BOTT-CHERN COHOMOLOGY AND THE HARTOGS EXTENSION THEOREM FOR PLURIHARMONIC FUNCTIONS

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Abstract. Let $X$ be a cohomologically $(n - 1)$-complete complex manifold of dimension $n \geq 2$. We prove a vanishing result for the Bott-Chern cohomology group of type $(1, 1)$ with compact support in $X$, which combined with the well-known technique of Ehrenpreis implies a Hartogs type extension theorem for pluriharmonic functions on $X$.

1. Introduction

As is well known, the Hartogs extension theorem for holomorphic functions is one of the striking results in complex analysis that distinguish the higher-dimensional case from the one-dimensional case. In its general form, the theorem states:

Theorem A. Let $X$ be a complex manifold of dimension $n \geq 2$ which is $(n - 1)$-complete in the sense of Andreotti-Grauert. Suppose $\Omega$ is a domain in $X$ and $K$ is a compact subset of $\Omega$ such that $\Omega \setminus K$ is connected. Then every holomorphic function on $\Omega \setminus K$ has a unique holomorphic extension to $\Omega$.

The theorem was proved by Andreotti-Hill [2] in 1972 and generalized to $(n - 1)$-complete normal complex spaces by Merker-Porten [12] in 2009; see [5] or [14] for a simple alternative proof. Coltoiu-Ruppenthal [5] also dealt with the more general cohomologically $(n - 1)$-complete case. We refer the reader to these works and the references therein for a more detailed account of Hartogs’ extension theorem, and to [16] for recent developments on related topics. It should also be mentioned that there is a metric analogue of Hartogs’ theorem under appropriate geometric conditions; see [9] for details.

On the other hand, Chen [4] recently proved the following very interesting result.

Theorem B. Let $\Omega$ be a domain in $\mathbb{C}^n$, $n \geq 2$, and let $K$ be a compact subset of $\Omega$ such that $\Omega \setminus K$ is connected. Then every pluriharmonic function on $\Omega \setminus K$ has a unique pluriharmonic extension to $\Omega$.

Inspired by this result and the work mentioned above, it seems natural to ask the following

Question. Does Theorem B still hold when $\mathbb{C}^n$ is replaced by a Stein (or more generally, cohomologically $(n - 1)$-complete) manifold of dimension $n \geq 2$?

Now recall that the Bott-Chern cohomology group of type $(1, 1)$ with compact support in a complex manifold $X$ is defined as

$$H_{BC,c}^{1,1}(X) = \frac{\text{Ker} d \cap D_{1,1}(X)}{\partial \bar{\partial} D(X)},$$
where $\mathcal{D}(X)$ (resp. $\mathcal{D}^{1,1}(X)$) denotes the set of all compactly supported smooth functions (resp. $(1,1)$-forms) on $X$. An equivalent formulation of Theorem B is $H^{1,1}_{BC,c}(\mathbb{C}^n) = 0$ for $n \geq 2$. Chen proved this cohomology-vanishing result by directly solving the $\partial\bar{\partial}$-equation, which depends heavily on an elegant trick of Lelong [11] and the Sobolev inequality (see [4, Section 3] for details). According to the celebrated work of Mok-Siu-Yau [13] and Croke [6], these two key ingredients apply only to complete Kähler manifolds of nonnegative holomorphic bisectional curvature.¹ For the question posed earlier, these seem to constitute only a rather restrictive class of complex manifolds, since by virtue of the well-known embedding theorem and the curvature-decreasing property for Kähler submanifolds, Stein manifolds—the objects of interest here—are usually those of nonpositive holomorphic bisectional curvature. In such a situation, to answer the previous question we instead use a (qualitative) cohomological argument and prove the following

**Theorem 1.1.** Let $X$ be a cohomologically $(n-1)$-complete complex manifold of dimension $n \geq 2$. Suppose $\Omega$ is a domain in $X$ and $K$ is a compact subset of $\Omega$ such that $\Omega \setminus K$ is connected. Then every pluriharmonic function on $\Omega \setminus K$ has a unique pluriharmonic extension to $\Omega$.

By adapting Ehrenpreis’ technique for the $\bar{\partial}$-operator (see [8, 10]) to the case of the $\partial\bar{\partial}$-operator, we reduce Theorem 1.1 to the following cohomology-vanishing result.

**Theorem 1.2.** Let $X$ be a cohomologically $(n-1)$-complete complex manifold of dimension $n \geq 2$. Then

$$H^{1,1}_{BC,c}(X) = 0.$$ 

This follows easily from the Serre duality [15], as we will show in Sect. 2. If $X$ is Stein, the result is previously known; see [3, Teorema 2.4].

We now conclude the introduction with some remarks on Theorem 1.1. Using the unique continuation property for holomorphic forms, one can easily show that Theorem 1.1 implies the Hartogs extension theorem for holomorphic functions. And as essentially pointed out in [12], the latter does not hold in general if the cohomological $(n-1)$-completeness assumption of $X$ is weakened to (cohomological) $n$-completeness, which is equivalent to requiring that $X$ have no compact connected components according to the Serre duality and a result of Greene-Wu (see [7, Chapter IX, Theorem 3.5]). These show that the cohomological completeness condition imposed on $X$ in Theorem 1.1 is sharp.

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## 2. PROOF OF THE RESULTS

We begin by recalling the notion of cohomological $q$-completeness, where $q$ is a positive integer.

**Definition.** A complex manifold $X$ is called cohomologically $q$-complete if its $k$-th cohomology group with coefficients in $\mathcal{F}$, $H^k(X, \mathcal{F})$, vanishes for every coherent analytic sheaf $\mathcal{F}$ over $X$ and integer $k \geq q$.

Recall also that an $n$-dimensional complex manifold is called $q$-complete in the sense of Andreotti-Grauert [1] if it possesses a smooth exhaustion function whose Levi form has at least $n - q + 1$ positive eigenvalues at every point. According to Grauert’s solution

¹Despite this, Chen’s method is attractive partly because it yields the desired solution in an efficient way, with a precise gradient estimate.
of the Levi problem (see, e.g., [10]), 1-complete complex manifolds are precisely Stein manifolds. In general, by an important result of Andreotti-Grauer [1] we know that $q$-complete complex manifolds are necessarily cohomologically $q$-complete, but the converse remains unclear except for the cases $q = 1$ and $q \geq \dim X$.

We are now ready to prove the results of this paper, starting with Theorem 1.2.

Proof of Theorem 1.2. Without loss of generality, we may assume that $X$ is connected. Let $\Omega^{p, q}_X$ denote the sheaf of germs of holomorphic $p$-forms on $X$, $p = 0, \ldots, n$. It is well-known that $\Omega^{p, q}_X$ is coherent. Since $n \geq 2$, the cohomological $(n-1)$-completeness of $X$ implies

$$H^q(X, \Omega^{p, q}_X) = 0 \quad \text{for} \quad p, q = n-1, n.$$ 

Then by Serre’s criterion [15, Proposition 6], we can invoke the Serre duality [15, Théorème 2] to get

$$H^1_c(X, \Omega^{p, q}_X) = 0 \quad \text{for} \quad p = 0, 1.$$ 

Equivalently, we have the corresponding vanishing result for the Dolbeault cohomology groups of $X$ with compact support:

$$H^0_c(X, \Omega^{p, q}_X) = 0 \quad \text{and} \quad H^1_c(X, \Omega^{p, q}_X) = 0,$$

in view of the Dolbeault isomorphism theorem (with compact support). Furthermore, using $H^n_c(X, \Omega^n_X) = 0$ and the Serre duality again we obtain

$$H^0_c(X, \Omega^0_X) = 0.$$ 

This means that $X$ is noncompact.

We now show that what we have obtained in turn implies

$$H_{BC}^{1,1}(X) = 0.$$ 

To see this, let $\mathcal{D}^{p, q}(X)$ denote the set of all compactly supported smooth $(p, q)$-forms on $X$. Then for every $d$-closed form $\alpha^{1,1} \in \mathcal{D}^{1,1}(X)$, the vanishing of $H_{BC}^{1,1}(X)$ implies that there exists a form $\alpha^{1,0} \in \mathcal{D}^{1,0}(X)$ such that $\bar{\partial}\alpha^{1,0} = \alpha^{1,1}$. Now

$$\bar{\partial}\alpha^{1,0} = -\partial\bar{\partial}\alpha^{1,0} = -\partial\alpha^{1,1} = 0.$$ 

It follows that $\partial\alpha^{1,0}$ is a compactly supported holomorphic 2-form on the noncompact manifold $X$, hence vanishes identically. Since $H_{BC}^{0,1}(X) = 0$, then there exists a function $\alpha \in \mathcal{D}(X)$ such that $\partial\alpha = -\alpha^{1,0}$. Consequently

$$\partial\bar{\partial}\alpha = \bar{\partial}\alpha^{1,0} = \alpha^{1,1},$$

and we are done. \qed

Proof of Theorem 1.1. Choose a cut-off function $\chi \in \mathcal{D}(\Omega)$ such that $\chi = 1$ on a neighborhood of $K$. Given a pluriharmonic function $f$ on $\Omega \setminus K$, we define

$$v := \partial\bar{\partial}((1-\chi)f).$$

Then $v$ is a smooth $d$-closed $(1, 1)$-form on $X$ satisfying

$$\text{supp } v \subset \text{supp } \chi \subset \Omega.$$ 

By Theorem 1.2, there exists a function $u \in \mathcal{D}(X)$ such that $\partial\bar{\partial}u = v$.

Now set

$$F := (1-\chi)f - u,$$

which is pluriharmonic on $\Omega$. We claim that $F|_{\Omega \setminus K} = f$. For this we may assume that $X$ itself is connected, since otherwise $\Omega$ would be contained in a single connected component of $X$ and we only need to replace $X$ with this connected component. Note that $u$ is
pluriharmonic (and hence real-analytic) on $X \setminus \text{supp} \chi$, which intersects $X \setminus \text{supp} u$ since $X$ is noncompact (as we pointed out in the proof of Theorem 1.2). Then by the unique continuation property, $u$ vanishes identically on some connected component, say $U$, of $X \setminus \text{supp} \chi$. On the other hand, the connectedness of $X$ implies that $U$ has a nonempty boundary in $X$, which is necessarily contained in $\text{supp} \chi \subset \Omega$ (by the fact that $U$ is a connected component of $X \setminus \text{supp} \chi$). We then conclude that

$$U \cap (\Omega \setminus \text{supp} \chi) = U \cap \Omega \neq \emptyset$$

and

$$F|_{U \cap (\Omega \setminus \text{supp} \chi)} = f|_{U \cap (\Omega \setminus \text{supp} \chi)}.$$ 

Consequently, $F|_{\Omega \setminus K} = f$ by the connectedness of $\Omega \setminus K$. □

**References**

[1] A. Andreotti and H. Grauert, *Théorème de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France 90 (1962), 193–259.

[2] A. Andreotti and C. Denson Hill, *E. E. Levi convexity and the Hans Lewy problem*, I and II, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) 26 (1972), 325–363 and 747–806.

[3] B. Bigolin, *Gruppi di Aeppli*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) 23 (1969), 259–287.

[4] B.-Y. Chen, *Hardy-Sobolev type inequalities and their applications*, arXiv:1712.02044v2.

[5] M. Coltoiu and J. Ruppenthal, *On Hartogs’ extension theorem on $(n−1)$-complete complex spaces*, J. Reine Angew. Math. 637 (2009), 41–47.

[6] C. B. Croke, *Some isoperimetric inequalities and eigenvalue estimates*, Ann. Sci. École Norm. Sup. 13 (1980), 419–435.

[7] J.-P. Demailly, “Complex Analytic and Differential Geometry”, available at https://www-fourier.ujf-grenoble.fr/~demailly/books.html.

[8] L. Ehrenpreis, *A new proof and an extension of Hartogs’ theorem*, Bull. Amer. Math. Soc. 67 (1961), 507–509.

[9] H. Gaussier and A. Zimmer, *A metric analogue of Hartogs’ theorem*, arXiv:2111.12029v2.

[10] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, 3rd ed., North-Holland Mathematical Library, vol. 7, North-Holland Publishing Co., Amsterdam, 1990.

[11] P. Lelong, *Fonctions entières $(n$ variables) et fonctions plurisousharmoniques d’ordre fini dans $\mathbb{C}^n$*, J. Anal. Math. 12 (1964), 365–407.

[12] J. Merker and E. Porten, *The Hartogs extension theorem on $(n−1)$-complete complex spaces*, J. Reine Angew. Math. 637 (2009), 23–39.

[13] N. Mok, Y.-T. Siu and S.-T. Yau, *The Poincaré-Lelong equation on complete Kähler manifolds*, Compositio Math. 44 (1981), 183–218.

[14] N. Øvrelid and S. Vassiliadou, *Semiglobal results for $\bar{\partial}$ on complex spaces with arbitrary singularities, Part II*, Trans. Amer. Math. Soc. 363 (2011), 6177–6196.

[15] J.-P. Serre, *Un théorème de dualité*, Comment. Math. Helv. 29 (1955), 9–26.

[16] V. Vîjîitu, *On Hartogs’ extension*, Ann. Mat. Pura Appl. 201 (2022), 487–498.