Relative Morita equivalence of Cuntz–Krieger algebras and flow equivalence of topological Markov shifts

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Abstract

In this paper, we will introduce notions of relative version of imprimitivity bimodules and relative version of strong Morita equivalence for pairs of $C^*$-algebras $(A, D)$ such that $D$ is a $C^*$-subalgebra of $A$ with certain conditions. We will then prove that two pairs $(A_1, D_1)$ and $(A_2, D_2)$ are relatively Morita equivalent if and only if their relative stabilizations are isomorphic. In particular, for two pairs $(O_A, D_A)$ and $(O_B, D_B)$ of Cuntz–Krieger algebras with their canonical masas, they are relatively Morita equivalent if and only if their underlying two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent. We also introduce a relative version of the Picard group Pic$(A, D)$ for the pair $(A, D)$ of $C^*$-algebras and study them for the Cuntz–Krieger pair $(O_A, D_A)$.

Contents

1. Introduction
2. Relative $\sigma$-unital $C^*$-algebras.
3. Relative imprimitivity bimodules and relative Morita equivalence.
4. Isomorphism of relative stabilizations.
5. Relative full corners.
6. Relative Morita equivalence in Cuntz–Krieger pairs.
7. Corner isomorphisms in Cuntz–Krieger pairs.
8. Relative Picard groups.
9. Relative Picard groups of Cuntz–Krieger pairs.
10. Appendix: Picard groups of Cuntz–Krieger algebras.

1 Introduction

In [28], M. Rieffel introduced the notion of imprimitivity bimodule for $C^*$-algebras as a Hilbert $C^*$-bimodule satisfying certain conditions from a viewpoint of representation theory of groups, so that he defined the notion of strong Morita equivalence in $C^*$-algebras. Let $A$ and $B$ be $C^*$-algebras. An $A$-$B$-bimodule $X$ means a Hilbert $C^*$-bimodule with
a left \( \mathcal{A} \)-module structure and an \( \mathcal{A} \)-valued inner product \( \langle \cdot | \cdot \rangle_\mathcal{A} \) and with a right \( \mathcal{B} \)-module structure and a \( \mathcal{B} \)-valued inner product \( \langle \cdot | \cdot \rangle_\mathcal{B} \) satisfying some comparability conditions (see [24], [28], [10], [25], etc.). It is said to be full if the ideals spanned by \( \langle \mathcal{A}(x | y) | x, y \in X \rangle \) and \( \langle x | y \rangle_\mathcal{B} | x, y \in X \rangle \) are dense in \( \mathcal{A} \) and in \( \mathcal{B} \), respectively. If a full \( \mathcal{A}-\mathcal{B} \)-bimodule \( X \) further satisfies the condition

\[
\mathcal{A}(x | y)z = x(y | z)_\mathcal{B} \quad \text{for } x, y, z \in X,
\]

it is called an \( \mathcal{A}-\mathcal{B} \)-imprimitivity bimodule. Two \( \mathcal{C}^* \)-algebras \( \mathcal{A} \) and \( \mathcal{B} \) are said to be strong Morita equivalent if there exists an \( \mathcal{A}-\mathcal{B} \)-imprimitivity bimodule, which means that \( \mathcal{A} \) and \( \mathcal{B} \) have same representation theory. Brown–Green–Rieffel in [3] have shown that two \( \sigma \)-unital \( \mathcal{C}^* \)-algebras \( \mathcal{A} \) and \( \mathcal{B} \) are strong Morita equivalent if and only if they are stably isomorphic, that is \( \mathcal{A} \otimes \mathcal{K} \) is isomorphic to \( \mathcal{B} \otimes \mathcal{K} \), where \( \mathcal{K} \) denotes the \( \mathcal{C}^* \)-algebra of compact operators on a separable infinite dimensional Hilbert space.

In this paper, we will study Morita equivalence of \( \mathcal{C}^* \)-algebras from a view point of symbolic dynamical systems. For an irreducible equivalence non-permutation matrix \( A = [A(i, j)]_{i,j=1}^N \) with entries in \{0, 1\}, two-sided topological Markov shift \((\bar{X}_A, \bar{\sigma}_A)\) are defined as a topological dynamical system on the shift space \( \bar{X}_A \) consisting of two-sided sequences \((x_n)_{n \in \mathbb{Z}}\) of \( x_n \in \{1, \ldots, N\} \) such that \( A(x_n, x_{n+1}) = 1 \) for all \( n \in \mathbb{Z} \) with the shift homeomorphism \( \bar{\sigma}_A((x_n)_{n \in \mathbb{Z}}) = (x_{n+1})_{n \in \mathbb{Z}} \) on the compact Hausdorff space \( \bar{X}_A \). J. Cuntz and W. Krieger introduced a \( \mathcal{C}^* \)-algebra \( \mathcal{O}_A \) associated to the matrix \( A \) ([7]). The \( \mathcal{C}^* \)-algebra is called the Cuntz–Krieger algebra, which is a universal unique \( \mathcal{C}^* \)-algebra generated by partial isometries \( S_1, \ldots, S_N \) subject to the relations:

\[
\sum_{j=1}^N S_j S_j^* = 1, \quad S_i^* S_i = \sum_{j=1}^N A(i, j) S_j S_j^*, \quad i = 1, \ldots, N. \quad (1.1)
\]

Since the stable isomorphism class of \( \mathcal{O}_A \) does not have complete informations about the underlying dynamical system \((\bar{X}_A, \bar{\sigma}_A)\), we need some extra structure to \( \mathcal{O}_A \) to study \((\bar{X}_A, \bar{\sigma}_A)\). In this paper, we consider the pair \((\mathcal{O}_A, \mathcal{D}_A)\) where \( \mathcal{D}_A \) is the \( \mathcal{C}^* \)-subalgebra of \( \mathcal{O}_A \) generated by the projections of the form: \( S_{i_1} \cdots S_{i_n} S_{i_n}^* \cdots S_{i_1}^* i_1, \ldots, i_n = 1, \ldots, N \). We call the pair \((\mathcal{O}_A, \mathcal{D}_A)\) the Cuntz–Krieger pair. As in [17], the isomorphism class of the pair \((\mathcal{O}_A, \mathcal{D}_A)\) is a complete invariant of the continuous orbit equivalence class of the underlying one-sided topological Markov shift \((X_A, \sigma_A)\). As one of the remarkable relationships between symbolic dynamics and Cuntz–Krieger algebras, Cuntz–Krieger showed in [7] that if topological Markov shifts \((\bar{X}_A, \bar{\sigma}_A)\) and \((\bar{X}_B, \bar{\sigma}_B)\) are flow equivalent, then there exists an isomorphism \( \Phi : \mathcal{O}_A \otimes \mathcal{K} \longrightarrow \mathcal{O}_B \otimes \mathcal{K} \) such that \( \Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C} \), where \( \mathcal{C} \) denotes the maximal commutative \( \mathcal{C}^* \)-subalgebra of \( \mathcal{K} \) consisting of the diagonal elements. Recently H. Matui and the author have proved that the converse implication also holds, so that \((\bar{X}_A, \bar{\sigma}_A)\) and \((\bar{X}_B, \bar{\sigma}_B)\) are flow equivalent if and only if there exists an isomorphism \( \Phi : \mathcal{O}_A \otimes \mathcal{K} \longrightarrow \mathcal{O}_B \otimes \mathcal{K} \) such that \( \Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C} \) ([21]). We call the pair \((\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C})\) the stabilized Cuntz–Krieger pair or the relative stabilization of \((\mathcal{O}_A, \mathcal{D}_A)\), so that the isomorphism class of the relative stabilization of \((\mathcal{O}_A, \mathcal{D}_A)\) is a complete invariant for the flow equivalence class of the underlying two-sided topological Markov shift \((\bar{X}_A, \bar{\sigma}_A)\).

In this paper, we will introduce notions of relative version of imprimitivity bimodules and of relative version of strong Morita equivalence for pairs of \( \mathcal{C}^* \)-algebras \((\mathcal{A}, \mathcal{D})\) such
that $D$ is a $C^*$-subalgebra of $A$ for which $D$ has an orthogonal countable approximate unit for $A$. Such a pair is said to be relative $\sigma$-unital. If $D$ contains the unit of $A$, the pair is relative $\sigma$-unital. Relative version of strong Morita equivalence is called the relative Morita equivalence. We will first show the following theorem for relative $\sigma$-unital pair $(A, D)$ of $C^*$-algebras:

**Theorem 1.1** (Lemma 3.9, Theorem 4.7 and Theorem 5.5). Let $(A_1, D_1)$ and $(A_2, D_2)$ be relative $\sigma$-unital pairs of $C^*$-algebras. Then the following assertions are mutually equivalent:

1. $(A_1, D_1)$ and $(A_2, D_2)$ are relatively Morita equivalent.
2. $(A_1 \otimes K, D_1 \otimes C)$ and $(A_2 \otimes K, D_2 \otimes C)$ are relatively Morita equivalent.
3. There exists an isomorphism $\Phi : A_1 \otimes K \to A_2 \otimes K$ of $C^*$-algebras such that $\Phi(D_1 \otimes C) = D_2 \otimes C$.
4. $(A_1, D_1)$ and $(A_2, D_2)$ are complementary relative full corners.

We will second apply the above theorem to the Cuntz–Krieger pair $(O_A, D_A)$ and clarify relationships between relative Morita equivalence and flow equivalence of underlying topological dynamical systems.

**Theorem 1.2** (Theorem 6.3 and Theorem 7.4, cf. [21, Corollary 3.8]). Let $A, B$ be irreducible non-permutation matrices with entries in $\{0, 1\}$. Let $(O_A, D_A), (O_B, D_B)$ be the associated Cuntz–Krieger pairs. Then the following assertions are mutually equivalent:

1. $(O_A, D_A)$ and $(O_B, D_B)$ are relatively Morita equivalent.
2. $(O_A \otimes K, D_A \otimes C)$ and $(O_B \otimes K, D_B \otimes C)$ are relatively Morita equivalent.
3. There exists an isomorphism $\Phi : O_A \otimes K \to O_B \otimes K$ of $C^*$-algebras such that $\Phi(D_A \otimes C) = D_B \otimes C$.
4. $(O_A, D_A)$ and $(O_B, D_B)$ are corner isomorphic.
5. The two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent.

By using J. Franks’ s Theorem [8] (cf. [11, 23]), the last assertion (5) is equivalent to the following (6):

6. The groups $\mathbb{Z}^N/(\text{id} - A)\mathbb{Z}^N$ and $\mathbb{Z}^M/(\text{id} - B)\mathbb{Z}^M$ are isomorphic and $\det(\text{id} - A) = \det(\text{id} - B)$,

where $N$ is the size of the matrix $A$ and $M$ is that of $B$. Hence we know that the group $\mathbb{Z}^N/(\text{id} - A)\mathbb{Z}^N$ with the value $\det(\text{id} - A)$ is a complete invariant of the relative Morita equivalence class of the Cuntz–Krieger pair $(O_A, D_A)$.

In [3], Brown–Green–Rieffel introduced the notion of the Picard group $\text{Pic}(A)$ for a $C^*$-algebra to study equivalence classes of imprimitivity bimodules of $C^*$-algebras. Natural isomorphism classes $[X]$ of imprimitivity bimodules $X$ over $A$ form a group under the relative tensor product $[X] \cdot [Y] = [X \otimes_A Y]$. The group is called the Picard group for the $C^*$-algebra $A$ and is written $\text{Pic}(A)$, that are considered as a sort of generalizations of
automorphism group $\text{Aut}(\mathcal{A})$ of $\mathcal{A}$. We will introduce relative version of the Picard group $\text{Pic}(\mathcal{A}, \mathcal{D})$ as the group of $(\mathcal{A}, \mathcal{D})$-relative imprimitivity bimodules and study their structure for the Cuntz–Krieger pairs $(\mathcal{O}_\mathcal{A}, \mathcal{D}_\mathcal{A})$. Let

$$\text{Aut}_o(\mathcal{O}_\mathcal{A}, \mathcal{D}_\mathcal{A}) = \{ \alpha \in \text{Aut}(\mathcal{O}_\mathcal{A}) \mid \alpha(\mathcal{D}_\mathcal{A}) = \mathcal{D}_\mathcal{A}, \alpha_* = \text{id on } K_0(\mathcal{O}_\mathcal{A}) \}.$$

Its quotient group $\text{Aut}_o(\mathcal{O}_\mathcal{A}, \mathcal{D}_\mathcal{A}) / \text{Int}(\mathcal{O}_\mathcal{A}, \mathcal{D}_\mathcal{A})$ by $\text{Int}(\mathcal{O}_\mathcal{A}, \mathcal{D}_\mathcal{A})$ is denoted by $\text{Out}_o(\mathcal{O}_\mathcal{A}, \mathcal{D}_\mathcal{A})$. Let $\text{Aut}_1(\mathbb{Z}^N/(\text{id} - A^t)\mathbb{Z}^N)$ be a subgroup of $\text{Aut}(\mathbb{Z}^N/(\text{id} - A^t)\mathbb{Z}^N)$ defined by

$$\text{Aut}_1(\mathbb{Z}^N/(\text{id} - A^t)\mathbb{Z}^N) = \{ \xi \in \text{Aut}(\mathbb{Z}^N/(\text{id} - A^t)\mathbb{Z}^N) \mid \xi([1]) = [1] \}$$

where $[1] \in \mathbb{Z}^N/(\text{id} - A^t)\mathbb{Z}^N$ denotes the class of the vector $(1, \ldots, 1)$ in $\mathbb{Z}^N$. It is well-known that there exists an isomorphism $\epsilon_\mathcal{A} : K_0(\mathcal{O}_\mathcal{A}) \to \mathbb{Z}^N/(\text{id} - A^t)\mathbb{Z}^N$ such that $\epsilon([1_{\mathcal{O}_\mathcal{A}}]) = [1]$ (16). We will obtain the following structure theorem for $\text{Pic}(\mathcal{O}_\mathcal{A}, \mathcal{D}_\mathcal{A})$.

**Theorem 1.3** (Theorem 9.3 and Theorem 10.5). Let $\mathcal{A}$ be an irreducible non-permutation matrix. Then there exist short exact sequences:

$$1 \to \text{Out}_o(\mathcal{O}_\mathcal{A}, \mathcal{D}_\mathcal{A}) \xrightarrow{\Psi} \text{Pic}(\mathcal{O}_\mathcal{A}, \mathcal{D}_\mathcal{A}) \xrightarrow{K} \text{Aut}(\mathbb{Z}^N/(\text{id} - A^t)\mathbb{Z}^N) \to 1,$$

$$1 \to \text{Out}(\mathcal{O}_\mathcal{A}, \mathcal{D}_\mathcal{A}) \xrightarrow{\Psi} \text{Pic}(\mathcal{O}_\mathcal{A}, \mathcal{D}_\mathcal{A}) \xrightarrow{K} \text{Aut}(\mathbb{Z}^N/(\text{id} - A^t)\mathbb{Z}^N)/\text{Aut}_1(\mathbb{Z}^N/(\text{id} - A^t)\mathbb{Z}^N) \to 1.$$

In Appendix of the paper, we refer to the ordinary Picard groups $\text{Pic}(\mathcal{O}_\mathcal{A})$ for Cuntz–Krieger algebras $\mathcal{O}_\mathcal{A}$, and especially the ordinary Picard groups $\text{Pic}(\mathcal{O}_N)$ for Cuntz algebras $\mathcal{O}_N$ (Theorem 10.4 and Corollary 10.5).

## 2 Relative $\sigma$-unital $C^*$-algebras

For a $C^*$-algebra $\mathcal{A}$, we denote by $M(\mathcal{A})$ its multiplier $C^*$-algebra (cf. [32]). The locally convex topology on $M(\mathcal{A})$ generated by the seminorms $x \to \|xa\|$, $x \to \|ax\|$ for $a \in \mathcal{A}$ is called the strict topology. Throughout the paper, we denote by $\{e_{ij}\}_{i,j \in \mathbb{N}}$ the matrix units on the separable infinite dimensional Hilbert space $\ell^2(\mathbb{N})$. The $C^*$-algebra generated by them is denoted by $\mathcal{K}$ which is the $C^*$-algebra of all compact operators on $\ell^2(\mathbb{N})$. The $C^*$-subalgebra of $\mathcal{K}$ generated by diagonal projections $\{e_{ii}\}_{i \in \mathbb{N}}$ is denoted by $\mathcal{C}$.

A $C^*$-algebra is said to be $\sigma$-unital if it has a countable approximate unit. We will first introduce a notion of relative version of a $\sigma$-unital $C^*$-algebra.

**Definition 2.1.** A pair $(\mathcal{A}, \mathcal{D})$ of $C^*$-algebras $\mathcal{A}, \mathcal{D}$ is called relative $\sigma$-unital if it satisfies the following conditions:

1. $\mathcal{D}$ is a $C^*$-subalgebra of $\mathcal{A}$.
2. $\mathcal{D}$ contains a countable approximate unit for $\mathcal{A}$.
3. There exists a sequence $a_n \in \mathcal{A}, n = 1, 2, \ldots$ of elements such that
   
   (a) $a_n^*a_n$, $a_n^*a_n^* \in \mathcal{D}$ for all $d \in \mathcal{D}$ and $n = 1, 2, \ldots$.
   
   (b) $\sum_{n=1}^\infty a_n^*a_n = 1$ in the strict topology of $M(\mathcal{A})$.
   
   (c) $a_n^*a_m = 0$ for all $d \in \mathcal{D}$ and $n, m \in \mathbb{N}$ with $n \neq m$. 

4.
We call the sequence \( \{a_n\}_{n \in \mathbb{N}} \) satisfying the three conditions (a), (b), (c) a relative approximate unit for the pair \((A, D)\).

**Remark 2.2.** By the above condition (2), we know that \(M(D)\) is a \(C^*\)-subalgebra of \(M(A)\) in natural way (cf. [32, p 46, 2G]).

**Lemma 2.3.** Assume that \((A, D)\) is a relative \(\sigma\)-unitar of \(C^*\)-algebras. Let \(\{a_n\}_{n \in \mathbb{N}}\) be a relative approximate unit for \((A, D)\). Then we have

(i) \(a_n^*a_n, a_na_n^* \in D\) for all \(n = 1, 2, \ldots\).

(ii) \(b_n = \sum_{k=1}^n a_k^*a_k\) belongs to \(D\) and the sequence \(\{b_n\}_{n \in \mathbb{N}}\) is a countable approximate unit for \(A\).

**Proof.** (i) Take and fix \(k \in \mathbb{N}\). Since \(\sum_{n=1}^\infty a_n^*a_n = 1\) in \(M(A)\), we have \(0 \leq a_k^*a_k \leq 1\) so that \(\|a_k\| \leq 1\). As \(D\) has an approximate unit for \(A\), for any \(\epsilon > 0\), there exists \(d \in D\) such that \(\|a_k - da_k\| < \epsilon\), so that \(\|a_k^*a_k - a_k^*da_k\| < \epsilon\). The condition \(a_k^*da_k \in D\) ensures us that \(a_k^*a_k\) belongs to \(D\). Similarly we know that \(a_k^*a_k\) belongs to \(D\).

(ii) Since \(b_n = \sum_{k=1}^n a_k^*a_k\) converges to 1 in the strict topology of \(M(A)\), \(\{b_n\}_{n \in \mathbb{N}}\) is an approximate unit for \(A\).

**Lemma 2.4.** Let \(D\) be a \(C^*\)-subalgebra of \(A\). Then \((A, D)\) is relative \(\sigma\)-unitar if and only if there exists a sequence \(d_n \in D, n = 1, 2, \ldots\) such that

(a) \(d_n \geq 0, n = 1, 2, \ldots\).

(b) \(\sum_{n=1}^\infty d_n = 1\) in the strict topology of \(M(A)\).

(c) \(d_n \sigma dd_m^* = 0\) for all \(d \in D\) and \(n, m \in \mathbb{N}\) with \(n \neq m\).

**Proof.** Suppose that \((A, D)\) is relative \(\sigma\)-unitar. Take a relative approximate unit \(\{a_n\}_{n \in \mathbb{N}}\) in \(A\). Put \(d_n = a_n^*a_n\). By the preceding lemma, \(d_n\) belongs to \(D\) and satisfies the desired properties. Conversely, suppose that there exists a sequence \(d_n\) in \(D\) satisfying the above three conditions. Put \(a_n = \sqrt{d_n}\), which becomes a relative approximate unit for \((A, D)\).

We call the sequence \(\{d_n\}_{n \in \mathbb{N}}\) in \(D\) satisfying the conditions (a), (b), (c) in Lemma 2.4 an orthogonal approximate unit for \((A, D)\).

**Example 2.5. 1.** If a \(C^*\)-subalgebra \(D\) of \(A\) contains the unit of \(A\), the pair \((A, D)\) is relative \(\sigma\)-unitar by putting \(d_1 = 1\) and \(d_n = 0\) for \(n = 2, 3, \ldots\).

2. Let \(A = K\) and \(D = C\). Then the pair \((A, D)\) is relative \(\sigma\)-unitar by putting \(d_n = e_{n,n}, n \in \mathbb{N}\) where \(\{e_{n,m}\}_{n,m \in \mathbb{N}}\) is the matrix units of \(K\).

More generally we know the following proposition.

**Proposition 2.6.** If \((A, D)\) is relative \(\sigma\)-unitar, so is \((A \otimes K, D \otimes C)\).

**Proof.** Take an orthogonal approximate unit \(\{d_n\}_{n \in \mathbb{N}}\) in \(D\) for the pair \((A, D)\). Put \(d_{(n,m)} = d_n \otimes e_{m,m}\) for \(n, m = 1, 2, \ldots\). It is straightforward to see that the sequence \(d_{(n,m)}, n, m = 1, 2, \ldots\) becomes an orthogonal approximate unit for the pair \((A \otimes K, D \otimes C)\).

We call the pair \((A \otimes K, D \otimes C)\) the relative stabilization for \((A, D)\).

**Corollary 2.7.** If a \(C^*\)-subalgebra \(D\) of \(A\) contains the unit of \(A\), both the pairs \((A, D)\) and \((A \otimes K, D \otimes C)\) are relative \(\sigma\)-unitar.
3 Relative imprimitivity bimodules and relative Morita equivalence

Let \( (A_1, D_1) \) and \( (A_2, D_2) \) be relative \( \sigma \)-unital pairs of \( C^* \)-algebras.

**Definition 3.1.** Let \( X \) be an \( A_1-A_2 \)-Hilbert \( C^* \)-bimodule. Put

\[
X_D = \{ x \in X \mid A_1(xd_2 \mid x) \in D_1 \text{ for all } d_2 \in D_2, \langle x \mid d_1 x \rangle_{A_2} \in D_2 \text{ for all } d_1 \in D_1 \}.
\]

The \( A_1-A_2 \)-Hilbert \( C^* \)-bimodule \( X \) is called an \( (A_1, D_1)-(A_2, D_2) \)-relative imprimitivity bimodule if it satisfies the following conditions:

1. \( X \) is an \( A_1-A_2 \)-imprimitivity bimodule.
2. There exists a sequence \( x_n \in X_D, n = 1, 2, \ldots \) such that
   a. \( \sum_{n=1}^{\infty} \langle x_n \mid x_n \rangle_{A_2} = 1 \) in the strict topology of \( M(A_2) \).
   b. \( A_1(xd_2 \mid x_m) = 0 \) for all \( d_2 \in D_2 \) and \( n, m \in \mathbb{N} \) with \( n \neq m \).
3. There exists a sequence \( y_n \in X_D, n = 1, 2, \ldots \) such that
   a. \( \sum_{n=1}^{\infty} A_1(y_n | y_n) = 1 \) in the strict topology of \( M(A_1) \).
   b. \( \langle y_n | d_1 y_m \rangle_{A_2} = 0 \) for all \( d_1 \in D_1 \) and \( n, m \in \mathbb{N} \) with \( n \neq m \).

**Remark 3.2.** (1) Since \( X \) is an \( A_1-A_2 \)-imprimitivity bimodule, norms on \( X \) defined by their inner products coincide each other, that is, \( \| A_1(x | x) \|^{\frac{1}{2}} = \| (x | x)_{A_2} \|^{\frac{1}{2}} \) for \( x \in X \) (cf. [25, Proposition 3.1]). We denote the norm by \( \| x \| \).

2. The above elements \( x_n, y_n \in X_D \) in Definition 3.1 satisfy the inequalities

\[
A_1(x_n | x_n) \leq 1, \quad \langle y_n | y_n \rangle_{A_2} \leq 1
\]

because of the inequality

\[
A_1(x_n | x_n) \leq \| A_1(x_n | x_n) \| = \| \langle x_n | x_n \rangle_{A_2} \| \leq \sum_{n=1}^{\infty} \langle x_n | x_n \rangle_{A_2} = 1
\]

and of a similar inequality for \( \langle y_n | y_n \rangle_{A_2} \).

3. Both the left action of \( A_1 \) and the right action of \( A_2 \) on \( X \) are non-degenerate, that is, \( A_1X = X = XA_2 \). More strongly we see that \( \overline{D_1X} = X = \overline{XD_2} \). In fact, for \( d_1 \in D_1 \) and \( x \in X \), the following inequalities hold

\[
\| x - d_1 x \| = \| A_1(x - d_1 x | x) \|
\]

\[
= \| A_1(x | x) - d_1 A_1(x | x) - A_1(x | x) d_1^* + d_1 A_1(x | x) d_1^* \|
\]

\[
\leq \| A_1(x | x) - d_1 A_1(x | x) \| + \| A_1(x | x) - d_1 A_1(x | x) \| d_1^*.
\]

As \( D_1 \) has a countable approximate unit for \( A_1 \), we have a sequence \( d_1(n) \) in \( D_1 \) such that \( \lim_{n \to \infty} \| x - d_1(n) x \| = 0 \) so that \( \overline{D_1X} = X \).
Lemma 3.3. For \( x \in X_D \) we have

(i) \( A_1(x \mid x) \in D_1 \).

(ii) \( \langle x \mid x \rangle_{A_2} \in D_2 \).

Proof. (i) Let \( x \in X_D \). For \( d_2 \in D_2 \), we have

\[
\langle x - xd_2 \mid x - xd_2 \rangle_{A_2} = \langle x \mid x \rangle_{A_2} - \langle x \mid x \rangle_{A_2} d_2 - d_2^* (x \mid x)_{A_2} + d_2^* (x \mid x)_{A_2} d_2. \tag{3.2}
\]

Now \( D_2 \) contains an approximate unit for \( A_2 \), the equality (3.2) shows that for any \( \epsilon > 0 \) there exists an element \( d_2 \in D_2 \) such that \( \| \langle x - xd_2 \mid x - xd_2 \rangle_{A_2} \| < \epsilon \). Since \( X \) is an \( A_1 \)-\( A_2 \)-imprimitivity bimodule, we see that \( \| A_1 \langle x - xd_2 \mid x - xd_2 \rangle \| < \epsilon \) by [25, Lemma 3.11]. By the Cauchy–Schwarz inequality (cf. [25, Lemma 2.5]), we have

\[
\| A_1 \langle x - xd_2 \mid x \rangle \|^2 = \| A_1 \langle x - xd_2 \mid x \rangle^* A_1 \langle x - xd_2 \mid x \rangle \| \leq \| A_1 \langle x - xd_2 \mid x - xd_2 \rangle \| \| A_1 \langle x \mid x \rangle \| \leq \epsilon \| A_1 \langle x \mid x \rangle \|. \tag{3.3}
\]

Hence we have

\[
\| A_1 \langle x \mid x \rangle - A_1 \langle xd_2 \mid x \rangle \|^2 = \| A_1 \langle x - xd_2 \mid x \rangle \|^2 < \epsilon \| A_1 \langle x \mid x \rangle \|. \tag{3.3}
\]

As \( A_1 \langle xd_2 \mid x \rangle \) belongs to \( D_1 \), we conclude that \( A_1 \langle x \mid x \rangle \) belongs to \( D_1 \).

(ii) is similarly shown to (i).

\( \Box \)

Lemma 3.4.

(i) We have \( z = \sum_{n=1}^{\infty} A_1(z \mid x_n) x_n \) for \( z \in X \) which converges in the norm of \( X \), and \( A_1(x_n \mid x_m) = 0 \) for \( n, m \in \mathbb{N} \) with \( n \neq m \).

(ii) We have \( z = \sum_{n=1}^{\infty} y_n (y_n \mid z)_{A_2} \) for \( z \in X \) which converges in the norm of \( X \), and \( \langle y_n \mid y_m \rangle = 0 \) for \( n, m \in \mathbb{N} \) with \( n \neq m \).

Proof. (i) As \( X = \overline{X D_2} \), for \( z \in X \) and \( \epsilon > 0 \) there exists \( d_2 \in D_2 \) such that \( \| z - zd_2 \| < \epsilon \). Since \( \sum_{n=1}^{\infty} \langle x_n \mid x_n \rangle_{A_2} = 1 \) in the strict topology of \( M(A_2) \), we may find \( K \in \mathbb{N} \) such that

\[
\| \sum_{n=1}^{K} d_2^* (x_n \mid x_n)_{A_2} - d_2 \| < \epsilon. \]

Therefore we have

\[
\| z - \sum_{n=1}^{K} A_1(z \mid x_n) x_n \|
\leq \| z - zd_2 \| + \| zd_2 - \sum_{n=1}^{K} d_2^* (x_n \mid x_n)_{A_2} \| + \| \sum_{n=1}^{K} d_2^* (x_n \mid x_n)_{A_2} - \sum_{n=1}^{K} z (x_n \mid x_n)_{A_2} \|
\leq \| z - zd_2 \| + \| z \| \| d_2 - \sum_{n=1}^{K} d_2^* (x_n \mid x_n)_{A_2} \| + \| (zd_2 - z) \sum_{n=1}^{K} (x_n \mid x_n)_{A_2} \|
= (2 + \| z \|) \epsilon,
\]
so that \( \sum_{n=1}^{\infty} A(x \mid x_n)x_n \) converges to \( z \) in the norm of \( X \).

As in the proof of Lemma 3.3 for \( n, m \in \mathbb{N} \) with \( n \neq m \), there exists \( d_2(k) \in \mathcal{D}_2 \) such that

\[
\lim_{n \to \infty} \|A(x_n \mid x_m) - A(x_n)\|_2 = 0.
\]

Since \( A(x_n) = 0 \), we have \( A(x_n \mid x_m) = 0 \). \( \square \)

The sequences \( \{x_n\}_{n \in \mathbb{N}} \), \( \{y_n\}_{n \in \mathbb{N}} \subset X_D \) satisfying the conditions (2), (3) in Definition 3.1 are called a relative left basis, a relative right basis, respectively. The pair \( \{\{x_n\}, \{y_n\}\} \) is called a relative basis for \( X \).

We arrive at our definition of relative version of strong Morita equivalence.

**Definition 3.5.** Two relative \( \sigma \)-unital pairs of \( C^* \)-algebras \( (A_1, D_1) \) and \( (A_2, D_2) \) are said to be relatively Morita equivalent if there exists an \( (A_1, D_1) \sim (A_2, D_2) \)-relative imprimitivity bimodule. In this case we write \( (A_1, D_1) \sim_{\text{RME}} (A_2, D_2) \).

**Lemma 3.6.** Let \( (A_1, D_1) \) and \( (A_2, D_2) \) be relative \( \sigma \)-unital pairs of \( C^* \)-algebras. If there exists an isomorphism \( \theta : A_1 \to A_2 \) of \( C^* \)-algebras such that \( \theta(D_1) = D_2 \), then we have \( (A_1, D_1) \sim_{\text{RME}} (A_2, D_2) \). In particular, for a relative \( \sigma \)-unital pair \( (A, D) \) of \( C^* \)-algebras, we have \( (A, D) \sim_{\text{RME}} (A, D) \).

**Proof.** Let \( a_n \in A_1, n \in \mathbb{N} \) be a relative approximate unit for \((A_1, D_1)\). Put \( X_\theta = A_1 \) as vector space having module structure and inner products given by

\[
a_1 \cdot x \cdot a_2 := a_1 x \theta^{-1}(a_2) \quad \text{for} \quad a_1, a_2 \in A_1, \quad x \in X_\theta,
\]

\[
A_1(x \mid y) = xy^*, \quad \langle x \mid y \rangle_{A_2} = \theta(x^*y) \quad \text{for} \quad x, y \in X_\theta.
\]

Put \( x_n = a_n, n \in \mathbb{N} \). We have for \( d_1 \in D_1, d_2 \in D_2 \)

\[
A_1(x_n d_2 \mid x_n) = a_n \theta^{-1}(d_2) a_n^* \in D_1, \quad \langle x_n \mid d_1 x_n \rangle_{A_2} = \theta(a_n^* d_1 a_n) \in D_2
\]

so that \( x_n \in (X_\theta)_D \). We also have

\[
\sum_{n=1}^{\infty} \langle x_n \mid x_n \rangle_{A_2} = \sum_{n=1}^{\infty} \theta(a_n^* a_n) = 1,
\]

and

\[
A_1(x_n d_2 \mid x_m) = a_n \theta^{-1}(d_2) a_m^* = 0 \quad \text{for all} \quad d_2 \in D_2 \quad \text{and} \quad n, m \in \mathbb{N} \quad \text{with} \quad n \neq m.
\]

Similarly by putting \( y_n = a_n^* \) we have

\[
A_1(y_n d_2 \mid y_n) = a_n \theta^{-1}(d_2) a_n^* \in D_1, \quad \langle y_n \mid d_1 y_n \rangle_{A_2} = \theta(a_n d_1 a_n^*) \in D_2
\]

so that \( y_n \in (X_\theta)_D \). We also have

\[
\sum_{n=1}^{\infty} A_1(y_n \mid y_n) = \sum_{n=1}^{\infty} a_n^* a_n = 1,
\]

8
and
\[ \langle y_n \mid d_1 y_m \rangle = \theta(a_n d_1 a_{m}^*) = 0 \quad \text{for all } d_1 \in D_1 \text{ and } n, m \in \mathbb{N} \text{ with } n \neq m. \]

Hence \( \{x_n\}, \{y_n\} \) is a relative basis for \( X_\theta \) so that \( X_\theta \) becomes an \((A_1, D_1)-(A_2, D_2)\)-relative imprimitivity bimodule to show \((A_1, D_1) \sim \text{RME} \) \((A_2, D_2)\).

We will next show that the relation \( \sim \text{RME} \) is an equivalence relation in relative \( \sigma \)-unital pairs of \( C^* \)-algebras.

**Lemma 3.7.** Suppose that \( X_{12} \) is an \((A_1, D_1)-(A_2, D_2)\)-relative imprimitivity bimodule and \( X_{23} \) is an \((A_2, D_2)-(A_3, D_3)\)-relative imprimitivity bimodule. Then the relative tensor product \( X_{12} \otimes_{A_2} X_{23} \) is an \((A_1, D_1)-(A_3, D_3)\)-relative imprimitivity bimodule.

**Proof.** Take relative bases \( \{x_n\}, \{y_n\} \) for \( X_{12} \) and \( \{z_n\}, \{w_n\} \) for \( X_{23} \). We will show that the pair \( \{x_n \otimes z_m\}_{n,m}, \{y_n \otimes w_m\}_{n,m} \) becomes a relative basis for \( X_{12} \otimes_{A_2} X_{23} \). For \( d_3 \in D_3, d_1 \in D_1 \), we have
\[
A_1((x_n \otimes z_m)d_3 \mid x_n \otimes z_m) = A_1(x_n \otimes (z_m d_3) \mid x_n \otimes z_m) = A_1(x_n A_2(z_m d_3) \mid z_m \mid x_n),
\]
\[
\langle x_n \otimes z_m \mid d_1(x_n \otimes z_m) \rangle_{A_3} = \langle x_n \otimes z_m \mid (d_1 x_n) \otimes z_m \rangle_{A_3} = \langle z_m \mid \langle x_n \mid d_1 x_n \rangle_{A_2} z_m \rangle_{A_3}.
\]
As \( A_2(z_m d_3) \mid z_m \in D_2 \), we have \( A_1(x_n A_2(z_m d_3) \mid z_m) \mid x_n) \in D_1 \) so that \( A_1((x_n \otimes z_m)d_3 \mid x_n \otimes z_m) \in D_1 \). Similarly we know that \( A_1(x_n A_2(z_m d_3) \mid z_m) \mid x_n) \in D_3 \).

We also have
\[
\sum_{n,m=1}^{\infty} \langle x_n \otimes z_m \mid x_n \otimes z_m \rangle_{A_3} = \sum_{n,m=1}^{\infty} \langle z_m \mid \langle x_n \mid x_n \rangle_{A_2} z_m \rangle_{A_3}
\]
\[
= \sum_{m=1}^{\infty} \langle z_m \mid \left( \sum_{n=1}^{\infty} \langle x_n \mid x_n \rangle_{A_2} z_m \right)_{A_3}
\]
\[
= \sum_{m=1}^{\infty} \langle z_m \mid z_m \rangle_{A_3} = 1.
\]

For \( d_3 \in D_3 \), we have
\[
A_1((x_n \otimes z_m)d_3 \mid x_l \otimes z_k) = A_1(x_n \otimes (z_m d_3) \mid x_l \otimes z_k) = A_1(x_n A_2(z_m d_3) \mid z_k) \mid x_l).
\]

If \( m \neq k \), then \( A_2(z_m d_3) \mid z_k = 0 \). If \( n \neq l \), then \( A_1(x_n A_2(z_m d_3) \mid z_k) \mid x_l = 0 \) because \( A_2(z_m d_3) \mid z_k \in D_2 \). Hence if \((n, m) \neq (l, k)\), we have \( A_1((x_n \otimes z_m)d_3 \mid x_l \otimes z_k) = 0 \) and know that the sequence \( \{x_n \otimes z_m\}_{n,m} \) is a relative left basis for \( X_{12} \otimes_{A_2} X_{23} \). By a similar argument, we know that \( \{y_n \otimes w_m\}_{n,m} \) is a relative right basis for \( X_{12} \otimes_{A_2} X_{23} \), so that \( \{x_n \otimes z_m\}_{n,m}, \{y_n \otimes w_m\}_{n,m} \) is a relative basis for \( X_{12} \otimes_{A_2} X_{23} \). \( \square \)

Therefore we have

**Proposition 3.8.** Relative Morita equivalence \( \sim \text{RME} \) is an equivalence relation in relative \( \sigma \)-unital pairs of \( C^* \)-algebras.
Proof. The reflexive law follows from Lemma 3.6. We will show the symmetric law. Suppose that \((A_1, D_1) \sim_{\text{RME}} (A_2, D_2)\) via relative imprimitivity bimodule \(X_{12}\). Then its conjugate module \(\overline{X}_{12}\) denoted by \(X_{21}\) becomes an \((A_2, D_2)-(A_1, D_1)\)-relative imprimitivity bimodule (see [28] Definition 6.17, cf. [11] p. 3443), so that \((A_1, D_2) \sim_{\text{RME}} (A_1, D_1)\). The transitive law follows from Lemma 3.7.

Lemma 3.9. Let \((A, D)\) be a relative \(\sigma\)-unital pair of \(C^*\)-algebras. Then we have

\[
(A, D) \sim_{\text{RME}} (A \otimes K, D \otimes C).
\]

Proof. Let \(a_n \in A, n \in \mathbb{N}\) be a relative approximate unit for \((A, D)\). Let \(\{e_{n,m}\}_{n,m \in \mathbb{N}}\) be the matrix units of \(K\). Define \(X = A \otimes e_{1,1}K\). By identifying \(A\) with \(A \otimes C_{1,1}\), \(X\) has a natural structure of \(A-A \otimes K\)-imprimitivity bimodule. Put \(x_{n,m} = a_n \otimes e_{1,m} \in X, n, m \in \mathbb{N}\). For \(d_1 \in D\) and \(d_2 = d \otimes f \in D \otimes C\), we have

\[
\langle x_{n,m} | d_1 x_{n,m} \rangle_{A \otimes K} = \sum_{n,m=1}^{\infty} (a_n \otimes e_{1,m})^* (a_n \otimes e_{1,m}) = \sum_{n,m=1}^{\infty} a_n^* a_n \otimes e_{1,m}^* e_{1,m} = 1 \otimes 1
\]

in \(M(A \otimes K)\). For \(d_2 = d \otimes f \in D \otimes C\), we have

\[
\langle x_{n,m} d_2 | x_{k,l} \rangle = (a_n \otimes e_{1,m})^* (a_k \otimes e_{1,l}) = a_n^* a_k \otimes e_{1,m} e_{1,l}.
\]

If \(n \neq k\), we have \(a_n^* a_k = 0\). If \(m \neq l\), we have \(e_{1,m} e_{1,l} = 0\). Hence if \((n, m) \neq (k, l)\), we have \(\langle x_{n,m} d_2 | x_{k,l} \rangle = 0\).

Put \(y_n = a_n^* \otimes e_{1,1}\). Then for \(d_1 \in D\) and \(d_2 = d \otimes f \in D \otimes C\), we have

\[
\langle y_n d_2 | y_n \rangle = (a_n^* \otimes e_{1,1})^* (a_n^* \otimes e_{1,1}) = a_n^* a_n \otimes e_{1,1} = 1 \otimes 1
\]

in \(M(A \otimes K)\). For \(d_2 = d \otimes f \in D \otimes C\), we have

\[
\langle y_n | d_1 y_n \rangle_{A \otimes K} = (a_n^* \otimes e_{1,1})^* (d \otimes e_{1,1}) (a_n^* \otimes e_{1,1}) = a_n^* a_n \otimes e_{1,1} = 1 \otimes 1
\]

so that \(y_n \in X_D\). We also have

\[
\sum_{n=1}^{\infty} \langle y_n | y_n \rangle = \sum_{n=1}^{\infty} a_n^* a_n \otimes e_{1,1} = 1 \otimes e_{1,1},
\]

and

\[
\langle y_n | d_1 y_m \rangle = (a_n^* \otimes e_{1,1})^* (d \otimes e_{1,1}) (a_m^* \otimes e_{1,1}) = a_n^* a_m^* \otimes e_{1,1} = 0 \text{ for } n \neq m.
\]

Therefore \(X\) becomes an \((A, D)-(A \otimes K, D \otimes C)\)-relative imprimitivity bimodule, so that \((A, D) \sim_{\text{RME}} (A \otimes K, D \otimes C)\).
Example 3.10. For \( m, k \in \mathbb{N} \), let \( A_1 = M_m(\mathbb{C}), D_1 = \text{diag}(M_m(\mathbb{C})) = \mathbb{C}^m \), and \( A_2 = M_k(\mathbb{C}), D_2 = \text{diag}(M_k(\mathbb{C})) = \mathbb{C}^k \). Then we have \((A_1, D_1) \sim \text{RME}_\ast (A_2, D_2)\).

We will present an \((A_1, D_1) - (A_2, D_2)\)-relative imprimitivity bimodule in the following way. Let \( A_0, D_0 \) be \( M_{m+k}(\mathbb{C}), \text{diag}(M_{m+k}(\mathbb{C})) \), respectively. Let \( p_1, p_2 \) be the projections in \( D_0 \) defined by
\[
\begin{aligned}
p_1 &= (1, \cdots, 1, 0, \cdots, 0), & p_2 &= (0, \cdots, 0, 1, \cdots, 1).
\end{aligned}
\]

We then have
\[
A_1 = p_1 A_0 p_1, \quad D_1 = D_0 p_1 \quad \text{and} \quad A_2 = p_2 A_0 p_2, \quad D_2 = D_0 p_2.
\]

Put \( X = p_1 A_0 p_2 \) with natural \( A_1 - A_2 \)-bimodule structure and inner products such that
\[
\begin{aligned}
\langle x | y \rangle_{A_1} &= x^* y \quad \text{for} \quad x, y \in X.
\end{aligned}
\]

It is not difficult to see that \( X \) becomes an \((A_1, D_1) - (A_2, D_2)\)-relative imprimitivity bimodule so that \((A_1, D_1) \sim \text{RME}_\ast (A_2, D_2)\).

4 Isomorphism of relative stabilizations

In this section, we devote to proving the following theorem, which is a relative version of a part of Brown–Green–Rieffel Theorem \([3, \text{Theorem 1.2}]\).

Theorem 4.1. Suppose \((A_1, D_1) \sim \text{RME}_\ast (A_2, D_2)\). Then there exists an isomorphism \( \Phi : A_1 \otimes K \rightarrow A_2 \otimes K \) of \( C^* \)-algebras such that \( \Phi(D_1 \otimes C) = D_2 \otimes C \).

Suppose that \( X \) is an \((A_1, D_1) - (A_2, D_2)\)-relative imprimitivity bimodule. Let \( \bar{X} \) be the conjugate bimodule of \( X \) \([28, \text{Definition 6.17}]\), cf. \([10, \text{p.3443}]\). The corresponding element in \( \bar{X} \) to \( y \in X \) is denoted by \( \bar{y} \). It is straightforward to see that \( \bar{X} \) is \((A_2, D_2) - (A_1, D_1)\)-relative imprimitivity bimodule. We define the relative linking pair \((A_0, D_0)\) by setting
\[
\begin{aligned}
A_0 &= \{ \begin{bmatrix} a_1 & x \\ \bar{y} & a_2 \end{bmatrix} | a_1 \in A_1, a_2 \in A_2, x \in X, \bar{y} \in \bar{X} \}, & (4.1)
\end{aligned}
\]
\[
\begin{aligned}
D_0 &= \{ \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} | d_1 \in D_1, d_2 \in D_2 \}. & (4.2)
\end{aligned}
\]

As in \([3, \text{p.350}]\), the products between two elements of \( A_0 \) is defined by
\[
\begin{aligned}
\begin{bmatrix} a_1 & x \\ \bar{y} & a_2 \end{bmatrix} \begin{bmatrix} b_1 & z \\ \bar{w} & b_2 \end{bmatrix} := \begin{bmatrix} a_1 b_1 + \langle x | w \rangle & a_1 z + x b_2 \\ \bar{y} b_1 + a_2 \bar{w} & \langle y | z \rangle_{A_2} + a_2 b_2 \end{bmatrix}.
\end{aligned}
\]

Let \( X \oplus A_2 \) be the Hilbert \( C^* \)-right module over \( A_2 \) with the natural right action of \( A_2 \) and \( A_2 \)-valued right inner product defined by
\[
\begin{aligned}
\langle \begin{bmatrix} x \\ a_2 \end{bmatrix} | \begin{bmatrix} y \\ b_2 \end{bmatrix} \rangle_{A_2} := \langle x | y \rangle_{A_2} + a_2 b_2.
\end{aligned}
\]
The algebra $A_0$ acts on $X \oplus A_2$ by

$$\begin{bmatrix} a_1 & x \\ y & a_2 \end{bmatrix} \begin{bmatrix} z \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 z + xb_2 \\ \langle y \mid z \rangle_{A_2} + a_2 b_2 \end{bmatrix}.$$  

As seen in [25 Lemma 3.20], $A_0$ itself is a $C^*$-subalgebra of all bounded adjointable operators on the Hilbert $C^*$-right module $X \oplus A_2$. We set

$$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \quad (4.3)$$

They satisfy $P_1 + P_2 = 1$ and

$$P_1 A_0 P_1 = A_1, \quad D_0 P_1 = D_1 \quad \text{and} \quad P_2 A_0 P_2 = A_2, \quad D_0 P_2 = D_2. \quad (4.4)$$

To prove Theorem 4.1 we provide several lemmas.

**Lemma 4.2.** Let $(\{x_n\}, \{y_n\}) \subset X_D$ be a relative bases for $X$.

(i) Put $U_n = \begin{bmatrix} 0 & x_n \\ y_n & 0 \end{bmatrix} \in A_0, n \in \mathbb{N}$. The sequence $U_n$ satisfies the following conditions:

(a) $P_2 = \sum_{n=1}^{\infty} U_n^* U_n$ which converges in the strict topology of $M(A_0)$.
(b) $U_n U_m^* P_1$ and $U_n U_m^* = 0$ for $n \neq m$.
(c) $U_n D_0 U_n^* \subset D_0 P_1 = D_1$.
(d) $U_n^* D_0 U_n \subset D_0 P_2 = D_2$.

(ii) Put $T_n = \begin{bmatrix} 0 & 0 \\ y_n & 0 \end{bmatrix} \in A_0, n \in \mathbb{N}$. The sequence $T_n$ satisfies the following conditions:

(a) $P_1 = \sum_{n=1}^{\infty} T_n^* T_n$ which converges in the strict topology of $M(A_0)$.
(b) $T_n T_m^* \leq P_2$ and $T_n T_m^* = 0$ for $n \neq m$.
(c) $T_n D_0 T_n^* \subset D_0 P_2 = D_2$.
(d) $T_n^* D_0 T_n \subset D_0 P_1 = D_1$.

Proof. (i) For $d_1 \in D_1, d_2 \in D_2$, we have

$$U_n^* \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} U_n = \begin{bmatrix} 0 & \langle x_n \mid d_1 x_n \rangle_{A_2} \\ 0 & 0 \end{bmatrix}. \quad (4.5)$$

Since $x_n \in X_D$ and $d_1 \in D_1$, we have $\langle x_n \mid d_1 x_n \rangle_{A_2} \in D_2$, so that $U_n^* D_0 U_n \subset D_0 P_2$, which shows (d). Since we have

$$U_n^* U_n = \begin{bmatrix} 0 & \langle x_n \mid x_n \rangle_{A_2} \\ 0 & 0 \end{bmatrix} \quad \quad (4.6)$$

the equality $\sum_{n=1}^{\infty} \langle x_n \mid x_n \rangle_{A_2} = 1$ implies $\sum_{n=1}^{\infty} U_n^* U_n = P_2$ which shows (a). And also for $d_1 \in D_1, d_2 \in D_2$, we have

$$U_n \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} U_n^* = \begin{bmatrix} A_1 \langle x_n d_2 \mid x_n \rangle & 0 \\ 0 & 0 \end{bmatrix}. \quad (4.7)$$

Proof. (ii)
Since \( x_n \in X_D \) and \( d_2 \in D_2 \), we have \( A_1(x_n d_2 \mid x_n) \in D_1 \) so that \( U_n D_0 U_n^* \subset D_0 P_1 \), which shows (c). Since we have

\[
U_n U_m^* = \begin{bmatrix}
A_1(x_n \mid x_m) & 0 \\
0 & 0
\end{bmatrix},
\tag{4.8}
\]

the inequality \( A_1(x_n \mid x_n) \leq 1 \) implies \( U_n U_n^* \leq P_1 \) and \( A_1(x_n \mid x_m) = 0 \) for \( n, m \in \mathbb{N} \) with \( n \neq m \). Which shows (b).

**Lemma 4.3.** The pair \((A_0, D_0)\) is relative \( \sigma \)-unital.

**Proof.** Keep the above notations. Put \( a_n = \begin{bmatrix} 0 & x_n \end{bmatrix} = U_n + T_n \). It then follows that

\[
\sum_{n=1}^{\infty} a_n^* a_n = \sum_{n=1}^{\infty} U_n^* U_n + \sum_{n=1}^{\infty} T_n^* T_n = P_2 + P_1 = 1.
\]

For \( d_1 \in D_1, d_2 \in D_2 \), we have

\[
a_n^* \begin{bmatrix} d_1 & 0 \\
0 & d_2
\end{bmatrix} a_n = U_n^* \begin{bmatrix} d_1 & 0 \\
0 & d_2
\end{bmatrix} U_n + T_n^* \begin{bmatrix} d_1 & 0 \\
0 & d_2
\end{bmatrix} T_n
= \begin{bmatrix}
\langle y_n \mid d_2 y_n \rangle & A_1 \\
0 & \langle x_n \mid d_1 x_n \rangle A_2
\end{bmatrix} \in D_1 \oplus D_2 = D_0.
\]

Similarly we have \( a_n \begin{bmatrix} d_1 & 0 \\
0 & d_2
\end{bmatrix} a_n^* \in D_1 \oplus D_2 \). We also have \( a_n d a_n^* = (U_n + T_n) d (U_m + T_m)^* = U_n d U_m^* + T_n d T_m^* = 0 \) for \( d = d_1 + d_2 \in D_1 \oplus D_2 \) and \( n \neq m \). Hence \{\(a_n\)\} is a relative approximate unit for \((A_0, D_0)\) to show \((A_0, D_0)\) is relative \( \sigma \)-unital.

Let us decompose the set \( \mathbb{N} \) of natural numbers into disjoint infinite subsets \( \mathbb{N} = \cup_{j=1}^{\infty} \mathbb{N}_j \), and decompose \( \mathbb{N}_j \) for each \( j \) once again into disjoint infinite sets \( \mathbb{N}_j = \cup_{k=0}^{\infty} \mathbb{N}_{jk} \). Let \( \{e_{i,j}\}_{i,j \in \mathbb{N}} \) be the set of matrix units which generate the algebra \( \mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N})) \). Put the projections \( f_j = \sum_{i \in \mathbb{N}_j} e_{i,i} \) and \( f_{jk} = \sum_{i \in \mathbb{N}_{jk}} e_{i,i} \). Take a partial isometry \( s_{jk,j} \) such that \( s_{jk,j}^* s_{jk,j} = f_j, s_{jk,j} s_{jk,j}^* = f_{jk} \) and put \( s_{j,jk} = s_{jk,j}^* \). We set for \( n = 1, 2, \ldots, \)

\[
u_n = \sum_{k=1}^{\infty} U_k \otimes s_{nk,n}, \quad w_n = P_1 \otimes s_{n0,n} + u_n,
\]

\[
\begin{array}{c}
t_n = \sum_{l=1}^{\infty} T_l \otimes s_{nl,n}, \\
z_n = P_2 \otimes s_{n0,n} + t_n.
\end{array}
\]

Then we have

**Lemma 4.4 (cf. [20] Lemma 3.3).** For each \( n \in \mathbb{N} \), we have

(i) \( w_n \) is a partial isometry in \( M(A_0 \otimes \mathcal{K}) \) such that

(a) \( w_n^* w_n = 1 \otimes f_n \).

(b) \( w_n w_n^* \leq P_1 \otimes f_n \).

(c) \( w_n (D_0 \otimes \mathcal{C}) w_n^* \subset D_1 \otimes \mathcal{C} \).
(d) \( w_n^*(D_0 \otimes C)w_n \subset D_2 \otimes C \).

(ii) \( z_n \) is a partial isometry in \( M(A_0 \otimes K) \) such that

(a) \( z_n^*z_n = 1 \otimes f_n \).
(b) \( z_nz_n^* \leq P_2 \otimes f_n \).
(c) \( z_n(D_0 \otimes C)z_n^* \subset D_2 \otimes C \).
(d) \( z_n^*(D_0 \otimes C)z_n \subset D_1 \otimes C \).

Proof. (i) Since \( u_n^*u_n = P_2 \otimes f_n \), we have

\[
w_n^*w_n = P_1 \otimes f_n + u_n^*u_n = P_1 \otimes f_n + P_2 \otimes f_n = 1 \otimes f_n.
\]

As \( u_n(P_2 \otimes s_{n,n_0}) = (P_2 \otimes s_{n,n_0})u_n^* = 0 \), we have

\[
w_n^*w_n = P_1 \otimes f_{n_0} + u_nu_n^* = P_1 \otimes f_{n_0} + \sum_{k=1}^{\infty} U_k U_k^* \otimes f_{n_k}.
\]

Since \( f_{n_0}, f_{n_k} \leq f_n \), we have \( w_n^*w_n \leq P_1 \otimes f_n \).

(ii) is similarly shown to (i). \( \square \)

We will construct and study the unitary \( V_1 \) in \( M(A_0 \otimes K) \) such that \( \Ad(V_1) : A_0 \otimes K \to A_1 \otimes K \) and \( \Ad(V_1)(D_0 \otimes C) = D_1 \otimes C \).

Let \( f_{n,m} \) be a partial isometry satisfying \( f_{n,m}^*f_{n,m} = f_m \), \( f_{n,m}f_{n,m}^* = f_n \). The following lemma is straightforward.

Lemma 4.5 (cf. [20] Lemma 3.4). We put

\[
v_1 = w_1 = P_1 \otimes s_{1,1} + u_1,
\]

\[
v_{2n} = (P_1 \otimes f_n - v_{2n-1}v_{2n-1}^*)(P_1 \otimes f_{n,n}) \quad \text{for } 1 \leq n \in \mathbb{N},
\]

\[
v_{2n-1} = w_n(1 \otimes f_n - v_{2n-2}v_{2n-2}) \quad \text{for } 2 \leq n \in \mathbb{N}.
\]

Then we have for \( n \in \mathbb{N} \)

(a) \( v_{2n-2}v_{2n-2}^* + v_{2n-1}v_{2n-1}^* = 1 \otimes f_n \).
(b) \( v_{2n-1}v_{2n-1}^* + v_{2n}v_{2n}^* = P_1 \otimes f_n \).
(c) \( v_n(D_0 \otimes C)v_n^* \subset D_1 \otimes C \).
(d) \( v_n^*(D_1 \otimes C)v_n \subset D_0 \otimes C \).

By the above lemma, one may see that the summation \( \sum_{n=1}^{\infty} v_n \) converges in \( M(A_0 \otimes K) \) to certain partial isometry written \( V_1 \) in the strict topology of \( M(A_0 \otimes K) \). Similarly we obtain a partial isometry \( V_2 \) in \( M(A_0 \otimes K) \) constructed from the preceding sequences \( t_n, z_n \) of partial isometries. As a consequence, we obtain the following proposition.

Proposition 4.6. Assume that \( (A_1, D_1) \sim_{\text{RME}} (A_2D_2) \). Let \( (A_0, D_0) \) be the relative linking pair defined in (4.1), (4.2).
(i) There exists an isometry $V_1$ in $M(A_0 \otimes K)$ such that

(a) $V_1^* V_1 = 1 \otimes 1$.
(b) $V_1 V_1^* = P_1 \otimes 1$.
(c) $V_1(D_0 \otimes C)V_1^* = D_1 \otimes C$.
(d) $V_1^*(D_1 \otimes C)V_1 = D_0 \otimes C$.

(ii) There exists an isometry $V_2$ in $M(A_0 \otimes K)$ such that

(a) $V_2^* V_2 = 1 \otimes 1$.
(b) $V_2 V_2^* = P_2 \otimes 1$.
(c) $V_2(D_0 \otimes C)V_2^* = D_2 \otimes C$.
(d) $V_2^*(D_2 \otimes C)V_2 = D_0 \otimes C$.

Therefore we reach the following theorem

**Theorem 4.7.** Let $(A_1, D_1)$ and $(A_2, D_2)$ be relative $\sigma$-unital pairs of $C^*$-algebras. Then $(A_1, D_1) \sim_{\text{RME}} (A_2, D_2)$ if and only if there exists an isomorphism $\Phi : A_1 \otimes K \rightarrow A_2 \otimes K$ of $C^*$-algebras such that $\Phi(D_1 \otimes C) = D_2 \otimes C$.

**Proof.** Suppose $(A_1, D_1) \sim_{\text{RME}} (A_2, D_2)$. Take isometries $V_1, V_2$ in $M(A_0 \otimes K)$ as in Proposition 4.6. Put $\Phi = \text{Ad}(V_2 V_1^*)$ which gives rise to an isomorphism $\Phi : A_1 \otimes K \rightarrow A_2 \otimes K$ of $C^*$-algebras such that $\Phi(D_1 \otimes C) = D_2 \otimes C$.

Converse implication comes from Lemma 3.6 and Lemma 3.9. $\square$

## 5 Relative full corners

It is well-known that two $C^*$-algebras are strong Morita equivalent if and only if they are complementary full corners of some $C^*$-algebra ([3, Theorem 1.1]). In this section, we will study a relative version of this fact.

**Definition 5.1.** For a relative $\sigma$-unital pair $(A, D)$ of $C^*$-algebra, a projection $P \in M(D)$ is said to be relative full for $(A, D)$ if it satisfies the following conditions

1. $Pd = dP$ for all $d \in D$.
2. There exists an sequence $a_n \in A, n = 1, 2, \ldots$ such that
   (a) $a_n^* d a_n \in D, a_n d a_n^* \in DP$ for all $d \in D$ and $n = 1, 2, \ldots$
   (b) $\sum_{n=1}^{\infty} a_n^* P a_n = 1 - P$ in the strict topology of $M(A)$.
   (c) $a_n d a_m^* = 0$ for all $d \in D$ and $n, m \in \mathbb{N}$ with $n \neq m$.

We call the sequence $\{a_n\}_{n \in \mathbb{N}}$ satisfying the three conditions (a), (b), (c) a relative full sequence for $P$. 
Remark 5.2. By the above condition (b), we know that

\[(b') \ a^*_n dP a_n \in \mathcal{D}(1 - P) \quad \text{for all } d \in \mathcal{D},\]

because we have

\[(a^*_n dP a_n)^* a^*_n dP a_n = a^*_n P d^* a_n a^*_n dP a_n \leq \|d^* a_n a^*_n d\| a^*_n P a_n \leq 1 - P.\]

Definition 5.3. Relative \(\sigma\)-unital pairs \((A_1, D_1)\) and \((A_2, D_2)\) of \(C^*\)-algebras are said to be complementary relative full corners if there exists a relative \(\sigma\)-unital pair \((A_0, D_0)\) of \(C^*\)-algebras such that there exist relative full projections \(P_1, P_2 \in M(D_0)\) such that

\[P_1 + P_2 = 1 \quad \text{and} \quad P_i A_0 P_i = A_i, \quad D_0 P_i = D_i, \quad i = 1, 2. \tag{5.1}\]

Proposition 5.4. Let \((A_1, D_1)\) and \((A_2, D_2)\) be relative \(\sigma\)-unital pairs of \(C^*\)-algebras. If they are complementary relative full corners, then we have \((A_1, D_1) \sim_{\text{RME}} (A_2, D_2)\).

Proof. Let \((A_0, D_0)\) and \(P_i \in M(D_0), i = 1, 2\) be a relative \(\sigma\)-unital pair of \(C^*\)-algebras and projections, respectively, satisfying Definition 5.3. Let \(\{a_n\}\) and \(\{b_n\}\) be relative full sequences for the projections \(P_1, P_2\), respectively. We set \(X = P_1 A_0 P_2\) and define two sequences by \(x_n = P_1 a_n P_2\) and \(y_n = P_1 b_n P_2\). For \(d \in D_0\), put \(d_i = d P_i, i = 1, 2\). It then follows that

\[A_1 \langle x_n d_2 | x_n \rangle = P_1 a_n P_2 d_2 P_2 a^*_n P_1 \in D_0 P_1 = D_1,\]
\[\langle x_n | d_1 x_n \rangle_{A_2} = P_2 a^*_n P_1 d_1 P_1 a_n P_2 \in D_0 P_2 = D_2.\]

Hence \(x_n\) belongs to \(X_D\). We also have

\[\sum_{n=1}^{\infty} \langle x_n | x_n \rangle_{A_2} = \sum_{n=1}^{\infty} P_2 a^*_n P_1 a_n P_2 = P_2,\]

and

\[A_1 \langle x_n d_2 | x_m \rangle = P_1 a_n P_2 d_2 P_2 a^*_m P_1 = 0 \quad \text{for } n \neq m.\]

Hence \(\{x_n\}\) is a relative left basis for \(X\). Similarly we have

\[A_1 \langle y_n d_2 | y_n \rangle = P_1 b^*_n P_2 d_2 P_2 b_n P_1 \in D_0 P_1 = D_1,\]
\[\langle y_n | d_1 y_n \rangle_{A_2} = P_2 b_n P_1 d_1 P_1 b^*_n P_2 \in D_0 P_2 = D_2.\]

Hence \(y_n\) belongs to \(X_D\). We also have

\[\sum_{n=1}^{\infty} A_1 \langle y_n | y_n \rangle = \sum_{n=1}^{\infty} P_2 b^*_n P_2 b_n P_1 = P_1,\]

and

\[\langle y_n | d_1 y_m \rangle_{A_2} = P_2 b_n P_1 d_1 b^*_m P_2 = 0 \quad \text{for } n \neq m.\]

Hence \(\{y_n\}\) is a relative right basis for \(X\). Therefore \(X\) is an \((A_1, D_1)-(A_2, D_2)\)-relative imprimitivity bimodule, so that we have \((A_1, D_1) \sim_{\text{RME}} (A_2, D_2)\). \(\square\)
We obtain the following theorem.

**Theorem 5.5.** Let \((A_1, D_1)\) and \((A_2, D_2)\) be relative \(\sigma\)-unital pairs of \(C^*\)-algebras. Then \((A_1, D_1) \sim_{\text{RMME}} (A_2, D_2)\) if and only if \((A_1, D_1)\) and \((A_2, D_2)\) are complementary relative full corners.

**Proof.** The if part has been proved in Proposition 5.4. To show the only if part, suppose \((A_1, D_1) \sim_{\text{RMME}} (A_2, D_2)\). Take \((A_0, D_0)\) the linking pair defined in (4.1), (4.2). Let \(P_1, P_2\) be the projections in \(M(D_0)\) defined by (4.3). Take the sequence \(U_n, T_n\) as in Lemma 4.2. The proof of Lemma 4.2 shows us that the sequences \(a_n := U_n\) and \(b_n := T_n\) are relative full sequences for \((A_0, D_0)\), respectively, so that \(P_1\) and \(P_2\) are relative full projections for \((A_0, D_0)\). Since \(P_1 + P_2 = 1\), the equalities (4.4) show that \((A_1, D_1)\) and \((A_2, D_2)\) are complementary relative full corners. \(\square\)

6 Relative Morita equivalence in Cuntz–Krieger pairs

In this section, we will study relative Morita equivalence particularly in Cuntz–Krieger algebras from a viewpoint of symbolic dynamical systems. For a nonnegative matrix \(A = [A(i, j)]_{i,j=1}^N\), the associated directed graph \(G_A = (V_A, E_A)\) consists of the vertex set \(V_A = \{v_1^A, \ldots, v_N^A\}\) of \(N\)-vertices and the edge set \(E_A = \{a_1, \ldots, a_{N_A}\}\) where there are \(A(i, j)\) edges from \(v_i^A\) to \(v_j^A\). For \(a_i \in E_A\), denote by \(t(a_i), s(a_i)\) the terminal vertex of \(a_i\) and the source vertex of \(a_i\), respectively. The graph \(G_A\) has the \(N_A \times N_A\) transition matrix \(A^G = [A^G(a_i, a_j)]_{i,j=1}^{N_A}\) of edges defined by

\[
A^G(a_i, a_j) = \begin{cases} 
1 & \text{if } t(a_i) = s(a_j), \\
0 & \text{otherwise}
\end{cases}
\]

(6.1)

for \(a_i, a_j \in E_A\). The Cuntz–Krieger algebra \(O_A\) for the matrix \(A\) is defined as the Cuntz–Krieger algebra \(O_{A^G}\) for the matrix \(A^G\) which is the universal \(C^*\)-algebra generated by partial isometries \(S_{a_i}\) indexed by edges \(a_i, i = 1, \ldots, N_A\) subject to the relations:

\[
\sum_{j=1}^{N_A} S_{a_j} S_{a_j}^* = 1, \quad S_{a_i} S_{a_i}^* = \sum_{j=1}^{N_A} A^G(a_i, a_j) S_{a_j} S_{a_j}^* \quad \text{for } i = 1, \ldots, N_A.
\]

(6.2)

The subalgebra \(D_A\) is defined as the algebra \(D_{A^G}\). The pair \((O_A, D_A)\) is called the Cuntz–Krieger pair for the matrix \(A\). Since \(1 \in D_A \subset O_A\), the pair \((O_A, D_A)\) is relative \(\sigma\)-unital.

As in [17], the isomorphism class of the pair \((O_A, D_A)\) is exactly corresponding to the continuous orbit equivalence class of the underlying one-sided topological Markov shift \((X_A, \sigma_A)\). Its complete classification result has been obtained in [21, Theorem 3.6].

Let \(A, B\) be irreducible square matrices with entries in nonnegative integers. In [34], R. F. Williams proved that the two-sided topological Markov shifts \((\tilde{X}_A, \tilde{\sigma}_A)\) and \((\tilde{X}_B, \tilde{\sigma}_B)\) are topologically conjugate if and only if the matrices \(A, B\) are strong shift equivalent. Two nonnegative matrices \(A, B\) are said to be elementary equivalent if there exist nonnegative rectangular matrices \(C, D\) such that \(A = CD, B = DC\). If there exists a finite sequence of nonnegative matrices \(A_0, A_1, \ldots, A_n\) such that \(A = A_0, B = A_n\) and \(A_i\) is elementary equivalent to \(A_{i+1}\) for \(i = 1, 2, \ldots, n - 1\), then \(A\) and \(B\) are said to be strong shift equivalent.
equivalent (34). Hence topological conjugacy of two-sided topological Markov shifts is generated by a finite sequence of elementary equivalence of underlying matrices. Let us denote by \( B_k(X_A) \) the set of admissible words with length \( k \) of the topological Markov shift \( (X_A, \sigma_A) \). Put \( B_0(X_A) = \bigcup_{k=0}^{\infty} B_k(X_A) \).

In this section we will first show the following proposition.

**Proposition 6.1.** Suppose that two nonnegative square matrices \( A \) and \( B \) are elementary equivalent such that \( A = CD \) and \( B = DC \). Then we have \( (O_A, D_A) \sim_{RME} (O_B, D_B) \).

**Proof.** Suppose that the size of \( A \) is \( N \) and that of \( B \) is \( M \) so that \( C \) is an \( N \times M \) matrix and \( D \) is an \( M \times N \) matrix, respectively. We set the square matrix \( Z = \begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix} \) as block matrix, and we see

\[
Z^2 = \begin{bmatrix} CD & 0 \\ 0 & DC \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.
\]

Let us consider the Cuntz–Krieger algebra \( O_Z \) for the matrix \( Z \). Since \( E_Z = E_C \cup E_D \), we may write the canonical generating partial isometries of \( O_Z \) as \( S_c, S_d, c \in E_C, d \in E_D \) so that

\[
\sum_{c \in E_C} S_c S_c^* + \sum_{d \in E_D} S_d S_d^* = 1
\]

for \( c \in E_C, d \in E_D \). Let us denote by \( S_a, a \in E_A \) (resp. \( S_b, b \in E_B \)) the canonical generating partial isometries of \( O_A \) (resp. \( O_B \)) satisfying the relations (1.1). As \( Z^2 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \), we have a bijective correspondence \( \varphi_{A,CD} \) from \( E_A \) to a subset of \( E_C \times E_D \) (resp. \( \varphi_{B,DC} \) from \( E_B \) to a subset of \( E_D \times E_C \)) such that \( S_c S_d \neq 0 \) (resp. \( S_d S_c \neq 0 \)) if and only if \( \varphi_{A,CD}(a) = cd \) (resp. \( \varphi_{B,DC}(b) = dc \)) for some \( a \in E_A \) (resp. \( b \in E_B \)). We may then write \( S_{cd} = S_a \) (resp. \( S_{dc} = S_b \)) where \( S_{cd} \) denotes \( S_c S_d \) (resp. \( S_{dc} \) denotes \( S_d S_c \)). Define the projections in \( O_Z \) by \( P_A = \sum_{c \in E_C} S_c S_c^* \) and \( P_B = \sum_{d \in E_D} S_d S_d^* \). Both of them belong to \( D_Z \) and satisfy \( P_A + P_B = 1 \). It has been shown in [16] (cf. [20]) that

\[
P_A O_Z P_A = O_A, \quad P_B O_Z P_B = O_B, \quad D_Z P_A = D_A, \quad D_Z P_B = D_B. \tag{6.3}
\]

We put \( X = P_A O_Z P_B \) which has a natural structure of \( O_A - O_B \) imprimitivity bimodule under the identification \( P_A O_Z P_A = O_A, P_B O_Z P_B = O_B \).

We will prove that \( X \) becomes \( (O_A, D_A) - (O_B, D_B) \)-relative imprimitivity bimodule. Put \( E_C = \{c_1, \ldots, c_{N_C}\} \) and \( E_D = \{d_1, \ldots, d_{N_D}\} \) for the matrices \( C \) and \( D \) respectively. For \( k = 1, \ldots, N_D \), take \( c(k) \in E_C \) such that \( c(k) d_k \in B_2(X_Z) \) so that we have

\[
S_{c(k)} S_{c(k)}^* \geq S_{d_k} S_{d_k}^*.
\]

Similarly for \( l = 1, \ldots, N_C \), take \( d(l) \in E_D \) such that \( d(l) c_l \in B_2(X_Z) \) so that we have

\[
S_{d(l)} S_{d(l)}^* \geq S_{c_l} S_{c_l}^*.
\]
We set
\[ x_k = S_{c(k)}S_{d_k}S_{d_k}^* \quad \text{for } k = 1, \ldots, N_D, \]
\[ y_l = S_{d(l)c_l}S_{c_l}^* \quad \text{for } l = 1, \ldots, N_D. \]

For \( d_1 \in \mathcal{D}_A, d_2 \in \mathcal{D}_B \), we have
\[
\sigma_A(x_kd_2 | x_k) = S_{c(k)}S_{d_k}S_{d_k}S_{d_k}^*d_2S_{d_k}S_{d_k}^*S_{c(k)}^* \in \mathcal{D}_A.
\]
\[
(x_k | d_1x_k)\sigma_B = S_{d_k}S_{d_k}^*S_{c(k)}d_1S_{c(k)}S_{d_k}S_{d_k}^* \in \mathcal{D}_B
\]
so that \( x_k \) belongs to \( X_D \) and similarly \( y_l \) belongs to \( X_D \). We also have
\[
\sum_{k=1}^{N_D} \langle x_k | x_k \rangle \sigma_B = \sum_{k=1}^{N_D} (S_{c(k)}S_{d_k}S_{d_k}^*)^*(S_{c(k)}S_{d_k}S_{d_k}^*) = \sum_{k=1}^{N_D} S_{d_k}S_{d_k}^*S_{c(k)}S_{d_k}S_{d_k}^* = P_B
\]
For \( n \neq m \), we have
\[
\sigma_A(x_n d_2 | x_m) = S_{c(n)}S_{d_n}S_{d_n}^*d_2S_{d_m}S_{d_m}^*S_{c(m)}^* = 0.
\]
Similarly we have \( \sum_{l=1}^{N_C} \sigma_A(y_l | y_l) = P_A \) and \( \langle y_n | d_1y_m \rangle \sigma_B = 0 \) for \( n \neq m \), so that \( X \) becomes \((\sigma_A, \mathcal{D}_A) - (\mathcal{O}_B, \mathcal{D}_B)\)-relative imprimitivity bimodule.

In [23], Parry–Sullivan proved that the flow equivalence relation of topological Markov shifts is generated by strong shift equivalences and expansions \( A \to \tilde{A} \) defined below.

For an \( N \times N \) matrix \( A = [A(i,j)]_{i,j=1}^N \) with entries in \( \{0,1\} \), put
\[
\bar{A} = \begin{bmatrix}
0 & A(1,1) & \cdots & A(1,N) \\
1 & 0 & \cdots & 0 \\
0 & A(2,1) & \cdots & A(2,N) \\
\vdots & \vdots & \ddots & \vdots \\
0 & A(N,1) & \cdots & A(N,N)
\end{bmatrix}, \tag{6.4}
\]
which is called the expansion of \( A \) at the vertex 1. The expansion of \( A \) at other vertices are similarly defined.

**Proposition 6.2.** \((\sigma_A, \mathcal{D}_A) \sim_{\text{RME}} (\sigma_{\bar{A}}, \mathcal{D}_{\bar{A}})\).

**Proof.** Let \( \{0,1,\ldots,N\} \) be the set of symbols for the topological Markov shifts \( (\tilde{X}_{\bar{A}}, \sigma_{\bar{A}}) \) defined by the matrix \( \bar{A} \). Let us denote by \( \tilde{S}_0, \tilde{S}_1, \ldots, \tilde{S}_N \) the canonical generating partial isometries of the Cuntz–Krieger algebra \( \mathcal{O}_{\bar{A}} \) satisfying \( \sum_{j=0}^N \tilde{S}_j \tilde{S}_j^* = 1, \tilde{S}_i^* \tilde{S}_j = \sum_{j=0}^N \bar{A}(i,j) \tilde{S}_j \tilde{S}_j^* \) for \( i, j = 0, 1, \ldots, N \). Put \( P = \sum_{i=1}^N \tilde{S}_i \tilde{S}_i^* \). The identities
\[
\tilde{S}_1^* P \tilde{S}_1 = \tilde{S}_1^* \tilde{S}_1 = \tilde{S}_0 \tilde{S}_0^*, \quad P + \tilde{S}_0 \tilde{S}_0^* = P + \tilde{S}_1^* P \tilde{S}_1 = 1 \tag{6.5}
\]
hold, so that we have
\[ PO_A P = O_A, \quad D_A P = D_A. \] (6.6)
We put \( X = PO_A \), which has a natural structure of \( O_A - O_A \) imprimitivity bimodule under the identification \( PO_A P = O_A, D_A P = D_A \). We will prove that \( X \) becomes \((O_A, D_A) - (O_A, D_A)\)-relative imprimitivity bimodule. We set \( x_1 = P, x_2 = P \tilde{S}_1 \) and \( y_1 = P \). For \( d_1 \in D_A, d_2 \in D_A \), we have
\[
\begin{align*}
O_A \langle x_1 d_2 \mid x_1 \rangle &= P d_2 P \in D_A, \\
O_A \langle x_2 d_2 \mid x_2 \rangle &= P \tilde{S}_1 d_2 \tilde{S}_1^* P \in D_A, \\
\langle d_1 x_1 \mid x_1 \rangle O_A &= P d_1 P \in D_A \subset D_A, \\
\langle d_1 x_2 \mid x_2 \rangle O_A &= \tilde{S}_1^* P d_1 P \tilde{S}_1 \in D_A,
\end{align*}
\]
so that \( x_1, x_2, y_1 \in X_D \). We also have
\[
\sum_{k=1}^{2} \langle x_k \mid x_k \rangle O_A = P^* P + (P \tilde{S}_1)^*(P \tilde{S}_1) = P + \tilde{S}_1^* P \tilde{S}_1 = 1,
\]
\[
O_A \langle x_1 d_2 \mid x_2 \rangle = P d_2 (P \tilde{S}_1)^* = P d_2 \tilde{S}_1^* P = 0,
\]
\[
O_A \langle x_2 d_2 \mid x_1 \rangle = P \tilde{S}_1 d_2 P^* = 0.
\]
Hence \( X \) becomes \((O_A, D_A) - (O_A, D_A)\)-relative imprimitivity bimodule. \hfill \Box

We have thus obtained the following theorem.

**Theorem 6.3.** If two-sided topological Markov shifts \((\bar{X}_A, \bar{\sigma}_A)\) and \((\bar{X}_B, \bar{\sigma}_B)\) are flow equivalent, then the Cuntz–Krieger pairs \((O_A, D_A)\) and \((O_B, D_B)\) are relatively Morita equivalent.

### 7 Corner isomorphisms in Cuntz–Krieger pairs

Let \( A, B, Z \) be square irreducible non-permutation matrices with entries in nonnegative integers.

**Definition 7.1.** Two Cuntz–Krieger pairs \((O_A, D_A)\) and \((O_Z, D_Z)\) are said to be *elementary corner isomorphic* if there exists a projection \( P \in D_Z \) such that
\[
PO_Z P = O_A, \quad D_Z P = D_A. \] (7.1)
Two Cuntz–Krieger pairs \((O_A, D_A)\) and \((O_B, D_B)\) are said to be *corner isomorphic* if there exists a finite chain of Cuntz–Krieger pairs \((O_{Z_i}, D_{Z_i}), i = 0, 1, \ldots, n\) such that \( Z_0 = A, Z_n = B \), and either \((O_{Z_i}, D_{Z_i})\) and \((O_{Z_{i+1}}, D_{Z_{i+1}})\) or \((O_{Z_{i+1}}, D_{Z_{i+1}})\) and \((O_{Z_i}, D_{Z_i})\) are elementary corner isomorphic for all \( i = 0, 1, \ldots, n \). That is, the equivalence relation generated by elementary corner isomorphisms in Cuntz–Krieger pairs is the corner isomorphism.

We will prove the following theorem.
Theorem 7.2. If two Cuntz–Krieger pairs \((\mathcal{O}_A, \mathcal{D}_A)\) and \((\mathcal{O}_B, \mathcal{D}_B)\) are corner isomorphic, then there exists an isomorphism \(\Phi : \mathcal{O}_A \otimes \mathcal{K} \rightarrow \mathcal{O}_B \otimes \mathcal{K}\) of \(C^*\)-algebras such that 
\(\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}\).

Proof. We use the notation \(Z\) of matrix instead of \(B\), so that we suppose that \((\mathcal{O}_A, \mathcal{D}_A)\) and \((\mathcal{O}_Z, \mathcal{D}_Z)\) are elementary corner isomorphic by a projection \(P \in \mathcal{D}_Z\) satisfying \((7.1)\). Although by showing that \(X = PO_Z\) is an \((\mathcal{O}_A, \mathcal{D}_A)\)–\((\mathcal{O}_Z, \mathcal{D}_Z)\)-relative imprimitivity bimodule, we know that \((\mathcal{O}_A, \mathcal{D}_A)\) and \((\mathcal{O}_Z, \mathcal{D}_Z)\) are relatively Morita equivalent, and hence there exists an isomorphism \(\Phi : \mathcal{O}_A \otimes \mathcal{K} \rightarrow \mathcal{O}_B \otimes \mathcal{K}\) of \(C^*\)-algebras such that 
\(\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}\), we will directly construct such an isomorphism \(\Phi\) in the following way.

We may assume that the projection \(Q = 1 - P\) is not zero. Let \(S_1, \ldots, S_{N_z}\) be the canonical generating partial isometries of the Cuntz–Krieger algebra \(\mathcal{O}_Z\). As \(Q \in \mathcal{D}_Z\), one may find a finite family of admissible words \(\mu(k) \in B_s(X_Z), k = 1, \ldots, N_1\) such that \(\vert \mu(1) \vert = \cdots = \vert \mu(N_1) \vert\) and 
\(Q = \sum_{k=1}^{N_1} S_{\mu(k)}S^*_{\mu(k)}\). Since \(Z\) is irreducible, we may find admissible words \(\nu(k) \in B_s(X_Z)\) for each \(\mu(k)\) such that \(\vert \nu(1) \vert = \cdots = \vert \nu(N_1) \vert\) and
\(P \geq S_{\nu(k)}S^*_{\nu(k)}; \quad S_{\nu(k)}S^*_{\mu(k)} \neq 0, \quad k = 1, \ldots, N_1.
\)

As \(\nu(k)\mu(k)\) is an admissible word in \(X_Z\), we know \(S^*_{\nu(k)}S_{\nu(k)} \geq S_{\mu(k)}S^*_{\mu(k)}\). Put
\(U_k = S_{\nu(k)}S^*_{\mu(k)}, \quad k = 1, \ldots, N_1.\)

Then we have
\[
\sum_{k=1}^{N_1} U_k^*U_k = \sum_{k=1}^{N_1} S^*_{\mu(k)}S_{\mu(k)}S^*_{\nu(k)}S_{\nu(k)}S^*_{\mu(k)}S_{\mu(k)} = \sum_{k=1}^{N_1} S^*_{\mu(k)}S_{\mu(k)} = Q,
\]
and
\[U_kU_k^* \leq S_{\nu(k)}S^*_{\nu(k)} \leq P.\]
The sequence further satisfies the following
\[U_kU_k^* = 0 \quad \text{for } k \neq l, \quad U_kD_ZU_k^* \subset D_ZP = D_A, \quad U_k^*D_ZU_k \subset D_ZQ \subset D_Z.\]

As in the proof of Theorem 4.1, by setting
\[u_n = \sum_{k=1}^{N_1} U_k \otimes s_{n_k,n_k}, \quad w_n = P \otimes s_{n_0,n} + u_n, \quad n = 1, 2, \ldots,\]
we have a sequence \(w_n, n \in \mathbb{N}\) of partial isometries in \(M(\mathcal{O}_Z \otimes \mathcal{K})\) such that
\begin{enumerate}
  \item \(w_n^*w_n = 1 \otimes f_n.\)
  \item \(w_nw_n^* \leq P \otimes f_n.\)
  \item \(w_n(D_Z \otimes \mathcal{C})w_n^* \subset D_ZP \otimes \mathcal{C}.\)
\end{enumerate}
(4) $w_n^*(\mathcal{D}Z \otimes \mathcal{C})w_n \subset \mathcal{D}ZQ \otimes \mathcal{C}$.

Let $f_{n,m}$ be a partial isometry satisfying $f_{n,m}^* f_{n,m} = f_m$, $f_{n,m} f_{n,m}^* = f_n$. We put

$$v_1 = w_1 = P \otimes s_{1,0,1} + u_1,$$

$$v_{2n} = (P \otimes f_n - v_{2n-1}v_{2n-1}) (p \otimes f_{n,n+1}) \text{ for } 1 \leq n \in \mathbb{N},$$

$$v_{2n-1} = w_n (1 \otimes f_n - v_{2n-2}v_{2n-2}) \text{ for } 2 \leq n \in \mathbb{N}.$$ 

By the same way as Lemma 4.5, we have for $n \in \mathbb{N}$

(1) $v_{2n-2}^* v_{2n-2} + v_{2n-1}^* v_{2n-1} = 1 \otimes f_n$.

(2) $v_{2n-1}^* v_{2n-1} + v_{2n} v_{2n} = P \otimes f_n$.

(3) $v_n (\mathcal{D}Z \otimes \mathcal{C})v_n^* \subset \mathcal{D}ZP \otimes \mathcal{C}$.

(4) $v_n^* (\mathcal{D}ZP \otimes \mathcal{C})v_n \subset \mathcal{D}Z \otimes \mathcal{C}$.

Hence the summation $\sum_{n=1}^{\infty} v_n$ converges in $M(OZ \otimes K)$ to certain partial isometry written $V_A$ in the strict topology of $M(OZ \otimes K)$, so that

(1) $V_A^* V_A = 1 \otimes 1$.

(2) $V_A V_A^* = P \otimes 1$.

(3) $V_A (\mathcal{D}Z \otimes \mathcal{C}) V_A^* = \mathcal{D}ZP \otimes \mathcal{C}$.

(4) $V_A^* (\mathcal{D}ZP \otimes \mathcal{C}) V_A = \mathcal{D}Z \otimes \mathcal{C}$.

Putting $\Phi_A = \text{Ad}(V_A) : OZ \otimes K \rightarrow OA \otimes K$ which is an isomorphism between $OZ \otimes K$ and $OA \otimes K$ such that $\Phi(\mathcal{D}Z \otimes \mathcal{C}) = \mathcal{D}A \otimes \mathcal{C}$. 

Therefore we have the following theorem.

**Theorem 7.3.** The Cuntz-Krieger pairs $(OA, DA)$ and $(OB, DB)$ are corner isomorphic if and only if there exists an isomorphism $\Phi : OA \otimes K \rightarrow OB \otimes K$ of $C^*$-algebras such that $\Phi(\mathcal{D}A \otimes \mathcal{C}) = \mathcal{D}B \otimes \mathcal{C}$.

**Proof.** The only if part follows from Theorem 7.2. We will show the if part. Suppose that there exists an isomorphism $\Phi : OA \otimes K \rightarrow OB \otimes K$ of $C^*$-algebras such that $\Phi(\mathcal{D}A \otimes \mathcal{C}) = \mathcal{D}B \otimes \mathcal{C}$. By [21], the two-sided topological Markov shifts $(\tilde{X}_A, \tilde{\sigma}_A)$ and $(\tilde{X}_B, \tilde{\sigma}_B)$ are flow equivalent, so that the two matrices are connected by a finite chain of strong shift equivalences and symbol expansions. In the proofs of Proposition 6.1 and Proposition 6.2, we know that $(OA, DA)$ and $(OB, DB)$ are corner isomorphic. 

Therefore we may summarize our discussions in the following way.

**Theorem 7.4.** Let $A, B$ be irreducible non-permutation matrices with entries in $\{0,1\}$. Let $OA, OB$ be the associated Cuntz-Krieger algebras. Then the following assertions are mutually equivalent:

(1) $(OA, DA) \sim_{RME} (OB, DB)$.
(2) \((O_A \otimes K, D_A \otimes C) \sim_{\text{RME}} (O_B \otimes K, D_B \otimes C)\).

(3) There exists an isomorphism \(\Phi : O_A \otimes K \to O_B \otimes K\) of \(C^*\)-algebras such that \(\Phi(D_A \otimes C) = D_B \otimes C\).

(4) \((O_A, D_A)\) and \((O_B, D_B)\) are corner isomorphic.

(5) The two-sided topological Markov shifts \((\bar{X}_A, \bar{\sigma}_A)\) and \((\bar{X}_B, \bar{\sigma}_B)\) are flow equivalent.

Proof. (1) \(\iff\) (2) comes from Lemma 3.9.
(1) \(\iff\) (3) comes from Theorem 4.7.
(3) \(\iff\) (4) comes from Theorem 7.3.
(5) \(\implies\) (1) comes from Theorem 6.3.
(3) \(\implies\) (5) comes from [21, Corollary 3.8].

We note that the implication (5) \(\implies\) (3) is seen in [7], and that the equivalence between (3) and (5) is seen in [21, Corollary 3.8].

8 Relative Picard groups

Let \((A_1, D_1)\) and \((A_2, D_2)\) be relative \(\sigma\)-unital pairs of \(C^*\)-algebras. Let \(X, Y\) be \((A_1, D_1)\)–\((A_2, D_2)\)-relative imprimitivity bimodule. Then \(X\) and \(Y\) are said to be equivalent if there exists an isomorphism \(\varphi : X \to Y\) of \(A_1\)–\(A_2\)-imprimitivity bimodule such that

\[
\langle \varphi(x_1) | \varphi(x_2) \rangle = \langle x_1 | x_2 \rangle \quad \text{for} \quad x_1, x_2 \in X
\]

for both left and right inner products. As \(\varphi : X \to Y\) preserves the bimodule structures and inner products of \(X\) and \(Y\), we know \(\varphi(X_D) = Y_D\). We denote by \([X]\) the equivalence class of relative imprimitivity bimodule. For a relative \(\sigma\)-unital pair \((A, D)\) of \(C^*\)-algebras, we define a relative version of Picard group as follows.

Definition 8.1. The relative Picard group \(\text{Pic}(A, D)\) for \((A, D)\) is defined by the group of equivalence classes \([X]\) of \((A, D)\)–\((A, D)\)-relative imprimitivity bimodule by the product

\[
[X] \cdot [Y] := [X \otimes_A Y].
\]

We remark that the identity element of the group \(\text{Pic}(A, D)\) is the class of the identity \((A, D)\)–\((A, D)\)-relative imprimitivity bimodule \(X = A\) defined by the module structure and the inner products:

\[
a \cdot x \cdot b = axb, \quad \langle x \mid y \rangle_A := xy^*, \quad \langle x \mid y \rangle_A := x^*y \quad \text{for} \quad a, b, x, y \in A. \tag{8.1}
\]

Since \((A, D)\) is relative \(\sigma\)-unital, the above \(X\) becomes an \((A, D)\)–\((A, D)\)-relative imprimitivity bimodule as seen in Lemma 3.6.

Lemma 8.2. If \((A_1, D_1) \sim_{\text{RME}} (A_2, D_2)\), we have \(\text{Pic}(A_1, D_1) = \text{Pic}(A_2, D_2)\). Hence we have \(\text{Pic}(A, D) = \text{Pic}(A \otimes K, D \otimes C)\) for every relative \(\sigma\)-unital pair \((A, D)\) of \(C^*\)-algebras.
Proof. Let $X$ be $(A_1, D_1) - (A_2, D_2)$-relative imprimitivity bimodule, and $\bar{X}$ its conjugate module, which is $(A_2, D_2) - (A_1, D_1)$-relative imprimitivity bimodule. It is easy to see that the correspondence

$$[Y] \in \text{Pic}(A_1, D_1) \rightarrow [\bar{X} \otimes_{A_1} Y \otimes_{A_1} X] \in \text{Pic}(A_2, D_2)$$

yields an isomorphism as groups, because $[\bar{X} \otimes_{A_1} X]$ is the unit of the group Pic$(A_2, D_2)$ and $[X \otimes_{A_2} \bar{X}]$ is the unit of the group Pic$(A_1, D_1)$.

If $\theta : A_1 \rightarrow A_2$ is an isomorphism of $C^*$-algebras such that $\theta(D_1) = D_2$, then we write $\theta : (A_1, D_1) \rightarrow (A_2, D_2)$ and call an isomorphism of relative $\sigma$-unital pairs of $C^*$-algebras. As in Lemma 8.6 any isomorphism $\theta : (A_1, D_1) \rightarrow (A_2, D_2)$ gives rise to a $(A_1, D_1) - (A_2, D_2)$-relative imprimitivity bimodule $X_\theta$. The following lemma is straightforward.

**Lemma 8.3.** Let $\theta_{12} : (A_1, D_1) \rightarrow (A_2, D_2)$ and $\theta_{23} : (A_2, D_2) \rightarrow (A_3, D_3)$ be isomorphisms of relative $\sigma$-unital pairs of $C^*$-algebras. Then we have

$$[X_{\theta_{12}} \otimes_{A_2} X_{\theta_{23}}] = [X_{\theta_{23} \circ \theta_{12}}].$$

Therefore we have a contravariant functor from the category of relative $\sigma$-unital $C^*$-algebras with isomorphisms $\theta : (A_1, D_1) \rightarrow (A_2, D_2)$ as morphisms into the category of relative $\sigma$-unital $C^*$-algebras with equivalence classes of relative imprimitivity bimodules.

Let Aut$(A, D)$ be the group of automorphisms $\theta$ on $A$ such that $\theta(D) = D$, that is,

$$\text{Aut}(A, D) := \{ \theta \in \text{Aut}(A) \mid \theta(D) = D \}$$

We denote by $U(A, D)$ the group of unitaries $u \in M(A)$ satisfying $uDu^* = D$. We denote by Ad$(u)$ the automorphism of $(A, D)$ defined by Ad$(u)(a) = uau^*$ for $a \in A$. Let us denote by Int$(A, D)$ the subgroup of Aut$(A, D)$ consisting of such automorphisms of $(A, D)$. Hence Int$(A, D)$ is a normal subgroup of Aut$(A, D)$. By the preceding lemma, we have an anti-homomorphism

$$\theta \in \text{Aut}(A, D) \rightarrow [X_\theta] \in \text{Pic}(A, D).$$

The following proposition and its corollary are achieved by a similar manner to Brown–Green–Rieffel’s argument [3 Proposition 3.1] and [3 Corollary 3.2].

**Proposition 8.4** (cf. [3 Proposition 3.1]). The kernel of the anti-homomorphism from Aut$(A, D)$ into Pic$(A, D)$ is exactly Int$(A, D)$. That is, we have an exact sequence:

$$1 \rightarrow \text{Int}(A, D) \rightarrow \text{Aut}(A, D) \rightarrow \text{Pic}(A, D).$$

**Corollary 8.5** (cf. [3 Corollary 3.2]). Let $(A_1, D_1)$ and $(A_2, D_2)$ be relative $\sigma$-unital pairs of $C^*$-algebras. Let $\alpha, \beta : (A_1, D_1) \rightarrow (A_2, D_2)$ be isomorphisms. If $X_\alpha$ and $X_\beta$ are equivalent, then there exists a unitary $u \in U(A, D)$ such that $\beta = \text{Ad}(u) \circ \alpha$.

The following lemma is also a relative version of [3 Lemma 3.3].

24
Lemma 8.6 (cf. [3 Lemma 3.3]). Let \( X \) be an \((A_1, D_1) - (A_2, D_2)\)-relative imprimitivity bimodule. Let \((A_0, D_0)\) be the linking pair of \( X \) defined by (4.1) and (4.2). Then \( X \) is equivalent to \( X_\theta \) for some isomorphism \( \theta : (A_1, D_1) \rightarrow (A_2, D_2) \) if and only if there exists a partial isometry \( v \in M(A_0) \) such that

\[
v^*v = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad vv^* = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\]  

(8.2)

and

\[
v D_0 v^* = D_0 vv^*, \quad v^* D_0 v = D_0 v^* v.
\]  

(8.3)

In this case, \( \theta \) is defined by \( \theta(a) = v a v^*, a \in A_1 \).

Remark 8.7. Under the equality (8.2), the second equality of (8.3) follows from the first equality of (8.3). Because the first one of (8.3) ensures us the equality

\[
v^*v D_0 v^* v = v^* D_0 vv^* v.
\]  

(8.4)

By (8.2), \( v^*v \) commutes with any elements of \( D_0 \) so that (8.4) goes to the second equality of (8.3).

Proof of Lemma 8.6. Although the proof basically follows the proof of [3 Lemma 3.3], we give it for the sake of completeness. Suppose that \( X \) is equivalent to \( X_\theta \) for some isomorphism \( \theta : (A_1, D_1) \rightarrow (A_2, D_2) \). By this isomorphism, the linking algebra \( A_0 \) of \( X \) is identified with that of \( X_\theta \). Hence \( X_\theta = A_1 \) and

\[
A_0 = \{ \begin{bmatrix} a_1 & x \\
\bar{y} & a_2 \end{bmatrix} \mid a_1 \in A_1, a_2 \in A_2, x \in X_\theta, \bar{y} \in X_\theta \}.
\]

We define operators \( v, v^* \) on \( X_\theta \oplus A_2 \) by

\[
v \begin{bmatrix} z \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \theta(z) \end{bmatrix}, \quad v^* \begin{bmatrix} z \\ c_2 \end{bmatrix} = \begin{bmatrix} \theta^{-1}(c_2) \\ 0 \end{bmatrix} \quad \text{for } z \in X_\theta, c_2 \in A_2
\]

where \( X_\theta = A_1 \) so that \( \theta(z) \in A_2 \). Put

\[
R_v \left( \begin{bmatrix} a_1 & x \\
\bar{y} & a_2 \end{bmatrix} \right) = \begin{bmatrix} a_1 & x \\
\bar{y} & a_2 \end{bmatrix} \quad \text{and} \quad L_v \left( \begin{bmatrix} a_1 & x \\
\bar{y} & a_2 \end{bmatrix} \right) = \begin{bmatrix} a_1 & x \\
\bar{y} & a_2 \end{bmatrix}.
\]

It is straightforward to see that

\[
R_v \left( \begin{bmatrix} a_1 & x' \\
\bar{y} & a_2' \end{bmatrix} \right) \begin{bmatrix} a_1' & x' \\
\bar{y}' & a_2' \end{bmatrix} = \begin{bmatrix} a_1 & x \\
\bar{y} & a_2 \end{bmatrix} L_v \left( \begin{bmatrix} a_1' & x' \\
\bar{y}' & a_2' \end{bmatrix} \right).
\]

Hence the pair \( (L_v, R_v) \) defines an element of \( M(A_0) \) as a double centralizer of \( A_0 \). Similarly \( (L_{v^*}, R_{v^*}) \) defines an element of \( M(A_0) \) such that \( (L_v, R_v)^* = (L_{v^*}, R_{v^*}) \), so that we may write \( (L_v, R_v) = v \). It then follows that

\[
v^*v \begin{bmatrix} z \\ c_2 \end{bmatrix} = \begin{bmatrix} z \\ 0 \end{bmatrix} \quad \text{and hence} \quad v^*v = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},
\]

\[
v v^* \begin{bmatrix} z \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ c_2 \end{bmatrix} \quad \text{and hence} \quad vv^* = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.
\]
It is direct to see that 
\[ v \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix} v^* = \begin{bmatrix} 0 & 0 \\ 0 & \theta(a_1) \end{bmatrix}. \]

This means that \( \theta(a_1) = va_1v^* \) for \( a_1 \in A_1 \) under the identification between \( a_1 \) and \( \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix} \) for \( a_1 \in A_1 \). Since \( \theta : A_1 \to A_2 \) satisfies \( \theta(D_1) = D_2 \) and \( D_1 = D_0v^*v, \ D_2 = D_0vv^* \), we have

\[ vD_0v^* = vD_1v^* = \theta(D_1) = D_2 = D_0vv^* \]

and \( v^*D_0v = D_0v^*v \).

Conversely suppose that a partial isometry \( v \in M(A_0) \) satisfies the equalities (8.2) and (8.3). It is easy to see that there exists an element \( \theta(a) \) in \( A_2 \) for each \( a \in A_1 \) such that

\[ v \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} v^* = \begin{bmatrix} 0 & 0 \\ 0 & \theta(a) \end{bmatrix} \]

and the correspondence \( a \in A_1 \mapsto \theta(a) \in A_2 \) gives rise to an isomorphism of \( C^* \)-algebras. The conditions (8.2) and (8.3) implies that \( vD_0v^* = D_0v^*v = D_2 \) and \( v^*D_0v = D_0v^*v = D_1 \) so that we have \( vD_1v^* = vv^*D_0v = D_2 \). This implies that \( \theta(D_1) = D_2 \).

We will next show that \( X \) is equivalent to \( X_\theta \). We identify \( A_1 \) with its image in \( A_0 \) and then we will define a map \( \eta : X \to A_1 (= X_\theta) \) by

\[ \eta(x) := \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} v \quad \text{for} \ x \in X. \]

Since 
\[ v^*v\eta(x)v^*v = v^*v\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} vv^*v = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} v \eta(x), \]

we see that \( \eta(x) \in A_1 \). By a routine calculation, we know that \( \eta \) is a bimodule homomorphism from \( X \) to \( X_\theta \) which preserves both inner products, and hence \( \eta \) gives rise to an isomorphism between \( X \) and \( X_\theta \).

The following theorem is also a relative version of a Brown-Green-Rieffel’s theorem. We will give its proof for the sake of completeness.

**Theorem 8.8** (cf. [34 Theorem 3.4]). Let \((A_1, D_1)\) and \((A_2, D_2)\) be relative \(\sigma\)-unital pairs of \(C^*\)-algebras. Let \(X\) be an \((A_1 \otimes K, D_1 \otimes C)\)-(\(A_2 \otimes K, D_2 \otimes C\))-relative imprimitivity bimodule. Then there exists an isomorphism \(\theta : A_1 \otimes K \to A_2 \otimes K\) satisfying \(\theta(D_1 \otimes C) = D_2 \otimes C\) such that \(X\) is equivalent to \(X_\theta\). Furthermore \(\theta\) is unique up to left multiplication by an element of \(\text{Int}(A_2 \otimes K, D_2 \otimes C)\), that is if \(X\) is equivalent to \(X_\varphi\) for some isomorphism \(\varphi : (A_1 \otimes K, D_1 \otimes C) \to (A_2 \otimes K, D_2 \otimes C)\), then there exists a unitary \(u \in U(A_2 \otimes K, D_2 \otimes C)\) such that \(\varphi = \text{Ad}(u) \circ \theta\).

**Proof.** The uniqueness follows immediately from Corollary 8.5.

Now let \(X\) be an \((A_1 \otimes K, D_1 \otimes C)\)-(\(A_2 \otimes K, D_2 \otimes C\))-relative imprimitivity bimodule. We put \(\bar{A}_i = A_i \otimes K, \bar{D}_i = D_i \otimes K\) for \(i = 1, 2\). Let \((\bar{A}_0, \bar{D}_0)\) be the linking pair for \(X\) defined from \(\bar{A}_i, \bar{D}_i, i = 1, 2\) and \(X\) by (4.1) and (4.2). By the assumption that \((A_1, D_1)\) is
(\bar{A}_2, \bar{D}_2) \) with Theorem 4.7. Proposition 4.6 tells us that there exist \( v_i \in M(\bar{A} \otimes \mathcal{K}), i = 1, 2 \) such that

\[
v_i^* v_i = 1 \otimes 1 \quad \text{in} \quad M(\bar{A}_0 \otimes \mathcal{K}), \quad i = 1, 2,
\]

\[
v_1 v_1^* = P_1 \otimes 1 \quad \text{where} \quad P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{in} \quad M(\bar{A}_0)
\]

\[
v_2 v_2^* = P_2 \otimes 1 \quad \text{where} \quad P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{in} \quad M(\bar{A}_0)
\]

and

\[
v_i(\bar{D}_0 \otimes C)v_i^* = \bar{D}_i \otimes C, \quad v_i^*(\bar{D}_i \otimes C)v_i = \bar{D}_0 \otimes C, \quad i = 1, 2.
\]

Put a partial isometry \( w = v_2 v_1^* \in M(\bar{A} \otimes \mathcal{K}) \) so that we have

\[
w^* w = P_1 \otimes 1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad w w^* = P_2 \otimes 1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{in} \quad M(\bar{A}_0 \otimes \mathcal{K}).
\]

and

\[
w(\bar{D}_1 \otimes C)w^* = \bar{D}_2 \otimes C, \quad w^*(\bar{D}_2 \otimes C)w = \bar{D}_1 \otimes C.
\]

Let \( p \in \mathcal{C} \) be the rank one projection \( p = e_{1,1} \), so that \( \bar{A}_1 \otimes p \otimes \mathcal{K} \) is a corner of \( \bar{A}_1 \otimes \mathcal{K} \otimes \mathcal{K} \). Hence by [2, Lemma 2.5] there exists a partial isometry \( \bar{v}_1 \in M(\bar{A}_1 \otimes \mathcal{K} \otimes \mathcal{K}) \) such that

\[
\bar{v}_1^* \bar{v}_1 = 1 \otimes 1 \otimes 1, \quad \bar{v}_1 \bar{v}_1^* = 1 \otimes p \otimes 1.
\]

By the construction of \( \bar{v}_1 \), we see that

\[
\bar{v}_1(\bar{D}_1 \otimes \mathcal{C} \otimes \mathcal{C})\bar{v}_1^* = \bar{D}_1 \otimes p \otimes \mathcal{C}, \quad \bar{v}_1^*(\bar{D}_1 \otimes \mathcal{C} \otimes \mathcal{C})\bar{v}_1 = \bar{D}_1 \otimes \mathcal{C} \otimes \mathcal{C}.
\]

We can identify \( \bar{A}_1 \) and \( \bar{D}_1 \) with \( \bar{A}_1 \otimes 1 \otimes \mathcal{K} \) and \( \bar{D}_1 \otimes 1 \otimes \mathcal{C} \), respectively, so that we have \( \bar{v}_1 \in M(\bar{A}_1 \otimes \mathcal{K}) \) and

\[
\bar{v}_1^* \bar{v}_1 = 1 \otimes 1, \quad \bar{v}_1 \bar{v}_1^* = 1 \otimes p, \quad \bar{v}_1(\bar{D}_1 \otimes \mathcal{C})\bar{v}_1^* = \bar{D}_1 \otimes p, \quad \bar{v}_1^*(\bar{D}_1 \otimes \mathcal{C})\bar{v}_1 = \bar{D}_1 \otimes \mathcal{C}.
\]

Similarly we have \( \bar{v}_2 \in M(\bar{A}_2 \otimes \mathcal{K}) \) and

\[
\bar{v}_2^* \bar{v}_2 = 1 \otimes 1, \quad \bar{v}_2 \bar{v}_2^* = 1 \otimes p, \quad \bar{v}_2(\bar{D}_2 \otimes \mathcal{C})\bar{v}_2^* = \bar{D}_2 \otimes p, \quad \bar{v}_2^*(\bar{D}_2 \otimes \mathcal{C})\bar{v}_2 = \bar{D}_2 \otimes \mathcal{C}.
\]

Define \( \bar{v} \in M(\bar{A}_0 \otimes \mathcal{K}) \) by

\[
\bar{v} = \begin{bmatrix} 0 & 0 \\ 0 & \bar{v}_2 \end{bmatrix} w \begin{bmatrix} \bar{v}_1^* & 0 \\ 0 & 0 \end{bmatrix} \quad \text{in} \quad M(\bar{A}_0 \otimes \mathcal{K}).
\]
We then have

\[
\bar{v}^* \bar{v} = \begin{bmatrix}
\bar{v}_1 & 0 \\
0 & 0 \\
\end{bmatrix} w^* \begin{bmatrix}
0 & 0 \\
0 & 1 \otimes 1 \\
\end{bmatrix} w \begin{bmatrix}
\bar{v}_1^* & 0 \\
0 & 0 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\bar{v}_1 & 0 \\
0 & 0 \\
\end{bmatrix} w^* w \begin{bmatrix}
\bar{v}_1^* & 0 \\
0 & 0 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 \otimes p & 0 \\
0 & 0 \\
\end{bmatrix}
\]

and

\[
\bar{v}\bar{v}^* = \begin{bmatrix}
0 & 0 \\
0 & \bar{v}_2 \\
\end{bmatrix} w \begin{bmatrix}
1 \otimes 1 & 0 \\
0 & 0 \\
\end{bmatrix} w^* \begin{bmatrix}
0 & 0 \\
0 & \bar{v}_2^* \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & 0 \\
0 & \bar{v}_2 w w^* \begin{bmatrix}
0 & 0 \\
0 & \bar{v}_2^* \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & 0 \\
0 & 1 \otimes p \\
\end{bmatrix}
\]

We will next show that \( \bar{v}(D_0 \otimes p)\bar{v}^* = (D_0 \otimes p)\bar{v}\bar{v}^* \). For \( \begin{bmatrix}
d_1 & 0 \\
0 & d_2 \\
\end{bmatrix} \in \mathcal{D}_0 \) with \( d_i \in \mathcal{D}_i, i = 1, 2 \), we have

\[
\bar{v} \begin{bmatrix}
d_1 \otimes p & 0 \\
0 & d_2 \otimes p \\
\end{bmatrix} \bar{v}^* = \begin{bmatrix}
0 & 0 \\
0 & \bar{v}_2 \\
\end{bmatrix} w \begin{bmatrix}
\bar{v}_1^* (d_1 \otimes p) \bar{v}_1 & 0 \\
0 & 0 \\
\end{bmatrix} w^* \begin{bmatrix}
0 & 0 \\
0 & \bar{v}_2^* \\
\end{bmatrix}.
\]

Since \( \bar{v}_1^* (d_1 \otimes p) \bar{v}_1 \in \bar{D}_1 \otimes \mathcal{C} \), we have \( w \begin{bmatrix}
\bar{v}_1^* (d_1 \otimes p) \bar{v}_1 & 0 \\
0 & 0 \\
\end{bmatrix} w^* \in w(\bar{D}_1 \otimes \mathcal{C})w^* = \bar{D}_2 \otimes \mathcal{C} \) so that

\[
\bar{v} \begin{bmatrix}
d_1 \otimes p & 0 \\
0 & d_2 \otimes p \\
\end{bmatrix} \bar{v}^* \in \bar{v}_2 (\bar{D}_2 \otimes \mathcal{C}) \bar{v}_2 = \bar{D}_2 \otimes p = (\bar{D}_0 \otimes p)\bar{v}\bar{v}^*.
\]

Therefore we have \( \bar{v}(D_0 \otimes p)\bar{v}^* \subset (D_0 \otimes p)\bar{v}\bar{v}^* \) and similarly \( \bar{v}^* (D_0 \otimes p) \bar{v} \subset (D_0 \otimes p)\bar{v}\bar{v}^* \) so that we have

\[
\bar{v}(D_0 \otimes p)\bar{v}^* = (D_0 \otimes p)\bar{v}\bar{v}^* \quad \text{and} \quad \bar{v}^*(D_0 \otimes p) \bar{v} = (D_0 \otimes p)\bar{v}^* \bar{v}.
\]

By the equalities

\[
\bar{v}^* \bar{v} = \begin{bmatrix}
1 \otimes p & 0 \\
0 & 0 \\
\end{bmatrix}, \quad \bar{v}\bar{v}^* = \begin{bmatrix}
0 & 0 \\
0 & 1 \otimes p \\
\end{bmatrix},
\]

we know that \( \bar{v} \) commutes with \( 1 \otimes p \) so that we can regard \( \bar{v} \) as an element of \( M(\bar{A}_0 \otimes p) = M(\bar{A}_0) \). Thus we obtain a partial isometry \( \bar{v} \) in \( M(\bar{A}_0) \) such that

\[
\bar{v}^* \bar{v} = \begin{bmatrix}
1 & 0 \\
0 & 0 \\
\end{bmatrix}, \quad \bar{v}\bar{v}^* = \begin{bmatrix}
0 & 0 \\
0 & 1 \otimes \bar{A}_2 \\
\end{bmatrix},
\]

and

\[
\bar{v}\bar{D}_0 \bar{v}^* = \bar{D}_0 \bar{v}\bar{v}^* \quad \text{and} \quad \bar{v}^* \bar{D}_0 \bar{v} = \bar{D}_0 \bar{v}^* \bar{v}.
\]

Therefore by Lemma 8.6 we conclude that \( X \) is equivalent to \( X_\theta \) for some isomorphism \( \theta : (\bar{A}_1, \bar{D}_1) \longrightarrow (\bar{A}_2, \bar{D}_2) \). \( \square \)
Recall that the subgroups $\text{Aut}(A \otimes K, D \otimes C)$ and $\text{Int}(A \otimes K, D \otimes C)$ of automorphism group $\text{Aut}(A \otimes K)$ are defined by

$$\text{Aut}(A \otimes K, D \otimes C) = \{ \beta \in \text{Aut}(A \otimes K) \mid \beta(D \otimes C) = D \otimes C \},$$

$$\text{Int}(A \otimes K, D \otimes C) = \{ \beta \in \text{Int}(A \otimes K) \mid \beta(D \otimes C) = D \otimes C \}.$$

**Corollary 8.9.** Let $(A,D)$ be a relative $\sigma$-unital pair of $C^*$-algebras. For any relative imprimitivity bimodule $[X] \in \text{Pic}(A \otimes K, D \otimes C)$, there exists an automorphism $\theta \in \text{Aut}(A \otimes K, D \otimes C)$ such that $[X] = [X_\theta]$. Thus we have a exact sequence

$$1 \longrightarrow \text{Int}(A \otimes K, D \otimes C) \longrightarrow \text{Aut}(A \otimes K, D \otimes C) \longrightarrow \text{Pic}(A \otimes K, D \otimes C) \longrightarrow 1.$$

Let us denote by $\text{Out}(A \otimes K, D \otimes C)$ the quotient group $\text{Aut}(A \otimes K, D \otimes C)/\text{Int}(A \otimes K, D \otimes C)$. We then have

**Corollary 8.10.** Let $(A,D)$ be a relative $\sigma$-unital pair of $C^*$-algebras. We have

$$\text{Pic}(A,D) = \text{Out}(A \otimes K, D \otimes C).$$

**Proof.** By Lemma 8.2, we see that $\text{Pic}(A,D) = \text{Pic}(A \otimes K, D \otimes C)$ so that we have the desired equality by the preceding corollary. 

### 9 Relative Picard groups of Cuntz–Krieger pairs

In this section, we will study the relative Picard group $\text{Pic}(A,D)$ for the Cuntz–Krieger pairs $(\mathcal{O}_A, D_A)$. By [13 Lemma 1.1], for a unitary $u \in M(\mathcal{O}_A \otimes K)$, the automorphism $\text{Ad}(u)$ acts trivially on $K_0(\mathcal{O}_A \otimes K)$. We will first show the following proposition which is a relative version of [13, Lemma 3.13] (Lemma 10.1 in Appendix).

**Proposition 9.1.** Let $\beta \in \text{Aut}(\mathcal{O}_A \otimes K)$ satisfy $\beta(D_A \otimes C) = D_A \otimes C$ and $\beta_s = \text{id}$ on $K_0(\mathcal{O}_A)$. Then there exists a unitary $u \in M(\mathcal{O}_A \otimes K)$ and an automorphism $\alpha \in \text{Aut}(\mathcal{O}_A)$ such that

$$\beta = \text{Ad}(u) \circ (\alpha \otimes \text{id}) \quad \text{and} \quad \alpha_s = \text{id} \quad \text{on} \quad K_0(\mathcal{O}_A),$$

$$u(D_A \otimes C)u^* = D_A \otimes C, \quad \alpha(D_A) = D_A.$$

To show the above proposition, we provide several lemmas.

**Lemma 9.2.** Let $\beta \in \text{Aut}(\mathcal{O}_A \otimes K)$ satisfy $\beta(D_A \otimes C) = D_A \otimes C$ and $\beta_s = \text{id}$ on $K_0(\mathcal{O}_A)$. Then for each $k \in \mathbb{N}$, there exists a partial isometry $w_k \in \mathcal{O}_A \otimes K$ such that

$$w_k^*w_k = 1 \otimes e_{kk}, \quad \beta(1 \otimes e_{kk}),$$

$$w_k(D_A \otimes C)w_k^* \subset D_A \otimes C, \quad w_k(D_A \otimes C)w_k \subset D_A \otimes C. \tag{9.1}$$

**Proof.** Let us denote by $N_\gamma(\mathcal{O}_A \otimes K, D_A \otimes C)$ the normalizer semigroup

$$\{ v \in \mathcal{O}_A \otimes K \mid v \text{ is a partial isometry}; v(D_A \otimes C)v^* \subset D_A \otimes C, \ v^*(D_A \otimes C)v \subset D_A \otimes C \}$$

of partial isometries in $\mathcal{O}_A \otimes K$. Denote by $K_0(\mathcal{O}_A \otimes K, D_A \otimes C)$ the Murray-von-Neumann equivalence classes of projections in $D_A \otimes C$ by partial isometries in $N_\gamma(\mathcal{O}_A \otimes K, D_A \otimes C)$. 

29
It has been proved in [18] that there exists a natural isomorphism between $K_0(O_A)$ and $K_0(O_A \otimes K, D_A \otimes C)$. Since $[\beta(1 \otimes e_{kk})] = \beta_*([1 \otimes e_{kk}]) = [1 \otimes e_{kk}]$, we have $\beta(1 \otimes e_{kk}) \sim 1 \otimes e_{kk}$ in $K_0(O_A \otimes K, D_A \otimes C)$. We may find a partial isometry $w_k \in O_A \otimes K$ satisfying the desired conditions. 

**Lemma 9.3.** Let $\beta \in \text{Aut}(O_A \otimes K)$ satisfy $\beta(D_A \otimes C) = D_A \otimes C$ and $\beta_* = \text{id}$ on $K_0(O_A)$. Then there exists a unitary $w \in M(O_A \otimes K)$ such that

$$\text{(Ad}(w^*) \circ \beta)(1 \otimes e_{kk}) = 1 \otimes e_{kk},$$

$$\text{(Ad}(w^*) \circ \beta)(O_A \otimes e_{kk}) = O_A \otimes e_{kk},$$

$$\text{Ad}(w^*) \circ \beta(D_A \otimes e_{kk}) = D_A \otimes e_{kk},$$

for all $k \in \mathbb{N}$.

**Proof.** Take $w_k \in O_A \otimes K$ be a partial isometry for each $k \in \mathbb{N}$ satisfying the conditions of Lemma 9.2. It is easy to see that the summation $\sum_{k=1}^{\infty} w_k$ converges in the strict topology of $M(O_A \otimes K)$. By (9.1) and (9.2), we have $w^*w = w w^* = 1$ and $w(D_A \otimes C)w^* = w^*(D_A \otimes C)w = D_A \otimes C$. We then see that

$$w(1 \otimes e_{kk})w^* = w w_k^* w w^* = w_k w_k^* = \beta(1 \otimes e_{kk})$$

so that $(\text{Ad}(w^*) \circ \beta)(1 \otimes e_{kk}) = 1 \otimes e_{kk}$. For $x \in O_A$, we have

$$\text{(Ad}(w^*) \circ \beta)(x \otimes e_{kk}) = (\text{Ad}(w^*) \circ \beta)((1 \otimes e_{kk})(x \otimes e_{kk})(1 \otimes e_{kk})) = (1 \otimes e_{kk})(1 \otimes e_{kk})$$

so that $(\text{Ad}(w^*) \circ \beta)(O_A \otimes e_{kk}) = O_A \otimes e_{kk}$. As $\beta(D_A \otimes C) = D_A \otimes C$ and $w(D_A \otimes C)w = D_A \otimes C$, we have $\text{Ad}(w^*) \circ \beta(D_A \otimes e_{kk}) = D_A \otimes e_{kk}$. 

**Proof of Proposition 9.1** Suppose that $\beta \in \text{Aut}(O_A \otimes K)$ satisfies $\beta(D_A \otimes C) = D_A \otimes C$ and $\beta_* = \text{id}$ on $K_0(O_A)$. Take a unitary $w \in M(O_A \otimes K)$ satisfying the conditions of Lemma 9.3. Put $\beta_w = \text{Ad}(w^*) \circ \beta \in \text{Aut}(O_A \otimes K)$. Since $(\text{Ad}(w^*) \circ \beta)(O_A \otimes e_{kk}) = O_A \otimes e_{kk}$, we may find an automorphism $\alpha_k \in \text{Aut}(O_A)$ for $k \in \mathbb{N}$ such that

$$\alpha_k(x) \otimes e_{kk} = \beta_w(x \otimes e_{kk}) \quad \text{for } x \in O_A.$$

By replacing $\beta$ with $\beta_w$, we may assume that $\beta_w(x \otimes e_{kk}) = \alpha_k(x) \otimes e_{kk}$. For $j, k \in \mathbb{N}$, we have

$$\beta(x \otimes e_{jk}) = \beta((1 \otimes e_{jk})(x \otimes e_{jk})) = \beta(1 \otimes e_{jk}) \cdot (\alpha_k(x) \otimes e_{jk}).$$

By putting $x = 1$, we see that

$$\beta(1 \otimes e_{jk}) = (1 \otimes e_{jj}) \beta(1 \otimes e_{jk}) (1 \otimes e_{kk})$$

so that there exists $w_{jk} \in O_A$ such that $w_{jk}^* = w_{kj}$ and $\beta(1 \otimes e_{jk}) = w_{jk} \otimes e_{jk}$. Since

$$w_{jk}^* w_{jk} \otimes e_{kk} = \beta(1 \otimes e_{jk})^* \beta(1 \otimes e_{jk}) = \beta(1 \otimes e_{kk}) = 1 \otimes e_{kk}$$

so that $w_{jk}^* w_{jk} = 1$ and similarly $w_{jk} w_{jk}^* = 1$. We also have for $a \in D_A$

$$w_{jk}^* a w_{jk} \otimes e_{jj} = \beta((1 \otimes e_{jk})(a \otimes e_{kk})(1 \otimes e_{kj})) = \beta(a \otimes e_{jj}) = \alpha_j(a) \otimes e_{jj}$$

30
so that \( w_{jk} D_A w_{jk}^* = D_A \). Since

\[
\beta(x \otimes e_{jk}) = \beta(1 \otimes e_{jk}) \cdot (\alpha_k(x) \otimes e_{kk}) = w_{jk} \alpha_k(x) \otimes e_{jk}
\]

and similarly \( \beta(x \otimes e_{jk}) = \alpha_j(x) w_{jk} \otimes e_{jk} \), we see \( w_{jk} \alpha_k(x) \otimes e_{jk} = \alpha_j(x) w_{jk} \otimes e_{jk} \) and hence \( \alpha_k(x) = w_{jk}^* \alpha_j(x) w_{jk} \) for \( x \in O_A \). Put \( u = \sum_{k=1}^{\infty} w_{1k} \otimes e_{kk} \) which is easily proved to be a unitary in \( M(O_A \otimes K) \). It then follows that

\[
\beta(x \otimes e_{jk}) = \beta((1 \otimes e_{j1}) (x \otimes e_{11}) (1 \otimes e_{1k}))
\]

\[
= (w_{j1} \otimes e_{j1})(\alpha_1(x) \otimes e_{11})(w_{1k} \otimes e_{1k})
\]

\[
= (w_{j1} \alpha_1(x) w_{1k}) \otimes e_{jk}
\]

\[
= u^*(\alpha_1(x) \otimes e_{jk}) u
\]

for \( x \in O_A \) so that \( \beta = \text{Ad}(u^*) \circ (\alpha_1 \otimes \text{id}) \). Since \( w_{1k} D_A w_{1k}^* = D_A \), we have \( u(D_A \otimes C) = D_A \otimes C \). By [13, Lemma 1.1], we know that \( \text{Ad}(u) = \text{id} \) on \( K_0(O_A) \) so that \( \alpha_0 = (\beta^{-1})^* = \text{id} \) on \( K_0(O_A) \).

We thus have the following theorem.

**Theorem 9.4.** Let \( \beta \in \text{Aut}(O_A \otimes K) \). Then \( \beta \) satisfies the following condition

\[
\beta(D_A \otimes C) = D_A \otimes C \quad \text{and} \quad \beta_* = \text{id} \text{ on } K_0(O_A) \quad (9.3)
\]

if and only if there exists an automorphism \( \alpha \in \text{Aut}(O_A) \) and a unitary \( u \in M(O_A \otimes K) \) such that

\[
\beta = \text{Ad}(u) \circ (\alpha \otimes \text{id}) \quad \text{and} \quad \alpha_* = \text{id} \text{ on } K_0(O_A), \quad (9.4)
\]

\[
u(D_A \otimes C) u^* = D_A \otimes C, \quad \alpha(D_A) = D_A. \quad (9.5)
\]

The following proposition shows that the expression \( \beta \) in the form (9.4) and (9.5) is unique up to inner automorphisms on \( O_A \) invariant globally \( D_A \).

**Proposition 9.5.** Suppose that \( \beta \in \text{Aut}(O_A \otimes K, D_A \otimes C) \) is of the form \( \beta = \text{Ad}(u) \circ (\alpha \otimes \text{id}) = \text{Ad}(u') \circ (\alpha' \otimes \text{id}) \) for some automorphisms \( \alpha, \alpha' \in \text{Aut}(O_A, D_A) \) and unitaries \( u, u' \in M(O_A \otimes K) \) satisfying both the conditions (9.4) and (9.5). Then there exists a unitary \( V \in O_A \) such that

\[
u = u'(V \otimes 1), \quad \alpha = \text{Ad}(V^*) \circ \alpha' \quad \text{and} \quad V D_A V^* = D_A. \quad (9.6)
\]

**Proof.** For \( x \otimes K \in O_A \otimes K \), we have \( u(\alpha(x) \otimes K) u^* = u'(\alpha'(x) \otimes K) u^* \). Put \( v = u^* u \in M(O_A \otimes K) \) which is a unitary satisfying \( v(\alpha(x) \otimes K) = (\alpha'(x) \otimes K) v \). We in particular see that \( v(1 \otimes e_{jk}) = (1 \otimes e_{jk}) v \) for all \( j, k \in \mathbb{Z}_+ \). Define \( V \in O_A \) by setting \( V \otimes e_{11} = (1 \otimes e_{11}) v(1 \otimes e_{11}) \). As \( v \) commutes with \( 1 \otimes e_{11} \), we know that \( V \) is a unitary in \( O_A \). We then have

\[
u^* u(1 \otimes e_{kk}) = v(1 \otimes e_{k1})(1 \otimes e_{1k})
\]

\[
= (1 \otimes e_{k1}) v(1 \otimes e_{1k})
\]

\[
= (1 \otimes e_{k1})(V \otimes e_{11})(1 \otimes e_{1k})
\]

\[
= (V \otimes 1)(1 \otimes e_{k1})(1 \otimes e_{11})(1 \otimes e_{1k})
\]

\[
= (V \otimes 1)(1 \otimes e_{kk})
\]

31
for all $k \in \mathbb{Z}_+$. Hence we have $u^*u = V \otimes 1$. As we have for $x \in \mathcal{O}_A$

$$\alpha(x) \otimes e_{11} = v^*(\alpha'(x) \otimes e_{11})v$$

$$= v^*(1 \otimes e_{11})(\alpha'(x) \otimes e_{11})(1 \otimes e_{11})v$$

$$= (V^* \otimes e_{11})(\alpha'(x) \otimes e_{11})(V \otimes e_{11})$$

$$= V^*\alpha'(x)V \otimes e_{11}$$

so that $\alpha(x) = V^*\alpha'(x)V$ for $x \in \mathcal{O}_A$. As $\alpha(D_A) = \alpha'(D_A) = D_A$, we have $V D_A V^* = D_A$. \hfill $\square$

**Corollary 9.6.** Let $\beta \in \text{Aut}(\mathcal{O}_A \otimes \mathcal{K})$. Let us denote by $1_A$ the unit of the $C^*$-algebra $\mathcal{O}_A$. Then $\beta$ satisfies the following condition

$$\beta(D_A \otimes \mathcal{C}) = D_A \otimes \mathcal{C} \quad \text{and} \quad \beta_*([1_A \otimes e_{11}]) = [1_A \otimes e_{11}] \quad \text{on} \quad K_0(\mathcal{O}_A \otimes \mathcal{K})$$

if and only if there exists an automorphism $\alpha \in \text{Aut}(\mathcal{O}_A)$ and a unitary $u \in M(\mathcal{O}_A \otimes \mathcal{K})$ such that

$$\beta = \text{Ad}(u) \circ (\alpha \otimes \text{id}) \quad \text{and} \quad \alpha_* = \beta_* \quad \text{on} \quad K_0(\mathcal{O}_A),$$

$$u(D_A \otimes \mathcal{C})u^* = D_A \otimes \mathcal{C}, \quad \alpha(D_A) = D_A.$$

**Proof.** The if part is clear. It suffices to show the only if part. Suppose that $\beta \in \text{Aut}(\mathcal{O}_A \otimes \mathcal{K})$ satisfies the following conditions

$$\beta(D_A \otimes \mathcal{C}) = D_A \otimes \mathcal{C} \quad \text{and} \quad \beta_*([1_A \otimes e_{11}]) = [1_A \otimes e_{11}] \quad \text{on} \quad K_0(\mathcal{O}_A \otimes \mathcal{K}).$$

Since $\beta \in \text{Aut}(\mathcal{O}_A \otimes \mathcal{K})$ satisfies $\beta_*([1_A \otimes e_{11}]) = [1_A \otimes e_{11}]$, By \[30\], there exists an automorphism $\alpha_0$ of $\mathcal{O}_A$ such that $\alpha_{0*} = \beta_*$ on $K_0(\mathcal{O}_A)$. Hence by using \[33\] Proposition 5.1, we may find an automorphism $\alpha_1$ of $\mathcal{O}_A$ such that $\alpha_1(D_A) = D_A$ and $\alpha_{1*} = \alpha_0$ on $K_0(\mathcal{O}_A)$. Put $\beta_1 := \beta \circ (\alpha_1^{-1} \otimes \text{id}) \in \text{Aut}(\mathcal{O}_A \otimes \mathcal{K})$. We have $\beta_1(D_A \otimes \mathcal{C}) = D_A \otimes \mathcal{C}$ and $\beta_{1*} = \beta_* \circ \alpha_1^{-1} = \text{id}$ on $K_0(\mathcal{O}_A)$. By Theorem 9.4 one may take an automorphism $\alpha_2 \in \text{Aut}(\mathcal{O}_A)$ and a unitary $u \in M(\mathcal{O}_A \otimes \mathcal{K})$ such that

$$\beta_1 = \text{Ad}(u) \circ (\alpha_2 \otimes \text{id}) \quad \text{and} \quad \alpha_{2*} = \text{id} \quad \text{on} \quad K_0(\mathcal{O}_A),$$

$$u(D_A \otimes \mathcal{C})u^* = D_A \otimes \mathcal{C}, \quad \alpha_2(D_A) = D_A.$$

Put $\alpha := \alpha_2 \circ \alpha_1 \in \text{Aut}(\mathcal{O}_A)$. We then have

$$\beta = \text{Ad}(u) \circ (\alpha \otimes \text{id}), \quad \alpha_* = \beta_* \quad \text{on} \quad K_0(\mathcal{O}_A), \quad \alpha(D_A) = D_A. \hfill \square$$

Let

$$\text{Aut}_o(\mathcal{O}_A, D_A) = \{ \alpha \in \text{Aut}(\mathcal{O}_A) \mid \alpha(D_A) = D_A, \alpha_* = \text{id} \quad \text{on} \quad K_0(\mathcal{O}_A) \}.$$ 

Since $\text{Int}(\mathcal{O}_A, D_A)$ is a subgroup of $\text{Aut}_o(\mathcal{O}_A, D_A)$, we may consider the quotient group $\text{Aut}_o(\mathcal{O}_A, D_A)/\text{Int}(\mathcal{O}_A, D_A)$ which we denote by $\text{Out}_o(\mathcal{O}_A, D_A)$. Thanks to Theorem 9.4
and Corollary [9.6] we know the following theorem on the relative Picard group \( \text{Pic}(\mathcal{O}_A, \mathcal{D}_A) \), which are relative versions of the results shown in Appendix after this section.

Let \( \Psi : \text{Aut}(\mathcal{O}_A, \mathcal{D}_A) \to \text{Aut}(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C}) \) be the homomorphism defined by \( \Psi(\alpha) = \alpha \otimes \text{id} \). Since \( \Psi(\text{Int}(\mathcal{O}_A, \mathcal{D}_A)) \subset \text{Int}(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C}) \), it induces a homomorphism from \( \text{Out}(\mathcal{O}_A, \mathcal{D}_A) \) to \( \text{Out}(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C}) \) written \( \bar{\Psi} \). The following is a corollary of Proposition 9.5.

**Corollary 9.7.** The homomorphism \( \bar{\Psi} : \text{Out}(\mathcal{O}_A, \mathcal{D}_A) \to \text{Out}(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C}) \) is injective.

**Proof.** Suppose that \( \alpha \in \text{Aut}(\mathcal{O}_A, \mathcal{D}_A) \) satisfies \( \alpha \otimes \text{id} = \text{Ad}(u') \) for some \( u' \in \mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C} \). Put \( \alpha' = \text{id} \) and \( u = 1 \) in the statement of Proposition 9.5 to have a unitary \( V \in \mathcal{O}_A \otimes \mathcal{D}_A \) such that \( u' = V \otimes 1 \) and \( \alpha = \text{Ad}(V) \).

By [13] Lemma 1.1], we may define a homomorphism \( K_* : \text{Out}(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C}) \to \text{Aut}(\mathcal{K}_0(\mathcal{O}_A \otimes \mathcal{K})) \) by setting \( K_*([\alpha]) = \alpha_* \) for \( [\alpha] \in \text{Out}(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C}) \).

**Theorem 9.8.** Let \( A \) be an irreducible non-permutation matrix. Then the following short exact sequence holds:

\[
1 \to \text{Out}_\circ(\mathcal{O}_A, \mathcal{D}_A) \xrightarrow{\bar{\Psi}} \text{Out}(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C}) \xrightarrow{K_*} \text{Aut}(\mathcal{K}_0(\mathcal{O}_A \otimes \mathcal{K})) \to 1.
\]

Hence there exists a short exact sequence:

\[
1 \to \text{Out}_\circ(\mathcal{O}_A, \mathcal{D}_A) \xrightarrow{\bar{\Psi}} \text{Pic}(\mathcal{O}_A, \mathcal{D}_A) \xrightarrow{K_*} \text{Aut}(\mathbb{Z}^N/(\text{id} - A^t)\mathbb{Z}^N) \to 1.
\]

**Proof.** We will show the exactness of (9.7). The injectivity of the homomorphism \( \bar{\Psi} : \text{Out}_\circ(\mathcal{O}_A, \mathcal{D}_A) \to \text{Out}(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C}) \) follows from Corollary 9.7. The inclusion relation \( \bar{\Psi}(\text{Out}_\circ(\mathcal{O}_A, \mathcal{D}_A)) \subset \text{Ker}(K_*) \) is clear. Conversely for any \( [\beta] \in \text{Ker}(K_*) \), we know \( \beta \in \text{Aut}(\mathcal{O}_A \otimes \mathcal{K}) \) satisfy \( \beta_* = \text{id} \) on \( \mathcal{K}_0(\mathcal{O}_A) \). By Theorem 9.4 there exist a unitary \( u \in \mathcal{M}(\mathcal{O}_A \otimes \mathcal{K}) \) and an automorphism \( \alpha \in \text{Aut}(\mathcal{O}_A, \mathcal{D}_A) \) such that \( \beta = \text{Ad}(u) \circ (\alpha \otimes \text{id}) \) and \( \alpha_* = \text{id} \) on \( \mathcal{K}_0(\mathcal{O}_A) \). Hence we have \( [\beta] = [\alpha \otimes \text{id}] = \bar{\Psi}([\alpha]) \) and \( [\alpha] \in \text{Aut}_\circ(\mathcal{O}_A, \mathcal{D}_A) \) such that we have \( \bar{\Psi}(\text{Out}_\circ(\mathcal{O}_A, \mathcal{D}_A)) = \text{Ker}(K_*) \).

For any \( \xi \in \text{Aut}(\mathcal{K}_0(\mathcal{O}_A \otimes \mathcal{K})) \), \( \xi \) gives rise to an automorphism of the abelian group \( \mathbb{Z}^N/(\text{id} - A^t)\mathbb{Z}^N \). The group \( \mathbb{Z}^N/(\text{id} - A^t)\mathbb{Z}^N \) is isomorphic to the Bowen–Franks group \( BF(A) = \mathbb{Z}^N/(\text{id} - A)\mathbb{Z}^N \) of the matrix \( A \). By Huang’s theorem [9 Theorem 2.15] and its proof, any automorphism of the Bowen–Franks group \( BF(A) \) comes from an flow equivalence of the underlying topological Markov shifts \( (X_A, \sigma_A) \). It implies that there exists an automorphism \( \psi \in \text{Aut}(\mathcal{O}_A \otimes \mathcal{K}) \) such that \( \psi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_A \otimes \mathcal{C} \) and \( \psi_* = \xi \) on \( \mathcal{K}_0(\mathcal{O}_A) \). Hence \( \psi \) belongs to \( \text{Aut}(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C}) \) such that \( K_*(\psi) = \xi \). Consequently the sequence (9.7) is exact.

Let \( \text{Aut}_1(\mathbb{Z}^N/(\text{id} - A^t)\mathbb{Z}^N) \) be a subgroup of \( \text{Aut}(\mathbb{Z}^N/(\text{id} - A^t)\mathbb{Z}^N) \) defined by

\[
\text{Aut}_1(\mathbb{Z}^N/(\text{id} - A^t)\mathbb{Z}^N) = \{ \xi \in \text{Aut}(\mathbb{Z}^N/(\text{id} - A^t)\mathbb{Z}^N) \mid \xi([1]) = [1] \}
\]

where \([1] \in \mathbb{Z}^N/(\text{id} - A^t)\mathbb{Z}^N \) denotes the class of the vector \((1, \ldots, 1)\) in \( \mathbb{Z}^N \).

33
Theorem 9.9. Let $A$ be an irreducible non-permutation matrix. Then there exists a short exact sequence:

$$1 \longrightarrow \text{Out}(O_A, D_A) \overset{\psi}{\longrightarrow} \text{Pic}(O_A, D_A) \overset{K_{\psi}}{\longrightarrow} \text{Aut}(\mathbb{Z}^N/(id - A^t)\mathbb{Z}^N)/\text{Aut}_1(\mathbb{Z}^N/(id - A^t)\mathbb{Z}^N) \longrightarrow 1.$$ 

Proof. It suffices to show the exactness at the middle. The inclusion relation $\Psi(\text{Out}(O_A)) \subset \text{Ker}(K_{\psi})$ is clear. Conversely, by Rørdam’s result [30] again, for any $\xi \in \text{Aut}(K_0(O_A \otimes K))$ with $\xi([1]) = [1]$, there exists an automorphism $\alpha_\xi$ of $O_A$ such that $\alpha_{\xi*} = \xi$ on $K_0(O_A)$. By [18] Proposition 5.1, we may find an automorphism $\alpha_1$ of $O_A$ such that $\alpha_1(D_A) = D_A$ and $\alpha_{1*} = \alpha_{\xi*}$ on $K_0(O_A)$. Hence $\alpha_1 \in \text{Aut}(O_A, D_A)$ such that $\Psi([\alpha_1]) = \xi$ so that $\Psi(\text{Out}(O_A)) = \text{Ker}(K_{\psi})$, and the sequence is exact. 

10 Appendix: Picard groups of Cuntz–Krieger algebras

In this appendix, we will refer to the Picard groups of Cuntz–Krieger algebras and especially Cuntz algebras. As examples of the Picard groups for some interesting class of $C^*$-algebras, K. Kodaka has studied the Picard groups for irrational rotation $C^*$-algebras $\text{Pic}(A_\theta)$ to show that $\text{Pic}(A_\theta)$ is isomorphic to $\text{Out}(A_\theta)$ if $\theta$ is not quadratic, and a semidirect product $\text{Out}(A_\theta) \rtimes \mathbb{Z}$ if $\theta$ is quadratic ([12], [13]). He also studied the Picard group of certain Cuntz algebras in [13]. He proved that $\text{Pic}(O_N) = \text{Out}(O_N)$ for $N = 2, 3$. He also showed that there exists a short exact sequence:

$$1 \longrightarrow \text{Out}(O_N) \overset{\Psi}{\longrightarrow} \text{Pic}(O_N) \overset{K_{\Psi}}{\longrightarrow} \text{Aut}(\mathbb{Z}/(1 - N)\mathbb{Z}) \longrightarrow 1 \quad (10.1)$$

for $N = 4, 6$. Since $\text{Aut}(\mathbb{Z}/(1 - N)\mathbb{Z})$ is trivial for $N = 2, 3$, the Kodaka’s results say that the exact sequence (10.1) holds for $N = 2, 3, 4, 6$.

We will show that the above exact sequence holds for all $1 < N \in \mathbb{N}$ (Theorem 10.4). As a corollary we know that the Picard group $\text{Pic}(O_N)$ of the Cuntz algebra $O_N$ is a semidirect product $\text{Out}(O_N) \rtimes \mathbb{Z}/(N - 2)\mathbb{Z}$ if $N - 1$ is a prime number.

We first refer to the Picard groups of Cuntz–Krieger algebras. Let $u \in M(A)$ be a unitary in the multiplier $C^*$-algebra $M(A)$ of a $C^*$-algebra $A$. The automorphism $\text{Ad}(u)$ on $A$ acts trivially on its K-group $K_0(A)$ by [13] Lemma 1.1.

Lemma 10.1 (Kodaka [13] Lemma 1.3). Let $\beta \in \text{Aut}(O_A \otimes K)$ satisfy $\beta_* = \text{id}$ on $K_0(O_A)$. Then there exists a unitary $u \in M(O_A \otimes K)$ and an automorphism $\alpha \in \text{Aut}(O_A)$ such that

$$\beta = \text{Ad}(u) \circ (\alpha \otimes \text{id}) \quad \text{and} \quad \alpha_* = \text{id} \text{ on } K_0(O_A).$$

For a $C^*$-algebra $A$, we put

$$\text{Aut}_0(A) = \{ \alpha \in \text{Aut}(A) \mid \alpha_* = \text{id} \text{ on } K_0(A) \}$$

which is a subgroup of $\text{Aut}(A)$. Since $\text{Ad}(u)_* = \text{id}$ on $K_0(A)$ for a unitary $u \in M(A)$, we see that $\text{Int}(A)$ is a subgroup of $\text{Aut}_0(A)$. The quotient group $\text{Aut}_0(A)/\text{Int}(A)$ is denoted by $\text{Out}_0(A)$. 

34
Let $A$ be a unital $C^*$-algebra. Let $\Psi : \text{Aut}(A) \rightarrow \text{Aut}(A \otimes K)$ be the homomorphism defined by $\Psi(\alpha) = \alpha \otimes \text{id}$ for $\alpha \in \text{Aut}(A)$. It induces a homomorphism $\bar{\Psi} : \text{Out}(A) \rightarrow \text{Out}(A \otimes K)$. If $\bar{\Psi}([\alpha]) = \text{id}$ for some $\alpha \in \text{Aut}(A)$, we have $\Psi(\alpha) = \text{Ad}(W)$ for some unitary $W \in M(A \otimes K)$. Hence we see that

$$\alpha(x) \otimes K = W(x \otimes K)W^* \quad \text{for all } x \in A, K \in K.$$  \quad (10.2)

Since

$$1 \otimes e_{11} = \alpha(1) \otimes e_{11} = W(1 \otimes e_{11})W^*,$$

the unitary $W$ commutes $1 \otimes e_{11}$ so that there exists a unitary $w \in A$ such that $w \otimes e_{11} = (1 \otimes e_{11})W(1 \otimes e_{11})$. We then have

$$\alpha(x) \otimes e_{11} = (1 \otimes e_{11})W(x \otimes e_{11})(1 \otimes e_{11}) = wxw^* \otimes e_{11} \quad \text{for all } x \in A.$$

Hence $\alpha = \text{Ad}(w) \in \text{Int}(A)$. This means that the map $\bar{\Psi} : \text{Out}(A) \rightarrow \text{Out}(A \otimes K)$ is injective. Any automorphism $\beta \in \text{Aut}(A \otimes K)$ induces an automorphism $\beta_* = K_*|_{\mathbf{Z}_1}$, which we denote by $K_*|_{\mathbf{Z}_1}$. By [3, Theorem 1.2] with [3, Corollary 3.5], we know $\text{Pic}(A) = \text{Pic}(A \otimes K) = \text{Out}(A \otimes K)$.

**Proposition 10.2.** Let $A$ be an irreducible non-permutation matrix. Then the following short exact sequence holds:

$$1 \rightarrow \text{Out}_*(A) \xrightarrow{\bar{\Psi}} \text{Out}(O_A \otimes K) \xrightarrow{K_*} \text{Aut}(K_0(O_A \otimes K)) \rightarrow 1.$$  \quad (10.3)

Hence there exists a short exact sequence:

$$1 \rightarrow \text{Out}_*(O_A) \xrightarrow{\Psi} \text{Pic}(O_A) \xrightarrow{K_*} \text{Aut}(\mathbf{Z}^N/(\text{id} - A^t)\mathbf{Z}^N) \rightarrow 1.$$  \quad (10.4)

**Proof.** We will show the exactness of (10.3). We have already known that the injectivity of $\bar{\Psi} : \text{Out}_*(O_A) \rightarrow \text{Out}(O_A \otimes K)$. By definition of the group $\text{Aut}_*(O_A)$, the inclusion relation $\bar{\Psi}(\text{Out}_*(O_A)) \subset \text{Ker}(K_*)$ is clear. Conversely for any $[\beta] \in \text{Ker}(K_*)$, we know that $\beta \in \text{Out}(O_A \otimes K)$ satisfy $\beta_* = \text{id}$ on $K_0(O_A)$. By Lemma 10.1 there exists a unitary $u \in M(O_A \otimes K)$ and an automorphism $\alpha \in \text{Aut}(O_A)$ such that

$$\beta = \text{Ad}(u) \circ (\alpha \otimes \text{id}) \quad \text{and} \quad \alpha_* = \text{id} \text{ on } K_0(O_A).$$

Hence we have $[\beta] = [\alpha \otimes \text{id}] = \bar{\Psi}([\alpha])$ and $[\alpha] \in \text{Aut}_*(O_A)/\text{Int}(O_A)$. Therefore we have $\bar{\Psi}(\text{Out}_*(O_A)) = \text{Ker}(K_*)$.

By Rørdam’s result [30], for any $\xi \in \text{Aut}(K_0(O_A \otimes K))$, there exists an automorphism $\beta$ of $O_A \otimes K$ such that $\beta_* = \xi$. Therefore the map $K_*$ is surjective to prove the exactness of the sequence (10.3). \hfill $\Box$

Let $\text{Aut}_1(\mathbf{Z}^N/(\text{id} - A^t)\mathbf{Z}^N)$ be a subgroup of $\text{Aut}(\mathbf{Z}^N/(\text{id} - A^t)\mathbf{Z}^N)$ defined by

$$\text{Aut}_1(\mathbf{Z}^N/(\text{id} - A^t)\mathbf{Z}^N) = \{\xi \in \text{Aut}(\mathbf{Z}^N/(\text{id} - A^t)\mathbf{Z}^N) \mid [\xi([1])] = [1]\}$$

where $[1] \in \mathbf{Z}^N/(\text{id} - A^t)\mathbf{Z}^N$ denotes the class of the vector $(1, \ldots, 1)$ in $\mathbf{Z}^N$. \hfill 35
**Proposition 10.3.** Let $A$ be an irreducible non-permutation matrix. Then there exists a short exact sequence:

$$1 \rightarrow \text{Out}(\mathcal{O}_A) \xrightarrow{\Psi} \text{Pic}(\mathcal{O}_A) \xrightarrow{K} \text{Aut}(\mathbb{Z}^N/(\text{id} - A^t)\mathbb{Z}^N) \rightarrow 1.$$ 

**Proof.** It suffices to show the exactness at the middle. The inclusion relation $\Psi(\text{Out}_o(\mathcal{O}_A)) \subset \text{Ker}(K)$ is clear. Conversely, by Rørdam’s result \cite{30} again, for any $\xi \in \text{Aut}(K_0(\mathcal{O}_A \otimes \mathcal{K}))$ with $\xi([1]) = [1]$, there exists an automorphism $\beta$ of $\mathcal{O}_A$ such that $\beta_* = \xi$. The sequence is exact. 

We will finally mention about the Picard groups of Cuntz algebras. By using Proposition \ref{prop:10.2}, we know the following theorem. For $N = 2, 3, 4, 6$, Kodaka has already shown in [12, Corollary 15, Remark 17].

**Theorem 10.4.** For each $1 < N \in \mathbb{N}$, there exists a short exact sequence:

$$1 \rightarrow \text{Out}(\mathcal{O}_N) \xrightarrow{\Psi} \text{Pic}(\mathcal{O}_N) \xrightarrow{K} \text{Aut}(\mathbb{Z}/(1 - N)\mathbb{Z}) \rightarrow 1 \quad (10.5)$$

**Proof.** Since $K_0(\mathcal{O}_N) = \mathbb{Z}/(1 - N)\mathbb{Z}$ by [5] and the unit 1 of the $C^*$-algebra $\mathcal{O}_N$ corresponds to the generator $[1]$ of the cyclic group $\mathbb{Z}/(1 - N)\mathbb{Z}$, the fact $\alpha(1) = 1$ for any automorphism $\alpha \in \text{Aut}(\mathcal{O}_N)$ ensures us that $\alpha_* = \text{id}$ on $K_0(\mathcal{O}_N)$. Hence we see that $\text{Aut}_o(\mathcal{O}_N) = \text{Aut}(\mathcal{O}_N)$ and hence $\text{Out}_o(\mathcal{O}_N) = \text{Out}(\mathcal{O}_N)$. Therefore the exact sequence (10.3) goes to (10.5). 

As a corollary, we have

**Corollary 10.5.** Suppose that $N - 1$ is a prime number. Then the Picard group $\text{Pic}(\mathcal{O}_N)$ of the Cuntz algebra $\mathcal{O}_N$ is a semidirect product $\text{Out}(\mathcal{O}_N) \rtimes \mathbb{Z}/(N - 2)\mathbb{Z}$ of the outer automorphism group by the cyclic group $\mathbb{Z}/(N - 2)\mathbb{Z}$:

$$\text{Pic}(\mathcal{O}_N) = \text{Out}(\mathcal{O}_N) \rtimes \mathbb{Z}/(N - 2)\mathbb{Z}.$$ 

**Proof.** As $N - 1$ is a prime number, an automorphism $\eta$ of the cyclic group $\mathbb{Z}/(1 - N)\mathbb{Z}$ is determined by $\eta(1)$ which can take its value in $\{1, 2, \ldots, N - 2\}$, so that we have $\text{Aut}(\mathbb{Z}/(1 - N)\mathbb{Z})$ is isomorphic to $\mathbb{Z}/(N - 2)\mathbb{Z}$. Since $N$ is not prime, by [12, Theorem 16], for any $k \in \mathbb{N}$ with $1 \leq k \leq N - 1$, there exists $\beta_k \in \text{Aut}(\mathcal{O}_N \otimes \mathcal{K})$ such that $(\beta_k)^* = k \cdot \text{id}$ on $K_0(\mathcal{O}_N)$. Hence the correspondence $k \in \{1, 2, \ldots, N - 1\} \rightarrow [\beta_k] \in \text{Pic}(\mathcal{O}_N)$ gives rise to a cross section for the exact sequence (10.5). Therefore the exact sequence (10.5) splits and yields a decomposition of $\text{Pic}(\mathcal{O}_N)$ into a semidirect product $\text{Out}(\mathcal{O}_N) \rtimes \mathbb{Z}/(N - 2)\mathbb{Z}$. 

**Remark 10.6.** After the first draft of the paper was completed, the following paper has appeared in arXiv.

Kazunori Kodaka, Tamotsu Teruya: The strong Morita equivalence for inclusions of $C^*$-algebras and conditional expectations for equivalence bimodules, \texttt{arXiv:1609.08263}.

In the above paper, Morita equivalence for pairs of $C^*$-algebras is defined. However, their definition of Morita equivalence is different from ours.

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