Spinor Representation of $O(3)$ for $S_4$

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All possible permutations in the discrete $S_4$ group are classified by three rotation angles associated with the orthogonal group $O(3)$. We construct a spinor representation $2_D$ of $O(3)$, which is transformed by four $4 \times 4$ matrices corresponding to four Pauli matrices in $SO(3)$. An irreducible decomposition of $2_D \otimes 2_D$ supplies a vector representation of $3$ of $O(3)$, thereby, of $S_4$.

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I. INTRODUCTION

Symmetries have taken a significant rôle in particle physics since the discovery of the $SU(3)$ symmetry in the hadron physics [1–8]. Nowadays, a possible rôles of symmetries is more important to discuss the current issues associated with neutrino oscillations, which contain the problem of neutrino masses. The origin of neutrino masses and their mass hierarchy inducing neutrino mixings, which are experimentally confirmed [9, 10], is not understood in the standard model and is expected to well understood by enlarging the standard model symmetry to include massive neutrinos. Modified standard models involve new flavor symmetries under which different species of neutrinos carry different flavor charges [11].

As a flavor symmetry of neutrinos, various non-Abelian discrete symmetries have been discussed to determine a mass spectrum of neutrinos as well as their mixings [12–14]. Among others, let us focus our attention to the discrete $S_4$ symmetry [13]. It is known that $S_4$ deals with all permutations among four objects $\{x_1, x_2, x_3, x_4\}$. Since $\xi_0 \propto x_1 + x_2 + x_3 + x_4$ is obviously invariant under any permutation of $S_4$, $\xi_0$ serves as a singlet $1$ of $S_4$. The remaining three degrees of freedom are described by a vector $\xi$:

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix},$$

where $\xi$'s are orthogonal to each other and also to $\xi_0$ and forms a triplet $3$ of $S_4$. All elements of $S_4$ acting on $3$ are, therefore, represented by $3 \times 3$ rotation matrices in the orthogonal group $O(3)$, which allows any dimensional representations other than three dimensional vector representation $[10, 17]$.

In this paper, we construct a spinor representation of $O(3)$ using two-component Pauli spinors of the special orthogonal group $SO(3)$, which turns out to be a four-component spinor to include a reflection in $O(3)$. This spinor is transformed by four $4 \times 4$ complex rotation matrices in $O(3)$. Our four-component spinor is essentially two-component one representing a doublet of $O(3)$, which is denoted by $2_D$. As a result, $S_4$ will contain two classes of doublets: one is the conventional one produced by $3 \otimes 3$ and the other is a new one generating $3$ out of $2_D \otimes 2_D$.

Mathematically speaking, there exists a double covering of $O(n)$, which is known as a $Pin(n)$ group and its subgroup is $Spin(n)$, which is a double covering of $SO(n)$. Some of physics-oriented studies are found in Ref. [18–21]. The spinor of $O(3)$ can be treated by the $Pin(3)$ group. However, we would like to develop more explicit and practical discussions on the spinor of $O(3)$ so that its applicability in particle physics becomes more visible.

This paper is organized as follows. In Sec. III we show a brief review of triplet representation $3$ in a vector space. In Sec. IV we present a new mathematical aspect of $S_4$ by constructing a doublet representation $2_D$ in a spinor space. To get a hint on a spinor representation of $O(3)$, we examine a “spinor” representation of $S_3$ associated with $O(2)$. Section V is devoted to a summary.

II. VECTOR REPRESENTATION

All permutations among four objects $\{x_1, x_2, x_3, x_4\}$ to the permuted state $\{x'_1, x'_2, x'_3, x'_4\}$ can be described by

$$x' = U x,$$

where

$$x' = \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \\ x'_4 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

and $U$ denotes a $4 \times 4$ matrix. The induced four objects of $\xi_{0,1,2,3}$ in Eq. (1) form $\xi$:

$$\xi = \begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix},$$

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix},$$

$$\xi = \begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix},$$

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix},$$

$$\xi = \begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix},$$

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix}.$$
which is also transformed into $\xi'$ by another $U$, $U'$, as

$$\xi' = U'\xi.$$  

(5)

Since $\xi$ is related to $x$ by an appropriate $4 \times 4$ matrix $W$:

$$\xi = Wx,$$  

(6)

$U'$ is given by

$$U' = WUW^{-1}.$$  

(7)

Because of $\xi'_0 = \xi_0$, a transformation $\xi \rightarrow \xi'$ is represented by a block diagonal matrix

$$\left(\begin{array}{c}
\xi_0' \\
\xi_1' \\
\xi_2' \\
\xi_3'
\end{array}\right) = \left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & * & * & * \\
0 & * & * & * \\
0 & * & * & *
\end{array}\right)
\left(\begin{array}{c}
\xi_0 \\
\xi_1 \\
\xi_2 \\
\xi_3
\end{array}\right),$$

(8)

and $\xi_{1,2,3}$ are mixed by three-dimensional rotations.

Since rotations in three dimensions are described in terms of three angles $\theta_{12}$, $\theta_{23}$ and $\theta_{13}$, we define the matrix $U'$ in Eq.(7) to be:

$$U'(\sigma, \theta_{12}, \theta_{13}, \theta_{23}) = \left(\begin{array}{c}
1 \\
0 \\
0 \\
0
\end{array}\right) T(\sigma, \theta_{12}, \theta_{13}, \theta_{23}),$$

(9)

where $\sigma = \text{det}(T)(= \pm 1)$ takes care of the reflection, and

$$T(\sigma, \theta_{12}, \theta_{13}, \theta_{23}) = \frac{1}{2} \left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & \cos \theta_{13} & 0 & \sin \theta_{13} \\
0 & 0 & 1 & 0 \\
\sigma \cos \theta_{12} & -\sin \theta_{12} & 0 & 0
\end{array}\right),$$

(10)

This $3 \times 3$ matrix $T$ acts on the triplet representation $3$ of $S_4$ in a vector space. Considering the arbitrariness of the inclusion of $\sigma$ in $T$, we may define

$$T_{23}(\sigma, \theta) = \left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & \sigma \cos \theta & -\sin \theta & 0 \\
0 & 0 & \sigma \sin \theta & \cos \theta \\
-\sin \theta & \sigma \cos \theta & 0 & 0
\end{array}\right),$$

$$T_{31}(\sigma, \theta) = \left(\begin{array}{cccc}
\cos \theta & 0 & \sigma \sin \theta & 0 \\
0 & 1 & 0 & 0 \\
-\sin \theta & \sigma \cos \theta & 0 & 0 \\
0 & 0 & \sigma \sin \theta & \sigma \cos \theta \\
0 & 0 & 1 & 0
\end{array}\right),$$

$$T_{12}(\sigma, \theta) = \left(\begin{array}{cccc}
\sigma \cos \theta & -\sin \theta & 0 & 0 \\
\sigma \sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right).$$

(11)

Any rotations in three dimensions can be generated by

$$T = T_{23}T_{31}T_{12}.$$  

(12)

It is noted that the matrices $T_{12,31,32}(\sigma, \theta)$ satisfy the relations of

$$T_{ij}(1, \theta_1)T_{ij}(1, \theta_2) = T_{ij}(1, \theta_1 + \theta_2),$$

$$T_{ij}(1, \theta_1)T_{ij}(-1, \theta_2) = T_{ij}(-1, \theta_1 + \theta_2),$$

$$T_{ij}(-1, \theta_1)T_{ij}(1, \theta_2) = T_{ij}(-1, \theta_1 - \theta_2),$$

$$T_{ij}(-1, \theta_1)T_{ij}(-1, \theta_2) = T_{ij}(1, \theta_1 - \theta_2),$$

(13)

where $(i, j) = (1, 2), (2, 3)$ and $(3, 1)$ (hereafter, we use $ij$ without referring the numbers). The tensor products of $S_4$ in a vector space are given by

$$3 \otimes 3 = 3' \otimes 3' = 1 \oplus 3 \oplus 2 \oplus 3',$$

$$3 \otimes 3' = 1' \oplus 3' \oplus 2 \oplus 3,$$

$$3 \otimes 2 = 3' \otimes 2 = 3 \oplus 3',$$

$$2 \otimes 2 = 1 \oplus 2 \oplus 1',$$

(14)

together with the obvious products of $1 \otimes 1 = 1' \otimes 1' = 1$, $1 \otimes 1' = 1'$, $1 \otimes 3 = 1' \otimes 3' = 3$, $1 \otimes 3' = 1' \otimes 3 = 3'$ and $1 \otimes 2 = 1' \otimes 2 = 2$, where prime stands for an antisymmetric representation [12, 22].

III. SPINOR REPRESENTATION

For $SO(3)$, the rotation matrices are restricted to the case of $\sigma = 1$. It is well known that the spinor representation uses the three Pauli matrices, $\tau_{1,2,3}$, which yield

$$S_{23}(1, \theta) = \exp(-\tau_3 \theta),$$

$$S_{31}(1, \theta) = \exp(-\tau_2 \theta),$$

$$S_{12}(1, \theta) = \exp(-\tau_1 \theta),$$

(15)

respectively, corresponding to $T_{23,31,12}$. For a two-component spinor of $SO(3)$, $\alpha = (\alpha_1, \alpha_2)^T$, if $\alpha$ is transformed into $\alpha'$ by $S_{23}(1, \theta)$, a vector $\vec{t}$ defined by

$$\vec{t} = \alpha^T \vec{r},$$

(16)

where

$$\vec{r} = \tau_1 \vec{i} + \tau_2 \vec{j} + \tau_3 \vec{k},$$

(17)

exhibits the same transformation property as that of $\vec{t}$. Namely, $\vec{t}' = T_{23}(1, \theta) \vec{t}$ for $\vec{t} = (t_1, t_2, t_3)^T$ and similarly for $S_{31,12}(1, \theta)$, respectively, giving rise to $\vec{t}' = T_{31}(1, \theta) \vec{t}$ and $\vec{t}' = T_{12}(1, \theta) \vec{t}$.

To get a hint to include the reflection for a spinor representation of $O(3)$, let us consider the simplest case of $O(2)$. For the vector representation, the rotation matrix $T(\sigma, \theta)$ is given by

$$T(\sigma, \theta) = \left(\begin{array}{cccc}
\sigma \cos \theta & -\sin \theta & 0 & 0 \\
\sigma \sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right),$$

(18)

acting on $(x_1, x_2)$, which is transformed into $(x_1', x_2')$. As a “spinor” representation, $z = x_1 + ix_2$ is transformed into $z' = e^{i\theta}z$ ($\sigma = 1$) and into $z' = -e^{i\theta}z^*$ ($\sigma = -1$), which suggest the use of

$$z = \left(\begin{array}{c}
z \\
z^*
\end{array}\right).$$

(19)
as a new basis. We obtain that \( z' = S(\sigma, \theta)z \) with the following \( S(\sigma, \theta) \):

\[
S(1, \theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix},
\]

\[
S(-1, \theta) = -\begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix}.
\] (20)

Since it can be further found that Eq. (20) is transformed into Eq. (18), nothing new arises from Eq. (20).

Although nothing new has been found in the case of \( O(2) \), it suggests that spinor representation matrices of \( O(3) \) may take the similar forms of

\[
\begin{pmatrix} a & 0 \\ 0 & a^* \end{pmatrix}, \quad \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}.
\] (21)

Since the entry of \( a \) can be one of the Pauli matrices, we expect that representation matrices are \( 4 \times 4 \) matrices to take care of the reflection in \( O(3) \). Furthermore, this construction implies that a 4-component spinor is something like \( (\alpha, \alpha^*) \).

We introduce a four-component spinor \( \psi \) defined by

\[
\psi = \begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix},
\] (22)

as a doublet representation \( 2_D \). The extended representation matrix of \( S_{ij}(1, \theta) \), now acting on \( \psi \), can be assumed to be the following form as suggested by the first form of Eq. (21):

\[
D_{ij}(1, \theta) = \begin{pmatrix} S_{ij}(1, \theta) & 0 \\ 0 & S_{ij}^*(1, \theta) \end{pmatrix},
\] (23)

which transforms the spinor \( \psi \) into \( \psi' \) as follows:

\[
\psi' = D_{ij}(1, \theta)\psi.
\] (24)

The extension of the Pauli matrices is straightforward and simply replaces \( \tau_i \) \((i = 1, 2, 3)\) with the following \( 4 \times 4 \) matrix \( \Sigma_i \):

\[
\Sigma_i = \begin{pmatrix} \tau_i & 0 \\ 0 & \tau_i^* \end{pmatrix}.
\] (25)

A vector \( \vec{s} \) can be defined by

\[
\vec{s} = \psi^\dagger \vec{\Sigma} \psi,
\] (26)

where

\[
\vec{\Sigma} = \Sigma_1 \vec{i} + \Sigma_2 \vec{j} + \Sigma_3 \vec{k}.
\] (27)

This vector \( \vec{s} \) is transformed under \( \psi \rightarrow \psi' = D_{ij}(1, \theta)\psi \) as

\[
\vec{s}' = \psi'^\dagger \vec{\Sigma} \psi'
= \psi'^\dagger \begin{pmatrix} u^\dagger(\theta) \tau u(\theta) & 0 \\ 0 & u^\dagger(\theta) \tau^* u^*(\theta) \end{pmatrix} \psi
= s_1' \vec{i} + s_2' \vec{j} + s_3' \vec{k},
\] (28)

where \( u(\theta) \) is either one of \( S_{ij}(1, \theta) \). It is readily found that the resulting transformation property of \( \vec{s} \) is the same as \( \vec{t} \) of Eq. (16), therefore, as \( \vec{\xi} \).

Let us proceed to construct appropriate matrices for the \( \sigma = -1 \) group of \( S_{ij}(-1, \theta) \). The constraints arise from the required transformation property given by \( T_{ij}(-1, \theta) \) acting on \( \vec{\xi} \). To find explicit forms of \( S_{ij}(-1, \theta) \), it is useful to employ the conventional definition of complex numbers arranged from a vector \( \vec{\xi} \):

\[
Z = \begin{pmatrix} \xi_3 & \xi_1 - i\xi_2 \\ \xi_1 + i\xi_2 & -\xi_3 \end{pmatrix},
\] (29)

from which Eq. (15) can be derived by requiring that

\[
Z' = u(\theta) Z u(\theta)^{-1}
\] (30)

induced by Eq. (11). The reflection can be implemented in the transformation of \( Z^* \) into \( Z' \) as suggested by \( z^* \) transformed into \( z' \) in the case of \( O(2) \):

\[
Z' = v(\theta) Z^* v(\theta)^{-1},
\] (31)

where \( v(\theta) \) is a \( 2 \times 2 \) matrix and is determined, up to possible phases, so as to reproduce \( \vec{\xi}' = T_{ij}(-1, \theta) \vec{\xi} \). Again, \( Z \) and \( Z^* \) are grouped into a block diagonal \( 4 \times 4 \) matrix of \( \text{diag}(Z, Z^*) \), which enables us to treat Eqs. (30) and (31) in a unified way.

If \( \psi \) is transformed by a matrix \( D_{ij}(-1, \theta) \):

\[
\psi' = D_{ij}(-1, \theta)\psi,
\] (32)

as a result of the action \( T_{ij}(-1, \theta) \), \( D_{ij}(-1, \theta) \) turns out to take the second form of Eq. (21) with \( a = v(\theta) \). Although, the computation is a bit involved, the forms of \( D_{ij}(-1, \theta) \) are very simple and described by \( S_{ij}(1, \theta) \) of \( SO(3) \). The obtained candidates of \( D_{ij}(-1, \theta) \) are as follows:

\[
D_{ij}(-1, \theta) = \begin{pmatrix} 0 & S_{ij}(-1, \theta) \\ S_{ij}^*(1, \theta) & 0 \end{pmatrix},
\] (33)

where

\[
S_{23}(-1, \theta) = S_{23}(1, \theta),
\]

\[
S_{31}(-1, \theta) = S_{31}(1, \theta) \tau_1,
\]

\[
S_{12}(-1, \theta) = S_{12}(1, \theta) \tau_3.
\] (34)

To be consistent, we should confirm that

\[
\vec{s}' = T_{ij}(-1, \theta) \vec{s},
\] (35)

where \( s = (s_1, s_2, s_3)^T \), is correctly derived by the action of Eq. (32). The vector \( \vec{s}' \), which is generated by \( D_{ij}(-1, \theta) \), is described by

\[
\vec{s}' = \psi'^\dagger \begin{pmatrix} v(\theta) & 0 \\ 0 & v(\theta)^\dagger \end{pmatrix} \psi,
\] (36)

where \( v(\theta) \) is either one of \( S_{ij}(-1, \theta) \). The derivations make the full use of \( S_{ij}(1, \theta) \) of \( SO(3) \):
\[ S_{23}^{T} (-1, \theta) = S_{23}^{T} (1, \theta) S_{23} (1, -\theta) \]
\[ = S_{23}^{T} (1, \theta) \left( \tau_{1} i - \tau_{2} j + \tau_{3} k \right) S_{23} (1, -\theta) \]
\[ = \tau_{1} i + (\tau_{2} \cos (-\theta) - \tau_{3} \sin (-\theta)) \left( -\bar{j} \right) \]
\[ + (\tau_{2} \sin (-\theta) + \tau_{3} \cos (-\theta)) \bar{k}, \]
which yields \( s' = T_{23}(-1, \theta)s; \)

\[ S_{31}^{T} (-1, \theta) = S_{31}^{T} (1, \theta) S_{31} (1, -\theta) \]
\[ = S_{31}^{T} (1, \theta) \left( \tau_{1} i - \tau_{2} j + \tau_{3} k \right) S_{31} (1, -\theta) \]
\[ = (\tau_{1} \cos (-\theta) + \tau_{3} \sin (-\theta)) i \]
\[ + \tau_{2} j + (\tau_{1} \sin (-\theta) + \tau_{3} \cos (-\theta)) \left( -\bar{k} \right), \]
which yields \( s' = T_{31}(-1, \theta)s; \)

\[ S_{12}^{T} (-1, \theta) = S_{12}^{T} (1, \theta) S_{12} (1, -\theta) \]
\[ = S_{12}^{T} (1, \theta) \left( \tau_{1} i - \tau_{2} j + \tau_{3} k \right) S_{12} (1, -\theta) \]
\[ = (\tau_{1} \cos (-\theta) - \tau_{2} \sin (-\theta)) \left( -\bar{i} \right) \]
\[ + (\tau_{1} \sin (-\theta) + \tau_{2} \cos (-\theta)) \bar{j} + \tau_{3} \bar{k}, \]
which yields \( s' = T_{12}(-1, \theta)s. \)

As a result, we find that Eq. (35) is reproduced. Finally, it should be noted that the relations
\[ D_{ij}(1, \theta_{1})D_{ij}(1, \theta_{2}) = D_{ij}(1, \theta_{1} + \theta_{2}), \]
\[ D_{ij}(1, \theta_{1})D_{ij}(-1, \theta_{2}) = D_{ij}(-1, \theta_{1} + \theta_{2}), \]
\[ D_{ij}(-1, \theta_{1})D_{ij}(1, \theta_{2}) = D_{ij}(-1, \theta_{1} - \theta_{2}), \]
\[ D_{ij}(-1, \theta_{1})D_{ij}(-1, \theta_{2}) = D_{ij}(1, \theta_{1} - \theta_{2}), \]
are satisfied for \( 2_D \), which are the same as those for \( 3 \) as shown in Eq. (35).

It is certain that our choice of \( D_{ij}(\pm 1, \theta) \) supplies correct representation matrices for the spinor of \( \psi \) as \( 2_D \).

The reflection on the \( SO(3) \) spinor causes the transformation owing to, for example, the 2-3 rotation as follows:
\[ \alpha' = \exp(-i\frac{\tau_{1}}{2}\theta)\alpha^*. \]

A new type of irreducible decomposition of
\[ 2_D \otimes 2_D = 1 + 3 \]
is added to \( S_{4i} \), where \( \psi^\dagger \psi = 1 \) and \( \psi^\dagger \Sigma_{1,2,3} \psi = 3 \). This is different from \( 2 \otimes 2 = 1 \oplus 2 \oplus 1^t \) for vector space.

**IV. SUMMARY**

We have succeeded to describe \( O(3) \) rotations acting on the four-component spinor \( \psi \), which contains the \( SO(3) \) spinor \( \alpha \) and its complex conjugate \( \alpha^* \): \( \psi = (\alpha, \alpha^*)^T \).

The four-component spinor behaves as a spinor doublet of \( O(3) \), \( 2_D \). The extended \( 4 \times 4 \) Pauli matrices can be given by the block diagonal form of \( \Sigma_{ij} \) \( (i=1,2,3) \). The transformation matrices describing the reflection consist of the well-known \( SO(3) \) rotation matrices \( S_{ij}(1, \theta) \) and are determined to be:
\[ S_{23}(-1, \theta) = S_{23}(1, \theta), \]
\[ S_{31}(-1, \theta) = S_{31}(1, \theta) \tau_{1} \]
\[ S_{12}(-1, \theta) = S_{12}(1, \theta) \tau_{3}. \]

We find that \( D_{ij}(\sigma, \theta) \) acting on \( \psi \) take the form:
\[ D_{ij}(1, \theta) = \begin{pmatrix} S_{ij}(1, \theta) & 0 \\ 0 & S_{ij}^*(1, \theta) \end{pmatrix}, \]
\[ D_{ij}(-1, \theta) = \begin{pmatrix} 0 & S_{ij}(-1, \theta) \\ S_{ij}^*(-1, \theta) & 0 \end{pmatrix}. \]

The effect of the reflection on \( \psi \) dictates from the transformation of the \( SO(3) \) spinor \( \alpha \). For example, under the \( 2 \)-\( 3 \) rotation with \( \sigma = -1 \), \( \alpha \) gets transformed according to \( \alpha' = \exp(-i\tau_{1}\theta/2)\alpha^* \).

The possible phenomenological application of the doublet representation \( 2_D \) to particle physics will be very limited because our spinor \( \psi \) involves the complex number and its complex conjugate, namely, a particle and an antiparticle. Detailed study of effects of \( 2_D \) on particle physics is left for our future study.

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