Branching Processes and Multi-Particle Production

S. G. Matinyan*

Department of Physics, Duke University
Durham, North Carolina 27708-0305

E. B. Prokhorenko

Theory Division, Yerevan Physics Institute
375036, Yerevan, Armenia

Abstract

The general theory of the branching processes is used for establishing the relation between the parameters \(k\) and \(\bar{n}\) of the negative binomial distribution. This relation gives the possibility to describe the overall data on multiplicity distributions in \(pp(p\bar{p})\)-collisions for energies up to 900 GeV and to make several interesting predictions for higher energies. This general approach is free from ambiguities associated with the extrapolation of the parameter \(k\) to unity.

1. Introduction

Theoretical description of the multiple production, based on so-called soft processes, today is beyond the limits of QCD, and the natural approach there is to look for empirical relations.

* On leave from Yerevan Physics Institute, 375036, Yerevan, Armenia
The most popular in this field was the KNO scaling for multiplicity distributions which was fulfilled very well for hadronic collisions up to ISR energies and for $e^+e^-$-annihilation. The evident violation of the KNO scaling at energies of the CERN collider [1] attracted much attention to the negative binomial distribution (NBD) which describes fairly well the overall features of the data on the multiplicity distribution (MD) of hadrons in different processes ($pp(\bar{p}p), e^+e^-, \nu p, AA$...), in different ranges of rapidity and in a wide interval of energies [2,3]. It is especially relevant to the $pp(\bar{p}p)$ interaction.

Taking into account the special role of NBD in describing the multiplicity distribution at high energy, it seems to be important to consider NBD on the basis of general assumptions about the character of the process of particle production in hadronic collisions without detailed specification of the dynamics. In reference [4] it was proposed as a basis for NBD to consider the multiple production as a random stationary branching process which is a rather general probabilistic model for the processes of the multiplication and transformation of the active particles. In this approach the transformation of each particle is independent of the history of the process and of the transformation of other particles, obeying the general probabilistic laws of Markov processes. The same refers to the fate of the generation of each particle.

The branching processes may find their realization in terms of the quark-gluonic cascades, corresponding to the microscopic description of the nonequilibrial evolution of the partonic system, e.g., in the rapidity space [5,6]. It is important to stress that for us there is no need to know the details of the dynamical laws governing these cascades.

It was realized that the system of produced hadrons may be considered as a result of the contribution from coherent and chaotic components (so called two-component model) and it is known that in $pp(\bar{p}p)$ collisions at high energy the chaotic component [7-10] dominates, which described by NBD, whereas in $e^+e^-$-annihilation the coherent (Poisson) part is essential.
The observation of dynamical chaos in the dynamics of nonabelian gauge fields (see e.g. [11]) raises the question about the role and origin of the chaotic component in hadronic collisions.

There exists an interesting practical observation [7,9], that in distinction to $e^+e^-$-annihilation where the addition of the small ($\approx 10\text{-}20\%$) chaotic (noise) amplitude essentially changes the multiplicity distribution, in $pp(\bar{p}p)$-collisions the addition of a small coherent component to the NBD does not change the shape of the distribution significantly.

Taking into consideration the above mentioned arguments and remarks we here consider the NBD as adequate for the description of the multiplicity distribution in $pp(\bar{p}p)$ collisions at high energy.\(^1\) The NBD

\[
P^{(k)}_n = \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)} \frac{((\bar{n}/k))^n}{(1 + \bar{n}/k)^{n+k}}
\]

has two parameters $\bar{n}$ and $k$. $\bar{n}$ is the average multiplicity. As for $k$, initially it was associated with the number of chaotically emitting cells. After the UA5 experiments [1,10,14] it is clear that such a meaning of $k$ in general is not necessary, because $Sp\bar{p}S$ collider data yield the empirical relation

\[
\frac{1}{k} = a + b \ln \sqrt{s} \quad (a \approx -0.1, \ b \approx 0.06)
\]

which is valid up to energy $\sqrt{s} = 900$ GeV, not showing a tendency for saturation. So, at such energies the KNO-like scaling continues to be violated, which is more clearly expressed in the observed strong rise of the moments

\[
C_q = \frac{\langle n^q \rangle}{\langle n \rangle^q}
\]

with $\sqrt{s}$ [14].

\(^1\) In principle on the basis of the quantum optics it is possible to generalize NBD to take into account the chaotic as well as the coherent components [7,12].
Of course, one cannot extrapolate the concrete form of the empirical relation (1) to higher energy since this would lead to contradiction. For instance, from (1) it would follow that the peak of the distribution would be at \( n = 0 \) at very high energy when \( k = 1 \). This means that saturation of \( k \) must take place at ultrahigh energies at a value larger than unity (see also [15]). It indicates the necessity to establish the relation between \( k \) and \( \sqrt{s} \) (or, at least, between \( k \) and \( \bar{n} \)) based on general theoretical considerations. We propose that such a basis could be a general theory of branching processes. As mentioned such an approach was developed in [4] where the idea of the stationarity of the branching process was used for establishing the relation between \( k \) and \( \bar{n} \). The result

\[
\frac{1}{k} = a + b \ln \frac{\bar{n}}{k} \quad (a \approx 0.12, \ b \approx 0.08)
\]  

(4)
gives the unconfined though weaker rise of \( \frac{1}{k} \) with \( \sqrt{s} \) leading to the above difficulty associated with the extrapolation of \( k \) to unity. Unfortunately, in deriving (4) the authors of [4] incorrectly used the conditions for stationarity of the branching process. In the paper [16] the condition for stationarity is used correctly though the authors missed the most interesting ansatz, in our opinion, of the relation between \( k \) and \( \bar{n} \). The resulting dependence of \( k \) on \( \bar{n} \),

\[
k = A \left( \frac{\bar{n}}{k} \right)^B \quad (A \approx 11, \ B \approx -0.5)
\]  

(5)

again did not avoid the problem resulting from the extrapolation of \( k \) to unity.

2. Relation between \( k \) and \( \bar{n} \)

Thus, we consider NBD as a result of the stationary branching process with one sort of multiplied particles (pions) and continuous evolution parameter \( t \). The generating function \( F \) for such a process satisfies the reverse Kolmogorov differential equation [17]:

\[
\frac{dF}{dt} = f(F, t).
\]  

(6)
For the generating function of the NBD

\[ F(x, t) = \sum_n P^{(k)}_n x^n = \left[ 1 + \frac{\bar{n}}{k} (1 - x) \right]^{-k} \]  

(7)

\[ f(F, t) \] equals

\[-F \ln F \dot{k} + F (1 - F^{1/k}) \dot{\bar{n}} \bar{m}, \]  

(8)

where \( m = \bar{n}/k \) and \( \dot{k} = \frac{dk}{dt} \), etc.

For a stationary branching process \( f(F, t) \) is factorized, \( f(F, t) = \varphi(F) \psi(t) \). Evidently the condition

\[ \frac{\dot{k}}{k} = \text{const.} \frac{\dot{\bar{n}}}{\bar{m}} \]  

(9)

[4] which leads to the relation (4) does not give such a factorization. For \( F \approx 1 \) factorization takes place [16] and this approximation is also good for \( k \) close to unity. More adequate here is a parameter

\[ \delta = \frac{\ln F}{k}, \]  

(10)

which is small at \( F \approx 1 \) and not too small \( k \). Expanding \( 1 - F^{1/k} \) up to \( \delta^2 \) in (8), it is easy to obtain the solution of the resulting differential equation which is a necessary and sufficient condition for factorization:

\[ k = \frac{a \bar{n}}{\bar{n} - b}, \]  

(11)

where \( a \) and \( b \) are the integration constants which one must find from comparison with experimental data. Thus it is possible to state that NBD with relation (11) between its parameters \( k \) and \( \bar{n} \) is the consequence of a stationary branching process.

The function \( k(\bar{n}) \) is very simple. At \( ab > 0 \) \( k \) is decreasing from \( a \) to \( -\infty \) and from \( +\infty \) to \( a \). But experimentally (at least for \( 10 < \sqrt{s} < 900 \text{ GeV} \) \( k \) is decreasing with \( \sqrt{s} \) [1,14,10], so physically interesting is a case \( ab > 0 \). But at the same time the case \( a < 0, b < 0 \) is also unphysical, because it corresponds to \( \bar{n} < 0 \), or to \( k < 0 \). By the same reason,
if we do not want to have negative $k$, we must discard the lower branch of (5) with $ab > 0$
 corresponding to decreasing $k$ in $(a, -\infty)$ interval.$^2$

Thus this simple analysis has shown that in the region of high energy the relation between $k$ and $\bar{n}$ is given by (5) with positive $a, b$, and it is necessary to confine to the branch of the hyperbola (5) in the first quadrant.

3. Consequences and predictions of the model

The relation (5) between $k$ and $\bar{n}$, in spite of its simplicity, is rather rich in content. Let us stress once more that this relation must be only used for $\bar{n} > b \approx 7$, i.e. for $\sqrt{s} > 14$ GeV; smaller energies should not be considered here. Our model ensures that $k > a > 0$ (from the fit follows $a = 3.06$), implying that the limit $k = 1$ never is achieved. Thus there does not exist the difficulty associated with $k = 1$ at very high energy that is characteristic of some other ansatze [1,4,16].

From (5) it follows that asymptotically, when $k$ goes to the saturation, KNO scaling is restored. The asymptotic distribution function at $\sqrt{s} >> 14$ GeV and $\bar{n} \gg k = a \approx 3.06$ has a form of $\Gamma$-distribution:

$$\psi(z) \equiv \bar{n}P_{\bar{n}}^{(k)} \approx \frac{k^{k-1}e^{-kz}}{\Gamma(k)} = 14.49z^{2.06}e^{-3.06z} \quad \left( z = \frac{n}{\bar{n}} \right)$$

(12)

Our model gives rather clear predictions for $C_q$-moments. In particular, Wroblewski’s relation here takes place well at high energies:

$$\frac{D_2}{\bar{n}} \equiv \frac{\sqrt{n^2 - \bar{n}^2}}{\bar{n}} = (C_2 - 1)^{\frac{1}{2}} = \left( \frac{1}{k} + \frac{1}{\bar{n}} \right)^{\frac{1}{2}} \approx 0.57 \left( 1 - \frac{1.92}{\bar{n}} \right).$$

(13)

$^2$ In this connection there is an interesting observation in [18] that the multiplicity data for small ($\sqrt{s} < 10$GeV) energies is possible to describe well with negative $k$. From this point of view it may be said that two branches of the hyperbola (5) naturally divide the large energy region ($\bar{n} > b > 0$) from small energies ($\bar{n} < b$). From our fit (see below) $b \approx 7$, so it means that we must not consider energies below $\sqrt{s} \approx 14$ GeV.
The second order correlation $g^{(2)} = \frac{n(n-1)}{n^2}$ which is increasing slowly and asymptotically equals 1.33 indicates not the presence of a coherent component as sometimes stated but just the fact that $k$ is always larger than unity. All high order moments $C_q$ are rising and saturate asymptotically, as is easy to understand from the relation ($q > 2$)

$$C_q = 1 + \sum_{m=0}^{q-2} P_m^{(q)} \left( \frac{1}{k} \right) (C_2 - 1)^{q-m-1}$$

(14)

where $P_m^{(q)} (\frac{1}{k})$ are polynomials of order $m$ with positive coefficients ($P_0^{(q)} = 1$) and $\frac{1}{k} = \frac{1}{a} - \frac{b}{a \bar{n}}$ is increasing. Asymptotically we have:

$$C_q \approx \frac{\Gamma(k + q)}{k^{q-1} \Gamma(k + 1)}$$

(15)

i.e. (up to $1/\bar{n}^2$):

$$C_2 \approx 1.33 - \frac{1.27}{\bar{n}}$$

$$C_3 \approx 2.21 - \frac{5.82}{\bar{n}}$$

$$C_4 \approx 4.39 - \frac{21.3}{\bar{n}}$$

$$C_5 \approx 10.19 - \frac{76.11}{\bar{n}}.$$  

(16)

The scaled peaks of the multiplicity distributions is moving to the left toward its asymptotic value:

$$z_{\text{peak}}(\sqrt{s}) = \frac{n_{\text{peak}}}{\bar{n}} = 1 - \frac{1}{a} + \frac{b}{a \bar{n}} = 0.67 + \frac{2.27}{\bar{n}}.$$  

(17)

Finally, before going into the comparison with experimental data, let us make one comment. By no means do we consider the limiting value of $k = a$ as an indication that corresponds to asymptotic value of the number of clusters, fireballs, minijets etc. in the multiple production.³

³ Note that some experiments (see[20]) are indicating that at sufficiently high energies
Let us stress only that the often used value of \( k = 1 \) is meaningless.\(^4\) In particular, in connection with this value of \( k \) in [16] it was made very strong and unusual statement that in the process of the multiple production the information entropy achieves its maximal value for \( k = 1 \) and as a result \( \bar{n} \) achieves its maximal value and thus does not depend on the energy at all. This statement is derived from the fact that this entropy for NBD near \( k = 1 \) behaves as \( \ln \bar{n} + 1 - \left( \frac{\pi^2}{6} - 1 \right) (k - 1)^2 \). But the lower bound on \( k \geq a \) in our model shows that such a statement is a result of the unphysical interpolation of \( k \) to unity.

4. Comparison with experiments

To obtain numerical values of constants \( a \) and \( b \) in (5) we used the results of the fit of parameters \( \bar{n} \) and \( k \) of NBD by experimental distributions of charged particles multiplicity in the range from \( \sqrt{s} = 19.5 \) GeV to \( \sqrt{s} = 900 \) GeV [10,14,19] (non-single diffractive events). The result of the fit of \( a \) and \( b \) in (5) on the basis of these data gives:

\[
a = 3.06 \pm 0.06, \quad b = 6.95 \pm 0.08.
\] (18)

In Fig. 1 is shown the function (5) obtained with these values for \( a \) and \( b \). We did not consider points corresponding to low energy (see footnote 2). Fig. 2 shows the dependence of \( 1/k \) on \( \bar{n} \), which is seen to saturate. Fig. 3 gives the curves for \( C_q (q = 2 - 5) \) for our \((E_L \approx 400 \) GeV\) the number of the clusters produced in \( pp \)-interactions is \( 4.2 \pm 1.7 \). If we continue such an interpretation of \( k \), then \( k^{-1} \) may be considered as the ratio of the probability for two particles to be emitted from one cluster to the probability of emission of these particles by two different clusters [6]. So the asymptotic “aggregation” degree is \( a^{-1} \approx 0.33 \).

\(^4\) The \( k = 1 \) in our model corresponds to the negative \( \bar{n} \) (lower branch of (5) from \( a \) to \(-\infty\)). Notice that \( k > a \simeq 3 \) shows that our expansion parameter of \( \delta = k^{-1} \ln F \) is adequate and selfconsistent.
model (solid line) and compares them with experimental data [10,14,19] for non-single-diffractive component of \( pp(\bar{p}p) \) reactions. It is seen that higher moments \( (q = 4,5) \) have not yet achieved their asymptotic values \( (C_q = 4.39 \) and \( C_5 = 10.19 \) at \( \sqrt{s} = 900 \) GeV.

In Fig. 4-7 are shown the distributions \( \psi(z,k) \) as a function of \( z = n/\bar{n} \) for energies 0.546, 1.8, 8, and 40 TeV, respectively. Solid lines show the asymptotic distribution (12). It is seen from these figures that scaling sets in at \( \sqrt{s} \approx 8 \) TeV. These curves show the systematic shift to the left of the peaks of \( \psi(z,k) \) with increasing \( \sqrt{s} \).

Finally, Fig. 8 shows the information entropy

\[
\begin{align*}
\nonumber w &= -\sum_n P_n \ln P_n \approx \ln \bar{n} - \int_0^\infty \psi(z,k) \ln \psi(z,k) dz \quad (19)
\end{align*}
\]

which is defined by the chaotic component only for \( k \) from (5). The figure also shows the “maximal” entropy \( w_{\text{max}} \) corresponding to \( k = 1 \) which is meaningless in our model.

5. Conclusions

In the present paper we have attempted to establish the relation between parameters of \( k \) and \( \bar{n} \) of NBD on the basis of the general theory of random branching processes. This relation seems to be rather interesting, selfconsistent and has a predictive power. It removes some contradictions which occured in the use of NBD for description of multiplicity distributions for high energy hadronic collisions.

On the whole the agreement of our model with the existing experimental data is good enough which, of course, is not surprising because of the coefficients \( a \) and \( b \) in (5) were derived from a fit to the experimental data for \( k \) and \( \bar{n} \). More important are the predictions for the behavior of \( C_q \left( \frac{D_n}{\bar{n}} \right) \) and \( \psi(z,k) \) at higher energies which may be checked at LHC and SSC: the restoration of the KNO scaling in the multi-TeV region, asymptotic constant values of \( C_q \), depending on \( q \), “explanation” of the Wroblewski rule at high energy, and the asymptotic value of the peak of \( \psi(z) \).
It is interesting to compare qualitatively our model of general branching processes with the detailed models of quark gluonic branching processes. If one neglects the quark branching the resulting parton distribution looks very similar to NBD and their conclusions qualitatively coincide with ours (limit of the widening of the distribution shape, increase and final saturation of $C_q$, etc. [21].) The dominant role of gluonic branching in comparison with quark branching is the characteristic feature of the detailed study of corresponding processes from the point of view of dynamical chaos [22], or from the approach based on the detailed consideration of branching of quarks and gluons at the formation of quark-gluon plasma [23,24].

There is at least one aspect which apparently necessitates the quark branching: the observed small oscillations in the high-multiplicity tail of $P_n$-distribution at Tevatron energy [25]. If we recall the very old prediction of such oscillations in the Regge-pole approach [26] which is connected with Pomeron cuts, then it seems reasonable that quarks may be responsible for these phenomena. (The “explanation” of these oscillations by the addition of two binomial distributions (five-parameter fit [25]) may also be the reflection of this two-Reggeon cut.

Finally in connection with the meaning of parameter $k$ of NBD and its asymptotic limit in our model ($k_{\text{min}} \approx 3$) it would be very interesting to apply our model to the multiplicity distribution of hadrons in $\pi p$, as well in $e^+ e^-$, $\nu p$ and $e p$ collisions at high energies.

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References

[1] G. J. Alner et al. Phys. Lett. 138B, 304 (1984).

[2] A. Giovannini and L. Van Hove, Acta Phys. Pol. B19, 495, 917, 931 (1988).

[3] A. Giovannini, Proc. Intern. Conf. on Physics in Collision, 6, ed. M. Derrick (World
Scientific, Singapore, 1987) p. 39.

[4] P. V. Chliapnikov and O. G. Tchikilev, Phys. Lett. 222B, 152 (1989).

[5] G. Altarelli and G. Parisi, Nucl. Phys. B126, 298 (1977).

[6] L. Van Hove and A. Giovannini, Proc. 25th Intern. Conf. High Energy Physics, V. II
Singapore p. 998, (1990).

[7] P. Carruthers and C.-C. Shih, Phys. Lett. 137B, 425 (1989).

[8] G. N. Fowler et al., Phys. Rev. Lett. 56, 14 (1986).

[9] P. A. Carruthers et al. Phys. Lett B309, 369 (1988).

[10] R. E. Ansorge, Z. Phys. C43, 357 (1989).

[11] S. G. Matinyan, Sov. J. Part. Nucl. 16, 226 (1985).

[12] B. A. Bambah and M. Venkata Satyanarayana, Phys. Rev. D37, 2202 (1988).

[13] A. Vourdas and R. M. Weiner, Phys. Rev. D38, 2209 (1988).

[14] G. J. Alner et al., Phys. Lett. 167B, 476 (1986).

[15] V. Gupta and N. Sarma, Z. Phys. C41, 415 (1988).

[16] A. K. Chakrabarti, Phys. Rev. D45, 4057 (1992).

[17] N. A. Dmitriev and A. N. Kolmogorov, Dokl. Acad. Nauk. SSSR, 56, 7 (1947) (in
Russian).

[18] R. Schwed, G. Wrochna and A. K. Wroblewski Acta Phys. Pol. 19B, 703 (1988).

[19] G. J. Alner et al., Phys. Rep. 154, 247 (1987).

[20] E. G. Boos et al., Proc. 25th Intern. Conf. High Energy Physics, Singapore, Vol. II,
p. 1018 (1990).
[21] I. Sarcevic, Mod Phys. Lett. A2, 513 (1987).

[22] B. Müller, Duke University preprint, Duke-TH-92-36.

[23] K. Geiger and B. Müller, Nucl. Phys. B369, 600 (1992).

[24] E. Shuryak, Phys. Rev. Lett. 68, 3270 (1992).

[25] C. S. Lindsey, Nucl. Phys. A 544, 343 (1992).

[26] V. A. Abramovski and O. V. Kancheli, JETP Lett. 15, 397 (1972); V. A. Abramovski, V. N. Gribov and O. V. Kancheli, Sov. J. Nucl. Phys. 18, 308 (1979).

[27] G. I. Alner et al., Phys. Lett. 160B, 199 (1985).
Figure Captions:

Fig. 1: The dependence of $k$ on $\bar{n}$ from eq. (5) with coefficients $a = 3.06$ $b = 6.95$ obtained by fit. Two points shown on Fig. 1 and corresponding to low energies ($\sqrt{s} < 19.5$ GeV), are not taken into account (see footnote 2).

Fig. 2: $\frac{1}{k}$ dependence on $\bar{n}$ from (5) ($a = 3.06$, $b = 6.95$).

Fig. 3: The $C_q$-moments ($q = 2 - 5$) as a function of $\bar{n}$ from (5) (solid lines) compared with experimental data on inelastic, non-single-diffractive component of $pp(\bar{p}p)$ reactions (see table 2 from [27] and [10,14]).

Fig. 4: The dependence of $\bar{n}P_n$ on $z = \frac{n}{\bar{n}}$. Dashed line for $\sqrt{s} = 546$ GeV, solid line is an asymptotic distribution.

Fig. 5: Same for $\sqrt{s} = 1800$ GeV.

Fig. 6: Same for $\sqrt{s} = 8$ TeV.

Fig. 7: Same for $\sqrt{s} = 40$ TeV.

Fig. 8: The information entropy as a function of $\bar{n}$. Solid line is for our model with $k$ dependence of (5). Dashed-dotted line corresponds to the “maximal” entropy ($k = 1$).