JOINT MOMENTS OF DERIVATIVES OF CHARACTERISTIC POLYNOMIALS

PAUL-OLIVIER DEHAYE

Pour Annie & Jean-Paul

Abstract. We investigate the joint moments of the $2k$-th power of the characteristic polynomial of random unitary matrices with the $2h$-th power of the derivative of this same polynomial. We prove that for a fixed $h$, the moments are given by rational functions of $k$, up to a well-known factor that already arises when $h = 0$.

We fully describe the denominator in those rational functions (this had already been done by Hughes experimentally), and define the numerators through various formulas, mostly sums over partitions.

We also use this to formulate conjectures on joint moments of the zeta function and its derivatives, or even the same questions for the Hardy function, if we use a “real” version of characteristic polynomials.

Our methods should easily be applicable to other similar problems, for instance with higher derivatives of characteristic polynomials.

More data is available online, either on the author’s web site or attached to the LaTeX source of this arXiv submission.

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1. INTRODUCTION

In Section 1.1, we merely define what is meant by joint moments of characteristic polynomials and state the results obtained in this paper. In Section 1.2, we motivate these Random Matrix Theory results by Number Theory questions and explain the interest of joint moments in the context of the Riemann $\zeta$-function. In Section 1.3, we discuss our techniques, which are essentially in Representation Theory and Algebraic Combinatorics. The organization of this paper is summarized in Section 1.4.

1.1. Presentation of results. We take for the characteristic polynomial of a $N \times N$ unitary matrix $U$

$$Z_U(\theta) := \prod_{j=1}^{N} \left( 1 - e^{i(\theta_j - \theta)} \right),$$

where the $\theta_j$s are the eigenangles of $U$.

We define

$$V_U(\theta) := e^{iN(\theta+\pi)/2} e^{-\frac{1}{2} \sum_{j=1}^{N} \theta_j} Z_U(\theta).$$

It is easily checked that for real $\theta$, $V_U(\theta)$ is real and $|V_U(\theta)| = |Z_U(\theta)|$.

In this paper, we will investigate the averages (with respect to Haar measure)

$$|M|_{N}(2k,r) := \left< |Z_U(0)|^{2k} \left( \frac{Z_U'(0)}{Z_U(0)} \right)^{r} \right>_{U(N)},$$

$$(M)_{N}(2k,r) := \left< |Z_U(0)|^{2k} \left( \frac{Z_U'(0)}{Z_U(0)} \right)^{r} \right>_{U(N)},$$

$$|V|_{N}(2k,r) := \left< |V_U(0)|^{2k} \left( \frac{V_U'(0)}{V_U(0)} \right)^{r} \right>_{U(N)}$$

and their asymptotics

$$|M|(2k,r) := \lim_{N \to \infty} \frac{|M|_{N}(2k,r)}{N^{2k+r}},$$

$$(M)(2k,r) := \lim_{N \to \infty} \frac{(M)_{N}(2k,r)}{N^{2k+r}},$$

$$|V|(2k,r) := \lim_{N \to \infty} \frac{|V|_{N}(2k,r)}{N^{2k+r}}.$$
Theorem 1. For $r \in \mathbb{N}$ and $k \in \mathbb{C}$, the moments $(\mathcal{M})(2k, r)$ are essentially given by rational functions, i.e. as meromorphic functions of $k$ we have

$$\mathcal{M}(2k, r) = \left(-\frac{i}{2}\right)^{r} \frac{G(k + 1)^2 X_r(2k)}{G(2k + 1)} Y_r(2k),$$

where $X_r$ and $Y_r$ are even monic polynomials with integer coefficients and with $\deg X_r = \deg Y_r$ and $G$ is the Barnes $G$-function [HKO00, Appendix].

Moreover

$$Y_r(u) = \prod_{1 \leq a \leq r-1 \text{ odd}} (u^2 - a^2)^{\alpha_a(r)},$$

with the $\alpha_a(\cdot)$ given by

$$\alpha_a(r) = \left\lfloor -a + \sqrt{a^2 + 4r^2} \right\rfloor.$$

We derive from this a similar result (Theorem 16, page 22) for $|\mathcal{M}|(2k, 2h)$ and $|\mathcal{V}|(2k, 2h)$ (for $h$ an integer). Finally, we have explicit expressions for $(\mathcal{M})(2k, r)$ given in Theorem 15, page 20 and Theorem 18, page 30 which allow us to compute the $X_r(u)$s, as given in Table 2, page 23, and additional data (available in Section 7).

This structure had been guessed a few years ago by Chris Hughes based on computational evidence [Hug05]. The author is deeply thankful to him for freely sharing and explaining all of his previous unpublished work.

This paper was initiated at Stanford University\footnote{with support from FRG DMS-0354662} while the author was finishing his Ph.D., mostly worked on at Merton College, University of Oxford and finalized at Institut des Hautes Etudes Scientifiques, Bures-sur-Yvette. The author wishes to thank his hosting institutions for their support as well as his Ph.D. adviser, Dan Bump, and Persi Diaconis, Masatoshi Noumi and Peter Neumann for helpful discussions.

1.2. Motivation. Ever since the works by Keating and Snaith [KS00b, KS00a], the Riemann $\zeta$-function can be (conjecturally but quantitatively) better understood through modelling by characteristic polynomials of unitary matrices. The classical example concerns moments. Let

$$g(k) := \frac{G(k + 1)^2}{G(2k + 1)},$$

$$a(k) := \prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right) \sum_{m=0}^{\infty} \left(\frac{\Gamma(m + k)}{m! \Gamma(k)}\right)^2 p^{-m}.$$

Then one can prove (fairly immediately, using the Selberg integral) that

$$|\mathcal{M}|(2k, 0) = g(k),$$

which according to the Keating-Snaith philosophy leads to the following conjecture (for $k > -1/2$):

$$\frac{1}{T} \int_{0}^{T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} dt \sim_T g(k)a(k) \left(\log \frac{T}{2\pi}\right)^{k^2}.$$

The main point is thus that $a(k)$ is obtained by looking at primes, while $g(k)$ is guessed at from the Random Matrix side.

Observe also that Equations (3) and then (4) can be analytically continued in $k$.\footnote{with support from FRG DMS-0354662}
Many of the authors cited above have now shown that this philosophy should be extended to derivatives of characteristic polynomials.

In particular, \(|\mathcal{M}|(2k, r)\) should show up as the RMT factor of\(^2\)

\[
I(2k, r) := \lim_{T \to \infty} \frac{1}{T \log \frac{T}{2\pi}} \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k-r} \left| \zeta'\left(\frac{1}{2} + it\right) \right|^r dt.
\]

and similarly \(|\mathcal{V}|(2k, r)\) is needed for

\[
J(2k, r) := \lim_{T \to \infty} \frac{1}{T \log \frac{T}{2\pi}} \int_0^T \left| \frac{1}{2} + it \right|^{-2k-r} \left| \zeta\left(\frac{1}{2} + it\right) \right| \left| \zeta'\left(\frac{1}{2} + it\right) \right|^r dt,
\]

where \(\mathfrak{S}\) is Hardy’s function (the relationship of \(\mathfrak{S}\) to \(\zeta\) is analogous to the relationship of \(\zeta\) to \(\mathfrak{Z}\), i.e. when \(t \in \mathbb{R}\), \(\mathfrak{S}(\frac{1}{2} + it) \in \mathbb{R}\) and \(\pm \mathfrak{S}(\frac{1}{2} + it) = |\zeta(\frac{1}{2} + it)|\). More precisely, it is expected that

\[
I(2k, r) = a(k) \left| \mathcal{M}\right|(2k, r) \quad \text{and} \quad J(2k, r) = a(k) \left| \mathcal{V}\right|(2k, r).
\]

Thus, Theorems 1 and 16 give us a conjectural handle on the moments of \(\zeta\) and \(\mathfrak{S}\).

One can compute some small cases (for integer \(k\) and \(r\)) and show that they agree with previous Number Theory (proved) results. This had already been done before and is repeated in Table 1.

However, while Keating and Snaith obtained a full conjecture for \(I(2k, 0)\) and \(J(2k, 0)\) by computing \(|\mathcal{M}|(2k, 0)\) and \(|\mathcal{V}|(2k, 0)\), for the case of joint moments this goal remains elusive. All the available formulas for \(|\mathcal{M}|(2k, r)\) or \(|\mathcal{V}|(2k, r)\) are rather inadequate. In particular, those formulas are limited to \(r := 2h\) (\(h\) an

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\(^2\)It is a conjecture of Hall [Hal04] and Hughes [Hug01] that this is the appropriate normalization with respect to \(T\).
integer), they are hard to compute for large values of \( k \) and \( h \), they obscure some of the structure in the results, and finally they cannot be analytically continued in \( h \).

The analytic continuation would be important, because Conrey and Ghosh have proved in [CG89] that (on RH)

\[
J(2, 1) = \frac{e^2 - 5}{4\pi}
\]

and hence effectively conjectured\(^3\)

\[
|V|(2, 1) = \frac{e^2 - 5}{4\pi}
\]

as well since \( a(1) = 1 \). In order to get this, we would need to have a sufficiently nice formula for \( |V|(2k, 2h) \) that would allow for analytic continuation in \( h \). We have simply been unable to do this but have no doubt that our results should be helpful for that goal (see the connection with Noumi’s work below).

On the other hand, the formulas obtained in Theorem 15, page 20 allow for much more effective computation than possible before, and we can compute longer tables for the different moments (see Section 7).

This numerical data is useful as well, as Hall has devised (around 2002) a method that uses \( J(2k, 2h) \) for all \( 0 \leq h \leq k \) to produce a lower bound \( \Lambda(k) \) on

\[
\Lambda := \limsup_{n \to \infty} \frac{t_{n+1} - t_n}{\log t_n},
\]

where the \( t_n \) is the \( n \)-th positive real zero of \( \zeta(1/2 + it) \). It is probably good to insist that this method does not depend on the Riemann Hypothesis, but only on values for moments! At the time of writing [Hal04], Hall only had the information he needed for \( k \) up to 2 (conjecturally, up to 6). In Section 7, we present our conjectural data for \( J(2k, 2h) \) as a direct function of \( k \) for \( h \) up to 15 (see [Deh07] or the source of this arXiv submission for data up to \( h = 30 \)). For a fixed \( h \), various conjectural formulas are also given in this paper for \( J(2k, 2h) \) as a function of \( k \). This, combined with Hall’s method, should lead to more (conjectural) lower bounds on \( \Lambda \). It is widely believed that \( \Lambda = \infty \) so potentially we could also see if Hall’s method has any hope to reach that, assuming only information on the \( J(2k, 2h) \), but not on the Riemann Hypothesis. In other words, it would also inform us on the relationship between moment conjectures, the Riemann Hypothesis and the conjecture \( \Lambda = \infty \). We leave this to a further paper.

Finally, Noumi in his book [Nou04] investigates the relationship between Painlevé equations and expressions similar to one of the expressions we obtain for \( (M)(2k, r) \), in Theorem 18. Connections of this sort have been uncovered before (see [FW06a, FW06b] and works of Borodin), but an approach through Noumi’s ideas would be original. One of our goals then would be to obtain analytic continuation for \( (M)(2k, r) \) in \( r \), which would again allow to compute \( |V|(2, 1) \). We also leave this for further study.

Our techniques are quite disconnected from the original motivation, so we discuss them separately.

1.3. Techniques. As mentioned earlier, our techniques lie mostly in Representation Theory and Algebraic Combinatorics. We look at the characteristic polynomials or the derivatives as symmetric functions of the eigenvalues of \( U \), and express them in that way. We eventually express those symmetric functions in the most natural basis to use, the Schur functions. This basis is particularly suitable since those functions are also (irreducible) characters of unitary groups \( U(N) \). We find

\[^3\]This is completely backwards from the usual flow of conjectures from Random Matrix Theory to Number Theory, and possibly an unique instance of a reversal of this type.
ourselves integrating irreducible characters over their support (groups), which is very enviable!

In order to express all the different functions in this basis of Schur functions, we use ideas present in a paper of Bump and Gamburd [BG06] and the author’s thesis [Deh06b]. We will introduce those ideas as we need them.

For a more thorough discussion of why a similar approach should always be attempted and other examples of its applications, please see the author’s thesis and the results in [Deh06a].

Once we have a concise expression for the various moments, we still have to evaluate it. This will involve sums over partitions of values of Schur functions. After reparametrizing those sums over the Frobenius coordinates of the partitions, results of El-Samra and King were immediately useful to obtain the Schur values, and results of Borodin to handle the combinatorics of the sums. We then obtain a very big sum for the moments (Theorem 13), but that can directly be evaluated on computer (and thus checked against small $N$ results). After taking asymptotics, our results start simplifying into Theorem 15, enough to prove Theorem 1 on the general shape of those moments. However, the best expression is probably obtained once we use Macdonald’s ninth variation of the Schur functions (Theorem 18).

### 1.4. Organization of this paper.

- In Section 2, we introduce all the non-standard notation we will be using.
- In Section 3, we present the basic relations satisfied by the integrands $|Z_U(0)|^{2k} \overline{Z_{V}(0)}^{rk}$, $|Z_U(0)|^{2k} \overline{Z_{V}(0)}^{rk}$ and $|V_U(0)|^{2k} \overline{V_{V}(0)}^{rk}$.
- In Section 4, we re-express the integrands as a sum in the Schur basis, in a way similar to Bump and Gamburd (via the Dual Cauchy Identity).
- In Section 5, we engage in a long computation to evaluate the result obtained in the previous section, mostly using results of El-Samra and King, and Borodin.
- Section 6 merely serves to tie what has been done in Sections 4 and 5 into the proof of Theorem 1.
- In Section 7 we present the data we are now able to compute, and particularly discuss the position of the roots of $|V|(2k; 2h)$ in Section 7.2.
- Section 8 describes two attempts to simplify our results further, one using Macdonald’s ninth variation of the Schur functions, and the second imitating a proof of the Cauchy identity.

The bulk of this paper is contained in Sections 4 and 5.

### 2. Notation

We let $\mathbb{N}_+$ be the set $\mathbb{N}\setminus 0$. To avoid confusion with the index $i$, we have $i^2 = -1$.

We use $\mathbf{v}$ for a generic vector (of integers) $(v_1, \cdots, v_d)$, and $\mathbf{v}$ for a sorted sequence of strictly decreasing integers $v_1 > v_2 > \cdots > v_d$, which we call a Frobenius sequence. Frobenius sequences are thus a special type of vectors.

Sequences of weakly decreasing positive integers amount to partitions, and we stick with classical notation for those, i.e. $\lambda = (\lambda_1, \cdots, \lambda_l(\lambda))$, which defines $l(\lambda)$. We also freely change our point of view to Young tableaux when discussing partitions.

Given a partition $\lambda$ of $|\lambda|$, we denote its conjugate by $\lambda'$. Define two sequences $p_i := \lambda_i - i$, $q_i := \lambda'_i - i$. They are strictly decreasing; $\lambda_i$ and $\lambda'_i$ are eventually 0, and hence $p_i = -i$ and $q_i = -i$ eventually. There exists $d$ such that $p_d \geq 0 > p_{d+1}$ and $q_d \geq 0 > q_{d+1}$. We call $d$ the rank of $\lambda$. We have that $\mathbf{p} = (p_1, \cdots, p_d)$ and $\mathbf{q} = (q_1, \cdots, q_d)$ are Frobenius sequences, and we call $\mathbf{p}$ and $\mathbf{q}$ the Frobenius coordinates of the partition $\lambda$. We write $\lambda = \{ \mathbf{p} \over \mathbf{q} \}$. 

Given \( \mathbf{p} \), we define \( \sigma_{\mathbf{p}} \in S_d \) such that \( \text{sort}(\mathbf{p}) := (p_{\sigma_p(i)}) \) is strictly decreasing (and hence a Frobenius sequence). This is thus not defined if \( p_i = p_j \) while \( i \neq j \). We set \( \text{sgn}(\mathbf{p}) := \text{sgn}(\sigma_{\mathbf{p}}) \), with the added convention that \( \text{sgn}(\mathbf{p}) := 0 \) if \( \sigma_{\mathbf{p}} \) is not defined.

If \( \lambda \) and \( \mu \) are partitions, \( \lambda \cup \mu \) is the partition obtained by taking the union of their parts. The partition \( \langle X^Y \rangle \) has a \( Y \times X \) rectangle for Young tableau.

We also use the notation \([1^R]\) for \( R \) copies of 1, used as argument to a (Schur) function.

3. Basic relations among the integrands

We logarithmically differentiate Equation (1) to obtain

\[
\frac{V_U'(\theta)}{V_U(\theta)} = \frac{iN}{2} + \frac{Z_U'(\theta)}{Z_U(\theta)}
\]

and hence, when \( \theta \) is real,

\[
\left| \frac{Z_U'(\theta)}{Z_U(\theta)} \right|^2 = \left| \frac{V_U'(\theta)}{V_U(\theta)} \right|^2 + \frac{N^2}{4},
\]

\[
= \left( \frac{V_U'(\theta)}{V_U(\theta)} \right)^2 + \frac{N^2}{4},
\]

\[
= \left( \frac{Z_U'(\theta)}{Z_U(\theta)} \right)^2 + iN \left( \frac{Z_U'(\theta)}{Z_U(\theta)} \right).
\]

These basic relations give

\[
|M|_{N}(2k, 2h) = \sum_{j=0}^{h} (iN)^{h-j} \binom{h}{j} (M)_{N}(2k, h + j),
\]

\[
|M|_{N}(2k, 2h) = \sum_{j=0}^{h} i^{h-j} \binom{h}{j} (M)(2k, h + j),
\]

\[
|V|_{N}(2k, 2h) = \sum_{j=0}^{h} \binom{h}{j} \left( -\frac{N^2}{4} \right)^{h-j} |M|_{N}(2k, 2j),
\]

\[
|V|_{N}(2k, 2h) = \sum_{j=0}^{h} \binom{h}{j} \left( -\frac{1}{4} \right)^{h-j} |M|(2k, 2j).
\]

These formulas are initially valid only when \( h \) is a non-negative integer, but the RHSs can be analytically continued by plugging in non-integer \( h \) and extending the sum to infinity\(^4\). We see thus that computing \( (M)_{N}(2k, r) \) would get us most of the way to \( |M|_{N}(2k, 2h) \) or \( |V|_{N}(2k, 2h) \), and we now focus on the integrand \( |Z_U(0)|^{2h} \left( \frac{Z_U'(0)}{Z_U(0)} \right)^r \).

\(^4\)Getting the correct analytic continuation can be tricky: The relation

\[
|V|_{N}(2k, 2h) = \sum_{j=0}^{2h} \binom{2h}{j} \left( \frac{1}{2} \right)^{j} (M)(2k, 2h - j)
\]

is also valid for integers \( h \) but here the RHS does not analytically continue in \( h \) to the LHS, since we exploit \( \left| \frac{V_U'(\theta)}{V_U(\theta)} \right|^{2h} = \left( \frac{V_U'(\theta)}{V_U(\theta)} \right)^{2h} \) where it is critical that \( h \) be an integer.
4. Derivation into the Schur basis

The goal here is to follow ideas similar to Bump and Gamburd’s [BG06] in order to prove Proposition 4, page 9. One of their main tools was the Dual Cauchy identity. We encourage the reader to look at their first Proposition and Corollary for the unitary group, since this is all we really exploit from that paper.

**Lemma 2** (Dual Cauchy identity). If \( \{x_i\} \) and \( \{y_j\} \) are finite sets of variables,

\[
\prod_i \prod_j (1 + x_i y_j) = \sum_\lambda s_\lambda'(x_i) s_\lambda(y_j),
\]

where the sum is over all partitions \( \lambda \) and \( s_\lambda \) is the Schur polynomial.

We (they) apply this Lemma setting \( \{x_j := e^{i \theta_j} : j \in [1, \cdots, N]\} \) to be the set of eigenvalues of \( U \), and \( \{y_j := j : j \in [1, \cdots, 2k]\} \). We chose the notation \( s_\lambda(U) := s_\lambda(e^{i \theta_1}, \cdots, e^{i \theta_N}) \). This gives

\[
\sum_\lambda s_\lambda'(U) s_\lambda(\{1^{2k}\}) = \det(\text{Id} + U)^{2k} = \frac{\det(U)}{s_{(k \cdot 1)}(U)} |\det(\text{Id} + U)|^{2k}
\]

or (replacing \( U \) by \( -U \))

\[
|Z_U(0)|^{2k} = |\det(\text{Id} - U)|^{2k} = (-1)^{kN} s_{(k \cdot 1)}(U) \sum_\lambda (-1)^{|\lambda|} s_{\lambda'}(U) s_\lambda(\{1^{2k}\}).
\]

We can also re-express

\[
\frac{Z_U(0)}{Z_U(0)} = \sum_{j=1}^{N} \frac{i e^{i \theta_j}}{1 - e^{i \theta_j}} = \sum_{j=1}^{N} i \lim_{z \to 1} \sum_{m=1}^{\infty} z^m e^{im \theta_j}
\]

\[
= i \lim_{z \to 1} \sum_{m=1}^{\infty} z^m p_m(U),
\]

where \( p_m(x_1, \cdots, x_N) \) is the \( m \)-th power sum \( x_1^m + \cdots + x_N^m \) and we have used the same convention as for \( s_\lambda(U) \) of inputting the eigenvalues. We will use the same convention soon for the power sums \( p_\lambda := \prod_i p_{\lambda_i} \).

In practice, we want the reader to just ignore the variable \( z \) and set it to 1. This will be justified a posteriori.

Putting everything together, we thus get for \( |Z_U(0)|^{2k} \left( \frac{Z_U'(0)}{Z_U(0)} \right)^r \)

\[
(-1)^{kN} s_{(k \cdot 1)}(U) \left( i \sum_{m=1}^{\infty} p_m(U) \right)^r \sum_\lambda (-1)^{|\lambda|} s_{\lambda'}(\{1^{2k}\}) s_\lambda(U).
\]

At this point, we will soon want to use the fact that the \( s_\lambda \) are characters of unitary groups.

Indeed, if \( U \in U(N) \) then when \( l(\lambda) > N \), we have\(^5\) \( s_\lambda(U) \equiv 0 \), but when \( l(\lambda), l(\mu) \leq N \), we have

\[
\left< s_\lambda(U), s_\mu(U) \right>_{U(N)} = \delta_{\lambda\mu},
\]

i.e. for large enough \( N \), \( s_\lambda \) is an irreducible character of \( U(N) \). This orthogonality is obviously good for our purposes, but the only obstacle is the need to express

\text{\(^5\)This is a consequence of the fact that \( s_\lambda(x_1, \cdots, x_n) \equiv 0 \) if \( l(\lambda) > n \).}
-contained in version of the M-N rule says that

\[ s_{(k^N)}(U) \left( \sum_{m=1}^{\infty} p_m(U) \right)^r \]

exclusively in terms of Schur functions. This can be done and will require the Murnaghan-Nakayama rule.

Let a ribbon be a connected Young skew-tableau not containing any 2 \times 2-block. If a ribbon contains \( m \) blocks, it is called a \( m \)-ribbon. A first approximation to one version of the M-N rule says that \( s_{\lambda} p_m \) is given by a signed sum of \( s_{\mu} \), where \( \mu \) runs through all partitions obtained by adding a \( m \)-ribbon to \( \lambda \).

If we average Expression (12) over \( U(N) \), we could thus see \( \lambda \) as running through all partitions obtained by adding \( r \) ribbons to the rectangle \( \langle N^k \rangle \) (this uses the fact that this lax version of the M-N rule is invariant under transpositions, since we have yet to discuss the signs). There are more conditions however. We also need \( l(\lambda') \leq N \) (since otherwise \( s_{\lambda}(U) \equiv 0 \), as in footnote 5), and we need \( l(\lambda) \leq 2k \) (since otherwise \( s_{\lambda}(\{1^{2k}\}) = 0 \), again just as in footnote 5). In other words, \( \lambda \) contains \( \langle N^k \rangle \) but is contained in \( \langle N^{2k} \rangle \). There are only finitely many (ways to obtain) such partitions, which will make the sum over \( \lambda s \) finite, and thus only finitely many sets of lengths of the \( r \) ribbons will contribute. This justifies a posteriori setting \( z \) to 1 in Equation (11), but only when we can apply the dominated convergence theorem. This will only occur if we know of a bound on the integrand independent of \( z \) that is itself integrable. We can pick \( |Z_U(0)|^{2k} \frac{Z_U(t)(0)}{z^{2k}(0)} \) wherever this is integrable, i.e. only when \( 2k - r > -1 \).

We now state a more precise version of the M-N rule.

**Theorem 3** (Murnaghan-Nakayama). Let \( \lambda \) be a partition and \( \overline{\rho} \) be a vector with \( |\lambda| = \sum \rho_i \). If \( \chi_{\overline{\rho}}^\lambda \) is the value of the irreducible character of \( S_{|\lambda|} \) associated to \( \lambda \) on the conjugacy class of cycle-type sort(\( \overline{\rho} \)), then

\[
\rho_{\overline{\rho}} = \sum_{\lambda} \chi_{\overline{\rho}}^\lambda S_{|\lambda|}
\]

and (more importantly)

\[
\chi_{\overline{\rho}}^\lambda = \sum_S (-1)^{ht(S)}
\]

summed over all sequences of partitions \( S = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(r)}) \) such that \( r := l(\lambda), 0 = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(r)} = \lambda \), and such that each \( \lambda^{(i)} - \lambda^{(i-1)} \) is a ribbon of length \( \rho_i \), and \( ht(S) = \sum_i ht(\lambda^{(i)} - \lambda^{(i-1)}) \).

We have not defined the height \( ht \) of a ribbon, but rather than doing so or detailing the computation here, we only expose the idea: Equation (13) tells us that \((\sum \rho_{\overline{\rho}})^r\) can be computed using the character values of symmetric groups, which can be evaluated by summing over sequences of partitions \( (\lambda^{(0)}, \ldots, \lambda^{(r)}) \). For each such sequence, the sequence \( (\lambda^{(0)}, \ldots, \lambda^{(r)}) \), with \( \lambda^{(i)} := \langle N^k \rangle \cup \lambda^{(i)} \), would be associated with the combinatorics of the expansion of the product in Expression (12). Indeed the combinatorics of ribbon is unchanged under translations (down by \( k \)) as long as the partitions are kept within a rectangle (actually, a horizontally bounded region).

If the computation is explicitly carried out, we get the following result:

**Proposition 4.** If \( 2k - r > -1 \), we have

\[
\langle M \rangle_N(2k, r) = (-i)^r \sum_{\overline{\rho} \in \mathbb{N}^r} \sum_{\lambda \text{ within } k \times N} \chi_{\overline{\rho}}^\lambda s_{\langle N^k \rangle \cup \lambda}(\{1^{2k}\}),
\]

with the understanding that \( \chi_{\overline{\rho}}^\lambda = 0 \) if \( |\lambda| \not\equiv \sum_i \mu_i \).

For this result, we have preferred to index all the partitions containing \( \langle N^k \rangle \) but contained in \( \langle N^{2k} \rangle \) as \( \langle N^k \rangle \cup \lambda \), for \( \lambda \subset \langle N^k \rangle \).
We are now left with the task of evaluating the RHS in Equation (15), which will turn out to be a tedious process.

5. Main Computation

We are left with two problems. The first one is due to the characters of the symmetric group. Those are of course desperately hard to evaluate directly and individually. We are helped here because we will actually only evaluate something close to

\[ \sum_{\pi \in \mathcal{S}_n} \chi_{\lambda, \pi} \]

for given \( \lambda \). This amounts to computing the sum of values of the character \( \chi_{\lambda} \) over permutations with \( l \) cycles. The second issue is evaluating \( s_{\langle N^k \rangle, \lambda} \left( \left[ 1^{2k} \right] \right) \). The author had previously used the Weyl Dimension Formula to do this (see [Deh06b]). A formula giving that dimension in terms of the Frobenius coordinates of \( \lambda \) is probably better adapted for our purposes.

In addition, both “problems” combine extremely well, in that both expressions should involve a sign, which turns out to be the same.

We will then sum our terms over all partitions, expressed in Frobenius coordinates. This amounts to summing over possible ranks \( 1 \leq d \) and then pairs of Frobenius sequences of length \( d \).

5.1. The value of the Schur function in Frobenius coordinates.

5.1.1. Dimension formula in Frobenius coordinates. El-Samra and King [ESK79] use the notation \( D_R \{ \frac{p}{q} \} \) for \( s_{\langle p \rangle} \left( \left[ 1^R \right] \right) \).

Assume \( \{ \frac{p}{q} \} \) has \( d \) Frobenius coordinates. They prove that

\[
\begin{align*}
    s_{\{ \frac{p}{q} \}} \left( \left[ 1^R \right] \right) &= \frac{(R + p_i)!}{(R - q_i - 1)!p_i!q_i!(p_i + q_i + 1)} \left| \begin{array}{l}
        1 \\
        p_i + q_i + 1
      \end{array} \right|_{d \times d}
    \\
    &= \prod_{i=1}^{d} \frac{(R + p_i)!}{(R - q_i - 1)!p_i!q_i!} \prod_{1 \leq i < j \leq d} (p_i - p_j) (q_i - q_j) \prod_{i,j=1}^{d} \frac{1}{p_i + q_j + 1}
\end{align*}
\]

where the first expression is also known as the reduced determinantal form (cf. Foulkes [Fou51], as cited in [ESK79]).

It is a consequence of Cauchy’s Lemma that the two expressions in Formula (16) are equivalent:

**Lemma 5** (Cauchy).

\[
\begin{align*}
    \left| \frac{1}{p_i + q_j + 1} \right|_{d \times d} &= \prod_{1 \leq i < j \leq d} (p_i - p_j) (q_i - q_j) \prod_{i,j=1}^{d} \frac{1}{p_i + q_j + 1}
\end{align*}
\]

Observe that Formula (16) is positive (as it should, given that it is also a dimension) because the \( p_i \) and \( q_i \) are strictly decreasing.

However, the RHS of Formula (16) still makes sense if we plug in unsorted vectors \( \{ \frac{p}{q} \} \) (with even the possibility of \( i \neq j \) but \( p_i = p_j \)). Hence this can be used to define \( s_{\{ \frac{p}{q} \}} \left( \left[ 1^R \right] \right) \) as well, which is then skew-symmetric in both the \( p_i \)s and the \( q_i \)s separately. This can be written

\[
\begin{align*}
    s_{\{ \frac{p}{q} \}} \left( \left[ 1^R \right] \right) &= \sgn(\{ p \}) \sgn(\{ q \}) s_{\{ \text{sort}(\{ p \}) \}} \left( \left[ 1^R \right] \right).
\end{align*}
\]
Observe that Formula (18) is still valid when sort$(\overline{p})$ or sort$(\overline{q})$ is not defined (this happens when two of the entries of $\overline{p}$ or $\overline{q}$ are equal) thanks to $\text{sgn}(\overline{p})\text{sgn}(\overline{q}) = 0$ (see conventions in Section 2).

Finally, it is helpful to remark that Formula (16) for $s_{\{\overline{p}\}}\left([1^{p}\right])$ can be seen as a product indexed by the sets $\overline{p} \cup \overline{q}$ and pairs in the set $(\overline{p} \times \overline{p}) \cup (\overline{q} \times \overline{q}) \cup (\overline{p} \times \overline{q})$.

5.1.2. Evaluation of $s_{\{\overline{p} \cup \lambda\}}\left([1^{2k}\right])$. We take $\lambda = \left\{\overline{p} \overline{q}\right\}$ to have $d$ Frobenius coordinates.

In total analogy with Equation (18), we first extend the definition of $s_{\{\overline{p} \cup \lambda\}}$ and set

$$s_{\{\overline{p} \cup \lambda\}}\left([1^{2k}\right]) := \text{sgn}(\overline{p})\text{sgn}(\overline{q})s_{\{\overline{p} \cup \lambda\}}\{|\text{sort}(\overline{p})\}$$

with the understanding (as before) that the value of the RHS should be 0 if $p_i = p_j$ (resp. $q_i = q_j$) for $i \neq j$. Again, this is skew-symmetric in the $p_i$s and separately in the $q_i$s.

We have the following Lemma

**Lemma 6.** Let $\overline{p}$, $\overline{q}$ be vectors with $d$ coordinates. Then

$$s_{\{\overline{p} \cup \lambda\}}\left([1^{2k}\right]) = s_{\{\overline{p} \cup \lambda\}}\left([1^{2k}\right]) \right) \prod_{i=1}^{d} \frac{(N-p_i)(k-q_i)}{(p_i+k+1)(N+q_i+1)(k)}.$$}

*Proof.* By skew-symmetry, we really only have to check this for $\left\{\overline{p} \overline{q}\right\}$. If we want to use Formula (16), we should look at the Frobenius coordinates of $\langle N^k \rangle \cup \lambda$. This would be rather unpleasant (particularly because the number of Frobenius coordinates would change for fixed $N$ and $k$ according to the $\lambda$ considered).

Let us look instead at:

$$\overrightarrow{\alpha} := (N+k-1, \cdots, N),$$
$$\overrightarrow{\beta} := (2k-1, \cdots, k),$$
$$\overrightarrow{\alpha} := \overrightarrow{\alpha} \cup \overrightarrow{\beta} \text{ (sorted)}$$

and $$\overrightarrow{\beta} := \overrightarrow{\alpha} \cup \overrightarrow{\beta} \text{ (sorted)}.$$ Then $\overrightarrow{\alpha}$ and $\overrightarrow{\beta}$ are strictly decreasing, so those are Frobenius coordinates. The partition corresponding to those coordinates is obtained geometrically by sticking a $\langle k^2 \rangle$ block to the left of $\langle N^k \rangle \cup \lambda$, or equivalently to shifting $\langle N^k \rangle \cup \lambda$ by $k$ spots to the right, while considering $\lambda = (\lambda_1, \cdots, \lambda_k)$ to have exactly $k$ parts (with some possibly empty).

Because of this, we have (as in [BG06, page 6]):

$$s_{\{\overrightarrow{\alpha} \overrightarrow{\beta}\}}\left([1^{2k}\right]) = e_{2k}^k\left([1^{2k}\right])s_{\{\overrightarrow{\alpha} \overrightarrow{\beta}\}}\left([1^{2k}\right]) = s_{\{\overrightarrow{\alpha} \overrightarrow{\beta}\}}\left([1^{2k}\right]).$$

Additionally, $\left\{\overrightarrow{\alpha} \overrightarrow{\beta}\right\}$ are the Frobenius coordinates of $\langle (N+k)^k \rangle \cup \langle k^k \rangle$. Hence, for the same reason as above, we have:

$$s_{\{\overrightarrow{\alpha} \overrightarrow{\beta}\}}\left([1^{2k}\right]) = e_{2k}^k\left([1^{2k}\right])s_{\{\overrightarrow{\alpha} \overrightarrow{\beta}\}}\left([1^{2k}\right]) = s_{\{\overrightarrow{\alpha} \overrightarrow{\beta}\}}\left([1^{2k}\right]).$$
When evaluating the product described in Equation (16) using the $→α$ and $→β$ coordinates, we have a big product taken over the sets $→α$, $→β$, $→α ×→α$, $→β ×→β$ and $→α ×→β$. We expand those index sets using $→α = →x ∪ →p$ and $→β = →y ∪ →q$.

One can see that the products indexed by $→p$, $→q$, $→p ×→p$, $→q ×→q$ and $→p ×→q$ together give $s(→p →q)([1^{2k}]) = s_λ([1^{2k}])$.

Similarly, the products indexed by $→x$, $→y$, $→x ×→x$, $→y ×→y$ and $→x ×→y$ give $s(→x →y)([1^{2k}]) = s_{(N*λ)}([1^{2k}])$.

We are left with only “cross-products” to evaluate, for the index sets $→x ×→p$, $→x ×→q$, $→y ×→p$ and $→y ×→q$. The definitions of $→x$ and $→y$ now give the result. □

5.2. Sums of characters over conjugacy classes with same number of cycles. Assume $f(\{→p →q\})$ is a function of pairs of vectors of the same length (say $d$).

One can set $f(λ) := f(\{→p →q\})$, where $λ = \{→p →q\}$.

The goal in this section is to evaluate sums of characters of the general form

$$\sum_{→p ∈ N^*_→} χ^{λ}_{→p} f(λ).$$

We will eventually take $f(λ) = s_{(N*λ)}([1^{2k}])$ but there is no reason to limit ourselves in that way for a while.

We rely on a few results of Borodin that give a slightly different version of the Murnaghan-Nakayama rule.

5.2.1. Definitions. This is based on [Bor00, around page 15] and [Bor98, around page 6]. The relevant definitions (not included here) are fragment, the different block types, the filling numbers, filled structure, sign of a structure.

Theorem 9 is almost in Borodin’s work, and his definitions are used in Proposition 10. Both of those results are used for Theorem 13, which can be read without looking at Borodin’s papers.

However, the first condition to have a fragment needs clarification in both papers: change

(1) there is exactly one hook block that precedes the others

to

(1) there is exactly one hook block in each fragment. That hook block precedes any other block in the fragment.

We also would like to correct a statement in [Bor00], in that linear horizontal or vertical blocks are positive, not just non-negative integers (in agreement with the other cited paper of Borodin [Bor98]).

We can highlight one of the definitions: Any filled structure $T$ with $d$ fragments produces a set of pairs

$$\{(p_1, q_1), \cdots, (p_d, q_d)\}$$

which consists of the filling $p$- and $q$- numbers of the fragments.

The sign of $T$ is defined as follows:

$$\text{sgn}(T) = \text{sgn}(→p) \text{sgn}(→q)(-1)^{\sum q_i + v(T)},$$

where, as a reminder, the sign inside the formula is 0 if $p_i = p_j$ (resp. $q_i = q_j$) for $i ≠ j$. 
5.2.2. Simplified Murnaghan-Nakayama rule. Although we haven’t defined anything, we state Proposition 4.3, taken from the first paper of Borodin:

**Proposition 7.** For any two partitions \( \lambda \) and \( \rho \) with \( |\lambda| = |\rho| \), we have

\[
\chi^\lambda_\rho = \sum_{T} \text{sgn} T,
\]

where the sum is taken over all filled structures of cardinality \( \rho = (\rho_1, \cdots, \rho_l) \) such that the sequences \((p_1, \cdots, p_d)\) and \((q_1, \cdots, q_d)\) of filling \( p \)-numbers and \( q \)-numbers of the structure \( T \) coincide, up to a permutation, with the Frobenius \( p \)-coordinates and \( q \)-coordinates of the partition \( \lambda \) (i.e. \( \lambda = \{ \text{sort}(\rho) \} \)).

The proof of this Proposition is quite simple: going back to the original presentation of the Murnaghan-Nakayama rule in terms of hooks, Borodin analyzes what happens to Frobenius coordinates when subtracting hooks/ribbons. Each such subtraction corresponds to a block. There are three cases to distinguish: the hook/ribbon can be above or below the “Frobenius diagonal” or even overlap it. Those cases correspond respectively to linear horizontal blocks, linear vertical blocks, and hook blocks.

This Proposition, as stated in Borodin’s work, is slightly restrictive: there is no need for \( \rho \) to be a partition. Let \( \overline{\rho} = (\rho_1, \cdots, \rho_l) \) to be a vector of positive integers and define (just as in Theorem 3) \( \chi^\lambda_{\overline{\rho}} := \chi^\lambda_{\text{sort}(\rho)} \). Then, by summing over all vectors \( \overline{\rho} \), we get

**Proposition 8.** For any partition \( \lambda \), we have

\[
\sum_{\overline{\rho} \in \mathbb{N}^+} \chi^\lambda_{\overline{\rho}} = \sum_{T} \text{sgn} T,
\]

where the sum is taken over all filled structures \( T \) of \( l \) blocks and with filling \( p \)-numbers \((p_1, \cdots, p_d)\) and \( q \)-numbers \((q_1, \cdots, q_d)\) such that \( \lambda = \{ \text{sort}(\overline{\rho}) \} \).

Observe that \( d \), the rank of \( \lambda \), has to be less or equal to \( l \) in order to have a structure.

We now state the main theorem we will use that is originated in Borodin’s work.

**Theorem 9.** Assume \( f \) is skew-symmetric within its two vector entries (separately), i.e. \( f \left( \text{sort}(\overline{\rho}) \right) = \text{sgn}(\overline{\rho}) \text{sgn} \left( \text{sort}(\overline{\tau}) \right) f \left( \overline{\tau} \right) \). Then,

\[
\sum_{\lambda \text{ within } k \times N} \sum_{\overline{\rho} \in \mathbb{N}^+} \chi^\lambda_{\overline{\rho}} f(\lambda) = \sum_{d=1}^{l} \sum_{\overline{\rho} \in [0,N-1]^d} f \left( \text{sort}(\overline{\rho}) \right) \sum_{\overline{\tau} \in [0,k-1]^d} (-1)^{\sum q_i + v(T)} T(\overline{\rho},\overline{\tau}),
\]

where \( T(\overline{\rho},\overline{\tau}) \) goes through all filled structures of \( d \) fragments, \( l \) blocks, \( v(T) \) vertical blocks with filling \( p \)-numbers \((p_1, \cdots, p_d)\) and \( q \)-numbers \((q_1, \cdots, q_d)\).

**Proof.** We start by summing Proposition 8 over \( \lambda \) as fitting inside a \( k \times N \) box:

\[
\sum_{\lambda \text{ within } k \times N} \sum_{\overline{\rho} \in \mathbb{N}^+} \chi^\lambda_{\overline{\rho}} f(\lambda) = \sum_{\lambda \text{ within } T(\overline{\rho},\overline{\tau})} \sum_{k \times N} \sum_{\overline{\rho} \in [0,N-1]^d} f(\lambda) = \sum_{\lambda \text{ within } T(\overline{\rho},\overline{\tau})} \sum_{k \times N} \sum_{\overline{\rho} \in [0,k-1]^d} f(\lambda),
\]

where the second sums in each RHS are taken over all filled structures \( T(\overline{\rho},\overline{\tau}) \) of \( l \) blocks and \( d \) fragments such that the sequences of filling \( p \)-numbers \((p_1, \cdots, p_d)\) and \( q \)-numbers \((q_1, \cdots, q_d)\) of the structure coincide, up to two permutations, with
the sequences of Frobenius $p$-coordinates and $q$-coordinates of the partition $\lambda$ (i.e. $\lambda = \{\text{sort}(\mathbf{p}), \text{sort}(\mathbf{q})\}$). Note that $d$ changes with $\lambda$.

We then obtain the final result by seeing the double sum over $\lambda$ then permuted Frobenius coordinates of $\lambda$ as a sum over all vectors of appropriate lengths.

We should not be concerned about vectors having two identical coordinates (say $p_i = p_j$), since the corresponding term in the RHS vanishes by skew-symmetry of $f$.

$\square$

5.2.3. Counting structures. We now need to compute the sum

$$\sum_{T(\mathbf{p}, \mathbf{q})} (-1)^{\sum q_i + v(T)},$$

which is taken over the structures described above, i.e. for given $l$, $d$, $\mathbf{p}$, $\mathbf{q}$, $v$.

It would help to know how many structures there are for each choices of those parameters. We prove the following Proposition.

Proposition 10. There are exactly

$$\begin{align*}
\#T(l, d, \mathbf{p}, \mathbf{q}, v) &= \sum_{s, t \in \mathbb{N}} d^{s_i + t_i} \prod_{i=1}^{d} (s_i + t_i + 1)^{d - 1 - i + \sum_{j=1}^{d} s_j + t_j} \\
& \times \prod_{i} (p_i)^{s_i} (q_i)^{t_i}
\end{align*}$$

structures with $d$ fragments, $l$ blocks, filling numbers $\mathbf{p} = (p_1, \cdots, p_d)$ and $\mathbf{q} = (q_1, \cdots, q_d)$ and $v$ vertical blocks. The indices in the sum $s_i$ (resp. $t_i$) count horizontal (resp. vertical) blocks in the $i$th fragment.

Proof. This is a purely combinatorial problem. Given the number of vertical blocks on each fragment, we essentially have a partial order on blocks that we want to extend to form a linear order (across fragments). Part of the rules in the initial partial order say that the hook-block in the $i$th fragment precedes any other block in that fragment. We then need to fill the structure (i.e. choose filling numbers for each block).

We can reverse this process:

- We first choose the numbers of horizontal and vertical blocks $s_i$ and $t_i$ on the $i$th fragment. We have the conditions that $\sum t_i = v(T)$ and $d + \sum s_i + t_i = l$ (i.e. there are $l$ blocks in total, $d$ hook, $s_i$ horizontal in the $i$th fragment and $t_i$ horizontal in the $i$th fragment).
- Starting from the $d$th fragment, we decide where to insert the horizontal and vertical blocks of the $i$th fragment in the partial order that is established so far on the set of fragments from the $i + 1$st to the $d$th one.
- We decide how to cut up the $i$th fragment into filled blocks, respecting the number of horizontal/vertical blocks decided upon earlier.

The equality in the statement is intended to reflect clearly the layering described above: the sum corresponds to the first layer, while the other two layers correspond to one square-bracketed factor each.

Observe that the relation $s_d + t_d + \cdots + s_1 + t_1 + d = l$ could be used to simplify the numerator in this expression.
The only hard part is to derive for the second step
\[
\frac{(s_d + t_d + \cdots + s_1 + t_1 + d)!}{\prod s_i! \prod t_i! \prod_{i=1}^d (d + 1 - i + \sum_{j=i}^d s_j + t_j)} = \frac{(s_d + t_d + \cdots + s_1 + t_1 + d)!}{\prod s_i! \prod t_i! \left( \frac{(s_d + t_d + \cdots + s_1 + t_1 + d) \times \cdots \times (s_d + t_d + s_{d-1} + t_{d-1} + 2) \times (s_d + t_d + s_{d-1} + t_{d-1} + \cdots + s_1 + t_1 + d)}{(s_d + t_d + s_{d-1} + t_{d-1} + \cdots + s_1 + t_1 + d)} \right)}.
\]
This is obtained by simplifying
\[
\prod_{i=0}^{d-1} \left( \frac{i + \sum_{j=d-i}^d s_j + t_j}{s_{d-i} + t_{d-i}} \right) \left( \frac{s_{d-i} + t_{d-i}}{s_{d-i}} \right),
\]
where the \(i\)th factor in the \(\prod_{i=0}^{d-1}\)-product counts the number of ways of choosing the linear order on the blocks of the \(d - i\)th fragment, knowing the linear order restricted on the blocks of the fragments \(d - i + 1\) to \(d\).

The first binomial factors intersperses the set of blocks of the \(d - i\)th fragment among the blocks of fragments \(d - i + 1\) to \(d\), while the second factor decides which blocks are horizontal and which are vertical.

We wish to insist on the fact that the summand in Equation (20) is not symmetric in the \(p_i\)s or the \(q_i\)s, because the factor in the denominator \(\prod_{i=1}^d (d + 1 - i + \sum_{j=i}^d s_j + t_j)\) is not symmetric in the \(s_j\)s or the \(t_j\)s. For instance, \(s_d\) appears \(d\) times while \(s_1\) appears only once.

5.2.4. Sum of determinants. We aim now to put together all the results obtained so far in this section, but we first need a quick lemma.

**Lemma 11.** Let \(\mathbf{s}\) and \(\mathbf{t}\) be vectors of integers. Then,
\[
\sum_{\sigma, \tau \in S_d} (\text{sgn} \sigma \text{sgn} \tau) \prod_{i=1}^d (d + 1 - i + \sum_{j=i}^d s_{\sigma(j)} + t_{\tau(j)}) = \prod_{1 \leq i < j \leq d} (s_i - s_j)(t_i - t_j) \prod_{1 \leq i, j \leq d} \frac{1}{1 + s_i + t_j}.
\]

**Proof.** The proof proceeds as for the classical computation for the Vandermonde determinant: the LHS is skew-symmetric in \(\mathbf{s}\) and \(\mathbf{t}\) separately, has obvious poles as prescribed in the RHS (when \(s_{t_0} + t_{p_0} = -1\), and the degrees in the RHS are appropriate. Up to a constant of proportionality, both sides are thus the same. This constant is shown to be 1 by looking at rates of decrease when \(s_1\) goes to infinity.

**Proposition 12.** Assume \(f\) is skew-symmetric within its two vector entries (separately), i.e. \(f\left( \left\{ \text{sort}(\mathbf{p}) \right\} \right) = \text{sgn}(\mathbf{p}) \text{sgn}(\mathbf{q}) f\left( \left\{ \mathbf{q} \right\} \right)\). Then,
\[
\sum_{\mathbf{p} \in [0,N]^d} \sum_{\lambda \text{ with } \lambda(N) = N} \chi_{\lambda}^{\mathbf{p}} f(\lambda) = l! \sum_{d=0}^l \sum_{\mathbf{p} \in [0,N-1]^d} \text{sgn}(\mathbf{p}) f\left( \left\{ \mathbf{p} \right\} \right) (-1)^{\sum q_i + v}
\]
\[
\sum_{\mathbf{p}, \mathbf{q} \in [0,N]^d} \prod_{i=1}^d \frac{p_i}{s_i} \frac{q_i}{t_i} \prod_{1 \leq i \leq l} \frac{1}{s_i t_i} \prod_{1 \leq i, j \leq d} (s_i - s_j)(t_i - t_j) \prod_{1 \leq i, j \leq d} \frac{1}{1 + s_i + t_j}.
\]
Proof. We go head first and combine Equation (15) and Theorem 9:

\[
\sum_{\pmb{p} \in \mathbb{N}^d} \sum_{\lambda \text{ within } k \times N} \chi^{\pmb{p}} f(\lambda) = l! \sum_{d=1}^{l} \sum_{\pmb{p} \in [0, N-1]^d} \sum_{\pmb{q} \in [0, k-1]^d} f\left(\left\{\frac{\pmb{p}}{\pmb{q}}\right\}\right) (-1)^{\sum q_i + v}
\]

\[
\sum_{\pmb{p} \in \mathbb{N}^d} \sum_{d=1}^{l} \sum_{\pmb{q} \in [0, k-1]^d} \frac{\prod_{i=1}^{d} \binom{p_i}{q_i}}{\prod s_{i}! \prod t_{i}! \prod_{j=1}^{d} (d + 1 - i + \sum_{j=1}^{d} s_j + t_j)}
\]

\[
= l! \sum_{d=1}^{l} \sum_{\pmb{p} \in [0, N-1]^d} \sum_{\pmb{q} \in [0, k-1]^d} f\left(\left\{\frac{\pmb{p}}{\pmb{q}}\right\}\right) (-1)^{\sum q_i + v}
\]

\[
\sum_{\pi, \sigma \in \mathcal{S}_d} \sum_{\tau, \theta \in \mathcal{S}_d} \frac{\left[\sum_{\pi, \theta \in \mathcal{S}_d} sgn(\pi) sgn(\theta) \prod_{i=1}^{d} \binom{p_{i}(\pi)}{s_i} \binom{q_{i}(\pi)}{t_i}\right]}{\prod s_{\pi(\sigma)}! \prod t_{\tau(\sigma)}! \prod_{j=1}^{d} (d + 1 - i + \sum_{j=1}^{d} s_{\sigma(j)} + t_{\tau(j)})}
\]

It is now crucial to observe that for fixed \(\pmb{p}, \pmb{q}, \mathcal{S}, \mathcal{T}\), the sign of the numerator of the summands (bracketed) will depend on the parity of \(\sigma\) and \(\tau\). Hence we obtain

\[
= l! \sum_{d=1}^{l} \sum_{\pmb{p} \in [0, N-1]^d} \sum_{\pmb{q} \in [0, k-1]^d} f\left(\left\{\frac{\pmb{p}}{\pmb{q}}\right\}\right) (-1)^{\sum q_i + v}
\]

\[
\sum_{\pi, \sigma \in \mathcal{S}_d} \sum_{\tau, \theta \in \mathcal{S}_d} sgn(\pi) sgn(\theta) \prod_{i=1}^{d} \binom{p_{i}(\pi)}{s_i} \binom{q_{i}(\pi)}{t_i}
\]

\[
\sum_{\sigma, \tau \in \mathcal{S}_d} sgn(\sigma) sgn(\tau)
\]

\[
= \prod s_{\sigma(\pi)}! \prod t_{\tau(\sigma)}! \prod_{j=1}^{d} (d + 1 - i + \sum_{j=1}^{d} s_{\sigma(j)} + t_{\tau(j)})
\]
For a fixed $k$, Theorem 13.

Theorem 13. For a fixed $k \in \mathbb{N}$, the two series

$$
\sum_{r > 0} (\mathcal{M}(2k, r) \frac{(iz)^r}{r!}) \text{ and } \sum_{d=1}^{\infty} \sum_{\Pi, \mathbf{t} \in \mathbb{N}^d \setminus \{0\}} \frac{z^{1+s_i+t_j}}{s!t! \prod_{i=1}^d (1+s_i+t_j)} d \times d \left| \sum_{\mathbf{p}, \mathbf{q} \in [0, N-1]^d} \frac{\prod_{i=1}^d (p_i + q_i)(k+p_i)(k+q_i)(N-p_i)(q_i)(-1)^q_i}{(N+q_j+1)^{k_j}(1+p_i+q_j)} \right|
$$

have equal coefficients of $z^r$ for $r < 2k + 1$.

Proof. A first necessary remark is that as a formal power series, the second series is well-defined: the sum to obtain the $r^{th}$ coefficient in that series reduces to a finite sum (because $s_i \leq p_i$ and $t_j \leq q_j$).

We know from Equation (18) that $s_{(N^k) \cup \{1^k\}}([1^k])$ is skew-symmetric in $\overline{\mathbf{p}}$ and $\overline{\mathbf{q}}$ (separately). Hence we can combine Equations (15), (16) and (19) and Proposition 12 to obtain a huge sum. The main statement then follows from recombinations of the main product into determinants, using Cauchy’s Lemma 5.

Remarks on Theorem 13

- This is a hypergeometric multisum (at least for fixed $d$), when we expand the determinants using Cauchy’s Lemma. However, not even small $ds$ seem tractable on computer.
- A definite advantage of this formula is that it can be tested at finite $N$ (by expanding the integral defining $(M_N)(2k, r)$ symbolically using the Haar measure). This is helpful to confirm the results obtained so far.
- We wish to insist on the idea behind this Theorem: initially we had a combinatorial problem on structures (see Formula (20) that had no symmetry
for its summands in the $s_i$s (resp. $t_i$s). We have exploited some skew-symmetry in the $a$s and $b$s in Formula (19) to change this. In particular, we have now switched from a sum over $\overline{p}, \overline{q}, \overline{s}, \overline{t}$ to a sum over $p, q, s, t$.

We have also simplified the denominator in Formula (20).

- As a consequence of the previous point, we can now assume that the $s_i$s are all different. The same is true for the $t_i$s.
- This has useful consequences, especially for computational purposes. It is interesting to compute a bound on $r$ such that partitions with $d$ fragments will have a non-zero contribution to the final sum in $(\mathcal{M})_{N}(2k, r)$. We have $r \geq d + \sum s_i + t_i$, and the $s_i$s (resp. $t_i$s) should be all different. We can take them to be $0, 1, \cdots, d - 1$. We thus have $r \geq d + 2 \frac{d(d-1)}{2} = d^2$.

We now define

$$H^{N,k,s,t} := s! \sum_{p \in [0,N-1]} \sum_{q \in [0,k-1]} \frac{k(N - p)^k(-1)^q}{(N + q + 1)^k(1 + p + q)} \binom{k + p}{p} \binom{k - 1}{q} \binom{q}{t},$$

where the RHS is taken to be similar to the entries in one of the determinants in Equation (22).

I have not been able to obtain a much better expression for this with Mathematica. Normally, the package MultiSum [Weg] should be able to deal with multiple hypergeometric series, but this particular one is too complicated. We will thus focus on an easier problem from now on, the problem of asymptotics (i.e. we switch from $(\mathcal{M})_{N}(2k, r)$ to $(\mathcal{M})(2k, r)$).

5.4. Asymptotics. We need to compute asymptotics for $H^{N,k,s,t}$ more precisely.

**Proposition 14.** For a fixed integer $k \geq 1$, when $k > t$,

$$H^{k,s,t} := \lim_{N \to \infty} \frac{H^{N,k,s,t}}{N^{1+t}}$$

$$= k \sum_{i=0}^{k-t-1} \frac{\Gamma(k+i)\Gamma(s+i+t+1)}{\Gamma(i+1)\Gamma(k+s+t+i+2)}$$

$$= \frac{1}{1+s+t} \frac{\prod_{i=k}^{2k-1} (i-t)}{\prod_{j=k+1}^{2k} (j+s)}$$

$$\frac{1}{1+s+t} \frac{\Gamma(2k-t)\Gamma(k+s+1)}{\Gamma(k-t)\Gamma(2k+s+1)}$$

**Proof.** Define

$$\tilde{H}^{N,k,s,t} := t! \sum_{p \in [0,N-1]} \sum_{q \in [0,k-1]} \frac{k(N - p)^k(-1)^q}{(N + q + 1)^k(1 + p + q)} \frac{p^k}{k!} \binom{k - 1}{q} p^t \binom{q}{t},$$

i.e. $H^{N,k,s,t}$ stripped of some of its terms of obviously lower order in $p, N$ and $q$ combined. We do this because we want to compute the leading order of $H^{N,k,s,t}$ and there will be lots of cancellation due to the sum over $q$ (as showed by Equation (28)).

We thus wish to compute $\lim_{N \to \infty} \tilde{H}^{N,k,s,t} / N^{1+s+t} = \lim_{N \to \infty} \tilde{H}^{N,k,s,t} / N^{1+s+t}$.

\footnote{Observe that Expression (27) is well defined, thanks to the bound $k > t$.}
The proof of the equality (24)-(25) essentially follows from two basic identities on formal series:

\[(1 - rX + r^2X^2 - \cdots)^k (1 - sX + s^2X^2 - \cdots) = \sum (-1)^j \sum_{i=0}^{j} \binom{k + i - 1}{i} r^i s^{j-i} X^j\]

and

\[\sum_{0 \leq j \leq k-1, 0 \leq q \leq k-1} (-1)^q \binom{k-1}{q} q^j X^j = (-1)^{k+1} (k-1)! X^{k-1}.\]

We expand the definition of \(\tilde{H}_{N,k,s,t}^{N,k,s,t}\) as a power series in \(q\). The first identity indicates that we should only look at the coefficient of \(q^{k-1}\), which we obtain by using the second identity (set \(r := 1/N, s := 1/(p+1)\)). We then let \(N\) tend to infinity, so the sum over \(p\) becomes a Riemann sum. Its limit is a \(\beta\)-integral, and thus a \(\beta\)-function appears, which can be expanded into a product of \(\Gamma\)-functions, giving the first equality.

The equality (26)-(27) is immediate and is the only one to require the bound \(k > t\).

For equality (25)-(26), we first define

\[H_{a}^{k,s,t} = k \sum_{i=0}^{\infty} \frac{\Gamma(k + a + i)\Gamma(s + a + i + t + 1)}{\Gamma(a + i + 1)\Gamma(k + s + t + a + i + 2)}\]

which satisfies \(H_{a}^{k,s,t} = H_{0}^{k,s,t} - H_{k-t}^{k,s,t}\). The second equality is merely a consequence of the definition of \(3F_2\).

Since (see [Mat]) \(3F_2\left(\frac{1, c, d}{c + e + d - e + 2}; 1\right) = \frac{c + d - e + 1}{c - e + 1} \frac{1 - e + \Gamma(c + d - e + 1)\Gamma(d)}{\Gamma(c)\Gamma(d)}\), we get

\[H_{a}^{k,s,t} = \frac{1}{1 + s + t} \left(1 - \frac{a\Gamma(a + k)\Gamma(a + s + t + 1)}{\Gamma(a + 1)\Gamma(a + k + s + t + 1)}\right),\]

which lets us prove equality (25)-(26) using the relation \(H_{a}^{k,s,t} = H_{0}^{k,s,t} - H_{k-t}^{k,s,t}\). \(\square\)

Let \(G(\cdot)\) be the Barnes \(G\)-function [HKO00, Appendix]. It is a quick consequence of the Weyl dimension formula (see [BG06, Equation (18)]) that

\[s_{t}(N\nu)([1^{2k}]) \sim_{N} \frac{G(k + 1)^2}{G(2k + 1)} N^{k^2}.\]

We use the previous Proposition to give a relatively concise expression for \((M)(2k, r)\).

\[\text{\textsuperscript{7}This equality was first proved using Mathematica. Paul Abbott observed that the hypergeometric function that appears is Saalschitzian and extracted the following proof by tracing Mathematica's output.}\]
Theorem 15. For a fixed $k \in \mathbb{N}$, the two series

$$\sum_{r>0} (\mathcal{M})(2k,r) \frac{(iz)^r}{r!}$$

and

$$\frac{G(k+1)^2}{G(2k+1)} \sum_{d=1}^{\infty} \sum_{\mathbf{s}, \mathbf{t} \in \mathbb{N}^d} \frac{1}{s_1!t_1!(1+s_1+t_1)} \frac{H^{k,s_i,t_j}}{s_i!t_j!} \left| d \times d \right|$$

have equal coefficients of $z^r$ for $r < 2k+1$. For a fixed $r$, the coefficients of $z^r$ for low values of $k$ can be meromorphically continued into each other.

Furthermore, by using Cauchy’s Lemma, one can switch to an expression involving products instead of determinants (i.e. a hypergeometric expression).

Proof. For Expression (29), we proceed essentially by substitution into Equation (22), and looking at terms of order $N^{k^2+r}$. Again, Cauchy’s Lemma is used repeatedly to reorganize determinants.

For Expressions (30) or (31), we reorganized yet again the determinants using Cauchy’s Lemma into a form corresponding to Formula (16). We also summed over partitions $\lambda$ instead of summing first over their rank $d$ then their Frobenius coordinates $s_i$, $t_j$.

For a fixed $r$, both sides indeed admit meromorphic continuations in $k$, which are equal by Carlson’s Theorem[AAR99, Theorem 2.8.1, p. 110]. Indeed, the LHS is shown to admit a meromorphic continuation in $k$ using a Pochhammer contour. The RHS’ meromorphic continuation is already written in Expression (30), if we admit that what is meant there is the value of the meromorphic continuation in $k$ evaluated at $k$. The difference of the two sides satisfies the hypotheses in Carlson’s Theorem, in that its value is 0 at integers, it is of exponential type, and type $< \pi$ along axes parallel to the imaginary axis. The author has shown similar statements in his thesis.

It is probably good to insist that the meromorphic continuation of $\Gamma(k-t_j)\Gamma(k+s_i+1)$ to the left has to be taken very carefully and cannot be obtained by just plugging in values of $k$, once $k \leq t$. We will discuss similar issues later, in Section 9.

We now aim to replace the determinant left in Equation (31) by a friendlier expression, a rational function of $k$.

6. General shape of $(\mathcal{M})(2k,r)$, $|\mathcal{M}|(2k,2h)$ and $|\mathcal{V}|(2k,2h)$

We now prove Theorem 1.

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8See also the author’s thesis[Deh06b].
Proof. By Equation (29), we know that (for fixed $r$ and as meromorphic functions of $k$)

$$\frac{iv_r}{r!} (\mathcal{M})(2k, r) = \frac{G(k + 1)^2}{G(2k + 1)} \sum_{1 \leq d \leq d' \in \mathbb{Q}} C(d, \bar{s}, \bar{t}) |H^{k,s,t}|_{d \times d},$$

with $C(d, \bar{s}, \bar{t}) \in \mathbb{Q}$, while for $s$ and $t$ fixed (and non-negative, of course), Equation (27) indicates that $H^{k,s,t}$ is a rational function of $k$:

$$H^{k,s,t} = \frac{1}{1 + s + t} \prod_{i=-t}^{s} \frac{k + i}{2k + i}$$

This already shows that we have a rational function of $k$ and that the degree of its numerator equals the degree of its denominator. Equations (32) and (29) together, along with the fact that $H^{k,s,t} = H^{-k,t,s}$ (a consequence of Equation (27)), explain why $X_r$ is even.

In order to determine the $Y_r$s a bit better, we need to investigate possible denominators in the terms of $|H^{k,s,t}|_{d \times d}$. If $a$ is positive, $|H^{k,s,t}|_{d \times d}$ will have a factor of $(2k + a)\alpha_a(r)$ in its denominator if and only if $a$ is odd (because there is cancellation in Formula (32)) and all of $s_1, \cdots, s_a$ are greater than $a$. For this to happen, we need

$$r = d + \sum s_i + \sum t_i \geq \alpha_a(r) + \sum_{i=1}^{a-1} (a + i - 1) + \sum_{i=1}^{a} (i - 1),$$

where the inequality is obtained by taking as small as possible values for $d$ (i.e. $\alpha_a(r)$), for the $s_i$’s (while requiring them to be different and greater or equal to $a$) and for the $t_i$’s (all different).

We turn this inequality around and get

$$\alpha_a(r) \leq \left\lfloor \frac{-a + \sqrt{a^2 + 4r}}{2} \right\rfloor.$$

The case of $a$ negative is the same, exchanging the roles played by $\bar{s}$ and $\bar{t}$.

Finally, the constant $D(r)$ ensuring that both $X_r$ and $Y_r$ are monic can be found, thanks to Equations (29) and (32), taking $\lim_{k \to \infty}$:

$$D(r) = \sum_{1 \leq d \leq d' \in \mathbb{Q}} \left| \frac{1}{s_i t_j (1 + s_i + t_j)} \right|_{d \times d} \frac{1}{2^{d + \sum (s_i + t_i)}} = \frac{1}{r!2^r},$$

where this last equality is left to the reader.

Actually, this last equality is enough to also guarantee that $X_r(u)$ and $Y_r(u)$ both have integer coefficients: just substitute for $H^{k,s,t}$ in Equation (29)

$$H^{k,s,t} = \frac{1}{1 + s + t} \prod_{i=-t}^{s} \frac{1}{2k + i} \left( k \sum_{i=0}^{s+t} h_i k^i \right)$$

for the appropriate (integer) $h_i$s (in particular, $h_{s+t} = 1$).

This proves Equation (2), at least for large $k$.

Meromorphic continuation has already been obtained in Theorem 15. \qed
Theorem 16. For $h \in \mathbb{N}$, there are polynomials $\tilde{X}_{2h}$, $\check{X}_{2h}$, with integer coefficients and $\deg \check{X}_{2h} = \deg X_{2h} > \deg \tilde{X}_{2h}$ such that as meromorphic functions of $k$,

$$|M|(2k, 2h) = \tilde{C}(h) \frac{G(k + 1)^2 X_{2h}(2k)}{G(2k + 1) Y_{2h}(2k)},$$

$$|V|(2k, 2h) = \check{C}(h) \frac{G(k + 1)^2 \check{X}_{2h}(2k)}{G(2k + 1) Y_{2h}(2k)},$$

where $Y_r(u)$ is as defined in Theorem 1.

Moreover (but this is conjectural), the numerators are additionally monic polynomials\(^9\) when $\check{C}(h) = \frac{1}{2\pi}$ and $\tilde{C}(h) = \left(\frac{2h}{\sqrt{\pi}}\right)^2$, and $\deg X_{2h} - \deg \check{X}_{2h} = 2h$.

Proof. For fixed integer $r$ and large integer $k$, most of this follows immediately from Equations (7) and (9), combined with Theorem 1.

The fact that $\deg \check{X}_{2h} < \deg X_{2h}$ for instance is a consequence of

$$(M)(2k, r) \sim_k \left(-\frac{i}{2}\right)^r \frac{G(k + 1)^2}{G(2k + 1)},$$

which we use in Equation (10):

$$\sum_{j=0}^{2h} \binom{2h}{j} \left(-\frac{i}{2}\right)^j \left(-\frac{i}{2}\right)^{2h-j} = 0.$$

We can similarly show that if it exists, $\check{C}(h) = \frac{1}{2\pi}$. The constant $\tilde{C}(h)$ is more mysterious, and involves lower order terms in $k$ of Equation (32).

The meromorphic continuation is obtained as in the proof of Theorem 1. \qed

Remark. Unfortunately, within their degree restrictions, the $X_r(u), \check{X}_{2h}(u)$ and $\tilde{X}_{2h}(u)$ polynomials still look utterly random. We merely have an expression for them as a sum of determinants of rank $d \leq \sqrt{r}$ (resp. $2h$). This expression is relatively quick and allows at least to compute a few of those polynomials.

7. Computational data

7.1. The polynomials $X_r(u)$, $\check{X}_{2h}(u)$ and $\tilde{X}_{2h}(u)$. We present our data for $(M)(2k, r)$ first, in Table 2, followed by data on $|M|(2k, 2h)$ in Table 3 and finally on $|V|(2k, 2h)$ in Table 4. Everything extends numerical results previously published, for instance in [Hal04, Hal02a] (but those rely on [Hug05]) or [CRS06] (which is limited to $k = h$). Extended versions of those tables are also made available in the source of this arXiv submission or (possibly more) at [Deh07].

To obtain those tables, we have implemented Equation (30), which is the most computationally accessible version of the formulas available in Theorem 15. A Magma implementation of this algorithm is also part of this arXiv submission.

7.2. The roots of $\tilde{X}_{2h}(u)$. It has been suggested before, based on limited numerical data, that the polynomials $\tilde{X}_{2h}(u)$ have only real roots. In fact we list in Table 5 the number of real roots and degree for each such polynomial. One quickly observes that $\tilde{X}_{42}(u)$ (of course!) is actually the first polynomial to break the initial fluke and have non-real roots. It is not clear at this point if this is related to a similar observation on the last line of [Hal02a] and throughout [Hal04].

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\(^9\)This is the normalization we will keep later, when discussing data about those polynomials.
Table 2. The first polynomials $X_r(u)$, i.e the numerators in $(\mathcal{M})(u,r)$. Data up to $r = 60$ available attached to the source of this arXiv submission or at [Deh07].

$$
\begin{array}{|c|c|}
\hline
r & X_r(u) \\
\hline
1 & 1 \\
2 & u^2 - 2 \\
3 & u^2 - 4 \\
4 & u^2 - 16u^2 + 66 \\
5 & u^2 - 20u^2 + 114 \\
6 & u^2 - 51u^2 + 864u^2 - 554u^2 + 4860 \\
7 & u^2 - 57u^2 + 1134u^2 - 8758u^2 + 8520 \\
8 & u^{10} - 113u^8 + 4620u^6 - 86332u^4 + 68244u^2 - 765660 \\
9 & u^{10} - 121u^8 + 5460u^6 - 115564u^4 + 1053964u^2 + 1457820 \\
10 & u^{14} - 220u^{12} + 18897u^{10} - 831010u^8 + 20196928u^6 - 260164440u^4 + 1428629724u^2 - 2060092440 \\
11 & u^{14} - 230u^{12} + 20997u^{10} - 996820u^8 + 26447168u^6 + 3274116640u^4 + 2707321464u^2 - 3257743024u^2 + 33957330400 \\
12 & u^{18} - 336u^{16} + 52929u^{14} - 4083011u^{12} + 135364922u^{10} - 4906031274u^8 + 37323636100u^6 - 51299431412u^4 + 1371835414728u^2 - 9272551213720 \\
13 & u^{18} - 375u^{16} + 57141u^{14} - 4663655u^{12} + 224398746u^{10} - 667410170u^8 + 105010072036u^6 - 806857605660u^4 + 246121847156u^2 - 1755890884400 \\
14 & u^{22} - 582u^{20} + 141344u^{18} - 18977780u^{16} + 1571817537u^{14} - 84399778978u^{12} + 2962887441370u^{10} - 66386724069396u^8 + 884609361264548u^6 - 6212383525692744u^4 + 19176051246319080u^2 - 13863690471430800 \\
15 & u^{22} - 596u^{20} + 149296u^{18} - 20838716u^{16} + 1807941481u^{14} - 102286957136u^{12} + 3809004157906u^{10} - 9081792433976u^8 + 129818810828836u^6 - 991780802141097u^4 + 3398674810883688u^2 - 2582872696544000 \\
16 & u^{24} - 836u^{22} + 295486u^{20} - 58491716u^{18} + 7245863641u^{16} - 593291868896u^{14} + 328861804018536u^{12} - 122708427320996u^{10} + 29900504376501736u^8 - 444180655702337856u^6 + 361603504484549600u^4 - 13500165816324763200u^2 + 10671549826596200 \\
17 & u^{24} - 852u^{22} + 308606u^{20} - 62999492u^{18} + 8101703961u^{16} - 692989945072u^{14} + 403215231654116u^{12} - 1589496896122752u^{10} + 4109820391053416u^8 - 652964167393180032u^6 + 5757854141713184000u^4 - 235900535012554056400u^2 + 19761261673907754000 \\
18 & u^{30} - 1216u^{28} + 641547u^{26} - 195081422892u^{24} + 38335209063u^{22} - 51718144228292u^{20} + 49575374207253u^{18} - 34535739682003042u^{16} + 172650770228490298u^{14} - 6229001763559681362u^{12} + 157525093809226179252u^{10} - 26868933066310680515376u^8 + 29359555373870511811056u^6 - 1882598606626601435313600u^4 + 58551243124714486977200u^2 - 46993557338039808278421000 \\
19 & u^{30} - 1234u^{28} + 663111u^{26} - 206226048u^{24} + 4162910907u^{22} - 5794171874298u^{20} + 57532061875577u^{18} - 41443936954862628u^{16} + 21719889339059952u^{14} - 81956498407017683696u^{12} + 217486587160406187416u^{10} - 3911133382221678672304u^8 + 453260970074045727302736u^6 - 308475628179498297260520u^4 + 1021901332105402434469800u^2 - 8861284072193198189544000 \\
20 & u^{16} - 1615u^{14} + 1140143u^{12} - 46722835u^{10} + 12459555742u^{8} - 22981207989952u^{6} + 30402375667531565u^{4} - 2946113821578635u^{2} + 21107532245629367310u^{18} - 11640547873874697410u^{16} + 4305831279563655000094u^{14} - 11830702476645259719035320u^{12} + 2237417297918849647921632u^{10} - 2776621834945030636885200u^8 + 209807174770807368884702224u^6 - 8269151494407104783899064u^4 + 12529695816537171356635200u^2 - 609918994091405005455848000 \\
\hline
\end{array}
$$
Table 3. The first polynomials $X^2_h(u)$, i.e. the numerators in $|M|$ up to $h = 30$, available attached to the source of this arXiv submission or at [Deh07].
Table 4. The first polynomials $\tilde{X}_2h(u)$, i.e. the numerators in $|\mathcal{V}|(u, 2h)$. Data up to $h = 30$ available attached to the source of this arXiv submission or at [Deh07].

| $r$ | $\tilde{X}_r(u)$ |
|-----|------------------|
| 2   | $1$              |
| 4   | $1$              |
| 6   | $u^2 - 9$        |
| 8   | $u^2 - 33$       |
| 10  | $u^4 - 90u^2 + 1497$ |
| 12  | $u^6 - 171u^4 + 6867u^2 - 27177$ |
| 14  | $u^8 - 316u^6 + 30702u^4 - 982572u^2 + 6973305$ |
| 16  | $u^8 - 484u^6 + 76902u^4 - 4461348u^2 + 67692705$ |
| 18  | $u^{12} - 766u^{10} + 215847u^8 - 27766980u^6 + 1653658695u^4 - 41530140126u^2 + 33796804585$ |
| 20  | $u^{14} - 1055u^{12} + 421093u^{10} - 79486155u^8 + 7242179715u^6 - 290444510205u^4 + 4099101803991u^2 - 831907513945$ |
| 22  | $u^{16} - 1496u^{14} + 892108u^{12} - 27218080u^{10} + 45430344630u^8 - 4121412379560u^6 + 189676636728876u^4 - 367492353427896u^2 + 1459253047899345$ |
| 24  | $u^{18} - 1961u^{16} + 1566628u^{14} - 658984788u^{12} + 157743552510u^{10} - 21750520014270u^8 + 1678578114026196u^6 - 67707100461703716u^4 + 123511033840081825u^2 - 6787361482472225$ |
| 26  | $u^{20} - 2610u^{18} + 2860437u^{16} - 1718473240u^{14} + 620475009522u^{12} - 139083336332460u^{10} + 19348398203611266u^8 - 1624499041247619480u^6 + 7719029457034594549u^4 - 181309531744966840101u^2 + 150094832620248466425$ |
| 28  | $u^{22} - 3243u^{20} + 4462647u^{18} - 3407674501u^{16} + 158634056782u^{14} - 466277764083726u^{12} + 86845227411024846u^{10} - 10042821279688179978u^8 + 68858208868174130469u^6 - 2569803795584549606727u^4 + 444470604942195922015755u^2 - 26541556083678039006605025$ |
| 30  | $u^{28} - 4190u^{26} + 7631083u^{24} - 7953124300u^{22} + 5258554468937u^{20} - 231332675869890u^{18} + 691451285514065259u^{16} - 141062102715758416040u^{14} + 1947709933654993586171u^{12} - 178110387265822723795970u^{10} + 10376447014337108680338137u^8 - 360731084573922422894990540u^6 + 66647887693999747894954784187u^4 - 51551421669410774166185623070u^2 + 65818312194409161806213714225$ |
Table 5. The degree and the number of real roots of $\tilde{X}_{2h}$. The $h$s for which there are non-real roots are highlighted.

| $h$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-----|----|----|----|----|----|----|----|----|----|----|
| deg($\tilde{X}_{2h}$) | 0  | 0  | 2  | 2  | 4  | 6  | 8  | 8  | 12 | 14 |
| # real roots | 0  | 0  | 2  | 2  | 4  | 6  | 8  | 8  | 12 | 14 |

| $h$ | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|-----|----|----|----|----|----|----|----|----|----|----|
| deg($\tilde{X}_{2h}$) | 16 | 18 | 20 | 22 | 28 | 28 | 30 | 34 | 36 | 38 |
| # real roots | 16 | 18 | 20 | 22 | 28 | 28 | 30 | 34 | 36 | 38 |

| $h$ | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
|-----|----|----|----|----|----|----|----|----|----|----|
| deg($\tilde{X}_{2h}$) | 44 | 46 | 48 | 50 | 52 | 54 | 56 | 62 | 64 | 66 |
| # real roots | 40 | 46 | 44 | 46 | 52 | 54 | 58 | 62 | 66 | 72 |

$X_{42}(u) = u^{44} - 12302u^{42} + 69239935u^{40} - 236610412148u^{38} + 5494541784913270486u^{36} + 114898096065687835213u^{34} - 109447219738494482680u^{32} + 8055331451433281755701013126u^{30} - 461541928978111025394237052u^{28} + 206514429127544387915748094513446u^{26} - 72119441183986979211215158707920u^{24} + 195779645769350260302671962834404502u^{22} - 4099121776759328236737053880626868012604u^{20} + 6541707279609370986861203148250462720819580u^{18} - 79212503734767115379758381371977926243880176u^{16} + 68369808080342487272269608814856434321006115519u^{14} - 4220982650886223501142365592089834534350710857462u^{12} + 1747680080497411049148752639441918166326024419531u^{10} - 4485403962926818267708804978029477583305447285620u^{8} + 6255269370120423323596984565590453401672539709730775u^{6} - 370130877562289934866287460643477762884558511248756u^{4} + 3621605245660597510164219597941635669472733838765625u^{2}.

This polynomial has four non-real roots ($\pm 18.8631835 \pm 0.0090603i$) that show up at once, since they would have to come in pairs of conjugate pairs by evenness of $\tilde{X}_{2h}(u)$. One could wonder why non-real roots show up so late, and if there is actually a good reason for this.

Fact. The polynomials $\tilde{X}_{2h}(u)$ tend to have many, but not all, of their roots real. For instance, for high $h$, $\tilde{X}_{2h}(u)$ has one root very close by to every odd integer between $h$ and $2h$.

We first present graphical clues for this fact in Figure 1, which depicts the position of the real roots for $h = 1$ to $h = 30$. It thus omits the complex roots.

We now explain the fact. It helps at this point to remember that $\tilde{X}_{2h}(u)$ is obtained by summing various $X_r(u)$ for $r \leq 2h$, which are themselves obtained from Equation (29), for instance. Furthermore, the summand in that Equation associated to $d, s, t$ (with $r = d + \sum_i s_i + t_i$) will have poles (as a function of $u = 2k$) at the odd integers $a$ such that $-s_1 \leq a \leq t_1$ (this uses Lemma 5 to expand the determinant in $H^{s_1,s_2,s_3,t_1,t_2}$). For each pole $a$, there are a few summand where this pole comes with multiplicity exactly $\alpha_d(r)$, but for most others the multiplicity is lower (see Equation (33)). So if we sum all of those terms, and multiply by $Y_{2h}(u)$ (the
Figure 1. The roots of $\tilde{X}_2(h)$. The line corresponding to $\tilde{X}_4(\alpha)$, i.e. where the first real zeroes go missing, has been indicated.
common denominator) to obtain $\tilde{X}_{2h}(u)$, a vast majority of terms factor a $(u - a)$ out. We thus have an expression of the form

$$\tilde{X}_{2h}(u) = (u - a)P_1(u) + P_2(u),$$

where the coefficients of $P_1(u)$ are expected to be much bigger than the coefficients of $P_2(u)$ (simply because much more terms are summed to obtain $P_1(u)$ than $P_2(u)$). Hence, we should expect $\tilde{X}_{2h}(u)$ to change sign when $u$ travels along the real axis from below $a$ to above $a$ (because $|P_1(a)| > |P_2(a)|$ and $(u - a)$ changes sign) and we know that a root will be around $u = a$. This is especially true if $a > r/2$, because the restrictions impose then $s_1 > a > s_2$, and as a consequence $\alpha_r(a) = 1$ and the phenomenon described just now is accentuated. We present in Table 6 some numerical data associated to this phenomenon.

It is obvious from Figure 1 that a lot is yet to be understood about the polynomials $\tilde{X}_{2h}(u)$. For instance, it is not clear if asymptotically in $h$ there is a positive proportion of real roots.

| $h$ | largest root of $\tilde{X}_{2h}(u)$ | difference with $2h - 1$ | log. difference |
|-----|------------------------------------|--------------------------|----------------|
| 1   | no root                            | no root                  | no root        |
| 2   | no root                            | no root                  | no root        |
| 3   | 3.0000000000000000000000           | 2.0000000000000000000000 | 0.69315        |
| 4   | 5.744562646380286598               | 1.255437354              | 0.22746        |
| 5   | 8.24482938491831987                | 0.7551076062             | -0.28900       |
| 6   | 10.56820444013080343              | 0.4310795560             | -0.84146       |
| 7   | 12.76945945674733521              | 0.2305405443             | -1.4673        |
| 8   | 14.8604829155973920               | 0.1139515708             | -2.1720        |
| 9   | 16.94550444560344620              | 0.05144955544            | -2.9672        |
| 10  | 18.97894377072905688              | 0.0210562913             | -3.8606        |
| 11  | 20.99206162055831068              | 0.00793837944            | -4.8544        |
| 12  | 22.997383184072186530             | 0.002616815928           | -5.9458        |
| 13  | 24.999198051064882757             | 0.0008019489351          | -7.1285        |
| 14  | 26.99977403173017860              | 0.0002259698270          | -8.3951        |
| 15  | 28.999941044846106152             | 5.895513589 × 10^{-5}    | -9.7388        |
| 16  | 30.999985671005722891             | 1.432899428 × 10^{-5}    | -11.153        |
| 17  | 32.99996783730003824              | 3.261269996 × 10^{-6}    | -12.633        |
| 18  | 34.99999301847217917             | 6.981527821 × 10^{-7}    | -14.175        |
| 19  | 36.99999858891343014              | 1.411086570 × 10^{-7}    | -15.774        |
| 20  | 38.99999972983353984              | 2.701664602 × 10^{-8}    | -17.427        |
| 21  | 40.9999995085836086              | 4.914163914 × 10^{-9}    | -19.131        |
| 22  | 42.9999999914859542               | 8.514045781 × 10^{-10}   | -20.884        |
| 23  | 44.99999999859167358              | 1.408326421 × 10^{-10}   | -22.683        |
| 24  | 46.999999997712180               | 2.228782021 × 10^{-11}   | -24.527        |
| 25  | 48.9999999996618870              | 3.381129731 × 10^{-12}   | -26.413        |
| 26  | 50.9999999999507453              | 4.925468142 × 10^{-13}   | -28.339        |
| 27  | 52.999999999930988               | 6.901186254 × 10^{-14}   | -30.304        |
| 28  | 54.99999999990686               | 9.313971788 × 10^{-15}   | -32.307        |
| 29  | 56.99999999998787               | 1.212486889 × 10^{-15}   | -34.346        |
| 30  | 58.99999999999847               | 1.524414999 × 10^{-16}   | -36.420        |
8. Alternative moments expressions

8.1. Using Macdonald’s ninth variation of Schur functions. Define, as in [NNSY01] and [Nou04], and similarly to [Mac92],
\begin{equation}
\tilde{s}_\lambda^{(R)} := \left| \begin{array}{cc}
\tilde{h}_{\lambda_1-i+j}^{(R-j+1)} & 1 \\
\tilde{h}_{\lambda_2-i+j}^{(R-j+1)} & 1 \\
\end{array} \right|_{|\lambda| \times |\lambda|},
\end{equation}
with
\begin{equation}
\tilde{h}_k^{(R)} := \frac{(R - 1)!}{(R + k - 1)!k!}.
\end{equation}
We first prove that this variation of Schur functions satisfies a Giambelli identity.

**Proposition 17.** Let \( \lambda \) be a partition and \( \{ \bar{\pi} \} \) its Frobenius coordinates, of rank \( d \). Then,
\begin{equation}
\tilde{s}_\lambda^{(R)} = \left| \frac{\tilde{s}(R)_{(s_i | t_j)}}{s!t_j!(1 + s_i + t_j)} \right|_{d \times d}.
\end{equation}

Note how this provides a second determinantal expression for this variation of Schur functions, but with a matrix of different rank.

**Proof.** We intend to use Exercise 3.21 in Macdonald’s book, but to show that the exercise applies, we need to prove:
\begin{equation}
\tilde{s}_\lambda^{(R)} := \det
\begin{pmatrix}
\tilde{h}_{p+1}^{(R)} & \tilde{h}_{p+2}^{(R-1)} & \cdots & \cdots & \cdots & \tilde{h}_{p+q}^{(R-q)} \\
\tilde{h}_{p+2}^{(R-1)} & \tilde{h}_{p+3}^{(R-2)} & \cdots & \cdots & \cdots & \tilde{h}_{p+q+1}^{(R-q)} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\tilde{h}_{p+q}^{(R-q)} & \cdots & \cdots & \cdots & \cdots & \tilde{h}_{p+q+1}^{(R-q)} \\
0 & 0 & \cdots & 0 & 1 & \tilde{h}_{p+q+1}^{(R-q)}
\end{pmatrix}.
\end{equation}

This can be shown by expanding the determinant along the last column to obtain
\begin{equation}
\tilde{s}_\lambda^{(R)} = (-1)^q \tilde{s}_{(p+q+1)}^{(R-q)} + \sum_{i=1}^{q} (-1)^{i+1} \tilde{h}_i^{(R-q)} \tilde{s}_{(p+q-i)}^{(R)}.
\end{equation}

Subtract the LHS from the RHS, proceed by induction on \( q \), factor out \( \frac{\Gamma(R-q)}{\Gamma(R+p+1)} \), and the result then follows from the following equalities, for \( p \) and \( q \) positive integers:
\begin{equation}
\frac{(-1)^q}{(p+q+1)!} - \sum_{i=1}^{q} \frac{(-1)^i}{(p+q-i)!(p+q-i+1)} - \frac{1}{p!q!(1 + p + q)} = \frac{(-1)^q}{(p+q+1)!} + \frac{p+q+q}{(p+q)(p+q+1)p!q!} - \frac{1}{p!q!(1 + p + q)}
\end{equation}
\begin{equation}
= \frac{(-1)^q}{(p+q+1)!} + \frac{q}{(p+q)(p+q+1)p!q!} (\frac{1}{1-p+q}) - \frac{1}{p!q!(1 + p + q)}
\end{equation}
\begin{equation}
= \frac{(-1)^q}{(p+q+1)!} + \frac{q}{(p+q)(p+q+1)p!q!} \frac{(q-1)!}{(1-p+q)(1-p+q+1)(q+1)!} = 0,
\end{equation}
the last one being a consequence of Gauss’s Hypergeometric theorem.

The theorem now results directly from Exercise 3.21 in Macdonald’s book. \( \square \)
In essence, this Proposition allows us to switch from a Giambelli-type expression to a Jacobi-Trudi expression. It immediately leads to a simplified version of Theorem 15.

**Theorem 18.** With $G(\cdot)$ the Barnes $G$-function, and $\tilde{s}_\lambda$ defined as in Equation (35),
\[
\sum_{r>0} (\mathcal{M})(2k, r) \frac{(iz)^r}{r!} = G(k+1)^2 \frac{G(2k+1)}{G(2k)} \sum_{\lambda} \tilde{s}_\lambda(2k) s_\lambda([1^k]) z^{\left\lfloor \lambda \right\rfloor},
\]
in the sense that their coefficients of $z^r$ are equal for fixed $r$ and large enough $k$ so the coefficient in the LHS is defined.

8.2. **Imitating the Cauchy identity.** We can also give an alternative for the expression in Equation (31), proceeding as in Gessel’s theorem in its lead up to the Cauchy identity (see [TW01]). This uses Theorem 18.

**Theorem 19.**
\[
\frac{G(k+1)^2}{G(2k+1)} \sum_{r>0} (\mathcal{M})(2k, r) \frac{(iz)^r}{r!} = \lim_{n \to \infty} \det \begin{pmatrix} (h_{j-i}([1^k]))_{n \times \infty} & (\tilde{h}_{l-j}([1^k]) z^{j-i})_{\infty \times n} \end{pmatrix}
\]
\[
= \lim_{n \to \infty} \det \begin{pmatrix} (h_{j-i}([1^k]) z^{j-i})_{n \times \infty} & (\tilde{h}_{l-j}([1^k]) z^{j-i})_{\infty \times n} \end{pmatrix}
\]
\[
= \lim_{n \to \infty} \det \left( \sum_{l \geq 0} h_{l-i}([1^k]) \tilde{h}_{l-j}([1^k]) z^{l-j} \right)_{n \times n}
\]
\[
= \lim_{n \to \infty} \det \left( \sum_{l \geq 0} \left( l - i + k - 1 \right) \frac{(2k - n - j - 1)!}{(l - j)! (2k - n + l - 1)!} z^{l-j} \right)_{n \times n},
\]
in the sense that their coefficients of $z^r$ are equal for fixed $r$ and large enough $k$ so the coefficient in the LHS is defined. The factorials on the last line should really be evaluated in groups, to give 0 if $l < j$, and $\Gamma(2k-n+j) \Gamma(2k-n+l) (l-j)!$ otherwise.

Note that this can be truncated significantly when we are after only
\[
\sum_{0<r \leq S} (\mathcal{M})(2k, r) \frac{(iz)^r}{r!}
\]
for a finite $S$ (i.e. when we are computing the head of the sequence of polynomials): we can drop the limit in $n$ and settle for a sufficiently big $n$ instead, and then cut the matrices in their infinite directions as well.

In Gessel’s Theorem, in order to get to the other side of the Cauchy identity, one would then observe that the matrix on the last line is Toeplitz, and then use Szegő’s theorem. Of course, that fails here because the matrix on the last line is not Toeplitz.

9. **The result of Conrey and Ghosh**

As explained in the introduction, Conrey and Ghosh’s theorem [CG89] that $\mathcal{J}(2, 1) = \frac{e^{2-5}}{4\pi}$ immediately leads to a conjecture that $|V|(2, 1) = \frac{e^{2-5}}{4\pi}$ as well. Our main concern is that we only know $|V|(2k, 2h)$, for integer $h$, through Equations (7) and (9) (while we would need $h = 1/2$).

We offer in Figure 2 one way to circumvent this problem. The idea is to compute for each fixed integer $h$ the values of the meromorphic continuation in $k$ of
Figure 2. The real part of the situation in the Conrey-Ghosh case. The circle at $(1, 1/2)$ indicates the point for which the value of $J'(2k, 2h)$ is coveted. The dots indicate the locations where Expression (30) applies, and the crosses indicate the points to which that expression is meromorphically continued (for a fixed $h$, i.e. horizontally) thanks to Expression (36). Note that for fixed integer $h$, this continuation hits a pole when crossing the dashed line (and many more before reaching $k = 1$, as $h$ increases: see Figure 1).

$(\mathcal{M})(2k, 2h)$ at $k = 1$ (i.e. at the crosses). This should be enough to know through Equation (7) any value of the form $|\mathcal{M}|(2, 2h)$, which could then finally be used to meromorphically continue $|\mathcal{V}|(2, 2h)$ to $h = 1/2$.

Getting the meromorphic continuation of Equation (30) to $k = 1$ is quite subtle.

**Proposition 20.** Define $(\mathcal{M})(2, r)$ as the meromorphic continuation in $k$ of $(\mathcal{M})(2k, r)$, evaluated at $k = 1$. Then, the exponential generating series of $(\mathcal{M})(2, r)$ is given
by
\[
\sum_{r>0} \frac{(\mathcal{M}(2,r))^{(1z)r}}{r!} = \sum_{d=1}^{\infty} \sum_{\tau, t \in \mathbb{N}^d} \frac{1}{s_i t_j (1 + s_i + t_j)} \left| \prod_{i,j=1}^{d} \frac{1 - s_i t_j}{2^{s_i t_j} (1 + s_i + t_j)} \right|^2 z^{d + \sum s_i + t_i},
\]
where \( \nu(0) = 0 \) when \( t = 0 \), \( \nu(t) = 1 \) when \( t \geq 2 \). The value \( \nu(1) \) is free to choose.

**Proof.** When looking for the analytic continuation in \( k \), most of the formulas we have found so far are misleading. For instance, in light of the remark in footnote 5, one could think that the sums over partitions \( \lambda \) in Expression (31) or Theorem 18 immediately reduce when \( k = 1 \) to sums over partitions \( \lambda \) of length 1, i.e. partitions indexed by a single variable. However, in those cases, the other factor in the summands (i.e. for instance \( \tilde{s}_{\lambda}^{(2k)} \) in Theorem 18) might actually be undefined if we take \( k = 1 \) (in that particular case, when \( l(\lambda) \geq 3^{10} \)).

We can get a better intuition through Expression (30), which we use as a basis of our proof. We are clearly required to find the meromorphic continuation to \( k = 1 \) and for fixed \( s,t \geq 0 \in \mathbb{N} \) of
\[
\frac{\Gamma(2k - t)}{\Gamma(k - t)} \cdot \frac{\Gamma(k + s + 1)}{\Gamma(k + s + 1)}.
\]
The second factor is certainly not a problem and immediately gives \( \frac{1}{s+2} \). For the first factor, we have to look at \( \lim_{k \to -1} \frac{\Gamma(2k - t)}{\Gamma(k - t)} \) for \( t \geq 0 \). Pick any integer \( a \) such that \( 1 + a - t \geq 0 \). Then, using the functional equation for \( \Gamma \), we have
\[
\lim_{k \to -1} \frac{\Gamma(2k - t)}{\Gamma(k - t)} = \lim_{k \to -1} \frac{\Gamma(2k + a - t)}{\Gamma(k + a - t + 1)} \cdot \frac{(k - t)(k - t + 1) \cdots (k - 1) \cdots (k + a - t)}{(2k - t) \cdots (2k - 2) \cdots (2k + a - t - 1)}.
\]
Note that the terms \( \frac{k+1}{2k-2} \) only appear if \( t \geq 2 \). In that case we get
\[
\lim_{k \to -1} \frac{\Gamma(2k - t)}{\Gamma(k - t)} = (1-t) \lim_{k \to -1} \frac{\Gamma(2k + a - t)}{\Gamma(k + a - t + 1)} \cdot \frac{k-1}{2k-2} = \frac{1}{2}(1-t),
\]
and in the case \( t \leq 2 \) the factor of 2 is missing. \( \square \)

One can also check that the values recovered using Proposition 20 agree with the values obtained using \( X_r(2) \) and thus Theorem 1.

For completeness, we give the beginning of the sequence of \( X_r(2) \)'s, for \( r = 1 \) to 15:
\[
1, 2, 0, 18, 50, -6540, -11760, 852180, 1228500, 590126040, 558613440,
-3927324760, 455842787400, 577511644337040, 14904865051876800.
\]

Unfortunately, we fall short of actually finding the full meromorphic continuation of \( (\mathcal{M})(2, r) \) and have to leave this for a further paper.

10. Conclusion

The initial goal was to compute the \( (\mathcal{M})(2k, r) \), \( |\mathcal{M}|(2k, 2h) \) and \( |\mathcal{V}|(2k, 2h) \) more effectively than previously done.

We feel that we have achieved this goal, since we have been able to shed some light (for instance in Theorem 1) on the structure of the results. This structure (rational functions with known denominators) underlines tables already available in [Hug05] or [CRS06]. We have also been able to use these results to obtain better
algorithms to compute those rational functions, thereby extending the data that was available. Much of that data is now available in the source of the arXiv submission, or at [Deh07]. As a corollary we have shown that for large(r) h the roots (in k) of |V|(2k, 2h) cease to all be real, a fluke only for the small-h cases available previously. However, we have not obtained a formula for all |V|(2k, r). In particular, we cannot recover the value of |V|(2, 1), which can be conjectured from Conrey and Ghosh’s result for J(2, 1).

Those methods should also give more general moments, for instance for expressions of the form

\[ \left\langle \left| Z_U(\theta_1) \right|^{2k} \frac{Z_U^r(\theta_2)}{Z_U(\theta_2)} \right\rangle_U(N) \]

or

\[ \left\langle \left| Z_U(\theta_1) \right|^{2k} \frac{Z_U^r(\theta_2)}{Z_U(\theta_2)} \right\rangle_U(N) \]

An expression for those two extensions in the shape of Equation (15) would definitely be available (for instance, in the case of Expression (37), we would most likely have to compute the equivalent of Equation (15) by summing over \( \mu \in (2N)_+^* \)). However, the second part of the computation, the part covered here by Proposition 10, would probably be significantly worsened.

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E-mail address: paul-olivier.dehaye@merton.ox.ac.uk

University of Oxford, Merton College, Merton Street, OX1 4JD, Oxford, United Kingdom