Advanced Analysis of Quantum Contextuality in a Psychophysical Double-Detection Experiment

Víctor H. Cervantes1,* and Ehtibar N. Dzhafarov1,†

1Purdue University, USA

The results of behavioral experiments typically exhibit inconsistent connectedness, i.e., they violate the condition known as “no-signaling,” “no-disturbance,” or “marginal selectivity.” This prevents one from evaluating these experiments in terms of quantum contextuality if the latter understood traditionally (as, e.g., in the Kochen-Specker theorem or Bell-type inequalities). The Contextuality-by-Default (CbD) theory separates contextuality from inconsistent connectedness. When applied to quantum physical experiments that exhibit inconsistent connectedness (due to context-dependent errors and/or signaling), the CbD computations reveal quantum contextuality in spite of this. When applied to a large body of published behavioral experiments, the CbD computations reveal no quantum contextuality: all context-dependence in these experiments is described by inconsistent connectedness alone. Until recently, however, experimental analysis of contextuality was confined to so-called cyclic systems of binary random variables. Here, we present the results of a psychophysical double-detection experiment that do not form a cyclic system: their analysis requires that we use a recent modification of CbD, one that makes the class of noncontextual systems more restricted. Nevertheless our results once again indicate that when inconsistent connectedness is taken into account, the system exhibits no contextuality.

KEYWORDS: contextuality, cyclic systems, double-detection, inconsistent connectedness, psychophysics.

In recent years there were many reports of behavioral experiments (Accardi, Khrennikov, Ohya, Tanaka, & Yamato, 2016; Aerts & Sozzo, 2014, 2015; Aerts, Sozzo, & Veloz, 2015; Asano, Hashimoto, Khrennikov, Ohya, & Tanaka, 2014; Bruza, Kitto, Ramm, & Sitbon, 2015; Cervantes & Dzhafarov, 2017; Dzhafarov, Zhang, & Kujala, 2015; Khrennikov, 2015; Sozzo, 2015; Wang, Solloway, Shifrin, & Busemeyer, 2014; Zhang & Dzhafarov, 2017) aimed at (or interpretable as aimed at) revealing contextuality of the kind predicted by and experimentally confirmed in quantum physics (Bell, 1964; Clauser, Horne, Shimony, & Holt, 1969; Fine, 1982; Hensen et al., 2015; Klyachko, Can, Binicioğlu, & Shumovsky, 2008; Kochen & Specker, 1967; Kurzyński, Ramanathan, & Kaszlikowski, 2012; Łapkiewicz et al., 2011). All known to us behavioral data, however, violate a certain condition that makes a direct application of the traditional quantum contextuality analysis impossible. This condition is variously called “no-signaling” or “no-disturbance” in quantum physics (Bacciagaluppi, 2015, 2016; Cerceda, 2000; Kofler & Brukner, 2013; Kurzyński, Cabello, & Kaszlikowski, 2014; Popescu & Rohrlich, 1994; Ramanathan, Soeda, Kurzyński, & Kaszlikowski, 2012) and “marginal selectivity” in psychology (Dzhafarov, 2003; Townsend & Schweickert, 1989; Zhang & Dzhafarov, 2015). It is a required condition for the traditional quantum contextuality analysis, even though it is often violated in quantum mechanical experiments as well (this issue was first systematically discussed in Adenier & Khrennikov, 2007; see also Adenier & Khrennikov, 2016; Łapkiewicz et al., 2011, 2013). The Contextuality-by-Default (CbD) theory (de Barros, Dzhafarov, Kujala, & Oas, 2015; Dzhafarov, 2016; Dzhafarov & Kujala, 2014a, 2014b, 2015, 2016a, 2016b, 2017a, in press; Dzhafarov, Kujala, & Cervantes, 2016; Dzhafarov, Kujala, & Larsen, 2015) overcomes this difficulty by proposing a principled way of separating contextuality proper from inconsistent connectedness (the CbD term for violations of the “no-signaling” or “marginal selectivity” condition).

This theory was used to reanalyze the behavioral experiments aimed at contextuality, with the conclusion that they provide no evidence for contextuality (Cervantes & Dzhafarov, 2017; Dzhafarov, Kujala, Cervantes, Zhang, & Jones, 2016; Dzhafarov, Zhang, & Kujala, 2015; Zhang & Dzhafarov, 2017): inconsistent connectedness is the only form of context-dependence that we have in them. By contrast, when CbD is used to reanalyze a quantum-mechanical experiment that exhibits inconsistent connectedness (Łapkiewicz et al., 2011), contextuality proper (on top of inconsistent connectedness) is established beyond doubt (Kujala, Dzhafarov, & Larsson, 2015).

Virtually all experiments aimed at revealing contextuality, both in quantum physics and in behavioral sciences, deal with a special kind of systems of random variables, called cyclic systems in CbD (Kujala et al., 2015). In these experiments each property is measured in precisely two different contexts, and each context contains two properties being measured together. If, in addition, all random variables in the system are binary (each indicating presence or absence of a certain property), then the system is amenable to complete and exhaustive contextuality analysis (Dzhafarov & Kujala, 2016a; Dzhafarov, Kujala, & Cervantes, 2016; Dzhafarov, Kujala, & Larsson, 2015; Kujala et al., 2015). In spite of their prominence in quantum theory, however, it is highly desirable to extend contextuality analysis beyond the class of cyclic systems. Many researchers (although not the present authors) find the lack of contextuality in behavioral data to be a disappointing negative result. What if this result is due to the fact that cyclic systems in human behavior are too simple? What if it is “too easy” for a cyclic system to be noncontextual? These are valid questions, and they will have no definite answers until we have a predictive theory of (at least certain types of) human behavior on a par with quantum mechanics.

*Email: cervantv@purdue.edu
†Email: ehtibar@purdue.edu
In the absence of a predictive theory, the only, admittedly imperfect way of dealing with these considerations is to expand the experimentation and contextuality analysis to progressively broader classes of systems. In this paper we make a first step in this direction by analyzing a psychophysical experiment whose results form a non-cyclic system of random variables. This experiment was reported previously (Cervantes & Dzhafarov, 2017), but its analysis was confined to extracting from it a large number of cyclic subsystems and showing all of them to be noncontextual. It is mathematically possible, however, that a system is contextual with all its cyclic subsystems being noncontextual.

A satisfactory way to expand the contextuality analysis beyond cyclic systems was proposed in a recent modification of CbD, dubbed “CbD 2.0” (Dzhafarov & Kujala, 2017a, in press): it is essentially the original CbD in which the measurements of the same property (say, responses to the same stimulus) are analyzed in pairs only. This modification has compelling reasons behind it. The main one is that in the modified theory a subsystem of a noncontextual system is always noncontextual. Another reason is that contextuality analysis is reduced to the problem of compatibility of two uniquely defined sets of distributions: the empirically known distributions of context-sharing random variables and the distributions of the “multimaximal couplings” of the random variables measuring the same property in different contexts. All of this is clarified below (Section 2). The modification in question does not affect the theory of cyclic systems, so the results mentioned earlier remain unchanged. However, when it comes to non-cyclic systems, the modification makes the requirements that a system should satisfy to be noncontextual more stringent.

The plan of the paper is as follows. In the next two sections we present the basics of the CbD theory, in the “CbD 2.0” version. The discussion is primarily confined to systems of binary random variables (dichotomic measurements), both for simplicity and because the double-detection experiment to be analyzed involves only dichotomic judgments. In Section 3 we apply this theory to the results of our double-detection experiment. Our conclusion is that in spite of the notion of noncontextuality we use being more restrictive than in the original version of the CbD theory, the double detection experiment does not exhibit any contextuality.

1. Introduction to Contextuality

Every experiment results in a system of random variables. In most physics experiments these random variables are interpreted as measurements of properties, in most behavioral experiments they are interpreted as responses to stimuli, such as questions. For brevity we will use the term “measurement” in both meanings (because responding to a stimulus can always be viewed as a form of measurement). What is being measured therefore is part of the identity of a random variable representing a measurement. It is referred to as the content of the random variable. The content, however, does not specify a random variable uniquely, because one and the same content can be measured under different conditions, referred to as contexts. For instance, if a content \( q \) is measured simultaneously with measurements of other contents, in some cases \( q' \) and in other cases \( q'' \), then in the former cases the content \( c = (q, q') \) and in the latter ones it is \( c' = (q, q'') \). As in Dzhafarov and Kujala (2016a, 2017a), we will write “conteXt” and “conteNt” to prevent their confusion in reading. The conteXt and conteNt of a random variable uniquely identify it within a given system of random variables. So each random variable in a system is double-indexed, \( R_{c\ell} \).

According to the CbD theory’s main principle (Dzhafarov, 2016; Dzhafarov & Kujala, 2014a, 2016a, 2016b, in press; Dzhafarov, Kujala, & Cervantes, 2016), two random variables \( R_{c'\ell'} \) and \( R_{c''\ell''} \) are jointly distributed if and only if \( c = c' \), i.e., if and only if they are recorded in the same conteXt. Otherwise they are stochastically unrelated, i.e., joint probabilities for them are undefined. This means, in particular, that any two \( R_{c\ell} \) and \( R_{q\ell} \) with the same conteNt in different conteXts are stochastically unrelated (which implies, among other things, that they can never be considered to be one and the same random variable). Their individual distributions may be the same but they need not be. If these distributions are different, the system exhibits a form of context-dependence. However, in CbD, this context-dependence by itself does not say that the system is contextual in the sense related to how this term is used in quantum mechanics. Rather the difference in the distributions is treated as manifestation of information/energy flowing to the measurements of conteNt \( q \) from elements of the conteXts \( c, c' \) other than \( q \). We will refer to this transfer of information/energy as direct cross-influences. Thus, if \( c = (q, q') \) and \( c' = (q, q'') \), the conteNt \( q \) does, of course, directly influence its measurement, but, with \( q \) fixed, the second conteNt in the pair can also affect this measurement. This can sometimes be attributed to some physical action of \( q' \) or \( q'' \) upon the process measuring \( q \), or (as another form of information transfer) it can be a form of contextual bias, a change in the procedure by which \( q \) is measured depending on what else is being measured.

\[
\begin{align*}
q & \quad q', q'' \\
\text{measurement of } q & \quad \text{(variable) direct cross-influence}
\end{align*}
\]

The difference between the distributions of \( R_{c\ell} \) and \( R_{c'\ell'} \) (equivalently, the strength of the direct cross-influences responsible for this difference) is measured in CbD by the probability with which \( R_{c\ell} \) and \( R_{c'\ell'} \) could be made to coincide if they were jointly distributed. This means that we consider all couplings of \( R_{c\ell}, R_{c'\ell'}, \) i.e., the jointly distributed pairs of random variables \( T_{c\ell}, T_{c'\ell'} \) whose respective individual distributions are the same as those of \( R_{c\ell}, R_{c'\ell'} \), and among these pairs we find the one(s) with the maximal possible probability of \( T_{c\ell} = T_{c'\ell'} \). The larger this maximal probability, the closer the two distributions to each other, and the weaker the direct cross-influences.
by conteXtSs \( c, c' \) upon the measurement of \( q \). This maximal probability is 1 if and only if the two distributions are identical, and it is 0 if an only if the two distributions have disjoint supports.

Consider now an experiment represented by a system of random variables \( R^c_i \) with varying \( c \) and \( q \), and suppose that we have computed the maximal probability just described for each pair of random variables that share a conteXt. And we know (or can empirically estimate) the joint distributions of all random variables that share a conteXt. Intuitively, quantum contextuality is about whether these computed maximal probabilities and these empirically defined joint distributions are mutually compatible. If they are not, then one can say that conteXtSs force the random variables sharing conteXtSs to be more dissimilar than they are made by direct cross-influences alone. The system then can be considered contextual.

To understand this without conceptual and technical complications, consider first a cyclic system of binary random variables (Dzhafarov, Kujala, & Larsson, 2015; Kujala & Dzhafarov, 2015; Kujala et al., 2015). It is depicted in Fig. 1. The conteXtSs and conteXtSs are such that, with appropriate enumeration, in conteXt \( c_i \) one measures precisely two cyclically-successive conteXtSs \( q_i, q_{i+1} \) (where \( i = 1, \ldots, n; i \oplus 1 = i + 1 \) for \( i < n \); and \( n \oplus 1 = 1 \)):

\[
\begin{align*}
q_1 & \rightarrow c_1 \leftarrow q_2 \rightarrow c_2 \leftarrow \cdots \leftarrow c_{n-2} \leftarrow q_{n-1} \leftarrow c_{n-1} \leftarrow q_n \rightarrow c_n \leftarrow q_1,
\end{align*}
\]

Each pair \( R^c_i, R^c_{i+1} \) (\( i = 1, \ldots, n \)) of random variables sharing a conteXt (within a row in Fig. 1) are jointly distributed. Since all the measurements in the system are binary (±1), the joint distribution of \( R^c_i \) is uniquely determined by three probabilities,

\[
\begin{align*}
p_i^c &= \Pr \left[ R^c_i = 1 \right], \quad p^c_{i+1} = \Pr \left[ R^c_{i+1} = 1 \right], \\
p^c &= \Pr \left[ R^c_i = R^c_{i+1} = 1 \right].
\end{align*}
\]

Random variables \( R^{c\oplus1}_i, R^c_i \) within a column share a conteXt, and we compute for each such a pair the magnitude of direct cross-influences, max \( \Pr \left[ T^c_i = T^{c\oplus1}_i \right] \), across all couplings \( (T^{c\oplus1}_i, T^c_i) \) of \( R^{c\oplus1}_i, R^c_i \): in this case the couplings are the pairs \( (T^{c\oplus1}_i, T^c_i) \) with all possible values of \( \Pr \left[ T^c_i = T^{c\oplus1}_i = 1 \right] \) and with

\[
\Pr \left[ T^c_i = 1 \right] = p_i^c, \quad \Pr \left[ T^{c\oplus1}_i = 1 \right] = p^{c\oplus1}_i.
\]

Here, \( i = 1, \ldots, n; i \oplus 1 = i - 1 \) for \( i > 1 \); and \( 1 \oplus 1 = n \). The coupling \( (T^{c\oplus1}_i, T^c_i) \) with this property is called maximal coupling. It is easy to show (Thorisson, 2000) that this maximal coupling always exists and is defined by complementing (2) with

\[
p_i = \Pr \left[ T^c_i = T^{c\oplus1}_i = 1 \right] = \min \left\{ p_i^c, p^{c\oplus1}_i \right\}.
\]

The probabilities (1) and (3) are shown in Fig. 2. Note that (2) and (3) uniquely define the joint distribution of the two random variables \( T^{c\oplus1}_i, T^c_i \) within each column of the matrix, in the same way as (1) uniquely define the joint distribution of \( R^c_i, R^c_{i+1} \) within each row of the matrix. The only difference is that the row-wise joint distributions are empirical reality, whereas the column-wise joint distributions are constructed artificially to depict the direct cross-influences. Contextuality in CbD is all about the compatibility of these column-wise and row-wise joint distributions: the system is considered noncontextual if all these probabilities can be achieved within a jointly distributed set of \( 2n \) random variables. In other word, we seek a set of jointly distributed random variables \( S^c_j \) replacing the star symbols in Fig. 1, such that

\[
\begin{align*}
(i) & \quad \Pr \left[ S^c_i = 1 \right] = p^c_i, \quad \Pr \left[ S^c_{i+1} = 1 \right] = p^c_{i+1}, \\
(ii) & \quad \Pr \left[ S^c_i = S^c_{i+1} = 1 \right] = p^c, \\
(iii) & \quad \Pr \left[ S^c_i = S^c_{i\oplus1} = 1 \right] = p_i = \min \left\{ p_i^c, p^{c\oplus1}_i \right\}
\end{align*}
\]

The equations (i) and (ii) in (4) tell us that the set of the \( S^c_j \)-variables we seek is a coupling of the original random variables \( R^c_i \) arranged row-wise in Fig. 1: in each row the variables \( R^c_i \) have a well-defined joint distribution, but different rows are stochastically unrelated, so the coupling “saws them together” in a single joint distribution. The equations (i) and (iii) in (4) tell us that the set of the \( S^c_j \)-variables is a coupling for the column-wise maximal couplings \( T^c_j \): in each of the columns the variables \( T^c_j \) have a well-defined joint distribution, but different columns are stochastically unrelated because the maximal couplings were computed for each column separately; so the coupling “saws the columns together” in a single joint distribution. It is easy to see that each of these two couplings (of the rows and of the columns) exists, because the random variables in the different rows do not overlap, and the same is true for different columns. In a typical case, each of the two couplings can be constructed in an infinity of ways, and the question is whether a jointly distributed set of \( 2n \) random variables can be simultaneously a coupling for the rows and for the columns. If the answer to this question is negative, the conteXtSs intervene beyond the effect of the direct cross-influences.

2. CONTEXTUALITY IN ARBITRARY SYSTEMS OF BINARY MEASUREMENTS

Let us discuss now how the analysis just presented extends beyond cyclic systems. We will continue to assume that all the random variables in play are binary.
Consider Fig. 3. The system \( X \) is not cyclic, as it has three random variables in the first row (conteXt \( c_1 \)) and three random variables in the fourth column (conteNt \( q_4 \)). The number and arrangement of the random variables in a row, however, is immaterial for the logic of the contextuality analysis. The joint distribution of \( R_1^{\star}, R_2^{\star}, R_3^{\star} \) in the first row of \( X \) is uniquely defined empirically. It simply requires more probabilities than in (1) to be described:

\[
p_1^{\star} = \Pr [R_1^{\star} = 1], \quad p_2^{\star} = \Pr [R_2^{\star} = 1], \\
p_3^{\star} = \Pr [R_3^{\star} = 1], \\
p_4^{\star} = \Pr [R_1^{\star} = R_2^{\star} = R_3^{\star} = 1].
\]

(5)

Nor does anything change in how one treats the pairs of the conteNt-sharing random variables in the first three columns: one computes the maximal coupling for each of these columns. One faces choices, however, when dealing with the three random variables in the fourth column. What is the right way of generalizing the maximal coupling in this case? There is a compelling reason (Dzhafarov & Kujala, 2017a, in press) to consider the three conteXt-sharing random variables one pair at a time, and to compute maximal couplings for them separately. This means finding a jointly distributed triple \((T_1^1, T_3^3, T_4^1)\) whose elements are distributional copies of, respectively, \( R_4, R_2, R_1 \), i.e.,

\[
\Pr [T_1^1 = 1] = p_1^1, \quad \Pr [T_3^3 = 1] = p_3^3, \\
\Pr [T_4^1 = 1] = p_4^1,
\]

(6)

such that \((T_1^1, T_3^3)\) is the maximal coupling of \( R_1, R_2 \), \((T_1^1, T_4^1)\) is the maximal coupling of \( R_1, R_4 \), and \((T_3^3, T_4^1)\) is the maximal coupling of \( R_3, R_4 \). In terms of probability values,

\[
\Pr [T_1^1 = T_3^3 = 1] = \min \{p_1^1, p_3^3\}, \\
\Pr [T_1^1 = T_4^1 = 1] = \min \{p_1^1, p_4^1\}, \\
\Pr [T_3^3 = T_4^1 = 1] = \min \{p_3^3, p_4^1\}.
\]

(7)

As shown in Dzhafarov and Kujala (2017a, in press), such a coupling (called multimaximal in CyBd) always exists, and it is unique (as all the random variables here are binary). The above-mentioned compelling reason for maximizing the couplings pairwise is that then, if the system is noncontextual, it will remain noncontextual after one deletes from it one or more random variables. In other words, any subsystem of a noncontextual system is noncontextual. This would not be true, for instance, if we only maximized the value of \( \Pr [T_1^1 = T_2^2 = T_3^3 = 1] \). At the same time, the maximization of \( \Pr [T_1^1 = T_2^2 = T_3^3 = 1] \) is achieved “automatically” if (7) is satisfied. Moreover, one of the equalities in (7) is redundant as it can be derived from the other two: if, e.g., \( p_1^1 \leq p_2^1 \leq p_4^1 \), then the redundant equality in (7) is the second one. Generalizing, we have the following theorem.

**Theorem 2.1** (Dzhafarov & Kujala, 2017a, in press). Let \( R_1^1, \ldots, R_k^k \), \( k > 1 \), be binary \((\pm 1)\) random variables with conteXts\(s\) enumerated so that

\[
p_i^q = \Pr [R_i^q = 1] = \ldots = \Pr [R_k^q = 1] = p_i^q.
\]

Then there is a unique set of jointly distributed \((T_1^1, \ldots, T_k^k)\) such that \((T_i^i, T_{i+1}^{i+1})\) is the maximal coupling of \( R_i^i, R_{i+1}^{i+1} \), for \( i = 1, \ldots, k-1 \). The coupling \((T_i^i, \ldots, T_k^k)\) has the following properties.

(i) For any subset \( \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, k\} \) \( m \leq k \), \((T_{i_1}^{i_1}, \ldots, T_{i_m}^{i_m})\) is the maximal coupling of \( R_{i_1}^{i_1}, \ldots, R_{i_m}^{i_m} \), i.e., \( \Pr [T_{i_1}^{i_1} = \ldots = T_{i_m}^{i_m} = 1] \) has the maximal possible value among all couplings of \( R_{i_1}^{i_1}, \ldots, R_{i_m}^{i_m} \). In particular, for any \( i, j \in \{1, \ldots, k\} \), \((T_i^i, T_j^j)\) is the maximal coupling of \( R_i^i, \ldots, R_j^j \).

(ii) The distribution of \((T_1^1, \ldots, T_k^k)\) is defined by

\[
\Pr [T_1^1 = \ldots = T_k^k = 1] = p_1^1, \\
\Pr [T_1^1 = \ldots = T_q^q = 1; T_{q+1}^{q+1} = \ldots = T_k^k = 1] = p_{q+1} - p_q, \\
\text{(for } l = 1, \ldots, k-1\text{)}
\]

(8)

with all other combinations of values having probability zero.

Now we can formulate the generalization of the definition of contextuality given in the previous section.

**Definition 2.2.** A system of binary random variables \( R_q^c \) is noncontextual if there exists a jointly distributed set of (correspondingly labeled) random variables \( S_q^c \) such that (i) for every conteXt \( c \), the joint distribution of all \( S_q^c \) with this value of \( c \) is identical to the joint distribution of the corresponding \( R_q^c \); and (ii) for every conteNt \( q \), the joint distribution of all \( S_q^c \) with this value of \( q \) forms (the unique) multimaximal coupling of the corresponding \( R_q^c \)
3. DOUBLE-DETECTION EXPERIMENT

We now apply the theory just described to the results of a double-detection experiment. We remind the reader that this experiment was previously described in Cervantes and Dzhafarov (2017), but to keep this paper self-sufficient we recapitulate the procedural details below. In Cervantes and Dzhafarov (2017) the system formed by the data was analyzed by extracting from it a multitude of cyclic subsystems. In this paper we analyze the system in its entirety.

The double-detection experiment is one of only two contextuality-aimed experiments known to us that uses a within-subject design, i.e., with probabilities estimated from the responses of a single person to multiple replications of stimuli. (The other such experiment is the psychophysical matching one described in Dzhafarov, Zhang, & Kujala, 2015, and Zhang & Dzhafarov, 2017.) Most experiments use aggregation of responses obtained from many persons. The double detection paradigm suggested in Dzhafarov and Kujala (2012) and Dzhafarov and Kujala (2017b) provides a framework where both (in)consistent connectedness and contextuality can be studied in a manner very similar to how they are studied in quantum-mechanical systems (or could be studied, because consistent connectedness in quantum physics is often assumed rather than documented).

3.1. Method

3.1.1. Participants

The participants were three volunteers, graduate students at Purdue University, two females and one male (the first author of this paper), aged around 30, with normal or corrected to normal vision. The experimental program was regulated by the Purdue University’s IRB protocol #1202011876. The participants are identified as P1 – P3 in the text below.

3.1.2. Equipment

A personal computer was used with an Intel® Core™ processor running Windows XP, a 24-in. monitor with a resolution of 1920 × 1200 pixels (px), and a standard US 104-key keyboard. The participant’s head was steadied in a chin-rest with forehead support at 90 cm distance from the monitor; at this distance a pixel on the screen subtended 62 sec arc.

3.1.3. Stimuli

The stimuli presented on the computer screen consisted of two brightly grey colored circles (RGB 100-100-100) on a black background, with their centers 320 px apart horizontally, each circle having the radius of 135 px and circumference 4 px wide. Each circle contained a dot of 4 px in diameter in its center or 4 px away from it, in the upward or downward direction. An example of the stimuli (in reversed contrast and scaled) is shown in Figure 4.

FIG. 4: An example of the stimulus in the double-detection experiment. In the left circle the dot is in the center, in the right one it is shifted 4 px upwards. The participant’s task was to say, for each of the two circles, whether the dot was in the center (the answer coded 1) or off-center (the answer coded -1), irrespective of whether it was shifted up or down.

3.1.4. Procedure

In each trial the participant was asked to indicate, for each circle, whether the dot was in its center or not in the center (irrespective of in what direction). The responses were given by pressing in any order and holding together two designated keys, one for each circle, and the stimuli were displayed until both keys were pressed. Then, the dots in each circle disappeared, and a “Press the space bar to continue” message appeared above the circles. Pressing the space bar removed the message, and the next pair of dots appeared 400 ms later. (Response times were recorded but not used in the data analysis.)

Each participant completed nine experimental sessions, each lasting 30 minutes and containing about 560 trials recorded and used for the analysis, preceded by several practice trials. In each practice trial the participants received feedback as to whether their response for each of the two circles was correct or not. No feedback
FIG. 5: Probabilities with which a trial was allocated to one of the 9 conteXts, with the notation used for the conteXts and the conteNts: c, u, and d denote that the dot is, respectively, in the center, shifted up, or shifted down. The 9 conteXts are denoted cc, cu, du, etc., the left (right) symbol indicating the location of the dot in the left (respectively, right) circle. To denote conteNts, the location of a dot is shown on the left or on the right with a dash filling the other side: thus, -c denotes the dot in the center of the left circle, -d denotes the dot shifted down in the right circle, etc.

was given in the non-practice trials. The experimental sessions were preceded by one to three training sessions, excluded from the analysis.

3.2. Experimental ConteXts and ConteNts

In each of two circles the dot presented could be located either at its center, or 4 px above the center, or else 4 px under the center. These pairs of locations produce a total of nine conteXts. During each session, excepting the practice trials, the dot was presented at the center in a half of the trials, above the center in a quarter of them, and below the center in the remaining quarter, for each of the circles. Figure 5 presents the proportions of allocations of trials to each of the 9 conditions.

For each session, each trial was randomly assigned to one of the conditions in accordance with Table 5. The number of experimental sessions was chosen so that the expected number of (non-practice) trials in the conditions with lowest probabilities was at least 300. This number of observations was chosen based on Copeda Cuervo et al. (2008), whose results show that coverage errors with respect to nominal values are below 1% for almost all confidence intervals for proportions with $n > 300$.

The system of random variables representing the data is shown in Figure 6.

3.3. Results

The results are shown in Figs. 7, 8, and 9, one for each of the three participants. Each row, together with its margins, specifies an empirical estimate of the joint distribution of the two random variables sharing the corresponding conteXt. This distribution is shown in the format

$$\text{Pr}[X = 1], \text{Pr}[Y = 1], \text{Pr}[X = Y = 1],$$

where $X$ and $Y$ are the two variables in the same row. Each column, together with its margins, shows an empirical estimate of the multimaximal coupling of the three random variables sharing the corresponding conteNt. The distribution of the coupling is shown in the format

$$\text{Pr}[A = 1], \text{Pr}[B = 1], \text{Pr}[C = 1], \text{Pr}[A = B = 1], \text{Pr}[B = C = 1], \text{Pr}[A = C = 1]$$

where $A, B, C$ are the three random variables in the same column listed from top down. The analysis of contextuality consists in considering a set of jointly distributed 18 binary random variables (corresponding to the star symbols in Fig. 6), and determining whether the 2$^{18}$ values of this set can be assigned probabilities that sum to the probabilities whose empirical estimates are shown in the data matrices (Figs. 7, 8, and 9). This is a standard linear programing task,

$$\mathbf{M} \cdot \mathbf{Q} = \mathbf{P} \cdot \mathbf{Q} > 0 \text{ (componentwise)}.$$

The number of the rows in $\mathbf{M}$ and $\mathbf{P}$ (i.e., the number of linear constraints imposed on $\mathbf{Q}$) is the number of the probability estimates shown in each of the data matrices (45) plus the constraint that ensures that all the 2$^{18}$ probabilities in $\mathbf{Q}$ sum to 1. (The number of the probability estimates could be reduced from 45 to 39, because one of the three marginal probabilities for each column could be eliminated. We did not, however, make use of this small reduction in our computations.) The linear programming was performed by using the GLPK (GNU Linear Programming Kit) package (version 4.6; Makhorn, 2012) and the R interface to the package (Rglpk, version 0.6-1; Theussl & Hornik, 2015).
FIG. 7: Empirical data (relative frequencies) for the context-context system in Fig. 6 for participant P1. For every context x and every context z measured in xy (either x- or y), the cell for \( R_{xy}^{x} \) contains the frequency estimate of \( \Pr[R_{xy}^{x} = 1] \); the right margins of the row for xy shows the frequency estimate of \( \Pr[R_{xy}^{y} = R_{xy}^{z} = 1] \) and the total number of measurements in this context. Since xy and z vary, the column for joint probabilities denotes the two random variables by \( x = R_{xy}^{x}, y = R_{xy}^{y} \). The bottom margins in the column for context z show the three frequency estimates of the maximal values of \( \Pr[R_{xy}^{x} = R_{xy}^{z} = 1], \Pr[R_{xy}^{y} = R_{xy}^{z} = 1], \) and \( \Pr[R_{xy}^{x} = R_{xy}^{y} = 1] \) (where xy, uv, st are three contexts in which z was measured). To make notation compact, the three random variables in each column are labeled A, B, C (from top down), and the three probabilities are shown as \( \Pr[A = B = 1], \Pr[B = C = 1], \) and \( \Pr[A = C = 1] \) (one of which is always redundant but shown for completeness).

FIG. 8: Empirical data (frequencies) for the context-context system in Fig. 6 for participant P2. The rest is as in Fig. 7.

FIG. 9: Empirical data (frequencies) for the context-context system in Fig. 6 for participant P3. The rest is as in Fig. 7.
4. CONCLUSION

The experiment presented in this paper illustrates the use of the double factorial paradigm in the search of contextuality in behavioral systems, namely in the responses of human observers in a double-detection task. This paradigm provides the closest analogue in psychophysical research to the Alice-Bob EPR/Bohm paradigm (Bell, 1964; Clauser et al., 1969; Fine, 1982). We have found that for the participants in the study there was no evidence of contextuality in their responses. These results add to the existing evidence that points towards lack of contextuality in behavioral data (Cervantes & Dzhafarov, 2017; Dzhafarov, Kujala, Cervantes, Zhang, & Jones, 2016; Dzhafarov, Zhang, & Kujala, 2015; Zhang & Dzhafarov, 2017). The present result is in fact stronger than the previous ones, as it uses a more stringent than before criterion of noncontextuality. This criterion is based on multimaximality rather than on the simple maximality of the couplings in cyclic systems. However, we should emphasize that in the absence of a predictive theory on a par with quantum mechanics, no failure to find contextuality in even a large number of experiments can be safely generalized: contextuality may very well be found under as yet unexplored modifications of experimental conditions. Consider, e.g., the Alice-Bob EPR/Bohm paradigm, and imagine that we have no theory that could guide us in choosing the specific axes along which Alice and Bob are to measure the spins in their respective particles. It would be rather unlikely to hit at a “right” combination of the angles by pure chance, and after numerous failures one could very well conclude, in this case wrongly, that contextuality is absent in this paradigm. More work is needed.

Acknowledgments. This research has been supported by AFOSR grant FA9550-14-1-0318.

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