On the solution of the Liouville equation

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Abstract

We give a short and rigorous proof of the existence and uniqueness of the solution of the Liouville equation with sources, both elliptic and parabolic, on the sphere and on all higher genus compact Riemann surfaces.

Keywords: Liouville theory, conformal field theory, solutions of Liouville equation

1. Introduction

Liouville theory is a subject of wide interest. It plays a key role in several chapters of conformal field theory, see [1, 2]. Liouville action appears in two-dimensional quantum gravity as the Faddeev–Popov term [3]. Another chapter where Liouville theory intervenes is $2+1$ dimensional gravity coupled to particles where the solution of the hamiltonian constraint is reduced to the solution of Liouville equation in presence of point sources [4, 5]. More recently the AGT [6] relationship has been discovered, which relates a class of four-dimensional gauge theories to Liouville theory. Also the problem of the analytic continuation of such a theory has come under attention [7].

Obviously the existence and uniqueness property of the solutions of the Liouville equation plays a key role both at the classical and quantum level. The first studies on Liouville theory go back, in addition to Liouville himself [8], to Picard [9, 10] and Poincaré [11].

When only elliptic singularities are present the existence proof of the solution of the equation

$$\Delta \phi = e^\phi$$

was first given by Picard in [9, 10] where also the uniqueness of the solution was proven. Picard’s method does not apply in presence of parabolic singularities (punctures).

The existence and uniqueness proof was extended to the presence of parabolic singularities by Poincaré [11]. Lichtenstein [12] reformulated the problem of finding the solution of the equation as a variational problem proving again the existence and uniqueness of the solution in presence of both elliptic and parabolic singularities. For the case of elliptic singularities...
McOwen [13] and Troyanov [14], also employing a variational procedure, proved more general results using Sobolev spaces techniques.

The proofs given in the quoted papers are lengthy in part due to the greater generality of the problem addressed by mathematicians (formulation on metric surfaces, achievement of some prescribed curvature function, etc).

The purpose of the present paper is to give a short and at the same time rigorous proof of the existence and uniqueness of the solution of the Liouville equation with sources both elliptic and parabolic, on the sphere and all higher genus compact Riemann surfaces, which is the usual setting in conformal theory both classical and quantum.

In order to make the paper more readable we shall outline here qualitatively the track of the proof and the methods used. The field \( \phi \) is singular at the sources and it is useful to decompose it as the sum of a singular background and a regular part. This can be done in several ways but the one introduced by Lichtenstein [12] turns out to be particularly useful. The reason is that with such a decomposition the original Liouville equation reduces to partial differential equations for the regular part, which naturally appears as the solution of the variational problem for an action which is bounded from below i.e. to a minimum problem. Thus one is faced with the simpler job of proving that such a minimum is really reached and that the solution is unique.

To reach the first result one builds a sequence of functions for which the action tends to the lower limit and then applying a simple compactness criterion one extracts a weak solution to the problem. Application of a standard result (Weyl’s lemma) allows to go over from the weak solution to the actual solution of the original differential equation.

This is done first for elliptic sources. The power of the Lichtenstein decomposition is that it can be given also for parabolic sources (punctures) providing a result which was out of reach in the original approach to the problem [9, 10]. The extension to all compact Riemann surfaces which is given in section 3, exploits the fact that for \( g \geq 2 \) they are represented by the quotient of the complex upper half plane, by a Fuchsian group and for \( g = 1 \) (the torus) by a parallelogram with periodic boundary conditions. The different topologies impose on the strength of the sources bounds which contain \( g \) and are the generalization of the bounds present on the sphere. Apart from such a difference the proof goes along the same lines as for the sphere.

Uniqueness never posed a real problem and in section 4 we give a particularly simple proof of it.

The sources are introduced by imposing on the field \( \phi \) the boundary conditions

\[
\phi + 2 \eta_k \log |z - z_k|^2 = \text{bounded}, \quad \eta_k < \frac{1}{2}
\]

(2)

in finite disks around the elliptic singularities \( z_k \) and

\[
\phi + \log |z - z_p|^2 + \log \log |z - z_p|^2 = \text{bounded}
\]

(3)

in finite disks around the parabolic singularities \( z_p \) and for the sphere \( (g = 0) \)

\[
\phi + 2 \log |z|^2 = \text{bounded}
\]

(4)

outside of a circle of sufficiently large radius. For \( g = 1 \) one imposes in addition to the conditions (2) and (3) periodic boundary conditions on the boundary of the fundamental parallelogram and for \( g \geq 2 \) periodicity of \( e^{i \phi} dz \wedge d\bar{z} \) on the boundary of the fundamental \( 2g \)-gon in the upper \( z \)-plane describing the compact Riemann surface.
2. Existence

We shall first deal with the sphere topology. To $z_K$ we associate non overlapping disks of radius $r_K$ excluding the other singularities and $R$ is chosen such that the disk of radius $R$ contains all singularities and the previously described disks.

When only elliptic singularities are present we construct following Lichtenstein [12] a smooth positive function $\beta$ such that in the above described disks we have

$$0 < \lambda_m < \beta |z - z_K|^4 \eta_K < \lambda_M$$

and for $|z| > R$

$$0 < \lambda_m < \beta |z|_1 < \lambda_M$$

and elsewhere

$$0 < \lambda_3 < \beta < \lambda_4.$$

Note that $\int \beta d^2 z < \infty$. In addition $\beta$ will be normalized as to have

$$- \sum K 2 \eta_K + \frac{1}{4\pi} \int \beta (z') d^2 z' = -2$$

which is possible due to the topological restriction (see section 3) $\sum K 2 \eta_K > 2(1 - g)$ where the sum extends to the sources.

We define

$$v = \phi_1 + \frac{1}{4\pi} \int \log |z - z'|^2 \beta (z') d^2 z'$$

with

$$\phi_1 = \sum K (-2 \eta_K) \log |z - z_K|^2$$

and $U$ by

$$\phi = v + U.$$ 

As $\phi$ behaves as $-2 \log |z|^2$ at infinity, $U$ behaves as a constant at infinity. The Liouville equation (1) becomes in $C \setminus \{z_K\}$

$$\Delta U + \beta = e^\phi = e^v e^U \equiv r \beta e^U$$

where we defined $e^v = r \beta$ and from the above we have that

$$0 < \lambda_1 < r < \lambda_2$$

for some $\lambda_1, \lambda_2$ all over the plane. Formally equation (12) solves the variational problem for the action functional

$$I[U] = \int (\frac{1}{2} \nabla U \cdot \nabla U - \beta U + r \beta e^U) d^2 z.$$ 

An important point is that, as we prove now, such a functional is bounded from below and thus the variational problem becomes a minimum problem which is much easier to deal with.

We start [13, 14] in the real pre-Hilbert space $H$ of $C^1$ functions $A$ with norm
\[ \langle A, A \rangle = \int \nabla A \cdot \nabla A d^2z + \int A^2 \beta d^2z \equiv (\nabla A, \nabla A)_1 + (A, A)_\beta. \]  

(15)

By \((A_1, A_2)_\beta\) we denote the scalar product with the measure \(\beta d^2z\) and with \(L^2_\beta\) the relative Hilbert space. A simple computation gives

\[ \int ( - A + re^A ) \beta d^2z \geq (1 + \log \lambda_1) \int \beta d^2z \]

(16)

showing that the functional \(I\) is lower bounded in \(H\). We have also

\[ I[0] = \int r \beta d^2z < \lambda_2 \int \beta d^2z. \]

(17)

Defined

\[ F = (re^A - A) \beta \quad \text{and} \quad L = \max (|\log \lambda_1|, |\log \lambda_2|) \]

(18)

one has

\[ \frac{\partial F}{\partial A} > 0 \quad \text{for} \quad A > L, \quad \frac{\partial F}{\partial A} < 0 \quad \text{for} \quad A < -L. \]

(19)

Thus given an \(A\) one can construct another \(\tilde{A} \in C^1\) with \(|\tilde{A}| \leq 2L\) and with the property \(I[\tilde{A}] \leq I[A]\). In fact consider a smooth always increasing function \(\sigma(x)\) with \(\sigma(-\infty) = -2L\), \(\sigma(\infty) = 2L\), \(\sigma(x) = x\) for \(-L \leq x \leq L\) and elsewhere \(0 < \sigma' < 1\). Then with \(\tilde{A}(z) = \sigma(A(z))\) we have that \(\int ( - \beta A + r \beta e^A ) d^2z\) is not increased and also

\[ \int \nabla \tilde{A} \cdot \nabla \tilde{A} d^2z = \int (\sigma'(A))^2 \nabla A \cdot \nabla A d^2z \leq \int \nabla A \cdot \nabla A d^2z. \]

(20)

This means that \(\inf(I[|A|])\) can be computed on the subset \(|A| \leq 2L\). Note also that \(\langle \tilde{A}, \tilde{A} \rangle \leq \langle A, A \rangle\). For \(A \in H, |A| \leq 2L\) we have

\[ I[\tilde{A}] = \frac{1}{2} \langle A, A \rangle - \frac{1}{2} (A, A)_{\beta} - \int A \beta d^2z + \int re^A \beta d^2z \geq \frac{1}{2} \langle A, A \rangle - 2(L^2 + L)(1, 1)_{\beta} \]

(21)

which shows that \(\inf(I[|A|])\) is obtained using elements in the subset of \(H\) with \(|A| \leq 2L\) and

\[ \frac{1}{2} \langle A, A \rangle \leq M + 1 \]

(22)

being

\[ M = I[0] + 2(L^2 + L)(1, 1)_{\beta}. \]

(23)

In fact if \(\frac{1}{2} \langle A, A \rangle > M + 1\) we have

\[ I[\tilde{A}] > M + 1 - 2(L^2 + L)(1, 1)_{\beta} = I[0] + 1 \]

(24)

and such \(A\) has to be discarded in the search of \(\inf I[|A|]\).

The subset \(S\) of \(H\), and thus also of \(L^2_{\beta}\), given by \(\frac{1}{2} \langle A, A \rangle \leq M + 2\) and \(|A| \leq 4L\) is relatively compact in \(L^2_{\beta}\), i.e. its closure is compact.

This is obtained by showing that the functions \(\sqrt{\beta} A\) with \(A \in S\) satisfy the three relative-compactness criteria [15] in the usual \(L^2\) norm which we shall write as \(\|\|\). To start we notice that denoting by \(\tau_h\) the operator which translates a function by \(h\) we have
\begin{equation}
\tau_h A(z) - A(z) = |h| \int_0^1 \hat{h} \cdot \nabla A(z + h\sigma)d\sigma
\end{equation}

and

\begin{equation}
\int |\tau_h A(z) - A(z)|^2d^2z \leq |h|^2 \int_0^1 d\sigma \int |\nabla A(z + \sigma h)|^2d^2z \leq |h|^2 \|\nabla A\|^2
\end{equation}

i.e.

\begin{equation}
\|\tau_h A - A\| \leq |h| \|\nabla A\| \leq |h| \sqrt{\langle A, A \rangle}.
\end{equation}

The criteria of relative compactness to be satisfied are [15]

(1) The boundedness on \( S \) of \( \|\sqrt{\beta A}\| \) which is immediate due to \( |A| \leq 4L \) and the integrability of \( \beta \).

(2) The uniformity on \( S \) of the limit \( \lim_{k\to\infty} \int_{|z|>k} (\sqrt{\beta A})^2d^2z = 0 \) which is true for the same reason.

(3) Finally we need to show that the limit

\begin{equation}
\lim_{h\to0} \|\tau_h(\sqrt{\beta A}) - \sqrt{\beta A}\| = 0
\end{equation}

is uniform on \( S \). This is easily obtained from

\begin{equation}
\|\tau_h(\sqrt{\beta A}) - \sqrt{\beta A}\| \leq \|(\tau_h \sqrt{\beta})(\tau_h A - A)\| + \|A(\tau_h \sqrt{\beta} - \sqrt{\beta})\|
\end{equation}

\begin{equation}
\leq \|(\tau_h \sqrt{\beta})(\tau_h A - A)\| + 4L \|\tau_h \sqrt{\beta} - \sqrt{\beta}\|
\end{equation}

and using equation (27), |A| \leq 4L and the integrability of \( \beta \).

Construct now in \( S \) a sequence \( A_m \) such that \( \lim_{m\to\infty} I[A_m] = \inf \). Actually due to the bounds given after equation (20) and the bound of equation (22) we can build this sequence with \( |A_m| \leq 2L \) and \( \frac{1}{2} \langle A_m, A_m \rangle \leq M + 1 \).

Due to the relative-compactness of \( S \) in \( L^2_\beta \) we can extract a sub-sequence \( A_n \) such that it converges in \( L^2_\beta \) to some \( U^* \in L^2_\beta \). We shall have \( |U^*| \leq 2L \) almost everywhere (a.e.).

Then due to the continuity in \( L^2_\beta \cap \{ |U| \leq 4L \} \) of \( \int re^\beta d^2z \) we have

\begin{equation}
\lim_{n\to\infty} I[A_n] = \lim_{n\to\infty} \int \frac{1}{2} \nabla A_n \cdot \nabla A_n d^2z - \int U^* \beta d^2z + \int re^{U^*} \beta d^2z = \inf.
\end{equation}

Given a \( \rho \in C_0^\infty \) the functions \( A_n + \epsilon \rho \) for sufficiently small \( |\epsilon| \) belong to \( S \) and thus

\begin{equation}
\lim_{n\to\infty} I[A_n + \epsilon \rho] \end{equation}

\begin{equation}
= \inf + \int (\frac{\epsilon^2}{2} \nabla \rho \cdot \nabla \rho - \epsilon \Delta \rho U^* - \epsilon \rho \beta + r(\epsilon^2 \rho - 1) e^{U^*} \beta) d^2z \geq \inf.
\end{equation}

Using \( e^x - 1 - x \leq \frac{1}{2} x^2 e^{x|l|} \) and the boundedness of \( \rho \) we have

\begin{equation}
0 = \int (-\Delta \rho U^* - \rho \beta + r \rho e^{U^*} \beta)d^2z \equiv (-\Delta \rho, U^*)_l + (\rho, r e^{U^*} - 1)_{l,\beta}
\end{equation}
for any $\rho \in C^\infty_0$. Equation (34) tells us that $U^*$ is a weak solution of equation (12). To reach a strong solution define

$$U_1(z) = \frac{1}{4\pi} \int \log |z - z'|^2 (r(z')e^{U^*(z')} - 1) \beta(z')d^2z'. \quad (35)$$

We have from (34)

$$0 = \int \Delta \rho (U^* - U_1)d^2z \quad (36)$$

whose most general solution is, due to Weyl lemma [16]

$$U^* = U_1 + h \quad \text{a.e.} \quad (37)$$

with $h$ harmonic function. Thus we can now replace in (35) $U^*$ with $U_1 + h$ obtaining

$$U_1(z) = \frac{1}{4\pi} \int \log |z - z'|^2 (r(z')e^{U_1(z')} + h(z')) \beta(z')d^2z'. \quad (38)$$

Being $r$ and $U^*$ bounded and satisfying $\beta$ the bounds (5)–(7), equations (35) and (38) imply that $U_1$ is continuous with its first and second derivatives and thus $W \equiv U_1 + h$ satisfies

$$\Delta W = (re^W - 1)\beta \quad (39)$$

which is equation (12) and this concludes the existence proof for the sphere with elliptic singularities.

We can also determine the harmonic function $h$. Being

$$\int (re^{U^*} - 1)\beta d^2z \quad (40)$$

convergent, $U_1$ grows at infinity not faster than $\log z\bar{z}$ and the boundedness of $U^*$ implies

$$h = c_1, \quad \int (re^{U^*} - 1)\beta d^2z = 0 \quad (41)$$

which fixes also the value of the constant $c_1$

$$e^{c_1} \int r\beta e^{U_1}d^2z = \int \beta d^2z. \quad (42)$$

In the presence of parabolic singularities the positive function $\beta$ in addition to the requirements (5)–(7) is chosen in finite domains $D_P$ around the parabolic singularities to be equal to

$$\beta = \frac{8}{|\zeta|^2 \log^2 |\zeta|^2} \quad (43)$$

with $\zeta = z - z_p$.

It is still possible to define a function $v$ such that

$$\Delta v = \beta, \quad v \approx -2\log |z|^2 \text{ for } z \to \infty. \quad (44)$$

First write

$$\phi_1 = \sum_k (-2\eta_k) \log |z - z_k|^2 - \sum_P \log |z - z_p|^2. \quad (45)$$
Then introduce \([12]\) a smooth function \(w_0\) with compact support which in each neighborhood \(D_P \) of \(z_P\) equals
\[
- \log \log^2 |ζ|^2
\]
and set
\[
v = φ_1 + w_0 + \frac{1}{4\pi} \int \log |z - z'|^2 (β(z') - Δw_0(z')) d^2z'.
\]
We normalize then \(β\) as to have
\[
\frac{1}{4\pi} \int (β - Δw_0) d^2z = \frac{1}{4\pi} \int β d^2z = -2 + \sum_k 2η_k + \sum_p 1
\]
which again is possible due to the topological inequality \(\sum_k 2η_k + \sum_p 1 > 2\).

Then everything follows as in the case where only elliptic singularities are present.

### 3. Extension to \(g \geq 1\)

We give now the extension of the previous results to the case of a compact Riemann surface of genus \(g \geq 1\).

From the viewpoint of the functional techniques and method, the case \(g \geq 1\) does not differ from the already considered case of the sphere. The main difference is that in the splitting of \(φ\) in a regular and a singular part, the singular background has to be constructed keeping into account the topology of the surface; moreover the source strengths are subject to a topological inequality which generalize the one holding for the sphere.

We deal first with \(g \geq 2\). In this case we can represent such a surface by a standard fundamental polygon in the upper \(z\)-plane \([16]\). This is a curvilinear \(2g\)-gon given by a sequence of arcs \(A_1B_1A^{-1}_2B_2 \ldots A^{-1}_nB^{-1}_n\) where such arcs are pairwise identified. The upper half plane is endowed with the metric
\[
e^{φ_\beta} dz \wedge d\bar{z} = \frac{8}{(z - \bar{z})(\bar{z} - z)} dz \wedge d\bar{z}
\]
and we have
\[
Δφ_\beta = e^{φ_\beta}.
\]
Applying the Gauss-Bonnet relation \([16]\)
\[
\int K e^{φ_\beta} dz \wedge d\bar{z} = 2\pi(2 - 2g) = 2\piχ_E
\]
and taking into account that the curvature \(K\) is given by
\[
K = -\frac{1}{2} \frac{Δφ_\beta}{e^{φ_\beta}} = -\frac{1}{2}
\]
we have for the area
\[
A = \int e^{φ_\beta} dz \wedge d\bar{z} = 4π(2g - 2).
\]
We split the field $\phi$ as
\[ \phi = \phi_B + \psi \] (54)
where $\psi$ obeys periodic boundary conditions. We have
\[ \Delta \psi + e^{\phi_B} = e^{\phi + \phi_B} \] (55)
which, integrated, implies due to (53) the topological restriction,
\[ \sum_k 2\eta_k + \sum_P 1 + 2g - 2 > 0. \] (56)
In analogy to what done for the sphere we set
\[ \psi = U + v \] (57)
with, when in presence of only elliptic singularities,
\[ v = 4\pi \sum_k -2\eta_k G(z, z_k) + \int G(z, z') \beta(z') d^2 z' \] (58)
where the positive function $\beta$ is chosen to satisfy around the singularities the properties given in the previous section and with $\beta \, dz \wedge d\bar{z}$ periodic at the boundary. $G(z, z')$ is the Green function of the Laplace–Beltrami operator $e^{-\phi_B} \Delta$ on the fundamental polygon satisfying
\[ 4\partial_z \partial_{\bar{z}} G(z, z') = \Delta G(z, z') = \delta(z, z') - \frac{e^{\phi_B(z)}}{A} \] (59)
and $A$ is given by equation (53). Equation (1) becomes
\[ \Delta U + \beta + \frac{e^{\phi_B}}{A} \left( 4\pi \sum_k 2\eta_k - \int \beta d^2 z \right) + e^{\phi_B} = e^{\phi_B} e^U \] (60)
and we normalize the positive function $\beta$ as
\[ \int \beta dz \wedge d\bar{z} \frac{1}{2} = 4\pi \left( \sum_k 2\eta_k + 2g - 2 \right) \] (61)
which is possible due to the topological inequality (56). For the torus we simply employ $\phi_B = 0$. Thus equation (60) takes the form
\[ \Delta U + \beta = e^{\phi_B} e^U \equiv e^{\phi_B} e^{U} \] (62)
and the functional (14) becomes
\[ I[U] = \int \left( \frac{1}{2} \nabla U \cdot \nabla U - \beta U + e^{\phi_B} e^U \right) d^2 z. \] (63)
We proceed now as for the sphere with $r$ replaced by $e^{\phi_B} r$. One starts from the real pre-Hilbert space $H$ of the $C^1$ functions satisfying periodic boundary conditions and with norm (15) where the integral is now extended to the fundamental polygon. The relative-compactness of the subset $S$ given by $|A| \leq 4L, \frac{1}{2}(A, A) \leq M + 2$ of $H$ is proven by multiplying the periodic field in a neighborhood of the fundamental polygon by a smooth positive function $\sigma(z)$ which is 1 inside the polygon and vanishes outside such neighborhood. The $L^2_\beta$ relative-compactness on such extended region induces the $L^2_\beta$ relative-compactness of the subset $S$ defined on the fundamental polygon. The analogue of the function $U_1$ is defined as in (35), replacing $r$ with $e^{\phi_B} r$. 

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and, in order to deal neatly with the points on the boundary, by extending the integration region to a finite domain containing the fundamental polygon. The sum $U_1 + h$ which equals $U^*$ does not depend on the choice of the domain. One then proceeds as in the case of the sphere.

When parabolic singularities are also present one acts again as in the case of the sphere.

4. Uniqueness

We know that the solution of equation (1) is locally equivalent to the solution of the ordinary differential equation in the complex plane (see e.g. [17])

$$f''(z) + Q(z)f(z) = 0$$

which is known as auxiliary differential equation.

In a neighborhood of an elliptic singularity we have, with $\phi = -2\eta_K \log(\zeta \bar{\zeta}) - 2\log[f(\zeta) f(\bar{\zeta}) - \kappa^4(g(\zeta)g(\bar{\zeta}))^{-1}]$ (65)

where $f(\zeta)$ and $g(\zeta)$ are given by a locally convergent power expansions with non zero constant terms and at infinity for the sphere $\phi = -2 \log \zeta \bar{\zeta} + h(1, 1)$ with $h$ analytic function in the two variables. Around parabolic singularities we have the expression [17]

$$\phi = -\log \zeta \bar{\zeta} - \log \log 2(\zeta \bar{\zeta}) - 2 \log \left[ g(\zeta)g(\bar{\zeta}) + \frac{f(\zeta)g(\bar{\zeta}) + f(\bar{\zeta})g(\zeta)}{\log \zeta \bar{\zeta}} \right].$$

Consider two solutions $\phi_1$ and $\phi_2$ of equation (1) satisfying equations (2)–(4). Then we have for the sphere, with $\partial f = \partial zf, \bar{\partial} f = \partial \bar{z}f$,

$$0 \leq i \int \partial(\phi_2 - \phi_1) \wedge \bar{\partial}(\phi_2 - \phi_1) = i \int (\phi_2 - \phi_1)\bar{\partial}(\phi_2 - \phi_1) - \frac{i}{2} \int (\phi_2 - \phi_1)\partial \bar{\partial}(\phi_2 - \phi_1)$$

where the contour integral is around the singularities $u_K$ and $u_P$ and at infinity and due to the behavior of $\phi_2 - \phi_1$ given by equation (65) and (66) it vanishes. Thus we have $\phi_2 = \phi_1$. For $g > 0$ one acts exactly in the same way using instead of $\phi$ the $\psi$ of equation (54). The usual uniqueness arguments [9–12] are more complicated because they do not use the information about the non leading terms appearing in the expansion of $\phi$ around the singularities and which we obtained from the expressions (65) and (66).

5. Conclusions

The Liouville equation plays a very important role in several chapters of theoretical physics. In this paper we gave a short, easily accessible and rigorous proof of the existence and uniqueness of the solutions of the Liouville equation in presence of both elliptic sources and parabolic sources (punctures), for all compact Riemann surfaces. These are the cases of pre-eminent interest in theoretical physics.

The tools employed are standard results of functional analysis, complemented in the case of higher genus by the usual representation of compact Riemann surfaces as quotients of the complex plane ($g = 1$) or of the complex upper half plane ($g \geq 2$).
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