HOMOGENEOUS REAL $(2,3,5)$ DISTRIBUTIONS WITH ISOTROPY

TRAVIS WILLSE

Abstract. We classify multiply transitive homogeneous real $(2,3,5)$ distributions up to local diffeomorphism equivalence.

Contents

1. Introduction 1
2. The geometry of $(2,3,5)$ distributions 2
3. Homogeneous distributions 2
4. Classification of multiply transitive homogeneous complex distributions 3
5. Classification of multiply transitive homogeneous real distributions 7
6. Identification algorithms 15
7. Realizations as rolling distributions 16
Appendix A. Classification tables 18
Appendix B. An explicit realization of $g_2$ 20
References 21

1. Introduction

The study of $(2,3,5)$ distributions, that is, tangent $2$-plane distributions $(M,D)$ on $5$-manifolds satisfying the genericity condition $[D,[D,D]] = TM$, dates to Cartan’s so-called “Five Variables Paper” [8]. That article (1) resolved the equivalence problem for these geometries and in doing so revealed a surprising connection between these geometries and the exceptional complex Lie algebra of type $G_2$, and (2) (nearly) locally classified complex $(2,3,5)$ distributions $D$ whose infinitesimal symmetry algebra $\text{aut}(D)$ has dimension at least $6$. Besides its historical significance and the connection with $G_2$, which mediates its relationship with other geometries [17], [22, § 5], this geometry is significant in part because of its appearance in simple, nonholonomic kinematic systems [4, 5], and it has enjoyed heightened attention in the last decade or so, owing in part to its realization in the class of parabolic geometries [6, § 4.3.2], a wide class of Cartan geometries for which many powerful results are available.

In the current article we classify (again, locally) all homogeneous real $(2,3,5)$ distributions with multiply transitive symmetry algebra, so again those for which $\dim \text{aut}(D) \geq 6$. Our motivation for carrying out this classification is twofold: First, it gives a canonical list of examples that additionally have favorable symmetry properties for work in progress about real $(2,3,5)$ distributions. But it is also independently interesting, in part because of the appearance of several distinguished rolling distributions (see § 7).

Our method is straightforward and is analogous to that, e.g., of [13]: Any multiply transitive homogeneous $(2,3,5)$-distribution $(M,D)$ can be encoded in an algebraic model $(\mathfrak{h},\mathfrak{t},\mathfrak{d})$ in the sense that the original distribution can (up to local diffeomorphism equivalence) be recovered and hence specified by the model data. Here, $\mathfrak{h} := \text{aut}(D)$ is the (infinitesimal) symmetry algebra of $D$, $\mathfrak{t}$ is the isotropy subalgebra of a point $u \in M$ (again, by hypothesis $\dim \mathfrak{t} \geq 1$), and $\mathfrak{d} \subset \mathfrak{h}$ is the subspace corresponding to $D_u$. Given any real algebraic model, its complexification $(\mathfrak{h} \otimes \mathbb{C},\mathfrak{t} \otimes \mathbb{C},\mathfrak{d} \otimes \mathbb{C})$ is a complex algebraic model, and conversely the given real algebraic model can be recovered from an appropriate antinvolution $\phi : \mathfrak{h} \otimes \mathbb{C} \to \mathfrak{h} \otimes \mathbb{C}$ admissible in the sense that it preserves the filtration $\mathfrak{t} \otimes \mathbb{C} \subset \mathfrak{d} \otimes \mathbb{C} \subset \mathfrak{h} \otimes \mathbb{C}$.

We thus briefly recall in § 4 (and summarize in Table 1 in Appendix A) the classification of complex $(2,3,5)$ distributions. For each distribution in the classification we give an explicit algebraic model in terms of abstract Lie algebra data. Most of these distributions were identified by Cartan [8], but Doubrov and Govorov found (much) later that Cartan’s list omitted the model we here call $N.6$ [10].
Then, in § 4 for each of the complex algebraic models (\( h, \varphi, \theta \)) we produced in § 3 with \( \dim h = 6 \) we classify the admissible anti-involutions of \( h \) up to a notion of equivalence that corresponds to diffeomorphism equivalence of the homogeneous distributions. Together with the real algebraic model \( O^R \) with maximal symmetry (unique up to equivalence) and the preexisting classification of the models with \( \dim h = 7 \) \cite{13} Theorem 2) (which are here denoted \( N, \mathcal{P}_n \)), this yields the main result of this article:

**Theorem A.** Any multiply transitive homogeneous real \((2, 3, 5)\) distribution is locally equivalent to exactly one distribution in Table 2 modulo the given equivalences of parameters.

In § 5 we give algorithms for identifying, in both the complex and real cases, a multiply transitive homogeneous \((2, 3, 5)\) distribution given in terms of an abstract algebraic model among the distributions in the classification; this amounts to constructing sufficiently many invariants to distinguish all of the models. Finally, in § 7 we identify many of the distributions in the real classification as rolling distributions, that is, 2-plane distributions on 5-manifolds defined on the configuration space of two surfaces rolling along one another by the kinematic no-slip and no-twist conditions, some of which are already well-known.

It is a pleasure to thank Boris Doubrov, both for several helpful conversations and for access to the unpublished notes \cite{11}. It is likewise a pleasure to thank Dennis The for an invaluable exchange about the classification of real forms of a given complex geometric structure. The author gratefully acknowledges support from the Austrian Science Fund (FWF), via project P27072-N25, the Simons Foundation, via grant 346300 for IMPAN, and the Polish Government, via the matching 2015-2019 Polish MNiSW fund.

## 2. The Geometry of \((2, 3, 5)\) Distributions

A \((2, 3, 5)\) distribution is a tangent 2-plane distribution \( D \) on a 5-manifold \( M \) satisfying the genericity condition

\[
[D, [D, D]] = TM. 
\]

Here, for distributions \( E, E' \) on \( M \), \([E, E'] := \bigcup_{u \in M} [E, E']_u \), where \([E, E']_u \subset T_u M \) is the vector subspace \([[X, Y]]_u : X \in \Gamma(E), Y \in \Gamma(E') \)\}. Implicit in the notation \([D, [D, D]] \) is the condition that \([D, D] \) has constant rank; for a \((2, 3, 5)\) distribution, \( \text{rank}[D, D] = 3 \). We will work in both the smooth and complex categories. In both cases we will always assume that \( M \) is connected.

### 2.1. Monge (quasi-)normal form.

One can construct 2-plane distributions via ordinary differential equations of the form

\[
z'(x) = F(x, y, y', y'', z): 
\]

We can prolong any solution \((x, y(x), z(x))\) to a curve \((x, y(x), y'(x), y''(x), z'(x))\) in the (partial) jet space \( J^{2,0}([\mathbb{R}, \mathbb{R}^2] \cong T^5 \) (\( F = \mathbb{R} \) or \( F = \mathbb{C} \)), and by construction any such curve is tangent to the 2-plane distribution \( D_F \subset T^5 \) defined in the respective jet coordinates \((x, y, p, q, z)\) as the common kernel of the canonical jet 1-forms \( dy - p\ dx \) and \( dp - q\ dz \) and the 1-form \( dz - F(x, y, p, q, z)\ dx \) defined by \( \{n\} \). Conversely, the projection of any integral curve of this distribution to which the pullback of \( dz \) vanishes nowhere defines a solution of this o.d.e. The distribution \( D_F \) is spanned by

\[
\partial_y \quad \text{and} \quad \partial_z + p\partial_y + q\partial_y + F(x, y, p, q, z)\partial_z 
\]

—the latter is the total derivative—and computing directly shows that \( D_F \) is a \((2, 3, 5)\) distribution iff \( \partial^2_y F \) vanishes nowhere.

Goursat showed that, in fact, every \((2, 3, 5)\) distribution arises locally this way and hence can be specified by some function \( F \) of five variables. Such an o.d.e. (or, by slight abuse of terminology, the function \( F \) itself) is called a Monge normal form of the distribution.

**Proposition 1.** \cite{10} § 76] Let \((M, D)\) be a \((2, 3, 5)\) distribution and fix a point \( u \in M \). There is a smooth neighborhood \( U \subset M \) of \( u \), a diffeomorphism \( h : U \to h(U) \subset J^{2,0}([\mathbb{R}, \mathbb{R}^2] \) and a smooth function \( F : h(U) \to \mathbb{R} \) for which \( Th \cdot D|_U = D_F \).

### 3. Homogeneous distributions

#### 3.1. Infinitesimal symmetries.

An infinitesimal symmetry of a \((2, 3, 5)\) distribution \((M, D)\) is a vector field \( \xi \in \Gamma(D) \) whose (local) flow preserves \( D \), or equivalently for which \( L_\xi \eta \in \Gamma(D) \) for all \( \eta \in \Gamma(D) \). We denote the set of infinitesimal symmetries, called the (infinitesimal) symmetry algebra by \( \mathfrak{aut}(D) \), and we say that \((M, D)\) is infinitesimally homogeneous if \( \mathfrak{aut}(D) \) is transitive, that is, if \( \{\xi_u : \xi \in \mathfrak{aut}(D)\} = T_u M \) for all \( u \in M \). In this article we are concerned with multiply transitive homogeneous distributions, that is, those for which the isotropy subalgebra \( \mathfrak{t}_u < \mathfrak{aut}(D) \) of infinitesimal symmetries vanishing at any \( u \in M \) is nontrivial, or equivalently, for which \( \dim \mathfrak{aut}(D) \geq 6 \).
3.2. Algebraic models for homogeneous distributions. Fix a homogeneous $(2,3,5)$ (real or complex) distribution $(M, D)$ with transitive symmetry algebra $\mathfrak{g} := \text{aut}(D)$, fix a point $u \in M$, and denote by $\mathfrak{t} \subset \mathfrak{g}$ the subalgebra of vector fields in $\mathfrak{g}$ vanishing at $u$ and by $\mathfrak{d} \subset \mathfrak{g}$ the subspace $\mathfrak{d} := \{ \xi \in \mathfrak{g} : \xi_u \in D_u \}$. Then, $\mathfrak{t} \subset \mathfrak{d}$, $[\mathfrak{t}, \mathfrak{d}] \subset \mathfrak{d}$ (we call this property $\mathfrak{t}$-invariance), and $\dim(\mathfrak{d}/\mathfrak{t}) = 2$. The fact that $D$ is a $(2,3,5)$ distribution implies the genericity condition $\mathfrak{d} + [\mathfrak{d}, \mathfrak{d}] + [\mathfrak{d}, [\mathfrak{d}, \mathfrak{d}]] = \mathfrak{g}$. We call the triple $(\mathfrak{g}, \mathfrak{t}, \mathfrak{d})$ a (real or complex) algebraic model of $(M, D)$.

Given an algebraic model, we can reconstruct $D$ up to local equivalence: For any groups $H, K$ respectively realizing $\mathfrak{g}, \mathfrak{t}$ and with $H > K$, invoke the canonical identification $T_{id, K}H/K \cong \mathfrak{g}/\mathfrak{t}$ to take $D \subset T(H/K)$ to be the distribution with fibers $D_{H, K} = T_{id, K}L_H \cdot (\mathfrak{d}/\mathfrak{t})$, where $L_H : H/K \to H/K$ is the map $L_H : g \cdot K \mapsto (hg) \cdot K$; by $\mathfrak{t}$-invariance this definition is independent of the coset representative, and by genericity $D$ is an $H$-invariant $(2,3,5)$ distribution. Via the above identification, $[\mathfrak{d}, \mathfrak{d}]_{id, K} = (\mathfrak{d} + [\mathfrak{d}, \mathfrak{d}])/\mathfrak{t}$.

We declare two algebraic models $(\mathfrak{g}, \mathfrak{t}, \mathfrak{d}), (\mathfrak{g}', \mathfrak{t}', \mathfrak{d}')$ to be equivalent iff there is a Lie algebra automorphism $\alpha : \mathfrak{g} \to \mathfrak{g}'$ satisfying $\alpha(\mathfrak{t}) = \mathfrak{t}'$ and $\alpha(\mathfrak{d}) = \mathfrak{d}'$. Unwinding definitions shows that equivalent algebraic models determine locally equivalent distributions.

4. Classification of multiply transitive homogeneous complex distributions

In his tour de force “Five Variables” paper \[9\], Cartan showed that for all $(2,3,5)$ distributions $D$, $\dim(\text{aut}(D)) \leq 14$, and that equality holds iff it is locally equivalent to the so-called flat distribution $\Delta$; his argument applies to both the real and complex settings. We call the corresponding (complex) algebraic model $\mathfrak{O}$ (see §3.1.1). In this case, $\text{aut}(D)$ is isomorphic to the simple complex Lie algebra of type $G_2$—we denote it by $\mathfrak{g}_2(\mathbb{C})$—and we say that $D$ is (locally) flat. Cartan furthermore claimed to classify up to local equivalence and (implicitly in the complex setting) all distributions $D$ with $\dim(\text{aut}(D)) \geq 6$\[4\] Doubrov and Goverov found (much) later, however, that Cartan’s classification misses a single distribution up to local equivalence \[10\]; this is model $\mathfrak{N.6}$ (see §3.3). See \[28\] for a short, expository account.

In this section, we summarize the classification of multiply transitive homogeneous complex $(2,3,5)$ distributions $D$ up to local equivalence. For each, we give (1) an explicit algebraic model $(\mathfrak{g}, \mathfrak{t}, \mathfrak{d})$ in terms of abstract Lie algebra data, (2) a complement of $\mathfrak{d}$ in $\mathfrak{g} + [\mathfrak{d}, \mathfrak{d}]$, (3) a Monge normal form function $F$ realizing the distribution, and (4) an explicit isomorphism $\mathfrak{h} \cong \text{aut}(D_F)$.

**Theorem 2.** \[8,10\] Any multiply transitive homogeneous complex $(2,3,5)$ distribution is locally equivalent to exactly one distribution in Table B, modulo the given equivalences of parameters.

We use this list of complex algebraic models to classify the real algebraic distributions in §4.4.

We use the convention that the undecorated Fraktur names $\mathfrak{g}_2, \mathfrak{g}_1, \mathfrak{sl}_m, \mathfrak{so}_m, \mathfrak{sp}_m$ refer to real Lie algebras, and we denote their complexifications by $\mathfrak{g}_1(\mathbb{C}), \mathfrak{sl}_m(\mathbb{C})$ and analogously. By mild abuse of notation, for any element $\mathfrak{v}$ of a real Lie algebra $\mathfrak{g}$ we also denote by $\mathfrak{v}$ the element $\mathfrak{v} \otimes 1 \in \mathfrak{g} \otimes \mathbb{C}$.

**Remark 3.** Our convention for labeling the nonflat distributions $D$ in the classification refers both to the dimension of $\text{aut}(D)$ and to a particular discrete invariant. The fundamental curvature quantity of a $(2,3,5)$ distribution $(M, D)$ is a section $A \in \Gamma(S^2D^* \otimes D)$ \[3,4,6\], and its nonvanishing is a complete local obstruction to local equivalence to the model $\mathfrak{O}$ (§6.4, 9 §33), hence the epithet flat. At each point $u \in M$ the Petrov type (or root type) of $A_u$ is the multiplicities of the roots of $A_u$; if $D$ is real, we instead use the multiplicities of the roots of $A_u \otimes \mathbb{C}$. If $D$ is infinitesimally homogeneous, the Petrov type is the same at all points, and among multiply transitive homogeneous distributions only Petrov types $\mathfrak{D}$ (two double roots), $\mathfrak{N}$ (a quadruple root), and $\mathfrak{O}$ ($A = 0$) occur.

4.1. The flat model $\mathfrak{O}$. Denote by $\mathfrak{g}_2$ the split real form of $\mathfrak{g}_2(\mathbb{C})$, take $q < \mathfrak{g}_2$ to be the (parabolic) subalgebra of elements fixing an isotropic line in the standard representation of $\mathfrak{g}_2$ (cf. \[4\] §4)), and denote by $q_\perp < q$ the orthogonal of $q$ with respect to the Killing form on $\mathfrak{g}_2$. Then, define the subspace

---

1 Conversely, a triple $(\mathfrak{g}, \mathfrak{t}, \mathfrak{d})$, where $\mathfrak{g}$ is a Lie algebra, $\mathfrak{t} \subset \mathfrak{g}$ is a Lie subalgebra, and $\mathfrak{d} \subset \mathfrak{g}$ is a subspace for which $\mathfrak{g} \geq \mathfrak{t}$ and $\dim(\mathfrak{d}/\mathfrak{t}) = 2$, together satisfying the $\mathfrak{t}$-invariance and genericity conditions, determines up to local equivalence a homogeneous distribution via this construction. The symmetry algebra of this distribution may be strictly larger than $\mathfrak{g}$, however; for example, in \[19\] $\dim \mathfrak{g} = 6$, but for the excluded value $\lambda = 9$, the resulting distribution $\Delta$ is flat, so $\dim(\text{aut}(\Delta)) = \dim (\mathfrak{g}_2) = 14$.

2 Cartan’s classification is restricted to distributions for which the Petrov type of the distribution is the same at all points; this condition holds automatically for locally homogeneous distributions. See Remark 3.

3 Cartan’s classification contains a family of distributions with symmetry algebra of dimension 6 whose symmetry algebra is not transitive, namely those in \[3,9,16\] with $J$ nonconstant.

4 Among (not necessarily multiply) transitive distributions, types I (four single roots) and II (a double root and two single roots) also occur, but type III (a triple root and a single root) does not.\[11\]
\[ g_2^{-1} := \{ \xi \in g_2 : [q, \xi] \subseteq q \}. \] The Killing form bracket identity and the Jacobi identity together give \([q(C), g_2(C)^{-1}] \subseteq g_2(C)^{-1}\) (in fact, equality holds), and inspecting the root diagram of \(g_2\)—or just using the explicit realization \(g_2 < g_\ell\) in Appendix [3]—gives that \(\dim(g_2^{-1}/q) = 2\) and \([g_2^{-1}, [g_2^{-1}, g_2^{-1}]]\) is isomorphic to 

\[ (g_2(C), q(C); g_2^{-1}(C)) \]

is an algebraic model, \(O\) of the complex flat distribution. Then, \([g_2^{-1}(C), g_2^{-1}(C)] = g_2^{-2}(C)\), where 

\[ g_2^{-2} := \{ \xi \in g_2 : [q, \xi] \subseteq q^{-1}\} \]

isomorphisms for which equality holds are sometimes called submaximal \[21\]. All submaximal complex distributions are homogeneous, and they fit together in a 1-parameter family, for which several convenient Monge normal forms have been found, including \[8\] \[IX (6)\]

\[
F_I(x, y, p, q, z) := q^2 + \frac{4s}{9}lp^2 + (l^2 + 1)yg^2, \quad I \in \mathbb{C}.
\]

The quantity \(I^2 \in \mathbb{C}\) is a complete invariant.

It is convenient for our purposes to use a generalization of this form studied by Doubrov and Zelenko in the context of control theory: Define \[13\]

\[
F_I, a(x, y, p, q, z) = q^2 + rp^2 + sy^2, \quad r, s \in \mathbb{C},
\]

and denote the distribution it determines by \(D_{r, s}\). Then, \(D_{r, s}\) is locally equivalent to the flat model iff the roots of the polynomial \(t^4 - rt^2 + s\) form an arithmetic sequence, that is, if \(9r^2 = 100s\); otherwise it is submaximal. In \[\S 6.1.1\] we recover the facts that (1) the distributions \(D_{r, s}\) and \(D_{r', s'}\) are locally equivalent iff there is a constant \(c \in \mathbb{C} - \{0\}\) such that \(r' = cr, s' = c^2s\), suggesting the invariant \(J = 4s/r^2\) and (2) \(D_{r, s}\) has Cartan invariant \(I^2 = 9r^2/(100s - 9r^2) = 9(25J - 9)\).

We realize the distributions \(D_{r, s}\) as abstract models as follows. Let \(n_5 = n_5^{-2} \oplus n_5^{-1}\) be the 5-dimensional real Heisenberg algebra endowed with its standard contact grading, and fix a standard basis \((u, s_1, s_2, t_1, t_2)\), so that \([s_1, t_1] = [s_2, t_2] = u\) (and all brackets of basis elements not determined by these identities are zero). Let \(E \in \text{Der}(n_5)\) denote the grading derivation, so that \([E, s_a] = -s_a\) and \([E, t_a] = -t_a\) for \(a = 1, 2\) and \([E, u] = -2u\). For each parameter value \((r, s)\) we choose a derivation \(F \in \text{Der}(n_5(\mathbb{C}))\) commuting with \(E\) and extend \(n_5(\mathbb{C})\) by these derivations to produce the Lie algebras

\[
h_{r, s} := n_5(\mathbb{C}) \ltimes (E, F) = n_5(\mathbb{C}) \ltimes C^2
\]

occurring in the respective abstract models. In each case, we give a Lie algebra isomorphism \(h_{r, s} \cong \text{aut}(D_{r, s})\) identifying 

\[
E \leftrightarrow y\partial_y + p\partial_p + q\partial_q + 2z\partial_z \quad \text{and} \quad F \leftrightarrow \partial_x
\]

and identifying \(u\) with a constant multiple of \(\partial_z\).

Following \[12\] \[\S 3\], choose \(a, b\) so that \(r = a^2 + b^2, s = a^2b^2\). Then, for \(\mu \neq 0, (a, b)\) and \((\mu a, \mu b)\) determine equivalent distributions, so we may specify a distribution by \([a : b] \in P^1\). (The condition that \(D_{r, s}\) is not flat is equivalent to the requirement that \(a \neq \pm 3b\) and \(b \neq \pm 3a\).) It is convenient to split cases according to whether \(s = 0\) (equivalently, whether \(ab = 0\)) and whether the discriminant \(r^2 - 4s\) of the auxiliary polynomial \(t^2 - rt + s\) is zero (equivalently, whether \(b = \pm a\)).

Generic case \((s \neq 0, r^2 - 4s \neq 0)\). Define \(F\) and then the algebraic model \(N, 7_{r, s}\) by 

\[
[F, s_1] = -as_1, \quad [F, s_2] = -bs_2, \quad [F, t_1] = at_1, \quad [F, t_2] = bt_2, \quad [F, u] = 0,
\]

\[
\mathfrak{l}_{r, s} := \langle s_1 + s_2 + \frac{a}{2}t_1 - \frac{a}{2}t_2, E \rangle, \quad \mathfrak{d}_{r, s} := \langle as_1 + bs_2 - t_1 + t_2, F \rangle \oplus \mathfrak{l}_{r, s},
\]

5This is a special case of a much more general construction \[21\] \[\S 3\] related to the realization of (2, 3, 5) distributions as so-called parabolic geometries \[22\] \[4.3.2\]. \[23\].

6As in Footnote 2, the bound in \[8\] is established for distributions with constant Petrov type; this assumption was removed in \[21\].

7Here the factor \(\frac{4}{9}\) corrects a numerical error of Cartan, first identified, to the author's knowledge, in \[26\].

8Alternatively one can choose \(\mathfrak{l}_{r, s}\) and \(\mathfrak{d}_{r, s}\) here in a way more symmetric in the coefficients of the basis elements \(s_i\) and \(t_i\), at the cost of introducing radicals in \(a\) and \(b\); Replace the first element in the definition of \(\mathfrak{l}_{r, s}\) to be \(\sqrt{a}s_1 - \sqrt{a}s_2 + \sqrt{b}t_1 + \sqrt{b}t_2\) and the first element in that of \(\mathfrak{d}_{r, s}\) by \(\sqrt{a}s_1 - \sqrt{a}s_2 - \sqrt{a}t_1 - \sqrt{a}t_2\).
Then, $\mathfrak{d}^C_{r,s} + [\mathfrak{d}^C_{r,s}, \mathfrak{d}^C_{r,s}] = \langle a^2 s_1 + b^2 s_2 + at_1 - bt_2 \rangle \otimes \mathfrak{d}^C_{r,s}$. We can identify $\mathfrak{h}^C_{r,s} \leftrightarrow \mathfrak{aut}(\mathbf{D}_{r,s})$ via

$$u \leftrightarrow 4(a^2 - b^2) \xi(a, b), \quad s_1 \leftrightarrow -\frac{1}{a} \xi(b, a), \quad s_2 \leftrightarrow \frac{1}{b} \xi(a, b), \quad t_1 \leftrightarrow \xi(b, a), \quad t_2 \leftrightarrow \xi(a, b),$$

where $\xi(u, v) := e^{uv}(\partial_y + v \partial_p + v^2 \partial_q + 2v(u^2 y + vp) \partial_z)$. 

**Case** $s = 0$. By relabeling we may assume $b = 0$ and thus $r = a^2$. Define the algebraic model $\mathbf{N}_{7,0}$ by

$$[F, s_1] = as_1, \quad [F, s_2] = 0, \quad [F, t_1] = -at_1, \quad [F, t_2] = as_2, \quad [F, u] = 0,$

$\mathfrak{t}^C_{r,0} := (s_1 - 2t_1 + 2t_2, \mathbf{E}), \quad \mathfrak{d}^C_{r,0} := (s_2 + 2t_1 - t_2, F) \oplus \mathfrak{t}_{r,0}$.

Then, $\mathfrak{d}^C_{r,0} + [\mathfrak{d}^C_{r,0}, \mathfrak{d}^C_{r,0}] = \langle t_2 \rangle \otimes \mathfrak{d}^C_{r,0}$. We can identify $\mathfrak{h}^C_{r,0} \leftrightarrow \mathfrak{aut}(\mathbf{D}_{r,0})$ via

$$u \leftrightarrow 8a^2 \partial_z, \quad s_1 \leftrightarrow -\frac{2}{a} \xi(0, a), \quad s_2 \leftrightarrow -\frac{2}{a} \partial_y, \quad t_1 \leftrightarrow -\frac{1}{a} \xi(0, -a), \quad t_2 \leftrightarrow \sqrt{2a}(x \partial_y + \partial_p + 2a^2 y \partial_z).$$

**Case** $r^2 - 4s = 0$. Equivalently, $s = r^2/4$, or $b = \pm a$ (without loss of generality, we take $b = a$). Define the algebraic model $\mathbf{N}_{7, r^2/4}$ by

$$[F, s_1] = as_1, \quad [F, s_2] = s_1 + as_2, \quad [F, t_1] = -at_1 + t_2, \quad [F, t_2] = -at_2, \quad [F, u] = 0,$

$\mathfrak{t}^C_{r^2/4} := (s_2 + at_1 - t_2, \mathbf{E}), \quad \mathfrak{d}^C_{r^2/4} := (s_2 - 2a^2 t_1 + at_2, F) \oplus \mathfrak{t}_{r^2/4}$.

Then, $\mathfrak{d}^C_{r^2/4} + [\mathfrak{d}^C_{r^2/4}, \mathfrak{d}^C_{r^2/4}] = \langle s_1 + t_2 \rangle \otimes \mathfrak{d}^C_{r^2/4}$. We can identify $\mathfrak{h}^C_{r^2/4} \cong \mathfrak{aut}(\mathbf{D}_{r^2/4})$ via

$$u \leftrightarrow -8a^2 \partial_z, \quad s_1 \leftrightarrow \sqrt{2a}(a, a), \quad s_2 \leftrightarrow \frac{a}{2a} [ax(x - 1) \partial_y + a^2 x \partial_p + a^2 (ax + 1) \partial_q + 2a^2 (a^2 x y + (xp - 2y)a + p) \partial_z], \quad t_1 \leftrightarrow \frac{a}{2a} [-ax(x - 1) \partial_y + a(a - 2x) \partial_q - 2a^2 (a^2 x y + (-xp + y)a + 2p) \partial_z], \quad t_2 \leftrightarrow \frac{1}{a} \xi(a, a).$$

**Remark 4.** One can also realize the submaximal distributions via Monge normal forms $F(x, y, p, q, z) = q^m$, $m \neq 0, 1$, (corresponding to $r = (2m - 1)^2 + 1$, $s = (2m - 1)^2$) and $F(x, y, p, q, z) = \log q$ ($r = 2, s = 1$).

### 4.3. The Doubrov–Govorov model $\mathbf{N}_6$. In [10] Doubrov & Govorov reported finding a homogeneous distribution $\mathbf{D}$ with $\dim \mathfrak{aut}(\mathbf{D}) = 6$ missing from Cartan’s ostensible classification and gave it in terms of a Monge normal form, $F(x, y, p, q, z) := q^{1/3} + y^3$.

We indicate briefly, using this distribution as an example, how to produce an algebraic model from a locally homogeneous distribution given in local coordinates (for example, in Monge normal form). From [10] the symmetry algebra $\mathfrak{h} = \mathfrak{aut}(\mathbf{D}_F)$ has basis

$$\xi_1 := -y \partial_x + p^2 \partial_y + 3pq \partial_q - \frac{1}{a^2} y^2 \partial_z \quad \xi_4 := \partial_z$$

$$\xi_2 := -(x \partial_y + \partial_p + \frac{1}{a^2} x^2 \partial_z) \quad \xi_5 := \partial_y$$

$$\xi_3 := -x \partial_x + y \partial_y + 2p \partial_p + 3q \partial_q \quad \xi_6 := \partial_y + x \partial_z,$$

but we can also compute $\mathfrak{h}$ with the Maple package DifferentialGeometry using the following routine.

```maple
with(DifferentialGeometry): with(GroupActions):
DGsetup([x, y, p, q, z], M);

F := q^(1/3) + y;
Q := D_q;
X := evalDG(D_x + p * D_y + q * D_p + F * D_z);
DF := [Q, X];

InfinitesimalSymmetriesOfGeometricObjectFields([DF], output = "list");
```

The subspace $\langle \xi_1, \xi_2, \xi_3 \rangle$ is a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{C})$, and we may identify $(\xi_1, \xi_2, \xi_3)$ with the complexification of a standard basis $(x, y, h)$ of $\mathfrak{sl}_2$, namely one satisfying $[x, y] = h, [h, x] = 2x, [h, y] = -2y$. The radical $\mathfrak{r}$ of $h$ is isomorphic to the complexification of the 3-dimensional real Heisenberg algebra $\mathfrak{n}_3$, and we may identify the basis $(f_4, f_5, f_6)$ of $\mathfrak{r}$ with the complexification of a standard basis $(u, s, t)$

[9] Other Monge normal forms for this distribution include $F(x, y, p, q, z) = q^{2/3} + p^2$ [11] and $F(x, y, p, q, z) = e^y [1 + e^{-2y/3}(q - \frac{2}{3} p^2)^{1/3}]$; the latter corrects an error in [20] Example 6.7.2.
thereof, namely one satisfying \([s, t] = u\) and \([u, s] = [u, t] = 0\). Computing brackets realizes \(\mathfrak{h}\) as the complexification of the semidirect product \(\mathfrak{sl}_2 \ltimes \mathfrak{n}_3\) specified by the bracket relations

\[
\begin{array}{c|ccc}
\cdot & s & u & t \\
\hline 
x & s & \cdot & \cdot \\
y & \cdot & t & \cdot \\
h & s & -t & \cdot \\
\end{array}
\]

At the basepoint \(u := (0, 0, 0, 1, 0) \in \mathbb{C}^5\), the isotropy subalgebra is \(\mathfrak{k} = \langle \xi_1 \rangle\), and \(\partial_\psi = \frac{1}{2} \xi_3\) and \(D_\lambda = - (\xi_2 - \xi_4 - \xi_5)\), so the resulting algebraic model \(N.6\) is

\[
\mathfrak{h} = \mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{n}_3(\mathbb{C}), \quad \mathfrak{k} = \langle x \rangle, \quad \mathfrak{d} = \langle h, y - u - s \rangle \oplus \mathfrak{k}.
\]

Then, \(\mathfrak{d} + [\mathfrak{d}, \mathfrak{d}] = \langle 2u + 3s \rangle \oplus \mathfrak{d}\).

### 4.4. The models \(D.6_\lambda\) with symmetry algebra \(\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})\).

For \(\lambda \in \mathbb{C} \setminus \{0, \frac{1}{2}, 1, 9\}\) define the models \(D.6_\lambda\) respectively by

\[
\mathfrak{h} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}), \quad \mathfrak{k} = \langle h \oplus h \rangle, \quad \mathfrak{d} := \langle x \oplus -\lambda x, y \oplus -y \rangle \oplus \mathfrak{k}.
\]

Then, \(\mathfrak{d} + [\mathfrak{d}, \mathfrak{d}] = \langle h \oplus \lambda h \rangle \oplus \mathfrak{d}\). These distributions were identified in [8] § 11, § 50.

For all parameter values \(\lambda\), the Lie algebra automorphism of \(\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})\) interchanging the summands is an isomorphism between \(D.6_\lambda\) and \(D.6_{\lambda/\lambda}\): It fixes \(\mathfrak{k}\) and \(\langle y \oplus -y \rangle\), and it maps \(\langle x \oplus -\lambda x \rangle\) to \(\langle -\lambda x \oplus x \rangle = \langle x \oplus -\lambda^{-1} x \rangle\). On the other hand, § 6.2.1 shows that \((\lambda^2 + 1)/\lambda\) is a complete invariant of the algebraic model, so parameter values \(\lambda, \lambda\)' determine the same algebraic model iff \(\lambda' = \lambda\) or \(\lambda' = 1/\lambda\).

These abstract models can be realized via the Monge normal forms \(F(x, y, p, q, z) = y^{3\alpha - 2}q^\alpha\), where \(\lambda = (2\alpha - 1)^2\) (the restriction on \(\lambda\) implies that \(\alpha \neq -1, 0, \frac{1}{2}, \frac{3}{2}, 1, 2\))\(^\text{10}\). We may identify \(\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \leftrightarrow \text{aut}(D_F)\) via

\[
\begin{align*}
x \oplus 0 & \mapsto \partial_x, \\
y \oplus 0 & \mapsto -x^2 \partial_x - xy \partial_y + (xp - y)p + 3xq\partial_q, \\
h \oplus 0 & \mapsto -2x\partial_x - y\partial_y + p\partial_p + 3q\partial_q, \\
0 \oplus x & \mapsto \partial_x, \\
0 \oplus y & \mapsto \frac{1}{2\alpha - 1} [\alpha y^{3\alpha - 1}q^{\alpha - 1} \partial_x + (\alpha y^{3\alpha - 1}pq^{\alpha - 1} - yz)\partial_y + [(\alpha - 1)y^{3\alpha - 1}q^\alpha - p^2]p + (\alpha \partial_x - 1)y^{3\alpha - 1}q^{\alpha - 1} - (2\alpha - 1)^2z\partial_z], \\
0 \oplus h & \mapsto \frac{1}{2\alpha - 1} [y\partial_y + p\partial_p + q\partial_q + 2(2\alpha - 1)z\partial_z],
\end{align*}
\]

### 4.5. The model \(D.6_\infty\) with symmetry algebra \(\mathfrak{so}_2(\mathbb{C}) \oplus (\mathfrak{so}_2(\mathbb{C}) \ltimes \mathbb{C}^2)\).

Realize \(\mathfrak{so}_{1,1} < \mathfrak{gl}_2 \cong \text{End}(\mathbb{R}^2)\) concretely as the Lie subalgebra preserving the bilinear form

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]

Then, the standard action on \(\mathbb{R}^{1,1}\) realizes the affine Lie algebra \(\mathfrak{so}_{1,1} \ltimes \mathbb{R}^{1,1}\): In terms of the standard generator \(z\) of \(\mathfrak{so}_{1,1}\) and the standard basis \((v_1, v_2)\) of \(\mathbb{R}^{1,1}\) it is characterized by

\[
[z, v_1] = v_1, \quad [z, v_2] = -v_2,
\]

and complexifying gives \(\mathfrak{so}_2(\mathbb{C}) \ltimes \mathbb{C}^2\). Define the model \(D.6_\infty\) by taking

\[
\mathfrak{h} = \mathfrak{sl}_2(\mathbb{C}) \oplus (\mathfrak{so}_2(\mathbb{C}) \ltimes \mathbb{C}^2), \quad \mathfrak{k} = \langle h + 2z \rangle, \quad \mathfrak{d} = \langle x + v_1, y + v_2 \rangle \oplus \mathfrak{k},
\]

in which case \(\mathfrak{d} + [\mathfrak{d}, \mathfrak{d}] = \langle \mathfrak{h} \rangle \oplus \mathfrak{d}\). This distribution occurs in [8] § 11 § 51.1.

\(^{10}\) Taking \(\lambda \in \{\frac{1}{2}, 0\}\) gives a flat distribution, and taking \(\lambda \in \{0, 1\}\) gives \(\mathfrak{d}\) that do not satisfy the genericity condition.

\(^{11}\) Other Monge normal forms for this distribution include \(F(x, y, p, q, z) = z^{m\alpha(1-2\alpha)}(q - \frac{1}{2}p^2)^\alpha\) for arbitrary \(m\) [26] Example 6.7.4 and \(F(x, y, p, q, z) = \frac{1}{4}(\lambda - 1)^{-1}p^{-2}(q - 2pz)^2 + z^2\) (this is the special case of [26] Example 6.8.1] with \(m = 0\) and coefficient \(\frac{1}{4}(\lambda - 1)^{-1}\).
One Monge normal form for this distribution is $F(x, y, p, q, z) = p^{-2}q^2$ and we may identify $\mathfrak{so}_2(\mathbb{C}) \circledast (\mathfrak{so}_2(\mathbb{C}) \ltimes \mathbb{C}^3) \leftrightarrow \operatorname{aut}(D_F)$ via

\[
\begin{align*}
x & \mapsto -\left[\frac{3}{2} y^2 \partial_y + y p \partial_p + (y q + p^2) \partial_q + 2p \partial_z\right], & x & \mapsto x \partial_x - p \partial_p - 2q \partial_q - z \partial_z, \\
y & \mapsto 2 \partial_y, & y & \mapsto 2 \partial_z, \\
h & \mapsto 2(y \partial_y + p \partial_p + q \partial_q), & h & \mapsto 2 \partial_x.
\end{align*}
\]

4.6. The model D.6, with complex Euclidean symmetry. Realize $\mathfrak{so}_{1,2} < \mathfrak{gl}_3 \cong \operatorname{End}(\mathbb{R}^3)$ concretely as the Lie algebra of endomorphisms preserving the nondegenerate, symmetric bilinear form

\[
\begin{pmatrix}
\frac{1}{2} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

on $\mathbb{C}^3$. We may realize the isomorphism $\mathfrak{sl}_2 \cong \mathfrak{so}_{1,2} < \mathfrak{gl}_3$ by identifying

\[
\begin{align*}
x & \leftrightarrow \begin{pmatrix} \cdot & \cdot & 2 \\
\cdot & \cdot & -1 \\
\cdot & \cdot & 1
\end{pmatrix}, & y & \leftrightarrow \begin{pmatrix} \cdot & \cdot & -2 \\
\cdot & \cdot & 1 \\
\cdot & \cdot & 0
\end{pmatrix}, & h & \leftrightarrow \begin{pmatrix} 2 & \cdot & \cdot \\
\cdot & -2 & \cdot \\
\cdot & \cdot & \cdot
\end{pmatrix}
\end{align*}
\]

The restriction of the standard action of $\mathfrak{gl}_3$ on $\mathbb{R}^3$ defines a semidirect product $\mathbb{R}^3 \ltimes \mathbb{R}^{6\times 3}$, which is characterized by

\[
\begin{pmatrix}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & 2
\end{pmatrix}
\]

and complexifying realizes $\mathfrak{so}_3(\mathbb{C}) \ltimes \mathbb{C}^3$. Define the model $\text{D.6.}$ by

\[
\mathfrak{h} = \mathfrak{so}_3(\mathbb{C}) \ltimes \mathbb{C}^3, \quad \mathfrak{t} = (x + y), \quad \mathfrak{d} = (h - 2w_3, x - y + 2w_1 - 2w_2) \oplus \mathfrak{t};
\]

in which case $\mathfrak{d} + [\mathfrak{d}, \mathfrak{d}] = (w_1 + w_2) \oplus \mathfrak{d}$. This distribution is described in [8, XI § 52].

One Monge normal form for this distribution is $F(x, y, p, q, z) = q \log q + p^2$ and we may identify $\mathfrak{so}_3(\mathbb{C}) \ltimes \mathbb{C}^3 \leftrightarrow \operatorname{aut}(D_F)$ via

\[
\begin{align*}
x & \mapsto (2y - \log q) \partial_x + [-p(\log q + 1) + z] \partial_y - (p^2 + q) \partial_p - 4pq \partial_q - [(p^2 + q) \log q + 3p^2 - q] \partial_z, \\
y & \mapsto 2[-x \partial_x + \partial_y + p \partial_p + 2q \partial_q + (2p + z) \partial_z], \\
h & \mapsto \mathfrak{w}_1 - \mathfrak{w}_2, \quad \mathfrak{w}_2 \leftrightarrow \partial_x, \quad \mathfrak{w}_3 \leftrightarrow \partial_y.
\end{align*}
\]

5. Classification of multiply transitive homogeneous real distributions.

We now ply the list in § 3 to classify multiply transitive homogeneous real $(2, 3, 5)$ distributions.

5.1. Real forms of complex algebraic models. Given a real algebraic model $(\mathfrak{h}, \mathfrak{t}, \mathfrak{d})$, the triple $(\mathfrak{h} \otimes \mathbb{C}, \mathfrak{t} \otimes \mathbb{C}; \mathfrak{d} \otimes \mathbb{C})$ is a complex algebraic model; we call the latter the complexification of the former.

Conversely, suppose that we have a complex algebraic model $(\mathfrak{h}, \mathfrak{t}, \mathfrak{d})$. Recall that a real form of $\mathfrak{h}$ is the fixed point algebra $\mathfrak{h}^o$ of an antiinvolution $\phi : \mathfrak{h} \to \mathfrak{h}$, that is, a complex-antilinear map satisfying $\phi^2 = \text{id}_\mathfrak{h}$ and $\phi([x, y]) = [\phi(x), \phi(y)]$ for all $x, y, \in \mathfrak{h}$. In analogy to [13, § 3] we call $\phi$ admissible iff it preserves $\mathfrak{t}$ and $\mathfrak{d}$, in which case $(\mathfrak{h}^o, \mathfrak{t}^o; \mathfrak{d}^o)$ is a real algebraic model, where $\mathfrak{t}^o := \mathfrak{t} \cap \mathfrak{h}^o$ and $\mathfrak{d}^o := \mathfrak{d} \cap \mathfrak{h}^o$.

By construction its complexification is $(\mathfrak{h}, \mathfrak{t}, \mathfrak{d})$, so we call $(\mathfrak{h}^o, \mathfrak{t}^o; \mathfrak{d}^o)$ a real form of $(\mathfrak{h}, \mathfrak{t}, \mathfrak{d})$.

We say that two admissible antiinvolutions $\phi, \psi$ are equivalent if $\psi = \alpha \circ \phi \circ \alpha^{-1}$ for some admissible automorphism $\alpha$ of $\mathfrak{h}$, that is, one preserving $\mathfrak{t}$ and $\mathfrak{d}$. Two admissible antiinvolutions are equivalent iff they determine equivalent real algebraic models (and hence locally equivalent real homogeneous distributions), so to classify the latter one can classify the former. Not all $\mathfrak{h}$ admit admissible antiinvolutions, and hence not all complex algebraic models admit real forms.

\[\text{12Other Monge normal forms for this distribution include } F(x, y, p, q, z) = p^{-2}q^2, F(x, y, p, q, z) = \sqrt{q^2} (\text{both given in [11])}, F(x, y, p, q, z) = \left( yz^2 + px - y \right)^{-1} \text{ (this is the special case } \epsilon = 1 \text{ of [26 Example 6.5.3])}, \text{ and } F(x, y, p, q, z) = \left(\frac{1}{2} x^2 - \frac{1}{2} p^2 \right)^{1/2} \text{ (the special case } m = 0, n = -\frac{1}{2} \text{ of [26 Example 6.7.4]).}\]

\[\text{13Another Monge normal form for this distribution is } F(x, y, p, q, z) = \log q + y \text{ [11].}\]
We thus classify the multiply transitive homogeneous real distributions as follows: Up to local equivalence there is only a single real flat distribution and hence only a single real form of $O$. For the submaximal models we appeal to the existing classification [14, Theorem 2] of those distributions.

For the remaining cases, $\dim \mathfrak{h} = 6$. For each such algebraic model $(\mathfrak{h}, \mathfrak{t}, \mathfrak{d})$ in the complex classification (Table XI), we fix a basis $(e_1, \ldots, e_9)$ of $\mathfrak{h}$ adapted to the filtration $\mathfrak{h} \supset \mathfrak{d} + [\mathfrak{d}, \mathfrak{d}] \supset \mathfrak{d} \supset \mathfrak{t}$ in the sense that
\begin{equation}
\mathfrak{t} = \langle e_9 \rangle, \quad \mathfrak{d} = \langle e_4, e_5 \rangle \oplus \mathfrak{t}, \quad \mathfrak{d} + [\mathfrak{d}, \mathfrak{d}] = \langle e_3 \rangle \oplus \mathfrak{d},
\end{equation}
so that with respect to $(e_9)$ any admissible antiinvolution $\phi$ of $\mathfrak{h}$ is commensurately block lower-triangular; in particular, any admissible antiinvolution $\phi$ satisfies $\phi(e_9) = \iota e_6$ with $|\iota| = 1$. Any such $\phi$ also preserves any other subspaces of $\mathfrak{h}$ constructed invariantly from the data $(\mathfrak{h}, \mathfrak{t}, \mathfrak{d})$, and in each case we are able to choose a basis well-adapted to some of these, which restricts further in a convenient way the form of $\phi$ with respect to the basis.

Next, for any automorphism $\alpha : \mathfrak{h} \to \mathfrak{h}$, by definition the constants $\sigma_{ij} := [\alpha(e_i), \alpha(e_j)] - \alpha([e_i, e_j])$ all vanish, and we impose those vanishing conditions to determine the admissible antiinvolutions $\phi$ of the complex algebraic model. After classifying them up to equivalence, we record a representative antiinvolution $\phi$, the corresponding real model $(\mathfrak{h}^0, \mathfrak{t}^0, \mathfrak{d}^0)$, and a complement of $\mathfrak{d}^0$ in $\mathfrak{d}^0 + [\mathfrak{d}^0, \mathfrak{d}^0]$.

5.1.1. **Local coordinate realizations.** For some purposes it is convenient to have local coordinate expressions of distributions. We give such forms for many of the real models in the classification, in some cases via Monge normal forms, and indicate procedures for producing them in others.

- For each complex algebraic model for which the data specifying the model can be interpreted as real, the first real form identified is the one given by interpreting thus, the Monge normal form function $F$ for the complex distribution can be regarded as one for that real model, and the given explicit isomorphism $\mathfrak{h} \leftrightarrow \text{aut}(D_F)$ can be regarded as an identification of the real data. This applies to all models without parameters (all models other than $N.7_{r,s}$ and $D.6_\lambda$) as well as to $N.7_{r,s}$, $r,s \geq 0$, $r^2 \geq 4s$, and $D.6_\lambda$, $\lambda > 0$.
- The submaximal real models, namely, the real forms $N.7_{r,s}$ of the models $N.7_{r,s}$ were classified in [12], and Monge normal forms were recorded there. See §5.3.
- Example 5 later in this section outlines, using the real form $D.6_1^\ast$ of model $D.6_1$ as an example, how to construct local coordinate realizations of a homogeneous distribution.
- Example 6 in §5 applies the algorithm in that section to show that a particular function $F$ defines a Monge normal form for the real form $N.6^\ast$ of model $N.6$.
- Section 7 realizes several of the real models in the classification as rolling distributions, from which one can readily construct coordinate realizations; this is carried out explicitly for the real forms $D.6_1^\ast$ of $D.6_1$, $\lambda > 0$.

5.2. **The flat model.** As in the complex case, up to local equivalence there is a unique flat distribution. Thus all admissible antiinvolutions of $\mathfrak{g}_2(\mathbb{C})$ are equivalent; taking complex conjugation gives the model
\begin{equation*}
(\mathfrak{g}_2, \mathfrak{q}; \mathfrak{g}_2^{-1}).
\end{equation*}
which we denote $\tilde{O}^5$.

The real flat model can be described as the rolling distribution (see §7 below) determined by a pair of spheres, one whose radius is thrice that of the other [11] and also as a canonical distribution determined by the algebra $\tilde{O}$ of split octonions on the null quadric in the projectivization $\mathbb{P}(\text{Im }\tilde{O})$ of the space of purely imaginary split octonions, [11, §6], [23]; see also [3].

5.3. **The submaximal models.** The classification of real submaximally symmetric $(2, 3, 5)$ distributions was established in [14, Theorem 2] using the geometry of curves distinguished by such distributions called abnormal extremals: Any such distribution can be written in the Monge normal form
\begin{equation*}
F_{r,s}(x, y, p, q, z) = q^2 + rp^2 + sy^2
\end{equation*}
for some $r, s \in \mathbb{R}$, and as in the complex case, the distribution $D_{r,s}$ so determined is locally equivalent to the flat model iff $sr^2 = 100s$, which again we henceforth exclude. Otherwise, $D_{r,s}$ and $D_{r',s'}$ are locally equivalent iff there is a constant $c \in \mathbb{R}_+$ such that $r' = cr, s' = c^2s$.

If $(\mathfrak{h}, \mathfrak{t}, \mathfrak{d})$ is a submaximal real algebraic model, then it follows from the algorithm in §6.1.1 that the Cartan invariant $J^2$ of its complexification in real, so reality of that invariant is a necessary condition for

14. Agrachev [11, §1] attributes this characterization to Bryant, who pointed out in a private note to Bor and Montgomery excerpted in the introduction of [4] that it can be derived from a characterization due to Cartan [3, XI §53].
existence of a real form of a complex algebraic model. The form of the equivalence relation implies that the triple \((I^2, \text{sign}(r), \text{sign}(s))\) is a complete invariant of the model.

Recall that the complex models \(N,7_{r,s}\) are specified using constants \(a, b\) such that \(r = a^2 + b^2, s = a^2b^2\) (see, e.g. \(\mathbb{C}\)); this has the advantage that we can give explicit algebraic models for all submaximal models without much case splitting, but \(a, b\) can only be taken to be real if \(r, s \geq 0, r^2 \geq 4s\). Thus, we record explicit real algebraic models

\[
\left(\mathfrak{h}_{r,s}, \mathfrak{t}_{r,s}; \mathfrak{d}_{r,s}\right)
\]

respectively realizing \(D_{r,s}\) for all \((r, s)\) in Table [3] in Appendix [A].

5.4. The real forms of the Doubrov–Govorov model. Take the basis

\[
e_1 := 3t, \quad e_2 := u, \quad e_3 := 2u + 3s, \quad e_4 := y - u - s, \quad e_5 := h, \quad e_6 := x,
\]

which is adapted in that it satisfies [7]. The center of \(\mathfrak{h}\) is \((u) = \langle e_2 \rangle\), so \(\phi(e_2) = \beta e_2\) for some \(\beta\). The radical of \(\mathfrak{r} = (s, t, u) = \langle e_1, e_2, e_3 \rangle \cong \mathfrak{n}_3(\mathbb{C})\), so \(\phi(e_1) = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3\), and \(t \cap (\mathfrak{d} + [\mathfrak{d}, 0]) = \langle e_3 \rangle\), so \(\phi(e_3) = \gamma e_3\). Finally, \(t^\perp \cap \mathfrak{d} = \langle x, h \rangle = \langle e_3, e_6 \rangle\) (here and henceforth, \(\perp\) denotes the orthogonal with respect to the Killing form on \(\mathfrak{h}\), so \(\phi(e_3) = e_5 e_5 + e_6 e_6\), and adaptation of the basis gives that \(\phi(e_3) = \delta e_4 + \epsilon e_5 + \delta e_6\). Nondegeneracy implies that \(\alpha_1, \beta, \gamma, \delta, e_5 \neq 0\).

Now, \(\sigma_{56} = 0\) implies that \(e_5 = 1\), and then \(\sigma_{35} = \sigma_{54} = 0\) implies \(\beta = \gamma = \delta = 1\). Next, \(\sigma_{13} = \sigma_{16} = \sigma_{34} = 0\) implies that \(\alpha_1 = 1, \delta_4 = \zeta = \pm 1\), then \(\sigma_{15} = 0\) implies \(\alpha_3 = -2\alpha_2 = e_6\), and then \(\sigma_{34} = \sigma_{46} = 0\) implies \(\delta_6 = \mp \delta_5, e_6 = \mp 2\delta_5\); in summary,

\[
\phi(e_1) = e_1 \mp \delta e_2 \pm \epsilon e_3, \quad \phi(e_2) = \pm e_2, \quad \phi(e_3) = \pm e_3,
\]

\[
\phi(e_4) = \pm e_4 \mp 3 \epsilon e_5 \pm \delta e_6, \quad \phi(e_5) = e_5 \mp 2 \delta e_6, \quad \phi(e_6) = \pm e_6,
\]

where \(\delta := \delta_5\). Conjugating one automorphism of this form by another preserves the sign \(\pm\), so admissible antiïnvolution with different choices of sign are inequivalent.

Case \(\pm 1 = +1\). Substituting gives that \(\phi^2 = \text{id}\) is equivalent to \(\text{Re} \delta = 0\), and conjugating by the admissible automorphism \((e_1, \ldots, e_6) \mapsto (e_1 - 2ie_2 + te_3, e_2, e_3, e_4 + t^2 e_6, e_5 - 2te_6, e_6)\) induces the transformation \(\delta \sim \delta + 2 \text{Im} t\). Setting \(t = -\frac{1}{2} \delta\) shows that all admissible antiïnvolution in this branch are equivalent to the one with \(\delta = 0\), for which the fixed point algebra \(h^0\) is the real span of \(\{e_1, \ldots, e_6\}\). The real form, which we denote \(N,7^+\), is

\[
\mathfrak{h}^0 = s_{12} \cap n_3, \quad \mathfrak{t}^0 = \langle x \rangle, \quad \mathfrak{d}^0 = \langle h, y - u - s \rangle \oplus t^0.
\]

and \(\mathfrak{d}^0 + [\mathfrak{d}^0, \mathfrak{d}^0] = \langle 2u + 3s \rangle \oplus \mathfrak{d}^0\).

Case \(\pm 1 = -1\). In this case \(\phi\) is an antiïnvolution iff \(\text{Im} \delta = 0\), and conjugating by the admissible automorphism \((e_1, \ldots, e_6) \mapsto (e_1 - 2ie_2 + te_3, -e_2, -e_3, -e_4 - te_5 + t^2 e_6, e_5 - 2te_6, e_6)\) induces \(\delta \sim \delta + 2 \text{Re} t\), so we may normalize to \(\delta = 0\). The fixed point algebra \(h^0\) is spanned over \(\mathbb{R}\) by \(\{e_1, ie_2, ie_3, ie_4, e_5, ie_6\}\), and we may identify

\[
\mathfrak{h}^0 = s_{12} \cap n_3
\]

via

\[
x \leftrightarrow ie_6, \quad y \leftrightarrow -\frac{i}{2}(e_2 + e_3 + 3e_4), \quad h \leftrightarrow e_5, \quad u \leftrightarrow ie_2, \quad s \leftrightarrow \frac{1}{2}i(-2e_2 + e_4), \quad t \leftrightarrow \frac{1}{2}e_1,
\]

where the semidirect product is the same as in the case \(\pm 1 = +1\). The model, which we denote \(N,7^−\), is

\[
\mathfrak{t}^0 = \langle x \rangle, \quad \mathfrak{d}^0 = \langle h, y + u + s \rangle \oplus t^0,
\]

and again \(\mathfrak{d}^0 + [\mathfrak{d}^0, \mathfrak{d}^0] = \langle 2u + 3s \rangle \oplus \mathfrak{d}^0\). In Example [6] we show that the function \(F(x, y, p, q, z) = y^{1/3} - y\) realizes this latter real model in Monge normal form.

5.5. The models with complexified symmetry algebra \(s_{12}(\mathbb{C}) \oplus s_{12}(\mathbb{C})\). This case is the most involved. We will see that (1) in some cases the qualitative features of the real forms (including the isomorphism types of the real forms \(h^0\)) depend on the sign of \(\lambda\), and (2) the case \(\lambda = -1\) is distinguished.

In § 6.2.1 we show that \(\lambda\) is an invariant of the distribution up to inversion, and it is manifestly real for distributions that are complexifications of real distributions. Thus, only the models \(D,6\) with real \(\lambda\) can admit real forms, and we therefore restrict to such \(\lambda\). (Since the expressions defining the models with real \(\lambda\) use only real coefficients, all such models do admit real forms.)
Before proceeding, we recall the fact, used below, that the real forms of $sl_2(C) \oplus sl_2(C) \cong so_4(C)$ are characterized up to isomorphism by the signatures of their Killing forms:\footnote{Here $sl_2(C)_R$ is the real Lie algebra underlying $sl_2(C)$.}

\[
\begin{align*}
(0, 6) & \quad so_3 \oplus so_3 \cong so_4 \\
(2, 4) & \quad sl_2 \oplus so_3 \cong so_4^* \\
(3, 3) & \quad sl_2(C)_R \cong so_{1,3} \\
(4, 2) & \quad sl_1 \oplus sl_2 \cong so_{2,2}.
\end{align*}
\]

We will see that all four forms occur in the classification.

Fix the adapted basis
\[
e_1 := x \oplus -\lambda^2 x, \quad e_2 := y \oplus -\lambda y, \quad e_3 := h \oplus \lambda h, \quad e_4 := x \oplus -\lambda x, \quad e_5 := y \oplus -y, \quad e_6 := h \oplus h,
\]
for which $e := t^1 \cap \mathfrak{d} = \langle e_4,e_5 \rangle$, $[e,c] = \langle e_3 \rangle$, and $[e,[e,c]] = \langle e_1,e_2 \rangle$. Since $\phi$ preserves each of these subspaces, it satisfies
\[
\begin{align*}
\phi(e_1) = \alpha_1 e_1 + \alpha_2 e_2, & \quad \phi(e_2) = \beta_1 e_1 + \beta_2 e_2, & \quad \phi(e_3) = \gamma e_3, \\
\phi(e_4) = \delta_4 e_4 + \delta_5 e_5, & \quad \phi(e_5) = \epsilon_4 e_4 + \epsilon_5 e_5, & \quad \phi(e_6) = \zeta e_6
\end{align*}
\]
for some coefficients, and nondegeneracy implies $\alpha_1 \beta_2 - \alpha_2 \beta_1, \gamma, \delta_5 \epsilon_5 - \delta_5 \epsilon_4 \neq 0$.

Now, $\sigma_{14} = \sigma_{35} = 0$ implies that $\alpha_1 = \gamma \delta_1, \alpha_2 = -\delta_5, \beta_1 = -\gamma e_1, \beta_2 = \gamma e_2$ and $\sigma_{16} = 0$ forces $\zeta = \pm 1$, and then $\sigma_{13} = 0$ implies $\gamma = \pm 1$. Conjugating one automorphism of this form by another shows that the respective signs $\pm, \pm'$ of $\gamma, \zeta$ are invariant under this conjugation, so admissible anti-involutions with different choices of $\pm, \pm'$ are inequivalent.

Forming a suitable linear combination of the $e_3$ and $e_6$ components of $\sigma_{15}$ gives that $(\lambda + 1)(\gamma - \zeta) = 0$, so $\gamma = \zeta$ or $\lambda = -1$. For each subcase, the conditions $\sigma_{16} = \sigma_{26} = 0$ imply the vanishing either of both $\delta_5$ and $e_4$ or of both $d_4$ and $e_5$, then $\sigma_{12} = 0$ implies that one of the two remaining quantities can be written in terms of the other, after which all of the conditions $\sigma_{1j}$ are satisfied.

**Case $\gamma = \zeta$.** We split cases according to the sign of $\gamma = \zeta = \pm 1$.

**Subcase $\gamma = \zeta = +1$.** We have $\delta_5 = e_4 = 0$ and $e_5 = \delta_5^{-1}$, so that $\phi(e_1) = \delta e_1, \phi(e_2) = \delta^{-1} e_2, \phi(e_3) = e_1, \phi(e_4) = \delta e_4, \phi(e_5) = \delta^{-1} e_5, \phi(e_6) = e_6$, where $\delta := \delta_4$, and the condition that $\phi$ is an anti-involution is $|\delta| = 1$. The admissible automorphism $(e_1, \ldots, e_6) \mapsto (\epsilon_1, t^{-1} e_2, \epsilon_3, e_4, t e_5, e_6)$ induces the transformation $\delta \sim \epsilon^2 t^2 |\delta|^2$, so we may normalize to $\delta = 1$, for which $h^0$ is the real span of $\{e_1, \ldots, e_6\}$. The gives a model, $D.6^\lambda_{-2}$:

\[
\begin{align*}
\mathfrak{h}^0 &= sl_2 \oplus sl_2, & \mathfrak{t}^0 &= (h \oplus h), & \mathfrak{d}^0 &= (x \oplus -\lambda x, y \oplus -y) \oplus \mathfrak{t}^0.
\end{align*}
\]

We have $[\mathfrak{d}^0, \mathfrak{d}^0] = (h \oplus \lambda h) \oplus \mathfrak{d}^0$. As in the complex case, the Lie algebra isomorphism exchanging the direct summands $sl_2$ is an isomorphism between $D.6^\lambda_{-2}$ and $D.6^\lambda_{-2}$; complexifying shows that this again exhausts the isomorphisms among these models.

**Subcase $\gamma = \zeta = -1$.** Now, $\delta_4 = e_5 = 0, e_4 = \delta_5^{-1}$, so $\phi(e_1) = \delta e_2, \phi(e_2) = \delta^{-1} e_1, \phi(e_3) = -e_3, \phi(e_4) = \delta e_5, \phi(e_5) = -\delta^{-1} e_4, \phi(e_6) = -e_6$, where $\delta := \delta_5$, and the anti-involution condition is $\Im \delta_5 = 0$. The admissible automorphism $(e_1, \ldots, e_6) \mapsto (ie_2, t^{-1} e_1, -e_3, ie_5, t^{-1} e_4, -ie_6)$ induces $\delta \sim |t|^2 |\delta|$, so we may normalize to $\delta = \pm 1$. The two anti-involutions these values determine give rise to real forms $h^0$ whose Killing forms have different signatures, so they cannot be equivalent.

**Subsubcase $\delta = +1$.** The fixed point algebra $h^0$ is the real span of $\{e_1 + e_2, i(e_1 - e_2), i e_3, e_4 + e_5, i(e_4 - e_5), i e_6\}$.

**Subsubcase $\lambda > 0$.** The signature of the Killing form of $h^0$ is $(4, 2)$, so

\[
h^0 = sl_2 \oplus sl_2.
\]

We can realize this identification via
\[
\begin{align*}
x \oplus 0 & \leftrightarrow \frac{1}{\sqrt{2(\lambda - 1)}} (e_1 + e_2 + ie_3 - \lambda e_4 - \lambda e_5 - i\lambda e_6), \\
y \oplus 0 & \leftrightarrow \frac{1}{\sqrt{2(\lambda - 1)}} ((-e_1 + e_2 - e_3 + \lambda e_4 - \lambda e_5 + \lambda e_6), \\
h \oplus 0 & \leftrightarrow \frac{1}{\sqrt{2\lambda - 1}} ((-1 - i)e_1 + (-1 + i)e_2 - ie_3 + (1 + i)\lambda e_4 + (1 - i)\lambda e_5 + i\lambda e_6), \\
0 \oplus x & \leftrightarrow \frac{1}{\sqrt{2(\lambda - 1)\sqrt{2\lambda}} ((-e_1 - e_2 - i\sqrt{\lambda} e_3 + e_4 + e_5 + i\sqrt{\lambda} e_6), \\
0 \oplus y & \leftrightarrow \frac{1}{\sqrt{2(\lambda - 1)\sqrt{2\lambda}} ((e_1 - e_2 + \sqrt{\lambda} e_3 - e_4 + e_5 - \sqrt{\lambda} e_6), \\
0 \oplus h & \leftrightarrow \frac{1}{\sqrt{2(\lambda - 1)\sqrt{2\lambda}} ((1 + i)e_1 + (1 - i)e_2 + i\sqrt{\lambda} e_3 + (-1 - i)e_4 + (-1 + i)e_5 - \sqrt{\lambda} e_6).
\]


and then
\[ \mathfrak{t}^\phi = (\tilde{h} \oplus h), \quad \mathfrak{d}^\phi = (\tilde{x} \oplus \sqrt{\lambda} x, \tilde{y} \oplus \sqrt{\lambda} y) \oplus \mathfrak{t}^\phi, \]

where
\[ \tilde{x} := \frac{1}{\sqrt{2}} x + \frac{1}{\sqrt{2}} h, \quad \tilde{y} := -\frac{1}{\sqrt{2}} y + \frac{1}{\sqrt{2}} h, \quad \tilde{h} := \frac{1}{\sqrt{2}} (x - y) + \frac{1}{\sqrt{2}} h, \]

and so \( \mathfrak{d}^\phi + [\mathfrak{d}^\phi, \mathfrak{d}^\phi] = (\tilde{h} \oplus \tilde{h}) \oplus \mathfrak{d}^\phi \); we denote this model \( \text{D.6}^{\lambda+}_1 \). Exchanging the direct summands \( \mathfrak{sl}_2 \) is an isomorphism between \( \text{D.6}^{\lambda+}_1 \) and \( \text{D.6}^{\lambda+}_1 \), and this exhausts the isomorphisms among these models.

**Subsubsubcase** \( \lambda < 0 \). The Killing form has signature \( (2, 4) \), so
\[ \mathfrak{h}^\phi = \mathfrak{sl}_2 \oplus \mathfrak{so}_3. \]

We can identify \( x, y, h \in \mathfrak{sl}_2 \) with the elements respectively identified with \( x \oplus 0, y \oplus 0, h \oplus 0 \) in the case \( \lambda > 0 \), and if we denote by \( (a, b, c) \) a standard basis of \( \mathfrak{so}_3(\mathbb{R}) \)—one characterized by \( [a, b] = c \) and its cyclic permutations—we can complete the identification via
\[ a \leftrightarrow \frac{1}{2(\lambda - 1) \sqrt{\lambda}} i(e_1 - e_2 - e_4 + e_5), \quad b \leftrightarrow \frac{1}{2(\lambda - 1) \sqrt{\lambda}} i(e_1 + e_2 - e_4 - e_5), \quad c \leftrightarrow \frac{1}{2(\lambda - 1)} i(-e_3 + e_6), \]
so
\[ \mathfrak{t}^\phi = (\tilde{h} + c), \quad \mathfrak{d}^\phi = (\tilde{x} + \sqrt{-\lambda} a, \tilde{y} + \sqrt{-\lambda} b) \oplus \mathfrak{t}^\phi, \]

and \( \mathfrak{d}^\phi + [\mathfrak{d}^\phi, \mathfrak{d}^\phi] = (\tilde{h} - c) \oplus \mathfrak{d}^\phi \). We denote this model \( \text{D.6}^{\lambda-}_1 \).

**Subsubcase** \( \delta = 1 \). Here, \( \mathfrak{h}^\phi \) is the real span of \( \{e_1 - e_2, i(e_1 + e_2), i(e_3 + e_4), i(e_3 - e_4), i\varepsilon_6\} \).

**Subsubcase** \( \lambda > 0 \). The Killing form of \( \mathfrak{h}^\phi \) is definite, so
\[ \mathfrak{h}^\phi \cong \mathfrak{so}_3 \oplus \mathfrak{so}_3. \]

We can realize the above identification via
\[ a \oplus 0 \leftrightarrow \frac{1}{2(\lambda - 1)} i(e_1 - e_2 - \lambda e_4 + \lambda e_5), \quad 0 \oplus a \leftrightarrow \frac{1}{2(\lambda - 1) \sqrt{\lambda}} (e_1 - e_2 - e_4 + e_5), \]
\[ b \oplus 0 \leftrightarrow \frac{1}{2(\lambda - 1)} i(e_1 + e_2 - \lambda e_4 - \lambda e_5), \quad 0 \oplus b \leftrightarrow \frac{1}{2(\lambda - 1) \sqrt{\lambda}} i(e_1 + e_2 - e_4 - e_5), \]
\[ c \oplus 0 \leftrightarrow \frac{1}{2(\lambda - 1)} i(-e_3 + \lambda e_6), \quad 0 \oplus c \leftrightarrow \frac{1}{2(\lambda - 1)} i(e_3 - e_6), \]
so
\[ \mathfrak{t}^\phi = (c \oplus c), \quad \mathfrak{d}^\phi = (a \oplus -\sqrt{\lambda} a, b \oplus -\sqrt{\lambda} b) \oplus \mathfrak{t}^\phi, \]

and \( \mathfrak{d}^\phi + [\mathfrak{d}^\phi, \mathfrak{d}^\phi] = (c \oplus -c) \oplus \mathfrak{d}^\phi \). We call this model \( \text{D.6}^{\lambda-}_3 \). Exchanging the direct summands \( \mathfrak{so}_3 \) is an isomorphism between \( \text{D.6}^{\lambda-}_1 \) and \( \text{D.6}^{\lambda-}_3 \), and this exhausts the isomorphisms among these models.

**Subsubcase** \( \lambda < 0 \). The Killing form of \( \mathfrak{h}^\phi \) has signature \( (2, 4) \), so
\[ \mathfrak{h}^\phi \cong \mathfrak{so}_3 \oplus \mathfrak{sl}_2, \]

and we can realize this isomorphism by identifying \( a \oplus 0, b \oplus 0, c \oplus 0 \) respectively with the elements identified with \( a, b, c \) in the \( \lambda > 0 \) case, and identifying
\[ 0 \oplus x \leftrightarrow \frac{1}{(\lambda - 1)^{1/2}} i(-e_1 - e_2 - \sqrt{-\lambda} e_3 + e_4 + e_5 + i\sqrt{-\lambda} e_6), \]
\[ 0 \oplus y \leftrightarrow \frac{1}{(\lambda - 1)^{1/2}} 2i(e_1 - e_2 - i\sqrt{-\lambda} e_3 - e_4 + e_5 + i\sqrt{-\lambda} e_6), \]
\[ 0 \oplus h \leftrightarrow \frac{1}{(\lambda - 1)^{1/2}} [(1 + i)e_1 + (1 - i)e_2 - i\sqrt{-\lambda} e_3 + (1 - i)e_4 + (1 + i)e_5 + i\sqrt{-\lambda} e_6]. \]
Then,
\[ \mathfrak{t}^\phi = (c + \tilde{h}), \quad \mathfrak{d}^\phi = (a + \sqrt{-\lambda} x, b + \sqrt{-\lambda} y) \oplus \mathfrak{t}^\phi, \]

and so \( \mathfrak{d}^\phi + [\mathfrak{d}^\phi, \mathfrak{d}^\phi] = (c - \tilde{h}) \oplus \mathfrak{d}^\phi \). The isomorphism \( \mathfrak{sl}_2 \oplus \mathfrak{so}_4 \to \mathfrak{so}_3 \oplus \mathfrak{sl}_2 \) given by reversing the order of the factors identifies the model with parameter value \( \lambda \) with the model \( \text{D.6}^{\lambda}/_1 \), so this branch contributes no new models.

---

16The basis \( (\tilde{x}, \tilde{y}, \tilde{h}) \) of \( \mathfrak{sl}_2 \cong \mathfrak{so}_{1, 2} \) is pseudo-orthonormal with respect to an appropriate multiple of the Killing form.
Case $\gamma \neq \zeta$. We have $\gamma = -\zeta = \mp 1$.

**Subcase $\gamma = -1, \zeta = \mp 1$.** Here, $\delta_5 = \epsilon_4 = 0$, $\epsilon_5 = -\delta_4^{-1}$, so $\phi(\epsilon_1) = -\delta\epsilon_1$, $\phi(\epsilon_2) = \delta^{-1}\epsilon_2$, $\phi(\epsilon_3) = -\epsilon_3$, $\phi(\epsilon_4) = \delta\epsilon_4$, $\phi(\epsilon_5) = -\delta^{-1}\epsilon_5$, $\phi(\epsilon_6) = \epsilon_6$, where $\delta := \delta_4$, and the antiinvolution condition is $|\delta| = 1$. The admissible automorphism $(e_1, \ldots, e_6) \mapsto (-te_1, t^{-1}e_2, e_3, te_4, -t^{-1}e_5, e_6)$ induces $\delta \mapsto t^2\delta/|t|^2$, so we may normalize to $\delta = 1$, for which $\mathfrak{h}^\phi$ is the real span of $\{i e_1, e_2, i e_3, e_4, i e_5, e_6\}$. The Killing form has signature $(3,3)$, so

$$\mathfrak{h}^\phi \cong \mathfrak{so}_{1,3}.$$  

If we fix a basis $(a, b, c, d_a, d_b, d_c)$ of $\mathfrak{so}_{1,3}$ satisfying the identities $[a, b] = c$, $[a, d_b] = -[b, d_a] = d_c$, and $[d_a, d_b] = c$ and their cyclic permutations, as well as $[a, d_a] = [b, d_b] = [c, d_c] = 0$, then we may realize this identification explicitly via

$$a \leftrightarrow \frac{1}{2}(-e_2 + e_4), \quad b \leftrightarrow \frac{1}{2}i(e_1 + e_5), \quad c \leftrightarrow \frac{1}{2}i e_3,$$

$$v_a \leftrightarrow \frac{1}{2}(ie_1 + e_3), \quad v_b \leftrightarrow \frac{1}{2}(e_2 + e_4), \quad v_c \leftrightarrow \frac{1}{2}e_6.$$

Then,

$$\mathfrak{t}^\phi = \langle d_a \rangle, \quad \mathfrak{d}^\phi = \langle a + d_b, b + d_a \rangle \oplus \mathfrak{t}^\phi,$$

and $\mathfrak{d}^\phi + [\mathfrak{d}^\phi, \mathfrak{d}^\phi] = \langle c \rangle \oplus \mathfrak{d}^\phi$. We denote this model $\mathbf{D.6}^{3\pm}$.

**Subcase $\gamma = +1, \zeta = -1$.** Here, $\delta_1 = \epsilon_5 = 0$ and $\epsilon_4 = -\delta_4^{-1}$, so $\phi(\epsilon_1) = -\delta\epsilon_1$, $\phi(\epsilon_2) = \delta^{-1}\epsilon_2$, $\phi(\epsilon_3) = e_3$, $\phi(\epsilon_4) = \delta e_4$, $\phi(\epsilon_5) = -\delta^{-1}\epsilon_5$, $\phi(\epsilon_6) = -\epsilon_6$, where $\delta := \delta_5$, and the antiinvolution condition is $\text{Re} \, \delta = 0$. The admissible automorphism $(e_1, \ldots, e_6) \mapsto (-te_2, t^{-1}e_1, e_3, te_4, -t^{-1}e_5, -e_6)$ induces $\delta \mapsto |t|^2/\delta$, so we may normalize to $\delta = i$, for which $\mathfrak{h}^\phi$ is the real span of $\{e_1 - ie_2, ie_2 - e_1, e_3, e_4 + ie_5, i e_4 + e_5, ie_6\}$. The Killing form again has signature $(3,3)$, so

$$\mathfrak{h}^\phi \cong \mathfrak{so}_{1,3},$$

and we may realize this identification via

$$a \leftrightarrow \frac{1}{2\sqrt{2}}(ie_1 - e_2 + e_4 + ie_5), \quad b \leftrightarrow \frac{1}{2\sqrt{2}}(e_1 + i e_2 + i e_4 + e_5), \quad c \leftrightarrow \frac{1}{2}i e_6,$$

$$v_a \leftrightarrow \frac{1}{2\sqrt{2}}(-ie_1 + e_2 + e_4 + ie_5), \quad v_b \leftrightarrow \frac{1}{2\sqrt{2}}(e_1 - i e_2 + i e_4 + e_5), \quad v_c \leftrightarrow \frac{1}{2}e_6.$$

Then,

$$\mathfrak{t}^\phi = \langle c \rangle, \quad \mathfrak{d}^\phi = \langle a + d_a, b + d_b \rangle \oplus \mathfrak{t}^\phi,$$

and $\mathfrak{d}^\phi + [\mathfrak{d}^\phi, \mathfrak{d}^\phi] = \langle d_c \rangle \oplus \mathfrak{d}^\phi$. We denote this model $\mathbf{D.6}^{1\mp}$.

### 5.6. The models with complexified symmetry algebra $\mathfrak{sl}_2(C) \oplus (\mathfrak{so}_2(C) \times \mathbb{C}^2)$. Fix

$$e_1 := x, \quad e_2 := y, \quad e_3 := h, \quad e_4 := x + v_1, \quad e_5 := y + v_2, \quad e_6 := h + 2z.$$

Proceeding as for the models $\mathbf{D.6}_\lambda$ in § 4.4 gives that $\phi$ has the form $\phi$. Then, $\sigma_{13} = \sigma_{16} = 0$ implies that $\zeta = \pm 1$. Next, $\sigma_{34} = \sigma_{35} = 0$ gives $\alpha_1 = \pm \delta_1, \alpha_2 = \mp \delta_2, \beta_1 = \mp \epsilon_4, \beta_2 = \pm \epsilon_5$. Conjugating any such automorphism by another fixes the sign $\pm$, so antiinvolutions with differing signs $\pm$ cannot be equivalent.

**Case $\zeta = +1$.** In this case, $\sigma_{26} = \sigma_{36} = 0$ implies that $\delta_5 = \epsilon_4 = 0$, and then $\sigma_{15} = 0$ implies that $\delta_5 = \epsilon_5 = \delta_4^{-1}$, so $\phi(\epsilon_1) = \delta_5 \epsilon_1$, $\phi(\epsilon_2) = \delta^{-1}_5 e_2$, $\phi(\epsilon_3) = e_3$, $\phi(\epsilon_4) = \delta_5 e_4$, $\phi(\epsilon_5) = -\delta^{-1}_5 e_5$, $\phi(\epsilon_6) = \epsilon_6$, where $\delta := \delta_4$, and the antiinvolution condition is $|\delta| = 1$. The admissible automorphism $(e_1, \ldots, e_6) \mapsto (te_1, t^{-1}e_2, e_3, te_4, t^{-1}e_5, e_6)$ induces $\delta \mapsto t^2\delta/|t|^2$, so we may normalize to $\delta = 1$, for which $\mathfrak{h}^\phi$ is the real span of $\{e_1, \ldots, e_6\}$. The model, which we denote $\mathbf{D.6}_1^{\infty}$, is

$$\mathfrak{h}^\phi = \mathfrak{sl}_2 \oplus (\mathfrak{so}_{1,1} \times \mathbb{R}^{1,1}), \quad \mathfrak{t}^\phi = (h + 2z), \quad \mathfrak{d}^\phi = (x + v_1, y + v_2) \oplus \mathfrak{t}^\phi,$$

and $\mathfrak{d}^\phi + [\mathfrak{d}^\phi, \mathfrak{d}^\phi] = \langle h \rangle \oplus \mathfrak{d}^\phi$.

**Case $\zeta = -1$.** Proceeding as before in the case $\zeta = +1$ gives $\delta_4 = \epsilon_5 = 0, \epsilon_4 = \delta_5^{-1}$, so that $\phi(\epsilon_1) = \delta_5 \epsilon_1$, $\phi(\epsilon_2) = \delta^{-1}_5 e_2$, $\phi(\epsilon_3) = -e_3$, $\phi(\epsilon_4) = \delta_5 e_4$, $\phi(\epsilon_5) = -\delta^{-1}_5 e_5$, $\phi(\epsilon_6) = -\epsilon_6$, where $\delta := \delta_5$, and the antiinvolution condition is $\text{Im} \, \delta = 0$. The admissible automorphism $(e_1, \ldots, e_6) \mapsto (te_2, t^{-1}e_1, -e_3, te_5, t^{-1}e_4, -e_6)$ induces $\delta \mapsto |t|^2/\delta$, so we may normalize to $\delta = \pm 1$. The two choices of sign determine fixed point algebras whose Killing forms have different signatures, so the resulting algebraic models are inequivalent.
Subcase $\delta = 1$. The real form $\mathfrak{h}^0$ is spanned by $\{e_1 + e_2, i(e_1 - e_2), ie_3, e_4 + e_5, i(e_4 - e_5), ie_6\}$. We may identify

$$\mathfrak{h}^0 = \mathfrak{sl}_2 \oplus (\mathfrak{so}_2 \times \mathbb{R}^2)$$

via

$$x \leftrightarrow \frac{1}{\sqrt{2}}[(1 + i)e_1 + (1 - i)e_2] + \frac{1}{2}ie_3, \quad z_+ \leftrightarrow \frac{1}{2}i(-e_3 + e_6),$$

$$y \leftrightarrow \frac{1}{\sqrt{2}}[(1 + i)e_1 + (1 - i)e_2] - \frac{1}{2}ie_3, \quad v_1 \leftrightarrow e_1 + e_2 - e_4 - e_5,$$

$$h \leftrightarrow \frac{1}{\sqrt{2}}[(-1 + i)e_1 + (-1 - i)e_2], \quad v_2 \leftrightarrow i(e_1 - e_2 - e_4 + e_5).$$

Here we realize $\mathfrak{so}_2$ as the Lie algebra preserving the standard inner product

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

on $\mathbb{R}^2$, and we take $z'$ to be its standard generator, so that its action is given by

$$[z', v_1] = v_2, \quad [z', v_2] = -v_1.$$ 

Then,

$$\mathfrak{t}^0 = (x - y + 2z'), \quad \mathfrak{d}^0 = (x + y - h - \sqrt{2}v_1, x + y + h - \sqrt{2}v_2) \oplus \mathfrak{t}^0,$$

and so $\mathfrak{d}^0 + [\mathfrak{d}^0, \mathfrak{d}^0] = (z') \oplus \mathfrak{t}^0$. We denote the model by $\textbf{D.6}^2_{\infty}$.

Subcase $\delta = -1$. The real form $\mathfrak{h}^0$ is spanned by $\{e_1 - e_2, i(e_1 + e_2), ie_3, e_4 - e_5, i(e_4 + e_5), ie_6\}$. We may identify

$$\mathfrak{h}^0 = \mathfrak{so}_3 \oplus (\mathfrak{so}_2 \times \mathbb{R}^2)$$

via

$$a \leftrightarrow \frac{1}{2}(e_1 - e_2), \quad z' \leftrightarrow \frac{1}{2}i(-e_3 + e_6),$$

$$b \leftrightarrow \frac{1}{2}i(e_1 + e_2), \quad v_1 \leftrightarrow \frac{1}{2}(-e_1 + e_2 + e_4 - e_5),$$

$$c \leftrightarrow \frac{1}{2}ie_3, \quad v_2 \leftrightarrow \frac{1}{2}i(-e_1 - e_2 + e_4 + e_5).$$

Then,

$$\mathfrak{t}^0 = (c + z'), \quad \mathfrak{d}^0 = (a + v_1, b + v_2) \oplus \mathfrak{t}^0,$$

and so $\mathfrak{d}^0 + [\mathfrak{d}^0, \mathfrak{d}^0] = (c) \oplus \mathfrak{t}^0$. We denote the model by $\textbf{D.6}^4_{\infty}$.

5.7. The models with (pseudo-)Euclidean symmetry. The the adapted basis

$$e_1 := -w_1 + w_2, \quad e_2 := w_3, \quad e_3 := w_1 + w_2, \quad e_4 := h - 2w_3, \quad e_5 := x - y + 2w_1 - 2w_2, \quad e_6 := x + y.$$ 

We have $c := \mathfrak{t}^0 \cap \mathfrak{d} = \langle e_4, e_5 \rangle$, $f := \text{rad}(\mathfrak{h}) \cap \langle (0, 0, 0) \rangle = \langle e_3 \rangle$, and $[c, f] = \langle e_1, e_2 \rangle$, so $\phi$ has the form $[\phi].$

Now, $\sigma_{34} \cap \mathfrak{d} = \langle e_4, e_5 \rangle$, $f := \text{rad}(\mathfrak{h}) \cap \langle (0, 0, 0) \rangle = \langle e_3 \rangle$, and $[c, f] = \langle e_1, e_2 \rangle$, so $\phi$ has the form $[\phi].$

Next, $\sigma_{45} \cap \mathfrak{d} = \langle e_4, e_5 \rangle$, $f := \text{rad}(\mathfrak{h}) \cap \langle (0, 0, 0) \rangle = \langle e_3 \rangle$, and $[c, f] = \langle e_1, e_2 \rangle$, so $\phi$ has the form $[\phi].$

Case $\zeta = +1$. The antiïnvolution condition is $\text{Re} \tau = 0$. The admissible automorphism $(e_1, \ldots, e_6) \mapsto (\text{cosh} t e_1 + \sinh t e_2, \sinh t e_1 + \cosh t e_2, e_3, \text{cosh} t e_4 + \sinh t e_5, \sinh t e_4 + \cosh t e_5, e_6)$ induces $\tau \mapsto \tau + 2 \text{Im} t$, so we may normalize to $\tau = 0$, for which $\mathfrak{h}^0$ is the real span of $\{e_1, \ldots, e_6\}$. The corresponding model, which we denote $\textbf{D.6}^1_{\infty}$, is

$$\mathfrak{h}^0 = \mathfrak{sl}_2 \times \mathbb{R}^{1,2}, \quad \mathfrak{t}^0 = \langle x + y \rangle, \quad \mathfrak{d}^0 = \langle h - 2w_3, x - y + 2w_1 - 2w_2 \rangle \oplus \mathfrak{t}^0.$$ 

$\mathfrak{h}^0$ is the affine Lorentzian algebra. Then, $\mathfrak{d}^0 + [\mathfrak{d}^0, \mathfrak{d}^0] = \langle w_1 + w_2 \rangle \oplus \mathfrak{d}^0$.

Case $\zeta = -1$. The antiïnvolution condition is that $\text{Im} \tau = k \pi i$ for some integer $k$. Conjugating by the admissible automorphism $(e_1, \ldots, e_6) \mapsto (\text{cosh} t e_1 + \sinh t e_2, \text{cosh} t e_1 + \sinh t e_2, e_3, \text{cosh} t e_4 - \sinh t e_4, \sinh t e_4 - \cosh t e_5, e_6)$ induces $\tau \mapsto \tau + 2 \text{Re} t$, so we may normalize to $\text{Re} \tau = 0$. 


Subcase $k$ even. In this case, $\mathfrak{h}^\phi$ is spanned by $\{i e_1, i e_2, i e_3, i e_4, i e_5, i e_6\}$. We may identify
\[
\mathfrak{h}^\phi = \mathfrak{sl}_2 \ltimes \mathbb{R}^3
\]
where the semidirect product is given by the same action as in the case $\zeta = +1$ via
\[
x \leftrightarrow i(e_1 + \frac{1}{2} e_5 + \frac{1}{2} e_6), \quad w_1 \leftrightarrow i \left( \frac{1}{2} e_1 - \frac{1}{2} e_3 \right),
\]
\[
y \leftrightarrow i(e_1 + \frac{1}{2} e_5 - \frac{1}{2} e_6), \quad w_2 \leftrightarrow i \left( \frac{1}{2} e_1 + \frac{1}{2} e_3 \right),
\]
\[
h \leftrightarrow 2 e_2 + e_4, \quad w_3 \leftrightarrow -e_2,
\]
in which case
\[
\mathfrak{t}^\phi = (x - y), \quad \mathfrak{d}^\phi = (h + 2w_4, x + y - 2w_1 - 2w_2) \oplus \mathfrak{t}^\phi,
\]
and $\mathfrak{d}^\phi + [\mathfrak{d}^\phi, \mathfrak{d}^\phi] = \langle w_1 - w_2 \rangle \oplus \mathfrak{d}^\phi$. We denote the model by $\textbf{D.6}_1^\phi$.

Subcase $k$ odd. Here, $\mathfrak{h}^\phi$ is the real span of $\{e_1, i e_2, i e_3, i e_4, i e_5, i e_6\}$. We may identify the real form with the real Euclidean algebra,
\[
\mathfrak{h}^\phi = \mathfrak{so}_3 \ltimes \mathbb{R}^3:
\]
Take the standard basis $(w_a, w_b, w_c)$ of $\mathbb{R}^3$, so that the action is characterized by $[a, w_b] = w_c$, its cyclic permutations in $a, b, c$, and $[a, w_a] = [b, w_b] = [c, w_c] = 0$. Then, we may identify
\[
a \leftrightarrow e_1 + \frac{1}{2} e_5, \quad b \leftrightarrow i(e_2 + \frac{1}{2} e_4), \quad c \leftrightarrow -\frac{1}{2} i e_6, \quad w_a \leftrightarrow e_1, \quad w_b \leftrightarrow i e_2, \quad w_c \leftrightarrow i e_3,
\]
in which case
\[
\mathfrak{t}^\phi = (c), \quad \mathfrak{d}^\phi = \langle a - w_a, b - w_b \rangle \oplus \mathfrak{t}^\phi,
\]
and $\mathfrak{d}^\phi + [\mathfrak{d}^\phi, \mathfrak{d}^\phi] = \langle w_c \rangle \oplus \mathfrak{d}^\phi$. We denote the model by $\textbf{D.6}_1^\phi$.

Example 5. We indicate, using $\textbf{D.6}_1^\phi$ as an example, how to realize a multiply transitive homogeneous distribution in local coordinates starting from an algebraic model.

First, choose a group realizing $\mathfrak{h}^\phi$, say,
\[
H := SO(3, \mathbb{R}) \ltimes \mathbb{R}^3 = \left\{ \begin{pmatrix} 1 & 0 \\ w \end{pmatrix} R : R \in SO(3, \mathbb{R}), w \in \mathbb{R}^3 \right\}.
\]
We can parameterize $SO(3, \mathbb{R})$ (and hence $H$) explicitly using Euler angles: Let $R_a(\lambda) \in SO(3, \mathbb{R})$ denote the anticlockwise rotation about the oriented $w_a$-axis through an angle $\lambda$, and define $R_b(\mu), R_c(\nu)$ analogously. Then, for appropriate restrictions on the domain, $R = R_a(\lambda)R_b(\mu)R_c(\nu)$ defines local coordinates $(\lambda, \mu, \nu)$ on $SO(3, \mathbb{R})$ and hence, using standard coordinates on $\mathbb{R}^3$, coordinates $(\lambda, \mu, \nu, r, s, t)$ on $H$. Reading a left-invariant coframe from appropriate components of the Maurer-Cartan form
\[
\left( \begin{array}{cc} 1 & 0 \\ w & R \end{array} \right)^{-1} d \left( \begin{array}{cc} 1 & 0 \\ w & R \end{array} \right) \in \Gamma(T^* SO(3, \mathbb{R}) \otimes \mathfrak{so}_3), \quad w = (r, s, t)^\top,
\]
and forming the dual, left-invariant frame gives the identifications
\[
a \leftrightarrow \sec \mu \cos \nu \partial_\lambda + \tan \mu \cos \nu \partial_\nu,
\]
\[
b \leftrightarrow -\sec \mu \sin \nu \partial_\lambda + \cos \mu \sin \nu \partial_\nu
\]
\[
c \leftrightarrow \partial_\nu
\]
\[
w_a \leftrightarrow \cos \mu \cos \nu \partial_r + (\cos \lambda \sin \nu + \sin \lambda \sin \mu \cos \nu) \partial_s + (\sin \lambda \sin \nu - \cos \lambda \sin \mu \cos \nu) \partial_t
\]
\[
w_b \leftrightarrow -\cos \mu \sin \nu \partial_r + (\cos \lambda \cos \nu - \sin \lambda \sin \mu \sin \nu) \partial_s + (\sin \lambda \cos \nu + \cos \lambda \sin \mu \sin \nu) \partial_t
\]
\[
w_c \leftrightarrow \sin \mu \partial_r - \sin \lambda \cos \mu \partial_s + \cos \mu \cos \lambda \partial_t.
\]
In these coordinates the fibers of $H \to H/K$ are the integral curves of $\partial_\lambda$, so we may use $(\lambda, \mu, \nu, r, s, t)$ as coordinates on the quotient space. Pulling back the 1-forms defining $\mathfrak{d}$ (viewed under this identification as a local distribution on $H$) by a suitable local section and computing the annihilator gives a coordinate expression for the distribution $\mathbf{D}$:
\[
\langle \partial_\lambda - \cos^2 \mu \partial_r - \sin \lambda \sin \mu \cos \partial_s + \cos \lambda \sin \mu \cos \partial_t, \partial_\mu - \cos \lambda \partial_s - \sin \lambda \partial_t \rangle.
\]
Following the procedure in [26, § 2] allows us to put this distribution in preferred forms convenient for other purposes. In coordinates $(x^i)$,
\[
x^1 = \lambda, \quad x^2 = s \cos \lambda + t \sin \lambda + \mu, \quad x^3 = -s \sin \lambda + t \cos \lambda, \quad x^4 = r, \quad x^5 = \tan \mu,
\]
D is the common kernel of the 1-forms
\[-x^3\,dx^1 + dx^2, \quad -f\,dx^1 - x^5\,dx^3 + dx^4, \quad -(\partial_x f)\,dx^1 + dx^3,
\]
where \( f := x^5 \arctan x^5 - x^2x^5 + 1 \); distributions in this form for some function \( f(x^1, \ldots, x^5) \) are said to be in Goursat normal form. For any distribution in that form, changing to coordinates \((x, y, p, q, z)\), where
\[ x = x^1, \quad y = x^2, \quad p = x^3, \quad q = \partial_x f, \quad z = x^4, \]
realizes the distribution in Monge normal form with the function \( F \) given by writing \( x^5\partial_x f - f \) in the coordinates \((x, y, p, q, z)\). In our case, \( \partial_x f = x^5/[(x^5)^2 + 1] + \arctan x^5 - x^2 \), and so one Monge normal form for \( D \) is given by \( F(x, y, p, q, z) = -\cos^2 q^{-1}(y + q) \); here \( q \) is the map \( u \mapsto u + \sin u \cos u \), which, up to composition with appropriate affine transformations, appears in Kepler’s equation and in the standard parameterization of the cycloid.

6. **Identification Algorithms**

We now present an algorithm for identifying the isomorphism type of a given abstract model \((\mathfrak{h}, \mathfrak{f}, \mathfrak{b})\) with an explicit model in the classification; this amounts to generating sufficiently many invariants to distinguish different distributions. We split cases according to \( \dim \mathfrak{h} \); up to isomorphism there is only one model with \( \dim \mathfrak{h} = 14 \) in both the complex and real settings.

6.1. **Models with** \( \dim \mathfrak{h} = 7 \). In this case, in both the complex and real settings, the models are determined up to equivalence by the underlying Lie algebra \( \mathfrak{h} \), so it is enough to distinguish those algebras.

6.1.1. **Complex models.** (Motivated by the discussion in §12) consider the maps \( \text{ad} \mathfrak{v}, \mathfrak{v} \in \mathfrak{h} \). Not all maps \( \text{ad} \mathfrak{v} \) are tracefree, so \( t := \{ \mathfrak{v} \in \mathfrak{h} : \text{tr} \text{ad} \mathfrak{v} = 0 \} \) has dimension 6, but the nilradical \( n < t \) is isomorphic to \( \mathfrak{n}_5(\mathbb{C}) \) and so has dimension 5. Thus, the spectrum of \( \text{ad} \mathfrak{v} \) depends only on the projection of \( \mathfrak{v} \) to the line \( \mathfrak{t}/\mathfrak{n} \), and so the conformal spectrum of \( \text{ad} \mathfrak{v} \) is independent of the choice of \( \mathfrak{v} \in \mathfrak{t} - \mathfrak{n} \). In particular, where \( \sigma_k \) denotes the \( k \)-th symmetric polynomial in the eigenvalues of \( \text{ad} \mathfrak{v} \),

\[
J := \frac{4\sigma_3}{\sigma_2^2}
\]
is an invariant of the conformal spectrum of \( \text{ad} \mathfrak{v} \) and hence an invariant of \( \mathfrak{h} \); here \( J \) is normalized to coincide with the invariant of the same name appearing in [20].

For the Lie algebras \( \mathfrak{h}_{r,s}^C = \mathfrak{n}_5(\mathbb{C}) \times \langle E, F \rangle \) in the abstract models, the nilradical is \( \mathfrak{n}_5(\mathbb{C}) \) and \( t = \mathfrak{n}_5(\mathbb{C}) \oplus \langle F \rangle \), so we need only compute the spectrum of \( \text{ad} F \). Consulting the formulae in §4.2 gives that the spectrum is \((+a, -a, +b, -b, 0, 0, 0)\), so \( \sigma_2 = -(a^2 + b^2) = -r \) and \( \sigma_4 = a^2b^2 = s \), and thus \( J = 4a^2b^2/(a^2 + b^2)^2 = 4s/r^2 \). Specializing to Cartan’s Monge normal form (\( r = \frac{10}{21} l, \; s = l^2 + 1 \)) gives \( J = 9(12^2 + 1)/(252)^2 \); thus, since \( 252^4 \) is a complete invariant, so is \( J \).

6.1.2. **Real models.** From §5.3 every submaximal complex model that admits a real form admits precisely two. So, to identify a submaximal real abstract model \((\mathfrak{h}, \mathfrak{f}, \mathfrak{b})\), we determine the complexification \((\mathfrak{h} \otimes \mathbb{C}, \mathfrak{f} \otimes \mathbb{C}; \mathfrak{b} \otimes \mathbb{C})\) and then determine which of the two real forms one started with.

If we define \( \mathfrak{t}, \mathfrak{n} \) as in the complex case, then replacing a choice \( \mathfrak{v}_0 \in \mathfrak{t} - \mathfrak{n} \) with \( \mu \mathfrak{v}_0 + \mathfrak{n}, \mu \in \mathbb{R}^*, \mathfrak{n} \in \mathfrak{n} \), respectively replaces \( \sigma_k \) by \( \mu^k \sigma_k \), so the signs of \( \sigma_2, \sigma_4 \) are independent of the choice of \( \mathfrak{v}_0 \). For real pairs \((r, s)\), the formulae derived in the complex case give for \( \mathfrak{h}_{r,s} \) and the choice \( \mathfrak{v}_0 = F \) that \( \text{sign}(\sigma_2) = -\text{sign}(r) \) and \( \text{sign}(\sigma_4) = \text{sign}(s) \), so for all \((r, s)\) the pair \((\text{sign}(\sigma_2), \text{sign}(\sigma_4))\) distinguishes between the real forms.

6.2. **Models with** \( \dim \mathfrak{h} = 6 \).

6.2.1. **Complex models.** The isomorphism type of \( \mathfrak{h} \) can be determined by examining the radical, \( \mathfrak{r} \): Consulting Table 11 we see that if \( \mathfrak{r} \neq \{0\} \), then \( \dim \mathfrak{r} = 3 \) and the derived algebra \([\mathfrak{r}, \mathfrak{r}]\) has dimension 0, 1, or 2, respectively, if \( \mathfrak{h} \) is isomorphic to \( \mathfrak{so}_3 \times \mathbb{C}^1, \mathfrak{so}_3 \times \mathfrak{m}_3 \), or \( \mathfrak{so}_3 \oplus (\mathfrak{so}_2 \times \mathbb{C}^2) \), in which case the model is isomorphic, respectively, to \( \mathbf{D}_6, \mathbf{N}_6 \), or \( \mathbf{D}_6^\infty \).

If instead \( \mathfrak{r} = \{0\} \), then \( \mathfrak{h} \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \), and the model is isomorphic to \( \mathbf{D}_6 \), for some parameter \( \lambda \) (which, recall, is only defined up to inversion, \( \lambda \mapsto 1/\lambda \)). Let \( \pi_1, \pi_2 : \mathfrak{h} \to \mathfrak{sl}_2(\mathbb{C}) \) denote the projections onto the two summands, and define \( \varepsilon := \partial \cap \mathfrak{r}^\perp \). Then, if \( Q \) is the quadratic form on \( \mathfrak{sl}_2(\mathbb{C}) \) induced by the Killing form, the restriction of the bilinear form \( \pi_2^*Q \) to \( \varepsilon \) is some constant multiple of the restriction of \( \pi_1^*Q \) to \( \varepsilon \). Reversing the assignment of the indices 1, 2 to the summands \( \mathfrak{sl}_2(\mathbb{C}) \) replaces the constant with its reciprocal, so up to this inversion the constant is an invariant of the algebraic model.
For the models $D,6_A$, we have $\mathcal{e} = (x \oplus -\lambda x, y \oplus -y)$, so for $\mathcal{v} := u(x \oplus -\lambda x) + v(y \oplus -y) \in \mathcal{e}$, we have $(\pi^*_1Q)(\mathcal{v}) = Q(u\lambda x + vy) = 8uv$ and $(\pi^*_2Q)(\mathcal{v}) = Q(-u\lambda x - vy) = 8uv$, and so the constant is just $\lambda$. The quantity

$$\begin{vmatrix} \lambda^2 + 1 \\ -\lambda \end{vmatrix}$$

is invariant under inversion in $\lambda$ and so is a (complete) invariant of the model.

6.2.2. **Real models.** As before, we first classify the complexification of the model, which reduces the real classification to distinguishing among the finitely many real forms thereof.

**N.6** We construct a basis $(f_\alpha)$ of $\mathfrak{h}$ adapted to the model: Fix $f_0 \in \mathfrak{k} - \{0\}$. With respect to the Killing form of $\mathfrak{h}$, $\mathfrak{k}$ is isotropic and $\mathfrak{d}$ is nondegenerate, so $\text{dim}(\mathfrak{k}^+ \cap \mathfrak{d}) = 2$. Pick an element $f_5$ in this intersection so that $[f_5, f_0] = 2f_0$, and choose $f_4$ so that $(f_4, f_5, f_0)$ is a basis of $\mathfrak{d}$.

As a vector space (not as a Lie algebra) $\mathfrak{h}$ decomposes as a direct sum $\mathfrak{rad}(\mathfrak{h}) \oplus \mathfrak{d}$. Denote by $f_3$ the projection of $[f_4, f_5]$ onto the first summand, and set $f_1 := [f_3, f_4]$ and $f_2 := [f_3, f_5] + f_3$. Now, $[f_1, f_3]$ is a multiple of $f_3$; this multiple is independent of the choices of $f_5$ and $f_4$, and scaling $f_6$ by a nonzero factor $\alpha$ scales the multiple by a factor $\alpha^2$, so the sign of the multiple is an invariant of the model. If the sign is $\pm$, the model is isomorphic to $N.6^\pm$.

**D.6_\lambda** The isomorphism type of $\mathbf{h}$ can be distinguished by its Killing form $\kappa$ (see §5.6). If $\mathbf{h} \cong \mathfrak{s}o_2 \oplus \mathfrak{s}l_2$ or $\mathbf{h} \cong \mathfrak{s}o_3 \oplus \mathfrak{s}l_1$, then the model is isomorphic to $D.6^1_\lambda$ or $D.6^6_\lambda$, respectively. In both remaining cases, $\mathbf{h} \cong \mathfrak{s}l_2 \oplus \mathfrak{s}l_2$ and $\mathbf{h} \cong \mathfrak{s}o_1 \oplus \mathfrak{s}l_3$, there are two real forms, and they can be distinguished by the restriction of $\kappa$ to $\mathfrak{k}$.

**D.6_\infty** The three possibilities are distinguished by the isomorphism type of $\mathfrak{h}$, or just as well by the signature of the Killing form of $\mathfrak{h}$. For each model $D.6^\infty_\lambda$ the signature is $(4 - s, s)$.

**D.6**. If the Killing form of $\mathbf{h}$ has signature $(0, 3)$, then the model is isomorphic to $D.6^1_\lambda$. Otherwise, $\mathbf{h} \cong \mathfrak{s}l_2 \times \mathbb{R}^3$, in which case the two possibilities are again distinguishable via restriction of the Killing form to $\mathfrak{k}$.

*Example 6.* Consider the real distribution given in Monge normal form by the function $F(x, y, p, q, z) := q^{1/3} - y$. Replacing the line $\mathbb{F} := \mathbb{F}^{*}(1 / 3) + y$; in the program in §4.3 with $\mathbb{F} := \mathbb{F}^{*}(1 / 3) - y$; and executing gives that $\mathfrak{h} := \text{aut}(D_F)$ has basis $(\xi_1)$, where $\xi_1 := -y\partial_x + p^2\partial_p + 3pq\partial_q + \frac{1}{2}y^2\partial_z$, $\xi_2 := -(x\partial_y + \partial_z - \frac{1}{2}x^2\partial_x)$, $\xi_3 := \xi_4$, $\xi_4 := \xi_5$, $\xi_5 := -\partial_y + x\partial_z$, where the $\xi_i$ were defined in §5; in particular, the distribution is locally homogeneous. The radical of $\mathfrak{h}$ is $(\xi_4, \xi_5, \xi_1)$, and its derived subalgebra $(\xi_5^*)$ has dimension 1, so the distribution is (locally equivalent to) a real form of N.6.

At the base point $(0, 0, 0, 1, 0)$, the isotropy subalgebra is $(\xi_1, \xi_5)$, so we may take $f_6 := \xi_1$, and $\mathfrak{d} = (\xi_1^*, \xi_2^* - \xi_4^*, \xi_4^*)$. Since $\kappa(f_0, f_6) = 0$ and $(\xi_1^*, f_0) = 2f_0$ we may take $f_0 := \xi_1$ and $f_3 = 2(\xi_1^* - \xi_4^* - \xi_2^*)$, and this determines $f_2 = 4\xi_1^* + 6\xi_5^*$, $f_1 = 12\xi_2^*$, and $f_3 = 4\xi_3^*$. Then, $[f_1, f_3] = -72\xi_4^* - 18f_2$, and this last coefficient is negative, so the model is isomorphic to $N.6^\pm$.

7. Realizations as rolling distributions

An important subclass of $(2, 3, 5)$ distributions are the so-called rolling distributions, which are defined kinematically by a system of two Riemannian surfaces $(\Sigma_1, g_1), (\Sigma_2, g_2)$ rolling along one another. Following §5, define the (5-dimensional) configuration space $C$ to be the set of triples $(x_1, x_2, \Phi)$ for which $\mathbf{x}_1 \in \Sigma_1, \mathbf{x}_2 \in \Sigma_2$ (so that $\mathbf{x}_1, \mathbf{x}_2$ are the points of tangency on the surfaces), and $\Phi : T_x\Sigma_1 \rightarrow T_x\Sigma_2$ is an isometry (encoding the relative rotation of $\Sigma_1, \Sigma_2$ about the point contact), and the projection $(x_1, x_2, \Phi) \mapsto (x_1, x_2)$ realizes $C$ as a principal $O(2)$-bundle $C \rightarrow \Sigma_1 \times \Sigma_2$. A smooth path $\gamma : I \rightarrow C, \gamma(t) = (x_1(t), x_2(t), \Phi(t))$, in this space encodes a rolling trajectory of the two surfaces along one another. The physical no-slip condition is that the point of contact moves with the same relative motion on each surface, that is, $x_2'(t) = \Phi(t) \cdot x_1'(t)$. The no-twist condition is that for any (equivalently every) parallel orthonormal frame field $(E_a(t))$ along $x_1(t)$, the frame field $(\Phi(t) \cdot E_a(t))$ along $x_2(t)$ must also be parallel. These conditions together impose three independent linear constraints on each tangent space $T(x_1, x_2, \Phi)C$, and so the space of admissible velocities $\gamma'$ is a 2-plane distribution $D$ on $C$. Direct computation shows that the restriction of $D$ to the open set of points $(x_1, x_2, \Phi) \in C$ where the respective Gaussian curvatures $\kappa_i(x_i)$ of $(\Sigma_i, g_i)$ at $x_i, i = 1, 2$, are unequal is a $(2, 3, 5)$ distribution. In this case, the 3-plane distribution $[D, D]$ is the space of velocities satisfying just the no-slip condition.
It is immediate that a pair of surfaces $\Sigma_1, \Sigma_2$ with constant (and unequal) Gaussian curvature give rise to a $(2, 3, 5)$ distribution $(C, D)$ with $\dim \mathfrak{aut}(D) \geq 6$: Since both metrics $g_i$ have constant curvature, the respective algebras $\mathfrak{aut}(g_i)$ of Killing fields both have dimension 3 and by functoriality lift to infinitesimal symmetries of $(C, D)$, which thus contains an isomorphic copy of $\mathfrak{aut}(g_1) \oplus \mathfrak{aut}(g_2)$. For any such surfaces and any positive constant $c$, the pair $(\Sigma_1, cg_1), (\Sigma_2, cg_2)$ of surfaces give rise to the same distribution $D$ as $(\Sigma_1, g_1), (\Sigma_2, g_2)$.

Many distributions occurring in the real classification are realizable this way.

$O^R$ (As mentioned in § 5.2) take two spheres whose radii have ratio $3 : 1$, or equivalently whose curvatures have ratio $9 : 1$.

$D.6^3_\lambda$ Take two 2-spheres normalized to have ratio $\kappa_1/\kappa_2 = \lambda > 0$, of Gaussian curvatures.

$D.6^2_\lambda$ Take a 2-sphere and a hyperbolic plane normalized to have Gaussian curvature ratio $\lambda < 0$.

$D.6^2_+ \lambda$ Take two hyperbolic planes normalized to have Gaussian curvature ratio $\lambda > 0$.

$D.6^2_\infty$ Take a 2-sphere and the Euclidean plane.

$D.6^2_\infty$ Take a hyperbolic plane and the Euclidean plane.

Up to the joint scaling of pairs of surfaces and local equivalence, this exhausts all of the pairs of constant curvature Riemannian surfaces of unequal curvature.

We can realize some of the remaining real forms with $\dim \mathfrak{h} = 6$ by extending our attention to rolling distributions generated by pairs of Lorentzian surfaces: We proceed as before, but instead take $(\Sigma_1, g_1), (\Sigma_2, g_2)$ to be Lorentzian surfaces, in which case $C \rightarrow \Sigma_1 \times \Sigma_2$ is a principal $O(1, 1)$-bundle.

The flat (zero-curvature) model of Lorentzian surfaces is the Lorentzian plane $\mathbb{R}^{1,1}$, the affine real plane equipped with the flat Lorentzian metric $\delta_{1,1}$ (see § 4.5). A Lorentzian space of negative constant scalar curvature is locally isometric to 2-dimensional anti de Sitter space: Consider 3-dimensional Lorentzian space $(\mathbb{R}^{1,2}, \delta_{1,2})$ (see § 4.5). Then, define $(AdS_2, g_{AdS})$ to be the hypersurface $\{w \in \mathbb{R}^{1,2} : g_{1,2}(w, w) = -\alpha^2 \}$ equipped with the (Lorentzian) pullback metric $g_{AdS}$; it has scalar curvature $-\alpha^2$. A Lorentzian space of constant positive scalar curvature is locally isometric to 2-dimensional de Sitter space, namely, $AdS_2$ equipped with the Lorentzian metric $g_{AdS} := -g_{AdS}$, which has scalar curvature $\alpha^2$. The algebras $\mathfrak{aut}(g_{AdS})$ and $\mathfrak{aut}(g_{AdS})$ of Killing fields are both isomorphic to $\mathfrak{so}_2 \cong \mathfrak{so}_{1,2}$.

With these objects in hand, we can realize the following distributions as follows (and in the first case, with the same restrictions on $\lambda$ as before):

$D.6^2_\lambda$ If $\lambda > 0$, take two copies of de Sitter space $dS_2$ (or, just as well, two copies of anti-de Sitter space $AdS_2$) with ratio $\kappa_1/\kappa_2 = \lambda$ of curvatures. If $\lambda < 0$, take a copy of de Sitter space and a copy of anti de Sitter space normalized to have curvature ratio $\lambda$. For $\lambda \neq -1$ there are two geometrically distinct possibilities: One for which the copy of de Sitter space has scalar curvature larger than the negative of the scalar curvature of the copy of anti-de Sitter space, and one for which the reverse is true.

$D.6^1_\infty$ Take de Sitter (or anti-de Sitter) space and the Lorentzian plane.

Example 7 (Two spheres). Realize the case of two spheres with curvature ratio $\lambda$ with spheres $\Sigma_1, \Sigma_2$ with radii $\sqrt{\lambda}, 1$. Writing the round metrics as $\lambda^{-1}[da \otimes da + (\sin \alpha) d\beta \otimes d\beta]$ and $d\tau \otimes d\tau + (\sin \gamma) d\zeta \otimes d\zeta$, taking respective orthonormal frames $(\sqrt{\lambda} \partial_\alpha, \sqrt{\lambda} \csc \alpha \partial_\beta)$ and $(\partial_\gamma, \csc \gamma \partial_\zeta)$, and applying the mentioned procedure gives the simple coordinate realization

\[
\sqrt{\lambda} \partial_\alpha + \cos \varphi \partial_\gamma + \csc \gamma \sin \varphi \partial_\zeta - \cot \gamma \sin \varphi \partial_\varphi, \quad \sqrt{\lambda} \csc \alpha \partial_\beta - \sin \varphi \partial_\varphi + \csc \gamma \cos \varphi \partial_\zeta + (\sqrt{\lambda} \cot \alpha - \cot \gamma \cos \varphi) \partial_\varphi
\]

of $D$; here, $\varphi$ is a standard coordinate on the fibers of $C \rightarrow \Sigma_1 \times \Sigma_2$.

Remark 8. Any two constant curvature surfaces (of the same signature) whose curvatures have ratio 9 : 1 determine a locally rolling distribution. Up to replacing both metrics with their negatives, the possibilities are: two spheres, two copies of the hyperbolic plane, and two copies of de Sitter space.

Remark 9. This poses a natural question: Which of the other distributions in the classification (namely, $N.7^7_{e,s}$, the real forms of $N.6$ and $D.6_+^4$, and $D.6_{11}^4$) are realizable as rolling distributions?

\footnote{This corrects a misstatement in [25], which asserts that the rolling distribution determined by these surfaces has symmetry algebra $\mathfrak{so}_3 \ltimes \mathbb{R}^3$, rather than $\mathfrak{so}_3 \oplus (\mathfrak{so}_2 \ltimes \mathbb{R}^2)$.}
### Table 1. Classification of multiply transitive homogeneous complex $(2,3,5)$ distributions

| dim $h$ | model $h$ | $\mathfrak{g}_2(\mathbb{C})$ | $\mathfrak{q}(\mathbb{C})$ | $\mathfrak{g}_2(\mathbb{C})^{-1}$ | parameters | equivalences | Monge $F$ | § |
|---|---|---|---|---|---|---|---|---|---|
| 14 | $O$ | $\mathfrak{g}_2(\mathbb{C})$ | $\mathfrak{q}(\mathbb{C})$ | $\mathfrak{g}_2(\mathbb{C})^{-1}$ | $(r,s) \in \mathbb{C}^2$ | $(r,s) \sim (cr,c^2s)$ | $c \in \mathbb{C} - \{0\}$ | $q^2$ | 1.1 |
| 7 | $N.7_{r,s}$ | $\mathfrak{g}_{r,s}^C$ | $\mathfrak{t}_{r,s}^C$ | $\mathfrak{b}_{r,s}^C$ | $\mathfrak{d}_{r,s}^C$ | $(r,s) \in \mathbb{C}^2$ | $7r^2 \neq 100$ | $q^2 + rp^2 + sy^2$ | 1.2 |

- **N.6** $\mathfrak{sl}_2(\mathbb{C}) \ltimes n_3(\mathbb{C})$  
  $(x)$  
  $(h,y-u-s) \oplus t$  
  $q^{1/3} + y$ | 1.3 |

- **D.6** $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$  
  $(h \oplus h)$  
  $(x \oplus -\lambda x, y \oplus -y) \oplus t$  
  $\lambda \in \mathbb{C} - \{0, \frac{1}{2}, 1, 9\}$  
  $\lambda \sim 1/\lambda$  
  $y^{3\alpha - 2}q^\alpha$ | 1.4 |

- **D.6** $\mathfrak{sl}_2(\mathbb{C}) \ltimes (\mathfrak{so}_2(\mathbb{C}) \ltimes \mathbb{C}^2)$  
  $(x+y-2z)$  
  $(h+y-h-\sqrt{2}v_1,x+y+h-\sqrt{2}v_2) \oplus t$  
  $p^{-1}q^2$ | 1.5 |

- **D.6** $\mathfrak{so}_3(\mathbb{C}) \ltimes \mathbb{C}^3$  
  $(x+y)$  
  $(h-2w_3,x-y+2w_1-2w_2) \oplus t$  
  $p^2 + q \log q$ | 1.6 |

| $h_{r,s}^C$ | $u$ | $s_1$ | $s_2$ | $t_1$ | $t_2$ | $E$ | $F$ | $s_{r,s}^C, b_{r,s}^C, \mathfrak{d}_{r,s}^C$: See § 1.2 |
|---|---|---|---|---|---|---|---|---|
| $s_{l,s}^C$ | $x$ | $y$ | $h$ | $n_3(\mathbb{C})$ | $u$ | $s$ | $t$ | $so_2(\mathbb{C}) \ltimes \mathbb{C}^2$ | $z$ | $v_1$ | $v_2$ |
| $2u$ | $x$ | $y$ | $h$ | $-2x$ | $u$ | $s$ | $u$ | $v_1$ | $-v_2$ |

| $sl_2(\mathbb{C}) \ltimes n_3(\mathbb{C})$ | $w_1$ | $w_2$ | $w_3$ | $sl_2(\mathbb{C}) \ltimes \mathbb{C}^3$ | $w_1$ | $w_2$ | $w_3$ | $sl_2(\mathbb{C}) \ltimes \mathbb{C}^3$ | $w_1$ | $w_2$ | $w_3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x$ | $-s$ | $x$ | $-w_3$ | $2w_1$ | $y$ | $-t$ | $y$ | $w_3$ | $-2w_2$ |
| $h$ | $s-t$ | $h$ | $2w_1$ | $-2w_2$ |
### Table 2. Classification of multiply transitive homogeneous real $(2, 3, 5)$ distributions

| dim $\mathfrak{h}$ | model $\mathfrak{g}$ | $\mathfrak{h}$ | $\mathfrak{k}$ | $\mathfrak{d}$ | parameters | equivalences | § |
|---------------------|----------------------|----------------|----------------|----------------|------------|-------------|---|
| 14                  | $O^\mathbb{R}$       | $\mathfrak{g}_2$ | $\mathfrak{q}$ | $g_2^{-1}$     | $(r, s) \in \mathbb{R}^2$ | $(r, s) \sim (cr, c^2s)$ | § 5.2 |
| 7                   | $N.7_{r,s}$          | $\mathfrak{h}_{r,s}$ | $\mathfrak{t}_{r,s}$ | $\mathfrak{d}_{r,s}$ | $0r^2 \neq 100s$ | $c \in \mathbb{R}^+$ | § 5.3 |

| N.6 | N.$6^\pm$ | $\mathfrak{s}_{12} \otimes \mathfrak{n}_3$ | $(\mathfrak{x})$ | $(\mathfrak{h}, \mathfrak{y} \mp (\mathfrak{u} + \mathfrak{s}))$ | $\pm$ | § 5.4 |

| D.$6_\lambda$ | D.$6^\dagger_\lambda$ | sl$_2$ $\oplus$ sl$_2$ | $(\mathfrak{h} \mp \mathfrak{b})$ | $(\mathfrak{x} \mp -\lambda \mathfrak{x}, \mathfrak{y} \mp -\mathfrak{y}) \oplus \mathfrak{t}$ | $\lambda \in \mathbb{R} - \{0, \frac{1}{2}, 1, 9\}$ | $\lambda \sim 1/\lambda$ | § 5.3 |
| D.$6_\infty$ | D.$6^\dagger_\infty$ | sl$_2$ $\oplus$ (so$_{1,1} \otimes \mathbb{R}^{1,1}$) | $(\mathfrak{x} \mp \mathfrak{y} \mp 2\mathfrak{z}) \oplus \mathfrak{t}$ | $(\mathfrak{x} \mp \mathfrak{v}_1, \mathfrak{y} \mp \mathfrak{v}_2) \oplus \mathfrak{t}$ | $\lambda \in \mathbb{R}^+$ | $\lambda \sim 1/\lambda$ | § 5.3 |

| D.$6_\ast$ | D.$6^\dagger_\ast$ | sl$_2$ $\otimes \mathbb{R}^{1,2}$ | $(\mathfrak{x} \mp \mathfrak{y}) \oplus \mathfrak{t}$ | $(\mathfrak{h} \mp 2\mathfrak{w}_3, \mathfrak{x} \mp (\mathfrak{y} - 2\mathfrak{v}_1) - 2\mathfrak{w}_2) \oplus \mathfrak{t}$ | § 5.3 |
| D.$6^\dagger_\ast$ | | so$_3 \otimes \mathbb{R}^3$ | $(\mathfrak{c})$ | $(\mathfrak{a} - \mathfrak{w}_3, \mathfrak{b} - \mathfrak{w}_3) \oplus \mathfrak{t}$ | § 5.3 |

### Notes:

- In models D.$6^\dagger_\lambda$ and D.$6^\dagger_\ast$, $\bar{x} := \frac{\sqrt{2}}{\sqrt{2}} x + \frac{1}{\sqrt{2}} h$, $\bar{y} := \frac{1}{\sqrt{2}} y + \frac{1}{\sqrt{2}} h$, $\bar{h} := \frac{1}{\sqrt{2}} (x - y) + \frac{1}{\sqrt{2}} h$. 

---

**References:**

- D.6$\lambda$, D.6$\infty$, D.$6_\ast$, D.$6^\dagger_\ast$, See Table 3.
Table 3. Explicit real algebraic models for the real submaximal distributions $D_{r,s}$

| $J$ | $r,s$ | $a,b$ | $F$-action | $t_{r,s}$ | $\partial_{r,s}$ | $\partial_{r,s} + [\partial_{r,s}, \partial_{r,s}]$ |
|-----|------|-------|-------------|---------|--------------|----------------|
| $J < 0$ | $\begin{cases} r = a^2 - b^2 \\
|\hline
|-----|------|-------|-------------|---------|--------------|----------------|
| $J = 0$ | $\begin{cases} r = -a^2 \\
|\hline
|-----|------|-------|-------------|---------|--------------|----------------|
| $J > 0$ | $\begin{cases} r = a^2 \\
|\hline
|-----|------|-------|-------------|---------|--------------|----------------|
| $0 < J < 1$ | $\begin{cases} r = -a^2 - b^2 \\
|\hline
|-----|------|-------|-------------|---------|--------------|----------------|
| $J = 1$ | $\begin{cases} r = -2a^2 \\
|\hline
|-----|------|-------|-------------|---------|--------------|----------------|
| $J > 0$ | $\begin{cases} r = 2(a^2 - b^2) \\
|\hline
|-----|------|-------|-------------|---------|--------------|----------------|
| $s < 0$ | $\begin{cases} r = 0 \\
|\hline
|-----|------|-------|-------------|---------|--------------|----------------|
| $s > 0$ | $\begin{cases} r = 0 \\

We split cases according to the complex invariant $J := 4s/r^2$ and, in some cases, the sign of $r$ or $s$; in each case, one can choose $a, b \in \mathbb{R}$ satisfying the given equations for $r, s$. Then, the specified action of $F$ on $n^+_L = \langle s_1, s_2, t_1, t_2 \rangle$ together with $[F, u] = 0$ defines $F$ and thus $\partial_{r,s} := n^+_L \otimes (E, F)$. For each entry: $t_{r,s} := \langle X, E \rangle$, where $X$ is the vector in the column labeled $t_{r,s}$, and $\partial_{r,s} := \langle Y, F \rangle \oplus t_{r,s}$, where $Y$ is the vector in the column labeled $\partial_{r,s}$. Also, $\partial_{r,s} + [\partial_{r,s}, \partial_{r,s}] = \langle Z \rangle \oplus \partial_{r,s}$, where $Z$ is the vector in the column labeled $\partial_{r,s} + [\partial_{r,s}, \partial_{r,s}]$. 


Appendix B. An explicit realization of \( \mathfrak{g}_2 \)

We give here an explicit basis \( \{Y_1, Y_2, r, X_1, X_2, A_{11}, A_{12}, A_{21}, A_{22}, Z_1, Z_2, s, W_1, W_2\} \) of \( \mathfrak{g}_2 \). If we realize \( q \) as the (parabolic) subalgebra fixing the line the line \( \{ (*, 0, \ldots, 0) \} \subset \mathbb{C}^r \) in the standard representation, then \( q = \langle A_{11}, A_{12}, A_{21}, A_{22}, Z_1, Z_2, s, W_1, W_2 \rangle, \) \( \mathfrak{g}_2^{-1} = (X_1, X_2) \oplus q, \) and \( \mathfrak{g}_2^2 = \mathfrak{g}_2^{-1} + [\mathfrak{g}_2^{-1}, \mathfrak{g}_2^{-1}] = (r) \oplus \mathfrak{g}_2^{-1}. \)

Realize \( \mathfrak{g}_2 \) as the subalgebra \[ \begin{pmatrix} -\text{tr} A & Z & s & W^T \\ X & - (\text{tr} A) I & \sqrt{2} Z^T & - s W \\ r & - \sqrt{2} X^T J & s & - W \\ r & - \sqrt{2} X^T J & s & - W \end{pmatrix} \subseteq \mathfrak{gl}_7, \]

where

\[ I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]

Then, write

\[ Y = (Y_1, Y_2), \quad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad Z = (Z_1, Z_2), \quad W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}, \]

set \( Y_1 \in \mathfrak{g}_2 \) to be the matrix with \( Y_1 = 1 \) and all other variables set to 0, and define the other 13 basis elements analogously.

For the Monge normal form \( F(x, y, p, q, z) = q^2 \) realizing \( \mathcal{O}^R \), we can identify \( \mathfrak{g}_2 \cong \text{aut}(\mathcal{D}^F) \) via

\[ Y_1 \leftrightarrow \frac{\partial}{\partial y}, \quad Y_2 \leftrightarrow \frac{\partial}{\partial x}, \quad r \leftrightarrow - \frac{1}{\sqrt{2}} (x \partial_y + \partial_p), \quad X_1 \leftrightarrow \frac{1}{2} x^2 \partial_y + x \partial_p + \partial_q + 2 p \partial_z, \quad X_2 \leftrightarrow \partial_x, \]

\[ A_{11} \leftrightarrow - (y \partial_y + p \partial_p + q \partial_q + 2 z \partial_z), \quad A_{12} \leftrightarrow - \frac{1}{2} x^3 \partial_y - \frac{1}{2} x^2 \partial_p - x \partial_q + (- 2 x p + 2 y) \partial_z, \]

\[ A_{21} \leftrightarrow - q \partial_x + (pq + \frac{1}{2} z) \partial_y - \frac{1}{2} q^2 \partial_p - \frac{1}{2} q \partial_q, \quad A_{22} \leftrightarrow -(x \partial_x + 2 y \partial_y + p \partial_p + z \partial_z), \]

\[ Z_0 \leftrightarrow (- 3 x q + 4 p) \partial_z + (- 3 x q + \frac{1}{2} x^2 + 2 p^2) \partial_y + (- \frac{1}{4} x q^2 + \frac{1}{2} x z) \partial_p - q^2 \partial_q - x q \partial_z, \]

\[ Z_2 \leftrightarrow - x^2 \partial_x - 3 x y \partial_y + (- x^2 y + 3 x^2 z + 8 x p^2) \partial_y + (3 x^2 z^2 + 6 x z + 4 p^2) \partial_p \]

\[ + (- 4 x q^2 + 4 p q + 6 x) \partial_q - (2 x^2 q^2 + 2 x z) \partial_z), \]

\[ W_1 \leftrightarrow (- x^3 q + 4 x^2 p - 6 x y) \partial_z + (- x^3 q + \frac{1}{2} x^3 z + 2 x^2 p^2 - 6 y^2) \partial_y + (\frac{1}{2} x^3 q^2 + \frac{1}{2} x^2 z^2 + 2 x^2 y + 2 x p y) \partial_p \]

\[ + (- x^2 q^2 + 2 x q y + x z - 4 y^2) \partial_q + (- \frac{1}{4} x^2 q^3 + 6 x p z - 6 y z - \frac{3}{2} y^3) \partial_z), \]

\[ W_2 \leftrightarrow (6 y q - 4 p^2) \partial_z + (6 y q - 3 y z - \frac{3}{2} y^3) \partial_y + (3 y q^2 - 3 p z) \partial_p + (2 q^2 - 3 q) \partial_q + (2 y q^2 - 3 z^2) \partial_z; \]

cf. [26, Table 7.3].

If we instead take all of the objects to be complex, this instead gives explicit realizations of \( \mathfrak{g}_2(\mathbb{C}) \) and \( (\mathfrak{g}_2, \mathfrak{g}(\mathbb{C}), q|\mathbb{C}|; \mathfrak{g}_2(\mathbb{C})^{-1}) \) (algebraic model \( \mathcal{O} \)).

References

[1] A.A. Agrachev, Rolling Balls and Octonions, Proc. Steklov Inst. Math. 258:1 (2007), 13–22. arXiv:math/0611812

[2] D. An, P. Nurevsky, Twistor space for rolling bodies, Comm. Math. Phys. 326 (2014), 393–414. arXiv:1410.7540

[3] J. Baez, J. Huerta, G2 and the rolling ball, Trans. Amer. Math. Soc. 366 (2014), 5257–5293. arXiv:1205.2447

[4] G. Bor, R. Montgomery, G2 and the rolling distribution, Enseign. Math. 55 (2009), 157–196. arXiv:math/0612469

[5] R. Bryant, L. Hsu, Rigidity of integral curves of rank two distributions, Invent. Math. 114 (1993), 435–461.

[6] A. Čap, J. Slovak, Parabolic geometries I: Background and general theory. Mathematical Surveys and Monographs 154, American Mathematical Society, Providence (2009), x+628 pp.

[7] É. Cartan, Sur la structure des groupes simples finis et continus, C. R. Acad. Sc. 116 (1893), 784–786.

[8] É. Cartan, Les systèmes de Pfaff a cinq variables et les équations aux dérivées partielles du second ordre, Ann. Ec. Normale 27 (1910), 109–192.

[9] É. Cartan, Sur l’équivalence absolue de certains systèmes d’équations différentielles et sur certaines familles de courbes, Bulletin de la S.M.F. 42 (1914), 12–48.

[10] B. Dubrovin, A. Gorovov, A new example of a generic 2-distribution on a 5-manifold with large symmetry algebra. arXiv:1305.7297

[11] B. Dubrovin, A. Gorovov, Unpublished notes.

[12] B. Dubrovin, and B. Kruglikov, On the models of submaximal symmetric rank 2 distributions in 5D, Differ. Geom. Appl. 35 (2013), 314–322. arXiv:1311.7057
[13] B. Doubrov, A. Medvedev, D. The, Homogeneous Levi non-degenerate hypersurfaces in \( \mathbb{C}^3 \), arXiv:1711.02389

[14] B. Doubrov, I. Zelenko, Geometry of rank 2 distributions with nonzero Wilczynski invariants and affine control systems with one input, J. Nonlinear Math. Phys. 21:2 (2014), 166–187. doi:10.1080/14029251.2014.900985 arXiv:1301.2797

[15] F. Engel, Sur un groupe simple à quatorze paramètres, C. R. Acad. Sc. 116 (1893), 786–788.

[16] É. Goursat, Leçons sur le problème de Pfaff, Librairie Scientifique J. Hermann, Paris (1922).

[17] A.R. Gover, R. Panai, T. Willse, Nearly Kähler geometry and (2, 3, 5)-distributions via projective holonomy, Indiana Univ. Math. J. 66 (2017), 1351–1416. doi:10.1512/iumj.2017.66.6089 arXiv:1403.1959

[18] M. Hammerl, K. Sagerschnig, Conformal Structures Associated to Generic Rank 2 Distributions on 5-Manifolds—Characterization and Killing-Field Decomposition, SIGMA 5 (2009), arXiv:0908.0483

[19] D. Hilbert, Über den begriff der klasse von differentialgleichungen, Math. Ann. 73 (1912), 95–108.

[20] B. Kruglikov, Lie theorem via rank 2 distributions (Integration of PDE of class \( \omega = 1 \)), J. Nonlinear Math. Phys. 19:2 (2012), 158–181. arXiv:1108.5854 doi:10.1142/S1402925112500118

[21] B. Kruglikov, D. The, The gap phenomenon in parabolic geometries, J. Reine Angew. Math. 723 (2014), 153–215. arXiv:1303.1307

[22] P. Nurowski, Differential equations and conformal structures, J. Geom. Phys. 55 (2005), 19–49. arXiv:math/0406400

[23] K. Sagerschnig, Split octonions and generic rank two distributions in dimension five, Arch. Math. (Brno) 42 (2006), suppl., 329–339.

[24] K. Sagerschnig, T. Willse, The Geometry of Almost Einstein (2,3,5) Distributions, SIGMA 13 (2017), 56 pp. arXiv:1606.01069

[25] K. Sagerschnig, T. Willse, The almost Einstein operator for (2,3,5) distributions, Arch. Math. (Brno) 53 (2017), 347–370. doi:10.5817/am2017-5-347 arXiv:1705.00996

[26] F. Strazzullo, Symmetry Analysis of General Rank-3 Pfaffian Systems in Five Variables, Ph.D. Thesis, Utah State University (2009). http://digitalcommons.usu.edu/etd/449/

[27] T. Willse, Highly symmetric 2-plane fields on 5-manifolds and 5-dimensional Heisenberg group holonomy, Differential Geom. Appl. 33 (2014), 81–111. arXiv:1302.7163

[28] T. Willse, Cartan’s incomplete classification and an explicit ambient metric of holonomy \( G_2 \), Eur. J. Math. (2017). doi:10.1007/s40879-017-0178-9 arXiv:1411.7172

T.W.: Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, 1090 Wien, AUSTRIA

E-mail address: travis.willse@univie.ac.at