A Note on the Symplectic Structure on the Dressing Group in the sinh–Gordon Model

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Abstract

We analyze the symplectic structure on the dressing group in the sinh–Gordon model by calculating explicitly the Poisson bracket \( \{ g \otimes g \} \) where \( g \) is the dressing group element which creates a generic one soliton solution from the vacuum. Our result is that this bracket does not coincide with the Semenov–Tian–Shansky one. The last induces a Lie–Poisson structure on the dressing group. To get the bracket obtained by us from the Semenov–Tian–Shansky bracket we apply the formalism of the constrained Hamiltonian systems. The constraints on the dressing group appear since the element which generates one solitons from the vacuum has a specific form.

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1 Introduction

The solitons are particle like solutions of the integrable non–linear equations \([1],[2]\). They describe elastic collision of solitary waves. After the interaction, the outgoing waves propagate with the same rapidities as the ingoing ones but with changed phases. The sine–Gordon theory is an example of an integrable model both at the classical and at the quantum level. The quantum scattering matrix was constructed in \([3]\) by using bootstrap methods. It is well known \([4]\) that the quantum sine –Gordon model for certain values of the coupling constant is a massive integrable perturbation of the minimal conformal models in two dimensions.

In \([5]\) the dressing group symmetry was proposed as an alternative approach to solve classical integrable models. It was also argued that the dressing group is a semiclasical limit of the quantum group symmetry of a quantum integrable model. The action of the dressing group is realized via gauge transformations which act on the components of the Lax connection. A non–trivial Poisson bracket, known as Semenov–Tian–Shansky bracket was introduced \([6]\) in order to ensure the covariance of the Poisson brackets on the phase space under the dressing group action. The dressing group together with the Semenov–Tian–Shansky bracket becomes a Lie–Poisson group.

In \([7]\) the dressing group elements which generate \(N\)–solitons from the vacuum in the sinh–Gordon model and in its conformally invariant extension \([8]\) are constructed explicitly. It was also shown that there is a relation between the dressing group and the vertex operator construction of the soliton solutions \([7],[9]\).

In the present note we calculate the Poisson bracket \(\{g \otimes g\}\) of the dressing group element \(g\) which generates an arbitrary one soliton solution in the sinh–Gordon model from the vacuum. We use the fact that the phase space of the one solitons is two dimensional \([1],[10]\). Surprisingly, the Poisson bracket found by us is not identical to the Semenov–Tian–Shansky expression which is proportional to \([r, g \otimes g]\) where \(r\) is the classical \(r\)–matrix \([1]\). Instead of the later we obtain the expressions \((3.9),(3.11)\). It is known \([7]\) that the dressing group element which produces one solitons from the vacuum has a special form. To reproduce it one has to impose constraints on the dressing group. We apply the formalism of the constrained Hamiltonian systems \([11]\) to show that our result for the bracket \(\{g \otimes g\}\) follows from the Semenov–Tian–Shansky bracket after a suitable reduction on the dressing group.

We outline the content of the paper. Sec. 2 is devoted to the one soliton solutions of the sinh–Gordon equation and to the construction of the dressing group element which produces the one solitons from the vacuum. In Sec. 3 we present our calculation of the Poisson bracket of the dressing group element and express it with the use of the classical \(r\)–matrix. In Sec. 4 we apply the general principles of the constrained dynamics in order to establish a relation between our bracket and the Semenov–Tian–Shansky bracket.
2 The sinh–Gordon equation, the one soliton solutions and the dressing group.

In this chapter we briefly review some basic facts concerning the sinh–Gordon model, its one soliton solutions and the dressing group \[1\], \[2\], \[5\], \[7\]. We start by recalling the sinh–Gordon equation in two dimensions

\[
\partial_+ \phi + \partial_- \phi = 2m^2 \sinh \phi \partial_\pm = \partial_{x_\pm} \quad (2.1)
\]

It is clear that it has a vacuum solution \( \phi = 0 \). The eq. (2.1) is equivalent to the zero curvature condition

\[
F_+ + F_- = \partial_+ A_- - \partial_- A_+ + [A_+, A_-] = 0
\]

of the connection

\[
A_\pm = \pm \partial_\pm \Phi + me^{+ad\Phi} E_\pm \quad (2.2a)
\]

\[
E_\pm = \lambda \mathbb{I} \pm \frac{1}{2} H 
\]

where \( E_\pm \) and \( H \) are the generators of the \( sl(2) \) Lie algebra

\[
[H, E_\pm] = \pm 2E_\pm \quad [E_+, E_-] = H \quad (2.3)
\]

and \( \lambda \) is the spectral parameter. The components of the Lax connection (2.2) belong to the loop algebra \( \tilde{sl}(2) \). In the principal gradation the last is generated by the elements \( E_n^\pm = \lambda^n E_\pm, \quad n \in 2\mathbb{Z} + 1 \) and \( H_n = \lambda^n H, \quad n \in 2\mathbb{Z} \). The commutation relations are the following

\[
[H_k, E_{i\pm}^\pm] = \pm 2E_{k+i}^\pm \quad [E_k^+, E_i^-] = H_{k+i} \quad (2.4)
\]

The flatness of the connection (2.2a) implies that there exists a solution of the linear system

\[
(\partial_+ + A_\pm) T(x^+, x^-, \lambda) = 0 \quad (2.5)
\]

which together with the initial condition \( T(0, 0, \lambda) = 1 \) is known in the literature as a normalized transport matrix.

The canonical symplectic structure \( \{ \partial_\tau \varphi(x,t), \varphi(y,t) \} = \delta(x-y) \) can be equivalently written in the form

\[
\{ A(x, t) \otimes A(y, t) \} = \frac{1}{4} [r, A(x, t) \otimes 1 + 1 \otimes A(y, t)] \delta(x-y) \quad (2.6a)
\]

\[
r = \frac{\lambda^2 + \zeta^2}{\lambda^2 - \zeta^2} H \otimes H
\]

\[
- 4 \frac{\lambda \zeta}{\lambda^2 - \zeta^2} (E^+ \otimes E^- + E^- \otimes E^+) \quad (2.6b)
\]
where $A = A_+ + A_-$ is the spatial component of the Lax connection; $\lambda$ and $\zeta$ are the spectral parameters corresponding the left and the right tensor factors respectively.

To introduce the one soliton solutions we consider the variable $\epsilon^+(x^+, x^-) = \epsilon^+(x)$ whose dependence on the light cone variables is dictated by the relation

$$\frac{\epsilon^+(x) + \mu}{\epsilon^+(x) - \mu} = a \exp\left\{2m\left(\mu x^+ + \frac{x^-}{\mu}\right)\right\}$$

$$a = \frac{\epsilon^+ + \mu}{\epsilon^+ - \mu}, \quad \epsilon^+ = \epsilon^+(0, 0) \quad (2.7)$$

The sinh–Gordon field is expressed in terms of $\epsilon^+$ and the soliton rapidity $\mu$ as follows

$$e^{-\varphi(x)} = -\frac{\epsilon^+(x)}{\mu} \quad (2.8)$$

We shall also need the variable $\epsilon^-(x)$ related to $\epsilon^+(x)$ and $\mu$ by $\epsilon^+(x)\epsilon^-(x) = \mu^2$ or equivalently

$$\frac{\epsilon^+(x) + \mu}{\epsilon^+(x) - \mu} = -\frac{\epsilon^-(x) + \mu}{\epsilon^-(x) - \mu} \quad (2.9)$$

In order to construct a solution of the linear problem (2.5) one observes that (2.7) is equivalent to

$$\psi_\pm(x, \mu) = \pm a \psi_\pm(x, -\mu) \quad (2.10)$$

where

$$\psi_\pm(x, \lambda) = \left(\epsilon^\pm(x) + \lambda\right) e^{-m(\lambda x^+ + \frac{x^-}{\lambda})} \quad (2.11)$$

It was shown in [12] that the matrix

$$\mathcal{T} = e^{\Phi(x)} \begin{pmatrix} \psi_+(x, \lambda) & \psi_+(x, -\lambda) \\ \psi_-(x, \lambda) & -\psi_-(x, -\lambda) \end{pmatrix} \quad (2.12)$$

satisfies the linear system (2.3) with $\varphi$ being the one soliton solution (2.8). Due to (2.10), the matrix $\mathcal{T}$ as a function on the spectral parameter $\lambda$ is degenerated at the points $\lambda = \pm \mu$. A direct calculation shows that

$$\det \mathcal{T} = 2(\lambda^2 - \mu^2) \quad (2.13)$$

The normalized transport matrix is

$$T(x, \lambda) = \mathcal{T}(x, \lambda) \cdot \mathcal{T}^{-1}(0, \lambda).$$

Starting from (2.12) one can construct algebraically solutions of the dressing problem. First we recall that the dressing group element which relates the vacuum to an one soliton solution is introduced by the equation

$$T(x, \lambda) = g(x, \lambda) \cdot T_0(x, \lambda) \cdot g^{-1}(0, \lambda) \quad (2.14)$$
where \( T_0(x, \lambda) = \exp\{-m(x^+ E_+ + x^- E_-)\} \) is the vacuum solution transport matrix. The last is expressed as \( T_0(x, \lambda) = T_0(x, \lambda) T_0^{-1}(0, \lambda) \); \( T_0(x, \lambda) \) has the same form as (2.12) but with \( \Phi = 0 \) and \( \psi_+(x, \lambda) = \psi_-(x, \lambda) = e^{-m(x^+ \lambda + \frac{x^-}{2\lambda})} \). The element \( g(x, \lambda) \) is expressed in terms of the non–normalized transport matrices \( T \) and \( T_0 \) as follows

\[
\begin{align*}
g(x, \lambda) &= T(x, \lambda) \cdot T_0^{-1}(x, \lambda) \cdot S(\lambda) \quad (2.15a) \\
S(\lambda) &= \begin{pmatrix} a(\lambda) & b(\lambda) \\
b(\lambda) & a(\lambda) \end{pmatrix} \\
det S(\lambda) &= \frac{1}{\mu^2 - \lambda^2} \quad (2.15b) \\
a(\lambda) &= a(-\lambda) \\
b(\lambda) &= -b(-\lambda) \quad (2.15c)
\end{align*}
\]

The form of the \( x^\pm \)–independent matrix \( S(\lambda) \) (2.15b) is fixed by the requirement that it should commute with the matrix \( T_0(x, \lambda) \). The equation (2.15c) guarantees that the dressing group element \( g(x, \lambda) \) has a unit determinant; the condition (2.15d) reflects the fact that it is represented in the principal gradation. Setting \( a(\lambda) + b(\lambda) = \lambda - \mu \) and \( a(\lambda) - b(\lambda) = -\lambda - \mu \) one recovers the solution constructed in [7].

\[
g(x, \lambda) = \begin{cases} e^{\Phi(x)} & \text{for } \lambda \to \infty \\
e^{-\Phi(x)} & \text{for } \lambda \to 0 \end{cases}
\]

We note that the above expression is not the unique which satisfies the following requirement: the matrix elements of \( g(x, \lambda) \) are meromorphic functions on the Riemann sphere with only simple poles at the points \( \lambda = \pm \mu \). There exist four solutions: \( a(\lambda) + b(\lambda) = (-)^p(\lambda - (-)^k\mu), a(\lambda) - b(\lambda) = (-)^{p-1}(\lambda + (-)^k\mu) \) where \( p, k = 0, 1 \). These solutions have the following asymptotic behaviour

\[
\begin{align*}
g(x, \lambda) &= (-)^p e^{\Phi(x)} + O(\lambda^{-1}) \quad \lambda \to \infty \\
g(x, \lambda) &= (-)^{p+k} e^{-\Phi(x)} + O(\lambda) \quad \lambda \to 0
\end{align*}
\]

The solution (2.16) is good since it turns to \( e^{\Phi} \) for \( \lambda \to \infty \) and to \( e^{-\Phi} \) when \( \lambda \to 0 \) [3]. More than that it permits to make a relation with the vertex operators.

### 3 Derivation of the Poisson bracket of the dressing group elements

In this paper we restrict ourselves to calculate the bracket \( \{g \otimes g\} \) for (2.16) only. The other choices for \( g \) mentioned above will be analyzed in [13]. We shall use the
coordinates \( \epsilon^+ \) and \( \mu \) to parametrize the phase space of the one solitons. It is clear that

\[ \{ g(\lambda) \otimes g(\zeta) \} = \left( \frac{\partial g(\lambda)}{\partial \epsilon^+} \otimes \frac{\partial g(\zeta)}{\partial \mu} - \frac{\partial g(\lambda)}{\partial \mu} \otimes \frac{\partial g(\zeta)}{\partial \epsilon^+} \right) \{ \epsilon^+, \mu \} \quad (3.1) \]

We recall that (2.16) contains dependence on the variable \( \epsilon^- \) but the last is a function on \( \epsilon^+ \) and \( \mu \) according to (2.9). To get an explicit expression for (3.1) we first obtain

\[ \frac{\partial g(\lambda)}{\partial \epsilon^+} g^{-1}(\lambda) = -\frac{1}{2\epsilon^+(\lambda^2 - \mu^2)} \left( (\lambda^2 + \mu^2)H + 2\lambda\mu(E^+ - E^-) \right) \]
\[ \frac{\partial g(\lambda)}{\partial \mu} g^{-1}(\lambda) = \left( \frac{\lambda^2 + \mu^2}{2\mu(\lambda^2 - \mu^2)} + \frac{(\epsilon^+)^2 - \mu^2}{\epsilon^+(\lambda^2 - \mu^2)^2} \right) H - \]
\[ -\frac{\lambda\mu\lambda^2 - (\epsilon^+)^2}{\epsilon^+(\lambda^2 - \mu^2)^2} E^+ - \frac{\epsilon^+}{\mu} \left( \frac{\lambda}{\lambda^2 - \mu^2} + 2 \frac{\lambda\mu}{\mu(\lambda^2 - (\epsilon^+)^2)} \right) \frac{\lambda^2 - (\epsilon^-)^2}{\lambda^2 - \mu^2} E^- \quad (3.2) \]

In what follows it will be convenient to introduce the following elements of the loop algebra \( \tilde{sl}(2) \)

\[ X^0(\lambda) = \frac{1}{\lambda^2 - \mu^2} \left( (\lambda^2 + \mu^2)H + 2\lambda\mu(E^+ - E^-) \right) \]
\[ X^\pm(\lambda) = \pm \frac{\lambda\mu}{\lambda^2 - \mu^2} \left( H + \left( \frac{\lambda}{\mu} \right)^\pm1 E^+ - \left( \frac{\mu}{\lambda} \right)^\pm1 E^- \right) \quad (3.3) \]

where we have omitted the dependence on \( \mu \) since we are going to keep the value of the rapidity fixed. The generators (3.3) are eigenvectors of the adjoint action of the element (2.16)

\[ g(\lambda)X^0(\lambda)g^{-1}(\lambda) = X^0(\lambda) \]
\[ g(\lambda)X^\pm(\lambda)g^{-1}(\lambda) = e^{\pm\varphi}X^\pm(\lambda) \quad (3.4) \]

We note that the expressions (3.3) provide two embeddings (one can consider the generators (3.3) as expansions on positive or negative powers on \( \lambda \)) of the \( sl(2) \) Lie algebra (2.3) in the loop algebra \( \tilde{sl}(2) \) which are given by

\[ H \rightarrow X^0(\lambda) \]
\[ E^\pm \rightarrow X^\pm(\lambda) \quad (3.5) \]

The fundamental two dimensional representation of the Lie algebra \( sl(2) \) in the realization (3.3) is spanned on the \( \lambda \)–depending vectors

\[ e_+(\lambda) = \left( \frac{1}{\mu} \right), \quad e_- (\lambda) = \left( \frac{-\mu}{\lambda} \right) \quad (3.6) \]
We also introduce the representation: \(\tilde{X}^0 = (X^0)^t, \tilde{X}^\pm = (X^\mp)^t\) where the upper index \(t\) stands for the matrix transposition. The corresponding basis dual to (3.6) is given by

\[
\tilde{e}_+ (\lambda) = \frac{\lambda^2}{\lambda^2 - \mu^2} \left( \frac{1}{\lambda} \right), \quad \tilde{e}_- (\lambda) = \frac{\lambda^2}{\lambda^2 - \mu^2} \left( \frac{\mu}{\lambda} \right)
\tag{3.7}
\]

Obviously \((\tilde{e}_a, e_b) = \tilde{e}_a^t \cdot e_b = \delta_{ab}\) for \(a, b = \pm\); \(X^+(\lambda)e_+ (\lambda) = \tilde{X}^+(\lambda)\tilde{e}_+ (\lambda) = 0\)

In terms of (3.3) the derivatives (3.2) are expressed as follows

\[
\frac{\partial g(\lambda)}{\partial \epsilon^+} g^{-1}(\lambda) = -\frac{1}{2\epsilon^+} X^0(\lambda)
\]

\[
\frac{\partial g(\lambda)}{\partial \mu} g^{-1}(\lambda) = \frac{1}{2\mu} X^0(\lambda) - \frac{\lambda}{\mu(\lambda^2 - \mu^2)} \left( (\mu + \epsilon^-)X^+(\lambda) + (\mu + \epsilon^+)X^-(\lambda) \right)
\tag{3.8}
\]

Substituing back the above expression into (3.1) we arrive at the expression

\[
\{g(\lambda) \otimes g(\zeta)\} \cdot g^{-1}(\lambda) \otimes g^{-1}(\zeta) = -\frac{1}{2} \left\{ \ln \epsilon^+, \mu \right\} \frac{\lambda}{\mu(\lambda^2 - \mu^2)} \left( (\mu + \epsilon^-)X^+(\lambda) + (\mu + \epsilon^+)X^-(\lambda) \right) \otimes X^0(\zeta) + \frac{1}{2} \left\{ \ln \epsilon^+, \mu \right\} \frac{\zeta}{\mu(\zeta^2 - \mu^2)} X^0(\lambda) \otimes \left( (\mu + \epsilon^-)X^+(\zeta) + (\mu + \epsilon^+)X^-(\zeta) \right)
\tag{3.9}
\]

In the basis (3.3) the \(r\)-matrix (2.6b) reads

\[
r = -\frac{\lambda\mu}{\lambda^2 - \mu^2} \left( X^+(\lambda) - X^-(\lambda) \right) \otimes X^0(\zeta) + 2\frac{\zeta\mu}{\zeta^2 - \mu^2} X^0(\lambda) \otimes \left( X^+(\zeta) - X^-(\zeta) \right) + \ldots
\tag{3.10}
\]

where we have omitted terms \(X^0 \otimes X^0\) and \(X^\pm \otimes X^\mp\) since they do not contribute to the commutator \([r, g \otimes g]\).

Taking into account (2.8), (3.4) and (3.10) we get the following expression for the bracket (3.3)

\[
\{g(\lambda) \otimes g(\zeta)\} = \frac{(Ad^{-1}g(\lambda) \otimes Ad^{-1}g(\zeta) - Adg(\lambda) \otimes Adg(\zeta))}{8} \cdot [r, g(\lambda) \otimes g(\zeta)]
\tag{3.11}
\]

In the derivation of the above equation we have also used the bracket

\[
\{\epsilon^+, \mu\} = -\frac{1}{2} \left( (\epsilon^+)^2 - \mu^2 \right) \tag{3.12}
\]
which was calculated in \[1\], \[10\].

The bracket (3.11) obviously differs from the Semenov–Tian–Shansky expression which reads

\[
\{ g(\lambda) \otimes g(\zeta) \}_{STS} = \frac{1}{4} [ r, g(\lambda) \otimes g(\zeta)]
\]

(3.13)

We point out that (3.13) was derived in \[5\] \[6\] from the requirement that the dressing group action is a Lie–Poisson action, i.e. the Poisson brackets of the model transform covariantly under the dressing transformations. On the other hand, the result (3.11) is a consequence of the bracket (3.12) which reflects the fact that the one–solitons can be represented as an one–body relativistic problem \[10\]. This observation suggests that (3.11) should be obtained from (3.13) by applying the theory of the constrained Hamiltonian systems. This will be done in the next section.

4 The Relation between the two brackets

A generic dressing group element \( f(\lambda) \in \tilde{SL}(2) \) is represented by a \( 2 \times 2 \) matrix with unit determinant, the elements of which \( f_{ij}(\lambda) \), \( i, j = 1, 2 \) are functions on \( \lambda \). The element (2.16) has a special form, and due to that we have to impose constraints on the dressing group in order to obtain it. To do that we first introduce the Gauss–like decomposition

\[
f(\lambda) = e^{a-(\lambda)X-}(\lambda)e^{\ln\kappa(\lambda)X^0(\lambda)}e^{a+(\lambda)X+}(\lambda)
\]

(4.1)

We note that, due to (3.3) for \( \lambda \to \infty \) one ends up with the decomposition of the Lie group \( SL(2) \): \( f = e^{a-(\infty)E^-}e^{\ln\kappa(\infty)H}e^{a+(\infty)E^+} \) while in the limit \( \lambda \to 0 \) the factorization \( f = e^{a-(0)E^-}e^{\ln\kappa(0)H}e^{a+(0)E^+} \) is valid. Therefore, in these limits we obtain two versions of the Gauss decomposition of the classical Lie group \( SL(2) \). We stress that (4.1) has nothing to do with the Gauss decomposition of the loop group.

Taking into account (3.3), (3.6) and (3.7) we obtain

\[
\kappa(\lambda) = \frac{\lambda^2}{\lambda^2 - \mu^2} \left( f_{11}(\lambda) - \frac{H}{\mu} (f_{12}(\lambda) - f_{21}(\lambda)) - \frac{\mu^2}{\lambda^2} f_{21}(\lambda) \right)
\]

(4.2a)

\[
a_+(\lambda) = \frac{f_{12}(\lambda) - \frac{H}{\mu} (f_{11}(\lambda) - f_{22}(\lambda)) - \frac{\mu^2}{\lambda^2} f_{21}(\lambda)}{f_{11}(\lambda) - \frac{H}{\mu} (f_{12}(\lambda) - f_{21}(\lambda)) - \frac{\mu^2}{\lambda^2} f_{21}(\lambda)}
\]

(4.2b)

\[
a_-(\lambda) = \frac{f_{21}(\lambda) + \frac{H}{\mu} (f_{11}(\lambda) - f_{22}(\lambda)) - \frac{\mu^2}{\lambda^2} f_{12}(\lambda)}{f_{11}(\lambda) - \frac{H}{\mu} (f_{12}(\lambda) - f_{21}(\lambda)) - \frac{\mu^2}{\lambda^2} f_{21}(\lambda)}
\]

(4.2c)

The inverse map is given by

\[
f_{ij}(\lambda) = (-)^i \frac{\lambda \mu}{\lambda^2 - \mu^2} \left( \frac{\lambda}{\mu} \right)^{-i-j+3}
\]
We next observe that the bracket \((3.13)\) can be explicitly written as follows

\[
\{f_{ij}(\lambda), f_{ij}(\zeta)\}_{\text{ST} S} = 0
\]  

\[
\{f_{ii}(\lambda), f_{ij}(\zeta)\}_{\text{ST} S} = -2\frac{\lambda^2 + \zeta^2}{\lambda^2 - \zeta^2} f_{ii}(\lambda) f_{ij}(\zeta) + 4\frac{\lambda \zeta}{\lambda^2 - \zeta^2} f_{ij}(\lambda) f_{ii}(\zeta)
\]  

\[
\{f_{ii}(\lambda), f_{ji}(\zeta)\}_{\text{ST} S} = 2\lambda^2 + \zeta^2 f_{ii}(\lambda) f_{ji}(\zeta) - 4\frac{\lambda \zeta}{\lambda^2 - \zeta^2} f_{ji}(\lambda) f_{ii}(\zeta)
\]  

\[
\{f_{ij}(\lambda), f_{ji}(\zeta)\}_{\text{ST} S} = 4\frac{\lambda \zeta}{\lambda^2 - \zeta^2} (f_{ii}(\lambda) f_{jj}(\zeta) - f_{jj}(\lambda) f_{ii}(\zeta))
\]  

\[
\{f_{ii}(\lambda), f_{jj}(\zeta)\}_{\text{ST} S} = 4\frac{\lambda \zeta}{\lambda^2 - \zeta^2} (f_{ij}(\lambda) f_{ji}(\zeta) - f_{ji}(\lambda) f_{ij}(\zeta))
\]  

It was shown in [10] that taking into account the bracket \((3.12)\) for the one soliton solutions, the Hamiltonians which generate the \(x^+\) and \(x^-\) flows, are proportional to \(\mu\) and \(\mu^{-1}\), respectively. In view of this remark we define the "canonical" Hamiltonian

\[
h(\lambda) = \frac{f_{11}(\lambda) - f_{22}(\lambda)}{f_{12}(\lambda) - f_{21}(\lambda)}
\]  

This is justified by the fact that when \(f(\lambda)\) is given by \((2.16)\) one immediately gets

\[
h(\lambda) = \frac{1}{2} \left( \frac{\lambda}{\mu} + \frac{\mu}{\lambda} \right)
\]  

Due to that for the dressing group elements \((2.16)\), the expression \((4.5)\) is a generating function of the \(x^\pm\) flows.

Denote by \(\bar{\Gamma}\) the abelian subgroup of the loop group \(\bar{\text{SL}}(2)\) characterized by the restrictions \(a_+(\lambda) = a_- (\lambda) = 0\) in the decomposition \((4.1)\). The element \((2.16)\) belongs to \(\bar{\Gamma}\). Alternatively \(\bar{\Gamma}\) is described by the primary constraints [11]

\[
c_1(\lambda) = f_{12}(\lambda) + f_{21}(\lambda) = 0
\]  

\[
c_2(\lambda) = f_{11}(\lambda) - f_{22}(\lambda) - \frac{\lambda^2 + \mu^2}{2\lambda \mu} (f_{12}(\lambda) - f_{21}(\lambda)) = 0
\]  

It is useful to use the Dirac's notation \(\approx\) (weak identity): functions on the loop group which coincide on \(\bar{\Gamma}\) are said to be weakly equal. Taking into account \((4.4a) - (4.4d)\) we get the weak identities

\[
\{h(\lambda), h(\zeta)\}_{\text{ST} S} \approx 0
\]  

\[
\{h(\lambda), c_1(\zeta)\}_{\text{ST} S} \approx 2\frac{\lambda^2 - \mu^2 \zeta}{\mu f_{12}(\lambda) (\lambda^2 - \zeta^2)} \det \left( \begin{array}{cc} \frac{\kappa(\lambda)}{\zeta^2 - \mu^2} f_{12}(\lambda) & \frac{\kappa(\zeta)}{\zeta^2 - \mu^2} f_{12}(\zeta) \\ \frac{\lambda^2 - \mu^2}{\zeta^2} f_{12}(\lambda) & \frac{\lambda^2 - \mu^2}{\zeta^2} f_{12}(\zeta) \end{array} \right)
\]  

\[
\{h(\lambda), c_2(\zeta)\}_{\text{ST} S} \approx 0
\]
Exploiting again (4.4b)–(4.4e) and the constraints (4.7a), (4.7b) we obtain
\[ \{c_1(\lambda), c_1(\zeta)\}_{ST S} = 0 \] (4.9a)
\[ \{c_1(\lambda), c_2(\zeta)\}_{ST S} \approx 4\frac{(\lambda^2 - \mu^2)\zeta}{(\lambda^2 - \zeta^2)\mu} \text{det} \begin{pmatrix} \frac{\kappa(\lambda)}{\lambda\mu} & \frac{\kappa(\zeta)}{\zeta\mu} \\ \frac{f_{12}(\lambda)}{\lambda^2} & \frac{f_{12}(\zeta)}{\zeta^2} \end{pmatrix} \] (4.9b)
\[ \{c_2(\lambda), c_2(\zeta)\}_{ST S} \approx 0 \] (4.9c)

The above equations together with the primary constraints (4.7a), (4.7b) allow us to state that the brackets \( \{h(\lambda), c_j(\zeta)\} \) and \( \{c_i(\lambda), c_j(\zeta)\} \) only vanish for the elements of \( \tilde{\Gamma} \) for which the parameter \( \kappa \) (4.1) does not depend on \( \lambda \). We also note that imposing on \( \tilde{\Gamma} \) the condition of \( \lambda \)-independence of \( \kappa \) one recovers (2.11) provided that
\[ \kappa = e^{\frac{\phi}{2}} \] (4.10)
as a consequence of (2.8), (2.16), (4.11), and (4.11a), (4.11b). That is why we impose as a secondary constraint
\[ c_3(\lambda) = \frac{d\kappa}{d\lambda}(\lambda) = 0 \] (4.11)

Following the Dirac’s algorithm we introduce the total Hamiltonian
\[ h_T(\lambda) = h(\lambda) + u^1(\lambda)c_1(\lambda) + u^2(\lambda)c_2(\lambda) \] (4.12)
where \( u^i(\lambda), i = 1, 2 \) are Lagrange multipliers. Obviously \( h_T(\lambda) \approx h(\lambda) \). Denote by \( \mathcal{F} \) the subgroup of \( \tilde{\Gamma} \) (4.1), (4.2) for which \( \kappa \) does not depend on \( \lambda \). In what follows we shall use the weak identity \( \approx \) for quantities which have equal values on \( \mathcal{F} \). The next step of the Dirac’s procedure is to try to fix the values of the Lagrange multipliers \( u^i(\lambda) \) by the conditions
\[ \{h_T(\lambda), c_i(\zeta)\} \approx 0, \ i = 1, 2, 3 \] (4.13)

The brackets \( \{c_i(\lambda), c_j(\lambda)\}, i, j = 1, 2, 3 \) (4.11a), (4.11b), (4.11d) vanish on \( \mathcal{F} \). Due to that \( c_i \) are first class constraints. Taking into account the weak identities
\[ \{h(\lambda), \kappa(\zeta)\} \approx 0 \]
\[ \{c_1(\lambda), \kappa(\zeta)\} \approx 4f_{12}(\lambda)\kappa(\zeta) \]
\[ \{c_2(\lambda), \kappa(\zeta)\} \approx 0 \]
\[ \{\kappa(\lambda), \kappa(\zeta)\} \approx 0 \] (4.14)

*More generally, instead of (4.12) one can consider the total Hamiltonian \( h_T(\lambda) = h(\lambda) + F^1[c_1(\lambda)] + F^2[c_2(\lambda)] \) where \( F^i \) are linear functionals. However, in what follows we shall demonstrate that (4.12) is sufficient to get a relation between the Semenov–Tian–Shansky bracket and the bracket (3.12).*
we verify that the Hamiltonian $h$ (4.6) has a vanishing Poisson brackets with all the constraints fixing $\mathcal{F}$. Therefore, the Lagrange multipliers in (4.12) can take arbitrary values. Imposing the relation
\[ \{ h_T(\lambda), \kappa(\zeta) \}_{STS} \approx \{ h(\lambda), \kappa \} \] (4.15)
we only fix
\[ u^1(\lambda) = \frac{1}{32} \left( \frac{\lambda^2 - \mu^2}{\lambda \mu} \right)^2 (\kappa + \kappa^{-1}) \] (4.16)
as consequence of (3.12) and (4.6).

We shall conclude by noting that since all the constraints are of the first class, our bracket (3.11) arises after a suitable "gauge fixing" from the Semenov–Tian–Shansky bracket.

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