The initial value problem for ordinary differential equations with infinitely many derivatives

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Abstract
We study existence, uniqueness and regularity of solutions for ordinary differential equations with infinitely many derivatives such as (linearized versions of) nonlocal field equations of motion appearing in particle physics, nonlocal cosmology and string theory. We develop a Lorentzian functional calculus via Laplace transform which allows us to interpret rigorously operators of the form \(f(\partial_t)\) on the half line, in which \(f\) is an analytic function. We find the most general solution to the equation

\[ f(\partial_t) \phi = J(t), \quad t \geq 0, \]

in the space of exponentially bounded functions, and we also analyze in full detail the delicate issue of the initial value problem. In particular, we state conditions under which the solution \(\phi\) admits a finite number of derivatives, and we prove rigorously that if a finite set of a priori data directly connected with our Lorentzian calculus is specified, then the initial value problem is well posed and it requires only a finite number of initial conditions.

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1. Introduction

In this paper we consider ordinary differential equations with an infinite number of derivatives. These equations have appeared recently as field equations of motion in particle physics [24–26], in string theory [8, 12, 14, 21, 22, 29, 31–34] and in gravity and cosmology [1, 3–6, 9, 23]. For instance, an important equation in this class is

\[ p^a \partial^a \phi = \phi^p, \quad a > 0. \]  (1)

Equation (1) describes the dynamics of the open \(p\)-adic string for the scalar tachyon field (see [1, 6, 12, 23, 31–33] and references therein), and it can be understood, at least formally, as an
equation in an infinite number of derivatives if we expand the left-hand side as a power series in $\partial_t^2$. This equation has been studied rigorously via integral equations of convolution type in [31, 32] (see also [1, 23]), and it has been also noted that in the limit $p \to 1$, equation (1) becomes the local logarithmic Klein–Gordon equation [7, 15].

In this paper we focus on analytic properties of nonlocal linear ordinary differential equations of the form

$$f(\partial_t)\phi = J(t)$$

for $t$ in the half line and investigate the initial value problem. We speak indistinctly of 'nonlocal equations' or 'equations in an infinite number of derivatives' in what follows. We remark that linear nonlocal equations appeared in mathematics already in the 1930s; see for instance [10] and further references appearing in [5]. More recently, they have been considered in connection with the modern theory of pseudo-differential operators; see for instance [13, 18, 30]. It appears to us, however, that a truly fundamental stimulus for their study has been the realization that linear and nonlinear nonlocal equations play a crucial role in contemporary physical theories.

A serious problem for the development of a rigorous theory for nonlocal equations has been the difficulties inherent in the understanding of the initial value problem for equations such as (1) and (2). These difficulties are carefully analyzed in the classical paper [14] by Eliezer and Woodard. One important remark appearing in [14] is that the natural idea of considering nonlocal ordinary equations

$$F(t, q', q'', \ldots, q^{(n)}, \ldots) = 0$$

as 'limits' of higher order equations of the form

$$G(t, q', q'', \ldots, q^{(n)}) = 0$$

is bound to be plagued with problems such as the phenomenon of Ostrogradsky instability known to appear in the (Lagrangian) study of equations such as (4); see [5, 14]. Another interpretative difficulty considered in [14] is the frequently stated argument (see for instance [3, 23]) that if an $n$th-order ordinary differential equation requires $n$ initial conditions, then the 'infinite-order' equation (3) requires infinitely many initial conditions, and therefore the solution $q$ would be determined a priori (via power series) without reference to the actual equation.

Our approach to these problems is, quite simply stated, to avoid thinking of equation (3) as a limit of finite-order equations. It seems to us that the Ostrogradsky instability should be investigated anew in this nonlocal world, and the same point of view should be taken with respect to the initial value problem.

In this paper we address the second difficulty mentioned above. Our point of view is to emphasize the fundamental role played by the Laplace transform—as a well-defined operator between Banach spaces, see [2, 11]—in the analysis of equation (2) via a 'Lorentzian functional calculus', and also in the correct formulation of the initial value problem. Using this approach we are able to generalize and formalize completely the interesting arguments advanced in [5, 6], in which the Laplace transform is explicitly mentioned as a natural and powerful tool for the study of nonlocal equations. Moreover, we can prove rigorously that if a finite set of a priori data directly connected with our Lorentzian calculus is specified, then the initial value problem is well posed and it requires only a finite number of initial conditions.

It is interesting that neither pseudo-differential operators nor generalized functions (see for instance [13, 18, 30]) appear explicitly in this work: our 'symbol' $f(s)$ appearing in equation (2) is (in principle) an arbitrary analytic function which may be beyond the reach of classical tools (such as the ones appearing in [20]) and, moreover, we stress once again that we wish
to solve nonlocal equations on the semi-axis \( t \geq 0 \), and not on the real line where perhaps we could use Fourier transforms and/or classical pseudo-differential analysis, see [20] and also [17]. The importance of studying nonlocal equations on a semi-axis has been recently stressed by Aref’eva and Volovich [1]: classical versions of Big Bang cosmological models contain singularities at the beginning of time, and therefore the time variable appearing in the field equations should vary over a half line. In [1], an alternative analysis of nonlocal equations also appears which applies (for instance) to linearized versions of nonlocal tachyon fields and \( p \)-adic strings, but not to linear equations with general analytic symbols. We also point out that a theory of pseudo-differential operators on the semi-axis has been recently developed by Hayashi and Kaikina [18], but the precise relation between this theory and our work remains to be investigated.

This paper is organized as follows. In section 2, we consider a rigorous interpretation of nonlocal ordinary differential equations via a Lorentzian functional calculus founded on the Laplace transform. For instance, our approach can be applied to the following equation:

\[
\sqrt{\partial_t^2 - m^2} \phi = J(t), \quad t \geq 0,
\]

(see [5, 13]) in the space \( L^\infty_{\omega}(\mathbb{R}_+) \), \( \omega \in \mathbb{R} \), of exponentially bounded functions. In section 3, we investigate (and propose a method for the solution of) initial value problems for linear equations of the form (2) and, finally, in section 4 we present some concluding remarks. This paper generalizes and refines our previous work [16], where we considered only entire functions \( f \), and we presented a preliminary version of our functional calculus.

2. Ordinary nonlocal differential equations

Motivated by the Barnaby–Kamran work [5, 6], and also by the interesting papers [3, 23, 31, 32], we are particularly interested in setting up a rigorous framework in which the initial value problem for nonlocal equations can be unambiguously understood. As stated in section 1, in this work we consider linear ordinary nonlocal equations of the form

\[
f(\partial_t)\phi(t) - J(t) = 0, \quad t > 0.
\]

(6)

This equation belongs to a class of equations studied long ago by Carmichael and others [10, 5]. In fact, Carmichael in [10] states and proves a theorem on the existence of solutions for a class of equations of the form (6) (see also [5]) using power series expansions and techniques from classical complex analysis. Our approach is certainly based on this earlier work, but we go beyond it in that we set up a rigorous ‘Lorentzian’ functional calculus based on the Laplace transform. As we will show, this approach effectively brings to the fore the ‘initial conditions’ issue.

We remark once more that solving (6) is not a simple application of the theory of classical pseudo-differential operators, since the ‘symbols’ \( f(s) \) appearing in (6) are in principle arbitrary (analytic or entire) functions and our aim is to solve nonlocal equations on the semi-axis \( t \geq 0 \).

2.1. Lorentzian functional calculus

We fix a number \( \omega \in \mathbb{R} \). The space \( L^\infty_{\omega}(\mathbb{R}_+) \) is the Banach space of all complex-valued and exponentially bounded functions on \( \mathbb{R}_+ \),

\[
L^\infty_{\omega}(\mathbb{R}_+) = \{ \phi \in L^1_{\text{loc}}(\mathbb{R}_+) : \| \phi \|_{\omega, \infty} = \text{ess sup}_{t \geq 0} | e^{-\omega t} \phi(t) | < \infty \}.
\]
and the Widder space $C_\infty^W(\omega, \infty)$ is

$$C_\infty^W(\omega, \infty) = \left\{ r : (\omega, \infty) \rightarrow \mathbb{C} / \| r \|_W = \sup_{n \in \mathbb{N}, r > \omega} \left| \frac{(s - \omega)^{n+1}}{n!} r^{(n)}(s) \right| < \infty \right\}.$$  

As proven in the monograph [2], functions in $C_\infty^W(\omega, \infty)$ extend analytically to the region $\text{Re}(s) > \omega$. Moreover, the Laplace transform

$$\phi \in L_\infty^\omega(\mathbb{R}_+) \mapsto \mathcal{L}(\phi)(s) = \int_0^\infty e^{-st} \phi(t) \, dt \in C_\infty^W(\omega, \infty)$$

is an isometric isomorphism from $L_\infty^\omega(\mathbb{R}_+)$ onto $C_\infty^W(\omega, \infty)$. We use this isomorphism to define the operator $f(\partial_t)$ appearing in equation (6). Specifically, we construct a functional calculus which allows us to consider $f(\partial_t)$ as a linear operator on $L_\infty^\omega(\mathbb{R}_+)$. 

First of all, we make a technical assumption on the function $f$, generalizing our previous work [16] where we considered only entire functions: we assume that there exist real numbers $R_f > 0$ and $\omega_f$ with $\omega_f < R_f$, such that the function $f$ is analytic in the domain

$$\{ s : |s| < R_f \} \cup \{ s : \text{Re}(s) > \omega_f \}.$$  

We note that it follows from this assumption that the Taylor series expansion of $f$ around zero converges absolutely for $|s| < R_f$ (see for instance [19, p 196]). For ease of reference we say that a function $f$ satisfying the above condition belongs to the class $\Gamma$. 

As motivation, let us perform a formal calculation of $f(\partial_t)\phi$: take $f$ in the class $\Gamma$ and $\phi \in L_\infty^\omega(\mathbb{R}_+)$, and let us write

$$f(\partial_t) = \sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!} \partial_t^n,$$

so that if $\phi$ is smooth, standard properties of the Laplace transform [11, 28] yield

$$\mathcal{L}(f(\partial_t)\phi) = \sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!} \mathcal{L}(\partial_t^n \phi)$$

$$= \sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!} \left( s^n \mathcal{L}(\phi) - s^{n-1} \phi(0) - s^{n-2} \phi'(0) - \ldots - \phi^{(n-1)}(0) \right)$$

$$= \sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!} \left( s^n \mathcal{L}(\phi) - \sum_{j=1}^n s^{n-j} \phi^{(j-1)}(0) \right)$$

$$= f(s)\mathcal{L}(\phi)(s) - \sum_{n=0}^\infty \sum_{j=1}^n \frac{f^{(n)}(0)}{n!} s^{n-j} \phi^{(j-1)}(0).$$

(7)

If we define the formal series

$$r(s) = \sum_{n=0}^\infty \sum_{j=1}^n \frac{f^{(n)}(0)}{n!} d_{j-1} s^{n-j},$$

(8)

in which $d = \{ d_j : j \geq 0 \}$ is a sequence of complex numbers, then we can write (7) as

$$\mathcal{L}(f(\partial_t)\phi) = f(s)\mathcal{L}(\phi)(s) - r,$$

where $d_j = \phi^{(j)}(0)$. 

Motivated by this computation, in our previous paper [16] we constructed a functional calculus using directly series of the form (8). A slightly improved version of our previous definition runs as follows.
Let \( f \) be a function belonging to the class \( \Gamma \), set

\[
\Lambda = \{ d = (d_j)_{j \geq 0} : \text{the series } r \text{ given by (8) converges for } \Re(s) > \omega_f, \}
\]

and let \( \mathcal{R} = \{ r : d \in \Lambda \} \). Then, the subspace \( Df \) of \( L^\infty(\mathbb{R}_+) \times \mathcal{R} \) consisting of all pairs \((\phi, r)\) such that

\[
\left( \phi, r \right) = f \mathcal{L}(\phi) - r
\]

belongs to the Widder space \( C^\infty_{\omega_f}(\omega_f, \infty) \) is a domain for \( f(\partial_\nu) \) with

\[
f(\partial_\nu)(\phi, r) = \mathcal{L}^{-1}\left( (\phi, r) \right) = \mathcal{L}^{-1}\left( f \mathcal{L}(\phi) - r \right).
\]

We note that with this definition, the operator \( f(\partial_\nu) \) is a linear operator on \( Df \), a fact on which we did not insist in [16].

This definition is reasonable in the sense that it is directly related to (7) and it generalizes the standard case of polynomial functions \( f \). Indeed, if \( f(s) = s^N \), then \( f(\partial_\nu) \) can be interpreted simply as \( \partial^N_\nu \). In this case, for each sequence \( d = (d_j : j \geq 0) \) the series \( r \) given by (8) has only \( N \) terms and, if we take a pair \((\phi, r) \in Df \) (as defined above), then equations (10) and (13) yield

\[
\mathcal{L}(f(\partial_\nu)(\phi, r)) = \left( \phi, r \right) = s^N \mathcal{L}(\phi) - d_0 s^{N-1} - d_1 s^{N-2} - \cdots - d_{N-1},
\]

while on the other hand, if \( \phi \) is a function of class \( C^N \), we have

\[
\mathcal{L}(\partial^N_\nu \phi) = s^N \mathcal{L}(\phi) - \phi(0)s^{N-1} - \phi^{(1)}(0)s^{N-2} - \cdots - \phi^{(N-1)}(0).
\]

Thus, we can make two observations. (a) Our definition is compatible with our intuition, that is, \( f(\partial_\nu)(\phi, r) = \partial^N_\nu \phi \), if \( d_j = \phi^{(j)}(0) \) for \( j = 1, \ldots, N \), but in principle other choices of \( d \) are possible. (b) The ‘remainder’ series \( r \) given by (8) really encodes all information on initial values of solutions to equations of the form \( f(\partial_\nu)\phi = J \).

Regrettably, it is technically difficult to operate directly with series \( r \) in the study of the initial value problem. This fact, and the observations above, have led us to considering a more abstract approach.

**Definition 2.1.** Let \( f \) be a function belonging to the class \( \Gamma \) and let \( \mathcal{R} \) be the space of analytic functions on \( \mathbb{C} \). We consider the subspace \( Df \) of \( L^\infty(\mathbb{R}_+) \times \mathcal{R} \) consisting of all the pairs \((\phi, r)\) such that

\[
\left( \phi, r \right) = f \mathcal{L}(\phi) - r
\]

belongs to the Widder space \( C^\infty_{\omega_f}(\omega_f, \infty) \). The domain of \( f(\partial_\nu) \) as a linear operator on \( L^\infty(\mathbb{R}_+) \times \mathcal{R} \) is \( Df \). If \((\phi, r) \in Df \), then

\[
f(\partial_\nu)(\phi, r) = \mathcal{L}^{-1}\left( (\phi, r) \right) = \mathcal{L}^{-1}\left( f \mathcal{L}(\phi) - r \right).
\]

Equipped with definition 2.1, we now study equation (6), generalizing our work reported in [16]. Before doing so, however, let us reconsider the example given after equation (11). We would like to precise the intuitions appearing in the physics literature [3–6, 14, 23] where operators of the form \( f(\partial_\nu) \) are defined (for appropriate functions \( f \)) by power series in \( \partial_\nu \), that is, as ‘operators in an infinite number of derivatives’. Following our previous papers [16, 17], we point out that this idea can be formalized using a natural extension of the classical theory of analytic vectors [27].

**Definition 2.2.** Let \( A \) be a linear operator on a Banach space \( X \) and let \( f \) be a complex-valued function such that \( f^{(n)}(0) \) exists for all \( n \geq 0 \). We say that \( \phi \in X \) is a \( f \)-analytic vector for \( A \) if \( \phi \) is in the domain of \( A^n \) for all \( n \geq 0 \) and the series

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} A^n(\phi)
\]

defines a vector in \( X \).
We observe that in our case $f$-analytic vectors exist.

**Lemma 2.1.** Fix a function $f$ as in definition 2.2. We have

(a) If $\psi$ is a polynomial function on $\mathbb{R}^+$, $\psi$ is an $f$-analytic vector for $\partial_t$ on $L^\infty_0(\mathbb{R}^+)$.

(b) If $\psi$ is a $C^\infty$ function on $\mathbb{R}^+$ such that $|e^{-\omega t} \psi^{(n)}(t)| \leq M t^n$ for $t < R_f$ and $\psi^{(n)}(t) = 0$ for $t \geq R_f$, then $\psi$ is an analytic $f$-vector for the operator $\partial_t$ on $L^\infty_0(\mathbb{R}^+)$.

**Proof.** Part (a) is trivial. For (b) we simply note that

$$\left| e^{-\omega t} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \partial_t^n \psi(t) \right| \leq \sum_{n=0}^{\infty} \frac{|f^{(n)}(0)|}{n!} |t|^n M$$

and the right-hand side of this inequality is finite by hypothesis. $\square$

We also have the following proposition, connecting $f$-analytic vectors with definition 2.1.

**Proposition 2.1.** Assume that $f$ is entire, let $\phi \in L^\infty_0(\mathbb{R}^+)$ be a smooth $f$-analytic vector for $\partial_t$ and consider the series $r$ given by (8) with $d_j = \phi^{(j)}(0)$. We assume that

$$|d_j| \leq CR^j,$$

in which $0 < R < 1$. Then, $(\phi, r) \in D_f$.

**Proof.** Condition (14) implies that the series $\sum d_{j-1}/s^j$ converges absolutely for $|s| > R$. It follows from lemma 2.1 in [16] that the series $r$ is in fact an entire function, and then proposition 2.1 of [16] allows us to conclude that $(\phi, r) \in D_f$. $\square$

We interpret this proposition as stating that, on smooth $f$-analytic vectors, the operator $f(\partial_t)$ can indeed be rigorously understood as an operator in an infinite number of derivatives, and that this fact is consistent with our definition 2.1.

2.2. Linear nonlocal equations

In this subsection we solve the nonlocal equation

$$f(\partial_t)(\phi, r) = J$$

using the Lorentzian functional calculus developed above. From now on we will assume that a function $r \in \mathcal{R}$ has been fixed. We understand equation (15) as an equation for $\phi \in L^\infty_0(\mathbb{R}^+)$ such that $(\phi, r) \in D_f$. We simply write $f(\partial_t)\phi = J$ instead of (15). First of all, we formalize what we mean by a solution.

**Definition 2.3.** Let us fix a function $r \in \mathcal{R}$. We say that $\phi \in L^\infty_0(\mathbb{R}^+)$ is a solution to equation $f(\partial_t)\phi = J$ if and only if

1. $\hat{\phi} = f \mathcal{L}(\phi) - r \in C^\infty_W(\omega_f, \infty)$ (i.e. $(\phi, r) \in D_f$);
2. $f(\partial_t)\phi = \mathcal{L}^{-1}(f \mathcal{L}(\phi) - r) = J$.

Our main theorem on existence and uniqueness of solutions to the linear problem (15) is as follows.

**Theorem 2.1.** Let us fix a function $f$ in $\Gamma$ and a function $J \in L^\infty_0(\mathbb{R}^+)$. We assume that the function $(\mathcal{L}(J) + r)/f$ is in the Widder space $C^\infty_W(\omega_f, \infty)$. Then, the linear equation

$$f(\partial_t)\phi = J$$

can be uniquely solved for $\phi \in L^\infty_0(\mathbb{R}^+)$. The solution is given by the explicit formula

$$\phi = \mathcal{L}^{-1}\left(\frac{\mathcal{L}(J) + r}{f}\right).$$
Corollary 2.1. Since $J \in L^\infty_0(\mathbb{R}_+)$, it follows that $(\phi, r)$, where $\phi = L^{-1}((L(J) + r)/f)$, is in the domain $D_f$ of the operator $f(\hat{\cdot})$: indeed, an easy calculation shows that $\hat{\phi} = L(\hat{J})$, which is an element of $C^\infty_0(\omega_J, \infty)$. We can then check directly that $\phi$ defined by (17) is a solution of (16). The isomorphism between $L^\infty_0(\mathbb{R}_+)$ and $C^\infty_0(\omega_J, \infty)$ given by the Laplace transform implies uniqueness.

Remark. We can prove uniqueness using only definition 2.3 as follows: let us assume that $\phi$ and $\psi$ are solutions to equation (16). Then, item 2 of definition 2.3 implies $f L(\phi - \psi) = 0$. Set $h = L(\phi - \psi)$ and suppose that $h(z_0) \neq 0$. By analyticity, $h(z) \neq 0$ in a neighborhood $U$ of $z_0$. But then $f = 0$ in $U$, so that (again by analyticity) $f$ is identically zero.

Theorem 2.1 can be considered as an abstract version of the Carmichael theorem on solutions to (16) as it appears in [10, 5]. It is important to note that the solution (17) does not need to be either analytic or even differentiable: at this stage all we know is that (17) is an element of $L^\infty_0(\mathbb{R}_+)$, so that in complete generality we cannot even formulate an initial value problem. In section 3, we impose further conditions on $J$ and $f$ which assure us that (17) is smooth at $s = 0$, and then, under these conditions, we study the initial value problem for (16). Previous discussions on the initial value problem appearing in the physics literature are, for instance, [3–5, 14, 23].

We recall that we have fixed an analytic function $r$. Now we assume the growth condition

$$|r(s)/f(s)| \leq C |s|^p \quad (18)$$

for all $|s|$ sufficiently large and some $p > 0$, and we examine three special cases of theorem 2.1: (a) the function $r/f$ has no poles; (b) the function $r/f$ has a finite number of poles; (c) the function $r/f$ has an infinite number of poles.

Corollary 2.1. Assume that the hypotheses of theorem 2.1 hold, that $L(J)/f$ is in $C^\infty_0(\omega_J, \infty)$ and that $r/f$ is an entire function such that (18) holds. Then, solution (17) to equation (16) is simply $\phi = L^{-1}(\frac{L(J)}{f})$. In addition, if $1/f$ belongs to $C^\infty_0(\omega_J, \infty)$, then solution (17) can be written as a convolution, $\phi = q * J$, in which $q$ is uniquely determined by $L(q) = 1/f$.

Proof. The growth condition (18) implies that there exists $M > 0$ such that $|r(s)/f(s)|$ is bounded for $|s| > M$. On the other hand, the function $|r(s)/f(s)|$ is bounded for $|s| \leq M$ simply by continuity. Thus, $r(s)/f(s)$ is an entire function with bounded module, and therefore $r(s)/f(s) = C_0$ for some constant complex number. But then (18) implies that $C_0 = 0$, and the result follows from the general formula (17). □

Corollary 2.2. Assume that the hypotheses of theorem 2.1 hold, and that $f$ has a finite number of poles $\omega_i$ ($i = 1, \ldots, N$) of order $r_i$ to the left of $\text{Re}(s) = \omega_J$, Suppose also that $L(J)/f$ is in $C^\infty_0(\omega_J, \infty)$, and that the growth condition $\left|\frac{r(s)}{f(s)}\right| \leq \frac{C}{|s|^p}$ holds for all $|s|$ sufficiently large and some $p > 0$. Then, solution (17) can be written in the form

$$\phi(t) = \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} e^{st} \left(\frac{L(J)}{f}\right) \, ds + \sum_{i=1}^{N} P_i(t) e^{\omega_i t}, \quad (19)$$

in which $P_i(t)$ are polynomials of degree $r_i - 1$. 7
Proof. We first note that the quotient $r/f \in C^\infty_W(\omega_j, \infty)$, since $(\mathcal{L}(f) + r)/f$ and $\mathcal{L}(f)/f$ are in $C^\infty_W(\omega_j, \infty)$, by assumption. We compute using the general solution (17) and the inversion formula for the Laplace transform [11, 28]:

$$
\phi(t) = \frac{1}{2\pi i} \int_{\omega_j - i\infty}^{\omega_j + i\infty} e^{s t} \mathcal{L}(f)/f \, ds + \frac{1}{2\pi i} \int_{\omega_j - i\infty}^{\omega_j + i\infty} e^{s t} f \, ds.
$$

(20)

Our growth condition on $r/f$ assures us that the assumptions for the evaluation of the inversion formula via the calculus of residues, [11, section 26], are verified. Thus, we can calculate the last integral on the right-hand side of (20) as

$$
\frac{1}{2\pi i} \int_{\omega_j - i\infty}^{\omega_j + i\infty} e^{s t} f \, ds = \sum_{n=1}^{N} \text{res}_i(t),
$$

in which $\text{res}_i(t)$ denotes the residue of $\frac{a(t)}{f(t)}$ at $\omega_i$. In order to compute $\text{res}_i(t)$ we consider the Laurent expansion around the pole $\omega_i$, that is

$$
\frac{r(s)}{f(s)} = \frac{a_{1,i}}{(s - \omega_i)} + \frac{a_{2,i}}{(s - \omega_i)^2} + \cdots + \frac{a_{r_i,i}}{(s - \omega_i)^{r_i}} + h_i(s),
$$

(21)

where $h_i$ is an analytic function inside a closed curve around $\omega_i$. We multiply (21) by $e^{st}/2\pi i$ and use Cauchy’s integral formula. We obtain (cf [11, loc. cit.])

$$
\text{res}_i(t) = P_i(t) e^{nt},
$$

where $P_i(t)$ is the polynomial of degree $r_i - 1$ given by

$$
P_i(t) = a_{1,i} + a_{2,i} \frac{t}{1!} + \cdots + a_{r_i,i} \frac{t^{r_i-1}}{(r_i-1)!}.
$$

(22)

The proof of this corollary is essentially in our previous paper [16] in the case $f$ entire. We present it here again because formula (19) for the solution $\phi(t)$ is precisely Carmichael’s formula appearing in [10], as quoted in [5], and also because explicit formulae such as (19) are crucial for the study of the initial value problem we carry out in section 3.

The last special case of theorem 2.1 is when the function $r(s)/f(s)$ has an infinite number of isolated poles $\omega_i$, located to the left of $\text{Re}(s) = \omega_f$, and such that $|\omega_0| \leq |\omega_1| \leq |\omega_2| \leq \cdots$.

Corollary 2.3. Assume that the hypotheses of theorem 2.1 hold, and that $f$ has an infinite number of poles $\omega_i$ of order $r_i$ to the left of $\text{Re}(s) = \omega_f$ satisfying $|\omega_0| \leq |\omega_{i+1}|$ for $i \geq 0$. We let $\sigma_i$ be curves in the half-plane $\text{Re}(s) \leq \omega_f$ connecting the points $\omega_f + i\omega_i$ and $\omega_f - i\omega_i$, such that $\sigma_i$ together with the segment of the line $\text{Re}(s) = \omega_f$ between these two points encloses exactly the first $n$ poles of $r(s)/f(s)$. Suppose that $\mathcal{L}(f)/f$ is in $C^\infty_W(\omega_f, \infty)$, that the curves $\sigma_n$ are chosen so that $\omega_n$ tends to infinity as $n$ tends to infinity, and that

$$
\lim_{n \to \infty} \int_{\sigma_n} e^{st} r(s)/f(s) \, ds = 0.
$$

Then, solution (17) to the linear equation (16) can be written in the form

$$
\phi(t) = \frac{1}{2\pi i} \int_{\omega_j - i\infty}^{\omega_j + i\infty} e^{s t} \left( \frac{\mathcal{L}(f)}{f} \right) \, ds + \sum_{i=1}^{\infty} P_i(t) e^{nt},
$$

(23)

in which $P_i(t)$ are polynomials of degree $r_i - 1$.

Proof. This corollary is proven as corollary 2.2: it follows from the analysis of the complex inverse Laplace transform appearing in [11]; see also [28, p 160]. In particular, it is explained in [11, p 170] why the series appearing in (23) indeed converges for $t \geq 0$. □
We note that the series appearing in corollary 2.3 is not necessarily differentiable. For instance, let us take 
\[ f = 1 + e^{at} \] where \( a > 0 \) and \( J = 1 \). We wish to solve the equation

\[ (1 + e^{at}) \phi = 1, \quad a > 0. \tag{24} \]

We take \( r = (2\phi_0 - 1)/s \), so that indeed \( r/f \) has an infinite number of poles, \( \omega = 0 \) and 
\[ \omega_n = \frac{2n-1}{a} \pi i, \quad n = 0, \pm 1, \pm 2, \ldots \] The solution \( \phi \) to equation (24) is

\[ \phi = \mathcal{L}^{-1} \left( \frac{\mathcal{L}(J) + r}{f} \right) = \mathcal{L}^{-1} \left( \frac{1/s + (2\phi_0 - 1)/s}{1 + e^{as}} \right) = 2\phi_0 \mathcal{L}^{-1} \left( \frac{1}{s(1 + e^{as})} \right), \tag{25} \]

and the inverse Laplace transform appearing in (25) is calculated as indicated in corollary 2.3. The answer is in [28]:

\[ \phi(t) = 2\phi_0 \left[ \frac{1}{2} - \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n-1} \sin \left( \frac{2n-1}{a} \pi t \right) \right]. \tag{26} \]

The function \( \phi \) satisfies \( \phi(0) = \phi_0 \), but it is not possible to take \( t \)-derivative of the series (26) in order to obtain \( \phi'(0) \). In fact, for \( \phi_0 = 1 \), the function \( \phi(t) \) is the periodic square wave function

\[ \phi(t) = \begin{cases} \phi_0, & 2na \leq t < (2n + 1)a, \\ 0, & (2n + 1)a \leq t < (2(n + 1)a, \end{cases} \]

\( n = 0, 1, 2, \ldots \), and the series expansion (26) corresponds to its Fourier series representation.

3. The initial value problem

In this section we discuss the initial value problem for equations of the form

\[ f(\partial_t)\phi = J, \quad t \geq 0, \tag{27} \]

in which \( f \) belongs to the class \( \Gamma \).

3.1. Generalized initial conditions

We note that our abstract formula (17) for the solution \( \phi \) to equation (27) tells us that—

expanding the analytic function \( r \) appearing in (17) as a power series (or considering \( r = rd \),

where \( rd \) is the series (8))—\( \phi \) depends in principle on an infinite number of arbitrary constants

given \( a \) priori. However, this fact does not mean that the equation itself is superfluous, as

formula (17) for \( \phi \) depends essentially on \( f \) and \( J \). We think of \( r \) as a sort of ‘generalized

initial condition’.

**Definition 3.1.** A generalized initial condition for the equation

\[ f(\partial_t)\phi = J \tag{28} \]

is an analytic function \( r_0 \) such that \( (\phi, r_0) \in D_f \) for some \( \phi \in L^\infty_{\text{loc}}(\mathbb{R}_+) \). A generalized initial value problem is an equation such as (28) together with a generalized initial condition \( r_0 \). A solution to a given generalized initial value problem \( (28), r_0 \) is a function \( \phi \) satisfying the conditions of definition 2.3 with \( r = r_0 \).
Thus, given a generalized initial condition, we find a unique solution for (27) using (17), much in the same way as given one initial condition we find a unique solution to a first-order linear ODE. We remark once more (see comment after the proof of theorem 2.1 and the example after corollary 2.3) that there is no reason to believe that (for a given \( r \)) the unique solution (17) to equation (28) will be analytic: within our general context, we can only conclude that the solution is an integrable exponentially bounded function. It follows that classical initial value problems do not even exist in full generality; definition 3.1 is what replaces them in the framework of our Lorentzian calculus. In the following subsection, we show that—provided \( f \) and \( J \) satisfy some extra technical conditions—we can consider classical initial value problems.

We present an explicit example where the foregoing analysis applies. In [3, section 5], Barnaby considers D-brane decay in a background de Sitter spacetime. Up to some parameters, the equation of motion is

\[
e^{-2\Box}(-\Box + 1)\phi = \alpha \phi^2,
\]

in which \( \alpha \) is a constant. In de Sitter spacetime we have \( \Box = -\partial_t^2 - \beta \partial_t \) for a constant \( \beta \), and so equation (29) becomes

\[
e^{2(\partial_t^2 + \beta \partial_t)}(\partial_t^2 + \beta \partial_t - 1)\phi = -\alpha \phi^2.
\]

Expanding the operator \( e^{2(\partial_t^2 + \beta \partial_t)} \) formally as a power series, we see that \( \phi_0 = 1/\alpha \) solves (30). We linearize about \( \phi_0 \); if \( \phi = \phi_0 + \tau \psi \), the small deformation \( \psi \) satisfies the equation

\[
e^{2(\partial_t^2 + \beta \partial_t)}(\partial_t^2 + \beta \partial_t - 1)\psi + 2\psi = 0.
\]

This equation can be solved using our Lorentzian calculus! Consider the entire function

\[
f(s) = e^{2(s^2 + \beta s)}(s^2 + \beta s - 1) + 2,
\]

and let \( r \) be a generalized initial condition, that is, \( r \) is an analytic function such that \( r/f \) belongs to Widder space. The most general solution to (31) is

\[
\psi = L^{-1}(r/f).
\]

Now, we can rewrite (31) as

\[
(\partial_t^2 + \beta \partial_t - 1)\psi = -2 e^{2\partial_t^2} \psi (t - 2\beta),
\]

an equation like the ones considered in [3]. We conclude that \( \psi = L^{-1}(r/f) \) solves (33), and we note that, as in Barnaby’s analysis, our solution depends on an essentially arbitrary function given a priori. However, as we show below, it is enough to assume a finite set of a priori data in order to set up initial value problems depending on a finite number of classical initial conditions.

### 3.2. Classical initial value problems

The main observation on which this subsection rests is that, if we can unravel the abstract formula (17) as in corollaries 2.1, 2.2 or 2.3, we can see that in fact \( r \) itself is not essential. The truly important information needed for formulating (and solving) initial value problems is encoded in the pole structure of \( r(s)/f(s) \).

**Definition 3.2.** A classical initial value problem for a nonlocal equation is an equation

\[
f(\partial_t)\phi = J
\]

together with a finite set of conditions

\[
\phi(0) = \phi_0, \quad \phi'(0) = \phi_1, \ldots, \phi^{(k)}(0) = \phi_k.
\]

A solution to a classical initial value problems (34) and (35) is a pair \((\phi, r_0) \in D_f \) satisfying the conditions of definition 2.3 with \( r = r_0 \) such that \( \phi \) is differentiable at zero and (35) holds.
We already remarked in the previous subsection that classical initial value problems do not exist in general. On the other hand, as we anticipated (rather roughly) in [16], following Barnaby and Kamran’s inspiring paper [5], it is possible to pose classical initial value problems if we consider, in addition to a finite number of initial conditions, some \textit{a priori} data directly related to our Laplace transform-based functional calculus. Intuitively, following Moeller and Zwiebach [23, section 2.3], a nonlocal equation such as the \textit{p}-adic string equation
\begin{equation}
\frac{1}{2} \ln p \Phi = \Phi^p
\end{equation}
imposes a large number of non-trivial constraints on the set of possible initial values. Can we describe a consistent set of initial conditions? In corollaries 2.1, 2.2 and 2.3 there are explicit formulas for solutions to the nonlocal equation (34). We would expect them to help us in setting up initial value problems. Now, corollary 2.1 fixes completely the solution using only \( f \) and \( J \), and therefore it leaves no room for a classical initial value problem. On the other hand, formula (23) of corollary 2.3 depends on an infinite number of parameters and, as the example after corollary 2.3 shows, this fact implies that we cannot insure differentiability of the solution (and hence existence of initial value problems) using only conditions on \( f \) and \( J \).

But, our explicit formula (19) tells us that a solution for the linear equation (34) is uniquely determined by \( f \), \( J \) and a finite number of parameters related to the singularities of the quotient \( r/f \). It is therefore not unreasonable to expect that, by using these finitely many parameters, we can set up consistent initial value problems, as conjectured in [23].

We use the following two technical lemmas on the differentiability of solutions:

\textbf{Lemma 3.1.} Assume that \( f \) and \( J \) are such that
\begin{equation}
y^n \left( L(J) \frac{\omega f + iy}{f} \right) \in L^1(\mathbb{R})
\end{equation}
for each \( n = 0, \ldots, M \), and some \( M \geq 0 \). Then, the function
\begin{equation}
t \mapsto \frac{1}{2\pi i} \int_{\omega f - i\infty}^{\omega f + i\infty} e^{st} \left( L(J) \frac{f}{f} \right)(s) \, ds
\end{equation}
is of class \( C^M \).

The proof of lemma 3.1 consists basically in realizing that function (38) can be viewed as the Fourier transform of the function
\begin{equation}
y \mapsto \left( L(J) \frac{\omega f}{f} \right)(\omega f + iy).
\end{equation}

\textbf{Lemma 3.2.} Assume that the conditions of corollary 2.2 hold, and that \( f \) and \( J \) satisfy (37). Then, solution (19) to the nonlocal equation
\begin{equation}
f(\partial_t)\phi(t) = J(t)
\end{equation}
is of class \( C^M \), and it satisfies
\begin{equation}
\phi^{(n)}(0) = L_n + \sum_{i=1}^{N} \sum_{k=0}^{n} \binom{n}{k} \omega^k \frac{d^{n-k}}{dt^{n-k}} \bigg|_{t=0} P_i(t), \quad n = 0, \ldots, M,
\end{equation}
for some numbers \( L_n \).

\textbf{Proof.} Indeed, let us consider equation (19) for the solution \( \phi \) to equation (39). Condition (37) implies that (19) defines a \( C^M \) function \( \phi \) on the semi-axis \( t \geq 0 \). The \( t \)-derivatives \( \phi^{(n)}(t) \) of the solution \( \phi(t) \) are given by
\begin{equation}
\phi^{(n)}(t) = \frac{d^n}{dt^n} \left( \frac{1}{2\pi i} \int_{\omega f - i\infty}^{\omega f + i\infty} e^{st} \left( L(J) \frac{f}{f} \right)(s) \, ds \right) + \sum_{i=1}^{N} c_{i}^{(n)} \sum_{k=0}^{n} \binom{n}{k} \omega^k \frac{d^{n-k}}{dt^{n-k}} P_i(t),
\end{equation}
and so equation (40) follows with

\[ L_n = \frac{d^n}{dr^n} \bigg|_{r=0} \left( \frac{1}{2\pi i} \int_{w-i\infty}^{w+i\infty} e^{r} \left( \frac{\mathcal{L}(J)}{f} \right) \, ds \right). \]  

(42)

This lemma was used by us already in [16], but we reproduce it here since it is crucial for the analysis that follows.

**Theorem 3.1.** Let \( f \) be a function on the class \( \Gamma \), and fix a function \( J \) in \( L_{\text{loc}}^\infty(\mathbb{R}_+) \) such that \( \mathcal{L}(J)/f \in C^\infty(\omega_f, \infty) \). Fix also a number \( N \geq 0 \), a finite number of points \( \omega_i \) to the left of \( \text{Re}(s) = \omega_f \) and (if \( N > 0 \)) a finite number of positive integers \( r_i, i = 1, \ldots, N \). Set \( K = \sum_{i=1}^{N} r_i \) and assume that condition (37) holds for all \( n = 0, \ldots, M, M \geq K \). Then, generically, given \( K \) initial conditions, \( \phi_0, \ldots, \phi_{K-1} \), there exists an analytic function \( r_0 \) such that

- (\( \alpha \)) has a finite number of poles \( \omega_i \) of order \( r_i \), \( i = 1, \ldots, N \) to the left of \( \text{Re}(s) = \omega_f \);
- (\( \beta \)) \( \frac{\mathcal{L}(J)/f}{f} \in C^\infty(\omega_f, \infty) \);
- (\( \gamma \)) \( \frac{n!}{r!} |f| \leq \frac{M}{r^p} \) for some \( p \geq 1 \) and \( |s| \) sufficiently large.

Moreover, the unique solution \( \phi \) to equation (34) given by (17) with \( r = r_0 \) is of class \( C^K \) and it satisfies \( \phi(0) = \phi_0, \ldots, \phi^{(K-1)}(0) = \phi_{K-1} \).

**Proof.** We take \( K \) arbitrary numbers \( \phi_n, n = 0, 1, \ldots, K-1 \), and, motivated by lemma 3.2, we set up the linear system

\[ \phi_n = L_n + \sum_{i=1}^{N} \sum_{k=0}^{n} \binom{n}{k} a_{i,k} \frac{d^{n-k}}{dr^{n-k}} P_i(t), \quad n = 0 \ldots K-1, \]  

(43)

in which

\[ L_n = \frac{d^n}{dr^n} \bigg|_{r=0} \left( \frac{1}{2\pi i} \int_{w-i\infty}^{w+i\infty} e^{r} \left( \frac{\mathcal{L}(J)}{f} \right) \, ds \right), \]

and

\[ P_i(t) = a_{1,i} t + a_{2,i} \frac{t^2}{1!} + \cdots + a_{r_i,i} \frac{t^{r_i}}{(r_i - 1)!} \]

are polynomials to be determined. Condition (37) guarantees us that the numbers \( L_n \) are well defined. The system (43) is a linear system for the coefficients of the polynomials \( P_i(t) \) which (generically, depending on the points \( \omega_i \)) can be solved uniquely in terms of the arbitrary data \( \phi_n \). These polynomials allow us to construct the solution to equation (39) as follows.

We set

\[ r_0(s) = f(s) \mathcal{L} \left( \sum_{i=1}^{N} P_i(t) e^{\omega_i t} \right)(s). \]  

(44)

We have the identity

\[ \mathcal{L}^{-1}(r_0/f) = \sum_{i=1}^{N} P_i(t) e^{\omega_i t}, \]  

(45)

and we easily conclude that \( r_0/f \) satisfies the conditions \( (\alpha), (\beta) \) and \( (\gamma) \) appearing in the enunciate of the theorem. Now we define

\[ \phi(t) = \frac{1}{2\pi i} \int_{w-i\infty}^{w+i\infty} e^{r} \left( \frac{\mathcal{L}(J)}{f} \right) \, ds + \sum_{i=1}^{N} P_i(t) e^{\omega_i t}, \]  

(46)
and we claim that this function is the solution to equation (39). In fact, the foregoing analysis implies that
\[ \phi(t) = \frac{1}{2\pi i} \int_{\omega-i\infty}^{\omega+i\infty} e^{s} \left( \frac{\mathcal{L}(J)}{f} \right) ds + \mathcal{L}^{-1}(r_0/f), \]
and this is precisely the unique solution to (39) appearing in corollary 2.2 for \( r = r_0 \).

Now we show that this solution satisfies \( \phi^{(n)}(0) = \phi_n \) for \( n = 0, \ldots, K - 1 \). Indeed, condition (37) tells us that \( \phi(t) \) is at least of class \( C^k \) and clearly
\[ \phi^{(n)}(0) = L_n + \sum_{i=1}^{N} \sum_{k=0}^{n} \binom{n}{k} \omega_i^k \frac{d^{n-k}}{d^{n-k}} \bigg|_{t=0} P_i(t) \]
in which
\[ L_n = \frac{d^n}{dn} \bigg|_{t=0} \left( \frac{1}{2\pi i} \int_{\omega-i\infty}^{\omega+i\infty} e^{s} \left( \frac{\mathcal{L}(J)}{f} \right) ds \right). \]
Comparing (43) and (47) we obtain \( \phi^{(n)}(0) = \phi_n \), \( n = 0, \ldots, K - 1 \). Thus, we can freely choose the first \( K \) derivatives \( \phi^{(n)}(0) \), \( n = 0, \ldots, K - 1 \). On the other hand, from \( n = K \) onward, equation (47) for the derivatives of \( \phi(t) \) and the foregoing analysis implies that the values of \( \phi^{(n)}(0) \), \( n \geq K \), are completely determined by the \( K \) initial conditions \( \phi_n \).

4. Concluding remarks

We have developed a Lorentzian functional calculus adequate for interpreting nonlocal operators appearing in models of particle physics, string theory and gravity. This calculus is directly related to the initial value problem for nonlocal equations: as seen in definition 2.1, the interpretation of an operator of the form \( f(\tilde{a}) \) depends on the choice of some \( a \) priori information (the analytic function \( r \) of definition 2.1). This information is considered as a generalized initial condition for an equation of the form \( f(\tilde{a})\phi = J(t) \). However, we can be much more specific. If \( f \) and \( J \) satisfy some technical conditions (see lemmas 3.1 and 3.2), we can take derivatives of the solution, at least a finite number of times, and we can consider a classical initial value problem for nonlocal equations (definition 3.2). As shown in theorem 3.1, we can solve this classical initial value problem explicitly: starting with a finite number of initial conditions and a finite set of extra data (see (49) and definition 3.2), we can obtain a unique regular solution to the given nonlocal equation.

It seems to us that the freedom in the choice of \( a \) priori data is an important feature of our approach, potentially of interest for applications. Thus, it does not appear to be obvious
how to generalize theorem 3.1 if $N$ is allowed to be infinite; besides the fact that if we wish to use the formulae appearing in corollary 2.3 we must solve an infinite linear system in order to obtain an appropriate function $r$ (see proof of theorem 3.1), we would need to give technical conditions on $f$, $J$ and on the a priori data (49) in order to assure that the solution be differentiable at $t = 0$.

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