Global Well-posedness of the Two Dimensional Beris–Edwards System with General Landau–de Gennes Free Energy

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Abstract

In this paper, we consider the Beris–Edwards system for incompressible nematic liquid crystal flows. The system under investigation consists of the Navier–Stokes equations for the fluid velocity \( u \) coupled with an evolution equation for the order parameter \( Q \)-tensor. One important feature of the system is that its elastic free energy takes a general form and in particular, it contains a cubic term that possibly makes it unbounded from below. In the two dimensional periodic setting, we prove that if the initial \( L^\infty \)-norm of the \( Q \)-tensor is properly small, then the system admits a unique global weak solution. The proof is based on the construction of a specific approximating system that preserves the \( L^\infty \)-norm of the \( Q \)-tensor along the time evolution.

Key words. Beris–Edwards system, liquid crystal flow, \( Q \)-tensor, global weak solution.

AMS subject classifications. 35Q35, 35Q30, 76D03, 76D05.

1 Introduction

The Landau–de Gennes theory is a fundamental continuum theory that describes the state of nematic liquid crystals [4, 16]. In this framework, the local orientation and degree of ordering for the liquid crystal molecules are modelled by a symmetric, traceless \( d \times d \) matrix \( Q \) in \( \mathbb{R}^d \) \( (d = 2, 3) \), known as the \( Q \)-tensor order parameter [5, 6]. The so-called Landau–de Gennes free energy functional is a nonlinear integral functional of the \( Q \)-tensor and its spatial derivatives [4]:

\[
E(Q) = \int_\Omega F(Q(x), \nabla Q(x)) \, dx,
\]

where \( Q \) is a matrix valued function defined on the spatial domain \( \Omega \subset \mathbb{R}^d \) that takes values in the configuration space

\[
S_0^{(d)} \equiv \{ M \in \mathbb{R}^{d\times d} : M = M^T, \, \text{tr}(M) = 0 \}.
\]

The energy density function \( F \) in [11] is composed of an elastic part and a bulk part [5, 33]:

\[
F(Q, \nabla Q) \equiv F_{\text{elastic}}(Q, \nabla Q) + F_{\text{bulk}}(Q).
\]

The bulk free-energy describes the isotropic-nematic phase transition. Its density function \( F_{\text{bulk}} \) is typically a truncated expansion in the scalar invariants of the \( Q \)-tensor, which in the simplest setting takes the following form [16]:

\[
F_{\text{bulk}}(Q) \equiv \frac{a}{2} \text{tr}(Q^2) - \frac{b}{3} \text{tr}(Q^3) + \frac{c}{4} \text{tr}^2(Q^2).
\]

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Here, \(a, b, c \in \mathbb{R}\) are material-dependent and temperature-dependent constants. On the other hand, the elastic free energy characterizes the distortion effect of the liquid crystal and its density function \(F_{\text{elastic}}\) gives the strain energy density due to spatial variations in the \(Q\)-tensor:

\[
F_{\text{elastic}}(Q, \nabla Q) \overset{\text{def}}{=} L_1 \partial_k Q_{ij} \partial_k Q_{ij} + L_2 \partial_j Q_{ik} \partial_k Q_{ij} + L_3 \partial_j Q_{ij} \partial_k Q_{ik} + L_4 \partial_k Q_{ij} \partial_k Q_{ij},
\]

for \(1 \leq i, j, k, l \leq d\). Throughout this paper, we use the Einstein summation convention over repeated indices. The coefficients \(L_1, L_2, L_3, L_4\) are material-dependent elastic constants. We note that \(F_{\text{elastic}}\) consists of three independent terms associated with \(L_1, L_2, L_3\) that are quadratic in the first order partial derivatives of the \(Q\)-tensor, plus a cubic term associated with the coefficient \(L_4\). In the literature, the case \(L_2 = L_3 = L_4 = 0\) is usually called isotropic, otherwise anisotropic if at least one of \(L_2, L_3, L_4\) does not vanish. In particular, the retention of the cubic term (i.e., \(L_4 \neq 0\)) is due to the physically relevant consideration that it allows a complete reduction of the Landau–de Gennes energy \(E(Q)\) to the classical Oseen–Frank energy for nematic liquid crystals [6, 31] (see also [24, Appendix B]). There exists a vast recent literature on the mathematical study of the Landau–de Gennes theory, and we refer interested readers to [2–7, 9–15, 17–24, 30–32, 34, 35, 38–40] as well as the references cited therein.

In this paper, we study a basic model for the evolution of an incompressible nematic liquid crystal flow, which was first proposed by Beris–Edwards [7]. The resulting PDE system consists of the Navier–Stokes equations for the fluid velocity \(u\) and nonlinear convection-diffusion equations of parabolic type for the \(Q\)-tensor (see, e.g., [21, 30–31]). To simplify the mathematical setting, throughout this paper we confine ourselves to the two dimensional periodic case such that \(d = 2\) and \(\Omega = T^2\), where \(T^2\) stands for the periodic box with period \(\ell_i\) in the \(i\)-th direction with \(O = (0, \ell_1) \times (0, \ell_2)\) being the periodic cell. Without loss of generality, we simply set \(O = (0, 1)^2\). Then the coupled system we are going to study takes the following form:

\[
\begin{align*}
\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla P &= \nabla \cdot (\sigma^a + \sigma^s), & \forall (x, t) \in \mathbb{T}^2 \times \mathbb{R}^+, \\
\nabla \cdot u &= 0, & \forall (x, t) \in \mathbb{T}^2 \times \mathbb{R}^+, \\
\partial_t Q + u \cdot \nabla Q + Q \omega - \omega Q &= H + \lambda I + \mu - \mu^T, & \forall (x, t) \in \mathbb{T}^2 \times \mathbb{R}^+.
\end{align*}
\]

The vector \(u(x, t) : \mathbb{T}^2 \times \mathbb{R}^+ \rightarrow \mathbb{R}^2\) denotes the velocity field of the fluid, \(Q(x, t) : \mathbb{T}^2 \times \mathbb{R}^+ \rightarrow S_0^{(2)}\) stands for the matrix-valued order parameter of liquid crystal molecules, and the scalar function \(P(x, t) : \mathbb{T}^2 \times \mathbb{R}^+ \rightarrow \mathbb{R}\) is the hydrostatic pressure. We assume that the system (1.5)–(1.7) is subject to the periodic boundary conditions

\[
\begin{align*}
\forall (x, t) \in \mathbb{T}^2 \times \mathbb{R}^+, \\
\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla P &= \nabla \cdot (\sigma^a + \sigma^s), & \forall (x, t) \in \mathbb{T}^2 \times \mathbb{R}^+, \\
\nabla \cdot u &= 0, & \forall (x, t) \in \mathbb{T}^2 \times \mathbb{R}^+, \\
\partial_t Q + u \cdot \nabla Q + Q \omega - \omega Q &= H + \lambda I + \mu - \mu^T, & \forall (x, t) \in \mathbb{T}^2 \times \mathbb{R}^+.
\end{align*}
\]

The first equation (1.5) is the Navier–Stokes equation for \(u\) with highly nonlinear anisotropic force terms given by the stresses \(\sigma^a, \sigma^s\), while the second equation (1.6) simply gives the incompressibility condition. The third equation (1.7) describes the evolution of the \(Q\)-tensor, in which the left-hand side stands for the upper convective derivative that represents the rate of change for a small particle of the material that is rotating and stretching with the fluid flow. We recall that when \(d = 2\) the general form of the upper convective derivative is given by

\[
\partial_t Q + u \cdot \nabla Q - S(\nabla u, Q),
\]

where the tensor \(S(\nabla u, Q)\) takes the following form

\[
S(\nabla u, Q) \overset{\text{def}}{=} (\xi A u + \omega)(Q + \frac{1}{2} I) + (Q + \frac{1}{2} I)(\xi A u - \omega) - 2\xi(Q + \frac{1}{2} I) \text{tr}(Q \nabla u),
\]

with \(I\) being the \(2 \times 2\) identity matrix and

\[
Au = \frac{\nabla u + (\nabla u)^T}{2}, \quad \omega = \frac{\nabla u - (\nabla u)^T}{2}
\]

(1.11)

2
being the symmetric and skew-symmetric parts of the strain rate, respectively. The parameter \( \xi \in \mathbb{R} \) in (1.10) depends on the molecular details of a given liquid crystal material and it measures the ratio between the tumbling and the aligning effects that a shear flow may exert over the liquid crystal directors \( \xi \). We note that \( \xi \) represents nontrivial interactions between the macro fluid and the micro molecule configuration \( \xi \). In this paper, we confine ourselves to the simple case with \( \xi = 0 \) (cf. (1.7)), which is referred to as the co-rotational case in the literature \( \xi \). On the right-hand side of equation (1.7), \( \mathcal{H} \) represents the molecular field that is defined to be minus the Fréchet derivative of the free energy \( \mathcal{E}(Q) \) with respect to \( Q \) (without any constraint):

\[
\mathcal{H} = -\frac{\delta \mathcal{E}(Q)}{\delta Q}.
\]  

(1.12)

while \( \lambda \in \mathbb{R} \) is a Lagrange multiplier corresponding to the traceless constraint \( \text{tr}(Q) = 0 \) and \( \mu \in \mathbb{R}^{2 \times 2} \) is the Lagrange multiplier corresponding to the constraint on symmetry \( Q = Q^T \). Through a straightforward calculation, we have (see Appendix)

\[
-\mathcal{H}_{ij} = -2L_1 \Delta Q_{ij} - 2(L_2 + L_3)Q_{ik,kj} - 2L_4 Q_{ij,lk} Q_{lk} + L_4 Q_{kl,lj} Q_{kl,j} + a Q_{ij} - b Q_{jk} Q_{ki} + c \text{tr}(Q^2) Q_{ij}
\]

(1.13)
as well as

\[
(\mathcal{H} + \lambda I + \mu - \mu^T)_{ij} = \zeta \Delta Q_{ij} + 2L_4 Q_{ij,lk} Q_{lk} - L_4 Q_{kl,lj} Q_{kl,j} + \frac{L_4}{2} |\nabla Q|^2 \delta_{ij} - a Q_{ij} - c \text{tr}(Q^2) Q_{ij},
\]

(1.14)

for \( 1 \leq i, j, k, l \leq 2 \), where we denote

\[
\zeta \overset{\text{def}}{=} 2L_1 + L_2 + L_3.
\]

(1.15)

Returning to equation (1.5) for \( u \), the two highly nonlinear stress terms \( \sigma^a \), \( \sigma^s \) on its right-hand side are referred to as the anti-symmetric viscous stress and the distortion stress, respectively. More precisely, they take the following form:

\[
\sigma^a = Q (\mathcal{H} + \lambda I + \mu - \mu^T) - (\mathcal{H} + \lambda I + \mu - \mu^T) Q,
\]

(1.16)

\[
\sigma^s_{ij} = -\frac{\partial \mathcal{E}(Q)}{\partial Q_{kl,i}} Q_{kl,j} = -2(L_1 Q_{kl,i} Q_{kl,j} + L_2 Q_{kj,l} Q_{kl,i} + L_3 Q_{kl,l} Q_{kj,i} + L_4 Q_{jm} Q_{kl,m} Q_{kl,i}),
\]

(1.17)

for \( 1 \leq i, j, k, l, m \leq 2 \).

Throughout this paper, we assume for simplicity that \( \nu \), the viscosity of the liquid crystal flow, is a positive constant. Moreover, we impose the following basic assumptions on the other coefficients:

\[
L_4 \neq 0,
\]

(1.18)

\[
\kappa \overset{\text{def}}{=} \min\{L_1 + L_2, L_1 + L_3\} > 0,
\]

(1.19)

\[
c > 0.
\]

(1.20)

We note that (1.19) can be viewed as a sufficient condition for the coercivity of the system (see [24, Appendix C]), while the condition (1.20) ensures that the bulk part of the free energy density \( \mathcal{F}_{\text{bulk}} \) is bounded from below (see [31, 32]). Moreover, from (1.15) and (1.19) we easily infer the following relation

\[
\zeta \geq 2\kappa > 0.
\]

(1.21)

Under the current assumption (1.18), the general system (1.5)–(1.7) differs from those that have been extensively studied in the literature, see for instance, [23, 10, 13, 15, 17, 21, 30, 33, 35, 38, 39]. An important feature is that its free energy \( \mathcal{E}(Q) \) now contains an unusual cubic term associated with the coefficient
Let $u \in H^1(0,T; (H^1_0(\Omega))^d) \cap C([0,T]; L^2_0(\Omega)) \cap L^2(0,T; H^1_0(\Omega))$, $Q \in H^1(0,T; L^2(\Omega)) \cap C([0,T]; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega))$, $Q \in L^\infty(0,T; L^\infty(\Omega))$, with $Q \in S_0^{(2)}$ a.e. in $\Omega \times (0,T)$ is a weak solution to the boundary problem (1.5)–(1.9), if

$$
\int_0^T (\partial_t u, \varphi)_{(H^1)^d} dt - \int_0^T (u \otimes u) : \nabla \varphi dx + \nu \int_0^T \int_\Omega \nabla u : \nabla \varphi dx dt
= - \int_0^T \int_\Omega (\sigma^0(Q) + \sigma^4(Q)) : \nabla \varphi dx dt,
$$

(1.22)

for every $\varphi \in L^2(0,T; H^1_0(\Omega))$, $Q \in H^1(0,T; L^2(\Omega))$, $\partial_t Q_{ij} + u_k Q_{ij,k} + Q_{ik} Q_{kj} - \omega_{ik} Q_{kj} = \zeta \Delta Q_{ij} + 2L_4(Q_{ij,l} Q_{li,k}) - L_4 Q_{kl,i} Q_{ki,j} - \frac{L_4}{2} (\nabla Q)^2 \delta_{ij} - a Q_{ij} - c (tr(Q^2) - Q_{ij})$, a.e. in $\Omega \times (0,T)$, and the initial condition (1.9) is satisfied. Moreover, for every $t \in [0,T]$, it holds

$$
E_{\text{total}}(t) + \nu \int_0^t \int_\Omega |\nabla u|^2 dxds + \int_0^t \int_\Omega tr^2(\mathcal{H} + \lambda \mathbb{I} + \mu - \mu T) dxds = E_{\text{total}}(0),
$$

(1.25)

where the total energy $E_{\text{total}}$ is given by

$$
E_{\text{total}}(t) \overset{\text{def}}{=} \frac{1}{2} \int_\Omega |u(t,x)|^2 dx + E(Q(t)).
$$

(1.26)

We are now in a position to state the main result of this paper.

**Theorem 1.1.** Let $T > 0$ be arbitrary. Under the assumptions (1.18)–(1.20), there exists a positive constant $\eta$ given by

$$
\eta = \min \left\{ K_1 \left( \frac{K_2}{L_4} \right)^2, K_2 \left( \frac{\zeta}{|L_4|} \right) \sqrt{\nu} \right\},
$$

(1.27)

where $K_1, K_2 > 0$ are two factors depending only on $\Omega$, such that if in addition, the coefficients of system (1.5)–(1.7) satisfy

$$
a \geq - c_1 \eta,
$$

(1.28)

then for any initial data $u_0 \in L^2_0(\Omega, \mathbb{R}^2)$, $Q_0 \in H^1(\Omega, S_0^{(2)}) \cap L^\infty(\Omega, S_0^{(2)})$ fulfilling

$$
\|Q_0\|_{L^\infty(\Omega)} < \sqrt{\eta},
$$

problem (1.5)–(1.9) admits a unique global weak solution on $\Omega \times [0,T]$ in the sense of Definition (1.2). Furthermore, we have the uniform estimate

$$
\|Q(t)\|_{L^\infty([0,T]; L^\infty(\Omega))} \leq \sqrt{\eta}.
$$
Remark 1.1. (1) The restriction on the size of \(|Q_0|\) is due to the appearance of the cubic term associated with \(L_4\). As we can see from the definition of \(\eta\), it is naturally determined by the ratios \(k/|L_4|\) and \(\zeta/|L_4|\), which measures the "good" part versus the "bad" part in the second order operator \(H + x + \mu - \mu^2\). Because the constants \(K_1, K_2\) are from the application of the Hölder and Young inequalities as well as the elliptic estimates, they may depend on \(T^2\) at most.

(2) Although the technical assumption (1.28) leads to a restriction on the coefficient \(a\) (that in general depends linearly on the temperature of the material), it is still able to partially capture the physically interesting regime of low temperature with \(a \leq 0\) (i.e., the "the deep nematic" regime), in which the isotropic state must lose stability, leaving only the uniaxial nematic states to be stable [23, 25].

The Beris–Edwards system \([15, 16, 17]\) has been studied by many authors in recent years. We recall some relevant results here. Concerning the simplified system with \(\xi = 0\) and the elastic energy taking the simplest form (i.e., \(L_2 = L_3 = L_4 = 0\)), in [35] the authors obtained the existence of global weak solutions to the Cauchy problem in \(\mathbb{R}^d\) with \(d = 2, 3\), and they proved results on global regularity as well as weak-strong uniqueness for \(d = 2\). Some improved results on the global well-posedness in two dimensions were established in [14], and we also refer to [12] for certain regularity criterion in \(\mathbb{R}^3\). Besides, long-time behavior of global weak solutions to the Cauchy problem in \(\mathbb{R}^3\) was obtained in [13] by the method of Fourier splitting. On the other hand, initial boundary value problems subject to various boundary conditions have been investigated by several authors, see for instance, [3, 10, 20], where the existence of global weak solutions, existence and uniqueness of local strong solutions as well as some regularity criteria were established. For the Beris–Edwards system with three elastic constants \(L_1, L_2, L_3\) but \(L_4 = \xi = 0\), in [21] the authors studied the Cauchy problem in \(\mathbb{R}^3\) and proved the existence of global weak solutions as well as the existence and uniqueness of global strong solutions provided that the fluid viscosity is sufficiently large. Next, concerning the full Beris–Edwards system with a general parameter \(\xi \in \mathbb{R}\), existence of global weak solutions for the Cauchy problem in \(\mathbb{R}^d\) with \(d = 2, 3\) was established in [34] for sufficiently small \(|\xi|\), while the uniqueness of weak solutions for \(d = 2\) was given in [15]. Quite recently, global well-posedness and long-time behavior of the system in a two dimensional periodic setting were established in [10], without any smallness assumption on \(|\xi|\). As far as the initial boundary value problem subject to inhomogeneous mixed Dirichlet/Neumann boundary conditions is concerned, in [2] the authors treated the case with \(\xi \neq 0\), \(L_2 = L_3 = L_4 = 0\) and proved the existence of global weak solutions as well as local well-posedness with higher time-regularity for \(d = 2, 3\). The local well-posedness result was recently improved and extended to the case with an anisotropic elastic energy in [30]. At last, we would like to mention that some modified versions of the Beris–Edwards system in terms of its free energy have been investigated in the literature. In [30], the bulk potential \([13]\) is replaced by a singular potential of Ball–Majumdar type (cf. [21]) that ensures the \(Q\)-tensor always stays in the "physical" region. There, in the co-rotational regime \(\xi = 0\), the author proved the existence of global weak solutions for \(d = 2, 3\) as well as global strong solutions for \(d = 2\). Besides, some non-isothermal variants of the Beris–Edwards system were recently derived and analyzed in [17, 18]. The authors proved existence of global weak solutions in the case of a singular potential under periodic boundary conditions for a general parameter \(\xi\) and \(d = 3\).

It is worth pointing out that all the results mentioned above are obtained under the crucial assumption \(L_4 = 0\). More related to our problem, the authors in [9, 11, 23, 24] investigated a gradient flow generated by the free energy \([1.1]\) with \(L_4 \neq 0\). The resulting parabolic equation may be considered as a fluid-free version of the Beris–Edwards system \([15, 17]\) by setting \(u = 0\) in \([1.7]\). Although the gradient flow admits a dissipative energy law, due to the fact that the free energy \(E(Q)\) can be unbounded from below, in general one cannot expect any useful information on a priori estimates of its solution. On the other hand, from \([12]\) we observe that the cubic term associated with \(L_4\) can be controlled by the other positive definite quadratic terms provided that \(|Q|_{L^\infty(T_2)}\) is suitably small. It was shown in [24] that the smallness of \(|Q|_{L^\infty(T_2)}\) can be preserved during the time evolution provided that its initial value is small enough. As a consequence, the usual \(H^1\)-level information provided by the energy dissipation in this gradient flow can be effectively used, and the authors of [24] were able to construct a suitable approximation scheme to prove its global well-posedness. Based on the same idea, a stable numerical
scheme was derived in [9], which also yields a different approach for the existence of global weak solutions for \( d = 2 \) under the smallness assumption of \( \|Q_0\|_{L^\infty(\mathbb{T}^2)} \).

For our Beris–Edward system (1.5)–(1.7) with \( L_4 \neq 0 \), the analysis turns out to be much more involved due to the complicated interactions between the fluid motion and the molecular alignment. Nevertheless, in the co-rotational case \( \xi = 0 \), we are able to show that the preservation of the smallness of \( \|Q_0\|_{L^\infty(\mathbb{T}^2)} \) as in [9,11,24] is still kept during the time evolution under additional effects of advection and rotation due to the fluid (see Lemma 3.1 below). Because of the highly nonlinear coupling between the macroscopic fluid flow and the microscopic orientational configuration of liquid crystal molecules in our system (1.5)–(1.7), there are several extra difficulties in the mathematical analysis, and none of those approximate schemes utilized in [9,11] can be applied to prove the existence of global weak solutions. Alternatively, we introduce a regularized system (see (4.1)–(4.3) below) where a higher-order regularizing term \( \delta (-\Delta)^k \) with \( \delta > 0 \), \( k \geq 4 \) is added to the Navier–Stokes equation (1.5). It yields certain improved regularity of the velocity field \( u \) and helps to guarantee the preservation of \( L^\infty \)-norm of the \( Q \)-tensor in this regularized system. Assuming a slightly more regular initial datum that belongs to \( L^2_0(\mathbb{T}^2) \times H^2(\mathbb{T}^2) \), local well-posedness of the regularized system can be proved by a nonstandard application of Banach’s fixed-point theorem, where a suitable nonlinear mapping \( \mathcal{Y} \) on the space \( L^2(0, T; H^1(\mathbb{T}^2)) \) is constructed in a delicate way (see (4.3) below). In this process, the smallness of \( \|Q_0\|_{L^\infty(\mathbb{T}^2)} \) plays an important role in proving the contractivity of \( \mathcal{Y} \). Then combining the key property on preservation of \( L^\infty \)-norm of the \( Q \)-tensor together with the energy method, we are able to derive uniform-in-time estimates that enable us to extend the local solution to be a global one that is defined on an arbitrary time interval \([0, T]\). Based on the dissipative energy law (see Proposition 2.1) and using the compactness argument, we are able to prove the existence of global weak solutions to our Beris–Edward system (1.5)–(1.7) by passing to the limit as \( \delta \to 0^+ \). Finally, global existence result for general initial data \( (u_0, Q_0) \in L^2_0(\mathbb{T}^2) \times (H^1(\mathbb{T}^2) \cap L^\infty(\mathbb{T}^2)) \) can be achieved by employing a density argument. Because of the highly nonlinear coupling terms related to the stress tensors \( \sigma^a \) and \( \sigma^s \), uniqueness of global weak solutions to system (1.5)–(1.7) is a nontrivial issue even in two spatial dimensions. Some higher-order terms cannot be controlled if we perform energy estimates for the difference of two solutions at the level of natural energy space, say \( (u, Q) \in L^2_0(\mathbb{T}^2) \times H^1(\mathbb{T}^2) \). Inspired by [25,26], we choose to prove a continuous dependence result with respect to the initial data in the lower-order energy space \( (H^1_0(\mathbb{T}^2))^k \times L^2(\mathbb{T}^2) \) (see Lemma 5.2), which immediately yields the uniqueness of global weak solutions. Again in the proof, the preservation of \( L^\infty \)-norm of the \( Q \)-tensor plays an essential role and furthermore, a special cancellation relation hidden in the coupling structure has to be exploited.

The rest of this paper is organized as follows. In Section 2, we state our notational conventions and present some technical lemmas. In Section 3, we derive the dissipative energy law satisfied by the global weak solution to problem (1.5)–(1.9) and a weak maximum principle for the \( Q \)-equation, which yields the preservation of \( \|Q\|_{L^\infty} \) along time evolution. In Section 4, we introduce the regularized problem and prove its global well-posedness with slightly more regular initial data, i.e., \( (u_0, Q_0) \in L^2_0(\mathbb{T}^2) \times H^2(\mathbb{T}^2) \). In the final Section 5, we complete the proof of Theorem 1.1 by passing to the limit as \( \delta \to 0^+ \) in the approximate problem together with a density argument for the initial data. Moreover, the continuous dependence result is established. In the Appendix, we present some detailed computations mentioned in the previous sections.

2 Preliminaries

2.1 Functional settings and notations

Let \( X \) be a real Banach space with norm \( \| \cdot \|_X \) and \( X' \) be its dual space. By \( \langle \cdot, \cdot \rangle_{X', X} \) we indicate the duality product between \( X \) and \( X' \). We denote by \( L^p(\mathbb{T}^2, M) \), \( W^{m,p}(\mathbb{T}^2, M) \), \( 1 \leq p \leq +\infty \), \( m \in \mathbb{N} \), the usual Lebesgue and Sobolev spaces defined on the torus \( \mathbb{T}^2 \) for \( M \)-valued functions (e.g., \( M = \mathbb{R} \), \( M = \mathbb{R}^2 \) or \( M = \mathbb{R}^{2 \times 2} \) that are in \( L^p_{loc}(\mathbb{R}^2) \) or \( W^{m,p}_{loc}(\mathbb{R}^2) \) and periodic in \( \mathbb{T}^2 \), with norms denoted by \( \| \cdot \|_{L^p}, \| \cdot \|_{W^{m,p}}, \) respectively. For \( p = 2 \), we simply denote \( H^m(\mathbb{T}^2) = W^{m,2}(\mathbb{T}^2) \) with norm \( \| \cdot \|_{H^m} \).
In particular for \( m = 0 \), we have \( H^0(T^2) = L^2(T^2) \) and the inner product on \( L^2(T^2) \) will be denoted by \((\cdot, \cdot)_{L^2}\). For simplicity, we shall not distinguish functional spaces when scalar-valued, vector-valued or matrix-valued functions are involved if they are clear from the context.

For arbitrary vectors \( u, v \in \mathbb{R}^2 \), we denote by \( u \cdot v \) the scalar product in \( \mathbb{R}^2 \), while for arbitrary matrices \( A, B \in \mathbb{R}^{2 \times 2} \), the Frobenius product between \( A \) and \( B \) is defined by \( A : B = \text{tr}(A^T B) \). For any matrix \( Q \in \mathbb{R}^{2 \times 2} \), its Frobenius norm is given by \( |Q| = \sqrt{\text{tr}(Q^T Q)} = |Q_{ij}| \). Concerning the norms for spatial derivatives of matrices, we denote \(|\nabla Q|^2(x) = \partial_1 Q_{ij}(x) \partial_1 Q_{ij}(x) \) and more generally for the multi-index \( \alpha = (\alpha_1, \alpha_2) \in (\mathbb{N} \cup \{0\})^2 \) with \(|\alpha| = \alpha_1 + \alpha_2 = l \in \mathbb{N} \), we denote \(|\partial^\alpha Q|^2(x) = \sum_{|\alpha|=l} |\partial^\alpha Q_{ij}(x)|^2 \partial^\alpha Q_{ij}(x) \), where \( \partial^\alpha = \frac{\partial^{(\alpha)}}{\partial x^1 \partial x^2} \) for \( \alpha \neq (0, 0) \) and we agree that \( \partial^{(0,0)}Q = Q \). Then Sobolev spaces for \( Q \)-tensors will be defined in terms of the above norms. For instance,

\[
L^2(T^2, S_0^{(2)}) = \left\{ Q: T^2 \to S_0^{(2)}, \int_{T^2} |Q(x)|^2 \, dx < \infty \right\},
\]

\[
H^m(T^2, S_0^{(2)}) = \left\{ Q: T^2 \to S_0^{(2)}, \sum_{l=0}^m \sum_{|\alpha|=l} \int_{T^2} |\partial^\alpha Q(x)|^2 \, dx < \infty \right\}, \quad m \in \mathbb{N}.
\]

For any \( 2 \times 2 \) differentiable matrix-valued function \( Q = (Q_{ij}) \), we denote the partial derivative of its \( ij \)-component by \( Q_{ij,k} = \partial_1 Q_{ij} \) and its divergence by \( (\nabla \cdot Q)_i = \partial_1 Q_{ij}, 1 \leq i, j \leq 2 \).

Next, we recall the well-established functional settings for periodic solutions to the Navier–Stokes equations (see e.g., [37]):

\[
L^2_q(T^2) = \left\{ v \in L^2(T^2, \mathbb{R}^2), \nabla \cdot v = 0 \right\}, \quad H^m = \left\{ v \in H^m(T^2, \mathbb{R}^2), \nabla \cdot v = 0 \right\},
\]

where \( L^2_q(T^2) \) denotes the usual space of solenoidal vector field. We note that in the spatial periodic setting, the Stokes operator is simply given by

\[
Su = -\Delta u, \quad \forall u \in D(S) \overset{\text{def}}{=} \{ u \in L^2_q(T^2), \Delta u \in L^2_q(T^2) \} = H^2_q(T^2).
\]

The operator \( S \) can be seen as an unbounded positive linear self-adjoint operator on \( L^2_q(T^2) \) and it becomes an isomorphism from \( H^2_q(T^2) \) onto \( L^2_q(T^2) \), where dot means a function space with the constraint of zero mean [37].

Throughout this paper, we denote by \( C \) a generic constant that may depend on \( \nu, L_i, a, b, c, T^2 \) and the initial data \( (u_0, Q_0) \), whose value is allowed to vary on occurrence. Specific dependence will be pointed out explicitly if necessary.

### 2.2 Useful lemmas

Below we present some preliminary results that will be used in the subsequent proofs. First of all, the following algebraic lemma turns to be useful:

**Lemma 2.1.** For any symmetric tensors \( Q, M \in \mathbb{R}^{2 \times 2} \) and a vector field \( u \in \mathbb{R}^2 \), it holds

\[
(QM - MQ) : \nabla u = (Q\omega - \omega Q) : M,
\]

where \( \omega \) is given by \((1.11)\).

**Proof.** Recall the definitions of \( Au \) and \( \omega \) given by \((1.1)\). Using the symmetry of \( Q, M, Au \), the anti-symmetry of \( \omega \) and property of the trace of matrix, we infer from a direct computation that

\[
(QM - MQ) : \nabla u = (QM - MQ) : (Au + \omega) = (QM - MQ) : \omega
\]

\[
= Q_{ij}M_{jk}\omega_{ik} - M_{ij}Q_{jk}\omega_{ik} = -Q_{ij}M_{jk}\omega_{ik} + M_{ij}Q_{jk}\omega_{ik}
\]

\[
= -\text{tr}(QM\omega) + \text{tr}(MQ\omega) = -\text{tr}(\omega QM) + \text{tr}(Q\omega M)
\]

\[
= (Q\omega - \omega Q) : M,
\]

which yields the required assertion. \(\square\)
Next, we recall some standard elliptic estimates in $\mathbb{T}^2$:

**Lemma 2.2.** It holds that

$$
\|Q\|_{H^2} \leq C(\|\Delta Q\|_{L^2} + \|Q\|_{L^2}), \quad \forall Q \in H^2(\mathbb{T}^2),
$$

(2.3)

$$
\|Q\|_{H^2} \leq C(\|\nabla \Delta Q\|_{L^2} + \|Q\|_{L^2}), \quad \forall Q \in H^3(\mathbb{T}^2),
$$

(2.4)

where the positive constant $C$ only depends on $\mathbb{T}^2$.

Besides, we present some interpolation inequalities.

**Lemma 2.3.** For any $f \in H^2(\mathbb{T}^2)$, the following inequalities are valid:

$$
\|\nabla f\|_{L^4}^2 \leq 3\|f\|_{L^\infty}\|\Delta f\|_{L^2},
$$

(2.5)

$$
\|\nabla f\|_{L^4}^2 \leq C\|\nabla f\|_{L^2}\(\|\Delta f\|_{L^2} + \|f\|_{L^2}\).
$$

(2.6)

Moreover, it holds

$$
\|\nabla f\|_{L^\infty} \leq C\|\nabla f\|_{L^2}(\|\Delta f\|_{L^2} + \|f\|_{L^2}), \quad \forall f \in H^3(\mathbb{T}^2).
$$

(2.7)

In the above inequalities, the positive constant $C$ only depends on $\mathbb{T}^2$. Besides, if $V \hookrightarrow H \hookrightarrow V'$ is a Gelfand-triple, then we have

$$
\|f\|_{C^2(0,T;H)} \leq 2(\|f\|_{H^1(0,T;V')}\|f\|_{L^2(0,T;V)} + \|f(\cdot,0)\|_H^2).
$$

(2.8)

**Proof.** Concerning (2.5), we observe the identity

$$
|\partial_j f|^4 = \partial_j(f\partial_j f|\partial_j f|^2) - 3f\partial_j\partial_j f|\partial_j f|^2, \quad \forall f \in H^1(\mathbb{T}^2),
$$

and

$$
\int_{\mathbb{T}^2} |\partial_1\partial_2 f|^2 dx = \int_{\mathbb{T}^2} (|\partial_1\partial_1 f| + |\partial_2\partial_2 f|) dx, \quad \forall f \in H^2(\mathbb{T}^2).
$$

(2.9)

Then by Hölder’s inequality, using integration by parts, we have that

$$
\|\nabla f\|_{L^4}^2 = \int_{\mathbb{T}^2} \sum_{j=1}^2 |\partial_j f|^4 dx = -3 \int_{\mathbb{T}^2} \sum_{j=1}^2 f\partial_j\partial_j f|\partial_j f|^2 dx
$$

$$
\leq 3\|f\|_{L^\infty}\|\nabla f\|_{L^2}^2\sqrt{\int_{\mathbb{T}^2} (|\partial_1\partial_1 f|^2 + |\partial_2\partial_2 f|^2) dx}
$$

$$
\leq 3\|f\|_{L^\infty}\|\nabla f\|_{L^2}^2\sqrt{\int_{\mathbb{T}^2} (|\partial_1\partial_1 f|^2 + 2|\partial_1\partial_2 f|^2 + |\partial_2\partial_2 f|^2) dx}
$$

$$
= 3\|f\|_{L^\infty}\|\nabla f\|_{L^2}^2\|\Delta f\|_{L^2},
$$

which yields the inequality (2.5). Inequalities (2.6) and (2.7) follow from the classical Ladyshenskaya and Agmon inequalities as well as the elliptic estimates in Lemma 2.2. For the last inequality (2.8), we refer to [11] Section 2.1 (see also [2] Lemma 2.6).

We end this section by deriving some estimates for an elliptic problem related to (1.14).

**Lemma 2.4.** Consider the problem

$$
\mathcal{H} + \lambda I + \mu - \mu^T = g, \quad \forall x \in \mathbb{T}^2,
$$

(2.10)

$$
Q(x + e_i, t) = Q(x, t), \quad \forall x \in \mathbb{T}^2, \quad i = 1, 2,
$$

(2.11)

where the left-hand side of (2.10) is given by (1.14). Then for any constant $\eta$ satisfying

$$
0 < \eta < \frac{1}{121}\left(\frac{\zeta}{L_d}\right)^2,
$$

we have

$$
\|\nabla f\|_{L^4}^2 \leq C\|\nabla f\|_{L^2}\(\|\Delta f\|_{L^2} + \|f\|_{L^2}\).
$$

(2.6)
if the solution $Q$ to problem (2.10)–(2.11) fulfills $\|Q\|_{L^\infty(T^2)} \leq \sqrt{\gamma}$, then the following estimate holds

$$\|Q\|_{H^2} \leq C(\|g\|_{L^2} + 1)$$

where the positive constant $C$ only depends on $a, c, L, \zeta, \eta$ and $T^2$.

Proof. Taking the inner product of equation (2.10) with $\Delta Q$ and integrating over $T^2$, using the expression (1.1), Hölder’s inequality as well as (2.5), we obtain that

$$\int_{T^2} |Q|_t \partial_t \Delta Q_{ij} |dx + \int_{T^2} |Q|_{ij} \partial_t \Delta Q_{ij} |dx$$

where the positive constant $\zeta$ testing (1.5) by $u$ yields that

$$\int_{T^2} \left( |Q|_t \partial_t \Delta Q_{ij} + |Q|_{ij} \partial_t \Delta Q_{ij} \right) |dx + \int_{T^2} \left( |a|Q_{ij} + c \operatorname{tr}(Q^2)Q_{ij} \right) \Delta Q_{ij} |dx$$

and integrating over $T^2$, we have

$$\int_{T^2} |Q|_t \partial_t \Delta Q_{ij} |dx + \int_{T^2} |Q|_{ij} \partial_t \Delta Q_{ij} |dx$$

where

$$|Q|_t \partial_t \Delta Q_{ij} + |Q|_{ij} \partial_t \Delta Q_{ij} \leq 2|L_4| \int_{T^2} \left( |Q|_t \partial_t \Delta Q_{ij} + |Q|_{ij} \partial_t \Delta Q_{ij} \right) |dx + 2|L_4| \int_{T^2} \left( |Q|_t \partial_t \Delta Q_{ij} + |Q|_{ij} \partial_t \Delta Q_{ij} \right) |dx$$

Using the assumption on $\eta$ and Young’s inequality, we get

$$\zeta \|\Delta Q\|_{L^2}^2 \leq 11|L_4|\sqrt{\gamma} \|\Delta Q\|^2 + \frac{1}{2}(\zeta - 11|L_4|\sqrt{\gamma}) \|\Delta Q\|_{L^2}^2$$

which together with Lemma (2.2) easily yields the conclusion. \qed

Remark 2.1. Obviously, $\eta$ can be chosen arbitrary large as $|L_4| \to 0^+$. Indeed when $L_4 = 0$, we see from the above proof that no smallness assumption on the $L^\infty$-norm of $Q$ is necessary in order to obtain the estimate for $\|Q\|_{H^2}$.

3 Basic Energy Law and Maximum Principle

In this section, we first derive the following dissipative energy law for global weak solutions of problem (1.5)–(1.9).

Proposition 3.1. Let $(u, Q)$ be a weak solution of problem (1.5)–(1.9) such that

$$u \in H^1(0, T; (H^1_0(T^2))' \cap C([0, T]; L^2_0(T^2)) \cap L^2(0, T; H^1_0(T^2))),$$

$$Q \in H^1(0, T; L^2(T^2)) \cap L^\infty(0, T; H^1(T^2)) \cap L^2(0, T; H^2(T^2)).$$

Then for any $t \in [0, T]$, we have

$$\mathcal{E}_{\text{total}}(t) + \nu \int_0^t \int_{T^2} |\nabla u|^2 dx ds + \int_0^t \int_{T^2} \theta^2(\mathcal{H} + \lambda \| + \mu - \mu^T) dx ds = \mathcal{E}_{\text{total}}(0),$$

where $\mathcal{E}_{\text{total}}$ is given by (1.20).

Proof. The regularity properties (3.1)–(3.2) guarantee that the solution $u$ and the quantity $\mathcal{H} + \lambda \| + \mu - \mu^T \in L^2(0, T; L^2(T^2))$ can be used as test functions for equations (1.5) and (1.7), respectively. Thus, testing (1.5) by $u$ yields that

$$\frac{1}{2} \frac{d}{dt} \int_{T^2} |u|^2 dx + \nu \int_{T^2} |\nabla u|^2 dx + \int_{T^2} \partial^a : \nabla u dx = \int_{T^2} \partial_j \epsilon_{j} \partial_i u_i dx, \quad \text{a.e. in } (0, T),$$

a.e. in $(0, T)$.
where one can verify the integrability of the right-hand side using \(3.1 \text{--} 3.2\) together with the Sobolev embedding theorem in two dimensions. Keeping the symmetry property of \(\mathcal{H} + \lambda I + \mu - \mu^T\) in mind, we deduce from (2.2) that

\[
\begin{align*}
\text{tr}^2(\mathcal{H} + \lambda I + \mu - \mu^T) & = (\mathcal{H} + \lambda I + \mu - \mu^T) : (Q_1 + u \cdot \nabla Q + Q\omega - \omega Q) \\
& = (\mathcal{H} + \lambda I + \mu - \mu^T) : (Q_1 + u \cdot \nabla Q) - (\mathcal{H} + \lambda I + \mu - \mu^T) : (Q\omega - \omega Q) \\
& = \mathcal{H} : (Q_1 + u \cdot \nabla Q) + \left[ Q(\mathcal{H} + \lambda I + \mu - \mu^T) - (\mathcal{H} + \lambda I + \mu - \mu^T)Q \right] : \nabla u,
\end{align*}
\]

where we have used the identity \((\lambda I + \mu - \mu^T) : (\partial_t Q + u \cdot \nabla Q) = 0\), since \(\mu - \mu^T\) is skew-symmetric and \(Q\) is traceless. Then it follows that for almost every \(t \in (0,T)\),

\[
\begin{align*}
\int_{\mathbb{T}^2} \text{tr}^2(\mathcal{H} + \lambda I + \mu - \mu^T) dx \\
& = -\frac{d}{dt} \mathcal{E}(Q) + \int_{\mathbb{T}^2} \mathcal{H} : (u \cdot \nabla Q) dx + \int_{\mathbb{T}^2} \sigma^\alpha : \nabla u dx. \quad (3.5)
\end{align*}
\]

Summing up the identities \(3.4\) and \(3.5\), we obtain

\[
\begin{align*}
\frac{d}{dt} \left( \int_{\mathbb{T}^2} |u|^2 dx + \mathcal{E}(Q) \right) & + \nu \int_{\mathbb{T}^2} |
abla u|^2 dx + \int_{\mathbb{T}^2} \text{tr}^2(\mathcal{H} + \lambda I + \mu - \mu^T) dx \\
& = \int_{\mathbb{T}^2} \mathcal{H} : (u \cdot \nabla Q) dx + \int_{\mathbb{T}^2} \sigma^\alpha \nabla u dx, \quad \text{a.e. in } (0,T), \quad (3.6)
\end{align*}
\]

It remains to show that the right-hand side of \(3.6\) vanishes. Since this step actually does not use the system \(1.3\)--\(1.7\), up to a suitable density argument, we can simply assume that \((u, Q)\) is smooth enough. Moreover, by some straightforward computations, we have the following identities:

\[
\begin{align*}
u_i \partial_j(Q_{kl,i}Q_{kl,j}) & = u_i Q_{kl,i}Q_{kl,j} + u_i Q_{kl,ij}Q_{kl,j} \\
& = u_i Q_{kl,i} \Delta Q_{kl} + \frac{1}{2} u_i \partial_i(Q_{kl,j}Q_{kl,j}),
\end{align*}
\]

\[
\begin{align*}
u_i \partial_j(Q_{kj,i}Q_{kl,i}) & = u_i Q_{kj,i}Q_{kj,l} + u_i Q_{kj,i}Q_{kl,i} \\
& = u_i Q_{kj,i}Q_{kj,l} + \frac{1}{2} u_i (Q_{kj,i}Q_{kl,ij} + Q_{kl,j}Q_{kj,i}) \\
& = u_i Q_{kj,i}Q_{kj,l} + \frac{1}{2} u_i \partial_i (Q_{kj,j}Q_{kl,j})
\end{align*}
\]

\[
\begin{align*}
u_i \partial_j(Q_{kl,i}Q_{kj,i}) & = u_i Q_{kj,i}Q_{kl,l} + u_i Q_{kj,i}Q_{kj,i} \\
& = u_i Q_{kj,i}Q_{kl,l} + \frac{1}{2} u_i (Q_{kj,i}Q_{kj,ij} + Q_{kj,j}Q_{kl,i}) \\
& = u_i Q_{kj,i}Q_{kl,l} + \frac{1}{2} u_i \partial_i (Q_{kl,j}Q_{kj,j})
\end{align*}
\]

\[
\begin{align*}
u_i \partial_j(Q_{jm}Q_{kl,m}Q_{kl,i}) & = u_i Q_{jm,j}Q_{kl,m}Q_{kl,i} + u_i Q_{jm,m}Q_{kl,i} + u_i Q_{jm,l}Q_{kl,i} \\
& = u_i Q_{jm,j}Q_{kl,m}Q_{kl,i} + u_i Q_{jm,m}Q_{kl,i} + \frac{1}{2} u_i Q_{jm} \partial_i (Q_{kl,m}Q_{kl,i}) \\
& = u_i Q_{jm,j}Q_{kl,m}Q_{kl,i} + u_i Q_{jm,m}Q_{kl,i} \\
& + \frac{1}{2} u_i \partial_i (Q_{jm}(Q_{kl,m}Q_{kl,j})) - \frac{1}{2} u_i Q_{jm,i}(Q_{kl,m}Q_{kl,j}).
\end{align*}
\]

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Inserting the above four identities into (1.17) and using the divergence-free condition of $u$, we deduce that
\[
\begin{align*}
\int_{\mathbb{T}^2} u_i \partial_j \sigma^i_j \, dx & = -2 \int_{\mathbb{T}^2} u_j \partial_j (L_1 Q_{kl,i} Q_{kl,j} + L_2 Q_{kj,i} Q_{kj,j} + L_3 Q_{kl,l} Q_{kj,j} + L_4 Q_{jm} Q_{kl,m} Q_{kl,j}) \, dx \\
& = -2 \int_{\mathbb{T}^2} (L_1 u_i Q_{kl,i} \Delta Q_{kl} + L_2 u_i Q_{kj,i} Q_{kj,j} + L_3 u_i Q_{kj,j} Q_{kl,l}) \, dx \\
& \quad - 2L_4 \int_{\mathbb{T}^2} (u_i Q_{jm} Q_{kl,m} Q_{kl,i} + u_i Q_{jm} Q_{kl,m} Q_{kl,i} - \frac{1}{2} u_i Q_{jm} Q_{kl,m} Q_{kl,j}) \, dx \\
& = -2 \int_{\mathbb{T}^2} [L_1 u_i Q_{kl,i} \Delta Q_{kl} + (L_2 + L_3) u_i Q_{kj,i} Q_{kl,j}] \, dx \\
& \quad - 2L_4 \int_{\mathbb{T}^2} (u_i Q_{jm} Q_{kl,m} Q_{kl,i} + u_i Q_{kl,i} Q_{jm} Q_{kl,m}) \, dx \\
& \quad + L_4 \int_{\mathbb{T}^2} u_i Q_{jm} Q_{kl,m} Q_{kl,j} \, dx \\
& \quad := - \int_{\mathbb{T}^2} \mathcal{H} : (u \cdot \nabla Q) \, dx.
\end{align*}
\]
It follows from (1.13) that $\mathcal{H} = H + aQ - bQ^2 + c \text{tr}(Q^2)Q$. On the other hand, by the incompressibility condition (1.6), we see that
\[
\int_{\mathbb{T}^2} [aQ - bQ^2 + c \text{tr}(Q^2)Q] : (u \cdot \nabla Q) \, dx = 0.
\]
As a consequence, it holds
\[
\int_{\mathbb{T}^2} u_i \partial_j \sigma^i_j \, dx = - \int_{\mathbb{T}^2} \mathcal{H} : (u \cdot \nabla Q) \, dx,
\]
which implies that the right-hand side of (3.6) will simply vanish. Then integrating (3.6) with respect to time, we arrive at our conclusion (3.3).

The basic energy law (3.3) will play an essential role in the analysis of problem (1.5)–(1.9). However, we note that when $L_4 \neq 0$, (3.3) fails to provide certain good a priori estimates for the weak solution $(u, Q)$, since the free energy $E(Q)$ may be unbounded from below. This is very different from the case that has been considered in the previous literature, where $L_4 = 0$ is always assumed. On the other hand, the above mentioned difficulty can be overcome if an estimate on $\|Q\|_{L^\infty_t L^\infty_x}$ is available. To this end, we consider the following evolution equation for $Q$ with advection and rotation effects due to the fluid:
\[
\begin{align*}
\partial_t Q + u \cdot \nabla Q + \sigma - \omega Q & = H + \lambda \Pi + \mu - \mu^T, \quad \forall (x, t) \in \mathbb{T}^2 \times \mathbb{R}^+, \\
Q(x + e_1, t) & = Q(x, t), \quad \forall (x, t) \in \mathbb{T}^2 \times \mathbb{R}^+, \\
Q(x, 0) & = Q_0(x), \quad \forall x \in \mathbb{T}^2,
\end{align*}
\]
where the right-hand side of (5.7) is given by (1.11). One interesting feature of the system (3.7)–(3.9) is that the $L^\infty$-norm of its solution $Q$ will be preserved during the time evolution, provided that it is suitably small at $t = 0$ (see [24] for the fluid free case, which is a gradient flow generated by the free energy $E(Q)$).

Lemma 3.1. Assume that $c > 0$, $\zeta > 0$ and $u \in C([0, T]; L^2(\mathbb{T}^2)) \cap L^2(0, T; H^2(\mathbb{T}^2))$ for some integer $k \geq 2$. Let $Q \in H^1(0, T; L^2(\mathbb{T}^2)) \cap C([0, T]; H^1(\mathbb{T}^2) \cap L^\infty(\mathbb{T}^2)) \cap L^2(0, T; H^2(\mathbb{T}^2))$ be a solution to problem (5.7)–(5.9). For any constant $\eta$ satisfying $0 < \eta < \frac{1}{4} \left( \frac{\zeta}{c} \right)^2$, if
\[
\|Q_0\|_{L^\infty_x} \leq \sqrt{\eta},
\]
(3.10)
and

$$a \geq -c\eta,$$

(3.11)

then it holds

$$\|Q\|_{L^\infty(0,T;L^2(\mathbb{T}^2))} \leq \sqrt{\eta}.$$  

(3.12)

**Proof.** We take the inner product of equation (3.11) with the test function $2(|Q|^2 - \eta)^+Q$, where $(\cdot)^+$ denotes the non-negative part of a function, and then integrate over $\mathbb{T}^2$. By the incompressibility condition $\nabla \cdot u = 0$, formula (1.14), and the facts that $\omega = -\omega^T$, $Q \in S_0^{(2)}$, we deduce, after integration by parts that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^2} \left| (|Q|^2 - \eta)^+ \right|^2 dx + \int_{\mathbb{T}^2} |\nabla (|Q|^2 - \eta)^+|^2 dx + 2\zeta \int_{\mathbb{T}^2} |\nabla Q|^2 (|Q|^2 - \eta)^+ dx$$

$$= -6L_4 \int_{\mathbb{T}^2} Q_{ij}\partial_i Q_{kl}\partial_j Q_{kl} (|Q|^2 - \eta)^+ dx - 2L_4 \int_{\mathbb{T}^2} Q_{ij}\partial_i (|Q|^2 - \eta)^+ \partial_j (|Q|^2 - \eta)^+ dx$$

$$- 2 \int_{\mathbb{T}^2} |Q|^2 (a + c|Q|^2) (|Q|^2 - \eta)^+ dx.$$

The first two terms on the right-hand side can be estimated as follows:

$$|I_1| \leq 6L_4 \int_{\mathbb{T}^2} |Q| \|\nabla Q\|^2 (|Q|^2 - \eta)^+ dx,$$

$$|I_2| \leq 2L_4 \int_{\mathbb{T}^2} |Q| \|\nabla (|Q|^2 - \eta)^+\|^2 dx.$$

As a consequence, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^2} \left| (|Q|^2 - \eta)^+ \right|^2 dx$$

$$\leq 2 \int_{\mathbb{T}^2} \left( 3L_4 |Q| - \zeta \right) |\nabla Q|^2 (|Q|^2 - \eta)^+ dx + \int_{\mathbb{T}^2} \left( 2L_4 |Q| - \zeta \right) \|\nabla (|Q|^2 - \eta)^+\|^2 dx$$

$$- 2 \int_{\mathbb{T}^2} |Q|^2 (a + c|Q|^2) (|Q|^2 - \eta)^+ dx.$$  

(3.13)

To finish the proof, we argue by contradiction. Denote

$$f(t) = \|Q(t)\|_{L^\infty},$$

which is a continuous function on $[0,T]$. According to the assumption $f(0) \leq \sqrt{\eta}$, if $f(t) \leq \sqrt{\eta}$ for any $t \in [0,T]$, then the conclusion automatic follows. Otherwise, there exists some $t_0 \in (0,T]$ and $s_0 \in [0,t_0)$ such that

$$\sqrt{\eta} < f(t) \leq \frac{\zeta}{3|L_4|} \text{ for any } t \in (s_0,t_0], \text{ and } f(s_0) = \sqrt{\eta}.$$

From (3.11), we have

$$\int_{\mathbb{T}^2} |Q|^2 (a + c|Q|^2) (|Q|^2 - \eta)^+ dx$$

$$\geq c \int_{\mathbb{T}^2} |Q|^2 (|Q|^2 - \eta) (|Q|^2 - \eta)^+ dx$$

$$\geq c \int_{\mathbb{T}^2} |Q|^2 \left[ (|Q|^2 - \eta)^+ \right]^2 dx \geq 0, \quad \forall t \in [s_0,t_0].$$

Then we infer from (3.13) that

$$\frac{d}{dt} \int_{\mathbb{T}^2} \left| (|Q|^2 - \eta)^+ \right|^2 dx \leq 0, \quad \forall t \in [s_0,t_0],$$
which together with the fact \( f(s_0) = \sqrt{\eta} \) yields that

\[
\int_{\mathbb{T}^2} \left( |(Q|^2 - \eta)^+ \right)^2(t) \, dx \leq 0, \quad \forall t \in [s_0, t_0).
\]

Hence, \( f(t) \leq \sqrt{\eta} \) for any \( t \in [s_0, t_0) \), which leads to a contradiction with the assumption \( f(t_0) > \sqrt{\eta} \) and the continuity of \( f(t) \). \( \square \)

4 Global Well-posedness of an Approximate System

In this section, we study an approximate system for our original problem (1.5)–(1.9), in which a higher-order dissipative term is added to the Navier–Stokes equation (1.5), for the sake of improving the regularity as well as integrability of the velocity field \( u \). Denote

\[
\mathcal{L}_\delta := \delta(\Delta)^k,
\]

for some given positive constant \( \delta \) and even integer \( k \geq 4 \). We consider the following regularized system

\[
\begin{align*}
\partial_t u + u \cdot \nabla u + \mathcal{L}_\delta u - \nu \Delta u + \nabla P &= \nabla \cdot (\sigma^a + \sigma^s), \quad \forall (x, t) \in \mathbb{T}^2 \times \mathbb{R}^+, \quad (4.1) \\
\nabla \cdot u &= 0, \quad \forall (x, t) \in \mathbb{T}^2 \times \mathbb{R}^+, \quad (4.2) \\
\partial_t Q + u \cdot \nabla Q + Q \omega - \omega Q &= \mathcal{H} + \lambda \| + \mu - \mu^T, \quad \forall (x, t) \in \mathbb{T}^2 \times \mathbb{R}^+, \quad (4.3)
\end{align*}
\]

subject to the periodic boundary conditions and initial conditions

\[
\begin{align*}
&u(x + e_i, t) = u(x, t), \quad Q(x + e_i, t) = Q(x, t), \quad \forall (x, t) \in \mathbb{T}^2 \times \mathbb{R}^+, \quad (4.4) \\
&u(x, 0) = u_0(x) \quad \text{with} \quad \nabla \cdot u_0 = 0, \quad Q(x, 0) = Q_0(x), \quad \forall x \in \mathbb{T}^2. \quad (4.5)
\end{align*}
\]

The main result of this section is as follows:

**Proposition 4.1.** Let \( T > 0 \) be arbitrary. Under the assumptions (4.8)–(4.9), there exists a positive constant

\[
\eta_1 = C_1 \left( \frac{\kappa}{L_4} \right)^2,
\]

where \( C_1 > 0 \) only depends on \( \mathbb{T}^2 \) such that if in addition, the coefficients \( a, c \) fulfill \( a \geq -c\eta_1 \), then for any initial data \((u_0, Q_0) \in L^2_0(\mathbb{T}^2, \mathbb{R}^2) \times H^2(\mathbb{T}^2, L^2(\mathbb{T}^2)) \)

\[
\|Q_0\|_{L^\infty} \leq \sqrt{\eta_1},
\]

the approximate system (4.1) admits a unique global solution such that

\[
\begin{align*}
&u \in H^1(0, T; (H^2_0(\mathbb{T}^2))') \cap C([0, T]; L^2(\mathbb{T}^2)), \cap L^2(0, T; H^2_0(\mathbb{T}^2)), \quad (4.8) \\
&Q \in H^1(0, T; H^1(\mathbb{T}^2)) \cap C([0, T]; H^2(\mathbb{T}^2)) \cap L^2(0, T; H^3(\mathbb{T}^2)). \quad (4.9)
\end{align*}
\]

Moreover, we have \( Q \in C([0, T]; L^\infty(\mathbb{T}^2)) \) with the uniform bound

\[
\|Q\|_{C([0, T]; L^\infty(\mathbb{T}^2))} \leq \sqrt{\eta_1}, \quad (4.10)
\]

as well as the following dissipative energy law

\[
\frac{1}{2} \int_{\mathbb{T}^2} |u(\cdot, t)|^2 \, dx + \mathcal{E}(Q(t)) + \int_0^t \int_{\mathbb{T}^2} (\sqrt{\eta} \nu \Delta u(s))^2 + \nu |\nabla u(s)|^2 + \text{tr}^2(\mathcal{H} + \lambda \| + \mu - \mu^T)) \, dxds \\
= \frac{1}{2} \int_{\mathbb{T}^2} |u_0|^2 \, dx + \mathcal{E}(Q_0), \quad \forall t \in [0, T]. \quad (4.11)
\]

The proof of Proposition 4.1 consists of several steps. Roughly speaking, we first prove the existence and uniqueness of a local-in-time solution via a suitable fixed point argument. Then using the uniform estimates provided by the dissipative energy law (4.11), we are able to extend the local solution to a global one.
4.1 Local well-posedness

We start with the local well-posedness of problem (1.1)–(1.3).

**Step 1.** Reformulation of the system.

Recalling the expression (4.14), we reformulate the system (4.1)–(4.3) into the following form:

\[
\begin{align*}
\partial_t u + u \cdot \nabla u + \mathcal{L}_\delta u - \nu \Delta u + \nabla P &= \nabla \cdot (\sigma^a + \sigma^e), \\
\nabla \cdot u &= 0, \\
\partial_t Q_{ij} - \Delta Q_{ij} - 2L_4(Q_{ij,l}(Q_0)_{ik})_{,k} &= 2L_4(Q_{ij,l}Q_{lk})_{,k} - 2L_4(Q_{ij,l}(Q_0)_{ik})_{,k} + \mathcal{G}_{ij}(u, Q),
\end{align*}
\]

where

\[\mathcal{G}_{ij}(u, Q) := -u \cdot \nabla Q_{ij} + \omega_{ik}Q_{kj} - Q_{ik}\omega_{kj} - aQ_{ij} - c tr(Q^2)Q_{ij} - L_4Q_{kl,j}Q_{kl,i} + \frac{L^2}{2}|\nabla Q|^2\delta_{ij}.\]

**Remark 4.1.** One can see that an extra term \(-2L_4(Q_{ij,l}(Q_0)_{ik})_{,k}\) is added to both sides of the equation (4.14). Since \(Q_0 \in H^2(T^2) \Rightarrow L^\infty(T^2)\) and \(||Q_0||_{L^\infty(T^2)}\) is assumed to be relatively small (see (4.7)), then the left-hand side of (4.14) simply behaves like a linear heat equation. On the other hand, an advantage of adding the extra term \(-2L_4(Q_{ij,l}(Q_0)_{ik})_{,k}\) on the right-hand side of (4.14) is that together with the nonlinear term \(2L_4(Q_{ij,l}Q_{lk})_{,k}\), it allows us to apply an interpolation argument that leads to the construction of a contraction mapping.

**Step 2.** Well-posedness of auxiliary problems for \(u\) and \(Q\).

In order to deal with the highly nonlinear system (1.12)–(1.14), we first present a preliminary result on the solvability of a higher-order Navier–Stokes type system.

**Lemma 4.1.** Let \(\delta > 0, k \geq 4\) be fixed. For any \(T > 0\), consider the following problem

\[
\begin{align*}
\partial_t u + u \cdot \nabla u + \mathcal{L}_\delta u - \nu \Delta u + \nabla P &= g, \quad (x, t) \in T^2 \times (0, T), \\
\nabla \cdot u &= 0, \quad (x, t) \in T^2 \times (0, T), \\
u(x + e_i, t) &= u(x, t), \quad (x, t) \in T^2 \times (0, T), \\
u|_{t=0} &= u_0(x), \quad \forall x \in T^2.
\end{align*}
\]

Then for any \(u_0 \in L^2_{\sigma}(T^2)\) and \(g \in L^2(0, T; (H^k_{\sigma}(T^2))^\prime)\), problem (4.16)–(4.19) admits a unique global weak solution such that

\[u \in H^1(0, T; (H^k_{\sigma}(T^2))^\prime) \cap C([0, T]; L^2_{\sigma}(T^2)) \cap L^2(0, T; H^k_{\sigma}(T^2)),\]

which satisfies

\[\|u\|_{H^1(0, T; (H^k_{\sigma}(T^2))^\prime) \cap C([0, T]; L^2_{\sigma}(T^2)) \cap L^2(0, T; H^k_{\sigma}(T^2))} \leq C\left(||g||_{L^2(0, T; (H^k_{\sigma}(T^2))^\prime)} + ||u_0||_{L^2_{\sigma}}\right).\]

Let \(u^i\) (\(i = 1, 2\)) be two weak solutions of problem (4.16)–(4.19) with external forces \(g^i \in L^2(0, T; L^2(T^2))\) and initial data \(u^i(x, 0) = u_0^i(x) \in L^2_{\sigma}(T^2)\). Then we have

\[\|u^1 - u^2\|_{C([0, T]; L^2(T^2)) \cap L^2(0, T; H^k_{\sigma}(T^2))} \leq C\left(||g^1 - g^2||_{L^2(0, T; (H^k_{\sigma}(T^2))^\prime)} + ||u_0^1 - u_0^2||_{L^2_{\sigma}(T^2)}\right),\]

where the constant \(C\) may depend on \(||u^i||_{L^2(0, T; L^\infty(T^2))}\), \(T\) and \(T^2\). Furthermore, if \(g \in L^2(0, T; L^2(T^2))\) and \(u_0 \in H^k_{\sigma}(T^2)\), then the solution is more regular and

\[\|u\|_{H^1(0, T; L^2(T^2)) \cap C([0, T]; H^k_{\sigma}(T^2)) \cap L^2(0, T; H^k_{\sigma}(T^2))} \leq C\left(||g||_{L^2(0, T; L^2(T^2))} + ||u_0||_{H^k_{\sigma}(T^2)}\right).\]

All the constants above may depend on the parameter \(\delta > 0\).
Proof. The regularized system (1.16)–(1.19) was used for the well-posedness of the Navier–Stokes equations (see e.g., [27]). Recalling that \( k \) is even, then based on the coerciveness of the bilinear form \( a(\cdot, \cdot) : H^k_0(\mathbb{T}^2) \times H^k_0(\mathbb{T}^2) \to \mathbb{R} \) (see e.g., [8]),

\[
a(u, v) = \int_{\mathbb{T}^2} \left( \frac{\partial}{\partial x^i} u \cdot \frac{\partial}{\partial x^j} v + u \cdot v \right) dx,
\]

the proof can be carried out in a similar manner as in [28] Proposition 4.1. Thus we omit the details here.

Next, we consider a linear parabolic problem associated with (4.14).

**Lemma 4.2.** Suppose that the assumptions in Proposition 4.1 are satisfied. For any \( T > 0 \) and \( G \in L^2(0, T; H^1(\mathbb{T}^2); \mathbb{S}^{(2)}_0) \), the following linear problem

\[
\begin{align*}
\partial_t Q_{ij} - \zeta \Delta Q_{ij} &- 2L_4(Q_{ij,l}(Q_0)_{lk})_{,k} = G_{ij}, \quad \forall (x, t) \in \mathbb{T}^2 \times (0, T), \\
Q(x + e_1, t) &= Q(x, t), \quad \forall (x, t) \in \mathbb{T}^2 \times (0, T), \\
Q(x, 0) &= Q_0(x), \quad \forall x \in \mathbb{T}^2,
\end{align*}
\]

admits a unique strong solution such that

\[
Q \in H^1(0, T; H^1(\mathbb{T}^2)) \cap C([0, T]; H^2(\mathbb{T}^2)) \cap L^2(0, T; H^3(\mathbb{T}^2)), \tag{4.27}
\]

and

\[
\|Q\|_{H^1(0, T; H^1(\mathbb{T}^2)) \cap C([0, T]; H^2(\mathbb{T}^2)) \cap L^2(0, T; H^3(\mathbb{T}^2))} \leq C(\|Q_0\|_{H^2} + \|G\|_{L^2(0, T; H^1(\mathbb{T}^2)))}. \tag{4.28}
\]

**Proof.** The assumptions (1.16)–(1.17) and the relation (1.21) guarantee the ellipticity of the second order operator in equation (4.24) when \( C_1 \) is small enough. Thus we can prove the existence of a unique strong solution that satisfies (4.24) by a standard Galerkin approximation. Below we only show the validity of necessary a priori estimates. Testing (4.24) with \( Q - \Delta Q \) and integrating over \( \mathbb{T}^2 \), using integration by parts, (2.5)–(2.6), Hölder's inequality and Young's inequality, we see that

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} (\|Q\|_{L^2}^2 + \|\nabla Q\|_{L^2}^2) + \zeta (\|\nabla Q\|_{L^2}^2 + \|\Delta Q\|_{L^2}^2) &= -2L_4 \int_{\mathbb{T}^2} (Q_0)_{ik} Q_{ij,l} Q_{ij,k} dx - 2L_4 \int_{\mathbb{T}^2} (Q_0)_{ikl} Q_{ij,k} \Delta Q_{ij} dx \\
&\quad - 2L_4 \int_{\mathbb{T}^2} (Q_0)_{ikl} Q_{ij,l} \Delta Q_{ij} dx + \int_{\mathbb{T}^2} G_{ij} (Q_{ij} - \Delta Q_{ij}) dx \\
&\leq 2|L_4||Q_0|_{L^\infty} \|\nabla Q\|_{L^2}^2 + C|L_4||Q_0|_{L^\infty} \|\Delta Q\|_{L^2} (\|\Delta Q\|_{L^2} + \|Q\|_{L^2}) \\
&\quad + 2|L_4||\nabla Q_0||_{L^1} \|\nabla Q\|_{L^1} \|\Delta Q\|_{L^2} + \|G\|_{L^2} \|\nabla Q\|_{L^2} + \|\Delta Q\|_{L^2} \\
&\leq \zeta \left( \|\nabla Q\|_{L^2}^2 + \|\Delta Q\|_{L^2}^2 \right) + C(\|Q_0\|_{L^2}^2 + \|\nabla Q\|_{L^2}^2 + \|\Delta Q\|_{L^2}^2) + C|L_4||Q_0|_{L^\infty} \|\nabla Q\|_{L^2} \\
&\quad \leq \frac{\zeta}{2} \left( \|\nabla Q\|_{L^2}^2 + \|\Delta Q\|_{L^2}^2 \right) + C(\|Q_0\|_{L^2}^2 + \|\nabla Q\|_{L^2}^2) + C|G|_{L^2}^2,
\end{align*}
\]

where in above estimate, we employed (4.24) with sufficiently small \( C_1 \) (but only depending on \( \mathbb{T}^2 \)). By Gronwall’s lemma, we get

\[
\|Q\|_{L^\infty(0, T; H^1(\mathbb{T}^2))} + \|Q\|_{L^2(0, T; H^2(\mathbb{T}^2))} \leq C(\|Q_0\|_{H^2} + \|G\|_{L^2(0, T; L^2(\mathbb{T}^2)))}. \tag{4.29}
\]

Next, testing (4.24) with \( \Delta^2 Q \) using (2.6), (2.7) and the elliptic estimates (2.6)–(2.7), we deduce that

\[
\frac{1}{2} \frac{d}{dt} \|\Delta Q\|_{L^2}^2 + \zeta \|\nabla \Delta Q\|_{L^2}^2 \leq C(\|Q_0\|_{H^2} + \|G\|_{L^2(0, T; L^2(\mathbb{T}^2)))}. \tag{4.29}
\]
Finally, keeping above estimates in mind, using the elliptic estimates and a comparison argument for
By virtue of (1.17), Hölder’s inequality and the Sobolev embedding theorem, we obtain
where in order to obtain the last inequality, we again require $C_1$ to be suitably small. Using Gronwall’s lemma and the estimate (1.29), we obtain
\[
\|\Delta Q\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla \Delta Q\|_{L^2(0,T;L^2(\Omega))} \leq C(\|Q_0\|_{H^2} + \|Q\|_{L^2(0,T;H^1(\Omega))}).
\]
Finally, keeping above estimates in mind, using the elliptic estimates and a comparison argument for $\partial_t Q$, we arrive at the conclusion (1.28).

As a corollary, we can easily derive some estimates on the induced nonlinear stress terms:

**Lemma 4.3.** Let $Q$ be the solution to problem (1.24)–(1.26) given by Lemma 4.2. Concerning the stress tensors $\sigma^s$, $\sigma^a$ that are defined via (1.17), we have
\[
\|\nabla \cdot (\sigma^a + \sigma^s)\|_{L^2(0,T;L^2(\Omega))} \leq C(\|Q_0\|_{H^2}, \|Q\|_{L^2(0,T;H^1(\Omega))}). \tag{4.30}
\]

**Proof.** By virtue of (1.17), Hölder’s inequality and the Sobolev embedding theorem, we obtain
\[
\|\nabla \cdot \sigma^a\|_{L^2} \leq 4(L_1 + |L_2| + |L_3|)\|\nabla Q\|_{L^\infty} \|Q\|_{H^2} + 2|L_4|\|\nabla Q\|_{L^2}^2
+ 4L_4\|\nabla Q\|_{L^\infty} \|\nabla Q\|_{L^2} \|Q\|_{H^2}
\leq C(\|Q\|_{H^2} \|Q\|_{H^3} + \|Q\|_{H^2}^2 + \|Q\|_{H^1} \|Q\|_{H^2}^2)
\leq C\|Q\|_{H^2} \|Q\|_{H^3} + C\|Q\|_{H^2}^3,
\]
which together with (1.28) yields that
\[
\|\nabla \cdot \sigma^a\|_{L^2(0,T;L^2(\Omega))} \leq C(\|Q_0\|_{H^2}, \|Q\|_{L^2(0,T;H^1(\Omega))}).
\]
Recalling the expression (1.14), we see that $\|\nabla \cdot \sigma^a\|_{L^2(0,T;L^2(\Omega))}$ can be estimated in a similar manner. The proof is complete. \hfill \Box

**Step 3.** Construction of a nonlinear mapping $\mathcal{Y}$.  

For arbitrary but fixed $T > 0$, we proceed to define a mapping $\mathcal{Y}$ on $L^2(0,T;L^2(\Omega))$ based on the results in Step 2. To this end, for any given matrix-valued function $G \in L^2(0,T;H^1(\Omega)^2)$, thanks to Lemma 4.2, problem (1.24)–(1.26) is uniquely solvable and its solution $Q = Q[G]$ satisfies (1.27). Then by Lemma 4.3, we also have $\nabla \cdot (\sigma^a(Q) + \sigma^s(Q)) \in L^2(0,T;L^2(\Omega))$, which enables us to apply Lemma 4.1 to conclude that problem (1.16)–(1.19) admits a unique global weak solution $u = u[G]$ with the regularity property (1.20). Besides, by Lemma 4.1, the solution $u$ of problem (1.16)–(1.19) fulfills
\[
\|u\|_{L^\infty(0,T;L^2(\Omega))} \leq C(\|\nabla \cdot (\sigma^a + \sigma^s)\|_{L^2(0,T;L^2(\Omega))} + \|u_0\|_{L^2}). \tag{4.31}
\]
Hence, it follows from Lemma 4.3, 4.28 and 4.29 that
\[
\|u\|_{L^\infty(0,T;L^2(\Omega))} + \|Q\|_{L^\infty(0,T;H^2(\Omega))} \leq C(\|G\|_{L^2(0,T;H^1(\Omega))}, \|u_0\|_{L^2}, \|Q_0\|_{H^2}). \tag{4.32}
\]
Next, we verify that
\[ G(u, Q) + 2L_4(Q_{ij}, Q_{ik})_t = 2L_4(Q_{ij}, (Q_0)_t)_t \in L^2(0, T; H^1(\mathbb{T}^2)) \]
where \( G \) is given by (4.15). First, since \( Q \in S_0^{(2)} \) a.e. in \( \mathbb{T}^2 \times (0, T) \), it is straightforward to check that \( G \in S_0^{(2)} \) a.e. in \( \mathbb{T}^2 \times (0, T) \). In view of (4.15), we deduce from (4.28), (4.32), Hölder’s inequality and the Sobolev embedding that
\[
\|G(u, Q)\|_{H^1} 
\leq C(\|u \cdot \nabla Q\|_{H^1} + \|\nabla u Q\|_{H^1} + \|1 + \text{tr}(Q^2)Q\|_{H^1} + \|\nabla \nabla Q\|_{H^1}) 
\leq C(\|u\|_{L^4} \|\nabla Q\|_{L^4} + C\|\nabla u\|_{L^4} \|Q\|_{W^{1,4}} + C\|u\|_{L^\infty} \|Q\|_{H^2} + C\|u\|_{H^2} \|Q\|_{L^\infty} 
+ C(1 + \|Q\|_{L^2}) \|Q\|_{H^1} + C\|\nabla Q\|_{L^2} \|Q\|_{W^{2,4}} + C\|\nabla Q\|_{L^4} \|Q\|_{W^{2,4}} 
\leq C(\|G\|_{L^2(0, T; H^1(\mathbb{T}^2))}, \|Q_0\|_{H^2}) \left( 1 + \|u\|_{W^{1,4}} + \|u\|_{L^\infty} + \|u\|_{H^2} + \|Q\|_{H^2}^\frac{1}{2} \right) 
\leq C(\|G\|_{L^2(0, T; H^1(\mathbb{T}^2))}, \|Q_0\|_{H^2}) \left( 1 + \|u\|_{H^2}^\frac{1}{2} + \|Q\|_{H^2}^\frac{1}{2} \right),
\]
\[ (4.33) \]
which together with (4.32) further implies \( G(u, Q) \in L^2(0, T; H^1(\mathbb{T}^2)) \).

It remains to estimate \( 2L_4(\|Q_{ij}\|_{L^2})_t - (\|Q_{ij}\|_{L^2}(Q_{ij})_t)_t ) \in L^2(0, T; H^1(\mathbb{T}^2))\). Note that
\[
2L_4(\|Q_{ij}\|_{L^2})_t - (\|Q_{ij}\|_{L^2}(Q_{ij})_t)_t ) \leq C(\|Q\|_{L^2(0, T; H^1(\mathbb{T}^2))} \|Q - Q_0\|_{L^\infty(0, T; L^2(\mathbb{T}^2))} 
+ C\|Q\|_{L^2(0, T; W^{2,4}(\mathbb{T}^2))} \|Q - Q_0\|_{L^\infty(0, T; W^{1,4}(\mathbb{T}^2))} 
+ C\|\nabla Q\|_{L^2(0, T; L^\infty(\mathbb{T}^2))} \|\nabla (Q - Q_0)\|_{L^\infty(0, T; H^1(\mathbb{T}^2))} 
:= \sum_{i=1}^3 J_i
\]
\[ (4.34) \]
with obvious notation. Using Agmon’s inequality in two dimensions and the interpolation inequality (4.28) (keeping in mind that \( (Q - Q_0)_{t=0} = 0 \)), we have
\[
J_1 \leq C\|Q\|_{L^2(0, T; H^1(\mathbb{T}^2))} \|Q - Q_0\|_{L^\infty(0, T; L^2(\mathbb{T}^2))} \|Q - Q_0\|_{H^1(0, T; L^2(\mathbb{T}^2))} 
\leq C(\|G\|_{L^2(0, T; H^1(\mathbb{T}^2))}, \|Q_0\|_{H^2}) \|Q - Q_0\|_{L^\infty(0, T; H^1(\mathbb{T}^2))} 
\leq C(\|G\|_{L^2(0, T; H^1(\mathbb{T}^2))}, \|Q_0\|_{H^2}) \|Q - Q_0\|_{L^\infty(0, T; H^1(\mathbb{T}^2))} \|Q - Q_0\|_{H^1(0, T; L^2(\mathbb{T}^2))} 
\leq C(\|G\|_{L^2(0, T; H^1(\mathbb{T}^2))}, \|Q_0\|_{H^2}) \|Q - Q_0\|_{L^\infty(0, T; H^1(\mathbb{T}^2))} T^\frac{1}{2}
\]
\[ (4.35) \]
Next, it follows from the interpolation inequality and (4.28) that
\[
J_2 \leq C\|Q\|_{L^\infty(0, T; H^2(\mathbb{T}^2))} \|Q\|_{L^1(0, T; L^2(\mathbb{T}^2))} \|Q - Q_0\|_{L^\infty(0, T; H^2(\mathbb{T}^2))} 
\leq C(\|G\|_{L^2(0, T; H^1(\mathbb{T}^2))}, \|Q_0\|_{H^2}) \|Q\|_{L^1(0, T; H^2(\mathbb{T}^2))} 
\leq C(\|G\|_{L^2(0, T; H^1(\mathbb{T}^2))}, \|Q_0\|_{H^2}) \|Q\|_{L^1(0, T; H^2(\mathbb{T}^2))} T^\frac{1}{2}
\]
\[ (4.36) \]
and in a similar manner,
\[
J_3 \leq C\|\nabla Q\|_{L^\infty(0, T; L^2(\mathbb{T}^2))} \|\nabla Q\|_{L^1(0, T; H^2(\mathbb{T}^2))} \|Q - Q_0\|_{L^\infty(0, T; H^2(\mathbb{T}^2))} 
\leq C(\|G\|_{L^2(0, T; H^1(\mathbb{T}^2))}, \|Q_0\|_{H^2}) T^\frac{1}{2},
\]
\[ (4.37) \]
As a consequence, we obtain
\[
2|L_4||(Q_{ij,l}Q_{lk})_{,k} - (Q_{ij,l}(Q_{0})_{,k})_{,k}\|_{L^2(0,T;H^1(\mathbb{T}^2))} \\
\leq C(||G||_{L^2(0,T;H^1(\mathbb{T}^2))}, ||Q_0||_{H^2})(T^\frac{3}{2} + T^\frac{1}{4}),
\] (4.35)
which implies 2|L_4|(Q_{ij,l}Q_{lk})_{,k} - 2|L_4|(Q_{ij,l}(Q_{0})_{,k})_{,k} \in L^2(0,T;H^1(\mathbb{T}^2)).

Therefore, we see that the following nonlinear mapping defined by
\[
Y: L^2(0,T;H^1(\mathbb{T}^2, S_0^{(2)})) \rightarrow L^2(0,T;H^1(\mathbb{T}^2, S_0^{(2)}))
\]
\[
G \rightarrow Y(G) = G(u, Q) + 2L_4(Q_{ij,l}Q_{lk})_{,k} - 2L_4(Q_{ij,l}(Q_{0})_{,k})_{,k}
\] (4.36)
is well defined.

**Step 4.** The mapping \(Y\) is a contraction for sufficiently small \(T > 0\).

To construct local-in-time solutions to problem (4.34)-(4.35), it suffices to show that \(Y\) is a contraction on certain closed ball
\[
B_{T,R} := \{ G \in L^2(0,T;H^1(\mathbb{T}^2, S_0^{(2)})) : ||G||_{L^2(0,T;H^1(\mathbb{T}^2))} \leq R \}.
\] (4.37)
Here, we simply take
\[
R = ||u_0||_{L^2} + ||Q_0||_{H^2} + 1.
\]

We first show that for certain sufficiently small time \(T_0 > 0\) (depending on \(R\)), \(Y\) maps \(B_{T_0,R}\) into itself. By (4.35), we can find \(T_1 > 0\) sufficiently small such that (recalling that \(k \geq 4\))
\[
||G(u, Q)||_{L^2(0,T_1;H^1(\mathbb{T}^2))} \\
\leq C(R, ||u_0||_{L^2}, ||Q_0||_{H^2})(T_1^\frac{3}{2} + ||u||_{L^2(0,T_1;H^1(\mathbb{T}^2))} + ||Q||_{L^2(0,T_1;H^1(\mathbb{T}^2))})
\]
\[
\leq C(R, ||u_0||_{L^2}, ||Q_0||_{H^2})(T_1^\frac{3}{2} + T_1^\frac{1}{4}||u||_{L^2(0,T_1;H^1(\mathbb{T}^2))} + T_1^\frac{1}{4}||Q||_{L^2(0,T_1;H^1(\mathbb{T}^2))}) \leq \frac{R}{2}.
\] (4.38)

On the other hand, due to (4.35), there exists a small \(T_2 > 0\) such that
\[
2|L_4||(Q_{ij,l}Q_{lk})_{,k} - (Q_{ij,l}(Q_{0})_{,k})_{,k}\|_{L^2(0,T_2;H^1(\mathbb{T}^2))} \\
\leq C(R, ||Q_0||_{H^2})(T_2^\frac{3}{2} + T_2^\frac{1}{4}) \leq \frac{R}{2}.
\] (4.39)
Thus, we conclude from (4.35) and (4.39) that for \(T_0 = \min\{T_1, T_2\}\), the nonlinear mapping \(Y\) maps \(B_{T_0,R}\) into itself.

Next, we show that \(Y\) is actually a contraction. For any \(G_i \in B_{T,R}(i = 1, 2)\), let \((u_i, Q_i)\) be the corresponding solution to the following problem with \(G = G_i\):
\[
\begin{align*}
\partial_t u + \nabla Q - \zeta \Delta Q - 2L_4(Q_{ij,l}(Q_{0})_{,k})_{,k} &= G, \\
\nabla \cdot u &= 0, \\
\n\partial_t Q - \zeta \Delta Q - 2L_4(Q_{ij,l}(Q_{0})_{,k})_{,k} &= G, \\
\n\nabla \cdot Q - \zeta \Delta Q - 2L_4(Q_{ij,l}(Q_{0})_{,k})_{,k} &= G, \\
\n\nabla \cdot u &= 0,
\end{align*}
\]
(4.40)
where \(\sigma^a\) and \(\sigma^s\) are given by (4.16), (4.17), respectively. Denote
\[
\hat{u} = u_1 - u_2, \quad \hat{Q} = Q_1 - Q_2, \quad \hat{P} = P_1 - P_2, \quad \hat{G} = G_1 - G_2.
\]
Then the difference functions \((\hat{u}, \hat{P}, \hat{Q})\) satisfy
\[
\begin{align*}
\partial_t \hat{u} + \nabla \hat{Q} - \nu \Delta \hat{u} + \nabla \hat{P} &= - \hat{u} \cdot \nabla \hat{u} - \hat{u} \cdot \nabla u_2 + \nabla \cdot (\hat{\sigma}^a + \hat{\sigma}^s), \\
\nabla \cdot \hat{u} &= 0, \\
\partial_t \hat{Q} - \zeta \Delta \hat{Q} - 2L_4(Q_{ij,l}(Q_{0})_{,k})_{,k} &= \hat{G}, \\
\n\hat{u}(x,0) &= u_0(x), \\
\hat{Q}(x,0) &= Q_0(x), \\
\n\hat{Q}(x,0) &= Q_0(x), \\
\n\hat{Q}(x,0) &= Q_0(x), \\
\end{align*}
\]
(4.41)
where\[
\dot{\sigma}^a = \sigma^a(Q_1) - \sigma^a(Q_2), \quad \dot{\sigma}^s = \sigma^s(Q_1) - \sigma^s(Q_2).
\]

First, by Lemma 4.22, we have the estimate for \( \dot{Q} \):
\[
\|\dot{Q}\|_{L^2(0,T;H^1(T^2))} \leq C\|\dot{Q}\|_{L^2(0,T;H^1(T^2))}.
\]

(4.42)

Denote for simplicity \( \tilde{\mathcal{H}} = \mathcal{H} + \lambda \mathcal{I} + \mu - \mu^T \). Then we deduce from the estimate (4.28) for \( Q_1, Q_2 \) that
\[
\| \nabla \cdot \dot{\sigma}^a \|_{L^2} \leq \nabla \cdot (\hat{\mathcal{H}}(Q_1)) + \nabla \cdot (Q_2(\tilde{\mathcal{H}}(Q_1) - \tilde{\mathcal{H}}(Q_2))) + C\|\nabla \hat{\mathcal{H}}(Q_1)\|_{L^2} + C\|\nabla \hat{\mathcal{H}}(Q_1)\|_{L^2} + C\|\nabla \hat{\mathcal{H}}(Q_1)\|_{L^2} + C\|\nabla \hat{\mathcal{H}}(Q_1)\|_{L^2}
\]

(4.43)

and henceforth
\[
\| \nabla \cdot \dot{\sigma}^a \|_{L^2(0,T;L^2(T^2))} \leq C(R, \|Q_0\|_{H^2})\|\dot{Q}\|_{L^2(0,T;H^1(T^2))} + C(R, \|Q_0\|_{H^2})\|\dot{Q}\|_{L^2(0,T;H^1(T^2))} \|\dot{Q}\|_{L^2(0,T;H^1(T^2))}
\]

Analogously, we have
\[
\| \nabla \cdot \dot{\sigma}^s \|_{L^2} \leq C(R, \|Q_0\|_{H^2})\|\dot{Q}\|_{L^2(0,T;H^1(T^2))} + C(R, \|Q_0\|_{H^2})\|\dot{Q}\|_{L^2(0,T;H^1(T^2))}
\]

which yields
\[
\| \nabla \cdot \dot{\sigma}^s \|_{L^2(0,T;L^2(T^2))} \leq C(R, \|Q_0\|_{H^2})\|\dot{Q}\|_{L^2(0,T;H^1(T^2))}.
\]

(4.44)

Back to the first equation of (4.41), we infer from Lemma 4.21 (i.e., (4.22)) together with the estimates (4.42) and (4.43) that
\[
\| \hat{u} \|_{C([0,T];L^2(T^2)) \cap L^2(0,T;H^1(T^2))} \leq C\|\nabla \cdot (\dot{\sigma}^a + \dot{\sigma}^s)\|_{L^2(0,T;L^2(T^2))}
\]

(4.45)
As a consequence, keeping in mind the assumption $k \geq 4$, we deduce that

\[
\|G(u_1, Q_1) - G(u_2, Q_2)\|_{H^{k}} \leq C(R, \|Q_0\|_{H^2}) \|\tilde{G}\|_{L^2(0,T;H^1(T^2))}. 
\]

(4.45)

which along with (4.32) and (4.33) further implies that

\[
\|G(u_1, Q_1) - G(u_2, Q_2)\|_{L^2(0,T;H^1(T^2))} \leq C(R, \|u_0\|_{L^2}, \|Q_0\|_{H^2}) \left( T^{\frac{k-2}{k}} \|\tilde{u}\|^{\frac{2}{k-2}}_{L^\infty(0,T;L^2(T^2))} + T^{\frac{k-2}{k}} \|\tilde{u}\|^{\frac{2}{k-2}}_{L^\infty(0,T;L^2(T^2))} \right). 
\]

(4.46)

Next, we note that

\[
2|L_4\| \|((Q_1)_{ij,l}(Q_1)_{lk})_k - ((Q_1)_{ij,l}(Q_0)_{lk})_k\|_{L^2(0,T;H^1(T^2))} 
\]

\[
\leq 2|L_4\| \|((Q_1)_{ij,l}(Q_1)_{lk} - (Q_0)_{lk})_k\|_{L^2(0,T;H^1(T^2))} 
\]

\[
+ 2|L_4\| \|((Q_2)_{ij,l}(Q_1)_{lk})_k\|_{L^2(0,T;H^1(T^2))} 
\]

\[
:= J_1 + J_2, 
\]

with obvious notation. Recalling that $(Q_1 - Q_0)_{i=0} = 0$ for $i = 1, 2$, then using (4.32) and a similar argument as for (4.34), we obtain

\[
J_1 \leq C\|\tilde{Q}\|_{L^2(0,T;H^1(T^2))} \|Q_1 - Q_0\|_{L^\infty(0,T;L^\infty(T^2))} 
\]

\[
+ C\|\tilde{Q}\|_{L^2(0,T;W^{1,4}(T^2))} \|Q_1 - Q_0\|_{L^\infty(0,T;W^{1,4}(T^2))} 
\]

\[
+ C\|\nabla \tilde{Q}\|_{L^2(0,T;L^2(T^2))} \|\nabla (Q_1 - Q_0)\|_{L^\infty(0,T;H^1(T^2))} 
\]

\[
\leq C\|Q_1 - Q_0\|_{L^2(0,T;H^2(T^2))} T^{\frac{k-2}{k}} \|\tilde{Q}\|_{L^2(0,T;H^1(T^2))} 
\]

\[
+ C\|Q_1 - Q_0\|_{L^\infty(0,T;H^2(T^2))} \|\nabla (Q_1 - Q_0)\|_{L^2(0,T;H^1(T^2))} T^{\frac{k-2}{k}} 
\]

\[
:= J_1 + J_2, 
\]

with obvious notation. Recalling that $(Q_1 - Q_0)_{i=0} = 0$ for $i = 1, 2$, then using (4.32) and a similar argument as for (4.34), we obtain

\[
J_1 \leq C\|\tilde{Q}\|_{L^2(0,T;H^1(T^2))} \|Q_1 - Q_0\|_{L^\infty(0,T;L^\infty(T^2))} 
\]

\[
+ C\|\tilde{Q}\|_{L^2(0,T;W^{1,4}(T^2))} \|Q_1 - Q_0\|_{L^\infty(0,T;W^{1,4}(T^2))} 
\]

\[
+ C\|\nabla \tilde{Q}\|_{L^2(0,T;L^2(T^2))} \|\nabla (Q_1 - Q_0)\|_{L^\infty(0,T;H^1(T^2))} 
\]

\[
\leq C\|Q_1 - Q_0\|_{L^2(0,T;H^2(T^2))} T^{\frac{k-2}{k}} \|\tilde{Q}\|_{L^2(0,T;H^1(T^2))} 
\]

\[
+ C\|Q_1 - Q_0\|_{L^\infty(0,T;H^2(T^2))} \|\nabla (Q_1 - Q_0)\|_{L^2(0,T;H^1(T^2))} T^{\frac{k-2}{k}} 
\]
problem (4.1)–(4.5) satisfies the following dissipative energy law:

\[
+ C\|Q_1 - Q_0\|_{L^\infty(0,T;H^2(T^2))}\|\nabla \hat{Q}\|_{L^\infty(0,T;L^2(T^2))}^\frac{3}{2} \|\nabla \hat{Q}\|_{L^2(0,T;H^2(T^2))}^\frac{1}{2}T^\frac{1}{2}
\leq CT^\frac{1}{2}(1 + T^\frac{1}{2})\|\hat{Q}\|_{L^2(0,T;H^2(T^2))} + \|\hat{Q}\|_{L^\infty(0,T;H^2(T^2))}
\leq C(R, \|Q_0\|_{H^2})T^\frac{1}{2}(1 + T^\frac{1}{2})\|\hat{G}\|_{L^2(0,T;H^2(T^2))},
\]

and

\[
J_2 \leq C\|Q_2\|_{L^2(0,T;H^2(T^2))}\|\hat{Q}\|_{L^\infty(0,T;L^\infty(T^2))} + C\|Q_2\|_{L^2(0,T;W^{2,4}(T^2))} + C\|\nabla Q_2\|_{L^2(0,T;L^\infty(T^2))}\|\nabla \hat{Q}\|_{L^\infty(0,T;H^1(T^2))}
\leq C\|\hat{Q}\|_{L^\infty(0,T;H^1(T^2))}T^\frac{1}{2}\|\hat{Q}\|_{H^1(0,T;L^1(T^2))}\|Q_2\|_{L^2(0,T;H^2(T^2))} + C\|\hat{Q}\|_{L^\infty(0,T;H^1(T^2))}\|Q_2\|_{L^2(0,T;H^2(T^2))}T^\frac{1}{2}
+ C\|\nabla Q_2\|_{L^\infty(0,T;L^2(T^2))} \|\nabla \hat{Q}_2\|_{L^\infty(0,T;L^2(T^2))}T^\frac{1}{2}
\leq CT^\frac{1}{2}(1 + T^\frac{1}{2})(\|\hat{Q}\|_{H^1(0,T;L^2(T^2))} + \|\hat{Q}\|_{L^\infty(0,T;H^2(T^2))})
\leq C(R, \|Q_0\|_{H^2})T^\frac{1}{2}(1 + T^\frac{1}{2})\|\hat{G}\|_{L^2(0,T;H^1(T^2))}.
\]

Then we have

\[
2|L_4|\left\| \left( (Q_1)_{ij,l}(Q_1)_{lk} \right)_{k} - \left( (Q_1)_{ij,l}(Q_0)_{lk} \right)_{k} \right\|_{L^2(0,T;H^1(T^2))}
- \left\| \left( (Q_2)_{ij,l}(Q_2)_{lk} \right)_{k} - \left( (Q_2)_{ij,l}(Q_0)_{lk} \right)_{k} \right\|_{L^2(0,T;H^1(T^2))}
\leq C(R, \|Q_0\|_{H^2})T^\frac{1}{2}(1 + T^\frac{1}{2})\|\hat{G}\|_{L^2(0,T;H^1(T^2))}.
\]

As a consequence, we conclude from (4.39), (4.46) and (4.47) that there exists a sufficiently small time \(T_0^* > 0\), it holds

\[
\|\mathcal{Y}(G_1) - \mathcal{Y}(G_2)\|_{L^2(0,T_0^*,H^2(T^2))} \leq \frac{1}{2}\|G_1 - G_2\|_{L^2(0,T_0^*,H^2(T^2))}, \quad \forall G_1, G_2 \in B_{T_0^*,R}.
\]

In summary, we can take \(T^* = \min\{T_0^*, T_0^\\}\) and apply Banach’s fixed point theorem to deduce that the nonlinear mapping \(\mathcal{Y}\) has a unique fixed point \(G^*\) in \(B_{T^*,R}\) such that \(G^* = \mathcal{Y}(G^*)\). By the definition of \(\mathcal{Y}\), this implies that problem (4.1)–(4.5) admits a unique local solution \((u, Q)\) (corresponding to the fixed point \(G^*)\) satisfying

\[
u \in H^1(0,T^*; (H^2_0(T^2))^\prime) \cap C([0,T^*]; L^2_0(T^2)) \cap L^2(0,T^*; H^2_0(T^2)),
\]

\[
Q \in H^1(0,T^*; H^2(T^2)) \cap C([0,T^*]; H^2(T^2)) \cap L^2(0,T^*; H^2(T^2)),
\]

and \(Q \in S_0^{(2)}\) a.e. in \(T^2 \times (0,T)\).

4.2 Global existence

In what follows, we proceed to extend the local-in-time solution \((u, Q)\) of problem (4.1)–(4.5) that was constructed above to be a global one. This goal can be achieved by deriving some uniform in time estimates.

First, it follows from Lemma 5.1 that

\[
\|Q\|_{C([0,T^*]; L^\infty(T^2))} \leq \sqrt{\eta},
\]

where we recall that the constant \(\eta_1\) in (4.3) is taken to be suitably small, but it only depends on \(T^2\). The above \(L^\infty\)-estimate will play an important role in deriving global estimates for the solution \((u, Q)\). Next, using an essentially identical argument for Proposition 3.1, we observe that the solution \((u, Q)\) problem (4.1)–(4.5) satisfies the following dissipative energy law:

\[
\frac{1}{2} \int_{T^2} |u(t)|^2 dx + \mathcal{E}(Q(t)) + \int_0^t \int_{T^2} (\nu |\nabla u|^2 + \delta |\mathcal{D}u|^2) dx dt
\]
Besides, by (2.5), (2.6) and the estimate (4.54), we see that

$$\int_{\mathbb{T}^2} (L_1 \partial_i Q_{ij} \partial_i Q_{ij} + L_2 \partial_j Q_{ik} \partial_i Q_{ij} + L_3 \partial_j Q_{ij} \partial_i Q_{ik}) \, dx \geq \kappa \int_{\mathbb{T}^2} |\nabla Q|^2 \, dx.$$  

Then choose the constant $C_1$ in (4.51) to be small enough (again only depending on $\mathbb{T}^2$, see also [24, Section 3.2]), we have

$$\mathcal{E}(Q(t)) \geq (\kappa - |L_4||Q(t)||L^\infty) \int_{\mathbb{T}^2} |\nabla Q(t)|^2 \, dx + \frac{c}{4} \int_{\mathbb{T}^2} \left[ (\tr(Q^2) + \frac{a^2}{c^2} - \frac{a^2}{c^2}) \right] \, dx$$

which implies that the free energy $\mathcal{E}(Q(t))$ is uniformly bounded from below. From (4.50)–(4.52), we infer that

$$\|u\|_{L^\infty(0,T^*;L^2(\mathbb{T}^2))} \leq C, \quad \|Q\|_{L^\infty(0,T^*;H^1(\mathbb{T}^2))} \leq C.$$

It follows from (4.51) that $\mathcal{H} + \lambda \mu - \mu^T \in L^2(0,T^*;L^2(\mathbb{T}^2))$. Hence, using (4.50), for sufficiently small $C_1$, we can apply Lemma (4.51) to conclude that

$$\|Q\|_{L^2(0,T^*;H^1(\mathbb{T}^2))} \leq C.$$  

Next, we derive necessary higher-order estimates for $Q$. Multiplying (4.1.3) by $\Delta^2 Q$, integrating over $\mathbb{T}^2$, after integration by parts and using the Cauchy–Schwarz inequality, we get

$$\frac{1}{2} \frac{d}{dt} \|\Delta Q\|_{L^2}^2 + \zeta \|\nabla \Delta Q\|_{L^2}^2
= \int_{\mathbb{T}^2} \partial_m \left( G_{ij}(u, Q) + 2L_4(Q_{ij}, Q_{ik}) \right) \partial_m \Delta Q_{ij} \, dx$$

$$\leq \frac{\zeta}{4} \|\nabla \Delta Q\|_{L^2}^2 + C|L_4||Q|_{L^\infty} \|Q\|_{H^3}^2 + C\|Q\|_{W^{2,1}}^2 \|\nabla Q\|_{L^2}^2$$

Choosing $C_1$ to be small enough, we have

$$C|L_4||Q|_{L^\infty} \|Q\|_{H^3}^2 \leq \frac{\zeta}{12 \left( \|\nabla \Delta Q\|_{L^2}^2 + \|Q\|_{L^2}^2 \right)}.$$

Besides, by (2.4), (2.6) and the estimate (4.51), we see that

$$C\|Q\|_{W^{2,1}}^2 \|\nabla Q\|_{L^2}^2$$
$$\leq C\|Q\|_{H^3} \|Q\|_{H^2} \|\nabla \Delta Q\|_{L^2} \|Q\|_{L^\infty}$$

$$\leq C \|\nabla \Delta Q\|_{L^2} \|\Delta Q\|_{L^2}^2 \|Q\|_{L^\infty} + C \|\nabla Q\|_{L^2} \|\Delta Q\|_{L^2} \|Q\|_{L^2} \|Q\|_{L^\infty}$$

$$+ C\|\Delta Q\|_{L^2}^2 \|Q\|_{L^2} \|Q\|_{L^\infty} + C \|\Delta Q\|_{L^2} \|Q\|_{L^2} \|Q\|_{L^\infty}$$

$$\leq \frac{\zeta}{12} \|\nabla \Delta Q\|_{L^2}^2 + C(1 + \|\Delta Q\|_{L^2}^2) \|\Delta Q\|_{L^2}^2 + C.$$  

Finally, it holds

$$\|\nabla G(u, Q)\|_{L^2}^2.$$
5.1 Global existence for initial data

In this section, we prove Theorem 1.1 on the existence and uniqueness of global weak solutions to the

Proof of the Main Result

Proof.

Lemma 5.1. Suppose that the assumptions in Proposition 4.1 are satisfied. For any $Q_0 \in L^2_0(T^2) \times H^2(T^2)$ and $\delta > 0$, we are able to extend the (unique) global solution $(u, Q)$ of problem (1.5)–(1.7) to arbitrary time interval $[0, T]$, i.e., it is indeed a global solution.

The proof of Proposition 1.1 is complete.

5.1 Global existence for initial data $(u_0, Q_0) \in L^2_0(T^2) \times H^2(T^2)$

Based on Proposition 4.1, we can pass to the limit as $\delta \to 0^+$ in the approximate problem (4.1)–(4.3) to show the existence of global weak solutions to problem (1.5)–(1.7) with a slightly more regular initial data, i.e., $Q_0 \in H^2(T^2)$.

Lemma 5.1. Suppose that the assumptions in Proposition 4.1 are satisfied. For any $(u_0, Q_0) \in L^2_0(T^2) \times H^2(T^2)$ with $\|Q_0\|_{L^\infty} \leq \sqrt{m}$, problem (1.5)–(1.7) admits a global weak solution $(u, Q)$ satisfying

$u \in H^1(0, T; (H^1_0(T^2))^d) \cap C([0, T]; L^2_0(T^2)) \cap L^2(0, T; H^1_0(T^2))$, $Q \in H^1(0, T; L^2(T^2)) \cap C([0, T]; H^1(T^2)) \cap L^2(0, T; H^2(T^2))$.

Besides, it holds

$$\|Q\|_{L^\infty(0, T; L^\infty(T^2))} \leq \sqrt{m}, \quad (5.1)$$

and now the energy identity (4.51) is satisfied with $\delta = 0$.

Proof. For the given initial data $(u_0, Q_0)$ and an arbitrary fixed $\delta > 0$, according to Proposition 4.1, problem (4.1)–(4.5) admits a unique global solution denoted by $(u^\delta, Q^\delta)$ that satisfies (4.8)–(4.9) and

$$\|Q^\delta\|_{L^\infty(0, T; L^\infty(T^2))} \leq \sqrt{m}. \quad (5.2)$$

Moreover, we have

$$\int_0^T (\partial_t u^\delta, v)_{(H^2_0(T^2))^d} dt - \int_0^T \int_{T^2} u^\delta \otimes u^\delta : \nabla v dx dt$$

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for any \( v \in L^2(0, T; H^2_b(\mathbb{T}^2)) \), and
\[
\partial_t Q^\delta_{ij} + u^\delta_{ij,k} Q^\delta_{ik,j} + Q^\delta_{ik,j} \omega^\delta_{ik,j} - \omega^\delta_{j}Q^\delta_{k,j} = \zeta \Delta Q^\delta_{ij} + 2L_4(Q^\delta_{ij,l}Q^\delta_{lk,j})_k - L_4 Q^\delta_{ik,l}Q^\delta_{kl,j} + \frac{L_4}{2} |\nabla Q^\delta_{ij}|^2 \delta_{ij} - a Q^\delta_{ij} - c \text{tr}((Q^\delta)^2)Q^\delta_{ij}\]
for a.e. \((x, t) \in \mathbb{T}^2 \times (0, T)\).

Since the approximate solution \((u^\delta, Q^\delta)\) satisfies the energy identity \((5.51)\), then by Lemma 2.7 and a similar argument in Section 4.2, we obtain the following estimates
\[
\|u^\delta\|_{L^\infty(0,T;L^2(\mathbb{T}^2)) \cap L^2(0,T;H^1(\mathbb{T}^2))} + \sqrt{\delta}\|D^h u^\delta\|_{L^2(0,T;L^2(\mathbb{T}^2))} \leq C,
\]
\[
\|Q^\delta\|_{L^\infty(0,T;H^1(\mathbb{T}^2)) \cap L^2(0,T;H^2(\mathbb{T}^2))} \leq C,
\]
where the constant \(C\) depends on \(\|u_0\|_{L^2}, \|Q_0\|_{H^1}, \eta_1, \nu, \mathbb{T}^2, L_1, a, c\), but it is independent of the parameter \(\delta\). The above estimates and the Sobolev embedding theorem yield that
\[
\|u^\delta\|_{L^4(0,T;L^4(\mathbb{T}^2))} + \|\nabla Q^\delta\|_{L^4(0,T;L^4(\mathbb{T}^2))} \leq C,
\]
which together with \((5.6)\) implies
\[
\|\sigma^a(Q^\delta)\|_{L^2(0,T;L^2(\mathbb{T}^2))} + \|\sigma^a(Q^\delta)\|_{L^2(0,T;L^2(\mathbb{T}^2))} \leq C,
\]
where \(C\) is again independent of \(\delta\). Then by comparison, we have
\[
\|\partial_t u^\delta\|_{L^2(0,T;H^2(\mathbb{T}^2))} + \|\partial_t Q\|_{L^2(0,T;L^2(\mathbb{T}^2))} \leq C.
\]

These uniform bounds imply that, up to the extraction of a subsequence, the following convergence results as \(\delta \to 0^+\):
\[
\begin{align*}
&u^\delta \rightharpoonup u \text{ weakly star in } L^\infty(0,T;L^2(\mathbb{T}^2)) \cap L^2(0,T;H^1(\mathbb{T}^2)), \\
&\partial_t u^\delta \rightharpoonup \partial_t u \text{ weakly in } L^2(0,T;H^2_b(\mathbb{T}^2)'), \\
&Q^\delta \rightharpoonup Q \text{ weakly star in } L^\infty(0,T;H^1(\mathbb{T}^2)) \cap L^2(0,T;H^2(\mathbb{T}^2)), \\
&\partial_t Q^\delta \rightharpoonup \partial_t Q \text{ weakly in } L^2(0,T;L^2(\mathbb{T}^2)), \\
&\sqrt{\delta} \left(\sqrt{\delta} D^h u^\delta\right) \rightharpoonup 0 \text{ strongly in } L^2(0,T;L^2(\mathbb{T}^2)).
\end{align*}
\]

Besides, using the well-known Aubin–Lions compactness lemma (see e.g., \([5]\)), we obtain the following strong convergence results (up to a subsequence)
\[
\begin{align*}
u^\delta &\to u \text{ strongly in } L^2(0,T;H^2_b(\mathbb{T}^2)), \\
Q^\delta &\to Q \text{ strongly in } L^2(0,T;H^2(\mathbb{T}^2)) \cap L^4(0,T;L^4(\mathbb{T}^2)).
\end{align*}
\]
for any \(\epsilon \in (0, \frac{1}{4})\). Concerning the convergence of nonlinear terms, we only need to treat the cubic term associated with \(L_4\) in \(\sigma^a\) (see \((1.17)\)). It follows from \((5.7)\) that \(Q\) and \(\nabla Q\) converge almost everywhere in \(\mathbb{T}^2 \times (0,T)\) (up to a subsequence). Then we infer from \((5.2)\) and \((5.5)\) that
\[
\lim_{\delta \to 0^+} \int_0^T \int_{\mathbb{T}^2} (Q^\delta)_{jm}(Q^\delta)_{kl,m}(Q^\delta)_{kl,i} M_{ij} dx dt = \int_0^T \int_{\mathbb{T}^2} Q_{jm} Q_{kl,m} Q_{kl,i} M_{ij} dx dt,
\]
for any \(M \in L^2(0,T;L^2(\mathbb{T}^2))\). The convergence of other nonlinear terms can be treated by using a similar argument like in \(\cite{2}, \text{Section 3}\) and the details are omitted here.
In conclusion, we are able to pass to the limit $\delta \to 0^+$ in \([5.4]\) and deduce that the limit function $Q$ satisfies \([1.23]\) almost everywhere in $\mathbb{T}^2 \times (0, T)$. Moreover, passing to the limit in \([5.3]\), we get
\[
\int_0^T \langle \partial_t u, v \rangle_{L^2_v(H^1_u)} dt - \int_0^T \int_{\mathbb{T}^2} u \otimes u : \nabla v dx dt + \nu \int_0^T \int_{\mathbb{T}^2} \nabla u : \nabla v dx dt
\]
for any $v \in L^2(0, T; H^1_u(L^2_v(\mathbb{T}^2)))$. By comparison, the above identity also implies that the time derivative of $u$ fulfills $\langle \partial_t u, v \rangle_{L^2_v(H^1_u)} \in L^2(0, T; (H^1_u(L^2_v(\mathbb{T}^2)))')$ and the weak formulation \([1.22]\) follows. Besides, we infer from a suitable interpolation inequality (recall \([2.8]\)) that $u \in C([0, T]; L^2_v(\mathbb{T}^2))$ and $Q \in C([0, T]; H^1(\mathbb{T}^2))$.

Finally, we easily deduce from \([1.50]\) and the lower semicontinuity property of the $L^\infty$-norm (see e.g., \([8, \text{Proposition 3.13}]\)) that the limit function $Q$ satisfies the uniform estimate \([5.1]\) associated with its $L^\infty$-norm. Furthermore, thanks to Proposition \([3.4]\) the energy identity \([1.51]\) (now with $\delta = 0$) is satisfied.

5.2 Continuous dependence with respect to initial data

Next, we aim to show the uniqueness of global weak solutions. For this purpose, we prove a continuous dependence result with respect to the initial data in a suitable topology.

A conventional approach is to estimate the difference between two solutions by energy estimates that are usually performed at the natural energy space level, namely, $(u, Q) \in L^\infty(0, T; L^2_v(\mathbb{T}^2)) \times L^\infty(0, T; H^1(\mathbb{T}^2))$ for our current problem \([1.50] - [1.54]\). However, due to the highly nonlinear stress tensors $\sigma^*, \sigma^e$, this goal cannot be achieved in such settings. This was also illustrated in \([35]\) where only a weak-strong uniqueness result was obtained in the simpler isotropic case $L_2 = L_3 = L_4 = 0$.

Inspired by \([25, 26]\), we shall perform the energy estimates at a lower-order energy space than that appeared in the basic energy law \([3.3]\). Roughly speaking, we use the $(H^1)'$ energy estimate for the velocity $u$ and $L^2$ energy estimate for the order parameter $Q$, respectively. As we shall see below, one advantage in such settings is that certain higher-order a priori estimates (compared with $(H^1)' \times L^2$) are automatically provided by the basic energy law.

**Lemma 5.2.** Let $(u_i, Q_i)$, $i = 1, 2$, be two global weak solutions to problem \([1.50] - [1.54]\) with initial data $(u_{0i}, Q_{0i}) \in L^2_v(\mathbb{T}^2) \times (H^1(\mathbb{T}^2) \cap L^\infty(\mathbb{T}^2))$. Let
\[
\eta_2 = \min \left\{ \frac{\sqrt{T}}{16C^* |L_4|}, \frac{1}{64} \left( \frac{\zeta}{L_4} \right)^2 \right\},
\]
where $C^*$ is the constant in the elliptic estimate \([2.3]\) depending only on $\mathbb{T}^2$. If
\[
\|Q_i\|_{L^\infty(0, T; L^\infty_v(\mathbb{T}^2))} \leq \sqrt{\eta_2},
\]
then we have
\[
\|w(t)\|_{H^1}^2 + \|Q(t)\|_{L^2}^2 + \int_0^t \left( \|\Delta w(s)\|_{L^2}^2 + \|\nabla Q(s)\|_{L^2}^2 \right) ds \leq C e^{Ct} (\|w_0\|_{H^1} + \|Q_0\|_{L^2}), \quad \forall t \in (0, T).
\]
Here,
\[
w \triangleq (-\Delta + I)^{-1}(u_1 - u_2), \quad Q \triangleq Q_1 - Q_2,
\]
\[
w_0 \triangleq (-\Delta + I)^{-1}(u_{01} - u_{02}), \quad Q_0 \triangleq Q_{01} - Q_{02}.
\]
$I$ stands for the identity operator and $C > 0$ is a constant that depends on $\|u_{0i}\|_{L^2}, \|Q_{0i}\|_{H^1}$ for $1 \leq i \leq 2$, $\eta_2$ and coefficients of the system.
Proof. Let
\[ \bar{u} \equiv u_1 - u_2, \quad \bar{P} \equiv P_1 - P_2, \quad \bar{\sigma} \equiv \sigma(Q_1) - \sigma(Q_2), \quad \bar{\sigma}^* \equiv \sigma^*(Q_1) - \sigma^*(Q_2), \]
\[ \bar{\mathcal{H}} \equiv \mathcal{H}(Q_1) + \lambda_1 \|\mu_1 - \mu_1^T\| - (\mathcal{H}(Q_2) + \lambda_2 \|\mu_2 - \mu_2^T\|). \]
First, we infer from the incompressibility condition (1.6) that
\[ \nabla \cdot \bar{w} = \nabla \cdot [(-\Delta + I)^{-1} \bar{u}] = (-\Delta + I)^{-1}(\nabla \cdot \bar{u}) = 0. \tag{5.11} \]
Besides, we see that \( \bar{w} \) satisfies the following equation
\[ \partial_t \bar{w} = (-\Delta + I)^{-1} \left[ u_2 \cdot \nabla u_2 - u_1 \cdot \nabla u_1 + \nu \Delta \bar{u} - \nabla \bar{P} + \nabla \cdot (\bar{\sigma} + \bar{\sigma}^*) \right]. \tag{5.12} \]
Multiplying equation (5.12) with \( \bar{w} - \Delta \bar{w} \), integrating over \( T^2 \), using (5.11), we obtain after integration by parts that
\[ \frac{1}{2} \frac{d}{dt} \int_{T^2} (|\bar{w}|^2 + |\nabla \bar{w}|^2) \, dx \]
\[ = \int_{T^2} ((I - \Delta) \bar{w}) \cdot (-\Delta + I)^{-1} \left[ u_2 \cdot \nabla u_2 - u_1 \cdot \nabla u_1 + \nu \Delta \bar{u} - \nabla \bar{P} + \nabla \cdot (\bar{\sigma} + \bar{\sigma}^*) \right] \, dx \]
\[ = - \int_{T^2} \nabla \bar{w} : (u_1 \otimes u_1 - u_2 \otimes u_2 - \nu \Delta \bar{u} + \nabla \bar{P} + \nabla \cdot (\bar{\sigma} + \bar{\sigma}^*)) \, dx \]
\[ = - \nu \int_{T^2} (|\nabla \bar{w}|^2 + |\Delta \bar{w}|^2) \, dx + \int_{T^2} \nabla \bar{w} : \left( \bar{u} \otimes u_1 + u_2 \otimes \bar{u} \right) \, dx \]
\[ - \int_{T^2} \nabla \bar{w} : \bar{\sigma}^* \, dx - \int_{T^2} \nabla \bar{w} : \bar{\sigma} \, dx. \tag{5.13} \]
Using the estimates for global weak solutions
\[ \|u_i\|_{L^\infty(0,T;L^2(T^2))} \leq C, \quad \|Q_i\|_{L^\infty(0,T;H^1(T^2))} \leq C, \quad 1 \leq i \leq 2 \tag{5.14} \]
and the \( L^\infty \)-estimate (5.20), we proceed to estimate the terms \( I_1 \) to \( I_3 \). By the Hölder and Young inequalities, we have
\[ I_1 \leq \int_{T^2} |\nabla \bar{w}|(|u_1| + |u_2|)(|\bar{w}| + |\Delta \bar{w}|) \, dx \]
\[ \leq C(\|u_1\|_{L^4} + \|u_2\|_{L^4})(\|\nabla \bar{w}\|_{L^4} \|\Delta \bar{w}\|_{L^2} + \|\nabla \bar{w}\|_{L^2} \|\bar{w}\|_{L^4}) \]
\[ \leq \frac{\nu}{32} \|\Delta \bar{w}\|_{L^2}^2 + C(\|u_1\|_{L^4}^2 + \|u_2\|_{L^4}^2)(\|\nabla \bar{w}\|_{L^4}^2 + C(\|u_1\|_{L^4} + \|u_2\|_{L^4})(\|\bar{w}\|_{L^4}^2), \]
\[ \leq \frac{\nu}{32} \|\Delta \bar{w}\|_{L^2}^2 + C(\|u_1\|_{L^4} + \|u_2\|_{L^4})(\|\bar{w}\|_{L^4}^2, \]
\[ + C(\|u_1\|_{L^2} + \|u_2\|_{L^2})(\|\nabla \bar{w}\|_{L^2} + \|\Delta \bar{w}\|_{L^2} + \|\bar{w}\|_{L^2}) \]
\[ \leq \frac{\nu}{16} \|\Delta \bar{w}\|_{L^2}^2 + C(1 + \|u_1\|_{H^1} + \|u_2\|_{H^1})(\|\bar{w}\|_{L^2}^2 + \|\nabla \bar{w}\|_{L^2}^2). \]
Next, for \( I_2 \), it holds
\[ I_2 \leq 2(\|L_1\| + \|L_2\| + \|L_3\|)(\|\nabla \bar{w}\|_{L^4})(\|\Delta Q_1\|_{L^4} + \|\nabla Q_2\|_{L^4}) \]
\[ + 2L_1(\|\nabla \bar{w}\|_{L^4})(\|Q_2\|_{L^\infty} \|\nabla \bar{Q}\|_{L^2}) + \|\nabla Q_2\|_{L^4}) \]
\[ + 2L_1(\|\nabla \bar{w}\|_{L^4})(\|Q_2\|_{L^\infty} \|\nabla \bar{Q}\|_{L^2}) \]
\[ \leq C(\|\nabla \bar{w}\|_{L^4} + \|\bar{w}\|_{L^2}^2) \|\nabla \bar{Q}\|_{L^2}(\|Q_1\|_{L^\infty} \|\Delta Q_1\|_{L^2} + \|Q_2\|_{L^\infty} \|\Delta Q_2\|_{L^2}). \]
where

Then by Young's inequality, we obtain that

Denote

is a constant depending only on \( \nu \).

\[
I = \frac{\nu}{16} \| \Delta w \|^2_{L^2} + C (1 + \| \Delta Q_1 \|^2_{L^2} + \| \Delta Q_2 \|^2_{L^2}) (\| w \|^2_{L^2} + \| \nabla w \|^2_{L^2} + \| Q \|^2_{L^2}).
\]

Observe that

\[
I_3 = -\int_{T^2} \nabla w : [Q (H(Q_1) + \lambda_1 \| + \mu_1 - \mu_1^T)] dx - \int_{T^2} \nabla w : (Q_2 H) dx
+ \int_{T^2} \nabla w : [(H(Q_1) + \lambda_1 \| + \mu_1 - \mu_1^T) Q] dx + \int_{T^2} \nabla w : (\bar{H} Q_2) dx
:= I_{3a} + I_{3b} + I_{3c} + I_{3d}
\]

with obvious notation. Concerning \( I_{3a} + I_{3c} \), we obtain from the Hölder and Young inequalities that

\[
I_{3a} + I_{3c} \leq \zeta \| \nabla w \|_{L^4} \| Q \|_{L^4} \| \Delta Q_1 \|_{L^4} + C \| L_4 \| \| \nabla w \|_{L^4} \| Q \|_{L^4} \| \Delta Q_1 \|_{L^4}
+ C \| L_4 \| \| \nabla w \|_{L^4} \| Q \|_{L^4} \| \Delta Q_1 \|_{L^4}
\leq C \| \nabla w \|^2_{L^2} \| \Delta w \|_{L^4} + \| w \|_{L^4})^{\frac{3}{2}} (\| Q \|^2_{L^4} \| \nabla Q \|^2_{L^2} + \| Q \|_{L^2}) (1 + \| \Delta Q_1 \|_{L^2})
+ C \| \nabla w \|^2_{L^2} (\| \Delta w \|_{L^4} + \| w \|_{L^4})^{\frac{3}{2}} (\| Q \|^2_{L^4} \| \nabla Q \|^2_{L^2} + \| Q \|_{L^2}) \| Q \|_{L^4} \| \Delta Q_1 \|_{L^4}
\leq \frac{\nu}{16} \| \Delta w \|^2_{L^2} + C (1 + \| \Delta Q_1 \|^2_{L^2}) (\| w \|^2_{L^2} + \| \nabla w \|^2_{L^2} + \| Q \|^2_{L^2}).
\]

Next, we rewrite \( I_{3b} + I_{3d} \) as

\[
I_{3b} + I_{3d} = \zeta \int_{T^2} \nabla w : (\Delta Q Q_2 - Q_2 \Delta Q) dx + I_{3c}.
\]

Then using Hölder and Young inequalities, we obtain that

\[
I_{3e} \leq 2 \| L_4 \| \| Q_1 \|_{L^\infty} \| \Delta Q_1 \|_{L^4} (\| Q \|_{H^2} + \| Q_2 \|_{L^\infty} + \| \nabla w \|_{L^4} \| \nabla Q_2 \|_{L^4}) \| \nabla Q \|_{L^2}
+ 2 \| L_4 \| \| \nabla w \|_{H^2} \| Q_2 \|_{L^\infty} + \| \nabla w \|_{L^4} \| \nabla Q_2 \|_{L^4}) \| \nabla Q \|_{L^2}
+ C \| L_4 \| \| \nabla w \|_{L^4} \| Q_2 \|_{L^\infty} \| \nabla Q \|_{L^2} (\| Q_1 \|_{L^4} + \| \nabla Q_2 \|_{L^4})
+ C \| \nabla w \|_{L^4} \| Q_2 \|_{L^\infty} (1 + \| Q_1 \|_{L^\infty} + \| Q_2 \|_{L^\infty}) \| Q \|_{L^2}
\leq 2 C^* \| L_4 \| \| Q_1 \|_{L^\infty} \| \Delta w \|_{L^4} + \| w \|_{L^4}) \| \nabla Q \|_{L^2}
+ C \| \nabla w \|^2_{L^2} (\| \Delta w \|_{L^4} + \| w \|_{L^4}) \| \nabla Q \|_{L^2}
+ C \| \Delta w \|_{L^4} \| \nabla Q_2 \|_{L^4} \| \nabla Q \|_{L^2} \| Q \|_{L^2}
+ C \| \nabla w \|^2_{L^2} (\| \Delta w \|_{L^4} + \| w \|_{L^4}) \| \nabla Q_2 \|_{L^4} \| \nabla Q \|_{L^2} \| Q \|_{L^2}
+ C \| \nabla w \|^2_{L^2} (\| \Delta w \|_{L^4} + \| w \|_{L^4}) \| \nabla Q \|_{L^2} \| \nabla Q_1 \|_{L^4} + \| \nabla Q_2 \|_{L^4} + \| Q \|_{L^4}
+ C \| \nabla w \|_{L^4} \| \nabla Q \|_{L^2},
\]

where \( C^* \) is a constant depending only on \( \mathbb{T}^2 \) (see \( (2.23) \)). By our choice of \( \eta_2 \) and \( (5.9) \), we have

\[
2 C^* \| L_4 \| \| Q_1 \|_{L^\infty} \| Q_2 \|_{L^\infty} \leq 2 C^* \| L_4 \| \eta_2^2 = \frac{\sqrt{3} \zeta}{8}.
\]

Then by Young's inequality, we obtain that

\[
I_{3e} \leq \frac{\nu}{16} \| \Delta w \|^2_{L^2} + C \| \nabla Q \|^2_{L^2} + C (1 + \| \Delta Q_1 \|^2_{L^2} + \| \Delta Q_2 \|^2_{L^2}) (\| w \|^2_{L^2} + \| \nabla w \|^2_{L^2} + \| Q \|^2_{L^2}).
\]

Denote

\[
\bar{\xi} = \frac{1}{2} (\nabla w - \nabla^T w).
\]
We infer from Lemma 2.1 that
\[
\zeta \int_{T^2} \nabla w : (\Delta Q_2 - Q_2 \Delta Q) \, dx
= \zeta \int_{T^2} \nabla w : (\Delta Q_2 - Q_2 \Delta Q) \, dx
= -\zeta \int_{T^2} (Q_2 \bar{\xi} - \bar{\xi} Q_2) : \Delta Q \, dx
= -\zeta \int_{T^2} (Q_2 \Delta \xi - \Delta (\xi Q_2)) : Q \, dx - 2\zeta \int_{T^2} (\partial_t Q_2 \partial_j \xi - \partial_j \xi \partial_t Q_2) : Q \, dx
- \zeta \int_{T^2} (\Delta Q_2 \xi - \xi \Delta Q_2) : Q \, dx
:= -\zeta \int_{T^2} (Q_2 \Delta \xi - \Delta (\xi Q_2)) : Q \, dx + I_{4a} + I_{4b},
\]
with obvious notation. Using Hölder and Young inequalities, we obtain
\[
I_{4a} + I_{4b} \leq C \|\nabla Q_2\|_{L^4} \|\nabla \xi\|_{L^4} \|\bar{Q}\|_{L^2} + C \|\Delta Q_2\|_{L^2} \|\xi\|_{L^4} \|\bar{Q}\|_{L^2}
\leq C \|Q_2\|_{L^4}^2 \|\nabla Q_2\|_{L^2}^2 \|\nabla w\|_{L^2}^2 (\|\Delta w\|_{L^2} + \|w\|_{L^2})^2 \|\bar{Q}\|_{L^2}
+ C \|\Delta Q_2\|_{L^2} (\|\nabla w\|_{L^2}^2 + \|w\|_{L^2}) (\|\bar{Q}\|_{L^2}^2 + \|\bar{Q}\|_{L^2})
\leq \frac{\nu}{4} \|\Delta w\|_{L^2}^2 + \frac{\zeta^2}{8} \|\nabla \bar{Q}\|_{L^2}^2 + C(1 + \|\Delta Q_2\|_{L^2}^2) (\|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \|\bar{Q}\|_{L^2}^2).
\]
Collecting the above estimates, we conclude that
\[
\frac{d}{dt} \|w\|_{H^1}^2 + 2\nu \|\nabla w\|_{L^2}^2 + \nu \|\Delta w\|_{L^2}^2
\leq C(1 + \|u_1\|_{L^2}^2 + \|u_2\|_{L^2}^2 + \|\Delta Q_1\|_{L^2}^2 + \|\Delta Q_2\|_{L^2}^2) (\|w\|_{H^1}^2 + \|\bar{Q}\|_{L^2}^2)
+ \zeta^2 \|\nabla \bar{Q}\|_{L^2}^2 - 2\zeta \int_{T^2} (Q_2 \Delta \xi - \Delta (\xi Q_2)) : Q \, dx. \tag{5.15}
\]
Now we investigate the equation for \(Q\):
\[
\partial_t Q_{ij} + u_i \partial_k (Q_1)_{kj} + (u_2)_k \partial_k Q_{ij} + (Q_1)_{ik} \partial_k \omega_{kj} + (Q_2)_{ik} \omega_{kj} - ((\omega_1)_{ik} \bar{Q}_{kj} + \omega_{ik} (Q_2)_{kj})
= \zeta \Delta Q_{ij} - a Q_{ij} - \epsilon \tau^2 (Q_1)_{ij} - c |\bar{Q} : (Q_1 + Q_2)(Q_2)_{ij}
+ 2L_4 \partial_k (Q_1)_{ik} \partial_k (Q_1)_{ij} + (Q_2)_{ik} \partial_k Q_{ij} - L_4 (\partial_t Q_{kl}) \partial_j (Q_{kl}) + \partial_t (Q_2)_{kl} \partial_j Q_{kl})
+ \frac{L_4}{2} \partial_m Q_{kl} \partial_m ((Q_1)_{kl} + (Q_2)_{kl}) \partial_j. \tag{5.16}
\]
Testing (5.10) with \(2Q_{ij}\), summing over \(i,j\) from 1 to 2, and then integrate over \(T^2\), by the incompressibility condition 1.6 and the fact \((\omega_1 Q - Q \omega_1) : Q = 0\) (recall Lemma 2.1), we deduce after integration by parts that
\[
\zeta \frac{d}{dt} \|Q\|_{L^2}^2 + 2\zeta^2 \|\nabla Q\|_{L^2}^2
= -2\zeta \int_{T^2} (\bar{u} \cdot \nabla Q_1) : Q \, dx + 2\zeta \int_{T^2} (\bar{\omega} Q_2 - Q_2 \bar{\omega}) : Q \, dx
- 2\zeta \int_{T^2} a |Q_1|^2 \, dx - 2\zeta c \int_{T^2} (\tau^2 (Q_1) |Q_1|^2 + |\bar{Q} : (Q_1 + Q_2)(Q_2) : Q|) \, dx
- 4\zeta L_4 \int_{T^2} ((Q_2)_{ik} \partial_k Q_{ij} \partial_k Q_{ij} \, dx - 4\zeta L_4 \int_{T^2} (Q_{ik} \partial_k (Q_1)_{ij} \partial_k Q_{ij} \, dx
- 2\zeta L_4 \int_{T^2} (\partial_t Q_{kl}) \partial_j (Q_{kl}) + \partial_t (Q_2)_{kl} \partial_j Q_{kl} \, dx
:= \sum_{i=1}^7 J_i. \tag{5.17}
\]
Let us estimate the terms $J_1$ to $J_7$ individually. First, for $J_1$, we have
\[
J_1 \leq \|u\|_{L^2} \|\nabla Q_1\|_{L^4} \|\bar{Q}\|_{L^4} \\
\leq (\|w\|_{L^2} + \|\Delta w\|_{L^2})\|Q_1\|_{L^4} \|\Delta Q_1\|_{L^2} (\|\bar{Q}\|_{L^4} \|\nabla Q\|_{L^2} + \|\bar{Q}\|_{L^2}) \\
\leq \frac{C^2}{8} \|\nabla Q\|_{L^2}^2 + \frac{\nu}{2} \|\Delta w\|_{L^2}^2 + C(1 + \|\Delta Q_1\|_{L^2})(\|w\|_{L^2}^2 + \|\bar{Q}\|_{L^2}^2).
\]

Concerning $J_2$, we see from the identity $(-\Delta + I)^{-1}\bar{\omega} = \bar{\xi}$ that
\[
J_2 = 2\zeta \int_{\mathbb{T}^2} \{[(-\Delta + I)\bar{\xi}]Q_2 - Q_2[(-\Delta + I)\bar{\xi}]\} : Q \, dx \\
\quad = 2\zeta \int_{\mathbb{T}^2} (Q_2\Delta \xi - \Delta \xi Q_2) : Q \, dx + 2\zeta \int_{\mathbb{T}^2} (\xi \nabla w) : Q \, dx \\
\quad \leq 2\zeta \int_{\mathbb{T}^2} (Q_2\Delta \xi - \Delta \xi Q_2) : Q \, dx + C\|Q_2\|_{L^\infty} \|\nabla w\|_{L^2} \|\bar{Q}\|_{L^2} \\
\quad \leq 2\zeta \int_{\mathbb{T}^2} (Q_2\Delta \xi - \Delta \xi Q_2) : Q \, dx + C(\|\nabla w\|_{L^2}^2 + \|\bar{Q}\|_{L^2}^2).
\]

By (5.5), it easily follows that
\[
J_3 + J_4 \leq C\|\bar{Q}\|^2,
\]
and
\[
J_5 \leq 4\zeta |L^4|\|Q_2\|_{L^\infty} \|\nabla Q\|_{L^2}^2 \leq \frac{C^2}{2} \|\nabla Q\|_{L^2}^2.
\]

Finally, we deduce that
\[
J_6 + J_7 \leq C \|\nabla Q\|_{L^2} \|\bar{Q}\|_{L^4} (\|\nabla Q_1\|_{L^4} + \|\nabla Q_2\|_{L^4}) \\
\quad \leq C \|\nabla Q\|_{L^2} \|\bar{Q}\|_{L^4} \|\nabla Q_1\|_{L^2} + \|Q_2\|_{L^\infty} \|\Delta Q_1\|_{L^2} + \|\Delta Q_2\|_{L^2} \\
\quad \leq \frac{C^2}{8} \|\nabla Q\|_{L^2}^2 + C(1 + \|\Delta Q_1\|_{L^2} + \|\Delta Q_2\|_{L^2}) \|\bar{Q}\|_{L^2}^2.
\]

Summing up the estimates for $J_1, \ldots, J_7$, we conclude from (5.17) that
\[
\zeta \frac{d}{dt} \|\bar{Q}\|_{L^2}^2 + \frac{3C^2}{2} \|\nabla Q\|_{L^2}^2 \\
\quad \leq \frac{\nu}{2} \|\Delta w\|_{L^2}^2 + 2\zeta \int_{\mathbb{T}^2} (Q_2\Delta \xi - \Delta \xi Q_2) : Q \, dx \\
\quad \quad + C(1 + \|\Delta Q_1\|_{L^2}^2 + \|\Delta Q_2\|_{L^2}^2)(\|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \|\bar{Q}\|_{L^2}^2). \quad (5.18)
\]

Finally, adding (5.18) with (5.18), we obtain the following inequality
\[
\frac{d}{dt}(\|w\|_{H^1}^2 + \zeta \|\bar{Q}\|_{L^2}^2) + \frac{\nu}{2} \|\Delta w\|_{L^2}^2 + \frac{C^2}{2} \|\nabla Q\|_{L^2}^2 \\
\quad \leq C(1 + \|u_1\|_{H^1}^2 + \|u_2\|_{H^2}^2 + \|\Delta Q_1\|_{L^2}^2 + \|\Delta Q_2\|_{L^2}^2)(\|w\|_{H^1}^2 + \|\bar{Q}\|_{L^2}^2). \quad (5.19)
\]

Then applying Gronwall’s inequality, we easily deduce the continuous dependence result (5.10) from (5.19).

As an immediate consequence of Lemma 5.7, we have

**Corollary 5.1.** Suppose the assumptions in Lemma 5.3 are satisfied. The global weak solution to problem (5.5) is unique, if there exists any.

**Remark 5.1.** When the spatial dimension is two, for the special case $L_2 = L_3 = L_4 = 0$, a weak-strong uniqueness result for the Cauchy problem of the Beris–Edwards system was given in [35] and later, the uniqueness of weak solutions was proved in [44]. Our Lemma 5.5 extends the previous results in the two dimensional periodic setting and it still works in the case of whole space $\mathbb{R}^2$ (cf. [37]).
Lemma 6.1. The right-hand side of equation (1.7) can be written as

\[
\begin{align*}
(H + \lambda I + \mu - \mu^T)_{ij} & = \zeta \Delta Q_{ij} + 2L_4(Q_{ij,Q_{ik}})_k - L_4 Q_{kl,ij} Q_{ki,l} + \frac{L_4}{2} |\nabla Q|^2 \delta_{ij} - aQ_{ij} - c \text{tr}(Q^2)Q_{ij} \\
\end{align*}
\]

for \(1 \leq i, j, k, l \leq 2\).

Proof. We notice the \(H\) is the minus variational derivative of \(E(Q)\) without imposing any constraints on \(Q\). A direct calculation yields that (see [23, Proposition A.1])

\[
\left( \frac{\delta E}{\delta Q} \right)_{ij} = -2L_1 \Delta Q_{ij} - 2(L_2 + L_3) Q_{ij,kj} - 2L_4 Q_{ij,sk} Q_{sk} - 2L_4 Q_{ij,qk} Q_{pk} + L_4 \partial_k Q_{kl} \partial_j Q_{kl} + aQ_{ij} - bQ_{jk} Q_{ki} + c \text{tr}(Q^2)Q_{ij},
\]

which implies (1.13).

Substituting the above relation in (1.7) and choosing \(\mu\) to enforce the symmetry constraint \(Q = Q^T\) yields that

\[
\mu_{ij} - \mu_{ji} = (L_2 + L_3) (\partial_i Q_{jk} - \partial_j Q_{ik}).
\]

Similarly, choosing \(\lambda\) to enforce the trace free constraint \(\text{tr}(Q) = 0\) forces

\[
\lambda = -\frac{b}{2} \text{tr}(Q^2) - (L_2 + L_3) \partial_i Q_{jk} + \frac{L_4}{2} |\nabla Q|^2.
\]

Combining the expressions of \(\lambda, \mu\) and \(H\) yields that

\[
(H + \lambda I + \mu - \mu^T)_{ij} = 2L_1 \Delta Q_{ij} + (L_2 + L_3)(\partial_j Q_{ik} + \partial_i Q_{jk}) + 2L_4 \partial_i Q_{ij} \partial_k Q_{lk}.
\]
\[ + 2L_4 \partial_i \partial_k Q_{ij} Q_{ik} - L_4 \partial_i Q_{ki} \partial_j Q_{kl} - aQ_{ij} + bQ_{jk} Q_{ki} - c \text{tr}(Q^2) Q_{ij} + \left( \frac{b}{2} \text{tr}(Q^2) - (L_2 + L_3) \partial_i \partial_k Q_{ik} + \frac{L_4}{2} \nabla Q^2 \right) \delta_{ij}. \] (6.1)

Since \( Q \) is a \( 2 \times 2 \) symmetric and traceless matrix, we can collect the terms in (6.1) with factor \( L_2 + L_3 \) and by a straightforward computation to show that
\[ (L_2 + L_3)(\partial_j \partial_k Q_{ik} + \partial_i \partial_k Q_{jk}) - (L_2 + L_3) \partial_i \partial_k Q_{ik} \delta_{ij} = (L_2 + L_3) \Delta Q_{ij}, \] (6.2)

This together with (6.1) implies
\[ \begin{aligned}
    (\mathcal{H} + \lambda I + \mu - \mu^T)_{ij} &= (2L_1 + L_2 + L_3) \Delta Q_{ij} + 2L_4 \partial_i Q_{ij} \partial_k Q_{ik} + 2L_4 \partial_i \partial_k Q_{ij} Q_{ik} - L_4 \partial_i Q_{ki} \partial_j Q_{kl} - aQ_{ij} + bQ_{jk} Q_{ki} - c \text{tr}(Q^2) Q_{ij} + \left( \frac{b}{2} \text{tr}(Q^2) + \frac{L_4}{2} \nabla Q^2 \right) \delta_{ij} \\
    \text{and the desired result follows from } & (1.15) \text{ and the identity}
\end{aligned} \]

\[ b \left( Q_{jk} Q_{ki} - \frac{1}{2} \text{tr}(Q^2) \right) \delta_{ij} = 0. \]

The proof is complete. \( \square \)

**Remark 6.1.** (1) Since \( \text{tr}(Q^3) = 0 \) holds for any \( Q \in \mathcal{S}_0^{(2)} \), the term \( \frac{b}{2} \text{tr}(Q^2) \) simply vanishes in \( \mathcal{E}(Q) \). Thus, in the two dimensional system (1.5)–(1.7) the term associated with \( b \) does not appear explicitly (see the expression of \( \mathcal{H} + \lambda I + \mu - \mu^T \)). Hence, we do not need to impose any condition on the coefficient \( b \).

(2) The specific relation (6.2) helps to derive the above simplified form of \( \mathcal{H} + \lambda I + \mu - \mu^T \), which however, only holds for \( Q \in \mathcal{S}_0^{(2)} \). In the three dimensional case, the terms associated with \( L_1, L_2, L_3 \) cannot be rewritten as \( (2L_1 + L_2 + L_3) \Delta Q \). Instead, they lead to an anisotropic elliptic operator that satisfies the strong Legendre condition (see e.g., [21, 30]). Whether the result on existence of global weak solutions obtained Theorem (1.4) can be extended to the three dimensional case remains an open question.

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