Breakdown Properties of the $M$-Estimators of Multivariate Scatter

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Summary

The $M$-estimates of multivariate scatter are known to have breakdown points no greater than $1/(p + 1)$, where $p$ is the dimension of the data. In high dimension, the breakdown points are usually considered to be disappointingly low. This paper studies the breakdown problem in more detail. The exact breakdown points for the $M$-estimates of scatter are obtained and it is shown that their low values are primarily due to contamination restricted to some plane. If such “coplanar” contamination is not present, then there exists $M$-estimates which have breakdown points close to $1/2$. The effect of “coplanar” contamination is further examined and is shown to be related to the singularity of the scatter matrix. Finally, the implications of the results of this paper on whether the low breakdown point is necessarily a bad feature and on multivariate outlier detection are briefly discussed.

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1. Introduction.

The affine invariant $M$-estimates of multivariate location and scatter were first proposed by Maronna (1976) as robust alternatives to the sample mean vector and covariance matrix. One feature of these estimators, though, which was noted by Maronna (1976) and has been a concern to others, e.g., Huber (1981), Stahel (1981), Donoho (1982), and Devlin et al. (1981), is their relatively low breakdown point, particularly in higher dimensions. Maronna (1976) obtains an upper bound for the breakdown point of an $M$-estimator and shows that none have a breakdown point greater than $1/(p+1)$, where $p$ represents the dimension of the data. Stahel (1981) obtains a general bound of $1/p$ for a slightly more general class of $M$-estimators. Although much work has appeared on properties and applications of the $M$-estimators of multivariate location and scatter, there has been no further theoretical results on their breakdown properties.

The aim of this paper is to study the breakdown problem in more detail and to address the question: Is the low breakdown point necessarily a bad feature? The notion of breakdown is viewed here more as a descriptive rather than an optimal property. Attention is restricted to the $M$-estimates of scatter in this paper since the low breakdown point of the multivariate $M$-estimates is due to the breakdown of the scatter component, as demonstrated by both Maronna (1976) and Stahel (1981).

Loosely summarizing, it is shown in Section 3 that the upper bounds given by Maronna (1976) for the breakdown point of the $M$-estimates of scatter are in fact the exact breakdown points. In Section 4, the cause of the low breakdown point is investigated and is shown to be primarily due to contamination restricted to some plane, a type of contamination unique to the multivariate setting. In fact, if “coplanar” contamination is not present, then there exist $M$-estimates with breakdown points close to $1/2$ (Theorem 4.1). Furthermore, some $M$-estimates of scatter are shown to breakdown under a small percent of “coplanar” contamination, even though no “outliers” or “inliers” are present (Theorem 4.2). Section 5 examines the effect of “coplanar” contamination, which as one might expect, is related to the singularity of the scatter matrix.

After formally presenting the aforementioned results, some brief concluding remarks concerning their implications are made in Section 6. To begin, some background on the $M$-estimates of scatter and on finite sample breakdown is given.
2. Background.

2.1 $M$-estimators of scatter. For $p$-dimensional data $x_1, x_2, \ldots, x_n$, Maronna (1976) defines the affine invariant $M$-estimator of scatter about some fixed center $t$ to be the positive definite symmetric (p.d.s.) matrix $V_n$ satisfying the equation

\begin{equation}
V_n = \text{ave} \left\{ u(s_i)(x_i - t)(x_i - t)' \right\}
\end{equation}

where $s_i = (x_i - t)'V_n^{-1}(x_i - t)$ and $u$ is some scalar valued function. The $M$-estimator $V_n$ can be viewed as an adaptively weighted covariance matrix whose weights depend on an adaptive Mahalanobis distance from the center. For future reference, multiplying (2.1) by $V_n^{-1}$ and taking the trace gives

\begin{equation}
p = \text{ave} \left\{ \psi(s_i) \right\},
\end{equation}

where $\psi(s) = su(s)$. Also, let $K = \sup_{s>0} \psi(s)$.

Some conditions on the function $u$ and on the empirical distribution are needed to insure the existence and uniqueness of $V_n$. The existence lemma given below is from Tyler (1985).

**Condition 2.1.**

(i) $u(s)$ is non-negative, non-increasing and continuous for $s > 0$.

(ii) $u(s)$ and $s$ are bounded.

(iii) $\psi(s)$ is non-decreasing for $x > 0$ and strictly increasing for $\psi(s) < K$.

(iv) $K > p$.

Let $n_0$ represent the size of the largest subset of $X = \{x_1, x_2, \ldots, x_n\}$ which is in general position about the center $t$, where a set of vectors from $\mathbb{R}^p$ is said to be in general position about a fixed vector $t$ if the plane generated by any subset of size $p$ together with $t$ is $\mathbb{R}^p$. Let $P_n$ be the empirical distribution function of $\{(x_i - t); 1 \leq i \leq n\}$.

**Condition 2.2.** For any subspace $S$ with $0 \leq \text{rank}(S) \leq m - 1$,

(i) $P_n(S) < 1 - p/K + \min[1, n_0 \text{rank}(S)/n]/K$ and $n_0 > p(p - 1)$.

(ii) $P_n(S) \leq 1 - \{p = \text{rank}(S)\}/K$. 

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LEMMA 2.1. Suppose $u$ satisfies Condition 2.1.

(i) If Condition 2.2.i holds, then there exists a unique p.d.s. solution $V_n$ to (2.1).

(ii) If a p.d.s. solution $V_n$ exists to (2.1), then Condition 2.2.ii must hold.

(iii) If a p.d.s. solution $V_n$ exists to (2.1) and $n_0 > p$, then it is unique.

Lemma 2.1 essentially states nonexistence of $V_n$ is due to too many data points being coplanar with the center $t$.

Maronna (1976) and Huber (1981) also give sufficient conditions for existence of $V_n$. Huber’s condition on $u$ is more general than Condition 2.1. Both Huber’s and Maronna’s condition of $P_n$ are more restrictive than Condition 2.1.i.

2.2 Finite sample breakdown. A number of different definitions of the breakdown point of an estimator have been proposed since Hampel (1971) formally introduced the concept. Recently, Donoho (1982) and Donoho and Huber (1983) define the notation of finite sample breakdown in the following manner. Let $m$ arbitrary data points $Y = \{y_1, y_2, \ldots, y_m\}$ augment the original data $X = \{x_1, x_2, \ldots, x_n\}$ producing an $\epsilon$-contaminated sample $Z = X \cup Y$ consisting of a fraction of $\epsilon = m/(n+m)$ bad values.

For a given $\epsilon$, a statistic is said to breakdown under $\epsilon$-contamination if the difference between the statistic defined on the original sample $X$ and the statistic defined on the contaminated sample $Z$ can be made arbitrarily large in some sense for varying choices of $Y$. The finite sample breakdown point of the statistic at the sample $X$ is $\epsilon^*(X)$, the infimum of all $\epsilon$ producing breakdown.

Let $V_n(X)$ and $V_{n+m}(Z)$ represent p.d.s. solutions to (2.1) for the original data $X$ and the contaminated data $Z$ respectively whenever they exist. For $\epsilon = m/(n+m)$ and $V_n(X)$ existing, define the maximum “bias” at $X$ caused by $\epsilon$-contamination to be

$$b(\epsilon; X) = \begin{cases} \sup \{\text{trace}\{V_n+m(Z)V_n^{-1}(X) + V_n(X)V_{n+m}^{-1}(Z)\} \}, & Z \in S_m(X), \\ \infty, & Z \not\in S_m(X), \end{cases}$$

where $S_m(X) = \{Z = X \cup Y|V_{n+m}(Z) \text{ exists}\}$ and the supremum is taken over all choices of $Y$ and all possible solutions for $V_n(X)$ and $V_{n+m}(Z)$. Breakdown occurs under $\epsilon$-contamination whenever $b(\epsilon; X) = \infty$. This implies either the statistic $V_{n+m}(Z)$ does not exist, trace{V_{n+m}(Z)} can be made arbitrarily large or $V_{n+m}(Z)$
can be made arbitrarily close to the zero matrix or some other singular matrix. This notion of breakdown for a p.d.s. statistic is in agreement with the notion used by Maronna (1976), Stahel (1981), and Donoho (1982). The finite sample breakdown point of $V_n(X)$ at $X$ is defined to be

$$\epsilon^*(X) = \min_m \{ \epsilon = m/(n + m) | b(\epsilon, X) = \infty \}.$$  

To simplify notation, the results of this paper are stated in terms of $\delta^*(X)$ where $\delta^*(X)$ is defined to be a fraction such that $b(\epsilon, X) = \infty$ if $\epsilon > \delta^*(X)$ and $b(\epsilon, X) < \infty$ if $\epsilon < \delta^*(X)$. Since the possible values of $\epsilon$ are discrete, $\delta^*(X)$ is not uniquely defined.

The relationship between $\delta^*(X)$ and $\epsilon^*(X)$ is easily shown to be

$$\delta^*(X) \leq \epsilon^*(X) < \left[ \delta^*(X) + \{ \delta^*(X) \} / n \right] / \left[ 1 + \{ 1 - \delta^*(X) \} / n \right].$$

3. The Breakdown Point of $V_n(X)$.

Hereafter, assume that the “good” data $X$ is in general position about $t$, which occurs almost surely when sampling from a continuous distribution in $\mathbb{R}^p$. This assumption concerning $X$ is also used by Donoho (1982) in studying the finite sample breakdown properties of projection pursuit based estimators of location and scatter. It is also assumed hereafter that $n > p(p - 1)$. By Lemmas 2.1.i and 2.1.ii, these assumptions assure the existence and uniqueness of $V_n(X)$ and the uniqueness of $V_{n+m}(Z)$ if it exists.

The general breakdown point of $V_n(X)$ is given in Theorem 3.1 below. Before presenting the theorem some lemmas concerning the existence and behavior of $V_{n+m}(Z)$ are given. The proofs of the lemmas are given in the appendix. For brevity, let $\epsilon_m = m/(n + m)$, and in all proofs assume without loss of generality that $t = 0$.

**Lemma 3.1.** $Z \in S_m(X)$ if either

(i) $\epsilon_m < 1 - p/K$ and $n \geq K$,

(ii) $\epsilon_m < 1 - np/[nK - (K - n)(p - 1)]$ and $n < K$, or

(iii) $\epsilon_m < 1 - \max\{np/[nK + (n - K)], n(p-1)/[(n-p+1)K]\}$, $n > K$ and $t \notin Z$. 


Lemma 3.2.

(i) If \( \epsilon_m < 1 - p/K \), then \( \{ \text{trace } V_{n+m}^{-1}(Z) | Z \in S_m(X) \} \) is bounded above.

(ii) If \( \epsilon_m < 1/K \), then \( \{ \text{trace } V_{n+m}(Z) | Z \in S_m(X) \} \) is bounded above.

(iii) If \( \epsilon_m < p/K \), then \( \{ \text{trace } V_{n+m}^{-1}(Z) | Z \in S_m(X) \} \) is bounded away from zero.

Theorem 3.1. \( \delta^*(X) = \min\{1/K, 1 - p/K\} \) for \( n + 1 > K \), and \( \delta^*(X) = 0 \) for \( n + 1 \leq K \).

Proof: If \( Y = \{0, 0, \ldots, 0\} \) and \( \epsilon_m > 1 - p/K \), then \( P_{n+m}(0) = \epsilon_m > 1 - p/K \).
This implies by Lemma 2.1.ii that \( V_{n+m}(Z) \) does not exist. If \( Y = \{y, y, \ldots, y\} \) and \( V_{n+m}(Z) \) exists, then \( y'Ay = \text{ave}\{u(z'Az)(y'Az)^2\} \) where \( A = \{V_{n+m}(Z)\}^{-1} \) and the average is over \( z \in Z \). This implies \( 1 - \text{ave}\{u(z'Az)(y'Az)^2/z'Az\} \) or
\[
1 = \epsilon_m \psi(y'Ay) + (n + m)^{-1} \sum_{1 \leq i \leq n} u(x_i'Ax_i) \times (x_i'Ay)^2/y'Ay.
\]
Express \( y = r\theta \) where \( \theta' \theta = 1 \) and let \( r \to \infty \). If \( V_{n+m}(Z) \) does not breakdown as \( r \to \infty \), the \( \psi(y'Ay) \to K \) and since \( X \) spans \( \mathbb{R}^p \), for some \( x \in X \),
\[
u(x'Ax)(x'Ay)^2/y'Ay = u(x'Ax)(x'A\theta)^2/\theta' A \theta \]
does not go to zero. This implies \( \epsilon_m < 1/K \) and when using \( m = 1, n + 1 > K \). Thus \( \delta^*(X) \leq \min(1 - p/K, 1/K) \) and if \( n + 1 \leq K \), \( \delta^*(X) = 0 \).

If \( \epsilon_m < \min(1/K, 1 - p/K) \), then by Lemma 3.1, \( z \in S_m(X) \). Application of Lemma 3.2 gives \( \delta^*(X) \geq \min(1/K, 1 - p/K) \). □

Maronna (1976) obtains \( \min\{1/K, 1 - p/K\} \) as an upper bound of the \( M \)-estimator of scatter at any continuous elliptically contoured distribution in \( \mathbb{R}^p \), and conjectures that the bound is the exact breakdown point. He uses the definition of breakdown at a model distribution rather than finite sample breakdown. The arguments given in the proof of Lemma 3.2 and Theorem 3.1 can be modified to show that for the case of known center the breakdown point of the \( M \)-estimate of scatter is equal to \( \min\{1/K, 1 - p/K\} \) at any continuous model in \( \mathbb{R}^p \). Maronna further states that this upper bound is obtained by letting a point mass contamination go to infinity. This is true for the \( 1/K \) term but not the \( 1 - p/K \) term, which is obtained by point mass contamination at the center. As noted by Maronna, the breakdown point is low for higher dimensions since \( K > p \) and so \( \min\{1/K, 1 - p/K\} \leq 1/(p + 1) \).
4. The Sources of Breakdown.

The objective of this section is to investigate what causes the \( M \)-estimate of multivariate scatter to breakdown. For univariate scale problems, breakdown is usually due to the existence of too many outliers or to the existence of too many inliers, that is, data points near the center. In the multivariate setting, though, breakdown may also occur because of too many data points lying in some lower dimensional plane containing the center of \( t \), which will be referred to as coplanar contamination. By examining the proof of Theorem 3.1, one can note that the low overall breakdown point of the \( M \)-estimates of scatter, that is the \( 1/K \) term, is obtained by outliers which are coplanar with the center. If coplanar contamination is not present, then it is shown in Theorem 4.1 below that some \( M \)-estimators of scatter can have breakdown points close to 1/2. Before formally presenting this result, some additional notation and definitions are needed.

Let \( C_m(X) \) be a subset of the product set \( \prod_{j=1}^{m} \mathbb{R}^p \), possibly dependent on \( X \). Define the finite sample breakdown point of \( V_n \) at \( X \) due to a sequence \( C_1(X), C_2(X), \ldots \) to be \( \epsilon^*(X) \) where \( \epsilon^*(X) \) is defined by (2.4) but with the restriction \( Y \in C_m(X) \) in the definition of \( b(\epsilon, X) \).

An element \( z \neq t \) from \( \mathbb{R}^p \) can be expressed as \( z = t + \theta \theta' \) where \( \theta = (z - t)/r \) and \( r = ((z - t)'(z - t))^{1/2} \). Using this representation, define for \( Z = X \cup Y = \{z_1, z_2, \ldots, z_{n+m}\} \)

\[
(4.1) \quad \rho_m(Z) = \min \lambda_p \left\{ \sum_{j=1}^{p} \theta_{i(j)} \theta'_{i(j)} \right\}
\]

where the minimum is taken over all subsets of size \( p \) from \( Z \) for which \( z \neq t \), and \( \lambda_p(\cdot) \) represents the smallest eigenvalue of the \( p \times p \) non-negative definite argument. The quantity \( \rho_m(Z) \neq 0 \) if and only if \( \{z|z \in Z, z \neq t\} \) is in general position about \( t \). Also, define for \( Y = \{y_1, y_2, \ldots, y_m\} \)

\[
(4.2) \quad r_m(Y) = \min \{(y_i - t)'(y_i - t), 1 \leq i \leq m\},
\]

and let \( C_{1,\rho,m}(X) = \{Y|\rho_m(X \cup Y) > \rho\}, C_{2,r,m}(X) = \{Y|r_m(Y) > r\} \) and \( C_{3,r,m} = \{Y|r_m(Y) < B\} \).
Some results concerning the behavior of $V_{n+m}(Z)$ when $Y$ is restricted to certain classes are given in the following lemma. The proof of the lemma is given in the appendix.

**Lemma 4.1.** Let $\rho > 0$, $r > 0$ and $B < \infty$.

(i) If $n + 1 > K$ and $\epsilon_m < p/K$, then $\{\text{trace } V_{n+m}(Z) | Z \in S_m(X) \text{ and } Y \in C_{1,p,m}(X)\}$ is bounded above.

(ii) For any $m$, $\{\text{trace } V_{n+m}(Z) | Z \in S_m(X) \text{ and } Y \in C_{3,B,m}(X)\}$ is bounded above.

(iii) For any $m$, $\{\text{trace } V_{n+m}(Z) | Z \in S_m(X) \text{ and } Y \in C_{2,r,m}(X)\}$ is bounded away from zero.

(iv) If $\epsilon_m < 1 - n(p-1)/\{(n-1)K\}$, then $\{\text{trace } V_{n+m}^{-1}(Z) | Z \in S_m(X) \text{ and } Y \in C_{2,r,m}(X)\}$ is bounded above.

(v) If $n + 1 > K$, then for any $m$, $\{\text{trace } V_{n+m}^{-1}(Z) | Z \in S_m(X) \text{ and } Y \in C_{1,p,m}(X) \cap C_{2,r,m}(X)\}$ is bounded above.

The following results concerning the breakdown of $V_n(X)$ whenever coplanar contamination is not present are similar to the breakdown results for univariate scale. Estimators which protect against outliers, i.e. $K$ near $p$, tend to breakdown in the presence of inliers. For the compromising choice $K = 2p$, the breakdown point given in Theorem 4.1.iii is approximately $1/2$.

**Theorem 4.1.** Let $\rho > 0$, $r > 0$ and $B < \infty$.

(i) For the sequence $C_{1,p,m}(x) \cap C_{2,r,m}(X)$, $\delta^*(X) = p/K$ if $n + 1 > K$ and $\delta^*(X) = 0$ if $n + 1 \leq K$.

(ii) For the sequence $C_{2,B,n}(X)$, $\delta^*(X) = 1 - p/K$ for $n \geq K$ and $1 - np/\{nK - (K-n)(p-1)\} \leq \delta^*(X) \leq 1 - p/K$ for $n < K$.

(iii) For the sequence $C_{1,p,m}(X)$, $\delta^*(X) = \min(p/K, 1 - p/K)$ if $(n+1) > K$ and $\delta^*(X) = 0$ if $n + 1 \leq K$.

**Proof:** (i) If $y \in C_{1,p,m}(X) \cap C_{2,r,m}(X)$, then it can be verified from Lemma 2.1.i that $Z \in S_m(X)$ for $n > p(p-1)$ after noting $P_{n+m}(S) \leq \text{rank}(S)/(n+m)$ and $(n+m)_0 = n + m$. For $n + 1 > K$, it follows from Lemmas 4.1.i and 4.1.v...
that $\delta^*(X) \geq p/K$. For (2.2), $p \geq (n + m)^{-1} \sum_{1 \leq i \leq m} \psi(y_i' V_{n+m}^{-1}(Z)y_i) \rightarrow e_m$ if $V_{n+m}(Z)$ does not breakdown as $y_i' y_i \rightarrow \infty$, $1 \leq i \leq m$, and so $\delta^*(X) \leq p/K$. For $n + 1 \leq K$, the proof that $\delta^*(X) = 0$ is analogous to the proof in Theorem 3.1.

(ii) As in the proof of Theorem 3.1, the upper bound for $\delta^*(X)$ is obtained by choosing $Y = \{0, 0, \ldots, 0\}$. The lower bound follows from Lemmas 3.1, 3.2.i and 4.1.i.

(iii) The lower bound follows from Lemmas 3.1.i, 3.2.i and 4.1.i. The upper bound follows from parts (i) and (ii) of this theorem. ☐

Theorem 4.1.i generalizes a statement made by Maronna (1976) in which he quotes $p/K$ as the breakdown point due to contaminating a spherically contoured model distribution by a long-tailed spherically contoured distribution.

An interesting aspect to the multivariate breakdown problem is that breakdown can occur because of coplanar contamination, even though the contamination contains no outliers or inliers. In fact, as seen in the next theorem, the breakdown point due to such contamination can be quite low.

**Theorem 4.2.** Let $a_1 = 1 - np/(nK - (K - n)(p - 1))$, $a_2 = 1 - np/(nK + n - K)$, $a_3 = 1 - n(p - 1)/\{(n - p + 1)K\}$, and $a_4 = 1 - n(p - 1)/\{(n - 1)K\}$. For the sequence $C_{2,r,m}(X) \cap C_{3,B,m}(X)$, with $r > 0$ and $B < \infty$, $\min(a_2, a_3) \leq \delta^*(X) \leq a_4$ when $n > K$, and $a_1 \leq \delta^*(X) \leq a_4$ when $n \leq K$.

**Proof:** The upper bound is obtained by letting $Y = \{x_1, x_1, \ldots, x_1\}$ and applying Lemma 2.1.ii. The lower bounds follow from Lemmas 3.1.ii, 3.1.iii, 4.1.ii and 4.1.iv. ☐

As $n$ goes to infinity, the bounds on $\delta^*(X)$ in Theorem 4.2 simplify to

\[(4.3) \quad 1 - p/(K + 1) \leq \delta^*(X) \leq 1 - (p - 1)/K.\]

For $K$ near $p$, the breakdown point is in the neighborhood of $1/(p + 1)$ to $1/p$.

It is interesting to note that for $K$ near $p$, the influence of non-coplanar outliers is essentially nonexistent; see Theorem 4.1.i. Furthermore, for such $K$ the breakdown points given in Theorem 4.2 do not differ greatly from $1/K$. Therefore, for such M-estimators of scatter the low breakdown point caused by coplanar outliers can be at-
tributed primarily to the coplanar aspect of the contamination rather than the outlier aspect. A brief heuristic explanation is helpful in understanding this phenomena. The defining equation (2.1) can be rewritten as

\[(4.4) \quad V_n = n^{-1} \sum_{i=1}^{n} \psi(s_i)(x_i - t)(x_i - t)' / s_i\]

where the summation is over \(x_i \neq t\). The function \(\psi\) can be viewed as measuring the influence of the distance of an observation from the center. The term \((x_i - t)(x_i - t)' / s_i\) is dependent only on the direction of the observation from the center and not on the distance. Since \(\psi\) is non-decreasing, if \(K\) is near \(p\), then (2.2) implies that \(\psi\) is roughly a constant function. Thus, outliers have little more influence than other data points and breakdown is primarily dependent on the interrelationships of the directions of the data points from the center.

5. The Effect of Coplanar Contamination.

The notion of contamination which is coplanar with the center distinguishes the multivariate breakdown problem from the univariate one. Intuitively, one might expect such contamination would be related in some way to the singularity of the estimate of scatter. In this section, this intuition is briefly but formally investigated.

The difference in the breakdown points in Theorem 3.1 and Theorem 3.2.iii can be attributed to the existence of outliers which are coplanar with the center, and by Theorem 4.2 cannot be attributed to coplanar contamination alone. For \(n + 1 > K\), and

\[(5.1) \quad \min(1/K, 1 - p/K) < \epsilon_m < \min(p/K, 1 - p/K),\]

Lemma 3.1.ii implies \(Z \in S_m(X)\), and furthermore Lemmas 3.2.i and 3.2.iii imply that there exists a nonzero non-negative definite symmetric matrix \(A_1\) and a positive definite symmetric matrix \(A_2\) such that for all \(Z \in S_m(X)\)

\[(5.2) \quad A_1 < V_{n+m}^{-1}(Z) < A_2,\]

where the ordering refers to the partial ordering of symmetric matrices. Thus, for \(n + 1 > K\) and (4.2) holding, “coplanar outliers” tend to make \(V_{n+m}^{-1}(Z)\) singular.
rather than “blowing up” or becoming strictly zero. A natural question which arises is
whether the limiting null space of \( V^{-1}_{n+m}(Z) \) and the contaminating plane coincide. For
the following case, which produces the \( 1/K \) term in Theorem 3.1, they do. The proof
is given in the appendix.

**Theorem 5.1.** For fixed \( m \), let \( Z_r = X \cap Y_r \) where \( Y_r = \{ y_r, y_r, \ldots, y_r \} \),
\( y_r = r \theta + t \) and \( \theta' \theta = 1 \). If \( n + 1 > K \) and (5.2) holds, then \( \theta' V^{-1}_{n+m}(Z_r) \theta \to 0 \) as
\( |r| \to \infty \), and if \( a \) is not proportional to \( \theta \), then \( \inf_r a' V^{-1}_{n+m}(Z_r) a > 0 \).

As shown in Theorem 4.2, contamination within some hyperplane containing the
center \( t \) can cause breakdown, even though no outliers or inliers are present. Break-
down by such contamination is due to either nonexistence or to the \( M \)-estimate of scatter
tending toward singularity. However, the estimate does not tend to zero nor does it
become arbitrarily large. To state this formally, let \( r > 0 \) and \( B < \infty \). Lemmas 4.1.ii
and 4.1.iii imply there exists a nonzero non-negative definite symmetric matrix \( W_1 \) and
a positive definite symmetric matrix \( W_2 \) such that for all \( Y \in C_{2,r,m}(X) \cap C_{3,B,m}(X) \)
\( (5.3) \)
\[ W_1 < V_{n+m}(Z) < W_2 \]
provided \( Z = X \cup Y \in S_m(X) \). Furthermore, if one considers a sequence \( Y_k \in C_{2,r,m}(X) \cap C_{3,B,m}(X) \) such that \( Z_k = X \cup Y_k \in S_m(X) \) and \( Z_k \to Z \in S_m(X) \),
then (5.3) and the continuity of \( u \) imply that the largest root of \( V_{n+m}(Z_k) \) is bounded
away from zero and infinity for all \( k \), and the smallest root tends to zero. Again,
a natural question which arises is whether the limiting range of \( V_{n+m}(Z_k) \) and the
contaminating plane coincide. For the following case, which produces the upper bound
in Theorem 3.3, they do. The condition \( \epsilon_m < 1 - n(p-2)/\{ K(n-2) \} \) is probably not
needed in the following theorem, but the author is not able to derive the result without
this condition. The proof is given in the appendix.

**Theorem 5.2.** For fixed \( m \), let \( Z_k = X \cup Y_k \) where \( Y_k = \{ x_1 + w_{1,k}, x_1 + w_{2,k}, \ldots, x_1 + w_{m,k} \} \) with \( w_{i,k} \to 0 \), \( 1 \leq i \leq m \), and \( Z_k \in S_m(X) \) for all \( k \). If
\[ 1 - n(p-1)/\{ K(n-1) \} < \epsilon_m < 1 - n(p-2)/\{ K(n-2) \}, \]
then \( \inf_k x_1' V_{n+m}(Z_k) x_1 > 0 \), and for any \( a \) such that \( a' x_1 = 0 \), \( a' V_{n+m}(Z_k) a \to 0 \).
6. Concluding Remarks.

Is the low breakdown point necessarily a weak feature of an $M$-estimate of multivariate scatter? One can respond yes if it is believed that contamination lying in or near some lower dimensional plane is feasible and no attempt is made to detect such contamination. Otherwise, $M$-estimators exist which have good breakdown properties.

An alternative or complimentary approach is to try to detect bad data points, particularly outliers. The results of Section 6 suggest that if a group of outliers lie in or near some lower dimensional plane, then the near singularity of an $M$-estimate of scatter can be used to help detect such systematic outliers, with the directions associated with the largest roots indicating where to search for the outliers. More research along these lines may be fruitful. If outliers exist which are not coplanar, then their detection may be more difficult. $M$-estimates of scatter exist, though, which are quite stable under such contamination.

Finally, if coplanar contamination is present, with or without outliers, it may be desirable to note this rather than simply attempt to summarize the data via a location and scatter statistics. Again the results of Section 6 suggest the near singularity of an $M$-estimate of scatter may indicate the existence of such systematic contamination, with the directions associated with the larger roots coinciding with the contaminating plane.

7. Appendix: Some Proofs.

The proofs of Lemmas 3.1, 3.2, 3.3 and 4.1 and Theorems 5.1 and 5.2 are given in this appendix. In all proofs, without loss of generality, $t$ is set equal to 0. Recall $X$ is assumed to be in general position about $t = 0$, and $n > p(p - 1)$.

**Proof of Lemma 3.1:** This lemma follows from Lemma 2.1.i after noting that $P_{n+m}(S) \leq \{m + \text{rank}(S)\}/(n + m) \geq n$. Further, if $t \notin Z$, then $P_{n+m}(0) = 0$. These above statements are true since $X$ is in general position about the center.

**Proof of Lemma 3.2:** Consider any sequence $V_{n+m}(Z_k)$ where $Z_k = X \cup Y_k \in S_m(X)$. Let $\alpha_k = \text{trace} \{V_{n+m}^{-1}(Z_k)\}$ and $\Gamma_k = V_{n+m}^{-1}(Z_k)/\alpha_k$. Since $\text{trace} \Gamma_k = 1$, there exists a convergent sub-sequence, say for $j \in J$, $\Gamma_j \rightarrow \Gamma$ a positive semi-definite symmetric (p.s.d.s.) matrix with $\text{trace} (\Gamma) = 1$. Let $Y_k = \{y_{i,k}; 1 \leq i \leq m\}$. 


\( r_{i,k} = (y_i^* y_{i,k})^{1/2} \) and for \( r_{i,k} \neq 0 \), let \( \theta_{i,k} = y_{i,k} / r_{i,k} \). If \( r_{i,k} = 0 \), define \( \theta_{i,k} = e \), a vector such that \( e^* e = 1 \) and \( \Gamma e \neq 0 \). Such a vector exists since \( \text{trace} \Gamma = 1 \) and so \( \Gamma \neq 0 \). Since \( \theta_{i,k}^* \theta_{i,k} = 1 \), the sub-sequence \( J \) can be chosen so that for each \( 1 \leq i \leq m \), \( \theta_{i,j} \rightarrow \theta_i \) with \( \theta_i^* \theta_i = 1 \). Pre- and post-multiply (2.1) by \( \Gamma_{i,k}^{1/2} \Gamma_{i,k}^{-1/2} \), where \( \Gamma_{i,k}^{-1/2} \) is the unique p.s.d.s. square root of \( \Gamma_i \), and then multiply by the orthogonal projection into the null space of \( \Gamma_i \), say \( P_i \). This gives

\[
(7.1) \quad P_i = \text{ave} \left\{ u(\alpha_k \Gamma_i \Gamma_k) \alpha_k P_i \Gamma_{i,k}^{1/2} \Gamma_k \Gamma_{i,k}^{1/2} P_i \right\}
\]

where the average is over \( z \in Z_k \). Taking the trace gives \( \text{rank} (P_i) = \text{ave} \{ \psi_k(z) \} \), where \( \psi_k(z) = u(\alpha_k \Gamma_i \Gamma_j) \alpha_k \Gamma_{i,k}^{1/2} \Gamma_k \Gamma_{i,k}^{-1/2} \Gamma_j \Gamma_{i,j} \). If \( \Gamma x_i \neq 0 \), then

\[
\psi_k(x_i) = \psi(\alpha_j x_i^* \Gamma_j x_i) \times x_i^* \Gamma_j \Gamma_{i,j} \Gamma_j \Gamma_{i,j} x_i / x_i^* \Gamma_j x_i,
\]

which goes to zero since \( \psi \) is bounded and \( P_i \Gamma_{i,k}^{-1/2} = 0 \). Likewise, if \( \Gamma \theta_i \neq 0 \), then

\[
\psi_j(y_{i,j}) = \psi(\alpha_j y_{i,j} \Gamma_j y_{i,j}) \theta_i \Gamma_j \Gamma_{i,j} \Gamma_j \Gamma_{i,j} / \theta_i^* \Gamma_j \Gamma_j \Gamma_{i,j} \theta_i \rightarrow 0,
\]

and so

\[
(7.2) \quad \lim_{j \rightarrow j} \frac{(n + m)^{-1}}{\sum z \in Z_{0,j}} \psi_j(z) = R_{\Gamma},
\]

where \( Z_{0,j} = \{ x_i | \Gamma x_i = 0 \} \cup \{ y_{i,j} | \Gamma \theta_i = 0 \} \) and \( R_{\Gamma} = \text{rank} (P_i) \).

From (2.2), \( p \geq (n + m)^{-1} \{ \sum z \in Z_{0,j} \psi(\alpha_j \Gamma_j z) + \sum_{\Gamma x_i \neq 0} \psi(\alpha_j x_i^* \Gamma_j x_i) \} \). If \( \Gamma x_i \neq 0 \) and \( \alpha_j \rightarrow \infty \) for \( j \in J \), then \( \psi(\alpha_j x_i^* \Gamma_j x_i) \rightarrow 0 \). Since \( P_i \) is idempotent, \( \psi_j(z) \leq u(\alpha_j \Gamma_j z) \alpha_j \Gamma_j z = \psi(\alpha_j \Gamma_j z) \). These results, together with (7.2) and the assumption that \( X \) is in general position about the origin, imply that if \( \alpha_j \rightarrow \infty \) for \( j \in J \), then \( p \geq R_{\Gamma} + (n - R_{\Gamma}) K/(n + m) \). This last inequality is equivalent to

\[
(7.3) \quad \epsilon_m \geq 1 - mn(p - R_{\Gamma})/(K(n - R_{\Gamma})�)
\]

(i) The right-hand side of (7.3) is an increasing function or \( R_{\Gamma} \) for \( 0 \leq R_{\Gamma} \leq p \). Thus, if \( \epsilon_m < 1 - p/K \), then \( \epsilon_k \) must be bounded above.

(ii) If \( \epsilon_k \) is not bounded above, then \( J \) can be chosen so that \( \alpha_j \rightarrow \infty \) for \( j \in J \). The right-hand side of (7.3) is greater than \( 1/K \) unless \( R_{\Gamma} = 0 \) and thus \( \Gamma \) is nonsingular. This implies \( \text{trace} \{ V_{n+m}(Z_j) \} = \text{trace} \Gamma_{j}^{-1} / \alpha_j \rightarrow 0 \).

If \( \alpha_j \) is bounded above, then \( J \) can be chosen so that \( \alpha_j \rightarrow \alpha \) for \( j \in J \). This implies for \( \Gamma x_i = 0 \), \( \psi_j(x_i) \leq \psi(\alpha_j x_i^* \Gamma_j x_i) \rightarrow \psi(0) = 0 \) and so from (7.2),
rank \((P_T) \leq \epsilon_m K\). This contradicts the condition on \(\epsilon_m\) unless rank \((P_T) = 0\) and thus \(\Gamma\) is nonsingular. This implies trace \(\{V_{n+m}(Z_j)\} \to \text{trace} \Gamma^{-1/\alpha}\) unless \(\alpha = 0\).

If \(\alpha = 0\), then \(\psi(\alpha_j x_j' \Gamma x_j) \to 0\) and so by (2.2), \(p \leq \epsilon_m K\), which contradicts the condition on \(\epsilon_m\).

(iii) If \(\alpha_j\) is not bounded away from zero, the \(J\) can be chosen so that \(\alpha_j \to 0\) for \(j \in J\). Using (7.2), this implies \(R_T = 0\) and so \(\Gamma\) is nonsingular. By (2.2), this implies \(p \leq \epsilon_m K\), a contradiction.

**Proof of Lemma 4.1:** The notation developed in the proof of Lemma 3.1 is used.

(i) Statement (7.2) implies \(R_T \leq R_T K/(n+m)\) since if \(Y_j \in C_{1,p,m}(X)\), then \(Z_{0,j}\) has at most \(R_T\) nonzero elements. This implies \(R_T = 0\), otherwise \(n+1 \leq n+m \leq K\).

The remainder of the proof is similar to the proof of Lemma 3.2.ii.

(ii) Since \(u\) is non-increasing, trace \(V_{n+m}(Z) \leq Bu(0)\).

(iii) If trace \(V_{n+m}(Z)\) is not bounded away from zero, then there exists a sequence \(V_{n+m}(Z_j) \to 0\). This implies \(p = \text{ave}\{\psi(x V_{n+m}^{-1}(Z_j) z)\} \to K > p\), a contradiction.

(iv) If trace \(V_{n+m}(Z)\) is not bounded above, then (7.3) holds. This contradicts the condition on \(\epsilon_m\) unless \(R_T = 0\). However, if \(R_T = 0\) and \(\alpha_j \to \infty\), then \(p = \text{ave}\{\psi(\alpha_j x_j' \Gamma_j z)\} \to K > p\), a contradiction.

(v) The same argument used for (i) implies \(R_T = 0\), then the same argument used for (iii) implies trace \(V_{n+m}^{-1}(Z)\) must be bounded.

**Proof of Theorem 5.1:** Lemma 3.1.ii insures the existence and uniqueness of \(V_{n+m}^{-1}(Z_k)\). Statement (5.2) implies a sequence \(J\) exists such that for \(j \in J\), \(A_n = V_{n+m}^{-1}(Z_{r(j)}) \to A > 0\). Arguments similar to those used in the proof of Theorem 3.1 give \(1 \leq \epsilon_m \psi(r(j) \theta' A_j \theta)\). Unless \(\theta' A \theta = 0\), \(\psi(r(j) \theta' A_j \theta) \to A\), which contradicts (5.1). Thus, \(\theta' V_{n+m}^{-1}(Z_r) \theta \to 0\) as \(r \to \infty\).

The notation developed in the proof of Lemma 3.2 is used in the remainder of this proof. Note that \(A = \alpha \Gamma\) with \(o < \alpha < \infty\).

By (2.2), \(\epsilon_m \psi(r(j) \alpha_j \theta' \Gamma_j \theta) \to c\) where \(c = p-(n+m)^{-1} \sum_{1 \leq i \leq n} \psi(\alpha x_i' \Gamma x_i)\).

The index set \(J\) can be chosen so that \(\Gamma_j^{1/2} \theta / (\theta' \Gamma_j \theta)^{1/2} \to \phi\) with \(\phi' \phi = 1\). Taking the limit in (7.1) over \(j \in J\) and recalling \(P_T \Gamma^{1/2} = 0\) implies \(P_T = cP_T \phi \phi' P_T\) and hence rank \((P_T) = 1\) or rank \((\Gamma) = p - 1\). Thus, for a not proportional to \(\theta\), \(\alpha' V_{n+m}^{-1}(Z_r) a\) is bounded away from zero.
Proof of Theorem 5.2: The notation developed in the proof of Lemma 3.2 is used. By (5.3), the subsequence $J$ can be chosen so that $V_{n+m}(Z_j) \to V$, a nonzero positive semi-definite matrix. The matrix $V$ must be singular, otherwise since $u$ is continuous Lemma 2.1.ii is contradicted when the limit is taken. This implies $\alpha_j \to \infty$. Since $x_i'\Gamma_j x_1 = \text{ave} \{u(\alpha_j x_i' \Gamma_j z) \alpha_j (x_i' \Gamma_j z)^2\}$, where the average is over $z \in Z_j$, if $\Gamma x_1 \neq 0$ then taking the limit in the above statement gives $x_i'\Gamma x_1 \geq (m+1)K(x_i'\Gamma x_1)^2/(n+m)$ or $n+m \geq (m+1)K$. This contradicts the lower bound on $\epsilon_m$ and thus $\Gamma x_1 = 0$. The upper bound on $\epsilon_m$ and (7.3) imply $\text{rank} (\Gamma) = p - 1$, and so since $X$ is in general position about the center, $\Gamma x_i \neq 0$ for $i \neq 1$. This implies $\alpha_j x_i'\Gamma_j x_i \to \infty$ or $u(\alpha_j x_i' \Gamma_j x_i) \to \infty$ for $1 \neq 1$, and thus if $a'x_1 = 0$, then $a'V_{n+m}(Z_j)a \to a'Va = 0$. Since $V$ is nonzero, $x_i'V x_1 > 0$. The theorem follows since the arguments can be applied to any convergence subsequence of $a'V_{n+m}(Z_j)a$ for $a'x_1 = 0$ or for $a = x_1$.

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