Global solutions and general decay for the dispersive wave equation with memory and source terms

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Received: 15 July 2020; Accepted: 6 November 2020; Published: 8 November 2020.

Abstract: This paper concerns with the global solutions and general decay to an initial-boundary value problem of the dispersive wave equation with memory and source terms.

Keywords: Dispersive wave equation, small data global solution, general decay.

MSC: 35L35, 35L82, 35B40.

1. Introduction

This paper deals with the initial boundary value problem of the dispersive wave equation with memory and source terms

\[ u_{tt} - \Delta u + \alpha \Delta^2 u - \int_0^t g(t - \tau) \Delta^2 u(\tau) d\tau + u_t = |u|^{p-1}u, \quad x \in \Omega, \ t > 0, \]  

where \( \Omega \) is a bounded domain in \( \mathbb{R}^d \) \( (d \geq 1) \) with a smooth boundary \( \partial \Omega \), \( \alpha \) is a positive constant and \( g(t) \) is a positive function that represents the kernel of the memory term, which will be specified in Section 2. Here, we understand \( \Delta^2 u \) to be the dispersive term. In the absence of the viscoelastic term and the dispersive term (that is, if \( g = \alpha = 0 \)), the model (1) reduces to the weakly damped wave equation

\[ u_{tt} - \Delta u + u_t = |u|^{p-1}u, \quad x \in \Omega, \ t > 0. \]  

The interaction between the weak damping term and the source term are considered by many authors. We refer the reader to, Haraux and Zuazua [1], Ikehata [2] and Levine [3,4]. If \( \alpha = 0 \) and \( g \) is not trivial on \( \mathbb{R} \), but replacing the fourth order memory term in (1) by a weaker memory of the form \( \int_0^t g(t - \tau) \Delta u(\tau) d\tau \), then (1) can be rewritten as follows

\[ u_{tt} - \Delta u + \int_0^t g(t - \tau) \Delta u(\tau) d\tau + u_t = |u|^{p-1}u, \quad x \in \Omega, \ t > 0, \]  

The Equation (3) has been considered by Wang et al [5]. Under some appropriate assumptions on \( g \), by introducing potential wells they obtained the existence of global solution and the explicit exponential energy decay estimates. Our main goal in the present paper is to discuss the global solutions and general decay to the following weakly damped wave equation with dispersive term, the fourth order memory term and the nonlinear source term

\[ u_{tt} - \Delta u + \Delta^2 u - \int_0^t g(t - \tau) \Delta^2 u(\tau) d\tau + u_t = |u|^{p-1}u \quad \text{in} \ \Omega \times \mathbb{R}^+, \]  

with simply supported boundary condition

\[ u = 0, \quad \frac{\partial u}{\partial v} = 0 \quad \text{on} \ \partial \Omega \times \mathbb{R}^+. \]
Remark 1. The relaxation function \( g(t) \) is a positive function that represents the kernel of the memory term, which will be specified in Section 2. We prove that Problem (4)-(6) has a global weak solution assuming small initial data.

2. Preliminaries

Before proceeding to our analysis, we use the following abbreviations \( || \cdot ||_q = || \cdot ||_{L^q(\Omega)} \) denotes usual \( L^q \) norm, \((\cdot, \cdot)\) denotes the \( L^2 \)-inner product, and consider the Sobolev spaces \( H^0_0(\Omega) \) and \( H^2_0(\Omega) \) with their usual scalar products and norms. We also use the embedding \( H^1_0(\Omega) \hookrightarrow L^q(\Omega) \) for \( 2 < q < \frac{2d}{d-2} \) if \( d \geq 3 \) or \( 2 < q < \infty \) if \( d = 1, 2 \). In this case, the embedding constant is denoted by \( C_* \), that is \( ||u||_q \leq C_* ||\nabla u||_2 \).

We define the polynomial \( Q \) by \( Q(z) = \frac{1}{2}z^2 - \frac{c_1}{p-1}z^{p-1} \), which is increasing in \( [0, z_0] \), where \( z_0 = C_1^{\frac{1}{p-1}} \) is its unique local maximum. Next, we give the assumptions for Problem (4)-(6).

\( \text{G1} \) The relaxation function \( g : \mathbb{R} \to \mathbb{R} \) is a bounded \( C^1 \) function such that \( g(0) > 0 \), \( 0 < \eta = 1 - \int_0^{\infty} g(\tau) d\tau \leq 1 - \int_0^{\infty} \eta(\tau) d\tau = \eta(t) \).

\( \text{G2} \) There exist positive constants \( \xi_1 \) and \( \xi_2 \) such that \( -\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t) \) \( \forall t \geq 0 \).

\( \text{G3} \) We also assume that \( 1 < p \leq \frac{d+2}{d-2} \) if \( d \geq 3 \) and \( p > 1 \) if \( d = 1, 2 \). where \( \lambda_1 \) is the first eigenvalue of the following problem

\[
\Delta^2 u = \lambda_1 u \quad \text{in } \Omega, \quad u = \frac{\partial u}{\partial \nu} = 0 \quad \text{in } \partial \Omega.
\]

(7)

Remark 1. [7] Assuming \( \lambda_1 \) is the first eigenvalue of the problem (7), we have

\[
||\Delta u||^2_2 \geq \lambda_1 ||\nabla u||^2_2.
\]

(8)

Now, we define the following energy function associated with a solution \( u \) of the Problem (4)-(6)

\[
E(t) = \frac{1}{2} ||u(t)||^2_2 + \frac{1}{2} \int_0^t \left( -\int_0^{\tau} g(\tau') d\tau' \right) ||\Delta u(t') ||^2_2 + \frac{1}{2} ||\nabla u(t)||^2_2 + \frac{1}{2} \left( g \circ \Delta u \right)(t) - \frac{1}{p+1} ||u||^p_{p+1}
\]

(9)

for \( u \in H^2_0(\Omega) \), and

\[
E(0) = \frac{1}{2} ||u_0||^2_2 + \frac{1}{2} ||\nabla u_0||^2_2 + \frac{1}{2} ||\Delta u_0||^2_2 - \frac{1}{p+1} ||u_0||^p_{p+1}
\]

(10)

is the initial total energy. To facilitate further on our analysis, we use the following notation

\[
(g \circ \Delta u)(t) = \int_0^t g(t-\tau) ||\Delta u(t) - \Delta u(\tau)||^2_2 d\tau.
\]

Now, we are in a position to state our main results.

3. Main results

Theorem 1. Assume that \( \text{G1} \) - \( \text{G3} \) hold, \( u_0 \in H^2(\Omega) \), \( u_1 \in L^2(\Omega) \). Further assume that \( ||\nabla u_0||^2_2 < z_0 \) and \( E(0) < Q(z_0) \), then the Problem (4)-(6) possesses a global weak solution satisfying \( u \in L^\infty(0, \infty; H^1_0(\Omega)) \), \( u_t \in L^\infty(0, \infty; L^2(\Omega)) \) for \( 0 \leq t < \infty \), and the energy identity

\[
E(t) + \int_0^t ||u_t(\tau)||^2_2 d\tau - \frac{1}{2} \int_0^t (g \circ \Delta u)(\tau) d\tau + \frac{1}{2} \int_0^t g(\tau) ||\Delta u(\tau)||^2_2 d\tau = E(0),
\]

(11)

holds for \( 0 \leq t < \infty \). Moreover, for \( \xi : \mathbb{R}^+ \to \mathbb{R}^+ \) a increasing \( C^2 \) function satisfying

\[
\xi(0) = 0, \quad \xi(t) > 0, \quad \lim_{t \to +\infty} \xi(t) = +\infty, \quad \xi_{tt}(t) < 0 \quad \forall t \geq 0,
\]

(12)
and, if \( \|g\|_{\ell^1(0,\infty)} \) is sufficiently small, we have for \( \kappa > 0 \); \( E(t) \leq E(0)e^{-\kappa t} \), \( \forall t \geq 0 \).

**Remark 2.** From (11) and (G2), we can easily obtain

\[
\frac{d}{dt} E(t) = -\|u_t(t)\|^2 + \frac{1}{2}(g' \circ \Delta u)(t) - \frac{1}{2}g(t)\|\Delta u(t)\|^2 \leq -\|u_t(t)\|^2 - \frac{1}{2}g_2(g \circ \Delta u)(t) - \frac{1}{2}g(t)\|\Delta u(t)\|^2 \leq 0.
\]

(13)

**Remark 3.** For \( \zeta(t) = t + \frac{1}{1+t} \), we can get the exponential decay rate \( E(t) \leq E(0)e^{-\kappa t} \), \( \forall t \geq 0 \). For \( \zeta(t) = \ln(1+t) \), we can get polynomial decay rate \( E(t) \leq E(0)(1+t)^{-\kappa} \), \( \forall t \geq 0 \).

4. Proof of main results

In this section, we shall divide the proof into two steps. In Step 1, we prove the global existence of weak solutions by using Galerkin’s approximations. In Step 2, we establish the general decay of energy employing the method used in [6].

**Step 1 Global existence of weak solutions**

Let \( \{\omega_j\}_{j=1}^{\infty} \) be an orthogonal basis of \( H_0^2(\Omega) \) with \( \omega_j \) being the eigenfunction of the problem \(-\Delta \omega_j = \lambda_j \omega_j, \ x \in \Omega, \ \omega_j = 0, \ x \in \partial \Omega \). Let \( V^n = \text{Span} \{\omega_1, \omega_2, \cdots, \omega_n\} \). By the standard method of ODE, we know that \( u^n(t) = \sum_{j=1}^{n} b_j(t)\omega_j(x) \) of the Cauchy problem as follows

\[
\int_\Omega \nabla u^n \cdot \nabla \omega dx + \int_\Omega \Delta u^n \cdot \Delta \omega dx - \int_0^t g(t-\tau) \int_\Omega \Delta u^n(\tau) \cdot \Delta \omega dx d\tau
\]

\[
+ \int_\Omega u^n \omega dx - \int_\Omega |u^n|^{p-1} u^n \omega dx = 0, \quad (14)
\]

\( u^n(0) = u_0^n \rightarrow u_0, \ \text{in} \ H_0^2(\Omega), \ u^n(0) = u_1^n \rightarrow u_1 \ \text{in} \ L^2(\Omega). \)

(15)

By the standard theory of ODE system, we prove the existence of solutions of Problem (14)-(15) on some interval \([0,t_0], 0 < t_0 < T \) for arbitrary \( T > 0 \), then, this solution can be extended to the whole interval \([0,T]\) using the first estimate given below. Taking \( \omega = u^n_t \) in (14), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|u^n_t\|^2 + \frac{1}{2} \frac{d}{dt} \|\nabla u^n\|^2 + \frac{1}{2} \frac{d}{dt} \|\Delta u^n\|^2 - \frac{1}{p+1} \frac{d}{dt} \|u^n_t\|^{p+1} + \|u^n_t\|_2^2
\]

\[
- \int_0^t g(t-\tau) \int_\Omega \Delta u^n(\tau) \cdot \Delta u^n(t) dx d\tau = 0. \quad (16)
\]

For the last term on the left hand side of (16) we have

\[
- \int_0^t g(t-\tau) \int_\Omega \Delta u^n(\tau) \cdot \Delta u^n(t) dx d\tau = \frac{1}{2} \frac{d}{dt} (g \circ \Delta u^n)(t) - \frac{1}{2} \frac{d}{dt} (\int_0^t g(\tau)d\tau) \|\Delta u^n(t)\|^2
\]

\[
- \frac{1}{2} (g' \circ \Delta u^n)(t) + \frac{1}{2} g(t) \|\Delta u^n(t)\|^2. \quad (17)
\]

Inserting (17) into (16) and integrating over \([0,t] \subset [0,T]\), we obtain

\[
\frac{1}{2} \|u^n_t\|^2 + \frac{1}{2} \int_\Omega \nabla u^n \cdot \nabla u^n dx + \frac{1}{2} \|\Delta u^n\|^2 - \frac{1}{p+1} \|u^n_t\|^{p+1} + \int_0^t \|u^n_t(\tau)\|^2 d\tau + \frac{1}{2} \int_0^t (g' \circ \Delta u^n)(t) d\tau
\]

\[
- \frac{1}{2} \int_0^t (g' \circ \Delta u^n)(t) d\tau + \frac{1}{2} \int_0^t g(t) \|\Delta u^n(t)\|^2 dx = E^n(0). \quad (18)
\]

Now from assumption (G3) and the Sobolev embedding, we have that

\[
\|u^n\|^{p+1}_{p+1} \leq C_{p+1}^{p+1} \|\nabla u^n\|^2_{2}, \quad (19)
\]
and then we have
\[
\frac{1}{2} \| u_n^\gamma \|_2^2 + \frac{\eta(t)}{2} \| \Delta u_n(t) \|_2^2 + Q(\| \nabla u_n \|_2^2) + \int_0^t \| u_n^\gamma (\tau) \|_2^2 d\tau + \frac{1}{2} (g \circ \Delta u_n)(t).
\]
\[
-\frac{1}{2} \int_0^t (g' \circ \Delta u_n)(\tau)d\tau + \frac{1}{2} \int_0^t g(\tau) \| \Delta u_n(\tau) \|_2^2 d\tau \leq E^n(0).
\] (20)

By using the fact that \(- \int_0^t (g' \circ \Delta u_n)(\tau)d\tau + \int_0^t g(\tau) \| \Delta u_n(\tau) \|_2^2 d\tau \geq 0,\) estimate (20) yields
\[
\frac{1}{2} \| u_n^\gamma \|_2^2 + \frac{\eta(t)}{2} \| \Delta u_n(t) \|_2^2 + \frac{1}{2} (g \circ \Delta u_n)(t) + \int_0^t \| u_n^\gamma (\tau) \|_2^2 d\tau \leq E^n(0).
\] (21)

From \(E(0) < Q(z_0)\) and (15), it follows that
\[
E^n(0) < Q(z_0),
\] (22)
for sufficiently large \(n.\) We claim that there exists an integer \(N\) such that
\[
\| \nabla u_n(t) \|_2 < z_0 \quad \forall t \in [0, t_n] \quad n > N.
\] (23)

Suppose the claim is proved. Then \(Q(\| \nabla u_n \|_2^2) \geq 0\) and from (21) and (22),
\[
\frac{1}{2} \| u_n^\gamma \|_2^2 + \frac{\eta(t)}{2} \| \Delta u_n(t) \|_2^2 + \frac{1}{2} (g \circ \Delta u_n)(t) + \int_0^t \| u_n^\gamma (\tau) \|_2^2 d\tau \leq E^n(0) < Q(z_0)
\] (24)
for sufficiently large \(n\) and \(0 \leq t < \infty.\)

**Proof of the claim.** Suppose that (23) false. Then for each \(n > N,\) there exists \(t \in [0, t_n]\) such that \(\| \nabla u_n(t) \|_2 \geq z_0.\) We note that from \(\| \nabla u_0 \|_2 < z_0 \) and (15) there exists \(N_0\) such that \(\| \nabla u_n(0) \|_2 < z_0 \quad \forall n > N_0.\) Then by continuity there exists a first \(t_n^* \in [0, t_n]\) such that
\[
\| \nabla u_n(t_n^*) \|_2 = z_0,
\] (25)
from where \(Q(\| \nabla u_n(t) \|_2) \geq 0 \quad \forall t \in [0, t_n^*].\) Now from \(E(0) < Q(z_0)\) and (24), there exists \(N > N_0\) and \(n \in (0, z_0)\) such that \(0 \leq \frac{1}{2} \| u_n^\gamma(t) \|_2^2 + \frac{\eta(t)}{2} \| \Delta u_n(t) \|_2^2 + \frac{1}{2} (g \circ \Delta u_n)(t) + Q(\| \nabla u_n(t) \|_2^2) \leq Q(\gamma) \quad \forall t \in [0, t_n^*] \quad \forall n > N.\) Then the monotonicity of \(Q\) in \([0, z_0]\) implies that \(0 \leq \| \nabla u_n(t) \|_2^2 \leq \gamma < z_0 \quad \forall t \in [0, t_n^*],\) and in particular, \(\| \nabla u_n(t) \|_2^2 < z_0,\) which is a contradiction to (24). From (24), we have
\[
\| \Delta u_n \|_2^2 < \frac{2Q(z_0)}{\eta}, \quad 0 \leq t < \infty,
\] (26)
\[
\| u_n^\gamma \|_2^2 < 2Q(z_0), \quad 0 \leq t < \infty,
\] (27)
\[
\int_0^t \| u_n^\gamma (\tau) \|^2_2 d\tau < Q(z_0), \quad 0 \leq t < \infty.
\] (28)

Using Sobolev inequality, (8) and (26), it follows that
\[
\| u_n \|_{p+1} \leq C_{\epsilon} \| \nabla u_n \|_2^2 \leq C_{\epsilon}^2 \lambda_1^{-1} \| \Delta u_n \|_2^2 < \frac{2C_{\epsilon}^2 \lambda_1^{-1} Q(z_0)}{\eta}, \quad 0 \leq t < \infty.
\] (29)

Furthermore, by (29), we get
\[
\| (u_n^p - u_n^p, u_n^p) \| \leq \| u_n^p \|_{p+1} < C_{\epsilon}^{p+1} \left( \frac{2C_{\epsilon}^2 \lambda_1^{-1} Q(z_0)}{\eta} \right)^{\frac{p+1}{2}}, \quad 0 \leq t < \infty.
\] (30)

The estimates (26)-(30) permit us to obtain a subsequences of \(\{ u_n \}\) which from now on will be also denoted by \(\{ u_n \}\) and functions \(u, \chi\) such that
\[ u_n \to u \text{ weak star in } L^\infty(0,\infty; H^0_0(\Omega)), \quad n \to +\infty, \tag{31} \]
\[ u^n_t \to u_t \text{ weak star in } L^\infty(0,\infty; L^2(\Omega)), \quad n \to +\infty, \tag{32} \]
\[ |u^n|^{p-1}u^n \to \chi \text{ weak star in } L^\infty(0,\infty; L^{\frac{p+1}{p}}(\Omega)), \quad n \to +\infty. \tag{33} \]

Besides, from Lions-Aubin Lemma we also have
\[ u^n \to u \text{ strongly in } L^2(0,\infty; L^2(\Omega)), \quad n \to +\infty, \tag{34} \]
and consequently, making use of the Lemma 1.3 in [8], we deduce
\[ |u^n|^{p-1}u^n \to \chi = |u|^{p-1}u \text{ weak star in } L^\infty(0,\infty; L^{\frac{p+1}{p}}(\Omega)), \quad n \to +\infty. \tag{35} \]

Thus, we obtain that \( u \) is a global weak of problem (4)-(6). Next, we shall prove that \( u \) satisfies (11). From the discussion above, we obtain for each fixed \( t > 0 \) that
\[ \lim_{n \to +\infty} (g \circ \Delta u^n)(t) = (g \circ \Delta u)(t), \quad \lim_{n \to +\infty} \|u^n\|^{p+1}_{p+1} = \|u\|^{p+1}_{p+1}. \tag{36} \]

We obtain for each fixed \( t > 0 \) that
\[
\begin{align*}
|(g \circ \Delta u)(t) - (g \circ \Delta u^n)(t)| & = \left| \int_0^t g(t-\tau)||\Delta u(\tau) - \Delta u^n(\tau)||_2 d\tau - \int_0^t g(t-\tau)||\Delta u^n(\tau) - \Delta u^n(t)||_2 d\tau \right| \\
& \leq \int_0^t g(t-\tau)||\Delta u(\tau) - \Delta u^n(\tau)||_2 ||\Delta u(t) + \Delta u^n(t)||_2 d\tau \\
& \quad + \int_0^t g(t-\tau)||\Delta u^n(\tau) - \Delta u^n(t)||_2 ||\Delta u(t) - \Delta u^n(t)||_2 d\tau \\
& \quad + \int_0^t g(t-\tau)||\Delta u(\tau) + \Delta u^n(\tau)||_2 ||\Delta u(t) - \Delta u^n(t)||_2 d\tau \\
& \leq C \int_0^t g(t-\tau)||\Delta u(\tau) - \Delta u^n(\tau)||_2 d\tau + C \int_0^t g(\tau)d\tau ||\Delta u(t) - \Delta u^n(t)||_2 \\
& \to 0,
\end{align*}
\tag{37}
\]
as \( n \to +\infty \), and
\[
\begin{align*}
\|u^n\|^{p+1}_{p+1} - \|u\|^{p+1}_{p+1} & \leq (p+1) \left| \int_\Omega |u + \theta_n u^n|^{p-1} (u + \theta_n u^n)(u^n - u) dx \right| \\
& \leq (p+1) \|u + \theta_n u^n\|^{p+1}_{p+1} \|u^n - u\|_{p+1} \leq C \|u^n - u\|_{p+1} \to 0,
\end{align*}
\tag{38}
\]
as \( n \to +\infty \), where \( 0 < \theta_n < 1 \). Hence, we have
\[
\lim_{n \to +\infty} (g \circ \Delta u^n)(t) = (g \circ \Delta u)(t), \quad \lim_{n \to +\infty} \|u^n\|^{p+1}_{p+1} = \|u\|^{p+1}_{p+1}. \tag{39}
\]

From (15), it follows that \( E^n(0) \to E(0) \) as \( n \to +\infty \). Finally, taking \( n \to +\infty \) in (18), we deduce that the energy identity (11) holds for \( 0 \leq t < \infty \). □

Step 2 General decay of the energy

Firstly, we state several Lemmas to prove the decay rate estimate of the energy.

**Lemma 1.** Let \( u \in L^\infty(0,\infty; H^0_0(\Omega)) \) be the solution of (4)-(6) and \( E(0) < Q(z_0), \|\nabla u_0\|_2 < z_0 \), then we have
\[ 0 \leq E(t) \leq \frac{1}{2} \|u_t\|^2_2 + C_1 \|\Delta u\|^2_2 + \frac{1}{2} (g \circ \Delta u)(t), \tag{40} \]
where \( C_1 = \frac{1}{2} + (2\lambda_1)^{-1} \).

**Proof.** From \( E(0) < Q(z_0) \) and \( \|\nabla u_0\|_2 < z_0 \), we can obtain \( Q(\|\nabla u(t)\|_2) \geq 0 \) for \( 0 \leq t < \infty \). Thus we have

\[
E(t) = \frac{1}{2} \|u_t\|^2_2 + \frac{1}{2} \left( 1 - \int_0^t g(\tau) d\tau \right) \|\Delta u\|^2_2 + \frac{1}{2} \|u\|^2_2 - \frac{1}{p+1} \|u\|^{p+1}_{p+1}
\]

\[
\geq \frac{1}{2} \|u_t\|^2_2 + \frac{1}{2} \|\Delta u\|^2_2 + \frac{1}{2} \|g \circ \Delta u\|_2 + \frac{1}{2} \|u\|^2_2 \geq 0,
\]

and

\[
E(t) \leq \frac{1}{2} \|u_t\|^2_2 + \frac{1}{2} \|\Delta u\|^2_2 + \frac{1}{2} \|g \circ \Delta u\|_2 + \frac{1}{2} \|u\|^2_2 \leq \|u_t\|^2_2 + C_1 \|\Delta u\|^2_2 + \frac{1}{2} \|g \circ \Delta u\|_2.
\]

\( \square \)

**Lemma 2.** The energy \( E(t) \) satisfies

\[
\frac{dE(t)}{dt} \leq - \|u_t(t)\|^2_2 - \frac{1}{2} \xi_2 (g \circ \Delta u)(t) - \frac{1}{2} \left[ g(0) - \xi_1 \|g\|_{L^1(0,\infty)} \right] \|\Delta u(t)\|^2_2 \forall t \geq 0.
\]

**Proof.** From (13), we have

\[
\frac{dE(t)}{dt} \leq - \|u_t(t)\|^2_2 - \frac{1}{2} \xi_2 (g \circ \Delta u)(t) - \frac{1}{2} \|g\|_{L^1(0,\infty)} \|\Delta u(t)\|^2_2.
\]

From assumptions (G2) and since \( \int_0^t g'(\tau) d\tau = g(t) - g(0) \), we obtain

\[
- \frac{1}{2} \xi_2 (g \circ \Delta u)(t) = - \frac{1}{2} \xi_2 g(0) \|\Delta u(t)\|^2_2 - \frac{1}{2} \left[ \int_0^t g'(\tau) d\tau \right] \|\Delta u(t)\|^2_2
\]

\[
\leq - \frac{1}{2} \xi_2 g(0) \|\Delta u(t)\|^2_2 + \frac{1}{2} \xi_2 \|g\|_{L^1(0,\infty)} \|\Delta u(t)\|^2_2
\]

\[
= - \frac{1}{2} \left[ g(0) - \xi_1 \|g\|_{L^1(0,\infty)} \right] \|\Delta u(t)\|^2_2.
\]

Then, Combining (45) and (44) our conclusion holds. Multiplying (43) by \( e^{\xi_1 t} \) (\( \kappa > 0 \)) and using (40), we have

\[
\frac{d}{dt} \left( e^{\xi_1 t} E(t) \right) \leq - \|u_t(t)\|^2_2 e^{\xi_1 t} E(t) - \frac{1}{2} \xi_2 (g \circ \Delta u)(t) e^{\xi_1 t} E(t)
\]

\[
- \frac{1}{2} \left[ g(0) - \xi_1 \|g\|_{L^1(0,\infty)} \right] \|\Delta u(t)\|^2_2 e^{\xi_1 t} E(t) \Delta u(t) E(t) + \kappa \xi_1 (t) e^{\xi_1 t} E(t)
\]

\[
\leq - \frac{1}{2} \left[ 2 - \kappa \xi_1(t) \right] \|u_t(t)\|^2_2 e^{\xi_1 t} E(t) - \frac{1}{2} \left[ \xi_2 - \kappa \xi_1(t) \right] (g \circ \Delta u)(t) e^{\xi_1 t} E(t)
\]

\[
- \frac{1}{2} \left[ g(0) - \xi_1 \|g\|_{L^1(0,\infty)} - 2C_1 \kappa \xi_1(t) \right] \|\Delta u(t)\|^2_2 e^{\xi_1 t} E(t).
\]

Using the fact that \( \xi_1(t) \) is decreasing by (12), we arrive at

\[
\frac{d}{dt} \left( e^{\xi_1 t} E(t) \right) \leq - \frac{1}{2} \left[ 2 - \kappa \xi_1(0) \right] \|u_t(t)\|^2_2 e^{\xi_1(t)} E(t) - \frac{1}{2} \left[ \xi_2 - \kappa \xi_1(0) \right] (g \circ \Delta u)(t) e^{\xi_1(t)} E(t)
\]

\[
- \frac{1}{2} \left[ g(0) - \xi_1 \|g\|_{L^1(0,\infty)} - 2C_1 \kappa \xi_1(t) \right] \|\Delta u(t)\|^2_2 e^{\xi_1(t)} E(t).
\]

Choosing \( \|g\|_{L^1(0,\infty)} \) sufficiently small so that \( g(0) - \xi_1 \|g\|_{L^1(0,\infty)} = B > 0 \) and defining \( \kappa_0 = \min \left\{ \frac{\xi_2}{\xi_1(0)}, \frac{\xi_1(0)}{2C_1(0)} \right\} \), we conclude by taking \( \kappa \in (0, \kappa_0) \) in (47) that \( \frac{d}{dt} \left( e^{\xi_1 t} E(t) \right) \leq 0 \), \( t > 0 \). Integrating the above inequality over \( (0, t) \), it follows that \( E(t) \leq E(0) e^{-\kappa_1 t} \), \( t > 0 \). \( \square \)

**Conflicts of Interest:** “The author declares no conflict of interest.”
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