Testing macroscopic local realism using cat-states and Bell inequalities in time

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We show how one may test macroscopic local realism using nonlinear beam splitters, where $N$ bosons are either transmitted or reflected, after an interaction. Such beam-splitters are realisable (at least mesoscopically) with a nonlinear Josephson interaction, in which case the transmission coefficient is determined by the time of evolution. By configuring the beam splitters according to the Ou-Mandel Bell experiment, we demonstrate how entangled cat-states can be created conditionally, enabling a test of Bell’s premises at the macroscopic level, for $N$ large. Here, different from conventional Bell tests, all relevant measurements give outcomes that are macroscopically distinguishable, as $N \to \infty$. The traditional Bell-choice between two spin measurements is replaced by a choice between two different times of evolution, at each of two sites. We explain how this leads to new tests involving mesoscopic nonlocality, irreversibility, and time.

Much effort has been devoted to testing quantum mechanics at a macroscopic level. Quantum superpositions of macroscopically distinguishable states, so-called Schrodinger cat states \cite{1}, have been created in a number of different physical scenarios \cite{2} \cite{11}, and recently two macroscopic objects have been demonstrated to be entangled \cite{12}. However, Leggett and Garg pointed out that a very strong test of macroscopic quantum mechanics would give a method to falsify all possible alternative theories satisfying the notion of macroscopic realism \cite{13}.

Motivated by this, Leggett and Garg formulated inequalities, similar to Bell inequalities \cite{14}, which if violated falsify a form of macroscopic realism (called macro-realism) \cite{13} \cite{15}. Leggett and Garg’s macro-realism combines two classical premises: The first premise is macroscopic realism per se (MR): a system which has two macroscopically distinguishable states available to it must be in one or other of those states at all times. Macroscopic realism assumes the existence of a hidden variable to predetermine (up to a macroscopic uncertainty) outcomes of measurements that are macroscopically distinct \cite{13}. The second premise is macroscopic noninvasive measurability: a measurement can in principle distinguish which of these states the system is in, with a negligible effect on the subsequent macroscopic dynamics of the system. There have been violations of Leggett-Garg inequalities reported in the literature \cite{16} \cite{25}, including experimentally for superconducting qubits \cite{23} and single atoms \cite{24}. A complication with the Leggett-Garg tests, which may allow loopholes \cite{25}, is the justification of the second “noninvasive measurability” premise for any practical measurement \cite{16} \cite{17} \cite{19}.

In this paper, we show how macroscopic realism may be tested using a Bell inequality involving time. This represents an advance because here the second Leggett-Garg premise is replaced by the premise of macroscopic locality (ML): a measurement made on a system at one location cannot instantly induce a macroscopic change to a second spatially-separated system. The combined premises of MR and MR constitute the premise of macroscopic local realism (MLR) \cite{29} \cite{31}. In this paper, we explain how it is possible to test the predictions of quantum mechanics against those of MLR. The test utilises the existence of a nonlinear beam splitter, where $N$ bosons can be either reflected or transmitted, for large $N$.

Here, we illustrate how the nonlinear beam splitter can be realised using a nonlinear Josephson interaction in an optimal regime, in which case the transmission coefficient is determined by the time of interaction. Such an interaction can be achieved with Bose-Einstein condensates or superconducting circuits \cite{32} \cite{39}. For this beam splitter, the analyzer angles $\theta, \theta'$ representing two alternative measurement settings in the original Bell inequality \cite{14} are replaced by two settings $t, t'$ in time. By considering nonlinear beam splitters at two sites, entangled cat-states can be created that, when $N$ is very large, violate macroscopic local realism. This is possible, because the outcomes of all relevant measurements are distinct by $N$.

The Bell tests of this paper differ from previous Bell tests for mesoscopic systems \cite{10}, including those for cat states \cite{10} \cite{41}, which almost invariably involve measurement outcomes that are not always macroscopically distinct, or else involve a continuous range of outcomes \cite{29} \cite{31} \cite{42}. These former tests are not in the spirit of Leggett and Garg, who considered only outcomes with macroscopic separations \cite{13} \cite{44}. The results of this paper provide a proof-of-principle that Bell tests can be performed in this regime. To the best of our knowledge, such tests have not been performed for $N > 1$.

It is a challenge to prepare entangled cat-states where the two cat-systems are spatially separated. The spatial separation is necessary to carry out a rigorous Bell test. We suggest how this might be achieved, using the Ou-Mandel version of a Bell experiment \cite{15} \cite{46}, but ex-
tended to \(N\) bosons, where path-entangled cat-states \(|0\rangle\) can be generated conditionally (refer \[47\]). In the framework of the Clauser-Horne inequalities \[48\], this configuration provides a rigorous test of local realism \[49, 50\].

**Nonlinear beam splitter (NBS):** We begin by considering a realisation of a nonlinear beam splitter. At a location \(A\), an incoming field (labelled \(a\)) interacts with a second incoming field (labelled \(a_2\)). The interaction is modelled by the nonlinear Josephson Hamiltonian \[32, 33\]

\[
H_{NL} = \kappa (\hat{a}^\dagger \hat{a}_2 + \hat{a} \hat{a}_2^\dagger) + g \hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger + g \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_2^\dagger
\]

Equation (1)

Here, \(\hat{a}\) and \(\hat{a}_2\) are the boson operators for the corresponding fields \(a\) and \(a_2\), modelled as single modes. This Hamiltonian can be realised for superconducting or atomic media \[33–39\]. For example, the Hamiltonian has been used to model Bose-Einstein condensates trapped in two potential wells, where the potential is sufficiently deep that a single-mode approximation is valid for each well \[33–39\]. In that case, \(\hat{a}\) and \(\hat{a}_2\) are operators for the creation of a particle in each mode, \(\kappa\) is the Josephson coupling between wells, and \(g\) the self-interaction term due to the nonlinearity of the medium in each well.

For certain choices of \(g\) and \(\kappa\), the interaction \[1\] acts as a macroscopic beam splitter, where the input \(|N\rangle_a|0\rangle_{a_2}\) after a time \(t\) gives, to a good approximation, the output \(|N\rangle_a|\pm\rangle_{a_2}\) \[27, 28, 31, 39\]

\[
|\psi(t_a)\rangle = e^{i\varphi(t_a)}(\cos t_a|N\rangle_a|0\rangle_{a_2} - i \sin t_a|0\rangle_a|N\rangle_{a_2})
\]

Equation (2)

We introduce a scaled time \(t_a = \omega_N t\) where \(\omega_N = 2g\sqrt{\frac{N}{\kappa}}\) \[34\]. Here \(|\psi(t_a)\rangle = U_A|N\rangle_a|0\rangle_{a_2}\), where \(U_A\) is the unitary time evolution governed by \(H_{NL}\), and \(|n\rangle_{a}\) is the number state for the mode denoted \(I\). The phase shift \(\varphi(t_a)\) depends on \(\kappa\) and \(g\). Figure 1 illustrates parameter regimes for the nonlinear beam-splitter oscillator given by \[2\].

**Bell inequality for macroscopic local realism:** For spatially separated sites \(A\) and \(B\), we consider the four two-qubit Bell states

\[
|\psi_{+,+}\rangle_{AB} = (|1\rangle_A |\pm\rangle_B + |\mp\rangle_A |\pm\rangle_B)/\sqrt{2}
\]

\[
|\psi_{-,-}\rangle_{AB} = (|1\rangle_A |\pm\rangle_B - |\mp\rangle_A |\pm\rangle_B)/\sqrt{2}
\]

Equations (3)

where \(|1\rangle_A = |N\rangle_a|0\rangle_{a_2}, \ |\pm\rangle_B = |N\rangle_b|0\rangle_{b_2}, \ |\mp\rangle_B = |0\rangle_a|N\rangle_{a_2}, |1\rangle_A = |0\rangle_a|N\rangle_{a_2}, \ |\mp\rangle_B = |0\rangle_b|N\rangle_{b_2}. \) Suppose that at site, \(A\), the nonlinear beam splitter (NBS) mixes the modes \(a\) and \(a_2\) according to (2), and that a similar interaction takes place at the second site \(B\), for modes \(b\) and \(b_2\). The unitary transformation for the NBS at \(B\) is thus \(|\psi_{B}(t_b)\rangle = U_B|N\rangle_b|0\rangle_{b_2} = e^{i\varphi(t_b)}(\cos t_b|N\rangle_b|0\rangle_{b_2} - i \sin t_b|0\rangle_b|N\rangle_{b_2}).\)

Taking the incoming state to be the Bell state \(|\psi_{+,+}\rangle_{AB}\), assuming the optimal parameters for the nonlinear beam splitters, the final state is

\[
U_A U_B |\psi_{AB}\rangle = e^{i\varphi(\cos t_\pm|\psi_{+,\pm}\rangle - i \sin t_\pm|\psi_{-,\pm}\rangle)}
\]

Equation (4)

where \(t_\pm = t_a \pm t_b\) and \(\varphi\) is a phase factor. Defining the “spin” at \(A\) (\(B\)) as \(\pm 1\) if the system is detected as \(|\pm 1\rangle_A\) (\(|\pm 1\rangle_B\)), the expectation value for the spin product is \(E(t_a, t_b) = \cos 2(t_\pm), \) which is the standard form known to violate the Clauser-Horne-Shimony-Holt Bell inequality for suitable choices of \(t_a\) and \(t_b\) \[48\]. For \(N\) large, it is clear the qubits \(|\pm 1\rangle_{A/B}\) correspond to macroscopically distinct outcomes for all choices of \(t_a\) and \(t_b\), and the violation of the Bell inequality will falsify MLR.

To demonstrate a specific method of preparation, we consider that the two separated modes \(a\) and \(b\) are pre-
pared in the NOON state \( \ket{\psi}_{ab} = \frac{1}{\sqrt{2}} (|N\rangle_a |0\rangle_b + e^{i\theta} |0\rangle_a |N\rangle_b) \) (Figure 2, top)

\[
|\psi\rangle_{ab} = \frac{1}{\sqrt{2}} (|N\rangle_a |0\rangle_b + e^{i\theta} |0\rangle_a |N\rangle_b)
\]

(5)

and that the modes \( a_2 \) and \( b_2 \) are prepared similarly, as \( |\psi\rangle_{a_2b_2} = U_{A}U_{B}|\psi\rangle_{ab}|\psi\rangle_{a_2b_2} \). We might alternatively prepare the modes \( a_2 \) and \( b_2 \) in the NOON state

\[
|\psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle_{a_2} |N\rangle_{b_2} + e^{i\theta} |N\rangle_{a_2} |0\rangle_{b_2})
\]

(6)

and choose \( \theta = -\pi/2 \). Such NOON states may be prepared using the nonlinear beamsplitter with a 50/50 mixing and the number state inputs \( |N\rangle_a |0\rangle_{a_2} |0\rangle_b |N\rangle_{b_2} \), as depicted in Figure 2 (lower), in which case we obtain a final state \( |\psi_{f-}\rangle \). We find \( (\phi_{\pm} \text{ is a phase factor}) \)

\[
|\psi_{f\pm}\rangle = \frac{\cos t_{\pm} (|\psi_{-\pm}\rangle + i \sin t_{\pm} |\psi_{+\pm}\rangle)}{\sqrt{2}} + \frac{1}{\sqrt{2}} (\varphi_{\pm} 2N)
\]

(7)

\( |\varphi_{\pm} 2N \rangle \) are states with all \( 2N \) bosons at site \( A \) or all \( 2N \) bosons at site \( B \). For \( N = 1 \), the configuration of Figure 2 (lower) is equivalent to that of Ou and Mandel \[45\, 46\].

To test local realism, the mode numbers at the final outputs \( a, a_2, b \) and \( b_2 \) are measured. At \( A \), we denote by + the outcome of detecting \( N \) bosons at location \( a \) and 0 bosons at \( a_2 \). A similar outcome + is defined for site \( B \). We thus define the joint probability \( P_{++} \) for the outcome + at both sites \( A \) and \( B \). We also specify \( P_A^0 \) as the marginal probability for the outcome + at site \( A \) and \( P_B^0 \) as the marginal probability for the outcome + at site \( B \). At each site \( A \) and \( B \), observers independently select a time of evolution \( t_a \) and \( t_b \) for the NBS interaction. This choice is made after the \( 2N \) bosons have separated into the regions \( A \) or \( B \).

We now suppose that the system is described by a local hidden variable theory. Locality is assumed between the sites \( A \) and \( B \), for which the measurement events can be spacelike separated if the distance between them is sufficiently great, taking into account the times \( t_a \) and \( t_b \) required for the NBS interactions. Such local hidden variable theories include all theories that allow for a predetermined determination of the outcomes for mode number at each site. Following Bell, Clauser and Horne (CH), the well-known CH-Bell inequality \( S \leq 1 \) \[48\] is predicted to hold for any such theory, where

\[
S = \frac{P_{++}(t_a, t_b) - P_{++}(t_a, t_b') + P_{++}(t_a', t_b) + P_{++}(t_a', t_b')}{P_A^0(t_a') + P_B^0(t_b')}
\]

(8)

Here we note there are two choices of interaction times at each location: \( t_a, t_a' \) at \( A \), and \( t_b, t_b' \) at \( B \). According to \[48\], the state \( |\psi_{f+}\rangle \) gives \( P_{++} = \frac{1}{2} \sin^2 t_a + t_b \) and

\[
\begin{align*}
S &= \frac{P_{++}(t_a, t_b) - P_{++}(t_a, t_b') + P_{++}(t_a', t_b) + P_{++}(t_a', t_b')}{P_A^0(t_a') + P_B^0(t_b')}
\end{align*}
\]

(8)

Figure 3. Top: Probability distribution \( P_N \) for the joint detection of \( N_A \) and \( N_B \) bosons at the sites \( A \) and \( B \) (refer Figure 2). Here \( t_a = 0, t_b = \varphi, t_a' = 2\varphi, t_b' = 3\varphi \) where \( \varphi = 1.93 \). \( N = 7, \kappa = 18.23, g = 47.85 \). The distribution is unchanged for settings \( (t_a', t_b), (t_a', t_b') \) and \( (t_a, t_b') \). Lower: The joint probability \( P(n, m) \) of detecting \( n \) bosons in mode \( a \) and \( m \) bosons in mode \( b \) (refer Figure 2b) and \( N \) bosons in total at site \( A \). The figures for the settings \( (t_a', t_b), (t_a', t_b') \) are unchanged from those of \( (t_a, t_b) \).

\[
P_A^0 = P_B^0 = \frac{1}{4}
\]

(9)

It is easy to verify that \( S \) maximizes at \( S = 1.207 \) for \( \varphi = \pi/16 \) (1.96), giving a violation of the CH-Bell inequality. For state \( |\psi_{f+}\rangle \), we introduce parameters \( \theta \) and \( \phi \), such that \( t_a = \theta, t_b = 2\pi - \phi \) where \( 0 < \phi < 2\pi \). Choosing angles \( \theta = \theta_0, \phi = \theta_0 + \varphi, \theta' = \theta_0 + 2\varphi \) and \( \phi' = \theta_0 + 3\varphi \), for which \( |\theta - \phi| = |\theta' - \phi| = |\theta' - \phi'| = \frac{1}{2}|\theta - \phi'| = \varphi \), the prediction for \( S \) becomes that of \( \ket{\psi_{f-}} \), given by \[48\]. Thus, the quantum predictions for both states \( |\psi_{\pm}\rangle \) violate the CH-Bell inequality (in fact by the maximum amount \[52\]).

The violation of the CH-Bell inequality defined by \[8\], in the context of the states \( |\psi_{\pm}\rangle \), is also a falsification of macroscopic local realism (MLR). This is evident from the expressions \( \ket{\psi_{\pm}} \), since it can be seen that the states of the superposition have mode numbers in each mode that are macroscopically distinguishable, as \( N \rightarrow \infty \). Specifically, the outcomes of a measurement of mode number at each detector are always one of \( 0, N \) or \( 2N \). The assumption of MLR thus implies the validity of a local hidden
variable theory, where the system at each site is predetermined to be in one of the states with mode number 0, N, and 2N. The Bell inequality $S \leq 1$ of eq. (5) will hold for such a theory, and its violation for large $N$ will therefore falsify MLR.

The nonlinear beam splitter is realised as the physical device modelled by the Hamiltonian $H_{NL}$, eqn (1), in the ideal regime (Figure 1). In practice, the ideal regime giving the precise solution (2) is unattainable, for $N > 1$, since probabilities for other than 0 or N bosons in each mode are not precisely zero. In Figures 3 and 4, we present the actual predictions for $S$, using the Hamiltonian $H_{NL}$. For large $gN/\kappa$, where care is taken to optimize for the NBS regime given by (2), the Bell violations are maintained, as shown in Figure 4. To test MLR, it is crucial to also establish that the outcomes of mode number are macroscopically distinct, for each of the joint probabilities $P(t_a, t_b)$ comprising $S$. One can test local realism at an $N$-scopic level (N-scopic local realism), by considering outcomes distinct by $N$. Figure 3 shows the distribution for outcomes, highlighting the correctness of this assumption in the optimal parameter space. The very small probabilities for mode numbers other than 0, N or 2N can be shown irrelevant to the test of $N$-scopic local realism, using the method implemented in Refs. 13 34.

Discussion: In this paper, we argue that violation of the Bell inequality falsifies MLR, because the outcomes measured at location $A$ ($B$) are macroscopically distinct. Small nonlocal effects cannot change the classification of the outcome at $A$ ($B$). It is concluded that the violation is due to a failure of macroscopic locality, or else of macroscopic realism (or both).

It is interesting to consider counter-arguments. A critic might claim that the violation of the Bell inequality is not due to a macroscopic nonlocal effect. The critic might argue that events at $B$ ($A$) cause a microscopic (of order $N = 1$) nonlocal effect to the system at $A$ ($B$), but that through the subsequent local evolution due to $H_{NL}$ at $A$ ($B$), the microscopic effect translates into a final macroscopic effect (of order $N$) registered by the detectors. This interpretation could be explored. The predictions for the Bell-inequality violation depend on the difference $t_a - t_b$ of the evolution times at each site, and hence the actual times used at locations $A$ and $B$ can be shifted against a shared clock. The distance between sites $A$ and $B$ needs to remain large enough to justify no causal effects between $A$ and $B$. The observer at each site makes the decision to cease evolution of the NBS at their location at some time, but the evolution is fully reversible, or can be later continued, up until the time of the final number detection, which is irreversible. The times can be adjusted so that the final irreversible detections at each site $A$ and $B$ are made simultaneously. Then the critic would have to argue that the nonlocal effect due to a detection at $B$ ($A$) acts back in time, if it is to be then locally amplified at $A$($B$). By increasing the time delay between turning off the NBS evolution and the final irreversible detection at each site, the test could be made stricter. Alternatively, the critic would have to argue that the reversible actions at $B$ ($A$) induced small nonlocal effects to $A$ ($B$) (which then leads to a larger change at $A$ ($B$) prior to detection).

In conclusion, while we have focused on NOON states, similar results apply to other cat-states. A practical problem is preparing entangled cat-states where $A$ and $B$ are spatially separated. This is addressed if the initial NOON state has separated modes (for $N = 2$, the Hong-Ou-Mandel effect might be useful 52), or if the outputs of the nonlinear beam splitters can be separated. For atomic modes with different hyperfine levels, as possible for a two-well BEC, the separation might be achieved by magnetic fields 55. A second difficulty is realisation of the evolution (2) for large $N$. For cold Rb atoms, timescales become inaccessibly long 34 35. Solitons may provide an alternative 56. The nonlinear beam splitter is likely to be achievable using superconducting circuits with high nonlinearities 9 30. A proof-of-principle test, even if without spatial separation, for moderate $N \geq 2$, is of interest.

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