Variable Modified Newtonian Mechanics I:
Single metric Universe

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Abstract

A few years ago Baker [1] proposed a metric which interpolates smoothly between the Schwarzschild metric at small scales and the Friedmann-Lemaître-Robertson-Walker (FLRW) metric at large scales which was rejected as incompatible with Newtonian solar system data. Based on the adiabatic approximation, we find a unique solution within the variants of the same construction scheme which avoids this problem. In this metric, a MOND-like non-Newtonian acceleration arises, which we have termed VMOND, where the MOND acceleration $a_0$ is replaced by a cosmological acceleration which depends on the background matter density. We give two circumstances in which VMOND mimics MOND without the need for additional dark matter.

1) We show that a reasonable overdensity at recombination, normally considered too small without additional mass, is enhanced by the effects of VMOND, sufficiently to permit turnaround and collapse to a tight central region before $z \sim 14$.

2) We consider a Milky Way mass overdensity at turnaround radius which picks up a systematic maximum angular momentum per unit mass. When the overdensity radius contracts to galactic disk size, we find that the rotating speed follows the Tully Fisher relation with an acceleration scale that is again close to the MOND acceleration $a_0$.

This is not a universal panacea to the problems encouraging dark matter but, we suggest, presents a new baseline for calculation. One of us (CCW) has already taken these two points of the model further. Here we are concentrating on the basics.

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1 Introduction

Serious mass discrepancy problems exist in cosmology and astrophysics which typically involve central gravitating masses in an expanding background. At galactic scales the anticipated Newtonian behaviour for rotation curves based on visible matter fails [2]-[5]. A popular resolution to this discrepancy preserving Newtonian laws, the presence of invisible Cold Dark Matter (CDM), has not been detected experimentally. It is also beset by a list of problems involving core-cusps [4]-[5], missing satellites [6]-[7], satellite-disks [8]-[9], a possible angular momentum catastrophe [10], so-called too-big-to-fail issues [11], dwarf galaxies without dark matter [12]-[13] and too-dense-to-be-satellite systems [14].

Another approach, of which this paper is part, is to construct variants of Newtonian gravity that change long-range behaviour to obviate the need for additional particles. The most familiar example of this approach is Milgrom’s Modified Newtonian Dynamics (MOND) in which [15] the Newtonian gravitational acceleration is modified when it takes values below some phenomenological scale $a_0$. In its original formulation MOND is a non-relativistic phenomenological model which is successful at galactic distances but becomes problematic at larger distances and in high acceleration epochs. The need for a relativistic framework has generated several MOND variants; EMOND [16], GMOND [17], Emergent gravity [18], MOG [19] and Relativistic MOND [20]-[21], all involving a non-Newtonian acceleration obtained by modifying General Relativity (GR). Even then, recent observations [22] shows that MOND as an universal physical law fails at Wide-Binaries scales.

In this paper we argue for a different starting point. Asymptotically, at short distances GR provides the Schwarzschild metric which passes all tests to date and the Friedmann-Lemaître-Robertson-Walker (FLRW) metric from GR seems to provide a good description at distances larger than 300Mpc. Our approach is very simple and consists of constructing a new metric which interpolates between these two regimes. As a result, there will be non-Newtonian gravitational effects at intermediate scales. Although we anticipate these effects to be small, in general, we will argue here, and have argued elsewhere, that there are circumstances in which they are important and observable [23]-[25]. In particular, they provide a stepping stone for the observed non-Newtonian galaxy rotation curves. They do not provide a complete answer but they provide a baseline metric for further study.

Finding ways to cross over from a Schwarzschild metric to the FLRW metric
has a long history. A well known solution for zero cosmological constant is the Einstein-Straus vacuole model [26] where the Schwarzschild solution can be matched smoothly to a spherical surface of the expanding background. For non-zero cosmological constant, an Einstein-Straus-de-Sitter solution has been constructed for a dust-like cosmological fluid [27]-[28]. This general approach has its problems [29]. McVittie [30]-[31] avoids these by allowing the mass of the Schwarzschild model to have scale factor dependence and match asymptotically to the FLRW background at large distances. However, for non-zero background pressure the McVittie metric has a pressure singularity at small distances which is at odds with observation.

The motivation for this paper is Baker’s derivation [1] of the Bona-Stella construction [32] in which a spherical patch containing a central mass is inserted in a flat FLRW background. Baker shows how this construction can be derived from a general Lemaître-Tolman metric that is ”interpolating” between the asymptotic solutions. The resulting metric possesses a non-Newtonian acceleration in its equation of motion. Baker rules out his particular choice of metric because of solar system data. In this work, our aim is to find a variant metric within the same family of metrics, which, while adopting Baker’s approach, avoids his specific problems while retaining non-Newtonian acceleration. [On occasion we shall cite from Baker using his equation numbers.]

We shall be brief in discussing the implications of VMOND for cosmology here. Our intention is to show that, although superficially the effects of VMOND are small (e.g. is solar system physics), at large scales and early time, when \( H(z) \gg H_0 \), VMOND can have a major effect on large-scale structure formation. More details can be found in the arXiv articles by one of us (CCW), yet to be submitted [23, 24, 25], to which we shall refer briefly.

2 The Model

We follow Baker’s [1] approach in adopting a Lemaître-Tolman (LT) metric

\[
ds^2 = c^2 d\tau^2 - e^{2\alpha(\varphi,\tau)} d\varphi^2 - e^{2\beta(\varphi,\tau)} d\Omega^2,\]

for a spatially isotropic ”flat” space to describe the motion of test particles in the vicinity of a single point-mass \( M \) placed at the origin in an expanding cosmological background, with coordinate time \( \tau \) and comoving distance \( \varphi \) \((d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2)\). Time-orthogonality requires \( e^\alpha = \beta' e^\beta \) where \( ' \) denotes differentiation with respect to \( \varphi \). That is, [1] takes the form

\[
ds^2 = c^2 d\tau^2 - \beta'^2 e^{2\beta(\varphi,\tau)} d\varphi^2 - e^{2\beta(\varphi,\tau)} d\Omega^2,\]

(2)
a function of $\beta$ only. Different solutions for $\beta$ represent different metrics. We stress that, rather than postulate a stress-momentum tensor for a particular cosmic environment and solve Einstein’s equations (in the presence of a cosmological constant), we adopt the contrary approach of constraining the metric to match the Schwarzschild and FLRW metrics at spatial extremes. Our approach, as Baker’s, is as follows:

Initially, we assume a single mass $M$ at the origin $r = 0$ in a universe which, at large distance from it, behaves like an FLRW fluid with cosmological constant $\Lambda$. With $e^\beta = r$ we look to interpolate between (in mixed coordinates)

- the Schwarzschild-Lemaître (SL) metric at short distances,

$$ds^2 = c^2 d\tau^2 - \frac{2GM}{c^2 r} dg^2 - r^2 d\Omega^2,$$

for which

$$e^\beta = r = [(3/2)\sqrt{2GM/c^2(\varrho - c\tau)}]^{2/3}. \tag{4}$$

- The Friedmann-Lemaître (FL) metric for scale factor $a(\tau)$ at large distances,

$$ds^2 = c^2 d\tau^2 - a^2 d\varphi^2 - r^2 d\Omega^2,$$

for which

$$e^\beta = r = a(\tau) \varrho. \tag{6}$$

Here index 1 denotes the radial coordinate and indices 2, 3 angular coordinates. To make comparison with [1] we note that Baker adopts the awkward convention that index 4 denotes temporal coordinates, whilst we adopt the convention that index 0 denotes temporal coordinate. Overdot denotes differentiation with respect to $\tau$.

Initially we adopt what is essentially Baker’s approach in making the simplest extension of the Schwarzschild- Lemaître metric commensurate with the Friedmann- Lemaître metric

$$r = e^{\beta(\varrho, \tau)} = a(\tau) L(\varrho, \tau) = a(\tau) [(3/2)\sqrt{2GM/c^2(\varrho - cT(\tau))}]^{2/3}, T(\tau) = \int_{\tau_0}^{\tau} \frac{dy}{a(y)^{3/2}}. \tag{7}$$

where $\eta$ is to be determined. The prefactor $a(\tau)$ is chosen to be consistent with the Friedmann-Lemaître metric at larger distance $r$. It follows directly that, with $H = \dot{a}/a$,

$$\dot{\tau} = \frac{1}{a^n(\tau)}; \quad \ddot{\tau} = -\eta\dot{T}H \tag{8}$$
For $a(\tau) = 1$ we recover the Schwarzschild-Lemaître metric.

3 Geodesic Equations: Choosing $\eta$

To choose $\eta$ compatible with observation we need both the geodesic and Einstein’s equations, which do not always sit comfortably with one another.

Beginning with the geodesic equations, the low-velocity acceleration equation for radial motion is given as

$$\frac{\ddot{r}}{r} = \ddot{\beta} + \dot{\beta}^2 = \frac{\ddot{a}}{a} - \frac{2\dot{T}^2 c^2}{9(\varrho - cT)^2} - \frac{2(2 - \eta)\dot{T}c}{3(\varrho - cT)} \left(\frac{\dot{a}}{a}\right).$$  \hspace{1cm} (9)

From (7)

$$\frac{2\dot{T}^2 c^2}{9(\varrho - cT)^2} = \frac{2c^2}{9a^{2n}(\varrho - cT)^2} = \frac{1}{a^{2n-3}} \frac{GM}{r^3}. \hspace{1cm} (10)$$

and

$$\frac{2(2 - \eta)\dot{T}c}{3(\varrho - cT)} \left(\frac{\dot{a}}{a}\right) = H(2 - \eta) \sqrt{\frac{2GM}{r^3 a^{2n-3}}}. \hspace{1cm} (11)$$

whence acceleration takes the general form

$$\frac{\ddot{r}}{r} = \frac{\ddot{a}}{a} - \frac{1}{a^{2n-3}} \frac{GM}{r^3} - H(2 - \eta) \sqrt{\frac{2GM}{r^3 a^{2n-3}}}. \hspace{1cm} (12)$$

This modifies Newtonian dynamics in two ways:

1. Modifying the Newtonian potential to

$$V_\eta(r) = \frac{1}{a^{2n-3}} \frac{GM}{r}. \hspace{1cm} (13)$$

2. Including an additional potential due to the expansion of the universe

$$\Delta V_\eta = -\frac{H(2 - \eta)}{a^{\eta - 3/2}} \sqrt{2GMr}, \quad \nabla(\Delta V_\eta) = -\frac{H(2 - \eta)}{a^{\eta - 3/2}} \sqrt{\frac{GM}{2r}} \hspace{1cm} (14)$$

and $\nabla$ denotes radial derivative. We see that, if

i) $\eta > 2$, we have repulsive non-Newtonian gravity from $\Delta V_\eta$, plus a $a(\tau)^{-2\eta+3}$ factor multiplied to the Newtonian gravity term.
ii) \( \eta < 2 \), we have attractive non-Newtonian gravity \( \Delta V_\eta \), plus a \( a(\tau)^{-2\eta+3} \) factor multiplied to the Newtonian gravity term.

iii) \( \eta = 3/2 \), we have attractive non-Newtonian gravity \( \Delta V_{3/2} \), plus exact Newtonian gravity.

iv) \( \eta = 2 \), we have a \( a(\tau)^{-1} \) factor multiplied to Newtonian gravity.

Baker’s insistence on isotropy (as an identity in Einstein’s equations) forces \( \eta = 3 \). This is turn leads to a modified Newtonian force \( 12 \)

\[
V_3(r) = \frac{GM}{a^3 r}, \quad \nabla V_3(r) = -\frac{GM}{a^3 r^2}.
\]  (15)

Phenomenologically, we know that Newtonian gravity works to a high precision \( O(10^{-9}) \) in the solar system. The extra powers of \( a(\tau) \) in (15) are inconsistent with the data on solar orbits and this choice was abandoned by Baker in favour of different approaches that preserved isotropy.

If we needed any further excuse not to take \( \eta = 3 \), we saw above that for \( \eta > 2 \) the additional non-Newtonian gravity is repulsive. This is contrary to the observation which motivated this analysis that, at above-galactic scales in the matter dominant epoch, we have observations of large non-Newtonian attractive acceleration (the origin of the dark matter paradigm). If we do not want repulsive gravity, we need to take \( \eta \leq 2 \). However, the problem of anisotropy will need to be addressed. The fact that isotropy is not guaranteed does not mean that it cannot be implemented, either exactly, or approximately so.

Perhaps the most important argument for the choice \( \eta = 3/2 \) is the following. Irrespective of solar system details, in gravitational systems evolving from an over-density since recombination, the scale factor \( a(\tau) \) has increased by a factor of 1000 from that at recombination, leading to significant changes in the quasi-Newtonian factor \( a^{-2\eta+3} \). Around recombination, the Dark matter effect is taken to be 4-6 times of the baryonic gravity. Therefore the \( a^{-2\eta+5} \) factor in front of the Newtonian acceleration needs to be very close to unity across the large expansion of the scale factor. This leaves us with a preferred choice of \( \eta = 3/2 \) where the Newtonian gravity term is exact, which we adopt henceforth in our analysis.

With the \( 1/r^2 \) Newtonian acceleration restored, the non-Newtonian additional potential \( \Delta V_{3/2} \), which we term VMOND, leads to an acceleration of
the form
\[ \nabla (\Delta V_{3/2}) = -\frac{H}{2} \sqrt{\frac{2GM}{r}} \] (16)

whence
\[ \dot{\beta} = H - \frac{2\dot{T}c}{3(\varrho - cT)} = H - \sqrt{\frac{2GM}{r^3}} \] (17)

Further,
\[ \beta' = \frac{2}{3} \frac{1}{(\varrho - cT)} = \sqrt{\frac{2GMa^3}{c^2 r^3}} \; ; \; \epsilon = \beta r = \sqrt{\frac{2GMa^3}{c^2 r}}. \] (18)

From Baker (Eq. 28) we then have
\[ \dot{r} = \left( \dot{\beta} + \beta' \dot{\varrho} \right)r = \left( H - \sqrt{\frac{2GM}{r^3}} + \frac{2\dot{\varrho}}{3(\varrho - cT)} \right)r. \] (19)

We note that, in particle free fall in the Schwarzschild-Lemaître metric, \( \varrho \) is taken as a constant and \( r \) reduces as the coordinate time \( T = \tau \) increases.

In the FLRW metric, \( \varrho \) is also taken to be constant in a comoving frame. In what follows we take \( \dot{\varrho} = 0 \), \( (\varrho = constant) \) and
\[ \frac{\dot{r}}{r} = \dot{\beta} = H - \sqrt{\frac{2GM}{r^3}}. \] (20)

This equation describes a free (zero energy) particle at a distance \( r \) from a central mass \( M \), following both the Hubble expansion at large distance and Newtonian gravity at small distances. In particular, it states that the free-fall speed of a particle in the presence of a central point mass and an expanding background is described by adding the (negative) particle free fall speed in the S-L metric and the (positive) particle free fall speed in an expanding FLRW universe. This is contrary to the normal procedure of adding (negative) particle free fall acceleration under point mass gravity to the (positive) point mass acceleration in an expanding FLRW universe. This is a highly non-trivial result. The condition \( \dot{\beta} = 0 \) defines the turnaround distance \( r_{TA} \),
\[ r_{TA}^3 = \frac{2GM}{H^2}, \] (21)
for which \( \dot{r}_{TA} = 0 \), where the two effects balance.
The gravitational potential (the second term of which we call the VMOND potential) now takes the form (in which we drop the suffix of 3/2)

$$V = \frac{GM}{r} - H\sqrt{2GMr} + \frac{1}{2}H^2 r^2 = \frac{1}{2}r^2 \dot{\beta}^2. \quad (22)$$

The acceleration equation (the second term of which we henceforth call the VMOND acceleration) is given by

$$\ddot{r} = -\frac{GM}{r^2} - H\sqrt{\frac{GM}{2r}} + \ddot{a}, \quad \frac{\ddot{a}}{a} = -\frac{1}{2}H^2 + \frac{c^2\Lambda}{3}, \quad (23)$$

where $H_m$ is the Hubble parameter in a matter only universe. Rewriting Eq. (23) in term of mass density gives

$$\ddot{r} = -\frac{4\pi G}{3} \rho_M(r) - \frac{4\pi G}{3} \sqrt{\rho_M(r)} \rho_H + \ddot{a}, \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \rho_m + \frac{c^2\Lambda}{3}, \quad (24)$$

where $\frac{4\pi}{3} \rho_M(r)r^3 = M$ and $H^2 = \frac{8\pi G}{3}\rho_H$.

For $\eta = 3/2$ the Lemaître-Tolman metric takes the form

$$ds^2 = c^2 d\tau^2 - \frac{2GMa^3}{c^2 r} dg^2 - r^2 d\Omega^2. \quad (25)$$

In terms of conformal time $\tau_c$ where $d\tau = ad\tau_c$, we have

$$ds^2 = a^2 \left[ c^2 d\tau_c^2 - \frac{2GM}{c^2 L} dg^2 - L^2 d\Omega^2 \right]. \quad (26)$$

The metric inside the bracket can be be rewritten in curvature coordinates $(t_c, L)$ with $\phi = -\frac{GM}{c^2 L}$, the Newtonian potential, as

$$ds^2 = a^2 \left[ (1 + 2\phi)c^2 dt_c^2 - \frac{dL^2}{(1 + 2\phi)} - L^2 d\Omega^2 \right]. \quad (27)$$

Note that, for early time scalar perturbation when $\phi \ll 1$, this is identical to the perturbed FLRW metric in conformal Newtonian gauge, see Mukhanov

$$ds^2 = a^2 \left[ (1 + 2\phi)c^2 dt_c^2 - (1 - 2\phi)(dx^2 + dy^2 + dz^2) \right]. \quad (28)$$

However, there is no requirement for $\phi \ll 1$ in our model. The metric (27) is applicable everywhere between the central mass to Hubble radius. In practice, we are required to use Eq. (20), Eq. (22) and Eq. (23) to describe the dynamics of a point particle around the central point mass in an expanding background.
3.1 Discussion:

The equations above are not difficult to derive but how we interpret them is a matter of debate. *A priori*, we are given a stress momentum tensor with a point mass and an expanding background with uniform matter density in which the metric around the point mass is specified by free falling speed $\dot{r} = -\sqrt{\frac{2GM}{r}}$, and the uniform density expanding background metric is specified by $\dot{r} = Hr$.

Either there is NO interpolation between these two asymptotic metrics, in that case, we are left with the choice of a bounded region around the central point mass with $\dot{r} = -\sqrt{\frac{2GM}{r}}$ (the SL metric), with an outside region that is described by the $\dot{r} = Hr$ (the FL metric). This is the Einstein-Straus vacuole solution. In this solution, a free falling point mass around $M$ can only experience the presence of the cosmological constant via the Schwarzschild de-Sitter metric, even at the high matter background density epoch after recombination, where the effect of high matter density is to shrink the vacuole size. However, the latest Jade GS z-14 observations from the JWST suggest that we need a much stronger gravity immediately after the recombination era to produce early large galaxies and to produce Oxygen in star which is not first generation [33].

Our approach assumes that there exists a physically and mathematically consistent way that $\dot{r}$ can morph between the two asymptotic solutions. By working with a single metric between the central mass and the Hubble radius and avoid the 2-metric vacuole model. We consider Baker’s approach as an attempt to find a mathematically consistent parametrisation (interpolation) between these known asymptotic solutions, despite its initial failure. Our analysis above indicates that such a parametrisation exists if we include both of the opposing free-falling speeds to the same point particle. This is a physically reasonable condition which we can postulate on physical grounds without going through Bakers’ formulation.

Thus, if the central mass is considered as a matter perturbation in an expanding background then, instead of keeping the free fall speed due to the matter perturbation inside a vacuole unaffected by the background expansion, we add the free-fall velocities due to the expanding background and the source of perturbation. This is the basis of the Linear and Newtonian perturbation. For non relativistic matter and for scales well inside the Hubble radius, Newtonian perturbation theory, which is more intuitive, produces similar overdensity evolution results to linear perturbation theory (see Mukhanov
In Newtonian perturbation theory one also uses $r = aq$ with

$$\rho(r, \tau) = \bar{\rho}(\tau) + \delta \rho(r, \tau), \quad u = \dot{r} = \dot{H}r + a\dot{q}, \quad \phi(r, \tau) = \phi_0 + \delta \phi(r, \tau),$$

(29)

where $\bar{\rho}(\tau)$, $\phi_0$ are the mean density and potential of the homogeneous background fluid and $\delta \rho(r, \tau)$, $\delta \phi(r, \tau)$, $a\dot{q} = \delta v(r, \tau)$ are the perturbed matter density, potential and velocity. $\dot{r}$ has a Hubble flow component and a peculiar velocity component and also a continuous prescription between a localised central matter density and the Hubble radius. The considerations are almost identical to our model. In Newtonian perturbation theory, the acceleration is taken implicitly to be

$$\dot{u} = \ddot{r} = \ddot{a}r - \frac{GM(r)}{r^2},$$

(30)

so that the perturbed potential in the Poisson equation reflects only the perturbed matter density

$$\nabla^2 \delta \phi(r, \tau) = 4\pi G \delta \rho(r, \tau).$$

(31)

One can regard our model as a variant of the linear perturbation theory in the Newtonian speed regime. In that context, our $\dot{r}$ interpolation defined in Eq.(20) is simply specifying $\delta v = a\dot{L}$, so that

$$\frac{L}{\dot{L}} = -\sqrt{\frac{2GM}{r}}.$$

Here the consequence of our specific choice Eq.(20) is the introduction of a non-Newtonian potential in Eq.(22) (or non-Newtonian acceleration in Eq.(23)), which admits a pure gravitational effect description. A crude way to interpret this result is to say "the Newtonian acceleration $g_N$ is modified to $\sqrt{g_N(\frac{1}{2}H^2 r)}$ when $g_N \ll \frac{1}{2}H^2 r$ ". This reproduces the MOND postulate if we can identify $\frac{1}{2}H^2 r = a_0$ with the canonical MOND acceleration $a_0$. Clearly, this physical condition will provide a significant effect for large size objects at high redshift where the Hubble constant is large. It could provide the strong gravity at high redshift that the JWST galactic results are alluding to.

This is reverse engineering. Having found an interpolating metric, unique in the Baker-induced single parameter family characterised by $\eta$, we then identify the mass-energy components. The non-Newtonian acceleration can be written in terms of a density $\rho_{MH} = \sqrt{\rho_M(r)\rho_H}$, which should appear in the context of the Poisson equation. However, the original stress momentum tensor only possesses the energy density of a point mass $M$ (vacuum) and
the background density $\rho_H$. The non-Newtonian density $\rho_{MH}$ does not come from the underlying stress momentum tensor but is a result of the dynamical condition Eq. (20).

We can interpret $\rho_{MH}$ as a pure gravitational effect as discussed above, which has no physical material content. The corresponding pressure will be zero, which means we assign its equation of state to be zero. In a stress momentum only approach, we will have to postulate a collisionless (pressureless) matter density $\rho_{MH}$ in Eq. (24) with its corresponding potential $\Delta V$.

The astrophysical and cosmological mass deficit problems invite modelling that provides a significant invisible gravity. For example, the $\Lambda CDM$ postulates a given amount of pre-existing collisionless dark matter and the phenomenological MOND program postulates a modification of acceleration to $\sqrt{\gamma N a_0}$ at low Newtonian acceleration, which essentially assumes the Tully-Fisher law at the outset. Here we argue that our model, on grounds of mathematical and physical reasons, could be an alternative baseline for calculations.

4 Einstein’s equations:

4.1 $\Lambda = 0$

We follow Baker’s approach [1] to Einstein’s equations. However, our metric in Eq. (27) is more easily understood as a variant of the perturbed metric in Eq. (28). We bear in mind that in comoving coordinates an overdensity mass shell can evolve adiabatically.

To understand the role of $\eta = 3/2$ better, we first consider Einstein’s equations without cosmological constant,

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \quad \kappa = \frac{8\pi G}{c^4},$$

(32)

where $T_{\mu\nu}$ is the stress-energy tensor of the perfect fluid used to mimic the matter content of the universe. We generalise our point mass $M$ at the origin to the spherically symmetric (inertial) gravitational mass $M(r)$ inside a sphere of radius $r$ where $M(r)$ satisfies

$$2GM(r) = r^3 \frac{\dot{\beta}^2}{\beta^2}.$$
Following Krasiński and Baker, we work with the rest frame of the observer obtained from the Einstein equation
\[ G_{\mu}^{\nu} = \kappa T_{\mu}^{\nu}. \]  
(34)

The Einstein equation for \( T_{0}^{0} \) is
\[ 8\pi G \rho = \frac{8\pi G T_{0}^{0}}{c^2} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( 2GM(r) \right). \]  
(35)

The Einstein equation for \( T_{1}^{1} \) (Baker Eq.(14)) is
\[ -8\pi G P = \frac{8\pi G T_{1}^{1}}{c^2} = \frac{1}{(r^2 \dot{r})} \frac{\partial}{\partial \tau} \left( 2GM(r) \right). \]  
(36)

\[ 8\pi GT_{2}^{2} = 8\pi GT_{3}^{3} = 8\pi GT_{1}^{1} + \frac{8\pi G}{2\beta} \frac{\partial T_{1}^{1}}{\partial \theta}. \]  
(37)

where \( \rho \) is the physical matter density and \( -P = T_{i}^{i} \) is pressure in the \( i \) direction. Here one does not assume isotropy at the outset but we now show how isotropy is recovered in the SL and FL extremes of (4) and (6).

In the Einstein equations, the stress-momentum tensor elements depend on the time and spatial variation of the enclosed mass \( M(r) \). Explicitly, for a central point mass \( M \) without cosmological expansion, based on the observed free falling (Newtonian) speed \( \dot{r} = -\sqrt{\frac{2GM}{r}} \), such that
\[ \dot{\beta} = -\sqrt{\frac{2GM}{r^3}}, \]  
(38)

we have the Schwarzschild-Lemaître metric. From Eq.(38), the enclosed mass term in \( 2GM(r) = r^3 \left( \frac{2GM}{r^3} \right) \) is a constant; there is no variation with respect to radius scale and time. From Eqns (35)-(36) we have \( T_{0}^{0} = T_{1}^{1} = 0 \) and therefore a vacuum solution outside the point mass.

Equivalently, when analysing the Schwarzschild metric, in terms of the flux of gravitational field coming through the surface of a 3-sphere, one considers a non-vanishing rest density of matter \( \rho_{M} \) with \( T_{0}^{0} = \rho_{M}c^2 \), see e.g. [37]. Here Eq. (35) can be written in terms of the Poisson equation (at fixed \( \theta, \varphi \))
\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \varphi(r) \right) = \nabla^2 \varphi = 4\pi G \rho_{M}. \]  
(39)
The end result is that, for a point mass scenario, in terms of perfect matter fluid outside the point mass, we have the vacuum solution. However, in terms of measuring the presence of a gravitational potential, there is an effective matter density \( \rho_M \) inside the sphere of radius \( r \). By direct calculation, we obtain the evolution equation for the gravitational matter density,
\[
\dot{\rho}_M = -3\rho_M \frac{\dot{r}}{r},
\]
and the mass inside \( r \) remains conserved. This is an example of the presence of a local matter density due to a point mass in an empty background solution of the Einstein equation.

On the other hand, in order to describe a uniform mass density in an expanding universe without cosmological constant (the FLRW metric), we note that both the cosmological background matter density \( \rho_m \) and pressure \( P \) are evaluated in the rest frame in Eq. (34). From Hubble’s Law \( \dot{r} = Hr \), the choice of \( \dot{\beta} \) is
\[
\dot{\beta} = H_m.
\]
The Einstein equation (35) yields
\[
3H_m^2 = 3\dot{\beta}^2 = 8\pi G \rho_m.
\]
The time evolution of \( \rho_m \) becomes
\[
\dot{\rho}_m = -3\left(\rho_m + \frac{P}{c^2}\right)H_m.
\]
In a matter only universe, \( P \ll \rho_m c^2 \), we take \( \rho_m = \rho_0 a^{-3} \) for a fixed \( \rho_0 \). This assumption leads to the gravitational mass \( M(r) \) such that \( 2GM(r) = r^3\dot{\beta}^2 = 2GM = \frac{8\pi G}{3} \rho_0 \), which is constant over time as the radius \( r \) expands. From Eq. (35) we have \( T_1^{\mu\nu} = -P = 0 \). In the rest frame of the Hubble flow such as a galactic cluster, one observes an uniform and isotropic universe that as \( H_m \) reduces, the cosmological matter density everywhere will decrease according to the Einstein equation (42). For a free falling test particle of physical radius \( r = a\bar{q} \) in the FLRW metric, in its comoving frame where the coordinate is given by \( \bar{q} \), the matter density remains constant. Therefore the mass enclosed by a 3-sphere of fixed radius \( \bar{q} \) remains constant and no mass crosses the sphere surface at fixed \( \bar{q} \) and is not creating pressure.

In our model, with an isolated mass in an otherwise uniform mass density, our Baker parametrisation leads to a free falling speed
\[
\frac{\dot{r}}{r} = \dot{\beta} = H_m - \sqrt{\frac{2GM}{r^3}}.
\]
Because of the competing nature of the free-fall speed from the central mass and the expanding universe, taking Baker’s Eq.(41) for $r^3 \dot{\beta}$ leads to

$$8\pi G T_0^0 = 3H \dot{\beta}, \quad (45)$$

where $T_0^0$ has a zero at $r_{TA}$. Eq.(45) describes a competition between a contracting mass density and an expanding mass density, but not the total gravitational mass within a sphere radius $r$ in the sense of Eq.(33). It would be difficult to establish the rate of mass floating through a sphere of radius $r$ to obtain the pressure in the sense of Einstein equation Eq.(36).

Instead, to capture the total mass representing the flux of gravitational field over a 3-sphere, we use the potential aspect of $r^2 \dot{\beta}^2$ in Eq.(22). The Einstein equation Eq.(35) for $\phi = -V$ leads to

$$4\pi G \rho = \nabla \cdot (\nabla \Phi) = 4\pi G \left( \rho_m + \rho_M + \sqrt{\rho_m \rho_M} \right), \quad (46)$$

where $\rho_M$ is due to the central mass $M$.

In Eq.(46), the matter density $\rho_{Mm} = \sqrt{\rho_m \rho_M(r)}$ corresponds to the non-Newtonian acceleration in Eq.(23), which we postulate as a gravitational effect (not made of physical particles and thereby pressureless).

The stress momentum tensor only possesses a point mass and a matter density and their corresponding pressure terms are zero. Next we consider the effect on the pressure due to time variation of $L$. At large distances, $L$ matches $\varrho$ (after rescaling by a constant) asymptotically, the mass contained inside $L$ remains constant when $a$ expands and there is no pressure. As $\tau$ increases and $L$ shrinks, the cosmological background mass contained in the comoving sphere radius $L/\varrho$ will reduce over time according to Eq.(7). The question is: Does this matter outflow (even if it is very small) as $L$ decreases constitute the presence of pressure?

This question has been asked in the context of early time matter overdensity evolution. A pragmatic answer is given in the adiabatic assumption which states that an overdensity will grow (and therefore contract in comoving scales) without experiencing pressure due to its background matter density. We will briefly describe this assumption below.

For early time scalar perturbation, where $|\phi| \ll 1$ is the comoving gravitational potential, the equivalence of (27) to the FLRW metric in conformal
Newtonian gauge [34] leads to the perturbed Einstein equation, where the Einstein equation of the cosmological background is taken out of the full Einstein equation.

\[ \delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu} \]  

(47)

with perfect fluid energy-momentum tensor, and setting \( c = 1 \) for convenience. For scalar perturbation, the \( G_0^0 \) component of the perturbed Einstein equation is given by

\[ \nabla^2 \phi - 3\mathcal{H}(\phi' + \mathcal{H}\phi) = 4\pi Ga^2\delta\rho \]  

(48)

where \( ' \) is the derivative by conformal time \( \tau_c \) and \( \mathcal{H} \) is Hubble’s parameter in conformal time.

The \( G_i^i \) component of the perturbed Einstein equation is

\[ \phi'' + 3\mathcal{H}\phi' + (2\mathcal{H}' + \mathcal{H}^2)\phi = 4\pi Ga^2(\delta P^a + \delta P^{na}), \]  

(49)

where \( \delta P^a \) and \( \delta P^{na} \) are the adiabatic and the non-adiabatic component of the pressure \( \delta P \) resulting from the perturbation. In Mukhanov [34] for pressureless baryonic matter perturbation after recombination, one assumes that the process is adiabatic \( (\delta P^{na} = 0) \) and pressureless \( (\delta P^a = 0) \). With \( 2\mathcal{H}' + \mathcal{H}^2 = 0 \) in a matter dominated universe where \( \mathcal{H} = 2/\tau_c \),

\[ \phi'' + 6\frac{\phi'}{\tau_c} = 0. \]  

(50)

Together with Eq.(48), one obtains the well known solution for shortwave length matter density perturbation [34].

\[ \bar{\delta} \equiv \frac{\delta\rho}{\rho} = \bar{C}_1\tau^{2/3} + \bar{C}_2\tau^{-1}, \]  

(51)

for constant \( \bar{C}_1 \) and \( \bar{C}_2 \). [We use \( \delta \) to denote differential changes and \( \bar{\delta} \) to denote ratios.] This is the same solution for the overdensity evolution equation in Newtonian perturbation theory at zero sound speed [34] [38],

\[ \ddot{\bar{\delta}} + 2\mathcal{H}\dot{\bar{\delta}} - 4\pi G\rho_m\bar{\delta} = 0, \]  

(52)

The overdensity evolves over time without coupling with the mean cosmological background density \( \rho_m \). This is a critical assumption for the overdensity evolution equation (52).
In a Newtonian perturbation scenario, for an uniform overdensity contracting in comoving coordinates, the pressure term due to entropy effect experienced by the overdensity becomes negligible if the entropy change is δS/S = 0. This is true if the process is isentropic (adiabatic). Although our parameterisation r = a(τ)L may differ from that used in the Newtonian perturbation theory, an adiabatic process would produce the same effect on the pressure term.

For a small overdensity $2GM/r^3 = \frac{8\pi G}{3} \delta \rho_m$ following Baker’s approach we have

$$\frac{P}{c^2} = \sqrt{\delta \rho_m}$$

which is still small for small $\delta \sim 10^{-5}$. (Note: in Newtonian perturbation theory, Eq.(52) is not justified for large $\delta$. However, in practice, Eq.(52) is still used to extrapolate to region where $\delta > 1$.) Adiabatic approximation ($\delta S/S = 0$) is a much stronger physical assumption which will render Eq.(53) ineffective.

After an overdensity turns around $\delta \gg 1$, we note that, during galactic evolution, the central matter distribution in a galactic cluster is also observed to evolve adiabatically, which leads to the postulate of the "pragmatic" Jeans Swindle. This Jean Swindle is used in large structure evolution simulations under MOND. Falco et al. points out that the Jeans Swindle which states that "an overdensity in an infinite homogeneous background, the gravitational potential is sourced by the fluctuations (overdensity) to this uniform background density, is vindicated by the right results it provides". This means that one can again separate the background density potential from the perturbation potential as in Linear or Newtonian perturbation theory, and there is no pressure effect on the perturbation due to its growth (and contraction) in the matter density background. Although one can argue that at late time, a free falling particle does not physically encounter any mean matter density and does not generate entropy and therefore the adiabatic approximation can still hold. This phenomenological Jeans Swindle has no formal justification.

We shall assume that the adiabatic approximation which is assumed at early time continues to hold at late time and there is no pressure term experienced by a free falling particle. Isotropy is preserved.

So far, the analysis above was for Λ = 0.
4.2 $\Lambda \neq 0$

As we have said, Baker \[1\] abandoned his model for $\eta \neq 3$ to preserve isotropy. The plausible assumption of adiabatic behaviour was sufficient to give no problems with $\Lambda = 0$. In reality $\Lambda$ is non-zero. Whether there is a problem or not in this case will depend even more strongly on the empirical behaviour of cosmological matter. The Einstein equation Eq.(52), including non-zero cosmological constant, is

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu},$$

(54)

where the metric components $g_{\mu\nu}$ are decided by a given $T_{\mu\nu}$. As long as the non-Newtonian density remains pressureless, the analysis above can be repeated.

Our starting point is

$$\frac{\dot{r}}{r} = \dot{\beta} = -\sqrt{\frac{2GM}{r^3}} + H,$$

(55)

where $H$ is the Hubble parameter, given by

$$H^2 = \frac{8\pi G}{3} \left( \rho_m + \rho_\Lambda \right) = \frac{8\pi G}{3} \rho_H = H_m^2 + \frac{c^2 \Lambda}{3} = H_m^2 + H_\Lambda^2.$$

(56)

The Einstein equation for $T_{00}$ is

$$8\pi G \rho = 8\pi G c^2 T_{00} - \Lambda c^2 = 8\pi G \left( \rho_m + \rho_M \right) + 8\pi G \sqrt{\rho_M} \sqrt{\rho_H},$$

(57)

where $\rho_{MH} = \sqrt{\rho_M \rho_H}$. Assuming $\rho_{MH}$ is pressureless, the energy density due to the presence of physical matter in the stress momentum tensor is the same. We continue to assume an adiabatic approximation such that a free falling particle will not experience pressure due to its mean matter density background. The (negative) pressure due to the cosmological constant background will continue to exist.

Most of the metrics in use in Astrophysical situations tend to treat the cosmological constant term as an independent background component. For example in the FLRW metric

$$\left(\frac{\dot{r}}{r}\right)^2 = \dot{\beta}^2 = H_m^2 + \frac{\Lambda c^2}{3}.$$

(58)

and in Schwarzschild de-Sitter metric

$$\left(\frac{\dot{r}}{r}\right)^2 = \dot{\beta}^2 = \frac{2GM}{r^3} + \frac{\Lambda c^2}{3}.$$

(59)
Therefore, rather than (55) we could make the choice (which is also Baker’s) that
\[ \dot{\beta}^2 = \left( \frac{\dot{r}}{r} \right)^2 = \left( H_m - \sqrt{\frac{2GM}{r^3}} \right)^2 + \frac{\Lambda c^2}{3}, \quad (60) \]
where the induced pressure-less energy density remains \( \rho_M = \sqrt{\rho_M \rho_m} \) which goes into the acceleration equation as in the \( \Lambda = 0 \) case, here the cosmological constant only plays the background role. The analysis for pressureless matter remains unchanged. In practice, the choice (60) and (55) give similar results apart from near the matter-cosmological constant equality epoch. The discrimination between these two choices is a matter for experimental observations. In what follows we continue to work with the choice (55).

We wish to point out that the problem of transforming a FLRW metric with general background fluid, such as a matter or a photon fluid to Curvature coordinates, remains an open problem [44]-[45]. We therefore would not attempt to transform our more involved metric into Curvature coordinates here. The closest we can get to a Curvature coordinates formulation is Eq.(27).

5 Implications for the creation of large-scale structure

As we said in the introduction, although we anticipate the effects of the interpolation to be small, in general, there are circumstances in which they are observable. We highlight two potentially important implications of our model in this and the next section: a) that baryonic overdensities at recombination can evolve fast enough to match observation and b) that, at late time and galactic distances the VMOND driven acceleration can match the phenomenological MOND acceleration \( a_0 \) within observational error bars.

We begin with the first of these. One of the major mass discrepancy problems that leads to the dark matter postulate is that from Eq.(52) the baryonic overdensity necessary for the creation of large scale structures due to a Cosmic Microwave Background temperature variation at recombination cannot evolve fast enough to match late time observations.

At recombination, for which \( z = 1080 \), a baryon perturbation arising from a source perturbation forms a comoving shell of order 150\( Mpc \). For is-entropic perturbation, we have radiation overdensity \( \delta_{rad} = \frac{4}{3} \delta_b \). Since \( \rho_{rad} \propto T^4 \)
(Stefan-Boltzmann law) whence \( \bar{\delta}_{\text{rad}} = 4 \frac{\delta T}{T} \), we have \( \bar{\delta}_b = 3 \frac{\delta T}{T} \). A CMB average temperature variation \( \frac{\delta T}{T} = 1 - 3 \times 10^{-5} \) corresponds to an initial baryon overdensity \( \delta_{\text{int}} = 3 \frac{\delta T}{T} = 3 - 9 \times 10^{-5} \). For \( z < 1080 \), the Newtonian overdensity alone evolves according to the equation

\[
\bar{\delta} = \bar{\delta}_{\text{int}} \left( \frac{1081}{1 + z} \right).
\]

At \( z = 0 \), we have \( \bar{\delta} \sim 3 - 9 \times 10^{-2} \), which is at variance with the very recent late time observation \( \sqrt{\langle \delta \rangle^2} = \sigma_8 \sim 0.745 \) [46]. [In more detail, this value is the root mean square of the amplitude of matter perturbations smoothed over \( 8h^{-1}\text{Mpc} \) where \( h \) is the Hubble constant in units of \( 100\text{km}\text{s}^{-1}\text{Mpc}^{-1} \). Even the inclusion of an additional biasing factor \( b_{\text{gal}} = 1.3 \pm 0.13 \) for clustered galaxies [47] is not enough to give agreement.] We need something more.

In fact, CDM of itself is not enough at galactic scales at very early time. The James Webb Space Telescope (JWST) recently observed [48]-[49] large galaxies at very high redshift \( (z > 11, \text{that is within } 500\text{Myr from recombination}) \) compared to the \( 2Gyr \) expected from small-halo merging process of the \( \Lambda CDM \) model. More recent JWST observations [50] of around 4000 galaxies show that for large mass galaxies \( (\geq 10^9 M_\odot) \), the fraction of spiral, spheroid and irregular galaxies are constant over the redshift range \( 1.5 < z < 6.5 \), which means that these large galaxies are already well developed by \( 850\text{Myr} \).

To account for these observations, an initially expanding overdensity cloud needs to turnaround and decouple from the cosmological background. This requires a turnaround redshift much higher than \( z = 6.5 \) for the galaxies studied in [50]. The dark matter potential at recombination is well known, persisting until \( z \sim 4 \) [51]. These early large galaxies therefore require an ”new mechanism” to increase the dark matter potential immediately after recombination and subsequently return the dark matter potential to the \( \Lambda CDM \) model at \( z \sim 4 \). This mechanism remains unknown.

Our resolution of the problem is as follows. In a matter dominant expanding universe, for a local baryon overdensity with mass \( M = \bar{\delta} \rho_m, \) radius \( r \),

\[
\frac{2GM}{r^3} = \frac{8\pi G}{3} \rho_M = \frac{8\pi G}{3} \bar{\delta} \rho_m.
\]

More specifically,

\[
M = \bar{\delta} \rho_m \frac{4\pi}{3} r^3, \quad H(z) = \sqrt{\frac{8\pi G}{3} \rho_m}, \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \rho_m.
\]
and, including an angular momentum term, Eq. (23) becomes

\[ \ddot{r} = \frac{\hbar^2}{r^3} - \left( \bar{\delta} + \bar{\delta}^{1/2} + 1 \right) \frac{4\pi G}{3} \rho_m(z) r. \]  

(64)

In Baker [1], the geodesic equation (29) has the full angular momentum term which takes the form

\[ \frac{h^2}{r^3} \left( \frac{1}{1 + \frac{v^2}{c^2}} \right), \quad v^2 = \frac{h^2}{r^2} \]  

(65)

which, at slow speed \( v^2 \ll c^2 \), gives the angular momentum term in Eq. (64).

The potential \( \phi \) in Eq. (22) due to the overdensity now includes an extra (dynamical) matter density \( \bar{\delta}^{1/2} \rho_m \). [We note that a ”phantom dark matter” density in addition to the Newtonian density is also used in providing MOND gravity by Milgrom [52], [53].]

In Eq. (48), this leads to the Einstein equation

\[ \nabla^2 \phi = 4\pi G \rho = 4\pi G (1 + \Delta) \rho_m, \quad \Delta = \bar{\delta} + \bar{\delta}^{1/2}. \]  

(66)

Following Mukanov [34], in the Newtonian perturbation approach to overdensity evolution based on the collisionless Boltzmann equation, the Euler equation for the overdensity for negligible spatial entropy gradient takes the form

\[ \frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\nabla \Phi, \]  

(67)

where \( \vec{v} \) is the radial velocity which is under the influence of both the Newtonian and the non-Newtonian potential.

The one also needs the continuity equation for \( \rho \) in Eq. (66)

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0. \]  

(68)

The end result is again Eq. (52) with \( \delta \) replaced by \( \Delta \),

\[ \ddot{\Delta} + 2H \dot{\Delta} - 4\pi G \rho_m \Delta = 0, \]  

(69)

and the growth mode for \( \Delta \) becomes

\[ \Delta \propto t^{2/3}, \quad \text{permitting} \quad \bar{\delta} \propto a^2 \propto t^{4/3}. \]  

(70)

for small \( \bar{\delta} \), which provides a much faster overdensity \( \bar{\delta} \) evolution after recombination without the need for introducing invisible matter.
5.1 Over-density evolution from VMOND

The question then is whether this initially faster overdensity evolution due to our mechanism is enough to give agreement with data. We shall suggest that it can be, but the argument is not straightforward. We have to make assumptions about galaxy formation, power spectra, simulations and energy redistribution that take us a long way beyond our original model, although they depend crucially upon its details.

From an initial overdensity $\bar{\delta}_{\text{int}}$ at recombination, we can calculate the overdensity $\bar{\delta}$ at redshift $z < 1080$ by

$$\bar{\delta} + \sqrt{\bar{\delta}} = \left(\bar{\delta}_{\text{int}} + \sqrt{\bar{\delta}_{\text{int}}}\right) \left(\frac{1081}{1+z}\right) = \frac{A_0}{1+z},$$  \quad(71)

Galaxy formation is favoured near the shell’s origin and at a radius of 150\,Mpc and at scales that are less than 2\,Mpc. In [54], the galaxy power spectrum is proposed to be $\delta^2 \propto k^n$ with $n > 3$ at large $k$. Separately, Nusser [55], uses a power law overdensity $\delta(r) \propto \delta r^{-S}$, $0 < S < 3$ to simulate overdensity growth under MOND. Therefore, we expect the overdensity to have an increasing power-law behaviour towards small length scales. In an uniform density approximation, the initial density for a galaxy should be much higher than the 150\,Mpc average of $10^{-5}$.

The baseline CMB average temperature variation $\delta T = 1 \times 10^{-5}$ corresponds to an initial baryon overdensity $\bar{\delta}_{\text{int}} = 3 \times 10^{-5}$ and $A_0 = 5.94$. To account for high redshift large galaxies, one needs an overdensity to turnaround at sufficiently higher redshift. In [42], in a spherical galaxy formation under MOND, Sanders takes $\bar{\delta}_{\text{int}} = 1.8 \times 10^{-3}$ which corresponds to $A_0 = 47.8$. [To better illustrate the effect of Eq.(71), we choose a slightly higher value $\bar{\delta}_{\text{int}} = 2.8 \times 10^{-3}$ ($A_0 = 60$), although Sander’s choice is sufficient for most of our purposes.] We obtain the turnaround redshift $z_{ta}$ where $\bar{\delta} = 1$ from Eq.(71),

$$1 + z_{ta} = \frac{A_0}{2} = 30, \quad z_{ta} = 29.$$ \quad(72)

We stress that this turnaround redshift (resulting from this choice of $\delta_{\text{int}}$) is well within an observationally viable redshift range $15 \lesssim z \lesssim 50$ [56], from 21cm radiation.

When comparing to the Newtonian gravity-only evolution Eq.(61)

$$1 + z_{ta} = 1081\bar{\delta}_{\text{int}} = 3.02, \quad z_{ta} = 2.02.$$ \quad(73)
We can see that the turnaround redshift based on Newtonian gravity-only evolution is too low to match the JWST observations, but the VMOND potential lifts $z_{ta}$ to a significantly higher value.

Although not directly related to this analysis of large-scale structure formation, we note that, in the Schwarzschild-de Sitter metric, the turnaround radius of galactic clusters and large galaxies has a theoretical upper limit. However, there are observations that violate this bound [57] and our modified metric provides a larger turnaround radius limit which can accommodate all existing observed turnaround radii [24].

5.2 The journey time of an outer mass shell in free fall to the mass centre

After $\delta \geq 1$, we can calculate the particle free fall time as follows. The $E = h = 0$ energy equation of a point around a central mass $M$ is Eq. (20)

$$\dot{r} = Hr - \sqrt{\frac{2GM}{r}} = \sqrt{\frac{2GM}{r}} \left( \sqrt{\frac{H^2r^3}{2GM}} - 1 \right).$$  (74)

In the matter dominant epoch, $H^2 = \frac{4}{9\pi}$. Setting

$$y = \frac{2}{3} \frac{r^{3/2}}{\sqrt{2GM}} , \quad y = xt,$$  (75)

Eq. (74) simplifies to

$$\frac{dy}{dt} = \frac{y}{t} - 1, \quad \left( \frac{dx}{dt} = -\frac{1}{t} \right).$$  (76)

Since $x$ is a dimensionless variable, one obtains a solution with a constant $t_0$ to be fixed.

$$\frac{y}{t} = x = \ln \left( \frac{t_0}{t} \right).$$  (77)

To fix $t_0$, we use the condition that, at the turnaround time $t = t_{ta}$ where $\dot{r} = 0 (\dot{y} = 0)$. Then, from Eq. (76)

$$\frac{dy}{dt} = 0, \quad \frac{y(r_{ta})}{t_{ta}} = 1 = \ln \left( \frac{t_0}{t_{ta}} \right),$$  (78)

so that $t_0 = et_{ta}$. From Eq. (77) as $r \to 0$ (we have $y \to 0$) where $t$ goes from $t_{ta}$ to reach its closest approach at $t_{ca} = et_{ta}$. Here the free fall time for a turnaround particle is $t_{ff} = (e - 1)t_{ta} = 1.72t_{ta}$. 22
The time for particle in the initial overdensity to reach the central mass’s closest approach \( t_{ca} \) is given by

\[
t_{ca} = t_{ta} + t_{ff} = \frac{2e}{3H(z_{ta})} = \frac{2}{3H(z_{ca})},
\]

using \( H^2(z) = H_0^2\Omega_b(1 + z)^3 \) (\( \Omega_b \) is the density parameter of baryon), one obtains a simple relation

\[
z_{ca} = 0.513(1 + z_{ta}) - 1 = 0.256A_0 - 1. \tag{80}
\]

For \( \delta_{int} = 2.8 \times 10^{-3} \), \( A_0 = 60 \), \( z_{ta} = 29 \), \( z_{ca} = 15.4 \).
For \( \delta_{int} = 1.8 \times 10^{-3} \), \( A_0 = 47.8 \), \( z_{ta} = 22.9 \), \( z_{ca} = 11.23 \).
For \( \delta_{int} = 6 \times 10^{-5} \), \( A_0 = 8.47 \), \( z_{ta} = 4.23 \), \( z_{ca} = 1.16 \).

After \( t_{ca} \), there is a central cloud with size much smaller than the overdensity cloud, where the Newtonian acceleration dominates. We can work with the Newtonian dynamical time \( t_{dyn} \) to consider the phase mixing and the assumed violent relaxation. However, this requires that we go beyond the equations with assumptions about galaxy formation, which we shall not do here. In [58], one of us (CCW) describes how a model elliptical galaxy can form from a spherical overdensity with \( \delta_{int} = 2.8 \times 10^{-3} \), which turns around at \( z_{ta} = 29 \) and reaches virialisation at Quasi-Stationary state (QSS) at \( z = 7.24 \). We find the Faber-Jackson relation in the virialised sphere. We evaluate the VMOND value at this redshift, which comes to \( a_0^V M = 0.895a_0 \) in remarkable qualitative agreement.

### 6 VMOND: the Tully-Fisher relation

At galactic scales, a key motivation for the postulate of dark particles and MOND is the need to explain the flat rotational curves outside a spiral galaxy’s central stellar disk, in the form of the phenomenological Tully-Fisher relation,

\[
v^4 = GMa_0, \tag{81}
\]

where \( v \) is the averaged particle’s (star) rotating speed around a central mass \( M \) and \( a_0 \) is the MOND acceleration which is also obtained statistically from observational data. MOND at large distance \( r \) (deep MOND region), gives

\[
\ddot{r} = 0; \quad 0 = \frac{h^4}{r^6} - \frac{GM}{r^2}a_0, \tag{82}
\]

\[
23
\]
which is the Tully Fisher relation for a single particle, so MOND is, effectively, reverse engineering from the phenomenological Tully Fisher relation to a new law of gravity for a point particle.

A recent result [22] shows that at Wide-binary scales, the gravitational acceleration is essentially Newtonian, rather than (81), which means if further confirmed that the MOND acceleration ($\ddot{r}^{MOND} = -\sqrt{\gamma N \alpha_0}$) is most likely not a canonical law. In that case, an alternative approach which can lead to the Tully-Fisher relation at galactic scales naturally without fine tuning will be required.

In the absence of MOND or dark particles, VMOND can play a similar role to MOND or dark matter halo. In the remainder of this paper we indicate how VMOND, which is applicable to a single point mass, could lead to the Tully Fisher relation which is a statement for the averaged rotating speed. To consolidate this requires a more extensive analysis than we can give here. This has been performed by one of us (CCW) elsewhere [23] and we refer the reader to this paper.

Since the primary objective of this paper is to find a new solution for the central mass in an expanding universe problem, it is sufficient to note that MOND has a built-in canonical $\alpha_0$ value, while for VMOND, the corresponding $\alpha_0(z, r)$ depends on $H(z)$ and $r$, which in general does not match the canonically fixed $\alpha_0$. Here we use a single mass shell to show how the Tully-Fisher relation for a single shell can arise.

We assume that an overdensity at turnaround radius $r_{ta}$ picks up a systematic maximum angular momentum per unit mass. As the radius shrinks from $r_{ta}$ to $R$, there is no energy loss. From the acceleration equation Eq. (64), at $\dot{R} \sim 0$, the rotational speed $v(R)$ at distance $R$ is given by

$$v^4 = \frac{h^4}{R^4} = GM \left( \frac{1}{2} H^2(z_{ta}) R \right) \left( 1 + \sqrt{\frac{2GM}{H^2(z_{ta}) R^3}} \right)^2. \quad (83)$$

Here the Newtonian acceleration term is also included. We test this idea by using the Milky way mass $M = 10^{10} M_\odot$, the edge of the outer-disk $R = 50 kpc$. At turnaround, we have $\delta = 1$ and

$$H^2(z_{ta}) = \frac{2GM}{r_{ta}^3}. \quad (84)$$
We assume that the overdensity radius has contracted from $r_{ta} = 2R$ and $z_{ta} = 29$ from an appropriate initial overdensity at recombination, we have

$$H^2(z_{ta}) = H_0^2(\Omega_b(1 + z_{ta})^3).$$

(85)

For baryon density parameter $\Omega_b = 0.05$ and $H_0 = 73$ $\text{km s}^{-1} (\text{Mpc})^{-1}$, we obtain

$$v^4 = GM(0.85 \times 10^{-10} \text{ms}^{-2}).$$

(86)

We note that in the literature the observed rotation curve follows the Tully-Fisher law

$$v^4 = GM a_0, \quad a_0 = 1.2 \pm 0.25 \times 10^{-10} \text{ms}^{-2}.$$  \hspace{1cm} (87)

where the $a_0$ value is a phenomenological result obtained at galactic scales. For such a sized shell the qualitative agreement is remarkable.

To go further again requires assumptions about galaxy formation [23]. We anticipate that a single mass shell after reaching its closest approach distance will oscillate back to its furtherest approach distance. For a large number of mass shells within an overdensity, violent relaxation occurs and most of the mass shells will virialise in the central region. In a more careful analysis [23], one needs to take into account the effect of virialisation, the overdensity’s dependence on radius, and most importantly the thin disk potential correction to the spherical symmetric potential, which tends to increase the rotational speed.

\section{Summary}

Given a central point mass in an expanding background, we have shown that it is possible to avoid the Einstein-Straus vacuole scenario and, thereby, introduce non-Newtonism mechanics in a natural way. In a Lemaître-Tolman metric formulation, by using a modified form of Baker’s parametrization, we find that the free falling speed can be defined continuously between the central mass and the Hubble radius, providing an exact Newtonian acceleration term in its geodesic equation. This free fall velocity is obtained by simply adding the (negative) Newtonian free fall velocity to the velocity of the Hubble flow, and not their accelerations. The resulting metric has both the Schwarzschild-Lemaître metric and FLRW metric as its asymptotic solutions. This solution avoids the Einstein-Straus vacuole scenario by modifying the geodesic equation to include a non-Newtonian acceleration which we have termed VMOND. Although the non-Newtonian potential leads to an apparent mass density in the Poisson equation, we expect that to be a
pure dynamical effect and therefore produces no pressure term.

In the FLRW metric, for an observer in the non-expanding comoving coordinate $\varrho$, a matter dominant universe will appear uniform and isotropic with constant density and zero pressure. Once a significant point mass at the observer is included, the underlying metric is modified. The new metric prescribes a physical distance $r = aL \neq a\varrho$ for a free falling particle shell. As the free-fall radial distance (also for $L$) reduces, the cosmological gravitational mass inside $L$ reduces, which could lead to additional pressure. However, for an overdensity evolution at early time and late time, we notice that either in linear or Newtonian perturbation theory and in the late time pragmatic "Jeans Swindle", the evolution process is assumed adiabatic (without entropy change) and this pressure term becomes zero. We shall follow this common assumption in our model.

Next, we explored whether this modified Newtonian equation is useful in mass deficit situations. In galactic overdensity evolution, we find that the non-Newtonian acceleration leads to an early time overdensity growth rate $\delta \propto a(t)^2$, which could lead to a high turnaround redshift $z_{ta}$ and the first closest approach redshift $z_{ca} \gg 6.5$. This could provide enough time for the overdensity to virialise before $z = 6.5$. (JWST observations indicate that a large number of galaxies are already formed by $z = 6.5$ and with no change of morphology after.)

Further, for a single mass shell with mass of the Milky Way, which takes up maximum systematic angular momentum per unit mass at turnaround radius, we show that when the mass shell comes down to the galactic disk scale, the rotating speed and central mass already exhibits a Tully-Fisher relation for a single particle with $a_0$ close to the MOND acceleration.

As far as we are aware, this is the only model which could produce significant non-Newtonian gravity without modifying Einstein’s Gravity or postulating invisible particles. Given the problems we have with our current cosmological models, we could see our model as providing a new baseline for calculation.

Data availability statement

Data Sharing not applicable to this article as no datasets were generated or analysed during the current study.
Competing Interests
The authors have no conflicts of interest to declare that are relevant to the content of this article.

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