Common fixed point theorems via implicit relations

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COMMON FIXED POINT THEOREMS VIA IMPLICIT RELATIONS

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Abstract. We prove common fixed point theorems for four mappings satisfying implicit relations without decreasing assumption which improve theorems of [1, 13, 14]. We also prove common fixed point theorems for four mappings satisfying implicit relations which generalize theorems of [3,4].

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1. INTRODUCTION

Let S and T be self-mappings of a metric space \((X,d)\). S and T are commuting if \(STx = TSx\) for all \(x \in X\). Sessa [15] defined S and T to be weakly commuting if, for all \(x \in X\),

\[
d(STx, TSx) \leq d(Tx, Sx).
\]

(1.1)

Jungck [7] defined S and T to be compatible, as a generalization of the weakly commuting property, if

\[
\lim_{n \to \infty} d(STx_n, TSx_n) = 0
\]

(1.2)

whenever \(\{x_n\}\) is a sequence in \(X\) such that \(\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t\) for some \(t \in X\).

It is easy to show that “commuting” implies “weakly commuting,” which, in turn, implies “compatible,” and there are examples in the literature justifying that the inclusions are proper (see [7, 15]).

Jungck et al. [6] defined S and T to be compatible mappings of type (A) if

\[
\lim_{n \to \infty} d(STx_n, T^2x_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} d(TSx_n, S^2x_n) = 0
\]

(1.3)

whenever \(\{x_n\}\) is a sequence in \(X\) such that \(\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t\) for some \(t \in X\).

Clearly, “weakly commuting” implies “compatible of type (A)”. By [6], the converse is not true, and the two concepts of compatibility are independent.

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Recently, Pathak and Khan [12] defined \( S \) and \( T \) to be compatible mappings of type (B), as a generalization of compatible mappings of type (A), if

\[
\lim_{n \to \infty} d(TSx_n, S^2x_n) \leq \frac{1}{2} \left[ \lim_{n \to \infty} d(TSx_n, Tt) + \lim_{n \to \infty} d(Tt, T^2x_n) \right],
\]

\[
\lim_{n \to \infty} d(STx_n, T^2x_n) \leq \frac{1}{2} \left[ \lim_{n \to \infty} d(STx_n, St) + \lim_{n \to \infty} d(St, S^2x_n) \right]
\]

whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some \( t \in X \).

Clearly, compatible mappings of type (A) are compatible mappings of type (B), but the converse is not true [9]. However, the notions of compatibility, compatibility of type (A), and compatibility of type (B) are equivalent to one another if \( S \) and \( T \) are continuous [12].

Pathak et al. [10] defined \( S \) and \( T \) to be compatible mappings of type (P) if

\[
\lim_{n \to \infty} d(S^2x_n, T^2x_n) = 0
\]

whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some \( t \in X \).

The notions of compatibility, compatibility of type (A), and compatibility of type (P) are mutually equivalent if \( S \) and \( T \) are continuous [10].

Pathak et al. [11] defined \( S \) and \( T \) to be compatible mappings of type (C) as a generalization of compatible mappings of type (A) if

\[
\lim_{n \to \infty} d(TSx_n, S^2x_n) \leq \frac{1}{3} \left( \lim_{n \to \infty} d(TSx_n, Tt) + \lim_{n \to \infty} d(Tt, T^2x_n) + \lim_{n \to \infty} d(Tt, T^2x_n) \right),
\]

\[
\lim_{n \to \infty} d(STx_n, T^2x_n) \leq \frac{1}{3} \left( \lim_{n \to \infty} d(STx_n, St) + \lim_{n \to \infty} d(St, S^2x_n) + \lim_{n \to \infty} d(St, S^2x_n) \right)
\]

whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some \( t \in X \).

Clearly, compatible mappings of type (A) are also compatible mappings of type (C), but the converse implication is not true [11]. However, the properties of compatibility of type (A) and compatibility of type (C) are mutually equivalent if \( S \) and \( T \) are continuous (see [11]).

2. Preliminaries

**Definition 1** ([5]). \( S \) and \( T \) are said to be weakly compatible if they commute at their coincidence points, i.e., the equality \( Su = Tu \) for some \( u \in X \) implies that \( STu = TSu \).
Lemma 1 ([6, 7, 10–12]). If \( S \) and \( T \) are compatible, or compatible of type (A), or compatible of type (P), or compatible of type (B), or compatible of type (C), then they are weakly compatible.

The converse is not true in general, see [2].

Definition 2 ([8]). \( S \) and \( T \) are said to be \( R \)-weakly commuting at a point \( x \in X \) if for some \( R > 0 \)

\[
d(STx, TSx) \leq Rd(Tx, Sx). \tag{2.1}
\]

Definition 3 ([9]). \( S \) and \( T \) are said to be pointwise \( R \)-weakly commuting on \( X \) if, given an \( x \in X \), there exists an \( R > 0 \) such that (2.1) holds.

It was proved in [9] that the \( R \)-weak commutativity is equivalent to commutativity at coincidence points, i.e., \( S \) and \( T \) are pointwise \( R \)-weakly commuting if and only if they are weakly compatible.

Let \( \mathbb{R}_+ \) be the set of all non-negative real numbers and \( K_6 \) the family of all continuous mappings \( K(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R} \) with \( t_3 + t_4 \neq 0 \) satisfying the following conditions:

1. \( K_1 \) \( K \) is decreasing in variables \( t_5 \) and \( t_6 \).
2. \( K_2 \) there exists \( 0 \leq h < 1 \) such that for all \( u, v \geq 0 \) with
   - \( K_3 \) \( K(u, v, u, u + v, 0) \leq 0 \) or
   - \( K_4 \) \( K(u, v, u, v, 0, u + v) \leq 0 \)
   we have \( u \leq hv \).

The following theorem was proved in [13].

Theorem 1. Let \( S, T, I \) and \( J \) be self-mappings of a metric space \((X, d)\) satisfying

(a) \( S(X) \subset J(X) \) and \( T(X) \subset I(X) \).
(b) the inequality

\[
F(\{d(Sx, Ty), d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), d(Ix, Ty), d(Sx, Jy)\}) \leq 0
\]

holds for all \( x, y \in X \) and \( F \in K_6 \) satisfies conditions \( K_1 \) and \( K_2 \) if \( d(Ix, Sx) + d(Jy, Ty) \neq 0 \), or

\[
d(Sx, Ty) = 0 \quad \text{if} \quad (Ix, Sx) + d(Jy, Ty) = 0.
\]

(c) the pairs \((S, I)\) and \((T, J)\) are weakly compatible.

If one of \( S(X) \), \( T(X) \), \( I(X) \) and \( J(X) \) is a complete subspace of \( X \), then, \( S, T, I \) and \( J \) have a unique common fixed point \( z \) in \( X \). Further, \( z \) is the unique common fixed point of \( S \) and \( I \) and \( T \) and \( J \).

It is our purpose in this paper to prove common fixed point theorems for weakly compatible mappings satisfying implicit relations without condition \( K_1 \), which improve results of Ali and Imdad [1] and Popa [13, 14]. We also prove a common fixed
point theorem for weakly compatible mappings satisfying implicit relations which generalizes results of Jeong and Rhoades [4] and Jeong [3].

3. IMPLICIT RELATIONS

Let $F_6$ be the family of all continuous mappings $F(t_1,t_2,t_3,t_4,t_5,t_6):\mathbb{R}_+^6 \to \mathbb{R}$ with $t_3 + t_4 \neq 0$ satisfying the following condition:

(F1) there exists $0 \leq h < 1$ such that for all $u,v,w \geq 0$ with

\begin{align*}
(F_a) & \quad F(u,v,v,u,w,0) \leq 0 \\
(F_b) & \quad F(u,v,u,v,0,w) \leq 0
\end{align*}

we have $u \leq hv$.

Example 1. $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a \frac{t_2 t_3}{t_3 + t_4} - b \frac{t_4 t_5}{t_5 + t_6 + 1}$, where $0 < a, b < 1$ and $a + b < 1$.

(F1) : Let $u,v,w \geq 0$ and $F(u,v,v,u,w,0) = u - a \frac{v^2}{u+v} - b \frac{uw}{w+1} \leq 0$. Then, $u \leq h v$, where $h_1 = \frac{a}{1-b} < 1$. Similarly, if $F(u,v,u,v,0,w) \leq 0$ then $u \leq h_2 v$, where $h_2 = a < 1$. We take $h = \max\{h_1, h_2\} < 1$.

Example 2. $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a \frac{t_2 t_3}{t_3 + t_4} - b \frac{t_4 t_5}{t_5 + t_6 + 1}$, where $0 < a, b < 1$ and $a + b < 1$. The condition (F1) can be verified as in Example 1.

Let $H_6$ be the family of all continuous mappings $H(t_1,t_2,t_3,t_4,t_5,t_6):\mathbb{R}_+^6 \to \mathbb{R}$ with $t_5 + t_6 \neq 0$ satisfying the following conditions:

(H1) there exists $0 \leq h < 1$ such that for all $u,v,w \geq 0$ with

\begin{align*}
(H_a) & \quad H(u,v,v,u,w,0) \leq 0 \\
(H_b) & \quad H(u,v,u,v,0,w) \leq 0
\end{align*}

we have $u \leq hv$.

(H2) $H(u,u,0,0,u,u) > 0$ for all $u > 0$.

Example 3. $H(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a \frac{t_2 t_3 + t_3 t_4}{t_3 + t_4} - bt_2$, where $a, b > 0$ and $a + b < 1$.

(H1) : Let $u,v,w \geq 0$ and $H(u,v,v,u,w,0) = u - au - bv \leq 0$. Then, $u \leq h v$, where $h = \frac{b}{1-a} < 1$. Similarly, if $H(u,v,u,v,0,w) \leq 0$ then $u \leq h v$.

(H2) : $H(u,u,0,0,u,u) = (1-b)u > 0$ for all $u > 0$.

Example 4. $H(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a \frac{t_2 t_3 + t_3 t_4}{t_3 + t_4} - ct_2$, where $a, b, c > 0$ and $a + b + c < 1$.

(H1) : Let $u,v,w \geq 0$ and $H(u,v,v,u,w,0) = u - av - cv \leq 0$. Then, $u \leq h_1 v$, where $h_1 = a + c < 1$. Similarly, if $H(u,v,u,v,0,w) \leq 0$ then $u \leq h_2 v$, where $h_2 = b + c < 1$. We take $h = \max\{h_1, h_2\} < 1$.

(H2) : $H(u,u,0,0,u,u) = (1-c)u > 0$ for all $u > 0$.

Let $G_6$ be the family of all continuous mappings $G(t_1,t_2,t_3,t_4,t_5,t_6):\mathbb{R}_+^6 \to \mathbb{R}$ with $t_2 + t_4 \neq 0$ satisfying the following conditions:
(G1) \( G \) is decreasing in variables \( t_5 \) and \( t_6 \).

(G2) there exists \( 0 \leq h < 1 \) such that for all \( u, v \geq 0 \) with
\[
(G_u) \quad G(u, v, u, u + v, 0) \leq 0 \quad \text{or} \\
(G_v) \quad G(u, v, u, 0, u + v) \leq 0
\]
we have \( u \leq hv \).

(G3) \( G(u, u, 0, u, u) > 0 \) for all \( u > 0 \).

Example 5. \( G(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a_1 \frac{t_2 t_5}{t_2 + t_5} - a_2 (t_3 + t_4) - a_3 (t_5 + t_6) - a_4 t_2 \),
where \( a_1, a_2, a_3, a_4 > 0 \) and \( a_1 + 2a_2 + 2a_3 + a_4 < 1 \).

(G1) : It is clear.

(G2) : Let \( u, v \geq 0 \) and \( G(u, v, u, u + v, 0) = u - a_1 \frac{v(u + v)}{u + v} - a_2 (u + v) - a_3 (u + v) - a_4 v \leq 0 \). Then \( u \leq h_1 v \), where \( h_1 = \frac{a_1 + a_2 + a_4 + a_4}{1 - a_2} < 1 \). Similarly, if \( G(u, v, u, 0, u + v) \leq 0 \) then \( u \leq h_2 v \), where \( h_2 = \frac{a_2 + a_3 + a_4}{1 - a_2 - a_3} < 1 \). We take \( h = \max\{h_1, h_2\} < 1 \).

(G3) : \( G(u, u, 0, u, u) = (1 - a_1 - 2a_3 - a_4)u > 0 \) for all \( u > 0 \).

Let \( C_6 \) be the family of all continuous mappings \( C(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R} \) with \( t_2 + t_4 \neq 0 \) satisfying the following conditions:

(C1) there exists \( 0 \leq h < 1 \) such that for all \( u, v, w \geq 0 \) with
\[
(C_u) \quad C(u, v, u, u, w, 0) \leq 0 \quad \text{or} \\
(C_v) \quad C(u, v, u, v, 0, w) \leq 0
\]
we have \( u \leq hv \).

(C2) \( C(u, u, 0, u, u) > 0 \) for all \( u > 0 \).

Example 6. \( C(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a \frac{t_2 t_4}{t_2 + t_4} - b \frac{t_3 t_5}{t_3 + t_5} + 1 \), where \( 0 < a, b < 1 \) and \( a + b < 1 \).

(C1) : Let \( u, v, w \geq 0 \) and \( C(u, v, u, u, w, 0) = u - a \frac{wv + v}{w + v} - b \frac{uv}{w + 1} \leq 0 \). Then \( u \leq h_1 v \), where \( h_1 = a + b < 1 \). Similarly, if \( C(u, v, u, v, 0, w) \leq 0 \) then \( u \leq h_2 v \), where \( h_2 = \frac{a}{2} < 1 \). We take \( h = \max\{h_1, h_2\} < 1 \).

(C2) : \( C(u, u, 0, u, u) = u > 0 \) for all \( u > 0 \).

Example 7. \( C(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a \frac{t_2 t_4}{t_2 + t_4} - b \frac{t_3 t_6}{t_3 + t_6} + 1 \), where \( 0 < a, b < 1 \) and \( a + 2b < 2 \). (C1) and (C2) follow as in Example 6.

4. Main results

Theorem 2. Let \( f, g, S \) and \( T \) be self-mappings of a metric space \( (X, d) \) satisfying the following conditions:
\[
S(X) \subset g(X), \quad T(X) \subset f(X), \quad (4.1)
\]
\[
F(d(Sx, Ty), d(fx, gy), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(Sx, gy)) \leq 0 \quad (4.2)
\]
for all $x, y \in X$ and $F \in F_6$ satisfies (F₁) if $d(fx, Sx) + d(gy, Ty) \neq 0$, or
\[ d(Sx, Ty) = 0 \text{ if } d(fx, Sx) + d(gy, Ty) = 0. \]  
(4.3)

Suppose that one of $S(X)$, $T(X)$, $f(X)$, and $g(X)$ is a complete subspace of $X$ and the pairs $(S, f)$ and $(T, g)$ are weakly compatible.

Then, $f$, $g$, $S$, and $T$ have a unique common fixed point $z$ in $X$. Further, $z$ is the unique common fixed point of $S$ and $f$ and $T$ and $g$.

Proof. Let $x_0$ be an arbitrary point in $X$. By (4.1), we can define inductively a sequence $\{y_n\}$ in $X$ such that
\[ y_{2n} = Sx_{2n} = gx_{2n+1}, \quad y_{2n+1} = fx_{2n+2} = Tx_{2n+1} \quad \text{for } n = 0, 1, 2, \ldots \]  
(4.4)

If
\[ d(fx_{2n}, Sx_{2n}) + d(gx_{2n+1}, Tx_{2n+1}) \neq 0, \]
then, using (4.2) and (4.4), we have
\[
F(d(Sx_{2n}, Tx_{2n+1}), d(fx_{2n}, gx_{2n+1}), d(fx_{2n}, Sx_{2n}), d(gx_{2n+1}, Tx_{2n+1}),
\]
\[
d(fx_{2n}, Tx_{2n+1}), d(Sx_{2n}, gx_{2n+1}))
\]
\[ = F(d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n+1}), d(y_{2n-1}, y_{2n+1}), 0) \leq 0 \]

By (F₆), we get
\[ d(y_{2n}, y_{2n+1}) \leq hd(y_{2n-1}, y_{2n}). \]

Similarly, if
\[ d(fx_{2n+2}, Sx_{2n+2}) + d(gx_{2n+1}, Tx_{2n+1}) \neq 0, \]
we obtain
\[ d(y_{2n+1}, y_{2n+2}) \leq hd(y_{2n}, y_{2n+1}). \]

Therefore,
\[ d(y_n, y_{n+1}) \leq hd(y_{n-1}, y_n). \]

Thus, $\{y_n\}$ is a Cauchy sequence in $X$ and, therefore, the subsequence $\{y_{2n}\} = \{gx_{2n+1}\} \subset g(X)$ is a Cauchy sequence in $g(X)$. Since $g(X)$ is complete, it converges to a point $z = gv$ for some $v \in X$.

Therefore, the sequence $\{y_n\}$ converges also to $z$ and the subsequences $\{Sx_{2n}\}$, $\{Tx_{2n+1}\}$, and $\{fx_{2n+2}\}$ converge to $z$.

If $z \neq Tv$, using (4.2), we obtain
\[
F((d(Sx_{2n}, Tv), d(fx_{2n}, gv), d(fx_{2n}, Sx_{2n}), d(gv, Tv), d(fx_{2n}, Tv),
\]
\[ d(Sx_{2n}, gv))) \leq 0 \]

Letting $n \to \infty$ and using the continuity of $F$, we obtain
\[ F(d(z, Tv), 0, 0, d(z, Tv), d(z, Tv), 0) \leq 0. \]
By \((F_a)\), we get \(z = T v = g v\). Since \(T(X) \subset f(X)\), there exists an \(u \in X\) such that \(z = f u = T v\).

If \(z \neq Su\), using \((4.2)\) we have

\[
F\left(d(Su, T v), d(f u, g v), d(f u, Su), d(g v, T v), d(f u, T v), d(Su, g v)\right) \\
= F\left(d(Su, z), 0, d(z, Su), 0, 0, d(Su, z)\right) \leq 0.
\]

By \((F_b)\), we get \(z = Su = f u\). Since the pairs \((S, f)\) and \((T, g)\) are weakly compatible, we get \(f z = S z\) and \(g z = T z\). Since \(d(f z, S z) + d(g v, T v) = 0\), in view of \((4.3)\), it follows that \(d(S z, T v) = 0\), i.e., \(z = S z = f z\).

Similarly, we can prove that \(z = g z = T z\). Hence, \(z\) is a common fixed point of \(f, g, S,\) and \(T\).

The proof is similar if we suppose that, instead of \(g(X)\), one of \(S(X), T(X),\) and \(f(X)\) is complete. The uniqueness of \(z\) follows from \((4.3)\).

**Remark 1.** As the function \(F\) in Theorem 2 is not decreasing in variables \(t_5\) and \(t_6\), theorems of [1, 13, 14] are not applicable in this case.

Considering Examples 1 and 2, we obtain two corollaries of Theorem 2.

**Theorem 3.** Let \(f, g, S,\) and \(T\) be self-mappings of a metric space \((X, d)\) satisfying \((4.1)\),

\[
H\left(d(Sx, Ty), d(fx, gy), d(fx, sx), d(gy, Ty), d(fx, Ty), d(sx, gy)\right) \leq 0
\]

for all \(x, y \in X\) and \(H \in H_6\) satisfies \((H_1)\) and \((H_2)\) if \(d(fx, Ty) + d(sx, gy) \neq 0\), or

\[
d(Sx, Ty) = 0 \quad \text{if} \quad d(fx, Ty) + d(sx, gy) = 0.
\]

Suppose that one of \(S(X), T(X), f(X),\) and \(g(X)\) is a complete subspace of \(X\) and the pairs \((S, f)\) and \((T, g)\) are weakly compatible.

Then \(f, g, S,\) and \(T\) have a unique common fixed point \(z\) in \(X\). Further, \(z\) is the unique common fixed point of \(S\) and \(f\) and \(T\) and \(g\).

**Proof.** Let \(x_0\) be an arbitrary point in \(X\). By \((4.1)\), we can define inductively a sequence \(\{y_n\}\) in \(X\) by \((4.4)\).

If

\[
d(fx_{2n}, Tx_{2n+1}) + d(sx_{2n}, gx_{2n+1}) = 0
\]

or

\[
d(fx_{2n+2}, Tx_{2n+1}) + d(gx_{2n+1}, sx_{2n+2}) = 0
\]

for some \(n\), the proof is similar to [14].

If

\[
d(fx_{2n}, Tx_{2n+1}) + d(sx_{2n}, gx_{2n+1}) \neq 0
\]

and

\[
d(fx_{2n+2}, Tx_{2n+1}) + d(gx_{2n+1}, sx_{2n+2}) \neq 0
\]
as in the proof of Theorem 2, we obtain \( z = f u = S u = g v = T v \).

Since the pairs \((S, f)\) and \((T, g)\) are weakly compatible, we get \( f z = S z \) and \( g z = T z \).

If \( z \neq S z \), using (4.2) we have
\[
H \left( d(S z, T v), d(f z, g v), d(f z, S z), d(g v, T v), d(f z, T v), d(S z, g v) \right)
= H \left( d(S z, z), d(S z, z), 0, 0, d(S z, z), d(S z, z) \right) \leq 0.
\]

which is a contradiction to \((H_2)\). Therefore, \( z = S z = f z \).

Similarly, we can prove that \( z = T z = g z \). Hence, \( z \) is a common fixed point of \( f, g, S, \) and \( T \). The uniqueness of \( z \) follows from (4.5) and \((H_2)\). \(\square\)

Theorem 3 generalizes [4, Theorem 5] and [3, Theorem 2].

**Theorem 4.** Let \( f, g, S, \) and \( T \) be self-mappings of a metric space \((X, d)\) satisfying (4.1),
\[
G \left( d(Sx, Ty), d(f x, gy), d(f x, Sx), d(gy, Ty), d(f x, Ty), d(Sx, gy) \right) \leq 0
\]
for all \( x, y \in X \) and \( G \in G_6 \) satisfies conditions \((G_1)\), \((G_2)\), and \((G_3)\) if \( d(f x, gy) + d(gy, Ty) \neq 0 \), or
\[
d(Sx, Ty) = 0 \quad \text{if} \quad d(f x, gy) + d(gy, Ty) = 0.
\]

Suppose that one of \( S(X), T(X), f(X), \) and \( g(X) \) is a complete subspace of \( X \) and the pairs \((S, f)\) and \((T, g)\) are weakly compatible.

Then, \( f, g, S, \) and \( T \) have a unique common fixed point \( z \) in \( X \). Further, \( z \) is the unique common fixed point of \( S \) and \( f \) and \( T \) and \( g \).

**Proof.** Let \( x_0 \) be an arbitrary point in \( X \). By (4.1), we can define inductively a sequence \( \{y_n\} \) in \( X \) by (4.4).

If
\[
d(f x_2n, T x_2n+1) + d(S x_2n, g x_2n+1) \neq 0
\]
using (4.2) and (4.4) we have
\[
G \left( d(S x_2n, T x_2n+1), d(f x_2n, g x_2n+1), d(f x_2n, S x_2n), d(g x_2n+1, T x_2n+1), d(f x_2n, T x_2n+1), d(S x_2n, g x_2n+1) \right)
= G \left( d(y_2n, y_2n+1), d(y_2n-1, y_2n), d(y_2n-1, y_2n), d(y_2n, y_2n+1), d(y_2n-1, y_2n+1), 0 \right) \leq 0.
\]

By \((G_1)\) we have
\[
G \left( d(y_2n, y_2n+1), d(y_2n-1, y_2n), d(y_2n-1, y_2n), d(y_2n, y_2n+1), d(y_2n-1, y_2n) + d(y_2n, y_2n+1), 0 \right) \leq 0.
\]
By virtue of $(G_a)$, we get
\[ d(y_{2n}, y_{2n+1}) \leq h d(y_{2n-1}, y_{2n}). \]

In the same manner, if
\[ d(f x_{2n+2}, S x_{2n+2}) + d(g x_{2n+1}, T x_{2n+1}) \neq 0 \]
we obtain
\[ d(y_{2n+1}, y_{2n+2}) \leq h d(y_{2n}, y_{2n+1}). \]
Therefore,
\[ d(y_n, y_{n+1}) \leq h d(y_{n-1}, y_n). \]
As in the proofs of Theorems 2 and 3, $z$ is a common fixed point of $f, g, S,$ and $T$. The uniqueness of $z$ follows from (4.6) and $(G_3)$. \qed

Remark 2. Theorem 4 generalizes [4, Theorem 9] and [3, Theorem 5].

Similarly to Theorem 3, we can prove the following statement which improves Theorem 4.

**Theorem 5.** Let $f, g, S,$ and $T$ be self-mappings of a metric space $(X, d)$ satisfying (4.1),
\[ C(d(Sx, Ty), d(fx, gy), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(Sx, gy)) \leq 0 \]
for all $x, y \in X$ and $C \in C_6$ satisfies $(C_1)$ and $(C_2)$ if $d(fx, gy) + d(gy, Ty) \neq 0$,
or
\[ d(Sx, Ty) = 0 \text{ if } d(fx, gy) + d(gy, Ty) = 0. \]
Suppose that one of $S(X), T(X), f(X),$ and $g(X)$ is a complete subspace of $X$ and the pairs $(S, f)$ and $(T, g)$ are weakly compatible.
Then, $f, g, S,$ and $T$ have a unique common fixed point $z$ in $X$. Further, $z$ is the unique common fixed point of $S$ and $f$ and $T$ and $g$.

If $S = T$ or $f = g$ or $S = T$ and $f = g$, or $f = g = I$, where $I$ denotes the identity mapping, or $S = T$ and $f = g = I$ in Theorems 2, 3, 4, and 5, one can obtain several corollaries.

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