DIRICHLET SERIES ANALOGUES OF $q$-SHIFTED FACTORIAL AND THE $q$-KUMMER SUM.

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Abstract. Recently, the concept of a $D$-analogue was introduced by the author. This is a Dirichlet series analogue for the already known and well researched hypergeometric $q$-series. We consider the $D$-analogues of the $q$-binomial coefficients, and a $D$-analogue of the $q$-Kummer (Bailey-Daum) sum.

1. Introduction

In a recent paper [23] the idea of a $D$-analogue was introduced. This is a Dirichlet series analogue for the already known and well researched hypergeometric $q$-series, often called the basic hypergeometric series. The $q$-series is itself an analogue for the ordinary hypergeometric series developed by Gauss in the early nineteenth century. We note that the ordinary hypergeometric series was introduced by Gauss [28] in 1813, while the $q$-series analogue’s are originally due to Heine [31, 32] in the mid 19th century.

Both types of hypergeometric series have been the topics of far-reaching development and application throughout the 20th century. The full range of applications would be exhaustive, but one only needs cite such works as those of Andrews (5 to 8), Askey (10 to 13) and Baxter (16 to 18) to affirm that the impact of $q$-series has been great.

The author proposes that the newly introduced $D$-series analogue should likewise turn out to be the subject of future development and application.

A superficial comparison of the simplest $\phi_1$ three types of analogue hypergeometric series is

\[ 2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{k!(c)_k} z^k, \tag{1.1} \]
\[ 2\phi_1(a, b; c; q, z) = \sum_{k=0}^{\infty} \frac{(a; q)_k(b; q)_k}{(q; q)_k(c; q)_k} z^k, \tag{1.2} \]
\[ 2\Theta_1(a, b; c; \gamma; z) = \sum_{k=1}^{\infty} \frac{\sigma_{-\gamma}(a; k)\sigma_{-\gamma}(b; k)}{\sigma_{-\gamma}(c; k)} \frac{1}{k^z}. \tag{1.3} \]

In these, each kind of series has its own factorial analogue function as follows.

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\begin{equation}
(\sigma)_k = \begin{cases} 1, & k = 0; \\
(a)(a+1)\cdots(a+k-1), & k = 1, 2, \ldots; 
\end{cases}
\end{equation}

\begin{equation}
(a; q)_k = \begin{cases} 1, & k = 0; \\
(1-a)(1-aq)\cdots(1-aq^{k-1}), & k = 1, 2, \ldots; 
\end{cases}
\end{equation}

\begin{equation}
\sigma_\gamma(a; k) = \begin{cases} 1, & k = 1; \\
\frac{\sigma_\gamma(k)\sigma_\gamma(k \prod_{p|k} p)\cdots\sigma_\gamma(k \prod_{p|k} p^{\gamma-2})}{\sigma_\gamma(1)\sigma_\gamma(\prod_{p|k} p)\cdots\sigma_\gamma(\prod_{p|k} p^{\gamma-2})}, & k = 2, 3, \ldots. 
\end{cases}
\end{equation}

In the case of (1.4) the similarity to a factorial, \(k!\), is clear, and \((\sigma)_k\) is known as the shifted factorial. In the case of (1.5), \((a; q)_k\) is called the \(q\)-shifted factorial function. In (1.6) \(\gamma\) must be an integer; a restriction that does not apply to (1.4) nor (1.5). For (1.6) the implied “\(D\)-shifted factorial” in relation to the factorial is more disguised. If \(k\) has a prime decomposition \(p_1^{\gamma_1}p_2^{\gamma_2}\cdots p_m^{\gamma_m}\), then some insight is gained from writing the fraction of (1.6) as

\begin{equation}
\frac{\sigma_\gamma(p_1^{\gamma_1}p_2^{\gamma_2}\cdots p_m^{\gamma_m})\sigma_\gamma(p_1^{\gamma_1+1}p_2^{\gamma_2+1}\cdots p_m^{\gamma_m+1})\cdots\sigma_\gamma(p_1^{\gamma_1+a-1}p_2^{\gamma_2+a-1}\cdots p_m^{\gamma_m+a-1})}{\sigma_\gamma(p_1^{\gamma_1}p_2^{\gamma_2}\cdots p_m^{\gamma_m})\sigma_\gamma(p_1^{\gamma_1+1}p_2^{\gamma_2+1}\cdots p_m^{\gamma_m+1})\cdots\sigma_\gamma(p_1^{\gamma_1+a-1}p_2^{\gamma_2+a-1}\cdots p_m^{\gamma_m+a-1})},
\end{equation}

in which the function \(\sigma_\gamma(k)\) is the sum of \(n\)th powers of the divisors of \(k\), whence (see [1], [9], [30] or [36])

\begin{equation}
\sigma_\gamma(k) = \sum_{d|k} d^{-\gamma} = k^{-\gamma}\sigma_\gamma(k) = \prod_{i=1}^{m} \frac{1-p^{-\gamma(a_i+1)}}{1-p^{-\gamma}}.
\end{equation}

We see from this that (1.7) can be rewritten as

\begin{equation}
\frac{\sigma_\gamma(p_1^{\gamma_1}p_2^{\gamma_2}\cdots p_m^{\gamma_m})\sigma_\gamma(p_1^{\gamma_1+1}p_2^{\gamma_2+1}\cdots p_m^{\gamma_m+1})\cdots\sigma_\gamma(p_1^{\gamma_1+a-1}p_2^{\gamma_2+a-1}\cdots p_m^{\gamma_m+a-1})}{\sigma_\gamma(p_1^{\gamma_1}p_2^{\gamma_2}\cdots p_m^{\gamma_m})\sigma_\gamma(p_1^{\gamma_1+1}p_2^{\gamma_2+1}\cdots p_m^{\gamma_m+1})\cdots\sigma_\gamma(p_1^{\gamma_1+a-1}p_2^{\gamma_2+a-1}\cdots p_m^{\gamma_m+a-1})k^{\gamma a},
\end{equation}

and an alternative form of (1.6) is

\begin{equation}
\sigma_\gamma(a; k) = \begin{cases} 1, & k = 1; \\
\frac{\sigma_\gamma(k)\sigma_\gamma(k \prod_{p|k} p)\cdots\sigma_\gamma(k \prod_{p|k} p^{\gamma-2})}{\sigma_\gamma(1)\sigma_\gamma(\prod_{p|k} p)\cdots\sigma_\gamma(\prod_{p|k} p^{\gamma-2})k^{\gamma a}}, & k = 2, 3, \ldots;
\end{cases}
\end{equation}

so then

\begin{equation}
\sigma_\gamma(a; k) = \frac{\sigma_\gamma(a; k)}{k^{\gamma a}}.
\end{equation}

A further comparison between the three kinds of analogue is shown by the three Gauss hypergeometric sum formulae in terms of gamma functions, \(q\)-shifted gamma functions as given in Askey [11, 12, 13] for example, and (perhaps) \(D\)-shifted gamma functions. These are, for various conditions given in respectively, Bailey [15] pages 2–3], Gasper and Rahman [27] pages 9–11, and the author’s paper [23], as
Theorem 1.1. (Campbell [23]) For positive integers \( n, \Re \beta > 1, \Re \gamma > 0, \)

\[
\sum_{k=1}^{\infty} \frac{\sigma_{\gamma}(n; k)}{k^{\beta}} = \prod_{k=0}^{n-1} \zeta(\beta + k\gamma),
\]

\[
\sum_{k \in S_m} \frac{\sigma_{\gamma}(n; k)\lambda(k)}{k^{\beta}} = \prod_{k=0}^{n-1} \frac{1}{\sigma^{-(\beta+k\gamma)}(\prod_{p \mid m} p^{\gamma})}.
\]
where \( \lambda(k) \) is the Liouville function,

\[
(1.18) \quad \lambda(k) = \prod_{i=1}^{t} (-1)^{a_i} \text{ for each } k = \prod_{i=1}^{t} p_i^{a_i}.
\]

See Apostol [9 page 37], or any of [30 36 38] for classical accounts describing this function. The Liouville function frequently arises in \( D \)-analogues where a \( q \)-series with a specific negative parameter is transformed.

Before stating the \( D \)-analogue for the Kummer theorem, we again examine briefly the function \( \sigma_{\gamma}(n; k) \). Theorem 1.1 gives us a good starting point. In [23] we stated the cases of (1.16), letting successively, \( n = 1, n = 2, n = 3, n = 4 \), with \( \beta \) mapped onto \( \beta + \gamma \),

\[
(1.19) \quad \sum_{k=1}^{\infty} \frac{\sigma_{\gamma}(1; k)}{k^{\beta+\gamma}} = \sum_{k=1}^{\infty} \frac{1}{k^{\beta+\gamma}} = \zeta(\beta + \gamma),
\]

\[
(1.20) \quad \sum_{k=1}^{\infty} \frac{\sigma_{\gamma}(2; k)}{k^{\beta+\gamma}} = \sum_{k=1}^{\infty} \frac{\sigma_{\gamma}(k)}{k^{\beta+\gamma}} = \zeta(\beta + \gamma)\zeta(\beta + 2\gamma),
\]

\[
(1.21) \quad \sum_{k=1}^{\infty} \frac{\sigma_{\gamma}(3; k)}{k^{\beta+\gamma}} = \sum_{k=1}^{\infty} \frac{\sigma_{\gamma}(k)\sigma_{\gamma}(k\prod p_k \ p)}{\sigma_{\gamma}(\prod p_k \ p)} \frac{1}{k^{\beta+\gamma}} = \zeta(\beta + \gamma)\zeta(\beta + 2\gamma)\zeta(\beta + 3\gamma),
\]

\[
(1.22) \quad \sum_{k=1}^{\infty} \frac{\sigma_{\gamma}(4; k)}{k^{\beta+\gamma}} = \sum_{k=1}^{\infty} \frac{\sigma_{\gamma}(k)\sigma_{\gamma}(k\prod p_k \ p)\sigma_{\gamma}(k\prod p_k \ p^2)}{\sigma_{\gamma}((\prod p_k \ p)\sigma_{\gamma}(\prod p_k \ p^2))} \frac{1}{k^{\beta+\gamma}} = \zeta(\beta + \gamma)\zeta(\beta + 2\gamma)\zeta(\beta + 3\gamma)\zeta(\beta + 4\gamma),
\]

and so on, valid where each of the Riemann zeta functions is in its range of absolute convergence. It is clear from (1.19) to (1.22) that \( \sigma_{\gamma}(n; k) \) simplifies when \( k \) is a prime, or a power of a prime. If \( k \) is a product of distinct primes, or an integer power of a product of distinct primes the function is also simpler to examine. The following is quite easy to ascertain from the above and from (1.6).

**Theorem 1.2.** For positive integers \( a \) and for any prime \( p \),

\[
(1.23) \quad \sigma_{\gamma}(a; p) = \sigma_{\gamma}(p^{a-1}) = \frac{1 - p^{-a\gamma}}{1 - p^{-\gamma}}.
\]

**Theorem 1.3.** If \( k \) is a product of distinct primes, then for positive integers \( a \),

\[
(1.24) \quad \sigma_{\gamma}(a; k) = \sigma_{\gamma}(k^{a-1}) = \prod_{p|k} \frac{1 - p^{-a\gamma}}{1 - p^{-\gamma}}.
\]

**Theorem 1.4.** If \( k \) is the square of prime \( p \), then for positive integers \( a \),

\[
(1.25) \quad \sigma_{\gamma}(a; k) = \begin{cases} 
\sigma_{\gamma}(p^a) = \frac{1 - p^{-(a+1)\gamma}}{1 - p^{-\gamma}}, & k = 2; \\
\frac{\sigma_{\gamma}(p^a)}{\sigma_{\gamma}(p)} = \frac{1 - p^{-(a+1)\gamma}}{1 - p^{-2\gamma}}, & k \neq 2.
\end{cases}
\]
Theorem 1.5. If \( k \) is the square of a product of distinct primes, then for positive integers \( a \),

\[
\begin{align*}
\sigma_{-\gamma}(a; k) &= \begin{cases} 
\sigma_{-\gamma}(k^a) = \prod_{p|k} \frac{1 - p^{-(a+1)\gamma}}{1 - p^{-\gamma}}, & k = 2; \\
\frac{\sigma_{-\gamma}(k^a)}{\sigma_{-\gamma}(\sqrt{k})} = \prod_{p|k} \frac{1 - p^{-(a+1)\gamma}}{1 - p^{-2\gamma}}, & k \neq 2.
\end{cases}
\end{align*}
\]

The general version of (1.19) to (1.22),

\[
\sum_{k=1}^{\infty} \frac{\sigma_{-\gamma}(a; k)}{k^{\beta+\gamma}} = \prod_{k=1}^{a} \zeta(\beta + k\gamma),
\]

shows upon equating Dirichlet coefficients, that

Theorem 1.6. If \( a \) and \( k \) are positive integers,

\[
\sigma_{-\gamma}(a; k) = k^{-\gamma} \sum_{k=k_1k_2\cdots k_a} \frac{1}{k_1^{\gamma}k_2^{2\gamma}\cdots k_a^{a\gamma}}
\]

This tiling pattern is enumerated recursively by a divisor function related explicitly to the coefficient terms that occur in the \( D \)-analogue of the binomial coefficients.

For a proof of this see Baake et al [14]. This fascinating new connection between the theory of tilings arising in the study of quasicrystals may be well worth further investigation.

2. Some further required notation.

In this section we restate and use the notation we defined for \( D \)-analogues in [23]. That choice of nomenclature resembled, where possible, the \( q \)-series notation. However it is also our intention to display the similarity of our new \( D \)-analogues with the ordinary hypergeometric series summations. Firstly though, we use the notation given in Gasper and Rahman [27] for a product of \( q \)-shifted factorials:
Definition 2.1.

(2.1) \((a_1, a_2, \ldots, a_r; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_r; q)_n.\)

Next we design the following definitions and notations. Firstly we enlarge the definition given previously in (1.15).

Definition 2.2. If \(\zeta(a)\) is the Riemann zeta function and \(\Re \gamma\) chosen such that the functions all exist, define for positive integers \(n,\)

\[
\zeta(a; \gamma)_n = \prod_{k=0}^{n-1} \zeta((a + k)\gamma)) = \prod_p \frac{1}{(p^{-a\gamma}; p^{-\gamma})_n},
\]

(2.3) \(\zeta(a_1, a_2, \ldots, a_r; \gamma)_n = \zeta(a_1; \gamma)\zeta(a_2; \gamma) \cdots \zeta(a_r; \gamma)_n.\)

This definition is intended to bring a required notation for the \(D\)-analogue of the normal gamma function \(\Gamma(z)\) or its \(q\)-analogue \(\Gamma_q(z)\) found in the \(q\)-series literature such as Gasper and Rahman [27]. However, we shall have to postpone our investigation of this feature to another paper.

Definition 2.3. If \(\sigma_k(a)\) is the sum of \(k\)th powers of the divisors of positive integer \(a\) as in (1.6) then for positive integers \(n,\)

\[
\sigma_{-\gamma}(a; k) = \prod_{j=0}^{a-2} \frac{\sigma_{-\gamma}(k \prod_{p|j} p^j)}{\sigma_{-\gamma}(\prod_{p|j} p^j)}, \text{ defined as 1 at } a = 1, \text{ and}
\]

(2.5) \(\sigma_{-\gamma}(a_1, a_2, \ldots, a_r; k) = \sigma_{-\gamma}(a_1; k)\sigma_{-\gamma}(a_2; k) \cdots \sigma_{-\gamma}(a_r; k).\)

With these definitions, we formed in [23] respectively, the following theorems.

Theorem 2.1. For positive integers \(a_i, b_i, c_i, d_i,\) with \(\Re z\) chosen for convergence, 

\[
\prod_p \phi_{r+s-1} \left[ \frac{p^{-a_1\gamma}, p^{-a_2\gamma}, \ldots, p^{-a_r\gamma}, -p^{-c_1\gamma}, -p^{-c_2\gamma}, \ldots, -p^{-c_s\gamma}; p^{-\gamma}, p^{-z}}{p^{-b_1\gamma}, p^{-b_2\gamma}, \ldots, p^{-b_{r-1}\gamma}, -p^{-d_1\gamma}, -p^{-d_2\gamma}, \ldots, -p^{-d_s\gamma}} \right]
\]

(2.6) \(= \sum_{k=1}^{\infty} \frac{\sigma_{-\gamma}(a_1, a_2, \ldots, a_r; k)\sigma_{-\gamma}(c_1, c_2, \ldots, c_s; k)\sigma_{-\gamma}(d_1, d_2, \ldots, d_s; k)}{\sigma_{-\gamma}(b_1, b_2, \ldots, b_{r-1}; k)\sigma_{-\gamma}(d_1, d_2, \ldots, d_s; k)} \quad 1
\]

and

\[
\prod_p \phi_{r+s-1} \left[ \frac{p^{-a_1\gamma}, p^{-a_2\gamma}, \ldots, p^{-a_r\gamma}, -p^{-c_1\gamma}, -p^{-c_2\gamma}, \ldots, -p^{-c_s\gamma}; p^{-\gamma}, p^{-z}}{p^{-b_1\gamma}, p^{-b_2\gamma}, \ldots, p^{-b_{r-1}\gamma}, -p^{-d_1\gamma}, -p^{-d_2\gamma}, \ldots, -p^{-d_s\gamma}} \right]
\]

(2.7) \(= \sum_{k \in S_m} \frac{\sigma_{-\gamma}(a_1, a_2, \ldots, a_r; k)\sigma_{-\gamma}(c_1, c_2, \ldots, c_s; k)\sigma_{-\gamma}(d_1, d_2, \ldots, d_s; k)}{\sigma_{-\gamma}(b_1, b_2, \ldots, b_{r-1}; k)\sigma_{-\gamma}(c_1, c_2, \ldots, c_s; k)} \quad 1
\]

Similarly, we have for \(-p^{-z}\) in place of \(p^{-z},\)

Theorem 2.2. For positive integers \(a_i, b_i, c_i, d_i,\) with \(\Re z\) chosen for convergence, 

\[
\prod_p \phi_{r+s-1} \left[ \frac{p^{-a_1\gamma}, p^{-a_2\gamma}, \ldots, p^{-a_r\gamma}, -p^{-c_1\gamma}, -p^{-c_2\gamma}, \ldots, -p^{-c_s\gamma}; p^{-\gamma}, -p^{-z}}{p^{-b_1\gamma}, p^{-b_2\gamma}, \ldots, p^{-b_{r-1}\gamma}, -p^{-d_1\gamma}, -p^{-d_2\gamma}, \ldots, -p^{-d_s\gamma}} \right]
\]

(2.8) \(= \sum_{k=1}^{\infty} \frac{\sigma_{-\gamma}(a_1, a_2, \ldots, a_r; k)\sigma_{-\gamma}(c_1, c_2, \ldots, c_s; k)\sigma_{-\gamma}(d_1, d_2, \ldots, d_s; k) \lambda(k)}{\sigma_{-\gamma}(b_1, b_2, \ldots, b_{r-1}; k)\sigma_{-\gamma}(d_1, d_2, \ldots, d_s; k)} \quad k^z, \text{ and}
\]
\[
\prod_{p|n} \phi_{r+s-1} \left[ p^{-a_1\gamma}, p^{-a_2\gamma}, \ldots, p^{-a_r\gamma}, -p^{-c_1\gamma}, -p^{-c_2\gamma}, \ldots, -p^{-c_s\gamma}, \ p^{-\gamma}, -p^{-z} \right] \\
\frac{\sigma_{-\gamma}(a_1, a_2, \ldots, a_r; k)\sigma_{-\gamma}(c_1, c_2, \ldots, c_s; k)\sigma_{-\gamma}(d_1, d_2, \ldots, d_s; k)}{\sigma_{-\gamma}(b_1, b_2, \ldots, b_{r-1}; k)\sigma_{-\gamma}(d_1, d_2, \ldots, d_s; k)\sigma_{-\gamma}(c_1, c_2, \ldots, c_s; k)} \lambda(k) \\
(2.9) = \sum_{k \in S_n} \gamma 
\]

Theorems 2.1 and 2.2 show how to take account of negative parameters arising in the q-series under the product operator in the context of this paper. Our notation always requires equal numbers of \(c_i\) and \(d_i\) terms in the q-series, due to the necessary pairing of numerator and denominator terms for cancellations. We next apply these theorems to known basic hypergeometric series summations and transforms to obtain new results as Dirichlet series analogues of the original q-summation formulæ. To do this we state the

**Definition 2.4.** Either side of (2.7), (2.7), (2.8) and (2.9) are defined respectively as:

\[
(2.10a) \quad r+s \Theta_{r+s-1} \left[ a_1, a_2, \ldots, a_r, -\gamma, z \right] \\
(2.10b) \quad r+s \Theta_{r+s-1} \left[ m! a_1, a_2, \ldots, a_r, -\gamma, z \right] \\
(2.10c) \quad r+s \Theta_{r+s-1} \left[ a_1, a_2, \ldots, a_r, -\gamma, -\gamma, \gamma, -\gamma \right] \\
(2.10d) \quad r+s \Theta_{r+s-1} \left[ m! a_1, a_2, \ldots, a_r, -\gamma, -\gamma, -\gamma \right] 
\]

where the \(-\) preceding any variable denotes that it comes from a negative valued parameter in the q-series.

In future use of the transforms of this section we will always delineate between the negative and positive values of a parameter even when this was previously not an issue of consideration in the q-series itself.

### 3. The Transform Applied to Known q-Series Identities.

This section restates two theorems from Campbell [23] as the basis of our results on the transformed q-series to D-series. They show that a valid q-series identity or transform maps onto a valid Dirichlet series identity or transform in many cases. We shall frequently employ definition 2.2 here, and a counterpart to this using the Jordan totient function, namely

**Definition 3.1.** Consider the Jordan totient function \(J_r(m) = m^\gamma \prod_{i=1}^{b_i} (1 - p_i^{-\gamma})\), where \(\gamma \geq 0\), and \(m = \prod_{i=1}^{b_i} p_i^{a_i}\) is the unique prime decomposition of \(m\). Define then for positive integers \(m, n, a_1, a_2, \ldots, a_r\),

\[
J(m| a_1, a_2, \ldots, a_r, \gamma)_n = \prod_{i=1}^{r} \prod_{j=0}^{n-1} \frac{J_{(a_i+j)\gamma}(m)}{m^{(a_i+j)\gamma}} = \prod_{i=1}^{r} \prod_{p|m} \frac{p^{-a_i\gamma}}{p^{-\gamma}}. 
\]
It is relatively easy to see from this and (1.8) that
\[
\frac{J(m|a_1,a_2,\ldots,a_r;\gamma)_n}{J(m|b_1,b_2,\ldots,b_r;\gamma)_n} = \prod_{i=1}^{r} \prod_{j=0}^{n-1} \frac{(1-p^{-\gamma(j+i)})}{(1-p^{-\gamma(j+b_i)})} = \prod_{i=1}^{r} \prod_{j=0}^{n-1} \frac{(p^{-a_i\gamma};p^{-\gamma})}{(p^{-b_i\gamma};p^{-\gamma})}
\]
(3.2)
and this will be used in simplifying the results from finite Euler product transforms.

**Theorem 3.1.** (see Campbell [23]) Suppose that for each prime \(p\) and positive integers \(a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i\), we have a \(q\)-series identity of the generic form
\[
(3.3a) \quad \sum_{r,s} \phi_{r+s-1} \left[ \begin{array}{cccc}
p^{-a_1\gamma}, & p^{-a_2\gamma}, & \ldots, & p^{-a_r\gamma}, \\
p^{-b_1\gamma}, & p^{-b_2\gamma}, & \ldots, & p^{-b_r\gamma}, \\
p^{-c_1\gamma}, & p^{-c_2\gamma}, & \ldots, & p^{-c_r\gamma}, \\
p^{-d_1\gamma}, & p^{-d_2\gamma}, & \ldots, & p^{-d_r\gamma}, \\
p^{-e_1\gamma}, & p^{-e_2\gamma}, & \ldots, & p^{-e_r\gamma}, \\
p^{-f_1\gamma}, & p^{-f_2\gamma}, & \ldots, & p^{-f_r\gamma}, \\
p^{-g_1\gamma}, & p^{-g_2\gamma}, & \ldots, & p^{-g_r\gamma}, \\
p^{-h_1\gamma}, & p^{-h_2\gamma}, & \ldots, & p^{-h_r\gamma}, \\
p^{-i_1\gamma}, & p^{-i_2\gamma}, & \ldots, & p^{-i_r\gamma}, \\
p^{-j_1\gamma}, & p^{-j_2\gamma}, & \ldots, & p^{-j_r\gamma},
\end{array} \right]
\]
(3.3b)
and the two Euler product transforms applied over primes to both sides of this yield respectively, the two \(D\)-analogue summation formulae
\[
(3.4a) \quad \sum_{r,s} \Theta_{r+s-1} \left[ \begin{array}{cccc}
a_1, & a_2, & \ldots, & a_r, \\
b_1, & b_2, & \ldots, & b_r, \\
c_1, & c_2, & \ldots, & c_r, \\
d_1, & d_2, & \ldots, & d_r, \\
\end{array} \right] \]
(3.4b)
\[
= \prod_{i=1}^{r} \frac{\zeta(f_1, f_2, \ldots, f_{s_i}; \gamma_{n_i}) \zeta(h_1, h_2, \ldots, h_{s_i}; \gamma_{2_{n_i}})}{\zeta(e_1, e_2, \ldots, e_{r_i}; \gamma_{m_i}) \zeta(g_1, g_2, \ldots, g_{r_i}; \gamma_{m_i})}, \quad \text{and}
\]
\[
(3.5a) \quad \sum_{r,s} \Theta_{r+s-1} \left[ \begin{array}{cccc}
m, & a_1, & a_2, & \ldots, & a_r, \\
b_1, & b_2, & \ldots, & b_r, \\
c_1, & c_2, & \ldots, & c_r, \\
d_1, & d_2, & \ldots, & d_r, \\
\end{array} \right] \]
(3.5b)
\[
= \frac{J(m|a_1, a_2, \ldots, a_r; \gamma)_m}{J(m|b_1, b_2, \ldots, b_r; \gamma)_m} = \frac{J(m|a_1, a_2, \ldots, a_r; \gamma)_m}{J(m|b_1, b_2, \ldots, b_r; \gamma)_m}.
\]

A simple restatement of this with appropriate negative parameter is

**Theorem 3.2.** (see Campbell [23]) Theorem 3.1 with \(-p^{-z}\) replacing \(p^{-z}\) in (3.3a), together with \(-\gamma\) replacing \(z\) in each of (3.4a) and (3.5a) is true.

These are the required theorems to enable us to write down at a glance, many \(D\)-series analogues of \(q\)-series. They assert no claims as to the convergence of the resultant series of functions, but formally give the \(D\)-analogues without deeper considerations. It may, for example, turn out that the question of convergence of the Euler product is non-trivial, and that various deeper analysis will be required to justify resulting formulae. Not at this stage concerning ourselves with the finer questions, we next show some simple examples, of which all appear to be new results.

4. The \(D\)-Kummer theorem.

In this section we apply the transforms in theorems 3.1 and 3.2 to the so-called "Kummer" theorems in ordinary and basic hypergeometric series. In later papers we may go into more detail as to the significance of these new results. But for now we shall content ourselves with statement of the summation formulae.
The ordinary Kummer theorem and its $q$-analogue theorem are shown in Gasper and Rahman [27, page 14] to be respectively,

\begin{equation}
2F_1(a, b; 1 + a - b; -1) = \frac{\Gamma(1 + a - b)\Gamma(1 + \frac{1}{2}a)}{\Gamma(1 + a)\Gamma(1 + \frac{1}{2}a - b)}.
\end{equation}

\begin{equation}
2\phi_1(a, b; aq/b; q, -q/b) = \frac{(-q; q)\infty(aq, aq^2/b^2; q^2)\infty}{(aq/b, -q/b; q)\infty}.
\end{equation}

For a proof of the ordinary hypergeometric series result (4.1) see Bailey [15, pages 9–10]. A straightforward application of theorem 3.2 to (4.2) yields the $D$-analogue formulae

**Proposition 4.1.** (The “would-be” D-Kummer Theorem) If each of $a$ and $b$ are positive integers and $\gamma > 0$ such that each of the Riemann zeta functions in (4.3) are defined from convergent Euler products,

\begin{equation}
2\Theta_1(a, b; 1 + a - b; \gamma, -\lambda(1 - b)\gamma) = \frac{\zeta(1, 1 + a - b; \gamma)\zeta(1, 1 - \gamma, \gamma)\infty}{\zeta(1, 1 - \gamma, \gamma)\zeta(1 + a, 1 + \frac{1}{2}a - b; 2\gamma)\infty};
\end{equation}

and

\begin{equation}
2\Theta_1(m | a, b; 1 + a - b; \gamma, -\lambda(1 - b)\gamma) = \frac{J(m | 1 - b; \gamma)\infty J(m | 1 + a, 1 + \frac{1}{2}a - b; 2\gamma)\infty}{J(m | 1 - b; \gamma)\infty J(m | 1 + a - b; 2\gamma)\infty}.
\end{equation}

According to theorem 2.2 applied to the left side of (4.2), the left sides of (4.3) and (4.4) are respectively

\begin{equation}
\sum_{k=1}^{\infty} \frac{\sigma_{-\gamma}(a, b; k)}{\sigma_{-\gamma}(1 + a - b; k)} \frac{\lambda(k)}{k^{(1-b)\gamma}}, \quad \text{and}
\end{equation}

\begin{equation}
\sum_{k \in S_m} \frac{\sigma_{-\gamma}(a, b; k)}{\sigma_{-\gamma}(1 + a - b; k)} \frac{\lambda(k)}{k^{(1-b)\gamma}}.
\end{equation}

A quick inspection of the oscillating series (4.5) indicates a general problem with its convergence. It is not at all obvious whether there is any case of the series that converges either absolutely or conditionally under the proposed conditions for the proposition. The term $k^{-(1-b)\gamma}$ for $k > 1$ with $b$ a fixed positive integer does not approach zero as $k$ increases. This means we will need to rely upon the coefficient terms in (4.5) and (4.6) to achieve convergence. Moreover, we know what the average order of $\sigma_{-\gamma}(k)$ is from the

**Theorem 4.1.** (see Apostol [9] pages 60-61) If $\gamma > 0$ and $x > 1$,

\begin{equation}
\sum_{k \leq x} \sigma_\gamma(k) = \begin{cases} \frac{1}{2}\zeta(2)x^2 + O(x \log x), & \gamma = 1; \\
\zeta(\gamma + 1)x^{\gamma+1} + O(x^\beta), & \gamma \neq 1, \beta = \max\{1, \gamma\}; \end{cases}
\end{equation}

\begin{equation}
\sum_{k \leq x} \sigma_{-\gamma}(k) = \begin{cases} \zeta(2)x + O(\log x), & \gamma = 1; \\
\zeta(\gamma + 1)x + O(x^\beta), & \gamma \neq 1, \beta = \max\{0, 1 - \gamma\}. \end{cases}
\end{equation}
Applying theorem 4.1 to the definitions of $\sigma_{-\gamma}(a; k)$ of (1.6) and (1.10) we can expect to obtain estimates of the behavior of the coefficients in a D-series, such as for those presented in proposition 4.1. Also, examining the coefficients of (4.5) using (1.23) tells us, for instance, that if $k$ is a prime

$$
\frac{\sigma_{-\gamma}(a; b; k)}{\sigma_{-\gamma}(1 + a - b; k)} = \prod_{p | k} \frac{(1 - p^{-a\gamma})(1 - p^{-b\gamma})}{(1 - p^{-\gamma})(1 - p^{-(1+a-b)\gamma})}.
$$

So, it seems per se that the series (4.5) and (4.6) will diverge for any given fixed positive integer $b$, for any positive integer $a$ chosen. However, this is not strictly the case, as the problem with divergence in (4.5) arises from the divergence of the Euler product over all primes. Since (4.6) comes from a finite Euler product, it is nonetheless true in Proposition 4.1, and therefore the D-series analogue (4.6) of the Kummer theorems could on its own perhaps be rated as a theorem. Let us state this therefore, simplifying using (3.2), to obtain our consolation prize,

**Theorem 4.2. (The “half-done” D-Kummer Theorem) If each of $\frac{1}{2}a$ and $b$ are positive integers and $\gamma > 0$,**

$$
2\Theta_1(m| a, b; 1 + a - b; \gamma, \gamma \setminus (1 - b)\gamma) = \frac{J(m| 1 - b; \gamma)\infty J(m| 1 + a, 1 + \frac{1}{2}a - b; 2\gamma)\infty}{J(m| 1 + a - b; \gamma)\infty J(m| 1 - b; 2\gamma)\infty} = \prod_{j=0}^{\infty} \frac{\sigma_{-\gamma}(\prod_{p | m} p^{j-b}) \sigma_{-2\gamma}(\prod_{p | m} p^{j+a}) \sigma_{-3\gamma}(\prod_{p | m} p^{j+a-b})}{\sigma_{-\gamma}(\prod_{p | m} p^{j}) \sigma_{-2\gamma}(\prod_{p | m} p^{j+a-b}) \sigma_{-3\gamma}(\prod_{p | m} p^{j+a}).
$$

(4.10)

Obviously, the convergence questions of this section will re-emerge in our sequel papers. Nevertheless it is clear that the theorems in section 3 will still yield many non-problematic D-analogues, such as the D-Dixon theorem, the paper for which is in preparation. In the case of the D-Dixon theorem there emerge many examples that are new, as was the case for the D-Gauss analogue formula in [23] given in (1.14).

Finally it may serve our purpose to present the first several cases of the theorem, namely the first few integer values of $a$ and $b$ substituted. We can use (4.5) and (4.6) whence the following coefficient terms are apt.

$$
\frac{\sigma_{-\gamma}(a, b; k)}{\sigma_{-\gamma}(1 + a - b; k)} = \prod_{p | k} \frac{(1 - p^{-a\gamma})(1 - p^{-b\gamma})}{(1 - p^{-\gamma})(1 - p^{-(1+a-b)\gamma})}.
$$

(4.11)

$$
\frac{\sigma_{-\gamma}(2, 1; k)}{\sigma_{-\gamma}(2; k)} = \prod_{p | k} \frac{(1 - p^{-2\gamma})(1 - p^{-\gamma})}{(1 - p^{-\gamma})(1 - p^{-2\gamma})} = 1.
$$

(4.12)

$$
\frac{\sigma_{-\gamma}(4, 2; k)}{\sigma_{-\gamma}(3; k)} = \prod_{p | k} \frac{(1 - p^{-4\gamma})(1 - p^{-2\gamma})}{(1 - p^{-\gamma})(1 - p^{-3\gamma})}.
$$

(4.13)

$$
\frac{\sigma_{-\gamma}(4, 3; k)}{\sigma_{-\gamma}(2; k)} = \prod_{p | k} \frac{(1 - p^{-4\gamma})(1 - p^{-3\gamma})}{(1 - p^{-\gamma})(1 - p^{-2\gamma})}.
$$

(4.14)

$$
\frac{\sigma_{-\gamma}(6, 1; k)}{\sigma_{-\gamma}(6; k)} = \prod_{p | k} \frac{(1 - p^{-6\gamma})(1 - p^{-\gamma})}{(1 - p^{-\gamma})(1 - p^{-6\gamma})} = 1.
$$

(4.15)
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(4.16) \[
\frac{\sigma_{-\gamma}(6, 2; k)}{\sigma_{-\gamma}(5; k)} = \prod_{p\mid k} \frac{(1-p^{-6\gamma})(1-p^{-2\gamma})}{(1-p^{-\gamma})(1-p^{-5\gamma})}.
\]

These coefficients when substituted give us the following particular cases of the $D$-Kummer theorem 4.2 above.

(4.17) \[
\frac{\sigma_{-\gamma}(a, b; k)}{\sigma_{-\gamma}(1+a-b; k)} = \prod_{p\mid k} \frac{(1-p^{-a\gamma})(1-p^{-b\gamma})}{(1-p^{-\gamma})(1-p^{-(1+a-b)\gamma})}.
\]

(4.18) \[
\frac{\sigma_{-\gamma}(2, 1; k)}{\sigma_{-\gamma}(2; k)} = \prod_{p\mid k} \frac{(1-p^{-2\gamma})(1-p^{-\gamma})}{(1-p^{-\gamma})(1-p^{-2\gamma})} = 1.
\]

(4.19) \[
\frac{\sigma_{-\gamma}(4, 2; k)}{\sigma_{-\gamma}(3; k)} = \prod_{p\mid k} \frac{(1-p^{-4\gamma})(1-p^{-2\gamma})}{(1-p^{-\gamma})(1-p^{-3\gamma})}.
\]

(4.20) \[
\frac{\sigma_{-\gamma}(4, 3; k)}{\sigma_{-\gamma}(2; k)} = \prod_{p\mid k} \frac{(1-p^{-4\gamma})(1-p^{-3\gamma})}{(1-p^{-\gamma})(1-p^{-2\gamma})}.
\]

(4.21) \[
\frac{\sigma_{-\gamma}(6, 1; k)}{\sigma_{-\gamma}(6; k)} = \prod_{p\mid k} \frac{(1-p^{-6\gamma})(1-p^{-\gamma})}{(1-p^{-\gamma})(1-p^{-6\gamma})} = 1.
\]

(4.22) \[
\frac{\sigma_{-\gamma}(6, 2; k)}{\sigma_{-\gamma}(5; k)} = \prod_{p\mid k} \frac{(1-p^{-6\gamma})(1-p^{-2\gamma})}{(1-p^{-\gamma})(1-p^{-5\gamma})}.
\]

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