Generalized Killing Tensors and Symmetry of Klein–Gordon–Fock Equations\textsuperscript{1}

Anatoly G. Nikitin and Oleksander I. Prylypko

Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivs'ka Str., Kyiv, 01601 Ukraine
E-mail: nikitin@imath.kiev.ua, URL: \url{http://www.imath.kiev.ua/~nikitin/}

Abstract

The paper studies non-Lie symmetry of the Klein–Gordon–Fock equation (KGF) in $(p + q)$-dimensional Minkowsky space. Full set of symmetry operators for the $n$-order KGF equation was explicitly calculated for arbitrary $n < \infty$ and $p + q \leq 4$.

Definition was given for generalized Killing tensors of rank $j$ and order $s$, and for generalized conformal Killing tensors of rank $j$ and order $s$ as a complete set of linearly independent solutions of some overdetermined systems of PDE. These tensors were found in explicit form for arbitrary fixed $j$ and $s$ in Minkowsky space of dimension $p + q \leq 4$. The received results can be used in investigation of higher symmetries of a wide class of systems of partial differential equations.

Introduction

Classical group theoretical analysis of differential equations whose foundations were laid by Sophus Lie over a hundred years ago finds increasing utilization in modern mathematical physics (see e.g. \cite{1, 2, 3}). At the same time certain limits of the classical Lie approach become obvious that nevertheless do not allow full description of the symmetry of an equation under study \cite{4, 5}. In particular, does not allow calculation of higher order symmetry operators that are widely used for calculation of reference frames admitting solution of equations in separated variables \cite{6, 7, 8}, in calculation of motion constants \cite{9} and in many other problems.

The present paper deals with investigation of non-Lie symmetry of the Klein–Gordon–Fock equation in $(p + q)$-dimensional Minkowsky space

\[ L\varphi \equiv \left(g^{\mu\nu} \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_\nu} - \kappa^2\right) \varphi = 0, \]  

where $\kappa$ is a real parameter,

\[ g^{\mu\nu} = \begin{cases} 0, & \mu \neq \nu, \\ 1, & \mu = \nu \leq p, \\ -1, & p < \mu = \nu \leq p + q, \end{cases} \]

\[ \varphi = \varphi(x_1, x_2, \ldots, x_{p+q}) \] is a function of $p + q$ variables.

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A symmetry operator of equation (1) is an arbitrary operator \( Q \) (linear, nonlinear, differential, integral) that transforms solutions of this equation into solutions, that is \( L(Q\varphi) = 0, \quad \text{if} \quad L\varphi = 0 \) (3)

(see Section 1 below for more rigorous definition).

We will call a differential operator of a finite order \( n \) being a symmetry operator of equation (1) a \( n \)-th order symmetry operator.

Description of the maximal (in the sense of Lie) symmetry of equation (1) may be reduced to finding of all linearly independent first order symmetry operators. Such operators are well-known, they form a basis of the Lie algebra of the generalized Poincaré group \( P(p,q) \) (for \( \kappa \neq 0 \)) or for the conformal group in \((p + q)\)-dimensional space (when \( \kappa = 0 \)).

One of the main results of the present paper is calculation in explicit form of a complete set of \( n \)-th order symmetry operators of equation (1) for arbitrary \( n < \infty \) and \( p + q \leq 4 \).

It is well-known that description of first order symmetry operators is based upon calculation of explicit form of the Killing vector \( [10, 2] \) that corresponds to the space of independent variables. We associate with higher order symmetry operators more complex fundamental objects that we call Killing tensors of rank \( j \) and order \( s \), with \( j, s = 1, 2, \ldots \) and conformal Killing tensors of rank \( j \) and order \( s \).

In this paper we give the definition of the mentioned tensors as a a complete set of linearly independent solutions of some overdetermined systems of PDE and find these tensors were found in explicit form for arbitrary fixed \( j \) and \( s \) in Minkowsky space of dimension \( p + q \leq 4 \). The results can be used in investigation of higher symmetries of a wide class of systems of partial differential equations of mathematical physics given in the same space, in particular, of relativistic and galilei-invariant wave equations.

Let us describe briefly arrangement of our presentation. Main definitions related to higher order symmetry operators, are adduced in Section 1, definition of Killing tensors of rank \( j \) and order \( s \) is given in Section 2, first order Killing tensors of rank \( j \) and order \( s \) in explicit form are found in Sections 3 and 4, conformal Killing tensors and Killing tensors of arbitrary rank and order are shown in Sections 6, 8 and 9. A complete set of \( n \)-th order symmetry operators for equation (1) with zero and non-zero “mass” \( \kappa \) are adduced in Sections 5 and 7.

1 Symmetry operators of order \( n \)

For the purpose of our study it is sufficient to consider only solutions of equation (1) defined on an open set \( D \) of the four-dimensional manifold \( \mathbb{R}_{p+q} \) consisting of points with co-ordinates \((x_1, x_2, \ldots, x_{p+q})\) and analytical with respect to real variables \( x_1, x_2, \ldots, x_{p+q} \). The set of all such solutions forms a complex vector space that we designate by the symbol \( \mathcal{F}_0 \). Setting \( D \) as fixed (e.g. assuming that \( D \) coincides with \( \mathbb{R}_{p+q} \)), we will call \( \mathcal{F}_0 \) the set of solutions of equation (1).

Let us designate with \( \mathcal{F} \) a vector space of all complex-valued functions defined on \( D \) and being real analytical, and with \( L \) a linear differential operator (1) defined on \( \mathcal{F} \). Then
Let $L \psi \in \mathcal{F}$ when $\psi \in \mathcal{F}$. At that $\mathcal{F}_0$ is such subspace of the vector space $\mathcal{F}$ that coincides with zero-space (kernel) of the operator $L$.

Let $\mathfrak{M}_n$ be a set (class) of differential operators of the order $n$ defined on $\mathcal{F}$. Then a symmetry operator $Q \in \mathfrak{M}_n$ of equation (1) is defined as follows.

**Definition.** A linear differential operator of order $n$

$$Q = \sum_{i=0}^{n} Q_i, \quad Q_i = H^{(a_1a_2...a_i)} \frac{\partial^i}{\partial x_{a_1} \partial x_{a_2} \cdots \partial x_{a_i}}, \quad H^{(a_1a_2...a_i)} \in \mathcal{F} \tag{4}$$

is called a symmetry operator of equation (1) in the class $\mathfrak{M}_n$ (or symmetry operator of order $n$) if

$$[Q, L] = \alpha_Q L, \quad \alpha_Q \in \mathfrak{M}_{n-1}, \tag{5}$$

where $[Q, L] =QL - LQ$ is the commutator of the operators $Q$ and $L$.

Relation (5) should be understood in the sense that operators in the right-hand and left-hand parts give the same acting on an arbitrary function $\varphi \in \mathcal{F}$. Functions $H^{a_1,a_2,...,a_i}$ to be determined are symmetric tensors of rank $i$. Hereinafter the parentheses enclose the set of symmetric indices.

It is easy to see that the relation (3) follows from (3) for each $\varphi \in \mathcal{F}_0$. The reverse statement is also true: if an operator (4) satisfies the relation (3) for arbitrary $\psi \in \mathcal{F}_0$ then the condition (5) is satisfied for such operator with some operator $\alpha_Q$. In the case $n = 1$ the symmetry operators defined above may be interpreted as generators of the symmetry group of the equation being considered [7]. We can show that the set of symmetry operators $Q \in \mathfrak{M}_1$ generates a Lie algebra, and corresponding finite transformations from the invariance group may be obtained by integration of the Lie equations [7, 9].

Symmetry operators of order $n > 1$ are not generators of a Lie algebra anymore and characterize generalized (non-Lie) symmetry of an equation under study. The problem of description of a complete set of $n$-th order for equation (1) be reduced to finding of the general solution of the operator equations (5).

### 2 Equations for coefficients of symmetry operators.

**Killing tensors of rank $j$ and order $s$**

For simplification of further calculations it is more convenient to present the operator (4) as the sum of $i$-multiple anticommutators

$$Q = \sum_{j=0}^{n} \hat{Q}_j, \tag{6}$$

where

$$\hat{Q}_j = \left[ \cdots \left[ F^{(a_1a_2...a_j)} \frac{\partial}{\partial x_{a_1}}, \frac{\partial}{\partial x_{a_2}}, \cdots \frac{\partial}{\partial x_{a_j}} \right] \right]. \tag{7}$$
\([A, B]_+ = AB + BA, F^{(a_1 a_2 \ldots a_i)}\) is a symmetric tensor of rank \(i\). Expanding anticommutators and transferring differentiation operators to the righthand side, it is possible to reduce the expression (6) for the operator \(Q\) to the form (4), and, vice versa, to write down any operator of the form (4) as (6).

We can use a similar representation for the operator \(\alpha_Q \in \mathcal{M}_{n-1}\)

\[
\alpha_Q = \sum_{i=n-2}^{n-1} \hat{\alpha}_i, \quad \hat{\alpha}_i = \left[ \ldots \left[ \left[ f^{a_1 a_2 \ldots a_i}, \frac{\partial}{\partial x_{a_1}} \right]_+, \frac{\partial}{\partial x_{a_2}} \right]_+, \ldots, \frac{\partial}{\partial x_{a_i}} \right]_+, \tag{8}
\]

and write the product \(\alpha_Q L\) as

\[
\alpha_Q L \equiv \frac{1}{4} \left[ \left[ \alpha_Q, \frac{\partial}{\partial x_\mu} \right]_+, \frac{\partial}{\partial x_\mu} \right]_+ + \frac{1}{2} \left[ \alpha_{Q_\mu}, \frac{\partial}{\partial x_\mu} \right]_+, \tag{9}
\]

where \(\alpha_{Q_\mu} = \frac{\partial \alpha_Q}{\partial x_\mu}\).

Using the representations (6)–(9) and taking into account that

\[
[Q, L] = \left[ Q_{\mu}, \frac{\partial}{\partial x_\mu} \right]_+, \quad Q_{\mu} = \frac{\partial Q}{\partial x_\mu}, \tag{10}
\]

it is possible to reduce the operator equation (5) to the system of equations for coefficients \(f^{a_1 a_2 \ldots a_j}\) and \(F^{a_1 a_2 \ldots a_j}\). In fact, substituting (6)–(10) into (5) and putting equal coefficients at identical degrees of operators of differentiation, we obtain

\[
\partial_{(a_j+1} F^{a_1 a_2 \ldots a_j)} = 0, \tag{11}
\]

\[
f^{(a_1 a_2 \ldots a_i)} = 0. \tag{12}
\]

Here \(\partial_{(a_j+1} = \frac{\partial}{\partial x_{a_j+1}}\), the round brackets contain symmetric indices (so symmetrization is implied (11)):

\[
\frac{1}{j!} \partial^{(a_j+1} F^{a_1 a_2 \ldots a_{j+1}} = \partial^{a_j+1} F^{a_1 a_2 \ldots a_j} + \partial^{a_1} F^{a_{j+1} a_2 \ldots a_j} + \partial^{a_2} F^{a_1 a_{j+1} \ldots a_j} + \ldots + \partial^{a_j} F^{a_1 a_2 \ldots a_{j+1}},
\]

\(F^{a_1 a_2 \ldots a_j}\) is a symmetric tensor of the rank \(j\).

If \(\kappa = 0\), the equation for coefficients of the symmetry operator takes the following form:

\[
\partial^{(a_j+1} F^{a_1 a_2 \ldots a_j)} = \delta^{(a_j a_{j+1} f^{a_1 a_2 \ldots a_{j-1}})}, \tag{13}
\]

where \(F^{a_1 a_2 \ldots a_j}\) and \(f^{a_1 a_2 \ldots a_{j-1}}\) are symmetric tensors with zero trace.

Convoluting equations (13) with respect to one pair of indices, we can eliminate the unknown functions \(f^{a_1 a_2 \ldots a_{j-1}}\). As a result we get

\[
\partial^{(a_j+1} F^{a_1 a_2 \ldots a_j)} - \frac{j}{m+j-1} \partial^b F^{b(a_2 a_3 \ldots a_j a_{j+1})} = 0, \tag{14}
\]
\[ f^{a_1a_2\ldots a_{j-1}} = \frac{j}{m + j - 1} \partial^b F^{b(a_1a_2\ldots a_{j-1}) g_{a_ja_{j+1}}}, \] (15)

where \( m = p + q \) is dimension of the space of independent variables.

We see that the problem of description of symmetry operators of order \( n \) for the equation (1) with \( \kappa = 0 \) appears to be equivalent to finding of the general solution of the system of partial differential equations given by the formula (11). This system is split with respect to the index \( j \), as it splits into independent subsystems corresponding to \( j = 0, 1, \ldots, n \). As it will be shown below, for complete description of the symmetry operators it is actually sufficient to solve only two such subsystems corresponding to \( j = n \) and \( j = n - 1 \).

In the case \( j = 1 \) the system (11) coincides with the Killing equations \[2, 10\], and for \( j = 2 \) it coincides with equations for the Killing tensor \[9\] in the flat de Sitter space. The corresponding equations (14) determine conformal Killing vector and Killing tensor in the \( p + q \)-dimensional Minkowsky space.

We shall call functions \( F^{a_1a_2\ldots a_j} \) satisfying equations (11) (or (14)) Killing tensors (or conformal Killing tensors) of rank \( j \) and order 1. The meaning of the term “order 1” (that we will omit sometimes) will be explained below.

The equations (11), (13) for a tensor of arbitrary rank were introduced (in the case \( j > 2 \), without relation to any particular problem) in the paper \[11\]. However, the general solution of these equations, as far as we are aware, was obtained in an explicit form only for \( j = 1 \) and \( j = 2 \) \[12\].

In the process of investigation of higher order symmetry operators admitted by systems of partial differential equations, we have to deal with more complicated equations for coefficients of such operators than those given by formulae (11) or (14). These equations include derivatives of the order \( s > 1 \) and have the form \[13\]

\[ \partial^{(a_{j+1}} \partial^{a_{j+2}} \ldots \partial^{a_{j+s}} F^{a_1a_2\ldots a_j)} = 0, \] (16)

where \( F^{a_1a_2\ldots a_j} \) is a symmetric tensor, and

\[ \left[ \partial^{a_{j+1}} \partial^{a_{j+2}} \ldots \partial^{a_{j+s}} F^{a_1a_2\ldots a_j} \right]_{SL} = 0, \] (17)

where \( \tilde{F}^{a_1a_2\ldots a_j} \) is a symmetric tensor with zero trace, and the symbol \( \left[ \ldots \right]_{SL} \) designates the zero trace part of the tensor inside the square brackets (in our case it is a symmetric tensor of the rank \( R = j + s \)):

\[ [G^{a_1a_2\ldots a_R}]_{SL} = G^{a_1a_2\ldots a_R} + \sum_{d=1}^{\{ \frac{R}{2} \}} (-1)^d K_d \left( \prod_{i=1}^{d} g^{a_{2i-1}a_{2i}} \right) \times F^{a_{2d+1}a_{2d+2}\ldots a_{R}b_1b_2b_3b_4\ldots b_{2d-1}b_{2d}} g_{b_1b_2b_3b_4\ldots b_{2d-1}b_{2d}}, \] (18)

where \( \{ \frac{R}{2} \} \) is the integer part of the number \( \frac{R}{2} \),

\[ K_d = \frac{n!}{(n - 2d)!2^{d-1}} \prod_{i=1}^{d} \frac{1}{2(n - i) + m - 2}, \] (19)
In the case $s = 1$ the equations (16) and (17) can be reduced to equations (11) and (14) respectively.

We will call a symmetric tensor $F^{a_1 a_2 \ldots a_j}$ satisfying equations (16) a Killing tensor of rank $j$ and order $s$. We will call a symmetric tensor $\tilde{F}^{a_1 a_2 \ldots a_j}$ with zero trace satisfying equations (17) a conformal Killing tensor of rank $j$ and order $s$.

In Sections 3–7 below we obtain the general solution of equations (11), (14) for arbitrary $j$ in the space of dimension $p + q \leq 4$. Equations (16), (17) are discussed in Sections 8, 9 where their general solution is found for $p + q \leq 4$ and arbitrary $j$ and $s$.

3 Reduction of equations for symmetry operators to a system of linear algebraic equations

Let us start investigation of the system of equations (11) describing the Killing tensor of rank $j$ and order 1.

The system (11) may be written in the following symbolic form:

$$F^{a_1 a_2 \ldots a_j+1} = 0,$$

where $F^{a_1 a_2 \ldots a_j+1}$ is a symmetric tensor of rank $j + 1$ in $m = p + q$-dimensional space, and unknown functions are components of symmetric tensor of rank $j$ in $m$-dimensional space. Whence we can see that the system under investigation is overdetermined, including $\binom{j+m}{j+1}$ equations for $\binom{j+m-1}{j}$ unknowns, $\binom{b}{a} = \frac{b!}{a!(b-a)!}$ designating binomial coefficients.

Following the general method for solving of overdetermined systems of partial differential equations [14], we consider the set of differential consequences of the system (11), obtained by differentiation of each term, $k$ times by $x_b$ ($i = 1, 2, \ldots, k$). For each fixed $k$ such differential consequences are systems of linear homogeneous algebraic equations for derivatives

$$\partial_1 \partial_2 \ldots \partial_k F^{a_1 a_2 \ldots a_j} \equiv F^{(a_1 a_2 \ldots a_j, a_{j+1})} b_1 b_2 \ldots b_k.$$

These systems have the form

$$F^{(a_1 a_2 \ldots a_j, a_{j+1})} b_1 b_2 \ldots b_k = 0.$$

The system of equations (22) determines condition for vanishing of the tensor of rank $j + k + 1$ symmetric with respect to $j + 1$ indices $a_1, a_2, \ldots, a_{j+1}$ and with respect to $k$ indices $b_1, b_2, \ldots, b_k$, with unknown components of the tensor (21) of rank $j + k + 1$ symmetric with respect to $j$ indices $a_1, a_2, \ldots, a_j$ and with respect to $k + 1$ indices $a_{j+1}, b_1, \ldots, b_k$. Whence we conclude that the corresponding numbers of equations ($N_e$) and of unknown variables ($N_u$) are given by the formulae

$$N_e = \binom{j+m}{j+1} \binom{k+m-1}{k}, \quad N_u = \binom{j+m-1}{j} \binom{k+m}{k+1},$$

where $m = p + q$ is dimension of the Minkowsky space where equations (11) are determined (that is the number of independent variables $x_1, x_2, \ldots$ of the function $F^{a_1 a_2 \ldots a_j}$).
According to (23)
\[ N_e < N_u, \quad k < j, \quad N_e = N_u, \quad k = j. \] (24)

The formulae (23) allow calculation of the number of linearly independent solutions of equations (22), as the following statement is true:

**Theorem 1.** The system of linear algebraic equations (22) is not degenerate.

Proof of Theorem 1 is adduced below in Appendix.

We conclude from (24) in virtue of Theorem 1 that for \( k = j \) the system of homogeneous linear algebraic equations (22) has only trivial solutions,

\[ F^{a_1a_2...a_j,a_{j+1}b_1b_2...b_j} \equiv 0. \]

Whence coefficients of the symmetry operator \( F^{a_1a_2...a_j} \) are polynomials on \( x_a \) \( (a = 1, 2, \ldots, m) \) of order \( j \). It follows from (23) that such polynomial contains \( N_j^m \) arbitrary parameters, where

\[ N_j^m = \sum_{k=0}^{j} (N_e^k - N_u^k) = \frac{1}{m} \binom{j + m - 1}{m - 1} j + m \] (25)

We see that equations (11) have \( N_j^m \) linearly independent solutions that form a complete system. To find these solutions in explicit form it is necessary to find the general solution of the system of linear homogeneous equations (22) for arbitrary given \( j, m \) and \( k < j \), and then reconstruct polynomials \( F^{a_1a_2...a_j} \) by found values of derivatives of the tensors \( F^{a_1...a_j,a_{j+1}b_1b_2...b_k} \) (let us remind that indices after the comma designate derivatives with respect to the corresponding arguments). The general solution of equations (11) is adduced in Section 4 below.

### 4 Explicit form of Killing tensor of rank \( j \)

According to the above proof, calculation of the explicit form of Killing tensor of rank \( j \) is reduced to finding of the general solution of non-degenerate system of linear homogeneous algebraic equations given by the formula (22). Actual solution of this system with arbitrary given \( j \) and \( m \) is a rather difficult task that may be circumvented using the following observation.

**Lemma 1.** Let \( F^{a_1a_2...a_{j_0}} \) be an arbitrary solution of the system (11) for \( j = j_0 \), and \( F^a \) be a solution of the same system for \( j = 1 \). Then the function

\[ F^{a_1a_2...a_{j_0+1}} = F^{a_1a_2...a_{j_0}F^{a_{j_0+1}}} \] (26)

is a solution of the system (11) for \( j = j_0 + 1 \).

**Proof** is elementary and can be done by direct check.

Lemma 1 given an efficient algorithm for construction of solutions of equations (11). In fact, solutions of these equations for \( j = 1 \) are well-known: they are Killing tensors
and a solution for arbitrary \( j \) may be obtained from a solution for \( j = 1 \) by successive application of the formula (26). If we manage to construct this way \( N_j^m \) linearly independent solutions where \( N_j^m \) is given by the formula (25), then such solutions form a complete system in virtue of Theorem 1.

Using the algorithm presented above we managed to obtain the general solution of equations (11) for \( m \leq 4 \) in the form

\[
F^{a_1a_2...a_j} = g^{(a_{j-1}a_j} F^{a_1a_2...a_{j-2})} + f^{a_1a_2...a_j},
\]

where \( F^{a_1a_2...a_{j-2}} \) is the general solution of equations (11) for \( j \to (j-2) \) depending on \( N_{j-2}^m \) arbitrary parameters, and \( f^{a_1a_2...a_j} \) is a solution of equations (11) depending on \( N_j^m - N_{j-2}^m \) arbitrary parameters.

The first addend in the right-hand part of the formula (27) corresponds to such symmetry operator (7) of order \( j \) that on the set of solutions of equation (1) can be reduced to a symmetry operator of order \( j - 2 \). Explicit expressions for \( f^{a_1a_2...a_j} \) corresponding to \( m \leq 4 \) are adduced below.

1. \( m = 1 \). The corresponding tensor \( f^{a_1a_2...a_j} \) can be reduced to a scalar not depending on the only variable.

2. \( m = 2 \). Tensors \( f^{a_1a_2...a_j} \) depend on two variables \( x_1 \) and \( x_2 \). The number of independent solutions, according to (25), is

\[
N = N_j^2 - N_{j-2}^2 = 2j + 1.
\]

Solutions are numbered by an integer number \( c \) satisfying the condition

\[
0 \leq c \leq j,
\]

and include for \( c = 0 \) one, and for each \( c > 0 \) two arbitrary parameters giving independent components of a symmetric zero trace tensor \( \lambda^{a_1a_2...a_{j-c}} \) of rank \( j - c \). The explicit form of the corresponding solution \( f_c^{a_1a_2...a_j} \) is given by the formula

\[
f_c^{a_1a_2...a_j} = \varepsilon \hat{f}^{a_1a_2...a_j} + (1 - \varepsilon) \hat{f}^{(a_1a_2...a_{j-1}a_j)} x_b,
\]

where \( \varepsilon^{ab} \) is the unit antisymmetric tensor, \( \varepsilon = \frac{1}{2} [1 + (-1)^c] \),

\[
\hat{f}^{a_1a_2...a_j} = \lambda^{(a_1a_2...a_{j-c})} \sum_{\mu=0}^{\min\{\frac{j}{2}, j-c+1\}} \left( \prod_{i=j-c+1}^{j-c+2\mu} x_i \right)^* \times \left( \prod_{k=\left(\frac{j-c+1}{2}\right)+\mu+1}^{\min\{\frac{j+c+1}{2}, j-c+1\}} g^{a_{2k+l}(a_{2k})} \right)^* (-1)^{\mu \left\lfloor \frac{\mu}{2} \right\rfloor (x^2)^{\left\lfloor \frac{j-c+1}{2} \right\rfloor - \mu},
\]

\[
\left( \prod_{\lambda=A}^{B} f_{\lambda} \right)^* = \begin{cases} 
\prod_{\lambda=A}^{B} f_{\lambda}, & B \geq A, \\
1, & B < A,
\end{cases}
\]

and symmetrization over the indices \( a_1, a_2, \ldots, a_j \) is implied.
3. $m = 3$. The tensor $f^{a_1a_2...a_j}$ depends on three variables $\vec{x} = (x_1, x_2, x_3)$. The number of independent solutions is equal to

$$N = N_j^3 - N_{j-2}^3 = \frac{1}{3}(j + 1)(2j^2 + 4j + 3).$$

(32)

The solutions are numbered with pairs of integers $c = (c_1, c_2)$ satisfying the conditions

$$0 \leq c_1 \leq 2 \left\{ \frac{j}{2} \right\}, \quad 0 \leq c_2 \leq j - i \left\{ \frac{c_1 + 1}{2} \right\}, \quad \varepsilon_a = \frac{1}{2}[1 + (-1)^a],$$

(33)

and include for each $c$ the set $2c_1 + 1$ of arbitrary parameters giving independent components of a symmetric zero trace tensor $\lambda^{a_1a_2...a_{c_1}}$ of rank $c_1$. Explicit forms of the corresponding solutions $f_c^{a_1a_2...a_j}$ are given by the formula

$$f_c^{a_1a_2...a_j} = \varepsilon_{c_2}f_{c_1c_2}^{a_1a_2...a_j} + (1 - \varepsilon_{c_2})f_{c_1c_2}^{b(a_1a_2...a_{j-1}b)c}x_c,$$

(34)

where $\varepsilon^{abc}$ is the unit antisymmetric tensor,

$$f_{c_1c_2}^{a_1a_2...a_j} = \sum_{\mu} K_{\mu} \lambda_{c_1a_\mu}^{\beta_\mu} \left( A_{\mu} + L_{\mu} \right)^* \left( \prod_{i=A_{\mu}+1}^{\min(\{f\}, \{j+1\})-1} x^a_i \right)^* \left( \prod_{k=\{l/2\}(A_{\mu}+L_{\mu})+1}^{g_{a_2k}a_{2k+l}} \right)^* (x^2)^{F_{\mu}}.$$  

(35)

Here

$$\mu = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_5), \quad x^2 = x_a x_b g^{ab},$$

$$K_{\mu} = (-1)^{\mu_1+\mu_3+\mu_5} 2^{\mu_2} \frac{\mu_2! \mu_3! \mu_4!}{\mu_1!},$$

$$B_\mu = 2\mu_2 + \mu_3 + \mu_5, \quad A_{\mu} = j - c_1 - B_{\mu}, \quad l = (-1)^{c_2+j+1},$$

$$L_{\mu} = c_1 + \mu_3 - 2\mu_1 - \mu_5, \quad F_{\mu} = \mu_1 + \mu_4,$$

(36)

and $\lambda_{B_{\mu}A_{\mu}}$ is an arbitrary symmetric zero trace tensor of rank $A_{\mu} + B_{\mu}$ convoluted with $B_{\mu}$ vectors $x_k$:

$$\lambda_{B_{\mu}A_{\mu}} = \lambda^{b_1b_2...b_{B_{\mu}}}a_1a_2...a_{A_{\mu}}x_{b_1}x_{b_2}...x_{b_{B_{\mu}}}.$$  

(37)

Summation in (35) is to be done over all possible nonnegative values of $\mu$ satisfying the conditions

$$0 \leq \mu_1 \leq \left\{ \frac{c_1}{2} \right\}, \quad \mu_2 + \mu_3 + \mu_4 = \left\{ \frac{c_2}{2} \right\}, \quad 0 \leq \mu_5 \leq \frac{1}{2}[1 + (-1)^{c_1}].$$

(38)

As well as in the formula (31) symmetrization over the indices $a_1, a_2, \ldots, a_j$ in the right-hand part of (35) is implied.

4. $m = 4$. The tensor $f^{a_1a_2...a_j}$ depends on four variables $x = (x_1, x_2, x_3, x_4)$. The number of independent solutions is equal to

$$N = N_j^4 - N_{j-2}^4 = \frac{1}{4!}(j + 1)(j + 2)(2j + 3)(j^2 + 3j + 4).$$

(39)
Solutions are numbered with triples of integers \( c = (c_1, c_2, c_3) \) satisfying conditions (33) and (40):

\[
0 \leq c_3 \leq j - 2 \left\lfloor \frac{c_1 + 1}{2} \right\rfloor - 2c_2,
\]

and include for each \( c \) a set of \( N_c \) arbitrary parameters, where

\[
N_c = \begin{cases} 
(c_2 + 2c_3 + 1)^2, & c_1 = c_2 + 2c_3, \\
2(c_2 + 2c_3 + 1)(2c_1 - c_2 - 2c_3 + 1), & c_1 \neq c_2 + 2c_3,
\end{cases}
\]

These parameters give independent components of an irreducible tensor

\[
\lambda_{a_1a_2...a_R_1[a_R_1+ib_1][a_R_1+2b_2]...[a_R_1+r_2b_{r_2}]},
\]

where \( R_1 = c_2 + 2c_3, \) \( R_2 = c_1 - c_2 - 2c_3 \) (let us remind that an irreducible tensor of rank \( R_1 + 2R_2 \) has \( R_1 \) symmetric indices and \( R_2 \) symmetric pairs of antisymmetric indices, and convolution by any pair of indices and any triple of indices with completely antisymmetric tensor \( \varepsilon_{\mu\nu\rho\sigma} \) vanishes). Explicit expressions for the respective solutions are given by the formula (42):

\[
f_{a_1a_2...a_j}^c = \sum_{\mu} K_\mu \lambda_{\beta_\mu,(A_\mu,D_\mu)} \left( \prod_{i=A_\mu+D_\mu+1}^{A_\mu+D_\mu+L_\mu} x^{a_i} \right)^* \\
\times \left( \prod_{k=\left\{ \frac{1}{2}(A_\mu+D_\mu+L_\mu) \right\}+1}^{\left\{ \frac{1}{2} \right\}} g^{a_2k+a_2k} \right)^* (x^2) F_{\mu}, \tag{42}
\]

where \( \mu, x^2, K_\mu, B_\mu, F_\mu \) are given by the formulae (36), (38) \( a, b = 1, 2, 3, 4, \) \( A_\mu = j - c_1 - B_\mu - D_\mu, D_\mu = c_3, l = (-1)^{n+1}, \)

\[
\lambda_{B_\mu,A_\mu,D_\mu} = \lambda_{b_1b_2...b_{B_\mu}a_1a_2...a_{A_\mu}[a_{A_\mu+1d_1}]...[a_{A_\mu+D_\mu}d_{D_\mu}]} \\
\times x_{b_1}x_{b_2}...x_{b_{B_\mu}}x_{d_1}x_{d_2}...x_{d_{D_\mu}}, \tag{43}
\]

symmetrization over the indices \( a_1, a_2, \ldots, a_j \) is implied in the right-hand side of (42).

So, we have obtained the general solution of equations (11) in the space of dimension \( m \leq 4. \) One can verify by a direct check that the found solutions satisfy equations (11) and are linearly independent (it is not difficult to prove the latter considering \( i \)-fold convolutions of the found solutions with \( g^{kl}, 0 \leq i \leq \left\{ \frac{1}{2} \right\} \)). On the other side, these solutions form a complete system, as the number of arbitrary parameters they include is in compliance with the formula (25).

Let us also mention that we can present the general solution of equations (11) also in the form

\[
F_{a_1a_2...a_j} = \sum_{l=0}^{j} \lambda_{a_1a_2...a_l[a_{l+1b_1}][a_{l+2b_2}][a_{l+3b_3}]...[a_{l+jb_{l+j}}]} x_{b_1}x_{b_2}...x_{b_{l+j}}, \tag{44}
\]
where $\lambda^{a_1a_2...a_l[a_{l+1}b_1]...[a_jb_{j-l}]}$ is a tensor, symmetric with respect to permutation of indices $a_1, \ldots, a_j$ and antisymmetric with respect to permutation of indices $a_{l+j}$ with $b_i$, $1 \leq i \leq j - l$, with vanishing convolution of this tensor over any three indices with $\varepsilon_{\mu\nu\rho\sigma}$. The latter means that a cyclic permutation with respect to any triple of indices $(a_k, a_{l+s}, b_s)$ gives zero, so the polynomial (44) admittedly satisfies equation (11). On the other side, the number of independent components of the tensor $\lambda^{a_1a_2...a_l[a_{l+1}b_1]...[a_jb_{j-l}]}$ for $0 \leq l \leq j$ is exactly $N^m_{m,j}$ (25), so the formula (44) gives the general solution of equations (11). Decomposing tensors $\lambda^{a_1a_2...a_l[a_{l+1}b_1]...[a_jb_{j-l}]}$, $0 \leq l \leq j$ into irreducible ones (that is having vanishing convolutions with respect to any pair of indices), we come to formulae (26)–(43).

We formulate the above results as the following theorem.

**Theorem 2.** Equations (11) in a space of dimension $m \leq 4$ have $N^m_{m,j}$ linearly independent solutions. These solutions are polynomials of $x_a$ of degree $j$ and are given in explicit form by relations (26)–(43).

The above theorem determines the explicit form of the Killing tensor of rank $j$ in a space of dimension $m \leq 4$.

## 5 Explicit form of symmetry operators $Q_n$ for $n \leq 4$

The above results allow presenting in explicit form of symmetry operators of order $n$ for equation (1) in $m$-dimensional space for arbitrary given $n < \infty$ and $m \leq 4$. For this purpose it is sufficient to look through all admissible values of $c$ given by the formulae (29), (33), (40) and construct in accordance to the formulae (26)–(43) the corresponding expressions for Killing tensors of rank $j$, $F^{a_1a_2...a_j}_c$ (following (27), it is sufficient to restrict oneself with construction of $f^{a_1a_2...a_j}$), then substitute obtained expressions into (6), (7) and sum up over $j$ from 0 to $n$.

In this Section we will realize this program for all $n \leq 4$ and $m \leq 4$, and write down in explicit form the corresponding symmetry operators.

Let us calculate the number of linearly independent symmetry operators of order $n$. It is equal, according to (27) to the number of linearly independent solutions of the system (11) for $j = n, n - 1$ or (see (25))

$$N(n, m) = N^m_n + N^m_{n-1} = \frac{2n^2 + 2mn + m(m-1)}{m(m-1)} \left( \begin{array}{c} n + m - 2 \\ m - 2 \end{array} \right) \left( \begin{array}{c} n + m - 1 \\ m - 2 \end{array} \right), \quad (45)$$

In particular, for $m = 2, 3, 4$

$$N(n, 2) = (n + 1)^2,$$

$$N(n, 3) = \frac{1}{6}(n + 1)(n + 2)(n^2 + 3n + 3),$$

$$N(n, 4) = \frac{1}{72}(n + 1)(n + 2)^2(n + 3)(n^2 + 4n + 6). \quad (46)$$

Values of these numbers for $n = 1, 2, 3, 4$ are adduced in Table 1.
Let us write down explicitly the corresponding solutions $F_m^{(n)} = F_m^{a_1...a_n}$ of equations (11).

$m = 1$

\[ n = 1, 2, 3, 4, \quad F_1^{(n)} = \lambda_{1n}; \]

$m = 2$

\[ n = 0, \quad F_2^{(0)} = \lambda_{20}; \]
\[ n = 1, \quad F_2^{a} = \lambda_{21} + \lambda_{21} \epsilon^{abc} x_b; \]
\[ n = 2, \quad F_2^{a_1a_2} = g^{a_1a_2} F_2^{(0)} + \lambda^{a_1a_2} + \lambda(a_1 \epsilon a_2) x_b + \lambda(x(a_1 x_2 - x_1 x_2)); \]
\[ n = 3, \quad F_2^{a_1a_2a_3} = g^{a_1a_2} F_2^{(0)} + \lambda^{a_1a_2a_3} + \lambda(a_1 \epsilon a_2 \epsilon a_3) x_b + \lambda(a_1 \epsilon a_2 \epsilon a_3) x_b; \]
\[ n = 4, \quad F_2^{a_1a_2a_3a_4} = g^{a_1a_2} F_2^{(0)} + \lambda^{a_1a_2a_3a_4} + \lambda(a_1 \epsilon a_2 \epsilon a_3 \epsilon a_4) x_b \]

$m = 3$

\[ n = 0, \quad F_3^{(0)} = \lambda_{3}; \]
\[ n = 1, \quad F_3^{a} = \lambda^{a} + \lambda \epsilon^{abc} x_c; \]
\[ n = 2, \quad F_3^{a_1a_2} = g^{a_1a_2} F_3^{(0)} + \lambda^{a_1a_2} + \lambda(b(a_1 \epsilon a_2) bc x_c + \lambda(x(a_1 x_2 - x_1 x_2)); \]
\[ n = 3, \quad F_3^{a_1a_2a_3} = g^{a_1a_2} F_3^{(0)} + \lambda^{a_1a_2a_3} + \lambda(b(a_1 \epsilon a_2 \epsilon a_3) bc x_c + \lambda(x(a_1 x_2 - x_1 x_2)); \]
\[ n = 4, \quad F_3^{a_1a_2a_3a_4} = g^{a_1a_2} F_3^{(0)} + \lambda^{a_1a_2a_3a_4} + \lambda(b(a_1 \epsilon a_2 \epsilon a_3 \epsilon a_4) bc x_c + \lambda(x(a_1 x_2 - x_1 x_2)); \]

### Table 1. Number of symmetry operators of order $n$ for equation (1) in $m$-dimensional space.

| $m$ | $n$ | 1 | 2 | 3 | 4 |
|-----|-----|---|---|---|---|
| 2   | 4   | 9 | 16| 25|   |
| 3   | 7   | 26| 70| 155|  |
| 4   | 11  | 60| 225| 665|  |

---

...
\[ \begin{align*}
&- 2\lambda b^{(01a^2} x^a x^e a) b c x_d x_d + \lambda a^{(0a} a x^a x^d + \lambda (0b) a g^{a_2 a x} x_c x_b x_b \\
&- 4\lambda (0a^{(0a} x^{a} a x^b x^b_x + 4\lambda b^{(0a} a \lambda b (0a) x^a x^a_x x_b x_b_x + 2\lambda (0a) b g^{a_3 a_3} x_b x_b x^2 \\
&- 4\lambda b^{(1a_3 a_3} x^e a x^b x_b x_b + \lambda (a^{0a} a x^a a) b c x_c \\
&+ \lambda b b^{(0a} a g^{a_2 a a_4} x_b x_b x_b - \lambda (0b) a g^{a_2 a x^a} + \lambda b^{(0a} a x^a a a_4 a) b c x_c \\
&- \lambda b^{0a_3 a_3 x^a a) x^d + \lambda (0a^{(1a_3} a x^a a x^2 - \lambda (0a_2) a g^{a_3 a} x^2 x_b^2 \\
&- 2\lambda b^{(0a_2} a x^a x^a a) x_b x_b + 3\lambda b^{(0a_2} a x^a a x^a a) x_b x_b + \lambda (a^{0a_2} a x^a a a_1 a) b c x_c \\
&- \lambda b^{(0a_1 a x^a a a_2 a^2} x^a x^2 + \lambda (a^{0a_2} a x^a a a_4 a) x^2 - 2\lambda b^{(0a_2} a x^a a a_4 a) x_b \\
&+ \lambda b^{(0a_2} a x^a x^a a) x^a - \lambda (a^{0a_2} a g^{a_3 a} a) x^a + 2\lambda b^{(0a_2} a x^a a a_4 a) x_b^2 \\
&+ \lambda (0a^{(0a_1} a x^a a a_2 a^2} x^a a^2 a^2) x^a x^a - \lambda (0a_2) a x^a a a_3 a - \lambda (a^{0a_2} a x^a a a_4 a) x^4 + \lambda (0a_2) a x^a a a_4 a x^2 \\
&- \lambda b^{(0a_1} a x^a x^a a) x_b + \lambda (0a_1) a x^a a a_2 a^2) x_b^2 + \lambda (a^{0a_2} a x^a a a_4 a) x_b x_b \\
&- 2\lambda (0a) x^a x^a a a_2 a a_4 a) x^2 + \lambda (0a_2) a x^a a a_4 a \\
&\end{align*} \]

\( m = 4 \)

\( n = 0, \quad F_4^{(0)} = \lambda; \)

\( n = 1, \quad F_4^{(a)} = \lambda^{(0,0)} + \lambda^{(d)} a^{(0,0)} x d; \)

\( n = 2, \quad F_4^{a^{(0)} a} = g^{a_1 a_2} F_4^{(0)} + \lambda^{a_2} a^{(0,0)} + \lambda (a_1 a_2 a_1 d) x d + \lambda^{(a_1 d)} a^{(a_2 d)} x d x d \\
+ \lambda b^{(0a_2} a x^a a a_2 a^2 a^2) x^a x^a + \lambda b^{(0a_2} a x^a a a_2 a^2 a^2) x^a x^a + \lambda^{(a_1 a_2 d)} x d x d \\
- \lambda b^{(0a_2} a x^a a a_2 a^2 a^2 x^2 - \lambda (a_2 d) a x^2 a^2 x^2; \)

\( n = 3, \quad F_4^{a a^{(0)} a} = g^{a_1 a_2} F_4^{(a)} + \lambda^{a a_2 a_2} a^{(0,0)} + \lambda (a_1 a_2 a_2 a_1 d) x d + \lambda^{(a_1 a_2 d)} a^{(a_2 d)} x d x d \\
+ \lambda b^{(a_2 a_3} x d x d x d x d - \lambda b^{(a_1 a_2 a_3} x d x d x d + \lambda^{a a_2 a_2 a_1 d} x d x d x d \\
+ \lambda b^{(a_2 a_2 a_1 d} a^{(0,11)} x^2 x^2 x^2 x^2 x^2 - 2\lambda b^{(a_1 a_2 a_2 a_1 d} x^2 x^2 x^2 x^2 x^2 \\
+ \lambda b^{(a_1 a_2 a_2 a_1 d} a^{(0,11)} x^2 x^2 x^2 x^2 x^2 - \lambda b^{(a_1 a_2 a_2 a_1 d} x^2 x^2 x^2 x^2 x^2 \\
- \lambda b^{(a_1 a_2 a_2 a_1 d} a^{(0,11)} x^2 x^2 x^2 x^2 x^2 - \lambda b^{(a_1 a_2 a_2 a_1 d} x^2 x^2 x^2 x^2 x^2; \)

\( n = 4, \quad F_4^{a a^{(0)} a} = g^{a_1 a_2} F_4^{a a_3} + \lambda^{a a_2 a_3 a_1 a a_4} x d + \lambda^{a a_2 a_3 a_1 a a_4} x d x d \\
+ \lambda^{(a_1 a_2 a_3 d)} x d x d + \lambda^{(a_1 a_2 a_3 d)} x d x d x d \\
+ \lambda^{(a_1 a_2 a_3 d)} x d x d x d x d + \lambda^{a a_2 a_3 a_1 a a_4} x^2 - 2\lambda b^{(a_1 a_2 a_3} x^2 x^2 x^2 x^2 x^2 \\
+ \lambda b^{(a_1 a_2 a_3 a_1 d} x d x d x d x d + \lambda^{a a_2 a_3 a_1 a a_4} x^2 - 2\lambda b^{(a_1 a_2 a_3} x^2 x^2 x^2 x^2 x^2 \\
+ \lambda b^{(a_1 a_2 a_3 d} x d x d x d x d + \lambda^{a a_2 a_3 a_1 a a_4} x^2 x^2 x^2 x^2 x^2 \\
- 2\lambda b^{(a_1 a_2 a_3 d} x d x d x d x d + \lambda^{a a_2 a_3 a_1 a a_4} x^2 x^2 x^2 x^2 x^2; \)
Substituting (47)–(54) into (6), (7) and carrying differentiation operators to the right, we obtain explicit form of the corresponding symmetry operators. For $n = 1$ we have a complete set of symmetry operators of the following form:

$$Q_1^a = P_a = i \frac{\partial}{\partial x^a}, \quad Q_{ab} = J_{ab} = x_a P_b - x_b P_a.$$  

(55)

We do not adduce explicit form of symmetry operators for $n > 1$ because of the corresponding formulae being extremely cumbersome (in fact, these expressions are given by relations (6), (7), (47)–(54)).

First order symmetry operators $Q_1$ adduced in (55), form a Lie algebra $AP(p,q)$, satisfying the following commutation relations:

$$[P_a, P_b] = 0, \quad [P_a, J_{bc}] = i(g_{ab} P_c - g_{ac} P_b),$$

$$[J_{ab}, J_{cd}] = i(g_{ac} J_{bd} + g_{bd} J_{ac} - g_{ac} J_{bd} - g_{bd} J_{ac}).$$  

(56)

It is easy to notice using the representation (36) that symmetry operators of arbitrary order are polynomials of the operators (55). In other words, all symmetry operators of finite order of equation (1) belong to the enveloping algebra of the algebra $AP(p,q)$.  

6  Explicit form of Killing tensor of arbitrary rank $j$

Calculation of conformal Killing tensors of rank $j$ (that is construction of the general solution of equation (14)) may be done similarly to what was presented above in Sections 3–5. Construction of such solution simplifies utilization of the result formulated in the following lemma.
Lemma 2. Let $F^{a_1a_2\ldots a_{j_0}}$ be an arbitrary solution of the system (14) for $j = j_0$, and $F^a$ be a solution of this system for $j = 1$. Then the function

$$F^{a_1a_2\ldots a_{j_0+1}} = [F^{a_1a_2\ldots a_{j_0} F^a_{j_0+1}}]^\text{SL}.$$ \hspace{1cm} (57)

where $[\cdot]^\text{SL}$ means the traceless part of the tensor in the square brackets (see (18)) is a solution of equations (14) for $j = j_0 + 1$.

Proof can be done by a direct check.

We adduce below without proof the general solution of equations (14) for $m \leq 4$ and arbitrary $j$.

By means of the reasoning similar to that in Section 3 we can show that in the two-dimensional space equations (14) are reduced to Cauchy–Riemann equations, and corresponding symmetry operators are determined up to arbitrary analytical functions determining independent components of a symmetric traceless tensor $F^{a_1a_2\ldots a_j}$ (there are two such components for $j \neq 0$ and one for $j = 0$ (that is for the case when then tensor $F^{a_1a_2\ldots a_j}$ is reduced to a scalar).

For $m = 3$ the number of independent solutions of equations (14) is equal to

$$N^3_j = \frac{1}{3}(j + 1)(2j + 1)(2j + 3).$$ \hspace{1cm} (58)

Solutions are numbered by the pair of integers $c = (c_1, c_2)$ satisfying the conditions

$$0 \leq c_1 \leq j, \quad 0 \leq c_2 \leq 2c_1,$$ \hspace{1cm} (59)

and for each $c_1$ contain $(2c_1 + 1)$ arbitrary parameters giving independent components of symmetric traceless tensor $\lambda^{a_1a_2\ldots a_{c_1}}$ of rank $c_1$. Explicit form of the corresponding solutions is given by the formula

$$F^{a_1a_2\ldots a_{c_1}}_{(c_1c_2)} = \left[\varepsilon_{c_2} f^{a_1a_2\ldots a_{c_1}}_{(c_1c_2)} + (1 - \varepsilon_{c_2}) f^{b(a_1a_2\ldots a_{c_1} - 1)\varepsilon a_j}_{(c_1c_2)} bc x_c \right]^\text{SL},$$ \hspace{1cm} (60)

where

$$f^{a_1a_2\ldots a_j}_{(c_1c_2)} = \sum_{m=0}^{\{\frac{2c_1}{2}\}} (-2)^m \binom{\{\frac{2c_1}{2}\}}{m} \lambda^{b_1b_2\ldots b_m(a_1a_2\ldots a_{c_1} - m)}_{(c_1c_2)} \times x^{a_1-m+1}x^{a_{c_1}-m+2}\ldots x^{a_j} x_{b_1} x_{b_2} \ldots x_{b_m} x^{2\{\frac{2c_1}{2}\}-m},$$ \hspace{1cm} (61)

and the symbol $[\cdot]^\text{SL}$ means the traceless part of the corresponding tensor; see (18) for $m = 3$.

For $m = 4$ the number of independent solutions of equations (14) is equal to

$$N^4_j = \frac{1}{12}(j + 1)^2(j + 2)^2(2j + 3).$$ \hspace{1cm} (62)

The solutions are numbered by triples of integers $c = (c_1, c_2, c_3)$ satisfying the conditions

$$0 \leq c_1 \leq j, \quad -c_1 \leq c_2 \leq c_1, \quad 0 \leq c_3 \leq \left\{ \frac{c_1 - |c_2|}{2} \right\},$$ \hspace{1cm} (63)
and for each \( c \) contain \( N_c \) arbitrary parameters where

\[
N_c = \begin{cases} 
(c_1 + 1)^2, & c_1 = |c_2|, \\
2(|c_2| + 2c_3 + 1)(2c_1 - |c_2| - 2c_3 + 1), & c_1 \neq |c_2|.
\end{cases}
\] (64)

These parameters determine independent components of an irreducible tensor of rank \( R = R_1 + 2R_2 \) where

\[
R_1 = |c_2| + 2c_3, \quad R_2 = c_1 - |c_2| - 2c_3,
\] (65)

and explicit expressions for the corresponding solutions have the form

\[
F_c^{(a_1a_2...a_j)} = \left[ \sum_{i=0}^{m+c_3} (-1)^i \binom{m + c_3}{i} (x^2)^i \right. \\
\times \lambda^{b_1b_2...b_{m-i+c_3}}(a_1a_2...a_{|c_2|-m+i+c_3}[a|c_2|+i-m+1+c_3d_1]...[a_{c_1-m+i-c_3}d_1-|c_2|-2c_3] \\
\times x^{a_1+c_1-m+i-c_3+1}x^{a_2+c_1-m+i-c_3+2}...x^{a_j}x_{b_1}x_{b_2}...x_{b_{m-i+c_3}} \\
\left. \times x_{d_1}x_{d_2}...x_{d_{|c_2|-2c_3}} \right]_{SL}.
\] (66)

Here \( \lambda^{b_1...b_{m-i+c_3}}a_{c_1}a_{c_2}...a_{c_3+c_1+c_3}d_1...d_{|c_2|-2c_3} \) is an arbitrary irreducible tensor of rank \( R_1 + 2R_2 \) \((R_1 \text{ and } R_2 \text{ are given in (65)),}\)

\[
m = \begin{cases} 
-c_2, & c_2 < 0, \\
0, & c_2 \geq 0,
\end{cases}
\] (67)

\( \binom{m+c_3}{i} \) is a binomial coefficient, and the symbol \( [\cdot]_{SL} \) means the traceless part of the corresponding tensor; see (18), (19) for \( m = 4 \). Symmetrisation is implied over the indices \( a_1, \ldots, a_j \) in the righthand part (the sum over all possible permutations).

Thus, we have found the explicit form of the conformal tensor of rank \( j \) for \( m \leq 4 \). The formula (66) determines the general form such tensor for arbitrary \( m > 3 \), but at that the total number of independent solutions of equations (14) cannot be determined, in general, by the relation (62), but requires special calculation for each value of \( m \).

Let us point out that the general solution of equations (14) for \( m > 2 \) can be presented in the form

\[
F^{a_1a_2...a_j} = \sum_{l,k=0}^{j} \sum_{i=0}^{j-l-k} \lambda^{b_1b_2...b_{j-i-k-l}}(a_1a_2...a_{i+i+1}[a_{i+i+1}d_1]...[a_{i+i+k}d_k] \\
\times (-1)^i \binom{j-l-k}{i} (x^2)^i x^{a_{i+i+k+1}}...x^{a_j}x_{d_1}x_{d_2}...x_{d_k}x_{b_1}x_{b_2}...x_{b_{j-l-k-1}} \right],
\] (68)

where \( \lambda^{b_1b_2...b_{j-l-k-l}}a_{i+i}a_{i+i+1}[a_{i+i+1}d_1]...[a_{i+i+k}d_k] \) is a tensor symmetric with respect to permutation of the indices \( b_1, \ldots, b_{j-l-k} \) and antisymmetric with respect to permutation of the indices \( a_{i+i+k} \) with \( d_f, f = 1, 2, \ldots, k, \) with convolution of this tensor by any three indices with an absolutely antisymmetric vanishing. Decomposing such tensor into irreducible tensors we come to formulae (58)–(67) giving solutions of equations (14) for \( m = 3, 4 \).
Examples of solutions and symmetry operators for \( n \leq 3 \)

Let us adduce an explicit form of the solutions obtained and corresponding symmetry operators for \( m \leq 4 \) and \( n \leq 3 \). Quantities of such solutions in accordance to (58) and (62) are adduced in Table 2.

**Table 2.** Quantities of independent solutions of equations (14).

| \( m \) | \( j \) | 1 | 2 | 3 |
|---|---|---|---|---|
| 3 | | 10 | 35 | 84 |
| 4 | | 15 | 84 | 300 |

Quantities of the corresponding symmetry operators of order \( n \) can be obtained by summation of quantities of solutions from \( j = 0 \) to \( j = n \). We adduce the result in Table 3.

**Table 3.** Quantities of order \( n \) symmetry operators of equation (1) with \( \kappa = 0 \) in \( m \)-dimensional space.

| \( m \) | \( n \) | 0   | 1   | 2   | 3   | 4   |
|---|---|-----|-----|-----|-----|-----|
| 3 | | 1 | 11 | 46 | 130 | 295 |
| 4 | | 1 | 16 | 100 | 400 | 1225 |

For \( m = 2 \) the number of solutions of equations (14) (and the number of the corresponding symmetry operators) is infinite as they are determined up to arbitrary functions.

Explicit expressions for all independent solutions of equations (14) for \( m \leq 4 \) and \( n \leq 3 \) are given by the following formulae (\( F^{(j)} = F^{(a_1 a_2 \ldots a_j)} \)):

\[
\begin{align*}
  & m = 2 \\
  & \quad j = 0, \quad F^{(0)} = \phi^0(x_1, x_2); \\
  & \quad j > 0, \quad F^{11 \ldots 1} = (\phi_j + \phi_j^*) + i(\xi_j + \xi_j^*); \\
  & \quad F^{11 \ldots 12} = i(\phi_j^* - \phi_j) + \xi_j - \xi_j^*. \\
  & \quad \text{(69)}
\end{align*}
\]

Here \( \phi_j \) and \( \xi_j \) are arbitrary analytical functions of two variables \( x_1, x_2 \), and other components of the tensor \( F^{a_1 a_2 \ldots a_j} \) are expressed through (69) using properties of of zero trace and symmetry.

\[
\begin{align*}
  & m = 3 \\
  & \quad j = 0, \quad F^{(0)} = \lambda; \\
  & \quad j = 1, \quad F^{a}_{(0,0)} = \lambda_{(0,0)} x^a, \quad F^{a}_{(1,0)} = \lambda^a_{(1,0)}; \quad F^{a}_{(1,1)} = \varepsilon_{abc} \lambda^b_{(1,1)} x^c; \\
  & \quad F^{a}_{(1,2)} = 2 \lambda^b_{(1,2)} x_b x^a - \lambda^a_{(1,2)} x^2; \\
  & \quad F^{a_1 a_2}_{(0,0)} = \lambda_{(0,0)} \left( x^{a_1} x^{a_2} - \frac{1}{3} g^{a_1 a_2} x^2 \right); \\
  & \quad F^{a_1 a_2}_{(1,0)} = \lambda^a_{(1,0)} x^{a_2} + \lambda^a_{(1,0)} x^{a_1} - \frac{2}{3} g^{a_1 a_2} \lambda^b_{(1,0)} x^b; \\
  & \quad F^{a_1 a_2}_{(1,1)} = (x^a \varepsilon^{a_2}_{bc} + x^{a_2} \varepsilon^{a_1}_{bc}) x^b \lambda^c_{(1,1)}; \\
  & \quad \text{(70)}
\end{align*}
\]
$$F_{(1,2)}^{a_1a_2} = (x^{a_1} \lambda^{a_2}_{(1,2)} + x^{a_2} \lambda^{a_1}_{(1,2)}) x^2 - 4x^{a_1} x^{a_2} \lambda^b_{(1,2)} x_b + \frac{2}{3} g^{a_1a_2} \lambda^b_{(1,2)} x_b x^2;$$

$$F_{(2,0)}^{a_1a_2} = \lambda^{a_1a_2}_{(2,0)};$$

$$F_{(2,1)}^{a_1a_2} = (\varepsilon^{a_1bc} \lambda^{b}_{(2,1)} + \varepsilon^{a_2bc} \lambda^{b}_{(2,1)}) x_c;$$

$$F_{(2,2)}^{a_1a_2} = \lambda^{a_1a_2}_{(2,2)} x^2 - (x^{a_1} \lambda^{a_2}_{(2,2)} + x^{a_2} \lambda^{a_1}_{(2,2)}) x_b + \frac{2}{3} g^{a_1a_2} \lambda^b_{(2,2)} x_b x_c;$$

$$F_{(2,3)}^{a_1a_2} = 2(x^{a_1} \varepsilon^{a_2b} + x^{a_2} \varepsilon^{a_1b}) \lambda^{k}_{(2,3)} x^b x_d - (\varepsilon^{a_1ck} \lambda^{b}_{(2,3)} + \varepsilon^{a_2ck} \lambda^{b}_{(2,3)}) x_c x^2;$$

$$F_{(2,4)}^{a_1a_2} = \lambda^{a_1a_2}_{(2,4)} x^4 - 2(x^{a_2} \lambda^c_{(2,4)} + x^{a_2} \lambda^{a_1c}_{(2,4)}) x^2 + 4x^{a_1} x^{a_2} \lambda^{ca}_{(2,4)} x_c x_d; \quad (71)$$

$$j = 3, \quad F_{(0,2x_3)}^{a_1a_2a_3} = \lambda^{(0,0)}(x^{a_1} x^{a_2} x^{a_3} - \frac{1}{10} g^{a_1a_2a_3} x^2);$$

$$F_{(1,0)}^{a_1a_2a_3} = \lambda^{a_1}_{(1,0)} x^{a_2} x^{a_3} - \frac{1}{5} g^{a_1a_2a_3} \lambda^{b}_{(1,0)} x_b + 2 g^{a_1a_2a_3} \lambda^{b}_{(1,0)} x_b;$$

$$F_{(1,1)}^{a_1a_2a_3} = x^{a_1} x^{a_2} x^{a_3} \lambda^{c}_{(1,1)} - \frac{1}{5} g^{a_1a_2a_3} \lambda^{c}_{(1,1)} x_b;$$

$$F_{(2,0)}^{a_1a_2a_3} = \lambda^{a_1a_2}_{(2,0)} x^{a_3} - \frac{2}{5} g^{a_1a_2a_3} \lambda^{b}_{(2,0)} x_b;$$

$$F_{(2,1)}^{a_1a_2a_3} = x^{a_1} x^{a_2} x^{a_3} \lambda^{c}_{(2,1)} x_b - \frac{1}{5} g^{a_1a_2a_3} \lambda^{c}_{(2,1)} x_b x_d;$$

$$F_{(2,2)}^{a_1a_2a_3} = \lambda^{a_1a_2}_{(2,2)} x^{a_3} x^2 - 2x^{a_1} x^{a_2} x^{a_3} x_b + \frac{4}{5} g^{a_1a_2a_3} \lambda^{b}_{(2,2)} x_b x_c;$$

$$F_{(2,3)}^{a_1a_2a_3} = \varepsilon^{a_1bc} x^{a_2} x^{a_3} \lambda^{bd}_{(2,3)} x_d - \lambda^{a_2b}_{(2,3)} x^2 x_c;$$

$$F_{(2,4)}^{a_1a_2a_3} = x^{a_1} \lambda^{a_2}_{(2,4)} x^2 - 4x^{a_2} \lambda^{a_1}_{(2,4)} x^2 + 4x^{a_2} x^{a_3} \lambda^{kl}_{(2,4)} x_k x_l - \frac{2}{5} g^{a_1a_2a_3} \lambda^{kl}_{(2,4)} x_k x_l x^2;$$

$$F_{(3,0)}^{a_1a_2a_3} = \lambda^{a_1a_2a_3}_{(3,0)};$$

$$F_{(3,1)}^{a_1a_2a_3} = \varepsilon^{a_1bc} \lambda^{a_2b}_{(3,1)} x_c;$$

$$F_{(3,2)}^{a_1a_2a_3} = \lambda^{a_1a_2a_3}_{(3,2)} x^2 - 2x^{a_1} x^{a_2} x^{a_3} x_b + \frac{4}{5} g^{a_1a_2a_3} \lambda^{b}_{(3,2)} x_b x_c;$$

$$F_{(3,3)}^{a_1a_2a_3} = \varepsilon^{a_1bc} \lambda^{a_2}_{(3,3)} x^2 - 2x^{a_1} x^{a_2} x^{a_3} x_b x_c + \frac{2}{5} g^{a_1a_2a_3} \lambda^{bcd}_{(3,3)} x_b x_c x_d;$$

$$F_{(3,4)}^{a_1a_2a_3} = \lambda^{a_1a_2a_3}_{(3,4)} x^4 - 4x^{a_1} (\lambda^{a_2}_{(3,4)} x^2 - x^{a_2} \lambda^{a_1}_{(3,4)} x_b x_c);$$

$$F_{(3,5)}^{a_1a_2a_3} = \frac{4}{5} g^{a_1a_2a_3} (2x^{a_3} \lambda^{b}_{(3,4)} x_d - \lambda^{a_3b}_{(3,4)} x^2 x_c);$$

$$F_{(3,6)}^{a_1a_2a_3} = \varepsilon^{a_1} (\lambda^{a_2}_{(3,5)} x^4 - 4x^{a_2} \lambda^{a_1}_{(3,5)} x^2 + 4x^{a_2} x^{a_3} \lambda^{bkl}_{(3,5)} x_k x_l) x_c;$$

$$m = 4 \quad j = 0, \quad F^{(0)} = \lambda; \quad j = 1, \quad F^{(1)}(0,0,0) = \lambda^x;$$

"
\( F_{(1,-1,0)}^{(a)} = \lambda^b x^a x_b - \lambda^a x^2; \)
\( F_{(1,0,0)}^{(a)} = \lambda^{[ad]} x_d; \)
\( F_{(1,1,0)}^{(a)} = \lambda^a; \)

\( j = 2, \quad F_{(0,0,0)}^{a_1 a_2} = \lambda_{(0,0,0)} \left( x^{a_1} x^{a_2} - \frac{1}{4} g^{a_1 a_2} x^2 \right); \)
\( F_{(1,-1,0)}^{a_1 a_2} = 2 \lambda^b_{(1,-1,0)} x_b \left( x^{a_1} x^{a_2} - \frac{1}{4} g^{a_1 a_2} x^2 \right) - \lambda^{a_1}_{(1,-1,0)} x^{a_2} x^2 \)
\( - \lambda^{a_2}_{(1,-1,0)} x^{a_1} x^2 + \frac{1}{2} g^{a_1 a_2} \lambda^c_{(1,-1,0)} x c d^2; \)
\( F_{(1,0,0)}^{a_1 a_2} = \lambda^{[a_1 d]}_{(1,0,0)} x^{a_2} x_d + \lambda^{a_2 d]}_{(1,0,0)} x^{a_1} x_d; \)
\( F_{(1,1,0)}^{a_1 a_2} = - \frac{1}{2} g^{a_1 a_2} \lambda^c_{(1,1,0)} x c + \lambda^{a_1}_{(1,1,0)} x^{a_2} + \lambda^{a_2}_{(1,1,0)} x^{a_1}; \)
\( F_{(2,-2,0)}^{a_1 a_2} = 2 \lambda^{b_{(2,-2,0)}}_{a_1 a_2} \lambda^{b_{(2,-2,0)}}_{a_1 a_2} x^{a_1} x^{a_2} x b_1 x b_2 - 2 \lambda^{b_{(2,-2,0)}}_{a_1 a_2} \lambda^{b_{(2,-2,0)}}_{a_1 a_2} x^{a_2} x^{a_1} x b_1 x b_2 \)
\( - 2 \lambda^{b_{(2,-2,0)}}_{a_1 a_2} x^{a_1} x^{a_2} x b_1 x b_2 + 2 \lambda^{a_1 a_2}_{(2,-2,0)} x^4 + \frac{1}{2} g^{a_1 a_2} \lambda^{b_{(2,-2,0)}}_{a_1 a_2} \lambda^{b_{(2,-2,0)}}_{a_1 a_2} x^{a_1} x^{a_2} x b_1 x b_2; \)
\( F_{(2,-1,0)}^{a_1 a_2} = \lambda^{b_{(2,-1,0)}}_{a_1 a_2} \lambda^{b_{(2,-1,0)}}_{a_1 a_2} x^{a_1} x^{a_2} x b_1 x d_1 + \lambda^{b_{(2,-1,0)}}_{a_1 a_2} \lambda^{b_{(2,-1,0)}}_{a_1 a_2} x^{a_1} x^{a_2} x b_1 x d_1 \)
\( - \lambda^{a_1 a_2}_{(2,-1,0)} x^2 x d_1 = \lambda^{a_2 a_1 d]}_{(2,-1,0)} x^2 x d_1; \)
\( F_{(2,0,1)}^{a_1 a_2} = \lambda^{b_{(2,0,1)}}_{a_1 a_2} x^{a_2} x b_1 + \lambda^{b_{(2,0,1)}}_{a_1 a_2} x^{a_1} x b_1 - 2 \lambda^{a_1 a_2}_{(2,0,1)} x^2 - \frac{1}{2} g^{a_1 a_2} \lambda^{b_{(2,0,1)}}_{a_1 a_2} \lambda^{b_{(2,0,1)}}_{a_1 a_2} x b_1 x c; \)
\( F_{(2,0,2)}^{a_1 a_2} = \lambda^{a_1 a_2}_{(2,0,0)} \lambda^{a_1 a_2}_{(2,0,0)} x d_1 x d_2; \)
\( F_{(2,1,0)}^{a_1 a_2} = \left( \lambda^{a_1 a_2}_{(2,1,0)} + \lambda^{a_2 a_1 d]}_{(2,1,0)} \right) x d; \)
\( F_{(2,2,0)}^{a_1 a_2} = \lambda^{a_1 a_2}_{(2,2,0)}; \)

\( j = 3, \quad F_{(0,0,0)}^{a_1 a_2 a_3} = \lambda_{(0,0,0)} \left( x^{(a_1} x^{a_2} x^{a_3)} - \frac{1}{2} g^{(a_1 a_2} x^{a_3)} x^2 \right); \)
\( F_{(1,-1,0)}^{a_1 a_2 a_3} = \lambda^b_{(1,-1,0)} \left( x^{(a_1} x^{a_2} x^{a_3)} - \frac{1}{2} g^{(a_1 a_2} x^{a_3)} x^2 \right) x_b - \lambda^{a_1}_{(1,-1,0)} x^{a_2} x^{a_3} x^2 \)
\( + \frac{1}{3} g^{(a_1 a_2} x^{a_3)} \lambda^b_{(1,-1,0)} x^2 + \frac{1}{6} g^{(a_1 a_2} \lambda^{a_3}_{(1,-1,0)} x^4; \)
\( F_{(1,0,0)}^{a_1 a_2 a_3} = \lambda^{(a_1 d]}_{(1,0,0)} x^{a_2} x^{a_3} x_d - \frac{1}{6} g^{(a_1 a_2} \lambda^{a_3 d)]_{(1,0,0)} x^2 x d; \)
\( F_{(1,1,0)}^{a_1 a_2 a_3} = \lambda^{(a_1}_{(1,1,0)} x^{a_2} x^{a_3}) - \frac{1}{3} g^{(a_1 a_2} x^{a_3}) \lambda^b_{(1,1,0)} x_b - \frac{1}{6} g^{(a_1 a_2} \lambda^{a_3}_{(1,1,0)} x^2; \)
\( F_{(2,-2,0)}^{a_1 a_2 a_3} = \lambda^{b_{(2,-2,0)}}_{a_1 a_2 a_3} \left( x^{(a_1} x^{a_2} x^{a_3)} - \frac{1}{2} g^{(a_1 a_2} x^{a_3)} x^2 \right) x b_1 x b_2 \)
\( - 2 \lambda^{b_{(2,-2,0)}}_{a_1 a_2 a_3} x^{a_2} x^{a_3} x b_1 x b_2 + \frac{2}{3} g^{(a_1 a_2} x^{a_3}) \lambda^{b c}_{(2,-2,0)} x b_1 x c x^2 + \frac{1}{3} g^{(a_1 a_2} \lambda^{a_3 b}_{(2,-2,0)} x b_1 x^4 \)
\( + \lambda^{a_1 a_2} x^3 x^4 + \frac{1}{3} g^{(a_1 a_2} \lambda^{a_3 b}_{(2,-2,0)} x b_1 x^4; \)
\( F_{(2,-1,0)}^{a_1 a_2 a_3} = \lambda^{b_{(2,-1,0)}}_{a_1 a_2 a_3} x^{a_2} x^{a_3} x b_1 x d - \lambda^{a_2 a_1 d]}_{(2,-1,0)} x d x^2. \)
To obtain an explicit form of the corresponding operators it is sufficient to substitute Formulae (69)–(75) give explicit form of all linearly independent conformal killing tensors in spaces of dimension $p+q = 2, 3, 4$ (for $p+q = 2j$ is arbitrary).

Here $\lambda_{(\ldots)}$, $\lambda_{(\ldots)}^0$, $\lambda_{(\ldots)}^{a_1a_2}$, $\lambda_{(\ldots)}^{a_1a_2a_3}$ are arbitrary symmetric tensors with zero trace.

Formulae (69)–(75) give explicit form of all linearly independent conformal killing tensors of rank $j \leq 3$ in spaces of dimension $p + q = 2, 3, 4$ (for $p + q = 2j$ is arbitrary). To obtain an explicit form of the corresponding operators it is sufficient to substitute (69)–(75) into (6), (7) that is write down $j$-multiple anticommutators of $F^{a_1a_2\ldots a_j}$ with

$$\frac{\partial}{\partial x_{a_1}}, \frac{\partial}{\partial x_{a_2}}, \ldots, \frac{\partial}{\partial x_{a_j}}.$$
8 Killing tensors of rank $j$ and order $s$

Until now we considered solutions of equations (11), (14) that define Killing tensors (and conformal Killing tensors) of arbitrary rank $j$, but only of the first order. In this section we obtain explicit form of Killing tensors of rank $j$ and of arbitrary order $s$. Such tensors are determined as general solutions of equations (16).

The system of equations (16) is overdetermined including $N_{js}^m$ equations for $\hat{N}_{js}^m$ unknown variables, where

$$N_{js}^m = \binom{j + s + m - 1}{m - 1}, \quad \hat{N}_{js}^m = \binom{j + m - 1}{m - 1}, \quad m = p + q. \quad \text{(76)}$$

In the same way as it was done above in Section 3, we consider the set of differential consequences of the system under consideration that are obtained by $k$-multiple differentiation of every term of the equation by $\frac{\partial}{\partial x_{a_1}}, \frac{\partial}{\partial x_{a_2}}, \ldots, \frac{\partial}{\partial x_{a_k}}$. This set is a system of linear homogeneous algebraic equations of the following form:

$$F(a_1a_2\ldots a_j, a_{j+1}a_{j+2}\ldots a_{j+s})b_1b_2\ldots b_k = 0, \quad \text{(77)}$$

where the following derivatives are unknown variables:

$$F_{a_1a_2\ldots a_j, a_{j+1}a_{j+2}\ldots a_{j+s}, b_1b_2\ldots b_k} = \partial_{a_{j+1}} \partial_{a_{j+2}} \ldots \partial_{a_{j+s}} \partial^{b_1} \partial^{b_2} \ldots \partial^{b_k} F(a_1a_2\ldots a_j). \quad \text{(78)}$$

The quantities of unknown variables $N_{u}^k$ and of equations $N_{e}^k$ are equal to

$$N_{u}^k = \binom{j + m - 1}{m - 1}\binom{k + s + m - 1}{m - 1}, \quad N_{e}^k = \binom{j + s + m - 1}{m - 1}\binom{j + m - 1}{m - 1}, \quad \text{(79)}$$

so conditions (24) are also fulfilled.

It can be shown (see attachment) that the system (77) is non-degenerate, so it follows from (24) that

$$F(a_1a_2\ldots a_j, a_{j+1}a_{j+2}\ldots a_{j+s}, b_1b_2\ldots b_j) = 0. \quad \text{(80)}$$

Whence we conclude that the Killing tensors of rank $j$ and order $s$ are polynomials of order $j + s - 1$. It follows from (77), (79) that such polynomial contains $n_{js}^m$ arbitrary parameters, where

$$n_{js}^m = \sum_{i=0}^{s-1} \binom{j + m - 1}{m - 1}\binom{j + m - 1}{m - 1} + \sum_{k=0}^{s-1} (N_{u}^k - N_{e}^k)$$

$$= \frac{s}{m} \binom{j + m - 1}{m - 1}\binom{j + s + m - 1}{m - 1}. \quad \text{(80)}$$

Here the first sum gives the number of independent solutions that have order by $x$ smaller than $s$. Such solutions can be written in the form

$$F_i^{a_1a_2\ldots a_j} = \lambda_{b_1b_2\ldots b_i}^{a_1a_2\ldots a_j} x_1^{b_1} x_2^{b_2} \cdots x_i^{b_i}, \quad i < s, \quad \text{(81)}$$
where $\lambda_{b_1b_2\ldots b_i}$ are numeric parameters with no limitations set by equations (77) (certainly the symmetry with respect to permutations of indices $a_\lambda \leftrightarrow a'_\lambda$, $b_\mu \leftrightarrow b'_\mu$, $\lambda, \lambda' = 1, 2, \ldots, j$, $\mu, \mu' = 1, 2, \ldots, i$.

In the case $s = 1$ the formula (80) is reduced to (25). In particular, for $m = 2, 3, 4$ we obtain from (80) that

$$
\begin{align*}
    n^2_{js} &= \frac{1}{2} s(j + 1)(j + s + 1), \\
n^3_{js} &= \frac{1}{12} s(j + 1)(j + 2)(j + s + 1)(j + s + 2), \\
n^4_{js} &= \frac{1}{3!4!} s(j + 1)(j + 2)(j + 3)(j + s + 1)(j + s + 2)(j + s + 3), 
\end{align*}
$$

Thus we have determined the number of linearly independent Killing tensors of rank $j$ and order $s$. To compute these tensors in explicit form we will use the following two lemmas.

**Lemma 3.** Let $F^{a_1a_2\ldots a_j}_i$ be Killing tensors of rank $j$ and order $s$, and $\varphi$ be a function satisfying the equation

$$
\partial^\mu \partial^\nu \varphi = 0, \quad \mu, \nu = 1, 2 \ldots m.
$$

Then the function

$$
\tilde{F}^{a_1a_2\ldots a_j} = \varphi F^{a_1a_2\ldots a_j}
$$

is a Killing tensor of rank $j$ and order $s$.

**Proof** is reduced to direct check of the lemma statement that is $(s + 1)$-multiple differentiation of (84) by $\partial^{a_j+1}, \partial^{a_j+1}, \ldots, \partial^{a_j+s+1}$ and subsequent symmetrization of the obtained expression by $a_1, a_2, a_{j+s+1}$ using relations (16), (83).

**Lemma 4.** Let $F^{a_1a_2\ldots a_j}$ be a Killing tensors of rank $j$ and order $s$. Then the convolution

$$
\tilde{F}^{a_1a_2\ldots a_{j-1}} = F^{a_1a_2\ldots a_j} x_{a_j}
$$

is a Killing tensor of rank $j + 1$ and order $s + 1$.

**Proof** is similar.

The adduced Lemmas provide an effective algorithm for construction of Killing tensors of order $s$ from Killing tensors of order 1 found above in Section 4. The only difficulty in application of this algorithm is the need to sort out all linearly independent solutions of the system (16) (whose number is determined by the formulae (80), (82), as, generally speaking, there are more solutions of the form (84) than we need).

The general solution of equations (16) is determined in the following theorem.

**Theorem 3.** Equations (16) in the space of dimension $m \leq 4$ have $n^m_{js}$ linearly independent solutions where $n^m_{js}$ is given by the formula (82). These solutions have the form

$$
F^{a_1a_2\ldots a_j}_{(s)} = g^{(a_j-1)a_j} F^{a_1a_2\ldots a_{j-2}}_{(s)} + \varepsilon_j \tilde{F}^{a_1a_2\ldots a_j}
$$
where $F^{a_1 a_2 \ldots a_j+d-1}_{(s)}$ are Killing tensors of rank $j+d-1$ and of order 1 whose explicit form is given by Theorem 2, $F^{a_1 a_2 \ldots a_{j-2}}_{(s)}$ are Killing tensors of rank $j-2$ and of order $s$;

$$
\hat{f}^{a_1 a_2 \ldots a_j} = \sum_{\mu=0}^{\frac{j}{2}-1} (-1)^\mu \left( \frac{\frac{j}{2} - 1}{\mu} \right) x^{(a_1 a_2 \ldots a_{2\mu+1}} \times g^{a_{2\mu+2} a_{2\mu+3} \ldots} g^{a_{j-2} a_{j-1}} \lambda^{[a_j],c] x_c},
$$

(87)

$\lambda^{[a_j,c]}$ is an arbitrary antisymmetric tensor of rank 2.

**Proof.** By virtue of Lemmas 3, 4 the function $F^{a_1 a_2 \ldots a_j}_{(s)}$ given by formula (86) is a Killing tensor of rank $j$ and order $s$; the first term — by definition, the second — in accordance to Lemma 3 (being the product of of a Killing tensor of order 1 and $\varphi = \lambda^\mu x_\mu$), the third — in accordance to Lemma 4 (each convolution with $x_\mu$ lowers the rank and increases the order of a Killing tensor, and we make the first convolution with the first-order tensors described above).

It is to some extent more difficult to make sure that formula (86) gives all linearly independent Killing tensors of order $s$. Proof of linear independence of all terms of the formula (86) is reduced to comparison of terms having the same order by $x_{a_i}$ using different convolutions by one, two etc. pairs of indices. Calculation of thenumber of independent solutions given by formula (86) can be done easily by sorting through independent solutions for first-order tensors $F^{a_1 a_2 \ldots a_{j+d-1}}$ entering the last term (such tensors are described in Theorem 2, but it is necessary to restrict consideration with solutions corresponding to $c_2 > d-1$, as others give zero input into convolutions of (86)), and adding the number of solutions of the form (87). The result obtained is in compliance with the formula

$$
N = n_{js}^m - n_{j-2s}^m,
$$

where $n_{js}^m$ is the total number of solutions given by the formulæ (82), $n_{j-2s}^m$ is the number of solutions of the form $g^{(a_{j-1} a_j} F^{a_1 a_2 \ldots a_{j-2})}_{(s)}$ that is also given in (82), $N$ is the total number of solutions under the summation sign and of solutions of the form (87). We omit the corresponding cumbersome calculations.

Formula (86) determines recurrent relations for calculation of explicit form of a Killing tensor of rank $j$ and order $s$ from a known tensor of order $s$ and rank $j-2$. Such calculations may be easily checked starting from known killing tensors of order 1 and arbitrary rank, see Theorem 2.

Let us adduce as an example explicit expressions for Killing vectors of order $s \leq 3$ in three-dimensional space received from general relations (86):

$$
\begin{align*}
    s = 1, & \quad F^{a}_{(1)} = \lambda^a + \varepsilon^{abc} \eta_b x_c, \\
    s = 2, & \quad F^{a}_{(2)} = F^{a}_{(1)} + \lambda^{ab} x_b + \lambda x^a + \varepsilon^{abc} \eta_{bd} x_c x^d + \xi^a x^2 - x^a \xi^b x_b, \\
    s = 3, & \quad F^{a}_{(3)} = F^{a}_{(2)} + \lambda^{abc} x_b x_c + \lambda^b x_b x^a + \varepsilon^{abc} \eta_{bd} x_c x_d x_l
\end{align*}
$$

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Here $\varepsilon^{abc}$ is a unit antisymmetric tensor, and other Greek letters designate arbitrary symmetric tensors with zero trace.

9 Conformal Killing tensors of rank $j$ and order $s$

Let us briefly discuss equations (17) describing conformal Killing tensors of rank $j$ and order $s$, and adduce without proof solutions of these equations for arbitrary $j$, $s$ and $m \leq 4$.

A constructive way for finding solutions of equations (17) is shown by the following statement that may be proven by direct check.

**Lemma 5.** Let $F_{a_1a_2...a_j}^s$ be a conformal Killing tensor of rank $j$ and order $s$, and $\varphi$ be an arbitrary function satisfying the equation

$$\partial^\mu \partial^\nu \varphi = g^{\mu\nu}\lambda, \quad \lambda = \text{const.} \tag{88}$$

Then the function

$$F_{s+1}^{a_1a_2...a_j} = \varphi F_{s}^{a_1a_2...a_j} \tag{89}$$

is a conformal Killing tensor of rank $j$ and order $s + 1$.

We can show that quantities of linearly independent solutions of equations (17) for arbitrary $j$, $s$ and $m = 3, 4$ are given by formulae

$$m = 3, \quad \tilde{N}_{js}^3 = \frac{s}{6}(2j + 1)(2j + 2s + 1)(2j + s + 1), \tag{90}$$

$$m = 4, \quad \tilde{N}_{js}^4 = \frac{s}{12}(j + 1)^2(j + s + 1)^2(2j + 2 + s).$$

Using Lemma 5 we managed to construct $N_{js}^m$ linearly independent Killing tensors of rank $j$ and order $s$ (giving full system of solutions of equations (17)) in the following form:

$$F_{s}^{a_1a_2...a_j} = \sum_{i=1}^{s} \left( F_{i}^{a_1a_2...a_j}(x^2)^{i-1} + \sum_{d=0}^{s-i} f_{i-1-d}^{a_1a_2...a_j}(x^2)^d \right). \tag{91}$$

Here $F_{i}^{a_1a_2...a_j}$ are conformal tensors of rank $j$ and order 1 given by formulae (60), (61) or (66) (the index “$i$” distinguishes independent solutions of (91) with various degrees of $x^2$), $f_{i-1-d}^{a_1a_2...a_j}$ are tensors of rank $j$ whose explicit form is adduced below.

In the case $m = 2$ the number of conformal Killing tensors of order 1 appears to be infinite, see (69). The same formulae (69) give a general form of a conformal Killing tensor of arbitrary order $s$.

In the case $m = 3$ functions $f_{i-1-d}^{a_1a_2...a_j}$ are characterized by an additional integer $c$,

$$0 \leq c \leq 2j, \tag{92}$$

$$+ \xi^{ab}x_bx^2 - x^a\xi^{bc}x_bx_c + \varepsilon^{abc}x_b\xi_cx^2.$$
and are determined up to arbitrary symmetric zero-trace tensor $\tilde{\lambda}^{a_1 a_2 \ldots a_R}$ of rank $R = 2(c + i) - 1$. Explicit form of these functions is given by formula (93):

$$f^{a_1 a_2 \ldots a_j}_{idc} = \left[ \varepsilon_c f^{a_1 a_2 \ldots a_j}_{ide} + (1 - \varepsilon_c) f^{b(a_1 a_2 \ldots a_j, -1 \varepsilon a_j)bc} x_c \right]^{SL},$$  \hspace{1cm} (93)

where

$$f^{a_1 a_2 \ldots a_j}_{idc} = \sum_{n=0}^{\lfloor \frac{j}{2} \rfloor} (-2)^n \binom{\lfloor \frac{j}{2} \rfloor}{n} \tilde{\lambda}^{b_1 b_2 \ldots b_{d+n}(a_1 a_2 \ldots a_j - n)} \times x^{a_j - n+1} x^{a_j - n + 2} \ldots x^{a_j} x_{b_1} x_{b_2} \ldots x_{b_{d+n}} x^{2(n - \frac{j}{2} - n)} \varepsilon_c = \frac{1}{2} \left[ 1 + (-1)^c \right] \hspace{1cm} (94)$$

with the symbol $[\cdot]^{SL}$ designating the zero trace part of the corresponding tensor, see (18), (19) for $m = 3$, and the index $d$ is introduced for numeration of linearly independent solutions of (91) with different degrees of $x^2$.

In the case $m = 4$ the functions $f^{a_1 a_2 \ldots a_j}_{idc}$ are characterized by a pair of additional indices $c = (c_1, c_2)$

$$-j \leq c_1 \leq j, \hspace{0.5cm} 0 \leq c_2 \leq \left\lfloor \frac{j - |c_1|}{2} \right\rfloor$$  \hspace{1cm} (95)

and is determined up to arbitrary irreducible tensor $\tilde{\lambda}^{a_1 a_2 \ldots a_R \{a_{R_1 + i b_1} \ldots [a_{R_1 + R_2} b_{R_2}] \}}$ of rank $R_1 + 2R_2$ where

$$R_1 = |c_1| + 2c_2 + i, \hspace{0.5cm} R_2 = j - |c_1| - 2c_2. \hspace{1cm} (96)$$

Explicit form of these functions is given by the formula (97):

$$f^{a_1 a_2 \ldots a_j}_{idc} = \sum_{d=0}^{n + c_2} (-1)^d \binom{n + c_2}{d} \left( x^2 \right)^d \lambda^{b_1 b_2 \ldots b_{n+c_2-i}(a_1 a_2 \ldots a_{c_1-1-n+d+c_2+1} d_1) \ldots [a_{j-n+i+d-c_2} d_j - |c_1| - 2c_2] x^{a_j - n + i + d - c_2 + 1} \ldots x^{a_j} x_{b_1} \ldots x_{b_{n+c_2-i-d-i}} x_{d_1} \ldots x_{d_j - |c_1| - 2c_2} \right]^{SL}, \hspace{1cm} (97)$$

where

$$n = \left\{ \begin{array}{ll} -c_1, & c_1 < 0, \\ 0, & c_1 \geq 0, \end{array} \right. $$  \hspace{1cm} (98)

and symmetrization is implied in the right-hand part by indices $a_1, a_2, \ldots, a_j$.

The formulae (91)–(98) give in explicit form all linearly independent conformal Killing tensors of rank $j$ and order $s$ in space of dimension $m = p + q \leq 4$. In particular, conformal vectors of order $s \leq 3$ in three-dimensional space, in accordance to (91), (94) have the following form:

$$s = 1, \hspace{0.5cm} F^a_{(1)} = \lambda^a_{(1)} + \varepsilon^{abc} \eta^b_{(1)} x_c + \xi^a_{(1)} x^2 - 2x^a \xi^b_{(1)} x_b + \mu x^a;$$
\[ s = 2, \quad F_{(2)}^a = F_{(1)}^a + F_{(1)}^a x^2 + \lambda^{ab} \eta_{(2)} x_b x_d + \xi^{ab}_{(2)} x^2 x_b - 2 x^a \xi^{bc}_{(2)} x_b x_c; \]

\[ s = 3, \quad F_{(3)}^a = F_{(2)}^a + x^4 \tilde{F}_{(1)}^a + x^2 (\lambda^{ab}_{(3)} x_b + \varepsilon^{abc} \eta_{(3)} x_c x_d + \xi^{ab}_{(3)} x^2 - 2 x^a \xi^{bc}_{(3)} x_b x_c) \]
\[ + \lambda^{abc}_{(3)} x_b x_c + \varepsilon^{abc} \eta_{(3)} x_c x_d x_k + \varepsilon^{abc}_{(3)} x_b x_c x^2 - 2 x^a \xi^{bcd}_{(3)} x_b x_c x_d. \]

Here \( \varepsilon^{abc} \) is the unit antisymmetric tensor, and other Greek letters designate arbitrary symmetric traceless tensors \( F_{(1)}^a, \tilde{F}_{(1)}^a \) and \( \approx F_{(1)}^a \) are first-order Killing vectors (generally speaking, different).

## 10 Conclusion

Let us sum up. We have defined the notion of Killing tensor of rank \( j \) and order \( s \) and of conformal Killing tensor of rank \( j \) and order \( s \). These tensors are defined as general solutions of equations (16) or (17) that in the case \( s = 1 \) coincide with generally accepted equations for Killing tensors and conformal Killing tensors, see e.g. \([11]\).

We limit ourselves with investigation of equations (16) and (17) in flat de Sitter space, and generalization of these equations for for the case of spaces with non-zero curvature requires replacement of \( \partial^a \) for covariant derivatives.

Equations (16) and (17) are natural generalizations of the Killing equations \([2, 10]\) and arise in description of higher-order symmetry operators. In the present paper we show relation of these equations (for first-order tensors) with higher-order symmetry operators of Klein–Gordon–Fock equation, see Sections 2, 5. Equations for Killing tensors and conformal Killing tensors of order \( s < 1 \) arise in problems of description of symmetry operators of order \( s \) for systems of partial differential equations — in particular, for the Maxwell equations \([13]\). We have found in explicit form all non-equivalent Killing tensors of rank \( j \) and order \( s \) in the space of dimension \( p + q \) for arbitrary \( j \) and \( s \) and \( p + q \leq 4 \). Limitation by dimension of space is based, on the one hand, on practical reasons (the absolute majority of equations of mathematical physics being the field of research interests of the authors, have dimension \( m \leq 4 \) with respect to independent variables), and, on the other side, on difficulties that had not been overcome to the moment in proof of non-degeneracy of systems of algebraic equations for coefficients of Killing tensors in spaces of arbitrary dimension, see attachment. At that formulae (42), (66), (86) giving solutions of equations (16), (17) for \( p + q = 4 \), probably give the general solution of these equations for arbitrary \( p + q \geq 4 \).

The found general solutions for Killing tensors and conformal Killing tensors of arbitrary order and rank may find quite large use in description of symmetry operators of systems of partial differential equations. In this paper using these solutions we found full set of symmetry operators of arbitrary finite order for Klein–Gordon–Fock equations with zero and non-zero mass.
A Non-degeneracy of systems of equations for coefficients of Killing tensors

We will adduce proof of Theorem 1 stating non-degeneracy of system of linear algebraic equations (22). As it is cumbersome we adduce it in abridged form.

The main difficulty of the analysis of system (22) is the need to do it for arbitrary value of \( j \), that is for a system of arbitrary fixed dimension given by formula (20).

Let us consider equations (22) for \( m = 4 \), at that equations for \( m < 4 \) will be included into the analysis as particular cases. Indices \( a_1, a_2, \ldots, a_j+1 \) and \( b_1, b_2, \ldots, b_k \) with \( k \leq j \) independently take values from 1 to 4. At that, as it is easy to notice, the system (22) splits at non-linked subsystems \( M(s_1, s_2, s_3, s_4) \), where \( s_l \ (l = 1, 2, 3, 4) \) gives the number of indices having the value \( l \). It is obvious that

\[
s_1 + s_2 + s_3 + s_4 = j + 1 + k,
\]

so \( 0 \leq s_i \leq j + k + 1 \).

System (22) is non-degeneracy iff all its subsystems \( M(s_1, s_2, s_3, s_4) \) are non-degeneracy. Without loss of generality, for arbitrary subsystem \( M(s_1, s_2, s_3, s_4) \) we can put

\[
s_1 \leq s_2 \leq s_3 \leq s_4,
\]

other cases can be reduced to (A.2) by renumeration of variables.

Let us prove non-degeneracy of an arbitrary subsystem \( M(s_1, s_2, s_3, s_4) \).

We designate by the symbol \( n_l \ (l = 1, 2, 3, 4) \) the number of indices of unknown variable \( F^{a_1a_2...a_j,a_{j+1}b_1b_2...b_k} \) present on the left of the comma and equal to \( l \), and by the symbol \( m_l \) — the number of indices after the comma that are equal to \( l \). Obviously, the following should be satisfied,

\[
m_c + n_c = s_c, \quad n_1 + n_2 + n_3 + n_4 = j, \quad m_1 + m_2 + m_3 + m_4 = k + 1,
\]

so out of eight numbers \( n_l \) and \( m_l \) only three will be linearly independent (see (A.1)). Let us choose the following numbers as independent: \( n_1, n_2 \) and \( n_3 \), then the triple \( (n_1, n_2, n_3) \) will completely determine a vector \( F^{a_1a_2...a_j,a_{j+1}b_1b_2...b_k} \) from the subsystem \( M(s_1, s_2, s_3, s_4) \). Using for such vector the designation \( F(n_1, n_2, n_3) \) and considering relations (A.1)–(A.3), we can write any equation (22) from the subsystem \( M(s_1, s_2, s_3, s_4) \) in one of the following forms:

\[
(n_3 + 1)F(0, 0, n_3) + (j - n_3)F(0, 0, n_3 + 1) = 0,
\]

\[
\max\{s_3 - k - 1, -1\} \leq n_3 \leq s_3 - 1;
\]

\[
(n_2 + 1)F(0, n_2, n_3) + n_3F(0, n_2 + 1, n_3 - 1) + (j - n_2 - n_3)F(0, n_2 + 1, n_3) = 0,
\]

\[
\max\{0, s_1 + s_2 - k - 1\} \leq n_2 \leq s_2 - 1,
\]

\[
\max\{0, s_1 + s_2 + s_3 - k - 1 - n_2\} \leq n_3 \leq \min\{s_3, j - n_2\};
\]

\[
(n_1 + 1)F(n_1, n_2, n_3) + n_2F(n_1 + 1, n_2 - 1, n_3) + n_3F(n_1 + 1, n_2, n_3 - 1) + (j - n_1 - n_2 - n_3)F(n_1 + 1, n_2, n_3) = 0,
\]

\[
\max\{0, s_1 - k - 1\} \leq n_1 \leq s_1 - 1, \quad \max\{0, s_1 + s_2 - k - 1 - n_1\} \leq n_2 \leq s_2,
\]
\[
\max\{0, s_1 + s_2 + s_3 - k - 1 - n_1 - n_2\} \leq n_3 \leq \min\{s_3, j - n_1 - n_2\}.
\]

(A.6)

When \(s_1 > 0\) (the case \(s_1 = 0\) is considered below) there are three possibilities:

1. \(s_1 + s_2 < k + 1\),
2. \(s_1 + s_2 \geq k + 1, \quad s_1 < k + 1\),
3. \(s_1 \geq k + 1\).

(A.7)

Let us consider these possibilities one by one.

In the case 1 the system under consideration is given by the formulae (A.4)–(A.6).

Let us present the vector \(F(n_1, n_2, n_3)\) in the form of a column whose components are numbered by the index

\[
F(n_1, n_2, n_3) = (F(n_1, n_2, \tilde{n}_3), F(n_1, n_2, \check{n}_3 + 1) \ldots F(n_1, n_2, \hat{n}_3))^T,
\]

(A.8)

where \(\tilde{n}_3\) and \(\hat{n}_3\) are minimal and maximal values of \(n_3\), and each vector \(F(n_1, n_2, \tilde{n}_3 + k)\), in its turn, will be regarded as a column whose components are numbered by the index \(n_2\), \(0 \leq n_2 \leq s_2\). Then it is possible to write equations (A.4)–(A.6) in the matrix form:

\[
AF = 0,
\]

(A.9)

where

\[
A = \begin{pmatrix}
B_0 & E_1 & B_1 & 2E_2 & B_2 & \cdots & s_1E & B_{s_1} \\
E_1 & 2E_2 & B_2 & \cdots & \cdots & \cdots & \cdots & \cdots \\
2E_2 & B_2 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \cdots & \cdots & \cdots & \cdots \\
s_1E & B_{s_1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix},
\]

(A.10)

\[
B_l = \begin{pmatrix}
D_l & E_{l1} & D_{l+1} & 2E_{l2} & D_{l+2} & \cdots & s_2E_{ls_2} & D_{l+s_2} \\
E_{l1} & D_{l+1} & 2E_{l2} & D_{l+2} & \cdots & \cdots & \cdots & \cdots \\
2E_{l2} & D_{l+2} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \cdots & \cdots & \cdots & \cdots \\
D_{l+s_2} & s_2E_{ls_2} & D_{l+s_2} & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}. 
\]

Here \(E_k\) and \(E_{lf}\) \((l, k = 1, 2, \ldots, s_1, f = 1, 2, \ldots, s_2)\) are unit matrices whose number of rows coincides with the number of rows of the adjacent matrices on the right (the number of columns of \(B_k\) \((D_l)\) coincides with the number of rows of \(B_{k+1}\) \((D_{l+1})\)), \(D_l\) being the matrices whose explicit form is determined below. Namely, with \(s_1 + s_2 + s_3 < k + 1\):

\[
D_l = \begin{pmatrix}
j - l + 1 & 1 & j - l \\
1 & 2 & j - l - 1 \\
\vdots & \vdots & \ddots \cdots \\
& & & s_3 & j - l + 1 - s_3 \\
\end{pmatrix}.
\]

(A.11)
with \( s_1 + s_2 + s_3 \geq k + 1 \)

\[
D_l = \begin{pmatrix}
    a_1 & j - a_1 - l + 1 \\
    a_1 + 1 & j - a_1 - l \\
    & \ddots & \ddots \\
    & & a_2 & j - a_2 - l + 1
\end{pmatrix}, \quad (A.12)
\]

where \( a_1 = \max\{0, s_1 + s_2 + s_3 - k - l\} \), \( a_2 = \min\{s_3, j - l + 1\} \).

At that in the cases \( a_1 = 0 \) or (and) \( s_3 \geq j - l + 1 \) in the matrix (A.12) the first or (and) last column should be crossed out.

Note A.1. It can be shown that there are always will be less of matrices \( D_l \) of the form (A.11) (or (A.12) \( a_1 \neq 0 \) \( j - a_1 - l + 1 \neq 0 \)) than of the matrices (A.12) with \( a_1 = j - a_2 - l + 1 \equiv 0 \).

Our task is to prove that all rows of the matrix \( A \) (A.10) are linearly independent.

Writing this matrix in the equivalent form

\[
A' = \begin{pmatrix}
    E_1 & B_1 \\
    2F_2 & B_2 \\
    & \ddots & \ddots \\
    B_0 & s_1E_{s_1} & B_{s_1}
\end{pmatrix}
\]

and subjecting \( A' \) to the transformation \( A' \rightarrow A'' = VA'W \) that does not change the rank, with \( V \) and \( W \) being reversible matrices of the form

\[
V = \begin{pmatrix}
    E_1 \\
    E_2 \\
    \vdots \\
    -B_0 & B_0B_1 & \cdots & (-1)^{s_1+1}\frac{1}{s_1!}B_0B_1B_2\cdots B_{s_1}E_0
\end{pmatrix},
\]

\[
W = \begin{pmatrix}
    E_1 & -B_1 & \frac{1}{2!}B_1B_2 & \cdots & (-1)^{s_1+1}\frac{1}{s_1!}B_0B_1B_2\cdots B_{s_1-1} \\
    E_2 & -B_2 & \frac{1}{2!}B_2B_3 & \cdots & (-1)^{s_1+1}\frac{1}{s_1!}B_0B_1B_2\cdots B_{s_1-1} \\
    \vdots & \vdots & \ddots & \ddots & \vdots \\
    & & & E_{s_1} & -B_{s_1-1} \\
    & & & \frac{1}{s_1!}B_0B_1B_2\cdots B_{s_1}
\end{pmatrix},
\]

we get

\[
A'' = \begin{pmatrix}
    E_1 \\
    E_2 \\
    \vdots \\
    E_{s_1} \\
    \frac{1}{s_1!}B_0B_1B_2\cdots B_{s_1}
\end{pmatrix}, \quad (A.13)
\]
and proof of linear independence of the rows of the matrix (A.10) is reduced to proof of linear independence of the rows of the matrix

\[ \hat{B}^{s_1} = B_0 B_1 \cdots B_{s_1}. \]  

(A.14)

**Lemma A.1.** The matrix (A.14) can be split into blocks \( \hat{B}^{s_1}_{ij} \) including the same number of rows as the matrix \( D_{s_1+1-l}, 1 \leq l \leq s_2+1, 1 \leq f \leq s_2+1 \). These blocks have the following form:

\[
B^{s_1}_{ij} = \begin{cases} 
\hat{0}, & -s_2 \leq l-f < 0 \text{ or } s_1+2 \leq l-f \leq s_2, \\
\left( \frac{s_1+1}{l-f} \right) P_{l-f}^{l-1} D_{l-1} D_l \cdots D_{s_1+f-1}, & 0 \leq l-f \leq s_1, \\
\left( \frac{s_1+1}{s_1+1} \right) P_{s_1+1}^{l-1} E, & l-f = s_1+1.
\end{cases}
\]  

(A.15)

Here \( D_{l-1}, D_l, \ldots \) are matrices from (A.10), the general form of which is given by the formulae (A.11), (A.12), \( \hat{0} \) and \( E_k \) are zero and unity matrices of corresponding dimensions, \( \binom{m}{k} = \frac{m!}{k!(m-k)!} \), \( P_m = \left( \frac{k!}{(k-m)!} \right)! \).

**Proof.** can be done by induction, by successive investigation of the products \( B_0 B_1, B_0 B_1 B_2, \ldots \).

**Lemma A.2.** The matrix (A.14) by finite number of elementary transformations can be reduced to the form

\[
\hat{B}^{s_1} = \begin{pmatrix}
D_0 D_1 \cdots D_{s_1+s_2} \\
D_{s_1-1} D_{s_1} \cdots D_{s_2+1} \\
D_{s_1} D_{s_1+1} \cdots D_{s_2} \\
E_{s_2-s_1} \\
\vdots \\
E_2 \\
E_1
\end{pmatrix}
\]  

(A.16)

**Proof.** We can describe elementary transformations mentioned in the lemma in the following way:

1. Let us go from the matrix (A.14) to the matrix \( \Pi_d \) (with \( d \) successively taking values \( 1, 2, \ldots, s_2-s_1 \)), using the following algorithm:

   1) Represent the matrix (A.14) in the block form (A.15) and operate with “rows” with the number \( l \), including blocks \( \hat{B}^{s_1}_{i_1}, \hat{B}^{s_1}_{i_2}, \ldots, \hat{B}^{s_1}_{i_{s_2+1}} \).

   2) Multiply \((t+1)\)-th “row” of the matrix \( \hat{B}^{s_1} \) by the matrix \( D_{l-1} \), and the “row” of the matrix \( \hat{B}^{s_1} \) with the number \( t \) — by the number \( s_1-t+2 \) and subtract the latter from the former. Write the obtained result instead of the “row” with the number \( t \). Perform this operation successively with all “rows” for \( t = 1, 2, \ldots, s_1 \).

   3) Multiply \((s_1+2)\)-th “row” of the matrix \( \hat{B}^{s_1} \) by \( D_{s_1} \cdots D_{s_2} \cdots D_{s_1+d-1} \) and subtract from the obtained result the “row” with the number \( s_1 + 1 \) multiplied by \( d \). Write the
obtained result instead of the “row” with the number \( s_1 \), leaving the remaining “rows” unchanged.

4) As a result of the described transformations \( \hat{B}^s \to \Pi'_d \), where \( \Pi'_d \) is the matrix having in the first column the only non-vanishing element \( (\Pi'_d)_{s_1+2} \). It is possible to get by means of elementary transformations that all elements of the \( s_1 + 2 \)-th “rows” would also vanish (except \( (\Pi'_d)_{s_1+2} \)).

5) Let us put the \( s_1 + 2 \)-th “row” to the lowest position. As a result we get the matrix \( \Pi_{s_1-s_2} \) of the following form:

\[
\Pi_{s_2-s_1} = \begin{pmatrix}
    & & & M \\
    & \cdots & E_{s_2-s_1} \\
    E_1 & E_2 \\
\end{pmatrix}, \tag{A.17}
\]

where \( M \) is a matrix that can be split into blocks of the following form:

\[
M_{lf} = \begin{cases}
    0, & -s_1 \leq l - f < 0, \\
    (s_2 + 1) & f \\
    (l - f) & l - f
\end{cases} \prod_{l-f} D_{l-1} D_l \cdots D_{f+s_2+1}, \quad 0 \leq l - f \leq s_1, \quad (f, l) = 1, 2, \ldots, s_1+1.
\]

2. Let us simplify the matrix \( M \) by using successively the adduced algorithm for \( q = 0, 1, \ldots, s_1 - 1 \), and for each value of \( q \) — for \( k = 1, 2, \ldots, s_1 - q \).

1) Let us multiply the \((k+1)\)-th “row” of the matrix \( M \) by the matrix \( D_{k-1} \), and the \( k \)-th column by the number \( s_1 - k + 2 \), and subtract the former from the latter, leaving other “rows” unchanged.

2) Perform this operation successively for all \( k = 1, 2, \ldots, s_1 - q \), simplifying the matrices obtained at each step by means of elementary transformations vanishing all elements of “rows” except one that was the only non-vanishing in its column.

3) Perform operations 1), 2) successively for all \( q = 0, 1, \ldots, s_1 - 1 \).

As a result we come to the matrix \( (A.16) \). The lemma is proved.

**Lemma A.3.** Let the matrix \( D = \|d_{ab}\|, a = 1, 2, \ldots, s, b = 1, 2, \ldots, r, s < r \) have the rang \( s \), with all minors \( D \) being positive, and the matrix \( B \) have the form

\[
B = \begin{pmatrix}
    b_{11} & b_{12} & \ldots & b_{1r} \\
    b_{21} & b_{22} & \ldots & b_{2r} \\
    \vdots & \vdots & \ddots & \vdots \\
    b_{r-1r-2} & b_{r-1r-1} & \ldots & b_{rr-1} \\
    b_{r+r-1} & b_{r+r-2} & \ldots & b_{rr}
\end{pmatrix}, \tag{A.18}
\]

where \( b_{kk} > 0 \) and \( b_{k+1k} > 0 \), \( k = 1, 2, \ldots, r - 1 \). Then the matrix \( C = DB \) also has the rank \( s \), and all minors \( C \) are positive.

**Proof** is reduced to direct utilization of the Binet–Cauchy formula \([15]\) representing spinors of the matrix product \( DB \) via sum of the products of minors of the matrix \( D \) by
minors of the matrix $B$. As a result each minor of the matrix $C$ can be represented as the sum of positive values. The Lemma is proved.

By virtue of the Lemma A.2 proof of linear independence of the matrix $A$ (A.10) is reduced to proof of linear independence of rows of the matrix $D^d$,

$$D^d \equiv D_{s_1-d}D_{s_1-d+1} \cdots D_{s_2+d}, \quad d = 0, 1, 2, \ldots, s_1,$$

(A.19)

where $D_{s_1-d}, D_{s_1-d+1}, \ldots$ are matrices of the form (A.11) or (A.12). By virtue of Note A.1 the number of rows of each matrix (A.19) does not exceed the number of its columns. Considering successively the products $D_{s_1-d}D_{s_1-d+1}, D_{s_1-d}D_{s_1-d+1}D_{s_1-d+2}, \ldots$ and using each time either the Silvester inequality [16] or Lemma A.3, it is not difficult to show that the rank of the matrix (A.19) coincides with the number of its rows, and, whence, all rows of the matrix $A$ (A.10) are linearly independent.

We have proved non-degeneracy of of the system (A.4)–(A.6) for the case 1 from (A.7). In the case 2 when $s_1 + s_2 \geq k + 1$, the system (A.4)–(A.6) is reduced to equations (A.5), (A.6) that can also be written in matrix form (A.9) where $A$ is given by (A.10), but the blocks $B_l$ have a new form. Namely, with $s_2 < k + 1$

$$B_l = \begin{pmatrix}
(s_1 + s_2 - k - l)E_0 & D_0 \\
(s_1 + s_2 - k - l + 1)E_1 & D_1 \\
& \ddots \\
& & \ddots \\
& & & s_2E_{k-s_1+l} & D_{k-s_1+l}
\end{pmatrix},$$

(A.20)

$l = 0, 1, \ldots, s_1 + s_2 - k - 1$, and

$$B_l = \begin{pmatrix}
D_{l+k-s_1-s_2} & E_1 & \cdots & \cdots & \cdots \\
E_1 & D_{l+k-s_1-s_2+1} & \cdots & \cdots & \cdots \\
& \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots \\
& & & & D_{l+k-s_1}
\end{pmatrix},$$

(A.21)

if $s_1 + s_2 - k \leq l \leq s_1$. If $s_2 \geq k$, then all matrices $B_l$, $0 \leq l \leq s_1$, are given by the formula (A.20).

Explicit form for the matrices $D_l$ is given by relations (A.22)–(A.23):

for $s_2 < k$

$$D_l = \begin{pmatrix}
a_1 & j - l - a_1 \\
a_{1+1} & j - l - a_{1+1} \\
& \ddots \\
a_2 & j - l - a_2
\end{pmatrix},$$

(A.22)

where $a_1 = \max\{0, s_1 + s_2 + s_3 - k - l - 1\}$, $a_2 = \min\{s_3, j - l\}$, $l = 0, 1, \ldots k - d_1$,

$$D_l = \begin{pmatrix}
a_1 & j - l + k + 1 - s_1 - s_2 - a_1 \\
a_1 + 1 & j - l + k - s_1 - s_2 - a_1 \\
& \ddots \\
& & \ddots \\
a_2 & j - l + k - s_1 - s_2 - a_2
\end{pmatrix},$$

(A.23)
where \( a_1 = \max\{0, s_3 - l\}, a_2 = \min\{s_3, j - l + k + 1 - s_1 - s_2\}, l = k - s_1 + 1, k - s_1 + 2, \ldots, k. \)

If \( a_1 = 0 \) then first columns in (A.22) and (A.23) should be crossed out, if \( s_3 \geq j - l \), then the last column in (A.22) should be crossed out, and with \( s_3 \geq j - l + k + 1 - s_1 - s_2 \) it is necessary to cross out the last column in (A.23).

In the case \( s_2 \geq k \) the matrices \( D_l \) are given by the formula (A.22) for all \( l. \)

Our task is to prove linear independence of rows of the matrix \( A \) determined by relations (A.10), (A.20)–(A.23). Transforming this matrix to the form (A.13) we reduce this problem again to investigation of the matrix (A.14) that in our case can be split into blocks of the form

\[
\begin{align*}
\hat{B}_{ij}^{S_l} = \begin{cases}
\hat{0}, & -s_2 \leq l - f < s_1 + s_2 - k \text{ or } k - s_2 + 2 \leq l - f \leq s_2, \\
\begin{pmatrix}
s_1 + 1 \\
l + s_1 + s_2 - k - f
\end{pmatrix} & P_{l+s_1+s_2-k-1}^{l+s_1+s_2-k-f} D_{l-1} D_l \cdots D_{k-s_2-1-f}, \\
-s_1 - s_2 + k \leq l - f \leq k - s_2, & (s_1 + 1) P_{s_1+1}^{l+s_1+s_2-k-1} E, \\
\end{cases} \\
\end{align*}
\]

(A.24)

where \( 1 \leq l \leq k + 1 - s_2, 1 \leq f \leq s_2 + 1. \)

Further proof is done in full analogy with the proof for the case 1.

Let us consider now the third case from (A.7). The corresponding system of equations (A.4)–(A.6) is reduced to equations (A.6). Writing these equations in the matrix form (A.9) we come to the corresponding matrix \( A \) of the following form:

\[
A = \begin{pmatrix}
(s_1 - k) E_0 & B_0 \\
(s_1 - k + 1) E_1 & B_1 \\
& \ddots & \ddots \\
& & s_1 E_k & B_k
\end{pmatrix},
\]

(A.25)

where

\[
B_r = \begin{pmatrix}
(s_2 - R) \hat{E}_0 & D_0 \\
(s_2 - R + 1) \hat{E}_1 & D_1 \\
& \ddots & \ddots \\
& & s_2 \hat{E}_R & D_R
\end{pmatrix}, \quad R = 0, 1, \ldots, k,
\]

\[
D_l = \begin{pmatrix}
(s_3 - l) j + k + 1 - s_1 - s_2 - s_3 \\
s_3 - l + 1 & j + k - s_1 - s_2 - s_3 \\
& \ddots & \ddots \\
& & s_3 & j + k + 1 - s_1 - s_2 - s_3 - l
\end{pmatrix}, \quad l = 0, 1, \ldots, k.
\]

All rows of the matrix \( A \) (A.25) are evidently linearly independent.

Thus we had proved that the system of equations (A.4)–(A.6) is non-degenerate in all cases listed in the formulae (A.7). Thus the system (72) in the case \( m = 4 \) is non-degenerate.
Considering only such systems of equations (A.4)–(A.6) that correspond to \( S_1 = 0 \) we get a full set of non-linked subsystems of the system (72) for \( m = 3 \), and in the case \( s_1 = s_2 = 0 \) we come to full set of non-linked subsystems of the system (72) for \( m = 2 \). Consequently non-degeneracy of the system (72) for \( m = 2 \) and \( m = 3 \) follows from the adduced proof as a particular case.

Similarly (but with involvement of somewhat more cumbersome calculations) it is possible to prove non-degeneracy of the systems of linear algebraic equations (74) for coefficients of Killing tensors of rank \( j \) and order \( s \).

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