New proposal for non-linear ghost-free massive $F(R)$ gravity: cosmic acceleration and Hamiltonian analysis

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We propose new version of massive $F(R)$ gravity which is natural generalization of convenient massive ghost-free gravity. Its Hamiltonian formulation in scalar-tensor frame is developed. We show that such $F(R)$ theory is ghost-free. The cosmological evolution of such theory is investigated. Despite the strong Bianchi identity constraint the possibility of cosmic acceleration (especially, in the presence of cold dark matter) is established. Ghost-free massive $F(R,T)$ gravity is also proposed.

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I. INTRODUCTION

The celebrated theory of the free massive gravity was established about seventy-five years ago in Ref. \cite{1} (for recent review, see \cite{2}). On the other hand, it has been known that the interacting or non-linear massive gravity contains the Boulware-Deser ghost \cite{3,4} and there appears so-called vDVZ discontinuity \cite{5} in the limit of $m \to 0$. This discontinuity can be screened by the Vainstein mechanism \cite{6} as shown, for example, in Ref. \cite{7} on the example of the DGP model \cite{8}.

Recently, there have been much progress in the non-linear massive gravity study and the ghost-free construction has been found for non-dynamical background metric in \cite{9–11} and for dynamical metric \cite{12}. General proof of absence of ghost in massive gravity has been given in Ref. \cite{13}. Especially, the minimal model was first treated in \cite{14}.

Cosmological evolution of such massive gravity models has been investigated in Refs. \cite{15–17}. In case of bimetric gravity, which contains two metrics (symmetric tensor fields) some cosmological solutions have been investigated and the solutions describing accelerating universe are known \cite{18–23}. In Ref. \cite{22,23}, we have proposed massive ghost-free $F(R)$ bigravity model which leads to rich variety of accelerating universes. $F(R)$ massive gravity without dynamical background metric was proposed recently in \cite{24}. In this paper, we propose another kind of $F(R)$ extension of the massive gravity. We show that due to the lack of the covariance, the Bianchi identity gives an equation which constrains the cosmic evolution very strongly. In spite of the constraint, there appears interesting solution which shows the self- acceleration when the cold dark matter exists. In this respect, the model is richer than that of Ref. \cite{24} where spatially-flat FRW expansion seems to be impossible as we show below.

For consistency of any massive gravity it is crucial to show the absence of Boulware-Deser ghost \cite{4}. The general proof of the absence of the ghost in non-linear massive gravity was presented in \cite{17} and in its St¨ uckelberg formulation in \cite{16,25}, see also \cite{26}. In case of the models studied in this paper, we also have to show that the Boulware-Deser ghost is absent. We proceed in the similar way as in \cite{17} and we prove that the Boulware-Deser ghost is absent in all models studied in this paper\footnote{For similar analysis, see \cite{27}.}. More precisely, we use the well-known equivalence between $f(R)$ theories and scalar-tensor theories to map the proposed model to the frame where the gravitational action has the canonical form and where the additional scalar field is present. It turns out, however, that the presence of this scalar does not modify the analysis performed in \cite{17} and we are able to show an existence of two second class constraints that are crucial for the elimination of the Boulware-Deser ghost and its conjugate momenta. Then we generalize given analysis to the case of $F(R,T)$ gravity \cite{28} with massive term. Introducing the appropriate number of auxiliary fields we can map given theory to the non-linear massive gravity coupled to the scalar field and to the trace of the stress-energy tensor of some matter field. The subsequent analysis depends on the fact whether we consider $T$ as fixed parameter or as dynamical quantity. In the second case we have to introduce the action for the matter as well when we consider concrete example of the scalar matter field. However, it turns out that again these theories are ghost free due to the
II. NON-LINEAR MASSIVE EXTENSION OF $F(R)$ GRAVITY

Let us propose new model of massive gravity which is an extension of $F(R)$ gravity (for recent general review, see [29, 30]). The action is given by

$$S_{\text{mg}} = M_g^2 \int d^4x \sqrt{-\det g} F \left( R^{(g)} + 2m^2 \sum_{n=0}^{4} \beta_n e_n \left( \sqrt{g^{-1}f} \right) \right) + S_{\text{matter}}. \tag{1}$$

Here $R^{(g)}$ is the scalar curvature for $g_{\mu\nu}$ and $f_{\mu\nu}$ is a non-dynamical reference metric. The action for matter is expressed by $S_{\text{matter}}$. The tensor $\sqrt{g^{-1}f}$ is defined by the square root of $g^{\mu\rho}f_{\rho\nu}$, that is, $\left( \sqrt{g^{-1}f} \right)^{\mu}_{\rho} \left( \sqrt{g^{-1}f} \right)^{\nu}_{\nu} = g^{\mu\nu}f_{\mu\nu}$. For general tensor $X^{\mu}_{\nu}$, $e_n(X)$’s are defined by

$$e_0(X) = 1, \quad e_1(X) = [X], \quad e_2(X) = \frac{1}{2}([X]^2 - [X^2]),$$
$$e_3(X) = \frac{1}{6}([X]^3 - 3[X][X^2] + 2[X^3]),$$
$$e_4(X) = \frac{1}{4}([X]^4 - 6[X]^2[X^2] + 3[X^2]^2 + 8[X][X^3] - 6[X]^4),$$
$$e_k(X) = 0 \quad \text{for} \quad k > 4. \tag{2}$$

Here $[X]$ expresses the trace of arbitrary tensor $X^{\mu}_{\nu}$: $[X] = X^{\mu}_{\mu}$.

In the following, for simplicity, we only consider the minimal case,

$$2m^2 \sum_{n=0}^{4} \beta_n e_n \left( \sqrt{g^{-1}f} \right) = 2m^2 \left( 3 - \text{tr} \sqrt{g^{-1}f} + \det \sqrt{g^{-1}f} \right). \tag{3}$$

Then the equation given by the variation of the metric has the following form:

$$0 = M_g^2 \left( \frac{1}{2} g_{\mu\nu} F' \left( \tilde{R}^{(g)} \right) - R^{(g)}_{\mu\nu} F' \left( \tilde{R}^{(g)} \right) + \nabla_{\nu} \nabla_{\mu} F' \left( \tilde{R}^{(g)} \right) - \nabla_{\mu} \nabla^{2} F' \left( \tilde{R}^{(g)} \right) \right)$$
$$+ m^2 M_g^2 F' \left( \tilde{R}^{(g)} \right) \left( \frac{1}{2} f_{\mu\rho} \left( \sqrt{g^{-1}f} \right)^{\mu}_{\rho} - g_{\mu\nu} \det \sqrt{g^{-1}f} \right) + \frac{1}{2} T_{\text{matter}, \mu\nu}. \tag{4}$$

Here

$$\tilde{R}^{(g)} \equiv R^{(g)} + 2m^2 \left( 3 - \text{tr} \sqrt{g^{-1}f} + \det \sqrt{g^{-1}f} \right), \tag{5}$$

and $\nabla_{\mu}$ is a covariant derivative given in terms of the Levi-Civita connection defined by the metric $g_{\mu\nu}$. In this paper, we do not use the covariant derivative with respect to the metric $f_{\mu\nu}$. We now have

$$\nabla^{\mu} \left( \frac{1}{2} g_{\mu\nu} F' \left( \tilde{R}^{(g)} \right) - R^{(g)}_{\mu\nu} F' \left( \tilde{R}^{(g)} \right) + \nabla_{\nu} \nabla_{\mu} F' \left( \tilde{R}^{(g)} \right) - g_{\mu\nu} \nabla^{2} F' \left( \tilde{R}^{(g)} \right) \right)$$
$$= m^2 F' \left( \tilde{R}^{(g)} \right) \partial_{\nu} \left( -\text{tr} \sqrt{g^{-1}f} + \det \sqrt{g^{-1}f} \right), \tag{6}$$

which can be explicitly shown as follows:

$$\nabla^{\mu} \left( \frac{1}{2} g_{\mu\nu} F' \left( \tilde{R}^{(g)} \right) - R^{(g)}_{\mu\nu} F' \left( \tilde{R}^{(g)} \right) + \nabla_{\nu} \nabla_{\mu} F' \left( \tilde{R}^{(g)} \right) - g_{\mu\nu} \nabla^{2} F' \left( \tilde{R}^{(g)} \right) \right)$$
$$= \frac{1}{2} g^{\mu\rho} \left( \nabla^{\rho} R^{(g)}_{\mu\nu} \right) F' \left( \tilde{R}^{(g)} \right) + m^2 F' \left( \tilde{R}^{(g)} \right) \partial_{\nu} \left( -\text{tr} \sqrt{g^{-1}f} + \det \sqrt{g^{-1}f} \right) - \left( \nabla^{\rho} R^{(g)}_{\mu\nu} \right) F' \left( \tilde{R}^{(g)} \right)$$
$$- R^{(g)}_{\mu\nu} \nabla^{\nu} F' \left( \tilde{R}^{(g)} \right) + \nabla^{\mu} \nabla_{\nu} F' \left( \tilde{R}^{(g)} \right) - g_{\mu\nu} \nabla^{2} F' \left( \tilde{R}^{(g)} \right)$$
$$= \left( \nabla^{\mu} \left( \frac{1}{2} g_{\mu\nu} R^{(g)} - R^{(g)}_{\mu\nu} \right) \right) F' \left( \tilde{R}^{(g)} \right) + m^2 F' \left( \tilde{R}^{(g)} \right) \partial_{\nu} \left( -\text{tr} \sqrt{g^{-1}f} + \det \sqrt{g^{-1}f} \right) - R^{(g)}_{\mu\nu} \nabla^{\nu} F' \left( \tilde{R}^{(g)} \right)$$
which further gives

\[- R^{(g)}_{\mu \nu} \nabla_\mu F' \left( \tilde{R}^{(g)} \right) + \nabla_\nu \nabla_\mu F' \left( \tilde{R}^{(g)} \right) - \nabla_\nu \nabla^2 F' \left( \tilde{R}^{(g)} \right) = m^2 F' \left( \tilde{R}^{(g)} \right) \partial_\nu \left( - \text{tr} \sqrt{g^{-1} f} + \det \sqrt{g^{-1} f} \right). \tag{7} \]

Here we have used the Bianchi identity \(0 = \nabla^\mu \left( \frac{1}{2} g_{\mu \nu} R^{(g)} - R^{(g)}_{\mu \nu} \right)\). Then by multiplying the covariant derivative \(\nabla^\mu\) with respect to the metric \(g\) with Eq. \([4]\) and using the conservation law \(0 = \nabla^\mu T^\text{matter}_{\mu \nu}\), we obtain

\[0 = F' \left( \tilde{R}^{(g)} \right) \partial_\nu \left( - \text{tr} \sqrt{g^{-1} f} + \det \sqrt{g^{-1} f} \right) + \left( \partial^\mu F' \left( \tilde{R}^{(g)} \right) \right) \left\{ \frac{1}{2} f_{\mu \rho} \left( \sqrt{g^{-1} f} \right)^{-1 \rho} + \frac{1}{2} f_{\nu \rho} \left( \sqrt{g^{-1} f} \right)^{-1 \rho} - g_{\mu \nu} \det \sqrt{g^{-1} f} \right\} \]

\[+ F' \left( \tilde{R}^{(g)} \right) \nabla^\mu \left( \frac{1}{2} f_{\rho \mu} \left( \sqrt{g^{-1} f} \right)^{-1 \rho} + \frac{1}{2} f_{\nu \rho} \left( \sqrt{g^{-1} f} \right)^{-1 \rho} - g_{\mu \nu} \det \sqrt{g^{-1} f} \right). \tag{8} \]

We now assume the FRW universe for the metrics \(g_{\mu \nu}\) and flat Minkowski space-time for \(f_{\mu \nu}\) and use the conformal time \(t\) for the universe with metric \(g_{\mu \nu}\):

\[ds_g^2 = \sum_{\mu, \nu = 0}^3 g_{\mu \nu} dx^\mu dx^\nu = a(t)^2 \left( -dt^2 + \sum_{i=1}^3 (dx^i)^2 \right), \quad ds_f^2 = \sum_{\mu, \nu = 0}^3 f_{\mu \nu} dx^\mu dx^\nu = -dt^2 + \sum_{i=1}^3 (dx^i)^2. \tag{9} \]

Then the \(\nu = i\) component in \([8]\) is trivially satisfied. On the other hand, the \(\nu = t\) component gives

\[0 = \partial_t \left( -4a^{-1} + a^{-4} \right) F' \left( \tilde{R}^{(g)} \right) + \left( a^{-1} - a^{-4} \right) \partial_t F' \left( \tilde{R}^{(g)} \right) = a^{-4} \partial \left\{ F' \left( \tilde{R}^{(g)} \right) (a^3 - 1) \right\}, \tag{10} \]

which further gives

\[F' \left( \tilde{R}^{(g)} \right) (a^3 - 1) = C, \tag{11} \]

with a constant \(C\). Eq. \([11]\) determines the form of \(F' \left( \tilde{R}^{(g)} \right)\). For a given time evolution of the scale factor \(a = a(t)\), we find the \(t\) dependence of \(\tilde{R}^{(g)}\): \(\tilde{R}^{(g)} = \tilde{R}^{(g)}(t)\), which can be solved with respect to \(t\) as a function of \(\tilde{R}^{(g)}\): \(t = t \left( \tilde{R}^{(g)} \right)\). Then Eq. \([11]\) gives the form of \(F' \left( \tilde{R}^{(g)} \right)\) as follows,

\[F' \left( \tilde{R}^{(g)} \right) = \frac{C}{a \left( t \left( \tilde{R}^{(g)} \right) \right)^3 - 1}, \tag{12} \]

As we will see soon, however, the time evolution of the scale factor \(a = a(t)\) cannot be arbitrary. We should also note that \(F' \left( \tilde{R}^{(g)} \right)\) diverges when the scale factor \(a\) goes to unity.

In the FRW metric with conformal time in \([9]\), the \((\mu, \nu) = (t, t)\) component in \([4]\) has the following form:

\[0 = -\frac{1}{2} a^{-2} F \left( \tilde{R}^{(g)} \right) + 3H F' \left( \tilde{R}^{(g)} \right) - 3H \partial_t F' \left( \tilde{R}^{(g)} \right) + \left( -a + a^{-3} \right) F' \left( \tilde{R}^{(g)} \right) + \frac{1}{2M_g^2} \rho_{\text{matter}}, \tag{13} \]

and the \((\mu, \nu) = (i, j)\) component gives

\[0 = \frac{1}{2} a^{-2} F \left( \tilde{R}^{(g)} \right) - \left( H + 2H^2 \right) F' \left( \tilde{R}^{(g)} \right) + \left( \partial_i^2 + H \partial_i \right) F' \left( \tilde{R}^{(g)} \right) - \left( -a + a^{-3} \right) F' \left( \tilde{R}^{(g)} \right) + \frac{1}{2M_g^2} \rho_{\text{matter}}. \tag{14} \]

By combining \([13]\) and \([14]\), one obtains

\[0 = 2 \left( H - H^2 \right) F' \left( \tilde{R}^{(g)} \right) + \left( \partial_i^2 - 2H \partial_i \right) F' \left( \tilde{R}^{(g)} \right) + \frac{1}{2M_g^2} \left( \rho_{\text{matter}} + p_{\text{matter}} \right). \tag{15} \]

Different from the Einstein gravity, Eqs. \([13]\), \([14]\), and conservation law

\[0 = \dot{\rho}_{\text{matter}} + 3H \left( \rho_{\text{matter}} + p_{\text{matter}} \right), \tag{16} \]
are independent equations. The form of the conservation law in terms of the conformal time is not changed from that of the cosmological time. Instead of Eqs. (13), (14), and (16), we may regard Eqs. (11), (15), and the conservation law (16) as independent equations. Eq. (11) gives
\[
\frac{\partial}{\partial t} F' \left( \tilde{R}^{(g)} \right) = -\frac{3H a^3}{a^3-1} \left( \frac{18H^2 a^6}{(a^3-1)^3} - \frac{(3H + 9H^2) a^3}{(a^3-1)^2} \right) C . \tag{17}
\]

Then Eq. (15) can be rewritten as
\[
0 = \left\{ \dot{H} \left( -a^6 - a^3 + 2 \right) + H^2 \left( 13a^6 + 7a^3 - 2 \right) \right\} C + \frac{1}{2M_g^2} \left( \rho_{\text{matter}} + p_{\text{matter}} \right) . \tag{18}
\]

Independent from the form of \( F \left( \tilde{R}^{(g)} \right) \), Eq. (18) describes the dynamics of the universe.

It is difficult to solve (18) explicitly. Then it is easier to consider the following three cases: a) \( a \to 1 \) case, b) \( a \gg 1 \) case, c) \( a \ll 1 \) case. In the following, for simplicity, we assume that the matter has a constant equation of state (EoS) parameter \( w \) and therefore
\[
p_{\text{matter}} = w \rho_{\text{matter}}, \quad \rho_{\text{matter}} = \rho_0 a^{-3(w+1)} . \tag{19}
\]

a) \( a \to 1 \) case. By putting \( a = 1 + \delta a \), from (18), we obtain
\[
0 \sim -9 \dot{H} + 18H^2 \sim -9\delta \dot{a} \delta a + 18 (\delta a)^2 = -9 (\delta a)^3 \frac{d}{dt} \left( \frac{\delta a}{(\delta a)^2} \right) , \tag{20}
\]
whose solution is given by
\[
\delta a = \frac{C_1}{t + C_2} . \tag{21}
\]

Here \( C_1 \) and \( C_2 \) are constants of the integration. Eq. (21) tells us that the limit \( a \to 1 (\delta a \to 0) \) is realized in the infinite past or future in conformal time, \( t \to \pm \infty \).

b) \( a \gg 1 \) case. In this case, Eq. (18) can be approximated as
\[
0 \sim C a^{-3} \left( -\dot{H} + 13H^2 \right) + \frac{1 + w}{2} \rho_0 a^{-3(w+1)} . \tag{22}
\]

1. \( \rho_0 = 0 \) case. The solution of (22) is given by
\[
H = \frac{1}{13 \left( t_0 - t \right)} , \tag{23}
\]
which describes the phantom universe which has a Big Rip singularity at \( t = t_0 \) since we assume \( a \gg 1 \).

2. \( \rho_0 \neq 0 \) case. If \( w \neq 0 \), we have a power law solution,
\[
a = a_0 t^{\frac{2}{3w}} . \tag{24}
\]
Here \( a_0 \) is given by solving the following equation
\[
0 = \frac{2C}{3w} \left( 1 + \frac{26}{3w} \right) + \frac{w + 1}{2} a_0^{-3w} \rho_0 . \tag{25}
\]

On the other hand, when \( w = 0 \), we obtain a solution describing de Sitter universe:
\[
H^2 = \frac{\rho_0}{26C} . \tag{26}
\]

This could be interesting since the accelerating expansion of the present universe can be realized by dust, which may be identified with cold dark matter. Then we find
\[
\frac{1}{26C} \rho_0 \sim (10^{-33} \text{ eV})^2 , \quad \rho_0 a^3 \sim (10^{-3} \text{ eV})^4 . \tag{27}
\]
c) $a \ll 1$ case. The approximated form of Eq. (18) in this case is given by

$$0 = -2C \left( \dot{H} - H^2 \right) + \frac{1 + w}{2} \rho_0 a^{-3(w+1)}. \quad (28)$$

1. $\rho_0 = 0$ case. The solution is given by

$$H = \frac{1}{t_0 - t}. \quad (29)$$

with a constant of integration $t_0$. The expression (29) is valid when $t \to \pm \infty$ because we are assuming $a \ll 1$. Therefore, in spite of the form in (29), there does not always occur a Big Rip singularity.

2. $\rho_0 \neq 0$ case. By solving (29), we find

$$a = a_0 t^{\frac{2}{3(w+1)}}. \quad (30)$$

Now $a_0$ is given by solving the following equation:

$$0 = -\frac{4C}{3(w+1)} \left( 1 - \frac{2}{3(w+1)} \right) + \frac{w+1}{2} a_0^{-3w} \rho_0. \quad (31)$$

The qualitative behavior is not changed from the Einstein gravity coupled with matter of a constant EoS parameter $w$.

Thus, we demonstrated the principal possibility to have accelerating cosmology within new non-linear massive $F(R)$ gravity. Nevertheless, the variety of possible cosmological solutions is not so wide as in convenient $F(R)$ gravity.

### III. FRW COSMOLOGY FROM ANOTHER MASSIVE $F(R)$ GRAVITY MODEL

Instead of the model in (1) for the purpose of comparison, we now consider an $F(R)$ extension of massive gravity proposed recently in [24]. The action is given by

$$S_{mg} = M_g^2 \int d^4 x \sqrt{-\det g} F \left( R^{(g)} \right) + 2 m^2 M_g^2 \int d^4 x \sqrt{-\det g} \sum_{n=0}^4 \beta_n e_n \left( \sqrt{g^{-1} f} \right) + S_{\text{matter}}. \quad (32)$$

In the following, just for simplicity, we only consider the minimal case, as in (3).

$$S_{mg} = M_g^2 \int d^4 x \sqrt{-\det g} F \left( R^{(g)} \right) + 2 m^2 M_g^2 \int d^4 x \sqrt{-\det g} \left( 3 - \text{tr} \sqrt{g^{-1} f} + \det \sqrt{g^{-1} f} \right) + S_{\text{matter}}, \quad (33)$$

Then by the variation over $g_{\mu \nu}$, we obtain

$$0 = M_g^2 \left( 1 \frac{1}{2} g_{\mu \nu} F \left( R^{(g)} \right) - R_{\mu \nu}^{(g)} F' \left( R^{(g)} \right) + \nabla_{\nu} \nabla_{\mu} F' \left( R^{(g)} \right) - g_{\mu \nu} \nabla^2 F' \left( R^{(g)} \right) \right)$$

$$+ m^2 M_g^2 \left( g_{\mu \nu} \left( 3 - \text{tr} \sqrt{g^{-1} f} \right) + \frac{1}{2} f_{\mu \rho} \left( \sqrt{g^{-1} f} \right)^{-1} \rho_{\nu} + \frac{1}{2} f_{\nu \rho} \left( \sqrt{g^{-1} f} \right)^{-1} \rho_{\mu} \right) + \frac{1}{2} T_{\text{matter} \mu \nu}. \quad (34)$$

Now, instead of (1), we have

$$0 = \nabla^\mu \left( 1 \frac{1}{2} g_{\mu \nu} F \left( R^{(g)} \right) - R_{\mu \nu}^{(g)} F' \left( R^{(g)} \right) + \nabla_{\nu} \nabla_{\mu} F' \left( R^{(g)} \right) - g_{\mu \nu} \nabla^2 F' \left( R^{(g)} \right) \right), \quad (35)$$

and

$$0 = -g_{\mu \nu} \nabla^\mu \left( \text{tr} \sqrt{g^{-1} f} \right) + \frac{1}{2} \nabla^\mu \left( f_{\mu \rho} \left( \sqrt{g^{-1} f} \right)^{-1} \rho_{\nu} + f_{\nu \rho} \left( \sqrt{g^{-1} f} \right)^{-1} \rho_{\mu} \right), \quad (36)$$

which corresponds to (3). Starting again from the FRW universe for the metric $g_{\mu \nu}$ and flat Minkowski space-time for $f_{\mu \nu}$ one can use the conformal time $t$ as in (3). Then $(t, t)$ component of (34) gives

$$0 = -3M_g^2 H^2 - 3m^2 M_g^2 (a^2 - a) + \rho_{\text{matter}}, \quad (37)$$
and \((i, j)\) components give
\[
0 = M_0^2 \left( 2\dot{H} + H^2 \right) + 3m^2 M_0^2 (a^2 - a) + p_{\text{matter}}. \tag{38}
\]
Here \(H = \dot{a}/a\). Eq. \(39\) gives the following constraint:
\[
\frac{\dot{a}}{a} = 0. \tag{39}
\]
Different from Eq. \(11\), the identity \(39\) shows that \(a\) should be a constant \(a = a_0\). This indicates that the only consistent solution for \(g_{\mu\nu}\) is the flat Minkowski space. Therefore we cannot obtain the expanding universe without extra fields and/or fluids.

In \([24]\), this model was studied in the FRW universe with non-vanishing spatial curvature. When the spatial curvature does not vanish, the scale factor is proportional to the spatial curvature and linear to the cosmological time (not conformal time as in this paper). In the limit that the spatial curvature vanishes, the scale factor becomes a constant, what is consistent with the result obtained here. Even in case that the spatial curvature does not vanish, Eq. \(40\) gives a strong constraint with the only possible solution, for which the scale factor must be linear in cosmological time.

IV. HAMILTONIAN FORMALISM

In this section, we perform Hamiltonian formulation of the model described by the action \(1\) and show that the model does not contain ghost. As the first step we introduce two auxiliary fields \(A, B\) and rewrite the action \(1\) into the following form
\[
S = M_0^2 \int d^4x \sqrt{-\det g} \left[ B \left( R^{(g)} + 2m^2 \sum_{n=0}^{4} \beta_n \epsilon_n (\sqrt{-g^{-1}} f) - A \right) + F(A) \right]. \tag{40}
\]
Using the Weyl transformation
\[
g'_{\mu\nu} = \Omega g_{\mu\nu} \tag{41}
\]
implies
\[
R^{(g)} [g] = \Omega \left( R^{(g)} [g'] - \frac{3}{2\Omega} g'^{\mu\nu} \nabla^\mu \Omega \nabla^\nu \Omega + 3g^{\mu\nu} \left( \frac{1}{\Omega} \nabla^\mu \Omega \nabla^\nu \Omega - \frac{1}{\Omega^2} \nabla^\mu g_{\nu\rho} \nabla^\rho \Omega \right) \right) \tag{42}
\]
Here \(\nabla^\mu\) is the covariant derivative with respect to \(g'_{\mu\nu}\). Now by choosing \(B = \Omega\), we find the theory with the canonical Einstein-Hilbert term. Further, the equation of motion with respect to \(A\) has the form
\[
F_A (A) = B. \tag{43}
\]
Here \(F_A (A) \equiv dF(A)/dA\). We presume that it can be solved for \(A = \Psi(B)\), where \(F_A (\Psi(x)) = x\). Then by introducing the scalar field \(\phi\) through the formula \(\Omega = \exp(\phi)\), we obtain the action in the form
\[
S = M_0^2 \int d^4x \sqrt{-\det g'} \left[ R^{(g')} - \frac{2}{3} g'^{\mu\nu} \nabla^\mu \phi \nabla^\nu \phi - V(\phi) \right] + 2m^2 \int d^4x \sqrt{-g} \sum_{n=0}^{4} \beta_n \epsilon_n \frac{\partial}{\partial \phi} \epsilon_n \left( \sqrt{g^{-1}} f \right). \tag{44}
\]
In what follows we omit \(\dot{\cdot}\) over metric variables.

Our goal is to find the Hamiltonian formulation of given theory and determine corresponding primary and the secondary constraints. As the first step we introduce \(3 + 1\) decomposition of both \(g_{\mu\nu}\) and \(f_{\mu\nu}\) \([31, 32]\)
\[
g_{00} = -N^2 + N_i g^{ij} N_j, \quad g_{0i} = N_i, \quad \dot{g}_{ij} = g_{ij}, \quad g^{00} = -\frac{1}{N^2}, \quad g^{0i} = \frac{N^i}{N^2}, \quad g^{ij} = g^{ij} - \frac{N^i N^j}{N^2}, \tag{45}
\]
and
\[
f_{00} = -M^2 + L_i f^{ij} L_j, \quad f_{0i} = L_i, \quad f_{ij} = f_{ij},
\]
\[ f^{00} = -\frac{1}{M^2}, \quad f^{0i} = -\frac{L^i}{M^2}, \quad f^{ij} = f^{ij} = f^{ij} = f^{ij} = f^{ij}, \quad L^i = L^j f^{ij}, \quad (46) \]

where we defined \( g^{ij} \) and \( f^{ij} \) as the inverse to \( g_{ij} \) and \( f_{ij} \), respectively \( g_{ik} g^{kj} = \delta^i_j, \quad f_{ik} f^{kj} = \delta^i_j \). By following [11] [14], we perform the following redefinition of the shift function

\[ N^i = M \dot{n}^i + L^i + N \ddot{D}^j \dot{n}^j, \quad (47) \]

so that the resulting action is linear in \( M \) and \( N \). Note that the matrix \( \ddot{D}^j \) obeys the equation [11] [14]

\[ \sqrt{x} \ddot{D}^j = \sqrt{(g^{ik} - \ddot{D}^i m \ddot{n}^m \ddot{D}^k \ddot{n}^m) f_{kj}} \quad (48) \]

and also following important property \( f_{ik} \ddot{D}^k_j = f_{jk} \ddot{D}^k_i \). Then after some calculations, we find that the action \( (44) \) has the form

\[ S = M_g^2 \int dtd^4x N \sqrt{(3) g} [K_{ij} G^{ijkl} K_{kl} + (3) R(g) + \frac{2}{3} \nabla_n \phi \nabla_n \phi - \frac{2}{3} g^{ij} \partial_i \phi \partial_j \phi - V(\phi)] \]

\[ + 2m^2 M_{\text{eff}}^2 \int dtd^4x \sqrt{(3) g} (\mathcal{M} \mathcal{U} + N \mathcal{V}), \quad (49) \]

where

\[ K_{ij} = \frac{1}{2} (\partial_t n_{ij} - \nabla_i n_j (\ddot{n}, \ddot{g}) - \nabla_j n_i (\ddot{n}, \ddot{g})), \quad \nabla_n \phi = \frac{1}{N} (\partial_t \phi - N^i (\ddot{n}, \ddot{g}) \partial_i \phi), \quad (50) \]

with

\[ N_i = M g_{ij} \ddot{n}^j + g_{ij} L^j + N g_{ik} \ddot{D}^k_j \ddot{n}^j, \quad L_i = f_{ij} L^j, \quad (51) \]

and where \( \nabla_i, (3) R(g) \) are the covariant derivative and scalar curvature calculated using \( g_{ij} \). We should also note that \( (3) g \) is the determinant of \( g_{ij} \). Furthermore, \( G^{ijkl} \) are the de Witt metrics defined as

\[ G^{ijkl} = \frac{1}{2} (g^{ik} g^{jl} + g^{il} g^{jk}) - g^{ij} g^{kl}, \quad (52) \]

with inverse

\[ G_{ijkl} G^{klmn} = \frac{1}{2} (\delta^m_{i} \delta^n_{j} + \delta^n_{i} \delta^m_{j}). \quad (54) \]

Finally, \( \mathcal{V} \) and \( \mathcal{U} \) introduced in \( (49) \) have the following form

\[ \mathcal{V} = \beta_0 + \beta_1 e^{2 \phi} \sqrt{x} \ddot{D}^i_i + \beta_2 e^{\phi} \frac{1}{2} \sqrt{x} \left( \ddot{D}^i_i \ddot{D}^j_j - \ddot{D}^j_i \ddot{D}^i_j \right) \]

\[ + \frac{1}{6} \beta_3 e^{2 \phi} \sqrt{x} \left[ \ddot{D}^i_i \ddot{D}^j_j \ddot{D}^k_k - 3 \ddot{D}^i_i \ddot{D}^j_j \ddot{D}^k_k + 2 \ddot{D}^j_j \ddot{D}^i_i \ddot{D}^k_k \right] \]

\[ + \beta_4 e^{2 \phi} \left[ \sqrt{x} (\dddot{D}^i_i \dddot{f}_{ij} \dddot{D}^i_i \dddot{D}^j_j \dddot{D}^i_i \dddot{D}^j_j n^k - \dddot{D}^i_i \dddot{n}^k \dddot{f}_{ij} \dddot{D}^i_i \dddot{D}^j_j \dddot{n}^k) + \frac{1}{2} \sqrt{x} (\dddot{D}^i_i \dddot{D}^j_j - \dddot{D}^j_i \dddot{D}^i_j) \right] + \beta_5 e^{2 \phi} \sqrt{(3) f} \frac{1}{\sqrt{(3) g}} \quad (55) \]

where \( \ddot{x} = 1 - \ddot{n}^i \ddot{n}^j \) and \( (3) f \) is the determinant of \( f_{ij} \). The action \( (49) \) is suitable for the Hamiltonian formalism. First we find the momenta conjugate to \( N, \ddot{n}^i \) and \( g_{ij} \)

\[ \pi_N \approx 0, \quad \pi_i \approx 0, \quad \pi^{ij} = M_g^2 \sqrt{(3) g} G^{ijkl} K_{kl}, \quad (56) \]

together with the momenta conjugate to \( \phi \)

\[ p_\phi = \frac{4}{3} M_g^2 \sqrt{(3) g} \nabla_n \phi. \quad (57) \]
Then after some calculations we find the following Hamiltonian

$$H = \int d^3x \left( \pi^{ij} \partial_i g_{ij} + p_\phi \partial_i \phi - \mathcal{L} \right) = \int d^3x \left( N C_0 + R_0 \right),$$ (58)

where

$$C_0 = \frac{1}{M^2 g} \pi^{ij} G_{ijkl} \pi^{kl} - M^2 \sqrt{(3)} g R \pi^{ij} + (R_k + p_\phi \partial_k \phi) \tilde{D}^i \tilde{n}^i - 2m^2 \sqrt{(3)} g V,$$

$$+ \frac{3}{8} \frac{1}{\sqrt{(3)} g M^2} p_\phi^2 + \frac{2}{3} M^2 \sqrt{(3)} g g^{ij} \partial_i \phi \partial_j \phi + M^2 \sqrt{(3)} g V(\phi),$$

$$\mathcal{R}_0 = (M \tilde{n}^i + L^i)(R_i + p_\phi \partial_i \phi) - 2m^2 M \sqrt{(3)} g M,$$ (59)

where we also denoted $R_i = -2 \partial_i \sqrt{g} \pi^i$. We see that the theory possesses four primary constraints

$$\pi_N \approx 0, \quad \pi_i \approx 0,$$ (60)

where $\pi_N$ and $\pi_i$ are momenta conjugate to $N$ and $\tilde{n}^i$, respectively with the following non-zero Poisson brackets

$$\{ N(\mathbf{x}), \pi_N(\mathbf{y}) \} = \delta(\mathbf{x} - \mathbf{y}), \quad \{ \tilde{n}^i(\mathbf{x}), \pi_j(\mathbf{y}) \} = \delta^i_j \delta(\mathbf{x} - \mathbf{y}).$$ (61)

To proceed further we need following relations

$$\frac{\delta \sqrt{x} \tilde{D}^k}{\delta \tilde{n}^i} = - \frac{1}{\sqrt{x}} \tilde{n}^p f_{pk} \frac{\delta}{\delta \tilde{n}^i}(\tilde{D}^k \tilde{n}^m).$$

$$\frac{\delta}{\delta \tilde{n}^i} \text{tr} \left( \sqrt{x} \tilde{D} \sqrt{x} \tilde{D} \right) = -2 \tilde{n}^j f_{jk} \frac{\delta}{\delta \tilde{n}^i}(\tilde{D}^l \tilde{n}^m),$$

$$\frac{\delta}{\delta \tilde{n}^i} \text{tr} \left( \sqrt{x} \tilde{D} \sqrt{x} \tilde{D} \sqrt{x} \right) = -3 \sqrt{x} \tilde{n}^j f_{jk} \tilde{D}^i \tilde{D}^j \frac{\delta}{\delta \tilde{n}^i}(\tilde{D}^m \tilde{n}^p),$$ (62)

that follow from (48) and also using the property $f_{ik} \tilde{D}^k_j = f_{jk} \tilde{D}^k_i$. Then we find

$$\frac{\delta \mathcal{R}_i}{\delta \tilde{n}^i} = M \left( R_i + p_\phi \partial_i \phi \right) + 2m^2 M \sqrt{(3)} g \left[ \frac{\beta_1}{\sqrt{x}} e^{\frac{2}{3} \phi} f_{ij} \tilde{n}^j + \beta_2 e^{\phi} \left( f_{ij} \tilde{n}^j \tilde{D}^i + f_{ij} \tilde{D}^i \tilde{n}^j \right) + \beta_3 \tilde{n}^p f_{pj} e^{\frac{2}{3}} \sqrt{x} \frac{1}{2} \delta^i \left( \tilde{D}^m \tilde{D}^n - \tilde{D}^m \tilde{D}^n \right) + \tilde{D}^i \tilde{D}^m \tilde{D}^m \tilde{D}^i \right]$$

$$= M C_i,$$ (63)

where $C_i$ is defined by

$$C_i = R_i + p_\phi \partial_i \phi + 2m^2 \sqrt{(3)} g \frac{f_{ij} \tilde{n}^i}{\sqrt{x}} \left[ \beta_1 e^{\frac{2}{3} \phi} \delta^i_j + \beta_2 e^{\phi} \sqrt{x} \left( \delta^i_j \tilde{D}^m \tilde{D}^n - \tilde{D}^n \tilde{D}^m \right) + \tilde{D}^i \tilde{D}^m \tilde{D}^m \tilde{D}^i \right].$$ (64)

In the same way, we find

$$\frac{\delta C_0}{\delta \tilde{n}^i} = C_i \frac{\delta \tilde{D}^i \tilde{n}^k}{\delta \tilde{n}^i}.$$ (65)

Now the requirement of the preservation of the primary constraints $\pi_N \approx 0, \pi_i \approx 0$ implies

$$\partial_i \pi_N = \{ \pi_N, H \} = -C_0 \approx 0,$$
\[
\partial_t \pi_i = \{\pi_i, H\} = -\frac{\delta H}{\delta \dot{n}^i} = -C_j \left( M \delta_i^j + \frac{\delta (\tilde{D}^j_i \tilde{n}^k)}{\delta n^i} \right) = 0, \tag{66}
\]

that implies an existence of the secondary constraints \( C_i \approx 0 \). As a result we find that the total Hamiltonian has the following form

\[
H_T = \int d^3x \left( H_0 + N C_0 + v_N \pi_N + v^i \pi_i + \Sigma^i C_i \right) \tag{67}
\]

where \( v_N, v^i, \Sigma^i \) are Lagrange multipliers corresponding to the constraints \( \pi_N \approx 0, \pi_i \approx 0, C_i \approx 0 \).

The next step is to analyze the requirement of the preservation of the secondary constraints \( C_0, C_i \). First of all note that the requirement of the preservation of the constraint \( \pi_i \approx 0 \) implies

\[
\partial_t \pi_i = \{\pi_i, H_T\} \approx \int d^3x \Sigma^j \{\pi_i, C_j\} = 0. \tag{68}
\]

It can be shown that \( \{\pi_i(x), C_j(y)\} \equiv \triangle_{ij} \delta(x - y) \) where \( \triangle_{ij} \) is a non-singular matrix so that the only possible solution of the equation above is \( \Sigma^j = 0 \).

Now we have to determine the time evolution of the constraint \( C_0 \):

\[
\partial_t C_0 = \{C_0, H_T\} = \int d^3x [N(x) \{C_0, C_0(x)\} + \{C_0, H(x)\}] = 0. \tag{69}
\]

To proceed further we determine the Poisson bracket \( \{C_0(x), C_0(y)\} \). Following [17], we easily find that this Poisson bracket has the following form:

\[
\{C_0(x), C_0(y)\} = -\mathcal{P}^i(y) \frac{\partial}{\partial y^i} \delta(x - y) + \mathcal{P}^i(x) \frac{\partial}{\partial x^i} \delta(x - y), \tag{70}
\]

where

\[
\mathcal{P}^i = C_0 \tilde{D}^j_i \tilde{n}^k + C_j g^{ji}. \tag{71}
\]

Since \( \mathcal{P}^i \) is given as the linear combination of the constraints, we find that \( \mathcal{P}^i \) vanishes on the constraint surface so that

\[
\{C_0(x), C_0(y)\} \approx 0. \tag{72}
\]

Now it is easy to see that the requirement of the preservation of the constraint \( C_0 \) implies following secondary constraint

\[
\int d^3x \{C_0, H(x)\} \equiv C^{(1)} \approx 0, \tag{73}
\]

with explicit form that it is not important for us. Finally the requirement of the preservation of the constraint \( C_i \) takes the form

\[
\partial_t C_i = \{C_i, H_T\} = \int d^3x \left( \{C_i, H(x)\} + v^j \{C_i, \pi_j(y)\} \right) \approx 0,
\]

that using the same arguments as in case of the preservation of the constraint \( \pi_i \approx 0 \) implies that given equation can be solved for \( v^j \) as functions of canonical variables. Note also that \( \pi_N \approx 0 \) is the first class constraint. Finally it can be easily shown that \( C, C^{(1)} \) has non-trivial Poisson bracket which implies that the are the second class constraints [17].

In summary we have following structure of constraints. We have six second class constraints \( \pi_i \approx 0, C_i \approx 0 \) that can be solved for \( \pi_i \) and for \( \tilde{n}^i \). Then we have one first class constraint \( \pi_N \) that can be gauge fixed by imposing the condition \( N = 1 \) (for example). Finally \( C, C^{(1)} \) are the second class constraints that can be solved for the Boulware-Deser ghost and its conjugate momenta. As a result we find that this theory possesses 12 degrees of freedom where 10 of them correspond to the massive gravity and 2 corresponds to \( \phi, p_\phi \).

We can generalize the ghost-free proposal [10] in several ways. For example, let us consider non-linear massive theory where \( F(R) \) depends on the trace of the stress energy tensor

\[
S = M_g^2 \int d^4x \sqrt{-\det g} F \left( R^{(g)}, T \right) + 2m^2 \int d^4x \sqrt{-\det g} \sum_{n=0}^4 \beta_n e_n \left( \sqrt{g^{-1}} f \right). \tag{74}
\]
Even in this model, we find Eq. (36) again. Let us introduce four auxiliary fields $A$, $B$, $C$, $D$ and rewrite the action into the following form

$$S = M_g^2 \int d^4x \sqrt{-\det g} \left[ F(A, C) + B(R^{(g)} - A) + D(T - C) \right] + 2m^2 \int d^4x \sqrt{-\det g} \sum_{n=0}^{4} \beta_n e_n \left( \sqrt{g^{-1}f} \right).$$ (75)

We again perform the Weyl transformation (41) in order to transform the action (75) to the action with Einstein-Hilbert term. Finally we can check the equation of motion with respect to $A$

$$F_A(A, C) = B,$$ (76)

and we presume that it can be solved for $A = \Psi(B, C)$. Putting all these results together we obtain the action in the form

$$S = M_g^2 \int d^4x \sqrt{-\det g} \left[ R^{(g)} - \frac{2}{3} g_{\mu\nu} \nabla_{\mu} \phi \nabla_{\nu} \phi - V(\phi, \tilde{T}[e^{\phi}]) \right] + 2m^2 \int d^4x \sqrt{-\det g} \sum_{n=0}^{4} \beta_n e_n \left( \sqrt{g^{-1}f} \right),$$ (77)

where $\tilde{T}[e^{\phi}]$ is the transformed trace of the stress energy tensor which is model dependent. Let us now consider the case when we treat $T$ as the external parameter. Then we can simply solve the equation of motion for $D$ which leads to the replacement $\tilde{T}[e^{\phi}] = C$ in the action so that we have

$$S = M_g^2 \int d^4x \sqrt{-\det g} \left[ R^{(g)} - \frac{2}{3} g_{\mu\nu} \nabla_{\mu} \phi \nabla_{\nu} \phi - V(\phi, \tilde{T}[e^{\phi}]) \right] + 2m^2 \int d^4x \sqrt{-\det g} \sum_{n=0}^{4} \beta_n e_n \left( \sqrt{g^{-1}f} \right).$$ (78)

This action has formally the same form as the action (41) so that we can quickly say that given theory is ghost free as well.

The situation will be more complicated in case when $T$ represents the dynamical quantity. In this case we should specify its explicit form in order to perform the Hamiltonian analysis of the coupled system of massive gravity and the matter that is represented by $T$. We should also consider the action for the matter field as well. Let us consider concrete example when the matter is represented by the scalar field $\psi$ with the action

$$S_{\text{matt}} = - \int d^4x \sqrt{-\det g} \left[ g^{\mu\nu} \partial_{\mu} \psi \partial_{\nu} \psi + \mathcal{F}(\psi) \right],$$ (79)

where $\mathcal{F}(\psi)$ is the potential for the scalar field $\psi$. Then the stress energy tensor has the form

$$T_{\mu\nu} = - \frac{1}{\sqrt{-\det g}} \frac{\delta S_{\text{matt}}}{\delta g^{\mu\nu}} = - \frac{1}{2} g_{\mu\nu} \left[ g^{\rho\sigma} \partial_{\rho} \psi \partial_{\sigma} \psi + \mathcal{F}(\psi) \right] + \partial_{\mu} \psi \partial_{\nu} \psi, \quad T = -\mathcal{F} - g^{\mu\nu} \partial_{\mu} \psi \partial_{\nu} \psi$$

As opposite to the case when $T$ is fixed parameter now we have to perform the Hamiltonian analysis of the system

$$S = M_g^2 \int d^4x dt \sqrt{(3)g} N \left[ K_{ij} G^{ijkl} K_{kl} + \frac{2}{3} (\nabla_n \phi)^2 - \frac{2}{3} g^{ij} \partial_i \phi \partial_j \phi - V(\phi, C) \right] + \frac{2m^2}{M_g^2} \sum_{n=0}^{4} \beta_n e_n \left( \sqrt{g^{-1}f} \right) - \frac{1}{M_g^2} e^{-2\phi} D(2\mathcal{F} + C) - \frac{e^{-\phi}}{M_g^2} \mathcal{F} + \frac{e^{-\phi}}{M_g^2} (D + 1) \nabla_n \psi \nabla_n \psi - \frac{e^{-\phi}}{M_g^2} (D + 1) g^{ij} \partial_i \psi \partial_j \psi \right].$$ (80)

This action is the starting point for the Hamiltonian formalism. We see that in case of massive gravity and the scalar field $\phi$ the Hamiltonian analysis is the same as in the previous model so that we will not repeat it here. There are additional terms that arise from the Hamiltonian analysis of the field $\psi$. We find that there are additional two primary constraints

$$P_C \approx 0, \quad P_D \approx 0,$$ (81)
that are variable conjugate to $C$ and $D$, respectively

$$\{C(x), P_D(y)\} = \delta(x - y), \quad \{D(x), P_D(y)\} = \delta(x - y).$$

(82)

The momentum conjugate to $\psi$ has the form

$$p_\psi = 2e^{-\phi}(D + 1)\sqrt{(3)g}n_\psi.$$

(83)

Then it is easy to perform the Legendre transformation with the resulting Hamiltonian in the form

$$H = \int d^4x \left(NC_0 + H_0 + v_Cp_C + v_Dp_D\right),$$

where

$$C_0 = \frac{1}{M_5^2\sqrt{(3)g}}\pi^{ij}\mathcal{G}_{ijkl}\pi^{kl} - M_5^2\sqrt{(3)g}\mathcal{R}^{(g)} + (\mathcal{R}_k + \pi_\phi \partial_k \phi + \pi_\psi \partial_k \psi)\tilde{D}^k\tilde{n}^l - 2m^2\sqrt{(3)g}\psi $$

$$+ \frac{3}{8}\frac{1}{\sqrt{(3)g}M_5^2}p_\phi^2 + \frac{2}{3}M_5^2\sqrt{(3)g}\delta^{ij}\partial_i \phi \partial_j \phi + M_5^2\sqrt{(3)g}V(\phi, C)$$

$$+ \frac{\phi}{4\sqrt{(3)g}(D + 1)}(p_\psi)^2 - \sqrt{(3)}ge^{-2\phi}(D + 1)g^{ij}\partial_i \phi \partial_j \psi + \sqrt{(3)}ge^{-2\phi}(D\mathcal{F} + C) + \sqrt{(3)}ge^{-2\phi}\mathcal{F}.$$

$$\mathcal{H}_0 = (M\tilde{n}^l + L^l)(\mathcal{R}_i + \pi_\phi \partial_i \phi + \pi_\psi \partial_i \psi) - 2m^2M\sqrt{(3)g}\mathcal{U}.$$

(85)

The preservation of the primary constraints $\pi_N \approx 0$, $\pi_\psi \approx 0$ again implies an existence of the secondary constraints $C_0, C_1$ when in $C_1$ we have an additional contribution $\partial_i \psi p_\psi$. On the other hand the preservation of the constraints $P_C \approx 0$, $P_D \approx 0$ implies following secondary constraints

$$G_C \equiv \frac{\delta C_0}{\delta C} \approx 0, \quad G_D \equiv \frac{\delta C_0}{\delta D} \approx 0.$$

(86)

These constraints together with $P_C \approx 0$, $P_D \approx 0$ form the second class constraints that can be solved for $C$ and $D$ as functions of canonical variables, at least in principle. Note that the presence of these constraints does not have consequence for the existence of the two constraints $C_0, C_1^{(i)}$ that are responsible for the elimination of the Boulware-Deser ghost. In other words, even the non-linear massive theory where the trace of the stress-energy tensor is dynamical variable is ghost-free.

V. DISCUSSION

In summary, using very clever ghost-free construction for massive generalization of General Relativity we propose the elegant way to formulate the class of ghost-free massive $F(R)$ gravities. The hamiltonian formulation of this theory is developed using its presentation in scalar-tensor form as well as analogy with the hamiltonian treatment of usual ghost-free massive gravity. Based on this analogy we prove that our theory turns out to be also ghost-free. The same strategy is applied for generalization of $F(R, T)$ gravity. Again, we demonstrate that its massive version turns out to be ghost-free.

Furthermore, the cosmological evolution in massive $F(R)$ gravity under consideration is studied. It turns out that Bianchi identity gives an equation which constrains the cosmic evolution quite strongly if compare with the case of convenient $F(R)$ theory where no such constraint appears. Nevertheless, the possibility of cosmic acceleration (especially, in the presence of cold dark matter) is established. Of course, the occurrence of cosmic acceleration in such theory is much restricted if compare even with massive $F(R)$ bigravity. Nevertheless, adding extra scalar fields in analogy with massive $F(R)$ bigravity may improve the occurrence of accelerated expansions. Using proposed strategy one can generate more massive extensions of modified gravity theories. This will be discussed elsewhere.

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Appendix A: Curvature in FRW metric with conformal time

We now give the explicit forms of connections and curvature in the FRW metric with conformal time in (9).

The non-vanishing components of the connections $\Gamma^i_{\mu\nu}$ are given by

$$\Gamma^i_{tt} = H, \quad \Gamma^i_{ij} = H \delta^i_j, \quad \Gamma^i_{tj} = \Gamma^j_{tt} = H \delta^i_j.$$  \hspace{1cm} (A1)

The non-vanishing components of the Ricci curvature and the scalar curvatures are given by

$$R_{yy} = -3\dot{H}, \quad R_{ij} = \left(\dot{H} + 2H^2\right) \delta_{ij}, \quad R = \frac{6}{a^2} \left(\dot{H} + H^2\right).$$  \hspace{1cm} (A2)
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