Multiplicity functions for tensor powers. $A_n$-case

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Abstract. For classical Lie algebras $\mathfrak{g} = A_r$, the tensor power $(L^{\omega_1})^\otimes p \mid p \in \mathbb{Z}_+$ of the first fundamental module $L^{\omega_1}$ is considered. We study the behavior of fusion coefficients in the decomposition $(L^{\omega_1})^\otimes p = \sum m(p, \nu) L^{\nu}$. Using symmetry properties of singular elements corresponding to tensor powers we find the coefficients $m(p, \nu)$. The multiplicity function $M(p; \nu)$ is constructed that shows explicitly how multiplicities of irreducible submodules $L^{\nu}$ depend on power $p$ and coordinates $\nu_i$ of their highest weight. This function provides an effective tool to study different properties of fusion.

1. Introduction

In physics tensors describe states of quantum mechanical systems. If a system has $n$ particles, its state is an element of $H_1 \otimes \cdots \otimes H_p$ with $H_j$ being Hilbert spaces. In many-body physics, solid state physics and in quantum field theory, one wants to deal with quantum states of thousands of particles, often arranged on a regular lattice (e.g., atoms in a crystal or spin chain). Due to the exponential growth of the dimension of tensor product (or tensor power) of spaces, direct methods of representing these tensors are often intractable even on a computer [1]. This awakes fundamental investigations of tensor product representation properties (see [2] and list of references therein).

There are numerous combinatorial studies of the problem [3, 4, 5] and also series of works dealing with fermionic formulas, some of them based on crystal basis approach [6],[7] and [8]. On this way important general results were obtained. Practical computations with the corresponding formulas are scarcely possible for all but the simplest examples [5]. As a rule the multiplicity formulas are connected with complicated path countings. They mostly describe combinatorial procedures necessary to evaluate multiplicity coefficients rather than multiplicities of irreducible modules in a product $H_1 \otimes \cdots \otimes H_p$.

Our approach is based on a full scale application of singular element properties (defined in a formal algebra $\mathcal{E}_\mathfrak{g}$). The latter describe economically highest weight modules of a Lie algebra $\mathfrak{g}$. Singular element $\Psi(L^{\mu})$ forms a bridge between Verma modules $V^{\mu}$ and finite-dimensional highest weight modules $L^{\mu}$. Usually this bridge is used to formulate the problem in terms of recurrent relations. In this paper we show how to avoid this complicated construction and to find an answer directly. We obtain a solution $M(p; \nu)$ that describes explicitly (in terms of algebraic functions) how multiplicities of irreducible submodules $L^{\nu}$ depend on $p$ and on coordinates $\nu_i$ of...
a highest weight. We call this solution the \textbf{multiplicity function} \( M (p; \nu) \). It is defined on the direct product \( \mathbb{Z}_{n \geq 0} \times P_{\mathfrak{g}} \) where \( P_{\mathfrak{g}} \) is the weight lattice. Our technique is closely related to an algorithm proposed in 1968 by A.U. Klimyk for tensor product decompositions. The difference is that transporting our problem from the dominant weight space \( P^{++}_{\mathfrak{g}} \) to the full lattice \( P_{\mathfrak{g}} \) we can use powerful tools of Weyl skew symmetry.

2. Basic instruments and notation

We consider simple Lie algebras \( \mathfrak{g} = A_r = \mathfrak{sl}(n) \).

The basic roots in \( e \)-basis are \( \alpha_1 = e_1 - e_2, \ldots, \alpha_r = e_r - e_{r+1} \);
the first fundamental weight: \( \omega_1 = e_1 \);
the fundamental module is \( L^{\omega_1} \);
\( \mathfrak{h}^* \) — the root space of \( \mathfrak{g} \);
\( r \) — the rank of \( \mathfrak{g} \);
\( \Delta \) — the root system; \( \Delta^+ \) — the positive root system (of \( \mathfrak{g} \));
\( S \) — the system of simple roots;
\( \alpha_i \) — the \( i \)-th simple root ; \( i = 0, \ldots, r \);
\( \rho \) — the Weyl vector;
\( L^\mu, L^\nu \) — the integrable \( \mathfrak{g} \)-modules with the highest weights \( \mu, \nu \);
\( N^\mu \) — the weight diagram of \( L^\mu \);
\( P \) — the weight lattice;
\( P^{++} \) — the dominant weight lattice;
\( C (\xi) \) — the congruence class of weights in \( P \) containing \( \xi \);
\( E \) — the formal algebra of weights;
\( m^\xi_\mu \) — the multiplicity of the weight \( \xi \in P \) in \( N^\mu \);
\( \text{ch} (L^\mu) \) — the formal character of \( L^\mu \);
\( R := \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha}) \) (resp. \( R_a := \prod_{\alpha \in \Delta^+_a} (1 - e^{-\alpha}) \)) — the Weyl denominator.

The Weyl formula [9]

\[
\text{ch} (L^\mu) = \frac{\sum_{w \in W} \epsilon(w) e^{w(\mu + \rho) - \rho}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})}
\]

connects the formal character with singular elements:

\[
\Psi^{(\mu)} := \sum_{w \in W} \epsilon(w) e^{w(\mu + \rho) - \rho}, \quad (1)
\]

\[
\text{ch} (L^\mu) = \frac{\Psi^{(\mu)}}{\Psi^{(0)}} = \frac{\Psi^{(\mu)}}{R}.
\]

We shall also use the unshifted singular elements:

\[
\Phi^{(\mu)} := \sum_{w \in W} \epsilon(w) e^{w(\mu + \rho)} \quad (2)
\]

Consider the tensor power \( (L^{\omega_1})^\otimes p \mid_{p \in \mathbb{Z}_+} \). Our aim is to find explicit dependence of multiplicities \( m (p, \nu) \) in the decomposition

\[
(L^{\omega_1})^\otimes p = \sum_{\nu} m (p, \nu) L^\nu \quad (3)
\]
on coordinates of \( \nu \) and the power \( p \).

One of the standard methods to solve such problems is to treat the tensor power \((L^\omega)_p\) as an irreducible module for a direct sum \( \oplus^p g \). In this approach one is to find a decomposition of the singular element

\[
\Psi((L^\omega)_p) = \sum_{\nu} m(p, \nu) \Psi^{(\nu)} = \sum_{\nu} m(p, \nu) \sum_{w \in W} \epsilon(w) e^{w(\lambda + p) - p},
\]

into a combination of corresponding elements \( \Psi^{(\nu)} \) for a diagonal subalgebra \( g^{\text{diag}} \rightarrow \oplus^p g \). Here the tensor power decomposition is treated as a reduction problem.

According to the recursive method [10] to find multiplicities \( m(\nu, p) \) in (4) we need both the singular element \( \Psi(L^\omega)_p \) and the so-called injection fan, in our case this is a fan \( \Gamma_{\text{diag}} \rightarrow \oplus^p A_n \).

2.1. Example

We shall illustrate the recursive method by a simple example: \( g = A_1 \) and \( p = 2 \). The tensor square module \((L^\omega)_2\) whose diagram is presented in Fig. 1

\[ \text{Figure 1. The weight diagram } \mathcal{N}(L^\omega \otimes L^\omega) \text{ in the weight lattice } P \text{ of the direct sum } A_1 \oplus A_1. \text{ The roots of the diagonal subalgebra are indicated by short arrows.} \]

is an irreducible \( A_1 \oplus A_1 \)-module. To use the injection fan technique we compose the singular element \( \Psi(L^\omega \otimes L^\omega) \), project it to the space \( P_\nu \) and consider the result as a formal element:

\[
\Psi(L^\omega \otimes L^\omega) \downarrow A_1^{\text{diag}} = \sum_{\xi \in P_\nu} \psi(\xi) e^\xi,
\]

\[ \text{Figure 2. Black points indicate the weights of} \] the singular element \( \Psi(L^\omega \otimes L^\omega) \) (the left part) and of its projection to the space of the diagonal subalgebra \( A_1^{\text{diag}} \) (the right part).

The injection fan here contains only one weight \( \gamma \) (the sum of \( A_1 \)-fundamental weights, see Fig. 3).

\[ \text{Figure 3. In the left part the black point indicates} \] the injection fan vector \( \gamma \in \Gamma(A_1 \rightarrow A_1 \oplus A_1) \). The sign function is \( s(\gamma) = + \). In the right part in the main Weyl chamber \( \mathcal{C}_{A_1^{\text{diag}}} \) we see two highest weights with multiplicities +1.

The recursion determined by \( \gamma \in \Gamma(A_1^{\text{diag}} \rightarrow A_1 \oplus A_1) \) [10] reduces to simple relations

\[ h^{(\mu)}(\xi) = \psi(\xi) + h^{(\mu)}(\xi + \gamma). \]

Their solution for the module \((L^\omega)_2 \downarrow A_1^{\text{diag}}\) is presented in the r.h.s. of Fig. 3. Two highest weights (in \( \mathcal{C}_{A_1^{\text{diag}}} \)) fix the desired decomposition \( \sum_{\nu} m(\nu, p) \Psi^{(\nu)} \).

For simple algebras with rank \( r > 1 \) the corresponding singular elements and injection fans can be constructed and solutions to the recurrent relations can be successively found. When the tensor power \( p \) is treated as an arbitrary parameter the calculations become cumbersome (see e.g. [11]).
3. Other approach to fusion.

In this section we shall show that for classical algebras of series $A_n$ the singular element $\Psi((L^{\omega_1})^{\otimes p})$ can be explicitly constructed as an element of formal algebra and its coordinates (in $E$) can be obtained as functions of weight coordinates and $p$.

Let $g = A_n$ and $L^{\omega_1}$ be its first fundamental module.

$$\text{ch} \left( L^{\omega_1} \right) = e^{\omega_1} + e^{\omega_1-\alpha_1} + \ldots + e^{\omega_1-\alpha_1-\ldots-\alpha_r}$$  \hspace{1cm} (5)

The weights $\eta_i \in N^{\omega_1}$ are subject to a condition $\sum_i^n \eta_i = 0$. The first $n - 1$ weights in $N^{\omega_1}$ can be used as a basis in the weight space. The coordinates of a weight $\lambda \in P$ in an ordinary $\omega_j$-basis will be denoted by a Latin index $j$: $\lambda = \sum_j^{n-1} \lambda_j \omega_j$. The coordinates in $\{\eta\}$-basis will be denoted by indices in brackets: $\{\lambda(j)\} \; j = 1, \ldots , n - 1$. In the latter case $\lambda \in P$ is often supplied by an additional coordinate $\lambda(n) = -\sum_i^{n-1} \lambda(j)$ so that $\sum_i^n \lambda(i) = 0, \; i = 1, \ldots , n$.

The minimal value of $\eta$-component of the Weyl vector is denoted by $m_\rho := \min(\rho(m))$. For a dominant weight $\nu \in P^{++}$ consider a shifted weight $l := \nu + \rho$, its $\omega$-coordinates will be denoted by $\{l_j\}$ and its $\eta$-coordinates by $\{l(i)\}$.

**Theorem 1.** Let $l = (\nu + \rho) \in C(\nu \omega_1 + \rho)$ and $\{l(i)\} = \{l(i)| i = 1, \ldots , n\}$ be its $\eta$-coordinates. Then the multiplicities $m(p; \nu)$ in the decomposition of a product $\Phi^{(0)}(\text{ch}(L^{\omega_1}))^{\otimes p} = \sum_{\nu \in P^{++}} m(p; \nu) \Phi^{(\nu)}$ are

$$m(p; \nu) = p! \prod_{i=1}^{n} \frac{(l(i) - l(\omega))}{(l(i) - m_\rho)!}.$$  \hspace{1cm} (6)

**Proof.** The singular element that we need (considered as an element of $E$) can be written as a product

$$\Phi^{(0)}(\text{ch}(L^{\omega}))^p = \sum_{w \in W} \epsilon(w) e^{w \rho p} \left( \text{ch}(L^{\omega}) \right)^p.$$  \hspace{1cm} (7)

According to (5) its $\omega$-basis decomposition can be expressed in terms of multinomial coefficients $C_p^{\nu} = \frac{p!}{k(1)k(2)\ldots k(n)}$:

$$\sum_{w \in W} \epsilon(w) e^{w \rho p} \sum_{k(1) + k(2) + \ldots + k(n) = p} C_p^{\nu} e^{(k(1) - k(2))\omega_1 + \ldots + (k(n-1) - k(n))\omega_{n-1}} = \sum_{\nu \in P} M(p; \nu + \rho) e^{\sum_j l_j \omega_j}.$$  \hspace{1cm} (8)

By applying the $\alpha_j$-specialization we get the set of relations

$$(w \circ \rho, \alpha_j) + k(j) - k(j+1) = l_j; \; j = 1, \ldots , r = n - 1.$$

$$k(1) + k(2) + \ldots + k(n) = p.$$

Notice that the integers $\{k(i)\}$ are the coordinates of $\nu$ in $\eta$-basis. Let us rewrite the obtained relations in a matrix form:

$$\begin{pmatrix} 1 & -1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & -1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} k(1) \\ k(2) \\ k(3) \\ \vdots \\ k(n-1) \\ k(n) \end{pmatrix} = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ \vdots \\ l_{n-1} \\ p \end{pmatrix} = \begin{pmatrix} (w \circ \rho, \alpha_1) \\ (w \circ \rho, \alpha_2) \\ (w \circ \rho, \alpha_3) \\ \vdots \\ (w \circ \rho, \alpha_r) \end{pmatrix}.$$
This equation can be solved and coordinates \( \{ k(i) \} \) can be found for any weight \( \nu \in P^{++} \) (with \( \nu + \rho = \sum_{j=1}^{n-1} l_j \omega_j \)):

\[
\frac{1}{n} \begin{pmatrix}
  n-1 & n-2 & \ldots & 1 & 1 \\
  -1 & n-2 & \ldots & 1 & 1 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  -1 & -2 & \ldots & 1 & 1 \\
  -1 & -2 & \ldots & -r & 1 \\
\end{pmatrix}
\begin{pmatrix}
  l_1 \\
  l_2 \\
  \vdots \\
  l_{n-1} \\
  p \\
\end{pmatrix}
- \begin{pmatrix}
  (w \circ \rho, \alpha_1) \\
  (w \circ \rho, \alpha_2) \\
  \vdots \\
  (w \circ \rho, \alpha_r) \\
  0 \\
\end{pmatrix} = \begin{pmatrix}
  k(1) \\
  k(2) \\
  \vdots \\
  k(n-1) \\
  k(n) \\
\end{pmatrix}.
\]

The transformation performed above is a transition from \( \omega \)-basis to \( \eta \)-basis. It can be written as

\[
k(i) = l(i) - (w \circ \rho)(i), \quad i = 1, \ldots, n.
\]

The coordinates in the r.h.s. have the properties: \( \sum l(i) = p \) and \( \sum (w \circ \rho)(i) = 0 \). Substituting the obtained solution in (8) we get

\[
M \left( p; \sum_j (l_j + 1) \omega_j \right) = \sum_{w \in W} \epsilon(w) \frac{p!}{\prod_i (l(i) - (w \circ \rho)(i))!}.
\]

The Weyl group acts on the \( \eta \)-coordinates of weights as a permutation group \( S_n \):

\[
M(p; \nu + \rho) = \sum_{s \in S_n} \epsilon(s) \frac{p!}{\prod_i (l(i) - (s \circ \rho)(i))!}.
\]

Let us rearrange the r.h.s. of (9)

\[
\frac{p!}{\prod_i (l(i) - m(\rho))!} \sum_{s \in S} \epsilon(s) \prod_{i=m(\rho)}^{n} (l(i) - t) \quad t = m(\rho), \ldots, (s \circ \rho)(i) - 1.
\]

The sum

\[
D := \sum_{s \in S} \epsilon(s) \prod_{j=1}^{n} \prod_{t=m(\rho)}^{l(j) - t} (l(j) - t)
\]

can be written as a determinant

\[
D = \det \begin{pmatrix}
  1 & \ldots & 1 \\
  (l(1) - m(\rho)) & \ldots & (l(n) - m(\rho)) \\
  \vdots & \ddots & \vdots \\
  \Pi_{t=m(\rho)}^{m(\rho)+n-2} (l(1) - t) & \ldots & \Pi_{t=m(\rho)}^{m(\rho)+n-2} (l(n) - t) \\
\end{pmatrix}.
\]

A successive transformation of rows in \( D \) performs it into the Vandermonde determinant. It becomes clear that \( D \) does not depend on values of \( t \):

\[
D = \det \begin{pmatrix}
  1 & 1 & 1 & 1 \\
  l(1) & l(2) & \ldots & l(n) \\
  (l(1))^2 & (l(2))^2 & \ldots & (l(n))^2 \\
  \vdots & \vdots & \ddots & \vdots \\
  (l(1))^{n-1} & (l(2))^{n-1} & \ldots & (l(n))^{n-1} \\
\end{pmatrix} = \prod_{1 \leq i < u \leq n} (l(i) - l(u)).
\]
This gives the final result:

\[ M(p; \nu + \rho) = m(p; \nu) = p! \prod_{i < u} (l(i) - l(u)) \prod_{j=1}^n (l(j) - m_p)! \]  \hspace{1cm} (10)

4. Properties

Here we want to indicate some important properties of multiplicities that can be obtained using the multiplicity function (10).

(i) **Connection with Weyl invariants.** The function \( D \) is a fundamental skew-invariant of the Weyl group in the algebra \( \text{Fun}(P, \mathbb{R}^n) \) (generated by the coordinate functions \( v(i) \)). It was demonstrated above that the multiplicity function \( M(p; \nu + \rho) \) is proportional to this skew-invariant.

(ii) **Multiplicities in \( \omega \)-coordinates.** Returning to \( \omega \)-coordinates (of the weight \( \nu + \rho \)) we get:

\[ M(p; \{l_i\}) = p! \frac{l_1 l_2 \ldots l_r \,(l_1 + l_2) \,(l_1 + l_2 + l_3) \ldots \,(l_1 + \ldots + l_r)}{\prod_{u=0}^{l_r} \left( p + \sum_{v=0}^{l_r-1} (r - v) l_{v+1} + \sum_{s=0}^u (-s) l_s + \frac{r(r+1)}{2} \right)!} \]  \hspace{1cm} (11)

In particular for algebra \( A_2 \) and the highest weight \( \nu + \rho = l_1 \omega_1 + l_2 \omega_2 \) we have the following compact expression:

\[ M_{A_2}(p; \{l_1, l_2\}) = \frac{p! l_1 l_2 \,(l_1 + l_2)}{\left( \frac{p - (l_1 + 2l_2 + 3)}{3} \right)! \left( \frac{p + (2l_1 + l_2 + 3)}{3} \right)! \left( \frac{p + (-l_1 + l_2 + 3)}{3} \right)!} \]  \hspace{1cm} (12)

(iii) **All the singular weights.** As it follows from relation (8) the multiplicity function describes the full set of singular weights in the decomposition of \( \Phi^p \) into a sum of singular elements \( \sum_{\nu \in P^+} \Phi^{(\nu)} \). One can consider the function \( M(p; \nu + \rho) \) not only for \( \nu \in C^{(0)} \) where its values are the multiplicity coefficients \( m(p; \nu + \rho) \) but for any weight \( \nu \in P \) (for a particular power \( p \) this means that \( \nu \in C(p \omega_1 + \rho) \)). To make this property more transparent notice that we can substitute the factorials in (10) by \( \Gamma \)-functions,

\[ M(p; \nu + \rho) = \Gamma(p + 1) \frac{\prod_{i < u} (l(i) - l(u))}{\prod_{j=1}^n \Gamma(l(j) - m_p + 1)} \]

(iv) **Piercing polynomials.** In (10)-(12) the multiplicity \( M(p; \nu + \rho) \) is a rational function of \( p \) and \( \{l_i\} \). In a special coordinate system the \( p \)-dependence of the multiplicity function becomes polynomial. We shall demonstrate this property for \( \mathfrak{g} = A_2 \). Let us place the center of the new system at the end of the highest weight \( (p + 1) \omega_1 + \omega_2 \). Put

\[ l_1 = \frac{1}{2} \,(2p - 3b_1 - b_2 + 2), \]

\[ l_2 = b_2 + 1. \]

In these new coordinates \( \{b_j\} \) the multiplicity function looks as follows,

\[ M_{A_2}(p; b_1, b_2) = \frac{p!}{(p - b_1 + 2)!} \left( \frac{2p - 3b_1 - b_2 + 2}{2p - 3b_1 + b_2 + 4} \right) \times \frac{(b_2 + 1)}{4 \left( \frac{1}{2} (b_1 - b_2) \right)! \left( \frac{1}{2} (b_1 + b_2) + 1 \right)!} \]

Inside the \( \Phi^{(p \omega_1)} \)-contour the coordinate \( b_1 \) is nonnegative. As a result the first three factors give a polynomial in \( p \).
(v) **Universality of piercing polynomials.** When \( \{b_j\} \) coordinates of the highest weight are fixed we obtain a unique \( p \)-polynomial that describe multiplicities for this type of modules. We call this the **piercing polynomial property.** For example, the multiplicity of the weight \( \lambda = (p + 1) \omega_1 - \alpha_1 \) is \( p - 1 \). Several piercing polynomials are explicitly exposed in Fig. 4. Notice that there the distance between the Weyl symmetry center and the highest weight of the initial module is not fixed (it depends on \( p \)). The congruence class of the weight \( \nu + \rho \) also depends on \( p \).

![Figure 4.](image)

**Figure 4.** The \( P_{A_2} \) root lattice with \( \Delta_{A_2} \) originating in the Weyl symmetry center, the corresponding axes go along the fundamental weights. With an arbitrary power \( p \) the size of \( \Phi^{(p \omega_1)} \)-contour is not fixed. Black lines encounter the chamber \( C^{(0)} \). Black dots inside it indicate the weights with multiplicities described by non-zero piercing polynomials. The polynomials indicated explicitly correspond to the highest weights \( \nu + \rho \) with coordinates \( (b_1, b_2) = \{(0, 0), (1, 1), (2, 2)\} \) in the first line and with \( (b_1, b_2) = (7, 1) \) in the tenth line.

(vi) **Asymptotic behavior.** With formula (11) it is possible to study asymptotic properties of multiplicities for \( p \to \infty \). For \( g = A_2 \) the multiplicities grow as \( p^{b_1} \). The multiplicity function in \( \{b_j\} \) coordinates has the following leading terms

\[
M_{A_2}(p; b_1, b_2)_{|p \to \infty} \sim \frac{b_2 + 1}{\left(\frac{1}{2} (b_1 + b_2) + 1\right)! \left(\frac{1}{2} (b_1 - b_2)\right)!} p^{b_1}.
\]

Notice that this property is valid only inside the \( \Phi^{(p \omega_1)} \)-contour.

5. **Conclusions**

Properties of singular elements were used to construct fusion rules for simple Lie algebras \( A_r \). It was proven that for the fundamental module \( L^{\omega_1} \) an arbitrary tensor power \( (L^{\omega_1})^{\otimes p} \) can be decomposed directly without implementing recurrent relations. An explicit expression for fusion coefficients were obtained. The multiplicity function \( M_{A_r}(p; \nu + \rho) \) provides a possibility to study different properties of fusion including asymptotic behavior of multiplicities. The latter offers numerous possibilities for applications in quantum physics.

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