Chern number and Berry curvature for Gaussian mixed states of fermions

Lukas Wawer and Michael Fleischhauer

Department of Physics and Research Center OPTIMAS, University of Kaiserslautern, 67663 Kaiserslautern, Germany

(Dated: June 15, 2021)

We generalize the concept of topological invariants for mixed states based on the ensemble geometric phase (EGP) to two-dimensional bandstructures. In contrast to the geometric Uhlmann phase for density matrices the EGP leads to a proper Chern number for Gaussian, finite-temperature or non-equilibrium steady states. The Chern number can be expressed as an integral of the ground-state Berry curvature of a fictitious lattice Hamiltonian, constructed from single-particle correlations. For the Chern number to be non-zero the fictitious Hamiltonian has to break time-reversal symmetry.

I. INTRODUCTION

Transitions between different phases of quantum matter are characterized either by a spontaneous breaking of symmetries or by changes of the topology of the many-body ground state [1-8]. Topologically different phases can be distinguished by invariants, which identify global properties of the system. The existence of these integer quantum numbers is also the origin of the robustness of characteristic features such as edge states and currents or quantized bulk transport [4-8]. Typical topological invariants are based on geometric phases such as the Berry or Zak phase [1] which characterize the parallel transport of the many-body ground-state upon cyclic changes of system parameters. They are thus applicable only to pure states. Several attempts have been made over the last years to extend the concept of topological invariants to mixed states of non-interacting fermions [9-20] with the aim of classifying finite-temperature or non-equilibrium steady states. Although some aspects of topology in open quantum systems can be captured by non-Hermitian Hamiltonians, [21-23], a proper classification must account for both dissipation and fluctuations and requires the discussion of density matrices.

A possible generalization of Berry’s phase to density matrices has been given by Uhlmann [24]. Based on Uhlmann’s construction a mixed-state topological invariant for one-dimensional (1D) systems was defined in [12] as the winding number of the Uhlmann phase upon cyclic parameter changes. For 1D lattice systems in a gapped ground state, the Uhlmann phase is identical to the Zak phase. In this limit its winding can thus be expressed as an integral of a Berry curvature over a 2D torus (lattice momentum and time) which is the well-known first Chern number. For mixed states the existence of a proper Berry connection is in general however not guaranteed. As shown in [16] the approach based on the Uhlmann phase fails when applied to two dimensions [13, 14]. The windings of the Uhlmann phases in $x$ and $y$ direction, $\int dk_x \partial_{k_x} \phi_{k_x}^\prime(k_x)$ and $\int dk_y \partial_{k_y} \phi_{k_y}^\prime(k_y)$, are for some parameters not the same, demonstrating that no proper Berry connection exists.

Recently, we have shown that a generalization of Resta’s many-body polarization [25] to mixed states, termed ensemble geometric phase (EGP) [17, 18], is an alternative way to define a topological invariant for Gaussian states of fermions in 1D. In the thermodynamic limit the EGP approaches the Zak phase of the lowest band of a single-particle Bloch Hamiltonian, termed fictitious Hamiltonian, which is defined by single-particle correlations and thus contains all properties of the Gaussian mixed state. The winding number of the EGP is a topological invariant characterizing this fictitious Hamiltonian. The EGP can be detected and a non-trivial topology has direct physical consequences such as quantized particle transport in an auxiliary system weakly coupled to the fermion chain [29-27]. It can also be extended to 1D models with interactions including systems with fractional topological charges [28].

As is the case for the Uhlmann phase, the EGP depends on the choice of the momentum direction, when considering two-dimensional lattice models. However, as will be shown here, and in contrast to the Uhlmann case, the EGP in any spatial direction can be expressed as a closed-loop integral over the corresponding component of a single Berry connection. The latter describes the ground-state wave function of the fictitious Bloch Hamiltonian. Thus based on the EGP one can define a unique Chern number for mixed states in 2D. To illustrate this we discuss the asymmetric Qi-Wu-Zhang model, which is a two-dimensional, topologically non-trivial lattice model with two bands. Due to the asymmetry of the band structure the winding of the Uhlmann phase is different in $x$ and $y$ direction in a certain range of temperatures [19]. The EGP winding, on the other hand, is always the same in all directions.

II. BERRY CURVATURE AND ZAK PHASE

To set the stage we start by shortly summarizing the topological classification of fermions in terms of the Berry curvature of Bloch states. To this end we consider insulating many-body states of non-interacting fermions on a two-dimensional lattice described by a number-conserving Hamiltonian. We set the lattice constant $a = 1$, consider a total number of $N^2$ unit cells with periodic boundary conditions in $x$ and $y$ directions and use
\( \hbar = 1 \). The operators \( \hat{c}_{j,\lambda}, \hat{c}_{j,\lambda}^\dagger \) describe the annihilation and creation of a fermion in the \( j \)th unit cell. \( j = (j_x, j_y) \) denotes the \( x \) and \( y \) coordinates of the unit cell, respectively, and the index \( \lambda \in \{1, \ldots, p\} \) labels a possible internal degree of freedom within a unit cell. Assuming translational invariance for simplicity, the Hamiltonian can be written in second quantization as

\[
H = \sum_k \sum_{\mu,\nu=1}^p \hat{c}_{\lambda}^\dagger(k) H_{\mu\nu}(k) \hat{c}_{\nu}(k). \tag{1}
\]

Here \( k = (k_x, k_y) \) is the lattice momentum and \( \hbar \) is the \( p \times p \) single-particle Hamiltonian matrix in momentum space. We assume that \( \hbar \) has multiple bands, separated by finite gaps, and we consider an insulator, i.e., assume that the chemical potential \( \mu \) lies within a band gap.

If \( \hbar \) breaks time-reversal symmetry, the topological properties of a gapped many-body state can be characterized by the Berry curvature \( \mathcal{F}(\lambda)(k) \) of all occupied Bloch bands \( n \)

\[
\mathcal{F}(\lambda)(k) = \left( \nabla_k \times \mathcal{A}(\lambda)(k) \right)_z \tag{2}
\]

\[
= i \left( \langle \partial_{k_x} u_n(k) | \partial_{k_y} u_n(k) \rangle - \langle \partial_{k_y} u_n(k) | \partial_{k_x} u_n(k) \rangle \right). \tag{3}
\]

Here \( |u_n(k)\rangle \) is the Bloch function of the \( n \)th band and \( \mathcal{A}(\lambda)(k) \) is the Berry connection \( A_{\lambda}(\lambda)(k) = i \langle u_n(k) | \partial_{k_y} u_n(k) \rangle \). While the Berry curvature is not gauge invariant its integral over the two-dimensional torus of the Brillouin zone is. It furthermore defines an integer-valued topological invariant of the band, the first Chern number

\[
C = \frac{1}{2\pi} \oint_{\mathcal{BZ}} dk \mathcal{F}(\lambda)(k). \tag{4}
\]

The Chern number can also be related to the geometric phase picked up by a Bloch state \( |u_n(k)\rangle \) upon parallel transport in momentum space through the Brillouin zone in either \( k_x \) or \( k_y \) direction. It can be written in terms of the winding of the Zak phases, defined as

\[
\phi_{x,y}^{\text{Zak}}(k_y) = i \oint_{\mathcal{BZ}} dk_x \langle u(k) | \partial_{k_x} u(k) \rangle = \oint_{\mathcal{BZ}} dk_x A_x(k). \tag{5}
\]

or \( \phi_{x,y}^{\text{Zak}}(k_x) \) respectively. Note that we have dropped the band index for simplicity. Most importantly the two Zak phases in \( x \) and \( y \) direction can be expressed as integrals of the components of a single vector, the Berry connection \( A = i \langle u(k) | \nabla_k u(k) \rangle \) as a consequence the windings of the two phases

\[
C_x = \frac{1}{2\pi} \oint_{\mathcal{BZ}} dk_y \frac{\partial}{\partial k_y} \phi_{x}^{\text{Zak}}(k_y), \tag{6}
\]

\[
C_y = \frac{1}{2\pi} \oint_{\mathcal{BZ}} dk_x \frac{\partial}{\partial k_x} \phi_{y}^{\text{Zak}}(k_x) \tag{7}
\]

must be identical and equal to the Chern number

\[
C_x = C_y = C. \tag{8}
\]

III. GEOMETRIC PHASE FOR DENSITY MATRICES: UHLRANNN CONSTRUCTION

Topological invariants for pure states, such as the Chern number, characterize how a gapped many-body state \( |\psi\rangle \) changes upon a parallel transport along a closed loop in parameter space. The parallel-transport requirement leads directly to the definition of the Berry phase, or for lattice systems to the Zak phase, eq.(4). The corresponding Berry connection transforms as a \( U(1) \) gauge field according to \( |u(k)\rangle \rightarrow e^{i \chi(k)} |u(k)\rangle \): \( A \rightarrow A - \nabla_k \chi(k). \)

A generalization of geometric phases to density matrices \( \rho \) has been introduced by Uhlmann \( [24] \), who pointed out that the decomposition \( \rho = w \cdot w^\dagger \) of an \( n \times n \) density matrix into matrices \( w \) contains a gauge freedom \( w \rightarrow w \cdot U \), where \( U \) is a \( U(n) \) unitary matrix. Since \( \rho \) is positive semi-definite, \( w \) can always be represented as

\[
w = \sqrt{\rho} U. \tag{9}
\]

Let \( \rho = \rho(\lambda) \) be a uniquely defined mixed state and let \( \lambda \in \{0, \Lambda\} \) parametrize a closed loop in parameter space such that \( \rho(0) = \rho(\Lambda) \). Requiring parallel transport of the density matrix in generalization of Berry’s construction, one can then define a \( U(n) \) Uhlmann holonomy

\[
\mathcal{H}_U = U(\lambda) U(0) = P e^{-i \oint_{\lambda} d\lambda A^U} \tag{10}
\]

Here \( P \) denotes path ordering and \( A^U = i \partial_\lambda U(\lambda) U(\lambda) \) is the \( U(n) \) Uhlmann connection. The \( U(n) \) gauge freedom can be reduced to \( U(1) \) by performing a trace which leads to the Uhlmann phase

\[
\phi^U = \text{Im} \ln \text{Tr}[\rho(0) \mathcal{H}_U]. \tag{11}
\]

For (pure) ground states of gapped fermionic models eq.(5) reduces to the well-known Zak phase, if \( \lambda \) is identified with the lattice momentum \( k \).

In \( [12] \), the winding of the Uhlmann phase \( \phi^U \) upon a cyclic change of an external parameter \( t \) has been proposed as a topological invariant to classify one-dimensional lattice models in a mixed state \( \rho \)

\[
\nu^U = \frac{1}{2\pi} \oint_{\mathcal{BZ}} dt \frac{\partial}{\partial t} \phi^U(t). \tag{12}
\]

Here the variable \( \lambda \) entering the definition of the Uhlmann phase is the quasi momentum \( k \) along the chain and the loop integral extends over the first Brillouin zone \( (\lambda \rightarrow k \in [-\pi, \pi]) \). While this defines a consistent invariant in 1D, its extension to two spatial dimensions as proposed in \( [13] \) \( [14] \) is problematic. As shown by Budich and Diehl in \( [15] \), the windings of the Uhlmann phase in \( x \) or \( y \) direction are in general not the same, i.e.

\[
C_x^U = \frac{1}{2\pi} \oint_{\mathcal{BZ}} dk_y \frac{\partial}{\partial k_y} \phi^U_x(k_y), \tag{13}
\]

\[
\neq \frac{1}{2\pi} \oint_{\mathcal{BZ}} dk_x \frac{\partial}{\partial k_x} \phi^U_y(k_x) = C_y^U. \tag{14}
\]
As an example they considered a finite-temperature state of a simple topological two-band model with asymmetric band structure

$$\hat{H}(k) = d(k) \cdot \sigma = \sum_{j=1}^{3} d_j(k) \sigma_j,$$

$$d(k) = \begin{pmatrix} \sin(k_x) \\ 3 \sin(k_y) \\ 1 - \cos(k_x) - \cos(k_y) \end{pmatrix},$$

which is a modification of the Qi-Wu-Zhang model [29]. Here $\sigma$ is the vector of Pauli matrices. Its band-spectrum is shown in Fig.3 While at temperatures $T = 0$ or $T = \infty$ the windings of the Uhlmann phase in both directions are the same, i.e. $C^U_y(T = 0) = C^U_x(T = 0) = 1$ and $C^U_y(T = \infty) = C^U_x(T = \infty) = 0$, there is a range of temperatures where $C^U_y(T) \neq C^U_x(T)$. This shows that in contrast to the full $U(n)$ Uhlmann holonomy, there is no proper $U(1)$ gauge structure underlying the Uhlmann phase.

IV. MANY-BODY POLARIZATION AND ENSEMBLE GEOMETRIC PHASE

In the following we want to show that in contrast to the Uhlmann phase, the ensemble geometric phase, introduced for one-dimensional lattice models in [17, 18], can be used to define a proper Berry curvature and Chern number for mixed states of 2D bandstructures.

A. Ground-state Zak phase and many-body polarization

A physical interpretation of the Zak phase can be picked up from its relation to the many-body polarization of insulating states, which in the form introduced by Resta reads

$$P = \frac{1}{2\pi} \text{Im} \log \left< e^{\frac{2\pi i}{N} \hat{X}} \right>.$$  \hspace{1cm} (13)

Here $\hat{X} = \sum_{j=1}^{N} \sum_{\lambda=1}^{p} j \hat{n}_{j,\lambda}$ is the position operator of all particles and the average is performed with respect to the insulating many-body ground state $|\psi_0\rangle$. Here $\hat{n}_{j,\lambda} = \hat{c}_{j,\lambda}^\dagger \hat{c}_{j,\lambda}$ is the number operator of fermions in the $\lambda$th site of the $j$th unit cell in a periodic system of size $L = pN$. We have disregarded the relative position of sites within the unit cell. The latter can straightforwardly be incorporated but does not affect the key properties of $P$. One immediately recognizes that the many-body polarization is the phase of a complex number, given by the expectation value of the collective momentum shift operator $e^{\frac{2\pi i}{N} \hat{X}}$ devided by $2\pi$. In fact, as shown by King-Smith and Vanderbilt [30], differences of $P$ are strictly related to differences of the Zak phase via

$$\Delta P = \frac{1}{2\pi} \Delta \phi^Zak.$$  \hspace{1cm} (14)

If the system Hamiltonian is time-dependent with period $T$, the polarization becomes time-dependent as well and its winding upon adiabatic changes in a period $(0, T)$ is the Chern number of a Berry connection on a 2D torus $(k, t)$ of lattice momentum and time

$$C = \frac{i}{2\pi} \int_0^T dt \oint_{\text{BZ}} dk \langle \partial_k u(k) | \partial_t u(k) \rangle.$$  \hspace{1cm} (15)

For translationally invariant 2D models one can introduce a polarization vector by mapping the 2D system to a set of independent 1D chains in either $x$ or $y$ direction, see Fig.2

$$P_x(k_y) = \frac{1}{2\pi} \text{Im} \log \left< e^{\frac{2\pi i}{N} \hat{X}_x(k_y)} \right>,$$

$$P_y(k_x) = \frac{1}{2\pi} \text{Im} \log \left< e^{\frac{2\pi i}{N} \hat{X}_y(k_x)} \right>,$$

with $\hat{X}_x(k_y) = \sum_{j_x=1}^{N} \sum_{\lambda=1}^{p} j_x \hat{c}_{j_x,\lambda}^\dagger (k_x) \hat{c}_{j_x,\lambda} (k_y)$, where $\hat{c}_{j_x,\lambda} (k_y)$ is the fermion annihilation operator in mixed position-momentum space. Similarly $\hat{Y}_x(k_x) = \sum_{j_y=1}^{N} \sum_{\lambda=1}^{p} j_y \hat{c}_{j_y,\lambda}^\dagger (k_x) \hat{c}_{j_y,\lambda} (k_x)$. Applying the King-Smith Vanderbilt relations to the individual components of the polarization vector

$$\Delta P_x(k_y) = \frac{1}{2\pi} \Delta \phi^Zak_x(k_y), \quad \Delta P_y(k_x) = \frac{1}{2\pi} \Delta \phi^Zak_y(k_x)$$

and taking into account eq. (5) shows that there are two equivalent representations of the lattice Chern number in the gapped ground state in terms of polarization components

$$C_0 = \oint_{\text{BZ}} dk_y \frac{\partial}{\partial k_y} P_x(k_y) \bigg|_{\psi_0} = -\oint_{\text{BZ}} dk_x \frac{\partial}{\partial k_x} P_y(k_x) \bigg|_{\psi_0}.$$  \hspace{1cm} (18)
the abbreviation \( k \) and are fully determined by a
in the form

eigenstates of non-interacting fermions. Gaussian states
Gaussian states, which are the analogue of pure many-body
physical implications such as quantized transport in a
be measured \[18\] and its non-trivial winding has direct
interacting bosons always lead to trivial winding numbers
mixed states of non-
conditions are fulfilled \[18\]. Note that while the EGP is
EGP upon an adiabatic, closed loop in parameter space
invariants in 1D lattices. Considering the change of the
many-body ground state
reduces in the thermodynamic limit to the Zak phase of
for several examples a \textit{finite} critical temperature was
found \[12, 14\].
If we consider an equilibrium state where the chemical
potential is within a band gap of \( h \), also \( h_{\text{fict}} \) has an en-
ergy gap which is centered around zero "energy". It was shown in \[18\] that for 1D bandstructures the EGP then
reduces in the thermodynamic limit to the Zak phase of
the many-body ground state \(|\Psi_0\rangle\) of the fictitious Hamiltonian via a mechanism termed gauge-reduction:

\[
\phi^{\text{EGP}}(\rho) = \phi^{\text{Zak}}(\langle \Psi_0 | \rho | \Psi_0 \rangle) + \mathcal{O}(N^{-\alpha}), \tag{23}
\]

with some \( \alpha > 0 \). Furthermore since the winding of \( \phi^{\text{EGP}} \)
as well as that of \( \phi^{\text{Zak}} \) upon an adiabatic parameter loop
must be a multiple of \( 2\pi \), the winding numbers for any
system size \( N \) are equal, \( \nu^{\text{Zak}} |_{\Psi_0} = \nu^{\text{EGP}} |_{\rho} \), and

\[
\nu^{\text{EGP}} |_{\rho} = \frac{1}{2\pi} \int_0^\Lambda d\lambda \frac{\partial}{\partial \lambda} \phi^{\text{EGP}} \tag{24}
\]

\[
= \frac{i}{2\pi} \int_0^\Lambda d\lambda \int_{\text{bzw}} dk \left( \langle \partial_\lambda u_f(k) | \partial_\lambda u_f(k) \rangle - c.c. \right)
\]

where \(|u_f(k)\rangle\) are the Bloch states of the negative energy
bands of the fictitious Hamiltonian.

\[
h^{\text{fict}}(k) | u^{(n)}_f(k) \rangle = \varepsilon_n(k) | u^{(n)}_f(k) \rangle. \tag{25}
\]
We note that while for thermal states the Bloch wavefunctions of $h^{\text{Harm}}$ are just those of the original Hamiltonian $h$, the $|u_f(k)\rangle$’s have a meaning of their own in the case of a non-equilibrium steady state.

We now argue that the same is true for a finite-temperature state in two spatial dimensions, if the system is translationally invariant, i.e. if a decomposition in independent one-dimensional systems as shown in Fig. 2 is possible. In such a case the density matrix at non-zero temperature can be decomposed in two different ways

$$\rho = \frac{1}{Z} \prod_{k_x} \exp \left\{ - \sum_{q} \hat{c}^\dagger(k_x, q) g(k_x, q) \hat{c}(k_x, q) \right\}$$

$$= \frac{1}{Z} \prod_{k_y} \exp \left\{ - \sum_{q} \hat{c}^\dagger(q, k_y) g(q, k_y) \hat{c}(q, k_y) \right\}.$$  

Then, following the lines of [18] the winding of the ensemble geometric phase in $x$ or $y$ direction upon moving $k_y$ or $k_x$ through the Brillouin zone is the same as that of the Zak phase in the ground state of the fictitious Hamiltonian

$$C_x^{\text{EGP}} = \frac{1}{2\pi} \int_{\text{BZ}} dk_y \frac{\partial}{\partial k_y} \phi_x^{\text{EGP}}(k_y)$$

$$= \frac{i}{2\pi} \iint_{\text{BZ}} dk_x dk_y \left( \langle \partial_{k_x} u_f(k) | \partial_{k_y} u_f(k) \rangle - \text{c.c.} \right)$$

$$C_y^{\text{EGP}} = -\frac{1}{2\pi} \int_{\text{BZ}} dk_x \frac{\partial}{\partial k_x} \phi_y^{\text{EGP}}(k_x)$$

$$= -\frac{i}{2\pi} \iint_{\text{BZ}} dk_x dk_y \left( \langle \partial_{k_y} u_f(k) | \partial_{k_x} u_f(k) \rangle - \text{c.c.} \right).$$

Obviously both expressions are the same and can be written as an integral of a Berry curvature $\mathcal{F}^{\text{EGP}}(k)$ over the two-dimensional Brillouin zone

$$C^{\text{EGP}} = C_x^{\text{EGP}} = C_y^{\text{EGP}}$$

$$= \frac{1}{2\pi} \iint_{\text{BZ}} \text{d}k \mathcal{F}^{\text{EGP}}(k).$$

$\mathcal{F}^{\text{EGP}}(k)$ is the Berry curvature of the ground state of the fictitious Hamiltonian

$$\mathcal{F}^{\text{EGP}}(k) = \left( \nabla_k \times A^{\text{EGP}}(k) \right)_z$$

$$= i \left( \langle \partial_{k_z} u_f(k) | \partial_{k_x} u_f(k) \rangle - \langle \partial_{k_x} u_f(k) | \partial_{k_z} u_f(k) \rangle \right).$$

We conclude that for the two-dimensional generalization of the ensemble geometric phase there exists always a proper Berry curvature. The corresponding Chern number can be non-zero only if the fictitious Hamiltonian $h^{\text{Harm}}(k)$ breaks time-reversal symmetry. For thermal states this is the case if the original Hamiltonian $h(k)$ breaks time-reversal symmetry.

The reduction of topological properties of Gaussian mixed states of fermions in 2D to the ground state of the fictitious Hamiltonian is fully consistent with the general finding that there are only 10 symmetry classes to classify steady states of open systems rather than 28 as expected, for example, for non-Hermitian Hamiltonians [35, 36].

To illustrate our findings, we calculated the EGP components of the asymmetric Qi-Wu-Zhang model, eq. (3), for a finite-temperature state for a finite system of $N \times N$ unit cells. The results are shown in Fig. 3 for $T = 0$ and a temperature much above the single-particle energy gap $T = 20\Delta_{\text{gap}}$. For the higher temperature we have also shown the results for different system sizes $N = 10, 50, 100$. One clearly recognizes the gauge reduction discussed in [18]: Even at a temperature much above the single-particle gap the EGP approaches the ground-state value for increasing $N$. Its winding, which determines the topological invariant is furthermore independent of system size. Moreover, in contrast to the Uhlmann phase, the winding of both components $\phi_x^{\text{EGP}}$ and $-\phi_y^{\text{EGP}}$ is always the same for all finite temperatures $T < \infty$. 

![Graph showing EGP of Qi-Wu-Zhang model](image-url)
V. SUMMARY

We have shown that the ensemble geometric phase which has been used to define topological winding numbers for mixed states of one-dimensional, Gaussian fermion systems can straightforwardly be extended to two spatial dimensions and defines a Chern number if there is translational invariance. Different from approaches based on other geometric phases for mixed states, such as the Uhlmann phase, this number is a true topological invariant as it is a two-dimensional integral over a proper Berry curvature. The latter is defined by the matrix of single-particle correlations in the Gaussian mixed state. Finite-temperature states of non-interacting fermion systems can straightforwardly be extended to the Gaussian state, such as the Uhlmann phase, this number is a true topological winding number for mixed states of one-dimensional, Gaussian fermions. While the discussion in the present paper relies on the assumption of translational invariance, allowing a mapping to decoupled one-dimensional chains (see Fig.2), we anticipate the results to hold also in the presence of disorder and with interactions. This will be subject of further studies.

acknowledgement

Financial support from the DFG through SFB TR 185, project number 277625399 is gratefully acknowledged.

[1] Di Xiao, Ming-Che Chang, and Qian Nu, Berry phase effects on electronic properties, Rev. Mod. Phys. 82, 1959 (2010).
[2] M. Z. Hazan and C.L. Kane, Colloquium: Topological insulators, Rev. Mod. Phys. 82, 3045 (2010).
[3] Xiao-Gang Wen Colloquium: Zoo of quantum-topological phases of matter, Rev. Mod. Phys. 89, 041004 (2017).
[4] K. V. Klitzing, G. Dorda, and M. Pepper, New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance, Phys. Rev. Lett. 45, 494 (1980).
[5] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Quantized Hall Conductance in a Two-Dimensional Periodic Potential, Phys. Rev. Lett. 49, 405 (1982).
[6] D. C. Tsui, H. L. Störmer, and A. C. Gossard, Two-Dimensional Magnetotransport in the Extreme Quantum Limit, Phys. Rev. Lett. 48, 1559 (1982).
[7] R. B. Laughlin, Anomalous Quantum Hall Effect: An Incompressible Quantum Fluid with Fractionally Charged Excitations, Phys.Rev.Lett. 50, 1395 (1983).
[8] D. Arovas, J. R. Schrieffer, and F. Wilczek, Fractional Statistics and the Quantum Hall Effect, Phys. Rev. Lett. 53, 722 (1984).
[9] J. E. Avron, M. Fraas, G. M. Graf, and O. Kenneth, Quantum response of dephasing open systems, New J. Phys. 13 053042 (2011).
[10] S. Diehl, E. Rico, M. A. Baranov, and P. Zoller, Topology by dissipation in atomic quantum wires, Nature Physics 7, 971 (2011).
[11] C.-E. Bardyn, M. A. Baranov, C. V. Kraus, E. Rico, A. Imamoglu, P. Zoller, and S. Diehl, Topology by dissipation, New J. Phys. 15, 085001 (2013).
[12] O. Viyuela, A. Rivas, and M. A. Martin-Delgado, Uhlmann Phase as a Topological Measure for One-Dimensional Fermion Systems, Phys. Rev. Lett. 112, 130401 (2014).
[13] Z. Huang and D. P. Arovas, Topological Indices for Open and Thermal Systems via Uhlmann’s Phase, Phys. Rev. Lett. 113, 076407 (2014).
[14] O. Viyuela, A. Rivas, and M. A. Martin-Delgado, Two-Dimensional Density-Matrix Topological Fermionic Phases: Topological Uhlmann Numbers, Phys. Rev. Lett. 113, 076408 (2014).
[15] E. P. L. van Nieuwenburg and S. D. Huber, Classification of mixed-state topology in one dimension, Phys. Rev. B 90, 075141 (2014).
[16] J.C. Budich and S. Diehl, Topology of density matrices, Phys. Rev. B 91, 165140 (2015).
[17] D. Linzner, L. Wawer, F. Grusdt, M. Fleischhauer, Reservoir-induced Thouless pumping and symmetry protected topological order in open quantum chains, Phys. Rev. B (R) 94, 201105 (2016).
[18] C. E. Bardyn, L. Wawer, A. Altland, M. Fleischhauer, S. Diehl, Probing the topology of density matrices, Phys. Rev. X 8, 011035 (2018).
[19] M. McGinley and N. R. Cooper, Topology of one-dimensional quantum systems out of equilibrium, Phys. Rev. Lett. 121, 090401 (2018).
[20] M. McGinley and N. R. Cooper, Classification of topological insulators and superconductors out of equilibrium, Phys. Rev. B 99, 075148 (2019).
[21] K. Kawabata, K. Shiozaki, M. Ueda, and M. Sato, Symmetry and topology in non-hermitian physics, Phys. Rev. X 9, 041015 (2019).
[22] H. Zhou and J. Y. Lee, Periodic table for topological bands with non-hermitian symmetries, Phys. Rev. B 99, 235112 (2019).
[23] Y. Ashida, Z. Gong, and M. Ueda, Non-hermitian physics (2020), [arXiv:2006.01837].
[24] A. Uhlmann, Parallel Transport and "Quantum Holonomy" along Density Operators, Rep. Math. Phys. 24, 229 (1986).
[25] R. Resta Quantum Mechanical Position Operator in Extended Systems, Phys. Rev. Lett. 80, 1800 (1998).
[26] L. Wawer, R. Li, and M. Fleischhauer, Quantized transport induced by topology transfer between coupled one-dimensional lattice systems, [arXiv:2009.04149].
[27] L. Wawer and M. Fleischhauer (in preparation)

[28] Razmik Unanyan, Maximilian Kiefer-Emmanouilidis, Michael Fleischhauer Finite-temperature topological invariant for interacting systems, Phys. Rev. Lett. 125, 215701 (2020).

[29] Xiao-Liang Qi, Yong-Shi Wu, and Shou-Cheng Zhang, Topological quantization of the spin Hall effect in two-dimensional paramagnetic semiconductors, Phys. Rev. B 74, 085308 (2006).

[30] R. D. King-Smith and David Vanderbilt Theory of polarization of crystalline solids, Phys. Rev. B 47, 1651 (1993).

[31] C. Mink, M. Fleischhauer, R.G. Unanyan, Absence of topology in Gaussian mixed states of bosons Phys. Rev. B 100, 014305 (2019).

[32] A. Altland, and M. Zirnbauer Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures, Phys. Rev. B 55, 1142 (1997).

[33] Andreas P. Schnyder, Shinsei Ryu, Akira Furusaki, and Andreas W. W. Ludwig Classification of topological insulators and superconductors in three spatial dimensions, Phys. Rev. B 78, 195125 (2008).

[34] Shinsei Ryu, Andreas P Schnyder, Akira Furusaki and Andreas W W Ludwig, Topological insulators and superconductors: ten-fold way and dimensional hierarchy, New J. of Phys. (2010).

[35] S. Lieu, M. McGinley, and N. R. Cooper, Tenfold way for quadratic Lindbladians, Phys.Rev.Lett. 124, 040401 (2020).

[36] Alexander Altland, Michael Fleischhauer, and Sebastian Diehl, Symmetry classes of open fermionic quantum matter, Phys. Rev. X 11,021037 (2021).