^2}, \]
\[ |T^8| \leq C(\| \partial_x \phi \partial_y \omega^p \|_{H^{z,0}} + \| \partial_y \phi \partial_y \omega^p \|_{H^{z,0}}) \| \omega \|_{X^2} \leq C \| \partial_x \phi \|_{X^2} \| \omega \|_{X^2} \leq C \| \partial_y \omega \|_{X^2} \| \omega \|_{X^2} \leq C \| \omega \|_{X^2}^2. \]

Here we use (7.3) and (7.5) in the last estimate.

**Estimate of \( T^9 \) and \( T^{10} \).** Integration by parts and boundary condition \( \omega|_{y=0,1} = 0 \) give that
\[ |T^9| = \int_S \langle D_\varepsilon \rangle^2 (N_{u^p}, \varepsilon \omega \phi) \cdot \nabla_\varepsilon \left( \frac{\langle D_\varepsilon \rangle^2 \omega \phi}{\partial_y \omega^p} \right) dx dy \leq C \| N_{u^p}, \varepsilon \omega \|_{X^2} (\| \omega \|_{X^2} + \| \nabla_\varepsilon \omega \|_{X^2}). \]

On the other hand, using H"older inequality, we get
\[ |T^{10}| \leq C \| \varepsilon^2 f_1 + f_2 - C(t) \partial_x \omega^p \|_{X^2} \| \omega \|_{X^2}. \]

Collecting \( T^0 - T^{10} \) together, we finally obtain
\[ \int_0^t |T^0| + \ldots + |T^{10}| ds \leq C \int_0^t \| N_{u^p}, \varepsilon \omega \|_{X^2} (\| \nabla_\varepsilon \omega \|_{X^2} + \| \omega \|_{X^2}) \]
\[ + \| \omega \|_{X^2}^2 + \| \nabla_\varepsilon \omega \|_{X^2}^2 + \| \varepsilon^2 f_1 + f_2 - C(t) \partial_x \omega^p \|_{X^2} ds \leq \frac{1}{10} \int_0^t \| \nabla_\varepsilon \omega \|_{X^2}^2 ds + C \int_0^t \| N_{u^p}, \varepsilon \omega \|_{X^2}^2 ds. \]
+ (C + \frac{\lambda}{4}) \int_0^t \|\omega\|_{X_{\frac{3}{2}}}^2 ds + \frac{C}{\lambda} \int_0^t \|\varepsilon^2 f_1 + f_2 - C(t)\partial_2 \varphi^p\|_{X_{\frac{3}{2}}}^2 ds.

Taking \lambda large enough, we deduce

(7.6) \quad \|\omega(t)\|_{X_{\frac{3}{2}}}^2 + \lambda \int_0^t \|\omega\|_{X_{\frac{3}{2}}}^2 ds + \int_0^t \|\nabla_x \omega\|_{X_{\frac{3}{2}}}^2 ds 
\leq C \int_0^t (\|N_x, \varepsilon N_v\|_{X_{\frac{3}{2}}}^2 + \frac{C}{\lambda} \int_0^t \|\varepsilon^2 f_1 + f_2 - C(t)\partial_2 \varphi^p\|_{X_{\frac{3}{2}}}^2 ds.

On the other hand, (7.5) gives

\|\nabla_x \varphi\|_{X_{\frac{3}{2}}} \leq C \|\omega\|_{X_{\frac{3}{2}}}.

Calderon-Zygmund inequality and Gagliardo-Nirenberg inequality (4.9) imply

\|\nabla_x \varphi\|_{X_{\frac{3}{2}}} \leq C \|\nabla_x \varphi\|_{X_{\frac{3}{2}}}^\frac{1}{2} (\|\nabla_x \varphi\|_{X_{\frac{3}{2}}}^\frac{1}{2} + \|\nabla_x \partial_1 \varphi\|_{X_{\frac{3}{2}}}^\frac{1}{2} \leq C \|\omega\|_{X_{\frac{3}{2}}}.

Along with (7.5) and (7.6), we get the desired result.

\[\square\]

8. Construction of the boundary corrector: Proof of Proposition 6.2

In the previous section, we construct a solution to the Orr-Sommerfeld equation with artificial boundary conditions: we replace condition \partial_y \varphi|_{y=0,1} = 0 by \Delta_x \varphi|_{y=0,1} = 0. To go back to the original system, we need to correct Neumann condition. Thus, we define \phi_{bc} satisfies the following system:

\[
\begin{cases}
(\partial_t - \Delta_x) \Delta_x \phi_{bc} + v^p \partial_x \Delta_x \phi_{bc} + v^p \partial_y \Delta_x \phi_{bc} + \partial_y \phi_{bc} \partial_x \varphi^p - \partial_x \phi_{bc} \partial_y \varphi^p = 0, \\
\phi_{bc}|_{y=0,1} = 0, \quad \partial_y \phi_{bc}|_{y=0,1} = -\partial_y \phi_{slip}|_{y=0,1} + C(t), \\
\phi|_{t=0} = 0,
\end{cases}
\]

(8.1)

To estimate \phi_{bc}, we use the following decomposition:

\[
\phi_{bc} = \phi_{bc,S} + \phi_{bc,T} + \phi_{bc,R},
\]

The definitions and estimates of \phi_{bc,S}, \phi_{bc,T} and \phi_{bc,R} are given in the following subsections.

8.1. The estimates of \phi_{bc,S}: Stokes equation. In this subsection, we deal with \phi_{bc,S}.

Because of two boundary \( y = 0 \) and \( y = 1 \), we define

\[
\phi_{bc,S} = \phi_{bc,S}^0 + \phi_{bc,S}^1,
\]

where \( \phi_{bc,S}^0 \) satisfies the following Stokes equation:

\[
\begin{cases}
(\partial_t - \Delta_x) \Delta_x \phi_{bc,S}^0 = 0, \quad (x, y) \in \mathbb{T} \times (0, +\infty) \\
\phi_{bc,S}^0|_{y=0} = 0, \quad \partial_y \phi_{bc,S}^0|_{y=0} = h^0, \\
\phi_{bc,S}^0|_{t=0} = 0,
\end{cases}
\]

(8.2)

and \( \phi_{bc,S}^1 \) satisfies the following Stokes equation

\[
\begin{cases}
(\partial_t - \Delta_x) \Delta_x \phi_{bc,S}^1 = 0, \quad (x, y) \in \mathbb{T} \times (-\infty, 1) \\
\phi_{bc,S}^1|_{y=1} = 0, \quad \partial_y \phi_{bc,S}^1|_{y=1} = h^1, \\
\phi_{bc,S}^1|_{t=0} = 0,
\end{cases}
\]

(8.3)
where \( t \in [0, T] \). Here \((h^0, h^1)\) is a given boundary data satisfying \((h^0(t), h^1(t)) = 0\) for \( t = 0 \) and \( t \geq T \). Here, we point out that \( h^i \) is defined by

\[
h^i = A(-\partial_y \phi_{\text{slip}}|_{y=0,1} + C(t)),
\]

where the operator \( A \) is a zero-order operator which is defined later.

In the following, we only give the process for \( \phi^0_{bc,S} \). The case of \( \phi^1_{bc,S} \) is almost the same and we leave details to readers.

At first, we give zero extension of \( \phi^0_{bc,S} \) and \( h^0 \) with \( t \leq 0 \) such that we can take Fourier transform in \( t \). Let \( \hat{\phi}^0_{bc,S} = \hat{\phi}^0_{bc,S}(\zeta, k, y) \) be the Fourier transform of \( \phi^0_{bc,S} \) on \( x \) and \( t \). Then \( (\hat{\phi}^0_{bc,S})_\Phi \) satisfies the ODE:

\[
\begin{align*}
- (\partial_y^2 - \varepsilon^2 |k|^2)(\hat{\phi}^0_{bc,S} )_\Phi + (i \zeta + \lambda(k) \frac{\varepsilon}{k}) (\hat{\phi}^0_{bc,S} )_\Phi &= 0, \\ (\hat{\phi}^0_{bc,S})_\Phi |_{y=0} &= 0, \\ \hat{\partial}_y (\hat{\phi}^0_{bc,S})_\Phi |_{y=0} &= \hat{h}^0_\Phi,
\end{align*}
\]

(8.4)

where \( \zeta \in \mathbb{R} \) and \( k \in \mathbb{Z} \). Assuming the decay of \((|k|\hat{\phi}^0_{bc,S}, \partial_y \hat{\phi}^0_{bc,S})\) and the boundedness of \( \partial_y \hat{\phi}^0_{bc,S} \), we obtain the formula:

\[
(\hat{\phi}^0_{bc,S})_\Phi(\zeta, k, y) = -e^{-\gamma y} e^{-\varepsilon |k| y} \frac{1}{\gamma - \varepsilon |k|} \hat{h}^0_\Phi(\zeta, k), \quad y > 0
\]

(8.5)

\[
\gamma = \gamma(\zeta, k, \varepsilon, \lambda) = \sqrt{\varepsilon^2 |k|^2 + \lambda(k) \frac{\varepsilon}{k} + i \zeta},
\]

(8.6)

where the square root is taken so that the real part is positive, and it follows that

\[
\varepsilon |k|, \frac{1}{2} \lambda(k)^{\frac{1}{2}} \leq \sqrt{\varepsilon^2 |k|^2 + \lambda(k) \frac{\varepsilon}{k} + i \zeta} \leq \text{Re}(\gamma) \leq |\gamma| \leq 2\text{Re}(\gamma).
\]

(8.7)

This inequality will be used frequently. It is easy to calculate that

\[
\partial_y (\hat{\phi}^0_{bc,S})_\Phi = -e^{-\gamma y} \hat{h}^0_\Phi - \varepsilon |k|(\hat{\phi}^0_{bc,S})_\Phi,
\]

(8.8)

\[
(\partial_y^2 - \varepsilon^2 |k|^2)(\hat{\phi}^0_{bc,S})_\Phi = (\gamma + \varepsilon |k|)e^{-\gamma y} \hat{h}^0_\Phi.
\]

(8.9)

The formula (8.8) will be used in estimating velocity and (8.9) will be used in estimating vorticity. With the same process above, we get the formula for \( (\hat{\phi}^1_{bc,S})_\Phi \):

\[
(\hat{\phi}^1_{bc,S})_\Phi(\zeta, k, y) = \frac{e^{-\gamma (1-y)} - e^{-\varepsilon |k|(1-y)}}{\gamma - \varepsilon |k|} \hat{h}^1_\Phi(\zeta, k), \quad y < 1,
\]

(8.10)

with \( \gamma \) given in (8.6). It is easy to see

\[
\begin{align*}
\partial_y (\hat{\phi}^1_{bc,S})_\Phi &= e^{-\gamma (1-y)} \hat{h}^1_\Phi + \varepsilon |k|(\hat{\phi}^1_{bc,S})_\Phi, \\ (\partial_y^2 - \varepsilon^2 |k|^2)(\hat{\phi}^1_{bc,S})_\Phi &= - (\gamma + \varepsilon |k|)e^{-\gamma (1-y)} \hat{h}^1_\Phi.
\end{align*}
\]

(8.11)

(8.12)

Remark 8.1. For \( \varepsilon = 0 \) in (8.2), \( \Delta_0 = \partial_y^2 \).

\[
(\hat{\phi}^0_{bc,S})_\Phi(\zeta, k, y) = \frac{\hat{h}^0_\Phi(\gamma_0 y)}{\gamma_0}(e^{-\gamma_0 y} - 1), \quad \gamma_0 = \sqrt{\lambda(k)^{\frac{1}{2}} + i \zeta}
\]

(8.13)

solves (8.4) with \( \varepsilon = 0 \) and \( (\hat{\phi}^0_{bc,S})_\Phi \) holds \( \lim_{y \to +\infty} = \frac{\hat{h}^0_\Phi}{\gamma_0} \). Though \( (\hat{\phi}^0_{bc,S})_\Phi \) don’t tend to zero as \( y \) tends to infinity, the solution \( (\hat{\phi}^0_{bc,S})_\Phi \) is only used to correct boundary condition near
we find these two terms are decay to zero as $y$ tends to infinity. By the same method, we can get another solution near $y = 1$:

$$
(\phi_{bc,S}^i)_{\Phi}(\zeta, k, y) = \frac{h_{\Phi}^0}{\gamma_0}(e^{-\gamma_0(1-y)} - 1).
$$

These constructions are main difference between $\varepsilon = 0$ and $\varepsilon \neq 0$, but they enjoy the same properties stated below.

**Lemma 8.2.** Let $\phi_{bc,S}^i$ be solution of (8.2). It holds that

$$
(8.16) \quad \sum_{k \in \mathbb{Z}} \|\varepsilon|k|(\phi_{bc,S}^i)_{\Phi}, \partial_y (\phi_{bc,S}^i)_{\Phi}\|_{L_y^2}^2 \leq \frac{C}{\lambda^{\frac{2}{z}}} \sum_{k \in \mathbb{Z}} \|k(\phi_{bc,S}^i)_{\Phi}\|_{L^2}^2,
$$

where $i = 0, 1$ and $L_y^2 \zeta \cdot y = l_y^2(L_y(0, +\infty))$ for $i = 0$ and $L_y^2 \zeta \cdot y = l_y^2(L_y(-\infty, 1))$ for $i = 1$.

It is also holds that

$$
(8.17) \quad \sum_{k \in \mathbb{Z}} \|k(\phi_{bc,S}^i)_{\Phi}\|_{L_y^2}^2 \leq \frac{C}{\lambda^{\frac{2}{z}}} \|k(\phi_{bc,S}^i)_{\Phi}\|_{L^2}^2,
$$

where $i = 0, 1$ and $L_y^2 \zeta \cdot y = l_y^2(L_y(0, 1))$.

**Proof.** We only give the proof for $i = 0$. The case $i = 1$ is almost the same and we omit details to readers.

(8.16) follows from (8.5), (8.8) and the Plancherel theorem, by observing the estimate for multipliers

$$
(8.18) \quad \|e^{-\text{Re}(\gamma)y}\|_{L_y^2(0, +\infty)} \leq \frac{C}{\lambda^{\frac{2}{z}}(k)^{\frac{2}{z}}},
$$

$$
(8.19) \quad \|\varepsilon|k| \cdot e^{-\varepsilon|k|y} \cdot \frac{1 - e^{-(\gamma - \varepsilon|k|)y}}{\gamma - \varepsilon|k|}\|_{L_y^2(0, +\infty)} \leq \frac{C}{\lambda^{\frac{2}{z}}(k)^{\frac{2}{z}}}.
$$

The estimate (8.18) is a direct consequence of

$$
(8.20) \quad \text{Re}(\gamma) \geq \frac{1}{\lambda^{\frac{2}{z}}(k)^{\frac{2}{z}}}.
$$

For (8.19), we divide it into two cases: 1. $\varepsilon|k| \leq \frac{1}{2}\lambda^{\frac{2}{z}}(k)^{\frac{1}{z}}$, and 2. $\varepsilon|k| \geq \frac{1}{2}\lambda^{\frac{2}{z}}(k)^{\frac{1}{z}}$. In case 1,

$$
|\gamma - \varepsilon|k|\| \geq \frac{\varepsilon|k| + \lambda^{\frac{2}{z}}(k)^{\frac{1}{z}}}{C},
$$

which implies

$$
\|\varepsilon|k| \cdot e^{-\varepsilon|k|y} \cdot \frac{1 - e^{-(\gamma - \varepsilon|k|)y}}{\gamma - \varepsilon|k|}\|_{L_y^2(0, +\infty)} \leq \frac{C}{\lambda^{\frac{2}{z}}(k)^{\frac{2}{z}}}(\varepsilon|k|)^{\frac{2}{z}} \|\varepsilon|k| \cdot e^{-\varepsilon|k|y}\|_{L_y^2(0, +\infty)} \leq \frac{C}{\lambda^{\frac{2}{z}}(k)^{\frac{2}{z}}}.
$$

In case 2, we use the bound

$$
\left|\frac{1 - e^{-z}}{z}\right| \leq C,
$$
for \( \text{Re}(z) > 0 \), which implies that

\[
\| \| | k | \cdot e^{- \varepsilon | k | y } \cdot \left| \frac{1 - e^{- (\gamma - \varepsilon | k |) y}}{\gamma - \varepsilon | k |} \right| \| L^2 (0, +\infty) \| \leq \| y \| e^{- \varepsilon | k | y } \| L^2 (0, +\infty) \| \leq \frac{C}{(\varepsilon | k |)^{\frac{3}{4}}} \leq \frac{C}{\lambda^\frac{3}{4} (k)^{\frac{3}{4}}}.
\]

Combining case 1-2 together, we complete (8.19), which yields (8.16). The estimate (8.17) is proved by using (8.16) and (8.17).

\[\text{Proof.} \quad \text{The proof is done by using (8.16) and (8.17).}\]

Next, we give the estimate to boundary term \( \phi^0_{bc,S} \) and \( \phi^1_{bc,S} \).

\[\text{Lemma 8.4.} \quad \text{For any } M \geq 0 \text{ and } i = 0, 1, \text{ it holds that}\]

\[\int_0^t | (\varepsilon \partial_x)^M \phi^i_{bc,S} | y = 1 - i \| X^{\frac{3}{4} + \frac{1}{4} } d s \leq \frac{C}{M} \int_0^t | h^i | X^{\frac{3}{4} } d s, \]

\[\text{and}\]

\[\int_0^t | (\varepsilon \partial_x)^M \partial_y \phi^i_{bc,S} | y = 1 - i \| X^{\frac{3}{4} + \frac{1}{4} } d s \leq \frac{C}{M} \int_0^t | h^i | X^{\frac{3}{4} } d s, \]

for any \( r \geq 0 \).
Proof. We only give the proof for the case \( i = 0 \), the case \( i = 1 \) is similar and we omit details to readers. Taking \( y = 1 \) in (8.35) and using

\[
(8.28) \quad \left| (\varepsilon|k|)^M \cdot e^{-\varepsilon|k|} \cdot \frac{e^{-(\gamma-\varepsilon)|k|}}{\gamma - \varepsilon|k|} \right| \leq \frac{C}{\lambda^{\frac{2}{3}}(k)^{\frac{3}{2}}},
\]

we get

\[
\int_0^t |(\varepsilon|k|)^M \phi_0^{bc,S}|_{y=1}|^2 \mathcal{X}^{r+\frac{1}{3}} \, ds \leq \frac{C}{\lambda} \int_0^t |h_0|^2_{\mathcal{X}^r} \, ds.
\]

On the other hand, we refer to (8.8) and take \( y = 1 \) in it by noticing

\[
\left| e^{-\text{Re}(\gamma)}(\varepsilon|k|)^M \right| \leq e^{-\frac{1}{2}|\varepsilon|k|(\varepsilon|k|)^M} e^{-\frac{1}{2}\lambda^{\frac{2}{3}}(k)^{\frac{3}{2}}} \leq C e^{-\frac{1}{2}\lambda^{\frac{2}{3}}(k)^{\frac{3}{2}}} \leq \frac{C}{(\lambda(k)^{\frac{2}{3}})^{N/2}},
\]

for any \( N \geq 0 \), and combining with (8.28) to deduce

\[
(8.29) \quad \int_0^t |(\varepsilon \partial_x)^M \partial_y \phi_0^{bc,S}|_{y=1}|^2 \mathcal{X}^{r+\frac{1}{3}} \, ds \leq \frac{C}{\lambda} \int_0^t |h_0|^2_{\mathcal{X}^r} \, ds.
\]

Thus, we finish our proof. \( \square \)

In the end of this subsection, we give some weight estimates of vorticity \( \omega_{bc,S}^i = \Delta_\varepsilon \phi_{bc,S}^i \). Denote

\[
(8.30) \quad \varphi^0(y) = y, \quad \varphi^1(y) = 1 - y.
\]

Proposition 8.5. It holds that

\[
(8.31) \quad |(\omega_{bc,S}^i)_{\varphi}(\zeta, k, y)| + |\varphi^1 \partial_y (\omega_{bc,S}^i)_{\varphi}(\zeta, k, y)| \leq C(|\gamma| + |\varepsilon|k|)e^{-\text{Re}(\gamma)\varphi^1}|h_{\Phi}^i(\zeta, k)|.
\]

As a consequence, we get for \( \theta' \in [-\frac{1}{3}, 2] \)

\[
\int_0^t \left( |(\varphi^i)^{1+\theta'} \omega_{bc,S}^i(2X^{\frac{2}{3}}+\frac{1}{3}(\theta'+\frac{1}{2})) + |(\varphi^i)^{2+\theta'}(\partial_y, \varepsilon|k|)\omega_{bc,S}^i(2X^{\frac{2}{3}}+\frac{1}{3}(\theta'+\frac{1}{2})) \right) \, ds \leq \frac{C}{\lambda^{\frac{2}{3}}(k)^{\frac{3}{2}}} \int_0^t |h_0|^2_{\mathcal{X}^{\frac{3}{2}}} \, ds.
\]

Proof. The result is obtained by using formula (8.4), (8.9) and (8.12), the Plancherel theorem and by observing that multiplier \( \varphi^i(y) \) gains \( \frac{1}{\lambda^{\frac{2}{3}}(k)^{\frac{3}{2}}} \). More precisely,

\[
|((\varphi^i)^{1+m}|\gamma|e^{-\text{Re}(\gamma)\varphi^1})|_{L^2(\mathcal{I}_t)} \leq \left( \frac{C}{\lambda^{\frac{2}{3}}(k)^{\frac{3}{2}}} \right)^{m+\frac{1}{2}}.
\]

Thus we complete the proof. \( \square \)

Based on the above proposition, we have more estimates on \( \omega_{bc,S}^i \):

Proposition 8.6. Let \( \theta \in [0, 2] \). It holds that

\[
(8.32) \quad \int_0^t \left( |(\varphi^i)^{1+\theta'} \Delta_\varepsilon \phi_{bc,S}^i(2X^{\frac{2}{3}}+\frac{1}{3}(\theta'+\frac{1}{2})) + |(\varphi^i)^{2+\theta'}(\partial_y, \varepsilon|k|)\Delta_\varepsilon \phi_{bc,S}^i(2X^{\frac{2}{3}}+\frac{1}{3}(\theta'+\frac{1}{2})) \right) \, ds \leq \frac{C}{\lambda^{\frac{2}{3}}(k)^{\frac{3}{2}}} \int_0^t |h_0|^2_{\mathcal{X}^{\frac{3}{2}}} \, ds,
\]

\[
(8.33) \quad \int_0^t \left( |\langle D_x \rangle^{\frac{2}{3}-\frac{1}{3}}(\varphi^i)^{1+\theta'}(\partial_x \Delta_\varepsilon \phi_{bc,S}^i))|_{L^2(\mathcal{I}_t)} \right) \, ds \leq \frac{C}{\lambda^{\theta+\frac{1}{2}}} \int_0^t |h_0|^2_{\mathcal{X}^{\frac{3}{2}}} \, ds,
\]

\[
(8.34) \quad \int_0^t \left( |\langle D_x \rangle^{\frac{2}{3}-\frac{1}{3}}(\varphi^i)^{1+\theta'}(\partial_y \Delta_\varepsilon \phi_{bc,S}^i))|_{L^2(\mathcal{I}_t)} \right) \, ds \leq \frac{C}{\lambda^{\theta+\frac{1}{2}}} \int_0^t |h_0|^2_{\mathcal{X}^{\frac{3}{2}}} \, ds.
\]
Proof. (8.32) is a direct result of Lemma 8.5 by taking \( \theta = 0 \). It is easy to check
\[
\| \langle k \rangle^{\frac{\theta}{3}} \cdot (\varphi^i)^{\theta+\frac{2}{3}} k \Delta_{\varepsilon} (\phi_{bc,S}^i) \phi \|_{L_2^0(\Omega_t)} \leq C \frac{\langle k \rangle^{\frac{\theta}{3}} \cdot \frac{2}{3}}{\lambda^{\frac{\theta}{3}} k^{\frac{\theta}{3}} + \frac{2}{3}} |h^\lambda_{\phi}| = \frac{C}{\lambda^{\frac{\theta}{3}} (k)^{\frac{\theta}{3}} + \frac{2}{3}} |h^\lambda_{\phi}|,
\]
by taking \( \theta' = \theta + \frac{1}{2} \) in Lemma 8.5 and complete (8.33). Similarly, we check
\[
\| \langle k \rangle^{\frac{\theta}{3}} \cdot (\varphi^i)^{\theta+\frac{2}{3}} \partial_y \Delta_{\varepsilon} (\phi_{bc,S}^i) \phi \|_{L_2^0(\Omega_t)} \leq C \frac{\langle k \rangle^{\frac{\theta}{3}} \cdot \frac{1}{2}}{\lambda^{\frac{\theta}{3}} k^{\frac{\theta}{3}} + \frac{1}{2}} |h^\lambda_{\phi}| \leq \frac{C}{\lambda^{\frac{\theta}{3}} (k)^{\frac{\theta}{3}} + \frac{1}{2}} |h^\lambda_{\phi}|,
\]
by taking \( \theta' = \theta - \frac{1}{2} \) in Lemma 8.5 to complete (8.34). \( \square \)

8.2. The estimates of \( \phi_{bc,T} \): Vorticity transport estimate. \( \phi_{bc,T} \) is defined by
\[
\phi_{bc,T} = \phi_{bc,T}^0 + \phi_{bc,T}^1,
\]
where \( \phi_{bc,T}^0 \) is defined by
\[
(\partial_t - \Delta_{\varepsilon}) \Delta_{\varepsilon} \phi_{bc,T}^0 + u^p \partial_x \Delta_{\varepsilon} \phi_{bc,T}^0 + v^p \partial_y \Delta_{\varepsilon} \phi_{bc,T}^0 = H^0, \quad (x, y) \in \mathbb{T} \times (0, +\infty)
\]
with \( \phi_{bc,T}^0 \) satisfying
\[
\phi_{bc,T}^0|_{y=0, t=0} = 0, \quad \Delta_{\varepsilon} \phi_{bc,T}^0|_{y=0, t=0} = 0, \quad \phi_{bc,T}^0|_{t=0} = 0.
\]
and \( \phi_{bc,T}^1 \) is defined by
\[
(\partial_t - \Delta_{\varepsilon}) \Delta_{\varepsilon} \phi_{bc,T}^1 + u^p \partial_x \Delta_{\varepsilon} \phi_{bc,T}^1 + v^p \partial_y \Delta_{\varepsilon} \phi_{bc,T}^1 = H^1, \quad (x, y) \in \mathbb{T} \times (0, +\infty)
\]
with \( \phi_{bc,T}^1 \) satisfying
\[
\phi_{bc,T}^1|_{y=1, t=0} = 0, \quad \Delta_{\varepsilon} \phi_{bc,T}^1|_{y=1, t=0} = 0, \quad \phi_{bc,T}^1|_{t=0} = 0.
\]
We need to emphasize that we extend \( (u^p, v^p) \) to \( y \in \mathbb{R} \) by zero which means that \( (u^p, v^p) = 0 \) when \( y \in \mathbb{R} \setminus [0, 1] \).

Before we give the estimates of \( \phi_{bc,T}^i \), using Proposition 8.6 and \( (u^p, v^p)|_{y=0, t=0} = 0 \) to get that
\[
\int_0^t \| (\varphi^i)^{\frac{\theta}{3}} + \theta H^i \|_{L_2^0(\Omega_t)} \leq C \int_0^t \| (\varphi^i)^{\frac{\theta}{3}} + \theta (\Delta_{\varepsilon} \phi_{bc,S}^i + \partial_y \Delta_{\varepsilon} \phi_{bc,S}^i) \|_{L_2^0(\Omega_t)} \leq C \int_0^t |h^\lambda_{\phi}| ds,
\]
where \( \theta = 0, 1, 2 \).

We are in the position to give the estimates of \( \phi_{bc,T}^i \):

**Proposition 8.7.** Let \( \theta = 0, 1, 2 \) and \( i = 0, 1 \). There exists \( \lambda_0 > 1 \) and \( 0 < T < \min \{ T_p, \frac{1}{2X} \} \) such that for all \( t \in [0, T] \), \( \lambda \geq \lambda_0 \), it holds that
\[
\| (\varphi^i)^{\theta} \omega_{bc,T}^i \|_{L_2^0(\Omega_t)}^2 + \lambda \int_0^t \| (\varphi^i)^{\theta} \omega_{bc,T}^i \|_{L_2^0(\Omega_t)}^2 ds + \int_0^t \| (\varphi^i)^{\theta} \nabla \omega_{bc,T}^i \|_{L_2^0(\Omega_t)}^2 ds \leq \frac{C}{\lambda^{\frac{\theta}{3}} (k)^{\frac{\theta}{3}} + \frac{2}{3}} |h^\lambda_{\phi}|^2 ds,
\]
where \( \Delta_{\varepsilon} \phi_{bc,T}^i = \omega_{bc,T}^i \) and \( \varphi^i \) is given in (8.30).
Proof. Acting $e^{\Phi(t,D_x)}$ on the first equation of (8.35), we obtain
\begin{equation}
(8.38) \quad (\partial_t + \lambda(D_x) \frac{3}{2} - \Delta_x)(\omega_{bc,T}^i)\Phi + u^p\partial_x(\omega_{bc,T}^i)\Phi + v^p\partial_y(\omega_{bc,T}^i)\Phi \\
+ \left( (u^p\partial_x\omega_{bc,T}^i)\Phi - u^p\partial_x(\omega_{bc,T}^i)\Phi \right) + \left( (v^p\partial_y\omega_{bc,T}^i)\Phi - v^p\partial_y(\omega_{bc,T}^i)\Phi \right) = H_\Phi^i.
\end{equation}

Then taking $L_0^2(I_i; H_0^{\frac{11}{6} + \frac{\theta}{3}})$ inner product with $(\varphi^i)^{20}(\omega_{bc,T}^i)\Phi$, we get by using $\omega_{bc,T}^i|_{y = i} = 0$, $\partial_x u^p + \partial_y v^p = 0$ and integrating by parts that
\begin{align*}
\frac{1}{2} \frac{d}{dt} \| (\varphi^i)^{\theta}(\omega_{bc,T}^i)\|^2_{X_{i}^{1/4} + \frac{\theta}{2}} + \lambda \| (\varphi^i)^{\theta}(\omega_{bc,T}^i)\|^2_{X_{i}^{1/4} + \frac{\theta}{2}} + \| (\varphi^i)^{\theta}\nabla(\omega_{bc,T}^i)\|^2_{X_{i}^{1/4} + \frac{\theta}{2}} & = - \int_{S_i} (\varphi^i)^{20}(D_x) \frac{11}{6} + \frac{\theta}{3} (u^p \partial_x + v^p \partial_y)(\omega_{bc,T}^i)\Phi (D_x) \frac{11}{6} + \frac{\theta}{3}(\omega_{bc,T}^i)\Phi dx dy \\
& \quad + \frac{1}{2} \int_{S_i} \partial_y((\varphi^i)^{20}) (D_x) \frac{11}{6} + \frac{\theta}{3} (\omega_{bc,T}^i)\Phi |(D_x) \frac{11}{6} + \frac{\theta}{3}(\omega_{bc,T}^i)\Phi|^2 dx dy \\
& \quad - \int_{S_i} (D_x) \frac{11}{6} + \frac{\theta}{3} \left( (u^p \partial_x\omega_{bc,T}^i)\Phi - u^p\partial_x(\omega_{bc,T}^i)\Phi \right) (D_x) \frac{11}{6} + \frac{\theta}{3}(\omega_{bc,T}^i)\Phi(\varphi^i)^{20} dx dy \\
& \quad - \int_{S_i} (D_x) \frac{11}{6} + \frac{\theta}{3} \left( (v^p \partial_y\omega_{bc,T}^i)\Phi - v^p\partial_y(\omega_{bc,T}^i)\Phi \right) (D_x) \frac{11}{6} + \frac{\theta}{3}(\omega_{bc,T}^i)\Phi(\varphi^i)^{20} dx dy \\
& \quad - \int_{S_i} \partial_y((\varphi^i)^{20}) (D_x) \frac{11}{6} + \frac{\theta}{3} \partial_y(\omega_{bc,T}^i)\Phi \ (D_x) \frac{11}{6} + \frac{\theta}{3}(\omega_{bc,T}^i)\Phi(\varphi^i)^{20} dx dy \\
& \quad + \frac{1}{6} \int_{S_i} (D_x) \frac{11}{6} + \frac{\theta}{3} H_\Phi^i \ (D_x) \frac{11}{6} + \frac{\theta}{3}(\omega_{bc,T}^i)\Phi(\varphi^i)^{20} dx dy \\
= & I_1^i + \cdots + I_6^i,
\end{align*}

where $S_i = T \times I_i$. Integrating on $[0, t]$ with $t \leq T$, we obtain
\begin{equation}
(8.39) \quad \| (\varphi^i)^{\theta}(\omega_{bc,T}^i(t))\|^2_{X_{i}^{1/4} + \frac{\theta}{2}} + 2\lambda \int_0^t \| (\varphi^i)^{\theta}(\omega_{bc,T}^i)\|^2_{X_{i}^{1/4} + \frac{\theta}{2}} ds + 2 \int_0^t \| (\varphi^i)^{\theta}\nabla(\omega_{bc,T}^i)\|^2_{X_{i}^{1/4} + \frac{\theta}{2}} ds \\
\leq 2 \int_0^t |I_1^i| + \cdots + |I_6^i| ds.
\end{equation}

Now, we estimate $I_j^i$, $j = 1, \cdots, 6$ term by term.

Estimate of $I_1^i$. It follows from Lemma 2.1 that
\begin{equation}
\| (D_x) \frac{11}{6} + \frac{\theta}{3} \partial_x + v^p \partial_y(\omega_{bc,T}^i)\Phi \|^2_{L_2^2} \leq C(\| (\varphi^i)^{\theta}(\omega_{bc,T}^i)\Phi \|^2_{H_{x}^{1/4} + \frac{\theta}{2}} + \| \partial_y(\omega_{bc,T}^i)\Phi \|^2_{H_{x}^{1/4} + \frac{\theta}{2}}),
\end{equation}

which deduces that
\begin{equation}
|I_1^i| \leq C\| (\varphi^i)^{\theta}(\omega_{bc,T}^i)\|^2_{X_{i}^{1/4} + \frac{\theta}{2}} + \| (\varphi^i)^{\theta}\partial_y(\omega_{bc,T}^i)\|^2_{X_{i}^{1/4} + \frac{\theta}{2}} \| (\varphi^i)^{\theta}(\omega_{bc,T}^i)\|^2_{X_{i}^{1/4} + \frac{\theta}{2}} \leq \frac{1}{10} \| (\varphi^i)^{\theta}(\omega_{bc,T}^i)\|^2_{X_{i}^{1/4} + \frac{\theta}{2}} + C \| (\varphi^i)^{\theta}(\omega_{bc,T}^i)\|^2_{X_{i}^{1/4} + \frac{\theta}{2}}.
\end{equation}

Estimate of $I_2^i$. Thanks to
\begin{equation}
|\partial_y((\varphi^i)^{20})v^p| = |20(\varphi^i)'(\varphi^i)^{20-1}v^p| \leq C\theta(\varphi^i)^{20}v^p \leq C\theta(\varphi^i)^{20},
\end{equation}
by using $v^p|_{y=i} = 0$, it is obvious to see
\[ |I_2^i| \leq C\theta \| \phi^\theta \omega_{bc,T}^i \|^2_{X^{\frac{11}{14} + \frac{\theta}{2}}} . \]

**Estimate of $I_3^i$.** Applying Lemma 2.3, we find
\[ |I_3^i| \leq \| \phi^\theta \| \langle D_x \rangle^\frac{1}{2} \left( (u^p \partial_x \omega_{bc,T}^i \Phi - u^p \partial_x (\omega_{bc,T}^i) \Phi) \right) \| L^2_2(I_i; L^2_1) \| \phi^\theta \omega_{bc,T}^i \|_{X^{\frac{11}{14} + \frac{\theta}{2}}} \leq C \| \phi^\theta \omega_{bc,T}^i \|^2_{X^{\frac{11}{14} + \frac{\theta}{2}}} . \]

**Estimate of $I_4^i$.** Applying Lemma 2.2, we get
\[ |I_4^i| \leq C \| \phi^\theta \partial_y \omega_{bc,T}^i \|_{X^{\frac{11}{14} + \frac{\theta}{2}}} \| \phi^\theta \omega_{bc,T}^i \|_{X^{\frac{11}{14} + \frac{\theta}{2}}} \leq \frac{1}{10} \| \phi^\theta \partial_y \omega_{bc,T}^i \|^2_{X^{\frac{11}{14} + \frac{\theta}{2}}} + C \| \phi^\theta \omega_{bc,T}^i \|^2_{X^{\frac{11}{14} + \frac{\theta}{2}}} . \]

**Estimate of $I_5^i$.** By the fact
\[ \partial_y ((\phi^\theta)^2) = 2 \theta (\phi^\theta)^{2\theta-1} , \]
we have
\[ |I_5^i| \leq C \theta \| \phi^\theta \partial_y \omega_{bc,T}^i \|_{X^{\frac{11}{14} + \frac{\theta}{2}}} \| \phi^\theta \omega_{bc,T}^i \|_{X^{\frac{11}{14} + \frac{\theta}{2}}} \leq \frac{1}{10} \| \phi^\theta \partial_y \omega_{bc,T}^i \|^2_{X^{\frac{11}{14} + \frac{\theta}{2}}} + C \theta^2 \| \phi^\theta \omega_{bc,T}^i \|^2_{X^{\frac{11}{14} + \frac{\theta}{2}}} . \]

**Estimate of $I_6^i$.** It follows from
\[ \| \langle D_x \rangle^\frac{1}{2} \phi (\phi^\theta)^{-\frac{1}{2}} \|_{L^2_2(I_i; L^2_1)} \leq \| \langle D_x \rangle^\frac{1}{2} \phi \|_{L^2_2(I_i; L^2_1)} \| \phi^\theta \|_{L^2_2(I_i; L^2_1)} \| \langle D_x \rangle^\frac{1}{2} \phi \|_{L^2_2(I_i; L^2_1)} \leq \| \phi^\theta \omega_{bc,T}^i \|_{X^{\frac{11}{14} + \frac{\theta}{2}}} \| \partial_y \omega_{bc,T}^i \|_{X^{\frac{11}{14} + \frac{\theta}{2}}} , \]
for $\theta = 1, 2$ and
\[ \| \langle D_x \rangle^\frac{1}{2} \phi (\phi^\theta)^{-\frac{1}{2}} \omega_{bc,T}^i \|_{L^2_2(I_i; L^2_1)} \leq \| \langle D_x \rangle^\frac{1}{2} \phi \|_{L^2_2(I_i; L^2_1)} \| \phi^\theta \omega_{bc,T}^i \|_{L^2_2(I_i; L^2_1)} \| \langle D_x \rangle^\frac{1}{2} \phi \|_{L^2_2(I_i; L^2_1)} \leq C \| \omega_{bc,T}^i \|_{X^{\frac{11}{14} + \frac{\theta}{2}}} \| \partial_y \omega_{bc,T}^i \|_{X^{\frac{11}{14} + \frac{\theta}{2}}} , \]
by using Hardy inequality for $\theta = 0$. Therefore, we get for $\theta = 0, 1, 2$ that
\[ |I_6^i| \leq C \| \phi^\theta \|_{X^{\frac{11}{14} + \frac{\theta}{2}}} \| \phi^\theta \omega_{bc,T}^i \|_{X^{\frac{11}{14} + \frac{\theta}{2}}} \times \left\{ \begin{array}{ll}
\| \phi^\theta \omega_{bc,T}^i \|_{X^{\frac{11}{14} + \frac{\theta}{2}}} & , \quad \theta = 1, 2, \\
\| \partial_y \omega_{bc,T}^i \|_{X^{\frac{11}{14} + \frac{\theta}{2}}} & , \quad \theta = 0,
\end{array} \right.
\]
\[ \leq \frac{1}{10} \| \partial_y \omega_{bc,T}^i \|^2_{X^{\frac{11}{14} + \frac{\theta}{2}}} + \lambda \frac{\theta^2}{2} \| \phi^\theta \|_{L^2_2(I_i; L^2_1)} ^2 \| \phi^\theta \omega_{bc,T}^i \|_{L^2_2(I_i; L^2_1)} ^2 + \frac{\lambda}{4} \| \phi^\theta \omega_{bc,T}^i \|_{L^2_2(I_i; L^2_1)} ^2 + \frac{\lambda}{8} \| \phi^\theta \omega_{bc,T}^i \|_{L^2_2(I_i; L^2_1)} ^2 \]
which implies $\lambda \geq \lambda_0$. Let $r = \min\{T, \frac{1}{2\lambda}\}$. Then we take $r > 0$ and use Proposition 8.7 to derive

$$\parallel (\varphi^i)^\theta \omega_{bc,T} \parallel^2_{X^{\frac{11}{6} + \frac{\theta}{2}}} \leq \frac{C}{\lambda^2} \int_0^t \parallel (\varphi^i)^\theta \omega_{bc,T} \parallel^2_{X^{\frac{11}{6} + \frac{\theta}{2}}} ds + \int_0^t \parallel (\varphi^i)^\theta \nabla \omega_{bc,T} \parallel^2_{X^{\frac{11}{6} + \frac{\theta}{2}}} ds$$

and using Hardy inequality to get

$$\parallel (\varphi^i)^\theta \omega_{bc,T} \parallel^2_{X^{\frac{11}{6} + \frac{\theta}{2}}} \leq \frac{C}{\lambda^2} \int_0^t \parallel (\varphi^i)^\theta \omega_{bc,T} \parallel^2_{X^{\frac{11}{6} + \frac{\theta}{2}}} ds$$

where we use (8.37) in the last step.

All we left is to control the last term of the above inequality. For that, we rewrite it as

$$\frac{\lambda}{8} \int_0^t \theta^2 \parallel (\varphi^i)^\theta \omega_{bc,T} \parallel^2_{X^{\frac{11}{6} + \frac{\theta}{2}}} ds \leq \frac{\lambda}{2} \sum_{\theta=0}^2 \int_0^t \parallel (\varphi^i)^\theta \omega_{bc,T} \parallel^2_{X^{\frac{11}{6} + \frac{\theta}{2}}} ds$$

Combing all the above estimates, we get the desired results.

Based on estimates of $\omega_{bc,T}$, we use the elliptic equation to get the estimates of $\phi_{bc,T}^i$.

**Corollary 8.8.** There exists $\lambda_0 > 1$ and $0 < T < \min\{T_p, \frac{1}{2\lambda}\}$ such that for all $t \in [0, T]$, $\lambda \geq \lambda_0$, it holds that

$$\int_0^t \left( \parallel \nabla \phi_{bc,T}^i \parallel^2_{X^{\frac{11}{6} + \frac{\theta}{2}}} + \parallel \partial_y \phi_{bc,T}^i \parallel^2_{X^{\frac{11}{6} + \frac{\theta}{2}}} + \parallel \partial_{\hat{\gamma}} \phi_{bc,T}^i \parallel^2_{X^{\frac{11}{6} + \frac{\theta}{2}}} + \parallel \phi_{bc,T} \parallel \parallel \nabla \omega_{bc,T} \parallel \parallel X^r \parallel \parallel \partial_y \phi_{bc,T} \parallel \parallel X^r \parallel \parallel \partial_{\hat{\gamma}} \phi_{bc,T} \parallel \parallel X^r \parallel \right) ds \leq C \lambda^2 \int_0^t \parallel \nabla \phi_{bc,T}^i \parallel^2_{X^{\frac{11}{6} + \frac{\theta}{2}}} ds$$

**Proof.** Here, we only give the proof of the case $i = 0$. The case $i = 1$ is the same. Recalling the elliptic equation

$$\Delta \phi_{bc,T} = \omega_{bc,T}, \quad \phi_{bc,T} |_{y=0} = 0,$$

for $y > 0$. Then we take $X^r$ inner product with $\phi_{bc,T}^0$ and using Hardy inequality to get

$$\parallel \nabla \phi_{bc,T}^0 \parallel^2_{X^r} \leq \parallel \phi_{bc,T}^0 \parallel \parallel \nabla \omega_{bc,T} \parallel \parallel X^r \parallel \parallel \phi_{bc,T}^0 \parallel \parallel X^r \parallel \leq C \parallel \phi_{bc,T}^0 \parallel \parallel \nabla \omega_{bc,T} \parallel \parallel X^r \parallel \parallel \phi_{bc,T}^0 \parallel \parallel X^r \parallel$$

which implies

$$\parallel \nabla \phi_{bc,T}^0 \parallel \parallel X^r \parallel \leq C \parallel \phi_{bc,T}^0 \parallel \parallel \nabla \omega_{bc,T} \parallel \parallel X^r \parallel$$

for $r \geq 0$. By Proposition 8.7, we get

$$\int_0^t \parallel \nabla \phi_{bc,T}^0 \parallel^2_{X^r} ds \leq \int_0^t \parallel \phi_{bc,T}^0 \parallel^2_{X^r} ds \leq C \lambda^2 \int_0^t \parallel h^0 \parallel^2_{X^\frac{11}{6} + \frac{\theta}{2}} ds$$

For the boundary term, using the interpolation inequality to get

$$\parallel \partial_y \phi_{bc,T}^0 \parallel \parallel X^r \parallel \leq C \parallel \partial_y \phi_{bc,T}^0 \parallel \parallel \omega_{bc,T} \parallel \parallel X^r \parallel \leq C \parallel \partial_y \phi_{bc,T}^0 \parallel \parallel \omega_{bc,T} \parallel \parallel X^r \parallel$$

$$\parallel \partial_{\hat{\gamma}} \phi_{bc,T}^0 \parallel \parallel X^r \parallel \leq C \parallel \partial_{\hat{\gamma}} \phi_{bc,T}^0 \parallel \parallel \omega_{bc,T} \parallel \parallel X^r \parallel \leq C \parallel \partial_{\hat{\gamma}} \phi_{bc,T}^0 \parallel \parallel \omega_{bc,T} \parallel \parallel X^r \parallel$$
\[ \leq C \| \varphi^0_{bc,T} \|_{X^\frac{1}{2}_0} \| \omega^0_{bc,T} \|_{X^\frac{1}{2}_0}, \]

where we use (8.43) and Calderon-Zygmund inequality in the last step. Along with Proposition 8.7, we arrive at

\[ \int_0^t | \partial_y \phi^i_{bc,T} |_{y=0,1} \|_{X^\frac{1}{2}_0} \leq C \int_0^t | h^i |_{X^\frac{1}{2}_0} ds. \]

Next, we deal with the term \( \| \partial_x \phi^0_{bc,T} \|_{X^\frac{1}{2}_0} \). Taking Fourier transform in \( x \) to (8.42), we write the solution

\[ \hat{\phi}^0_{bc,T}(k, y) = \int_0^y e^{-\varepsilon |k| (y-y')} \int_{y'}^{\infty} e^{-\varepsilon |k| (y''-y')} \hat{\omega}^0_{bc,T}(k, y'') dy'' dy'. \]

Then we have

\[ \| \hat{\phi}^0_{bc,T}(k, y) \|_{L^2} \leq \int_0^y \int_{y'}^{\infty} | \hat{\omega}^0_{bc,T}(k, y'') | dy'' dy'. \]

Decomposing the integral \( \int_0^y \) into \( \int_{y''=0}^{\min \{ y, (k)^{-\frac{1}{4}} \} } \) and \( \int_{y''=0}^{\min \{ y, (k)^{-\frac{1}{4}} \} } \), it follows from the H"older inequality that

\[ \sup_{y \geq 0} \| \hat{\phi}^0_{bc,T}(k, y) \|_{L^2} \leq C \langle k \rangle^{-\frac{1}{4}} \| \hat{\omega}^0_{bc,T} \|_{L^2} + C \langle k \rangle^{\frac{1}{2}} \| \hat{\omega}^0_{bc,T} \|_{L^2}. \]

We take summation \( \sum_{k \in \mathbb{Z}} \) and use the Plancherel theorem to deduce

\[ \sup_{y \geq 0} \| \phi_{bc,T}(\cdot, y) \|_{L^2} \leq C \langle D_x \rangle^{-\frac{1}{4}} \| \omega_{bc,T} \|_{L^2} + \| \langle D_x \rangle^{\frac{1}{2}} \hat{\omega}_{bc,T} \|_{L^2}. \]

Thus, we get that

\[ \int_0^t \left( \| \partial_x \phi^i_{bc,T} \|_{X^\frac{1}{2}_0}^2 + \| \phi^i_{bc,T} \|_{y=1-1}^2 \right) ds \leq C \int_0^t \| \varphi^i \omega_{bc,T} \|_{X^\frac{1}{2}_0}^2 ds + C \int_0^t \| (\varphi^i)^2 \omega_{bc,T} \|_{X^\frac{1}{2}_0}^2 ds \]

\[ \leq C \int_0^t | h^i |_{X^\frac{1}{2}_0} ds, \]

Collecting (8.44), (8.45) and (8.48) together, we get the corollary proved.

\[ \square \]

### 8.3. The estimates of \( \phi_{bc,R} \): Full construction of boundary corrector.

All we left is the term \( \phi_{bc,R} \). Like previous argument, we define

\[ \phi_{bc,R} = \phi_{bc,R}^0 + \phi_{bc,R}^1, \]

where \( \phi_{bc,R} \) satisfies that

\[ \begin{align*}
(\partial_t - \Delta_\varepsilon) \Delta_\varepsilon \phi_{bc,R}^0 + u^p \partial_x \Delta_\varepsilon \phi_{bc,R}^0 + v^p \partial_y \Delta_\varepsilon \phi_{bc,R}^0 + \partial_y \phi_{bc,R}^0 \partial_x \omega^p - \partial_x \phi_{bc,R}^0 \partial_y \omega^p &= -\partial_y (\phi_{bc,S}^0 + \phi_{bc,T}^0) \partial_x \omega^p + \partial_x (\phi_{bc,S}^0 + \phi_{bc,T}^0) \partial_y \omega^p, & t > 0, \ x \in \mathbb{T}, \ y \in (0,1), \ \\
\phi_{bc,R}^0 |_{y=0} &= 0, \ \phi_{bc,R}^0 |_{y=1} = - (\phi_{bc,S}^0 + \phi_{bc,T}^0) |_{y=1}, \ \Delta_\varepsilon \phi_{bc,R}^0 |_{y=0,1} = 0, \ \\
\phi_{bc,R}^1 |_{t=0} &= 0.
\end{align*} \]
Firstly, we give some elliptic estimates.

For simplicity, denote $\omega_{bc,R}^1 = \Delta \phi_{bc,R}^1$ who has the following relationship

\begin{equation}
\begin{cases}
\Delta \phi_{bc,R}^0 = \omega_{bc,R}^0, \\
\phi_{bc,R}^0|_{y=0} = 0, \quad \phi_{bc,R}^0|_{y=1} = f^0,
\end{cases}
\end{equation}

and

\begin{equation}
\begin{cases}
\Delta \phi_{bc,R}^1 = \omega_{bc,R}^1, \\
\phi_{bc,R}^1|_{y=0} = f^1, \quad \phi_{bc,R}^1|_{y=1} = 0,
\end{cases}
\end{equation}

where

\begin{align}
f^0 &= f^0(t, x) = - (\phi_{bc,S}^0 + \phi_{bc,T}^0)|_{y=1}, \\
f^1 &= f^1(t, x) = - (\phi_{bc,S}^1 + \phi_{bc,T}^1)|_{y=0}.
\end{align}

In order to homogenize boundary condition, we introduce

\begin{equation}
\tilde{\phi}_{bc,R}^0 = \phi_{bc,R}^0 + g^0, \quad g^0 = y(\phi_{bc,S}^0 + \phi_{bc,T}^0),
\end{equation}

where $\tilde{\phi}_{bc,R}^0$ satisfies

\begin{equation}
\begin{cases}
\Delta \tilde{\phi}_{bc,R}^0 = \omega_{bc,R}^0 + \Delta \phi_{bc,R}^0, \\
\tilde{\phi}_{bc,R}^0|_{y=0,1} = 0,
\end{cases}
\end{equation}

where

\begin{equation}
\Delta \phi_{bc,R}^0 = y(\Delta \phi_{bc,S}^0 + \Delta \phi_{bc,T}^0) + 2(\partial_y \phi_{bc,S}^0 + \partial_y \phi_{bc,T}^0).
\end{equation}

Similarly, we introduce

\begin{equation}
\tilde{\phi}_{bc,R}^1 = \phi_{bc,R}^1 + g^1, \quad g^1 = (1-y)(\phi_{bc,S}^1 + \phi_{bc,T}^1),
\end{equation}

and $\tilde{\phi}_{bc,R}^1$ satisfies

\begin{equation}
\begin{cases}
\Delta \tilde{\phi}_{bc,R}^1 = \omega_{bc,R}^1 + \Delta \phi_{bc,R}^1, \\
\tilde{\phi}_{bc,R}^1|_{y=0,1} = 0,
\end{cases}
\end{equation}

where

\begin{equation}
\Delta \phi_{bc,R}^1 = (1-y)(\Delta \phi_{bc,S}^1 + \Delta \phi_{bc,T}^1) - 2(\partial_y \phi_{bc,S}^1 + \partial_y \phi_{bc,T}^1).
\end{equation}
Lemma 8.9. Let \((f^0, f^1), (g^0, g^1)\) introduced in (8.53), (8.54), (8.55) and (8.58). It holds that

\[
\int_0^t \left( |f|^2_{X^\frac{3}{2}} + \|\nabla g_\varepsilon^1\|^2_{X^\frac{3}{2}} + \|\Delta g_\varepsilon^1\|^2_{X^\frac{3}{2}} \right) ds \leq \frac{C}{\lambda^2} \int_0^t |h|^2_{X^\frac{3}{2}} ds
\]

for \(i = 0, 1\).

Moreover, \(\phi_{bc,R}^i (i = 0, 1)\) has the following estimates:

\[
\int_0^t \|\nabla \phi_{bc,R}^i\|^2_{X^\frac{5}{2}} ds \leq \frac{C}{\lambda^2} \int_0^t |h|^2_{X^\frac{3}{2}} ds + C \int_0^t \|\omega_{bc,R}\|^2_{X^\frac{3}{2}} ds.
\]

Proof. Here we only prove the case \(i = 0\). The case \(i = 1\) is almost the same and we omit details to readers. We first give proof for \(f^0\). By the definition of \(f^0\), we get

\[
\int_0^t |f_0|^2_{X^\frac{3}{2}} ds \leq \int_0^t |\phi_{bc,S}^0|_{y=1}^2_{X^\frac{3}{2}} ds + \int_0^t |\phi_{bc,T}^0|_{y=1}^2_{X^\frac{3}{2}} ds \leq \frac{C}{\lambda} \int_0^t |h_0|^2_{X^\frac{3}{2}} ds.
\]

where we used Lemma 8.3 and Corollary 8.8.

For \(g^0\), by Corollary 8.8, Proposition 8.3, we have

\[
\int_0^t \|\nabla g_\varepsilon^1\|^2_{X^\frac{3}{2}} ds \leq \int_0^t \left( \|\nabla \phi_{bc,S}^0\|_{X^\frac{5}{2}} + \|\nabla \phi_{bc,T}^0\|_{X^\frac{5}{2}} + \|\phi_{bc,S}^0\|_{X^\frac{3}{2}} + \|\phi_{bc,T}^0\|_{X^\frac{3}{2}} \right) ds 
\]

\[
\leq \frac{C}{\lambda^2} \int_0^t |h|^2_{X^\frac{3}{2}} ds.
\]

On one hand, using Proposition 8.3, Proposition 8.5, Corollary 8.8 and Proposition 8.7 we get

\[
\int_0^t \left( \|y \Delta \phi_{bc,S}^0\|^2_{X^\frac{3}{2}} + \|\partial_y \phi_{bc,S}^0\|^2_{X^\frac{3}{2}} + \|y \Delta \phi_{bc,T}^0\|^2_{X^\frac{3}{2}} + \|\partial_y \phi_{bc,T}^0\|^2_{X^\frac{3}{2}} \right) ds \leq \frac{C}{\lambda^2} \int_0^t |h|^2_{X^\frac{3}{2}} ds,
\]

which implies that

\[
\int_0^t \|\Delta g_\varepsilon^1\|^2_{X^\frac{3}{2}} ds \leq \frac{C}{\lambda^2} \int_0^t |h|^2_{X^\frac{3}{2}} ds.
\]

At last, we prove (8.62). Taking \(X^\frac{3}{2}\) inner product with \(\phi_{bc,R}^0\) toward (8.56), we use integration by parts and then integrate time from 0 to \(t\) that

\[
\int_0^t \|\nabla \phi_{bc,R}^0\|^2_{X^\frac{3}{2}} ds = -\int_0^t \langle \omega_{bc,R}, \phi_{bc,R}^0 \rangle_{X^\frac{3}{2}} ds + \int_0^t \langle \Delta g_\varepsilon^1, \phi_{bc,R}^0 \rangle_{X^\frac{3}{2}} ds.
\]

Due to \(\phi_{bc,R}^0|_{y=0,1} = 0\), we use Poincaré inequality to imply

\[
\int_0^t \langle \omega_{bc,R}, \phi_{bc,R}^0 \rangle_{X^\frac{3}{2}} ds \leq \frac{1}{10} \int_0^t \|\partial_y \phi_{bc,R}^0\|^2_{X^\frac{3}{2}} ds + C \int_0^t \|\omega_{bc,R}\|^2_{X^\frac{3}{2}} ds.
\]

According to (8.61), we get

\[
\int_0^t \langle \Delta g_\varepsilon^1, \phi_{bc,R}^0 \rangle_{X^\frac{3}{2}} ds \leq \int_0^t \|\Delta g_\varepsilon^1\|^2_{X^\frac{3}{2}} \|\phi_{bc,R}^0\|_{X^\frac{3}{2}} ds 
\]

\[
\leq \frac{1}{10} \int_0^t \|\partial_y \phi_{bc,R}^0\|^2_{X^\frac{3}{2}} ds + \frac{C}{\lambda^2} \int_0^t |h|^2_{X^\frac{3}{2}} ds.
\]

Combining (8.63) and (8.64), it deduces

\[
\int_0^t \|\nabla \phi_{bc,R}^0\|^2_{X^\frac{3}{2}} ds \leq \frac{C}{\lambda^2} \int_0^t |h|^2_{X^\frac{3}{2}} ds + C \int_0^t \|\omega_{bc,R}\|^2_{X^\frac{3}{2}} ds.
\]
Bringing $\nabla_\varepsilon \phi^0_{bc,R} = \nabla_\varepsilon \tilde{\phi}^0_{bc,R} - \nabla_\varepsilon g^0$ into above inequality, we obtain

$$
\int_0^t \|\nabla_\varepsilon \phi^0_{bc,R}\|^2_{X^2} ds \leq \int_0^t \|\nabla_\varepsilon \tilde{\phi}^0_{bc,R}\|^2_{X^2} ds + \int_0^t \|\nabla_\varepsilon g^0\|^2_{X^2} ds
\leq \frac{C}{\lambda^2} \int_0^t |h|^2_{X^2} ds + C \int_0^t \|\omega_{bc,R}\|^2_{X^2} ds.
$$

By now, we finish the proof.

In order to estimate the right hand side of (8.49) and (8.50) and boundary term, we need the following results:

**Lemma 8.10.** For $i = 0, 1$, we have that

$$
\int_0^t \left( \|\partial_x (\phi^i_{bc,S} + \phi^i_{bc,T})\|^2_{X^2} + \|\partial_y (\phi^i_{bc,S} + \phi^i_{bc,T})\|^2_{X^2} \right) ds \leq \frac{C}{\lambda^2} \int_0^t |h|^2_{X^2} ds,
$$

(8.65)

$$
\int_0^t |\partial_y \phi^i_{bc,R}|_{y=0,1}^2_{X^2} ds \leq C \int_0^t \|\omega^i_{bc,R}\|^2_{X^2} ds + \frac{C}{\lambda^2} \int_0^t |h|^2_{X^2} ds,
$$

(8.66)

$$
\int_0^t \left\langle \partial_x (\phi^i_{bc,R})_\Phi, \partial_y (\phi^i_{bc,R})_\Phi \right\rangle_{H^2_y} |y=0,1|_{X^2} ds \leq C \int_0^t \|\omega^i_{bc,R}\|^2_{X^2} ds + \frac{C}{\lambda^2} \int_0^t |h|^2_{X^2} ds.
$$

(8.67)

**Proof.** Here we only prove the case $i = 0$. The case $i = 1$ is almost the same and we omit details to readers.

By Proposition 8.3 and Corollary 8.8 we get the (8.65) proved.

Next, we deal with the boundary term. A direct calculation gives that

$$
\partial_y \phi^0_{bc,R}|_{y=0,1} = (\partial_y \tilde{\phi}^0_{bc,R} - \partial_y g^0)|_{y=0,1}
$$

$$
= \partial_y \tilde{\phi}^0_{bc,R}|_{y=0,1} - (\partial_y \phi^0_{bc,S} + \partial_y \phi^0_{bc,T})|_{y=0,1} - (\phi^0_{bc,S} + \phi^0_{bc,T})|_{y=0,1},
$$

and

$$
\partial_y \phi^0_{bc,R}|_{y=0} = (\partial_y \tilde{\phi}^0_{bc,R} - \partial_y g^0)|_{y=0} = \partial_y \phi^0_{bc,R}|_{y=0},
$$

due to $\phi^0_{bc,S}|_{y=0} = \phi^0_{bc,T}|_{y=0} = 0$.

By Corollary 8.8 we get implies

$$
\int_0^t |\partial_y \tilde{\phi}^0_{bc,R}|_{y=0,1}^2_{X^2} ds \leq \frac{C}{\lambda^2} \int_0^t |h|^2_{X^2} ds + C \int_0^t \|\omega^0_{bc,R}\|^2_{X^2} ds
\leq C \int_0^t \|\omega^0_{bc,R}\|^2_{X^2} ds + \frac{C}{\lambda^2} \int_0^t |h|^2_{X^2} ds,
$$

where we use elliptic estimate and Calderon-Zygmund inequality

$$
\|\partial_y \tilde{\phi}^0_{bc,R}\|_{X^2} + \|\partial_y^2 \phi^0_{bc,R}\|_{X^2} \leq C \|\omega^0_{bc,R}\|_{X^2} + C \|\Delta_\varepsilon g^0\|_{X^2}.
$$

For the last estimate, we use (8.66) and (8.61) to imply

$$
\int_0^t \left\langle \partial_x (\phi^0_{bc,R})_\Phi, \partial_y (\phi^0_{bc,R})_\Phi \right\rangle_{H^2_y} |y=0,1|_{X^2} ds \leq C \int_0^t \|\phi^0_{bc,S} + \phi^0_{bc,T}\|_{y=0,1}^2_{X^2} ds
\leq C \int_0^t |f|^2_{X^2} ds.
$$
\[
\leq C \int_0^t \| \omega_{bc,R}^0 \|_{X^\frac{3}{2}}^2 ds + \frac{C}{\lambda^2} \int_0^t |h|^2_{X^\frac{3}{2}} ds.
\]

Here, we complete this lemma.

We are coming to the main part of this section. We shall give the estimate for the system (8.39) and (8.50).

**Proposition 8.11.** Let \( \phi_{bc,R}^i \) and \( \phi_{bc,R}^1 \) be the solution of (8.39) and (8.50) respectively, and \( \omega_{bc,R}^i = \Delta \phi_{bc,R}^i \) for \( i = 0, 1 \). Then, for every \( i = 0, 1 \), it holds that

\[
\| \omega_{bc,R}^i(t) \|_{X^\frac{3}{2}}^2 + \int_0^t \| \omega_{bc,R}^i \|_{X^\frac{3}{2}}^2 ds + \int_0^t (|\nabla \phi_{bc,R}^i|_{X^\frac{3}{2}} + |\partial_y \phi_{bc,R}^i|_{y=0,1}|^2_{X^\frac{3}{2}}) ds + \int_0^t |\nabla \omega_{bc,R}^i|_{X^\frac{3}{2}}^2 ds,
\]

where \( 0 < T < \min \{ T_0, \frac{1}{2\lambda} \} \).

**Proof.** The result mainly comes from the process of Propoposition 7.1. Here we take \( (N_u, \varepsilon N_y) = 0 \) and \( \varepsilon^2 f_1 + f_2 - C(t) \partial_x \omega^p \) is replaced by \( G^i = -\partial_y (\phi_{bc,S}^i + \phi_{bc,T}^i) \partial_x \omega^p + \partial_x (\phi_{bc,S}^i + \phi_{bc,T}^i) \partial_y \omega^p \) for \( i = 0, 1 \). In order to estimate the source term \( \int_0^t \| G^i \|_{X^\frac{3}{2}}^2 ds \), using Lemma 8.10 and product estimate in Lemma 2.2, we get

\[
\int_0^t \| G^i \|_{X^\frac{3}{2}}^2 ds \leq C \int_0^t \| \partial_y (\phi_{bc,S}^i + \phi_{bc,T}^i) \|_{X^\frac{3}{2}}^2 ds + C \int_0^t \| \partial_x (\phi_{bc,S}^i + \phi_{bc,T}^i) \|_{X^\frac{3}{2}}^2 ds
\]

The only difference comes from boundary condition

\[
\phi_{bc,R}^i|_{y=i} = 0, \quad \phi_{bc,R}^i|_{y=1-i} = -(\phi_{bc,S} + \phi_{bc,T})|_{y=1-i},
\]

which are not zero compared with equation (6.2). We review \( T_0 \) in Propoposition 7.1. After integration by parts, the boundary term is left. More precisely, we need to estimate \( \int_0^t \left \langle \partial_x (\phi_{bc,R}^i) \Phi, \partial_y (\phi_{bc,R}^i) \Phi \right \rangle_{H^2_y} \). According to Lemma 8.10 we have

\[
\int_0^t \left \langle \partial_x (\phi_{bc,R}^i) \Phi, \partial_y (\phi_{bc,R}^i) \Phi \right \rangle_{H^2_y} \|^2_{H^2_y} \leq \int_0^t \| \omega_{bc,R}^i \|_{X^\frac{3}{2}}^2 ds + \frac{C}{\lambda^2} \int_0^t |h|^2_{X^\frac{3}{2}} ds.
\]

Here, we take \( \lambda \) large enough to complete the proof.

\[ \square \]

8.4. **Proof of Proposition 6.2.** In this subsection, we combine all above estimates to finish the proof of Proposition 6.2. Recalling the definition of \( \phi_{bc} \):

\[
\phi_{bc} = \phi_{bc,S} + \phi_{bc,T} + \phi_{bc,R},
\]

we get that

\[
\begin{cases}
(\partial_t - \Delta \varepsilon) \Delta \phi_{bc} + u^p \partial_x \Delta \phi_{bc} + \varepsilon \partial_y \Delta \phi_{bc} + \partial_y \phi_{bc,T} \partial_x \omega^p - \partial_x \phi_{bc,T} \partial_y \omega^p = 0, \\
\phi_{bc}|_{y=0,1} = 0, \quad \partial_y \phi_{bc}|_{y=0} = h^0 + R_{bc}^{00} + R_{bc}^{01}, \quad \partial_y \phi_{bc}|_{y=1} = h^1 + R_{bc}^{10} + R_{bc}^{11},
\end{cases}
\]

(8.68)
Here $R_{bc}^{ji}$ $(j = 0, 1, \ i = 0, 1)$ are linear operators and are defined by

\[
R_{bc}^{00} = \left( \partial_y \phi_{bc,T}^0 + \partial_y \phi_{bc,R}^0 \right) |_{y=0},
R_{bc}^{01} = (\partial_y \phi_{bc,S}^1 + \partial_y \phi_{bc,T}^1 + \partial_y \phi_{bc,R}^1) |_{y=0},
R_{bc}^{10} = (\partial_y \phi_{bc,S}^0 + \partial_y \phi_{bc,T}^0 + \partial_y \phi_{bc,R}^0) |_{y=1},
R_{bc}^{11} = (\partial_y \phi_{bc,T}^1 + \partial_y \phi_{bc,R}^1) |_{y=1}.
\]

Compared with the system (8.1), we need to find $(h^0, h^1)$ such that

\[
\begin{align*}
R_{bc}^{00} + R_{bc}^{01} &= -\partial_y \phi_{\text{slip}} |_{y=0} + C(t), \\
R_{bc}^{10} + R_{bc}^{11} &= -\partial_y \phi_{\text{slip}} |_{y=1} + C(t),
\end{align*}
\]

holds. To do that, we define an operator $R_{bc}[h^0, h^1]$, which is defined by

\[
R_{bc}[h^0, h^1] = \begin{pmatrix} R_{bc}^{00} & R_{bc}^{01} \\ R_{bc}^{10} & R_{bc}^{11} \end{pmatrix}
\]

is a $2 \times 2$ matrix operator and is well-defined on the Banach space

\[
Z_{bc} = \{(h^0, h^1) \in L^2(0, t; L^2) | \int_0^t |(h^0, h^1)|^2_{X_T^2} ds < +\infty\}.
\]

**Proposition 8.12.** There exists $\lambda_0 \geq 1$ such that if $\lambda \geq \lambda_0$, the map $R_{bc} : Z_{bc} \to Z_{bc}$ defined by (8.71) satisfies

\[
\int_0^t |R_{bc}[h^0, h^1]|^2_{X_T^2} ds \leq \frac{C}{\lambda^2} \int_0^t |(h^0, h^1)|^2_{X_T^2} ds.
\]

Hence, the operator $I + R_{bc}$ is invertible in $Z_{bc}$. Moreover, there exists $(h_0, h_1) \in Z_{bc}$ such that (8.70) holds and $(h_0, h_1)$ is defined by

\[
(h_0, h_1) = (I + R_{bc})^{-1}(-\partial_y \phi_{\text{slip}} |_{y=0} + C(t), -\partial_y \phi_{\text{slip}} |_{y=1} + C(t)).
\]

**Proof.** First, by Lemma 8.4, Proposition 8.8, Corollary 8.7 and Proposition 8.11 it is easy to get

\[
\int_0^t |R_{bc}[h^0, h^1]|^2_{X_T^2} ds \leq \frac{C}{\lambda^2} \int_0^t |(h^0, h^1)|^2_{X_T^2} ds.
\]

Taking $\lambda$ large enough, we get that the operator $I + R_{bc}$ is invertible in $Z_{bc}$. Thus, there exists $(h_0, h_1) \in Z_{bc}$ such that (8.70) holds.

Let’s continue to prove Proposition 8.12. According to Proposition 8.3, Proposition 8.6, Corollary 8.8 and Proposition 8.11, we get by (8.68) that

\[
\int_0^t \|\nabla \phi_{bc}|_X^2 \|_{X_T^2} ds \leq \int_0^t \|\nabla \phi_{bc,S} |_X^2 \|_{X_T^2} ds + \int_0^t \|\nabla \phi_{bc,T} |_X^2 \|_{X_T^2} ds + \int_0^t \|\nabla \phi_{bc,R} |_X^2 \|_{X_T^2} ds
\]

\[
\leq \frac{C}{\lambda^2} \int_0^t |(h^0, h^1)|^2_{X_T^2} ds,
\]

and

\[
\int_0^t \|\varphi \Delta \phi_{bc}|_X^2 \|_{X_T^2} ds \leq \int_0^t \|\varphi \Delta \phi_{bc,S} |_X^2 \|_{X_T^2} ds + \int_0^t \|\varphi \Delta \phi_{bc,T} |_X^2 \|_{X_T^2} ds + \int_0^t \|\Delta \phi_{bc,R} |_X^2 \|_{X_T^2} ds
\]
\[ \leq \frac{C}{\lambda^2} \int_0^t \frac{1}{X_{\frac{1}{2}}^{\frac{3}{2}}} \left( (h^0, h^1)^2 \right) ds, \]

which imply

\[ \int_0^t \left( \| \nabla \phi_{bc} \|_{X_{\frac{1}{2}}}^2 + \| \Delta \phi_{bc} \|_{X_{\frac{1}{2}}}^2 \right) ds \leq \frac{C}{\lambda^2} \int_0^t \frac{1}{X_{\frac{1}{2}}^{\frac{3}{2}}} \left( (h^0, h^1)^2 \right) ds. \]

Due to Proposition 8.12 and taking \( A = (I + R_{bc})^{-1} \), we know \( A \) is a zero-order bounded operator in \( Z_{bc} \) and obtain

\[ \int_0^t \left( (h^0, h^1)^2 \right) \frac{1}{X_{\frac{1}{2}}} ds = \int_0^t \left( |A(\partial_y \phi_{slip}|_{y=0} + C(s) - \partial_y \phi_{slip}|_{y=1} + C(s))|^2 \right) \frac{1}{X_{\frac{1}{2}}} ds \]

\[ \leq C \int_0^t \left( |\nabla \phi_{slip}|_{y=0,1}^2 + |C(s)|^2 \right) ds, \]

which finish this proposition.

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**References**

[1] R. Alexandre, Y. Wang, C.-J. Xu and T. Yang, *Well-posedness of the Prandtl Equation in Sobolev Spaces*, J. Amer. Math. Soc., 28(2015), 745–784.

[2] H. Bahouri, J. Y. Chemin and R. Danchin, *Fourier analysis and nonlinear partial differential equations*, Grundlehren der mathematischen Wissenschaften 343, Springer-Verlag Berlin Heidelberg, 2011.

[3] Y. Brenier, *Homogeneous hydrostatic flows with convex velocity profiles*. Nonlinearity, 12(3): 495–512, 1999.

[4] Y. Brenier, *Remarks on the derivation of the hydrostatic Euler equations*. Bull. Sci. Math., 127(7):585–595, 2003.

[5] C. Cao, S. Ibrahim, K. Nakanishi, and E. Titi, *Finite-time blowup for the inviscid Primitive equations of oceanic and atmospheric dynamics*. Comm. Math. Phys., 337(2):473–482, 2015.

[6] D. Chen, Y. Wang and Z. Zhang, *Well-posedness of the linearized Prandtl equation around a non-monotonic shear flow*, Ann. Inst. H. Poincare Anal. Non Lineaire, 35(2018), 1119–1142.

[7] Q. Chen, D. Wu, and Z. Zhang, *On the \( L^\infty \) stability of prandtl expansions in gevrey class*. arXiv:2004.09755.

[8] H. Dietert and Gérard-Varet, *Well-posedness of the Prandtl equation without any structural assumption*, Ann. PDE, 5(2019), Art. 8, 51 pp.

[9] M. Fei, T. Tao and Z. Zhang, *On the zero-viscosity limit of the Navier-Stokes equations in \( R^3 \) without analyticity*. J. Math. Pures Appl. (9), 112 (2018), 170–229.

[10] D. Gérard-Varet and E. Dormy, *On the ill-posedness of the Prandtl equation*. J. Amer. Math. Soc., 23 (2010), 591–609.

[11] D. Gérard-Varet, S. Iyer and Y. Maekawa, *Improved well-posedness for the Triple-Deck and related models via concavity*. arXiv: 2205.15829.

[12] D. Gérard-Varet, Y. Maekawa and N. Masmoudi, *Gevrey stability of Prandtl expansions for 2-dimensional Navier-Stokes flows*. Duke Math. J., 167(2018), 2531–2631.

[13] D. Gérard-Varet, Y. Maekawa and N. Masmoudi, *Optimal Prandtl expansion around concave boundary layer*. arXiv:2005.05022.

[14] D. Gérard-Varet and N. Masmoudi, *Well-posedness for the Prandtl system without analyticity or monotonicity*. Ann. Sci. Ec. Norm. Super., 48(2015), 1273–1325.

[15] D. Gérard-Varet, N. Masmoudi and Y. Vicol, *Well-posedness of the hydrostatic Navier-Stokes equations*, Anal. PDE 13 (2020), no. 5, 1417–1455.
[16] E. Grenier, *On the derivation of homogeneous hydrostatic equations*. M2AN Math. Model. Numer. Anal., 33(5):965–970, 1999.

[17] E. Grenier, Y. Guo and T. Nguyen, *Spectral instability of general symmetric shear flows in a two-dimensional channel*. Adv. Math., 292 (2016), 52-110.

[18] E. Grenier and T. Nguyen, *L^∞ instability of Prandtl layers*. Ann. PDE, 5 (2019), Paper No. 18, 36 pp.

[19] I. Kukavica, N. Masmoudi, V. Vicol and T. Wong, *On the local well-posedness of the Prandtl and the hydrostatic Euler equations with multiple monotonicity regions*. SIAM J. Math. Anal., 46(6):3865–3890, 2014.

[20] I. Kukavica, R. Temam, V. Vicol and M. Ziane, *Local existence and uniqueness for the hydrostatic Euler equations on a bounded domain*. J. Differential Equations, 250(3): 1719–1746, 2011.

[21] P.-Y. Lagrée and S. Lorthois, *The RNS/Prandtl equations and their link with other asymptotic descriptions: application to the wall shear stress scaling in a constricted pipe*, Int. J. Eng. Sci., 43(2005), 352–378.

[22] W. Li and T. Yang, *Well-posedness in Gevrey space for the Prandtl equations with non-degenerate critical points*. J. Eur. Math. Soc., 22(2020), 717–775.

[23] M. C. Lombardo, M. Cannone and M. Sammartino, *Well-posedness of the boundary layer equations*. SIAM J. Math. Anal., 35 (2003), 987–1004.

[24] O. Oleinik, *On the mathematical theory of boundary layer for an unsteady flow of incompressible fluid*, J. Appl. Math. Mech., 30(1966), 951–974(1967).

[25] N. Masmoudi and T. Wong, *On the H^s theory of hydrostatic Euler equations*. Arch. Ration. Mech. Anal., 204(1):231–271, 2012.

[26] M. Piauci, P. Zhang and Z. Zhang, *On the hydrostatic approximate of the Navier-Stokes equations in a thin strip*, arXiv:1904.04438.

[27] M. Renardy, *Ill-posedness of the hydrostatic Euler and Navier-Stokes equations*, Arch. Ration. Mech. Anal., 194(2009), 877–886.

[28] M. Sammartino and R. E. Caflisch, *Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. I. Existence for Euler and Prandtl equations*. Comm. Math. Phys., 192 (1998), 433–461.

[29] M. Sammartino and R. E. Caflisch, *Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. II. Construction of the Navier-Stokes solution*. Comm. Math. Phys., 192 (1998), 463–491.

[30] C. Wang, Y. Wang and Z. Zhang, *Zero-viscosity limit of the Navier-Stokes equations in the analytic setting*. Arch. Ration. Mech. Anal., 224 (2017), 555–595.

[31] C. Wang, Y. Wang and Z. Zhang, *Gevrey stability of hydrostatic approximate for the Navier-Stokes equations in a thin domain*. Nonlinearity 34 (2021), no. 10, 7185–7226.

[32] L. Wang, Z. Xin and A. Zang, *Vanishing viscous limits for 3D Navier-Stokes equations with a Navier slip boundary condition*. J. Math. Fluid Mech., 14(2012), 791–825.

[33] X. Wang, *A Kato type theorem on zero viscosity limit of Navier-Stokes flows*. Indiana Univ. Math. J., 50(2001), 223–241.

[34] T. Wong, *Blowup of solutions of the hydrostatic Euler equations*. Proc. Amer. Math. Soc., 143(3):1119–1125, 2015.

[35] Y. Xiao and Z. Xin, *On the vanishing viscosity limit for the 3D Navier-Stokes equations with a slip boundary condition*. Comm. Pure Appl. Math., 60(2007), 1027–1055.

[36] Z. Xin and L. Zhang, *On the global existence of solutions to the Prandtl system*. Adv. Math., 181 (2004), 88–133.