A SPECIAL FORM OF SOLUTION TO HALF-WAVE EQUATIONS

HYUNGIN HUH*

Department of Mathematics
Chung-Ang University
Seoul 156-756, Republic of Korea

(Communicated by Vladimir Georgiev)

Abstract. We investigate a special form of solution to the one-dimensional half-wave equations with particular forms of nonlinearities. Using the special form of solution involving Hilbert transform, the half-wave equations reduce to nonlocal nonlinear transport equation which can be solved explicitly.

1. Introduction. In this paper we are interested in a special form of solutions to the one-dimensional half-wave equations:

\[ i\partial_t \varphi + |D| \varphi = \varphi^3, \]  

(1)

and

\[ i\partial_x \varphi + |D| \varphi = \lambda \varphi \bar{\psi}, \]

\[ i\partial_t \psi - |D| \psi = \mu \bar{\varphi}^2, \]

(2)

where \( \varphi, \psi : \mathbb{R}^{1+1} \rightarrow \mathbb{C} \) and \( \bar{\varphi} \) is a complex conjugate of \( \varphi \), \( \lambda \) and \( \mu \) are complex constants. The operator \( |D| \) is defined by

\[ \mathcal{F}(\langle D \rangle f)(\xi) = |\xi| \mathcal{F}(f)(\xi), \]

where \( \mathcal{F} \) is Fourier transform. We propose the one-dimensional half-wave equations (1) and (2) and will show that an special algebraic structure of nonlinearities implies that one-dimensional half-wave equations may be reduced to some complex transport equations.

Several authors studied the following cubic half-wave equation:

\[ i\partial_t \varphi + |D| \varphi = |\varphi|^2 \varphi, \]

\[ \varphi(x, 0) = \varphi_0(x). \]

(3)

The associated conserved quantities of the equation (3) are

\[ \mathcal{N}(\varphi(t)) = \int_{\mathbb{R}} |\varphi|^2(x, t) dx = \mathcal{N}(\varphi_0), \]

\[ \mathcal{E}(\varphi(t)) = \int_{\mathbb{R}} \left( \langle J \varphi, \varphi \rangle - \frac{1}{2} |\varphi|^4 \right)(x, t) dx = \mathcal{E}(\varphi_0), \]

2020 Mathematics Subject Classification. Primary: 35L45; Secondary: 35F25.

Key words and phrases. Half-wave equation, Hilbert transform, nonlocal nonlinear transport equation.

This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government (MSIT) (2020R1F1A1A01072197).

* Corresponding author: Hyungjin Huh.
where $\int_\mathbb{R} \langle J\phi, \phi \rangle \, dx = \int_\mathbb{R} |\xi| |\mathcal{F}(\phi)|^2 \, d\xi$. The initial value problem of the equation (3) was studied in [1, 2, 10, 13]. The existence of ground state and its stability was investigated in [2, 4]. The asymptotic behaviors of the solution was studied in [8, 9]. It is not known that equation (1) has the similar conservation law.

The following semirelativistic equations with nongauge invariant power type nonlinearity was studied.

$$i \partial_t \phi + |D| \phi = |\phi|^3. \quad (4)$$

In particular, nonexistence of global solution was proved in [5, 11].

Well-posedness of the Cauchy problem for the following system of semirelativistic equations was studied in [6, 7].

$$i \partial_t \phi + |D| \phi = \lambda \bar{\phi} \psi,$$
$$i \partial_t \psi - |D| \psi = \mu \phi^2, \quad (5)$$

with initial data $\phi(x, 0) = \phi_0(x)$, $\psi(x, 0) = \psi_0(x)$. They proved the local wellposedness of (5) for the initial data $\phi_0, \psi_0 \in W^{s,2} \times W^{s,2}$ ($s \geq 0$). Moreover the global existence of solution was proved for $\lambda$ and $\mu$ satisfying $\lambda = c \mu$ with some constant $c > 0$ under which the system (5) has the following conservation law of charge:

$$\|\phi(\cdot, t)\|_{L^2}^2 + c \|\psi(\cdot, t)\|_{L^2}^2 = \|\phi_0\|_{L^2}^2 + c \|\psi_0\|_{L^2}^2.$$

Here we consider the equations (1) and (2) which have different algebraic structure for nonlinearity from (3), (4) and (5). Especially we are looking for solutions to (1) and (2) of the form

$$\phi = u + i Hu, \quad \psi = v - i Hv, \quad (6)$$

where $u$, $v$ are real-valued functions and $H$ is the Hilbert transform. Note that the ansatz (6) can not be applied to (5).

We will show that the equation (1) reduces, under the condition (6), to the following one:

$$u_t - u_x = 3u^2 Hu - (Hu)^3,$$
$$u(x, 0) = u_0(x). \quad (7)$$

We note that (7) is a transport equation with nonlocal nonlinear term. Using the property of Hilbert transform, we can reduce (7) to complex valued nonlinear transport equation without nonlocal term to obtain an explicit solution formula of (7). The detailed analysis of (7) can be found in section 2. We also note that the assumption (6) is not compatible with the equation (5).

**Theorem 1.1.** Let $u$ be the solution of the equation (7). Then we have the following explicit solution formula.

$$(u + iHu)^2(x - t, t) = \frac{(u_0(x) + i(Hu_0)(x))^2}{1 + i2t(u_0(x) + i(Hu_0)(x))^2}.$$ 

We will show in section 3 that the system (2) reduces, under the condition (6), to the following one.

$$u_t - u_x = \lambda_1(u Hv + v Hu) - \lambda_2(Hu H v - uv),$$
$$v_t - v_x = -2\mu_1 u Hu - \mu_2((Hu)^2 - u^2),$$
$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x). \quad (8)$$
where \( \lambda = \lambda_1 + i\lambda_2 \) and \( \mu = \mu_1 + i\mu_2 \). We can solve the nonlocal nonlinear equations (8) by reducing them to complex valued transport equations.

**Theorem 1.2.** Let \( u, v \) be the solution of the equation (8). We define
\[
\omega(x,t) = |\lambda|e^{-i(\frac{3\pi}{2} + \arg \lambda)}(v - iHu)(x,t) \quad \text{and} \quad z(x,t) = \sqrt{\lambda\mu}e^{i\frac{1}{2}(\arg \lambda - \arg \mu)}(u + iHu)(x,t).
\]
Then \( \omega \) satisfies, for a given \( x \), the following ODE
\[
\frac{d\omega}{dt}(x-t,t) = \omega^2(x-t,t) - \omega^2(x,0) + \bar{z}^2(x,0),
\]
and \( z \) is obtained by
\[
z(x-t,t) = z_0(x) \exp \left( \int_0^t \tilde{\omega}(x-s,s) \, ds \right),
\]
where \( \omega_0(x) = |\lambda|e^{-i(\frac{3\pi}{2} + \arg \lambda)}(v_0 - iHv_0)(x) \) and \( z_0(x) = \sqrt{\lambda\mu}e^{i\frac{1}{2}(\arg \lambda - \arg \mu)}(u_0 + iHu_0)(x) \).

We will find an explicit solution formula (33) using that ODE (9) is Riccati’s equation.

We prove Theorem 1.1 and 1.2 and observe several behaviors of solutions in section 2 and 3 respectively.

2. **Proof of Theorem 1.1.** Let us introduce the Hilbert transform [12].
\[
H(f)(x) := \frac{1}{\pi} \text{P.V.} \int_{-\infty}^{\infty} \frac{f(y)}{x-y} \, dy.
\]
Note that
\[
\mathcal{F}(Hf_x)(\xi) = |\xi|\mathcal{F}(f)(\xi).
\]
Then we can rewrite the equations (1) as
\[
i\phi_t + H\phi_x = \phi^3. \quad (10)
\]
Let us recall the formula related with the Hilbert transform [3, 12].
\[
H(Hf) = -f,
\]
\[
H(fHg + gHf) = (Hf)(Hg) - fg,
\]
\[
(Hf)_x = H(f_x).
\]
We are interested in the solution to (10) of the form
\[
\phi = u + iHu, \quad (12)
\]
where \( u \) is a real-valued function. Substituting the ansatz (12) in (10) and taking real and imaginary parts, we have
\[
u_t - u_x = 3u^2Hu - (Hu)^3, \quad (13)
\]
\[
(Hu)_t - (Hu)_x = 3u(Hu)^2 - u^3. \quad (14)
\]
We need to check the compatibility of the equations (13) and (14). Applying (11), we have \( H(2uHu) = (Hu)^2 - u^2 \), which is equivalent to
\[
H \left( (Hu)^2 - u^2 \right) = -2uHu.
\]
Then we have
\[ uH \left( (Hu)^2 - u^2 \right) + Hu \left( (Hu)^2 - u^2 \right) = (Hu)^3 - 3u^2Hu. \] (15)

On the other hand, taking \( H \) on the left side of (15) and applying (11), we obtain
\[
H \left( uH((Hu)^2 - u^2) + Hu((Hu)^2 - u^2) \right) = HuH((Hu)^2 - u^2) - u((Hu)^2 - u^2)
\]
\[ = Hu(2Hu) - u((Hu)^2 - u^2)
\]
\[ = u^3 - 3u(Hu)^2. \] (16)

Combining (15) and (16), we obtain
\[ H \left( (Hu)^3 - 3u^2Hu \right) = u^3 - 3u(Hu)^2. \] (17)

taking \( H \) on both sides of (13) and considering (17), we have the equation (14). Therefore the equations (14) is compatible with (13).

Considering
\[ H\phi = H(u + iHu) = Hu - iu = -i(u + iHu) = -i\phi, \]
and using the notation \( z(x,t) = (u + iHu)(x,t) \), we can rewrite (10) as
\[ iz_t - iz_x = z^3, \] (18)

which is equivalent to (13) and (14). Note that (18) is a complex valued transport equation. More precisely, we consider \( z(x-t,t) = (u + iHu)(x-t,t) \) for a given \( x \). Then we can rewrite (18) as
\[ \frac{dz}{dt} = -iz^3, \]

which can be solved explicitly by
\[ z^2(x-t,t) = \frac{z_0^2(x)}{1 + i2tz_0^2(x)}, \]
where \( z_0(x) = u_0(x) + i(Hu_0)(x) \). Then we have an explicit solution formula for the equation (13).
\[ (u + iHu)^2(x-t,t) = \frac{(u_0(x) + i(Hu_0)(x))^2}{1 + i2t(u_0(x) + i(Hu_0)(x))^2}. \] (19)

Applying (19), we can observe several behaviors of solutions. For the initial data satisfying \( u_0(a) = (Hu_0)(a) \neq 0 \) for some \( a \), we have
\[ (u + iHu)^2(a-t,t) = \frac{i2u_0^2(a)}{1 - 4tu_0^2(a)}, \]
from which we derive \( (u^2 - (Hu)^2)(a-t,t) = 0 \). Then we have
\[ (uHu)(a-t,t) = u^2(a-t,t) = \frac{u_0^2(a)}{1 - 4tu_0^2(a)}, \]
which blows up in finite time as \( t \to 1/4u_0^2(a) \). Note that
\[ H \left( \frac{1}{x^2 + 1} \right) = \frac{x}{x^2 + 1} \] (20)

from which we can choose \( a = 1 \) for the initial data \( u_0(x) = 1/x^2 + 1 \).
For the other example, we consider the initial data satisfying \( u_0(b) \neq 0 \) and \((Hu_0)(b) = 0\) for some \( b \). Then the formula (19) implies
\[
(u + iHu)^2(b - t, t) = \frac{u_0^2(b)}{1 + i2t u_0^2(b)},
\]
from which we can derive \(|(u + iHu)^2|(b - t, t) = (u^2 + (Hu)^2)(b - t, t) \to 0\) as \( t \to \infty\). Note the identity (20) from which we can choose \( b = 0 \) for the initial data \( u_0(x) = \frac{1}{x^2 + 1} \).

The above two examples can be generalized by
\[
\frac{(u_0(x) + i(Hu_0)(x))^2}{1 + i2t(u_0(x) + i(Hu_0)(x))^2} = \begin{cases} 
\text{blows up as } t \to 1/2\alpha \text{ if } (u_0 + iHu_0)^2 = \alpha i (\alpha > 0), \\
\text{converges to zero as } t \to \infty \text{ otherwise.}
\end{cases}
\]

In the remaining part of this section, we will show that the assumption (12) is not compatible with (3). Substituting the ansatz (12) in (3) and taking real and imaginary parts, we have
\[
u_t - u_x = (u^2 + (Hu)^2) Hu,
\]
\[(Hu)_t - (Hu)_x = -(u^2 + (Hu)^2) u.
\]
To show the compatibility, it is enough to show that
\[
H \left( (u^2 + (Hu)^2) Hu \right) = -(u^2 + (Hu)^2) u. \tag{21}
\]
However, considering the identity (20), we can check that
\[
H \left( (u^2 + (Hu)^2) Hu \right) = H \left( \frac{1}{x^2 + 1} H \left( \frac{1}{x^2 + 1} \right) \right) = \frac{1}{2} \left( \frac{x^2 - 1}{(x^2 + 1)^2} \right)' = -\frac{1}{(x^2 + 1)^2},
\]
where we applied (11) with \( f = g = \frac{1}{x^2 + 1} \). Therefore the relation (21) is not satisfied in general.

3. **Proof of Theorem 1.2.** As in section 2, we can rewrite the equations (2) as
\[
i\phi_t + H\phi_x = \lambda\phi\bar{\psi},
\]
\[
i\psi_t - H\psi_x = \mu\bar{\phi}^2. \tag{22}
\]
We are interested in the solution to (22) of the form
\[
\phi = u + iHu \quad \text{and} \quad \psi = v - i Hv,
\]
where \( u \) and \( v \) are real-valued functions. Let \( \lambda = \lambda_1 + i\lambda_2 \) and \( \mu = \mu_1 + i\mu_2 \). Taking real and imaginary parts, we have
\[
u_t - u_x = \lambda_1(u Hv + v Hu) - \lambda_2(Hu Hv - uv), \tag{23}
\]
\[
Hu_t - Hu_x = \lambda_1(Hu Hv - uv) + \lambda_2(u Hv + v Hu), \tag{24}
\]
\[
v_t - v_x = -2\mu_1 u Hu - \mu_2((Hu)^2 - u^2), \tag{25}
\]
\[
Hu_t - Hv_x = -\mu_1((Hu)^2 - u^2) + 2\mu_2 u Hu. \tag{26}
\]
Taking into account
\[
H(u Hv + v Hu) = Hu Hv - uv \quad \text{and} \quad H(2u Hu) = (Hu)^2 - u^2,
\]
we can check that the equation (24) and (26) are compatible with (23) and (25) respectively.
Taking into account
\[ H\phi = H(u + iHu) = Hu - iu = -i(u + iHu) = -i\phi, \]
\[ H\psi = H(v - iHv) = Hv + iu = i(v - iHv) = i\psi, \]
the system (22) can be written as
\[ i\phi_t - i\phi_x = \lambda\phi, \]
\[ i\psi_t - i\psi_x = \mu\psi, \]
which is equivalent to (23)–(26). To make (27) simpler, we define
\[ \phi(x, t) = e^{\frac{i}{2}(\arg\mu - \arg\lambda)} \frac{z(x, t)}{\sqrt{|\lambda|}} \quad \text{and} \quad \psi(x, t) = e^{i\frac{3\pi}{2} + \arg\lambda} \frac{\omega(x, t)}{|\lambda|}. \]
Then the system (27) becomes
\[ \frac{dz}{dt} = z\omega \quad \text{and} \quad \frac{d\omega}{dt} = \bar{z}^2. \]
with initial data \( z(x, 0) \) and \( \omega(x, 0) \) which are related with the initial data \( \phi(x, 0), \psi(x, 0) \) through the relation (28).
Note that the system (29) is coupled ODEs along the characteristic line for a given \( x \). More precisely, we consider \( z(x - t, t) \) and \( \omega(x - t, t) \) for a given \( x \). Then we can rewrite (29) as
\[ \frac{d\omega}{dt} = \omega^2 - C \quad \text{with} \quad \omega(0) = \omega_0, \]
which is known as Riccati's equation. We will find an explicit solution of (32).
For \( C = 0 \), we have
\[ \omega(t) = \frac{\omega_0}{1 - \omega_0 t}. \]
For \( C = |C|e^{i\theta} \) (\( C \neq 0, 0 \leq \theta < 2\pi \)), letting \( \omega = -\frac{h'}{\pi} \), we have \( h'' - Ch = 0 \). Then we obtain
\[ \omega = \frac{a_1\gamma_1 e^{\gamma_1 t} + a_2\gamma_2 e^{\gamma_2 t}}{a_1 e^{\gamma_1 t} + a_2 e^{\gamma_2 t}} = \gamma_2 \frac{a_1 e^{\gamma_1 t} - a_2 e^{\gamma_2 t}}{a_1 e^{\gamma_1 t} + a_2 e^{\gamma_2 t}}, \]
where
\[ \gamma_1 = -\sqrt{|C|e^{i\frac{\theta}{2}}}, \quad \gamma_2 = \sqrt{|C|e^{i\frac{\theta}{2}}}, \quad a_1 = \frac{\gamma_2 + \omega_0}{\gamma_2 - \gamma_1}, \quad a_2 = \frac{\gamma_1 + \omega_0}{\gamma_1 - \gamma_2}. \]
Note that we choose \( a_1, a_2 \) satisfying
\[ \omega_0 = -\frac{a_1\gamma_1 + a_2\gamma_2}{a_1 + a_2} \quad \text{and} \quad a_1 + a_2 = 1. \]
Depending on \( C \) and \( \omega_0 \), the solution (33) has different behaviors. For the case of \( C = -1 \), we have
\[
\omega(t) = \frac{(1 + i \omega_0)e^{-it} + (1 + i \omega_0)e^{it}}{\omega_0 + i})e^{-it} + (i - \omega_0)e^{it}.
\]
We can derive the behavior of \( \omega \) as follows.
\[
\omega(t) = \begin{cases}
\text{periodic solution} & \text{if } (\omega_0 + i)e^{-it} + (i - \omega_0)e^{it} \neq 0 \text{ for all } t, \\
\text{blows up} & \text{if } (\omega_0 + i)e^{-it} + (i - \omega_0)e^{it} = 0 \text{ for some } t_0.
\end{cases}
\]
In particular, we have a static solution \( \omega(t) = \pm i \) for \( \omega_0 = \pm i \). We can check that \( \omega(t) = \tan t \) for \( \omega_0 = 0 \) which blows up as \( t \to \frac{\pi}{2} \).
For \( C = i \), we have \( \gamma_1 = -(\sqrt{2} \beta + i \frac{\sqrt{2}}{2}) \), \( \gamma_2 = \sqrt{2} \beta + i \frac{\sqrt{2}}{2} \), \( a_1 = \frac{1 + i + \sqrt{2} \omega_0}{2(1 + i)} \) and \( a_2 = \frac{1 + i - \sqrt{2} \omega_0}{2(1 + i)} \). Especially, we have for \( \omega_0 = 0 \)
\[
\omega(t) = \left(\sqrt{2} \beta + i \frac{\sqrt{2}}{2}\right) e^{-\sqrt{2} \beta t} e^{-i \sqrt{2} \beta t} - e^{-\sqrt{2} \beta t} e^{-i \sqrt{2} \beta t} + e^{\sqrt{2} \beta t} e^{i \sqrt{2} \beta t}
\]
go to \( -\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \) as \( t \to \infty \).
For a given \( \omega \) in (33), \( z \) can be obtained from (30)
\[
z(x - t, t) = z_0(x) \exp \left( \int_0^t \omega(x - s, s) ds \right),
\]
where \( z_0(x) = z(x, 0) \).

REFERENCES

[1] J. Bellazzini, V. Georgiev, E. Lenzmann and N. Visciglia, On traveling solitary waves and absence of small data scattering for nonlinear half-wave equations, Comm. Math. Phys., 372 (2019), 713–732.
[2] J. P. Borgna and D. F. Rial, Existence of ground states for a one-dimensional relativistic Schrödinger equation, J. Math. Phys., 53 (2012), 062301, 19 pp.
[3] D. Chae, A. Córdoba, D. Córdoba and M. A. Fontelos, Finite time singularities in a 1D model of the quasi-geostrophic equation, Adv. Math., 194 (2005), 203–223.
[4] Y. Cho, H. Hajaiej, G. Hwang and T. Ozawa, On the orbital stability of fractional Schrödinger equations, Commun. Pure Appl. Anal., 13 (2014), 1267–1282.
[5] K. Fujiwara, A note for the global nonexistence of semirelativistic equations with nongauge invariant power type nonlinearity, Math. Methods Appl. Sci., 41 (2018), 4955–4966.
[6] K. Fujiwara, S. Machihara and T. Ozawa, Well-posedness for the Cauchy problem for a system of semirelativistic equations, Comm. Math. Phys., 338 (2015), 367–391.
[7] K. Fujiwara, S. Machihara and T. Ozawa, On a system of semirelativistic equations in the energy space, Commun. Pure Appl. Anal., 14 (2015), 1343–1355.
[8] P. Gérard and S. Grellier, Effective integrable dynamics for a certain nonlinear wave equation, Anal. PDE, 5 (2012), 1139–1155.
[9] P. Gérard, E. Lenzmann, O. Pocovnicu and P. Raphael, A two-soliton with transient turbulent regime for the cubic half-wave equation on the real line, Ann. PDE, 4 (2018), paper no. 7, 166 pp.
[10] K. Hidano and C. Wang, Fractional derivatives of composite functions and the Cauchy problem for the nonlinear half wave equation, Selecta Math. (N.S.), 25 (2019), paper no. 2, 28 pp.
[11] T. Inui, Some nonexistence results for a semirelativistic Schrödinger equation with nongauge power type nonlinearity, Proc. Amer. Math. Soc., 144 (2016), 2901–2909.
[12] J. N. Pandey, The Hilbert Transform of Schwartz Distributions and Applications, Wiley, New York, 1996.
[13] Q. Shi and C. Peng, Wellposedness for semirelativistic Schrödinger equation with power-type nonlinearity, *Nonlinear Anal.*, **178** (2019), 133–144.

Received April 2021; 1st revision August 2021; 2nd revision August 2021; early access October 2021.

*E-mail address: huh@cau.ac.kr*