A NEW BLOWUP CRITERION FOR STRONG SOLUTIONS OF 
THE COMPRESSIBLE NEMATIC LIQUID CRYSTAL FLOW

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ABSTRACT. In this paper, we establish a new blowup criterion for the strong solution to the simplified Ericksen-Leslie system modeling compressible nematic liquid crystal flows in a bounded domain Ω ⊂ R³. Specifically, we obtain the blowup criterion in terms of ∥P∥₁,L∞(BMOₓ) and ∥∇d∥₁,L∞(Lₓ), for any s > 3. The appearance of vacuum could be allowed.

1. Introduction. This paper aims to study the blowup criterion for a simplified version of Ericksen-Leslie system modeling the flow of compressible nematic liquid crystals in a domain Ω ⊂ R³. Mathematically, the model can be written as follows:

ρt + ∇ · (ρu) = 0,
ρu₁ + ρu₂ · ∇u + ∇(P(ρ)) = Lu - ∇d · ∆d,
d₁ + u · ∇d = ∆d + |∇d|²d,

where ρ, u = (u₁, u², u³) are functions of x ∈ Ω and t ≥ 0, representing density and velocity of the fluid respectively; P = P(ρ) is the pressure function; d : Ω × [0, +∞) → S² represents the macroscopic average of the nematic liquid crystal orientation field and L denotes the Lamé operator:

Lu = μΔu + (μ + λ)∇div u.

Here μ and λ are viscous constants, satisfying:

μ > 0, 2μ + 3λ ≥ 0. (1.4)

Throughout this paper, we assume that P ∈ C¹[0, ∞) ∩ C²₀(0, ∞) and that for any given γ ≥ 1 there exist some positive constants Cᵢ for i = 1, 2, 3, 4, 5, such that

\[ \begin{cases} 
C₁ρ^γ ≤ P(ρ) ≤ C₂(ρ^γ + 1), \\
C₃ρ^{γ-1} ≤ |P'(ρ)| ≤ C₄ρ^{γ-1}, \\
|P''(ρ)| ≤ C₅ρ^{γ-2},
\end{cases} \] (1.5)

for any ρ ≥ 0. Note that the usual γ-law, i.e., P(ρ) = ργ for γ ≥ 1, is included in (1.5).

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In this paper, we will consider the following initial-boundary conditions:

\[(\rho, u, d)|_{t=0} = (\rho_0, u_0, d_0), \quad (1.6)\]

\[(u, \frac{\partial d}{\partial n})|_{\partial \Omega} = 0, \quad (1.7)\]

where \(\nu\) is the unit outer normal vector of \(\partial \Omega\).

To begin with, let us briefly review some previous works. In dimension one, Ding-Lin-Wang-Wen [4] and Ding-Wang-Wen [5] have proven the existence of global strong solution and weak solution respectively. In dimension two, Liu-Zheng-Li-Liu [15] obtained the local existence of strong solution to the Cauchy problem. In dimension two or three, Jiang-Jiang-Wang [11] has proven the global existence of weak solution to the initial-boundary problem with large initial energy. Liu-Qing [14], Wang-Yu [17] and Lin-Lai-Wang [12] also studied the existence of weak solution to the three-dimensional case. Under the condition that the initial data is close to a constant equilibrium state in \(H^N(\mathbb{R}^N)\) \((N \geq 3)\), Gao-Tao-Yao [6] obtained the global existence and uniqueness of classical solution. While Huang-Wang-Wen [8] proved the local existence of strong solution provided that the initial data \((\rho_0, u_0, d_0)\) was sufficiently regular.

But, whether the strong solutions could exist globally in two or more dimensions is still an outstanding open problem. It is necessary to study the blowup mechanism of the nematic liquid crystal flow system (1.1)-(1.3). For three dimension, Huang-Wang-Wen [8] built up the following blowup criterion (1.8) under the assumption \(7\mu > 9\lambda\),

\[
\limsup_{T \uparrow T^*} \left( \|\rho\|_{L^\infty(0,T;L^\infty)} + \|\nabla d\|_{L^3(0,T;L^\infty)} \right) = \infty, \quad (1.8)
\]

where \(0 < T^* < +\infty\) is the maximum existence time for strong solution. At the same time, Huang-Wang [10] obtained a Serrin-type blowup criterion

\[
\limsup_{T \uparrow T^*} \left( \|D(u)\|_{L^1(0,T;L^{\infty})} + \|\nabla d\|_{L^2(0,T;L^{\infty})} \right) = \infty, \quad (1.9)
\]

where \(D(u) = \frac{\nabla u + (\nabla u)^T}{2}\) is the deformation tensor. While Huang-Wang [10] obtained a Serrin-type blowup criterion

\[
\limsup_{T \uparrow T^*} \left( \|\rho\|_{L^\infty(0,T;L^\infty)} + \|u\|_{L^{s_1}(0,T;L^{r_1})} + \|\nabla d\|_{L^{r_2}(0,T;L^{r_2})} \right) = \infty, \quad (1.10)
\]

where \(\frac{2}{s_i} + \frac{n}{r_i} \leq 1, n < r_i \leq \infty, i = 1, 2\) and \(n\) is the spatial dimension. For two dimensions, Gao-Tao-Yao [7] established a Serrin-type blowup criterion for initial-boundary value problem of (1.1)-(1.3) as follows,

\[
\limsup_{T \uparrow T^*} \left( \|\rho\|_{L^\infty(0,T;L^\infty)} + \|\nabla d\|_{L^2(0,T;L^{r_2})} \right) = \infty, \quad \frac{2}{s} + \frac{2}{r_2} \leq 1, \quad 2 < r_2 \leq \infty. \quad (1.11)
\]

More recently, Liu-Wang [13] gave a blowup criterion of the strong solution to the 2D Cauchy problem in terms of density only

\[
\limsup_{T \uparrow T^*} \left( \|\rho\|_{L^\infty(0,T;L^\infty)} \right) = \infty. \quad (1.12)
\]

Inspired by the papers [8] and [9] on nematic liquid crystal flows and the paper [2] on the MHD system and the paper [3] on a viscous liquid-gas two-phase flow model, we will establish in this paper a new blowup criterion for the strong solution to the
initial and boundary problem (1.1)-(1.3) subject to (1.6) and (1.7). The blowup criterion we obtain is in terms of $\|P\|_{L^\infty_t(BMO)}$ and $\|\nabla d\|_{L^s_tL^\infty}$, for any $s > 3$.

Before stating our main result, let us introduce some notations and the definition of strong solution.

Notations.

1. $A \lesssim B$: $A \leq CB$, for some constant generic $C > 0$.
2. $\int f \, dx = \int_\Omega f \, dx$.
3. $L^l = L^l(\Omega)$, for $1 \leq l \leq \infty$.
4. $H^k = W^{k,2}$, $H^k_0 = W^{k,2}_0$.
5. $Q_T = \Omega \times [0,T]$ ($T > 0$).

Definition 1.1. For $T > 0$, $(\rho, u, d)$ is called a strong solution to the compressible nematic liquid crystal flow (1.1)-(1.3) in $\Omega \times (0, T)$, if for some $q \in (3, 6]$,

$$
0 \leq \rho \in C([0,T];W^{1,q}(\Omega)), \quad \rho_t \in C([0,T];L^q(\Omega));
$$

$$
\rho \in C([0,T];H^2(\Omega) \cap H^1_0(\Omega)) \cap L^2(0,T;W^{2,q}(\Omega)),
$$

$$
u_t \in L^2(0,T;H^1_0(\Omega)), \quad \sqrt{\rho}u_t \in L^\infty(0,T;L^2(\Omega));
$$

$$
\nabla d \in C([0,T];H^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),
$$

$$
d_t \in C([0,T];H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)), \quad |d| = 1 \text{ in } Q_T;
$$

and $(\rho, u, d)$ satisfies (1.1)-(1.3) a.e. in $\Omega \times (0, T]$.

Our main results are as follows.

Theorem 1.2. Assume that $P$ satisfies (1.5), the initial data $\rho_0 \in W^{1,q}(\Omega)$ for some $q \in (3, 6]$, $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$, $\nabla d_0 \in H^2(\Omega)$ with $|d_0| = 1$ in $\Omega$, and $(\rho_0, u_0, d_0)$ satisfies the compatibility condition (2.1). Let $(\rho, u, d)$ be a strong solution of the initial boundary problem (1.1)-(1.3), together with (1.6) and (1.7). If $0 < T_* < +\infty$ is the maximum time of existence and $\gamma \mu > 9\lambda$, then, for any $s > 3$

$$
\lim_{T \uparrow T_*} \left( \|P\|_{L^\infty(0,T;BMO(\Omega))} + \|\nabla d\|_{L^s(0,T;L^\infty)} \right) = \infty.
$$

Remark 1. When $d$ is a constant, the model (1.1)-(1.3) is reduced to the compressible isentropic Navier-Stokes equations, and the blowup criterion (1.14) becomes

$$
\lim_{T \uparrow T_*} \|P\|_{L^\infty(0,T;BMO(\Omega))} = \infty,
$$

where $P$ can be chosen as $\rho^\gamma$ for $\gamma \geq 1$. It is new compared with some previous works on the same topics. We refer to [2, 3, 16].

Remark 2. The blowup criterion (1.14) can be replaced by

$$
\lim_{T \uparrow T_*} \left( \|\rho\|_{L^\infty(0,T;BMO(\Omega))} + \|\nabla d\|_{L^s(0,T;L^\infty)} \right) = \infty, \text{ for any } s > 3,
$$

if in addition $P : [0, \infty) \to \mathbb{R}$ is a globally Lipschitz continuous function that satisfies $P(0) = 0$. In fact, it suffices to prove that

$$
[P]_{BMO} \leq C[\rho]_{BMO},
$$

for some known constant $C > 0$. 


Proof of (1.15). With the fact that $P$ is globally Lipschitz continuous, we have

$$[P]_{BMO} = \sup_{x \in \Omega, r \in (0, d)} \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} |P(\rho(y)) - P_{\Omega_r(x)}| dy$$

\[
\leq \sup_{x \in \Omega, r \in (0, d)} \frac{1}{|\Omega_r(x)|^2} \int_{\Omega_r(x)} \int_{\Omega_r(x)} |P(\rho(y)) - P(\rho(z))| dz dy
\]

\[
\leq C \sup_{x \in \Omega, r \in (0, d)} \frac{1}{|\Omega_r(x)|^2} \int_{\Omega_r(x)} \int_{\Omega_r(x)} |\rho(y) - \rho(z)| dz dy
\]

\[
\leq C \sup_{x \in \Omega, r \in (0, d)} \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} |\rho(y) - \rho_{\Omega_r(x)}| dy
\]

\[
\leq C [\rho]_{BMO}.
\]

Therefore, we finish the proof of (1.15). \qed

We will prove Theorem 1.2 by the contradiction argument. Assume that (1.14) were false, that is, there is some $s_0$, which can be infinitely close to but bigger than 3, and a positive constant $M$ such that

$$\limsup_{T \uparrow T_0} \left( \|P\|_{L^\infty(0,T;BMO(\Omega))} + \|\nabla d\|_{L^{s_0}(0,T;L^\infty)} \right) \leq M < +\infty. \quad (1.16)$$

Since $\Omega$ is bounded, (1.16) implies that, there is some $s_0$, which can be infinitely close to but bigger than 3 such that

$$\limsup_{T \uparrow T_0} \left( \|P\|_{L^\infty(0,T;L^p)} + \|\nabla d\|_{L^{s_0}(0,T;L^\infty)} \right) \leq M_1 < \infty, \quad (1.17)$$

for any $1 \leq p < \infty$ and some positive constant $M_1$.

We aim to prove that

$$\limsup_{T \uparrow T_0} \left( \|\rho\|_{L^\infty(0,T;L^\infty)} + \|\nabla d\|_{L^3(0,T;L^\infty)} \right) \leq C < +\infty, \quad (1.18)$$

where $C$ is a generic positive constant which may depend on $\mu, \lambda, \Omega, \rho_0, u_0, d_0, M, M_1, T_0$ and $C_i (i = 1, 2, 3, 4, 5)$. (1.18) together with Theorem 1.3 in Huang-Wang-Wen [8] implies that $T_0$ is not the maximum existence time, which is the desired contradiction.

The paper is organized as follows. In Section 2, we will present two important fundamental results. In Section 3, we will prove Theorem 1.3.

2. Preliminaries. For completeness, we will firstly present the existence and uniqueness of local strong solution to the initial and boundary value problem (1.1)-(1.3), (1.6) together with (1.7) in the following theorem. The proof can be found in Huang-Wang-Wen [8].

**Theorem 2.1.** Assume that $P : [0, +\infty) \to \mathbb{R}$ is a locally Lipschitz continuous function, $\rho_0 \in W^{1,q}(\Omega)$ for some $q \in (3, 6)$, $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$, $\nabla d_0 \in H^2(\Omega)$ and $|d_0| = 1$ in $\overline{\Omega}$. If, in additions, the following compatibility condition

$$Lu_0 - \nabla (P(\rho_0)) - \Delta d_0 \cdot \nabla d_0 = \sqrt{\rho_0} g \text{ for some } g \in L^2(\Omega) \quad (2.1)$$

holds, then there exists a positive time $T_0 > 0$ and a unique strong solution $(\rho, u, d)$ of problem (1.1)-(1.3), together with (1.6) and (1.7) in $\Omega \times (0, T_0)$. 
Next we would like to introduce a variant of the Brezis-Waigner’s inequality which will play a crucial role when we establish the $L^\infty_t L^q_x$-estimate of $\nabla P$. (see [16])

**Lemma 2.2.** Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^N$ and $f \in W^{1,p}(\Omega)$ with $p \in (3, \infty)$. Then there exists a constant $C$ depending on $p$ and the Lipschitz property of the domain $\Omega$ such that

$$\|f\|_{L^\infty(\Omega)} \leq C(1 + \|f\|_{BMO(\Omega)} \ln(e + \|\nabla f\|_{L^p(\Omega)})).$$

Here $BMO(\Omega)$ denotes the John-Nirenberg’s space of bounded mean oscillation whose norm is defined by

$$\|f\|_{BMO(\Omega)} = \|f\|_{L^2(\Omega)} + [f]_{BMO(\Omega)},$$

with the semi-norm

$$[f]_{BMO(\Omega)} = \sup_{x \in \Omega, r \in (0,d)} \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} |f(y) - f_{\Omega_r(x)}| \, dy,$$

where $\Omega_r(x) = B_r(x) \cap \Omega$, $B_r(x)$ is the ball with center $x$ and radius $r$ and $d$ is the diameter of $\Omega$. For a measurable subset $E$ of $\mathbb{R}^N$, $|E|$ denotes its Lebesgue measure and

$$f_{\Omega_r(x)} = \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} f(y) \, dy.$$

3. **Proof of Theorem 1.2.** The proof of Theorem 1.2 is based on several lemmas.

**Corollary 1.** As a corollary for (1.5) and (1.17), we have

$$\limsup_{T \uparrow T_*} \|\rho\|_{L^\infty(0,T;L^p)} \leq C,$$

for any $1 \leq p < \infty$.

With (1.17), we can obtain the standard energy estimate as follows. The proof can be found in Huang-Wang-Wen [8], for simplicity, we omit the detail.

**Lemma 3.1.** Under the conditions of Theorem 1.2, (1.16) and (3.1), we have for all $0 \leq t < T_*$,

$$\int (\rho |u|^2 + |\nabla d|^2) \, dx + \int_0^t \int \left[ |\nabla u|^2 + |\Delta d + |\nabla d|^2 d|^2 \right] \, dx \, dt \leq C. \quad (3.2)$$

Then we can derive the following estimate of $\|\nabla^2 d\|_{L^2(0,T;L^2)}$.

**Lemma 3.2.** Under the conditions of Theorem 1.2 and (1.16), we have

$$\int_0^{T_*} \int_\Omega |\nabla^2 d|^2 \, dx \, dt \leq C. \quad (3.3)$$

**Proof.** Applying (3.2) and (1.16), we see that

$$\int_0^{T_*} \int |\Delta d|^2 \, dx \, dt = \int_0^{T_*} \int |\Delta d + |\nabla d|^2 d|^2 \, dx \, dt + \int_0^{T_*} \int |\nabla d|^4 \, dx \, dt$$

$$\leq C + \sup_{0 \leq t < T_*} \int |\nabla d|^2 \, dx \int_0^{T_*} \|\nabla d\|_{L^\infty} \, dt$$

$$\leq C.$$
Taking $H^2$ regularity estimate for $d$ with Neumann boundary condition, we have
\[
\int_0^{T^*} \int_\Omega |\nabla^2 d|^2 \, dx \, dt \leq C \int_0^{T^*} (\|\triangle d\|_{L^2}^2 + \|d\|_{H^1}^2) \, dt.
\]
Therefore, the proof is completed.

In order to obtain a high order estimate of $u$, we would adopt the approach by decomposing the velocity into two parts (see [16]). Specifically, we denote $u = w + v$, where $v = \mathcal{L}^{-1} \nabla (P(\rho))$ is the solution of the Lamé system:
\[
\begin{cases}
\mathcal{L} v = \nabla (P(\rho)), \\
v|_{\Omega} = 0
\end{cases}
\tag{3.4}
\]
and $w$ satisfies
\[
\begin{cases}
\rho w_t - \mathcal{L} w = \rho F - \nabla d \cdot \Delta d, \\
w|_{t=0} = w_0 = u_0 - v_0, \\
w|_{\Omega} = 0
\end{cases}
\tag{3.5}
\]
where
\[
F = -u \cdot \nabla u - \mathcal{L}^{-1} \nabla (\partial_t (P(\rho)))
= -u \cdot \nabla u + \mathcal{L}^{-1} \nabla \text{div} (P(\rho)u) - \mathcal{L}^{-1} \nabla ((P - P'(\rho))\text{div} u).
\]
We can use Proposition 2.1 of [16] and (1.17) to obtain that
\[
\|\nabla v\|_{L^q} \leq C \|P(\rho)\|_{L^q} \leq C, \quad 1 < q \leq 6.
\tag{3.6}
\]
Moreover from [1] and (1.16), we find
\[
\|\nabla v\|_{BMO} \leq C \|P(\rho)\|_{BMO} \leq C.
\tag{3.7}
\]
Next, we aim to prove Lemma 3.3 in the following, which plays an important role in proving Theorem 1.2.

**Lemma 3.3.** Under the assumptions of Theorem 1.2 and (1.17), if $\lambda < \frac{7\mu}{\mu + \lambda}$, there exists some $r > 5$, such that for any $0 \leq t < T^*$,
\[
\int (\rho |u|^r + |\nabla w|^2 + |\nabla d|_{L^2}^{2(\frac{q}{r})} + |\nabla^2 d|^2) \, dx + \int_0^t \int (|\nabla^3 d|^2 + |\nabla d_t|^2) \, dx \, ds \leq C.
\tag{3.8}
\]
Since the proof for Lemma 3.3 is somewhat complicated, for better illustration, we will first prove Lemmas 3.4-3.6 and based on these three lemmas we can finally prove Lemma 3.3.

**Lemma 3.4.** Under the assumptions of Theorem 1.2 and (1.17), given any $r > 5$, which can be infinitely close to 5, there is a constant $C$ such that for any $0 \leq t < T^*$,
\[
\frac{d}{dt} \int (\mu |\nabla w|^2 + (\mu + \lambda)|\text{div} w|^2) \, dx + \frac{1}{2} \int \rho |w_t|^2 \, dx \\
\leq 2 \frac{d}{dt} \int (\nabla d \otimes \nabla d - \frac{1}{2} |\nabla d|^2 \delta_3) : \nabla w \, dx + C\|\nabla d\|_{L^\infty}^2 (\|\nabla w\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) + C\|\rho \tilde{u}\|_{L^r}^{r} \|\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 + C\|\rho \tilde{u}\|_{L^r}^{r} + C.
\tag{3.9}
\]
Proof. Multiplying (3.5) by \( w_\varepsilon \), integrating over \( \Omega \), and using integration by parts and Cauchy’s inequality, we have
\[
\frac{d}{dt} \int (\mu |\nabla w|^2 + (\mu + \lambda) |\text{div} w|^2) \, dx + \int \rho |w_t|^2 \, dx \\
\leq ||\nabla F||_{L^2}^2 + 2 \frac{d}{dt} \int (\nabla d \otimes \nabla d - \frac{1}{2} |\nabla d|^2 I_3) : \nabla w \, dx + C \int |\nabla d||\nabla d_i||\nabla w| \, dx \\
= \sum_{i=1}^3 I_i. \tag{3.10}
\]

We then estimate \( I_1 \) and \( I_3 \) respectively. First of all, we can easily use Cauchy’s inequality to show that
\[
I_3 \leq \frac{1}{2} \int |\nabla d_i|^2 \, dx + C \int |\nabla d|^2 |\nabla w|^2 \, dx \\
\leq \frac{1}{2} \int |\nabla d_i|^2 \, dx + C \|\nabla d\|_{L^\infty}^2 \int |\nabla w|^2 \, dx. \tag{3.11}
\]
As for \( I_1 \), different from what are shown in Huang-Wang-Wen [8], we need to overcome the difficulty arising from the lack of the assumption for \( ||\rho||_{L^\infty} \). By Minkowski inequality, we obtain that
\[
I_1 \leq ||\sqrt{\rho} u \cdot \nabla u||_{L^2} + ||\sqrt{\rho} L^{-1} \nabla \text{div}(P(\rho) u)\|_{L^2} \\
+ ||\sqrt{\rho} L^{-1}(P(\rho) - P'(\rho) \rho)\text{div} u||_{L^2} = \sum_{j=1}^3 I_{1j}. \tag{3.12}
\]
Before estimating \( I_{11} \), we want to emphasize that given any \( r > 5 \), we can choose \( p_1 \in (\frac{5p_2}{r-2}, 6) \) and \( p_2 \in (\frac{3p_1}{3+p_1}, 2) \) such that \( 2/(1-\alpha) < r \) with \( \alpha = (\frac{2}{r} - \frac{1}{p_1})/(\frac{2}{6} - \frac{1}{p_2}) \).

Thanks to this observation, we can prove that
\[
I_{11} \leq ||\rho||_{L^\infty}^{\frac{2}{2\alpha}} ||\frac{\rho^{\frac{5}{2}} u}{\rho_f^{\frac{5}{2}}}||_{L^2} \|\nabla u\|_{L^{p_1}}^2 \\
= C ||\rho^{\frac{5}{2}} u||_{L^2}^2 \left( ||\nabla u||_{L^2}^2 + ||\nabla v||_{L^2}^2 \right) \\
\leq C ||\rho^{\frac{5}{2}} u||_{L^2}^2 \left( ||\nabla u||_{L^2}^2 + ||\nabla^2 w||_{L^2}^{2(1-\alpha)} ||\nabla w||_{L^2}^2 \right) \tag{3.13}
\]
\[
\leq \varepsilon ||\nabla^2 w||_{L^2}^2 + C ||\rho^{\frac{5}{2}} u||_{L^2}^2 ||\nabla w||_{L^2}^2 + C ||\rho^{\frac{5}{2}} u||_{L^2}^2 ||\nabla w||_{L^2}^2 + C ||\rho^{\frac{5}{2}} u||_{L^2}^2, \\
\leq \varepsilon ||\nabla^2 w||_{L^2}^2 + C ||\rho^{\frac{5}{2}} u||_{L^2} \left( ||\nabla u||_{L^2}^2 + 1 \right),
\]
where we have used Hölder’s inequality, Young’s inequality, interpolation inequality, (3.1) and (3.6). Since \( p_2 < 2 \), we can then use Hölder’s inequality, Proposition 2.1 in [16] and (3.5) to obtain
\[
||\nabla^2 w||_{L^2} \leq C \left( ||\rho w_1||_{L^p} + ||\rho F||_{L^p} + ||\nabla d \cdot \Delta d||_{L^2} \right) \\
\leq C \left( ||\sqrt{\rho w_1||}_{L^2} + ||\sqrt{\rho F||}_{L^2} + ||\nabla d \cdot \Delta d||_{L^2} \right). \tag{3.14}
\]
Substituting (3.14) into (3.13), we find that for any \( \varepsilon > 0, \)
\[
I_{11} \leq \varepsilon (||\sqrt{\rho w_1||}_{L^2}^2 + ||\sqrt{\rho F||}_{L^2}^2) \\
+ C \left( ||\rho^{\frac{5}{2}} u||_{L^2}^2 ||\nabla u||_{L^2}^2 + ||\nabla d||_{L^\infty}^2 ||\Delta d||_{L^2}^2 + ||\rho^{\frac{5}{2}} u||_{L^2}^2 \right) \tag{3.15}
\]
Now let us turn to the estimates for \( I_{12} \) and \( I_{13} \). We choose \( 2 < p_3 < 6 \), then by Hölder’s inequality, Sobolev’s inequality, Proposition 2.1 in [16], (1.17) and (3.1) we
have
\[
I_{12} \leq C \|\rho\|_{L^{p_1}_{t,r}} \|\mathcal{L}^{-1} \nabla \div (P(\rho)u)\|_{L^{p_{r,3}}}^{2} \\
\leq C \|P(\rho)u\|_{L^{p_{r,3}}}^{2} \leq C \|P(\rho)\|_{L^{\frac{3p_{a}}{p_{a}+3}}}^{2} \|u\|_{L^{6}}^{2} \tag{3.16}
\]
and
\[
I_{13} \leq \|\rho\|_{L^{p_{1}}_{t,r}} \|\mathcal{L}^{-1} \nabla ((P(\rho) - P'(\rho)\rho) \div u)\|_{L^{p_{r,3}}}^{2} \\
\leq C \|\nabla \mathcal{L}^{-1} \nabla ((P(\rho) - P'(\rho)\rho) \div u)\|_{L^{\frac{3p_{a}}{p_{a}+3}}}^{2} \tag{3.17}
\]
Putting (3.15), (3.16) and (3.17) into (3.12), we finish the estimate for some constant \(r \geq 0\).

Lemma 3.5. Under the assumptions of Theorem 1.2 and (1.17), there is some constant \(c_0 > 0\) depending on \(r\) such that for any \(0 \leq t < T_{s}\),

\[
\frac{d}{dt} \int \rho |u|^{r} dx + c_{0} \int |u|^{r-2} |\nabla |u|^{2} dx \\
\leq C \int \rho |u|^{r} dx + C \|\nabla d\|_{L^\infty} \|\nabla d\|_{L^{\frac{4r}{4r+7}}}^{4} + C,
\tag{3.19}
\]

where \(m(r) = \frac{2(2r-1)(r-2)}{3(r+1)}\) satisfying \(3 < m(r) \leq s_0\) given in (1.17).

Proof. Multiplying \((1.2)\) by \(r |u|^{r-2} u\), integrating it over \(\Omega\), and using integration by parts, \((1.5)\) as well as Cauchy’s inequality, we have

\[
\frac{d}{dt} \int \rho |u|^{r} dx + \int r |u|^{r-2} (\mu |\nabla u|^{2} + (\mu + \lambda) |\div u|^{2} + (r - 2) \mu |\nabla |u|^{2}) dx \\
= \int r P(\rho) \div (|u|^{r-2} u) dx + \int r \left( \nabla d \otimes \nabla d - \frac{1}{2} |\nabla d|^{2} I_{3} \right) \div (|u|^{r-2} u) dx \\
- \int r (r - 2) (\mu + \lambda) (\div u) |u|^{r-3} u \cdot \nabla |u| dx \\
\leq C \int (\rho^{\gamma} + 1) |u|^{r-2} \nabla u |u| dx + \int |\nabla d|^{2} |u|^{r-2} |\nabla u| |dx| \\
+ \int r (\mu + \lambda) |u|^{r-2} |\div u|^{2} dx + \int \frac{r(r - 2)^{2}}{4} (\mu + \lambda) |u|^{r-2} |\nabla |u|^{2} dx \\
\leq C \int (\rho^{\gamma} + 1) |u|^{r-2} dx + C \int |\nabla d|^{4} |u|^{r-2} dx + r \varepsilon \int |u|^{r-2} |\nabla u|^{2} dx \\
+ \int r (\mu + \lambda) |u|^{r-2} |\div u|^{2} dx + \int \frac{r(r - 2)^{2}}{4} (\mu + \lambda) |u|^{r-2} |\nabla |u|^{2} dx 
\]
\[ \leq C \int \rho^{2\gamma}|u|^{-2} \, dx + C \int (|\nabla d|^4 + 1)|u|^{-2} \, dx + r \varepsilon \int |u|^{-2} |\nabla u|^2 \, dx \]
\[ + \int r(\mu + \lambda)|u|^{-2} \text{div } u^2 \, dx + \int \frac{r(r-2)^2}{4}(\mu + \lambda)|u|^{-2} |\nabla u|^2 \, dx. \]

By rearranging the above inequality and using Hölder's inequality, Sobolev's inequality, Young's inequality and (3.1), we have
\[ \frac{d}{dt} \int \rho|u|^r \, dx + \int r|u|^{r-2}(\mu - \varepsilon)|\nabla u|^2 \, dx \]
\[ + \int (r-2)(\mu - \frac{(r-2)(\mu + \lambda)}{4})r|u|^{r-2} |\nabla u|^2 \, dx \leq C \int \rho^{2^*}|u|^{2^*} \, dx + C \int (|\nabla d|^4 + 1)|u|^{-2} \, dx \]
\[ \leq C \left( \int \rho|u|^r \, dx \right)^{\frac{r-2}{r}} \left( \int \rho^{r+1-\frac{2}{r}} \, dx \right)^{\frac{2}{r}} \]
\[ + C\|\nabla (|u|^{\frac{2}{r}})\|_{L^2}^{2(r-2)} \left( \int (|\nabla d|^{\frac{6r}{4r-1}} + 1) \, dx \right)^{\frac{2(r+1)}{3r}} \leq r \varepsilon \int |u|^{-2} |\nabla u|^2 \, dx + C \int \rho|u|^r \, dx + C \left( \int |\nabla d|^{\frac{6r}{4r-1}} \, dx \right)^{\frac{r+1}{3r}} + C. \]

By Kato's inequality \(|\nabla u|^2 \geq |\nabla u|^2| \) and (3.2), we choose \( \varepsilon < \mu \) to obtain
\[ \frac{d}{dt} \int \rho|u|^r \, dx + \int r|u|^{r-2} \left( (r-1)\mu - \frac{(r-2)(\mu + \lambda)}{4} - 2\varepsilon \right)|\nabla u|^2 \, dx \leq C \int \rho|u|^r \, dx + C \left( \int |\nabla d|^{\frac{6r}{4r-1}} \, dx \right)^{\frac{r+1}{3r}} + C. \]

(3.20)

Observe that
\[ \left( \int |\nabla d|^{\frac{6r}{4r-1}} \, dx \right)^{\frac{r+1}{3r}} \leq C\|\nabla d\|_{L^m}^{m(r)} \|\nabla d\|_{L^{\frac{6r}{4r-1}}}^{\frac{6r}{4r-1}}, \]

where \( m(r) = \frac{2(r-1)(r-2)}{3r+1} \). Clearly \( m(5) = 3 \) and \( m'(r) > 0 \) for \( r > 5 \), which means that for the given constant \( s_0 > 3 \) in (1.17), we have \( 3 < m(r) \leq s_0 \) when \( r \) is sufficiently close to 5. On the other hand, given \( \lambda < \frac{7\mu}{4r} \), we find that if \( r \) is close enough to 5, then
\[ (r-1)\mu - \frac{(r-2)^2(\mu + \lambda)}{4} > 0. \]
(3.21)

It turns out that we can choose some \( r > 5 \), which can be infinitely close to 5, such that \( 3 < m(r) \leq s_0 \) and the above inequality holds. Meanwhile we let \( \varepsilon \) sufficiently small such that
\[ c_0 := (r-1)\mu - \frac{(r-2)^2(\mu + \lambda)}{4} - 2\varepsilon > 0. \]
(3.22)

Putting (3.22) into (3.20), we get (3.19).
Lemma 3.6. Under the assumptions of Theorem 1.2, given any \( r > 5 \), which can be infinitely close to 5, we have that for any \( 0 \leq t < T \),

\[
\frac{d}{dt} \int (|\nabla d|^n + |\Delta d|^2) dx + \int (n|\Delta d|^2|\nabla d|^{n-2} + |\nabla d_t|^2) dx \\
\leq C(|\nabla d|_L^{2-\infty} + ||\nabla d_t||_L^2)(||\nabla^2 d||_L^2 + ||\nabla w||_L^2) + C||\nabla d||_L^{n-2} \\
+ C||\nabla d_t||_L^n ||\nabla d||_L^n + C||\nabla d||_L^2 + C,
\]

(3.23)

where we could take \( n = \frac{6r}{r+1} \).

Proof. The proof of this lemma can be divided into two steps. The first is to estimate \( \int |\nabla d|^n dx \) and the second is to estimate \( \int |\nabla^2 d|^2 dx \).

Firstly we differentiate (1.3) with respect to \( t \) to find that

\[
\nabla d_t - \nabla \Delta d + \nabla (u \cdot \nabla d) = \nabla (|\nabla d|^2 d).
\]

(3.24)

Then we multiply (3.24) by \( u|\nabla d|^{n-2}\nabla d \) and integrate the resulting equation over \( \Omega \) to obtain

\[
\frac{d}{dt} \int |\nabla d|^n dx + n \int |\Delta d|^2 |\nabla d|^{n-2} dx \\
= n \int \left[ \nabla (|\nabla d|^2 d) - \nabla (u \cdot \nabla d) \right] \cdot |\nabla d|^{n-2} \nabla d dx - n \int \Delta d \cdot \nabla (|\nabla d|^{n-2}) \cdot \nabla d dx \\
\leq C \int (|\nabla d|^2 |\nabla^2 d| + |\nabla d|^{n+2} + |\nabla u| |\nabla d|^n + |\nabla d|^{n-2} |\nabla^2 d|^2) dx.
\]

Since \( |d| = 1 \), it is easy to show \( |\nabla d|^2 = -d \cdot \Delta d \). Consequentially,

\[
\frac{d}{dt} \int |\nabla d|^n dx + n \int |\Delta d|^2 |\nabla d|^{n-2} dx \\
\leq C \int (|\nabla d|^{n-2} |\nabla^2 d|^2 + |\nabla d|^{n+2} + |\nabla u| |\nabla d|^{n-2} |\nabla^2 d|) dx \\
\leq C \left( ||\nabla d||_L^{2-\infty} ||\nabla^2 d||_L^2 + ||\nabla d||_L^n ||\nabla d||_L^n + ||\nabla d_t||_L^{n-2} ||\nabla u||_L^2 ||\nabla^2 d||_L^2 \right) \\
\leq C ||\nabla d||_L^{n-2} (||\nabla^2 d||_L^2 + ||\nabla u||_L^2) + C ||\nabla d_t||_L^{n-2} ||\nabla d||_L^n \\
\leq C ||\nabla d||_L^{n-2} (||\nabla^2 d||_L^2 + ||\nabla w||_L^2) + C ||\nabla d||_L^{n-2} + C ||\nabla d||_L^{n-2} ||\nabla d||_L^n,
\]

(3.25)

where we have used \( u = w + v \) and (3.6).

Next let us turn to the estimate of \( \int |\nabla^2 d|^2 dx \). Again thanks to \( |\nabla d|^2 = -d \cdot \Delta d \), we have

\[
\int |\nabla d|^2 dx \leq C ||\nabla d||_L^2 \int |\nabla^2 d|^2 dx.
\]

(3.26)

Then we multiply (3.24) by \( \nabla d_t \), integrate it by parts over \( \Omega \) and use (3.26) to obtain

\[
\frac{1}{2} \frac{d}{dt} \int |\Delta d|^2 dx + \int |\nabla d_t|^2 dx \\
= \int (\nabla (|\nabla d|^2 d) - \nabla (u \cdot \nabla d)) \cdot \nabla d_t dx
\]

(3.27)
Taking use of Hölder’s inequality and Nirenberg’s interpolation inequality, we have
\[
\leq \varepsilon \| \nabla d_t \|_{L^2}^2 + C \int \left( |\nabla d|^6 + |\nabla d|^2 |\nabla^2 d|^2 + |\nabla u|^2 |\nabla d|^2 + |u|^2 |\nabla^2 d|^2 \right) dx
\]
\[
\leq \varepsilon \| \nabla d_t \|_{L^2}^2 + C \left[ \| \nabla d \|_{L^\infty}^2 \left( \| \nabla^2 d \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 \right) + \int |u|^2 |\nabla^2 d|^2 dx \right].
\]

What left is to estimate the last term on the right hand side of the above inequality. Before doing so, we need to find the upper bound of \( \| \nabla^3 d \|_{L^2}^2 \). By applying \( H^2 \)-estimate of elliptic equations to (1.3) and taking use of (3.26), we have
\[
\| \nabla^3 d \|_{L^2}^2 \lesssim \| \nabla d_t \|_{L^2}^2 + \| \nabla (u \cdot \nabla d) \|_{L^2}^2 + \| \nabla (|\nabla d|^2) \|_{L^2}^2
\]
\[
\lesssim \| \nabla d_t \|_{L^2}^2 + \| \nabla d \|_{L^\infty}^2 \left( \| \nabla u \|_{L^2}^2 + \| \nabla^2 d \|_{L^2}^2 \right) + \| \nabla d \|_{L^6}^6 + \int |u|^2 |\nabla^2 d|^2 dx \tag{3.28}
\]
\[
\leq C \| \nabla d_t \|_{L^2}^2 + C \| \nabla d \|_{L^\infty}^2 \left( \| \nabla u \|_{L^2}^2 + \| \nabla^2 d \|_{L^2}^2 \right) + C \int |u|^2 |\nabla^2 d|^2 dx.
\]

Now we come back to estimate the last term on the right hand side of (3.27). Taking use of Hölder’s inequality and Nirenberg’s interpolation inequality, we have
\[
\int |u|^2 |\nabla^2 d|^2 dx \leq C \left( \| u \|_{L^6}^{\frac{3}{2}} \| \nabla^2 d \|_{L^\infty}^{\frac{3}{2}} \leq \varepsilon \| \nabla u \|_{L^2}^2 + C \| \nabla^2 d \|_{L^2}^2 \right)
\]
\[
\leq \varepsilon \int |u|^{r-2} |\nabla u|^2 dx + C \| \nabla^2 d \|_{L^\infty}^{\frac{2r}{2r-5}} \| \nabla d \|_{H^{\frac{2r-5}{2r}}}^{\frac{2r}{r-5}} \| \nabla d \|_{H^{\frac{2r-5}{2r}}}^{\frac{2r}{r-5}} \| \nabla^2 d \|_{L^2}^2 + C \left( \| \nabla^2 d \|_{L^2}^2 + \| \nabla^2 d \|_{L^2}^2 + 1 \right)
\]
\[
\leq \varepsilon \int |u|^{r-2} |\nabla u|^2 dx + \varepsilon \| \nabla^3 d \|_{L^2}^2 + C \left( \| \nabla^2 d \|_{L^2}^2 + \| \nabla^2 d \|_{L^2}^2 + 1 \right)
\]
\[
\leq \varepsilon \int |u|^{r-2} |\nabla u|^2 dx + \varepsilon \| \nabla^3 d \|_{L^2}^2 + C \left( \| \nabla^2 d \|_{L^2}^2 + \| \nabla^2 d \|_{L^2}^2 + 1 \right),
\]

where we have used the fact that \( \frac{2r}{r-5} < 2, \frac{2r}{2r-5} < 4 \) due to \( r > 5 \). Putting (3.28) into (3.29) and choosing \( \varepsilon \) sufficiently small, we have
\[
\int |u|^2 |\nabla^2 d|^2 dx \leq \varepsilon \int |u|^{r-2} |\nabla u|^2 dx + C \| \nabla^2 d \|_{L^2}^2 + \varepsilon \| \nabla d_t \|_{L^2}^2
\]
\[
+ C \| \nabla^3 d \|_{L^2}^2 \left( \| \nabla u \|_{L^2}^2 + \| \nabla^2 d \|_{L^2}^2 \right) + C. \tag{3.30}
\]

Substituting (3.30) into (3.27), using (3.6), and choosing \( \varepsilon \) sufficiently small, we obtain
\[
\frac{d}{dt} \int |\Delta d|^2 dx + \int |\nabla d_t|^2 dx \leq C \| \nabla d \|_{L^\infty}^2 \left( \| \nabla^2 d \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 \right) + \varepsilon \int |u|^{r-2} |\nabla u|^2 dx + C \| \nabla^2 d \|_{L^2}^2 + C \tag{3.31}
\]
\[
\leq C \| \nabla d \|_{L^\infty}^2 \left( \| \nabla^2 d \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 \right) + C \| \nabla d \|_{L^\infty}^2 + \varepsilon \int |u|^{r-2} |\nabla u|^2 dx
\]
\[
+ C \| \nabla^2 d \|_{L^2}^4 + C.
\]

Combining (3.25) and (3.31), we prove (3.23).

Based on Lemmas 3.4-3.6, we will prove Lemma 3.3 in the following.
Proof of Lemma 3.3. We choose \( r > 5 \), which is given in Lemma 3.5, such that 3 < \( m(r) \leq s_0 \). Then we add (3.9), (3.19) and (3.23) together, and choose \( \varepsilon \) sufficiently small to obtain

\[
\frac{d}{dt} \int \left( \rho |u|^r + \mu |\nabla w|^2 + (\mu + \lambda) |\text{div} w|^2 + |\nabla d|^\frac{6r}{r+4} + |\Delta d|^2 \right) dx \\
+ \frac{1}{2} \int \rho |w_t|^2 dx + \frac{1}{2} \int |\nabla d_t|^2 dx + \frac{c_0}{2} \int |u|^{r-2}|\nabla |u|^2|^2 dx \\
\leq 2 \frac{d}{dt} \int (\nabla d \otimes \nabla d - \frac{1}{2} |\nabla d|^2 I_3) : \nabla w dx + C (\|\nabla u\|_{L^2}^2 + 1) \left( \int \rho |u|^r dx + 1 \right) \\
+ C \|\nabla d\|_{L^\infty}^{m(r)} + C \|\nabla d\|_{L^\infty}^{m(r)} \left( \|\nabla d\|_{L^\infty}^{\frac{6r}{r+4}} + \|\nabla^2 d\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 \right) \\
+ C \|\nabla d\|_{L^\infty}^2 \left( \|\nabla d\|_{L^\infty}^{\frac{6r}{r+4}} + \|\nabla^2 d\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 \right) + C \|\nabla d\|_{L^2}^2 + C \|\nabla^2 d\|_{L^2}^4 \\
\leq 2 \frac{d}{dt} \int (\nabla d \otimes \nabla d - \frac{1}{2} |\nabla d|^2 I_3) : \nabla w dx + C \|\nabla u\|_{L^2}^2 + C \|\nabla d\|_{L^\infty}^{m(r)} + C \\
+ C K(t) \left( \|\rho^\frac{1}{2} u\|_{L^\infty}^{\frac{6r}{r+4}} + \|\nabla d\|_{L^\infty}^{\frac{6r}{r+4}} + \|\nabla^2 d\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 \right),
\]

where \( K(t) = \|\nabla u(t)\|_{L^2}^2 + \|\nabla^2 d(t)\|_{L^2}^2 + \|\nabla d(t)\|_{L^\infty}^{m(r)} + 1 \).

It follows from (1.17), (3.2) and (3.3) that

\[
\int_0^t K(s) ds \leq C.
\]

Integrating (3.32) over \( (0, t) \), and using (1.17), (3.2) and Young’s inequality, we have

\[
\int \left( \rho |u|^r + |\nabla w|^2 + |\nabla d|^\frac{6r}{r+4} + |\nabla^2 d|^2 \right) dx + \int_0^t \int (\rho |w_t|^2 + |\nabla d_t|^2) dx ds \\
+ \int_0^t \int |u|^{r-2}|\nabla |u|^2|^2 dx ds \\
\leq C \int_0^t K(s) \left( \|\rho^\frac{1}{2} u\|_{L^\infty}^{\frac{6r}{r+4}} + \|\nabla w\|_{L^2}^2 + \|\nabla d\|_{L^\infty}^{\frac{6r}{r+4}} + \|\nabla^2 d\|_{L^2}^2 \right) ds + C \\
+ C \int |\nabla d|^2 |\nabla w| dx
\]
\[ \leq C \int_0^t K(s) \left( \| \rho^{1/2} u \|_{L^p}^2 + \| \nabla w \|^2_{L^2} + \| \nabla d \|^2_{L^2} \right) ds + C + \frac{1}{2} \int |\nabla w|^2 \, dx + C \int |\nabla d|^4 \, dx \]

Thus we have

\[ \leq C \int_0^t K(s) \left( \| \rho^{1/2} u \|_{L^p}^2 + \| \nabla w \|^2_{L^2} + \| \nabla d \|^2_{L^2} \right) ds + C + \frac{1}{2} \int |\nabla w|^2 \, dx. \]

(3.33), (3.34) and Gronwall’s inequality yield that for any \( 0 \leq t < T_\ast \),

\[ \int \left( \rho |u|^r + |\nabla w|^2 + |\nabla d|^2 \right) dx + \int_0^t \int \left( \rho |w_i|^2 + |\nabla d_i|^2 \right) dx \, ds \]

(3.35) and \( \nabla \| \nabla d \|^2_{L^2} \) \( ds \)

On the other hand we combine (3.28), (3.30) and (3.35) to get

\[ \nabla d \|_{L^2(0,T; L^2)} \leq C. \]

Thus we finish the proof of Lemma 3.3. \( \square \)

In what follows, we will estimate \( \| \sqrt{\rho} \dot{u} \|_{L^\infty(0,T; L^2)} \) and \( \nabla \dot{u} \|_{L^2(0,T; L^2)} \). Before that, we will present Corollary 2, which will be used in the later proof.

**Corollary 2.** Under the same assumptions of Theorem 1.2 and (1.17), we have that for any \( p_4 \in \left[ \frac{12}{7}, 2 \right), 2 \leq q \leq 6, \)

\[ \sup_{0 \leq t < T_\ast} \left( \| u \|_{L^5} + \| \nabla u \|_{L^2} + \| \nabla d \|_{L^5} + \| d_i \|_{L^2} + \| \nabla^2 w \|_{L^2(0,T; L^{p_4})} \right) \]

(3.36)

\[ + \| \nabla u \|_{L^2(0,T; L^q)} \leq C. \]

**Proof.** It follows from (3.6) and (3.8) that

\[ \| \nabla u(t) \|_{L^2} \leq \| \nabla w(t) \|_{L^2} + \| \nabla v(t) \|_{L^2} \leq C, \]

(3.37)

which together with Sobolev’s inequality yields the bound of \( \sup_{0 \leq t < T_\ast} \| u \|_{L^5} \). (3.8) and interpolation inequality imply the bound of \( \sup_{0 \leq t < T_\ast} \| \nabla d \|_{L^5} \). The bound of \( \| \nabla^2 w \|_{L^2(0,T; L^{p_4})} \) follows from (3.8), (3.14), (3.18) and (3.35). For the last term of (3.36), by Sobolev’s inequality, (3.6) and (3.8), we have

\[ \| \nabla u \|_{L^2(0,T; L^q)} \leq \| \nabla w \|_{L^2(0,T; L^q)} + \| \nabla v \|_{L^2(0,T; L^q)} \]

\[ \leq \| \nabla d \|_{L^2(0,T; L^q)} \leq C. \]

\[ \leq \| \nabla^2 w \|_{L^2(0,T; L^{p_4})} + \| \nabla w \|_{L^2(0,T; L^{p_4})} + 1 \leq C. \]
By equation (1.3), (3.8) and Hölder’s inequality, we have
\[
\sup_{0 \leq t < T_*} \|d_t\|_{L^2} \lesssim \sup_{0 \leq t < T_*} \left( \|\Delta d\|_{L^2} + \|\nabla d\|_{L^4}^2 + \|u \cdot \nabla d\|_{L^2} \right) \\
\lesssim \sup_{0 \leq t < T_*} \left( \|u\|_{L^6} \|\nabla d\|_{L^3} \right) + 1 \leq C.
\]
This completes the proof. \(\Box\)

**Lemma 3.7.** Under the same assumptions of Theorem 1.2 and (1.17), we have that for any \(0 \leq t < T_*\),
\[
\int (\rho|\dot{u}(t)|^2 + |\nabla d_t|^2) (t) \, dx + \int_0^t \int (|\nabla \dot{u}|^2 + |d_{tt}|^2) \, dx \, ds \leq C,
\]
where \(\dot{f} = f_t + u \cdot \nabla f\).

**Proof.** (1.2) can be written as
\[
\rho \ddot{u} + \nabla (P(\rho)) = \mathcal{L}u - \nabla d \cdot \Delta d.
\]
Differentiating (3.39) with respect to \(t\), we obtain
\[
\rho \dddot{u}_t + \rho u \cdot \nabla \dot{u} + \nabla (P(\rho)_t) + (\nabla d \cdot \Delta d)_t \\
= \mathcal{L}\dot{u} - \mathcal{L}(u \cdot \nabla u) + \nabla \left[ Lu \otimes u - \nabla (P(\rho)) \otimes u - (\nabla d \cdot \Delta d) \otimes u \right].
\]

Testing (3.40) by \(\dot{u}\), we derive
\[
\frac{1}{2} \frac{d}{dt} \int \rho|\dot{u}|^2 \, dx + \int \left( (\mu + \lambda)|\nabla \dot{u}|^2 + (\mu + \lambda)|\nabla \dot{u}|^2 \right) \, dx \\
= \int \left( (P(\rho)_t) \, \nabla \dot{u} + u \otimes \nabla (P(\rho)) : \nabla \dot{u} \right) \, dx \\
+ (\mu + \lambda) \int \left( \nabla \cdot (\nabla \dot{u} \otimes u) - \nabla \cdot (\dot{u} \cdot \nabla u) \right) \, dx + \mu \int \left( \nabla \cdot (\Delta u \otimes u) - \Delta (u \cdot \nabla u) \right) \, dx + \int (u \otimes (\Delta d \cdot \nabla d)) : \nabla \dot{u} \, dx
\]
\[
= \int \left( \nabla d_t \otimes \nabla d + \nabla d \otimes \nabla d_t - \nabla d \cdot \nabla d_t \right) : \nabla \dot{u} \, dx = \sum_{i=1}^5 J_i.
\]

By the same calculations as that in Huang-Wang-Wen [8], we have
\[
J_2 \lesssim \|\nabla \dot{u}\|_{L^2} \|\nabla u\|_{L^4}^2; \\
J_3 \lesssim \|\nabla \dot{u}\|_{L^2} \|\nabla u\|_{L^4}^2; \\
J_4 \lesssim \|\nabla \dot{u}\|_{L^2} \|\Delta d\|_{L^2}; \\
J_5 \lesssim \|\nabla \dot{u}\|_{L^2} \|\nabla d_t\|_{L^2} \|\nabla d\|_{L^\infty}.
\]

As for \(J_1\), we should be careful due to the lack of the assumption about the upper bound of \(\rho\). Taking use of Equation (1.1), (1.5) and (3.1), we find that
\[
J_1 = \int \left( - \div (P(\rho) u) \div \dot{u} - (P'(\rho) \rho - P(\rho)) \div u \div \dot{u} \right) \, dx \\
+ \int u \otimes \nabla (P(\rho)) : \nabla \dot{u} \, dx \\
= \int \left( P(\rho) u \cdot \nabla \div \dot{u} + (P(\rho) - P'(\rho) \rho) \div u \div \dot{u} \right) \, dx
\]
\[- \int (P(\rho)(\nabla u)^t : \nabla \dot{u} + P(\rho) u \cdot \nabla \text{div} \, \dot{u}) \, dx \]

\[= \int (P(\rho) - P'(\rho) \rho) \text{div} \, u \, d\mu \, dx - \int P(\rho)(\nabla u)^t : \nabla \dot{u} \, dx \]

\[\lesssim \|\rho^2 + 1\|_{L^4} \|\nabla u\|_{L^4} \|\nabla \dot{u}\|_{L^2} \lesssim \|\nabla u\|_{L^4} \|\nabla \dot{u}\|_{L^2}.\]

Substituting the aforementioned inequalities for \(J_i, i = 1, 2, 3, 4, 5\) into (3.41) and using Cauchy inequality as well as Sobolev’s inequality, we have

\[\frac{1}{2} \frac{d}{dt} \int \rho \lvert \dot{u} \rvert^2 \, dx + \int (\mu \lvert \nabla \dot{u} \rvert^2 + (\mu + \lambda) \text{div} \, \dot{u}^2) \, dx \]

\[\lesssim \lvert \nabla u \rvert_{L^4} \lvert \nabla \dot{u} \rvert_{L^2} + \lvert \nabla u \rvert_{L^4}^2 \lvert \nabla \dot{u} \rvert_{L^2} + \lvert \nabla \dot{u} \rvert_{L^2} \lvert \Delta d \rvert_{L^4} + \lvert \nabla \dot{u} \rvert_{L^2} \| \nabla d \|_{L^2} \| \nabla d \|_{L^\infty} \]

\[\lesssim \frac{\mu}{2} \lvert \nabla \dot{u} \rvert_{L^2}^2 + C \left( \lvert \nabla u \rvert_{L^4}^2 + \lvert \nabla u \rvert_{L^4}^2 + \lvert \Delta d \rvert_{H^1}^2 + \lvert \nabla d \|_{L^2}^2 \| \nabla d \|_{L^\infty}^2 \right) \]

\[\lesssim \frac{\mu}{2} \lvert \nabla \dot{u} \rvert_{L^2}^2 + C \left( \lvert \nabla u \rvert_{L^4}^2 + \lvert \nabla d \|_{L^2}^2 \right) + \lvert \nabla d \|_{L^\infty}^2 + 1, \]

which applies

\[\frac{d}{dt} \int \rho \lvert \dot{u} \rvert^2 \, dx + \mu \int \lvert \nabla \dot{u} \rvert^2 \, dx \]

\[\lesssim \lvert \nabla d \rvert_{L^4}^2 + \lvert \nabla u \rvert_{L^4}^2 + \lvert \nabla d \|_{L^2}^2 \| \nabla d \|_{L^\infty}^2 + 1. \tag{3.42} \]

In order to derive the estimate of \(\lvert \nabla d \rvert_{L^2}^2\), we differentiate (1.3) with respect to \(t\), multiply the resulting equation by \(d_{tt}\) and then integrate it over \(\Omega\) to obtain

\[\frac{1}{2} \frac{d}{dt} \int \lvert \nabla d \rvert^2 \, dx + \int \lvert d_{tt} \rvert^2 \, dx = \int \partial_t \left( \lvert \nabla d \rvert^2 \, d - u \cdot \nabla d \right) \, d \, dx \]

\[\lesssim \int \left( \lvert \nabla d \rvert^2 \lvert d_t \rvert + \lvert \nabla d \rvert \lvert \nabla d_t \rvert \right) \, d \, dx + \int \left( \lvert u \rvert \lvert \nabla d \rvert + \lvert u \rvert \lvert \nabla d_t \rvert \right) \, d \, dx \tag{3.43} \]

\[= K_1 + K_2, \]

where we have used \(\frac{\partial d}{\partial \nu} \bigg|_{\partial \Omega} = 0\). By the same calculations as that in Huang-Wang-Wen [8], we can easily estimate \(K_1\), which is given by

\[K_1 \lesssim \frac{1}{8} \lvert \lvert d_{tt} \rvert_{L^2}^2 + C \left( \lvert \lvert d_{tt} \rvert_{L^2}^2 + \lvert \nabla \rvert_{L^\infty}^2 \right) \lvert \lvert d_{tt} \rvert_{L^2}^2 + 1. \]

When we come to estimate \(K_2\), we can not just follow the steps as that in Huang-Wang-Wen [8]. In Corollary 2, difficulty arises because we can not derive the estimate for \(\lvert \nabla u \rvert_{L^2(0, T; L^6)}\). By the definition of \(\dot{u}\), Hölder’s inequality, Sobolev’s inequality, Corollary 2 and Young’s inequality, we estimate \(K_2\) as follows

\[K_2 \lesssim \int \left[ \lvert \lvert \dot{u} \rvert + \lvert u \rvert \lvert \nabla u \rvert \rvert \nabla d \rvert + \lvert u \rvert \lvert \nabla d_t \rvert \right] \, d \, dx \]

\[\lesssim \lvert d_{tt} \rvert_{L^2} \| \dot{u} \|_{L^6} \| \nabla d \|_{L^3} + \lvert d_{tt} \rvert_{L^2} \| u \|_{L^\infty} \| \nabla u \|_{L^4} \| \nabla d \|_{L^4} \]

\[+ \lvert d_{tt} \rvert_{L^2} \| u \|_{L^\infty} \| \nabla d_t \|_{L^2} \]

\[\lesssim \lvert d_{tt} \rvert_{L^2} \| \dot{u} \|_{L^6} \| \nabla d \|_{L^3} + \lvert d_{tt} \rvert_{L^2} \| u \|_{W^{1,4}} \| \nabla u \|_{L^4} \| \nabla d \|_{L^4} \]

\[+ \lvert d_{tt} \rvert_{L^2} \| u \|_{W^{1,4}} \| \nabla d_t \|_{L^2} \]

\[\lesssim \lvert d_{tt} \rvert_{L^2} \| \dot{u} \|_{L^6} + \lvert d_{tt} \rvert_{L^2} \left( \| u \|_{L^6}^2 + \| \nabla u \|_{L^4} \right) \]

\[+ \| d_{tt} \rvert_{L^2} (\| \nabla u \|_{L^4} + \| u \|_{L^6}) \| \nabla d_t \|_{L^2} \]
Applying Gronwall’s inequality, we establish the conclusion of Lemma 3.7. Putting the estimates of $K_1$ and $K_2$ into (3.43), and then combining the resulting inequality together with (3.42), we have

$$\frac{d}{dt} \int (\rho |\dot{u}|^2 + |\nabla d_t|^2) \, dx + \int (|\nabla \dot{u}|^2 + |d_{tt}|^2) \, dx \lesssim \frac{C}{4} \|\nabla^3 d\|_{L^2}^2 + \|\nabla u\|_{L^4}^4 + \|\nabla d_t\|_{L^2}^2 (\|\nabla d\|_{L^\infty}^2 + \|\nabla u\|_{L^4}^2 + 1) + 1.$$

What left is to estimate the first two terms on the right hand side of (3.42). Firstly by applying $H^3$-estimate of elliptic equations to (1.3) as well as taking use of Lemma 3.3, Corollary 2 and Nirenberg’s interpolation inequality, we have

$$\|\nabla^3 d\|_{L^2} \lesssim \|\nabla d_t\|_{L^2} + \|\nabla u\|_{L^4} + \|\nabla u\|_{L^2} + \|\nabla d\|_{L^2} + \|\nabla d^2\|_{L^2} + \|\nabla d^3\|_{L^2}$$

which yields

$$\|\nabla^3 d\|_{L^2} \lesssim \|\nabla d_t\|_{L^2} + \|\nabla u\|_{L^4} + 1.$$  \hspace{1cm} (3.45)

Next let us get started to estimate $\|\nabla u\|_{L^4}^4$. Observe that $w$ satisfies

$$Lw = \rho \ddot{u} + \Delta d \cdot \nabla d.$$  \hspace{1cm} (3.46)

Then we have

$$\|\nabla^2 w\|_{L^4}^2 \lesssim \|\rho \ddot{u}\|_{L^4}^2 + \|\Delta d \cdot \nabla d\|_{L^4}^2 \lesssim \|\rho \frac{1}{2} \ddot{u}\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 \|\nabla d\|_{L^6}^2 \lesssim \|\rho \frac{1}{2} \ddot{u}\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 \|\Delta d\|_{H^1}^2 \lesssim \|\rho \frac{1}{2} \ddot{u}\|_{L^2}^2 + \varepsilon \|\nabla^3 d\|_{L^2}^2 + 1 \hspace{1cm} (3.47)$$

where we have used (3.1), Corollary 2, Nirenberg’s interpolation inequality and (3.45). And thus

$$\|\nabla u\|_{L^4}^4 \lesssim \|\nabla w\|_{L^4}^2 + \|\nabla v\|_{L^4}^2 \lesssim \|\nabla^2 w\|_{L^4}^2 + 1 \hspace{1cm} (3.48)$$

which implies that

$$\|\nabla u\|_{L^4}^2 \lesssim \|\rho \frac{1}{2} \ddot{u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 + 1 \hspace{1cm} (3.49)$$

Putting (3.49) and (3.45) into (3.44), we have

$$\frac{d}{dt} \int (\rho |\dot{u}|^2 + |\nabla d_t|^2) \, dx + \int (|\nabla \dot{u}|^2 + |d_{tt}|^2) \, dx \lesssim \|\nabla u\|_{L^4}^4 + \|\rho \frac{1}{2} \ddot{u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 (\|\nabla d\|_{L^\infty}^2 + \|\nabla u\|_{L^4}^2 + 1) + \|\nabla u\|_{L^4}^2 + 1. \hspace{1cm} (3.50)$$

Applying Gronwall’s inequality, we establish the conclusion of Lemma 3.7.  \hspace{1cm} \square

By the equation (3.46) and Lemma 3.7, we obtain the following Corollary.
We complete the proof.

Integrating (3.54) over Ω, by (1.5), we obtain
\[ \sup_{0 \leq t < T} (\|\nabla^3 d\|_{L^2} + \|\nabla d\|_{L^\infty}) + \|\nabla w\|_{L^2(0,T;L^\infty)} + \|\nabla^2 w\|_{L^2(0,T;L^6)} \leq C. \quad (3.51) \]

**Proof.** It follows from (3.45), (3.49) and Lemma 3.7 that
\[ \sup_{0 \leq t < T} \|\nabla^3 d\|_{L^2} \leq C. \]

By Sobolev’s inequality, we see that
\[ \sup_{0 \leq t < T} \|\nabla d\|_{L^\infty} \leq C. \]

From (3.46) and (3.1), we obtain
\[ \|\nabla^2 w\|_{L^q} \leq C \rho \|\bar{u}\|_{L^q} + C \|\Delta d \cdot \nabla d\|_{L^q} \]
\[ \leq C \rho \|\partial_{x_i}^{\alpha \alpha'} \bar{u}\|_{L^q} + \|\Delta d\|_{H^1} \|\nabla d\|_{L^\infty} \]
\[ \leq C \|\nabla \bar{u}\|_{L^2} + \|\Delta d\|_{H^1} \|\nabla d\|_{L^\infty}. \]

Therefore, by Lemma 3.7 and Sobolev’s inequality, we have
\[ \|\nabla^2 w\|_{L^2(0,T;L^q)} + \|\nabla w\|_{L^2(0,T;L^\infty)} \leq C. \quad (3.52) \]

We complete the proof. \( \square \)

With Corollary 3, we are ready to prove the upper bound for \( \|\nabla P\|_{L^q} \) with \( q > 3 \), which is important to derive the desired contradiction. Unlike the proof for lemma 3.7 of [2] where the upper bound for \( \|\rho\|_{L^q} \) was obtained due to the given boundedness for \( \|\rho\|_{L^\infty} \) in the contradiction arguments, here we have to derive the upper bound in terms of the pressure gradient. It turns out that the Brezis-Wainger’s inequality (2.2) and the BMO estimate for the Lamé system play important parts.

**Lemma 3.8.** Under the same assumptions of Theorem 1.2, (1.5) and (1.16), we have that for \( q \in (3,6) \),
\[ \sup_{0 \leq t < T} \|\nabla P\|_{L^q} \leq C. \quad (3.53) \]

**Proof.** Multiplying the equation (1.1) by \( P'(\rho) \) and then differentiating with respect to \( x_j \), and then multiplying both sides of the resulting equation by \( q|\nabla P|^{q-2} \partial_j P \), we get
\[ \partial_t (|\nabla P|^q) + \partial_i (|\nabla P|^q u_i) + (q-1)|\nabla P|^q \partial_i u_i + q|\nabla P|^{q-2} \partial_i P \partial_j P \partial_j u_i \]
\[ + q \rho P'|\nabla P|^{q-2} \partial_j P \partial_j u_i + q \rho P'' \frac{1}{P'} |\nabla P|^q \partial_i u_i = 0. \quad (3.54) \]

Integrating (3.54) over Ω, by (1.5), we obtain
\[ \frac{d}{dt} \int |\nabla P|^q \ dx \leq C \int (1 + \rho |P'|^2) |\nabla u| |\nabla P|^q + C \int \rho |P'| |\nabla \text{div} u| |\nabla P|^{q-1} \]
\[ \leq C \|\nabla u\|_{L^\infty} \|\nabla P\|_{L^q}^q + C \|P \nabla \text{div} u\|_{L^q} \|\nabla P|^{q-1} \]
\[ \leq C \|\nabla u\|_{L^\infty} \|\nabla P\|^q_{L^q} + C \|P \nabla^2 u\|_{L^q} \|\nabla P|_{L^q}^{q-1}. \quad (3.55) \]
Thus, we have
\[
\frac{d}{dt} \|\nabla P\|_{L^q} \\
\leq C(\|\nabla u\|_{L^\infty}) \|\nabla P\|_{L^q} + C\|\nabla^2 u\|_{L^q}
\leq C(\|\nabla w\|_{L^\infty} + \|\nabla v\|_{L^\infty}) \|\nabla P\|_{L^q} + C\|P\|_{L^\infty}(\|\nabla^2 w\|_{L^q} + \|\nabla^2 v\|_{L^q})
\leq C(\|w\|_{W^{2,q}} + \|\nabla v\|_{L^\infty}) \|\nabla P\|_{L^q} + C\|P\|_{L^\infty}(\|w\|_{W^{2,q}} + \|\nabla P\|_{L^q})
\leq C(\|w\|_{W^{2,q}} + \|\nabla v\|_{L^\infty} + \|P\|_{L^\infty}) \|\nabla P\|_{L^q} + C\|P\|_{L^\infty}\|w\|_{W^{2,q}}
\leq C(\|w\|_{W^{2,q}} + \|\nabla v\|_{L^\infty} + \|P\|_{L^\infty}) \|\nabla P\|_{L^q} + C\|P\|_{L^\infty}\|w\|_{W^{2,q}},
\]
where we have used Proposition 2.1 of [16] and Sobolev’s inequality.

Taking use of (1.16), (2.2) and (3.7), we have
\[
\|\nabla v\|_{L^\infty} \leq C(1 + \|\nabla v\|_{BMO(\Omega)} \ln(e + \|\nabla P\|_{L^q}))
\leq C(1 + \|P\|_{BMO(\Omega)} \ln(e + \|\nabla P\|_{L^q}))
\leq C(1 + \ln(e + \|\nabla P\|_{L^q})),
\]
and
\[
\|P\|_{L^\infty} \leq C(1 + \|P\|_{BMO(\Omega)} \ln(e + \|\nabla P\|_{L^q}))
\leq C(1 + \ln(e + \|\nabla P\|_{L^q})).
\]

Therefore, we have
\[
\frac{d}{dt}(e + \|\nabla P\|_{L^q})
\leq C(\|w\|_{W^{2,q}} + 1 + (\|w\|_{W^{2,q}} + 1) \ln(e + \|\nabla P\|_{L^q}))(e + \|\nabla P\|_{L^q}).
\]

Let \(G(t) = \ln(e + \|\nabla P\|_{L^q})\), we have
\[
\frac{d}{dt}G(t) \leq C(\|w\|_{W^{2,q}} + 1) + C(\|w\|_{W^{2,q}} + 1)G(t).
\]

Combining (3.51) with Gronwall’s inequality, we can obtain
\[
G(t) \leq C.
\]

Thus, we have
\[
\sup_{0 \leq t < T_*} \|\nabla P\|_{L^q} \leq C.
\]

We complete the proof. \(\square\)

Finally, according to Lemma 3.8 and Sobolev’s inequality, we can obtain
\[
\limsup_{T \uparrow T_*} \|P\|_{L^\infty(0,T;L^\infty)} \leq C.
\]

Further, by (1.5) and (3.63), we can get
\[
\limsup_{T \uparrow T_*} \|\rho\|_{L^\infty(0,T;L^\infty)} \leq C.
\]

On the other hand, since \(\Omega\) is bounded, (1.16) implies that
\[
\limsup_{T \uparrow T_*} \|\nabla d\|_{L^3(0,T;L^\infty)} \leq C.
\]

Combining (3.64) with (3.65), we can yield (1.18). Based on the blowup criterion shown in Huang-Wang-Wen [8], (1.18) implies that \(T_*\) is not the maximum existence time, which is the desired contradiction. Then, we finish the proof of Theorem 1.2.
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