We discuss how the global geometry and topology of manifolds depend on different group actions of their fundamental groups, and in particular, how properties of a non-trivial compact 4-dimensional cobordism $M$ whose interior has a complete hyperbolic structure depend on properties of the variety of discrete representations of the fundamental group of its 3-dimensional boundary $\partial M$. In addition to the standard conformal ergodic action of a uniform hyperbolic lattice on the round sphere $S^{n-1}$ and its quasiconformal deformations in $S^n$, we present several constructions of unusual actions of such lattices on everywhere wild spheres (boundaries of quasisymmetric embeddings of the closed $n$-ball into $S^n$), on non-trivial $(n-1)$-knots in $S^{n+1}$, as well as actions defining non-trivial compact cobordisms with complete hyperbolic structures in its interiors. We show that such unusual actions always correspond to discrete representations of a given hyperbolic lattice from “non-standard” components of its varieties of representations (faithful or with large kernels of defining homomorphisms).

1. Geometric structures and varieties of representations

In this survey we discuss how different hyperbolic group actions determine the global geometry and topology of corresponding quotient spaces having conformally flat structures (locally modelled on the geometry of the sphere $S^n, n \geq 3$); in particular how a non-trivial compact 4-dimensional cobordism $M$ whose interior has a hyperbolic structure depends on properties of the variety of discrete representations of the (hyperbolic) fundamental group of its 3-dimensional boundary $\partial M$. In addition to the standard conformal ergodic
action of a uniform hyperbolic lattice on the round sphere $S^{n-1}$ and its quasiconformal deformations in $S^n$, we present several constructions of unusual actions of such lattices on everywhere wild spheres (boundaries of quasisymmetric embeddings of the closed $n$-ball into $S^n$, with a dense subset of points with wild knotting, see Section 3), on non-trivial $(n-1)$-knots in $S^{n+1}$ (Section 2), as well as actions defining non-trivial compact cobordisms with complete hyperbolic structures in their interiors (Sections 3 and 4). We shall show that such unusual actions of a given hyperbolic lattice always correspond to its discrete representations from “non-standard” connected components of its varieties of representations (faithful or with large kernels of defining homomorphisms, see Sections 4 and 5).

For a given smooth $n$-manifold/orbifold $M$ with fundamental group $\pi_1(M)$, one can use its Teichmüller space $T(M) = T(M; X)$ to describe all possible geometric structures on $M$ modeled on some $(X, G)$-geometry. In other words, we have a pair $(M, s)$ where $s$ is an $(X, G)$-structure on $M$ determined by a maximal $(X, G)$-atlas on $M$. Two $(X, G)$-structures $(M, s_1)$ and $(M, s_2)$ on $M$ are equivalent if there exists an $(X, G)$-diffeomorphism $f : (M, s_1) \to (M, s_2)$ homotopic to the identity. An equivalence class of $(X, G)$-structures on $M$ is called a marked $(X, G)$-structure. Then $T(M) = T(M; X)$ is the space of marked $(X, G)$-structures on $M$. The space $T(M; X)$ is covered by the space $D(M) = D(M; X)$ of the development maps $d : \tilde{M} \to X$ of $(X, G)$-structures (defined as extensions of $(X, G)$-atlas charts on $M$), with a metrizable topology defined by uniform convergence of the developments on compact sets. So we have a topology on $T(M)$ induced by $D(M)$-topology. In particular, if our $(X, G)$-geometry is hyperbolic the above space $T(M) = T(M; H^n)$ becomes the Teichmüller space of marked complete hyperbolic structures on $M$. In the case of conformal geometry, $(X, G) = (S^n, \text{Möb}(n))$, we have the Teichmüller space $T(M; S^n)$ of marked conformal (i.e. conformally flat) structures on $M$.

Due to the uniqueness of the development map $d : \tilde{M} \to X$ (up to composition with $g \in G$), each deck transformation $T_\alpha \in G(\tilde{M}, M)$ of the universal covering $\tilde{M}$ corresponds to a unique (up to conjugation by $g \in G$) element $g_\alpha \in G$ such that $g_\alpha d = dT_\alpha$. This defines the holonomy representation

$$d_\ast : \pi_1 M \to G$$

and thus defines the holonomy map

$$\text{hol}: T(M, X) \to \text{Hom}(\pi_1(M), G)/G, \quad (1.1)$$

where the group $G$ acts on the variety of representations $\text{Hom}(\pi_1(M), G)$ by conjugation, and the image $\text{hol}(M, s)$ of a structure $(M, s) \in T(M, X)$ is the equivalence class $G \cdot d_\ast$ consisting of representations $gd_\ast g^{-1}$, $g \in G$. Here $\text{Hom}(\pi_1(M), G)/G$ has the quotient topology induced by the algebraic convergence topology on the representation variety $\text{Hom}(\pi_1(M), G)$, where two representations of a group are close if they are close on generators of the group.

The holonomy representation alone does not necessarily determine an $(X, G)$-structure on $M$. In particular, starting with a hyperbolic $n$-manifold $M$,
\( n \geq 2 \), one can obtain different conformal structures with the same holonomy (cf. [22, 15, 4]). One of them is uniformizable, i.e. it is conformally equivalent to \( \Omega/\Gamma \) where \( \Omega = d(\tilde{M}) \subset S^n \) is a connected component of the discontinuity set of its holonomy group \( \Gamma = d_{\ast} (\pi_1 (M)) \subset \text{M"ob}(n) \). The other structure is non-uniformizable, with surjective development map \( d: \tilde{M} \rightarrow S^n, \ d(\tilde{M}) = S^n \). However these different structures have different developments due to the Lok-Thurston-Goldman holonomy theorem ([4], Theorem 7.1) for closed \((X,G)\)-manifolds \( M \). This theorem shows that the holonomy map \( \text{hol} \) in (1.1) is an open map which lifts to a local homeomorphism \( \tilde{\text{hol}} \) of the space of development maps to the variety of representations, \( \text{hol}: D(M, X) \rightarrow \text{Hom}(\pi_1(M), G) \).

The map \( \text{hol} \) is not necessarily a local homeomorphism, especially in neighborhoods of structures with unstable holonomy representations. However, in neighborhoods of structures with good holonomies \( d_{\ast} \), the map \( \text{hol} \) itself is a local homeomorphism, see Johnson-Millson [18]. In the case of conformal structures, such a property follows from the Sullivan stability theorem (Apanasov [4], Theorem 7.2):

**Theorem 1.1.** Let \( G \subset \text{M"ob}(n) \) be a non-elementary convex-cocompact discrete group, and \( \{G_m\} \) a sequence of discrete groups \( G_m \subset \text{M"ob}(n) \) isomorphic to \( G \) which algebraically (on generators) converge to \( G \). Then, for sufficiently large \( m \), the groups \( G_m \) are convex co-compact, and their actions on the sphere \( S^n \) are quasi-conformally conjugate to the action of the group \( G \).

In contrast to Riemann surfaces whose Teichmüller spaces of hyperbolic and conformal structures are isomorphic, in dimensions \( n \geq 3 \) one has a different situation due to the hyperbolic rigidity of closed or finite-volume hyperbolic \( n \)-manifolds \( M = H^n/\Gamma, n \geq 3, \pi_1(M) \cong \Gamma \subset \text{Isom}(H^n) \). In fact the Teichmüller space \( T(M,H^n) \) of hyperbolic structures on \( M \) degenerates into a point while its Teichmüller space \( T(M,S^n) \) of conformal structures and corresponding variety \( \mathcal{R}_n(\Gamma) \) of conjugacy classes of representations of the hyperbolic lattice \( \pi_1(M) \cong \Gamma \subset \text{Isom}(H^n) \):

\[
(\mathcal{R}_n)(\Gamma) = \text{Hom}(\Gamma, \text{M"ob}(n))/\text{M"ob}(n) = \text{Hom}(\Gamma, \text{Isom}(H^{n+1}))/\text{Isom}(H^{n+1})
\]

is often non-degenerate. On the infinitesimal level, the non-triviality of this variety for \( n = 3 \) (i.e. non-triviality of its tangent bundle represented by the group cohomology with coefficients in the Lie algebra of the group \( \text{M"ob}(3) \)) was first questioned by Borel-Wallach [12], p.221-224. The affirmative answer to this question was given by Apanasov [3] who presented the first construction of a smooth curve in this variety defining a smooth nontrivial quasiconformal deformation of a uniform hyperbolic lattice \( \Gamma \) - see also Apanasov [4], Johnson-Millson [18], Kourouniotis [19], Lafontaine [21] and Sullivan [24]. Among such conformal structures (different from the hyperbolic one) on our hyperbolic \( n \)-manifold \( M \) we have the so called quasi-Fuchsian structures which correspond to quasi-Fuchsian representations \( \rho \in \mathcal{R}_n(\Gamma) \). The action of a quasi-Fuchsian
group $\rho \Gamma$ on the $n$-sphere $S^n = \partial H^{n+1}$ is quasi-conformally conjugate to the action of the Fuchsian group $\rho_0 \Gamma \subset \text{M"ob}(n-1) \cong \text{Isom} H^n$. Due to Theorem 1.1 the subvariety $R_{qf}(\Gamma) \subseteq R_n(\Gamma)$ of all quasi-Fuchsian representations is an open connected component of the variety of discrete representations $\rho \in R_n(\Gamma)$.

Equivalently, one can consider a conformal $n$-manifold (orbifold) $M$ and its Teichmüller space $T'(M) \subset T(M, S^n)$ of equivalence classes of marked conformal structures $c$ with faithful holonomy $d_* : \pi_1(M) \to \text{M"ob}(n)$. Let also $T^c(M) \subset T(M)$ be a subset of (classes of) conformal structures on $M$ whose development maps are non-surjective, i.e. $T^c(M)$ is a subset of almost uniformizable conformal structures (see [4], Theorem 6.63). On the base of the stability Theorem 1.1 one can describe the subspaces $T^c(M)$ and $T'(M)$ as follows (see [4], Cor. 7.4).

**Corollary 1.2.** Let $M$ be a closed manifold/orbifold with a uniformizable conformal structure. Then the subset

$$T_c(M) = \{[M,c] \in T'(M) : d_* \pi_1(M) = G \text{ is convex cocompact, } |Z_G| < \infty\}$$

is open in the Teichmüller space $T'(M)$, and $T^c(M)$ is closed in $T(M)$.

Applying this result to the case of a conformal manifold with a convex cocompact holonomy group $\Gamma$ and with a faithful representation $d_* : \Gamma \to \text{M"ob}(n)$, one can see that the connected components of the subvariety of $R_n(\Gamma)$ consisting of (classes of) discrete representations can be characterized by the property that any two representations in such a component are conjugate by an equivariant quasiconformal map $f : S^n \to S^n$. In particular if our conformal structure on $M$ is given by a complete hyperbolic metric (i.e. a hyperbolic lattice $\pi_1(M) = \Gamma$ acts on a round $n$-ball $B^n \subset S^n$), one has the space $T_c(M)$ of equivalence classes of marked conformal structures on $M$ whose holonomy representations $d_* : \Gamma \to \text{M"ob}(n)$ are faithful and $d_* \Gamma$ are convex co-compact. Let also $T_{qf}(M)$ be the subspace of $T_c(M)$ consisting of classes of quasi-Fuchsian structures. Such a quasi-Fuchsian structure on $M$ is characterized by the property that its development $d : \tilde{M} = H^n \to S^n$ is the composition of the inclusion $H^n \cong B^n(0,1) \subset \mathbb{R}^n = S^n$ and a $\Gamma$-equivariant quasiconformal map $f : S^n \to S^n$ so that the holonomy group $d_* \Gamma \subset \text{M"ob}(n)$ is a quasi-Fuchsian group, $d_* \Gamma = f \Gamma f^{-1}$.

We note that here we have an ergodic action of our hyperbolic lattice $\pi_1(M) \cong \Gamma$ on topologically trivially embedded spheres $S^{n-1} \hookrightarrow S^n$ (these quasi-spheres split $S^n$ into two quasi-balls). Later we shall consider such ergodic actions on non-trivially embedded spheres as well as actions given by non-faithful representations.
GROUP ACTIONS 5

2. Hyperbolic action on non-trivial knots

Now, instead of Teichmüller spaces $T(M)$ of closed hyperbolic $n$-orbifolds $M = H^n/\Gamma$ and corresponding varieties $R_n(\Gamma)$ for their fundamental groups $\pi_1(M) \cong \Gamma \subset \text{Isom } H^n$ in [1.2], we may consider conformal manifolds fibered over $M$ and associated varieties of representations $R_m(\Gamma), m \geq n$, of their holonomy groups. It is possible to interpret these varieties of representations as the spaces of uniformizable $m$-dimensional conformal structures on the fundamental group $\Gamma \subset \text{Isom } H^n, n \geq 2$, of the base $M$ of such a fibration, see Apanasov [4], Section 7.6.

Namely, a conformal $m$-dimensional structure on the group $\Gamma \subset \text{Isom } H^n, m \geq n$, is determined by a pair $\{N, \varphi\}$ where $N$ is a conformal $m$-manifold with a non-surjective development $d: \tilde{N} \to S^m$ of its universal covering space $\tilde{N}$ onto $d(\tilde{N}) = \Omega \subset S^m$ and $\varphi$ is a monomorphism, $\varphi: \Gamma \to \pi_1(N)$, corresponding to the short exact sequence:

$$0 \to \pi_1(\Omega) \to \pi_1(N) \to \Gamma \to 0.$$  (2.1)

Here (due to uniformizable structures [4]) we may assume that the development map $d$ and the natural projection $\pi: \Omega \to \Omega/G = N$ are covering maps which factor through the universal projection $\tilde{N} \to N$. We say that two pairs $\{N_0, \varphi_0\}$ and $\{N_1, \varphi_1\}$ determine the same conformal structure on $\Gamma$ if there is an orientation preserving conformal homeomorphism $f: N_0 \to N_1$ such that $f_*\varphi_0$ and $\varphi_1$ differ (up to the isotropy subgroup $Z(\rho_0)$ of the inclusion $\rho_0: \Gamma \subset \text{Möb}(m)$) by an inner automorphism of $\Gamma$. The set $T_m(\Gamma)$ of equivalence classes $[N, \varphi]$ is called the space of conformal $m$-dimensional structures on a given group $\Gamma$, and its corresponding variety of conjugacy classes of group representations is $R_m(\Gamma)$.

As an example, we consider the fundamental group $\Gamma = \pi_1(S_g)$ of a hyperbolic surface $S_g$ of genus $g > 1$. Then the space $T_3(\Gamma)$ of conformal 3-structures on $\Gamma$ is determined by Seifert fibrations $N$ over the surface $S_g$ (for fibrations with nontrivial Euler numbers, see Gromov-Lawson-Thurston [16] and [4], Corollary 6.77). Such conformal 3-manifolds $N$ have either $H^2 \times \mathbb{R}$- or $S\mathbb{L}_2\mathbb{R}$-geometries. Here we only notice that the Teichmüller spaces $T_3(\Gamma)$ have many connected components. This fact is based on the existence of Seifert fibrations with non-zero Euler classes and on the topology of $\Gamma$-actions on knots in $S^3$.

Now for a uniform hyperbolic lattice $\Gamma \subset \text{Isom } H^n$, we consider the problem of connectedness of the variety $R_m(\Gamma), m > n$, of conjugacy classes of faithful discrete representations $\rho: \Gamma \to \text{Möb}(m)$. As we observed, this variety of discrete representations contains the space $T_m(\Gamma)$ of $m$-dimensional uniformizable conformal structures on $\Gamma$ which satisfy the exact sequence (2.1). Obviously, the Teichmüller space $T_n(\Gamma)$ is a subspace of $T_m(\Gamma)$. Nevertheless, the possible non-connectedness of the variety $T_n(\Gamma)$ (see the next section) does not imply non-connectedness of the bigger variety $T_m(\Gamma)$. In particular the...
topological obstruction for connectedness of $T_3(\Gamma)$ (nontrivial knotting of the
limit 2-sphere $\Lambda(G) \subset S^3$ - see the next section) is not an obstruction in di-
mensions $m \geq 4$. In fact, the nerve of that knotting $S^2 \hookrightarrow S^3$ is 1-dimensional,
and hence the topological 2-sphere $\Lambda(G) \subset S^3$ nontrivially knotted in $S^3$ is
unknotted in $S^m$, $m \geq 4$.

However, we shall show that in general the varieties $T_n(\Gamma)$ are non-connected.
To do that we shall first establish a link between components of these vari-
eties and $(n - 2)$-dimensional knots in the n-sphere $S^n$ which carry conser-
vative dynamics of conformal actions of a given uniform hyperbolic lattice
$\Gamma \subset \text{Isom} H^{n-1}$, see Apanasov [4], Theorems 7.55 and 7.60:

**Theorem 2.1.** For a given nontrivial ribbon $(n - 2)$-knot $K \subset S^n$, $n \geq 4$,
there exists a discrete faithful representation $\rho : \Gamma \rightarrow \text{Möb}(n)$ of a uniform
hyperbolic lattice $\Gamma \subset \text{Isom} H^{n-1}$ such that the Kleinian group $G = \rho \Gamma$ acts
ergodically on the everywhere wild $(n - 2)$-knot $K_\infty = \Lambda(G) \subset S^n$ obtained
as an infinite compounding of the knot $K$, $K_\infty = \ldots \# K \# K \# K \# \ldots$. Moreover,
the varieties $T_n(\Gamma)$ of conformal structures on $\Gamma$ and $R_n(\Gamma)$ of conjugacy
classes of discrete faithful representations of $\Gamma$ are not connected.

**Proof:** Here we give main steps of the proof (see details in Apanasov [4],
pp. 447-457), considering for simplicity the case $n = 4$ where a nontrivial two-
knot $K \subset S^4$ (not equivalent to the natural inclusion $S^2 \subset S^4$) is given by an
embedding $K : S^2 \hookrightarrow S^4$ of the oriented 2-sphere into the oriented 4-sphere (up
to orientation preserving homeomorphisms $f : S^4 \rightarrow S^4$). Simplest examples
of such 2-knots can be obtained by using the so-called suspensions and spins
of classical knots in $S^3$. The latter spun 2-knots $K \subset S^4$ can be obtained as
ribbon 2-knots. Such $(n - 2)$-knots in $S^n$ generalize classical ribbon knots in
$S^3$ and can be obtained as follows, see Suzuki [26] and Figure 1.
Let \( S_0 \cup \ldots \cup S_m \subset \mathbb{R}^4 \) be a trivial 2-link with \((m+1)\) components (which are trivial non-linked 2-knots) and \( f_i : [0,1] \times B^2 \hookrightarrow \mathbb{R}^4, \ i = 1, \ldots, m, \) be appropriate embeddings of 3-balls, which make \( m \) fusions of the 2-link. Each of the embeddings \( f_i \) is such that

\[
(2.2) \quad f_i([0,1] \times B^2) \cap (S_0 \cup \ldots \cup S_m) = f_i([0,1] \times B^2)
\]

has an orientation coherent with that of the 2-link, and the disks \( f_i(\{0\} \times B^2) \) and \( f_i(\{1\} \times B^2) \) are contained in different components of the link. Then the connected sum of the sphere \( f_i(\partial([0,1] \times B^2)) \) and the spheres \( S_0, \ldots, S_m \) represented by the homological sum

\[
(2.3) \quad (S_0 \cup \ldots \cup S_m) + f_i(\partial([0,1] \times B^2)) = S_0^3 \cup \ldots \cup S_m^3
\]

is a trivial 2-link with \((m-1)\) components. Continuing this process of fusions on the link, we finally obtain a ribbon 2-knot with \( m \) fusions, see Figure 1.

The proof of Theorem 2.1 uses the stability Theorem 1.1 and is based on our block-building method (see Apanasov [4], Section 5.4) and geometrically controlled PL-approximations of smooth ribbon \((n-2)\)-knots \( K \subset S^n \) (in the conformal category). This means that all spheres involved in the definition of a given ribbon knot are round (conformal) spheres in \( S^n \), and each image \( f_i(B^{n-1}) \) is contained in the union of finitely many round \((n-1)\)-balls \( B_j \) in \( S^n \), \( 1 \leq j \leq j_i \), such that the boundary spheres of any two adjacent balls intersect each other along a round \((n-3)\)-sphere, see Figure 2.

In other words, the \((n-1)\)-dimensional ribbon \( f_i(B^{n-1}) \) (the union of spherical annuli) can be obtained from a flat ribbon in \( \mathbb{R}^{n-1} \) by sequential bendings along \((n-2)\)-planes (defining bendings of corresponding discrete groups). Here each round \((n-1)\)-ball \( B_j \) in the definition of our ribbon knot \( K \) (either a ball from one of the ribbons \( f_i(B^{n-1}) \) or one of the balls bounded by spheres \( S_k, \ 0 \leq k \leq m \)) has a discrete action of a hyperbolic group \( G_j \subset \text{Isom} H^{n-1} = \text{Möb}(B_j) \). Up to isotopy of the knot \( K \) and the family \( \Sigma \) of \((n-1)\)-balls \( B_j \), we may assume that the groups \( G_j \) have bending hyperbolic \((n-2)\)-planes whose boundaries at infinity \( \partial B_j \) are the intersection spheres \( \delta_j = \partial B_j \cap \partial B_{j+1} \) for the adjacent balls \( B_j \) and \( B_{j+1} \), and that the stabilizers of \( \delta_j \) in \( G_j \) and \( G_{j+1} \) coincide. We denote such stabilizers by \( \Gamma_j = G_j \cap G_{j+1} \). This property guarantees that the amalgamated free product

\[
(2.4) \quad G = \cdots *_{\Gamma_j} G_j *_{\Gamma_j} G_{j+1} *_{\Gamma_{j+1}} \cdots \subset \text{Möb}(n)
\]

is a Kleinian group isomorphic to a uniform hyperbolic lattice \( \Gamma \subset \text{Isom} H^{n-1} \).

This construction represents our ribbon \((n-2)\)-knot \( K \subset S^n \) as the union \( K_0 \) of disjoint \((n-2)\)-dimensional cylinders corresponding to the ribbons \( f_1, \ldots, f_m \) and \((m+1)\) disjoint round \((n-1)\)-spheres with deleted disjoint round \((n-1)\)-balls (corresponding to \( m \) fusions). The disjoint cylinders are the unions of spherical \((n-2)\)-dimensional annuli with disjoint interiors which
Figure 2. Block-building of a hyperbolic lattice action on \((n - 2)\)-knot.

Figure 3. Spun 2-knot of the trefoil as PL-ribbon knot

lie on the boundary spheres \(\partial B_j\) of the round \((n - 1)\)-balls \(B_j\) in the construction, see Fig.2.

Using our block-groups \(G_j \subset \text{Möb}(B_j) \subset \text{Möb}(n)\) in \((2.4)\) and sequential bendings, we obtain a uniform hyperbolic lattice \(\Gamma \subset \text{Isom} H^{n-1}\) having the same amalgamated free product structure as the group \(G\) in \((2.4)\). The limit set \(\Lambda(G)\) of the group \(G \cong \Gamma \subset \text{Isom} H^{n-1}\) is the desired everywhere wild \((n - 2)\)-knot \(K_\infty \subset S^n\) (infinite compounding of the knot
Figure 4. Big and small cube sizes and ball covering

$K_0 = K), K_\infty = \ldots \# K \# K \# K \# \ldots$. The proof of this fact is based on calculations of the Alexander invariant of the knot $K_\infty = \Lambda(G)$ with the complement $\Omega = S^n \setminus K_\infty$, i.e. the $\Lambda$-module $H_\ast(\widehat{\Omega})$, the integral homology $H_\ast(\widehat{\Omega}) = H_\ast(\widehat{\Omega}; \mathbb{Z})$ with $\Lambda$-module structure where $\Lambda$ denotes the ring of finite Laurent polynomials with integer coefficients, and $\widehat{\Omega}$ is an infinite cyclic covering space of the discontinuity set $\Omega(G)$, see Apanasov [4], Lemmas 7.57, 7.58 and Cor. 7.58.

We illustrate how our construction works in the simplest case of the spun 2-knot of the classical trefoil knot $k \subset \mathbb{R}^3$. This 2-knot can be also represented as a PL-ribbon knot $K \subset \mathbb{R}^4$ obtained by one fusion from two unlinked 2-spheres $S_0$ and $S_1$ which are the boundaries of 3-dimensional cubes $Q_0$ and $Q_1$ in 3-planes $\mathbb{R}^3 \times \{0\}, \mathbb{R}^3 \times \{54\} \subset \mathbb{R}^4$. As the ribbon $f_1 : B^4 \hookrightarrow \mathbb{R}^4$ we shall use the union of 3-dimensional (smaller) cubes $Q_j$, $2 \leq j \leq m$ in $\mathbb{R}^4$ as it is indicated in Fig.3. Here the plus signs show the character of intersections (in our 3-dimensional projection of $K$) of the boundaries $\partial Q_0$ and $\partial Q_1$ with the boundary of 3-dimensional tube which is the union $\bigcup_{2 \leq j \leq m} Q_j$ of small cubes. Edges of all cubes are parallel to the coordinate axes in $\mathbb{R}^4$. Later we shall explain what is a relation between sizes of small and big cubes, as well as our choice of parallel 3-planes $\mathbb{R}^3 \times \{-27\}, \mathbb{R}^3 \times \{0\}, \mathbb{R}^3 \times \{54\}$ and $\mathbb{R}^3 \times \{81\}$ in $\mathbb{R}^4$ which contain some of the cubes $Q_j$ and are orthogonally joined by tubes that are boundaries of the union of the remaining small cubes.
Now we define discrete block-groups $G_j$ associated with the cubes $Q_j$. Although these $G_j$ are isomorphic to hyperbolic isometry groups in $H^3$, it is more convenient to use quasi-Fuchsian bending deformations of these hyperbolic groups (cf. Apanasov [3, 4]) so that the obtained groups $G_j$ match the cubes $Q_j$ in the following sense. Assuming $K = K_0$, we cover the 2-knot $K \subset \bigcup_{0 \leq j \leq m} \partial Q_j$ by a family $\Sigma = \{b_{ji}\}$ of closed round 4-balls $b_{ji}$ whose boundary spheres $\partial b_{ji}$ are orthogonal to $K$. Namely, in the first step, we take 4-balls $b_{ji}$ centered at the vertices of the cubes $Q_j$, $0 \leq j \leq m$, whose radii $r_{ji}$ are equal to each other if either $j = 0, 1$ or $2 \leq j \leq m$. One more condition on these radii $r_{ji}$ is that $b_{ji} \cap b_{kl} \neq \emptyset$ only if the centers of the different balls $b_{ji}$ and $b_{kl}$ are the ends of a common 1-edge of one of the cubes. In the latter case, the magnitude of the exterior dihedral angle bounded by the spheres $\partial b_{ji}$ and $\partial b_{kl}$ should equal $\pi/3$. These 4-balls $b_{ji}$ do not cover the entire knot $K$. On each square 2-side $X \subset \partial Q_j \cap K$, we have uncovered 4-gon bounded by circular arcs. We cover such a 4-gon by five additional 4-balls $b_{ji}$ centered at $X$ and whose boundary spheres $\partial b_{ji}$ intersect (orthogonally) only those previously constructed balls that are centered at the vertices of $X$. Among these five new balls, the first four sequentially intersect each other with external dihedral angles $\pi/3$. The fifth ball is centered at the center of $X$ and (orthogonally) intersects only the last new four balls, see Fig.4. After that, we still have uncovered those two 2-sides $X_0$ and $X_1$ of the big cubes $Q_0$ and $Q_1$ that are (orthogonally) joined by the tube $\bigcup_{2 \leq j \leq m} Q_j$. Here we assume that $Q_0 \cap Q_2$ and $Q_1 \cap Q_m$ are small squares centered at the centers of $X_0$ and $X_1$, respectively. Furthermore, we choose the size of the cubes $Q_j$ so that $Q_j$, $j \geq 2$, are unit cubes and the cubes $Q_0$ and $Q_1$ have the size which matches the covering family $\{b_{ji}\}$. In fact, the boundary spheres of the four additional balls centered at $\text{int}(X_0)$ orthogonally intersect the corresponding spheres centered at the four vertices of the small cube $Q_2$, see Fig.4. This completes the construction of the family $\Sigma = \{b_{ji}\}$ of 4-balls that cover the knot $K$. The union of these balls, $\bigcup_{i,j} \text{int} b_{ji} = N(K)$, is a regular neighborhood $N(K)$ of the PL-ribbon knot $K$.

We can take the size of cubes $Q_j$, $j \geq 2$, arbitrarily smaller than the size of the cubes $Q_0$ and $Q_1$. To do that, we repeat the above process of covering the sides $X_0$ and $X_1$ by balls $b_{ji}$ where, instead of the vertices of $X_0$ (and $X_1$), we take the centers of the four new small balls. Then each of the annuli in $X_0 \setminus Q_2$ and $X_1 \setminus Q_m$ will be covered by $(4 + 8k)$ additional balls $b_{ji}$ (for sufficiently large integer $k \geq 0$) instead of the above four additional balls corresponding to $k = 0$. This allows us to take the ribbon $f_1 : B^3 \hookrightarrow \mathbb{R}^4$ as thin as we need. This observation makes it possible to apply our construction of the group $G \subset \text{Möb}(4)$ in [2, 3] from block-groups $G_j \subset \text{Möb}(4)$ to represent an arbitrary ribbon 2-knot $K \subset S^4$ as the knot which lie on the boundary of the union of 3-cubes similar the above cubes $Q_j$. 
Now we define a discrete block-group $G_j$ in (2.4) associated with a cube $Q_j$, $0 \leq j \leq m$, as the group generated by reflections with respect to all spheres $\partial b_{ji}$, that is, with respect to all spheres $\partial b_{ji}$ that intersect the cube $Q_j$. Obviously, $G_j$ is discrete because all spherical dihedral angles with edges $\partial b_{ji} \cap \partial b_{jl}$ are either $\pi/3$ or $\pi/2$. Furthermore, $G_j$ preserves each of the (coordinate) 3-planes $\mathbb{R}^3 \subset \mathbb{R}^4$ that contain the cube $Q_j$. In such a 3-plane $\mathbb{R}^3$, the group $G_j$ can be deformed by bendings to a Fuchsian group $G_j'$ acting in a 3-ball $B^3 \subset \mathbb{R}^3$, see Apanasov [4]. This is why we can consider the groups $G_j$ as discrete subgroups in $\text{Isom} \, H^3$, $G_j \cong G_j' \subset \text{Isom} \, H^3$. Amalgama subgroups $\Gamma_j = G_j \cap G_{j+1}$ in (2.4) are common subgroups of groups associated with any two adjacent cubes $Q_j$ and $Q_{j+1}$. Such a group $\Gamma_j$ is generated by four reflections with respect to the spheres centered at the vertices of the square $Q_j \cap Q_{j+1}$. Now the Maskit combinations (see [4], Theorem 5.17) produce a Kleinian group $G \subset \text{Möb}(4)$ as the free amalgamated product in (2.4). We remark that for each amalgamated free product $G_j \ast_{\Gamma_j} G_{j+1}$, we can use a bending deformation along the hyperbolic 2-plane $H_j$ whose boundary circle $\partial H_j$ is the limit circle of the amalgama subgroup $\Gamma_j$. As a result, we get a new hyperbolic isometry group $G_j' \subset \text{Isom} \, H^3$ which is isomorphic to $G_j \ast_{\Gamma_j} G_{j+1}$. Applying this process $m$ times, we obtain a cocompact discrete group (a uniform hyperbolic lattice) $\Gamma \subset \text{Isom} \, H^3$ isomorphic to the group $G$.

In dimension $n = 4$, there is another (non-algorithmical) way to get such a unique hyperbolic lattice $\Gamma$ by using the Andreev [1] classification of hyperbolic compact polyhedra in $H^3$. Namely, for the constructed group $G \subset \text{Möb}(4)$, we can take the complement of our regular neighborhood $N(K)$ of the knot $K$ to be its fundamental polyhedron $P = P(G) \subset S^4$:

\begin{equation}
(2.5) \quad P = P(G) = \mathbb{R}^4 \setminus N(K), \quad N(K) = \bigcup_{i,j} \text{int } b_{ji}.
\end{equation}

The boundary $\partial P$ of the polyhedron in (2.5) has the combinatorial type of $S^2 \times S^1$ where the 2-sphere $S^2$ is decomposed into the union of spherical polygons. In fact, $\partial P$ is the union of 3-sides each of which is the annulus on a sphere $\partial b_{ji}$, i.e. each 3-side is the product of a spherical 2-polygon $D_{ji}$ and the circle $S^1$. The dihedral angles between such 3-sides are determined by the corresponding 3-dimensional dihedral angles bounded by 2-spheres $\partial b_{ji} \cap \mathbb{R}^3$ in the corresponding 3-planes $\mathbb{R}^3 \subset \mathbb{R}^4$, so they are either $\pi/3$ or $\pi/2$, and the Andreev conditions apply, see [1] and [4], Theorem 2.41. It follows that the combinatorial type of the 4-polyhedron $P$ determines the combinatorial type of a 3-dimensional compact hyperbolic polyhedron $P' \subset H^3$, with the same magnitudes of dihedral angles as those for $P$. Thus the group $\Gamma \subset \text{Isom} \, H^3$ generated by reflections in sides of $P'$ is a uniform hyperbolic lattice isomorphic to our group $G \subset \text{Möb}(4)$.\]
3. Hyperbolic 4-cobordisms

In this section we shall show how to change the standard conformal ergodic action of a uniform hyperbolic lattice $\Gamma \subset \text{Isom} H^n$ on the round sphere $S^{n-1} \subset S^n$ to its unusual actions on everywhere wild spheres (boundaries of quasisymmetric embeddings of the closed $n$-ball into $S^n$). Such actions define non-trivial compact ($n + 1$)-cobordisms with complete hyperbolic structures in their interiors. Such unusual actions always correspond to discrete representations of our hyperbolic lattice representing “non-standard” connected components of its variety of discrete faithful representations in $\mathcal{R}_n(\Gamma)$ in [12].

We shall start with hyperbolic cobordisms.

A compact $(n + 1)$-dimensional cobordism $M$ whose interior has a hyperbolic structure can be identified with a Kleinian $(n + 1)$-manifold $M(G) = (H^{n+1} \cup \Omega(G)) / G$, where $G \subset \text{Isom} H^{n+1}$ is a convex co-compact group of hyperbolic isometries acting in the $n$-sphere at infinity $S^n = \partial H^{n+1}$ by Möbius transformations and whose discontinuity set $\Omega(G) \subset S^n$ is the union of two invariant connected components $\Omega_0$ and $\Omega_1$, $\partial M = \Omega(G)/G$, cf. [4]. Since similar Kleinian 3-manifolds are always homeomorphic to surface layers $S_g \times [0, 1]$ of genus $g$, it is natural to ask: to what extent holds the analogy with the surface layer for such $(n + 1)$-cobordisms given by Kleinian $(n + 1)$-manifolds $M(G)$? One can consider such analogies of the surface layer with various degrees of generality:

1. The product of an $n$-dimensional manifold $N_0 = \Omega_0/G$ and the segment $[0, 1]$.
2. An $h$-cobordism $(M; N_0, N_1)$ with trivial relative homotopy groups for both boundary components $N_0$ and $N_1$, $\pi_s(M, N_0) = \pi_s(M, N_1) = 0$.
3. A homology cobordism $(M; N_0, N_1)$ with trivial relative homology groups for both boundary components $N_0$ and $N_1$, $H_s(M, N_0) = H_s(M, N_1) = 0$.

It follows from Apanasov-Tetenov [10] that homology cobordisms are very natural for Kleinian manifolds:

**Theorem 3.1.** Suppose a Kleinian group $G \subset \text{Isom} H^{n+1}$ is convex cocompact and has an invariant contractible component $\Omega_0 \subset \Omega(G) \subset S^n$. Then its compact Kleinian manifold $M(G)$ has two boundary components $N_0 = \Omega_0/G$ and $N_1 = (\Omega(G) \setminus \Omega_0)G$, and the triple $(M(G); N_0, N_1)$ is a homology cobordism.

It follows from the proof of this theorem (see also Cor.5.42 and Theorem 5.43 in [4]) that an additional condition of contractibility of two invariant components $\Omega_0, \Omega_1 \subset \Omega(G)$ of the group $G$ turns the above homology cobordism into an $h$-cobordism:

**Theorem 3.2.** Let a torsion free Kleinian group $G$ acting on $S^n$, $n \geq 2$, have two invariant contractible components $\Omega_0, \Omega_1 \subset \Omega(G)$ with compact quotient manifolds $N_0 = \Omega_0/G$ and $N_1 = \Omega_1/G$. Then the Kleinian manifold $M(G)$ is
also compact, has exactly two boundary components $N_0 \cup N_1 = \partial M(G)$, and the triple $(M(G); N_0, N_1)$ is an $h$-cobordism, $\pi_*(M(G), N_0) = \pi_*(M(G), N_1) = 0$.

On the other hand Apanasov-Tetenov [10] and Apanasov [6] constructions show that there are homotopically non-trivial homology cobordisms $M = M(G)$:
Theorem 3.3. There exist compact Kleinian 4-manifolds $M(G)$ (both orientable or not, and with convex cocompact hyperbolic structures in their interiors $\text{int} M(G) = H^4/G$) which are homotopically non-trivial homology cobordisms.

Convex cocompact hyperbolic isometry groups $G \subset \text{Isom} H^4$ in these constructions in Theorem 3.3 have two $G$-invariant components $\Omega_0, \Omega_1 \subset \Omega(G) \subset S^3 = \partial H^4$ with compact quotient manifolds $N_0 = \Omega_0/G$ and $N_1 = \Omega_1/G$. Nevertheless these boundary manifolds $N_0$ and $N_1$ differ very much from each other. While $N_0$ is homotopy equivalent to the 4-manifold $M(G)$ and is universally covered by a quasiconformal 3-ball $\Omega_0 \subset S^3$, the second boundary component $N_1$ is not homotopy equivalent to $M(G)$, and is covered by the non-simply connected $G$-invariant component $\Omega_1 = S^3 \setminus \overline{\Omega_0}$ which is the complement in $S^3$ of a wildly knotted closed 3-ball $\overline{\Omega_0} \subset S^3$ shown in Fig.5. This wildly knotted closed 3-ball can be obtained by a quasisymmetric embeddings of a closed round 3-ball inextensible in neighborhoods of any boundary points (see Apanasov [5]) which can be constructed by a sequence of bendings of the initial hyperbolic 3-manifold $H^3/\Gamma \approx \Omega_0$, along its finitely many disjoint totally geodesic surfaces. Results of such bendings of the original hyperbolic lattice $\Gamma$ are indicated in Fig.5 by sequentially intersecting balls $B_i$ forming a period of the initial Fox-Artin wild 3-ball.

So one may come to the following conjecture:

Conjecture 3.4. If one had a hyperbolic 4-cobordism $M$ whose boundary components are highly (topologically and geometrically) symmetric to each other it would be in fact an $h$-cobordism, possibly not trivial, i.e. not homeomorphic to the product of $N_0$ and the segment $[0,1]$.

In the next section we negatively answer this Conjecture 3.4. Namely we construct such hyperbolic 4-cobordisms $M = M(G)$ whose boundary components are covered by the discontinuity set $\Omega(G) \subset S^3$ with two connected components $\Omega_0$ and $\Omega_1$, where the fundamental group action $\Gamma$ is symmetric and has contractible fundamental polyhedra of the same combinatorial type allowing to realize them as a compact polyhedron in the hyperbolic 3-space, i.e. the dihedral angle data of these polyhedra satisfy the Andreev's conditions [1]. Nevertheless we show that a geometric symmetry of boundary components of our hyperbolic 4-cobordism $M(G))$ are not enough to ensure that the group $G = \pi_1(M)$ is quasi-fuchsian and our 4-cobordism $M$ is trivial. This is related to the non-connectedness of the variety of discrete representations of $\Gamma$ (Teichmüller space) and homomorphisms $\Gamma \rightarrow G$ with infinite kernels.

4. Teichmüller Spaces and Reflection Groups in the 3-sphere

If the fundamental group $\Gamma = \pi_1(N_0)$ of a boundary component of our hyperbolic 4-cobordism $(M; N_0, N_1)$, acts in the hyperbolic 3-space $H^3$ as a
uniform hyperbolic lattice $\Gamma \subset \text{Isom} H^3$, one may consider the natural conformal (conformally flat) structure on $N_0 = \Omega_0 / G$ either as a point of the Teichmüller space $\mathcal{T}(H^3 / \Gamma)$ of marked conformal structures on $H^3 / \Gamma$, or a point of the variety $\mathcal{R}_3(\Gamma)$ of conjugacy classes of discrete representations of the group $\Gamma$ into $\text{Isom} H^4$ in (1.2).

Among such conformal structures on our hyperbolic 3-manifold/orbifold $H^3 / \Gamma$ we have the so called quasi-Fuchsian structures which correspond to quasi-Fuchsian representations of the inclusion $\pi_1(N_0) \cong \Gamma \subset \text{Isom} H^3 \subset \text{Isom} H^4$ (deformations in $\mathcal{R}_3(\Gamma)$). Here the Fuchsian group $\Gamma$ preserves a round ball $B^3 \subset S^3 = \partial H^4$ and conformally acts on this ball as a cocompact discrete group of isometries of $H^3$. Due to the Sullivan structural stability, the space of such quasi-Fuchsian conformal structures and the space of classes of quasi-Fuchsian representations of the hyperbolic lattice $\Gamma$ into $\text{Isom} H^4$ are open connected components of the Teichmüller space $T(H^3 / \Gamma)$ and of the variety of conjugacy classes of discrete representations $\rho : \Gamma \rightarrow \text{Isom} H^4$, $\mathcal{R}(\Gamma, \text{Isom} H^4)$, respectively. Obviously points in these (quasi-Fuchsian) components (faithful representations $\rho : \Gamma \rightarrow \text{Isom} H^4$) correspond to trivial hyperbolic 4-cobordisms $(M(G), N_0, N_1)$ where $G = \rho(\Gamma) \subset \text{Isom} H^4$ has the discontinuity set $\Omega(G) = \Omega_0 \cup \Omega_1 \subset S^3 = \partial H^4$ and $N_i = \Omega_i / G$, $i = 1, 2$, i.e. $M(G)$ is homeomorphic to the product of $N_0$ and the closed interval $[0, 1]$.

On the other hand, our non-trivial homology cobordisms from Theorem 3.3 correspond to non-quasi-Fuchsian discrete representations $\rho$ of $\Gamma$ which are points of another connected components consisting of discrete representations in the variety $\mathcal{R}(\Gamma, \text{Isom} H^4)$, see Apanasov [6]. These discrete representations $\rho$ can be connected to the inclusion representation $i : \Gamma \subset \text{Isom} H^3 \subset \text{Isom} H^4$ in the variety (1.2) by a continuous curve $\beta : [0, 1] \rightarrow \mathcal{R}_3(\Gamma)$, $\beta(0) = i$, $\beta(1) = \rho$, obtained by several deformations of the original closed hyperbolic 3-manifold $H^3 / \Gamma$. These deformations (showing non-triviality of Teichmüller space for a hyperbolic $n$-manifold, $n \geq 3$) were introduced for the first time by Apanasov [3] and they later (after W.Thurston’s Mickey Mouse example) became known as “bendings” (of a hyperbolic 3-manifold $H^3 / \Gamma$ along disjoint totally geodesic surfaces) - see a detailed description of this curve $\beta$ in Apanasov [5, 6, 4]. However it is important that this curve $\beta([0, 1] \subset \mathcal{R}_3(\Gamma))$ must contain some points $\beta(t_0)$, $0 < t_0 < 1$, corresponding to non-discrete representations $\Gamma \rightarrow \text{Isom} H^4$.

Now we consider the situation of hyperbolic 4-cobordisms $M(\rho(\Gamma))$ corresponding to uniform hyperbolic lattices $\Gamma \subset \text{Isom} H^3$ generated by reflections (or cobordisms related to their finite index subgroups). Natural inclusions of these lattices into $\text{Isom} H^4$ act at infinity $\partial H^4 = S^3$ as Fuchsian groups $\Gamma \subset \text{Möb}(3)$ preserving a round ball in the 3-sphere $S^3$. In this case the Conjecture 3.3 can be reformulated as the following question on the Möbius action of corresponding reflection groups $G = \rho(\Gamma) \subset \text{Isom} H^4$ on the 3-sphere $S^3 = \partial H^4$:...
**Question 4.1.** Is any discrete Möbius group $G$ generated by finitely many reflections with respect to spheres $S^2 \subset S^3$ and whose fundamental polyhedron $P(G) \subset S^3$ is the union of two contractible polyhedra $P_1, P_2 \subset S^3$ of the same combinatorial type (with equal corresponding dihedral angles) quasiconformally conjugate in the sphere $S^3$ to some Fuchsian group preserving a round ball $B^3 \subset S^3$?

In the next section we answer this question negatively by constructing a corresponding discrete representation $\rho : \Gamma \to \text{Isom } H^4 \cong \text{Möb}(3)$ of a cocompact hyperbolic lattice $\Gamma \subset \text{Isom } H^3$ whose image $\rho(\Gamma) = G \subset \text{Isom } H^4 \cong \text{Möb}(3)$ will have all the properties of the reflection group $G \subset \text{Möb}(3)$ in Question 4.1.

Necessary and sufficient conditions on a group $G$ which guarantee an affirmative answer to Question 4.1 can be found in [11].

5. Homomorphisms of hyperbolic groups with infinite kernels

In this section our goal is to answer Question 4.1 (and Conjecture 3.4). In other words, one would like to decide whether two Möbius groups on the 3-sphere $S^3$ generated by reflections and having combinatorially similar fundamental polyhedra (with equal corresponding dihedral angles) are quasiconformally conjugate, that is, whether they lie in the same component of the discrete representation variety (or Teichmüller space of conformal structures).

Our answer to that question is negative. In fact we present a construction which implies the following (see Apanasov [8] for detailed proof):

**Theorem 5.1.** There exists a discrete Möbius group $G \subset \text{Möb}(3)$ on the 3-sphere $S^3$ generated by finitely many reflections such that:

1. Its discontinuity set $\Omega(G)$ is the union of two invariant components $\Omega_0, \Omega_1$;
2. Its fundamental polyhedron $P \subset S^3$ has two contractible components $P_i \subset \Omega_i, i = 1, 2$, having the same combinatorial type (of a compact hyperbolic polyhedron $P_0 \subset H^3$);
3. For the uniform hyperbolic lattice $\Gamma \subset \text{Isom } H^3$ generated by reflections in sides of the hyperbolic polyhedron $P_0 \subset H^3$ and acting on the sphere $S^3 = \partial H^4$ as a discrete Fuchsian group $i(\Gamma) \subset \text{Isom } H^4 = \text{Möb}(3)$ preserving a round ball $B^3$ (where $i : \text{Isom } H^3 \subset \text{Isom } H^4$ is the natural inclusion), the group $G$ is its image under a homomorphism $\rho : \Gamma \to G$ but it is not quasiconformally (topologically) conjugate in $S^3$ to $i(\Gamma)$.

**Proof:** Omitting some parts of the proof (see Apanasov [8] for our detailed proof), here we concentrate on the basic for our proof construction of the desired Möbius group $G \subset \text{Möb}(3)$ generated by reflections which are defined by a finite collection $\Sigma$ of reflecting 2-spheres $S_i \subset S^3, 1 \leq i \leq N$. As the first four spheres we consider mutually orthogonal spheres centered at the vertices...
of a regular tetrahedron in \( \mathbb{R}^3 \). Let \( B = \bigcup_{1 \leq i \leq 4} B_i \) be the union of the closed balls bounded by these four spheres, and let \( \partial B \) be its boundary (a topological 2-sphere) having four vertices which are the intersection points of four triples of our spheres. Applying a Möbius transformation in \( S^3 \cong \mathbb{R}^3 \cup \{ \infty \} \), we may assume that the first three spheres \( S_1, S_2 \) and \( S_3 \) correspond to the coordinate planes \( \{ x \in \mathbb{R}^3 : x_i = 0 \} \), and \( S_4 = S^2(0, R) \) is the round sphere of some radius \( R > 0 \) centered at the origin. The value of the radius \( R \) will be determined later.

On the topological 2-sphere \( \partial B \) with four vertices we consider a simple closed loop \( \alpha \subset \partial B \) which does not contain any of our vertices and which symmetrically separates two pairs of these vertices from each other as the white loop does on the tennis ball shown in Figure 6. This loop \( \alpha \) can be considered as the boundary of a topological 2-disc \( \sigma \) embedded in the complement \( D = S^3 \setminus B \) of our four balls. Our geometric construction needs a detailed description of such a 2-disc \( \sigma \) and its boundary loop \( \alpha = \partial \sigma \) obtained as it is shown in Figure 7.

The desired disc \( \sigma \subset D = S^3 \setminus B \) can be described as the boundary in the domain \( D \) of the union of a finite chain of adjacent blocks \( Q_i \) (regular cubes) with disjoint interiors whose centers lie on the coordinate planes \( S_1 \) and \( S_2 \) and whose sides are parallel to the coordinate planes. This chain starts from the unit cube whose center lies in the second coordinate axis, in \( e_2 \cdot \mathbb{R}^+ \subset S_1 \cap S_3 \). Then our chain goes up through small adjacent cubes centered in the coordinate plane \( S_1 \), at some point changes its direction to the horizontal one toward the third coordinate axis, where it turns its horizontal direction by a right angle again (along the coordinate plane \( S_2 \)), goes toward the vertical line passing through the second unit cube centered in \( e_1 \cdot \mathbb{R}^+ \subset S_2 \cap S_3 \), then goes down along that vertical line and finally ends at that second unit cube, see Figure 7. We will define the size of small cubes \( Q_i \) in our block chain and the distance of the centers of two unit cubes to the origin in the next step of our construction. Let us consider one of our cubes \( Q_i \), i.e. a block of our chain, and let \( f \) be its square side having a nontrivial intersection with our 2-disc \( \sigma \subset D \). For that side \( f \) we consider spheres \( S_j \) centered at its vertices and having a radius such that each two spheres centered at the ends of an edge of \( f \) intersect each other with angle \( \pi/3 \). In particular, for the unit cubes such spheres have radius \( \sqrt{3}/3 \). From such defined spheres we select those spheres that have centers in our domain \( D \) and then include them in the collection \( \Sigma \) of reflecting spheres. Now we define the distance of the centers of our big (unit) cubes to the origin. It is determined by the condition that the sphere \( S_4 = S^2(0, R) \) is orthogonal to the sphere \( S_j \in \Sigma \) centered at the vertex of such a cube closest to the origin.

As in Figure 4 let \( f \) be a square side of one of our cubic blocks \( Q_i \) having a nontrivial intersection \( f_\sigma = f \cap \sigma \) with our 2-disc \( \sigma \subset D \). We consider a ring of four spheres \( S_i \) whose centers are interior points of \( f \) which lie outside of the
four previously defined spheres $S_j$ centered at vertices of $f$ and such that each sphere $S_i$ intersects two adjacent spheres $S_{i-1}$ and $S_{i+1}$ (we numerate spheres $S_i \mod 4$) with angle $\pi/3$. In addition these spheres $S_i$ are orthogonal to the previously defined ring of bigger spheres $S_j$, see Figure 4. From such defined spheres $S_i$ we select those spheres that have nontrivial intersections with our domain $D$ outside the previously defined spheres $S_j$, and then include them in the collection $\Sigma$ of reflecting spheres. If our side $f$ is not the top side of one of the two unit cubes we add another sphere $S_k \in \Sigma$. It is centered at the center of this side $f$ and is orthogonal to the four previously defined spheres $S_i$ with centers in $f$, see Figure 4.

Now let $f$ be the top side of one of the two unit cubes of our chain. Then, as before, we consider another ring of four spheres $S_k$. Their centers are interior points of $f$, lie outside of the four previously defined spheres $S_i$ closer to the center of $f$ and such that each sphere $S_k$ intersects two adjacent spheres $S_{k-1}$ and $S_{k+1}$ (we numerate spheres $S_k \mod 4$) with angle $\pi/3$. In addition these new four spheres $S_k$ are orthogonal to the previously defined ring of bigger spheres $S_i$, see Figure 4. We note that the centers of these four new spheres $S_k$ are vertices of a small square $f_s \subset f$ whose edges are parallel to the edges of $f$, see Figure 4. We set this square $f_s$ as the bottom side of the small cubic box adjacent to the unit one. This finishes our definition of the family of twelve round spheres whose interiors cover the square ring $f \setminus f_s$ on the top side of one of the two unit cubes in our cube chain and tells us which two spheres among the four new defined spheres $S_k$ were already included in the collection.
Σ of reflecting spheres (as the spheres $S_j \in \Sigma$ associated to small cubes in the first step).

This also defines the size of small cubes in our block chain. Now we can vary the remaining free parameter $R$ (which is the radius of the sphere $S_4 \in \Sigma$) in order to make two horizontal rows of small blocks with centers in $S_1$ and $S_2$, correspondingly, to share a common cubic block centered at a point in $e_3 \cdot \mathbb{R}_+ \subset S_1 \cap S_2$, see Figure 7.
The constructed collection $\Sigma$ of reflecting spheres $S_j$ bounding round balls $B_j$, $1 \leq j \leq N$, has the following properties:

1. The closure of our 2-disc $\sigma \subset D$ is covered by balls $B_j$: $\bar{\sigma} \subset \text{int} \bigcup_{j \geq 5} B_j$;
2. Any two spheres $S_j, S_{j'} \in \Sigma$ either are disjoint or intersect with angle $\pi/2$ or $\pi/3$;
3. The complement of all balls, $S^3 \setminus \bigcup_{j=1}^N B_j$ is the union of two contractible polyhedra $P_1$ and $P_2$ of the same combinatorial type.

Therefore we can use the constructed collection $\Sigma$ of reflecting spheres to define a discrete group $G = G_\Sigma \subset \text{Möb}(3)$ generated by $N$ reflections in spheres $S_j \in \Sigma$. The fundamental polyhedron $P = P_1 \cup P_2 \subset S^3$ for the action of this discrete reflection group $G$ on the sphere $S^3$ is the union of two connected polyhedra $P_1$ and $P_2$ which are disjoint topological balls. So the discontinuity set $\Omega(G) \subset S^3$ of $G$ consists of two invariant connected components $\Omega_0$ and $\Omega_1$:

$$\Omega(G) = \bigcup_{g \in G} g(\bar{P}) = \Omega_0 \cup \Omega_1, \quad \Omega_i = \bigcup_{g \in G} g(\bar{P}_i), \quad i = 1, 2.$$  

In fact, despite the contractibility of polyhedra $P_1$ and $P_2$ both components $\Omega_0$ and $\Omega_1$ are not simply connected and even are mutually linked:

**Lemma 5.2.** The splitting of the discontinuity set $\Omega \subset S^3$ of our discrete reflection group $G = G_\Sigma \subset \text{Möb}(3)$ into $G$-invariant components $\Omega_0$ and $\Omega_1$ in (5.1) defines a Heegaard splitting of the 3-sphere $S^3$ of infinite genus with ergodic word hyperbolic group $G$ action on the separating boundary $\Lambda(G)$ which is quasi-self-similar in the sense of Sullivan.

This fact is illustrated by Figure 8 where one can see a handlebody obtained from our initial chain of building blocks in Figure 7 by the union of the images of this block chain by first generating reflections in the group $G$ (in $S_1, S_2$ and $S_3$). Then one has a non-contractible loop $\beta_0 \subset \Omega_0$ (represents a non-trivial element of the fundamental group $\pi_1(\Omega_0)$) which lies inside of this handlebody in Figure 8 and is linked with the second loop $\beta_1 \subset \Omega_1$ in the unbounded component $\Omega_1$ which goes around $\tilde{P}_1 \cup g_3(\tilde{P}_1)$, i.e. around one of the handles of the handlebody in Figure 8. This loop $\beta_1$ represents a non-trivial element of the fundamental group $\pi_1(\Omega_1)$. The resulting handlebodies $\Omega_0$ and $\Omega_1$ are the unions of the corresponding images $g(\tilde{P}_i)$ of the polyhedra $\tilde{P}_0$ and $\tilde{P}_1$, so they have infinitely many mutually linked handles. Their fundamental groups $\pi_1(\Omega_0)$ and $\pi_1(\Omega_1)$ have infinitely many generators, and some of those generators correspond to the group $G$-images of the linked loops $\beta_0 \subset \Omega_0$ and $\beta_1 \subset \Omega_1$. The limit set $\Lambda(G)$ is the common boundary of $\Omega_0$ and $\Omega_1$. Since the group $G \subset \text{Möb}(3)$ acts on the hyperbolic 4-space $H^4$, $\partial H^4 = S^3$, as a
Figure 8. Handlebody obtained by the first 3 reflections of the cub chain.
convex cocompact isometry group, its action on the limit set $\Lambda(G)$ is ergodic. Moreover, the common boundary $\Lambda(G)$ of the handlebodies $\Omega_0$ and $\Omega_1$ is quasi-self-similar in the sense of Sullivan [25] (see Apanasov [8] for details).

Another important observation in our proof of Theorem 5.1 is that the combinatorial type (with magnitudes of dihedral angles) of the bounded component $P_1$ of the fundamental polyhedron $P \subset S^3$ coincides with the combinatorial type of its unbounded component $P_2$. Applying Andreev’s theorem on 3-dimensional hyperbolic polyhedra [1], one can see that there exists a compact hyperbolic polyhedron $P_0 \subset H^3$ of the same combinatorial type with the same dihedral angles ($\pi/2$ or $\pi/2$). So one can consider a uniform hyperbolic lattice $\Gamma \subset \text{Isom} H^3$ generated by reflections in sides of the hyperbolic polyhedron $P_0$. This 3-hyperbolic lattice $\Gamma$ acts in the sphere $S^3$ as a discrete co-compact Fuchsian group $i(\Gamma) \subset \text{Isom} H^4$ (i.e. as the group $i(\Gamma) \subset \text{Isom} H^4$ where $i: \text{Isom} H^3 \subset \text{Isom} H^4$ is the natural inclusion) preserving a round ball $B^3$ and having its boundary sphere $S^2 = \partial B^3$ as the limit set. Obviously there is no self-homeomorphism of the sphere $S^3$ conjugating the action of the groups $G$ and $i(\Gamma)$ because the limit set $\Lambda(G)$ is not a topological sphere. So the constructed group $G$ is not a quasi-Fuchsian group.

One can construct a natural homomorphism $\rho: \Gamma \to G$, $\rho \in R_3(\gamma)$, between these two Gromov hyperbolic groups $G \subset \text{Isom} H^4$ and $\Gamma \subset \text{Isom} H^3$ defined by the correspondence between sides of the hyperbolic polyhedron $P_0 \subset H^3$ and reflecting spheres $S_i$ in the collection $\Sigma$ bounding the fundamental polyhedra $P_1$ and $P_2$.

The homomorphism $\rho$ cannot be an isomorphism since its kernel $\rho^{-1}(e_G)$ is not trivial:

**Proposition 5.3.** The homomorphism $\rho \in R_3(\gamma)$, $\rho: \Gamma \to G$, in Theorem 5.1 is not an isomorphism. Its kernel $\ker(\rho) = \rho^{-1}(e_G)$ is a free rank 3 subgroup $F_3 \triangleleft \Gamma$.

In fact the kernel $\rho^{-1}(e_G)$ is a free rank 3 group $F_3 = \langle x, y, z \rangle$ generated by three hyperbolic translations $x, y, z \in \Gamma$. The first hyperbolic translation $x = a_1b_1$ in $H^3$ is the composition of reflections $a_1$ and $b_1$ in two disjoint hyperbolic planes $H_1, H'_1 \subset H^3$ containing those two 2-dimensional faces of the hyperbolic polyhedron $P_0$ that correspond to two sides of the polyhedron $P_1$ which are disjoint parts of the sphere $S_1$. The second hyperbolic translation $y = a_2b_2$ in $H^3$ is the composition of reflections $a_2$ and $b_2$ in two disjoint hyperbolic planes $H_2, H'_2 \subset H^3$ containing those two 2-dimensional faces of the hyperbolic polyhedron $P_0$ that correspond to two sides of the polyhedron $P_1$ which are disjoint parts of the sphere $S_3$. And the third generator $z$ is a hyperbolic translation in $H^3$ which is $a_1$-conjugate of $y$, $z = a_1ya_1$. The fact that these hyperbolic 2-planes $H_1$ and $H'_1$ (correspondingly, the 2-planes $H_2$ and $H'_2$) are disjoint follows from Andreev’s result [2] on sharp angled
hyperbolic polyhedra. Restricting our homomorphism \( \rho \) to the subgroup of \( \Gamma \) generated by reflections \( a_1, a_2, b_1, b_2 \in \Gamma \), we can formulate its properties as the following statement in combinatorial group theory (see Apanasov for its detailed proof):

**Lemma 5.4.** Let \( A = \langle a_1, a_2 \mid a_1^2, a_2^2, (a_1a_2)^2 \rangle \cong B = \langle b_1, b_2 \mid b_1^2, b_2^2, (b_1b_2)^2 \rangle \cong C = \langle c_1, c_2 \mid c_1^2, c_2^2, (c_1c_2)^2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \), and let \( \varphi : A \ast B \to C \) be a homomorphism of the free product \( A \ast B \) into \( C \) such that \( \varphi(a_1) = \varphi(b_1) = c_1 \) and \( \varphi(a_2) = \varphi(b_2) = c_2 \). Then the kernel \( \ker(\varphi) = \varphi^{-1}(e_C) \) of \( \varphi \) is a free rank 3 subgroup \( F_3 \triangleleft A \ast B \) generated by elements \( x = a_1b_1, y = a_2b_2 \) and \( z = a_1a_2b_2a_1 = a_1ya_1 \).

Therefore the configuration of reflecting spheres \( S_j \subset \Sigma \) shows that one may deform our discrete co-compact Fuchsian group \( i(\Gamma) \subset \text{Isom}\, H^4 = \text{Möb}(3) \) preserving a round ball \( B^3 \subset S^3 \) into the group \( G \subset \text{Isom}\, H^4 \) by continuously moving two pairs of reflecting 2-spheres of the Fuchsian group \( i(\Gamma) \) corresponding to the pairs of hyperbolic planes \( H_1, H_1' \subset H^3 \) and \( H_2, H_2' \subset H^3 \) into the reflecting spheres \( S_4 \) and \( S_5 \) while keeping all dihedral angles unchanged.

Also our construction shows that the subvariety \( \mathcal{R}_{\text{dis}}(\Gamma, \text{Isom}\, H^4) \subset \mathcal{R}_3(\Gamma) \) consisting of conjugacy classes of discrete representations of a hyperbolic lattice \( \Gamma \subset \text{Isom}\, H^3 \) generated by reflections may have several connected components. Indeed as we showed our representation \( \rho : \Gamma \to \text{Isom}\, H^4 \) from Proposition 5.3 whose image is our convex cocompact discrete reflection group \( G = \rho(\Gamma) \subset \text{Isom}\, H^4 \) in Theorem 5.1 is not quasi-Fuchsian. The stability Theorem 1.1 implies that the conjugacy class defined by this representation \( \rho \) does not belong to the connected (quasi-Fuchsian) component of our subvariety \( \mathcal{R}_{\text{dis}}(\Gamma, \text{Isom}\, H^4) \) consisting of conjugacy classes of all quasi-Fuchsian representations quasiconformally conjugate to the natural inclusion \( i : \Gamma \subset \text{Isom}\, H^3 \subset \text{Isom}\, H^4 \).

Moreover, counting the reflecting spheres \( S_j \) of the generators of our group \( G = \rho(\Gamma) \subset \text{Isom}\, H^4 \) and the generators of the corresponding hyperbolic lattice \( \Gamma \subset \text{Isom}\, H^3 \) and using Theorem 7.36 in [4] (for the co-finite case, see Huling [17]), one can see that dimensions of these two connected components of \( \mathcal{R}_{\text{dis}}(\Gamma, \text{Isom}\, H^4) \) one of which contains the representation \( \rho \) and the other one contains the inclusion \( i \) are different:

**Proposition 5.5.** Let \( \Gamma \subset \text{Isom}\, H^3 \) and \( G = \rho(\Gamma) \subset \text{Isom}\, H^4 \) be a uniform hyperbolic lattice and its reflection group image in Theorem 5.1 and let \( \mathcal{R}_G \) and \( \mathcal{R}_{af} \) be two connected components of the subvariety \( \mathcal{R}_{\text{dis}}(\Gamma, \text{Isom}\, H^4) \subset \mathcal{R}(\Gamma, \text{Isom}\, H^4) \) consisting of conjugacy classes of discrete representations quasiconformally conjugate in \( S^3 \) to, correspondingly, the representation \( \rho : \Gamma \to G \) and the inclusion \( i : \Gamma \subset \text{Isom}\, H^3 \subset \text{Isom}\, H^4 \). Then, near the inclusion \( i \), the variety \( \mathcal{R}_i \) is a smooth manifold of dimension 775 while the variety \( \mathcal{R}_G \) has dimension 773.
Remark 5.6. Our Theorem 5.1 shows that a hyperbolic 4-cobordism \( M \) whose boundary components are highly (topologically and geometrically) symmetric to each other is not necessarily a trivial 4-cobordism, i.e. not homeomorphic to the product. This fact is related to the Novikov conjecture [23] on the homotopy invariance of characteristic numbers of a manifold derived from the fundamental group. Also, since hyperbolic manifolds are of \( K(\pi_1, 1) \) type (their universal covers are simply connected), it indicates that such hyperbolic 4-cobordisms (and hyperbolic groups) could be among classes for which the conjecture has been verified, see [13, 15, 20].

Remark 5.7. We refer to [5, 9] for applications of our constructions of non-trivial 4-dimensional cobordisms \( M \) to geometric function theory, in particular to quasiconformal, quasisymmetric and quasiregular mappings. This is related to the M.A.Lavrentiev problem, the Zorich map with an essential singularity at infinity and a quasiregular analogue of domains of holomorphy in complex analysis.

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