EVERY GROUP IS A MAXIMAL SUBGROUP OF THE FREE IDEMPOTENT GENERATED SEMIGROUP OVER A BAND

IGOR DOLINKA AND NIK RUŠKUC

Abstract. Given an arbitrary group $G$ we construct a semigroup of idempotents (band) $B_G$ with the property that the free idempotent generated semigroup over $B_G$ has a maximal subgroup isomorphic to $G$. If $G$ is finitely presented then $B_G$ is finite. This answers several questions from recent papers in the area.

1. Introduction

Let $S$ be a semigroup. The set $E = E(S)$ of all idempotents of $S$ carries a structure of a partial algebra, called the biordered set of $S$, by retaining the products of the so-called basic pairs: these are pairs of idempotents $\{e, f\}$ such that $\{ef, fe\} \cap \{e, f\} \neq \emptyset$. It should be noted that if $ef \in \{e, f\}$ then $fe$ is also an idempotent, possibly different from $e, f$ and $ef$. Also, if $S$ is an idempotent semigroup (i.e. a band) then its biordered set is in general different from $S$ itself, since not every pair is necessarily basic. The term ‘biordered set’ comes from an alternative (but equivalent) approach, where one considers $E(S)$ as a relational structure equipped with two partial pre-orders; here we shall not pursue this approach, directing instead to [4, 5, 6, 11, 16] for further background.

The class of idempotent generated semigroups is of prime importance in semigroup theory, with a host of natural examples, such as the semigroups of singular (non-bijective) transformations of a finite set (Howie [12]) or singular $n \times n$ matrices over a field (Erdos [7]). It is not difficult to show that the category of all idempotent generated semigroups with a fixed biordered set $E$ has an initial object $IG(E)$, called the free idempotent generated semigroup over $E$ (we shall also say ‘over $S$’ when $E = E(S)$). This semigroup is defined by the presentation

$$IG(E) = \langle E \mid e \cdot f = ef \ (\{e, f\} \text{ is a basic pair}) \rangle.$$  

Here $e \cdot f$ stands for a word of length 2 in the free semigroup $E^+$, while $ef$ is the element of $E$ to which the product equals in $S$. Unsurprisingly, $IG(E)$ plays a crucial role in understanding the structure of semigroups with a prescribed biordered set of idempotents.

For reasons that are intrinsic to basic structure theory of semigroups [14, 15], this in turn depends upon the knowledge of maximal subgroups of $IG(E)$. It was conjectured for a long time that the maximal subgroups of $IG(E)$ are always free; this conjecture was widely circulated back in the 1980s, and was explicitly recorded in [15]. The conjecture was proved in a number of particular cases, see e.g. [15, 17, 19]. In 2009, Brittenham, Margolis and Meakin [1] disproved the conjecture by means of

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an explicit 72-element semigroup $S$ such that $IG(E(S))$ has a maximal subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, the free abelian group of rank 2. This was followed by Gray and Ruškuc [9] who proved that every group arises as a maximal subgroup of $IG(E(S))$ for a suitably chosen semigroup $S$; if the group in question is finitely presented then a finite $S$ will suffice. Further ensuing work such as [10, 3, 8] investigates maximal subgroups of $IG(S)$ for some specific natural semigroups $S$, and the first author [2] initiates the study of $IG(B)$, where $B$ is a band.

The aim of the present note is to prove the result announced in the title:

**Theorem 1.** Let $G$ be a group. Then there exists a band $B_G$ such that $IG(B_G)$ has a maximal subgroup isomorphic to $G$. Furthermore, if $G$ is finitely presented, then $B_G$ can be constructed to be finite.

This single construction provides an alternative, simpler proof of all the main results of [9] (Theorems 1–4), resolves [9, Problem 1] which asks whether every finitely presented group is a maximal subgroup of $IG(S)$ for some finite regular semigroup $S$, and solves [2, Problem 2] which calls for a characterisation of maximal semigroups of free idempotent generated semigroups over bands.

### 2. Presentation for maximal subgroups

A general presentation for maximal subgroups of $IG(S)$ in terms of parameters that depend only on the structure of $S$ has been exhibited in [9, Theorem 5]. Since we are interested here only in the case of bands, we utilise the particular form of this theorem, deduced in [2, Corollary 5].

First of all, recall [13, Theorem 4.4.1] that any band $B$ decomposes into a semilattice of rectangular bands, which are the $R$-classes of $B$. Thus a $R$-class $D$ of $S$ can be viewed as an $I \times J$ ‘table’ of idempotents $e_{ij}$ ($i \in I, j \in J$), where \{ $R_i : i \in I$ \} and \{ $L_j : j \in J$ \} are the $R$- and $L$-classes in $D$ respectively. For $i, k \in I$ and $j, l \in J$ we refer to the tuple $(e_{ij}, e_{il}, e_{kj}, e_{kl})$ as the $(i, k; j, l)$ square.

Suppose now we have an element $f \in B$ belonging to a $R$-class above $D$. From the basic theory of bands (see, for example, [13, Section 4.4]) we know that $f$ induces idempotent mappings $\sigma : I \to I, i \mapsto \sigma(i)$, and $\tau : J \to J, j \mapsto (j)\tau$, such that for all $i \in I, j \in J$ we have

$$fe_{ij} = e_{\sigma(i), j}, e_{ij}f = e_{i, (j)\tau}.$$  

We say that the square $(i, k; j, l)$ is *singular* induced by $f$ if one of the following holds:

1. $\sigma(i) = i, \sigma(k) = k$ and $(j)\tau = (l)\tau \in \{j, l\}$; or
2. $\sigma(i) = \sigma(k) \in \{i, k\}$ and $(j)\tau = j, (l)\tau = l$.

We talk of a *left-right* or *up-down* singular square depending on whether (a) or (b) applies.

With the above conventions the general presentation we need is as follows:

**Proposition 2 ([9,2]).** The maximal subgroup $H$ of $IG(B)$ containing $e_{11} \in D$ is presented by

$$\langle f_{ij} \mid (i, j) \in I \rangle \quad f_{11} = f_{1j} = 1 \quad (i \in I, j \in J),$$

$$f_{ij}^{-1}f_{il} = f_{kj}^{-1}f_{kl} \quad ((i, k; j, l) \text{ a singular square in } D).$$
3. Construction of $B_G$

Let $G$ be any group. Let us choose and fix a presentation $\langle A \mid R \rangle$ for $G$ in which every relation has the form $ab = c$ for some $a, b, c \in A$. It is clear that $G$ has such a presentation – for instance the Cayley table would do. What is less obvious, but nonetheless still true, is that if $G$ is finitely presented then it has a finite presentation of this form. One way of seeing this is as follows: A relation $a_1 \ldots a_k = b_1 \ldots b_l = c$ can be replaced by two relations of the form $a_1 \ldots a_k = c$, $b_1 \ldots b_l = c$, at the expense of introducing a new generator $c$. Furthermore, the relation $a_1 \ldots a_k = c$ can be replaced by $k - 1$ relations $a_1 a_2 = d_2, d_2 a_3 = d_3, \ldots, d_{k-2}a_{k-1} = d_{k-1}, d_{k-1}a_k = c$ of the desired form, with new generators $d_2, \ldots, d_{k-1}$.

Define sets

\[ A_0 = A \cup \{0\}, \quad A'_0 = \{a' : a \in A_0\}, \quad I = A_0 \cup A'_0, \quad J = A_0 \cup \{\infty\}, \]

where 0, $\infty$ and $a'$ ($a \in A_0$) are symbols distinct from each other and those already in $A$. Consider the semigroup $T = T_I^{(l)} \times T_J^{(r)}$, where $T_I^{(l)}$ (respectively $T_J^{(r)}$) is the semigroup of all mappings $I \to I$ (resp. $J \to J$) written on the left (resp. right). The semigroup $T$ has a unique minimal ideal $K$ consisting of all $(\sigma, \tau)$ with both $\sigma$ and $\tau$ constant. This ideal is naturally isomorphic to the rectangular band $I \times J$, and we will identify the two. We will visualise $K$ as in Figure 1.

![Figure 1](image)

**Figure 1.** A visual representation of $K = I \times J$, highlighting the partition $I = A_0 \cup A'_0$, as well as the four distinguished rows and columns.

We now define a set $L \subseteq T$. All elements $(\sigma, \tau) \in L$ will have

\[ \sigma^2 = \sigma, \quad \tau^2 = \tau, \quad \ker(\sigma) = \{A_0, A'_0\}, \quad \im(\tau) = A_0. \]

Recall that $\ker(\sigma)$ is the equivalence on $I$ defined by $(i, i') \in \ker(\sigma)$ if and only if $\sigma(i) = \sigma(i')$, and that it can be identified with the resulting partition of $I$ into equivalence classes. Therefore, each $(\sigma, \tau)$ will be uniquely determined by $\im(\sigma)$ which must be a two-element set that is a cross-section of $\{A_0, A'_0\}$, and the value $(\infty)\tau \in A_0$. The elements of $L$ come in four groups: $Z$ – the initial pair; $G, \overline{G}$ – the elements arising from the generators $A$; $R$ – the elements arising from the relations $R$:
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
Type & Notation & Indexing & im(\sigma) \quad (\infty) \tau \\
\hline
\hline
\Z & (\sigma_0, \tau_0) & a \quad \{0, 0^t\} & 0 \\
\G & (\sigma_a, \tau_a) & a \in A \quad \{0, a^t\} & a \\
\G & (\sigma_a, \tau_a) & a \in A \quad \{a, a^t\} & 0 \\
R & (\sigma_r, \tau_r) & r = (ab, c) \in R \quad \{b, c^t\} & a \\
\hline
\end{tabular}
\end{table}

These elements can be visualised as shown in Figure 2.

\begin{figure}[h]
\centering
\begin{tabular}{ccc}
0 & $\infty$ & 0 & $\infty$ & 0 & $\infty$ & a & $\infty$ \\
0' & & a' & & a' & & b & \\
a' & & & & & & & \\
a & & & & & & & \\
\Z : (\sigma_0, \tau_0) & \G : (\sigma_a, \tau_a) & \T : (\sigma_a, \tau_a) & \R : (\sigma_r, \tau_r)
\end{tabular}
\caption{The elements (\sigma_0, \tau_0), (\sigma_a, \tau_a), (\sigma_a, \tau_a) \quad (a \in A), (\sigma_r, \tau_r) \quad (r = (ab, c) \in R) of L. For each (\sigma, \tau) shaded are the two rows corresponding to im(\sigma) and one column corresponding to (\infty)\tau. They all have ker(\sigma) = \{A_0, A_0^t\} and im(\tau) = A_0.}
\end{figure}

Because ker(\sigma) and im(\tau) are the same for all (\sigma, \tau) \in L it follows that L is a left zero semigroup (i.e. xy = x for all x, y \in L). Furthermore, since K is an ideal in \T (i.e. xy, yx \in K for all x \in K, y \in \T), the set B_G = K \cup L is a subsemigroup of \T. We remark that, strictly speaking, B_G depends not only on G, but crucially on the chosen presentation for G.

4. Proof of Theorem 1

We will now use the presentation given in Proposition 2 to compute the maximal subgroup H of IG(B_G) containing the idempotent e_0 = (0, 0) \in K. Relations (1) in our context read

f_{0j} = f_{i0} = 1 \quad (i \in I, j \in J).

(4)

The remaining relations (2) arise from the singular squares induced by the elements of L acting on K. Each up-down singular square is of one of the following forms:

(a_1, a_2; c_1, c_2), \quad (a'_1, a'_2, c_1, c_2) \quad (a_1, a_2, c_1, c_2 \in A_0).

The square (a_1, a_2; c_1, c_2) yields the relation

f_{a_1, c_1}f_{a_2, c_2} = f_{a_2, a_1}f_{a_2, c_2} \quad (a_1, a_2, c_1, c_2 \in A_0).

(5)

Putting a_1 = c_1 = 0, a_2 = a, c_2 = c and using (4) yields

f_{a, c} = 1 \quad (a, c \in A_0).

(6)
clearly, all the remaining relations (5) are consequences of (6). Similarly, the squares 
\((a'_1, a'_2, c_1, c_2)\) yield the relations
\[
f_{a'_i, c} = f_{0', c} \quad (a, c \in A_0).
\] (7)
(Note that we do not necessarily have \(f_{0', c} = 1\), and so cannot deduce \(f_{a'_i, c} = 1\).)

Turning to the left-right singular squares, each \((\sigma, \tau) \in L\) induces precisely one. Below we list respectively the squares introduced by \((e_0', t_0)\) of type \(Z\), \((e_a', t_a)\) of type \(G\), \((T_a, T_a)\) of type \(\overline{G}\), and \((e_r, e_r)\) of type \(R\), together with the relations they yield:

- \((0, 0'; 0, \infty)\) : \(f_{0', 0}^{-1} f_{0, \infty} = f_{0', 0}^{-1} f_{0', \infty}\) (8)
- \((0, a'; a, \infty)\) : \(f_{0', a}^{-1} f_{0, \infty} = f_{0', a}^{-1} f_{0', \infty}\) \((a \in A)\) (9)
- \((a, a'; 0, \infty)\) : \(f_{a', a}^{-1} f_{a, \infty} = f_{a', a}^{-1} f_{a', \infty}\) \((a \in A)\) (10)
- \((b, c'; a, \infty)\) : \(f_{b, c}^{-1} f_{b, \infty} = f_{b, c}^{-1} f_{b, \infty}\) \((r = (ab, c) \in R)\). (11)

Using the relations (4), (6), (7), we can transform (8)–(11) into:

- \(f_{0', \infty} = 1\) (12)
- \(f_{a', \infty} = f_{0', a}\) \((a \in A)\) (13)
- \(f_{a, \infty} = f_{a', \infty} = f_{0', a}\) \((a \in A)\) (14)
- \(f_{a', b} = f_{0', a} f_{0', c}\) \((r = (ab, c) \in R)\). (15)

So, the group \(H\) is defined by the generators \(f_{i, j}\) \((i \in I, j \in J)\) and relations (4), (6), (7), (12)–(15). The relations (4), (6), (7), (12)–(14) can be used simply to eliminate all the generators except \(f_{0', a}\) \((a \in A)\). Replacing each symbol \(f_{0', a}\) by the symbol \(a\), the remaining relations (15) become

\[ab = c \quad (r = (ab, c) \in R)\]

In other words, we obtain the original presentation for \(G\). This proves that \(H \cong G\).

Finally note that if \(\langle A \mid R\rangle\) is a finite presentation, the semigroup \(B_G\) is also finite, with

\[|B_G| = (2|A| + 2)(|A| + 2) + 1 + 2|A| + |R|,\]

and this completes the proof of our theorem.

5. AN EXAMPLE, TWO REMARKS AND AN OPEN PROBLEM

It may be instructive to follow in a specific example the sequence of Tietze transformations constituting the brunt of the above proof. Let us take \(G = Q_8\), the quaternion group, with the well known Fibonacci \(F(2, 3)\) presentation (see [14, Section 7.3]):

\[\langle a, b, c \mid ab = c, bc = a, ca = b \rangle\]

The dimension of \(K\) in this case is \(8 \times 5\), and Proposition[2] gives a presentation in terms of 40 generators. This is then simplified by a sequence of generator eliminations, using relations (1), up-down singular squares, and left-right singular squares induced by the elements of \(L\) of types \(Z, G, \overline{G}\). In the final step further singular squares are revealed, giving back the original presentation.

If we record the original generators in a natural \(8 \times 5\) grid, this process may be encapsulated as shown in Figure[3].
Remark 3. It is possible to describe completely the structure of the free idempotent generated semigroup $IG(B_G)$. By known results (see e.g.\cite{9} (IG1)-(IG4)) $IG(B_G)$ has precisely two regular $\mathcal{D}$-classes. The ‘upper’ one is a left zero semigroup $\mathcal{L}$ isomorphic to $L$ (as all products in $L$ are basic), while the ‘lower’ one $\mathcal{R}$, the completely simple minimal ideal, has a Rees matrix representation with structure group $G$ and (normalised) sandwich matrix $(a_{ji}^{-1})$, where $(a_{ij})$ is the $|I| \times |J|$ table that is the end-product of Tietze transformations performed in the proof of Theorem\footnote{1}. In our example this is the last table in Figure\footnote{3}. We claim that in fact $IG(B_G) = \mathcal{L} \cup \mathcal{R}$. To confirm this, and see that the structure is completely determined, we need to show how to write products $ef$ and $fe$ with $e \in L$, $f \in K$ as products of idempotents from $K$ in $IG(B)$. For the product $ef$ note that there exists $g \in K$ such that $fRg$ and $ge = g$; both pairs $\{e,g\}$ and $\{f,g\}$ are critical and we have $ef = egf = hfe$, where $h = eg \in K$. The product $fe$ can be treated similarly.

Remark 4. Associated to the bider $E$ of idempotents of a regular semigroup $S$ there is another free idempotent generated object $RIG(E)$, the free regular idempotent generated semigroup on $E$. It is the largest regular semigroup with the bider of idempotents $E$, and its presentation can be obtained by adding further relations to the defining presentation for $IG(E)$. For definition and references we refer the
reader to [9]. In particular, from (IG1)–(IG4), (RIG1), (RIG2) in [9] it follows that
\[ \text{IG}(B_G) = \text{RIG}(B_G). \]

One way of interpreting Remarks 3, 4 is to say that the word problem for \( \text{IG}(B_G) \) is
decidable if and only if the word problem for \( G \) is decidable. It is the authors’ belief that the
next stage in the ongoing exploration of free idempotent generated
semigroups is precisely an analysis of the word problem for \( \text{IG}(S) \). This at present
seems a daunting task, even in the case where \( S \) is finite. Nonetheless, we propose
the following problem which may just be within reach at this stage:

**Question 1.** Let \( B \) be a finite band such that all maximal subgroups of \( \text{IG}(B) \) have
recursively soluble word problems. Is the word problem of \( \text{IG}(B) \) necessarily recursively soluble?

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**DEPARTMENT OF MATHEMATICS AND INFORMATICS, UNIVERSITY OF NOVI SAD, TRG DOSITEJA OBRADOVIĆA 4, 21101 NOVI SAD, SERBIA**  
*E-mail address: dockie@dmi.uns.ac.rs*

**SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF ST ANDREWS, ST ANDREWS KY16 9SS, SCOTLAND, UK**  
*E-mail address: nik@mcs.st-and.ac.uk*