A Revisit to Optimal Control of Forward-Backward Stochastic Differential System with Observation Noise

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Abstract

This paper revisits the partial information optimal control problem considered by Wang, Wu and Xiong [13], where the system is derived by a controlled forward-backward stochastic differential equation with correlated noises between the system and the observation. For this type of partial information optimal control problem, one necessary and one sufficient (a verification theorem) conditions of optimality are derived using a unified way. We improve the $L^p$-bounds on the control from $L^8$ in [13] to $L^4$ in this paper.

Keywords Maximum Principle, Forward-Backward Stochastic Differential Equation, Partial Information, Girsanovs Theorem

1 Introduction

It is well known that forward-backward stochastic differential equations (FBSDEs in short) consist of a forward stochastic differential equation (SDE in short) of Itô type and a backward stochastic differential equation (BSDE in short) of Pardoux-Peng for details see [11, 5].

FBSDEs are not only encountered in stochastic optimal control problems when applying the stochastic maximum principle but also used in mathematical finance (see Antonelli [1], Duffie and Epstein [3], El Karoui, Peng and Quenez [5] for example). It now becomes more clear that certain important problems in mathematical economics and mathematical finance, especially in the optimization problem, can be formulated to be FBSDEs.

There are two important approaches to the general stochastic optimal control problem. One is the Bellman dynamic programming principle, which results in the Hamilton-Jacobi-Bellman equation. The other is the maximum principle. Now the maximum principle of forward-backward stochastic systems driven by Brownian motion have been studied extensively in the literature. We refer to [12, 11, 4, 6] and references therein.

In recent years, there have been growing interests on stochastic optimal control problems under partial information, partly due to the applications in mathematical finance. For the partial information optimal control problem, the objective is to find an optimal control for which the controller has less information than the complete information filtration. In particular, sometimes an economic model in which there are information gaps among economic agents can be formulated as a partial information optimal control problem (see Øksendal [10], Kohlmann and Xiong [9]).

Recently, Baghery and Øksendal [2] established a maximum principle of forward systems with jumps under partial information. In 2009, Meng [7] studied a partial information stochastic optimal control problem of continuous fully coupled forward-backward stochastic systems driven by a Brownian motion. As in [2], the author established one sufficient (a verification theorem) and one necessary conditions of optimality. The main limitation of [2] and [7] is that the stochastic systems do not contain observation noise and the partial information filtration is too general to have more practical application. In 2013, Wang, Wu and Xiong [13] studied a partial information optimal control problem derived by forward-backward stochastic systems with correlated noises.
between the system and the observation. Utilizing a direct method, an approximation method, and a Malliavin derivative method, they established three versions of maximum principle (i.e., necessary condition) for optimal control, where the following $L^8-$ bounds is imposed on the admissible controls:

$$
E\left[\sup_{0\leq t\leq T} |u(t)|^8\right].
$$

(1)

The present paper revisits the partial information optimal control problem considered by Wang, Wu and Xiong [13]. Its one of the two main contributions is that we improve the $L^p-$ bounds on the control from $L^8-$ to the following $L^4-$ bounds

$$
E\left[\left(\int_0^T |u(t)|^2 dt\right)^2\right] < \infty.
$$

(2)

Another contribution of this paper is that we will establish necessary and sufficient conditions for an optimal control in a unified way. The main idea is to get directly a variation formula in terms of the Hamiltonian and the associated adjoint system which is a linear forward-backward stochastic differential equation and neither the variational systems nor the corresponding Taylor type expansions of the state process and the cost functional will be considered.

The paper is organized as follows. In section 2, we formulate the problem and give various assumptions used throughout the paper. Section 3 is devoted to derive necessary as well as sufficient optimality conditions in the form of stochastic maximum principles in a unified way.

2 Formulation of Problem

In this section, we introduce some basic notations which will be used in this paper. Let $T := [0, T]$ denote a finite time index, where $0 < T < \infty$. We consider a complete probability space $(\Omega, \mathcal{F}, P)$ equipped with two one-dimensional standard Brownian motions $\{W(t), t \in T\}$ and $\{Y(t), t \in T\}$, respectively. Let $\{\mathcal{F}_t^W\}_{t \in T}$ and $\{\mathcal{F}_t^Y\}_{t \in T}$ be $P$-completed natural filtration generated by $\{W(t), t \in T\}$ and $\{Y(t), t \in T\}$, respectively. Set $\{\mathcal{F}_t\}_{t \in T} := \{\mathcal{F}_t^W\}_{t \in T} \vee \{\mathcal{F}_t^Y\}_{t \in T}, \mathcal{F} = \mathcal{F}_T$. Denote by $E[\cdot]$ the expectation under the probability $P$. Let $E$ be a Euclidean space. The inner product in $E$ is denoted by $(\cdot, \cdot)$, and the norm in $E$ is denoted by $|\cdot|$. Let $A_\Gamma$ denote the transpose of the matrix or vector $A$. For a function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$, $\psi_x$ is the corresponding $k \times n$-Jacobian matrix. By $\mathcal{P}$ we denote the predictable $\sigma$-field on $\Omega \times [0, T]$ and by $\mathcal{B}(\Lambda)$ the Borel $\sigma$-algebra of any topological space $\Lambda$. In the follows, $K$ represents a generic constant, which can be different from line to line. Next we introduce some spaces of random variable and stochastic processes. For any $\alpha, \beta \in [1, \infty)$, denote by $M^\beta_{\mathcal{F}}(0, T; E)$ the space of all $E$-valued and $\mathcal{F}_t$-adapted processes $f = \{f(t, \omega), (t, \omega) \in T \times \Omega\}$ satisfying $\|f\|_{M^\beta_{\mathcal{F}}(0, T; E)} \triangleq \left(\mathbb{E}\left[\int_0^T |f(t)|^\beta dt\right]\right)^{\frac{1}{\beta}} < \infty$, by $S^\beta_{\mathcal{F}}(0, T; E)$ the space of all $E$-valued and $\mathcal{F}_t$-adapted c\'adl\'ag processes $f = \{f(t, \omega), (t, \omega) \in T \times \Omega\}$ satisfying $\|f\|_{S^\beta_{\mathcal{F}}(0, T; E)} \triangleq \left(\mathbb{E}\left[\sup_{t \in T} |f(t)|^\beta\right]\right)^{\frac{1}{\beta}} < \infty$, by $L^\beta(\Omega, \mathcal{F}, P; E)$ the space of all $E$-valued random variables $\xi$ on $(\Omega, \mathcal{F}, P)$ satisfying $\|\xi\|_{L^\beta(\Omega, \mathcal{F}, P; E)} \triangleq \left(\mathbb{E}[|\xi|^\beta]\right)^{\frac{1}{\beta}} < \infty$, by $M^\beta_{\mathcal{F}}(0, T; L^\alpha(0, T; E))$ the space of all $L^\alpha(0, T; E)$-valued and $\mathcal{F}_t$-adapted processes $f = \{f(t, \omega), (t, \omega) \in [0, T] \times \Omega\}$ satisfying $\|f\|_{M^\beta_{\mathcal{F}}(0, T; L^\alpha(0, T; E))} \triangleq \left\{\mathbb{E}\left[\left(\int_0^T |f(t)|^\alpha dt\right)^{\frac{\beta}{\alpha}}\right]\right\}^{\frac{1}{\beta}} < \infty$.

Consider the following forward-backward stochastic differential equation

$$
\begin{align*}
dx(t) &= b(t, x(t), u(t))dt + \sigma_1(t, x(t), u(t))dW(t) + \sigma_2(t, x(t), u(t))dW^u(t), \\
nu(t) &= f(t, x(t), y(t), z_1(t), z_2(t), u(t))dt + z_1(t)dW(t) + z_2(t)dW^u(t), \\
x(0) &= x, \\
y(T) &= \phi(x(T))
\end{align*}
$$

(3)

with one observation processes $Y(\cdot)$ driven by the following stochastic differential equation
\[ \begin{align*}
\{dY(t) &= h(t, x(t), u(t))dt + dW^u(t), \\
Y(t) &= 0,
\end{align*} \tag{4}\]

where \( b : \mathcal{T} \times \Omega \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n, \sigma_1 : \mathcal{T} \times \Omega \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n, \sigma_2 : \mathcal{T} \times \Omega \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n, f : \mathcal{T} \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times U \rightarrow \mathbb{R}^m, \phi : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) and \( h : \mathcal{T} \times \Omega \times \mathbb{R}^n \times U \rightarrow \mathbb{R} \) are given random mapping with \( U \) being a nonempty convex subset of \( \mathbb{R}^k \). In the above equations, \( u(\cdot) \) is our admissible control process defined as follows.

**Definition 2.1.** An admissible control process is defined as an \( \mathcal{F}_t^U \)-adapted process valued in a nonempty convex subset \( U \) in \( \mathbb{R}^K \) such that

\[ \mathbb{E} \left[ \left( \int_0^T |u(t)|^2 dt \right)^2 \right] < \infty. \]

The set of all admissible controls is denoted by \( \mathcal{A} \).

Now we make the following standard assumptions on the coefficients of the equations \( \mathbf{(i)} \) and \( \mathbf{(ii)} \):

**Assumption 2.1.** (i) The coefficients \( b, \sigma_1, \sigma_2 \) and \( h \) are \( \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(U) \)-measurable. For each \( (x, u) \in \mathbb{R}^n \times U, b(\cdot, x, u), \sigma_1(\cdot, x, u), \sigma_2(\cdot, x, u) \) and \( h(\cdot, x, u) \) are all \( \{\mathcal{F}_t\}_{t \in \mathcal{T}} \)-adapted processes. For almost all \((t, \omega) \in \mathcal{T} \times \Omega \), the mapping

\[ (x, u) \mapsto \psi(t, \omega, x, u) \]

is continuous differentiable with respect to \((x, u)\) with appropriate growths, where \( \psi = b, \sigma_1, \sigma_2 \) and \( h \). More precisely, there exists a constant \( C > 0 \) such that for all \( x \in \mathbb{R}^n, u \in U \) and a.e. \((t, \omega) \in \mathcal{T} \times \Omega, \)

\[
\begin{align*}
(1 + |x| + |u|^{-1} &\leq |\alpha(t, x, u)|, |\alpha_1(t, x, u)|, |\alpha_2(t, x, u)| \leq C, \alpha = b, \sigma_1, \\
|\beta(t, x, u)| &\leq \beta_2(t, x, u) \leq C, \beta = h, \sigma_2.
\end{align*}
\]

(ii) The coefficient \( f \) is \( \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(U) \)-measurable. For each \( (x, y, z_1, z_2, u) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times U, f(\cdot, x, y, z_1, z_2, u) \) is \( \{\mathcal{F}_t\}_{t \in \mathcal{T}} \)-adapted processes. For almost all \((t, \omega) \in \mathcal{T} \times \Omega \), the mapping

\[ (x, y, z_1, z_2, u) \mapsto f(t, \omega, x, y, z_1, z_2, u) \]

is continuous differentiable with respect to \((x, y, z_1, z_2, u)\) with appropriate growths. More precisely, there exists a constant \( C > 0 \) such that for all \((x, y, z_1, z_2, u) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times U \) and a.e. \((t, \omega) \in \mathcal{T} \times \Omega, \)

\[
\begin{align*}
(1 + |x| + |y| + |z_1| + |z_2| + |u|^{-1} &\leq |f(t, x, y, z_1, z_2, u)|, |f_1(t, x, y, z_1, z_2, u)|, |f_2(t, x, y, z_1, z_2, u)|, |f_3(t, x, y, z_1, z_2, u)|, |f_4(t, x, y, z_1, z_2, u)|, \leq C.
\end{align*}
\]

(iii) The coefficient \( \phi \) is \( \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^n) \)-measurable. For almost all \((t, \omega) \in [0, T] \times \Omega \), the mapping

\[ x \mapsto \phi(\omega, x) \]

is continuous differentiable with respect to \( x \) with appropriate growths, respectively. More precisely, there exists a constant \( C > 0 \) such that for all \( x \in \mathbb{R}^n \) and a.e. \( \omega \in \Omega, \)

\[
(1 + |x|)^{-1} |\phi(x)| + |\phi_2(x)| \leq C.
\]

Now we begin to discuss the well-posedness of \( \mathbf{(i)} \) and \( \mathbf{(ii)} \). Indeed, putting \( \mathbf{(ii)} \) into the state equation \( \mathbf{(i)} \), we get that

\[
\begin{align*}
\begin{cases}
\begin{align*}
\frac{dx(t)}{dt} &= (b - \sigma_2 h)(t, x(t), u(t))dt + \sigma_1(t, x(t), u(t))dW(t) + \sigma_2(t, x(t), u(t))dY(t), \\
\frac{dy(t)}{dt} &= (f(t, x(t), y(t), z_1(t), z_2(t), u(t)) - z_2(t)h(t, x(t), u(t)))dt + z_1(t)dW(t) + z_2(t)dY(t), \\
x(0) &= x,
\end{align*}
\end{cases}
\end{align*}
\tag{7}
\]

Under Assumption \( \mathbf{2.1} \) for any admissible control \( u(\cdot) \in \mathcal{A} \), we have the following basic result.
Lemma 2.1. Let Assumption 2.1 be satisfied. Then for any admissible control \( u(\cdot) \in \mathcal{A} \), the equation (7) admits a unique strong solution \((x(\cdot), y(\cdot), z_1(\cdot), z_2(\cdot)) \in S^4_\mathcal{F}(0,T;\mathbb{R}^m) \times S^4_\mathcal{F}(0,T;\mathbb{R}^m) \times M^2_\mathcal{F}(0,T;L^2(0,T;\mathbb{R}^m)) \times M^2_\mathcal{F}(0,T;L^2(0,T;\mathbb{R}^m)) \). Moreover, we have the following estimate:

\[
E \left[ \sup_{t \in T} |x(t)|^4 \right] + E \left[ \sup_{t \in T} |y(t)|^4 \right] + E \left[ \left( \int_0^T |z_1(t)|^2 \, dt \right)^2 \right] \leq K \left\{ 1 + |x|^4 + E \left[ \left( \int_0^T |u(t)|^2 \, dt \right)^2 \right] \right\}.
\] (8)

Further, if \((\bar{x}(\cdot), \bar{y}(\cdot), \bar{z}_1(\cdot), \bar{z}_2(\cdot))\) is the unique strong solution corresponding to another admissible control \( \bar{u}(\cdot) \in \mathcal{A} \), then the following estimate holds:

\[
E \left[ \sup_{t \in T} |x(t) - \bar{x}(t)|^4 \right] + E \left[ \sup_{t \in T} |y(t) - \bar{y}(t)|^4 \right] + E \left[ \left( \int_0^T |z_1(t) - \bar{z}_1(t)|^2 \, dt \right)^2 \right] + E \left[ \left( \int_0^T |z_2(t) - \bar{z}_2(t)|^2 \, dt \right)^2 \right] \leq K E \left[ \int_0^T |u(t) - \bar{u}(t)|^2 \, dt \right]^2.
\] (9)

Proof. The proof can be directly obtained by combining Proposition 2.1 in [8] and Lemma 2 in [6].

For the strong solution \((x^u(\cdot), y^u(\cdot), z_1^u(\cdot), z_2^u(\cdot))\) of the equation (7) associated with any given admissible control \( u(\cdot) \in \mathcal{A} \), we introduce a process

\[
\rho^u(t) = \exp \left\{ \int_0^t h(s, x^u(s), u(s)) \, dY(s) - \frac{1}{2} h^2(s, x^u(s), u(s)) \, ds \right\},
\] (10)

which is obviously the solution to the following SDE

\[
\begin{align*}
\frac{d\rho^u(t)}{\rho^u(t)} &= h(s, x^u(s), u(s)) \, dY(s), \\
\rho^u(0) &= 1.
\end{align*}
\] (11)

For the stochastic process \( \rho^u(\cdot) \), we have the following basic result.

Lemma 2.2. Let Assumption 2.1 holds. Then for any \( u(\cdot) \in \mathcal{A} \), we have for any \( \alpha \geq 2 \),

\[
E \left[ \sup_{t \in T} |\rho^u(t)|^\alpha \right] \leq K.
\] (12)

Further, if \( \tilde{\rho}(\cdot) \) is the process defined by (10) or (11) corresponding to another admissible control \( \bar{u}(\cdot) \in \mathcal{A} \), then the following estimate holds

\[
E \left[ \sup_{t \in T} |\rho^u(t) - \tilde{\rho}(t)|^2 \right] \leq K \left\{ E \left[ \int_0^T |u(t) - \bar{u}(t)|^2 \, dt \right]^2 \right\}^\frac{1}{2}.
\] (13)

Proof. The proof can be directly obtained by combining Proposition 2.1 in [8].

Under Assumption 2.1 \( \rho^u(\cdot) \) is an \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in T}, \mathbb{P})\) martingale. Define a new probability measure \( \mathbb{P}^u \) on \((\Omega, \mathcal{F})\) by

\[
d\mathbb{P}^u = \rho^u(1) \, d\mathbb{P}.
\] (14)

Then from Girsanov’s theorem and (11), \((W(\cdot), W^u(\cdot))\) is an \(\mathbb{R}^2\)-valued standard Brownian motion defined in the new probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P}^u)\). So \((\mathbb{P}^u, x^u(\cdot), y^u(\cdot), z_1^u(\cdot), z_2^u(\cdot), \rho^u(\cdot), W(\cdot), W^u(\cdot))\) is a weak solution on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in T})\) of (3) and (11).

The cost functional is given by

\[
J(u(\cdot)) = E^u \left[ \int_0^T l(t, x(t), y(t), z_1(t), z_2(t), u(t)) \, dt + \Phi(x(T)) + \gamma(y(0)) \right].
\] (15)

where \( E^u \) denotes the expectation with respect to the probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P}^u)\) and \( l : T \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times U \rightarrow \mathbb{R} \), \( \Phi : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R} \) and \( \gamma : \Omega \times \mathbb{R}^m \rightarrow \mathbb{R} \) are given random mappings satisfying the following assumption:
Problem 2.1. Find an admissible control \( \bar{u}(\cdot) \) such that

\[
J(\bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{A}} J(u(\cdot)),
\]
subject to the state equation (3), the observation equation (4) and the cost functional (15).

Obviously, according to Bayes’ formula, the cost functional (15) can be rewritten as

\[
J(u(\cdot)) = \mathbb{E} \left[ \int_0^T \rho^u(t)l(t, x(t), y(t), z_1(t), z_2(t), u(t))dt + \rho^u(T)\Phi(x(T)) + \gamma(y(0)) \right].
\]  

(17)

Therefore, we can translate Problem 2.1 into the following equivalent optimal control problem in its strong formulation, i.e., without changing the reference probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})\), where \(\rho^u(\cdot)\) will be regarded as an additional state process besides the state process \((x^u(\cdot), y^u(\cdot), z_1^u(\cdot), z_2^u(\cdot))\).

Problem 2.2. Find an admissible control \( \bar{u}(\cdot) \) such that

\[
J(\bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{A}} J(u(\cdot)),
\]
subject to the cost functional \( J(u(t)) \) and the following state equation

\[
\begin{align*}
\frac{dx(t)}{dt} &= (b - \sigma_2 h)(t, x(t), u(t)) + \sigma_1(t, x(t), u(t))dW(t) + \sigma_2(t, x(t), u(t))dY(t), \\
\frac{dy(t)}{dt} &= (f(t, x(t), y(t), z_2(t), u(t)) - z_2(t)h(t, x(t), u(t)))dt + z_1(t)dW(t) + z_2(t)dY(t), \\
\frac{dp_2(t)}{dt} &= \rho_{\nu}(t)\mu(s, x^\nu(s), u(s))dY(s), \\
\rho_{\nu}(0) &= 1, \\
x(0) &= x, \\
y(T) &= \Phi(x(T)).
\end{align*}
\]

Any \( \bar{u}() \in \mathcal{A} \) satisfying above is called an optimal control process of Problem 2.2 and the corresponding state process \((\bar{x}(\cdot), \bar{y}(\cdot), \bar{z}_1(\cdot), \bar{z}_2(\cdot), \bar{\rho}(\cdot))\) is called the optimal state process. Correspondingly \((\bar{u}(\cdot), \bar{x}(\cdot), \bar{y}(\cdot), \bar{z}_1(\cdot), \bar{z}_2(\cdot), \bar{\rho}(\cdot))\) is called an optimal pair of Problem 2.2.

Remark 2.1. The present formulation of the partially observed optimal control problem is quite similar to a completely observed optimal control problem; the only difference lies in the admissible class \( \mathcal{A} \) of controls.

### 3 Stochastic Maximum Principle

This section is devoted to establishing the stochastic maximum principle of Problem 2.2 or Problem 2.1, i.e., establishing the necessary and sufficient optimality conditions of Pontryagin’s type for an admissible control to be optimal. To this end, for the state equation (18), we first introduce the corresponding adjoint equation.

Define the Hamiltonian \( H : \Omega \times \mathcal{T} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{U} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R} \) as follows:

\[
H(t, x, y, z_1, z_2, u, p, q_1, q_2, k, R_2) = l(t, x, y, z_1, z_2, u) + \langle b(t, x, u), p \rangle + \langle \sigma_1(t, x, u), q_1 \rangle + \langle \sigma_2(t, x, u), q_2 \rangle + \langle f(t, x, y, z_1, z_2, u), k \rangle + \langle R_2, h(t, x, u) \rangle.
\]

For any given admissible control pair \((\tilde{u}(\cdot), \tilde{x}(\cdot), \tilde{y}(\cdot), \tilde{z}_1(\cdot), \tilde{z}_2(\cdot))\), the corresponding adjoint equation is defined as follows.

\[
\begin{align*}
\frac{d\bar{v}(t)}{dt} &= -l(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t)) + \bar{\mu}_1(t)dw(t) + \bar{\mu}_2(t)dw(t), \\
\frac{d\bar{p}(t)}{dt} &= -\bar{H}_x(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t))dt + \bar{q}_1(t)dw(t) + \bar{q}_2(t)dw(t), \\
\frac{d\bar{k}(t)}{dt} &= -\bar{H}_y(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t))dt - \bar{H}_{z_1}(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t))dw(t) \\
&+ \bar{H}_{z_2}(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t))dw(t), \\
\bar{p}(T) &= \Phi_x(\bar{x}(T)) - \phi_x^*(\bar{x}(T))\bar{k}(T), \\
\bar{r}(T) &= \Phi(\bar{x}(T)), \\
\bar{k}(0) &= -\gamma_y(\bar{y}(0)).
\end{align*}
\]

Here we have used the following notation:

\[
\begin{align*}
\bar{H}_a(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t)) &= \bar{H}_a(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t)), \\
J(u(\cdot)) &= \mathcal{H}(u(\cdot), \bar{x}(\cdot), \bar{y}(\cdot), \bar{z}_1(\cdot), \bar{z}_2(\cdot), \bar{u}(\cdot)), \\
J(\tilde{u}(\cdot)) &= \mathcal{H}(\tilde{u}(\cdot), \tilde{x}(\cdot), \tilde{y}(\cdot), \tilde{z}_1(\cdot), \tilde{z}_2(\cdot), \tilde{u}(\cdot)).
\end{align*}
\]

where \( a = x, y, z_1, z_2, u \).

Note the adjoint equation (20) is a forward-backward stochastic differential equation whose solution consists of a 7-tuple process \((\bar{p}(\cdot), \bar{q}_1(\cdot), \bar{q}_2(\cdot), \bar{k}(\cdot), \bar{r}(\cdot), \bar{R}_1(\cdot), \bar{R}_2(\cdot))\). Under Assumptions 2.1 and 2.2 by Proposition 2.1 in [8] and Lemma 2 in [6], it is easy to see that the adjoint equation (20) admits a unique solution \((\bar{p}(\cdot), \bar{q}_1(\cdot), \bar{q}_2(\cdot), \bar{k}(\cdot), \bar{r}(\cdot), \bar{R}_1(\cdot), \bar{R}_2(\cdot)) \in S^4_{\mathbb{F}}(0, T; \mathbb{R}^n) \times M^2_{\mathbb{F}}(0, T; L^2(0, T; \mathbb{R}^n)) \times M^2_{\mathbb{F}}(0, T; L^2(0, T; \mathbb{R}^n)) \times S^4_{\mathbb{F}}(0, T; \mathbb{R}^n) \times M^2_{\mathbb{F}}(0, T; L^2(0, T; \mathbb{R}^n)) \times M^2_{\mathbb{F}}(0, T; L^2(0, T; \mathbb{R}^n)) \times M^2_{\mathbb{F}}(0, T; L^2(0, T; \mathbb{R}^n)) \times M^2_{\mathbb{F}}(0, T; L^2(0, T; \mathbb{R}^n))
\]

also called the adjoint process corresponding the admissible pair \((u(\cdot), \bar{x}(\cdot), \bar{y}(\cdot), \bar{z}_1(\cdot), \bar{z}_2(\cdot), \bar{\rho}(\cdot))\).

Let \((u(\cdot), x^u(\cdot), y^u(\cdot), z^u_1(\cdot), z^u_2(\cdot), \rho^u(\cdot))\) and \((\bar{u}(\cdot), \bar{x}(\cdot), \bar{y}(\cdot), \bar{z}_1(\cdot), \bar{z}_2(\cdot), \bar{\rho}(\cdot))\) be two admissible pairs. Next we represent the difference \( J(u(\cdot)) - J(\bar{u}(\cdot)) \) in terms of the Hamiltonian \( H \) and the adjoint process \((\bar{p}(\cdot), \bar{q}_1(\cdot), \bar{q}_2(\cdot), \bar{k}(\cdot), \bar{r}(\cdot), \bar{R}_1(\cdot), \bar{R}_2(\cdot)) \) as well as other relevant expressions.
To unburden our notation, we will use the following abbreviations:

\[
\begin{align*}
\gamma^n(0) &= \gamma(y^n(0)) , \quad \bar{\gamma}(0) = \gamma(\bar{y}(0)), \\
\phi^n(T) &= \phi(x^n(T)), \quad \bar{\phi}(T) = \phi(\bar{x}(T)), \\
\Phi^n(T) &= \Phi(x^n(T)), \quad \bar{\Phi}(T) = \Phi(\bar{x}(T)), \\
\alpha^n(t) &= \alpha(t, x^n(t), u(t)), \\
\bar{\alpha}(t) &= \alpha(t, \bar{x}(t), \bar{u}(t)), \quad \alpha = b, \sigma_1, \sigma_2, h, \\
\beta^n(t) &= \alpha(t, x^n(t), y^n(t), z^n_1(t), z^n_2(t), u(t)), \\
\bar{\beta}(t) &= \alpha(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t)), \quad \beta = f, l.
\end{align*}
\]

(22)

Lemma 3.1. Let Assumptions [27] and [24] be satisfied. Using the notations (21) and (22), we have

\[
J(u(\cdot)) - J(\bar{u}(\cdot)) = \mathbb{E}^u \left[ \int_0^T \mathcal{H}(t, x^n(t), y^n(t), z^n_1(t), z^n_2(t), u(t)) - \mathcal{H}(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t)) dt \right]
\]

\[
+ \mathbb{E}^\bar{u} \left[ \Phi^n(T) - \bar{\Phi}(T) - \langle x^n(T) - \bar{x}(T), \Phi_x(T) \rangle \right]
\]

\[
- \mathbb{E}^\bar{u} \left[ \langle \phi^n(T) - \bar{\phi}(T), \bar{k}(T) \rangle - \langle \bar{\phi}_x^n(T) \bar{k}(T) - \bar{x}(T) - \bar{\Phi}_x(T) \rangle \right]
\]

\[
+ \mathbb{E}^\bar{u} \left[ \gamma^n(0) - \bar{\gamma}(0) - \langle y^n(0) - \bar{y}(0), \bar{\gamma}_y(0) \rangle \right]
\]

\[
+ \mathbb{E} \left[ \int_0^T \bar{R}_2(t)(\rho^n(t) - \bar{\rho}(t))(h^n(t) - \bar{h}(t)) dt \right]
\]

\[
+ \mathbb{E} \left[ \int_0^T (l^n(t) - \bar{l}(t))(\rho^n(t) - \bar{\rho}(t)) dt \right]
\]

\[
+ \mathbb{E} \left[ (\rho^n(T) - \bar{\rho}(T))(\Phi^n(T) - \bar{\Phi}(T)) \right].
\]

(23)

Proof. From [4], it is easy to check that the \((x^n(\cdot), y^n(\cdot), z^n(\cdot))\) satisfies the following FBSDE:

\[
\begin{align*}
&dx(t) = \left[ b^n(t) + \sigma^n_1(t)(\bar{h}(t) - h^n(t)) \right] dt + \sigma^n_1(t) dW(t) + \sigma^n_2(t) dW^\bar{u}(t) \\
&dy(t) = \left[ f^n(t) + z_2(t)(\bar{h}(t) - h^n(t)) \right] dt + z_1(t) dW(t) + z_2(t) dW^\bar{u}(t) \\
&x(0) = x, \\
&y(T) = \phi(x(T))
\end{align*}
\]

(24)

Therefore \((x^n(t) - \bar{x}(t), y^n(t) - \bar{y}(t), z^n(t) - \bar{z}(t))\) satisfies the following FBSDE:

\[
\begin{align*}
&dx(t) - \bar{dx}(t) = \left[ b^n(t) - \bar{b}(t) + \sigma^n_1(t)(\bar{h}(t) - h^n(t)) \right] dt + \sigma^n_1(t) dW(t) + \bar{\sigma}_1(t) dW^\bar{u}(t)

&dy(t) - \bar{dy}(t) = \left[ f^n(t) - \bar{f}(t) + z_2(t)(\bar{h}(t) - h^n(t)) \right] dt + z_1(t) dW(t) + \bar{z}_2(t) dW^\bar{u}(t)

&x(0) - \bar{x}(0) = 0, \\
&y(T) - \bar{y}(T) = \phi(x(T)) - \phi(\bar{x}(T)).
\end{align*}
\]

(25)

From [24], we know that \((\bar{\rho}(\cdot), \bar{q}_1(\cdot), \bar{q}_2(\cdot), \bar{k}(\cdot))\) satisfies the following FBSDE
Applying Itô formula to

\[
\begin{align*}
  d\bar{\rho}(t) &= -\mathcal{H}_x(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t)) dt + \bar{q}_1(t) dW(t) + \bar{q}_2(t) dW^\mathcal{U}(t), \\
  d\bar{\kappa}(t) &= -\mathcal{H}_y(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t)) dt - \mathcal{H}_{z_1}(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t)) dW(t) \\
  &\quad - \mathcal{H}_{z_2}(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t)) dW^\mathcal{U}(t), \\
  \bar{\rho}(T) &= \Phi_x(T) - \bar{\phi}_x(T) \bar{\kappa}(T), \\
  \bar{\kappa}(0) &= -\gamma_y(0),
\end{align*}
\]

It is easy to check that \((\bar{r}(\cdot), \bar{R}_1(\cdot), \bar{R}_2(\cdot))\) satisfying the following BSDE

\[
\begin{align}
  d\bar{r}(t) &= -[\bar{r}(t) + \bar{R}_2(t) \bar{h}(t)] dt + \bar{R}_1(t) dW(t) + \bar{R}_2(t) dY(t), \\
  \bar{r}(T) &= \Phi(T), \tag{27}
\end{align}
\]

From the definition of the cost function \(J(u(\cdot))\), we have

\[
J(u(\cdot)) - J(\bar{u}(\cdot)) = \mathbb{E}^u \left[ \int_0^T l^u(t) dt + \Phi^u(T) + \gamma^u(0) \right] - \mathbb{E}^\bar{u} \left[ \int_0^T \bar{l}(t) dt + \Phi(T) + \bar{\gamma}(0) \right] \\
= \mathbb{E} \left[ \int_0^T \left( \rho^u(t) l^u(t) - \bar{\rho}(t) \bar{l}(t) \right) dt \right] + \mathbb{E} \left[ \rho^u(T) \Phi^u(T) - \bar{\rho}(T) \Phi(T) + \mathbb{E} \left[ \gamma^u(0) - \bar{\gamma}(0) \right] \right] \\
= \mathbb{E}^u \left[ \int_0^T \left( l^u(t) - \bar{l}(t) \right) dt \right] + \mathbb{E}^u \left[ \Phi^u(T) - \Phi(T) \right] + \mathbb{E} \left[ \int_0^T \left( \rho^u(t) - \bar{\rho}(t) \right) l^u(t) dt \right] \\
+ \mathbb{E} \left[ \mathbb{E} \left[ \gamma^u(0) - \bar{\gamma}(0) \right] \right] \tag{28}
\]

From the definition of \(\mathcal{H}\), we get that

\[
\mathbb{E}^u \left[ \int_0^T \left( l^u(t) - \bar{l}(t) \right) dt \right] = \mathbb{E}^u \left[ \int_0^T \left( \mathcal{H}(t, x^u(t), y^u(t), z^u_1(t), z^u_2(t), u(t)) - \mathcal{H}(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t)) \right) dt \right] \\
- \mathbb{E}^u \left[ \int_0^T \left( \langle \bar{\rho}(t), b^u(t) - \bar{b}(t) \rangle + \langle \bar{q}_1(t), \sigma^u_1(t) - \bar{\sigma}_1(t) \rangle + \langle \bar{q}_2(t), \sigma^u_2(t) - \bar{\sigma}_2(t) \rangle \right) \right] \\
+ \langle \bar{\kappa}(t), f^u(t) - \bar{f}(t) \rangle + \langle \bar{R}_2(t) - \bar{\sigma}_2(t), \bar{\rho}(t) - \bar{\sigma}_2(t), \bar{\kappa}(t), h^u(t) - \bar{h}(t) \rangle \right) dt \right] \tag{29}
\]

Applying Itô formula to \(\langle \bar{\rho}(t), x^u(t) - \bar{x}(t) \rangle + \langle \bar{\kappa}(t), y^u(t) - \bar{y}(t) \rangle\) and taking expectation under \(P^\bar{u}\), we have

\[
\mathbb{E}^u \left[ \langle \Phi_x(T) - \bar{\phi}_x(T) \bar{\kappa}(T), x^u(T) - \bar{x}(T) \rangle \right] + \mathbb{E}^u \left[ \langle \bar{\kappa}(T), \phi^u(T) - \bar{\phi}(T) \rangle \right] \\
= \mathbb{E}^u \left[ \int_0^T \left( \langle \bar{\rho}(t), b^u(t) - \bar{b}(t) \rangle + \langle \bar{q}_1(t), \sigma^u_1(t) - \bar{\sigma}_1(t) \rangle + \langle \bar{q}_2(t), \sigma^u_2(t) - \bar{\sigma}_2(t) \rangle \right) \right] \\
+ \langle \bar{\kappa}(t), f^u(t) - \bar{f}(t) \rangle + \langle \bar{R}_2(t) - \bar{\sigma}_2(t), \bar{\rho}(t) - \bar{\sigma}_2(t), \bar{\kappa}(t), h^u(t) - \bar{h}(t) \rangle \right) dt \right] \\
- \mathbb{E}^u \left[ \int_0^T \langle \mathcal{H}_x(t, \bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{u}(t)), x^u(t) - \bar{x}(t) \rangle dt \right] \\
- \mathbb{E}^u \left[ \int_0^T \langle \mathcal{H}_y(t, \bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{u}(t)), y^u(t) - \bar{y}(t) \rangle dt \right] \\
- \mathbb{E}^u \left[ \int_0^T \langle \mathcal{H}_{z_1}(t, \bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{u}(t)), z^u_1(t) - \bar{z}_1(t) \rangle dt \right] \\
- \mathbb{E}^u \left[ \int_0^T \langle \mathcal{H}_{z_2}(t, \bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{u}(t)), z^u_2(t) - \bar{z}_2(t) \rangle dt \right] \\
- \mathbb{E} \left[ \langle y^u(0) - \bar{y}(0), \gamma_y(0) \rangle \right] \tag{30}
\]
which implies that
\[
\mathbb{E}^u \left[ \int_0^T \left( \bar{p}(t), b^u(t) - \bar{b}(t) \right) + \langle \bar{q}_1(t), \sigma_1^u(t) - \bar{\sigma}_1(t) \rangle + \langle \bar{q}_2(t), \sigma_2^u(t) - \bar{\sigma}_2(t) \rangle + \langle \bar{k}(t), f^u(t) - \bar{f}(t) \rangle \right] dt
\]
\[
= \mathbb{E}^u \left[ \langle \bar{\Phi}_x(T), x^u(T) - \bar{x}(T) \rangle \right] + \mathbb{E}^u \left[ \langle \bar{k}(T), \phi^u(T) - \bar{\phi}(T) \rangle \langle x^u(T) - \bar{x}(T) \rangle \right]
\]
\[
+ \mathbb{E}^u \left[ \int_0^T \langle \bar{p}(t), \sigma_2^u(t) (h^u(t) - \bar{h}(t)) \rangle + \langle \bar{k}(t), z_2^u(t) (h^u(t) - \bar{h}(t)) \rangle \right] dt
\]
\[
+ \mathbb{E}^u \left[ \int_0^T \langle H_x(t), \bar{x}(t), \bar{y}(t), z(t), \bar{u}(t) \rangle, x^u(t) - \bar{x}(t) \rangle \right] dt
\]
\[
+ \mathbb{E}^u \left[ \int_0^T \langle H_y(t), \bar{x}(t), \bar{y}(t), z(t), \bar{u}(t) \rangle, y^u(t) - \bar{y}(t) \rangle \right] dt
\]
\[
+ \mathbb{E}^u \left[ \int_0^T \langle H_{z1}(t), \bar{x}(t), \bar{y}(t), z(t), \bar{u}(t) \rangle, z_1^u(t) - \bar{z}_1(t) \rangle \right] dt
\]
\[
+ \mathbb{E}^u \left[ \int_0^T \langle H_{z2}(t), \bar{x}(t), \bar{y}(t), z(t), \bar{u}(t) \rangle, z_2^u(t) - \bar{z}_2(t) \rangle \right] dt
\]
\[
+ \mathbb{E}^u \left[ \langle y^u(0) - \bar{y}(0), \gamma_y(0) \rangle \right]
\]
(31)

Applying Itô formula to \((\rho^u(t) - \bar{\rho}(t)) \bar{r}(t)\), we have
\[
\mathbb{E}[\langle \rho^u(T) - \bar{\rho}(T) \rangle \bar{\Phi}(T)] = - \mathbb{E} \left[ \int_0^T (\rho^u(t) - \bar{\rho}(t)) \langle \bar{l}(t) + \bar{R}_2(t) \bar{h}(t) \rangle dt \right] + \mathbb{E} \left[ \int_0^T \bar{R}_2(t) (\rho^u(t) \bar{h}(t) - \bar{\rho}(t) \bar{h}(t)) dt \right]
\]
(32)

which implies that
\[
\mathbb{E}[\langle \rho^u(T) - \bar{\rho}(T) \rangle \bar{\Phi}(T)] + \mathbb{E} \left[ \int_0^T (\rho^u(t) - \bar{\rho}(t)) \bar{l}(t) dt \right] = \mathbb{E} \left[ \int_0^T \bar{R}_2(t) \rho^u(t) \bar{h}(t) dt \right]
\]
(33)

Putting (31) into (29), we have
\[
\mathbb{E}^u \left[ \int_0^T (l^u(t) - \bar{l}(t)) dt \right] = \mathbb{E}^u \left[ \int_0^T \left( H(t, x^u(t), y^u(t), z_1^u(t), z_2^u(t), u(t)) - H(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t)) \right) \right.
\]
\[
- \langle H_x(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t)), x^u(t) - \bar{x}(t) \rangle \right)
\]
\[
- \langle H_y(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t)), y^u(t) - \bar{y}(t) \rangle \right)
\]
\[
- \langle H_{z1}(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t)), z_1^u(t) - \bar{z}_1(t) \rangle \right)
\]
\[
- \langle H_{z2}(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t)), z_2^u(t) - \bar{z}_2(t) \rangle \right)
\]
\[
- \langle (\sigma_2^u(t) - \bar{\sigma}_2(t)) (h^u(t) - \bar{h}(t)) \rangle \bar{p}(t) \rangle \right) dt
\]
\[
- \mathbb{E}^u \left[ \langle x^u(T) - \bar{x}(T), \bar{\Phi}_x(T) \rangle \right] - \mathbb{E} \left[ \langle y^u(0) - \bar{y}(0), \gamma_y(0) \rangle \right]
\]
\[
- \mathbb{E}^u \left[ \langle \phi^u(T) - \bar{\phi}(T), \bar{\Phi}(T) \rangle \right] - \langle \bar{\phi}_x^u(T) \bar{k}(T), x^u(T) - \bar{x}(T) \rangle \right]
\]
(34)

Then putting (33) and (34) into (29), we get (23). The proof is complete.

Since the control domain \( U \) is convex, for any given admissible controls \( u(\cdot) \in \mathcal{A} \), the following perturbed control process \( u^\varepsilon(\cdot) \):
\[
u^\varepsilon(\cdot) := \bar{u}(\cdot) + \varepsilon (u(\cdot) - \bar{u}(\cdot)), \quad 0 \leq \varepsilon \leq 1,
\]
is also in \( \mathcal{A} \). We denote by \((\bar{x}(\cdot), \bar{y}(\cdot), \bar{z}_1(\cdot), \bar{z}_2(\cdot))\) and \((x^\varepsilon(\cdot), y^\varepsilon(\cdot), z_1^\varepsilon(\cdot), z_2^\varepsilon(\cdot))\) the corresponding state processes associated with \( \bar{u}(\cdot) \) and \( u^\varepsilon(\cdot) \), respectively. Denote by \((\bar{p}(\cdot), \bar{q}_1(\cdot), \bar{q}_2(\cdot), \bar{k}(\cdot), \bar{r}(\cdot), \bar{R}_1(\cdot), \bar{R}_2(\cdot))\) the adjoint process associated with the admissible pair \((\bar{u}(\cdot); \bar{x}(\cdot), \bar{y}(\cdot), \bar{z}_1(\cdot), \bar{z}_2(\cdot))\).
Lemma 3.2. Let Assumptions 2.1 and 2.2 be satisfied. Then we have
\[ E \left[ \sup_{t \in T} |r'(t) - \bar{x}(t)|^4 \right] + E \left[ \sup_{t \in T} |v'(t) - \bar{y}(t)|^4 \right] + E \left[ \left( \int_0^T |z_1'(t) - \bar{z}_1(t)|^2 dt \right)^2 \right] + E \left[ \left( \int_0^T |z_2'(t) - \bar{z}_2(t)|^2 dt \right)^2 \right] = O(c^4) . \]
and
\[ E \left[ \sup_{t \in T} |\rho'(t) - \bar{\rho}(t)|^2 \right] = O(c^2) . \]

Proof. The proof can be obtained directly by Lemmas 2.4 and 2.5.

Now we are in the position to use Lemma 3.1 and Lemma 3.2 to derive the variational formula for the cost functional \( J(u(\cdot)) \) in terms of the Hamiltonian \( \mathcal{H} \).

Theorem 3.1. Let Assumptions 2.1 and 2.2 be satisfied. Then for any admissible control \( u(\cdot) \in A \), the directional derivative of the cost functional \( J(u(\cdot)) \) at \( \bar{u}(\cdot) \) in the direction \( u(\cdot) - \bar{u}(\cdot) \) exists and is given by
\[ \frac{d}{d\epsilon} J(\bar{u}(\cdot) + \epsilon(u(\cdot) - \bar{u}(\cdot)))|_{\epsilon=0} = \mathbb{E}^\bar{u} \left[ \int_0^T \mathcal{H}_u(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t), u(t) - \bar{u}(t)) dt \right] . \]

Proof. For notational simplicity, write
\[ \beta^\epsilon := \mathbb{E}^\bar{u} \left[ \int_0^T \left( \mathcal{H}(t, x^{u^\epsilon}(t), y^{u^\epsilon}(t), z_1^{u^\epsilon}(t), z_2^{u^\epsilon}(t), u(t)) - \mathcal{H}(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t)) \right) \right. \]
\[ - \langle \mathcal{H}_x(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t)), x^{u^\epsilon}(t) - \bar{x}(t) \rangle - \langle \mathcal{H}_y(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t)), y^{u^\epsilon}(t) - \bar{y}(t) \rangle \]
\[ - \langle \mathcal{H}_{x_1}(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t)), z_1^{u^\epsilon}(t) - \bar{z}_1(t) \rangle - \langle \mathcal{H}_{x_2}(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t)), z_2^{u^\epsilon}(t) - \bar{z}_2(t) \rangle \]
\[ - \langle (z_2^{u^\epsilon}(t) - \bar{z}_2(t))(h^{u^\epsilon}(t) - \bar{h}(t)), \bar{k}(t) dt \rangle \]
\[ + \mathbb{E}^\bar{u} \left[ \langle \phi^{u^\epsilon}(T) - \bar{\Phi}(T), - 1 \rangle - \langle \phi^{u^\epsilon}_x(T) \bar{k}(T), x^{u^\epsilon}(T) - \bar{x}(T) \rangle \right] \]
\[ - \mathbb{E}^\bar{u} \left[ \langle \phi^{u^\epsilon}(T) - \bar{\Phi}(T), \bar{k}(T) \rangle - \langle \phi^{u^\epsilon}_x(T) \bar{k}(T), x^{u^\epsilon}(T) - \bar{x}(T) \rangle \right] \]
\[ + \mathbb{E} \left[ \gamma^{u^\epsilon}(0) - \bar{\gamma}(0) - \langle y^{u^\epsilon}(0) - \bar{y}(0), \bar{\gamma}_y(0) \rangle \right] \]
\[ + \mathbb{E} \left[ \int_0^T R_2(t)(\rho^{u^\epsilon}(t) - \bar{\rho}(t))(h^{u^\epsilon}(t) - \bar{h}(t)) dt \right] \]
\[ + \mathbb{E} \left[ \int_0^T (l^{u^\epsilon}(t) - \bar{l}(t))(\rho^{u^\epsilon}(t) - \bar{\rho}(t)) dt \right] \]
\[ + \mathbb{E} \left[ \langle \rho^{u^\epsilon}(T) - \bar{\rho}(T), \phi^{u^\epsilon}(T) - \bar{\Phi}(T) \rangle \right] \]
\[ (35) \]
By Lemma 3.3, we have
\[ J(u^\epsilon(\cdot)) - J(\bar{u}(\cdot)) = \beta^\epsilon + \epsilon \mathbb{E}^\bar{u} \left[ \int_0^T \mathcal{H}_u(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t)), u(t) - \bar{u}(t) \right] dt . \]
Under Assumptions 2.1 and 2.2 combining the Taylor Expansions, Lemma 3.2 and the dominated convergence theorem, we have
\[ \beta^\epsilon = o(c) . \]
(38)
Plugging (38) into (37) gives
\[ \lim_{\epsilon \to 0^+} \frac{J(u^\epsilon(\cdot)) - J(\bar{u}(\cdot))}{\epsilon} = \mathbb{E}^\bar{u} \left[ \int_0^T \mathcal{H}_u(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t)), u(t) - \bar{u}(t) \right] dt . \]
This completes the proof.
Now we derive the necessary condition and sufficient maximum principles for Problem 2.1 or 2.2. We first give the necessary condition of optimality for the existence of an optimal control.

**Theorem 3.2 (Necessary Stochastic Maximum principle).** Let Assumptions 2.1 and 2.2 be satisfied. Let \((\bar{u}(); \bar{x}(\cdot), \bar{y}(\cdot), \bar{z}_1(\cdot), \bar{z}_2(\cdot), \bar{\rho}(\cdot))\) be an optimal pair of Problem 2.2. Then

\[
\langle \mathbb{E}[\mathcal{H}_u(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t))|\mathcal{F}_t^Y], v - \bar{u}(t) \rangle \geq 0, \quad \forall v \in U, \ a.e. \ a.s.,
\]

*Proof.* Since all admissible controls are \(\{\mathcal{F}_t^Y\}_t\) adapted processes, from the property of conditional expectation, Theorem 3.1 and the optimality of \(\bar{u}(\cdot)\), we deduce that

\[
\mathbb{E}\left[\int_0^T \langle \mathbb{E}[\check{\mathcal{H}}_u(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t))|\mathcal{F}_t^Y], u(t) - \bar{u}(t) \rangle dt \right] = \mathbb{E}\left[\int_0^T \langle \check{\mathcal{H}}_u(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t))|\mathcal{F}_t^Y], u(t) - \bar{u}(t) \rangle dt \right] = \mathbb{E}\left[\int_0^T \langle \mathcal{H}_u(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t))|\mathcal{F}_t^Y], u(t) - \bar{u}(t) \rangle dt \right] = \lim_{\epsilon \to 0^+} \frac{J(\bar{u}(\cdot) + \epsilon(u(\cdot) - \bar{u}(\cdot))) - J(\bar{u}(\cdot))}{\epsilon} \geq 0,
\]

which implies that

\[
\langle \mathbb{E}[\check{\mathcal{H}}_u(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t))|\mathcal{F}_t^Y], v - \bar{u}(t) \rangle \geq 0, \quad \forall v \in U, \ a.e. \ a.s.,
\]

On the other hand, since \(\check{\rho}(t) > 0\),

\[
\langle \mathbb{E}[\check{\mathcal{H}}_u(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t))|\mathcal{F}_t^Y], v - \bar{u}(t) \rangle = \frac{1}{\mathbb{E}[\check{\rho}(t)|\mathcal{F}_t^Y]} \langle \mathbb{E}[\check{\mathcal{H}}_u(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t))|\mathcal{F}_t^Y], v - \bar{u}(t) \rangle \geq 0.
\]

The proof is complete. \(\Box\)

Next we give the sufficient condition of optimality for the existence of an optimal control of Problem 2.2 in the case when the observation process is not affected by the control process. Suppose that

\[
h(t, x, u) = h(t)
\]

is an \(\mathcal{F}_t^Y\) adapted bounded process. Define a new probability measure \(Q\) on \((\Omega, \mathcal{F})\) by

\[
dQ = \rho(1)dP, \quad (42)
\]

where

\[
\left\{ \begin{array}{l}
d\rho(t) = \rho(t)h(s)dY(s) \\
\rho(0) = 1. \end{array} \right. (43)
\]

**Theorem 3.3. [Sufficient Maximum Principle]** Let Assumptions 2.1 and 2.2 be satisfied. Let \((\bar{u}(); \bar{x}(\cdot), \bar{y}(\cdot), \bar{z}_1(\cdot), \bar{z}_2(\cdot), \bar{\rho}(\cdot))\) be an admissible pair with \(\phi(x) = \phi x\), where \(\phi\) is \(\mathcal{F}_T\) measurable bounded random variable. If the following conditions are satisfied,

(i) \(\Phi\) and \(\gamma\) is convex in \(x\) and \(y\), respectively,

(ii) the Hamiltonian \(\mathcal{H}\) is convex in \((x, y, z_1, z_2, u)\),

(iii)

\[
\mathbb{E}\left[\mathcal{H}(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t))|\mathcal{F}_t^Y\right] = \min_{u \in U} \mathbb{E}\left[\mathcal{H}(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), u)|\mathcal{F}_t^Y\right], \ a.e. \ a.s.,
\]

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then \((\bar{u}(\cdot), \bar{x}(\cdot), \bar{y}(\cdot), \bar{z}_1(\cdot), \bar{z}_2(\cdot))\) is an optimal pair of Problem \([2]\).

Proof. Let \((u(\cdot); x^u(\cdot), y^u(\cdot), z_1^u(\cdot), z_2^u(\cdot), \rho^u(\cdot))\) be an arbitrary admissible pair. By Lemma \([3.1]\) we can represent the difference \(J(u(\cdot)) - J(\bar{u}(\cdot))\) as follows

\[
\begin{align*}
J(u(\cdot)) - J(\bar{u}(\cdot)) &= \mathbb{E}^Q \left[ \int_0^T \left( \mathcal{H}(t, x^u(t), y^u(t), z^u(t), u(t)) - \mathcal{H}(t, \bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{u}(t)) ight. \
&\quad - \left. \langle \mathcal{H}_x(t, \bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{u}(t)), x^u(t) - \bar{x}(t) \rangle ight) - \langle \mathcal{H}_y(y(t), \bar{x}(t), y^u(t), \bar{z}(t), \bar{u}(t)), y^u(t) - \bar{y}(t) \rangle \
&\quad - \langle \mathcal{H}_z(z(t), \bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{u}(t)), z^u(t) - \bar{z}(t) \rangle dt \right] \
&\quad + \mathbb{E}^Q \left[ \Phi(T) - \bar{\Phi}(T) - \langle x^u(T) - \bar{x}(T), \Phi_x(T) \rangle \right] \
&\quad + \mathbb{E} \left[ \gamma^u(0) - \bar{\gamma}(0) - \langle y^u(0) - \bar{y}(0), \gamma_\rho(0) \rangle \right].
\end{align*}
\]

By the convexity of \(\mathcal{H}, \Phi\) and \(\gamma\) (i.e. Conditions (i) and (ii)), we have

\[
\begin{align*}
\mathcal{H}(t, x^u(t), y^u(t), z_1^u(t), z_2^u(t), u(t)) - \mathcal{H}(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t)) \\
&\geq \langle \mathcal{H}_x(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t)), x^u(t) - \bar{x}(t) \rangle + \langle \mathcal{H}_y(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t)), y^u(t) - \bar{y}(t) \rangle \\
&\quad + \langle \mathcal{H}_z(z(t), \bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{u}(t)), z^u(t) - \bar{z}(t) \rangle + \langle \mathcal{H}_z(z(t), \bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{u}(t)), z_1^u(t) - \bar{z}_1(t) \rangle \\
&\quad + \langle \mathcal{H}_z(z(t), \bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{u}(t)), z_2^u(t) - \bar{z}_2(t) \rangle \\
&\quad + \langle \mathcal{H}_u(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t)), u(t) - \bar{u}(t) \rangle,
\end{align*}
\]

and

\[
\Phi^u(T) - \bar{\Phi}(T) \geq \langle X^u(T) - \bar{X}(T), \Phi_x(T) \rangle
\]

Furthermore, from the optimality condition (iii) and the convex optimization principle (see Proposition 2.21 of \([1]\)), we have

\[
\left\langle u(t) - \bar{u}(t), \mathbb{E} \left[ \mathcal{H}_u(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t)) | \mathcal{F}_t^Y \right] \right\rangle \geq 0.
\]

which imply that

\[
\mathbb{E}^Q \left[ \langle u(t) - \bar{u}(t), \mathcal{H}_u(t, \bar{x}(t), \bar{y}(t), \bar{z}_1(t), \bar{z}_2(t), \bar{u}(t)) \rangle \right] \geq 0.
\]

Putting (15), (16), (17) and (49) into (14), we have

\[
J(u(\cdot)) - J(\bar{u}(\cdot)) \geq 0.
\]

Due to the arbitrariness of \(u(\cdot)\), we can conclude that \(\bar{u}(\cdot)\) is an optimal control process and thus \((\bar{u}(\cdot), \bar{x}(\cdot), \bar{y}(\cdot), \bar{z}_1(\cdot), \bar{z}_2(\cdot))\) is an optimal pair. The proof is completed. \(\square\)

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