Layer methods for Navier-Stokes equations with additive noise

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Abstract

We propose and study a number of layer methods for stochastic Navier-Stokes equations (SNSE) with spatial periodic boundary conditions and additive noise. The methods are constructed using conditional probabilistic representations of solutions to SNSE and exploiting ideas of the weak sense numerical integration of stochastic differential equations. We prove some convergence results for the proposed methods. Results of numerical experiments on two model problems are presented.

Keywords Navier-Stokes equations, Oseen-Stokes equations, Helmholtz-Hodge-Leray decomposition, conditional Feynman-Kac formula, weak approximation of stochastic differential equations layer methods.

AMS 2000 subject classification. 65C30, 60H15, 60H35

1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(w(t), \mathcal{F}_t^w) = ((w_1(t), \ldots, w_q(t))^\top, \mathcal{F}_t^w)$ be a $q$-dimensional standard Wiener process, where $\mathcal{F}_t^w$, $0 \leq t \leq T$, is an increasing family of $\sigma$-subalgebras of $\mathcal{F}$ induced by $w(t)$. We consider the system of stochastic Navier-Stokes equations (SNSE) with additive noise for velocity $v$ and pressure $p$ in a viscous incompressible flow:

\[
dv(t) = \left[ \frac{a^2}{2} \Delta v - (v, \nabla) v - \nabla p + f(t, x) \right] dt + \sum_{r=1}^{q} \gamma_r(t, x) dw_r(t),
\]

\[0 \leq t \leq T, \quad x \in \mathbb{R}^n,\]

\[
\text{div } v = 0,
\]

with spatial periodic conditions

\[
v(t, x + Le_i) = v(t, x), \quad p(t, x + Le_i) = p(t, x),
\]

\[0 \leq t \leq T, \quad i = 1, \ldots, n,
\]

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and the initial condition

\[ v(0, x) = \varphi(x). \]  

(1.4)

In (1.1)-(1.2) we have \( v \in \mathbb{R}^n \), \( p \) is a scalar, \( f \in \mathbb{R}^n \), \( \gamma_r \in \mathbb{R}^n \); \( \{e_i\} \) is the canonical basis in \( \mathbb{R}^n \) and \( L > 0 \) is the period (for simplicity in writing, the periods in all the directions are taken the same). The functions \( f = f(t, x) \) and \( \gamma_r(t, x) \) are supposed to be spatial periodic as well. Further, we require that \( \gamma_r(t, x) \) are divergence free:

\[ \text{div} \gamma_r(t, x) = 0, \quad r = 1, \ldots, q. \]  

(1.5)

SNSE can be useful for explaining the turbulence phenomenon (see [6, 12, 20] and references therein). They have complicated dynamics and some interesting properties (e.g., ergodicity of solutions [14, 15, 18]). At the same time, rather little has been done in numerics for SNSE. Let us cite [13], where algorithms based on Wiener Chaos expansion are considered, and quite recent works [2, 4, 11], where splitting schemes with finite element or Galerkin approximations are applied. Here we suggest to exploit some probabilistic representations of solutions to SNSE for constructing numerical methods of the layer type. The proposed methods are promised to be effective, reliable numerical methods for studying SNSE. Layer methods for deterministic semilinear and quasilinear partial differential equations of parabolic type were proposed in [22, 24] (see also [23, 10]), and for deterministic NSEs they were first considered in [1] and further developed in [27, 28]. Layer methods for linear and semilinear stochastic partial differential equations (SPDE) were constructed and analyzed in [26].

The rest of the paper is organized as follows. In Section 2 we introduce additional notation and write down probabilistic representations for linearized SNSE (i.e., stochastic Oseen-Stokes equations) and for the SNSE (1.1)-(1.4) which we use in Section 3 for constructing layer methods for the SNSE. Three layer methods are given in Section 3 together with discussion of their implementation. Numerical error analysis is done in Section 4. Results of numerical experiments on two test models are presented in Section 5.

2 Preliminaries

In this section we recall the required function spaces [5, 33, 34, 20, 21] and write probabilistic representations of solutions to linearized SNSE and to SNSE resting on results from [16, 17, 29, 31].

2.1 Function spaces, the Helmholtz-Hodge-Leray decomposition, and notation

Let \( \{e_i\} \) be the canonical basis in \( \mathbb{R}^n \). We shall consider spatial periodic \( n \)-vector functions \( u(x) = (u^1(x), \ldots, u^n(x))^\top \) in \( \mathbb{R}^n : u(x + Le_i) = u(x), \quad i = 1, \ldots, n \), where \( L > 0 \) is the period in \( i \)th direction. Denote by \( Q = (0, L)^n \) the cube of the period (of course, one may consider different periods \( L_1, \ldots, L_n \) in the different directions). We denote by \( L^2(Q) \) the Hilbert space of functions on \( Q \) with the scalar product and the norm

\[ (u, v) = \int_Q \sum_{i=1}^n u^i(x)v^i(x)dx, \quad \|u\| = (u, u)^{1/2}. \]
We keep the notation \(| \cdot |\) for the absolute value of numbers and for the length of \(n\)-dimensional vectors, for example,
\[
|u(x)| = [(u^1(x))^2 + \cdots + (u^n(x))^2]^{1/2}.
\]

We denote by \(H^m_p(Q)\), \(m = 0, 1, \ldots\), the Sobolev space of functions which are in \(L^2(Q)\), together with all their derivatives of order less than or equal to \(m\), and which are periodic functions with the period \(Q\). The space \(H^m_p(Q)\) is a Hilbert space with the scalar product and the norm
\[
(u, v)_m = \int_Q \sum_{i=1}^{n} \sum_{[\alpha]} \leq m D^{\alpha i}_n u^i(x) D^{\alpha i}_n v^i(x) dx, \quad \|u\|_m = [(u, u)_m]^{1/2},
\]
where \(\alpha^i = (\alpha_1^i, \ldots, \alpha_n^i)\), \(\alpha_j \in \{0, \ldots, m\}\), \([\alpha^i] = \alpha_1^i + \cdots + \alpha_n^i\), and
\[
D^{\alpha^i} = D_1^{\alpha_1^i} \cdots D_n^{\alpha_n^i} = \frac{\partial^{[\alpha^i]}}{\partial x_1^{\alpha_1^i} \cdots \partial x_n^{\alpha_n^i}}, \quad i = 1, \ldots, n.
\]

Note that \(H^0_p(Q) = L^2(Q)\).

Introduce the Hilbert subspaces of \(H^m_p(Q)\):
\[
V^m_p = \{ v : v \in H^m_p(Q), \ \text{div} v = 0\}, \quad m > 0,
\]
\[
V^0_p = \text{the closure of } V^m_p, \quad m > 0 \text{ in } L^2(Q).
\]

Clearly,
\[
V^{m_1}_p = \text{the closure of } V^{m_2}_p \text{ in } H^{m_1}_p(Q) \text{ for any } m_2 \geq m_1.
\]

Denote by \(P\) the orthogonal projection in \(H^m_p(Q)\) onto \(V^m_p\) (we omit \(m\) in the notation \(P\) here). The operator \(P\) is often called the Leray projection. Due to the Helmholtz-Hodge-Leray decomposition, any function \(u \in H^m_p(Q)\) can be represented as
\[
u = Pu + \nabla g, \ \text{div} Pu = 0,
\]
where \(g = g(x)\) is a scalar \(Q\)-periodic function such that \(\nabla g \in H^m_p(Q)\). It is natural to introduce the notation \(P^\perp u := \nabla g\) and hence write
\[
u = Pu + P^\perp u
\]
with
\[
P^\perp u \in (V^m_p)^\perp = \{ v : v \in H^m_p(Q), \ v = \nabla g\}.
\]

Let
\[
u(x) = \sum_{n \in \mathbb{Z}^n} u_n e^{i(2\pi/L)(n,x)}, \ g(x) = \sum_{n \in \mathbb{Z}^n} g_n e^{i(2\pi/L)(n,x)}, \ g_0 = 0,
\]

\[
P\nu(x) = \sum_{n \in \mathbb{Z}^n} (P\nu)_n e^{i(2\pi/L)(n,x)}, \ P^\perp u(x) = \nabla g(x) = \sum_{n \in \mathbb{Z}^n} (P^\perp u)_n e^{i(2\pi/L)(n,x)}
\]
be the Fourier expansions of \(u, g, Pu,\) and \(P^\perp u = \nabla g\). Here \(u_n, (P\nu)_n,\) and \((P^\perp u)_n = (\nabla g)_n\) are \(n\)-dimensional vectors and \(g_n\) are scalars. We note that \(g_0\) can be any real
number but for definiteness we set \( g_0 = 0 \). The coefficients \((Pu)_n\), \((P^\perp u)_n\), and \(g_n\) can be easily expressed in terms of \(u_n\):

\[
(Pu)_n = u_n - \frac{u_n^\top n}{|n|^2} n, \quad (P^\perp u)_n = i\frac{2\pi}{L} g_n n = \frac{u_n^\top n}{|n|^2} n, \quad (2.2)
\]

\[g_n = -i\frac{L}{2\pi} \frac{u_n^\top n}{|n|^2}, \quad n \neq 0, \quad g_0 = 0.
\]

We have

\[\nabla e^{i(2\pi/L)(n,x)} = ne^{i(2\pi/L)(n,x)} \cdot \frac{2\pi}{L},\]

hence \(u_ne^{i(2\pi/L)(n,x)} \in V^m_p\) only if \((u_n, n) = 0\). We obtain from here that the orthogonal basis of the subspace \((V^m_p)^\perp\) consists of \(ne^{i(2\pi/L)(n,x)}, n \in \mathbb{Z}^n, n \neq 0\); and an orthogonal basis of \(V^m_p\) consists of \(ku_ne^{i(2\pi/L)(n,x)}, k = 1, \ldots, n - 1, n \in \mathbb{Z}^n\), where under \(n \neq 0\) the vectors \(ku_n\) are orthogonal to \(n\): \((ku_n, n) = 0, k = 1, \ldots, n - 1\), and they are orthogonal among themselves: \((ku_n, m u_n) = 0, k, m = 1, \ldots, n - 1, m \neq k\), and finally, for \(n = 0\), the vectors \(ku_0, k = 1, \ldots, n\), are orthogonal.

In what follows we suppose that the above assumptions hold.

**Assumptions 2.1.** We assume that the coefficients \(f(t, x)\) and \(\gamma_r(s, x), r = 1, \ldots, q\), are sufficiently smooth and the problem \((1.1)-(1.4)\) has a unique classical solution \(v(t, x), p(t, x), (t, x) \in [0, T] \times \mathbb{R}^n\), which has continuous derivatives in the space variable \(x\) up to some order, and the solution and the derivatives have uniformly in \((t, x)\) bounded moments of a sufficiently high order \(m, 2 \leq m < m_0\), where \(m_0 > 2\) is a positive number or \(m_0 = \infty\).

The solution \(v(t, x), p(t, x), (t, x) \in [0, T] \times \mathbb{R}^n\), to \((1.1)-(1.4)\) is \(F^m_t\)-adaptive, \(v(t, \cdot) \in V^m_p\) and \(\nabla p(t, \cdot) \in (V^m_p)^\perp\) for every \(t \in [0, T]\) and \(\omega \in \Omega\).

Assumptions of this kind are rather usual for works dedicated to numerics. They are rested on results concerning regularity of solutions (see, e.g., the corresponding theory for deterministic NSE in [33, 34]). Unfortunately, we could not find explicit results on the classical solution for SNSE in literature. At the same time, the question about existence of the unique sufficiently regular (with respect to \(x\)) solution of the SNSE \((1.1)-(1.4)\) on a time interval \([0, T]\) is analogous to the one in the deterministic case. Indeed, the following remark reduces this problem of regularity for the SNSE to regularity of solutions to NSE with random coefficients which is close to the theory of deterministic NSE treated in [33, 34].

**Remark 2.1** Let \(\Gamma(t, x) = \sum_{r=1}^q \int_0^t \gamma_r(s, x)dw_r(s)\). Then \(V(t, x) = v(t, x) + \Gamma(t, x)\) together with \(p(t, x)\) solves the following ‘usual’ NSE with random coefficients:

\[
\frac{\partial}{\partial t} V = \frac{\sigma^2}{2} \Delta V - (V - \Gamma(t, x), \nabla)(V - \Gamma(t, x)) - \nabla p + f(t, x) - \frac{\sigma^2}{2} \Delta \Gamma(t, x),
\]

\[0 \leq t \leq T, \quad x \in \mathbb{R}^n,\]

\[
\text{div } V = 0,
\]

with spatial periodic conditions

\[
V(t, x + Le_i) = V(t, x), \quad p(t, x + Le_i) = p(t, x),
\]

\[0 \leq t \leq T, \quad i = 1, \ldots, n,
\]
and the initial condition

\[ V(0, x) = \varphi(x). \]

### 2.2 Probabilistic representations of solutions to linearized SNSE

We start with considering a linearized version of the SNSE (1.1)-(1.4), i.e., the stochastic Oseen-Stokes equations (see [19]):

\[
\begin{align*}
     dv_a(t) &= \left[ \frac{\sigma^2}{2} \Delta v_a - (a, \nabla) v_a - \nabla p_a + f(t, x) \right] dt + \sum_{r=1}^{q} \gamma_r(t, x) dw_r(t), \quad 0 \leq t \leq T, \quad x \in \mathbb{R}^n, \\
     \text{div } v_a &= 0,
\end{align*}
\]

with spatial periodic conditions

\[ v_a(t, x + Le_i) = v_a(t, x), \quad p_a(t, x + Le_i) = p_a(t, x), \quad 0 \leq t \leq T, \quad i = 1, \ldots, n, \]

and the initial condition

\[ v_a(0, x) = \varphi(x), \]

where \( a = a(t, x) \) is an \( n \)-dimensional vector \( a = (a^1, \ldots, a^n)^\top \) with \( a^i \) being \( Q \)-periodic deterministic functions which have continuous derivatives with respect to \( x \) up to some order; and the rest of the notation is the same as in (1.1)-(1.4).

We re-write the problem (2.3)-(2.6) with positive direction of time into the problem with negative direction of time which is more convenient for making use of probabilistic representations. To this end, introduce the new time variable \( s = T - t \) and the functions \( u_a(s, x) := v_a(T - s, x), \tilde{a}(s, x) := a(T - s, x), \tilde{f}(s, x) := f(T - s, x), \tilde{\gamma}_r(s, x) := \gamma_r(T - s, x), \) and \( \tilde{p}_a(s, x) := p_a(T - s, x). \)

Further, we recall the definition of a backward Ito integral [31]. Introduce the “backward” Wiener processes

\[ \tilde{w}_r(t) := w_r(T) - w_r(T - t), \quad r = 1, \ldots, q, \quad 0 \leq t \leq T, \]

and a decreasing family of \( \sigma \)-subalgebras \( \mathcal{F}^{\tilde{w}}_{T-t} \), \( 0 \leq t \leq T \), induced by the increments \( w_r(T) - w_r(t') \), \( r = 1, \ldots, q, \) \( t' \geq t \). The increasing family of \( \sigma \)-subalgebras \( \mathcal{F}^\tilde{w}_t \) induced by \( \tilde{w}_r(s') \), \( s' \leq t \), coincides with \( \mathcal{F}^{\tilde{w}}_{T-t,t} \), while \( \mathcal{F}^\tilde{w}_{t,T} \) is induced by the increments \( \tilde{w}_r(T) - \tilde{w}_r(t') \), \( r = 1, \ldots, q, \) \( t' \geq t \), and coincides with \( \mathcal{F}_T^{w_{T-t}} \). The backward Ito integral with respect to \( \tilde{w}_r(s) \) is defined as the Ito integral with respect to \( w_r(s) \):

\[
\int_t^{t'} \psi(t'') d\tilde{w}_r(t'') := \int_{T-t}^{T-t'} \psi(T - t'') dw_r(t''), \quad 0 \leq t \leq t' \leq T,
\]

where \( \psi(T - t), \ t \leq T, \) is an \( \mathcal{F}^\tilde{w}_t \)-adapted square-integrable function and \( \psi(t) \) is \( \mathcal{F}^{\tilde{w}}_t \)-adapted. Note that \( w_r(t) = \tilde{w}_r(T) - \tilde{w}_r(T - t), \) \( r = 1, \ldots, q, \) \( 0 \leq t \leq T. \)

The backward stochastic Oseen-Stokes equations can be written as

\[
\begin{align*}
     -du_a(s) &= \left[ \frac{\sigma^2}{2} \Delta u_a - (\tilde{a}, \nabla) u_a - \nabla \tilde{p}_a + \tilde{f}(s, x) \right] ds + \sum_{r=1}^{q} \tilde{\gamma}_r(s, x) * d\tilde{w}_r(s), \quad 0 \leq s \leq T, \quad x \in \mathbb{R}^n, \\
     \text{div } u_a &= 0,
\end{align*}
\]
with spatial periodic conditions

\[ u_a(s, x + L e_i) = u_a(s, x), \quad \tilde{p}_a(s, x + L e_i) = \tilde{p}_a(s, x), \quad \text{for } 0 \leq s \leq T, \quad i = 1, \ldots, n, \]

and the terminal condition

\[ u_a(T, x) = \varphi(x). \]

We note that (2.8) implies

\[ \int_s^T \gamma_r(s', x) * d\tilde{w}_r(s') = \int_0^{T-s} \gamma_r(s', x) dw_r(s'). \]

The processes \( u_a(s, x) \) and \( \tilde{p}_a(s, x) \) are \( \mathcal{F}_{s,T}^\omega \)-adapted (and \( \mathcal{F}_{T-s}^\omega \)-adapted), they depend on \( \tilde{w}_r(T) - \tilde{w}_r(s') = w_r(T - s'), \quad s \leq s' \leq T \).

Let \( u_a(s, x) \) and \( \tilde{p}_a(s, x) \) be a solution of the problem (2.9)-(2.12). For the function \( u_a(s, x) \), one can use the following probabilistic representation of solutions to the Cauchy problem for linear SPDE of parabolic type (the conditional Feynman-Kac formula or the averaging over characteristics formula, see, e.g., [31] and [26]):

\[ u_a(s, x) = E^\tilde{w} [\varphi(X_{s,x}(T))Y_{s,x,1}(T) + Z_{s,x,1,0}(T)], \quad 0 \leq s \leq T, \]

where \( X_{s,x}(s'), Y_{s,x,y}(s'), Z_{s,x,y,z}(s'), \quad s' \geq s \), solves the system of Ito stochastic differential equations:

\[ dX = (-a(s', X) - \sigma \mu(s', X)) ds' + \sigma dW(s'), \quad X(s) = x, \]

\[ dY = \mu^1(s', X) Y dW(s'), \quad Y(s) = y, \]

\[ dZ = (-\nabla \tilde{p}_a(s', X) + \bar{f}(s', X)) Y ds' + F(s', X) Y dW(s') \]

\[ + \sum_{r=1}^{q} \tilde{\gamma}_r(s', X) Y d\tilde{w}_r(s'), \quad Z(s) = z. \]

In (2.13)-(2.16), \( W(s) \) is a standard \( n \)-dimensional Wiener process independent of \( \tilde{w}_r(s) \) on the probability space \((\Omega, \mathcal{F}, P)\); \( Y \) is a scalar, and \( Z \) is an \( n \)-dimensional column-vector; \( \mu(s, x) \) is an arbitrary \( n \)-dimensional spatial periodic vector function and \( F(s, x) \) is an arbitrary \( n \times n \)-dimensional spatial periodic matrix function, which are sufficiently smooth in \( s, x \); the expectation \( E^\tilde{w} \) in (2.13) is taken over the realizations of \( W(s), \quad t \leq s \leq T \), for a fixed \( \tilde{w}_r(s') \), \( r = 1, \ldots, q, \quad s \leq s' \leq T \), in other words, \( E^\tilde{w}(\cdot) \) means the conditional expectation:

\[ E(\cdot | \tilde{w}_r(s') - \tilde{w}_r(s), \quad r = 1, \ldots, q, \quad s \leq s' \leq T). \]

The probabilistic representation like (2.13)-(2.16) for the Cauchy problem (2.9), (2.12) is obtained (see, e.g., [31]) for linear SPDEs with deterministic coefficients. However here \( \tilde{p}_a(s, x) \) is a part of solution of problem (2.9)-(2.12) and it is random (more precisely it is \( \mathcal{F}_{s,T}^\omega \)-adapted). In this case the representation (2.13)-(2.16) can be rigorously justified in the following way. The solution \( u_a \) of (2.9), (2.12) can be represented in the form of the sum

\[ u_a = u^{(0)}_a + u^{(1)}_a, \]
where \( u_a^{(0)} \) satisfies the Cauchy problem for the backward deterministic linear parabolic PDE with random parameters:

\[
-\frac{\partial u_a^{(0)}}{\partial s} = \frac{\sigma^2}{2} \Delta u_a^{(0)} - (\bar{a}, \nabla) u_a^{(0)} - \nabla \tilde{p}_a, \tag{2.17}
\]

\[ u_a^{(0)}(T, x) = 0, \]

and \( u_a^{(1)} \) satisfies the Cauchy problem for the backward stochastic linear parabolic PDE with deterministic parameters:

\[
-du_a^{(1)}(s) = \left[ \frac{\sigma^2}{2} \Delta u_a^{(1)} - (\bar{a}, \nabla) u_a^{(1)} + \bar{f}(s, x) \right] ds + \sum_{r=1}^q \gamma_r(s, x) * d\tilde{w}_r(s), \tag{2.18}
\]

\[ u_a^{(1)}(T, x) = \varphi(x). \]

Clearly,

\[
u_a^{(0)}(s, x) = E^\omega \left[ Z_{s,x,1,0}^{(0)}(T) \right] = -E^\omega \int_s^T \nabla \tilde{p}_a(s', X_{s,x}(s')) Y_{s,x,1}(s') ds'.
\]

The Feynman-Kac formula for \( u_a^{(1)} \) coincides with \((2.13)-(2.16)\) under \( \nabla \tilde{p}_a(s, x) = 0 \).

Let \( \mathcal{F}_{s,t}^W \) be a \( \sigma \)-algebra induced by \( W_r(s') - W_r(s), r = 1, \ldots, n, s \leq s' \leq t \). We note that \( \nabla \tilde{p}_a(s', X_{s,x}(s')) \) in \((2.16)\) is \( \mathcal{F}_{s,s'}^W \mathcal{F}_{s',t}^\omega \)-adapted, where the family of \( \sigma \)-algebras \( \mathcal{F}_{s,s'}^W \mathcal{F}_{s',t}^\omega \) is neither increasing nor decreasing in \( s' \). Consequently, \( Z_{s,x,1,0}(s') \) is measurable with respect to \( \mathcal{F}_{s,s'}^W \mathcal{F}_{s',t}^\omega \) for every \( s' \in [s, T] \). Since \( \gamma_r(s', X_{s,x}(s')) Y(s') \) are independent of \( \tilde{w}_r \), the Ito integral in \((2.16)\) is well defined.

**Remark 2.2** We remark that within the non-anticipating stochastic calculus the probabilistic representation \((2.13)-(2.16)\) for the linear problem \((2.9)-(2.12)\) cannot be carried over to the backward SNSE problem by changing the coefficient \( \bar{a}(s, x) \) to \( u(s, x) \) since then the integrand \( \gamma_r(s', X_{s,x}(s')) Y(s') \) would be \( \mathcal{F}_{s,s'}^W \mathcal{F}_{s',t}^\omega \)-measurable. Nevertheless, the representation \((2.13)-(2.16)\) allows us to derive layer methods for the stochastic Oseen-Stokes equations \((2.9)-(2.12)\), and then, using them as a guidance, one can obtain layer methods for the SNSE \((1.1)-(1.4)\) as well (see Sections 3.1 and 3.2).

For deriving layer methods, we also use some direct probabilistic representations for solutions of the SNSE. In Sections 2.3 and 2.4 we give two such representations. The first one follows from a specific probabilistic representation for a linear SPDE which differs from \((2.13)-(2.16)\) and the second one uses backward doubly stochastic differential equations [30].

### 2.3 A direct probabilistic representation for solutions of SNSE

As in the case of the stochastic Oseen-Stokes equations, we re-write the SNSE problem \((1.1)-(1.4)\) with positive direction of time into the problem with negative direction of time. Again introduce the new time variable \( s = T - t \) and the functions \( u(s, x) := v(T - s, x) \),
\( \tilde{f}(s, x) := f(T-s, x) \), \( \tilde{\gamma}_r(s, x) := \gamma_r(T-s, x) \), and \( \tilde{p}(s, x) := p(T-s, x) \). The corresponding backward SNSE take the form:

\[ -du = \left( \frac{\sigma^2}{2} \Delta u - (u, \nabla) u - \nabla \tilde{p} + \tilde{f} \right) ds + \sum_{r=1}^{q} \tilde{\gamma}_r(s, x) * d\tilde{\omega}_r(s), \ u(T, x) = \varphi(x), \]  

(2.19)

\[ \text{div} u = 0, \]  

(2.20)

with spatial periodic conditions for \( u \) and \( \tilde{p} \).

Introduce \( F(s, x, u, \nabla u) := -(u, \nabla) u - \nabla \tilde{p} + \tilde{f} \) and write (2.19) as

\[ -du = \left( \frac{\sigma^2}{2} \Delta u + F(s, x, u, \nabla u) \right) ds + \sum_{r=1}^{q} \tilde{\gamma}_r(s, x) * d\tilde{\omega}_r(s), \ u(T, x) = \varphi(x). \]  

(2.21)

Let us assume that the solution \( u(s, x) = u(s, x, \omega) \) to (2.19)-(2.20) is known. We substitute it in the \( F(s, x, u, \nabla u) \) which becomes a function \( \tilde{F}(s, x, \omega) \) depending on \( \omega \) as a parameter. Hence (2.21) can be considered as a linear parabolic SPDE. For solutions of this linear SPDE, we can write the following probabilistic representation analogously to (2.13)-(2.16) (we take \( Y \equiv 1 \)):

\[ u(s, x) = E^{\tilde{\omega}} \varphi(X_{s,x}(T)) \]  

(2.22)

\[ -E^{\tilde{\omega}} \left[ \int_{s}^{T} \{ \nabla \tilde{p}(s', X_{s,x}(s')) - \tilde{f}(s', X_{s,x}(s')) \} \right. \]

\[ + \left. (u(s', X_{s,x}(s')), \nabla) u(s', X_{s,x}(s')) \} ds' \right] \]

\[ + \sum_{r=1}^{q} E^{\tilde{\omega}} \left[ \int_{s}^{T} \tilde{\gamma}_r(s', X_{s,x}(s')) d\tilde{\omega}_r(s') \right], \]

where \( X_{s,x}(s'), s' \geq s \), solves the system of stochastic differential equations

\[ dX = \sigma dW(s'), \ X(s) = x, \]  

(2.23)

\( W \) is a standard \( n \)-dimensional Wiener process independent of \( \tilde{\omega}_r \) on the probability space \( (\Omega, \mathcal{F}, P) \).

### 2.4 A probabilistic representation for solution of SNSE using backward doubly stochastic differential equations

In connection with the backward SNSE (2.19)-(2.20), we introduce the system of backward doubly stochastic differential equations [30]:

\[ dX = \sigma dW(s'), \ X(s) = x, \]  

(2.24)

\[ dU = (\nabla \tilde{p}(s', X) - \tilde{f}(s', X) + \frac{1}{\sigma} ZU) ds' + ZdW(s') - \sum_{r=1}^{q} \tilde{\gamma}_r(s', X) * d\tilde{\omega}_r(s'), \]  

(2.25)

\[ U(T) = \varphi(X_{s,x}(T)). \]  

(2.26)
In (2.24)-(2.26) $X, U, W$ are column vectors of dimension $n$ and $Z$ is a matrix of dimension $n \times n$, $W(s)$ and $\tilde{w}(s)$, $0 \leq s \leq T$, are mutually independent standard Wiener processes on the probability space $(\Omega, F, P)$. We recall that the triple \{$X_{s,x}(s'), U_{s,x}(s'), Z_{s,x}(s')$, $s \leq s' \leq T$\} is a solution of (2.24)-(2.26) if $X_{s,x}(s')$ satisfies (2.24), $(U_{s,x}(s'), Z_{s,x}(s'))$ for each $s'$ is $\mathcal{F}_{s',T}^{\tilde{w}}$-measurable, and

\[
U_{s,x}(s') = \varphi(X_{s,x}(T)) - \int_{s'}^{T} (\nabla \tilde{p}(s'', X_{s,x}(s'')) - \tilde{f}(s'', X_{s,x}(s''))) \, ds'' + \frac{1}{\sigma} Z_{s,x}(s'') U_{s,x}(s'')) \, ds''
\]

(2.27)

\[- \int_{s'}^{T} Z_{s,x}(s'') dW(s'') + \int_{s'}^{T} \sum_{r=1}^{q} \tilde{\gamma}_r(s'', X_{s,x}(s'')) \ast d\tilde{w}_r(s''), \quad s \leq s' \leq T.
\]

Let $u(s, x)$ be a solution of the problem (2.19), i.e.,

\[
u(s, x) = \varphi(x) + \int_{s}^{T} \left( \frac{\sigma^2}{2} \Delta u(s', x) - (u, \nabla) u(s', x) - \nabla \tilde{p}(s', x) + \tilde{f}(s', x) \right) ds' + \sum_{r=1}^{q} \int_{s}^{T} \tilde{\gamma}_r(s', x) \ast d\tilde{w}_r(s').
\]

(2.28)

It is known (see [30]) that then

\[
X(s') = X_{s,x}(s'), \quad U(s') = U_{s,x}(s') = u(s', X_{s,x}(s')),
\]

(2.29)

\[
Z(s') = Z_{s,x}(s') = \{Z_{k,j}(s') \} = \sigma \cdot \left\{ \frac{\partial u^k}{\partial x^j}(s', X_{s,x}(s')) \right\}, \quad k, j = 1, \ldots, n,
\]

is a solution of (2.24)-(2.26). Conversely, if $X_{s,x}(s'), U_{s,x}(s'), Z_{s,x}(s')$ is a solution of the system of backward doubly stochastic differential equations (2.24)-(2.26) then it can be verified that

\[
u(s, x) = U_{s,x}(s)
\]

(2.30)

is the solution of (2.19) (see [30]). The condition (2.20) is satisfied by choosing an appropriate pressure $\tilde{p}$.

We note that $u(s, x)$ is $\mathcal{F}_{s,T}^{\tilde{w}}$-measurable and then using (2.27) we get

\[
u(s, x) = U_{s,x}(s) = E[U_{s,x}(s)|\mathcal{F}_{s,T}^{\tilde{w}}] = E^{\tilde{w}} U_{s,x}(s)
\]

(2.31)

\[
= E^{\tilde{w}} \varphi(X_{s,x}(T))
\]

\[- E^{\tilde{w}} \int_{s}^{T} (\nabla \tilde{p}(s', X_{s,x}(s'))) - \tilde{f}(s', X_{s,x}(s')) + \frac{1}{\sigma} Z_{s,x}(s'') U_{s,x}(s'')) ds''
\]

\[+ \sum_{r=1}^{q} E^{\tilde{w}} \int_{s}^{T} \tilde{\gamma}_r(s', X_{s,x}(s')) \ast d\tilde{w}_r(s').
\]

Due to smoothness of $\tilde{\gamma}_r(s, x)$ in $s$ and independence of $X$ and $\tilde{w}$, the equality

\[
\int_{s}^{T} \tilde{\gamma}_r(s', X_{s,x}(s')) \ast d\tilde{w}_r(s') = \int_{s}^{T} \tilde{\gamma}_r(s', X_{s,x}(s')) d\tilde{w}_r(s')
\]

holds. Hence the right-hand side of (2.31) coincides with the right-hand side of the probabilistic representation (2.22).
3 Layer methods

In this section we construct three layer methods based on the probabilistic representations from Sections 2.2 and 2.3. In the case of deterministic NSE (i.e., when \( \gamma_r = 0 \) in the SNSE (1.1)-(1.4)) these methods coincide with the ones presented in [27].

On the basis of the probabilistic representation (2.13)-(2.16) we, first, construct layer methods for the stochastic Oseen-Stokes equations and, second, using the obtained methods as a guidance, we construct the corresponding methods for the SNSE (this way of deriving numerical methods for nonlinear SPDEs was proposed in [26]). This is done in Sections 3.1 and 3.2. We underline that derivation of these methods does not rely on direct probabilistic representations for the SNSE themselves that would require the anticipating stochastic calculus (see Remark 2.2) which is not developed satisfactorily from the numerical point of view. That is why we prefer to use the mimicry approach here.

In Section 3.3 we derive a layer method based on the direct probabilistic representation for the SNSE from Section 2.3.

In Sections 3.1, 3.2 and 3.3 we deal with approximation of velocity \( v(t, x) \) (i.e., a part of the solution \( v(t, x), p(t, x) \) to the SNSE) only. Since we consider here the spatial-periodic problem (1.1)-(1.4), we can separate approximation of velocity \( v(t, x) \) and pressure \( p(t, x) \) in a constructive way. Approximation of pressure is considered in Section 3.4.

Let us introduce a uniform partition of the time interval \([0, T]\) : \( 0 = t_0 < t_1 < \cdots < t_N = T \) and the time step \( h = T/N \) (we restrict ourselves to the uniform partition for simplicity only).

3.1 A layer method based on the standard probabilistic representation

Each choice of \( \mu(s, x) \) and \( F(s, x) \) in (2.13)-(2.16) gives us a particular probabilistic representation for the solution of the stochastic Oseen-Stokes equations (2.9)-(2.12) which can be used for deriving the corresponding layer method. In this and the next section we derive layer methods based on two of such probabilistic representations which can be, in a sense, viewed as limiting cases of (2.13)-(2.16). If we put \( \mu(s, x) = 0 \) and \( F(s, x) = 0 \) in (2.13)-(2.16), we obtain the standard probabilistic representation for the solution to the backward linear SPDE (2.9)-(2.12) [31]. This case is considered in this section. The case of \( F(s, x) = 0 \) and \( \mu(s, x) \) turning the equation (2.14) for \( X(s) \) into pure diffusion is treated in the next section.

Analogously to (2.13)-(2.16) with \( \mu(s, x) = 0 \) and \( F(s, x) = 0 \), we get the following local probabilistic representation of the solution to (2.9)-(2.12):

\[
\begin{align*}
\quad u_a(t_k, x) &= E^y \left[ u_a(t_{k+1}, X_{t_k, x}(t_{k+1})) - \int_{t_k}^{t_{k+1}} \nabla \bar{p}_a(s, X_{t_k, x}(s)) ds \right. \\
&+ \left. \int_{t_k}^{t_{k+1}} f(s, X_{t_k, x}(s)) ds + \sum_{r=1}^{q} \int_{t_k}^{t_{k+1}} \tilde{\gamma}_r(s, X_{t_k, x}(s)) d\tilde{w}_r(s) \right], \\
\quad dX &= -\bar{a}(s, X) ds + \sigma dW(s), \quad X(t_k) = x.
\end{align*}
\]
A slightly modified explicit Euler scheme with the simplest noise simulation applied to \((3.2)\) gives
\[
X_{t_k,x}(t_{k+1}) \simeq \dot{X}_{t_k,x}(t_{k+1}) = x - \alpha(t_{k+1}, x)h + \sigma\sqrt{h}\xi, \tag{3.3}
\]
where \(\xi = (\xi^1, \ldots, \xi^n)\) and \(\xi^1, \ldots, \xi^n\) are i.i.d. random variables with the law \(P(\xi^i = \pm 1) = 1/2\). We substitute \(\dot{X}_{t_k,x}(t_{k+1})\) from \((3.3)\) in \((3.1)\) instead of \(X_{t_k,x}(t_{k+1})\), evaluate the expectation exactly, and thus obtain (recall that \(\text{div} \, \bar{\xi} = 0\) and \(\nabla \bar{p}_a(s,x) \in (V_m)^\perp\) : \(\bar{u}_a(t_k,x) = \bar{u}_a(t_{k+1}, x) - \nabla \bar{p}_a(t_{k+1}, x)h + \tilde{f}(t_{k+1}, x)h + \sum_{r=1}^{q} \bar{\gamma}_r(t_{k+1}, x) (\tilde{w}_r(t_{k+1}) - \tilde{w}_r(t_k)) + \rho \)
\[
= P\bar{u}_a(t_{k+1}, x) + P\tilde{f}(t_{k+1}, x)h + P \perp \bar{u}_a(t_{k+1}, x) + P \perp \tilde{f}(t_{k+1}, x)h - \nabla \bar{p}_a(t_{k+1}, x)h + \sum_{r=1}^{q} \bar{\gamma}_r(t_{k+1}, x) \Delta_k \tilde{w}_r + \rho,
\]
where \(\Delta_k \tilde{w}_r = \tilde{w}_r(t_{k+1}) - \tilde{w}_r(t_k), r = 1, \ldots, q; \rho = \rho(t_k, x)\) is a remainder, and
\[
\bar{u}_a(t_{k+1}, x) = E\bar{u}_a(t_{k+1}, \dot{X}_{k+1}) = 2^{-n} \sum_{j=1}^{\Sigma_2^n} u_a(t_{k+1}, x - \alpha(t_{k+1}, x)h + \sigma\sqrt{h}\xi) + \rho. \tag{3.4}
\]
with \(\xi_1 = (1,1, \ldots, 1)^{\top}, \ldots, \xi_2 = (-1, -1, \ldots, -1)^{\top}\). Taking into account that \(u_a(t_k, x)\) in \((3.4)\) is divergence free, we get
\[
u_a(t_k, x) = Pu_a(t_{k+1}, x) + P\tilde{f}(t_{k+1}, x)h + \sum_{r=1}^{q} \gamma_r(t_{k+1}, x) \Delta_k \tilde{w}_r + P\rho. \tag{3.5}
\]
Neglecting the remainder, we get the one-step approximation for \(u_a(t_k, x)\):
\[
\dot{u}_a(t_k, x) = Pu_a(t_{k+1}, x) + P\tilde{f}(t_{k+1}, x)h + \sum_{r=1}^{q} \gamma_r(t_{k+1}, x) \Delta_k \tilde{w}_r. \tag{3.6}
\]
Re-writing \(\dot{u}_a(t_k, x)\) of \((3.6)\) in the positive direction of time, we obtain the one-step approximation for the velocity \(v_a(t_k, x)\) of the forward-time stochastic Oseen-Stokes equations \((2.3)-(2.6)\) :
\[
\dot{v}_a(t_{k+1}, x) = Pv_a(t_k, x) + Pf(t_k, x)h + \sum_{r=1}^{q} \gamma_r(t_k, x) \Delta_k w_r, \tag{3.7}
\]
where \(\Delta_k w_r = w_r(t_{k+1}) - w_r(t_k), r = 1, \ldots, q, \) and
\[
\dot{v}_a(t_k, x) = 2^{-n} \sum_{j=1}^{\Sigma_2^n} v_a(t_k, x - \alpha(t_k, x)h + \sigma\sqrt{h}\xi) + \rho. \tag{3.8}
\]
Now let us turn our attention from the stochastic Oseen-Stokes equation to the stochastic NSE \((1.1)-(1.4)\).
Using the one-step approximation (3.8)-(3.9) for the stochastic Oseen-Stokes equations (2.3)-(2.6) as a guidance, we construct the one-step approximation for the SNSE (1.1)-(1.4) by substituting $a(t_k, x)$ with $v(t_k, x)$:

$$
\hat{v}(t_{k+1}, x) = P\hat{v}(t_k, x) + Pf(t_k, x)h + \sum_{r=1}^{q} \gamma_r(t_k, x)\Delta_kw_r,
$$

(3.10)

where

$$
\hat{v}(t, x) = 2^{-n} \sum_{j=1}^{2^n} v(t, x - v(t, x)h + \sigma \sqrt{h} \xi_j).
$$

(3.11)

It is easy to see that under Assumptions 2.1 $\text{div} \hat{v}(t_{k+1}, x) = 0$.

The corresponding layer method for the SNSE (1.1)-(1.4) has the form

$$
\bar{v}(0, x) = \varphi(x), \quad \bar{v}(t_{k+1}, x) = P\bar{v}(t_k, x) + Pf(t_k, x)h + \sum_{r=1}^{q} \gamma_r(t_k, x)\Delta_kw_r,
$$

(3.12)

$$
k = 0, \ldots, N - 1,
$$

where

$$
\bar{v}(t, x) = 2^{-n} \sum_{j=1}^{2^n} \bar{v}(t, x - \bar{v}(t, x)h + \sigma \sqrt{h} \xi_j).
$$

(3.13)

We note that we use the same notation $\hat{v}(t_k, x)$ for the functions appearing in the one-step approximation (3.11) and in the layer method (3.13) but this does not cause any confusion.

Knowing the expansions

$$
\hat{v}(t, x) = \sum_{n \in \mathbb{Z}^n} \hat{v}_n(t_k)e^{i(2\pi/L)(n,x)}, \quad f(t, x) = \sum_{n \in \mathbb{Z}^n} f_n(t_k)e^{i(2\pi/L)(n,x)},
$$

(3.14)

$$
\gamma_r(t, x) = \sum_{n \in \mathbb{Z}^n} \gamma_{r,n}(t_k)e^{i(2\pi/L)(n,x)},
$$

it is not difficult to find $\bar{v}(t_{k+1}, x)$. Indeed, using (2.1) and (2.2), we obtain from (3.12)-(3.13):

$$
\bar{v}(t_{k+1}, x) = \sum_{n \in \mathbb{Z}^n} \bar{v}_n(t_{k+1})e^{i(2\pi/L)(n,x)},
$$

(3.15)

$$
\bar{v}_n(t_{k+1}) = \bar{v}_n(t_k) + f_n(t_k)h - \frac{\bar{v}_n^\top(t_k)n}{|n|^2}n - h\frac{f_n^\top(t_k)n}{|n|^2}n + \sum_{r=1}^{q} \gamma_{r,n}(t_k)\Delta_kw_r.
$$

We note that turning the layer method (3.12)-(3.13) into a numerical algorithm requires to complement it with an interpolation in order to compute the terms $\bar{v}(t_k, x - \bar{v}(t_k, x)h + \sigma \sqrt{h} \xi_j)$ in (3.13) used for finding $\bar{v}_n(t_k)$ from (3.14), see the corresponding discussion in the case of deterministic NSE in [27].
3.2 Layer methods based on the probabilistic representation with simplest characteristics

If we put \( \mu(s, x) = -\bar{a}(s, x)/\sigma \) and \( F(s, x) = 0 \) in (2.13)-(2.16), we can obtain the following local probabilistic representation for the solution to the backward stochastic Oseen-Stokes equation (2.9)-(2.12):

\[
\begin{align*}
&\quad u_a(t_k, x) = E^{\bar{a}}[u_a(t_{k+1}, X_{t_k,x}(t_{k+1}))Y_{t_k,x,1}(t_{k+1})] \\
&+ E^{\bar{a}}\left[ - \int_{t_k}^{t_{k+1}} \nabla \bar{p}_a(s, X_{t_k,x}(s))Y_{t_k,x,1}(s)ds + \int_{t_k}^{t_{k+1}} \tilde{f}(s, X_{t_k,x}(s))Y_{t_k,x,1}(s)ds \\
&\quad + \sum_{r=1}^{q} \int_{t_k}^{t_{k+1}} \tilde{\gamma}_r(s, X_{t_k,x}(s))Y_{t_k,x,1}(s)d\tilde{w}_r(s) \right],
\end{align*}
\]

where \( X_{t,x}(s), Y_{t,x,1}(s), s \geq t, \) solve the system of stochastic differential equations

\[
\begin{align*}
dX &= \sigma dW(s), \quad X(t) = x, \\
dY &= -\frac{1}{\sigma}Y a^\top(s, X)dW(s), \quad Y(t) = 1.
\end{align*}
\]

We apply a slightly modified explicit Euler scheme with the simplest noise simulation to (3.17)-(3.18):

\[
\begin{align*}
\bar{X}_{t_k,x}(t_{k+1}) &= x + \sigma \sqrt{h} \xi, \quad \bar{Y}_{t_k,x,1}(t_{k+1}) = 1 - \frac{1}{\sigma}a^\top(t_{k+1}, x)\sqrt{h} \xi, \\
\end{align*}
\]

where \( \xi \) is the same as in (3.3). Approximating \( X_{t_k,x}(t_{k+1}) \) and \( Y_{t_k,x,1}(t_{k+1}) \) in (3.20) by \( \bar{X}_{t_k,x}(t_{k+1}) \) and \( \bar{Y}_{t_k,x,1}(t_{k+1}) \) from (3.19), we obtain

\[
\begin{align*}
u_a(t_k, x) = E^{\bar{a}}[u_a(t_{k+1}, x + \sigma \sqrt{h} \xi)(1 - \frac{1}{\sigma}a^\top(t_{k+1}, x)\sqrt{h} \xi)] - \nabla \bar{p}_a(t_{k+1}, x)h \\
+ \tilde{f}(t_{k+1}, x)h + \sum_{r=1}^{q} \tilde{\gamma}_r(t_{k+1}, x)\Delta_k \tilde{w}_r + \rho \\
= 2^{-n} \sum_{q=1}^{2^n} u_a(t_{k+1}, x + \sigma \sqrt{h} \xi_q) - \frac{\sqrt{h}}{\sigma} \tilde{a}(t_{k+1}, x) - \nabla \bar{p}_a(t_{k+1}, x)h \\
+ \tilde{f}(t_{k+1}, x)h + \sum_{r=1}^{q} \tilde{\gamma}_r(t_{k+1}, x)\Delta_k \tilde{w}_r + \rho,
\end{align*}
\]

where

\[
\tilde{a}_a(t_{k+1}, x) = E^{\bar{a}}[u_a(t_{k+1}, x + \sigma \sqrt{h} \xi)^\top] \tilde{a}(t_{k+1}, x) \]

and \( \rho = \rho(t_k, x) \) is a remainder.

Using the Helmholtz-Hodge-Leray decomposition and taking into account that

\[
\text{div } u_a(t_{k+1}, x + \sigma \sqrt{h} \xi) = 0, \quad \text{div } \gamma_r = 0,
\]

13
we get from (3.20)-(3.21):

\[ u_a(t_k, x) = 2^{-n} \sum_{j=1}^{2^n} u_a(t_{k+1}, x + \sigma \sqrt{h} \xi_j) - \frac{\sqrt{h}}{\sigma} P \tilde{u}_a(t_{k+1}, x) + P \tilde{f}(t_{k+1}, x) h \]

\[ -\frac{\sqrt{h}}{\sigma} P \hat{u}_a(t_{k+1}, x) + P \hat{f}(t_{k+1}, x) h - \nabla \tilde{p}_a(t_{k+1}, x) \]

\[ + \sum_{r=1}^{q} \tilde{\gamma}_r(t_{k+1}, x) \Delta_k \tilde{w}_r + \rho, \]

whence we obtain after applying the operator \( P \):

\[ u_a(t_k, x) = 2^{-n} \sum_{j=1}^{2^n} u_a(t_{k+1}, x + \sigma \sqrt{h} \xi_j) - \frac{\sqrt{h}}{\sigma} P \tilde{u}_a(t_{k+1}, x) + P \tilde{f}(t_{k+1}, x) h \]

\[ + \sum_{r=1}^{q} \tilde{\gamma}_r(t_{k+1}, x) \Delta_k \tilde{w}_r + P \rho. \]

Dropping the remainder in (3.22) and re-writing the obtained approximation in the one with positive direction of time, we obtain the one-step approximation for the forward-time stochastic Oseen-Stokes equation (2.3)-(2.6):

\[ \hat{v}_a(t_{k+1}, x) = 2^{-n} \sum_{j=1}^{2^n} v_a(t_k, x + \sigma \sqrt{h} \xi_j) - \frac{\sqrt{h}}{\sigma} P \hat{v}_a(t_k, x) + Pf(t_k, x) h \]

\[ + \sum_{r=1}^{q} \gamma_r(t_k, x) \Delta_k w_r, \]

where

\[ \hat{v}_a(t_k, x) = 2^{-n} \sum_{j=1}^{2^n} v_a(t_k, x + \sigma \sqrt{h} \xi_j) \xi_j^\top a(t_k, x). \]

Using (3.23)-(3.24) as a guidance, we arrive at the one-step approximation for the SNSE (1.1)-(1.4):

\[ \hat{v}(t_{k+1}, x) = 2^{-n} \sum_{q=1}^{2^n} v(t_k, x + \sigma \sqrt{h} \xi_q) - \frac{\sqrt{h}}{\sigma} P \hat{v}(t_k, x) \]

\[ + Pf(t_k, x) h + \sum_{r=1}^{q} \gamma_r(t_k, x) \Delta_k w_r, \]

where

\[ \hat{v}(t_k, x) = 2^{-n} \sum_{j=1}^{2^n} v(t_k, x + \sigma \sqrt{h} \xi_j) \xi_j^\top v(t_k, x). \]

It is easy to see that under Assumptions 2.1 \( \text{div} \hat{v}(t_{k+1}, x) = 0 \). The corresponding layer method for the SNSE (1.1)-(1.4) has the form

\[ \hat{v}(0, x) = \varphi(x), \hspace{1cm} \hat{v}(t_{k+1}, x) = 2^{-n} \sum_{j=1}^{2^n} \hat{v}(t_k, x + \sigma \sqrt{h} \xi_j) - \frac{\sqrt{h}}{\sigma} P \hat{v}(t_k, x) \]
\[ + P f(t_k, x) h + \sum_{r=1}^{q} \gamma_r(t_k, x) \Delta_k w_r, \quad k = 0, \ldots, N - 1, \]

where

\[ \bar{v}(t_k, x) = 2^{-n} \sum_{j=1}^{2^n} \tilde{v}(t_k, x + \sigma \sqrt{h} \xi_j) \xi_j^\top \bar{v}(t_k, x). \] (3.28)

Practical implementation of the layer method (3.27)-(3.28) is straightforward and efficient. Let us write the corresponding numerical algorithm for simplicity in the two-dimensional \((n = 2)\) case. We choose a positive integer \(M\) as a cut-off frequency and write the approximate velocity at the time \(t_{k+1}\) as the partial sum:

\[ \bar{v}(t_{k+1}, x) = \sum_{n_1=-M}^{M-1} \sum_{n_2=-M}^{M-1} \bar{v}_n(t_{k+1}) e^{i(2\pi/L)(n,x)} , \] (3.29)

where \(\mathbf{n} = (n_1, n_2)^\top\).

We note that we use the same notation \(\bar{v}(t_{k+1}, x)\) for the partial sum in (3.29) instead of writing \(\bar{v}_M(t_{k+1}, x)\) while in (3.27) \(\bar{v}(t_{k+1}, x)\) denotes the approximate velocity containing all frequencies but this should not lead to any confusion.

Further, we have

\[ \frac{1}{4} \sum_{j=1}^{4} \bar{v}(t_k, x + \sigma \sqrt{h} \xi_j) = \sum_{n_1=-N}^{M-1} \sum_{n_2=-N}^{M-1} \bar{v}_n(t_k) e^{i(2\pi/L)(n,x)} \frac{1}{4} \sum_{j=1}^{4} e^{i(2\pi \sigma \sqrt{h}/L)(n,\xi_j)} . \] (3.30)

Then

\[ \bar{v}(t_k, x) = \frac{1}{4} \sum_{j=1}^{4} \bar{v}(t_k, x + \sigma \sqrt{h} \xi_j) \xi_j^\top \bar{v}(t_k, x) \]

\[ = \sum_{n_1=-M}^{M-1} \sum_{n_2=-M}^{M-1} \bar{v}_n(t_k) e^{i(2\pi/L)(n,x)} \frac{1}{4} \sum_{j=1}^{4} e^{i(2\pi \sigma \sqrt{h}/L)(n,\xi_j)} \xi_j^\top \bar{v}(t_k, x) \]

\[ = \sum_{n_1=-M}^{M-1} \sum_{n_2=-M}^{M-1} V_n(t_k) e^{i(2\pi/L)(n,x)} \bar{v}(t_k, x), \]

where

\[ V_n(t_k) = \bar{v}_n(t_k) \cdot \frac{1}{4} \sum_{j=1}^{4} e^{i(2\pi \sigma \sqrt{h}/L)(n,\xi_j)} \xi_j^\top. \]

Note that \(V_n(t_k)\) is a \(2 \times 2\)-matrix. Let

\[ V(t_k, x) := \sum_{n_1=-M}^{M-1} \sum_{n_2=-M}^{M-1} V_n(t_k) e^{i(2\pi/L)(n,x)} \] (3.31)

then

\[ \bar{v}(t_k, x) = V(t_k, x) \bar{v}(t_k, x). \]
We obtain the algorithm:

\[
\tilde{v}_n(0) = \varphi_n, \\
\tilde{v}_n(t_{k+1}) = \tilde{v}_n(t_k) - \frac{\sqrt{h}}{\sigma} \left( \tilde{v}_n(t_k) - \frac{\tilde{v}_n^T(t_k) n}{|n|^2} n \right) + f_n(t_k) h - h \frac{f_n^T(t_k) n}{|n|^2} n \\
+ \sum_{r=1}^{q} \gamma_{r,n}(t_k) \Delta_k w_r,
\]

where

\[
\tilde{v}_n(t_k) = (\tilde{v}(t_k, x))_n = (V(t_k, x)\tilde{v}(t_k, x))_n.
\]

To find \(\tilde{v}_n(t_k)\) one can either multiply two partial sums of the form (3.29) and (3.31) or exploit fast Fourier transform in the usual fashion (see, e.g. [3]) to speed up the algorithm. The algorithm (3.32) can be viewed as analogous to spectral methods. It is interesting that the layer method (3.27)-(3.28) is, on the one hand, related to a finite difference scheme (see below) and on the other hand, to spectral methods.

Let us discuss a relationship between the layer method (3.27)-(3.28) and finite difference methods. For simplicity in writing, we give this illustration in the two-dimensional case. It is not difficult to notice that the two-dimensional analog of the layer approximation (3.27) can be re-written as the following finite difference scheme for the SNSE (1.1)-(1.4):

\[
\frac{\tilde{v}(t_{k+1}, x) - \tilde{v}(t_k, x)}{h} = \frac{\tilde{v}(t_k, x^1 + \sigma \sqrt{h}, x^2 + \sigma \sqrt{h}) + \tilde{v}(t_k, x^1 - \sigma \sqrt{h}, x^2 + \sigma \sqrt{h}) - 4\tilde{v}(t_k, x^1, x^2)}{4h} \\
+ \frac{\tilde{v}(t_k, x^1 + \sigma \sqrt{h}, x^2 - \sigma \sqrt{h}) + \tilde{v}(t_k, x^1 - \sigma \sqrt{h}, x^2 - \sigma \sqrt{h})}{4h} \\
- \frac{1}{\sigma \sqrt{h}} Pf(t_k, x) + \sum_{r=1}^{q} \gamma_r(t_k, x) \frac{\Delta w_r(t_{k+1})}{h}
\]

with

\[
\frac{\tilde{v}(t_k, x)}{\sigma \sqrt{h}} = \tilde{v}^1(t_k, x) \frac{\tilde{v}(t_k, x^1 + \sigma \sqrt{h}, x^2 + \sigma \sqrt{h}) - \tilde{v}(t_k, x^1 - \sigma \sqrt{h}, x^2 + \sigma \sqrt{h})}{4\sigma \sqrt{h}} \\
+ \tilde{v}^1(t_k, x) \frac{\tilde{v}(t_k, x^1 + \sigma \sqrt{h}, x^2 - \sigma \sqrt{h}) - \tilde{v}(t_k, x^1 - \sigma \sqrt{h}, x^2 - \sigma \sqrt{h})}{4\sigma \sqrt{h}} \\
+ \tilde{v}^2(t_k, x) \frac{\tilde{v}(t_k, x^1 + \sigma \sqrt{h}, x^2 + \sigma \sqrt{h}) - \tilde{v}(t_k, x^1 + \sigma \sqrt{h}, x^2 - \sigma \sqrt{h})}{4\sigma \sqrt{h}} \\
+ \tilde{v}^2(t_k, x) \frac{\tilde{v}(t_k, x^1 - \sigma \sqrt{h}, x^2 + \sigma \sqrt{h}) - \tilde{v}(t_k, x^1 - \sigma \sqrt{h}, x^2 - \sigma \sqrt{h})}{4\sigma \sqrt{h}}.
\]

As one can see, \(\tilde{v}(t_k, \cdot)\) in the right-hand side of (3.34) is evaluated at the nodes \((x^1, x^2), (x^1 \pm \sigma \sqrt{h}, x^2 \pm \sigma \sqrt{h})\), which is typical for a standard explicit finite difference scheme with the space discretization step \(h_x\) taken equal to \(\sigma \sqrt{h}\) and \(h\) being the time-discretization step. We also note that if in the approximation (3.19) we choose a different random
vector $\xi$ than in (3.3) then we can obtain another layer method for the SNSE which can be again re-written as a finite difference scheme (see such a discussion in the case of the deterministic NSE in [27]).

We recall [22, 23, 26] that convergence theorems for layer methods (in comparison with the theory of finite difference methods) do not contain any conditions on stability of their approximations. In layer methods we do not need to a priori prescribe space nodes: they are obtained automatically depending on choice of a probabilistic representation and a numerical scheme. We note that our error analysis for the layer methods (see Section 4) immediately implies the same error estimates for the corresponding finite difference scheme (3.34).

**Remark 3.1** It is not difficult to see from (3.35) that

$$(\bar{v}(t_k, x), \nabla)\bar{v}(t_k, x) \approx \frac{\ddot{v}(t_k, x)}{\sqrt{\sigma h}}.$$ (3.36)

If we put the exact $v(t_k, x)$ in (3.36) (both in its left and right-hand sides) instead of the approximate $\bar{v}(t_k, x)$ then the accuracy of the approximation in (3.36) is of order $O(h)$. This observation is helpful for understanding a relationship between the layer methods from this and the next section (see Remark 3.2 at the end of the next section).

### 3.3 A layer method based on the direct probabilistic representation

The local version of probabilistic representation (2.22)-(2.23) for the solution to the backward SNSE (2.19)-(2.20) has the form:

$$u(t_k, x) = E\bar{w} u(t_{k+1}, X_{t_k,x}(t_{k+1}))$$

$$- E\bar{w} \left[ \int_{t_k}^{t_{k+1}} \{ \nabla \bar{p}(s', X_{t_k,x}(s')) - \tilde{f}(s', X_{t_k,x}(s')) \\ + (u(s', X_{t_k,x}(s')), \nabla) u(s', X_{t_k,x}(s')) \} ds' \right]$$

$$+ \sum_{r=1}^{q} E\bar{w} \left[ \int_{t_k}^{t_{k+1}} \bar{\gamma}_r(s', X_{t_k,x}(s')) d\bar{w}_r(s') \right].$$ (3.37)

Using (3.37), we construct the one-step approximation of the solution to the backward SNSE (2.19)-(2.20):

$$u(t_k, x) = E\bar{w} u(t_{k+1}, X_{t_k,x}(t_{k+1})) - h\{ \nabla \bar{p}(t_{k+1}, x) - \tilde{f}(t_{k+1}, x)$$

$$+ (u(t_{k+1}, x), \nabla) u(t_{k+1}, x) \} + \sum_{r=1}^{q} \bar{\gamma}_r(t_{k+1}, x) \Delta_k \bar{w}_r + \rho$$

$$= 2^{-n} \sum_{j=1}^{2^n} u(t_{k+1}, x + \sigma \sqrt{h} \xi_j)$$

$$- h\{ \nabla \bar{p}(t_{k+1}, x) - \tilde{f}(t_{k+1}, x) + (u(t_{k+1}, x), \nabla) u(t_{k+1}, x) \}$$
where $\rho = \rho(t_k, x)$ is a remainder.

Using the Helmholtz-Hodge-Leray decomposition and taking into account that \(\text{div} \, u(t_{k+1}, x + \sigma \sqrt{h} \xi_q) = 0\) and \(\text{div} \, \gamma_r = 0\), we get from (3.38):

\[
\begin{align*}
\sum_{r=1}^{q} \tilde{\gamma}_r(t_{k+1}, x) \Delta_k \tilde{w}_r + \rho, \\
\end{align*}
\]

whence we obtain after applying the operator \(P\):

\[
\begin{align*}
\sum_{r=1}^{q} \tilde{\gamma}_r(t_{k+1}, x) \Delta_k \tilde{w}_r + \rho.
\end{align*}
\]

We re-write (3.39)-(3.40) for the forward-time SNSE (1.1)-(1.4):

\[
\begin{align*}
\sum_{j=1}^{2^n} u(t_{k+1}, x + \sigma \sqrt{h} \xi_j) - P[(u(t_{k+1}, x), \nabla)u(t_{k+1}, x)]h \\
+ P \tilde{f}(t_{k+1}, x)h - P^\perp[(u(t_{k+1}, x), \nabla)u(t_{k+1}, x)]h + P^\perp \tilde{f}(t_{k+1}, x)h \\
- \nabla \tilde{p}(t_{k+1}, x)h + \sum_{r=1}^{q} \tilde{\gamma}_r(t_{k+1}, x) \Delta_k \tilde{w}_r + \rho.
\end{align*}
\]

and

\[
\begin{align*}
\sum_{j=1}^{2^n} v(t_{k+1}, x + \sigma \sqrt{h} \xi_j) - P[(v(t_{k+1}, x), \nabla)v(t_{k+1}, x)]h \\
+ P f(t_k, x)h - P^\perp[(v(t_{k+1}, x), \nabla)v(t_{k+1}, x)]h + P^\perp f(t_k, x)h \\
- \nabla p(t_k, x)h + \sum_{r=1}^{q} \gamma_r(t_k, x) \Delta_k w_r + \rho.
\end{align*}
\]

Dropping the remainder in (3.42), we obtain the one-step approximation for the velocity \(v(t_{k+1}, x)\) in (1.1)-(1.4):

\[
\begin{align*}
\hat{v}(t_{k+1}, x) = 2^{-n} \sum_{j=1}^{2^n} v(t_k, x + \sigma \sqrt{h} \xi_j) - P[(v(t_k, x), \nabla)v(t_k, x)]h \\
+ P f(t_k, x)h + \sum_{r=1}^{q} \gamma_r(t_k, x) \Delta_k w_r.
\end{align*}
\]
It is easy to see that under Assumptions 2.1 $\text{div} \, \bar{v}(t_{k+1}, x) = 0$. The corresponding layer method for the velocity of the SNSE (1.1)-(1.4) has the form

$$\bar{v}(0, x) = \varphi(x),$$

(3.44)

$$\bar{v}(t_{k+1}, x) = 2^{-n} \sum_{j=1}^{2^n} \bar{v}(t_k, x + \sigma \sqrt{h} \xi_j) - P[(\bar{v}(t_k, x), \nabla)\bar{v}(t_k, x)] h + P f(t_k, x) h + \sum_{r=1}^{q} \gamma_r(t_k, x) \Delta_k w_r, \quad k = 0, \ldots, N - 1.$$

This method can be turned into a numerical algorithm analogously to how we constructed the numerical algorithm (3.32) based on the layer method (3.27) in Section 3.2.

**Remark 3.2** It is interesting to note (see also (3.35) and (3.36)) the relationship between the methods (3.27) and (3.44): $\sqrt{h} \bar{v}(t_k, x) / \sigma$ from (3.25)-(3.26) is a finite-difference approximation of the term $(\bar{v}(t_k, x), \nabla)\bar{v}(t_k, x) h$ in (3.44). We remark that this finite difference naturally arises via the probabilistic approach. It is useful to have both methods in the arsenal of layer methods for SNSE: while the method (3.44) has a smaller one-step error than (3.27), it requires evaluation of spatial derivatives of $\bar{v}(t_k, x)$.

### 3.4 Approximation of pressure

In the previous sections we constructed numerical methods for velocity $v(t, x)$, in this section we propose approximations for pressure $p(t, x)$.

Applying the projection operator $P^\perp$ to SNSE (1.1)-(1.4), we get (see also (1.5)):

$$\nabla p(t, x) = -P^\perp [(v(t, x), \nabla)v(t, x)] + P^\perp f(t, x).$$

(3.45)

Based on (3.45), we complement the layer method (3.44) for the velocity by the approximation of pressure as follows:

$$\nabla \bar{p}(t_{k+1}, x) = -P^\perp [(\bar{v}(t_{k+1}, x), \nabla)\bar{v}(t_{k+1}, x)] + P^\perp f(t_{k+1}, x).$$

(3.46)

As a result, we obtain the layer method (3.44), (3.46) for the solution of SNSE (1.1)-(1.4).

It is clear that the numerical error $\nabla \bar{p}(t_{k+1}, x) - \nabla p(t, x)$ is of the same order as the global errors of $\bar{v}(t_{k+1}, x)$ and $\nabla \bar{v}(t_{k+1}, x)$. We note that in (3.46) to evaluate pressure at time $t_{k+1}$ we use velocity at time $t_{k+1}$, i.e., the updated velocity.

**Remark 3.3** We observe that $\rho$ in (3.41) is such that $P^\perp \rho = 0$. Indeed, it follows from (3.41)-(3.42) (with $t_{k+1}$ instead of $t_k$) that

$$\nabla p(t_{k+1}, x) = -P^\perp [(v(t_{k+1}, x), \nabla)v(t_{k+1}, x)] + P^\perp f(t_{k+1}, x) + P^\perp \rho.$$  

(3.47)

Comparing (3.45) and (3.47), we get $P^\perp \rho = 0$.

Let us now return to the layer method (3.27) for velocity. We have to complement it with an approximation of pressure. To this end, we approximate (see Remark 3.2 and
the term $(\bar{\nu}(t_{k+1}, x), \nabla)\bar{\nu}(t_{k+1}, x)$ in (3.46) by $\bar{\nu}(t_{k+1}, x)/\sigma \sqrt{h}$ with $\nu(t_{k+1}, x)$ from (3.28) (with $t_{k+1}$ instead of $t_k$). We obtain
\[
\nabla \tilde{p}(t_{k+1}, x) = -\frac{1}{\sigma \sqrt{h}} P^\perp \nu(t_{k+1}, x) + P^\perp f(t_{k+1}, x),
\]
(3.48)
where $\nu(t_{k+1}, x)$ is from (3.28). Note that in the velocity approximation (3.27) we use $\nu(t_k, x)$ while in the pressure approximation (3.48) we use $\nu(t_{k+1}, x)$.

As a result, we obtain the layer method (3.27)-(3.28), (3.48) for the solution of SNSE (1.1)-(1.4). For definiteness, we consider the layer method (3.12). Analogous results can be obtained for the other two layer methods proposed in Sections 3.2 and 3.3.

In this section we provide theoretical support for the numerical methods from the previous section. For definiteness, we consider the layer method (3.12). Analogous results can be obtained for the other two layer methods proposed in Sections 3.2 and 3.3.

As before, $||u(\cdot)|| = ||u(x)||$ denotes the $L^2$-norm of a function $u(x), x \in Q$. In this section we use the same letter $K$ for various deterministic constants and $C = C(\omega)$ for various positive random variables.

We start with analysis of the local mean-square error.

**Theorem 4.1** Let Assumptions 2.1 hold with $m_0 > 6$. The one-step error
\[
\rho(t_{k+1}, x) = \bar{\nu}(t_{k+1}, x) - \nu(t_{k+1}, x)
\]
(4.1)
of the one-step approximation (3.10)-(3.11) for the SNSE (1.1)-(1.4) is estimated as
\[
||E(\rho(t_{k+1}, x)F^w_{\nu(t_{k+1})})|| \leq C(\omega)h^2,
\]
(4.2)
and for $1 \leq p < p_0$
\[
(E||\rho(t_{k+1}, \cdot)||^{2p})^{1/2p} \leq Kh^{3/2},
\]
(4.3)
where a random constant $C(\omega) > 0$ with $EC^2 < \infty$ does not depend on $h$ and $k$, a deterministic constant $K > 0$ does not depend on $h$ and $k$ but depends on $p$, and $p_0 = p_0(m_0) > 1$ is a positive number or $p_0 = \infty$. 

4 Error analysis
Proof. Using Assumptions 2.1, we expand the right-hand side of (3.11), substitute the outcome in (3.10), and obtain

$$
\dot{v}(t_{k+1}, x) = v(t_k, x) - hP [(v(t_k, x), \nabla v(t_k, x)] + \frac{\sigma^2}{2} h \Delta v(t_k, x) \tag{4.4}
$$

$$
+ Pf(t_k, x) h + \sum_{r=1}^{q} \gamma_r(t_k, x) \Delta r + r_1(t_k, x),
$$

where the remainder $r_1(t_k, x)$ has the form

$$
r_1(t_k, x) = \frac{h^2}{2} \sum_{i,j=1}^{n} P \left[ v^i(t_k, x)v^j(t_k, x) \frac{\partial^2}{\partial x^i \partial x^j} v(t_k, \Theta) \right]
$$

$$
+ \frac{\sigma^2 h^2}{2} \sum_{i,j=1}^{n} P \left[ v^i(t_k, x) \frac{\partial^2}{(\partial x^i)^2} v(t_k, \tilde{\Theta}) \right]
$$

$$
+ \frac{\sigma^4 h^2}{24} 2^{-n} \sum_{j=1}^{2^n} \sum_{i=1}^{n} P \left[ \frac{\partial^4}{\partial x^i_1 \partial x^i_2 \partial x^i_3 \partial x^i_4} v(t_k, \Xi_j) \xi^{i_1}_j \xi^{i_2}_j \xi^{i_3}_j \xi^{i_4}_j \right],
$$

and $\Theta$ and $\tilde{\Theta}$ are some intermediate points between $x$ and $-v(t_k, x)h$, and $\Xi_j$ are some intermediate points between $x - v(t_k, x)h$ and $x - v(t_k, x)h + \sigma \sqrt{h} \xi_j$ (we note that $r_1$ is a vector and the intermediate points depend on the component of $r_1$ but we do not reflect this in the notation). It is not difficult to estimate that this remainder satisfies the inequalities

$$
||E(r_1(t_k, x)|F_{t_k}^w)|| \leq C(\omega) h^2, \quad (E||r_1(t_k, \cdot)||^{2p})^{1/2p} \leq Kh^2. \tag{4.5}
$$

We write the solution $v(s, x)$, $s \geq t_k$, of (1.1)-(1.4) as

$$
v(s, x) = v(t_k, x) + \int_{t_k}^{s} \left[ \frac{\sigma^2}{2} \Delta v(s', x) - (v(s', x), \nabla v(s', x) + f(s', x) \right] ds' \tag{4.6}
$$

$$
- \int_{t_k}^{s} \nabla p(s', x) ds' + \sum_{r=1}^{q} \int_{t_k}^{s} \gamma_r(s', x) dw_r(s')
$$

and, in particular,

$$
v(t_{k+1}, x) = v(t_k, x) + \int_{t_k}^{t_{k+1}} \left[ \frac{\sigma^2}{2} \Delta v(s, x) - (v(s, x), \nabla v(s, x) + f(s, x) \right] ds \tag{4.7}
$$

$$
- \int_{t_k}^{t_{k+1}} \nabla p(s, x) ds + \sum_{r=1}^{q} \int_{t_k}^{t_{k+1}} \gamma_r(s, x) dw_r(s).
$$

Substituting $v(s, x)$ from (4.6) in the integrand of the first integral in (4.7) and expanding $\gamma_r(s, x)$ at $(t_k, x)$, we obtain

$$
v(t_{k+1}, x) = v(t_k, x) + h \frac{\sigma^2}{2} \Delta v(t_k, x) - h(v(t_k, x), \nabla v(t_k, x) + hf(t_k, x) \tag{4.8}
$$

$$
- \int_{t_k}^{t_{k+1}} \nabla p(s, x) ds + \sum_{r=1}^{q} \gamma_r(t_k, x) \Delta w_r + r_2(t_k, x),
$$

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where
\[ r_2(t_k, x) = r_2^{(1)}(t_k, x) + r_2^{(2)}(t_k, x) \]
and
\[
r_2^{(1)}(t_k, x) = \frac{\sigma^2}{2} \int_{t_k}^{t_{k+1}} \left[ \int_{t_k}^{s} \Delta \left( \frac{\sigma^2}{2} \Delta v(s', x) - (v(s', x), \nabla) v(s', x) \right) + f(s', x) \right] ds \, ds' - \frac{\sigma^2}{2} \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \Delta \nabla p(s', x) ds' \, ds \]
\[
- \int_{t_k}^{t_{k+1}} (v(s, x), \nabla) \left[ \int_{t_k}^{s} \left( \frac{\sigma^2}{2} \Delta v(s', x) - (v(s', x), \nabla) v(s', x) \right) + f(s', x) \right] ds' \, ds + (t_{k+1} - s) \frac{\partial}{\partial s} f(s, x) ds,
\]
\[
r_2^{(2)}(t_k, x) = \frac{\sigma^2}{2} \sum_{r=1}^{q} \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \Delta \gamma_r(s', x) dw_r(s') \, ds
\]
\[
- \sum_{r=1}^{q} \int_{t_k}^{t_{k+1}} \left[ (v(s, x), \nabla) \int_{t_k}^{s} \gamma_r(s', x) dw_r(s') \right] \, ds
\]
\[
- \sum_{r=1}^{q} \int_{t_k}^{t_{k+1}} \left( \int_{t_k}^{s} \gamma_r(s', x) dw_r(s'), \nabla \right) v(s, x) \, ds
\]
\[
+ \sum_{r=1}^{q} \int_{t_k}^{t_{k+1}} (w_r(t_{k+1} - w_r(s)) \frac{\partial}{\partial s} \gamma_r(s, x) \, ds.
\]

We see that the remainder \( r_2(t_k, x) \) consists of 1) \( r_2^{(1)}(t_k, x) \) with terms of mean-square order \( h^2 \) and 2) \( r_2^{(2)}(t_k, x) \) with terms containing \( \mathcal{F}_{t_{k+1}}^w \)-measurable Ito integrals of mean-square order \( h^{3/2} \) which expectations with respect to \( \mathcal{F}_t^w \) equal zero. Further, using Assumptions 2.1, one can show that
\[
|E (r_2(t_k, x)|\mathcal{F}_t^w) | \leq C(\omega)h^2, \quad (E |r_2(t_k, x)|^{2p})^{1/2p} \leq Kh^{3/2}, \tag{4.9}
\]
where \( C(\omega) > 0 \) and \( K > 0 \) do not depend on \( k, x, \) and \( h \). Based on the second inequality
in (4.9), we obtain

\[ E \| r_2(t_k, \cdot) \|^p = E \left( \int_Q |r_2(t_k, x)|^2 dx \right)^p \leq KE \int_Q |r_2(t_k, x)|^{2p} dx \quad (4.10) \]

\[ \leq K \int Q E |r_2(t_k, x)|^{2p} dx \leq K\hbar^{2p\times 3/2}. \]

Applying the projector operator \( P \) to the left- and right-hand sides of (4.8), we arrive at

\[ v(t_{k+1}, x) = v(t_k, x) + h \sigma^2 2 \Delta v(t_k, x) - hP[(v(t_k, x), \nabla)v(t_k, x)] + hf(t_k, x) \quad (4.11) \]

\[ + \sum_{r=1}^q \gamma_r(t_k, x) \Delta w_r + r_3(t_k, x), \]

where the new remainder \( r_3(t_k, x) = Pr_2(t_k, x) \). Using (4.10), we get

\[ E \| r_3(t_k, \cdot) \|^p = E \| Pr_2(t_k, \cdot) \|^p \leq E \| r_2(t_k, \cdot) \|^p \leq K\hbar^{2p\times 3/2}. \quad (4.12) \]

Hence from here, (4.5) and (4.4), (4.11), we obtain (4.3).

Observing that expectation of projection \( P \) of Ito integrals remains equal to zero, we get \( E \left( P r_2^{(2)}(t_k, x)|F_{t_k}^w \right) = 0 \). Since \( r_2^{(1)}(t_k, x) \) consists of terms of mean-square order \( h^2 \), we obtain

\[ \| E \left( r_3(t_k, x)|F_{t_k}^w \right) \|^2 = \| E \left( Pr_2^{(1)}(t_k, x)|F_{t_k}^w \right) \|^2 \]

\[ = \int Q \left( E \left( Pr_2^{(1)}(t_k, x)|F_{t_k}^w \right) \right)^2 dx \]

\[ \leq \int Q E \left( \left[ Pr_2^{(1)}(t_k, x) \right]^2 |F_{t_k}^w \right) dx \]

\[ = E \left( \int Q \left[ Pr_2^{(1)}(t_k, x) \right]^2 dx |F_{t_k}^w \right) \]

\[ \leq E \left( \int Q \left[ r_2^{(1)}(t_k, x) \right]^2 dx |F_{t_k}^w \right) \leq C(\omega)h^4 \]

whence

\[ \| E \left( r_3(t_k, x)|F_{t_k}^w \right) \| \leq C(\omega)h^2. \quad (4.13) \]

Then the estimate (4.2) follows from (4.5), (4.13) and (4.4), (4.11). \( \square \)

**Remark 4.1** We recall that in Assumptions 2.1 we require existence of moments of order \( m \), \( 2 \leq m < m_0 \), of the solution and its spatial derivatives. The higher the \( m_0 \), the higher \( p \), \( 1 \leq p < p_0 \), can be taken in (4.3). In particular, to guarantee (4.3) with \( p = 1 \), we need existence of moments of up to the order \( m = 6 \), while if the moments of any order \( m \) (i.e., \( m_0 = \infty \)) are finite then (4.3) is valid for any \( p \). We also note that the smoothness conditions on the SNSE solution (see Assumptions 2.1) required for proving Theorem 4.1 are so that \( v(t, x) \) should have continuous spatial derivatives up to order four and \( p(t, x) \) — up to order three.
**Corollary 4.1** Let Assumptions 2.1 hold with the bounded moments of any order $m \geq 2$. Then for almost every trajectory $w(\cdot)$ and any $0 < \varepsilon < 3/2$ there exists a constant $C(\omega) > 0$ such that the one-step error from (4.1) is estimated as

$$||\rho(t_{k+1}, \cdot)|| \leq C(\omega)h^{3/2 - \varepsilon},$$

(4.14)
i.e., the layer method (3.12) has the one-step error of order $3/2 - \varepsilon$ a.s.

**Proof.** Here we follow the recipe used in [13, 25, 26]. The Markov inequality together with (4.3) implies

$$P(||\rho(t_{k+1}, \cdot)|| > h^\gamma) \leq \frac{E||\rho(t_{k+1}, \cdot)||^{2p}}{h^{2p\gamma}} \leq Kh^{2p(3/2 - \gamma)}.$$  

Then for any $\gamma = 3/2 - \varepsilon$ there is a sufficiently large $p \geq 1$ such that (recall that $h = T/N$)

$$\sum_{N=1}^\infty P\left(||\rho(t_{k+1}, \cdot)|| > \frac{T^\gamma}{N^\gamma}\right) \leq KT^{2p(3/2 - \gamma)} \sum_{N=1}^\infty \frac{1}{N^{2p(3/2 - \gamma)}} < \infty.$$  

Hence, due to the Borel-Cantelli lemma, the random variable

$$\varsigma := \sup_{h>0} h^{-\gamma} ||\rho(t_{k+1}, \cdot)||$$

is a.s. finite which implies (4.14). □

**Remark 4.2** Since it is desirable for the order of the one-step error $||\rho(t_{k+1}, \cdot)||$ to be greater than one, we should impose the restriction on $\varepsilon$ in (4.14) to be in $(0, 0.5)$. If we restrict ourselves to fulfilment of the inequality (4.14) with $\varepsilon_0 < \varepsilon < 1/2$, where $\varepsilon_0$ is some positive number, then the conditions of Corollary 4.1 can be weakened since for such $\varepsilon$ it is sufficient to take $p_0 = 1/(2\varepsilon_0)$.

The intuition built on numerics for ordinary stochastic differential equations (see, e.g. [23]) and also based on layer methods for SPDEs [25, 26] together with convergence results for layer methods for deterministic NSE [1, 27] suggests that the one-step error properties proved in Theorem 4.1 should lead to mean-square convergence of the layer method (3.12) with order one, i.e.,

$$\left(E||\bar{v}(t_k, \cdot) - v(t_k, \cdot)||^{2p}\right)^{1/2p} \leq Kh.$$  

(4.15)

However, we have not succeeded in proving such a result. Below we prove an almost sure (a.s.) convergence of the method (3.12) with lower order of $1/2 - \varepsilon$ for arbitrary $\varepsilon > 0$ than the $1 - \varepsilon$ a.s. order which should follow from (1.15) and the Borel-Cantelli-type of arguments (see, e.g. [25, 26] and also the proof of Corollary 4.1 above). In our numerical experiments (see Section 5) we observed the first order (both mean-square and a.s.) convergence of a layer method on test examples.

Since we assumed in Assumptions 2.1 that the problem (1.1)-(1.4) has a unique classical solution $v(t, x), p(t, x)$ which has continuous derivatives in the space variable $x$ up to some order and since we are considering the periodic case, then $v(t, x), p(t, x)$ and their derivatives are a.s. finite on $[0, T] \times Q$.

To prove the below a.s. convergence Theorem 4.2 we make the following assumptions on the approximate solution $\bar{v}(t_k, x)$ from (3.12).
Assumptions 4.1. Let \( \bar{v}(t_k, x), k = 0, \ldots, N \), have continuous first-order spatial derivatives and

\[
|\bar{v}(t_k, x)| \leq C(\omega),
\]

\[
|\partial \bar{v}(t_k, x)/\partial x^i| \leq C(\omega), \quad i = 1, \ldots, n,
\]

where \( C(\omega) > 0 \) is an a.s. finite constant independent of \( x, h, k \).

The first inequality in (4.16) is necessary for a.s. convergence of the layer method (3.12). The second inequality is also necessary if one expects convergence of spatial derivatives of \( \bar{v}(t, x) \). We note that even in the case of deterministic NSE [1, 27] it turns out to be problematic to derive the inequalities (4.16) for the approximate solutions. At the same time, verifying Assumptions 4.1 in numerical experiments is straightforward. We also note that in the case of Oseen-Stokes equations we succeeded in deriving such estimates for approximate solutions and their spatial derivatives.

Theorem 4.2 Let Assumptions 2.1 hold with the bounded moments of any order \( m \geq 2 \) and Assumptions 4.1 also hold. For almost every trajectory \( w(\cdot) \) and any \( 0 < \varepsilon < 1/2 \) there exists a constant \( C(\omega) > 0 \) such that

\[
||\bar{v}(t_k, \cdot) - v(t_k, \cdot)|| \leq C(\omega)h^{1/2 - \varepsilon},
\]

i.e., the layer method (3.12) for the SNSE (1.1)-(1.4) converges with order \( 1/2 - \varepsilon \) a.s..

Proof. First, we note that it is easy to see that under Assumptions 2.1 and 4.1:

\[
\text{div } \bar{v}(t_k, x) = 0.
\]

Denote the error of the method (3.12)-(3.13) on the \( k \)th layer by

\[
\varepsilon(t_k, x) = \bar{v}(t_k, x) - v(t_k, x).
\]

Due to (3.12) and (3.13), we obtain

\[
\varepsilon(t_{k+1}, x) + v(t_{k+1}, x) = \bar{v}(t_{k+1}, x)
\]

\[
= 2^{-n} \sum_{j=1}^{2^n} P\bar{v}(t_k, x - \bar{v}(t_k, x)h + \sigma\sqrt{h}\xi_j) + Pf(t_k, x)h
\]

\[
+ \sum_{r=1}^{q} \gamma_r(t_k, x)\Delta_k w_r,
\]

\[
= 2^{-n} \sum_{j=1}^{2^n} P\bar{v}(t_k, x - \bar{v}(t_k, x)h + \sigma\sqrt{h}\xi_j)
\]

\[
+ 2^{-n} \sum_{j=1}^{2^n} P\varepsilon(t_k, x - \bar{v}(t_k, x)h + \sigma\sqrt{h}\xi_j)
\]

\[
+ Pf(t_k, x)h + \sum_{r=1}^{q} \gamma_r(t_k, x)\Delta_k w_r.
\]
Using Assumptions 2.1, we obtain
\begin{equation}
v(t_k, x - \bar{v}(t_k, x)h + \sigma \sqrt{n} \xi_j) = v(t_k, x - v(t_k, x)h + \sigma \sqrt{n} \xi_j) + r_j(t_k, x),
\end{equation}
where
\begin{equation}
|r_j(t_k, x)| \leq C(\omega)|\varepsilon(t_k, x)|h
\end{equation}
and $C(\omega)$ is an a.s. finite random variable. Hence
\begin{align*}
\varepsilon(t_{k+1}, x) + v(t_{k+1}, x) &= 2^{-n} \sum_{j=1}^{2^n} P v(t_k, x - v(t_k, x)h + \sigma \sqrt{n} \xi_j) \\
&+ 2^{-n} \sum_{j=1}^{2^n} P r_j(t_k, x) + 2^{-n} \sum_{j=1}^{2^n} P \varepsilon(t_k, x - \bar{v}(t_k, x)h + \sigma \sqrt{n} \xi_j) \\
&+ P f(t_k, x)h + \sum_{r=1}^{q} \gamma_r(t_k, x) \Delta_k w_r.
\end{align*}
Then we get
\begin{equation}
\varepsilon(t_{k+1}, x) = 2^{-n} \sum_{j=1}^{2^n} P \varepsilon(t_k, x - \bar{v}(t_k, x)h + \sigma \sqrt{n} \xi_j) + 2^{-n} \sum_{j=1}^{2^n} P r_j(t_k, x) + \rho(t_{k+1}, x),
\end{equation}
where $\rho(t_{k+1}, x)$ is the error (see (4.11)) of the one-step approximation (3.10)-(3.11) and this one-step error satisfies the inequality (4.14) from Corollary 4.1. It follows from (4.21), (4.20) and (4.14) that
\begin{equation}
\|\varepsilon(t_{k+1}, \cdot)\| \leq 2^{-n} \sum_{j=1}^{2^n} \|P \varepsilon(t_k, \cdot - \bar{v}(t_k, \cdot)h + \sigma \sqrt{n} \xi_j)\| + 2^{-n} \sum_{j=1}^{2^n} \|P r_j(t_k, \cdot)\| + \|\rho(t_{k+1}, \cdot)\|
\end{equation}
\begin{align*}
&\leq 2^{-n} \sum_{j=1}^{2^n} \|\varepsilon(t_k, \cdot - \bar{v}(t_k, \cdot)h + \sigma \sqrt{n} \xi_j)\| + 2^{-n} \sum_{j=1}^{2^n} \|r_j(t_k, \cdot)\| \\
&+ \|\rho(t_{k+1}, \cdot)\|
\end{align*}
\begin{align*}
&\leq 2^{-n} \sum_{j=1}^{2^n} \|\varepsilon(t_k, \cdot - \bar{v}(t_k, \cdot)h + \sigma \sqrt{n} \xi_j)\| + C(\omega)\|\varepsilon(t_k, \cdot)\|h \\
&+ C(\omega)h^{3/2-\varepsilon}.
\end{align*}
Consider $\delta(x) = \varepsilon(t_k, x - \bar{v}(t_k, x)h + \sigma \sqrt{n} \xi_j)$. Due to Assumptions 4.1, the function $y(x) = x - \bar{v}(t_k, x)h + \sigma \sqrt{n} \xi_j$ is a differentiable function with continuous partial derivatives. Furthermore, using Assumptions 4.1, one can show that for sufficiently small $h > 0$ the function $y(x) = x - \bar{v}(t_k, x)h + \sigma \sqrt{n} \xi_j$ is injective. Then, taking into account the $Q$-periodicity of $\bar{v}(t_k, x)$ and $\varepsilon(t_k, x)$, we obtain
\begin{align*}
&\|\delta(\cdot)\|^2 = \int_0^1 \sum_{i=1}^n \left[ \varepsilon^i(t_k, x - \bar{v}(t_k, x)h + \sigma \sqrt{n} \xi_j) \right]^2 dx \\
&= \int_0^1 \sum_{i=1}^n \left[ \varepsilon^i(t_k, y) \right]^2 \frac{D(x^1 \ldots x^n)}{D(y^1 \ldots y^n)} dy.
\end{align*}
Due to Assumptions 4.1 and due to (4.18), we get

\[
D(y^1 \ldots y^n) = \begin{vmatrix} 1 - h \frac{\partial \nu^i(t_k,x)}{\partial x^1} & \ldots & -h \frac{\partial \nu^i(t_k,x)}{\partial x^n} \\ \vdots & \ddots & \vdots \\ -h \frac{\partial \nu^i(t_k,x)}{\partial x^1} & \ldots & 1 - h \frac{\partial \nu^i(t_k,x)}{\partial x^n} \end{vmatrix}
\]

(4.23)

\[
= 1 + C(\omega)h^2,
\]

where \(C(\omega)\) is an a.s. finite random variable. Then, we also have

\[
\frac{D(x^1 \ldots x^n)}{D(y^1 \ldots y^n)} = 1 + C(\omega)h^2.
\]

We obtain from (4.22) and (4.23):

\[
||\varepsilon(t_{k+1},\cdot)|| \leq ||\varepsilon(t_k,\cdot)|| + C(\omega)||\varepsilon(t_k,\cdot)||h + C(\omega)h^{3/2-\varepsilon},
\]

whence (4.17) follows. □

**Remark 4.3** We recall that we have proved in Theorem 4.1 that the mean and mean-square one-step errors of the layer method (3.12) (and analogously of the other two layer methods from Section 3) are of orders \(O(h^2)\) and \(O(h^{3/2})\), respectively. This has given us the basis to argue that the methods from Section 3 are of global mean-square order one (see (4.15)). The same intuition implies that if we incorporate terms of mean-square order \(O(h^{3/2})\) and of mean order \(O(h^2)\) in these first order methods (and thus make the mean-square one-step errors to be of order \(O(h^2)\) and the mean errors of order \(O(h^3)\)) then they become of global mean-square order 3/2. The required Ito integrals of mean-square order \(O(h^{3/2})\) can be simulated in the constructive way (and hence these methods of order 3/2 are constructive). In the case of deterministic NSE (i.e., when \(\gamma_T = 0\)) such a method of global mean-square order 3/2 becomes of order two and coincides with the corresponding layer method derived in [27].

Let us now consider the error of the approximations of pressure considered in Section 3.4. In the next proposition we prove convergence of pressure evaluated by (3.46), (3.12). Analogously, one can prove convergence of the other approximations of pressure derived in Section 3.4.

**Proposition 4.1** Let assumptions of Theorem 4.1 hold. In addition assume that second-order spatial derivatives of the approximate solution are a.s. finite: \(|\partial^2 \nu(t_k,x)/\partial x^i \partial x^j| \leq C(\omega)\). Then for almost every trajectory \(w(\cdot)\) and any \(0 < \varepsilon < 1/3\) there exists a constant \(C(\omega) > 0\) such that the approximate pressure \(\bar{p}(t_k,x)\) from (3.46), (3.12) satisfies the following inequality

\[
||\bar{p}(t_k,\cdot) - p(t_k,\cdot)|| \leq C(\omega)h^{1/3-\varepsilon}.
\]

**Proof.** We have

\[
\frac{\partial \nu^i}{\partial x^j}(t_k,x) = \frac{v^i(t_k,x + \delta e_j) - v^i(t_k,x - \delta e_j)}{2\delta} + O(\delta^2),
\]

(4.26)
Remark 4.5

Subtracting (3.45) with $t = t_k$ from (3.46) with $t_k$ instead of $t_{k+1}$, we get

$$\left\| \nabla p(t_k, \cdot) - \nabla p(t_k, \cdot) \right\| = \left\| P^\perp \left[ (v(t_k, \cdot), \nabla) v(t_k, \cdot) \right] - P^\perp \left[ (\bar{v}(t_k, \cdot), \nabla) \bar{v}(t_k, \cdot) \right] \right\|$$

$$\leq \left\| P^\perp \left[ (v(t_k, \cdot), \nabla) v(t_k, \cdot) \right] \right\| + \left\| P^\perp \left[ (v(t_k, \cdot) - \bar{v}(t_k, \cdot), \nabla) \bar{v}(t_k, \cdot) \right] \right\|$$

$$\leq \left\| (v(t_k, \cdot), \nabla) (v(t_k, \cdot) - \bar{v}(t_k, \cdot)) \right\| + \left\| (v(t_k, \cdot) - \bar{v}(t_k, \cdot), \nabla) \bar{v}(t_k, \cdot) \right\| .$$

Due to Assumptions 2.1 and (4.28),

$$\left\| (v(t_k, \cdot), \nabla) (v(t_k, \cdot) - \bar{v}(t_k, \cdot)) \right\| \leq C(\omega)h^{1/3-\varepsilon} \text{ a.s. .} \quad (4.30)$$

Due to Assumptions 4.1 and Theorem 4.2,

$$\left\| (v(t_k, \cdot) - \bar{v}(t_k, \cdot), \nabla) \bar{v}(t_k, \cdot) \right\| \leq C(\omega)h^{1/2-\varepsilon} \text{ a.s. .} \quad (4.31)$$

Thus, (4.29)-(4.31) imply (4.25). □

**Remark 4.4** To prove the estimate

$$\left\| \frac{\partial v}{\partial x_j} (t_k, x) - \frac{\bar{v}(t_k, x + \delta e_j) - \bar{v}(t_k, x - \delta e_j)}{2\delta} \right\| \leq C(\omega)h^{1/3-\varepsilon} \text{ a.s. .} \quad (4.32)$$

we do not need in the assumption on boundedness of second-order spatial derivatives of the approximate solution. Then, under the conditions of Theorem 4.2 (without the additional assumption on second-order spatial derivatives of the approximate solution), we can analogously prove convergence with a.s. order $1/3 - \varepsilon$ of the approximate pressure $\bar{p}(t_k, x)$ from (3.48) with $\bar{v}(t_{k+1}, x)$ from (3.28) in which we substitute $\bar{v}(t_{k+1}, x)$ found due to (3.12).

**Remark 4.5** As we discussed earlier in this section, though we proved $1/2 - \varepsilon$ a.s. convergence order for the velocity approximation in Theorem 4.2, we are expecting that the actual a.s. convergence order is $1 - \varepsilon$ which was observed in our numerical experiments in Section 3. Analogously, we expect that spatial derivatives of the approximate velocity converge with a.s. order $1 - \varepsilon$ instead of $1/3 - \varepsilon$ shown in (4.28). It is not difficult to see from the proof of Proposition 4.1 that a.s. convergence of both velocity and its first-order spatial derivatives with order $1 - \varepsilon$ implies a.s. convergence of pressure with order $1 - \varepsilon$. In our numerical experiments (see Section 3) we observed convergence (both mean-square and a.s.) of pressure with order one.
5 Numerical examples

In this section we test the numerical algorithm (3.32) from Section 3.2 on two model problems. The experiments indicate that the algorithm has the first order mean-square convergence.

5.1 Model problems

We introduce two model examples of SNSE (1.1)-(1.4) which solutions can be written in an analytic form. Both examples are generalizations of the deterministic model of laminar flow from [32] to the stochastic case.

First model problem. Let
\[ f(t, x) = 0, \quad \varphi(x) = 0, \quad q = 1, \]
\[ \gamma_1(t, x) = A \sin \frac{2\pi \kappa x^1}{L} \cos \frac{2\pi \kappa x^2}{L} \exp \left( -\sigma^2 \left( \frac{2\pi \kappa L}{L} \right)^2 t \right), \]
\[ \gamma_2(t, x) = -A \cos \frac{2\pi \kappa x^1}{L} \sin \frac{2\pi \kappa x^2}{L} \exp \left( -\sigma^2 \left( \frac{2\pi \kappa L}{L} \right)^2 t \right), \quad \kappa \in \mathbb{Z}, \quad A \in \mathbb{R}, \]
then it is easy to check that the problem (1.1)-(1.4), (5.1)-(5.2) has the following solution
\[ v^1(t, x) = A \sin \frac{2\pi \kappa x^1}{L} \cos \frac{2\pi \kappa x^2}{L} \]
\[ \times \exp \left( -\sigma^2 \left( \frac{2\pi \kappa L}{L} \right)^2 t \right) w(t), \]
\[ v^2(t, x) = -A \cos \frac{2\pi \kappa x^1}{L} \sin \frac{2\pi \kappa x^2}{L} \]
\[ \times \exp \left( -\sigma^2 \left( \frac{2\pi \kappa L}{L} \right)^2 t \right) w(t), \]
\[ p(t, x) = \frac{A^2}{4} \left( \cos \frac{4\pi \kappa x^1}{L} + \cos \frac{4\pi \kappa x^2}{L} \right) \]
\[ \times \exp \left( -2\sigma^2 \left( \frac{2\pi \kappa L}{L} \right)^2 t \right) (w(t))^2. \]

Second model problem. To construct this example, we recall the following proposition from [15].

Proposition 5.1 Let \( V(t, x), P(t, x) \) be a solution of the deterministic NSE with zero forcing (i.e., of (1.1)-(1.4) with all \( \gamma_r = 0 \) and \( f(t, x) = 0 \)) then the solution \( v(t, x), p(t, x) \)
of (1.1)-(1.4) with constant \( \gamma_r(t, x) = \gamma_r \) and \( f(t, x) = 0 \) is equal to

\[
v(t, x) = V(t, x - \int_0^t \sum_{r=1}^q \gamma_r w_r(s)ds) + \sum_{r=1}^q \gamma_r w_r(t),
\]

(5.4)

\[
p(t, x) = P(t, x - \int_0^t \sum_{r=1}^q \gamma_r w_r(s)ds).
\]

(5.5)

Combining this proposition with the deterministic model of laminar flow from [32], we obtain that if

\[
f(t, x) = 0, \quad \varphi(x) = \left(A \sin \frac{2\pi \kappa}{L} x^1 \cos \frac{2\pi \kappa}{L} x^2, -A \cos \frac{2\pi \kappa}{L} x^1 \sin \frac{2\pi \kappa}{L} x^2 \right)^T,
\]

(5.6)

\[\kappa \in \mathbb{Z}, \quad A \in \mathbb{R},\]

and

\[q = 1, \quad \gamma_1^1(t, x) = \gamma^1, \quad \gamma_1^2(t, x) = \gamma^2.\]

(5.7)

then the problem (1.1)-(1.4), (5.6)-(5.7) has the following solution

\[
v^1(t, x) = A \sin \frac{2\pi \kappa}{L} (x^1 - \gamma^1 I(t)) \cos \frac{2\pi \kappa}{L} (x^2 - \gamma^2 I(t)) \exp \left(-\sigma^2 \left(\frac{2\pi \kappa}{L}\right)^2 t + \gamma^1 w(t)\right),
\]

(5.8)

\[
v^2(t, x) = -A \cos \frac{2\pi \kappa}{L} (x^1 - \gamma^1 I(t)) \sin \frac{2\pi \kappa}{L} (x^2 - \gamma^2 I(t)) \exp \left(-\sigma^2 \left(\frac{2\pi \kappa}{L}\right)^2 t + \gamma^2 w(t)\right),
\]

\[
p(t, x) = \frac{A^2}{4} \left(\cos \frac{4\pi \kappa}{L} (x^1 - \gamma^1 I(t)) + \cos \frac{4\pi \kappa}{L} (x^2 - \gamma^2 I(t))\right) \exp \left(-2\sigma^2 \left(\frac{2\pi \kappa}{L}\right)^2 t\right),
\]

where

\[I(t) = \int_0^t w(s)ds, \quad w(s) = w_1(s).\]

5.2 Results of numerical experiments

In our numerical experiments we test the algorithm (3.32)-(3.33), (3.49) which is a realization of the layer method (3.27)-(3.28), (3.48). This algorithm possesses the following properties.

Proposition 5.2 1. The approximate solution of the problem (1.1)-(1.4), (5.1)-(5.2) obtained by the algorithm (3.32)-(3.33), (3.49) contains only those modes which are present in the coefficient \( \gamma_1(t, x) \) from (5.2), i.e., which are present in the exact solution (5.3).
2. The approximate solution of the problem \((1.1)-(1.4), (5.6)-(5.7)\) obtained by the algorithm (3.32)-(3.33), (3.49) contains only those modes which are present in the initial condition \(\varphi(x)\) from (5.6) and the zero mode, i.e., which are present in the exact solution (5.8).

The proof of this proposition is analogous to the proof of a similar result in the deterministic case [27] and it is omitted here.

We measure the numerical error in the experiments as follows. First, we consider the relative mean-square error defined as

\[
err_{msq}^v = \frac{\sqrt{E \sum_n (\bar{v}_n(T) - v_n(T))^2}}{\sqrt{E \sum_n |v_n(T)|^2}}, \quad err_{msq}^p = \frac{\sqrt{E \sum_n (\bar{p}_n(T) - p_n(T))^2}}{\sqrt{E \sum_n |p_n(T)|^2}}. \tag{5.9}
\]

Analysis of this error provides us with information about mean-square convergence of the numerical algorithm considered. To evaluate this error in the experiments, we use the Monte Carlo technique for finding the expectations in (5.9) by running \(K\) independent (with respect to realizations of the Wiener process \(w(t)\)) realizations of \(\bar{v}_n(T), v_n(T), \bar{p}_n(T), p_n(T)\). Second, we consider the relative \(L^2\)-error for a fixed trajectory of \(w(t)\):

\[
err^v = \frac{\sqrt{\sum_n (\bar{v}_n(T) - v_n(T))^2}}{\sqrt{\sum_n |v_n(T)|^2}}, \quad err^p = \frac{\sqrt{\sum_n (\bar{p}_n(T) - p_n(T))^2}}{\sqrt{\sum_n |p_n(T)|^2}}. \tag{5.10}
\]

Analysis of this error provides us with information about a.s. convergence of the numerical algorithm. To evaluate this error in the tests, we fix a trajectory \(w(t), 0 \leq t \leq T\), which is obtained with a small time step.

We note that in the case of the considered examples and the tested algorithm (see Proposition 5.2) \(v_n(T)\) are nonzero only for \(|\mathbf{n}^1| = |\mathbf{n}^2| = |\kappa|\) and \(p_n(T)\) are nonzero only for \(|\mathbf{n}^1| = 2|\kappa|, \mathbf{n}^2 = 0\) and \(|\mathbf{n}^1| = 0, |\mathbf{n}^2| = 2|\kappa|\). Hence, the sums in (5.9) and (5.10) are finite here. This also implies that it is sufficient here to take the cut-off parameter \(M\) in the algorithm (3.32)-(3.33), (3.49) to be equal to \(2|\kappa|\).

The test results for the algorithm (3.32)-(3.33), (3.49) applied to the first model problem (1.1)-(1.4), (5.1)-(5.2) are presented in Tables 5.1 and 5.2. In Table 5.1 the “±” reflects the Monte Carlo errors in evaluating of \(err_{msq}^v\) and \(err_{msq}^p\), they give the confidence intervals for the corresponding values with probability 0.95.

We can conclude from Table 5.1 that both velocity and pressure found due to the algorithm (3.32)-(3.33), (3.49) demonstrate the mean-square convergence with order 1. We also see from Table 5.2 that both velocity and pressure converge with order 1 for a particular, fixed trajectory of \(w(t)\). We note that we repeated the experiment for other realizations of \(w(t)\) and observed the same behavior. The observed first order convergence of the algorithm is consistent with our prediction (see (1.15), the discussion after it, and Remark 4.5).

The test results for the algorithm (3.32)-(3.33), (3.49) applied to the second model problem (1.1)-(1.4), (5.6)-(5.7) are presented in Table 5.3. In these tests we limit ourselves to simulation for a particular, fixed trajectory of \(w(t)\) and observation of a.s. convergence. We note that evaluation of the exact solution (5.8) requires simulation of the integral \(I(t)\). This was done in the following way. At each time step \(k + 1, k = 0, \ldots, N - 1\), we simulate a Wiener increment \(\Delta_k w\) as i.i.d. Gaussian \(\mathcal{N}(0, h)\) random variables (and we
Table 5.1: Mean-square relative errors $\text{err}^v_{\text{msq}}$ and $\text{err}^p_{\text{msq}}$ from (5.9) at $T = 3$ in simulation of the problem (1.1)-(1.4), (5.1)-(5.2) with $\sigma = 0.1$, $A = 1$, $\kappa = 1$, $L = 1$ by the algorithm (3.32)-(3.33), (3.49) with $M = 2$ and various time steps $h$. The “±” reflects the Monte Carlo error in evaluating $\text{err}^v_{\text{msq}}$ and $\text{err}^p_{\text{msq}}$ via the Monte Carlo technique with $K = 4000$ independent runs. The exact values (up to 5 d.p.) of the denominators in (5.9) are 0.37470 and 0.12159, respectively.

| $h$  | velocity     | pressure    |
|------|--------------|-------------|
| 0.2  | 0.0537 ± 0.0012 | 0.0710 ± 0.0038 |
| 0.1  | 0.0263 ± 0.0006  | 0.0337 ± 0.0016  |
| 0.05 | 0.0130 ± 0.0003  | 0.0170 ± 0.0009  |
| 0.02 | 0.0052 ± 0.0001  | 0.0066 ± 0.0003  |
| 0.01 | 0.0025 ± 0.00006 | 0.0031 ± 0.0001  |

Table 5.2: Relative errors $\text{err}^v$ and $\text{err}^p$ from (5.10) at $T = 3$ in simulation of the problem (1.1)-(1.4), (5.1)-(5.2) with $\sigma = 0.1$, $A = 1$, $\kappa = 1$, $L = 1$ for a fixed trajectory of the Wiener process $w(t)$ by the algorithm (3.32)-(3.33), (3.49) with $M = 2$ and various time steps $h$. The exact values (up to 5 d.p.) of the denominators in (5.10) are 0.43950 and 0.09658, respectively.

| $h$  | velocity     | pressure    |
|------|--------------|-------------|
| 0.2  | 0.0485       | 0.0585      |
| 0.1  | 0.0237       | 0.0284      |
| 0.05 | 0.0117       | 0.0141      |
| 0.02 | 0.0047       | 0.0056      |
| 0.01 | 0.0023       | 0.0028      |

Again, the observed first order convergence of the algorithm in Table 5.3 is consistent with our prediction (see the discussion after (4.15) and Remark 4.5).

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Table 5.3: Relative errors $err^v$ and $err^p$ from (5.10) at $T = 3$ in simulation of the problem (1.1)-(1.4), (5.6)-(5.7) with $\sigma = 0.1$, $A = 1$, $\kappa = 1$, $L = 1$, $\gamma^1 = 0.5$, $\gamma^2 = 0.2$ for a fixed trajectory of the Wiener process $w(t)$ by the algorithm (3.32)-(3.33), (3.49) with $M = 2$ and various time steps $h$. The exact values (up to 6 d.p.) of the denominators in (5.10) are 0.505620 and 0.000548, respectively.

| $h$  | velocity | pressure |
|------|----------|----------|
| 0.01 | 0.166    | 0.973    |
| 0.005| 0.068    | 0.384    |
| 0.002| 0.024    | 0.134    |
| 0.001| 0.0118   | 0.0645   |
| 0.0005| 0.0058  | 0.0313   |

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