PARTIALLY GAUSSIAN STATIONARY STOCHASTIC PROCESSES IN DISCRETE TIME

K. R. PARTHASARATHY

Abstract. We present here an elementary example, for every fixed positive integer $k$, of a strictly stationary non-gaussian stochastic process in discrete time, all of whose $k$-marginals are gaussian.

1. Introduction

It is well known that, for every positive integer $n > 1$, there exists a probability distribution in $\mathbb{R}^n$ which is not gaussian but has all its $(n-1)$-dimensional marginal distributions gaussian. (See, for example, Section 10.3 in Stoyanov [2].) Using the finer theory of pathwise stochastic integrals and martingale methods, Föller, Wu and Yor [1] have shown that, for every positive integer $k$, there exists a variety of non-gaussian stochastic processes with continuous trajectories in the interval $[0, 1)$ which have the same $k$-dimensional marginals as the standard brownian motion process. Here we present an elementary example, for every fixed positive integer $k > 1$, a discrete time stationary stochastic process which is not gaussian but has all its $(k-1)$-marginals gaussian. However, we do not know how to construct such processes in continuous time.

2. The Basic Construction

Let $k > 1$ be any fixed positive integer and let $\nu$ be a probability distribution in $\mathbb{R}^k$, which is not gaussian but has all its $(k-1)$-marginals gaussian with mean 0 and covariance matrix identity. For example, we may choose $\nu$ to have the probability density function

$$
\psi(x) = (2\pi)^{-\frac{k}{2}} \left( 1 + x_1 x_2 \cdots x_k \right) e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}j^2x^2}
$$

where $x = (x_1, x_2, \ldots, x_k)$. If $(X_1, X_2, \ldots, X_k)$ is an $\mathbb{R}^k$-valued random variable with distribution $\nu$, then the sequence $X_1, X_2, \ldots, X_k$ with the $i$-th term omitted consists of i.i.d. $N(0, 1)$ random variables, for each $i$.

Now consider a bilateral sequence $\{(X_{n1}, X_{n2}, \ldots, X_{nk}), -\infty < n < \infty\}$ of i.i.d $\mathbb{R}^k$-valued random variables with the common distribution $\nu$ as described in the preceding paragraph. Define

$$Y_n = X_{nk} + X_{n+1, k-1} + X_{n+2, k-2} + \cdots + X_{n+k-1, 1}, -\infty < n < \infty.$$
It is to be noted that the sum of the two suffixes in each summand on the right hand side is equal to \( n + k \).

**Theorem 2.1.** The sequence \( \{Y_n, -\infty < n < \infty\} \) is strictly stationary, \((k - 1)\)-step independent with every \((k - 1)\)-dimensional marginal being gaussian with mean 0 and covariance matrix \( kI \), \( I \) being the identity matrix of order \( k - 1 \). In particular, \( \{Y_n\} \) is ergodic.

**Proof.** Fix an integer \( m \) and consider the two sets \( \{Y_n, n \leq m\} \) and \( \{Y_n, n \geq m + k\} \). Since \( Y_m = X_{mk} + X_{m+1,k-1} + \cdots + X_{m+k-1} \) and \( Y_{m+k} = X_{m+k,k} + X_{m+k+1,k-1} + \cdots + X_{m+k+2,k-1} \) and the first suffix in the last summand in the definition of \( Y_m \) is less than the first suffix in the first summand in the definition of \( Y_{m+k} \) it follows that the two sets \( \{Y_n, n \leq m\} \) and \( \{Y_n, n \geq m + k\} \) are independent. In other words \( \{Y_n\} \) is a \((k - 1)\)-step independent process.

We now look at the column vector-valued random variable

\[
\begin{bmatrix}
Y_{n+1} \\
Y_{n+2} \\
\vdots \\
Y_{n+m}
\end{bmatrix} = \begin{bmatrix}
X_{n+1,k} + X_{n+2,k-1} + \cdots + X_{n+k,1} \\
X_{n+2,k} + X_{n+3,k-1} + \cdots + X_{n+k+1,1} \\
\vdots \\
X_{n+m,k} + X_{n+m+1,k-1} + \cdots + X_{n+m+k-1,1}
\end{bmatrix}
\]

and express it as \( S_1 + S_2 + S_3 \) where

\[
S_1 = \begin{bmatrix}
X_{n+1,k} \\
0 \\
\vdots \\
0
\end{bmatrix} + \begin{bmatrix}
X_{n+2,k} \\
0 \\
\vdots \\
0
\end{bmatrix} + \cdots + \begin{bmatrix}
X_{n+k-1,1} \\
0 \\
\vdots \\
0
\end{bmatrix},
\]

\[
S_2 = \begin{bmatrix}
X_{n+k,1} \\
X_{n+k+1,2} \\
\vdots \\
X_{n+m,k}
\end{bmatrix} + \begin{bmatrix}
0 \\
X_{n+k+1,1} \\
\vdots \\
X_{n+m+1,k}
\end{bmatrix} + \cdots + \begin{bmatrix}
0 \\
X_{n+m,1} \\
\vdots \\
X_{n+m+k,1}
\end{bmatrix},
\]

\[
S_3 = \begin{bmatrix}
0 \\
\vdots \\
0 \\
X_{n+m+1,1} + X_{n+m+2,1}
\end{bmatrix} + \begin{bmatrix}
0 \\
\vdots \\
0 \\
X_{n+m+2,2}
\end{bmatrix} + \cdots + \begin{bmatrix}
0 \\
\vdots \\
0 \\
X_{n+m+k-1,1}
\end{bmatrix}.
\]
In each column on the right hand side of $S_1$ or $S_3$ there are at most $k - 1$ nonzero entries whereas in each column on the right hand side of $S_2$ there are exactly $k$ entries. All the column vectors appearing in $S_1, S_2, S_3$ together are mutually independent. By the choice of the measure $\nu$, $S_1$ and $S_3$ are gaussian random vectors. Denote by $\mu([i,j])$ the $(j-i+1)$-dimensional standard normal distribution imbedded in $\mathbb{R}^m$ so that the first $i-1$ and the last $m-j$ coordinates are $0$ when $1 \leq i \leq j \leq m$. Similarly, denote by $\nu([j,k+j-1])$ the $k$-dimensional distribution $\nu$ imbedded in $\mathbb{R}^m$ with the first $j-1$ and the last $m-k-j+1$ coordinates $0$ for $1 \leq j \leq m - k + 1$, assuming $m \geq k$. Then it follows that the random variables $Y_{n+1}, Y_{n+2}, \ldots, Y_{n+m}$ expressed as a single column vector has the $m$-dimensional distribution $\nu_m$ (in $\mathbb{R}^m$) given by
\[
\nu_m = \mu([1,1]) \ast \mu([1,2]) \ast \cdots \ast \mu([1,k-1]) \\
\ast \nu([1,k]) \ast \nu([2,k+1]) \ast \cdots \ast \nu([m-k+1,m]) \\
\ast \mu([m-k+2,m]) \ast \mu([m-k+3,m]) \ast \cdots \ast \mu([m,m]),
\]
for every $m \geq k$. Since $\nu_m$ is independent of $n$ it follows that $\{Y_n\}$ is a strictly stationary process. Since $\nu([1,k])$ is nongaussian it is clear that $\nu_m$ is not gaussian for every $m \geq k$.

We now observe that $Y_n$, being a sum of $k$ independent $N(0,1)$ random variables, is an $N(0,k)$ variable with mean $0$ and variance $k$. Now consider the pair $(Y_0, Y_m)$. If $m \geq k$ we have already seen that $Y_0$ and $Y_m$ are independent. If $m < k$, we write
\[
\begin{bmatrix}
Y_0 \\
Y_m
\end{bmatrix} = \begin{bmatrix}
X_{0k} + X_{1,k-1} + \cdots + X_{m-1,k-m+1} \\
0
\end{bmatrix} + \begin{bmatrix}
X_{m-k-m} \\
X_{m-k-1}
\end{bmatrix} + \cdots + \begin{bmatrix}
X_{k-1,1} \\
X_{k-1,m-1}
\end{bmatrix} + \begin{bmatrix}
0 \\
X_{km} + X_{k+1,m-2} + \cdots + X_{k+m-1,1}
\end{bmatrix}.
\]
Now the special choice of $\nu$ implies that $Y_0$ and $Y_m$ are independent $N(0,k)$ random variables. Stationarity of the process $\{Y_n\}$ implies that $Y_{n_1}$ and $Y_{n_2}$ are independent $N(0,k)$ random variables for any $n_1, n_2$.

Now consider, for any $n_1 < n_2 < \cdots < n_{k-1}$ the random vector
\[
\mathbf{Y} = \begin{bmatrix}
Y_{n_1} \\
Y_{n_2} \\
\vdots \\
Y_{n_{k-1}}
\end{bmatrix} = \begin{bmatrix}
X_{n_1,k} + X_{n_1+1,k-1} + \cdots + X_{n_1+k-1} \\
X_{n_2,k} + X_{n_2+1,k-1} + \cdots + X_{n_2+k-1} \\
\vdots \\
X_{n_{k-1},k} + X_{n_{k-1}+1,k-1} + \cdots + X_{n_{k-1}+k-1}
\end{bmatrix}.
\]
The right hand side can be expressed as a sum of column vectors in which the entries in each column are either $0$ or an $X_{rs}$ where the first suffix $r$ is fixed and the second suffix takes at most $k - 1$ values from the set $\{1, 2, \ldots, k\}$. The different column vectors are independent and by the choice of $\nu$ each column has a multivariate gaussian distribution. Thus $\mathbf{Y}$ is gaussian. Since any two $Y_i$ and $Y_j$ are independent where $k > 2$, it follows that $Y_{i_1}, Y_{i_2}, \ldots, Y_{i_{k-1}}$ are i.i.d $N(0,k)$ random variables. This completes the proof. 
\[\square\]
Remark 2.2. From the proof of Theorem 2.1 it is clear that any $k - 1$ of the random variables $\{Y_n\}$ are i.i.d $N(0, k)$. This motivates the introduction of the following notion of limited exchangeability. We say that a stationary random process $\{Z_n, -\infty < n < \infty\}$ is $k$-exchangeable if any $Z_{n_1}, Z_{n_2}, \ldots, Z_{n_k}$ has the same distribution for any $k$-point set $\{n_1, n_2, \ldots, n_k\} \subset \mathbb{Z}$. The probability measures of all such $k$-exchangeable stationary processes constitute a convex set. One wonders what are the extreme points of this convex set.

References

1. Föllmer, H., Wu, C. T., and Yor, M.: On weak brownian motions of arbitrary order, *Ann. Inst. Henri Poincaré* 36 (2000) 447–487.
2. Stoyanov, J.: *Counterexamples in Probability*. Wiley, 1987.