GENUS FIELDS OF KUMMER EXTENSIONS OF RATIONAL FUNCTION FIELDS

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ABSTRACT. In this paper we obtain the genus field of a general Kummer extension of a global rational function field. We study first the case of a general Kummer extension of degree a power of a prime. Then we prove that the genus field of a composite of two abelian extensions of a global rational function field with relatively prime degrees is equal to the composite of their respective genus fields. Our main result, the genus of a general Kummer extension of a global rational function field, is a direct consequence of this fact.

1. INTRODUCTION

The theory of genus fields for number fields has been around in several presentations for more than two hundred years. Nowadays it may be used to study the “easy” part of the Hilbert Class Field (HCF) of a finite extension of the field of rational numbers. The genus field of a number field $K$ (over $\mathbb{Q}$) is a subfield of $K^H$, the HCF of $K$. In this setting, $K^H$ is canonically defined as the maximal unramified abelian extension of $K$ so the definition of the genus field $K_{ge}$ is also canonically given.

We are interested in global function fields. In this case, there are several possible definitions for the HCF of a global function field $K$, depending on which aspect of $K$ we are interested in. The simple generalization of defining the HCF as the maximal unramified extension of $K$, has the inconvenient of being of infinite degree due to extensions of constants. Since every prime in a global function field becomes eventually inert in the extensions of constants, a convenient way to define the HCF of $K$ is to require full decomposition of at least one prime. When $K^H$ is finally defined, the genus field $K_{ge}$ of $K$ over a subfield $K_0$ is defined as the composite $KK'$ so that $KK' \subseteq K^H$ and $K'/K_0$ is the maximal abelian extension satisfying this property. In other words, $K'$ is the maximal abelian extension of $K_0$ contained in $K^H$.

Our setting is the following. We fix a rational function field $k = \mathbb{F}_q(T)$ and we consider a finite Kummer extension $K$ of $k$. We set $S$ as the set of the infinite primes of $K$ and let $K_H$ be the maximal unramified abelian extension of $K$ such that the elements of $S$ decompose fully. Our main result is Theorem 4.8, where we give a complete explicit expression of the genus field $K_{ge}$ of $K$ over $k$.

The tools we will be using in this paper are the cyclotomic function fields given by the Carlitz module. In this setting we found in [5] a general expression of the genus field of any field cyclotomic function field $E$ using the ramification theory of
Dirichlet characters given by Leopoldt in [4]. Once we had the general cyclotomic case, we were able to give a general expression for a finite abelian extension of $K$. In the same paper, using these general descriptions, we gave explicit expressions for Kummer cyclic extensions of prime degree and for Artin–Schreier extensions of $k$. In the first case it was previously obtained by Peng [7] and in the second one by Hu and Li [3]. We also gave the explicit expression of $K_{ge}$ for cyclic extensions of degree $p^n$, where $p$ is the characteristic of $k$, using Witt vectors.

In [1] we presented another description of genus fields of finite abelian extensions of $k$, which is much more transparent and general than the one in [5]. In particular we gave general explicit expressions of genus fields of arbitrary abelian $p$–extensions. The case of cyclic extensions of degree $l^n$ with $l$ a prime number $l \neq p$ has not been solved in general. When the extension of degree $l^n$ is a Kummer extension, the explicit description of $K_{ge}$ is given in [8]. For an extension that is not a Kummer extension, we still do not have an explicit description of $K_{ge}$.

The main goal of this paper is to give the explicit expression of $K_{ge}$ when $K/k$ is a finite Kummer extension. The main difficulty is that in general, for two finite extensions $K_1$, $K_2$ of $k$, we have $(K_1)_{ge}(K_2)_{ge} \subseteq (K_1K_2)_{ge}$, but the equality does not always hold. We will prove that, when the degrees of $K_i/k$ are relatively prime, we have equality. This is Theorem 4.1. The explicit expression of the genus field of a general Kummer extension of $k$, is now a direct consequence of this equality and the case of prime power degree.

In our study of genus fields, the main obstructions are the appearance of inertia in the composite of fields and in the proper contention of genus fields mentioned above. These are the main reasons why the $p$–case has been solved but the $l$–case, with $l \neq p$ has not. To conclude the finite abelian case, it remains to study the general cyclic case of prime power degree, not necessarily Kummer, and the composite of several of these cases.

This paper can be considered as the end of the problem first studied by Peng [7] where he found the genus field of a cyclic Kummer extension of a rational global function field of degree $l$. The next step was the case of a Kummer cyclic extension of prime power degree. This problem was first studied in [2] under the condition that the field was contained in a cyclotomic function field and a strong restriction. In [8] the general Kummer cyclic extension of prime power degree was settled. Finally, here we obtained explicitly the genus field of an arbitrary finite Kummer extension of a global rational function field. This is Theorem 4.8. This result is a consequence of Theorems 3.4 and 3.6 where the general Kummer extension of prime power degree is established, and of Theorem 4.1.

2. Notations and general results

For the general Carlitz–Hayes theory of cyclotomic function fields, we refer to [10, Ch. 12] and [9, Cap. 9]. For the results on genus fields of function fields we refer to [1, 5, 6] and [9, Cap. 14].

We will be using the following notation. Let $k = \mathbb{F}_q(T)$ be a global rational function field, where $\mathbb{F}_q$ is the finite field of $q$ elements. Let $R_T = \mathbb{F}_q[T]$ and let $R_T^*$ denote the set of the monic irreducible elements of $R_T$. For $N \in R_T$, $k(\Lambda_N)$ denotes the $N$–th cyclotomic function field where $\Lambda_N$ is the $N$–th torsion of the Carlitz module. For $D \in R_T$ we denote $D^* := (-1)^{\deg D} D$. 


We will call a field $F$ a cyclotomic function field if there exists $N \in R_F$ such that $F \subseteq k(\Lambda_N)$.

Let $N \in R_F$. The Dirichlet characters $\chi \bmod N$ are the group homomorphisms $\chi : (R_T/(N))^* \rightarrow \mathbb{C}^*$. Given a group $X$ of Dirichlet characters modulo $N$, the field associated to $X$ is the fixed field $F = k(\Lambda_N)^H$ where $H = \bigcap_{\chi \in X} \ker \chi$. We say that $F$ corresponds to the group $X$ and that $X$ corresponds to $F$. We have that $X \cong \text{Hom}(\text{Gal}(F/k), \mathbb{C}^*)$. When $X$ is a cyclic group generated by $\chi$, we have that the field associated to $X$ is equal to $F = k(\Lambda_N)^{\ker \chi}$ and we say that $F$ corresponds to $\chi$.

Given a cyclotomic function field $F$ with group of Dirichlet characters $X$, the maximal cyclotomic extension of $F$ unramified at the finite prime divisors is the field that corresponds to $Y := \prod_{P \in R_F} X_P$ where $X_P = \{ \chi_P \mid \chi \in X \}$ and $\chi_P$ is the $P$-th component of $\chi$, see [5]. We have that the ramification index of $P \in R_F$ in $F/k$ equals $|X_P|$.

When $X$ is a cyclic group generated by $\chi$, the maximal cyclotomic extension unramified at the finite primes of $F$ is given by $M = F_1 \cdots F_r$, where $\chi = \prod_{i=1}^r \chi_i$, and $F_i := k(\Lambda_N)^{\chi_i \neq \chi}$. The only finite ramified prime in each $F_i/k$ is $P_i$, $1 \leq i \leq r$.

We denote the infinite prime of $k$ by $p_\infty$. That is, $p_\infty$ is the pole divisor of $T$ and $1/T$ is a uniformizer for $p_\infty$. We have that the inertia group and the decomposition group of $p_\infty$ in $k(\Lambda_N)/k$ are both equal to $\mathbb{F}_q^* \subseteq (R_T/(N))^*$. In particular the ramification index of $p_\infty$ in $k(\Lambda_N)/k$ is equal to $q - 1$ and we define the maximal real subfield of $k(\Lambda_N)$ by the fixed field $k(\Lambda_N)^+ := k(\Lambda_N)^{\mathbb{F}_q^*}$. We have that $p_\infty$ decomposes fully in $k(\Lambda_N)^+/k$ and the inertia degree of $p_\infty$ in every cyclotomic function field is always equal to 1.

We define the maximal real subfield of a cyclotomic function field $F$ as $F^+ := F \cap k(\Lambda_N)^+$, where $F \subseteq k(\Lambda_N)$. We have that $p_\infty$ decomposes fully in $F^+/k$ and $p_\infty$ is totally ramified in $F/F^+$.

Given a finite extension $K/k$ and a finite non-empty set $S$ of prime divisors of $K$, the Hilbert Class Field $K_{H,S}$ of $K$ with respect to $S$ is defined as the maximal unramified extension of $K$ such that every prime in $S$ decomposes fully in $K_{H,S}/K$. We will always take $S$ as the set of prime divisors dividing $p_\infty$ and we will denote $K_{H,S}$ simply as $K_H$. The genus field $K_{ge}$ of $K$ with respect to $k$ is the maximal extension of $K$ contained on $K_H$ that is of the form $Kk^*$, where $k^*/k$ is an abelian extension. We will always choose $k^*$ as the maximal extension with respect to this property. In other words, $K_{ge}$ is equal to $Kk^*$, where $k^*$ is the maximal abelian extension of $k$ contained in $K_H$. In the particular case when $K/k$ is abelian, $K_{ge}$ is the maximal abelian extension of $k$ containing $K$ such that $K_{ge}/K$ is unramified and each element of $S$ decomposes fully in $K_{ge}/K$.

Let $F$ be any cyclotomic function field. Then $F_{ge} = M^+ F$, where $M$ is the maximal cyclotomic extension of $F$ unramified at the finite primes. That is, $M$ is the field associated to $Y = \prod_{P \in R_F^+} X_P$, where $X$ is the group of Dirichlet characters associated to $F$ (see [1]). We denote $M = F_{ge}$ and then $F_{ge} = F_{ge}^+ F$. We have that $F_{ge}/F$ is totally ramified at every element of $S$ and if $e_\infty(F_{ge}/F)$ denotes the ramification index of every element of $S$ in $F_{ge}/F$, then $[F_{ge} : F_{ge}^+] = e_\infty(F_{ge}/F) = e_\infty(F_{ge}/F_{ge})$. Therefore, to obtain $F_{ge}$ we need to compute a subextension of $F_{ge}$ of degree $e_\infty(F_{ge}/F)$.

We will use both notations: $e_P(F|k)$ or $e_{F/k}(P)$ to denote the ramification index of the prime $P$ of $k$ in $F$. 

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When $K/k$ is a finite abelian extension, it follows from the Kronecker–Weber Theorem that there exist $N \in R_T$, $n \in \mathbb{N} \setminus \{0\}$ and $m \in \mathbb{N}$ such that $K \subseteq _n k(\Lambda_N)_{m}$, where for any $F, F_m := F\mathbb{F}_q^m(T)$ and $nF = FL_m$, with $L_m$ the maximal subfield of $k(\Lambda_1/T^n)$, where $p_\infty$ is totally and wildly ramified. Then we define $E := K\mathcal{M} \cap k(\Lambda_N)$, where $\mathcal{M} = L_n k_m$. Then $K_{ge} = E^{1}_{ge} k$, where $H$ is the decomposition group of the infinite primes in $KE_{ge}/K$ (see [1]).

One important result on ramification of tamely ramified extensions, is the following.

**Theorem 2.1 (Abhyankar’s Lemma).** Let $L/K$ be a separable extension of global function fields. Assume that $L = K_1 K_2$ with $K \subseteq K_i \subseteq L$, $1 \leq i \leq 2$. Let $p$ be a prime divisor of $K$ and $\mathfrak{p}$ a prime divisor in $L$ above $p$. Let $\mathfrak{p}_i := \mathfrak{p} \cap K_i$, $i = 1, 2$. If at least one of the extensions $K_i/K$ is tamely ramified at $p$, then

$$e_{L/K}(\mathfrak{p}|p) = \text{lcm}[e_{K_1/K}(\mathfrak{p}_1|p), e_{K_2/K}(\mathfrak{p}_2|p)],$$

where $e_{L/K}(\mathfrak{p}|p)$ denotes the ramification index.

**Proof.** See [10, Theorem 12.4.4]. \[\square\]

**Remark 2.2.** Abhyankar’s Lemma is valid for function fields, not only for global function fields.

### 3. KUMMER EXTENSIONS OF PRIME POWER DEGREE

Let $l$ be a prime with $l^n$ dividing $q - 1$ and let $K/k$ be a Kummer extension of exponent $l^n$. Then from Kummer theory we have that $K$ is the composite $K = K_1 \cdots K_s$ of linearly disjoint cyclic Kummer extensions. More precisely, $K$ can be written as

$$K = k(i^{\sqrt{\gamma_1}D_1}, \cdots, i^{\sqrt{\gamma_s}D_s}) = K_1 \cdots K_s$$

where

$$K_\varepsilon = k(i^{\sqrt{\gamma_\varepsilon D_\varepsilon}}), \quad 1 \leq \varepsilon \leq s.$$ 

We also have

$$\text{Gal}(K/k) \cong \text{Gal}(K_1/k) \times \cdots \times \text{Gal}(K_s/k) \cong C_{l^{n_1}} \times \cdots \times C_{l^{n_s}}$$

with $n = n_1 \geq n_2 \geq \cdots \geq n_s$, $\gamma_\varepsilon \in \mathbb{F}_q^*$, $D_\varepsilon \in R_T$ monic, and $K_\varepsilon = k(i^{\sqrt{\gamma_\varepsilon D_\varepsilon}})$ a cyclic extension of $k$ of degree $l^n$ for $1 \leq \varepsilon \leq s$.

Let $P_1, \ldots, P_r$ be the set of finite primes of $k$ ramified in $K$ with $P_1, \ldots, P_r \in R_T^+$ distinct. Then we may assume that

$$D_\varepsilon = P_1^{\alpha_{1,\varepsilon}} \cdots P_r^{\alpha_{r,\varepsilon}} \quad \text{with} \quad 0 \leq \alpha_{j,\varepsilon} \leq l^{n_\varepsilon} - 1, \quad 1 \leq j \leq r, \quad 1 \leq \varepsilon \leq s.$$ 

In fact, $\alpha_{j,\varepsilon} = 0$ if and only if $P_j$ is unramified in $K_\varepsilon/k$.

Let $\alpha_{j,\varepsilon} = b_{j,\varepsilon} l^{a_{j,\varepsilon}}$ with $\gcd(b_{j,\varepsilon}, l) = 1$ when $\alpha_{j,\varepsilon} \neq 0$ and let $\deg P_j = c_j l^{d_j}$ with $\gcd(c_j, l) = 1, 1 \leq j \leq r$.

For $x \in \mathbb{Z}_l$, $v_l(x)$ denotes the valuation of $x$ at $l$. That is, $v_l(x) = \gamma$ if $l^\gamma | x$ and $l^{\gamma+1} \not| x$. We write $v_l(0) = \infty$. 

3.1. The cyclotomic case. First, we assume that \( K \) is contained in a cyclotomic function field, more precisely in \( k(\Lambda_{D_1, \ldots, D_s}) \), and this is so if and only if \( \gamma_s \equiv (-1)^{\deg D_s} \mod (\mathbb{F}_p^*)^{l_{D_s}} \) for \( 1 \leq s \leq s \) (see [9, Corolario 9.5.12]). When \( K_\varepsilon \) is contained in a cyclotomic function field, we may assume that \( K_\varepsilon = k(i^\varepsilon \sqrt{D_{\varepsilon}}) \), \( 1 \leq \varepsilon \leq s \). Note that if \( l_{D_\varepsilon} \mid \deg D_\varepsilon \) then \( k(i^\varepsilon \sqrt{D_{\varepsilon}}) = k(i^\alpha \sqrt{D}) \).

First consider \( F = k(i^\varepsilon \sqrt{D}) \), with \( D = P_1^{\alpha_1} \cdots P_r^{\alpha_r} \), a Kummer cyclic extension of \( k \). Let \( X = \langle \chi \rangle \) be the group of Dirichlet characters associated to \( F \). Note that for any \( \nu \in \mathbb{N} \) relatively prime to \( l \), the field associated to \( \chi^\nu \) is \( F \) since \( X = \langle \chi^\nu \rangle \). This corresponds to the fact that \( F = k(i^\alpha \sqrt{D^\nu}) \).

When \( D = P \in R_T^+ \) we have that the character associated to \( F \) is \( (\frac{N}{P})_{l_n} \), the Legendre symbol which is defined as follows: if \( P \) is of degree \( d \), then for any \( N \in R_T \) with \( P \nmid N \) mod \( P \in (R_T/(P))^* \cong \mathbb{F}_q^* \). Then \( (\frac{N}{P})_{l_n} \) is defined as the unique element of \( \mathbb{F}_q^* \) such that \( N^{\frac{\delta P}{l_n}} \equiv (\frac{N}{P})_{l_n} \mod P \). We have that \( (\frac{N}{P})_{l_n} \) is the character associated to \( k(i^\alpha \sqrt{P}) \) (see [9, Proposici \( \acute{\o} \) n 9.5.16]). Let us denote \( \chi_P = (\frac{N}{P})_{l_n} \). Then, for any \( \nu \in \mathbb{Z} \), \( \chi_P^\nu \) is the character associated to \( k(i^\alpha \sqrt{P^\nu}) \).

Hence, if \( \chi_D \) is the character associated to \( k(i^\alpha \sqrt{D}) \), then \( \chi_D = \prod_{j=1}^r \chi_{P_j}^\alpha \).

In general, for a radical extension, we have:

**Theorem 3.1.** Let \( F = k(i^\alpha \sqrt{D}) \) be a geometric separable extension of \( k \), with \( \gamma \in \mathbb{F}_q^* \) and let \( D = P_1^{\alpha_1} \cdots P_r^{\alpha_r} \in R_T \). Then
\[
e_F/k(P_j) = \frac{m}{\gcd(\alpha_j, m)}, \quad 1 \leq j \leq r \quad \text{and} \quad e_{F/k}(P) := e_{F/k}(P_\infty) = \frac{m}{\gcd(\deg D, m)}.
\]

**Proof.** See [6, \S 5.2]. \( \square \)

As a consequence we obtain the following result for a cyclic cyclotomic Kummer extension \( F = k(i^\alpha \sqrt{D}) \). Let \( X \) be the group of Dirichlet characters associated to \( F \) and let \( Y = \prod_{P \in R_T^+} X_P \) be the group associated to \( M \), the maximal cyclotomic extension of \( F \) unramified at the finite primes. Let \( P = P_j \), \( X = X_P = \langle \chi_P \rangle \) and let \( F_0 \) be the field associated to \( X_P \). Then \( F_0 \) is cyclotomic, \( P \) is the only ramified finite prime in \( F_0/k \) and \( P \) is tamely ramified in \( F_0/k \). This implies that \( F_0 \subseteq k(\Lambda_P) \) and \( \text{Gal}(k(\Lambda_P)/k) \cong C_{q^{d_P}} \) with \( d_P := \deg P \). Therefore \( F_0 \) is the only field of degree \( \text{ord}(\chi_P) = l^{d_P} \) over \( k \). Since \( F_0/k \) is a Kummer extension, it follows that \( F_0 = k(i^\alpha \sqrt{P}) \). Then, we have:

**Theorem 3.2.** The maximal unramified at the finite primes cyclotomic extension of \( F = k(i^\alpha \sqrt{D}) \) is \( M := k(i^{\alpha_1}(P_1^{\alpha_1})^\nu, \ldots, i^{\alpha_r}(P_r^{\alpha_r})^\nu) \).

**Remark 3.3.** Let \( \alpha = l^ab \) with \( \gcd(b, l) = 1 \) and \( 0 \leq a < n \). Then \( k(i^{\alpha}(P_\nu)^\nu) = k(i^{\alpha}(P^{\nu})) \). In particular, if \( \alpha_\varepsilon = l^{a_\varepsilon}b_\varepsilon \) with \( \gcd(l, b_\varepsilon) = 1, 1 \leq \varepsilon \leq r \), then
\[
M = k(i^{\alpha_\varepsilon}(P_1^{\alpha_\varepsilon})^\nu, \ldots, i^{\alpha_r}(P_r^{\alpha_r})^\nu) = F_1 \cdots F_r.
\]
with $F_\varepsilon = k\left(\sqrt[n]{P_\varepsilon}\right)$, $1 \leq \varepsilon \leq r$.

We give another proof of Theorem 3.2 using Abhyankar’s Lemma. On the one hand we have that

$$[M : k] = \prod_{P \in R_T^+} |X_P| = \prod_{j=1}^{r} |X_{P_j}| = \prod_{j=1}^{r} e_{M/k}(P_j) = \prod_{j=1}^{r} l^{n-a_j}.$$ 

On the other hand if $F_j = k\left(\sqrt[n]{(P_j)^e}\right)$, by Abhyankar’s Lemma, we have $FF_j/F$ is unramified at every finite prime, so $FF_1 \cdots F_r/F$ is unramified at the finite primes and $F \subseteq F_1 \cdots F_r$. Hence $F_1 \cdots F_r \subseteq F_{\geq}$ and $[F_1 \cdots F_r : k] = [M : k]$. Therefore $M = F_1 \cdots F_r$ as claimed.

Let $\alpha_j = b_j l_\varepsilon$ with $\gcd(l, b_j) = 1$ and $\deg P_j = l^{d_j} c_j$ with $\gcd(l, C_j) = 1, 1 \leq j \leq r$. Then

$$e_{P_j}(F|k) = l^{a_j},$$

$$e_\infty(F|k) = \frac{l^n}{\gcd(l^n, \deg D)} = l^t,$$ (3.1)

$$e_\infty(F_j|k) = \frac{l^{n-a_j}}{\gcd(l^{n-a_j}, \deg P_j)} = \frac{l^{n-a_j}}{\min\{n-a_j, d_j\}} = l^{n-a_j - \min\{n-a_j, d_j\}}.$$ 

From Abhyankar’s Lemma we obtain that

$$e_\infty(M|k) = \frac{l^n}{\gcd(l^n, \deg D)} = l^m.$$ 

Therefore $[M : F_{\geq}] = l^{m-t}$. To find $F_{\geq}$ we must find a subfield $F \subseteq L$ of $M$ such that $L/F$ is unramified and $[M : L] = l^{m-t}$. If $L$ is such field, we have $L \subseteq F_{\geq}$ since $p_\infty$ is unramified at $L/F$ and $L$, being cyclotomic, satisfies that $p_\infty$ decomposes fully in $L/F$. Since $[M : L] = [M : F_{\geq}]$ it follows that $L = F_{\geq}$.

Going back to a general cyclotomic Kummer extension $K$ of $k$, where $K = k\left(\sqrt[n_1]{D_1}, \cdots, \sqrt[n_s]{D_s}\right) = K_1 \cdots K_s$. Let $P_1, \ldots, P_r$ be the ramified finite primes in $K/k$. From Abhyankar’s Lemma and (3.1), we have, for $P_j \in R_T^+$

$$e_{P_j}(K|k) = \frac{\gcd(l^{n-a_j}, \deg D)}{\gcd(l^{n-a_j}, \deg D)} = l^{\beta_j}$$

with

$$\beta_j = \max_{1 \leq \varepsilon \leq s} \left\{ n_\varepsilon - v_\varepsilon(\alpha_j, \varepsilon) \right\} = \max_{1 \leq \varepsilon \leq s} \left\{ n_\varepsilon - a_j, \varepsilon \right\},$$ (3.2)

and

$$l^t = e_\infty(K|k) = \frac{l^{n_\varepsilon}}{\gcd(l^{n_\varepsilon}, \deg D)} = \frac{l^{n_\varepsilon}}{\gcd(l^{n_\varepsilon}, \deg D)}$$

(3.3)

that is, $t = \max_{1 \leq \varepsilon \leq s} \left\{ n_\varepsilon - \min\{n_\varepsilon, v_\varepsilon(\deg D)\} \right\}$.

Let $X_\varepsilon = \langle \chi_\varepsilon \rangle$ be the group of Dirichlet characters corresponding to $K_\varepsilon$, $1 \leq \varepsilon \leq s$. Let $\chi_\varepsilon = \prod_{P \in R_T^+} \chi_{\varepsilon, P}$ be the product of $\chi_\varepsilon$ into its $P$-components, $e_P(K_\varepsilon|k) = \text{ord}(\chi_{\varepsilon, P})$. 


In this way, we obtain that \( X = (\chi_1, \ldots, \chi_s) \) is the group of Dirichlet characters associated to \( K, X = X_1 \cdots X_s \). We have
\[
X_P = (X_1)_P \cdots (X_s)_P = (\chi_1, P) \cdots (\chi_s, P) = (\chi, P)
\]
with \( \operatorname{ord}(\chi, p) = \max_{1 \leq i \leq s} \{ \operatorname{ord}(\chi_i, p) \} = e_P(K|k) \) for \( P \in \mathbb{P}_R \).

Let \( M \) be the field associated to \( Y := \prod_{P \in \mathbb{P}_R^+} X_P \), the maximal cyclotomic extension of \( K \) unramified at every finite prime. From (3.2) we obtain
\[
e_{P_j}(K|k) = l^{\beta_j}.
\]

Then \( M = F_1 \cdots F_r \) with \( F_j = k(\sqrt[\ell^{j}]{P_j}) \) and from (3.1) we obtain
\[
l_m := e_{\infty}(M|k) = \max_{1 \leq j \leq r} \{ e_{\infty}(F_j|k) \} = \max_{1 \leq j \leq r} \left\{ \frac{l^{\beta_j}}{\gcd(l^{\beta_j}, \deg P_j)} \right\} = \max_{1 \leq j \leq r} \{ l^{\beta_j} - \min(l^{\beta_j}, d_j) \},
\]
so that
\[
m = \max_{1 \leq j \leq r} \{ l^{\beta_j} - \min(l^{\beta_j}, d_j) \}.
\]

The procedure to obtain \( K_{ge} \) is the following. We order \( P_1, \ldots, P_r \) so that \( n = \beta_1 \geq \beta_2 \geq \cdots \geq \beta_r \geq 1 \), that is, we fix \( P_1, \ldots, P_r \) in decreasing order of their ramification indexes in \( K/k \). There is at least one \( F_i \) such that \( e_{\infty}(F_i|k) = l^m \). We choose \( i \) as the largest index with this property. Then we will show that there exist some powers \( z_{ij}, 1 \leq j \leq i - 1 \) such that \( p_{\infty} \) is unramified in \( k((\sqrt[l_j^{\beta_j}]{P_j})^*)/k \).

For \( j > i \) we have two cases, \( Q_j = P_j^{y_j} P_i^{\varepsilon_j} \) or \( Q_j = P_j^{y_j} P_i^{r_j} \) for some \( y_j, \varepsilon_j \in \mathbb{Z} \) such that if \( F_j = k((\sqrt[l_j^{\beta_j}]{P_j})^*) \) for some \( \beta_j \), satisfies that the ramification index of \( F_j/k \) at \( P_j \) is \( l^{\beta_j} \), at \( P_i \) is less than or equal to \( l^{\beta_i} \) and \( p_{\infty} \) is unramified. The rest will follow taking the composite of all these fields and one of the form \( k((\sqrt[l_j^{\beta_j}]{P_j})^*) \) for some \( \xi_j \).

The result for the cyclotomic prime power degree case is the following.

**Theorem 3.4.** Let \( K/k \) be a finite cyclotomic Kummer \( l \)-extension of \( K = K_1 \cdots K_s, K_1 = k((\sqrt[l_1^{\beta_1}]{P_1})^*), \ldots, K_s = k((\sqrt[l_s^{\beta_s}]{P_s})^*), D_2 \in R, \) monic, \( 1 \leq s \leq 5 \) and \( \operatorname{Gal}(K/k) \cong C_{l_1^n} \times \cdots \times C_{l_s^n} \) with \( n = n_1 \geq n_2 \geq \cdots \geq n_s \) and \( l^n - 1 \) is a prime. Then \( K \subseteq k(\Lambda_{D_1, \ldots, D_s}) \). Let \( P_1, \ldots, P_r \) be the finite primes in \( k \) ramified in \( K \) with \( P_1, \ldots, P_r \in \mathbb{P}_R^+ \) distinct. Let
\[
e_{P_j}(K|k) = l^{\beta_j}, \quad 1 \leq \beta_j \leq n, \quad 1 \leq j \leq r, \quad \text{and} \quad e_{\infty}(K|k) = l^t, \quad 0 \leq t \leq n
\]
given by (3.2) and (3.3) and let \( \deg P_j = c_j l^{d_j} \) with \( \gcd(c_j, l) = 1, 1 \leq j \leq r \).

We order \( P_1, \ldots, P_r \) so that \( n = \beta_1 \geq \beta_2 \geq \cdots \geq \beta_r \).

The maximal cyclotomic extension \( M \) of \( K \), unramified at the finite primes, is given by
\[
M = K_{ge} = k((\sqrt[l_1^{\beta_1}]{P_1})^*, \ldots, \sqrt[l_r^{\beta_r}]{P_r}).
\]

Let \( t^m = e_{\infty}(M|k) \) be given by (3.4).

Choose \( i \) such that \( m = \beta_i - \min(\beta_i, d_i) \) and such that for \( j > i \) we have \( m > \beta_j - \min(\beta_j, d_j) \). That is, \( i \) is the largest index obtaining \( t^m \) as the ramification index of \( p_{\infty} \).

In case \( m = t \) we have \( M = K_{ge} = \prod_{j=1}^{r+1} k((\sqrt[l_j^{\beta_j}]{P_j})^*) \).

In case \( m > t \geq 0 \), we have \( \min(\beta_i, d_i) = d_i \) and \( m = \beta_i - d_i \). Let \( a, b \in \mathbb{Z} \) be such that \( a \deg P_j + b l^{n+d_i} = d_i, \deg P_i \). Set \( x_j = a, y_j = b l^{n+d_i} = d_i, z_j = -a l^{n+d_i} = -d_i, \in \mathbb{Z} \) for \( 1 \leq j \leq i - 1 \). For \( j > i \), consider \( y_j \in \mathbb{Z} \) with \( y_j \equiv -c_j^{-1} z_j \mod l^n \) and \( y_j \equiv -c_j \mod l^n \).
Let
\[
E_j = \begin{cases}
  k\left(\sqrt{P_j P_i^2}\right) & \text{if } j < i, \\
  k\left(\sqrt{P_i^2}\right) & \text{if } j = i, \\
  k\left(\sqrt{P_j P_i^{\beta_j-d_j}}\right) & \text{if } j > i \text{ and } d_j \geq d_i, \\
  k\left(\sqrt{P_i^{d_i-d_j} P_j^{\beta_j}}\right) & \text{if } j > i \text{ and } d_i > d_j.
\end{cases}
\]

Then \(K_{\mathfrak{g}} = E_1 \cdots E_{i-1} E_i E_{i+1} \cdots E_r\).

Proof. When \(m = t\) it follows that \(K_{\mathfrak{g}} = M = \prod_{j=1}^{r} F_j\), where \(F_j = k\left(\sqrt{P_j}\right)\).

Assume \(m > t \geq 0\). Then \(d_i < \beta_i\) and \(\beta_i - d_i = m\).

For \(j < i\) we have \(\beta_j \geq \beta_i\) and \(\beta_i - d_j = \beta_j - \min\{\beta_j, d_j\} \leq \beta_i - d_i\).

Therefore \(d_j \geq \beta_j - m = \beta_j - (\beta_i - d_i) = \beta_j - \beta_i + d_i \geq d_i\). In particular \(d_i \mid \deg P_j\), \(1 \leq j \leq i - 1\). Since \(\gcd(l^{n+d_i}, \deg P_i) = l^{d_i}\), there exist \(a, b \in \mathbb{Z}\), such that \(a \deg P_i + b l^{n+d_i} = l^d\). Multiplying by \(\deg P_j\) and dividing by \(l^d\), \(j < i\), we obtain
\[
a \frac{\deg P_j}{l^d} \deg P_i + b \deg P_j l^n = \deg P_j.
\]

That is, \(\deg P_j + z_j \deg P_i = b \deg P_j l^n\), where \(z_j = -a \frac{\deg P_i}{l^d} = -a c_j l^{d_j-d_i}\). Note that \(\gcd(a, l) = 1\). We have that \(l^n \mid \deg(P_j P_i^{z_j})\). Let \(E_j := k\left(\sqrt{P_j P_i^{z_j}}\right)\). It follows that
\[
e_{\infty}(E_j|k) = 1, \quad e_{P_j}(E_j|k) = l^{\beta_j} = e_{P_i}(K|k) \quad \text{and} \quad e_{P_i}(E_j|k) = l^{\beta_j - v_l(z_j)} l^{\beta_i} = e_{P_i}(K|k),
\]

since \(v_l(z_j) = d_j - d_i\) and \(\beta_j - v_l(z_j) = -d_j + d_i \leq m + d_i = \beta_i\). In particular \(E_j \subseteq K_{\mathfrak{g}}\).

Now consider \(j > i\). Let \(y_j \in \mathbb{Z}\) such that \(y_j \equiv -c_i^{-1} c_j \mod l^n\). Since \(ac_i \equiv 1 \mod l^n\) we have \(c_j^{-1} l^{d_i} = a \mod l^n\). This is possible since \(\gcd(c_i c_j, l) = 1\). Note that \(\gcd(y_j, l) = 1\).

First assume \(d_j \geq d_i\). Let \(Q_j = P_j P_i^{y_j l^{d_j-d_i}}\) and \(E_j := k\left(\sqrt{Q_j}\right)\).

We have
\[
\deg Q_j = \deg P_j + y_j l^{d_j-d_i}, \quad \deg P_i = c_i l^{d_j} + y_j l^{d_j-d_i} c_i l^{d_i} = l^{d_i} (c_j + y_j c_i).
\]

It follows that \(l^n \mid \deg Q_j\) and \(e_{\infty}(E_j|k) = 1\). On the other hand
\[
e_{P_j}(E_j|k) = l^{\beta_j} = e_{P_j}(K|k) \quad \text{and} \quad e_{P_i}(E_j|k) = l^{d_j-d_i} l^{-v_l(y_j)} = e_{P_i}(K|k)\).
\]

Since \(\gcd(y_j, l) = 1\), we have \(\beta_j - d_j + d_i \leq m + d_i = \beta_i\) and \(e_{P_i}(E_j|k) e_{P_i}(K|k)\). It follows that \(E_j \subseteq K_{\mathfrak{g}}\).

Now consider \(d_i \geq d_j\). Set \(Q_j = P_j^{d_i-d_j} P_i^{y_j}\) and let \(E_j = k\left(\sqrt{Q_j}\right)\). We have
\[
\deg Q_j = l^{d_i-d_j} \deg P_j + y_j \deg P_i = l^{d_i-d_j} c_j l^{d_j} + y_j c_i l^{d_i} = l^{d_i} (c_j + y_j c_i).
\]

Therefore \(l^n \mid \deg Q_j\) and \(e_{\infty}(E_j|k) = 1\). Next, we have
\[
e_{P_j}(E_j|k) = l^{\beta_j + d_i-d_j - d_i} = l^{\beta_j} = e_{P_j}(E_j|k).
\]
and
\[ e_{P_j}(E_j|k) = l^{\beta_j + d_i - d_j}|l^{d_i} = e_{P_j}(K|k). \]

It follows that \( E_j \subseteq K_{gt}. \)

Therefore \( L := E_1\cdots E_{i-1}E_{i+1}\cdots E_r \subseteq K_{gt}. \) Note that from Abhyankar’s Lemma, we have
\[ e_\infty(L|k) = 1, \]
\[ e_{P_j}(L|k) = e_{P_j}(K|k) = l^{\beta_j}, j \neq i \]
and
\[ e_{P_j}(L|k)e_{P_j}(K|k) = l^{\beta_j}. \]

In fact, we can give a direct argument to show that actually \( e_{P_j}(L|k) = e_{P_j}(K|k) = l^{\beta_j} \) (see Remark 3.5).

For any \( 1 \leq j \leq r \) let \( I_j \) denote the inertia group of \( P_j \) in \( M'/k \) where \( M' \) is any subfield of \( M \) containing \( E_j \). For any such \( M' \) we have \( |I_j| = l^{\beta_j}. \)

Let \( J := \{ j > i \mid d_i > d_j \} \) and \( I := \{ 1, 2, \ldots, i-1, i+1, \ldots, r \} \setminus J. \) In other words, if \( j \in I \) then \( E_j = k\left(\sqrt[\mu]{P_j^{x_j}}\right) \) for some \( x_j \in \mathbb{Z}. \)

Set \( I = \{ m_1, \ldots, m_u \}. \) Let \( F := E_{m_1} \cap E_{m_2}. \) Since we have that \( P_{m_1} \) is fully ramified in \( E_{m_1} \) and is unramified in \( E_{m_2} \), it follows that \( F = k \) and therefore \( [E_{m_1}, E_{m_2} : k] = [E_{m_1} : k][E_{m_2} : k]. \)

Furthermore, \( \text{Gal}(E_{m_1}, E_{m_2}/k) \cong \text{Gal}(E_{m_1}/k) \times \text{Gal}(E_{m_2}/k) \cong I_{m_1} \times I_{m_2} \) and \((E_{m_1}, E_{m_2})^{I_{m_1}I_{m_2}} = k. \) By induction, we obtain for \( 1 \leq v \leq u: \)

1. \( (E_{m_1}, \ldots, E_{m_{v-1}}) \cap E_{m_v} = k, \)
2. \( E_{m_1}, \ldots, E_{m_v} : k = [E_{m_1} : k] \cdots [E_{m_v} : k], \)
3. \( (E_{m_1}, \ldots, E_{m_v})^{I_{m_1}, \ldots, I_{m_v}} = k, \)
4. \( I_{m_1}, \ldots, I_{m_v} \cong I_{m_1} \times \cdots \times I_{m_v}. \)

That is, for any \( \mu \in I \) we have that \( \left( \prod_{j \in I \setminus \{ \mu \}} E_j \right) \cap E_\mu = k \) since for any non-trivial subfield of \( A := \prod_{j \in I \setminus \{ \mu \}} E_j \) at least one \( P_j \) with \( j \in I \setminus \{ \mu \} \) is ramified in this subfield and \( P_j \) is unramified in \( E_\mu. \) In particular we have
\[ \left[ \prod_{j \in I} E_j : k \right] = \prod_{j \in I} [E_j : k]. \tag{3.5} \]

We also have that
\[ \left( \prod_{j \in I} E_j \right) \cap \left( \prod_{j \in J} E_j \right) = k \tag{3.6} \]

since in any non-trivial subfield of \( \prod_{j \in I} E_j \) at least one \( P_\mu \) with \( \mu \in I \) is ramified and \( P_\mu \) is unramified in \( \prod_{j \in J} E_j. \) In other words,
\[ [L : k] = \left[ \prod_{j \notin I} E_j : k \right] = \left[ \prod_{j \in I} E_j : k \right] \left[ \prod_{j \in J} E_j : k \right]. \]
To compute \( \prod_{j \in \mathcal{J}} E_j : k \) we order \( \mathcal{J} \) as follows. Write \( \mathcal{J} = \{j_1, \ldots, j_s\} \) with \( d_i - d_{j_1} \leq d_i - d_{j_2} \leq \cdots \leq d_i - d_{j_s} \). We have that \( E_j^{d_j} = k(\sqrt[\sqrt{d_j}]{P_{j}}) = k(\sqrt[\sqrt{d_j}]{P_{j}}) \).

First we consider \( E_{j_1}E_{j_2} \). We have \( E_{j_1} \cap E_{j_2} = C_1 \) where we denote \( C_u := k(\sqrt[\sqrt{d_j}]{P_{j}}) \), \( j_u \in \mathcal{J}, 1 \leq u \leq s \). In fact, if \( \Lambda := E_{j_1} \cap E_{j_2} \), then \( P_{j_1} \) is not ramified in \( E_{j_2} \) and \( P_{j_2} \) is not ramified in \( E_{j_1} \), thus, the only ramified prime in \( \Lambda/k \) is \( P_i \).

Furthermore, \( C_1 \subseteq E_{j_1} \) and \( C_2 \subseteq E_{j_2} \) and \( C_1 = C_1 \cap C_2 \subseteq E_{j_1} \cap E_{j_2} \). Now, \( P_{j_1} \) is fully ramified in \( E_{j_1}/C_1 \). In particular, if \( C_1 \subseteq C' \subseteq F_{j_1}, P_{j_1} \) is ramified in \( C'/C_1 \) and since \( P_{j_1} \) is unramified in \( (E_{j_1} \cap E_{j_2})/C_1 \), it follows that \( E_{j_1} \cap E_{j_2} = C_1 \).

Consider the following diagram

\[
\begin{array}{ccc}
E_{j_1} & \longrightarrow & E_{j_1}E_{j_2} \\
\downarrow & & \downarrow \\
C_1 & \longrightarrow & E_{j_2}
\end{array}
\]

We have \([E_{j_1} E_{j_2} : k] = [E_{j_1} : C_1][E_{j_2} : C_1][C_1 : k] = \frac{[E_{j_1}, k][E_{j_2}, k]}{[C_1, k]} \).

Now we consider \( E_{j_1}E_{j_2}E_{j_3} \). With an argument similar to the one in the previous case, we have \( E_{j_1}E_{j_2} \cap E_{j_3} = C_2 \). We consider the following diagram.

\[
\begin{array}{ccc}
E_{j_1}E_{j_2} & \longrightarrow & E_{j_1}E_{j_2}E_{j_3} \\
\downarrow & & \downarrow \\
C_2 & \longrightarrow & E_{j_3}
\end{array}
\]

Thus \([E_{j_1} E_{j_2} E_{j_3} : k] = [E_{j_1} E_{j_2} : C_2][E_{j_3} : C_2][C_2 : k] = [E_{j_1} E_{j_2} : k][E_{j_3} : C_2] = \frac{[E_{j_1}, k][E_{j_2}, k][E_{j_3}, k]}{[C_2, k]} \).

By induction, we obtain for \( 1 \leq v \leq s \)

1. \( (E_{j_1} \cdots E_{j_{v-1}}) \cap E_{j_v} = C_{v-1}, v \geq 2 \),
2. \( [E_{j_1} \cdots E_{j_v} : C_1] \cdots [E_{j_v} : C_v][C_v : k] = (\prod_{u=1}^{v} l^{b_{j_u}})^{d_i-d_{j_v}} \),
3. \( (E_{j_1} \cdots E_{j_v}) I_{j_1} \cdots I_{j_v} = k_v \),
4. \( I_{j_1} \cdots I_{j_v} \cong I_{j_1} \times \cdots \times I_{j_v} \).

That is,

\[
\prod_{j \in \mathcal{J}} E_j : k = \left(\prod_{j \in \mathcal{J}} E_j \right) : k = \left(\prod_{u=1}^{s} [E_{j_u} : C_u]\right) [C_s : k] = (\prod_{j \in \mathcal{J}} l^{b_j})^{d_i-d_s}. \tag{3.7}
\]

From (3.5), (3.6) and (3.7) we obtain

\[
[L : k] = (\prod_{j=1}^{r} l^{b_j})^{d_i-d_s} \quad \text{and} \quad L \cap E_i = k\left(\sqrt[\sqrt{d_j}]{P_i}\right) = C_s,
\]
where $E_i := k\left(\sqrt[\beta_j]{P_j}\right)$. Now, we have that
\[ e_{P_i}(E_i|k) = t^{d_i+\varepsilon} = t^{d_i+m} = e_{P_i}(K|k) \]
and
\[ e_\infty(F_i|k) = \frac{t^{d_i+\varepsilon}}{\gcd(t^{d_i+\varepsilon})} = t^{d_i+t-d_i} = t^t = e_\infty(K|k). \]

It follows that $LE_i \subseteq K_{ge}$ and
\[ [LE_i : k] = [L : C_s][E_i : C_s][C_s : k] = [L : C_s][E_i : k] = \prod_{j=1}^{\beta_j} \frac{[M : k]}{[M : K_{ge}]} = [K_{ge} : k]. \]

Therefore $K_{ge} = LE_i = E_1 \cdots E_{i-1} E_i E_{i+1} \cdots E_r$. □

**Remark 3.5.** In the notation of Theorem 3.4, we have that in the case $m > t$, $K \subseteq K_{ge} = E_1 \cdots E_{i-1} E_i E_{i+1} \cdots E_r$. Therefore there exists $j \neq i$ such that $e_{P_i}(E_j|k) = t^{d_i}$. However one wonders why. Here we give a direct proof. For $j < i$ we have $E_j = k(\sqrt[\beta_j]{P_j}P_j^{\varepsilon})$ with $z_j = -acl^{d_j-d_i}$. Now, $d_j > d_i$, and $\beta_j > \beta_i$ if $j < i$ and $\nu_l(z_j) = d_j - d_i$ since $\gcd(ac, j) = 1$. Then $e_{P_i}(E_j|k) = t^{d_j-\nu_l(z_j)} = t^{d_j-d_i}$. So we require that for some $j < i$ it holds $\beta_j - d_j + d_i = \beta_i$, equivalently, $\beta_j - d_j = \beta_i - d_i$.

From the definition of the index $i$ we have $\beta_j - d_j \leq m = \beta_i - d_i$, and for $j > i$ we have $\beta_j - d_j < m$.

Assume that $\beta_j - d_j < m$ for all $j \neq i$. From (3.2) we obtain that $\beta_j = \max_{1 \leq s \leq \delta}\{n_{\varepsilon} - d_j\}$, where for convenience we choose $a_{\varepsilon} = n$ in case $a_{\varepsilon} = b_{\varepsilon} = 0$, that is, when $b_{\varepsilon} = 0$ because in this way $n_{\varepsilon} - d_j \leq 0$ and the maximum can not be obtained in $\varepsilon$ since $1 \leq \beta_j \leq m$.

Let $1 \leq \mu \leq s$ be such that $\beta_i = \max_{1 \leq \mu \leq s}\{n_{\varepsilon} - d_i\}$ so that $m = \beta_i - d_i = \nu_l(\deg D_{\mu}) = d_0$. Since $D_{\mu} = \prod_{j=1}^{\beta_j} P_j^{\delta_j+\mu}$ we have
\[ \deg D_{\mu} = \sum b_{j,\mu}^{\delta_j+\mu} = \sum b_{j,\mu}^{\delta_j+\mu} + \sum b_{j,\mu}^{\delta_j+\mu} + d_j. \]

Fix $j \neq i$ and let $\beta_j = \max_{1 \leq s \leq \delta}\{n_{\varepsilon} - d_j\} \geq n_{\varepsilon} - a_{j,\varepsilon}$. Therefore $a_{j,\mu} + d_j = n_{\varepsilon} - \beta_j$ and $a_{j,\mu} + d_j = n_{\varepsilon} - \beta_j$. Thus from (3.8) we obtain
\[ a_{j,\mu} + d_j > n_{\varepsilon} - m \quad \text{for} \quad j \neq i \quad \text{and} \quad a_{j,\mu} + d_j = n_{\varepsilon} - m. \]

From (3.9) and (3.10) it follows that
\[ d_0 = a_{i,\mu} + d_i = n_{\varepsilon} - m. \]

On the other hand we have
\[ t = \max_{1 \leq \mu \leq s}\{n_{\varepsilon} - \min\{n_{\varepsilon}, \nu_l(\deg D_{\mu})\}\} \geq n_{\varepsilon} - \min\{n_{\mu}, d_0\} = n_{\mu} - \min\{n_{\mu}, n_{\varepsilon} - m\} = n_{\mu} - (n_{\mu} - m) = m, \]
that is, \( t \geq m \) contrary to our assumption: \( t < m \). Therefore, there exists \( j < i \) such that \( \beta_j - d_j = m \) and \( e_P(E_j|k) = l^{\beta_j} = e_P(K|k) \).

### 3.2. The general case of prime power degree.

Now consider \( K/k \) a Kummer extension of degree a power of the prime number \( l \). If \( \text{Gal}(K/k) \) is of exponent \( l^n \), we have \( \text{Gal}(K/k) \cong C_{l^n_1} \times \cdots \times C_{l^n_r} \) with \( n_1 \geq \cdots \geq n_r \) and \( l^n|q - 1 \). We have that \( K \) is of the form \( K = k(\sqrt[l^n\gamma_1D_1], \ldots, \sqrt[l^n\gamma_sD_s]) \) with \( D_\varepsilon \in R_P \) monic, \( \gamma_\varepsilon \in F_q^* \), \( 1 \leq \varepsilon \leq s \) and \( K_\varepsilon = k(\sqrt[l^n\gamma_\varepsilon D_\varepsilon}) \).

Let \( E = Kk^\infty \cap k(\Delta_{D_1} \cdots \Delta_{D_s}) = k(\sqrt[l^n\gamma_1D_1], \ldots, \sqrt[l^n\gamma_sD_s]) \). From [1] we have that \( K_{ge} = E_{ge}^H \) where \( H \) is the decomposition group of the infinite primes in either \( KE_{ge}/KE_{ge} \) or \( KE_{ge}/K \). We also have that \( KE/K \) is an extension of constants where \( H_1 = H|E \) is the decomposition group of the infinite primes in \( KE/K \) (see [1]).

We have that \( KE = E(\sqrt[l^n\gamma_1D_1], \ldots, \sqrt[l^n\gamma_sD_s]) \) is an extension of constants and the inertia degree of \( p_\infty \) in \( KE/k \) is \( f = l^n \) where \( \mathbb{F}_q^\gamma = \mathbb{F}_q(\sqrt[l^n\gamma_1D_1], \ldots, \sqrt[l^n\gamma_sD_s]) \) since \( E \) being cyclotomic satisfies that the inertia degree of \( p_\infty \) in \( E/k \) is 1.

It follows that

\[
|H| = \frac{|\mathbb{F}_q(\sqrt[l^n\gamma_1D_1], \ldots, \sqrt[l^n\gamma_sD_s]) : \mathbb{F}_q|}{\deg_k p_\infty} = l^n.
\]

Now \( H_1 = H|E \subseteq I_\infty(E|k) \) where \( I_\infty(E|k) \) denotes the inertia group of \( p_\infty \) in \( E/k \) and also denotes the inertia group of \( p_\infty \) in \( KE_{ge}/k \). In particular \( E_{ge}^{H_1} = E_{ge}^{I_\infty(E|k)} \). We denote by \( \mathcal{H}_1 \) the inertia group of \( p_\infty \) in \( E_{ge}/k \). From Theorem 3.4 we have that \( p_\infty \) is unramified in \( L = E_1 \cdots E_{i-1}E_{i+1} \cdots E_r \) and totally ramified in \( E_{ge}/Lk(\sqrt[l^n\gamma_1D_1]) \). Therefore \( E_{ge}^{I_\infty(E_{ge}/k)} = Lk(\sqrt[l^n\gamma_1D_1]) \) and

\[
I_\infty(E_{ge}|k) = \text{Gal}(E_{ge}/Lk(\sqrt[l^n\gamma_1D_1])).
\]

The group \( \mathcal{H}_1 \) is the subgroup of order \( l^n \) of \( I_\infty(E_{ge}|k) \).

It follows that \( E_{ge}^{H_1} = Lk(\sqrt[l^n\gamma_1D_1]) \) and therefore

\[
K_{ge} = E_{ge}^{H_1}K = E_1 \cdots E_{i-1}k(\sqrt[l^n\gamma_1D_1])E_{i+1} \cdots E_r K.
\]

Therefore we have proved the main theorem on Kummer extensions of prime power degree over \( k \).

**Theorem 3.6.** Let \( K = k(\sqrt[l^n\gamma_1D_1], \ldots, \sqrt[l^n\gamma_sD_s]) \) be a Kummer extension of \( K \) of prime power degree with \( D_\varepsilon \in R_P \) monic, \( \gamma_\varepsilon \in F_q^* \), \( 1 \leq \varepsilon \leq s \) and \( K_\varepsilon = k(\sqrt[l^n\gamma_\varepsilon D_\varepsilon}) \). Let \( E = k(\sqrt[l^n\gamma_1D_1], \ldots, \sqrt[l^n\gamma_sD_s]) \).

With the notations of Theorem 3.4, with \( E \) in the place of \( K \), we have that

\[
K_{ge} = Lk(\sqrt[l^n\gamma_1D_1])K = E_1 \cdots E_{i-1}E_{i+1} \cdots E_r k(\sqrt[l^n\gamma_1D_1])K
\]

where \( l^n = \frac{|\mathbb{F}_q(\sqrt[l^n\gamma_1D_1]) : \mathbb{F}_q|}{\deg_k p_\infty} \) and \( \varepsilon_\varepsilon = (-1)^{\deg_{D_\varepsilon} \gamma_{j_\varepsilon}}, 1 \leq j \leq s \). \( \square \)
4. General Kummer extensions

Let $K/k$ be a Kummer extension. Let $G := \text{Gal}(K/k)$ be the Galois group of $K/k$. If $S_1, \ldots, S_h$ are the various Sylow subgroups of $G$, $G \cong S_1 \times \cdots \times S_h$ and each $S_j$ is a group of prime power order, say $|S_j| = l_j^{m_j}$, $1 \leq j \leq h$.

We may write $K = K_1 \cdots K_h$ with $\text{Gal}(K_j/k) \cong S_j$, $1 \leq j \leq h$.

From Theorem 3.6, we know each $(K_j)_{ge}, 1 \leq j \leq h$. The knowledge of $(K_j)_{ge}$ would follow if we had $K_{ge} = \prod_{j=1}^{h} (K_j)_{ge}$. However in general for any two fields $L_1$ and $L_2$, we only have that $(L_1)_{ge}(L_2)_{ge} \subseteq (L_1 L_2)_{ge}$ (see [6] for an example where $(L_1)_{ge}(L_2)_{ge} \subsetneq (L_1 L_2)_{ge}$).

Now, in our case we have $[K_i : k] = l_i^{m_i}$ and $\gcd([K_i : k], [K_j : k]) = 1$ for all $i \neq j$. The knowledge of $(K_j)_{ge}$ is an immediate consequence of the following result.

**Theorem 4.1.** Let $L_i/k, i = 1, 2$ be two finite abelian extensions with $\gcd([L_1 : k], [L_2 : k]) = 1$. Then $(L_1)_{ge}(L_2)_{ge} = (L_1 L_2)_{ge}$.

To prove Theorem 4.1, we first prove:

**Proposition 4.2.** Let $L/k$ be a finite abelian extension and let $l$ be any prime number. Then $l|[L : k]$ if and only if $l|[L_{ge} : k]$.

**Proof.** It is clear that if $l|[L : k]$ then $l|[L_{ge} : k]|$, because $L \subseteq L_{ge}$.

Now let $l|[L_{ge} : k]$. First assume that $L \subseteq k(\Lambda_X)$. Let $X$ be the group of Dirichlet characters associated to $L$ and let $Y = \prod_{P \in R^+_F} X_P$. Let $L_{ge}$ be the field associated to $Y$. We have $L_{ge} \subseteq L_{ge}$. In fact $L_{ge} = L_{ge}^+ L$. Then $l|[L_{ge} : k]$. Therefore there exists $\chi \in Y$ of order $l$. Let $\chi = \prod_{P \in R^+_F} \chi_P$ with $\chi_P \in X_P$ and $\chi \neq 1, \chi^l = 1$.

In particular there exists $P \in R^+_F$ with $\text{ord}(\chi_P) = l$. Let $\varphi \in X$ with $\varphi_P = \chi_P$. Now if $l | [L : k]$, $\gcd([L : k], l) = 1$. Let $m = \text{ord}(\varphi)$. Then $l | m$ and $\varphi^m = \prod_{P \in R^+_F} \varphi_P^m = 1$. Hence $\varphi_P^m = \chi_P^m = 1$ and $\chi_P^l = 1$. It follows that $\chi_P = 1$. This contradicts that $\text{ord}(\chi_P) = l$. It follows that $l|[L : k]$.

For the general case, with the notation as above, we have $L_{ge} = E_{ge} H L$. If $l|[L_{ge} : k]$ then $l|[E_{ge} H : k]$ or $l|[L : k]$. If $l|[E_{ge} H : k]$ then $l|[E_{ge} : k]$ so, from the cyclotomic case, we obtain that $l|[E : k]| [L : k]$. Hence $l|[L : k]$. □

**Corollary 4.3.** When $L$ is cyclotomic, we have $l|[L : k]$ if and only if $l|[L_{ge} : k]$.

**Proof.** It follows immediately from the proof of Proposition 4.2. □

**Lemma 4.4.** Let $L_i, i = 1, 2$ be two cyclotomic fields. Then, if $\gcd([L_1 : k], [L_2 : k]) = 1$, we have $(L_1 L_2)^+ = L_1^+ L_2^+$.

**Proof.** Since $L_i^+ \subseteq L_i, i = 1, 2$, it follows that $\gcd([L_1^+ : k], [L_2^+ : k]) = 1$. □
We have that $e_i := e_\infty(L_i | k) = [L_i : L_1^+]$, $i = 1, 2$. Therefore $\gcd(e_1, e_2) = 1$. It follows that $e_\infty(L_1/L_2 | \bigcup_{i=1}^2 L_i^+ | L_2) = e_1 e_2$ and $[L_1 L_2 : L_1^+ L_2^+] = e_1 e_2$.

That is, $(L_1 L_2)^+ / L_1 + L_2^+$ is the maximal subfield of $L_1 L_2$ where $\mathfrak{p}_\infty$ decomposes totally in $(L_1 L_2)^+/k$ and $L_1 L_2/(L_1 L_2)^+$ is totally ramified at $p_\infty$. It follows that $e_\infty(L_1 L_2 | (L_1 L_2)^+ e_1 e_2 = e_\infty(L_1 L_2 | L_1^+ L_2^+)$. Thus $(L_1 L_2)^+ \subseteq L_1^+ L_2^+ \subseteq (L_1 L_2)^+$ and $(L_1 L_2)^+ = L_1^+ L_2^+$.

**Remark 4.5.** If $L_1$ and $L_2$ are cyclotomic it is possible that $\gcd([L_1^+ : k], [L_2^+ : k]) = 1$ but $\gcd([L_1 : k], [L_2 : k]) \neq 1$.

For instance, let $P, Q \in \mathbb{F}_1^\infty$ be distinct of degree 1. Then $[k(\Lambda_P) : k] = q - 1 = [k(\Lambda_Q) : k]$ and $k(\Lambda_P)^+ = k(\Lambda_Q)^+ = k$. Thus, if $L_1 = k(\Lambda_P)$ and $L_2 = k(\Lambda_Q)$ we have $\gcd([L_1^+ : k], [L_2^+ : k]) = \gcd(1, 1) = 1$ but $\gcd([L_1 : k], [L_2 : k]) = \gcd(q - 1, q - 1) = q - 1 > 1$ for $q > 2$.

In the cyclotomic case, the possibility $(L_1)_{\text{ger}}(L_2)_{\text{ger}} \subseteq (L_1 L_2)_{\text{ger}}$ is related to the fact that it may happen that $(L_1)_{\text{ger}}^+(L_2)_{\text{ger}} \subseteq (L_1 L_2)_{\text{ger}}$. We will show that in our case we have equality.

**Corollary 4.6.** If $\gcd([L_1 : k], [L_2 : k]) = 1$ with $L_i \subseteq k(\Lambda_{N_i})$, $i = 1, 2$, then $(L_1 L_2)_{\text{ger}} = (L_1)_{\text{ger}}(L_2)_{\text{ger}}$.

**Proof.** From Corollary 4.3 we obtain that $\gcd([L_1)_{\text{ger}} : k], [(L_2)_{\text{ger}} : k]) = 1$. Now we have $(L_1)_{\text{ger}} = (L_1)_{\text{ger}}^+ L_1$ and $(L_2)_{\text{ger}} = (L_2)_{\text{ger}}^+ L_2$. Thus

$$(L_1)_{\text{ger}}(L_2)_{\text{ger}} = (L_1)_{\text{ger}}^+ L_1 (L_2)_{\text{ger}}^+ L_2 = (L_1)_{\text{ger}}^+ (L_2)_{\text{ger}}^+ L_1 L_2$$

$$= ((L_1)_{\text{ger}} L_2)_{\text{ger}}^+(L_1 L_2) = ((L_1 L_2)_{\text{ger}}^+(L_1 L_2) = (L_1 L_2)_{\text{ger}}$.

The proof of the following result is straightforward.

**Proposition 4.7.** Let $A, B$ and $C$ be global function fields such that $B/A$ and $C/A$ are finite Galois extensions with $\gcd([B : A], [C : A]) = 1$. Then $B \cap C = A$. Let $D = BC$.

If $p_A$ is a prime divisor of $A$ and $p_B, p_C, p_D$ satisfy $p_B \cap A = p_C \cap A = p_D \cap A = p_A$, then

$$e_{p_A}(B|A) = e_{p_C}(D|C), \quad f_{p_A}(B|A) = f_{p_C}(D|C), \quad h_{p_A}(B|A) = h_{p_C}(D|C),$$

$$e_{p_A}(C|A) = e_{p_D}(D|B), \quad f_{p_A}(C|A) = f_{p_D}(D|B), \quad h_{p_A}(C|A) = h_{p_D}(D|B).$$
where e, f and h denote the ramification index, the inertia degree and the decomposition degree respectively.

Proof of Theorem 4.1. Let $L_i/k$ be two finite abelian extensions, $i = 1, 2$ such that $\gcd([L_1 : k], [L_2 : k]) = 1$. Let $E_i = L_i \cap k(\Lambda_N)$, $i = 1, 2$ where $L_i \subseteq k(\Lambda_N)_m$, $i = 1, 2$. Let $L = L_1 L_2$ and $E = E_1 E_2$. Then $E = LM \cap k(\Lambda_N)$.

Now $(L_i)_{ge} = L_i \cap (E_i)_{ge}$, $i = 1, 2$ with $|H_i| = f_{\infty}(L_i|E_i|L_i)$, $i = 1, 2$ and $L_{ge} = LE_{ge}$ such that $|H| = f_{\infty}(LE|L)$.

We have $[E : k] = [LM : M]$ and $[LM : M] = [L : L \cap M]|L : k|$. Therefore $[E : k]|L : k|$. Analogously we have $[E_i : k]|[L_i : k]$, $i = 1, 2$.

It follows that $\gcd([E_1 : k], [E_2 : k]) = 1$. From Proposition 4.7 we obtain that $|H| = |H_1||H_2|$. Then from Corollary 4.6 we obtain that

$$[E_{ge} : E_{ge}^H] = |H| = |H_1||H_2| = [(E_1)_{ge} : (E_1)^{H_1}_{ge}][E_2)_{ge} : (E_2)^{H_2}_{ge}]$$

and we have, with $a = |H_1|$, $b = |H_2|$ and $ab = |H_1||H_2| = |H|$, that

$$\gcd(a, b) = 1$$

It follows that

$$[(E_1 E_2)_{ge} : (E_1)^{H_1}_{ge} (E_2)^{H_2}_{ge}] = [(E_1)_{ge} : (E_1)^{H_1}_{ge}][E_2)_{ge} : (E_2)^{H_2}_{ge}] = |H_1||H_2| = |H|$$

since $(E_1 E_2)_{ge} = (E_1)_{ge} (E_2)_{ge}$. On the other hand, we have that $[(E_1 E_2)_{ge} : (E_1 E_2)^H_{ge}] = |H|$. Since $(E_1)^{H_1}_{ge} (E_2)^{H_2}_{ge} \subseteq (E)^{H}_{ge}$ we obtain that $(E_1)_{ge}^{H_1} (E_2)_{ge}^{H_2} = (E_1 E_2)^H_{ge}$. Therefore

$$(L_1 L_2)_{ge} = (E_1 E_2)^{H}_{ge} (L_1 L_2) = (E_1)^{H_1}_{ge} (E_2)_{ge}^{H_2} L_1 L_2$$

As a consequence, we obtain the main result of the paper.
Theorem 4.8. Let $K/k$ be a finite Kummer extension of order $n = l_1^{m_1} \cdots l_s^{m_s}$ with $l_1, \ldots, l_s$ different primes. Let $K = K_1 \cdots K_s$ with $[K_j : k] = l_j^{m_j}$, $1 \leq j \leq s$. Then

$$K_{ge} = \prod_{j=1}^{s} (K_j)_{ge}$$

where each $(K_j)_{ge}$, $1 \leq j \leq s$ is computed in Theorems 3.4 and 3.6.

Proof. From Theorem 4.1 we have $K_{ge} = (K_1)_{ge} \cdots (K_s)_{ge}$. The explicit description of each $(K_j)_{ge}$ is the content of Theorems 3.4 and 3.6. $\blacksquare$

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