The real nonnegative inverse eigenvalue problem is NP-hard *†‡

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Abstract

A list of complex numbers is realizable if it is the spectrum of a nonnegative matrix. In 1949 Suleimanova posed the nonnegative inverse eigenvalue problem (NIEP): the problem of determining which lists of complex numbers are realizable. The version for reals of the NIEP (RNIEP) asks for realizable lists of real numbers. In the literature there are many sufficient conditions or criteria for lists of real numbers to be realizable. We will present an unified account of these criteria. Then we will see that the decision problem associated to the RNIEP is NP-hard and we will study the complexity for the decision problems associated to known criteria.

1 Introduction

A matrix is nonnegative if all its entries are nonnegative numbers. The Real Nonnegative Inverse Eigenvalue Problem (which we will denote as RNIEP) asks for the characterization of all possible real spectra of nonnegative matrices. A list \( \Lambda = (\lambda_1, \ldots, \lambda_n) \) of \( n \) real numbers is said to be realizable if there exists some nonnegative matrix \( A \geq 0 \) of order \( n \) with spectrum \( \sigma(A) = \{\lambda_1, \ldots, \lambda_n\} \). With some abuse of notation, from now on we will use the expression \( \sigma(A) = \Lambda \) or \( \sigma(A) = (\lambda_1, \ldots, \lambda_n) \).

For \( \Lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \) define

\[ \rho(\Lambda) = \max\{|\lambda_1|, \ldots, |\lambda_n|\} \quad \text{and} \quad \Sigma(\Lambda) = \lambda_1 + \cdots + \lambda_n. \]

We will restrict to lists of monotonically nonincreasing real numbers, that is, elements of the sets

\[ \mathbb{R}_+^n \equiv \{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : \lambda_1 \geq \cdots \geq \lambda_n \}. \]

If \( \Lambda \in \mathbb{R}_+^n \) is the spectrum of a nonnegative matrix \( A \) then \( \Sigma(\Lambda) \) is the trace of \( A \) (which implies that \( \Sigma(\Lambda) \geq 0 \)) and \( \rho(\Lambda) \) is the Perron eigenvalue of \( A \) (which implies that \( \rho(\Lambda) = \lambda_1 \)). So the candidates to be a real spectrum of some nonnegative matrix belong to the set \( \Pi = \Pi_1 \cup \Pi_2 \cup \cdots \) where

\[ \Pi_1 = \{ \Lambda \in \mathbb{R}_+^n : \Sigma(\Lambda) \geq 0; \ \rho(\Lambda) = \lambda_1 \}. \]

The set of all real spectra of nonnegative matrices is \( \Pi_{\text{RNIEP}} = \Pi_1 \cup \Pi_2 \cup \cdots \) where

\[ \Pi_{\text{RNIEP}} = \{ \Lambda \in \mathbb{R}_+^n : \exists \text{ a nonnegative matrix } A \text{ of order } n \text{ with } \sigma(A) = \Lambda \}. \]

The RNIEP asks for the characterization of \( \Pi_{\text{RNIEP}} \). The complete characterization of \( \Pi_{\text{RNIEP}} \) is only known for \( n \leq 4 \). Indeed this seems to be an intractable problem for large \( n \). Nevertheless several subsets

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of $\Pi_{RNIEP}$ are known. These partial solutions are presented in the literature as criteria, so that if $\Lambda \in \Pi_{RNIEP}$ satisfies the conditions that define the criterion $C$ then $\Lambda \in \Pi_{RNIEP}$. For each criterion $C$ we define the set $\Pi_C = \Pi^1_C \cup \Pi^2_C \cup \cdots$ where

$$\Pi^a_C \equiv \{ \Lambda \in \Pi^a_R : \Lambda \text{ satisfies the condition of the criterion } C \} \subset \Pi^a_{RNIEP}.$$  

The RNIEP has associated the following decision problem: the input is a list $\Lambda \in \Pi_R$ and the output is ‘yes’ if $\Lambda \in \Pi_{RNIEP}$ or ‘no’ if $\Lambda \not\in \Pi_{RNIEP}$. Similarly, each criterion $C$ has associated a decision problem where the input is a list $\Lambda \in \Pi_R$ and the output is ‘yes’ if $\Lambda \in \Pi_C$ or no’ if $\Lambda \not\in \Pi_C$. The aim of this paper is to study the complexity of these decision problems.

2 A review of the main criteria for the RNIEP

For each criteria $C$ we will explicitly present the set $\Pi_C$. We have divided this sets into four different groups depending on the type of conditions:

Group 1. Sets $\Pi_C$ whose lists are defined by a collection of linear inequalities.

We introduce the following notation associated to a given $\Lambda = (\lambda_1, \ldots, \lambda_n) \in \Pi_R$:

- $p(\Lambda)$ is the number of nonnegative elements of $\Lambda$.
- $q(\Lambda)$ is the number of negative elements of $\Lambda$.
- $\Psi(\Lambda) = \{ i \in \{1, \ldots, \min\{p(\Lambda), q(\Lambda)\} : \lambda_i + \lambda_{n+1-i} < 0 \}$.
- $\psi_k(\Lambda) = \sum_{i \in \Psi(\Lambda), i < k} (\lambda_i + \lambda_{n+1-i}) + \lambda_{n+1-k}$ for each $k \in \Psi(\Lambda)$.
- $\psi(\Lambda) = \sum_{i \in \Psi(\Lambda)} (\lambda_i + \lambda_{n+1-i}) + \sum_{j=p(\Lambda)+1}^{q(\Lambda)} \lambda_{n+1-j}$ (the last summation appears if $q(\Lambda) > p(\Lambda)$).

And now we present, in chronological order, the sets that belong to this group:

(a) The Suleimanova criterion [23] gives rise to the set

$$\Pi_{Su} \equiv \{ (\lambda_1, \ldots, \lambda_n) \in \Pi_R : \lambda_1 \geq 0 > \lambda_2 \geq \cdots \geq \lambda_n \}.$$  

(b) The Ciarlet criterion [6] gives rise to the set

$$\Pi_{Ci} \equiv \{ (\lambda_1, \ldots, \lambda_n) \in \Pi_R : |\lambda_i| \leq \frac{\lambda_1}{n} \text{ for } i = 2, \ldots, n \}.$$  

(c) The Kellogg criterion [13] gives rise to the set

$$\Pi_{Ke} \equiv \{ (\lambda_1, \ldots, \lambda_n) \in \Pi_R : \lambda_1 \geq -\psi(\Gamma) \text{ and } \lambda_1 \geq -\psi_k(\Gamma) \forall k \in \Psi(\Gamma) \}.$$  

(d) The Salzmann criterion [18] gives rise to the set

$$\Pi_{Sa} \equiv \left\{ \Lambda = (\lambda_1, \ldots, \lambda_n) \in \Pi_R : \frac{\Sigma(\Lambda)}{n} \geq \frac{\lambda_i + \lambda_{n-i+1}}{2} \text{ for } i = 2, \ldots, \lfloor \frac{n+1}{2} \rfloor \right\}.$$  

(e) The Fiedler criterion [9] gives rise to the set

$$\Pi_{Fi} \equiv \left\{ \Lambda = (\lambda_1, \ldots, \lambda_n) \in \Pi_R : \lambda_1 + \lambda_n + \Sigma(\Lambda) \geq \sum_{2 \leq i \leq n-1} \frac{|\lambda_i + \lambda_{n-i+1}|}{2} \right\}.$$  

2
Group 2. Sets \( \Pi_C \) whose lists contain a partition that satisfies some conditions.

It is necessary to introduce some notation. If \( \Lambda = (\lambda_1, \ldots, \lambda_n) \in \Pi_\mathbb{R} \) for \( i = 1, \ldots, k \) then \( \Lambda_1 \cup \cdots \cup \Lambda_k \) denotes the list of \( \mathbb{R}^{n_1+\cdots+n_k} \) that contains all the reals of the lists \( \Lambda_1, \ldots, \Lambda_k \). For example

\[
(9, -1) \cup (5, 3, -4) \cup (3, 3, -1, -1, -4, -7) = (9, 5, 3, 3, 3, -1, -1, -4, -7).
\]

If \( \Lambda = \Lambda_1 \cup \cdots \cup \Lambda_k \) then we say that \( \Lambda_1 \cup \cdots \cup \Lambda_k \) is a partition of \( \Lambda \).

For any \( \Lambda = (\lambda_1, \ldots, \lambda_n) \in \Pi_\mathbb{R} \) we will denote by \( \Lambda^+ \) the sublist that contains all nonnegative values of \( \Lambda \) and we will denote by \( \Lambda^- \) the sublist that contains all negative values of \( \Lambda \). Note that \( \Lambda = \Lambda^+ \cup \Lambda^- \).

Now we write, in chronological order, the sets that belong to this group:

(a) The Suleimanova-Perfect criterion \([23, 16]\) gives rise to the set

\[
\Pi_{SP} \equiv \left\{ \Lambda \in \Pi_\mathbb{R} : \exists \text{ a partition } \Lambda = \Lambda_1 \cup \cdots \cup \Lambda_k \text{ such that } \Lambda_1, \ldots, \Lambda_k \in \Pi_{Su} \right\}.
\]

(b) The Perfect-1 criterion \([16]\) gives rise to the set

\[
\Pi_{Pe1} \equiv \left\{ \Lambda \in \Pi_\mathbb{R} : \exists \text{ a partition } \Lambda = (\alpha, \beta) \cup \Lambda_1 \cup \cdots \cup \Lambda_k \text{ such that } \alpha = \rho(\Lambda), \beta \leq 0, \text{ and }
\right. \\
\left. \text{ for } i = 1, \ldots, k, \quad \Lambda_i = (\lambda_i, \lambda_{i1}, \ldots, \lambda_{ik}) \text{ with } \Sigma(\Lambda_i) \leq 0, \lambda_{i1}, \ldots, \lambda_{ik} \leq 0 \leq \lambda_i, \text{ and } \lambda_i + \beta \leq 0 \right\}.
\]

(c) The Borobia criterion \([3]\) gives rise to the set

\[
\Pi_{Bo} \equiv \left\{ \Lambda \in \Pi_\mathbb{R} : \exists \text{ a partition } \Lambda^- = \Lambda_1 \cup \cdots \cup \Lambda_k \text{ such that } \Lambda^+ \cup (\Sigma(\Lambda_1)) \cup \cdots \cup (\Sigma(\Lambda_k)) \in \Pi_{Ke} \right\}.
\]

Group 3. Sets \( \Pi_C \) whose lists are defined recursively.

On this category appear the subsets of \( \Pi_{RNIEP} \) associated to four different criteria whose authors are: (a) Soules \([22]\) (we consider the extended Soules criterion as presented in Section 2.1 of \([8]\)); (b) Borobia, Moro and Soto \([4]\); (c) Soto \([21]\); and (d) Šmigoc and Ellard \([19, 8]\). The exposition of these criteria is quite elaborate. So we will refer to \([8]\) where it is made a detailed presentation of each criterion and it is proved that all the four criteria are equivalent, that is, that the four sets associated to the criteria are equal: \( \Pi_{Su} = \Pi_{BMS} = \Pi_{SO} = \Pi_{SE} \). Therefore here we only need to expose one of them. We have chosen the criterion given by Borobia, Moro and Soto since it has the simplest recursive definition. Consider the following assertions:

(i) If \( \Lambda_1 \in \Pi_{RNIEP} \) and \( \Lambda_2 \in \Pi_{RNIEP} \) then \( \Lambda_1 \cup \Lambda_2 \in \Pi_{RNIEP} \).

(ii) If \( (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Pi_{RNIEP} \) then \( (\lambda_1 + \epsilon, \lambda_2, \ldots, \lambda_n) \in \Pi_{RNIEP} \) for any \( \epsilon > 0 \).

(iii) If \( (\lambda_1, \ldots, \lambda_i, \ldots, \lambda_n) \in \Pi_{RNIEP} \) then \( (\lambda_1 + \epsilon, \ldots, \lambda_i \pm \epsilon, \ldots, \lambda_n) \in \Pi_{RNIEP} \) for any \( \epsilon > 0 \).

That (i) and (ii) are true is well known, and the proof of (iii) is due to Guo \([10]\). A list \( (\lambda_1, \ldots, \lambda_n) \) is \emph{C-realizable} if it may be obtained by starting with the \( n \) trivially realizable lists \((0),(0),\ldots,(0)\) and then using (i), (ii) and (iii) any number of times in any order. Associated to this recursive construction is the set

\[
\Pi_{BMS} \equiv \left\{ \Lambda \in \Pi_\mathbb{R} : \Lambda \text{ is C-realizable} \right\}.
\]

An interesting and open question about \( \Pi_{BMS} \) is if the subsets \( \Pi_{BMS}^n = \Pi_{BMS} \cap \Pi_\mathbb{R}^n \) could be characterized by a list of linear inequalities for any \( n \).
Group 4. Sets $\Pi_C$ whose lists are dependent on the existence of a nonnegative matrix with prescribed diagonal and spectrum.

Again it is necessary to introduce some notation. If $\Lambda_2 \in \mathbb{R}^{n_2}_+$ is a sublist of $\Lambda_1 \in \mathbb{R}^{n_1}_+$ then $\Lambda_1 \setminus \Lambda_2 \in \mathbb{R}^{n_1-n_2}_+$ denotes the sublist of $\Lambda_1$ such that $\Lambda_1 = \Lambda_2 \cup (\Lambda_1 \setminus \Lambda_2)$. For example

$$(8,6,3,3,3,-4,-4,-6,-7) \setminus (8,3,-4,-4,-7) = (6,3,3,-6).$$

The Perfect-2+ criterion [17] gives rise to the set

$$\Pi_{\text{Pe}_+} = \{ \Lambda \in \Pi_R : \Lambda = \left( [\Lambda_1 \setminus (\rho(\Lambda_1))] \cup (\alpha_1) \right) \cup \cdots \cup \left( [\Lambda_k \setminus (\rho(\Lambda_k))] \cup (\alpha_k) \right) \}$$

for some $\Lambda_1, \ldots, \Lambda_k \in \Pi_{\text{Su}}$ and some $\alpha_1 = \rho(\Lambda), \alpha_2, \ldots, \alpha_k \geq 0$, and $\exists A = \left[ \begin{array}{cccc} \rho(\Lambda_1) & \ast & \cdots & \ast \\ & \ddots & \ddots & \ast \\ & & \ddots & \rho(\Lambda_k) \end{array} \right] \geq 0$ with $\sigma(A) = (\alpha_1, \ldots, \alpha_k)$.

Marijuan, Pisonero and Soto [15] analyzed the relationships between the sets $\Pi_C$. Recently, Ellard and Šmigoc [8] have completed the analysis. Figure 1 reflects all relationships and updates the original map found in [15].

Figure 1: $\Pi_{\text{SP}} \subset \Pi_{\text{Bo}} \subset \Pi_{\text{Su}} = \Pi_{\text{BMS}} = \Pi_{\text{So}} = \Pi_{\text{SE}} \subset \Pi_{\text{Pe}_2+} \subset \Pi_{\text{RNIEP}}$

3 The $(\Pi_1, \Pi_2)$–Problem

Complexity classes (P, NP, NP-complete, NP-hard) are collections of decision problems: problems whose inputs can be answered by ‘yes’ or ‘no’. Actually a decision problem is completely described by the inputs for which the answer is ‘yes’. To be more precise, consider two sets $\Pi_1$ and $\Pi_2$ such that $\Pi_1 \subset \Pi_2$. The $(\Pi_1, \Pi_2)$–Problem is the decision problem in which the input is an element $\Lambda \in \Pi_2$ and the output is ‘yes’ if $\Lambda \in \Pi_1$ or ‘no’ if $\Lambda \notin \Pi_1$. The set $\Pi_1$ will be the set we are interested in, and the set $\Pi_2$ can be thought as the context of the problem (that is, a set that imposes some minimum requirements to its elements to discard trivial non-elements of $\Pi_1$). Now we will present the decision problems that will appear in this work.
The \((\Pi_{\text{RNIEP}}, \Pi_\mathbb{R})-\text{Problem}\)

We are interested in the decision version of the RNIEP, that is, in the \((\Pi_{\text{RNIEP}}, \Pi_\mathbb{R})-\text{Problem}\) for which the input is an element \(\Lambda \in \Pi_\mathbb{R}\) and the output is ‘yes’ if \(\Lambda \in \Pi_{\text{RNIEP}}\) or ‘no’ if \(\Lambda \not\in \Pi_{\text{RNIEP}}\).

Example 3.1. The following are two instances of the \((\Pi_{\text{RNIEP}}, \Pi_\mathbb{R})-\text{Problem}\):

(i) \((6, -3) \in \Pi_\mathbb{R}\) is a yes-instance of the \((\Pi_{\text{RNIEP}}, \Pi_\mathbb{R})-\text{Problem}\) since it is the spectrum of \([\frac{1}{2} \ 3]\).

(ii) \(\Lambda = (3, 3, -2, -2, -2) \in \Pi_\mathbb{R}\) is a no-instance of the \((\Pi_{\text{RNIEP}}, \Pi_\mathbb{R})-\text{Problem}\). Suppose \(A\) is a nonnegative matrix of order 5 with spectrum \(\Lambda\). By the Perron-Frobenius Theorem, \(A\) is reducible and \(\Lambda\) can be partitioned into two nonempty lists each one being the spectrum of a nonnegative matrix with Perron eigenvalue equal to 3. This is not possible since one of the sublists must contain numbers with negative sum. So \(\Lambda\) is a no-instance.

The \((\Pi_\mathbb{C}, \Pi_\mathbb{R})-\text{Problem where } \mathbb{C} \text{ is a criterion of realizability}\)

For each criterion \(\mathbb{C}\) of realizability we have constructed the set \(\Pi_\mathbb{C}\) where \(\Pi_\mathbb{C} \subseteq \Pi_{\text{RNIEP}} \subseteq \Pi_\mathbb{R}\). So, associated to \(\mathbb{C}\) we have the \((\Pi_\mathbb{C}, \Pi_\mathbb{R})-\text{Problem}\), a decision problem for which the input is an element \(\Lambda \in \Pi_\mathbb{R}\) and the output is ‘yes’ if \(\Lambda \in \Pi_\mathbb{C}\) or ‘no’ if \(\Lambda \not\in \Pi_\mathbb{C}\).

Example 3.2. We will consider the same instance for two different \((\Pi_\mathbb{C}, \Pi_\mathbb{R})-\text{Problems}\):

(i) \((4, 2, -3, -3) \in \Pi_\mathbb{R}\) is a no-instance of the \((\Pi_{\text{SP}}, \Pi_\mathbb{R})-\text{Problem}\) since it can not be partitioned in Suleimanova sets.

(ii) \((4, 2, -3, -3) \in \Pi_\mathbb{R}\) is a yes-instance of the \((\Pi_{\text{BMS}}, \Pi_\mathbb{R})-\text{Problem}\) because of the sequence:

1. \((0), (0), (0), (0)\).
2. \((0, 0), (0), (0)\).
3. \((3, -3), (0), (0)\).
4. \((3, -3), (0, 0)\).
5. \((3, -3), (2, -2)\).
6. \((3, 2, -2, -3)\).
7. \((4, 2, -3, -3)\).

Remark 3.3. In the \((\Pi_{\text{RNIEP}}, \Pi_\mathbb{R})-\text{Problem}\) and in the \((\Pi_\mathbb{C}, \Pi_\mathbb{R})-\text{Problem}\) the context is \(\Pi_\mathbb{R}\). To analyze the complexity of a decision problem it is important that the context has as few restrictions as possible, otherwise we are hiding part of the complexity since the instances are already preselected. In our case the restrictions are the following: \(i\) the numbers are ordered, \(ii\) \(\Sigma(\Lambda) \geq 0\), and \(iii\) \(\rho(\Lambda) = \lambda_1\).

What happens if the context is the set \(\mathbb{R} \cup \mathbb{R}^2 \cup \cdots\) of lists of real numbers without further restrictions? For a list with \(n\) elements the cost of ordering its elements is \(n \log(n)\), the cost of checking condition \(ii\) is \(n\), and the cost of checking condition \(iii\) is unitary since it is only necessary to check that \(\lambda_1 \geq |\lambda_n|\). So the overall process is \(n \log(n)\). This is the hidden part of the complexity when the context is \(\Pi_\mathbb{R}\).

Having said that, the reason of considering \(\Pi_\mathbb{R}\) as the context is because it makes the exposition clearer.

The Partition Problem

Let \(\Pi_\mathbb{N}\) be the set of lists of non-increasing positive integers, that is,

\[
\Pi_\mathbb{N} = \Pi_\mathbb{N}_1 \cup \Pi_\mathbb{N}_2 \cup \cdots \text{ where } \Pi_\mathbb{N}_n \equiv \{(i_1, \ldots, i_n) : i_1, \ldots, i_n \in \mathbb{N}; i_1 \geq \cdots \geq i_n > 0\}.
\]

And consider the set

\[
\Pi_{\text{PP}} \equiv \{I \in \Pi_{\mathbb{N}} : \exists \text{ a partition } I = J \cup K \text{ such that } \Sigma(J) = \Sigma(K)\}.
\]

As \(\Pi_{\text{PP}} \subset \Pi_{\mathbb{N}}\) then it makes sense to consider the \((\Pi_{\text{PP}}, \Pi_{\mathbb{N}})-\text{Problem}\). Indeed this is a well known decision problem that in the literature is known as the Partition Problem\(^4\). The input of the Partition Problem

\(^4\)For an interesting and nontechnical presentation of the Partition Problem see [1].
is usually a list of unordered positive integers, but the restriction to ordered list does not change the complexity of the Partition Problem.

In what follows we will use $(\Pi_{\text{PP}}, \Pi_{\text{P}})$–Problem or Partition Problem interchangeably.

**Example 3.4.** The following are two instances of the Partition Problem:

(i) $(9, 6, 4, 4, 2, 1) \in \Pi_\text{N}$ is a yes-instance of the $(\Pi_{\text{PP}}, \Pi_\text{N})$–Problem since

$$(9, 6, 4, 4, 2, 1) = (9, 4) \cup (6, 4, 2, 1) \quad \text{with} \quad 9 + 4 = 6 + 4 + 2 + 1.$$ 

(ii) $(8, 6, 4, 1) \in \Pi_\text{N}$ is a no-instance of the $(\Pi_{\text{PP}}, \Pi_\text{N})$–Problem since the sum of its integers is odd.

### 4 NP-hardness of the RNIEP

A decision problem is in the class **NP-hard** when every decision problem in the class **NP** (nondeterministic polynomial-time) can be reduced in polynomial time to it\(^2\). We will prove that the $(\Pi_{\text{RNIEP}}, \Pi_{\text{P}})$–Problem is NP-hard. Actually we will see that for any arbitrary set $X$ such that $\Pi_{\text{SP}} \subseteq X \subseteq \Pi_{\text{RNIEP}}$ the $(X, \Pi_{\text{P}})$–Problem is NP-hard. This will be done by using the technique of reducing a problem that is known to be NP-hard, the Partition Problem, to our decision problem.

**Lemma 4.1.** The $(\Pi_{\text{PP}}, \Pi_{\text{N}})$–Problem is reducible to the $(X, \Pi_{\text{P}})$–Problem for $\Pi_{\text{SP}} \subseteq X \subseteq \Pi_{\text{RNIEP}}$.

**Proof.** Define the function

$$
\phi : \Pi_{\text{N}} \quad I = (i_1, \ldots, i_n) \quad \mapsto \quad \phi(I) = (\frac{\Sigma(I)}{2}, \frac{\Sigma(I)}{2}, -i_n, \ldots, -i_1)
$$

It is clear that the transformation of $I$ into $\phi(I)$ is done in linear time with respect to the size of the input. It remains to prove that $I \in \Pi_{\text{N}}$ is a yes-instance for the $(\Pi_{\text{PP}}, \Pi_{\text{N}})$–Problem if and only if $\phi(I) \in \Pi_{\text{P}}$ is a yes-instance for the $(X, \Pi_{\text{P}})$–Problem. That is, to prove for $I \in \Pi_{\text{N}}$ that $I \in \Pi_{\text{PP}}$ if and only if $\phi(I) \in X$:

- If $I \in \Pi_{\text{PP}}$ then $\phi(I) \in X$.

  If $I \in \Pi_{\text{PP}}$ then there exist a partition $I = J \cup K = (j_1, \ldots, j_p) \cup (k_1, \ldots, k_q)$ with $\Sigma(J) = \Sigma(K) = \Sigma(I)/2$. Thus

  $$
  \phi(I) = \left(\frac{\Sigma(I)}{2}, -j_p, \ldots, -j_1\right) \cup \left(\frac{\Sigma(I)}{2}, -k_q, \ldots, -k_1\right)
  $$

  with $(\frac{\Sigma(I)}{2}, -j_p, \ldots, -j_1) \in \Pi_{\text{Su}}$ and $(\frac{\Sigma(I)}{2}, -k_q, \ldots, -k_1) \in \Pi_{\text{Su}}$. Then $\phi(I) \in \Pi_{\text{SP}} \subseteq X$.

- If $\phi(I) \in X$ then $I \in \Pi_{\text{PP}}$.

If $\phi(I) \in X$ then $\phi(I) \in \Pi_{\text{RNIEP}}$ since $X \subseteq \Pi_{\text{RNIEP}}$. Then there exists a nonnegative matrix $A$ whose spectrum is $\phi(I)$. The Perron root of an irreducible nonnegative matrix is its spectral radius and has algebraic multiplicity one. As the spectral radius of $A$ is $\Sigma(I)/2$ and it appears twice in $\phi(I)$ then $A$ is reducible. This implies that there exists a permutation matrix $P$ such that

$$
P^T A P = \begin{bmatrix}
A_1 & * \\
0 & A_2
\end{bmatrix}
$$

where $A_1$ and $A_2$ have spectral radius $\Sigma(I)/2$. So

$$
\sigma(A) = \sigma(A_1) \cup \sigma(A_2) = \left(\frac{\Sigma(I)}{2}, -\alpha_1, \ldots, -\alpha_r\right) \cup \left(\frac{\Sigma(I)}{2}, -\beta_1, \ldots, -\beta_s\right) = \phi(I).
$$

Therefore $I = (\alpha_r, \ldots, \alpha_1) \cup (\beta_s, \ldots, \beta_1)$. As $A \geq 0$ and the trace of $A$ is zero (this is because $\Sigma(\phi(I)) = 0$) then all the entries on the diagonal of $A$ are equal to zero, and so all the entries on the diagonals of $A_1$ and $A_2$ are equal to zero. So the traces of $A_1$ and $A_2$ are equal to zero and then

$$
\frac{\Sigma(I)}{2} = \alpha_1 + \cdots + \alpha_r = \beta_1 + \cdots + \beta_s
$$

and so $I \in \Pi_{\text{PP}}$.\(^2\)

\(^2\) A good reference for computational complexity theory is [1].
When a problem that is known to be NP-hard (like the famous list of Karp’s 21 NP-complete problems [12], which includes the partition problem) is reducible to a new problem, then automatically, the new problem becomes NP-hard because of a standard argument that we reproduce below.

**Theorem 4.2.** The \((X, \Pi_{\mathcal{R}})\)–Problem is NP-hard for \(\Pi_{\mathcal{SP}} \subseteq X \subseteq \Pi_{\mathcal{RNIEP}}\).

*Proof.* All NP problems are reducible in polynomial time to the Partition Problem (see Karp [12]). On the other hand, in Lemma 4.1 we have seen that the Partition Problem is reducible in polynomial time to the \((X, \Pi_{\mathcal{R}})\)–Problem. Therefore, by the transitivity of the reduction relation, every problem in the class NP is reducible in polynomial time to the \((X, \Pi_{\mathcal{R}})\)–Problem. Thus the \((X, \Pi_{\mathcal{R}})\)–Problem is NP-hard.

**Corollary 4.3.** (i) The \((\Pi_{\mathcal{RNIEP}}, \Pi_{\mathcal{R}})\)–Problem is NP-hard.
(ii) The \((\Pi_{\mathcal{C}}, \Pi_{\mathcal{R}})\)–Problem is NP-hard for \(C=\mathcal{SP}, \mathcal{Bo}, \mathcal{Sou}, \mathcal{BMS}, \mathcal{So}, \mathcal{SE}\) and \(\mathcal{Pe}_{2+}\).

*Proof.* It is important to notice that all criteria \(C=\mathcal{SP}, \mathcal{Bo}, \mathcal{Sou}, \mathcal{BMS}, \mathcal{So}, \mathcal{SE}\) and \(\mathcal{Pe}_{2+}\) contain \(\mathcal{SP}\) as Figure 1 shows. The result follows from Theorem 4.2.

The criteria \(\mathcal{Pe}_{1}\) did not fit well into the scheme of Lemma 4.1 so we will treat it separately.

**Theorem 4.4.** The \((\Pi_{\mathcal{Pe}_{1}}, \Pi_{\mathcal{R}})\)–Problem is NP-hard.

We outline the proof. First we prove that the Partition Problem is reducible to the \((\Pi_{\mathcal{Pe}_{1}}, \Pi_{\mathcal{R}})\)–Problem. The function that gives rise to the reduction is the function

\[
\phi : \Pi_{\mathcal{N}} \rightarrow \Pi_{\mathcal{R}}
\]

\[
I = (i_1, \ldots, i_n) \rightarrow \phi(I) = \left(\frac{\Sigma(I)}{2}, \frac{\Sigma(I)}{2}, -i_n, \ldots, -i_1, -\frac{\Sigma(I)}{2}\right)
\]

Note that

\[
\phi(I) = \left(\frac{\Sigma(I)}{2}, -\frac{\Sigma(I)}{2}\right) \cup \left(\frac{\Sigma(I)}{2}, -j_p, \ldots, -j_1\right) \cup \left(\frac{\Sigma(I)}{2}, -k_q, \ldots, -k_1\right)
\]

satisfies the conditions that define \(\Pi_{\mathcal{Pe}_{1}}\). To finish the proof we argue as in the proof of Theorem 4.2.

5 The RNIEP and the decision problems for rationals

In Section 4 we have established the NP-hardness of several decision problems: the ones corresponding to RNIEP and Groups 2, 3 and 4. The following natural question is to ask if we can be more specific and prove that all these decision problems are NP-complete (a decision problem is NP-complete if it is NP-hard and NP). For the rest of the decision problems not treated (the ones corresponding to Group 1), we will see if they are in the class \(\mathcal{P}\) (polynomial-time solvable).

We require a specification. To determine if a decision problem belongs to the class \(\mathcal{P}\) or to determine if it belongs to the seemingly broader class NP implies typically that we deal with discrete problems, over the integers or rationals, about graphs, etc. This inadequateness of complexity theory to treat problems for real and complex numbers is well explained in [2]. The computational problems that arise in the RNIEP have as domain the reals. To *discretize* the \((\Pi_{\mathcal{RNIEP}}, \Pi_{\mathcal{R}})\)–Problem (but remain as faithful to the original problem as possible) in what follows we will consider its rational version. Namely, if

\[
\Pi_{\mathcal{Q}} = \{ (\lambda_1, \ldots, \lambda_n) \in \Pi_{\mathcal{R}} : \lambda_1, \ldots, \lambda_n \in \mathbb{Q} \}
\]

and

\[
\Pi_{\mathcal{QNIIEP}} = \Pi_{\mathcal{RNIEP}} \cap \Pi_{\mathcal{Q}}
\]

then the \((\Pi_{\mathcal{QNIIEP}}, \Pi_{\mathcal{Q}})\)–Problem is the *rational version* of the \((\Pi_{\mathcal{RNIEP}}, \Pi_{\mathcal{R}})\)–Problem. In a similar way, for any given criterion \(C\) if

\[
\Pi_{\mathcal{C}(\mathcal{Q})} = \{ (\lambda_1, \ldots, \lambda_n) \in \Pi_{\mathcal{C}} : \lambda_1, \ldots, \lambda_n \in \mathbb{Q} \} = \Pi_{\mathcal{C}} \cap \Pi_{\mathcal{Q}}
\]

then the \((\Pi_{\mathcal{C}(\mathcal{Q})}, \Pi_{\mathcal{Q}})\)–Problem is the *rational version* of the \((\Pi_{\mathcal{C}}, \Pi_{\mathcal{R}})\)–Problem.

If we reproduce the content of Section 4, considering rationals instead of reals, then we conclude that Theorems 4.2 and 4.4 are also valid in the following rational version.

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Theorem 5.1. The \((X, \Pi_Q)\)–Problem is NP-hard for \(\Pi_{SP(Q)} \subseteq X \subseteq \Pi_{QNIEP}\).

Theorem 5.2. The \((\Pi_{Pe_1(Q)}, \Pi_Q)\)–Problem is NP-hard.

Finally, we are ready to review all the decision problems and determine their complexity class:

The complexity of the QNIEP

We do not know how to decide if the \((\Pi_{QNIEP}, \Pi_Q)\)–Problem belongs to the class NP. The difficulty is that a certificate of membership in NP is apparently the solution to the problem. That is, for a yes-instance \(\Lambda \in \Pi_{QNIEP}\) of the \((\Pi_{QNIEP}, \Pi_Q)\)–Problem the certificate would be a nonnegative matrix \(A \geq 0\) with \(\sigma(A) = \Lambda\). As the entries of \(A\) are real numbers then to check that \(\Lambda\) is the spectrum of \(A\) can not be done, in general, in polynomial time.

The complexity of the criteria of Group 1

Theorem 5.3. The \((\Pi_{C(Q)}, \Pi_Q)\)–Problem is in the class P for \(C=Su, Ci, Ke, Sa, Fi\) and \(So_1\).

Proof. If \(\Lambda = (\lambda_1, \ldots, \lambda_n) \in \Pi_Q\) then \(\Lambda \in \Pi_{C(Q)}\) if and only if \(\Lambda \in \Pi_{C(Q)}^n\). That is, \(\Lambda \in \Pi_{C(Q)}\) if and only if \(\lambda_1, \ldots, \lambda_n\) satisfies the linear inequalities that defines \(\Pi_{C(Q)}^n\). Observe that in all the cases \(\Pi_{C(Q)}^n\) is defined by a collection of at most \(n\) inequalities. Therefore, the overall process to check that \(\Lambda \in \Pi_{C(Q)}\) will employ at most quadratic time with respect to the size of the input \(\Lambda\). We conclude that the \((\Pi_{C(Q)}, \Pi_Q)\)–Problem belongs to the class P.

The complexity of the criteria of Groups 2 and 3

Theorem 5.4. The \((\Pi_{C(Q)}, \Pi_Q)\)–Problem is NP-complete for \(C=SP, Bo, Sou, BMS, So, SE\) and \(Pe_1\).

Proof. All these decision problems are NP-hard by Theorems 5.1 and 5.2.

Let us see that they are also NP. A decision problem belongs to the class NP if for each yes-instance there exists a certificate that can be checked in polynomial time.

Let \(\Lambda \in \Pi_Q\) be a yes-instance for the \((\Pi_{SP(Q)}, \Pi_Q)\)–Problem. Take as certificate for \(\Lambda\) any partition \(\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_k\) such that \(\Lambda_1, \ldots, \Lambda_k \in \Pi_{Su(Q)}\). Checking that \(\Lambda_1, \ldots, \Lambda_k \in \Pi_{Su(Q)}\) can be done in linear time. So the \((\Pi_{SP(Q)}, \Pi_Q)\)–Problem belongs to the class NP.

Let \(\Lambda \in \Pi_Q\) be a yes-instance for the \((\Pi_{Bo(Q)}, \Pi_Q)\)–Problem. Take as certificate for \(\Lambda\) any partition \(\Lambda^- = \Lambda_1 \cup \cdots \cup \Lambda_k\) such that \(\Lambda^+ \cup (\Sigma(\Lambda_1)) \cup \cdots \cup (\Sigma(\Lambda_k)) \in \Pi_{Ke(Q)}\). Checking that \(\Lambda^+ \cup (\Sigma(\Lambda_1)) \cup \cdots \cup (\Sigma(\Lambda_k)) \in \Pi_{Ke(Q)}\) can be done in polynomial time. So the \((\Pi_{Bo(Q)}, \Pi_Q)\)–Problem belongs to the class NP.

Let \(\Lambda = (\lambda_1, \ldots, \lambda_n) \in \Pi_Q\) be a yes-instance for the \((\Pi_{BMS(Q)}, \Pi_Q)\)–Problem. Take as certificate for \(\Lambda\) any sequence of the three allowed moves that transform the \(n\) trivially realizable lists \((0),(0),\ldots,(0)\) into the list \((\lambda_1, \ldots, \lambda_n)\). Checking that the moves perform this transformation can be done in polynomial time. So the \((\Pi_{BMS(Q)}, \Pi_Q)\)–Problem belongs to the class NP.

Consider \(C=Sou, So\) or \(SE\). As \(\Pi_{BMS(Q)} = \Pi_{Sou(Q)} = \Pi_{So(Q)} = \Pi_{SE(Q)}\) then take as certificate for the \((\Pi_{C(Q)}, \Pi_Q)\)–Problem the same certificate than for the \((\Pi_{BMS(Q)}, \Pi_Q)\)–Problem. So the \((\Pi_{C(Q)}, \Pi_Q)\)–Problem belongs to the class NP.

Let \(\Lambda \in \Pi_Q\) be a yes-instance for the \((\Pi_{Pe_1(Q)}, \Pi_Q)\)–Problem. Take as certificate for \(\Lambda\) any partition \(\Lambda = (\alpha, \beta) \cup \Lambda_1 \cup \cdots \cup \Lambda_k\) that satisfies the condition in the definition of \(\Pi_{Pe_1(Q)}\). Checking those conditions can be done in linear time. So the \((\Pi_{Pe_1(Q)}, \Pi_Q)\)–Problem belongs to the class NP.

The complexity of the criterion of Group 4

That the \((\Pi_{Pe_2(Q)}, \Pi_Q)\)–Problem is NP-hard is an immediate consequence of Theorem 5.1. But as in the case of the \((\Pi_{QNIEP}, \Pi_Q)\)–Problem, for the \((\Pi_{Pe_2(Q)}, \Pi_Q)\)–Problem we do not know if it belongs to the class NP, since a certificate of membership in NP includes apparently a nonnegative matrix with prescribed diagonal and spectrum.
Discussion

The situation described in the complexity for the QNIEP and for the criterion of Group 4 lead us to ask the following question:

Let $A$ be a square nonnegative matrix whose eigenvalues are rational numbers. Does there always exist a rational nonnegative matrix $B$ with the same spectrum than $A$?

This question resembles the one posed by Cohen and Rothblum [7] related to the nonnegative matrix factorization of a rational nonnegative matrix. This question is restated by Vavasis [24] as follows: “Suppose an $m \times n$ rational matrix $A$ has nonnegative rank $k$ and a corresponding nonnegative factorization $A = WH$, $W \in \mathbb{R}^{m \times k}$, $H \in \mathbb{R}^{k \times n}$. Is it guaranteed that there exist rational $W, H$ with the same properties?” Interestingly, Vavasis proves that the nonnegative matrix factorization is NP-hard.

Now let us consider a similar question focused on the coefficients of the characteristic polynomial. We will use the result of Kim, Ormes and Roush [14] who proved the Spectral Conjecture of Boyle and Hendelman [5] for rationals. Suppose that $A$ is a positive matrix (no necessarily with rational entries) of order $n$ whose characteristic polynomial $f(\lambda)$ has rational coefficients. The Spectral Theorem for rationals implies that there exists a nonnegative matrix with rational entries of a sufficient large order $N$ whose characteristic polynomial is $\lambda^{N-n}f(\lambda)$. This lead us to also ask the following:

Let $A$ be a square nonnegative matrix whose characteristic polynomial has rational coefficients. Does there always exist a rational nonnegative matrix $B$ with the same characteristic polynomial than $A$?

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