Abstract. We deduce using the Ringel-Hall algebra approach explicit formulas for the cardinalities of some Grassmannians over a finite field associated to the Kronecker quiver. We realize in this way a quantification of the formulas obtained by Caldero and Zelevinsky for the Euler characteristics of these Grassmannians. We also present a recursive algorithm for computing the cardinality of every Kronecker quiver Grassmannian over a finite field.

Key words. Kronecker algebra, Hall algebra, Grassmannian

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Introduction

Let $K$ be the Kronecker quiver $K : 1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 1$, $kK$ be the Kronecker algebra over the finite field $k = \mathbb{F}_q$ with $q$ elements and mod-$kK$ the category of its finite dimensional right modules (called Kronecker modules). We consider the rational Ringel-Hall algebra $\mathcal{H}(kK)$ of the Kronecker algebra, with a $\mathbb{Q}$-basis formed by the isomorphism classes $[M]$ from mod-$kK$ and multiplication

$$[N_1][N_2] = \sum_{[M]} F^M_{N_1N_2}[M].$$

The structure constants $F^M_{N_1N_2} = |\{M \supseteq U | U \cong N_2, M/U \cong N_1\}|$ are called Ringel-Hall numbers.

For any module $M \in \text{mod-}kK$, and any $\underline{e} = (a,b)$ in $\mathbb{N}^2$, we denote by $Gr_{\underline{e}}(M)_{\mathbb{F}_q}$ the Grassmannian of submodules of $M$ with dimension vector $\underline{e}$:

$$Gr_{\underline{e}}(M)_{\mathbb{F}_q} = \{N \in \text{mod-}kK | N \leq M, \dim(N) = \underline{e}\}.$$

Then we have that

$$|Gr_{\underline{e}}(M)_{\mathbb{F}_q}| = \sum_{[X],[Y]} F^M_{XY} \text{ dim}^Y = \underline{e}.$$

The Grassmannian cardinalities above play an important role in the theory of cluster algebras. In [3] Caldero and Reineke show for affine quivers that these cardinal numbers are given by integral polynomials in $q$ with positive coefficients. So in our case there is an integral polynomial $p_{\underline{e}}M$ such that $|Gr_{\underline{e}}(M)_{\mathbb{F}_q}| = p_{\underline{e}}M(q)$. Moreover the Euler characteristics $\chi(Gr_{\underline{e}}(M)_C) = p_{\underline{e}}M(1)$.

In [4] Caldero and Zelevinsky describe explicit combinatorial formulas for the Euler characteristics $\chi(Gr_{\underline{e}}(M)_C) = p_{\underline{e}}M(1)$ whenever $M$ is indecomposable.

Using specific recursions obtained by the Ringel-Hall algebra approach and the use of reflection functors we deduce in this paper explicit combinatorial formulas for the cardinalities (polynomials) $|Gr_{\underline{e}}(M)_{\mathbb{F}_q}| = p_{\underline{e}}M(q)$ whenever $M$ is indecomposable. We obtain in this way a quantification of the formulas by Caldero and Zelevinsky. Moreover our recursions provide a
recursive algorithm for computing the cardinality of every Kronecker quiver Grassmannian over a finite field.

1. Facts on Kronecker modules and Ringel-Hall algebras

The indecomposables in mod-$kK$ are divided into three families: the preprojectives, the regulars and the preinjectives (see [1], [2], [8]).

The preprojective (respectively preinjective) indecomposable modules are up to isomorphism uniquely determined by their dimension vectors. For $n \in \mathbb{N}$ we will denote by $P_n$ (respectively with $I_n$) the indecomposable preprojective module of dimension $(n+1,n)$ (respectively the indecomposable preinjective module of dimension $(n,n+1)$). So $P_0, P_1$ are the projective indecomposable modules ($P_0 = S_1$ being simple) and $I_0, I_1$ the injective indecomposable modules ($I_0 = S_2$ being simple).

The regular indecomposables (up to isomorphism) are $R_p(t)$ for $t \geq 1$ and $p \in \mathbb{P}_k^1$ of dimension vector $(td_p, td_p, d_p)$ standing for the degree of the point $p$. The module $R_p(t)$ has regular length $t$ and regular socle the regular simple $R_p(1)$. Suppose that $R_p(0) = 0$. Note that $R_p(t)$ is regular uniserial meaning that the only regular submodule series of $R_p(t)$ is $0 \subset R_p(1) \subset \ldots \subset R_p(t)$.

We will denote by $R_p(\lambda)$ (where $\lambda$ is a partition) the module $\bigoplus R_p(\lambda_i)$ and by $P$ (respectively $I, R$) a module with all its indecomposable components preprojective (respectively preinjective, regular).

Denote by $cM = M \oplus \ldots \oplus M$ $c$-times.

The following lemma is well known.

**Lemma 1.1.** a) $\text{Hom}(R,P) = \text{Hom}(I,P) = \text{Hom}(I,R) = \text{Ext}^1(P,R) = \text{Ext}^1(P,I) = \text{Ext}^1(R,I) = 0$.

b) There are no nontrivial morphisms and extensions between regular modules from different tubes, i.e. if $p \neq p'$, then $\text{Hom}(R_p(t), R_{p'}(t')) = \text{Ext}^1(R_p(t), R_{p'}(t')) = 0$.

c) For $n \leq m$, we have $\dim_k \text{Hom}(P_n, P_m) = m-n+1$ and $\text{Ext}^1(P_n, P_m) = 0$; otherwise $\text{Hom}(P_n, P_m) = 0$ and $\dim_k \text{Ext}^1(P_n, P_m) = n-m-1$. In particular $\text{End}(P_n) \cong k$ and $\text{Ext}^1(P_n, P_n) = 0$.

d) For $n \geq m$, we have $\dim_k \text{Hom}(I_n, I_m) = m-n+1$ and $\text{Ext}^1(I_n, I_m) = 0$; otherwise $\text{Hom}(I_n, I_m) = 0$ and $\dim_k \text{Ext}^1(I_n, I_m) = n-m-1$. In particular $\text{End}(I_n) \cong k$ and $\text{Ext}^1(I_n, I_n) = 0$.

e) $\dim_k \text{Hom}(P_n, I_m) = n+m$ and $\dim_k \text{Ext}^1(I_m, P_n) = m+n+2$.

f) $\dim_k \text{Hom}(P_n, R_p(t)) = \dim_k \text{Hom}(R_p(t), I_n) = d_p t$ and $\dim_k \text{Ext}^1(R_p(t), P_n) = \dim_k \text{Ext}^1(I_n, R_p(t)) = d_p t$.

g) $\dim_k \text{Hom}(R_p(t_1), R_p(t_2)) = \dim_k \text{Ext}^1(R_p(t_1), R_p(t_2)) = d_p \min (t_1, t_2)$.

Let now $\tilde{K}$ be the quiver obtained by reversing the arrows in $K$. The category mod-$k\tilde{K}$ can be identified with the category mod-$kK$ after a formal relabeling of the vertices. In general we will denote by $-M \in \text{mod-}k\tilde{K}$ (respectively by $-M \in \text{mod-}kK$) the relabeled version of $M \in \text{mod-}kK$ (respectively of $M \in \text{mod-}k\tilde{K}$). So we have $-M = M$.

For $i = 1, 2$ denote by mod-$k\tilde{K}\langle i \rangle$ (respectively by mod-$kK\langle i \rangle$) the full subcategory of modules not containing the simple $S_i$ (respectively the simple $\tilde{S}_i$) as a direct summand. Notice that using the formal relabeling mentioned above mod-$k\tilde{K}\langle 1 \rangle$ can be identified with mod-$kK\langle 2 \rangle$ and mod-$k\tilde{K}\langle 2 \rangle$ with mod-$kK\langle 1 \rangle$. Since the vertex 1 is a sink in the quiver $K$ and a source in $\tilde{K}$.
the restriction of the corresponding reflection functors to the above mentioned full subcategories will give us the following inverse pair of equivalences

$$-R^+_1 : \text{mod-}kK(1) \rightarrow \text{mod-}kK(2) \text{ and } R^-_1 : \text{mod-}kK(2) \rightarrow \text{mod-}kK(1).$$

We refer to [5] for all notions and properties related to the reflection functors. Notice that we have

$$-R^+_1 (P_n) = P_{n-1} \text{ for } n \in \mathbb{N}^*, -R^+_1 (I_n) = I_{n+1} \text{ for } n \in \mathbb{N}.$$ Moreover for each $d$ there is a permutation $\sigma_d$ of the set $\{p \in \mathbb{P}_1 | d_p = d\}$ such that for each $p$ with $d_p = d$ we have

$$-R^+_1 (R_p(t)) = R_{\sigma_d(p)}(t) \text{ for all } t \in \mathbb{N}^*.$$

**Remark 1.2.** Notice that if $M \in \text{mod-}kK(1)$ with $\text{dim}M = (m,n)$, then $m - n$ is at most $n$ and $\text{dim}(-R^+_1 (M)) = (m - (m - n), n - (m - n)) = (n, 2n - m)$. Also if $M \in \text{mod-}kK(2)$ with $\text{dim}M = (m,n)$, then $m - n$ is at most $m$ and $\text{dim}(R^-_1 (M)) = (m + (m - n), n + (m - n)) = (2m - n, m)$.

Related with the Ringel-Hall algebra we will need the following properties (see [9]):

**Lemma 1.3.** (Associativity of the Ringel-Hall multiplication) $\sum_{[X]} F^X_A F^M_X F^X_C = \sum_{[X]} F^M_AX F^X_BC$.

**Lemma 1.4.** For $N_1, N_2 \in \text{mod-}kK$ with $\text{Ext}^1(N_1, N_2) = 0$ and $\text{Hom}(N_2, N_1) = 0$ we have $[N_1][N_2] = [N_1 \oplus N_2].$

**Lemma 1.5.** a) If $M,N$ and $L$ are in $\text{mod-}kK(1)$, then $F^L_{MN} = F^{-R^+_1 (L)}_{R^+_1 (M) - R^+_1 (N)}$.

b) If $M,N$ and $L$ are in $\text{mod-}kK(2)$, then $F^L_{MN} = F^{-R^-_1 (L)}_{R^-_1 (M) - R^-_1 (N)}$.

2. Identities for Gaussian coefficients

For $l, a \in \mathbb{Z}$, $l > 0$ we will denote by $G^l_a(q) = \frac{(q^a - 1) \cdots (q^{a-l+1} - 1)}{(q-1) \cdots (q-l+1)}$ the Gaussian $q$-binomial coefficients. By definition $G^0_a(q) = 1$ and $G^{-1}_a(q) = 0$. The following properties of the Gaussian coefficients are well known

**Lemma 2.1.** a) $G^l_a(q) = 0$ for $0 \leq a < l$. Also $G^l_a = G^{a-l}_a$ for $a, l \geq 0$.

b) (Cross product) For all $a, l, j \in \mathbb{Z}$ we have $G^l_a(q) G^j_l(q) = G^l_{a+j}(q)$.

c) (q-Vandermonde) For all $l, a, b \in \mathbb{Z}$ we have $G^{l+a}_a(q) = \sum_{j \in \mathbb{Z}} q^{l(a-j)} G^j_a(q) G^l_a(q)$. Notice that the sums are finite.

Finally we will prove a $q$-analogue of the so called Nanjundiah identity (see [7])

**Proposition 2.2.** For all $m, p, \mu, \nu \in \mathbb{Z}$ we have

$$\sum_{r \in \mathbb{Z}} q^{(m-\mu+\nu-r)(p-r)} G^r_{m-\mu+\nu}(q) G^{p-r}_{p+\mu-\nu}(q) G^{m+r}_{\mu+r}(q) = G^m_{\nu}(q) G^p_{\mu}(q)$$

**Proof.** Denote by $A$ the left expression and by $B$ the right one.

One can immediately see that for $p < 0$ we have $A = B = 0$.

Applying 3 times Lemma 2.1. c) and 2 times Lemma 2.1. b) we have

$$A = \sum_{r \in \mathbb{Z}} q^{(m-\mu+\nu-r)(p-r)} G^r_{m-\mu+\nu}(q) G^{p-r}_{p+\mu-\nu}(q) G^{m+r}_{\mu+r}(q)$$
One can see from here that for \( m < 0 \) we have \( A = 0 \) and trivially also \( B = 0 \).

Consider now the case \( m, p \geq 0 \). Then using Lemma 2.1. a),b) notice that
\[
A = \sum_{s \in \mathbb{Z}} q^{s(m-m-p+s)} G_{m-m+\nu}(q) G_{m+p-s}(q) G_{m+p-s}(q)
\]
\[
= \sum_{s=0}^{p} q^{s(m-m-p+s)} G_{m-m+\nu}(q) G_{m+p-s}(q) G_{m+p-s}(q)
\]
\[
= \sum_{s \in \mathbb{Z}} q^{s(m-m-p+s)} G_{m-m+\nu}(q) G_{m+p-s}(q) G_{m+p-s}(q)
\]
\[
= G_{m}(q) \sum_{s \in \mathbb{Z}} q^{s(m-m-p+s)} G_{m-m+\nu}(q) G_{m+p-s}(q)
\]
\[
= G_{m}(q) G_{p}(q) = B
\]

\[\square\]

3. The recursions

Let \( a, b \in \mathbb{Z} \). We introduce the following notations.

For \( M \in \text{mod-} k \cdot K \)
\[
A_{a,b}^{M} := |Gr_{(a,b)}(M)| = \sum_{[X], [Y]} F_{XY}^{M}
\]

For \( M \in \text{mod-} k \cdot K(1) \)
\[
B_{a,b}^{M} := \sum_{[X], [Y]} F_{XY}^{M}
\]

\( X, Y \in \text{mod-} k \cdot K(1) \)
For $M \in \text{mod-}kK(2)$

$$C_{a,b}^M := \sum_{\substack{\{X,Y\} \in \text{dim}M \cap \text{mod-}kK(2), \dim Y = (a,b)}} F_{XY}^M$$

The sums $A_{a,b}^M, \ B_{a,b}^M, \ C_{a,b}^M$ are considered to be 0 if they are empty. In particular they are 0 if $a < 0$ or $b < 0$. Also notice that if $\dim M = (m,n)$ then $A_{a,b}^M = B_{a,b}^M = C_{a,b}^M = 0$ for $a > m$ or $b > n$.

**Proposition 3.1.** Suppose up to isomorphism $M = sS_1 \oplus M' \oplus tS_2$ with $M' \in \text{mod-}kK(1) \cap \text{mod-}kK(2)$. Let $a, b \in \mathbb{Z}$, $l = a - b$ and $\dim M = (m,n)$.

a) We have that

$$A_{a,b}^M = \sum_{a \in \mathbb{Z}} G_{m-a+c}^c B_{a-c,b}^{M' \oplus tS_2} = \sum_{a \in \mathbb{Z}} G_{m-a+c}^c C_{a-l,b-l+c}^{R_1(M' \oplus tS_2)}$$

the sum being finite.

b) We have that

$$A_{a,b}^M = \sum_{a \in \mathbb{Z}} G_{b+d}^d C_{a,b+d}^{sS_1 \oplus M'} = \sum_{a \in \mathbb{Z}} G_{b+d}^d B_{a+l-b,l+c}^{R_1(sS_1 \oplus M')}$$

the sum being finite.

**Proof.** a) If $b < 0$ then trivially

$$A_{a,b}^M = \sum_{a \in \mathbb{Z}} G_{m-a+c}^c B_{a-c,b}^{M' \oplus tS_2} = \sum_{a \in \mathbb{Z}} G_{m-a+c}^c C_{b,b-l+c}^{R_1(M' \oplus tS_2)} = 0.$$ 

If $a < 0$ then $A_{a,b}^M = 0$. The sum $\sum_{c \in \mathbb{Z}} G_{m-a+c}^c B_{a-c,b}^{M' \oplus tS_2} = 0$ because for $c < 0$ $G_{m-a+c}^c(q) = 0$ and for $c \geq 0$ we have $a-c < 0$ so $B_{a-c,b}^{M' \oplus tS_2} = 0$. We also have $\sum_{c \in \mathbb{Z}} G_{m-a+c}^c C_{b,2b-a+c}^{R_1(M' \oplus tS_2)} = 0$ because for $c < 0$ $G_{m-a+c}^c(q) = 0$ and for $c \geq 0$ there is no $Y \in \text{mod-}kK(2)$ with dimension $(b, 2b - a + c)$ (see Remark 1.2.) so $C_{a-l,b-l+c}^{R_1(M' \oplus tS_2)} = 0$.

Consider now the case $a, b \geq 0$. Firstly notice that if $Y_c \in \text{mod-}kK(1)$ then by Lemma 1.1. and Lemma 1.4. we have $[cS_1[Y_c] = [cS_1 \oplus Y_c]$, so $F_{cS_1Y_c}^Z = 1$ for $[Z] = [cS_1 \oplus Y_c]$ and $F_{cS_1Y_c}^Z = 0$ in all the other cases.

Using Lemma 1.3. we obtain:

$$A_{a,b}^M = \sum_{\dim Y_c = (a-c, b)} F_{cS_1Y_c}^Z \cdot \sum_{\dim Y_c = (a-c, b)} F_{cS_1Y_c}^Z = \sum_{\dim Y_c = (a-c, b)} F_{XcS_1Y_c}^Z \cdot \sum_{\dim Y_c = (a-c, b)} F_{XcS_1Y_c}^Z$$

$$= \sum_{c \geq 0} \sum_{\dim Y_c = (a-c, b)} F_{XcS_1Y_c}^Z \cdot \sum_{\dim Y_c = (a-c, b)} F_{XcS_1Y_c}^Z = \sum_{c \geq 0} G_{m-a+c}^c(q) \sum_{\dim Y_c = (a-c, b)} F_{ZYc}^M$$

$$= \sum_{c \geq 0} \sum_{\dim Y_c = (a-c, b)} \sum_{\dim Y_c = (a-c, b)} F_{XcS_1Y_c}^Z \cdot F_{ZYc}^M = \sum_{c \geq 0} G_{m-a+c}^c(q) \sum_{\dim Y_c = (a-c, b)} F_{ZYc}^M$$

$$= \sum_{c \geq 0} G_{m-a+c}^c(q) \sum_{\dim Y_c = (a-c, b)} F_{ZYc}^M.$$
Here we have used the following: if \( F'_{Z,Y_c} \neq 0 \) then since \( Y_c \in \text{mod}-kK(1) \) it follows that \( Y_c \) embeds only in \( M' \oplus tS_2 \) (see Lemma 1.1.), so \( Z = sS_1 \oplus Z' \) with \( 0 \rightarrow Y_c \rightarrow M' \oplus tS_2 \rightarrow Z' \rightarrow 0 \) exact, \( Z' \in \text{mod}-kK(1) \) (because \( M' \oplus tS_2 \) does not project on \( S_1 \)) and in this way \( F_{Z,Y_c}^{sS_1 \oplus M' \oplus tS_2} \).

To prove the other identity we will use reflection functors. Using Remark 1.2. and Lemma 1.5. we have

\[
A_{a,b}^M = \sum_{c \geq 0} G^c_{m-a+c} q \sum_{[Y_c],[Z'] \in \text{mod}-kK(1)} F'_{Z,Y_c}^{M' \oplus tS_2} = \sum_{c \in \mathbb{Z}} G^c_{m-a+c} q B_{a-c,b}^{M' \oplus tS_2}
\]

b) dual of a). \( \square \)

We can state now the recursion theorem for the numbers \( A_{a,b}^M \).

**Theorem 3.2.** Suppose up to isomorphism \( M = sS_1 \oplus M' \oplus tS_2 \) with \( M' \in \text{mod}-kK(1) \cap \text{mod}-kK(2) \). Let \( a, b \in \mathbb{Z}, l = a - b \) and \( \dim M = (m,n) \). We have the following recursions

a) 
\[
A_{a,b}^M = \sum_{c \in \mathbb{Z}} q^{c(b-l+c)} G^c_{m-a+c} A_{a-l,b-l+c}^{sS_1 \oplus M'}
\]

the sum being finite.

b) 
\[
A_{a,b}^M = \sum_{d \in \mathbb{Z}} q^{d(2m-n+t-a-l+d)} G^d_{2a-2m+n-t} A_{a+l-d,b+l}^{sS_1 \oplus M'}
\]

the sum being finite.

**Proof.** a) Using the previous proposition and the fact that \( -R^+_1 (M' \oplus tS_2) \in \text{mod}-kK(2) \) we have

\[
A_{a,b}^M = \sum_{c \in \mathbb{Z}} G^c_{m-a+c} C_{a-l,b-l+c}^{sS_1 \oplus M'}
\]

b) Let \( u = c + d \). Using Lemma 2.1. c)

\[
\sum_{c \in \mathbb{Z}} q^{c(b-l+c)} G^c_{m-a+c} A_{a-l,b-l+c}^{sS_1 \oplus M'} = \sum_{c,d \in \mathbb{Z}} q^{c(b-l+c)} G^c_{m-2b} G^d_{b-l+c+d} C_{a-l,b-l+c+d}^{sS_1 \oplus M'}
\]

\[
A_{a-l,b-l+c}^{sS_1 \oplus M'} = \sum_{d_1 \in \mathbb{Z}} q^{d_1(b-l+c)} G^{d_1}_{d_1(c+b-l+c)} C_{a-l,b-l+c+d_1}^{sS_1 \oplus M'}
\]

The sum being finite.
Theorem 4.1. \(A_{a,b} = |\text{Gr}_{(a,b)}(M)|\) with \(M\) indecomposable

Using the recurrences from the previous section we will provide closed formulas for \(A_{a,b}^n, A_{a,b}^\nu\)
(with \(n \in \mathbb{N}, a, b \in \mathbb{Z}\) and \(A_{a,b}^\nu(t)\) (with \(t \in \mathbb{N}^*, a, b \in \mathbb{Z}\) and \(p \in \mathbb{P}^1_k\) of degree 1).

\[\begin{align*}
\text{Theorem 4.1.} & \quad A_{a,b}^n = |\text{Gr}_{(a,b)}(P_n)| = \begin{cases} 0 & \text{for } a < 0 \text{ or } b < 0 \\ G_{n+1-a}^{a-b-1}(q) & \text{for } a = b = 0 \\ G_{n+1-a}^{a-b-1}(q) & \text{otherwise} \end{cases} \\
\end{align*}\]

Remark 4.2. Using the definitions and Lemma 2.1. a) notice that \(G_{n+1-a}^{n+1-a}(q)G_{a-1}^{a-b-1} = 0\) for \(0 < a \leq b, \text{ for } a > n + 1, \text{ for } b > n, \text{ for } a > 0 \text{ and } b < 0.\)

Proof. Induction on \(n\). For \(n = 0\) we have that \(A_{a,b}^0 = 1\) when \((a, b) = (1, 0)\) or \((a, b) = (0, 0)\)
and 0 otherwise so using the previous remark we can see that the formula is true.

Suppose now \(n \geq 1\). Then trivially \(A_{a,b}^0 = 0\) for \(a < 0\) or \(b < 0\) and \(A_{0,0}^0 = 1\) so we only need to look at the case \(a, b > 0, a^2 + b^2 \neq 0\). Using Theorem 3.2. a) we obtain the recursion

\[A_{a,b}^n = \sum_{c \in \mathbb{Z}} q^c \cdot (b-l+c) \cdot (a-b-l+c) \cdot G_{n-2b+1}^{a-b-1}(q) A_{a-l,b-l+c}^{n-1},\]

the sum being finite.

Using Remark 4.2. and the induction hypothesis notice that if \(b > 0\) (and \(a \geq 0\)) then

\[A_{a-l,b-l+c}^{n-1} = A_{b-2b+a-c}^{n-1} = G_{n-2b+a-c}^{a-b-c-1}(q) G_{b-1}^{a-b-c-1}(q)\]

so denoting by \(u = a - b - c - 1\), using the previous recursion and Proposition 2.2. with the entries \(p = a - b - 1, m = n + 1 - a, \mu = n + 1 - b\) and \(\nu = a - 1\)

\[A_{a,b}^n = \sum_{c \in \mathbb{Z}} q^c \cdot (b-l+c) \cdot (a-b-l+c) A_{a-l,b-l+c}^{n-1},\]
\[
A^n_{a,b} = \sum_{d \in \mathbb{Z}} q^{d(n-1-a-b+d)} G^d_{2a-n+1}(q) A^{l_{n-1}}_{a+l-d,b+l},
\]
the sum being finite.

Using Remark 4.4. and the induction hypothesis notice that if \(a < n, \) and \(b \leq n+1\) then
\[
A^n_{a+1-d,b+1} = A^{l_{n-1}}_{2a-b-d,a} = G^a_{n-a-1}(q) G^b_{2a-b-d+1}(q)
\]
so denoting by \(u = a - b - d,\) using the previous recursion and Proposition 2.2. with the entries \(p = a - b,\) \(m = b,\) \(\mu = a + 1\) and \(\nu = n - b\)
\[
A^n_{a,b} = \sum_{d \in \mathbb{Z}} q^{d(n-1-a-b+d)} G^d_{2a-n+1}(q) A^{l_{n-1}}_{a+l-d,b+l}
\]
\[
= \sum_{d \in \mathbb{Z}} q^{d(n-2a+b+d)} G^d_{2a-n+1}(q) G^a_{n-a-1}(q) G^b_{2a-b-d+1}(q)
\]
\[
= \sum_{u \in \mathbb{Z}} q^{(a-b-u)(n-1-a-u)} G^{a-b-u}_{2a-n+1}(q) G^u_{n-a-1}(q) G^a_{a+u+1}(q) = G^{a-b}_{n-b}(q) G^b_{a+1}(q).
\]
If now \(a = n\) (and \(0 \leq b < n+1\)) then trivially
\[
A^n_{a,b} = \sum_{[X]} F^{l_{n+1-b}}_{(n+1-b)S_2 X} = G^{n+1-b}_{n+1}(q) = G^{a-b}_{n-b}(q) G^b_{n+1}(q).
\]
If \(a = n\) and \(b < 0\) then trivially \(A^n_{a,b} = 0\) (see Remark 4.4.).

**Lemma 4.5.** Let \(t \in \mathbb{N}^+,\ a, b \in \mathbb{Z}\) and \(p \in \mathbb{P}^1_k\) of degree 1. Then we have
\(a)\ A^n_{a,a} = 1\) for \(0 \leq a \leq t,\)
\(b)\ A^n_{a,b} = 0\) for \(0 \leq a < b \leq t,\)
\(c)\ For two points \(p, p' \in \mathbb{P}^1_k\) of the same degree 1 we have, that \(A^n_{a,p} = A^n_{a,p'}\).
Proof. a) Suppose \( \dim Y = (a, a) \), \( a > 0 \) (so the defect is 0) and \( Y \) embeds into \( R_p(t) \). Then using Lemma 1.1. and the uniseriality of the regulars one can see that \( Y \) must be of the form \( R_p(t') \) with \( 0 < t' \leq t \). So it follows that for \( 0 < a \leq n \), we have \( A^{R_p(t)}_{a, b} = F^{R_p(t)}_{t'(t)} = 1 \). The rest of the statement follows easily.

b) If for \( 0 \leq a < b \leq t \) \( A^{R_p(t)}_{a, b} > 0 \) this would mean that there is a module \( Y \) of dimension \((a, b)\) which embeds into \( R_p(t) \). But \( a < b \) means that \( Y \) must have a preinjective component. Using Lemma 1.1. one can notice that we can’t embed a preinjective into a regular module.

c) Using Lemma 1.1. and the uniseriality of regulars, observe that for \( F^{R_p(t)}_{X, Y} \neq 0 \) the modules \( X, Y \) can contain at most a single regular direct component which is of the form \( R_p(t') \). Permuting the points \( \{p \in \mathbb{P}^1_k | d_p = d\} \) the assertion follows.

\[ \square \]

**Theorem 4.6.** Let \( t \in \mathbb{N}^* \), \( a, b \in \mathbb{Z} \) and \( p \in \mathbb{P}^1_k \) of degree 1. Then we have

\[ A^{R_p(t)}_{a, b} = |Gr_{(a, b)}(R_p(t))| = \begin{cases} 0 & \text{for } a < 0 \text{ or } b < 0 \\ G_{t-b}^t(q)G_a^{a-b}(q) & \text{otherwise} \end{cases} \]

**Remark 4.7.** Using the definitions and Lemma 2.1. a) notice that \( G_{t-b}^t(q)G_a^{a-b} = 0 \) for \( a < b \), for \( a > t \), for \( b > t \) and \( G_{t-b}^t(q)G_a^{a-b} = 1 \) for \( 0 \leq a = b \leq t \).

**Proof.** Using Remark 4.7. observe that the formula is trivially true whenever \( a < 0 \) or \( b < 0 \) or \( a > t \) or \( b > t \). Also when \( b = 0 \) and \( 0 \leq a < t \) then trivially

\[ A^{R_p(t)}_{a, 0} = \sum_{|X|} F^{R_p(t)}_{X, aB_1} = G_t^0(q) = G_{t-0}(q)G_a^{a-0}(q). \]

Using Lemma 4.5. one can see that the formula is true in the cases \( 0 \leq a = b \leq t \) and \( 0 \leq a < b \leq t \).

So we only need to consider the case \( 0 < b < a \leq t \). Using Theorem 3.2. a) and Lemma 4.5. c) we obtain the recursion

\[ A^{R_p(t)}_{a, b} = \sum_{c \in \mathbb{Z}} q^{c(b-l+c)}G_{t-2b}^c(q)A^{R_p(t)}_{a-l, b-l+c}, \]

the sum being finite.

We proceed by induction on \( a \). Using the recursion and the considerations above for \( a = 2 \leq t \) we have

\[ A^{R_p(t)}_{2, 1} = \sum_{c \in \mathbb{Z}} q^cG_{t-2}^c(q)A^{R_p(t)}_{1, c} = G_{t-2}^0G_1^t + qG_1^{t-2} = G_{t-1}^1G_2^t \]

Let now \( 3 \leq a \leq t \) and \( 0 < b < a \). Using Remark 4.7. and the induction hypothesis notice that

\[ A^{R_p(t)}_{a-1, b-l+c} = A^{R_p(t)}_{b-2a+b+c} = G_{t-2b}^{t-b-a+c}(q) \]

so denoting by \( u = a - b - c \), using the previous recursion and Proposition 2.2. with the entries \( p = a - b, m = t - a, \mu = t - b \) and \( \nu = a \)

\[ A^{R_p(t)}_{a, b} = \sum_{c \in \mathbb{Z}} q^{c(b-l+c)}G_{t-2b}^c(q)A^{R_p(t)}_{a-l, b-l+c} = \sum_{c \in \mathbb{Z}} q^{c(2b-a+c)}G_{t-2b}^{t-b}(q)G_{t-2b-a-c}^{t-b}(q)G_b^{a-b-c}(q) \]

\[ = \sum_{u \in \mathbb{Z}} q^{(a-b-u)(b-u)}G_{t-2b}^{a-b-u}(q)G_{t-b+u}^{t-b}(q)G_b^{a-b}(q) = G_{t-b}(q)G_a^{a-b} \]

\[ \square \]
We can see that in the cases above \(|Gr_e(M)_{\mathbb{Z}}|\) is an integer polynomial \(p_{\mathbb{Z}M}(q)\). Using that \(\chi(Gr_e(M)c) = p_{\mathbb{Z}M}(1)\) and \(G^n_a(1) = \begin{pmatrix} a \\ n \end{pmatrix}\) we obtain

**Corollary 4.8.**

1. \(\chi(Gr_{(a,b)}(P_n)c) = \begin{cases} 0 & \text{for } a < 0 \text{ or } b < 0 \\ 1 & \text{for } a = b = 0 \\ \left( \begin{array}{cc} n + 1 - b \\ n + 1 - a \end{array} \right) \left( \begin{array}{c} a - 1 \\ a - b - 1 \end{array} \right) & \text{otherwise} \end{cases}\)
2. \(\chi(Gr_{(a,b)}(I_n)c) = \begin{cases} 0 & \text{for } a > n \text{ or } b > n + 1 \\ 1 & \text{for } a = n, b = n + 1 \\ \left( \begin{array}{cc} n - b \\ a - b \end{array} \right) \left( \begin{array}{c} a + 1 \\ b \end{array} \right) & \text{otherwise} \end{cases}\)
3. \(\chi(Gr_{(a,b)}(R_p(t))c) = \begin{cases} 0 & \text{for } a < 0 \text{ or } b < 0 \\ \left( \begin{array}{cc} t - b \\ t - a \end{array} \right) \left( \begin{array}{c} a \\ a - b \end{array} \right) & \text{otherwise} \end{cases}\)

**Remark 4.9.** Notice that there is no closed formula for \(A^{R_p(t)}_{a,b}\) with \(t \in \mathbb{N}^+, a, b \in \mathbb{Z}\) and \(p \in \mathbb{P}_k^1\) of degree \(d_p > 1\). This because \(A^{R_p(t)}_{a,b} = 1\) only for \(0 \leq a \leq d_p t\) with \(d_p | a\). However this case will not appear over \(\mathbb{C}\).

5. **A recursive algorithm for the cardinalities** \(A^{M}_{a,b} = |Gr_{(a,b)}(M)|\) **with** \(M\) **arbitrary**

Let \(M \in \text{mod-}kK\) arbitrary and suppose \(\text{dim}M = (m, n)\). We know that up to isomorphism \(M = P \oplus R \oplus I\) where \(P\) (respectively \(I, R\)) is a module with all its indecomposable components preprojective (respectively preinjective, regular). We also know that \(A^{M}_{a,b} = 0\) for \(a < 0\) or \(b < 0\) or \(a > m\) or \(b > n\).

Applying the recursion from Theorem 3.2. a) after a finite number of steps \(A^{P \oplus R \oplus I}_{a,b}\) is reduced to knowing some numbers of the form \(A^{R \oplus I}_{a',b'}\). Applying the recursion from Theorem 3.2. b) after a finite number of steps \(A^{R \oplus I}_{a',b'}\) is reduced to knowing some numbers of the form \(A^{R}_{a'',b''}\).

Using the arguments from the proof of Lemma 4.5. b) we can see that \(A^{R}_{a'',b''} = 0\) for \(a'' < b''\) so applying the recursion from Theorem 3.2. a) after a finite number of steps \(A^{R}_{a'',b''}\) with \(a'' \geq b'' \geq 0\) is reduced to knowing some numbers of the form \(A^{R''}_{a'''',b''''}\). (Here ”some” means of course “a finite number”).

Suppose \(R'' = \bigoplus_{i=1}^m R_{p_i}(\lambda^i)\), where \(\lambda^i\) are partitions, \(p_i \in \mathbb{P}_k^1\) different points with degree \(d_{p_i}\) and \(\sum_{i=1}^m d_{p_i} | \lambda^i | = n\) so \(\text{dim}R'' = (n, n)\). Denote \(a''\) simply by \(a\) and suppose \(0 \leq a \leq n\). For partitions \(\lambda, \mu, \nu\) we will denote by \(g^{\lambda}_{\mu, \nu}(q^{d_p}) = F^{R_{p_i}(\lambda)}_{R_{p_i}(\mu)}\) the classical Hall polynomial (see [6] for details). We know that \(g^{\lambda}_{\mu, \nu} = g^{\lambda}_{\nu, \mu}\) and \(g^{\lambda}_{\mu, \nu} = 0\) unless \(|\lambda| = |\mu| + |\nu|\) and \(\mu, \nu \subseteq \lambda\).

Using Lemma 1.1. b) and Lemma 1.4. we have that

\[
\left[ \bigoplus_{i=1}^m R_{p_i}(\nu^i) \right] \left[ \bigoplus_{i=1}^m R_{p_i}(\mu^i) \right] = \prod_{i=1}^m [R_{p_i}(\nu^i)][R_{p_i}(\mu^i)]
\]

so

\[
F^{\bigoplus_{i=1}^m R_{p_i}(\lambda^i)}_{\bigoplus_{i=1}^m R_{p_i}(\nu^i) \oplus \bigoplus_{i=1}^m R_{p_i}(\mu^i)} = \prod_{i=1}^m F^{R_{p_i}(\lambda^i)}_{R_{p_i}(\nu^i) R_{p_i}(\mu^i)}
\]
Using the considerations above and the arguments from the proof of Lemma 4.5. a) we will have

\[ A^{R_{\omega}}_{a,a} = \sum_{\nu^i, \mu^i \subseteq \lambda^i, \sum_{i=1}^{m} d_{\nu^i} |\nu^i| = a, \sum_{i=1}^{m} d_{\mu^i} |\mu^i| = n - a} \prod_{i=1}^{m} g_{\nu^i |\mu^i} \left( q^{d_{\nu^i}} \right) \]

\[ \prod_{i=1}^{m} g_{\nu^i |\mu^i} \left( q^{d_{\nu^i}} \right) \]

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