Linear complexity of generalized cyclotomic sequences of period $2p^m$

Yi Ouyang · Xianhong Xie

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Abstract
In this paper, we construct two generalized cyclotomic binary sequences of period $2p^m$ based on the generalized cyclotomy and compute their linear complexity, showing that they are of high linear complexity when $m \geq 2$.

Keywords Binary sequence · Linear complexity · Cyclotomy · Generalized cyclotomic sequence

Mathematics Subject Classification 11B50 · 94A55 · 94A60

1 Introduction

A sequence $s^\infty = \{s_0, s_1, s_2, \ldots\}$ is called a binary sequence of period $N$ if $s_i \in \mathbb{F}_2$ and $s_i = s_{i+N}$ for all $i \geq 0$. The linear complexity (LC) of a periodic binary sequence $s^\infty$, denoted by $\text{LC}(s^\infty)$, is the length of shortest linear feedback shift register (LFSR) that generates the sequence [10], i.e., the smallest positive integer $l$ such that $s_i = c_l s_{i-l} + \cdots + c_2 s_{i-2} + c_1 s_{i-1}$ for $i \geq l$ and constants $c_0 = 1, c_1, \ldots, c_l \in \mathbb{F}_2$. For $s^\infty$ a sequence of period $N$, the characteristic power series/polynomial of $s^\infty$ and $s^N = \{s_0, s_1, \ldots, s_{N-1}\}$ are defined respectively as $c^\infty(x) = s_0 + s_1 x + \cdots$ and $c^N(x) = s_0 + s_1 x + \cdots + s_{N-1} x^{N-1}$, the
minimal polynomial [3] of $s^\infty$ is

$$m(x) = (x^N - 1) / \gcd\left(c^N(x), x^N - 1\right).$$

Then we have the following classical relation

$$\text{LC}(s^\infty) = \deg(m(x)) = N - \deg\left(\gcd\left(x^N - 1, c^N(x)\right)\right). \tag{1}$$

The linear complexity of a sequence is an important criteria of its quality. As we all know, sequences with high linear complexity (such that $\text{LC}(s^\infty) > \frac{N}{2}$) have important applications in cryptography.

Cyclotomic generators based on cyclotomy can generate sequences with large linear complexity. Generalized cyclotomic classes with respect to $pq$ and $p^2$ were introduced by Whiteman and Ding for the purposes of searching for residue difference sets [19] and cryptography [4] respectively. Based on Whiteman’s generalized cyclotomy of order 2, Ding [5] constructed a class of generalized cyclotomic sequences of period $pq$ and determined their linear complexity. Autocorrelation and linear complexity of period $p^2$ and $p^3$ were studied in [18,22]. The linear complexity of generalized cyclotomic sequences of period $p^m$ were investigated in [14,15]. In addition, the generalized cyclotomy of order 2 was extended to the case of period $p_1^{e_1} \cdots p_m^{e_m}$, which is not consistent with the classical cyclotomy [7]. Subsequently, new generalized cyclotomic sequences of period $p_1^{e_1} \cdots p_m^{e_m}$ that include the classical ones as special cases were presented in [6], and the linear complexity of such sequences of period $pq$ were calculated in [1]. Furthermore, new classes of generalized cyclotomic sequences of period $2p^m$ were proposed in [8], which included the sequence presented in [12] as a special case, and they were shown to have high linear complexity. For recent development of the linear complexity of generalized cyclotomic sequences with different periods, the reader is referred to [2,11–13,16,17,21,23].

In this paper, we construct two new classes of generalized cyclotomic binary sequences of period $2p^m$ and compute their linear complexity, showing that they are of high linear complexity when $m \geq 2$.

## 2 Generalized binary cyclotomic sequences of period $2p^m$

Let $p$ be an odd prime and $g$ be a primitive root module $p^m$. Replace $g$ by $g + p^m$ if necessary, without loss of generality, we may assume that $g$ is an odd integer, and thus $g$ is a common primitive root module $p^j$ and $2p^j$ for all $1 \leq j \leq m$. For a decomposition $p - 1 = ef$, write $d_j = \frac{\varphi(p^j)}{e} = p^{j-1}f$ for each $j$ where $\varphi(\cdot)$ is Euler’s totient function. For $i \in \mathbb{Z}$, $s = p^j$ or $2p^j$, define

$$D_i^{(s)} := \left\{ g^{i+djt} \pmod{s} : 0 \leq t < e \right\} = g^i D_0^{(s)}. \tag{2}$$

One can see immediately $D_i^{(s)}$ depends only on the congruence class $i \pmod{d_j}$. By abuse of notation we say an integer $n \in D_i^{(s)}$ if $n \pmod{s} \in D_i^{(s)}$.

For $(s, a) = (p^j, p^{m-j})$, $(p^j, 2p^{m-j})$ or $(2p^j, p^{m-j})$, we define

$$aD_i^{(s)} := \left\{ ag^{i+djt} \pmod{as} : 0 \leq t < e \right\}. \tag{3}$$
It is well known that \( \{D_0^{(p^j)}, D_1^{(p^j)}, \ldots, D_{d_j-1}^{(p^j)} \} \) forms a partition of \( \mathbb{Z}_{p^j}^* \) (see [24]), which we call the generalized cyclotomic class of order \( d_j \) with respect to \( p^j \), and

\[
\mathbb{Z}_{p^m} = \bigcup_{j=1}^{m} \bigcup_{i=0}^{d_j-1} p^{m-j} D_i^{(p^j)} \cup \{0\},
\]

\[
\mathbb{Z}_{2p^m} = \bigcup_{j=1}^{m} \bigcup_{i=0}^{d_j-1} p^{m-j} \left( 2D_i^{(p^j)} \cup D_i^{(2p^j)} \right) \cup \{0, p^m\}.
\]

From now on, take

\[
f = 2^r (r \geq 1), \ b \in \mathbb{Z}, \ \frac{d_j}{2} = \frac{p^{j-1}f}{2}.
\]

In the following we define two families of generalized cyclotomic sequences of period \( 2p^m \). The ideal of construction comes from Xiao et al. [20], where generalized cyclotomic sequences of period \( p^m \) were constructed and studied.

(i) The generalized cyclotomic binary sequence of period \( 2p^m \) is defined as \( s^\infty = \{s_i\}_{i \geq 0} \) with

\[
s_i = \begin{cases} 
1, & \text{if } i \pmod{2p^m} \in C_1, \\
0, & \text{if } i \pmod{2p^m} \in C_0.
\end{cases}
\]

where

\[
C_0 = \bigcup_{j=1}^{m} \bigcup_{i=\delta_j}^{d_j-1} p^{m-j} \left( 2D_{i+b}^{(p^j)} \cup D_{i+b}^{(2p^j)} \right) \cup \{p^m\},
\]

\[
C_1 = \bigcup_{j=1}^{m} \bigcup_{i=0}^{\delta_j-1} p^{m-j} \left( 2D_{i+b}^{(p^j)} \cup D_{i+b}^{(2p^j)} \right) \cup \{0\}.
\]

For the above sequence \( s^\infty \), the following theorem holds.

**Theorem 1** For the generalized cyclotomic sequence defined by (6) of period \( 2p^m \),

1. if \( 2^e \equiv \pm 1 \pmod{p} \) or \( 2^e \equiv \mp 1 \pmod{p} \) but \( 2^e \not\equiv \pm 1 \pmod{p^2} \), then \( \text{LC}(s^\infty) = 2p^m \);
2. if \( 2^e \equiv -1 \pmod{p} \) but \( 2^e \not\equiv -1 \pmod{p^2} \), then \( 2p^m - 2(p - 1) \leq \text{LC}(s^\infty) \leq 2p^m - (p - 1) \).

(ii) The modified generalized cyclotomic binary sequence of period \( 2p^m \) is defined as \( ˜s^\infty = \{˜s_i\}_{i \geq 0} \) with

\[
˜s_i = \begin{cases} 
1, & \text{if } i \pmod{2p^m} \in ˜C_1, \\
0, & \text{if } i \pmod{2p^m} \in ˜C_0,
\end{cases}
\]

where

\[
˜C_0 = \bigcup_{j=1}^{m} p^{m-j} \left( \bigcup_{i=0}^{\delta_j-1} 2D_{i+b}^{(p^j)} \bigcup_{i=\delta_j}^{d_j-1} D_{i+b}^{(2p^j)} \right) \cup \{p^m\},
\]

\[
˜C_1 = \bigcup_{j=1}^{m} p^{m-j} \left( \bigcup_{i=\delta_j}^{d_j-1} 2D_{i+b}^{(p^j)} \bigcup_{i=0}^{\delta_j-1} D_{i+b}^{(2p^j)} \right) \cup \{0\}.
\]

For the above sequence \( ˜s^\infty \), the following theorem holds.
Theorem 2  For the modified generalized cyclotomic sequence defined by (7) of period $2p^m$,

1. if $2^e \not\equiv 1 \pmod{p}$, then $\text{LC}(s^\infty) = 2p^m$;
2. if $2^e \equiv 1 \pmod{p}$ but $2^e \not\equiv 1 \pmod{p^2}$, then $2p^m - 2(p - 1) \leq \text{LC}(s^\infty) \leq 2p^m - (p - 1)$.

We give two remarks about our main results.

Remark  (1) The two theorems covers all non-Wieferich primes, as in this case, $2^{p-1} \not\equiv 1 \pmod{p^2}$ implies $2^e \not\equiv \pm 1 \pmod{p^2}$. Consequently the case that $2^e \equiv \pm 1 \pmod{p^a}$ but $\not\equiv \pm 1 \pmod{p^{a+1}}$ for $a > 1$ is rare.

(2) A key argument of our computation follows from the work of Edemskiy et al. [9]. Based on our computation, a new (but essentially the same) proof of the conjecture by Xiao et al. in [20] can be achieved.

The inequalities in Theorems 1(2) and 2(2), arising from the inseparability of the polynomial $x^2p^m - 1$ over $\mathbb{F}_2$, are strong enough to deduce that the two generalized sequences are of high linear complexity if $m \geq 2$. For the exact values there, based on numerical evidence, we have the following conjecture:

Conjecture  If $2^e \equiv -1 \pmod{p}$ but $2^e \not\equiv -1 \pmod{p^2}$, then $\text{LC}(s^\infty) = 2p^m - (p - 1)$.

Remark  If $2^e \equiv 1 \pmod{p}$ but $2^e \not\equiv 1 \pmod{p^2}$, we expected that $\text{LC}(s^\infty) = 2p^m - (p - 1) - e$ and checked many examples. However, as pointed out by the referee, if $p = 73$, $m = 1$ and $f = 4$, then $\text{LC}(s^\infty) = 38 \not= p + 1 - e = 56$. So the prediction is false and we now expect $\text{LC}(s^\infty) \leq 2p^m - (p - 1) - e$.

3 Proof of the main results

Let $\beta = \beta_m$ be a fixed primitive $p^m$-th root of unity, then the field $\mathbb{F}_2(\beta) = \mathbb{F}_{2^n}$ where $n$ is the order of 2 module $p^m$. For $j < m$, $\beta_j = \beta_m^{p^{m-j}}$ is a primitive $p^j$-th root of unity.

We fix the decomposition $p - 1 = ef$, $f = 2^r$ for $r \geq 1$, $\delta_j = \frac{d_j}{2} = \frac{p^{r-1}f}{2}$ for $1 \leq j \leq m$ and $b \in \mathbb{Z}$. Note that $\delta_1 = \frac{f}{2}$ and $d_1 = f$. For $v \in \mathbb{Z}_n$, set

$$H_{m,v}^{(p^j)} := \bigcup_{i=0}^{\delta_j-1} p^{m-j} D_{i+v}^{(p^j)}, \quad H_{m,v}^{(2p^j)} := 2H_{m,v}^{(p^j)}, \quad H_{m,v}^{(2p^j)} := \bigcup_{i=0}^{\delta_j-1} p^{m-j} D_{i+v}^{(2p^j)}$$

and

$$H_{m,v}^{(p^j)}(x) := \sum_{t \in H_{m,v}^{(p^j)}} x^t, \quad H_{m,v}^{(2p^j)}(x) := \sum_{t \in H_{m,v}^{(2p^j)}} x^t = H_{m,v}^{(p^j)}(x^2), \quad H_{m,v}^{(2p^j)}(x) := \sum_{t \in H_{m,v}^{(2p^j)}} x^t.$$

The characteristic polynomials of $s^\infty$ and $\tilde{s}^\infty$ are

$$s(x) := \sum_{t \in C_1} x^t = 1 + \sum_{j=1}^{m} \left( H_{m,b}^{(p^j)}(x) + H_{m,b}^{(2p^j)}(x) \right),$$

$$\tilde{s}(x) := \sum_{t \in \tilde{C}_1} x^t = 1 + \sum_{j=1}^{m} \left( H_{m,b+\delta_j}^{(p^j)}(x) + H_{m,b}^{(2p^j)}(x) \right).$$
To study the linear complexity of $s^\infty$ and $\tilde{s}^\infty$, note that there is some subtlety here: the polynomial $x^{2p^n} - 1$ is inseparable, each root $\beta^a$ $(a \in \mathbb{Z}_{p^n})$ is of multiplicity 2, so by Eq. (1), we have the inequalities

$$2p^m - 2|\{a \in \mathbb{Z}_{p^m} \mid s(\beta^a) = 0\}| \leq \text{LC}(s^\infty) \leq 2p^m - |\{a \in \mathbb{Z}_{p^m} \mid s(\beta^a) = 0\}|. \quad (8)$$

$$2p^m - 2|\{a \in \mathbb{Z}_{p^m} \mid \tilde{s}(\beta^a) = 0\}| \leq \text{LC}(\tilde{s}^\infty) \leq 2p^m - |\{a \in \mathbb{Z}_{p^m} \mid \tilde{s}(\beta^a) = 0\}|. \quad (9)$$

Since the polynomial is valued over a field of characteristic 2, for $p \in \mathbb{Z}$, we have

$$H_{m,v}^{(p)}(\beta^a) = H_{m,v}^{(p)}(\beta^{2a}) = \alpha(H_{m,v}^{(p)}(\beta^a))^2,$$  

(10)

$$H_{m,v}^{(2p)}(\beta^a) = H_{m,v}^{(p)}(\beta^a).$$  

(11)

To study $s(\beta^a)$ and $\tilde{s}(\beta^a)$, it suffices to evaluate $H_{m,b}^{(p)}(\beta^a)$ for each $j \leq m$.

**Lemma 1** ([20], Lemma 4) For $v \in \mathbb{Z}$, we have

$$H_{m,v}^{(p)}(\beta) + H_{m,v+\frac{1}{f}}^{(p)}(\beta) = \sum_{t \in p^{m-l}\mathbb{Z}_p} \beta^t = 1,$$  

(12)

$$H_{m,v}^{(p)}(\beta) + H_{m,v+\delta}^{(p)}(\beta) = \sum_{t \in p^{m-j}\mathbb{Z}_p} \beta^t = 0 \text{ if } 2 \leq j \leq m.$$  

(13)

**Lemma 2** Let $a = p^l u \in p^l D_k^{(p^{m-l})}$ where $0 \leq l \leq m - 1$. Then for $j = 1, 2, \ldots, m$,

1. if $j \leq l$, $H_{m,b}^{(p)}(\beta^a) = \frac{p^{j-1}(p-1)}{2}$;
2. if $j = l + 1$, $H_{m,b}^{(p)}(\beta^a) = \frac{p^{j-1}}{2} + H_{m,b+k}^{(p)}(\beta)$;
3. if $j > l + 1$, $H_{m,b}^{(p)}(\beta^a) = H_{b+k}^{(p^{j-l})}(\beta)$.

**Proof** First note the computation here is carried out in $\mathbb{F}_2(\beta)$. By definition,

$$H_{m,b}^{(p)}(\beta^a) = \sum_{r \in H_{m,b}^{(p)}} \beta^{at} = \sum_{i=0}^{\delta_j-1} \sum_{t \in p^{m-j}D_i^{(p)}} \beta^{tp^l u} = \sum_{i=0}^{\delta_j-1} \sum_{t \in p^{m+l-j}D_i^{(p)}} \beta^{tu}. \quad (14)$$

If $j \leq l$, each term in $H_{m,b}^{(p)}(\beta^a)$ defined in (14) equals to 1, hence

$$H_{m,b}^{(p)}(\beta^a) = \delta_j \cdot |D_{i+b}^{(p)}| = \delta_j p^{j-1} \frac{p-1}{p^{j-1} f} = \frac{p^{j-1}(p-1)}{2}.$$  

If $j > l$, let $s = j - l$, then

$$H_{m,b}^{(p)}(\beta^a) = \sum_{i=0}^{\delta_j-1} \sum_{t \in p^{m+l-j}D_i^{(p)}} \beta^{tu} = \sum_{i=0}^{\delta_j-1} \sum_{t \in D_i^{(p)}} \beta^{p^{n-tu}}. \quad (15)$$

Note that when $i$ passes through $\{0, 1, \ldots, \delta_j - 1\}$, $i \pmod{d_5}$ takes value $\frac{p^{j-1}}{2}$ times on each element in $\{0, 1, \ldots, d_5 - 1\}$ and one additional time on elements in $\{0, 1, \ldots, \delta_5 - 1\}$. Hence the multiset...
\[
\left\{ tu \mod p^s \mid t \in D_{i+b}^{(p^j)}, \ 0 \leq i \leq \delta_j - 1 \right\}
\]

passes \( \frac{p^l-1}{2} \) times through \( \mathbb{Z}_{p^r}^* \), and one additional time over the union of \( D_{i+k+b}^{(p^j)} \) for \( 0 \leq i \leq \delta_j - 1 \). Since \( \beta^{p^{m-s}} \) is a primitive \( p^s \)-th root of unity, by (15), we have

\[
H_{m,b}^{(p^{j+1})}(\beta^a) = \frac{p^l-1}{2} \sum_{a \in \mathbb{Z}_{p^r}} \beta^{p^{m-s}a} + H_{m,b+k}^{(p^j)}(\beta),
\]

which is \( \frac{p^l-1}{2} + H_{m,b+k}^{(p^j)}(\beta) \) if \( s = 1 \) and \( H_{m,b+k}^{(p^j)}(\beta) \) if \( s \geq 2 \) by Lemma 1.

For \( 1 \leq j \leq m \) and \( v \in \mathbb{Z} \), set

\[
A_{m,j,v}(x) := \sum_{s=1}^{j} H_{m,v}^{(p^s)}(x).
\]

Note that \( H_{m,v}^{(p^s)}(\beta_m) = H_{j,v}^{(p^s)}(\beta_j) \) for \( s \leq j \), then

\[
A_{m,j,v}(\beta_m) = \sum_{s=1}^{j} H_{m,v}^{(p^s)}(\beta_m) = \sum_{s=1}^{j} H_{j,v}^{(p^s)}(\beta_j) = A_{j,v}(\beta_j).
\]

Set

\[
A_{j,v} := A_{j,v}(\beta_j) \in \mathbb{F}_2(\beta_j).
\]

By Lemma 2 and Eqs. (10)–(11), for \( a \in p^l D_k^{(p^{m-l})}, 0 \leq l < m \), let \( t = m - l \), then

\[
s(\beta^a) = 1 + A_{t,b+k} + A_{t,b+k}^2, \quad \tilde{s}(\beta^a) = 1 + A_{t,b+k+\delta_l} + A_{t,b+k}^2.
\]

By Lemma 1, \( 1 + A_{t,b+k+\delta_l} = A_{t,b+k} \). In conclusion, then we have:

**Proposition 1** For \( a = 0 \), one has \( s(1) = \tilde{s}(1) = 1 \). For \( a \in p^l D_k^{(p^{m-l})}, 0 \leq l < m \), let \( t = m - l \), then

\[
s(\beta^a) = 1 + A_{t,b+k} + A_{t,b+k}^2, \quad (18)
\]

\[
\tilde{s}(\beta^a) = A_{t,b+k} + A_{t,b+k}^2.
\]

It now suffices to study the values of \( A_{j,v} \) for \( j \geq 1 \) and \( v \in \mathbb{Z} \). We first list three key identities about \( A_{j,v} \):

**Lemma 3** For each \( j \geq 1 \) and \( v \in \mathbb{Z} \), one has

(1) \( A_{j,v} = A_{j,v+d_j} \).

(2) \( A_{j,v} + A_{j,v+\delta_j} = 1 \).

(3) If \( 2 \in D_h^{(p^j)} \), then \( A_{j,v}^2 = A_{j,v+h} \).

**Proof** (1) is trivial. (2) follows immediately from Lemma 1.

For (3), if \( 2 \in D_h^{(p^j)} \), then \( 2 \in D_h^{(p^j)} \) for all \( s \leq j \). For any \( i \), we have \( \{ 2t \mid t \in D_{i+h}^{(p^j)} \} = D_{i+h}^{(p^j)} \), hence \( H_{j,v}^{(p^s)}(\beta_j)^2 = H_{j,v}^{(p^s)}(\beta_j^2) = H_{j,v+h}^{(p^s)}(\beta_j) \) and (3) follows. \( \square \)

Following the proof of [9, Proposition 2], we have the following essential result.

\( \square \) Springer
Lemma 4 Suppose \([\mathbb{F}_2(\beta_j) : \mathbb{F}_2(\beta_{j-1})] = p\). Then \(A_{j,v} + A_{j,v+f/2} \notin \mathbb{F}_2(\beta_{j-1})\). In particular, for \(0 < t < j\), set

\[ A_{j,v}^t := A_{j,v} - A_{t,v} = \sum_{s=t+1}^{j} H_{j,v}^{(p^s)}(\beta_j). \]

Then \(A_{j,v}^t + A_{j,v+f/2} \notin \mathbb{F}_2(\beta_{j-1})\), and consequently, \(A_{j,v}^t \neq A_{j,v+f/2}^t\).

Proof Note that in our case \(j \geq 2\) as \([\mathbb{F}_2(\beta_1) : \mathbb{F}_2(\beta_0)] \leq p - 1 < p\). Let \(\xi = H_{j,v}^{(p^s)}(\beta_j) + H_{j,v+f/2}^{(p^s)}(\beta_j)\). If \(A_{j,v} + A_{j,v+f/2} \in \mathbb{F}_2(\beta_{j-1})\), then

\[ \xi = (A_{j,v} + A_{j,v+f/2}) - (A_{j-1,v} + A_{j-1,v+f/2}) \in \mathbb{F}_2(\beta_{j-1}). \]

On the other hand, by definition we have \(\xi = \sum_{k \in \mathcal{D}} \beta_j^k\), where

\[ \mathcal{D} = \bigcup_{i=0}^{f/2-1} \left( D_{i+v}^{(p^i)} \cup D_{i+b_j+v}^{(p^i)} \right) \]

is the same \(\mathcal{D}\) (with translation by \(v\)) in the proof of [9, Proposition 2]. Note that if \(k_1 \neq k_2 \in \mathcal{D}\), then \(k_1 \pmod{p} \neq k_2 \pmod{p}\), and the set \(\mathcal{D} \pmod{p}\) is nothing but the set \(\mathbb{Z}_p^*\). We have

\[ \xi = \sum_{i=1}^{p-1} c_i \beta_j^i, \quad 0 \neq c_i \in \mathbb{F}_2(\beta_{j-1}). \]

Thus the minimal polynomial of \(\beta_j\) over \(\mathbb{F}_2(\beta_{j-1})\) is of degree \([\mathbb{F}_2(\beta_j) : \mathbb{F}_2(\beta_{j-1})] < p\), which leads to a contradiction. \(\square\)

Lemma 5 For \(j \geq 1\), suppose \(2 \in D_{h_j}^{(p^j)}\). Then one of the following holds:

1. \(2^e \neq \pm 1 \pmod{p}\), equivalently, \(\delta_1 = \frac{f}{2} \nmid h\).
2. \(2^e \equiv 1 \pmod{p^a}\) and \(2^e \neq 1 \pmod{p^{a+1}}\), equivalently, \(2 \in D_0^{(p^i)}\) for \(j \leq a\) and \(2 \notin D_0^{(p^i)}\) for \(j > a\).
3. \(2^e \equiv -1 \pmod{p^a}\) and \(2^e \neq -1 \pmod{p^{a+1}}\), equivalently, \(2 \in D_{b_j}^{(p^i)}\) for \(j \leq a\) and \(2 \notin D_{b_j}^{(p^i)}\) for \(j > a\).

Furthermore,

4. If (2) holds, then \(\mathbb{F}_2(\beta_1) = \mathbb{F}_2(\beta_0)\) and \([\mathbb{F}_2(\beta_j) : \mathbb{F}_2(\beta_{j-1})] = p\) for \(j > a\).
5. If (3) holds, then \(\mathbb{F}_2(\beta_1) = \mathbb{F}_2(\beta_0)\) and \([\mathbb{F}_2(\beta_j) : \mathbb{F}_2(\beta_{j-1})] = p\) for \(j > a\).

Proof The equivalence of different descriptions of each condition is easy to get. (4) and (5) can be proved in the same way. We only show (5) here.

Let \(\tau_j\) be the order of \(2\) \(\pmod{p^j}\) and \(\tau = \tau_1\). It is well-known \(\mathbb{F}_2(\beta_j) = \mathbb{F}_2^{\tau_j}\). It suffices to show \(\tau_a = \tau\) and \(\tau_j = \tau p^{j-a}\) for \(j > a\).

On one hand \(\tau_j \mid \tau_{j+1}\). On the other hand, \(2^{\tau_j} \equiv 1 \pmod{p^j}\), then \(2^{\tau_j/p^k} \equiv 1 \pmod{p^{j+k}}\), hence \(\tau_{j+k} \mid \tau_j p^k\). The condition (3) means \(\tau_j\) is a factor of \(2e\) for \(j \leq a\), thus \(\tau_a \mid \gcd(\tau p^{a-1}, 2e) = \tau\), and \(\mathbb{F}_2(\beta_a) = \mathbb{F}_2(\beta_1)\).
Now we have $2^2 \equiv 1 \mod p^a$ and $2^2 \not\equiv 1 \mod p^{a+1}$ (otherwise $2^{2^e} \equiv 1 \mod p^{a+1}$ and $2^e \equiv -1 \mod p^{a+1}$). Write $2^2 = 1 + \lambda p^a$, then $p \nmid \lambda$. For $j > a$,

$$2^{j}p^{j-a-1} = (1 + \lambda p^a)p^{j-a-1} \equiv 1 + \lambda p^{j-1} \not\equiv 1 \pmod{p^j}.$$ 

Hence $\tau_j \nmid \tau_p^{j-a-1}$. Along with $\tau \mid \tau_j \mid \tau_p^{j-a}$, one must have $\tau_j = \tau p^{j-a}$.

\[ \square \]

**Proposition 2** For any $v \in \mathbb{Z}$, we have

1. If $2^e \equiv 1 \pmod{p^j}$, then $A_{j,v} \in \mathbb{F}_2$. If $2^e \not\equiv 1 \pmod{p}$, then $A_{j,v} \not\in \mathbb{F}_2$ for $j \geq 1$.
2. If $2^e \equiv 1 \pmod{p}$ but $2^e \not\equiv 1 \pmod{p^2}$, then $A_{1,v} \in \mathbb{F}_2$ and $A_{j,v} \not\in \mathbb{F}_2$ for $j \geq 2$.
3. If $2^e \equiv -1 \pmod{p}$ but $2^e \not\equiv -1 \pmod{p^2}$, then $A_{1,v} \in \mathbb{F}_4 - \mathbb{F}_2$ and $A_{j,v} \not\in \mathbb{F}_4$ for $j \geq 2$.
4. If $2^e \not\equiv \pm 1 \pmod{p}$, then $A_{j,v} \not\in \mathbb{F}_4$ for any $j \geq 1$.

**Proof** Suppose $2 \in D_h^{(p_j)}$. We may assume $0 \leq h < d_j$.

1. The condition $2^e \equiv 1 \pmod{p^j}$ means $h = 0$. Then Lemma 3(3) implies $A^2_{v} = A_{v}$, hence $A_{v} \in \mathbb{F}_2$.

The condition $2^e \not\equiv 1 \pmod{p}$ means $2 \not\in D_0^{(p)}$, hence $f \nmid h$, there exists $x_1 > 0$ such that $hx_1 \equiv \delta_j \pmod{d_j}$. By Lemma 3(2), we have

$$A_{j,v+hx_1} = A_{j,v+\delta_j} = A_{j,v} + 1.$$

On the other hand, if $A_v \in \mathbb{F}_2$, by Lemma 3(3), for all $n \in \mathbb{Z}$, we have

$$A_{j,v} = A_{j,v+n} = \cdots = A_{j,v+nh} \in \mathbb{F}_2.$$

This is a contradiction.

2. The condition means $2 \in D_0^{(p)}$ but $2 \not\in D_0^{(p^2)}$. That $A_{1,v} \in \mathbb{F}_2$ follows from (1). For $j \geq 2$, the assumption means $\gcd(h, d_j) = d_1 = f$ and hence $\gcd(h, \delta_j) = \delta_1 = f/2$. For $A_{j,v} = A_{j,v} - A_{1,v}$, by Lemma 3(2),

$$A_{j,v} = A_{j,v+\delta_j} = \cdots = A_{j,v+n\delta_j}, \quad n \in \mathbb{Z}.$$

If $A_{j,v} \in \mathbb{F}_2$, then $A_{j,v} = A_{j,v+\delta_j} = \cdots = A_{j,v+nh} \in \mathbb{F}_2$.

Hence $A_{j,v} = A_{j,v+n_1h+n_2\delta_j}$ for any $n_1, n_2 \in \mathbb{Z}$, and $A_{j,v} = A_{j,v+n\delta_1}$ for $n \in \mathbb{Z}$. In particular, $A_{j,v} = A_{j,v+\delta_1}$ and $A_{j,v+f/2}$. By Lemma 5(4), $[\mathbb{F}_2(\beta_j) : \mathbb{F}_2(\beta_{j-1})] = p$ for $j \geq 2$. Then Lemma 4 implies $A_{j,v} \not\in A_{j,v+f/2}$, a contradiction. Hence $A_{j,v} \not\in \mathbb{F}_2$.

If $A_{j,v} \in \mathbb{F}_4 - \mathbb{F}_2$, then $A_{j,v} \in \mathbb{F}_4 - \mathbb{F}_2$, we have $A_{j,v+h} = (A_{j,v})^2 = A_{j,v} + 1$ and $A_{j,v+2h} = A_{j,v}$; and $(A_{j,v-h})^2 = A_{j,v} = (A_{j,v} + 1)^2$. As $A_{j,v-h} = A_{j,v} + 1$ and $A_{j,v-h} = A_{j,v}$. Again we get $A_{j,v} = A_{j,v+n\delta_1}$, which is impossible by Lemma 4.

3. The condition means $2 \in D_0^{(p)}$ but $2 \not\in D_0^{(p^2)}$. Hence

$$A_{j,v} = A_{j,v+\delta_1} = A_{j,v} + 1$$

and $A_{j,v} \in \mathbb{F}_4$. For $j \geq 2$, then $(A_{j,v})^2 = A_{j,v+h}$. If $A_{j,v} \in \mathbb{F}_2$, we have $A_{j,v+h} = A_{j,v}$. If $A_{j,v} \in \mathbb{F}_4 - \mathbb{F}_2$, we have $A_{j,v+2h} = A_{j,v}$. Since by assumption, $\gcd(h, \delta_j) = \gcd(2h, \delta_j) =$
$\delta_1$, we get $A_{j,v}^{[1]} = A_{j,v+n\delta_1}^{[1]}$. By Lemma 5(5), $[F_2(\beta_j) : F_2(\beta_{j-1})] = p$, and by Lemma 4, $A_{j,v}^{[1]} \neq A_{j,v+n\delta_1}^{[1]}$. We get a contradiction.

(4) The condition means $\frac{f}{2} \mid h$, in particular $\frac{f}{2} = 2^{r-1}$ is even and there exists an even integer $x_1 > 0$ such that $hx_1 \equiv \frac{f}{2} \pmod{f}$. If $A_{j,v} \in F_2$, by the proof of (1), we may assume $A_{j,v} = \epsilon_0 \notin F_2$, thus $\epsilon_0^2 + \epsilon_0 + 1 = 0$. By Lemma 3(2),

$$\epsilon_{p^{j-1}h x_1} := A_{j,v+p^{j-1}h x_1} = A_{j,v} + 1 = \epsilon_0 + 1.$$ 

By Lemma 3(3), we have $\epsilon_1 = A_{j,v+h} = \epsilon_0^2 = \epsilon_0 + 1$, $\epsilon_2 = A_{j,v+2h} = \epsilon_1^2 = \epsilon_0$, hence $\epsilon_0 = \epsilon_2 = \cdots = \epsilon_{p^{j-1}hx_1}$. This is a contradiction. \hfill $\square$

**Remark** For the case $2^e \equiv \pm 1 \pmod{p^a}$ but $\delta \neq \pm 1 \pmod{p^{a+1}}$ for $a > 1$, if $j \geq 2a$, we can imitate the proof of Lemma 4 and Proposition 2 (i.e., the method in the proof of [9, Proposition 2]) to show $A_{j,v} \notin F_4$. However, we don’t know how to treat the case $a < j < 2a$.

We are now ready to prove our main results by applying Propositions 1 and 2.

**Proof of Theorem 1** If $2^e \equiv 1 \pmod{p}$ but $2^e \neq 1 \pmod{p^2}$, then $A_{1,v} \in F_2$ and $A_{j,v} \notin F_4$ for $j \geq 2$, in both cases, $s(\beta^a) = 1 \neq 0$. If $2^e \neq \pm 1 \pmod{p}$, then $\delta_1 \mid h$ and $A_{j,v} \notin F_4$, hence $s(\beta^a) \neq 0$. Therefore $\text{LC}(s^{\infty}) = 2p^m$.

If $2^e \equiv -1 \pmod{p}$ but $2^e \neq -1 \pmod{p^2}$, then $A_{1,v} \in F_4 - F_2$ and $A_{j,v} \notin F_4$ for $j \geq 2$. Hence $s(\beta^a) = 0$ for $a \in p^{m-1}\mathbb{Z}_p^*$ and $s(\beta^a) \neq 0$ for all other $a$’s. Hence $2p^m - 2(p - 1) \leq \text{LC}(s^{\infty}) \leq 2p^m - (p - 1)$. \hfill $\square$

**Proof of Theorem 2** If $2^e \neq 1 \pmod{p}$, then $2 \notin D_0^{(p)}$. Hence $A_{j,v} \notin F_2$ for all $j$ and $\tilde{s}(\beta^a) \neq 0$. Therefore $\text{LC}(\tilde{s}^{\infty}) = 2p^m$.

If $2^e \equiv 1 \pmod{p}$ but $2^e \neq 1 \pmod{p^2}$, then only $A_{1,v} \in F_2$ and $\tilde{s}(\beta^a) = 0$ for $a \in p^{m-1}\mathbb{Z}_p^*$. For all other $a$, $\tilde{s}(\beta^a) \neq 0$. Hence $2p^m - 2(p - 1) \leq \text{LC}(\tilde{s}^{\infty}) \leq 2p^m - (p - 1)$. \hfill $\square$

### 4 Numerical evidence

By using Magma, we compute the following examples to check our results.

**Example 1** Let $p = 7$, $m = 2$ and $g = 3$. Take $f = 2$ and $e = 3$, then $2^3 \equiv 1 \pmod{p}$ and $2^3 \neq 1 \pmod{p^2}$. For $b = 0$,

$$s^{\infty} = \hat{1}11110111011100111001000000111111010001101010101010$$

$$0101010101010011101000000011111101110011001000100\hat{0},$$

$$\tilde{s}^{\infty} = \hat{1}101110111001110100001101010100011011111011111111111$$

$$0000000000001101111010101001101110010001001010.$$

Then $\text{LC}(s^{\infty}) = 98 = 2p^m$ and $\text{LC}(\tilde{s}^{\infty}) = 89 = 2p^m - (p - 1) - e$, consistent with Theorems 1(1) and 2(2).

**Example 2** Let $p = 5$, $m = 2$ and $g = 3$. Then $f$ can be taken either 2 or 4.
Table 1. LC($s^\infty$) for $2^e \equiv -1 \pmod{p}$ but $\not\equiv -1 \pmod{p^2}$

| $p$ | $m$ | $e$ | $g$ | $b$ | LC($s^\infty$) | $2p^m - (p - 1)$ |
|-----|-----|-----|-----|-----|-------------|-----------------|
| 5   | 2   | 2   | 3   | 0, 1, 3 | 46          | 46              |
| 11  | 2   | 5   | 7   | 2, 19  | 232         | 232             |
| 13  | 2   | 6   | 7   | 6, 11  | 326         | 326             |
| 17  | 1   | 4   | 3   | 0, 3   | 18          | 18              |
| 5   | 2   | 3   | 0, 2 | 562    | 562         |                 |
| 19  | 2   | 9   | 3   | 1, 6   | 704         | 704             |
| 13  |     |     |     | 3, 22  |             |                 |

(i) If one takes $f = 2$, then $e = 2$, $2^2 \equiv -1 \pmod{p}$ and $2^2 \not\equiv -1 \pmod{p^2}$. For $b = 0$,

$s^\infty = i11111111000100000000001001010100110100001011111111\bar{i}$,

$\tilde{s}^\infty = i10101000110000101101011011011111111000001100101010101\bar{i}$.

Then LC($s^\infty$) = 46 = $2p^m - (p - 1)$ and LC($\tilde{s}^\infty$) = 50 = $2p^m$, consistent with Theorems 1(2) and 2(1).

(ii) If one takes $f = 4$, then $e = 1$, $2 \not\equiv 1 \pmod{p}$. For $b = 0$,

$s^\infty = i1111111111100100010000100001000100011010101001001000000000\bar{0}$,

$\tilde{s}^\infty = i101010001010100110000110001001000110001101101011011101101\bar{0}$.

Then LC($s^\infty$) = LC($\tilde{s}^\infty$) = 50 = $2p^m$, consistent with Theorems 1(1) and 2(1) respectively.

**Example 3** Let $p = 31$, $m = 1$, $g = 3$ and $e = 15$. Then $2^{15} \equiv 1 \pmod{31}$ and $2^{15} \not\equiv 1 \pmod{31^2}$. For $b = 0$,

$s^\infty = i110110111110001010111100010001000101110000100110111001000000\bar{0}$,

$\tilde{s}^\infty = i001101011010010000000100010111100100010001011111111011100010000\bar{0}$.

Then LC($s^\infty$) = 62 = $2p$ and LC($\tilde{s}^\infty$) = 17 = $2p - (p - 1) - e$, consistent with Theorems 1(1) and 2(2).

Because of the above examples, we form our conjecture and try more examples in Table 1.

**5 Conclusion**

In this paper, we introduced two generalized cyclotomic binary sequences of period $2p^m$, which include the sequences in [13,25] as special cases. We computed their linear complexity.
in most cases (all cases for \( p \) a non-Wieferich odd prime) and showed each of our sequences is of high linear complexity if \( m \geq 2 \).

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