Scattered Factor Universality - The Power of the Remainder

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Abstract Scattered factor (circular) universality was firstly introduced by Barker et al. in 2020. A word $w$ is called $k$-universal for some natural number $k$, if every word of length $k$ of $w$’s alphabet occurs as a scattered factor in $w$; it is called circular $k$-universal if a conjugate of $w$ is $k$-universal. Here, a word $u = u_1 \cdots u_n$ is called a scattered factor of $w$ if $u$ is obtained from $w$ by deleting parts of $w$, i.e. there exists (possibly empty) words $v_1, \ldots, v_{n+1}$ with $w = v_1 u_1 v_2 \cdots v_n u_n v_{n+1}$. In this work, we prove two problems, left open in the aforementioned paper, namely a generalisation of one of their main theorems to arbitrary alphabets and a slight modification of another theorem such that we characterise the circular universality by the universality. On the way, we present deep insights into the behaviour of the remainder of the so called arch factorisation by Hebrard when repetitions of words are considered.

1 Introduction

By deleting letters from a word one obtains another word, a so called scattered factor (also known as subword or subsequence). More formally, a word $u = u_1 \cdots u_n$ is a scattered factor of $w$ if there exist (possibly empty) words $v_1, \ldots, v_{n+1}$ with $w = v_1 u_1 v_2 \cdots v_n u_n v_{n+1}$. For instance, Latin is a scattered factor of dalmatian but lama is not. Scattered factors are a fundamental concept in mathematics and computer science: whenever data are transmitted via a lossy channel or in aligning DNA-sequences, scattered factors are the formal model to describe the incomplete data (e.g. \cite{9}). Thus, it is not astonishing that scattered factors are strongly related to partial words \cite{3}. Moreover, automatic sequences can be in part understood by their subword complexity, a measure of complexity on words that is defined by scattered factors \cite{1}. Parikh matrices and subword histories use scattered factors to encode numerical properties of words into matrices, thus connecting the world of words and languages with the world of vectors and matrices (see \cite{25,27,28}). From an algorithmic point of view, scattered factors are crucial in some classical problems: the longest common subsequence, the shortest common supersequence, and the string-to-string correction problem \cite{23,4,30}. On the other hand, scattered factors are also used in logic-theories and have applications in formal software verification \cite{31,14,21}.

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The line of research that lead to this work began by Higman [16], who showed that in any infinite set of words there are always two words such that one is a scattered factor of the other, albeit only as an application of a more general theorem about partial orderings on an abstract algebra and without explicitly defining scattered factors. Later in 1967 Haines [13] explicitly introduced scattered factors and rediscovered Higman’s result. In the seminal work [29] from 1975, Simon used this partial ordering to define the equivalence relation \( \sim_k \), now known as Simon congruence, where \( x \sim_k y \) iff \( x \) and \( y \) have the same set of scattered factors of a fixed length \( k \). In 1991, Hebrard introduced the arch factorisation which is a very powerful tool in investigating the scattered factors of a word [15]. A very profound overview from a mathematical point of view can be found in [22, Chapter 6], where Simon and Sakarovitch expand Simon’s previous work.

In this work we focus on a special \( \sim_k \)-class of words. A word \( w \in \Sigma^* \) is called \( k \)-universal if its set of scattered factors of length \( k \) is \( \Sigma^k \). For instance, anana is 2-universal over \( \{a,n\} \) but banana is only 1-universal over \( \{a,b,n\} \). Notice, that this notion is equivalent to the notion of richness introduced in [18,19] in the context of piecewise testable languages; as in [2] we prefer the notion of universality for avoiding confusion with the richness w.r.t. palindromes. While the classical universality problem, which asks whether a given language \( L \subseteq \Sigma^* \) is equal to \( \Sigma^* \), and many variants of it, as well as the universality problem for (partial) words, which asks given an \( \ell \) whether there exists a word \( w \in \Sigma^* \) that contains all words of length \( \ell \) exactly once as a factor are well studied (see [17,11,26,20] and [24,8,5,12] and the references therein), the universality problem for scattered factors just got recently attention (see [7,2,6,10] and the references therein).

Our contribution. Following the line of research started in [2], we investigate the (circular) universality of words. In particular, we study the universality of repetitions, which leads to several characterisations of its growth by the remainder of the arch factorisation. We present that intervals, on which the universality of repetitions is constant, correspond to either ascending or descending chains of the remainders of those repetitions. These deep insights into the behaviour of the remainder of the arch factorisation are linked to the circular universality such that we are able to present results on two open problems of [2]. As a consequence, we also get an efficient algorithm to compute the circular universality of a word.

Structure of the work. In Section 2 we present the basic notions and in Section 3 we study the remainder of the arch factorisation and especially its growth behaviour on repetitions. Afterwards, in Section 4 we connect the previous results and define ascending and descending chains of the remainders which leads to our main results, the generalisations of Theorem 22 and Theorem 23 from [2].
2 Preliminaries

Let $\mathbb{N} = \{1, 2, \ldots \}$ denote the natural numbers. Set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N}_{\geq k} = \{n \in \mathbb{N} \mid n \geq k\}$ for a $k \in \mathbb{N}_0$. We also define the discrete interval $[i, j] = \{i, i + 1, \ldots, j\}$ for $i, j \in \mathbb{N}_0$. Define for a function $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ the backward difference in $x \in \mathbb{N}$ by $\nabla f(x) = f(x) - f(x - 1)$ and call $\nabla$ the backward difference operator and $\nabla f(x)$ the growth of $f$ in $x$.

An alphabet $\Sigma$ is a finite set of symbols, called letter. A word $w$ is a finite sequence of letters from a given alphabet and its length $|w|$ is the number of $w$’s letters. For $i \in [1, |w|]$ let $w[i]$ denote the $i^{th}$ letter of $w$. The set of all finite words over the alphabet $\Sigma$, denoted by $\Sigma^*$, is the free monoid generated by $\Sigma$ with concatenation (product) as operation and the neutral element is the empty word $\varepsilon$, i.e. the word of length $0$. Set $\Sigma^+ = \Sigma^* \setminus \{\varepsilon\}$ and $\Sigma^k = \{w \in \Sigma^* \mid |w| = k\}$ for some $k \in \mathbb{N}$. Let $u, w \in \Sigma^*$ be words. Then $u$ is called a factor of $w$, if $w = xuy$ for some words $x$ and $y$ over $\Sigma$. If $x = \varepsilon$ (resp. $y = \varepsilon$) then $u$ is called a prefix (resp. suffix) of $w$. A factor (resp. prefix, suffix) $u$ of $w$ is called a proper factor (resp. prefix, suffix), if $u \notin \{\varepsilon, w\}$. If $w = xy$ then we define $x^{-1}w = y$ and $wy^{-1} = x$. Furthermore $u$ and $w$ are called prefix-compatible (resp. suffix-compatible) if one is a prefix (resp. suffix) of the other. Two words $w$ and $u$ are said to be conjugate to each other if there exist words $x, y \in \Sigma^*$ such that $w = xy$ and $u = yx$. We denote the reversal of a word by $w^R$, i.e. if $|w| = n$ then $w^R = w[n] \cdot w[n - 1] \cdots w[1]$. We say that a letter $a \in \Sigma$ occurs in $w$, if $a$ is a factor of $w$. We denote the set of all letters that occur in $w$ by alph$(w)$.

If we have words $w_i \in \Sigma^*$ for all $i \in [1, n]$ and some $n \in \mathbb{N}$, then we define $\prod_{i=1}^n w_i = w_k \cdots w_n$. In the special case $w_1 = w_2 = \ldots = w_n = : w$ we also write $w^n = \prod_{i=1}^n w$ and call $w^n$ the $n^{th}$ power of $w$.

Now, we introduce the basic notions around scattered factor universality, firstly introduced in [2].

**Definition 1.** A word $u \in \Sigma^*$ is called a scattered factor (or subword) of a word $w \in \Sigma^*$ if there exist $x_1, \ldots, x_{|u|+1} \in \Sigma^*$ such that $w = x_1u[1]x_2 \cdots x_{|u|}u[|u|]x_{|u|+1}$. We denote the set of all scattered factors of a given word $w \in \Sigma^*$ by ScatFact$(w)$ and the set of all scattered factors of a given length $k \in \mathbb{N}_0$ by ScatFact$_k(w)$.

**Definition 2.** A word $w \in \Sigma^*$ is called $k$-universal (w.r.t $\Sigma$) for some $k \in \mathbb{N}_0$ if ScatFact$_k(w) = \Sigma^k$. A word $w \in \Sigma^*$ is called circular $k$-universal (w.r.t. $\Sigma$) for some $k \in \mathbb{N}_0$, if there exists a conjugate $\nu$ of $w$ such that $\nu$ is $k$-universal. We define the universality index of $w$ as the maximal $k$ such that $w$ is $k$-universal and denote it by $\iota(w)$; analogously defined, $\zeta(w)$ denotes the circular universality index of $w$.

**Remark 3.** By definition, the universality is w.r.t. to a given alphabet $\Sigma$. Notice, that we have immediately $\iota(w) = 0 = \zeta(w)$ if alph$(w) \subset \Sigma$. Therefore, we implicitly assume w.l.o.g. $\Sigma = \text{alph}(w)$ from now on. Thus, we also assume $\iota(w) > 0$ at any time without mentioning it. For abbreviation, we set $\sigma = |\Sigma|$.

Since this work focuses on the (circular) universality index of powers of $w \in \Sigma^*$, we introduce the following parametrisation.

**Definition 4.** For \( w \in \Sigma^* \) define \( \iota_w : \mathbb{N}_0 \to \mathbb{N}_0; s \mapsto i(w^s) \) and \( \zeta_w : \mathbb{N}_0 \to \mathbb{N}_0; s \mapsto \zeta(w^s) \).

The following remark captures properties of \( \iota_w \) and \( \zeta_w \) for a given \( w \in \Sigma^* \).

**Remark 5.** Consider \( w \in \Sigma^* \) with \( k = i(w) \). Then we have, for all \( s \in \mathbb{N}_0 \), first, \( i(w^s) = \iota_w(s) = \sum_{i=1}^s \nabla \iota_w(i) \), second \( k \leq \nabla \iota_w(s) \leq k + 1 \), and finally \( \nabla \iota_w(1) = k \). Therefore, \( sk \leq \iota_w(s) \leq sk + s - 1 \). There are equivalent ways to represent these two extreme cases, namely we have
1. \( \iota_w(s) = sk \) iff \( \nabla \iota_w(i) = k \) for all \( i \in [1, s] \) and
2. \( \iota_w(s) = sk + s - 1 \) iff \( \nabla \iota_w(i) = k + 1 \) for all \( i \in [2, s] \).

Next we recall the arch factorisation introduced by Hebrard in [15], which is a powerful tool in investigating the universality.

**Definition 6.** Let \( w \in \Sigma^* \). The factorisation \( w = a r_1(w) \cdot \ldots \cdot a r_k(w) \cdot r(w) \), for some \( k \in \mathbb{N}_0 \), is called arch factorisation if for all \( i \in [1, k] \) we have \( a r_i(w) = u_i a_i \) for some \( u_i \in \Sigma^* \) and \( a_i \in \Sigma \), \( \text{alph}(u_i) \neq \Sigma \), \( \text{alph}(a_i) = \Sigma \), and \( \text{alph}(r(w)) \neq \Sigma \). Furthermore, we define \( m(w) = a_1 \cdot \ldots \cdot a_k \). We call \( a r_i(w) \) the \( i \)-th arch, \( r(w) \) the remainder, and denote the set of letters that occur in the remainder by \( \mathcal{R}(w) = \text{alph}(r(w)) \).

**Remark 7.** Note, that for all \( w \in \Sigma^* \) and appropriate \( i \in \mathbb{N} \), \( m(w)[i] \) is unique in \( a r_i(w) \) and the number of arches of a word \( w \) is \( i(w) \). In [2, Proposition 10] the arch factorisation was computed recursively by: \( a r_1(w) \) as the shortest prefix of \( w \) such that \( \text{alph}(a r_1(w)) = \Sigma \) and then \( a r_i(w) = a r_i(a r_i(w) \cdots a r_{i-1}(w))^{-1}w \).

In examples we will visualise the arch factorisation with the use of brackets. For example, we will write \( (abc) \cdot (cbba) \cdot (caab) \cdot a \) to indicate the three arches \( abc, cbba, \) and \( caab \) and the remainder (without brackets) \( a \).

In [2] the dynamic between universality and circular universality is studied. Naïvely, one would expect that the universality of powers of \( w \) grows linearly with its universality, i.e. \( \nabla i(w^s) = i(w) \) for all \( s \in \mathbb{N} \). But this is not always the case. Instead, its actual growth is related to the circular universality of \( w \). The following statements about (circular) universality from [2] are fundamental and the basis for our work.

**Lemma 8 ([2]).** For \( w \in \Sigma^* \), we have \( i(w) = i(w^R) \) and \( i(w) \leq \zeta(w) \leq i(w) + 1 \).

**Theorem 9 ([2]).** Let \( w \in \Sigma^* \) and \( k = i(w) \). For all \( s \in \mathbb{N} \), if \( \zeta(w) = k + 1 \) then \( i(w^s) = sk + s - 1 \).

**Theorem 10 ([2]).** Let |\( \Sigma \)| = 2, \( w \in \Sigma^* \) with \( k = i(w) \), and \( s \in \mathbb{N} \). Then \( i(w^s) = sk + s - 1 \) if \( \zeta(w) = k + 1 \) and \( i(w^s) = sk \) otherwise.

As stated in [2], neither Theorem 10 nor the converse of Theorem 9 hold for ternary alphabets witnessed by the following example: considering \( w = (bac) \cdot (caab) \cdot c \), we have \( \zeta(w) = i(w) \) and \( i(w^w) = 2i(w) + 1 \).

We finish the preliminaries with a lemma that follows the line of arguments repeatedly used in [2].
Lemma 11. Let \( w \in \Sigma^* \) and \( k = \iota(w) \). If \( \nabla_{\iota w}(s) = k \) for all \( s \in \mathbb{N} \) then also \( \nabla_{\iota w}(s) = k \) for all \( s \in \mathbb{N} \).

Proof. Let \( \nabla_{\iota w}(s) = k \) for all \( s \in \mathbb{N} \). Now suppose that \( \nabla_{\iota w}(s_0) = k + 1 \) for some \( s_0 \in \mathbb{N} \) and choose \( s_0 \) to be minimal with this property. Then we have \( \iota_w(s_0) = s_0 k + 1 \). This implies that \( \iota_w(2s_0) \geq 2s_0k + 2 \). Therefore we get \( \iota_w(2s_0) \geq 2s_0 + 1 \) by Lemma 8. This is a contradiction to our assumption that \( \nabla_{\iota w}(s) = k \) for all \( s \in \mathbb{N} \). Thus there cannot be such an \( s_0 \).

3 The Growth Behaviour of the Universality

In this section, our goal is to find a characterisation for the previous example: for \( w = \text{babccabc} \), we have \( k = \iota(w) = 2 \). Then we obtain \( \iota(w^2) = 2k + 1 \) and \( \iota(w^3) = 3k + 1 \) again. This means \( \nabla_{\iota w}(2) = k + 1 \), but \( \nabla_{\iota w}(3) = k \).

Thus, we seek criteria to determine whether \( \nabla_{\iota w}(s_0) \) is \( k \) or \( k + 1 \) for a given \( s_0 \). Having this purpose in mind, we now look at the arch factorisation of \( w^2 \): the first \( k \) arches of \( w^2 \) are those of \( w \) but the next arch begins with the remainder \( r(w) \) (which may be empty) and ends with a non-empty prefix \( p \) of \( w \). The remaining arches are those of \( p^{-1}w \). Thus \( \iota(w^2) = k + 1 + \iota(p^{-1}w) \) or, equivalently, \( \nabla_{\iota w}(2) = 1 + \iota(p^{-1}w) \). Clearly, by increasing the length of \( p \), we decrease \( \iota(p^{-1}w) \). So we get \( \nabla_{\iota w}(2) = k + 1 \) iff \( p \) is not too long. Thus, how long does \( p \) have to be, and what is the longest prefix \( p \) that we can remove from \( w \) without reducing its universality? Answering these questions will yield the characterisation that we are striving to. Regarding the first question, \( p \) must be long enough for the equation \( \alpha p(h(w)p) = \Sigma \) to hold, since only then \( r(w)p \) is an arch of \( w^2 \). The second question is answered by the following lemma.

Lemma 12. For \( w \in \Sigma^* \), the word \( p = r(w^R)R \) is the longest prefix of \( w \) such that \( \iota(p^{-1}w) = \iota(w) \) holds.

Proof. By reversing the arch factorisation of \( w^R \) we can write \( w \) as

\[
w = r(w^R)R \cdot \prod_{i=1}^{k} \ar_{k+1-i}(w^R)^R.
\]

There one can see that \( \iota(p^{-1}w) = k \) still holds. Next we have to show that the universality becomes smaller when taking a longer prefix, i.e. \( \iota(q^{-1}w) < k \) for any prefix \( q \) of \( w \) with \( |q| > |p| \). It suffices to show this for the prefix \( q \) with length \( |p| + 1 \). Let \( c \) be the first letter of the arch \( \ar_{k}(w^R)^R \) and \( y_k \in \Sigma^* \) the suffix such that

\[
\ar_{k}(w^R)^R = cy_k.
\]

Then \( q = pc \) is the prefix of \( w \) of length \( |p| + 1 \). Now we show that \( \iota(q^{-1}w) = k - 1 \) by building its arch factorisation. Since \( c \) is the last letter of \( \ar_{k}(w^R) \) it is unique in this arch. Thus, the letter \( c \) does not occur in \( y_k \), which implies that \( \iota(y_k) = 0 \). If now \( w \) had only one arch then

\[
\iota(q^{-1}w) = \iota(y_k) = 0 < k.
\]
would follow and thus our claim holds. So let us assume that \( k > 1 \). Then \( y_k \) still needs a non-empty prefix of \( \text{ar}_{k-1}(w^R)^R \) as suffix to build a full arch. Hence, there exist \( x_{k-1} \in \Sigma^+ \) and \( y_{k-1} \in \Sigma^* \) such that

\[
\text{ar}_{k-1}(w^R)^R = x_{k-1}y_{k-1},
\]

\[
\text{ar}_1(q^{-1}w) = y_kx_{k-1}.
\]

But then again it follows that \( \iota(y_{k-1}) = 0 \). Consequently, by iterating this process of building the arches of \( q^{-1}w \) we get for all \( i \in [1, k-1] \) some factors \( x_{k-i} \in \Sigma^+ \) and \( y_{k-i} \in \Sigma^* \) such that

\[
\text{ar}_{k-i}(w^R)^R = x_{k-i}y_{k-i},
\]

\[
\text{ar}_i(q^{-1}w) = y_{k-i+1}x_{k-i},
\]

\[
\iota(y_{k-i}) = 0.
\]

It follows that first \( r(q^{-1}w) = y_1 \) and second \( q^{-1}w \) has exactly \( k - 1 \) arches, and hence \( \iota(q^{-1}w) = k - 1 \), which was to be shown. \( \square \)

For instance, if \( w = nabanabanab \), we have \( r(w^R) = an \), and indeed, if we remove \( na \) from the beginning of \( w, \iota(bananab) = 2 \) still holds.

In our considerations above we looked at the arch factorisation of \( w^2 \). This can be generalised to the concatenation of two arbitrary words \( w \) and \( u \). The answer to the question, whether \( \iota(wu) \) equals \( \iota(w) + \iota(u) + 1 \) or \( \iota(w) + \iota(u) \), is answered by a slightly more general version of [2, Proposition 18].

**Proposition 13.** Let \( u, w \in \Sigma^* \) with \( k = \iota(w) \) and \( \ell = \iota(u) \). Then \( \iota(wu) = k + \ell + 1 \) if and only if \( \text{alph}(r(w)r(u^R)) = \Sigma \).

**Proof.** From the arch factorisations of \( w \) and \( u^R \) respectively follows immediately that if

\[
\text{alph}(r(w)r(u^R)) = \Sigma
\]

then \( \iota(wu) = k + \ell + 1 \).

Now on the other hand assume that

\[
\text{alph}(r(w)r(u^R)) \neq \Sigma.
\]

We can factorise \( wu \) as

\[
wu = w(u^R)^R = \prod_{i=1}^{k} \text{ar}_i(w) \cdot r(w) \cdot r(u^R)^R \cdot \prod_{i=1}^{\ell} \text{ar}_{\ell+1-i}(u^R)^R,
\]

i.e. as the product of the arch factorisation of \( w \) and the reversal of the arch factorisation of \( u^R \). Then one can see that

\[
\iota(wu) = k + \iota(r(w)u).
\]
Now it is left to show that \( i(r(w)u) < \ell + 1 \). We will determine \( i(r(w)u) \) by looking as usual at its arch factorisation. First, there exists a prefix \( p \) of \( u \) such that the first arch of \( r(w)u \) is

\[
\text{ar}_1(r(w)u) = r(w)p.
\]

Then it follows from Equation (2) that \( p \) has to be longer than \( r(u^R)^R \), i.e. \(|p| > |r(u^R)^R|\). And so, Lemma [12] tells us that \( i(p^{-1}u) < \ell \) holds. Therefore, finally,

\[
i(r(w)u) = i(r(w)p) + i(p^{-1}u) < 1 + \ell. \square
\]

Proposition [13] implies immediately the desired characterisation of the growth of \( \ell_w \), i.e. \( \nabla\ell_w \) using \( ws = ws^{-1}\cdot w \).

**Corollary 14.** Let \( w \in \Sigma^* \), \( k = i(w) \) and \( s \in \mathbb{N} \). Then we have \( \nabla\ell_w(s) = k + 1 \) if and only if \( \text{alph}(r(w^s-1)r(w^R)) = \Sigma \).

*Proof.* The claim follows by Proposition [13] applied on \( w^s-1 \) and \( w \). \( \square \)

**Corollary 14** is fundamental to the remainder of this chapter. It gives us a useful tool to investigate the universality of repetitions \( w^s \). It implies that the growth \( \nabla\ell_w \) and the remainder mapping \( s \mapsto r(w^s) \) of repetitions depend on each other. Thus we can gain insight into the behaviour of \( \nabla\ell_w \) by studying \( s \mapsto r(w^s) \) and vice versa. Especially, we are interested in question under which circumstances we gain eventual periodicity in the growth. Notice, that \( r(w^s) \) depends recursively on \( r(w^{s-1}) \) by the equation \( r(w^s) = r(w^{s-1}\cdot w) \). However, we can slightly refine this notion: since removing whole arches from a word does not change its remainder, we have the following observation.

**Remark 15.** Let \( w \in \Sigma^* \) and \( s \in \mathbb{N} \). Then \( r(w^s) = r(r(w^{s-1})w) \). Thus, there is always a word \( u \in \Sigma^* \) with \( \text{alph}(u) \not\subseteq \Sigma \) such that \( r(w^s) = r(uw) \).

Now, we investigate when \( r(uw) \) and \( r(w) \) differ (or more general \( r(uw) \) and \( r(vw) \) for some \( v \in \Sigma^* \{u\} \) with \( \text{alph}(v) \not\subseteq \Sigma \)). Consider for motivation, \( w = \text{abc} \cdot \text{caab} \cdot \text{caa} \). If \( u \in \{a,b\}^* \) then in \( uw \), \( u \) simply adds to the first arch of \( w \) without bringing any significant change into the arch factorisation. This happens because the last letter of an arch is unique and \( c \) does not occur in \( u \) leading to \( r(uw) = r(w) \). Now consider \( u = c \). This time, regarding \( uw \), \( u \) causes the factor \( bc \) to get released from \( w \)’s first arch and \( ab \) gets released from \( w’s \) second arch and \( \text{binds} \ c \), the first letter of \( w’s \) remainder, to build a third arch. Thus by prepending \( u \) to \( w \), changes in the arch factorisation get carried through the whole word. Finally, let \( u = \text{caa} \). The arch factorisations of \( cw \) and \( uw \) do not differ significantly from each other since the last letter of \( \text{ar}_1(aw) \) is \( b \) and adding any number of the letter \( a \) to \( c \) has no effect. Notice, that in all cases, we only argued by \( \text{alph}(u) \) but neither on the number of letters nor their position. In fact, for any \( u \in \Sigma^* \) with \( \text{alph}(u) \not\subseteq \Sigma \), \( \text{alph}(u) \) is the only relevant information about \( u \) for the computation of \( r(uw) \), since we assumed \( i(w) > 0 \).
Lemma 16. If \( u, v, w \in \Sigma^* \) with \( \text{alph}(u) = \text{alph}(v) \subseteq \Sigma \) then \( r(uw) = r(vw) \) holds.

Proof. The first arch of \( uw \) is formed by \( u \) and the shortest prefix \( p \) of \( w \) such that \( \text{alph}(u) \cup \text{alph}(p) = \Sigma \). Thus \( p \) depends solely on \( \text{alph}(u) \). This makes \( p \) also the shortest prefix such that \( \text{alph}(v) \cup \text{alph}(p) = \Sigma \). Hence
\[
r(uw) = r(vw) = r^i(w) \tag{16}
\]

Now, we return our attention to the equation \( r(w^0) = r(r(w^0) \cdot w) \) for some \( s_0 \in \mathbb{N}_0 \). If the letters occurring in \( r(w^0) \) and \( r(w^0) \cdot w \) are the same, Lemma 16 implies that \( s \rightarrow r(w^i) \) stays constant beginning at \( s_0 \). And even more general, we get in the same way that, if for some \( t_0 \in \mathbb{N}_0 \) the letters occurring in \( r(w^0) \) and \( r(w^0) \cdot w \) are the same, then \( r(w^{0+1}) = r(w^{0+1}) \), i.e. the mapping \( s \rightarrow r(w^i) \) is periodic beginning at \( \min \{ s_0, t_0 \} \). The following lemma proves this observation. Recall that we defined \( \mathcal{R}(w) = \text{alph}(r(w)) \).

Lemma 17. Let \( w \in \Sigma^* \) and let \( s, t \in \mathbb{N}_0 \). If \( \mathcal{R}(w^s) = \mathcal{R}(w^t) \) then \( r(w^{s+i}) = r(w^{t+i}) \) for all \( i \in \mathbb{N} \).

Proof. Let \( \mathcal{R}(w^s) = \mathcal{R}(w^t) \). We show by induction that \( r(w^{s+i}) = r(w^{t+i}) \) for all \( i \in \mathbb{N} \). For \( i = 1 \), we get by Lemma 16 that
\[
r(w^{s+1}) = r(r(w^s)w) = r(r(w^t)w) = r(w^{t+1}) \tag{17}
\]

For \( i > 1 \), assume that \( r(w^{s+i}) = r(w^{t+i}) \). Then we get
\[
r(w^{s+i+1}) = r(r(w^{s+i})w) = r(r(w^{t+i})w) = r(w^{t+i+1}) \tag{17}
\]

The converse of Lemma 17 is not necessarily true: considering \( w = (\text{a} \text{c} \text{a} \text{b}) \cdot \text{b} \), we have \( r(w^2) = r(w^2) \) for all \( s \geq 2 \), but \( \mathcal{R}(w) \neq \mathcal{R}(w^2) \). Notice also, that Lemma 17 does not hold for \( i = 0 \). For \( w = (\text{a} \text{b}) \cdot \text{b} \) we have \( \mathcal{R}(w) = \mathcal{R}(w^2) \), but \( r(w) \neq r(w^2) \). But indeed, if we only care about the letters in the remainders, then we can extend Lemma 17 to all \( i \in \mathbb{N}_0 \) and the converse immediately holds as well.

Lemma 18. Let \( w \in \Sigma^* \) and let \( s, t \in \mathbb{N}_0 \). Then we have \( \mathcal{R}(w^{s+i}) = \mathcal{R}(w^{t+i}) \) for all \( i \in \mathbb{N} \) if and only if \( \mathcal{R}(w^s) = \mathcal{R}(w^t) \).

Proof. Let \( \mathcal{R}(w^s) = \mathcal{R}(w^t) \). Then Lemma 17 implies that \( r(w^{s+i}) = r(w^{t+i}) \) for all \( i \in \mathbb{N} \). Hence \( \mathcal{R}(w^{s+i}) = \mathcal{R}(w^{t+i}) \) for all \( i \in \mathbb{N} \).

The other direction follows immediately by \( i = 0 \).

Lemma 18 implies the desired periodicity property of the growth.

Proposition 19. The growth of the universality index, \( \nabla_{uw} \), is eventually periodic.

Proof. Since the mapping \( s \rightarrow \mathcal{R}(w^s) \) can only have finitely many values (\( \Sigma \) is finite), there exist \( s_0, t_0 \in \mathbb{N}_0 \) with \( s_0 \neq t_0 \) such that \( \mathcal{R}(w^{s_0}) = \mathcal{R}(w^{t_0}) \) and thus Lemma 18 implies that \( s \rightarrow \mathcal{R}(w^s) \) is periodic. The claim follows by Corollary 14. \( \square \)
The following lemma shows that \( s \) and \( t \) can be bound by \( \sigma \) and leads to a theorem capturing the above considerations.

**Lemma 20.** For all \( w \in \Sigma^* \) we have \(| \{ \mathcal{R}(w^s) \mid s \in \mathbb{N}_0 \} | \leq \sigma \).

**Proof.** For all \( s \in \mathbb{N}_0 \) the remainder \( r(w^s) \) is a suffix of \( w \). Therefore these remainders are pairwise suffix-compatible with each other, and consequently also the sets \( \mathcal{R}(w^s) \) are pairwise comparable regarding inclusion, i.e. the set \( \{ \mathcal{R}(w^s) \mid s \in \mathbb{N}_0 \} \) is totally ordered. Thus we can order them by some bijective mapping \( \pi : \mathbb{N}_0 \to \mathbb{N}_0 \) such that
\[
\mathcal{R}(w^{\pi(0)}) \supseteq \mathcal{R}(w^{\pi(1)}) \supseteq \ldots
\]
But since also each set contains less than \( \sigma \) elements, i.e. \( 0 \leq |\mathcal{R}(w^s)| < \sigma \) for all \( s \in \mathbb{N}_0 \), it follows that \( s \mapsto \mathcal{R}(w^s) \) has at most \( \sigma \) different values. \( \square \)

**Theorem 21.** For all \( w \in \Sigma^* \) there exist \( s, t \in [0, \sigma] \) with \( s < t \) such that
1. \( r(w^{s+i}) = r(w^{t+i}) \) for all \( i \in \mathbb{N} \),
2. \( \mathcal{R}(w^{s+i}) = \mathcal{R}(w^{t+i}) \) for all \( i \in \mathbb{N}_0 \), and
3. \( \nabla_{tw}(s+i) = \nabla_{tw}(t+i) \) for all \( i \in \mathbb{N} \).

**Proof.** By Lemma 20 there exist \( s < t \in [0, \sigma] \) such that \( \mathcal{R}(w^s) = \mathcal{R}(w^t) \). Then the first claim follows by Lemma 17 and the second by Proposition 18. The third claim follows by combining the second claim with Corollary 14. \( \square \)

Theorem 21 states that beginning at \( s+1 \), \( \nabla_{tw} \) is periodic of length \( t - s \). So, given an \( n \in \mathbb{N} \) we can divide \([1, n]\) into \([1, s]\) (before \( \nabla_{tw} \) is periodic) and \([s+1, n]\) (containing the periodic part). By division with remainder, we obtain \( \ell, m \in \mathbb{N}_0 \) such that \( n - s = \ell \cdot (t - s) + m \) and thus, the subinterval \([s+1, s+\ell(t-s)]\) contains \( \ell \) full periods while \([s+\ell(t-s)+1, n]\) is the rest. This observation motivates the following Proposition.

**Proposition 22.** Let \( w \in \Sigma^* \) and let \( n \in \mathbb{N} \) with \( n \geq \sigma \). Then there exist \( \ell, m, s, t \in \mathbb{N}_0 \) with \( s < t \leq \sigma \), \( m < t - s \), and \( n - s = \ell \cdot (t - s) + m \) such that \( \nabla_{tw}(n) = \nabla_{tw}(s+m) + \ell \cdot (\nabla_{tw}(t) - \nabla_{tw}(s)) \).

**Proof.** Let \( s \) and \( t \) be minimal such that \( s < t \) and \( \mathcal{R}(w^s) = \mathcal{R}(w^t) \). By Theorem 21 we have \( s, t \leq \sigma \). Then there are by division with remainder \( m, \ell \in \mathbb{N}_0 \) such that \( n - s = \ell \cdot (t - s) + m \) with \( m < t - s \). It follows by Theorem 21 that
\[
\nabla_{tw}(n) = \sum_{i=1}^{n} \nabla_{tw}(i)
= \left[ \sum_{i=1}^{s} \nabla_{tw}(i) \right] + \ell \cdot \left[ \sum_{i=s+1}^{t} \nabla_{tw}(i) \right] + \left[ \sum_{i=s+1}^{s+m} \nabla_{tw}(i) \right]
= \nabla_{tw}(s+m) + \ell \cdot \left[ \sum_{i=s+1}^{t} \nabla_{tw}(i) \right]
= \nabla_{tw}(s+m) + \ell \cdot (\nabla_{tw}(t) - \nabla_{tw}(s)),
\]
which was to be shown. \( \square \)
Proposition 23. Given \( w \in \Sigma^* \), we can compute \( t_w(n) \) for all \( n \in \mathbb{N}_0 \) in constant time with a preprocessing time of \( O(|\sigma| |w|) \).

Proof. We only have to compute the minimal \( s, t \in \mathbb{N}_0 \) with \( R(w^s) = R(w^t) \) and \( t_w(i) \) for all \( i \in [0, \sigma] \) once, taking in total \( O(|\sigma| |w|) \) time. Afterwards, by Proposition 22 we can compute \( t_w(n) \) for all \( n \in \mathbb{N}_0 \) in constant time. \( \square \)

In the rest of this work we will usually only need the notion of constancy and not of periodicity. Thus we restate Lemma 17 and Proposition 18 for the case \( s = t - 1 \). The claims follow directly by the lemma and the proposition.

Lemma 24. Let \( w \in \Sigma^* \) and \( s_0 \in \mathbb{N} \). If \( R(w^{s_0 - 1}) = R(w^{s_0}) \) then \( r(w^s) = r(w^{s_0}) \) for all \( s \geq s_0 \).

Proposition 25. Let \( w \in \Sigma^* \) and \( s_0 \in \mathbb{N} \). Then we have \( R(w^s) = R(w^{s_0}) \) for all \( s \geq s_0 - 1 \) if and only if \( R(w^{s_0 - 1}) = R(w^{s_0}) \).

Hence, if \( R(w^{s_0 - 1}) = R(w^{s_0}) \) holds for any \( s_0 \in \mathbb{N} \) then the mappings \( s \mapsto r(w^s) \) and \( s \mapsto R(w^s) \) are both eventually constant.

4 Chaining the Remainder

Section 3 established a correspondence between the growth and the remainder mapping \( s \mapsto r(w^s) \). We achieved this corollary by interpreting \( w^s \) recursively as \( w^{s-1} \cdot w \). But if we interpret it as \( w \cdot w^{s-1} \) instead then we find another useful relationship that we capture in the following lemma.

Lemma 26. Let \( w \in \Sigma^* \) and \( s \in \mathbb{N}_0 \). Then the following two statements hold:
1. If \( \nabla t_w(s + 1) = i(w) \) then \( r(w^s) \) is a suffix of \( r(w^{s+1}) \).
2. If \( \nabla t_w(s + 1) = i(w) + 1 \) then \( r(w^{s+1}) \) is a suffix of \( r(w^s) \).

Proof. Set \( k = i(w) \). First note that \( r(w^{s+1}) = r(r(w)w^s) \) holds. Second, let \( \ell = i(w^s) \) be the universality of \( w^s \). Now we examine the arch factorisation of \( r(w)w^s = r(w) \prod_{i=1}^{\ell} ar_i(w^s) \).

For all \( i \in [1, \ell] \) there exist factors \( x_i, y_i \in \Sigma^* \) such that the arches of \( w^s \) are factorised by

\[
ar_i(w^s) = x_i y_i
\]

and such the arches of \( r(w)w^s \) are factorised by

\[
ar_i(r(w)w^s) = \begin{cases} r(w)x_i, & \text{if } i = 1, \\ y_{i-1}x_i, & \text{if } i \geq 2. \end{cases}
\]

Now the question is whether the remaining factor \( y_1 r(w^s) \) contains yet another arch or whether it is already the remainder of \( r(w)w^s \).
**Case 1:** Assume that it is already the remainder, i.e. that
\[ r(r(w)w^s) = y_\ell r(w^s) \]
holds. Then \( r(w)w^s \) has exactly \( \ell \) arches and hence \( w^{s+1} \) has \( k + \ell \) arches. In other words, \( \nabla_{tw}(s + 1) = k \). But, since \( r(w^{s+1}) = r(w)w^s \), it also follows that \( r(w^s) \) is a suffix of \( r(w^{s+1}) \).

**Case 2:** Now assume that \( y_\ell r(w^s) \) contains yet another arch. Then there exist factors \( x_{\ell+1}, y_{\ell+1} \in \Sigma^* \) such that
\[ r(w^s) = x_{\ell+1}y_{\ell+1}r(w^s) \]
\[ a r_{\ell+1}(r(w)w^s) = y_{\ell+1} x_{\ell+1} r(w)w^s = y_{\ell+1}. \]

Then with similar arguments as in the previous case it follows that \( \nabla_{tw}(s + 1) = k + 1 \) and also that \( r(w^{s+1}) = y_{\ell+1} \) is a suffix of \( r(w^s) \).

And so we have either the case that \( \nabla_{tw}(s + 1) = k \) and \( r(w^s) \) is a suffix of \( r(w^{s+1}) \), or the case that \( \nabla_{tw}(s + 1) = k + 1 \) and \( r(w^{s+1}) \) is a suffix of \( r(w^s) \). \( \square \)

Combining Corollary 14 and Lemma 26 implies that if \( r(w^s) \) is long enough then \( \nabla_{tw}(s + 1) = k + 1 \), which implies that \( r(w^{s+1}) \) is a suffix of \( r(w^s) \). Thus, now \( r(w^{s+1}) \) may have become short that in the next step we get \( \nabla_{tw}(s + 2) = k \). This is exactly what happens for \( w = (bac) \cdot (caab) \cdot c. \) The other case is symmetrical: if \( r(w^s) \) is not long enough then \( \nabla_{tw}(s + 1) = k \), which implies that \( r(w^s) \) is a suffix of \( r(w^{s+1}) \). Thus, now \( r(w^{s+1}) \) may have become long enough for \( \nabla_{tw}(s + 2) = k + 1 \). Also note, that the converses of Lemma 26 do not necessarily hold if \( r(w^s) = r(w^{s+1}) \). Considering \( w = (ab) \cdot a, \) we get \( \nabla_{tw}(2) = 1 \) and \( r(w) = a = r(w^2) \). On the other hand, if \( w = (aab) \cdot b \) then \( r(w) = 1, \) but \( \nabla_{tw}(2) = 2 \) and \( r(w) = r(w^2) \). But as soon as two successive remainders are equal or even if only their set of occurring letters are equal, then \( s \Rightarrow r(w^s) \) is eventually constant. Thus, it seems useful to consider the cases, where these conditions are excluded, explicitly. We state them in the following two corollaries.

**Corollary 27.** Let \( w \in \Sigma^* \) and \( s \in \mathbb{N}_0 \). If \( r(w^s) \neq r(w^{s+1}) \), then
1. \( \nabla_{tw}(s + 1) = k \) iff \( r(w^s) \) is a (proper) suffix of \( r(w^{s+1}) \) and
2. \( \nabla_{tw}(s + 1) = k + 1 \) iff \( r(w^{s+1}) \) is a (proper) suffix of \( r(w^s) \).

**Corollary 28.** Let \( w \in \Sigma^* \) and \( s \in \mathbb{N}_0 \) with \( \mathcal{R}(w^s) \neq \mathcal{R}(w^{s+1}) \). Then we obtain
1. \( \nabla_{tw}(s + 1) = k \) iff \( r(w^s) \) is a (proper) suffix of \( r(w^{s+1}) \) iff \( \mathcal{R}(w^s) \subseteq \mathcal{R}(w^{s+1}) \) and
2. \( \nabla_{tw}(s + 1) = k + 1 \) iff \( r(w^{s+1}) \) is a (proper) suffix of \( r(w^s) \) iff \( \mathcal{R}(w^s) \supseteq \mathcal{R}(w^{s+1}) \).

However, usually we do not know whether \( s \mapsto r(w^s) \) is eventually constant, i.e. whether Corollary 28 is applicable. So now the following lemma gives a criterion to decide whether this condition is satisfied.

**Lemma 29.** For all \( w \in \Sigma^* \), \( s \mapsto r(w^s) \) is eventually constant iff \( \nabla_{tw} \) is.
Proof. From Corollary 14 follows immediately that if \( s \mapsto r(w^s) \) is eventually constant then \( \nabla t_w \) is, too.

So now assume that \( \nabla t_w \) is eventually constant. Then there is an \( s_0 \in \mathbb{N} \) such that \( \nabla t_w(s) = \ell \) for all \( s \geq s_0 \), where \( \ell \) is either \( k \) or \( k + 1 \).

Case \( \ell = k \): Then Lemma 26 implies that there is an infinite ascending chain

\[
R(w^{s_0}) \subseteq R(w^{s_0+1}) \subseteq \ldots.
\]

But since \( R(w^{s}) \subseteq \Sigma \) for all \( s \in \mathbb{N} \) and \( \Sigma \) is finite, it follows that the chain is not strictly increasing. Hence there is some \( t \geq s_0 \) such that \( R(w^{t}) = R(w^{t+1}) \) and thus \( s \mapsto r(w^s) \) is eventually constant.

Case \( \ell = k + 1 \): This case is analogous to the previous one. \( \square \)

Applying Lemma 29 gives a new insight regarding the characterisation when \( \zeta(w) = i(w) + 1 \) holds.

Corollary 30. Let \( w \in \Sigma^* \) and \( k = i(w) \). If \( \zeta(w) = k + 1 \) then \( s \mapsto r(w^s) \) is eventually constant.

We will usually apply Lemma 26 and Corollary 28 in the following way: an interval \( [\ell + 1, n] \) on which we have \( \nabla t_w(s) = k \) implies that there is an ascending chain \( R(w^{\ell}) \subseteq R(w^{\ell+1}) \subseteq \ldots \subseteq R(w^n) \). But notice that we cannot follow that \( \nabla t_w(s) = k \) holds on \( [\ell + 1, n] \) by the existence of such a chain since there may be equality in some steps. However, if we exclude \( R(w^{s}) = R(w^n) \) for all \( s \geq n - 1 \), i.e. \( s \mapsto R(w^s) \) is not yet constant then we know that the chain from \( R(w^{\ell}) \) to \( R(w^n) \) is strict, and such a strictly ascending chain implies \( \nabla t_w(s) = k \) on \( [\ell + 1, n] \) (the case where \( \nabla t_w(s) = k + 1 \) on \( [\ell + 1, n] \) is symmetrical with descending chains). This way Lemma 26 and Corollary 28 can be used to translate questions about \( t_w \) and \( \zeta_w \) into questions about chains of sets.

In the following two subsections we investigate ascending and descending chains in more detail. By improvements of Lemma 29 we are able to generalise the two aforementioned results from 2.

4.1 Ascending Chains. So far, we established that an interval \( [\ell + 1, n] \) on which \( \nabla t(w^s) = k \) holds implies an ascending chain \( R(w^{\ell}) \subseteq \ldots \subseteq R(w^n) \) and if that chain is strict then the implication holds also in the other direction. The following proposition gives us a structural property about a strictly ascending chain of length exactly \( \sigma \), where \( \sigma \) is the size of the alphabet.

Lemma 31. Let \( w \in \Sigma^* \) and \( \ell \in \mathbb{N}_0 \). If \( R(w^{\ell}) \subseteq \ldots \subseteq R(w^{\ell+\sigma-1}) \) is a strictly ascending chain then we have \( |R(w^s)| = s - \ell \) for all \( s \in [\ell, \ell + \sigma - 1] \).

Proof. By the definition of the remainder we have \( 0 \leq |R(u)| < \sigma \) for all \( u \in \Sigma^* \). Thus by

\[
0 \leq |R(w^{\ell})| < \ldots < |R(w^{\ell+\sigma-1})| < \sigma
\]

the claim follows. \( \square \)
By Lemma 31 strictly ascending chains of length $\sigma + 1$ cannot exist.

**Corollary 32.** Let $w \in \Sigma^*$, $\ell, n \in \mathbb{N}_0$, and $R(w^f) \subseteq \ldots \subseteq R(w^{f+n-1})$ be a strictly ascending chain of length $n$. Then $n \leq \sigma$.

**Proof.** If there was a strictly ascending chain with length $\sigma + 1$ then Lemma 31 would imply $\sigma - 1 = \left| R(w^{f+\sigma-1}) \right| < \left| R(w^{f+\sigma}) \right|$ and thus $\left| R(w^{f+\sigma}) \right| = \sigma$. This is a contradiction since not all letters can occur in the remainder. \hfill \Box

**Remark 33.** In fact, there actually exists a strictly ascending chain of length $\sigma$. For $\Sigma = \{a_1, \ldots, a_\rho\}$ set $w = \prod_{i=1}^{\rho} a_i^2$. Then we have $\iota(w) = 1$, $\nabla_{tw}(s) = 1$ for all $s \in [1, \sigma - 1]$ and $\nabla_{tw}(\sigma) = 2$. Furthermore $R(w^0) \subseteq \ldots \subseteq R(w^{\rho-1})$ is a strictly ascending chain of length $\sigma$.

Corollary 32 leads to the following proposition, which states that if we have $\nabla_{tw}(s) = k$ for the first $\sigma - 1$ repetitions then $\nabla_{tw}$ is already constant.

**Proposition 34.** Let $w \in \Sigma^*$, $k = \iota(w)$. If $\nabla_{tw}(s) = k$ for all $s \in [1, \sigma]$ then $\nabla_{tw}(s) = k$ for all $s \in \mathbb{N}$.

**Proof.** Let $\nabla_{tw}(s) = k$ for all $s \in [1, \sigma]$. Then Lemma 26 gives us the ascending chain $R(w^0) \subseteq \ldots \subseteq R(w^\sigma)$.

By Corollary 32 this chain is not strict. Therefore there exists an $s_0 \in [1, \sigma]$ such that $R(w^{s_0-1}) = R(w^{s_0})$. Then Proposition 25 implies that $s \rightarrow R(w^s)$ is eventually constant with $R(w^s) = R(w^{s_0-1})$ for all $s \geq s_0 - 1$. Consequently it follows from Corollary 14 that $\nabla_{tw}(s) = \nabla_{tw}(s_0) = k$ for all $s \geq s_0$. \hfill \Box

Even though the bound $\sigma$ in Proposition 34 is tight, we can still improve the statement in another way. If we consider the circular universality $\zeta_w$ instead of the plain universality $\iota_w$ then we can lower the bound to $\sigma - 1$. Before we present the corresponding proposition, we prove two auxiliary lemmata.

**Lemma 35.** Let $w \in \Sigma^*$ and $k = \iota(w)$. If each letter $a \in \Sigma$ occurs only once in each arch of $w$ and at most once in the remainder $r(w)$ then $\nabla_{\zeta_w}(s) = k$ for all $s \in \mathbb{N}$.

**Proof.** Let $a \in \Sigma$ such that $a$ does not occur in $r(w)$. Then by assumption $a$ occurs $k$ times in $w$ and therefore $sk$ times in $w^s$. So $\zeta_w(s)$ is bounded by $sk$. However, it is also at least $sk$. Hence $\zeta_w(s) = sk$. \hfill \Box

**Lemma 36.** Let $w \in \Sigma^*$, $k = \iota(w)$. Let there be a word $y \in \Sigma^*$ and a letter $a \in \Sigma$ such that $|\text{alph}(y)| \leq \sigma - 2$ and such that $ay$ is a factor of some conjugate $v$ of $w$. If $\nabla_{\zeta_w}(s) = k$ holds for all $s \in [1, \sigma - 1]$ then we have $\nabla_{tw}(s) = k$ for all $s \in \mathbb{N}$.

**Proof.** Because $ay$ is a factor of $v$, there exist factors $x,z \in \Sigma^*$ such that $v = xayaz$. Then $u = azay$.
is a conjugate of \( w \), too. Now suppose that \( \nabla_{t_w} \) grew in the \( \sigma \)th step by \( k + 1 \), i.e.
\[
\nabla_{t_w}(\sigma) = k + 1.
\]
(6)

Then the same holds for the circular universality of \( u \) in at least some step, i.e. \( \nabla_{t_u}(t) = k + 1 \) for some \( t \in \mathbb{N} \). This now implies by Lemma\[11\] that the same must hold for its plain universality, i.e. we have \( \nabla_{t_u}(t') = k + 1 \) for some \( t' \in \mathbb{N} \). However, Proposition\[34\] states that any such growth must have already occurred in the interval \([1, \sigma]\), i.e. there is some \( t'' \in [1, \sigma] \) with that property.

But since by assumption \( \nabla_{t_u}(s) = k \) for all \( s \in [1, \sigma - 1] \), the only possible value is \( t'' = \sigma \). Thus far we have shown that
\[
\nabla_{t_u}(s) = \begin{cases} 
  k & \text{if } s \in [1, \sigma - 1] \\
  k + 1 & \text{if } s = \sigma.
\end{cases}
\]
(7)

The first case gives us by Lemma\[26\] the ascending chain
\[
\mathcal{R}(u^0) \subseteq \ldots \subseteq \mathcal{R}(u^{\sigma - 1})
\]
and the second case implies that this chain is strict. Then Lemma\[31\] is applicable, resulting in \( |\mathcal{R}(u^s)| = s \) for all \( s \in [1, \sigma - 1] \). Now note that by Lemma\[8\] these arguments can be analogously applied to \( u^R \) as well. Consequently we can follow the same way that \( |\mathcal{R}((u^R)^s)| = s \) for all \( s \in [1, \sigma - 1] \). And so we have in particular
\[
|\mathcal{R}(u^{\sigma - 1})| = \sigma - 1 \quad \text{and} \quad |\mathcal{R}(u^R)| = 1.
\]
(8)

Therefore, since by assumption \( |\text{alph}(ay)| \leq \sigma - 1 \), it follows from the construction of \( u \) that both
\[
a \in \mathcal{R}(u^{\sigma - 1}) \quad \text{and} \quad \mathcal{R}(u^R) = \{ a \}.
\]

However, this implies that
\[
\text{alph}(r(u^{\sigma - 1})r(u^R)) = \mathcal{R}(u^{\sigma - 1}) \neq \Sigma
\]
and hence we have \( \nabla_{t_u}(\sigma) = k \) by Corollary\[14\] This is a contradiction to Equation\[7\] and so the supposition \( \nabla_{t_w}(\sigma) = k + 1 \) must be false. Thus we have \( \nabla_{t_w}(s) = k \) not only on the interval \([1, \sigma - 1]\), but on \([1, \sigma]\). Now the claim follows by Proposition\[34\] \( \square \)

**Proposition 37.** Let \( w \in \Sigma^* \). Then we have \( \nabla_{t_w}(s) = k \) for all \( s \in \mathbb{N} \) if and only if \( \nabla_{t_w}(s) = k \) for all \( s \in [1, \sigma - 1] \).

**Proof.** First let \( \nabla_{t_w}(s) = k \) for all \( s \in [1, \sigma - 1] \). If each letter occurs only once in each arch of \( w \) and at most once in the remainder \( r(w) \), then the claim already follows by Lemma\[33\] So now assume that there exists some letter \( a \in \Sigma \) that occurs at least twice in an arch or the remainder of \( w \). Since the last letter of an arch is unique this implies that the conditions of Lemma\[33\] are met. Then the claim follows.

The other direction follows immediately by Lemma\[11\] \( \square \)
With Proposition 37 we can finally achieve our first main goal of generalising Theorem 10 to alphabets of arbitrary size.

**Theorem 38.** Let \( w \in \Sigma^* \) with \( k = \iota(w) > 0 \) and let \( s \in \mathbb{N} \).
1. If \( \zeta(w) = k + 1 \) then \( \iota(w^s) = sk + s - 1 \).
2. If \( \nabla_{\zeta w}(t) = k \) for all \( t \in [1, \sigma - 1] \) then \( \iota(w^s) = sk \).

**Proof.** The claim follows by combining Theorem 9 and Proposition 37. □

**Remark 39.** Considering again \( w = \prod_{i=1}^{\sigma} a_i^2 \) shows that the bound \( \sigma - 1 \) in Proposition 37 is tight. The word \( u = a_1 \prod_{i=2}^{\sigma} a_i^2 \) is a conjugate of \( w \) with \( \nabla_{tu}(s) = 1 \) for all \( s \leq \sigma - 2 \) and \( \nabla_{tu}(\sigma - 1) = 2 \). Thus we have \( \nabla_{\zeta w}(s) = 1 \) for all \( s \leq \sigma - 2 \) and \( \nabla_{\zeta w}(\sigma - 1) = 2 \).

### 4.2 Descending Chains

Now, we discuss descending chains instead of ascending chains. We begin by searching for the longest strictly descending chain that is possible. The following proposition gives us a structural property about such a chain of length exactly \( \sigma \) and is symmetrical to Lemma 31.

**Lemma 40.** Let \( w \in \Sigma^* \) and \( \ell \in \mathbb{N}_0 \). If \( \mathcal{R}(w^\ell) \supseteq \ldots \supseteq \mathcal{R}(w^{\ell+\sigma-1}) \) is a strictly descending chain of length \( \sigma \) then \( |\mathcal{R}(w^s)| = \sigma + \ell - s \) for all \( s \in [\ell + 1, \ell + \sigma] \).

**Proof.** The proof is symmetrical to the proof of Lemma 31. □

Analogously to Corollary 32 there cannot be a strictly descending chain of length \( \sigma + 1 \). Surprisingly, such a chain of length \( \sigma \) leads to a contradiction as well. First, we present an auxiliary lemma.

**Lemma 41.** Let \( w \in \Sigma^* \) and \( \ell, n \in \mathbb{N}_0 \). If \( \mathcal{R}(w^\ell) \supseteq \ldots \supseteq \mathcal{R}(w^{\ell+\sigma-1}) \) is a strictly descending chain of length \( \sigma \) then \( \mathcal{R}(w) \supseteq \ldots \supseteq \mathcal{R}(w^n) \) is one as well.

**Proof.** We can apply Lemma 40. It implies that \( \mathcal{R}(w^\ell) = \sigma - 1 \), and hence \( \ell \neq 0 \). It also implies that \( s \mapsto \mathcal{R}(w^s) \) assumes \( \sigma \) different values on the interval \( [\ell, \ell + \sigma - 1] \), i.e. all possible values by Lemma 20. Therefore \( \mathcal{R}(w^{\ell+1}) \) is one of them, i.e. we have \( \mathcal{R}(w^{\ell+1}) = \mathcal{R}(w^{\ell+m}) \) for some \( m \in [0, \sigma - 1] \). However, by Corollary 14 their successors are also equal, i.e. \( \mathcal{R}(w^{\ell}) = \mathcal{R}(w^{\ell+m+1}) \). Hence, since the chain is strict, \( m = \sigma - 1 \) is the only possible value for \( m \). Thus, since \( |\mathcal{R}(w^{\ell+\sigma-1})| = 0 \) by Lemma 40 it follows that \( \mathcal{R}(w^{\ell+1}) = \emptyset \), and consequently \( r(w^{\ell+1}) = \epsilon = r(w^0) \).

Then Lemma 17 implies that
\[
 r(w^s) = r(w^{\ell+1+s})
\]
for all \( s \in \mathbb{N}_0 \), and therefore \( \mathcal{R}(w) \supseteq \ldots \supseteq \mathcal{R}(w^n) \) is a strictly descending chain of length \( \sigma \), too. □
Proposition 42. Let $w \in \Sigma^*$ and $\ell, n \in \mathbb{N}_0$. If $\mathcal{R}(w^\ell) \supseteq \ldots \supseteq \mathcal{R}(w^{\ell+n-1})$ is a strictly descending chain of length $n$ then $n \leq \sigma - 1$.

Proof. We lead the proof by contradiction. It is structured as follows. First we argue that one can assume $\ell = 1$. Second we argue that the conjugates of $w$ obtained by cyclic shifts of whole arches give us strictly descending chains, too. Third we follow that each arch and the remainder end with the same letter. Last we show that this leads to a contradiction.

Suppose that $n > \sigma - 1$. Then in particular there is a strictly descending chain of length $\sigma$. By Lemma 41 we can assume w.l.o.g. that $\ell = 1$. So we have

\[ \mathcal{R}(w) \supseteq \ldots \supseteq \mathcal{R}(w^{\sigma}) \]  

is a strictly descending chain.

Now let $j \in [0, k]$ and

\[ w_j = \left[ \prod_{i=j+1}^{k} \text{ar}_i(w) \right] r(w) \left[ \prod_{i=1}^{j} \text{ar}_i(w) \right]. \]  

(10)

In other words, $w_j$ is a conjugate of $w$ obtained by cyclic shifts of full arches of $w$ and in particular $w_0 = w$. Then for all $s \in \mathbb{N}$ the arch factorisation

\[ w_j^s = \left[ \prod_{i=j+1}^{\iota(w^s)} \text{ar}_i(w^s) \right] r(w^s) \left[ \prod_{i=1}^{j} \text{ar}_i(w^s) \right] \]

contains at least $\iota(w^s)$ arches and thus

\[ \iota(w_j^s) \geq \iota(w^s). \]  

(11)

Next, since $\mathcal{R}(w^{\sigma-1})$ is empty by Lemma 40, it follows that $\nabla t_w(\sigma) = k$, and so by Theorem 9 we must have $\zeta(w) = k$. Hence by Remark 5

\[ \iota(w^s) \leq \zeta(w^s) \leq sk + s - 1. \]

for all $s \in \mathbb{N}$. However, because the chain is strict, Corollary 28 implies that $\nabla t_w(s) = k + 1$ on the interval $[2, \sigma]$ and thus $\iota(w^s) = sk + s - 1$ for all $s \in [1, \sigma]$. Consequently we get for all $s \in [1, \sigma]$ that

\[ \iota(w_j^s) \leq \iota(w^s). \]  

(12)

But moreover note that, since $r(w^\sigma) = \varepsilon$, we have $\iota(w_j^{\sigma+1}) = \iota(w^{\sigma+1})$ by definition of $w_j$. Thus far we have shown that

\[ \iota(w_j^s) = \iota(w^s) \]  

(13)

for all $s \in [1, \sigma + 1]$. Then it follows for all $s \in [2, \sigma]$ that $\nabla \iota(w_j^s) = k + 1$ and also that $\nabla \iota(w_j^{\sigma+1}) = k$. And so Corollary 28 implies that

\[ \mathcal{R}(w_j) \supseteq \ldots \supseteq \mathcal{R}(w_j^\sigma) \]  

(14)
is also a strictly descending chain.

Now we can apply Lemma 40 on this chain, too, and get

\[ |R(w_j)| = \sigma - 1 \quad \text{and} \quad |R(w_j^{\sigma-1})| = 1. \]  \hspace{1cm} (15)

Since, moreover, \( \nabla \iota(w_j^{\sigma}) = k + 1 \) by Corollary 14 implies that

\[ \text{alph}(r(w_j^{\sigma-1})r(w_j^R)) = \Sigma, \]  \hspace{1cm} (16)

it follows that

\[ |R(w_j^R)| = \sigma - 1 \quad \text{and} \quad R(w_j^{\sigma-1}) \cap R(w_j^R) = \emptyset. \]  \hspace{1cm} (17)

Next let \( a_j \in R(w_j^{\sigma-1}) \) be the letter that occurs in \( r(w_j^{\sigma-1}) \). Note that \( a_j \) is also the last letter of \( w_j \). Then, since \( R(w_j^{\sigma-1}) \) and \( R(w_j^R) \) are disjunct, we have

\[ a_j \notin R(w_j^R). \]  \hspace{1cm} (18)

And so, since in \( r(w_j^R) \) occur all letters of \( \Sigma \) except \( a_j \) and it is also a prefix of the first arch of \( w_j \), i.e. \( ar_1(w_j) \), the last letter of \( ar_1(w_j) \) must be \( a_j \), because the last letter of an arch is unique. However, by construction of \( w_j \) we have that \( ar_1(w_j) \) is a suffix of \( w_{j+1} \) for all \( j < k \). This implies that \( a_j \) is the last letter of \( w_{j+1} \) and hence \( a_j = a_{j+1} \) for all \( j < k \). Therefore it follows inductively that

\[ a_j = a_0 \]

for all \( j \leq k \). In other words, every arch and also the remainder of \( w \) ends with the same letter \( a_0 \).

Now note that

\[ w_k = r(w) \left( \prod_{i=1}^{k} ar_i(w) \right). \]

Then, since \( r(w_k^R) \) and \( r(w) \) are both prefixes of \( w_k \) and in both occur exactly \( \sigma - 1 \) different letters, it follows that

\[ R(w_k^R) = R(w). \]

But we have on the one hand \( a_0 \in R(w) \), since \( a_0 \) is the last letter of \( w \), and on the other hand also \( a_0 \notin R(w_k^R) \) by (18) This is a contradiction. Therefore the supposition \( n > \sigma - 1 \) is wrong.

\[ \Box \]

Remark 43. The word \( w = a_n \left( \prod_{i=1}^{\sigma} a_i^2 \right)^{n-2} \left( \prod_{i=1}^{\sigma-1} a_i \right) \) over \( \Sigma = \{ a_1, \ldots, a_{\sigma} \} \) witnesses that a strictly descending chain of length \( \sigma - 1 \) actually exists: we have \( \iota(w) = \sigma - 1 \) and \( \nabla \iota(w)(s) = \sigma \) for all \( s \in [1, n-1] \) as well as \( \nabla \iota(w)(\sigma) = \sigma - 1 \) and \( \xi(w) = \sigma - 1 \). Furthermore, \( R(w) \supseteq \ldots \supseteq R(w^{\sigma-1}) \) is a strictly descending chain of length \( \sigma - 1 \).
With Proposition 42 we can achieve our second main goal: the following theorem is a reasonable modification of Theorem 9 such that its converse holds.

**Theorem 44.** For \( w \in \Sigma^* \) the following statements are equivalent:
1. \( \nabla_{tw}(s) = k + 1 \) for all \( s \in [2, \sigma] \),
2. \( \nabla_{tw}(s) = k + 1 \) for all \( s \in \mathbb{N}_{\geq 2} \),
3. \( \zeta(w) = k + 1 \).

**Proof.** Firstly, let \( \nabla_{tw}(s) = k + 1 \) for all \( s \in [2, \sigma] \). Then Lemma 26 gives us the descending chain \( \mathcal{R}(w) \supseteq \ldots \supseteq \mathcal{R}(w^3) \). By Lemma 32 this chain is not strict and thus, \( \nabla_{tw}(s) = k + 1 \) for all \( s \in \mathbb{N}_{\geq 2} \). This proves the first implication. Now let \( \nabla_{tw}(s) = k + 1 \) for all \( s \geq 2 \). Then, again, the chain \( \mathcal{R}(w) \supseteq \mathcal{R}(w^2) \supseteq \ldots \) is not strict. Hence the mapping \( s \mapsto r(w^s) \) is eventually constant. Thus there exists \( t \geq 2 \) such that \( r(w^t) = r(w^{t+1}) \). Therefore, we have \( r(r(w^t)w) = r(w^{t+1}) = r(w^t) \). Note that, since \( \nabla_{tw}(t+1) = k + 1 \), we have \( \iota(r(w^t)w) = k + 1 \). Since, moreover, removing the remainder does not change the universality of a word, \( \iota(r(w^t) \cdot w \cdot r(w^{t+1})) = k + 1 \) follows. Because the word \( r(w^t) \cdot w \cdot r(w^{t+1}) \) is a conjugate of \( w \), we have \( \zeta(w) = k + 1 \). This proves the second implication. Finally, let \( \zeta(w) = k + 1 \). Then Theorem 9 implies immediately that \( \nabla_{tw}(s) = k + 1 \) for all \( s \in [2, \sigma] \).

Theorem 44 provides an algorithm to compute the circular universality of a word \( w \) in \( O(|\sigma||w|) \), which is, if \( \sigma < |w| \) holds, better than the naïve approach by computing \( \iota(v) \) for every conjugate \( v \) of \( w \).

**Proposition 45.** Given a word \( w \in \Sigma^* \), we can compute \( \zeta(w) \) in time \( O(|\sigma||w|) \).

**Proof.** By [2, Proposition 10] we can compute \( \iota(w^\sigma) \) in \( O(|\sigma||w|) \). Let \( k = \iota(w) \). If \( \iota(w^\sigma) = \sigma k + \sigma - 1 \), then \( \zeta(w) = k + 1 \), else \( \zeta(w) = k \).

## 5 Conclusion

The main goal of this work was to improve certain results from [2] on the connection between the universality of repetitions and the circular universality, namely Theorem 9 and Theorem 10.

At first we focused our investigation on repetitions. In Section 3 we showed that the growth of the universality of repetitions can be characterised by their remainders and that the growth is eventually periodic beginning its periodicity latest after \( \sigma \) repetitions. Thus, the universality of all other repetitions can be computed in constant time. In Section 4 we found that one can translate questions about the universality of repetitions into questions about ascending or descending chains of the remainders of those repetitions. The investigation of strictly ascending chains led to a tight bound on the length of the longest possible strictly ascending chain and the connection of such chains with the circular universality, gives the extension of Theorem 10 to alphabets of arbitrary size. On the other hand, on investigating strictly descending chains, we found a tight bound on the length of such chains, which is surprisingly one step shorter.
than the ascending pendant. This lead to a modification of Theorem 9 such that its converse holds, too, and also to an efficient algorithm to compute the circular universality of a word.

It remains an interesting open problem to characterise the class of words, for which the remainder of some proper repetition is the empty word. We propose to call such words perfect $k$-universal. Furthermore, one could extend the study of $k$-universality from finite words to infinite words, e.g. one could study the universality of the sequence of finite prefixes of aperiodic infinite words.

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