THE COHOMOLOGY OF EXOTIC
2–LOCAL FINITE GROUPS

JELENA GRBIĆ

Abstract. There exist spaces $B\text{Sol}(q)$ which are the classifying
spaces of a family of 2–local finite groups based on certain fusion
system over the Sylow 2–subgroups of $\text{Spin}_7(q)$. In this paper we
calculate the cohomology of $B\text{Sol}(q)$ as an algebra over the Steen-
rod algebra $A_2$. We also provide the calculation of the cohomology
algebra over $A_2$ of the finite group of Lie type $G_2(q)$.

1. Introduction

In 1974, working on the classification of finite groups, Solomon [16]
looked at fusion systems over a finite group $S$, that is, a category whose
objects are the subgroups of $S$ and whose morphisms are monomor-
phisms of groups which include all those induced by conjugation by
elements of $S$. In particular, he considered groups in which all invo-
lutions are conjugate, and such that the centralizer of each involution
contains a normal subgroup isomorphic to $\text{Spin}_7(q)$ with odd index,
where $q$ is an odd prime power. Solomon showed that a group with
such a fusion system does not exist. Later on, in 1996 Benson [1],
inspired by Solomon’s work, constructed a family of spaces that he
informally entitled “2–completed classifying spaces of the nonexistent
Solomon finite groups”, and denoted them by $B\text{Sol}(q)$. Chronologically
first Dwyer and Wilkerson [3] came up with the celebrated discovery
of the exotic finite loop space $\text{DI}(4)$. They constructed a space $B\text{DI}(4)$
which realises the mod 2 Dickson invariants of rank 4, that is,

$$H^\ast(B\text{DI}(4); \mathbb{F}_2) \cong P[u_8, u_{12}, u_{14}, u_{15}],$$

with the action of the Steenrod algebra $A_2$ given by $Sq^4u_8 = u_{12},
Sq^2u_{12} = u_{14}$ and $Sq^1u_{14} = u_{15}$. Soon after the experts became aware
of the existence of self homotopy equivalences $\Psi^q: B\text{DI}(4) \to B\text{DI}(4)$
for each odd prime power $q$, which are analogues of the unstable Adams
operations on classifying spaces of compact Lie groups completed away
from $q$. The first publish proof of this fact appeared in Notbohm’s work

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Benson observed that Dwyer-Wilkerson’s finite loop space $DI(4)$ carries a 2–local structure closely related to Solomon’s fusion systems. For each odd prime power $q$, Benson defined a space $BSol(q)$ as the homotopy pullback space of the diagram

\[ \begin{array}{ccc}
BSol(q) & \to & BDI(4) \\
\downarrow & & \downarrow \\
BDI(4) & \overset{(1,\Psi^q)}{\to} & BDI(4) \times BDI(4).
\end{array} \]

Benson claimed that the space $BSol(q)$ realises the 2–fusion pattern over the Sylow 2–subgroup of $\text{Spin}_7(q)$ studied by Solomon and shown by him not to correspond to any genuine finite group.

In Levi and Oliver have been interested in the fusion patterns studied by Solomon in the context of saturated fusion systems and $p$–local finite groups. They have shown that the fusion system studied by Solomon is saturated in the sense of Puig and that each one of these fusion systems gives rise to a unique 2–local finite group, which they named $\text{Sol}(q)$. Their approach to the subject enabled them to show that if $n|m$, then $\text{Sol}(q^n)$ can be thought of as a subgroup of $\text{Sol}(q^m)$ and that the resulting limiting term, which they denoted by $\text{Sol}(q^{\infty})$, has a classifying space homotopy equivalent to $BDI(4)$. This in turn made it possible to give an alternative definition of the unstable Adams operations $\Psi^q$ on $BDI(4)$ and to show that the space $BSol(q)$ constructed by Benson is homotopy equivalent to the classifying space of the 2–local finite group $\text{Sol}(q)$, for each odd prime power $q$.

The object of this paper is the calculation of the cohomology algebra over the Steenrod algebra $A_2$ of “Solomon’s groups”. We use Benson’s notation, denoting the classifying space of Solomon’s groups by $BSol(q)$. The cohomology of a $p$–local finite group is given in the appropriate sense as the subalgebra of “stable elements” in the cohomology of its Sylow $p$–subgroup. However, as in the case of ordinary finite groups, calculating cohomology by stable elements is quite a difficult task. The fact that $BSol(q)$ is given by a rather simple pullback diagram makes the calculation of the cohomology as a module over the mod 2 Dickson invariants of rank 4, done initially by Benson, straightforward. The difficulties arise when one wants to determine the algebra structure of the cohomology of $BSol(q)$ with $\mathbb{F}_2$ coefficients as an algebra over $A_2$.

The main result of the paper is given by the following theorem.

**Theorem 1.1.** Fix an odd prime power $q$. Then

\[ H^*(BSol(q); \mathbb{F}_2) \cong \mathbb{F}_2[u_8, u_{12}, u_{14}, u_{15}, y_7, y_{11}, y_{13}]/I, \]
where $I$ is the ideal generated by the polynomials
\[
y^2_1 + u_8y^2_7 + u_{15}y_7, \\
y^2_3 + u_{12}y^2_7 + u_{15}y_11, \\
y^2_7 + u_{14}y^2_7 + u_{15}y_{13}.
\]
The action of the Steenrod algebra $A_2$ is determined by
\[
Sq^1(u_8) = u_{12}, \quad Sq^1(u_{12}) = 0, \quad Sq^2(u_{14}) = u_{14}, \quad Sq^1(u_{14}) = u_{15}, \\
Sq^1(y_7) = 0, \quad Sq^4(y_7) = y_{11}, \quad Sq^1(y_{11}) = 0, \quad Sq^2(y_{11}) = y_{13}, \quad Sq^1(y_{13}) = y^2_7
\]
and the Steenrod algebra axioms.

The second page of the Bockstein spectral sequence is given by
\[
B_2(BSol(q)) \cong P[u_8, u_{12}, u_{14}^2] \otimes E[y_7, y_{11}, y^2_7y_{13}].
\]
The higher Bockstein operators are given as follows. Let $k = \nu_2(q^2 - 1)$, which is at least 3. Then
\[
\beta_{k-1}(y_7) = u_8, \quad \beta_k(y_{11}) = u_{12}, \quad \beta_k(y_7u_8u_{12}) = u^2_8u_{12}, \quad \beta_{k-1}(y^2_7y_{13} + \epsilon u^2_8y_{11}) = u^2_{14}
\]
where $\epsilon = 0$ or 1.

As regards the cohomology of the classifying space of the finite group of Lie type $G_2(q)$ with $\mathbb{F}_2$ coefficients, Kleinerman [6] computed its module structure, while Milgram [9] described it as an algebra over the Steenrod algebra. Although Milgram’s work is elementary in a sense, it is quite lengthy and technical. The method that has been used to calculate the cohomology algebra of $BSol(q)$ can be also applied to the calculation of the cohomology algebra over the Steenrod algebra $A_2$ of $BG_2(q)$. As it is based in deeper results (Friedlander’s fibre square) it produces a short and elegant proof that we shall present in the Appendix. The result is as follows.

**Theorem 1.2.** Fix an odd prime power $q$. Then
\[
H^*(BG_2(q); \mathbb{F}_2) \cong \mathbb{F}_2[d_4, d_6, d_7, y_3, y_5]/I
\]
where $I$ is the ideal generated by the polynomials
\[
y^2_3 + y_3d_7 + y^2_5d_4, \\
y^2_7 + y_5d_7 + y^2_3d_6.
\]
The action of $A_2$ is determined by
\[
Sq^1d_4 = 0, \quad Sq^2d_4 = d_6, \quad Sq^1d_6 = d_7, \quad Sq^1y_3 = 0, \quad Sq^2y_3 = y_5, \quad Sq^1y_5 = u^2_3
\]
and the Steenrod algebra axioms.

For the sake of completeness, we state Kleinerman’s result [6] on the higher Bockstein relations in $H^*(BG_2(q))$. The second page of the Bockstein spectral sequence is given by
\[
B_2(BG_2(q)) \cong P[d_4, d_6^2] \otimes E[y_3, y_3y_5].
The higher Bockstein operators are given as follows. Let \( k = \nu_2(q^2 - 1) \). Then
\[
\beta_k(y_3) = d_4 \quad \text{and} \quad \beta_{k-1}(y_3^2y_5) = d_0^2 + \epsilon d_4^3
\]
for \( \epsilon = 0 \) or 1.

Through the rest of the paper, if not specified differently, we work over the field \( \mathbb{F}_2 \) and denote by \( H^*(X) \) the cohomology of a topological space \( X \) with \( \mathbb{F}_2 \) coefficients.

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2. The Cohomology of \( BSol(q) \) as a Module over \( A_2 \)

In this section we apply the Eilenberg–Moore spectral sequence to pullback diagram (1) to derive the cohomology of \( BSol(q) \) as a module over the mod 2 Steenrod algebra \( A_2 \).

Since the homotopy fibre of the diagonal map \( \Delta: X \rightarrow X \times X \) is \( \Omega X \), and \( BSol(q) \) is defined by homotopy pullback diagram (1), the homotopy fibre of \( p: BSol(q) \rightarrow BDI(4) \) is homotopy equivalent to \( BDI(4) \) and there is a map of homotopy fibrations

\[
\begin{array}{c}
\text{DI}(4) \downarrow \downarrow \\
\text{BSol}(q) \downarrow p \\
\text{BDI}(4) \downarrow \Delta \\
\text{BDI}(4) \times \text{BDI}(4)
\end{array}
\]

where \( \text{DI}(4) \) is the loop space \( \Omega BDI(4) \). The following proposition states the \( H^*(BDI(4)) \)–module structure of \( H^*(BSol(q)) \).

**Proposition 2.1.** The mod 2 cohomology of \( BSol(q) \) as a \( P[u_8, u_{12}, u_{14}, u_{15}] \)–module is given as
\[
H^*(BSol(q); \mathbb{F}_2) \cong P[u_8, u_{12}, u_{14}, u_{15}] \otimes E[v_{11}, v_{13}] \otimes P[v_7]/v_7^4.
\]

The action of the Steenrod algebra \( A_2 \) is determined by
\[
\begin{align*}
Sq^2(u_8) &= u_{12}, & Sq^1(u_{12}) &= 0, & Sq^2(u_{12}) &= u_{14}, & Sq^1(u_{14}) &= u_{15}, \\
Sq^1(u_7) &= 0, & Sq^4(v_7) &= v_{11}, & Sq^1(v_{11}) &= 0, & Sq^2(v_{11}) &= v_{13}, & Sq^1(v_{13}) &= v_7^2
\end{align*}
\]
and the Steenrod algebra axioms.
Proof. We look at the Eilenberg–Moore spectral sequence of pullback diagram (1). The above spectral sequence satisfies the hypothesis of Smith’s “big collapse theorem” [14, Theorem II.3.1] and hence collapses at the $E_2$ term. Namely, one obtains
\[ E_\infty \cong E_2 \cong P[u_8, u_{12}, u_{14}, u_{15}] \otimes E[a_7, a_{11}, a_{13}, a_{14}] \]
with $P[u_8, u_{12}, u_{14}, u_{15}] \cong H^*(BDI(4))$ in filtration degree 0 and $a_i$'s in filtration degree $-1$ with the Steenrod algebra action $Sq^1(a_7) = a_{11}, Sq^2(a_{11}) = a_{13}$ and $Sq^1(a_{13}) = a_{14}$. If we take $v_7 \in H^7(BSol(q))$ representing $a_7$ and then $v_{11} = Sq^3(a_7), v_{13} = Sq^2(v_{11})$ and $v_{14} = Sq^1(v_{13})$, then the Adem relations show that $v_{14} = v_2^7$. That proves the proposition. □

We want to make the statement of the last proposition more geometrical. Namely, we want to show that
\[ H^*(BSol(q)) \cong H^*(BDI(4)) \otimes H^*(DI(4)), \]
as $H^*(BDI(4))$–modules and that the action of the Steenrod algebra splits.

By the following proposition, that will have a crucial role later on in determining the algebra structure of $H^*(BSol(q))$, we investigate the interaction of the action of the Steenrod algebra on $H^*(BDI(4))$ and $H^*(DI(4))$ in the category of modules over $A_2$.

**Proposition 2.2.** The factor $H^*(BDI(4))$ is a split summand of $H^*(BSol(q))$ in the sense that the natural map $H^*(BDI(4)) \to H^*(BSol(q))$ has a left inverse in the category of unstable algebras over $A_2$.

**Proof.** From the construction of $DI(4)$ in [3], there is a unique, up to homotopy, map $f: B(Z/2)^4 \to BDI(4)$ such that $f^*: H^*(BDI(4)) \to H^*(B(Z/2)^4)$ as a map of $A_2$ modules, is the inclusion of mod 2 Dickson invariants of rank 4. As mentioned before, $\Psi^q$ is the identity in mod 2 cohomology, so $\Psi^q \circ f \simeq f$ by the uniqueness property of $f$. Thus $(1, \Psi^q) \circ f$ lifts through $\Delta$, that is, $(1, \Psi^q) \circ f \simeq \Delta \circ f$. Now, using the universal property of pullback diagram (2), there is a map $g: B(Z/2)^4 \to BSol(q)$ such that $p \circ g \simeq f$. Thus in cohomology the images of both maps $f^*$ and $g^* \circ p^*$ are the mod 2 Dickson invariants of rank 4 sitting inside $H^*(B(Z/2)^4)$. Therefore the natural map $H^*(BDI(4)) \to H^*(BSol(q))$ has a left inverse in the category of unstable algebras over $A_2$. □

Now consider the path-loop fibration sequence
\[ DI(4) \to PBDI(4) \to BDI(4). \]
Applying the Eilenberg–Moore spectral sequence to this fibration, we obtain the module structure of $H^*(\text{DI}(4))$ that is isomorphic to $E[x_7, x_{11}, x_{13}, x_{14}]$. Since in this case the Eilenberg–Moore spectral sequence is compatible with $\mathcal{A}_2$, one can also determine the action of $\mathcal{A}_2$ as $Sq^4 x_7 = x_{11}$, $Sq^2 x_{11} = x_{13}$ and $Sq^1 x_{13} = x_{14}$. A Steenrod squares manipulation now implies that

$$H^*(\text{DI}(4)) \cong \mathcal{P}[x_7]/x_7^4 \otimes E[v_{11}, v_{13}],$$

as an algebra.

This shows that

$$H^*(B\text{Sol}(q)) \cong H^*(\text{BDI}(4)) \otimes H^*(\text{DI}(4)),$$

as an $H^*(\text{BDI}(4))$–module; that the action of the Steenrod algebra splits and that the cohomology classes $v_i$ in $H^*(B\text{Sol}(q))$ are detected in $H^*(\text{DI}(4))$.

3. THE COHOMOLOGY OF $B\text{Sol}(q)$ AS AN ALGEBRA OVER $\mathcal{A}_2$

This section is devoted to the calculation of the algebra structure of $H^*(B\text{Sol}(q))$. By the previous section,

$$H^*(B\text{Sol}(q)) \cong \mathcal{P}[u_8, u_{12}, u_{14}, u_{15}] \otimes \mathcal{P}[v_7]/(v_7^4) \otimes \mathcal{E}[v_{11}, v_{13}]$$

as modules over $\mathcal{P}[u_8, u_{12}, u_{14}, u_{15}]$. Thus, let $y_7 \in H^*(B\text{Sol}(q))$ denote the unique generator which maps to $v_7 \in H^*(\text{DI}(4))$ under the map induced by the fibre inclusion. Let $y_{11}, y_{13} \in H^*(B\text{Sol}(q))$ denote $Sq^4(y_7)$ and $Sq^2 Sq^4(y_7)$ respectively. The structure given above is also the algebra structure of $H^*(B\text{Sol}(q))$ if and only if the subalgebra of $H^*(B\text{Sol}(q))$ generated by $y_7, y_{11}$ and $y_{13}$ is isomorphic to $\mathcal{P}[v_7]/(v_7^4) \otimes \mathcal{E}[v_{11}, v_{13}]$. As we shall show, whether or not this is the case depends only on a single parameter $A \in \mathbb{F}_2$, the value of which determines all the relations in $H^*(B\text{Sol}(q))$. We shall then proceed by showing that the cohomology algebra cannot split as a tensor product, thus determining the value of $A$ to be 1. The full structure of $H^*(B\text{Sol}(q))$ as an algebra over the Steenrod algebra will then follow at once.

Consider the possible multiplications on $H^*(B\text{Sol}(q))$. Looking at the action of the Steenrod algebra $\mathcal{A}_2$, we have $y_{11}^2 = Sq^8(y_7^2)$. Because of dimensional reasons, $y_{11}^2$ can be presented as

$$y_{11}^2 = Ay_7^2 u_8 + By_7 u_{15} + Cu_8 u_{14}. \tag{3}$$

On the one hand we have $Sq^4 y_{11}^2 = 0$, and on the other hand applying $Sq^2$ to relation (3), we have $Sq^2 y_{11}^2 = Cu_8 u_{15}$. This implies that $C = 0$. Next

$$y_{13}^2 = Sq^4 y_{11}^2 = Ay_7^2 u_{12} + By_{11} u_{15}.$$
Notice that $Sq^1 y_{13} = y_7$ as $y_{13} = Sq^2 Sq^4 y_7$. Hence
\[ y_7^4 = Sq^2 y_{13}^2 = Ay_7^2 u_{14} + By_{13} u_{15} \]
and
\[ 0 = Sq^1 y_7^4 = (A + B)y_7^2 u_{15}. \]
This implies that $A = B$ and so one derives the relations claimed in the main theorem if $A = 1$. Notice also that if $A = 0$, then there is a ring homomorphism $H^*(\text{DI}(4)) \to H^*(B\text{Sol}(q))$ and the composite
\begin{equation}
H^*(B\text{DI}(4)) \otimes H^*(\text{DI}(4)) \to H^*(B\text{Sol}(q)) \otimes^2 \to H^*(B\text{Sol}(q))
\end{equation}
is an isomorphism of vector spaces and a map of algebras. Hence composite (4) is an isomorphism of algebras over the Steenrod algebra. Thus to prove the first part of Theorem 1 (not including the information on the Bockstein spectral sequence) it remains to show the following lemma.

**Lemma 3.1.** The cohomology $H^*(B\text{Sol}(q))$ cannot split as a tensor product of $H^*(B\text{DI}(4))$ and $H^*(\text{DI}(4))$ in the category of algebras over $\mathcal{A}_2$.

**Proof.** Let $V = (\mathbb{Z}/2)^4$, $f : BV \to B\text{DI}(4)$ the maximal elementary abelian 2–subgroup of $\text{DI}(4)$ and $g : BV \to B\text{Sol}(q)$ the lift constructed in the proof of Proposition 2.2.

Applying the functor $\text{Map}(BV, -)$ to the homotopy pullback diagram
\[
\begin{array}{ccc}
B\text{Sol}(q) & \longrightarrow & B\text{DI}(4) \\
\downarrow & & \downarrow \Delta \\
B\text{DI}(4) & \overset{(1, \Psi^q)}{\longrightarrow} & B\text{DI}(4) \times B\text{DI}(4)
\end{array}
\]
and fixing the component of $f$, we obtain the homotopy pullback diagram
\begin{equation}
\begin{array}{ccc}
\text{Map}(BV, B\text{Sol}(q))_F & \longrightarrow & \text{Map}(BV, B\text{DI}(4))_f \\
\downarrow & & \downarrow \Delta \\
\text{Map}(BV, B\text{DI}(4))_f & \overset{\Delta}{\longrightarrow} & \text{Map}(BV, B\text{DI}(4))_f \times^2
\end{array}
\end{equation}
where $F$ stands for all of the connected components of all lifts (up to homotopy) of $f : BV \to B\text{DI}(4)$ to $B\text{Sol}(q)$. It is shown in [8] that there are more than just one faithful representation of an elementary abelian 2–group of rank 4 in Solomon 2–local finite group.

Let us assume that $H^*(B\text{Sol}(q)) \cong H^*(B\text{DI}(4)) \otimes H^*(\text{DI}(4))$ as algebras over the Steenrod algebra. Then there is only one lift $g$ of $f$
contradicting the existence of pullback diagram (5). This is based on
the fact that
\[
\begin{align*}
[BV, BSol(q)] & \cong \text{Hom}_A(H^*(BSol(q)), H^*(BV)) \\
& \cong \text{Hom}_A(H^*(BDI(4))) \otimes H^*(DI(4)), H^*(BV)) \\
& \cong \text{Hom}_A(H^*(BDI(4)), H^*(BV))
\end{align*}
\]
where the first isomorphism is given in [7], and the last isomorphism
follows since \(H^*(DI(4))\) is a finite vector space. Further on, Dwyer and
Wilkerson [3] showed that \(\text{Hom}_A(H^*(BDI(4)), H^*(BV))\) contains only
one non-identity homotopy class. That proves the assertion that there
is at most one lifting \(g\) of \(f\) and finishes the proof of the lemma. \(\square\)

4. The Bockstein Spectral Sequence for \(BSol(q)\)

In this section we describe the Bockstein operators of \(H^*(BSol(q))\).
As it has been seen, only the module structure is needed for the
\(Sq^1\) calculation as the algebra splits the same way as a module over
\(Sq^1\) (Proposition 2.1). Thus the second page of the Bockstein spectral
sequence looks like
\[
B_2(BSol(q)) \cong P[u_8, u_{12}, u_{14}] \otimes E[v_7, v_{11}, v_7^2v_{13}].
\]
A Serre spectral sequence calculation in cohomology with coefficient s
in the 2–adic integers shows that \(v_7, v_{11}, u_8u_{12}v_7\) and \(v_7^2v_{13}\) support
higher Bocksteins.

Notbohm [11] showed that the effect of the Adams operation \(\Psi^q\)
from diagram (2) on \(H^{2n}(BDI(4); \hat{\mathbb{Z}}_2) \otimes \mathbb{Q}\) is multiplication by \(q^n\). By
[11] Proposition 3.3], the Adams operation \(\Psi^q\) is an equivalence and
therefore it induces an isomorphism of the algebra \(H^*(BDI(4); \mathbb{F}_2)\).
For dimensional reasons \((\Psi^q)^*(u_8) = u_8\), so \((\Psi^q)^*\) is the identity on
mod 2 cohomology. Now, apply the Serre spectral sequence with 2–
adic coefficients to the fibration
\[
\begin{align*}
\text{DI}(4) & \longrightarrow BDI(4) \longrightarrow BDI(4) \times BDI(4).
\end{align*}
\]
Notice first that \(\Delta^*\) is epimorphic, thus all classes above the bottom line
of the spectral sequence are annihilated by a differential, that is, \(E_{\infty}^{p,q} = 0\)
for \(q > 0\). Furthermore the classes \(E_2^{p,0}\) that are in the image of some
differential are those that belong to the kernel of \(\Delta^*\). Let \(\rho: \hat{\mathbb{Z}}_2 \longrightarrow \mathbb{F}_2\)
be the reduction mod 2 map. Denote by \(y_i\) classes in \(H^*(\text{DI}(4); \hat{\mathbb{Z}}_2)\)
such that \(\rho(y_i) = v_i\) and by \(\omega_j\) classes in \(H^*(BDI(4); \hat{\mathbb{Z}}_2)\) such that
\(\rho(\omega_j) = u_j\). For degree reasons \(y_7 \in H^*(\text{DI}(4); \hat{\mathbb{Z}}_2)\) transgresses to
\(\omega_7 - \omega_8 \in H^*(BDI(4); \hat{\mathbb{Z}}_2)^{\otimes 2}\). Further on using pullback diagram (2),
the class \(\omega_8 - \omega_8'\), under the effect of the map \((1, \Psi^q)^*\), maps on \((q^4 -
\[ 1) \omega_8 \in H^*(BDI(4); \hat{\mathbb{Z}}_2) \]. Now using the naturality of the Serre spectral sequence, in the homotopy fibration

\[ (7) \quad \text{DI}(4) \longrightarrow B\text{Sol}(q) \longrightarrow B\text{DI}(4) \]

the integral cohomology class \( y_7 \in H^*(\text{DI}(4); \hat{\mathbb{Z}}_2) \) transgresses onto the class \( (q^4 - 1)\omega_8 \in H^*(BDI(4); \hat{\mathbb{Z}}_2) \).

In an analogue way, the class \( y_{11} \) transgresses onto the class \( (q^6 - 1)\omega_{12} \in H^*(BDI(4); \hat{\mathbb{Z}}_2) \).

For degree reasons, in the Serre spectral sequence for fibration \( (3) \) the first non-trivial differential on the class \( y_7\omega_8\omega_{12} = E_2^{20,7} \) is \( d_8 \) so that

\[ d_8(y_7\omega_8\omega_{12}) = d_8(y_7)\omega_8\omega_{12} = \omega_8^2\omega_{12}. \]

Further on using pullback diagram \( (2) \), the class \( \omega_8^2\omega_{12} \) under the effect of the map \( (1,\Psi^q)^* \), maps on \( (q^4 - 1)\omega_8^2\omega_{12} \in H^*(BDI(4); \hat{\mathbb{Z}}_2) \). Therefore in the Bockstein spectral sequence for \( B\text{Sol}(q), \beta_{v_2(q^4 - 1)}(v_7u_8u_{12}) = u_8^2u_{12}. \)

Now we want to know what happens with the integral class \( v_7^2v_{13} \) of \( H^*(\text{DI}(4)) \). Denote by \( y_{27} \) the class in \( H^*(\text{DI}(4); \hat{\mathbb{Z}}_2) \) such that \( \rho(y_{27}) = v_7^2v_{13} \). In the Serre spectral sequence of fibration \( (6) \), the first possibly non-trivial differential on \( y_{27} \) is \( d_{14} \). In particular,

\[ E_2^{14,14} = H^{14}(BDI(4) \times BDI(4); H^{14}(\text{DI}(4))) = H^{14}(BDI(4) \times BDI(4); \mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \]

having two 2–torsion classes \( y_{13}^2(\omega_{14} \otimes 1) \) and \( y_{14}^2(1 \otimes \omega_{14}) \). These classes can be only hit by a differential coming from \( E_{15}^{0,27} = y_{27}\hat{\mathbb{Z}}_2 \) and \( E_{16}^{29,0} \) contains only the extension class \( \omega_{25}\mathbb{Z}_2 = \text{Ext}(\omega_{14}\mathbb{Z}_2, \omega_{14}\mathbb{Z}_2) \) it follows that \( d_{14}(y_{27}) \) is nontrivial and \( d_{15}: E_{15}^{14,14} \longrightarrow E_{15}^{29,0} \) is an isomorphism. The above argument shows that in page 28 we are left with \( 2y_{27}\hat{\mathbb{Z}}_2 \) in \( E_{28}^{0,27} \), hence \( d_{28}(2y_{27}) = \omega_{14}^2 \otimes 1 - 1 \otimes \omega_{14}^2 \).

Using the information obtained in the above paragraphs, we write the Serre spectral sequence for \( \text{DI}(4) \longrightarrow B\text{Sol}(q) \longrightarrow B\text{DI}(4) \) with 2–adic coefficients. The pullback diagram \( (2) \) provides a maps of Serre spectral sequences. First notice that there is a 2–torsion class in \( E_2^{14,14} \) which must survive to \( E_{\infty} \) as it is the only class in total degree 28, and according to the calculation of the Bockstein spectral sequence, there is a 2–torsion class in \( H^{28}(B\text{Sol}(q); \hat{\mathbb{Z}}_2) \). Thus all previous differentials are trivial on \( y_{27} \), and therefore

\[ d_{28}(y_{27}) = \frac{q^{14} - 1}{2}\omega_{14}^2. \]
So there must be \( v_{27} \in H^* (BSol(q); \mathbb{Z}/2) \) with \( \beta_{\nu_2(q^{14} - 1) - 1} (v_{27}) = \omega_1^2 \). It follows that \( v_{27} = v_7^2 a_3 + \epsilon u_2^2 a_1 \), where \( \epsilon = 0 \) or 1.

The following remark is kindly suggested by the referee in order to underline the variety of approaches that could be taken to solve the above problem.

**Remark.** Since we have the transgression \( d_{28} (v_{27}) = \frac{q^{14} - 1}{2} \omega_1^2 \), the method of universal example (construct a fibration of Eilenberg-MacLane spaces detecting the relevant cohomology classes), shows that \( \beta_{\nu_2(q^{14} - 1) - 1} (v_7^2 a_3 + \omega_2^2 a_1) = \omega_1^2 \). In fact, also \( \beta_{\nu_2(q^{14} - 1) - 1} (v_7^2 a_3 + \omega_2^2 a_1) = \omega_1^2 \).

We finish the proof by showing that in \( H^* (BSol(q)) \) for every \( k \), \( d_k (v_7^2) = 0 \). First of all, direct calculation shows that \( d_8 (v_7^2) = 0 \) because \( d_8 \) is an antiderivation. Since \( \omega_{15} \) is a 2–torsion element and \( \Psi^q \) is an isomorphism, it follows that \( \Psi^q (\omega_{15}) = \omega_{15} \). As in the Serre spectral sequence of fibration \( (7) \) with coefficients in \( \mathbb{F}_2 \) the class \( \omega_{15} \) survives, we have \( d_{15} (v_7^2) = 0 \).

Therefore we have determined that in the mod 2 Bockstein spectral sequence the pairs \((v_7^2 u_8), (v_{11}^2, u_{12}), (v_{15} u_8 u_{12}, u_8^2 u_{12}) \) and \((v_7^2 a_3 + \epsilon u_2^2 a_1, u_7^2 a_1) \) are connected by higher Bocksteins operations of orders \( \nu_2 (q^4 - 1), \nu_2 (q^8 - 1), \nu_2 (q^{14} - 1) \) and \( \nu_2 (q^{14} - 1) \), respectively.

5. Appendix

In this Appendix we give the cohomology of the finite group of Lie type \( G_2 (q) \) as an algebra over the Steenrod algebra \( A_2 \) using the methods developed for the cohomology calculation of \( Sol(q) \). The module structure of the cohomology of \( BG_2 (q) \) is well-known (see for example \( [13] \)). As already mentioned in the introduction Milgram calculated the algebra structure of \( H^* (BG_2 (q)) \) using completely different methods to ours.

**Theorem 5.1.** Fix an odd prime power \( q \). Then

\[
H^* (BG_2 (q); \mathbb{F}_2) \cong \mathbb{F}_2 [d_4, d_6, d_7, y_3, y_5] / I
\]

where \( I \) is the ideal generated by the polynomials

\[
y_5^2 + y_3 d_7 + y_4 d_4
\]

\[
y_3^3 + y_5 d_7 + y_3^2 d_6.
\]

The action of \( A_2 \) is determined by

\[
S q^2 d_4 = d_6, \quad S q^1 d_6 = d_7, \quad S q^1 y_3 = 0, \quad S q^2 y_3 = y_5, \quad S q^1 y_5 = u_3.
\]

To specify a finite (untwisted) group of Lie type \( G(q) \) we need to know a compact connected Lie group \( G \) and a finite field \( \mathbb{F}_q \). The following theorem of Quillen \( [13] \) and Friedlander \( [3] \) relates the classifying space of the finite group \( G(q) \) to the classifying space of \( G \).
Theorem 5.2 (Quillen, Friedlander). For every prime $l$ not dividing $q$ there is a homotopy pullback diagram

$$\begin{array}{ccc}
BG(q)_l^\wedge & \rightarrow & BG_l^\wedge \\
\downarrow & & \downarrow \Delta \\
BG_l^\wedge \times BG_l^\wedge & \rightarrow & \Delta
\end{array}$$

Consider the fibration sequence ($\Omega BG_2)_3^\wedge \rightarrow (BG_2(q))_2^\wedge \rightarrow (BG_2)_2^\wedge$. The cohomology $H^*(BG_2)$ is the mod 2 Dickson algebra of rank 3, while the cohomology of the fibre at the prime two is $P[u_3]/u_3^4 \otimes E[u_5]$ where $u_5 = Sq^2 u_3$ (see for example [10, Theorem 6.2]). Applying Smith’s Big Collapse Theorem [14] to the Eilenberg–Moore spectral sequence calculation, we have

$$H^*(BG_2(q)) \cong P[d_4, d_6, d_7] \otimes P[u_3]/u_3^4 \otimes E[u_5]$$

as $P[d_4, d_6, d_7]$–modules, with the action of $A_2$ given by $Sq^2 d_4 = d_6$, $Sq^1 d_6 = d_7$, $Sq^2 u_3 = u_5$, $Sq^1 u_5 = u_5^2$.

Lemma 5.3. The natural map $H^*(BG_2) \rightarrow H^*(BG_2(q))$ has a left inverse in the category of unstable algebras over $A_2$.

Proof. To show that the natural map $H^*(BG_2) \rightarrow H^*(BG_2(q))$ has a left inverse in the category of unstable algebras over $A_2$ we proceed as in the proof of Proposition 2.2.

There is a bijection [7]

$$[B(Z/2)^3, BG_2] \cong \text{Hom}_{A_2}(H^*(BG_2), H^*(B(Z/2)^3))$$

which assigns to the inclusion of the mod 2 Dickson invariants of rank 3 into $H^*((Z/2)^3)$ a geometric map $f: B(Z/2)^3 \rightarrow BG_2$. Now proceed exactly as in the proof of Proposition 2.2 using the pullback diagram

$$\begin{array}{ccc}
BG_2(q) & \rightarrow & BG_2 \\
\downarrow p & & \downarrow \Delta \\
BG_2 & \rightarrow & BG_2 \times BG_2
\end{array}$$

in place of pullback diagram [2].

As we shall show, there are two possible algebra extensions

$$\mathbb{F}_2 \rightarrow H^*(BG_2) \rightarrow H^*(BG_2(q)) \rightarrow H^*(\Omega BG_2) \rightarrow \mathbb{F}_2$$

determined by the value of a single parameter $A \in \mathbb{F}_2$.

Let us consider possible multiplications on $H^*(BG_2(q))$. We denote by $D(3)$ the loop space $\Omega BG_2 \simeq G_2$. Denote by $y_3 \in H^*(BG_2(q))$ the unique generator which maps to $u_3 \in H^*(D(3))$ under the map
induced by the fibre inclusion. Let \( y_5 \in H^*(BG_2(q)) \) denote \( Sq^2(y_3) \). The algebra structure of \( H^*(BG_2(q)) \) splits in the same way as a module structure if and only if the subalgebra of \( H^*(BG_2(q)) \) generated by \( y_3 \) and \( y_5 \) is isomorphic to \( P[u_3]/(u_3^4) \otimes E[u_5] \).

Looking at the action of the Steenrod algebra \( A_2 \), we have \( y^2_5 = Sq_4^1(y_3) \). Because of dimensional reasons, \( y^2_5 \) can be presented as

\[
y^2_5 = Au_3d_7 + By_3^2d_4 + Cd_4d_6.
\]

On the one hand we have \( Sq^1 y^2_5 = 0 \), and on the other hand applying \( Sq^1 \) to relation (9), we have \( Sq^1 y^2_5 = C d_4d_6 \). This implies that \( C = 0 \).

Next \( y^4_3 = Sq^2 y^2_5 = Ay_5d_7 + By_3^2d_6 \) and

\[
0 = Sq^1 y^4_3 = (A + B)y_3^2d_7.
\]

This implies that \( A = B \). Therefore all relations in cohomology of \( G_2(q) \) depend on a single parameter \( A \in \mathbb{F}_2 \). Notice that if \( A = 0 \), then there is a ring homomorphism \( H^*(DI(3)) \to H^*(BG_2(q)) \) and the composite

\[
(10) \quad H^*(BDI(3)) \otimes H^*(DI(3)) \to H^*(BG_2(q))^{\otimes 2} \to H^*(BG_2(q))
\]

is an isomorphism of vector spaces and a map of algebras. Hence composite (10) is an isomorphism of algebras over the Steenrod algebra. The same argument as in the case of \( H^*(BSol(q)) \) in Lemma 3.1 shows that the cohomology algebra cannot be split as the tensor product of \( H^*(BG_2) \) and \( H^*(\Omega BG_2) \) since \( H^*(DI(3)) \not\cong H^*((\mathbb{Z}/2)^3) \). Therefore \( A \) must be 1. This proves the algebra structure of the mod 2 cohomology of \( BG_2(q) \) stated in Theorem 5.1.

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