Positive Solutions for Slightly Subcritical Elliptic Problems Via Orlicz Spaces

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Abstract. This paper concerns semilinear elliptic equations involving sign-changing weight function and a nonlinearity of subcritical nature understood in a generalized sense. Using an Orlicz–Sobolev space setting, we consider superlinear nonlinearities which do not have a polynomial growth, and state sufficient conditions guaranteeing the Palais–Smale condition. We study the existence of a bifurcated branch of classical positive solutions, containing a turning point, and providing multiplicity of solutions.

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1. Introduction

In this paper we study the classical positive solutions to the Dirichlet problem for a class of semilinear elliptic equations whose nonlinear term is of subcritical nature in a generalized sense and involves indefinite nonlinearities. More precisely, given \( \Omega \subset \mathbb{R}^N, N > 2 \), a bounded, connected open subset, with \( C^2 \) boundary \( \partial \Omega \), we look for positive solutions to:

\[
- \Delta u = \lambda u + a(x)f(u), \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial \Omega, \quad (1.1)
\]

where \( \lambda \in \mathbb{R} \) is a real parameter, \( a \in C^1(\bar{\Omega}) \) changes sign in \( \Omega \),

\[
f(s) := g(s) + h(s), \quad \text{with } h(s) := \frac{|s|^{2^*-2}s}{[\ln(e + |s|)]^\alpha}, \quad (1.2)
\]

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\( 2^* = \frac{2N}{N-2} \) is the critical Sobolev exponent, \( \alpha > 0 \) is a fixed exponent, and \( f, g \in C^1(\mathbb{R}) \) satisfy

\[
\begin{align*}
(H)_0 & \quad \lim_{s \to 0} \frac{f(s)}{|s|^{p-2}s} = L_1, \quad \text{for some } L_1 > 0, \text{ and } p \in \left( 2, \frac{2N}{N-2} \right] \\
(H)_1 & \quad \lim_{s \to \infty} \frac{f(s)}{|s|^{p-2}s} = L_2, \quad \text{for some } L_2 \geq 0, \text{ and } q \in \left( 2, \frac{2N}{N-2} \right] \\
(H)_{g'} & \quad |g'(s)| \leq C (1 + |s|^{q-2}), \quad \text{for } s \in \mathbb{R}.
\end{align*}
\]

We will say that \( f \) satisfies hypothesis (H) whenever \((H)_0, (H)_1, \) and \((H)_g'\) are satisfied. Since we are interested in positive solutions, we redefine \( f \) to be zero on \((-\infty, 0]\),

\[
\text{(1.3)}
\]

note that, since \((H)_0\), \( f(0) = 0 \) and that

\[
\lim_{s \to 0^+} \left( \frac{f(s)}{s} - L_1|s|^{p-2} \right) = 0.
\]

(1.4)

When \( \lambda = 0 \), \( a(x) \equiv 1 \) and \( g(s) \equiv 0 \), this kind of nonlinearity has been studied in \([5–7,16]\), and in \([11]\) for the case of the \( p \)-laplacian operator, with \( \alpha > \frac{p}{N-p} \). It is known the existence of uniform \( L^\infty \) a-priori bounds for any positive classical solution, and as a consequence, the existence of positive solutions. When \( \alpha \to 0 \), there is a positive solution blowing up at a non-degenerate point of the Robin function as \( \alpha \to 0 \), see \([9]\) for details.

Let \((\lambda_1, \varphi_1)\) stands for the first eigen-pair of the Dirichlet eigenvalue problem

\[-\Delta \varphi = \lambda \varphi \] in \( \Omega \), \( \varphi = 0 \) on \( \partial \Omega \). From \([10]\) it is known that \((\lambda_1, 0)\) is a bifurcation point of positive solutions \((\lambda, u_\lambda)\) to the equation \((1.1)\). If \( f \) behaves like \( |u|^{p-2}u \) at zero with \( 2 \leq p \leq 2^* \), the influence of the negative part of the weight \( a \) is displayed under the sign of \( \int_\Omega a(x) \varphi_1(x)^p \, dx \), where \( \varphi_1 \) is the first positive eigenfunction for \(-\Delta \) in \( H^1_0(\Omega) \). Specifically, whenever

\[
\text{(1.5)}
\]

the bifurcation of positive solutions from the trivial solution set is 'on the right' of the first eigenvalue, in other words, for values of \( \lambda > \lambda_1 \). When

\[
\int_\Omega a(x) \varphi_1(x)^p \, dx > 0
\]

the bifurcation from the trivial solution set is 'on the left' of the first eigenvalue, in other words, for values of \( \lambda < \lambda_1 \).

Inspired by the work of Alama and Tarantello in \([1]\), we will focus our attention to the case of \( a(x) \) changing sign and \((1.5)\) is being satisfied, and, among other things, we will prove the existence of a turning point for a value of the parameter \( \Lambda > \lambda_1 \), and in particular the existence of solutions when \( \lambda = \lambda_1 \). We will use local bifurcation and variational techniques.

All throughout the paper, for \( v: \Omega \to \mathbb{R}, v = v^+ - v^- \) where

\[
v^+(x) := \max\{v(x), 0\} \quad \text{and} \quad v^-(x) := \max\{-v(x), 0\}.
\]
Let us also define
\[ \Omega^\pm := \{ x \in \Omega : \pm a(x) > 0 \}, \quad \Omega^0 := \{ x \in \Omega : a(x) = 0 \}, \]
and assume that both \( \Omega^+ \), \( \Omega^- \) are non empty sets.

For this nonlinearity the Palais–Smale condition of the energy functional becomes a delicate issue, needing Orlicz spaces and a Orlicz–Sobolev embedding theorem.

In order to prove (PS) condition, Alama and Tarantello ([1]) assume that the zero set \( \Omega^0 \) has a non empty interior. This is also a common hypothesis for other authors when dealing with changing sign superlinear nonlinearities [8,20,23]. But this is a technical hypothesis. (PS)-condition will be proved in Proposition 3.1 without assuming that hypothesis. We neither use Ambrosetti-Rabinowitz condition.

Let us now denote
\[ C_0 = \inf\{ C \geq 0 : f'(s) + C \geq 0 \text{ for all } s \geq 0 \}, \quad (1.6) \]
and remark that hypothesis (H) implies that \( C_0 < +\infty \). Observe also that
\[ f(s) + C_0 s \geq 0, \text{ for all } s \geq 0; \quad f(s)s + C_0 s^2 \geq 0, \text{ for all } s \in \mathbb{R}. \quad (1.7) \]
Let \( u \) be a weak solution to (1.1). By a regularity result, see Lemma 2.1, \( u \in C^2(\Omega) \cap C^1,\mu(\Omega) \). So by a solution, we mean a classical solution.

Assume that \( u \) is a non-negative nontrivial solution. It is easy to see that the solution is strictly positive. Indeed, adding \( \pm C_0 a(x) u \) to the r.h.s. of the equation, splitting \( a = a^+ - a^- \), taking into account (1.4) and (1.7), and letting in each side the nonnegative terms, we can write
\[
\left(-\Delta + a^-(x) \left[ \frac{f(u)}{u} + C_0 \right] + C_0 a(x)^+ \right) u
= \lambda u + a(x)^+ \left[ f(u) + C_0 u \right] + C_0 a(x)^- u, \quad \text{in } \Omega. \quad (1.8)
\]
Now, the strong Maximum Principle implies that \( u > 0 \) in \( \Omega \), and \( \frac{\partial u}{\partial \nu} < 0 \) on \( \partial \Omega \).

Our main result is the following theorem.

**Theorem 1.1.** Assume that \( g \in C^1(\mathbb{R}) \) satisfies hypothesis (H). Let \( C_0 > 0 \) be defined by (1.6). If a changes sign in \( \Omega \), and (1.5) holds, then there exists a \( \Lambda \in \mathbb{R} \),
\[
\lambda_1 < \Lambda < \min \left\{ \lambda_1(\text{int } (\Omega^0)), \quad \lambda_1(\text{int } (\Omega^+ \cup \Omega^-)) + C_0 \sup a^+ \right\}
\]
and such that (1.1) has a classical positive solution if and only if \( \lambda \leq \Lambda \).

Moreover, there exists a continuum (a closed and connected set) \( \mathcal{C} \) of classical positive solutions to (1.1) emanating from the trivial solution set at the bifurcation point \( (\lambda, u) = (\lambda_1, 0) \) which is unbounded. Furthermore,

(a) For every, \( \lambda \in (\lambda_1, \Lambda) \), (1.1) admits at least two classical ordered positive solutions.

(b) For \( \lambda = \Lambda \), problem (1.1) admits at least one classical positive solution.

(c) For every \( \lambda \leq \lambda_1 \), problem (1.1) admits at least one classical positive solution.
The paper is organized in the following way. Section 2 contains a regularity result and a non existence result. (PS)-condition and an existence of solutions result for \( \lambda < \lambda_1 \) based in the Mountain Pass Theorem will be proved in Sect. 3. A bifurcation result for \( \lambda > \lambda_1 \) is developed in Sect. 4. The main result is proved in Sect. 5. Appendix A contains some useful estimates. Orlicz spaces, and a Orlicz–Sobolev embeddings theorems, will be treated in Appendix B.

2. A Regularity Result and a Non Existence Result

Next, we recall a regularity Lemma stating that any weak solution is in fact a classical solution.

**Lemma 2.1.** If \( u \in H^1_0(\Omega) \) weakly solves (1.1) with a continuous function \( f \) with polynomial critical growth

\[
|f(x, s)| \leq C(1 + |s|^{2^* - 1}),
\]

then, \( u \in C^2(\Omega) \cap C^{1,\mu}(\overline{\Omega}) \) and

\[
\|u\|_{C^{1,\mu}(\overline{\Omega})} \leq C \left(1 + \|u\|_{L^{2^* - 1,r}(\Omega)}^{2^* - 1}ight),
\]

for any \( r > N \) and \( \mu = 1 - N/r \). Moreover, if \( \partial \Omega \in C^{2,\mu} \), then \( u \in C^{2,\mu}(\Omega) \).

**Proof.** Due to an estimate of Brézis-Kato [3], based on Moser’s iteration technique [17], \( u \in L^r(\Omega) \) for any \( r > 1 \); and by elliptic regularity \( u \in W^{2,r}(\Omega) \), for any \( r > 1 \) (see [22, Lemma B.3] and comments below).

Moreover, by Sobolev embeddings for \( r > N \) and interior elliptic regularity \( u \in C^{1,\alpha}(\overline{\Omega}) \cap C^2(\Omega) \). Furthermore, if \( \partial \Omega \in C^{2,\alpha} \), then \( u \in C^{2,\alpha}(\overline{\Omega}) \). \( \square \)

**Proposition 2.2.** Let \( f \) satisfy hypothesis \( (H) \) and let \( C_0 \) be defined in (1.6). Assume that \( a \) changes sign in \( \Omega \).

1. Problem (1.1) does not admit a positive solution \( u \in H^1_0(\Omega) \) for any

\[
\lambda \geq \lambda_1(\text{int} (\Omega^+ \cup \Omega^0)) + C_0 \sup a^+.
\]

2. If \( \text{int} (\Omega^0) \neq \emptyset \), then \( \lambda_1(\text{int} (\Omega^0)) < +\infty \) and (1.1) does not admit a positive solution for any

\[
\lambda \geq \lambda_1(\text{int} (\Omega^0)).
\]

**Proof.** 1. Let \( \lambda \geq \lambda_1(\text{int} (\Omega^+ \cup \Omega^0)) + C_0 \sup a^+ \), and assume by contradiction that there exists a non-negative non-trivial solution \( u \in H^1_0(\Omega) \) to (1.1) for the parameter \( \lambda \). Since the Maximum Principle \( u > 0 \) in \( \Omega \), see (1.8).

Let \( \hat{\varphi} \) be the positive eigenfunction of \( (-\Delta, H^1_0(\text{int} (\Omega^+ \cup \Omega^0))) \) of \( L^2 \)-norm equal to 1. For simplicity, we will also denote by \( \hat{\varphi} \) the extension by 0 of \( \hat{\varphi} \) in all \( \Omega \). By Hopf’s maximum principle, we have \( \frac{\partial \hat{\varphi}}{\partial \nu} < 0 \) on \( \partial(\text{int} (\Omega^+ \cup \Omega^0)) \), where \( \nu \) is the outward normal.

Again, if we multiply the equation (1.1) by \( \hat{\varphi} \) and integrate along \( \text{int} (\Omega^+ \cup \Omega^0) \) we find, after integrating by parts,

\[
0 > \int_{\partial(\text{int} (\Omega^+ \cup \Omega^0))} \frac{\partial \hat{\varphi}}{\partial \nu} d\sigma
\]
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\[ + \int_{\Omega^+} \left[ \lambda \left( \int (\Omega^+ \cup \Omega^0) \right) - \lambda + C_0 a^+(x) \right] u \tilde{\varphi} \, dx \]

\[ = \int_{\Omega^+} a^+(x) \left[ f(u) + C_0 u \right] \tilde{\varphi} \, dx > 0, \]

a contradiction.

2. Let \( \lambda \geq \lambda_1 (\int (\Omega^0)) \) and, by contradiction, assume the existence of a positive solution \( u \in H^1_0(\Omega) \) of problem (1.1) for the parameter \( \lambda \). Let \( \tilde{\varphi} \) be a positive eigenfunction associated to \( \lambda_1 (\int (\Omega^0)) < +\infty \). For simplicity, we will also denote by \( \tilde{\varphi} \) the extension by 0 in all \( \Omega \). If we multiply equation (1.1) by \( \tilde{\varphi} \) and integrate along \( \Omega^0 \) we find, after integrating by parts,

\[ \int_{\Omega^0} \nabla u \cdot \nabla \tilde{\varphi} \, dx = \lambda \int_{\Omega^0} u \tilde{\varphi} \, dx. \]

On the other hand

\[ \int_{\Omega^0} \nabla u \cdot \nabla \tilde{\varphi} \, dx = \lambda_1 (\int (\Omega^0)) \int_{\Omega^0} \tilde{\varphi} u \, dx + \int_{\partial (\Omega^0)} u \frac{\partial \tilde{\varphi}}{\partial \nu} \, d\sigma. \]

Hence

\[ 0 > \int_{\partial (\Omega^0)} u \frac{\partial \tilde{\varphi}}{\partial \nu} \, d\sigma = \left( \lambda - \lambda_1 (\int (\Omega^0)) \right) \int_{\Omega^0} u \tilde{\varphi} \, dx \geq 0, \]

a contradiction. \( \square \)

3. An Existence Result for \( \lambda < \lambda_1 \)

In this section, we prove the existence of a nontrivial solution to equation (1.1) for \( \lambda < \lambda_1 \), through the Mountain Pass Theorem.

3.1. On Palais–Smale Sequences

In this subsection, we define the framework for the functional \( J_\lambda \) associated to the problem (1.1)_\lambda. Hereafter, we denote by \( \| \cdot \| \) the usual norm of \( H^1_0(\Omega) \):

\[ \| u \| = \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{1/2}. \]

Given \( f(s) = h(s) + g(s) \) defined by (1.2), let us denote by \( F(s) := \int_0^s f(t) \, dt \). Observe that (1.7) implies the following

\[ F(s) + \frac{1}{2} C_0 s^2 \geq 0, \text{ for all } s \geq 0. \] (3.1)

Consider the functional \( J_\lambda : H^1_0(\Omega) \to \mathbb{R} \) given by

\[ J_\lambda[v] := \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, dx - \lambda \int_{\Omega} (v^+)^2 \, dx - \int_{\Omega} a(x) F(v^+) \, dx. \]

Take note that for all \( v \in H^1_0(\Omega) \), \( J_\lambda[v^+] \leq J_\lambda[v] \).

The functional \( J_\lambda \) is well defined and belongs to the class \( C^1 \) with

\[ J_\lambda'[v] \psi = \int_{\Omega} \nabla v \nabla \psi \, dx - \lambda \int_{\Omega} v^+ \psi \, dx - \int_{\Omega} a(x) f(v^+) \psi \, dx, \]
for all $\psi \in H_0^1(\Omega)$. As a result, non-negative critical points of the functional $J_\lambda$ correspond to non-negative weak solutions to (1.1).

The next Proposition establishes that *Palais–Smale sequences* are bounded whenever $\lambda < \lambda_1(\text{int } \Omega^0)$, where $\lambda_1(\text{int } \Omega^0)$ may be infinite.

**Proposition 3.1.** Assume that $g \in C^1(\mathbb{R})$ fulfills hypothesis (H) and that $\lambda < \lambda_1(\text{int } \Omega^0) \leq +\infty$.

Then any (PS) sequence, that is, a sequence satisfying the conditions

$(J_1)$\, $J_\lambda[u_n] \leq C$,

$(J_2)$\, $|J_\lambda'[u_n]| \leq \varepsilon_n \|\psi\|$, where $\varepsilon_n \to 0$ as $n \to +\infty$

is a bounded sequence.

**Proof.** 1. Let $\{u_n\}_{n \in \mathbb{N}}$ be a (PS) sequence in $H_0^1(\Omega)$ and, in contradiction, assume that $\|u_n\| \to +\infty$. Let us first prove the following claim:

**Claim.** Let $v \in H_0^1(\Omega)$ be the weak limit of $v_n = \frac{u_n}{\|u_n\|}$ and assume that $v_n \to v$, strongly in $L^{2^* - 1}(\Omega)$ and a.e. Then $v = 0$ a.e. in $\Omega$.

Assume that $v \not\equiv 0$ and write $\gamma_n = \|u_n\|$. Let $\omega_n := \{x \in \Omega : v_n^+(x) > 1\}$, then for any $\psi \in C_0^1(\Omega)$,

$$\frac{\ln(e + \gamma_n)^\alpha}{\gamma_n^{2^* - 1}} \frac{(u_n^+(x))^{2^* - 1}}{\ln(e + \gamma_n v_n^+(x))}\|\psi\| \leq \|v_n^+(x)\|^{2^* - 1} \|\psi\|_\infty, \quad \forall x \in \omega_n.$$ 

Let $x \in \Omega \setminus \omega_n$, based on the estimates (A.1),

$$\frac{\ln(e + \gamma_n)^\alpha}{\gamma_n^{2^* - 1}} \frac{(u_n^+(x))^{2^* - 1}}{\ln(e + \gamma_n v_n^+(x))}\|\psi\| \leq \left(\|v_n^+(x)\|^{2^* - 2}\right) \|\psi\|_\infty \leq \|\psi\|_\infty$$

Besides, by the reverse of the Lebesgue dominated convergence theorem, see for instance [2, Theorem 4.9, p. 94], there exists $h_i \in L^1(\Omega)$, $1 \leq i \leq 3$ such that, up to a subsequence,

$$|v_n^+|^{2^* - 1} \leq h_1, |v_n^+|^{p - 1} \leq h_2, |v_n^+|^{2^* - 2} \leq h_3, \quad \text{a.e. } x \in \Omega,$$

for all $n \in \mathbb{N}$, and therefore

$$\frac{\ln(e + \gamma_n)^\alpha}{\gamma_n^{2^* - 1}} f(u_n^+)\psi \leq C (h_1 + h_2 + h_3 + 1) \|\psi\|_\infty \in L^1(\Omega).$$

By Lebesgue’s dominated convergent theorem, we have

$$\frac{\ln(e + \gamma_n)^\alpha}{\gamma_n^{2^* - 1}} a(\cdot)(u_n^+)\psi \to a(\cdot)(v^+)^{2^* - 1}\psi \quad \text{strongly in } L^1(\Omega).$$

We have used here that if $v^+(x) \neq 0$, then

$$\lim_{n \to +\infty} \frac{\ln(e + \gamma_n)}{\ln(e + \gamma_n v_n^+(x))} = 1,$$

and if $v^+(x) = 0$, then

$$\lim_{n \to +\infty} \left(\frac{\ln(e + \gamma_n)}{\ln(e + \gamma_n v_n^+(x))}\right)^\alpha |v_n^+(x)|^{2^* - 1} \leq \lim_{n \to +\infty} |v_n^+(x)|^{2^* - 2} = 0.$$
On the other hand
\[ \frac{\ln(e + \gamma n)^{\alpha}}{\gamma n^{2^* - 1}} \int_{\Omega} \nabla u_n \cdot \nabla \psi \, dx \to 0. \]

Hence, using \((J_2)\) for an arbitrary test function \(\psi\), multiplying by \(\frac{\ln(e + \gamma n)^{\alpha}}{\gamma n^{2^* - 1}}\) and passing to the limit we find
\[ \int_{\Omega} a(x)(v^+)^{2^* - 1} \psi \, dx = 0 \quad \forall \psi \in C_0^1(\Omega). \]

In particular \(v^+ = 0\) a.e. in \(\Omega \setminus \Omega^0\).

Assume that \(\text{int } \Omega^0 \neq \emptyset\), and that \(\lambda < \lambda_1(\text{int } \Omega^0)\). Thus, for any \(\psi \in C_0^1(\text{int } \Omega^0)\) we have from \((J_2)\)
\[ \int_{\text{int } \Omega^0} \nabla u_n \cdot \nabla \psi \, dx - \lambda \int_{\text{int } \Omega^0} u_n^+ \psi \, dx = o(1). \]

Dividing by \(\|u_n\|\) and passing to the limit we have
\[ \int_{\text{int } \Omega^0} \nabla v \cdot \nabla \psi \, dx = \lambda \int_{\text{int } \Omega^0} v^+ \psi \, dx. \]

From the Maximum Principle, \(v \geq 0\) in \(\text{int } \Omega^0\). Since \(\lambda < \lambda_1(\text{int } \Omega^0)\) then it must be \(v^+ \equiv 0\) in \(\text{int } \Omega^0\). Hence \(v^+ \equiv 0\) in \(\Omega\).

On the other hand, taking \(u_n^-\) as a test function in the condition \((J_2)\),
\[ \left| - \int_{\Omega} |\nabla u_n^-|^2 \, dx - \int_{\Omega} a(x)f(u_n^+)u_n^- \, dx \right| = \int_{\Omega} |\nabla u_n^-|^2 \, dx \leq \epsilon_n \|u_n^-\| \]
so \(\|u_n^-\| \to 0\) and then \(v^- \equiv 0\), and we conclude the proof of the claim.

2. In order to achieve a contradiction, we use a Hölder inequality, and properties on convergence into an Orlicz space, cf. Appendix B.

To this end, the analysis of Lemma A.2 gives us the existence of \(\alpha^* > 0\) such that the function \(s \rightarrow \frac{s^{2^* - 1}}{[\ln(e + s)]^\alpha}\) is increasing along \([0, +\infty]\) if \(\alpha \leq \alpha^*\). In this case, we will denote
\[ m(s) = \frac{s^{2^* - 1}}{[\ln(e + s)]^\alpha} \quad (3.2) \]

If \(\alpha > \alpha^*\) the function \(s \rightarrow \frac{s^{2^* - 1}}{[\ln(e + s)]^\alpha}\) possesses a local maximum \(s_1\) in \([0, +\infty[.\) Let us denote by \(s_1\) the unique solution \(s > s_1\) such that
\[ \frac{s_1^{2^* - 1}}{[\ln(e + s_1)]^\alpha} = \frac{s^{2^* - 1}}{[\ln(e + s)]^\alpha} \]
and define the non-decreasing function
\[ m(s) := \begin{cases} \frac{s^{2^* - 1}}{[\ln(e + s)]^\alpha} & \text{if } s \notin [s_1, s_1], \\ \frac{s_1^{2^* - 1}}{[\ln(e + s_1)]^\alpha} & \text{if } s \in [s_1, s_1]. \end{cases} \quad (3.3) \]

It follows that
\[ s \to M(s) = \int_0^s m(t) \, dt \quad \text{is a } N \text{- function in } [0, +\infty[. \quad (3.4) \]
By using
\[ \lim_{s \to +\infty} \frac{\ln(e + s)}{\ln(e + 2s)} = 1 \quad \text{and} \quad \lim_{s \to 0} \frac{\ln(e + s)}{\ln(e + 2s)} = 1, \]
we get that
\[ \lim_{s \to +\infty} \frac{m(2s)}{m(s)} < +\infty \quad \text{and} \quad \lim_{s \to 0^+} \frac{m(2s)}{m(s)} < +\infty, \]
which implies that there exists \( K > 0 \) such that \( m(2s) \leq K m(s) \) for all \( s \geq 0 \) and consequently \( M \) satisfies the \( \Delta_2 \)-condition (B.1).

Since \( v_n \to 0 \) in \( H^1_0(\Omega) \) and strongly in \( L^2(\Omega) \), it follows from \((J_2)\) applied to \( \psi = u_n \) that
\[
\lim_{n \to -\infty} \int_{\Omega} a(x) \frac{f(u_n^+)}{\|u_n\|^2} \, dx = \lim_{n \to -\infty} \int_{\Omega} a(x) \frac{f(u_n^+)}{\|u_n\|} \, v_n^+ \, dx = 1. \tag{3.5}
\]

Since the Hölder inequality into Orlicz spaces, see Proposition B.11.(ii),
\[
\int_{\Omega} \left| a(x) \frac{f(u_n^+)}{\|u_n\|} \right| v_n^+ \, dx \leq \|a\|_{\infty} \|f(u_n^+)\|_{M^*} \|v_n^+\|_M \tag{3.6}
\]
By Theorem B.3 and Theorem B.12 we have
\[
\|v_n - v\|_M \to 0. \tag{3.7}
\]
Moreover, since there exists \( C > 0 \) such that \( m(s) \leq Cs^{2^* - 1} \), \( M(s) \leq Cs^{2^*} \) for all \( s \geq 0 \), and the sequence \( \{u_n\}_{n \in \mathbb{N}} \subset H^1_0(\Omega) \), then, for each \( n \in \mathbb{N} \), there exists a \( C_n \) such that
\[
\int_{\Omega} \left| u_n^+ \right| m\left( |u_n^+| \right) \leq C_n, \quad \int_{\Omega} M\left( |u_n^+| \right) \leq C_n.
\]
By using definition B.8 of \( M^* \) and identities of Proposition B.9 we have
\[
M^* \left( m\left( |u_n^+| \right) \right) = |u_n^+| m\left( |u_n^+| \right) - M\left( |u_n^+| \right)
\]
then, for each \( n \in \mathbb{N} \),
\[
\int_{\Omega} M^* \left( m\left( |u_n^+| \right) \right) \, dx \leq 2C_n.
\]
Observe that \( |f(s)| \leq C(1 + m(s)) \), so then
\[
\|f(u_n^+)\|_{M^*} \leq C \|1 + m(u_n^+)\|_{M^*} \leq C \left[ 1 + \int_{\Omega} M^* \left( m\left( |u_n^+| \right) \right) \right] \leq C'_n,
\]
see Proposition B.11.(iii) and (i), concluding that the l.h.s. is bounded for each \( n \).

Consequently, \( a(x) \frac{f(u_n^+)}{\|u_n\|} \in L_{M^*}(\Omega) \), which is the dual of \( L_M(\Omega) \) (see [15], Theorem 14.2).

On the other hand, from \( J_2 \), for all \( \psi \in C_0^\infty(\Omega) \),
\[
\left| \int_{\Omega} \nabla v_n \nabla \psi \, dx - \lambda_n \int_{\Omega} v_n \psi \, dx - \int_{\Omega} a(x) \frac{f(u_n^+)}{\|u_n\|} \psi \, dx \right| \leq \frac{\varepsilon_n}{\|u_n\|} \|\psi\|. \tag{3.8}
\]
Taking the limit, and since \( C_0^\infty(\Omega) \) is dense in \( L_M(\Omega) \) (see [13]),
\[
\lim_{n \to -\infty} \int_{\Omega} a(x) \frac{f(u_n^+)}{\|u_n\|} \psi \, dx = 0, \quad \text{for all} \quad \psi \in L_M(\Omega). \tag{3.9}
\]
Moreover, since (3.7), $v_n \to v = 0$ in $L_M(\Omega)$, \cite[Proposition 3.13 (iv)]{2}, and (3.9) imply
\[
\lim_{n \to \infty} \int_{\Omega} a(x) \frac{f(u_n^+)}{\|u_n\|} v_n \, dx = 0,
\]
which contradicts (3.5). This concludes the proof. \hfill \Box

**Theorem 3.2.** Assume the hypothesis of Proposition 3.1 and let $\{u_n\}_{n \in \mathbb{N}}$ be a (PS) sequence in $H^1_0(\Omega)$.

Then, there exists a subsequence, denoted by $\{u_n\}_{n \in \mathbb{N}}$, such that
\[
\begin{align*}
  u_n &\to u \quad \text{in} \quad H^1_0(\Omega), \\
  \int_{\Omega} a(x) g(u_n)|u_n - u| \, dx &\to 0, \\
  u_n &\to u \quad \text{a.e. (3.12)}
\end{align*}
\]

By testing $(J_2)$ against $\psi = u_n - u$ and using (3.10), and (3.11) we get
\[
\|u_n - u\|^2 = \int_{\Omega} \nabla u_n \cdot \nabla (u_n - u) \, dx + o(1)
\leq \|a\|_{\infty} \int_{\Omega} \frac{|u_n|^{2^* - 1}}{[\ln(e + |u_n|)]^{\alpha}} |u_n - u| \, dx + o(1).
\]

**Claim.**
\[
\int_{\Omega} \frac{|u_n|^{2^* - 1}}{[\ln(e + |u_n|)]^{\alpha}} |u_n - u| \, dx = o(1),
\]

In order to prove this claim, we use, as in the above proposition, a Hölder inequality and a compact embedding into some Orlicz space, c.f. Appendix B.

By Theorem B.3 and Theorem B.12 we have
\[
\|u_n - u\|_M \to 0,
\]
where $m$, and $M$ are defined by (3.2)–(3.4), as in the above proposition. On the other hand, because there exists $C > 0$ such that $m(s) \leq Cs^{2^* - 1}$ and $M(s) \leq Cs^{2^*}$ for all $s \geq 0$, and the sequence $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $H^1_0(\Omega)$, then
\[
\|u_n m(|u_n|)\|_{L^1(\Omega)} \leq C, \quad \|M(|u_n|)\|_{L^1(\Omega)} \leq C \quad \text{for all} \quad n \in \mathbb{N}
\]
By using definition B.8 of $M^*$ and identities of Proposition B.9 we have
\[
M^* (m(|u_n|)) = |u_n| m(|u_n|) - M(|u_n|)
\]
then
\[
\int_{\Omega} M^* (m(|u_n|)) \, dx \leq C
\]
for all $n \in \mathbb{N}$. Finally, by inequality (B.5) of Proposition B.12 we get
\[
\sup \left\{ \|m(|u_n|)\|_{M^*}, \ n \in \mathbb{N} \right\} \leq C + 1.
\]
Now, using Holder’s inequality (B.6) and that \( \frac{s^{2^* - 1}}{\ln(e + s)^{\alpha}} \leq m(s) \) for all \( s \geq 0 \), we get
\[
\int_{\Omega} \frac{|u_n|^{2^*-1}}{\ln(e + |u_n|)} |u_n - u| \, dx \leq \|u_n - u\|_M \|m(|u_n|)\|_{M^*} \leq (C + 1)\|u_n - u\|_M
\]
and it follows from (3.13) that \( \|u_n - u\| \to 0 \). \( \square \)

3.2. An Existence Result for \( \lambda < \lambda_1 \)

The next theorem provides a solution to (1.1) for \( \lambda < \lambda_1 \) based on the Mountain Pass Theorem.

**Theorem 3.3.** Assume that \( \Omega \subset \mathbb{R}^N \) is a bounded domain with \( C^2 \) boundary. Assume that the nonlinearity \( f \) defined by (1.2) satisfies (H), and that the weight \( a \in C^1(\Omega) \). Then, the boundary value problem (1.1) has at least one classical positive solution for any \( \lambda < \lambda_1 \).

**Proof.** We verify the hypothesis of the Mountain Pass Theorem, see [14, Theorem 2, Section 8.5]. Observe that the derivative of the functional \( J_\lambda : H^1_0(\Omega) \to \mathbb{R} \) is Lipschitz continuous on bounded sets of \( H^1_0(\Omega) \); also the (PS) condition is satisfied, see Proposition 3.1. Clearly \( J_\lambda(0) = 0 \).

1. Let now \( u \in H^1_0(\Omega) \) with \( \|u\| = r \), for \( r > 0 \) to be chosen below. Then,
\[
J_\lambda[u] = \frac{r^2}{2} - \frac{\lambda}{2} \int_{\Omega} (u^+)^2 \, dx - \int_{\Omega} a(x) F(u^+) \, dx.
\]
From hypothesis (H) we have
\[
\left| \int_{\Omega} a(x) G(u^+) \, dx \right| \leq C \int_{\Omega} (|u|^p + |u|^q) \, dx \leq C (r^p + r^q).
\]
where \( G(s) := \int_0^s g(t) \, dt \). Now, definition (1.2) implies that
\[
\left| \int_{\Omega} a(x) F(u^+) \, dx \right| \leq C \left( r^p + r^q + r^{2^*} \right).
\]
In view of (3.14), and as a result of the Poincaré inequality, we get
\[
J_\lambda[u] \geq \frac{1}{2} \left( 1 - \frac{\lambda}{\lambda_1} \right) r^2 - C \left( r^p + r^q + r^{2^*} \right) \geq C_1 r^2,
\]
taking \( \lambda < \lambda_1 \), \( r > 0 \) small enough, and using that \( p, q, 2^* > 2 \).

2. Now, fix some element \( 0 \leq u_0 \in H^1_0(\Omega) \), \( u_0 > 0 \) in \( \Omega^+ \), \( u_0 \equiv 0 \) in \( \Omega^- \). Let \( v = tu_0 \) for a certain \( t = t_0 > 0 \) to be selected a posteriori. Since
\[
f(tu_0) = |t|^{2^*-2} f(u_0) \left( \frac{\ln(e + |u_0|)}{\ln(e + |tu_0|)} \right)^\alpha + g(tu_0),
\]
then \( f(tu_0)/t \to +\infty \) as \( t \to +\infty \) in \( \Omega^+ \).

From definition, and integrating by parts,
\[
F(s) = \int_0^s \left( \frac{t^{2^*-1}}{\ln(e + t)^\alpha} + g(t) \right) \, dt
\]
Positive Solutions for Slightly Subcritical

\[ \frac{1}{2^*} sh(s) + G(s) + \frac{\alpha}{2^*} \int_0^s \left( \frac{1}{\ln(e + t)} \right)^{\alpha + 1} \frac{t^{2^*}}{e + t} dt. \]

It can be easily seen that \( \lim_{s \to +\infty} \frac{G(s)}{s} = 0. \)

Therefore, using l'Hôpital's rule we can write

\[ \lim_{s \to +\infty} \frac{F(s)}{s} = \frac{1}{2^*} \in \left( 0, \frac{1}{2} \right), \]

hence

\[ \lim_{t \to +\infty} \frac{F(\nu_0)}{\nu_0 f(\nu_0)} = \frac{1}{2^*} \in \left( 0, \frac{1}{2} \right) \text{ in } \Omega^+. \] (3.17)

Let \( C_0 \geq 0 \) be such that \( F(s) + \frac{1}{2} C_0 s^2 \geq 0 \) for all \( s \geq 0 \) (see (1.7)), and let

\[ \tilde{\Omega}_\delta^+ := \{ x \in \Omega^+ : a(x) = a^+(x) > \delta \}. \] (3.18)

By definition, \( u_0 \equiv 0 \) in \( \Omega^- \), so, introducing \( \pm \frac{1}{2} C_0 (\nu_0)^2 \), splitting the integral, and using (3.17)-(3.18) we obtain

\[ -\int_\Omega a(x) F(\nu_0) \, dx = -\int_{\Omega^+} a^+(x) F(\nu_0) \, dx \]
\[ \leq \frac{C_0 t^2}{2} \int_{\Omega^+} a^+(x) \nu_0^2 \, dx - \int_{\tilde{\Omega}_\delta^+} a^+(x) \left[ \frac{1}{2} C_0 (\nu_0)^2 + F(\nu_0) \right] \, dx \]
\[ \leq C + \frac{C_0 t^2}{2} \int_{\Omega^+} a^+(x) \nu_0^2 \, dx - \frac{\delta t^2}{2} \int_{\tilde{\Omega}_\delta^+} \left[ C_0 \nu_0^2 + \frac{\nu_0 f(\nu_0)}{2^* t} \right] \, dx. \]

Hence, there exists a positive constant \( C > 0 \) such that

\[ J_\lambda[\nu_0] = \frac{t^2}{2} \| \nu_0 \|_2^2 - t^2 \frac{\lambda}{2} \| \nu_0 \|_{L^2(\Omega)}^2 - \int_{\Omega^+} a^+(x) F(\nu_0) \]
\[ \leq C(1 + t^2) - \frac{\delta t^2}{2} \int_{\tilde{\Omega}_\delta^+} \left[ C_0 (\nu_0)^2 + \frac{\nu_0 f(\nu_0)}{2^* t} \right] \, dx < 0 \]

for \( t = t_0 > 0 \) big enough.

Step 3. We have at last checked that all the hypothesis of the Mountain Pass Theorem are accomplished. Let

\[ \Gamma := \{ g \in C([0, 1]; H^1_0(\Omega)) : g(0) = 0, g(1) = t_0 u_0 \}, \]

then, there exists \( c \geq C_1 t^2 > 0 \) such that

\[ c := \inf_{g \in \Gamma} \max_{0 \leq t \leq 1} J_\lambda[g(t)] \]

is a critical value of \( J_\lambda \), that is, the set \( \mathcal{K}_c := \{ v \in H^1_0(\Omega) : J_\lambda[v] = c, J'_\lambda[v] = 0 \} \neq \emptyset. \) Thus there exists \( u \in H^1_0(\Omega), u \geq 0, u \neq 0 \) such that for each \( \psi \in H^1_0(\Omega) \), we have

\[ \int_\Omega \nabla u \cdot \nabla \psi \, dx = \int_\Omega \left[ \lambda u^+ + a(x) f(u^+) \right] \psi \, dx. \] (3.19)

and thereby, \( u \) is a nontrivial weak solution to (3.19). By Lemma 2.1, \( u \) is a classical solution, and by (1.8), \( u > 0 \) in \( \Omega. \) \( \square \)
4. A Bifurcation Result for $\lambda > \lambda_1$

Next Proposition uses Crandall-Rabinowitz’s local bifurcation theory, see [10], and Rabinowitz’s global bifurcation theory, see [19].

**Proposition 4.1.** Let us define

$$\Lambda := \sup \{ \lambda > 0 : (1.1)_\lambda \text{ admits a positive solution} \}.$$ 

If (1.5) holds then,

$$\lambda_1 < \Lambda < \min \left\{ \lambda_1 (\text{int} (\Omega^0)) , \quad \lambda_1 (\text{int} (\Omega^+ \cup \Omega^0)) + C_0 \sup a^+ \right\}$$

where $C_0 > 0$ is such that $f(s) + C_0 s \geq 0$ for all $s \geq 0$, (see definition (1.6)).

Moreover, there exists an unbounded continuum (a closed and connected set) $\mathcal{C}$ of classical positive solutions to (1.1) emanating from the trivial solution set at the bifurcation point $(\lambda, u) = (\lambda_1, 0)$.

**Proof.** Proposition 2.2 establishes the upper bounds for $\Lambda$. Next, we concentrate our attention in proving that $\Lambda > \lambda_1$. Choosing $\lambda$ as the bifurcation parameter, we check that the conditions of Crandall - Rabinowitz’s Theorem [10] are satisfied. For $r > N$, we define the set $W^{2, r}_{+} := \{ u \in W^{2, r}(\Omega) : u > 0 \text{ in } \Omega \}$, and consider $W_{+}^{2, r}(\Omega) \cap W_{0}^{1, r}(\Omega)$ endowed with the topology of $W^{2, r}(\Omega)$. If $r > N$, we have that $W_{+}^{2, r}(\Omega) \cap W_{0}^{1, r}(\Omega) \rightarrow C^{1, \mu}_{0}(\Omega)$ for $\mu = 1 - \frac{N}{r} \in (0, 1)$. Moreover, from Hopf’s lemma, we know that if $\tilde{u}$ is a positive solution to (1.1) then $\tilde{u}$ lies in the interior of $W_{+}^{2, r}(\Omega) \cap W_{0}^{1, r}(\Omega)$.

We consider the map $\mathcal{F} : \mathbb{R} \times W^{2, r}_{+}(\Omega) \cap W^{1, r}_{0}(\Omega) \rightarrow L^{r}(\Omega)$ for $r > N$,

$$\mathcal{F} : (\lambda, u) \rightarrow -\Delta u - \lambda u - a(x)f(u)$$

The map $\mathcal{F}$ is a continuously differentiable map. Since hypothesis (i), $g(0) = 0$, and so $a(x)F(0) = 0$, $\mathcal{F}(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$, and since $F_u(x, 0) = 0$,

$$D_u \mathcal{F}(\lambda_1, 0) w := -\Delta w - \lambda_1 w,$$

$$D_{\lambda, u} \mathcal{F}(\lambda_1, 0) w := -w.$$ 

Observe that

$$N(D_u \mathcal{F}(\lambda_1, 0)) = \text{span}[\varphi_1], \quad \text{codim } R(D_u \mathcal{F}(\lambda_1, 0)) = 1,$$

$$D_{\lambda, u} \mathcal{F}(\lambda_1, 0) \varphi_1 = -\varphi_1 \notin R(D_u \mathcal{F}(\lambda_1, 0)),$$

where $N(\cdot)$ is the kernel, and $R(\cdot)$ denotes the range of a linear operator.

Hence, the hypotheses of Crandall-Rabinowitz’s Theorem are satisfied and $(\lambda_1, 0)$ is a bifurcation point. Thus, decomposing

$$C^{1, \mu}_0(\overline{\Omega}) = \text{span}[\varphi_1] \oplus Z,$$

where $Z = \text{span}[\varphi_1]^\perp$, there exists a neighborhood $\mathcal{U}$ of $(\lambda_1, 0)$ in $\mathbb{R} \times C^{1, \mu}_0(\overline{\Omega})$, and continuous functions $\lambda(s), \tilde{w}(s), s \in (-\varepsilon, \varepsilon)$, $\lambda : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$, $\tilde{w} : (-\varepsilon, \varepsilon) \rightarrow Z$ such that $\lambda(0) = \lambda_1$, $\tilde{w}(0) = 0$, with $\int_{\Omega} \tilde{w} \varphi_1 \, dx = 0$, and the only nontrivial solutions to (1.1) in $\mathcal{U}$ are

$$\{(\lambda(s), s \varphi_1 + s \tilde{w}(s)) : s \in (-\varepsilon, \varepsilon) \}. \quad (4.1)$$
Set $u = u(s) = s\varphi_1 + s\tilde{w}(s)$. Note that by continuity $\tilde{w}(s) \to 0$ as $s \to 0$, which guarantees that $u(s) > 0$ in $\Omega$ for all $s \in (0, \varepsilon)$ small enough.

Next, we show that $\lambda(s) > \lambda_1$ for all $s$ small enough. Since (3.15), and hypothesis (H)$_0$ on $f$, note that \( \frac{a(x)f(su)}{sp^{-1}u^{p-1}} \to L_1a(x) \) as $s \to 0$. In fact, as $\tilde{w}(s) \to 0$ uniformly as $s \to 0$, hypothesis (H)$_0$ yields

\[
\frac{a(x)f(s\varphi_1 + s\tilde{w}(s))}{sp^{-1}(\varphi_1 + \tilde{w}(s))^{p-1}} \to L_1a(x) \text{ uniformly in } \Omega \quad \text{as } s \to 0.
\]

Hence, multiplying and dividing by $(\varphi_1 + \tilde{w}(s))^{p-1}$, we deduce

\[
\frac{1}{sp^{-1}} \int_{\Omega} a(x)f(u(s))\varphi_1 \to L_1 \int_{\Omega} a(x)\varphi_1^p.
\]

Now we prove that $\lambda(s) > \lambda_1$ arguing by contradiction. Assume that there is a sequence $(\lambda_n, u_n) = (\lambda(s_n), u(s_n))$ of bifurcated solutions to (1.1) in $\mathcal{W}$, with $\lambda(s_n) \leq \lambda_1$. Multiplying (1.1)$_{\lambda_n}$ by $\varphi_1$ and integrating by parts

\[
0 \leq \frac{(\lambda_1 - \lambda(s_n))}{sp_n^{-1}} \int_{\Omega} u(s_n)\varphi_1 = \frac{1}{sp_n^{-1}} \int_{\Omega} a(x)f(u(s_n))\varphi_1 \to L_1 \int_{\Omega} a(x)\varphi_1^p < 0
\]

which yields a contradiction, and consequently, $\Lambda > \lambda_1$.

Finally, Rabinowitz’s global bifurcation Theorem [19] states that, in fact, the set $\mathcal{C}$ of positive solutions to (1.1) emanating from $(\lambda_1, 0)$ is a continuum (a closed and connected set) which is either unbounded, or contains another bifurcation point, or contains a pair of points $(\lambda, u)$, $(\lambda, -u)$ with $u \neq 0$. Since (1.8), any non-negative non-trivial solution is strictly positive, and moreover $(\lambda_1, 0)$ is the only bifurcation point to positive solutions, so $\mathcal{C}$ cannot reach another bifurcation point. Since (1.3), neither $\mathcal{C}$ contains a pair of points $(\lambda, u)$, $(\lambda, -u)$ with $u \neq 0$, which states that $\mathcal{C}$ is unbounded, ending the proof. \(\square\)

5. Proof of Theorem 1.1

First we prove an auxiliary result.

**Proposition 5.1.** For each $\lambda \in (\lambda_1, \Lambda)$, the following holds:

(i) Problem (1.1)$_{\lambda}$ admits a positive solution

$$u_{\lambda} = \inf \left\{ u(x) : u > 0 \text{ solving } (1.1)_{\lambda} \right\},$$

in other words $u_{\lambda}$ is minimal.

(ii) Moreover, the map $\lambda \to u_{\lambda}$ is strictly monotone increasing, that is, if $\lambda < \mu < \Lambda$, then $u_{\lambda}(x) < u_{\mu}(x)$ for all $x \in \Omega$, and $\partial u_{\lambda}/\partial \nu(x) > \partial u_{\mu}/\partial \nu(x)$ for all $x \in \partial\Omega$.

(iii) Furthermore, $u_{\lambda}$ is a local minimum of the functional $J_{\lambda}$.

**Proof.** (i.a) Step 1. Existence of positive solutions for any $\lambda \in (\lambda_1, \Lambda)$.

Let $\lambda \in (\lambda_1, \Lambda)$ be fixed. By definition of $\Lambda$, there exists a $\lambda_0 \in (\lambda, \Lambda)$ such that the problem (1.1)$_{\lambda_0}$ admits a positive solution $u_0$. It is easy to verify that $u_0 > 0$ is a
supersolution to (1.1). Indeed, for any \( \psi \in H_0^1(\Omega) \) with \( \psi \geq 0 \) in \( \Omega \)
\[
\int_\Omega \nabla u_0 \cdot \nabla \psi \, dx - \lambda \int_\Omega u_0 \psi \, dx - \int_\Omega a(x) f(u_0) \psi \, dx = (\lambda_0 - \lambda) \int_\Omega u_0 \psi \, dx \geq 0.
\]
Moreover, for every \( \delta > 0 \) satisfying
\[
0 < \delta < \left( \frac{\lambda - \lambda_1}{2L_1 \| a^- \|_\infty} \right)^{\frac{1}{p-2}} \frac{1}{\| \varphi_1 \|_\infty}
\]
the function \( u = \delta \varphi_1 \) is a subsolution for (1.1) whenever \( \lambda > \lambda_1 \). Let \( \delta > 0 \) satisfying (5.1) and such that \( g(s) \geq 0 \) for any \( s \in [0, \delta \| \varphi_1 \|_{L^\infty(\Omega)}] \). For any \( \psi \in H_0^1(\Omega) \), \( \psi > 0 \) with in \( \Omega \) we deduce
\[
d \int_\Omega \nabla \varphi_1 \cdot \nabla \psi \, dx - \lambda \delta \int_\Omega \varphi_1 \psi \, dx - \int_\Omega a(x) f(\delta \varphi_1) \psi \, dx
\]
\[
= - (\lambda - \lambda_1) \delta \int_\Omega \varphi_1 \psi \, dx - \int_\Omega a(x) f(\delta \varphi_1) \psi \, dx
\]
\[
= - (\lambda - \lambda_1) \delta \int_\Omega \varphi_1 \psi \, dx - \int_\Omega a(x) \left[ \frac{(\delta \varphi_1)^{2^*-1}}{\ln(e + \delta \varphi_1)} + g(\delta \varphi_1) \right] \psi \, dx
\]
\[
\leq - (\lambda - \lambda_1) \delta \int_\Omega \varphi_1 \psi \, dx + \| a^- \|_\infty \int_\Omega [h(\delta \varphi_1) + g(\delta \varphi_1)] \psi \, dx < 0.
\]
This allows us to take \( u = \delta \varphi_1 \) as a subsolution for (1.1) with \( u \leq u_0 \). The sub- and supersolution method now guarantees a positive solution \( u \) to (1.1), with \( u \leq u \leq u_0 \).

(i.b) Step 2. Existence of a minimal positive solution \( u_\lambda \) for any \( \lambda \in (\lambda_1, \Lambda) \).

To show that there is in fact a minimal solution, for each \( x \in \Omega \) we define
\[
\underline{u}_\lambda(x) := \inf \{ u(x) : u > 0 \text{ solving (1.1)} \}.
\]
Firstly, we claim that \( \underline{u}_\lambda \geq 0 \), \( \underline{u}_\lambda \neq 0 \). Assume that \( \underline{u}_\lambda \equiv 0 \) by contradiction. This would yield a sequence \( u_n \) of positive solutions to (1.1) such that \( \| u_n \|_{C(\overline{\Omega})} \to 0 \) as \( n \to \infty \), or in other words, \( (\lambda, 0) \) is a bifurcation point from the trivial solution set to positive solutions. Set \( v_n := \frac{u_n}{\| u_n \|_{C(\overline{\Omega})}} \). Observe that \( v_n \) is a weak solution to the problem
\[
- \Delta v_n = \lambda v_n + a(x) f(u_n) / \| u_n \|_{C(\overline{\Omega})} \quad \text{in } \Omega; \quad v_n = 0 \text{ on } \partial \Omega.
\]
It follows from (H)\(_0\) that \( \frac{a(x) f(u_n)}{\| u_n \|_{C(\overline{\Omega})}} \to 0 \) in \( C(\overline{\Omega}) \) as \( n \to \infty \). Therefore, the right-hand side of (5.2) is bounded in \( C(\overline{\Omega}) \). Hence, by the elliptic regularity, \( v_n \in W^{2,r}(\Omega) \) for any \( r > 1 \), in particular for \( r > N \). Then, the Sobolev embedding theorem implies that \( \| v_n \|_{C^{1,\alpha}(\overline{\Omega})} \) is bounded by a constant \( C \) that is independent of \( n \). Then, the compact embedding of \( C^{1,\mu}(\overline{\Omega}) \) into \( C^{1,\beta}(\overline{\Omega}) \) for \( 0 < \beta < \mu \) yields, up to a subsequence, \( v_n \to \Phi \geq 0 \) in \( C^{1,\beta}(\overline{\Omega}) \). Since \( \| v_n \|_{C(\overline{\Omega})} = 1 \), we have that \( \| \Phi \|_{C(\overline{\Omega})} = 1 \). Hence, \( \Phi \geq 0, \Phi \neq 0 \).

Using the weak formulation of equation (5.2), passing to the limit, and taking into account that \( \lambda \) is fixed and \( v_n \to \Phi \), we obtain that \( \Phi \geq 0, \Phi \neq 0 \), is a weak solution to the equation
\[
- \Delta \Phi = \lambda \Phi \quad \text{in } \Omega, \quad \Phi = 0 \text{ on } \partial \Omega.
\]
Then, by the maximum principle, it follows that \( \Phi = \varphi_1 > 0 \), the first eigenfunction, and \( \lambda = \lambda_1 \) is its corresponding eigenvalue, which contradicts that \( \lambda > \lambda_1 \).

Secondly, we show that \( u_\lambda \) solves \((1.1)_\lambda \). We argue on the contrary. Observe that the minimum of any two positive solutions to \((1.1)_\lambda \) furnishes a supersolution to \((1.1)_\lambda \). Assume that there are a finite number of solutions to \((1.1)_\lambda \), then \( u_\lambda(x) := \min\{u(x) : u > 0 \text{ solves } (1.1)_\lambda \} \) and \( u_\lambda \) is a supersolution. Choosing \( \varepsilon_0 \) small enough so that \( \varepsilon_0 \varphi_1 < u_\lambda \), the sub- supersolution method provides a solution \( \varepsilon_0 \varphi_1 \leq v \leq u_\lambda \). Since \( v \) is a solution and \( u_\lambda \) is not, then \( v \leq u_\lambda \), \( v \neq u \), contradicting the definition of \( u_\lambda \), and achieving this part of the proof.

Assume now that there is a sequence \( u_n \) of positive solutions to \((1.1)_\lambda \) such that, for each \( x \in \Omega \), \( \inf u_n(x) = u_\lambda(x) \geq 0 \), \( u_\lambda \neq 0 \). Let \( u_1 := \min\{u_1, u_2\} \). Choosing \( \varepsilon_1 \) small enough so that \( \varepsilon_1 \varphi_1 < u_1 \), the sub- supersolution method provides a solution \( \varepsilon_1 \varphi_1 \leq v_1 \leq u_1 \). We reason by induction.

Let \( u_n := \min\{v_{n-1}, u_{n+1}\} \). Choosing \( \varepsilon_n \) small enough so that \( \varepsilon_n \varphi_1 < u_n \), the sub- supersolution method provides a solution \( \varepsilon_n \varphi_1 \leq v_n \leq u_n \leq v_{n-1} \). With this induction procedure, we build a monotone sequence of solutions \( v_n \), such that

\[
0 < v_n \leq u_n \leq v_{n-1} \leq u_{n-1} \leq \cdots \leq v_1.
\]

(5.3)

Since monotonicity and Lemma 2.1, \( \|v_n\|_{C^1(\overline{\Omega})} \leq \|v_1\|_{C^1(\overline{\Omega})} \), by elliptic regularity, \( \|v_n\|_{C^{1,\beta}(\Omega)} \leq C \) for any \( \mu < 1 \), and by compact embedding \( v_n \to v \) in \( C^{1,\beta}(\Omega) \) for any \( \beta < \alpha \). Using the weak formulation of equation \((1.1)_\lambda \), passing to the limit, and taking into account that \( \lambda \) is fixed, we obtain that \( v \) is a weak solution to the equation \((1.1)_\lambda \). Hence \( v(x) \geq u_\lambda > 0 \). Moreover, since (5.3), \( v(x) \downarrow v(x) \) pointwise for \( x \in \Omega \), so \( \inf v_n(x) = v(x) \). Also, and due to (5.3), \( u_n(x) \downarrow v(x) \) pointwise for \( x \in \Omega \), and \( \inf u_n(x) = v(x) \).

On the other hand, by construction \( u_n \leq u_{n+1} \), so, for each \( x \in \Omega \), \( v(x) = \inf u_n(x) \leq u_\lambda(x) \). Therefore, and by definition of \( u_\lambda \), necessarily \( v = u_\lambda \), proving that \( u_\lambda \) solves \((1.1)_\lambda \), and achieving the proof of step 2.

(ii) The monotonicity of the minimal solutions is concluded from a sub- supersolution method. Reasoning as in step 1, \( u_\mu \) is a strict supersolution to \((1.1)_\lambda \), so \( w := u_\mu(x) - u_\lambda(x) \geq 0 \), \( w \neq 0 \). Moreover, \( w = 0 \) on \( \partial \Omega \), and we can always choose \( c_0 := C_0 \|a\|_{\infty} > 0 \) where \( C_0 \) is defined by (1.6), so that \( a^{-}(x)f'(s) + c_0 \geq 0 \) and \( a^{+}(x)f'(s) + c_0 \geq 0 \) for all \( s \geq 0 \), then

\[
\left( -\Delta + a^{-}(x)f'(\theta u_\mu + (1-\theta)u_\lambda) + c_0 \right) w = (\mu - \lambda)u_\mu + \lambda w
\]

\[
+ \left[a^{+}(x)f'(\theta u_\mu + (1-\theta)u_\lambda) + c_0 \right] w > 0 \quad \text{in } \Omega,
\]

finally, the Maximum Principle implies that \( w > 0 \) in \( \Omega \), and \( \frac{\partial w}{\partial \nu} < 0 \) on \( \partial \Omega \), ending the proof of step 3.

(iii) Since [4, Theorem 2] if there exists an ordered pair of \( L^\infty \) bounded sub and supersolution \( u \leq \overline{v} \) to \((1.1)_\lambda \), and neither \( u \) nor \( \overline{v} \) is a solution to \((1.1)_\lambda \), then there exist a solution \( u < u < \overline{v} \) to \((1.1)_\lambda \) such that \( u \) is a local minimum of \( J_\lambda \) at \( H_0^1(\Omega) \).

Reasoning as in (i), \( \overline{u} := u_\mu \) with \( \mu > \lambda \) is a strict supersolution to \((1.1)_\lambda \), and \( u := \delta \varphi_1 \) is a strict sub-solution for \( \delta > 0 \) small enough, such that \( u(x) < \overline{u}(x) \) for each \( x \in \Omega \). This achieves the proof. \( \square \)
Proof of Theorem 1.1. Theorem 3.3 provides the existence of positive solutions for \( \lambda < \lambda_1 \), and Proposition 5.1 provide the existence of minimal positive solutions for \( \lambda \in (\lambda_1, \Lambda) \).

(a) Step 1. Existence of a second positive solution for \( \lambda \in (\lambda_1, \Lambda) \).
Fix an arbitrary \( \lambda \in (\lambda_1, \Lambda) \), and let \( u_\lambda \) be the minimal solution to (1.1)\( \lambda \) given by Proposition 5.1, minimizing \( J_\lambda \). A second solution follows seeking a solution through variational arguments [12, Theorem 5.10] and the Mountain Pass procedure shown below.

First, reasoning as in Proposition 5.1(iii), we get a local minimum \( \tilde{u}_\lambda > 0 \) of \( J_\lambda \). If \( \tilde{u}_\lambda \neq u_\lambda \), then \( \tilde{u}_\lambda \) is the second positive solution, ending the proof. Assume that \( \tilde{u}_\lambda = u_\lambda \).

Now we reason as in [12, Theorem 5.10] on the nature of local minima. Thus, either

(i) there exists \( \varepsilon_0 > 0 \), such that \( \inf \{ J_\lambda(u) : \|u - \tilde{u}_\lambda\| = \varepsilon_0 \} > J_\lambda(\tilde{u}_\lambda) \), in other words, \( \tilde{u}_\lambda \) is a strict local minimum, or

(ii) for each \( \varepsilon > 0 \), there exists \( u_\varepsilon \in H^1_0(\Omega) \) such that \( J_\lambda \) has a local minimum at a point \( u_\varepsilon \) with \( \|u_\varepsilon - \tilde{u}_\lambda\| = \varepsilon \) and \( J_\lambda(u_\varepsilon) = J_\lambda(\tilde{u}_\lambda) \).

Let us assume that (i) holds, since otherwise case (ii) implies the existence of a second solution.

Consider now the functional \( I_\lambda : H^1_0(\Omega) \rightarrow \mathbb{R} \) given by \( I_\lambda[v] = J_\lambda[u_\lambda + v] - J_\lambda[u_\lambda] \), more specifically

\[
I_\lambda[v] := \frac{1}{2} \int_\Omega |\nabla v|^2 \, dx - \frac{\lambda}{2} \int_\Omega (v^+)^2 \, dx - \int_\Omega \tilde{G}_\lambda(x, v^+) \, dx.
\]

where

\[
\tilde{G}_\lambda(x, s) := a(x) \left[ F(u_\lambda(x) + s) - F(u_\lambda(x)) - f(u_\lambda(x))s \right]
\]

\[
= a(x) \left[ \frac{1}{2} f'(u_\lambda(x))s^2 + o(s^2) \right].
\]

Obviously \( I_\lambda[v^+] \leq I_\lambda[v] \), and observe that \( I'_\lambda[v] = 0 \iff J'_\lambda[u_\lambda + v] = 0 \).

Fix now some element \( 0 \leq v_0 \in H^1_0(\Omega) \cap L^\infty(\Omega) \), \( v_0 > 0 \) in \( \Omega^+ \), \( v_0 \equiv 0 \) in \( \Omega^- \). Let \( v = tv_0 \) for a certain \( t = t_0 > 0 \) to be selected a posteriori, and evaluate

\[
I_\lambda[tv_0] = \frac{1}{2} t^2 \left( \|\nabla v_0\|_{L_2(\Omega)}^2 - \lambda \|v_0\|_{L_2(\Omega)}^2 \right) - \int_\Omega \tilde{G}_\lambda(x, tv_0) \, dx.
\]

Reasoning as in the proof of Theorem 3.3 for large positive \( t \), since \( F(t)/t^2 \rightarrow \infty \) as \( t \rightarrow \infty \), and using also (3.1) we obtain that

\[
I_\lambda[tv_0] \leq C(1 + t + t^2) - \int_{\Omega^+} a^+(x) \left[ F(u_\lambda + tv_0) + \frac{1}{2} C_0(u_\lambda + tv_0)^2 \right] - \delta \int_{\Omega_3^+} \left[ F(u_\lambda + tv_0) + \frac{1}{2} C_0(u_\lambda + tv_0)^2 \right] \, dx,
\]

so

\[
I_\lambda[tv_0] < 0
\]
for $t = t_0$ big enough, and where $\tilde{\Omega}_t^+$ is defined by (3.18). Thus, the Mountain Pass Theorem implies that if

$$\Gamma := \{ g \in C([0,1];H^1_0(\Omega)) : g(0) = 0, \ I_\lambda[g(1)] < 0 \},$$

then, there exists $c > 0$ such that

$$c := \inf_{g \in \Gamma} \max_{0 \leq t \leq 1} I_\lambda[g(t)]$$

is a critical value of $I_\lambda$, and thereby $\mathcal{K}_c := \{ v \in H^1_0(\Omega) : I_\lambda[v] = c, \ I'_\lambda[v] = 0 \}$ is non-empty.

Since for any $g \in \Gamma$ we have $I_\lambda[g^+(t)] \leq I_\lambda[g(t)]$ for all $t \in [0,1]$, it follows that $g^+ \in \Gamma$, and we derive the existence of a sequence $v_n$ such that

$$I_\lambda[v_n] \to c, \quad \|I'_\lambda[v_n]\| \to 0, \quad v_n \geq 0.$$ 

On the other hand, $u_n := u_\lambda + v_n$ is a (PS) sequence for the original functional $J_\lambda$. Since Theorem 3.2, if $\lambda < \lambda_1(\text{int} \Omega^0)$, $v_n \to v_\lambda$ in $H^1_0(\Omega)$, so $I'_\lambda[v] = 0$ and $I_\lambda[v] = c > 0$, hence $v_\lambda \geq 0$ is a nontrivial critical point of $I_\lambda$. Consequently, $w_\lambda := u_\lambda + v_\lambda$ is a positive critical point of $J_\lambda$, such that, for each $\psi \in H^1_0(\Omega)$, we have

$$\int_\Omega \nabla w_\lambda \cdot \nabla \psi dx = \int_\Omega \left( \lambda w_\lambda + a(x)f(w_\lambda) \right) \psi dx,$$

and thereby $w_\lambda := u_\lambda + v_\lambda \geq u_\lambda$, $w_\lambda \neq u_\lambda$ is a second positive solution to (1.1)$_\lambda$.

(b) Step 2. Existence of a classical positive solution for $\lambda = \Lambda$.

We prove the existence of a solution for $\lambda = \Lambda$. For each $\lambda \in (\lambda_1, \Lambda)$, problem (1.1) admits a minimal positive weak solution $u_\lambda$ and $\lambda \to u_\lambda$ is increasing, see Proposition 5.1. Taking the monotone pointwise limit, let us define

$$u_\Lambda(x) := \lim_{\lambda \to \Lambda} u_\lambda(x).$$

We next see that $\|u_\Lambda\| < +\infty$, reasoning on the contrary. Assume that there exists a sequence of solutions $u_n := u_{\lambda_n}$ such that $\|u_{\lambda_n}\| \to +\infty$ as $\lambda_n \to \Lambda$. Set $v_n := u_n/\|u_n\|$, then there exists a subsequence, again denoted by $v_n$ such that $v_n \to v$ in $H^1_0(\Omega)$, and $v_n \to v$ in $L^p(\Omega)$ for any $p < 2^*$ and a.e. Arguing as in the claim of Proposition 3.1, $v \equiv 0$. Moreover

$$\lim_{n \to \infty} \int_\Omega a(x) \frac{f(u_n)}{\|u_n\|} v_n dx = 1. \quad (5.4)$$

On the other hand, from the weak formulation, for all $\psi \in C^\infty_c(\Omega),$

$$\int_\Omega \nabla v_n \cdot \nabla \psi dx = \lambda_n \int_\Omega v_n \psi dx + \int_\Omega a(x) \frac{f(u_n)}{\|u_n\|} \psi dx. \quad (5.5)$$

Taking the limit, and since $C^\infty_c(\Omega)$ is dense in $L^2(\Omega)$

$$\lim_{n \to \infty} \int_\Omega a(x) \frac{f(u_n)}{\|u_n\|} \psi dx = 0, \quad \text{for all } \psi \in L^2(\Omega). \quad (5.6)$$
Since Lemma 2.1, \( u \in C^2(\Omega) \cap C^{1,\mu}(\overline{\Omega}) \) and so \( a(x) \frac{f(u_n)}{\|u_n\|} \in L^2(\Omega) \). Moreover \( v_n \to v = 0 \) in \( L^2(\Omega) \). Hence [2, Proposition 3.13 (iv)], and (5.6) imply
\[
\lim_{n \to \infty} \int_{\Omega} a(x) \frac{f(u_n)}{\|u_n\|} v_n \, dx = 0,
\]
which contradicts (5.4) and yields \( \|u_\Lambda\| < +\infty \).

By Sobolev embedding and the Lebesgue dominated convergence theorem, \( u_n \to u_\Lambda \) in \( L^2(\Omega) \).

Now, by substituting \( \psi = u_n \) in (5.5), using H"older inequality and Sobolev embeddings we obtain
\[
\left[ \|u_n\| \leq \Lambda \|v_n\|_{L^2(\Omega)} \|u_n\| + C, \quad \text{with} \quad \|v_n\|_{L^2(\Omega)} \to 0 \right] \Rightarrow \|u_n\| \leq C.
\]
By compactness, for a subsequence again denoted by \( u_n, u_n \to u^* \) in \( H^1_0(\Omega) \), \( u_n \to u^* \) in \( L^p(\Omega) \) for any \( p < 2^* \) and a.e. By uniqueness of the limit, \( u_\Lambda = u^* \). Finally, by taking limits in the weak formulation of \( u_n \) as \( \lambda_n \to \Lambda \), we get
\[
\int_{\Omega} \nabla u_\Lambda \cdot \nabla \psi = \Lambda \int_{\Omega} u_\Lambda \psi + \int_{\Omega} a(x) f(u_\Lambda) \psi.
\]
Hence \( u_\Lambda \) is a positive weak solution to (1.1)\(_\Lambda\). Lemma 2.1 yields that \( u_\Lambda \in C^2(\Omega) \cap C^{1,\mu}(\overline{\Omega}) \) is a classical solution.

(c) Step 3. Existence of a classical positive solution for \( \lambda \leq \lambda_1 \).

The existence of a classical positive solution for \( \lambda < \lambda_1 \) is done in Theorem 3.3. Let’s look for a solution when \( \lambda = \lambda_1 \).

Since step 1, for any \( \lambda \in (\lambda_1, \Lambda) \) there exists a second positive solution to (1.1)\(_\lambda\). Let’s denote it by \( \tilde{u}_\lambda \neq u_\lambda \). Now, define the pointwise limit
\[
\tilde{u}_{\lambda_1}(x) := \limsup_{\lambda \to \lambda_1} \tilde{u}_\lambda(x). \quad (5.7)
\]
Reasoning as in step 2, \( \|\tilde{u}_{\lambda_1}\| < +\infty \) and \( \tilde{u}_{\lambda_1} \in C^2(\Omega) \cap C^{1,\mu}(\overline{\Omega}) \) is a classical solution to (1.1)\(_{\lambda_1}\).

Moreover, \( \tilde{u}_{\lambda_1} > 0 \). Assume on the contrary that \( \tilde{u}_{\lambda_1} = 0 \). By the Crandall-Rabinowitz’s Theorem [10], the only nontrivial solutions to (1.1) in a neighbourhood of the bifurcation point \( (\lambda_1, 0) \) are given by \((4.1)\). Since Proposition 5.1, those are the minimal solutions \( u_\lambda \), and due to \( \tilde{u}_\lambda \neq u_\lambda \), \( \tilde{u}_\lambda \) are not in a neighbourhood of \( (\lambda_1, 0) \), contradicting the definition of \( \tilde{u}_{\lambda_1}(x), (5.7) \)

Hence, \( \tilde{u}_{\lambda_1} \geq 0 \), and reasoning as in (1.8), the Maximum Principle implies that \( \tilde{u}_{\lambda_1} > 0 \).

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Positive Solutions for Slightly Subcritical

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### A. Some Estimates

First, we prove an useful estimate of \( \frac{\ln(e+s)}{\ln(e+as)} \).

**Lemma A.1.** Let \( 0 < a \leq 1 \) be fixed. Then for all \( s \geq 0 \),

\[
\frac{\ln(e+s)}{\ln(e+as)} \leq \ln \left( \frac{e}{a} \right) \leq \frac{1}{a}.
\]  

**(A.1)**

**Proof.** Denote \( \ell(s) = \frac{\ln(e+s)}{\ln(e+as)} \) for all \( s \geq 0 \). Then \( 1 \leq \ell(s) \leq \ell(s_0) \) where \( s_0 > 0 \) is the unique value where \( \ell'(s) = 0 \). When computing \( s_0 \) we find

\[
\ell'(s_0) = 0 \iff (e+as_0) \ln(e+as_0) - a(e+s_0) \ln(e+s_0) = 0
\]

and therefore

\[
\max \ell = \ell(s_0) = \frac{\ln(e+s_0)}{\ln(e+as_0)} = \frac{e+as_0}{a(e+s_0)}.
\]

Notice that we have \( \ell(s_0) \leq \frac{1}{a} \). In order to find a better upper bound of \( \ln(\frac{e+as_0}{e+s_0}) \) let us denote for all \( s \geq 0 \)

\[
\theta(s) = (e+as) \ln(e+as) - a(e+s) \ln(e+s).
\]

Then, there exists \( \chi \in (0, s_0) \) such that

\[
0 - e(1-a) = \theta(s_0) - \theta(0) = \theta'(\chi)s_0 \implies \frac{e(1-a)}{s_0} = -\theta'(\chi).
\]

Then

\[
-\theta'(s) = a \ln \left( \frac{e+s}{e+as} \right) \leq a \ln \left( \frac{1}{a} \right),
\]

and

\[
\frac{e(1-a)}{s_0} \leq a \ln \left( \frac{1}{a} \right) \implies s_0 \geq \frac{e(1-a)}{a \ln \left( \frac{1}{a} \right)}.
\]
Since \( \frac{e + as}{a(e + s)} \) is decreasing,

\[
\max_{s \geq 0} \ell(s) = \ell(s_0) = \frac{e + as_0}{a(e + s_0)} \leq \frac{e + \frac{e(1-a)}{\ln\left(\frac{e}{2}\right)}}{ae + \frac{e(1-a)}{\ln\left(\frac{e}{2}\right)}}
\]

\[
= \frac{\ln(1/a) + 1 - a}{a \ln(1/a) + 1 - a} \leq \ln(1/a) + 1,
\]

and the first inequality of (A.1) is achieved. The second one is obvious. \( \square \)

Next lemma is about the variations of \( h(s) = s^2 - 1 - \left( \ln\left( e + s \right) \right)^{\frac{\alpha}{2}} \) for \( s \geq 0 \).

**Lemma A.2.** There exists \( \alpha^* > 2(2^* - 1) \) such that \( h \) is an increasing function on \( ]0, +\infty[ \) if and only if \( \alpha \leq \alpha^* \). Moreover, if \( \alpha > \alpha^* \) there exists \( s_1 < s_2 \) such that \( h \) is increasing in \( ]0, +\infty[ \setminus s_1, s_2 \).

**Proof.** We have

\[
h'(s) = \frac{s^{2^* - 2}}{[\ln(e + s)]^{\alpha + 1}} \left( (2^* - 1) \ln(e + s) - \frac{\alpha s}{s + e}, \right)
\]

Let us define for \( s \geq 0 \),

\[
\theta(s) := \ln(e + s) - \frac{\alpha s}{2^* - 1 s + e},
\]

so

\[
h'(s) \geq 0 \iff \theta(s) \geq 0.
\]

We have:

\[
\begin{cases}
\theta(0) = 1, \\
\theta(s) \to +\infty \quad \text{as} \quad s \to +\infty, \\
\theta'(s) = \frac{s + e(1 - \frac{\alpha}{2^* - 1})}{(e + s)^2}.
\end{cases}
\]

Hence:

1. If \( \frac{\alpha}{2^* - 1} \leq 1 \) then \( \theta'(s) \geq 0 \) for all \( s \geq 0 \) and in particular \( \theta(s) \geq 0 \) and therefore \( h'(s) \geq 0 \) for all \( s \geq 0 \);
2. If \( \frac{\alpha}{2^* - 1} > 1 \) then

\[
\theta'(s_0) = 0 \quad \text{for} \quad s_0 = e \left( \frac{\alpha}{2^* - 1} - 1 \right).
\]

Let us compute \( \theta(s_0) \):

\[
\theta(s_0) = \ln\left( \frac{\alpha}{2^* - 1} \right) - \frac{\alpha}{2^* - 1} + 2,
\]

and hence:

1. If \( \theta(s_0) \geq 0 \) then \( \theta(s) \geq 0 \) for all \( s \geq 0 \) and therefore \( h'(s) \geq 0 \) for all \( s \geq 0 \);
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(ii) if \( \theta(s_0) < 0 \) then there exists \( s_1 < s_2 \) such that

\[
\theta(s) > 0 \quad \forall s \in [0, +\infty[, s_1, s_2] \implies h'(s) > 0 \quad \forall s \in [0, +\infty[, s_1, s_2].
\]

Notice that \( t \to \ln t \) is greater than \( t \to t - 2 \) somewhere between some \( t_1 < 1 \) and the value \( t^* \) = the unique solution > 2 of the equation

\[
\ln t^* = t^* - 2.
\]

Finally the statement of the lemma holds for \( \alpha^* = t^*(2^* - 1) \). □

B. A Compact Embedding Using Orlicz Spaces

For references on Orlicz spaces see [15, 21]. Throughout \( \Omega \subset \mathbb{R}^N \) is an bounded open set. We will denote

\[
\mathcal{L}(\Omega) = \{ \varphi : \Omega \to \mathbb{R} : \varphi \text{ is Lebesgue measurable} \}.
\]

**Definition B.1.** We will say that a function \( M : [0, +\infty[ \to [0, +\infty[ \) is a N-function if and only if

(N1) \( M \) is convex, increasing and continuous,

(N2) \( \lim_{s \to 0^+} \frac{M(s)}{s} = 0 \),

(N3) \( \lim_{s \to +\infty} \frac{M(s)}{s} = +\infty \).

The proof of the following property is trivial, we just quoted it for the sake of completeness.

**Proposition B.2.** Any N-function \( M \) admits a representation of the form

\[
M(s) = \int_0^s m(t)dt
\]

where \( m : [0, +\infty[ \to [0, +\infty[ \) is a non-decreasing right-continuous function satisfying \( m(0) = 0 \) and

\[
\lim_{s \to +\infty} m(s) = +\infty.
\]

Thus, \( m \) is the right-derivative of \( M \).

Our first aim is to prove the following result:

**Theorem B.3.** Let \( M : [0, +\infty[ \to \mathbb{R} \) be a N-function such that

\[
\lim_{s \to +\infty} \frac{s^{2^*}}{M(s)} = +\infty.
\]

Assume also that \( M \) satisfies the \( \Delta_2 \)-condition, that is,

\[
\exists K > 0, \quad \forall s \in [0, +\infty[, \quad M(2s) \leq KM(s). \quad (B.1)
\]

Let \( \{ u_n \}_{n \in \mathbb{N}} \) in \( H^1_0(\Omega) \) be a sequence satisfying

1. \( \sup_{n \in \mathbb{N}} \| u_n \|_{2^*} < \infty \),
2. there exists \( u \in H^1_0(\Omega) \) such that \( \lim_{n \to +\infty} u_n(x) = u(x) \) a.e.
Then there exists a subsequence \( \{ u_{n_k} \}_{k \in \mathbb{N}} \) such that
\[
\lim_{k \to \infty} \int_{\Omega} M\left( |u_{n_k}(x) - u(x)| \right) dx = 0. \tag{B.2}
\]

In order to prove this theorem we need some definitions.

**Definition B.4.** Let \( K \subset L(\Omega) \). We say that \( K \) has equi-absolutely continuous integrals if and only if \( \forall \varepsilon > 0 \) there exists \( h > 0 \) such that
\[
\forall \varphi \in K, \forall A \subset \Omega \text{ mesurable}, \ |A| < h \implies \int_A |\varphi(x)| dx < \varepsilon.
\]

**Lemma B.5.** Let \( M : [0, +\infty[ \to \mathbb{R} \) be a \( N \)-function satisfying the \( \Delta_2 \) condition (B.1). Let \( \{ u_n \}_{n \in \mathbb{N}} \) be a sequence of measurable functions converging a.e. to some function \( u \) and such that the set
\[
\left\{ M(|u_n|) : n \in \mathbb{N} \right\}
\]
has equi-absolutely continuous integrals. Then (B.2) holds.

**Proof.** Let fix \( \varepsilon > 0 \) and let \( \delta > 0 \) be such that
\[
\forall n \in \mathbb{N}, \forall A \subset \Omega \text{ mesurable}, \ |A| < \delta \implies \int_A M(|u_n|) dx \leq \varepsilon.
\]

Using Fatou’s lemma we infer that also
\[
\forall A \subset \Omega \text{ mesurable}, \ |A| < \delta \implies \int_A M(|u|) dx \leq \varepsilon.
\]

Let \( \Omega_n = \{ x \in \Omega : |u_n(x) - u(x)| > M^{-1}(\varepsilon) \} \). As a consequence of Egoroff’s theorem, the sequence \( (u_n)_{n \in \mathbb{N}} \) converge in measure to \( u \) so there exists \( n_0 \in \mathbb{N} \) such that
\[
|\Omega_n| < \delta.
\]

Then, using the convexity of \( M \) and (B.1) it comes
\[
\int_{\Omega} M(|u_n - u|) dx = \int_{\Omega_n} M(|u_n - u|) dx + \int_{\Omega \setminus \Omega_n} M(|u_n - u|) dx
\]
\[
\leq \frac{1}{2} \left( \int_{\Omega_n} (M(2|u_n|) + M(2|u|)) dx \right) + |\Omega| M(M^{-1}(\varepsilon))
\]
\[
\leq \frac{K}{2} \left( \int_{\Omega_n} (M(|u_n|) + M(|u|)) dx \right) + |\Omega|\varepsilon \leq (K + |\Omega|)\varepsilon.
\]

\( \square \)

In order to prove that, for the sequence of our theorem, the set
\[
\left\{ M(|u_n|) : n \in \mathbb{N} \right\}
\]
has equi-absolutely continuous integrals we are going to use the following lemma:
Lemma B.6. Let $\mathcal{K} \subset \mathcal{L}(\Omega)$ and let $\Phi : [0, +\infty] \rightarrow [0, +\infty]$ be an increasing function satisfying

$$\lim_{s \to +\infty} \frac{\Phi(s)}{s} = +\infty. \quad (B.3)$$

Suppose that there exists $D > 0$ such that

$$\sup_{u \in \mathcal{K}} \int_{\Omega} \Phi(|u|) \, dx \leq D. \quad (B.4)$$

Then all the functions $u \in \mathcal{K}$ are integrable and $\mathcal{K}$ has equi-absolutely continuous integrals (Valle Poussin’s theorem). Moreover, if $M : [0, +\infty] \rightarrow [0, +\infty]$ is a continuous increasing function satisfying

$$\lim_{s \to +\infty} \frac{M(s)}{s} = +\infty \quad \text{and} \quad \lim_{s \to +\infty} \frac{\Phi(s)}{M(s)} = +\infty,$$

then the family $\mathcal{K}_1 = \{M(|u|) : u \in \mathcal{K}\}$ has equi-absolutely continuous integrals.

Proof. For the Valle Poussin’s theorem see [18] page 159. To prove the second statement remark that the function $\tilde{\Phi} = \Phi \circ M^{-1}$ satisfies (B.3). Here $M^{-1}$ stand for the right-hand inverse. \hfill \Box

Proof of theorem B.3. Let us take $\Phi(s) = |s|^2$. From hypothesis (1) of the theorem, the set $\mathcal{K} = \{u_n : n \in \mathbb{N}\}$ satisfies (B.4) for some $D > 0$. Then the conclusion follows from Lemma B.5 and Lemma B.6. \hfill \Box

Remark B.7. Whenever (B.2) is satisfied we say that the sequence $\{u_{nk}\}_{k \in \mathbb{N}}$ converges in $M$-mean to $u$.

One can formulate Theorem B.3 as a compact embedding of $H^1_0(\Omega)$ in some vector space endowed of the Luxembourg norm associate to $M$ (see [15, 21]). Instead, we are going to use the Orlicz-norm which is more suitable to our purposes. We will see later in Theorem B.12 that the convergence in $M$-mean implies the convergence with respect to the Orlicz-norm, provided that the $\Delta_2$-condition is satisfied.

Definition B.8. Let $M$ be a $N$-function. The complementary of $M$ defined for all $s \geq 0$ is the function

$$M^*(s) := \max \{st - M(t) : t \geq 0\}.$$  

As before, we give the following trivial result for the sake of completeness:

Proposition B.9. If $m$ is the right derivative of $M$ then

$$m^*(s) = \sup \{t : m(t) \leq s\}$$

is the right derivative of $M^*$ and $M^*$ is a $N$-function. Furthermore, for all $s \geq 0$ we have

$$sm(s) = M(s) + M^*(m(s)), \quad sm^*(s) = M(m^*(s)) + M^*(s).$$

Next, let us introduce the Orlicz norm associated to $M$:
Definition B.10. Let $M$ be a $N$-function and let $M^*$ be its complementary. Let us denote for any $v \in \mathcal{L}(\Omega)$

$$\rho(v, M^*) = \int_{\Omega} M^*(|v|) \, dx$$

and define the Orlicz norm of any $u \in \mathcal{L}(\Omega)$ by

$$\|u\|_M := \sup \left\{ \int_{\Omega} uv \, dx : v \in \mathcal{L}(\Omega), \rho(v, M^*) \leq 1 \right\}.$$ 

$\| \cdot \|_M$ is a norm in the real vector space $L_M(\Omega) = \{ u \in \mathcal{L}(\Omega) : \|u\|_M < +\infty \}$.

(see [15] for the details). Let us prove the following less trivial properties:

Proposition B.11. (i) For all $u \in \mathcal{L}(\Omega)$,

$$\|u\|_M \leq \int_{\Omega} M(|u|) \, dx + 1. \quad \text{(B.5)}$$

(ii) For any $u$ and $v$ in $\mathcal{L}(\Omega)$ it holds

$$\left| \int_{\Omega} uv \, dx \right| \leq \|u\|_M \|v\|_{M^*} \quad \text{ (Holder’s inequality).} \quad \text{(B.6)}$$

(iii) For any $u$ and $v$ in $\mathcal{L}(\Omega)$ we have $\|u\|_M \leq \|v\|_M$ if $|u| \leq |v|$ a.e.

Proof. (i) This follows from the definition of $\| \cdot \|_M$ and the inequality $|uv| \leq M(|u|) + M^*(|v|)$.

(ii) The divide the proof in 3 steps.

Step 1: For all $v \in \mathcal{L}(\Omega)$,

$$\left| \int_{\Omega} uv \, dx \right| \leq \left\{ \begin{array}{ll} \|u\|_M \|v\|_{M^*} & \text{if } \rho(v, M^*) \leq 1 \\ \rho(v, M^*) \|u\|_M & \text{if } \rho(v, M^*) > 1 \end{array} \right.$$ 

Indeed, the first case follows directly from the definition. If $\rho(v, M^*) > 1$ then by convexity

$$M^*(\frac{|v|}{\rho(v, M^*)}) \leq \frac{M^*(|v|)}{\rho(v, M^*)}$$

and therefore

$$\rho\left( \frac{|v|}{\rho(v, M^*)}, M^* \right) \leq \frac{1}{\rho(v, M^*)} \int_{\Omega} M^*(|v|) \, dx = 1$$

and

$$\left| \int_{\Omega} u \frac{v}{\rho(v, M^*)} \, dx \right| \leq \|u\|_M.$$ 

Step 2: If $\|u\|_M \leq 1$ then $\rho(m(|u|), M^*) \leq 1$.

Set $u_n = u \chi_{\{|u| \leq n\}}$ for all $n \in \mathbb{N}$. Since $u_n$ is bounded then $\rho(m(|u_n|), M^*) < +\infty$. Assume by contradiction that $\int_{\Omega} M^*(m(|u|)) \, dx > 1$ and let $n_0 \in \mathbb{N}$ be such that $\int_{\Omega} M^*(m(|u_{n_0}|)) \, dx > 1$. We have

$$M^*(m(|u_{n_0}|)) < M(|u_{n_0}|) + M^*(m(|u_{n_0}|)) = |u_{n_0}|m(|u_{n_0}|)$$
and therefore, by (i),
\[ \rho(m(|u_{n_0}|), M^*) < \int_{\Omega} |u_{n_0}| m(|u_{n_0}|) \, dx \leq \|u_{n_0}\|_M \rho(m(|u_{n_0}|), M^*) \]
which contradicts \( \|u_{n_0}\|_M \leq \|u\|_M \leq 1 \).

This is trivial from the definition of \( \|u\|_M \), step 1 and the fact that \( |u|m(|u|) = M(|u|) + M^*(m(|u|)) \).

Step 3: If \( \|u\|_M \leq 1 \) then \( \rho(u, M) \leq \|u\|_M \).

Let us remark that for all \( s \geq 0 \)
\[ M^*(m(s)) + M(s) = sm(s). \]

Set \( v_0 = m(|u|) \). From step 2, \( \rho(v_0, M^*) \leq 1 \) and then
\[ \rho(u, M) \leq \rho(u, M) + \rho(v_0, M^*) = \int_{\Omega} uv_0 \, dx \leq \|u\|_M. \]

Now we prove Holder’s inequality. From step 2 applied to \( M^* \) and \( \frac{v}{\|v\|_{M^*}} \) we have \( \rho\left(\frac{v}{\|v\|_{M^*}}, M^*\right) \leq 1 \), so then
\[ \left| \int_{\Omega} \frac{uv}{\|v\|_{M^*}} \, dx \right| \leq \|u\|_M \]
and Holder’s inequality follows.

The proof of (iii) is trivial.

Finally, we give the following compact embedding result:

**Theorem B.12.** Let \( M \) be a \( N \)-function satisfying the \( \Delta_2 \)-condition (B.1) and let \( \{u_n\}_{n \in \mathbb{N}} \) be a sequence in \( \mathcal{L}^1(\Omega) \) such that
\[ \lim_{n \to \infty} \rho(u_n, M) = 0. \]

Then
\[ \lim_{n \to \infty} \|u_n\|_M = 0. \]

Thus, the convergence in \( M \)-mean implies the converge with respect to the \( \| \cdot \|_M \)

**Proof.** Let \( \epsilon > 0 \) and take \( m \in \mathbb{N} \) such that \( \frac{1}{2^{m-1}} < \epsilon \). Using condition (B.1) we also have
\[ \lim_{n \to \infty} \int_{\Omega} M(2^m|u_n|) \, dx = 0. \]
Let \( n_0 \in \mathbb{N} \) be such that for all \( n \geq n_0 \) we have
\[ \int_{\Omega} M(2^m|u_n|) \, dx < 1. \]

From step 1 of the proof in the previous proposition we have that for all \( n \geq n_0 \)
\[ \|2^m u_n\|_M \leq \rho(2^m |u_n|, M) + 1 < 2, \]
which implies that
\[ \|u_n\|_M < \frac{1}{2^{m-1}} < \epsilon. \]
\[ \square \]
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