Scales

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Abstract

We introduce the notions of scale for sets and measures on metric space by generalizing the usual notions of dimension. Several versions of scales are introduced such as Hausdorff, packing, box, local and quantization. They are defined for different growth, allowing a refined study of infinite dimensional spaces. We prove general theorems comparing the different versions of scales. They are applied to describe geometries of ergodic decompositions, of the Wiener measure and from functional spaces. The first application solves a problem of Berger on the notions of emergence (2020); the second lies in the geometry of the Wiener measure and extends the work of Dereich-Lifshits (2005); the last refines Kolmogorov-Tikhomirov (1958) study on finitely differentiable functions.

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1 Introduction and results

Dimension theory was popularized by Mandelbrot in the article How long is the coast of Britain? \cite{Man67} and shed light on the general problem of measuring how large a natural object is. The category of objects considered are metric spaces possibly endowed with a measure. Dimension theory encompasses not only smooth spaces such as manifolds, but also wild spaces such as fractals, so that the dimension may be any non-negative real number. There are several notions of dimension: for instance Hausdorff \cite{Hau18}, packing \cite{Tri82} or box dimensions \cite{Bou28}. Also, when the space is endowed with a measure, there are moreover the local and the quantization dimensions. These different versions of dimension are bi-Lipschitz invariants. They are in general not equal, so that they reveal different aspects of the underlying space. Seminal works by Hausdorff, Frostman, Tricot, Fan, Tamashiro, Pötzelberger, Graf-Luschgy, and Dereich-Lifshits have described the relationships between these notions and provided conditions under which they coincide.

Obviously, these invariants do not give much information on infinite dimensional spaces. However, such spaces are the subject of many studies. As motivations, Kolmogorov-Tikhomirov in \cite{KT93} gave asymptotics of the covering numbers of some functional spaces. Dereich-Lifshits gave asymptotics of the mass of the small balls for the Wiener measure and exhibited their relationship with the quantization problem, see \cite{DFMS03,DL05,CM44,Chu47,BR92,KL93}. Also Berger and Bochi \cite{Ber20} gave estimates on the covering number and quantization number of the ergodic decomposition of some smooth dynamical systems. See also \cite{BR92,Klo15,BB21}.

These results lead to the following:

**Question.** Are there infinite-dimensional counterparts of the various versions of dimension that maintain similar relationships?

To address this question, we introduce the notion of scale. The key idea involves considering a scaling, which is a one-parameter family of gauge functions that satisfies mild assumptions, dictating at which ’scale’ the size of the space is examined. For instance the families for the \(dim\) for the dimension and \(ord\) for the order given in Example 1.1 are scalings. Given a scaling, different versions of scales are defined. In particular, Hausdorff dimension, packing dimension or box dimension are scales.

Given a scaling, we will generalize comparison theorems between the different kinds of dimensions to all the different kinds of scales in **Theorem A**, **B** and **C**. The definition of scaling is crafted so that the proofs for **Theorem A** and **Theorem B** are nearly direct extensions of established results from dimension theory (see Section 1.2). The main difficulty will be then to prove **Theorem C**, which enables us to compare the quantization scales with both the local and the box scales. Also even for the specific case of dimension, new inequalities between quantization dimension of a measure and box dimension of the set of positive mass are proven in **Theorem C** (inequalities (f) and (h)). Main novelties from **Theorem C** are reformulated in Theorems 3.10 and 3.11.

In the next Section 1.1 we recall usual definitions of dimension and introduce the notions of scaling and scales. The theorems comparing the different versions of scales are stated in Section 1.2. Precise definitions of the involved
scales are given in Section 2 and in Section 3 for measures. Then, Section 1.3 is devoted to applications of the main
results. In Section 1.3.1, a first application of Theorem C together with Dereich-Lifshits estimate in [DL05] implies the
coincidence of local, Hausdorff, packing, quantization and box orders of the Wiener measure for the $L^p$-norm, for any
$p \in [1, \infty]$. Then in Section 1.3.2, we apply Theorem A to show the coincidence of the box, Hausdorff and packing
orders for finitely regular spaces; refining the Kolmogorov-Tikhomirov study in [KT93 Thm XV]. Lastly in
Section 1.3.3, a consequence of Theorem C is that the local order of the ergodic decomposition is at most its quantization
order. This solves a problem set by Berger in [Ber20].

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structure of this article.

1.1 From dimension to scale

Let us first recall some classical definitions from dimension theory and see how they could be naturally extended to
define finite invariants for infinite dimensional spaces. The Hausdorff, packing and box dimensions of a totally bounded
metric space $(X, d)$ are defined by looking at families of subsets of $X$. First consider the box dimension. Recall that,
given an error $\epsilon > 0$, the covering number $N_\epsilon(X)$ is the minimal cardinality of a covering of $X$ by balls of radius $\epsilon$.
Then the lower and upper box dimensions of $(X, d)$ are given by:

\[
\dim_B X := \sup \left\{ \alpha > 0 : N_\epsilon(X) \cdot \phi_\alpha(\epsilon) \to +\infty \right\} \quad \text{and} \quad \overline{\dim}_B X := \inf \left\{ \alpha > 0 : N_\epsilon(X) \cdot \phi_\alpha(\epsilon) \to 0 \right\},
\]

where $(\phi_\alpha)_{\alpha > 0}$ is the family of functions on $(0, 1)$ given for $\alpha > 0$ by $\phi_\alpha : \epsilon \mapsto \epsilon^\alpha$.

Box dimensions, also called box counting dimensions or Minkovski-Bouligand dimensions were introduced by Bouli-
gand in [Bou28]. In general, upper and lower box dimensions do not coincide (see e.g. [Fal97, Fal04]). However, when
$X$ is a smooth manifold endowed with the Euclidean metric, these two dimensions coincide with the usual definition of
dimension. Basic properties of box dimensions are revealed when looking at subsets of a metric space with the induced
metric. Notably, box dimensions are non decreasing for the inclusion and are invariant by topological closure. In general
they are not $\sigma$-stable, i.e. the box dimensions of a countable union of subsets of a metric space are a priori not equal to
the suprema of the corresponding dimensions of the subsets. The most popular version of dimension that enjoys the
property of $\sigma$-stability is Hausdorff dimension. Let us recall its definition. Given an error $\epsilon > 0$, consider:

\[
\mathcal{H}_\epsilon^\alpha(X) := \inf_{C \in C_H(\epsilon)} \sum_{B(x, \delta) \in C} \phi_\alpha(\delta),
\]

where $C_H(\epsilon)$ is the set of countable coverings of $X$ by balls of radius at most $\epsilon$. Then, the Hausdorff dimension of $(X, d)$ is
given by:

\[
\dim_H X = \sup \left\{ \alpha > 0 : \mathcal{H}_\epsilon^\alpha(X) \xrightarrow[\epsilon \to 0]{} +\infty \right\} = \inf \left\{ \alpha > 0 : \mathcal{H}_\epsilon^\alpha(X) \xrightarrow[\epsilon \to 0]{} 0 \right\}.
\]

Lastly, another interesting dimension that enjoys $\sigma$-stability is the packing dimension. Its construction is analogous to
that of Hausdorff dimension and was introduced by Tricot in his thesis [Tri82]. It is actually linked to the upper box
dimension by the following characterization that we will use for the moment as a definition:

$$\dim_P X := \inf \sup_{n \geq 1} \dim_B E_n ,$$

where the infimum is taken over countable coverings $\{E_n\}_{n \geq 1}$ by subsets of $X$. These four versions of dimension are bi-Lipschitz invariants; they quantify different aspects of the geometry of the studied metric space since they a priori do not coincide. However, it always holds:

$$\dim_H X \leq \dim_B X \leq \overline{\dim} B X \quad \text{and} \quad \dim_H X \leq \dim_P X \leq \overline{\dim} B X .$$

See e.g. [Fal97], [Fal04] for detailed proofs.

Let us now introduce scales. A simple observation is that all of the above versions of dimension involve a specific parameterized family $$(\phi_\alpha)_{\alpha > 0} = (\epsilon \mapsto \epsilon^\alpha)_{\alpha > 0}$$ of gauge functions with polynomial behavior. In dimension theory, gauge functions generalize the measurement of ball diameters, enabling more control over the definition of the Hausdorff measure in the finite dimensional case. For scales, the objective is different. Roughly speaking, we will allow gauge functions to exhibit behaviors that are far from being polynomial. Let us precise the discussion. If a space $(X, d)$ is infinite dimensional then its covering number $N_\epsilon(X)$ grows faster than any polynomial in $\epsilon^{-1}$ as $\epsilon$ decreases to $0$. In order to define finite invariants for infinite dimensional spaces we must allow other gauge functions that decrease faster than any polynomial when the radius of the involved balls decreases to $0$. Consequently, we propose to replace the family $$(\phi_\alpha)_{\alpha > 0} = (\epsilon \in (0, 1) \mapsto \epsilon^\alpha)_{\alpha > 0}$$ in all the above definitions of dimensions, by other families of gauge functions that encompass the following examples of growth:

**Example 1.1.**

1. The family $\dim = (\epsilon \in (0, 1) \mapsto \epsilon^\alpha)_{\alpha > 0}$ which is used in the definitions of dimensions,

2. the family $\ord = (\epsilon \in (0, 1) \mapsto \exp(-\epsilon^{-\alpha}))_{\alpha > 0}$ which is called order. It fits with the growth of the covering number of spaces of finitely regular functions studied by Kolmogorov-Tikhomirov [KT93], see Theorem 1.11, or with the one of the space of ergodic measures spaces by Berger-Bochi [Ber20], as we will see in Theorem F,

3. the family $$(\epsilon \in (0, 1) \mapsto \exp(-\log \epsilon^{-1})^\alpha)_{\alpha > 0}$$ which fits with the growth of the covering number of holomorphic functions estimated by Kolmogorov-Tikhomirov [KT93], as we will see in Theorem 2.5.

To properly extend definitions and comparison theorems among different scales, i.e. the generalized box, Hausdorff, and packing dimensions; the family of functions $(\phi_\alpha)_{\alpha > 0}$ must satisfy certain mild assumptions. This requirement leads us to introduce the notion of scaling:

**Definition 1.2 (Scaling).** A family $\scl = (\scl_\alpha)_{\alpha > 0}$ of positive non-decreasing functions on $(0, 1)$ is a scaling when for every $\alpha > \beta > 0$ and any $\lambda > 1$ close enough to $1$, it holds:

$$\scl_\alpha(\epsilon) = o\left(\scl_\beta(\epsilon^\lambda)\right) \quad \text{and} \quad \scl_\alpha(\epsilon) = o\left(\scl_\beta(\epsilon)^\lambda\right) \quad \text{as} \ \epsilon \to 0 . \quad (*)$$

**Remark 1.3.** The left hand side condition is used in all the proofs of the theorems represented on Fig. 1. The right hand side condition is only used to prove the equalities between packing and upper local scales in Theorem B and to compare upper local scales with upper box and upper quantization scales in Theorem C inequalities $(c)\&(g)$. It also allows us to characterize packing scale with packing measure.

**Remark 1.4.** There are scalings that allow one to study $0$-dimensional spaces, for instance the family:

$$(\epsilon \mapsto \log(\epsilon^{-1})^{-\alpha})_{\alpha > 0} .$$
We will show in Proposition 2.4 that the families in Example 1.1 are scalings. Scalings enable defining scales that generalize packing dimension, Hausdorff dimension, box dimensions, quantization dimensions and local dimensions. For each scaling, the different kinds of scales do not a priori coincide on a generic space. Nevertheless, in Section 1.3 as a direct application of our comparison theorems, we bring examples of metric spaces and measures where all those definitions coincide. In these examples, equalities between the different scales are linked to some underlying "homogeneity" of the space that is provided by the existence of an equilibrium state.

Now for a metric space \((X, d)\), replacing the specific family \(\text{dim}\) in the definition of box dimensions by a given scaling \(\text{scl} = (\text{scl}_\alpha)_{\alpha > 0}\) gives the following:

**Definition 1.5 (Box scales).** Lower and upper box scales of a metric space \((X, d)\) are defined by:

\[
\text{scl}_B X = \sup \left\{ \alpha > 0 : N_\alpha(X) \cdot \text{scl}_\alpha(\epsilon) \xrightarrow{\epsilon \to 0} +\infty \right\}
\]

and

\[
\text{scl}^{\text{loc}}_B X = \inf \left\{ \alpha > 0 : N_\alpha(X) \cdot \text{scl}_\alpha(\epsilon) \xrightarrow{\epsilon \to 0} 0 \right\}.
\]

Moreover, we will generalize the notion of Hausdorff and packing dimensions to the Hausdorff scale denoted \(\text{scl}_H X\) (see Definition 2.13) and packing scale denoted \(\text{scl}_P X\) (see Definition 2.14). The constructions are fully detailed in the next section. Let us now state the main results on the comparison of scales of metric spaces.

### 1.2 Results on the comparison of scales

In this section, we introduce other kinds of scales and Theorems A, B and C that state the inequalities between them as illustrated in Fig. 1.

First, we bring the following generalization of classical inequalities comparing dimensions of metric spaces to the frame of scales:

**Theorem A.** Let \((X, d)\) be a metric space and \(\text{scl}\) a scaling, the following inequalities hold:

\[
\text{scl}_H X \leq \text{scl}_P X \leq \text{scl}_B X \quad \text{and} \quad \text{scl}_H X \leq \text{scl}_B X \leq \text{scl}_B X.
\]

In the specific case of dimension, these inequalities are well known and presented for instance by Tricot [Tri82] or Falconer [Fal97, Fal04]. The proof of this theorem will be done in Section 2.5. The key part is to show that Hausdorff scales and packing scales are well defined quantities. Then we will follow the lines of Falconer’s proof to show Theorem A.

The relationship between the dimensions of \((X, d)\) and its measures was first studied by Frostman [Fro35], who used equilibrium states to describe the Hausdorff dimension of the underlying space. Conversely, the dimensions of sets can characterize the asymptotic behavior of the mass of small balls of a measure. This concept was introduced by Fan as local dimension [Fan94], leading to seminal studies by Fan, Lau, and Rao [FLR02], Pötzelberger [Pot99], and Tamashiro [Tam95]. Similarly we introduce local scales that extend the notion of local dimensions of a measure:

**Definition 1.6 (Local scales).** Let \(\mu\) be a Borel measure on a metric space \((X, d)\) and \(\text{scl}\) a scaling. The lower and upper scales of \(\mu\) are the functions that map a point \(x \in X\) to:

\[
\text{scl}_\text{loc} \mu(x) = \sup \left\{ \alpha > 0 : \frac{\mu(B(x, \epsilon))}{\text{scl}_\alpha(\epsilon)} \xrightarrow{\epsilon \to 0} 0 \right\}
\]
Figure 1: Diagram presenting results of Theorems A, B and C.

Each arrow is an inequality, the scale at the starting point of the arrow is at least the one at its ending point:

\[ \rightarrow \Rightarrow \geq \]

and

\[ \text{scl}_{\text{loc}}\mu(x) = \inf \left\{ \alpha > 0 : \frac{\mu(B(x, \epsilon))}{\text{scl}_\alpha(\epsilon)} \rightarrow 0 \right\} \rightarrow +\infty \].

As in dimension theory, we should not compare the local scales with the scales of \( X \) but to those of its subsets with positive mass. This observation leads to considering the following:

**Definition 1.7 (Hausdorff, packing and box scales of a measure).** Let \( \text{scl} \) be a scaling and \( \mu \) a non-null Borel measure on a metric space \((X, d)\). For any \( \text{scl} \in \{ \text{scl}_H, \text{scl}_P, \text{scl}_B, \text{scl}_B' \} \) we define lower and upper scales of the measure \( \mu \) by:

\[
\text{scl}_\bullet \mu = \inf_{E \in \mathcal{B}} \{ \text{scl}_E : \mu(E) > 0 \} \quad \text{and} \quad \text{scl}_\bullet^* \mu = \inf_{E \in \mathcal{B}} \{ \text{scl}_E : \mu(X \setminus E) = 0 \},
\]

where \( \mathcal{B} \) is the set of Borel subsets of \( X \).

In the case of dimension Fan [Fan94, FLR02] and Tamashiro [Tam95] exhibited the relationships between the Hausdorff and packing dimensions of measures and their local dimensions that we generalize as:

**Theorem B.** Let \( \mu \) be a Borel measure on a metric space \((X, d)\), then for any scaling \( \text{scl} \), Hausdorff and packing scales of \( \mu \) are characterized by:

\[
\text{scl}_H \mu = \text{ess inf} \text{scl}_{\text{loc}} \mu, \quad \text{scl}_B^* \mu = \text{ess sup} \text{scl}_{\text{loc}} \mu, \quad \text{scl}_P \mu = \text{ess inf} \text{scl}_{\text{loc}} \mu, \quad \text{scl}_B^* \mu = \text{ess sup} \text{scl}_{\text{loc}} \mu,
\]

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where \( \text{ess sup} \) and \( \text{ess inf} \) denote the essential suprema and infima of a function.

The proof of Theorem B is provided in Section 3.1. It is inspired by the proofs of Fan [Fan94] and Tamashiro [Tam95] in the dimensional case.

Let us introduce a last kind of scale, the quantization scale. It generalizes the quantization dimension. This definition is motivated by the following works [GL07, Pöt99, DFMS03, DL05, Ber17, BB21, Ber20].

**Definition 1.8 (Quantization scales).** Let \((X, d)\) be a metric space and \(\mu\) a Borel measure on \(X\). Given an error \(\epsilon > 0\), the quantization number \(Q_\mu(\epsilon)\) of \(\mu\) is the minimal cardinality of a finite set of points that is \(\epsilon\)-close on average to any point in \(X\):

\[
Q_\mu(\epsilon) = \inf \left\{ N \geq 0 : \exists \{c_i\}_{i=1}^N \subset X, \int_X d(x, \{c_i\}_{1 \leq i \leq N}) \, d\mu(x) \leq \epsilon \right\}.
\]

Then lower and upper quantization scales of \(\mu\) for a given scaling \(scl\) are defined by:

\[
scl_{Q\mu}(\epsilon) = \sup \left\{ \alpha > 0 : scl_\alpha(\epsilon) \cdot Q_\mu(\epsilon) \xrightarrow{\epsilon \to 0} \infty \right\}
\]

and

\[
scl_{Q\mu}(\epsilon) = \inf \left\{ \alpha > 0 : scl_\alpha(\epsilon) \cdot Q_\mu(\epsilon) \xrightarrow{\epsilon \to 0} 0 \right\}.
\]

The following gives relationships between the different kinds of scales of measures:

**Theorem C (Main).** Let \((X, d)\) be a metric space. Let \(\mu\) be a Borel measure on \(X\). For any scaling \(scl\), the following inequalities on the scales of \(\mu\) hold:

\[
\text{ess inf}\ scl_{\text{loc}} \mu \leq\n\text{(a)}\ scl_B \mu \leq scl_{Q\mu} \leq\n\text{(b)}\ scl_{\text{loc}B} \mu \leq scl_B \mu \leq\n\text{(c)}\ scl_{\text{loc}Q\mu} \leq scl_{Q\mu} \leq scl_{\text{loc}B} \mu \leq\n\text{(d)}\ scl_{B\mu} \leq scl_{Q\mu} \mu.
\]

Inequalities (b) and (d) are part of Theorem 3.10 and rely mainly on the use of the Borel-Cantelli lemma. Even in the specific case of dimension, these inequalities are new, as far as we know. Inequalities (e) and (g) were shown by Pützelberger in [Pöt99] for dimension and in \([0, 1]^d\). A new approach for the general case is brought in Theorem 3.11. We deduce the inequality (a) from (e) and (f) and the inequality (c) from (g) and (h). The proof of inequalities (f) and (h) is straightforward, see Lemma 3.8.

As a direct application, inequality (e) allows us to answer a problem set by Berger in [Ber20] (see Section 1.3.3). We will give in Section 4.1 examples of spaces for which the different versions of orders do not coincide.

### 1.3 Applications

Let us see how our main theorems easily imply the equality between the different scales of some natural infinite dimensional spaces.
1.3.1 Wiener measure

The first example is the computation of the orders of the Wiener measure $W$ that describes uni-dimensional standard Brownian motion on $[0, 1]$. Recall that $W$ is the law of a continuous process $(B_t)_{t \in [0, 1]}$ with independent increments. It is such that for any $t \geq s$ the law of the random variable $B_t - B_s$ is $\mathcal{N}(0, t - s)$. Computation of the local scales of the Wiener measure relies on small ball estimates which received much interest [CM44, Chu47, BR92, KL93]. These results gave asymptotics on the measure of small balls centered at 0 for $L^p$ norms and Hölder norms. Moreover, for a random ball, Dereich-Lifshits made the following estimate for $L^p$-norms:

**Theorem 1.9** (Dereich-Lifshits [DL05][3.2, 5.1, 6.1, 6.3]). For the Wiener measure on $C^0([0, 1], \mathbb{R})$ endowed with the $L^p$-norm, for $p \in [1, \infty)$, there exists a constant $\kappa > 0$ such that for $W$-almost any $\omega \in C^0([0, 1], \mathbb{R})$:

$$-\epsilon^2 \cdot \log W(B(\omega, \epsilon)) \rightarrow \kappa, \text{ when } \epsilon \rightarrow 0,$$

and moreover the quantization number of $W$ verifies:

$$\epsilon^2 \cdot \log Q_W(\epsilon) \rightarrow \kappa, \text{ when } \epsilon \rightarrow 0.$$

As a direct consequence of Theorem [B] and Theorem [C] we get that the new invariants we introduced for a measure with growth given by $\text{ord}_{\text{all}}$ coincide:

**Theorem D** (Orders of the Wiener measure). The Wiener measure on $C^0([0, 1], \mathbb{R})$ endowed with the $L^p$-norm, for $p \in [1, \infty]$, verifies for $W$ almost every $\omega \in C^0([0, 1])$ :

$$2 = \text{ord}_{\text{loc}}(\omega) = \text{ord}_H W = \text{ord}_P W = \text{ord}_Q W = \text{ord}_B W = \text{ord}_H W = \text{ord}_P W = \text{ord}_Q W.$$

In particular, the framework of scales allows us to define Hausdorff, packing and box orders which are equal to 2 for the Wiener measure. This indicates some kind of constant geometric property of subsets of maps with positive Wiener measure.

**Proof.** By Theorem 1.9 for $W$-almost $\omega$ and for any $p \in [1, \infty]$, in the $L^p$-norm it holds:

$$\text{ord}_{\text{loc}} W(\omega) = \overline{\text{ord}}_{\text{loc}} W(\omega) = 2 = \text{ord}_Q W = \overline{\text{ord}}_Q W.$$

Now by Theorem [B] it holds:

$$\text{ord}_H W = \text{ord}_{\text{loc}} W(\omega) = \text{ord}_P W \quad \text{and} \quad \text{ord}_P W = \overline{\text{ord}}_{\text{loc}} W(\omega) = \text{ord}_P W.$$

Finally, since by Theorem [C] we have:

$$\overline{\text{ord}}_Q W \geq \overline{\text{ord}}_B W \geq \text{ord}_P W \geq \text{ord}_H W,$$

the desired result comes by combining the three above lines of equalities and inequalities.

**Remark 1.10.** Since $(C^0([0, 1], \mathbb{R}), L^p)$ is not (totally) bounded, as well as any of its subsets with total mass, it holds $\text{ord}_B = +\infty$.

\footnote{Note that for $p < \infty$, the constant $\kappa$ does not depend on the value of $p.$}
1.3.2 Functional spaces endowed with the $C^0$-norm

Let $d$ be a positive integer. For any integer $k \geq 0$ and for any $\alpha \in (0,1]$ denote:

\[ F_{d,k,0} := \{ f \in C^k([0,1]^d, [-1,1]) : \|f\|_{C^k} \leq 1 \} \]

and

\[ F_{d,k,\alpha} := \{ f \in C^k([0,1]^d, [-1,1]) : \|f\|_{C^k} \leq 1 \text{ and } D^k f \text{ is } \alpha\text{-Hölder with constant } 1 \} . \]

We endow these spaces with the $C^0$ uniform norm (see Section 4.2 for details).

Kolmogorov and Tikhomirov gave the following asymptotics:

**Theorem 1.11** (Kolmogorov-Tikhomirov, [KT93][Thm XV]). Let $d$ be a positive integer. For any integer $k$ and for any $\alpha \in [0,1]$, there exist two constants $C_1 > C_2 > 0$ such that for every $\epsilon > 0$, the covering number $N_\epsilon(F_{d,k,\alpha})$ of the space $(F_{d,k,\alpha}, \| \cdot \|_\infty)$ verifies:

\[ C_1 \cdot \epsilon^{-\frac{d}{k+\alpha}} \geq \log N_\epsilon(F_{d,k,\alpha}) \geq C_2 \cdot \epsilon^{-\frac{d}{k+\alpha}} . \]

In Section 4.2 we will embed an infinite-dimensional Kantor set into $F_{k,d,\alpha}$ via an expanding map. This embedding will enable us to prove:

**Lemma 1.12.** Let $d$ be a positive integer. For any integer $k$ and for any $\alpha \in [0,1]$, it holds:

\[ \text{ord}_H F_{d,k,\alpha} \geq \frac{d}{k + \alpha} . \]

The above lemma together with **Theorem A** gives the following consequence of the theorem of Kolmogorov-Tikhomirov:

**Theorem E.** Let $d$ be a positive integer. For any integer $k$ and for any $\alpha \in [0,1]$, it holds:

\[ \text{ord}_H \overline{F}_{d,k,\alpha} = \text{ord}_P \overline{F}_{d,k,\alpha} = \overline{\text{ord}_B F_{d,k,\alpha}} = \overline{\text{ord}_B \overline{F}_{d,k,\alpha}} = \frac{d}{k + \alpha} . \]

**Proof.** First, by **Theorem A** it holds:

\[ \text{ord}_H \overline{F}_{d,k,\alpha} \leq \text{ord}_B \overline{F}_{d,k,\alpha} \leq \overline{\text{ord}_B F_{d,k,\alpha}} \text{ and } \text{ord}_H F_{d,k,\alpha} \leq \text{ord}_P F_{d,k,\alpha} \leq \overline{\text{ord}_B F_{d,k,\alpha}} . \]

From there, by Theorem 1.11 and Lemma 1.12 it holds:

\[ \frac{d}{k + \alpha} \leq \text{ord}_H F_{d,k,\alpha} \leq \overline{\text{ord}_B F_{d,k,\alpha}} \leq \overline{\text{ord}_B F_{d,k,\alpha}} = \frac{d}{k + \alpha} , \]

and

\[ \frac{d}{k + \alpha} \leq \text{ord}_H F_{d,k,\alpha} \leq \text{ord}_P F_{d,k,\alpha} \leq \overline{\text{ord}_B F_{d,k,\alpha}} = \frac{d}{k + \alpha} . \]

From there, all of the above inequalities are indeed equalities, which gives the desired result. \qed
1.3.3 Local and global emergence

The framework of scales moreover allows us to answer a problem set by Berger in [Ber20] on the largeness of ergodic decomposition for wild dynamical systems. We now consider a compact metric space \((X, d)\) and a measurable map \(f : X \to X\). We denote \(\mathcal{M}\) the set of probability Borel measures on \(X\) and \(\mathcal{M}_f\) the subset of \(\mathcal{M}\) containing \(f\)-invariant measures. The space \(\mathcal{M}\) is endowed with the Wasserstein distance \(W_1\) defined by:

\[
W_1(\nu_1, \nu_2) = \sup_{\phi \in \text{Lip}^1(X)} \int \phi(d\nu_1 - d\nu_2),
\]

inducing the weak *- topology for which \(\mathcal{M}\) is compact. A way to measure the wildness of a dynamical system is to measure how far from being ergodic an invariant measure \(\mu\) is. Then by Birkhoff’s theorem, given a measure \(\mu \in \mathcal{M}_f\), for \(\mu\)-almost every \(x \in X\) the following measure is well defined:

\[
e^f(x) := \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)},
\]

and also this limit measure is ergodic. The notion of emergence, introduced by Berger, describes the size of the subset of ergodic measures that can be obtained by limits of empirical measures, given an \(f\)-invariant probability measure on \(X\).

**Definition 1.13 (Emergence, [Ber17, BB21]).** The emergence of a measure \(\mu \in \mathcal{M}_f\) at \(\epsilon > 0\) is defined by:

\[
E_\mu(\epsilon) = \min \{ N \in \mathbb{N} : \exists \nu_1, \ldots, \nu_N \in \mathcal{M}_f, \int_X W_1(e^f(x), \{\nu_i\}_{1 \leq i \leq N}) d\mu(x) \leq \epsilon \}.
\]

Note that the emergence is the quantization number of the ergodic decomposition of \(\mu\). The case of high emergence corresponds to dynamics where the considered measure is far from being ergodic. The following result shows us that the order is an adapted scaling in the study of the ergodic decomposition.

**Theorem 1.14 (BGV07, Klo15, BB21).** Let \((X, d)\) be a metric compact space of finite dimension, then:

\[
\dim_B X \leq \overline{\text{ord}}_B(\mathcal{M}) \leq \text{ord}_B(\mathcal{M}) \leq \underline{\text{dim}}_B X.
\]

Given a measure \(\mu \in \mathcal{M}_f\) we define its emergence order by:

\[
\overline{\text{ord}}_\mu = \limsup_{\epsilon \to 0} \frac{\log \log E_\mu(\epsilon)}{-\log \epsilon} = \inf \left\{ \alpha > 0 : E_\mu(\epsilon) \cdot \exp(-\epsilon^{-\alpha}) \to 0 \right\}.
\]

We denote \(e^f \cdot \mu\) the ergodic decomposition of \(\mu\); i.e. the push forward by \(e^f\) of \(\mu\). A local analogous quantity to the emergence order is the local order of the ergodic decomposition of \(\mu\), for \(\nu \in \mathcal{M}_f\) it is defined by:

\[
\overline{\text{ord}}_\mu^\text{loc}(\nu) := \limsup_{\epsilon \to 0} \frac{\log \log (e^f(\nu, \epsilon))}{-\log \epsilon}.
\]

Berger asked if the following comparison between the asymptotic behavior of the mass of the balls of the ergodic decomposition of \(\mu\) and the asymptotic behavior of its quantization holds.

**Problem 1.15 (Berger, [Ber20 Pbm 4.22]).** Let \((X, d)\) be a compact metric space, \(f : X \to X\) a measurable map and \(\mu\) an \(f\)-invariant probability measure on \(X\). Does the following hold?

\[
\int_{\mathcal{M}_f} \overline{\text{ord}}_\mu^\text{loc} d\mu \leq \overline{\text{ord}}_\mu.
\]
We propose here a stronger result that answers to the latter problem as a direct application of Theorem C.

**Theorem F.** Let $(X, d)$ be a compact metric space, $f : X \to X$ a measurable map and $\mu$ an $f$-invariant probability measure on $X$. For $\mu_{e,f}$-almost every $\nu \in M$, it holds:

$$\overline{\text{ord}}_{\mu}^{\text{loc}}(\nu) \leq \overline{\text{ord}}_{\mu}^{\text{Q}}.$$

**Proof.** Note that $\overline{\text{ord}}_{\mu}^{\text{loc}} = \overline{\text{ord}}_{\mu_{e,f}}^{\text{loc}}$ and $\overline{\text{ord}}_{\mu}^{\text{Q}} = \overline{\text{ord}}_{\mu_{e,f}}^{\text{Q}}$. Now by Theorem C, it holds $\mu_{e,f}$-almost surely that $\overline{\text{ord}}_{\mu_{e,f}}^{\text{loc}} \leq \overline{\text{ord}}_{\mu_{e,f}}^{\text{Q}}$ which is the desired result.

## 2 Metric scales

Metric scales will be bi-Lipschitz invariants generalizing Hausdorff, packing and box dimensions of metric spaces. Before defining and comparing metric scales we show a handful of basic properties of scalings and present some relevant examples.

### 2.1 Scalings

We first recall that a family $\text{scl} = (scl_\alpha)_{\alpha \geq 0}$ of positive non-decreasing functions on $(0, 1)$ is a scaling when for any $\alpha > \beta > 0$ and any $\lambda > 1$ close enough to $1$, it holds:

$$scl_\alpha(\epsilon) = o(\lambda^{\epsilon}) \quad \text{and} \quad scl_\alpha(\epsilon) = o(\lambda^{\epsilon}) \quad \text{when} \ \epsilon \to 0. \quad (\star)$$

An immediate consequence of the latter definition is the following:

**Fact 2.1.** Let $\text{scl}$ be a scaling then for any $\alpha > \beta > 0$ and for any constant $C > 0$, for $\epsilon > 0$ small enough, it holds:

$$\text{scl}_\alpha(\epsilon) \leq \text{scl}_\beta(C \cdot \epsilon).$$

A consequence of the latter fact is the following which will allow us to compare the different versions of scales:

**Lemma 2.2.** Let $f, g : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be two functions such that $f \leq g$ near $0$. For any constant $C > 0$, it holds:

$$\inf \left\{ \alpha > 0 : f(C \cdot \epsilon) \cdot \text{scl}_\alpha(\epsilon) \to \epsilon \to 0 \right\} \leq \inf \left\{ \alpha > 0 : g(\epsilon) \cdot \text{scl}_\alpha(\epsilon) \to \epsilon \to 0 \right\}$$

and

$$\sup \left\{ \alpha > 0 : f(C \cdot \epsilon) \cdot \text{scl}_\alpha(\epsilon) \to +\infty \right\} \leq \sup \left\{ \alpha > 0 : g(\epsilon) \cdot \text{scl}_\alpha(\epsilon) \to +\infty \right\}.$$

**Proof.** It suffices to observe that, by Fact 2.1, for any $\alpha > \beta > 0$, and $\epsilon > 0$ small, it holds:

$$f(\epsilon) \cdot \text{scl}_\alpha(\epsilon) \leq g(\epsilon) \cdot \text{scl}_\beta(C^{-1} \cdot \epsilon) = g(C \cdot \tilde{\epsilon}) \cdot \text{scl}_\beta(\tilde{\epsilon}),$$

with $\tilde{\epsilon} = C \cdot \epsilon$.

The following will provide a sequential characterization of scales:
Lemma 2.3 (Sequential characterization of scales). Let \( \text{scl} \) be a scaling and \( f: \mathbb{R}_+^* \to \mathbb{R}_+^* \) be a non-increasing function. Let \((r_n)_{n \geq 1}\) be a sequence of positive real numbers decreasing to 0 and such that \(\log r_{n+1} \sim \log r_n \) as \(n \to +\infty\), then it holds:
\[
\inf \left\{ \alpha > 0 : f(\epsilon) \cdot \text{scl}_\alpha(\epsilon) \xrightarrow{\epsilon \to 0} 0 \right\} = \inf \left\{ \alpha > 0 : f(r_n) \cdot \text{scl}_\alpha(r_n) \xrightarrow{n \to +\infty} 0 \right\}
\]
and
\[
\sup \left\{ \alpha > 0 : f(\epsilon) \cdot \text{scl}_\alpha(\epsilon) \xrightarrow{\epsilon \to +\infty} +\infty \right\} = \sup \left\{ \alpha > 0 : f(r_n) \cdot \text{scl}_\alpha(r_n) \xrightarrow{n \to +\infty} +\infty \right\}.
\]

Proof. Fix \(\alpha > 0\) and \(\epsilon > 0\). If \(\epsilon\) is sufficiently small, there exists an integer \(n > 0\) such that \(r_{n+1} < \epsilon \leq r_n\). Since \(f\) is non-increasing and \(\text{scl}_\alpha\) is increasing, we have the inequalities:
\[
f(r_n) \cdot \text{scl}_\alpha(r_{n+1}) \leq f(\epsilon) \cdot \text{scl}_\alpha(\epsilon) \leq f(r_{n+1}) \cdot \text{scl}_\alpha(r_n).
\]
(2.1)

Now, let \(\beta\) and \(\gamma\) be positive real numbers such that \(0 < \beta < \alpha < \gamma\). For \(\lambda\) sufficiently close to 1 and for sufficiently small \(\epsilon\), by Eq. (2.1) it holds:
\[
\text{scl}_\lambda(r_n) \leq \text{scl}_\alpha(r_n) \quad \text{and} \quad \text{scl}_\alpha(r_n) \leq \text{scl}_\beta(r_n).
\]
(2.2)

As \(\lambda \geq \frac{\log r_{n+1}}{\log r_n}\) for large \(n\), it holds:
\[
r_n^\lambda \leq r_{n+1}.
\]

This together with Eq. (2.2) implies:
\[
\text{scl}_\gamma(r_n) \leq \text{scl}_\alpha(r_{n+1}) \quad \text{and} \quad \text{scl}_\alpha(r_n) \leq \text{scl}_\beta(r_{n+1}).
\]
(2.3)

By combining Eqs. (2.1) and (2.3), we obtain:
\[
f(r_n) \cdot \text{scl}_\gamma(r_n) \leq f(\epsilon) \cdot \text{scl}_\alpha(\epsilon) \quad \text{and} \quad f(\epsilon) \cdot \text{scl}_\alpha(\epsilon) \leq f(r_{n+1}) \cdot \text{scl}_\beta(r_{n+1}).
\]

Thus, it follows that:
\[
\limsup_{n \to +\infty} f(r_n) \cdot \text{scl}_\gamma(r_n) \leq \limsup_{\epsilon \to 0} f(\epsilon) \cdot \text{scl}_\alpha(\epsilon) \leq \limsup_{n \to +\infty} f(r_n) \cdot \text{scl}_\beta(r_n),
\]
and similarly:
\[
\liminf_{n \to +\infty} f(r_n) \cdot \text{scl}_\gamma(r_n) \leq \liminf_{\epsilon \to 0} f(\epsilon) \cdot \text{scl}_\alpha(\epsilon) \leq \liminf_{n \to +\infty} f(r_n) \cdot \text{scl}_\beta(r_n).
\]

Since this holds for every positive \(\alpha\), and since \(\beta\) and \(\gamma\) can be taken arbitrarily close to \(\alpha\), we obtain the desired result.

The following provides a whole class of scalings. It shows in particular that the families brought in Example 1.1 are indeed scalings.

Proposition 2.4. For any integers \(p, q \geq 1\), the family \(\text{scl}^{p,q} = (\text{scl}^{p,q}_\alpha)_{\alpha > 0}\) defined for any \(\alpha > 0\) by:
\[
\text{scl}^{p,q}_\alpha : \epsilon \in (0,1) \mapsto \frac{1}{\exp^{p}(\alpha \cdot \log_+^{\alpha}(\epsilon^{-1}))}
\]
is a scaling; where \(\log_+ : t \in \mathbb{R} \mapsto \log(t) \cdot 1_{t > 1}\) is the positive part of the logarithm.
We prove this proposition below. Note in particular that:
\[ \text{scl}^{1,1} = \dim = (\epsilon \in (0, 1) \mapsto \epsilon^\alpha)_{\alpha > 0} \quad \text{and} \quad \text{scl}^{2,1} = \text{ord} = (\epsilon \in (0, 1) \mapsto \exp(-\epsilon^{-\alpha}))_{\alpha > 0} \]
are both scalings. Let us give an example of a space that has finite box scales for the scaling \( \text{scl}^{2,2} \) as defined in Proposition 2.4. Consider the space \( A \) of holomorphic functions on the disk \( \mathbb{D}(R) \subset \mathbb{C} \) of radius \( R > 1 \) which are bounded by 1:

\[
A = \left\{ \phi = \sum_{n \geq 0} a_n z^n \in C^\infty(\mathbb{D}(R), \mathbb{C}) : \sup_{D(R)} |\phi| \leq 1 \right\}
\]

endowed with the norm \( \|\phi\|_\infty := \sup_{z \in D(1)} |\phi(z)| \).

Kolmogorov and Tikhomirov gave the following estimate of its covering number:

**Theorem 2.5** (Kolmogorov, Tikhomirov [KT93] [Equality (129)]). The following estimate on the covering number of \( (A, \| \cdot \|_\infty) \) holds:

\[
\log N_\epsilon(A) = (\log R)^{-1} \cdot |\log |\epsilon|^2 + O(\log |\epsilon|^{-1} \cdot \log \log |\epsilon|^{-1}), \text{ when } \epsilon \text{ tends to } 0.
\]

In the framework of scales, the above translates as:

\[
\text{scl}^{2,2}_R A = \text{scl}^{2,2}_R A = 2.
\]

Let us now give a proof of Proposition 2.4.

**Proof of Proposition 2.4**. The proof is based on the following two facts:

**Fact 2.6.** For every \( \nu > 1 \), for every \( d \geq 1 \) and for \( y > 0 \) large enough, it holds:

\[
\log^{\nu d}(y^\nu) \leq \nu \cdot \log^{\nu d}(y).
\]

**Fact 2.7.** For any \( \gamma > 0 \) and \( \nu > 1 \) close to 1, it holds for \( \epsilon > 0 \) small:

\[
\text{scl}^{p,q}_{\nu,\gamma}(\epsilon) \leq \text{scl}^{p,q}(\epsilon^\nu) \quad \text{and} \quad \text{scl}^{p,q}_{\nu,\gamma}(\epsilon) \leq \text{scl}^{p,q}(\epsilon^\nu).
\]

Actually **Fact 2.7** will be proved using **Fact 2.6**. First let us show recursively:

**Proof of Fact 2.6**. We prove this fact by induction on \( d \geq 1 \). For \( d = 1 \) note that the inequality is obvious as the equality holds. Then we conclude by induction on \( d \) based on the following:

\[
\log^{\nu(d+1)}(y^\nu) = \log(\log^{\nu d}(y^\nu)) \leq \log(\nu \log^{\nu d} y) = \log \nu + \log^{\nu(d+1)} y,
\]

where the inequality is given by the induction hypothesis.

We are now ready to show:

**Proof of Fact 2.7**. We apply **Fact 2.6** with \( d = q \) and \( y = \epsilon^{-1} \) for sufficiently small values of \( \epsilon \) to obtain for every \( \nu > 1 \):

\[
\log^{\nu q}(\epsilon^{-\nu}) \leq \nu \cdot \log^{\nu q}(\epsilon^{-1}).
\]

Multiplying by \( \gamma \) and composing by \( t \mapsto 1/\exp^{\nu p}(t) \) which is decreasing, yields the first inequality in **Fact 2.7**.
To show the second inequality, we apply again Fact 2.6 with \( d = p \) and \( y = \frac{1}{\text{scl}_{\gamma}(\epsilon)} \) which is large for small values of \( \epsilon > 0 \) to obtain:

\[
\log^{op}(y') \leq \nu \cdot \log^{op}(y).
\]

As \( y = \exp^{op}(\gamma \cdot \log^{sq}(\epsilon^{-1})) \), the above inequality translates as \( \log^{op}(y') \leq \nu \cdot \gamma \cdot \log^{sq}(\epsilon^{-1}) \). It follows that:

\[
y' \leq \exp^{op}(\nu \cdot \gamma \cdot \log^{sq}(\epsilon^{-1})) = \frac{1}{\text{scl}_{\nu,\gamma}(\epsilon)}.
\]

In other words, it holds \( \text{scl}_{\nu,\gamma}(\epsilon) \leq y^{-\nu} \) which is exactly the second inequality of Fact 2.7.

We are now ready to prove the two estimates of Eq. (1) in the definition of scaling for the family \( \text{scl}^{p,q} \). Let us fix \( \alpha > \beta > 0 \) and \( \lambda > 1 \) such that \( \alpha > \lambda^2 \cdot \beta \). As \( \text{scl}_{\alpha}^{p,q} \) is decreasing with \( \alpha \), it holds:

\[
\text{scl}_{\alpha}^{p,q}(\epsilon) \leq \text{scl}_{\lambda^2 \cdot \beta}^{p,q}(\epsilon).
\]

On the other hand, by Fact 2.7 we have:

\[
\text{scl}_{\lambda^2 \cdot \beta}^{p,q}(\epsilon) \leq \text{scl}_{\lambda \cdot \beta}^{p,q}(\epsilon^\lambda) \leq \left( \text{scl}_{\beta}^{p,q}(\epsilon^\lambda) \right)^\lambda \quad \text{and} \quad \text{scl}_{\lambda^2 \cdot \beta}^{p,q}(\epsilon) \leq \left( \text{scl}_{\beta}^{p,q}(\epsilon) \right)^{\lambda^2}.
\]

The above Eqs. (2.4) and (2.5) imply:

\[
\frac{\text{scl}_{\alpha}^{p,q}(\epsilon)}{\text{scl}_{\beta}^{p,q}(\epsilon^\lambda)} \leq \left( \frac{\text{scl}_{\beta}^{p,q}(\epsilon)}{\text{scl}_{\beta}^{p,q}(\epsilon^\lambda)} \right)^{\lambda^{-1}} \quad \text{and} \quad \frac{\text{scl}_{\alpha}^{p,q}(\epsilon)}{\text{scl}_{\beta}^{p,q}(\epsilon)^\lambda} \leq \left( \frac{\text{scl}_{\beta}^{p,q}(\epsilon)}{\text{scl}_{\beta}^{p,q}(\epsilon)^\lambda} \right)^{\lambda \cdot (\lambda^{-1})},
\]

which both converge to 0 as \( \epsilon \) goes to 0, providing the desired result.

2.2 Box scales

As introduced in Definition 1.5, lower and upper box scales of a metric space \((X, d)\) are defined by:

\[
\text{scl}_B X = \sup \left\{ \alpha > 0 : \mathcal{N}_\epsilon(X) \cdot \text{scl}_\alpha(\epsilon) \xrightarrow{\epsilon \to 0} +\infty \right\}
\]

and

\[
\overline{\text{scl}}_B X = \inf \left\{ \alpha > 0 : \mathcal{N}_\epsilon(X) \cdot \text{scl}_\alpha(\epsilon) \xrightarrow{\epsilon \to 0} 0 \right\},
\]

where the covering number \( \mathcal{N}_\epsilon(X) \) is the minimal cardinality of a covering of \( X \) by balls with radius \( \epsilon > 0 \).

In general, the upper and lower box scales must not coincide. We will give such examples for the order in Example 4.4. Now we list a few properties of box scales that are well known in the specific case of dimension.

Fact 2.8. Let \((X, d)\) be a metric space. The following properties hold true:

1. if \( \text{scl}_B(X) < +\infty \), then \((X, d)\) is totally bounded,

2. for every subset \( E \subset X \) it holds \( \text{scl}_B E \leq \text{scl}_B X \) and \( \overline{\text{scl}}_B E \leq \overline{\text{scl}}_B X \),

3. for every subset \( E \subset X \) it holds \( \text{scl}_B E = \text{scl}_B \text{cl}(E) \) and \( \overline{\text{scl}}_B E = \overline{\text{scl}}_B \text{cl}(E) \).

Observe that 1. and 2. are direct consequences of the definitions.

To see 3. it is enough to observe that \( \mathcal{N}_\epsilon(E) \leq \mathcal{N}_\epsilon(\text{cl}(E)) \leq \mathcal{N}_{\epsilon/2}(E) \) for every \( \epsilon > 0 \).

As for box dimensions, box scales can also be defined by replacing the covering number by packing number:
Definition 2.9 (Packing number). For $\epsilon > 0$ let $\tilde{N}_\epsilon(X)$ be the packing number of the metric space $(X,d)$. It is the maximum cardinality of an $\epsilon$-separated set of points in $X$ for the distance $d$:

$$\tilde{N}_\epsilon(X) = \sup\{N \geq 0 : \exists x_1, \ldots x_N \in X, d(x_i, x_j) \geq \epsilon \text{ for every } 1 \leq i < j \leq N \}.$$ 

A well known comparison between packing and covering numbers is the following:

Lemma 2.10. Let $(X, d)$ be a metric space. For every $\epsilon > 0$, it holds:

$$\tilde{N}_{2\epsilon}(X) \leq N_\epsilon(X) \leq \tilde{N}_\epsilon(X).$$

In virtue of the basic properties of scalings, the covering number can be replaced by the packing number in the definitions of box scales:

Lemma 2.11. Let $(X, d)$ be a metric space and $\text{scl}$ a scaling, then box scales of $X$ can be written as:

$$\text{scl}_B X = \sup \left\{ \alpha > 0 : \tilde{N}_\epsilon(X) \cdot \text{scl}_\alpha(\epsilon) \rightarrow +\infty \right\}$$

and

$$\overline{\text{scl}}_B X = \inf \left\{ \alpha > 0 : \tilde{N}_\epsilon(X) \cdot \text{scl}_\alpha(\epsilon) \rightarrow 0 \right\}.$$

The proof is provided by direct application of Lemmas 2.2 and 2.10.

Remark 2.12. Another property for the scaling $\text{scl}^{p,q}$ from Proposition 2.4, with $p, q \geq 1$, is that the upper and lower box scales for a metric space $(X, d)$ can be written as:

$$\text{scl}_B^{p,q}(X) = \liminf_{\epsilon \to 0} \frac{\log(\tilde{N}_\epsilon(X))}{\log(\epsilon^{-1})}$$

and

$$\overline{\text{scl}}_B^{p,q}(X) = \limsup_{\epsilon \to 0} \frac{\log(\tilde{N}_\epsilon(X))}{\log(\epsilon^{-1})}.$$

In particular, for dimension and order:

$$\overline{\dim}_B(X) = \limsup_{\epsilon \to 0} \frac{\log(\tilde{N}_\epsilon(X))}{-\log \epsilon}, \quad \overline{\dim}_B(X) = \liminf_{\epsilon \to 0} \frac{\log(\tilde{N}_\epsilon(X))}{-\log \epsilon},$$

and

$$\overline{\text{ord}}_B(X) = \liminf_{\epsilon \to 0} \frac{\log \log(\tilde{N}_\epsilon(X))}{-\log \epsilon}, \quad \overline{\text{ord}}_B(X) = \limsup_{\epsilon \to 0} \frac{\log \log(\tilde{N}_\epsilon(X))}{-\log \epsilon}.$$ 

The above equalities coincide with the most usual definitions of box dimensions and orders.

2.3 Hausdorff scales

The definition of Hausdorff scales, generalizing Hausdorff dimension, is introduced here using the definition of Hausdorff outer measure as given for instance by Tricot in [Tri82]. We still consider a metric space $(X, d)$. Given a non-decreasing function $\phi \in C(\mathbb{R}^+_+, \mathbb{R}^+_+)$, such that $\phi(\epsilon) \to 0$ when $\epsilon \to 0$, we recall:

$$\mathcal{H}_\phi^\epsilon(X) := \inf_{J \text{ countable set}} \left\{ \sum_{j \in J} \phi(|B_j|) : X = \bigcup_{j \in J} B_j, \forall j \in J : |B_j| \leq \epsilon \right\},$$
where $|B|$ is the radius of a ball $B \subset X$. A countable family $(B_j)_{j \in J}$ of balls with radius at most $\epsilon > 0$ such that $X = \bigcup_{j \in J} B_j$ will be called an $\epsilon$-cover of $X$. Since the set of $\epsilon$-cover is non-decreasing for inclusion as $\epsilon$ decreases to 0, the following limit does exist:

$$\mathcal{H}^{\phi}(X) := \lim_{\epsilon \to 0} \mathcal{H}^{\phi}_\epsilon(X).$$

Now replacing $(X, d)$ in the previous definitions by any subset $E$ of $X$ endowed with the same metric $d$, we observe that $\mathcal{H}^{\phi}$ defines an outer-measure on $X$. It is usually called the $\phi$-Hausdorff measure on $X$. We now introduce the following:

**Definition 2.13 (Hausdorff scale).** The Hausdorff scale of a metric space $(X, d)$ is defined by:

$$\text{scl}_H X = \sup \{ \alpha > 0 : \mathcal{H}^{\text{scl}_\alpha}(X) = +\infty \} = \inf \{ \alpha > 0 : \mathcal{H}^{\text{scl}_\alpha}(X) = 0 \}.$$

Note that the above definition gives us two quantities that are a priori not equal. However, the mild assumptions in the definition of scaling allow us to verify that they indeed coincide and allow us to use the machinery of Hausdorff outer measure to define metric invariants generalizing Hausdorff dimension. Thus scalings allow us to have some consistent extension of the definition of Hausdorff dimension.

**Proof of the equality in Definition 2.13.** It is clear from its definition that $\alpha \mapsto \mathcal{H}^{\text{scl}_\alpha}(X)$ is non-increasing. It is then enough to check that if there exists $\alpha > 0$ such that $0 < \mathcal{H}^{\text{scl}_\alpha}(X) < +\infty$ then, for any positive $\delta < \alpha,$ it holds:

$$\mathcal{H}^{\text{scl}_{\alpha-\delta}}(X) = +\infty.$$

Let us fix $\eta > 0,$ by Definition 1.2, for $\epsilon > 0$ small it holds:

$$\text{scl}_\alpha \left( \epsilon \right) \leq \eta \cdot \text{scl}_\alpha \left( \epsilon \right) \quad \text{and} \quad \text{scl}_\alpha \left( \epsilon \right) \leq \eta \cdot \text{scl}_{\alpha-\delta} \left( \epsilon \right).$$

Since $\epsilon$ is small, it holds:

$$0 < \frac{1}{2} \mathcal{H}^{\text{scl}_\alpha}(X) \leq \mathcal{H}^{\text{scl}_{\alpha-\delta}}(X) < +\infty.$$

Given $(B_j)_{j \in J}$ an $\epsilon$-cover of $X$, the following holds:

$$\frac{1}{2} \mathcal{H}^{\text{scl}_\alpha}(X) \leq \mathcal{H}^{\text{scl}_{\alpha-\delta}}(X) \leq \sum_{j \in J} \text{scl}_\alpha (|B_j|),$$

and then:

$$\frac{1}{2\eta} \mathcal{H}^{\text{scl}_\alpha}(X) \leq \frac{1}{\eta} \sum_{j \in J} \text{scl}_\alpha (|B_j|) \leq \sum_{j \in J} \text{scl}_{\alpha-\delta} (|B_j|).$$

Since this holds for every $\epsilon$-cover, the latter inequality leads to:

$$\frac{1}{2\eta} \mathcal{H}^{\text{scl}_\alpha}(X) \leq \mathcal{H}^{\text{scl}_{\alpha-\delta}}(X),$$

and so:

$$\frac{1}{2\eta} \mathcal{H}^{\text{scl}_\alpha}(X) \leq \mathcal{H}^{\text{scl}_{\alpha-\delta}}(X). \quad (2.6)$$

\[\text{Note that the historical construction of the Hausdorff measures uses subsets of } X \text{ with diameter at most } \epsilon \text{ instead of the balls with radius at most } \epsilon. \text{ However, both these constructions lead to the same definitions of Hausdorff scales.}\]
On the other side, there exists an \( \epsilon \)-cover \( (B_j)_{j \in J} \) of \( E \) such that:
\[
\sum_{j \in J} \text{scl}_\alpha(|B_j|) \leq 2 \mathcal{H}^{\text{scl}}_{\epsilon}(X) .
\]
Now since \( \mathcal{H}^{\text{scl}}_{\epsilon}(X) \leq \mathcal{H}^{\text{scl}}(X) \), this leads to:
\[
\sum_{j \in J} \text{scl}_{\alpha+\delta}(|B_j|) \leq \eta \cdot \sum_{j \in J} \text{scl}_\alpha(|B_j|) \leq 2 \eta \cdot \mathcal{H}^{\text{scl}}(X) .
\]
From there:
\[
\mathcal{H}^{\text{scl}}_{\epsilon+\delta}(X) \leq 2 \eta \cdot \mathcal{H}^{\text{scl}}(X) , \tag{2.7}
\]
and this holds for every small \( \epsilon \). Taking the limit as \( \epsilon \to 0 \) in Eqs. (2.6) and (2.7) gives:
\[
\frac{1}{2\eta} \mathcal{H}^{\text{scl}}(X) \leq \mathcal{H}^{\text{scl}}_{\epsilon-\delta}(X) \quad \text{and} \quad \mathcal{H}^{\text{scl}}_{\epsilon+\delta}(X) \leq 2 \eta \cdot \mathcal{H}^{\text{scl}}(X) .
\]
To conclude, note that as the latter holds for \( \eta \) arbitrarily small, it follows that \( \mathcal{H}^{\text{scl}}_{\epsilon-\delta}(X) = +\infty \) and \( \mathcal{H}^{\text{scl}}_{\epsilon+\delta}(X) = 0 \).

As box scales, Hausdorff scales are non-decreasing for inclusion. We will see a stronger property of Hausdorff scales in Lemma 2.21.

### 2.4 Packing scales

#### 2.4.1 Packing scales through modified box scales

The original construction of packing dimension relies on the packing measure introduced by Tricot in [Tri82]. We first define packing scales by modifying upper box scales and we show later how they are related to packing measures.

**Definition 2.14 (Packing scale).** Let \((X, d)\) be a metric space and \(\text{scl}\) a scaling. The packing scale of \(X\) is defined by:
\[
\text{scl}_P X = \inf \left\{ \sup_{n \geq 1} \text{cl}_B E_n : (E_n)_{n \geq 1} \in X^{\mathbb{N}^*} \text{ s.t. } \bigcup_{n \geq 1} E_n = X \right\} .
\]

The following comes directly from the definition of packing scale:

**Proposition 2.15.** Let \((X, d)\) be a metric space and let \(\text{scl}\) be a scaling. It holds:
\[
\text{scl}_P X \leq \text{cl}_B X .
\]

#### 2.4.2 Packing measures

In this paragraph we show the relationship between packing measures and packing scales. Let us first recall a few definitions.

Given \(\epsilon > 0\), an \(\epsilon\)-pack of a metric space \((X, d)\) is a countable collection of disjoint balls of \(X\) with radii at most \(\epsilon\). As for Hausdorff outer measure, consider \(\phi : \mathbb{R}^*_+ \to \mathbb{R}^*_+\) an non-decreasing function such that \(\phi(\epsilon) \to 0\) as \(\epsilon \to 0\). For \(\epsilon > 0\), put:
\[
\mathcal{P}^{\phi}_\epsilon(X) := \sup \left\{ \sum_{i \in I} \phi(|B_i|) : (B_i)_{i \in I} \text{ is an } \epsilon\text{-pack of } X \right\} .
\]
Since $\mathcal{P}^\phi_\epsilon(X)$ is non-increasing when $\epsilon$ decreases to 0, the following quantity is well defined:

$$\mathcal{P}^\phi_0(X) := \lim_{\epsilon \to 0} \mathcal{P}^\phi_\epsilon(X).$$

The idea of Tricot is to build an outer measure from this quantity:

**Definition 2.16 (Packing measure).** For every subset $E$ of $X$ endowed with the same metric $d$, the packing $\phi$-measure of $E$ is defined by:

$$\mathcal{P}^\phi(E) = \inf \left\{ \sum_{n \geq 1} \mathcal{P}^\phi_0(E_n) : E = \bigcup_{n \geq 1} E_n \right\}.$$  

Note that $\mathcal{P}^\phi$ is an outer-measure on $X$ and can eventually be infinite or null. The following shows the equivalence of Tricot’s counterpart definition of the packing scale; this will be useful to show the equality between the upper local scale and the packing scale of a measure given by Theorem C, eq. (c&g).

**Proposition 2.17.** The packing scale of a metric space $(X, d)$ verifies:

$$\sup \left\{ \alpha > 0 : \mathcal{P}^{sc\lambda}_\alpha(X) = +\infty \right\} = \text{scl}_B X = \inf \left\{ \alpha > 0 : \mathcal{P}^{sc\lambda}_0(X) = 0 \right\}.$$  

**Proof.** Let $(E_n)_{n \geq 1}$ be a family of subsets of $X$. Since each map $\alpha \mapsto \mathcal{P}^{sc\lambda}_0 (E_n)$ is non-increasing and non negative, we have:

$$\inf \left\{ \alpha > 0 : \sum_{n \geq 1} \mathcal{P}^{sc\lambda}_0 (E_n) = 0 \right\} = \inf \left\{ \alpha > 0 : \mathcal{P}^{sc\lambda}_0 (E_n) = 0 \right\}.$$  

We decompose the proof into two intermediary steps given by the following lemmas:

**Lemma 2.18.** Given $\alpha > 0$, if $\mathcal{P}^{sc\lambda}_0 (E)$ is finite, then for every $\delta \in (0, \alpha)$, it holds:

$$\mathcal{P}^{sc\lambda}_{\alpha + \delta}(E) = 0 \quad \text{and} \quad \mathcal{P}^{sc\lambda}_{\alpha - \delta}(E) = +\infty.$$

**Lemma 2.19.** For every $E \subset X$, it holds:

$$\sup \left\{ \alpha > 0 : \mathcal{P}^{sc\lambda}_0 (E) = +\infty \right\} = \text{scl}_B E = \inf \left\{ \alpha > 0 : \mathcal{P}^{sc\lambda}_0 (E) = 0 \right\}. $$

**Lemma 2.18** will allow proving Lemma 2.19. Before proving these lemmas let us see how they allow us to conclude the proof of Proposition 2.17 First note that the second equality of Lemma 2.18 implies:

$$\sup \left\{ \alpha > 0 : \sum_{n \geq 1} \mathcal{P}^{sc\lambda}_0 (E_n) = +\infty \right\} = \sup_{n \geq 1} \sup \left\{ \alpha > 0 : \mathcal{P}^{sc\lambda}_0 (E_n) = +\infty \right\}. $$

Consequently by Lemma 2.19 it holds:

$$\sup \left\{ \alpha > 0 : \sum_{n \geq 1} \mathcal{P}^{sc\lambda}_0 (E_n) = +\infty \right\} = \sup \text{scl}_B E_n = \inf \left\{ \alpha > 0 : \sum_{n \geq 1} \mathcal{P}^{sc\lambda}_0 (E_n) = 0 \right\}. $$

Taking the infimum over families $(E_n)_{n \geq 1}$ that cover $X$, we obtain the desired result. 

It remains to show Lemmas 2.18 and 2.19.
Given $\eta > 0$, by Definition 1.2 of scaling, for $\epsilon > 0$ small enough, it holds:

\[ \text{scl}_{\alpha + \delta}(\epsilon) \leq \eta \cdot \text{scl}_{\alpha}(\epsilon) \quad \text{and} \quad \text{scl}_{\alpha}(\epsilon) \leq \eta \cdot \text{scl}_{\alpha - \delta}(\epsilon). \] (2.9)

Let $(B_j)_{j\geq1}$ be an $\epsilon$-pack of $E$. Then, by the above Eq. (2.9) it holds:

\[ \eta^{-1} \cdot \sum_{j\geq1} \text{scl}_{\alpha + \delta}(|B_j|) \cdot |B_j| \leq \sum_{j\geq1} \text{scl}_{\alpha}(|B_j|) \leq \eta \cdot \sum_{j\geq1} \text{scl}_{\alpha - \delta}(|B_j|). \]

As this holds for every $\delta$-pack of $E$, it follows:

\[ \eta^{-1} \cdot P_{\epsilon}^{\text{scl}_{\alpha + \delta}}(E) \leq P_{\epsilon}^{\text{scl}_{\alpha}}(E) \leq \eta \cdot P_{\epsilon}^{\text{scl}_{\alpha - \delta}}(E). \] (2.10)

Taking the limit as $\epsilon$ goes to 0 and $\eta$ arbitrary small allows us to conclude.

Proof of Lemma 2.19

By Lemma 2.18, it suffices to show that:

\[ \sup \left\{ \alpha > 0 : P_{\epsilon}^{\text{scl}_{\alpha}}(E) = +\infty \right\} \leq \text{scle}_{B} E \leq \inf \left\{ \alpha > 0 : P_{\epsilon}^{\text{scl}_{\alpha}}(E) = 0 \right\}. \] (2.11)

We start with the second inequality. If $P_{0}^{\text{scl}_{\alpha}}(E) > 0$ for every $\alpha > 0$, there is nothing to prove. Thus consider $\alpha > 0$ such that $P_{0}^{\text{scl}_{\alpha}}(E) = 0$. Then for every $\epsilon > 0$ sufficiently small it holds $P_{\epsilon}^{\text{scl}_{\alpha}}(E) < 1$. In particular, the packing number (see def. 2.9) satisfies $\tilde{N}_{\epsilon}(E) \cdot \text{scl}_{\alpha}(\epsilon) < 1$. By Lemma 2.11 we obtain $\text{scle}_{B} E \leq \alpha$ which gives the second inequality by taking the infima of such $\alpha > 0$.

To prove the first inequality, assume that there exists $\alpha > 0$ such that $P_{0}^{\text{scl}_{\alpha}}(E) = +\infty$, otherwise there is nothing to prove. For such an $\alpha$ and for every $\epsilon > 0$ there exists an $\epsilon$-pack $(B_j)_{j\geq1}$ such that:

\[ \sum_{j\geq1} \text{scl}_{\alpha}(B_j) > 1. \] (2.12)

For an integer $k \geq 1$, denote:

\[ n_k := \text{Card} \left\{ j \geq 1 : 2^{-(k+1)} \leq \text{scl}_{\alpha}(|B_j|) < 2^{-k} \right\}. \] (2.13)

Since $\text{scl}_{\alpha}$ is non-decreasing, it holds:

\[ \sum_{k \geq 1} n_k \cdot 2^{-k} \geq \sum_{j \geq 1} \text{scl}_{\alpha}(|B_j|). \]

Thus by Eq. (2.11), it holds:

\[ \sum_{k \geq 1} n_k \cdot 2^{-k} > 1. \] (2.14)

Note that, as $|B_j| \leq \delta$ for every $j \geq 1$, it holds $n_k = 0$ for every $k < -\log_2 \text{scl}_{\alpha}(\delta)$. Then for $\delta$ small it holds:

Fact 2.20. There exists an integer $j \geq 2$ such that:

\[ n_j > j^{-2} \cdot 2^j. \]

Proof. Otherwise we would have:

\[ \sum_{k \geq 1} n_k \cdot 2^{-k} \leq \sum_{k \geq 2} \frac{1}{k^2} < 1, \]

as $n_0 = n_1 = 0$ for small $\delta$, and this contradicts Eq. (2.14).
This latter fact translates as: there exist at least \( n_j \) disjoint balls with radii at least \( \text{scl}_n^{-1}(2^{-(j+1)}) \). This implies:

\[
\tilde{N}_{\text{scl}_n^{-1}(2^{-(j+1)})}(E) \geq n_j > j^{-2}2^j,
\]

and moreover:

\[
j \geq -\log_2 \text{scl}_n(\delta) .
\]

Since these inequalities hold true for \( \delta \) arbitrarily small, there exists an increasing sequence of integers \( (j_n)_{n \geq 1} \) such that:

\[
\tilde{N}_{\epsilon_n}(E) > j_n^{-2}2^{j_n} \quad \text{where} \quad \epsilon_n := \text{scl}_n^{-1}(2^{-(j_n+1)}) .
\]  

(2.14)

Given a positive \( \beta < \alpha \), by Definition 1.2 of scaling, for \( \lambda > 1 \) close to 1, it holds:

\[
s_{\beta}(\epsilon) \cdot (\text{scl}_\alpha(\epsilon))^{-\lambda^{-1}} \xrightarrow[\epsilon \to 0]{} +\infty .
\]  

(2.15)

On the other hand, given such a \( \lambda > 1 \), for \( n \) large enough, it holds:

\[
j_n^{-2}2^{j_n} \geq 2^{\lambda^{-1}(j_n+1)} .
\]

It follows by Eq. (2.14):

\[
\tilde{N}_{\epsilon_n}(E) \geq \left(2^{-(j_n+1)}\right)^{-\lambda^{-1}} = (\text{scl}_\alpha(\epsilon_n))^{-\lambda^{-1}} .
\]

This together with Eq. (2.15) gives:

\[
s_{\beta}(\epsilon_n) \cdot \tilde{N}_{\epsilon_n}(E) > s_{\beta}(\epsilon) \cdot (\text{scl}_\alpha(\epsilon))^{-\lambda^{-1}} \xrightarrow[\epsilon \to 0]{} +\infty .
\]

By Lemma 2.11 we deduce \( \overline{\text{scl}}_B E \geq \beta \). Taking \( \beta \) close to \( \alpha \) ends the proof.

\[\square\]

### 2.5 Properties and comparison of scales of metric spaces

We first give a few basic properties of scales that would allow us to compare them. Since both packing and Hausdorff scales are defined via measures, they both are \( \sigma \)-stable as shown in the following:

**Lemma 2.21.** Let \((X, d)\) be a metric space. Let \( I \) be a countable set and \((E_i)_{i \in I}\) a covering of \(X\), then for any scaling \( \text{scl} \):

\[
\text{scl}_H X = \sup_{i \in I} \text{scl}_H E_i \quad \text{and} \quad \text{scl}_P X = \sup_{i \in I} \text{scl}_P E_i .
\]

**Proof.** The equality on packing scales is clear by definition. Let us prove the equality on Hausdorff scales. By monotonicity of the Hausdorff measure, it holds \( \text{scl}_H X \geq \sup_{i \in I} \text{scl}_H E_i \). For the reverse inequality, consider \( \alpha > \sup_{i \in I} \text{scl}_H E_i \), then for any \( i \in I \) it holds \( \mathcal{H}^{\text{scl}_\alpha}(E_i) = 0 \). Thus, it holds:

\[
\mathcal{H}^{\text{scl}_\alpha}(X) \leq \sum_{i \in I} \mathcal{H}^{\text{scl}_\alpha}(E_i) = 0 ,
\]

and then \( \text{scl}_H X \leq \alpha \). Since this is true for any \( \alpha > \sup_{i \in I} \text{scl}_H E_i \), the desired result comes.

\[\square\]

Note that \( \sigma \)-stability is not a property of box scales. To see that, it suffices to consider a countable dense subset of a metric space \((X, d)\) with positive box scales. This is actually a basic known fact for the specific case of dimension that naturally still holds there.

The following lemma shows in particular that the above scales are bi-Lipschitz invariants.
**Lemma 2.22.** Let $(X, d)$ and $(Y, d)$ be two metric spaces such that there exists a Lipschitz map $f : (X, d) \to (Y, d)$. Then for any scaling $\text{scl}$, the scales of $f(X)$ are at most the ones of $X$:

$$\text{scl}_H f(X) \leq \text{scl}_H X; \quad \text{scl}_P f(X) \leq \text{scl}_P X; \quad \text{scl}_H f(X) \leq \text{scl}_H X; \quad \overline{\text{scl}}_B f(X) \leq \overline{\text{scl}}_B X.$$

**Proof.** Let us fix $\epsilon > 0$. Let $K > 0$ be a Lipschitz constant for $f$. We first show the inequalities on box and packing scales.

Consider a minimal covering $(B(x_j, \epsilon))_{1 \leq j \leq N}$ of $\epsilon$-balls centered in $X$, it holds:

$$f(X) = \bigcup_{j=1}^{N} B(x_j, \epsilon_j) \subset \bigcup_{j=1}^{N} B(f(x_j), K \cdot \epsilon_j).$$

Then $(B(f(x_j), K \cdot \epsilon))_{1 \leq j \leq N}$ is a covering by $K \cdot \epsilon$-balls of $f(X)$. Then $N_{K \cdot \epsilon}(f(X)) \leq N_\epsilon(X)$ and all the inequalities on the box and packing scales are immediately deduced from Eq. (2.2). Now for Hausdorff scales, consider a countable set $J$ and $(B(x_j, \epsilon_j))_{j \in J}$ an $\epsilon$-cover of $X$. Then it holds:

$$f(X) \subset \bigcup_{j \in J} B(f(x_j), K \cdot \epsilon_j).$$

For any $\alpha > \beta > 0$ and $\delta > 0$ small enough, by **Fact 2.1**, it holds:

$$\text{scl}_\alpha(\delta) \leq \text{scl}_\beta(K^{-1} \cdot \delta).$$

Hence for $\epsilon$ small, it holds:

$$\mathcal{H}_{K \cdot \epsilon}^{\text{scl}}(f(X)) \leq \sum_{j \in J} \text{scl}_\alpha(K \cdot \epsilon_j) \leq \sum_{j \in J} \text{scl}_\beta(\epsilon_j).$$

As $\beta > \text{scl}_H X$, the $\epsilon$-cover $(B(x_j, \epsilon_j))_{j \in J}$ can be chosen such that $\sum_{j \in J} \text{scl}_\beta(\epsilon_j)$ is arbitrarily small. Consequently, it holds $\mathcal{H}_{K \cdot \epsilon}^{\text{scl}}(f(X)) = 0$, and so $\text{scl}_H f(X) \leq \alpha$. As $\alpha$ is arbitrarily close to $\text{scl}_H X$, it holds:

$$\text{scl}_H f(X) \leq \text{scl}_H X.$$

\[\square\]

As a direct application, we obtain the following:

**Corollary 2.23.** Let $(X, d)$ and $(Y, d)$ be two metric spaces. Assume that there exists an embedding $g : (Y, \delta) \to (X, d)$ such that $g^{-1}$ is Lipschitz on $g(X)$. Then for every scaling $\text{scl}$, the scales of $Y$ are at most the ones of $X$:

$$\text{scl}_H Y \leq \text{scl}_H X; \quad \text{scl}_P Y \leq \text{scl}_P X; \quad \text{scl}_H Y \leq \text{scl}_H X; \quad \overline{\text{scl}}_B Y \leq \overline{\text{scl}}_B X.$$

**Proof.** By Lemma 2.22 we have $\text{scl}_\bullet Y \leq \text{scl}_\bullet g(Y)$ for any $\text{scl}_\bullet \in \{\text{scl}_H, \text{scl}_P, \text{scl}_B, \overline{\text{scl}}_B\}$. As $g(Y) \subset X$, we have also $\text{scl}_\bullet g(Y) \leq \text{scl}_\bullet X$. \[\square\]

Remark that Lemma 2.22 and Corollary 2.23 hold even for scalings that have sub-polynomial behaviors.

The end of this section consists of comparing the different scales introduced and proving **Theorem A**. We start by comparing the Hausdorff with lower box scales. The following proposition generalizes well known facts on dimension. See e.g. [Fal04] [(3.17)].
Proposition 2.24. Let \((X, d)\) be a metric space and \(\text{scl}\) a scaling, its Hausdorff scale is at most its lower box scale:

\[
\text{scl}_H X \leq \text{scl}_B X .
\]

Proof. We can assume without any loss that \((X, d)\) is totally bounded. If \(\text{scl}_H X = 0\) the inequality obviously holds, thus consider a positive number \(\alpha < \text{scl}_H X\). For \(\delta > 0\) small enough, \(\mathcal{H}_\delta^{\text{scl}_\alpha}(X) > 1\). Also there exists a \(\delta\)-cover \((B_j)_{1 \leq j \leq N_\delta(X)}\). It verifies:

\[
1 < \sum_{1 \leq j \leq N_\delta(F)} \text{scl}_\alpha(|B_j|) = N_\delta(X) \cdot \text{scl}_\alpha(\delta) .
\]

From there, it holds \(\text{scl}_B X \geq \alpha\). We conclude by taking \(\alpha\) arbitrarily close to \(\text{scl}_H X\). \(\square\)

We have compared Hausdorff and packing scales with their corresponding box scales. It remains to compare each other with the following:

Proposition 2.25. Let \((X, d)\) be a metric space and \(\text{scl}\) a scaling. It holds:

\[
\text{scl}_H X \leq \text{scl}_P X .
\]

Proof. By Lemma 2.21 Hausdorff scale is \(\sigma\)-stable:

\[
\text{scl}_H X = \inf_{\bigcup_{n \geq 1} E_n = X} \sup_{n \geq 1} \text{scl}_H E_n ,
\]

where the infimum is taken over countable coverings of \(X\). Moreover, by Proposition 2.24 we have:

\[
\text{scl}_H E \leq \text{scl}_B E \leq \overline{\text{scl}}_B E ,
\]

for any subset \(E\) of \(X\). It follows then:

\[
\text{scl}_H X \leq \inf_{\bigcup_{n \geq 1} E_n = X} \sup_{n \geq 1} \text{scl}_B E_n = \text{scl}_P X .
\]

\(\square\)

For the sake of completeness we will resume:

Proof of Theorem A Let \((X, d)\) be a metric space and \(\text{scl}\) a scaling. By Proposition 2.24 Proposition 2.25 and Proposition 2.15 it holds respectively:

\[
\text{scl}_H X \leq \text{scl}_B X , \quad \text{scl}_H X \leq \text{scl}_P X \quad \text{and} \quad \text{scl}_H X \leq \overline{\text{scl}}_B X .
\]

Now since \(\text{scl}_B X \leq \overline{\text{scl}}_B X\) obviously holds, we deduce the desired result:

\[
\text{scl}_H X \leq \text{scl}_P X \leq \overline{\text{scl}}_B X \quad \text{and} \quad \text{scl}_H X \leq \text{scl}_B X \leq \overline{\text{scl}}_B X .
\]

\(\square\)
3 Scales of measures

In this section we recall the different versions of scales of measures we introduced and show the inequalities and equalities comparing them. In particular, we provide proofs of Theorem B and Theorem C. They generalize known results from dimension theory to any scaling and moreover bring new comparisons (see Theorem 3.10) between quantization and box scales that were not shown yet for even the case of dimension.

3.1 Hausdorff, packing and local scales of measures

Let us recall the definition of local scales. Let \( \mu \) be a Borel measure on a metric space \((X, d)\) and \( \text{scl} \) a scaling. The lower and upper scales of \( \mu \) are the functions that map a point \( x \in X \) to:

\[
\text{scl}_{\text{loc}}\mu(x) = \sup \left\{ \alpha > 0 : \frac{\mu(B(x, \epsilon))}{\text{scl}_\alpha(\epsilon)} \to 0 \right\}
\]

and

\[
\overline{\text{scl}}_{\text{loc}}\mu(x) = \inf \left\{ \alpha > 0 : \frac{\mu(B(x, \epsilon))}{\text{scl}_\alpha(\epsilon)} \to +\infty \right\}.
\]

We shall compare local scales with the followings:

**Definition 3.1 (Hausdorff scales of a measure).** Let \( \text{scl} \) be a scaling and \( \mu \) a non-null Borel measure on a metric space \((X, d)\). We define the Hausdorff and \(+\)-Hausdorff scales of the measure \( \mu \) by:

\[
\text{scl}_H \mu = \inf_{E \in \mathcal{B}} \{ \text{scl}_H E : \mu(E) > 0 \} \quad \text{and} \quad \text{scl}_+ \mu = \inf_{E \in \mathcal{B}} \{ \text{scl}_H E : \mu(X \setminus E) = 0 \},
\]

where \( \mathcal{B} \) is the set of Borel subsets of \( X \).

**Definition 3.2 (Packing scales of a measure).** Let \( \text{scl} \) be a scaling and \( \mu \) a non-null Borel measure on a metric space \((X, d)\). We define the packing and \(+\)-packing scales of \( \mu \) by:

\[
\text{scl}_P \mu = \inf_{E \in \mathcal{B}} \{ \text{scl}_P E : \mu(E) > 0 \} \quad \text{and} \quad \text{scl}_+ P \mu = \inf_{E \in \mathcal{B}} \{ \text{scl}_P E : \mu(X \setminus E) = 0 \}.
\]

**Remark 3.3.** In order to avoid excluding the null measure 0, we set as a convention:

\[
\text{scl}_H 0 = \text{scl}_+ H 0 = \text{scl}_P 0 = \text{scl}_+ P 0 = 0.
\]

The lemma below will allow us to compare local scales with the other scales of measures.

**Lemma 3.4.** Let \( \mu \) be a Borel measure on \( X \). Then for any Borel subset \( F \) of \( X \) such that \( \mu(F) > 0 \), the restriction \( \sigma \) of \( \mu \) to \( F \) verifies:

\[
\text{ess inf} \text{scl}_{\text{loc}}\mu \leq \text{ess inf} \text{scl}_{\text{loc}}\sigma \quad \text{and} \quad \text{ess inf} \text{scl}_{\text{loc}}\mu \leq \text{ess inf} \text{scl}_{\text{loc}}\sigma.
\]

Moreover, if there exists \( \alpha > 0 \) such that \( F \subset \{ x \in X : \text{cl}_{\text{loc}}\mu(x) > \alpha \} \), it holds then:

\[
\text{ess inf} \text{scl}_{\text{loc}}\sigma \geq \alpha,
\]

and similarly if \( F \subset \{ x \in X : \text{cl}_{\text{loc}}\mu(x) > \alpha \} \), it holds:

\[
\text{ess inf} \text{scl}_{\text{loc}}\sigma \geq \alpha.
\]
Proof. Consider a point \( x \in X \), then for any \( \epsilon > 0 \), one has \( \sigma(B(x, \epsilon)) \leq \mu(B(x, \epsilon)) \), thus by definition of local scales:

\[
\overline{\text{scl}}_{\text{loc}} \mu \leq \overline{\text{scl}}_{\text{loc}} \sigma \quad \text{and} \quad \underline{\text{scl}}_{\text{loc}} \mu \leq \underline{\text{scl}}_{\text{loc}} \sigma.
\]

Now if there exists \( \alpha > 0 \) such that \( F \subset \{ x \in X : \overline{\text{scl}}_{\text{loc}} \mu(x) > \alpha \} \), as \( \overline{\text{scl}}_{\text{loc}} \mu(x) \geq \alpha \) for \( \mu \)-almost every \( x \) in \( F \), it follows from the above inequality that \( \overline{\text{scl}}_{\text{loc}} \sigma(x) \geq \alpha \) for \( \sigma \)-almost every \( x \) in \( F \), and thus for \( \sigma \)-almost every \( x \in X \). It follows \( \text{ess inf} \overline{\text{scl}}_{\text{loc}} \sigma \geq \alpha \). And the same holds for lower local scales. \( \square \)

The following lemma corresponds to part of the results of Theorem B. We will prove this lemma later in Section 3.4. First, we will use it to prove Theorem C in Section 3.3. This lemma states that the lower and upper local scales of a measure are, respectively, not greater than the Hausdorff and packing scales of the underlying space:

**Lemma 3.5.** Let \((X, d)\) be a metric space and \(\mu\) a Borel measure on \(X\). Let \(\text{scl}\) be a scaling. Then it holds:

\[
\text{ess sup} \overline{\text{scl}}_{\text{loc}} \mu \leq \text{scl}_{H} X \quad \text{and} \quad \text{ess sup} \underline{\text{scl}}_{\text{loc}} \mu \leq \text{scl}_{P} X.
\]

Note that in the above we can replace \(X\) by any of its subsets with total mass. This observation directly implies:

**Corollary 3.6.** Let \((X, d)\) be a metric space and \(\mu\) a Borel measure on \(X\). Let \(\text{scl}\) be a scaling. It holds:

\[
\text{ess sup} \overline{\text{scl}}_{\text{loc}} \mu \leq \text{scl}_{H}^{\ast} \mu \quad \text{and} \quad \text{ess sup} \underline{\text{scl}}_{\text{loc}} \mu \leq \text{scl}_{P}^{\ast} \mu.
\]

To prove Theorem C, we need to study quantization scales of measures.

### 3.2 Quantization and box scales of measures

Let us first recall the definition of quantization scales. Let \((X, d)\) be a metric space and \(\mu\) a Borel measure on \(X\). Given \(\epsilon > 0\), the \(\epsilon\)-quantization number \(Q_\epsilon(\mu)\) of \(\mu\) is the minimal cardinality of a set of points that is on average \(\epsilon\)-close to any point in \(X\):

\[
Q_\epsilon(\mu) = \inf \left\{ N \geq 0 : \exists \{c_i\}_{i=1}^N \subset X, \int_X d(x, \{c_i\}_{1 \leq i \leq N}) d\mu(x) < \epsilon \right\}.
\]

Then lower and upper quantization scales of \(\mu\) for a given scaling \(\text{scl}\) are defined by:

\[
\text{scl}_{Q} \mu = \sup \left\{ \alpha > 0 : Q_\epsilon(\mu) \cdot \text{scl}_{\alpha}(\epsilon) \xrightarrow[\epsilon \to 0]{} +\infty \right\}
\]

and

\[
\overline{\text{scl}}_{Q} \mu = \inf \left\{ \alpha > 0 : Q_\epsilon(\mu) \cdot \text{scl}_{\alpha}(\epsilon) \xrightarrow[\epsilon \to 0]{} 0 \right\}.
\]

Quantization scales of a measure are compared in Theorem C with box scales of measures:

**Definition 3.7** (Box scales of a measure). Let \(\text{scl}\) be a scaling and \(\mu\) a positive Borel measure on a metric space \((X, d)\). We define the lower box scale and the \(\ast\)-lower box scale of \(\mu\) by:

\[
\text{scl}_{B} \mu = \inf_{E \in \mathcal{B}} \{ \text{scl}_{B} E : \mu(E) > 0 \} \quad \text{and} \quad \text{scl}_{B}^{\ast} \mu = \inf_{E \in \mathcal{B}} \{ \text{scl}_{B} E : \mu(X \setminus E) = 0 \},
\]

where \(\mathcal{B}\) is the set of Borel subsets of \(X\). Similarly, we define the upper box scale and the \(\ast\)-upper box scale of \(\mu\) by:

\[
\overline{\text{scl}}_{B} \mu = \inf_{E \in \mathcal{B}} \{ \overline{\text{scl}}_{B} E : \mu(E) > 0 \} \quad \text{and} \quad \overline{\text{scl}}_{B}^{\ast} \mu = \inf_{E \in \mathcal{B}} \{ \overline{\text{scl}}_{B} E : \mu(X \setminus E) = 0 \}.
\]
As for Hausdorff scales of measures, we choose that all box scales of the null measure are equal to 0 as a convention. The following is straightforward:

**Lemma 3.8.** Let \((X, d)\) be a metric space and \(\mu\) a Borel measure on \(X\). Given \(\text{scl}\) a scaling, it holds:

\[
\text{scl}_Q \mu \leq \text{scl}_B^* \mu \quad \text{and} \quad \text{scl}_Q \mu \leq \overline{\text{scl}}_B \mu .
\]

**Proof.** We can assume without loss of generality that \(\text{scl}_B^* \mu\) and \(\text{scl}_B^* \mu\) are finite. Let \(E\) be a Borel set with total mass such that \(\text{scl}_B \mu\) is finite, then \(E\) is totally bounded by **Fact 2.8**. Now for \(\epsilon > 0\), consider a minimal covering by \(\epsilon\)-balls centered at some points \(x_1, \ldots, x_N\) in \(E\). Since \(\mu(X\setminus E) = 0\), it comes:

\[
\int_X d(x, \{x_i\}_{1 \leq i \leq N}) d\mu(x) = \int_E d(x, \{x_i\}_{1 \leq i \leq N}) d\mu(x) \leq \epsilon < 2\epsilon .
\]

It follows that \(Q_\mu(2\epsilon) \leq N_\epsilon(E)\), and thus by **Lemma 2.2** we obtain:

\[
\text{scl}_Q \mu \leq \text{scl}_B E \quad \text{and} \quad \overline{\text{scl}}_Q \mu \leq \overline{\text{scl}}_B E .
\]

Since this holds true for any Borel set \(E\) with total mass, the desired results come. \(\square\)

The following lemma will allow us to compare quantization scales with box scales.

**Lemma 3.9.** Let \(\mu\) be a Borel measure on \((X, d)\) such that \(Q_\mu(\epsilon) < +\infty\) for any \(\epsilon > 0\). Let us fix \(\epsilon > 0\) and an integer \(N \geq \overline{Q}_\mu(\epsilon)\). Thus consider \(x_1, \ldots, x_N \in X\) such that:

\[
\int_X d(x, \{x_i\}_{1 \leq i \leq N}) d\mu(x) < \epsilon .
\]

For any \(r > 0\), with \(E_r := \bigcup_{i=1}^N B(x_i, r)\), it holds:

\[
\mu(X\setminus E_r) < \frac{\epsilon}{r} .
\]

**Proof.** Since \(X\setminus E_r\), the complement of \(E_r\) in \(X\) is the set of points with distance at most \(r\) from the set \(\{x_1, \ldots, x_N\}\), it holds:

\[
r \cdot \mu(X\setminus E_r) \leq \int_{X\setminus E_r} d(x, \{x_i\}_{1 \leq i \leq N}) d\mu(x) < \epsilon ,
\]

which gives the desired result by dividing both sides by \(r\). \(\square\)

The following result exhibits the relationship between quantization scales and box scales. As far as we know, this result has not yet been proved, even for the specific case of dimension. It is a key element in the answer to **Problem 1.15**.

**Theorem 3.10.** Let \(\mu\) be a non null Borel measure on a metric space \((X, d)\). For any scaling \(\text{scl}\), there exists a Borel set \(F\) with positive mass arbitrarily close to \(\mu(X)\) such that:

\[
\text{scl}_B F \leq \text{scl}_Q \mu \quad \text{and} \quad \overline{\text{scl}}_B F \leq \overline{\text{scl}}_Q \mu .
\]

Consequently, it holds:

\[
\text{scl}_B \mu \leq \text{scl}_Q \mu \quad \text{and} \quad \overline{\text{scl}}_B \mu \leq \overline{\text{scl}}_Q \mu .
\]

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Proof. If $Q_\mu(\epsilon)$ is not finite for some $\epsilon > 0$ we have $\text{scl}_Q\mu = +\infty = \overline{\text{scl}}_Q\mu$ and nothing to prove. So we can assume that $Q_\mu(\epsilon) < +\infty$ for every $\epsilon > 0$. We consider two sequences of positive numbers $\epsilon_n := \exp(-n)$ and $r_n := n^2 \cdot \exp(-n) = n^2 \cdot \epsilon_n$ for $n \geq 1$. Then for each $n \geq 1$, consider a finite set $C_n \subset X$ with minimal cardinality such that:

$$\int_X d(x, C_n) d\mu(x) < \epsilon_n.$$ 

By Lemma [3.9] it holds:

$$\mu(X \setminus E_n) < \frac{\epsilon_n}{r_n} = \frac{1}{n^2} \quad \text{with} \quad E_n := \bigcup_{x \in C_n} B(x, r_n).$$

By Borell-Cantelli lemma, it holds:

$$\mu(\lim inf E_n) = \mu \left( \bigcup_{m \geq 1} \bigcap_{n \geq m} E_n \right) = \mu(X) > 0.$$ 

Hence, for sufficiently large $m$, we have $\mu(F) > 0$, where $F := \bigcap_{n \geq m} E_n$. Moreover if $\mu$ is finite by taking $m$ even larger we can have $\mu(X \setminus F)$ arbitrarily small; and if $\mu$ is infinite, we can take $\mu(F)$ arbitrarily large. We now fix such a value of $m$. Observe that $F \subset E_n$ for every $n \geq m$. As $E_n$ is a union of balls of radius $r_n$, we have trivially:

$$\forall n \geq m : N_{r_n}(F) \leq \text{Card } C_n = Q_\mu(\epsilon_n).$$

Finally by Lemma [2.3] and the fact that $\log r_n \sim \log \epsilon_n \sim \log r_{n+1}$, we obtain:

$$\text{scl}_B F = \sup \{ \alpha > 0 : N_{r_n}(F) \cdot \text{scl}_\alpha(r_n) \xrightarrow{n \to +\infty} +\infty \}$$

$$\leq \sup \{ \alpha > 0 : Q_\mu(\epsilon_n) \cdot \text{scl}_\alpha(\epsilon_n) \xrightarrow{n \to +\infty} +\infty \}$$

$$= \text{scl}_Q \mu.$$ 

Similarly, we can prove $\overline{\text{scl}}_B F \leq \overline{\text{scl}}_Q \mu$. Then the last two inequalities stated in Theorem [3.10] follow directly from the definitions of $\text{scl}_B \mu$ and $\overline{\text{scl}}_B \mu$. \hfill \Box

3.3 Comparison between local and global scales of measures and proof of Theorem C

To prove Theorem C we will need:

**Theorem 3.11.** Let $(X, d)$ be a separable metric space and $\mu$ a finite Borel measure on $X$. Let $\text{scl}$ be a scaling. It holds:

$$\text{ess sup } \text{scl}_{\text{loc}} \mu \leq \text{scl}_Q \mu \quad \text{and} \quad \text{ess sup } \overline{\text{scl}}_{\text{loc}} \mu \leq \overline{\text{scl}}_Q \mu.$$ 

**Proof.** If the lower (respectively upper) local scale of $\mu$ is zero at almost every point then obviously it is not greater than the lower (respectively upper) quantization scale of $\mu$. Otherwise, consider $\alpha, \beta > 0$ such that:

$$\alpha < \text{ess sup } \text{scl}_{\text{loc}} \mu \quad \text{and} \quad \beta < \text{ess sup } \overline{\text{scl}}_{\text{loc}} \mu. \quad (3.1)$$

Note that $E := \{ x \in X : \text{scl}_{\text{loc}} \mu(x) > \alpha \ \text{and} \ \overline{\text{scl}}_{\text{loc}} \mu(x) > \beta \}$ has positive mass. Applying Theorem [3.10] to the restriction of $\mu$ to $E$ provides a Borel subset $F \subset E$ with $\mu(F) > 0$ such that:

$$\text{scl}_B F \leq \text{scl}_Q \mu \quad \text{and} \quad \overline{\text{scl}}_B F \leq \overline{\text{scl}}_Q \mu.$$
Yet by Proposition 2.24 and Proposition 2.15 it holds respectively:

\[ \text{scl}_H F \leq \text{scl}_B F \quad \text{and} \quad \text{scl}_P F \leq \text{scl}_B F. \]  

(3.2)

Now, by setting \( \sigma \) as the restriction of \( \mu \) to \( F \), we obtain by Lemma 3.4 and Eq. (3.1):

\[ \alpha \leq \text{ess inf } \text{scl}_\text{loc} \sigma \quad \text{and} \quad \beta \leq \text{ess inf } \text{scl}_\text{loc} \sigma. \]  

(3.3)

By Lemma 3.5 it holds:

\[ \text{ess inf } \text{scl}_\text{loc} \sigma \leq \text{scl}_H F \quad \text{and} \quad \text{ess inf } \text{scl}_\text{loc} \sigma \leq \text{scl}_P F. \]  

(3.4)

Finally, combining Eqs. (3.2) to (3.4) gives:

\[ \alpha \leq \text{ess inf } \text{scl}_\text{loc} \sigma \leq \text{scl}_H F \leq \text{scl}_B F \leq \text{scl}_Q \mu \quad \text{and} \quad \beta \leq \text{ess inf } \text{scl}_\text{loc} \sigma \leq \text{scl}_P F \leq \text{scl}_B F \leq \text{scl}_Q \mu. \]

Since this holds true for any \( \alpha \) and \( \beta \) arbitrarily close to \( \text{ess sup } \text{scl}_\text{loc} \mu \) and \( \text{ess sup } \text{scl}_\text{loc} \mu \) we obtain the desired result.

We are now able to show:

Proof of Theorem C

By Theorem 3.10 and Lemma 3.8 it holds:

\[ \text{scl}_B \mu \leq \text{scl}_Q \mu \leq \text{scl}_B^* \mu \quad \text{and} \quad \text{scl}_B \mu \leq \text{scl}_Q \mu \leq \text{scl}_B^* \mu. \]

By Theorem 3.11 it holds:

\[ \text{ess sup } \text{scl}_\text{loc} \mu \leq \text{scl}_Q \mu \quad \text{and} \quad \text{ess sup } \text{scl}_\text{loc} \mu \leq \text{scl}_Q \mu. \]

Thus it remains only to show:

\[ \text{ess inf } \text{scl}_\text{loc} \mu \leq \text{scl}_B \mu \quad \text{and} \quad \text{ess inf } \text{scl}_\text{loc} \mu \leq \text{scl}_B \mu. \]  

(3.5)

To prove this, consider a subset \( E \subset X \) with positive mass. Set \( \sigma \) as the restriction of \( \mu \) to \( E \). By Lemma 3.4 it holds:

\[ \text{ess inf } \text{scl}_\text{loc} \mu \leq \text{ess inf } \text{scl}_\text{loc} \sigma \quad \text{and} \quad \text{ess inf } \text{scl}_\text{loc} \mu \leq \text{ess inf } \text{scl}_\text{loc} \sigma. \]  

(3.6)

By Theorem 3.11 it holds:

\[ \text{ess sup } \text{scl}_\text{loc} \sigma \leq \text{scl}_Q \sigma \quad \text{and} \quad \text{ess sup } \text{scl}_\text{loc} \mu \leq \text{scl}_Q \sigma. \]  

(3.7)

Moreover, by Lemma 3.8 we have:

\[ \text{scl}_Q \sigma \leq \text{scl}_B E \quad \text{and} \quad \text{scl}_Q \sigma \leq \text{scl}_B E. \]  

(3.8)

Combining Eqs. (3.6) to (3.8) provides:

\[ \text{ess inf } \text{scl}_\text{loc} \mu \leq \text{scl}_B E \quad \text{and} \quad \text{ess inf } \text{scl}_\text{loc} \mu \leq \text{scl}_B E. \]

Taking the infima over such subsets \( E \subset X \) with positive mass gives Eq. (3.5).
3.4 Proof of Theorem \( \textbf{B} \)

This subsection contains the proof of Theorem \( \textbf{B} \), we recall its statement below. We will use Vitali covering lemma to compare local scales with Hausdorff and packing scales. This lemma was first used for the dimensional case by Tamashiro [Tam95].

**Lemma 3.12** (Vitali covering lemma). Let \((X, d)\) be a separable metric space. Given \( \delta > 0 \), \( B \) a family of open balls in \( X \) with radii at most \( \delta \) and \( F \) the union of these balls. There exists a countable set \( J \) and a \( \delta \)-pack \( (B(x_j, r_j))_{j \in J} \subset B \) of \( F \) such that:

\[
F \subset \bigcup_j B(x_j, 5r_j) .
\]

For a proof of this version of the Vitali covering lemma, see [EG15][1.5.1, p. 35]. Despite they stated it for the Euclidean case, their proof adapts straightforwardly for separable metric spaces. We are now ready to prove:

**Proof of Lemma 3.5**

**Proof of the first inequality:** If \( \text{ess sup } \text{scl}_{\text{loc}} \mu = 0 \) or \( \text{scl}_H X = +\infty \), it obviously holds \( \text{ess sup } \text{scl}_{\text{loc}} \mu \leq \text{scl}_H X \). Otherwise, \( X \) is separable and consider a positive \( \alpha < \text{ess sup } \text{scl}_{\text{loc}} \mu \). There exists \( r_0 > 0 \) such that the set:

\[
A := \{ x \in X : \mu(B(x, r)) \leq \text{scl}_\alpha(r), \forall r \in (0, r_0) \}
\]

has positive measure. Consider \( \delta \leq r_0 \) and a \( \delta \)-cover \((B_j)_{j \in J} \) of \( A \). It holds:

\[
0 < \mu(A) \leq \sum_{j \in J} \mu(B_j) \leq \sum_{j \in J} \text{scl}_\alpha(|B_j|) .
\]

Taking the infimum over \( \delta \)-covers provides:

\[
0 < \mu(A) \leq H^{\text{scl}_\alpha}(A) .
\]

Now as \( \delta \) goes to 0 we obtain:

\[
0 < \mu(A) \leq H^{\text{scl}_\alpha}(A) .
\]

It follows:

\[
\text{scl}_H X \geq \text{scl}_H A \geq \alpha ,
\]

which allows us to conclude the proof of that first inequality.

**Proof of the second inequality:** If \( \text{ess sup } \text{scl}_{\text{loc}} \mu = 0 \) or \( \text{scl}_P X = +\infty \), it obviously holds \( \text{ess sup } \text{scl}_{\text{loc}} \mu \leq \text{scl}_P X \). Otherwise, \( X \) is separable and consider a positive \( \alpha < \text{ess sup } \text{scl}_{\text{loc}} \mu \). Put:

\[
F = \{ x \in X : \text{scl}_{\text{loc}} \mu(x) > \alpha \} .
\]

Let \((F_N)_{N \geq 1}\) be a covering of \( X \) by Borel subsets. Given \( 0 < \beta < \alpha \), by **Fact 2.1** there exists \( \delta_0 > 0 \) such that for any \( r \leq \delta_0 \), it holds:

\[
\text{scl}_\alpha(5r) \leq \text{scl}_\beta(r) . \tag{3.9}
\]

Fix \( \delta \in (0, \delta_0) \) and an integer \( N \geq 1 \). For each \( x \) in \( F_N \), by Lemma 2.3 there exists a minimal integer \( n(x) \) such that:

\[
\mu(B(x, 5r(x))) \leq \text{scl}_\alpha(5r(x)) \quad \text{where } r(x) := \exp(-n(x)) \leq \delta . \tag{3.10}
\]
We now set:
\[ F = \{ B(x, r(x)) : x \in F_N \} . \]

Thus by Vitali covering Lemma 3.12 there exists a countable subset \( J \subset F_N \) such that \( (B(x, r(x)))_{x \in J} \) is a \( \delta \)-pack of \( F_N \) and \( (B(x, 5r(x)))_{x \in J} \) is a covering of \( F_N \). From there, by Eqs. (3.9) and (3.10) it holds:
\[
\mu(F_N) \leq \sum_{x \in J} \mu(B(x, 5r(x))) \leq \sum_{x \in J} \text{scl}_\alpha(5r(x)) \leq \sum_{x \in J} \text{scl}_\beta(r(x)) .
\]

Since this holds true for any \( \delta \)-pack, we have:
\[
P^{\text{scl}}_\beta(F_N) \geq \mu(F_N) ,
\]
and then taking \( \delta \) arbitrarily close to 0 leads to:
\[
P^{\text{scl}}_0(F_N) \geq \mu(F_N) .
\]

Summing over \( N \geq 1 \) provides:
\[
\sum_{N \geq 1} P^{\text{scl}}_\beta(F_N) \geq \sum_{N \geq 1} \mu(F_N) \geq \mu(F) > 0 .
\]

Recall that \( (F_N)_{N \geq 1} \) is an arbitrary covering by Borel sets of \( F \), thus by definition of the packing measure and the latter equation, it holds:
\[
P^{\text{scl}}_\beta(F) \geq \mu(F) > 0 .
\]

It holds then \( \text{scl}_P F \geq \beta \) for any \( \beta < \alpha < \text{ess sup} \text{scl}_{\text{loc}} \mu \), which allows us to conclude the proof.

We deduce then:

**Proposition 3.13.** Let \( (X, d) \) be a metric space and \( \mu \) a Borel measure on \( X \), then:
\[
\text{ess inf} \text{scl}_{\text{loc}} \mu \leq \text{scl}_H \mu \quad \text{and} \quad \text{ess inf} \overline{\text{scl}}_{\text{loc}} \mu \leq \text{scl}_P \mu ,
\]
and
\[
\text{ess sup} \text{scl}_{\text{loc}} \mu \leq \text{scl}^*_H \mu \quad \text{and} \quad \text{ess sup} \overline{\text{scl}}_{\text{loc}} \mu \leq \text{scl}^*_P \mu .
\]

**Proof.** The second line of inequalities is given by Corollary 3.6. It remains to show the first line of inequalities. Let \( E \) be a Borel subset of \( X \) with \( \mu \)-positive mass. With \( \sigma \) the restriction of \( \mu \) to \( E \), it holds by Lemma 3.5:
\[
\text{ess sup} \text{scl}_{\text{loc}} \sigma \leq \text{scl}_H E \quad \text{and} \quad \text{ess sup} \overline{\text{scl}}_{\text{loc}} \sigma \leq \text{scl}_P E .
\]

Then, by Lemma 3.4 it follows:
\[
\text{ess inf} \text{scl}_{\text{loc}} \mu \leq \text{scl}_H E \quad \text{and} \quad \text{ess inf} \overline{\text{scl}}_{\text{loc}} \mu \leq \text{scl}_P E .
\]
Taking the infima over \( E \) with positive mass ends the proof.

Explicit links between packing scales, Hausdorff scales and local scales of measures can now be established by proving **Theorem B**. Let us first recall its statement: Let \( (X, d) \) be a metric space and \( \mu \) a Borel measure on \( X \), then it holds:
\[
\text{scl}_H \mu = \text{ess inf} \text{scl}_{\text{loc}} \mu \leq \text{scl}_P \mu = \text{ess inf} \overline{\text{scl}}_{\text{loc}} \mu
\]
and
\[
\text{scl}^*_H \mu = \text{ess sup} \text{scl}_{\text{loc}} \mu \leq \text{scl}^*_P \mu = \text{ess sup} \overline{\text{scl}}_{\text{loc}} \mu .
\]
**Proof of Theorem B**  By Proposition 3.13 it remains only to show four inequalities:

\[
scl_H \mu \leq \text{ess inf}_{\alpha} \text{scl}_{\alpha} \mu \quad ; \quad \text{scl}_P \mu \leq \text{ess inf}_{\alpha} \text{scl}_{\alpha} \mu
\]

and

\[
scl^*_H \mu \leq \text{ess sup}_{\alpha} \text{scl}_{\alpha} \mu \quad ; \quad \text{scl}_P^* \mu \leq \text{ess sup}_{\alpha} \text{scl}_{\alpha} \mu .
\]

Note that for each of the above inequalities, if the right-hand side quantity is infinite, there is nothing to prove. In each of the following proofs, we will assume that this quantity is finite.

**Proof of (i):** Fix \( \alpha > \text{ess inf}_{\alpha} \text{scl}_{\alpha} \) and \( \beta > \alpha \). By definition of scaling, there exists \( \delta > 0 \) such that for any \( r \in (0, \delta) \) it holds

\[
scl_{\beta}(5r) \leq \text{scl}_{\alpha}(r) .
\]  

(3.11)

Denote:

\[
F := \{ x \in X : \text{scl}_{\alpha}(x) < \alpha \} .
\]

Then it holds \( \mu(F) > 0 \) and by Lemma 2.3 for any \( x \) in \( F \) there exists a minimal integer \( n(x) \) such that:

\[
\mu \left( B(x, r(x)) \right) \geq \text{scl}_{\alpha}(r(x)) \quad \text{where } r(x) := \exp(-n(x)) \leq \delta .
\]  

(3.12)

Now set:

\[
F := \{ B(x, r(x)) : x \in F \} .
\]

By Vitali covering Lemma 3.12 there exists a countable subset \( J \subset F \) such that \( (B(x, r(x)))_{x \in J} \) is a \( \delta \)-pack of \( F \) and \( F \subset \bigcup_{x \in J} B(x, 5r(x)) \). Then, by Eqs. (3.11) and (3.12) it holds:

\[
\sum_{x \in J} \text{scl}_{\beta}(5r(x)) \leq \sum_{x \in J} \text{scl}_{\alpha}(r(x)) \leq \sum_{x \in J} \mu(B(x, r(x))) \leq \mu(F) .
\]

It follows that \( H_{\delta}^{\text{scl}_{\beta}}(F) \leq \mu(F) \) and thus \( H^{\text{scl}_{\alpha}}(F) \leq \mu(F) \) as \( \delta \) goes to 0. It follows that \( \text{scl}_H \mu \leq \text{scl}_H F \leq \beta \). Taking \( \beta > \alpha \) close to \( \text{ess inf}_{\alpha} \text{scl}_{\alpha} \), we obtain inequality (i).

**Proof of (ii):** Consider \( \alpha > \text{ess sup}_{\alpha} \text{scl}_{\alpha} \). Set \( E = \{ x \in X : \text{scl}_{\alpha}(x) < \alpha \} \). Observe that \( \mu(X \setminus E) = 0 \) and:

\[
E = \bigcup_{i \geq 1} E_i \quad \text{where } E_i := \{ x \in E : \forall r \leq 2^{-i}, \mu(B(x, r)) \geq \text{scl}_{\alpha}(r) \} .
\]

By \( \sigma \)-stability of packing scales provided by Lemma 2.21 it holds \( \text{scl}_P E = \sup_{i \geq 1} \text{scl}_P E_i \). It is then enough to show that for any \( i \geq 1 \), we have \( \text{scl}_P E_i \leq \alpha \). Indeed, then taking \( \alpha \) arbitrarily close to \( \text{ess sup}_{\alpha} \text{scl}_{\alpha} \) allows us to conclude. In that way, let us fix \( i \geq 1 \). Fix also \( \delta \in (0, 2^{-i}) \) and consider a countable set \( J \) and \( (B_j)_{j \in J} \) a \( \delta \)-pack of \( E_i \). It follows:

\[
\sum_{j \in J} \text{scl}_{\alpha}(|B_j|) \leq \sum_{j \in J} \mu(B_j) \leq 1 .
\]

Since this holds true for any \( \delta \)-pack, we have:

\[
\mathcal{P}_{\delta}^{\text{scl}_{\alpha}}(E_i) \leq 1 .
\]

As \( \delta \) goes to 0 we obtain:

\[
\mathcal{P}^{\text{scl}_{\alpha}}(E_i) \leq 1 .
\]
It follows that we indeed have $\text{scl}_{P_E} \leq \alpha$.

**Proof of (iii):** Fix $\alpha > \text{ess sup } \text{scl}_{\text{loc}}$. For $\beta > \alpha$, consider $\delta > 0$ such that for any $r \in (0, \delta)$ it holds:

$$\text{scl}_\beta(5r) \leq \text{scl}_\alpha(r). \quad (3.13)$$

Denote:

$$F := \{ x \in X : \text{scl}_{\text{loc}} \mu(x) < \alpha \}.$$

Note that $F$ has total mass and by Lemma 2.3 for every $x \in F$ there exists a minimal integer $n(x)$ such that:

$$\mu(B(x, r(x))) \geq \text{scl}_\alpha(r(x)) \quad \text{where } r(x) := \exp(-n(x)) \leq \delta. \quad (3.14)$$

Now put:

$$F := \{ B(x, r(x)) : x \in F \}.$$

By Vitali’s Lemma 3.12, there exists a countable subset $J \subset F$ such that $B(x, r(x)) \in J$ is a $\delta$-pack and $F = \bigcup_{j \in J} B(x, 5r(x))$. Thus, by Eqs. (3.13) and (3.14) it holds:

$$\sum_{x \in J} \text{scl}_\beta(5r(x)) \leq \sum_{x \in J} \text{scl}_\alpha(r(x)) \leq \sum_{x \in J} \mu(B(x, r(x))) \leq \mu(F).$$

It follows that $H^{\text{scl}_\beta}_\delta(F) \leq \mu(F)$ and thus $H^{\text{scl}_\beta}(F) \leq \mu(F)$ as $\delta$ goes to 0. It follows that $\text{scl}_H^* \mu \leq \text{scl}_H F \leq \beta$. Taking $\beta > \alpha$ close to $\text{ess sup } \text{scl}_{\text{loc}} \mu$, we obtain inequality (iii).

**Proof of (iv):** Fix $\alpha > \text{ess inf } \text{scl}_{\text{loc}} \mu$. Consider the set $E := \{ x \in X : \text{scl}_{\text{loc}} \mu < \alpha \}$. Then observe that $\mu(X \setminus E) = 0$ and that we can write:

$$E = \bigcup_{i \geq 1} E_i \quad \text{where } E_i = \{ x \in E : \forall r \leq 2^{-i}, \mu(B(x, r)) \geq \text{scl}_\alpha(r) \}.$$

By Lemma 2.21 we have $\text{scl}_P E = \sup_{i \geq 1} \text{scl}_P E_i$, it is then enough to show that for any $i \geq 1$, we have $\text{scl}_P E_i \leq \alpha$. We indeed can take $\alpha$ arbitrarily close to $\text{ess inf } \text{scl}_{\text{loc}} \mu$. We then fix $i \geq 1$. Fix $\delta \in (0, 2^{-i})$. We consider $J$ a countable set and $(B_j)_{j \in J}$ a $\delta$-pack of $E_i$. Then it holds:

$$\sum_{j \in J} \text{scl}_\alpha(|B_j|) \leq \sum_{j \in J} \mu(B_j) \leq 1.$$

Since this holds true for any $\delta$-pack, it follows:

$$P^{\text{scl}_\alpha}_\delta(E_i) \leq 1.$$

When $\delta$ tends to 0, the latter inequality leads to:

$$P^{\text{scl}_\alpha}(E_i) \leq P^{\text{scl}_\alpha}_0(E_i) \leq 1.$$

From there, we deduce $\text{scl}_P E_i \leq \alpha$, which concludes the proof of $\text{scl}_P^* \mu \leq \text{ess sup } \text{scl}_{\text{loc}} \mu$ and thus the one of Theorem B. \qed
4 Examples and applications of theorems of comparison of scales

4.1 Scales of infinite products of finite sets

Natural toy models in the study of scales are given by products $Z = \prod_{n \geq 1} Z_k$ of finite sets. To define the metric $\delta$ on this set, we fix a sequence $(\epsilon_n)_{n \geq 1}$ decreasing to 0 and verifying $\log \epsilon_{n+1} \sim \log \epsilon_n$ as $n \to +\infty$. We then define the distance between $x = (x_n)_{n \geq 1} \in Z$ and $x' = (x'_n)_{n \geq 1} \in Z$ by:

$$\delta(x, x') := \epsilon_m,$$

where $m = \nu(x, x') := \inf \{n \geq 1 : x_n \neq x'_n\}$ is the minimal index such that the sequences $x$ and $x'$ differ. We endow each $Z_n$ with the discrete topology, thus $\delta$ provides the product topology on $Z$.

A natural measure on $Z$ is the following product measure:

$$\mu := \bigotimes_{n \geq 1} \mu_n,$$

where $\mu_n$ is the equidistributed measure on $Z_n$ for $n \geq 1$. Note that the metric is highly dependent on the choice of the sequence $(\epsilon_n)_{n \geq 1}$. We will tune both values of the sequences $(\epsilon_n)_{n \geq 1}$ and $(\text{Card } Z_n)_{n \geq 1}$ to reach different values of scales for $Z$ and $\mu$. First, box scales of $Z$ coincide with local scales of $\mu$ according to the following:

**Proposition 4.1.** For any scaling $\text{ scl }$, it holds for every $x \in Z$:

$$\text{ scl}_{\text{loc}} \mu(x) = \text{ scl}_{B} Z = \sup \left\{ \alpha > 0 : \text{ scl}_{\alpha} (\epsilon_n) \cdot \prod_{k=1}^{n} \text{ Card } Z_k \xrightarrow{n \to +\infty} +\infty \right\}$$

and

$$\overline{\text{ scl}}_{\text{loc}} \mu(x) = \overline{\text{ scl}}_{B} Z = \inf \left\{ \alpha > 0 : \text{ scl}_{\alpha} (\epsilon_n) \cdot \prod_{k=1}^{n} \text{ Card } Z_k \xrightarrow{n \to +\infty} 0 \right\}.$$  \hspace{1cm} (4.1)

**Proof.** To prove this, first note that for $x \in X$ and $n \geq 1$, it holds:

$$B(x, \epsilon_n) = \{x' \in Z : x'_1 = x_1, \ldots, x'_n = x_n\}.$$  \hspace{1cm} (4.2)

Such a ball is usually called a $n$-cylinder. By definition of $\mu$ it holds:

$$\mu (B(x, \epsilon_n)) = \prod_{k=1}^{n} \mu_k(x_k) = \prod_{k=1}^{n} \frac{1}{\text{ Card } Z_k}.$$  \hspace{1cm} (4.3)

Moreover as two balls with the same radius are either equal or disjoint, there are exactly $\prod_{k=1}^{n} \text{ Card } Z_k$ different balls of radius $\epsilon_n$ and in particular $\mathcal{N}_{\epsilon_n}(Z) = \prod_{k=1}^{n} \text{ Card } Z_k$.

We just proved:

$$\mathcal{N}_{\epsilon_n}(Z) = \prod_{k=1}^{n} \text{ Card } Z_k = \mu (B(x, \epsilon_n))^{-1}.$$  \hspace{1cm} (4.4)

Since $\log \epsilon_{n+1} \sim \log \epsilon_n$ as $n \to +\infty$, by Lemma 2.3 and Eq. (4.3), we obtain:

$$\text{ scl}_{B} Z = \sup \left\{ \alpha > 0 : \prod_{k=1}^{n} \text{ Card } Z_k \cdot \text{ scl}_{\alpha} (\epsilon_n) \xrightarrow{n \to +\infty} 0 \right\} = \text{ scl}_{\text{loc}} \mu(x),$$

which is actually Eq. (4.1). We obtain similarly Eq. (4.2).
Proposition 4.1 together with Theorem A and C directly implies:

**Corollary 4.2.** We have moreover:

\[
\text{scl}_H Z = \text{scl}_Q \mu = \text{scl}_B Z \quad \text{and} \quad \text{scl}_P Z = \text{scl}_Q \mu = \text{scl}_B Z .
\]

Also, for \( \text{scl} = \text{ord} \) and if \( \frac{-\log \epsilon_n}{n} \) is moreover converging to some positive finite constant, we have:

**Corollary 4.3.** Suppose further that \( \frac{-\log \epsilon_n}{n} \) converges to \( C > 0 \) when \( n \to +\infty \) (a typical choice is \( \epsilon_n = \exp(C \cdot n) \)). Then for any scaling \( \text{scl} \), it holds:

\[
\text{ord}_H Z = \text{ord}_B Z = \liminf_{n \to +\infty} \frac{1}{C \cdot n} \log \left( \sum_{k=1}^{n} \log \left( \text{Card} Z_k \right) \right)
\]

and

\[
\text{ord}_P Z = \text{ord}_B Z = \limsup_{n \to +\infty} \frac{1}{C \cdot n} \log \left( \sum_{k=1}^{n} \log \left( \text{Card} Z_k \right) \right) .
\]

**Proof.** Recall that the scaling defining order is given for \( \alpha > 0 \) and \( \epsilon \in (0, 1) \) by \( \text{scl}^{2,1}_\alpha = \exp(-\epsilon - \alpha) \). The first equality in Eq. (4.4) reads as:

\[
\text{ord}_B(X) = \sup \left\{ \alpha > 0 : -\epsilon^{-\alpha} + \sum_{k=1}^{n} \log \text{Card} Z_k \xrightarrow{n \to +\infty} +\infty \right\} .
\]

As \( \epsilon^{-\alpha} = e^{-\alpha \log \epsilon} \) and \( -\log \epsilon_n \sim Cn \), we obtain the first announced formula. The second formula is deduced similarly.

Such examples of products of finite sets allow us to exhibit compact metric spaces with arbitrarily high orders:

**Example 4.4.** For any \( \alpha \geq \beta > 0 \), there exists a compact metric probability space \((Z, \delta, \mu)\) such that for any \( z \in Z \):

\[
\beta = \text{ord}_{loc} \mu(z) = \text{ord}_H Z = \text{ord}_Q \mu = \text{ord}_B Z
\]

and

\[
\alpha = \text{ord}_{loc} \mu(z) = \text{ord}_P Z = \text{ord}_Q \mu = \text{ord}_B Z.
\]

In particular, with \( \alpha > \beta \) we obtain examples of metric spaces with finite order such that the Hausdorff and packing orders do not coincide. Moreover, for a countable dense subset \( F \) of \( X \), it holds:

\[
\text{ord}_H F = \text{ord}_P F = 0 \quad \text{and} \quad \text{ord}_B F = \beta < \alpha = \text{ord}_B F .
\]

It follows that none of the inequalities of Theorem A for the case of order is an equality in the general case.

**Construction of Example 4.4.** Let \((u_k)_{k \geq 0}\) be the sequence defined by:

\[
u_k = \begin{cases} \exp(\exp(\beta \cdot k)) & \text{if} \quad c^{2j} \leq k < c^{2j+1} \\ \exp(\exp(\alpha \cdot k)) & \text{if} \quad c^{2j+1} \leq k < c^{2j+2} \end{cases}
\]

where \( e = \lceil \frac{\alpha}{\beta} \rceil + 1 \). We denote \( Z := \prod_{n \geq 1} Z / u_k Z \) endowed with the metric \( \delta \) defined by:

\[
\delta(z, w) := \exp(-\inf \{ n \geq 1 : z_n \neq w_n \})
\]
for $z = (z_n)_{n \geq 1}$ and $w = (w_n)_{n \geq 1}$ in $Z$. Let us denote $\lambda_n = \frac{1}{n} \log \sum_{k=1}^{n} \log u_k$. Thus by Corollary 4.3 it follows:

\[
\text{ord}_H Z = \text{ord}_B Z = \lim_{n \to +\infty} \inf \lambda_n \quad \text{and} \quad \text{ord}_P Z = \overline{\text{ord}}_B Z = \lim_{n \to +\infty} \sup \lambda_n .
\]

It remains to show that $\lambda^- := \lim_{n \to +\infty} \inf \lambda_n = \beta$ and $\lambda^+ := \lim_{n \to +\infty} \sup \lambda_n = \alpha$ in order to show that $(Z, \delta)$ satisfies the desired properties. First notice that $\exp(\exp(\beta \cdot n)) \leq u_n \leq \exp(\exp(\alpha \cdot n))$ for every integer $n$. It follows that $\lambda^- \geq \beta$ and $\lambda^+ \leq \alpha$. Denote $n_j = c^{2j+1}$ and observe that:

\[
\lambda_{n_j} \geq \frac{1}{n_j} \log \log(u_{n_j}) \xrightarrow{j \to +\infty} \alpha .
\]

Thus we have $\lambda^+ \geq \alpha$. Moreover, denote $m_j = c^{2j+1} - 1$. We have the following:

**Lemma 4.5.** For any $j \geq 1$ and for any $1 \leq k \leq m_j$, it holds:

\[
u_k \leq u_{m_j} .
\]

**Proof.** If $c^{2j} \leq k \leq m_j$, then $u_k = \lfloor \exp(\exp(\beta \cdot k)) \rfloor \leq \lfloor \exp(\exp(\beta \cdot m_j)) \rfloor = u_{m_j}$. Otherwise, we have $k < c^{2j}$, and then $u_k < \lfloor \exp(\exp(\alpha \cdot c^{2j})) \rfloor < \lfloor \exp(\exp(\beta \cdot c^{2j}+1)) \rfloor = u_{m_j}$, since $\alpha < \beta \cdot c$. \hfill \Box

From the above lemma, we have:

\[
\lambda_{n_j} \geq \frac{1}{m_j} \log m_j \log(u_{m_j}) \xrightarrow{j \to +\infty} \beta ,
\]

and so $\lambda^- \leq \beta$ which ends the construction of Example 4.4. \hfill \Box

### 4.2 Functional spaces

This last sub-section consists in showing Lemma 1.12 that allows us to show Theorem E. Denote by $\| \cdot \|_{C^k}$ the $C^k$-uniform norm on $C^k([0,1]^d, \mathbb{R})$:

\[
\|f\|_{C^k} := \sup_{0 \leq j \leq k} \|D^j f\|_\infty .
\]

**Definition 4.6.** For integers $d \geq 1$ and $k \geq 0$, let us denote:

\[
\mathcal{F}^{d,k,0} := \{ f \in C^k([0,1]^d, [-1,1]) : \|f\|_{C^k} \leq 1 \}
\]

the $C^k$-unit ball; and for $\alpha \in (0,1]$ let us define:

\[
\mathcal{F}^{d,k,\alpha} := \{ f \in C^k([0,1]^d, [-1,1]) : \|f\|_{C^k} \leq 1 \text{ and } D^k f \text{ is } \alpha\text{-Hölder with constant } 1 \} .
\]

Recall that for $\alpha > 0$, the map $D^k f$ is $\alpha$-Hölder with constant 1 if for any $x, y \in [0,1]^d$ it holds:

\[
\|D^k f(x) - D^k f(y)\|_\infty \leq \|x - y\|^\alpha .
\]

Let us recall the asymptotic given by Kolmogorov-Tikhomirov [KT93] [Thm XV] on the covering number of $(\mathcal{F}^{d,k,\alpha}, \| \cdot \|_\infty)$ stated in Theorem 1.11

\[
C_1 \cdot e^{-\frac{d}{k+\alpha}} \geq \log \mathcal{N}(\mathcal{F}^{d,k,\alpha}) \geq C_2 \cdot e^{-\frac{d}{k+\alpha}} ,
\]

where $C_1 > C_2 > 0$ are two constants depending on $d$, $k$ and $\alpha$. In order to prove Theorem E which states that box, packing and Hausdorff scales of $\mathcal{F}^{d,k,\alpha}$ are all equal to $\frac{d}{k+\alpha}$, by Theorem E it remains to prove Lemma 1.12. The latter states:

\[
\text{ord}_H \mathcal{F}^{d,k,\alpha} \geq \frac{d}{k+\alpha} .
\]
Proof of Lemma 1.12. We first assume \( \alpha > 0 \). The case \( \alpha = 0 \) will be deduced at the end of the proof. We consider the following product of finite sets:
\[
\Lambda = \prod_{n \geq 1} \{0, 1\}^{\mathcal{R}_n}.
\]
where:
\[
\mathcal{R}_n := \left\{ \left( \frac{i_1}{R^n}, \ldots, \frac{i_d}{R^n} \right) : i_1, \ldots, i_d \in \{0, \ldots, R^n - 1\} \right\} \text{ with } R := \lceil 3^{\frac{k+\alpha}{\alpha}} \rceil + 1. \tag{4.5}
\]
Observe that \( \mathcal{R}_n \) is a meshgrid of step \( R^{-n} \) of \([0, 1]^d\) with cardinal \( R^{nd} \).

We endow \( \Lambda \) with the ultrametric distance \( \delta \) defined by:
\[
\delta(\lambda, \lambda') = \epsilon_m, \quad \text{with } m \text{ the minimal index such that the sequences } \lambda \text{ and } \lambda' \text{ differ and } (\epsilon_n)_{n \geq 1} \text{ a decreasing sequence of positive real numbers.}
\]
The following will enable us to conclude the proof of Lemma 1.12:

Lemma 4.7. We can choose the sequence \( (\epsilon_n)_{n \geq 1} \) such that:
\[
- \log \epsilon_n \sim n \cdot \log R^{k+\alpha} \quad \text{as } n \to +\infty. \tag{4.6}
\]
and such that there exists an expanding embedding \( I : (\Lambda, \delta) \to (\mathcal{F}^d, k, \alpha, \| \cdot \|_\infty) \). More precisely, for every \( \lambda, \lambda' \in \Lambda \) it holds:
\[
\| I(\lambda) - I(\lambda') \|_\infty \geq \frac{1}{2} \delta(\lambda, \lambda'). \tag{4.7}
\]

The choice of \( (\epsilon_n)_{n \geq 1} \) will be given in Eq. (4.9). Lemma 4.7 is proven below. Let us see how it allows us to finish the proof of Lemma 1.12. Since \( \log \epsilon_{n+1} \sim \log \epsilon_n \), by Corollary 4.3 it holds:
\[
\text{ord}_H \Lambda = \liminf_{n \to +\infty} \frac{1}{n \log R^{k+\alpha}} \log \left( \sum_{j=1}^n \log \text{Card} \left( \{0, 1\}^{\mathcal{R}_j} \right) \right) = \liminf_{n \to +\infty} \frac{1}{n \log R^{k+\alpha}} \log \left( \sum_{j=1}^n R^{dj} \cdot \log 2 \right) = \frac{d}{k+\alpha}.
\]
Now Lemma 4.7 together with Corollary 2.23 implies:
\[
\text{ord}_H \mathcal{F}^d, k, \alpha \geq \text{ord}_H \Lambda = \frac{d}{k+\alpha},
\]
which is the desired inequality from Lemma 1.12.

We now show Lemma 4.7 using two intermediary results Lemmas 4.8 and 4.9. We will actually deduce Lemma 4.9 from Lemma 4.8.

Proof of Lemma 4.7. Let us denote \( q := k + \alpha \) so that \( R = \lfloor 3^{\frac{k+\alpha}{\alpha}} \rfloor + 1 \). For \( f \in \mathcal{F}^d, k, \alpha \), let \( \| f \|_q \) be the \( \alpha \)-Hölder constant of \( D^k f \), i.e the minimal constant \( C > 0 \) such that for any \( x, y \in [0, 1]^d \) it holds:
\[
\| D^k f(x) - D^k f(y) \|_\infty \leq C \cdot \| x - y \|_\alpha^n.
\]
Note that \( \| \cdot \|_q \) is a semi-norm on \( \mathcal{F}^d, k, \alpha \) and moreover we have:
\[
\mathcal{F}^d, k, \alpha = \{ f \in \mathcal{F}^d, 0, \alpha : \| f \|_q \leq 1 \}.
\]
We consider the following function defined on \( \mathbb{R} \):
\[
\phi(t) := 4^q \cdot t^q \cdot (1 - t)^q \cdot \mathbb{I}_{0 < t < 1}.
\]
It is clear that $\text{supp} \phi = [0, 1]$, $\phi(0) = 0 = \phi(1)$, $\|\phi\|_{\infty} = \phi(1/2) = 1$. Observe also that $D^k \phi(0) = 0 = D^k \phi(1)$.

Let us denote:

$$\Phi : x \in \mathbb{R}^d \mapsto \phi(2\|x\|).$$

Note that $\Phi$ is of class $C^k$ with support ($\|x\| \leq 1/2$) and $\|\Phi\|_{\infty} = 1$. Also $D^k \Phi$ equals 0 outside ($\|x\| < 1/2$) and is $\alpha$-Hölder with constant:

$$0 < \|\Phi\|_q < +\infty.$$  \hfill (4.8)

We now proceed to the construction of the embedding. We first fix $n$ and denote $\mathcal{R} := \mathcal{R}_n$. To each $\lambda = (\lambda_r)_{r \in \mathcal{R}} \in \{0, 1\}^\mathcal{R}$ we associate the following map:

$$f_\lambda : x \in [0, 1]^d \mapsto \epsilon_n \cdot \sum_{r \in \mathcal{R}} \lambda_r \cdot \Phi(R^n(x - r)),$$

with:

$$\epsilon_n := \frac{3}{\pi^2 \cdot n^2 \cdot R^{q_n} \cdot \|\Phi\|_q}. \hfill (4.9)$$

First note that $(\epsilon_n)_{n \geq 1}$ obviously verifies Eq. (4.6). We have the following result:

**Lemma 4.8.** For every $\lambda \neq \lambda'$ in $\{0, 1\}^\mathcal{R}$, it holds:

$$\| f_\lambda - f_{\lambda'} \|_{\infty} = \epsilon_n \quad \text{and} \quad \| f_\lambda - f_{\lambda'} \|_q \leq \frac{6}{\pi^2 \cdot n^2}.$$  

**Proof.** For any $x \in [0, 1]^d$, there exists at most one point $r \in \mathcal{R}$ such that $\|x - r\| < R^{-n}/2$. Consequently, the maps $x \mapsto \Phi(R^n(x - r))$ for $r \in \mathcal{R}$ have supports with disjoint interiors. It comes then:

$$\| f_\lambda - f_{\lambda'} \|_{\infty} = \epsilon_n \cdot \left\| \sum_{r \in \mathcal{R}} (\lambda_r - \lambda'_r) \cdot \Phi(R^n(\cdot - r)) \right\|_{\infty} = \epsilon_n \cdot \sup_{r \in \mathcal{R}} |\lambda_r - \lambda'_r| \cdot \|\Phi\|_{\infty} = \epsilon_n.$$

This provides the equality of Lemma 4.8.

Now note similarly that the maps $x \mapsto D^k \Phi(R^n \cdot (x - r))$ for $r \in \mathcal{R}$ also have supports with disjoint interiors and moreover they are equal to 0 on the boundary of their support. Thus for every $x \in [0, 1]^d$ there exists at most one value of $r \in \mathcal{R}$ such that $D^k \Phi(R^n(x - r))$ is not null. It follows:

$$\| f_\lambda - f_{\lambda'} \|_q = \epsilon_n \cdot \left\| \sum_{r \in \mathcal{R}} (\lambda_r - \lambda'_r) \cdot \Phi(R^n(\cdot - r)) \right\|_q \leq \epsilon_n \cdot \sup_{r, s \in \mathcal{R}} \sup_{x \neq y \in \mathbb{R}^d} \left\| (\lambda_r - \lambda'_r) \cdot R^{kn} D^k \Phi(R^n(x - r)) - (\lambda_s - \lambda'_s) \cdot R^{kn} D^k \Phi(R^n(y - s)) \right\|_{\infty} \leq \epsilon_n \cdot R^{q_n} \cdot \sup_{r, s \in \mathcal{R}} \sup_{x \neq y \in \mathbb{R}^d} \left\| (\lambda_r - \lambda'_r) \cdot D^k \Phi(X - R^n r) - (\lambda_s - \lambda'_s) \cdot D^k \Phi(Y - R^n s) \right\|_{\infty} \leq \epsilon_n \cdot R^{q_n} \cdot \sup_{r \in \mathcal{R}} \left\| \Phi(\cdot - R^n r) \right\|_q = 2 \epsilon_n \cdot R^{q_n} \|\Phi\|_q.$$

This allows us to conclude as the latter term is equal to $\frac{6}{\pi^2 n^2}$ by Eq. (4.9).

$$\square$$

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To conclude the proof, it remains to show Eq. (4.7) of Lemma 4.7. Consider $R_k := 3\nu \lambda_n$.

Since $\epsilon_n > 0$ implies $X_n \geq 3\nu \lambda_n$.

Now Lemma 4.8 provides:

$$||f_{\lambda_n}||_{\infty} \leq ||f_{\lambda_n}||_{q} \leq \frac{6}{n^2} \cdot ||f_{\lambda_n}||_q.$$  \hspace{1cm} (4.10)

To embed $\Lambda$ into $F_{d,k,\alpha}$, we use the following that we will prove using the above Lemma 4.8.

**Lemma 4.9.** For every $\Lambda = (\lambda_n)_{n \geq 1} \subset \Lambda$, the function series $\sum_{n \geq 1} f_{\lambda_n}$ converges in $C^0([0, 1]^d, [-1, 1])$ and moreover its limit lies in $F_{d,k,\alpha}$.

**Proof.** By the equality in Eq. (4.10) the normal convergence of the series $\sum f_{\lambda_n}$ holds for the $C^0$-norm as $\sum \epsilon_n < +\infty$. Thus its limit $g$ is continuous. Let us show that $g$ indeed lies in $F_{d,k,\alpha}$. First note that for any $n \geq 1$ and for any $1 \leq l \leq k$ it holds $D^lf_{\lambda_n}(0) = 0$, thus by Taylor integral formula, it holds:

$$||D^lf_{\lambda_n}||_{\infty} \leq ||D^lf_{\lambda_n}||_{q}.$$  \hspace{1cm} (4.11)

Still by Eq. (4.10), it holds:

$$\sum_{n \geq 1} ||f_{\lambda_n}||_{q} \leq \sum_{n \geq 1} \frac{6}{n^2} = 1.$$  \hspace{1cm} (4.12)

Note moreover that as $D^lf_{\lambda_n}(0) = 0$, it holds $||D^lf_{\lambda_n}||_{\infty} \leq ||f_{\lambda_n}||_{q}$ thus by Lemma 4.8 it follows:

$$\sum_{n \geq 1} ||f_{\lambda_n}||_{C^k} \leq \sum_{n \geq 1} ||f_{\lambda_n}||_{q} \leq 1.$$  \hspace{1cm} (4.13)

Consequently, the partial sums of the series lie in $F_{d,k,\alpha}$ and so does $g$ as $F_{d,k,\alpha}$ is closed for the $C^0$-norm. \hfill \Box

We will now finish the proof of the case $\alpha > 0$ of Lemma 4.7 using Lemmas 4.8 and 4.9. First, by the above Lemma 4.9 the following map is well defined:

$$I : \Lambda = (\lambda_n)_{n \geq 1} \in (\Lambda, \delta) \mapsto \lim_{n \to +\infty} \sum_{n \geq 1} f_{\lambda_n} \in (F_{d,k,\alpha}, || \cdot ||_{\infty}).$$

To conclude the proof, it remains to show Eq. (4.7) of Lemma 4.7. Consider $\Lambda = (\lambda_n)_{n \geq 1}$ and $\Lambda' = (\lambda'_n)_{n \geq 1} \in \Lambda$. Denote $k := \nu(\Lambda, \Lambda')$ the first index such that the sequences $\lambda$ and $\lambda'$ differ. Then it holds:

$$||I(\Lambda) - I(\Lambda')||_{\infty} = \left| \sum_{n \geq k} f_{\lambda_n} - f_{\lambda'_n} \right|_{\infty} \geq ||f_{\lambda_k} - f_{\lambda'_k}||_{\infty} - \sum_{n > k} ||f_{\lambda_n} - f_{\lambda'_n}||_{\infty}.$$  \hspace{1cm} (4.14)

Now Lemma 4.8 provides:

$$||f_{\lambda_k} - f_{\lambda'_k}||_{\infty} = \epsilon_k \quad \text{and} \quad \sum_{n > k} ||f_{\lambda_n} - f_{\lambda'_n}||_{\infty} \leq \sum_{n > k} \epsilon_n.$$  \hspace{1cm} (4.15)

Note that Eq. (4.9) implies:

$$\sum_{n > k} \epsilon_n \leq \frac{\epsilon_k}{R^{\nu - 1}} \leq \frac{1}{2}.$$  \hspace{1cm} (4.16)

As $R^\nu > 3$, it holds then $\frac{1}{R^{\nu - 1}} \leq \frac{1}{2}$ and consequently combining Eqs. (4.15) to (4.16) leads to:

$$||I(\Lambda) - I(\Lambda')||_{\infty} \geq \frac{1}{2} \epsilon_k (d(\Lambda, \Lambda')).$$

Since $\epsilon_k (d(\Lambda, \Lambda')) = \delta(\Lambda, \Lambda')$ by definition of $\delta$, the desired result comes. \hfill \Box
It remains to deduce the case $\alpha = 0$ from that previous one. For any $\beta > 0$, it holds $F_{d,k,\beta} \subset F_{d,k,0}$. From there, since Hausdorff scales are non-decreasing for inclusion, it holds then $\text{ord}_H F_{d,k,0} \geq \frac{d}{k+\beta}$. Since we can take $\beta > 0$ arbitrarily small, it indeed holds $\text{ord}_H F_{d,k,0} \geq \frac{d}{k}$.

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