Hermitian Geometry and Complex Space-Time

Ali H. Chamseddine

Center for Advanced Mathematical Sciences (CAMS) and Physics Department, American University of Beirut, Lebanon.

Abstract

We consider a complex Hermitian manifold of complex dimensions four with a Hermitian metric and a Chern connection. It is shown that the action that determines the dynamics of the metric is unique, provided that the linearized Einstein action coupled to an antisymmetric tensor is obtained, in the limit when the imaginary coordinates vanish. The unique action is of the Chern-Simons type when expressed in terms of the Kähler form. The antisymmetric tensor field has gauge transformations coming from diffeomorphism invariance in the complex directions. The equations of motion must be supplemented by boundary conditions imposed on the Hermitian metric to give, in the limit of vanishing imaginary coordinates, the low-energy effective action for a curved metric coupled to an antisymmetric tensor.

*email: chams@aub.edu.lb
1 Introduction

The idea of complexifying space-time in general relativity was put forward in the early sixties. It appeared in different but related lines of research. These include complexifying the four-dimensional manifold and equipping it with a holomorphic metric, asymptotically complex null surfaces and theory of twistors [1], [2], [3], [4], [5], [6], [7]. More recently, Witten [8] considered string propagation on complexified space-time where he presented some evidence that the imaginary part of the complex coordinates enters in the study of the high-energy behavior of scattering amplitudes [9]. In this string picture it is assumed that the imaginary parts of the coordinates are small at low-energies. At a fundamental level the complex coordinates $X^\mu, \mu = 1, \cdots, d$ with complex conjugates $\overline{X}^\mu \equiv X^{\overline{\nu}}$ are described by the topological $\sigma$ model action [8]

$$I = \int d\sigma d\overline{\sigma} g_{\mu\nu}(X(\sigma, \overline{\sigma}), \overline{X}(\sigma, \overline{\sigma})) \partial_\sigma X^\mu \partial_{\overline{\sigma}} X^{\overline{\nu}},$$

where the world-sheet coordinates are denoted by $\sigma$ and $\overline{\sigma}$, and where the background metric for the complex $d$-dimensional manifold $M$ is Hermitian so that

$$\overline{g_{\mu\nu}} = g_{\nu\mu}, \quad g_{\mu\nu} = g_{\overline{\nu}\overline{\mu}} = 0.$$ 

Decomposing the metric into real and imaginary components

$$g_{\mu\nu} = G_{\mu\nu} + iB_{\mu\nu},$$

the hermiticity condition implies that $G_{\mu\nu}$ is symmetric and $B_{\mu\nu}$ is antisymmetric. The low-energy effective string action is given by the Einstein-Hilbert action coupled to the field strength of the antisymmetric tensor. This can be related to the invariance of the sigma model under complex transformations $X^\mu \to X^\mu + \zeta^\mu(X), X^{\overline{\nu}} \to X^{\overline{\nu}} + \zeta^{\overline{\nu}}(X)$.

A related phenomena was observed in noncommutative geometry [10] where the space-time coordinates are deformed and become noncommuting, $[x^\mu, x^\nu] = i\theta^{\mu\nu}$ [11]. Furthermore, It was found that in the effective action of open-string theory, the inverse of the combinations $(G_{\mu\nu} + B_{\mu\nu})^{-1}$ does appear [12]. This was taken as a motivation to study the dynamics of a complex Hermitian metric on a real manifold [13], considered first by Einstein and Strauss [14]. In [13] it was shown that the invariant action constructed have the required behavior for the propagation of the fields $G_{\mu\nu}$ and $B_{\mu\nu}$ at
the linearized level, but problems do arise when non-linear interactions are taken into account. This is due to the fact that there is no gauge symmetry to prevent the ghost components of $B_{\mu\nu}$ from propagating. It is then important to address the question of whether it is possible to have consistent interactions in which the field $B_{\mu\nu}$ appears explicitly in analogy with $G_{\mu\nu}$ and not only through the combination of derivatives

$$H_{\mu\nu\rho} = \partial_{\mu}B_{\nu\rho} + \partial_{\nu}B_{\rho\mu} + \partial_{\rho}B_{\mu\nu}.$$  

This suggests that the gauge parameters for the transformation $B_{\mu\nu} \to B_{\mu\nu} + \partial_{\mu}\Lambda_{\nu} - \partial_{\nu}\Lambda_{\mu}$ that keep $H_{\mu\nu\rho}$ invariant must be combined with the diffeomorphism parameters on the real manifold. For this to happen there must be diffeomorphism invariance of the Hermitian manifold $M$ of complex dimensions $d$, with complex coordinates $z^\mu = x^\mu + iy^\mu$, $\mu = 1, \cdots, d$. The line element is then given by [15]

$$ds^2 = 2g_{\mu\sigma}dz^\mu d\bar{z}^\sigma,$$

where we have denoted $\bar{z}^\mu = z^\bar{\mu}$. The metric preserves its form under infinitesimal transformations

$$z^\mu \to z^\mu - \zeta^\mu(z),$$

$$\bar{z}^\mu \to \bar{z}^\mu - \bar{\zeta}^\mu(\bar{z}),$$

as can be seen from the transformations

$$0 = \delta g_{\mu\nu} = \partial_{\mu}\zeta^\lambda g_{\nu\lambda} + \partial_{\nu}\bar{\zeta}^\lambda g_{\mu\lambda},$$

$$0 = \delta g_{\mu\sigma} = \partial_{\mu}\zeta^\lambda g_{\sigma\lambda} + \partial_{\sigma}\bar{\zeta}^\lambda g_{\mu\lambda},$$

$$\delta g_{\mu\nu} = \partial_{\mu}\zeta^\lambda g_{\nu\lambda} + \partial_{\nu}\bar{\zeta}^\lambda g_{\mu\lambda} + \zeta^\lambda \partial_{\lambda}g_{\mu\nu} + \bar{\zeta}^\lambda \partial_{\lambda}g_{\mu\nu}.$$  

It is instructive to express these transformations in terms of the fields $G_{\mu\nu}(x, y)$ and $B_{\mu\nu}(x, y)$ by writing

$$\zeta^\mu(z) = \alpha^\mu(x, y) + i\beta^\mu(x, y),$$

$$\bar{\zeta}^\mu(\bar{z}) = \alpha^\mu(x, y) - i\beta^\mu(x, y).$$

The holomorphicity conditions on $\zeta^\mu$ and $\bar{\zeta}^\mu$ imply the relations

$$\partial^\mu_\mu \beta^\nu = \partial_\mu^\nu \alpha^\nu,$$

$$\partial^\mu_\mu \alpha^\nu = -\partial^\nu_\mu \beta^\nu.$$
where we have denoted
\[ \partial_\mu = \frac{\partial}{\partial y^\mu}, \quad \partial_\mu^x = \frac{\partial}{\partial x^\mu}. \]

The transformations of \( G_{\mu\nu}(x, y) \) and \( B_{\mu\nu}(x, y) \) are then given by
\[
\delta G_{\mu\nu}(x, y) = \partial_\mu^x \alpha^\lambda G_{\lambda\nu} + \partial_\nu^x \alpha^\lambda G_{\mu\lambda} + \alpha^\lambda \partial_\lambda^x G_{\mu\nu} \\
- \partial_\mu^x \beta^\lambda B_{\lambda\nu} + \partial_\nu^x \beta^\lambda B_{\mu\lambda} + \beta^\lambda \partial_\lambda^x G_{\mu\nu},
\]
\[
\delta B_{\mu\nu}(x, y) = \partial_\mu^x \beta^\lambda G_{\lambda\nu} - \partial_\nu^x \beta^\lambda G_{\mu\lambda} + \alpha^\lambda \partial_\lambda^x B_{\mu\nu} \\
+ \partial_\mu^x \alpha^\lambda B_{\lambda\nu} + \partial_\nu^x \alpha^\lambda B_{\mu\lambda} + \beta^\lambda \partial_\lambda^x B_{\mu\nu}.
\]

One readily recognizes that in the vicinity of small \( y^\mu \) the fields \( G_{\mu\nu}(x, 0) \) and \( B_{\mu\nu}(x, 0) \) transform as symmetric and antisymmetric tensors with gauge parameters \( \alpha^\mu(x) \) and \( \beta^\mu(x) \) where
\[
\alpha^\mu(x, y) = \alpha^\mu(x) - \partial_\nu^x \beta^\mu(x) y^\nu + O(y^2),
\]
\[
\beta^\mu(x, y) = \beta^\mu(x) + \partial_\nu^x \alpha^\mu(x) y^\nu + O(y^2),
\]
as implied by the holomorphicity conditions.

The purpose of this work is to investigate the dynamics of the Hermitian metric \( g_{\mu\nu} \) on a complex space-time with complex dimensions four, such that in the limit of vanishing imaginary values of the coordinates, the action reduces to that of a symmetric metric \( G_{\mu\nu} \) and an antisymmetric field \( B_{\mu\nu} \). The plan of this paper is as follows. In section two we summarize the essentials of Hermitian geometry. In section three we construct the most general action which gives, in the linearized limit, the correct equations of motion for a symmetric metric \( G_{\mu\nu} \) and an antisymmetric field \( B_{\mu\nu} \) and show that the action is unique. In section four we impose constraints on the torsion and curvature in the four dimensional limit where the imaginary values of the coordinates vanish and study the equations of motion. Section five is the conclusion.

2 Hermitian Geometry

The Hermitian manifold \( M \) of complex dimensions \( d \) is defined as a Riemannian manifold with real dimensions \( 2d \) with Riemannian metric \( g_{ij} \) and
complex coordinates \( z^i = \{ z^\mu, \zeta^\overline{\nu} \} \) where Latin indices \( i, j, k, \cdots \), run over the range \( 1, 2, \cdots, d, \overline{\overline{1}}, \overline{\overline{2}}, \cdots, \overline{\overline{d}} \). The invariant line element is then [16]

\[
 ds^2 = g_{ij} dz^i dz^j,
\]
where the metric \( g_{ij} \) is hybrid

\[
 g_{ij} = \left( \begin{array}{cc}
 0 & g_{i\sigma} \\
 g_{\nu\sigma} & 0 
\end{array} \right) .
\]

It has also an integrable complex structure \( F^j_i \) satisfying

\[
 F^k_i F^j_k = -\delta^j_i,
\]
and with a vanishing Nijenhuis tensor

\[
 N_{ji}^h = F^t_j \left( \partial_t F^h_i - \partial_i F^h_t \right) - F^t_i \left( \partial_t F^h_j - \partial_j F^h_t \right) .
\]

The complex structure has components

\[
 F^j_i = \left( \begin{array}{cc}
 i\delta^i_
u & 0 \\
 0 & -i\delta^\overline{\nu}_\overline{\tau} 
\end{array} \right) .
\]

The affine connection with torsion \( \Gamma_{ij}^h \) is introduced so that the following two conditions are satisfied

\[
 \nabla_k g_{ij} = \partial_k g_{ij} - \Gamma_{ik}^h g_{hj} - \Gamma_{jk}^h g_{ih} = 0,
\]

\[
 \nabla_k F^j_i = \partial_k F^j_i - \Gamma_{ik}^h F^j_h + \Gamma_{jk}^i F^h_i = 0.
\]

These conditions do not determine the affine connection uniquely and there exists several possibilities used in the literature. We shall adopt the Chern connection, which is the one most commonly used, . It is defined by prescribing that the \((2d)^2\) linear differential forms

\[
 \omega^j_i = \Gamma^i_{jk} dz^k,
\]
be such that \( \omega^\mu_\nu \) and \( \omega^\overline{\nu}_{\overline{\mu}} \) are given by [15]

\[
 \omega^\mu_\nu = \Gamma^\mu_{\nu \rho} dz^\rho, \\
 \omega^\overline{\nu}_{\overline{\mu}} = \omega^\overline{\nu}_{\overline{\rho}} = \Gamma^\overline{\nu}_{\overline{\rho \overline{\tau}}} dz^\overline{\tau} ,
\]

4
with the remaining \((2d)^2\) forms set equal to zero. For \(\omega^\mu_\nu\) to have a metrical connection the differential of the metric tensor \(g\) must be given by

\[ dg_{\mu\nu} = \omega^\rho_\mu g_{\rho\nu} + \omega^\rho_\nu g_{\mu\rho}, \]

from which we obtain

\[ \partial_\lambda g_{\mu\nu} dz^\lambda + \partial_\sigma g_{\mu\sigma} dz^\sigma = \Gamma^\rho_\mu g_{\rho\sigma} dz^\lambda + \Gamma^\rho_\sigma g_{\mu\rho} dz^\lambda, \]

so that

\[ \Gamma^\rho_\mu_\lambda = g^{\sigma\rho} \partial_\lambda g_{\mu\sigma}, \]
\[ \Gamma^{\rho\sigma}_\mu_\lambda = g^{\sigma\mu} \partial_\lambda g_{\mu\sigma}, \]

where the inverse metric \(g^{\mu\rho}\) is defined by

\[ g^{\mu\rho} g_{\rho\sigma} = \delta^\mu_\sigma. \]

The condition \(\nabla_\kappa F^j_i = 0\) is then automatically satisfied and the connection is metric. The torsion forms are defined by

\[ \Theta^\mu \equiv -\frac{1}{2} T^\mu_\nu\rho dz^\nu \wedge dz^\rho = \omega^\mu_\nu dz^\nu = -\Gamma^\mu_\nu_\rho dz^\nu \wedge dz^\rho, \]

which implies that

\[ T^\mu_\nu\rho = \Gamma^\mu_\nu_\rho - \Gamma^\mu_\rho_\nu = g^{\mu\sigma} (\partial_\rho g_{\nu\sigma} - \partial_\nu g_{\rho\sigma}). \]

The torsion form is related to the differential of the Hermitian form

\[ F = \frac{1}{2} F_{ij} dz^i \wedge dz^j, \]

where

\[ F_{ij} = F^k_i g_{kj} = -F_{ji}, \]

is antisymmetric and satisfy

\[ F_{\mu\nu} = 0 = F_{\pi\pi}, \]
\[ F_{\mu\sigma} = ig_{\mu\sigma} = -F_{\sigma\mu}, \]
so that
\[ F = ig_{\mu \pi} dz^\mu \wedge dz^\pi. \]
The differential of \( F \) is then
\[ dF = \frac{1}{6} F_{ijk} dz^i \wedge dz^j \wedge dz^k, \]
so that
\[ F_{ijk} = \partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij}. \]
The only non-vanishing components of this tensor are
\[ F_{\mu \nu \rho} = \partial_\rho g_{\mu \nu} - \partial_\mu g_{\nu \rho} = -iT_{\mu \nu \rho} \]
\[ F_{\nu \mu \rho} = -i (\partial_\rho g_{\mu \nu} - \partial_\nu g_{\mu \rho}) = iT_{\mu \nu \rho} g_{\mu \rho} = iT_{\mu \nu \rho}. \]
The curvature tensor of the metric connection is constructed in the usual manner
\[ \Omega^i_j = d\omega^i_j - \omega^i_k \wedge \omega^k_j, \]
with the only non-vanishing components \( \Omega^\nu_{\mu} \) and \( \Omega^\nu_{\mu}. \) These are given by
\[ \Omega^\nu_{\mu} = -R^\nu_{\mu \kappa \lambda} dz^\kappa \wedge dz^\lambda - R^\nu_{\mu \kappa} dz^\kappa \wedge d\bar{z}^\lambda \]
\[ = (\partial_\kappa \Gamma^\nu_{\mu \lambda} - \Gamma^\rho_{\mu \kappa} \Gamma^\nu_{\rho \lambda}) dz^\kappa \wedge dz^\lambda - \partial_\lambda \Gamma^\nu_{\mu \kappa} dz^\kappa \wedge d\bar{z}^\lambda. \]
Comparing both sides we obtain
\[ R^\nu_{\mu \kappa \lambda} = \partial_\lambda \Gamma^\nu_{\mu \kappa} - \partial_\kappa \Gamma^\nu_{\mu \lambda} + \Gamma^\rho_{\mu \kappa} \Gamma^\nu_{\rho \lambda} - \Gamma^\rho_{\mu \lambda} \Gamma^\nu_{\rho \kappa}, \]
\[ R^\nu_{\mu \kappa} = \partial_\lambda \Gamma^\nu_{\mu \kappa}. \]
One can easily show that
\[ R^\nu_{\mu \kappa \lambda} = 0, \]
\[ R^\nu_{\mu \kappa} = g^\rho_\nu \partial_\kappa \partial_\lambda g_{\mu \rho} + \partial_\lambda g^\rho_\nu \partial_\kappa g_{\mu \rho}. \]
Transvecting the last relation with \( g_{\mu \pi} \) we obtain
\[ -R_{\mu \pi \kappa \lambda} = \partial_\kappa \partial_\lambda g_{\mu \pi} + g_{\mu \pi} \partial_\kappa g^\rho_\nu \partial_\lambda g_{\mu \rho}. \]
Therefore the only non-vanishing covariant components of the curvature tensor are
\[ R_{\mu \pi \kappa \lambda}, \quad R_{\mu \nu \kappa \lambda}, \quad R_{\mu \pi \nu \lambda}, \quad R_{\nu \pi \nu \lambda}. \]
which are related by
\[ R_{\mu\nu\kappa\lambda} = -R_{\nu\mu\kappa\lambda} = -R_{\mu\nu\lambda\kappa}, \]
and satisfy the first Bianchi identity [15]
\[ R^{\nu}_{\mu\kappa\lambda} - R^{\nu}_{\mu\kappa\lambda} = \nabla_{\mu} T_{\nu\kappa}. \]
The second Bianchi identity is given by
\[ \nabla_{\rho} R_{\mu\nu\kappa\lambda} - \nabla_{\kappa} R_{\mu\nu\rho\lambda} = R_{\mu\nu\sigma\lambda} T_{\sigma\rho\kappa}, \]
together with the conjugate relations. There are three possible contractions for the curvature tensor which are called the Ricci tensors
\[ R_{\mu\nu} = -g^{\lambda\kappa} R_{\mu\lambda\nu\kappa}, \quad S_{\mu\nu} = -g^{\lambda\kappa} R_{\mu\nu\lambda\kappa}, \quad T_{\mu\nu} = -g^{\lambda\kappa} R_{\kappa\lambda\mu\nu}. \]
Upon further contraction these result in two possible curvature scalars
\[ R = g^{\rho\mu} R_{\mu\rho\nu}, \quad S = g^{\rho\mu} S_{\mu\rho\nu} = g^{\rho\mu} T_{\mu\rho}.\]
Note that when the torsion tensors vanishes, the manifold \( M \) becomes Kähler. We shall not impose the Kähler condition as we are interested in Hermitian non-Kählerian geometry. We note that it is also possible to consider the Levi-Civita connection \( \Gamma_{ij}^k \) and the associated Riemann curvature \( \hat{K}_{kij}^h \) where
\[ \hat{\Gamma}_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij}), \]
\[ K_{kij}^h = \partial_k \hat{\Gamma}_{ij}^h - \partial_i \hat{\Gamma}_{kj}^h + \hat{\Gamma}_{ki}^h \hat{\Gamma}_{ij}^t - \hat{\Gamma}_{ti}^h \hat{\Gamma}_{kj}^t. \]
The relation between the Chern connection and the Levi-Civita connection is given by
\[ \Gamma_{ij}^k = \hat{\Gamma}_{ij}^k + \frac{1}{2} (T_{ij}^k - T_{ij}^k - T_{ji}^k). \]
The Ricci tensor and curvature scalar are \( K_{ij} = K_{ij}^l F_l^i \) and \( K = g^{ij} K_{ij} \). Moreover \( H_{kij} = K_{kij}^l F_l^i \) and \( H = g^{ij} H_{kij} \). The two scalar curvatures \( K \) and \( H \) are related by [17]
\[ K - H = \hat{\nabla}^h F_{ij} \hat{\nabla}^i F_{hi} - \hat{\nabla}^k F_{ki} \hat{\nabla}^h F_{hi} - 2 F_{ji} \hat{\nabla}^j \hat{\nabla}^k F_{ki}. \]
There are also relations between curvatures of the Chern connection and those of the Levi-Civita connection, mainly [17]

\[
\frac{1}{2}K = S - \nabla^\mu T_\mu - \nabla^\nu T_\nu T_\mu g^{\mu\nu},
\]

where \( T_\mu = T_{\mu \nu} \). There are two natural conditions that can be imposed on the torsion. The first is \( T_\mu = 0 \) which results in a semi-Kähler manifold. The other is when the torsion is complex analytic so that \( \nabla_\chi T_{\mu \nu} = 0 \) implying that the curvature tensor has the same symmetry properties as in the Kähler case. In this work we shall not impose any conditions on the torsion tensor.

### 3 An invariant action

We now specialize to the realistic case of a complexified four dimensional space-time. To construct invariants up to second order in derivatives we write the following possible terms

\[
I = \int_{M^4} d^4 z d^4 \bar{z} g \left( a R + b S + c T_{\mu \nu} T_{\rho \sigma \lambda} g^{\mu \rho} g^{\nu \sigma} g^{\lambda \kappa} + d T_{\mu \nu} T_{\rho \sigma \lambda} g^{\mu \rho} g^{\sigma \lambda} g^{\kappa \nu} + e \right).
\]

The density factor is \( |\det g_{ij}|^{\frac{1}{2}} = \det g_{\mu \nu} \equiv g \). We shall set the cosmological term to zero \( (e = 0) \). The above action can equivalently be written in terms of the Riemannian metric \( g_{ij} \) in the form

\[
I = \int_{M} d^4 z d^4 \bar{z} |\det g_{ij}|^{\frac{1}{2}} \left( a' K + b' H + c' F_{ijk} F^{ijk} + d' F_i F^i \right),
\]

where \( F_i = F_{ijk} F^{ijk} \) and \( a', b', c', d' \) are parameters linearly related to the parameters \( a, b, c, d \). We shall now impose the requirement that the linearized action, in the limit of \( y \to 0 \) gives the correct kinetic terms for \( G_{\mu \nu}(x) \) and \( B_{\mu \nu}(x) \). Therefore writing

\[
G_{\mu \nu}(x, y) = \eta_{\mu \nu} + h_{\mu \nu}(x),
\]

\[
B_{\mu \nu}(x, y) = B_{\mu \nu}(x),
\]
and keeping only quadratic terms in the action, we obtain, after integrating by parts, the quadratic $h_{\mu\nu}$ terms,

\[ I = \int d^4x d^4y \left( 2c \partial^\kappa h_{\mu\nu} \partial^\kappa h_{\mu\nu} + (a - 2c + d) \partial^{x\mu} h_{\mu\nu} \partial^{x\lambda} h^{\mu\lambda} + (a - b + 2d) \partial^{x\nu} h_{\mu\nu} \partial^{x\lambda} h^{\lambda} + (d - b) \partial^{\mu\nu} h_{\mu\nu} \partial^{x\lambda} h^{\lambda} \right). \]

Comparing with the linearized Einstein action we obtain the following conditions

\[ 2c = 1, \quad a - 2c + d = -2, \quad -a + b - 2d = 2, \quad d - b = -1, \]

which are equivalent to

\[ b = -a, \quad c = \frac{1}{2}, \quad d = -1 - a. \]

With this choice of coefficients, the quadratic $B$ contributions simplify to

\[ \int d^4x d^4y \left( \partial^\mu B_{\nu\rho} \partial^{x\mu} B^{\nu\rho} - 2\partial^{x\mu} B_{\mu\lambda} \partial^{x\nu} B^{\nu\lambda} \right), \]

which is identical to the term

\[ \frac{1}{3} \int d^4x d^4y H_{\mu\nu\rho} H^{\mu\nu\rho}, \]

where $H_{\mu\nu\rho} = \partial^\mu B_{\nu\rho} + \partial^\nu B_{\rho\mu} + \partial^\rho B_{\mu\nu}$. The action can then be regrouped into the form

\[ I = \int d^4z d^4z g \left( a \left( R - S - T_{\mu\nu\pi\rho\lambda} g^{\mu\rho} g^{\nu\lambda} g^{\pi\sigma} \right) + \frac{1}{2} T_{\mu\nu\pi\rho\lambda} g^{\mu\rho} g^{\nu\lambda} g^{\pi\sigma} - 2g^{\mu\rho} g^{\nu\lambda} g^{\pi\sigma} \right). \]

Using the first Bianchi identity we have

\[ \int d^4z d^4z g \left( R - S \right) = \int d^4z d^4z g^\mu g^{\nu} \partial^\mu T_{\mu\nu} \quad = \int d^4z d^4z g T_{\mu\nu\pi\rho\lambda} g^{\mu\rho} g^{\nu\lambda} g^{\pi\sigma}, \]
where we have integrated by parts and ignored a surface term. This implies that the group of terms with coefficient $a$ drop out, and the action becomes unique:

$$I = \frac{1}{2} \int_M d^4z d^4\bar{z} g T_{\mu \nu \sigma \lambda} (g^{\bar{\mu}} g^{\bar{\nu}} g^{\bar{\sigma}} g^{\bar{\lambda}} - 2g^{\bar{\mu}} g^{\bar{\nu}} g^{\bar{\sigma}} g^{\bar{\lambda}}).$$

Substituting for the torsion tensor in terms of the metric $g_{\mu \nu}$, the above action reduces to

$$I = \frac{1}{2} \int_M d^4z d^4\bar{z} g X^{\bar{\kappa} \bar{\lambda} \sigma \mu \nu \rho} \partial_{\nu} g_{\mu \sigma} \partial_{\lambda} g_{\sigma \rho},$$

where

$$X^{\bar{\kappa} \bar{\lambda} \sigma \mu \nu \rho} = g_{\sigma \rho} \left( g^{\bar{\kappa} \mu} g^{\bar{\lambda} \nu} - g^{\bar{\kappa} \nu} g^{\bar{\lambda} \mu} \right) + g_{\bar{\nu} \rho} \left( g^{\bar{\sigma} \nu} g^{\bar{\kappa} \rho} - g^{\bar{\sigma} \rho} g^{\bar{\kappa} \nu} \right) + g_{\bar{\sigma} \nu} \left( g^{\bar{\kappa} \rho} g^{\bar{\lambda} \mu} - g^{\bar{\kappa} \mu} g^{\bar{\lambda} \rho} \right),$$

which is completely antisymmetric in the indices $\mu \nu \rho$ and in $\bar{\kappa} \bar{\lambda} \sigma$

$$X^{\bar{\kappa} \bar{\lambda} \sigma \mu \nu \rho} = X^{[\bar{\kappa} \bar{\lambda} \sigma] [\mu \nu \rho]}.$$

This is remarkable because the simple requirement that the linearized action for $G_{\mu \nu}$ should be recovered determines the action uniquely. This form of the action is valid in all complex dimensions $d$, however, when $d = 4$, we can write

$$X^{\bar{\kappa} \bar{\lambda} \sigma \mu \nu \rho} = -\frac{1}{g} \epsilon^{\bar{\kappa} \bar{\lambda} \sigma \eta} \epsilon^{\mu \nu \rho \tau} g_{\tau \eta},$$

and the action takes the very simple form

$$I = -\frac{1}{2} \int_M d^4z d^4\bar{z} \epsilon^{\bar{\kappa} \bar{\lambda} \sigma \eta} \epsilon^{\mu \nu \rho \tau} g_{\tau \eta} \partial_{\mu} g_{\nu \sigma} \partial_{\lambda} g_{\rho \kappa}.$$

The above expression has the advantage that the action is a function of the metric $g_{\mu \nu}$ and there is no need to introduce the inverse metric $g^{\mu \nu}$. This suggests that the action could be expressed in terms of the Kähler form $F$. Indeed, we can write

$$I = \frac{i}{2} \int_M F \wedge \partial F \wedge \overline{\partial F}.$$
The equations of motion are given by
\[ \epsilon^{\lambda \sigma \kappa} \chi_{\rho \mu}^{\tau} \left( g_{\nu \tau} \partial_\mu \partial_\kappa g_{\rho \chi} + \frac{1}{2} \partial_\mu g_{\nu \tau} \partial_\kappa g_{\rho \chi} \right) = 0. \]

Notice that the above equations are trivially satisfied when the metric \( g_{\mu \sigma} \) is Kähler
\[ \partial_\mu g_{\nu \tau} = \partial_\nu g_{\mu \tau}, \quad \partial_\kappa g_{\nu \tau} = \partial_\tau g_{\nu \kappa}, \]
where these conditions are locally equivalent to \( g_{\mu \sigma} = \partial_\mu \partial_\sigma K \) for some scalar function \( K \).

4 Four dimensional limit with vanishing imaginary part

To study the spectrum of the action we have to assume that although the coordinates are complex, the imaginary parts are small in low-energy experiments. The action is a function of the fields \( G_{\mu \nu} (x, y) \) and \( B_{\mu \nu} (x, y) \) which depend continuously on the coordinates \( y^\mu \) implying a continuous spectrum with an infinite number of fields depending on \( x^\mu \). To obtain a discrete spectrum a certain physical assumption should be made that forces the imaginary coordinates to be small. One way, suggested by Witten, [8] is to suppress the imaginary parts by constructing an orbifold space \( M' = M/G \) where \( G \) is the group of imaginary shifts
\[ z^\mu \rightarrow z^\mu + i (2 \pi k^\mu), \]
where \( k^\mu \) are real. To maintain invariance under general coordinate transformation we must have \( k^\mu (x, y) \). It is not easy, however, to deal with such an orbifold in field theoretic considerations. To determine what is needed we proceed by first expressing the full action in terms of the fields \( G_{\mu \nu} (x, y) \) and \( B_{\mu \nu} (x, y) \). We write
\[ \partial_\mu g_{\nu \tau} \partial_\kappa g_{\rho \chi} = \frac{1}{4} \left( A_{\kappa \lambda \sigma \mu \nu \rho} + i B_{\kappa \lambda \sigma \mu \nu \rho} \right), \]
where
\[ A_{\kappa\lambda\mu\nu\rho} = \left( \partial^\mu G_{\nu\sigma} + \partial^\nu B_{\mu\sigma} \right) \left( \partial^\rho G_{\kappa\lambda} - \partial^\sigma B_{\rho\lambda} \right) \]
\[ - \left( \partial^\mu B_{\nu\sigma} - \partial^\nu G_{\mu\sigma} \right) \left( \partial^\rho B_{\kappa\lambda} + \partial^\sigma G_{\rho\lambda} \right), \]
\[ B_{\kappa\lambda\mu\nu\rho} = \left( \partial^\mu G_{\nu\sigma} + \partial^\nu B_{\mu\sigma} \right) \left( \partial^\rho B_{\kappa\lambda} + \partial^\sigma G_{\rho\lambda} \right) \]
\[ + \left( \partial^\mu B_{\nu\sigma} - \partial^\nu G_{\mu\sigma} \right) \left( \partial^\rho G_{\kappa\lambda} - \partial^\sigma B_{\rho\lambda} \right). \]

The equations of motion split into real and imaginary parts. These are given by
\[ 0 = \epsilon^{\kappa\lambda\sigma\eta} \epsilon^{\mu\nu\rho\tau} \left( G_{\nu\sigma} \left( \left( \partial^\rho \partial^\kappa + \partial^\kappa \partial^\rho \right) G_{\mu\lambda} - \left( \partial^\rho \partial^\mu + \partial^\mu \partial^\rho \right) B_{\nu\lambda} \right) \right. \]
\[ - B_{\nu\sigma} \left( \left( \partial^\rho \partial^\kappa - \partial^\kappa \partial^\rho \right) G_{\mu\lambda} + \left( \partial^\rho \partial^\mu + \partial^\mu \partial^\rho \right) B_{\nu\lambda} \right) \]
\[ + \left( \frac{1}{2} A_{\kappa\lambda\sigma\mu\nu\rho} \right), \]
\[ 0 = \epsilon^{\kappa\lambda\sigma\eta} \epsilon^{\mu\nu\rho\tau} \left( G_{\nu\sigma} \left( \left( \partial^\rho \partial^\kappa + \partial^\kappa \partial^\rho \right) B_{\mu\lambda} + \left( \partial^\rho \partial^\mu + \partial^\mu \partial^\rho \right) G_{\nu\lambda} \right) \right. \]
\[ + B_{\nu\sigma} \left( \left( \partial^\rho \partial^\kappa + \partial^\kappa \partial^\rho \right) G_{\mu\lambda} - \left( \partial^\rho \partial^\mu + \partial^\mu \partial^\rho \right) B_{\nu\lambda} \right) \]
\[ + \left( \frac{1}{2} B_{\kappa\lambda\sigma\mu\nu\rho} \right). \]

We are interested in evaluating this action and equations of motion for small values of the imaginary coordinates \( y^\mu \). The above expressions contain terms which are at most quadratic in \( \partial^\mu \) derivatives, it is then enough to expand the fields to second order in \( y^\mu \) and take the limit \( y \to 0 \). We therefore write
\[ G_{\mu\nu}(x, y) = G_{\mu\nu}(x) + G_{\mu\nu\rho}(x) y^\rho + \frac{1}{2} G_{\mu\nu\rho\sigma}(x) y^\rho y^\sigma + O(y^3), \]
\[ B_{\mu\nu}(x, y) = B_{\mu\nu}(x) + B_{\mu\nu\rho}(x) y^\rho + \frac{1}{2} B_{\mu\nu\rho\sigma}(x) y^\rho y^\sigma + O(y^3). \]

What is needed is a principle that determines the fields \( G_{\mu\nu\rho}(x), B_{\mu\nu\rho}(x), G_{\mu\nu\rho\sigma}(x) \) and \( B_{\mu\nu\rho\sigma}(x) \) and all higher terms as functions of \( G_{\mu\nu}(x), B_{\mu\nu}(x) \). For our purposes it will be enough to determine the expansions only to second order. This can be achieved by imposing boundary conditions in the limit \( y \to 0 \) on the first and second derivatives of the Hermitian metric. The invariances of the string action given in the introduction suggests that the equations
of motion in the \( y \to 0 \) limit reproduce the low-energy limit of the string equations

\[
0 = G^{\eta \tau} \left( R(G) + \frac{1}{6} H_{\mu \nu \rho} H^{\mu \nu \rho} \right) - 2 \left( R^{\eta \tau}(G) + \frac{1}{4} H^{\eta \nu \rho} H^{\tau \nu \rho} \right),
\]

\[
0 = \nabla^{\mu(G)} H_{\nu \tau},
\]

In the absence of a principle that reduces the continuous spectrum, we shall impose the boundary conditions on the Hermitian metric \( g_{\mu \nu} (x, y) \) to be such that

\[
T_{\mu \nu \rho} \big|_{y \to 0} = 2i B_{\mu \nu \rho} (x),
\]

\[
[R_{\mu \sigma \kappa \lambda} - R_{\kappa \sigma \mu \lambda}] \big|_{y \to 0} = -2 \left( R_{\mu \kappa \sigma \lambda} (G) + i \left( \nabla^{G} H_{\mu \kappa \sigma} - \nabla^{G} H_{\mu \kappa \lambda} \right) \right).
\]

The solution of the torsion constraint gives, to lowest orders,

\[
G_{\mu \nu \rho} (x) = \partial_{\nu} B_{\mu \rho} (x) + \partial_{\mu} B_{\nu \rho} (x),
\]

\[
B_{\mu \nu \rho} (x) = -G_{\mu \nu, \rho} (x) + G_{\nu \rho, \mu} (x),
\]

where all derivatives are with respect to \( x^\mu \). Substituting these into the curvature constraints yield

\[
G_{\mu \sigma \kappa \lambda} (x) = \partial_\sigma \partial_\lambda G_{\mu \kappa} (x) + \partial_\mu \partial_\lambda G_{\sigma \kappa} (x) + \partial_\sigma \partial_\kappa G_{\mu \lambda} (x)
+ \partial_\mu \partial_\kappa G_{\sigma \lambda} (x) - \partial_\kappa \partial_\lambda G_{\mu \sigma} (x) + O (\partial G, \partial B),
\]

\[
B_{\mu \sigma \kappa \lambda} (x) = \partial_\sigma \partial_\lambda B_{\mu \kappa} (x) - \partial_\mu \partial_\lambda B_{\sigma \kappa} (x) + \partial_\sigma \partial_\kappa B_{\mu \lambda} (x)
- \partial_\mu \partial_\kappa B_{\sigma \lambda} (x) - \partial_\kappa \partial_\lambda B_{\mu \sigma} (x) + O (\partial G, \partial B),
\]

where \( O (\partial G, \partial B) \) are terms of second order. To write the equations of motion in component form, we substitute the \( G_{\mu \nu} (x, y) \) and \( B_{\mu \nu} (x, y) \) expansions into \( A_{\kappa \lambda \sigma \mu \rho} \) and \( B_{\kappa \lambda \sigma \mu \rho} \) using the above solutions to obtain

\[
A_{\kappa \lambda \sigma \mu \rho} = \Gamma_{\mu \nu \sigma} (G) \Gamma_{\kappa \lambda \rho} (G)
- (\partial_\mu B_{\nu \sigma} + \partial_\sigma B_{\mu \nu} - \partial_\nu B_{\sigma \mu}) (\partial_\kappa B_{\rho \lambda} + \partial_\lambda B_{\rho \kappa} + \partial_\rho B_{\kappa \lambda}) + O(y),
\]

\[
B_{\kappa \lambda \sigma \mu \rho} = \Gamma_{\mu \nu \sigma} (G) (\partial_\kappa B_{\rho \lambda} + \partial_\lambda B_{\rho \kappa} + \partial_\rho B_{\kappa \lambda})
+ (\partial_\mu B_{\nu \sigma} + \partial_\sigma B_{\mu \nu} - \partial_\nu B_{\sigma \mu}) \Gamma_{\kappa \lambda \rho} (G) + O(y),
\]

where

\[
\Gamma_{\mu \nu \sigma} (G) = \partial_\nu G_{\mu \sigma} + \partial_\mu G_{\nu \sigma} - \partial_\sigma G_{\mu \nu}.
\]
In terms of components, the equations of motion take the form

\[
0 = \varepsilon^{\kappa\lambda\sigma\eta} \varepsilon_{\mu\nu\rho\tau} \left( G_{\nu\sigma} \left( \partial_\mu \partial_\kappa G_{\rho\lambda} + G_{\rho\lambda\mu\kappa} - \partial_\mu B_{\rho\lambda\kappa} + \partial_\kappa B_{\rho\lambda\mu} \right) - B_{\nu\sigma} \left( \partial_\mu \partial_\kappa B_{\rho\lambda} + B_{\rho\lambda\mu\kappa} + \partial_\mu G_{\rho\lambda\kappa} - \partial_\kappa G_{\rho\lambda\mu} \right) + \frac{1}{2} A_{\kappa\lambda\sigma\mu\nu\rho} \right),
\]

\[
0 = \varepsilon^{\kappa\lambda\sigma\eta} \varepsilon_{\mu\nu\rho\tau} \left( G_{\nu\sigma} \left( \partial_\mu \partial_\kappa B_{\rho\lambda} + B_{\rho\lambda\mu\kappa} + \partial_\mu G_{\rho\lambda\kappa} - \partial_\kappa G_{\rho\lambda\mu} \right) - B_{\nu\sigma} \left( \left( \partial_\mu \partial_\kappa G_{\rho\lambda} + G_{\rho\lambda\mu\kappa} - \partial_\mu B_{\rho\lambda\kappa} + \partial_\kappa B_{\rho\lambda\mu} \right) \right) + \frac{1}{2} B_{\kappa\lambda\sigma\mu\nu\rho} \right).
\]

After substituting the solutions of the constraints these take the form

\[
e^{\kappa\lambda\sigma\eta} \varepsilon_{\mu\nu\rho\tau} \left( G_{\nu\sigma} R_{\mu\lambda\rho\kappa} (G) - \frac{1}{4} \partial_{\sigma} B_{\mu\nu} \partial_{\rho} B_{\kappa\lambda} - 2 B_{\nu\sigma} \partial_{\lambda} \partial_{\mu} B_{\rho\kappa} \right) = 0,
\]

\[
e^{\kappa\lambda\sigma\eta} \varepsilon_{\mu\nu\rho\tau} \left( G_{\nu\sigma} \left( \partial_\lambda \partial_\mu B_{\rho\kappa} \right) - B_{\nu\sigma} R_{\mu\lambda\rho\kappa} (G) \right) = 0.
\]

Using the identity

\[
e^{\kappa\lambda\sigma\eta} \varepsilon_{\mu\nu\rho\tau} G_{\nu\sigma} = 6 \det (G_{\mu\nu}) G^{\mu[\kappa} G^{\lambda] \rho} G^{\eta]} \tau,
\]

these equations reduce to the correct equations of motion, up to terms of the form \( O (\partial G, \partial B) \) which were neglected in the derivation.

## 5 Conclusions

In this work we have investigated the structure of a complexified space-time. The geometry is taken to be that of a Hermitian manifold with complex metric given by \( g_{\mu\nu} (z, \overline{z}) = G_{\mu\nu} (x, y) + i B_{\mu\nu} (x, y) \). After studying the properties of Hermitian geometry, we find that there is a unique action, up to boundary terms, that gives the correct linearized kinetic energies for \( G_{\mu\nu} (x) \) and \( B_{\mu\nu} (x) \) in the limit when the metric is restricted to depend only the variables \( x^\mu \). The unique action is of the Chern-Simons type when expressed in terms of the Kähler form. We have shown that the diffeomorphism invariance in the complex coordinates protect both fields \( G_{\mu\nu} (x) \) and \( B_{\mu\nu} (x) \) keeping them massless. The physical requirement that the imaginary parts of the coordinates are small at low energies, must be imposed in such a way
as to reduce the continuous spectrum of $G_{\mu\nu}(x, y)$ and $B_{\mu\nu}(x, y)$ to a discrete spectrum. In the absence of information about the spectrum arising at high energies where the imaginary coordinates are expected to play a role, it is enough for our purposes to impose conditions on first and second derivatives of the Hermitian metric, which allows us to solve for the lowest order terms in the expansion in terms of $y^\mu$. These constraints are imposed on the torsion and curvature of the Hermitian geometry in the limit $y^\mu \to 0$. We have solved the constraints and shown that the equations of motion for the Hermitian metric results in the low-energy string equations in the limit $y^\mu \to 0$. The results obtained so far, give circumstantial evidence that space-time might be enlarged to become complex. Much more work is needed to determine the principle that restricts the form of the hermitian metric to give a discrete spectrum and fixes the dependence on the imaginary coordinates to all orders. This will be necessary in order to understand the contributions of the imaginary parts of the coordinates at high energies. One would expect that $B_{\mu\nu}(x)$ would also enter in the higher order terms of the action and not only through their derivatives, in analogy with the field $G_{\mu\nu}(x)$.

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