Nonlinear wave interactions in quantum magnetoplasmas

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Abstract

Nonlinear interactions involving electrostatic upper-hybrid (UH), ion-cyclotron (IC), lower-
hybrid (LH), and Alfvén waves in quantum magnetoplasmas are considered. For this purpose,
the quantum hydrodynamical equations are used to derive the governing equations for nonlinearly
coupled UH, IC, LH, and Alfvén waves. The equations are then Fourier analyzed to obtain nonlinear
dispersion relations, which admit both decay and modulational instabilities of the UH waves
at quantum scales. The growth rates of the instabilities are presented. They can be useful in
applications of our work to diagnostics in laboratory and astrophysical settings.

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I. INTRODUCTION

Quantum plasma physics is a new and rapidly emerging subfield of plasma physics. It has received a great deal of attention due to its wide range of applications [1, 2, 3, 4, 5]. Quantum plasmas can be composed of the electrons, positrons, holes, and ions. They are characterized by low temperatures and high particle number densities. Quantum plasmas and collective effects play an important role in microelectronic components [1], dense astrophysical systems (in particular white dwarf and neutron star environments) [2], intense laser-matter experiments [3], and nonlinear quantum optics [4, 5]. It is well known that when the thermal de Broglie wavelength of the charged particles is equal to or larger than the average inter-particle distance \( d = n^{-1/3} \), where \( n \) is a typical plasma density, the quantum mechanical effects play a significant role in the behaviour of the charged particles. There are two well-known mathematical formulations, the Wigner-Poisson and the Schrödinger-Poisson approaches, that have been widely used to describe the statistical and hydrodynamic behavior of the plasma particles at quantum scales in quantum plasmas. These formulations are the quantum analogues of the kinetic and the fluid models in classical plasma physics. Manfredi [6] has studied these approaches, taking into account the quantum effects in a collisionless quantum plasma. In particular, the quantum hydrodynamic model (QHD) has attracted much interest in studies of the negative differential resistance [7] in the tunnelling diode. Several collective processes [8, 9, 10, 11, 12, 13, 14] have been analyzed both analytically and numerically in plasmas with quantum corrections.

Haas et al. [15] studied a quantum multi-stream model for one- and two-stream plasma instabilities, presented a new purely quantum branch, and investigated the stationary states of the nonlinear Schrödinger-Poisson system. Anderson et al. [16] used a Wigner-Poisson formulation showing that Landau-like damping due to phase noise can suppress the instabilities. Furthermore, a detailed study of the linear and nonlinear properties of ion acoustic waves (IAW) in an unmagnetized quantum plasma has been presented by Haas et al. [17]. For this purpose, they employed the QHD equations containing a non-dimensional quantum parameter \( H \). The latter is the ratio between the plasmon and thermal energies. For a weakly nonlinear quantum IAW, a modified Korteweg-de Vries (KdV) equation was analyzed for \( H \to 2, \ H < 2 \) and \( H > 2 \), connected with a shock wave, as well as bright and dark solitons, respectively. Finally, they also observed a coherent, periodic pattern for a
fully nonlinear IAW in a quantum plasma. Such a pattern cannot exist in classical plasmas. The formation and dynamics of dark solitons and vortices in quantum electron plasmas has also been reported by Shukla and Eliasson [18].

Recently, Haas [19] extended the QHD equations for quantum magnetoplasmas and presented a magnetohydrodynamic model by using the Wigner-Poisson system. He pointed out the importance of the external magnetic field, by establishing the conditions for equilibrium in ideal quantum magnetohydrodynamics. Garcia et al. [20] derived the quantum Zakharov equations by considering a one-dimensional quantum system composed of electrons and singly charged ions. They also investigated the decay and four-wave instabilities for the nonlinear coupling between high-frequency Langmuir waves and low-frequency IAWs. Marklund [21] considered the statistical aspect and solved the Zakharov system at quantum scales, and analyzed the modulational instability both analytically and numerically. Recently, Shukla and Stenflo [22] investigated parametric and modulational instabilities due to the interaction of large amplitude electromagnetic waves and low-frequency electron and ion plasma waves in quantum plasmas. Drift modes in quantum plasmas [23], as well as new modes in quantum dusty plasmas [24, 25], have also been considered.

In the past, Yu and Shukla [26] studied the nonlinear coupling of UH waves with low-frequency IC waves and obtained near-sonic UH cusped envelope solitons in a classical magnetoplasma. The nonlinear dispersion relations [27] were also derived for three wave decay interactions and modulational instabilities due to nonlinear interactions of mode-converted electron Bernstein and low-frequency waves, such as IAWs, electron-acoustic waves (EAWs), IC waves, quasimodes, magnetosonic waves, and Alfvén waves. Murtaza and Shukla [28] illustrated the nonlinear generation of electromagnetic waves by UH waves in a uniform magnetoplasma. Kaufman and Stenflo [29] considered the interaction between UH waves and magnetosonic modes, and showed that UH solitons could be formed.

In the present paper, we consider the nonlinear interactions between UH waves, IC waves, LH waves, and Alfvén waves in a quantum magnetoplasma, by using the one-dimensional QHD equations. Both decay and modulational instabilities will be analyzed in quantum settings. The manuscript is organized in the following fashion: In Sec. II, we derive the governing equations for nonlinearly coupled UH waves, IC waves, LH waves, and Alfvén waves in quantum plasmas. The coupled equations are then space-time Fourier transformed to obtain the dispersion relations. The latter admit a class of parametric instabilities of the
UHs. Details of the decay and modulational instabilities in quantum plasmas are presented in Sec. III. Section IV summarizes our main results.

II. NONLINEAR DISPERSION RELATIONS

In this section, we derive the governing equations and dispersion relations for nonlinearly coupled UH, IC, LH, and Alfvén waves in a quantum magnetoplasma by using the one-dimensional QHD equations [19].

A. UH waves

Let us consider the nonlinear propagation of an electrostatic UH wave in a cold quantum plasma embedded in an external magnetic field $B_0\hat{z}$, where $B_0$ is the strength of the magnetic field and $\hat{z}$ is the unit vector along the z-axis in a Cartesian coordinates system. The UH wave electric field is $E \approx \hat{x} E_{x0} \exp(i k_0 \cdot r - i \omega_0 t) + $ complex conjugate, where $k_0$ is the wave vector and $\omega_0$ is the wave frequency. We then assume that the parallel electric field is small, i.e. $E_z \ll E_x$. In the presence of the electron density fluctuation $n_{e1}$ ($n_{e1} \ll n_{e0}$, where $n_{e0}$ is the unperturbed electron number density) of the electrostatic IC and LH waves, as well as of the magnetic field fluctuation of the Alfvén waves, the UH wave dynamics is here governed by the continuity equation

$$\frac{\partial n_{e1}}{\partial t} + n_{e0} \frac{\partial}{\partial x} (1 + N_s) U_{ex} = 0 ,$$

the $x$- and $y$-components of the electron momentum equation

$$\frac{\partial U_{ex}}{\partial t} = - \frac{e}{m_e} E_x - \omega_{ce} \left(1 + \frac{B_1}{B_0}\right) U_{ey} + \frac{\hbar^2}{4 m_e^2 n_{e0}} \frac{\partial}{\partial x} \nabla^2 n_{e1} ,$$

$$\frac{\partial U_{ey}}{\partial t} = \omega_{ce} \left(1 + \frac{B_1}{B_0}\right) U_{ex} ,$$

and the Poisson equation

$$\frac{\partial E_x}{\partial x} = - 4\pi e n_{e1} .$$
where \( \omega_{ce} = eB_0/m_e c \) is the electron gyro frequency, \( e \) is the magnitude of the electron charge, \( c \) is the speed of light in vacuum, \( m_e \) is the electron mass, and \( \hbar \) is the Planck constant divided by \( 2\pi \). Furthermore, \( \nabla^2 = \partial^2_x + \partial^2_z \), \( N_s = n_{e1}^s/n_{e0} \) is the relative electron number density perturbation associated with the plasma slow motion, and \( B_1(\ll B_0) \) is the compressional magnetic field perturbation associated with the Alfvén wave. In addition, \( U_{ex} \) and \( U_{ey} \) are the \( x \)- and \( y \)-components of the perturbed electron fluid velocity associated with the \( UH \) wave, respectively. The origin of the last term in the right-hand side of Eq. (2) is the quantum correlation due to the electron density fluctuations \[6\] in dense quantum plasmas. We have also assumed that the electron pressure term is much smaller than the electron quantum diffraction term, i.e., \( V^2 F_e n_{e1} \ll \hbar^2/(4m_e^2) \nabla^2 n_{e1} \), where \( V_F \) is the Fermi speed of the electrons.

Combining (1)-(4), we obtain

\[
\left[ \frac{\partial^2}{\partial t^2} + \omega_H^2 + 2 \omega_{ce}^2 \left( \frac{B_1}{B_0} \right) + N_s \omega_{pe}^2 + (1 + N_s) \frac{\hbar^2}{4m_e^2} \frac{\partial^2}{\partial x^2} \nabla^2 \right] E_x = 0 ,
\]

where \( \omega_H = \sqrt{\omega_{pe}^2 + \omega_{ce}^2} \) is the \( UH \) resonance frequency, and \( \omega_{pe} = \sqrt{4\pi n_{e0} e^2/m_e} \) is the electron plasma frequency. In the absence of electron density and magnetic field fluctuations, Eq. (5) reduces to \([\partial^2_t + \omega_H^2 + (h^2/4m_e^2)\partial^2_x \nabla^2]E_{x0} = 0 \), i.e. the pump wave frequency is \( \omega_0 = \sqrt{\omega_{pe}^2 + \omega_{ce}^2 + (h^2/4m_e^2)k_z^2} \), where \( k_0 = \sqrt{k_{x0}^2 + k_{z0}^2} \) is the magnitude of the wavevector. As \( k_{z0} \) here is much smaller than \( k_{x0} \), we can write the pump wave frequency as \( \omega_0 = \sqrt{\omega_{pe}^2 + \omega_{ce}^2 + (h^2/4m_e^2)k_{x0}^2} \).

\section*{B. Electrostatic IC waves}

In the quasi-neutral approximation \( (n_{e1}^s \approx n_{i1}^s) \), we now derive the expression for the electrostatic potential associated with the IC waves in the presence of the \( UH \) ponderomotive force. We assume that the electrons are inertialess, and obtain from the parallel component of the electron momentum equation

\[
0 = -\frac{e^2 \omega_H^2}{4m_e^2 \omega_{pe}^4} \frac{\partial}{\partial z} \langle |E_x|^2 \rangle + e \frac{\partial \phi}{\partial z} + \frac{\hbar^2}{4m_e} \frac{\partial}{\partial z} \nabla^2 N_s
\]

or

\[
E_x = 0
\]
\[
\phi = \frac{e \omega_H^2}{4m_e \omega_{pe}^4} \langle |E_x|^2 \rangle - \frac{\hbar^2}{4m_e} \nabla^2 N_s \tag{7}
\]

The first term in the right-side of (6) is the parallel (to \( \hat{z} \)) component of the ponderomotive potential of the UH waves. The ion dynamics associated with the electrostatic IC waves are governed by the equation of continuity

\[
\frac{\partial N_s}{\partial t} + \frac{\partial}{\partial x} U_{ix} = 0 , \tag{8}
\]

and the \( x \)- and \( y \)-components of the ion-momentum equation

\[
\frac{\partial U_{ix}}{\partial t} = -\frac{e}{m_i} \frac{\partial \phi}{\partial x} + \omega_{ci} U_{iy} + \frac{\hbar^2}{4m_i^2} \frac{\partial}{\partial x} \nabla^2 N_s , \tag{9}
\]

and

\[
\frac{\partial U_{iy}}{\partial t} = -\omega_{ci} U_{ix} . \tag{10}
\]

We have here ignored the ponderomotive force acting on the ions, since it is smaller (in comparison with the electron ponderomotive force) by the electron to ion mass ratio. Furthermore, \( U_{ix} \) and \( U_{iy} \) are the \( x \)-and \( y \)-components of the perturbed ion fluid velocity associated with the plasma slow motion, respectively, \( \omega_{ci} = eB_0/m_i c \) is the ion gyrofrequency, and \( m_i \) is the ion mass.

Solving (8)-(10), we obtain

\[
\left( \frac{\partial^2}{\partial t^2} + \omega_{ci}^2 \right) N_s = \frac{e}{m_i} \frac{\partial^2 \phi}{\partial x^2} \tag{11}
\]

Eliminating \( \phi \) from (7) and (11), and invoking the quasi-neutrality condition, we then have

\[
\left( \frac{\partial^2}{\partial t^2} + \Omega_{IC}^2 \right) N_s = \frac{e^2 \omega_H^2}{4m_e m_i \omega_{pe}^4} \frac{\partial^2}{\partial x^2} \langle |E_x|^2 \rangle , \tag{12}
\]

where \( \Omega_{IC} = \left[ \omega_{ci}^2 + \left( \frac{\hbar^2}{4m_e m_i} \right) \frac{\partial^2}{\partial x^2} \nabla^2 \right]^{1/2} \) is the ion-cyclotron wave gyrofrequency including quantum diffraction effects. In deriving Eq. (12), we have assumed

\[
\frac{\partial^2}{\partial t^2} N_s \gg \frac{\hbar^2}{m_i^2} \frac{\partial^2}{\partial x^2} \nabla^2 N_s .
\]

Equation (12) is the driven (by the UH ponderomotive force) IC wave equation. In the absence of the UH waves and using \( N_s = \hat{N}_s \exp(-i\Omega t + i\mathbf{k} \cdot \mathbf{r}) \) in Eq. (12), we obtain the frequency \( \Omega \) of the IC waves in a quantum magnetoplasma.
\[ \Omega^2 = \omega^2_{ci} + \frac{\hbar^2}{4m_e m_i} k_z^2 k_x^2 \equiv \Omega^2_{IC}, \]  

which shows the dispersion due to quantum electron density correlations. Here, \( k = \sqrt{k_x^2 + k_z^2} \) is the wavenumber of the electrostatic IC waves. By neglecting the quantum diffraction effects (\( \hbar \to 0 \)), the dispersion relation of the usual IC wave in a cold magnetoplasma is obtained. Equation (5) with \( B_1 = 0 \) and Eq. (12) are the desired set for the nonlinearly coupled electrostatic UH and IC waves in a quantum magnetoplasma.

C. Electrostatic LH waves

For the electrostatic LH waves, we assume \( \omega_{ci} \ll \Omega \ll \omega_{ce} \), so that the ions (electrons) are unmagnetized (magnetized). The electron dynamics is then governed by the continuity equation, the momentum equation including the UH ponderomotive potential and the electron quantum diffraction effects under the approximation \( \Omega \ll \omega_{ce} \). We have, respectively,

\[ \frac{\partial N_s}{\partial t} + \frac{\partial}{\partial x} U_{ex} = 0 , \]  

and

\[ U_{e\perp} = \frac{c}{\omega_{ce} B_0} \frac{\partial}{\partial t} \nabla_{\perp} \varphi_e + \frac{c}{B_0} (\hat{z} \times \nabla_{\perp}) \varphi_e . \]  

Since the second term in the right-hand side of Eq. (15) does not contribute to the x-component of the perturbed electron fluid velocity, we have

\[ U_{ex} = \frac{c}{\omega_{ce} B_0} \frac{\partial^2 \varphi_e}{\partial t \partial x} , \]  

with

\[ \varphi_e = \phi + \frac{\hbar^2}{4m_e c^2 \omega_{ce}^2} \nabla^2 N_s - \phi_{p\perp} , \]  

where \( \phi_{p\perp} = \epsilon \omega_H^2 \langle |E_x|^2 \rangle / 4m_e \omega_{pe}^4 \) is the perpendicular (to \( \hat{z} \)) component of the UH wave ponderomotive potential. Combining Eqs. (14) and (16) we obtain

\[ \left( 1 + \lambda_{qe}^4 \frac{\partial^2}{\partial x^2} \right) N_s + \left( \frac{c}{\omega_{ce} B_0} \right) \frac{\partial^2 \phi}{\partial x^2} \phi = \frac{\lambda_e^2 \omega_H^2}{4B_0^2 \omega_{pe}^2} \frac{\partial^2}{\partial x^2} \langle |E_x|^2 \rangle , \]  

where \( \lambda_{qe} = (\hbar^2 / 4m_e c^2 \omega_{ce}^2)^{1/4} \) is the quantum wavelength of the electrons and \( \lambda_e = c / \omega_{pe} \) is the electron skin depth.
In the electrostatic LH field, the ions are unmagnetized and their dynamics in the quasi-neutrality approximation is governed by Eqs. (8) and (9). Assuming \( \omega_{ci} \ll \Omega \) as well as ignoring the ion quantum diffraction effects, we obtain

\[
\frac{\partial^2}{\partial t^2} N_s - \frac{c \omega_{ci}}{B_0} \frac{\partial^2}{\partial x^2} \phi = 0 .
\]

Eliminating \( \phi \) from Eqs. (17) and (18), we have

\[
\left( \frac{\partial^2}{\partial t^2} + \Omega_{LH}^2 \right) N_s = \frac{\lambda_e^2}{4B_0^2} \frac{\omega_H^2 \omega_{LH}^2}{\omega_{pe}^2} \frac{\partial^2}{\partial x^2} \left\langle |E_x|^2 \right\rangle,
\]

which is the driven (by the perpendicular component of the UH ponderomotive force) electrostatic LH wave equation. Here \( \Omega_{LH} = \omega_{LH} \left(1 + \lambda_{qe}^4 \partial^2 / \partial x^2 \nabla^2 \right)^{1/2}, \) and \( \omega_{LH} = \sqrt{\omega_{ce} \omega_{ci}} \) is the LH resonance frequency. In the absence of the UH waves, Eq. (19) gives the electrostatic LH wave frequency

\[
\Omega^2 = \omega_{LH}^2 \left(1 + \lambda_{qe}^4 k_x^2 k_z^2 \right) \equiv \Omega_{LH}^2 ,
\]

which exhibits a dispersion due to quantum electron density correlations. By neglecting the quantum electron wavelength (\( \lambda_{qe} \rightarrow 0 \)), we obtain the usual LH resonance frequency. Equations (5) with \( B_1 = 0 \), (12), and (19) are the desired set for nonlinearly coupled UH and LH waves in a quantum magnetoplasma.

D. Alfvén waves

Finally, we present the driven Alfvén wave equation in a magnetized quantum plasma. For this purpose, we use the momentum equations for the inertialless electrons and mobile ions, respectively,

\[
0 = -e \left( \mathbf{E} + \frac{\mathbf{U}_{e1} \times \mathbf{B}_0}{c} \right) + \frac{\hbar^2}{4m_e n_{e0}} \nabla \nabla^2 n_{e1} - \mathbf{x} \frac{c^2}{4m_e} \frac{\partial}{\partial x} \omega_H^2 \frac{\partial^2}{\partial x^2} \left\langle |E_x|^2 \right\rangle ,
\]

and

\[
m_i \frac{\partial \mathbf{U}_{i1}}{\partial t} = e \left( \mathbf{E} + \frac{\mathbf{U}_{i1} \times \mathbf{B}_0}{c} \right) .
\]

We have here ignored the quantum diffraction effects and the ponderomotive force on the ions. Here \( \mathbf{U}_{e1} \) (\( \mathbf{U}_{i1} \)) is the electron (ion) perturbed fluid velocity. Adding Eqs. (21) and
(22), and introducing the total current density $\mathbf{J} = e(n_i n_i - n_e n_e)$ from the Maxwell equation $\nabla \times \mathbf{B} = 4\pi \mathbf{J}/c$, and using $n_{e0} \approx n_{i0}$, we obtain

$$\frac{\partial \mathbf{U}_{ix}}{\partial t} = \frac{1}{4\pi m_i n_{i0}} (\nabla \times \mathbf{B}_1) \times \mathbf{B}_0 + \frac{\hbar^2}{4m_e m_i n_{e0}} \nabla \nabla^2 n_{e1} - \hat{x} \frac{e^2}{4m_e m_i} \frac{\partial}{\partial x} \omega_{pe}^2 \langle |E_x|^2 \rangle, \quad (23)$$

From (23) we obtain

$$\frac{\partial U_{ix}}{\partial t} = -\frac{V_A^2}{B_0} \frac{\partial}{\partial x} B_1 + \frac{\hbar^2}{4m_e m_i n_{e0}} \nabla^2 n_{e1} - \frac{e^2}{4m_e m_i} \frac{\partial}{\partial x} \frac{\omega_{pe}^2}{\omega_H^2} \langle |E_x|^2 \rangle, \quad (24)$$

where $V_A = B_0/\sqrt{4\pi m_i n_{i0}}$ is the Alfvén speed. By using the frozen-in field condition $(B_1/B_0) = (n_{i1}/n_{i0})$ in Eq. (24) and combining it with Eq. (8), we have

$$\left( \frac{\partial^2}{\partial t^2} - V_a^2 \frac{\partial^2}{\partial x^2} \right) N_s = \frac{e^2}{4m_e m_i} \frac{\omega_H^2}{\omega_{pe}^2} \frac{\partial^2}{\partial x^2} \langle |E_x|^2 \rangle. \quad (25)$$

where $V_a = [V_A^2 - (\hbar^2/4m_e m_i) \nabla^2]^{1/2}$ is the Alfvén speed including the quantum diffraction effects. In the absence of the UH waves, we have

$$\Omega^2 = k_x^2 \left( V_A^2 + \frac{\hbar^2 k_x^2}{4m_e m_i} \right) \equiv k_x^2 V_a^2 \quad (26)$$

Ignoring the electron quantum diffraction effects $\hbar \to 0$, we obtain from (26) the frequency of the usual Alfvén waves in an electron ion plasma. Equations (5) and (25) are the desired set for investigating the parametric interactions between the UH and Alfvén waves in a quantum magnetoplasma.

In the following, we shall study the decay and modulational instabilities of an UH wave involving the IC, LH, and Alfvén waves in a quantum magnetoplasma.

### III. NONLINEAR DISPERSION RELATIONS AND GROWTH RATES

In this section, we shall derive the nonlinear dispersion relations for three-wave decay and modulational instabilities.

#### A. Coupling of UH and IC waves

To derive the nonlinear dispersion relation for parametric instabilities in a quantum magnetoplasma, we write the UH electric field as the sum of the pump wave and the upper
and lower UH sideband fields. The latter arise due to the coupling of the pump $E_{x0} \exp(i k_0 \cdot r - i \omega_0 t) + c.c.$ with low-frequency IC, LH and Alfvénic perturbations. Specifically, the high-frequency UH pump ($\omega_0, k_0$) interacts with the low-frequency electrostatic IC waves ($\Omega, k$) having $N_s = \hat{N}_s \exp(i k \cdot r - i \Omega t)$, and produces two UH sidebands $E_{x \pm} \exp(i k_\pm \cdot r - i \omega_\pm t)$, with frequencies $\omega_\pm = \Omega \pm \omega_0$ and wavenumbers $k_\pm = k \pm k_0$. By using the Fourier transformation, and matching phasors, we obtain from Eq. (5) with $B_1 = 0$, and Eq. (12)

$$D_{\pm} E_{x \pm} = \omega_{pe}^2 \hat{N}_s E_{x0 \pm},$$

(27)

where $E_{x0+} = E_{x0}$ and $E_{x0-} = E^*_{x0}$, and

$$\left(\Omega^2 - \Omega_{IC}^2\right) \hat{N}_s = \frac{k_x^2}{16\pi n_0 m_i} (E^*_{x0} E_{x0+} + E_{x0} E_{x0-}),$$

(28)

where the asterisk denotes the complex conjugate. The upper and lower sidebands can be written as

$$D_\pm = \omega^2_{\pm} - \omega_{H}^2 - \frac{\hbar^2}{4m_e^2} k_x^2 \pm k_0^2.$$  

(29)

For $\Omega \ll \omega_0$, (29) reduces to

$$D_\pm = \pm 2\omega_0 (\Omega \mp \Delta - \delta),$$

(30)

where $\omega_0 = \sqrt{\omega_H^2 + (\hbar^2/4m_e^2) k_x^2 k_0^2}$ is the UH wave frequency modified by the quantum effects, $\Delta = (\hbar^2/8m_e^2 \omega_0) (k_x^2 k_0^2 + k_x^2 k_0^2 + k_0^2 k_0^2 + 4k_x k_0 k_x \cdot k_0)$, and $\delta = (\hbar^2/4m_e^2 \omega_0) \{ k_x k_x (k_x^2 + k_0^2) + k \cdot k_0 (k_x^2 + k_0^2) \}$ are the frequency shifts arising from the nonlinear coupling between the UH and IC waves. Eliminating $E_{x+}$ and $E_{x-}$ from Eq. (27) and Eq. (28), we have

$$\Omega^2 - \Omega_{IC}^2 = \frac{\omega_{pe}^2 k_x^2 |E_{x0}|^2}{16\pi n_0 m_i} \sum_{\pm} \frac{1}{D_\pm}.$$  

(31)

Equation (31) is the dispersion relation for parametrically coupled UH and IC waves in a quantum magnetoplasma.

For three-wave decay interaction, we consider the lower sideband $D_-$ to be resonant, while the upper sideband $D_+$ is assumed off-resonant. We then obtain from (31)

$$\left(\Omega^2 - \Omega_{IC}^2\right) (\Omega + \Delta - \delta) = -\frac{\omega_{pe}^2 k_x^2 |E_{x0}|^2}{32\pi n_0 m_i \omega_0}.$$  

(32)
Letting $\Omega = \Omega_{IC} + i\gamma_{IC}$ and $\Omega = \delta - \Delta + i\gamma_{IC}$ with $\Omega_{IC} \sim \delta - \Delta$, we obtain from (32) for $\gamma_{IC} \ll \Omega_{IC}$, the growth rate

$$\gamma_{IC} \simeq \frac{\omega_{pe} k_x |E_{x0}|}{8\sqrt{\pi n_0 m_i \omega_0} \Omega_{IC}}$$

(33)

For the modulational instability, both the lower and upper sidebands $D_{\pm}$ are resonant. Thus, Eq. (31) gives

$$\left(\Omega^2 - \Omega_{IC}^2\right) \left[(\Omega - \delta)^2 - \Delta^2\right] = \frac{\omega_{pe}^2 k_x^2 |E_{x0}|^2}{16\pi n_0 m_i \omega_0} \Delta .$$

(34)

Assuming $\Omega \gg \delta$, we obtain

$$\Omega^4 - \left(\Delta^2 + \Omega_{IC}^2\right) \Omega^2 + \Delta^2 \Omega_{IC}^2 = \frac{\omega_{pe}^2 k_x^2 |E_{x0}|^2}{16\pi n_0 m_i \omega_0} \Delta = 0 .$$

(35)

The solutions of Eq. (35) are

$$\Omega^2 = \frac{1}{2} \left[\Delta^2 + \Omega_{IC}^2 \pm \sqrt{(\Omega_{IC}^2 - \Delta^2)^2 + \Omega_{m1}^4}\right] ,$$

(36)

where

$$\Omega_{m1} = \left(\frac{\omega_{pe}^2 k_x^2 \Delta}{4\pi n_0 m_i \omega_0}\right)^{1/4} |E_{x0}|^{1/2} .$$

(37)

The growth rate of the modulational instability is

$$\gamma_{m1} = \left(\frac{\omega_{pe}^2 k_x^2 |\Delta|}{16\pi n_0 m_i \omega_0}\right)^{1/4} |E_{x0}|^{1/2} .$$

(38)

**B. Coupling of UH and LH waves**

In this case, the UH pump wave interacts with the low-frequency electrostatic LH waves $(k, \Omega)$. By using Fourier transformations and matching phasors, we obtain from Eq. (5) with $B_1 = 0$, and Eq. (19)

$$D_{\pm} E_{x\pm} = \omega_{pe} \tilde{N}_\alpha E_{x0\pm} ,$$

(39)

and
\[
\left( \Omega^2 - \Omega_{LH}^2 \right) \hat{N}_s = \frac{k_x^2 \lambda_c^2 \omega_H^2 \omega_{LH}^2}{4 B_0^2 \omega_{pe}^2} (E_{x0}^+ E_{x0} + E_{x0} E_{x-}) ,
\]

where \( D_\pm = \pm 2 \omega_0 (\Omega \mp \Delta - \delta) \) for \( \Omega \ll \omega_0 \), \( \Delta = \left( \hbar^2 / 8 m_e^2 \omega_0 \right) (k_x^2 k_0^2 + k_{x0}^2 k^2 + k_x^2 k^2 + 4 k_{x0} k_x k \cdot k_0) \), and \( \delta = \left( \hbar^2 / 4 m_e^2 \omega_0 \right) \{ k_{x0} k_x (k^2 + k_0^2) + k \cdot k_0 (k_x^2 + k_{x0}^2) \} \) are the frequency shifts arising from the nonlinear coupling of the UH waves with the LH waves. Inserting the expressions for \( E_{x+} \) and \( E_{x-} \) from Eq. (39) into Eq. (40), we find the nonlinear dispersion relation

\[
\Omega^2 - \Omega_{LH}^2 = \frac{k_x^2 \lambda_c^2 \omega_H^2 \omega_{LH}^2}{4 B_0^2 \omega_0} \left| E_{x0} \right|^2 \sum_{\pm} \frac{1}{D_\pm} .
\]

(41)

Since for three-wave decay interactions, the lower and upper sidebands \( D_- \) \( (D_+) \) are resonant \( (\text{off-resonant}) \), we obtain from (41)

\[
\left( \Omega^2 - \Omega_{LH}^2 \right) \left( \Omega + \Delta - \delta \right) = -\frac{k_x^2 \lambda_c^2 \omega_H^2 \omega_{LH}^2}{8 B_0^2 \omega_0} \left| E_{x0} \right|^2 .
\]

(42)

Letting \( \Omega = \Omega_{LH} + i \gamma_{LH} \) and \( \Omega = \delta - \Delta + i \gamma_{LH} \), with \( \Omega_{LH} \sim \delta - \Delta \), we obtain the growth rate from Eq. (42), under the approximation \( \gamma_{LH} \ll \Omega_{LH} \),

\[
\gamma_{LH} \simeq \frac{k_x \lambda_c \omega_H \omega_{LH} \left| E_{x0} \right|}{4 B_0 \sqrt{\omega_0 \Omega_{LH}}} .
\]

(43)

Since for the modulational instability, both the sidebands \( D_\pm \) are resonant, we have from (41)

\[
\left( \Omega^2 - \Omega_{LH}^2 \right) \left[ \left( \Omega - \delta \right)^2 - \Delta^2 \right] = \frac{k_x^2 \lambda_c^2 \omega_H^2 \omega_{LH}^2}{4 B_0^2 \omega_0} \left| E_{x0} \right|^2 \Delta .
\]

(44)

Simplifying Eq. (44) for \( \Omega \gg \delta \), we have

\[
\Omega^4 - \left( \Delta^2 + \Omega_{LH}^2 \right) \Omega^2 + \Delta^2 \Omega_{LH}^2 = \frac{k_x^2 \lambda_c^2 \omega_H^2 \omega_{LH}^2}{4 B_0^2 \omega_0} \left| E_{x0} \right|^2 \Delta = 0 .
\]

(45)

Equation (45) admits the solutions

\[
\Omega^2 = \frac{1}{2} \left( \Delta^2 + \Omega_{LH}^2 \right) \pm \frac{1}{2} \left[ \left( \Delta^2 - \Omega_{LH}^2 \right)^2 + \Omega_m^4 \right]^{1/2} ,
\]

(46)

where

\[
\Omega_m = \left( \frac{k_x^2 \lambda_c^2 \omega_H^2 \omega_{LH}^2}{4 B_0^2 \omega_0} \Delta \right)^{1/4} \left| E_{x0} \right|^{1/2} .
\]

(47)
C. Coupling of UH and Alfvén waves

Finally, we consider the nonlinear interaction of the UH pump wave with Alfvén waves \((\Omega, k)\). We follow the same procedure as described above, and obtain

\[
D_{\pm} E_{x\pm} = (\omega_{pe}^2 + 2\omega_{ce}^2) \hat{N}_s E_{x0\pm},
\]

and

\[
\left(\Omega^2 - k_x^2 V_a^2\right) \hat{N}_s = \frac{e^2 k_x^2}{4 m_e m_i} \frac{\omega_H^2}{\omega_{pe}^4} \left( E_{x0}^* E_{x+} + E_{x0} E_{x-} \right),
\]

where \(D_{\pm} = \pm 2\omega_0 (\Omega \mp \Delta - \delta)\) with \(\Delta = \left(h^2/8m_e^2\omega_0\right)(k_x^2 k_0^2 + k_x^2 k_0^2 + k_x^2 + 4k_x k_0 k \cdot k_0)\) and \(\delta = \left(h^2/4m_e^2\omega_0\right)\{k_x k_x (k_x^2 + k_0^2) + k \cdot k_0 (k_x^2 + k_0^2)\}\) are the frequency shifts arising from the nonlinear coupling of the UH waves with the Alfvén waves. Combining Eqs. (48) and (49), we have the nonlinear dispersion relation

\[
\Omega^2 - k_x^2 V_a^2 = \frac{e^2 k_x^2}{4 m_e m_i} \frac{(\omega_{pe}^2 + 2\omega_{ce}^2) \omega_H^2}{\omega_{pe}^4} |E_{x0}|^2 \sum_{\pm,-} \frac{1}{D_{\pm}}.
\]

Proceeding as before, Eq. (50) yields, respectively,

\[
\gamma_{AL} \simeq \frac{e(\omega_{pe}^2 + 2\omega_{ce}^2)^{1/2} \omega_H}{4\omega_{pe}^2} \sqrt{\frac{k_x}{m_e m_i \omega_0 V_a}} |E_{x0}|^{1/2},
\]

and

\[
\gamma_{m3} = \left( \frac{e^2 k_x^2 (\omega_{pe}^2 + 2\omega_{ce}^2) \omega_H^2}{4 m_e m_i \omega_0 \omega_{pe}^4} |\Delta|^4 \right)^{1/4} |E_{x0}|^{1/2}
\]

for the growth rates of the three-wave decay and modulational instabilities in quantum magnetoplasmas when the UH and Alfvén waves are nonlinearly coupled.

IV. SUMMARY

In summary, we have considered the nonlinear couplings between UH, IC, LH, and Alfvén waves in a quantum magnetoplasma. We have derived the governing nonlinear equations and the appropriate dispersion relations by employing the one-dimensional quantum magnetohydrodynamical equations. It is found that the wave dispersion is due to the quantum correction arising from the strong electron density correlations at quantum scales. The dispersion relations have been analyzed analytically to obtain the growth rates for both the
decay and modulational instabilities involving dispersive IC, LH and Alfvén waves. Since the frequencies of the latter are significantly modified due to the quantum corrections, the growth rates are accordingly affected in quantum magnetoplasmas. The present results can be important for diagnostic purposes in magnetized quantum systems, such as those in dense astrophysical objects, intense laser-matter experiments, and in dense semiconductor devices in an external magnetic field.

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