AVERAGING OF HAMILTON-JACOBI EQUATIONS ALONG
DIVERGENCE-FREE VECTOR FIELDS

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Abstract. We study the asymptotic behavior of solutions to the Dirichlet
problem for Hamilton-Jacobi equations with large drift terms, where the drift
terms are given by divergence-free vector fields. This is an attempt to under-
stand the averaging effect for fully nonlinear degenerate elliptic equations. In
this work, we restrict ourselves to the case of Hamilton-Jacobi equations. The
second author has already established averaging results for Hamilton-Jacobi
equations with convex Hamiltonians (G below) under the classical formulation
of the Dirichlet condition. Here we treat the Dirichlet condition in the viscos-
ity sense and establish an averaging result for Hamilton-Jacobi equations with
relatively general Hamiltonian G.

1. Introduction. In this paper, we consider the Dirichlet problem for the Hamilton-
Jacobi equation

$$\lambda u^\varepsilon - \frac{1}{\varepsilon} b \cdot Du^\varepsilon + G(x, Du^\varepsilon) = 0 \quad \text{in } \Omega, \quad \text{(HJ)}$$
$$u^\varepsilon = g \quad \text{on } \partial \Omega. \quad \text{(BC)}$$

Here $\lambda > 0$ and $\varepsilon > 0$ are constants, $\Omega \subset \mathbb{R}^2$ is an open and bounded set, $u^\varepsilon : \overline{\Omega} \to \mathbb{R}$
is the unknown function, and $b : \overline{\Omega} \to \mathbb{R}^2$, $G : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$, and $g : \partial \Omega \to \mathbb{R}$ are
given functions.

Our primary purpose is to investigate the behavior, as $\varepsilon \to 0+$, of the solution
$u^\varepsilon$ of (HJ) and (BC).
In problem (HJ\(\varepsilon\)) and (BC\(\varepsilon\)), our primary hypothesis is that \(b\) is a smooth, divergence-free, vector field on \(\mathbb{R}^2\). As is well-known in vector calculus, there exists a real-valued function \(H\) on \(\mathbb{R}^2\) such that
\[
b(x_1, x_2) = (H_{x_2}(x_1, x_2), -H_{x_1}(x_1, x_2)) \quad \text{for all } x = (x_1, x_2) \in \mathbb{R}^2.
\]
We fix throughout such a function \(H\) and call it the Hamiltonian generating the vector field \(b\). Note that the system of differential equations \(\dot{x}_1(t) = H_{x_2}(x_1(t), x_2(t))\) and \(\dot{x}_2(t) = -H_{x_1}(x_1(t), x_2(t))\) is a Hamiltonian system.

Our choice of the domain \(\Omega\) and the vector field \(b\) features as follows: first of all, we assume that the function \(H: \mathbb{R}^2 \to \mathbb{R}\) has the properties (H1)–(H3) described below. Let \(N\) be an integer such that \(N \geq 2\). Set \(I_0 := \{0, 1, \ldots, N-1\}\) and \(I_1 := \{1, \ldots, N-1\}\).

(H1) \(H \in C^2(\mathbb{R}^2)\) and \(\lim_{|z| \to \infty} H(z) = \infty\).
(H2) \(H\) has exactly \(N\) critical points \(z_i \in \mathbb{R}^2\), with \(i \in I_0\), and attains a local minimum at every \(z_i\), with \(i \in I_1\). Moreover \(z_0 = 0\) and \(H(0) = 0\).

The geometry of \(H\) is stated as follows (see also [14]). The set \(D_0 = \{x \in \mathbb{R}^2 \mid H(x) > 0\}\) is open and connected, and the open set \(\{x \in \mathbb{R}^2 \mid H(x) < 0\}\) has exactly \(N-1\) connected components \(D_i\), with \(i \in I_1\), such that \(z_i \in D_i\) (see Figure 1). Furthermore, it follows that \(\partial D_0 := \{x \in \mathbb{R}^2 \mid H(x) = 0\}\), \(\partial D_0 = \bigcup_{i \in I_1} \partial D_i\), and \(\partial D_i \cap \partial D_j = \{0\}\) if \(i, j \in I_1\) and \(i \neq j\).

![Figure 1. \(N = 6\)](image)

We choose \(h_i \in \mathbb{R}\), with \(i \in I_0\), so that
\[
h_0 > 0 \quad \text{and} \quad H(z_i) < h_i < 0 \quad \text{for } i \in I_1,
\]
and define
\[
\Omega_0 = \{x \in D_0 \mid H(x) < h_0\}, \quad \Omega_i = \{x \in D_i \mid H(x) > h_i\} \quad \text{for } i \in I_1,
\]
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and

$$\partial_i \Omega = \{ x \in \Omega_i \mid H(x) = h_i \} \quad \text{for } i \in \mathcal{I}_0.$$ 

Finally, the set $$\Omega$$ is given by

$$\Omega = \{ x \in \mathbb{R}^2 \mid H(x) = 0 \} \cup \bigcup_{i \in \mathcal{I}_0} \Omega_i,$$

and the drift vector $$b : \mathbb{R}^2 \to \mathbb{R}^2$$ is given by the Hamiltonian vector field of $$H$$, that is,

$$b = (H_{x_2}, -H_{x_1}).$$

Note that

$$\partial \Omega = \bigcup_{i \in \mathcal{I}_0} \partial_i \Omega.$$ 

(H3) There exist constants $$m \geq 0$$, $$n > 0$$, $$A_1 > 0$$, $$A_2 > 0$$ and a neighborhood $$V \subset \mathbb{R}^2$$ of 0 such that $$n < m + 2$$ and

$$|H_{x_i x_j}(x)| \leq A_1 |x|^m \quad \text{for all } x \in V \text{ and } i, j \in \{1, 2\},$$

and

$$A_2 |x|^n \leq |DH(x)| \quad \text{for all } x \in V.$$ 

Condition (H3) quantifies a “weak non-degeneracy” of $$H$$ near the critical point 0, and, in the following argument, it has an essential role to ensure the equi-continuity of the solutions $$u^\varepsilon$$ of (HJ$$^\varepsilon$$) and (BC$$^\varepsilon$$) near the level set $$\{ x \mid H(x) = 0 \}$$ (for more details, see the estimates (2) and (3) as well as Lemmas 7.2, 7.3, and 7.4).

An example of $$H$$ which satisfies (H1)–(H3) is given by

$$H(x_1, x_2) = \Re (x_1 + \sqrt{-1} x_2)^{N-1} + |x_1 + \sqrt{-1} x_2|^N$$

$$= \sum_{k=0}^{|(N-1)/2|} (-1)^k \binom{N-1}{2k} x_1^{N-1-2k} x_2^{2k} + (x_1^2 + x_2^2)^{N/2}$$

$$= r^{N-1} \cos (N-1) \theta + r^N \quad \text{(in polar coordinates).}$$

Here, $$|(N-1)/2|$$ denotes the largest integer less than or equal to $$(N-1)/2$$. It is easily seen that $$H$$ satisfies (H1)–(H3), with the $$z_i$$, $$i \in \mathcal{I}_1$$, being given by the points $$r = N/(N-1)$$ and $$\theta = (2i+1)\pi/(N-1)$$, $$i \in \mathcal{I}_0$$, in polar coordinates, and with $$(m, n) = (N-3, N-2)$$.

Our primary interest in this work is to generalize fully the averaging results obtained by Freidlin-Wentzell [6] and Ishii-Souganidis [12] for stochastic processes to those for controlled stochastic processes. The analysis of the averaging of stochastic processes can be phrased, in terms of partial differential equations, as the study of the asymptotic behavior of solutions to linear second-order elliptic partial differential equations, with the large Hamiltonian drift term $$-b \cdot Du^\varepsilon/\varepsilon$$, while for controlled stochastic processes, fully nonlinear second-order degenerate elliptic equations, of the form

$$\lambda u^\varepsilon - \frac{1}{\varepsilon} b \cdot Du^\varepsilon + G(x, Du^\varepsilon, D^2u^\varepsilon) = 0 \quad \text{in } \Omega,$$  \quad (1)

take over the role of linear elliptic equations.

However, by the technical reasons, we restrict ourselves to the case where the function $$G$$ of $$(x, u, Du, D^2u)$$ in (1) does not depend on $$D^2u$$. That is, we treat here the first-order equation (HJ$$^\varepsilon$$). In other words, we deal with deterministic control or differential games processes. The second author has already studied the asymptotic problem for such deterministic processes by analyzing (HJ$$^\varepsilon$$) and (BC$$^\varepsilon$$).
A crucial difference of this work from [13, 14] is that $G$ is not anymore convex so that the results cover the differential games processes. Another critical point here is that we treat the Dirichlet boundary condition in the viscosity sense, which makes the statement of our results transparent.

There are two difficulties to be dealt with here beyond those in [13, 14]. One is that the optimal control interpretation is not available anymore of the problem, and the second is how to deal with the boundary layer and to determine the effective boundary data. The bottom line to solve these difficulties is that the perturbed Hamiltonian $-\varepsilon^{-1}b(x) \cdot p + G(x, p)$ is coercive in the direction of $DH(x)$ although it is not coercive in the other directions when $\varepsilon$ is very small.

Our result is stated in Theorem 3.1, which claims that the effective problem is identified with the Dirichlet problem for a Hamilton-Jacobi equation on a graph. Indeed, the large Hamiltonian drift term, as $\varepsilon \to 0+$, makes $u^\varepsilon$ nearly constant along the level sets of $H$. If we identify every $h$-level set of $H$ in $\Omega_i$ with a point $h$ in the intervals

$$J_0 = (0, h_0) \quad \text{and} \quad J_i = (h_i, 0) \quad \text{for} \ i \in I_1$$

and the zero level set of $H$ with point 0 connecting all the intervals $J_i$, then we obtain a graph consisting of one node 0 and $N$ edges $J_i$. These suggest that the limit problem should be posed naturally and effectively on the graph.

Various definitions of viscosity solutions on graphs have been introduced in the literature, and we refer for these to [2, 9, 8, 16, 17], although those cannot be adopted to our effective problem. Our effective Hamiltonians in the edges are not well-defined at the node and their coercivities break down near the node. In our result, we identify the limit function of $u^\varepsilon$ with a maximal continuous viscosity solution of the effective problem posed on the graph. We also refer to [1, 16, 7] for asymptotic problems related to ours, in which Hamilton-Jacobi equations on graphs appear as effective problems.

This paper is organized as follows. In the next section, we give some assumptions on $G$ and a basic existence result for (HJ$^\varepsilon$) and (BC$^\varepsilon$) as well as a typical example of $G$ satisfying the assumptions. In Section 3, we present the main results. Section 4 makes fundamental observations concerning the effective problem in the edges. Section 5 outlines the proof of the main theorem based on three propositions and proves one of these propositions. The other two propositions are shown in Sections 6 and 7, respectively. In the appendix a basic proposition is presented together with its proof.

**Notation.** For a function $f : X \to \mathbb{R}^m$, we write $\|f\|_\infty = \|f\|_{\infty, X} : = \sup \{|f(x)| \mid x \in X\}$. For $r_1, r_2 \in \mathbb{R}$, we write $r_1 \wedge r_2 := \min\{r_1, r_2\}$ and $r_1 \vee r_2 := \max\{r_1, r_2\}$.

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2. The problem \((HJ^\varepsilon)\) and \((BC^\varepsilon)\). This section concerns the problem \((HJ^\varepsilon)\) and \((BC^\varepsilon)\). We set
\[
\bar{h} = \min_{i \in I_0} |h_i| \quad \text{and} \quad \Omega(s) = \{x \in \mathbb{R}^2 \mid |H(x)| < s\} \quad \text{for} \ s \in (0, \bar{h}),
\]
and denote the closure of \(\Omega(s)\) by \(\overline{\Omega}(s)\).

We need the following assumptions.
\begin{itemize}
  \item [(G1)] \(G \in C(\overline{\Omega} \times \mathbb{R}^2)\) and \(g \in C(\partial \Omega)\).
  \item [(G2)] There exists a continuous nondecreasing function \(m_1 : [0, \infty) \rightarrow [0, \infty)\) satisfying \(m_1(0) = 0\) such that
  \[
  |G(x,p) - G(y,p)| \leq m_1(|x-y|)(1 + |p|)
  \]
  for all \(x,y \in \overline{\Omega}\) and \(p \in \mathbb{R}^2\).
  \item [(G3)] There exists a continuous nondecreasing function \(m_2 : [0, \infty) \rightarrow [0, \infty)\) satisfying \(m_2(0) = 0\) such that
  \[
  |G(x,p) - G(x,q)| \leq m_2(|p-q|)
  \]
  for all \(x \in \overline{\Omega}\) and \(p,q \in \mathbb{R}^2\).
  \item [(G4)] There exists \(\gamma \in (0, \bar{h})\) such that, for each \(x \in \Omega(\gamma) \setminus c_0(0)\), the function \(\mathbb{R} \ni q \mapsto G(x,qDH(x))\) is convex.
  \item [(G5)] There exist \(\nu > 0\) and \(M > 0\) such that
  \[
  G(x,p) \geq \nu |p| - M \quad \text{for all} \ (x,p) \in \overline{\Omega} \times \mathbb{R}^2.
  \]
\end{itemize}

As already mentioned in the introduction, in this paper, we deal with solutions satisfying Dirichlet boundary conditions in the sense of viscosity solutions. We now recall the definition (see e.g. \([3, 11]\)) of viscosity solutions to \((HJ^\varepsilon)\) as well as those to \((HJ^\varepsilon)\) and \((BC^\varepsilon)\).

In what follows, we always assume \((G1)\).

**Definition 2.1.** A function \(u : \Omega \rightarrow \mathbb{R}\) is called a viscosity subsolution (resp., supersolution) of \((HJ^\varepsilon)\) if \(u\) is locally bounded in \(\Omega\) and, for any \(\phi \in C^1(\Omega)\) and \(z \in \Omega\) such that \(u^* - \phi\) attains a local maximum (resp., \(u_* - \phi\) attains a local minimum) at \(z\),
\[
\lambda u^*(z) - \varepsilon^{-1}b(z) \cdot D\phi(z) + G(z,D\phi(z)) \leq 0
\]
(resp., \(\lambda u_*(z) - \varepsilon^{-1}b(z) \cdot D\phi(z) + G(z,D\phi(z)) \geq 0\)),
where \(u^*\) and \(u_*\) denote, respectively, the upper and lower semicontinuous envelope of \(u\). A function \(u : \Omega \rightarrow \mathbb{R}\) is called a viscosity solution of \((HJ^\varepsilon)\) if \(u\) is both a viscosity sub- and supersolution of \((HJ^\varepsilon)\).

**Definition 2.2.** A function \(u : \overline{\Omega} \rightarrow \mathbb{R}\) is called a viscosity subsolution (resp., supersolution) of \((HJ^\varepsilon)\) and \((BC^\varepsilon)\) if \(u\) is bounded on \(\overline{\Omega}\) and the following two conditions (i), (ii) hold: (i) \(u\) is a viscosity subsolution (resp., supersolution) of \((HJ^\varepsilon)\), (ii) for any \(\phi \in C^1(\overline{\Omega})\) and \(z \in \partial \Omega\) such that \(u^* - \phi\) attains a local maximum (resp., \(u_* - \phi\) attains a local minimum) at \(z\),
\[
\min\{\lambda u^*(z) - \varepsilon^{-1}b(z) \cdot D\phi(z) + G(z,D\phi(z)), u^*(z) - g(z)\} \leq 0
\]
(resp., \(\max\{\lambda u_*(z) - \varepsilon^{-1}b(z) \cdot D\phi(z) + G(z,D\phi(z)), u_*(z) - g(z)\} \geq 0\)).

A function \(u : \overline{\Omega} \rightarrow \mathbb{R}\) is called a viscosity solution of \((HJ^\varepsilon)\) and \((BC^\varepsilon)\) if \(u\) is both a viscosity sub- and supersolution of \((HJ^\varepsilon)\) and \((BC^\varepsilon)\).

Let \(\mathcal{S}_{\varepsilon}\) (resp., \(\mathcal{S}_{\varepsilon}^-\)) denote the set of all viscosity solutions (resp., subsolutions) of \((HJ^\varepsilon)\) and \((BC^\varepsilon)\).
Proposition 2.1. For each \( \varepsilon > 0 \), there exists a viscosity solution \( u^\varepsilon \) of \((HJ^\varepsilon)\) and \((BC^\varepsilon)\), that is, \( S^\varepsilon \neq \emptyset \). Furthermore, the set \( \bigcup_{\varepsilon > 0} S^\varepsilon \) is uniformly bounded on \( \overline{\Omega} \).

Proof. Fix any \( \varepsilon > 0 \). We choose a constant \( C > 0 \) so that
\[
\max_{x \in \overline{\Omega}} |G(x,0)| \leq \lambda C \quad \text{and} \quad \max_{x \in \partial \Omega} |g(x)| \leq C,
\]
and observe that \( C \) and \( -C \) are, respectively, a viscosity super- and subsolution of \((HJ^\varepsilon)\) and \((BC^\varepsilon)\). Set
\[
u^\varepsilon(x) = \sup \{ v(x) \mid v \in S^-_\varepsilon, \ |v| \leq C \ \text{on} \ \overline{\Omega} \} \text{ for } x \in \overline{\Omega},
\]
and conclude by [10] that \( \nu^\varepsilon \) is a viscosity solution of \((HJ^\varepsilon)\) and \((BC^\varepsilon)\). Thus, \( S^\varepsilon \neq \emptyset \).

Next, let \( \varepsilon > 0 \) and \( v \in S^\varepsilon \). Let \( \hat{x} \in \overline{\Omega} \) be a maximum point of \( v^\varepsilon \). If \( x \in \Omega \), then we have
\[
\lambda v^\varepsilon(\hat{x}) + G(\hat{x},0) \leq 0.
\]
If, otherwise, \( \hat{x} \in \partial \Omega \), then, either,
\[
\lambda v^\varepsilon(\hat{x}) + G(\hat{x},0) \leq 0 \quad \text{or} \quad v^\varepsilon(\hat{x}) \leq g(\hat{x}).
\]
Hence, we get
\[
\sup_{\overline{\Omega}} v = v^\varepsilon(\hat{x}) \leq \max \{-\lambda^{-1} \max_{x \in \overline{\Omega}} G(x,0), \max_{\partial \Omega} g\}.
\]
Similarly, we obtain
\[
\min_{\overline{\Omega}} v_* = \min \{-\lambda \min_{x \in \overline{\Omega}} G(x,0), \min_{\partial \Omega} g\}.
\]
Thus, we have
\[
\sup_{\overline{\Omega}} |v| \leq \max \{\lambda^{-1} \max_{x \in \overline{\Omega}} |G(x,0)|, \max_{\partial \Omega} |g|\},
\]
which shows that \( \bigcup_{\varepsilon > 0} S^\varepsilon \) is uniformly bounded on \( \overline{\Omega} \). \( \Box \)

The following example shows that, in general, viscosity solutions of \((HJ^\varepsilon)\) and \((BC^\varepsilon)\) do not satisfy the Dirichlet condition in the classical sense. Moreover, the uniqueness of the viscosity solutions of \((HJ^\varepsilon)\) and \((BC^\varepsilon)\) does not hold.

Example 2.1. Let \( G \) and \( g \) be the functions defined by \( G(x,p) = |p| \) for \((x,p) \in \overline{\Omega} \times \mathbb{R}^2 \) and \( g(x) = 1 \) for \( x \in \partial \Omega \), respectively. Then \( u(x) = 0 \) is a viscosity solution of \((HJ^\varepsilon)\) and \((BC^\varepsilon)\). However, it does not satisfy \( u = 1 \) on \( \partial \Omega \). If we set \( v(x) = 0 \) for \( x \in \Omega \) and \( v(x) = 1 \) for \( x \in \partial \Omega \), then the function \( v \) is another viscosity solution of \((HJ^\varepsilon)\) and \((BC^\varepsilon)\).

The following comparison theorem is a direct consequence of [11, Theorem 2.1].

Proposition 2.2. Assume (G1)–(G3). Let \( u \) and \( v \) be a viscosity sub- and supersolution of \((HJ^\varepsilon)\) and \((BC^\varepsilon)\), respectively. If both \( u \) or \( v \) are continuous at the points of \( \partial \Omega \), then \( u \leq v \) on \( \overline{\Omega} \). Also, if \( u \) (resp., \( v \)) is continuous at the points of \( \partial \Omega \) and \( u \leq g \) (resp., \( v \geq g \)) on \( \partial \Omega \), then \( u \leq v \) on \( \overline{\Omega} \).

We remark here that assumption (G5) does not ensure that \( -\varepsilon^{-1}b(x) \cdot p + G(x,p) \) is coercive when \( \varepsilon > 0 \) is very small. Condition (G4) is assumed for technical reasons, and we do not know if such a convexity assumption on \( G \) is needed or not to get the convergence result in our main theorem. In our treatment below, condition (G4) actually matters in Theorems 3.1 and 5.3 (more specifically, in the observation that the functions \( w_i \) in (36) have the subsolution property), where the
limit functions on $J_i$ of the solutions $u^\varepsilon$ of (HJ$^\varepsilon$) and (BC$^\varepsilon$) consist of a maximal solution of the limit problem (HJ).

**Example 2.2.** Consider the function $G$ defined by

$$G(x, p) = \theta |p| - |p \cdot b(x)| - f(x) \quad \text{for } (x, p) \in \overline{\Omega} \times \mathbb{R}^2,$$

where $f \in C(\overline{\Omega})$ and $\theta > 0$ is chosen so that $\theta > \|DH\|_{\infty, \overline{\Omega}}$. It is easy to check that $G$ satisfies (G1)–(G5) and that, if $x \neq 0$, $G(x, \cdot)$ is not convex.

3. **Main result.** For $i \in I_0$, we set

$$c_i(h) = \{x \in \overline{\Omega} \mid H(x) = h\} \quad \text{for } h \in J_i,$$

and define the function $\overline{G}_i : (J_i \setminus \{0\}) \times \mathbb{R} \to \mathbb{R}$ by

$$\overline{G}_i(h, q) = \frac{1}{T_i(h)} \int_{\{x \in \overline{\Omega} \mid H(x) = h\}} \frac{G(x, qDH(x))}{|DH(x)|} \, dl,$$

where

$$T_i(h) = \int_{\{x \in \overline{\Omega} \mid H(x) = h\}} \frac{1}{|DH(x)|} \, dl$$

and $dl$ denotes the line element. We call the functions $\overline{G}_i$ the effective Hamiltonians.

For another type of representations of $\overline{G}_i$ and $T_i(h)$, see (10) and (9), respectively, where the line integrals are parametrized by the “time” $t$ generating the flow by $b$.

Our main result, Theorem 3.1 below, claims that the limit function of $u^\varepsilon$, as $\varepsilon \to 0+$, is characterized by the maximal viscosity solution $(u_0, u_1, \ldots, u_N-1)$ to

$$\begin{cases}
(HJ_i) & \lambda u_i + \overline{G}_i(h_i, u_i') = 0 \quad \text{in } J_i, \\
(BC_i) & u_i(h_i) = \min_{\partial \Omega} g, \\
(NC) & u_0(0) = u_1(0) = \cdots = u_{N-1}(0),
\end{cases}$$

We recall the definition of viscosity solution of (HJ$_i$) and (BC$_i$).

**Definition 3.1.** A function $u : J_i \to \mathbb{R}$ is called a viscosity subsolution (resp., supersolution) of (HJ$_i$) if $u$ is locally bounded in $J_i$ and, for any $\phi \in C^1(J_i)$ and $z \in J_i$ such that $u^* - \phi$ attains a local maximum (resp., $u_* - \phi$ attains a local minimum) at $z$,

$$\lambda u^*(z) + \overline{G}_i(z, \phi'(z)) \leq 0 \quad \text{(resp., } \lambda u_*(z) + \overline{G}_i(z, \phi'(z)) \geq 0).$$

A function $u : J_i \to \mathbb{R}$ is called a viscosity solution of (HJ$_i$) if $u$ is both a viscosity sub- and supersolution of (HJ$_i$).

**Definition 3.2.** A function $u : J_i \setminus \{0\} \to \mathbb{R}$ is called a viscosity subsolution (resp., supersolution) of (HJ$_i$) and (BC$_i$) if $u$ is locally bounded in $J_i \setminus \{0\}$ and the following two conditions hold: (i) $u$ is a viscosity subsolution (resp., supersolution) of (HJ$_i$), (ii) for any $\phi \in C^1(J_i \setminus \{0\})$ such that $u^* - \phi$ attains a local maximum (resp., $u_* - \phi$ attains a local minimum) at $h_i$,

$$\min \{\lambda u^*(h_i) + \overline{G}_i(h_i, \phi'(h_i)), u^*(h_i) - \min_{\partial \Omega} g\} \leq 0 \quad \text{(resp., } \max \{\lambda u_*(h_i) + \overline{G}_i(h_i, \phi'(h_i)), u_*(h_i) - g(h_i)\} \geq 0).$$

A function $u : J_i \setminus \{0\} \to \mathbb{R}$ is called a viscosity solution of (HJ$_i$) and (BC$_i$) if $u$ is both a viscosity sub- and supersolution of (HJ$_i$) and (BC$_i$).
We give the definition of (maximal) viscosity solutions of \((HJ)\).

**Definition 3.3.** We say that \((u_0, u_1, \ldots, u_{N-1}) \in \prod_{i \in I_0} \mathcal{C}(\bar{J}_i)\) is a viscosity solution (resp., subsolution) of \((HJ)\) if (NC) holds and, for each \(i \in I_0\), \(u_i\) is a viscosity solution (resp., subsolution) of \((HJ_i)\) and \((BC_i)\). Also, we say that \((u_0, u_1, \ldots, u_{N-1})\) is a maximal viscosity solution of \((HJ)\) provided it is a viscosity solution of \((HJ)\) and that, if \((v_0, v_1, \ldots, v_{N-1})\) is a viscosity solution of \((HJ)\), then \(u_i \geq v_i\) on \(\bar{J}_i\) for all \(i \in I_0\).

We write \(S\) (resp., \(S^-\)) for the set of all viscosity solutions (resp., subsolutions) \((u_0, \ldots, u_{N-1}) \in \prod_{i \in I_0} \mathcal{C}(\bar{J}_i)\) of \((HJ)\).

For any viscosity solution \((u_0, \ldots, u_{N-1})\) of \((HJ)\), we write \(d(u_0, \ldots, u_{N-1}) := u_0(0) = \cdots = u_{N-1}(0)\).

It is clear that a maximal viscosity solution defined above is unique if it exists.

The main result in this paper is stated as follows.

**Theorem 3.1.** Assume that \((G1)–(G5)\) hold. (i) There exists a maximal viscosity solution \((u_0, \ldots, u_{N-1})\) of \((HJ)\). (ii) Define the function \(u \in \mathcal{C}(\bar{\Omega})\) by \(u(x) = u_i \circ H(x)\) for \(x \in \bar{\Omega}_i\) and \(i \in I_0\).

Then the set \(S_\varepsilon\) converges to the function \(u\) as \(\varepsilon \to 0^+\) in the sense that for any compact subset \(K\) of \(\Omega\),

\[
\lim_{\varepsilon \to 0^+} \sup\{\|v - u\|_{\infty, K} \mid v \in S_\varepsilon\} = 0.
\]

The proof of this theorem is presented in Section 5.

4. **Effective problem \((HJ_i)\) and \((BC_i)\) in the edge \(J_i\).** Hereafter, we always assume \((G1)–(G5)\). We study here some properties of the effective Hamiltonians \(\overline{G}_i\) and the functions \(T_i\) as well as viscosity subsolutions of the effective problem \((HJ_i)\) and \((BC_i)\) in the edge \(J_i\).

**Lemma 4.1.** Let \(i \in I_0\).

(i) \(T_i \in C^1(\bar{J}_i \setminus \{0\})\).

(ii) \(T_i(h) = O(|h|^{-\frac{n}{m}})\) as \(J_i \ni h \to 0\).

We do not give here the proof of the lemma above, and refer for it to the proof of [14, Lemmas 3.2 and 3.3].

Since \(n < m + 2\), we see by (ii) of Lemma 4.1 that

\[
T_i \in L^p(J_i) \quad \text{if} \; 1 \leq p < \frac{m+2}{n} \quad \text{and} \quad \text{for all} \; i \in I_0.
\]

**Lemma 4.2.** Let \(i \in I_0\).

(i) \(\overline{G}_i \in C(\bar{J}_i \setminus \{0\} \times \mathbb{R})\).

(ii) For any \(h \in \bar{J}_i \setminus \{0\}\) and \(q, q' \in \mathbb{R}\),

\[
|\overline{G}_i(h, q) - \overline{G}_i(h, q')| \leq m_2(\max_{\Omega} |DH||q - q'|),
\]

where \(m_2\) is the function from \((G3)\).

(iii) Let \(\gamma\) be the positive number from \((G4)\). For each \(h \in J_i \cap (-\gamma, \gamma)\), the function \(q \mapsto \overline{G}_i(h, q)\) is convex.
(iv) For every \((h, q) \in \bar{J}_i \setminus \{0\} \times \mathbb{R}\),

\[
\overline{G}_i(h, q) \geq \frac{\nu L_i(h)}{T_i(h)} |q| - M,
\]

where \(\nu, M\) are the constants from (G5) and \(L_i(h)\) denotes the length of \(c_i(h)\), that is,

\[
L_i(h) = \int_{c_i(h)} dl.
\]

**Proof.** We give an outline of the proof, and we leave it to the reader to check the details. Assertions (i), (ii), (iii), and (iv) follow from (G1) and (i) of Lemma 4.1, (G3), (G2), and (G5), respectively.

We note that \(G_i\) are locally coercive in \(\bar{J}_i \setminus \{0\}\) in the sense that, for any closed interval \(I\) of \(\bar{J}_i \setminus \{0\}\),

\[
\lim_{r \to \infty} \inf \{G_i(h, q) \mid h \in I, \ |q| \geq r\} = \infty.
\]

This is an easy consequence of the fact that \(L_i(h) \geq l_0\) for all \((h, i) \in \bar{J}_i \setminus \{0\}\) and some constant \(l_0 > 0\), Lemma 4.1, and (3).

The next lemma is taken from [14, Lemma 3.6].

**Lemma 4.3.** We have

\[
\lim_{J_i \ni h \to 0} \min_{q \in \mathbb{R}} G_i(h, q) = \lim_{J_i \ni h \to 0} \overline{G}_i(h, 0) = G(0, 0) \quad \text{for all } i \in I_0.
\]

**Lemma 4.4.** Let \(i \in I_0\) and \(v \in \text{USC}(\bar{J}_i)\) be a viscosity subsolution of \((HJ_i)\). Then \(u\) is uniformly continuous in \(J_i\) and, hence, it can be extended uniquely to \(\bar{J}_i\) as a continuous function on \(\bar{J}_i\). Furthermore the extended function is also locally Lipschitz continuous in \(\bar{J}_i \setminus \{0\}\).

**Lemma 4.5.** Let \(i \in I_0\) and \(F\) be a family of viscosity subsolutions of \((HJ_i)\). Assume that \(F \cap C(\bar{J}_i)\) is uniformly bounded on \(\bar{J}_i\). Then \(F \cap C(\bar{J}_i)\) is equi-continuous on \(\bar{J}_i\).

These two lemmas are easy consequences of (2) and (3). We refer to [13, Lemmas 3.2–3.4] for the detail of the proof.

The local coercivity of \(\overline{G}_i\) ensures that the classical inequalities hold at \(h_i\) for any viscosity sub-solutions of \((\text{HJ}_i)\) and \((\text{BC}_i)\).

**Lemma 4.6.** Let \(i \in I_0\) and \(v \in C(\bar{J}_i)\) be a viscosity subsolution of \((\text{HJ}_i)\) and \((\text{BC}_i)\). Then we have \(v(h_i) \leq \min_{h_i \in \partial \Omega} g\).

Thanks to Lemmas 4.2 and 4.6, the comparison principle is valid for \((\text{HJ}_i)\) and \((\text{BC}_i)\), as stated in the next lemma.

**Lemma 4.7.** Let \(i \in I_0\) and let \(v \in C(\bar{J}_i)\) and \(w \in \text{LSC}(\bar{J}_i)\) be, respectively, a viscosity sub- and supersolution of \((\text{HJ}_i)\) and \((\text{BC}_i)\). Assume that \(v(0) \leq w(0)\). Then \(v(h) \leq w(h)\) for all \(h \in \bar{J}_i\).

5. **Proof of the main theorem.** We present the proof in two parts.

**Proof of (i) of Theorem 3.1.** In view of Lemmas 4.2 and 4.3, we may choose a constant \(C > 0\) so that

\[
|\overline{G}_i(h, 0)| \leq \lambda C \quad \text{for all } h \in J_i, i \in I_0, \text{ and } |g(x)| \leq C \quad \text{for all } x \in \partial \Omega.
\]
It is obvious that the $N$-tuple of the constant function $C$ and that of $-C$ are viscosity super- and sub-solution of (HJ), respectively. We may assume that $\lambda C \geq M$, where $M$ is the constant from (G5).

Let $\mathcal{S}_\infty$ denote the set of $(v_0, \ldots, v_{N-1}) \in \prod_{i=0}^{N-1} C(\bar{J}_i)$ such that $v_i$ is a viscosity sub-solution of (HJ) in $J_i$ for any $i \in \mathcal{I}_0$, $v_0(0) = \ldots = v_{N-1}(0)$, and $|v_i| \leq C$ in $J_i$ for all $i \in \mathcal{I}_0$.

According to Lemma 4.5, the family $\mathcal{S}_\infty$ is equi-continuous in the sense that for every $i \in \mathcal{I}_0$, the family \{ $v_i \in C(\bar{J}_i) \mid (v_0, \ldots, v_{N-1}) \in \mathcal{S}_\infty$ \} is equi-continuous on $\bar{J}_i$. Hence, setting

$$u_i(h) = \sup \{ v_i(h) \mid (v_0, \ldots, v_{N-1}) \in \mathcal{S}_\infty \} \quad \text{for } h \in \bar{J}_i, \ i \in \mathcal{I}_0,$$

we see that $u := (u_0, \ldots, u_{N-1}) \in \prod_{i \in \mathcal{I}_0} C(\bar{J}_i)$ and $u_0(0) = \ldots = u_{N-1}(0)$. Moreover, in view of the Perron method, we find that $u \in \mathcal{S}$.

To see the maximality of $(u_0, \ldots, u_{N-1})$, let $(v_0, \ldots, v_{N-1})$ be a viscosity solution of (HJ). Note by (iv) of Lemma 4.2 that for any $i \in \mathcal{I}_0$, we have, in the viscosity sense,

$$0 \geq \lambda v_i + \frac{\nu L_i(h)}{T_i(h)} |v_i'| - M \geq \lambda v_i - \lambda C \quad \text{in } J_i,$$

which implies that $v_i \leq C$ on $\bar{J}_i$. We set

$$w_i := v_i \vee (-C) = \min \{ v_i, -C \} \quad \text{on } \bar{J}_i, \ i \in \mathcal{I}_0.$$

It is easily seen that $(w_0, \ldots, w_{N-1}) \in \mathcal{S}_\infty$, and consequently, $v_i \leq w_i \leq u_i$ on $\bar{J}_i, i \in \mathcal{I}_0$. Thus, $u$ is a maximal viscosity solution of (HJ).

We need some preliminary observations before going into the proof of (ii) of Theorem 3.1.

Since the set $\bigcup_{r>0} \mathcal{S}_r$ is uniformly bounded on $\bar{\Omega}$ by Proposition 2.1, and hence, the half relaxed-limits $v^+$ and $v^-$ of $\mathcal{S}_r$, as $r \to 0+$,

$$
\begin{align*}
v^+(x) &= \lim_{r \to 0^+} \sup \{ u(y) \mid u \in \mathcal{S}_r, \ y \in B_r(x) \cap \Omega, \ \varepsilon \in (0, r) \}, \\
v^-(x) &= \lim_{r \to 0^+} \inf \{ u(y) \mid u \in \mathcal{S}_r, \ y \in B_r(x) \cap \Omega, \ \varepsilon \in (0, r) \}
\end{align*}
$$

are well-defined, bounded and, respectively, upper and lower semicontinuous on $\bar{\Omega}$.

For $i \in \mathcal{I}_0$, we set

$$v_i^+(h) = \max_{c_i(h)} v^+ \quad \text{for } h \in J_i \setminus \{ h_i \} \quad \text{and} \quad v_i^-(h) = \min_{c_i(h)} v^- \quad \text{for } h \in J_i,$$

and $v_i^+(h_i) = \limsup_{J_i \ni h \uparrow h_i} v_i^+(h)$. It is easily seen that $v_i^+ \in \text{USC}(J_i)$ and $v_i^- \in \text{LSC}(J_i)$ for all $i \in \mathcal{I}_0$.

For the proof of (ii) of Theorem 3.1, the following three propositions are crucial.

**Proposition 5.1.** For any $i \in \mathcal{I}_0$ and $h \in J_i$,

$$v^+(x) = v_i^+(h) \quad \text{and} \quad v^-(x) = v_i^-(h) \quad \text{for all } x \in c_i(h).$$

**Theorem 5.2.** For every $i \in \mathcal{I}_0$, the functions $v_i^+$ and $v_i^-$ are, respectively, a viscosity sub- and supersolution of (HJ) and (BC).

**Theorem 5.3.** For any $i \in \mathcal{I}_0$,

$$v_i^+(0) = v_i^-(0) = d(u_0, \ldots, u_{N-1}),$$

where $(u_0, \ldots, u_{N-1})$ is the maximal viscosity solution of (HJ).
Once these three propositions are in hand, the completion of the proof of (ii) of Theorem 3.1 is easily  

**Proof of (ii) of Theorem 3.1.** By the definition of $v^\pm_i$, we have $v^-_i(h) \leq v^+_i(h)$ for all $h \in J_i$ and $i \in I_0$. Hence, we deduce by Theorem 5.3 and the semicontinuities of $v^\pm_i$ that the functions $v_i^+$ and $v_i^-$ are continuous at $h = 0$. Now, since $v_i^+(0) = v_i^-(0) = u_i(0)$ by Theorem 5.3, Lemma 4.7 ensures that $v_i^+ \leq u_i \leq v_i^-$ on $J_i$ for all $i \in I_0$, which implies that $v_i^+ = v_i^- = u_i$ on $J_i$ for all $i \in I_0$. By the standard compactness argument together with Proposition 5.1, we conclude that for any compact subset $K$ of $\Omega$, we have

$$
\lim_{\varepsilon \to 0^+} \sup \{ \|u - w\|_{\infty, K} \mid w \in S_{\varepsilon} \} = 0. 
$$

We remark that the proof above shows that

$$
\lim_{\varepsilon \to 0^+} \sup \{ \|(w - u) - \|_{\infty, K} \mid w \in S_{\varepsilon} \} = 0,
$$

where $a_-$ denotes the negative part $\max\{0, -a\}$ for $a \in \mathbb{R}$.

It remains to prove Proposition 5.1, Theorems 5.2, and 5.3, and we give the proof of Proposition 5.1, Theorems 5.2, and 5.3, respectively, in this section, Sections 6, and 7.

We consider the flow generated by the vector field $b$:

$$
\dot{X}(t) = b(X(t)) \quad \text{and} \quad X(0) = x \in \mathbb{R}^2,
$$

and write $X(t, x)$ for the solution of (8), which has a basic property:

$$
H(X(t, x)) = H(x) \quad \text{for all} \ (t, x) \in \mathbb{R} \times \mathbb{R}^2.
$$

In particular, if $x \in c_i(h)$, with $h \in J_i$ and $i \in I_0$, then

$$
X(t, x) \in c_i(h) \quad \text{for} \ t \in \mathbb{R}.
$$

It follows from $(H1)$ and $(H2)$ that the curve $c_i(h)$ is $C^1$-diffeomorphic to circle $S^1$ for any $h \in J_i \setminus \{0\}$ and $i \in I_0$. Moreover, if $h \in J_i \setminus \{0\}$ and $i \in I_0$, then $b(x) \neq 0$ for all $x \in c_i(h)$ and $t \mapsto X(t, x)$ has a finite period for any $x \in c_i(h)$. Let $x \in c_i(h)$, with $i \in I_0$ and $h \in J_i \setminus \{0\}$ and let $\tau_i > 0$ denote the minimal period of $t \mapsto X(t, x)$. Observe that

$$
\tau_i = \int_0^{\tau_i} dt = \int_0^{\tau_i} \frac{|X(t, x)| dt}{|DH(X(t, x))|} = \int_{c_i(h)} \frac{dl}{|DH(x)|} = T_i(h). 
$$

Thus, if $h \in J_i \setminus \{0\}$, with $i \in I_0$, and if $x \in c_i(h)$, then $T_i(h)$ equals to the minimal period of $t \mapsto X(t, x)$.

We note here that $G_i$ can be rewritten as

$$
G_i(h, q) = \frac{1}{T_i(h)} \int_0^{T_i(h)} G(X(t, x), qDH(X(t, x))) dt \quad \text{if} \ h \neq 0,
$$

where $x \in c_i(h)$ is an arbitrary point. This representation says that $G_i(h, q)$ is the average value of the periodic function $t \mapsto G(X(t, x), qDH(X(t, x)))$ for $x \in c_i(h)$ over the period $T_i(h)$.

**Proof of Proposition 5.1.** We see immediately that $v^+$ and $v^-$ are a viscosity suband supersolution of

$$
-b \cdot Du = 0 \quad \text{in} \ \Omega,
$$

which, moreover, implies that $-v^-$ is a viscosity subsolution of (11). This observation ensures together with Proposition A.1 in the appendix (or [4, Theorem
I.14) that for any $x \in \Omega$, the functions $t \mapsto v^+(X(t,x))$ and $t \mapsto -v^-(X(t,x))$ are nondecreasing in $\mathbb{R}$. Hence, by the periodicity of $t \mapsto X(t,x)$, with $x \in c_i(h)$, $h \in J_i$, and $i \in I_0$, we infer that the functions $v^+$ and $v^-$ are constant on $c_i(h)$ for $h \in J_i$, $i \in I_0$. It is now clear that (7) holds.

6. Viscosity properties of the functions $v^+_i$ and $v^-_i$. We prove Theorem 5.2 in this section.

Let $M$ be the positive constant from (G5), and in view of Proposition 2.1, we define a positive number $C_M$ by

$$C_M = \max\{M, \sup\{\|u\|_{\infty, \Omega} \mid u \in \mathcal{S}_\varepsilon, \varepsilon > 0\}\}. \quad (12)$$

The next lemma is a quantitative version of Proposition 5.1.

**Lemma 6.1.** There is a constant $C > 0$ such that for any $\varepsilon > 0$, $u \in \mathcal{S}_\varepsilon$, $i \in I_0$, and $h \in J_i$,

$$|u^*(x) - u^*(y)| \leq \varepsilon CT_i(h) \quad \text{for all } x, y \in c_i(h).$$

**Proof.** Fix any $\varepsilon > 0$, $u \in \mathcal{S}_\varepsilon$, $i \in I_0$, $h \in J_i$, and $x, y \in c_i(h)$. The trajectory $t \mapsto X(t,x)$ stays in $c_i(h)$ and for some $\tau \in (0,2T_i(h)]$, it meets $y$ at $t = \tau$, that is, $X(\tau,x) = y$. Since $u^*$ is a viscosity subsolution of $\lambda u^* - \varepsilon^{-1}b \cdot Du^* - M = 0$ in $\Omega$ by (G5), and $Y(t) = X(\varepsilon^{-1}t,x)$ satisfies

$$\dot{Y}(t) = \frac{1}{\varepsilon} b(Y(t)) \quad \text{for all } t \in \mathbb{R},$$

we deduce by Proposition A.1 that

$$u^*(x) \leq e^{-\varepsilon \lambda \tau} u^*(Y(\varepsilon \tau)) + \int_0^{\varepsilon \tau} e^{-\lambda t} M \, dt$$

$$\leq u^*(y) + C_M(1 - e^{-\varepsilon \lambda \tau}) + M\varepsilon \tau \leq u^*(y) + 2\varepsilon C_M(\lambda + 1)T_i(h).$$

Thus, by the symmetry in $x$ and $y$, we obtain

$$|u^*(x) - u^*(y)| \leq 2\varepsilon C_M(\lambda + 1)T_i(h). \quad \Box$$

**Lemma 6.2.** For any $\varepsilon > 0$, $u \in \mathcal{S}_\varepsilon$, and $y \in \partial \Omega$, we have $u^*(y) \leq g(y)$.

**Proof.** Fix any $\varepsilon > 0$, $u \in \mathcal{S}_\varepsilon$, and $y \in \partial \Omega$. Choose $i \in I_0$ so that $y \in \partial \Omega = c_i(h_i)$. Note by (G5) that $u^*$ is a viscosity subsolution of

$$\lambda u^* - \varepsilon^{-1}b \cdot Du^* + \nu|Du^*| - M = 0 \quad \text{in } \Omega \quad \text{and } u^* = g \text{ on } \partial \Omega, \quad (13)$$

For $\alpha > 0$ and $\beta > 0$, we set

$$\phi_{\alpha,\beta}(x) = \alpha |x - y|^2 + \beta |H(x) - h_i| \quad \text{for } x \in \overline{\Omega}_i.$$

Let $x_{\alpha,\beta} \in \overline{\Omega}_i$ be a maximum point of the function $u^* - \phi_{\alpha,\beta}$ on $\overline{\Omega}_i$. It is easily seen that

$$\lim_{\alpha,\beta \to \infty} x_{\alpha,\beta} = y \quad \text{and} \quad \lim_{\alpha,\beta \to \infty} u^*(x_{\alpha,\beta}) = u^*(y).$$

Fix $r > 0$ so that $\text{dist}(B_r(y), c_i(0)) > 0$ and hence, $\inf_{B_r(y) \cap \Omega_i} |DH| > 0$. We fix $\alpha_0 > 0$ so that if $\alpha, \beta \in (\alpha_0, \infty)$, then $x_{\alpha,\beta} \in B_r(y)$.
For $x \in B_r(y) \cap \overline{\Omega_1}$, we compute that
\[
\lambda u^*(x) - \varepsilon^{-1} b(x) \cdot D\phi_{\alpha,\beta}(x) + \nu |D\phi_{\alpha,\beta}(x)| - M
= \lambda u^*(x) - \varepsilon^{-1} b(x) \cdot \left(2(\alpha(x-y) + \beta \frac{H(x)-h_i}{H(x)-h_{i,1}} \cdot DH(x)\right)
+ \nu \left(2\alpha(x-y) + \beta \frac{H(x)-h_i}{H(x)-h_{i,1}} \cdot DH(x)\right) - M
= \lambda u^*(x) - 2\alpha \varepsilon^{-1} b(x) \cdot (x-y) + \nu \left(2\alpha(x-y) + \beta \frac{H(x)-h_i}{H(x)-h_{i,1}} \cdot DH(x)\right) - M
= \lambda u^*(x) - 2\alpha \varepsilon^{-1} r \|b\|_{\infty,\Omega} + \beta \nu \inf_{B_r(y) \cap \Omega_1} |DH| - 2\alpha \nu r - M.
\]
Hence, for any $\alpha > \alpha_0$, we may choose $\beta = \beta(\alpha) > \alpha$ so that
\[
\lambda u^*(x_{\alpha,\beta}) - \varepsilon^{-1} b(x_{\alpha,\beta}) \cdot D\phi_{\alpha,\beta}(x_{\alpha,\beta}) + \nu |D\phi_{\alpha,\beta}(x_{\alpha,\beta})| - M > 0.
\]
Now, we deduce from (13) that for any $\alpha > \alpha_0$,
\[
x_{\alpha,\beta}(\alpha) \in \partial \Omega \text{ and } u^*(x_{\alpha,\beta}(\alpha)) \leq g(x_{\alpha,\beta}(\alpha)).
\]
Sending $\alpha \to \infty$, we conclude that $u^*(y) \leq g(y)$.

Lemma 6.3. For every $i \in I_0$,
\[
v_i^+(h_i) \leq \min_{\partial \Omega} g, \tag{14}
\]

Proof. We give the proof of (14) only for $i = 0$ since we can prove the others similarly.

Fix any $h \in J_0$ and $y \in c_0(h)$. By Proposition 5.1, we have $v_0^+(h) = v^+(y)$. We select sequences of $\varepsilon_k > 0$, $y_k \in \Omega_0$, and $v_k \in \mathcal{S}_{\varepsilon_k}$, with $k \in \mathbb{N}$, so that
\[
\lim_{k \to \infty} (\varepsilon_k, y_k, u_k^+(y_k)) = (0, y, v^+(y)).
\]
We set $\gamma_k = H(y_k) \in J_0$ for $k \in \mathbb{N}$.

Let $z \in \partial_0 \Omega$ be a minimum point of $g$ over $\partial_0 \Omega$, and fix $k \in \mathbb{N}$. Consider the initial value problem
\[
\dot{Z}(t) = \frac{1}{\varepsilon_k} b(Z(t)) + \nu F(Z(t)) \quad \text{and} \quad Z(0) = z, \tag{15}
\]
where $F(x) := DH(x)/|DH(x)|$. This problem has a unique solution $Z(t)$ as long as $Z(t)$ is away from any of critical points of $H$. Let $I$ be the maximal existence interval of the solution $Z(t)$.

Note that
\[
\frac{d}{dt} |DH(Z(t))| = DH(Z(t)) \cdot \dot{Z}(t) = \nu |DH(Z(t))| > 0 \quad \text{for all } t \in I, \tag{16}
\]
and hence the function $t \mapsto H(Z(t))$ is increasing in $I$. Since the origin is the only critical point of $H$ in $\overline{\Omega_0}$ and $H(0) = 0$, we deduce that there is $\sigma \in I$, with $\sigma < 0$, such that $0 < H(Z(\sigma)) = \gamma_k$. Moreover, we have $Z(t) \in \Omega_0$ for all $t \in (\sigma, 0)$.

We may assume, by reselecting the sequence $\{(\varepsilon_k, y_k, u_k)\}_{k \in \mathbb{N}}$ if necessary, that
\[
\gamma_k > h_0/2 \text{ for all } k \in \mathbb{N}. \text{ There exists a constant } \delta > 0 \text{ such that } |DH(x)| > \delta \text{ for all } x \in \Omega_0 \text{ satisfying } H(x) > h_0/2. \text{ It follows from (16) that}
\]
\[
h_0 - \gamma_k \geq \nu \delta |\sigma|. \tag{17}
\]
Note that \( u_k^* \) is a viscosity subsolution of
\[
\lambda u_k^* - \left( \frac{b}{\varepsilon_k} + \nu F \right) \cdot Du_k^* - M = 0 \quad \text{in } \Omega.
\]
Set \( z_k = Z(\sigma) \). By Proposition A.1, we obtain
\[
e^{-\lambda \sigma} u_k^*(z_k) \leq e^{-\lambda \sigma} u_k^*(Z(t)) + \int_{\sigma}^t e^{-\lambda \sigma} M \, ds \quad \text{for all } t \in (\sigma, 0),
\]
which implies, in the limit as \( t \to 0^- \), that
\[
u_k(z_k) \leq e^{\lambda \sigma} (u_k^*(z) + M|\sigma|) \leq u_k^*(z) + C_M (1 - e^{-\lambda|\sigma|}) + M|\sigma|.
Combining this with Lemmas 6.1 and 6.2, we get
\[
u_k(y_k) \leq \varepsilon_k CT_0(\gamma_k) + u_k^*(z_k) \leq \varepsilon_k C_0(\gamma_k) + g(z) + (\lambda C_M + M)|\sigma|
for some constant \( C \), and moreover, by (17),
\[
u_k(y_k) \leq \varepsilon_k C_0(\gamma_k) + g(z) + (\lambda C_M + M)\sigma^{-1}(h_0 - \gamma_k),
\]
Sending \( k \to \infty \) yields
\[
v_0^+(h) = v^+(y) \leq g(z) + (\lambda C_M + M)\sigma^{-1}(h_0 - h).
Consequently,
\[
v_0^+(h_0) = \lim_{\delta \to h_0} v_0^+(h) \leq g(z) = \min_{\partial \Omega} g.
\]
The next lemma is proved in the proof of [13, Theorem 3.6]. For any \( \alpha < \beta \) and \( i \in I_0 \), we write \( \Omega_i(\alpha, \beta) \) and \( \overline{\Omega_i}(\alpha, \beta) \) for the sets \( \{ x \in \Omega_i \mid \alpha < H(x) < \beta \} \) and \( \{ x \in \overline{\Omega_i} \mid \alpha \leq H(x) \leq \beta \} \), respectively.

**Lemma 6.4.** Let \( i \in I_0 \), \( h \in \overline{J}_i \setminus \{0\} \), and \( q \in \mathbb{R} \). For any \( \delta > 0 \), there exist an interval \( [\alpha, \beta] \subset \overline{J}_i \setminus \{0\} \) and \( \psi \in C^1(\overline{\Omega_i}(\alpha, \beta)) \) such that \( [\alpha, \beta] \) is a neighborhood of \( h \), relative to \( \overline{J}_i \setminus \{0\} \), and
\[
| -b(x) \cdot D\psi(x) + G(x, qDH(x)) - \overline{G_0}(H(x), q) | \leq \delta \quad \text{for all } x \in \overline{\Omega_i}(\alpha, \beta).
\]

**Proof of Theorem 5.2.** We follow the proof of [13, Theorem 3.6], which is based on the perturbed test function method due to [5]. We show that \( v_0^- \) is a viscosity supersolution of (HJ0) and (BC0). A parallel argument shows that \( v_i^- \), with \( i \in I_1 \), is a viscosity supersolution of (HJi) and (BCi), the detail of which we omit presenting here.

Let \( \phi \in C^1(\overline{J}_0 \setminus \{0\}) \) and assume that \( v_0^- - \phi \) has a strict minimum at \( \hat{h} \). Since the treatment for the case when \( \hat{h} < h_0 \) is similar to and easier than the case when \( \hat{h} = h_0 \), we, henceforth, consider only the case when \( \hat{h} = h_0 \).

We need to show that either
\[
\lambda v_0^- (\hat{h}) + \overline{G_0}(\hat{h}, \phi'(\hat{h})) \geq 0 \quad \text{or} \quad v_0^- (\hat{h}) \geq \min_{\partial \Omega} g.
\]
For this, we suppose that
\[
v_0^- (\hat{h}) < \min_{\partial \Omega} g, \quad (18)
\]
and prove that
\[
\lambda v_0^- (\hat{h}) + \overline{G_0}(\hat{h}, \phi'(\hat{h})) \geq 0. \quad (19)
\]
Fix any \( \delta > 0 \) and set \( q = \phi'(\hat{h}) \). By Lemma 6.4, there exist \( \alpha \in (0, \hat{h}) \) and \( \psi \in C^1(\overline{\Omega_0}(\alpha, \hat{h})) \) such that
\[
- b(x) \cdot D\psi(x) + G(x, qDH(x)) - \overline{G_0}(H(x), q) < \delta \quad \text{for all } x \in \overline{\Omega_0}(\alpha, \hat{h}).
\]
Recalling that $v_0^-(\hat{h}) = \min_{x \in c_0(h)} v^-(x)$, we select $\hat{x} \in c_0(h)$ so that $v_0^-(\hat{h}) = v^-(\hat{x})$. We next select sequences of $\varepsilon_k > 0$, $x_k \in \bar{\Omega}_0(\alpha, \hat{h})$, and $u_k \in \mathcal{S}_{\varepsilon_k}$, with $k \in \mathbb{N}$, so that
\[
\lim_{k \to \infty} (\varepsilon_k, x_k, (u_k)_*(x_k)) = (0, \hat{x}, v^-(\hat{x})).
\]
For $k \in \mathbb{N}$, we consider the function
\[
\Phi_k(x) := (u_k)_*(x) - \phi(H(x)) - \varepsilon_k \psi(x) \quad \text{on} \quad \bar{\Omega}_i(\alpha, \hat{h}).
\]
This function is lower semicontinuous and has a minimum at some point $y_k$. We may assume, by relabeling the sequences if needed, that $\{y_k\}_{k \in \mathbb{N}}$ converges to some point $y_0 \in \bar{\Omega}_0(\alpha, \hat{h})$.

Noting that $\Phi_k(x_k) \geq \Phi_k(y_k)$ for all $k \in \mathbb{N}$,
\[
\lim_{k \to \infty} \Phi_k(x_k) = v^-(\hat{x}) - \phi(H(\hat{x})) = (v_0^- - \phi)(\hat{h}),
\]
and
\[
\liminf_{k \to \infty} \Phi_k(y_k) \geq v^-(y_0) - \phi(H(y_0)) \geq (v_0^- - \phi)(\hat{h}) = (\Phi_k(y_k) + \varepsilon_k - \phi)(y_k),
\]
we deduce that
\[
\lim_{k \to \infty} ((u_k)_*(y_k) - \phi(H(y_k))) = (v_0^- - \phi)(\hat{h}), \quad \lim_{k \to \infty} (u_k)_*(y_k) = v_0^-(\hat{h}) \quad \text{and} \quad y_0 \in c_0(\hat{h}).
\]
Thanks to (18), we may assume without loss of generality that
\[
(u_k)_*(y_k) < \min_{\partial_d \Omega} g,
\]
and, by the viscosity property of $(u_k)_*$ and by choice of $\psi$, we obtain
\[
0 \leq \lambda(u_k)_*(y_k) - \frac{1}{\varepsilon_k} b(y_k) \cdot (\phi'(H(y_k))DH(y_k) + \varepsilon_k D\psi(y_k))
+ G(y_k, \phi'(H(y_k))DH(y_k) + \varepsilon_k D\psi(y_k))
= \lambda(u_k)_*(y_k) - b(y_k) \cdot D\psi(y_k)) + G(y_k, \phi'(H(y_k))DH(y_k) + \varepsilon_k D\psi(y_k))
\leq \lambda(u_k)_*(y_k) + \delta - G(y_k, qDH(y_k)) + \mathcal{G}_0(H(y_k), q)
+ G(y_k, \phi'(H(y_k))DH(y_k) + \varepsilon_k D\psi(y_k)).
\]
Hence, in the limit as $k \to \infty$, we obtain
\[
-\delta \leq \lambda v_0^- (\hat{h}) + \mathcal{G}_0(\hat{h}, q),
\]
which proves (19).

According to Lemma 6.3, we have $v_i^+(h_i) \leq \min_{\partial_d \Omega} g$ for all $i \in I_0$. Hence, it remains to show that $v_i^+$, with $i \in I_0$, is a viscosity subsolution of $(H_i)$. The argument presented above is easily adapted to show this, the detail of which we leave it to the reader to check.

7. The maximality of the viscosity solution $(v_0^+ \ldots, v_{N-1}^+)$. Due to Theorem 5.2 and Lemma 4.4, the functions $v_i^+$, with $i \in I_0$, are continuous on $I_0 \setminus \{0\}$ and have the limit $\lim_{I_0, \hat{h} \to 0} v_i^+(\hat{h}) \in \mathbb{R}$. We set
\[
d(v_i^+) = \lim_{I_0, \hat{h} \to 0} v_i^+(\hat{h}) \quad \text{for} \quad i \in I_0.
\]

Lemma 7.1. For any $i \in I_1$,
\[
\inf_{x \in c_i(0)} v^+(x) \geq \max\{d(v_i^+), d(v_0^+)\}.
\]
Proof. Fix any $i \in \mathcal{I}$ and $x \in c_i(0)$. Fix any $\delta > 0$, and choose $r > 0$ so that 
\[ v^+(x) + \delta > \sup \{ u(y) \mid u \in \mathcal{S}_\varepsilon, \ y \in \overline{B}_r(x), \ 0 < \varepsilon < r \}. \]

We choose $h_{i,\delta} \in J_i$ and $h_{0,\delta} \in J_0$ so that 
\[ B_r(x) \cap c_i(h_{i,\delta}) \neq \emptyset \text{ and } B_r(x) \cap c_0(h_{0,\delta}) \neq \emptyset, \]
and that 
\[ v^+_i(h_{i,\delta}) + \delta > d(v^+_i) \text{ and } v^+_0(h_{0,\delta}) + \delta > d(v^+_0). \]

By Proposition 5.1, we have 
\[ v^+_i(h_{i,\delta}) = v^+(x) \text{ for all } x \in c_i(h_{i,\delta}). \]
Hence, we may choose $x_{\delta} \in B_r(x)$ and $u_{\delta} \in \mathcal{S}_{\varepsilon_{\delta}},$ with $0 < \varepsilon_{\delta} < r,$ such that 
\[ u_{\delta}(x_{\delta}) + \delta > v^+_i(h_{i,\delta}). \]
Combining these observations, we obtain 
\[ v^+(x) + 3\delta > u_{\delta}(x_{\delta}) + 2\delta > v^+_i(h_{i,\delta}) + \delta > d(v^+_i), \]
from which we conclude that 
\[ \inf_{x \in c_i(0)} v^+(x) \geq d(v^+_i). \]

An argument similar to the above yields 
\[ \inf_{x \in c_i(0)} v^+(x) \geq d(v^+_0), \]
which completes the proof. \hfill \box

**Lemma 7.2.** We have 
\[ \max_{x \in c_0(0)} v^+(x) \leq d(v^+_0). \quad (20) \]

We need the following two lemmas for the proof of Lemma 7.2.

**Lemma 7.3.** There exists a constant $A_0 > 0$ such that 
\[ |DH(x)| \geq A_0 |H(x)|^\alpha \text{ for all } x \in \Omega, \]
where $\alpha := n/(m + 2) \in (0, 1)$ and the constants $n, m$ are from (H3).

**Proof.** Let $m, n, A_1, A_2,$ and $V$ be the constants and neighborhood of the origin from (H3), respectively. We may assume that $V = B_R$ for some $R > 0$. Since $H(0) = 0$ and $DH(0) = 0,$ we deduce by (H3) that 
\[ |H(x)| \leq C|x|^{m+2} \text{ for all } x \in B_R \]
and some constant $C > 0$, and consequently, 
\[ |DH(x)| \geq A_2 |x|^n \geq A_2 \left( \frac{|H(x)|}{C} \right)^{\frac{n}{m+2}} = \frac{A_2}{C^{\alpha}} |H(x)|^\alpha \text{ for all } x \in B_R. \]
Noting that 
\[ \min_{x \in \partial B_R} \frac{|DH(x)|}{|H(x)|^\alpha} > 0, \]
we conclude that for some constant $A_0 > 0,$ 
\[ |DH(x)| \geq A_0 |H(x)|^\alpha \text{ for all } x \in \Omega. \] \hfill \box
Hence, we obtain each $h$ and that, since $\rho > 1$ and $A_3 > 0$ such that

$$m_H(r) \geq A_3 r^\rho \quad \text{for all } r \in (0, R).$$

(21)

**Proof of Lemma 7.2.** Fix any $\rho > 1$ and $A_3 > 0$ so that $m_H(r) > m_H(s)$ for all $r > s$, and choose a point $x_s \in \overline{\Omega} \cap \overline{B}_s$. Let $H(x) = H(x_s)$.

Since $DH(x) \neq 0$ for $x \in \Omega \setminus \{0\}$, it follows that $H$ does not take a local maximum at any point in $\Omega \setminus \{0\}$ and hence, $m_H(r) > m_H(s)$. More generally, the function $m_H$ is increasing in $(0, R)$.

Solve the initial value problem

$$\dot{Y}(t) = F(Y(t)) \quad \text{and} \quad Y(0) = x_s,$$

where $F$ is the function given by $F(x) := DH(x)/|DH(x)|$. We note that

$$\frac{d}{dt} H(Y(t)) = |DH(Y(t))| > 0 \quad \text{for all } t \geq 0$$

(22)

as far as $Y(t)$ exists, and we infer that $H(Y(t)) \geq m_H(s)$ for all $t \geq 0$, and that there exists $\tau > 0$ such that $H(Y(\tau)) = m_H(r)$. From these, we deduce, together with the strict monotonicity of $m_H$, that $|Y(t)| \geq s$ for all $t \geq 0$, and $|Y(\tau)| = r$.

Noting by Lemma 7.3 that $|DH(x)| \geq A_0 |H(x)|$ for all $x \in \Omega$ and some constants $A_0 > 0$ and $\alpha \in (0, 1)$, we compute by (22) that

$$m_H(r)^{1-\alpha} - m_H(s)^{1-\alpha} = H(Y(\tau))^{1-\alpha} - H(Y(0))^{1-\alpha}$$

$$= (1 - \alpha) \int_0^\tau H(Y(t))^{-\alpha} \frac{d}{dt} H(Y(t)) dt \geq (1 - \alpha) A_0 \tau.$$

and that, since $|\dot{Y}(t)| = 1$,

$$r - s \leq |Y(\tau) - Y(0)| \leq |Y(\tau) - Y(0)| \leq \int_0^\tau |\dot{Y}(t)| dt = \tau.$$

Hence, we obtain

$$m_H(r)^{1-\alpha} - m_H(s)^{1-\alpha} \geq (1 - \alpha) A_0 (r - s).$$

Sending $s \to 0+$ yields

$$m_H(r) \geq ((1 - \alpha) A_0)^{1/\rho} = A_3 r^\rho,$$

where $\rho := 1/(1 - \alpha)$ and $A_3 := ((1 - \alpha) A_0)^\rho$, which completes the proof.

**Proof of Lemma 7.2.** Fix any $\eta > 0$ and choose $\delta_0 \in J_0 = (0, h_0)$ so that

$$d(v_{i0}^+) + \eta > v_{i0}^+(h) \quad \text{for all } h \in (0, \delta_0).$$

We may assume that $\delta_0 < \eta$ and $\delta_0 < \bar{h}$. By the definition of $v_{i0}^+$, we infer that for each $h \in (0, \delta_0)$, there is $\varepsilon(h) > 0$ such that if $h \in (0, \delta_0)$, then

$$d(v_{i0}^+) + \eta > \sup\{u^*(x) \mid u \in S_\varepsilon, 0 < \varepsilon < \varepsilon(h), x \in c_0(h)\}.$$
Fix any $\delta \in (0, \delta_0)$. We choose a continuous nondecreasing function $f : (-\infty, \delta) \to \mathbb{R}$ so that
\[ f(r) = 1 \quad \text{for } r < \delta/2 \quad \text{and} \quad \lim_{r \to \delta^-} f(r) = \infty. \]
Define $g : (-\infty, \delta) \to \mathbb{R}$ by
\[ g(r) = \int_0^r f(t) \, dt. \]
Observe that
\[ g(r) = r \quad \text{for } r \leq \delta/2 \quad \text{and} \quad |r| \leq |g(r)| \leq g'(r)|r| \quad \text{for } r \in (-\infty, \delta). \]
According to Lemma 7.3, there are constants $\alpha \in (0, 1)$ and $A_0 > 0$ such that
\[ |DH(x)| \geq A_0 |H(x)|^\alpha \quad \text{for all } x \in \Omega. \tag{24} \]
Let $\beta \in (0, 1)$ be a constant to be fixed later. We define the function $w \in C(\Omega(\delta) \cup c_0(\delta))$ by
\[ w(x) = g(-H(x))|H(x)|^{\beta-1} + \delta^\beta + d(v_0^+) + \eta. \]
Observe that
\[ w \in C^1(\Omega(\delta) \setminus c_0(0)), \]
\[ \partial \Omega_0(\delta) = c_0(0) \cup \bigcup_{j \in I_1} c_j(-\delta), \]
\[ w(x) = d(v_0^+) + \eta \quad \text{for all } x \in c_0(\delta), \]
\[ \lim_{\Omega(\delta) \ni y \to x} w(y) = \infty \quad \text{uniformly for } x \in \bigcup_{j \in I_1} c_j(-\delta). \]
Compute that for $x \in \Omega(\delta) \setminus c_0(0)$,
\[ Dw(x) = \left[ -g'(-H)|H|^{\beta-1} + (\beta - 1)g(-H)|H|^{\beta-2} \right] DH, \]
\[ = \left[ g'(-H)(-H) + (\beta - 1)g(-H) \right] |H|^{\beta-3} HDH \]
and moreover,
\[ |Dw(x)| \geq (g'(-H)|H| - (1 - \beta)g(-H)) |H|^{-\beta-2} |DH| \]
\[ = \beta g(-H)|H|^{\beta-2} |DH|. \]
Combining this with (24) yields
\[ |Dw(x)| \geq \beta A_0 |g(-H(x))| |H(x)|^{\alpha + \beta - 2} \geq \beta A_0 |H(x)|^{\alpha + \beta - 1}. \]
Moreover, using (G5), we compute
\[ \lambda w(x) - \varepsilon^{-1} b(x) \cdot Dw(x) + G(x, Dw(x)) \geq \lambda d(v_0^+) + \nu \beta A_0 |H(x)|^{\alpha + \beta - 1} - M \tag{25} \]
for all $x \in \Omega(\delta) \setminus c_0(0)$.
We assume in what follows that $\beta > 0$ is sufficiently small so that
\[ \alpha + \beta - 1 < 0. \tag{26} \]
In view of (25), by choosing $\delta \in (0, \delta_0)$ sufficiently small, we may assume that
\[ \lambda w - \varepsilon^{-1} b(x) \cdot Dw(x) + G(x, Dw(x)) > 0 \quad \text{for all } x \in \Omega(\delta) \setminus c_0(0). \tag{27} \]
By Lemma 7.4, we have
\[ m_H(r) \geq A_3 r^\alpha \quad \text{for all } r \in (0, R), \tag{28} \]
where $\rho > 1$, $A_3 > 0$, and $R > 0$ are constants. In addition to (26), we assume hereafter that $\beta < 1/\rho$. That is, we fix $\beta > 0$ so that
$$\beta < \min\{\rho^{-1}, 1 - \alpha\}.$$ 

We claim that
$$D^- w(x) = \emptyset \quad \text{for all } x \in c_0(0),$$
where $D^- w(x)$ denotes the subdifferential of $w$ at $x$.

To see this, we fix any $x \in c_0(0)$. By contradiction, we suppose that $D^- w(x) \neq \emptyset$. Let $\phi \in C^1(\Omega(\delta))$ be a function such that $w - \phi$ attains a minimum at $x$. If $x \neq 0$, then
$$x + tDH(x) \in \Omega_0 \cap \Omega(\delta) \quad \text{for all } t \in (0, t_0)$$
and some $t_0 > 0$, and consequently, we have for $t \in (0, t_0)$,
$$(w - \phi)(x) \leq (w - \phi)(x + tDH(x)).$$
For sufficiently small $t > 0$, this reads
$$t^{-\beta} (\phi(x) - \phi(x + tDH(x))) \geq t^{-\beta} (w(x) - w(x + tDH(x)))$$
$$= t^{-\beta} H(x + tDH(x))^\beta,$$
which yields, in the limit as $t \to 0+$,
$$0 \geq |DH(x)|^{2\beta}.$$ 
This is a contradiction. Otherwise, we have $x = 0$ and, for any $y \in \Omega(\delta)$,
$$w(x) - w(y) \leq \phi(x) - \phi(y).$$
Moreover, for any $y \in \Omega(\delta) \cap \Omega_0$, we have
$$H(y)^\beta \leq \phi(x) - \phi(y),$$
and for any $r \in (0, \delta \wedge R)$,
$$m_H(r)^\beta \leq \max_{y \in B_r \cap \Omega_0} (\phi(x) - \phi(y)).$$
Since $m_H(r)^\beta \geq A_3^\beta r^{\beta \rho}$ by (28) and $\beta \rho < 1$, we obtain from the above
$$A_3^\beta \leq \lim_{r \to 0^+} r^{-\beta \rho} \max_{y \in B_r \cap \Omega_0} (\phi(x) - \phi(y)) = 0,$$
which is a contradiction. Thus, we conclude that (29) is valid, and moreover from (27) and (29) that $w$ is a viscosity supersolution of
$$\lambda w - \varepsilon^{-1}b \cdot Dw + G(x, Dw) \geq 0 \quad \text{in } \Omega(\delta).$$
Recalling (23), we deduce by the comparison theorem that for any $\varepsilon \in (0, \varepsilon(\delta))$ and $u \in S_\varepsilon$, we have
$$u^*(x) \leq w(x) \quad \text{for all } x \in \Omega(\delta),$$
which yields
$$u^+(x) \leq w(x) = \delta^\beta + d(v_0^+) + \eta \quad \text{for all } x \in c_0(0).$$
This ensures that $u^+(x) \leq d(v_0^+)$ for all $x \in c_0(0)$.

**Lemma 7.5.** For every $i \in I_1$,
$$d(v_0^+) \leq d(v_i^+).$$
Proof. Fix $i \in I_1$, $z \in c_i(0) \setminus \{0\}$, and $\delta > 0$ so that $\delta < h_0 \wedge |h_i|$. We choose sequences of $\varepsilon_k > 0$, $u_k \in S_{z_k}$, and $x_k \in \Omega_0$ such that as $k \to \infty$,

$$(\varepsilon_k, H(x_k), u_k^*(x_k)) \to (0, \delta, v_0^*(\delta)).$$

We set $\gamma_k = H(x_k)$ and, by relabeling the sequences if needed, we may assume that $\gamma_k < 2\delta$ for all $k \in \mathbb{N}$.

Fix $k \in \mathbb{N}$ and consider the initial value problem

$$\dot{Y}_k(t) = \frac{1}{\varepsilon_k} b(Y_k(t)) - \nu F(Y_k(t)) \quad \text{and} \quad Y_k(0) = z,$$

where the function $F$ is given by $F(x) := DH(x)/|DH(x)|$. Let $I_k$ denote the maximal interval of existence of the solution $Y_k(t)$. Noting that

$$\frac{d}{dt} H(Y_k(t)) = -\nu |DH(Y_k(t))| \quad \text{for } t \in I_k,$$

we deduce that there exist $\sigma_k, \tau_k \in I_k$ such that $\sigma_k < 0 < \tau_k$,

$$H(Y_k(\sigma_k)) = \gamma_k \quad \text{and} \quad H(Y_k(\tau_k)) = -\delta.$$

According to Lemma 7.3, there are constants $\alpha \in (0, 1)$ and $A_0 > 0$ such that $|DH(x)| \geq A_0|H(x)|^\alpha$ for $x \in \Omega$.

Noting that $Y_k(t) \in \Omega$ for $t \in [\sigma_k, \tau_k]$, we compute that for $t \in (\sigma_k, 0)$,

$$\frac{d}{dt} H(Y_k(t))^{1-\alpha} = (1 - \alpha) H(Y_k(t))^{-\alpha} \frac{d}{dt} H(Y_k(t)) \leq -(1 - \alpha) \nu A_0,$$

and, after integration over $(\sigma_k, 0)$,

$$-\gamma_k^{1-\alpha} \leq -(1 - \alpha) \nu A_0 |\sigma_k|,$$

which ensures that

$$- \sigma_k = |\sigma_k| \leq \frac{\gamma_k^{1-\alpha}}{(1 - \alpha) \nu A_0} \leq \frac{(2\delta)^{1-\alpha}}{(1 - \alpha) \nu A_0}. \quad (30)$$

Similarly, we deduce that

$$\tau_k \leq \frac{\delta^{1-\alpha}}{(1 - \alpha) \nu A_0}. \quad (31)$$

Since $|F(x)| = 1$ and, hence, $u_k^*$ is a viscosity subsolution of

$$\lambda u_k^* = \left(\frac{b}{\varepsilon_k} - \nu F\right) \cdot Du_k^* - M = 0 \quad \text{in } \Omega \setminus \{0\}$$

by (G5), we may apply Proposition A.1, to obtain

$$u_k^*(Y_k(\sigma_k)) \leq e^{\lambda \sigma_k} \left(e^{-\lambda \tau_k} u_k^*(Y_k(\tau_k)) + \int_{\tau_k}^{\sigma_k} M e^{-\lambda t} dt \right).$$

Recalling that $\gamma_k = H(x_k) = H(Y_k(\sigma_k))$, we combine the above with Lemma 6.1, to get

$$u_k^*(x_k) \leq C\varepsilon_k T_0(\gamma_k) + e^{\lambda(\sigma_k - \tau_k)} u_k^*(Y_k(\tau_k)) + \lambda^{-1} M \left(1 - e^{\lambda(\sigma_k - \tau_k)}\right)$$

$$\leq C\varepsilon_k T_0(\gamma_k) + u_k^*(Y_k(\tau_k)) + C_M(1 + \lambda^{-1}) \left(1 - e^{\lambda(\sigma_k - \tau_k)}\right),$$

and, moreover, by (30) and (31),

$$u_k^*(x_k) \leq C\varepsilon_k T_0(\gamma_k) + \max_{c_i(\cdot, \cdot)} u_k^* + C_M(1 + \lambda^{-1}) \left\{1 - \exp \left(-\lambda \left(\frac{(2\delta)^{1-\alpha} + \delta^{1-\alpha}}{(1 - \alpha) \nu A_0}\right)\right)\right\}.$$
Lemma 7.7. \( \text{For any} \ k \to \infty \ \text{yields} \)

\[
v_i^+ (\delta) \leq v^+_i (-\delta) + C_M (1 + \lambda^{-1}) \left\{ 1 - \exp \left( -\lambda \left( \frac{(2\delta)^{1-\alpha} + \delta^{1-\alpha}}{(1-\alpha)\nu A_0} \right) \right) \right\},
\]

and hence, \( d(v_i^+ ) \leq d(v_i^+) \). \( \square \)

**Corollary 7.6.** \( \text{For every} \ \ i \in \mathcal{I}_0, \)

\[
v^+ (x) = v_i^+ (0) = d(v_i^+) \quad \text{for all} \ x \in c_0(0).
\]

**Proof.** Combining Lemmas 7.1, 7.2, and 7.5 yields

\[
\max \{ d(v_i^+ ), d(v_i^+) \} \leq \inf_{c_i(0)} v^+ \leq \max v^+ \leq d(v_0^+) \leq d(v_i^+) \quad \text{for all} \ i \in \mathcal{I}_1,
\]

which shows that

\[
\inf_{c_i(0)} v^+ = \max v^+ = d(v_i^+) = d(v_i^+) \quad \text{for all} \ i \in \mathcal{I}_1.
\]

Since \( c_0(0) = \bigcup_{i \in \mathcal{I}_1} c_i(0) \), we conclude that

\[
v^+ (x) = d(v_i^+) \quad \text{for all} \ x \in c_0(0), \ i \in \mathcal{I}_0,
\]

and, by the definition of \( v_i^+(0) \),

\[
v_i^+ (0) = d(v_i^+) \quad \text{for all} \ i \in \mathcal{I}_0. \quad \square
\]

For the proof of Theorem 5.3, we argue below as in the proof of [13, Lemma 3.8]. We need the following lemma, the proof of which we refer to [13, Lemma 4.4].

**Lemma 7.7.** \( \text{For any} \ \eta > 0, \ \text{there exist a constant} \ \delta \in (0, \bar{h}) \ \text{and a function} \ \psi \in C^1(\Omega(\delta)) \ \text{such that} \)

\[
- b \cdot D\psi + G(x, 0) < G(0,0) + \eta \quad \text{in} \ \Omega(\delta).
\]

**Proof of Theorem 5.3.** We set \( d = d(u_0, \ldots, u_{N-1}) \), and note by the maximality of \( (u_0, \ldots, u_{N-1}) \) Corollary 7.6, and Theorem 5.2 that

\[
v^- (x) \leq v^+ (x) = d(v_i^+) = \cdots = d(v_N^+) \leq d \quad \text{for all} \ x \in c_0(0).
\]

It remains to show that

\[
v^- (x) \geq d \quad \text{for all} \ x \in c_0(0). \quad (32)
\]

To prove (32), we argue by contradiction, and suppose that \( \min_{c_0(0)} v^- < d \). We set \( \kappa := \min_{c_0(0)} v^- \).

For any \( i \in \mathcal{I}_0 \), we have

\[
\lambda u_i (h) + \min_{q \in \mathbb{R}} \overline{G}_i (h, q) \leq 0 \quad \text{for all} \ h \in J_i,
\]

and, hence, by Lemma 4.3,

\[
\lambda \kappa + \lim_{J_i \ni h \to 0} \overline{G}_i (h,0) = \lambda \kappa + G(0,0) < 0. \quad (33)
\]

Combining this and the equi-continuity (see (ii) of Lemma 4.2) of \( q \mapsto \overline{G}_i (h, q) \), with \( h \in J_i \), we deduce that there exists \( \delta > 0 \) such that for all \( i \in \mathcal{I}_0 \) and \( h \in [-\delta, \delta] \cap J_i \),

\[
\lambda (\kappa + \delta^2) + \overline{G}_i (h, \delta) < -\delta \quad \text{and} \quad u_i (h) \geq \kappa + \delta^2,
\]
where \( \delta_i = \delta \) if \( i = 0 \) and \( = -\delta \) otherwise, which implies that for all \( i \in \mathcal{I}_0 \) and \( h \in [-\delta, \delta] \cap J_i \),

\[
\lambda(\kappa + \delta_i h) + \overline{G}(h, \delta_i) < -\delta \quad \text{and} \quad u_i(h) \geq \kappa + \delta_i h
\]  

(34)

By (33), we may assume as well that

\[
\lambda \kappa + G(0, 0) < -\delta.
\]

According to Lemma 7.7, we may choose, after replacing \( \delta > 0 \) by a smaller number if necessary, a function \( \psi \in C(\overline{\Omega}(\delta)) \) such that

\[
-b \cdot D\psi + G(x, 0) < G(0, 0) + \delta \quad \text{in } \Omega(\delta).
\]

This yields

\[
\lambda \kappa - b \cdot D\psi + G(x, 0) < 0 \quad \text{in } \Omega(\delta).
\]  

(35)

For each \( i \in \mathcal{I}_0 \), we define the function \( w_i \) on \( \bar{J}_i \) by

\[
w_i(h) = \begin{cases} 
\delta_i h + \kappa & \text{for } h \in \bar{J}_i \cap [-\delta, \delta], \\
u_i(h) - u_i(\delta_i) + \delta^2 + \kappa & \text{for } h \in \bar{J}_i \setminus [-\delta, \delta].
\end{cases}
\]  

(36)

By Lemma 4.4, the function \( u_i \) is locally Lipschitz continuous in \( \bar{J}_i \setminus \{0\} \) and, hence, \( w_i \) is Lipschitz continuous on \( \bar{J}_i \). Moreover, thanks to the convexity of \( \overline{G}(h, q) \) in \( q \), i.e., (iii) of Lemmas 4.2, the function \( w_i \) is a viscosity subsolution of \((\text{HJ}_i)\) and \((\text{BC}_i)\). Note that \( v_i^- \) is a viscosity supersolution of \((\text{HJ}_i)\) and \((\text{BC}_i)\) and satisfies \( \liminf_{I_i \ni h \to 0} v_i^-(h) \geq \kappa \). Hence, by applying Lemma 4.7, we obtain

\[
w_i(h) \leq v_i^-(h) \quad \text{for all } h \in J_i \text{ and } i \in \mathcal{I}_0.
\]

Fix any \( \mu \in (0, \delta^2) \). The inequality above allows us to choose \( \varepsilon_0 > 0 \) so that for any \( \varepsilon \in (0, \varepsilon_0) \) and \( u \in S_\varepsilon \),

\[
\sigma^2 + \kappa - \mu < u_*(x) \quad \text{for all } x \in \partial \Omega(\delta).
\]  

(37)

We next choose a constant \( a \in (\kappa, \delta^2 + \kappa - \mu) \), define the function \( z^\varepsilon \) on \( \overline{\Omega}(\delta) \) by

\[
z^\varepsilon(x) = a + \varepsilon \psi(x),
\]

and compute by (35) that for any \( x \in \Omega(\delta) \),

\[
\lambda z^\varepsilon(x) - \frac{1}{\varepsilon} b(x) \cdot Dz^\varepsilon(x) + G(x, Dz^\varepsilon(x)) = \lambda(a - \kappa) + \lambda \varepsilon \psi(x) + G(x, \varepsilon D\psi(x)) - G(x, 0).
\]

Reselecting \( \varepsilon_0 > 0 \) small enough if needed, we see that for any \( \varepsilon \in (0, \varepsilon_0) \), the function \( z^\varepsilon \) is a viscosity subsolution of \((\text{HJ}_i^\varepsilon)\) in \( \Omega(\delta) \). Moreover, we may assume that for any \( \varepsilon \in (0, \varepsilon_0) \),

\[
z^\varepsilon(x) \leq \delta^2 + \kappa - \mu \quad \text{on } \overline{\Omega}(\delta).
\]

Hence, by the comparison principle for \((\text{HJ}_i^\varepsilon)\) on \( \overline{\Omega}(\delta) \), we get

\[
z^\varepsilon(x) \leq u_*(x) \quad \text{for all } u \in S_\varepsilon \text{ and } x \in \overline{\Omega}(\delta),
\]

which yields a contradiction:

\[
\kappa < a \leq u^-(x) \quad \text{for all } x \in c_0(0).
\]

This completes the proof. \( \square \)
REFERENCES

[1] Y. Achdou and N. Tchou, Hamilton-Jacobi equations on networks as limits of singularly perturbed problems in optimal control: dimension reduction, Comm. Partial Differential Equations, 40 (2015), 652–693.

[2] Y. Achdou, F. Camilli, A. Cutrì and N. Tchou, Hamilton-Jacobi equations constrained on networks, NoDEA Nonlinear Differential Equations Appl., 20 (2013), 413–445.

[3] M. G. Crandall, H. Ishii and P.-L. Lions, User’s guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. (N.S.), 27 (1992), 1–67.

[4] M. G. Crandall and P.-L. Lions, Viscosity solutions of Hamilton-Jacobi equations, Trans. Amer. Math. Soc., 277 (1983), 1–42.

[5] L. C. Evans, The perturbed test function method for viscosity solutions of nonlinear PDE, Proc. Roy. Soc. Edinburgh Sect. A, 111 (1989), 359–375.

[6] M. I. Freidlin and A. D. Wentzell, Random perturbations of Hamiltonian systems, Mem. Amer. Math. Soc., 109 (1994), viii+82pp.

[7] G. Galise, C. Imbert and R. Monneau, A junction condition by specified homogenization and application to traffic lights, Anal. PDE, 8 (2015), 1891–1929.

[8] C. Imbert and R. Monneau, Flux-limited solutions for quasi-convex Hamilton-Jacobi equations on networks, Ann. Sci. Éc. Norm. Supér. (4), 50 (2017), 357–448.

[9] C. Imbert, R. Monneau and H. Zidani, A Hamilton-Jacobi approach to junction problems and application to traffic flows, ESAIM Control Optim. Calc. Var., 19 (2013), 129–166.

[10] H. Ishii, Perron’s method for Hamilton-Jacobi equations, Duke Math. J., 55 (1987), 369–384.

[11] H. Ishii, A boundary value problem of the Dirichlet type for Hamilton-Jacobi equations, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 16 (1989), 105–135.

[12] H. Ishii and P. E. Souganidis, A pde approach to small stochastic perturbations of Hamilton flows, J. Differential Equations, 252 (2012), 1748–1775.

[13] T. Kumagai, A perturbation problem involving singular perturbations of domains for Hamilton-Jacobi equations, Funkcial. Ekvac., 61 (2018), 377–427.

[14] T. Kumagai, An asymptotic analysis for Hamilton-Jacobi equations with large Hamiltonian drift terms, Adv. Calc. Var., 2017.

[15] T. Kumagai, A Study of Hamilton-Jacobi Equations with Large Hamiltonian Drift Terms, Ph.D, Waseda University, Tokyo, Japan, 2018.

[16] P.-L. Lions and P. Souganidis, Viscosity solutions for junctions: Well posedness and stability, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl., 27 (2016), 535–545.

[17] P.-L. Lions and P. Souganidis, Well-posedness for multi-dimensional junction problems with Kirchhoff-type conditions, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl., 28 (2017), 807–816.

Appendix.

Proposition A.1. Let $m \in \mathbb{N}$ be such that $m \geq 2$, $U$ an open subset of $\mathbb{R}^m$ and $E: U \to \mathbb{R}^m$ a Lipschitz continuous vector field. Let $v \in \text{USC}(U)$ be a viscosity subsolution of

$$\lambda v - E \cdot Dv - f = 0 \quad \text{in } U,$$

where $\lambda \geq 0$ is a given constant and $f \in C(U)$ be a given function. Let $c, d \in \mathbb{R}$ be such that $c < d$ and let $X: (c, d) \to U$ be a $C^1$-curve such that

$$\dot{X}(t) = E(X(t)) \quad \text{for all } t \in (c, d).$$

Set $w(t) = v(X(t))$ and $g(t) = f(X(t))$ for $t \in (c, d)$. Let $\sigma, \tau$ be real numbers such that $c < \sigma < \tau < d$. Then

$$e^{-\lambda \sigma} w(\sigma) \leq e^{-\lambda \tau} w(\tau) + \int_{\sigma}^{\tau} e^{-\lambda t} g(t) \, dt.$$

Proof. Set $I = (c, d)$. It is obvious that $w \in \text{USC}(I)$. We show first that $w$ is a viscosity subsolution of

$$\lambda w - w' - g = 0 \quad \text{in } I.$$

(A.1)
For this, let $\phi \in C^1(I)$ and assume that $w - \phi$ has a strict maximum at $\hat{t} \in I$. Set $\hat{x} = X(\hat{t})$ and choose $\delta > 0$ so that
\[ [\hat{t} - \delta, \hat{t} + \delta] \subset I \quad \text{and} \quad \overline{B}_\delta(\hat{x}) \subset U. \]

Fix any $\alpha > 0$ and consider the function
\[ \Phi_\alpha(t, x) := v(x) - \phi(t) - \alpha|x - X(t)|^2 \quad \text{on} \quad K := [\hat{t} - \delta, \hat{t} + \delta] \times \overline{B}_\delta(\hat{x}). \]

Let $(t^\alpha, x^\alpha) \in K$ be a maximum point of $\Phi_\alpha$. It is easily seen that, as $\alpha \to \infty$,
\[ (t^\alpha, x^\alpha) \to (\hat{t}, \hat{x}) \quad \text{and} \quad \alpha|x^\alpha - X(t^\alpha)|^2 \to 0. \]

Accordingly, by assuming $\alpha$ large enough, we may assume that $(t^\alpha, x^\alpha) \in (\hat{t} - \delta, \hat{t} + \delta) \times B_\delta(\hat{x})$, and, by the viscosity property of $v$, we have
\[ \lambda v(x^\alpha) - E(x^\alpha) \cdot 2\alpha(x^\alpha - X(t^\alpha)) - f(x^\alpha) \leq 0. \]

Also, since $t \mapsto \Phi_\alpha(t, x^\alpha)$ has a local maximum at $t^\alpha$, we have
\[ -\phi'(t^\alpha) - 2\alpha(X(t^\alpha) - x^\alpha) \cdot \dot{X}(t^\alpha) = 0. \]

Adding these two yields
\[ \lambda v(x^\alpha) - \phi'(t^\alpha) - 2\alpha(x^\alpha - X(t^\alpha)) \cdot (E(x^\alpha) - E(X(t^\alpha))) - f(x^\alpha) \leq 0. \]

Hence, letting $C$ be the Lipschitz constant of the function $E$, we obtain
\[ \lambda v(x^\alpha) - \phi'(t^\alpha) - 2\alpha C|x^\alpha - X(t^\alpha)|^2 - f(x^\alpha) \leq 0. \]

Sending $\alpha \to \infty$ in the above, we get $\lambda v(X(\hat{t})) - \phi'(\hat{t}) - f(X(\hat{t})) \leq 0$, and conclude that $w$ satisfies (A.1) in the viscosity sense.

To complete the proof, we fix any $\tau \in I$. The function
\[ z(t) := e^{\lambda t} \left( e^{-\lambda \tau} w(\tau) + \int_\tau^t e^{-\lambda s} g(s) \, ds \right) \]

is a classical solution of (A.1) and satisfies the condition that $z(\tau) = w(\tau)$. Fix any $\sigma \in (c, \tau)$, choose $a \in (c, \sigma)$, and, for $\varepsilon > 0$, set
\[ \chi_\varepsilon(t) = \frac{\varepsilon}{t - a} \quad \text{for} \quad t \in (a, \tau]. \]

The function $\zeta_\varepsilon := z + \chi_\varepsilon$ on $(a, \tau]$ satisfies in the classical sense
\[ \lambda \zeta_\varepsilon - \zeta_\varepsilon' - g > 0 \quad \text{in} \quad (a, \tau) \quad \text{and} \quad \zeta_\varepsilon(\tau) > w(\tau). \]

If $w - \zeta_\varepsilon$ has a maximum at some point in $(a, \tau)$, then the first inequality above yields a contradiction. On the other hand, since $\lim_{t \to a^+} (w - \zeta_\varepsilon)(t) = -\infty$ and $(w - \zeta_\varepsilon)(\tau) < 0$, the function $w - \zeta_\varepsilon$ has a maximum at a point $t_0 \in (a, \tau]$ and, moreover, $t_0 = \tau$, which implies that
\[ (w - \zeta_\varepsilon)(t) \leq (w - \zeta_\varepsilon)(\tau) < 0 \quad \text{for all} \quad t \in (a, \tau]. \]

Sending $\varepsilon \to 0$, we see that
\[ w(\sigma) \leq z(\sigma) = e^{\lambda \sigma} \left( e^{-\lambda \tau} w(\tau) + \int_\sigma^\tau e^{-\lambda s} F(s) \, ds \right), \]

which finishes the proof. \qed

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