Root subgroups on affine spherical varieties

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Abstract
Given a connected reductive algebraic group $G$ and a Borel subgroup $B \subseteq G$, we study $B$-normalized one-parameter additive group actions on affine spherical $G$-varieties. We establish basic properties of such actions and their weights and discuss many examples exhibiting various features. We propose a construction of such actions that generalizes the well-known construction of normalized one-parameter additive group actions on affine toric varieties. Using this construction, for every affine horospherical $G$-variety $X$ we obtain a complete description of all $G$-normalized one-parameter additive group actions on $X$ and show that the open $G$-orbit in $X$ can be connected with every $G$-stable prime divisor via a suitable choice of a $B$-normalized one-parameter additive group action. Finally, when $G$ is of semisimple rank 1, we obtain a complete description of all $B$-normalized one-parameter additive group actions on affine spherical $G$-varieties having an open orbit of a maximal torus $T \subseteq B$.

Keywords
Additive group action · Toric variety · Spherical variety · Demazure root · Locally nilpotent derivation

Mathematics Subject Classification
14R20 · 14M27 · 14M25 · 13N15

1 Introduction

Let $\mathbb{K}$ be an algebraically closed field $\mathbb{K}$ of characteristic zero. If the additive group $\mathbb{G}_a := (\mathbb{K}, +)$ acts nontrivially on an irreducible algebraic variety $X$, its image $H$ in the automorphism group $\text{Aut}(X)$ is called a $\mathbb{G}_a$-subgroup. Moreover, if $X$ is equipped

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with a regular action of a linear algebraic group $F$ and $H$ is normalized by $F$, then we call $H$ an $F$-\textit{root subgroup} on $X$. In this case, the action of $F$ on $H$ by conjugation is controlled by a character of $F$, called the \textit{weight} of $H$.

When $F = T$ is a torus, $T$-root subgroups on $T$-varieties are used to study the automorphism groups and rationality questions [3, 7, 19, 29, 30, 36], transitivity properties for automorphism groups [5, 6], equivariant group embeddings [4, 8], and affine algebraic monoids [1, 20]. The simplest in this setting is the classical case of toric varieties, i.e. normal irreducible $T$-varieties containing an open $T$-orbit. Toric varieties admit a complete combinatorial description in terms of objects of convex geometry called fans [17, 22, 37], and the $T$-root subgroups on a given toric $T$-variety are described in the following simple way: every such subgroup is uniquely determined by its weight and the set of all possible weights is the collection of so-called Demazure roots of the associated fan [16, 18, 29, 37].

A natural intention is to extend the above picture to algebraic varieties equipped with an action of an arbitrary connected reductive algebraic group $G$. In this setting, a proper generalization of toric varieties is given by \textit{spherical} varieties, i.e. normal irreducible $G$-varieties containing an open orbit of a Borel subgroup $B \subseteq G$. These varieties possess many remarkable properties and admit a complete combinatorial description in terms of so-called colored fans, which generalize the fans from the toric case; see [25, 41]. The problem of describing all $G$-root subgroups on affine spherical $G$-varieties was raised in the preprint [28]. It is shown there that such a subgroup is uniquely determined by its weight, and the set of weights is described in some particular cases. However, the set of $G$-root subgroups on an affine spherical $G$-variety seems to be quite restricted; in particular, it is empty whenever $G$ is semisimple.

In this paper, we initiate a systematic study of $B$-root subgroups on affine spherical $G$-varieties. One of our aims is to demonstrate that $B$-root subgroups are more natural and suitable generalizations of $T$-root subgroups on $T$-varieties.

By now, $B$-root subgroups on affine spherical $G$-varieties have already appeared in the literature. In [40], $B$-root subgroups are used to prove that a smooth affine spherical $G$-variety not isomorphic to a torus is uniquely determined by its automorphism group in the category of smooth affine irreducible varieties. The only $B$-root subgroups appearing there are a central $G_a$-subgroup of the unipotent radical $U \subseteq B$ (see Construction 5.7) and all its replicas (see Construction 5.8). In [9], $B$-root subgroups on affine spherical $G$-varieties naturally arise and play an important role in the study of certain combinatorial invariants of spherical subgroups; see Sect. 5.9 for details.

As was already mentioned above, every $T$-root subgroup on an affine toric $T$-variety $Z$ is uniquely determined by its weight and the set of weights is described in terms of Demazure roots. In many applications, the following property of $T$-root subgroups on affine toric $T$-varieties is of particular importance: every $T$-root subgroup on $Z$ moves a unique $T$-stable prime divisor on $Z$. Moreover, there is a bijection between the $T$-stable prime divisors on $Z$ and the equivalence classes of $T$-root subgroups on $Z$ under which each $T$-stable prime divisor $D \subseteq X$ corresponds to all $T$-root subgroups on $Z$ that move $D$. See Sect. 3 for details on these results.

It turns out that in the general spherical case the situation is much more complicated. First, examples show that a $B$-root subgroup on an affine spherical $G$-variety $X$ is not uniquely determined by its weight. Second, $B$-root subgroups on $X$ are naturally
divided into two types: vertical and horizontal. In geometrical terms, a $B$-root subgroup on $X$ is vertical if it preserves the open $B$-orbit and horizontal otherwise. We note that all $T$-root subgroups on affine toric $T$-varieties are horizontal in this terminology. The set of weights of vertical $B$-root subgroups is rather elusive and we cannot say much about it apart from some straightforward observations. On the other hand, horizontal $B$-root subgroups admit a certain reduction to the toric case, which imposes rigid restrictions on the set of their weights and hence makes it much more observable, though still far from a complete understanding. Next, it is easy to see that vertical $B$-root subgroups cannot move any $B$-stable prime divisor on $X$. On the other hand, we show that every horizontal $B$-root subgroup moves a unique such divisor. In the general theory of spherical varieties, $B$-stable prime divisors in $X$ that are not $G$-stable are called colors of $X$. We prove that every color in $X$ that is not of type $a$ (see Definition 4.3) cannot be moved by a $B$-root subgroup. On the other hand, examples show that some colors of type $a$ can be moved by such subgroups. At last, we conjecture that every $G$-stable prime divisor in $X$ can be moved by an appropriate $B$-root subgroup (see Conjecture 5.31).

The main contribution of our paper is a general construction of horizontal $B$-root subgroups on an affine spherical $G$-variety $X$; we call such subgroups standard. In a certain sense, this construction is a generalization of that for $T$-root subgroups on affine toric $T$-varieties. As a first application of standard $B$-root subgroups, we obtain a complete description of the $G$-root subgroups on $X$ whose weights belong to the lattice of weights of $G$-semiinvariant rational functions on $X$. When $X$ has no colors of type $a$, this is in fact a description of all $G$-root subgroups on $X$. The other applications of standard $B$-root subgroups concern the case where $X$ is horospherical (see the definition in Sect. 4.4). First, in the horospherical case there are no colors of type $a$, hence by the above discussion we get a complete description of all $G$-root subgroups on $X$. Second, we obtain a complete description of the set of weights of horizontal $B$-root subgroups on $X$. Third, we prove the above-mentioned conjecture on moving $G$-stable prime divisors by $B$-root subgroups (see Theorem 6.16).

We also study in detail a particular situation where $G$ is of semisimple rank 1 (that is, up to a finite covering, $G$ is isomorphic to the direct product of $SL_2$ with a torus) and $X$ is an affine spherical $G$-variety having an open orbit of a maximal torus $T \subseteq G$. Then $X$ is an affine toric $T$-variety and the known description of $T$-root subgroups on $X$ enables us to describe completely all $B$-root subgroups on $X$, both vertical and horizontal (see Theorem 7.8). In this case, $X$ is automatically horospherical, so our conjecture also holds for $X$.

This paper is organized as follows. In Sect. 2 we discuss preliminary notions and results needed in this paper. In Sect. 3 we present the well-known combinatorial description of $T$-root subgroups on affine toric $T$-varieties via Demazure roots. In Sect. 4 we gather all the necessary material on spherical varieties. In Sect. 5 we discuss basic properties of $B$-root subgroups on affine spherical $G$-varieties and present many examples exhibiting various features. In Sect. 6 we introduce standard $B$-root subgroups and present their applications, including our description of $G$-root subgroups. In Sect. 7 we work out the case of an affine toric variety acted on by a connected reductive group of semisimple rank 1.
While this paper was under review, a proof of our Conjecture 5.31 in the general case appeared in the preprint [12]. That proof is different from our proof in the horospherical case.

2 Preliminaries

In this section we recall basic facts on algebraic transformation groups used in this paper and provide a framework for the study of root subgroups on affine spherical varieties.

2.1 General notation and conventions

Throughout this paper, we work over an algebraically closed field $\mathbb{K}$ of characteristic zero. The notation $\mathbb{K}^\times$ stands for the multiplicative group $(\mathbb{K}, \times)$. The additive group $(\mathbb{K}, +)$ is denoted by $\mathbb{G}_a$ and regarded as a one-dimensional linear algebraic group.

If $G$ is an algebraic group then $\chi(G)$ denotes the character group of $G$ (in additive notation).

If $X$ is an irreducible algebraic variety then $\text{Aut}(X)$ denotes its automorphism group and $\mathbb{K}[X]$ (resp. $\mathbb{K}(X)$) stands for the algebra of regular functions (resp. field of rational functions) on $X$. If in addition $X$ is equipped with a regular action of an algebraic group $H$ then $\mathbb{K}[X]^H$ (resp. $\mathbb{K}(X)^H$) denotes the subalgebra (resp. subfield) of $H$-invariant functions in $\mathbb{K}[X]$ (resp. $\mathbb{K}(X)$).

Given a regular action $G \times X \to X$ of an algebraic group $G$ on an algebraic variety $X$, we say that the image of $G$ in $\text{Aut}(X)$ is an algebraic subgroup of $\text{Aut}(X)$.

If the group $\text{Aut}(X)$ itself has a structure of an algebraic group such that the natural action $\text{Aut}(X) \times X \to X$ is regular, this definition agrees with the standard definition of an algebraic subgroup in an algebraic group.

2.2 $\mathbb{G}_a$-actions and locally nilpotent derivations

Given an algebraic variety $X$, by a $\mathbb{G}_a$-action on $X$ we mean a regular action $\mathbb{G}_a \times X \to X$. If such an action is nontrivial, it defines a nontrivial algebraic subgroup of $\text{Aut}(X)$, which is called a $\mathbb{G}_a$-subgroup.

A derivation $\partial$ of an algebra $A$ is said to be locally nilpotent (LND for short) if for every $a \in A$ there exists $k \in \mathbb{Z}_{>0}$ such that $\partial^k(a) = 0$. If $A = \mathbb{K}[X]$ for an affine algebraic variety $X$, for any LND $\partial$ on $A$ the map

$$\varphi_\partial: \mathbb{G}_a \times A \to A, \quad (s, a) \mapsto \exp(s \partial)(a), \quad (2.1)$$

defines a rational $\mathbb{G}_a$-algebra structure on $A$, hence induces a $\mathbb{G}_a$-action on $X$. In fact, by [21,Section 1.5] any $\mathbb{G}_a$-action on $X$ arises this way, which yields

**Proposition 2.1** Given an affine variety $X$, the map $\partial \mapsto \varphi_\partial$ induces a bijection between the nonzero LNDs on $\mathbb{K}[X]$ modulo proportionality and the $\mathbb{G}_a$-subgroups on $X$. 
2.3 Equivalence of $\mathbb{G}_a$-subgroups

Let $X$ be an irreducible variety. We say that two algebraic subgroups $H_1$ and $H_2$ in $\text{Aut}(X)$ are equivalent if there exists a nonempty open subset $W \subseteq X$ such that

$$H_1x \cap W = H_2x \cap W$$

for all $x \in W$.

By Rosenlicht’s Theorem (see, e.g., [39, Theorem 2.3]), rational invariants separate orbits of general position in $X$. This implies that two subgroups $H_1$ and $H_2$ are equivalent if and only if the fields $\mathbb{K}(X)^{H_1}$ and $\mathbb{K}(X)^{H_2}$ of rational invariants coincide.

If $X$ is affine and $H_1$, $H_2$ are unipotent, the fields of rational invariants are the quotient fields of the algebras of regular invariants $\mathbb{K}[X]^{H_1}$ and $\mathbb{K}[X]^{H_2}$, respectively; see [39, Theorem 3.3]. Since $\mathbb{K}[X]^{H_i} = \mathbb{K}[X] \cap \mathbb{K}(X)^{H_i}$ for $i = 1, 2$, we conclude that $H_1$ and $H_2$ are equivalent if and only if $\mathbb{K}[X]^{H_1} = \mathbb{K}[X]^{H_2}$.

**Proposition 2.2** Suppose $X$ is an irreducible affine variety, $H_1$, $H_2$ are two $\mathbb{G}_a$-subgroups in $\text{Aut}(X)$, and $\partial_1$, $\partial_2$ are the corresponding LNDs on $\mathbb{K}[X]$. Then the following conditions are equivalent.

1. $H_1$ and $H_2$ are equivalent.
2. $\text{Ker} \, \partial_1 = \text{Ker} \, \partial_2$.
3. There are $a_1, a_2 \in \mathbb{K}[X] \setminus \{0\}$ such that $a_1 \partial_1 = a_2 \partial_2$.

**Proof** Observe that $\mathbb{K}[X]^{H_i} = \text{Ker} \, \partial_i$ for $i = 1, 2$; then the equivalence $(1) \iff (2)$ follows from the above discussion. The equivalence $(2) \iff (3)$ is implied by [21, Principle 12].

\[\square\]

2.4 Root subgroups

Let $X$ be an irreducible variety and let $F$ be a linear algebraic subgroup of $\text{Aut}(X)$.

**Definition 2.3** An $F$-root subgroup on $X$ is a $\mathbb{G}_a$-subgroup in $\text{Aut}(X)$ normalized by $F$.

Given an $F$-root subgroup $H$ on $X$, the corresponding group homomorphism $\varphi: \mathbb{G}_a \to \text{Aut}(X)$ satisfies

$$g\varphi(s)g^{-1} = \varphi(\chi(g)s)$$

for all $g \in F$, $s \in \mathbb{K}$ and some $\chi \in \mathcal{X}(F)$. The character $\chi = \chi_H$ is called the weight of $H$.

Now assume that $X$ is affine. An LND $\partial$ on $\mathbb{K}[X]$ is said to be $F$-normalized if

$$g \cdot \partial(g^{-1} \cdot f) = \chi(g)\partial(f)$$

for all $g \in F$, $f \in \mathbb{K}[X]$ and some character $\chi \in \mathcal{X}(F)$, which is called the weight of $\partial$.

The next result is a direct consequence of Proposition 2.1.
Proposition 2.4 Given an affine variety \( X \), the map \( \partial \mapsto \varphi_\partial \) in (2.1) induces a weight-preserving bijection between the nonzero \( F \)-normalized LNDs on \( \mathbb{K}[X] \) modulo proportionality and the \( F \)-root subgroups on \( X \).

Remark 2.5 If \( X \) is affine and \( F = T \) is a torus then the algebra \( \mathbb{K}[X] \) has the natural grading
\[
\mathbb{K}[X] = \bigoplus_{u \in \mathfrak{X}(T)} \mathbb{K}[X]_u
\]
where each homogeneous component \( \mathbb{K}[X]_u \subseteq \mathbb{K}[X] \) is the \( T \)-weight subspace of weight \( u \), i.e. \( \mathbb{K}[X]_u = \{ f \in \mathbb{K}[X] \mid t \cdot f = u(t) f \text{ for all } t \in T \} \). It is easy to see that an LND \( \partial \) on \( \mathbb{K}[X] \) is \( T \)-normalized if and only if \( \partial \) is homogeneous, i.e. sends homogeneous elements to homogeneous ones; see [21, Section 3.7].

2.5 Orbits of root subgroups

Let \( X \) and \( F \) be as in Sect. 2.4 (\( X \) is not necessarily affine) and let \( H \) be an \( F \)-root subgroup on \( X \).

Take an \( H \)-orbit \( Y \subseteq X \) and let \( F_Y \) denote the stabilizer in \( F \) of the subvariety \( Y \). Since \( H \) is \( F \)-normalized, \( gY \) is again an \( H \)-orbit in \( X \) for all \( g \in F \). In particular, the stabilizer in \( F \) of a point in \( Y \) is contained in \( F_Y \).

Assume that \( Y \) is not a point. Then \( Y \) is isomorphic to the affine line \( \mathbb{A}^1 \), whose automorphisms are well known to have the form \( x \mapsto ax + b \) for some \( a, b \in \mathbb{K}, a \neq 0 \). Thus we have the following three possibilities.

Case 1: \( F_Y \) acts on \( Y \) transitively. In this case, \( Y \) is contained in a single \( F \)-orbit \( O \), which is automatically preserved by \( H \).

Case 2: \( F_Y \) acts on \( Y \) as a one-dimensional torus. In this case, \( Y \) meets precisely two \( F \)-orbits in \( X \), say \( O_1 \) and \( O_2 \). The stabilizers in \( F \) of a point in \( O_1 \) and of a point in \( O_2 \) differ by a one-dimensional torus. So, up to renumbering, we have \( \dim O_1 = \dim O_2 + 1 \). \( O_2 \) meets \( Y \) at a single point \( P \), and \( O_1 \) meets \( Y \) in the open subset \( Y \setminus \{ P \} \). Then the \( H \)-orbit of any other point in \( O_1 \) meets \( O_2 \) in a single point whose complement is contained in \( O_1 \), and we say that the \( F \)-orbits \( O_1 \) and \( O_2 \) are connected by \( H \).

Case 3. \( F_Y \) acts on \( Y \) as a finite group. In this case, \( Y \) meets infinitely many \( F \)-orbits in \( X \).

Note the following observations:

• if \( F \) acts on \( X \) with finitely many orbits then Case 3 is excluded;
• if \( F_Y \) is a torus then Case 1 does not occur.

We say that \( H \) moves an \( F \)-stable prime divisor \( D \subseteq X \) if \( D \) is not \( H \)-stable.

Proposition 2.6 Under the above assumptions suppose that \( F \) is connected and acts on \( X \) with an open orbit \( O_F \). If \( H \) does not preserve \( O_F \) then there is exactly one \( F \)-stable prime divisor on \( X \) moved by \( H \); moreover, this divisor contains an open \( F \)-orbit.
Proof Let $Y \subseteq X$ be an $H$-orbit that meets $O_F$. Since the intersection $Y \cap O_F$ is open in $Y$, Case 3 does not occur. Since $O_F$ is not preserved by $H$, Case 1 is excluded. So Case 2 holds for $Y$ and hence there is an $F$-orbit $O' \subseteq X$ with $\dim O' = \dim O_F - 1$ such that $O_F$ and $O'$ are connected by $H$. Now the closure of $O'$ in $X$ is an $F$-stable prime divisor on $X$ moved by $H$. Clearly, all other $F$-stable prime divisors on $X$ are $H$-stable. \hfill \Box

3 Demazure roots and root subgroups on affine toric varieties

In this section we present the well-known description of root subgroups on affine toric varieties.

3.1 Demazure roots of a cone

Let $M$ be a lattice of finite rank and consider the dual lattice $N = \text{Hom}_\mathbb{Z}(M, \mathbb{Z})$ along with the natural pairing $(\cdot, \cdot): N \times M \to \mathbb{Z}$. Consider also the rational vector spaces $M_\mathbb{Q} = M \otimes_{\mathbb{Z}} \mathbb{Q}$ and $N_\mathbb{Q} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ and extend the pairing to a bilinear map $(\cdot, \cdot): N_\mathbb{Q} \times M_\mathbb{Q} \to \mathbb{Q}$.

Let $G \subseteq M_\mathbb{Q}$ be a finitely generated (or, equivalently, polyhedral) convex cone and recall some terminology related to it. The cone $G$ is said to be strictly convex if $G \cap (-G) = \{0\}$, i.e. $G$ contains no nonzero subspaces of $M_\mathbb{Q}$. The dimension of $G$ is that of its linear span. The dual cone of $G$ is

$$G^\vee := \{ q \in N_\mathbb{Q} \mid (q, v) \geq 0 \text{ for all } v \in G \};$$

this is a finitely generated convex cone in $N_\mathbb{Q}$. Note that one automatically has

$$G = \{ v \in M_\mathbb{Q} \mid (q, v) \geq 0 \text{ for all } q \in G^\vee \},$$

so that the cone $G$ is dual to $G^\vee$. A face of $G$ is a subset $\mathcal{F} \subseteq G$ of the form

$$\mathcal{F} = \{ v \in G \mid (q, v) = 0 \}$$

for some $q \in G^\vee$. Every face of $G$ is itself a finitely generated convex cone. Faces of $G^\vee$ are defined in a similar way. A face of codimension one is called a facet. A face of dimension one of a strictly convex cone is called a ray.

Now put $\mathcal{E} := G^\vee$ and assume in addition that $G$ has full dimension, so that $\mathcal{E}$ is strictly convex. Let $\mathcal{E}^1$ be the set of primitive elements $\rho$ of the lattice $N$ such that $Q_{\geq 0} \rho$ is a ray of $\mathcal{E}$. For every $\rho \in \mathcal{E}^1$, let $\mathcal{F}_\rho = G \cap \text{Ker } \rho$ be the facet of $G$ defined by $\rho$. Note that the map $\rho \mapsto \mathcal{F}_\rho$ is a bijection between $\mathcal{E}^1$ and the facets of $G$. For every $\rho \in \mathcal{E}^1$, we define

$$\mathcal{R}_\rho(\mathcal{E}) := \{ e \in M \mid (\rho, e) = -1 \text{ and } (\rho', e) \geq 0 \text{ for all } \rho' \in \mathcal{E}^1 \setminus \{ \rho \} \}.$$
One easily checks that for \( \text{rk } M \geq 2 \) the set \( \mathcal{R}_\rho(\mathcal{E}) \) is infinite for each \( \rho \in \mathcal{E}^1 \). The elements of the set \( \mathcal{R}(\mathcal{E}) := \bigcup_{\rho \in \mathcal{E}^1} \mathcal{R}_\rho(\mathcal{E}) \) are called the Demazure roots of the cone \( \mathcal{E} \).

For future reference, we mention the following consequence of the construction.

**Remark 3.1** One has \( \mathcal{R}(\mathcal{E}) \subseteq M \).

Below in this paper we shall need the following simple result.

**Lemma 3.2** Let \( \mathcal{G}, \mathcal{G}' \) be two finitely generated convex cones of full dimension in \( M_\mathbb{Q} \) and let \( \mathcal{E} := \mathcal{G}' \), \( \mathcal{E}' := \mathcal{G}'^\vee \) be the respective dual cones. Suppose that \( \mathcal{G} \subseteq \mathcal{G}' \) (and hence \( \mathcal{E}' \subseteq \mathcal{E} \)). Then for every \( \rho \in \mathcal{E}^1 \backslash \mathcal{E}' \) the set \( \mathcal{R}_\rho(\mathcal{E}) \cap \mathcal{G}' \) contains infinitely many elements.

**Proof** As \( \mathbb{Q}_{\geq 0} \rho \) is a ray of \( \mathcal{E} \), there exists an element \( v \in \mathcal{G} \) such that \( \langle \rho, v \rangle = 0 \) and \( \langle \rho', v \rangle > 0 \) for all \( \rho' \in \mathcal{E}^1 \backslash \{\rho\} \). It follows that \( \langle q, v \rangle > 0 \) for all \( q \in \mathcal{E}' \backslash \{0\} \). Rescaling \( v \) if necessary we may assume in addition that \( v \in M \). Then for every \( e \in \mathcal{R}_\rho(\mathcal{E}) \) all elements of the form \( e + kv \) with \( k \geq k_0 \) yield the desired infinite set of Demazure roots for a sufficiently large \( k_0 \in \mathbb{Z}_{> 0} \). \( \square \)

### 3.2 Root subgroups on affine toric varieties

Let \( T \) be a torus. For every \( u \in \mathcal{X}(T) \), let \( \chi^u \) be the regular function on \( T \) representing the character \(-u\). Then \( \chi^u \) is \( T \)-semiinvariant of weight \( u \), \( \chi^{u_1 \cdot u_2} = \chi^{u_1 + u_2} \) for all \( u_1, u_2 \in \mathcal{X}(T) \), and there is the decomposition

\[
\mathbb{K}[T] = \bigoplus_{u \in \mathcal{X}(T)} \mathbb{K} \chi^u.
\]

A \( T \)-variety \( X \) is said to be toric if it is irreducible, normal, and has an open \( T \)-orbit. It is well known that affine toric \( T \)-varieties are parametrized by pairs \((M, \mathcal{G})\) where \( M \) is a sublattice of \( \mathcal{X}(T) \) and \( \mathcal{G} \) is a finitely generated convex cone of full dimension in \( M_\mathbb{Q} = M \otimes_\mathbb{Z} \mathbb{Q} \). More precisely, given such a pair \((M, \mathcal{G})\), the corresponding affine toric \( T \)-variety is \( X = \text{Spec } A \) where \( A = \bigoplus_{u \in M \cap \mathcal{G}} \mathbb{K} \chi^u \). The monoid \( \Gamma = M \cap \mathcal{G} \) is said to be the weight monoid of \( X \), it is characterized as the set of \( T \)-weights of the algebra \( \mathbb{K}[X] \). Note that \( M = \mathbb{Z} \Gamma \) and \( \mathcal{G} = \mathbb{Q}_{\geq 0} \Gamma \), which gives a direct way to recover the pair \((M, \mathcal{G})\) from \( X \).

**Remark 3.3** In the definition of a toric \( T \)-variety it is often additionally required that the action of \( T \) be effective. This corresponds to \( M = \mathcal{X}(T) \) in our notation.

Let \( X \) be a toric \( T \)-variety corresponding to a pair \((M, \mathcal{G})\) as above. Put \( N = \text{Hom}_\mathbb{Z}(M, \mathbb{Z}) \) and retain the notation of Sect. 3.1.

Let \( \rho \in \mathcal{E}^1 \) and \( e \in \mathcal{R}_\rho(\mathcal{E}) \). One defines a \( T \)-normalized LND \( \partial_e \) of weight \( e \) on \( A \) by the rule

\[
\partial_e(\chi^u) = \langle \rho, u \rangle \chi^{u+e}.
\] (3.1)
Let $H_e$ denote the $T$-root subgroup on $X$ corresponding to $\partial_e$; see Proposition 2.1. It is known from [29,Theorem 2.7] that every nonzero $T$-normalized LND on $\mathbb{K}[X]$ has the form $c\partial_e$ for some $e \in \mathcal{R}(\mathcal{E})$ and $c \in \mathbb{K}^\times$, which yields

**Theorem 3.4** The following assertions hold.

(a) The map $e \mapsto \partial_e$ is a bijection between $\mathcal{R}(\mathcal{E})$ and the nonzero $T$-normalized LNDs on $\mathbb{K}[X]$ modulo proportionality.

(b) The map $e \mapsto H_e$ is a bijection between $\mathcal{R}(\mathcal{E})$ and the $T$-root subgroups on $X$.

By construction, $\text{Ker} \partial_e = \bigoplus_{u \in \mathcal{F}_\rho} \mathbb{K}\chi^u$. Combining this with Proposition 2.2 we find that, given $e_1, e_2 \in \mathcal{R}(\mathcal{E})$, the $T$-root subgroups $H_{e_1}$ and $H_{e_2}$ are equivalent if and only if the associated elements $\rho_1, \rho_2 \in \mathcal{E}^1$ coincide.

Recall from [17,Theorem 3.2.6] that $T$-stable prime divisors on $X$ are in bijection with $\mathcal{E}^1$. Under this bijection, an element $\rho \in \mathcal{E}^1$ corresponds to a $T$-stable prime divisor $D_\rho$ whose ideal $I(D_\rho)$ in $\mathbb{K}[X]$ equals $\bigoplus_{u \in \Gamma \setminus \mathcal{F}_\rho} \mathbb{K}\chi^u$. Now for every $e \in \mathcal{R}_\rho(\mathcal{E})$ we have $\mathbb{K}[X] = \text{Ker} \partial_e \oplus I(D_\rho)$, therefore $I(D_\rho)$ is $\partial_e$-unstable and hence $H_e$-unstable. Thus $H_e$ moves the divisor $D_\rho$.

We come to the following result, which can be extracted from [5,Section 2].

**Theorem 3.5** Let $X$ be an affine toric $T$-variety. Then equivalence classes of $T$-root subgroups on $X$ are in bijection with $T$-stable prime divisors on $X$. More precisely, in the notation introduced above, the equivalence class corresponding to a divisor $D_\rho$ consists of all $T$-root subgroups that move $D_\rho$, and these subgroups are defined by all Demazure roots in $\mathcal{R}_\rho(\mathcal{E})$.

We finish this subsection with an example illustrating all the concepts introduced above.

**Example 3.6** (*The affine space as a toric variety*) Consider the affine space $X = \mathbb{K}^n$ equipped with the standard diagonal action of the torus $T = (\mathbb{K}^\times)^n$ given by $t \cdot (x_1, \ldots, x_n) = (t_1x_1, \ldots, t_nx_n)$ for all $t = (t_1, \ldots, t_n) \in T$. Then $X$ is a toric $T$-variety with an effective action of $T$, so that $M = \mathcal{X}(T)$. In what follows, we identify $M$ with $\mathbb{Z}^n$ by choosing the basis $\chi_1, \ldots, \chi_n$ in $\mathcal{X}(T)$ where $\chi_i(t) = t_i^{-1}$ for all $i = 1, \ldots, n$ and $t = (t_1, \ldots, t_n) \in T$. The dual lattice $N$ will be also identified with $\mathbb{Z}^n$ via the dot product. Modulo the above identifications, we have $M = N = \mathbb{Z}^n$, $M_\mathbb{Q} = N_\mathbb{Q} = \mathbb{Q}^n$, $\Gamma = \mathbb{Z}_{\geq 0}^n$, $\mathcal{G} = \mathcal{E} = \mathbb{Q}^n$, $\mathcal{E}^1 = \{\rho_1, \ldots, \rho_n\}$ with $\rho_1 = (1, 0, \ldots, 0), \ldots, \rho_n = (0, \ldots, 0, 1)$. Then

$$\mathcal{R}_{\rho_i}(\mathcal{E}) = \{(c_1, \ldots, c_{i-1}, -1, c_{i+1}, \ldots, c_n) \mid c_j \in \mathbb{Z}_{\geq 0}\}$$

for all $i = 1, \ldots, n$. For $n = 2$, the picture looks like
Let $x_1, \ldots, x_n$ be the coordinate functions on $X$, so that $\mathbb{K}[X] = \mathbb{K}[x_1, \ldots, x_n]$. It is easy to see that the $T$-normalized LND on $\mathbb{K}[X]$ corresponding to a Demazure root $e = (c_1, \ldots, c_{i-1}, -1, c_{i+1}, \ldots, c_n) \in \mathfrak{R}_\rho$ is

$$\partial_e = x_1^{c_1} \cdots x_{i-1}^{c_{i-1}} x_i^{-1} x_{i+1}^{c_{i+1}} \cdots x_n^{c_n} \frac{\partial}{\partial x_i}.$$ 

This LND gives rise to the $\mathbb{G}_a$-action

$$(s, (x_1, \ldots, x_n)) \mapsto (x_1, \ldots, x_{i-1}, x_i + sx_1^{c_1} \cdots x_{i-1}^{c_{i-1}} x_{i+1}^{c_{i+1}} \cdots x_n^{c_n}, x_{i+1}, \ldots, x_n)$$

on $X$. The corresponding $T$-root subgroup $H_e$ moves the divisor $D_i = \{x_i = 0\}$.

### 4 Generalities on affine spherical varieties

#### 4.1 Notation for reductive groups

In this subsection we introduce some general notation for reductive groups that will be used in the remaining part of this paper.

In what follows, $G$ denotes a connected reductive algebraic group and $(G, G)$ stands for the derived subgroup of $G$. Fix a Borel subgroup $B \subseteq G$ along with a maximal torus $T \subseteq B$. There is a unique Borel subgroup $B^{-}$ of $G$ such that $B \cap B^{-} = T$, it is said to be opposite to $B$. Let $U$ (resp. $U^{-}$) be the unipotent radical of $B$ (resp. $B^{-}$); both $U$ and $U^{-}$ are maximal unipotent subgroups of $G$. We also put $u = \text{Lie } U$.

We identify the groups $X(B)$ and $X(T)$ via restricting characters from $B$ to $T$. Similarly, we regard $\mathfrak{X}(G)$ as a subgroup of $\mathfrak{X}(T)$.

Let $\Delta \subseteq \mathfrak{X}(T)$ be the root system of $G$ with respect to $T$ and let $\Pi \subseteq \Delta$ be the set of simple roots with respect to $B$. For every $\alpha \in \Delta$, we let $\alpha^\vee \in \text{Hom}_\mathbb{Z}(\mathfrak{X}(T), \mathbb{Z})$ be the corresponding dual root and let $U_\alpha \subseteq G$ be the corresponding one-parameter unipotent subgroup.

Let $\Lambda^+ \subseteq \mathfrak{X}(T)$ be the monoid of dominant weights with respect to $B$. Recall that $\Lambda^+$ is in bijection with the (isomorphism classes of) simple finite-dimensional $G$-modules. Under this bijection, every $\lambda \in \Lambda^+$ corresponds to the simple $G$-module with highest weight $\lambda$.

#### 4.2 Spherical varieties and related notions

A $G$-variety $X$ is said to be spherical if $X$ is irreducible, normal, and possesses an open $B$-orbit.

Recall that a rational $G$-module $W$ is said to be multiplicity free if each simple $G$-module occurs in $W$ with multiplicity at most one.

**Theorem 4.1** ([42, Theorem 2]) Let $X$ be a normal irreducible $G$-variety. The following assertions hold.

(a) If $X$ is spherical then the $G$-module $\mathbb{K}[X]$ is multiplicity free.
(b) If the $G$-module $\mathbb{K}[X]$ is multiplicity free and $X$ is quasi-affine then $X$ is spherical.

In what follows we let $X$ be a spherical $G$-variety. The weight lattice (resp. weight monoid) of $X$ is the set $M = M(X)$ (resp. $\Gamma = \Gamma(X)$) consisting of weights of $B$-semiinvariant functions in $\mathbb{K}(X)$ (resp. $\mathbb{K}[X]$). Clearly, $M$ is a sublattice of $\mathfrak{X}(T)$ and $\Gamma$ is a submonoid of $M \cap \Lambda^+$. Thanks to Theorem 4.1, for every $\lambda \in \Gamma$ there is a unique simple $G$-submodule $\mathbb{K}[X]_\lambda \subseteq \mathbb{K}[X]$ with highest weight $\lambda$, and one has the decomposition

$$\mathbb{K}[X] = \bigoplus_{\lambda \in \Gamma} \mathbb{K}[X]_\lambda. \quad (4.1)$$

Since $X$ has an open $B$-orbit, for every $\lambda \in M$ there is a unique up to proportionality $B$-semiinvariant function $f_\lambda \in \mathbb{K}(X)$ of weight $\lambda$. Note that for $\lambda \in \Gamma$ the function $f_\lambda$ is a highest-weight vector in $\mathbb{K}[X]_\lambda$. Requiring each function $f_\lambda$ to take the value 1 at a fixed point of the open $B$-orbit in $X$, we may assume $f_\lambda \cdot f_\mu = f_{\lambda+\mu}$ for all $\lambda, \mu \in \Gamma$.

Let $M_\mathbb{Q}, N, N_\mathbb{Q}$ be as in Sect. 3.1. Put also $\mathcal{G} = \mathbb{Q}_{\geq 0} \Gamma \subseteq M_\mathbb{Q}$ and $\mathcal{E} = \mathcal{G}^\vee \subseteq N_\mathbb{Q}$. Every discrete $\mathbb{Q}$-valued valuation $v$ of the field $\mathbb{K}(X)$ vanishing on $\mathbb{K}^\times$ determines an element $\varphi(v) \in N_\mathbb{Q}$ such that $\varphi(v)(\lambda) = v(f_\lambda)$ for all $\lambda \in M$. It is known (see [32,Section 7.4] or [25,Corollary 1.8]) that the restriction of the map $v \mapsto \varphi(v)$ to the set of $G$-invariant discrete $\mathbb{Q}$-valued valuations of $\mathbb{K}(X)$ vanishing on $\mathbb{K}^\times$ is injective; we denote its image by $\mathcal{V} = \mathcal{V}(X)$. Moreover, $\mathcal{V} \subseteq N_\mathbb{Q}$ is a finitely generated convex cone of full dimension containing the image of the antidominant Weyl chamber; see [15,Proposition 3.2 and Corollary 4.1, i]) or [25,Corollary 5.3]. The cone $\mathcal{V}$ is called the valuation cone of $X$. Elements of the set

$$\Sigma = \Sigma(X) := -\langle \mathcal{V}^\vee \rangle^1 \quad (4.2)$$

are called spherical roots of $X$. The above discussion implies that every spherical root is a nonnegative linear combination of simple roots.

Since the group $B$ is solvable, the open $B$-orbit in $X$ is affine, therefore its complement is a union of a finite number of prime divisors (which are automatically $B$-stable). We note that the open $G$-orbit in $X$ need not be affine and in general its complement may contain irreducible components of codimension $\geq 2$.

Let $\mathcal{D}^B = \mathcal{D}^B(X)$ (resp. $\mathcal{D}^G = \mathcal{D}^G(X)$) be the set of $B$-stable (resp. $G$-stable) prime divisors on $X$ and put $\mathcal{D} = \mathcal{D}(X) = \mathcal{D}^B \setminus \mathcal{D}^G$. Elements of $\mathcal{D}$ are called colors of $X$. For every $D \in \mathcal{D}^B$, let $v_D$ be the valuation of $\mathbb{K}(X)$ defined by $D$, i.e. $v_D(f) = \text{ord}_D(f)$ for every $f \in \mathbb{K}(X) \setminus \{0\}$. We define the map

$$\nu: \mathcal{D}^B \to N, \quad D \mapsto \varphi(v_D), \quad (4.3)$$

so that $\langle \nu(D), \lambda \rangle = \text{ord}_D f_\lambda$ for all $D \in \mathcal{D}^B$ and $\lambda \in M$. It follows from the definitions that

$$\nu(\mathcal{D}^G) \subseteq \mathcal{V}. \quad (4.4)$$
Since $X$ is normal, a regular function on the open $B$-orbit in $X$ is regular on the whole $X$ if and only if it has no poles along all the $B$-stable prime divisors on $X$, therefore

$$
\Gamma = \{ \lambda \in M \mid \langle \kappa(D), \lambda \rangle \geq 0 \text{ for all } D \in \mathcal{D}^B \}. \quad (4.5)
$$

In particular, it follows that the monoid $\Gamma$ is finitely generated and saturated, where the latter means that $\Gamma$ is the intersection of a lattice with a cone. An equivalent form of (4.5) is as follows:

$$
\mathcal{E} = \mathbb{Q}_{\geq 0}\{ \kappa(D) \mid D \in \mathcal{D}^B \}. \quad (4.6)
$$

In particular, for every element $\rho \in \mathcal{E}^1$ there is $D \in \mathcal{D}^B$ such that $\kappa(D)$ is a positive multiple of $\rho$.

Let $O$ be the open $G$-orbit in $X$, which is a homogeneous spherical $G$-variety. Then

$$
\Gamma(O) = \{ \lambda \in M \mid \langle \kappa(D), \lambda \rangle \geq 0 \text{ for all } D \in \mathcal{D} \}. \quad (4.7)
$$

Observe that the algebra $\mathbb{K}[X]$ is identified with the subalgebra $\bigoplus_{\lambda \in \Gamma} \mathbb{K}[O]_{\lambda}$ of $\mathbb{K}[O]$.

To finish this subsection, we single out a certain subclass of colors of $X$. For every $\alpha \in \Pi$, let $P_\alpha \supseteq B$ be the minimal parabolic subgroup of $G$ containing $U_{-\alpha}$ and put $\mathcal{D}_\alpha := \{ D \in \mathcal{D} \mid D \text{ is moved by } P_\alpha \}$. Then $\mathcal{D} = \bigcup_{\alpha \in \Pi} \mathcal{D}_\alpha$. The next result is extracted from [31,Sections 2.7, 3.4]; see also [41,Section 30.10].

**Proposition 4.2** For every $\alpha \in \Pi$, the following assertions hold.

(a) $|\mathcal{D}_\alpha| \leq 2$ and the equality is attained if and only if $\alpha \in \Sigma$.

(b) If $|\mathcal{D}_\alpha| = 2$ and $\mathcal{D}_\alpha \cap \mathcal{D}_\beta \neq \emptyset$ for some $\beta \in \Pi$ then $|\mathcal{D}_{\beta}| = 2$.

(c) If $|\mathcal{D}_\alpha| = 1$ and $\mathcal{D}_\alpha = \{ D \}$ then there is $c \in \{1, \frac{1}{2}\}$ such that $\langle \kappa(D), \lambda \rangle = c \langle \alpha^{\vee}, \lambda \rangle$ for all $\lambda \in M$.

In view of parts (a), (b) of the above proposition, the following notion is well defined.

**Definition 4.3** A color $D \in \mathcal{D}$ is of type $a$ if $D \in \mathcal{D}_\alpha$ for some $\alpha \in \Pi \cap \Sigma$.

**Remark 4.4** It follows from Definition 4.3 and Proposition 4.2(a) that $X$ has colors of type $a$ if and only if $\Pi \cap \Sigma \neq \emptyset$.

Combining Proposition 4.2(a, c) with (4.7) we obtain

**Proposition 4.5** Suppose that $\Pi \cap \Sigma = \emptyset$. Then $\Gamma(O) = M \cap \Lambda^+$.

**Remark 4.6** As follows from the definitions, for given $D \in \mathcal{D}$ and $\alpha \in \Pi$ the condition $D \in \mathcal{D}_\alpha$ holds if and only if $D$ is moved by $U_{-\alpha}$, which induces a $T$-root subgroup on $X$ that is not a $B$-root subgroup. If in addition $\alpha \in \Sigma$ then we know from Proposition 4.2(a) that $U_{-\alpha}$ moves exactly two colors; compare with Proposition 2.6.
4.3 Affine spherical varieties

In this subsection we present several notions and results specific to affine spherical varieties. Until the end of this subsection we assume that $X$ is an affine spherical $G$-variety and retain the notation of Sect. 4.2.

The following result is well known; see, e.g., [41,Proposition 5.14] for a proof. It implies that the cone $\mathcal{G}$ has full dimension in $M_\mathbb{Q}$ and hence the cone $\mathcal{E}$ is strictly convex, which will be important in our subsequent considerations.

**Proposition 4.7** One has $M = \mathbb{Z}\Gamma$.

The algebra $\mathbb{K}[X]^U$ decomposes as
\[
\mathbb{K}[X]^U = \bigoplus_{\lambda \in \Gamma} \mathbb{K}f_{\lambda}. \tag{4.8}
\]

By [39,Theorem 3.13], $\mathbb{K}[X]^U$ is finitely generated, hence we can consider the variety $Z := \text{Spec} \mathbb{K}[X]^U$. Observe that $Z$ is a toric $T$-variety with weight monoid $\Gamma$. The inclusion $\mathbb{K}[X]^U \subseteq \mathbb{K}[X]$ gives rise to a dominant $T$-equivariant morphism $\pi_U : X \to Z$. \tag{4.9}

**Proposition 4.8** A generic fiber of $\pi_U$ is a single $U$-orbit.

**Proof** The discussion in Sect. 2.3 implies that the functions in $\mathbb{K}[X]^U$ separate generic $U$-orbits in $X$. Since all $U$-orbits in $X$ are closed (see [39,Section 1.3] or [41,Lemma 3.4]), this yields the assertion. \hfill \Box

Since $\mathbb{K}[X]^G = \mathbb{K}$, it follows that $X$ contains a unique closed $G$-orbit. Then similarly to [25,Lemma 2.4] one proves the following result.

**Proposition 4.9** The restriction of $\kappa$ to $\mathcal{D}^G$ is an injective map to $\mathcal{E}^1$ and its image is
\[
\{ \rho \in \mathcal{E}^1 \mid \mathbb{Q}_{\geq 0} \rho \cap \kappa(\mathcal{D}) = \emptyset \}. \tag{4.10}
\]

Given $\lambda, \mu, \nu \in \Gamma$ such that the linear span of $\mathbb{K}[X]_\lambda \cdot \mathbb{K}[X]_\mu$ contains $\mathbb{K}[X]_\nu$, the expression $\lambda + \mu - \nu$ is said to be a tail of $X$. The cone $\mathcal{T} = \mathcal{T}(X) \subseteq M_\mathbb{Q}$ generated by all tails of $X$ is said to be the tail cone of $X$.

**Proposition 4.10** (see [26,Lemma 6.6, iii])) Both cones $\mathcal{T}$ and $\mathcal{T}(O)$ coincide with $-\mathcal{V}^\vee$.

In view of (4.2) and (4.4) we obtain

**Corollary 4.11** The following assertions hold.

(a) $\Sigma = T^1$.

(b) For all $D \in \mathcal{D}^G$ and $\tau \in T$, one has $\langle \kappa(D), \tau \rangle \leq 0$. 
4.4 Affine horospherical varieties

An irreducible $G$-variety $X$ is said to be horospherical if the stabilizer of a point in general position in $X$ contains a maximal unipotent subgroup of $G$. A normal horospherical $G$-variety is spherical if and only if it contains an open $G$-orbit. By abuse of terminology, from now on and till the end of this paper all horospherical $G$-varieties are assumed to be spherical.

Affine horospherical varieties were studied in detail in [43] under the name “$S$-varieties”. The following well-known result is deduced essentially from [43, Theorem 6].

**Theorem 4.12** Given an affine spherical $G$-variety $X$, the following conditions are equivalent.

1. $X$ is horospherical.
2. $\mathcal{T}(X) = \{0\}$ or, equivalently, $\mathbb{K}[X]_{\lambda} \cdot \mathbb{K}[X]_{\mu} \subseteq \mathbb{K}[X]_{\lambda+\mu}$ for all $\lambda, \mu \in \Gamma(X)$.

Moreover, the map $X \mapsto \Gamma(X)$ is a bijection between affine horospherical $G$-varieties, considered up to $G$-equivariant isomorphisms, and submonoids in $\Lambda^+$ that are finitely generated and saturated.

**Remark 4.13** Condition (2) of Theorem 4.12 means that decomposition (4.1) is a grading.

**Remark 4.14** Under the conditions of Theorem 4.12, one has $\Sigma = \emptyset$ by Corollary 4.11(a).

**Remark 4.15** Given a finitely generated and saturated submonoid $\Gamma \subseteq \Lambda^+$, the affine horospherical $G$-variety $X$ with $\Gamma(X) = \Gamma$ can be constructed in the following simple way; see [43, Section 3]. Choose a finite generating set $E$ for $\Gamma$. For every $\lambda \in E$, let $V_\lambda$ be the simple $G$-module with highest weight $\lambda$ and choose a highest-weight vector $v_\lambda$ in the dual $G$-module $V_\lambda^*$. Put

$$V = \bigoplus_{\lambda \in E} V_\lambda^* \quad \text{and} \quad v = \sum_{\lambda \in E} v_\lambda \in V.$$  

Then $X$ can be realized as the closure of the orbit $G \cdot v$ in $V$.

4.5 Classification results for affine spherical varieties

The following result was first proved in [33, Theorem 1.2]; see also [10, Corollary 4.16] for a different proof.

**Theorem 4.16** Up to a $G$-equivariant isomorphism, an affine spherical $G$-variety $X$ is uniquely determined by the pair $(\Gamma(X), \Sigma(X))$.

Given a finitely generated and saturated submonoid $\Gamma \subseteq \Lambda^+$, we already know from Theorem 4.12 that there exists an affine horospherical $G$-variety $X_0$ with $\Gamma(X_0) = \Gamma$, in which case $\Sigma(X_0) = \emptyset$ by Remark 4.14. A complete description of all possible
sets $\Sigma$ for which there exists an affine spherical $G$-variety $X$ with $\Gamma(X) = \Gamma$ and $\Sigma(X) = \Sigma$ was obtained in [11,Theorem 6.9] and [13,Proposition 2.13]; there are always only finitely many such sets $\Sigma$. The above-cited sources also explain how to recover $D(X)$ as an abstract set equipped with the map $\varkappa: D(X) \to N$ starting from the pair $(\Gamma(X), \Sigma(X))$. The set $D^G(X)$ along with the map $\varkappa: D^G(X) \to N$ is then recovered by Proposition 4.9.

4.6 Notation for various objects assigned to an affine spherical variety

For convenience of the reader and future reference, in this subsection we list the notation for various objects assigned to an affine spherical $G$-variety $X$. This notation will be systematically used in the rest of our paper.

- $\Gamma$ is the weight monoid of $X$
- $O$ is the open $G$-orbit in $X$
- $Z = \text{Spec } \mathbb{K}[X]^U$; this is the affine toric $T$-variety with weight monoid $\Gamma$
- $M$ is the weight lattice of $X$; $M = \mathbb{Z}\Gamma$ by Proposition 4.7
- $M_\mathbb{Q} = M \otimes_{\mathbb{Z}} \mathbb{Q}$ is the rational vector space spanned by $M$
- $N = \text{Hom}_\mathbb{Z}(M, \mathbb{Z})$ is the dual lattice of $M$
- $N_\mathbb{Q} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ is the dual vector space of $M_\mathbb{Q}$
- $\iota: \text{Hom}_\mathbb{Z}(X(T), \mathbb{Z}) \to N$ is the restriction map
- $\mathcal{E} = \mathbb{Q}_{\geq 0}\Gamma \subseteq M_\mathbb{Q}$ is the cone in $M_\mathbb{Q}$ generated by $\Gamma$
- $\mathcal{E}^1$ is the set of primitive elements $\rho \in N$ such that $\mathbb{Q}_{\geq 0}\rho$ is a ray of $\mathcal{E}$
- $D^B$ is the set of $B$-stable prime divisors in $X$
- $D^G \subseteq D^B$ is the set of $G$-stable prime divisors in $X$
- $\varkappa: D^B \to N$ is the map given by (4.3)
- $\Sigma \subseteq M$ is the set of spherical roots of $X$
- $T \subseteq M_\mathbb{Q}$ is the tail cone of $X$ and also of $O$
- $T^\perp = \{\rho \in N_\mathbb{Q} | \langle \rho, \tau \rangle = 0 \text{ for all } \tau \in T\}$
- $f_\lambda, f_\mu \in \mathbb{K}(X)$ is a fixed $B$-semiinvariant function of weight $\lambda \in M$; we assume $f_\lambda \cdot f_\mu = f_{\lambda + \mu}$ for all $\lambda, \mu \in M$

5 Basic properties of $B$-root subgroups

Let $G$ be a connected reductive algebraic group, fix a Borel subgroup $B \subseteq G$, and retain the notation of Sect. 4.1. Throughout this section, unless otherwise specified, $X$ denotes an affine spherical $G$-variety. For objects related to $X$, we use the notation listed in Sect. 4.6.

In all examples presented in this section, $X$ is a spherical $G$-module, i.e. a finite-dimensional $G$-module that is spherical as a $G$-variety. A plenty of useful information on spherical $G$-modules, including their classification and combinatorial invariants, can be found in [27]. In all our examples where $X = \mathbb{K}^n$ for some $n$ we use the


following conventions: $G$ is a subgroup of $GL_n$ acting linearly on $X$; the group $B$ (resp. $U$, $T$) consists of all upper triangular (resp. upper unitriangular, diagonal) matrices contained in $G$; $x_i$ is the $i$th coordinate function on $X$.

5.1 First properties of $B$-root subgroups

Throughout this subsection, $X$ is an affine irreducible $G$-variety (not necessarily spherical). Let $H$ be a $B$-root subgroup on $X$ of weight $\chi_H$ and let $\partial$ be the corresponding $B$-normalized LND of $K[X]$.

Proposition 5.1 The following assertions hold.

(a) $\chi_H \in \Lambda^+$. 
(b) $H$ is $G$-normalized (and hence a $G$-root subgroup on $X$) if and only if $\chi_H \in \mathcal{X}(G)$.

Proof Consider the $G$-module $\text{Der}(K[X])$ of all derivations of the algebra $K[X]$. Since the $G$-module $K[X]$ is rational (see [39, Lemma 1.4]) and any derivation is uniquely determined by its values on some finite-dimensional $G$-invariant generating subspace in $K[X]$, we conclude that the $G$-module $\text{Der}(K[X])$ is rational as well. Now $\partial$ is a $B$-semiinvariant vector in this module, which yields (a). Clearly, $\partial$ is $G$-normalized if and only if it generates a one-dimensional $G$-submodule in $\text{Der}(K[X])$. Since the latter condition is equivalent to $\chi_H \in \mathcal{X}(G)$, we get (b). $\square$

Proposition 5.2 For every $G$-stable subspace $V \subseteq K[X]$ generating $K[X]$ as an algebra, the derivation $\partial$ is uniquely determined by its restriction to $V^{U^-}$. In particular, $\partial$ is uniquely determined by its restriction to $K[X]^{U^-}$ and also to any system of generators of $K[X]^{U^-}$.

Proof Clearly, $\partial$ is uniquely determined by its restriction to $V$. In turn, the action of $\partial$ on $V$ is uniquely determined by that on $V^{U^-}$ because $\partial$ commutes with $u = \text{Lie } U$ and every simple $G$-module is the linear span of elements obtained from a lowest-weight vector via the action of $u$. $\square$

5.2 Vertical and horizontal $B$-root subgroups

Since the group $U$ has no nontrivial characters, every $B$-root subgroup $H$ on $X$ commutes with $U$. In particular, the action of $H$ on $K[X]$ preserves the subalgebra $K[X]^{U}$ and hence induces an action of $H$ on $Z = \text{Spec } K[X]^{U}$ such that the morphism $\pi_U$ in (4.9) is $H$-equivariant.

Definition 5.3 A $B$-root subgroup $H$ on $X$ is called vertical if $H$ acts trivially on $K[X]^U$ and horizontal otherwise.

The above notions naturally extend to $B$-normalized LNDs on $K[X]$. Namely, such an LND is vertical if and only if it vanishes on $K[X]^U$ and horizontal otherwise.

Clearly, a $B$-root subgroup $H$ on $X$ is vertical if and only if it acts trivially on $Z$ and horizontal otherwise. It follows that every vertical $B$-root subgroup on $X$ preserves all
fibers of $\pi_U$ and every horizontal one permutes them, which explains the terminology. Let $O_B$ be the open $B$-orbit in $X$. The following proposition provides a geometrical criterion for a $B$-root subgroup on $X$ to be vertical or horizontal.

**Proposition 5.4** Let $H$ be a $B$-root subgroup on $X$.

(a) $H$ is vertical if and only if it preserves $O_B$. In this case, $H$ moves no $B$-stable prime divisors on $X$.

(b) $H$ is horizontal if and only if it does not preserve $O_B$. In this case, $H$ moves precisely one $B$-stable prime divisor on $X$.

**Proof** Thanks to Proposition 2.6, it suffices to prove part (a).

Suppose that $H$ is vertical. Since a generic fiber of $\pi_U$ is a single $U$-orbit, it follows that $H$ preserves generic $U$-orbits in $X$. Then generic $H$-orbits in $X$ are contained in $O_B$, hence $H$ preserves $O_B$; see Case 1 in Sect. 2.5.

Conversely, suppose that $H$ preserves $O_B$. Then every $H$-orbit $Y \subseteq O_B$ fits into Case 1 of Sect. 2.5 and hence coincides with an orbit of a $G_a$-subgroup in $B_Y$. Such a $G_a$-subgroup is contained in $U$, therefore $Y$ is contained in a single $U$-orbit. It follows that all functions in $K[X]^U$ are constant along generic $H$-orbits, hence $H$ acts trivially on $K[X]^U$ and $H$ is vertical. $\Box$

**Remark 5.5** All root subgroups on affine toric varieties are horizontal.

**Proposition 5.6** Let $H_1, H_2$ be two equivalent $B$-root subgroups on $X$.

(a) $H_1, H_2$ are either both vertical or both horizontal.

(b) If $H_1, H_2$ are horizontal then they move the same $B$-stable prime divisor on $X$.

**Proof** If one of $H_1, H_2$ is vertical then so is the other one thanks to condition (2.2) of Proposition 2.2, which yields (a). Now suppose $H_1$ is horizontal and moves a divisor $D \in D^B$. Let $W \subseteq X$ be a nonempty open subset such that $H_1x \cap W = H_2x \cap W$ for all $x \in W$. Since $H_i$-orbits in $X$ are closed for $i = 1, 2$ (see [39, Section 1.3] or [41, Lemma 3.4]), it follows that $H_1x = H_2x$ for all $x \in W$. Then for every $x \in W \cap O_B$ the orbit $H_1x$ meets $D$ (see Case 2 of Sect. 2.5) and coincides with $H_2x$, therefore $D$ is $H_2$-unstable. Applying Proposition 5.4(b) completes the proof of (b). $\Box$

## 5.3 Simplest constructions of $B$-root subgroups and/or $B$-normalized LNDs

In this subsection we discuss several basic constructions providing a number of nontrivial examples of $B$-root subgroups on affine spherical $G$-varieties or, equivalently, $B$-normalized LNDs on their algebras of regular functions.

**Construction 5.7** (Central subgroups) Suppose that the derived subgroup $(G, G)$ is nontrivial and acts on $X$ with a finite kernel (for example, effectively). Then the image in $\text{Aut}(X)$ of every $T$-normalized central $G_a$-subgroup of $U$ is a vertical $B$-root subgroup on $X$. Such a central $G_a$-subgroup in $U$ always exists; for simple $(G, G)$ it is unique and corresponds to the highest root of $B$. 


Construction 5.8 (Replicas) Suppose $\partial$ is a $B$-normalized LND on $K[X]$ of weight $\chi$ and $\lambda \in \Gamma$ is such that $f_\lambda \in \text{Ker} \, \partial$. Then $f_\lambda \partial$ is a $B$-normalized LND on $K[X]$ of weight $\chi + \lambda$. Moreover, $\partial$ and $f_\lambda \partial$ are equivalent by Proposition 2.2.

Construction 5.9 (Group extensions) Suppose that the action of $G$ on $X$ extends to an action of a bigger connected reductive algebraic group $G_1 \supseteq G$ and the derived subgroup of $G_1$ is nontrivial and acts on $X$ with a finite kernel. If $B_1 \supseteq T_1$ are a Borel subgroup and a maximal torus of $G_1$ satisfying $B_1 \supseteq B$, $T_1 \supseteq T$ and $U_1$ is the unipotent radical of $B_1$ then the image in $\text{Aut}(X)$ of every $T_1$-normalized central $G_a$-subgroup of $U_1$ is a $B$-root subgroup on $X$.

The following example shows that $B$-root subgroups arising in Construction 5.9 may be not only vertical but also horizontal.

Example 5.10 (Horizontal $B$-root subgroups arising from group extensions) Take $X = K^6$ and let $G \subseteq \text{GL}_6$ be the subgroup of all matrices of the form

$$\begin{pmatrix} sA & 0 \\ 0 & sA^{-1} \end{pmatrix}$$

where $s \in K^\times$, $A \in \text{GL}_3$, and $A^\sigma$ stands for the transpose of $A^{-1}$ with respect to the antidiagonal. Then

$$T = \{ \text{diag}(st_1, st_2, st_3, st_3^{-1}, st_2^{-1}, st_1^{-1}) \mid s, t_1, t_2, t_3 \in K^\times \}$$

and a basis of $\mathcal{X}(T)$ is given by the characters $\chi, \chi_1, \chi_2, \chi_3$ where $\chi(t) = s^2, \chi_1(t) = st_1, \chi_2(t) = st_2, \chi_3(t) = st_3$ for all $t = \text{diag}(st_1, st_2, st_3, st_3^{-1}, st_2^{-1}, st_1^{-1}) \in T$. The algebra $K[X]^U$ is freely generated by the functions

$$f_1 = x_3, \ f_2 = x_6, \text{ and } f_3 = x_1x_6 + x_2x_5 + x_3x_4$$

of weights $-\chi_3, \chi_1 - \chi$, and $-\chi$, respectively. Now let $G_1 \subseteq \text{GL}_6$ be the subgroup consisting of all matrices of the form $sA_1$ with $s \in K^\times$ and $A_1 \in \text{SO}_6$ where $\text{SO}_6$ preserves the quadratic form $f_3$. Put also $G_2 = \text{GL}_6$ and note that $G \subseteq G_1 \subseteq G_2$. For $i = 1, 2$ we choose $B_i$ (resp. $U_i, T_i$) to be the subgroup of all upper-triangular (resp. upper unitriangular, diagonal) matrices in $G_i$. Let $H, H_1, H_2 \subseteq \text{GL}_6$ be the one-parameter unipotent subgroups consisting of all matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -x \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & x & 0 \\ 0 & 1 & 0 & 0 & -x \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 & x \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

respectively. Then $H, H_1, H_2$ are central subgroups in $U, U_1, U_2$, respectively, normalized by $B$. It is easy to check that all the functions $f_1, f_2, f_3$ are $H$-invariant and $H_1$-invariant and $f_3$ is not $H_2$-invariant, thus the $B$-root subgroups induced by $H$ and $H_1$ are vertical and that induced by $H_2$ is horizontal. Moreover, a simple analysis of orbits in general position for $H$ and $H_1$ shows that $H$ and $H_1$ are not equivalent.
**Remark 5.11** Even for simple $G$ a vertical $B$-root subgroup on $X$ need not be equivalent to a central $B$-root subgroup. Indeed, in the situation of Example 5.10, taking $X' = \{ f_3 = 0 \} \subseteq X$, $G' = (G, G) \cong \text{SL}_3$, and $B' = B \cap G'$, we would find that $X'$ is a spherical $G'$-variety and $H, H_1$ still induce nonequivalent vertical $B'$-root subgroups on $X'$.

**Construction 5.12** (Partial derivations) Let $V$ be a spherical $G$-module, choose a simple $G$-submodule $V_0 \subseteq V$ with highest weight $\lambda$, and fix a $G$-module decomposition $V = V_0 \oplus W$. Fix a lowest-weight vector $f \in V_0^*$ and regard it as a coordinate function on $V$ via the chain $V_0^* \subseteq V_0^* \oplus W^* \cong V^* \subseteq \mathbb{K}[V]$. (The asterisk always denotes the dual $G$-module.) Let $R \subseteq V^*$ be the $T$-invariant complement of the line $\mathbb{K} f$. Then the derivation $\partial/\partial f$ with respect to the decomposition $V^* = \mathbb{K} f \oplus R$ is locally nilpotent and $B$-normalized of weight $\lambda$.

**Remark 5.13** The above construction can yield both vertical and horizontal $B$-root subgroups. For instance, in the situation of Example 5.10, $\partial/\partial x_1$ is horizontal (resp. horizontal, vertical) as a $B$-normalized (resp. $B_1$-normalized, $B_2$-normalized) LND on $X$ regarded as a spherical $G$-module (resp. $G_1$-module, $G_2$-module).

### 5.4 $B$-root subgroups and the open $G$-orbit

While a vertical $B$-root subgroup preserves the open $B$-orbit in $X$ (see Proposition 5.4(a)), it may not preserve the open $G$-orbit.

**Example 5.14** (A vertical $B$-root subgroup not preserving the open $G$-orbit) Take $X = \mathbb{K}^2$, $G = \text{SL}_2$ and consider the $B$-normalized LND $\partial/\partial x_1$ on $X$ (see Construction 5.12). Then the corresponding $B$-root subgroup $H$ on $X$ acts as $(s, (x_1, x_2)) \mapsto (x_1 + s, x_2)$. Since the points $(1, 0)$ and $(0, 0)$ lie in the same $H$-orbit, the open $G$-orbit in $X$ is not $H$-stable. Note that the image of $U$ in $\text{Aut}(X)$, which acts as $(s, (x_1, x_2)) \mapsto (x_1 + sx_2, x_2)$, is a replica of $H$ (see Construction 5.8).

The above example is a manifestation of the following general result.

**Proposition 5.15** Let $G$ be a connected semisimple algebraic group acting on a smooth affine variety $Y$ (not necessarily spherical) with an open orbit. Assume that the $G$-action on $Y$ is not transitive. Then there is a $B$-root subgroup $H$ in $\text{Aut}(Y)$ that does not preserve the open $G$-orbit.

**Proof** It is shown in the proof of [2, Theorem 5.6] that there exists a finite-dimensional $G$-module $V$ such that the $G$-action on $Y$ extends to a transitive action of the bigger group $G_V := G \ltimes V$ where the multiplication is given by

$$(g_1, v_1) \cdot (g_2, v_2) = (g_1 g_2, g_2^{-1} v_1 + v_2).$$

Let $v$ be a $B$-eigenvector in $V$ and let $H$ denote the corresponding $B$-root subgroup acting on $Y$. We claim that at least one subgroup $H$ arising this way does not preserve the open $G$-orbit in $Y$. Assume the converse. Note that $V$ is the linear span of $G$-orbits of $B$-eigenvectors in $V$. This implies that the subgroups conjugate to subgroups
coming from $B$-eigenvectors by elements of $G$ generate the unipotent radical $V$ of $G_V$. So the group $G_V$ preserves the open $G$-orbit in $Y$, a contradiction. \hfill $\square$

### 5.5 Multiple $B$-root subgroups of the same weight

The following example shows that, unlike the toric case, a $B$-normalized LND $\partial$ on $\mathbb{K}[X]$ is not uniquely determined by its weight (up to proportionality).

**Example 5.16 (B-root subgroups of the same weight)** Take $X = \mathbb{K}^3$ and let $G \subseteq \text{GL}_3$ be the subgroup of all matrices of the form $egin{pmatrix} sA & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{pmatrix}$ where $s \in \mathbb{K}^\times$ and $A \in \text{SL}_2$. It is easy to check (e.g., using Remark 4.15) that $X$ is horospherical. We have

$$T = \{ \text{diag}(ss_1, ss_1^{-1}, s_2^2) \mid s, s_1 \in \mathbb{K}^\times \}$$

and let $\chi_1, \chi_2 \in \mathcal{X}(T)$ be the characters defined by $\chi_1(t) = ss_1$, $\chi_2(t) = ss_1^{-1}$ for all $t = \text{diag}(ss_1, ss_1^{-1}, s_2^2) \in T$. The algebra $\mathbb{K}[X]^U$ is freely generated by the coordinate functions $x_2, x_3$ of weights $-\chi_2$, $-\chi_1 - \chi_2$, respectively. Consider the LNDs $\partial_1 = \partial/\partial x_1$ and $\partial_2 = x_2\partial/\partial x_3$ on $\mathbb{K}[V]$. Both $\partial_1$ and $\partial_2$ commute with $x_2\partial/\partial x_1$, which corresponds to $U$, and hence are $B$-normalized. It is easy to see that $\partial_1$ and $\partial_2$ are not proportional and have the same weight $\chi_1$. Note also that $\partial_1$ is vertical and $\partial_2$ is horizontal.

**Remark 5.17** In the above example any nontrivial linear combination $\partial = c_1\partial_1 + c_2\partial_2$ with $c_1, c_2 \in \mathbb{K}$ is again a $B$-normalized LND of weight $\chi_1$ (it is locally nilpotent because every $T$-semiinvariant function in $\mathbb{K}[X]$ has weight $k_1\chi_1 + k_2\chi_2$ with $k_1, k_2 \leq 0$); hence we get a two-dimensional space of $B$-normalized LNDs on $\mathbb{K}[X]$ of the same weight. Note that $\partial$ is horizontal whenever $c_2 \neq 0$.

The next example shows that even a vertical $B$-root subgroup may be not uniquely determined by its weight.

**Example 5.18 (Vertical B-root subgroups of the same weight)** Take $X = \mathbb{K}^4$ and let $G \subseteq \text{GL}_4$ be the subgroup of all matrices of the form $egin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ where $A_1, A_2 \in \text{SL}_2$. Define characters $\chi_1, \chi_2 \in \mathcal{X}(T)$ by $\chi_1(t) = s_1$, $\chi_2(t) = s_2$ for all $t = \text{diag}(s_1, s_1^{-1}, s_2, s_2^{-1}) \in T$. The algebra $\mathbb{K}[X]^U$ is freely generated by the coordinate functions $x_2, x_4$ of weights $\chi_1, \chi_2$, respectively. Consider the LNDs $\partial_1 = x_4\partial/\partial x_1$ and $\partial_2 = x_2\partial/\partial x_3$ on $\mathbb{K}[X]$. Both $\partial_1$ and $\partial_2$ commute with $x_2\partial/\partial x_1$ and $x_4\partial/\partial x_3$, which correspond to the root subgroups in $U$, and hence are $B$-normalized. Moreover, $\partial_1$ and $\partial_2$ vanish on $\mathbb{K}[X]^U$ and hence are vertical. It is easy to see that $\partial_1$ and $\partial_2$ are not proportional and have the same weight $\chi_1 + \chi_2$. We also note that any nontrivial linear combination of $\partial_1$ and $\partial_2$ is again a vertical $B$-normalized LND on $\mathbb{K}[X]$.

**Remark 5.19** Another example of two vertical $B$-root subgroups of the same weight can be obtained in the situation of Remark 5.11 by taking appropriate replicas of $H$ and $H_1$. 
5.6 Weights of B-root subgroups

In this subsection we present straightforward necessary conditions for the set of weights of vertical B-root subgroups and that of horizontal ones.

**Proposition 5.20** Suppose that $\chi$ is the weight of a vertical B-root subgroup on $X$. Then each element of the set $\chi + \Gamma$ is again the weight of a vertical B-root subgroup on $X$.

**Proof** Let $\partial$ be a vertical $B$-normalized LND on $\mathbb{K}[X]$ of weight $\chi$. By definition, $\mathbb{K}[X]^U \subseteq \text{Ker } \partial$, hence it remains to apply Construction 5.8.

**Remark 5.21** When $\chi$ is the weight of a $T$-normalized central subgroup in $U$ (see Construction 5.7), the assertion of Proposition 5.20 was observed in [40, Proposition 8.1].

**Proposition 5.22** Suppose that $H$ is a horizontal B-root subgroup on $X$. Then $\chi_H \in \mathcal{R}(\mathcal{E}) \cap \Lambda^+$. In particular, $\chi_H \in M$.

**Proof** We have $\chi_H \in \Lambda^+$ by Proposition 5.1(a). As $H$ is horizontal, its action induces a $T$-root subgroup on $Z$, hence $\chi_H \in \mathcal{R}(\mathcal{E})$.

The following example shows that the set of weights of horizontal B-root subgroups on $X$ may be a proper subset of $\mathcal{R}(\mathcal{E}) \cap \Lambda^+$.

**Example 5.23** (“Missing” B-root subgroups) Take $X = \mathbb{K}^3$ and let $G \subseteq \text{GL}_3$ be the subgroup of all matrices of the form $sA$ where $s \in \mathbb{K}^\times$ and $A \in \text{SO}_3$ with $\text{SO}_3$ preserving the quadratic form $f = x_2^2 + 2x_1x_3$ on $X$. Then $\Lambda^+ = \mathbb{Z}_{\geq 0} \alpha \oplus \mathbb{Z} \chi_1$ where the characters $\chi, \alpha \in \mathcal{X}(T)$ are defined by $\chi(t) = s$, $\alpha(t) = s_1$ for all $t = \text{diag}(ss_1, s, ss_1^{-1}) \in T$; note that $\alpha$ is the unique root in $\Pi$. The algebra $\mathbb{K}[X]^U$ is freely generated by the two functions $x_3$ and $f$ of weights $\alpha - \chi$ and $-2\chi$, respectively, and so $\Gamma = \mathbb{Z}_{\geq 0}(\alpha - \chi, -2\chi)$. It is not hard to see that $\mathcal{R}(\mathcal{E}) \cap \Lambda^+ = 2\chi + \mathbb{Z}_{\geq 0}(\alpha - \chi)$. We now show that there is no nonzero $B$-normalized LND on $\mathbb{K}[X]$ of weight $2\chi$. Indeed, by Proposition 5.2 such an LND is uniquely determined by the image of $x_1$. Since $x_1$ is of weight $-\alpha - \chi$, its image should be a nonzero $T$-semiinvariant function of weight $-\alpha + \chi$. However there are no such functions because every $T$-semiinvariant function in $\mathbb{K}[X]$ has weight of the form $k\alpha + l\chi$ with $l \leq 0$. Note also that for every $k \geq 0$ the formula $x_3^k \partial / \partial x_1$ defines a horizontal $B$-normalized LND of weight $\alpha + \chi + k(\alpha - \chi)$ on $X$, hence the set of weights of horizontal B-root subgroups on $X$ equals $(\mathcal{R}(\mathcal{E}) \cap \Lambda^+) \setminus \{2\chi\}$.

5.7 B-stable prime divisors moved by B-root subgroups

In this subsection we discuss various conditions under which a given B-stable prime divisor in $X$ is moved or not moved by a B-root subgroup.

**Proposition 5.24** Suppose $H$ is a horizontal B-root subgroup on $X$ and $\rho \in \mathcal{E}^1$ is such that $\chi_H \in \mathcal{R}_\rho(\mathcal{E})$. Given $D \in \mathcal{D}^B$, the following conditions are equivalent.

1. $\chi(D)$ is a positive multiple of $\rho$. 

(2) $D$ is moved by $H$.

In particular, there is exactly one $D \in \mathcal{D}^B$ such that $\varepsilon(D)$ is a positive multiple of $\rho$.

**Proof** (1) $\Rightarrow$ (2) Let $I \subseteq \mathbb{K}[X]$ be the ideal of $D$. Then for every $\lambda \in \Gamma$ with $\langle \rho, \lambda \rangle > 0$ one has $f_\lambda \in I$ but $\vartheta(\rho, \lambda) f_\lambda \notin I$ where $\vartheta$ is the LND corresponding to $H$. Consequently, $I$ is $\vartheta$-unstable and hence $D$ is $H$-unstable.

(2) $\Rightarrow$ (1) In view of (4.6) there is $D' \in \mathcal{D}^B$ such that $\varepsilon(D')$ is a positive multiple of $\rho$. Then $D'$ is moved by $H$ by the above argument. Now Proposition 5.4(b) yields $D' = D$.

The last claim is also implied by Proposition 5.4(b). \hfill \Box

Combining Proposition 5.24 with (4.7) we obtain

**Corollary 5.25** Let $H$ be a horizontal $B$-root subgroup on $X$ and let $D \in \mathcal{D}^B$ be moved by $H$. If $\chi_H \in \Gamma(O)$ then $D \in \mathcal{D}^G$.

The next proposition shows that some colors of $X$ cannot be moved by $B$-root subgroups.

**Proposition 5.26** Suppose that $D \in \mathcal{D}^B$ satisfies one of the following conditions:

(a) $\varepsilon(D)$ is not proportional to any $\rho \in \mathcal{E}^1$;
(b) $\varepsilon(D) = c \varepsilon(D')$ for some $D' \in \mathcal{D}^B \setminus \{D\}$ and $c > 0$;
(c) $\langle \varepsilon(D), \lambda \rangle \geq 0$ for all $\lambda \in M \cap \Lambda^+$.

Then there is no $B$-root subgroup on $X$ that moves $D$. Moreover, $D \in \mathcal{D}$, i.e. $D$ is necessarily a color of $X$.

**Proof** (a,b) In both cases, the first claim follows directly from Proposition 5.24 and the second one is implied by Proposition 4.9.

(c) In view of part (a) it remains to consider the case where $\varepsilon(D)$ is a positive multiple of some $\rho \in \mathcal{E}^1$. Assume that a $B$-root subgroup $H$ on $X$ moves $D$. Then $\chi_H \in \mathfrak{R}_p(\mathcal{E})$ by Proposition 5.24, and so $\langle \varepsilon(D), \chi_H \rangle < 0$. On the other hand, $\chi_H \in M \cap \Lambda^+$ by Remark 3.1 and Proposition 5.1(a), therefore $\langle \varepsilon(D), \chi_H \rangle \geq 0$, a contradiction. Next, since $\Gamma(O) \subseteq M \cap \Lambda^+$, it follows from (4.7) that $\varepsilon(D) \in \mathbb{Q} \geq 0 \{\varepsilon(D') \mid D' \in \mathcal{D}\}$. Thanks to Proposition 4.9, the latter is possible only if $D \in \mathcal{D}$.

Recall colors of type $a$ on $X$; see Definition 4.3.

**Corollary 5.27** If a color $D \in \mathcal{D}$ is not of type $a$ then there is no $B$-root subgroup on $X$ that moves $D$.

**Proof** This follows from Propositions 4.2(c) and 5.26(c). \hfill \Box

The next corollary indicates a large class of affine spherical $G$-varieties containing no colors that can be moved by a $B$-root subgroup. By Proposition 4.5 and Remarks 4.4, 4.14, this class includes all affine spherical $G$-varieties having no colors of type $a$ and in particular all affine horospherical $G$-varieties. Moreover, $X$ belongs to this class when $O$ is a symmetric space; see [41,Definition 26.1 and Proposition 26.24].
**Corollary 5.28** Suppose that $\Gamma(O) = M \cap \Lambda^+$. Then no color in $X$ can be moved by a $B$-root subgroup.

**Proof** The claim is implied by Proposition 5.26(c) along with (4.7). \qed

**Remark 5.29** It follows from (4.7) that $\mathcal{D} \neq \emptyset$ whenever $M \cap \Lambda^+ \neq M$.

In view of Corollary 5.28 it is natural to ask whether it happens at all that a color on an affine spherical $G$-variety is moved by a $B$-root subgroup. The next example shows that such situations do occur.

**Example 5.30** (Colors moved by $B$-root subgroups) Let $G_1 = G_2 = \text{GL}_2$, $X = \text{Mat}_{2 \times 2}(\mathbb{K})$ and consider the action of $G_1 \times G_2$ on $X$ given by $((g_1, g_2), x) \mapsto g_1 x g_2^{-1}$.

Restrict this action to the subgroup $G = F_1 \times F_2$ where $F_1 = \text{SL}_2 \subseteq G_1$ and $F_2 \cong (\mathbb{K}^\times)^2$ is the diagonal torus in $G_2$. Choose the Borel subgroup $B = B_1 \times F_2 \subseteq G$ where $B_1$ consists of all upper-triangular matrices in $F_1$. Then $X$ is an affine spherical $G$-variety and $\mathcal{D} = \{D_1, D_2\}$ with $D_1 = \{x_{21} = 0\}$, $D_2 = \{x_{22} = 0\}$ where $x_{ij}$ stands for the $(ij)$th matrix element of $X$. Note also that $\mathcal{D}^B = \{D_1, D_2, D_3\}$ with $D_3 = \{\det = 0\}$. Now the action of the subgroup of all lower (resp. upper) unitriangular matrices in $G_2$ induces a $B$-root subgroup $H_1$ (resp. $H_2$) on $X$ that moves $D_1$ (resp. $D_2$) and preserves the open $G$-orbit $O = \text{GL}_2 \subseteq X$. Of course, both colors $D_1, D_2$ are of type $a$; moreover, $\mathcal{D} = \mathcal{D}_a$ where $a$ is the unique simple root of $G$. Observe that both $H_1, H_2$ are normalized by the whole $G$, so in fact they are $G$-root subgroups on $X$. Besides, it is worth mentioning that the $B$-normalized LNDs $\partial/\partial x_{11}$ and $\partial/\partial x_{12}$ (see Construction 5.12) do not vanish on det, therefore the respective $B$-root subgroups $H_3, H_4$ on $X$ move $D_3$.

Let $T_1$ be the subgroup of diagonal matrices in $F_1$, so that $T = T_1 \times F_2$ is a maximal torus in $G$. Let $\omega = a/2 \in \mathfrak{X}(T_1)$ be the fundamental weight of $F_1$, so that $\omega(t_1) = s$ for all $t_1 = \text{diag}(s, s^{-1}) \in T_1$. Choose also the basis $\chi_1, \chi_2 \in \mathfrak{X}(F_2)$ such that $\chi_i(t_2) = s_i$ for $i = 1, 2$ and all $t_2 = \text{diag}(s_1, s_2) \in F_2$. The algebra $\mathbb{K}[X]^U$ is freely generated by the functions $x_{21}, x_{22}, \det$ of weights $\omega + \chi_1, \omega + \chi_2, \chi_1 + \chi_2$, respectively, therefore

$$\Gamma = \mathbb{Z}_{\geq 0}(\omega + \chi_1, \omega + \chi_2, \chi_1 + \chi_2).$$

We note also that $\Gamma(O) = \mathbb{Z}_{\geq 0}(\omega + \chi_1, \omega + \chi_2) \oplus \mathbb{Z}(\chi_1 + \chi_2)$. The weights of $H_1, H_2, H_3, H_4$ are $\chi_2 - \chi_1, \chi_1 - \chi_2, \omega - \chi_1, \omega - \chi_2$, respectively.

Despite the above-discussed negative results on moving colors by $B$-root subgroups, we propose the following

**Conjecture 5.31** For every $D \in \mathcal{D}^G$ there is a $B$-root subgroup on $X$ that moves $D$.

In Theorem 6.16 below we prove this conjecture when $X$ is horospherical.

### 5.8 The case of $\text{G}$-saturated $\Gamma$

The monoid $\Gamma$ is said to be $G$-saturated if $\Gamma = \mathbb{Z}\Gamma \cap \Lambda^+$. In other words, $\Gamma$ equals the intersection of $\Lambda^+$ with a sublattice of $\mathfrak{X}(T)$.
Remark 5.32 $\Gamma$ is $G$-saturated if and only if $E = \mathbb{Q}_{\geq 0}\{\ell(\alpha^\vee) \mid \alpha \in \Pi\}$.

Proposition 5.33 Suppose that $\Gamma$ is $G$-saturated. Then every $B$-root subgroup on $X$ is vertical.

Proof Recall from Remark 3.1 that $\mathcal{R}(E) \subseteq \mathbb{Z}\Gamma$. Then

$$\mathcal{R}(E) \cap \Lambda^+ = \mathcal{R}(E) \cap \mathbb{Z}\Gamma \cap \Lambda^+ = \mathcal{R}(E) \cap \Gamma = \emptyset,$$

where the latter equality holds because $\mathcal{R}(E) \cap \mathcal{G} = \emptyset$. Now the claim follows from Proposition 5.22.

Remark 5.34 If $\Gamma$ is $G$-saturated then, combining Remark 5.32 with Propositions 4.9 and 5.26(c), we find that $D^G = \emptyset$. So Proposition 5.33 does not contradict Conjecture 5.31.

Corollary 5.35 Suppose that $\text{rk} \ \Gamma = 1$ and $(G, G)$ is nontrivial and acts on $X$ with a finite kernel. Then every $B$-root subgroup on $X$ is vertical.

Proof The hypotheses imply that $\Gamma = \mathbb{Z}_{\geq 0}\lambda$ for a dominant weight $\lambda \in \Lambda^+$ that restricts nontrivially to $(G, G)$. Then $-\lambda \notin \Lambda^+$ and hence $\Gamma = \mathbb{Z}\Gamma \cap \Lambda^+$, which yields the claim thanks to Proposition 5.33.

Corollary 5.36 Suppose that $G = \text{SL}_2$ and $G$ acts nontrivially on $X$. Then every $B$-root subgroup on $X$ is equivalent to $U$.

Proof Clearly, the conditions of Corollary 5.35 are satisfied, therefore every $B$-root subgroup on $X$ is vertical and hence preserves generic $U$-orbits in $X$. But the group $U$ is one-dimensional, so every $B$-root subgroup on $X$ is equivalent to $U$ in this case.

Remark 5.37 Along with the case $G = \text{SL}_2$, Corollary 5.35 may be applied to other natural classes of spherical varieties. Recall that an $HV$-variety is the closure of the orbit of a highest weight vector in a simple module $V$ of a semisimple group $G$. It is known that every $HV$-variety is a normal cone consisting of two orbits—the open orbit and the origin—and the weight monoid of such a variety is generated by the dual of the highest weight of $V$; see [43,Section 1] for details. By Corollary 5.35, there are only vertical $B$-root subgroups in this case. An important example of $HV$-varieties is given by Grassmann cones, which are cones of highest weight vectors in the fundamental representations of the group $\text{SL}_n$. Another example is the nondegenerate quadratic cone with the action of the group $\text{SO}_n$.

5.9 $B$-root subgroups arising in the study of spherical subgroups

This subsection serves as a complement to Sect. 5.3. Here we present one more setting that yields nontrivial examples of affine spherical $G$-varieties equipped with $B$-root subgroups on them. Besides, we state an open problem in this setting, which provides an extra motivation for studying $B$-root subgroups.
Let $P \subseteq G$ be a parabolic subgroup such that $P \supseteq B^-$ and let $P_u$ denote the unipotent radical of $P$. Let $L$ be the unique Levi subgroup of $P$ containing $T$ and let $\Pi_L \subseteq \Pi$ be the set of simple roots of $L$. Let $Q \subseteq P$ be a closed subgroup with unipotent radical $Q_u$ and a Levi subgroup $K$. Suppose that $K \subseteq L$ and $Q_u \subseteq P_u$.

It is known from [14,Proposition 1.1] and [38,Theorem 1.2] that the following conditions are equivalent:

1. $Q$ is a spherical subgroup of $G$, i.e. $G/Q$ is a spherical $G$-variety;
2. $P/Q$ is a spherical $L$-variety (which is automatically smooth and affine);
3. $P_u/Q_u$ is a spherical $S$-variety (which is an affine space) for a certain reductive subgroup $S \subseteq K$ uniquely determined up to conjugacy by the pair $(L, K)$.

Put $B_L = B \cap L$ and $B_S = B \cap S$, then $B_L$ is a Borel subgroup of $L$ and the connected component of the identity $B^0_S \subseteq B_S$ is a Borel subgroup of $S$ under an appropriate choice of $S$ within its conjugacy class in $K$.

Now for every $\alpha \in \Pi \setminus \Pi_L$ the group $U_{-\alpha} \subseteq P$ naturally acts on $P/Q$ and is normalized by $B_L$. When this action is nontrivial, it provides a $B_L$-root subgroup on $P/Q$. Similarly, $U_{-\alpha} \subseteq P_u$ naturally acts on $P_u/Q_u$ and is normalized by $B^0_S$. Again, if this action is nontrivial then it provides a $B^0_S$-root subgroup on $P_u/Q_u$. We note that in the case $P = B^-$ both groups $L$ and $S^0$ are tori and the above-mentioned root subgroups were considered in [24].

Thanks to [34,Lemma 1.4], there is a $K$-equivariant (and hence $S$-equivariant) isomorphism $P_u/Q_u \simeq p_u/q_u$ where $p_u = \text{Lie } P_u$ and $q_u = \text{Lie } Q_u$. Thus we obtain a $B^0_S$-normalized action of $U_{-\alpha}$ on the spherical $S$-module $p_u/q_u$. A criterion for this action to be nontrivial is given by [9,Lemma 6.6]. When $U_{-\alpha}$ acts nontrivially on $p_u/q_u$, it follows from [9,Lemma 6.7] that the induced $B^0_S$-root subgroup is horizontal if and only if $\alpha$ is a spherical root of $G/Q$. As discussed in [9], the set $\Pi \cap \Sigma(G/Q)$ of simple spherical roots of $G/Q$ plays an essential role in computing several key combinatorial invariants of $G/Q$. Thus, it is important to find methods for computing $\Pi \cap \Sigma(G/Q)$ itself, which is equivalent to the following open problem in our setting.

Problem 5.38 Under the above notation and assumptions suppose that $U_{-\alpha}$ acts nontrivially on $P_u/Q_u$. Determine whether the induced $B^0_S$-root subgroup on $P_u/Q_u$ is vertical or horizontal.

6 Standard $B$-root subgroups

In this section, we present and discuss a general construction of horizontal $B$-root subgroups on affine spherical $G$-varieties, which we call standard, and provide several applications.

Let $X$ be an affine spherical $G$-variety and retain all the notation from Sect. 4.6. Suppose $\delta$ is a horizontal $B$-normalized LND on $\mathbb{K}[X]$ of weight $\mu$ and $\rho \in \mathcal{E}^L$ is such that $\mu \in \mathfrak{N}_\rho(\mathcal{E})$. Recall that the restriction of $\delta$ to $\mathbb{K}[X]^U$ is $T$-normalized. Thanks to Theorem 3.4(a) and formula (3.1), up to rescaling, we have

$$\delta(f_\lambda) = \langle \rho, \lambda \rangle f_\mu f_\lambda$$

(6.1)
for all $\lambda \in \Gamma$. Our construction naturally generalizes formula (6.1) to the whole $\mathbb{K}[X]$ when $\mu \in \Gamma(O)$.

### 6.1 Description of the construction

Take any $\mu \in \mathcal{R}(E) \cap \Gamma(O)$ and let $\rho \in \mathcal{E}^1$ be such that $\mu \in \mathcal{R}_\rho(E)$. We define a linear map $\partial_\mu : \mathbb{K}[X] \rightarrow \mathbb{K}[O]$ as follows. For every $\lambda \in \Gamma$ and $g \in \mathbb{K}[X]_\lambda$, we put

$$\partial_\mu(g) = \langle \rho, \lambda \rangle f_\mu g.$$  

(6.2)

As $f_\mu$ is a highest-weight vector, $\partial_\mu$ commutes with $u$ and hence is $B$-normalized.

**Proposition 6.1** The image of $\partial_\mu$ is contained in $\mathbb{K}[X]$, so that $\partial_\mu$ is a well-defined linear map of $\mathbb{K}[X]$ to itself.

**Proof** Recall that $\langle \rho, \mu \rangle = -1$ and $\langle \rho', \mu \rangle \geq 0$ for all $\rho' \in \mathcal{E}^1 \setminus \{\rho\}$. If $\langle \rho, \lambda \rangle = 0$ then $\partial_\mu(g) = 0$ for all $g \in \mathbb{K}[X]_\lambda$, so we assume in what follows that $\langle \rho, \lambda \rangle \geq 1$. Take $\nu \in \Gamma(O)$ such that $\mathbb{K}[O]_\nu$ is contained in the linear span of $\mathbb{K}[O]_\lambda \cdot \mathbb{K}[O]_\mu$ and consider the corresponding tail $\tau = \lambda + \mu - \nu \in \mathcal{T}$. Let $\rho' \in \mathcal{E}^1$ be an arbitrary element; we need to show that

$$\langle \rho', \nu \rangle \geq 0.$$  

(6.3)

We know from (4.6) that there is $D \in \mathcal{D}^B$ such that $\chi(D)$ is a positive multiple of $\rho'$. If $D \in \mathcal{D}$ then (6.3) holds by (4.7), so in what follows we assume $D \in \mathcal{D}^G$. Then Proposition 4.10 and Corollary 4.11(b) yield $\langle \rho', \tau \rangle \leq 0$, therefore

$$\langle \rho', \nu \rangle = \langle \rho', \lambda + \mu - \tau \rangle \geq \langle \rho', \lambda \rangle + \langle \rho', \mu \rangle,$$

and it remains to prove that the latter expression is nonnegative. If $\rho' \neq \rho$ then both summands are nonnegative. If $\rho' = \rho$ then $\langle \rho', \lambda \rangle \geq 1$ and $\langle \rho', \mu \rangle = -1$, and we are done. $\Box$

**Proposition 6.2** The map $\partial_\mu$ is a derivation of $\mathbb{K}[X]$ if and only if $\rho \in \mathcal{T}^\perp$. Moreover, under these conditions $\partial_\mu$ is locally nilpotent.

**Proof** Take arbitrary $\lambda, \lambda' \in \Gamma$ and $g \in \mathbb{K}[X]_\lambda, g' \in \mathbb{K}[X]_{\lambda'}$. Then $gg' = h + \sum_{i=1}^{k} h_i$ where $h \in \mathbb{K}[X]_{\lambda + \lambda'}$ and $h_i \in \mathbb{K}[X]_{v_i} \setminus \{0\}$ for some pairwise distinct weights $v_1, \ldots, v_k \in \Gamma \setminus \{\lambda + \lambda'\}$. Observe that $\lambda + \lambda' - v_i \in \mathcal{T}$ for all $i = 1, \ldots, k$. We have

$$\partial_\mu(gg') - g\partial_\mu(g') - g'\partial_\mu(g)$$

$$= \langle \rho, \lambda + \lambda' \rangle f_\mu h + \sum_{i=1}^{k} \langle \rho, v_i \rangle f_\mu h_i - \langle \rho, \lambda' \rangle f_\mu gg' - \langle \rho, \lambda \rangle f_\mu gg'$$
\[
\begin{align*}
    &\langle \rho, \lambda + \lambda' \rangle f_\mu h + \sum_{i=1}^{k} \langle \rho, \nu_i \rangle f_\mu h_i - \langle \rho, \lambda + \lambda' \rangle f_\mu (h + \sum_{i=1}^{k} h_i) \\
    &= -\sum_{i=1}^{k} \langle \rho, \lambda + \lambda' - \nu_i \rangle f_\mu h_i.
\end{align*}
\]

(6.4)

If \( \rho \in T^\perp \) then the last expression vanishes and hence \( \partial_\mu \) is a derivation.

Conversely, suppose that \( \partial_\mu \) is a derivation of \( \mathbb{K}[X] \). Then the last expression in (6.4) vanishes; dividing it by \( f_\mu \) we obtain \( \sum_{i=1}^{k} \langle \rho, \lambda + \lambda' - \nu_i \rangle h_i = 0 \), which implies \( \langle \rho, \lambda + \lambda' - \nu_i \rangle = 0 \) for all \( i = 1, \ldots, k \). Since every tail \( \tau \in T \) can be realized as \( \lambda + \lambda' - \nu_i \) for an appropriate choice of \( \lambda, \lambda', g, g' \), \( i \), we obtain \( \langle \rho, \tau \rangle = 0 \).

Finally, consider the decomposition \( \mathbb{K}[X] = \bigoplus_{i=0}^{\infty} \mathbb{K}[X]^{(i)} \) where \( \mathbb{K}[X]^{(i)} = \bigoplus_{\lambda \in \Gamma: (\rho, \lambda) = i} \mathbb{K}[X]_{\lambda} \). Then it is easy to see that \( \partial_\mu (\mathbb{K}[X]^{(0)}) = 0 \) and \( \partial_\mu (\mathbb{K}[X]^{(i+1)}) \subseteq \mathbb{K}[X]^{(i)} \) for all \( i \in \mathbb{Z}_{\geq 0} \), so \( \partial_\mu \) is locally nilpotent. \( \square \)

**Corollary 6.3** Under the conditions of Proposition 6.2, \( \partial_\mu \) is a horizontal \( B \)-normalized LND of \( \mathbb{K}[X] \) and the corresponding \( B \)-root subgroup moves a \( G \)-stable prime divisor on \( X \).

**Proof** Formula (6.2) implies that \( \partial_\mu \) does not vanish on \( \mathbb{K}[X]^U \), hence \( \partial_\mu \) is horizontal. The last assertion follows from Corollary 5.25. \( \square \)

**Definition 6.4** A nonzero \( B \)-normalized LND on \( \mathbb{K}[X] \) is called standard if it is proportional to the LND \( \partial_\mu \) given by (6.2) for some \( \rho \in E^1 \cap T^\perp \) and \( \mu \in R_\rho(\mathcal{E}) \cap \Gamma(O) \). A \( B \)-root subgroup on \( X \) is called standard if it corresponds to a standard \( B \)-normalized LND on \( \mathbb{K}[X] \).

**Remark 6.5** In general, not all horizontal \( B \)-root subgroups on affine spherical \( G \)-varieties that move a \( G \)-stable prime divisor are standard. Moreover, this remains valid even for affine horospherical \( G \)-varieties. For instance, in the situation of Example 5.16 and Remark 5.17, where \( X \) is horospherical, the LND \( \partial_2 + c\partial_1 \) is standard if and only if \( c = 0 \). In Example 5.23, the LND \( x_k^2 \partial/\partial x_1 \) is not standard for all \( k \geq 0 \).

### 6.2 \( G \)-stable prime divisors moved by standard \( B \)-root subgroups

The next proposition provides a necessary and sufficient combinatorial condition for a \( G \)-stable prime divisor on \( X \) to be moved by a standard \( B \)-root subgroup.

**Proposition 6.6** Given \( D \in \mathcal{D}^G \), the following conditions are equivalent.

1. There is a standard \( B \)-root subgroup on \( X \) that moves \( D \).
2. \( \kappa(D) \in T^\perp \).

**Proof** (1)\( \Rightarrow \)(2) Suppose \( D \) is moved by a standard \( B \)-root subgroup of weight \( \mu \) and let \( \rho \in E^1 \) be such that \( \mu \in R_\rho(\mathcal{E}) \). Then \( \kappa(D) \) is a positive multiple of \( \rho \) by Proposition 5.24, which implies \( \kappa(D) \in T^\perp \) by Proposition 6.2.
(2)⇒(1) Let \( \tilde{E} \) be the cone in \( N_\mathbb{Q} \) dual to \( \mathbb{Q}_{\geq 0} \Gamma(O) \); observe that \( \tilde{E} \subseteq E \). It follows from (4.7) that \( \tilde{E} \) is generated by the set \( \{ \kappa(D') \mid D' \in D \} \). Put \( \rho = \kappa(D) \); then Proposition 4.9 yields \( \rho \in E_1 \setminus \tilde{E} \). Note that Lemma 3.2 is applicable in this situation, hence the set \( \mathcal{R}_\rho(E) \cap \Gamma(O) \) contains infinitely many elements. We choose such an element \( \mu \) and consider the map \( \partial_\mu \) given by (6.2). As \( \rho \in T^\perp \), Proposition 6.2 implies that \( \partial_\mu \) is a standard \( B \)-normalized LND on \( K[X] \). Thanks to Proposition 5.24, the corresponding \( B \)-root subgroup on \( X \) moves \( D \).

6.3 G-root subgroups

In this subsection we apply the construction of standard \( B \)-root subgroups to obtain a partial description of \( G \)-root subgroups on \( X \). Recall from Sect. 2.4 that the weight of every \( G \)-root subgroup on \( X \) belongs to \( \mathcal{X}(G) \), which is identified with a subgroup of \( \mathcal{X}(T) \).

Proposition 6.7 (compare with [28, Lemma 5.1]) Suppose \( H \) is a \( G \)-root subgroup on \( X \). Then \( H \) is horizontal as a \( B \)-root subgroup and uniquely determined by its weight among the \( B \)-root subgroups on \( X \). Moreover, \( \chi_H \in \mathcal{R}(E) \cap \mathcal{X}(G) \) and in particular \( \chi_H \in M \).

Proof Clearly, \( H \) is \( B^- \)-normalized, therefore by Proposition 5.2 the corresponding \( B^- \)-normalized LND \( \partial \) on \( K[X] \) is uniquely determined by its restriction to \( K[X]^U \), which in turn is \( T \)-normalized. By Theorem 3.4(a), up to proportionality, the latter restriction is uniquely determined by its weight. Note that \( \partial \) acts nontrivially on \( K[X]^U \), so \( H \) is horizontal. By Proposition 5.1(b), there are no other \( B \)-root subgroups on \( X \) of weight \( \chi_H \). The last claim is implied by Proposition 5.22. □

Corollary 6.8 If \( G \) is semisimple then there are no \( G \)-root subgroups on \( X \).

Proof The claim follows from \( \mathcal{X}(G) = \{ 0 \} \) and \( 0 \notin \mathcal{R}(E) \). □

The next result shows that all \( G \)-normalized LNDs on \( K[X] \) act in a rather simple way.

Proposition 6.9 Let \( \partial \) be a \( G \)-normalized LND on \( K[X] \) of weight \( \mu \) and let \( \rho \in E_1 \) be such that \( \mu \in \mathcal{R}_\rho(E) \). Then there is \( c \in K^\times \) such that, for every \( \lambda \in \Gamma \), \( \partial \) acts on \( K[X]_\lambda \) as follows:

- if \( \langle \rho, \lambda \rangle = 0 \) then \( \partial(K[X]_\lambda) = 0 \);
- if \( \langle \rho, \lambda \rangle > 0 \) then the restriction of \( \partial \) to \( K[X]_\lambda \) is a \( (G, G) \)-equivariant isomorphism \( K[X]_\lambda \simto K[X]_{\lambda+\rho} \) such that \( \partial(f_\lambda) = c\langle \rho, \lambda \rangle f_{\lambda+\rho} \).

Proof The assertion is implied by formula (6.1) for the restriction of \( \partial \) to \( K[X]^U \) and the \( (G, G) \)-invariance of \( \partial \). □

Let \( \tilde{\mathcal{M}} \subseteq \mathcal{X}(G) \) be the lattice of weights of \( G \)-semiinvariant functions in \( K(X) \). Since every such function is automatically regular on \( O \), we have

\[ \tilde{\mathcal{M}} = \mathcal{X}(G) \cap \Gamma(O). \] (6.5)
The next result provides a description of all weights of $G$-root subgroups on $X$ that belong to $\tilde{M}$.

**Theorem 6.10** For a weight $\mu \in \tilde{M}$, the following conditions are equivalent.

1. $\mu$ is the weight of a $G$-root subgroup on $X$.
2. $\mu \in \mathcal{R}_\rho(\mathcal{E})$ for some $\rho \in \mathcal{E}^1 \cap T^\perp$.

Moreover, under these conditions the $G$-root subgroup of weight $\mu$ on $X$ is automatically standard as a $B$-root subgroup.

**Proof** (1)$\Rightarrow$(2) Let $H$ be a $G$-root subgroup on $X$ of weight $\mu \in \tilde{M}$ and let $\vartheta$ be the corresponding $G$-normalized LND on $\mathbb{K}[X]$. We know from Proposition 6.7 that $H$ is horizontal as a $B$-root subgroup and there is $\rho \in \mathcal{E}^1$ such that $\mu \in \mathcal{R}_\rho(\mathcal{E})$. In view of (6.5), the function $f_\mu$ belongs to $\mathbb{K}[O]$ and is $G$-semiinvariant, therefore for every $\lambda \in \Gamma$ with $\langle \rho, \lambda \rangle > 0$ the map $g \mapsto f_\mu g$ induces a $(G, G)$-equivariant isomorphism $\mathbb{K}[X]_\lambda \rightarrow \mathbb{K}[X]_{\lambda + \mu}$. Now Proposition 6.9 yields $\vartheta = c\vartheta_\mu$ for some $c \in \mathbb{K}^\times$, hence $\rho \in T^\perp$ by Proposition 6.2.

(2)$\Rightarrow$(1) By (6.5) and Proposition 6.2, $\vartheta_\mu$ is a $B$-normalized LND on $\mathbb{K}[X]$ of weight $\mu$. As $f_\mu$ is $G$-semiinvariant, $\vartheta_\mu$ is in fact $G$-normalized.

**Remark 6.11** Describing all weights of $G$-root subgroups on $X$ that do not belong to $\tilde{M}$ remains an open problem. Such $G$-root subgroups may exist: in Example 5.30 we have seen $G$-root subgroups on $X$ that move a color; their weights do not belong to $\Gamma(O)$ (and hence to $\tilde{M}$) by Corollary 5.25.

Combining Theorem 6.10 with (6.5) we obtain the following complete description of weights of all $G$-root subgroups for a large class of affine spherical $G$-varieties.

**Corollary 6.12** Suppose that $\Gamma(O) = M \cap \Lambda^+$. Then, for a weight $\mu \in \mathcal{X}(G)$, the following conditions are equivalent.

1. $\mu$ is the weight of a $G$-root subgroup on $X$.
2. $\mu \in \mathcal{R}_\rho(\mathcal{E})$ for some $\rho \in \mathcal{E}^1 \cap T^\perp$.

Moreover, all $G$-root subgroups on $X$ are standard as $B$-root subgroups.

A refined version of Corollary 6.12 in the horospherical case is provided by Proposition 6.13(a) below.

### 6.4 The horospherical case

Let $X$ be an affine horospherical $G$-variety (see Sect. 4.4) and recall from Theorem 4.12, Remark 4.14, and Proposition 4.5 that $\mathcal{T} = \{0\}$ and $\Gamma(O) = M \cap \Lambda^+$. In particular, we automatically get $\mathcal{E}^1 \subseteq T^\perp$.

**Proposition 6.13** The following assertions hold.

(a) The set of weights of $G$-root subgroups on $X$ is $\mathcal{R}(\mathcal{E}) \cap \mathcal{X}(G)$. Moreover, all $G$-root subgroups on $X$ are standard as $B$-root subgroups.
(b) The set of weights of horizontal B-root subgroups on $X$ is $\mathcal{R}(E) \cap \Lambda^+$. Moreover, for every $\mu \in \mathcal{R}(E) \cap \Lambda^+$ there is a standard B-root subgroup on $X$ of weight $\mu$.

**Proof** (a) The claim follows directly from Corollary 6.12.

(b) Recall from Remark 3.1 that $\mathcal{R}(E) \subseteq M$, which yields $\mathcal{R}(E) \cap \Lambda^+ = \mathcal{R}(E) \cap \Gamma(O)$ by Proposition 4.5. Now Proposition 6.2 and Corollary 6.3 imply that every $\mu \in \mathcal{R}(E) \cap \Lambda^+$ is the weight of a standard B-root subgroup on $X$. It remains to apply Proposition 5.22.

**Example 6.14** (G-root subgroups on affine horospherical G-varieties) Take $G = SL_2 \times \mathbb{K}^\times$ and let $\alpha$ (resp. $\chi$) be the unique positive root of $SL_2$ (resp. a basis character of $\mathbb{K}^\times$), so that $\Lambda^+ = \mathbb{Z}_{\geq 0}^2 \oplus \mathbb{Z}_\chi$ and $\mathcal{X}(G) = \mathbb{Z}_\chi$. Suppose that $X$ is the affine horospherical $G$-variety with $\Gamma = \mathbb{Z}_{\geq 0}\{a\alpha + \chi, b\alpha - \chi\}$ for fixed $a, b \in \mathbb{Z}_{>0}$. Then easy calculations of the set $\mathcal{R}(E) \cap \mathcal{X}(G)$ show that there are no $G$-root subgroups on $X$ if none of the numbers $a, b$ is divisible by the other, there are exactly two such subgroups when $a = b = 1$, and such subgroup is unique otherwise. We note that there are infinitely many horizontal B-root subgroups on $X$ regardless of the values of $a, b$.

**Example 6.15** (A horizontal B-root subgroup on an affine horospherical variety of a simple group) Take $G = SL_n$ ($n \geq 3$) and choose $B$ (resp. $T$) to be the subgroup of all upper-triangular (resp. diagonal) matrices in $G$. Consider the $G$-module $W = \text{Mat}_{n \times n}(\mathbb{K}) \oplus \mathbb{K}^n$ with the action given by $(g, (A, v)) \mapsto (gA^{-1}, gv)$ and let $X$ be the closure in $W$ of the orbit of the pair of highest-weight vectors

\[
\begin{pmatrix}
0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix}.
\]

In other words, $X = \{(A, v) \in W \mid \text{tr}(A) = 0, \text{rk}(A \mid v) \leq 1\}$. In view of Remark 4.15, $X$ is an affine horospherical $G$-variety and $\Gamma = \mathbb{Z}_{\geq 0}\{\omega_1 + \omega_{n-1}, \omega_{n-1}\}$ where $\omega_i \in \mathcal{X}(T)$ is the $i$th fundamental weight of $G$, so that $\omega_i(t) = t_1 \cdots t_i$ for all $t = \text{diag}(t_1, \ldots, t_n) \in T$.

For $i, j = 1, \ldots, n$ let $a_{i j}$ (resp. $x_i$) denote the restriction to $X$ of the $(ij)$th coordinate function on $\text{Mat}_{n \times n}(\mathbb{K})$ (resp. $i$th coordinate function on $\mathbb{K}^n$). Then the algebra $\mathbb{K}[X]^U$ is freely generated by the two functions $a_{i 1}$ and $x_n$ of weights $\omega_1 + \omega_{n-1}$ and $\omega_{n-1}$, respectively.

The weight $\mu = \omega_1$ is a Demazure root of the cone $E$. Applying formula (6.2), we see that the standard LND $\partial_\mu$ on $\mathcal{E}$ annihilates all functions $a_{i 1}$ and sends $x_i$ to $a_{i 1}$ for all $i = 1, \ldots, n$. This shows that the corresponding B-root subgroup $H$ on $X$ acts as

\[(s, (A, v)) \mapsto (A, v + sA_1)\]

where $A_1$ is the first column of the matrix $A$. This subgroup moves the $G$-stable prime divisor $\{v = 0\}$ on $X$.  

The following theorem confirms Conjecture 5.31 for affine horospherical $G$-varieties.

**Theorem 6.16** For every $D \in \mathcal{D}^G$, there is a $B$-root subgroup on $X$ that moves $D$.

**Proof** Since $T = \{0\}$, every $D \in \mathcal{D}^G$ satisfies $\kappa(D) \in T^\perp$, which implies the assertion by Proposition 6.6. $\square$

**Remark 6.17** It is shown in [23] that, for an affine horospherical $G$-variety $X$ satisfying $\Gamma \cap (-\Gamma) = \{0\}$ (i.e. the cone $\mathcal{G}$ is strictly convex), the subgroup in $\text{Aut}(X)$ generated by all $G_a$-subgroups acts on the regular locus of $X$ transitively. This implies that every smooth point on a $G$-stable prime divisor in $X$ can be moved to a point in the open $G$-orbit by an appropriate sequence of (not necessary root) $G_a$-subgroups. The same transitivity property for an arbitrary affine spherical $G$-variety $X$ with strictly convex cone $\mathcal{G}$ is an open problem.

### 7 Reductive groups of semisimple rank one acting on toric varieties

Throughout this section, $G$ is a connected reductive linear algebraic group of semisimple rank one. Replacing $G$ by a finite covering, we assume that $G = \text{SL}_2 \times S$ where $S$ is an algebraic torus. Let $T_0$, $U \subseteq \text{SL}_2$ be a maximal torus and a maximal unipotent subgroup normalized by $T_0$, respectively. Then $T = T_0 \times S$ and $B = TU$ are a maximal torus and a Borel subgroup of $G$, respectively. Let $\alpha \in \mathcal{X}(T)$ be the unique positive root of $G$ with respect to $B$ and let $\alpha^\vee \in \text{Hom}(\mathcal{X}(T), \mathbb{Z})$ be the corresponding dual root, so that $\langle \alpha^\vee, \alpha \rangle = 2$ and $\langle \alpha^\vee, \chi \rangle = 0$ for all $\chi \in \mathcal{X}(S)$. As before, $X$ denotes an affine spherical $G$-variety, and we retain all the notation of Sect. 4.6.

Our main goal in this section is to obtain a complete description of the $B$-root subgroups on $X$ in the case where $X$ is toric as a $T$-variety. To this end, we first obtain a complete description of the $T$-root subgroups on $X$ in terms of the weight monoid $\Gamma$ and then determine which of the $T$-root subgroups are in fact $B$-root subgroups on $X$.

#### 7.1 A criterion for the existence of an open $T$-orbit

For every $\lambda \in \Lambda^+$, we put $d_\lambda = \langle \alpha^\vee, \lambda \rangle$ for short.

**Proposition 7.1** The following conditions are equivalent.

1. $X$ is toric as a $T$-variety.
2. $\alpha \notin \mathbb{Q}\Gamma$.

**Proof** It follows from the representation theory of $\text{SL}_2$ that for every $\lambda \in \Gamma$ there is a decomposition $\mathbb{K}[X]_\lambda = \bigoplus_{i=0}^{d_\lambda} \mathbb{K}[X]_{\lambda,i}$ where $\mathbb{K}[X]_{\lambda,i} \subseteq \mathbb{K}[X]_\lambda$ is a one-dimensional $T$-submodule of weight $\lambda - i\alpha$. Then we have a $T$-module decomposition

\[
\mathbb{K}[X] = \bigoplus_{\lambda \in \Gamma} \bigoplus_{i=0}^{d_\lambda} \mathbb{K}[X]_{\lambda,i}.
\]
Recall from Theorem 4.1 that $X$ is toric as a $T$-variety if and only if $\mathbb{K}[X]$ is a multiplicity-free $T$-module.

If $\alpha \in \mathbb{Q}\Gamma$ then $k\alpha = \lambda - \mu$ for some $k \in \mathbb{Z}_{>0}$ and $\lambda, \mu \in \Gamma$. Then $\mathbb{K}[X]_{\lambda, k} \simeq \mathbb{K}[X]_{\mu, 0}$ as $T$-modules and thus the $T$-module $\mathbb{K}[X]$ is not multiplicity free.

Conversely, if $\mathbb{K}[X]$ is not multiplicity free as a $T$-module then there are $\lambda, \mu \in \Gamma$ and $k, l \in \mathbb{Z}$ such that $\lambda \neq \mu$, $0 \leq k \leq d_\lambda$, $0 \leq l \leq d_\mu$, and $\mathbb{K}[X]_{\lambda, k} \simeq \mathbb{K}[X]_{\mu, l}$ as $T$-modules. It follows that $\lambda - k\alpha = \mu - l\alpha$, which implies $\alpha \in \mathbb{Q}\Gamma$ as $k \neq l$.

**Corollary 7.2** If $X$ is toric as a $T$-variety then $X$ is horospherical as a $G$-variety.

**Proof** Thanks to Theorem 4.12, it suffices to prove that $\mathbb{K}[X]_{\lambda} \cdot \mathbb{K}[X]_{\mu} \subseteq \mathbb{K}[X]_{\lambda + \mu}$ for all $\lambda, \mu \in \Gamma$. The multiplication of $\mathbb{K}[X]_{\lambda}$ and $\mathbb{K}[X]_{\mu}$ induces a $G$-module homomorphism $\varphi : \mathbb{K}[X]_{\lambda} \otimes \mathbb{K}[X]_{\mu} \to \mathbb{K}[X]$, and it suffices to show that $\operatorname{Im} \varphi = \mathbb{K}[X]_{\lambda + \mu}$. As $f_\lambda \cdot f_\mu = f_{\lambda + \mu}$, we have $\mathbb{K}[X]_{\lambda + \mu} \subseteq \operatorname{Im} \varphi$. If $\mathbb{K}[X]_{\nu} \subseteq \operatorname{Im} \varphi$ for some $\nu \in \Gamma \setminus \{\lambda + \mu\}$ then $\lambda + \mu - \nu$ is a positive multiple of $\alpha$, which is impossible by Proposition 7.1. □

### 7.2 Auxiliary results

In what follows we assume that $X$ is toric as a $T$-variety and the derived subgroup $(G, G) = \text{SL}_2$ acts nontrivially on $X$, which is equivalent to $\Gamma \not\subseteq \mathcal{X}(S)$. Now $X$ can be regarded both as a spherical $G$-variety and as a toric $T$-variety. Each of these points of view involves its own combinatorial data attached to $X$, and in this subsection we clarify relations between them.

Let $w$ be the nontrivial element of the Weyl group $N(T)/T$, where $N(T)$ is the normalizer of $T$ in $G$. Then $w$ naturally acts on $\mathcal{X}(T)$ in such a way that $w(\alpha) = -\alpha$ and $w(\chi) = \chi$ for all $\chi \in \mathcal{X}(S)$.

Put $\bar{M} = M \oplus \mathbb{Z}\alpha \subseteq \mathcal{X}(T)$, $\bar{N} = \text{Hom}_\mathbb{Z}(\bar{M}, \mathbb{Z})$, $\bar{M}_Q = \bar{M} \otimes_\mathbb{Z} Q$, $\bar{N}_Q = \bar{N} \otimes_\mathbb{Z} Q$ and extend the pairing $\langle \cdot, \cdot \rangle$ to a bilinear map $\bar{M}_Q \times \bar{N}_Q \to Q$. For every cone $\mathcal{C} \subseteq M_Q$, let $\bar{\mathcal{C}}$ denote the cone in $\bar{M}_Q$ generated by $\mathcal{C} \cup w(\mathcal{C})$. It follows from (7.1) that the weight monoid of $X$ as a toric $T$-variety is $\mathcal{F} := \bar{\mathcal{C}} \cap \bar{M}$. Let $\bar{\mathcal{E}} \subseteq \bar{N}_Q$ be the cone dual to $\bar{\mathcal{C}}$.

To describe the $T$-root subgroups on $X$, we need to know the facets of $\bar{\mathcal{C}}$ or, equivalently, the set $\bar{\mathcal{E}}^{\perp}$. It is easy to see that $\bar{G}$, $w(\mathcal{G})$ are facets of $\bar{\mathcal{C}}$, and we let $\delta, \delta' \in \bar{\mathcal{E}}^{\perp}$ be the corresponding elements. Observe that

\[
\langle \delta, M \rangle = 0, \quad \langle \delta, \alpha \rangle = -1; \quad (7.2)
\]

\[
\langle \delta', w(M) \rangle = 0, \quad \langle \delta', \alpha \rangle = 1. \quad (7.3)
\]

(The precise values of $\langle \delta, \alpha \rangle$ and $\langle \delta', \alpha \rangle$ are implied by the primitivity of $\delta, \delta'$ in $\bar{N}$.)

**Lemma 7.3** One has $\iota(\delta') = \iota(\alpha')$.

**Proof** Take any $\lambda \in M$. Since $w(\lambda) = \lambda - \langle \alpha', \lambda \rangle \alpha$, by (7.3) one has $\langle \delta', \lambda - \langle \alpha', \lambda \rangle \alpha \rangle = 0$ and therefore $\langle \delta', \lambda \rangle = \langle \delta', \alpha \rangle \langle \alpha', \lambda \rangle = \langle \alpha', \lambda \rangle$ as required. □

Let $Q \subseteq M_Q$ be the subspace spanned by $M_Q \cap \mathcal{X}(S)$; one has $M_Q \not\subseteq Q$ by our assumptions. The next result provides an explicit relation between the facets of $\mathcal{G}$ and $\bar{\mathcal{C}}$ and also between the sets $\mathcal{E}^{\perp}$ and $\bar{\mathcal{E}}^{\perp}$.


Lemma 7.4 The following assertions hold.

(a) If \( F \) is a facet of \( G \) contained in \( Q \) and \( \rho \in E^1 \) corresponds to \( F \) then \( \iota(\delta') \) is a positive multiple of \( \rho \). In particular, \( F \) is unique if exists.

(b) The map \( F \mapsto F \) yields a bijection

\[
\{ \text{facets of } G \text{ not contained in } Q \} \rightarrow \{ \text{facets of } G \text{ different from } G \text{ and } w(G) \}.
\]

Moreover, if \( \rho \in E^1 \) (resp. \( \overline{\rho} \in \overline{E}^1 \)) corresponds to \( F \) (resp. \( \overline{F} \)) then \( \overline{\rho} \) is the extension of \( \rho \) to \( N \) defined by \( \langle \rho, \alpha \rangle = 0 \).

Proof Choose a finite generating set \( E \) for \( \Gamma \).

(a) Since \( M_Q \not\subseteq Q \), it follows that \( \rho \) and \( \iota(\delta') \) have the same kernel equal to \( QF \), therefore they are proportional. It remains to observe that both \( \rho \) and \( \iota(\delta') \) take positive values on any element in \( E \setminus Q \).

(b) Let \( F \) be a facet of \( G \) not contained in \( Q \), let \( \rho \in E^1 \) be the corresponding element, and extend \( \rho \) to an element \( \overline{\rho} \in \overline{N} \) by setting \( \langle \overline{\rho}, \alpha \rangle = 0 \). Then \( \overline{F} = \overline{G} \cap \text{Ker} \overline{\rho} \), and so \( \overline{F} \) is a face of \( \overline{G} \). Clearly, \( \overline{F} \) has codimension one in \( \overline{G} \) and \( \overline{F} \notin \{ G, w(G) \} \).

Conversely, take any \( \overline{\rho} \in \overline{E}^1 \setminus \{ \delta, \delta' \} \) and let \( \overline{F} \) be the corresponding facet of \( \overline{G} \). If \( \langle \overline{\rho}, \alpha \rangle < 0 \) then \( \langle \overline{\rho}, \lambda \rangle > 0 \) for all \( \lambda \in w(E) \setminus E \), hence \( \overline{F} \) is contained in \( G \), hence \( \overline{F} = G \) and \( \overline{\rho} = \delta \), which is excluded. If \( \langle \overline{\rho}, \alpha \rangle > 0 \) then we similarly obtain \( \overline{F} = w(G) \) and \( \overline{\rho} = \delta' \), which is also excluded. Thus \( \langle \overline{\rho}, \alpha \rangle = 0 \) and in particular \( \overline{F} \) is \( w \)-stable. Let \( \rho \) be the restriction of \( \overline{\rho} \) to \( N \). Then \( \rho \in E \) and therefore \( F = G \cap \text{Ker} \rho \) is a face of \( G \). Now observe that \( F = G \cap \overline{F} \), hence \( F \) is generated by the set \( E \cap \overline{F} \). Since \( \overline{F} \) is \( w \)-stable, it follows that \( \overline{F} = F \). By dimension reasons, \( F \) is a facet of \( G \) not contained in \( Q \).

Combining (7.2), (7.3) with Lemma 7.4(b) we obtain

Corollary 7.5 Suppose \( \overline{\rho} \in \overline{E}^1 \); then

\[
\langle \overline{\rho}, \alpha \rangle = \begin{cases} 
-1 & \text{if } \overline{\rho} = \delta; \\
1 & \text{if } \overline{\rho} = \delta'; \\
0 & \text{otherwise.}
\end{cases}
\]

In particular, \( \alpha \in R_\delta(\overline{E}) \) and \(-\alpha \in R_{\delta'}(\overline{E}) \).

The next proposition will be a key ingredient for converting the description of \( T \)-root subgroups on \( X \) in terms of \( \Gamma \) into a description of \( B \)-root subgroups on \( X \) in terms of \( \Gamma \).

Lemma 7.6 The following assertions hold.

(a) \( \mathcal{R}(E) \cap \Lambda^+ \subseteq \mathcal{R}(\overline{E}) \).

(b) Suppose \( e \in \mathcal{R}(\overline{E}) \) and \( \delta_e \in \overline{E}^1 \) is the corresponding element. Then \( e \in \mathcal{R}(E) \cap \Lambda^+ \) if and only if \( \langle \delta, e \rangle = \langle \delta_e, \alpha \rangle = 0 \).
Proposition 7.7
Given \( \alpha \) ∈ \( \mathfrak{R}(\mathcal{E}) \) and \( \rho \) ∈ \( \mathcal{E}^1 \) be the corresponding element, so that \( \langle \rho, e \rangle = -1 \). Remark 3.1 yields \( e \in M \), therefore \( \delta, e \) = 0 by (7.2). Next, \( \langle \delta', e \rangle \geq 0 \) by Lemma 7.3. Let \( \mathcal{F} \) be the facet of \( \mathcal{G} \) corresponding to \( \rho \). Applying Lemma 7.4(a) we find that \( \mathcal{F} \not\in Q \). Then Lemma 7.4(b) implies that \( \rho \) extends to an element \( \overline{\rho} \in \overline{\mathcal{E}}^1 \) with \( \langle \overline{\rho}, \alpha \rangle = 0 \) and every \( \overline{\rho} \in \overline{\mathcal{E}}^1 \setminus \{ \delta, \delta', \overline{\rho} \} \) is an extension of an element in \( \mathcal{E}^1 \setminus \{ \rho \} \), which implies \( \langle \overline{\rho}', e \rangle \geq 0 \). It follows that \( e \in \mathfrak{R}(\mathcal{E}) \), and we have proved (a) and the ‘only if’ part of (b).

It remains to prove the ‘if’ part of (b). Let \( e, \delta \) be as in the hypothesis and suppose \( \langle \delta, e \rangle = \langle \delta, \alpha \rangle = 0 \). Then \( \delta \notin \{ \delta, \delta' \} \) in view of (7.2) and (7.3). As \( \langle \delta, e \rangle = 0 \), one has \( e \in M \). Let \( \mathcal{F}_e \) be the facet of \( \mathcal{G} \) corresponding to \( \delta \) and let \( \rho_e \) be the restriction of \( \delta \) to \( M \), so that \( \langle \rho_e, e \rangle = -1 \). By Lemma 7.4(b), the condition \( \langle \delta, \alpha \rangle = 0 \) implies \( \rho_e \in \mathcal{E}^1 \) and \( \mathcal{F}_e = \mathcal{F} \) for a facet \( \mathcal{F} \) of \( \mathcal{G} \) not contained in \( Q \). Now take any \( \rho \in \mathcal{E}^1 \setminus \{ \rho_e \} \) and let \( \mathcal{F}' \) be the corresponding facet of \( \mathcal{G} \). If \( \mathcal{F}' \subseteq Q \) then Lemma 7.4(a) yields \( \langle \rho, e \rangle \geq 0 \). If \( \mathcal{F}' \not\subseteq Q \) then, by Lemma 7.4(b), \( \rho \) extends to an element of \( \overline{\mathcal{E}}^1 \setminus \{ \delta \} \), hence again \( \langle \rho, e \rangle \geq 0 \), and we thus obtain \( e \in \mathfrak{R}(\mathcal{E}) \). Lemma 7.3 implies \( \langle \alpha', e \rangle \geq 0 \), and so \( e \in \Lambda^+ \).

\[ \square \]

7.3 Main results
Retain the notation and assumptions of Sect. 7.2. Recall from Sect. 3.2 that there is a bijection \( \overline{\rho} \mapsto D_{\overline{\rho}} \) between \( \overline{\mathcal{E}}^1 \) and the \( T \)-stable prime divisors on \( X \).

Proposition 7.7 Given \( \overline{\rho} \in \overline{\mathcal{E}}^1 \), the divisor \( D_{\overline{\rho}} \) is \( B \)-stable (resp. \( B^- \)-stable, \( G \)-stable) if and only if \( \overline{\rho} \in \overline{\mathcal{E}}^1 \setminus \{ \delta \} \) (resp. \( \overline{\rho} \in \overline{\mathcal{E}}^1 \setminus \{ \delta' \} \), \( \overline{\rho} \in \overline{\mathcal{E}}^1 \setminus \{ \delta, \delta' \} \)). In particular, \( D_{\delta'} \) is the unique color in \( X \).

Proof Clearly, \( D_{\overline{\rho}} \) is \( B \)-stable if and only if it is \( U \)-stable. Since \( U \) is a \( T \)-root subgroup on \( X \) of weight \( \alpha \) and \( \alpha \in \mathfrak{R}_\delta(\mathcal{E}) \) by Corollary 7.5, it follows from Theorem 3.5 that \( D_{\overline{\rho}} \) is \( U \)-stable if and only if \( \overline{\rho} \neq \delta \). Similarly, \( D_{\overline{\rho}} \) is \( B^- \)-stable if and only if \( \overline{\rho} \neq \delta' \). It remains to notice that \( D_{\overline{\rho}} \) is \( G \)-stable if and only if it is simultaneously \( B \)-stable and \( B^- \)-stable. \( \square \)

The following theorem provides a complete combinatorial description of all \( B \)-root subgroups on \( X \). Part (c) of this theorem can be deduced from Corollary 7.2 and Proposition 6.13(b), but we provide a direct proof based on Lemma 7.6.

Theorem 7.8 The following assertions hold.
(a) Every \( B \)-root subgroup on \( X \) is uniquely determined by its weight.
(b) Every vertical \( B \)-root subgroup on \( X \) is equivalent to \( U \) and the set of weights of such subgroups is \( \mathfrak{R}_{\delta}(\mathcal{E}) \).
(c) Every horizontal \( B \)-root subgroup on \( X \) is standard and the set of weights of such subgroups is \( \mathfrak{R}(\mathcal{E}) \cap \Lambda^+ \).

Proof Since every \( B \)-root subgroup on \( X \) is a \( T \)-root subgroup and \( X \) is toric as a \( T \)-variety, part (a) is implied by Theorem 3.4(b). In what follows we simultaneously prove (b) and (c).
Lemma 7.10
For every $e \in \mathcal{R}(\mathcal{E})$, consider the corresponding element $\delta_e \in \mathcal{E}_1^1$, and let $H_e$ be the $T$-root subgroup on $X$ of weight $e$. Clearly, $H_e$ is a $B$-root subgroup on $X$ if and only if $H_e$ commutes with $U$, which in turn is a $T$-root subgroup on $X$ of weight $\alpha \in \mathcal{R}_\beta(\mathcal{E})$ (see Corollary 7.5). Thanks to [8, Lemma 3.2], $H_e$ and $U$ commute if and only if one of the following two cases occurs.

Case 1: $e \in \mathcal{R}_\beta(\mathcal{E})$. By Theorem 3.5, this means that $H_e$ is equivalent to $U$, and so $H_e$ is vertical.

Case 2: $\langle \delta, e \rangle = \langle \delta, \alpha \rangle = 0$. By Lemma 7.6(b), this happens if and only if $e \in \mathcal{R}(\mathcal{E}) \cap \Lambda^\perp$. Since $\delta_e \neq \delta$ by (7.2), we find from Proposition 7.7 that the divisor $D_{\delta_e}$ moved by $H_e$ is in fact $B$-stable, hence $H_e$ is horizontal. Moreover, from (3.1) and $\langle \delta_e, \alpha \rangle = 0$ we conclude that $H_e$ is standard.

As $\mathcal{R}(\mathcal{E}) \cap \Lambda^\perp \subseteq \mathcal{R}(\mathcal{E})$ by Lemma 7.6(a), the proof of (b) and (c) is completed. □

Remark 7.9 A $B$-root subgroup $H$ on $X$ is $G$-normalized if and only if it commutes with $U^-$, which is a $T$-root subgroup of weight $-\alpha \in \mathcal{R}_\beta(\mathcal{E})$. Applying again [8, Lemma 3.2] and using Theorem 7.8 we find that $H$ is a $G$-root subgroup if and only if $\chi_H \in \mathcal{R}(\mathcal{E}) \cap \Lambda^\perp \cap \text{Ker} \, \delta'$. By Theorem 7.3, the latter set equals $\mathcal{R}(\mathcal{E}) \cap \mathcal{X}(G)$, which yields a direct proof of Proposition 6.13(a) for our $X$.

As follows from Theorem 7.8(b), there is only one equivalence class of vertical $B$-root subgroups on $X$: all such subgroups are equivalent to $U$. Our last goal is to determine the equivalence classes of horizontal $B$-root subgroups on $X$. Before stating and proving the result (see Theorem 7.11), we need the following

Lemma 7.10 For every $\bar{\rho} \in \mathcal{E}_1^1 \setminus \{\delta, \delta'\}$, the set $\mathcal{R}_\mathcal{E}(\mathcal{E}) \cap \mathcal{R}(\mathcal{E}) \cap \Lambda^\perp$ is nonempty.

Proof Fix any $\bar{\rho} \in \mathcal{E}_1^1 \setminus \{\delta, \delta'\}$ and $e' \in \mathcal{R}(\mathcal{E})$. Then $\langle \bar{\rho}, e' \rangle = -1$ and $\langle \bar{\rho}', e' \rangle \geq 0$ for all $\bar{\rho}' \in \mathcal{E}_1^1 \setminus \{\bar{\rho}\}$. Put $e = e' + q\alpha$ where $q = \langle \delta, e' \rangle \geq 0$. Corollary 7.5 yields $\langle \bar{\rho}, e \rangle = -1, \langle \bar{\rho}', e \rangle \geq 0$ for all $\bar{\rho}' \in \mathcal{E}_1^1 \setminus \{\delta, \delta', \bar{\rho}\}$, $\langle \delta, e \rangle = 0$, and $\langle \delta', e \rangle = \langle \delta', e' \rangle + q \geq 0$, which implies $e \in \mathcal{R}(\mathcal{E})$. As $\langle \delta, e \rangle = 0$ and $\langle \bar{\rho}, \alpha \rangle = 0$, Lemma 7.6(b) yields $e \in \mathcal{R}(\mathcal{E}) \cap \Lambda^\perp$. □

Theorem 7.11 The equivalence classes of horizontal $B$-root subgroups on $X$ are in bijection with the $G$-stable prime divisors on $X$. More precisely, under this bijection every $G$-stable prime divisor $D \subseteq X$ corresponds to all $B$-root subgroups that move $D$.

Proof Let $H$ be a horizontal $B$-root subgroup on $X$ and let $\bar{\rho} \in \mathcal{E}_1^1$ be such that $\chi_H \in \mathcal{R}(\mathcal{E})$, so that $H$ moves the divisor $D_{\bar{\rho}}$. By Theorem 7.8(c) and Lemma 7.6(b), we have $\langle \bar{\rho}, \alpha \rangle = 0$. Then Corollary 7.5 yields $\bar{\rho} \notin \{\delta, \delta'\}$, and so $D_\bar{\rho}$ is $G$-stable by Proposition 7.7. On the other hand, thanks to Proposition 7.7, Theorem 7.8(c), and Lemma 7.10, for every $G$-stable prime divisor $D$ on $X$ there exists a horizontal $B$-root subgroup that moves $D$. The rest is implied by Theorem 3.5. □

Example 7.12 (An illustration of the main results) Retain the situation and notation of Example 5.30 and take $X' = D_3 = \{\text{det} = 0\}$. Then $X'$ is toric as a $T$-variety and $\Gamma(X') = \mathbb{Z}_{\geq 0}(\omega + \chi_1, \omega + \chi_2)$. There are two $G$-stable prime divisors $D'_1 = \{x_{11} = x_{21} = 0\}, D'_2 = \{x_{12} = x_{22} = 0\}$ and one color $D' = \{x_{21} = x_{22} = 0\}$. The rest is implied by Theorem 3.5. □
in $X'$. By Corollary 5.28, there is no $B$-root subgroup on $X'$ moving $D'$. For $i = 1, 2$, the set of weights of (the equivalence class of) $B$-root subgroups on $X'$ moving $D'_i$ is $\chi_j - \chi_i + \mathbb{Z}_{\geq 0}(\omega + \chi_j)$ (where $j = 3 - i$); among these subgroups there is a unique $G$-root subgroup, which is induced by the action of $H_i$ and has the weight $\chi_j - \chi_i$. The set of weights of vertical $B$-root subgroups on $X'$ is $\alpha + \Gamma(X')$, and so all such subgroups are replicas of $U$.

**Remark 7.13** The general strategy in this section is to take an affine spherical $G$-variety $X$ and impose the restriction that $T$ acts on $X$ with an open orbit. We could also start with an affine toric $T$-variety $X$ and impose the restriction that the $T$-action on $X$ extends to a $G$-action; this situation is discussed in [7,Section 2.2]. In this way one can obtain an alternative proof of Theorem 7.11.

**Remark 7.14** It is an interesting problem to study $B$-root subgroups on affine spherical varieties of a connected reductive group $G$ of semisimple rank one such that the maximal torus $T$ in $G$ has no open orbit in $X$. In this case $T$ acts on $X$ with complexity one and a description of $B$-root subgroups can be obtained using the results of [35].

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