On universality in penalisation problems with multiplicative weights

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Abstract

We give a general framework for the universality classes of \( \sigma \)-finite measures in penalisation problems with multiplicative weights. We discuss penalisation problems for Brownian motions, Lévy processes and Langevin processes in our framework.

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1 Introduction

For a measure \( \mu \) and a non-negative measurable function \( f \), we write \( \mu[f] \) for the integral \( \int f \, d\mu \).

For a probability space \( (\Omega, \mathcal{F}, P) \) equipped with a filtration \( (\mathcal{F}_s)_{s \geq 0} \), and for a non-negative process \( \Gamma = (\Gamma_t)_{t \geq 0} \) called a weight, we mean by a penalisation a problem of finding a limit probability \( P^\Gamma \) on \( (\Omega, \mathcal{F}) \) called the penalised probability such that

\[
\frac{P[F_s \Gamma_t]}{P[\Gamma_t]} \xrightarrow{t \to \infty} P^\Gamma[F_s]
\]

is satisfied for all \( s \geq 0 \) and all bounded \( \mathcal{F}_s \)-measurable functional \( F_s \). Under the penalised probability \( P^\Gamma \), the process \( (\Gamma_t)_{t \geq 0} \) is prevented from taking small values; this is why Roynette–Vallois–Yor [14] (see also [15]) called this problem the penalisation. Conditioning a process to stay in a domain \( D \) may be regarded as a special case of the penalisation, as we take the weight \( \Gamma_t = 1_{\{\tau_D > t\}} \) where \( \tau_D \) denotes the exit time of \( D \).

Although the penalised probability \( P^\Gamma \) depends upon the weight \( \Gamma \), we can often find a \( \sigma \)-finite measure \( \mathcal{P} \) on \( (\Omega, \mathcal{F}) \) independent of a particular weight such that

\[
P^\Gamma(A) = \frac{\mathcal{P}[\Gamma_t = \infty; A]}{\mathcal{P}[\Gamma_t = \infty]}, \quad A \in \mathcal{F}
\]

holds with a suitable limit \( \Gamma_\infty \) of \( \Gamma_t \) in a certain class of weights \( \Gamma \). In this case we say that \( \Gamma \) belongs to the universality class of \( \mathcal{P} \). The aim of this paper is to gain a clear

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insight into the universality classes in penalisation problems. For this purpose, we confine ourselves to multiplicative weights.

Let \( \{B = (B_t)_{t \geq 0}, W_x\} \) denote the canonical representation of the one-dimensional Brownian motion with \( W_x(B_0 = x) = 1 \) and let \( \mathcal{F}_t^B = \sigma(B_s : s \leq t) \) denote the natural filtration of the coordinate process \( B \). Let \( \tau_D = \inf\{t \geq 0 : B_t = 0\} \) denote the exit time of \( B \) from the non-zero real \( D = \mathbb{R} \setminus \{0\} \). Let \( x \in D \) be fixed. It is then well-known that

\[
W_x[F_s|\tau_D > t] \xrightarrow{t \to \infty} W_x^{\pm 3B}[F_s] = \frac{1}{|x|} W_x[F_s|B_s|_{1\{\tau_D > t\}}] \tag{1.3}
\]

for all bounded \( \mathcal{F}_s^B \)-measurable functional \( F_s \), where \( W_x^{\pm 3B} \) denotes the law of \( \pm \) times 3-dimensional Bessel process starting from \( x \). This conditioning to avoid zero may be regarded as a special case of the penalisation with the weight being given by \( \Gamma_t = 1\{\tau_D > t\} \). Note that \( W_x^{\pm 3B} \) is locally absolutely continuous with respect to \( W_x \), i.e. \( W_x^{\pm 3B}|_{\mathcal{F}_s^B} \) is absolutely continuous with respect to \( W_x|_{\mathcal{F}_s^B} \) for all \( s \geq 0 \). But \( W_x^{\pm 3B} \) and \( W_x \) are mutually singular on \( \mathcal{F}_\infty := \sigma(B) \), because \( W_x^{\pm 3B}(\tau_D = \infty) = W_x(\tau_D < \infty) = 1 \). While the original process \( \{B, W_x\} \) is recurrent, the penalised process \( \{B, W_x^{\pm 3B}\} \) is transient.

Roynette–Vallois–Yor ([13] and [12]) have studied the penalisation problems for the one-dimensional Brownian motion. They determined the penalised probabilities for \( \Gamma_t \) mutually singular on \( \mathcal{F}_s^B \) for all bounded \( \mathcal{F}_s^B \)-measurable functional \( F_s \). Although \( W_0^T \) is locally absolutely continuous with respect to \( W_0 \), the two measures \( W_0^T \) and \( W_0 \) are mutually singular on \( \mathcal{F}_\infty^B \), because \( W_0^T(L_\infty < \infty) = W_0(L_\infty = \infty) = 1 \). While the original process \( \{B, W_0\} \) is recurrent, the penalised process \( \{B, W_0^T\} \) is transient.

Najnudel–Roynette–Yor ([8]) have introduced the \( \sigma \)-finite measure \( \mathcal{W}_0 \) defined by

\[
\mathcal{W}_0 = \int_0^\infty \frac{du}{\sqrt{2\pi u}} \Pi^{(u)} \bullet W_0^{3B}, \tag{1.5}
\]

where \( \Pi^{(u)} \) stands for the law of the Brownian bridge from 0 to 0 of length \( u \), \( W_0^{3B} \) for the law of the symmetrised Bessel process, and \( \bullet \) for the law of the concatenated path of two independent paths. They proved that the penalised probability \( W_0^T \) for any weight \( \Gamma \) in the previous paragraph is absolutely continuous on \( \mathcal{F}_\infty^B \) with respect to \( \mathcal{W}_0 \):

\[
W_0^T[F] = \frac{\mathcal{W}_0[F\Gamma_\infty]}{\mathcal{W}_0[\Gamma_\infty]} \tag{1.6}
\]

for all bounded \( \mathcal{F}_\infty^B \)-measurable functional \( F \). Moreover, if we define \( \mathcal{W}_x(\cdot) = \mathcal{W}_0(x + B \in \cdot) \), we have

\[
W_x^{\pm 3B}[F] = \frac{\mathcal{W}_x[F; \tau_D = \infty]}{\mathcal{W}_x(\tau_D = \infty)} \tag{1.7}
\]
for all $x > 0$ and all bounded $\mathcal{F}_\infty^B$-measurable functional $F$. In other words, all the weights belong to the universality class of $\mathcal{W}_x$.

K.Yano–Y.Yano–Yor [20, 21], Y.Yano [22] and recently Takeda–K.Yano [16] studied the penalisation problems for one-dimensional stable Lévy processes and found out that there are two different universality classes. In this paper, we would like to give a general framework to characterise universality classes, where we will give some new results.

Groeneboom–Jongbloed–Wellner [6] studied the conditioning to stay positive for the Langevin process. Profeta [10] studied penalisation problems with several kinds of weights. In this paper, we shall discuss universality classes for those penalisation problems.

This paper is organized as follows. In Section 2 we develop a general study on penalised probabilities with multiplicative weights. In Section 3 we define the unweighted measures and discuss the subsequent Markov property of them. In Section 4 we state and prove our main theorems on universality classes. In Section 5 we give a general discussion on penalisation problems with multiplicative weights. In Sections 6, 7 and 8, we look at some known results of penalisation problems for Brownian motions, Lévy processes and Langevin processes in our framework. In Section 9 as an appendix, we discuss extension of the transformed probability measures given by local absolute continuity.

## 2 Penalised probability

For a measure $\mu$ and a non-negative measurable function $f$, we write $f \cdot \mu$ for the transformed measure defined by $(f \cdot \mu)(A) = \int_A f d\mu$ for all measurable set $A$. Let $(\mathcal{F}_s)_{s \geq 0}$ be a filtration. For two measures $\mu$ and $\nu$, we say that $\mu$ is locally absolutely continuous with respect to $\nu$ if $\mu|_{\mathcal{F}_s}$ is absolutely continuous with respect to $\nu$. We say the two measures are locally equivalent if they are locally absolutely continuous with respect to each other. For a parameterised family $(\mu_\lambda)_{\lambda}$ of finite measures and a finite measure $\mu$, we say that

$$\lim_{\lambda} \mu_\lambda = \mu \text{ along } (\mathcal{F}_s)_{s \geq 0}$$

if

$$\lim_{\lambda} \mu_\lambda[F_s] = \mu[F_s]$$

holds for all $s \geq 0$ and all bounded measurable functional $F_s$.

Let $S$ be a locally compact separable metric space and let $\mathbb{D}$ denote the space of càdlàg paths from $[0, \infty)$ to $S$. Let $X = (X_t)_{t \geq 0}$ denote the coordinate process: $X_t(\omega) = \omega(t)$ for $t \geq 0$ and $\omega \in \mathbb{D}$. Let $\mathcal{F}_t^X = \sigma(X_s : s \leq t)$ denote the natural filtration of $X$ and set $\mathcal{F}_t = \bigcap_{s \geq 0} \mathcal{F}_s^X$ so that $(\mathcal{F}_t)_{t \geq 0}$ is a right-continuous filtration. We write $\mathcal{F}_\infty = \sigma(\bigcup_{t \geq 0} \mathcal{F}_t) = \sigma(X)$. For $t \geq 0$, let $\theta_t$ denote the shift operator of $\mathbb{D}$: $\theta_t \omega(s) = \omega(t + s)$ for $s \geq 0$.

Let $\{X, \mathcal{F}_\infty, (P_x)_{x \in S}\}$ denote the canonical representation of a strong Markov process taking values in $S$ with respect to the augmented filtration $(\mathcal{G}_t)_{t \geq 0}$ of $(\mathcal{F}_t)_{t \geq 0}$. A process
Γ = (Γ_t)_{t \geq 0} is called a weight if it is a non-negative càdlàg process. A weight Γ is called multiplicative if Γ is adapted to \((\mathcal{F}_t)_{t \geq 0}\) and
\[
Γ_t = Γ_s \cdot (Γ_{t-s} \circ θ_s), \text{P}_x\text{-a.s. for all } 0 \leq s \leq t < \infty \text{ and all } x ∈ S.
\]
(2.3)

Let Γ be a multiplicative weight. Since Γ_0 = Γ_0 \cdot (Γ_0 \circ θ_0) = Γ_0^2, we note that for any x ∈ S we have either P_x(Γ_0 = 1) = 1 or P_x(Γ_0 = 0) = 1. (2.4)

We set
\[
S^Γ = \{x ∈ S : P_x(Γ_0 = 1) = 1\}.
\]
(2.5)

It is easy to see that
\[
τ^Γ := \inf\{t ≥ 0 : X_t /∈ S^Γ\} = \inf\{t ≥ 0 : Γ_t = 0\} \text{P}_x\text{-a.s. for all } x ∈ S,
\]
(2.6)

since \([Γ_0 = 0 \implies Γ_t = 0 \text{ for all } t ≥ t_0]\) because of the multiplicativity.

We introduce the following assumptions:

(A1) There is a Borel function ϕ^Γ on S such that ϕ^Γ > 0 on S^Γ and
\[
P_x[Γ_t ϕ^Γ(X_t)] = ϕ^Γ(x) \text{ for all } x ∈ S \text{ and } t ≥ 0.
\]
(2.7)

(A2) It holds that
\[
P_x[Γ_{e(q)}] → 0 \text{ as } q ↓ 0 \text{ for all } x ∈ S^Γ,
\]
(2.8)

where we abuse P_x for the extended probability measure of P_x supporting a standard exponential variable e independent of \(\mathcal{F}_∞\) and we set e(q) = e/q for q > 0.

Note that, by the dominated convergence theorem, the condition (A2) follows from the following condition:

(A2') It holds that
\[
P_x[Γ_t] → 0 \text{ as } t → \infty \text{ for all } x ∈ S^Γ.
\]
(2.9)

By the multiplicativity, the condition (2.7) is equivalent to the condition that
\[
(Γ_t ϕ^Γ(X_t))_{t ≥ 0} \text{ is a right-continuous } (\mathcal{G}_t)_{t ≥ 0}, P_x\text{-martingale for all } x ∈ S
\]
(2.10)

(for right-continuity, see, e.g., [5, Theorem 5.8]). Under (A1), for x ∈ S^Γ, we may define a probability measure P^Γ_x on \(\mathbb{D}, \mathcal{F}_∞\), which we call the penalised probability of P_x for Γ, by the following (see Section 9):
\[
P^Γ_x |_{\mathcal{F}_t} = \frac{Γ_t ϕ^Γ(X_t)}{ϕ^Γ(x)} \cdot P_x |_{\mathcal{F}_t} \text{ for all } t ≥ 0.
\]
(2.11)

It is then immediate that the penalised process \(\{X, \mathcal{F}_∞, (P^Γ_x)_{x ∈ S}\}\) is a Markov process with respect to \((\mathcal{F}_t)_{t ≥ 0}\).

We write \(\overset{P}{→}\) for convergence in probability. In addition to (A1) and (A2), we also introduce the following assumptions:
(A3) There is a non-negative finite $\mathcal{F}_{\infty}$-measurable functional $\Gamma_{\infty}$ such that

$$P_x^\Gamma \left( \Gamma_t \underset{t \to \infty}{\longrightarrow} \Gamma_{\infty} > 0 \right) = 1 \text{ for all } x \in S^\Gamma. \quad (2.12)$$

Note that in many examples we have (A3) and $P_x(\liminf_{t \to \infty} \Gamma_t = 0) = 1$, which implies that the two measures $P_x^\Gamma$ and $P_x$ are mutually singular on $\mathcal{F}_{\infty}$.

The following is a routine argument.

**Proposition 2.1.** Let $\Gamma$ be a multiplicative weight. Then the following hold.

(i) Under (A1), it holds that

$$P_x^\Gamma (\tau^\Gamma = \infty) = 1 \text{ for all } x \in S^\Gamma. \quad (2.13)$$

(ii) Under (A1), (A2) and (A3), it holds that

$$P_x^\Gamma \left( \varphi^\Gamma(X_t) \underset{t \to \infty}{\longrightarrow} \infty \right) = 1 \text{ for all } x \in S^\Gamma. \quad (2.14)$$

**Proof.** (i) We apply the optional stopping theorem to the $(\mathcal{G}_t)_{t \geq 0}, P_x)$-martingale $M_t := \Gamma_t \varphi^\Gamma(X_t)/\varphi^\Gamma(x)$ (by (A1)) to see that

$$P_x^\Gamma (\tau^\Gamma > t) = P_x \left[ M_t; \tau^\Gamma > t \right] = P_x \left[ M_t \wedge \tau^\Gamma; \tau^\Gamma \leq t \right] = P_x [M_0] - P_x \left[ M_\tau^\Gamma; \tau^\Gamma \leq t \right] = 1, \quad (2.15)$$

which implies that $P_x^\Gamma (\tau^\Gamma = \infty) = 1$.

(ii) Let $0 \leq s \leq t < \infty$ and $A_s \in \mathcal{F}_s$. We then have

$$P_x^\Gamma \left[ \frac{1}{\Gamma_t \varphi^\Gamma(X_t)}; A_s \right] = \frac{1}{\varphi^\Gamma(x)} P_x (A_s, \tau^\Gamma > t) \leq \frac{1}{\varphi^\Gamma(x)} P_x (A_s, \tau^\Gamma > s) = P_x^\Gamma \left[ \frac{1}{\Gamma_s \varphi^\Gamma(X_s)}; A_s \right]. \quad (2.18)$$

This shows that $N_t := 1/\{\Gamma_t \varphi^\Gamma(X_t)\}$ is a non-negative $P_x^\Gamma$-supermartingale with respect to the completed filtration $(\mathcal{F}_t^x)_{t \geq 0}$ of $(\mathcal{F}_t)_{t \geq 0}$, and consequently it converges $P_x^\Gamma$-a.s. as $t \to \infty$ to some random variable $N_{\infty}$. By (A3), we see that

$$\frac{1}{\varphi^\Gamma(x)} = \Gamma_t N_t \underset{t \to \infty}{\longrightarrow} \Gamma_{\infty} N_{\infty} \quad P_x^\Gamma\text{-a.s.,} \quad (2.19)$$

which implies $1/\varphi^\Gamma(X_{e(q)}) \underset{q \downarrow 0}{\longrightarrow} \Gamma_{\infty} N_{\infty}$. Using Fatou’s lemma, we obtain

$$P_x^\Gamma [\Gamma_{\infty} N_{\infty}] \leq \liminf_{q \downarrow 0} P_x^\Gamma \left[ \frac{1}{\varphi^\Gamma(X_{e(q)})} \right] = \frac{1}{\varphi^\Gamma(x)} \lim_{q \downarrow 0} P_x \left[ \Gamma_{e(q)} \right] = 0 \quad (2.20)$$

by (A2). Hence we obtain (2.14). □
3 Subsequent Markov property

Let $\Gamma$ be a multiplicative weight satisfying (A1), (A2) and (A3). For $x \in S^\Gamma$, we may define a measure $\mathcal{P}_x^\Gamma$ on $(\mathbb{D},\mathcal{F}_\infty)$, which we call the unweighted measure of $P_x^\Gamma$, by

$$\mathcal{P}_x^\Gamma = \varphi^\Gamma(x)\Gamma_\infty^{-1} \cdot P_x^\Gamma$$ on $\mathcal{F}_\infty$. (3.1)

Note that $\mathcal{P}_x^\Gamma$ is $\sigma$-finite on $\mathcal{F}_\infty$, because $\mathbb{D} = \bigcup_{n \in \mathbb{N}} \{\Gamma_\infty > 1/n\}$, $\mathcal{P}_x^\Gamma$-a.e. and

$$\mathcal{P}_x^\Gamma(\Gamma_\infty > 1/n) \leq n\varphi^\Gamma(x) < \infty \quad \text{for all } n \in \mathbb{N}. \quad (3.2)$$

The family of the unweighted measures satisfies the following property.

**Theorem 3.1.** Let $\Gamma$ be a multiplicative weight satisfying (A1)-(A3). Then, for any $x \in S^\Gamma$, any non-negative $\mathcal{F}_t$-measurable functional $F_t$ and any non-negative $\mathcal{F}_\infty$-measurable functional $G$, it holds that

$$\mathcal{P}_x^\Gamma[F_t(G \circ \theta_t)] = P_x[F_t; P_{X_t}^\Gamma [G]; \tau^\Gamma > t].$$ (3.3)

**Proof.** By definition of $\mathcal{P}_x^\Gamma$, we have

$$\mathcal{P}_x^\Gamma[(F_t \Gamma_t)((G \Gamma_\infty) \circ \theta_t)] = \mathcal{P}_x^\Gamma[F_t(G \circ \theta_t) \Gamma_\infty] \quad (3.4)$$

$$= \varphi^\Gamma(x) P_x^\Gamma[F_t(G \circ \theta_t)]. \quad (3.5)$$

By the Markov property for $X$ under $P_x^\Gamma$, by the local equivalence between $P_x^\Gamma$ and $P_x$, and by the global equivalence between $P_x^\Gamma$ and $\mathcal{P}_x^\Gamma$, we obtain

$$\mathcal{P}_x^\Gamma[F_t \Gamma_t(G \Gamma_\infty) \circ \theta_t] = P_x[F_t \varphi^\Gamma(X_t) \Gamma_t P_{X_t}^\Gamma [G]] \quad (3.6)$$

$$= P_x[F_t \varphi^\Gamma(X_t) \Gamma_t P_{X_t}^\Gamma [G]] \quad (3.7)$$

$$= P_x[F_t \Gamma_t \mathcal{P}_{X_t}^\Gamma (G \Gamma_\infty)] \quad (3.8)$$

where we used the fact obtained from Proposition 2.1 that $X_t \in S^\Gamma$, $P_x$-a.s. on $\{\Gamma_t > 0\}$. Thus we obtain

$$\mathcal{P}_x^\Gamma[F_t \Gamma_t(G \Gamma_\infty) \circ \theta_t] = P_x[F_t \Gamma_t \mathcal{P}_{X_t}^\Gamma (G \Gamma_\infty)]. \quad (3.9)$$

Replacing $F_t$ by $F_t \Gamma_t^{-1} 1_{\{\tau^\Gamma > t\}}$ and $G$ by $G \Gamma_{\infty}^{-1} 1_{\{\Gamma_\infty > 0\}}$, we obtain the desired identity, since $\tau^\Gamma = \infty$ and $\Gamma_\infty > 0$, $\mathcal{P}_x^\Gamma$-a.e. The proof is now complete. \(\square\)

Theorem 3.1 asserts that, the process under $\mathcal{P}_x^\Gamma$ behaves until a fixed time $t$ as the process under $P_x$ killed upon leaving $S^\Gamma$, and it starts afresh at time $t$ to behave as the process under $\mathcal{P}_{X_t}^\Gamma$. In this sense, we may call this property (3.3) the subsequent Markov property.
4 Universality class

Let $E$ be a particular multiplicative weight satisfying $(A1)$-$(A3)$. We would like to give a sufficient condition for existence of a positive function $c(x)$ such that

$$S^\Gamma \subset S^E \quad \text{and} \quad \mathcal{P}_x^\Gamma = c(x)1_{\{\Gamma > 0\}} \cdot \mathcal{P}_x^E \quad \text{for all} \quad x \in S^\Gamma. \quad (4.1)$$

We note that $[\mathcal{P}_x^\Gamma = c(x)1_{\{\Gamma > 0\}} \cdot \mathcal{P}_x^E]$ yields $[\Gamma \text{ belongs to the universality class of } \mathcal{P}_x^E]$ in the sense we mentioned in Introduction.

Theorem 4.1 (Universality theorem). Let $E$ and $\Gamma$ be two multiplicative weights satisfying $(A1)$-$(A3)$. Suppose there exists a positive function $c(x)$ such that

$$P_x^E \left( \Gamma_t \xrightarrow{t \to \infty} \Gamma_\infty \right) = 1, \quad \frac{\varphi^\Gamma(x_t)}{\varphi^E(x_t)} \xrightarrow{t \to \infty} c(x) \quad \text{for all} \quad x \in S^\Gamma \quad (4.2)$$

and

$$P_x^\Gamma \left( E_t \xrightarrow{t \to \infty} E_\infty > 0 \right) = 1, \quad \frac{\varphi^\Gamma(x_t)}{\varphi^E(x_t)} \xrightarrow{t \to \infty} c(x) \quad \text{for all} \quad x \in S^\Gamma. \quad (4.3)$$

(Notice that these assumptions do not follow from $(A3)$.) Then $(4.1)$ holds.

Proof. Let $x \in S^\Gamma$ be fixed. Since $P_x = P_x^\Gamma$ on $\mathcal{F}_0$, we have

$$P_x(E_0 = 1) = P_x^\Gamma(E_0 = 1) \geq P_x^\Gamma(E_\infty > 0) = 1, \quad (4.4)$$

which shows $x \in S^E$. By the assumptions, we have

$$R_t \xrightarrow{t \to \infty} R_\infty \quad \text{and} \quad R_t \xrightarrow{t \to \infty} R_\infty \quad \text{with} \quad R_t = \frac{\Gamma_t \varphi^\Gamma(x_t)}{\varphi^E(x_t)} \quad \text{and} \quad R_\infty = c(x) \Gamma_\infty. \quad (4.5)$$

Let $s > 0$ and let $F_s$ be a non-negative $\mathcal{F}_s$-measurable functional. For $t > s$, we have

$$P_x^E \left[ F_s \cdot \frac{R_t}{1 + R_t + E_t} \right] = \frac{\varphi^\Gamma(x)}{\varphi^E(x)} P_x^\Gamma \left[ F_s \cdot \frac{R_t}{1 + R_t + E_t} \cdot E_t \varphi^E(x_t) \right] \quad (4.6)$$

$$= \frac{\varphi^\Gamma(x)}{\varphi^E(x)} P_x^\Gamma \left[ F_s \cdot \frac{R_t}{1 + R_t + E_t} \cdot E_t \varphi^E(x) \right] \quad (4.7)$$

$$= \frac{\varphi^\Gamma(x)}{\varphi^E(x)} P_x^\Gamma \left[ F_s \cdot \frac{E_t}{1 + R_t + E_t} \right]. \quad (4.8)$$

Letting $t \to \infty$ and applying the dominated convergence theorem, we obtain

$$P_x^E \left[ F_s \cdot \frac{R_\infty}{1 + R_\infty + E_\infty} \right] = \frac{\varphi^\Gamma(x)}{\varphi^E(x)} P_x^\Gamma \left[ F_s \cdot \frac{E_\infty}{1 + R_\infty + E_\infty} \right]. \quad (4.9)$$

Since $s > 0$ and $F_s$ are arbitrary, we obtain

$$c(x) \Gamma_\infty \cdot P_x^E = E_\infty \cdot P_x^\Gamma, \quad (4.10)$$

which yields

$$c(x)1_{\{\Gamma_\infty > 0\}} \cdot \mathcal{P}_x^E = 1_{\{E_\infty > 0\}} \cdot \mathcal{P}_x^\Gamma = \mathcal{P}_x^\Gamma, \quad (4.11)$$

since $P_x^\Gamma(E_\infty > 0) = 1$. We thus obtain the desired result.
5 Penalties problems

We give two systematic methods of ensuring the conditions (A1) and (A2) in penalisation problems.

5.1 Constant clock

We give a general framework for penalisation problems with constant clock.

**Proposition 5.1.** Let $\Gamma$ be a multiplicative weight. Let $\rho(t)$ be a function such that

\[
\rho(t) \xrightarrow{t \to \infty} \infty \quad \text{and} \quad \frac{\rho(t)}{\rho(t-s)} \xrightarrow{t \to \infty} 1 \quad \text{for all } s > 0,
\]

or in other words, $r(\log t)$ is divergent and slowly varying at $t = \infty$. Suppose there exists a process $(M_s)_{s \geq 0}$ such that $P_x(M_0 > 0) = P_x(\Gamma_0 = 1)$ for all $x \in S$ and

\[
\rho(t)P_x[\Gamma_t | F_s] \xrightarrow{s \to \infty} M_s \quad \text{in } L^1(P_x) \quad \text{for all } x \in S \quad \text{and all } s \geq 0.
\]

Then the weight $\Gamma$ satisfies (A1) and (A2') with

\[
\varphi^\Gamma(x) = \lim_{t \to \infty} \rho(t)P_x[\Gamma_t],
\]

and the following penalisation limit with constant clock holds:

\[
\frac{\Gamma_t \cdot P_x}{P_x[\Gamma_t]} \xrightarrow{t \to \infty} P^\Gamma_x \quad \text{along } (F_s)_{s \geq 0} \quad \text{for all } x \in S^\Gamma.
\]

**Proof.** The convergence (5.2) for $s = 0$ becomes (5.3). By the multiplicativity $\Gamma_t = \Gamma_s \cdot (\Gamma_{t-s} \circ \theta_s)$ and by the Markov property, we have

\[
\rho(t)P_x[\Gamma_t | F_s] = \frac{\rho(t)}{\rho(t-s)}\Gamma_s \cdot \rho(t-s)P_x[\Gamma_{t-s}] \xrightarrow{t \to \infty} \Gamma_s \varphi^\Gamma(X_s) \quad \text{in } P_x\text{-a.s..}
\]

which yields $M_s = \Gamma_s \varphi^\Gamma(X_s)$. Hence we have

\[
P_x[\Gamma_t \varphi^\Gamma(X_t)] = \lim_{u \to \infty} \rho(u)P_x[\Gamma_u [F_t]] = \lim_{u \to \infty} \rho(u)P_x[\Gamma_u] = \varphi^\Gamma(x),
\]

which shows that (A1) is satisfied. As $\rho(t) \to \infty$, we obtain (A2'). For $s > 0$ and for a bounded $F_s$-measurable functional $F_s$, we obtain

\[
\rho(t)P_x[F_s \Gamma_t] = P_x[F_s \rho(t)P_x[\Gamma_t | F_s]] \xrightarrow{t \to \infty} P_x[F_s M_s] = \varphi^\Gamma(x)P^\Gamma_x[F_s].
\]

This shows (5.4).
5.2 Exponential clock

Conditioning and penalisation problems with exponential clock have been widely studied; see [3], [4], [9], [19] and [11]. We give a general framework for them.

**Proposition 5.2.** Let \( r(q) \) be a function defined for small \( q > 0 \) such that \( r(q) \to \infty \) as \( q \downarrow 0 \). We abuse \( P_x \) for the extended probability measure of \( P_x \) supporting a standard exponential variable \( e \) independent of \( (F_t)_{t \geq 0} \) and set \( e(q) = e/q \) for \( q > 0 \). Suppose there exists a process \( (M_s)_{s \geq 0} \) such that \( P_x(M_0 > 0) = P_x(\Gamma_0 = 1) \) for all \( x \in S \) and

\[
\lim_{q \downarrow 0} r(q) P_x[\Gamma_{e(q)} | F_s] = \lim_{q \downarrow 0} r(q) P_x[\Gamma_{e(q)} 1_{\{e(q) > s\}} | F_s] = M_s \quad \text{in } L^1(P_x)
\]

for all \( x \in S \) and all \( s \geq 0 \).

Then the weight \( \Gamma \) satisfies \( (A1) \) and \( (A2) \) with

\[
\varphi^\Gamma(x) = \lim_{q \downarrow 0} r(q) P_x[\Gamma_{e(q)}],
\]

and the following penalisation limit with exponential clock holds:

\[
\lim_{q \downarrow 0} \frac{\Gamma_{e(q)} \cdot P_x}{P_x[\Gamma_{e(q)}]} = \lim_{q \downarrow 0} \frac{\Gamma_{e(q)} 1_{\{e(q) > s\}} \cdot P_x}{P_x[\Gamma_{e(q)}; e(q) > s]} = P^r_x \quad \text{along } (F_s)_{s \geq 0} \text{ for all } x \in S^\Gamma. \tag{5.10}
\]

**Proof.** The convergence (5.8) for \( s = 0 \) becomes (5.9). By the multiplicativity \( \Gamma_t = \Gamma_s \cdot (\Gamma_{t-s} \circ \theta_s) \), by the Markov property and by the memoryless property

\[
P(e(q) - s \in \cdot | e(q) > s) = P(e(q) \in \cdot), \tag{5.11}
\]

we have

\[
r(q) P_x[\Gamma_{e(q)} 1_{\{e(q) > s\}} | F_s] = e^{-qs} r(q) P_x[\Gamma_{e(q) + s} | F_s]
\]

\[
= e^{-qs} \Gamma_s r(q) P_{X_s}[\Gamma_{e(q)}] \xrightarrow{q \downarrow 0} \Gamma_s \varphi^\Gamma(X_s) \quad P_x\text{-a.s.}, \tag{5.13}
\]

which yields \( M_s = \Gamma_s \varphi^\Gamma(X_s) \). Hence we obtain

\[
P_x[\Gamma_t \varphi^\Gamma(X_t)] = P_x[M_t] = \lim_{q \downarrow 0} r(q) P_x[P_x[\Gamma_{e(q)} | F_t]] = \lim_{q \downarrow 0} r(q) P_x[\Gamma_{e(q)}] = \varphi^\Gamma(x), \tag{5.14}
\]

which shows that \( (A1) \) is satisfied. As \( r(q) \to \infty \), we obtain \( (A2) \). For \( s > 0 \) and for a bounded \( F_s \)-measurable functional \( F_s \), we obtain

\[
r(q) P_x[F_s \Gamma_{e(q)}] = P_x[F_s r(q) P_x[\Gamma_{e(q)} | F_s]] \xrightarrow{q \downarrow 0} P_x[F_s M_s] = \varphi^\Gamma(x) P^F_x[F_s]. \tag{5.15}
\]

This shows (5.10). \( \Box \)
6 Brownian penalisation revisited

Let us look at some results of Roynette–Vallois–Yor [13, 12] and Najnudel–Roynette–Yor [8] in our framework.

Let \( \{ B = (B_t)_{t \geq 0}, (W_x)_{x \in \mathbb{R}} \} \) denote the canonical representation of the one-dimensional Brownian motion with \( W_x(B_0 = x) = 1 \). Set \( B_s = \sup_{s \leq t} B_s \) and let \( L_t \) denote the local time of \( B \) at 0. For the shift operator on the path space, we have

\[
B_{t+s} = B_t \circ \theta_s, \quad B_t^\ast = B_t \lor (B_t \circ \theta_s), \quad L_{t+s} = L_s + (L_t \circ \theta_s). \tag{6.1}
\]

For a technical reason, we set

\[
S = \{ (x, y, l) \in \mathbb{R}^3 : y \geq x, \ l \geq 0 \} \tag{6.2}
\]

as the state space and consider the coordinate process \( X = (X_t)_{t \geq 0} = (X^B_t, X^\sup_t, X^R_t)_{t \geq 0} \) on the space of càdlàg paths from \([0, \infty)\) to \( S \). Writing \( a \lor b = \max\{a, b\} \), we define \( P_{(x,y,l)} \) by the law on \( D \) of \( (B, y \lor B, l + L) \) under \( W_x \), and adopt the notation of Section 2. By the identities (6.1), we see that the process \( \{ X, F_\infty, (P_{(x,y,l)})_{(x,y,l) \in S} \} \) is a strong Markov process with respect to the augmented filtration.

(1) Supremum penalisation. For an integrable function \( f : \mathbb{R} \to [0, \infty) \) such that for some \(-\infty < y_0 \leq \infty \) we have \( f(y) > 0 \) for \( y \leq y_0 \) and \( f(y) = 0 \) for \( y > y_0 \), we set

\[
\Gamma^\sup_{x, f} = \frac{f(X^\sup_t)}{f(X^\sup_0)} 1_{X^\sup_0 \leq y_0}, \quad S^\sup_{x, f} = \{ (x, y, l) \in S : y \leq y_0 \}. \tag{6.3}
\]

Then we see that \( S^\sup_{x, f} \) is a multiplicative weight with \( S^{\Gamma^\sup_{x, f}} = S^\sup_{x, f} \) (in what follows we will omit similar remarks). By Roynette–Vallois–Yor [12, Theorem 3.6], we see that all the assumptions of Proposition 5.1 are satisfied with \( \rho(t) = \sqrt{\pi t / 2} \) and

\[
\varphi^\sup_{x, f}(x, y, l) = y - x + \frac{1}{f(y)} \int_y^{y_0} f(u) \, du, \quad (x, y, l) \in S^\sup_{x, f}, \tag{6.4}
\]

so that \( (A1) \) and \( (A2') \) are satisfied. By the discussion of Roynette–Vallois–Yor [12, Subsection 1.4], we can derive that

\[
P^t_{(x,y,l)}(X^\sup_\infty > a) = \frac{\int_y^{y_0} f(u) \, du}{(y - x)f(y) + \int_y^{y_0} f(u) \, du}, \quad y \leq a < \infty, \tag{6.5}
\]

and hence that \( [X^\sup_t = X^\sup_\infty \text{ for large } t] \) and \( [\Gamma^\sup_{x, f} \to \Gamma^\sup_{x, f} > 0] \) \( P^t_{(x,y,l)} \)-a.s., which shows \( (A3) \). By (ii) of Proposition 2.1, we obtain the following known results:

\[
P^t_{(x,y,l)}(X^B_t \to -\infty, \ \frac{\varphi^\sup_{x, f}(X_t)}{|X^B_t|} \to 1) = 1. \tag{6.6}
\]
(2) Local time penalisation. For an integrable function $f : [0, \infty) \to [0, \infty)$ such that for some $0 \leq l_0 \leq \infty$ we have $f(l) > 0$ for $l \leq l_0$ and $f(l) = 0$ for $l > l_0$, we set

$$
\Gamma_{lt}^{f} = \frac{f(X_{lt}^B)}{f(X_{lt}^B)} 1_{\{X_{lt}^B \leq l_0\}}, \quad S_{lt}^{f} = \{(x, y, l) \in S : l \leq l_0\}. \quad (6.7)
$$

By Roynette–Vallois–Yor [12, Theorem 3.13 and Lemma 3.15], we see that all the assumptions of Proposition 5.1 are satisfied with $\rho(t) = \sqrt{\pi t/2}$ and

$$
\varphi_{lt}^{f}(x, y, l) = |x| + \frac{1}{f(l)} \int_{l}^{l_0} f(u)du, \quad (x, y, l) \in S_{lt}^{f},
$$

so that (A1) and (A2') are satisfied. Moreover, (A3) is also satisfied and

$$
P_{(x, y, t)}^{lt, f}(X_{lt}^B \to \pm \infty) = \frac{x^\pm f(l) + \frac{1}{2} \int_{l}^{l_0} f(u)du}{|x| f(l) + \int_{l}^{l_0} f(u)du},
$$

with $x^\pm = \max\{\pm x, 0\}$. It is then obvious that

$$
P_{(x, y, t)}^{lt, f}(|X_{lt}^B| \to \infty, \frac{\varphi_{lt}^{f}(X_{lt}^B)}{|X_{lt}^B|} \to 1) = 1. \quad (6.10)
$$

Note that the conditioning to avoid zero, which we have mentioned in Introduction, can be regarded as a special case of the local time penalisation with the weight $1_{\{X_{lt}^B = 0\}} = \Gamma_{lt}^{f}$ for $f(l) = 1_{\{l=0\}}$.

(3) Kac killing penalisation with integrable potential. For an integrable function $v : \mathbb{R} \to [0, \infty)$ satisfying

$$
0 < \int_{\mathbb{R}} (1 + |x|)v(x)dx < \infty, \quad (6.11)
$$

we set

$$
\Gamma_{t}^{Kac, v} = \exp\left(- \int_{0}^{t} v(X_{s}^B)ds\right), \quad S^{Kac, v} = S. \quad (6.12)
$$

By Roynette–Vallois–Yor [13, Theorem 4.1], we see that all the assumptions of Proposition 5.1 are satisfied with $\rho(t) = \sqrt{\pi t/2}$ and $\varphi_{Kac}^{v}(x, y, l) = \varphi_{v}(x)$ where $\varphi_{v}$ is the unique solution to the Sturm–Liouville equation

$$
\frac{1}{2} \frac{d^2 \varphi_{v}}{dx^2}(x) = v(x)\varphi_{v}(x), \quad \lim_{x \to \pm \infty} \frac{d\varphi_{v}}{dx}(x) = \pm 1.
$$

so that (A1) and (A2') are satisfied. Moreover, (A3) is also satisfied and

$$
P_{(x, y, t)}^{Kac, v}(X_{lt}^B \to -\infty) = \frac{1}{C_{v}} \int_{x}^{\infty} \frac{dy}{\varphi_{v}(y)^2}, \quad P_{(x, y, t)}^{Kac, v}(X_{lt}^B \to \infty) = \frac{1}{C_{v}} \int_{-\infty}^{x} \frac{dy}{\varphi_{v}(y)^2} \quad (6.14)
$$
with \( C_v = \int_{\mathbb{R}} \frac{d\nu}{\varphi_v(y)} \). By (6.13) it is obvious that

\[
P_{(x,y,l)}^{Kac,v} (|X_t^B| \to \infty, \frac{\varphi^{Kac,v}(X_t)}{|X_t^B|} \to 1) = 1.
\] (6.15)

(4) Kac killing penalisation with Heviside potential. For \( \lambda > 0 \), we set

\[
\Gamma_{\text{Hev},\lambda} = \exp \left( -\lambda \int_0^t 1_{\{X_s^B > 0\}} ds \right), \quad S^{Kac,v} = S.
\] (6.16)

By Roynette–Vallois–Yor [13, Theorem 5.1 and Example 5.4], we see that all the assumptions of Proposition 5.1 are satisfied with \( \rho(t) = \sqrt{\pi t/2} \) and

\[
\varphi_{\text{Hev},\lambda}(x, y, l) = \begin{cases} 
\frac{1}{\sqrt{2\lambda}} e^{-\sqrt{2\lambda}x} & (x \geq 0), \\
\frac{1}{\sqrt{2\lambda}} - x & (x < 0),
\end{cases}
\] (6.17)

so that (A1) and (A2') are satisfied. Moreover, (A3) is also satisfied and

\[
P_{(x,y,l)}^{\text{Hev},\lambda} (X_t^B \to -\infty, \frac{\varphi_{\text{Hev},\lambda}(X_t)}{|X_t^B|} \to 1) = 1.
\] (6.18)

(*) The universality class of Brownian penalisation. Take \( E_t = \exp(-X_t^l) \) as a special case of (2) with \( f(l) = e^{-l} \). (Note that, by Najnudel–Roynette–Yor [8, Theorem 1.1.2], the corresponding unweighted measure \( \mathcal{P}_x^E \) coincides with \( \mathcal{Y}_x \) given in Introduction.) By the above argument, we see that all the assumptions of Theorem 4.1 are satisfied with \( \mathcal{E} \) and \( \Gamma = \Gamma^{f,\text{sup}}, \Gamma^{f,\text{inf}}, \Gamma^{Kac,v} \) or \( \Gamma^{\text{Hev},\lambda} \), so that we obtain the following known result:

\[
\mathcal{P}^{\Gamma}_{(x,y,l)} = 1_{\{\Gamma_\infty > 0\}} \cdot \mathcal{P}^{\mathcal{E}}_{(x,y,l)} \quad \text{for all } (x, y, l) \in S^\Gamma.
\] (6.19)

We remark the following obvious facts: It holds up to \( \mathcal{P}^{\mathcal{E}}_{(x,y,l)} \)-null sets that

\[
\mathbb{D} = \{ X_t^B \to \infty \text{ or } X_t^B \to -\infty \},
\] (6.20)

and that the event \( \{ \Gamma_\infty > 0 \} \) becomes

\[
\{ \Gamma_{\text{sup},\lambda}^{\text{sup}} > 0 \} = \{ X_t^B \to -\infty \text{ and } X_{\text{inf}}^{\text{sup}} \leq y_0 \},
\] (6.21)

\[
\{ \Gamma_{\text{inf},\lambda}^{\text{inf}} > 0 \} = \{ [X_t^B \to \infty \text{ or } X_t^B \to -\infty] \text{ and } X^{\text{inf}}_t \leq l_0 \},
\] (6.22)

\[
\{ \Gamma_\infty^{Kac,v} > 0 \} = \{ X_t^B \to \infty \text{ or } X_t^B \to -\infty \},
\] (6.23)

\[
\{ \Gamma_\infty^{\text{Hev},\lambda} > 0 \} = \{ X_t^B \to -\infty \}.
\] (6.24)

7 Lévy penalisation revisited

Let us look at some results of K.Yano–Y.Yano–Yor [20, 21], Y.Yano [22] and Takeda–K.Yano [16] in our framework.
Let \( \{Z = (Z_t)_{t \geq 0}, (P^Z_x)_{x \in \mathbb{R}}\} \) denote the canonical representation of one-dimensional strictly \( \alpha \)-stable process of index \( 1 < \alpha < 2 \), skewness \(-1 \leq \beta \leq 1\) and scaling parameter \( c_\theta > 0\):

\[
P^Z_0[e^{i\lambda Z_t}] = \exp\left(-c_\theta |\lambda|^\alpha \left(1 - i\beta \text{sgn}(\lambda) \tan \frac{\pi \alpha}{2}\right)\right), \quad \lambda \in \mathbb{R}.
\]  

(7.1)

(For the facts in this paragraph, see e.g. [2, Section VIII].) We assume that \( 1 < \alpha < 2 \) so as to exclude the Brownian case and to assure that zero is regular for itself: Writing \( T_0 = \inf\{t > 0 : Z_t = 0\} \) for the hitting time of zero, we have

\[
P^Z_0(T_0 > 0) = 1.
\]  

(7.2)

Set \( \overline{Z}_t = \sup_{s \leq t} Z_s \) and let \( L_t \) denote the local time of \( Z \) at 0. Let

\[
\rho := \frac{1}{2} + \frac{1}{\pi \alpha} \arctan \left(\beta \tan \frac{\pi \alpha}{2}\right) \in [1 - 1/\alpha, 1/\alpha]
\]  

(7.3)

and let \( k \) denote the positive constant such that

\[
\lim_{y \to \infty} y^\alpha P^Z_0(\overline{Z} > y) = k.
\]  

(7.4)

We set

\[
S = \{(x, y, l) \in \mathbb{R}^3 : y \geq x, \ l \geq 0\}
\]  

(7.5)

as the state space and consider the coordinate process \( X = (X_t)_{t \geq 0} = (X^Z_t, X^{\sup}_t, X^l_t)_{t \geq 0}\) on the space of càdlàg paths from \([0, \infty)\) to \( S\). We define \( P_{(x,y,l)} \) by the law on \( \mathbb{D} \) of \((Z, y \vee \overline{Z}, l + L)\) under \( P^Z_x \), and adopt the notation of Section 2.

1) Supremum penalisation. For a non-increasing function \( f : \mathbb{R} \to [0, \infty)\) such that for some \(-\infty < y_0 \leq \infty\) we have \( f(y) > 0 \) for \( y \leq y_0 \) and \( f(y) = 0 \) for \( y > y_0 \), and

\[
\int_0^{y_0} x^{\alpha \rho - 1} f(y) dy < \infty,
\]  

we set

\[
\Gamma^{sup,f}_t = \frac{f(X^{sup}_t)}{f(X^{sup}_0)} 1_{\{X^{sup}_t \leq y_0\}}, \quad S^{sup,f} = \{(x, y, l) \in S : y \leq y_0\}.
\]  

(7.7)

By K.Yano–Y.Yano–Yor [21, Theorem 5.1], we see that all the assumptions of Proposition 5.1 are satisfied with \( \rho(t) = t^\rho/k \) and

\[
\varphi^{sup,f}(x, y, l) = (y - x)^{\alpha \rho} + \frac{\alpha \rho}{f(y)} \int_y^{y_0} f(u)(u - x)^{\alpha \rho - 1} du, \quad (x, y, l) \in S^{sup,f},
\]  

(7.8)

so that \((A1)\) and \((A2')\) are satisfied. In the same way as that of deducing (6.5), we see that \([X^{sup}_t = X^{\sup}_\infty \text{ for large } t]\) and \([\Gamma^{sup,f}_t \to \Gamma^{sup,f}_\infty > 0]\) \( P_{(x,y,l)}^{sup,f} \)-a.s., which shows \((A3)\). By
(ii) of Proposition 2.1 and by the dominated convergence theorem, we obtain the following known results:

\[ p_{\{x,y,l\}}^{sup,f}(X_t^Z \to -\infty, \frac{\varphi^{sup,f}(X_t)}{(-X_t^Z)^\alpha} \to 1) = 1. \quad (7.9) \]

Note that the special case of the supremum penalisation with the weight \(1_{\{X_t^{sup}=0}\} = \Gamma_t^{sup,f}\) for \(f(l) = 1_{\{y=0\}}\) corresponds to the conditioning to stay negative.

(2) Local time penalisation. For an integrable function \(f : [0, \infty) \to [0, \infty)\) such that for some \(0 \leq l_0 \leq \infty\) we have \(f(l) > 0\) for \(l \leq l_0\) and \(f(l) = 0\) for \(l > l_0\), we set

\[ \Gamma_t^{lt,f} = \frac{f(X_t^Z)}{f(X_0^Z)} 1_{\{X_t^Z \leq l_0\}}, \quad S_t^{lt,f} = \{(x, y, l) \in S : l \leq l_0\}. \quad (7.10) \]

By Takeda–K.Yano [16] and by certain computation in [18, Section 5], we see that all the assumptions of Proposition 5.2 are satisfied with \(r(t) = c_r q^{1/\alpha - 1}\) for a certain constant \(c_r > 0\) and

\[ \varphi^{lt,f}(x, y, l) = C_{\alpha, \beta}(1 - \beta \operatorname{sgn}(x))|x|^{\alpha - 1} + \frac{1}{f(t)} \int_{l_0}^l f(u)du, \quad (x, y, l) \in S_t^{lt,f}. \quad (7.11) \]

with a certain constant \(C_{\alpha, \beta} > 0\), so that \((A1)\) and \((A2)\) are satisfied. In the same way as that of deducing (6.5), we see that \([X_t^{lt} = X_t^Z\) for large \(t\)] and \([\Gamma_t^{lt,f} \to \Gamma_\infty^{lt,f} > 0]\) \(P_{\{x,y,l\}}^{lt,f}\) a.s., which shows \((A3)\). By (ii) of Proposition 2.1, we obtain

\[ p_{\{x,y,l\}}^{lt,f}\left((1 - \beta \operatorname{sgn}(X_t^Z))|X_t^Z|^{\alpha-1} \to \infty, \frac{\varphi^{lt,f}(X_t)}{C_{\alpha, \beta}(1 - \beta \operatorname{sgn}(X_t^Z))|X_t^Z|^{\alpha-1}} \to 1\right) = 1; \quad (7.12) \]

in particular,

\[ p_{\{x,y,l\}}^{lt,f}\left(X_t^Z \to -\infty, \frac{\varphi^{lt,f}(X_t)}{(-X_t^Z)^\alpha} \to 2C_{\alpha,1}\right) = 1 \quad (\text{if } \beta = 1), \quad (7.13) \]

\[ p_{\{x,y,l\}}^{lt,f}\left(X_t^Z \to \infty, \frac{\varphi^{lt,f}(X_t)}{(X_t^Z)^\alpha} \to 2C_{\alpha,-1}\right) = 1 \quad (\text{if } \beta = -1). \quad (7.14) \]

In the case of \(-1 < \beta < 1\), we have a stronger convergence result in Takeda–K.Yano [16]:

\[ p_{\{x,y,l\}}^{lt,f}\left(\lim X_t^Z = \lim \sup X_t^Z = \lim \sup (-X_t^Z) = \infty\right) = 1 \quad \text{if } -1 < \beta < 1. \quad (7.15) \]

Note that the special case of the local time penalisation with the weight \(1_{\{X_t=0\}} = \Gamma_t^{lt,f}\) for \(f(l) = 1_{\{l=0\}}\) corresponds to the conditioning to avoid zero. See [17] for comparison of two types of conditionings for Lévy processes.

(*) The universality classes of Lévy penalisation. By (7.9), it holds that

\[ \{\Gamma_\infty^{sup,f} > 0\} = \{X_t^Z \to -\infty \text{ and } X_\infty^{sup} \leq y_0\} \quad \text{up to } \mathcal{G}_{\{x,y,l\}}^{sup,f}\text{ null sets} \quad (7.16) \]
in any case of $-1 \leq \beta \leq 1$.

(*1) Consider the case of $-1 < \beta < 1$. By (7.15), it holds that
\[
\{\Gamma_\infty^{lt,g} > 0\} = \{\lim X_t^Z = \limsup X_t^Z = \limsup (-X_t^Z) = \infty \text{ and } X_\infty^{lt} \leq y_0\}
\]
up to $\mathcal{P}^{lt,g}_{(x,y,l)}$-null sets. (7.17)

This shows that the two $\sigma$-finite measures $\mathcal{P}^{sup,f}_{(x,y,l)}$ and $\mathcal{P}^{lt,g}_{(x,y,l)}$ are singular to each other. Note that (7.9) and (7.15) imply
\[
\mathcal{P}^{sup,f}_{(x,y,l)} \left( \frac{\varphi^{lt,g}(X_t)}{\varphi^{sup,f}(X_t)} \to 0 \right) = 1
\]
because $\alpha \rho > \alpha - 1$, so that the assumption of Theorem 4.1 is not satisfied.

(*2) Consider the case of $\beta = 1$, the spectrally positive case. Take $\mathcal{E}_t = \exp(X_0^{sup} - X_t^{sup})$ as a special case of (1) with $f(y) = e^{-y}$. Then, since $\alpha \rho = \alpha - 1$, all the assumptions of Theorem 4.1 are satisfied with $\mathcal{E}$ and $\Gamma = \Gamma^{sup,f}$ or $\Gamma^{lt,g}$, so that we conclude as a new result that
\[
\mathcal{P}^{\Gamma}_{(x,y,l)} = 1_{\{\Gamma_\infty > 0\}} \cdot \mathcal{P}^{\mathcal{E}}_{(x,y,l)} \text{ for all } (x, y, l) \in S^\Gamma.
\]
(7.19)

It holds up to $\mathcal{P}^{\mathcal{E}}_{(x,y,l)}$-null sets that
\[
\mathbb{D} = \{X_t^Z \rightarrow -\infty\},
\]
(7.20)
and that the event $\{\Gamma_\infty > 0\}$ becomes
\[
\{\Gamma_\infty^{lt,g} > 0\} = \{X_t^Z \rightarrow -\infty \text{ and } X_\infty^{lt} \leq l_0\}. \quad (7.21)
\]

(*3) Consider the case of $\beta = -1$, the spectrally negative case. Then
\[
\{\Gamma_\infty^{lt,g} > 0\} = \{X_t^Z \rightarrow \infty \text{ and } X_\infty^{lt} \leq l_0\} \text{ up to } \mathcal{P}^{lt,g}_{(x,y,l)}\text{-null sets,}
\]
(7.22)
which shows that $\mathcal{P}^{sup,f}_{(x,y,l)}$ and $\mathcal{P}^{lt,g}_{(x,y,l)}$ are singular to each other.

**8 Langevin penalisation revisited**

Let us look at some results of Profeta [10] in our framework.

Let $\{(B, A), (W_{(b,a)})_{(b,a)\in \mathbb{R}^2}\}$ denote the canonical representation of the two-dimensional diffusion $(B, A) = (B_t, A_t)_{t \geq 0}$ where $B$ is a Brownian motion starting from $b$ and
\[
A_t = a + \int_0^t B_u du. \quad (8.1)
\]
This two-dimensional diffusion is a special case of the Langevin process and the process \( A \) is called the integrated Brownian motion. Set \( \overline{A}_t := \sup_{s \leq t} A_s \).

We set
\[
S = \{(b, a, y) \in \mathbb{R}^3 : y \geq a\}
\] (8.2)
as the state space and consider the coordinate process
\[
X = (X_t)_{t \geq 0} = (X_t^B, X_t^A, X_t^{\sup})_{t \geq 0}
\] (8.3)
on the space of càdlàg paths from \([0, \infty)\) to \( S \). We define \( P_{(b,a,y)} \) by the law on \( \mathcal{D} \) of \((B, A, y \vee \overline{A})\) under \( W_{(b,a)} \), and adopt the notation of Section 2.

We recall the confluent hypergeometric function (see [1, Chapter 13]):
\[
U(\alpha, \beta, z) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-zu} u^{\alpha-1} (1 + u)^{\beta-\alpha-1} du, \quad \alpha > 0, \ \beta \in \mathbb{R}, \ z > 0.
\] (8.4)

It is easy to see that
\[
\frac{d}{dz}(z^\alpha U(\alpha, \beta, z)) = -\alpha(\beta - \alpha - 1)z^{\alpha-1}U(\alpha + 1, \beta, z).
\] (8.5)

The following asymptotics are taken from [1, Formulae 13.5.2 and 13.5.8]:
\[
\lim_{z \to \infty} z^\alpha U(\alpha, \beta, z) = 1 (\beta \in \mathbb{R}), \quad \lim_{z \downarrow 0} z^{\beta-1} U(\alpha, \beta, z) = \frac{\Gamma(\beta - 1)}{\Gamma(\alpha)} (1 < \beta < 2).
\] (8.6)

(1) Conditioning to stay negative. We write \( \tau^A = \inf\{t > 0 : X^A_t \geq 0\} \) for the exit time from \((-\infty, 0)\) for the process \( X^A \) and set
\[
\Gamma^A_t = 1_{\{\tau^A > t\}}, \quad S^A = \{(b, a, y) \in S : y < 0\} = \{(b, a, y) \in \mathbb{R}^3 : a \leq y < 0\}.
\] (8.7)

By modifying Profeta [10, Theorem 5], we see that all the assumptions of Proposition 5.1 are satisfied with \( \rho(t) = c_1 t^{1/4} \) for a certain constant \( c_1 > 0 \) and
\[
\varphi^A(b, a, y) = h(-a, -b), \quad (b, a, y) \in S^A,
\] (8.8)
with a continuous function \( h : (0, \infty) \times \mathbb{R} \to (0, \infty) \) given as
\[
h(x, y) = \begin{cases} \left(\frac{9}{2}x\right)^{1/6} y^{1/3} U\left(\frac{1}{6}, \frac{4}{3}, z\right) = y^{1/2} z^{1/6} U\left(\frac{1}{6}, \frac{4}{3}, z\right) & (y > 0), \\
\frac{1}{6}\left(\frac{9}{2}x\right)^{1/6} z^{1/3} U\left(\frac{2}{6}, \frac{4}{3}, z\right) e^{-z} = \frac{1}{6} |y|^{1/2} z^{1/6} U\left(\frac{2}{6}, \frac{4}{3}, z\right) e^{-z} & (y < 0), \end{cases}
\] (8.9)
for \( x > 0 \) and \( z = \frac{2 \log y}{y} \), so that \((A1)\) and \((A2')\) are satisfied. Moreover, \((A3)\) is also satisfied and
\[
P^A_{(b,a,y)}(X^B_t \to -\infty \text{ and } X^A_t \to -\infty) = 1.
\] (8.10)
Let us prove this fact, as the part \([X_t^B \to -\infty]\) was not mentioned in [10]. By the formulae (8.6), we see that both \(z^{1/6}U(\frac{1}{6}, \frac{4}{3}, z)\) and \(z^{1/6}U(\frac{2}{6}, \frac{4}{3}, z)e^{-z}\) are bounded in \(z > 0\), we obtain \(h(x, y) \leq c_2|y|^{1/2}\) for some constant \(c_2 > 0\). It holds \(P_{(b,a,y)}^A\)-a.s. that, by (ii) of Proposition 2.1,

\[
\varphi^A(X_t) = h(-X_t^A, -X_t^B) \to \infty,
\]

which yields \([|X_t^B| \to \infty]\). But \([P_{(b,a,y)}^A(X_t^B \to \infty) = 0]\), since \([X_t^B \to \infty]\) implies \([X_t^A = a + \int_0^t X_s^Bds \to \infty]\), which contradicts the fact that \(X_0^A = a < 0\) and \(\tau^A = \infty\) by (i) of Proposition 2.1. Hence we obtain (8.10).

(2) Supremum penalisation. Let \(f: \mathbb{R} \to [0, \infty)\) be a continuous function such that for some \(-\infty < y_0 \leq 0\), we have \(f(y) > 0\) for \(y \leq y_0\) and \(f(y) = 0\) for \(y > y_0\). Set

\[
\Gamma_t^{\sup,f} = \frac{f(X_t^A)}{f(X_0^A)}1\{X_t^A \leq y_0\}, \quad S_t^{\sup,f} = \{(b, a, y) \in S : y \leq y_0\} = \{(b, a, y) \in \mathbb{R}^3 : a \leq y < y_0\}.
\]

By Profeta [10, Proposition 18 and Theorem 19], we see that all the assumptions of Proposition 5.1 are satisfied with \(\rho(t) = ct^{1/4}\) and

\[
\varphi^{\sup,f}(b, a, y) = h(y - a, -b) + \frac{1}{f(y)} \int_y^{y_0} f(w) \frac{\partial}{\partial w}h(w - a, -b)dw, \quad (b, a, y) \in S^{\sup,f},
\]

so that (A1) and (A2') are satisfied. By a similar argument to that deducing (6.5), we see that \([X_t^{\sup} = X_\infty^{\sup}\) for large \(t\)] \(P_{(b,a,y)}^{(\sup,f)}\)-a.s., and that \([\Gamma_t^{\sup,f} \to \Gamma_\infty^{\sup,f} > 0] P_{(b,a,y)}^{(sup,f)}\)-a.s., which shows (A3). By the fact that \(\frac{\partial}{\partial w}h \geq 0\), we have

\[
\varphi^{\sup,f}(b, a, y) \leq \left( \sup_{y \leq w \leq y_0} f(w) \right) h(y_0 - a, -b).
\]

By a similar argument after (8.11), and by (ii) of Proposition 2.1, we can deduce

\[
P_{(b,a,y)}^{(\sup,f)}(X_t^B \to -\infty \text{ and } X_t^A \to -\infty) = 1.
\]

(*) The universality class of Langevin penalisation. We would like to compare the three unweighted measures \(\mathcal{P}_{(b,a,y)}^A\), \(\mathcal{P}_{(b,a,y)}^{\sup,f}\) and \(\mathcal{P}_{(b,a,y)}^B\). Here we write \(\tau^B = \inf\{t > 0 : X_t^B \geq 0\}\) for the exit time from \((\infty, 0)\) for the Brownian motion \(X^B\) and set

\[
\Gamma_t^B = 1_{\{\tau^B > t\}}, \quad S_t^B = \{(b, a, y) \in S : b < 0\}.
\]

The penalisation for the weight \(\Gamma^B\) is nothing else but the conditioning to stay negative for the Brownian motion, so that we obtain \(\varphi^B(b, a, y) = -b\). The penalized probability \(P_{(b,a,y)}^B\) is the minus times 3-dimensional Bessel process and the corresponding unweighted measure is given as \(\mathcal{P}_{(b,a,y)}^B = (-b)P_{(b,a,y)}\). Since \(X_t^A = a + \int_0^t X_u^Bdu\), we obtain

\[
P_{(b,a,y)}^B(X_t^B \to -\infty \text{ and } X_t^A \to -\infty) = 1.
\]

We prove the following proposition with conjectured assumptions.
Proposition 8.1. Set \( Z_t = \frac{(-X_t^B)^3}{(-X_t^A)} \). Then the following assertions hold:

(i) Suppose the following conjecture is true:

\[
Z_t \xrightarrow{P_{(b,a,y)}^{A}} \infty \text{ and } Z_t \xrightarrow{P_{(b,a,y)}^{sup,f}} \infty \text{ for } (b,a,y) \in S^{sup,f}.
\]  

Then \( \mathcal{P}^{sup,f}_{(b,a,y)} \) and \( \mathcal{P}^A_{(b,a,y)} \) coincide for \((b,a,y) \in S^{sup,f}(\subset S^A)\).

(ii) Suppose the following conjecture is true:

\[
Z_t \xrightarrow{P_{(b,a,y)}^{A}} \infty \text{ and } Z_t \xrightarrow{P_{(b,a,y)}^{B}} \infty \text{ for } (b,a,y) \in S^A \cap S^B.
\]

Then \( \mathcal{P}^A_{(b,a,y)} \) and \( \mathcal{P}^B_{(b,a,y)} \) are singular to each other for \((b,a,y) \in S^A \cap S^B\).

Proof. (i) Set \( Z_t^{sup} = \frac{(-X_t^B)^3}{(X_t^A - X_t^B)} \). Then \( Z_t \xrightarrow{P} \infty \) both for \( P = P_{(b,a,y)}^{A} \) and for \( P = P_{(b,a,y)}^{sup,f} \). Since \( X_t^B < 0 \) for large \( t \), we have

\[
\frac{h(X_t^{sup} - X_t^A, -X_t^B)}{h(-X_t^A, -X_t^B)} = \frac{(Z_t^{sup})^{1/6}U(\frac{1}{6}, \frac{4}{3}, Z_t^{sup})}{(Z_t^{sup})^{1/6}U(\frac{4}{6}, \frac{4}{3}, Z_t)} \xrightarrow{P} 1 (8.20)
\]

by the assumption. Noting that (8.5) implies

\[
\frac{\partial}{\partial x} h(x, y) = c_3 x^{-5/6} \cdot z^{7/6} U(\frac{7}{6}, \frac{4}{3}, z) \leq c_4 x^{-5/6}, \quad x, y > 0, Z = \frac{2|y|^3}{9} x
\]

for some constants \( c_3, c_4 > 0 \), we obtain

\[
\frac{\varphi^{sup,f}(X_t)}{\varphi^A(X_t)} \xrightarrow{P} 1 (8.22)
\]

both for \( P = P_{(b,a,y)}^{A} \) and for \( P = P_{(b,a,y)}^{sup,f} \). We may now apply Theorem 4.1 for \( \mathcal{E} = \Gamma^A \) and \( \Gamma = \Gamma^{sup,f} \), and thus we obtain the desired result.

(ii) By the assumption, we have

\[
R_t := \frac{\Gamma_t^A \varphi^A(X_t)}{\varphi^B(X_t)} = \frac{\Gamma_t^A \cdot (-X_t^B)^{1/2} \cdot (Z_t)^{1/6} U(\frac{1}{6}, \frac{4}{3}, Z_t)}{(-X_t^B)} \xrightarrow{P} 0 (8.23)
\]

both for \( P = P_{(b,a,y)}^{A} \) and for \( P = P_{(b,a,y)}^{B} \). By the same argument of Theorem 4.1 with \( \mathcal{E} = \Gamma^B \) and \( \Gamma = \Gamma^A \), we obtain

\[
P_{(b,a,y)}^{B} F_s \cdot \frac{R_t}{1 + R_t + \Gamma_t} = \frac{\varphi^A(b, a, y) P_{(b,a,y)}^{A}}{\varphi^B(b, a, y) P_{(b,a,y)}^{B}} \left[ F_s \cdot \frac{\Gamma_t}{1 + R_t + \Gamma_t^B} \right]. (8.24)
\]

Letting \( t \to \infty \), we obtain \( P_{(b,a,y)}^{A}(\Gamma_t^B > 0) = 0 \). Since \( P_{(b,a,y)}^{B}(\Gamma_t^B > 0) = 1 \), we obtain the desired result.

\[\square\]
9 Appendix: Extension of transformed probability measures

We discuss in general extension of the transformed probability measures given by local absolute continuity like (2.11). Recall that $\mathbb{D}$ is the space of càdlàg paths from $[0, \infty)$ to a locally compact separable metric space $S$ and $X$ is the coordinate process on $\mathbb{D}$.

**Theorem 9.1.** Let $P$ be a probability measure on $(\mathbb{D}, \sigma(X))$ and let $(M_t)_{t \geq 0}$ be a non-negative martingale such that $P[M_t] = 1$ for all $t \geq 0$. Then there exists a unique probability measure $Q$ on $(\mathbb{D}, \sigma(X))$ such that

$$Q|_{\mathcal{F}^X_t} = M_t \cdot P|_{\mathcal{F}^X_t}, \quad t \geq 0,$$

where $\mathcal{F}^X_t = \sigma(X_s : s \leq t)$ is the natural filtration of $X$.

**Proof.** Since $\bigcup_{t \geq 0} \mathcal{F}^X_t$ is a $\pi$-system generating $\sigma(X)$, uniqueness of $Q$ follows immediately from Dynkin’s $\pi$-$\lambda$ theorem.

Let us prove existence of $Q$. For $n \in \mathbb{N}$, let $\mathbb{D}_n$ denote the space of càdlàg paths from $[n-1, n)$ to $S$, equipped with the $\sigma$-field $\mathcal{B}_n$ generated by the coordinate process on $\mathbb{D}_n$. We thus see that $\mathbb{D}$ is the product space of $\{\mathbb{D}_n\}$:

$$\mathbb{D} = \prod_{n=1}^{\infty} \mathbb{D}_n, \quad \sigma(X) = \sigma \left( \prod_{k=1}^{n} B_k \times \prod_{k=n+1}^{\infty} \mathbb{D}_k : B_1 \in \mathcal{B}_1, \ldots, B_n \in \mathcal{B}_n; \ n \in \mathbb{N} \right).$$

Let $\mu_n$ denote the law on $\mathbb{D}_1 \times \cdots \times \mathbb{D}_n$, the space of càdlàg paths from $[0, n)$ to $S$, of $(X_t)_{0 \leq t < n}$ under $M_n \cdot P|_{\mathcal{F}^X_n}$. We then see that $\{\mu_n\}$ is a projective sequence:

$$\mu_{n+1}(\cdot \times \mathbb{D}_{n+1}) = \mu_n, \quad n \in \mathbb{N}.$$

We may apply Daniell’s extension theorem (cf. [7, Theorem 6.14]) to see that there exists a sequence of random variables $\{\xi_n\}$ defined on a probability space $(\Omega', \mathcal{F}', P')$ such that $\xi_n$ for each $n$ takes values in $\mathbb{D}_n$ and the joint distribution of $(\xi_1, \ldots, \xi_n)$ under $P'$ for each $n$ coincides with $\mu_n$.

We now define $Q$ by the law on $\mathbb{D}$ of $(\xi_1, \xi_2, \ldots)$ under $P'$. For any $A \in \mathcal{F}^X_n$ for each $n \in \mathbb{N}$, we can find $B \subset \mathbb{D}_1 \times \cdots \times \mathbb{D}_n$ which belongs to $\sigma(\prod_{k=1}^{n} B_k : B_1 \in \mathcal{B}_1, \ldots, B_n \in \mathcal{B}_n)$ such that $A = \{(X_t)_{0 \leq t < n} \in B\}$, so that we obtain

$$Q(A) = P'(\{(\xi_1, \ldots, \xi_n) \in B\}) = \mu_n(B) = P[M_n; (X_t)_{0 \leq t < n} \in B] = P[M_n; A].$$

We thus conclude that $Q$ is as desired.

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