Abstract. We address an interesting question raised by Dos Santos Ferreira, Kenig and Salo [11] about regions $\mathcal{R}_g \subset \mathbb{C}$ for which there can be uniform $L^{\frac{2n}{n-2}} \to L^{\frac{2n}{n+2}}$ resolvent estimates for $\Delta_g + \zeta$, $\zeta \in \mathcal{R}_g$, where $\Delta_g$ is the Laplace-Beltrami operator with metric $g$ on a given compact boundaryless Riemannian manifold of dimension $n \geq 3$. This is related to earlier work of Kenig, Ruiz and the third author [18] for the Euclidean Laplacian, in which case the region is the entire complex plane minus any disc centered at the origin. Presently, we show that for the round metric on the sphere, $S^n$, the resolvent estimates in [11], involving a much smaller region, are essentially optimal. We do this by establishing sharp bounds based on the distance from $\zeta$ to the spectrum of $\Delta_{S^n}$. In the other direction, we also show that the bounds in [11] can be sharpened logarithmically for manifolds with nonpositive curvature, and by powers in the case of the torus, $T^n = \mathbb{R}^n/\mathbb{Z}^n$, with the flat metric. The latter improves earlier bounds of Shen [21]. The work of [11] and [21] was based on Hadamard parametrices for $(\Delta_g + \zeta)^{-1}$. Ours is based on the related Hadamard parametrices for $\cos \sqrt{-\Delta_g}$, and it follows ideas in [25] of proving $L^p$-multiplier estimates using small-time wave equation parametrices and the spectral projection estimates from [24]. This approach allows us to adapt arguments in Béard [3] and Hlawka [15] to obtain the aforementioned improvements over [11] and [21]. Further improvements for the torus are obtained using recent techniques of the first author [5] and his work with Guth [7] based on the multilinear estimates of Bennett, Carbery and Tao [2]. Our approach also allows us to give a natural necessary condition for favorable resolvent estimates that is based on a measurement of the density of the spectrum of $\sqrt{-\Delta_g}$, and, moreover, a necessary and sufficient condition based on natural improved spectral projection estimates for shrinking intervals, as opposed to those in [24] for unit-length intervals. We show that the resolvent estimates are sensitive to clustering within the spectrum, which is not surprising given Sommerfeld’s original conjecture [31] about these operators.

1. Introduction.

The purpose of this paper is to address a question of Dos Santos Ferreira, Kenig and Salo [11] about the regions $\mathcal{R}_g \subset \mathbb{C}$ for which there can be uniform $L^p$-resolvent bounds of the form

$$\|u\|_{L^{\frac{2n}{n-2}}(M)} \leq C_R \|\Delta_g + \zeta\|_{L^{\frac{2n}{n+2}}(M)} \|u\|_{L^{\frac{2n}{n+2}}(M)}, \quad \zeta \in \mathcal{R}_g,$$

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if \( \Delta_g \) is the Laplace-Beltrami operator on a compact boundaryless Riemannian manifold \((M,g)\) of dimension \(n \geq 3\). We shall be able to obtain sharp results in some cases (Zoll manifolds) and improvements over the known results in others (nonpositive curvature). The results that we obtain are related to known bounds for the remainder term in the sharp Weyl formula for \( \Delta_g \), which measures how uniformly its spectrum is distributed.

Our results also are related to Sommerfeld’s original conjecture and reasoning regarding the resolvent operators that are associated with \((1.1)\), \([31]\) (see \([1]\)). Recall that solutions of the Helmholtz equation, 
\[
\Delta_g u(x) + \lambda^2 u(x) = F(x),
\]
give rise to solutions \(w(t,x) = u(x) \sin t\lambda\) of the dynamic equation of forced vibration, 
\[
(\partial_t^2 - \Delta_g)w(t,x) = \sin \lambda t F(x).
\]
A classical problem proposed by Sommerfeld is to determine how solutions of the forced vibration equation are related to solutions of the stationary free vibration equation \((\Delta_g + \lambda^2_j)e_j(x) = 0\) (eigenfunctions) and the possible \(\lambda_j\) (the frequencies). Theorems 1.2 and 1.3 below address this issue. We are also able to give a complete answer to the current variant, \((1.1)\), of this problem for the standard sphere, and an essentially sharp one for Zoll manifolds, due to the tight clustering of the eigenvalues in these cases. Also, in many ways, the pointwise estimates that we employ in the proof of our estimates, as well as the negative results that we obtain, are in accordance with Sommerfeld’s reasoning. Note also how we have introduced a square, \(\zeta = \lambda^2\), into the problem, which will turn out to be a matter of bookkeeping that will simplify the analysis.

Regarding a related problem for Euclidean space, involving the standard Laplacian, \(\Delta_{\mathbb{R}^n}\), on \(\mathbb{R}^n\), \(n \geq 3\), it was shown by Kenig, Ruiz and the third author (KRS) \([18]\) that for each \(\delta > 0\) one has the uniform estimates
\[
\|v\|_{L^2_{\mathbb{R}^n}} \leq C\delta\|((\Delta_{\mathbb{R}^n} + \zeta)v\|_{L^2_{\mathbb{R}^n}}), \quad \text{if } \zeta \in \mathbb{C}, \ |\zeta| \geq \delta, \text{ and } v \in \mathcal{S}(\mathbb{R}^n).
\]
In particular, the bound even holds when \(\zeta\) is in the spectrum of \(-\Delta_{\mathbb{R}^n}\), but of course \((1.1)\) cannot hold if \(\mathcal{R}\) intersects the spectrum of \(-\Delta_g\), \(\text{Spec}(-\Delta_g)\), since the latter is discrete.

For the manifold case, the interesting question is how close \(\mathcal{R}\) can come to \(\text{Spec}(-\Delta_g)\) near infinity and still have \((1.1)\). Given \(\delta > 0\) it is very easy to see (see \(\S 2\)) that \((1.1)\) is valid if \(\mathcal{R}\) is
\[
(1.3) \quad \mathcal{R}_\delta^- = \{\zeta \in \mathbb{C} : |\zeta| \geq \delta \text{ and } \text{Re } \zeta \leq \delta\}.
\]
This essentially follows from elliptic regularity estimates, and one can prove the assertion (see \(\S 2\)) using the spectral projection estimates of the third author \([24]\).

A very nontrivial result due to Shen \([21]\) (see also \([22]\)) for the flat torus, \(\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n\), \(n \geq 3\), and DKS \([11]\) for general compact manifolds of these dimensions is that it also holds when \(\mathcal{R}\) is
\[
(1.4) \quad \mathcal{R}_{\text{DKSS}} = \{\zeta \in \mathbb{C} : (\text{Im } \zeta)^2 \geq \delta \text{Re } \zeta, \text{ Re } \zeta \geq \delta\} \cup \mathcal{R}_\delta^-.
\]
The boundary of the nontrivial part of this region is the curve \(\gamma_{\text{DKSS}}\) in Figure 1 below.

In part following \([24]\), Dos Santos Ferreira, Kenig and Salo \([11]\) used the Hadamard parametrix for \((\Delta_g + \zeta)^{-1}\) and Stein’s \([32]\) oscillatory integral theorem. In his work for the torus Shen \([21]\) was also able to make use of identities akin to the Poisson summation...
formula. Once the parametrix was established the main estimates were similar to the earlier ones in (1.2) for the Euclidean case of KRS [18].

In addition to proving the aforementioned results Dos Santos Ferreira, Kenig and Salo in [11] also asked whether (1.1) can hold for larger regions than the one in (1.4), and in particular if $\mathcal{R}$ is the region

\begin{equation}
\mathcal{R}_{\text{opt}} = \{ \zeta \in \C : |\text{Im } \zeta| \geq \delta \} \cup \mathcal{R}_{\delta}^-.
\end{equation}

Clearly the condition that $|\text{Im } \zeta|$ be bounded below is needed since $\Delta_g$ has discrete spectrum. This region is the one bounded by the curve $\gamma_{\text{opt}}$ in the figure below.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Earlier results and the problem}
\end{figure}

To motivate our improvements over previous results and our progress on this question, let us return to the analogous bounds (1.2) for the Euclidean case. Taking $\zeta = 1 + i\varepsilon$, $0 < \varepsilon \leq 1$, we see that this estimate implies that the multiplier operators

\[ f \to (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} (|\xi|^2 - 1 + i\varepsilon)^{-1} \hat{f}(\xi) d\xi \]

are uniformly bounded from $L^{\frac{2n}{n+2}}(\mathbb{R}^n)$ to $L^{\frac{2n}{n-2}}(\mathbb{R}^n)$. Considering the imaginary part of this operator, one can see (cf. [18], p. 342) that this implies the following variant of a special case of the Stein-Tomas restriction theorem [34] (written in a "TT*" fashion):

\begin{equation}
\left\| (2\pi)^{-n} \int_{\{\xi \in \mathbb{R}^n : ||\xi|-1| \leq \varepsilon \}} e^{ix \cdot \xi} \hat{f}(\xi) d\xi \right\|_{L^{\frac{2n}{n+2}}(M)(\mathbb{R}^n)} \leq C\varepsilon \left\| f \right\|_{L^{\frac{2n}{n+2}}(M)(\mathbb{R}^n)}, \quad 0 < \varepsilon \leq 1.
\end{equation}

The issue of whether (1.1) can hold for regions larger than the one in (1.4) is tied closely to what extent bounds like this one are possible on a given $(M, g)$, which is an interesting question in its own right. To state things more clearly, we need to introduce some notation. First, we label and count with multiplicity the eigenvalues of

\[ P = \sqrt{-\Delta_g} \]
as follows

\[ 0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots. \]

We associate a real orthonormal basis of eigenfunctions \( \{e_j(x)\} \), i.e., \( -\Delta_g e_j(x) = \lambda_j^2 e_j(x) \), and let \( E_j \) denote projection onto the \( j \)th eigenspace:

\[ E_j f(x) = \left( \int_M f(y) e_j(y) \, dV_g \right) e_j(x). \]

Then for a given \( 0 < \varepsilon \leq 1 \), we consider the \( \varepsilon \)-scale spectral projection operators

\[ \chi_{\lambda-\varepsilon, \lambda+\varepsilon} f = \sum_{|\lambda_j - \lambda| \leq \varepsilon} E_j f. \]

From [24], it follows that for \( \varepsilon = 1 \), we have

\[ \left\| \chi_{\lambda-1, \lambda+1} f \right\|_{L^p(M)} \leq C \lambda^{\sigma(p)} \| f \|_{L^{p+1}(M)}, \]

if \( p \geq \frac{2(n+1)}{n-1} \), and \( \sigma(p) = 2n \left( \frac{1}{2} - \frac{1}{p} \right) - 1 \).

The relevant issue for improving the resolvent bounds on a given Riemannian manifold \( (M, g) \) is whether there is a function \( \varepsilon(\lambda) \) taking on values in \((0, 1]\) that decreases to 0 as \( \lambda \to +\infty \), i.e., \( \varepsilon(\lambda) = o(1) \), for which

\[ \left\| \chi_{\lambda-\varepsilon(\lambda), \lambda+\varepsilon(\lambda)} f \right\|_{L^p(M)} \leq C_p \varepsilon(\lambda)^{\sigma(p)} \| f \|_{L^{p+1}(M)}, \]

with a uniform constant \( C_p \), for some \( p \geq \frac{2(n+1)}{n-1} \). The relevant case for improvements of (1.1) beyond (1.4) of course is \( p = \frac{2n}{n-2} \), but we are stating things this way to help motivate negative results as well.

The latter are based on the fact that, as is well known, if (1.9) holds for a given exponent \( p_0 \in \left[ \frac{2(n+1)}{n-1}, \infty \right) \), then it must hold for all larger exponents, including \( p = \infty \). This assertion follows from Sobolev-type inequalities for finite exponents and a Bernstein-type inequality for \( p = \infty \) (see the remark after Lemma 3.1 below). The estimate (1.9) for \( p = \infty \) is equivalent to the statement that the kernel of the operator satisfies the following bounds along the diagonal:

\[ \chi_{\lambda-\varepsilon(\lambda), \lambda+\varepsilon(\lambda)}(x, x) = \sum_{|\lambda_j - \lambda| \leq \varepsilon(\lambda)} |e_j(x)|^2 \leq C \varepsilon(\lambda) \lambda^{n-1}. \]

As a result, the trace of the operator then would have to satisfy the following bounds if (1.9) were valid for some exponent \( \frac{2(n+1)}{n-1} \leq p \leq \infty \),

\[ N(\lambda + \varepsilon(\lambda)) - N((\lambda - \varepsilon(\lambda))-) = \int_M \chi_{\lambda-\varepsilon(\lambda), \lambda+\varepsilon(\lambda)}(x, x) \, dV_g \leq C \varepsilon(\lambda) \lambda^{n-1}, \]

with

\[ N(\lambda) = \# \{ j : \lambda_j \leq \lambda \} \]

being the Weyl counting function. Here \( N(\lambda-) = \lim_{\tau \to \lambda} N(\tau) \), and thus the quantity in (1.10) is \( \# \{ j : \lambda - \varepsilon(\lambda) \leq \lambda_j \leq \lambda + \varepsilon(\lambda) \} \). Consequently, a necessary condition for (1.9) is that the number of eigenvalues in bands of width \( \varepsilon(\lambda) \) about \( \lambda \) should be comparable to the size of the corresponding \( \varepsilon(\lambda) \)-annulus about the sphere of radius \( \lambda \) in \( \mathbb{R}^n \).
No such results are possible on the sphere or Zoll manifolds\footnote{Recall that a Zoll manifold is one for which the geodesic flow is periodic with a common minimal period $\ell$. These manifolds are also sometimes called $P\ell$ manifolds for this reason. See \cite{[4]}.} and so \eqref{1.9} cannot hold in this case with $\varepsilon(\lambda) \to 0$. Correspondingly, the earlier resolvent bounds cannot be improved in this case:

**Theorem 1.1.** Let $(M, g)$ be a Zoll manifold of dimension $n \geq 3$. Then if 

$$\mathbb{R}^+ \ni \tau \to \tau^2 + i\varepsilon(\tau) \tau = \zeta(\tau) \in \mathbb{C}$$

is a curve for which $\varepsilon(\tau) > 0$ for all $\tau$ and $\varepsilon(\tau) \to 0$ as $\tau \to +\infty$, it follows that

\begin{equation}
\sup_{1 \leq \tau \leq \lambda} \left\| (\Delta_g + \varepsilon(\tau))^{-1} \right\|_{L^{2\pi\mathbb{Z}_{\ell}}(M) \to L^{2\pi\mathbb{Z}_{\ell}}(M)} \to +\infty, \text{ as } \lambda \to +\infty. \tag{1.11}
\end{equation}

Moreover, for the Laplacian on the standard sphere, we have

\begin{equation}
\left\| (\Delta_{S^n} + \zeta)^{-1} \right\|_{L^{2\pi\mathbb{Z}_{\ell}}(S^n) \to L^{2\pi\mathbb{Z}_{\ell}}(S^n)} \approx \max(\text{dist}\left(\sqrt[\lambda]{\zeta}, \text{Spec}(-\Delta_{S^n})^{-1}\right), 1) \quad \text{for } \zeta \in \mathbb{C}, \Re \zeta \geq 1, \text{ and } (\Im \zeta)^2 \leq |\Re \zeta|.
\end{equation}

These bounds also hold for any Zoll manifold $(M, g)$ in the subset of this region where $|\Im \sqrt[\lambda]{\zeta} \geq C_M/\Re \sqrt[\lambda]{\zeta}$, with $C_M$ depending on $M$.

We are able to make these precise estimates on the norms in \eqref{1.12} since we know how the eigenvalues are distributed with a great deal of precision. In the case of the standard sphere, $S^n$, the distinct eigenvalues of $-\Delta_{S^n}$ are $k(k + n - 1)$ repeating with multiplicity $\approx k^{n-1}$. If $(M, g)$ is a Zoll manifold with $2\pi$-periodic geodesic flow, then, by a theorem of Weinstein \cite{[35]}, there is a number $\alpha = \alpha(M)$ so that, for large $k$, there are $\approx k^{n-1}$ eigenvalues of $-\Delta_g$ counted with multiplicity in the intervals $[(k+\alpha)^2-C_M, (k+\alpha)^2+C_M]$, $k = 1, 2, \ldots$. The tight clustering of the spectrum is what accounts for \eqref{1.12}.

Although we are only able to compute the resolvent norms in these special cases, we are able to obtain the following result which gives a necessary condition to improvements of the earlier results in terms of the density of eigenvalues on small intervals.

**Theorem 1.2.** Suppose that $0 \leq \tau_k \to +\infty$ as $k \to \infty$ and suppose further that there exist $\varepsilon(\tau_k) > 0$ satisfying $\varepsilon(\tau_k) \to 0$ as $k \to \infty$ and

$$\varepsilon(\tau_k)^{n-1} \left\langle N(\tau_k + \varepsilon(\tau_k)) - N(\tau_k - \varepsilon(\tau_k)) \right\rangle \to +\infty.$$

Then

\begin{equation}
\left\| (\Delta_g + \tau_k^2 + i\varepsilon(\tau_k))^{-1} \right\|_{L^{2\pi\mathbb{Z}_{\ell}}(M) \to L^{2\pi\mathbb{Z}_{\ell}}(M)} \to +\infty.
\end{equation}

Based on this, for there to be positive results for the problem posed in \cite{[11]} for a given $(M, g)$, one would need that $N(\lambda + 1/\lambda) - N(\lambda - 1/\lambda) = O(\lambda^{n-2})$, as $\lambda \to +\infty$, which appears to be an almost impossibly strong condition (cf. \cite{[17], [33]}). This is because the curve $\gamma_{\text{opt}}$, which is the boundary of the region in the problem raised in \cite{[11]} corresponds to $\varepsilon(\lambda) = c/\lambda$.

In the other direction, there are many cases where for $p = \infty$ \eqref{1.9} and hence \eqref{1.10} are valid. The first place where this occurs for \eqref{1.9} seems to be in a paper of Toth, Zelditch and the third author \cite{[28]}, who following earlier work of Zelditch and this author \cite{[29]}.
showed that for any $M$ we have (1.9) with $\varepsilon(\lambda) = \varepsilon_\gamma(\lambda) \to 0$ for a generic choice of metrics. (See (1.8) in \[28\] and §6 in \[29\].)

One of our main results says that improvements for (1.9) for $p = \frac{2n}{n-2}$ provide a necessary and sufficient condition for improvements of the earlier resolvent estimates:

**Theorem 1.3.** Let $(M, g)$ be a compact Riemannian manifold of dimension $n \geq 3$. Suppose that $0 < \varepsilon(\lambda) \leq 1$ decreases monotonically to zero as $\lambda \to +\infty$ and that $\varepsilon(2\lambda) \geq \frac{1}{2} \varepsilon(\lambda)$, $\lambda \geq 1$. Then one has the uniform spectral projection estimates

$$\sum_{|\lambda - \lambda_j| \leq \varepsilon(\lambda)} E_j f \leq C\varepsilon(\lambda)\|f\|_{L^{\frac{2n}{n-2}}(M)}, \quad \lambda \geq 1,$$

(1.13)

if and only if one has the uniform resolvent estimates in the region where $|\Im \zeta| \geq (\Re \zeta)^{\frac{1}{2}} \varepsilon(\Re \zeta)$, $\Re \zeta \geq 1$, i.e.,

$$\|u\|_{L^{\frac{2n}{n-2}}(M)} \leq C\|((\Delta_g + (\lambda + i\mu)^2)u\|_{L^{\frac{2n}{n-2}}(M)},$$

(1.14)

$$\lambda, \mu \in \mathbb{R}, \quad \lambda \geq 1, \quad |\mu| \geq \varepsilon(\lambda), \quad u \in C^\infty(M).$$

It is not difficult to see that in the case of $\mathbb{T}^n$ the estimate (1.13) is valid for certain negative powers of $\lambda$. One can use the lattice point counting argument of Hlawka \[15\] (see §3.5 in \[27\]) to see that for $p = \infty$ (1.9) is valid with $\varepsilon(\lambda) = \lambda^{-\frac{n}{n+1}}$. Using these bounds and a simple interpolation argument one can see that (1.13) and hence (1.14) is valid with $\varepsilon(\lambda) = \lambda^{-\frac{1}{n+2}}$. Using the multilinear estimates of Bennett, Carbery and Tao \[2\] and recent techniques of the first author and Guth \[7\] as well as techniques from his recent work \[3\]-\[4\] one can do better and therefore obtain the following power improvements over the earlier ones of Shen \[21\].

**Theorem 1.4.** If $\Delta_{\mathbb{T}^n}$ is the Laplacian on the flat torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ of dimension $n \geq 3$, then (1.13) is valid with

$$\varepsilon(\lambda) = \lambda^{-\varepsilon_n},$$

for some $\varepsilon_n > 0$. In fact for $n = 3$ one may take any $\varepsilon_3$ satisfying

$$\varepsilon_3 < \frac{85}{252}.$$  

For higher odd dimensions one may take any $\varepsilon_n$ satisfying

$$\varepsilon_n < \frac{2(n - 1)}{n(n + 1)},$$

and for even dimensions $n \geq 4$ one may take any $\varepsilon_n$ satisfying

$$\varepsilon_n < \frac{2(n - 1)}{n^2 + 2n - 2}.$$  

Consequently, for this choice of $\varepsilon_n$ we have (1.1) with $R$ equal to

$$R_{\mathbb{T}^n} = \{\zeta \in \mathbb{C} : |\Im \zeta| \geq \delta(\Re \zeta)^{\frac{1}{2}} \varepsilon_n, \Re \zeta \geq \delta\} \cup R_{\mathbb{T}^n}^-.$$  

We remark that even though the argument of Hlawka \[15\] shows that (1.9) is valid for $p = \infty$ with $\varepsilon(\lambda) = \lambda^{-\frac{n}{n+1}}$, the best we are able to do for $p = \frac{2n}{n-2}$ is $\varepsilon(\lambda) = \lambda^{-\varepsilon_n}$, with $\varepsilon_n \to 0$ as $n \to \infty$. It would be interesting whether for the torus there can be further improvements. In the case of 2-dimensions, this may be related to a theorem
of Zygmund [36], which says that $L^2(T^2)$-normalized eigenfunctions have bounded $L^4$-norms, while in higher dimensions further improvements might be related to recent work of the first author [6].

Besides the torus, it is known that for $p = \infty$ the arguments of Bérard [3] show that one may take $\varepsilon(\lambda) = (\log(2 + \lambda))^{-1}$ in (1.9) (see §3.6 in [27]). Correspondingly, we have the following

**Theorem 1.5.** Fix a Riemannian manifold $(M, g)$ of dimension $n \geq 3$ with nonpositive sectional curvatures. Then given $\delta > 0$ we have (1.1) with $R$ equal to

$$R_{\text{neg}} = \{\zeta \in \mathbb{C} : |\text{Im } \zeta| \geq \delta (\text{Re } \zeta)^2 / (\log(2 + \text{Re } \zeta)), \; \text{Re } \zeta \geq \delta\} \cup R^-_{\delta}.$$ 

Our proof requires the special case of $p = \frac{2n}{n-2}$ of a recent result of Hassell and Tacy [14] showing that under the assumption of negative curvature (1.9) is valid for all $p > \frac{2(n+1)}{n-1}$ if $\varepsilon(\lambda) = (\log(2 + \lambda))^{-1}$ (with constants depending on $p$). For the sake of completeness, we shall present the proof in §5 where Theorem 1.5 will be proved. Related results to those of [14] in 2-dimensions for the complementary range of exponents $2 < p < 6$ were also recently obtained by Zelditch and the third author [30], and related results for eigenfunction restriction bounds were also obtained by X. Chen [10].

In the figure below we draw the boundary of the regions described by the above results.

![Figure 2](image_url)

**Figure 2.** Current results for $(\Delta_g + \zeta)^{-1}$

Our techniques are similar to the earlier ones in [24], [18], [11], [21] and [22] in many ways, but there are some important differences. First, instead of using a Hadamard parametrix for $(\Delta_g + \zeta)^{-1}$, we use short-time ones for $\cos tP$, $P = \sqrt{-\Delta_g}$, defined for $f \in C^\infty(M)$ by

$$(\cos tP)f = \sum_{j=0}^\infty \cos t\lambda_j E_j f.$$ (1.15)

Doing so allows us to adopt arguments of the third author [25] that were originally used to prove certain sharp results for Riesz means on compact manifolds. The approach is
also simplified by writing $\zeta = z^2$ so that we can write $(\Delta_g + z^2)^{-1}$ in a simple formula that involves $\cos tP$ and the extension of the Fourier transform of the Poisson integral kernel to $\mathbb{C} \setminus \mathbb{R}$. These modifications allow us to obtain our main estimates just by using stationary phase and not appealing to oscillatory integral theorems, which is useful in proving the improved results in Theorems 1.4 and 1.5.

When we write $\zeta = z^2$, we shall take $z = \lambda + i\mu$ where $\lambda \in \mathbb{R}$ and (usually) $\mu \in \mathbb{R} \setminus \{0\}$. As we shall review in the beginning of §2, it is easy to prove the resolvent estimates when $|\Re z|$ is bounded above by a fixed constant. Thus, we shall concern ourselves with the situation where it is large and since we squaring $z$, it suffices to treat the case where $\Re z \gg 1$. Then the question is how small can $|\mu|$ be so that we can obtain uniform estimates. In Theorem 1.1, $\mu$ then essentially becomes $\varepsilon(\lambda)$, and so, as stated in Theorem 1.2, we cannot take $\mu(\lambda) = o(1)$ as $\lambda \to \infty$ for the sphere or Zoll manifolds.

For the positive results, Theorem 1.4 says that we can take $\mu(\lambda) = \lambda^{-\varepsilon_n}$ for $\mathbb{T}^n$, and Theorem 1.5 says that under the assumption of nonpositive curvature we can take it to be $1/\log \lambda$. Finally, the curve $\gamma_{\text{opt}}$ which corresponds to the region in the problem raised by Dos Santos Ferreira, Kenig and Salo [11] corresponds to $\mu(\lambda) = 1/\lambda$ in this notation, and we already indicated why Theorem 1.2 indicates how difficult it would be to obtain results curves close to this one. In the following figure, we show the regions involving uniform estimates for $(\Delta_g + (\lambda + i\mu)^2)^{-1}$.

This paper is organized as follows. In the next section, we prove estimates for a localized version of $(\Delta_g + z^2)^{-1}$ that are uniform in $z \in \mathbb{C} \setminus \mathbb{R}$. This reduces bounds for $(\Delta_g + z^2)^{-1}$ to proving remainder estimates for the difference of the actual inverse and the localized version. These remainder terms are handled by (1.9). After proving the localized estimates, we give the simple proof of Theorem 1.3. We then successively treat the results for the sphere $S^n$, the torus $\mathbb{T}^n$ and manifolds with nonpositive curvature. Also, in what follows, $C$ denotes a finite positive constant which might change at each occurrence, although we shall be careful with issues of uniformity.

![Figure 3. Boundary of regions for $((\Delta_g + (\lambda + i\mu)^2)^{-1}$](image_url)
2. Uniform bounds for a local term.

The purpose of this section is to prove uniform estimates for a localized version of the resolvents $\left( \Delta_g + \zeta \right)^{-1}$, $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$. These are the natural analogs for the current setting of the uniform Sobolev estimates for the Euclidean case of Kenig, Ruiz and the third author [18].

To state these let us derive a natural formula for the Sommerfeld-Green’s function (cf. [1]) for the resolvent

$$S(x, y) = \sum_j e_j(x) e_j(y) \frac{1}{\zeta - \lambda_j^2}, \quad x, y \in M, \quad \zeta \in \mathbb{C} \setminus \mathbb{R}_+.$$  

Since the formula involves $\lambda_j^2$, and since the map $\mathbb{C} \ni z \to z^2 \in \mathbb{C}$ is onto, it is natural to write $\zeta$ as a square

$$\zeta = (\lambda + i\mu)^2, \quad \lambda, \mu \in \mathbb{R}, \quad \mu \neq 0$$

so that we can use the following

**Lemma 2.1.** For all $\mu \in \mathbb{R}$, $\mu \neq 0$

$$\int_{-\infty}^{\infty} \frac{e^{-\tau t}}{\mu - i\tau} \, d\tau = 2\pi \text{sgn}\mu H(\mu t)e^{-\mu t},$$

where $H(t)$ is the Heaviside function which equals one for $t \geq 0$ and 0 for $t < 0$. Also, for all $\lambda, \mu \in \mathbb{R}$, $\mu \neq 0$,

$$\int_{-\infty}^{\infty} \frac{e^{-i\tau t}}{\tau^2 - (\lambda + i\mu)^2} \, d\tau = \frac{i\pi \text{sgn}\mu}{\lambda + i\mu} e^{i(\text{sgn}\mu)|\lambda|} e^{-|\mu|},$$

**Proof.** To prove (2.1), we may assume that $\mu > 0$ since the other case follows from reflection. Then if $0 < \varepsilon < \mu$, Cauchy’s integral formula yields

$$\int_{-\infty}^{\infty} \frac{e^{-\tau t}}{\mu - i\tau} \, d\tau = e^{-(\mu - \varepsilon)t} \int_{-\infty}^{\infty} e^{-i\tau t} \frac{1}{\varepsilon - i\tau} \, d\tau.$$  

Letting $\varepsilon \searrow 0$, the right side tends to $e^{-\mu t}$ times the Fourier transform of $i(\tau + i)^{-1}$, and since the latter is $2\pi H(t)$, we get (2.1).

Since

$$\frac{1}{\tau^2 - (\lambda + i\mu)^2} = \frac{1}{2(\lambda + i\mu)} \left( \frac{1}{\tau - \lambda - i\mu} - \frac{1}{\tau + \lambda + i\mu} \right),$$

(2.1) implies (2.2). Alternatively, since for $\mu > 0$ the right side of (2.2) is

$$\frac{\pi i}{\lambda + i\mu} e^{i(\lambda + i\mu)|\lambda|} e^{-|\mu|},$$

we conclude that for $\lambda = 0$ and $\mu > 0$, (2.2) is just the formula for the Fourier transform of the Poisson kernel:

$$\frac{1}{\pi} \int_{-\infty}^{\infty} e^{-i\tau t} \frac{\varepsilon}{\tau^2 + \varepsilon^2} \, d\tau = e^{-\varepsilon|t|}, \quad \varepsilon > 0.$$  

Since both sides of (2.2) are analytic in the half-plane $\{z = \lambda + i\mu, \mu > 0\}$, we conclude that (2.2) must be valid when $\mu > 0$. The formula for $\mu < 0$ follows from this and the fact that the left side of (2.2) is an even function of $z = \lambda + i\mu$. □
Using the formula \( (2.2) \) we get from Fourier’s inversion formula
\[
(\lambda_j^2 - (\lambda + i\mu)^2)^{-1} = \frac{\pi i \text{sgn} \mu}{2\pi(\lambda + i\mu)} \int_{-\infty}^{\infty} e^{i(\text{sgn} \mu)\lambda t} e^{-|\mu|t} e^{i\lambda_j t} dt
\]
Consequently, if, as before, we set,
\[
P = \sqrt{-\Delta_g},
\]
then we have for \( f \in C^\infty \) the formula
\[
(2.3)
\]
\[
(\Delta_g + (\lambda + i\mu)^2)^{-1} f = \frac{\text{sgn} \mu}{i(\lambda + i\mu)} \int_0^\infty e^{i(\text{sgn} \mu)\lambda t} e^{-|\mu|t} \cos tP dt.
\]
In other words, the Sommerfeld-Green kernel for the resolvent, \( (\Delta_g + (\lambda + i\mu)^2)^{-1} \), is given by the formula
\[
(2.4)
\]
\[
S(x,y) = \frac{\text{sgn} \mu}{i(\lambda + i\mu)} \int_0^\infty e^{i(\text{sgn} \mu)\lambda t} e^{-|\mu|t} \sum_{j=0}^\infty \cos t\lambda_j e_j(x)e_j(y) dt.
\]
Here \( \cos tP \) is as in (1.15). Put another way, \( u(t,x) = (\cos tP f)(x) \) is the solution of the Cauchy problem for the wave operator on \( (M,g) \),
\[
\begin{cases}
(\partial_t^2 - \Delta_g)u(t,x) = 0 \\
u(0,\cdot) = f, \quad \partial_t u(0,\cdot) = 0.
\end{cases}
\]
To define the localized resolvent operators, we fix a function \( \rho \in C^\infty(\mathbb{R}) \) satisfying
\[
\rho(t) = 1, \quad t \leq \delta_0/2, \quad \rho(t) = 0, \quad t \geq \delta_0,
\]
where
\[
\delta_0 = \min(1, \text{Inj } M/2),
\]
with \( \text{Inj } M \) denoting the injectivity radius of \( (M,g) \). The localized versions of resolvent operators then are given by
\[
(2.6)
\]
\[
\mathcal{S}_{\text{loc}}(\lambda,\mu) = \frac{\text{sgn} \mu}{i(\lambda + i\mu)} \int_0^\infty \rho(t)e^{i(\text{sgn} \mu)\lambda t} e^{-|\mu|t} \cos tP dt,
\]
and we have the following uniform KRS-type estimates for them:

**Theorem 2.2.** Fix a compact boundaryless Riemannian manifold \( (M,g) \) of dimension \( n \geq 3 \). Then, given \( \delta > 0 \), there is a constant \( C_5 \) so that
\[
(2.7)
\]
\[
\|\mathcal{S}_{\text{loc}}(\lambda,\mu)f\|_{L^{\frac{2n}{n+2}}(M)} \leq C_5\|f\|_{L^{\frac{2n}{n+2}}(M)},
\]
for all
\[
(2.8)
\]
\[\lambda, \mu \in \mathbb{R}, \quad \mu \neq 0, \quad |\lambda + i\mu| \geq \delta.\]

At the end of this section we shall give a simple argument showing how the estimates of Dos Santos Ferreira, Kenig and Salo [11] are an immediate corollary of this result and the spectral projection bounds in [24]. Later we shall also use Theorem 2.2 to obtain the improvements of the resolvent bounds in [11] and [21], mentioned in the introduction.
Let us first realize that it is easy to estimate the operators in (2.6) when \(|\lambda|\) is bounded by a fixed constant \(A\), i.e., \(|\lambda| \leq A\). We note a simple integration by parts argument shows that the function

\[
\mathbb{R} \ni \tau \rightarrow \mathcal{G}_{\lambda, \mu}(\tau) = \frac{\text{sgn} \mu}{i(\lambda + i\mu)} \int_0^\infty \rho(t) e^{i(\text{sgn} \mu) \lambda t} e^{-|\mu| t} \cos \tau dt,
\]

is uniformly bounded by \(C(1 + \tau^2)^{-1}\), under this assumption if \(|\lambda + i\mu| \geq \delta\), with \(C = C_{\delta, A}\) depending on \(A\) and \(\delta > 0\). Consequently, we can obtain (2.7) in this case by appealing to the following easy consequence of the spectral projection estimates of the third author.

**Lemma 2.3.** Given a fixed compact Riemannian manifold of dimension \(n \geq 3\) there is a constant \(C\) so that whenever \(\alpha \in C(\mathbb{R}_+^n)\) we have

\[
\|\alpha(P)f\|_{L^{\frac{2n}{n+2}}(M)} \leq C \left( \sup_{\tau \in \mathbb{R}_+} (1 + \tau^2)|\alpha(\tau)| \right) \|f\|_{L^{\frac{2n}{n+2}}(M)},
\]

if \(\alpha(P)\) is the operator defined for \(f \in C^\infty(M)\) by

\[
\alpha(P)f = \sum_{j=0}^\infty \alpha(\lambda_j)E_j f.
\]

Also, if we let

\[
\alpha_k(P)f = \sum_{\lambda_j \in [k-1, k)} \alpha(\lambda_j)E_j f, \quad k = 1, 2, 3, \ldots,
\]

then if \(p \geq \frac{2(n+1)}{n-1}\) and \(\sigma(p)\) is as in (1.8)

\[
\|\alpha_k(P)f\|_{L^p(M)} \leq Ck^{\sigma(p)} \left( \sup_{\tau \in [k-1, k)} |\alpha(\tau)| \right) \|f\|_{L^{\frac{2n}{n+2}}(M)}.
\]

**Proof.** Let us start with the second inequality. If \(\chi_{\lambda}\) are the spectral projection operators

\[
\chi_{\lambda} f = \sum_{\lambda_j \in [\lambda-1, \lambda)} E_j f,
\]

then it was shown in [24] that for \(p \geq \frac{2(n+1)}{n-1}\) we have the following estimates for their operator norms

\[
\|\chi_{\lambda}\|_{L^2(M) \rightarrow L^p(M)} = \|\chi_{\lambda}\|_{L^{\frac{2n}{n+2}}(M) \rightarrow L^p(M)} \leq C\lambda^{\frac{p}{2}}, \quad \lambda \geq 1.
\]

Therefore since \(\alpha_k(P) = \chi_k \circ \alpha_k(P) \circ \chi_k\), for \(k = 1, 2, 3, \ldots\), we have

\[
\|\alpha_k(P)f\|_{L^p(M)} \leq Ck^{\sigma(p)} \|\alpha_k(P)\chi_k f\|_{L^2(M)} \\
\leq Ck^{\sigma(p)} \left( \sup_{\tau \in [k-1, k]} |\alpha(\tau)| \right) \|\chi_k f\|_{L^2(M)} \\
\leq Ck^{\sigma(p)} \left( \sup_{\tau \in [k-1, k]} |\alpha(\tau)| \right) \|f\|_{L^{\frac{2n}{n+2}}(M)},
\]

giving us the last part of the lemma.

To prove the \(L^{\frac{2n}{n+2}}(M) \rightarrow L^{\frac{2n}{n+2}}(M)\) bounds for the non-truncated operator, since \(\frac{2n}{n+2} < 2 < \frac{2n}{n-2}\), we can use Littlewood-Paley theory to reduce to the case where the
Fourier multiplier $\alpha$ has dyadic support (see [26]). In other words, it is enough to prove the first inequality under the additional assumption that for some $j = 0, 1, 2, \ldots$

$$\alpha(\tau) = 0 \quad \text{if} \quad \tau \notin [2^{j-1}, 2^j].$$

But since then

$$\|\alpha(P)f\|_{L^{2n/(2n-2)}(M)} \leq \sum_{k \in [2^{j-1}, 2^j]} \|\alpha_k(P)f\|_{L^{2n/(2n-2)}(M)},$$

this desired estimate follows from what we have just done, since $\sigma(p) = 1$ when $p = 2^{n-2}$.

□

Returning to the proof of Theorem 2.2, by what we have just done, we are left with proving the nontrivial part of (2.7), which would be to show that the estimate holds when $|\lambda| \geq 1$. After possibly multiplying $\lambda + i\mu$ by $-1$, we can simplify the notation further, and note that our task is reduced to proving the uniform estimates under the assumption that $\lambda \geq 1$ and $\mu \neq 0$.

To prove the bounds for this case, we shall use the Hadamard parametrix for the wave equation and the following stationary phase estimates that are essentially in the paper of Kenig, Ruiz and the third author [18]:

**Proposition 2.4.** Let $n \geq 2$ and assume that $a \in C^\infty(\mathbb{R}_+)$ satisfies the Mihlin-type condition that for each $j = 0, 1, 2, \ldots$

$$\left| \frac{d^j}{ds^j}a(s) \right| \leq A_j s^{-j}, \quad s > 0.$$  \hspace{1cm} (2.10)

Then there is a constant $C$, which depends only on the size of finitely many of the constants $A_j$ so that for every $w \in \mathbb{C} \setminus \mathbb{R}$

$$\left| \int_{\mathbb{R}^n} \frac{a(|\xi|)e^{ix \cdot \xi}}{|\xi| - w} \, d\xi \right| \leq C \left( |x|^{1-n} + \frac{|w|/|x|^{n+1}}{n+1} \right).$$  \hspace{1cm} (2.11)

Note that this estimate immediately implies the following bounds which will be useful later on

$$\left| \int_{\mathbb{R}^n} \frac{a(|\xi|)e^{ix \cdot \xi}}{|\xi| + w_1(|\xi| + w_2)} \, d\xi \right| \leq \frac{C}{|w_1 - w_2|} \left( |x|^{1-n} + \frac{|w_1| + |w_2|}{|x|^{n+1}} \right), \quad w_0, w_1 \in \mathbb{C} \setminus \mathbb{R}.$$  \hspace{1cm} (2.12)

As in (2.11), the constant $C$ depends only on $n$ and finitely many of the constants in (2.10).

**Proof of Proposition 2.4** Recall that if $b \in C^\infty(\mathbb{R}_+)$ satisfies

$$\left| \frac{d^j}{ds^j}b(s) \right| \leq B_j s^{-j}, \quad s > 0,$$

then we have that the Fourier transform of $b(|\xi|)$ is a function satisfying

$$\left| \int_{\mathbb{R}^n} e^{-ix \cdot \xi}b(|\xi|) \, d\xi \right| \leq B|x|^{1-n},$$
where for each $n$, the constant $B$ depends only on the size of finitely many of the constants $B_j$. Therefore since the dilates of $a(|\xi|)$, $\xi \mapsto a(\varepsilon|\xi|)$ satisfy exactly the same bounds as in (2.10), we conclude from a change of scale that there must be a uniform constant $C$ so that the left side of (2.11) is $\leq C|x|^{-n}$ if $w = |w|e^{i\theta}$ and $\theta \in (-\pi, \pi)\setminus[-\pi/4, \pi/4]$.

As a result, by a change of scale, we would have (2.11) if we could show that

$$\left| \int_{\mathbb{R}^n} \frac{a(|\xi|)e^{ix\cdot \xi}}{|\xi| - 1 + i\varepsilon} \, d\xi \right| \leq C \left( |x|^{-n} + |x|^{-\frac{n+1}{2}} \right), \quad 0 < \varepsilon < 1.$$ 

If $\beta \in C_0^\infty(\mathbb{R})$ equals one when $s \in [1/2, 2]$ and is supported in $[1/4, 4]$, by the above argument, if we replace $a(|\xi|)$ by $(1 - \beta(|\xi|))a(|\xi|)$ then the resulting terms are bounded by a uniform constant times $|x|^{-n}$.

If we let $\alpha(|\xi|) = \beta(|\xi|)a(|\xi|)$, we conclude that it suffices to show that if $\alpha \in C_0^\infty(\mathbb{R})$ is supported in $\{s \in \mathbb{R} : s \in (1/4, 4)\}$, then

$$\left| \int_{\mathbb{R}^n} \frac{\alpha(|\xi|)e^{ix\cdot \xi}}{|\xi| - 1 + i\varepsilon} \, d\xi \right| \leq C(1 + |x|)^{-\frac{n+1}{2}}, \quad 0 < \varepsilon < 1. \tag{2.13}$$

To prove this we note that since

$$\frac{1}{|\xi| - 1 + i\varepsilon} = \frac{|\xi| - 1}{(|\xi| - 1)^2 + \varepsilon^2} - \frac{i\varepsilon}{(|\xi| - 1)^2 + \varepsilon^2},$$

the estimate trivially holds if $|x| \leq 1$. To handle the remaining case, we recall that we have the following stationary phase formula for the Fourier transform of the Euclidean surface measure on $S^{n-1}$, $n \geq 2$,

$$\int_{S^{n-1}} e^{ix \cdot \omega} \, ds(\omega) = |x|^{-\frac{n-1}{2}} c_+(|x|) e^{i|x|} + |x|^{-\frac{n+1}{2}} c_-(|x|) e^{-i|x|}, \tag{2.14}$$

where for, say, $r \geq 1/4$, the coefficients satisfy

$$\left| \frac{d^j}{dr^j} c_+(r) \right| + \left| \frac{d^j}{dr^j} c_-(r) \right| \leq C_r r^{-j}, \quad j = 0, 1, 2, \ldots. \tag{2.15}$$

Therefore, the integral in the left side of (2.13) is the sum of the two terms

$$|x|^{-\frac{n-1}{2}} \int_{1/4}^4 \frac{\alpha(\sigma) c_+(\sigma|x|) e^{\pm i|\sigma||x|}}{\sigma - 1 + i\varepsilon} \, d\sigma.$$ 

Since the Fourier transforms of the functions $\sigma \mapsto \alpha(\sigma) c_\pm(\tau|x|) e^{\pm i|\sigma||x|}$ in $C_0^\infty((1/4, 4))$, have uniformly bounded $L^1$-norms when $|x| \geq 1$, we get (2.13) for such $x$ by Lemma 2.1.

Let us turn to the proof of the missing uniform bounds

$$\|\mathcal{S}_{\text{loc}}(\lambda, \mu)f\|_{L^{\frac{2n}{n+2}}(M)} \leq C\|f\|_{L^{\frac{2n}{n+2}}(M)}, \quad \lambda \geq 1, \mu \in \mathbb{R}\setminus\{0\}. \tag{2.16}$$

Note that the function $\mathcal{S}_{\text{loc}}(\lambda, \mu)(\tau)$ in (2.9) satisfies

$$|\mathcal{S}_{\text{loc}}(\lambda, \mu)(\tau)| \leq C_\tau^{-2}, \quad \text{if } \tau \geq 2\lambda \geq 1,$$

for some uniform constant $C$. This just follows from an integration by parts argument using (2.9) or one can use (2.38) below and (2.9). On account of this, if we fix a function $b \in C_0^\infty(\mathbb{R})$ satisfying $b(s) = 1$ for $s \leq 2$ and $b(s) = 0$ for $s \geq 4$, by Lemma 2.3 we have the following uniform bounds

$$\left\| (I - b(P/\lambda)) \circ \mathcal{S}_{\text{loc}}(\lambda, \mu)f \right\|_{L^{\frac{2n}{n+2}}(M)} \leq C\|f\|_{L^{\frac{2n}{n+2}}(M)}.$$
Thus, in order to prove (2.16), it suffices to show that we have the uniform bounds
\[
\|b(P/\lambda) \circ \mathcal{S}_{loc}(\lambda, \mu)f\|_{L^\frac{2n}{n-2}(M)} \leq C\|f\|_{L^\frac{2n}{n-2}(M)}; \quad \lambda \geq 1, \mu \in \mathbb{R}\setminus\{0\}.
\]

We shall do this by dyadically breaking up the $t$-integral in the definition of $\mathcal{S}_{loc}(\lambda, \mu)$, estimating the pieces in $L^2$ and $L^\infty$ and interpolating. To this end, let us fix a function $\beta \in C^\infty_0(\mathbb{R})$ satisfying
\[
\beta(t) = 0, \quad t \notin [1/2, 2], \quad |\beta(t)| \leq 1, \quad \text{and} \quad \sum_{j=0}^\infty \beta(2^{-j}t) = 1, \quad t > 0,
\]
Then for a given $\lambda \geq 1$ and $\mu \in \mathbb{R}\setminus\{0\}$, we define operators
\[
S_jf = \frac{1}{i(\lambda + i\mu)} \int_0^\infty \beta(\lambda 2^{-j})\rho(t)e^{i(\text{sgn} \mu)\lambda t}e^{-|\mu|t} \cos tPf \, dt, \quad j = 1, 2, 3, \ldots,
\]
and
\[
S_0f = \frac{1}{i(\lambda + i\mu)} \int_0^\infty \tilde{\rho}(\lambda t)\rho(t)e^{i(\text{sgn} \mu)\lambda t}e^{-|\mu|t} \cos tPf \, dt,
\]
with
\[
\tilde{\rho}(t) = (1 - \sum_{j=0}^\infty \beta(2^{-j}t)) \in C^\infty(\mathbb{R}),
\]
and consequently, $\tilde{\rho}(t) = 0$ if $|t| \geq 4$.

By Corollary 4.3.2 in [26] or multiplier theorems in [20],
\[
\sup_{\lambda \geq 1} \|b(P/\lambda)\|_{L^\frac{2n}{n-2}(M) \rightarrow L^\frac{2n}{n-2}(M)} < \infty,
\]
and therefore, since $\mathcal{S}_{loc}(\lambda, \mu) = \text{sgn} \mu \sum_{j=0}^\infty S_j$, we would obtain (2.17) if we could show that there is a uniform constant $C$ so that for $\lambda \geq 1$ and $\mu \in \mathbb{R}\setminus\{0\}$ we have
\[
\|b(P/\lambda) \circ S_0f\|_{L^\frac{2n}{n-2}(M)} \leq C\|f\|_{L^\frac{2n}{n-2}(M)},
\]
as well as
\[
\|S_jf\|_{L^\frac{2n}{n-2}(M)} \leq C2^{-j/4}\|f\|_{L^\frac{2n}{n-2}(M)}, \quad j = 1, 2, \ldots.
\]
Note that $S_j = 0$ if $2^j/\lambda \geq 2$.

Let us start with the first estimate since it is fairly trivial. Since the kernel of $b(P/\lambda) \circ \cos tP$ is
\[
\sum_{\lambda_k \leq 4\lambda} b(\lambda_j/\lambda) \cos t\lambda_j e_j(x)e_j(y),
\]
and by the local Weyl law (see e.g., (4.2.2) in [26]), $\sum_{\lambda_k \leq 4\lambda} |e_k(x)|^2 \leq C\lambda^n$, $x \in M$, we deduce that
\[
|\{b(P/\lambda) \circ \cos tP\}(x, y)| \leq C\lambda^n.
\]
Since $\rho(\lambda t) = 0$ for $t \geq 4/\lambda$ we deduce from (2.20) that the kernel $K_0(x, y)$ of $b(P/\lambda) \circ S_0$ can be uniformly bounded as follows,
\[
|K_0(x, y)| \leq C\lambda^{n-2}.
\]
Since, by the finite propagation speed for the wave operator, we have \((\cos tP)(x, y) = 0\) if \(d_g(x, y) > t\), where \(d_g(x, y)\) is the geodesic distance between \(x\) and \(y\), we also have that \(K_0(x, y) = 0\), if \(d_g(x, y) \geq 4/\lambda\).

These two facts about \(K_0\) lead to (2.21) after an application of Young’s inequality.

We are now left with proving (2.22). By interpolation, we would get this estimate if we could establish the uniform bounds

\[
\|S_j f\|_{L^2(M)} \leq C \lambda^{-1}(2^j/\lambda)\|f\|_{L^2(M)},
\]

and

\[
\|S_j f\|_{L^\infty(M)} \leq C \lambda^{-2}2^{-\frac{\cdot - 1}{2}}\|f\|_{L^1(M)},
\]

since \(\frac{n-2}{2n} = \frac{1}{2} \cdot \frac{n-2}{n}\) and

\[
2^{-\frac{n}{2}} = (\lambda^{-1}(2^j/\lambda))^{\frac{n-2}{n}} (\lambda^{-2}2^{\frac{n-1}{2}})^{\frac{2}{n}}.
\]

The first estimate is easy. If we use (2.19), the fact that \(\cos tP\) is bounded on \(L^2\) with norm 1 and (2.19), we get

\[
\|S_j f\|_{L^2} \leq \lambda^{-1} \int_{2^{i-1}/\lambda \leq t \leq 2^{i+1}/\lambda} \|\cos tP f\|_{L^2} dt \leq 4\lambda^{-1}(2^j/\lambda)\|f\|_{L^2(M)},
\]

as asserted.

This leaves us with proving (2.24), which is the same as showing the kernel \(K_j\) of \(S_j\) satisfies

\[
|K_j(x, y)| \leq C \lambda^{-2}2^{-\frac{\cdot - 1}{2}}\|f\|_{L^1(M)},
\]

if, by (2.19),

\[
K_j(x, y) = \frac{1}{i(\lambda + i\mu)} \int_0^\infty \beta(\lambda 2^{-j})\rho(t)e^{i(\text{sgn} \mu)\lambda t} e^{-|\mu| t} (\cos tP)(x, y) dt.
\]

To proceed, we need to use the Hadamard parametrix. Since \(\rho(t) = 0\) when \(t\) is larger than half the injectivity radius, the Hadamard parametrix says, that, modulo a smooth error, on the support of \(\rho\), we have

\[
(\cos tP)(x, y) = \sum_{\pm} \int_{\mathbb{R}^n} e^{i\kappa(x, y)\xi} e^{\pm i|\xi|\alpha_\pm(t, x, y, |\xi|)} d\xi,
\]

where \(\kappa(x, y)\) denotes local geodesic coordinates of \(x\) about \(y\) so that

\[
|\kappa(x, y)| = d_g(x, y),
\]

and the symbol satisfies

\[
|D_{x,y}^\gamma D_{x,y}^\gamma \alpha_\pm(t, x, y, |\xi|)| \leq C_{\gamma_1, \gamma_2} (1 + |\xi|)^{-|\gamma_1|},
\]

for all multi-indices \(\gamma_j, j = 1, 2\). See [13], [16], [19] or [27]. If we replace \((\cos tP)(x, y)\) by the smooth error in the Hadamard parametrix, the resulting expression will be \(O(\lambda^{-1}(2^j/\lambda))\) by (2.26), which is better than the desired bounds for \(K_j\), assuming as we are that \(j \geq 1\) and \(2^j/\lambda \leq 2\).

Thus, if we set

\[
\varepsilon = 2^j/\lambda, \quad \lambda^{-1} \leq \varepsilon \leq 2,
\]
it suffices to show that the kernels $K_j^\pm(x, y)$ given by

\[
\frac{1}{i(\lambda + i\mu)} \int_{\mathbb{R}_n^+} \beta(t/\varepsilon) \rho(t) e^{i(\text{sgn} \mu)\lambda t} e^{-|\mu|t} \alpha_\pm(t, x, y, |\xi|) e^{i\xi(\pm x + it)/\varepsilon} d\xi dt
\]

\[
= \frac{\varepsilon^{1-n}}{i(\lambda + i\mu)} \int_{\mathbb{R}_n^+} \beta(t) \rho(\varepsilon t) e^{i(\text{sgn} \mu)\lambda \varepsilon t} e^{-|\mu|t} \alpha_\pm(\varepsilon t, x, y, |\xi|/\varepsilon) e^{i\xi(\pm x + \varepsilon t)/\varepsilon} d\xi dt
\]

satisfy

\[
(2.29) \quad |K_j^\pm(x, y)| \leq C\lambda^{n-3} \varepsilon^{-n-1}, \quad \text{if } \lambda^{-1} \leq \varepsilon \leq 2.
\]

If $t \approx 1$ then using (2.28) and a simple integration by parts argument we see that we have the uniform bounds

\[
\left| \int_{\mathbb{R}_n^+} e^{iv \cdot \xi} |\alpha_\pm(\varepsilon t, x, y, |\xi|/\varepsilon) d\xi \right| \leq C, \quad \text{if } |v|/t \notin [1/2, 2].
\]

From this and the support properties of $\beta$ we deduce that

\[
|K_j^\pm(x, y)| \leq C\varepsilon^{1-n} \lambda^{-1}, \quad \text{if } |v_\varepsilon| = |\kappa(x, y)|/\varepsilon \notin [1/4, 4],
\]

which is better than the bounds in (2.29) since we are assuming that $\varepsilon \geq \lambda^{-1}$.

Consequently, we have reduced matters to proving (2.29) under the assumption that $1/4 \leq |v_\varepsilon| \leq 4$. To this, we note that if $a_\varepsilon^\pm(\tau, x, y, |\xi|)$ is the inverse Fourier transform of

\[
t \rightarrow \beta(t) \rho(\varepsilon t) \alpha_\pm(\varepsilon t, x, y, |\xi|/\varepsilon),
\]

which, by (2.28) satisfies

\[
(2.30) \quad |D_\tau^\gamma a_\varepsilon^\pm(\tau, x, y, \xi)| \leq C_{N, \gamma}(1 + |\tau|)^{-N} |\xi|^{-|\gamma|},
\]

for all $\varepsilon$ and every $N$ and $\gamma$, then by (2.2),

\[
\text{sgn} \mu K_j^\pm(x, y) = \varepsilon^{1-n} \int_{-\infty}^{\infty} \left( \int_{\mathbb{R}_n^+} e^{iv \cdot \xi} \frac{e^{-1} a_\varepsilon^\pm(\tau, x, y, |\xi|)}{(\pm |\xi| - \tau/\varepsilon^2 - (\lambda + i\mu)^2)} d\xi \right) d\tau
\]

\[
= \varepsilon^{2-n} \int_{-\infty}^{\infty} \left( \int_{\mathbb{R}_n^+} e^{iv \cdot \xi} \frac{a_\varepsilon^\pm(\tau, x, y, |\xi|)}{(\pm |\xi| - \tau - \varepsilon \lambda - i\varepsilon \mu)(\pm |\xi| - \tau + \varepsilon \lambda + i\varepsilon \mu)} d\xi \right) d\tau.
\]

Since we are assuming now that $|v_\varepsilon| = |\kappa(x, y)|/\varepsilon \approx 1$, it follows from (2.12) and (2.30) that for each $N \in \mathbb{N}$ there must be a constant $C_N$ so that

\[
\left| \int_{\mathbb{R}_n^+} e^{iv \cdot \xi} \frac{a_\varepsilon^+(\tau, x, y, |\xi|)}{(\pm |\xi| - \tau - \varepsilon \lambda - i\varepsilon \mu)(\pm |\xi| - \tau + \varepsilon \lambda + i\varepsilon \mu)} d\xi \right| \leq C_N(1 + |\tau|)^{-N} |\varepsilon \lambda|^{-1} (1 + |\tau| + |\varepsilon \lambda|)^{n-1},
\]

Thus, if we choose $N > \frac{n-2}{2} + 2$, then by the previous inequality and our assumption that $\varepsilon \lambda \geq 1$

\[
|K_j^\pm(x, y)| \leq C \varepsilon^{2-n} (\varepsilon \lambda)^{-1} (\varepsilon \lambda)^{\frac{n-1}{2}} = C \lambda^{\frac{n-3}{2}} \varepsilon^{-\frac{n-3}{2}},
\]

which is (2.29) for this remaining case.

This completes the proof of Theorem 2.2.

\[\square\]

Global resolvent estimates at the unit scale

Let us now see how our localized estimates imply the following unit scale estimates.

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17. JEAN BOURGAIN, PENG SHAO, CHRISTOPHER D. SOGGE, AND XIAOHUA YAO
Theorem 2.5. Fix a compact boundaryless Riemannian manifold of dimension \( n \geq 3 \). Then given \( \delta > 0 \) there is a constant \( C \) so that if

\[
z = \lambda + i\mu \quad \text{with} \quad \lambda, \mu \in \mathbb{R}, \ \lambda, |\mu| \geq \delta,
\]

then

\[
\|u\|_{L^{2\pi n} \Delta^n(L^2(M))} \leq C \|(\Delta_g + z^2)u\|_{L^{2\pi n} \Delta^n(L^2(M))}, \quad u \in C^\infty(M).
\]

Additionally, if we let

\[
\chi_{[\lambda-\delta, \lambda+\delta]} f = \sum_{\lambda_j \in [\lambda-\delta, \lambda+\delta]} E_j f,
\]

then for fixed \( 0 < \delta \leq 1 \) we have the uniform bounds

\[
\|(I - \chi_{[\lambda-\delta, \lambda+\delta]}) \circ (\Delta_g + z^2)^{-1} f\|_{L^{2\pi n} \Delta^n(L^2(M))} \leq C \|f\|_{L^{2\pi n} \Delta^n(L^2(M))},
\]

for \( z = \lambda + i\mu, \ \lambda \geq 1, \ \mu \in \mathbb{R} \setminus \{0\} \).

Inequality (2.32) implies the earlier unit-scale resolvent estimates of [11] and [21], since, as noted before Proposition 2.4 yields uniform resolvent estimates for \( \Delta + \zeta \) when \( \text{Re} \ \zeta \leq \delta \) and \( |\zeta| \geq \delta \). Also, after scaling the metric, we see that in proving (2.32), we may assume that \( \delta = 1 \).

Proof of Theorem 2.5. Let us start by proving (2.32). As we just noted, we may assume that

\[
z = \lambda + i\mu, \ \lambda, \mu \in \mathbb{R}, \ \lambda, |\mu| \geq 1.
\]

We then wish to show that there is a uniform constant \( C \) so that

\[
\|(\Delta_g + z^2)^{-1} f\|_{L^{2\pi n} \Delta^n(L^2(M))} \leq C \|f\|_{L^{2\pi n} \Delta^n(L^2(M))},
\]

for \( z = \lambda + i\mu, \ \lambda \geq 1, \ \mu \in \mathbb{R} \setminus \{0\} \).

By (2.7), if we let

\[
r_{\lambda, \mu}(P) = (\Delta_g + z^2)^{-1} - \Theta_{loc}(\lambda, \mu), \quad z = \lambda + i\mu,
\]

then we would obtain this estimate if we could establish the following uniform bounds for this remainder term

\[
\|r_{\lambda, \mu}(P)f\|_{L^{2\pi n} \Delta^n(L^2(M))} \leq C \|f\|_{L^{2\pi n} \Delta^n(L^2(M))}, \quad \lambda, \mu \in \mathbb{R}, \ \lambda, |\mu| \geq 1.
\]

To prove this, we use (2.6) to see that the multiplier that is associated to \( r_{\lambda, \mu}(P) \) is just the function

\[
r_{\lambda, \mu}(\tau) = \frac{\text{sgn} \mu}{\bar{t}(\lambda + i\mu)} \int_0^\infty (1 - \rho(t)) e^{i(\text{sgn} \mu)\lambda t} e^{-|\mu| t} \cos \tau \ dt.
\]

Consequently, since \( (1 - \rho(t)) \) vanishes near the origin, if we use Euler’s formula to write \( \cos \tau = \frac{1}{2}(e^{i\tau} + e^{-i\tau}) \) and apply a simple integration by parts argument, we derive that for every \( N = 1, 2, 3, \ldots \) there is a uniform constant \( C_N \) so that

\[
|r_{\lambda, \mu}(\tau)| \leq C_N \lambda^{-1} \left( (1 + |\lambda - \tau|)^{-N} + (1 + |\lambda + \tau|)^{-N} \right),
\]

for \( \tau, \lambda, \mu \in \mathbb{R}, \ \lambda, |\mu| \geq 1 \).
From this we deduce (2.36), since by the second part of Lemma 2.3
\[ \| r_{\lambda,\mu}(P) \|_{L^{\frac{2}{n+2}} - L^{\frac{2}{n-2}}} \leq C \sum_{k=1}^{\infty} k\lambda^{-1}(1 + |k|)^{-3} \leq C'. \]

To complete the proof of Theorem 2.5 we need to prove (2.33). We could do so by adapting the proof of the unit-scale estimates (2.32). However, it is very straightforward to do so just by combining (2.32) with the spectral projection estimates of the third author [24].

Note that, by Lemma 2.3,
\[ \| \chi_{[\lambda - \delta, \lambda + \delta]} \circ (\Delta_g + (\lambda + i\mu)^2)^{-1} \|_{L^{\frac{2}{n+2}} - L^{\frac{2}{n-2}}} \leq C, \quad \text{if } \lambda \geq 1 \text{ and } |\mu| \geq 1, \]
which, along with (2.32), implies (2.33) for \( \lambda, |\mu| \geq 1 \). Since the second part of Lemma 2.3 also yields the uniform bounds
\[ \left\| (I - \chi_{[\lambda - \delta, \lambda + \delta]} \circ ((\Delta_g + (\lambda + i)^2)^{-1} - (\Delta_g + (\lambda + i\mu)^2)^{-1})) \right\|_{L^{\frac{2}{n+2}} - L^{\frac{2}{n-2}}} \leq C\delta, \]
\( \lambda \geq 1, \mu \in [-1,1] \{0\} \),
we therefore get (2.33), for the remaining range where \( \mu \in (-1,1) \{0\} \).

**Improved global estimates from small-scale spectral projection estimates:**

**Proof of Theorem 1.3**

We shall first see how (1.13) implies (1.14). In view of Theorem 2.5 to prove (1.14), it suffices to just consider the case where
\[ \varepsilon(\lambda) \leq \mu \leq 1. \]

Also, in view of Theorem 2.2 if \( \rho \in C^\infty(\mathbb{R}) \) satisfies
\[ \rho(t) = 1, \ t \leq \delta_0/2, \quad \rho(t) = 0, \ t \geq \delta_0, \]
then it suffices to verify that
\[ \int_0^\infty e^{i\lambda t} (\cos t \sqrt{-\Delta_g} f) (1 - \rho(t)) e^{-\mu t} dt \leq C\lambda \| f \|_{L^{\frac{2}{n+2}}(M)}, \quad \varepsilon(\lambda) \leq \mu \leq 1. \]

To use (1.13) to prove the resolvent estimates, we need the following simple lemma.

**Lemma 2.6.** Suppose that \( 0 < \mu \leq 1 \) and that \( \rho \) is as in (2.40). Then for every \( N = 1, 2, 3, \ldots \) there is a constant \( C_N \) so that
\[ \int_0^\infty e^{i\lambda t \pm i\tau t} (1 - \rho(t)) e^{-\mu t} dt \leq C_N \left[ (1 + |\lambda \pm \tau|)^{-N} + \mu^{-1}(1 + \mu^{-1}|\lambda \pm \tau|)^{-N} \right]. \]
Proof. If $|\lambda \pm \tau| \leq \mu$, the result is trivial. So we may assume $|\lambda \pm \tau| \geq \mu$. If we then integrate by parts and use Leibnitz’s rule, we find that the left side of (2.42) is majorized by

$$|\lambda \pm \tau|^{-N} \sum_{j+k=N} \int_0^\infty \mu^j e^{-\mu t} \left| \frac{d^k}{dt^k} (1 - \rho(t)) \right| dt.$$ 

If $k \neq 0$, the summand is dominated by the first term in the right side of (2.42), in view of (2.40). For the remaining case where $j = N$ and $k = 0$, it is clearly dominated by the second term in the right side of (2.42).

Proof of Theorem 1.3. To show that (1.13) implies (1.14) we need to verify (2.41). By Lemma 2.6, the operator in the left side of this inequality is of the form

$$\sum_{j=0}^\infty m_{\lambda, \mu}(\lambda_j) E_j f,$$

where for every $N = 1, 2, 3, \ldots$ there is a constant $C_N$ so that for $0 < \mu \leq 1$

$$|m_{\lambda, \mu}(\lambda_j)| \leq C_N \left[ (1 + |\lambda - \lambda_j|)^{-N} + \mu^{-1}(1 + \mu^{-1}|\lambda - \lambda_j|)^{-N} \right].$$

Since our assumption (3.2) implies that there is a uniform constant $C$ so that

$$\left\| \sum_{|\lambda - \lambda_j| \leq \mu} E_j f \right\|_{L^{\frac{2n}{n+2}}(M)} \leq C \lambda \mu \|f\|_{L^{\frac{2n}{n+2}}(M)} , \quad \lambda \geq 1, \quad \varepsilon(\lambda) \leq \mu \leq 1,$$

(2.41) follows from the proof of Lemma 2.3.

To prove the converse, we note that if (1.14) were valid, then we would have the uniform bounds

(2.43) $\varepsilon(\lambda) |\varepsilon(\lambda)| \sum_{j=0}^\infty \left( (\lambda_j^2 - \lambda^2 + \varepsilon(\lambda)^2)^2 + (2\varepsilon(\lambda) \lambda)^2 \right)^{-1} E_j f \left\|_{L^{\frac{2n}{n+2}}(M)} \leq C \|f\|_{L^{\frac{2n}{n+2}}(M)}$, due to the fact that

$$\frac{4i \varepsilon(\lambda) \lambda}{(\lambda_j^2 - \lambda^2 + \varepsilon(\lambda)^2)^2 + (2\varepsilon(\lambda) \lambda)^2} = \frac{1}{\lambda_j^2 - (\lambda + i\varepsilon(\lambda))^2} - \frac{1}{\lambda_j^2 - (\lambda - i\varepsilon(\lambda))^2}.$$

This and a $T^*T$ argument in turn implies that

$$\sqrt{\varepsilon(\lambda) \lambda} \sum_{j=0}^\infty \left( (\lambda_j^2 - \lambda^2 + \varepsilon(\lambda)^2)^2 + (2\varepsilon(\lambda) \lambda)^2 \right)^{-\frac{1}{2}} E_j f \left\|_{L^2(M)} \leq C \|f\|_{L^{\frac{2n}{n+2}}(M)}.$$

As

$$\sqrt{\varepsilon(\lambda) \lambda} \left( (\lambda_j^2 - \lambda^2 + \varepsilon(\lambda)^2)^2 + (2\varepsilon(\lambda) \lambda)^2 \right)^{-\frac{1}{2}} \geq \frac{1}{10} (\varepsilon(\lambda) \lambda)^{-\frac{1}{2}}, \quad \text{if} \quad |\lambda - \lambda_j| \leq \varepsilon(\lambda),$$

orthogonality and the preceding inequality imply that

(2.44) $\left\| \sum_{|\lambda_j - \lambda| \leq \varepsilon(\lambda)} E_j f \right\|_{L^2(M)} \leq C \sqrt{\varepsilon(\lambda) \lambda} \|f\|_{L^{\frac{2n}{n+2}}(M)}$.

Since by another $T^*T$ argument (2.44) is equivalent to (1.13), the proof is complete. □
3. Saturation of certain resolvent norms.

In this section we shall prove Theorems 1.1 and 1.2. For each it will be convenient to use the following simple lemma.

Lemma 3.1. Suppose \( \beta \in C^\infty_0(\mathbb{R}) \) satisfies
\[
\beta(\tau) = 0, \quad \tau \notin [1/4, 4].
\]
Then if \( 1 \leq q \leq r \leq \infty \), there is a constant \( C = C(r, q) \) so that
\[
(3.1) \quad \|\beta(P/\lambda)f\|_{L^r(M)} \leq C\lambda^{n(\frac{1}{q} - \frac{1}{r})}\|f\|_{L^q(M)}, \quad \lambda \geq 1.
\]
We remark that by using the lemma we can verify our assertion that if (1.9) is valid for some finite exponent then it must be valid for \( p = \infty \). We just choose a \( \beta \) as in the lemma satisfying \( \beta(\tau) = 1 \) for \( \tau \in [1/2, 2] \). It then follows that for large \( \lambda \) we have
\[
\chi_{[\lambda^{-\varepsilon}(\lambda), \lambda^{1+\varepsilon}(\lambda)]} = \beta(P/\lambda) \circ \chi_{[\lambda^{-\varepsilon}(\lambda), \lambda^{1+\varepsilon}(\lambda)]} \circ \beta(P/\lambda)
\]
and so applying the lemma twice with exponents \( (q, r) \) being equal to \((p, \infty)\) and \((1, p)/(p-1)\), where \( p \) as in (1.9) yields
\[
\|\chi_{[\lambda^{-\varepsilon}(\lambda), \lambda^{1+\varepsilon}(\lambda)]}\|_{L^1(M) \to L^{\infty}(M)} \leq C\lambda^{2p} \|\chi_{[\lambda^{-\varepsilon}(\lambda), \lambda^{1+\varepsilon}(\lambda)]}\|_{L^{p\infty}(M) \to L^p(M)}.
\]
Since if \( \sigma(p) \) is as in (1.8), we have \( \sigma(p) + 2n/p = (n - 1) = \sigma(\infty) \), we conclude that (1.9) implies that
\[
\|\chi_{[\lambda^{-\varepsilon}(\lambda), \lambda^{1+\varepsilon}(\lambda)]}\|_{L^1(M) \to L^{\infty}(M)} \leq C\varepsilon\lambda^{-n},
\]
as we asserted before.

Proof of Lemma 3.1. Let \( K_\lambda(x, y) \) denote the kernel of \( \beta(P/\lambda) \). Then the proof of Theorem 4.3.1 in [26] shows that for every \( N = 1, 2, 3, \ldots \) there is a constant \( C_N \) which is independent of \( \lambda \) so that
\[
|K_\lambda(x, y)| \leq C_N \lambda^n(1 + \lambda d_q(x, y))^{-N}, \quad \lambda \geq 1.
\]
Consequently, (3.1) follows from an application of Young’s inequality. \( \square \)

Proof of Theorem 1.1. Since (1.11) is a special case of Theorem 1.2, we shall only prove the second assertion in the theorem.

Let us start by handling the sphere with the standard metric. We need to see that if \( \zeta = (\lambda + i\mu)^2 \) with \( |\mu| \geq 1 \) and \( |\mu| \leq 1 \) then
\[
\|\left(\Delta S^n + (\lambda + i\mu)^2\right)^{-1}\|_{L^{\frac{2n}{n+2}(S^n)} \to L^{\frac{2n}{n+2}(S^n)}} \approx \text{dist}(\lambda + i\mu, \text{Spec}(-\Delta S^n))^{-1},
\]
since the bounds for the remaining cases are a consequence of Theorem 2.5 and Lemma 2.3.

We also know from (2.33) with \( \delta = 1/10 \) that if \( \text{dist}(\lambda, \text{Spec}(-\Delta S^n)) \geq 1/10 \) then \( (\Delta S^n + (\lambda + i\mu)^2)^{-1} \) is bounded with norm independent of \( \lambda \) and \( \mu \). Since the spectrum of \( \sqrt{-\Delta S^n} \) is \( \{\sqrt{k(k+n-1)}\}, k = 0, 1, 2, \ldots \), the remaining case which we need to handle is where for some large \( k_0 \in \mathbb{N} \) we have \( |\lambda - \sqrt{k_0(k_0+n-1)}| < 1/10 \) and then by (2.33), we need to show that
\[
(3.2) \quad \|\chi_{[-1/10, 1/10]} \circ (\Delta S^n + (\lambda + i\mu)^2)^{-1}\|_{L^{\frac{2n}{n+2}(S^n)} \to L^{\frac{2n}{n+2}(S^n)}} \approx |\sqrt{k_0(k_0+n-1)} - (\lambda + i\mu)|^{-1}, \quad \text{if } |\mu| \leq 1 \text{ and } \lambda \gg 1.
\]
Because $\sqrt{\lambda_0}(k_0 + n - 1)$ is the unique eigenvalue lying in $[\lambda - 1/10, \lambda + 1/10]$, it follows that $\chi_{[\lambda - 1/10, \lambda + 1/10]}$ just equals the projection operator, $H_{k_0}$ onto the space of spherical harmonics of degree $k_0$. Thus the operator occurring in the left side of (3.2) is just
\[ (-k_0(k_0 + n - 1) + (\lambda + i\mu)^2)^{-1} H_{k_0}. \]

It was shown by the third author in [23] that
\[ \|H_{k_0}\|_{L^{\frac{2n}{n-2}}(S^n) \to L^2(S^n)} \approx k_0^{1/2}, \]
and since, by a $TT^*$ argument, we know from this that the $L^{\frac{2n}{n-2}}(S^n) \to L^{\frac{2n}{n-2}}(S^n)$ norm of $H_{k_0}$ is comparable to $k_0$, we conclude that the left side of (3.2) is comparable to
\[ | -k_0(k_0 + n - 1) + (\lambda + i\mu)^2|^{-1} k_0. \]

Finally, since, by our assumptions, for large $\lambda$ and for $|\mu| \leq 1$ this is comparable to
\[ |\sqrt{k_0}(k_0 + n - 1) - (\lambda + i\mu)|^{-1}, \]
we have proven the second part of Theorem 1.2 for the sphere.

Let us now handle the case of Zoll manifolds. These are manifolds whose geodesics are all periodic with a common minimal period, which we may assume to be equal to $2\pi$ after possibly multiplying the metric by a constant. Then, by a theorem of Weinstein [35], there is a constant $\alpha = \alpha_M \geq 0$ so that all of the nonzero eigenvalues of $-\Delta_g$ cluster around the values $(k + \alpha)^2$, $k = 1, 2, 3, \ldots$. Specifically, for each $k$ there is a cluster of dimension $d_k \approx k^{n-1}$ of eigenvalues $\lambda_{k,j}^2$, $j = 1, 2, \ldots, d_k$ for which $|(k + \alpha) - \lambda_{k,j}| \leq A/k$, $k = 1, 2, 3, \ldots$ for a fixed constant $A$, and all of the nonzero eigenvalues of $-\Delta_g$ are in one of the clusters.

Repeating the arguments for the sphere, we need to show that if $\lambda \gg 1$ and for some $k \in \mathbb{N}$ we have $|\lambda - (k + \alpha)| < 1/20$, then
\[ (3.3) \|\chi_{[\lambda - 1/10, \lambda + 1/10]} \circ (\Delta_g + (\lambda + i\mu)^2)^{-1}\|_{L^{\frac{2n}{n-2}}(M) \to L^{\frac{2n}{n-2}}(M)} \approx |(k + \alpha) - (\lambda + i\mu)|^{-1}, \]
assuming that $|\mu| \leq 1$ and, because of the additional assumption in the Zoll case,
\[ (3.4) \quad |\lambda + i\mu - (k + \alpha)| \geq C/k, \]
with $C$ being a constant depending on the constant $A$ for the spectral gaps.

By the argument in the remark following Lemma 3.1 this would follow from showing that under the above assumptions
\[ (3.5) \quad k^{-(n-2)} \|\chi_{[\lambda - 1/10, \lambda + 1/10]} \circ (\Delta_g + (\lambda + i\mu)^2)^{-1}\|_{L^1(M) \to L^{\infty}(M)} \approx |\lambda + i\mu - (k + \alpha)|^{-1}, \]
assuming (3.4) with $C$ sufficiently large. Since $|\lambda - (k + \alpha)| < 1/20$, it follows that if $\lambda$ is sufficiently large then $\lambda_{l,m} \notin [\lambda - 1/10, \lambda + 1/10]$ unless $l = k$. Therefore, assuming $\lambda$ is sufficiently large, it follows that if $\{e_{k,j}(x)\}$, $j = 1, \ldots, d_k$ is an orthonormal basis of real eigenfunctions for the cluster, then the kernel of the above operator is
\[ \sum_{j=1}^{d_k} (-\lambda_{k,j}^2 + (\lambda + i\mu)^2)^{-1} e_{k,j}(x)e_{k,j}(y). \]
Since the $L^1(M) \to L^\infty(M)$ norm is the supremum of the kernel, we deduce that the left side of (3.5) majorizes
\[ k^{-(n-2)} \sum_{j=1}^{d_k} (\lambda_j^2 + (\lambda + i\mu)^2)^{-1} |e_{k,j}(x)|^2. \]

Let $z_0 = (-(k + \alpha)^2 + (\lambda + i\mu)^2)^{-1}$ and $z_j = (\lambda_j^2 + (\lambda + i\mu)^2)^{-1}, j = 1, 2, \ldots, d_k$. Then since $|\lambda_j - (k + \alpha)| \leq A/k$, it follows that if the constant $C$ in (3.4) is fixed to be sufficiently large, then $|z_0 - z_j| \leq \frac{1}{2} |z_0|, j = 1, 2, \ldots, d_k$. As a result, the left side of (3.5) dominates
\[
|-(k + \alpha)^2 + (\lambda + i\mu)^2|^{-1} k^{-(n-2)} \sum_{j=1}^{d_k} |e_{k,j}(x)|^2 
\geq \frac{|-(k + \alpha)^2 + (\lambda + i\mu)^2|^{-1} k^{-(n-2)} \sum_{j=1}^{d_k} |e_{k,j}(x)|^2}{\Vol_g(M)} \int_M \sum_{j=1}^{d_k} |e_{k,j}(x)|^2 dV_g
\]
\[ = |-(k + \alpha)^2 + (\lambda + i\mu)^2|^{-1} k^{-(n-2)} d_k / \Vol_g(M). \]

Since $d_k \approx k^{n-1}$ and $|-(k + \alpha)^2 + (\lambda + i\mu)^2|^{-1} \approx k^{-1}|(\lambda + i\mu) - (k + \alpha)|^{-1}$, we conclude that the left side of (3.3) dominates the right side. Since a similar argument using the second part of Lemma 2.3 implies the opposite inequality, the proof is complete. \hfill \Box

Proof of Theorem 1.2. We need to show that if there is a sequence $\tau_k \to \infty$ and $\varepsilon(\tau_k) \searrow 0$ with $\varepsilon(\tau_k) > 0$ and
\[
(\varepsilon(\tau_k) \tau_k^{n-1})^{-1} \left[ N(\tau_k + \varepsilon(\tau_k)) - N(\tau_k - \varepsilon(\tau_k)) \right] \to \infty,
\]
then
\[
\| (\Delta_g + \tau_k^2 + i\tau_k \varepsilon(\tau_k))^{-1} \|_{L^{\frac{2n}{n-2}}(M) \to L^{\frac{2n}{n-2}}(M)} \to \infty.
\]

We know from (2.33) that $(I - \chi_{[\tau_k - 1, \tau_k + 1]}) \circ (\Delta_g + \tau_k^2 + i\tau_k \varepsilon(\tau_k))^{-1}$ has $L^{\frac{2n}{n-2}}(M) \to L^{\frac{2n}{n-2}}(M)$ norm bounded by a uniform constant. Consequently, we would have (3.7) if we could show that
\[
\| \chi_{[\tau_k - 1, \tau_k + 1]} \circ (\Delta_g + \tau_k^2 + i\tau_k \varepsilon(\tau_k))^{-1} \|_{L^{\frac{2n}{n-2}}(M) \to L^{\frac{2n}{n-2}}(M)} \to \infty.
\]

Arguing as in the remark following Lemma 3.1 shows that the operator norm in the left is bounded from below by $c \tau_k^{-(n-2)}$ times the $L^1(M) \to L^\infty(M)$ operator norm for some positive constant $c$. Consequently, we would be done if we could show that
\[
\tau_k^{-(n-2)} \left\| \sum_{|\lambda_j - \tau_k| \leq 1} (-\lambda_j^2 + \tau_k^2 + i\tau_k \varepsilon(\tau_k))^{-1} e_j(x) e_j(y) \right\|_{L^1(M) \to L^\infty(M)} \to \infty.
\]

The kernel of the operator is
\[
\sum_{|\lambda_j - \tau_k| \leq 1} (-\lambda_j^2 + \tau_k^2 + i\tau_k \varepsilon(\tau_k))^{-1} e_j(x) e_j(y).
\]
and since the \( L^1(M) \rightarrow L^\infty(M) \) norm is the supremum of the kernel, we deduce that the left side of (3.8) majorizes
\[
\tau_k^{-(n-2)} \sup_{x \in M} \left| \sum_{|\lambda_j - \tau_k| \leq 1} (-\lambda_j^2 + \tau_k^2 + i\tau_k \varepsilon(\tau_k))^{-1} |e_j(x)|^2 \right|.
\]
Since the imaginary part of \((-\lambda_j^2 + \tau_k^2 + i\tau_k \varepsilon(\tau_k))^{-1}\) is \(-\tau_k \varepsilon(\tau_k)/(\tau_k^2 - \lambda_j^2)^2 + \tau_k^2 \varepsilon(\tau_k)^2\), we deduce that for large \( k \) we have
\[
\tau_k^{-(n-2)} \| \chi_{[\lambda-1,\lambda+1]} \circ (\Delta_g + \tau_k^2 + i\tau_k \varepsilon(\tau_k))^{-1} \|_{L^1(M) \rightarrow L^\infty(M)} 
\geq \tau_k^{-(n-2)} \sup_{x \in M} \sum_{|\lambda_j - \tau_k| \leq 1} \chi_k \varepsilon(\tau_k) \left( \frac{\tau_k^2 - \lambda_j^2)^2 + \tau_k^2 \varepsilon(\tau_k)^2}{(\tau_k^2 - \lambda_j^2)^2 + \tau_k^2 \varepsilon(\tau_k)^2} \right) |e_j(x)|^2
\geq \frac{1}{10} \tau_k^{-(n-1)} (\varepsilon(\tau_k))^{-1} \sup_{x \in M} \sum_{|\lambda_j - \tau_k| \leq 1} |e_j(x)|^2
\geq \frac{1}{10\text{Vol}_g(M)} \tau_k^{-(n-1)} (\varepsilon(\tau_k))^{-1} \int_M \sum_{|\lambda_j - \tau_k| \leq 1} |e_j(x)|^2 dV_g
= \frac{1}{10\text{Vol}_g(M)} \tau_k^{-(n-1)} (\varepsilon(\tau_k))^{-1} [N(\tau_k + \varepsilon(\tau_k)) - N(\tau_k - \varepsilon(\tau_k)) -],
\]
and since, by assumption, the last quantity tends to \( \infty \) as \( k \to \infty \), we get (3.8), which completes the proof.

\[\Box\]

4. Improved bounds for the torus.

We now consider the torus \( \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n \). To prove Theorem 1.3, we need to show, in view of Theorem 1.4, that if \( \varepsilon_n \) is as in the statement of Theorem 1.4, then
\[
(4.1) \quad \left\| \sum_{|\lambda - \lambda_j| \leq \varepsilon(\lambda)} E_j f \right\|_{L^\infty(\mathbb{T}^n)} \leq C \lambda \varepsilon(\lambda) \| f \|_{L^\infty(\mathbb{T}^n)}, \quad \varepsilon(\lambda) = \lambda^{-\varepsilon_n}, \quad \lambda \geq 1.
\]
Writing \( \mathbb{T}^n \cong (-\frac{1}{2}, \frac{1}{2})^n \), the eigenfunctions of \( \sqrt{-\Delta_{\mathbb{T}^n}} \) are \( e^{2\pi i k \cdot x} \), \( k \in \mathbb{Z}^n \), with eigenvalues \( 2\pi |k| \). Thus, if
\[
(4.2) \quad \hat{f}(k) = \int_{(-\frac{1}{2}, \frac{1}{2})^n} f(y) e^{-2\pi i k \cdot y} dy,
\]
an equivalent way of writing (4.1) is
\[
(4.3) \quad \left\| \sum_{k \in \mathbb{Z}^n: |k| - \lambda \leq \varepsilon(\lambda)} \hat{f}(k) e^{2\pi i k \cdot x} \right\|_{L^\infty(\mathbb{T}^n)} \leq C \lambda \varepsilon(\lambda) \| f \|_{L^\infty(\mathbb{T}^n)}, \quad \varepsilon(\lambda) = \lambda^{-\varepsilon_n}, \quad \lambda \geq 1.
\]
4. Model argument for $\mathbb{T}^3$.

As a first step, to motivate the refinements to follow, and also the arguments for the case of general manifolds of nonpositive curvature to follow, let us prove a weaker result than the one stated in Theorem 1.2 when $n = 3$, by giving a simple proof that (4.1) holds for the 3-torus when $\varepsilon_3 = \frac{1}{4}$, i.e.

\begin{equation}
(4.1.1) \quad \left\| \sum_{|\lambda - \lambda_j| \leq \lambda^{-\frac{1}{4}}} E_j f \right\|_{L^6(\mathbb{T}^3)} \leq C \lambda^{\frac{2}{3}} \|f\|_{L^{\frac{6}{5}}(\mathbb{T}^3)}, \quad \lambda \geq 1.
\end{equation}

In order to do this, we fix an even nonnegative function $a \in \mathcal{S}(\mathbb{R})$ satisfying $a(0) = 1$ and having Fourier transform supported in $(-1, 1)$. We then claim that to prove (4.1.1), it suffices to see that the operators

\begin{equation}
(4.1.2) \quad a(\lambda^{\frac{1}{2}}(\lambda - \sqrt{-\Delta_{\mathbb{T}^3}})) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lambda^{-\frac{1}{4}} \hat{a}(t/\lambda^{\frac{1}{2}}) e^{it\lambda} e^{-it\sqrt{-\Delta_{\mathbb{T}^3}}} dt
\end{equation}

satisfy

\begin{equation}
(4.1.3) \quad \|a(\lambda^{\frac{1}{2}}(\lambda - \sqrt{-\Delta_{\mathbb{T}^3}}))f\|_{L^6(\mathbb{T}^3)} \leq C \lambda^{\frac{2}{3}} \|f\|_{L^{\frac{6}{5}}(\mathbb{T}^3)}.
\end{equation}

To see this, we note that if $\sqrt{a}(\tau)$ denotes the square root function $\sqrt{a(\tau)}$, then by a $TT^*$ argument, (4.1.3) would imply that

$$
\|\sqrt{a}(\lambda^{\frac{1}{2}}(\lambda - \sqrt{-\Delta_{\mathbb{T}^3}}))f\|_{L^2(\mathbb{T}^3)} \leq C \lambda^{\frac{5}{6}} \|f\|_{L^{\frac{6}{5}}(\mathbb{T}^3)}.
$$

Since $a(0) = 1$, it follows by continuity that $a(\tau) \geq \frac{1}{2} \mathbb{1}_{[-\delta, \delta]}$ for some $\delta > 0$, and, therefore, by orthogonality, the preceding estimate implies that

$$
\left\| \sum_{|\lambda - \lambda_j| \leq \delta \lambda^{-\frac{1}{4}}} E_j f \right\|_{L^2(\mathbb{T}^3)} \leq C \lambda^{\frac{2}{3}} \|f\|_{L^{\frac{6}{5}}(\mathbb{T}^3)}.
$$

After adding up $O(\delta^{-1})$ of such estimates, we conclude that we may take $\delta = 1$. Consequently, the operator in (4.1.1) maps $L^{\frac{6}{5}}(\mathbb{T}^3) \to L^2(\mathbb{T}^3)$ with norm the square root of that posited in (4.1.1). By duality, the same is true for the $L^2(\mathbb{T}^3) \to L^{\frac{6}{5}}(\mathbb{T}^3)$ operator norm, which gives us (4.1.1), since the operator there is a projection operator.

To prove (4.1.3), we note that since the operators $a(\lambda^{\frac{1}{2}}(\lambda + \sqrt{-\Delta_{\mathbb{T}^3}}))$ are trivially bounded between any Lebesgue spaces with norm $O(\lambda^{-N})$, it suffices to verify

\begin{equation}
(4.1.4) \quad \left\| \int_{-\infty}^{\infty} \hat{a}(t/\lambda^{\frac{1}{2}}) e^{it\lambda} \cos t \sqrt{-\Delta_{\mathbb{T}^3}} f dt \right\|_{L^6(\mathbb{T}^3)} \leq C \lambda \|f\|_{L^{\frac{6}{5}}(\mathbb{T}^3)}.
\end{equation}

To do this, we choose an even function $b \in C^\infty_0(\mathbb{R})$ satisfying $b(t) = 1$, $|t| \leq 1$, and $b(t) = 0$, $|t| \geq 2$.

We then split the operator in (4.1.4) as $A_0 + A_1$, where

$$
A_0 f = \int_{-\infty}^{\infty} b(t) \hat{a}(t/\lambda^{\frac{1}{2}}) e^{it\lambda} \cos t \sqrt{-\Delta_{\mathbb{T}^3}} f dt,
$$

and

$$
A_1 f = \int_{-\infty}^{\infty} (1 - b(t)) \hat{a}(t/\lambda^{\frac{1}{2}}) e^{it\lambda} \cos t \sqrt{-\Delta_{\mathbb{T}^3}} f dt.
$$
By Lemma 2.3, we know that
\[ \|A_0 f\|_{L^6(T^3)} \leq C\lambda \|f\|_{L^{\frac{3}{2}}(T^3)}, \]
\[ \|A_0 f\|_{L^4(T^3)} \leq C\lambda \|f\|_{L^{\frac{3}{2}}(T^3)}, \]
\[ \|A_1 f\|_{L^1(T^3)} \leq C\lambda \|f\|_{L^{\frac{3}{2}}(T^3)}. \]
\[ \|A_1 f\|_{L^\infty(T^3)} \leq C\lambda \|f\|_{L^{\frac{3}{2}}(T^3)}. \]

To finish the proof of (4.1.1), by showing that \( A_1 \) enjoys similar bounds, by interpolation, it suffices to show that
\[ \|A_1 f\|_{L^4(T^3)} \leq C\lambda \|f\|_{L^{\frac{3}{2}}(T^3)} \]
and
\[ \|A_1 f\|_{L^\infty(T^3)} \leq C\lambda \|f\|_{L^{\frac{3}{2}}(T^3)}. \]

To prove the first estimate, we note that
\[ |\alpha_1(\tau)| \leq C_N \lambda \left((1 + |\lambda - \tau|)^{-N} + (1 + |\lambda + \tau|)^{-N}\right) \]
for any \( N \). Consequently, (4.1.5) follows from the second part of Lemma 2.3 as \( \sigma(4) = \frac{1}{2} \) when \( n = 3 \).

To prove (4.1.6), we need to show that the kernel of \( A_1 \) is \( O(\lambda^2) \). To do this, we shall use an argument of Hlawka [15]. We first recall that if we identify \( T^3 \) with its fundamental domain \( Q = (-\frac{1}{2}, \frac{1}{2}]^3 \), then
\[ (\cos t \sqrt{-\Delta_{T^3}})(x, y) = \sum_{j \in \mathbb{Z}^3} (\cos t \sqrt{-\Delta_{T^3}})(x - y + j), \quad x, y \in Q, \]
Since, by sharp Huygens principle, \( (\cos t \sqrt{-\Delta_{T^3}})(x) \) is supported on the set where \(|x| = |t|, \hat{a}(t) = 0 \) for \(|t| \geq 1 \) and \((1 - b(t)) = 0 \) for \(|t| \leq 1 \) it follows that the kernel \( K_1 \) of \( A_1 \) is
\[ K_1(x, y) = (2\pi)^{-3} \sum_{j \in \mathbb{Z}^3} \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} e^{i(x-y+j)} \xi (1 - b(t))\hat{a}(t/\lambda^{\frac{1}{2}}) e^{it\lambda} \cos t |\xi| \ d\xi dt. \]

Therefore, in order to show that \( |K_1(x, y)| \leq C\lambda^2 \) and thus obtain (4.1.6), it suffices to show that for \( x, y \in Q \) we have
\[ \sum_{j \in \mathbb{Z}^3} \left| \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} e^{i(x-y+j)} \xi (1 - b(t))\hat{a}(t/\lambda^{\frac{1}{2}}) e^{it\lambda} \cos t |\xi| \ d\xi dt \right| \leq C\lambda^2. \]

If we replace \((1 - b(t))\) by \( b(t) \) then, by Huygens principle, all summands with \(|j| \geq 10 \) vanish since \( b(t) = 0 \) for \(|t| \geq 2 \). If then \( \Psi_{\lambda}(\tau) \) is the Fourier transform of \( t \to b(t)\hat{a}(t/\lambda^{\frac{1}{2}}) \) then, by Euler’s formula, the resulting nonzero summands equal
\[ \frac{1}{2} \int_{\mathbb{R}^3} e^{i(x-y+j)} \xi \left( \Psi_{\lambda}(\lambda - |\xi|) + \Psi_{\lambda}(-\lambda + |\xi|) \right) d\xi, \]
and since the \( \Psi_{\lambda} \) belong to a bounded subset of \( S(\mathbb{R}) \) for \( \lambda \geq 1 \), by (2.14)-(2.15), each of these is \( O(\lambda) \), which is better than (4.1.7). Thus, it suffices to show that if in each
summand \((1 - b(t))\) is replaced by 1, then the bound still holds, which is the same thing as showing that for \(x, y \in Q\), we have
\[
\sum_{j \in \mathbb{Z}^3: |j| \leq \lambda^{\frac{4}{3}} + 1} \left| \int_{\mathbb{R}^3} e^{i(x-y+j) \cdot \xi} \lambda^\frac{1}{6} \left[ a(\lambda^{\frac{1}{2}} (\lambda - |\xi|)) + a(\lambda^{\frac{1}{2}} (\lambda + |\xi|)) \right] d\xi \right| \leq C\lambda^\frac{2}{3}.
\
If we use (2.14)-(2.15) again, we find that each summand is bounded by a fixed multiple of \(\lambda(1 + |j|)^{-1}\), which of course implies (4.1.7) as
\[
\sum_{\{j \in \mathbb{Z}^3: |j| \leq \lambda^{\frac{4}{3}}\}} (1 + |j|)^{-1} \leq C\lambda^\frac{5}{6}.
\
This completes the proof of (4.1.1). The same argument works in higher dimensions and yields
\[
(4.1.8) \quad \left\| \sum_{|\lambda - \lambda_j| \leq \lambda^{\frac{4}{3}} + \frac{1}{\sqrt{n}}} E_j f \right\|_{L^2_{n+1}(T^n)} \leq C\lambda^{-\frac{1}{n+1}} \|f\|_{L^2_{n+3}(T^n)}.
\]
i.e., (4.1) with \(\varepsilon_n = \frac{1}{n+1}\). In this case one would use an interpolation argument involving \(L^{2(n+1)}_{n+3} \rightarrow L^{2(n+1)}_{n+1}\) and, as before, \(L^1 \rightarrow L^\infty\).

4.2. Improved restriction estimates for \(T^3\).

To get the improvements over these results that are stated in Theorem 1.4, we shall have to improve the estimates that arise in the interpolations. Let us start out by first handling the 3-dimensional case and then turn to the case of \(T^n, n \geq 4\) in the next subsection.

The most significant improvements over those just obtained earlier (with \(\varepsilon_3 = \frac{1}{4}\)) will come from improving the \(L^\frac{6}{5}(T^3) \rightarrow L^4(T^3)\) estimates used in the interpolation arguments. To obtain these we first require an estimate of the first author and Guth [7] (see also [6]), which involves the Fourier extension operator for the sphere \(S^2 \subset \mathbb{R}^3\) (or any compact smooth positively curved hypersurface), which is given by
\[
Tf(x) = \int_{S^2} e^{i x \cdot \omega} f(\omega) d\sigma(\omega).
\]
The estimate that we require then is the following (see p. 1259 in [7])

**Lemma 4.2.1.** Let \(R \gg 1\) and \(\frac{1}{\sqrt{R}} < \delta_0 \leq 1\). For \(x \in B_R\), we have
\[
(4.2.1) \quad |Tf| \lesssim_R e^R \sum_{\delta \text{dyadic}} \left( \sum_{\delta_0 \leq \delta \leq 1} (\phi_\tau |Tf_{\tau_1}|^{1/3} |Tf_{\tau_2}|^{1/3} |Tf_{\tau_3}|^{1/3})^2 \right)^{1/2}
\
\[
+ R e^R \left( \sum_{\tau, \delta_0 \text{cap}} (\phi_\tau |Tf_{\tau}|^2) \right)^{1/2},
\]
where $f_\tau$ denotes the restriction of $f \in L^2(S^2, d\sigma)$ to $\tau \subset S^2$ and

$$\tau_1, \tau_2, \tau_3 \subset \tau \text{ are } 3 \text{ transversal (in the sense of } [2]) \frac{\delta}{K} \text{ caps (with } K \text{ a large constant)}$$

(4.2.4) For each $\tau, \phi_\tau \geq 0$ is a function on $\mathbb{R}^3$ satisfying

(4.2.5) $$\int_Q \phi_\tau \ll R^c$$

for all $Q$ taken in a tiling of $\mathbb{R}^3$ by translates of $\hat{\tau}$, the polar of the convex hull of $\tau$. Also, here $f_Q$ denotes the average over $Q$.

In the above $B_R$ denotes the ball of radius $R$ centered at the origin, and $\lesssim_{c'}$ denotes inequality with a constant depending on $c'$. Also, the polar of a $\delta^{1/2} \times \delta^{1/2} \times \delta$ rectangle with short side of length $\delta$ pointing in the $\nu \in S^2$ direction is the $\delta^{-1/2} \times \delta^{-1/2} \times \delta^{-1}$ rectangle centered at the origin with long side pointing in the same direction. Finally, $R \gg 1$ will play the role of $\lambda$ in (4.1) or (4.3), and we have decided to use this parameter instead of $\lambda$ since this was the notation used in [7], [5] and [6] and we need to use results from these works.

Since the $Tf_\tau$ may be essentially be viewed as constant on the $\hat{\tau}$-tiles and (4.2.5) is valid we get

**Lemma 4.2.2.** With the above notation

(4.2.6) $$\|Tf\|_{L^4(B_R)} \lesssim_{c'} R^{c'} \sum_{\delta \text{ dyadic} \ \delta_0 \leq \delta \leq 1} \left( \sum_{\tau \delta - \text{cap}} \|Tf_\tau\|_{L^1(B_R)} 1/3 |Tf_\tau|^{1/3} \right)^2 \
+ R^{c'} \left( \sum_{\tau \delta_0 - \text{cap}} \|Tf_\tau\|_{L^4(B_R)}^2 \right)^{1/2}$$

Next, parabolic scaling and the trilinear $L^3$-inequality from Bennett, Carbery and Tao [2] gives

(4.2.8) $$\|Tf_\tau\|_{L^3(B_R)} \lesssim_{c'} R^{c'} \delta^{-1/2} \|f\|_{L^2}.$$ 

Therefore, by interpolation of $L^4$ between $L^3$ and $L^6$

(4.2.9) $$\|Tf_\tau\|_{L^3(B_R)} \lesssim_{c'} R^{c'} \delta^{-1/2} \left( \prod_{i=1}^3 \|Tf_\tau\|_{L^6(B_R)} \right)^{1/6} \|f\|_{L^2(B_R)}.$$

Fix a subset $\Omega \subset S^2$ (which is a union of $O(\frac{1}{R})$-caps). For a spherical cap $\tau \subset S^2$, we denote

(4.2.10) $$B_p(\tau) = \max \{ \|Tf\|_{L^p(B_R)} : \text{ supp } f \subset \tau \cap \Omega \text{ and } \|f\|_{L^2(\sigma)} \leq 1 \},$$

and for $\delta > 0$, let

(4.2.11) $$B_p(\delta) = \max_{\tau \delta - \text{cap}} B_p(\tau).$$
Then, by (4.2.6), (4.2.7) and (4.2.9) we clearly have the following.

**Lemma 4.2.3.** With the above notation, for $\delta_0 \leq \delta \leq 1$,

\[(4.2.12) \quad B_4(\delta) \lesssim \epsilon^c B_4(\delta_0) + R^{\epsilon} \max_{\{\delta_i: \delta_0 \leq \delta_i \leq \delta\}} \delta_i^{-\frac{1}{2}} B_0(\delta_i)^{\frac{1}{2}}.\]

As we shall see in the next subsection, similar manipulations are possible in higher dimensions.

To prove estimates for the torus, $T^3 \cong (-\frac{1}{2}, \frac{1}{2})^3$, we need to reformulate (4.2.12) for the sphere of radius $R$, $RS^2 \subset \mathbb{R}^3$ (using the same notation). So we now fix $\Omega \subset RS^2$ a union of $O(1)$-caps and define for $\rho < R$

\[(4.2.13) \quad B_p(\rho) = \max\{\|Tf\|_{L^p(\mathbb{Z}^3)} : \text{ supp } f \subset \tau \cap \Omega, \|f\|_{L^2(\mathbb{R}^3)} \leq 1\}.\]

Then from Lemma 4.2.3 and scaling, we obtain the following.

**Lemma 4.2.4.** With the notation (4.2.13), we have for $\sqrt{R} \leq \rho_0 < \rho < R$

\[(4.2.14) \quad B_4(\rho) \lesssim \epsilon^c R^\epsilon B_4(\rho_0) + R^{\epsilon} \max_{\{\rho_i: \rho_0 < \rho_i < \rho\}} \rho_i^{-\frac{1}{2}} B_0(\rho_i)^{\frac{1}{2}}.\]

Fix $\frac{1}{\sqrt{R}} < \epsilon \ll 1$. Then if $\tau \subset RS^2$ is a $\rho$-cap, $\rho > \sqrt{R}$, let us set

\[A_{\epsilon}(\tau) = \{x \in \mathbb{R}^3 : R \frac{x}{|x|} \in \tau, |x| \in [R-\epsilon, R+\epsilon]\}.\]

Let us also fix a nonnegative function $\beta \in C^\infty_0(\mathbb{R})$ with $\beta(s) = 1$ for $|s| \leq 1/10$ and $\beta(s) = 0$ for $|s| > 1/4$, and let

\[(4.2.15) \quad K_{p}^{(\epsilon)}(\tau) = K_{p}(\tau) = \max_{\tau \rho\text{-cap}} K_{p}(\tau).\]

If

\[(4.2.16) \quad K_{p}^{(\epsilon)}(\rho) = K_{p}(\rho) = \max \{K_p(\tau) : \tau \subset RS^2, |x, \mathcal{L}_{\epsilon}| < \frac{1}{100}\},\]

then if we let $\Omega$ as above be

\[(4.2.17) \quad \Omega = \{x \in RS^2 : \text{ dist } (x, \mathcal{L}_{\epsilon}) < \frac{1}{100}\},\]

it follows from (4.2.14) that for $\sqrt{R} < \rho_0 < \rho < R$ we have

\[(4.2.18) \quad K_4(\rho) \lesssim \epsilon^c R^\epsilon K_4(\rho_0) + R^{\epsilon} \max_{\{\rho_i: \rho_0 < \rho_i < \rho\}} \rho_i^{-\frac{1}{2}} K_0(\rho_i)^{\frac{1}{2}}.\]

To see this, we note that if $f$ denotes the restriction to $RS^2$ of

\[y \to \sum_{k \in \mathbb{Z}^3 \cap A_{\epsilon}(\tau)} \beta(\epsilon^{-1}(|k| - R))\beta(100|y - k|) a_k\]

then if $\rho_0 < \rho < R$ we have

\[(4.2.19) \quad K_4(\rho) \lesssim \rho^{\epsilon} K_4(\rho_0) + R^{\epsilon} \max_{\{\rho_i: \rho_0 < \rho_i < \rho\}} \rho_i^{-\frac{1}{2}} K_0(\rho_i)^{\frac{1}{2}}.\]
then for \( x \in [-\frac{1}{2}, \frac{1}{2}]^3 \),
\[
Tf(x) = \sum_{k \in \mathbb{Z}^3 \cap A_x(\tau)} \alpha(k, x) \beta(\varepsilon^{-1}(|k| - R)) a_k e^{2\pi i k \cdot x},
\]
where \( |\alpha(x, k)| \approx 1 \) and \( D^\gamma \alpha(k, x) = O(1) \).

To estimate \( K_p(\rho) \) it is convenient to involve a comparable quantity involving smoothed out multipliers. Specifically, if \( \rho \subset RS^2 \) is a \( \rho \)-cap with center \( \omega_\tau \in RS^2 \), let us define for \( f \in L^2(\mathbb{T}^3) \)
\[
(4.2.19) \quad T^\tau f = \sum_{k \in \mathbb{Z}^3} \beta(\varepsilon^{-1}(|k| - R)) \beta(\rho^{-1}|k - \omega_\tau|) \hat{f}(k) e^{2\pi i k \cdot x}.
\]
If we then put for \( p > 2 \)
\[
\tilde{K}_p^\varepsilon(\rho) = \tilde{K}_p(\rho) = \max_{\tau \rho \text{-cap}} \| T^\tau f \|_{L^2(\mathbb{T}^3)} \rightarrow L^p(\mathbb{T}^3),
\]
then the simple orthogonality arguments used before yield
\[
(4.2.20) \quad K_p(\rho) \approx \tilde{K}_p(\rho).
\]
If
\[
m_{\tau,\varepsilon}(\xi) = \left( \beta(\varepsilon^{-1}(|\xi| - R)) \beta(\rho^{-1}(|\xi - \omega_\tau|)) \right)^2,
\]
and we let
\[
Mf(x) = \sum_{k \in \mathbb{Z}^3} m_{\tau,\varepsilon}(k) \hat{f}(k) e^{2\pi i k \cdot x},
\]
then of course
\[
(4.2.21) \quad (\tilde{K}_p(\tau))^2 = \| M \|_{L^p'(\mathbb{T}^3) \rightarrow L^p(\mathbb{T}^3)},
\]
where \( p' = p/(p - 1) \). To estimate \( M \), we split it into two parts in a similar way to how the operator in \((4.1.2)\) was split into \( A_0 + A_1 \). In this case, let us fix a \( C_0^\infty(\mathbb{R}^3) \) bump function \( \eta \) supported in the unit ball satisfying \( \int_{\mathbb{R}^3} \eta = 1 \). We then put
\[
M_0 f = \sum_{k \in \mathbb{Z}^3} m_{0,\varepsilon}(k) \hat{f}(k) e^{2\pi i k \cdot x},
\]
where
\[
m_{0,\varepsilon}(\xi) = (m_{\tau,\varepsilon} \ast \eta)(\xi).
\]
We then can estimate the \( L^2(\mathbb{T}^3) \rightarrow L^6(\mathbb{T}^3) \) norm of \( M \) as follows
\[
(4.2.22) \quad \| M \|_{2 \rightarrow 6} \leq \| M_0 \|_{2 \rightarrow 6} + \left( \| M \|_{\frac{3}{2} \rightarrow 4} + \| M_0 \|_{\frac{3}{2} \rightarrow 4} \right)^\frac{\frac{3}{2}}{2} \| M - M_0 \|_{\frac{1}{2} \rightarrow \infty}.
\]
From Lemma 2.3 we deduce that
\[
(4.2.23) \quad \| M_0 \|_{\frac{3}{4} \rightarrow 4} \leq \varepsilon R^{1/2},
\]
and interpolation with \( \| M_0 \|_{1 \rightarrow \infty} \lesssim \varepsilon \rho^2 \) gives
\[
(4.2.24) \quad \| M_0 \|_{\frac{4}{3} \rightarrow 6} \lesssim \varepsilon R^{\frac{1}{2}} \rho^{\frac{2}{3}}.
\]
To evaluate the last factor in \((4.2.22)\) we note that
\[
\| M - M_0 \|_{1 \rightarrow \infty} = \| \sum_{k \in \mathbb{Z}^3} (m_{\tau,\varepsilon}(k) - m_{0,\varepsilon}(k)) e^{2\pi i k \cdot x} \|_{L^\infty(\mathbb{T}^3)},
\]
Taking \( x \in [-\frac{1}{2}, \frac{1}{2}]^3 \), Poisson summation gives
\[
\left| \sum_{k \in \mathbb{Z}^3} (m_{\tau, \varepsilon}(k) - m_{\tau, \varepsilon}^0(k)) e^{2\pi i k \cdot x} \right| = \left| \sum_{k \in \mathbb{Z}^3} (\hat{m}_{\tau, \varepsilon}(1 - \hat{\eta}))(k + x) \right|,
\]
where here for \( g \in \mathcal{S}(\mathbb{R}^3), \hat{g}(\xi) = \int_{\mathbb{R}^3} e^{-2\pi i x \cdot \xi} g(x) \, dx \). Consequently,
\[
(4.2.25) \quad \|M - M_0\|_{1 \to \infty} \leq \sup_{x \in [-\frac{1}{2}, \frac{1}{2}]^3} |\hat{m}_{\tau, \varepsilon}(x)| |1 - \hat{\eta}(x)| + \sup_{x \in [-\frac{1}{2}, \frac{1}{2}]^3} |(\hat{m}_{\tau, \varepsilon} \hat{\eta})(k + x)| + \sup_{x \in [-\frac{1}{2}, \frac{1}{2}]^3} |\hat{m}_{\tau, \varepsilon}(k + x)|.
\]
Recall that \( \rho > \sqrt{R} \), therefore, by using stationary phase, one finds that for every \( N = 1, 2, 3, \ldots \)
\[
(4.2.26) \quad |\hat{m}_{\tau, \varepsilon}(y)| \leq C_N \frac{\varepsilon R}{|y|} (1 + \varepsilon |y|)^{-N},
\]
which implies that the first two terms in the right side of (4.2.25) are \( O(\varepsilon R) \), since \( \hat{\eta}(0) = 1 \). To estimate the third term in the right side we note that there is a constant \( A \) so that if \( C \) is the conic region with vertex at the center \( \omega_\tau \) of \( \tau \) and angle \( A R \), then for every \( N = 1, 2, 3, \ldots \) we have
\[
(4.2.27) \quad |\hat{m}_{\tau, \varepsilon}(y)| \leq C_N \varepsilon^2 \rho^2 \left( 1 + \frac{\rho^2}{R} |y| \right)^{-N}, \quad x \notin C,
\]
which, together with (4.2.26), means that \( \hat{m}_{\tau, \varepsilon} \) is essentially supported in \( B_{O(\frac{1}{2})} \cap (C \cup B_{O(\frac{1}{2})}) \). Using these two bounds we find the third term in the right side of (4.2.25) is majorized by
\[
\varepsilon R \sum_{0 \neq k \in C} |k|^{-1}(1 + \varepsilon |k|)^{-3} + \varepsilon \rho^2 \sum_{k \neq 0} (1 + \rho^2 |k|/R)^{-4} \lesssim \varepsilon R \left( \frac{\rho}{\varepsilon R} \right)^2 + \varepsilon R.
\]
If we combine this with our earlier estimates for the first two terms in the right side of (4.2.25), we conclude that
\[
(4.2.28) \quad \|M - M_0\|_{1 \to \infty} \lesssim \varepsilon R + \frac{\rho^2}{\varepsilon R}
\]
Therefore, by (4.2.23), (4.2.24) and (4.2.22), we have
\[
\|M\|_{2 \to 6} \lesssim \varepsilon R \frac{\rho^2}{\varepsilon R} + (\|M\|_{\frac{1}{2} \to 4} + \varepsilon R \frac{\rho^2}{\varepsilon R}) \frac{1}{2} (\varepsilon R + \frac{\rho^2}{\varepsilon R}) \frac{1}{2},
\]
implying by (4.2.20) and (4.2.21) that
\[
(4.2.29) \quad K_6(\rho) \lesssim \sqrt{\varepsilon R} \frac{\rho^2}{\varepsilon R} + (K_4(\rho) + \sqrt{\varepsilon R}) \frac{1}{2} (\varepsilon R + \frac{\rho^2}{\varepsilon R}) \frac{1}{2}.
\]
Also, since \( \|M - M_0\|_{2 \to 2} = O(1) \), by using (4.2.23) and (4.2.28) we find
\[
\|M\|_{\frac{1}{2} \to 4} \leq \|M_0\|_{\frac{1}{2} \to 4} + \|M - M_0\|_{1 \to \infty} \lesssim \varepsilon R \frac{\rho^2}{\varepsilon R} + (\varepsilon R \frac{\rho^2}{\varepsilon R} + \frac{\rho^2}{\varepsilon R}) \frac{1}{2},
\]
and, therefore by (4.2.20),
\[
K_4(\rho) \lesssim \varepsilon^{\frac{1}{4}} R^{\frac{1}{4}} + \left(\varepsilon^{\frac{1}{4}} R^{\frac{1}{4}} + \frac{\rho^{\frac{1}{2}}}{(\varepsilon R)^{\frac{1}{2}}}\right).
\]

Set \( \rho_0 = \left(\frac{R}{\varepsilon}\right)^{\frac{1}{2}} \). From (4.2.30), we obtain
\[
K_4(\rho) \lesssim \varepsilon^{\frac{1}{4}} R^{\frac{1}{4}} \quad \text{for} \quad \rho \leq \rho_0,
\]
provided that
\[
\varepsilon > R^{-\frac{1}{2}}.
\]
In order to prove that (4.2.31) holds for all \( \rho \), we combine (4.2.18) and (4.2.29). Denoting
\( K_4 \equiv K_4^{(\varepsilon)}(R) \), it follows that
\[
K_4 \lesssim \varepsilon^{\frac{1}{4}} R^{\frac{1}{4}} + \varepsilon^{\frac{1}{4}} R^{\frac{1}{4}} + \max_{\rho_1 > \rho_0} \left\{ \varepsilon^{\frac{1}{4}} R^{\frac{1}{4}} \rho_1^{\frac{1}{2}} + K_4^{(\varepsilon)} \left(\frac{\rho_1^{\frac{1}{2}}}{(\varepsilon R)^{\frac{1}{2}}} + \frac{\rho^{\frac{1}{2}}}{(\varepsilon R)^{\frac{1}{2}}}\right) \right\}
\]
\[
\lesssim \varepsilon^{\frac{1}{4}} R^{\frac{1}{4}} + K_4^{(\varepsilon)} \left(\frac{\varepsilon R}{\rho_0} + \frac{\rho_1^{\frac{1}{2}}}{(\varepsilon R)^{\frac{1}{2}}} + \frac{\rho^{\frac{1}{2}}}{(\varepsilon R)^{\frac{1}{2}}}\right),
\]
and hence, by (4.2.32),
\[
K_4 \lesssim \varepsilon^{\frac{1}{4}} R^{\frac{1}{4}},
\]
as claimed.

Since \( K_4 = K_4^{(\varepsilon)} \) is obviously an increasing function of \( \varepsilon \) we have the following

**Lemma 4.2.5.**
\[
K_4^{(\varepsilon)} \lesssim \varepsilon^{\frac{1}{4}} R^{\frac{1}{4}} + \max \left\{ \varepsilon^{\frac{1}{4}} R^{\frac{1}{4}} \rho_1^{\frac{1}{2}} + K_4^{(\varepsilon)} \left(\frac{\rho_1^{\frac{1}{2}}}{(\varepsilon R)^{\frac{1}{2}}} + \frac{\rho^{\frac{1}{2}}}{(\varepsilon R)^{\frac{1}{2}}}\right) \right\}
\]
\[
\lesssim \varepsilon^{\frac{1}{4}} R^{\frac{1}{4}} + K_4^{(\varepsilon)} \left(\frac{\varepsilon R}{\rho_0} + \frac{\rho_1^{\frac{1}{2}}}{(\varepsilon R)^{\frac{1}{2}}} + \frac{\rho^{\frac{1}{2}}}{(\varepsilon R)^{\frac{1}{2}}}\right),
\]

and hence, by (4.2.32),
\[
K_4 \lesssim \varepsilon^{\frac{1}{4}} R^{\frac{1}{4}},
\]
as claimed.

Since \( K_4 = K_4^{(\varepsilon)} \) is obviously an increasing function of \( \varepsilon \) we have the following

**Lemma 4.2.5.**
\[
K_4^{(\varepsilon)} \lesssim \varepsilon^{\frac{1}{4}} R^{\frac{1}{4}} \quad \text{if} \quad \varepsilon \geq R^{-\frac{1}{3}}
\]
\[
K_4^{(\varepsilon)} \lesssim R^{\frac{1}{3}} \quad \text{if} \quad \varepsilon \leq R^{-\frac{1}{3}}.
\]

**Remarks:**
(i) Considering \( R \in \mathbb{Z} \) and the points
\[
\left\{ (z_1, z_2, R) \in \mathbb{Z}^3 : \max(|z_1|, |z_2|) < \frac{1}{10} \sqrt{R \varepsilon} \right\} \subset L_\varepsilon
\]
shows that (4.2.32) is essentially optimal.
(ii) Possibly (4.2.35) holds for all \( \varepsilon > R^{-\frac{1}{2}} \). Note that if we assume \( R^2 \in \mathbb{Z} \)
\[
K_4^{(\frac{1}{4})} \sim \max \left\{ \left\| \sum_{\left| \xi^2 - R^2 \right|} a_\xi e^{ix\cdot\xi} \right\|_4, \sum |a_\xi|^2 \leq 1 \right\} \ll R^{0+},
\]
since for \( z \in B_R \) the number of representations
\[
\{(\xi, \eta) \in \mathbb{Z}^3 \times \mathbb{Z}^3; |\xi| = |\eta| = R, \xi + \eta = 2z\}
\]
amounts to the number of lattice points in a planar section of the \( R \)-sphere, which is bounded by \( R^\varepsilon \) (see §2 in [8]).
Applying (4.2.22) with \( \tau = RS^2 \) it follows from Lemma 4.2.5, 4.2.23, and (4.2.28) that if we set
\[
Mf = M_zf = \sum_{k \in \mathbb{Z}^3} (\beta(\varepsilon^{-1}|k| - R))^2 \hat{f}(k)e^{ik \cdot x},
\]
then
\[
(4.2.38) \quad ||M_z||_2 \lesssim \varepsilon R + R^{\alpha^+} (\varepsilon^{\frac{1}{3}} R^\frac{1}{2} + R^\frac{2}{3}) ||M - M_0||_{1 \to \infty} \lesssim \varepsilon R + (\varepsilon^{\frac{1}{3}} R^\frac{1}{2} + R^\frac{2}{3}) R^\frac{1}{2} + \varepsilon^{-\frac{1}{3}} \lesssim \varepsilon R,
\]
provided that \( \varepsilon > R^{-\frac{1}{3}+} \). Consequently, we have (4.3) for \( n = 3 \) with \( \varepsilon(\lambda) = \lambda^{-\frac{1}{3}+} \), which improves our earlier results of \( \lambda^{-\frac{1}{3}} \).

We can make a further improvement of what was done in the last step by a less rough evaluation of \( ||M - M_0||_{1 \to \infty} \). Fixing \( x \in [-\frac{1}{2}, \frac{1}{2}]^3 \), Poisson summation leads to bounding exponential sums of the type
\[
(4.2.39) \quad \varepsilon R \sum_{j \in \mathbb{Z}^3, 0 < |j| \leq \frac{1}{\varepsilon}} \frac{e^{iRj \cdot x}}{|j + x|}.
\]
Breaking up summation ranges, we need to evaluate for \( M \lesssim \frac{1}{\varepsilon} \)
\[
(4.2.40) \quad \frac{1}{M} \left| \sum_{j_{1} \in I_{1}, j_{2} \in I_{2}, j_{3} \in I_{3}} e^{iR(j_{1} + j_{2})} \right|
\]
with \( I_{1} \times I_{2} \times I_{3} \subset B_{M} \setminus B_{\frac{M}{10}} \) a cube of size \( \frac{M}{10} \). Hence, fixing one of the variables, say, \( j_{3} \), and making a coordinate change \( (j_{1}, j_{2}) \to (j_{1} + j_{2}, j_{2}) \) or \( (j_{1}, j_{2}) \to (j_{1}, j_{2} + j_{1}) \) if needed, we get
\[
(4.2.41) \quad \left| \sum_{(j_{1}, j_{2}) \in D} e^{i(f(j_{1}, j_{2})} \right|
\]
where \( D \) is a quadrangle such that \( |j_{1}|, |j_{2}| \approx M \) for \( (j_{1}, j_{2}) \in D \) and \( f(\alpha, \beta) \) is one of the functions
\[
R[(x_{1} + \alpha)^2 + (x_{2} + \beta)^2 + (x_{3} + j_{3})^2]^{\frac{1}{2}}
\]
\[
R[(x_{1} + \alpha - \beta)^2 + (x_{2} + \beta)^2 + (x_{3} + j_{3})^2]^{\frac{1}{2}}
\]
or
\[
R[(x_{1} + \alpha)^2 + (x_{2} + \beta - \alpha)^2 + (x_{3} + j_{3})^2]^{\frac{1}{2}}.
\]

At this point, we may invoke the exponential sum bound (stated as Lemma 2) in the paper [12], the above function \( f \) satisfying the required conditions (with \( \Lambda = R \) and any \( k \)).

It follows that
\[
(4.2.42) \quad \frac{4.2.40}{4.2.42} \lesssim \log R M^2 \left( \frac{R}{M^{k+1}} \right)^{\frac{1}{48}} \text{ for any } k \in \mathbb{Z}_.
\]
Since (4.2.42) is increasing in \( M \), it follows that
\[
(4.2.43) \quad \frac{4.2.39}{4.2.43} \lesssim R \log R \left( \frac{1}{\varepsilon} R^{k+1} \right)^{\frac{1}{48}} \text{ for any } k \in \mathbb{Z}_+.
\]
which bounds $\|M - M_0\|_{1 \rightarrow \infty}$.

Taking $k = 3$, substitution in (4.2.38) gives
\begin{equation}
\|M_\epsilon\|_{\frac{5}{6}} \leq \epsilon R + R^\frac{5}{6} + \frac{1}{36} + \epsilon^{- \frac{1}{3}} + \frac{1}{66} \leq \epsilon R \quad \text{for } \epsilon > R^{- \frac{85}{252}},
\end{equation}
hence we have (4.3) for any $\epsilon < \frac{85}{252}$
\[
\left\| \sum_{\{k \in \mathbb{Z}^3 : |k| - R(\leq \epsilon)\}} \hat{f}(j)e^{2\pi ik \cdot x} \right\|_{L^{6}(\mathbb{T}^3)} \leq C\epsilon R \left\| f \right\|_{L^{6}(\mathbb{T}^3)}, \quad \text{for } \epsilon > R^{- \frac{85}{252}}, \frac{85}{252} = 0.337..., \]
which is the first part of Theorem 1.4.

4.3. Improved restriction estimates for $\mathbb{T}^n$, $n > 3$.

Let $n > 3$. We require the following result which follows from the arguments in §3 and §4 of [7] which is a higher dimensional version of Lemma 4.2.1

Lemma 4.3.1. Fix $3 \leq k \leq n$. Let $R \gg 1$ and $\frac{1}{\sqrt{R}} < \delta_0 < 1$. On $B_R$, we have
\begin{align}
\left\| T f \right\|_{L^2(S^{n-1}, d\sigma)} &\leq C R \left[ \sum_{\delta \text{ dyadic}} \left( \phi_\tau \prod_{j=1}^{k} \left| T f_\tau \right|^{\frac{1}{p_1}} \right)^{\frac{1}{2}} \right]^{\frac{1}{2}} \\
&+ R^{\epsilon} \left[ \sum_{\delta < \delta_0} \left( \phi_\tau \left| T f_\tau \right| \right)^{\frac{1}{2}} \right]^{\frac{1}{2}} \tag{4.3.1}
\end{align}
for $f \in L^2(S^{n-1}, d\sigma)$, where
(i) $\tau_1, \ldots, \tau_k \subset \tau$ are $k$-transversal caps of size $\rho$;
(ii) For each $\tau, \phi_\tau \geq 0$ is a function on $\mathbb{R}^n$ satisfying
\begin{equation}
\int_Q \phi_\tau \frac{2(k-1)}{k-2} \ll R^\epsilon, \tag{4.3.3}
\end{equation}
for all $Q$ taken in a tiling of $\mathbb{R}^n$ by translates of $\frac{\tau}{2}$.

We choose $k$ such that
\[
\frac{2(k-1)}{k-2} \geq \frac{2(n+1)}{n-1} > \frac{2k}{k-1}
\]
thus
\begin{equation}
\frac{n+1}{2} < k \leq \frac{n+3}{2}. \tag{4.3.4}
\end{equation}
We may then state the analogue of Lemma 4.2.2 denoting $p_0 = \frac{2(n+1)}{n-1}$, $p_1 = \frac{2n}{n-2}$, then
Lemma 4.3.2.

\[
\|T f\|_{L^p(B_R)} \lesssim \epsilon' \sum_{\delta \text{ dyadic}} \sum_{\delta_0 < \delta < 1} \left( \frac{1}{\delta} \sum_{\text{cap}} \| \prod_{j=1}^{k} |T f_{\tau_j}| \|_{L^p(B_R)} \right)^{1/2}.
\]

From parabolic rescaling and the \(k\)-linear inequality from [2] we have

\[
\| \prod_{j=1}^{k} |T f_{\tau_j}| \|_{L^{2n/k}(B_r)} \lesssim \epsilon' \delta_0^{2n/k}.
\]

Interpolating \(L^p\) between \(L^{2n/k}\) and \(L^1\) gives

\[
\| \prod_{j=1}^{k} |T f_{\tau_j}| \|_{L^p(B_R)} \lesssim \epsilon' \delta_0^{2n/k}.
\]

with

\[
\theta = \frac{2n/k - 1}{2 - 1/k}.
\]

Using same notation, Lemma 4.2.3 becomes

\[
B_{p_0}(\delta) \lesssim R^{0+} B_{p_0}(\delta_0) + R^{0+} \max_{\delta_0 < \delta_1 < \delta} \delta_1^{-\frac{\theta}{2}} \epsilon' K_{p_1}(\delta_1)^{\theta}.
\]

Rescaling, we get for caps on \(RS^{n-1}\)

\[
B_{p_0}(\rho) \lesssim R^{0+} B_{p_0}(\rho_0) + R^{0+} \max_{\rho_0 < \rho_1 < \rho} \rho_1^{-\frac{\theta}{2}} B_{p_1}(\rho_1)^{\theta}.
\]

Inequality (4.2.18) becomes

\[
K_{p_0}(\rho) \lesssim R^{0+} K_{p_0}(\rho_0) + R^{0+} \max_{\rho_0 < \rho_1 < \rho} \rho_1^{-\frac{\theta}{2}} K_{p_1}(\rho_1)^{\theta}.
\]

Let us distinguish between the cases where \(n\) is odd and even.

- \(n\) is odd. Then \(k = \frac{n+3}{2}, \theta = \frac{2}{3} \cdot \frac{n}{n+1}\) and (4.3.12) gives

\[
K_{p_0}(\rho) \lesssim R^{0+} K_{p_0}(\rho_0) + R^{0+} \max_{\rho_0 < \rho_1 < \rho} \rho_1^{-\frac{1}{2(n+1)}} K_{p_1}(\rho_1)^{\frac{2n}{2(n+1)}}.
\]

- \(n\) is even. Then \(k = \frac{n}{2} + 1, \theta = \frac{n}{2(n+1)}\) and

\[
K_{p_0}(\rho) \lesssim R^{0+} K_{p_0}(\rho_0) + R^{0+} \max_{\rho_0 < \rho_1 < \rho} \rho_1^{-\frac{1}{2(n+1)}} K_{p_1}(\rho_1)^{\frac{n}{2(n+1)}}.
\]

We will use Proposition 1 from [3], which gives, after rescaling
Lemma 4.3.5. Let $1 \ll \rho < \sqrt{R}$ and $\{\tau_\alpha(\rho)\}$ a partition of $RS^{n-1}$ in cells of size $\rho$. For $r > C(x)\frac{R}{\rho^4}$ and $q = \frac{2n}{n-1}$, one has the inequality

$$
(4.3.15) \quad \left\| \sum_{\alpha} \int_{\tau_\alpha} g(\xi) e^{ix \cdot \xi} \sigma_R(d\xi) \right\|_{L^\kappa_\rho(\tau)} \lesssim R^\kappa \left\{ \sum_{\alpha} \left\| \int_{\tau_\alpha} g(\xi) e^{ix \cdot \xi} \sigma_R(d\xi) \right\|_{L^2_\rho(\tau)}^2 \right\}^{\frac{1}{2}}
$$

where $L^q_\rho(\tau)$ denotes $L^q(\eta(\xi), dx)$, where $0 \leq \eta \leq 1$ is some rapidly decreasing bump function on $\mathbb{R}^n$. Also $\kappa > 0$ is an arbitrarily small, fixed constant.

Exploiting Lemma 4.3.5 in the content of lattice points requires the following simple observation that was also crucial in [5].

Let $\rho = \varepsilon^\frac{1}{2} R^\frac{1}{4}$ in Lemma 4.3.5 so that we may take $r = o(\frac{1}{\varepsilon})$. Let $\mathcal{L}_\varepsilon' \subset RS^{n-1}$ be a collection of points obtained by $\varepsilon$-perturbation of the points of $\mathcal{L}_\varepsilon \subset \mathbb{Z}^n$. Applying (4.3.15) in a discretized version gives

$$
(4.3.16) \quad \left\| \sum_{\xi \in \mathcal{L}_\varepsilon'} a_{\xi} e^{ix \cdot \xi} \right\|_{L^\kappa_\rho(\tau)} \lesssim R^{0+} \left[ \sum_{\alpha} \left\| \sum_{\xi \in \mathcal{L}_\varepsilon(\tau_\alpha)} a_{\xi} e^{ix \cdot \xi} \right\|_{L^2_\rho(\tau)}^2 \right]^{\frac{1}{2}}
$$

and since $|\xi - \xi'| < \varepsilon$ and $r \ll 1/\varepsilon$, a perturbation argument permits us to deduce from (4.3.16) that we also have

$$
(4.3.17) \quad \left\| \sum_{\xi \in \mathcal{L}_\varepsilon} b_{\xi} e^{ix \cdot \xi} \right\|_{L^\kappa_\rho(\tau)} \lesssim R^{0+} \left[ \sum_{\alpha} \left\| \sum_{\xi \in \mathcal{L}_\varepsilon(\tau_\alpha)} b_{\xi} e^{ix \cdot \xi} \right\|_{L^2_\rho(\tau)}^2 \right]^{\frac{1}{2}}
$$

for some coefficients $\{b_{\xi}\}$, $|b_{\xi}| \leq |a_{\xi}|$.

Now, since the functions $\sum_{\xi \in \mathcal{L}_\varepsilon} a_{\xi} e^{ix \cdot \xi}$ and $\sum_{\xi \in \mathcal{L}_\varepsilon} b_{\xi} e^{ix \cdot \xi}$ are 1-periodic, (4.3.17) is equivalent to

$$
(4.3.18) \quad \left\| \sum_{\xi \in \mathcal{L}_\varepsilon} a_{\xi} e^{ix \cdot \xi} \right\|_{L^\kappa(\mathbb{T}^n)} \lesssim R^{0+} \left[ \sum_{\alpha} \left\| \sum_{\xi \in \mathcal{L}_\varepsilon(\tau_\alpha)} b_{\xi} e^{ix \cdot \xi} \right\|_{L^2(\mathbb{T}^n)}^2 \right]^{\frac{1}{2}}.
$$

Hence

$$
K_q^{(\varepsilon)}(R) \lesssim R^{0+} K_q^{(\varepsilon)}(R^{0+} \sqrt{\varepsilon R})
$$

and therefore we have the following

Lemma 4.3.6.

$$
(4.3.19) \quad K_q^{(\varepsilon)}(R) \lesssim (\varepsilon R)^{\frac{2n-1}{n+1}}, \quad \text{where} \quad q = \frac{2n}{n-1}.
$$

Turning to the second part of the argument, (4.2.22) gets replaced by

$$
(4.3.20) \quad \|M\|_{p_1 \to p_1} \leq \|M_0\|_{p_1 \to p_1} + (\|M\|_{p_0 \to p_0} + \|M_0\|_{p_0 \to p_0})^{1-\frac{2}{n(n-1)}} \|M-M_0\|_{1 \to \infty}^{\frac{2}{n(n-1)}}
$$

and interpolation of $L^{p_0}$ between $L^2$ and $L^\infty$ together with (4.3.19) implies

$$
(4.3.21) \quad \|M\|_{p_0 \to p_0} \leq \|M_0\|_{p_0 \to p_0} + \|M-M_0\|_{1 \to \infty} (\varepsilon R)^{\frac{2n-1}{n+1}} + \|M_0\|_{q' \to q}^{\frac{2n}{n-1}}.
$$
Next, we have
\begin{equation}
||M_0||_{p'_0 \to p_0} \lesssim \varepsilon R^{\frac{n-1}{p_0+1}}, \quad ||M_0||_{q' \to q} \lesssim \varepsilon R^{\frac{n-1}{p'}},
\end{equation}
and interpolating with $L^\infty$ gives
\begin{equation}
||M_0||_{p'_1 \to p_1} \lesssim \varepsilon R^{1-\frac{2}{n}p}. \tag{4.3.23}
\end{equation}

Similarly, the $n$-dimensional version of (4.2.26) is
\begin{equation}
|m_{\tau,\epsilon}(y)| \leq C_N \frac{\varepsilon R^{\frac{n-1}{2}}}{|y|^{\frac{n-1}{2}}} (1+\varepsilon|y|)^{-N}, \quad N = 1, 2, 3, \ldots,
\end{equation}
and, also, as before we get better estimates if $y$ is not in a conic region $C$ centered at the center of $\tau$ and of angle $O\left(\frac{p}{q}\right)$, namely,
\begin{equation}
|m_{\tau,\epsilon}(y)| \leq C_N \varepsilon^{p-1} (1 + \frac{\rho^2}{R}|y|)^{-N}, \quad y \notin C, \quad N = 1, 2, 3, \ldots.
\end{equation}

Therefore, the earlier arguments involving the Poisson summation formula yield
\begin{equation}
||M - M_0||_{1 \to \infty} \lesssim \varepsilon R^{\frac{n-1}{p}} + \left(\frac{\rho^2}{\varepsilon R}\right)^{\frac{n-1}{2}}. \tag{4.3.24}
\end{equation}

It follows from (4.3.21), (4.3.22), (4.3.24) that
\begin{equation}
||M||_{p'_0 \to p_0} \lesssim \varepsilon R^{\frac{n-1}{p_0+1}} + \left[\varepsilon R^{\frac{n-1}{p_0+1}} + \left(\frac{\rho^2}{\varepsilon R}\right)^{\frac{n-1}{2}}\right] \left(\varepsilon R\right)^{\frac{n-1}{p_0+1}}, \tag{4.3.25}
\end{equation}
and from (4.3.20), (4.3.22), (4.3.23), (4.3.24)
\begin{equation}
||M||_{p'_1 \to p_1} \lesssim \varepsilon R^{1-\frac{2}{n}p} + \left(\varepsilon R^{\frac{n-1}{p_0+1}} + ||M||_{p'_0 \to p_0}\right)^{1-\frac{2}{n(n-1)}} \left(\varepsilon R^{\frac{n-1}{p_0+1}} + \left(\frac{\rho^2}{\varepsilon R}\right)^{\frac{1}{2}}\right). \tag{4.3.26}
\end{equation}

Hence
\begin{equation}
K_{p_1}(\rho) \lesssim \sqrt{\varepsilon R}^{1-\frac{1}{p}p} + \left(\varepsilon R^{\frac{n-1}{2p_0+1}} + K_{p_0}(\rho)\right)^{1-\frac{2}{n(n-1)}} \left(\varepsilon R^{\frac{n-1}{2p_0+1}} + \left(\frac{\rho^2}{\varepsilon R}\right)^{\frac{1}{2}}\right). \tag{4.3.27}
\end{equation}

Let
\begin{equation}
\rho_0 = \varepsilon^{\frac{n-1}{n(n-1)}} R. \tag{4.3.28}
\end{equation}

It follows from (4.3.25) that for $\rho \leq \rho_0$
\begin{equation}
K_{p_0}(\rho) \lesssim \varepsilon^\frac{1}{2} R^{\frac{n-1}{2p_0+1}}. \tag{4.3.29}
\end{equation}

Case of $n$ odd

From (4.3.29), (4.3.13) and (4.3.27) we have
\begin{equation}
K_{p_0} \lesssim \varepsilon^\frac{1}{2} R^{\frac{n-1}{2n+1}} + R^{\frac{1}{2}} \max_{\rho_0 < \rho_1} \rho_1^{-\frac{2n}{n(n+1)}} K_{p_1}(\rho_1) R^{\frac{2n}{n(n+1)}}
\end{equation}
\begin{equation}
\lesssim \varepsilon^\frac{1}{2} R^{\frac{n-1}{2n+1}} + \varepsilon R^{\frac{n-1}{2n+1}} R^{\frac{n-1}{2n+1}} + R^{\frac{1}{2}} \max_{\rho_1 > \rho_0} \rho_1^{-\frac{2n}{n(n+1)}} K_{p_0} \left(1 - \frac{2}{n(n-1)}\right) R^{\frac{2n}{n(n+1)}} \left(\frac{\rho_1^2}{\varepsilon R}\right)^{\frac{1}{2}}
\end{equation}
\begin{equation}
\lesssim \varepsilon^\frac{1}{2} R^{\frac{n-1}{2n+1}} + R \frac{1}{n(n+1)} \varepsilon^{-\frac{1}{n(n-1)}} K_{p_0} \left(1 - \frac{2}{n(n-1)}\right) R^{\frac{2n}{n(n+1)}}
\end{equation}
from which we deduce that
\begin{equation}
K_{p_0}^{(\epsilon)} \lesssim \varepsilon^\frac{1}{2} R^{\frac{n-1}{2n+1}} \quad \text{for} \quad \varepsilon > R^{-\frac{1}{n(n+1)}}. \tag{4.3.30}
\end{equation}
Assuming (4.3.30) and applying (4.3.26), \( \rho = R \), gives
\[
\| M \|_{p' \to p_1} \lesssim \varepsilon R + \left( \varepsilon^{2} R^{\frac{n}{n+1}} + \right) R^{\frac{n}{n+1}} \left( \frac{R}{\varepsilon} \right)^{\frac{1}{n}} \lesssim \varepsilon R
\]
provided that moreover
\[
(4.3.31) \quad \varepsilon \gtrsim R^{-\frac{2(n-1)}{n(n+1)} +}
\]
which supercedes the condition (4.3.30). Thus, we have obtained the higher dimensional results in Theorem 1.4 in the case of odd dimensions.

Case of \( n \) even

From (4.3.14) we have
\[
K_{p_0} \lesssim \varepsilon^{\frac{1}{2}} R^{\frac{n}{n+1}} + \varepsilon^{\frac{n}{n+1}} R^{\frac{n}{n+1}} +
\]
\[
+ R^{\frac{n}{n+1}} \max_{p_1, \rho_1 > p_0} R^{\frac{n}{n+1}} K_{p_0}^{\frac{n}{n+1}} \left( 1 - \varepsilon^{\frac{1}{2}} R^{\frac{n}{n+1}} \right)^{\frac{1}{n}} \left( \frac{\rho_1}{\sqrt{\varepsilon}} \right)^{\frac{1}{n+1}} \]
\[
\lesssim \varepsilon^{\frac{n}{n+1}} R^{\frac{n}{n+1}} + R^{\frac{n}{n+1}} \varepsilon^{-\frac{1}{2}} R^{\frac{n}{n+1}} K_{p_0}^{\frac{n}{n+1}} - \varepsilon^{\frac{n}{n+1}} K_{p_0}^{\frac{n}{n+1}}
\]
\[
\lesssim \varepsilon^{\frac{n}{n+1}} R^{\frac{n}{n+1}} + R^{\frac{n}{n+1}} \varepsilon^{-\frac{1}{2}} R^{\frac{n}{n+1}} K_{p_0}^{\frac{n}{n+1}}
\]
which implies that
\[
(4.3.32) \quad K_{p_0}^{(c)} \lesssim \varepsilon^{\frac{n}{n+1}} R^{\frac{n}{n+1}} , \quad \text{provided that} \quad \varepsilon > R^{-\frac{2(n-1)}{n(n+2)}}.
\]

Assuming (4.3.32), application of (4.3.26) with \( \rho = R \) gives
\[
\| M \|_{p' \to p_1} \lesssim \varepsilon R + \left( \varepsilon^{\frac{n}{n+1}} R^{\frac{n}{n+1}} + \right) R^{\frac{n}{n+1}} \left( \frac{R}{\varepsilon} \right)^{\frac{1}{n}} \lesssim \varepsilon R,
\]
meaning that we have obtained the results in Theorem 1.4 for even dimensions.

This completes the proof of Theorem 1.4. Note that we disregarded here the additional savings from non-trivial estimates on the exponential sum (cf (4.2.39)), which will be small in above range for \( \varepsilon \).

5. Improved bounds for manifolds with nonpositive sectional curvatures.

To prove Theorem 1.5 in view of Theorem 1.3, our task is equivalent to showing that if \((M, g)\) is a fixed compact manifold of dimension \( n \geq 3 \) with nonpositive sectional curvatures then
\[
(5.1) \quad \left\| \sum_{|\lambda_j - \lambda| \leq 1/ \log \lambda} E_{\lambda} f \right\|_{L^{\frac{2n}{2n-1}}(M)} \leq C(\log \lambda)^{-1} \| f \|_{L^{\frac{2n}{2n-1}}(M)}, \quad \lambda \gg 1.
\]
This is a special case of a recent unpublished estimate of Hassell and Tacey, which is an \( L^p \) variant of earlier bounds of Bérrard [3]. For the sake of completeness, we shall present a proof which is based on a slightly different interpolation scheme than the one employed by Hassell and Tacey.
Let us now present the proof of (5.1). If, as in (4.1.2), $a \in \mathcal{S}(\mathbb{R})$ is an even nonnegative function satisfying $a(0) = 1$ and $	ext{supp } \hat{a} \subset (-1,1)$, then by the proof of (4.1.1), we would obtain (5.1) if we could show that for a small fixed $\varepsilon_1$ to be determined later we have

$$
(5.2) \quad \left\| \int_{-\infty}^{\infty} \hat{a}(t/\varepsilon_1 \log \lambda) e^{it\lambda} \cos t \sqrt{-\Delta_g} f \, dt \right\|_{L^2(M^\infty)} \leq C \lambda \| f \|_{L^2(M^\infty)}.
$$

To do this, as in § 4.1, we choose an even function $b \in C_0^\infty(\mathbb{R})$ satisfying

$$
b(t) = 1, \quad |t| \leq 1, \quad \text{and } b(t) = 0, \quad |t| \geq 2.
$$

If we then, exactly as we did in the beginning of § 4.1, split the operator into two parts, $A_0$ and $A_1$ where we multiply the integrand by $b(t)$ and $(1 - b(t))$, respectively, then just as before $A_0$ satisfies the analog of (5.2) by virtue of Lemma 2.3.

Consequently, for the proof of (5.2), we are left with proving that for appropriate $\varepsilon_1 > 0$

$$
A_1 f = \int_{-\infty}^{\infty} (1 - b(t)) \hat{a}(t/\varepsilon_1 \log \lambda) e^{it\lambda} \cos t \sqrt{-\Delta_g} f \, dt.
$$

satisfies

$$
(5.3) \quad \| A_1 f \|_{L^2(M^\infty)} \leq C \lambda \| f \|_{L^2(M^\infty)}.
$$

We can even do a bit better than this, by interpolation, if we can show that given $\delta_1 > 0$ we can choose $\varepsilon_1 > 0$ small but fixed so that we have

$$
(5.4) \quad \| A_1 f \|_{L^{2(n+1)/(n+1)}(M)} \leq C \lambda^{n+1} \log \lambda \| f \|_{L^{2(n+1)/(n+1)}(M)},
$$

and

$$
(5.5) \quad \| A_1 f \|_{L^\infty(M)} \leq C \lambda^{n+1} \log \lambda \| f \|_{L^1(M)},
$$

for, together, if $\delta_1 < \frac{n+1}{n}$, they imply an improvement of (5.3) involving a smaller power of $\lambda$ in the right side.

Repeating the proof of (4.1.5) shows that (5.4) follows immediately from Lemma 2.3 as $\sigma(p)$ there equals $\frac{n+1}{n}$ when $p = \frac{2(n+1)}{n-1}$. Therefore, the proof of (5.1) would be complete if we could verify (5.5).

To prove this inequality, we shall use the fact that, like for $\mathbb{T}^n$, because of our assumption of nonpositive sectional curvatures, there is a Poisson-type formula that relates the kernel of $\cos t \sqrt{-\Delta_g}$ to a periodic sum of wave kernels on the universal cover of $(M, g)$, which is $\mathbb{R}^n$. If $p : \mathbb{R}^n \to M$ is a covering map, we shall let $\tilde{g}$ denote the pullback of $g$ via $p$. If $\Delta_{\tilde{g}}$ denotes the associated Laplace-Beltrami operator on the universal cover $(\mathbb{R}^n, \tilde{g})$, the formula we require is

$$
(5.6) \quad (\cos t \sqrt{-\Delta_{\tilde{g}}}) (x, y) = \sum_{\gamma \in \Gamma} (\cos t \sqrt{-\Delta_{\tilde{g}}}) (\gamma x, \gamma y), \quad x, y \in D,
$$

where $D \subset \mathbb{R}^n$ is a fixed fundamental domain, which we identify with $M$ via the covering map $p$, and $\Gamma$ denotes the group of deck transformations for the covering. The latter is the group of homeomorphisms $\gamma : \mathbb{R}^n \to \mathbb{R}^n$ for which $p = p \circ \gamma$, and $\gamma y$ denotes the image of $y$ under $\gamma$. Note then that if $(M, \tilde{g}) = (\mathbb{R}^n, \tilde{g})$, then $M \simeq M/\Gamma$.

To use this formula let us first note that, by Huygens principle, $(\cos t \sqrt{-\Delta_{\tilde{g}}}) (x, y) = 0$ if $d_{\tilde{g}}(x, y) > t$, with $d_{\tilde{g}}$ denoting the Riemannian distance with respect to the metric $\tilde{g}$. 


Consequently, since \( \hat{a}(t) = 0 \) for \( |t| \geq 1 \), in order to prove (5.5) it would be enough to show that for \( x, y \in D \) we have

\[
\sum_{\{\gamma \in \Gamma: d_{\hat{g}}(x, y) \leq \varepsilon_1 \log \lambda \}} \left| \int_{-\infty}^{\infty} (1 - b(t))\hat{a}(t/\varepsilon_1 \log \lambda)e^{it\lambda}\cos t\sqrt{-\Delta_{\hat{g}}}(x, y) \, dt \right| \leq C\lambda^{n-1/2}.
\]

Since \((\mathbb{R}^n, \hat{g})\) has no conjugate points, following Bérard \cite{3}, we can use the Hadamard parametrix for large times to prove estimates like (5.7). We no longer have uniform bounds on the amplitudes as we did in (2.27) and (2.28) for short times. On the other hand, for \( |t| \geq 1 \), by writing the Fourier integral terms in the Hadamard parametrix in an equivalent way to those in \cite{3} (cf. e.g., \cite{27}, Remark 1.2.5), we see that if \( \tilde{g} \) is as above then there is a constant \( c_0 = c_0(\tilde{g}) > 0 \), which is independent of \( T > 1 \) so that

\[
(\cos t\sqrt{-\Delta_{\tilde{g}}})(x, y) = \sum_{\pm} \int_{\mathbb{R}^n} \alpha_{\pm}(t, x, y, |\xi|)e^{i\xi(x,y)}e^{\pm it|\xi|} \, d\xi + R(t, x, y),
\]

where, as before, \( \kappa(x, y) \) denotes the geodesic normal coordinates (now with respect to \( \tilde{g} \)) of \( x \) about \( y \), but now (2.28) has to be replaced by

\[
\frac{d^j}{dt^j} \frac{d^k}{dx^k} \alpha_{\pm}(t, x, y, r) \leq A_{jk}e^{c_0T}(1 + r)^{-k},
\]

if \( 1 \leq |t| \leq T, \quad r > 0, \quad \text{and} \quad j, k \in \{0, 1, 2, \ldots \} \).

The remainder term can be taken to be continuous, but it also satisfies bounds that become exponentially worse with time:

\[
|R(t, x, y)| \leq Ae^{c_0T}, \quad \text{if} \quad 1 \leq |t| \leq T.
\]

To use these we shall require the following simple stationary phase lemma.

**Lemma 5.1.** Suppose that \( \alpha(t, r) \in C^\infty(\mathbb{R} \times \mathbb{R}_+) \) satisfies

\[
\alpha(t, r) = 0 \quad \text{if} \quad |t| \notin [1, T],
\]

and for every \( j = 0, 1, 2, \ldots \) and \( k = 0, 1, 2, \ldots \)

\[
\frac{d^j}{dt^j} \frac{d^k}{dr^k} \alpha(t, r) \leq A_{jk}e^{cT}(1 + r)^{-k},
\]

for a fixed constant \( c > 0 \). Then there is a constant \( B \) depending only on \( n \) and the size of finitely many of the constants in (5.11) so that

\[
\int_{-\infty}^{\infty} \left| \int_{\mathbb{R}^n} \alpha(t, |\xi|)e^{it\lambda|\xi|}e^{i\nu|\xi|} \, d\xi dt \right| \leq BTe^{cT}\lambda^{n-1}, \quad \lambda, T > 1.
\]

**Proof.** If \( |v| \leq 1/2 \), then one can integrate by parts in the \( \xi \)-variable to see that better bounds hold where in the right side \( \lambda^{n-1/2} \) is replaced by one. The result for \( |v| \geq 1/2 \) follows from (2.14) and (2.15) after noting that if \( \hat{\alpha}(\tau, |\xi|) \) denotes the partial Fourier transform in the \( t \)-variable then our assumptions (5.11) and an integration by parts argument imply that

\[
\int_{\mathbb{R}^n} |\hat{\alpha}(\lambda \pm |\xi|, |\xi|)| |\xi|^{n-1/2} \, d\xi \leq BTe^{cT}\lambda^{n-1}.
\]
We can now finish the proof of (5.7). If we use the lemma along with (5.8)-(5.10), we conclude that each term in the sum in the left side of (5.7) is bounded by
\[(5.12)\quad BT\varepsilon_1 T \lambda^{-\frac{n-1}{2}}, \quad T = \varepsilon_1 \log \lambda,\]
for some constant $B$ when $\lambda \gg 1$. Furthermore, as noted by Bérard classical volume growth estimates imply that there are $O(e^{c_1 T})$ nonzero terms in the sum in (5.7) for a fixed constant $c_1$. Thus, after possibly increasing the constant $c_0$ in (5.12), the whole sum is bounded by this quantity. Since given $\delta_1 > 0$ we can choose a fixed $\varepsilon_1 > 0$, $\varepsilon_1 = \varepsilon_1(c_0, \delta_1)$ so that the quantity in (5.12) is $O(\lambda^{-\frac{n-1}{2} + \delta_1})$ as $\lambda \to +\infty$, we obtain (5.7), and hence (5.1).

\[\Box\]

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