ASYMPTOTIC BEHAVIOR OF NON-AUTONOMOUS RANDOM GINZBURG-LANDAU EQUATION DRIVEN BY COLORED NOISE

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Abstract. This paper investigates mainly the long term behavior of the non-autonomous random Ginzburg-Landau equation driven by nonlinear colored noise on unbounded domains. Due to the noncompactness of Sobolev embeddings on unbounded domains, pullback asymptotic compactness of random dynamical system associated with such random Ginzburg-Landau equation is proved by the tail-estimates method. Moreover, it is proved that the pullback random attractor of the non-autonomous random Ginzburg-Landau equation driven by a linear multiplicative colored noise converges to that of the corresponding stochastic system driven by a linear multiplicative white noise.

1. Introduction. Ginzburg-Landau equations can be found in many fields such as wave propagation and fluid mechanics. In this paper, we study the long term behavior of the non-autonomous random Ginzburg-Landau equation driven by a colored noise on unbounded domain

\[ \frac{\partial u}{\partial t} - (1 + i\lambda) \Delta u + \rho u = f(u) + g(t, x) + h(t, x, u) G_\delta(\theta_t \omega), \quad t > \tau, \quad x \in \mathbb{R}, \]

with initial condition

\[ u(\tau, x) = u_\tau(x), \quad x \in \mathbb{R}, \]

where \( u(t, x) \) is an unknown complex-valued function, \( i \) is the imaginary unit, \( \lambda \in \mathbb{R} \), \( \rho > 0 \), \( g(t, x) \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R})) \), the nonlinear terms \( f(u) = -(1 + i\mu) |u|^2 u \) and \( h \) are complex-valued functions, the process \( G_\delta(\theta_t \omega) \) is colored noise which is the unique stationary solution of the stochastic differential equation

\[ dG_\delta + \frac{1}{\delta} G_\delta dt = \frac{1}{\delta} dW. \]

\( W(t) \) is a two-sided real-valued Wiener process on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), where \( \Omega = C_0(\mathbb{R}, \mathbb{R}) := \{ \omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0 \} \) with the compact topology, \( \mathcal{F} \) is its Borel \( \sigma \)-algebra, \( \mathbb{P} \) is the probability measure.

As for deterministic Ginzburg-Landau equations, there are a lot of works were researched to show the well posedness of solutions, see [3, 5, 7, 8, 9, 10, 11, 20]. The concept of random attractors was introduced to random dynamical systems in [6].

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The existence of random attractors for stochastic Ginzburg-Landau equations with additive noise or linear multiplicative noise has been investigated in [15, 37, 32, 36, 38] on bounded domains. Moreover, there are results about the existence of random attractors and asymptotic compactness for stochastic Ginzburg-Landau equations on unbounded domains [4, 16, 24]. However, there are few papers investigating random non-autonomous complex Ginzburg-Landau equations driven by nonlinear colored noise on unbounded domains.

In general, the Wiener process $W$ can be chosen as a stochastic process to represent the position of the Brownian particle, but the velocity of the particle cannot be obtained from the Wiener process because of the nowhere differentiability of the sample paths of $W$. In such a case, the O-U process was originally constructed in [27, 33] to approximately describe the stochastic behavior of the velocity, hence it can be further used to determine the position of the particle. On the other hand, as demonstrated in [26], in many complex systems, stochastic fluctuations are actually correlated and hence should be modeled by colored noise rather than white noise. The colored noise has been extensively used in the literature to investigate the solutions of random equations, see [1, 12, 13, 17, 18, 34] and the references therein. However, for the random Ginzburg-Landau equation driven by nonlinear multiplicative colored noise \( (1) \), we find that there is no result available to the existence of random attractors. In this paper, we will prove that complex equation \( (1) \) is pathwise well-posed and generates a continuous non-autonomous cocycle, and the cocycle possesses a unique tempered random attractor. This is different from the corresponding stochastic system driven by white noise

\[
\frac{\partial u}{\partial t} - (1 + i\lambda)\Delta u + \rho u = f(u) + g(t, x) + h(t, x, u) \circ dW, \quad t > \tau, \, x \in \mathbb{R},
\]

where the symbol $\circ$ indicates that the equation is understood in the sense of Stratonovich integration. For \( (3) \), one can define a random dynamical system when $h(\cdot, \cdot, u)$ is a linear function. But for a general nonlinear function $h$, a random dynamical system associated with \( (3) \) cannot be defined due to the absence of appropriate transformation, hence asymptotic behavior of such stochastic equations has not been investigated until now by the random dynamical system approach. This paper indicates that the colored noise is much easier to handle than the white noise for studying pathwise dynamics of such stochastic equations.

Specially, in the case of $h(t, x, u) = cu$ in \( (1) \) and \( (3) \), this paper will show the solution of \( (1) \) converges to that of \( (3) \) in $L^2(\mathbb{R})$ as $\delta \to 0^+$, and proves the random attractor of \( (1) \) converges to that of \( (3) \) in $L^2(\mathbb{R})$ as $\delta \to 0^+$. These mean that \( (1) \) is closely related to \( (3) \) for sufficiently small $\delta > 0$. We also remark that the Wong-Zakai approximations can be used to investigate the long term behavior of stochastic equations, see [14, 22, 23, 25, 28, 35].

Since complex equation \( (1) \) has not only random term, but also non-autonomous deterministic term, thus in order to study random attractor of \( (1) \), we need adapt the concept of random attractor which was introduced in [30]. On the other hand, the noncompactness of Sobolev embeddings on unbounded domains gives rise to difficulty in showing the pullback asymptotic compactness of solutions in $L^2(\mathbb{R})$. To get through of it, we use the tail-estimates method to obtain the pullback asymptotic compactness, from which the existence of random attractor can be proved.

This paper is organized as follows. In Section 2, we recall some basic concepts and results on random dynamical systems and introduce some assumptions. In Section 3, we prove problem \( (1) \) with initial datum \( (2) \) has a unique solution and a unique
random attractor in $L^2(\mathbb{R})$. In Section 4, we consider the case that $h(t, x, u) = cu$ for $t \in \mathbb{R}$, $x \in \mathbb{R}$, and then show the convergence of solutions and the upper semicontinuity of random attractor $A_\delta$ as $\delta \to 0^+$.

Throughout this paper, let $\| \cdot \|$ and $(\cdot, \cdot)$ denote the norm and the inner product of $H = L^2(\mathbb{R})$, respectively. Denote by $H^k(\mathbb{R})$ ($k \in \mathbb{N}$) the Sobolev space consisting of all $u \in L^2(\mathbb{R})$ whose weak derivatives up to order $k$ belong to $L^2(\mathbb{R})$. $H^k(\mathbb{R})$ is a separable Banach space with norm

$$
\|u\|_{H^k(\mathbb{R})} := \left( \sum_{|\alpha| \leq k} \int_{\mathbb{R}} |D^\alpha u(x)|^2 \, dx \right)^{\frac{1}{2}},
$$

and $H^{-1}(\mathbb{R})$ is the dual of $H^1(\mathbb{R})$. The letters $c$, $c'$ and $c_i$ ($i = 1, 2, \ldots$) are used to denote positive constants whose values are not significant in the context.

2. Preliminaries. In this section, we first recall some basic concepts on non-autonomous random dynamical systems which is can be found in [30], and then introduce some main results and assumptions.

Let $(X, \| \cdot \|_X)$ be a complete separable metric space, and $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ an ergodic metric dynamical system, see Arnold [2].

**Definition 2.1.** A continuous random dynamical system on $X$ over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ is a $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable mapping:

$$
\varphi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X, (s, \tau, \omega, u) \mapsto \varphi(s, \tau, \omega, u)
$$

such that $\varphi$ satisfies:

1. $\varphi(0, \tau, \omega, u)$ is identity on $X$;
2. $\varphi(t + s, \tau, \omega, u) = \varphi(t, \tau + s, \theta_s \omega, \varphi(s, \tau, \omega, u))$ for all $s, t \geq 0$;
3. $\varphi(t, \tau, \omega, u) : X \to X$ is continuous for all $t \geq 0$.

**Definition 2.2.** A set-valued mapping $D(\tau, \cdot) : \Omega \to 2^X$, $\omega \mapsto D(\tau, \omega)$ is said to be a random set, if for every $\tau \in \mathbb{R}$, the mapping $\omega \mapsto d(u, D(\tau, \omega))$ is measurable for any $u \in X$, moreover, $\{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is also closed for each $\omega \in \Omega$.

**Definition 2.3.** A random set $\{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is called tempered, if for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\beta > 0$

$$
\lim_{t \to +\infty} e^{-\beta t} d(D(\tau - t, \theta_{-t} \omega)) = 0,
$$

where $d(D) = \sup \{||b||_X : b \in D\}$.

**Definition 2.4.** A random set $\{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is said to be a random pullback absorbing set if for any tempered random set $\{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$, every $\tau \in \mathbb{R}$, $\omega \in \Omega$, there exists $t_0$ such that for each $t \geq t_0$,

$$
\varphi(t, \tau - t, \theta_{-t} \omega, D(\tau - t, \theta_{-t} \omega)) \subset B(\tau, \omega).
$$

**Definition 2.5.** Let $D$ be a collection of random subset of $X$, $\varphi$ is said to be $D$-pullback asymptotically compact in $X$, if for all $\tau \in \mathbb{R}$, $\omega \in \Omega$, $\{\varphi(t_n, \tau - t_n, \theta_{-t_n} \omega, x_n)\}_{n=1}^\infty$ has a convergent subsequence in $X$ whenever $t_n \to +\infty$, and $x_n \in B(\tau - t_n, \theta_{-t_n} \omega)$ with $\{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$. 

Definition 2.6. A family \( A = \{ A(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in D \) is called a \( D \)-pullback attractor for \( \varphi \) if for every \( \tau \in \mathbb{R}, \omega \in \Omega \),

1. \( A(\tau, \omega) \) is compact in \( X \) and for every \( \tau \in \mathbb{R}, \omega \mapsto d(X, A(\tau, \omega)) \) is measurable;
2. \( A \) is invariant: \( \varphi(t, \tau, \omega, A(\tau, \omega)) = A(t + \tau, \theta_t \omega) \), for every \( \tau \in \mathbb{R}, \omega \in \Omega \);
3. \( A \) attracts all member of \( D \): for every \( B \in D, \tau \in \mathbb{R} \) and \( \omega \in \Omega \), we have
   \[
   \lim_{t \to +\infty} d_X(\varphi(t, \tau - t, \theta_{-t} \omega, B(t - t, \theta_{-t} \omega)), A(\tau, \omega)) = 0,
   \]
where \( d_X \) is Hausdorff semi-distance in \( X \).

The following result is the theorem on existence and uniqueness of random attractors for non-autonomous random dynamical system.

Theorem 2.7 ([30] Theorem 2.23). Let \( \varphi \) be a continuous random dynamical system on \( X \) over \( (\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}}) \), if there exists a closed random tempered absorbing set \( \{ B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) of \( \varphi \) and \( \varphi \) is asymptotically compact in \( X \), then the \( D \)-pullback attractor \( A \) is unique and is given by,
   \[
   A(\tau, \omega) = \bigcap_{t \geq 0} \bigcup_{\tau \geq t} \varphi(t, \tau - t, \theta_{-t} \omega, B(t - t, \theta_{-t} \omega)), \quad \tau \in \mathbb{R}, \omega \in \Omega.
   \]

Recall that there exists a \( \theta_t \)-invariant subset of full measure (see [2]), which is still denoted by \( \Omega \), such that for \( \omega \in \Omega \),
   \[
   \lim_{t \to \pm \infty} \frac{\omega(t)}{t} = 0.
   \]
(4)

For every \( \omega \in \Omega \) and \( \delta \in (0, 1] \), we write
   \[
   G_\delta(\omega) = \frac{1}{\delta} \int_{-\infty}^{0} e^{\frac{s}{\delta}} dW = -\frac{1}{\delta^2} \int_{-\infty}^{0} e^{\frac{s}{\delta}} \omega(s) ds.
   \]
Then \( G_\delta(\theta_t \omega) \) is the so-called O-U process.

Lemma 2.8 ([12]). (1). For every \( \omega \in \Omega \), the mapping \( t \mapsto G_\delta(\theta_t \omega) \) is continuous, and for every \( 0 < \delta \leq 1 \),
   \[
   \lim_{t \to \pm \infty} \frac{|G_\delta(\theta_t \omega)|}{t} = 0.
   \]
(2). For every \( \omega \in \Omega \),
   \[
   \lim_{t \to \pm \infty} \frac{1}{t} \int_{0}^{t} G_\delta(\theta_s \omega) ds = 0 \quad \text{uniformly for } 0 < \delta \leq 1.
   \]

Lemma 2.9 ([12]). Let \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( T > 0 \). Then for every \( \varepsilon > 0 \), there exists \( \delta_0 = \delta_0(\tau, \omega, T, \varepsilon) > 0 \) such that for all \( 0 < \delta < \delta_0 \) and \( t \in [\tau, \tau + T] \),
   \[
   \left| \int_{0}^{t} G_\delta(\theta_s \omega) ds - \omega(t) \right| < \varepsilon.
   \]

Lemma 2.10 ([12]). Let \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( T > 0 \). Then there exist \( \delta_0 = \delta_0(\tau, \omega, T) > 0 \) and \( M = M(\tau, \omega, T) > 0 \) such that for all \( 0 < \delta < \delta_0 \) and \( t \in [\tau, \tau + T] \),
   \[
   \left| \int_{0}^{t} G_\delta(\theta_s \omega) ds \right| \leq M.
   \]
We introduce the following hypotheses to complete the uniform estimates.

\((H_1)\) We will assume that for every \(\tau \in \mathbb{R}\),
\[
\int_{-\infty}^{0} e^{\frac{i}{4}t \omega} \|g(s + \tau, \cdot)\|^2 ds < \infty,
\]
and for every \(\alpha > 0, \tau \in \mathbb{R}\),
\[
\lim_{t \to +\infty} e^{-\alpha t} \int_{-\infty}^{0} e^{\frac{i}{4}t \omega} \|g(s + \tau - t, \cdot)\|^2 ds = 0.
\]

\((H_2)\) Let \(h : \mathbb{R} \times \mathbb{R} \times \mathbb{C} \to \mathbb{C}\) be continuous such that for all \(t \in \mathbb{R}, x \in \mathbb{R}\) and \(s \in \mathbb{C}\),
\[
|h(t,x,s)| \leq \beta_1(t,x)|s| + \beta_2(t,x),
\]
and
\[
|h(t,x,s_1) - h(t,x,s_2)| \leq L|s_1 - s_2|,
\]
where \(\beta_1 \in L^{2}_{\text{loc}}(\mathbb{R}, L^{2}(\mathbb{R}))\) and \(\beta_2 \in L^{2}_{\text{loc}}(\mathbb{R}, L^{4}(\mathbb{R}))\) and \(L\) is a constant independent of \(s_1\) and \(s_2\).

3. Random attractors of Ginzburg-Landau equation driven by nonlinear colored noise. In this section, we consider the following random Ginzburg-Landau equation with initial condition

\[
\begin{cases}
\frac{\partial u}{\partial t} - (1 + i\lambda) \Delta u + pu = f(u) + g(t,x) + h(t,x,u)G_{\delta}(\theta_t \omega), & t > \tau, x \in \mathbb{R}, \\
u(\tau,x) = u_{r}(x), & x \in \mathbb{R}.
\end{cases}
\]

We define \(A : H^{1}(\mathbb{R}) \to H^{-1}(\mathbb{R})\) by
\[
(Au, v) = \int \nabla u \cdot \nabla v dx, \quad \text{for all } u, v \in H^{1}(\mathbb{R}),
\]
which is a self-adjoint operator and the infinitesimal generator of a strongly continuous semigroup \(\{e^{-tA} t > 0\}\).

**Definition 3.1.** Given \(\tau \in \mathbb{R}, \omega \in \Omega\) and \(u_{\delta, \tau} \in L^{2}(\mathbb{R})\), a continuous function \(u_{\delta} \sim (\tau, \omega, u_{\delta, \tau}) : [\tau, \infty) \to L^{2}(\mathbb{R})\) is called a solution of problem \((10)\) if \(u_{\delta} \sim (\tau, \omega, u_{\delta, \tau}) = u_{\delta, \tau}\) and
\[
u \delta \in C ([\tau, \infty); L^{2}(\mathbb{R})) \cap L^{2}_{\text{loc}}([\tau, \infty); H^{1}(\mathbb{R})) \cap L^{4}_{\text{loc}}([\tau, \infty); L^{4}(\mathbb{R})),
\]
and \(u_{\delta}\) satisfies, for every test function \(\eta \in C_{c}^{\infty}(\mathbb{R})
\[
(u_{\delta}, \eta) + (1 + i\lambda) \int_{\tau}^{t} (Au_{\delta}, \eta) ds + p \int_{\tau}^{t} (u_{\delta}, \eta) ds
\]
\[
= (u_{\delta, \tau}, \eta) - (1 + i\mu) \int_{\tau}^{t} (|u_{\delta}|^{2} u_{\delta}, \eta) ds + \int_{\tau}^{t} (g, \eta) ds + \int_{\tau}^{t} (hG_{\delta}(\theta_{s}\omega), \eta) ds
\]
in the sense of distribution on \([\tau, +\infty)\).

In order to investigate the long-time dynamics, we are now ready to prove the existence and uniqueness of solutions of \((10)\).
Let $O_n = \{ x \in \mathbb{R}, |x| < n \}$ for each $n \in \mathbb{N}$ and consider the following equation with initial and boundary conditions defined in bounded domain $O_n$:
\[
\begin{array}{l}
\frac{d}{dt} u_\delta + (1 + i\lambda)Au_\delta + \rho u_\delta = f(u_\delta) + g(t, x) \\
\quad + h(t, x, u_\delta)G_\delta(\theta_{i\omega}), \quad t > \tau, x \in O_n,
\end{array}
\]
(13)
\[
\begin{array}{l}
u_\delta(t, x) = 0, \quad t > \tau, |x| = n, \\
u_\delta(\tau, x) = u_{\delta\tau}(x), \quad x \in O_n.
\end{array}
\]

**Theorem 3.2.** Suppose $(H_2)$ holds. Then for every $\tau \in \mathbb{R}, \omega \in \Omega$ and $u_{\delta\tau} \in L^2(\mathbb{R})$, problem (10) has a unique solution $u_\delta(t, \tau, \omega, u_{\delta\tau})$ in the sense of Definition 3.1. This solution is $(\mathcal{F}, \mathcal{B}(H))$-measurable in $\omega$ and continuous in initial datum $u_{\delta\tau}$ in $L^2(\mathbb{R})$. Moreover, the solution $u_\delta$ satisfies the energy equation:
\[
\|u_\delta(t, \tau, \omega, u_{\delta\tau})\|^2 + 2 \int_\tau^t \|\nabla u_\delta(s, \tau, \omega, u_{\delta\tau})\|^2 ds + 2\rho \int_\tau^t \|u_\delta(s, \tau, \omega, u_{\delta\tau})\|^2 ds \\
= \|u_{\delta\tau}\|^2 - 2 \text{Re}(1 + i\mu) \int_\tau^t \int_\mathbb{R} |u_\delta(s, \tau, \omega, u_{\delta\tau})|^4 dx ds \\
+ 2 \text{Re} \int_\tau^t \int_\mathbb{R} g(s, x)\overline{u_\delta} dx ds + 2 \text{Re} \int_\tau^t \int_\mathbb{R} G_\delta(\theta_{i\omega}) h(s, x, u_\delta)\overline{u_\delta} dx ds.
\]
(14)

**Proof.** The proof will be divided into three steps. We first construct a sequence of approximate solutions and derive uniform estimates, and then take the limit of those approximate solutions, finally prove the uniqueness of solutions.

**Step 1:** Uniform estimates

Given $n \in \mathbb{N}$, let $\{e_1, e_2, \ldots, e_n, \ldots\}$ be an orthogonal basis of $L^2(O_n)$, $H_n(O_n)$ be the subspace spanned by $\{e_j : j = 1, \ldots, n\}$ and $P_n : L^2(O_n) \to H_n(O_n)$ be the orthogonal projection given by
\[
P_n u = \sum_{j=1}^n (u, e_j) e_j, \quad \text{for all } u \in L^2(O_n).
\]

Note that $P_n$ can be extended to $H^{-1}(O_n)$ and $L^\delta(O_n)$ by
\[
P_n u = \sum_{j=1}^n (u, e_j) e_j, \quad \text{for all } u \in H^1_0(O_n) \text{ or } u \in L^\delta(O_n).
\]

We consider the finite dimensional projection problem of (13) as follows
\[
\begin{array}{l}
\frac{d}{dt} u_{\delta n} + (1 + i\lambda)P_n Au_{\delta n} + \rho u_{\delta n} = P_n f(u_{\delta n}) + P_n g(t, x) \\
\quad + P_n h(t, x, u_{\delta n})G_\delta(\theta_{i\omega}), \quad t > \tau, x \in O_n,
\end{array}
\]
(15)
\[
\begin{array}{l}
u_{\delta n}(t, x) = 0, \quad t > \tau, x \in \partial O_n, \\
u_{\delta n\tau} = P_n u_{\delta\tau}, \quad x \in O_n.
\end{array}
\]

From the standard Picard existence theory [19] for ordinary differential equations and $(H_2)$, it follows that for every $\tau \in \mathbb{R}, \omega \in \Omega$ and $u_{\delta\tau} \in L^2(\mathbb{R})$, problem (15) has a solution $u_{\delta n}(t, \tau, \omega, u_{\delta n\tau}) \in C^1([\tau, \tau + T]; H_n(O_n))$ for some $T > 0$, which is measurable in $\omega \in \Omega$. Next, we derive uniform estimate on $u_{\delta n}$.
Taking the inner product of (15) with $u_{\delta n}$ and then taking real parts, we have
\[
\frac{1}{2} \frac{d}{dt} \|u_{\delta n}\|^2 + \|\nabla u_{\delta n}\|^2 + \rho \|u_{\delta n}\|^2 \\
= - \text{Re}(1 + i\mu) \int_{O_n} P_n |u_{\delta n}|^2 u_{\delta n} \overline{u_{\delta n}} \, dx + \text{Re} \int_{O_n} P_n g(t, x) \overline{u_{\delta n}} \, dx 
\]
(16)

\[+ \text{Re} \int_{O_n} P_n h(t, x, u_{\delta}) \mathcal{G}_{\delta}(\theta_t \omega) \overline{u_{\delta n}} \, dx.
\]

The first term on the right-hand side of (16) can be bounded by
\[
\text{Re}(1 + i\mu) \int_{O_n} P_n |u_{\delta n}|^2 u_{\delta n} \overline{u_{\delta n}} \, dx = \int_{O_n} |u_{\delta n}|^4 \, dx = \|u_{\delta n}\|^4_{L^4(O_n)},
\]
(17)

By Young’s inequality and (8), we have
\[
\text{Re} \int_{O_n} P_n g(t, x) \overline{u_{\delta n}} \, dx \leq \frac{\rho}{4} \|u_{\delta n}\|^2 + \frac{1}{\rho} \|g(t, x)\|^2,
\]
(18)

\[
\text{Re} \int_{O_n} P_n h(t, x, u_{\delta}) \mathcal{G}_{\delta}(\theta_t \omega) \overline{u_{\delta n}} \, dx \\
\leq |\mathcal{G}_{\delta}(\theta_t \omega)| \int_{O_n} |\beta_1(t, x)| u_{\delta n}|^2 + |\beta_2(t, x)| u_{\delta n} \|dx \\
\leq \frac{3}{4} \int_{O_n} \|u_{\delta n}\|^4 \, dx + c \int_{O_n} |\mathcal{G}_{\delta}(\theta_t \omega) | \beta_1(t, x)|^2 \, dx \\
+ c \int_{O_n} |\mathcal{G}_{\delta}(\theta_t \omega) | \beta_2(t, x)|^2 \, dx.
\]
(19)

By (16)-(19), we obtain
\[
\frac{d}{dt} \|u_{\delta n}\|^2 + 2 \|\nabla u_{\delta n}\|^2 + \frac{3}{2} \rho \|u_{\delta n}\|^2 + \frac{1}{2} \|u_{\delta n}\|^4_{L^4(O_n)} \\
\leq \frac{2}{\rho} \|g(t, x)\|^2 + 2c(\|\mathcal{G}_{\delta}(\theta_t \omega) | \beta_1(t, x)|^2 \|^2_{L^2(O_n)} + \|\mathcal{G}_{\delta}(\theta_t \omega) | \beta_2(t, x)|^2 \|^2_{L^4(O_n)}).
\]
(20)

Multiplying (20) with $e^{\frac{2}{\rho} \rho t}$ and then integrating the inequality on $(\tau, t)$, we have
\[
\|u_{\delta n}\|^2 + 2 \int_{\tau}^{t} e^{\frac{2}{\rho} \rho (r-\tau)} \|\nabla u_{\delta n}\|^2 \, dr + \frac{1}{2} \int_{\tau}^{t} e^{\frac{2}{\rho} \rho (r-\tau)} \|u_{\delta n}\|^2_{L^4(O_n)} \, dr \\
\leq e^{-\frac{2}{\rho} \rho (t-\tau)} \|u_{\delta n}(\tau)\|^2 + \frac{2}{\rho} \int_{\tau}^{t} e^{\frac{2}{\rho} \rho (r-\tau)} \|g(r, x)\|^2 \, dr \\
+ 2c \int_{\tau}^{t} e^{\frac{2}{\rho} \rho (r-\tau)} \|\mathcal{G}_{\delta}(\theta_r \omega) |^2 + \|\mathcal{G}_{\delta}(\theta_r \omega) |^2 \| \, dr.
\]
(21)

By (H1) and (5), it follows that
\[
\{u_{\delta n}\}_{n=1}^\infty \text{ is bounded in } L^2(\tau, \tau + T; H^1_0(O_n)) \cap L^4(\tau, \tau + T; L^4(O_n)) \cap L^\infty(\tau, \tau + T; L^2(O_n)).
\]
(22)

By (11) and (22), we obtain
\[
\{P_n A u_{\delta n}\}_{n=1}^\infty \text{ is bounded in } L^2(\tau, \tau + T; H^{-1}(O_n)).
\]
(23)
Multiplying (15) with a test function $\eta(x)$, and then integrating over $O_n t = O_n \times [\tau, t]$, it follows that

$$
\int_{\tau}^{t} \left( \frac{du_{\delta n}}{dt}, \eta \right) ds = - \int_{\tau}^{t} (1 + i\lambda)(P_nAu_{\delta n}, \eta) ds - \int_{\tau}^{t} \rho(u_{\delta n}, \eta) ds
$$

$$
- (1 + i\mu) \int_{\tau}^{t} (P_n |u_{\delta n}|^2 u_{\delta n}, \eta) ds + \int_{\tau}^{t} (P_n g(s, x), \eta) ds + \int_{\tau}^{t} (P_n h(s, x, u_{\delta n}) G_\delta(\theta, \omega), \eta) ds.
$$

(24)

Applying the Hölder inequality, we obtain

$$
\int_{\tau}^{t} (P_n |u_{\delta n}|^2 u_{\delta n}, \eta) ds \leq c \| u_{\delta n} \|_{L^4(O_n t)}^2 \| \eta \|_{L^4(O_n t)}^2,
$$

(25)

$$
\int_{\tau}^{t} (P_n g(s, x), \eta) ds \leq c \| g \|_{L^2(O_n t)} \| \eta \|_{L^2(O_n t)}^2,
$$

(26)

$$
\leq 2^\frac{3}{4} \left( \int_{\tau}^{t} \int_{O_n} |P_n h(s, x, u_{\delta n}) G_\delta(\theta, \omega) \beta_1(s, x) u_1 |^\frac{3}{4} ds dx \right) \| \eta \|_{L^4(O_n t)}^2
$$

$$
\leq 2^\frac{3}{4} \left( \int_{\tau}^{t} \int_{O_n} |G_\delta(\theta, \omega) \beta_1(s, x) |^\frac{3}{4} ds dx \right) \left( \int_{\tau}^{t} \int_{O_n} |u_1 |^4 ds dx \right)^\frac{1}{4}
$$

$$
\leq c \| \beta_1 \|_{L^2(O_n t)} \| u_1 \|_{L^4(O_n t)}^2 + \| \beta_2 \|_{L^4(O_n t)} \| \eta \|_{L^4(O_n t)}^2.
$$

(27)

By (H2), (25)-(27) imply that

$$
\{ P_n f(u_{\delta n}) \}_{n=1}^\infty \text{ is bounded in } L^4(\tau, \tau + T; L^4(O_n)),
$$

(28)

$$
\{ P_n h(t, x, u_{\delta n}) G_\delta(\theta, \omega) \}_{n=1}^\infty \text{ is bounded in } L^4(\tau, \tau + T; L^4(O_n)),
$$

(29)

which along with (22)-(24) show that

$$
\left\{ \frac{du_{\delta n}}{dt} \right\}_{n=1}^\infty \text{ is bounded in } L^4(\tau, \tau + T; L^4(O_n)) + L^2(\tau, \tau + T; H^{-1}(O_n)).
$$

(30)

**Step2:** Existence of solutions

Extend $u_{\delta n}$ to the entire space $\mathbb{R}$ by setting $u_{\delta n} = 0$ on $\mathbb{R} \setminus O_n$ and denote the extension still by $u_{\delta n}$. It follows from (22), (23) and (28)-(30) that there exist $u_\delta \in L^2(\tau, \tau + T; H^1(\mathbb{R})) \cap L^4(\tau, \tau + T; L^4(\mathbb{R})) \cap L^\infty(\tau, \tau + T; L^2(\mathbb{R}))$, $f_1 \in L^2(\tau, \tau + T; H^{-1}(\mathbb{R}))$, $f_2 \in L^4(\tau, \tau + T; L^4(\mathbb{R}))$, $f_3 \in L^2(\tau, \tau + T; L^2(\mathbb{R}))$, $f_4 \in L^2(\tau, \tau + T; H^{-1}(\mathbb{R})) + L^4(\tau, \tau + T; L^2(\mathbb{R}))$ such that, up to a subsequence,

$$
u_{\delta n} \to u_\delta \text{ weakly-star in } L^\infty(\tau, \tau + T; L^2(\mathbb{R})),
$$

(31)

$$
u_{\delta n} \to u_\delta \text{ weakly in } L^2(\tau, \tau + T; H^1(\mathbb{R})) \cap L^4(\tau, \tau + T; L^4(\mathbb{R})),
$$

(32)
\[ P_n Au_{\delta n} \to f_1 \] weakly in \( L^2(\tau, \tau + T; H^{-1}(\mathbb{R})) \), \hspace{1cm} (33)\n
\[ P_n f(u_{\delta n}) \to f_2 \] weakly in \( L^\frac{4}{3}(\tau, \tau + T; L^\frac{4}{3}(\mathbb{R})) \), \hspace{1cm} (34)\n
\[ P_n h(t, x, u_{\delta n}) G_\delta(\theta_\omega) \to f_3 \] weakly in \( L^\frac{4}{3}(\tau, \tau + T; L^\frac{4}{3}(\mathbb{R})) \), \hspace{1cm} (35)\n
\[ \frac{du_{\delta n}}{dt} \to f_4 \] weakly in \( L^2(\tau, \tau + T; H^{-1}(\mathbb{R})) + L^\frac{4}{3}(\tau, \tau + T; L^\frac{4}{3}(\mathbb{R})) \). \hspace{1cm} (36)\n
From (30), it follows that \( \{u_{\delta n}\}_{n=1}^\infty \) is compact in \( L^2(\tau, \tau + T; L^2(O_n)) \), \hspace{1cm} (37)\n
see Theorem 5.1 in [21].

We claim that \( f_1 = Au_{\delta} \), \( f_2 = f(u_{\delta}) \), \( f_3 = h(t, x, u_{\delta}) G_\delta(\theta_\omega) \).

For every \( \eta \in C_c^\infty(\mathbb{R}) \),

\[
\int_\tau^t (P_n Au_{\delta n} - Au_{\delta}, \eta) \, ds = \int_\tau^t (A(u_{\delta n} - u_{\delta}), \eta) \, ds = \int_\tau^t (\nabla (u_{\delta n} - u_{\delta}), \nabla \eta) \, ds - \int_\tau^t (u_{\delta n} - u_{\delta}, \Delta \eta) \, ds \rightarrow 0,
\]

which follows from (32).

\[
|\int_\tau^t ([P_n h(s, x, u_{\delta n}) - h(s, x, u_{\delta})] G_\delta(\theta_\omega), \eta) \, ds| = \left| \int_\tau^t [P_n h(s, x, u_{\delta n}) - P_n h(s, x, u_{\delta}) + P_n h(s, x, u_{\delta}) - h(s, x, u_{\delta})] G_\delta(\theta_\omega), \eta) \, ds \right|
\leq \int_\tau^t |G_\delta(\theta_\omega)||[P_n h(s, x, u_{\delta n}) - P_n h(s, x, u_{\delta})], \eta) \, ds
\leq L \int_\tau^t |G_\delta(\theta_\omega)||[u_{\delta n} - u_{\delta}], \eta) \, ds \rightarrow 0,
\]

which follows from (32) and \( h(t, x, u) \in L^2(\mathbb{R}) \).

Applying the interpolation inequality to \( u_{\delta n} - u_{\delta} \) follows

\[
\|u_{\delta n} - u_{\delta}\|_{L^3(\mathbb{R} \times [\tau, \tau + T])}^3 \leq \varepsilon \|u_{\delta n} - u_{\delta}\|_{L^4(\mathbb{R} \times [\tau, \tau + T])}^4 + c(\varepsilon) \|u_{\delta n} - u_{\delta}\|_{L^2(\mathbb{R} \times [\tau, \tau + T])}^2
\]

for any \( \varepsilon > 0 \).

Since \( \{u_{\delta n} - u_{\delta}\}_{n=1}^\infty \) is bounded in \( L^4(\mathbb{R} \times [\tau, \tau + T]) \), by the arbitrariness of \( \varepsilon > 0 \), (37) and \( u_{\delta n} = 0 \) on \( \mathbb{R} \setminus O_n \), we have

\[ u_{\delta n} \to u_{\delta} \] strongly in \( L^3(\tau, \tau + T; L^3(\mathbb{R})) \).
Then we can obtain
\[ |u_{\delta n}|^2 u_{\delta n} \to |u_\delta|^2 u_\delta \text{ weakly in } L^1(\tau, \tau + T; L^1(\mathbb{R})). \]

So we have
\[ \int_{\tau}^{t} (P_n|u_{\delta n}|^2 u_{\delta n}, \eta) ds \to \int_{\tau}^{t} (|u_{\delta}|^2 u_{\delta}, \eta) ds. \]

Thus we get the claim. By (32), (38)-(40), we get the limit \( u_\delta \) satisfies the random Ginzburg-Landau equation in the sense of (12).

We now prove \( u_\delta : [\tau, \infty) \to L^2(\mathbb{R}) \) is weakly continuous. Because \( u_\delta \in L^2(\tau, \tau + T; H^1(\mathbb{R})) \cap L^4(\tau, \tau + T; L^4(\mathbb{R})) \cap L^{\infty}(\tau, \tau + T; L^2(\mathbb{R})) \) and \( \{|\frac{d u_\delta}{dt}\}| \in L^4(\tau, \tau + T; L^4(\mathbb{R})) + L^2(\tau, \tau + T; H^{-1}(\mathbb{R})) \), it follows from [29] that \( u_\delta \in C([\tau, \tau + T]; \omega - L^2(\mathbb{R})) \) and \( \frac{1}{2} \frac{d}{dt} ||u_\delta||^2 = (\frac{4}{\mu} u_{\delta}, u_\delta) \) for almost every \( t \in [\tau, \tau + T] \). So we get (14).

**Step 3:** Uniqueness of solutions

Let \( u_{\delta 1}, u_{\delta 2} \) be solutions of (10), and \( \tilde{u}_\delta = u_{\delta 1} - u_{\delta 2} \), then we obtain
\[
\frac{d}{dt} ||\tilde{u}_\delta||^2 + 2 ||\nabla \tilde{u}_\delta||^2 + 2 \rho ||\tilde{u}_\delta||^2 \\
= -2 \text{Re}(1 + i\mu) \int_{\mathbb{R}} (|u_{\delta 1}|^2 u_{\delta 1} - |u_{\delta 2}|^2 u_{\delta 2}) \tilde{u}_\delta dx \\
+ 2 \text{Re} G_\delta (\theta, \omega) \int_{\mathbb{R}} (h(t, x, u_{\delta 1}) - h(t, x, u_{\delta 2})) \tilde{u}_\delta dx.
\]

For the first term on the right-hand side of (41), we can get
\[
|\text{Re}(1 + i\mu) \int_{\mathbb{R}} (|u_{\delta 1}|^2 u_{\delta 1} - |u_{\delta 2}|^2 u_{\delta 2}) \tilde{u}_\delta dx| \\
\leq |\text{Re}(1 + i\mu) \int_{\mathbb{R}} |u_{\delta 1}|^2 (u_{\delta 1} - u_{\delta 2}) + (|u_{\delta 1}|^2 - |u_{\delta 2}|^2) u_{\delta 2} \tilde{u}_\delta dx| \\
\leq \int_{\mathbb{R}} |u_{\delta 1}|^2 |\tilde{u}_\delta|^2 dx + |\text{Re}(1 + i\mu) \int_{\mathbb{R}} (|u_{\delta 1}|^2 - |u_{\delta 2}|^2) u_{\delta 2} \tilde{u}_\delta dx| \\
\leq \int_{\mathbb{R}} |u_{\delta 1}|^2 |\tilde{u}_\delta|^2 dx + \sqrt{1 + \mu^2} \int_{\mathbb{R}} (|u_{\delta 1}|^2 + |u_{\delta 2}|^2) |\tilde{u}_\delta|^2 dx \\
\leq (\int_{\mathbb{R}} |u_{\delta 1}|^4 dx)^{\frac{1}{2}} ||\tilde{u}_\delta||^2_{L^2(\mathbb{R})} + 2 \sqrt{1 + \mu^2} \int_{\mathbb{R}} (|u_{\delta 1}|^2 + |u_{\delta 2}|^2) ||\tilde{u}_\delta||^2 dx \\
\leq (\int_{\mathbb{R}} |u_{\delta 1}|^4 dx)^{\frac{1}{2}} ||\tilde{u}_\delta||^2_{L^2(\mathbb{R})} \\
+ 2 \sqrt{1 + \mu^2} (\int_{\mathbb{R}} (|u_{\delta 1}|^4 + |u_{\delta 2}|^4) dx)^{\frac{1}{2}} ||\tilde{u}_\delta||^2_{L^2(\mathbb{R})} \\
\leq (1 + 2 \sqrt{1 + \mu^2}) ||u_{\delta 1}||^2_{L^2(\mathbb{R})} + ||u_{\delta 2}||^2_{L^2(\mathbb{R})} ||\tilde{u}_\delta||^2_{L^2(\mathbb{R})} \\
\leq (1 + 2 \sqrt{1 + \mu^2}) ||u_{\delta 1}||^2_{L^2(\mathbb{R})} + ||u_{\delta 2}||^2_{L^2(\mathbb{R})} ||\tilde{u}_\delta||^2_{L^2(\mathbb{R})}.
which applied the Hölder inequality and Young’s inequality. By Gronwall Lemma, we get

\begin{equation}
\leq (1 + 2\sqrt{2}\sqrt{1 + \mu^2}) \varepsilon \| \bar{u}_\delta \|_{H^1(\mathbb{R})}^2 + \frac{3}{4} (4\varepsilon)^{-\frac{1}{2}} \left[ \| u_{\delta 1} \|_{L^4(\mathbb{R})}^2 + \| u_{\delta 2} \|_{L^4(\mathbb{R})}^2 \right] \| \bar{u}_\delta \|_{L^2(\mathbb{R})}^2
\end{equation}

(42)

which applied the Hölder inequality and Young’s inequality.

The second term on the right-hand side of (41) can be bounded by

\begin{align*}
|\text{Re} G_\delta (\theta_i\omega) & \int_\mathbb{R} (h(t, x, u_{\delta 1}) - h(t, x, u_{\delta 2})) \bar{u}_\delta dx | \\
& \leq |G_\delta (\theta_i\omega)| \int_\mathbb{R} L(u_{\delta 1} - u_{\delta 2}) \bar{u}_\delta dx | \\
& = L|G_\delta (\theta_i\omega)| \| \bar{u}_\delta \|_{L^2(\mathbb{R})}^2.
\end{align*}

Set \((1 + 2\sqrt{2}\sqrt{1 + \mu^2}) \varepsilon = \min\{1, \frac{\rho}{2}\}\), then (41) together with (42) and (43) implies that

\[
\frac{d}{dt} \| \bar{u}_\delta \|_{L^4(\mathbb{R})}^2 \leq -\rho \| \bar{u}_\delta \|_{L^4(\mathbb{R})}^2 + \frac{3}{2} \left( 1 + 2\sqrt{2}\sqrt{1 + \mu^2} \right)^{\frac{1}{2}} (4\min\{1, \frac{\rho}{2}\})^{-\frac{1}{2}} 2\delta
\]

\[
\left[ \| u_{\delta 1} \|_{L^4(\mathbb{R})}^2 + \| u_{\delta 2} \|_{L^4(\mathbb{R})}^2 \right] \| \bar{u}_\delta \|_{L^2(\mathbb{R})}^2 + 2L|G_\delta (\theta_i\omega)| \| \bar{u}_\delta \|_{L^2(\mathbb{R})}^2.
\]

By the Hölder inequality,

\[
\int_\tau^t \| \bar{u}_\delta \|_{L^4(\mathbb{R})}^2 ds \leq \left( \int_\tau^t \| \bar{u}_\delta \|_{L^4(\mathbb{R})}^2 ds \right)^{\frac{1}{2}} (t - \tau)^{\frac{1}{2}}.
\]

By Gronwall Lemma, we get

\[
\| u_{\delta 1}(t, \tau, \omega, u_{\delta 1}(\tau)) - u_{\delta 2}(t, \tau, \omega, u_{\delta 2}(\tau)) \|
\leq e^{-\rho(t - \tau) + c(\int_\tau^t \| u_{\delta 1} \|_{L^4(\mathbb{R})}^2 ds)^{\frac{1}{2}} + (\int_\tau^t \| u_{\delta 2} \|_{L^4(\mathbb{R})}^2 ds)^{\frac{1}{2}}(t - \tau)^{\frac{1}{2}}}
\]

\[
eq 2L \int_\tau^t |G_\delta (\theta_i\omega)| ds \| u_{\delta 1}(\tau) - u_{\delta 2}(\tau) \|,
\]

which implies the uniqueness of solutions on initial datum in \(L^2(\mathbb{R})\).

Finally, we prove the measurability of solution in \(\omega \in \Omega\). Note that (37) and the uniqueness of solutions indicate that the sequence \(u_{\delta n}(t, \tau, \omega, \cdot) \to u_\delta(t, \tau, \omega, \cdot)\) weakly in \(L^2(\mathbb{R})\) for any \(t \geq \tau\) and \(\omega \in \Omega\). Since \(u_{\delta n}(t, \tau, \omega, \cdot)\) is measurable in \(\omega \in \Omega\), we find that the weak limit \(u_\delta(t, \tau, \omega, \cdot)\) is also measurable in \(\omega \in \Omega\). The proof is completed.

We now define a mapping \(\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times L^2(\mathbb{R}) \to L^2(\mathbb{R})\) such that for all \(t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega\) and \(u_\tau \in L^2(\mathbb{R})\)

\[
\Phi (t, \tau, \omega, u_\tau) = u_\delta(t + \tau, \tau, \theta_{-\tau}\omega, u_{\delta \tau}),
\]

(44)

where \(u_\delta\) is a solution of problem (10). Then \(\Phi\) is a continuous cocycle on \(L^2(\mathbb{R})\) over \((\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})\).

Later in this section, we assume \(\beta_1 \in L^\infty (\mathbb{R}, L^2 (\mathbb{R}))\) and \(\beta_2 \in L^\infty (\mathbb{R}, L^\frac{1}{2} (\mathbb{R}))\).
Lemma 3.3. Suppose \((H_1)-(H_2)\) hold. Then for every \(\sigma \in \mathbb{R}, \tau \in \mathbb{R}, \omega \in \Omega\) and \(B_\delta = \{B_\delta(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}\), there exists \(T = T(\tau, \omega, B_\delta, \sigma) \geq 0\) such that for all \(t \geq T\), the solution \(u_\delta\) of problem (10) satisfies

\[
\begin{align*}
&\|u_\delta(\sigma, \tau - t, \theta_\tau \omega, u_\delta(\tau - t))\|^2 \\
&+ \int_0^\sigma e^{\frac{1}{2} \rho(s - \sigma)} \|\nabla u_\delta(s, \tau - t, \theta_\tau \omega, u_\delta(\tau - t))\|^2 \, ds \\
&\leq M \int_{-\infty}^{\sigma - \tau} e^{\frac{1}{2} \rho(s + \tau - \sigma)} \{\|g(s + \tau)\|^2 + |G_\delta(\theta_s \omega)|^2 + |G_\delta(\theta_s \omega)|^\frac{2}{s+\tau}\} \, ds,
\end{align*}
\]

where \(u_\delta(\tau - t) \in B_\delta(\tau - t, \theta_\tau \omega)\) and \(M\) is a positive constant independent of \(\sigma, \tau, \omega,\) and \(B_\delta\).

Proof. First we have

\[
\text{Re}(1 + i\mu) \int_{\mathbb{R}} |u_\delta|^2 u_\delta \overline{u_\delta} \, dx = \int_{\mathbb{R}} |u_\delta|^4 \, dx = \|u_\delta\|_{L^4(\mathbb{R})}^4
\]

Using (8) and Young’s inequality, we get

\[
\text{Re} \int_{\mathbb{R}} g(t, x) \overline{u_\delta} \, dx \leq \frac{3}{4} \|u_\delta\|^2 + \frac{1}{\rho} \|g(t, x)\|^2.
\]

Moreover, applying (8) and Young’s inequality, we can get

\[
\text{Re} \int_{\mathbb{R}} h(t, x, u_\delta) G_\delta(\theta_s \omega) \overline{u_\delta} \, dx \\
\leq \frac{1}{2} \int_{\mathbb{R}} |u_\delta|^4 \, dx + c \left[ \int_{\mathbb{R}} |G_\delta(\theta_s \omega) \beta_1(t, x)|^2 \, dx + \int_{\mathbb{R}} |G_\delta(\theta_s \omega) \beta_2(t, x)|^\frac{2}{s} \, dx \right].
\]

It follows from (46)-(48) that

\[
\frac{d}{dt} \|u_\delta\|^2 + 2 \|\nabla u_\delta\|^2 + \frac{3}{2} \rho \|u_\delta\|^2 + \|u_\delta\|_{L^4(\mathbb{R})}^4 \leq \frac{2}{\rho} \|g(t, x)\|^2 + 2c \left[ \int_{\mathbb{R}} |G_\delta(\theta_s \omega) \beta_1(t, x)|^2 \, dx + \int_{\mathbb{R}} |G_\delta(\theta_s \omega) \beta_2(t, x)|^\frac{2}{s} \, dx \right].
\]

Multiplying (49) with \(e^{\frac{1}{2} \rho s}\), integrating the inequality on \((\sigma - t, \sigma)\) with \(\sigma > t - \tau\) and replacing \(\omega\) by \(\theta_\tau \omega\), we get

\[
\begin{align*}
&\|u_\delta(\sigma, \tau - t, \theta_\tau \omega, u_\delta(\tau - t))\|^2 \\
&+ 2 \int_{\tau - t}^\sigma e^{\frac{1}{2} \rho(s - \sigma)} \|\nabla u_\delta(s, \tau - t, \theta_\tau \omega, u_\delta(\tau - t))\|^2 \, ds \\
&+ \rho \int_{\tau - t}^\sigma e^{\frac{1}{2} \rho(s - \sigma)} \|u_\delta(s, \tau - t, \theta_\tau \omega, u_\delta(\tau - t))\|^2 \, ds \\
&\leq e^{\frac{1}{2} \rho(\tau - t)} \|u_\delta(\tau - t)\|^2 + \frac{2}{\rho} \int_{-\infty}^{\sigma - \tau} e^{\frac{1}{2} \rho(s + \tau - \sigma)} \|g(s + \tau)\|^2 \, ds \\
&+ 2c \int_{-\infty}^{\sigma - \tau} e^{\frac{1}{2} \rho(s + \tau - \sigma)} \{\|G_\delta(\theta_s \omega)\|^2 + |G_\delta(\theta_s \omega)|^\frac{2}{s+\tau}\} \, ds.
\end{align*}
\]

Since \(u_\delta(\tau - t) \in B_\delta(\tau - t, \theta_\tau \omega)\) and \(B_\delta\) is tempered, we find that

\[
\limsup_{t \to +\infty} e^{\frac{1}{2} \rho(\tau - t)} \|u_\delta(\tau - t)\|^2 \leq \limsup_{t \to +\infty} e^{\frac{1}{2} \rho(\tau - t)} \|B_\delta(\tau - t, \theta_\tau \omega)\|^2 = 0,
\]
which shows that there exists \( T = T(\sigma, \tau, \omega, B_0) > 0 \) such that for all \( t \geq T \),
\[
e^{\frac{1}{2} \rho (t-s)} \| u_{\delta(t-s)} \| \leq \int_{-\infty}^{\sigma-t} e^{\frac{1}{2} \rho (s+t-\sigma)} \| \mathcal{G}_0 (\theta_s \omega) \|^2 + \| \mathcal{G}_0 (\theta_s \omega) \|^\frac{1}{2} \| g \| ds.
\] (51)

By (50) and (51), we obtain (45). \( \square \)

Based on Lemma 3.3, we will prove that system (10) has a \( D \)-pullback absorbing set in \( L^2(\mathbb{R}) \).

**Lemma 3.4.** Suppose \( (H_1) \) and \( (H_2) \) hold. Then the continuous cocycle \( \Phi \) has a \( D \)-pullback absorbing set \( B_0 = \{ B_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in D \), which is given by
\[
B_0(\tau, \omega) = \{ u \in L^2(\mathbb{R}) : \| u \|^2 \leq R_0(\tau, \omega) \},
\] (52)
where
\[
R_0(\tau, \omega) = M \int_{-\infty}^{0} e^{\frac{1}{2} \rho s} \| g(s+\tau) \|^2 + \| \mathcal{G}_0 (\theta_s \omega) \|^2 + \| \mathcal{G}_0 (\theta_s \omega) \|^\frac{1}{2} \| g \| ds.
\] (53)

**Proof.** As a special case of Lemma 3.3 with \( \sigma = \tau \), we obtain there exists \( T = T(\tau, \omega, B_0) > 0 \) such that for all \( t \geq T \),
\[
u_0 (\tau, \tau-t, \theta_{-\tau} \omega, u_0(t-s)) \in B_0(\tau, \omega).
\]
Now we need only prove that for any \( \beta > 0 \),
\[
\lim_{t \to +\infty} e^{-\beta t} R_0(\tau-t, \theta_{-t} \omega) = 0.
\] (54)

From (53) we have
\[
e^{-\beta t} R_0(\tau-t, \theta_{-t} \omega) = Me^{-\beta t} \int_{-\infty}^{0} e^{\frac{1}{2} \rho s} \| g(s+\tau-t) \|^2 + \| \mathcal{G}_0 (\theta_{s-t} \omega) \|^2 + \| \mathcal{G}_0 (\theta_{s-t} \omega) \|^\frac{1}{2} \| g \| ds.
\]
Since
\[
Me^{-\beta t} \int_{-\infty}^{0} e^{\frac{1}{2} \rho s} \| \mathcal{G}_0 (\theta_{s-t} \omega) \|^2 + \| \mathcal{G}_0 (\theta_{s-t} \omega) \|^\frac{1}{2} \| g \| ds \\
\leq M \int_{-\infty}^{0} e^{\min (\beta, \frac{1}{2} \rho) s} \| \mathcal{G}_0 (\theta_{s-t} \omega) \|^2 + \| \mathcal{G}_0 (\theta_{s-t} \omega) \|^\frac{1}{2} \| g \| ds \\
\leq M \int_{-\infty}^{-t} e^{\min (\beta, \frac{1}{2} \rho) s} \| \mathcal{G}_0 (\theta_{s-t} \omega) \|^2 + \| \mathcal{G}_0 (\theta_{s-t} \omega) \|^\frac{1}{2} \| g \| ds.
\]
By Lemma 2.8 (1), we can get
\[
\lim_{t \to +\infty} Me^{-\beta t} \int_{-\infty}^{0} e^{\frac{1}{2} \rho s} \| \mathcal{G}_0 (\theta_{s-t} \omega) \|^2 + \| \mathcal{G}_0 (\theta_{s-t} \omega) \|^\frac{1}{2} \| g \| ds = 0.
\]
By (7), we can get
\[
\lim_{t \to +\infty} Me^{-\beta t} \int_{-\infty}^{0} e^{\frac{1}{2} \rho s} \| g(s+\tau-t) \|^2 ds = 0.
\]
So we get (54), as desired. \( \square \)

Next, we will prove the asymptotic compactness of solutions of problem (10).
Lemma 3.5. Suppose \((H_1)\) and \((H_2)\) hold. Then for every \(\tau \in \mathbb{R}, \omega \in \Omega, B_\delta = \{B_\delta(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}\) and for any \(\varepsilon > 0\), there exists \(T = T(\tau, \omega, B_\delta, \varepsilon) \geq 1\) and \(N = N(\tau, \omega, \varepsilon) > 0\) such that for all \(t \geq T\) and \(s \in [\tau - 1, \tau]\), the solution \(u_\delta\) of problem (10) with \(\omega\) replaced by \(\theta - \tau \omega\) satisfies

\[
\int_{|x| \geq N} |u_\delta(\sigma, \tau - t, \theta - \tau \omega, u_\delta(\tau - t))|^2 \, dx \leq \varepsilon, \tag{55}
\]

where \(u_\delta(\tau - t) \in B_\delta(\tau - t, \theta - t \omega)\).

Proof. Let \(\beta\) be a smooth function defined on \(\mathbb{R}^+\) such that \(0 \leq \beta(s) \leq 1\) for all \(s \in \mathbb{R}^+\), and

\[
\beta(s) = \begin{cases} 
0, & \text{for } 0 \leq s \leq 1, \\
1, & \text{for } s \geq 2.
\end{cases}
\]

Let \(n\) be a fixed positive integer which will be specified later. First multiplying (10) by \(\beta \left(\frac{|x|^2}{n^2}\right) u_\delta\) and then taking the integral over \(\mathbb{R}\), taking real parts, we have

\[
\frac{d}{dt} \int_{\mathbb{R}} \beta \left(\frac{|x|^2}{n^2}\right) |u_\delta|^2 \, dx - 2\text{Re}(1 + i\lambda) \int_{\mathbb{R}} \beta \left(\frac{|x|^2}{n^2}\right) \Delta u_\delta \overline{u_\delta} \, dx \\
+ \rho \int_{\mathbb{R}} \beta \left(\frac{|x|^2}{n^2}\right) |u_\delta|^2 \, dx \\
= -2 \int_{\mathbb{R}} \beta \left(\frac{|x|^2}{n^2}\right) |u_\delta|^4 \, dx + 2\text{Re} \int_{\mathbb{R}} \beta \left(\frac{|x|^2}{n^2}\right) g(t, x) \overline{u_\delta} \, dx \\
+ 2\text{Re} \mathcal{G}_\delta(\theta, \omega) \int_{\mathbb{R}} \beta \left(\frac{|x|^2}{n^2}\right) h(t, x, u_\delta) \overline{u_\delta} \, dx. \tag{56}
\]

We infer from integration by parts and Young’s inequality that

\[
\text{Re}(1 + i\lambda) \int_{\mathbb{R}} \beta \left(\frac{|x|^2}{n^2}\right) \Delta u_\delta \overline{u_\delta} \, dx \\
= - \text{Re}(1 + i\lambda) \int_{\mathbb{R}} \beta \left(\frac{|x|^2}{n^2}\right) |\nabla u_\delta|^2 \, dx \\
+ \int_{n \leq |x| \leq \sqrt{2}n} \beta' \left(\frac{|x|^2}{n^2}\right) \frac{2x}{n^2} u_\delta \cdot \nabla u_\delta \, dx \\
\leq - \int_{\mathbb{R}} \beta \left(\frac{|x|^2}{n^2}\right) |\nabla u_\delta|^2 \, dx \\
+ \frac{2\sqrt{2}}{n} \sqrt{1 + \lambda^2} \int_{n \leq |x| \leq \sqrt{2}n} \beta' \left(\frac{|x|^2}{n^2}\right) |u_\delta \nabla u_\delta| \, dx \\
\leq - \int_{\mathbb{R}} \beta \left(\frac{|x|^2}{n^2}\right) |\nabla u_\delta|^2 \, dx + \frac{C}{n} (\|u_\delta\|^2 + \|u_\delta\|^2), \tag{57}
\]

\[
\text{Re} \int_{\mathbb{R}} \beta \left(\frac{|x|^2}{n^2}\right) g(t, x) \overline{u_\delta} \, dx \\
\leq \frac{1}{\rho} \int_{\mathbb{R}} \beta \left(\frac{|x|^2}{n^2}\right) |g(t, x)|^2 \, dx + \frac{\rho}{4} \int_{\mathbb{R}} \beta \left(\frac{|x|^2}{n^2}\right) |u_\delta|^2 \, dx, \tag{58}
\]

\[
\text{Re} \int_{\mathbb{R}} \beta \left(\frac{|x|^2}{n^2}\right) h(t, x, u_\delta) \overline{u_\delta} \, dx.
\]
\[
\text{ReG}_\delta (\theta \omega) \int \beta \left( \frac{|x|^2}{n^2} \right) h(t, x, u_\delta) \, dx \\
\leq \frac{3}{4} \int \beta \left( \frac{|x|^2}{n^2} \right) |u_\delta|^4 \, dx + C \int \beta \left( \frac{|x|^2}{n^2} \right) |G_\delta (\theta \omega) \beta_1 (t, x)|^2 \, dx \\
\quad + \int \beta \left( \frac{|x|^2}{n^2} \right) |G_\delta (\theta \omega) \beta_2 (t, x)|^\frac{1}{2} \, dx.
\] (59)

It follows from (56)-(59) that, there is \( N_1 = N_1(\varepsilon) > 0 \) such that for all \( n \geq N_1 \),

\[
\frac{d}{dt} \int \beta \left( \frac{|x|^2}{n^2} \right) |u_\delta|^2 \, dx + 2 \int \beta \left( \frac{|x|^2}{n^2} \right) \nabla u_\delta^2 \, dx \\
+ \frac{3}{2} \rho \int \beta \left( \frac{|x|^2}{n^2} \right) |u_\delta|^2 \, dx + \frac{1}{2} \beta \left( \frac{|x|^2}{n^2} \right) |u_\delta|^4 \, dx \\
\leq \varepsilon \|u_\delta\|^2_{H^1(\mathbb{R})} + \frac{2}{\rho} \int_{|x| \geq n} \beta \left( \frac{|x|^2}{n^2} \right) |g(t, x)|^2 \, dx \\
+ 2C \int \beta \left( \frac{|x|^2}{n^2} \right) |G_\delta (\theta \omega) \beta_1 (t, x)|^2 \, dx \\
+ \int \beta \left( \frac{|x|^2}{n^2} \right) |G_\delta (\theta \omega) \beta_2 (t, x)|^\frac{1}{2} \, dx.
\] (60)

Multiplying (60) with \( e^{\frac{1}{2} \rho t} \), integrating the inequality on \( (\tau - t, \sigma) \) with \( \sigma \in [\tau - 1, \tau] \) and replacing \( \omega \) by \( \theta_{-\tau} \omega \), we get

\[
\int \beta \left( \frac{|x|^2}{n^2} \right) |u_\delta (\sigma, \tau - t, \theta_{-\tau} \omega, u_{\delta(\tau-t)})|^2 \, dx \\
\leq e^{\frac{1}{2} \rho (\tau - t - \sigma)} \|u_{\delta(\tau-t)}\|^2 + \varepsilon \int_{\tau-t}^\sigma e^{\frac{1}{2} \rho (s-\sigma)} \|u_\delta\|^2_{H^1(\mathbb{R})} \, ds \\
+ \frac{2}{\rho} \int_\tau^{\sigma} e^{\frac{1}{2} \rho (s-\sigma)} \int_{|x| \geq n} |g(s, x)|^2 \, dx \, ds + 2C \int_\tau^{\sigma} e^{\frac{1}{2} \rho (s-\sigma)} \\
\int_{|x| \geq n} \left[ |G_\delta (\theta_{s-\tau} \omega) \beta_1 (s, x)|^2 + |G_\delta (\theta_{s-\tau} \omega) \beta_2 (s, x)|^\frac{1}{2} \right] \, dx \, ds \\
\leq e^{\frac{1}{2} \rho (\tau - t - \sigma)} \|u_{\delta(\tau-t)}\|^2 + \varepsilon \int_{-\infty}^{\sigma-\tau} e^{\frac{1}{2} \rho (s+\tau-\sigma)} \|u_\delta\|^2_{H^1(\mathbb{R})} \, ds \\
+ \frac{2}{\rho} \int_{-\infty}^{\sigma-\tau} e^{\frac{1}{2} \rho (s+\tau-\sigma)} \int_{|x| \geq n} |g(s + \tau, x)|^2 \, dx \, ds \\
+ 2C \int_{-\infty}^{\sigma-\tau} e^{\frac{1}{2} \rho (s+\tau-\sigma)} \int_{|x| \geq n} \left[ |G_\delta (\theta_{s} \omega) \beta_1 (s + \tau, x)|^2 \\
+ |G_\delta (\theta_{s} \omega) \beta_2 (s + \tau, x)|^\frac{1}{2} \right] \, dx \, ds.
\] (61)

Since \( u_{\delta(\tau-t)} \in B_\delta (\tau - t, \theta_{-\tau} \omega) \) and \( B_\delta \) is tempered, we find that

\[
\limsup_{t \to +\infty} e^{\frac{1}{2} \rho (\tau - t - \sigma)} \|u_{\delta(\tau-t)}\|^2 \leq \limsup_{t \to +\infty} e^{\frac{1}{2} \rho (\tau - t - \sigma)} \|B_\delta (\tau - t, \theta_{-\tau} \omega)\|^2 = 0,
\]

which shows that there exists \( T_1 = T_1 (\sigma, \tau, \omega, B_\delta, \varepsilon) > 0 \) such that for all \( t \geq T_1 \),

\[
e^{\frac{1}{2} \rho (\tau - t - \sigma)} \|u_{\delta(\tau-t)}\|^2 \leq \frac{\varepsilon}{5}.
\] (62)
By (H1), there is $N_2 = N_2(\tau, \rho, \varepsilon) \geq N_1$ such that for all $n \geq N_2$,
\[
\int_{-\infty}^{T} e^{\frac{1}{2}\rho(s+\tau-\sigma)} \int_{|x| \geq n} |g(s,x)| dx ds \leq \frac{\varepsilon}{5}.
\] (63)

By Lemma 2.8 (1), we find
\[
\int_{-\infty}^{\sigma-\tau} e^{\frac{1}{2}\rho(s+\tau-\sigma)} \left| \int_{\mathbb{R}} |G_{\delta}(\theta_t, \omega) \beta_1(s+\tau,x)|^2 dx \right| ds < +\infty,
\]
which implies that there is $N_3 = N_3(\tau, \rho, \omega, \varepsilon) \geq N_2$ such that for all $n \geq N_3$,
\[
\int_{-\infty}^{\sigma-\tau} e^{\frac{1}{2}\rho(s+\tau-\sigma)} \left| \int_{|x| \geq n} |G_{\delta}(\theta_t, \omega) \beta_1(s+\tau,x)|^2 dx \right| ds < \frac{\varepsilon}{5}.
\] (64)

Therefore, it follows from (61)-(64) and Lemma 3.3 that for all $\sigma \in [\tau - 1, \tau]$, $t \geq T_1(\tau, \omega, B_3, \varepsilon)$ and $n \geq N_3$,
\[
\int_{|x| \geq \sqrt{2n}} |u_{\delta}(\sigma, \tau - t, \theta_{-\tau}\omega, u_{\delta}(\tau - t))|^2 dx \\
\leq \int_{\mathbb{R}} \beta \left( \frac{|x|^2}{n^2} \right) |u_{\delta}(\sigma, \tau - t, \theta_{-\tau}\omega, u_{\delta}(\tau - t))|^2 dx \\
\leq \varepsilon,
\]
which completes the proof. \hfill \Box

**Lemma 3.6.** Suppose (H1) and (H2) hold. Then the continuous cocycle $\Phi$ is asymptotically compact in $L^2(\mathbb{R})$.

**Proof.** By Lemma 3.3, we find there exists $T = T(\tau, \omega, B_3) > 0$ and $C = C(\tau, \omega) > 0$ such that for all $t \geq T$ and $u_0 \in B_3(\tau - t, \theta_{-t}\omega)$,
\[
||u_{\delta}(\tau - 1, \tau - t, \theta_{-t}\omega, u_0)|| \leq C(\tau, \omega).
\] (65)

When $t_k \to +\infty$ ($k \to +\infty$) and $u_{0,k} \in B_3(\tau - t_k, \theta_{-t_k}\omega)$, there is $K_1 = K_1(\tau, \omega, B_3) > 0$ such that for all $k \geq K_1$,
\[
||u_{\delta}(\tau - 1, \tau - t_k, \theta_{-t_k}\omega, u_{0,k})|| \leq C(\tau, \omega).
\]

This shows that
\[
\{u_{\delta}(\tau - 1, \tau - t_k, \theta_{-t_k}\omega, u_{0,k})\} \text{ is bounded in } L^2(\mathbb{R}).
\] (66)

By (22), (30), and $H^1(O_n) \hookrightarrow L^2(O_n)$ is compact, we can prove that $u_{\delta}(\tau - 1, \tau - t, \omega, \cdot) : L^2(O_n) \to L^2(O_n)$ is compact. Then there exists $\tilde{u}_{\delta} \in L^2(\tau - 1, \tau; L^2(\mathbb{R}))$ such that, up to a subsequence and for all $n \in \mathbb{N}$ as $k \to +\infty$
\[
u_{\delta}(\cdot, \tau - t_k, \theta_{-t_k}\omega, u_{0,k}) = u_{\delta}(\cdot, \tau - 1, \theta_{-t_k}\omega, u_{\delta}(\tau - 1, \tau - t_k, \theta_{-t_k}\omega, u_{0,k}))
\]
\[
\to \tilde{u}_{\delta}(\cdot) \text{ in } L^2(\tau - 1, \tau; L^2(O_n)).
\] (67)

By choosing a further subsequence (which we not relabel), it follows from (67) that
\[
u_{\delta}(s, \tau - t_k, \theta_{-t_k}\omega, u_{0,k}) \to \tilde{u}_{\delta}(s) \text{ in } L^2(O_n) \text{ for almost all } s \in [\tau - 1, \tau].
\] (68)

Since $\tilde{u}_{\delta}(s) \in L^2(\mathbb{R})$, for any given $\varepsilon > 0$, there exists $N_1 = N_1(\varepsilon) > 0$ such that for all $n \geq N_1$,
Theorem 3.7. Suppose

\[ \int_{|x| \geq n} |\tilde{u}_\delta|^2 dx \leq \varepsilon. \]  \hspace{1cm} (69)

On the other hand, by Lemma 3.5, there exists \( K_2 = K_2(\tau, \omega, \varepsilon) \geq 1 \) and \( N_2 = N_2(\tau, \omega, B_\delta, \varepsilon) \geq N_1 \) such that for all \( k \geq K_2 \) and \( n \geq N_2 \),

\[ \int_{|x| \geq n} |u_\delta(s, \tau - t_k, \theta_\tau \omega, u_{0,k})|^2 dx \leq \varepsilon. \]  \hspace{1cm} (70)

Finally, by (69), we find that there exists \( K_3 = K_3(\tau, \omega, B_\delta, \varepsilon) \geq K_2 \) such that for all \( k \geq K_3 \),

\[ \int_{|x| < N_2} |u_\delta(s, \tau - t_k, \theta_\tau \omega, u_{0,k}) - \tilde{u}_\delta(s)|^2 dx \leq \varepsilon. \]  \hspace{1cm} (71)

Then we have

\[
\begin{align*}
&\|u_\delta(\tau, s, \theta_\tau \omega, u_\delta(s, \tau - t_k, \theta_\tau \omega, u_{0,k})) - u_\delta(\tau, s, \theta_\tau \omega, \tilde{u}_\delta(s))\|^2 \\
&\leq c \left\{ \int_{|x| < N_2} + \int_{|x| \geq N_2} \right\} |u_\delta(s, \tau - t_k, \theta_\tau \omega, u_{0,k}) - \tilde{u}_\delta(s)|^2 dx \\
&\leq c \int_{|x| < N_2} |u_\delta(s, \tau - t_k, \theta_\tau \omega, u_{0,k}) - \tilde{u}_\delta(s)|^2 dx \\
&\quad + c \int_{|x| \geq N_2} (|u_\delta(s, \tau - t_k, \theta_\tau \omega, u_{0,k})|^2 + |\tilde{u}_\delta(s)|^2) dx,
\end{align*}
\]

which together (69)-(71) implies that

\[ u_\delta(\tau, \tau - t_k, \theta_\tau \omega, u_{0,k}) \to u_\delta(\tau, s, \theta_\tau \omega, \tilde{u}_\delta(s)) \text{ in } L^2(\mathbb{R}). \]

Since \( \Phi(t_k, \tau - t_k, \theta_\tau \omega, u_{0,k}) = u_\delta(\tau, \tau - t_k, \theta_\tau \omega, u_{0,k}) \), the proof is completed. \( \square \)

By Theorem 2.7, Lemma 3.4 and Lemma 3.6, we will present the following existence of \( D \)-pullback attractors for \( \Phi \).

**Theorem 3.7.** Suppose (H1) and (H2) hold. Then the continuous cocycle \( \Phi \) of problem (10) has a unique \( D \)-pullback random attractor \( A_\delta = \{ A_\delta(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in D \) in \( L^2(\mathbb{R}) \).

4. Random Ginzburg-Landau equation driven by a linear multiplicative colored noise. In this section, we first consider the stochastic Ginzburg-Landau equation driven by a multiplicative white noise:

\[
\begin{cases}
\frac{\partial u}{\partial t} - (1 + i\lambda)\Delta u + \rho u = f(u) + g(t, x) + cu \circ dW, & t > \tau, x \in \mathbb{R}, \\
u(\tau, x) = u_\tau(x), & x \in \mathbb{R}.
\end{cases}
\]  \hspace{1cm} (72)

By a standard transformation

\[ v(t, \tau, \omega, v_\tau) = e^{-c_\omega(t)} u(t, \tau, \omega, u_\tau), \quad v_\tau = e^{-c_\omega(\tau)} u_\tau, \]

then (72) is converted into a pathwise random equation

\[
\begin{cases}
\frac{dv}{dt} - (1 + i\lambda)\Delta v + \rho v = e^{-c_\omega(t)} f(e^{c_\omega(t)} v) + e^{-c_\omega(t)} g(t, x), \\
v(\tau, x) = v_\tau(x)
\end{cases}
\]  \hspace{1cm} (73)

for \( t > \tau, x \in \mathbb{R} \). Similar to Theorem 3.2 in the previous section, one can prove that problem (73) has a unique solution \( v(\cdot, \tau, \omega, v_\tau) \in L^\infty([\tau, \infty); L^2(\mathbb{R})) \cap L^2_{\text{loc}}([\tau, \infty); H^1(\mathbb{R})) \cap L^4_{\text{loc}}([\tau, \infty); L^4(\mathbb{R})) \).
Based on the solution of (73), one can define a continuous cocycle $\Phi_0$ for the stochastic system (72)

$$
\Phi_0(t, \tau, \omega, u_\tau) = u(t + \tau, \tau, \theta_{-\tau} \omega, u_\tau).
$$

We will prove that the system (72) has a unique $D$-pullback random attractor $A_0$ in $L^2(\mathbb{R})$. First, we derive the following uniform estimates of solutions in $L^2(\mathbb{R})$.

**Lemma 4.1.** Suppose $\text{(H}_1\text{)}$ holds. Then for every $\sigma \in \mathbb{R}$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$, there exists $T = T(\tau, \omega, B) \geq 0$ such that for all $t \geq T$, the solution $u$ of problem (72) satisfies

$$
e^{-2c(\omega(\sigma-\tau)-\omega(-\tau))} \|u(\sigma-t, \theta_{-\tau} \omega, u_{t-\tau})\|^2 + \int_{-t}^{\sigma-t} e^{\frac{1}{2} \rho(s+\tau-\sigma)-2c(\omega(s)-\omega(-\tau))} \|u(s+\tau-t, \theta_{-\tau} \omega, u_{t-\tau})\|_{H^1(\mathbb{R})} ds \leq M \int_{-\infty}^{\sigma-t} e^{\frac{1}{2} \rho(\tau+s-\sigma)-2c(\omega(s)-\omega(-\tau))} \|g(s+\tau)\|^2 ds,
$$

where $u_{t-\tau} \in B(\tau-t, \theta_{-\tau} \omega)$ and $M$ is a positive constant independent of $\sigma$, $\tau$, $\omega$, and $B$.

**Proof.** Taking the inner product of (73) with $v$ and then taking real parts, we have

$$
\frac{d}{dt} \|v\|^2 + 2\|\nabla v\|^2 + 2\rho \|v\|^2 = 2e^{-c\omega(t)} \text{Re}(f(e^{c\omega(t)} v), v) + 2e^{-c\omega(t)} \text{Re}(g(t, x), v).
$$

Notice that, by Young’s inequality, we first have

$$
2e^{-c\omega(t)} \text{Re} \int_{\mathbb{R}} g(t, x) v dx \leq \frac{\rho}{2} \|v\|^2 + \frac{2}{\rho} e^{-2c\omega(t)} \|g\|^2.
$$

On the other hand, we find

$$
2e^{-c\omega(t)} \text{Re} \int_{\mathbb{R}} f(e^{c\omega(t)} v) v dx = -2e^{2c\omega(t)} \int_{\mathbb{R}} |v|^4 dx.
$$

It follows from (75)-(77) that

$$
\frac{d}{dt} \|v\|^2 + 2\|\nabla v\|^2 + \frac{3}{2} \rho \|v\|^2 + 2e^{2c\omega(t)} \|v\|_L^4(\mathbb{R}) \leq \frac{2}{\rho} e^{-2c\omega(t)} \|g\|^2.
$$

Multiplying (78) by $e^{\frac{1}{2} \rho t}$ and then integrating over $(\tau-t, \sigma)$ with $\sigma \geq \tau-t$, we get for every $\omega \in \Omega$,

$$
\|v(\sigma, \tau-t, \omega, v_{\tau-t})\|^2 + \int_{\tau-t}^{\sigma} e^{\frac{1}{2} \rho(s-\sigma)} \|\nabla v(s, \tau-t, \omega, v_{\tau-t})\|^2 ds + \rho \int_{\tau-t}^{\sigma} e^{\frac{1}{2} \rho(s-\sigma)} \|v(s, \tau-t, \omega, v_{\tau-t})\|^2 ds \leq e^{\frac{1}{2} \rho(\tau-t-\sigma)} \|v_{\tau-t}\|^2 + \frac{2}{\rho} \int_{\tau-t}^{\sigma} e^{\frac{1}{2} \rho(s-\sigma)-2c\omega(s)} \|g(s)\|^2 ds.
$$

Replacing $\omega$ by $\theta_{\tau-t} \omega$ and then changing variables, we obtain the following result
\[
\|v(\sigma, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^2
+ 2 \int_{-t}^{\tau - t} e^{\frac{1}{2} \rho(s + \tau - \sigma)} \|\nabla v(s + \tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 ds
+ \rho \int_{-t}^{\tau - t} e^{\frac{1}{2} \rho(s + \tau - \sigma)} \|v(s + \tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 ds
\leq e^{\frac{1}{2} \rho(t - \sigma)} \|v_{\tau-t}\|^2 \\
+ \frac{2}{\rho} \int_{-t}^{\tau - t} e^{\frac{1}{2} \rho(s + \tau - \sigma) - 2c(\omega(s) - \omega(\tau))} g(s + \tau)\|g(s + \tau)\|^2 ds. 
\]  

By definition we have
\[
u(s, \tau - t, \theta_{-\tau}\omega, u_{\tau-t}) = e^{c(\omega(s) - \omega(\tau))} v(s, \tau - t, \theta_{-\tau}\omega, v_{\tau-t}), 
\]  
with
\[
v_{\tau-t} = e^{-c(\omega(t) - \omega(\tau))} u_{\tau-t}, 
\]  
which along with (79) yields
\[
e^{-2c(\omega(s - \tau) - \omega(\tau))} \|u(\sigma, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 \\
+ 2 \int_{-t}^{\tau - t} e^{\frac{1}{2} \rho(s + \tau - \sigma) - 2c(\omega(s) - \omega(\tau))} \|\nabla u(s + \tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 ds \\
+ \rho \int_{-t}^{\tau - t} e^{\frac{1}{2} \rho(s + \tau - \sigma) - 2c(\omega(s) - \omega(\tau))} \|u(s + \tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 ds \\
\leq e^{\frac{1}{2} \rho(t - \sigma) - 2c(\omega(t) - \omega(\tau))} \|u_{\tau-t}\|^2 \\
+ \frac{2}{\rho} \int_{-t}^{\tau - t} e^{\frac{1}{2} \rho(s + \tau - \sigma) - 2c(\omega(s) - \omega(\tau))} g(s + \tau)\|g(s + \tau)\|^2 ds. 
\]  

By (4), we see that there exists \(T_1 = T_1(\omega) > 0\) such that for all \(t \geq T_1\),
\[
-\omega(-t) \leq \frac{\rho t}{16c}. 
\]  

Since \(u_{\tau-t} \in B(\tau - t, \theta_{-\tau}\omega)\) and \(B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}\) is tempered, there exists \(T_2 = T_2(\tau, \omega, B) \geq T_1\) such that for all \(t \geq T_2\),
\[
\limsup_{t \to +\infty} e^{\frac{1}{2} \rho(t - s - \tau) - 2c(\omega(s) - \omega(\tau))} \|u_{\tau-t}\|^2 \\
\leq c' \limsup_{t \to +\infty} e^{-\frac{1}{2} \rho t} \|B(\tau - t, \theta_{-\tau}\omega)\|^2 = 0. 
\]  

For the second term on the right-hand side of (81), we get for all \(t \geq T_2\),
\[
\frac{2}{\rho} \int_{-t}^{\tau - t} e^{\frac{1}{2} \rho(s + \tau - s - \sigma) - 2c(\omega(s) - \omega(\tau))} \|g(s + \tau)\|^2 ds < +\infty, 
\]  
which shows that for all \(t \geq T_2\),
\[
e^{\frac{1}{2} \rho(t - \sigma) - 2c(\omega(t) - \omega(\tau))} \|u_{\tau-t}\|^2 \\
\leq \frac{2}{\rho} \int_{-t}^{\tau - t} e^{\frac{1}{2} \rho(s + \tau - s - \sigma) - 2c(\omega(s) - \omega(\tau))} \|g(s + \tau)\|^2 ds. 
\]  

Then (81), (83), (84) imply the desired estimates. \(\Box\)
Lemma 4.2. Suppose \((H_1)\) holds. Then the continuous cocycle \(\Phi_0\) has a \(D\)-pullback absorbing set \(B = \{B(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D\), which is given by
\[
B(\tau,\omega) = \{u \in L^2(\mathbb{R}) : \|u\|^2 \leq R(\tau,\omega)\},
\]
where
\[
R(\tau,\omega) = M \int_0^\infty e^{\frac{1}{2} \rho s - 2\epsilon w(s)} \|g(s + \tau)\|^2 ds.
\]

Proof. By Lemma 4.1, we see that there exists \(T = T(\tau,\omega,B) > 0\) such that for all \(t \geq T\),
\[
u(\tau,\tau - t, \theta_{-\tau}, u_{\tau-t}) \in B(\tau,\omega).
\]
By (4), there exists \(T_1 = T_1(\tau,\omega) \geq T\) such that for all \(t \geq T_1\),
\[
\lim_{t \rightarrow +\infty} e^{-\beta t} R(\tau - t, \theta_{-\tau}) = \lim_{t \rightarrow +\infty} M e^{-\beta t} \int_{-\infty}^0 e^{\frac{1}{2} \rho s - 2\epsilon \omega(-t+s) - \omega(-t)} \|g(s + \tau - t)\|^2 ds \\
\leq M e^{-\frac{1}{2} \beta t} \int_{-\infty}^0 e^{\frac{1}{2} \rho s} \|g(s + \tau - t)\|^2 ds.
\]
By (7), we get
\[
\lim_{t \rightarrow +\infty} e^{-\beta t} R(\tau - t, \theta_{-\tau}) = 0,
\]
as desired. \(\Box\)

Lemma 4.3. Suppose \((H_1)\) holds. Then for every \(\tau \in \mathbb{R}, \omega \in \Omega\), \(B = \{B(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}\) and for any \(\epsilon > 0\), there exists \(T = T(\tau,\omega,B,\epsilon) \geq 1\) and \(N = N(\tau,\omega,\epsilon) > 0\) such that for all \(t \geq T\) and \(\sigma \in [\tau - 1, \tau]\), the solution \(u\) of problem (72) with \(\omega\) replaced by \(\theta_{-\tau}\omega\) satisfies
\[
\int_{|x| \geq N} |u(\sigma,\tau - t, \theta_{-\tau} \omega, u_{\tau-t})|^2 dx \leq \epsilon,
\]
where \(u_{\tau-t} \in B(\tau - t, \theta_{-\tau}\omega)\).

Proof. Let \(\beta\) be the smooth function as defined in Lemma 3.5. Let \(n\) be a fixed positive integer which will be specified later. First multiplying (73) by \(\beta\left(\frac{|x|^2}{n^2}\right) v\) and then taking the integral over \(\mathbb{R}\), taking real parts, we can get
\[
\frac{d}{dt} \int_{\mathbb{R}} \beta\left(\frac{|x|^2}{n^2}\right) |v|^2 dx - 2\Re(1 + i\lambda) \int_{\mathbb{R}} \beta\left(\frac{|x|^2}{n^2}\right) \Delta v \bar{\nu} dx \\
+ 2\rho \int_{\mathbb{R}} \beta\left(\frac{|x|^2}{n^2}\right) |v|^2 dx
= 2e^{-c\omega(1)\Re} \int_{\mathbb{R}} \beta\left(\frac{|x|^2}{n^2}\right) (f(e^{c\omega(t)}v),v) dx + \int_{\mathbb{R}} \beta\left(\frac{|x|^2}{n^2}\right) g(t,x) \bar{\nu} dx.
\]
Similar to Lemma 3.5, we have
\[
\Re(1 + i\lambda) \int_{\mathbb{R}} \beta\left(\frac{|x|^2}{n^2}\right) \Delta v \bar{\nu} dx \\
\leq - \int_{\mathbb{R}} \beta\left(\frac{|x|^2}{n^2}\right) |\nabla v|^2 dx + \frac{C}{n} (\|\nabla v\|^2 + |v|^2).
\]
By Young's inequality, we see that
\[
e^{-cw(t)} \int_{\mathbb{R}} \beta \left( \frac{|x|^2}{n^2} \right) g(t, x) v dx
\leq \frac{1}{\rho} e^{-cw(t)} \int_{\mathbb{R}} \beta \left( \frac{|x|^2}{n^2} \right) |g(t, x)|^2 dx + \frac{\rho}{4} \int_{\mathbb{R}} \beta \left( \frac{|x|^2}{n^2} \right) |v|^2 dx. \tag{90}
\]

On the other hand, we find
\[
2e^{-cw(t)} \int_{\mathbb{R}} \beta \left( \frac{|x|^2}{n^2} \right) (f(e^{cw(t)} v), v) dx
= -2 e^{2cw(t)} \int_{\mathbb{R}} \beta \left( \frac{|x|^2}{n^2} \right) |v|^4 dx. \tag{91}
\]

It follows from (88)-(91) that, there is \(N_1 = N_1(\varepsilon) > 0\) such that for all \(n \geq N_1\),
\[
\frac{d}{dt} \int_{\mathbb{R}} \beta \left( \frac{|x|^2}{n^2} \right) |v|^2 dx + \frac{3}{2} \rho \int_{|x| \geq n} \beta \left( \frac{|x|^2}{n^2} \right) |v|^2 dx
\leq \varepsilon |v|_{H^1(\mathbb{R})}^2 + \frac{2}{\rho} e^{-2cw(t)} \int_{|x| \geq n} \beta \left( \frac{|x|^2}{n^2} \right) |g(t, x)|^2 dx. \tag{92}
\]

Multiplying (92) by \(e^{\frac{\rho}{2} \tau} \) and then integrating over \((\tau - t, \sigma)\) with \(\sigma \in [\tau - 1, \tau]\), we get
\[
\int_{\mathbb{R}} \beta \left( \frac{|x|^2}{n^2} \right) |v(\sigma, \tau - t, \theta_{-\tau \omega}, v_{\tau - t})|^2 dx
\leq e^{\frac{\rho}{2} \tau} \rho(\tau - \sigma) \|v_{\tau - t}\|^2
+ \varepsilon \int_{-\tau}^{\sigma - \tau} e^{\frac{\rho}{2} \rho(s + \tau - \sigma)} \|v(s + \tau, \tau - t, \theta_{-\tau \omega}, v_{\tau - t})\|_{H^1(\mathbb{R})}^2 ds
+ \frac{2}{\rho} \int_{-\tau}^{\sigma - \tau} e^{\frac{\rho}{2} \rho(s + \tau - \sigma) - 2c(\omega(s) - \omega(-\tau))} \int_{|x| \geq n} \beta \left( \frac{|x|^2}{n^2} \right) |g(s + \tau, x)|^2 dx ds. \tag{93}
\]

It follows from (93) and (80) that
\[
e^{-2c(\omega(s) - \omega(-\tau))} \int_{\mathbb{R}} \beta \left( \frac{|x|^2}{n^2} \right) |u(s, \tau - t, \theta_{-\tau \omega}, u_{\tau - t})|^2 dx
\leq e^{\frac{\rho}{2} \rho(\tau - \sigma) - 2c(\omega(\tau) - \omega(-\tau))} \|u_{\tau - t}\|^2
+ \varepsilon \int_{-\tau}^{\sigma - \tau} e^{\frac{\rho}{2} \rho(s + \tau - \sigma) - 2c(\omega(s) - \omega(-\tau))} \|u(s + \tau, \tau - t, \theta_{-\tau \omega}, u_{\tau - t})\|_{H^1(\mathbb{R})}^2 ds
+ \frac{2}{\rho} \int_{-\tau}^{\sigma - \tau} e^{\frac{\rho}{2} \rho(s + \tau - \sigma) - 2c(\omega(s) - \omega(-\tau))} \int_{|x| \geq n} \beta \left( \frac{|x|^2}{n^2} \right) |g(s + \tau, x)|^2 dx ds. \tag{94}
\]

By (4), we see that there exists \(T_1 = T_1(\omega) > 0\) such that for all \(t \geq T_1\),
\[\omega(-t) \leq \frac{\rho t}{16c}.\]

Since \(u_{\tau - t} \in B(\tau - t, \theta_{-\tau \omega})\) and \(B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}\) is tempered, there exists \(T_2 = T_2(\tau, \omega, B, \varepsilon) \geq T_1\) such that for all \(t \geq T_2\),
\[e^{2c(\omega(s) - \omega(-\tau))} e^{\frac{\rho}{2} \rho(s + \tau - \sigma) - 2c(\omega(s) - \omega(-\tau))} \|u_{\tau - t}\|^2 \leq \frac{\varepsilon}{3}. \tag{95}\]
By Lemma 4.1, there is $T_3 = T_3(\tau, \omega, B) \geq T_2$ such that for all $t \geq T_3$, we can get
\[
e^{2\epsilon(\omega(\sigma-\tau)-\omega(-\tau))} \int_{-t}^{\sigma-\tau} e^{\frac{1}{2} \rho(\sigma-s-\sigma)+2\epsilon(\omega(s)-\omega(-\tau))} \|u(s, \tau, t, \theta_{-t} \omega, u_{-t})\|^2_{L^2(\mathbb{R})} ds \\
\leq M \int_{-\infty}^{\sigma-\tau} e^{\frac{1}{2} \rho(\sigma-s-\sigma)+2\epsilon(\omega(s)-\omega(-\tau))} \|g(s, \tau)\|^2 ds \\
< + \infty.
\]
By (H$_1$), there is $N_2 = N_2(\tau, \epsilon) \geq N_1$ such that for all $n \geq N_2$,
\[
\int_{-\infty}^{\sigma-\tau} e^{\frac{1}{2} \rho(\sigma-s-\sigma)} \int_{|x| \geq n} |g(s, \tau, x)| dx ds \leq \frac{\epsilon}{3}.
\]
Therefore, it follows from (94), (95)-(97) that for all $\sigma \in [\tau-1, \tau], \ t \geq T_3$ and $n \geq N_2$,
\[
\int_{|x| \geq \sqrt{n}} |u(\sigma, \tau-t, \theta_{-t} \omega, u_{-t})|^2 d\omega \\
\leq \int_{\mathbb{R}} \beta \left(\frac{|x|^2}{n^2}\right) |u(\sigma, \tau-t, \theta_{-t} \omega, u_{-t})|^2 dx \leq \epsilon,
\]
which completes the proof. \hfill \Box

Similar to Lemma 3.6, we can obtain the following lemma.

**Lemma 4.4.** Suppose (H$_1$) holds. Then the cocycle $\Phi_0$ associated with the stochastic equation (72) is $\mathcal{D}$-pullback asymptotically compact in $L^2(\mathbb{R})$.

By Theorem 2.7, Lemma 4.2 and Lemma 4.4, we will present the following existence and uniqueness of $\mathcal{D}$-pullback attractors for $\Phi_0$.

**Theorem 4.5.** Suppose (H$_1$) holds. Then the continuous cocycle $\Phi_0$ of problem (72) has a unique $\mathcal{D}$-pullback random attractor $A_0 = \{A_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ in $L^2(\mathbb{R})$.

We now consider the random Ginzburg-Landau equation driven by a linear multiplicative colored noise
\[
\begin{cases}
\frac{\partial u_\delta}{\partial t} - (1 + i\lambda) \Delta u_\delta + \rho u_\delta = f(u_\delta) + g(t, x) + cu_\delta \mathcal{G}_\delta(\theta \omega), \ t > \tau, \ x \in \mathbb{R}, \\
u_\delta(\tau, x) = u_{\delta t}(x), \ x \in \mathbb{R}.
\end{cases}
\]

(98)

Note that we write the solution of equation (98) as $u_\delta$ in order to indicate its dependence on $\delta$. We know (98) is a deterministic equation, as we did in Section 3, which can define a continuous cocycle $\Phi_\delta$ in $L^2(\mathbb{R})$ and possess a unique $\mathcal{D}$-pullback attractor $A_\delta$. In what follows, we will establish the relations between the solutions of systems (72) and (98) and show that the limiting behavior of system (98) as $\delta \to 0^+$.

To prove the convergence of solution, a similar transformation for (98) is given as we did for (72)
\[
v_\delta(t, \tau, \omega, \delta t) = e^{-c \int_{0}^{t} \mathcal{G}_\delta(\theta \omega) ds} u_\delta(t, \tau, \omega, \delta t), \ v_{\delta t} = e^{-c \int_{0}^{t} \mathcal{G}_\delta(\theta \omega) ds} u_{\delta t}.
\]
Then we get
\[
\begin{aligned}
\frac{\partial v_\delta}{\partial t} - (1 + i \lambda) \Delta v_\delta + \rho v_\delta = e^{-c \int_0^t \mathcal{G}_\delta(\theta, \omega) ds} f(e^{c \int_0^t \mathcal{G}_\delta(\theta, \omega) ds} v_\delta) \\
&+ e^{-c \int_0^t \mathcal{G}_\delta(\theta, \omega) ds} g(t, x),
\end{aligned}
\tag{99}
\]
for \( t > \tau, x \in \mathbb{R} \). For the solution of (99) we have the following estimates.

**Lemma 4.6.** Suppose \((H_1)\) holds. Then for every \( t \in \mathbb{R}, \omega \in \Omega \), there exists \( T = T(\tau, \omega, B_\delta) > 0 \) such that for all \( t \geq T \), the solution \( u_\delta \) of problem (98) satisfies
\[
e^{-2c \int_0^t \mathcal{G}_\delta(\theta, \omega) ds} \left\| u_\delta \left( \sigma, t - \tau, \theta_\tau \omega, u_\delta(\tau - t) \right) \right\|^2
\]
\[
+ \int_{-t}^{\tau \omega} e^{\frac{1}{2} \rho (s + \tau - \sigma) - 2c \int_0^t \mathcal{G}_\delta(\theta, \omega) ds} \left\| u_\delta \left( \sigma, t - \tau, \theta_\tau \omega, u_\delta(\tau - t) \right) \right\|^2_{H^1(\mathbb{R})} ds
\]
\[
\leq M \int_{-t}^{\tau \omega} e^{\frac{1}{4} \rho (s + \tau - \sigma) - 2c \int_0^t \mathcal{G}_\delta(\theta, \omega) ds} \| g(s + \tau) \|^2 ds,
\]
where \( u_\delta(\tau - t) \in B_\delta(\tau - t, \theta_\tau \omega) \) and \( M \) is a positive constant independent of \( \sigma, \tau, \omega, \) and \( B_\delta \).

**Proof.** Taking the inner product of (99) with \( v_\delta \) and then taking real parts, we have
\[
\frac{d}{dt} \| v_\delta \|^2 + 2 \left\| \nabla v_\delta \right\|^2 + 2\rho \| v_\delta \|^2
\]
\[
= 2e^{-c \int_0^t \mathcal{G}_\delta(\theta, \omega) ds} \text{Re} \left( f(t, x, e^{c \int_0^t \mathcal{G}_\delta(\theta, \omega) ds} v_\delta), v_\delta \right) + 2e^{-c \int_0^t \mathcal{G}_\delta(\theta, \omega) ds} \text{Re} \left( g(t, x), v_\delta \right).
\tag{101}
\]
By Young’s inequality, we obtain
\[
2e^{-c \int_0^t \mathcal{G}_\delta(\theta, \omega) ds} \text{Re} \int_{\mathbb{R}} g(t, x) \nabla v_\delta dx \leq \frac{\rho}{2} \| v_\delta \|^2 + 2e^{-2c \int_0^t \mathcal{G}_\delta(\theta, \omega) ds} \| g \|^2,
\tag{102}
\]
on the other hand, we find
\[
2e^{-c \int_0^t \mathcal{G}_\delta(\theta, \omega) ds} \text{Re} \int_{\mathbb{R}} f(e^{c \int_0^t \mathcal{G}_\delta(\theta, \omega) ds} v_\delta, v_\delta) \nabla v_\delta dx
\]
\[
= -2e^{-2c \int_0^t \mathcal{G}_\delta(\theta, \omega) ds} \int_{\mathbb{R}} |v_\delta|^4 dx.
\tag{103}
\]
It follows from (101)-(103) that
\[
\frac{d}{dt} \| v_\delta \|^2 + 2 \left\| \nabla v_\delta \right\|^2 + 3\left\| v_\delta \right\|^2 + 2e^{2c \int_0^t \mathcal{G}_\delta(\theta, \omega) ds} \| v_\delta \|^2_{L^4(\mathbb{R})}
\]
\[
\leq 2e^{-2c \int_0^t \mathcal{G}_\delta(\theta, \omega) ds} \| g \|^2.
\tag{104}
\]
Multiplying (104) by \( e^{\frac{1}{2} \rho t} \) and then integrating over \((\tau - t, \sigma)\) with \( \sigma \geq \tau - t \), we get for every \( \omega \in \Omega \),
\[
\left\| v_\delta \left( \sigma, t - \tau, \omega, v_\delta(\tau - t) \right) \right\|^2
\]
\[
+ 2 \int_0^{\sigma} e^{\frac{1}{2} \rho (s - \sigma)} \left\| \nabla v_\delta \left( s, t - \tau, \omega, v_\delta(\tau - t) \right) \right\|^2 ds
\]
Suppose Lemma 4.7.

Replacing \( \omega \) by \( \theta_{-t}\omega \) and then changing variables, we obtain the following result

\[
\| v_\delta (\sigma, \tau - t, \theta_{-\tau} \omega, v_\delta(\tau-t)) \|^2 
+ 2 \int_{\tau-t}^{\tau} e^{\frac{1}{2} \rho(s-\sigma)} \| \nabla v_\delta (s, \tau - t, \theta_{-\tau} \omega, v_\delta(\tau-t)) \|^2 ds 
+ \rho \int_{\tau-t}^{\tau} e^{\frac{1}{2} \rho(s-\sigma)} \| v_\delta (s, \tau - t, \theta_{-\tau} \omega, v_\delta(\tau-t)) \|^2 ds 
\leq e^{\frac{1}{2} \rho(\tau-t-\sigma)} \| v_\delta(\tau-t) \|^2 + \frac{2}{\rho} \int_{\tau-t}^{\tau} e^{\frac{1}{2} \rho(s-\sigma)-2c \int_{s}^{t} \mathcal{G}_s(\theta,\omega)dr} \| g(s) \|^2 ds.
\]

By definition we have

\[
u_\delta(\tau-t) = e^{-c \int_{0}^{t} \mathcal{G}_s(\theta,\omega)ds} v_\delta(\tau-t, \theta_{-\tau} \omega, v_\delta(\tau-t)), \quad (106)
\]

which along with (105) yields

\[
e^{-2c \int_{s}^{t} \mathcal{G}_s(\theta,\omega)ds} \| u_\delta (\sigma, \tau - t, \theta_{-\tau} \omega, u_\delta(\tau-t)) \|^2 
+ 2 \int_{-t}^{\sigma-\tau} e^{\frac{1}{2} \rho(s-\sigma)} \| \nabla v_\delta (s + \tau, \tau - t, \theta_{-\tau} \omega, u_\delta(\tau-t)) \|^2 ds 
+ \rho \int_{-t}^{\sigma-\tau} e^{\frac{1}{2} \rho(s-\sigma)} \| v_\delta (s + \tau, \tau - t, \theta_{-\tau} \omega, u_\delta(\tau-t)) \|^2 ds 
\leq e^{\frac{1}{2} \rho(\tau-t-\sigma)} \| v_\delta(\tau-t) \|^2 + \frac{2}{\rho} \int_{-t}^{\sigma-\tau} e^{\frac{1}{2} \rho(s-\sigma)-2c \int_{s}^{t} \mathcal{G}_s(\theta,\omega)dr} \| g(s + \tau) \|^2 ds.
\]

Then by (4), Lemma 2.8 (2), Lemma 2.10 and (107) we get the desired estimates. \( \square \)

**Lemma 4.7.** Suppose \((H_1)\) holds. Then the continuous cocycle \( \Phi_\delta \) has a \( \mathcal{D} \)-pullback absorbing set \( B_\delta = \{ B_\delta(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D} \), which is given by

\[
B_\delta(\tau, \omega) = \{ u \in L^2(\mathbb{R}) : \| u \|^2 \leq R_\delta(\tau, \omega) \},
\]

where

\[
R_\delta(\tau, \omega) = M \int_{-t}^{0} e^{\frac{1}{2} \rho s - 2c \int_{s}^{t} \mathcal{G}_s(\theta,\omega)dr} \| g(s + \tau) \|^2 ds.
\]

In addition,

\[
\lim_{\delta \to 0^+} R_\delta(\tau, \omega) = M \int_{-\infty}^{0} e^{\frac{1}{2} \rho s - 2c \omega(s)} \| g(s + \tau) \|^2 ds.
\]

**Proof.** By Lemma 4.6, we see that there exists \( T = T(\tau, \omega, B_\delta) > 0 \) such that for all \( t > T \),

\[
u_\delta (\tau, \tau - t, \theta_{-\tau} \omega, u_\delta(\tau-t)) \in B_\delta(\tau, \omega).
\]
Since
\[ e^{-\beta t} R_{\delta}(\tau - t, \theta_{-t}\omega) \]
\[ = e^{-\beta t} M \int_{-t}^{0} e^{\frac{1}{2} \rho s - 2c s} f_{\rho}(\theta_{s}\omega) dr \| g(s + \tau - t) \|^2 ds. \]

By Lemma 2.9 (2) and (7), we get that there exists \( T_1 = T_1(\tau, \omega) > T \) such that for all \( t > T_1 \),
\[ \lim_{t \to +\infty} e^{-\beta t} R_{\delta}(\tau - t, \theta_{-t}\omega) = 0, \]
as desired.

Given \( s \in \mathbb{R}, \delta > 0, \tau \in \mathbb{R} \) and \( \omega \in \Omega \), let
\[ \tilde{R}_{\delta}(\tau, \omega) = Me^{\frac{1}{2} \rho s - 2c s} f_{\rho}(\theta_{s}\omega) dr \| g(s + \tau) \|^2. \]

By Lemma 4.8, we see that
\[ \lim_{\delta \to 0^+} \tilde{R}_{\delta}(\tau, \omega) = Me^{\frac{1}{2} \rho s - 2c \omega(s)} \| g(s + \tau) \|^2. \]  \((111)\)

By (109), we know that
\[ R_{\delta}(\tau, \omega) = \int_{-\infty}^{0} \tilde{R}_{\delta}(\tau, \omega) ds = \int_{-\infty}^{-T_1} \tilde{R}_{\delta}(\tau, \omega) ds + \int_{-T_1}^{0} \tilde{R}_{\delta}(\tau, \omega) ds. \]

For all \( s \leq -T_1 \),
\[ \tilde{R}_{\delta}(\tau, \omega) \leq Me^{\frac{1}{2} \rho s} \| g(s + \tau) \|^2. \]  \((112)\)

By (H1), we find that the right-hand side of (112) is integrable over \((-\infty, T_1)\), which along with (111), (112) and the Lebesgue dominated convergence theorem yields
\[ \lim_{\delta \to 0^+} \int_{-\infty}^{-T_1} \tilde{R}_{\delta}(\tau, \omega) ds = M \int_{-\infty}^{-T_1} e^{\frac{1}{2} \rho s - 2c \omega(s)} \| g(s + \tau) \|^2 ds. \]  \((113)\)

On the other hand, by Lemma 2.9 we know that \( \int_{-T_1}^{0} G_{\delta}(\theta_{s}\omega) ds \) converges to \( \omega(s) \) uniformly on \([-T_1, 0]\) as \( \delta \to 0^+ \), and hence one can verify
\[ \lim_{\delta \to 0^+} \int_{-T_1}^{0} \tilde{R}_{\delta}(\tau, \omega) ds = M \int_{-T_1}^{0} e^{\frac{1}{2} \rho s - 2c \omega(s)} \| g(s + \tau) \|^2 ds. \]  \((114)\)

Finally, by (109), (113) and (114), we get (110). The proof is completed. \( \square \)

**Lemma 4.8.** Suppose (H1) holds. Then for every \( \tau \in \mathbb{R}, \omega \in \Omega \), \( B_{\delta} = \{B_{\delta}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \) and for any \( \varepsilon > 0 \), there exists \( T = T(\tau, \omega, B_{\delta}, \varepsilon) \geq 1 \) and \( N = N(\tau, \omega, \varepsilon) > 0 \) such that for all \( t \geq T \), the solution \( u_{\delta} \) of problem (98) with \( \omega \) replaced by \( \theta_{-\tau}\omega \) satisfies
\[ \int_{|x| \geq N} \left| u_{\delta}(\tau, \tau - t, \theta_{-\tau}\omega, u_{\delta}(\tau - t)) \right|^2 dx \leq \varepsilon, \]  \((115)\)
where \( u_{\delta}(\tau - t) \in B_{\delta}(\tau - t, \theta_{-\tau}\omega) \).
Proof. Let $\beta$ be the smooth function as defined in Lemma 3.5. First multiplying (99) by $\beta \left( \frac{|x|^2}{n^2} \right) v_\delta$ and then taking the integral over $\mathbb{R}$, taking real parts, we get

$$
\frac{d}{dt} \int_{\mathbb{R}} \beta \left( \frac{|x|^2}{n^2} \right) |v_\delta|^2 dx - 2\text{Re}(1 + i\lambda) \int_{\mathbb{R}} \beta \left( \frac{|x|^2}{n^2} \right) \Delta v_\delta \xi dx
+ 2\rho \int_{\mathbb{R}} \beta \left( \frac{|x|^2}{n^2} \right) |v_\delta|^2 dx
= 2e^{-c} \int_{0}^{|\tau-t|} G_\delta(\theta, \omega) d\xi \text{Re} \int_{\mathbb{R}} \beta \left( \frac{|x|^2}{n^2} \right) (f(e^{-c} \int_{0}^{|\tau-t|} G_\delta(\theta, \omega) d\xi) v_\delta) dx
+ 2\text{Re}e^{-c} \int_{0}^{|\tau-t|} G_\delta(\theta, \omega) d\xi \int_{\mathbb{R}} \beta \left( \frac{|x|^2}{n^2} \right) g(t, x, \xi) dx.
$$

By (116) and the process to derive (94) we obtain

$$
e^{-c} \int_{0}^{|\tau-t|} G_\delta(\theta, \omega) d\xi \int_{\mathbb{R}} \beta \left( \frac{|x|^2}{n^2} \right) |v_\delta|^2 dx
\leq e^{-c} \int_{0}^{|\tau-t|} G_\delta(\theta, \omega) d\xi \int_{\mathbb{R}} \beta \left( \frac{|x|^2}{n^2} \right) |v_\delta|^2 dx
+ 2\rho \int_{0}^{|\tau-t|} G_\delta(\theta, \omega) d\xi \int_{\mathbb{R}} \beta \left( \frac{|x|^2}{n^2} \right) g(t, x, \xi) dx.
$$

By Lemma 2.8 (2), we see that there exists $T_1 = T_1(\omega) > 0$ such that for all $t \geq T_1$,

$$-(\int_{|\tau-t|}^{-t} G_\delta(\theta, \omega) d\xi \leq \frac{\rho t}{16c}.
$$

Since $u_\delta(\tau-t) \in B_\delta(\tau-t, \theta-\tau \omega)$ and $B_\delta = \{B_\delta(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is tempered, there exists $T_2 = T_2(\tau, \omega, B_\delta, \epsilon) \geq T_1$ such that for all $t \geq T_2$,

$$e^{-c} \int_{0}^{|\tau-t|} G_\delta(\theta, \omega) d\xi \leq \frac{\epsilon}{3}.
$$

By Lemma 4.6, there is $T_3 = T_3(\tau, \omega, B_\delta) \geq T_2$ such that for all $t \geq T_3$, we can get

$$\int_{0}^{|\tau-t|} e^{-c} \int_{0}^{|\tau-t|} G_\delta(\theta, \omega) d\xi \geq M \int_{0}^{|\tau-t|} e^{-c} \int_{0}^{|\tau-t|} G_\delta(\theta, \omega) d\xi \leq \frac{\epsilon}{3}.
$$

By (H_1), there is $N_2 = N_2(\tau, \epsilon) \geq N_1$ such that for all $n \geq N_2$,

$$\int_{0}^{N_2} e^{-c} \int_{|\tau-t|}^{-t} g(t, x, \xi) d\xi \leq \frac{\epsilon}{3}.
$$
Therefore, it follows from (117), (119)-(121) that for all \( t \geq T_3 \) and \( n \geq N_2 \),
\[
\int_{|x| \geq \sqrt{2n}} |u_\delta (\tau, \tau - t, \theta_{-\tau} \omega, u_\delta (\tau - t))|^2 dx \\
\leq \int_{\mathbb{R}} \beta \left( \frac{|x|^2}{n^2} \right) |u_\delta (\tau, \tau - t, \theta_{-\tau} \omega, u_\delta (\tau - t))|^2 dx \leq \varepsilon,
\]
which completes the proof.

Next, we will establish the convergence of solution of (98) as \( \delta \to 0^+ \).

**Lemma 4.9.** Suppose (H1) holds. If \( u_\delta \) and \( u \) are the solutions of problem (98) and (72), respectively. Then for all \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( T > 0 \), there exists \( c = c(\tau, \omega, T) \) such that for all \( t \in [\tau, \tau + T] \),
\[
\|u_\delta - u\|^2 \leq c\|u_{s\tau} - u_{\tau}\|^2 + c\varepsilon(\|u_{\tau}\|^2 + \int_{\tau}^{t} \|g(s, x)\|^2 + \|u\|_{L^4(\mathbb{R})} ds).
\]

**Proof.** By (99) and (73), we have
\[
\frac{d}{dt}(v_\delta - v) - (1 + i\lambda)\Delta (v_\delta - v) + \rho(v_\delta - v) \\
= [e^{-c\int_0^t \mathcal{G}_0 (\theta_{s}, \omega) ds} f(e^{c\int_0^t \mathcal{G}_0 (\theta_{s}, \omega) ds} v_\delta) - e^{-c\omega(t)} f(e^{c\omega(t)} v)] \\
+ [e^{-c\int_0^t \mathcal{G}_0 (\theta_{s}, \omega) ds} - e^{-c\omega(t)}] g(t, x).
\]
Taking the inner product of (123) with \( v_\delta - v \) and then taking real parts, we have
\[
\frac{1}{2} \frac{d}{dt} \|v_\delta - v\|^2 + \|\nabla (v_\delta - v)\|^2 + \rho \|v_\delta - v\|^2 \\
= \text{Re} [e^{-c\int_0^t \mathcal{G}_0 (\theta_{s}, \omega) ds} f(e^{c\int_0^t \mathcal{G}_0 (\theta_{s}, \omega) ds} v_\delta) - e^{-c\omega(t)} f(e^{c\omega(t)} v)] \\
+ \text{Re} ([e^{-c\int_0^t \mathcal{G}_0 (\theta_{s}, \omega) ds} - e^{-c\omega(t)}] g(t, x, v_\delta - v) \\
:= J_1 + J_2.
\]
Now we need only solve \( J_1 \) and \( J_2 \).

\[ J_1 = \text{Re} [e^{-c\int_0^t \mathcal{G}_0 (\theta_{s}, \omega) ds} f(e^{c\int_0^t \mathcal{G}_0 (\theta_{s}, \omega) ds} v_\delta) - e^{-c\omega(t)} f(e^{c\omega(t)} v)] \\
= \text{Re} [e^{-c\int_0^t \mathcal{G}_0 (\theta_{s}, \omega) ds} f(e^{c\int_0^t \mathcal{G}_0 (\theta_{s}, \omega) ds} v_\delta) - f(e^{c\int_0^t \mathcal{G}_0 (\theta_{s}, \omega) ds} v)] \\
+ [e^{-c\int_0^t \mathcal{G}_0 (\theta_{s}, \omega) ds} - e^{-c\omega(t)}] f(e^{c\int_0^t \mathcal{G}_0 (\theta_{s}, \omega) ds} v) \\
+ e^{-c\omega(t)} [f(e^{c\int_0^t \mathcal{G}_0 (\theta_{s}, \omega) ds} v) - f(e^{c\omega(t)} v)], v_\delta - v) \\
= \text{Re} (1 + i\mu) \int_{\mathbb{R}} e^{2c\int_0^t \mathcal{G}_0 (\theta_{s}, \omega) ds} (|v_\delta|^2 v_\delta - |v|^2 v) \tilde{v} \tilde{r} dx \\
+ \int_{\mathbb{R}} [e^{-c\int_0^t \mathcal{G}_0 (\theta_{s}, \omega) ds} - e^{-c\omega(t)}] e^{3c\int_0^t \mathcal{G}_0 (\theta_{s}, \omega) ds} |v|^2 v \tilde{v}_\delta - \tilde{v} dx \\
+ \int_{\mathbb{R}} e^{-c\omega(t)} (e^{3c\int_0^t \mathcal{G}_0 (\theta_{s}, \omega) ds} - e^{3c\omega(t)}) |v|^2 v \tilde{v}_\delta - \tilde{v} dx \\
\leq \sqrt{1 + \mu^2} \int_{\mathbb{R}} e^{2c\int_0^t \mathcal{G}_0 (\theta_{s}, \omega) ds} (|v_\delta|^2 v_\delta - v)^2 dx \\
+ 2 \int_{\mathbb{R}} e^{2c\int_0^t \mathcal{G}_0 (\theta_{s}, \omega) ds} (|v_\delta|^2 + |v|^2)(v_\delta - v)^2 dx
\]
By Lemma 2.9, we find for every \( \varepsilon > 0 \), there exists \( \delta_1 = \delta_1(\varepsilon, \tau, \omega, T) > 0 \), such that for all \( 0 < \| \delta \| < \delta_1 \) and \( t \in [\tau, \tau + T] \),
\[
\left| \int_0^t G_\delta(\theta_s, \omega)ds - \omega(t) \right| < \varepsilon \quad \text{and} \quad \left| e^{2c\int_0^t G_\delta(\theta_s, \omega)ds} - e^{2c\omega(t)} \right| < \varepsilon. \tag{126}
\]

So we have
\[
J_1 \leq \min\left\{ \frac{\rho}{2}, \| v_\delta - v \|_{L^4(\mathbb{R})}^2 + c'(\omega)(\| v \|_{L^4(\mathbb{R})}^2 + \| v_\delta \|_{L^4(\mathbb{R})}^2)\| v_\delta - v \|^2
+ c'\varepsilon(\| v \|_{L^4(\mathbb{R})}^2 + \| v_\delta - v \|^2) \right\}. \tag{127}
\]

\[
J_2 = \text{Re}[e^{c\int_0^t G_\delta(\theta_s, \omega)ds} - e^{c\omega(t)}]g(t, x, \varepsilon, \tau, \omega) - \| v_\delta - v \|^2.
\tag{128}
\]

Combining (124)-(128), we have
\[
\frac{d}{dt}\| v_\delta - v \|^2 \leq \| v_\delta - v \|_{L^4(\mathbb{R})}^2(\| v \|_{L^4(\mathbb{R})}^2 + \| v_\delta \|_{L^4(\mathbb{R})}^2)\| v_\delta - v \|^2
+ \varepsilon\| g(t, x) \|^2 + c'\varepsilon\| v \|_{L^4(\mathbb{R})}^2.
\]

By Gronwall Lemma, we get that
\[
\| v_\delta - v \|^2
\leq e^{(t-\tau)+c'(\omega)\int_\tau^t(\| v \|_{L^4(\mathbb{R})}^2 + \| v_\delta \|_{L^4(\mathbb{R})}^2)ds}\| v_\delta - v_\tau \|^2
+ \int_\tau^t e^{(t-s)+c'(\omega)\int_\tau^s(\| v \|_{L^4(\mathbb{R})}^2 + \| v_\delta \|_{L^4(\mathbb{R})}^2)ds} (\varepsilon\| g(s, x) \|^2 + c'\varepsilon\| v \|_{L^4(\mathbb{R})}^2)ds.
\]

Then
\[
\| u_\delta - u \|^2 = \| e^{c\int_0^t G_\delta(\theta_s, \omega)ds}v_\delta - e^{c\omega(t)}v \|^2
\leq 2\| e^{c\int_0^t G_\delta(\theta_s, \omega)ds}(v_\delta - v) \|^2 + 2\| (e^{c\int_0^t G_\delta(\theta_s, \omega)ds} - e^{c\omega(t)})v \|^2
\leq c\| u_\delta - u_\tau \|^2 + c\varepsilon(\| u_\tau \|^2 + \int_\tau^t \| g(s, x) \|^2 + \| u \|_{L^4(\mathbb{R})}^2)ds).
\]

By Theorem 4.9, we obtain the following convergence of solution of (98) as \( \delta \to 0^+ \).

Lemma 4.10. Suppose (H1) holds and \( \delta_n \to 0^+ \). Let \( u_{\delta_n} \) and \( u \) be the solutions of problem (98) and (72) with initial datum \( u_{\delta_n, \tau} \) and \( u_\tau \), respectively. If \( u_{\delta_n, \tau} \to u_\tau \) in \( L^2(\mathbb{R}) \) as \( n \to +\infty \), then for every \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( t > \tau \),
\[
u_{\delta_n}(t, \tau, \omega, u_{\delta_n, \tau}) \to u(t, \tau, \omega, u_\tau) \quad \text{in} \quad L^2(\mathbb{R}).
\]

For the attractor \( A_\delta \) of \( \Phi_\delta \), we have the following uniform compactness.

Lemma 4.11. Suppose (H1) holds. Then for every \( \tau \in \mathbb{R}, \omega \in \Omega \), if \( \delta_n \to 0^+ \) and \( u_{\delta_n} \in A_{\delta_n}(\tau, \omega) \), Then \( \{ u_{\delta_n} \}_{n=1}^\infty \) is precompact in \( L^2(\mathbb{R}) \).
Proof. By Lemma 4.8, there exists $T = T(\tau, \omega, \varepsilon) > 0$, and $N = N(\tau, \omega, \varepsilon) > 0$ such that for all $t \geq T$,

$$
\int_{|x| \geq N} |u_\delta(t, \tau - t, \theta_{-\tau} \omega, u_\delta(\tau - t))|^2 \, dx \leq \frac{\varepsilon}{2},
$$

(129)

for any $u_\delta(\tau - t) \in B_\delta(\tau - t, \theta_{-\tau} \omega)$. By (129) and the invariance of $A_\delta$, we get

$$
\int_{|x| \geq N} |u_\delta|^2 \, dx \leq \frac{\varepsilon}{2},
$$

(130)

for any $u_\delta \in A_\delta(\tau, \omega)$.

On the other hand, from the case of bounded domains, we find the sequence $\{u_{\delta_n}\}_{n=1}^\infty$ is precompact in $L^2(O_N)$ with $O_N = \{x \in \mathbb{R}: |x| < N\}$. This together with (130) completes the proof. \qed

Now we present the upper semicontinuity of random attractors as $\delta \to 0^+$.

**Theorem 4.12.** Suppose (H1) holds. Then for all $\tau \in \mathbb{R}, \omega \in \Omega$,

$$
\lim_{\delta \to 0^+} d_{L^2(\mathbb{R})}(A_\delta(\tau, \omega), A_0(\tau, \omega)) = 0.
$$

**Proof.** Let $\delta_n \to 0^+$ and $u_{\delta_n, \tau} \to u_\tau$ in $L^2(\mathbb{R})$. Then by Lemma 4.10, we find that for all $\tau \in \mathbb{R}, \omega \in \Omega$ and $t \geq 0$,

$$
\Phi_{\delta_n}(t, \tau, \omega, u_{\delta_n, \tau}) \to \Phi_0(t, \tau, \omega, u_\tau).
$$

(131)

By (110), we get that for all $\tau \in \mathbb{R}, \omega \in \Omega$,

$$
\lim_{\delta \to 0^+} \|B_\delta(\tau, \omega)\|^2 \leq R(\tau, \omega),
$$

(132)

which along with (131) and Lemma 4.11, we prove this theorem from Theorem 3.1 in [31]. \qed

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