Mixing in thermally stratified nonlinear spin-up with sources and sinks

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Abstract

Stratified spin-up experiments in enclosed cylinders have reported the presence of small pockets of well-mixed fluids but quantitative measurements of the mixedness of the fluid has been lacking. Previous numerical simulations have not addressed these measurements. Here we present numerical simulations that address how the combined effect of spin-up and thermal boundary conditions enhances or hinders mixing of a fluid in a cylinder. Measurements of efficiency of mixing are based on the variance of temperature and explained in terms of the potential energy available. The numerical simulations of the Navier–Stokes equations for the problem with different sets of thermal boundary conditions at the horizontal walls helped shed some light on the physical mechanisms of mixing, for which a clear explanation was lacking.

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I. INTRODUCTION

Stratified spin-up flow is a classical fluid mechanics problem that has received considerable attention in recent years. The transient flow is created when the fluid, either at rest or in the state of solid body rotation, experiences an increase in the rotation rate and results in the propagation of stresses into the interior. The dynamics of spin-up/down is particularly relevant to large-scale geophysical flows, for example in situations where wind stresses in the open ocean and coastal regions generate ocean gyres and can result in baroclinic motions that distort the temperature field, turbulent mixing and redistribution of heat fluxes [1, 8, 10, 11, 13–16].

The study of mechanisms leading to efficient mixing has long been appreciated in the context of stratified shear flows [23] and thermal convection [3, 12, 27, 30, 31, 37]. For example, shear can increase mixing at stratified interfaces by triggering Kelvin-Helmholtz (K-H) instabilities and can produce turbulence via interaction of Reynolds stresses [5, 26]. Turbulence in the ocean can also be generated by another mechanisms, including mean velocity shear, breaking of surface or internal waves and surface cooling.

Motions associated with upwelling are known to cause localized mixing [14, 35]. Since most of the time new water masses are formed at the surface by cooling, and their spin-up is clearly of utility in determining ensuing flow patterns, it will be helpful to understand how the spin of water masses in basins subjected to different thermal boundary conditions affect the mixing. Laboratory experiment of salt-stratified spin-up in a cylinder have shown qualitative measures of mixing [6, 7, 9, 22], and recent three-dimensional simulations have demonstrated how different sets of thermal boundary conditions at the horizontal walls (adiabatic or fixed temperatures) affect the time of formation of columnar baroclinic vortices [24]. Nevertheless, quantitative measurements of mixing and the physical mechanisms controlling its efficiency in spin-up has remained relatively unexplored.

In this paper, we study the spin-up of a thermally stratified flow in a cylindrical container in a numerical setting. In addition to the two sets of thermal boundary conditions already considered in [21, 24], we include a combination of (i) prescribed temperature at the bottom wall and adiabatic at the top, and (ii) prescribed temperature at the top wall and adiabatic at the bottom. The quest here is for a quantitative measure of mixing for a variety of thermal boundary conditions potentially relevant to ocean flows. Our procedure for determining the
FIG. 1: Schematic of the spin-up seven rotations after the cylinder is accelerated from the initial rotation rate $\Omega_i = \Omega(1 - \epsilon)$ to $\Omega$. The left and right quadrants show the vortex core and the accumulation of cold fluid at the bottom corner respectively.

quality of mixing is based on the variance of temperature [17, 19, 28] and on the available potential energy [36, 37]. Quantifying mixing in the initial-value decaying problem must be interpreted very differently when sources and sinks are present. Common belief assumes that the best stirring to create mixing is either turbulent or exhibits chaotic trajectories. However this depends on the source-sink configuration, so a straight forward answer is not possible. We will address these features in the next sections.

II. GOVERNING EQUATIONS AND THE NUMERICAL SCHEME

Consider a Newtonian fluid of kinematic viscosity $\nu$, thermal diffusivity $\kappa$, and coefficient of volumetric expansion $\alpha$, confined in a cylinder of radius $R$ and height $h$ where the gravity and rotation vectors are colinear, as shown schematically in figure 1. Initially, the fluid is thermally stratified in the vertical direction, with a temperature difference of $\Delta T$ over $h$. The flow is spun-up by the sudden change of background rotation by the amount $\Delta \Omega$ to a new rotation rate $\Omega$ from its initial state $\Omega_i = \Omega(1 - \epsilon)$ where $\epsilon = \Delta \Omega / \Omega$. The system is non-dimensionalized using the flow depth $h$ as the length scale, the inertial time $\Omega^{-1}$ as
the time scale and $\Delta T$ as the temperature scale. There are six non-dimensional parameters in this problem:

- Aspect ratio: $\Gamma = R/h$,
- Ekman number: $E = \nu/\Omega h^2$,
- Froude number: $F = \Omega^2 h/g$,
- Burger number: $B = N/\Omega$,
- Prandtl number: $Pr = \nu/\kappa$,
- Rossby number: $\epsilon = \Delta \Omega/\Omega$,

where $N = (\alpha g \Delta T / h)^{1/2}$ is the buoyancy frequency. The non-dimensional governing equations are

$$
(\partial_t + \mathbf{u} \cdot \nabla) \mathbf{u} = \nabla p + B^2 \Theta e_z + 2 \mathbf{u} \times e_z - F B^2 \Theta r e_r + E \nabla^2 \mathbf{u},
$$

$$
(\partial_t + \mathbf{u} \cdot \nabla) \Theta = Pr^{-1} E \nabla^2 \Theta, \quad \nabla \cdot \mathbf{u} = 0,
$$

where $\mathbf{u}$ is the velocity field in the rotating frame, $(u, v, w)$ are the components of $\mathbf{u}$, the cylindrical coordinates $(r, \theta, z)$ are the components of $\mathbf{r}$, $p$ is the pressure (including gravitational and centrifugal contributions), $\Theta$ is the non-dimensional temperature. The unit vectors in the radial and vertical directions are $e_r$ and $e_z$ respectively. The initial conditions in the rotating frame are $u = w = 0, v = -\epsilon r$, and $\Theta = z$, the side-wall is no-slip and adiabatic, the top boundary is shear-free and the bottom wall non-slip, and the lateral wall is insulated. We shall focus our efforts on four sets of thermal boundary conditions applied to the horizontal walls, listed in table I. Two of these (PB_PT and AB_AT) were used in the analysis of [21] to explain how different sets of boundary conditions affect the time of formation of baroclinic vortices.

The governing equations (1)–(2) are discretized on a staggered grid with the velocities at the faces and all the scalars in the center of the computational cell; the resulting system of equations is solved by a fractional-step method. The finite-difference solver is based on that described by [34] and has been tested in a wide variety of enclosed cylindrical flows [18, 20, 24, 25, 32, 33], establishing resolution requirements over a wide range of parameters.

The grid is evenly spaced in the azimuthal direction while it is non uniform in the radial and vertical direction in such a way to cluster more computational points close to the solid
(no–slip) boundaries where the largest gradients occur. At least ten grid points were placed inside the bottom Ekman and side-wall boundary layers respectively, with \( n_\theta \times n_r \times n_z = 96 \times 351 \times 151 \). Details about the experimental and numerical test problems used for verification of the numerical code and selection of number of grid points can be found in [21].

We split the variables into axisymmetric and nonaxisymmetric parts and employ the energy equation to quantify the azimuthal perturbations. The axisymmetric part represents the mean flow (averaged quantities on the azimuth), while the non-axisymmetric part corresponds to the flow perturbations. For example, the velocity in (2) can be expressed as

\[
u(r, \theta, z) = \bar{\nu}(r, z) + \nu'(r, \theta, z),
\]

(3)

Substituting (3) in the momentum equation (2), taking the dot product with \( \nu' \), and integrating over the entire domain \( V \), yields the energy equation for the azimuthal disturbances

\[
\frac{dt}{t} = \frac{d}{dt} \int_V \frac{1}{2} |\nu'|^2 dV = - \int_V \nu' \cdot (\nu' \cdot \nabla \bar{\nu}) dV - B^2 \int_V \Theta \nu_z' dV + FB^2 \int_V \Theta r \nu'_r dV - E \int_V |\nabla \nu'|^2 dV = \sum_{i=1}^{4} h_i.
\]

(4)

The left-hand-side of (4) represents the kinetic energy growth rate of the azimuthal disturbance due to \( (h_1) \) shear of the mean axisymmetric flow (barotropic production); \( (h_2) \) conversion of gravitational potential energy (baroclinic production); \( (h_3) \) conversion of centrifugal potential energy; and \( (h_4) \) viscous dissipation [33]. A norm that is commonly used to quantify the mixing of the fluid is given by the magnitude of the variance of the scalar \( \Theta \),

\[
\sigma = \frac{\langle \Theta^2(r, \theta, z, t) \rangle - \langle \Theta(r, \theta, z, t) \rangle^2}{\langle \Theta^2(r, \theta, z, 0) \rangle - \langle \Theta(r, \theta, z, 0) \rangle^2},
\]

(5)
where \( \langle \cdot \rangle = 1/V \int_V \cdot \, dV \). In the presence of sources and sinks, the norm (5) would reach an asymptotic limit, and normalizing the global measure by the value it would have in the absence of stirring, instead of the initial value, would be more helpful, i.e.

\[
\hat{\sigma} = \frac{\langle \Theta^2(r, \theta, z, t) \rangle}{\langle \hat{\Theta}^2(r, \theta, z, t) \rangle},
\]

where \( \hat{\Theta} \) is the temperature due to diffusion only [4]. Efficient mixing implies \( \hat{\sigma} < 1 \) if the stirring decreases the variance relative to molecular diffusion alone, which is not always the case when sources or sinks are present. We will describe the time-evolution of the solutions in terms of the number of rotation \( \tau (= t/2\pi) \) instead of the normalized time \( t \).

We also quantify the available potential energy for mixing \((PE_A)\) by computing the difference between the total potential energy \((PE_T)\) and the potential energy of a reference state \((PE_R)\) [2, 3, 29], that is the minimum potential energy that can be obtained through an adiabatic redistribution of temperature (density),

\[
PE_A = PE_T - PE_R = \int_V (1 - z) \Theta dV - \int_V (1 - z_R) \Theta dV.
\]

Here, \( z_R(\Theta, \tau) \) is the vertical coordinate of the reference state (where all the temperature surfaces are horizontal). The vertical height of the reference state \( z_R \) can be computed in different manners, for example, by reorganizing the vertical position of layers in the reference state according to their density with the Heaviside step function \( H \),

\[
z_R(r, \theta, z, \tau) = \frac{1}{\pi T^2} \int_V H[\Theta(r, \theta, z, \tau) - \Theta(r', \theta', z', \tau)] dV',
\]

or by computing the probability density function \( \lambda(\Theta) \) of the temperature,

\[
\lambda(\hat{\Theta}) = \frac{1}{V} \int_V \delta(\hat{\Theta} - \Theta) dV.
\]

We evaluated numerically the probability density function \( \lambda(\Theta) \) by scanning the temperature field and placing its values into a bin and by normalizing the number of control volumes in each bin. The reference position \( z_R(\Theta) \) is obtained using the probability density function from (9) from

\[
z_R(\Theta) = 1 - \int_\Theta^{\Theta_M} \lambda(\hat{\Theta}) d\hat{\Theta},
\]

where the nondimensional height of the domain is 1 and \( \Theta_M \) is the maximum value of the temperature at time \( \tau \). The potential energy of the reference state \( PE_R \) can now be obtained.
from

\[ PE_R = \pi R^2 \int_0^1 (1 - z_R) \Theta \, dz_R. \] (11)

The parametric studies of [6, 21] suggest that for \( \Gamma < 1 \) the spin-up is less prone to become non-axisymmetric, therefore we restricted the values of the Rossby numbers to \( \epsilon \in [0.5, 1] \), and fixed the aspect ratio at \( \Gamma = 3.3 \), the Ekman number at \( E = 7.2 \times 10^{-4} \), the Froude number at \( F = 9.0 \times 10^{-4} \), the Burger number at \( B = 2.52 \) and the Prandtl number at \( Pr = 6.85 \).

III. RESULTS AND DISCUSSION

Before discussing how different boundary conditions affect the mixing, it is useful to briefly review the flow dynamics addressed in [21, 24]. Spin-up is a typical example of baroclinicity whose dynamics is dictated by the equation for absolute vorticity \( \omega \). Taking the curl of (1) yields

\[ (\partial_t + u \cdot \nabla) \omega = \omega \cdot \nabla u + B^2 \nabla \Theta \times \mathbf{e}_z + E \nabla^2 \omega. \] (12)

The first term on the right-hand side is responsible for vortex stretching and tilting, the second accounts for baroclinic vorticity and the third term represents vorticity diffusion. The production of barotropic vorticity can be expressed in its components as

\[ \omega_b = B^2 \nabla \Theta \times \mathbf{e}_z = B^2 \left( \frac{1}{r} \frac{\partial}{\partial \theta} e_r - \frac{\partial}{\partial r} e_\theta \right) \Theta. \] (13)

When the motion of the flow is initiated by the sudden increase in rotation rate, Ekman transport along the bottom boundary layer pushes fluid radially outwards and forms well-mixed corner regions that rotate faster than the interior. The stable stratification causes the azimuthal flow to develop vertical shear owing to the strong deformation of the isotherms developing an unstable system that can convert potential energy into kinetic energy. The kinetic energy dissipates through friction and reduces the temperature contrast through temperature advection. This is a common feature of a stratified spin-up flow regardless of the thermal boundary conditions imposed on the horizontal walls.

The numerical simulations of [24] for nonlinear spin-up and large aspect ratios demonstrated that after the initial phase of motion, the resulting stratification originating from different boundary conditions triggered different instabilities. When the cylinder walls were
FIG. 2: (Color online) Contours of temperature $\Theta$ on the planes $\theta = 0 - \pi$ at $\epsilon = 1$. At $\tau = 0$ there are 10 linearly spaced contour-levels in the range $\Theta = [0, 1]$. The figures in the left column correspond to PB.AT, and those in the right column to AB.PT. See the supplementary movies for animations.
adiabatic (AB\textsubscript{AT}), the vortex-core became baroclinically unstable, breaking up into different lenses. For isothermal boundary conditions (PB\textsubscript{PT}) the baroclinic perturbation decayed, the vortex-core began to oscillate and after several tens of rotations the flow broke into several columnar vortex structures as the flow returned to a state of linear stratification.

How and if the unstable system develops columnar vortices was addressed in the parametric study of [21]. They explained that when the temperatures are prescribed, the flow becomes three-dimensional due to small azimuthal variations of temperature leading to an increase in the baroclinic vorticity through (13). The baroclinic vorticity in the radial component \( r^{-1} \partial \Theta / \partial \theta \) excites the kinetic energies in the asymmetric azimuthal Fourier components, and compensates for the decrease in \( \partial \Theta / \partial r \). The growth of baroclinic vorticity produced by the azimuthal variations of temperature enhances the axial vorticity, and this in turn increases the vertical shear around the boundary of the central vortex-core, further deforming the isotherms on the \( z \)-plane. This local advective heat transport enhances azimuthal temperature gradients completing the feedback cycle causing the core vortex to break.

The typical evolution of the flow for PB\textsubscript{AT} and AB\textsubscript{PT} at \( \epsilon = 1 \) is shown in figures 2 and 3. The sequence of images in figure 2 demonstrates the flow change on the planes \( \theta = 0 - \pi \). At ten rotations (\( \tau = 10 \)), pockets of well-mixed cold fluid accumulate at the bottom corner separated from the core by a vortex. The left quadrant of figure 1 illustrates the vortex core (front) using the \( Q \)-criterion. At 30 rotations, the flow AB\textsubscript{PT} is three-dimensional with evidence of internal waves. For PB\textsubscript{AT} and the same number of rotations, the accumulation of well-mixed fluid at the corner regions is still visible. The flush back of cold fluid, after the Ekman transport shuts down, occurs about 20 rotations later for PB\textsubscript{AT} than for AB\textsubscript{PT}. The delay is influenced mainly by the boundary condition at the bottom wall. This event is clearly seen from the spatio-temporal evolution of temperature along a vertical line at three fixed radii of figure 3. From this figure, we can also see the quality of mixing in the interior (\( r = 0.5 \)), around the interface of the core vortex during upwelling (\( r = 1.7 \)), and near the lateral wall (\( r = 3.2 \)). At early times the isotherms are compressed near the bottom wall (\( r = 0.5 \)) and near the top wall (\( r = 3.2 \)) as the cold fluid moves through the Ekman layer pushing cold fluid to the corner. When the Ekman pumping ceases, the secondary circulation reverses direction and the cold fluid from the corner regions moves back to replace the warm fluid in the core. Near the adiabatic walls, the fluid that is replaced is nearly homogeneous, whereas near the wall with prescribed temperatures the
FIG. 3: (Color online) Spatio-temporal evolution of the temperature along a vertical line at $\theta = 0$ and $r$ as indicated. The horizontal axis indicates time in the range $0 \leq \tau \leq 160$ and the vertical axis the location of the probes in the range $0 \leq z \leq 1$. At $\tau = 0$ there are 10 linearly spaced contour-levels in the range $\Theta = [0, 1]$. The figures in the left column correspond to PB_AT, and those in the right column to AB_PT.

The fluid remains stratified (figure 3). The appearance of baroclinic waves is shown in figure 3(b). Notice that the baroclinic instability propagates from the vortex core to both, the interior and to the outer wall.

The flow behavior for $\epsilon = \{0.5, 0.73\}$ is similar to that of $\epsilon = 1$. The main difference is the time at which the flow becomes three-dimensional, with the transition occurring later for smaller Rossby numbers. This is better appreciated from the history profile of azimuthal disturbances $h_i$. For comparison, we only show values for $\epsilon = \{0.73, 1\}$ in figure 4. The left-hand-side of the figure shows the azimuthal disturbances for PB_AT, and this demonstrates a remarkable similarity with PB_PT, with two distinct states in the flow development. The first is an increase in the energy of perturbation followed by a decay due to the horizontal realignment of the isotherms in the $\theta$-plane as the Ekman transport shuts down. The second is characterized by an increase in the energy of perturbation due to the baroclinic vorticity contribution in the radial component, which compensates for the decrease in $\partial \Theta / \partial r$ as
Case: PB\_AT

Case: AB\_PT

FIG. 4: (Color online) Time evolution of $h_i$-terms in the rate of change of kinetic energy of azimuthal perturbations at $\epsilon$ as indicated. The figures in the left correspond to PB\_AT, and those in the right to AB\_PT. Barotropic term $h_1$ ( — black); baroclinic term $h_2$ ( – red); centrifugal term $h_3$ (–·· green); viscous dissipation term $h_4$ ( · · blue).

explained by [21].

The path to three-dimensionality of the flow for AB\_PT is very similar to AB\_AT as well, with the baroclinic disturbance remaining positive until it reaches a global maximum. The barotropic term initially contributes to the instability, and then extracts energy from the mean flow. The viscous dissipation as expected is negative and the centrifugal term negligible.

The probability density function $\lambda(\Theta)$ is a good indicator of how the temperature is spatially distributed during spin-up. This is evaluated numerically by scanning the temperature field, placing its values into bins and normalizing the values by the number of control
FIG. 5: (Color online) (a) Contours of probability density $\lambda(\Theta)$ as function of number of rotations $\tau$ and (b) cross-sections of $\lambda$ at: — (black) $\tau = 0$; -- (red) $\tau = 10$; --- -- (green) $\tau = 125$; -- · -- (blue) $\tau = 160$; -- · (brown) $\tau = 200$. The Rossby number is $\epsilon = 1$.

volumes in each bin. Contours of $\lambda(\Theta)$ and cross-sections at various numbers of rotation are shown in figures 5-6 at $\epsilon = 1$ to show the spatio-temporal distribution of temperature for the different sets of boundary conditions considered in this study. At late times, the linear stratification for PB_PT is almost recovered, whereas for AB_AT, the distribution of temperature is bi-modal, with the asymptotic values of temperature concentrating around the mean $\langle \Theta \rangle = 0.5$. The time evolution of $\lambda(\Theta)$ for PB_AT and AB_PT at $\epsilon = 1$ is also shown in figure 6. For PB_AT the asymptotic temperature distribution will be $\Theta = 0$ whereas for AB_PT will be $\Theta = 1$.

One of the main objectives of this study is the quantification of mixing for several types of thermal boundary conditions on the horizontal walls. For PB_PT the flow mixes locally, but asymptotically, the flow returns to a state of linear stratification, therefore we can expect
that this flow will not produce any global mixing. The typical case of an initial value-decaying problem is \( \text{AB}_{-}\text{AT} \) where the final state of mixing is equal to the initial mean. For this case, it is custom to use the variance of temperature to quantify the mixing efficiency. A uniform measure of mixing when sinks and sources are present is obtained by normalizing the variance by the value it would have in the absence of flow motion. This ‘efficiency’ measures how much mixing is increased by stirring, i.e. if stirring decreases the variance compared to the value based purely on diffusion then the flow is mixed efficiently.

Figure 7 shows the variance for \( \text{PB}_{-}\text{AT} \) and \( \text{AB}_{-}\text{PT} \) at \( \epsilon = 0.5 \) and 1. These norms are bounded below by a solid line (\( \text{AB}_{-}\text{AT} \)) and above by circles (\( \text{PB}_{-}\text{PT} \)) representing the best and worst mixing efficiencies (at \( \epsilon = 1 \)) respectively. For prescribed temperature at the bottom wall, the variance increases from its initial value to a maximum and then
decreases. The variance is larger in the higher $\epsilon$ case due to the more energetic spin-up that pushes more well-mixed cold fluid to the corner regions (compared to the smaller $\epsilon$ value, generating a higher temperature contrast with the core. The opposite effect is seen when the bottom wall is adiabatic, i.e. the variance of temperature is lower for $\epsilon = 1$ than for $\epsilon = 0.5$. This is also expected, since the the amount of fluid and its temperature (carried to the corner region through the Ekman layer) is larger for the higher value of $\epsilon$. The mixing features mentioned above seem to agree with the flow similarities of PB_AT with PB_PT and AB_PT with AB_AT. The modified variance $\hat{\sigma}$ in figure 8 demonstrates how well the fluid mixes compared to the purely diffusive case for the same conditions as figure 7, where $\hat{\sigma} < 1$ corresponds to efficient mixing. Notice however, that after several tens of rotations the mixing generated by PB_AT is unexpectedly smaller than AB_PT. Surprisingly, at $\tau = 300$ the flow AB_PT generates as much mixing as the pure diffusion case, and excluding AB_AT, only PB_AT at $\epsilon = 1$ generates $\hat{\sigma} < 1$ for $\tau > 300$.

Figure 9 shows the time evolution of the potential energy available for mixing at $\epsilon = \{0.5, 1\}$, for PB_AT and AB_PT. As expected, the available potential energy is larger in the higher Rossby number due to more energetic stirring, and higher for PB_AT than for AB_PT at the same $\epsilon$. If a system has more potential energy available for mixing than
another, then the system will mix better globally. This confirms our findings that if PB\_AT has more potential energy available than AB\_PT (for the same \(\epsilon\)), then asymptotically, PB\_AT will mix better than AB\_PT. The reason for PB\_AT to have more available potential energy than AB\_PT can be explained as follows: for PB\_AT, the bottom wall is a sink of temperature, and during upwelling, the masses of fluid transported radially outwards through the Ekman layer cool down and accumulate at the corner regions (figure 2). The corner regions are well mixed, but only locally. These regions are separated from the core flow which remains in nearly solid body rotation. The stirring caused by the upwelling also increases the temperature gradients and the potential energy that will be released when the Ekman transport shuts down. The effect of prescribed bottom wall temperature also deteriorates the mixing but only during upwelling, creating sharp gradients of temperatures among pockets of cold and relatively warm temperature. These gradients are higher for PB\_AT than for AB\_PT, because for AB\_PT, the bottom wall does not cool down the fluid and thus the potential energy available for mixing for PB\_AT is larger than for AB\_PT. Once the available potential energy is released, mixing will be generated by transforming the potential energy to kinetic energy. Thus the higher the \(PE_A\) the better the mixing.
IV. CONCLUSIONS

In this paper we have studied numerically the mixing efficiency of spin-up stratified by temperature. Four different combinations of boundary conditions were considered at the bottom/top walls, prescribed but fixed temperatures, adiabatic or a combination of these two. The kinetic energy growth rate of the azimuthal disturbance was used to determine when the baroclinic instability occurred. We found that the spin-up with prescribed temperature at the bottom wall and adiabatic top wall was remarkably similar to the flow generated when the temperatures at the horizontal walls were prescribed (PB<sub>PT</sub>). We focused on the quantifying the mixing using the variance of temperature and a ratio of the variance to the value it would have without stirring. When the temperatures are prescribed on the horizontal walls the asymptotic state recovers its initial stratification, thus the effect of spin-up worsens the global norm of mixing. When the walls were adiabatic, the flow achieved the highest efficiency of mixing. The mixing efficiency for a flow with prescribed temperature on one wall and adiabatic on the other yielded a mixing efficiency higher than PB<sub>PT</sub> but
lower than AB\_PT.

Since the flow features for AB\_PT resembled those of AB\_AT, and the latter yielded the highest degree of mixing, we expected that the combination of bottom adiabatic wall and prescribed temperature at the top would render better mixing than PB\_AT. This was true only for intermediate times, but asymptotically, PB\_AT always performed better than AB\_PT (for the same $\epsilon$). This was confirmed by evaluating the potential energy available for mixing for the two flows. During spin-up, the prescribed bottom-wall temperature cooled down the fluid moving radially through the Ekman layer towards the corner regions, creating pockets of cold, but well-mixed fluid, keeping the potential energy available for mixing at a higher level than that obtained through the bottom adiabatic wall. This in turn created higher gradients of temperature, and therefore better mixing for large times.

There are many aspects of non-linear spin-up flows that remain unexplored, and this study provides the framework for further investigations. One of them is how the mixing is affected by the thermal diffusivity. The thermal diffusion effect seems relevant for a period longer than the Ekman spin-up time interval. If the thermal diffusion is large, the azimuthal variations of temperature will decay quickly and the the baroclinic term is likely to produce less vorticity. But whether or not a small thermal diffusion will render better mixing, is an open question. Further investigation is also needed on the effects of salt-stratification. These two effects are currently being investigated.

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