Dynamical analysis of Hilfer–Hadamard type fractional pantograph equations via successive approximation

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ABSTRACT
In this note, we consider a nonlinear pantograph equation with Hilfer–Hadamard fractional derivative. We investigate the existence and continuous dependence results by using successive approximations and generalized Gronwall inequality.

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1. Introduction
Fractional differential equations (FDEs) are appeared in mathematical modelling of processes and phenomena of science and engineering. Hence the theory of FDEs is an area intensively developed during last few decades. The monographs of Hilfer [1], Kilbas et al. [2], Miller and Ross [3] and Abbas et al. [4], include a study of techniques of solving which are an extension of procedures from differential equations theory. Recently, considerable attention has been given to the Hilfer fractional derivative which was introduced by R. Hilfer in [1]. The numerous results on existence and uniqueness of solution of FDEs with Hilfer and Hilfer–Hadamard fractional derivative have been studied in [5–10] by different methods.

Functional differential equations with proportional delays are usually referred to as pantograph equations. It arises in rather different fields of pure and applied mathematics, such as electrodynamics, control systems, number theory, probability, and quantum mechanics. Therefore, the problems have attracted a great deal of attention, one can refer to [11–14]. Recently, Dhaigude and Bhairat [8] used the Hilfer fractional derivative for investigating the existence and uniqueness of solution of Cauchy-type problem for Hilfer FDEs. Inspired by the discussion, our main concern is to prove the existence and continuation dependence of solution to a pantograph equation with Hilfer–Hadamard fractional derivative.

In this note, we consider the pantograph equation with Hilfer–Hadamard fractional derivative

\[ H^{\alpha, \beta}_{1+} x(t) = H(t, x(t), x(\mu t)), \quad \mu \in (0, 1), \]

\[ H^{1-\gamma}_{1+} x(1) = x_0, \quad \gamma = \alpha + \beta(1 - \alpha), \]

where, \( H^{\alpha, \beta}_{1+} \) is the Hilfer–Hadamard fractional derivative of order \( \alpha \) and type \( \beta \) such that \( \alpha \in (0, 1), \beta \in [0, 1] \) and \( H^{1-\gamma}_{1+} \) is the left-sided mixed Hadamard integral of order \( 1 - \gamma \).

It is easy to see that Equations (1)–(2) is equivalent to the following integral equation

\[ x(t) = x_0 \frac{(\log t)^{\gamma-1}}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} H(s, x(s), x(\mu s)) ds. \]

Here we remark that the fixed point technique does not indicate the interval of existence of solution, which is a necessary aspect for application purpose. This drawback of fixed point technique is removed by using Picard’s iterative technique and existence of a solution is confirmed on \([1, T]\) as done in [15]. The fundamental results, existence results and the dependence of solutions on order are studied in the subsequent sections.
2. Preliminaries

Let $-\infty < T < +\infty$. Let $C[1,T]$, $AC[1,T]$, and $C^\infty[1,T]$ be the spaces of continuous, absolutely continuous, $n$-times continuous and continuous differentiable functions on $[1,T]$ respectively. Here $L^p(1,T)$, $p \geq 1$ is the space of Lebesgue integrable functions on $(1,T)$. Furthermore, we recall the following weighted space

$$C_{\gamma,log}[1,T] = \{ \mathcal{H} : (1,T) \to X : (\log t)^y \mathcal{H} (t) \in C[1,T], \gamma \in [0,1), \}$$

and $C_{1,log}[1,T]$ is a complete metric space with the metric $d$ defined by

$$d(x_1,x_2) = \| x_1 - x_2 \|_{C_{\gamma,log}[1,T]} : \max_{t \in [1,T]} (| \log t |)^{-1} \| x_1 (t) - x_2 (t) \|_{C_{\gamma,log}[1,T]}.$$

**Definition 2.1** ([16]): The Riemann–Liouville fractional order derivative of order $\alpha$ of function $\mathcal{H}$ is defined by

$$D^\alpha \mathcal{H} (t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathcal{H} (s) \, ds,$$

provided that the integral on the right side exits over $(0,\infty)$.

**Definition 2.2 ([16]):** The Caputo fractional order derivative of order $\alpha$ of function $\mathcal{H}$ is defined by

$$C D^\alpha \mathcal{H} (t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \mathcal{H}^{(n)} (s) \, ds,$$

provided that integral on the right is pointwise defined on $(0,\infty)$, where $n = [\alpha] + 1$ and $[\alpha]$ denotes the greatest integer which is less than or equal to the real number $\alpha$.

**Definition 2.3 ([16]):** The Riemann–Liouville fractional order derivative of order $\alpha$ of function $\mathcal{H}$ is defined by

$$\mathcal{H}^\alpha (t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathcal{H} (s) \, ds,$$

provided that integral on the right is pointwise defined on $(0,\infty)$, where $n = [\alpha] + 1$ and $[\alpha]$ denotes the greatest integer which is less than or equal to the real number $\alpha$.

**Definition 2.4 ([17]):** Let $\Omega = (1,T)$ and $\mathcal{H} : (1,\infty) \to X$ is a real valued continuous function. Then the Riemann–Liouville Hadamard fractional integral of a function of order $\alpha \in \mathbb{R}^+$ is denoted as $H I_1^\alpha \mathcal{H}$ and defined by

$$H I_1^\alpha \mathcal{H} (t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} \mathcal{H} (s) \, ds, \quad t > 1,$$

where $\Gamma(\alpha)$ is the Euler’s Gamma function.

**Definition 2.5 ([17]):** Let $\Omega = (1,T)$ and $\mathcal{H} : (1,\infty) \to X$ is a real valued continuous function. The Riemann–Liouville Hadamard fractional derivative of function $\mathcal{H}$ of order $\alpha \in \mathbb{R}^+$ is denoted as $H D_1^\alpha \mathcal{H}$ and defined by

$$H D_1^\alpha \mathcal{H} (t) = \frac{1}{\Gamma(n-\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{n-\alpha-1} \mathcal{H} (s) \, ds, \quad t > 1,$$

where $n = [\alpha] + 1$, and $[\alpha]$ means the integral part of $\alpha$, provided the right-hand side is point wise defined on $(1,\infty)$.

**Definition 2.6 ([10]):** The Hilfer–Hadamard fractional derivative $H D_1^{\alpha,\beta}$ of function $\mathcal{H} \in L^1(1,T)$ of order $1 < \alpha < n$ and type $0 \leq \beta \leq 1$ is defined by

$$H D_1^{\alpha,\beta} \mathcal{H} (t) = H I_1^{\beta-\alpha} H D_1^\beta H I_1^{1-\beta} \mathcal{H} (t),$$

where $H D_1^{\alpha,\beta}$ and $H I_1^{\alpha,\beta}$ are Riemann–Liouville Hadamard fractional integral and derivative defined by (6) and (7), respectively.

**Definition 2.7:** Assume that $\mathcal{H} (t,x(t),x(\mu t))$ is defined on the set $(1,T) \times \mathcal{G}$ such that $\mathcal{G} \subset X$. A function $\mathcal{H} (t,x(t),x(\mu t))$ satisfies Lipschitz condition with respect to $x$, if for all $t \in (1,T)$ and for $x, \tilde{x} \in \mathcal{G}$, one has

$$|\mathcal{H} (t,x(t),x(\mu t)) - \mathcal{H} (t,\tilde{x}(t),\tilde{x}(\mu t))| \leq \lambda \left( |x(t) - \tilde{x}(t)| + |x(\mu t) - \tilde{x}(\mu t)| \right),$$

where $\lambda > 0$ is Lipschitz constant.
Definition 2.8 ([10]): Let \( \alpha \in (0, 1) \), \( \beta \in [0, 1] \), the weighted space \( C_{1-\gamma, \log}^{\alpha, \beta} \) is defined by
\[
C_{1-\gamma, \log}^{\alpha, \beta} = \left\{ \mathcal{H} \in C_{1-\gamma, \log}^{\alpha, \beta} : H^{\alpha, \beta} \in C_{1-\gamma, \log}^{\alpha, \beta}, \; \gamma = \alpha + \beta (1 - \alpha) \right\}.
\]
(10)

Lemma 2.9: If \( \alpha > 0 \) and \( \mu \in [0, 1] \), then \( H^{\alpha, \mu}_{1-\gamma} \) is bounded from \( C_{\mu, \log}^{\alpha, \beta} \) into \( C_{1-\gamma, \log}^{\alpha, \beta} \). In addition, if \( \mu \leq \alpha \), then \( H^{\alpha, \mu}_{1-\gamma} \) is bounded from \( C_{1-\gamma, \log}^{\alpha, \beta} \) into \( C[1, T] \).

Lemma 2.10 ([10]): For \( t > 1 \), we have
\[
(1) \; H^{\alpha, \beta}_{1-\gamma} \left( \log t \right)^{-\beta - 1} = \left( \Gamma(\beta) / \Gamma(\beta + \alpha) \right) \left( \log t \right)^{\beta + \alpha - 1}, \; \alpha \geq 0, \beta > 0; \\
(2) \; H^{\alpha, \beta}_{1-\gamma} \left( \log t \right)^{-\alpha - 1} = 0, \alpha \in (0, 1).
\]

Lemma 2.11 ([10]): Let \( \alpha \in (0, 1), \mu \in [0, 1] \). If \( \mathcal{H} \in C_{\mu, \log}^{\alpha, \beta} \) and \( H^{\alpha, \mu}_{1-\gamma} \in C_{\mu, \log}^{\alpha, \beta} \). Then
\[
H^{\alpha, \beta}_{1-\gamma} \mathcal{H} (t) = H^{\alpha, \beta}_{1-\gamma} \mathcal{H} (1) \left( \log t \right)^{\alpha - 1}, \; \forall t \in (1, T).
\]

3. Existence results

In this section, we prove the existence and uniqueness of solution of Hilfer–Hadamard type pantograph equation (1)–(2) in \( C_{1-\gamma, \log}^{\alpha, \beta} \). We need the following lemma.

Lemma 3.1: If \( \gamma = \alpha + \beta (1 - \alpha) \), \( \alpha \in (0, 1) \), \( \beta \in [0, 1] \), then the Hadamard type Riemann–Liouville fractional integral operator \( H^{\alpha, \beta}_{1-\gamma} \) is bounded from \( C_{1-\gamma, \log}^{\alpha, \beta} \) to \( C_{1-\gamma, \log}^{\alpha, \beta} \):
\[
\left\| H^{\alpha, \beta}_{1-\gamma} \mathcal{H} \right\|_{C_{1-\gamma, \log}^{\alpha, \beta}} \leq \mathcal{M} \frac{\Gamma(\gamma)}{\Gamma(\gamma + \alpha)} (\log t_1)^{\alpha},
\]
where, \( \mathcal{M} \) is the bound of a bounded function \( \mathcal{H} \).

Proof: From Lemma 2.9, the result follows. Now we prove the estimate (11). By the weighted space given in (4), we have
\[
\left\| H^{\alpha, \beta}_{1-\gamma} \mathcal{H} \right\|_{C_{1-\gamma, \log}^{\alpha, \beta}} \leq \mathcal{M} \frac{\Gamma(\gamma)}{\Gamma(\gamma + \alpha)} (\log t)^{\alpha}.
\]
Using Lemma 2.10, we get
\[
\left\| H^{\alpha, \beta}_{1-\gamma} \mathcal{H} \right\|_{C_{1-\gamma, \log}^{\alpha, \beta}} = \mathcal{M} \frac{\Gamma(\gamma)}{\Gamma(\gamma + \alpha)} (\log t)^{\alpha}.
\]
Thus the proof is complete.

Theorem 3.2: Let \( \gamma = \alpha + \beta (1 - \alpha) \) where \( \alpha \in (0, 1) \) and \( \beta \in [0, 1] \). Let \( \mathcal{H} : [1, T] \times X \times X \to X \) be a function such that \( \mathcal{H}(t, x(t), x(\mu t)) \in C_{1-\gamma, \log}^{\alpha, \beta} \) for any \( x \in C_{1-\gamma, \log}^{\alpha, \beta} \) and satisfies the Lipschitz condition (9) with respect to \( x \). Then, there exists a unique solution \( x(t) \) for the problem (1)–(2) in \( C_{1-\gamma, \log}^{\alpha, \beta} \).

Proof: The integral equation (3) makes sense in any interval \([1, t_1] \subset [1, T] \). Choose \( t_1 \) such that
\[
\frac{\mathcal{M}}{\Gamma(\gamma + \alpha)} \left( \log t_1 \right)^{\alpha} < 1
\]
holds and first we prove the existence of unique solution \( x \in C_{1-\gamma, \log}^{\alpha, \beta} \). We proceed by taking Picard’s sequence as described by the functions
\[
x_0(t) = \frac{x_0}{\Gamma(\gamma)} (\log t)^{\gamma - 1}, \; \gamma = \alpha + \beta (1 - \alpha),
\]
\[
x_m(t) = x_0(t) + \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha - 1} \mathcal{H}(s, x_{m-1}(s), x_{m-1}(\mu s)) \frac{ds}{s}, \; m \in \mathbb{N}.
\]
We now show that \( x_m(t) \in C_{1-\gamma, \log}^{\alpha, \beta} \). From Equation (13), it follows that \( x_0(t) \in C_{1-\gamma, \log}^{\alpha, \beta} \). By Lemma 3.1, \( H^{\alpha, \beta}_{1-\gamma} \mathcal{H} \) is bounded from \( C_{1-\gamma, \log}^{\alpha, \beta} \) to \( C_{1-\gamma, \log}^{\alpha, \beta} \) which gives \( x_m(t) \in C_{1-\gamma, \log}^{\alpha, \beta} \). Now we consider \( t \in (1, T] \). By Equations (13) and (14), we have
\[
\| x_1(t) - x_0(t) \|_{C_{1-\gamma, \log}^{\alpha, \beta}} \leq \mathcal{M} \frac{\Gamma(\gamma)}{\Gamma(\gamma + \alpha)} (\log t_1)^{\alpha}.
\]
Further we obtain
\[
\| x_2(t) - x_1(t) \|_{C_{1-\gamma, \log}^{\alpha, \beta}} \leq \mathcal{M} \frac{\Gamma(\gamma)}{\Gamma(\gamma + \alpha)} (\log t_1)^{\alpha} \left( 2^{m-1} \frac{\Gamma(\gamma)}{\Gamma(\gamma + \alpha)} (\log t_1)^{\alpha} \right)^{m-1}.
\]
Continuing in this way, \( m \)-times, we obtain
\[
\| x_m(t) - x_{m-1}(t) \|_{C_{1-\gamma, \log}^{\alpha, \beta}} \leq \mathcal{M} \frac{\Gamma(\gamma)}{\Gamma(\gamma + \alpha)} (\log t_1)^{\alpha} \left( 2^{m-1} \frac{\Gamma(\gamma)}{\Gamma(\gamma + \alpha)} (\log t_1)^{\alpha} \right)^{m-1}.
\]
By Equation (12), we get
\[
\| x_m(t) - x_{m-1}(t) \|_{C_{1-\gamma, \log}^{\alpha, \beta}} \to 0 \quad \text{as} \quad m \to +\infty.
\]
Again by Lemma 3.1, it follows that
\[ \| & \mathcal{L}^\alpha_{+} & \mathcal{H}(t, x(t), x(\mu t)) \|_{C_{1-\gamma, \log}[1, T]} \\
- & \mathcal{L}^\alpha_{+} & \mathcal{H}(t, x(t), x(\mu t)) \|_{C_{1-\gamma, \log}[1, T]} \\
& & \leq 2 \alpha^\gamma (\log t)^\alpha \Gamma(\gamma) \Gamma(\gamma + \alpha) \| x(t) - x(t) \|_{C_{1-\gamma, \log}[1, T]} \]
and hence by Equation (18)
\[ \| & \mathcal{L}^\alpha_{+} & \mathcal{H}(t, x(t), x(\mu t)) \|_{C_{1-\gamma, \log}[1, T]} \\
- & \mathcal{L}^\alpha_{+} & \mathcal{H}(t, x(t), x(\mu t)) \|_{C_{1-\gamma, \log}[1, T]} \\
& & \to 0 \text{ as } m \to +\infty. \] (19)

From Equations (18) and (19), it follows that \( x(t) \) is the solution of integral equation (3) in \( C_{1-\gamma, \log}[1, T] \).

Now to show that the solution \( x(t) \) is unique, consider there exists two solutions \( x(t) \) and \( y(t) \) of the integral equation (3) on \([1, t_1] \). Substituting them into (3) and using Lemma 2.9 with Lipschitz condition (9), we get
\[ \| x(t) - y(t) \|_{C_{1-\gamma, \log}[1, T]} \]
\[ = \| \mathcal{L}^\alpha_{+} & \mathcal{H}(t, x(t), x(\mu t)) \\
- & \mathcal{L}^\alpha_{+} & \mathcal{H}(t, y(t), y(\mu t)) \|_{C_{1-\gamma, \log}[1, T]} \]
\[ \leq 2 \alpha^\gamma (\log t_1)^\alpha \Gamma(\gamma) \Gamma(\gamma + \alpha) \| x(t) - y(t) \|_{C_{1-\gamma, \log}[1, T]} \]
(20)

This yields \( 2 \alpha^\gamma (\log t_1)^\alpha (\Gamma(\gamma)/\Gamma(\gamma + \alpha)) \geq 1 \), which contradicts to condition (12). Thus there exists \( x(t) = x_1(t) \in C_{1-\gamma}[1, t_1] \) as a unique solution on \([1, t_1] \).

Next, consider the interval \([t_1, t_2] \), where \( t_2 = t_1 + h_1 \), \( h_1 > 1 \) such that \( t_2 < T \). Now the integral equation (3) takes the form
\[ x(t) = \frac{x_0}{\Gamma(\gamma)} (\log t)^{\gamma - 1} + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} \left( \log \frac{t}{s} \right)^{\alpha - 1} \mathcal{H}(s, x(s), x(\mu s)) \frac{ds}{s}, \]
\[ + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} \left( \log \frac{t_1}{s} \right)^{\alpha - 1} \mathcal{H}(s, x(s), x(\mu s)) \frac{ds}{s}, \]
\[ t \in [t_1, t_2]. \] (21)

Since the function \( x(t) \) is uniquely defined on \([1, t_1] \), the last integral is known function and therefore the integral equation (21) can be written in the form
\[ x(t) = x_0(t) + \frac{1}{\Gamma(\gamma)} \int_{t_1}^{t} \left( \log \frac{t}{s} \right)^{\alpha - 1} \mathcal{H}(s, x(s), x(\mu s)) \frac{ds}{s}, \]
\[ t \in [t_1, t_2], \] (22)

where
\[ x_0(t) = \frac{x_0}{\Gamma(\gamma)} (\log t)^{\gamma - 1} + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} \left( \log \frac{t_1}{s} \right)^{\alpha - 1} \mathcal{H}(s, x(s), x(\mu s)) \frac{ds}{s} \] (23)
is known function. Using the same arguments as above, we deduce that there exists a unique solution \( x(t) = x_2(t) \in C_{1-\gamma, \log}[1, T] \) on \([t_1, t_2] \). Taking interval \([t_2, t_3] \), where \( t_3 = t_2 + h_2, h_2 > 1 \) such that \( t_3 < T \), and repeating the above process, we obtain a unique solution \( x(t) = x_3(t) \in C_{1-\gamma, \log}[1, T] \), for \( j = 1, 2, \ldots, l \) and \( l = t_0 < t_2 < \cdots < t_l = T \).

Using Equation (1) and Lipschitz condition (9), we obtain
\[ \| & \mathcal{L}^\alpha_{+} & \mathcal{H}(t, x(t), x(\mu t)) \\
- & \mathcal{L}^\alpha_{+} & \mathcal{H}(t, x(t), x(\mu t)) \|_{C_{1-\gamma, \log}[1, T]} \]
\[ = \| \mathcal{L}^\alpha_{+} & \mathcal{H}(t, x(t), x(\mu t)) \\
- & \mathcal{L}^\alpha_{+} & \mathcal{H}(t, x(t), x(\mu t)) \|_{C_{1-\gamma, \log}[1, T]} \]
\[ \leq 2 \alpha^\gamma (\log t_1)^\alpha \Gamma(\gamma) \Gamma(\gamma + \alpha) \| x(t) - x(t) \|_{C_{1-\gamma, \log}[1, T]} \]
(24)
Clearly, (18) and (24) implies that \( \| \mathcal{L}^\alpha_{+} & \mathcal{H}(t, x(t), x(\mu t)) \|_{C_{1-\gamma, \log}[1, T]} \) and results follows.

4. Continuous dependence

Consider Equations (1)–(2), for \( \alpha \in (0, 1), \beta \in [0, 1), \]
\[ t \in [1, T], \]
\( T \leq +\infty \) and \( \mathcal{H} : [1, T] \times X \times X \to X \). To present dependence of solution on the order, let us consider the solutions of two initial value problem for pantograf equations with the neighbouring orders. We need the following lemma.

Lemma 4.1 ([17]): Let \( v, w : [1, T] \to [1, +\infty) \) be continuous functions. If \( w \) is non-decreasing and there are constants \( k \geq 0 \) and \( 0 < \alpha < 1 \) such that
\[ v(t) \leq w(t) + k \int_{t_1}^{t} \left( \log \frac{t}{s} \right)^{\alpha - 1} v(s) \frac{ds}{s}, \]
\[ t \in J := [1, T], \]
then
\[ v(t) \leq w(t) + \int_{t}^{T} \left[ \sum_{n=1}^{\infty} \left( \kappa \Gamma(\alpha) \right)^n \left( \log \frac{t}{s} \right)^{\alpha n - 1} w(s) \right] \frac{ds}{s}, \]
\[ t \in J. \]

Theorem 4.2: Let \( \alpha > 0, \delta > 0 \) such that \( 0 < \alpha - \delta < \alpha \leq 1 \). Let \( \mathcal{H} \) is continuous function satisfying Lipschitz condition (9) in \( X \). For \( 1 \leq t < h < T \), assume that \( x \) is the solution of Equations (1)–(2) and \( \tilde{x} \) is the solution of the following problem
\[ \mathcal{L}^\alpha_{+} & \mathcal{H}(t, \tilde{x}(t), \tilde{x}(\mu t)), \]
\[ \alpha \in (0, 1), \beta \in [0, 1), \mu \in (0, 1), \]
\[ \mathcal{L}^\alpha_{+} & \mathcal{H}(1, \tilde{x}(1)) = x_0, \gamma = \alpha + \beta(1 - \alpha). \]
Then for \( 1 < t < T \)
\[ |\tilde{x}(t) - x(t)| \leq \mathcal{R}(t) + \int_{t}^{T} \left[ \sum_{n=1}^{\infty} \left( \log \frac{t}{s} \right)^{n(\alpha - \delta) - 1} \mathcal{R}(s) \right] \frac{ds}{s} \]
\[ \frac{1}{\Gamma(n(\alpha - \delta))} \left( \log \frac{t}{s} \right)^{n(\alpha - \delta) - 1} \mathcal{R}(s) \frac{ds}{s}, \]
hold, where
\[
\mathcal{B}(t) = \left| \frac{x_0}{\Gamma(\gamma + \delta(\beta - 1) - 1)} (\log t)^{\gamma + \delta(\beta - 1) - 1} - \frac{x_0}{\Gamma(\gamma)} (\log t)^{\gamma - 1} \right|
\]
\[
+ \left\| \mathcal{H} \right\| \left| \frac{1}{\Gamma(\alpha - \delta + 1)} (\log t)^{\alpha - \delta} \right|
\]
\[
- \frac{1}{\Gamma(\alpha - \delta)} \mathcal{H}(s, x(s), x(\lambda s)) \frac{ds}{s} \left| \mathcal{H}(s, x(s), x(\mu s)) \right| \frac{ds}{s},
\]
and
\[
\left\| \mathcal{H} \right\| = \max_{t \in [1, T]} |\mathcal{H}(t, x(t), x(\mu t))|.
\]

Proof: The equivalent integral solutions of Equations (1)–(2) and (25)–(26) are
\[
x(t) = \frac{x_0}{\Gamma(\gamma)} (\log t)^{\gamma - 1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left( \frac{\log s}{s} \right)^{\alpha - 1} \mathcal{H}(s, x(s), x(\lambda s)) \frac{ds}{s},
\]
and
\[
\hat{x}(t) = \frac{x_0}{\Gamma(\gamma + \delta(\beta - 1))} (\log t)^{\gamma + \delta(\beta - 1) - 1}
\]
\[
+ \frac{1}{\Gamma(\alpha - \delta)} \int_1^t \left( \frac{\log s}{s} \right)^{\alpha - \delta - 1} \mathcal{H}(s, \hat{x}(s), \hat{x}(\lambda s)) \frac{ds}{s},
\]
respectively. It follows that
\[
|\hat{x}(t) - x(t)| \leq \left| \frac{x_0}{\Gamma(\gamma + \delta(\beta - 1))} (\log t)^{\gamma + \delta(\beta - 1) - 1} - \frac{x_0}{\Gamma(\gamma)} (\log t)^{\gamma - 1} \right|
\]
\[
+ \left\| \mathcal{H} \right\| \left| \frac{1}{\Gamma(\alpha - \delta + 1)} (\log t)^{\alpha - \delta} \right|
\]
\[
- \frac{1}{\Gamma(\alpha - \delta)} \mathcal{H}(s, x(s), x(\mu s)) \frac{ds}{s} \left| \mathcal{H}(s, x(s), x(\mu s)) \right| \frac{ds}{s},
\]
the result is proved.

Next, we consider Equation (1) with the small change in the initial condition (2),
\[
\frac{x_1}{\Gamma(\gamma)} = x_0 + \epsilon, \quad \gamma = \alpha + \beta(1 - \alpha),
\]
where \(\epsilon\) is arbitrary constant. We state and prove the result as follows:

Theorem 4.3: Suppose that assumption of Theorem 3.2 hold. Suppose \(x(t)\) and \(\hat{x}(t)\) are solutions of (1)–(2) and (1)–(28), respectively. Then,
\[
|x(t) - \hat{x}(t)| \leq |\epsilon| (\log t)^{\gamma - 1} E_{\alpha, \gamma} (2 \alpha / (\log t)^{\alpha - 1}) \mathcal{B}(s) \frac{ds}{s},
\]
holds, where \(E_{\alpha, \gamma} = \sum_{k=0}^{\infty} \left( z^k / \Gamma(k \alpha + \gamma) \right) \) is the Mittag-Leffler function.

Proof: In accordance with Theorem 3.2, we have \(x(t) = \lim_{\gamma \to \infty} x_m(t)\), with \(x_0(t)\) and \(x_m(t)\) are as defined in Equations (13) and (14), respectively. Clearly, we can write \(\hat{x}(t) = \lim_{\gamma \to \infty} \hat{x}_m(t)\), and
\[
\hat{x}(t) = \frac{x_0 + \epsilon}{\Gamma(\gamma)} (\log t)^{\gamma - 1},
\]
\[
\hat{x}_m(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_1^t \left( \frac{\log s}{s} \right)^{\alpha - 1} \mathcal{H}(s, \hat{x}_m(s), \hat{x}_m(\mu s)) \frac{ds}{s},
\]
From (13) and (30), we have
\[
|x_0(t) - \hat{x}_0(t)| = \left| \frac{x_0}{\Gamma(\gamma)} (\log t)^{\gamma - 1} - \frac{x_0 + \epsilon}{\Gamma(\gamma)} (\log t)^{\gamma - 1} \right|,
\]
\[
|x_0(t) - \hat{x}_0(t)| \leq |\epsilon| (\log t)^{\gamma - 1} \mathcal{B}(s) \frac{ds}{s}.
\]
By subsequent relation (14) and (31), the Lipschitz condition (9) and the inequality (32), we obtain

$$|x_1(t) - \tilde{x}_1(t)| \leq |e| \left(\frac{1}{\Gamma(\gamma)} (\log t)^{\gamma-1} + \frac{2}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} ds \right) \frac{2^j \alpha_j \log(t)^{\alpha_j}}{\Gamma(\alpha_j + \gamma)}.$$  

Similarly, by using (33), it directly follows that

$$|x_2(t) - \tilde{x}_2(t)| \leq |e| \left(\frac{1}{\Gamma(\gamma)} (\log t)^{\gamma-1} + \frac{2}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} ds \right) \frac{2^j \alpha_j \log(t)^{\alpha_j}}{\Gamma(\alpha_j + \gamma)}.$$  

By the induction, we obtain

$$|x_m(t) - \tilde{x}_m(t)| \leq |e| \left(\frac{1}{\Gamma(\gamma)} (\log t)^{\gamma-1} + \frac{2}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} ds \right) \frac{2^j \alpha_j \log(t)^{\alpha_j}}{\Gamma(\alpha_j + \gamma)}.$$  

Taking limit as $m \to +\infty$ in (34), we have

$$|x(t) - \tilde{x}(t)| \leq |e| \left(\frac{1}{\Gamma(\gamma)} (\log t)^{\gamma-1} + \frac{2}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} ds \right) \frac{2^j \alpha_j \log(t)^{\alpha_j}}{\Gamma(\alpha_j + \gamma)}.$$  

which completes the proof of Theorem 4.3.

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