Chapter 25:

Quantum Phase Transitions in Dense QCD$^1$

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Chapter 25

Quantum Phase Transitions in Dense QCD

Quantum chromodynamics (QCD) at finite temperature, $T$, and quark chemical potential, $\mu$, has a rich phase structure: at low $T$ and low $\mu$, the Nambu-Goldstone (NG) phase with nearly massless pions is realized by the dynamical breaking of chiral symmetry through condensation of quark–anti-quark pairs, while, at low $T$ and high $\mu$, a Fermi liquid of deconfined quarks is expected to appear as a consequence of asymptotic freedom. Furthermore, in such a cold quark matter, condensation of quark–quark pairs leads to the color superconductivity (CSC). At high $T$ for arbitrary $\mu$, all the condensates melt away and a quark-gluon plasma (QGP) is realized. The experimental exploration of thermal phase transition from the NG phase to QGP is being actively pursued in ultrarelativistic heavy ion collisions at RHIC (Relativistic Heavy Ion Collider), and will be continued in the future at LHC (Large Hadron Collider). The quantum phase transition from the NG phase to the CSC at low $T$ is also relevant to heavy-ion collisions at moderate energies, and is of interest in the interiors of neutron stars and possible quark stars.

In this Chapter, after a brief introduction to the basic properties of QCD, the current status of the QCD phase structure and associated quantum phase transitions will be summarized with particular emphasis on the symmetry realization of each phase. Possible connection between the physics of QCD and that of ultracold atoms is also discussed.

25.1 Introduction to QCD

The color SU(3)$_C$ gauge theory of quarks and gluons [1] is now called the quantum chromodynamics (QCD) and is established as the fundamental theory of strong interaction. The Lagrangian density of QCD reads

\[
\mathcal{L}_{\text{QCD}} = \bar{q}_L i \gamma_\mu \partial^\mu q_L + \bar{q}_R i \gamma_\mu \partial^\mu q_R - \frac{1}{4} G^{\alpha \mu \nu} G^\alpha_{\mu \nu} + \bar{q}_L m_q q_L + \bar{q}_R m_q q_R ,
\]  

(25.1.1)
Figure 25.1: The QCD fine-structure constant $\alpha_s$ determined from the $\tau$ decay, the $\Upsilon$ decay, the deep inelastic scattering, the $e^+e^-$ annihilation and the $Z$-boson resonance shape and width [4].

where the covariant derivative is defined as $D_\mu \equiv \partial_\mu + ig t^a A_\mu^a$ with $g$ being the QCD coupling constant, $t^a$ the SU(3)$_C$ group generator and $A_\mu^a$ the gluon field belonging to the adjoint representation of SU(3)$_C$. The gluon field-strength tensor is defined as $G_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{a\beta\gamma} A_\mu^\beta A_\nu^\gamma$ with $f^{a\beta\gamma}$ being the structure constant of the SU(3)$_C$ group.

The quark field $q$ belongs to the fundamental representation of SU(3)$_C$. The right (left) handed quark $q_R = \frac{1}{2}(1 + \gamma_5)q$ ($q_L = \frac{1}{2}(1 - \gamma_5)q$) is an eigenstate of the chirality operator $\gamma_5$ with the eigenvalue $+1$ ($-1$). Although quarks have six flavors (u,d,c,s,t,b) in the real world [2], we focus only on three light quarks (u,d,s) in this Chapter, so that the quark mass matrix is $m = \text{diag}(m_u, m_d, m_s)$.

As is evident from Eq.(25.1.1), only the mass term can mix the left-handed quark and the right-handed quark in the QCD Lagrangian.

The running coupling constant $g(\kappa)$ is defined as an effective coupling strength at the energy scale $\kappa$. Due to the asymptotic freedom of QCD [3], $g(\kappa)$ becomes small when $\kappa$ increases as seen explicitly in the two-loop perturbation,

$$\alpha_s(\kappa) = \frac{g^2(\kappa)}{4\pi} \simeq \frac{1}{4\pi\beta_0 \ln(\kappa^2/\Lambda_{\text{QCD}}^2)} \left[ 1 - \frac{\beta_1}{\beta_0} \frac{\ln(\ln(\kappa^2/\Lambda_{\text{QCD}}^2))}{\ln(\kappa^2/\Lambda_{\text{QCD}}^2)} \right], \tag{25.1.2}$$

where $\beta_0 = (11 - \frac{2}{3}N_f)/(4\pi)^2$ and $\beta_1 = (102 - \frac{28}{3}N_f)/(4\pi)^4$ with $N_f$ being the number of flavors. Here, $\Lambda_{\text{QCD}} \approx 200 \text{ MeV} = 2 \times 10^8 \text{ eV}$ is called the QCD scale parameter which is determined from the comparison of Eq.(25.1.2) with the experimental data in high energy processes satisfying $\kappa \gg \Lambda_{\text{QCD}}$ (see Fig.25.1).

Equation (25.1.2) implies that $\alpha_s(\kappa \sim \Lambda_{\text{QCD}}) \sim O(1)$, so that the QCD per-
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Figure 25.2: The masses of u, d and s quarks at the scale $\kappa = 2 \text{ GeV} = 2 \times 10^9 \text{ eV}$ determined from various observables and methods [4].

perturbation theory breaks down. This leads to various non-perturbative phenomena such as the confinement of quarks and gluons and the dynamical breaking of chiral symmetry [5, 6] at low energies, $\kappa < \Lambda_{\text{QCD}}$. These are responsible for the formation of hadrons ($q\bar{q}$ mesons and $qqq$ baryons), and also for the origin of their masses. On the other hand, at extremely high temperature and/or high baryon density where $\alpha_s(\kappa \gg \Lambda_{\text{QCD}}) \ll 1$, the system may be treated as a weakly interacting matter of quarks and gluons. Thus, there must be a phase transition from the hadronic matter composed of confined quarks and gluons at low energies to the deconfined quark-gluon matter at high energies.

Due to quantum corrections, the quark mass $m$ also becomes $\kappa$ dependent. As seen from Fig. 25.2, the current determination of the u and d quark masses at $\kappa = 2 \text{ GeV}$ indicates that they are about 50 to 100 smaller than $\Lambda_{\text{QCD}}$, while s quark mass is comparable to $\Lambda_{\text{QCD}}$. Therefore, it is legitimate to treat $m_u/\Lambda_{\text{QCD}}$ and $m_d/\Lambda_{\text{QCD}}$ as small expansion parameters, while the expansion by $m_s/\Lambda_{\text{QCD}}$ does not necessarily work. Systematic expansion in terms of the quark masses is called the chiral perturbation theory and has been successfully applied to a wide variety of QCD phenomena [7].

25.1.1 Symmetries in QCD

Let us consider the following transformations of the quark fields,

$$
q_L \rightarrow e^{-i\theta_B} e^{-i\theta_A} V_L \ V_C \ q_L, \quad q_R \rightarrow e^{-i\theta_B} e^{+i\theta_A} V_R \ V_C \ q_R,
$$

(25.1.3)

where $V_C$ (gauge rotation) is a local SU(3)$_C$ transformation in the color space, while $V_{L(R)}$ (chiral rotation) is a global SU(3)$_{L(R)}$ transformation in the flavor space. The $\theta_B$ and $\theta_A$ are phases associated with a global U(1)$_B$ transformation (baryon-number rotation) and the global U(1)$_A$ transformation (axial rotation), respectively. For $m_u, m_d = 0$ (the flavor-SU(3) chiral limit), QCD Lagrangian Eq. (25.1.1) is invariant under Eq. (25.1.3) together with the SU(3)$_C$ gauge tran-
Figure 25.3: Light hadron spectra obtained from lattice QCD Monte Carlo simulations with dynamical u, d, s quarks in the Wilson fermion formalism. The spatial lattice volume \( V \) and the lattice spacing \( a \) are \((2.9 \text{ fm})^3\) and 0.09 fm, respectively. Horizontal bars denote the experimental values [10].

The formation of the gluons, so that the full continuous symmetries of QCD become

\[
\mathcal{G} = \left[ \text{SU}(3)_C \right]_{\text{local}} \otimes \left[ \text{SU}(3)_L \otimes \text{SU}(3)_R \right]_{\text{global}} \otimes \left[ \text{U}(1)_B \right]_{\text{global}}. \tag{25.1.4}
\]

Although the \( \text{U}(1)_A \) looks like a symmetry of Eq. (25.1.1), it is explicitly broken by quantum effect known as the axial anomaly [8] which reduces \( \text{U}(1)_A \) down to its discrete subgroup \( Z(2N_f)_A = Z(6)_A \). The masses of light quarks \( m_u, m_d, m_s \) act as small external fields to break the global chiral symmetry \( \left[ \text{SU}(3)_L \otimes \text{SU}(3)_R \right]_{\text{global}} \).

In the past few years, there arises a remarkable progress in lattice gauge theory [9] particularly in calculating the hadron spectra on the basis of the QCD Monte Carlo simulations with light dynamical u, d, s quarks. This has been achieved partly due to the growth of the supercomputer speed and partly due to the new algorithms: Simulations with quark masses very close to the physical values are now possible in the Wilson fermion formalism [10, 11]. Shown in Fig. 25.3 is an example of such calculations for meson and baryon masses extrapolated to the physical quark masses using the simulation data taken in the interval, \( \frac{1}{2}(m_u + m_d) = 3.5 \text{ MeV} - 67 \text{ MeV} \) at \( \kappa = 2 \text{ GeV} \). The experimental values are reproduced in 3% accuracy at present.
25.1.2 Dynamical breaking of chiral symmetry

Although QCD Lagrangian in the flavor-SU(3) chiral limit has the symmetry $G$ in Eq. (25.1.4), the ground state of the system breaks some of the symmetries dynamically. Consider the QCD vacuum $|0\rangle$ at zero temperature and zero baryon density. Taking into account the fact that the QCD does not allow dynamical breaking of parity and vector symmetries in the vacuum [12], the following is one of the possible symmetry breaking patterns,

$$G \rightarrow SU(3)_C \otimes SU(3)_{L+R} \otimes U(1)_B,$$

(25.1.5)

where simultaneous transformation of the left-handed and right-handed quarks (the vector rotation, $V_L = V_R$) as indicated by $SU(3)_{L+R}$ remains as a symmetry of the vacuum. An order parameter to characterize this “dynamical breaking of chiral symmetry” would be the “chiral condensate”,

$$\langle \bar{q}q \rangle_0 \equiv \langle 0 | \bar{q}q | 0 \rangle,$$

(25.1.6)

The non-vanishing chiral condensate, $\langle \bar{q}q \rangle_0 = \langle \bar{q}_L q_R + \bar{q}_R q_L \rangle_0 \neq 0$, implies that the quark–anti-quark pairs are Bose-Einstein condensed. It also implies that a non-perturbative mixing between the left-handed and the right-handed quarks takes place in the QCD vacuum: In other words, an effective quark mass called the chiral gap is dynamically generated. Indeed, there are phenomenological evidences that the u, d quarks and s quark have effective masses $M_u \sim 350$ MeV and $M_s \sim 550$ MeV inside hadrons [6].

The Nambu-Goldstone (NG) bosons associated with the dynamical breaking of the flavor-SU(3) chiral symmetry are nothing but the pions, kaons and the $\eta$-meson. Moreover, one can derive a spectral sum rule, called the Gell-Mann–Oakes–Renner relation [14], which relates the pion mass to the chiral condensate as $f_\pi^2 m_\pi^2 = -\hat{m}(\bar{u}u + \bar{d}d)_0 + O(\hat{m}^2)$. Here $\hat{m} \equiv (m_u + m_d)/2$, $f_\pi (= 92.4$ MeV) is the pion decay constant, and $m_{e^\pm} (\simeq 140$ MeV) are the charged pion masses. Similar relation holds also for the neutral pion $\pi^0$. In the limit $m_u, d \to 0$, the pion mass vanishes as it should be from the Nambu-Goldstone theorem.

25.2 QCD matter at high temperature

As the temperature $T$ of the system increases, the condensed $q\bar{q}$ pairs in the QCD vacuum are melted away by thermal fluctuations. This is analogous to the phase transition in metallic superconductors with the electron pairing $\langle e_\uparrow e_\downarrow \rangle$ as an order parameter.
Figure 25.4: Normalized chiral condensate $R_{\bar{q}q} = [(\bar{u}u) - (\hat{m}/m_u)(\bar{s}s)]/[(\bar{u}u)_0 - (\hat{m}/m_u)(\bar{s}s)_0]$ as a function of $T$ for two different lattice spacings, $a = 0.24$ fm and $a = 0.17$ fm, calculated by the lattice QCD simulations with dynamical $u$, $d$, $s$ quarks in the staggered fermion formalism. Vertical band in the middle indicates the pseudo-critical temperature $T_{pc}$ [15]. To see that $\langle \bar{q}q \rangle$ vanishes at extreme high $T$, let us consider the QCD partition function at zero baryon density, $Z_{\text{QCD}} = \text{Tr} \left[ e^{-\hat{H}_{\text{QCD}}/T} \right] \equiv e^{P(T)V/T}$, which leads to $\langle \bar{q}q \rangle = -\frac{\partial P(T)}{\partial m_q}$. If we have a situation where $T \gg \Lambda_{\text{QCD}}$, the system is approximated by the Stefan-Boltzmann gas of free quarks and gluons because of the asymptotic freedom. Since the quark-gluon vertex does not change chirality (the transition between $q_L$ and $q_R$ is not allowed in perturbation theory) and thus the expectation value of $\bar{q}q = \bar{q}_L q_R + \bar{q}_R q_L$ vanishes in any finite order of the perturbation as long as $m_q = 0$.

As shown in Fig. 25.4 lattice QCD Monte Carlo simulations at finite $T$ with zero baryon density indeed indicate a sudden drop of the chiral condensate around the pseudo-critical temperature determined from the susceptibility peak [16, 17].

$$T_{pc} \approx (150 - 200) \text{ MeV} = (1.7 - 2.3) \times 10^{12} \text{ K.} \quad (25.2.7)$$

Note that the phase transition from the low $T$ phase (Nambu-Goldstone phase) to the high $T$ phase (quark-gluon plasma) is first order in the flavor-SU(3) chiral limit ($m_{u,d,s} = 0$) and is second order in the flavor-SU(2) chiral limit ($m_{u,d} = 0, m_s = \infty$) [18]. On the other hand, in the presence of the finite quark masses $m_{u,d,s}$ acting as external fields, the transition is crossover as seen in Fig. 25.4. For more details on the QCD thermodynamics, see the review [19].

The high temperature quark-gluon plasma (QGP) is believed to be present in the early universe with its age younger than $10^{-5}$ sec. Attempts to create...
such extremely hot system by the relativistic heavy-ion collisions in the laboratory have been started from 2000 at RHIC (Relativistic Heavy Ion Collider) in Brookhaven national laboratory and will be pursued at LHC (Large Hadron Collider) in CERN. RHIC has already produced a plenty of data showing not only the evidence of QGP but also strongly interacting characters of QGP [20]. (See the recent reviews [21] on this rapidly developing subject.) In the following, we will focus more on QCD matter with finite baryon density at low $T$. Such a system may undergo successive quantum phase transitions which are relevant to the physics of neutron stars and of possible quark stars.

25.3 QCD matter at high baryon density

Soon after the discovery of the asymptotic freedom of QCD, a possible transition from hadronic matter to quark matter in the core of the neutron stars has been pointed out [22]. A strange quark star entirely made of deconfined $u$, $d$, $s$ quarks, yet undiscovered, was also proposed [23] as a modern version of the early idea of the quark star [24]. Although there has been no observational evidence of quark stars yet, nature may be strange enough to accommodate such compact object in our universe and we should prepare for the future discovery.

Shown in Fig. 25.5 is a schematic view of various forms of compact stars, from the neutron star to the quark star. Typical radius of the neutron star is about 10 km, while its mass is comparable to the solar mass $M_\odot (\approx 2 \times 10^{30}$ kg). Since neutrons cannot be bound by the strong interaction only, the presence
of gravitational force is essential to hold the neutron star. This implies that the radius increases as the mass decreases. The masses of the neutron stars in the binary systems are centered around $1.35M_\odot$. Observed upper limit of the surface temperature of neutron stars is less than $10^9$ K after 1 year from their births. In the early stage, the cooling occurs through the neutrino emissions, while in the later stages it is dominated by surface photon emissions.

For pulsars (rotating neutron stars), the measured rotational frequency is ranged from milli second to several seconds. The surface magnetic field is typically $10^{12}$ gauss for ordinary pulsars with rotational period $P \sim 1$ s and $dP/dt \sim 10^{-15}$ s. There are also stars with much larger (smaller) $dP/dt$ and larger (smaller) magnetic field $\sim 10^{15}(10^9)$ gauss. Sudden spin up of the rotation associated with a subsequent relaxation to the normal rotation has been observed and is called the glitch. This phenomena should be related to the internal structure of neutron stars, in particular the superfluidity of the neutron liquid.

The outer crust of the neutron stars is a solid composed of heavy nuclei forming a Coulomb lattice in the sea of degenerate electrons. As the pressure and the density increase toward the inner region, electrons tend to be captured by nuclei and at the same time neutrons drip out from the nuclei, so that the system is composed of neutron-rich heavy nuclei in the Fermi sea of the neutrons and electrons. Eventually, the nuclei dissociate into neutron liquid and the system becomes a degenerate Fermi system composed of superfluid neutrons together with a small fraction of superconducting protons and normal electrons.

When the baryon number density ($\rho$) of the core of the neutron stars exceeds a few times of the central density of heavy atomic nuclei ($\rho_0 = 0.16$ fm$^{-3} = 0.16 \times 10^{39}$ cm$^{-3}$), one may expect exotic components such as the hyperons (baryons with s quarks), Bose-Einstein condensates of pions and kaons, and the deconfined quark matter, which can contribute to the acceleration of the neutron-star cooling. For more details of the physics of high density matter and compact stars, see e.g. [25, 26].

### 25.3.1 Neutron-star matter and hyperonic matter

Although the neutron star is mainly composed of the degenerate neutrons, other species are also present as a result of the chemical equilibrium conditions. Indeed, the matter made of only neutrons is unstable against the $\beta$-decay, $n \rightarrow p + e^- + \bar{\nu}_e$. After the decay, the electron-neutrino leaves the star without much interactions if the neutron star is cold enough. On the other hand, the protons and the electrons remain in the star and form degenerate Fermi liquid together with the neutrons. The equilibrium configuration of $n$, $p$ and $e^-$, which we call the standard neutron-star matter (see Fig. 25.6(a)), is determined by the three conditions: chemical equilibrium, charge neutrality and the baryon-number conservation: $\mu_n = \mu_p + \mu_e$, $\rho_p = \rho_e$, $\rho = \rho_n + \rho_p$, where $\rho_i$ denotes the number density of $i$-species.

If we assume the non-interacting degenerate fermions for simplicity, it is
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Figure 25.6: Compositions of matters under chemical equilibrium and charge neutrality conditions in the Fermi gas model. Shaded areas show the occupied states. (a) The neutron-star matter with n, p, and e\(^{-}\), (b) the hyperon matter with n, Σ\(^{-}\) and Λ, (c) the u-d-s quark matter with finite strange quark mass \(m_s\).

It is easy to find the analytic solution of the above conditions; e.g., the proton fraction in a neutron star for a given baryon number density \(\rho\) reads \(\rho_p/\rho_n \approx \frac{1}{8} \left[1 + \left(\frac{m_n^3}{3\pi^2\rho_n}\right)^{2/3}\right]^{-3/2}\). This is a monotonically increasing function of \(\rho_n\) and approaches the asymptotic limit 1/8 from below. As the neutron density further increases and the electron chemical potential exceeds the muon mass, \(\mu_e > m_\mu = 105\) MeV, the system composed of n, p, e\(^{-}\) and \(\mu^-\) is realized.

As the baryon density increases further, hyperons enter into the game. This is because the Fermi energy of the neutron exceeds the threshold of the neutron-decay into hyperons. See Fig. 25.6(b). Hyperons such as Σ\(^{-}\) and Λ may appear for \(\rho > (2 - 3)\rho_0\). Which hyperon appears first depends on the still uncertain hyperon-nucleon interactions.

### 25.3.2 Quark matter

As the baryon number density \(\rho\) of the system exceeds \((3 - 5)\rho_0\), the neutrons, protons and hyperons start to percolate each other due to their finite sizes. The quark number density \(\rho_q\) for each flavor is related to \(\rho\) as \(\rho_q = (N_c/N_f)\rho\) with \(N_c = 3\) being the number of colors and \(N_f\) the active number of flavors so that the critical quark chemical potential, above which the percolation to quark matter takes place, is estimated as

\[
\mu_c = \left(\pi^2\rho_q\right)^{1/3} \approx (380 - 450)\text{ MeV.}
\]

Here we have assumed a non-interacting and degenerate quark matter composed of massless u and d quarks.

Let us now consider the quark matter composed of only u, d and e\(^{-}\). The condition of chemical equilibrium (d ↔ u + e\(^{-}\)), charge neutrality and the baryon
number conservation read \( \mu_d = \mu_u + \mu_e, \frac{2}{3} \rho_u - \frac{1}{3} \rho_d - \rho_e = 0 \), and \( \frac{1}{3} (\rho_u + \rho_d) = \rho \). The factors \( \frac{2}{3} \) and \( -\frac{1}{3} \) originate from the electric charges of the quarks and \( \frac{1}{3} \) from the baryon number of a quark. If we assume non-interacting quarks at high density \( \mu_q \gg m_q \), one immediately finds \( \mu_u \simeq 0.80 \mu_d \). Thus the Fermi energy of the \( d \) quark is slightly higher than that of the \( u \) quark, which is different from the situation of neutron matter where \( n \) and \( p \) have quite a different Fermi energies due to non-relativistic kinematics as shown in Fig. 25.6(a).

If the quark matter is composed of \( u, d, s \) and \( e^- \), the chemical equilibration is achieved through the processes, \( d \leftrightarrow u + e^- \), \( s \leftrightarrow u + e^- \), and \( d + u \leftrightarrow u + s \). Then the equilibrium conditions read \( \mu_d = \mu_u + \mu_e, \mu_s = \mu_d, -\frac{1}{3} (\rho_d + \rho_s) + \frac{2}{3} \rho_u - \rho_e = 0 \), and \( \frac{1}{3} (\rho_u + \rho_d + \rho_s) = \rho \). For \( \mu_q \gg m_q \), they lead to \( \mu_u = \mu_d = \mu_s \) and \( \mu_e = 0 \). Namely the massless \( u-d-s \) quark matter is charge neutral by itself without electrons. If we have finite \( \rho_s \), then \( \rho_s \) is reduced relative to \( \rho_u, d \) and the electrons become necessary to make the system charge neutral, as shown in Fig. 25.6(c).

### 25.4 Superfluidity in neutron-star matter

If there exists attractive channel between the fermions near the Fermi surface, the system undergoes a transition to superfluidity or superconductivity in three spatial dimensions [28]. This is indeed the case in the neutron-star matter where the attraction between the protons due to spin-independent nuclear force in the \((S, L, J)=(\text{spin, orbital angular momentum, total angular momentum})=(0,0,0)\) channel leads to the condensation of \( ^2S^+1L_J=^1S_0 \) Cooper pairs (the proton superconductivity) and the attraction between the neutrons due to spin-orbit nuclear force in the \((S, L, J)=(1,1,2)\) channel leads to the condensation of \( ^3P_2 \) Cooper pairs (the neutron superfluidity) [29].

Other than the superfluidity and superconductivity of the nucleons, the condensation of pions \((\pi^0 \text{ and } \pi^-)\) and that of kaons \((K^-)\) have been studied extensively. For more details on nucleon superfluidity, meson condensation and its implication to the physics of compact stars, see the reviews [30] [31] [32].

### 25.5 Color superconductivity in quark matter

The quark matter exhibits the color superconductivity (CSC) which originates from the formation of Cooper pairs of quarks near the Fermi surface (see the recent reviews [33] [34] and references therein). Dominant attractive interaction responsible for the quark–quark pairing at high density is the color-magnetic interaction mediated by the gluon.

There are some characteristic differences between CSC and the standard BCS-type superconductivity:

(i) The quark matter at high density is a relativistic system where the quark chemical potential \( \mu \) is comparable or larger than the quark mass \( m_q \). In such a case, the velocities of quarks near the Fermi surface are close to
light velocity and the magnetic interaction is not any more suppressed in comparison to the electric interaction.

(ii) The color-magnetic interaction is screened only dynamically by Landau-damping, while the color-electric interaction is Debye screened as usual. Therefore, the collinear quark–quark scattering on the Fermi surface is dominated by the color-magnetic interaction, which leads to an unconventional form of the fermion gap $\Delta \propto \mu e^{-c/\sqrt{\alpha_s}}$ \[35\]. Because of this non-BCS form where the coupling strength enters as $\sqrt{\alpha_s}$ instead of $\alpha_s$, $\Delta/\mu$ becomes sizable even in the weak coupling.

(iii) Due to color and flavor indices of the quarks, the CSC gap acquires color-flavor matrix structure, which leads to various different phases depending on $\mu$ and $T$; they include the 2SC (2-flavor color superconducting) phase, the CFL (color-flavor locked) phase, the FFLO phase, the crystalline phase, and so on as reviewed in \[34\].

25.5.1 The gap equation

Let us illustrate the role of the above color-magnetic interaction by considering a simplified situation where quark matter is composed of only massless u and d quarks with equal Fermi energies at $T = 0$. In this case, the u quark with red-color and the d quark with green-color in flavor-singlet and color anti-triplet combinations are paired, while all the others quarks are unpaired \[36\]. This is called the 2SC phase.

Using the standard Nambu-Gor’kov field $\Psi = (q, \bar{q})_t$, the Schwinger-Dyson equation for the quark self-energy $\Sigma$ in the ladder approximation is written as

$$\Sigma(k) = -i \int \frac{d^4p}{(2\pi)^4} \frac{g^2}{(2\pi)^4} \Gamma_\mu \mathcal{S}(p,k) \Gamma_\nu D^{ab}_{\mu\nu}(p-k), \quad (25.5.9)$$

where $D^{ab}_{\mu\nu}$ is the in-medium gluon propagator and $\Gamma_\mu$ is the bare quark-gluon vertex. The off-diagonal (anomalous) component of $\Sigma$ in the Nambu-Gor’kov space is directly related to the gap function $\Delta(k) = \Delta_+(k)\Lambda_+ + \Delta_-(k)\Lambda_-$, where $\Lambda_\pm$ are the projection operators to the positive and negative energy states. Therefore, $\Delta_+(k)$ and $\Delta_-(k)$ are interpreted as the quark gap and anti-quark gap, respectively. As for $g(q,k)$, the Higashijima-Miransky ansatz (a momentum-dependent QCD coupling with phenomenological infrared regulator) may be adopted \[37\]. As for the gluon propagator, we take the screened propagator in the Landau gauge,

$$D^{ab}_{\mu\nu}(k) = \left(-\frac{P^{L}_{\mu\nu}}{k^2 + m_D^2} - \frac{P^{T}_{\mu\nu}}{k^2 + i\frac{\pi}{2}m_D^2|k_0|/|k|}\right)\delta^{ab}, \quad (25.5.10)$$

where $P^{L,T}_{\mu\nu}$ are the longitudinal and transverse projection operators. The longitudinal (electric) part of the propagator has static screening by the Debye mass $m_D^2 = (1/\pi^2)g^2\mu^2$, so that the static interaction between quarks in the
Figure 25.7: (a) The quark gap $\Delta_+ (k)$ as a function of $|k|/\mu$ for various quark chemical potentials, $\mu = 2^n \times \Lambda$ with $\Lambda = 400$ MeV and $n = 1, 2, 3, 12$, obtained by solving the gap equation Eq. (25.5.9) numerically [38]. (b) Ratio of the coherence length $\xi_c$ and the average inter-quark distance $d_q$ as a function of $\mu$.

coordinate space is the Yukawa-type short range potential. On the other hand, the transverse (magnetic) part has only dynamical screening due to Landau damping, so that the static interaction between quarks in the coordinate space is the Coulomb-type long range potential.

In the high density limit (or equivalently the weak coupling limit due to asymptotic freedom), only the interactions near the Fermi surface become relevant and the gap equation can be simplified to give the form [35]

$$\Delta_+ (|k| = \mu) \approx 2b \mu e^{-(3\pi^2/\sqrt{2})/g(\mu)}, \quad (25.5.11)$$

with $b = 256\pi^4/g^2(\mu)$ and $g(\mu) = \sqrt{4\pi\alpha_s(\kappa = \mu)}$ [34]. The characteristic form of the gap, $e^{-c/\sigma}$, in Eq. (25.5.11) originates from the long-range color-magnetic interaction in Eq. (25.5.10) and is different from the BCS form $e^{-c/\sigma^2}$ as we have mentioned.

At low densities, sizable diffusion of the Fermi surface occurs, and the weak-coupling approximation in the asymptotic high density leading to Eq. (25.5.11) is not justified. Therefore, we solve the gap equation Eq. (25.5.9) numerically with both magnetic and electric interactions [38]. In Fig. 25.7(a), we show such numerical solution of $\Delta_+ (k)$ as a function of $|k|/\mu$ for wide range of density. The figure shows that (i) the ratio of the gap and the quark chemical potential $\Delta_+ / \mu$ can be sizable in magnitude of order 0.03 even for baryon density as high as $\rho \sim 10\rho_0$, and that (ii) the gap function $\Delta_+ (k)$ has non-trivial momentum dependence with a peak near the Fermi surface.
25.5.2 Tightly bound Cooper pairs

The color superconductivity is a strongly coupled system partly due to the large value of coupling constant $\alpha_s(\kappa = \mu)$ and partly due to long-range nature of the color-magnetic interaction. To clarify this point further, let us consider the coherence length $\xi_c$ defined as a root-mean-square radius of the Cooper-pair wave function, $\varphi_+(r) \propto \langle q(r)q(0) \rangle \propto \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{r}} \frac{\Delta_+(k)}{2\sqrt{(|k| - \mu)^2 + \Delta_+^2}}$.

Shown in Fig. 25.7(b) is the ratio of $\xi_c$ and the average inter-quark distance $d_q = (\pi^2/2)^{1/3} \mu^{-1}$ in the u-d quark matter. At high density, this ratio is very large (about $10^5$ at $\mu \sim 10^6$ MeV), while at low densities, the ratio becomes small (about $10$ at $\mu = 800$ MeV) and may even become less than 1 at lower densities. This situation is quite similar to the BCS-BEC crossover phenomenon [39] which was recently observed in ultracold atomic systems [40]: The result here suggests that the quark matter possibly realized in the core of neutron stars ($\mu \sim 400$ MeV) may be rather like the BEC of tightly bound Cooper pairs. For further studies of the BCS-BEC crossover in the relativistic system, see e.g. [41].

25.6 QCD phase structure

The continuous QCD symmetry in the flavor-SU(3) chiral limit ($m_u, m_d, m_s = 0$), Eq. (25.1.4), exhibits various symmetry breaking patterns depending on the temperature $T$ and the quark chemical potential $\mu$. In Table 25.1, three examples of the symmetry realizations are shown: the QGP (quark-gluon plasma) phase, the NG (Nambu-Goldstone) phase and the CFL (color-flavor locked) phase. (CFL is one of the color superconducting phases to be discussed later in more details.) Each phase would appear in the $T - \mu$ phase diagram as illustrated in Fig. 25.8. In the intermediate values of $\mu \sim 400$ MeV at low $T$, a variety of quantum phases have been proposed [30, 34].

As we have discussed in Sec. 25.1.2, the NG phase is characterized by the dynamical breaking of chiral symmetry due to nonvanishing chiral condensate $\langle \bar{q}q \rangle$. On the other hand, in the color superconductivity at high chemical potential, the diquark condensate $\langle q\bar{q} \rangle$ is formed as discussed in Sec. 25.5. In the presence of these condensates, the light quarks and anti-quarks acquire the Dirac type mass $M$ and the Majorana type mass $\Delta_{\pm}$, which leads to the relativistic quasi-particle spectrum of a quark near the Fermi surface,

$$\omega(p) = \sqrt{(\sqrt{p^2 + M^2} - \mu)^2 + |\Delta_+|^2}. \quad (25.6.12)$$

25.6.1 Ginzburg-Landau potential for hot/dense QCD

In this subsection, we focus our attention on the QCD phase structure at intermediate value of the chemical potential ($\mu \sim 400$ MeV) where there would be an interplay between the quark–anti-quark pairing characterized by the chiral condensate $\langle \bar{q}q \rangle$ and the quark–quark pairing characterized by the diquark condensate $\langle q\bar{q} \rangle$ [42, 43]. Study of this region is not only important to understand
Table 25.1: Symmetry breaking patterns of QCD in the flavor-SU(3) chiral limit ($m_u, d, s = 0$). The QGP phase, the NG phase, and the CFL phase denote the quark-gluon plasma phase, the Nambu-Goldstone phase, and the color-flavor locked phase, respectively.

| phase       | region                        | unbroken continuous symmetries                               |
|-------------|-------------------------------|--------------------------------------------------------------|
| QGP phase   | $\frac{T}{\Lambda_{QCD}} \gg 1$ | $\text{SU}(3)_C \otimes \text{SU}(3)_L \otimes \text{SU}(3)_R \otimes U(1)_B$ |
| NG phase    | $\frac{T}{\Lambda_{QCD}} \ll 1, \frac{\mu}{\Lambda_{QCD}} \ll 1$ | $\text{SU}(3)_C \otimes \text{SU}(3)_{L+R} \otimes U(1)_B$ |
| CFL phase   | $\frac{T}{\Lambda_{QCD}} \ll 1, \frac{\mu}{\Lambda_{QCD}} \gg 1$ | $\text{SU}(3)_{C+L+R}$ |

Figure 25.8: The three basic phases of QCD in the $T$-$\mu$ plane for flavor-SU(3) chiral limit.

the quantum phase transition to quark matter in the deep interior of the neutron stars, but also interesting in relation to similar phenomena in other systems such as the interplay between magnetically ordered phases and metallic superconductivity [44] and that between superfluidity and magnetism in ultracold atoms [45].

To analyze such interplay in QCD in a model-independent manner, let us construct a Ginzburg-Landau (GL) potential $\Omega$ on the basis of the QCD symmetry Eq. (25.1.4) as $\Omega(\Phi, d_L, d_R) = \Omega_\chi(\Phi) + \Omega_d(d_L, d_R) + \Omega_{\chi d}(\Phi, d_L, d_R)$. Here the chiral field $\Phi$, which has a 3×3 matrix structure in the flavor space, is defined with the transformation property under Eq. (25.1.3) as,

$$\Phi_{ij} = \langle \bar{q}_R^{[j}[q_L^{i]} \rangle, \quad \Phi \rightarrow e^{-2i\theta_A} V_L \Phi V_R^\dagger. \quad (25.6.13)$$

On the other hand, the diquark field $d_L$, which has a 3×3 matrix structure in
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the flavor-color space, is defined with the transformation property as

$$[d_{L}^{i}]_{ai} = \epsilon_{ijk} \epsilon_{abc} ([g_{L}]_{b}^{j} C g_{L}^{k}_{c}], \quad d_{L} \rightarrow e^{2i\theta_{a} e^{2i\theta_{b}} V_{L} d_{L} V_{L}^{\dagger}}, \quad (25.6.14)$$

where $C = i\gamma^{5}\gamma^{0}$ is the charge conjugation matrix. Similar definition holds for $d_{R}$ too. By definition, the $3 \times 3$ matrix $[d_{L(R)}]_{ai}$ belongs to the fundamental representation of $SU(3)_{C}$ and $SU(3)_{L(R)}$.

Most general form of the GL potential which is invariant under $G$ in Eq. $(25.6.14)$, written in terms of the chiral field up to $O(\Phi^{4})$, reads $[18]$

$$\Omega_{\chi} = \frac{a_{0}}{2} \text{Tr}\Phi^{4} + \frac{b_{1}}{4!} \left( \text{Tr}\Phi^{4} \right)^{2} + \frac{b_{2}}{4!} \text{Tr}\Phi^{4} - \frac{c_{0}}{2} \left( \det\Phi + \det\Phi^{\dagger} \right), \quad (25.6.15)$$

where “Tr” and “det” are taken over the flavor indices, $i$ and $j$. The first three terms in the right hand side are invariant under $G \otimes U(1)_{A}$, while the last term represents the axial anomaly which breaks $U(1)_{A}$ down to $Z(6)_{\chi}$. The potential $\Omega_{\chi}$ is bounded from below for $b_{1} + b_{2}/3 > 0$ and $b_{2} > 0$. If these conditions are not satisfied, we need to introduce terms in $O(\Phi^{6})$ to stabilize the potential, a situation we will indeed encounter. We assume $c_{0}$ to be positive so that the chiral condensate at low temperature is positive. Also, we assume that $a_{0}$ changes its sign at a certain temperature to drive the chiral phase transition.

Most general form of the GL potential which is invariant under $G$, written in terms of the diquark field up to $O(d^{4})$, reads $[46, 47]$

$$\Omega_{d} = a_{0} \text{Tr}[d_{L}^{d_{L}} d_{R}^{d_{R}}] \quad + \beta_{1} \left( \text{Tr}[d_{L}^{d_{L}}]^{2} + \text{Tr}[d_{R}^{d_{R}}]^{2} \right) + \beta_{2} \left( \text{Tr}[d_{L}^{d_{L}}]^{2} + \text{Tr}[d_{R}^{d_{R}}]^{2} \right) \quad + \beta_{3} \text{Tr}[d_{L}^{d_{L}} d_{L}^{d_{L}}] + \beta_{4} \text{Tr}[d_{L}^{d_{L}} d_{R}^{d_{R}}] \quad (25.6.16)$$

The transition from the normal state to color superconductivity is driven by $a_{0}$ changing sign. Unlike det $\Phi$ in $\Omega_{\chi}$, terms such as det $d_{L(R)}$ are not allowed in $\Omega_{d}$, since det $d_{L(R)}$ carries baryon number and is not invariant under $U(1)_{B}$.

Finally, the interaction potential which is invariant under $G$, written in terms of both chiral and diquark fields to fourth order, reads $[43, 48]$

$$\Omega_{\chi d} = \gamma_{1} \text{Tr}[d_{L}^{d_{L}}] + \gamma_{2} \text{Tr}[d_{R}^{d_{R}}] \quad + \lambda_{1} \text{Tr}[d_{L}^{d_{L}} d_{L}^{d_{L}}] + \lambda_{2} \text{Tr}[d_{L}^{d_{L}} + d_{R}^{d_{R}}] + \lambda_{3} \left( \det\Phi \cdot \text{Tr}[d_{L}^{d_{L}}]^{2} \Phi^{-1} \right) + h.c. \quad (25.6.17)$$

The term with the coefficient $\gamma_{1}$ originates from the axial anomaly which imposes the sign of $\gamma_{1}$ in Eq. $(25.6.17)$ and that of $c_{0}$ in Eq. $(25.6.15)$ being the same.

Equations $(25.6.15)$, $(25.6.16)$ and $(25.6.17)$ constitute the most general form of the GL potential under the conditions that the phase transition is not strongly first order (i.e., the magnitudes of $\Phi, d_{L(R)}$ are sufficiently smaller than those at zero temperature) and that the condensed phases are spatially homogeneous. To proceed analytically for the flavor-$SU(3)$ chiral limit, we restrict ourselves to maximally symmetric condensates of the form:

$$\Phi = \text{diag}(\sigma, \sigma, \sigma), \quad d_{L} = -d_{R} = \text{diag}(d, d, d), \quad (25.6.18)$$
where \(\sigma\) and \(d\) are assumed to be real and spatially uniform. We have chosen the relative sign between \(d_L\) and \(d_R\) in Eq. (25.6.18) so that the ground state has positive parity, as is indeed favored by the axial anomaly together with finite quark masses. The above ansatz for the diquark condensate has residual symmetry SU(3)\(_{C+L+R}\) \(\otimes\) Z(2) and is called the color-flavor locking (CFL) because of its symmetry realization [49]. (Note that Z(2) corresponds to the reflection, \(q_{L(R)} \rightarrow -q_{L(R)}\).)

The reduced GL potential with Eq. (25.6.18) is

\[
\Omega_{3F} = \left(\frac{a}{2}\sigma^2 - \frac{c}{3}\sigma^3 + \frac{b}{4}\sigma^4 + \frac{f}{6}\sigma^6\right) + \left(\frac{\alpha}{2}d^2 + \frac{\beta}{4}d^4\right) - \gamma d^2\sigma + \lambda d^2\sigma^2. \tag{25.6.19}
\]

Here the axial anomaly leads to \(c > 0\) and \(\gamma > 0\), while microscopic calculation based on the Nambu–Jona-Lasinio model as well as the weak-coupling QCD suggests that \(\lambda\) is positive and plays a minor role in comparison to \(\gamma\) [48]. Note that we have introduced \(f\)-term (\(f > 0\)) in case that \(b\) becomes negative. This system can have four phases with the following dynamical breaking patterns of continuous symmetries:

- **QGP phase**: \(\sigma = 0, d = 0\)
- **NG phase**: \(\sigma \neq 0, d = 0\) : \(G \rightarrow SU(3)_C \otimes SU(3)_{L+R} \otimes U(1)_B\)
- **CFL phase**: \(\sigma = 0, d \neq 0\) : \(G \rightarrow SU(3)_{C+L+R}\)
- **COE phase**: \(\sigma \neq 0, d \neq 0\) : \(G \rightarrow SU(3)_{C+L+R}\). \tag{25.6.20}

The COE (coexistence) phase is favored by the axial anomaly, since the simultaneous presence of \(d\) and positive \(\sigma\) makes the GL potential lower because of the \(\gamma\)-term with \(\gamma > 0\). Note that even the unbroken discrete symmetry Z(2) is common between CFL and COE phases, so that they cannot be distinguishable from the symmetry point of view.

For flavor-SU(2) chiral limit with \(m_u, d = 0\) and \(m_s = \infty\), the condensates with the s quark disappear, so that we have \(\Phi = \text{diag}(\sigma, \sigma, 0)\) and \(d_L = -d_R = \text{diag}(0, 0, d)\). Then the reduced GL potential becomes \(\Omega_{2F} = \left(\frac{a}{2}\sigma^2 + \frac{4}{3}\sigma^3 + \frac{b}{4}\sigma^4 + \frac{f}{6}\sigma^6\right) + \left(\frac{\alpha}{2}d^2 + \frac{\beta}{4}d^4\right) + \gamma d^2\sigma^2 + \lambda d^2\sigma^2\). Thus, the coexistence of \(d\) and \(\sigma\) is disfavored in this case, because of the \(\Lambda\)-term with \(\lambda > 0\).

### 25.6.2 Possible phase structure for realistic quark masses

The mapping of the phase diagrams obtained from the GL potentials, \(\Omega_{3F}\) and \(\Omega_{2F}\), in the \(a-\alpha\) plane to the \(T-\mu\) plane is a dynamical question which cannot be addressed within the phenomenological GL theory. Nevertheless, we can draw a speculative phase structure of QCD for \(m_s \sim \Lambda_{QCD} \gg m_u, d \neq 0\) by interpolating the phase structures obtained from \(\Omega_{3F}\) and \(\Omega_{2F}\) as shown in Fig. 25.9 [43]. In this figure, the double line indicates the first order phase transition driven by the negative \(b\) in Eq. (25.6.19). The single lines indicate the second order phase transitions (within the analysis of the GL potential without fluctuations) which
Figure 25.9: Schematic phase structure with two light (up and down) quarks and a medium heavy (strange) quark \[43\]. The double line indicates the first order transition. AY and HTYB are the second-order critical points at which the first-order line terminates.

separate the \(d \neq 0\) and \(d = 0\) phases. We draw two critical points at which the first order phase transition turns into crossover; the one near the vertical axis indicated as “AY” (Asakawa-Yazaki critical point \[50\]) and the other one near the horizontal axis indicated as “HTYB” \[13\]. The latter is driven by the axial anomaly with positive \(\gamma\) in Eq. (25.6.19).

The existence of the AY critical point implies that the transition from the NG phase to the QGP phase on the \(\mu = 0\) axis is a crossover. Indeed, the lattice QCD Monte Carlo simulations at finite \(T\) with the finite-size scaling analyses indicate that the thermal phase transition at \(\mu = 0\) is likely to be crossover \[51\]. We note here that the AY critical point has special importance to the fluctuation observables in relativistic heavy-ion collisions \[52\] and the determination of its location is highly called for both theoretically and experimentally.

On the other hand, the existence of the HTBY critical point implies that the hadronic matter (characterized by \(\sigma > d > 0\) and the quark matter characterized by \(d > \sigma > 0\) are continuously connected with each other and both are classified into the COE phase. This is intimately related to the idea of hadron-quark continuity, i.e. smooth transition from the superfluid/superconducting hadronic matter to the superconducting quark matter \[53, 54\]. Indeed, there are evidences of the continuity not only for the ground state but also for the excitation spectra: Typical example is the continuity of the flavor-octet vector mesons in hadronic matter at low \(\mu\) and the color-octet gluons in quark matter at high \(\mu\) \[53, 55\]. Unfortunately, the lattice QCD simulations have difficulty to treat the matter with \(\mu/T \gg 1\) because of the severe sign problem originating from the complex fermion determinant in the presence of \(\mu\) \[56\]. Therefore, the quantitative study in this region is still an open issue.
25.7 Simulating dense QCD with ultracold atoms

Ultracold atomic systems and high density QCD matter, although differing by some twenty orders of magnitude in energy scales, share certain analogous physical aspects, e.g., BEC-BCS crossovers \[57\]. Motivated by phenomenological studies of QCD that indicate a strong spin-singlet diquark correlation inside the nucleon \[58\], we focus here on modeling the transition from the 2-flavor quark matter at high density to the nuclear matter at low density in terms of a boson-fermion system, in which small size diquarks are the bosons, unpaired quarks the fermions, and the extended nucleons are regarded as composite boson-fermion particles \[59\]. This would be a starting point to understand the quantum phase transition at \(\mu \sim \mu_c\) between the hadronic superfluid discussed in Sec. 25.4 and the color superconductivity discussed in Sec. 25.5.

Recent advances in atomic physics have made it possible indeed to realize boson-fermion mixture in the laboratory. In particular, tuning the atomic interaction via a Feshbach resonance allows formation of heteronuclear molecules, as recently observed in a mixture of \(^{87}\text{Rb}\) and \(^{40}\text{K}\) atomic vapors in a 3D optical lattice \[60\], and in an optical dipole trap \[61\].

Let us start from a non-relativistic boson-fermion mixture with Hamiltonian density,

\[
\mathcal{H} = \frac{1}{2m_b} |\nabla \phi(x)|^2 - \mu_b |\phi(x)|^2 + \frac{1}{2} \bar{g}_{bb} |\phi(x)|^4
+ \sum_\sigma \left( \frac{1}{2m_f} |\nabla \psi_\sigma(x)|^2 - \mu_\sigma |\psi_\sigma(x)|^2 \right) + \bar{g}_{ff} |\psi_\uparrow(x)|^2 |\psi_\downarrow(x)|^2
+ \sum_\sigma \bar{g}_{bf} |\phi(x)|^2 |\psi_\sigma(x)|^2, \tag{25.7.21}\]

where \(\phi\) is the boson and \(\psi\) the fermion field. The two internal states of the fermions are labeled by spin indices \(\sigma = \{\uparrow, \downarrow\}\). For simplicity, we consider an equally populated mixture of \(n\) bosons and \(n\) fermions with \(n_\uparrow = n_\downarrow = n/2\).

The bare boson-fermion coupling \(\bar{g}_{bf}\) is related to the renormalized coupling \(g_{bf}\) and to the s-wave scattering length \(a_{bf}\) by

\[
\frac{m_b}{2\pi a_{bf}} = \frac{1}{\bar{g}_{bf}} + \int_{|k| \leq \Lambda} \frac{d^3k}{(2\pi)^3} \frac{1}{\epsilon_b(k) + \epsilon_f(k)}, \tag{25.7.22}\]

where \(\epsilon_i(k) = k^2/2m_i\) \((i = b, f)\) is the single-particle kinetic energy, \(m_b\) is the boson-fermion reduced mass, and \(\Lambda = \pi/(2r_0)\) is a high momentum cutoff with \(r_0\) being a typical atomic scale. We assume an attractive bare b-f interaction \((\bar{g}_{bf} < 0)\), tunable in magnitude, with \(\Lambda\) fixed, so that the scattering length \(a_{bf}\) can change sign: \(a_{bf} \rightarrow \bar{g}_{bf} m_b / (2\pi)\) for small negative \(\bar{g}_{bf}\), while \(a_{bf} \rightarrow r_0\) for large negative \(\bar{g}_{bf}\). We keep the bare boson-boson and fermion-fermion interactions fixed and repulsive \((\bar{g}_{bb} > 0, \bar{g}_{ff} > 0)\) for the stability of this system.

In the regime of the weak bare b-f coupling where the dimensionless parameter \(\eta \equiv -1/(n^{1/3}a_{bf})\) is large and positive, the system at low temperature is
Table 25.2: Correspondence between the boson-fermion mixture in ultracold atoms and the diquark-quark mixture in high density QCD.

| cold atoms | dense QCD |
|------------|-----------|
| b (bosonic atom) | d (diquark), |
| f↑, f↓ (fermionic atom) | q↑, q↓ (unpaired quark) |
| N↑, N↓ (boson-fermion molecule) | Ν↑, Ν↓ (nucleon) |
| b-f attraction | gluonic attraction |
| b-BEC | 2-flavor color superconductivity |
| N-BCS | nucleon superfluidity |

a weakly interacting mixture of BEC of the b-bosons (b-BEC) and degenerate f-fermions. The induced interaction through the density fluctuation of b-BEC may also lead to the pairing of f’s (f-BCS).

On the other hand, in the regime of strong bare b-f coupling where the η is large and negative, bound molecules of b-bosons and f-fermions called composite fermions, N = (bf), are formed with a kinetic mass $m_N = m_b + m_f$. The s-wave scattering length of two N’s of opposite spins can be estimated by the exchange of constituent b or f [59],

$$a_{NN} \simeq -\frac{m_{N}}{2m_{bf}}a_{bf}.$$  (25.7.23)

This is the same in magnitude but is opposite in sign from the scattering length between difermion molecules due to different statistics. It can be shown that this result is the leading order term in an extension of the present model to large internal degrees of freedom.

Eq. (25.7.23) implies that the low energy effective interaction between composite fermions in the spin-singlet channel is weakly attractive; the stronger the bare b-f attraction the weaker the N-N attraction. Such an effective interaction causes composite fermions to become BCS-paired (N-BCS) below a transition temperature,

$$T_c(N-BCS) = \frac{e^7}{\pi} \left( \frac{2}{e} \right)^{7/3} \varepsilon_N e^{\pi/(2k_Fa_{NN})}.$$  (25.7.24)

where $\varepsilon_N = k_F^2/2m_N$ is the Fermi energy of the N.

The above analyses for large $|\eta|$ suggest a possible phase structures of boson-fermion mixtures in the $T - \eta$ plane as shown in Fig. 25.10(a). At intermediate bare b-f coupling ($\eta \sim 0$) where a transition from the b-BEC phase to N-BCS takes place, the phase diagram would have complex structure depending on the relative magnitudes of $g_{bb}$, $g_{ff}$, and $g_{bf}$. The f-BCS phase possibly occurs for $\eta > 0$ is not shown in this figure. For more detailed analyses of the phase diagram of the present model, see [59].
Figure 25.10: (a) A possible phase structure of the boson-fermion mixture (such as $^{87}$Rb and $^{40}$K) in ultracold atoms with attractive b-f interaction and repulsive b-b and f-f interactions. Large and positive (large and negative) $\eta$ corresponds to the weak (strong) b-f attraction. (b) A possible phase structure of QCD. Large (small) chemical potential $\mu$ corresponds to the weak (strong) coupling due to asymptotic freedom.

The phase boundary in the region $\eta \sim 0$ may be classified by the realization of internal symmetry. If we focus only on the continuous symmetries, the Hamiltonian density, Eq. \[25.7.21\], has $U(1)_b \otimes U(1)_f^{\uparrow} \otimes U(1)_f^{\downarrow}$ symmetry corresponding to independent phase rotations of $\phi$, $\psi^{\uparrow}$ and $\psi^{\downarrow}$. On the other hand, b-BEC and N-BCS break $U(1)_b$ and $U(1)_b^{\uparrow}\otimes (f^{\uparrow}+f^{\downarrow})$, respectively. The difference in such symmetry breaking patterns implies the existence of a well-defined phase boundary between b-BEC and N-BCS as indicated in Fig. 25.10(a). This is in contrast to the continuous BEC-BCS crossover in two-component Fermi systems.

The phase structure we find for boson-fermion mixture of ultracold atoms displays features of that in QCD with equal numbers of u and d quarks. The ground state of such system at high density is the 2-flavor color superconductivity (2SC) discussed in Sec. 25.5.1. The order parameter for color-symmetry breaking is the diquark condensate $\langle d_3 \rangle$ with the diquark operator $d_v = \epsilon_{ij} \epsilon_{abc} [q_i^{a} C \gamma_5 q_j^{b}]$. The gap is of order a few tens of MeV; remaining quarks are unpaired and form degenerate Fermi seas. On the other hand, the ground state of QCD at low density is the nuclear matter with equal numbers of protons and neutrons denoted by $N_{\uparrow,\downarrow}^z$, a superfluid state with a pairing gap of a few MeV \[32\]; the order parameter for the spontaneous breaking of baryon-number symmetry $U(1)_b$ is the six-quark condensate $\langle N_{\uparrow}^0 N_{\downarrow}^0 \rangle = \langle d_{\alpha}^{a} [q_{\alpha}^{a} C \gamma_5 q_{\beta}^{b}] \rangle$. If we model the nucleon, of radius $r_N \sim 0.86$ fm, as a bound molecule of a diquark (of radius $r_d \sim 0.5$ fm) and an unpaired quark, we can make the correspondence between boson-fermion mixture of cold atoms and the diquark-quark mixture in QCD as shown in Table 25.2.
Such correspondence can be also found between the phase diagram of ultracold atoms in Fig. 25.10(a) and that of dense QCD in Fig. 25.10(b). In particular, the BCS-like superfluidity of composite fermion (N) with a small gap is a natural consequence of the strong b-f attraction as shown in Eq. (25.7.23), which may explain why the fermion gap in nucleon superfluidity is order of magnitude smaller than the gap in BEC-like color superconductivity. It is thus quite interesting to carry out the experiments of boson-fermion mixture in ultracold atoms for wide range of the boson-fermion attraction.

Note, however, that tuning the coupling strength at fixed density is not possible in dense QCD matter because of the running coupling $\alpha_s(\kappa = \mu)$; furthermore, dynamical breaking of chiral symmetry and its interplay with the color superconductivity have an important role in the quantum phase transition in QCD as discussed in Sec. 25.6.1. With these reservations in mind, we suggest that fuller understanding, both theoretical and experimental, of the boson-fermion mixture [59] as well as a mixture of three species of atomic fermions [45, 62] can reveal properties of high density QCD not readily observable in laboratory experiments.

25.8 Conclusion

In this Chapter, we have discussed thermal and quantum phase transitions in QCD. The former is relevant to the physics of hot matter in early universe right after the big bang, while the latter is relevant to the physics of dense matter in the interiors of neutron stars and quark stars. There are three fundamental QCD phases in the $T - \mu$ plane: the NG phase, the QGP phase and the CSC phase (Fig. 25.8).

We have shown an interesting possibility of hadron-quark continuity in which the superfluid hadronic phase and the CSC phase are continuously connected with each other at low temperature due to the QCD axial anomaly (Fig. 25.9). We have also discussed that the existence of the nucleon superfluidity may be a logical consequence of the tightly bound diquarks interacting with the unpaired quarks in the CSC phase. Although such a system with $\mu/T \gg 1$ is difficult to be treated in lattice QCD Monte Carlo simulations at present, a mixture of ultracold atoms with different masses, different statistics and different internal degrees of freedom would provide us with an exciting new tool to study the essential features of quantum phase transitions in dense QCD (Fig. 25.10).

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