Sparse direction of arrival estimation of co-prime MIMO radar using sparse aperture completion

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Abstract: In this study, the authors consider the problem of direction of arrival (DOA) estimation in compressive sensing multiple-input and multiple-output (CS-MIMO) radars with co-prime receive arrays. A sparse aperture completion scheme is proposed to fill the ‘holes’ that in the difference co-array, achieving the full virtual receive aperture. Structured measurement matrices are devised in order that the output data of match filters can be seen as space–time compressed signal of a virtual uniform linear array. By employing the space–time recovery scheme, the sparse target scene can be accurately recovered while the amount of samples is further reduced. Numerical results demonstrate that the proposed scheme can achieve accurate DOA estimation with space–time compressed data and outperform the CS-based difference co-array methods.

1 Introduction

To solve the problem of a large amount of data to be processed in multiple-input and multiple-output (MIMO) radars [1–3], the application of compressive sensing (CS) to MIMO radars has been studied for years. In [4], the CS was applied to co-located MIMO radar, where CS was employed to reduce the number of samples in receive elements. The estimation of targets’ parameters as direction of arrival/departure (DOA/DOD) and Doppler shift from the output data of match filters can be concisely represented by $\theta$.

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The resulting CS-based difference co-array methods in [8, 9] cannot exploit the full virtual aperture since the existence of the ‘holes’. It only has an effective aperture with a contiguous set of elements between $(MN + M - 1)d_v$ and $(MN + M - 1)d_v$.

In order to fill the ‘holes’ in difference co-array and achieve the whole virtual aperture of CS-MIMO radar, a sparse aperture completion method based on measurement matrix design is proposed. Furthermore, we compress the number of pulses to expanding the sparse aperture completion to a space–time compressing problem. Hence, the performance of the entire virtual uniform linear array (ULA) can be achieved with fewer receive elements and snapshots. The least absolute shrinkage and selection operator (LASSO) algorithm is used in this paper for sparse recovery. Conventional method that used multiple signal classification (MUSIC) algorithm and the CS-based difference co-array methods in [8, 9] are also applied for comparison.

Throughout the paper, we use capital italic bold letters to represent matrices, and lowercase italic bold letters to represent vectors.

2 Signal model

Co-prime CS-MIMO radar, as illustrated in Fig. 2, consists of a transmit array with $N_T$ sensors and a co-prime pair of uniform linear receive sub-arrays, where one sub-array uses $2M$ sensors with an inter-element spacing of $N$ units, whereas the other uses $N$ elements with an inter-element spacing of $N$ units [7]. The $M$ and $N$ are chosen to be co-prime. The unit inter-element spacing $d$ is set as half-wavelength $\lambda/2$.

The resulting $N_R = 2M + N - 1$ receive sensors are positioned at

$$Z = \{Nmd, 0 \leq n \leq N - 1\} \cup \{Nmd, 0 \leq m \leq 2M - 1\}$$

Let $s_j(t)$ denote the waveform transmitted by the $j$th transmit antenna, then the matrix of transmitted signal is denoted as $S \triangleq [s_1(t), s_2(t), \ldots, s_{N_T}(t)]$. Suppose that the location of the $j$th

Fig. 1 Effective aperture of difference co-array
sensor is $Z_i$ and $Z_i \in Z, \forall i$. We assume there are $K$ targets located at $\phi_1, \phi_2, \ldots, \phi_K$, and

$$a_k(\phi_k) = [1, e^{j2\pi Z_1 \sin \phi_k}, \ldots, e^{j(2\pi Z_{N-1} \sin \phi_k)}]^T \quad (2)$$

is the transmit steering vector. Define the matrix of the received signal as $R = [r(t), r(t), \ldots, r_{Nt}]$. Let

$$a_k(\phi_k) = [1, e^{j2\pi Z_1 \sin \phi_k}, \ldots, e^{j(2\pi Z_{M-1} \sin \phi_k)}]^T \quad (3)$$

denote the receive steering vector. Thus, the signal received by the co-prime array relative to the $p$th pulse can be expressed as

$$R_p(t) = \sum_{k=1}^{K} \eta_{kp} a_k(\theta_k) a_k^T(\phi_k) S(t) + e(t) \quad (4)$$

where $\eta_{kp}$ is the complex amplitude proportional to the radar cross sections of the $k$th target relative to the $p$th pulse. The target's positions are assumed constant over the observation interval of $P$ pulses. We assume that the target gains $\{\eta_{kp}\}$ follow a Swerling Case II model, meaning that they are fixed during the pulse repetition interval $T$, and vary independently from pulse to pulse [10].

Let $x_p$ denote the matched filtered signal relative to the $p$th pulse. Assuming the length of $x_p$ is $L$, then the match filtered signal relative to the $p$th pulse can be expressed as

$$x_p = \text{vec}(R_p S^H L) \quad (5)$$

where $\text{vec}(\cdot )$ is to stack the columns of a matrix on top of each other. According to the theory of MIMO radar [1–3], $x_p$ is a $N_t N_r \times 1$ vector and it can be expressed as

$$x_p(t) = A s(t) + n(t) \quad (6)$$

where the $k$th column of steering matrix $A$ is $a_k(\theta_k) \otimes a_k(\phi_k)$ and $s(t) = \eta_{kp} e^{j2\pi \phi_k}$. $n(t)$ is an additive Gaussian white noise vector with zero mean and covariance $\sigma^2$.

Then the match filtered signal of $P$ pulses can be expressed as a $N_t N_r \times P$ matrix

$$X = [x_1, x_2, \ldots, x_P] \quad (7)$$

3 Proposed approach

Consider the scenario in which the receivers locations support the Nyquist array (virtual ULA) geometry with $(2M-1)N$ receive elements. We assume the received match filtered signal of the virtual ULA relative to the $p$th pulse is $X_p$. Neglecting the discretisation error, it is assumed that the target possible locations $\theta$ comply with a grid of $Q$ points. Then the receive steering vector of the virtual ULA is

$$e(\phi_k) = [1, e^{j2\pi Z_1 \sin \phi_k}, \ldots, e^{j(2\pi Z_{MN-1} \sin \phi_k)}]^T$$

let $\Psi_q$ denote the $q$th vector of the redundant dictionary

$$\Psi_q = e(\phi_q) \otimes a_k(\phi_q) \quad (8)$$

then the redundant dictionary $\Psi$ can be expressed as

$$\Psi = [\Psi_1, \Psi_2, \ldots, \Psi_Q] \quad (9)$$

We define a $Q \times 1$ vector $\theta_p = [\theta_1, \theta_2, \ldots, \theta_Q]$. If the $k$th target is located at $\phi_k$, we define $\theta_k = \eta_{kp}$. Otherwise, $\theta_k = 0$. Then the match filtered signal of the virtual ULA relative to the $p$th pulse can be expressed as

$$\tilde{X}_p = \Psi \Theta + E \quad (10)$$

where the sparse matrix $\Theta = [\theta_1, \theta_2, \ldots, \theta_P]$. The received noise $E = [e_1, e_2, \ldots, e_P]$.

The shortcoming of the Nyquist array setup is that the number of targets must scale linearly with the array aperture. The concept of co-prime array provides us a solution to overcome the shortcoming of the Nyquist array. The match-filtered signal of the co-prime receive array can be regarded as the space–time compression of $\tilde{X}$.

We define a spatial domain compression matrix $\Phi_r$ of size $(2M + N - 1) \times (2MN - N)$, which transforms the virtual ULA of dimension $(2M - 1)N$ to a real array of dimension $2M + N - 1$. The structure of $\Phi_r$ is decided by the positions of the receive sensors of the co-prime arrays. We define a $1 \times (2M - 1)N$ vector $U^r$.

$$u^r(x) = \begin{cases} 1, & x = z \\ 0, & \text{else} \end{cases} \quad (11)$$

then $\Phi_r$ can be arranged by $u^r$ corresponding to the position of sensors

$$\Phi_r = [(u^r)^T_1, (u^r)^T_2, \ldots, (u^r)^T_{2M+N-1}]^T \quad (12)$$

Then the channel compressing measurement matrix can be expressed as

$$\Phi_{\text{channel}} = I_{N_r} \otimes \Phi_r \quad (13)$$

We then define a temporal domain compression matrix $\Phi_{\text{pulse}}$ of size $P \times (P < P)$, which represents pulse compression at $P \times P$. $\Phi_{\text{pulse}}$ contains coefficients drawn i.i.d. from a random distribution. Thus the match filtered received signal of the co-prime receive array can be expressed as

$$\hat{X} = \Phi_{\text{channel}} \Phi_{\text{pulse}} \quad (14)$$

Equation (15) can be regarded as a two-dimensional projection problem. According to [11], it can be equivalently shown as
\[ x = (\Phi_{\text{pulse}} \otimes \Phi_{\text{chirp}}) \hat{x} = \Phi \hat{x} = \Phi \Psi \theta + e' \]  \tag{17}

where \( x = \text{vec}(X) \), \( \hat{x} = \text{vec}(\hat{X}) \) and \( \theta = \text{vec}(\Theta) \). \( \Psi = I_p \otimes \Psi \) and \( e' \) is the measured noise. In the expression of the above measurement vector, \( \Psi \) and \( \Phi \) are known and only \( \theta \) depends on the actual targets present in the illuminated area. The non-zero entries of \( \theta \) represent the target attenuation values and the corresponding indices represent the DOAs of targets.

Recovery of \( \theta \) from the measurement vector \( x \) can be achieved by solving the non-convex combination \( \ell_1\)-norm problem

\[
\min_{\theta} \| \theta \|_0 \quad \text{s.t.} \quad \| x - \Phi \Psi \theta \|_2 \leq \epsilon \quad \tag{18}
\]

A variety of polynomial complexity algorithms have been proposed for obtaining an approximate solution to (18). One family of methods is matching pursuit (MP). Among the matching pursuit algorithms, the most notable are orthogonal matching pursuit (OMP) [12] and CoSaMP [13]. Another family of methods is known as basis pursuit (BP). The BP strategy relaxes the \( \ell_1\)-norm in (15) with the \( \ell_2\)-norm [4]. The reformulation is known as LASSO, which is a convex problem and a global solution can be found in polynomial time. In this paper, the batch LASSO method is employed for sparse vector recovery.

Employing the aforementioned space–time measurement matrices, a virtual receive aperture equivalent to a ULA that has \( 2M - 1 \) elements can be achieved. Nevertheless, the difference co-array can only exploit a virtual receive aperture of \( MN + M - 1 \). Therefore, the proposed method can provide better detection performance of co-prime CS-MIMO radar, with more degree of freedom.

Here, we give a brief computation complexity analysis of the proposed method, the MUSIC-based method and the difference co-array method in [8, 9]. When applied to CS-MIMO radar, the MUSIC-based method process the complexity of \( O(M^2N^2 + MN) \) while the difference co-array method requires \( O(M^4N^2) \). Without computing the second-order information, the proposed method needs the complexity of \( O(KMN) \). It is obvious that the proposed method has low computation complexity and the comparison of the runtime will be shown in the numerical results.

According to [14], the Cramer–Rao bound (CRB) for DOA estimation in MIMO radar can be derived as

\[
\text{CRB} = \frac{\sigma^2}{\sum_j \text{Re}(\Phi^H(\Omega_j \Lambda_j D) \otimes P_j^H)^{-1}} \tag{19}
\]

where \( \otimes \) stands for Hadamard product and \( \sigma^2 \) is the noise level. \( \Omega_j = I - A(A^H A)^{-1}A^H \), \( D = [\partial \phi / \partial \theta_1, \ldots, \partial \phi / \partial \theta_N] \) and \( P = (1/L) \sum_{l=1}^L \mathbf{X}_l^H \mathbf{X}_l^H(l) \).

4 Numerical results

We consider a co-prime CS-MIMO radar with 35 transmitters and 12 physical receive sensors. The first layer of receive sensors is located at positions \( 0, 4, 8, 12, 16d \), and the second layer of receive sensors is located at positions \( 0, 5, 10, 15, 20, 25, 30, 35d \), with \( d \) taken as \( 2d / \sqrt{2} \). That is \( M = 4 \) and \( N = 5 \) in this case. These two layers generate a virtual array with consecutive lags from \( 0d \) to \( 35d \). We take the grid from \(-90^\circ \) to \( 90^\circ \) with step size \( 0.1^\circ \). We perform batch LASSO and compare the results with that of the CS based difference co-array method and MUSIC algorithm. We demonstrate that the proposed compressive sensing method can detect the DOA more accurately than the other two methods when they are applied to the co-prime CS-MIMO radar, with both higher resolution and lower estimated error.

Fig. 3 Spatial spectrum for all three methods with SNR = 0 dB

Fig. 4 Resolution performance (SNR = 0 dB)

4.1 Detection performance

In this first simulation, we demonstrate that for the co-prime CS-MIMO radar application the proposed method can detect targets more accurately than the difference co-array scheme and MUSIC method. We assume that there are 25 targets uniformly distributed between \(-60^\circ \) and \( 60^\circ \). The pulses number \( P = 10 \) and the compressed pulses number \( P = 5 \). The sampling length \( L = 128 \). The signal-to-noise ratio (SNR) is set to be 0 dB. The original DOAs are denoted by dashed lines in the figures.

We can see from Fig. 3 that the proposed CS method has better DOA estimation performance than the other two methods. In this experiment, the proposed CS method with LASSO detects all the targets correctly while both difference co-array scheme and MUSIC method miss some targets. On the other hand, the proposed method takes less time for computation. Here, the proposed method takes 0.75 s, the difference co-array scheme takes 1.03 s and MUSIC takes 1.51 s.

4.2 Resolution tests

In this numerical experiment, we test the resolution ability by detecting two closely located targets. The two targets are located at \( 30.3^\circ \) and \( 35.2^\circ \). The pulse number \( P = 10 \) and the compressed pulses number \( P = 5 \). The number of time samples is 128, and we perform the experiment with SNR equal to 0 and \(-10 \) dB.

We can see from Figs. 4 and 5 that the proposed method has better resolution performance than the other two methods. Difference co-array method and MUSIC method fail even when SNR equals to \(-10 \) dB in Fig. 5.

4.3 Estimation accuracy

In the last simulation, we test the estimation accuracy of all three methods. The CRB of the estimated error is also included. We
assume that there is one target randomly located in the range from −90° to 90°, with the SNR from −10 to 10 dB. We run 100 times Monte Carlo simulations for each SNR. Then the root mean squared errors of all three methods are shown in Fig. 6. We can see that for one target estimation, the proposed method has the best performance when SNR is lower since the number of pulses is compressed that the covariance matrix does not have enough second-order information for accurately target parameter estimation.

5 Conclusion

In this work, we apply co-prime array techniques to compressive sensing MIMO radar and reduce the number of receive elements. A sparse aperture completion scheme by designing structured measurement matrices is proposed to overcome the problem of ‘holes’ in the difference co-array. The proposed scheme is employed to fully utilise the virtual aperture and achieve a larger virtual aperture than that of CS-based difference co-array method. Numerical results show that the proposed method provides better DOA estimation performance than the CS-based difference co-array method and MUSIC algorithm with space–time compressed data. Future work can include a modified co-prime array configuration for further performance improvement.

6 Acknowledgment

This work was supported in part by the National Natural Science Foundation of China (61471191, 61071163, 61501233), Natural Science Foundation of Jiangsu Province (BK20181032) and scientific research foundation of Changshu Institute of Technology (KY2017105Z).

7 References

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Fig. 5 Resolution performance (SNR = −10 dB)

Fig. 6 Estimation accuracy for single target