Phase-dependent thermoelectricity in short Josephson junctions at helical edge states

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Thermoelectric currents in Josephson junctions are tunable by the superconducting phase difference $\phi$ across junctions. We show that the phase-dependent thermoelectric effect is achievable in short Josephson junctions with two normal metal leads attached at opposite ends, formed at helical edge states of two-dimensional topological insulators (TIs). For all finite phases, an asymmetry appears around the zero energy in the transmission spectra except for $\phi = n\phi_0$, where $n$ is an half-integer and $\phi_0$ is the flux quantum. This asymmetry plays a key role in inducing the charge and heat current through the thermally biased junction. The finite-size topological junction hosts topological Andreev bound states, tunable by the phase, which mediate the current through the junction. However, the current amplitudes are sensitive to the junction size. The optimization condition for a good thermoelectric junction is achieved when the superconductors are of the order of coherence length. The present work demonstrates that TI-based short Josephson junctions are promising systems for superconductor-based phase-dependent thermoelectricity.

I. INTRODUCTION

Study of thermoelectric effects in superconductors and superconducting junctions has been rejuvenated in recent years, breaking the concepts of poor thermoelectricity in superconductors 1–9. The thermoelectric effects in ordinary superconductors were expected to be low or even vanishing, primarily because of the superconducting gap in the density of states. The particle-hole symmetry in the energy spectrum is responsible for low thermoelectricity in the linear regime 9. On top of that, thermally induced current interferences with the superflow, and this causes the separation of those tricky. For these reasons, conventional Bardeen-Cooper-Schrieffer (BCS) superconductors were not considered as active thermoelectric materials for several years 10. In contrast, unconventional superconductors were studied in few works to enhance the thermoelectric effects in them 11,12.

Recently, some efforts have been put to enhance thermoelectricity in superconducting junctions instead of bare superconductors. Breaking the particle-hole symmetry separately for each spin by applying a local spin-splitting field using ferromagnetic elements 1 and forming ferromagnet/superconductor 2–8,13 or antiferromagnet/superconductor hybrid structures 14 have been predicted as promising pathways to enhance the thermoelectricity in superconducting systems. Generating thermoelectric currents in superconducting junctions in this way has drawn more attention after the experimental verification in 2016 2, where an excellent agreement with the theoretical prediction 1 was confirmed. In most of the above-mentioned works, only the linear regime was considered. Very recently, it has been predicted that a nonlinear thermoelectric effect can occur in the presence of spontaneously broken particle-hole symmetry 15.

For the thermoelectricity in superconducting structures, topological materials have been considered in very few works 16,17 as the combination of global topology and local superconducting order has been established to host exotic transport properties in the literature 18–22. Particularly, superconducting junctions involving two-dimensional (2D) topological insulators (TI) 23 have drawn great attention because of its potential to influence scattering processes 24–26 and most importantly to host Majorana fermions 23,24,27–30. The one-dimensional (1D) helical edge states make 2D TIs 31 more effective by preventing all the backscatterings but admitting only two processes: (i) Andreev reflections and (ii) electron transmissions through the junction 20. Topological Andreev bound states (ABS) appear in these junctions, which have drawn special attention 20,32.

A natural question arises whether it is possible to detect these ABS via thermoelectricity and utilize the topological properties of the junctions in enhancing the thermoelectric effects in them. In a recent work, Kalenkov et al. have shown that depending on the topology and the temperature gradient it is possible to generate a large phase-coherent thermoelectric voltage in a Josephson junction (JJ) 33. Also, there are recent predictions for the thermoelectric detection of ABS in ferromagnet/unconventional superconductor junction 34 and in JJ 17. Notably, among various superconducting junctions, JJ is no exception for thermoelectricity. There is an advantage of using JJ to enhance the thermoelectricity in it. One can avoid using external elements like non-magnetic or magnetic impurity 35, or any engineering like creating vacancy 36, which have been utilized to enhance thermoelectric effects in other junctions. A non-trivial thermoelectric voltage can be achieved just by tuning the superconducting phase of JJ 15,33,37.

Motivated by this, we study the thermoelectric effects in short JJ (sJJ) when it is formed at the 1D helical edges of 2D TI in proximity to ordinary superconductor. The phase-tunable topological ABS formed in our normal metal/short Josephson junction/normal metal (N-sJJ-N) junction at the edge of 2D TI help generate charge current in the thermally biased junction. The appearance of the asymmetry around the zero energy in the transmission spectra plays the key role here. The heat currents flow-
ing through the junction are also tunable by the phase of the junction. However, the amplitudes of the currents in our $s$JJ are sensitive to the lengths of the superconductors. We demonstrate that the charge and heat current can be optimized when the superconductors’ lengths are of the order of coherence length. Our work thus predicts topological Josephson junction as a potential thermoelectric system where the thermoelectric effects are smoothly controllable by the phase of the junction.

II. MODEL AND HAMILTONIAN

![FIG. 1. N-sJJ-N junction at the edge of 2D TI with two ends at different temperatures. The red and blue lines represent the helical edge states of TI.](image)

We consider a sJJ where two finite size superconductors, each having length $L_S$, are coupled via a tiny insulating region. We take this insulator region as tiny just to simplify the calculation. A finite width of the insulator region will not affect our results qualitatively. The junction is formed at the edge of a 2D TI and attached to two normal metal leads on opposite sides to form N-sJJ-N set-up. The superconductivity is proximity induced by using a traditional BCS superconductor as presented in Fig. 1. The lengths of the two superconductors are set exactly equal to each other (denoted by $L_S$) for simplicity. A small difference between them will not affect our results qualitatively. The phase difference between two superconductors of the sJJ can be tuned by external magnetic flux $\phi$. We describe each part of the N-sJJ-N junction by Bogoliubov-de Gennes (BdG) Hamiltonian in the basis $\Psi(x) = (\psi_\uparrow(x), \psi_\downarrow(x), \psi^\dagger_\uparrow(x), -\psi^\dagger_\downarrow(x))$ as $^{20}$,

$$H_{\text{BdG}} = \begin{pmatrix} H & \Delta \\ \Delta^\dagger & -H \end{pmatrix}, \tag{1}$$

where the normal part Hamiltonian is given by

$$H = -i\alpha F \partial_x \sigma_z - \mu \sigma_0. \tag{2}$$

The first term of Eq. (2) is the kinetic energy term following the linear dispersion relation of the 1D metallic edge states of 2D TI. The second term includes the chemical potential $\mu$. The Pauli matrices $\sigma_i$, act in spin space and $\psi^\dagger_\uparrow(x)$ ($\psi_\downarrow(x)$) is the creation (annihilation) operator for an electron with spin $\sigma \in \{\uparrow, \downarrow\}$ at position $x$. The off-diagonal matrices of Eq. (1) are responsible for the proximity-induced superconductivity described by the pair potential as: $\Delta(x) = \Delta(x) \delta(x)$, where $\Delta(x)$ is the superconducting order parameter. We set it as:

$\Delta(x) = $ for the left ($0 < x < L_S$) and $\Delta(x) = \Delta, e^{ix}$ for the right superconductor ($L_S < x < 2L_S$) to have a finite phase difference in our sJJ, otherwise $\Delta(x) = 0$ in all normal regions. We set the Fermi velocity $v_F = 1$ and $\Delta = 1$ so that for the symmetric junction where $\Delta_L = \Delta_R = \Delta$, the superconducting coherence length is $\xi = h v_F / \Delta = 1$ considering $\hbar = 1$. We show all the results for $\mu = 0$ (for normal regions) and $\mu = 2$ (for superconducting regions) but our results are insensitive to the chemical potential qualitatively.

For the symmetric junction, we take $\Delta_L = \Delta_R$ determined by the system temperature $T$, following the relation $\Delta(T) = \Delta_0 \tanh(1.74 \sqrt{T_c / T - 1})$. We show all the results for the symmetric junction at $T/T_c = 0.3$. To model the gap asymmetry, we consider $\Delta_L = \Delta(0.7)$ i.e., $T/T_c = 0.7$ and $\Delta_R = \Delta_0$. We take the difference between the two gaps ($\Delta_L - \Delta_R$) much higher to maximize the effect of the gap asymmetry on the transport properties for the sake of understanding.

III. THEORETICAL FORMALISM

We consider a temperature gradient across the junction without any bias voltage. The temperatures of the two leads are maintained at $T + \Delta T/2$ and $T - \Delta T/2$ (as shown in Fig. 1) to set the temperature difference across the junction as $\Delta T$. Note that, $T$ is scaled by the superconducting transition temperature $T_c$. The applied temperature gradient acts in two ways: (i) it tunes the gaps in the density of states of the two superconductors of the sJJ, and (ii) it also affects the quasiparticles’ occupation factors in the junction $^{38}$. Consequently, there appear two different types of currents: charge current and heat current. The thermally induced charge current consists of a dissipative and a non-dissipative component. The variation in superconducting gaps affects the usual Josephson current, which is non-dissipative. It can be expressed in terms of the variation of the gap as: $\delta I_c = \sum I_c \partial I_c / \partial \Delta_i \delta \Delta_i$ considering the contribution by each lead $i$ connected to the superconductor with gap $\Delta_i$. On the other hand, the occupation factor affects the charge current induced by the thermal gradient, which is dissipative. The dissipative and non-dissipative parts of the charge current can be separated by reversing the sign of the superconducting phase $\phi$. The dissipative part is even in phase $\phi$, i.e., $I(\phi) = -I(-\phi)$, whereas the non-dissipative component is odd in $\phi$: $I(\phi) = -I(\phi)$. $^{38}$

**Charge current:** To evaluate the charge current induced by the temperature gradient $\Delta T$, we employ the Landauer transport theory. It can be written as the difference between the currents flowing in the opposite directions (coming from opposite leads) as $^{3,38}$ $I_c = I_L^c - I_R^c$

where

$$I_L^c = \frac{2e}{\hbar} \int_0^\infty d\varepsilon [\tilde{v}_{\uparrow}^\dagger(\varepsilon) - \tilde{v}_{\downarrow}^\dagger(\varepsilon)] f(\varepsilon / T_L) \tag{3}$$
where $e$ is the electronic charge, $h$ is the Planck’s constant, $\varepsilon$ is the incoming electron energy, and $f$ is the Fermi distribution function. Here, $l$ stands for L or R to represent the left or right normal metal leads, respectively, and $i_{l}^{\varepsilon(h)}$ denotes the contributions by the electrons (holes) in $l$-th lead according. Keeping in mind the initial condition that $T_{L}^{R} = T_{R}^{R}$ for $\Delta T = 0$, Eq.(4) transforms into the form involving only one lead as,

$$I^{c} = \frac{2e}{h} \Delta T \int_{0}^{\infty} d\varepsilon \left[ i_{L}^{e}(\varepsilon) - i_{L}^{h}(\varepsilon) \right] \frac{\partial f(\varepsilon/T)}{\partial T}.$$  \hspace{1cm} (4)

The lower limit of the integration in Eq.(4) is to be replaced by the maximum among $\Delta_{L}$ and $\Delta_{R}$ if $T_{e}^{R} = T_{h}^{R} = 0$ within the subgap energy in our case.

Following the current conservation, the charge current should be continuous and we can find it out using the BdG wavefunctions and finally express it in terms of the transmission probabilities given by,

$$i_{L}^{\eta} = T_{e}^{RL} - T_{h}^{RL}$$  \hspace{1cm} (5)

with $\eta \in \{e,h\}$ and $T_{e}^{ll'}_{\eta \eta'} = |t_{ll'}^{\eta}_{\eta'}|^{2}$ where $T_{e}^{ll'}_{\eta \eta'}$ ($t_{ll'}^{\eta}_{\eta'}$) is the probability (amplitude) of the transmission of $\eta'$ type particles from $l'$-th to $l$-th lead as $\eta$. In our case, $t_{ll'}^{\eta}_{\eta'} = 0$ when $l \neq l'$ for $\eta \neq \eta'$. The quasiparticles' transmissions take part in the dissipative part of the thermally induced charge current. The expressions for the transmission amplitudes are mentioned in the Appendix A. From now on, we will use the notation $T_{e}^{RL}$ in place of $T_{e}^{RL}$ throughout the rest of the manuscript for simplicity.

Now, in absence of any bias voltage, the charge current per unit temperature difference is denoted by thermoelectric coefficient as

$$L_{12} = \frac{I^{c}}{\Delta T}.$$  \hspace{1cm} (6)

Note that, the conventional Seebeck current is always associated with the condensate flow in JJs. The phase can help in separating the non-dissipative current from the usual Josephson current. Reversing the phase can help in separating the non-dissipative charge current from the usual Josephson current.

**Heat current:** To calculate the heat current, we follow the same prescription considering the contributions by the individual leads as $I^{\eta} = I_{L}^{\eta} - I_{R}^{\eta}$ where

$$I^{\eta} = \int_{0}^{\infty} \varepsilon d\varepsilon \left[ i_{L}^{e}(\varepsilon) + i_{L}^{h}(\varepsilon) \right] f(\varepsilon/T).$$  \hspace{1cm} (7)

Using the initial condition for $\Delta T = 0$ similar to the charge current, we finally arrive at

$$I^{\eta} = \frac{2e}{h} \Delta T \int_{0}^{\infty} d\varepsilon \left[ i_{L}^{e}(\varepsilon) + i_{L}^{h}(\varepsilon) \right] \frac{\partial f(\varepsilon/T)}{\partial T}.$$  \hspace{1cm} (8)

where the contributions by the electron-like and hole-like quasiparticles are give by Eq.(5). The heat current per unit temperature difference is defined as the thermal conductance and it is given by

$$K = \frac{I^{\eta}}{\Delta T}.$$  \hspace{1cm} (9)

We calculate thermoelectric coefficient $L_{12}$ and thermal conductance $K$ for our sJJ considering small temperature gradient i.e., $\Delta T \ll T/T_{c}$ within the linear response regime.

**IV. RESULTS AND DISCUSSIONS**

To investigate thermoelectric properties of our sJJ, we compute the charge and heat current and present the results in this section. For the sake of understanding of the behaviors of the currents, we also investigate the quasiparticle transmissions throughout our N-sJJ-N junction. Since the temperatures of the two normal regions are different due to the temperature gradient across the N-sJJ-N junction, it is expected that it will affect the nearby superconductors accordingly to have different superconducting gaps. To explore the effect of the gap asymmetry in detail, we discuss both junctions with symmetrical and asymmetrical superconducting gaps called symmetric and asymmetric junctions, respectively, in the following subsections.

**A. Symmetric junction ($\Delta_{L} = \Delta_{R}$)**

We start by considering the simplest scenario where both the superconductors of the sJJ have the same gaps determined by the system temperature $T/T_{c}$.

**1. Transmission probability**

In order to understand the behaviors of the charge and heat currents flowing through the N-sJJ-N junction, we analyze the behaviors of the transmission spectra at first. We employ the scattering matrix method to calculate the transmission probability and present them in Fig. 2. For the details of the formalism and expression of the transmission probability, $T_{e}^{RL}$, we refer to Appendix A.

In Fig. 2, we show the results for two values of the phase difference ($\phi = \phi_{0}/4$ and $3\phi_{0}/4$) across the junction and various lengths $L_{S}$ of the superconductors. We see that the spectra is asymmetric with respect to $\omega=0$ for both phases. This asymmetry exists as long as the phase is
neither zero i.e., $\phi \neq 0$, nor half-integer multiples of $\phi_0$ i.e., $\phi/\phi_0 \neq n$ where $n$ is an half-integer and $\phi_0 = 2\pi$. In absence of any phase difference between the two superconductors i.e., $\phi = 0$, the transmission amplitude is zero throughout the energy gap window with coherence peaks at the edges of the gap\cite{32}, similar to what we get in any ordinary transparent normal metal/superconductor junction\cite{33}. Because of the helical nature of the edge states of the TI, there is no ordinary reflection to take place in the junction. The zero transmission is compensated by the unity Andreev reflection following the unitarity relation. However, the situation becomes dramatic when we tune the superconducting phase. By tuning the phase, the transmission peaks associated by reduction in Andreev reflection (guaranteed by the unitarity relation between them) are found to exist due to the formation of ABS at the junction as discussed in Ref. \cite{32}. In the present work, we are only interested in other phases which are not discussed in Ref. \cite{32}. For any finite phase other than the time-reversal symmetric point set by $\phi/\phi_0 = n$, the transmission peaks are asymmetrically positioned around $\omega = 0$. The symmetry breaking around $\omega = 0$ is true for any finite $L_S$, but the transmission peaks get flattened with the decrease in the size of the two superconductors. Specifically, when $L_S \ll \xi$, the transmission amplitude is close to unity throughout the energy window. It shows prominent peaks when $L_S \sim \xi$ and the transmission peaks get more sharpen when $L_S > \xi$. Naively, for any particular $L_S$, the behaviors of the spectra for a particular phase within the range $0 < \phi < \phi_0/2$, the peak position gets almost inverted to its mirror image with respect to $\omega = 0$ when we tune the phase to another symmetrically chosen value within the range $\phi_0/2 < \phi < \phi_0$. We explore the role of this asymmetry present in the transmission spectra about $\omega = 0$ in inducing both charge and heat currents through the junction.

2. Charge current

With the understanding of the transmission probability, we present the results for thermoelectric function i.e., charge current per unit temperature gradient, when the junction is only subjected to the temperature gradient without any bias voltage. To see the effect of the symmetry breaking around $\omega = 0$ for $\phi/\phi_0 \neq n$ (where $n = 0, 1/2, 3/2, \ldots$) on the charge current, we plot the thermoelectric coefficient defined in Eq. (6) as a function of $\phi/\phi_0$ and $L_S$ in Fig. 3.

In Fig. 3(a) we see that the behavior of the charge current is oscillatory with the change in $\phi/\phi_0$ maintaining zero amplitudes at $\phi/\phi_0 = n$. There remains perfect symmetry around $\omega = 0$ in the transmission spectra when $\phi/\phi_0 = n$ as seen in the previous subsection. Tuning the phase difference to other finite values results in the symmetry breaking around $\omega = 0$. The role of the symmetry can be confirmed from the nodes in the current profiles which exist for $\phi = 0$ and $\phi/\phi_0 = n$. The phase-tunable asymmetry in the transmissions of the quasiparticles within the subgap regime causes an imbalance between the left and right moving charges and as a consequence, a net charge current flows through the junction for all finite values of the phases except half-integer multiples of $\phi_0$. However, beyond the subgap limit, the transmission probabilities of the quasiparticles are finite for all phases of the sJJ.

The change in the peak positions, naively mirror inversion about $\omega = 0$, by tuning the phase, in the transmission spectra reflects in the behavior of the charge current as well. It can be understood as follows. As soon as we tune the phase from zero to a positive finite value, the charge current starts increasing in amplitude (but with negative sign) from zero and then again drops to zero at $\phi_0/2$. With further increase in $\phi$, the phase of the current reverses. The current amplitudes start increasing with positive sign and the profiles get inverted when $\phi_0/2 < \phi < \phi_0$ compared to the profiles found for phases in the range $0 < \phi < \phi_0/2$. This phase reversal corresponds to the naive mirror inversion of the transmission peaks around $\omega = 0$ described in the previous subsection.

The finite transmission of the quasiparticles via ABS in the subgap regime play a major role in the charge current through our junction. However, it is subjected to the asymmetry around $\omega = 0$ present in the transmission spectra. To investigate the role of the phase-tunable ABS, we provide some further results in Appendix B. Remarkably, the appearance of ABS is not accidental. It is protected by the topology of the 2D TI and thus, enhances the possibility of utilizing our sJJ as a good thermoelectric junction. We refer to Appendix C for the discussions on the effect of the base temperature of the system.

Next, we discuss the sensitivity of the charge current to the junction size. We observe that when $L_S \ll \xi$, the amplitude of $L_{12}$ increases with the increase in $L_S$. In contrast, when $L_S > \xi$, the behavior of the thermoelectric function changes. The current amplitude is decreasing with the rise in $L_S$. To investigate the behavior of $L_{12}$ in more detail, we present the density plot of $L_{12}$ as a function of both $\phi/\phi_0$ and $L_S/\xi$ in Fig. 3(b). It is clear that the charge current amplitude is maximum when $L_S/\xi \sim 1$. It shows opposite behavior, either increasing or decreasing with $L_S$ in the two regimes define
by $L_S/\xi \ll 1$ and $L_S/\xi \gg 1$, respectively. Behavior of the charge current with $L_S$ can also be explained by looking at the transmission spectra. When $L_S \ll \xi$, there is almost uniform transmission throughout the subgap regime. With the increase in $L_S$, the asymmetry around $\omega = 0$ starts to appear in the spectra and that leads to increasing charge current with $L_S$. On the other hand, the peak widths get much smaller when $L_S/\xi \gg 1$ resulting in decreasing behavior of the current through the junction.

3. Heat current

In general, it is not compulsory to break the particle-hole symmetry to generate heat current. However, it is possible to tune the thermal current by introducing asymmetry in the junction. To see the effect of the symmetry breaking around $\omega = 0$ on the heat current, we present the results of thermal conductance i.e., heat current per unit temperature gradient as a function of $\phi/\phi_0$ in Fig. 4(a).

We observe that similar to the charge current the behavior of the thermal conductance is also oscillatory with the phase of the junction. The heat current may have maximum values at $\phi/\phi_0 = n$ with $n$ being either zero or half-integer. This behavior is completely different from the behavior of the charge current. This happens because the energy carried by the electrons and holes are additive and they do not cancel with each other even when there exists symmetry around $\omega = 0$ in the transmission spectra. However, the maxima of the heat current profiles for $L_S/\xi \ll 1$ turns into minima when we increase the system size in the limit $L_S/\xi \gg 1$. This can be explained by the presence of sharp peaks in the transmission probability spectrum. With the increase in the system size, the almost flat close to unity profiles change to have a few peaks and that effectively reduces the total transmission probability of the quasiparticles within the subgap regime and that further reduces the heat current. Also, unlike the behavior of the charge current, the behavior of the heat current is monotonic with $L_S$. The heat current amplitude continuously decreases with the increase in $L_S$. However, the rate of decrease of the heat current in the regime $L_S/\xi \gg 1$ is much lower than the

![FIG. 4. Thermal conductance $\mathcal{K}$ for symmetric junction as a function of (a) $\phi/\phi_0$ and (b) $\phi/\phi_0$ and $L_S/\xi$.](image1)

Note that, the heat current induced by the temperature gradient is higher for extremely short junction. We reconfirm the same from the density plot of $\mathcal{K}$ as shown in Fig. 4(b). We see that the heat current amplitude is highest for the smallest size of the junction. It decreases when $L_S/\xi \gtrsim 1$ being oscillatory with the phase across the junction. Similar to the charge current, the heat current is also phase-tunable. To increase the efficiency of any thermoelectric system, it is always recommended to minimize the thermal conductance and maximize the Seebeck coefficient\(^5,21\). We can optimize this condition for our JJ based thermoelectric system in the limit $L_S/\xi \sim 1$ and it is externally controllable by the phase of the junction. To understand the behaviors of $\mathcal{K}$ as a function of $T/T_c$, we refer to Appendix B.

B. Asymmetric junction ($\Delta_L \neq \Delta_R$)

Till now, our discussions are restricted to the symmetric junction where both the superconductors of our sJJ have similar gaps. In reality, as soon as we apply a temperature gradient between the normal regions, it is highly possible that the gaps of the two superconductors are modified accordingly since the superconductors are directly attached to the normal regions. To investigate the effect of this gap asymmetry on the thermoelectricity in sJJ, we present the results of transmission amplitudes, thermoelectric function and the heat conductance for the condition of asymmetric superconducting gaps.

We refer to Fig. 5 for the results of the probability of transmission of the quasiparticles in the asymmetric junction. We notice that in the transmission spectra some new kinks appear when we take the gaps of the two superconductors different. For $L_S/\xi \ll 1$, it gets more flattened with some additional kinks. For other limits of the lengths, the heights or widths of the peaks get reduced. Note that, we show the results for two different values of the phase difference of the sJJ. The qualitative behaviors of the transmission amplitudes, particularly the asymmetry around $\omega = 0$ remain similar for all finite values except $\phi/\phi_0 = n$.

Now, we discuss the behaviors of the charge and heat current induced by temperature difference for the asym-
metric junction as shown in Fig. 6. We observe that the charge current amplitude decreases for all limits of the lengths. The behavior of the charge current with the length and the phase is similar to the those in symmetric junction. This behavior of the charge current can be explained in terms of the transmission probability following the similar prescription as mentioned for symmetric junction. In contrast to the charge current, the heat current amplitude shows different behavior in the asymmetric junction. It shows smaller magnitude for lower superconductor size compared to that in symmetric junction. This will help in optimizing the current amplitudes for our short thermoelectric junction in the limit \( L_S/\xi \sim 1 \).

V. SUMMARY AND CONCLUSIONS

To summarize, we have explored the thermoelectric properties of sJJ formed at the helical edges of 2D TI in terms of the thermoelectric function and thermal conductance. In order to understand the behavior of thermoelectric currents, we have investigated the transmission probability as well. The topological ABS formed in the junction helps optimize the thermally induced charge and heat current by preventing the loss due to ordinary reflection, allowing only quasiparticles’ transmissions through the junction in the subgap regime. The asymmetry in the transmission spectra around zero energy, achieved by tuning the phase difference of the sJJ, induces the charge current and tunes the heat current through the junction. The phase-dependent thermoelectricity is sensitive to the lengths of two finite sized superconductors of the junction. To optimize the thermoelectricity in the sJJ, we recommend to consider two superconductors of our short junction in the regime \( L_S/\xi \sim 1 \). Note that, we have taken only the electronic contributions into account neglecting the phonon part for both symmetric and asymmetric junctions. It is justified since we are in the low-temperature regime. For realization of the TI based sJJ, HgCd/HgTe and InAs/GaSb are good candidates for the TI as shown in Refs. [42 and 43]. For the proximity-induced superconductivity, any ordinary BCS superconductor e.g., Nb \((T_c \sim 9.2K)\) can be used. Our anticipation for the phase-dependent thermoelectricity in sJJ thus enhances the potential of JJs from the perspective of application in thermoelectric devices.

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Appendix A: Calculation of scattering amplitudes: scattering matrix formalism

In this Appendix, we describe the scattering matrix formalism which is employed to find the scattering amplitudes for our N-sJJ-N junction. The general form of the scattering states at the different regions of the N-sJJ-N junction is given below. For symmetric junction, expressions for the scattering amplitudes drop down to the expressions
mentioned in Ref. [32].

\[\Psi_1(x) = \psi_1^N e^{ik_{xL}} + \frac{1}{v_F} \psi_2^N e^{ik_{xR}} + \frac{1}{v_F} \psi_2^N e^{-ik_{xL}} x < 0 \]
\[= a_1 \psi_1^S e^{ik_{xL}} + b_1 \psi_1^S e^{-ik_{xR}} + c_1 \psi_1^S e^{ik_{xR}} + d_1 \psi_1^S e^{-ik_{xL}} \quad \text{for } 0 < x < L_S \]
\[= p_1 \psi_1^S e^{ik_{xL}} + q_1 \psi_1^S e^{-ik_{xR}} + r_1 \psi_1^S e^{ik_{xR}} + s_1 \psi_1^S e^{-ik_{xL}} \quad \text{for } L_S < x < 2L_S \]
\[= i_R \psi_1^N e^{ik_{xL}} + i_R \psi_1^N e^{-ik_{xL}} \quad \text{for } x > 2L_S \]
\[\Psi_2(x) = \psi_2^N e^{ik_{xL}} + \frac{1}{v_F} \psi_2^N e^{-ik_{xR}} + \frac{1}{v_F} \psi_2^N e^{ik_{xR}} x < 0 \]
\[= a_2 \psi_2^S e^{ik_{xL}} + b_2 \psi_2^S e^{-ik_{xR}} + c_2 \psi_2^S e^{ik_{xR}} + d_2 \psi_2^S e^{-ik_{xL}} \quad \text{for } 0 < x < L_S \]
\[= p_2 \psi_2^S e^{ik_{xL}} + q_2 \psi_2^S e^{-ik_{xR}} + r_2 \psi_2^S e^{ik_{xR}} + s_2 \psi_2^S e^{-ik_{xL}} \quad \text{for } L_S < x < 2L_S \]
\[= i_R \psi_2^N e^{ik_{xL}} + i_R \psi_2^N e^{-ik_{xL}} \quad \text{for } x > 2L_S \]
\[\Psi_3(x) = \psi_3^N e^{ik_{xL}} + d_3 \psi_1^S e^{ik_{xL}} \quad \text{for } x < 0 \]
\[= a_3 \psi_3^S e^{ik_{xL}} + b_3 \psi_3^S e^{-ik_{xL}} + c_3 \psi_3^S e^{ik_{xR}} + d_3 \psi_3^S e^{-ik_{xL}} \quad \text{for } 0 < x < L_S \]
\[= p_3 \psi_3^S e^{ik_{xL}} + q_3 \psi_3^S e^{-ik_{xL}} + r_3 \psi_3^S e^{ik_{xR}} + s_3 \psi_3^S e^{-ik_{xL}} \quad \text{for } L_S < x < 2L_S \]
\[= \psi_3^N e^{-ik_{xL}} + a_3 \psi_3^N e^{ik_{xL}} + b_3 \psi_3^N e^{-ik_{xL}} \quad \text{for } x > 2L_S \]
\[\Psi_4(x) = c_4 \psi_4^N e^{ik_{xL}} + d_4 \psi_4^N e^{-ik_{xL}} \quad \text{for } x < 0 \]
\[= a_4 \psi_4^S e^{ik_{xL}} + b_4 \psi_4^S e^{-ik_{xL}} + c_4 \psi_4^S e^{ik_{xR}} + d_4 \psi_4^S e^{-ik_{xL}} \quad \text{for } 0 < x < L_S \]
\[= p_4 \psi_4^S e^{ik_{xL}} + q_4 \psi_4^S e^{-ik_{xL}} + r_4 \psi_4^S e^{ik_{xR}} + s_4 \psi_4^S e^{-ik_{xL}} \quad \text{for } L_S < x < 2L_S \]
\[= \psi_4^N e^{ik_{xL}} + a_4 \psi_4^N e^{ik_{xL}} + b_4 \psi_4^N e^{-ik_{xL}} \quad \text{for } x > 2L_S \]

where

\[\psi_1^N = (1, 0, 0, 0)^T, \psi_2^N = (0, 1, 0, 0)^T, \psi_3^N = (0, 0, 1, 0)^T, \psi_4^N = (0, 0, 0, 1)^T, \]
\[\psi_1^S = (u_i, 0, v_i, 0)^T, \psi_2^S = (0, u_i, 0, v_i)^T, \psi_3^S = (v_i, 0, u_i, 0)^T, \psi_4^S = (0, v_i, 0, u_i)^T. \]

Note that, here we use the notation N for normal regions, both left and right, since they have exactly similar parameters except the gaps in the asymmetric case, as mentioned in the main text. The wave vectors in the normal regions are given by

\[k_{e(h)}(\omega) = \frac{\mu \pm \omega}{v_F}. \]

Within the superconductors the wave vectors take the form as

\[k_{e(h)}^{S, (i)}(\omega, \Delta_i) = \frac{\mu \pm \sqrt{\omega^2 - \Delta_i^2}}{v_F}. \]

with \(i\) denoting L or R. The coherence factors for the two superconductors are given by

\[u_i, v_i = \left[ \frac{\omega \pm \sqrt{\omega^2 - \Delta_i^2}}{2\omega} \right]^{1/2}. \]

To solve the equations, the wave functions are matched at the three interfaces of the junction, at \(x = 0, x = L_S\), and \(x = 2L_S\). The ordinary reflection and crossed Andreev reflection are prevented due to the helicity of the edge states of 2DTI [30], allowing only two processes: (1) Andreev reflections where an incident electron (a hole) is reflected as a hole (an electron) at the left normal region and (2) electron (hole) transmission at the right normal region. The corresponding transmission amplitudes are given as follow.

For the transmission of electron in the right normal region after injecting an electron from the left normal region, the amplitude reads as
It follows the unitarity relation: \( R_{\text{LL}}^{hh} + T_{\text{RL}}^{hh} = 1 \) where \( R_{\text{LL}}^{hh} = |r_{\text{LL}}^{hh}|^2 \) and \( T_{\text{RL}}^{hh} = |t_{\text{RL}}^{hh}|^2 \). Note that, the following relations hold for our sJJ formed at the helical states: 
\[ v_{\text{LL}}^{ee} = v_{\text{LL}}^{hh}, r_{\text{RR}}^{ee} = r_{\text{RR}}^{hh}, t_{\text{RL}}^{ee} = t_{\text{RL}}^{hh}, t_{\text{LR}}^{ee} = t_{\text{LR}}^{hh} = 0. \]

**Appendix B: The role of ABS in charge current**

To unveil the role of the phase-tunable ABS, we plot the charge current through the symmetric junction in Fig. 7 by dividing the limit of the integration of Eq. (4) into two parts: \([0, \Delta_0]\), \([\Delta_0, \infty]\). The total current is found by setting the limit as \([0, \infty]\). This total charge current is the same as the charge current mentioned in the main text and in the following subsection. We show this charge current breakups for various sizes of the superconductors because of the sensitivity of the charge current to the superconductor size.

In Fig. 7 we observe that for \( L_S/\xi \ll 1 \), the total charge current in the junction is mostly dominated by the transmissions above the superconducting gap. The contributions by the subgap and supergap transmissions are in phase giving rise to the additive total charge current for an extremely short junction. However, the scenario changes when we increase the lengths of the two superconductors. The contributions by the subgap quasi-particles’ transmissions increase with the increase of \( L_S \). When \( L_S/\xi \sim 1 \), the major contribution to the charge current comes from the ABS formed in the junction. Comparing all the sub-figures in Fig. 7, we observe that the total charge current is increasing with \( L_S \) initially but falls down when \( L_S/\xi > 1 \). In fact, when \( L_S/\xi \gg 1 \), the total charge current decreases and is lower than the contributions by the current carried by ABS.
It happens because of the phase change between the contributions by the subgap and supergap transmissions resulting in the enhancement of the suppression of the total charge current. For $L_S/\xi \ll 1$, both subgap and super-gap contributions are in-phase resulting into the higher total charge current. The scenario changes when $L_S/\xi \sim 1$ and more prominently when $L_S/\xi \gg 1$. The total current goes down as the contributions by the subgap and supergap quasiparticles are completely out of phase but comparable in amplitudes. As a consequence, we see that the contributions by the subgap ABS to the thermally induced total charge current is highest and in phase when $L_S/\xi$. Beyond this regime of $L_S/\xi$, the contributions by ABS are majorly compensated by the contributions from the supergap states.

Appendix C: Effects of temperature on charge current

In the main text, we have only presented the behaviors of the charge current at a particular temperature. In the present section, we discuss the behaviors of the charge current at different temperatures. Note that, unless we mention specifically, we always consider the whole range of the integration to calculate the charge current throughout the study. Fixing the base temperature of the system $T/T_c$ to various values, we present all the results by applying a small gradient around that base temperature.

In Fig. 8, we show the density plots of $L_{12}$ as a function of temperature and phase difference of the junction. We see that the oscillatory behavior of the charge current as a function of $\phi/\phi_0$ as discussed in the main text. For $L_S/\xi \ll 1$, the current amplitude becomes large when $T/T_c \sim 0.5$. At this regime of the superconductor length, the asymmetry around $\omega = 0$ in the transmission spectra is much smaller. To enhance the current in this scenario, the system temperature has to be increased sufficiently. Increasing the temperature beyond this value will result in smaller gaps and thus reduces the ABS contributions to the current. This is confirmed when we take larger superconductors i.e., $L_S/\xi \sim 1$. At this limit, the asymmetry around $\omega = 0$ in the transmission probability profile is much higher and we can get enhanced current even in the very low temperature limit. However, the picture changes when we take larger size superconductors in the limit $L_S/\xi \gg 1$. At this limit of the size, the higher amplitude of the current is constrained to the very small regime of the temperature. This can be explained following the similar prescription mentioned for other limits of the size of the two superconductors.

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