POINTWISE GRADIENT BOUNDS FOR A CLASS OF VERY SINGULAR QUASILINEAR ELLIPTIC EQUATIONS

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Abstract. A pointwise gradient bound for weak solutions to Dirichlet problem for quasilinear elliptic equations $-\text{div}(A(x,\nabla u)) = \mu$ is established via Wolff type potentials. It is worthwhile to note that the model case of $A$ here is the non-degenerate $p$-Laplacian operator. The central objective is to extend the pointwise regularity results in [Q.-H. Nguyen, N. C. Phuc, Pointwise gradient estimates for a class of singular quasilinear equations with measure data, J. Funct. Anal. 278(5) (2020), 108391] to the very singular case $1 < p \leq \frac{3n-2}{2n-1}$, where the data $\mu$ on right-hand side is assumed belonging to some classes that close to $L^1$. Moreover, a global pointwise estimate for gradient of weak solutions to such problem is also obtained under the additional assumption that $\Omega$ is sufficiently flat in the Reifenberg sense.

1. Introduction and main results. Our objective of this paper is to study the pointwise gradient bounds for solutions to the general quasilinear elliptic equations of the type

$$-\text{div}(A(x,\nabla u)) = \mu \text{ in } \Omega; \quad u = 0 \text{ on } \partial \Omega,$$

by using the Wolff type nonlinear potentials, where the domain $\Omega$ is open and bounded in $\mathbb{R}^n$ with $n \geq 2$; the functional data $\mu$ belongs to an appropriate Lebesgue space. Our detailed assumptions on the problem are imposed as following.

(A1) Basic assumption on $A$. The continuous vector field $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is Carathéodory function (i.e. $A$ is measurable with respect to $x$ on $\Omega$ for every $\zeta \in \mathbb{R}^n$ and is $C^1$ with respect to $\zeta$ on $\mathbb{R}^n$ for almost every $x \in \Omega$). Here, the nonlinearity $A$ also satisfies the following growth, ellipticity assumptions: there exist $p > 1$, $c_A \geq 1$, $\tau \in [0,1]$, and $\gamma \in (0,2-p)$ such that

$$|A(x,\zeta)| \leq c_A (\tau^2 + |\zeta|^2)^{-\frac{1}{2}}, \quad |\partial_\zeta A(x,\zeta)| \leq c_A (\tau^2 + |\zeta|^2)^{\frac{3}{2} - 1},$$

$$\langle \partial_\zeta A(x,\zeta) \nu, \nu \rangle \geq c_A^{-1} (\tau^2 + |\zeta|^2)^{\frac{3}{2} - 1} |\nu|^2,$$

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\[ |\partial_A(x, \zeta) - \partial_A(x, \nu)| \leq c_A (\tau^2 + |\zeta|^2)^{1-\frac{2}{n}} (\tau^2 + |\nu|^2)^{1-\frac{2}{n}} \cdot (\tau^2 + |\zeta|^2 + |\nu|^2)^{\frac{2-n}{2-n}} |\zeta - \nu|^\gamma, \quad (4) \]
whenever \( x \in \Omega \) and every \( \zeta, \nu \in \mathbb{R}^n; \partial_A \) is standard notation of gradient of \( A \) with respect to variable \( \zeta \).

It remarks that in this paper, in assumptions (2)-(4) of operator \( A \), we shall treat with \( \tau \in [0,1] \), and the parameter \( \tau \) appears to distinguish both degenerate and non-degenerate cases. When \( \tau = 0 \), our results regard the degenerate case, for instance the \( p \)-Laplace equation \(-\text{div}(|\nabla u|^{p-2}\nabla u) = \mu \), and that also being a prototype of a class of (1). This equation associated with the elliptic operator \( A(\cdot, \zeta) = |\zeta|^{p-2}\zeta \), and obviously, assumption (A1) is satisfied. It is well known that regularity results and potential estimates for general non-degenerate case were obtained in many recent studies by Mingione and his collaborators [12, 14, 26, 34]; the particular case \( \tau = 1 \) for non-degenerate \( p \)-Laplacian equations presented in [3,7] for example. And there has been a substantial amount of contributions for degenerate case [3,7, 37, 40, 41, 44–46], etc.

(A2) Assumption on \( p \). The growth exponent \( p \) will be a number in \( \mathbb{R} \), and in the present work, we confine ourselves to the very singular case, where
\[ 1 < p \leq \frac{3n-2}{2n-1}. \quad (5) \]

Significant progresses have been made for the pointwise estimates via linear and nonlinear potentials through the years. In particular, many contributions to quasilinear elliptic problems with a \( L^1 \) or measure data have been intensively studied for various range of \( p \). The study of pointwise estimates employing linear and nonlinear potentials has been developed in a series of succeeding articles by Mingione and his coworkers in [12–14, 22–27, 35]. In these works, authors obtained local pointwise bounds not only for solutions but also gradient of solutions and their fractional derivatives to the nonlinear equations/systems according to the range of \( p \). For instance, two cases where \( p > 2 - \frac{1}{n} \) and \( 2 - \frac{1}{n} < p \leq 2 \) are separately considered and the pointwise results are obtained in terms of linear and nonlinear potentials; via Riesz, Wolff or Havin-Maz’ya type potentials. For instance, when \( p \geq 2 \), gradient estimates via Wolff potentials obtained in [14], and Riesz potential gradient estimates for \( p \)-Laplacian type equations in [23], or \( C^1 \) regularity criterion for \( p \)-Laplace type equations presented in [13]. For \( 2 - \frac{1}{n} < p \leq 2 \), gradient Riesz potential estimates were done in [12]. In some independent works, the singular range of \( p \) for which \( 1 < p \leq 2 - \frac{1}{n} \) started to get attention in the last years. Especially, we refer to an interesting paper by Nguyen and Phuc in [38], where they have dealt with the pointwise gradient estimates for the case where \( \frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n} \) by using the means of Wolff type potentials. Motivated by the contributions above-mentioned and by the ideas recently studied in [38], in this paper we are devoted to the study of pointwise gradient bounds for (1) when \( 1 < p \leq \frac{3n-2}{2n-1} \).

(A3) Assumption on datum \( \mu \). Due the the ‘very singular’ case (5), in this paper we are concerned with the right-hand side from classes that close to \( L^1 \). More precisely, the given datum is considered to be a function belonging to \( L^\sigma(\Omega) \) for some \( \sigma \in (\sigma^*, \sigma^{**}) \), where
\[ \sigma^* := \frac{n}{2n(p-1) + 2 - p} \quad \text{and} \quad \sigma^{**} := \frac{n}{n(p-1) + 1}. \quad (6) \]
The very nature of the problem investigated in our work here requires the following fact

\[ p \leq \frac{3n - 2}{2n - 1} \iff \sigma^* \geq 1, \]

and in this case, generally speaking, based on methods given in [5, 37], one cannot expect to obtain comparison results with right-hand side as a function in \( L^1 \).

At this stage, let us discuss further these assumptions (A2), (A3) and some previous related works on the same topic. As far as we know, the nonlinear elliptic equations/systems with the general case of \( \mu \) have attracted a lot of attention through the years, where the inhomogeneous term \( \mu \) in (1) is merely a Radon measure with bounded total variation. Roughly speaking, one devotes special attention to the regularity of solutions when \( \mu \in L^1 \). For equations involving measure data, it is noteworthy that there have been many regularity results for solutions and gradient of solutions obtained for \( p > 2 - \frac{1}{n} \) (see [2, 12, 14, 22, 23, 26, 33, 34]), and for the singular case when \( \frac{3n - 2}{2n - 1} < p \leq 2 - \frac{1}{n} \) (see [37, 44]). Recently, ourselves; Nguyen and Phuc were trying to deal with the ‘very singular’ case as \( 1 < p \leq \frac{2n - 2}{2n - 1} \) in [39, 47].

It is worth noting that when considering such the ‘very singular’ case, ones have to pay a price by the additional assumptions on measure datum \( \mu \), or the solution itself. Particularly, in our previous work [47], assumption (A3) was imposed, meanwhile in [39], when \( \mu \in L^1 \), authors restricted their study to deal only with solutions \( u \) satisfying \( \nabla u \in L^{2-p}(\Omega) \) (the notion of renormalized solutions was introduced). In some sense, when \( p \) is very close to 1, the additional assumption \( \nabla u \in L^{2-p}(\Omega) \) seems to be an extremely strong requirement on regularity of \( u \). However, in general, these extra assumptions are flexible enough to prove the regularity results or gradient estimates in appropriate function spaces. Therefore, it can be seen that the problem of finding the conditions on \( \mu \) (as a measure) is still open in regularity and potential theories.

**(A4) Dini assumption on \( A \).** In addition, we shall further assume that \( A \) satisfies Dini continuity condition: there is a non-decreasing and concave mapping \( \omega : \mathbb{R}^+ \to [0, 1] \) and \( \omega(0) = 0 = \lim_{\varrho \searrow 0} \omega(\varrho) \) such that

\[ |A(x, \zeta) - A(\xi, \zeta)| \leq c_A \omega(|x - \xi|)(\tau^2 + |\zeta|^2)^{\frac{p-1}{2}}, \]

holds for any \( x, \xi \in \Omega \) and \( \zeta \in \mathbb{R}^n \setminus \{0\} \). It is worth mentioning here that the function \( \omega \) in (7) is modulus of continuity of \( A \) that required to satisfy Dini condition

\[ \int_0^1 [\omega(\varrho)]^\theta \frac{d\varrho}{\varrho} = D_\omega < +\infty, \]

for some

\[ \theta \in \left( \frac{n}{2n - 1}, \frac{n\sigma(p - 1)}{n - \sigma} \right). \]

It emphasizes that the condition (7), traced back to the work by Dini [11] in 1902, plays a limit case of the Hölder continuity, that usually applied to potential theory, nonlinear PDEs, and so on (for e.g., see [30, 43]).

Apart from regularity results and/or global gradient estimates of solutions to quasilinear elliptic equations (1), the study of pointwise estimates via linear/nonlinear potentials has been extensively investigated in recent years. Starting from the pioneering works by Kilpeläinen and Malý in [19, 20], where authors proved pointwise potential estimates in terms of Wolff potentials, there has been heightened interest in
pointwise estimates for solutions and their gradients to quasilinear elliptic equations via potentials. Some important results can be found lately, for instance a different method proposed by Trudinger and Wang in [48, 49]; or Wolff potentials estimates to nonlinear Hessian type operators obtained by Labutin in [28]; pointwise Wolff potentials estimates for elliptic equations involving measure in [21]; pointwise bounds for the systematic $p$-Laplace equations in [10]. Besides, the line of research has been subsequently enriched, for example, with nonlinear parabolic equations in [24, 25], with non-standard growth problems, $p(\cdot)$-Laplacian type problems in [4, 6, 9], etc.

Following the previous results and being motivated by some intensive studies from Calderón-Zygmund estimates and global regularity results under the certain range of $p$, our interest here is to further establish these pointwise estimates for gradient of solutions to quasilinear elliptic equations in terms of Wolff type potentials of the source term $\mu$, under ‘very singular’ case for which $1 < p \leq \frac{n+2}{n-2}$. Our main approach based on the technique presented by Nguyen and Phuc in [38] and our pointwise gradient potential estimates obtained in this paper extend such kind of results under assumptions (A1), (A2), (A3), (A4). Moreover, in order to obtain the global pointwise bounds for the gradient of solutions to problem (1) (the second main result in this paper), it is important for us to impose an additional assumption on the boundary of domain $\Omega$. And here, we would like to discuss about the geometric result in this paper, it is important for us to impose an additional assumption on $\partial \Omega$ to ensure the validity of results in $\Omega$ geometrically. For further reading of the Reifenberg flat domain, we recommend the reader to [8, 18, 29] and references therein.

(A5) Assumption on $\partial \Omega$. We assume that $\Omega$ is $(\delta, R_0)$-Reifenberg for $R_0 > 0$ and an appropriate small number $\delta \in (0, 1)$. It means at every $z \in \partial \Omega$ and $\varrho \in (0, R_0]$, one can find a coordinate system $\{\xi_1, \xi_2, \ldots, \xi_n\}$ with origin at $z$ such that

$$B^+_{\delta, \varrho}(z) \subset B_{\varrho}(z) \cap \Omega \subset B^-_{\delta, \varrho}(z),$$

where $B^+_{\delta, \varrho}(z) := B_{\varrho}(z) \cap \{\xi_n > \delta \varrho\}$ and $B^-_{\delta, \varrho}(z) := B_{\varrho}(\bar{z}) \cap \{\xi_n > -\delta \varrho\}$.

Before stating the main results in our study here, for the readers’ convenience, in the entirety of the article we shall use the following compact notation $\text{data}$

$$\text{data} = (\epsilon_0, n, p, \gamma, \tau, \theta, \sigma, D),$$

to highlight the set of given parameters on which the generic constant depends. For the sake of stating our main theorems, we need to present some notation and definitions that will be needed throughout this article.

Notation. In what follows, $\Omega \subset \mathbb{R}^n$ denotes the bounded domain. We also write $B_R(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < R\}$ in place of ball in $\mathbb{R}^n$ with radius $R$ and center $x_0$. By the capital letter $C$, we denote a universal positive constant whose relevant dependencies will be clarified and not necessarily the same from line to line. Besides, if $B_R \Subset \Omega$ and $h \in L^{1}_{\text{loc}}(\Omega)$, the mean value of $h$ will be denoted by

$$[h]_{B_R} := \int_{B_R} h(\zeta) d\zeta = \frac{1}{|B_R|} \int_{B_R} h(\zeta) d\zeta,$$

where $|B_R|$ denotes the Lebesgue measure of $B_R$ in $\mathbb{R}^n$. For reasons of brevity, for any $h \in L^2(\Omega)$, $g \in W^{1,p}(\Omega)$ and an open subset $\mathcal{O} \subset \Omega$, let us write $v \in$
that

\[ \text{Theorem 1.4} \quad (\text{Local gradient bounds via truncated Wolff potentials}) \]

in \( \Omega \) via the use of truncated Wolff potentials. As a consequence of Theorem 1.4, the second result in Theorem 1.5 states a nonlinear potential gradient bound for solutions at every point \( x \in \Omega \). In the first result, Theorem 1.4, we employ the truncated Wolff potentials, there are two types of pointwise gradient bounds of solutions to (1) included in this paper. In the first result, Theorem 1.4, we

Moreover, the truncated Wolff potential at level \( R > 0 \) is defined by

\[ W_{\alpha,\beta}^R(|h|)(x) = \int_0^R \left( \varrho^{\alpha\beta} \int_{B_{\varrho}(x)} |h(\zeta)| d\zeta \right)^{\frac{1}{\alpha\beta}} \frac{d\varrho}{\varrho}, \quad x \in \mathbb{R}^n. \] (12)

Moreover, the truncated Wolff potential at level \( R > 0 \) is given by

\[ W_{\alpha,\beta}^R(|h|)(x) = \int_0^R \left( \varrho^{\alpha\beta} \int_{B_{\varrho}(x)} |h(\zeta)| d\zeta \right)^{\frac{1}{\alpha\beta}} \frac{d\varrho}{\varrho}. \] (13)

Here, it is also remarkable that \( \mathbf{I}_\alpha(|h|) = W_{\frac{\alpha\beta}{2},\frac{\alpha\beta}{2}}(|h|) \).

**Results.** Let us now present our main results in two following theorems. Employing the truncated Wolff potentials, there are two types of pointwise gradient bounds of solutions to (1) included in this paper. In the first result, Theorem 1.4, we state a nonlinear potential gradient bound for solutions at every point \( x \) inside balls contained in \( \Omega \). As a consequence of Theorem 1.4, the second result in Theorem 1.5 infers that the behavior of \( \nabla u \) can be pointwise controlled for almost everywhere \( x \) in \( \Omega \) via the use of truncated Wolff potentials.

**Theorem 1.4** (Local gradient bounds via truncated Wolff potentials). Suppose that \( u \in C^1(\Omega) \) solves (1) under assumptions (A1)-(A4). Then we have

\[ |\Phi_{\tau,\theta}(\nabla u(x))| \leq C \left\{ \left[ W_{\frac{\alpha\beta}{2},\frac{\alpha\beta}{2}}^R(|u|^\alpha)(x) \right]^{\frac{1}{\alpha\beta}} + \int_{B_R(x)} (\tau + |\nabla u(\xi)|)^{\theta} d\xi \right\}. \] (14)

for each ball \( B_R(x) \subset \Omega \), where \( C = C(\text{data}) > 0 \). Here the truncated Wolff potential \( W_{\frac{\alpha\beta}{2},\frac{\alpha\beta}{2}}^R \) is given by (13) and the vector field \( \Phi_{\tau,\theta} \) is defined by

\[ \Phi_{\tau,\theta}(\xi) := (\tau^2 + |\xi|^2)^{\frac{\alpha\beta-1}{2}} \xi, \quad \xi \in \mathbb{R}^n. \] (15)
Here, we remark that the result in Theorem 1.4 applies to the degenerate equations, when \( \tau = 0 \), leading to the local pointwise gradient bound for solutions in terms of Wolff potentials. It can be found in the following corollary.

**Corollary 1** (Local gradient bounds for the degenerate problem). Under assumptions of Theorem 1.4 for the degenerate problem (1) with \( \tau = 0 \), there holds

\[
|\nabla u(x)| \leq C \left\{ \left[ W^{R}_{\frac{1}{\tau} + \sigma + \theta}(|\mu|^\sigma)(x) \right]^{\frac{1}{\tau - 1}} + \left( \int_{B_R(x)} |\nabla u(\xi)|^\theta d\xi \right)^{\frac{1}{\theta}} \right\},
\]

for each ball \( B_R(x) \subset \Omega \).

**Theorem 1.5** (Global gradient bounds via truncated Wolff potentials). Suppose that \( u \) is a weak solution to (1) under assumptions (A1)-(A4). Then for each \( \kappa \in \left( 0, \frac{n}{p} \right) \), there exits \( \delta = \delta(\kappa, \text{data}) > 0 \) such that if the assumption (A5) is satisfied, then we have

\[
|\Phi_{\tau, \theta}(\nabla u(x))| \leq Cd(x)^{-\kappa \theta} \left\{ \left[ W^{D_0}_{\frac{1}{\tau} + \sigma + \theta}(|\mu|^\sigma)(x) \right]^{\frac{1}{\tau - 1}} + \tau^\theta \right\},
\]

for almost everywhere \( x \in \Omega \) and \( D_0 = 2\text{diam}(\Omega) \). Here \( d(x) := \text{dist}(x, \partial\Omega) \) denotes the distance from \( x \) to \( \partial\Omega \).

The organization of this paper is as follows. In Section 2, some interior pointwise gradient estimates will be presented, in which some comparison procedures between the original solution \( u \) and the unique solution \( w \) to homogeneous Dirichlet problem will be established. In this section we also focus on the proof of Theorem 1.4. Next, combining with local estimates near the boundary of our domain, Section 3 will be established. In this section we also focus on the proof of Theorem 1.4, as a direct consequence of Theorem 1.4.

2. Interior pointwise gradient bounds. This section is devoted to prove some pointwise gradient estimates for solutions in the interior of domain \( \Omega \). Before proceeding to these gradient estimates, some comparison procedures between original solutions to (1) and to the corresponding homogeneous problems are presented.

2.1. Related to interior homogeneous problems. Let us firstly recall the following technical lemma, Lemma 2.1, a result related to the comparison estimate between solutions to both problems (1) and the homogeneous one. This result was proved by Q.-H. Nguyen in his earlier unpublished work [36] and an initial version of the result has been proved in [37] when \( \mu \in L^1(\Omega) \) and \( \frac{3n-2}{n-1} < p \leq 2 - \frac{1}{\lambda} \), see also [47, Appendix A] for a detail proof. It is worth noting that the proof of this comparison result in [47] is still valid for the non-degenerate case \( \tau > 0 \).

Let us fix \( x_0 \in \Omega \) and \( 0 < 2R \leq r_0 \) such that \( B_{2r}(x_0) \subset \Omega \). Unless other stated, we shall denote \( B_\rho := B_\rho(x_0) \) for any \( \rho \in (0, 2R] \).

**Lemma 2.1.** Let \( 1 < p < \frac{3n-2}{2n-1} \) and \( \mu \in L^p_{\text{loc}}(\Omega) \) for some \( \sigma \in (\sigma^*, \sigma^{**}) \). Suppose that \( u \in W^{1,p}_{\text{loc}}(\Omega) \) is a solution to (1) and \( w = \text{sol}_\lambda(u, B_{2R}) \). Then for any \( \frac{n}{2n-1} < \theta < \frac{np(p-1)}{n-\sigma} \) and \( \lambda > 0 \), there holds

\[
\int_{B_{2R}} |\nabla u - \nabla w|^\theta d\xi \leq \lambda \int_{B_{2R}} (\tau + |\nabla u|^\theta d\xi + C \left( R^\sigma \int_{B_{2R}} |\mu|^\sigma d\xi \right)^{\frac{\theta}{\sigma(p-1)}} ,
\]

with \( C = C(\lambda, \text{data}) > 0 \).
In the next lemma, we establish the comparison estimates between \( w = \text{sol}_{A}(u, B_{2R}) \) and the unique solution \( v \) the the homogeneous problem corresponding to the operator \( A_{0} := A(x_{0}, \cdot) \) in the smaller ball \( B_{R}(x_{0}) \), i.e. \( v = \text{sol}_{A_{0}}(w, B_{R}) \).

**Lemma 2.2.** Let \( 1 < p \leq \frac{3n-2}{2n-1} \) and \( \mu \in L^{p}_{\text{loc}}(\Omega) \) for some \( \sigma \in (\sigma^{*}, \sigma^{**}) \). Then, for any \( \frac{n}{2n-1} < \theta < \frac{n(\rho-1)}{n-\sigma} \), there holds

\[
\int_{B_{R}} |\nabla v - \nabla w|^{\theta}d\xi \leq C[\omega(R)]^{\theta} \int_{B_{2R}} (\tau + |\nabla w|)^{\theta}d\xi,
\]

with \( C = C(\text{data}) > 0 \).

**Proof.** Let us first recall an estimate for the difference \( \nabla v - \nabla w \) as follows

\[
\int_{B_{R}} |\nabla v - \nabla w|^{p}d\xi \leq C[\omega(R)]^{p} \int_{B_{2R}} (\tau + |\nabla w|)^{p}d\xi.
\]

The proof of this fact can be found in [12, Equ.(4.35)]. We recall here the reverse Hölder’s inequality applied to solution of the local homogeneous problem, that is

\[
\left( \int_{B_{r}(\xi)} (\tau + |\nabla w|)^{\theta_{0}}d\xi \right)^{\frac{1}{\theta_{0}}} \leq C \left( \int_{B_{2r}(\xi)} (\tau + |\nabla w|)^{s}d\xi \right)^{\frac{1}{s}},
\]

for any \( s > 0 \) and \( B_{2r}(\xi) \subset B_{2R} \), where \( C = C(c_{A}, n, p) \) and \( \Theta_{0} = C_{0}(\text{data}) > p \). This inequality is well-known as a version of interior Gehring’s lemma (see [15] or [16, Theorem 6.7]). Using (20) and Hölder’s inequality, we get (19).

The next lemma involves the comparison estimate related to the following potential \( E_{\tau, \theta} \), for a given \( \tau \in [0, 1] \) and \( \theta \in (0, p) \), defined by

\[
E_{\tau, \theta}(\varphi, B) := \int_{B} |\Phi_{\tau, \theta}(\nabla \varphi) - [\Phi_{\tau, \theta}(\nabla \varphi)]_{B}|d\xi,
\]

for \( \varphi \in W^{1,p}_{\text{loc}}(\Omega) \) and for any ball \( B \subset \Omega \), where \( \Phi_{\tau, \theta} \) defined as in (15). Here, we also remark that

\[
E_{\tau, \theta}(\varphi, B) \leq 2 \int_{B} |\Phi_{\tau, \theta}(\nabla \varphi)|d\xi.
\]

**Lemma 2.3.** Let \( 1 < p \leq \frac{3n-2}{2n-1} \) and \( \mu \in L^{p}_{\text{loc}}(\Omega) \) for some \( \sigma \in (\sigma^{*}, \sigma^{**}) \). Suppose that \( u \in W^{1,p}_{\text{loc}}(\Omega) \) is a solution to equation (1). Then there is \( \beta \in (0, 1) \) such that for any \( \varepsilon \in (0, 1) \), \( \lambda > 0 \) and \( \frac{n}{2n-1} < \theta < \frac{n(\rho-1)}{n-\sigma} \) there holds

\[
E_{\tau, \theta}(u, B_{\varepsilon R}) \leq C\varepsilon^{\beta\theta}E_{\tau, \theta}(u, B_{R}) + (\lambda + C\varepsilon^{-n}\omega(R)^{\theta}) \int_{B_{2R}} (\tau + |\nabla u|)^{\theta}d\xi
\]

\[
+ C(\varepsilon, \lambda) \left( R^{\beta} \int_{B_{2R}} |\mu|^{\sigma}d\xi \right)^{\frac{\theta}{\sigma(p-1)}},
\]

with \( C = C(\text{data}) > 0 \) and \( C(\varepsilon, \lambda) = C(\varepsilon, \lambda, \text{data}) > 0 \). Here the potential \( E_{\tau, \theta} \) is defined by (21).

**Proof.** Let us set \( w = \text{sol}_{A}(u, B_{2R}) \) and \( v = \text{sol}_{A_{0}}(w, B_{R}) \) with \( A_{0} := A(x_{0}, \cdot) \). For simplicity we use the following notations

\[
U(\xi) := \Phi_{\tau, \theta}(\nabla u(\xi)) \quad \text{and} \quad V(\xi) := \Phi_{\tau, \theta}(\nabla v(\xi)), \quad \text{for} \ \xi \in \Omega, \ \zeta \in B_{R}.
\]
where $\Phi_{\tau, \theta}$ defined by (15). Let us first decompose as follows

$$
E_{\tau, \theta}(u, B_{\varepsilon R}) = \int_{B_{\varepsilon R}} |U(\xi) - [U]_{B_{\varepsilon R}}| \, d\xi.
$$

Combining with the following fact

$$
\int_{B_{\varepsilon R}} |[U]_{B_{\varepsilon R}} - [V]_{B_{\varepsilon R}}| \, d\xi = \int_{B_{\varepsilon R}} |U(\xi) - V(\xi)| \, d\xi,
$$

from (24) it yields that

$$
E_{\tau, \theta}(u, B_{\varepsilon R}) \leq E_{\tau, \theta}(v, B_{\varepsilon R}) + 2 \int_{B_{\varepsilon R}} |U(\xi) - V(\xi)| \, d\xi.
$$

Thanks to [38, Theorem 1.2], one can find $\beta = \beta(\text{data}) \in (0, 1)$ and $C = C(\text{data}) > 1$ such that

$$
E_{\tau, \theta}(v, B_{\varepsilon R}) \leq C\varepsilon^\beta \varepsilon^{\theta} E_{\tau, \theta}(v, B_{R}),
$$

which follows that

$$
E_{\tau, \theta}(v, B_{\varepsilon R}) \leq C\varepsilon^\beta \varepsilon^{\theta} E_{\tau, \theta}(v, B_{R}) \leq C\varepsilon^\beta \varepsilon^{\theta} E_{\tau, \theta}(u, B_{R}).
$$

(26)

Here we remark that the parameter $\sigma \in (\sigma^*, \sigma^{**})$ with $\sigma^*$ and $\sigma^{**}$ given as in (6), it allows us to arrive

$$
\theta \in \left( \frac{n}{2n-1}, \frac{n\sigma(p-1)}{n-\sigma} \right) \subset \left( \frac{1}{2}, 1 \right).
$$

To estimate the last term of (25), let us recall the classical and fundamental inequality (see [1, Lemma 2.2] for instance) with $\theta < 1$ above. More precisely, there is $C = C(n, \theta) > 0$ satisfying

$$
|U - V| = \left( \frac{2+n}{2} \right)^{\theta} \nabla u - \left( \frac{2+n}{2} \right)^{\theta} \nabla v \leq C|\nabla u - \nabla v|^{\theta},
$$

which leads to

$$
\int_{B_{\varepsilon R}} |U(\xi) - V(\xi)| \, d\xi \leq C \int_{B_{\varepsilon R}} |\nabla u - \nabla v|^{\theta} \, d\xi
$$

$$
\leq C\varepsilon^{-n} \int_{B_{R}} |\nabla u - \nabla v|^{\theta} \, d\xi.
$$

(27)

Moreover, thanks to inequality (19) in Lemma 2.2, it gives

$$
\int_{B_{\varepsilon R}} |\nabla u - \nabla v|^{\theta} \, d\xi \leq C \int_{B_{\varepsilon R}} |\nabla u - \nabla w|^{\theta} \, d\xi + C \int_{B_{\varepsilon R}} |\nabla w - \nabla v|^{\theta} \, d\xi
$$

$$
\leq C \int_{B_{2R}} |\nabla u - \nabla w|^{\theta} \, d\xi + C\omega(R)^{\theta} \int_{B_{2R}} (\tau + |\nabla w|)^{\theta} \, d\xi
$$

$$
\leq C \int_{B_{2R}} |\nabla u - \nabla w|^{\theta} \, d\xi + C\omega(R)^{\theta} \int_{B_{2R}} (\tau + |\nabla u|)^{\theta} \, d\xi.
$$

(28)
Substituting (28) into (27) and combining with estimation (18) in Lemma 2.1 for every \( \lambda > 0 \), one obtains that
\[
\int_{B_{2R}} |U(\xi) - V(\xi)| \, d\xi \leq \left[ \frac{\lambda}{2} + C\varepsilon^{-n}\omega(R)^{\theta} \right] \int_{B_{2R}} (\tau + |\nabla u|)^{\theta} \, d\xi + C(\varepsilon, \lambda) \left( R^\sigma \int_{B_{2R}} |\mu|^\sigma \, d\xi \right)^{\frac{\theta}{\sigma(p-1)}} \tag{29}
\]
Finally one may conclude (23) to complete the proof by taking two estimations (29) and (26) into account of (25).

2.2. Interior estimate. We are now interested in the proof of the interior pointwise estimate result in Theorem 1.4. The key technique bases on the previous relations to interior homogeneous problems.

Proof of Theorem 1.4. Let \( u \in C^1(\Omega) \) be a solution to (1) with \( \mu \in L^\sigma(\Omega) \) for some \( \sigma \in (\sigma^*, \sigma^*) \). For every \( \xi \in \Omega \), let us consider the vector field
\[
U(\xi) := \Phi_{\tau, \theta}(\nabla u(\xi)) = (\tau^2 + |\nabla u(\xi)|^{2})^{\frac{p-1}{2}} \nabla u(\xi).
\]
With \( \theta \in \left( \frac{n}{2n-1}, \frac{n\sigma(p-1)}{n-\sigma} \right) \subset \left( \frac{1}{2}, 1 \right) \), it ensures that
\[
|U(\xi)| \leq |\nabla u(\xi)|^{\theta} \leq (\tau + |\nabla u(\xi)|)^{\theta}, \quad \text{for all } \xi \in \Omega. \tag{30}
\]
For \( R > 0 \) and \( x \in \Omega \) satisfying \( B_{R}(x) \subset \Omega \), we can see that it is enough to prove (14) by showing
\[
|U(x)| \leq C \left\{ \left[ \frac{W_{R}^{n\sigma(p-1)+\theta}}{\frac{pn}{n+\theta} \cdots \frac{p}{n+\theta}} (|\mu|^\sigma)(x) \right]^{\frac{1}{\theta}} + \int_{B_{R}(x)} (\tau + |\nabla u(\xi)|)^{\theta} \, d\xi \right\}. \tag{31}
\]
For every \( k \in \mathbb{N} \) and \( \varepsilon \) chosen later, we shall use the following notations
\[
R_{k} := \varepsilon^{k} R, \quad B_{k} := B_{2R_{k}}(x), \quad \mathcal{E}_{k} := \mathcal{E}_{\tau, \theta}(u, B_{R_{k}}(x)),
\]
with \( \mathcal{E}_{\tau, \theta} \) defined as in (21), and
\[
ak_k := \left( R_{k}^\sigma \int_{B_{k}} |\mu|^\sigma \, d\xi \right)^{\frac{\theta}{\sigma(p-1)}}, \quad bk := \int_{B_{k}} (\tau + |\nabla u(\xi)|)^{\theta} \, d\xi.
\]
Thanks to (23) in Lemma 2.3 with the choice \( \lambda = \frac{2^{-k}}{\varepsilon^{n}} \), one can find constants \( \beta \in (0, 1) \) and \( C_0 = C_0(\text{data}) > 0, \ C_1 = C_1(\text{data}) > 0 \) such that
\[
\mathcal{E}_{k+1} \leq C_{0} \varepsilon^{\beta \theta} \mathcal{E}_{k} + C_{1}(\varepsilon)ak_k + \left( \frac{2^{-k} \varepsilon^{n}}{4} + C_{0} \varepsilon^{-n} \omega(R_{k})^{\theta} \right) bk, \tag{32}
\]
for all \( k \in \mathbb{N} \). Let us choose \( \varepsilon \in (0, \frac{1}{2}) \) small enough such that \( C_{0} \varepsilon^{\beta \theta} \leq \frac{1}{2} \), where \( C_{0} \) is the constant in (32). Under assumption (8), there exists a constant \( k_0 = k_0(n, \varepsilon, C_{0}, D_{\omega}) > 1 \) large enough such that
\[
2C_{0} \varepsilon^{-2n} \sum_{k=k_0}^{\infty} \omega(R_{k})^{\theta} \leq \frac{1}{4}. \tag{33}
\]
For every \( N > k_0 \), summing (32) up over \( k \in \{ k_0, k_0+1, k_0+2, \ldots, N-1 \} \), one obtains that
\[
\sum_{k=k_0}^{N} \mathcal{E}_{k} \leq 2\mathcal{E}_{k_0} + 2C_{1}(\varepsilon) \sum_{k=k_0}^{N-1} ak_k + \sum_{k=k_0}^{N-1} \left( \frac{2^{-k} \varepsilon^{n}}{2} + 2C_{0} \varepsilon^{-n} \omega(R_{k})^{\theta} \right) bk. \tag{34}
\]
Moreover, it is not difficult to see that

\[
\left|\{U\}_{B_{k+1}} - \{U\}_{B_k}\right| \leq \int_{B_k} |U(\xi) - \{U\}_{B_k}| \, d\xi \\
\leq \varepsilon^{-n} \int_{B_k} |U(\xi) - \{U\}_{B_k}| \, d\xi = \varepsilon^{-n} E_k,
\]

which deduces to

\[
\varepsilon^{-n} \sum_{k=k_0}^{N} E_k \geq \sum_{k=k_0}^{N} \left|\{U\}_{B_{k+1}} - \{U\}_{B_k}\right| \\
\geq \left|\{U\}_{B_{N+1}} - \{U\}_{B_{k_0}}\right| \geq \left|\{U\}_{B_{N+1}} - \{U\}_{B_{k_0}}\right|.
\]

Combining between the above inequality with (32) and (34), it yields that

\[
\left|\{U\}_{B_{N+1}}\right| + E_{N+1} \leq \left|\{U\}_{B_{k_0}}\right| + (1 + 2\varepsilon^{-n}) E_{k_0} + 2C_1(\varepsilon)\varepsilon^{-n} \sum_{k=k_0}^{N-1} a_k \\
+ \sum_{k=k_0}^{N-1} \left(\frac{2^{-k}}{2} + 2C_0\varepsilon^{-2n}\omega(R_k)^\theta\right) b_k.
\]

(35)

On the other hand, for \( p \in \left(1, \frac{3n-2}{2n-1}\right) \subset (1, 2) \) we note that

\[
\sum_{k=k_0}^{N} a_k \leq C_2 \left[\int_{0}^{2R_{k_0}-1} \left(\varepsilon^\sigma \int_{B_{u(\xi)}} |\mu|^\sigma \, d\xi\right) \frac{d\varrho}{\varrho}\right]^{\frac{1}{p-1}}.
\]

(36)

Thanks to (30), to obtain (31) it is sufficient to show that

\[
|U(x)| \leq C b_{k_0} + C \left[\int_{0}^{2R_{k_0}-1} \left(\varepsilon^\sigma \int_{B_{u(\xi)}} |\mu|^\sigma \, d\xi\right) \frac{d\varrho}{\varrho}\right]^{\frac{1}{p-1}}.
\]

(37)

It is clearly that (37) is trivial if \(|U(x)| \leq b_{k_0}\). Otherwise, let us consider two cases which correspond to the existence and non-existence of \( k_1 \in \mathbb{N} \) satisfying

\[
b_k \leq |U(x)| \leq b_{k_1+1}, \quad \forall k_0 \leq k \leq k_1.
\]

(38)

In the first case, we assume that one can find \( k_1 \in \mathbb{N} \) validating (38). Then thanks to (30) again one has

\[
|U(x)| \leq \int_{B_{k_1+1}} (\tau + |\nabla u|^\theta) \, d\xi \\
\leq \tau^\theta + \int_{B_{k_1+1}} |\nabla u| \, d\xi \leq \tau^\theta + E_{k_1+1} + \left|\{U\}_{B_{k_1+1}}\right|.
\]
Let us now apply (35) with $N = k_1$ and take into account (36) and (38) to arrive
\[
|U(x)| \leq \tau^\theta + (1 + 2\epsilon^{-n}) E_{k_0} + \left| \left[ U \right]_{B_{k_0}} \right|
\]
\[
+ C_3(\epsilon) \left[ \int_0^{2^{R_{k_0}-1}} \left( g^\sigma \int_{B_{k_0}(x)} \left| \mu \right|^\sigma d\xi \right)^{\frac{\hat{s}}{2}} \frac{d\theta}{\theta} \right]^{\frac{1}{\hat{s}}} + \frac{1}{2} |U(x)|
\]
\[
+ \sum_{k=k_0}^{k_1} \left( \frac{2^{-k}}{2} + 2C_0\epsilon^{-2n}\omega(R_k)^\theta \right) |U(x)|.
\] (39)

Hence using (33) to the last term in (39) we can find
\[
|U(x)| \leq \tau^\theta + (1 + 2\epsilon^{-n}) E_{k_0} + \left| \left[ U \right]_{B_{k_0}} \right|
\]
\[
+ C_3(\epsilon) \left[ \int_0^{2^{R_{k_0}-1}} \left( g^\sigma \int_{B_{k_0}(x)} \left| \mu \right|^\sigma d\xi \right)^{\frac{\hat{s}}{2}} \frac{d\theta}{\theta} \right]^{\frac{1}{\hat{s}}} + \frac{1}{2} |U(x)|
\]
\[
\leq C_\epsilon b_{k_0} + C_\epsilon \left[ \int_0^{2^{R_{k_0}-1}} \left( g^\sigma \int_{B_{k_0}(x)} \left| \mu \right|^\sigma d\xi \right)^{\frac{\hat{s}}{2}} \frac{d\theta}{\theta} \right]^{\frac{1}{\hat{s}}} + \frac{1}{2} |U(x)|,
\]
which guarantees (37) from (22).

In the remain case, one may suppose that $b_k < |U(x)|$ for all $k \geq k_0$. Then from (35) for any $N > k_0$, there holds
\[
\left| \left[ U \right]_{B_{N+1}} \right| \leq \tau^\theta + (1 + 2\epsilon^{-n}) E_{k_0} + \left| \left[ U \right]_{B_{k_0}} \right|
\]
\[
+ C_3(\epsilon) \left[ \int_0^{2^{R_{k_0}-1}} \left( g^\sigma \int_{B_{k_0}(x)} \left| \mu \right|^\sigma d\xi \right)^{\frac{\hat{s}}{2}} \frac{d\theta}{\theta} \right]^{\frac{1}{\hat{s}}} + \frac{1}{2} |U(x)|
\]
\[
+ \sum_{k=k_0}^{N} \left( \frac{2^{-k}}{2} + 2C_0\epsilon^{-2n}\omega(R_k)^\theta \right) |U(x)|.
\] (40)

As the previous case, let us now take into account (33) to the last term of (40), it yields that
\[
\left| \left[ U \right]_{B_{N+1}} \right| \leq \tau^\theta + (1 + 2\epsilon^{-n}) E_{k_0} + \left| \left[ U \right]_{B_{k_0}} \right|
\]
\[
+ C_3(\epsilon) \left[ \int_0^{2^{R_{k_0}-1}} \left( g^\sigma \int_{B_{k_0}(x)} \left| \mu \right|^\sigma d\xi \right)^{\frac{\hat{s}}{2}} \frac{d\theta}{\theta} \right]^{\frac{1}{\hat{s}}} + \frac{1}{2} |U(x)|.
\]

Passing $N \to \infty$ in the above inequality, one obtains that
\[
|U(x)| \leq Cb_{k_0} + C \left[ \int_0^{2^{R_{k_0}-1}} \left( g^\sigma \int_{B_{k_0}(x)} \left| \mu \right|^\sigma d\xi \right)^{\frac{\hat{s}}{2}} \frac{d\theta}{\theta} \right]^{\frac{1}{\hat{s}}} + \frac{1}{2} |U(x)|,
\]
which ensures (37) and completes the proof. \(\square\)
3. Boundary pointwise gradient bounds. In this section, we give a detailed proof of Theorem 1.5, in which the global pointwise gradient estimate will be established. Here, under additional assumption (A5), local estimates at any point \(x \in \Omega\) near the boundary are mainly treated. Without loss of generality, we may assume that
\[
d(x) \leq r_1 \leq R_0/100 \leq \text{diam}(\Omega)/1000.
\]

For any point \(x\) that is closed to the boundary of \(\Omega\), it enables us to take \(\bar{x} \in \partial \Omega\) such that \(|\bar{x} - x| = d(x)|\). For any \(R \in (0, 2r_1]\), let us set \(w = \text{sol}_{\lambda_1}(u, B_{2R}(\bar{x}) \cap \Omega)\) and \(v = \text{sol}_{\lambda_1}(w, B_R(\bar{x}) \cap \Omega)\) with \(\lambda_1 := \lambda(\bar{x}, \cdot)\). We may consider the solutions \(u, w, v\) defined over \(\mathbb{R}^n\) by extend \(u = 0\) in \(\mathbb{R}^n \setminus \Omega\), \(w = u\) in \(\mathbb{R}^n \setminus (B_{2R}(\bar{x}) \cap \Omega)\) and \(v = w\) in \(\mathbb{R}^n \setminus (B_R(\bar{x}) \cap \Omega)\). The following lemma can be obtained by using the comparison estimates in previous section and the higher integrability up to the boundary under the regularity assumption on \(\partial \Omega\).

**Lemma 3.1.** Let \(1 < p \leq \frac{3n - 2}{2n - 1}\) and \(\mu \in L^\sigma_\text{loc}(\Omega)\) for some \(\sigma \in (\sigma^*, \sigma^{**})\). Then for every \(\eta \in (0, \frac{p-1}{p})\) and \(\frac{n}{2n - 1} < \theta < \frac{n\sigma(p-1)}{n-\sigma}\), there exists \(\delta = \delta(\theta, \eta) > 0\) such that if (A5) holds then
\[
\int_{B_{2R}(\bar{x})} |\nabla v|^\theta d\xi \leq C \left( \frac{\theta}{R} \right)^{-\eta \theta} \int_{B_{2R}(\bar{x})} (\tau + |\nabla v|)^\theta d\xi,
\]
for every \(0 < \theta < R/800\).

**Proof.** Since \(w = \text{sol}_{\lambda_1}(u, B_{2R}(\bar{x}) \cap \Omega)\) and \(v = \text{sol}_{\lambda_1}(w, B_R(\bar{x}) \cap \Omega)\), one can find \(\delta_1 > 0\) such that if (A5) is valid then there holds
\[
\int_{B_{2R}(\bar{x})} |\nabla u - \nabla w|^\theta d\xi \leq \lambda \int_{B_{2R}(\bar{x})} (\tau + |\nabla u|)^\theta d\xi
\]
\[
+ C \left( R^\sigma \int_{B_{2R}(\bar{x})} |\mu|^\sigma d\xi \right)^{\frac{\theta}{n\sigma - 1}},
\]
for all \(\lambda > 0\) and
\[
\int_{B_{R}(\bar{x})} |\nabla v - \nabla w|^\theta d\xi \leq C \omega(R)^\theta \int_{B_{2R}(\bar{x})} (\tau + |\nabla w|)^\theta d\xi.
\]
Using (42) and (43) and performing the same method as in the proof of [38, Lemma 4.2] for the case \(\mu \in L^1(\Omega)\) and \(\frac{3n - 2}{2n - 1} < p \leq 2 - \frac{1}{n}\), for every \(\epsilon \in (0, 1/800)\) one has
\[
\int_{B_{R}(\bar{x})} |\nabla v|^\theta d\xi \leq C \epsilon^{-\eta \theta} \int_{B_{R}(\bar{x})} (\tau + |\nabla v|)^\theta d\xi,
\]
which implies to (41).

Let us recall the fundamental lemma in [17, Lemma 1.4] as below.

**Lemma 3.2.** Let \(0 \leq \alpha_2 < \alpha_1, \varsigma \in (0, 1)\) and \(\varphi : [0, R_0] \to \mathbb{R}^+\) be a non-decreasing mapping. Assume that
\[
\varphi(\theta) \leq Q_1 \left[ \left( \frac{\theta}{R} \right)^{\alpha_1} + \epsilon \right] \varphi(R) + Q_2 R^{\alpha_2},
\]
for all \(0 < \theta \leq \varsigma R < R_0\) and \(\epsilon > 0\) small enough, with two constants \(Q_1, Q_2 > 0\). Then, there is a constant \(\epsilon_0 = \epsilon_0(Q_1, \alpha_1, \alpha_2) > 0\) such that if \(\epsilon \in (0, \epsilon_0)\) then
\[
\varphi(\theta) \leq C \left[ \left( \frac{\theta}{R} \right)^{\alpha_1} \varphi(R) + Q_2 R^{\alpha_2} \right],
\]
for all $0 \leq \rho \leq R \leq R_0$, where $C = C(Q_1, \alpha_1, \alpha_2) > 0$.

Proof of Theorem 1.5. Let $x \in \Omega$ and a small number $r_1$ such that $0 < r_1 \leq R_0/100 \leq D_0/1000$. One can find $\bar{x} \in \partial \Omega$ satisfying $|\bar{x} - x| = d(x) \leq r_1$. For every $R \in (0, 2r_1]$ and $\rho$ satisfying $0 < \rho \leq R/800$, let $w = \text{sol}_\rho(u, B_{2R})$ and $v = \text{sol}_{\rho_1}(w, B_R)$ with $\rho_1 := \rho(x, \cdot)$. Thanks to (42) and (43) in Lemma 3.1, one has

\[
\int_{B_{2R}(\bar{x})} |\nabla u - \nabla w|^\theta \, d\xi \leq \epsilon \int_{B_{2R}(\bar{x})} (\tau + |\nabla u|)^\theta \, d\xi + C_\epsilon \left( R^\frac{\theta}{\pi(p-1)} \int_{B_{2R}(\bar{x})} |\mu|^{\sigma} \, d\xi \right),
\]  

(44)

for every $\epsilon \in (0, 1)$ and

\[
\int_{B_R(\bar{x})} |\nabla v - \nabla w|^\theta \, d\xi \leq C \omega(R)^\theta \int_{B_{2R}(\bar{x})} (\tau + |\nabla w|)^\theta \, d\xi
\]

\[
\leq C \omega(R)^\theta \int_{B_{2R}(\bar{x})} (\tau + |\nabla u|)^\theta \, d\xi.
\]  

(45)

On the other hand, for every $\kappa \in (0, \frac{\mu}{p})$, let us take $\eta \in (0, \kappa)$. Lemma 3.1 gives us the existence of $\delta = \delta(\theta, \eta) > 0$ such that if (A5) holds then

\[
\int_{B_{\rho}(\bar{x})} |\nabla v|^\theta \, d\xi \leq C \left( \frac{\rho}{R} \right)^{-\eta \theta} \int_{B_R(\bar{x})} (\tau + |\nabla v|)^\theta \, d\xi
\]

\[
\leq C \left( \frac{\rho}{R} \right)^{-\eta \theta} \int_{B_{2R}(\bar{x})} (\tau + |\nabla u|)^\theta \, d\xi.
\]  

(46)

Using the following fundamental inequality, one has

\[
\int_{B_{\rho}(\bar{x})} |\nabla u|^\theta \, d\xi \leq C \left( \int_{B_{\rho}(\bar{x})} |\nabla u - \nabla w|^\theta \, d\xi + \int_{B_{\rho}(\bar{x})} |\nabla w - \nabla v|^\theta \, d\xi \right)
\]

\[
+ \int_{B_{\rho}(\bar{x})} |\nabla v|^\theta \, d\xi,
\]

which implies from (46) that

\[
\int_{B_{\rho}(\bar{x})} |\nabla u|^\theta \, d\xi \leq C \left( \int_{B_{2R}(\bar{x})} |\nabla u - \nabla w|^\theta \, d\xi + \int_{B_R(\bar{x})} |\nabla w - \nabla v|^\theta \, d\xi \right)
\]

\[
+ \left( \frac{\rho}{R} \right)^{n-\eta \theta} \int_{B_{2R}(\bar{x})} (\tau + |\nabla u|)^\theta \, d\xi.
\]  

(47)

Substituting two previous estimations in (44) and (45) into (47), one obtains that

\[
\int_{B_{\rho}(\bar{x})} |\nabla u|^\theta \, d\xi \leq C \left[ \epsilon + \omega(R)^\theta + \left( \frac{\rho}{R} \right)^{n-\eta \theta} \right] \int_{B_{2R}(\bar{x})} (\tau + |\nabla u|)^\theta \, d\xi
\]

\[
+ C_\epsilon R^n \left( R^\frac{\theta}{\pi(p-1)} \int_{B_{2R}(\bar{x})} |\mu|^{\sigma} \, d\xi \right)
\]  

(48)
Using Hölder’s inequality into (48), it yields that
\[
\varphi(\rho) \leq C \left[ \varepsilon + \omega(R)^\theta + \left( \frac{\theta}{R} \right)^{n-\eta\theta} \right] \varphi(R)
\]
\[+ C \varepsilon R^{n-\kappa\theta} r_1 \left\{ \left[ W_{D_0, \frac{n+\theta}{\alpha+n+\theta}, \frac{n\sigma}{n+\theta}} (|\mu|^{\sigma})(x) \right]^{\frac{1}{\sigma'}} + \tau^\theta \right\}. \tag{49}
\]
where the function \( \varphi \) defined by
\[
\varphi(\rho) := \int_{B_{\rho}(\bar{x})} |\nabla u|^{\theta} d\xi, \quad \rho \in (0, 2r_1].
\]
As the setting at the beginning of the proof, we remind that \( \varphi \) satisfies (49) for all \( 0 < \rho \leq R/800 < 2r_1 \). Hence we can apply Lemma 3.2 with
\[
0 < \alpha_2 := n - \kappa\theta < \alpha_1 := n - \eta\theta \quad \text{and} \quad \varepsilon = \epsilon + \omega(R)^\theta,
\]
to arrive
\[
\varphi(\rho) \leq C \left( \frac{\theta}{R} \right)^{n-\kappa\theta} \varphi(R) + C \varepsilon R^{n-\kappa\theta} r_1 \left\{ \left[ W_{D_0, \frac{n+\theta}{\alpha+n+\theta}, \frac{n\sigma}{n+\theta}} (|\mu|^{\sigma})(x) \right]^{\frac{1}{\sigma'}} + \tau^\theta \right\}. \tag{50}
\]
for all \( 0 < \rho \leq R \leq 2r_1 \). Let us choose \( \rho = 2d(x) \) and \( R = 2r_1 \), we may rewrite (50) as follows
\[
\int_{B_{2d(x)}(\bar{x})} |\nabla u|^{\theta} d\xi \leq C \left( \frac{r_1}{d(x)} \right)^{n-\kappa} \int_{B_{2r_1}(\bar{x})} |\nabla u|^{\theta} d\xi
\]
\[+ \left[ W_{D_0, \frac{n+\theta}{\alpha+n+\theta}, \frac{n\sigma}{n+\theta}} (|\mu|^{\sigma})(x) \right]^{\frac{1}{\sigma'}} + \tau^\theta \right\}. \tag{51}
\]
Moreover, for \( \frac{n}{2n-1} < \theta < \frac{n\sigma(p-1)}{n-\sigma} \) we can using Hölder’s inequality to get
\[
\int_{B_{2r_1}(\bar{x})} |\nabla u|^{\theta} d\xi \leq C r_1^{-n} \int_{B_{D_0}(\bar{x})} |\nabla u|^{\theta} d\xi
\]
\[\leq C r_1^{-n} D_0^{-\frac{(n-\sigma\theta)}{\sigma(p-\sigma\theta)}} \left( \int_{\Omega} |\nabla u|^{\frac{n\sigma(p-1)}{n-\sigma}} d\xi \right)^{\frac{\theta}{\sigma(p-\sigma\theta)}}
\]
\[\leq C r_1^{-n} D_0^{-\frac{(n-\sigma\theta)}{\sigma(p-\sigma\theta)}} \left( \int_{\Omega} |\mu|^{\sigma} d\xi \right)^{\frac{\theta}{\sigma(p-\sigma\theta)}} + C \tau^\theta
\]
\[\leq C (D_0, r_1) \left[ W_{D_0, \frac{n+\theta}{\alpha+n+\theta}, \frac{n\sigma}{n+\theta}} (|\mu|^{\sigma})(x) \right]^{\frac{1}{\sigma'}} + C \tau^\theta,
\]
which with (51) allows us to get
\[
\int_{B_{d(x)}(x)} |\nabla u|^{\theta} d\xi \leq C \left( \frac{r_1}{d(x)} \right)^{\kappa} \left\{ \left[ W_{D_0, \frac{n+\theta}{\alpha+n+\theta}, \frac{n\sigma}{n+\theta}} (|\mu|^{\sigma})(x) \right]^{\frac{1}{\sigma'}} + \tau^\theta \right\}. \tag{52}
\]
Finally, Theorem 1.4 gives us
\[
|\Phi_{\tau, \theta}(\nabla u(x))| \leq C \left\{ \left[ W_{D_0, \frac{n+\theta}{\alpha+n+\theta}, \frac{n\sigma}{n+\theta}} (|\mu|^{\sigma})(x) \right]^{\frac{1}{\sigma'}} + \int_{B_{d(x)}(x)} |\nabla u|^{\theta} d\xi + \tau^\theta \right\}. \tag{53}
\]
We may conclude (17) by combining between two estimations (52) and (53). \( \square \)
REFERENCES

[1] E. Acerbi and N. Fusco, Regularity for minimizers of non-quadratic functionals: The case $1 < p < 2$, J. Math. Anal. Appl., 140 (1989), 115–135.

[2] B. Avelin, T. Kuusi and G. Mingione, Nonlinear Calderón-Zygmund theory in the limiting case, Arch. Rational. Mech. Anal., 227 (2018), 663–714.

[3] P. Baroni, Lorentz estimates for degenerate and singular evolutionary systems, J. Differential Equations, 255 (2013), 2927–2951.

[4] P. Baroni and J. Habermann, Elliptic interpolation estimates for non-standard growth operators, Ann. Acad. Sci. Fenn. Math., 39 (2014), 119–162.

[5] P. Benilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre and J. L. Vázquez, An $L^{1}$-theory of existence and uniqueness of solutions of nonlinear elliptic equations, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (IV), 22 (1995), 241–273.

[6] V. Bögelein and J. Habermann, Gradient estimates via non standard potentials and continuity, Ann. Acad. Sci. Fenn. Math., 35 (2010), 641–678.

[7] S.-S. Byun and L. Wang, $L^{p}$-estimates for general nonlinear elliptic equations, Indiana Univ. Math. J., 56 (2007), 3193–3221.

[8] S.-S. Byun and L. Wang, Elliptic equations with BMO nonlinearity in Reifenberg domains, Adv. Math., 219 (2008), 1937–1971.

[9] S.-S. Byun and Y. Youn, Optimal gradient estimates via Riesz potentials for $p(\cdot)$-Laplacian type equations, Quart. J. Math., 68 (2017), 1071–1115.

[10] A. Cianchi and S. Schwarzacher, Potential estimates for the $p$-Laplace system with data in divergence form, J. Differential Equations, 265 (2018), 478–499.

[11] U. Dini, Sur la méthode des approximations successives pour les équations aux dérivées partielles du deuxième ordre, Acta Math., 25 (1902), 185–230.

[12] F. Duzaar and G. Mingione, Gradient estimates via linear and nonlinear potentials, J. Funct. Anal., 259 (2010), 2961–2998.

[13] F. Duzaar and G. Mingione, Gradient continuity estimates, Calc. Var. and PDE, 39 (2010), 379–418.

[14] F. Duzaar and G. Mingione, Gradient estimates via non-linear potentials, Amer. J. Math., 133 (2011), 1093–1149.

[15] F. W. Gehring, The $L^{p}$ integrability of partial derivatives of a quasiconformal mapping, Bull. Amer. Math. Soc., 79 (1973), 465–466.

[16] E. Giusti, Direct Methods in the Calculus of Variations, World Scientific Publishing Co., Inc., River Edge, 2003.

[17] Q. Han and F. Lin, Elliptic Partial Differential Equations, 2nd edn. American Mathematical Society, Providence, RI, 2011.

[18] C. Kenig and T. Toro, Poisson kernel characterization of Reifenberg flat chord arc domains, Ann. Sci. Éc. Norm. Supér., 36 (2003), 323–401.

[19] T. Kilpeläinen and J. Malý, Degenerate elliptic equations with measure data and nonlinear potentials, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (IV), 19 (1992), 591–613.

[20] T. Kilpeläinen and J. Malý, The Wiener test and potential estimates for quasilinear elliptic equations, Acta Math., 172 (1994), 137–161.

[21] R. Korte and T. Kuusi, A note on the Wolff potential estimate for solutions to elliptic equations involving measures, Adv. Calc. Var., 3 (2010), 99–113.

[22] T. Kuusi and G. Mingione, Universal potential estimates, J. Funct. Anal., 262 (2012), 4205–4289.

[23] T. Kuusi and G. Mingione, Linear potentials in nonlinear potential theory, Arch. Ration. Mech. Anal., 207 (2013), 215–246.

[24] T. Kuusi and G. Mingione, The Wolff gradient bound for degenerate parabolic equations, J. Eur. Math. Soc., 16 (2014), 835–892.

[25] T. Kuusi and G. Mingione, Riesz potentials and nonlinear parabolic equations, Arch. Ration. Mech. Anal., 212 (2014), 727–780.

[26] T. Kuusi and G. Mingione, Guide to nonlinear potential estimates, Bull. Math. Sci., 4 (2014), 1–82.

[27] T. Kuusi and G. Mingione, Vectorial nonlinear potential theory, J. Eur. Math. Soc., 20 (2018), 929–1004.

[28] D. Labutin, Potential estimates for a class of fully nonlinear elliptic equations, Duke Math. J., 111 (2002), 1–49.
[29] A. Lemenant, E. Milakis and L. V. Spinolo, On the extension property of Reifenberg flat domains, *Ann. Acad. Sci. Fenn. Math.*, **39** (2014), 51–71.

[30] G. Lieberman, Higher regularity for nonlinear oblique derivative problems in Lipschitz domains, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (V)*, **1** (2002), 111–152.

[31] P. Lindqvist, Notes on the $p$-Laplace equation, *Univ. Jyväskylä* Report, **102** (2006).

[32] T. Mengesha and N. C. Phuc, Weighted and regularity estimates for nonlinear equations on Reifenberg flat domains, *J. Differential Equations*, **250** (2011), 1485–2507.

[33] G. Mingione, The Calderón-Zygmund theory for elliptic problems with measure data, *Ann. Scuola. Norm. Super. Pisa Cl. Sci. (V)*, **6** (2007), 195–261.

[34] G. Mingione, Gradient estimates below the duality exponent, *Math. Ann.* **346** (2010), 571–627.

[35] G. Mingione, *Gradient potential estimates*, *J. Eur. Math. Soc.*, **13** (2011), 459–486.

[36] Q.-H. Nguyen, Gradient estimates for singular quasilinear elliptic equations with measure data, preprint, [arXiv:1705.07440](https://arxiv.org/abs/1705.07440).

[37] Q.-H. Nguyen and N. C. Phuc, Good-$\lambda$ and Muckenhoupt-Wheeden type bounds, with applications to quasilinear elliptic equations with gradient power source terms and measure data, *Math. Ann.*, **374** (2019), 67–98.

[38] Q.-H. Nguyen and N. C. Phuc, Pointwise gradient estimates for a class of singular quasilinear equations with measure data, *J. Funct. Anal.*, **278** (2020), 108391, 35pp.

[39] Q.-H. Nguyen and N. C. Phuc, Existence and regularity estimates for quasilinear equations with measure data: The case $1 < p \leq \frac{3n-2}{2n-1}$, preprint, [arXiv:2003.03725](https://arxiv.org/abs/2003.03725).

[40] T.-N. Nguyen and M.-P. Tran, Lorentz improving estimates for the $p$-Laplace equations with mixed data, *Nonlinear Anal.*, **200** (2020), 111960, 23pp.

[41] T.-N. Nguyen and M.-P. Tran, Level-set inequalities on fractional maximal distribution functions and applications to regularity theory, *J. Funct. Anal.*, **280** (2021), 108797, 47pp.

[42] E. Reifenberg, Solutions of the plateau problem for $m$-dimensional surfaces of varying topological type, *Acta Math.*, **104** (1960), 1–92.

[43] M. E. Taylor, *Tools for PDE. Pseudodifferential Operators, Paradifferential Operators, and Layer Potentials*, Mathematical Surveys and Monographs, **81** (2000), Providence, RI: American Mathematical Society.

[44] M.-P. Tran, Good-$\lambda$ type bounds of quasilinear elliptic equations for the singular case, *Nonlinear Anal.*, **178** (2019), 266–281.

[45] M.-P. Tran and T.-N. Nguyen, Lorentz-Morrey global bounds for singular quasilinear elliptic equations with measure data, *Commun. Contemp. Math.*, **22** (2020), 1950033, 30pp.

[46] M.-P. Tran and T.-N. Nguyen, New gradient estimates for solutions to quasilinear divergence form elliptic equations with general Dirichlet boundary data, *J. Differential Equations*, **268** (2020), 1427–1462.

[47] M.-P. Tran and T.-N. Nguyen, Global gradient estimates for very singular nonlinear elliptic equations with measure data, preprint, [arXiv:1909.06991](https://arxiv.org/abs/1909.06991).

[48] N. S. Trudinger and X. J. Wang, On the weak continuity of elliptic operators and applications to potential theory, *Amer. J. Math.*, **124** (2002), 369–410.

[49] N. S. Trudinger and X. J. Wang, Quasilinear elliptic equations with signed measure data, *Disc. Cont. Dyn. Systems A*, **23** (2009), 477–494.

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