Robin-Dirichlet alternating iterative procedure for solving the Cauchy problem for Helmholtz equation in an unbounded domain

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Abstract

We consider the Cauchy problem for the Helmholtz equation with a domain in $\mathbb{R}^d$, $d \geq 2$ with $N$ cylindrical outlets to infinity with bounded inclusions in $\mathbb{R}^{d-1}$. Cauchy data are prescribed on the boundary of the bounded domains and the aim is to find solution on the unbounded part of the boundary. In 1989, Kozlov and Maz’ya [14] proposed an alternating iterative method for solving Cauchy problems associated with elliptic, self-adjoint and positive-definite operators in bounded domains. Different variants of this method for solving Cauchy problems associated with Helmholtz-type operators exists. We consider the variant proposed by Mpinganzima et.al [5] for bounded domains and derive the necessary conditions for the convergence of the procedure in unbounded domains. For the numerical implementation, a finite difference method is used to solve the problem in a simple rectangular domain in $\mathbb{R}^2$ that represent a truncated infinite strip. The numerical results shows that by appropriate truncation of the domain and with appropriate choice of the Robin parameters $\mu_0$ and $\mu_1$, the Robin-Dirichlet alternating iterative procedure is convergent.

Key words. Helmholtz equation; Cauchy problem; Inverse problem; Ill-posed problem;

1 Introduction

Let $\Omega$ be a domain in $\mathbb{R}^d$, $d \geq 2$, with $C^2$ boundary and with $N$ cylindrical outlets to infinity, i.e. for sufficiently large $|x|$ the domain $\Omega$ coincides with the union of $N$ disjoint cylinders $C^{(j)}$, $j = 1, \ldots, N$, which can be described in a certain cartesian coordinates $x^{(j)} = (y^{(j)}, z^{(j)})$, as

$$C^{(j)} = \{x^{(j)} : y^{(j)} \in \omega^{(j)}, z^{(j)} \in \mathbb{R}\},$$

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where the cross-sections $\omega^{(j)}$ are bounded domains in $\mathbb{R}^{d-1}$ with $C^2$ boundaries. We denote the boundary of $\Omega$ by $\Gamma$. We assume that a certain bounded open set $\Gamma_0$ is chosen on the boundary $\Gamma$ and the boundary of this set is of class $C^2$ also. Let also $\Gamma_1$ is the interior of $\Gamma \setminus \Gamma_0$.

We consider the following Cauchy problem for the Helmholtz equation

$$(\Delta + k^2)u = 0 \text{ in } \Omega$$

and

$$u = f_0 \text{ on } \Gamma_0, \quad \partial_\nu u = g_0 \text{ on } \Gamma_0,$$

where $k$ is a real scalar, $\nu$ is the outward unit normal to $\Gamma$, $\partial_\nu$ is the normal derivative and $(f_0, g_0)$ is a prescribed Cauchy data.

The Cauchy problem for the Helmholtz equation in bounded and unbounded domains arises in many important physical applications, for instance in capacity problems or scattering of acoustics or electromagnetic waves, see [7, 9, 10, 8, 13].

The Cauchy problem for the Helmholtz equation is an inverse problem and it is ill-posed. Small perturbation in the Cauchy data $f_0$ and $g_0$ results into a big error in the solution and as a result classical numerical methods cannot be used to solve this problem. Regularization methods are instead used to solve inverse problems.

Over the years, much theoretical and numerical studies have been done on the Cauchy problem associated with the Helmholtz equation on bounded domains. These include both Tikhonov type regularization methods and iterative regularization methods. Lesnic et al. [18] and Marin [17], have solved the Cauchy problem associated with the Helmholtz equation using the conjugate gradient method (CGM) and the boundary element-minimal error method, respectively. Wei et al. [20, 27] solved the Cauchy problem associated with the Helmholtz-type equations by transforming the Cauchy problem into a moment problem and then applied a Tikhonov type regularization method. Zhang et al. [16] have solved the Cauchy problem for the Helmholtz equation using a Fourier-Bessel method. Numerical methods for solving the Cauchy problem for two and three dimensional Helmholtz-type equations have also been studied by Lesnic et at. [21] and Marin [20] respectively. They used the method of fundamental solutions (MFS) in conjunction with Tikhonov regularization method.

Kozlov and Maz’ya [14] developed the alternating iterative procedure for solving linear elliptic partial differential equations. The alternating iterative procedure is applicable to equations where the operator is symmetric and positive in a certain sense. The regularizing character is achieved by appropriately changing the boundary conditions and one advantage of this procedure is that it preserves the original operator. Kozlov et al. [15] used this procedure to solve the Cauchy problem for the Laplace equation and the Lame’ system. Chapko et al. [13] further applied the alternating iterative procedure to solve the Cauchy

\footnote{This is a set where measurements are taken and it is reasonable to assume it bounded}
problem for the Laplace equation in a bounded domain with a cut. It has also been demonstrated that the alternating iterative procedure does not only work for linear elliptic partial differential equations but also for nonlinear elliptic partial differential equations, see [23, 2]. However, as mentioned above, the alternating iterative procedure converge if the operator is self-adjoint and positive-definite. Example of operators which do not fulfil this requirements are Helmholtz-type operators. Marin et.al [19] used the alternating iterative procedure, numerically implemented using the boundary element method (BEM) to solve the Cauchy problem for the Helmholtz equation with purely imaginary wavenumber, \( k \) i.e they considered the equation \((\Delta - k^2)u = 0\) which is in fact a modified version of the Helmholtz equation. They noticed that the alternating iterative procedure applied to the Helmholtz equation does not always converge. Kozlov et al. [11] modified the alternating iterative procedure to accommodate second order elliptic operators which are self-adjoint but does not fulfil the condition of positivity. Mpinganzima et al. [4, 3] also presented other modifications of the alternating iterative procedure for Cauchy problem associated to the Helmholtz equation by an introduction of artificial boundary and boundary conditions which allow to treat all values of \( k \) i.e the positivity condition introduced in [14, 15] is essentially selected. The latter, [3], involves employing an operator equation formulation of the Robin-Dirichlet algorithm and using the conjugate gradient method in order to accelerate the slow convergence achieved in [4]. In [5], Mpinganzima et al. further presented a simpler modification of the alternating iterative procedure for Cauchy problem associated to the Helmholtz equation by replacing the Neumann-Dirichlet iterations by the Robin-Dirichlet iterations. In [1], we presented an analysis of Robin-Dirichlet alternating iterative procedure. We prove that the Robin-Dirichlet alternating iterative procedure is in fact convergent for general elliptic operators provided that the parameters in the Robin conditions are appropriately chosen. The precise behaviour of \( k \) in the Helmholtz equation is also numerically investigated.

The aim of this paper is to derive the necessary conditions for the convergence of the Robin-Dirichlet alternating iterative procedure for solving the Cauchy problem for the Helmholtz equation in unbounded domains. In unbounded domains, for example in the cylinder \( C^{(j)} \) considered in our problem, the continuous spectrum of the Dirichlet-Laplacian in \( \Omega \) coincides with \([\min_j \lambda_0^{(j)}, \infty)\) where \( \lambda_0^{(j)} \) is the first eigenvalue of the Dirichlet-Laplacian in the cross-section \( \omega^{(j)} \), see [12, 25]. If there is no discrete spectrum below \( \min_j \lambda_0^{(j)} \), then we prove the convergence of the Robin-Dirichlet alternating iterative procedure if \( k^2 < \lambda_0^{(j)} \) for all \( j \). However, if there are eigenvalues below \( \min_j \lambda_0^{(j)} \) then we prove the convergence of the Robin-Dirichlet alternating iterative procedure if \( k^2 < \Lambda_0 \) where \( \Lambda_0 \) is the smallest eigenvalue in the discrete spectrum. The convergence analysis of the Robin-Dirichlet alternating iterative procedure is based on an analysis of the spectrum of the Laplacian operator in \( \Omega \) with Dirichlet and Robin boundary conditions.
1.1 Alternating iterative procedure

As usual the notation $H^1(\Omega)$ corresponds to the Sobolev space of functions in $\Omega$ with finite norm

$$||u||_{H^1(\Omega)} = \left( \int_\Omega (|\nabla u|^2 + |u|^2) dx \right)^{1/2}.$$ 

Our main assumption concerning the parameter $k$ is the following: there exist a positive constant $\epsilon$ such that

$$\int_\Omega (|\nabla u|^2 - k^2 |u|^2) dx \geq \epsilon ||u||^2_{H^1(\Omega)}$$

for all $u \in H^1(\Omega, \Gamma)$, (1.3)

In Lemma 2.3, we give an equivalent version of the assumption (1.3). Namely, it is equivalent to the following: there exist positive constants $\mu_0$, $\mu_1$ and $\delta$ such that

$$\int_\Omega (|\nabla u|^2 - k^2 |u|^2) dx + \mu_0 \int_{\Gamma_0} |u|^2 dS + \mu_1 \int_{\Gamma_1} |u|^2 dS \geq \delta ||u||^2_{H^1(\Omega)}$$

for all $u \in H^1(\Omega)$. (1.4)

In order to describe the Robin-Dirichlet alternating iterative procedure, let us introduce two mixed boundary value problems:

$$\begin{align*}
\Delta u + k^2 u &= 0 & \text{in } \Omega, \\
u \partial_{\nu} u + \mu_1 u &= \eta & \text{on } \Gamma_1,
\end{align*}$$

and

$$\begin{align*}
\Delta u + k^2 u &= 0 & \text{in } \Omega, \\
\partial_{\nu} u + \mu_0 u &= g & \text{on } \Gamma_0, \\
u \partial_{\nu} u + \mu_1 u &= \phi & \text{on } \Gamma_1,
\end{align*}$$

Here, $f \in H^2(\Gamma_0)$, $\eta \in H^{-1/2}(\Gamma_1)$, $g \in H^{-1/2}(\Gamma_0)$ and $\phi \in H^2(\Gamma_1)$. From the assumption (1.3), or equivalently from (1.4), it follows well-posedness of these problems, see section 3.2.

The algorithm for solving (1.1), (1.2) is described as follows. We take $f = f_0$ and $g = g_0 + \mu_0 f_0$ where $f_0$ and $g_0$ are the Cauchy data given in (1.2) then:

1. The first approximation $u_0$ is obtained by solving (1.5) where $\eta \in H^{-1/2}(\Gamma_1)$ is an arbitrary initial approximation of the Robin condition on $\Gamma_1$.

2. Having constructed $u_{2n}$, we obtain $u_{2n+1}$ by solving (1.6) with $\phi = u_{2n}$ on $\Gamma_1$.

3. We then obtain $u_{2n+2}$ by solving (1.5) with $\eta = \partial_{\nu} u_{2n+1} + \mu_1 u_{2n+1}$

In section 3.3, we present a theorem on convergence of this algorithm. The proof basically follows the same lines as in the case of bounded domains, see §1.
2 About condition (1.3)

We denote by \( \lambda_0^{(j)} \) the first eigenvalue of the operator \(-\Delta_y^{(j)}\) in the cross-section \(\omega^{(j)}\) and \(\lambda^{(j)}(\mu)\) the first eigenvalue of the operator \(-\Delta_y^{(j)}\) in \(\omega^{(j)}\) with the Robin boundary condition \((\partial_\nu + \mu) u = 0\) on \(\partial\omega^{(j)}, \ j = 1, \ldots, N\).

As is known the continuous spectrum of \(-\Delta\) in \(\Omega\) lies in \([\min_j \lambda_0^{(j)}, \infty)\) and of the operator \(-\Delta\) in \(\Omega\) with the Robin boundary condition \((\partial_\nu + \mu) u = 0\) on \(\Gamma\) is located in \([\min_j \lambda^{(j)}(\mu), \infty)\). It can happen that there is also a discrete spectrum for both operators lying in \((\infty, \min_j \lambda_0^{(j)})\) and \((\infty, \min_j \lambda^{(j)}(\mu))\) respectively.

**Lemma 2.1** The function \(\lambda^{(j)}(\mu)\) is monotonically increasing with respect to \(\mu\) and

\[
\lambda^{(j)}(\mu) \rightarrow \lambda_0^{(j)} \quad \text{as} \quad \mu \rightarrow \infty, \quad \text{for} \quad j = 1, \ldots, N.
\]

**Proof.** The eigenvalue \(\lambda^{(j)}(\mu)\) of the operator \(-\Delta_y^{(j)}\) in \(\omega^{(j)}\) with the Robin boundary condition \((\partial_\nu + \mu) u = 0\) on \(\omega^{(j)}\) is given by

\[
\lambda^{(j)}(\mu) = \min_{u \in H^1(\Omega)} \frac{\mu \int_{\partial\omega^{(j)}} u^2 dS + \int_{\omega^{(j)}} |\nabla u|^2 \ dx}{\int_{\omega^{(j)}} u^2 \ dx}.
\]

From (2.8), we see that the function \(\lambda^{(j)}(\mu)\) is non-negative and monotonically increasing with respect to \(\mu\). Then it remains to prove that \(\lim_{\mu \rightarrow \infty} \lambda^{(j)}(\mu) = \lambda_0^{(j)}\) which follows from Lemma 2.1 in [4].

**Lemma 2.2** If condition (1.3) holds then

\[
k^2 < \lambda_0^{(j)}, \quad j = 1, \ldots, N.
\]

**Proof.** Let \(k^2 \geq \lambda_0^{(j)}\) for a certain \(j\). We choose a test functions in (1.3) in the form \(u(x^{(j)}) = \phi^{(j)}(y^{(j)}) \eta(z^{(j)})\), where \(\phi^{(j)}\) is an eigenfunction of the Dirichlet-Laplacian in \(\omega^{(j)}\) corresponding to \(\lambda_0^{(j)}\) with the norm \(||\phi^{(j)}||_{L^2(\omega^{(j)})} = 1\). Substituting \(u(x^{(j)})\) to the left-hand side of (1.3) we obtain

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( |\nabla (\phi^{(j)} \eta)|^2 - k^2 (\phi^{(j)})^2 (\eta|^2) \right) dy^{(j)} dz^{(j)}
\]

\[
= \int_{-\infty}^{\infty} (\eta')^2 + (\lambda_0^{(j)} - k^2) \eta^2 dz^{(j)}
\]

Let the test function \(\eta\) be equal to 1 for \(\tau + 1 < z^{(j)} < \tau + T - 1\), \(\eta(z^{(j)}) = z^{(j)} - \tau\) for \(\tau < z^{(j)} < \tau + 1\), \(\eta(z^{(j)}) = -z^{(j)} + \tau + T\) for \(\tau + T - 1 < -z^{(j)} < \tau + T\) and
0 otherwise. Then
\[\int_{-\infty}^{\infty} (\eta')^2 dz^{(j)} = 2 < \frac{2}{T-2} \int_{-\infty}^{\infty} \eta^2 dz^{(j)} \]
and therefore the inequality (1.3) can not be true for all such test functions in the case \(k^2 \geq \lambda_0^{(j)}\). This prove this lemma.

From Lemma 2.1 and inequality (2.9), it follows that
\[k^2 < \lambda^{(j)}(\mu), \ j = 1, \ldots, N, \tag{2.11}\]
for large \(\mu\).

Now we can prove the equivalence of the conditions (1.3) and (1.4).

**Lemma 2.3** Condition (1.3) is equivalent to existence of positive constants \(\mu\) and \(\delta\) such that
\[\int_{\Omega} \left( |\nabla u|^2 - k^2 |u|^2 \right) dx + \mu \int_{\Gamma} |u|^2 dS \geq \delta ||u||_{H^1(\Omega)}^2 \text{ for all } u \in H^1(\Omega). \tag{2.12}\]

**Proof.** Clearly (1.3) follows from (2.12). Let us prove the opposite implication.

We introduce
\[\Lambda_0 = \inf_{u \in H^1(\Omega)} \int_{\Omega} |\nabla u|^2 dx.\]

Then
\[\epsilon + k^2 \leq \Lambda_0 \leq \min_j \lambda_0^{(j)} . \tag{2.13}\]

Furthermore, if we put
\[\Lambda(\mu) = \inf_{u \in H^1(\Omega)} \left( \int_{\Omega} |\nabla u|^2 dx + \mu \int_{\Gamma} |u|^2 dS \right), \tag{2.14}\]
where \(\mu\) is non-negative, then on can see that
\[0 \leq \Lambda(\mu) \leq \min_j \lambda^{(j)}(\mu) \leq \Lambda_0. \tag{2.15}\]

Moreover \(\Lambda(\mu)\) is a monotonically increasing function with respect to \(\mu\). Then the required assertion will follow from
\[\Lambda(\mu) \to \Lambda_0 \text{ as } \mu \to \infty. \tag{2.16}\]

Let us prove (2.16).

First, consider the case when there is a sequence \(\{\mu_k\}_{k=1}^{\infty}\) such that \(\mu_k \to \infty\) as \(k \to \infty\) and \(\Lambda(\mu_k) = \min_j \lambda^{(j)}(\mu_k)\). Since \(\lambda^{(j)}(\mu) \to \lambda_0^{(j)}\) as \(\mu \to \infty\), we get
\[\min_j \lambda^{(j)}(\mu) \to \min_j \lambda_0^{(j)} \text{ as } \mu \to \infty. \]
Therefore $\Lambda(\mu_k) \to \min_j \lambda_0^{(j)}$ and we get \((2.14)\) due to monotonicity of $\Lambda(\mu)$ and because of the right inequality in \((2.13)\) (in this case $\Lambda_0 = \min_j \lambda_0^{(j)}$).

Second, suppose that $\Lambda(\mu) < \min_j \lambda_0^{(j)}(\mu)$ for large $\mu$. Then $\Lambda(\mu)$ is an eigenvalue of the Laplacian in $\Omega$ with the Robin boundary condition $\partial_\nu u + \mu u = 0$ on $\Gamma$. We denote by $u_\mu$ a corresponding eigenfunction normalized by $||u_\mu||_{L^2(\Omega)} = 1$. The function $u_\mu$ satisfies

$$\Delta u_\mu = -\Lambda(\mu) u_\mu \text{ in } \Omega \text{ and } \partial_\nu u_\mu + u_\mu = 0 \text{ on } \Gamma$$

and hence

$$\int_\Omega |\nabla u_\mu|^2 dx + \mu \int_\Gamma |u_\mu|^2 dS = \Lambda_\mu \leq \Lambda_0. \quad (2.17)$$

We represent solution as $u_\mu = v + w$, where $v$ solves the problem

$$\Delta v = 0 \text{ in } \Omega \text{ and } v = u_\mu \text{ on } \Gamma.$$ 

The function $v$ belongs to $H^1(\Omega)$ and due to \((2.17)\) satisfies the estimate

$$||v||_{L^2(\Omega)} \leq C ||u_\mu||_{L^2(\Gamma)} \leq C\mu^{-1/2}. \quad (2.18)$$

Therefore $w \in H^1(\Omega, \Gamma)$ and satisfies $-\Delta w = \Lambda(\mu) u_\mu$ in $\Omega$. Multiplying this equation by $w$ and integrating over $\Omega$, we get

$$\int_\Omega |\nabla w|^2 dx = \Lambda(\mu) \int_\Omega (w + v)w dx.$$ 

Using the definition of the constant $\Lambda_0$ we derive from the last identity the following estimate

$$(\Lambda_0 - \Lambda(\mu)) \int_\Omega |w|^2 dx \leq \Lambda(\mu)||w||_{L^2(\Omega)}||v||_{L^2(\Omega)}.$$ 

Therefore

$$||w||_{L^2(\Omega)} \leq \frac{\Lambda(\mu)}{\Lambda_0 - \Lambda(\mu)} ||v||_{L^2(\Omega)}$$

or

$$||v||_{L^2(\Omega)} + ||w||_{L^2(\Omega)} \leq \frac{\Lambda_0}{\Lambda_0 - \Lambda(\mu)} ||v||_{L^2(\Omega)}.$$ 

Since the left-hand side $\geq 1$ by using \((2.18)\), we obtain

$$\Lambda_0 - \Lambda(\mu) \leq C\mu^{-1/2},$$

which implies \((2.16)\).

**Example 2.4** Let $\Omega$ be a strip in $\mathbb{R}^2$ i.e $\Omega = \{ (x, y) : x \in \mathbb{R}, \ 0 < y < L \}$.

Consider the following spectral boundary value problem in the cross-section $[0, L]$. Find $Y$ such that

$$-Y'' = \lambda Y \quad (2.19)$$
and
\[ Y''(0) - \mu_0 Y'(0) = Y'(L) + \mu_1 Y(L) = 0, \quad (2.20) \]
where \( \mu_0, \mu_1 \) are non-negative and \( \mu_0 + \mu_1 > 0 \).

The first eigenvalue of (2.19) with homogeneous Dirichlet boundary conditions in the cross-section \([0, L]\) is \( \pi^2 L^2 \) and we denote it by \( \lambda_0 \). Our aim is to evaluate the first eigenvalue, \( \lambda(\mu) \) of problem (2.19), (2.20) and to demonstrate that this eigenvalue is close to \( \lambda_0 \). Eigenvalues of (2.19) and (2.20) are positive.

Multiplying both sides of (2.19) by \( Y \), integrating by parts and applying the boundary conditions (2.20) gives,
\[ \int_0^L Y'(y)^2 + \mu_0 Y^2(0) + \mu_1 Y^2(L) = \lambda \int_0^L Y^2(y) \, dy \quad (2.21) \]
Therefore \( \lambda \) must be positive.

To evaluate the first eigenvalue \( \lambda(\mu) \) of (2.19) and (2.20), let \( \lambda = \alpha^2, \alpha > 0 \). The general solution to (2.19) is
\[ Y(y) = A \cos(\alpha y) + B \sin(\alpha y) \quad (2.22) \]
Using (2.20), we obtain
\[ \alpha B - \mu_0 A = 0 \quad (2.23) \]
and
\[ A(\mu_1 \cos(\alpha L) - \alpha \sin(\alpha L)) + B(\mu_1 \sin(\alpha L) + \alpha \cos(\alpha L)) = 0 \quad (2.24) \]
For existence of non-trivial solution and for \( \mu = \mu_0 = \mu_1 \) we have
\[ \cot(\alpha L) = \frac{(\alpha^2 - \mu^2)}{2\alpha \mu} \quad (2.25) \]
or
\[ \cot(\beta) = \frac{(\beta^2 - \mu^2 L^2)}{2L \beta \mu} \quad (2.26) \]
where \( \beta = \alpha L \). Let
\[ f(\beta) = \frac{(\beta^2 - \mu^2 L^2)}{2L \beta \mu} \quad (2.27) \]
The smallest root of (2.26) is located in the interval \((0, \pi)\), see Figure [1]. We denote this root by \( \beta_1 \). If \( \mu L < \frac{\pi}{2} \), then \( \beta_1 < \frac{\pi}{2} \), see Figure [1]. If \( \mu L > \frac{\pi}{2} \), then \( \beta_1 > \frac{\pi}{2} \), see Figure [2] and \( \beta_1 \) tends to \( \pi \) as \( \mu \) tends to infinity.

We denote the first root of (2.25) by \( \alpha_1 = \frac{\beta_1}{L} \). Therefore the first eigenvalue \( \lambda(\mu) < \left( \frac{\alpha_1^2}{2\pi^2 L^2} \right)^2 \) when \( \alpha_1 > \mu \), and \( \lambda(\mu) > \left( \frac{\alpha_1^2}{2\pi^2 L^2} \right)^2 \) whenever \( \alpha_1 < \mu \). For large \( \mu \), we have,
\[ f(\beta, \mu) = \frac{\beta}{2L \mu} - \frac{\mu L}{2\beta} \quad (2.28) \]
and \( f(\beta, \mu) \) tends to \(-\infty\) as \( \mu \to \infty \). The leading term of the root of (2.26) is \( \pi \), so \( \beta_1 = \pi - \delta \) where \( \delta \) is a small number for large \( \mu \). Substituting \( \beta_1 \) into (2.26) and using (2.28) for \( \mu \to \infty \), we have

\[
\frac{\cos(\pi - \delta)}{\sin(\pi - \delta)} = -\frac{1}{\delta} \left( 1 + O(\delta^2) \right) = -\frac{\mu L}{2\beta}
\]

and therefore

\[
\delta = \frac{2\beta}{\mu L} + O\left( \frac{1}{\mu^3} \right)
\]

As a result, we have that \( \lambda(\mu) \to \lambda_0 \) as \( \mu \to \infty \).

The first eigenvalue of problem (2.19) with boundary conditions (2.20) is given by the formula \( \lambda(\mu) = \frac{\mu^2}{\pi^2} \) and is evaluated from equation (2.26). In Table 1 we present some numerically computed values of the first eigenvalue for some increasing values of \( \mu \). By Inequality (2.11) and Theorem 3.3 we can obtain exponential decay of the solution at infinity and convergence for the iterations presented in Section 1.1, respectively if \( k^2 < \lambda(\mu) \).

| \( \mu \) | 2 | 4 | 6 | 8 | 10 | 12 | 14 |
|---|---|---|---|---|---|---|---|
| \( \lambda \) | 10.6 | 15.6 | 22.6 | 26.3 | 29.7 | 32.2 | 34.8 |

Table 1: This table presents the first eigenvalue \( \lambda(\mu) \) of problem (2.19) with homogenous Robin boundary conditions in the cross-section [0,0.4].

Example 2.5 Consider the following spectral boundary value problem in a bounded domain \( \omega \) in \( \mathbb{R}^{d-1} \) with \( C^3 \) boundary \( \partial \omega \).

\[
\begin{cases}
-\Delta u_\mu = \lambda_\mu u_\mu & \text{in } \omega \\
\partial_\nu u_\mu + \mu u_\mu = 0 & \text{on } \partial \omega
\end{cases}
\]

where \( \mu \geq 0 \), \( \lambda_\mu \) is the least positive eigenvalue and \( u_\mu \) is a corresponding eigenfunction.

Lemma 2.6 The following formula holds

\[
\lambda_\mu = \lambda_0 - \frac{\lambda_1}{\mu} + O\left( \frac{1}{\mu^2} \right),
\]

where \( \lambda_0 \) is the least positive eigenvalue of the Dirichlet-Laplacian in \( \omega \),

\[
\lambda_1 = \frac{\int_{\partial \omega} |\partial_\nu u_0|^2 \, dS}{\int_\omega |u_0|^2 \, dx},
\]

and \( u_0 \) is the eigenfunction corresponding to \( \lambda_0 \).
Proof. We normalize the sequence in (2.29) by $\|u_\mu\|_{L^2(\omega)} = 1$ then from Lemma 3.1 in [1], it follows that

$$\int_\omega |\nabla u_\mu|^2 dx + \mu \int_{\partial\omega} u_\mu^2 dS \leq C,$$

(2.32)

where $C$ does not depend on $\mu$. This implies that $u_\mu$ is weakly convergent to $u_D$ in $H^1(\omega)$, $u_\mu$ is convergent to $u_D$ in $L^2(\omega)$ and $\lambda_\mu$ converges to $\lambda_D$.

Let us now construct an approximate solution to (2.29), $\tilde{u} = u_D + \frac{1}{\mu} u_1$ and $\tilde{\lambda} = \lambda_D + \frac{1}{\mu} \lambda_1$, where the function $u_1$ and the number $\lambda_1$ are found from the following problem

$$\begin{cases}
(\Delta + \tilde{\lambda}) \tilde{u} = \mu^{-2} \tilde{f} & \text{in } \omega \\
\partial_\nu \tilde{u} + \mu \tilde{u} = \mu^{-1} \tilde{g} & \text{on } \partial\omega
\end{cases}$$

(2.33)

by equating coefficients in $\mu^{-1}$ in the equation and in $\mu^0$ in the boundary condition. As the result we get the following equation for $\lambda_1$ and $u_1$:

$$\begin{cases}
-\Delta u_1 = \lambda_0 u_1 + \lambda_1 u_0 & \text{in } \omega \\
u_1 + \partial_\nu u_0 = 0 & \text{on } \partial\omega
\end{cases}$$

(2.34)

Multiplying the first equation in (2.34) by $u_0$ and integrating over $\omega$ we obtain the following solvability criterion for the problem (2.34), which is considered as
a problem with respect to $u_1$,

$$ - \int_\omega \Delta u_1 u_0 dx = \lambda_1 \int_\omega u_0^2 dx + \lambda_0 \int_\omega u_1 u_0 dx $$

$$ = \int_{\partial \omega} (-\partial_{\nu} u_1 u_0 + u_1 \partial_{\nu} u_0) dS - \int_\omega u_1 \Delta u_0 dx $$

which implies that

$$ \lambda_1 \int_\omega u_0^2 dx = \int_{\partial \omega} u_1 \partial_{\nu} u_0 dS = - \int_{\partial \omega} |\partial_{\nu} u_0|^2 dS $$

or (2.31).

The function $u_1$ is determined by solving the problem (2.34). Clearly, $u_1 \in H^1(\omega)$ and

$$ \tilde{f} = \lambda_1 u_1, \quad \tilde{g} = \partial_{\nu} u_1. $$

Multiplying the first equation in (2.33) by $u_\mu$, integrating by parts over $\omega$ and applying the boundary conditions in (2.33) together with (2.29) we obtain

$$ \int_\omega (\Delta + \tilde{\lambda}) \tilde{u}_\mu = \int_{\partial \omega} (\partial_{\nu} \tilde{u}_\mu - \tilde{u} \partial_{\nu} u_\mu) dS + (\tilde{\lambda} - \lambda_\mu) \int_\omega \tilde{u}_\mu dx = \frac{1}{\mu^2} \int_\omega \tilde{f} u_\mu dx $$
which implies that
\[(\tilde{\lambda} - \lambda_\mu) \int_\omega \tilde{u} u_\mu dx = \frac{1}{\mu^2} \int_\omega \tilde{f} u_\mu dx - \frac{1}{\mu} \int_{\partial\omega} \tilde{g} u_\mu dS\]
Since
\[\int_\omega \tilde{u} u_\mu dx = 1 + \mathcal{O} \left( \frac{1}{\mu} \right)\]
and using (2.32) we get
\[|\tilde{\lambda} - \lambda_\mu| \leq \mathcal{O} \left( \frac{1}{\mu^2} \right) + C \frac{1}{\sqrt{\mu}} \|\tilde{g}\|_{L^2(\Omega)} \leq \frac{C}{\mu^2}\]
which proves (2.30) for \(C^3\) boundary \(\partial\omega\).

Remark 2.7 We note that inequalities (2.9) and (2.11) implies that solution from \(L^2(\Omega)\) to the Helmholtz equation with Dirichlet and Robin boundary conditions and with compactly supported right-hand sides exponentially decay at infinity.

3 Solvability of problems (1.5), (1.6) and convergence of the alternating iterative procedure

In this section we describe the function spaces involved in problems (1.5) and (1.6), define the weak solutions for the problems and state their solvability results. We also state without proof the theorem on convergence of the alternating iterative procedure described in Section 1.1.

3.1 Function spaces

The Sobolev space \(H^1(\Omega)\) consists of all functions in \(L^2(\Omega)\) whose first order weak derivatives belong to \(L^2(\Omega)\). As an inner product in \(H^1(\Omega)\), we have
\[a_\mu(u, v) = \int_\Omega (\nabla u \cdot \nabla v - k^2 u v) dx + \mu_0 \int_{\Gamma_0} u v dS + \mu_1 \int_{\Gamma_1} u v dS, \quad (3.35)\]
The corresponding norm we denote by \(\|u\|_\mu = a_\mu(u, u)^{1/2}\) and by Assumption 1.3, this norm is equivalent to the standard norm in \(H^1(\Omega)\). We denote by \(H^{1/2}(\Gamma)\), the space of traces of functions in \(H^1(\Omega)\) on \(\Gamma\). Also, \(H^{1/2}(\Gamma_0)\), the space of restrictions of functions belonging to \(H^{1/2}(\Gamma)\) to \(\Gamma_0\) and \(H^{1/2}(\Gamma_0)\), the subspace of \(H^{1/2}(\Gamma)\) consisting of functions with supports contained in \(\Gamma_0\). The dual spaces of \(H^{1/2}(\Gamma_0)\) is denoted by \(H^{-1/2}(\Gamma_0)\). Similarly, we can define the spaces \(H^{1/2}(\Gamma_1), H^{1/2}(\Gamma_1)\) and \(H^{-1/2}(\Gamma_1)\), see [22, 24].

We also define the following subspaces of \(H^1(\Omega)\). \(H^1(\Omega, \Gamma)\) is the space of functions from \(H^1(\Omega)\) vanishing in \(\Gamma\). \(H^1(\Omega, \Gamma_0)\) and \(H^1(\Omega, \Gamma_1)\) are the spaces of functions from \(H^1(\Omega)\) vanishing in \(\Gamma_0\) and \(\Gamma_1\) respectively.
3.2 Well-posedness of problems (1.5), (1.6)

Similar to [1], Section 3.3, we can introduce the weak solution of the Helmholtz equation, \((\Delta + k^2)u = 0\) as a function \(u\) satisfying the following identity

\[
\int_{\Omega} (\nabla u \cdot \nabla v - k^2 uv) dx = 0, \tag{3.36}
\]

for every function \(v \in H^1(\Omega, \Gamma)\).

We denote the set of weak solutions to the Helmholtz equation by \(\mathbb{H}\). Clearly \(\mathbb{H}\) is a closed subspace of \(H^1(\Omega)\). We define the normal derivative of \(u\) as a function \(\partial_{\nu} u\) in the space \(H\) satisfying the following inequality,

\[
||\partial_{\nu} u||_{H^{-1/2}(\Gamma)} \leq C||u||_{H^1(\Omega)}. \tag{3.37}
\]

Consequently, \(\partial_{\nu} u\big|_{\Gamma_0}\) and \(\partial_{\nu} u\big|_{\Gamma_1}\) are well defined on \(H^{-1/2}(\Gamma_0)\) and \(H^{-1/2}(\Gamma_1)\) respectively and satisfy

\[
||\partial_{\nu} u\big|_{\Gamma_0}||_{H^{-1/2}(\Gamma_0)} + ||\partial_{\nu} u\big|_{\Gamma_1}||_{H^{-1/2}(\Gamma_1)} \leq C||\partial_{\nu} u||_{H^{-1/2}(\Gamma)}.
\]

Again, similar to [1], Section 3.4, we define weak solutions for the two boundary value problems (1.5) and (1.6) and show that they are well-posed. We will state the results without proofs since the proofs follow similar arguments as in the case of elliptic equation in bounded domains in [1].

**Definition 3.1** The weak solutions for the two boundary value problems (1.5) and (1.6) are defined as follows.

1. Let \(f \in H^\frac{1}{2}(\Gamma_0)\) and \(\eta \in H^{-1/2}(\Gamma_1)\). A function \(u \in H^1(\Omega)\) is a weak solution to (1.5) if

\[
a_0(u, v) = \int_{\Gamma_1} \eta v ds, \tag{3.38}
\]

for every function \(v \in H^1(\Omega, \Gamma_0)\), \(u = f\) on \(\Gamma_0\) and \(a_0(u, v)\) denote the restrictions of \(a_\mu(u, v)\) to \(H^1(\Omega, \Gamma_0)\).

2. Let \(\phi \in H^\frac{1}{2}(\Gamma_1)\) and \(g \in H^{-1/2}(\Gamma_0)\). A function \(u \in H^1(\Omega)\) is a weak solution to (1.6) if

\[
a_1(u, v) = \int_{\Gamma_0} g v ds, \tag{3.39}
\]

for every function \(v \in H^1(\Omega, \Gamma_1)\), \(u = \phi\) on \(\Gamma_1\) and \(a_1(u, v)\) denote the restrictions of \(a_\mu(u, v)\) to \(H^1(\Omega, \Gamma_1)\)

The solvability results are presented in the following proposition.

**Proposition 3.2** Uniqueness conditions for the solutions to the two boundary value problems (1.5) and (1.6) are defined as follows.
1. Let $f \in H^{1/2}(\Gamma_0)$ and $\eta \in H^{-1/2}(\Gamma_1)$ and assume that condition (1.4) holds for $u \in H^1(\Omega)$ then there exist a unique weak solution $u \in H^1(\Omega)$ to problem (1.5) such that
\[ \|u\|_{H^1(\Omega)} \leq C \left( \|f\|_{H^{1/2}(\Gamma_0)} + \|\eta\|_{H^{-1/2}(\Gamma_1)} \right), \] (3.40)
where the constant $C$ is independent of $f$ and $\eta$.

2. Let $g \in H^{-1/2}(\Gamma_0)$ and $\phi \in H^{{1/2}}(\Gamma_1)$ and assume that condition (1.4) holds for $u \in H^1(\Omega)$ then there exist a unique weak solution $u \in H^1(\Omega)$ to problem (1.6) such that
\[ \|u\|_{H^1(\Omega)} \leq C \left( \|g\|_{H^{-1/2}(\Gamma_0)} + \|\phi\|_{H^{1/2}(\Gamma_1)} \right), \] (3.41)
where the constant $C$ is independent of $g$ and $\phi$.

3.3 Convergence of the alternating iterative procedure

The alternating iterative algorithm described in section 1.1 is linearly dependent on the functions $f$, $g$ and $\eta$. This alternating iterative procedure gives a convergent approximation of $u$ in $H^1(\Omega)$ as stated in the following theorem provided that in unbounded domains, condition (1.4) holds.

**Theorem 3.3** Let $f_0 \in H^{1/2}(\Gamma_0)$ and $g_0 \in H^{-1/2}(\Gamma_0)$, and let $u \in H^1(\Omega)$ be the solution to the problems (1.1), (1.2). Then for any $\eta \in H^{-1/2}(\Gamma_1)$, the sequence $\{u_n\}_{n=0}^{\infty}$, obtained using the algorithm described in section 1.1 converges to $u$ in $H^1(\Omega)$.

The proof of this theorem follows the same lines as the proof of convergence of the alternating iterative procedure in bounded domains presented in Section 4 in [1] since solutions defined in unbounded domains converges to solutions defined in a bounded cross-section of the unbounded domain.

4 Numerical Experiments

In this section, we present numerical experiments and results that illustrate the convergence of the alternating iterative procedure presented in Section 1.1. We specify the geometry and implement a finite difference method to solve the two well-posed boundary problems presented in Section 1.2.

For the test we choose a simple rectangular domain that represents an infinite strip truncated at the point where the solution has its support. Let $a, b, A$ and $L$ be real scalars and consider the domain,
\[ \Omega = (-A, A) \times (0, L) \]
with
\[ \Gamma_0 = (a, b) \times \{0\} \quad \text{and} \quad \Gamma_1 = ((-A, a) \times \{0\}) \cup ((b, A) \times \{0\}) \cup ((-A, A) \times \{L\}), \]
Figure 3: Description of the domain considered in the test problem.

We choose $A$ in such a way that the solution is supported in $(-A, A) \times (0, L)$ and decay exponentially. In the finite difference implementation we introduce a uniform grid on the domain $\Omega$ of size $N \times M$, such that the step size is $h = 2AN^{-1}$, and thus $M = \text{round}(Lh^{-1})$, and use a standard $O(h^2)$ accurate finite difference scheme. Our finite difference code solves the Helmholtz equation in the domain $\Omega' = (0, 1) \times (0, L')$ and thus we apply the change of variable $x = 2A(x' - \frac{1}{2})$ and $y = 2Ay'$ before computing the numerical solution. Note that the change of variable alters the frequency $k^2$ in the Helmholtz equation and also the Robin boundary conditions since the robin parameters $\mu_0$ and $\mu_1$ are also altered. After having solved the problem on the domain $\Omega'$, we undo the change of variables and display the results in the original domain $\Omega$.

In order to obtain the test problem used in our examples, we first set $N = 1601$ and $A = 4$. We then pick two functions $u(x, 0)$ and $u(x, L)$ which have support in the interval $[-1, 1]$. We use these functions, $A$ and $N$ for all the tests conducted. By solving the Dirichlet problem for the Helmholtz equation in $\Omega$ we obtain a function $u(x, y)$ for $(x, y) \in \Omega$. In Figure 4 we show both
Figure 4: Data $u(x,0)$ (left, red curve), $u(x,L)$ (left, blue curve) and the numerical solution $u(x,y)$ (right) used to set up the problem. The graphs are plotted for $-1 \leq x \leq 1$.

the Dirichlet data $u(x,0)$ on $\Gamma_0$, $u(x,L)$ on $\Gamma_1$ and the computed solution for $k^2 = 5$ and $L = 0.4$. For this computation, $M = 41$. We display the result in the interval $-1 \leq x \leq 1$.

To illustrate the Robin-Dirichlet alternating iterative procedure we choose the initial approximations $\eta^0(x) = \phi^0(x) = 0$ and compute a sequence of approximations $\phi^k(x)$ for different values of $k^2$, $\mu_0$, $\mu_1$ and $L$. We conduct several tests as in the following examples.

**Example 4.1** For the first test, we set $L = 0.4$, $\mu_0 = \mu_1 = 2$. We investigate the convergence of the alternating iterative procedure presented in Section 1.1 with respect to the wavenumber $k^2$. We also investigate how truncation of the domain affect convergence of the procedure. We observe that for the domain truncated at $A = 2$, the alternating iterative procedure produces a convergent sequence for $k^2 < 13.2$ and divergent sequence for $k^2 > 13.2$ while for $A = 4$ the procedure produces a convergent sequence for $k^2 < 12.8$ and divergent sequence for $k^2 > 12.8$. $A = 6$ and $A = 8$ have no significant differences compared to $A = 4$, see Table 2. This motivates our choice for using $A = 4$ in all the tests.

We compare the results obtained in test one by the results computed in Table 1 of Example 2.4. This is because according to Theorem 3.3 and by Inequality 2.11 convergence of iterations and exponential decay of solution at infinity is achieved if $k^2 < \lambda(\mu)$, the first eigenvalue of the Robin-Laplacian. Therefore, in unbounded domains, this estimate should determine how the domain is truncated. From Table 1 for $L = 0.4$ and $\mu = \mu_0 = \mu_1 = 2$, $\lambda(\mu) = 10.6$. The variation of this result from the result obtained in test one is possibly due to the choice of A and other errors.

We also noticed that the convergence is quite slow. See Figure 5 and Figure 6 for illustration of convergence and divergence of the procedure.
Table 2: This table presents the minimum values of $k^2$ needed for convergence of the Robin-Dirichlet alternating iterative procedure for the domain truncated at different points, A and for fixed $L = 0.4$ and $\mu = \mu_0 = \mu_1 = 2$.

| $k^2$ | 13.2 | 12.8 | 12.7 | 12.7 |
|-------|------|------|------|------|

Figure 5: The error $||\phi^{(k)}(x) - u(x,L)||_2$ during the Robin-Dirichlet iterations for $L = 0.4$ and $\mu_0 = \mu_1 = 2.0$. The case $k^2 = 9.5$ (left) represents a case when the iterations converge and the case $k^2 = 13.0$ (right) represent when the iterations diverge. The convergent iterations also demonstrate that the convergence is quite slow.

Example 4.2 For the second test, we investigate for the minimum $\mu = \mu_0 = \mu_1$ needed for convergence of the Robin-Dirichlet alternating iterative procedure for different values of $L$ and $k^2$. We set $L = 0.2, 0.4$ and $0.6$ and $k^2$ ranging from 5.0 to 50.0 as shown in Table 3. From the table, we observe that the minimum $\mu$ needed for convergence increase as $k^2$ increases for all the three different values of $L$. Moreover, our problem is ill-posed and the degree of ill-posedness depends on $L$ i.e at $L = 0.2$ we have a better solution compared to solution obtained at $L = 0.6$ which also explains why we need small values of $\mu$ at $L = 0.2$ in order to obtain convergence as compared to the values of $\mu$ at $L = 0.6$ needed to obtain convergence.

In Table 3 ($-$) means that for $L = 0.2$ and $k^2 < 15.1$ there is no minimum $\mu$ needed for convergence of the alternating iterative procedure, $\mu = 0$ is sufficient. This is as a result of the truncation of the domain so that the Neumann-Dirichlet alternating iterative procedure works for small values of $k^2$ in the Helmholtz equation in a bounded domain. In principle Neumann-Dirichlet alternating iterative procedure should not work in unbounded domains.

Also from Table 3 ($\ast$) represent cases where for large values of $k^2$, we obtain solutions which are not supported in $(-1,1) \times (0,L)$ and which do not decay.
Figure 6: The exact function $u(x, L)$ (left, solid line) and reconstructed function $\varphi^j(x)$ for $j = 500$ (left, dashed line) obtained by the alternating iterative procedure. We also display the approximate numerical solution $u(x, y)$ (right). Here, $L = 0.4$, $\mu_0 = \mu_1 = 2.0$ and $k^2 = 5$. We plot the graphs for $-1 \leq x \leq 1$.

exponentially at infinity hence do not solve the test problem (4.42). In Figure 7 we present four solutions which shows the transition of the solutions as $k^2$ increases.

| $k^2$ | $L$ | 0.2 | 0.4 | 0.6 |
|-------|-----|-----|-----|-----|
| 5.0   | -   | 0.2 | 1.1 |
| 10.0  | -   | 1.3 | 3.4 |
| 15.0  | -   | 2.6 | 7.2 |
| 20.0  | 0.5 | 4.2 | 15.7|
| 25.0  | 1.0 | 6.1 | 54.0|
| 30.0  | 1.6 | 8.4 | *   |
| 35.0  | 2.1 | 11.9| *   |
| 40.0  | 2.7 | 16.5| *   |
| 50.0  | 3.9 | 36.8| *   |

Table 3: This table presents the minimum $\mu$ needed for convergence of the Robin-Dirichlet alternating iterative procedure for different values of $L$ and $k^2$.

5 Conclusion

It was proved in [5] that the Robin-Dirichlet alternating iterative procedure converges even for large values of $k^2$ in the Helmholtz equation in bounded domains if the Robin parameter $\mu$ is appropriately chosen. In this paper, we derive
Figure 7: Top left is the solution obtained for $k^2 = 25$ which is supported in $(-1, 1) \times (0, 0.6)$ hence solves problem (4.42). Top right is a sine solution for $k^2 = 27$ which is just above the limit point where the transition takes place. It is clearly not supported in $(-1, 1) \times (0, 0.6)$ hence does not solve problem (4.42). Bottom left and bottom right are solutions obtained for $k^2 = 28$ and $k^2 = 35$ respectively which are not supported in $(-1, 1) \times (0, 0.6)$ and which illustrate the transition of the solution as $k^2$ increases. All the four solutions are computed for $L = 0.6$.

the necessary conditions for the convergence of the Robin-Dirichlet alternating iterative procedure in unbounded domains.

Estimate for values of $k^2$ in the Helmholtz equation in terms of positivity of a certain quadratic form which guarantees convergence of the Robin-Dirichlet alternating iterative procedure is given. We analyse this condition and present some explicit estimates for $k^2$ in terms of eigenvalues of certain auxiliary problems.

For the numerical experiments, we choose a rectangular domain in $\mathbb{R}^2$ that represents a truncated infinite strip and test for convergence of the procedure. We know that for the Cauchy problem for the Helmholtz equation in a finite do-
main, the Neumann-Dirichlet alternating procedure converges for small values of $k^2$ in the Helmholtz equation while in an infinite domain the Neumann-Dirichlet procedure does not converge at all. By appropriate truncation of the infinite domain and with the introduction of the Robin parameters $\mu_0$ and $\mu_1$, we achieve convergence of the Robin-Dirichlet alternating iterative procedure and the solution decay exponentially at infinity. However we noticed that the convergence is generally slow. We further investigated dependence of the procedure on the parameter $\mu = \mu_0 = \mu_1$ for different values of $k^2$ and $L$ (the distance between the boundaries). For $\mu = \mu_0 = \mu_1$, the Robin-Dirichlet alternating iterative procedure is divergent for small values of $\mu$ and convergence for large values of $\mu$ and grows as $k^2$ and $L$ increases. However for large values of $k^2$, we obtain solutions that do not solve the Helmholtz equation in an infinite domain i.e solutions that do not decay exponentially at infinity.

In our future work, we will investigate the effect of the size of the bounded inclusion on the convergence of the alternating iterative procedure. We will also investigate the procedure with inexact Cauchy data and add regularization. Finally we will seek to find areas of application.

References

[1] P. Achieng, F. Berntsson, J. Chepkorir, and V.A. Kozlov. Analysis of Dirichlet-Robin iterations for solving the Cauchy Problem for Elliptic equations. *Bulletin of the Iranian Mathematical Society* (in press), 2020.

[2] Sergei Avdonin, Vladimir Kozlov, D Maxwell, and M Truffer. Iterative methods for solving a nonlinear boundary inverse problem in glaciology. *Journal of inverse and ill-posed problems*, 17(3):239–258, 2009.

[3] F. Berntsson, V.A. Kozlov, L. Mpinganzima, and B.O. Turesson. An accelerated alternating procedure for the Cauchy problem for the Helmholtz equation. *Computers & Mathematics with Applications*, 68(1-2):44–60, 2014.

[4] Fredrik Berntsson, VA Kozlov, Lydie Mpinganzima, and Bengt-Ove Turesson. An alternating iterative procedure for the Cauchy problem for the Helmholtz equation. *Inverse Problems in Science and Engineering*, 22(1):45–62, 2014.

[5] Fredrik Berntsson, Vladimir Kozlov, Lydie Mpinganzima, and Bengt-Ove Turesson. Robin–Dirichlet algorithms for the Cauchy problem for the Helmholtz equation. *Inverse Problems in Science and Engineering*, 26(7):1062–1078, 2018.

[6] R Chapko and BT Johansson. An alternating potential-based approach to the Cauchy problem for the laplace equation in a planar domain with a cut. *Computational Methods in Applied Mathematics*, 8(4):315–335, 2008.
[7] J.T. Chen and F.C. Wong. Dual formulation of multiple reciprocity method for the acoustic mode of a cavity with a thin partition. *Journal of Sound and Vibration*, 217(1):75–95, 1998.

[8] D. Colton and R. Kress. *Inverse Acoustic and Electromagnetic Scattering Theory*. Springer-Verlag, 2nd edition, 1998.

[9] T. Delillo, V. Isakov, N. Valdivia, and L. Wang. The detection of the source of acoustical noise in two dimensions. *SIAM J. Appl. Math.*, 61(6):2104–2121, 2001.

[10] T. Delillo, V. Isakov, N. Valdivia, and L. Wang. The detection of surface vibrations from interior acoustical pressure. *Inverse Problems*, 19:507–524, 2003.

[11] B Tomas Johansson and Vladimir A Kozlov. An alternating method for Cauchy problems for Helmholtz-type operators in non-homogeneous medium. *IMA journal of applied mathematics*, 74(1):62–73, 2009.

[12] D.S. Jones. The eigenvalues of $\nabla^2 u + \lambda u = 0$ when the boundary conditions are given on semi-infinite domains. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 49, pages 668–684. Cambridge University Press, 1953.

[13] D.S. Jones. *Acoustic and Electromagnetic Waves*. Clarendon Press, 1986.

[14] V.A. Kozlov and V.G. Maz’ya. Iterative procedures for solving ill-posed boundary value problems that preserve the differential equations. *Algebra i Analiz*, 1(5):144–170, 1989. translation in Leningrad Math. J. 1(1990), no. 5, pp. 1207–1228.

[15] V.A. Kozlov, V.G. Maz’ya, and A.V. Fomin. An iterative method for solving the cauchy problem for elliptic equations. *Comput. Maths. Math. Phys.*, 31(1):46–52, 1991.

[16] Minghui Liu, Deyue Zhang, Xu Zhou, and Feng Liu. The Fourier–Bessel method for solving the Cauchy problem connected with the Helmholtz equation. *Journal of Computational and Applied Mathematics*, 311:183–193, 2017.

[17] L. Marin. Boundary element–minimal error method for the Cauchy problem associated with Helmholtz–type equations. *Comput. Mech.*, 44(2):205–219, 2009.

[18] L. Marin, L. Elliott, P. J. Heggs, D. B. Ingham, D. Lesnic, and X. Wen. Conjugate gradient-boundary element solution to the Cauchy problem for Helmholtz-type equations. *Comput. Mech.*, 31(3-4):367–377, 2003.
[19] L Marin, L Elliott, PJ Heggs, DB Ingham, D Lesnic, and X Wen. An alternating iterative algorithm for the Cauchy problem associated to the Helmholtz equation. *Computer methods in applied mechanics and engineering*, 192(5-6):709–722, 2003.

[20] Liviu Marin. A meshless method for the numerical solution of the Cauchy problem associated with three-dimensional Helmholtz-type equations. *Applied Mathematics and Computation*, 165(2):355–374, 2005.

[21] Liviu Marin and Daniel Lesnic. The method of fundamental solutions for the Cauchy problem associated with two-dimensional Helmholtz-type equations. *Computers & Structures*, 83(4-5):267–278, 2005.

[22] David Maxwell. Kozlov-maz’ya iteration as a form of landweber iteration. *arXiv preprint arXiv:1107.2194*, 2011.

[23] David Maxwell, Martin Truffer, Sergei Avdonin, and Martin Stuefer. An iterative scheme for determining glacier velocities and stresses. *Journal of Glaciology*, 54(188):888–898, 2008.

[24] W McLean. *Strongly Elliptic Systems and Boundary Integrated Equations*. Cambridge University Press, New York, 2000.

[25] Sergei Aleksandrovich Nazarov. Variational and asymptotic methods for finding eigenvalues below the continuous spectrum threshold. *Siberian mathematical journal*, 51(5):866–878, 2010.

[26] H.H. Qin, T. Wei, and R. Shi. Modified Tikhonov regularization method for the Cauchy problems of the Helmholtz equation. *J. Comput. Appl. Math.*, 24:39–53, 2009.

[27] H.H Qin and DW Wen. Tikhonov type regularization method for the Cauchy problem of the modified Helmholtz equation. *Applied mathematics and computation*, 203(2):617–628, 2008.