The synchrotron radiation of first excited state electrons

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Abstract. We give the precise analysis of first excited state electron synchrotron radiation (SR) power using quantum theory methods.

1. Introduction

The precise analytical expressions for the basic synchrotron radiation (SR) characteristics such as spectral-angular distribution, spectral and angular distribution, total output, radiation polarization are entirely formulated in terms of classical theory. These expressions could be found in monographic and educational literature [1 - 5].

As for the quantum theory the only expressions which could be represented analytically at any quantum factor value are the expressions for the spectral-angular and spectral distributions and for the total SR polarization components output in ultrarelativistic limit [1-5] when we can successfully use the quasi-classical approximation method modification. However even in ultrarelativistic limit the precise analytical expression for the angular distribution at the arbitrary quantum factor value is unknown. Still we can obtain the precise analytical expressions for SR characteristics in terms of quantum theory e. g. in case of initial state charge being in low energy level. This kind of analysis in non-relativistic limit was taken in [6, 7]. The results of the precise analytical and numerical analysis of the first excited state electron SR characteristics are represented below.

2. Basic theoretical expressions

We study the synchrotron radiation (SR) angular distribution structure of an electron in the first excited state (charge \(-e, e > 0\)).

In this case the only possible quantum transition is the transition from the first excited state to the ground state. So, the electron radiates one harmonic with the frequency

\[
\omega = \frac{mc^2}{h} \frac{\gamma^2 - 1}{\gamma + \sqrt{\gamma^2 \cos^2 \theta + \sin^2 \theta}} = \frac{E}{h} \frac{\beta^2}{1 + \sqrt{1 - \beta^2 \sin^2 \theta}}, \quad 0 \leq \theta \leq \pi.
\]

(1)

Here \(m\) - is the rest mass of an electron, \(c\) is the velocity of light, \(h\) is the Planck constant, \(\gamma = E/mc^2 = (1 - \beta^2)^{-1/2}\) is the relativistic factor, \(E\) is the electron energy, \(\beta = v/c, v\) is the
particle velocity, \( \theta \) is the angle between the photon’s impulse direction and the magnetic field. For the first excited state we have the relation

\[
\gamma = \sqrt{1 + 2 \frac{H}{H_0}}, \quad H_0 = \frac{m^2 c^3}{e \hbar} - \text{the Schwinger field,}
\]

where \( H > 0 \) is the magnetic field strength.

Let the quantum number \( \zeta = \pm 1 \) specify the initial state electron spin orientation with respect to the magnetic field direction, \( \zeta = 1 \) means that electron spin and the magnetic field do have the same direction, \( \zeta = -1 \) signifies that they have opposite directions. The final (ground) state of the electron is distinguished by that the electron being in it has the spin opposite to the field. So, the jump from the initial state \( \zeta = 1 \) to the ground state occurs with a spin reorientation (spin - flip) only, the transition from the initial state \( \zeta = -1 \) to the ground state happens without the spin-flip.

In the case under consideration the SR power \( W \) and its polarization components angular distribution can be represented as follows

\[
\frac{dW_i}{d\Omega} = Q M G_i, \quad Q = \frac{e^2 m^2 c^3}{\hbar^2}, \quad d\Omega = \sin \theta d\theta,
\]

\[
M = \frac{\beta^2 (1 + \sqrt{1 - \beta^2})}{64 (1 - \beta^2)} = \frac{2}{(1 - x_0^2)^2} \left( \frac{x_0}{1 + x_0} \right)^3 = \frac{(\gamma + 1)(\gamma^2 - 1)^3}{64 \gamma^5},
\]

\[
x_0 = \frac{1 - \sqrt{1 - \beta^2}}{1 + \sqrt{1 - \beta^2}} = \frac{\gamma - 1}{\gamma + 1}, \quad 0 \leq x_0 < 1.
\]

where the index \( i \) denotes the radiation polarization: \( i = 2 \) corresponds to the radiation power of the "\( \sigma \)" - linear polarization component, \( i = 3 \) is for the "\( \pi \)" - linear polarization component, \( i = 0 \) is for the total (summed over polarizations) output, \( i = l \) \( (l = \pm 1) \) is for the right \( (l = 1) \) and left \( (l = -1) \) circular polarizations. The angular distribution structure is completely defined by the functions \( G_i = G_i(\zeta, \beta, \theta) \), given below

\[
G_2(\zeta, \beta, \theta) = (1 - \zeta) R_1(\beta, \theta) + (1 + \zeta) x_0 R_2(\beta, \theta),
\]

\[
G_3(\zeta, \beta, \theta) = (\gamma - \zeta)(\gamma + \zeta)^{-1} G_2(-\zeta, \beta, \theta),
\]

\[
G_0(\zeta, \beta, \theta) = 2[1 + x_0 - \zeta (1 - x_0)] R_0(\beta, \theta),
\]

\[
2 R_0(\beta, \theta) = R_1(\beta, \theta) + R_2(\beta, \theta),
\]

\[
G_l(\zeta, \beta, \theta) = [1 + x_0 - \zeta (1 - x_0)] S_l(\beta, \theta),
\]

\[
S_l(\beta, \theta) = R_0(\beta, \theta) + l S(\beta, \theta) \cos \theta.
\]

Here we introduce the notations

\[
R_1(\beta, \theta) = \frac{(1 - x_0)(1 + x)^3}{1 - x} e^{-x}, \quad R_2(\beta, \theta) = \frac{(x_0 - x)(1 + x)^3}{x_0(1 - x)} e^{-x},
\]

\[
R_0(\beta, \theta) = \frac{2 x_0 - (1 + x_0^2)(1 + x)^3}{2 x_0 (1 - x)} e^{-x}, \quad S(\beta, \theta) = \frac{(1 + x)^4}{1 - x} e^{-x},
\]

\[
x = \frac{1 - \sqrt{1 - \beta^2} \sin^2 \theta}{1 + \sqrt{1 - \beta^2} \sin^2 \theta} = \frac{\gamma - \sqrt{\gamma^2 \cos^2 \theta + \sin^2 \theta}}{\gamma + \sqrt{\gamma^2 \cos^2 \theta + \sin^2 \theta}} \quad 0 \leq x \leq x_0 < 1.
\]
Thus we can see that in our case the synchrotron radiation angular distribution structure is completely defined by the functions \( R_s(\beta, \theta) \), \( s = 0, 1, 2 \); \( S_l(\beta, \theta) \). The following properties are easy to check

\[
G_k(\zeta, \beta, \theta) = G_k(\zeta, \beta, \pi - \theta), \quad k = 0, 2, 3; \quad G_l(\zeta, \beta, \theta) = G_{-l}(\zeta, \beta, \pi - \theta),
\]

\[
R_s(\beta, \theta) = R_s(\beta, \pi - \theta), \quad s = 0, 1, 2; \quad S_l(\beta, \theta) = S_{-l}(\beta, \pi - \theta).
\] (6)

It follows from (6) that it is quite enough to consider the \( R_s(\beta, \theta) \) function behavior on the segment \( 0 \leq \theta \leq \pi/2 \). Besides it is sufficient to investigate one of the functions \( S_l(\beta, \theta) \) (say, \( S_1(\beta, \theta) \)).

3. SR polarization components angular distribution structure of the first excited state electron

At first we mention that the functions \( R_s(\beta, \theta) \), \( S_l(\beta, \theta) \) are finite at any argument value \( 0 \leq \beta \leq 1 \) (including the point \( \beta = 1 \)). In particular, at the boundary points of the segment \( 0 \leq \beta \leq 1 \) we have

\[
R_1(0, \theta) = 1, \quad R_2(0, \theta) = \cos^2 \theta, \quad R_0(0, \theta) = \frac{1}{2}(1 + \cos^2 \theta), \quad S_l(0, \theta) = \frac{1}{2}(1 + l \cos \theta)^2.
\] (7)

\[
R_s(1, \theta) = (1 + p)^3 e^{-p}, \quad S_l(1, \theta) = \left(1 + l \frac{\cos \theta}{|\cos \theta|}\right) (1 + p)^3 e^{-p}; \quad p = \frac{1 - |\cos \theta|}{1 + |\cos \theta|}.
\] (8)

It is not difficult to obtain the relations

\[
R_s(\beta, 0) = R_s(\beta, \pi) = 1, \quad S_l(\beta, 0) = 1 + l, \quad S_l(\beta, \pi) = 1 - l;
\]

\[
R_1(\beta, \pi/2) = 2R_0(\beta, \pi/2) = 2S_l(\beta, \pi/2) = (1 + x_0)^2 e^{-x_0}, \quad R_2(\beta, \pi/2) = 0.
\] (9)

It follows from (4) and (8) that in the ultrarelativistic limit \( (\gamma \to \infty) \) the functions \( G_l = G_{l1}(\zeta, \beta, \theta) \) do not longer depend on the spin orientation \( \zeta \). It means that in ultrarelativistic limit the spin-flip quantum jumps and the jumps without spin reorientation become equiprobable.

The second expression in (4) implies that the angular distribution structure of the ”\( \pi \)”-linear polarization component for an electron with spin \( \zeta \) and the ”\( \sigma \)”-linear polarization component for an electron with spin \(-\zeta\) are the same (the corresponding expressions differs only by the factors which do not depend on \( \theta \)).

The ”\( \pi \)”-linear polarization component does not vanish at \( \theta = \pi/2 \) for an electron with spin \( \zeta = 1 \) and this is an essentially quantum effect.

As it ensues from (8) in the ultrarelativistic case \( (\gamma \to \infty) \) the only right circular polarization is radiated in the upper half plane \( (\theta < \pi/2) \), while the lower half plane \( (\theta > \pi/2) \) contains a pure left circular polarization. It is a characteristic feature of the transitions to the ground state.

The analysis of the functions \( R_s(\beta, \theta) \), \( S_l(\beta, \theta) \) behavior shows up that the function \( R_1(\beta, \theta) \) is a monotone increasing function of \( \theta \) on \( 0 < \theta < \pi/2 \) at any \( \beta \).

The function \( R_2(\beta, \theta) \) is a monotone decreasing function of \( \theta \) on \( 0 < \theta < \pi/2 \) and while \( \beta^2 < \frac{3}{4} \) it has its maximum at some inner point \( \theta = \theta^{(\max)}_2(\beta) \) of this interval.

The functions \( R_0(\beta, \theta) \) on \( 0 < \theta < \pi/2 \) and \( S_l(\beta, \theta) \) on \( 0 < \theta < \pi \) at \( \beta^2 < 1/2 \) are monotone decreasing functions of \( \theta \). At \( \beta^2 > 1/2 \) they reach maxima on \( 0 < \theta < \pi/2 \) at the points \( \theta = \theta^{(\max)}_0(\beta) \) and \( \theta = \theta^{(\max)}_1(\beta) \), respectively.

The figures illustrate the graphs of the functions \( R_s(\beta, \theta) \), \( S_l(\beta, \theta) \) for different \( \beta \) and the graphs of \( \theta^{(\max)}_n(\beta) \), \( n = 0, 2 \)
Figure 1. This functions define the character of the polarization components angular distribution $R_0(\theta)$ (a), $R_1(\theta)$ (b), $R_2(\theta)$ (c) and $S_l(\theta)$ ($l = 1$) (d).

Figure 2. Here the functions $\theta_{\text{max}}^{(1)}(\beta)$ (a) and $\theta_{\text{max}}^{(2)}(\beta)$ (b) are represented.
4. The total output

For the total output radiated in the upper half plane (integrated over $\theta$ from $0 \leq \theta \leq \pi/2$) we obtain

$$W_i = Q \overline{M} P_1(\zeta, \beta), \quad \overline{M} = \frac{\beta^6}{32(1-\beta^2)(1+\sqrt{1-\beta^2})} = \frac{1}{(1-x_0)^2} \left( \frac{x_0}{1+x_0} \right)^3 = \frac{(\gamma^2-1)^3}{32\gamma^3(\gamma+1)},$$

$$P_2(\zeta, \beta) = (1-\zeta)f_1(\beta) + (1+\zeta)x_0 f_2(\beta),$$

$$P_3(\zeta, \beta) = (\gamma-\zeta)(\gamma+\zeta)^{-1} P_2(-\zeta, \beta),$$

$$P_0(\zeta, \beta) = [1 + x_0 - \zeta(1-x_0)] f_0(\beta), \quad f_0(\beta) = f_1(\beta) + f_2(\beta),$$

$$2P_l(\zeta, \beta) = [1 + x_0 - \zeta(1-x_0)] [f_0(\beta) + 2l \phi(\beta)],$$

In (10) the following notations are used

$$f_1(\beta) = \int_0^1 (1 + x_0 y) \exp(-x_0 y) \sqrt{\frac{1-x_0^2}{1-y}} dy, \quad f_2(\beta) = \int_0^1 (1 + x_0 y) \exp(-x_0 y) \sqrt{\frac{1-y}{1-x_0^2 y}} dy, \quad \phi(\beta) = \frac{2-(2+x_0)\exp(-x_0)}{x_0}.$$  \hspace{1cm} (11)

Substituting $l \rightarrow -l$ in (10) we get the total output radiated in the lower half plane (integrated over $\theta$ from $\pi/2 \leq \theta \leq \pi$).

The functions defined by (11) are positive and bounded, the functions $f_0(\beta), f_1(\beta)$ and $\phi(\beta)$ are monotone decreasing and $f_2(\beta)$ is monotone increasing on the segment $0 \leq \beta \leq 1$. It is easy to calculate the values of the function $f_0(\beta)$ at the segment boundary points

$$f_1(0) = 3 \beta(0) = 3 \phi(0) = 2, \quad f_1(1) = f_2(1) = \phi(1) = 2 - \frac{3}{e}. \hspace{1cm} (12)$$

At $\beta \ll 1, (x_0 \ll 1)$ we find for the functions $f_n(\beta)$

$$f_1(\beta) \approx 2 \left( 1 - \frac{13}{15} x_0^2 \right), \quad f_2(\beta) \approx \frac{2}{3} \left( 1 + \frac{11}{35} x_0^2 \right), \quad f_0(\beta) \approx \frac{8}{3} \left( 1 - \frac{4}{7} x_0^2 \right), \hspace{1cm} (13)$$

and for $\phi(\beta)$ we have the convergent at any $x_0$ series expansion

$$\phi(\beta) = \sum_{n=0}^{\infty} \frac{(-1)^n (1-n)}{(n+1)!} x_0^n \approx 1 - \frac{x_0^2}{6}. \hspace{1cm} (14)$$

For the functions $f_n(\beta)$ ($s = 0, 1, 2$) and $\phi(\beta)$ at $\beta \approx 1 (\gamma \gg 1)$ we find

$$f_0(\beta) \approx 2 \left( 2 - \frac{3}{e} + \frac{2\ln\gamma}{\gamma} \right) + O\left( \frac{1}{\gamma} \right), \quad f_1(\beta) \approx 2 - \frac{3}{e} + \frac{8\ln\gamma}{\gamma} + O\left( \frac{1}{\gamma} \right),$$

$$f_2(\beta) \approx 2 - \frac{3}{e} + \frac{4\ln\gamma}{\gamma} + O\left( \frac{1}{\gamma} \right), \quad \phi(\beta) \approx 2 - \frac{3}{e} + \left( 2 - \frac{5}{e} \right) \frac{1}{\gamma} + O\left( \frac{1}{\gamma^2} \right). \hspace{1cm} (15)$$

The precise analytical expression (G.A. Schott formula) for the total SR output in terms of classical theory is known [1-5] and we can rewrite it using our own designation

$$W^d = \frac{2}{3} Q \left( \frac{H}{H_0} \right)^2 (\gamma^2 - 1) = \frac{2}{3} Q \left( \frac{H}{H_0} \right)^2 (\beta \gamma)^2. \hspace{1cm} (16)$$
In the case under consideration in accordance with (2) there is a relation between the field tension $H$ and Lorentz - factor $\gamma$,

$$\frac{H}{H_0} = \frac{1}{2} (\gamma^2 - 1),$$  

(17)

which let us rewrite (16) as follows

$$W^{cl} = \frac{1}{6} Q(\gamma^2 - 1)^3 = \frac{1}{6} Q(\beta \gamma)^6,$$  

(18)

and therefore

$$W^{cl} \approx \frac{1}{6} Q \beta^6 \text{ at } \beta \ll 1; \ W^{cl} \approx \frac{1}{6} Q \gamma^6 \text{ at } \gamma \gg 1.$$  

(19)

For the upper half plane radiation $W^{cl}_{(+)}$ we finally obtain

$$W^{cl}_{(+)} = \frac{1}{2} W^{cl} = \frac{1}{12} Q(\gamma^2 - 1)^3 = \frac{1}{12} Q(\beta \gamma)^6;$$

$$W^{cl}_{(+)} \approx \frac{1}{12} Q \beta^6 \text{ at } \beta \ll 1; \ W^{cl}_{(+)} \approx \frac{1}{12} Q \gamma^6 \text{ at } \gamma \gg 1.$$  

Comparing the classical expressions (19) with the results obtained at $\beta \ll 1$ for an electron with spin $\zeta = -1$ (the transitions without spin-flip) and taking (10) and (12) into account we find

$$W_i(\zeta = -1) = W^{cl}_{(+)} q_i; \ q_0 = 1, \ q_2 = \frac{3}{4}, \ q_3 = \frac{1}{4}, \ q_l = \frac{(4 + 3l)}{16} \ (l = \pm 1).$$  

(20)

Thus, here we have the total coincidence with the classical theory (the value of $q_l$ was for the first time obtained in [8]).

If the electron spin $\zeta = 1$ (the spin-flip transitions) then we gain

$$W_i(\zeta = 1) = \frac{\beta^2}{4} W^{cl}_{(+)} p_i; \ p_0 = 1, \ p_2 = \frac{1}{4}, \ p_3 = \frac{3}{4}, \ p_l = \frac{(4 + 3l)}{16}.$$  

(21)

Thus, here the radiated output is small ($\approx \beta^2/4$) with respect to the case $\zeta = -1$ and the linear polarization components switch their places.

Averaging on the initial spin we gain

$$W_i = \frac{1}{2} W_i(\zeta = -1)$$  

(22)

- nonpolarized nonrelativistic electron radiates the half of classical output (that coincides with [6,7]).

At $\beta \approx 1 (\gamma \gg 1)$ from (10) and (15) we gain

$$W_i = Q \gamma^2 \frac{2 e^{-3}}{8e} r_i, \ r_0 = 1 + \frac{2}{2e-3} \frac{\ln \gamma}{\gamma}, \ r_2 = \frac{1}{2} \left( 1 + \frac{2 - 6\zeta}{2e-3} \frac{\ln \gamma}{\gamma} \right), \ r_3 = \frac{1}{2} \left( 1 + \frac{2 + 6\zeta}{2e-3} \frac{\ln \gamma}{\gamma} \right),$$

$$r_l = \frac{1}{2} \left( 1 + \frac{2 + 6\zeta}{2e-3} \frac{\ln \gamma}{\gamma} \right).$$  

(23)
Here with $\gamma$ rising the quantum output increases $\approx \gamma^2$ that is much slower than the classical output growth $\approx \gamma^6$. The left circular polarization ($l = -1$) in the upper half plane is strongly suppressed ($\approx \ln \gamma / \gamma$) in comparison with the right one ($l = 1$). And there is no prior linear polarization: the difference between $"\sigma"$ and $"\pi"$ linear polarization components is small ($\approx \ln \gamma / \gamma$) with respect to the primary terms. The SR power dependance on spin disappears. The spin dependance shows up in the amendments $\approx \ln \gamma$.

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