Variable Modified Newtonian Mechanics II: Baryonic Tully Fisher Relation

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Recently we find a single metric solution for a point mass residing in an expanding universe [1], which apart from the Newtonian acceleration, gives rise to an additional MOND-like acceleration in which the MOND acceleration $a_0$ is replaced by the cosmological acceleration. We study a Milky Way size protogalactic cloud in this acceleration, in which a source of angular momentum can lead to an end of the overdensity growth. Within realistic redshifts, the overdensity stops growing at a value where the MOND-like acceleration dominates over Newton and the largest mass shell rotational velocity obeys the Baryonic Tully Fisher Relation (BTFR) with a smaller MOND acceleration. The rotational velocity BTFR persists as the largest mass shell shrinks to a few scale length distances. Due to the conservation of angular momentum, the MOND acceleration grows to the phenomenological MOND acceleration value $a_0$ at late time.

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INTRODUCTION

Gas rich galaxies have non-Newtonian rotational velocities whose asymptotic behaviour can be parametrised approximately $2$ by the empirical Baryonic Tully-Fisher Relation (BTFR)

$$V = (a_0 GM_{Gal})^{1/4}.$$  \hspace{1cm} (1)

A common response is to retain Newtonian behaviour by postulating the existence of electromagnetically invisible cold Dark Matter (DM) to bridge the gap, although this has its difficulties [6–10]. Another approach, of which this paper is part, is to construct variants of Newtonian gravity to change long-range behaviour to modify the need for additional particles. The most familiar example of this approach is Modified Newtonian Dynamics (MOND) in which Milgrom [11] modifies the gravitational acceleration when the Newtonian acceleration is below some phenomenological scale $a_0$. This reproduces the asymptotic BTFR of Eq. (1) [11–13]. As it stands, this simplest MOND remains a limited phenomenological model restricted by its non-relativistic formulation [13, 14]. Even then, large scale structure formation simulations need a significantly smaller $a_0$ than the canonical value from BTFR to reproduce structure similar to the DM results [15] whereas a MOND acceleration treatment of the high temperature X-ray region of galaxy clusters [16, 17] shows that either more invisible mass is required or a fourfold increase of $a_0$ to produce the observed temperature is necessary.

To return to the data, emerging studies of Early Type Galaxy (ETG) rotational curves open up a new perspective on gravitational acceleration of large structures during cosmic evolution. Recently, Genzel et.al. [18] and Lang [19] show that early type galaxies have Newtonian rotational curves outside its scale length $r_0$, which in fact fall off faster than the (Newtonian) exponential disk from $2r_0 \sim 5r_0$, where $5r_0$ is the galactic disk size. Lang [19] suggests that ”at $z \sim 2$ there is no need for dark matter inside the galactic disk up to $5r_0$”. Between $z \sim 2$ to $z \sim 0.2$, galactic rotational speed outside the scale length starts to grow alongside with mass growth [20–22]. This corroborates with the time when the non-Newtonian HI gas profile enters the galactic disk in DiskMass Survey [23–25]. Even more specifically, Cappellari [26] finds that for ETG at small radii outside $r_0$, the amount of DM required is small ($\sim 10\%$). However, for large radii outside $5r_0$, the rotational curves follows a BTFR with very tight scatter Heijer et al. [27]. This BTFR matches the late time BTFR of spiral galaxies at $60 – 70\%$ of maximal mass to light ratio. In MOND term where the central mass is fixed, this is equivalent to requiring a smaller MOND acceleration $a_0$ at higher redshift and at larger distances. In terms of scale, a dense central protogalaxy at $z \sim 2.279$ is observed to have an external cloud spanning up to a distance of $130kpc$ [28].

These difficulties and others have led to a proliferation of variants of the original model; in particular, EMOND [29], GMOND [30], Emergent Gravity [31] and MOG [32] and Relativistic MOND [33]. Whatever the case, a viable MOND-like theory seems to require the MOND acceleration $a_0$ to vary in strength according to the environment where mass discrepancies appear up to 1 Gpc [34].

Given the successes of General Relativity (GR) to date, the question we address in this paper is that within GR, whether such scale and redshift dependent MOND-like acceleration can be found naturally without the need to postulate it in advance. The basic idea is simple. We look for gravitational metrics that interpolate between a Schwarzschild behaviour at short distances to FRW behaviour at large distances. The hope is that the behaviour of
the metric in the transition region permits an interpretation in a DM/MOND framework. We shall argue that, in some circumstances it can. While giving no complete solution to the problems that we have just stated, it provides a new platform upon which to model.

In studies involving a galaxy in an expanding background, the background metric is generally assumed to be the Einstein-Straus (Swiss-cheese) metric [35] or the McVittie metric [36]. However, neither of these solutions is without criticism [37]-[38] nor they are necessarily the only choices. Some years ago Baker [39] proposed an interpolating metric in Lemaître-Tolman (LT) coordinates. As it stands, by implementing the Bona-Stela construction, its predictions are incompatible with solar system data. In a previous work [1], we reexamined Baker’s [39] implementation to find a unique single metric which is compatible with solar system data. From this metric, a MOND-like (non-Newtonian) acceleration arises at slow speeds. Combining Newtonian and non-Newtonian acceleration (which we name ”VMOND” acceleration) amounts to the MOND scheme if the MOND acceleration $a_0$ is replaced by the cosmological acceleration $H_m(z)r$, where $H_m(z)$ is the Hubble parameter for a matter only universe at redshift $z$. Therefore our question is whether the late time BTFR has its origin from the dynamics of a protogalaxy in a VMOND potential.

In section 2, we recall the main features of the metric, calculate the overdensity growth rate under VMOND acceleration and find the orbital equation of a particle after its turnaround at high redshift. In section 3, we study in the VMOND potential a Milky Way model protogalaxy that drops out from its cosmological background due to a large angular momentum and calculate the corresponding MOND acceleration $a_0$ for the outermost mass shell. In section 4, we discuss briefly the formation of the central disk and the low MOND acceleration for Low Surface Brightness spiral galaxies. The last section is a summary and discussion.

2: THE MODEL

In a previous work [1], we follow Baker’s construction [39] of a metric that interpolates between Schwartzschild metric and FRW metric. The Bona-Stela solution was ignored by Baker because it does not match solar system data. We reexamine the construction and find a solution that avoids his problem. In Lemaitre-Tolman coordinates this metric takes the form

$$ds^2 = -\frac{2GMa^3}{c^2r}d\eta^2 - r^2d\Omega^2 + c^2d\tau^2$$

(2)

where $G$, $M$, $c$, $r$ are the Newton’s constant, the central point mass, the speed of light and the radial distance respectively. $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, where $\theta$ and $\phi$ are the polar angles in the spherical symmetric coordinates. The radial acceleration is given as

$$\ddot{r} = \frac{h^2}{r^3} - \frac{GM}{r^2} - H_m\sqrt{\frac{GM}{2r}} - \frac{\dot{a}}{a}r,$$

(3)

where $H_m(z)$ is the Hubble constant due to the matter component of the universe at red-shift $z$ and $h$ is the angular momentum per unit mass. The effect of the cosmological constant $\Lambda$ only affects the acceleration term $\dot{a}/a$. We term the sum of the Newtonian and the non-Newtonian gravitational acceleration the ”VMOND” acceleration. The slow speed energy equation (without background radiation) is given by integrating Eq.(3) by

$$E = \frac{1}{2}\dot{r}^2 + \frac{h^2}{2r^2} - \frac{1}{2}\left(\frac{2GM}{r} - H_m(z)r\right)^2 - \frac{\Lambda c^2}{6}r^2,$$

(4)

where $E$ is the energy of the particle. Without cosmological constant, in particle free fall ($h = 0$ and $E = 0$) the energy equation reduces to the simple relation

$$\dot{r} = H_m r - \sqrt{\frac{2GM}{r}},$$

(5)

which describes a particle speed that follows the Hubble expansion at large distances but will follow a Newtonian path at small distances. The distance from the centre

$$r = \left(\frac{2GM}{H_m^2}\right)^{1/3}$$

(6)

describes a turnaround radius of the particle.
2.1 Density perturbation evolution: Spherical Top-hat model

The Hubble parameter at redshift $z$ for a flat universe where radiation is negligible is given by

$$H^2(z) = \frac{8\pi G}{3} \rho_H = \frac{8\pi G}{3} \left( \rho_m + \rho_\Lambda \right) = H_0^2 \left( \Omega_m (1 + z)^3 + \Omega_\Lambda \right), \quad H_0^2 = \frac{8\pi G}{3} \rho_c,$$

where $\rho_m$ and $\rho_\Lambda$ are cosmological background densities of baryonic matter and dark energy respectively. $H_0$ and $\rho_c$ are the Hubble parameter and the critical density of the present epoch. $\Omega_m$ and $\Omega_\Lambda$ are the density parameter for matter and $\Lambda$ respectively.

Early time matter density fluctuation can be modelled by a spherical top hat, where the cosmological background is in the matter dominant epoch (approximated by the Einstein de-Sitter universe) with mean density

$$\rho_m = \frac{1}{6\pi G t^2}. \quad (8)$$

The baryon density perturbation $\delta$ is specified by

$$\delta = \frac{\rho - \rho_m}{\rho_m} = \delta \rho_m.$$ 

where $\rho$ is the total mass density of the overdense region inside radius $r$. The baryon matter overdensity is given by $\rho_b = \delta \rho_m$.

Under Newtonian gravity, the acceleration equation for a point at $r$ on the overdensity surface is given by

$$\frac{\ddot{r}}{r} = -\frac{4\pi G}{3} (1 + \delta) \rho_m = -\frac{GM_t}{r^3}, \quad M_t = M + m,$$ 

where $M$ is the overdensity mass, $m$ is the mass of background matter density and $M_t$ is the total mass within the sphere radius $r$, which is assumed to be conserved over time. Eq.(10) also describes a particle orbit under Newtonian gravity with central mass $M_t$.

Following the derivation of Binney & Tremaine [40] in which a parametrisation of $r = a(1 + \cos \theta)$ and $t = b(\theta - \sin \theta)$ with $a^3 = GM_t b^2$, where $\theta$ is the "eccentricity anomaly". For $t$ small and $\theta$, solving for $r$ (upto 6th order in $\theta$) one obtains

$$r = \left(\frac{GM_t}{2} \frac{1}{r_H^3}\right)^{1/3} \left(1 - \frac{1}{20} \left(\frac{6t}{b}\right)^{2/3} \right) = r_H \left(1 + \frac{\delta r}{r_H} \right) \quad (11)$$

where $r_H$ is the radius if no overdensity is present. The density $\rho$ inside the overdensity changes as

$$\rho = \frac{M_t}{4\pi r^3} = \rho_m \left(1 - 3 \frac{\delta r}{r_H} \right). \quad (12)$$

As the cosmological background density decreases, the sphere’s comoving volume decreases leading to the overdensity increase. From Eq.(11) and Eq.(12) we obtain

$$\delta = -3 \frac{\delta r}{r_H} = 3 \frac{6t}{20} \left(\frac{b}{b} \right)^{2/3}. \quad (13)$$

This produces the well known density perturbation growth result $\delta \propto r^{2/3} \propto a$. For a typical overdensity at recombination $\delta \sim 5 \times 10^{-4}$, the growth rate under Newtonian gravity leads to insufficient overdensity growth compared to observations at late times. One needs to postulate an additional input such as DM particles.

With our new metric, the acceleration equation Eq.(10) takes the form

$$\frac{\ddot{r}}{r} = -\frac{4\pi G}{3} \left(1 + \delta \right) \rho_m - \frac{4\pi G}{3} \frac{\delta^{1/2}}{r^3} \rho_m = -\frac{GM_t}{r^3} - \frac{GM_1}{r^3}, \quad M_1 = \sqrt{Mm}$$ 

$$\quad (14)$$
The density component $\delta^{1/2}\rho_m$ is driven by the non-Newtonian acceleration, which can be regarded as a Newtonian acceleration from a dynamical mass $M_1$. For a small variation of the initial mass values $M$ and $m$ to $M_i = M + \delta M$, $m_i = m + \delta m$, the conservation of $M_i = M + m$ gives us $\delta M = -\delta m$. For $M \ll m$ we have

$$M_i m_i = M m \left(1 + \delta_M\right); \quad \delta_M = \frac{\delta M}{M}$$

(15)

For small $\delta_M \ll 1$ and $M_{iT} = M_i + m_i + 2\sqrt{m_iM_i}$,

$$\frac{M_{iT}}{M_T} = 1 + \sqrt{\frac{mM\delta_M}{M_T}} < 1 + \frac{1}{2} \delta^{1/2} \delta_M.$$  

(16)

For small $\delta \ll 1$, $M_i + M_1 = M_T$ remains approximately constant. We can adopt the Newtonian picture above with dynamical mass $M_T$ (where the dynamical overdensity is $\delta + \delta^{1/2}$).

The treatment of an orbit under Newtonian gravity for $M_T > M_i$ remains a good approximation and Eq.(12) is valid for $M_T = M_i + M_1$ and we have deviation of $r$ from $r_H$ driven by the total dynamical mass $M + M_1$ due to the overdensity, so that

$$-3\frac{\delta r}{r_H} = \frac{3}{20} \left(\frac{6t}{b}\right)^{2/3} = \delta + \delta^{1/2}.$$  

(17)

For $\delta \sim 10^{-4}$, as $\delta^{1/2} \gg \delta$, it is the term $\delta^{1/2}$ that dominates the R.H.S. of Eq.(17), we have $\delta^{1/2} \propto t^{2/3}$ and thus $\delta \propto t^{4/3} \propto a^2$. This overdensity growth rate from recombination to the present time is greater than $10^5$ and is therefore sufficient to account for observations in the large scale structures at late time. We note that this $\delta \propto a^2$ growth rate can also be obtained in the MOND scheme [41].

### 2.2: Density Perturbation Evolution-Linear Perturbation

We look at the density perturbation growth in small $\delta$ regime using the linear perturbation approach. For our purpose, we start with physical radius with 3-velocity field $\vec{v}$. The Euler equation for negligible spatial entropy gradient over the overdensity and radial velocity is simply the acceleration equation

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\nabla \Phi,$$  

(18)

where $\Phi$ is the gravitational potential, and the Poisson equation is given by

$$\nabla^2 \Phi = 4\pi G \rho.$$  

(19)

Also from the Gauss theorem

$$\oint_{\partial V} \nabla \Phi \cdot dA = 4\pi GM$$  

(20)

Under Newtonian gravity, $\Phi = -\frac{GM}{r}$, the enclosed mass $M_i$ in a volume $V$ is the ”physical” mass $M_i$ of the baryons, and density $\rho = M_i/V$. Starting $\frac{dM_i}{dt}$, one can obtain the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0.$$  

(21)

Eq.(18)-(21) work for an overdensity $\delta \rho_m$ on top of a background matter density $\rho_m$ with

$$\rho = \rho_m (1 + \delta).$$  

(22)

However, in VMOND the gravitational potential $\Phi$ is different from the Newtonian potential. We have for the radial velocity, the Euler equation

$$\frac{d\vec{v}}{dt} = -\nabla \Phi = -\frac{GM}{r^2} - H_m \sqrt{\frac{GM}{2r}} - \frac{4\pi G}{3 \rho_m r} - \frac{4\pi G}{3 \rho_m r} (\delta + \delta^{1/2} + 1).$$  

(23)
The Poisson equation becomes
\[ \nabla^2 \Phi = 4\pi G\rho = 4\pi G\rho_m (1 + \delta + \delta^{1/2}). \] (24)

The Gauss theorem gives
\[ \oint_{\partial V} \nabla \Phi \cdot dA = 4\pi GM_T = 4\pi G(1 + \delta + \delta^{1/2})\rho_m V. \] (25)

As discussed in top-hat model above, there is an additional dynamical mass density \( \delta^{1/2}\rho_m \) term that enters Eq.(24) and Eq.(25). From Eq.(16) the \( M_T \) is nearly conserved during small radius expansion,

\[ \frac{dM_T}{dt} = \int_V \frac{\partial \rho}{\partial t} dV = -\oint_{\partial V} \rho V dA = -\int_V \nabla (\rho \vec{v}) dV, \] (26)

one obtains the continuity equation for \( \rho = \frac{M_T}{V} = \rho_m (1 + \delta + \delta^{1/2}) \)

\[ \frac{\partial \rho}{\partial t} = \nabla (\rho \vec{v}). \] (27)

which is the more relevant continuity equation in VMOND potential with an enclosed density described in Eq.(24).

Working in comoving coordinates where \( \rho \) is the dynamical mass density within a comoving volume of the overdensity and \( \vec{v} \) is peculiar velocity, one can follow [40] p.722 to obtain the density perturbation evolution equation for \( \Delta(\delta) = \delta + \delta^{1/2} \),

\[ \frac{\partial^2}{\partial t^2} \Delta(\delta) + 2H_m \frac{\partial}{\partial t} \Delta(\delta) = 4\pi G\rho_m \Delta(\delta). \] (28)

in a flat matter dominant epoch with \( H_m = \frac{2}{3t} \)

\[ \frac{\partial^2}{\partial t^2} \Delta(\delta) + \frac{4}{3t} \frac{\partial}{\partial t} \Delta(\delta) - \frac{2}{3t^2} \Delta(\delta) = 0, \] (29)

with a growth solution given by

\[ \Delta(\delta) \propto t^{2/3} \propto a(t); \quad a(t) \propto \frac{1}{1+z}. \] (30)

With VMOND we obtain a new \( \delta \) growth rate \( \delta \propto t^{4/3} \propto a(t)^2 \). In MOND, the same overdensity growth rate \( \delta \propto a^2 \) is used in large scale structure simulation [15] and in elliptical galaxy formation simulation [42].

To make contact with MOND, from Eq.(29) the non-Newtonian acceleration can be rewritten as

\[ \ddot{r} = -H_m(z) \sqrt{\frac{GM}{2r}} = -\sqrt{\frac{GM}{r^2}} \sqrt{\frac{H_m(z)^2r}{2}} = -\sqrt{g_N a_0(z,r)} \] (31)

where \( g_N \) is the Newtonian acceleration and \( a_0(z,r) \) is a cosmological acceleration at distance \( r \) from the mass centre. At small \( g_N \ll a_0 \) limit, the acceleration in MOND is given by \( \ddot{r} = -\sqrt{g_N a_0} \) where \( a_0 = 1.2 \times 10^{-10} m/s^2 \) is a fixed phenomenological MOND acceleration.

Similar to MOND, if \( g_N \ll a_0(z,r) \), we will enter into a non-Newtonian acceleration dominant region. For late time galaxies, a quick check using a high value of \( H_0 = 75 \text{km}s^{-1}(\text{Mpc})^{-1} \) and a large distance outside the stellar disk of the Milky Way at \( r = 50 \text{kpc} \), the cosmological acceleration is \( a_0(0,r) \sim 1.02 \times 10^{-15} m/s^2 \). This \( a_0(0,r) \) is too weak to match the MOND acceleration \( a_0 \). However, as observed in ETG, if a non-Newtonian rotating gas appears at an epoch with larger \( H(z) \) (e.g. \( \rho_H \sim 10^3 \times \rho_c \)) and at a distance \( r \gg 50 \text{kpc} \), \( a_0(z,r) \) can be much larger than \( a_0(0,r) \). To see whether it is possible to reproduce the phenomenological MOND acceleration, we shall consider a model protogalactic cloud.
2.3: Effect of a large Angular momentum at small $\delta$

We shall study uniform density spherical mass shells moving under the effect of cosmic expansion. The mass shells move independently without friction between neighbouring shells. One point on the mass shell can be represented by a point particle at a distance $r$ from the centre in a VMOND potential.

In the strongly matter dominant epoch, a spherical shell initially without angular momentum follows Eq. (4)

$$E = \frac{1}{2} \dot{r}^2 - \frac{1}{2} \left( \sqrt{\delta} - 1 \right)^2 H^2_m(z) r^2 = \frac{1}{2} \dot{r}^2 - \frac{GM(r)}{r} \left( 1 - \frac{1}{\sqrt{\delta}} \right)^2 ,$$

(32)

where $M(r)$ is the enclosed mass inside a sphere at radius $r$ and small $\delta$ grows as $\delta \propto a^2$. For $\delta = 0$ and $E = 0$, a shell at radius $r$ will simply follow the Hubble expansion. In a Newtonian potential, for $\delta > 0$, a bounded orbit requires $E < 0$. However, Eq. (32) admits an $E = 0$ solution, so that $\dot{r} \to 0$ as $\delta \to 1$. Since the starting radial velocity $\dot{r} > 0$ is outward moving, when $\dot{r}$ reaches $\dot{r} = 0$, it could turnaround and the overdensity breaks away from its cosmological background. As we see below, for negligible angular momentum and realistic overdensity at recombination, the breakaway redshift is $z \sim 13$ which is outside the probable galaxy formation redshift range $15 < z < 50$, see Barkana and Loeb [43].

When including angular momentum, the radial acceleration equation Eq. (3) in terms of densities is

$$\ddot{r} = \frac{h^2}{r^3} - \left( 1 + \delta + \sqrt{\delta} \right) \frac{4\pi G \rho_m(z) r}{3} .$$

(33)

During the over-density growth, the overdense region receives an angular momentum due to tidal fields from its neighbours. A common assumption is that angular momentum behaves as a small perturbation and does not affect the over-density growth (in this case $h^2 = 0$ is kept in Eq. (33)). In the small $\delta$ regime, the angular momentum growth based on tidal fields of neighbours is modelled by Peebles [44] in which the time dependent angular momentum per unit mass is given by the second order perturbation which grows as

$$h(t) \propto t^{5/3} .$$

(34)

so that the angular speed squared $\dot{\varphi}^2$ grows as

$$\frac{h^2}{r^4} \propto a(t) .$$

(35)

White [43] obtains a first order perturbation from a flatten sphere which grows as

$$h(t) \propto t ,$$

(36)

and that $\dot{\varphi}^2$ grows as

$$\frac{h^2}{r^4} \propto \frac{1}{a(t)} .$$

(37)

Casuso and Burkert in [46]-[47] propose that an expanding void can also provide a source of the protogalaxy’s angular momentum. In this scenario, the angular momentum transfer from the void causes an overdensity to gravitationally collapse and breaks-away from its cosmological background.

In this study we consider the scenario that a source of angular momentum catches up to the gravitational acceleration, which ends the over-density growth as the acceleration driving perturbation growth in Eq. (28) is no longer in effect. Angular momentum source from an expanding void is an attractive option. We want to mention the condition in which the White model can also provide a large angular momentum source. Once the angular momentum is introduced, we work from the mid-plane of the shell, perpendicular to the rotational axis. At some $\delta$, the protogalactic mass shell picks up an angular speed squared $h^2 r^{-4}$ which for second order perturbation starts to grow as $a$. As the expanding shell flattens, it starts to pick up larger first order perturbations which fall as $a^{-1}$. As
the background density $\rho_m \propto a^{-3}$ continues to fall, the gravitational acceleration due to central density perturbation in Eq. (33) changes as

$$\left(\delta + \delta^{1/2}\right) \frac{4\pi G}{3} \rho_m = \left(\delta_{\text{int}} a^{-1} + \delta_{\text{int}}^{1/2} a^{-2}\right) \frac{4\pi G}{3} \rho_{\text{int}}, \tag{38}$$

where $\delta_{\text{int}}$ and $\rho_{\text{int}}$ are initial overdensity and background density at recombination respectively. Initially, at small $a(t)$ the gravitational acceleration driving overdensity growth dominates over the angular acceleration. As $a(t)$ grows, the large gravitational attraction ($a^{-2}$) term in Eq. (33) drops faster than $\varphi^2 \propto a^{-1}$ obtained from the White’s model, here we assume that the first order perturbation is dominant. Based on the White model, an accumulation of angular momentum so that $\dot{\varphi}^2$ is much larger than $\frac{4\pi G}{3} \rho_{\text{int}}$ will suffice for the angular acceleration to catch up to the central gravitational acceleration at some $\delta < 1$. The collapse of initial inhomogeneities into filaments, walls and voids provides an environment of matter concentration which is favourable for angular momentum and energy transfer.

Whatever the source, for spiral galaxies, we assume that there exists a source of angular momentum which for the largest shell of an overdense cloud at radius $r_i$ and redshift $z_i$ leads to

$$\dot{\varphi}^2 = \frac{h_i^2}{r_i^4} = \left(\delta + \sqrt{\delta}\right) \frac{4\pi G \rho_m(z_i)}{3}, \tag{39}$$

where the angular momentum $h_i$ is now fixed and the mass inside $r_i$, $M(r_i) = \int_0^{r_i} r^2 \delta \rho_m(z_i) dz$ is also fixed. After this point, the overdensity drops out of the cosmic matter density background. This is similar to what is assumed in MOND [12].

### 2.4: The particle orbit in the VMOND potential after $\delta$ stops growing

After the overdensity stops growing, the energy equation for a shell with fixed mass $M$ becomes

$$\frac{1}{2} \dot{r}^2 + \frac{h^2}{2r^2} = \frac{1}{2} \left(\frac{2GM}{r} - H_m(z)r\right)^2 + E = \alpha \frac{GM}{r} + E, \tag{40}$$

where

$$\alpha = \left(\frac{\rho_m(z)}{\rho_m(r)} - 1\right)^2. \tag{41}$$

$\rho_b(r)$ is no longer linked to the background matter density. At this time, $\dot{r} > 0$ remains outgoing from the mass centre and $\alpha = (1 - \frac{r_i}{r})^2 > 0$. $\rho_m(z)$ decreases with $z$ and $\alpha$ decreases. From Eq. (41), both $\dot{r}$ and $\frac{h^2}{r^2}$ decrease. For $E > 0$, there exists a solution with a turnaround radius $r_{ta}$ where

$$\dot{r}_{ta} = 0, \quad \alpha = 0, \quad E = \frac{1}{2} \frac{h^2}{r_{ta}^2}. \tag{42}$$

As redshift reduces further, $\alpha$ increases from zero towards unity, the mass shell starts to move away from the turnaround radius towards the mass centre.

After the turnaround, the particle’s orbit with planar polar coordinates ($r, \varphi$) satisfies

$$\varphi = \int_{r_0}^r \frac{r^{-1} dr}{\sqrt{\frac{2}{\mu} (E - \Phi)r^2 - 1}} = \int_{r_0}^r \frac{r^{-1} dr}{\sqrt{\frac{2}{\mu} (E + \alpha \frac{GM}{r}) - 1}}. \tag{43}$$

We now work in the range $0 \leq \alpha \leq 1$ and in the matter dominant epoch. We follow the usual parametrisations by taking a length scale $l$ with $\frac{h^2}{2GM} = \frac{l}{l} \cdot E_0 = E_0 = \frac{E}{2GM}$ and eccentricity $e^2 = 1 + 4E_0$.

Rearranging Eq. (43) in terms of $\epsilon$ we obtain

$$\varphi = \int_{r_0}^r \frac{lr^{-2} dr}{\epsilon \sqrt{1 + \frac{1}{4\epsilon} (\alpha^2 - 1) - \frac{1}{4\epsilon} \left(\alpha - \frac{2\varphi}{r}\right)^2}} = \int_{r_0}^r \frac{lr^{-2} dr}{\epsilon \sqrt{1 - \frac{1}{4\epsilon} - \frac{1}{4\epsilon} \left(\frac{2\varphi}{r}\right)^2 + \left(\frac{2\varphi}{r}\right) \left(\frac{2\alpha}{r}\right)}}. \tag{44}$$
Setting
\[ \epsilon x = -\frac{l}{r}, \]
we obtain
\[ \varphi = \int \frac{dx}{\sqrt{\left(1 - \frac{l^2}{r^2}\right) - \frac{2\alpha x - x^2}{r^2}}} . \]

The variation of \( \alpha \) is based on a cosmological time scale, we approximate \([16]\) by taking \( \alpha \) constant in the integral, and the linear term in \( x \) is finite as long as \( \epsilon > 0 \). Eq. (46) is then a standard integral, we obtain
\[
\left(1 + \frac{\alpha^2 - 1}{\epsilon^2}\right)^{1/2} \epsilon \sin \varphi = \frac{l}{r},
\]
That is, in Keplerian form
\[ r = \frac{l}{\alpha - (\epsilon^2 + \alpha^2 - 1)^{1/2} \sin \varphi} = \frac{l'}{1 - \epsilon' \sin \varphi}, \]
where
\[ l' = \frac{l}{\alpha}, \quad \epsilon' = \frac{\epsilon}{\alpha} \left(1 + \frac{\alpha^2 - 1}{\epsilon^2}\right)^{1/2} . \]

For a particle with positive energy \( E > 0 \) at high redshift, the particle orbit is hyperpolic. A large angular momentum leads to a large \( l \). To have a closest approach much smaller than \( l \) one will need both a large eccentricity \( \epsilon \) and maximum \( \alpha = 1 \). At late time, when \( \alpha \sim 1 \), the effect of the cosmological constant is much less than Newtonian gravity, the above orbital equation remains a good approximation. From Eq. (46), at small \( \alpha \), one can regard the particle orbit as having a large eccentricity \( \epsilon' \) with a large \( l' \).

### 3: \( a_0 \) IN A MODEL PROTOGALAXY IN VMOND POTENTIAL

After recombination, in a Newtonian potential, it is generally assumed that when the baryonic density perturbation reaches \( O(1) \) the protogalaxy will start to break away from the expanding universe. It is expected that massive galaxies begin to form at a narrow range of redshifts \( z \sim 15 \text{ to } 50 \), see Barkana-Loeb [43]. To date, the oldest observed galaxy GN-z11 at \( z \sim 11.09 \) [48] at mass \( \sim 10^9 \text{M}_\odot \) and HD1 at \( z \sim 13.27 \) [49] with a similar mass, provide important signposts on the plausible break-away epoch for galaxies with mass \( 10^9 \text{M}_\odot \sim 10^{11} \text{M}_\odot \).

To simulate the monolithic formation of a spiral galaxy with mass \( 10^9 \text{M}_\odot \sim 10^{11} \text{M}_\odot \), a Milky Way formation simulation under a pure Newtonian potential is done using an initial cloud density at \( \rho_{\text{int}} = 7 \times 10^{-23} \text{kg/m}^3 \) [50]. In a MOND background acceleration, the authors in [51] use an initial uniform cloud density \( \rho_{\text{int}} = 1.11 \times 10^{-23} \text{kg/m}^3 \).

We pick a model protogalaxy using the observed masses \( M(r) \) in Milky Way (MW) as a guard-rail, assuming that the mass distribution \( M(r) \) is not altered by a past encounter with Andromeda in the local group. Within the Milky-Way scale length \( r_0 = 8.5 \text{kpc} \), the observed stellar mass is found to be \( M(r_0) = 6.07 \times 10^{10} \text{M}_\odot \) [52]-[53]. Assuming pure Newtonian potential and using the relation \( h^2 = 1.1GMr_0 \), the observed rotational speeds at \( r_0 \) gives \( M(r_0) = 8.4 \times 10^{10} \text{M}_\odot \). In [54], outside the Milky Way stellar disk, an exponential disk of HI gas is observed extended up to 60 kpc and the simulation work, [55] uses a dark gas ring of \( \sim 2.4 \times 10^{10} \text{M}_\odot \) (where the baryonic gas content is \( 1.2 \times 10^{10} \text{M}_\odot \)). We therefore take the Milky Way baryonic mass estimate at distance \( r \), called \( M_r \) to be
\[ M_r = 8.4 \times 10^{10} \text{M}_\odot, \quad M_{50 \text{kpc}} = 50 = 9.5 \times 10^{10} \text{M}_\odot. \]

The model protogalaxy mass is then chosen to be that of \( M_{50 \text{kpc}} \). Our primary interest is the BTFR in the region of \( 2r_0 \sim 6r_0 \), where the present galactic baryonic mass density \( \rho_b \) at \( r_0 \) and \( r_{50} = 50 \text{kpc} \) respectively are given by
\[
\rho_b(r_0) = 2.42 \times 10^{-21} \text{kg/m}^3, \quad \rho_b(r_{50}) = 1.34 \times 10^{-23} \text{kg/m}^3 .
\]

Although \( \rho_b(r_{50}) \) is much higher than the critical density, as long as its protogalactic cloud at early time has a density lower than the background matter density, the non-Newtonian gravity will become important.
3.1: Early time Baryonic Tully Fisher Relation

We consider the largest mass shell intersecting the mid-plane, Eq. (39) in the ($\delta \ll \sqrt{\delta}$) region leads to the rotational speed $V(r_i)$

$$V(r_i) = \frac{h^4}{r_i^4} = \left(\frac{4\pi}{3}\rho_m(z_i)r_i(1 + \sqrt{\delta})^2\right)GM(r_i) = a_0(z_i, r_i)GM(r_i). \quad (51)$$

Here the $\rho_m(z_i)$ is set at redshift $z_i$ when the overdensity growth stops and the Newtonian acceleration part is represented by the factor $\sqrt{\delta}$. Eq. (51) already shows that the mass shell rotational speed follows the BTFR, with an effective MOND acceleration given by

$$a_0(z_i, r_i) = \left(\frac{4\pi}{3}\rho_m(z_i)r_i(1 + \sqrt{\delta})^2\right). \quad (52)$$

We note that in Eq. (51), a small variation of $\delta$ does not change the overall power law behaviour between $V$ and $M$. It is also possible to write Eq. (51) in the "variable" Newtonian form

$$V(r_i)^2 = \left(1 + \frac{1}{\sqrt{\delta}}\right)\frac{GM(r_i)}{r_i}. \quad (53)$$

For large $\delta$, a variation of $\delta$ maintains the Newtonian nature of the acceleration with some varying invisible matter effect. However, for small $\delta$, the $\delta^{-1/2}$ term dominates the bracket and disrupts the Newtonian power-law behaviour, i.e. with a strongly varying dark matter mass profile. Thus variations of a small $\delta$ is better described by the power law in Eq. (51). As $\delta$ varies from a small value to a large value, we expect to see the relation $V^\beta \propto M$ with $\beta$ varies from $\sim 4$ to $\sim 2$.

The mass shell at $r_i$, which is usually far away from $r_{50}$, will eventually shrink to galactic disk scale $r_{50}$ by a contracting factor $n$ such that

$$r_i = nr_{50} \quad (54)$$

for $n > 1$. The conservation of the angular momentum of the mass shell dictates that the rotational speed will increase as its radius shrinks. After mass shell radius contracts to $50kpc$, from Eq. (51) the rotational speed continues to obey the BTFR but will take higher values given by

$$V^4(r_{50}) = n^4V^4(r_i) = n^4a_0(z_i, r_i)GM(r_i) = a_0^{VM}(r_{50})GM(r_i), \quad (55)$$

so that the MOND acceleration at $r_{50}$ by our model is

$$a_0^{VM}(r_{50}) = a_0(z_i, r_i)n^4; \quad a_0^{VM}(r_{50}) = (1 + \sqrt{\delta})^2\left(\frac{4\pi G}{3}\rho_m(z_i)\right) \times 50kpc \times n^5. \quad (56)$$

To calculate $a_0^{VM}(r_{50})$, we need to specify $\delta$, $\rho_m(z_i)$ and $n$.

After the mass shell radius $r_i$ shrinks to the size $r_{50}$, we have

$$M_{50kpc} = \frac{4}{3}\rho_b(r_{50})r_{50}^3, \quad \rho_b(r_{50}) = n^3(\delta\rho_m(z_i)) \quad (57)$$

For our MW model, $\rho_b(r_{50})$ is given in Eq. (50). Using the present day critical density $\rho_c = 1.11 \times 10^{-26}kg/m^3$ and $\Omega_m = 0.05$, the background matter density at redshift $z_i$ is given by

$$\rho_m(z_i) = (0.05)(1 + z_i)^3\rho_c. \quad (58)$$

(This is the same assumption used in Sanders [12] to simulate elliptical galaxy formation in MOND acceleration. In [12], the overdensity is assumed to stop growing and condenses out of the expanding background when the "MOND acceleration" is twice the size of the cosmological acceleration.)
At recombination $z = 1080$, we consider a realistic initial $\delta_{\text{int}} \sim 1.65 \times 10^{-4}$ ($\Delta T \sim 5.5 \times 10^{-5}$). Since $\delta$ grows as $a(t)^2$, we can calculate the overdensity $\delta$ at redshift $z_i$ by the scale factor growth to obtain

$$\delta = \delta_{\text{int}} \left( \frac{1081}{1 + z_i} \right)^2, \quad \sqrt{\delta} = \frac{13.88}{1 + z_i}$$

(59)

so that for a range of $z_i$, we can fix $n$ and $\rho_m(z_i)$ to obtain $a_0^{VM}(r_{50})$ from Eq. (59). We notice that with negligible angular momentum, a protogalaxy turnaround at $\delta \to 1$ where the redshift is $z_i = 12.88$. This $z_i$ is outside the probable galaxy breakaway redshift $15 < z < 50$ range.

(Note: The Hubble tension and the KBC void question could incur up to 10% uncertainty in $H_0$ value but it should not invalidate the physical picture of this analysis.)

We focus on an observationally viable redshift range $15 \lesssim z \lesssim 50$ in which the $\delta$ growth may stop and the overdensity breaks away from the background. Write the phenomenological MOND acceleration in terms of $H_0$ and $r_{50}$ as

$$a_0 = \frac{1}{6} cH_0 = \left( \frac{1}{2} H_0^2 r_{50} \right) \gamma; \quad \gamma = \frac{c/H_0}{3 \times 50 \text{ kpc}} = 2.67 \times 10^4,$$

(60)

where the factor $\gamma$ specifies the MOND observed value. Eq. (59) takes the form

$$a_0^{VM}(r_{50}) = \left( \frac{1}{2} H_0^2 r_{50} \right) \gamma(z_i), \quad \gamma(z_i) = (1 + \sqrt{\delta})^2 \Omega_m (1 + z_i)^3 n^5 = \left( 1.45 \times 10^2 \times (1 + z_i)^{4/3} \right)(1 + \sqrt{\delta})^2.$$

(61)

If $\delta$ freezes at redshift $z_i = 17$, we have $\sqrt{\delta} = 0.771 (\delta = 0.594)$, $n = 1.90$ and $\rho_m(z_i) = 3.23 \times 10^{-24} \text{ kg/m}^3$ so that $\gamma(z_i) = 2.14 \times 10^4$

$$a_0^{VM}(r_{50}) = 0.803 a_0.$$

(62)

If $\delta$ freezes at higher redshift $z_i = 38$, $\sqrt{\delta} = 0.355 (\delta = 0.126)$, $n = 1.47$ and $\rho_m(z_i) = 3.29 \times 10^{-23} \text{ kg/m}^3$, $\gamma(z_i) = 3.51 \times 10^4$ so that

$$a_0^{VM}(r_{50}) = 1.31 a_0.$$

(63)

At $z_i = 23$, we have

$$a_0^{VM}(r_{50}) = a_0.$$

(64)

The above work demonstrates that given a realistic $\delta_{\text{int}}$ value at recombination, within a realistic but narrow range of redshifts ($17 \lesssim z \lesssim 38$) where a protogalaxy could stop growing in mass after picking up enough angular momentum. This happens at the small $\delta$ regime where the non-Newtonian gravitational acceleration dominates and the rotational velocity of the largest mass shell is naturally constrained by a BTFR. As its radius shrinks to galactic disk scale, the VMOND acceleration $a_0^{VM}$ grows to the phenomenological MOND value $a_0$ within observable errors. Within this paradigm, the MOND acceleration has a natural cosmological origin.

In galaxy formation simulation of Newtonian or MOND using mass shells, the correct angular momentum and energy are introduced as necessarily conditions. For simplicity, consider specifically a protogalactic overdensity which stops growing at $z \sim 17$. After the largest mass shell of the model galaxy contracts to $r_{50}$, the rotating velocity is $V = (GM a_0)^{1/4} = 181 \text{ km/s}$ and $\sqrt{\frac{2GM}{r_{50}}} = 91.5 \text{ km/s}^2$. We have the orbital length scale $l \sim 4r_{50}$. For the largest mass shell to reach $r_{50}$, we need $\alpha = 1$ and $\epsilon > 3$. Taking the particle energy $E = \frac{1}{2} \frac{L^2}{r_E^2}$,

$$\epsilon^2 - 1 = \frac{2EL}{GM} = 16 \left( \frac{r_{50}}{r_E} \right),$$

(65)

so that to obtain $\epsilon > 3$, one needs $r_E < \sqrt{2} r_{50}$. From Eq. (65), at turnaround, $r_E = r_{ta} > 1.905 r_{50}$, so that after turnaround extra energy input is needed from the environment. The higher the redshift $z_i$, the less energy increase is required. Energy increases that does not involve changing angular momentum will not affect the BTFR discussed above. When a protogalaxy interacts with other protogalaxies and its environment at early time, it could lead to some non-Newtonian shells receiving enough energy to come inside the galactic disk while others will not. It would not be a surprise that some spiral galaxies may not have non-Newtonian shells outside its scale length.
3.2: other non-Newtonian mass shells

We consider the non-Newtonian mass shells initially at \( r_j < r_i \), with the angular speed \( \dot{\varphi} \) from the Eq. (69)

\[
\dot{\varphi}^2 = \frac{h_j^2}{r_j^3} = \left( \delta + \sqrt{\delta} \right) \frac{4\pi G \rho_m(z_i)}{3}
\]

(66)

The spherical shell collapse in the "z" direction onto a thin disk along the mid-plane happens in a time scale similar to Newtonian gravitational collapse. For constant \( \delta \), \( h(r_j) = \dot{\varphi}r_j^2 \) is fixed when the overdensity stops growing. Mestel [51] argues that if an uniform density sphere with uniform angular velocity collapses onto a thin spheroid without turbulent mixing, the distribution of angular momentum \( h(r_j) \) remains the same as that of the sphere. This idea is supported by findings in a group of disk galaxies [51]. Using Mestel’s assumption for the outer non-Newtonian mass shells, with disk mass \( \Sigma(r_j) \) at \( r_j \), the acceleration takes the form

\[
\dot{r}_j = \frac{h_j^2}{r_j^3} \frac{3\pi G \Sigma(r_j)}{4 \dot{\varphi}^2}.
\]

(67)

For small angular momentum, the disk will reach equilibrium at \( \dot{\varphi}r_j^2 \) under gravity [56]. For large angular momentum, the orbit is hyperbolic and the incoming particle orbit is decided by its energy. Smaller non-Newtonian shell could reach the stellar disk and establish a dynamical equilibrium as observed [58]–[62], while larger shell could maintain its early time dynamics in its orbit towards its closest approach.

The rotational velocity of a mass shell on the disk arrived at \( r = 35 \text{kpc} \) can be described by Eq. (55)

\[
V^4(r_{35}) = \frac{4\pi G \rho_m(z_i)(r_{35})}{3}(1 + \sqrt{\delta})^2 n^5 GM_{35} = a_0^{\text{V}}(r_{35})GM_{35}.
\]

(68)

The mass lying between \( M_{r_0} \sim M_{50} \) is about 10% of the overall mass, so that \( M_{35} \sim 0.97 M_{50} \) is fair estimate.

Take an overdensity freeze value at \( \delta = 0.381 \) (\( z_i = 21.5 \)) where \( r_i = 93 \text{kpc} \) (\( n = 1.86 \)), and the largest shell will have travelled a distance of \( 43 \text{kpc} \) to reach \( 50 \text{kpc} \). Assuming an uniform radial speed for large non-Newtonian shells, the late time shell at \( r_{35} \) comes from a shell \( r_j = 78 \text{kpc} \) with a contraction factor \( n_j = 78/35 \) and one obtains

\[
a_0^{\text{V}}(r_{35}) = a_0^{\text{V}}(r_{50}) \left( \frac{n_i}{n} \right)^5 \left( \frac{r_{35}}{r_{50}} \right) = 1.20 \times 10^{-10} \text{m/s}^2.
\]

(69)

The large non-Newtonian shells share similar rotating speed BFTR as the largest shell.

4: CENTRAL DISK

Similar to elliptical galaxies, spiral galaxies also has a Newtonian dominant core which appears as an exponential disk upto scale length distances. From an uniform density, uniform angular speed spherical cloud in a Newtonian potential, the exponential disk can be produced analytically by Mestel [56], Freeman [63] or using simulation [51]. At the time when the largest overdensity stops growing at \( \delta < 1 \), there needs to be a central region within which the mass shells have either turnaround or turnaround very quickly. This ensures that \( \alpha > 0 \) and guarantees an un-interrupted Newtonian gravitational collapse. To split the overdensity cloud into Newtonian and non Newtonian regions, one scenario is suggested by Nusser [64] that the magnitude of the overdensity has a power law dependence on \( r \) such that

\[
\delta(r) = \delta \left( \frac{r_i}{r} \right)^S, \quad \ddot{r} = \frac{h_i^2}{r^3} - \left[ \delta \left( \frac{r_i}{r} \right)^S + \delta^{1/2} \left( \frac{r_i}{r} \right)^{S/2} \right] \frac{4\pi}{3} \rho_m r,
\]

(70)

where \( r_i \) is the maximum size of the cloud, and the exponent \( S > 0 \) is a constant. Here \( \delta(r) \) plays the role of an effective \( \delta \) at \( r \). The factor \( \delta(r) = \delta(R_{0})^{S} > 1 \) leads to a central Newtonian dominant region inside a radius \( r_M \) region, where

\[
r_M \leq \delta^{1/S} r_i.
\]

(71)
We see that $R_M$ starts to increase as $\delta$ increases from $\delta_{nf}$. This could lead to Newtonian gravitational collapse in the small $r$ region at very high redshifts. As the outer overdensity drops out from the cosmic background at $\delta < 1$, the entire overdensity also stops growing. Inside $r_M$ the mass density is given by

$$\delta(r)\rho_m(z_i) = \delta_{nf}\left(\frac{r_i}{r}\right)^S$$  \hspace{1cm} (72)

and the central mass up to shell at $r$ is

$$M(r) = 4\pi \int_0^r r^2 \rho(r)dr = 4\pi \rho_m \int_0^r \delta(r)r^2dr = \frac{4\pi\delta_{nf}\rho_m}{(3 - S)} r_i^S r^{(3-S)}.$$  \hspace{1cm} (73)

From observations that a Newtonian central mass $M(r_M)$ is up to $\sim 90\%$ of the total protogalaxy mass at $r_i$. As a first approximation, we neglect the contribution to the central bulge from larger shell’s collapse, we have

$$\frac{M(r_i)}{M(r_M)} = \frac{10}{9} = \left(\frac{r_i}{r_M}\right)^{3-S} = \delta^{-\frac{3-S}{3}}.$$  \hspace{1cm} (74)

For a low $z = 17.8$, $\delta = 0.5$ we obtain $S \approx 2.6$ and the initial central Newtonian cloud radius is given by $r_M = 0.768 r_i = 74.88 kpc$. The outer non-Newtonian gas shells span upto $23\%$ of the total spherical cloud radius. The decreasing power law proposal in [64] comes from the observed late time galactic matter density profiles. Primordial black-holes or cosmological defects [63], [65] proposed to account for the Supermassive Black Holes (SMBHs) at high redshifts ($z \sim 6$) could be important but we would not consider them here. Take a protogalaxy that stops growing at the low $z = 6$, with a cloud uniform density $\rho \geq 3.23 \times 10^{-24} km^{-3}$, the free-fall time of the Newtonian core is $t_{ff} \sim 1.1$ Gyr, where central region collapse ends at redshift $z \sim 7.7$. This redshift is still much earlier than that of most observed Early Type galaxies.

Similar power-law behaviour in the angular momentum can also provide a central Newtonian region at overdensity freeze. That is

$$h^2(r) = h^2\left(\frac{r}{r_i}\right)^S, \quad \dot{r} = \frac{h^2}{r^3}\left(\frac{r}{r_i}\right)^S - \left(\delta + \delta^{1/2} + 1\right)\frac{4\pi}{3}\rho_m r.$$  \hspace{1cm} (75)

This power law behaviour can be due to the finite speed and inefficiency of angular momentum transport from the outer shells. As the angular momentum decreases as $r$ decreases, there is also a region $r < R_M$ that the angular momentum is negligible. This $r < R_M$ core also increases as $r_i$ increases ($z$ decreases). We can estimate the time needed for these mass shells to gravitationally collapse. The energy equation of an unit mass around a central mass $M$ at $E = h^2 = 0$ is

$$\dot{r} = H_m r - \sqrt{\frac{2GM}{r}} = \sqrt{\frac{2GM}{r}} \left(\sqrt{\frac{H_m^2 r^4}{2GM}} - 1\right).$$  \hspace{1cm} (76)

In matter dominant epoch, $H_m^2 = \frac{1}{3}\rho_m$. Setting

$$y = \frac{2}{3} \frac{r^{3/2}}{\sqrt{2GM}}, \quad y = xt.$$  \hspace{1cm} (77)

Eq. (76) simplifies to

$$\frac{dy}{dt} = \frac{y}{t} - 1, \quad \left(\frac{dx}{dt} = -\frac{1}{t}\right).$$  \hspace{1cm} (78)

Since $x$ is dimensionless variable, one obtains a solution for a constant $t_0$ such that

$$\frac{y}{t} = x = \ln\left(\frac{t_0}{t}\right).$$  \hspace{1cm} (79)

At the turnaround time $t = t_{ta}$ where $\dot{r} = 0$ ($\dot{y} = 0$), we have from Eq. (78)

$$\frac{dy}{dt} = 0, \quad \frac{y(t_{ta})}{t_{ta}} = 1 = \ln\left(\frac{t_0}{t_{ta}}\right).$$  \hspace{1cm} (80)
so that we have $t_0 = et_{ta}$. If we take that the overdensity freezes at $z = 17$, the corresponding (freeze) time $t_i$ from $H^2 = 4/9t^2$ is given as

$$t_i = \frac{2}{3H(z)} = 0.5 \text{Gyr.} \quad (81)$$

At $t_i$ from Eq. (77)

$$\frac{y}{t_i} = \frac{1}{\sqrt{\delta}} = \ln \left(\frac{et_{ta}}{t_i}\right). \quad (82)$$

From subsection 3.1 above $\sqrt{\delta} = 0.771$, so that we can obtain the turnaround time

$$t_{ta} = e^{0.3}t_i = 0.675 \text{Gyr.} \quad (83)$$

From the turnaround radius $r_{ta}$ at $t = t_{ta}$, to go to $r \to 0$ ($y \to 0$) where $t$ goes to $t = et_{ta}$, with a dynamical time

$$t_{dyn} = et_{ta} - t_{ta} = 1.718 t_{ta}. \quad (84)$$

The total time to reach this point is at $z \sim 6.6$. The collapse of the central cloud can be studied using simulation [50], [51]. Qualitatively one expects the Newtonian dominant gas to follow a violent relaxation [67], which produces a virialised uniform density cloud with constant angular momentum. This gas can then relax to an exponential disk [56].

4.1: A quick look at $a_0$ in LSB galaxies

It is found that the rotational curves of low surface brightness (LSB) galaxies tend to produce a lower MOND acceleration $a_0 = 0.7 \times 10^{-10} \text{ms}^{-2}$ [68]. For a Giant LSB galaxy having the same scale length as that of a HSB spiral galaxy, the low surface brightness is due to its smaller stellar core inside the scale length [69]. In the above paradigm where initial overdensity has the power-law dependence on radius, $\delta(r) \propto r^{-S}$ (or $h^2 \propto r^{-S}$) that the Newtonian core grows as redshift decreases. In this case, a LSB galaxy can be produced by a protogalaxy at less total mass than its HSB counterpart, but both protogalaxies stops growing at similar redshifts. A good example is Giant LSB galaxy UGC11861 listed in [68]. This LSB galaxy has a scale length and outer HI gas mass similar to the MW model galaxy. Assuming that the overdensity of UGC11861 stops growing at $z = 38$ as in Eq. (63), its stellar disk surface brightness is observed to be 1 magnitude smaller than our MW model galaxy. From [68], this LSB galaxy’s central disk mass $M_\ast$ would be $25\times$ that of the model MW galaxy. The total mass of the LSB galaxy including outer gases will be 0.47 times of our MW model. At $z = 38$, a smaller mass cloud spans a smaller overdensity size at $r_i = 1.27r_{50}$, comparing to our MW model overdensity size $r_i = 1.47r_{50}$. In this case, smaller contraction ratio $n$ leads to smaller $a_0$. From Eq. (61) the corresponding MOND acceleration for UGC11861 comes to

$$a_0^M = \left(1 + \sqrt{\delta}\right)^2 \frac{4\pi G}{3} \rho_m(z_i)r_{50}n^5 = 0.75 \times 10^{-10} \text{ms}^{-2}. \quad (85)$$

This paradigm provides a simple explanation for the lower $a_0$ value obtained in LSB galaxies in [68]. The Tidal Dwarf Galaxies (TDG) are formed by debris at late time. Within our model, the underlying force law is always Newtonian, so that TDGs should have only Newtonian acceleration as observed [71].

SUMMARY AND DISCUSSION

We study some aspects of the evolution of an overdensity under a MOND-like (VMOND) acceleration. In the matter dominant epoch, where overdensity $\delta \ll 1$, the overdensity growth rate is $\delta \propto a^2$. Using a Milky Way scale protogalaxy and the assumption that an uniform angular acceleration catches up to the overdensity’s gravitational acceleration at some $\delta < 1$ and redshift $z_i$. This leads to the end of the overdensity growth where overdensity mass
is fixed. The angular momentum of the largest mass shell of radius $r_i$ is specified by the VMOND acceleration $a_0(z_i, r_i)$ and the BTFR holds automatically. As this mass shell’s radius shrinks to galactic disk size at $50kpc$, angular momentum is conserved and the VMOND acceleration $a_0(z, r)$ grows towards a value $a_0^{VM}$. From a realistic initial $\delta$ value at recombination and an observationally acceptable range of redshifts $17 < z < 38$, the predicted $a_0^{VM}$ at late time matches the phenomenological MOND acceleration value $a_0$ within its observable bounds. This is the main result and that the phenomenological MOND acceleration $a_0$ scale could have a natural and cosmological origin. In this model, the non-Newtonian rotational curve is the relic of the particle dynamics attained at high redshift epoch. We explore the scenario that the overdensity has a power law dependence on its radius, we find that the low $a_0$ value in Low Surface Brightness spiral galaxies can be readily explained by this paradigm.

References

[1] C. C. Wong, "Variable Modified Newtonian Mechanics I: Single metric universe", arXiv: 1601.00376.
[2] S. S. McGaugh, APJ, 609, 652 (2004); ApJ, 632, 859 (2005); AJ, 143, 40 (2012).
[3] F. Lelli, S. S. McGaugh, J. M. Schombert, M. S. Pawlowski, "One Law to Rule Them All: The Radial Acceleration Relation of Galaxies" M. S. 2017, ApJ, 836, 152.
[4] M Milgrom, R.H. Sanders,"MOND rotation curves of very low mass spiral galaxies", arXiv:0611494, AJ Lett. 658, L17 (2007).
[5] R. H. Sanders, "The prediction of rotation curves in gas-dominated dwarf spiral galaxies with modified dynamics", arXiv:1811.0926.
[6] Van den Bosch, Burket and Swaters, MBRAS, 326, 1205-1215 (2001).
[7] O. Müller, et al. "A whirling plane of satellite galaxies around Centaurus A challenges cold dark matter cosmology", arXiv:1802.00081 Müller et al., Science 359, 534, 2018.
[8] P. Kroupa, "Galaxies as simple dynamical systems: observational data disfavor dark matter and stochastic star formation", Canadian Journal of Physics, 2015, 93(2): 169-202.
[9] J. D. Bowman, A. E. Rogers, R. A. Monslave, T. J. Mozden, & N. Mahesh, Nature, 555, 67 (2018);arXiv: 1810.05912.
[10] S. McGaugh, "Strong Hydrogen Absorption at Cosmic Dawn: the Signature of a Baryonic Universe", [arXiv:1803.02365.

To appear in RNAAS.
[11] M. Milgrom, APJ, vol. 270, 371-383 (1983).
[12] B. Famaey, S. McGaugh, "Challenges for Lambda-CDM and MOND", arXiv: 1301.0623, Proceedings of the Meeting of the International Association for Relativistic Dynamics, IARD 2012, Florence.
[13] S. McGaugh, "A tale of two paradigms: the mutual incommensurability of ΛCDM and MOND", arXiv:1404.7525. Canadian Journal of Physics 93,250 (2015).
[14] B. Famaey, S. McGaugh, Living Reviews in Relativity, 15, 2012, 10; [arXiv:1112.3960.
[15] A. Nusser, "Modified Newtonian dynamics of large-scale structure", arXiv:0109016, MNRAS, 331,909, (2002).
[16] A. Aguirre, J. Schaye, E. Quataert, ApJ, 561, 550 (2001).
[17] R. H. Sanders, MNRAS, 342, 901, 2003.
[18] R. Genzel et al., Nature 543, 397-401 (2017).
[19] P. Lang et al., arXiv: 1703.05491, submitted to Astrophysical Journal.
[20] Simons et al., "An Epoch of Disk Assembly" arXiv:1705.03474 "z ~ 2.
[21] Kassin et al., "The Epoch of disk settling z ~ 1 to now" arXiv:1207.7072 submitted to APJ.
[22] S. Sachdeva, K. Saha, "Survival of the pure disk galaxies over the last 8 billion years,[arXiv:1602.08912]•
[23] R. Swaters et al., "The link between the Baryonic Mass Distribution and the Rotation Curve Shape", arXiv: 1207.2729. accepted for publication in MNRAS.
[24] T. Martinsson et al., " The DiskMass Survey, X. Radio synthesis imaging of spiral galaxies" arXiv:1510.7666 A & A, 585, A99(2016).
[25] G. Angus et al., " Mass Models of disk galaxies from the DiskMass Survey in MOND", arXiv:1505.05522 accepted for publication in MNRAS.
[26] M. Cappellari, Ann. Rev. Astron. Astrophys. 54:597-665, 2016; arXiv:1602.04267.
[27] M. Den Heijer et al., A&A, 581 (2015) A98; arXiv:1509.05236.
[28] D. C. Martin et al., "A giant protogalactic disk linked to the cosmic web", Nature, 524, 192-195 (2015).
[29] H. Zhao, B. Famaey, Phys. Rev. D, 86, 067301 (2012).
[30] J. Khoury, Phys. Rev. D, 91, 024022 (2015).
[31] E. P. Verlinde, "Emergent Gravity and the Dark Universe", SciPost Phys. 2, 016 (2017). arXiv:1611.02269.
[32] J. W. Moffat, "Acceleration in Modified Gravity (MOG) and the Mass-Discrepancy Baryonic Relation", arXiv:1610.06909v2.
C. Skordis, T. Zlosnik, "A general class of gravitational theories as alternatives to dark matter where the speed of gravity always equals the speed of light", arXiv:1905.00465.

M. Haslbauer, I. Banik, P. Kroupa, "The KBC void and Hubble tension contradict CDM on a Gpc scale a' Milgromian dynamics as a possible solution", arXiv:2009.11292v1, Published in the Monthly Notices of the Royal Astronomical Society.

A. Einstein, E. G. Straus, "The Influence of the Expansion of Space on the Gravitation Fields Surrounding the Invidual Stars.", Reviews of Modern Physics, 17(2-3), 120124. (1945); A. Einstein, E. G. Straus, "Corrections and Additional Remarks to our Paper: The Influence of the Expansion of Space on the Gravitation Fields Surrounding the Invidual Stars.", Reviews of Modern Physics, 18(1), 148149 (1946).

G. C. McVittie, "The Mass-particle in an Expanding Universe.", Monthly Notices of the Royal Astronomical Society, 93(5), 325339 (1933).

A. Oeftiger, "On the Effect of Global Cosmological Expansion on Local Dynamics", http://www.oeftiger.net/global-cosmic-expansion.pdf.

M. Carrera, D. Giulini, "On the influence of the global cosmological expansion on the local dynamics in the Solar System", arXiv:gr-qc/0602098.

G. A, Baker Jr., "Effects on the structure of the universe of an accelerating expansion" General Relativity and Gravitation June 2002, volume 34, issue 6, pp 767-791.

J. Binney, S. Tremaine, "Galactic Dynamics" Princeton University Press, (2008).

A. Nusser, MNRAS, 325, 1397-1401(2001).

R. H. Sanders, "Forming galaxies with MOND", arXiv:0712.2576 accepted for publication in MNRAS.

R. Barkana, A. Loeb, "In the Beginning: The First Sources of Light and the Reionization of the Universe", Phys. Rept. 349, 125-238, 2001; arXiv: 0010468.

P. E. Peebles, APJ. 155, 393 (1969).

S. D. M. White, "Angular Momentum Growth in Protogalaxies", APJ. 296:38-41 (1984).

E. Casuso, J. E. Beckman, "On the Origin of the Angular momentum of Galaxies: Cosmological tidal torques supplemented by the Coriolis force", MNRAS, 1-9, 2015.

A. Burkert, E O'Nighia, "Galaxy formation and the cosmological angular momentum problem", arXiv:0409540.

P. A. Oesch, G. Brammer, P. van Dokkum et al., "A Remarkably Luminous Galaxy at z=11.1 Measured with Hubble Space Telescope Grism Spectroscopy", APJ. 819 (2) 129 (2016); arXiv:1603.00461.

Y. Harikane et al. "A Search for H-Dropout Lyman Break Galaxies at z 12-16", arXiv:2112.09141.

J. I. Davies, "A Heavy Baryonic Galactic Disc", arXiv:1204.4649.

N. Wittenburg, P. Kroupa, B. Famaey,"The formation of exponential disk galaxies in MOND", arXiv:2002.10941 accepted for publication in ApJ 18.01.2020.

T.Licquia, J. Newman, APJ 806, 96 2(015); arXiv:1407.1078v3.

T. Licquia, J. Newman, ApJ 831. 71 (2016); arXiv:1607.05281.

P. M. W. Kalberla, L. Dedes, "Global properties of the HI distribution in the outer Milky Way", arXiv:0804.4831.

P. M. W Kalberla et al. " Dark matter in the Milky Way, II. the HI gas distribution as a tracer of the gravitational potential ", A&A 2006-6326 (2018); arXiv 0704.3925.

L. Mestel,"On the Galactic law of rotation", MNRAS, 126, 553, 1963.

D. Crampin, F. Hoyle, APJ, 74, 186, 1964.

G. Pezzulli, F. Fraternali, "Accretion, radial flow and abundance gradients in spiral galaxies", arXiv:1510.04289.

E. Pitts, R. Taylor 1989, MNRAS, 240, 373.

E. Pitt, R.Taylor 1996, MNRAS, 280, 1101.

Mayor and Vignroux (1981).

Lacey and Fall (1985).

K. Freeman, "On the disks of spiral and SO galaxy" APJ, vol 160, June 1970.

A. Nusser, E. Pointecouteau, "Modeling the formation of galaxy clusters in MOND", arXiv:0505095, MNRAS. 366, 969-976 (2006).

S. F. Bramberger et al., "Cosmic String Loops as the Seeds of Super-Massive Black Holes", arXiv:1503.02317.

R. Brandenberger, H. Jiao, "Cosmic Textures and Global Monopoles as seeds for Super-Massive Black Holes", arXiv:1908.04585.

D. Lydden Bell, "Statistical Mechanics of Violent Relaxation in Stellar Systems", MNRAS (1967) 136, 101-121.

R. A. Swaters, R. H. Sanders, S. S. McGaugh, "Testing Modified Newtonian Dynamics with Rotation Curves of Dwarf and Low Surface Brightness Galaxies", arXiv:1005.5456 ApJ 718, 380-391 (2010).

M. Das, "Giant Low Surface Brightness Galaxies : Evolution in Isolation", arXiv:1301.6495, 2013, JApA, 34, 19.

F. Lelli et al., "Tidal Dwarf Galaxies" Disc Formation at z = 0, Galaxies 2015, 2, 1-x; arXiv:1511.00689.