A posteriori virtual element method for the acoustic vibration problem

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Received: 26 July 2022 / Accepted: 8 December 2022 / Published online: 13 February 2023
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Abstract
In two dimensions, we propose and analyze an a posteriori error estimator for the acoustic spectral problem based on the virtual element method in $\mathbb{H}(\text{div}; \Omega)$. Introducing an auxiliary unknown, we use the fact that the primal formulation of the acoustic problem is equivalent to a mixed formulation, in order to prove a superconvergence result, necessary to despise high order terms. Under the virtual element approach, we prove that our local indicator is reliable and globally efficient in the $L^2$-norm. We provide numerical results to assess the performance of the proposed error estimator.

Keywords Virtual element method · Acoustic vibration problem · Polygonal meshes · A posteriori error estimates · Superconvergence

Mathematics Subject Classification (2010) 65N30 · 65N25 · 70J30 · 76M25

Communicated by: Lourenco Beirao da Veiga

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1 Introduction

One of the most important subjects in the development of numerical methods for partial differential equations is the a posteriori error analysis, since it allows dealing with singular solutions that arise due to, for instance, geometrical features of the domain or some particular boundary conditions, among others. In this sense, and in particular for eigenvalue problems arising from problems related to solid and fluid mechanics and electromagnetism, just to mention some possible applications, the a posteriori analysis has taken relevance in recent years (see [2, 11, 12, 14, 21, 22, 33, 35, 38, 39] and the references therein).

The virtual element method (VEM), introduced in [4], has shown remarkable results in different problems, and particularly for solving eigenproblems, showing great accuracy and flexibility in the approximation of eigenvalues and eigenfunctions. We mention [18, 19, 24–28, 30–34, 36] as recent works on this topic.

The acoustic vibration problem appears in important applications in engineering. In fact, it can be used to design of structures and devices for noise reduction in aircraft or cars mainly related with solid-structure interaction problems, among other important applications. In the last years, several numerical methods have been developed in order to approximate the eigenpairs of the associated spectral problem. In particular, a virtual element discretization has been proposed in [9]. It is well known that one of the most important features of the virtual element method is the efficient computational implementation and the flexibility on the geometries for meshes, where precisely adaptivity strategies can be implemented in an easy way. In fact, the hanging nodes that appear in the refinement of some element of the mesh, can be treated as new nodes since adjacent non matching element interfaces are acceptable in the VEM. Recent research papers report interesting advantages of the VEM in the a posteriori error analysis and adaptivity for source problems. We refer to [7, 16, 17, 37] and the references therein, for instance, for a further discussion. On the other hand, a posteriori error analysis for eigenproblems by VEM has been recently introduced in [33, 35, 40], where primal formulations in $H^1$ have been considered.

The contribution of our work is the design and analysis of an a posteriori error estimator for the acoustic spectral problem, by means of a VEM. The VEM that we consider in our analysis is the one introduced in [9] for the a priori error analysis of the acoustic eigenproblem. We stress that the VEM presented in [9] may be preferable to more standard finite elements even in the case of triangular meshes in terms of dofs (cf. [9, Remark 3]). The formulation for the acoustic problem is written only in terms of the displacement of the fluid, which leads to a bilinear form with divergence terms, implying that the analysis for the a posteriori error indicator is not straightforward. This difficulty produced by the $H(\text{div})$ formulations leads to analyze, in first place, an equivalent mixed formulation which provides suitable results in order to control the so-called high order terms that naturally appear. This analysis depending on an equivalent mixed formulation has been previously considered in [13, 14] for the a posteriori analysis for the Maxwell’s eigenvalue problem, inspired by the superconvergence results of [29] for mixed spectral formulations. We will follow the same techniques for the present $H(\text{div})$ framework. However, due to the nature of the
VEM, the local indicator that we present contains an extra term depending on the virtual projector which needs to be analyzed carefully.

The organization of our paper is the following: in Section 2 we present the acoustic problem and the mixed equivalent formulation for it. We recall some properties of the spectrum of the spectral problem and regularity results. In Section 3 we found the core of the analysis of our paper, where we introduce the virtual element method for our spectral problem and technical results that will be needed to establish a super-convergence result, with the aid of mixed formulations. Section 4 is dedicated to the a posteriori error analysis, where we introduce our local and global indicators which, as is customary in the a posteriori error analysis, will be reliable and efficient. In Section 5, we report numerical tests where we assess the performance of our estimator. We end the article with some concluding remarks.

1.1 Notations and preliminaries

Throughout this work, \( \Omega \) is a generic Lipschitz bounded domain of \( \mathbb{R}^2 \). Moreover, we use the following notation for any 2D vector field \( \mathbf{r} \), and any scalar field \( v \):

\[
\begin{align*}
\text{div } \mathbf{r} &:= \partial_1 r_1 + \partial_2 r_2, \\
\nabla v &:= \begin{pmatrix} \partial_1 v \\ \partial_2 v \end{pmatrix}, \\
\text{rot } \mathbf{r} &:= \partial_1 r_2 - \partial_2 r_1, \\
\text{curl } v &:= \begin{pmatrix} \partial_2 v \\ -\partial_1 v \end{pmatrix}.
\end{align*}
\]

Next, for \( s \geq 0 \), \( \| \cdot \|_{s, \Omega} \) stands indistinctly for the norm of the Hilbertian Sobolev spaces \( H^s(\Omega) \) or \( [H^s(\Omega)]^2 \) with the convention \( H^0(\Omega) := L^2(\Omega) \). We also define the Hilbert space \( H(\text{div}; \Omega) := \{ \mathbf{r} \in [L^2(\Omega)]^2 : \text{div } \mathbf{r} \in L^2(\Omega) \} \), whose norm is given by \( \| \mathbf{r} \|^2_{\text{div}, \Omega} := \| \mathbf{r} \|^2_{0, \Omega} + \| \text{div } \mathbf{r} \|^2_{0, \Omega} \). For \( s \geq 0 \), we define the Hilbert space \( H^s(\text{div}; \Omega) := \{ \mathbf{r} \in [H^s(\Omega)]^2 : \text{div } \mathbf{r} \in H^s(\Omega) \} \), whose norm is given by \( \| \mathbf{r} \|^2_{H^s(\text{div}; \Omega)} := \| \mathbf{r} \|^2_{s, \Omega} + \| \text{div } \mathbf{r} \|^2_{s, \Omega} \).

Finally, we employ \( \mathbf{0} \) to denote a generic null vector and the relation \( a \lesssim b \) indicates that \( a \leq Cb \), with a positive constant \( C \) which is independent of \( a, b, \) and the size of the elements in the mesh. The value of \( C \) might change at each occurrence. We remark that we will write the constant \( C \) only when is needed.

2 The spectral problem

We consider the free vibration problem for an acoustic fluid within a bounded rigid cavity \( \Omega \subset \mathbb{R}^2 \) with polygonal boundary \( \Gamma \) and outward unit normal vector \( \mathbf{n} \):

\[
\begin{align*}
-\omega^2 \rho \mathbf{w} &= -\nabla p \quad \text{in } \Omega, \\
p &= -\rho c^2 \text{div } \mathbf{w} \quad \text{in } \Omega, \\
\mathbf{w} \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma,
\end{align*}
\]

(1)

where \( \mathbf{w} \) is the fluid displacement, \( p \) is the pressure fluctuation, \( \rho \) the density, \( c \) the acoustic speed and \( \omega \) the vibration frequency. For simplicity on the forthcoming analysis, we consider \( \rho \) and \( c \) equal to one.
Multiplying the first equation in (1) by a test function \( \tau \in H_0(\text{div}; \Omega) \), where
\[
H_0(\text{div}; \Omega) := \{ \tau \in H(\text{div}; \Omega) : \tau \cdot n = 0 \quad \text{on} \; \Gamma \},
\]
integrating by parts, using the boundary condition of \( w \), and replacing the definition of \( p \) in the first equation of (1), we arrive at the following weak formulation.

**Problem 2.1** Find \( (\lambda, w) \in \mathbb{R} \times H_0(\text{div}; \Omega), \ w \neq 0 \), such that
\[
\int_{\Omega} \text{div} w \, \text{div} \tau = \lambda \int_{\Omega} w \cdot \tau \quad \forall \tau \in H_0(\text{div}; \Omega),
\]
where \( \lambda := \omega^2 \). It is well known that the spectrum of Problem 2.1 consists in a sequence of eigenvalues \( \{0\} \cup \{\lambda_k\}_{k \in \mathbb{N}} \), such that

i) \( \lambda = 0 \) is an infinite-multiplicity eigenvalue and its associated eigenspace is \( H_0(\text{div}^0; \Omega) := \{ \tau \in H_0(\text{div}; \Omega) : \text{div} \tau = 0 \; \text{in} \; \Omega \} \);

ii) \( \{\lambda_k\}_{k \in \mathbb{N}} \) is a sequence of finite-multiplicity eigenvalues which satisfy \( \lambda_k \rightarrow \infty \).

To perform an a posteriori error analysis for spectral problems, we need the so-called *superconvergence result*, in order to neglect high order terms as has been proved in [29] and already applied in, for instance, the Maxwell’s eigenvalue problem [13, 14]. In order to obtain this superconvergence result, we begin by introducing an equivalent mixed formulation for Problem 2.1. For \( \lambda \neq 0 \) let us introduce the unknown
\[
u := -\frac{\text{div} w}{\lambda} \in L^2(\Omega).
\]
To remain consistent with the notations, we will denote by \((\cdot, \cdot)_{0,\Omega}\) the \( L^2(\Omega) \) inner-product.

With the aid of (2) we write the following mixed eigenproblem:

**Problem 2.2** Find \( (\lambda, w, u) \in \mathbb{R} \times H_0(\text{div}; \Omega) \times L^2(\Omega), \) with \((w, u) \neq (0, 0)\), such that
\[
\begin{cases}
\int_{\Omega} w \cdot \tau + \int_{\Omega} u \, \text{div} \tau = 0 & \forall \tau \in H_0(\text{div}; \Omega), \\
\int_{\Omega} v \, \text{div} w = -\lambda \int_{\Omega} u & \forall v \in L^2(\Omega).
\end{cases}
\]

It is easy to check that the spectral Problems 2.1 and 2.2 are equivalent, except for \( \lambda = 0 \) on the following sense:

- If \( (\lambda, w) \) is a solution of Problem 2.1, with \( \lambda \neq 0 \), then \( (\lambda, w, -\text{div} w/\lambda) \) is solution of Problem 2.2.
- If \( (\lambda, w, u) \) is a solution of Problem 2.2, with \( \lambda \neq 0 \), then \( (\lambda, w) \) is solution of Problem 2.1 and \( u \) is defined as in (2).

We introduce the bounded and symmetric bilinear forms \( a : H_0(\text{div}; \Omega) \times H_0(\text{div}; \Omega) \rightarrow \mathbb{R} \) and \( b : H_0(\text{div}; \Omega) \times L^2(\Omega) \rightarrow \mathbb{R} \), defined as
\[
a(w, \tau) := \int_{\Omega} w \cdot \tau, \quad b(\tau, v) := \int_{\Omega} v \, \text{div} \tau.
\]
for all \( \mathbf{w}, \mathbf{r} \in H_0(\text{div}; \Omega) \), and for all \( \mathbf{v} \in L^2(\Omega) \), which allows us to rewrite Problem 2.2 as follows:

**Problem 2.3** Find \((\lambda, \mathbf{w}, \mathbf{u}) \in \mathbb{R} \times H_0(\text{div}; \Omega) \times L^2(\Omega)\), with \((\mathbf{w}, \mathbf{u}) \neq (\mathbf{0}, 0)\), such that

\[
\begin{align*}
a(\mathbf{w}, \mathbf{r}) + b(\mathbf{r}, \mathbf{u}) &= 0 \quad \forall \mathbf{r} \in H_0(\text{div}; \Omega), \\
b(\mathbf{w}, \mathbf{v}) &= -\lambda(\mathbf{u}, \mathbf{v})_0, \quad \forall \mathbf{v} \in L^2(\Omega).
\end{align*}
\]

**Remark 2.1** It is easy to check that if \((\lambda, \mathbf{w}, \mathbf{u})\) is a solution of Problem 2.3, then

\[ \mathbf{w} = \nabla \mathbf{u} \quad \text{and} \quad \text{div } \mathbf{w} = -\lambda \mathbf{u}. \]

Let \( \mathcal{K} \) be the kernel of bilinear form \( b(\cdot, \cdot) \) defined by:

\[ \mathcal{K} := \{ \mathbf{r} \in H_0(\text{div}; \Omega) : \text{div } \mathbf{r} = 0 \text{ in } \Omega \}. \]

It is well known that bilinear form \( a(\cdot, \cdot) \) is elliptic in \( \mathcal{K} \) and that \( b(\cdot, \cdot) \) satisfies the following inf-sup condition (see [10])

\[ \sup_{\mathbf{r} \in H_0(\text{div}; \Omega)} \frac{b(\mathbf{r}, \mathbf{v})}{\| \mathbf{r} \|_{\text{div}, \Omega} \| \mathbf{v} \|_{0, \Omega}} \geq \beta \Omega \| \mathbf{v} \|_{0, \Omega}, \quad \forall \mathbf{v} \in L^2(\Omega), \tag{3} \]

where \( \beta \Omega \) is a positive constant.

**Remark 2.2** The eigenvalues of Problem 2.3 are positive. Indeed, taking \( \mathbf{r} = \mathbf{w} \) and \( \mathbf{v} = \mathbf{u} \) in Problem 2.3 and subtracting the resulting forms, we obtain

\[ \lambda = \frac{a(\mathbf{w}, \mathbf{w})}{\| \mathbf{u} \|_{0, \Omega}^2} \geq 0. \]

In addition, \( \lambda = 0 \) implies \((\mathbf{w}, \mathbf{u}) = (\mathbf{0}, 0)\).

Let us introduce the following source problem: For a given \( g \in L^2(\Omega) \), the pair \((\mathbf{\tilde{w}}, \mathbf{\tilde{u}}) \in H_0(\text{div}; \Omega) \times L^2(\Omega)\) is the solution of the following well posed problem

\[
\begin{align*}
a(\mathbf{\tilde{w}}, \mathbf{r}) + b(\mathbf{r}, \mathbf{\tilde{u}}) &= 0 \quad \forall \mathbf{r} \in H_0(\text{div}; \Omega), \\
b(\mathbf{\tilde{w}}, \mathbf{v}) &= -(g, \mathbf{v})_0, \quad \forall \mathbf{v} \in L^2(\Omega). \tag{4}
\end{align*}
\]

According to [1], the regularity for the solution of system (4) and (5), (the associated source problem to Problem 2.3) is the following: there exists a constant \( \mathcal{R} > 1/2 \) depending on \( \Omega \) such that the solution \( \mathbf{\tilde{u}} \in H^{1+\mathcal{R}}(\Omega) \), where \( \mathcal{R} \) is at least 1 if \( \Omega \) is convex and \( \mathcal{R} \) is at least \( \pi/\omega - \varepsilon \), for any \( \varepsilon > 0 \) for a non-convex domain, with \( \omega < 2\pi \) being the largest re-entrant angle of \( \Omega \). Hence we have the following well-known additional regularity result for the source problem (4) and (5)

\[ \| \mathbf{\tilde{w}} \|_{\mathcal{R}, \Omega} + \| \mathbf{\tilde{u}} \|_{1+\mathcal{R}, \Omega} \lesssim \| g \|_{0, \Omega}. \tag{6} \]

Also, the eigenvalues are well characterized for this problem as is stated in the following result (see [3] for instance).
Lemma 2.1 The eigenvalues of Problem 2.2 consist in a sequence of positive eigenvalues \( \{\lambda_n : n \in \mathbb{N}\} \), such that \( \lambda_n \to \infty \) as \( n \to \infty \). In addition, the following additional regularity result holds true for eigenfunctions

\[
\|w\|_{r, \Omega} + \|\text{div } w\|_{1+r, \Omega} + \|u\|_{1+r, \Omega} \leq C(\lambda)\|u\|_{0, \Omega},
\]

with \( r > 1/2 \) and \( C(\lambda) \) is a constant depending on the eigenvalue.

3 The virtual element discretization

We begin this section recalling the mesh construction and the assumptions considered to introduce the discrete virtual element space. Then, we will introduce a virtual element discretization of Problem 2.1 and provide a spectral characterization of the resulting discrete eigenvalue problem.

Let \( \{\mathcal{T}_h\}_h \) be a sequence of decompositions of \( \Omega \) into polygons \( K \). Let \( h_K \) denote the diameter of the element \( K \) and let \( h := \max_{K \in \Omega} h_K \) be the mesh size. For the analysis, the following standard assumptions on the meshes are considered (see [5, 15]): there exists a positive real number \( C_T \) such that, for every \( K \in \mathcal{T}_h \) and for every \( h \), there holds.

\[ A_1: \quad \text{the ratio between the shortest edge and the diameter } h_K \text{ of } K \text{ is larger than } C_T, \]

\[ A_2: \quad K \in \mathcal{T}_h \text{ is star-shaped with respect to every point of a ball of radius } C_T h_K. \]

For any subset \( S \subseteq \mathbb{R}^2 \) and nonnegative integer \( k \), we denote by \( \mathbb{P}_k(S) \) the space of polynomials of degree up to \( k \) defined on \( S \). To keep the notation simpler, we denote by \( n \) a general normal unit vector, its precise definition will be clear from the context.

We consider now a polygon \( K \) and define the following local finite dimensional space for \( 0 \) (see [5, 15]):

\[ H^K_h := \{\tau_h \in H(\text{div}; K) \cap H(\text{rot}; K) : (\tau_h \cdot n) \in \mathbb{P}_k(\ell) \forall \ell \in \partial K, \text{ div } \tau_h \in \mathbb{P}_k(K), \text{ rot } \tau_h = 0 \text{ on } K\}, \]

We define the following degrees of freedom (dof) for functions \( \tau_h \) in \( H^K_h \):

\[
\int_{\ell} (\tau_h \cdot n) q \, ds \quad \forall q \in \mathbb{P}_k(\ell) \quad \forall \text{ edge } \ell \in \partial K, \tag{7}
\]

\[
\int_{K} \tau_h \cdot \nabla q \quad \forall q \in \mathbb{P}_k(K)/\mathbb{P}_0(K), \tag{8}
\]

which are unisolvent (see [9, Proposition 1]).

For every decomposition \( \mathcal{T}_h \) of \( \Omega \) into polygons \( K \), we define the global space

\[ H_h := \left\{ \tau_h \in H_0(\text{div}; \Omega) : \tau_h|_K \in H^K_h \right\}. \]

In agreement with the local choice we choose the following degrees of freedom:

\[
\int_{\ell} (\tau_h \cdot n) q \, ds \quad \forall q \in \mathbb{P}_k(\ell) \quad \text{for all internal edges } \ell \in \mathcal{T}_h, \]

\[
\int_{K} \tau_h \cdot \nabla q \quad \forall q \in \mathbb{P}_k(K)/\mathbb{P}_0(K) \quad \text{in each element } K \in \mathcal{T}_h.
\]
Observe that from the aforementioned degrees of freedom, for all \( q \in P_k(K) \), the integral

\[
\int_K \nabla q \cdot \nabla v_h
\]

is explicitly computable.

On the other hand, in order to construct the discrete scheme, we need some preliminary definitions. For each element \( K \in \mathcal{T}_h \), we define the orthogonal projector \( \Pi^K_h : [L^2(K)]^2 \rightarrow \nabla (P_{k+1}(K)) \subseteq H_K^K \) by

\[
\int_K \Pi^K_h \tau \cdot \nabla q = \int_K \tau \cdot \nabla q \quad \forall q \in P_{k+1}(K),
\]

and we point out that \( \Pi^K_h \tau_h \) is explicitly computable for every \( \tau_h \in H_K^K \) using only its degrees of freedom (7) and (8). In fact, it is easy to check that, for all \( \tau_h \in H_K^K \) and for all \( q \in P_{k+1}(K) \),

\[
\int_K \Pi^K_h \tau_h \cdot \nabla q = \int_K \tau_h \cdot \nabla q = -\int_K q \nabla \tau_h + \int_{\partial K} (\tau_h \cdot n) q \, ds.
\]

On the other hand, let \( S^K(\cdot, \cdot) \) be any symmetric positive definite (and computable) bilinear form that satisfies

\[
c_0 \int_K \tau_h \cdot \tau_h \leq S^K(\tau_h, \tau_h) \leq c_1 \int_K \tau_h \cdot \tau_h \quad \forall \tau_h \in H_K^K,
\]

for some positive constants \( c_0 \) and \( c_1 \) depending only on the shape regularity constant \( C_T \) from mesh assumptions \( A_1 \) and \( A_2 \). Then, we define on each \( K \) the following bilinear form:

\[
a^K_h(u_h, \tau_h) := \int_K \Pi^K_h u_h \cdot \Pi^K_h \tau_h + S^K(u_h - \Pi^K_h u_h, \tau_h - \Pi^K_h \tau_h) \quad u_h, \tau_h \in H_K^K.
\]

One way of defining \( S^K(\cdot, \cdot) \) is as follows: let \( \{\phi_i\}_{i=1, \ldots, N_{dof}^K} \) be the base of \( H_K^K \), such that

\[
S^K(\phi_i, \phi_j) := \sum_{r=1}^{N_{dof}^K} dof_r(\phi_i) dof_r(\phi_j).
\]

Also, in a natural way, we have

\[
a_h(u_h, \tau_h) := \sum_{K \in \mathcal{T}_h} a^K_h(u_h, \tau_h) \quad u_h, \tau_h \in H_h.
\]

The following properties of the bilinear form \( a^K_h(\cdot, \cdot) \) are easily derived (by repeating, in our case, the arguments from [15, Proposition 4.1]),

- **Consistency:**

\[
a^K_h(\nabla q, \tau_h) = \int_K \nabla q \cdot \tau_h \quad \forall q \in P_{k+1}(K) \quad \forall \tau_h \in H_K^K, \quad \forall K \in \mathcal{T}_h.
\]
Stability: There exist two positive constants $\alpha_*$ and $\alpha^*$, independent of $K$, such that:

$$\alpha_* \int_K \tau_h \cdot \tau_h \leq a_h^K(\tau_h, \tau_h) \leq \alpha^* \int_K \tau_h \cdot \tau_h \quad \forall \tau_h \in H_h^K, \quad \forall K \in T_h.$$  

Now, we are in position to write the virtual element discretization of Problem 2.1.

**Problem 3.1** Find $(\lambda_h, w_h) \in \mathbb{R} \times H_h$, with $w_h \neq 0$ such that

$$(\text{div} w_h, \text{div} \tau_h)_0 = \lambda_h a_h(w_h, \tau_h) \quad \forall \tau_h \in H_h.$$  

We have the following spectral characterization of the discrete eigenvalue Problem 3.1 (see [9]).

**Remark 3.1** There exist $M_h := \dim(H_h)$ eigenvalues of Problem 3.1 repeated according to their respective multiplicities, which are $\{0\} \cup \{\lambda_{h,k}\}_{k=1}^{N_h}$, where:

i) the eigenspace associated with $\lambda_h = 0$ is $K_h := \{v_h \in H_h : \text{div} v_h = 0\}$;

ii) $\lambda_{h,k} > 0$, $k = 1, \ldots, N_h := M_h - \dim(K_h)$, are non-defective eigenvalues repeated according to their respective multiplicities.

Now, we introduce the virtual element discretization of Problem 2.3.

**Problem 3.2** Find $(\lambda_h, w_h, u_h) \in \mathbb{R} \times H_h \times Q_h$, with $(w_h, u_h) \neq (0, 0)$, such that

$$\begin{cases}
a_h(w_h, \tau_h) + b(\tau_h, u_h) = 0 & \forall \tau_h \in H_h, \\
b(w_h, v_h) = -\lambda_h (u_h, v_h)_0 & \forall v_h \in Q_h,
\end{cases}$$

where $Q_h := \{q \in L^2(\Omega) : q|_K \in P_k(K) \quad \forall K \in T_h \}, k \geq 0$.

We also introduce the $L^2(\Omega)$-orthogonal projection

$$P_k : L^2(\Omega) \rightarrow Q_h,$$

and the following approximation result (see [5]): if $0 \leq s \leq k + 1$, it holds

$$\|v - P_k v\|_{0,\Omega} \lesssim h^s \|v\|_{s,\Omega} \quad \forall v \in H^s(\Omega).$$

The next two technical results establish the approximation properties for $\tau_I$ and their proofs can be found in [9, Appendix].

**Lemma 3.1** Let $\tau \in H_0(\text{div}; \Omega)$ be such that $\tau \in [H^t(\Omega)]^2$ with $t > 1/2$. There exists $\tau_I \in H_h$ that satisfies:

$$\text{div} \tau_I = P_k(\text{div} \tau) \quad \text{in} \ \Omega.$$  

Consequently, for all $K \in T_h$

$$\|\text{div} \tau_I\|_{0,K} \leq \|\text{div} \tau\|_{0,K},$$

and, if $\text{div} \tau|_K \in H^s(K)$ with $s \geq 0$, then

$$\|\text{div} \tau - \text{div} \tau_I\|_{0,K} \lesssim h_K^{\min\{s,k+1\}} |\text{div} \tau|_{s,K}.$$
Lemma 3.2 Let \( \tau \in H_0(\text{div}; \Omega) \) be such that \( \tau \in [H^t(\Omega)]^2 \) with \( t > 1/2 \). Then, there exists \( \tau_I \in H_h \) such that, if \( 1 \leq t \leq k + 1 \), there holds
\[
\|\tau - \tau_I\|_{0,K} \lesssim h_K^t |\tau|_{r,K},
\]
where the hidden constant is independent of \( h \). Moreover, if \( 1/2 < t \leq 1 \), then
\[
\|\tau - \tau_I\|_{0,K} \lesssim h_K^t |\tau|_{r,K} + h_K \|\text{div} \tau\|_{0,K}.
\]

Let \( \Pi_h \) be defined in \([L^2(\Omega)]^2\) by \( (\Pi_h \tau)|_K := \Pi^K_h \tau \) for all \( K \in T_h \), where \( \Pi^K_h \) is the operator defined in (9), and that satisfies the following result proved in [9, Lemma 8].

Lemma 3.3 For every \( q \in H^{1+t}(\Omega) \) with \( 1/2 < t \leq k + 1 \), there holds
\[
\|\nabla q - \Pi_h(\nabla q)\|_{0,\Omega} \lesssim h^t \|\nabla q\|_{r,\Omega}.
\]

As a consequence of the previous result we have the following estimate.

Lemma 3.4 For all \( r > 1/2 \) as in Lemma 2.1, the following error estimate holds
\[
\|w - \Pi_h w\|_{0,\Omega} \lesssim h^{\min\{r,k+1\}}.
\]

Proof The proof follows directly from Remark 2.1, Lemmas 2.1 and 3.3. \( \square \)

The following result give us the error estimates between the eigenfunctions and eigenvalues of Problems 3.2 and 2.3.

Theorem 3.1 For all \( r > 1/2 \) as in Lemma 2.1, the following error estimates hold
\[
\|w - w_h\|_{\text{div},\Omega} + \|u - u_h\|_{0,\Omega} \lesssim h^{\min\{r,k+1\}},
\]
where the hidden constants are independent of \( h \).

Proof The proof follows by repeating the arguments in [27, Theorems 4.2, 4.3 and 4.4]. \( \square \)

For the a posteriori error analysis that will be developed in Section 4, we will need the following auxiliary results, which have been adapted from [13, 14].

In what follows, let \( (\lambda, w, u) \) be a solution of Problem 2.3, where we assume that \( \lambda \) is a simple eigenvalue and we normalize the associated eigenfunction by taking \( \|u\|_{0,\Omega} = 1 \). Then, for each mesh \( T_h \), there exists a solution \( (\lambda_h, w_h, u_h) \) of Problem 3.2 such that \( \lambda_h \) converges to \( \lambda \), as \( h \) goes to zero, \( \|u_h\|_{0,\Omega} = 1 \), and Theorem 3.1 holds true.
Let us introduce the following well posed source problem with data $(\lambda, u)$: Find $(\mathbf{\tilde{w}}_h, \mathbf{\tilde{u}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$, such that

$$
\begin{align*}
\begin{cases}
  a_h(\mathbf{\tilde{w}}_h, \mathbf{\tau}_h) + b(\mathbf{\tau}_h, \mathbf{\tilde{u}}_h) = 0 & \forall \mathbf{\tau}_h \in \mathbf{H}_h, \\
  b(\mathbf{\tilde{w}}_h, \mathbf{v}_h) = -\lambda(\mathbf{u}, \mathbf{v}_h)_{0, \Omega} & \forall \mathbf{v}_h \in \mathbf{Q}_h.
\end{cases}
\end{align*}
$$

(13)

With this mixed problem at hand, we will prove the following technical lemmas with the goal of deriving a superconvergence result for our VEM. To make matters precise, the forthcoming analysis is inspired by [14], where the authors have generalized the results previously obtained by [10, 20, 23]. We begin proving a higher order approximation between $\mathbf{\tilde{u}}_h$ and $P_k \mathbf{u}$.

**Lemma 3.5** Let $(\lambda, \mathbf{w}, \mathbf{u})$ be a solution of Problem 2.3 and $(\mathbf{\tilde{w}}_h, \mathbf{\tilde{u}}_h)$ be a solution of the mixed formulation (13). Then, there holds

$$
\| \mathbf{\tilde{u}}_h - P_k \mathbf{u} \|_{0, \Omega} \lesssim h^\tau \left( \| \mathbf{w} - \mathbf{\tilde{w}}_h \|_{\text{div}, \Omega} + \| \mathbf{w} - \Pi_h \mathbf{w} \|_{0, \Omega} \right),
$$

where $\tau \in \left( \frac{1}{2}, 1 \right]$ and the hidden constant is independent of $h$.

**Proof** Let $(\mathbf{\tilde{w}}, \mathbf{\tilde{u}}) \in H_0(\text{div}; \Omega) \times L^2(\Omega)$ be the unique solution of the following well posed mixed problem

$$
\begin{align*}
\begin{cases}
  a(\mathbf{\tilde{w}}, \mathbf{\tau}) + b(\mathbf{\tau}, \mathbf{\tilde{u}}) = 0 & \forall \mathbf{\tau} \in H_0(\text{div}; \Omega), \\
  b(\mathbf{\tilde{w}}, \mathbf{v}) = -\left( \mathbf{\tilde{u}}_h - P_k \mathbf{u}, \mathbf{v} \right)_{0, \Omega} & \forall \mathbf{v} \in L^2(\Omega).
\end{cases}
\end{align*}
$$

(14)

Notice that (14) is exactly problem (4) and (5) with datum $\mathbf{\tilde{u}}_h - P_k \mathbf{u}$. Hence, since this problem is well posed, we have the following regularity result, consequence of (6)

$$
\| \mathbf{\tilde{w}} \|_{\tau, \Omega} + \| \mathbf{\tilde{u}} \|_{1+\tau, \Omega} \lesssim \| \mathbf{\tilde{u}}_h - P_k \mathbf{u} \|_{0, \Omega}.
$$

(15)

Observe that, thanks to the definition of $P_k$, the first equation of Problem 2.3, and the first equation of (13), we have

$$
\| \mathbf{\tilde{u}}_h - P_k \mathbf{u} \|_{0, \Omega}^2 = -b(\mathbf{\tilde{w}}, \mathbf{\tilde{u}}_h - P_k \mathbf{u}) = -b(\mathbf{\tilde{w}}_I, \mathbf{\tilde{u}}_h - P_k \mathbf{u}) = -b(\mathbf{\tilde{w}}_I, \mathbf{\tilde{u}}_h - \mathbf{u})
\begin{align*}
&= -b(\mathbf{\tilde{w}}_I, \mathbf{\tilde{u}}_h) + b(\mathbf{\tilde{w}}_I, \mathbf{u})
&= a_h(\mathbf{\tilde{w}}_h, \mathbf{\tilde{u}}_h) - a(\mathbf{w}, \mathbf{\tilde{w}}_I).
\end{align*}

(16)

In the last two terms of (16), we add and subtract $\Pi_h \mathbf{w}$ in order to obtain

$$
a_h(\mathbf{\tilde{w}}_h, \mathbf{\tilde{w}}_I) - a(\mathbf{w}, \mathbf{\tilde{w}}_I) = a_h(\mathbf{\tilde{w}}_h - \Pi_h \mathbf{w}, \mathbf{\tilde{w}}_I) + a(\Pi_h \mathbf{w} - \mathbf{\tilde{w}}_h, \mathbf{\tilde{w}}_I) + a(\mathbf{\tilde{w}}_h - \mathbf{w}, \mathbf{\tilde{w}}_I)
\begin{align*}
&= A_I + a(\mathbf{\tilde{w}}_h - \mathbf{w}, \mathbf{\tilde{w}}_I - \mathbf{\tilde{w}}_I) + a(\mathbf{\tilde{w}}_h - \mathbf{w}, \mathbf{\tilde{w}}_I)
&= A_I + A_{\Pi} - b(\mathbf{\tilde{w}}_h - \mathbf{w}, \mathbf{\tilde{u}})
\end{align*}

A_{\Pi}

\begin{align*}
&= A_I + A_{\Pi} - b(\mathbf{\tilde{w}}_h - \mathbf{w}, \mathbf{\tilde{u}} - P_k \mathbf{\tilde{u}}) - b(\mathbf{\tilde{w}}_h - \mathbf{w}, P_k \mathbf{\tilde{u}}),
\end{align*}

where we have used the first equation of system (14). Moreover, testing the second equation in Problem 2.3 and (13) with $P_k \mathbf{\tilde{u}} \in Q_h$, we have $b(\mathbf{\tilde{w}}_h - \mathbf{w}, P_k \mathbf{\tilde{u}}) = 0$, and hence

$$
\| \mathbf{\tilde{u}}_h - P_k \mathbf{u} \|_{0, \Omega}^2 = A_I + A_{\Pi} - b(\mathbf{\tilde{w}}_h - \mathbf{w}, \mathbf{\tilde{u}} - P_k \mathbf{\tilde{u}}).
$$

(17)
Now, our next task is to estimate each term on the right-hand side of the identity above. We begin with $A_1$:

$$A_1 = a_h(\tilde{w}_h - \Pi_h w, \tilde{w}_I - \Pi_h \tilde{w}) + a(\tilde{w}_h - \Pi_h w, \Pi \tilde{w} - \tilde{w}_I)$$

$$\lesssim \|\tilde{w}_h - \Pi_h w\|_{0, \Omega} \|\tilde{w}_I - \Pi_h \tilde{w}\|_{0, \Omega}$$

$$\lesssim (\|\tilde{w}_h - w\|_{0, \Omega} + \|w - \Pi_h w\|_{0, \Omega})(\|\tilde{w}_I - \tilde{w}\|_{0, \Omega} + \|\tilde{w} - \Pi_h \tilde{w}\|_{0, \Omega})$$

$$\lesssim h^r (\|\tilde{w}_h - w\|_{0, \Omega} + \|w - \Pi_h w\|_{0, \Omega}) \|\tilde{w}_h - P_k u\|_{0, \Omega}. \quad (17)$$

where we have applied Lemmas 3.1 and 3.3 and (15). Now, for $A_\Pi$ we have

$$A_\Pi = a(\tilde{w}_h - w, \tilde{u} - \tilde{w}) \lesssim h^r \|w - \tilde{w}_h\|_{0, \Omega} \|\tilde{u} - P_k u\|_{0, \Omega}, \quad (18)$$

and finally, invoking (15), we obtain

$$b(\tilde{w}_h - w, \tilde{u} - P_k \tilde{u}) \lesssim h^r \|\text{div}(\tilde{w}_h - w)\|_{0, \Omega} \|\tilde{u} - P_k u\|_{0, \Omega}. \quad (19)$$

Collecting (17), (18) and (19), we have

$$\|\tilde{u}_h - P_k u\|_{0, \Omega} \lesssim h^r (\|w - \tilde{w}_h\|_{\text{div}, \Omega} + \|w - \Pi_h w\|_{0, \Omega}),$$

concluding the proof.

The following auxiliary result shows that the term $\|w - \tilde{w}_h\|_{\text{div}, \Omega}$ is bounded.

**Lemma 3.6** Let $(\lambda, w, u)$ be a solution of Problem 2.3 and $(\tilde{w}_h, \tilde{u}_h)$ be a solution of (13). Then, there holds

$$\|w - \tilde{w}_h\|_{\text{div}, \Omega} \lesssim h^r,$$

with $r > 1/2$ as in Lemma 2.1 and the hidden constant is independent of $h$.

**Proof** Let $(\lambda_h, w_h, u_h)$ be solution of Problem 3.2. Now, subtracting (13) from Problem 3.2 we obtain

$$\begin{cases}
\lambda_h w_h - \tilde{w}_h, \forall \tau_h \in H_h,
\\tau_h \in H_h,
\lambda u - \lambda_h u_h, \forall u_h \in Q_h.
\end{cases}$$

First, using the inf-sup condition for bilinear form $b(\cdot, \cdot)$ (cf. (3)) we have

$$\|u_h - \tilde{u}_h\|_{0, \Omega} \leq \|w_h - \tilde{w}_h\|_{0, \Omega}.$$  

Thus,

$$\|w_h - \tilde{w}_h\|_{0, \Omega} \lesssim \|\lambda_h u_h - \lambda u\|_{0, \Omega}.$$  

Moreover from the second equation, we get

$$\|\text{div}(w_h - \tilde{w}_h)\|_{0, \Omega} \leq \|\lambda_h u_h - \lambda u\|_{0, \Omega}.$$  

On the other hand, from the triangle inequality and the above estimates, we obtain

$$\|w - \tilde{w}_h\|_{\text{div}, \Omega} \leq \|w - w_h\|_{\text{div}, \Omega} + \|w_h - \tilde{w}_h\|_{\text{div}, \Omega}$$

$$\lesssim \|w - w_h\|_{\text{div}, \Omega} + \|\lambda_h u_h - \lambda u\|_{0, \Omega}$$

$$\lesssim \|w - w_h\|_{\text{div}, \Omega} + |\lambda_h - \lambda| \|u_h\|_{0, \Omega} + |\lambda| \|u_h - u\|_{0, \Omega},$$

where using (12) we conclude the proof.  

\[ Springer \]
We have the following essential identity to conclude the superconvergence result presented in Lemma 3.7.

\[ -\lambda_h(\widehat{u}_h, u_h) = -\lambda_h(u_h, \widehat{u}_h) = b(w_h, \widehat{u}_h) \]
\[ = -a_h(w_h, w_h) = a_h(w_h, \widehat{w}_h) = b(w_h, u_h) = -\lambda(u, u_h). \]  

**Lemma 3.7** Let \( (\lambda, w, u) \) and \( (\lambda_h, w_h, u_h) \) be solutions of Problems 2.3 and 3.2, respectively, with \( \|u\|_{0,\Omega} = \|u_h\|_{0,\Omega} = 1 \). Then, there holds

\[ \|P_k u - u_h\|_{0,\Omega} \lesssim h^{2\overline{r}}, \]
where \( \overline{r} \in (1/2, 1] \) and the hidden constant are independent of \( h \).

**Proof** Let \( (\widehat{w}_h, \widehat{u}_h) \) be the solution of (13). From the triangle inequality we have

\[ \|P_k u - u_h\|_{0,\Omega} \leq \|P_k u - \widehat{u}_h\|_{0,\Omega} + \|\widehat{u}_h - u_h\|_{0,\Omega}. \]  

(21)

Now, adapting the arguments of [14, Lemma 11] and using (20), we derive the following estimate

\[ \|\widehat{u}_h - u_h\|_{0,\Omega}^2 \lesssim \|\widehat{u}_h - P_k u\|_{0,\Omega}^2 + \left[ \|w - w_h\|_{0,\Omega}^2 + \|u - u_h\|_{0,\Omega}^2 + \|w - \Pi_h w\|_{0,\Omega}^2 \right]^2. \]

Finally, from (21), the above estimate, together with Lemmas 3.4, 3.5, 3.6, and Theorem 3.1, we conclude the proof.  \( \square \)

**4 A posteriori error analysis**

In this section, we develop an a posteriori error estimator for the acoustic eigenvalue problem (1). The a posteriori error estimator that we will propose is of residual type and our goal is to prove that is reliable and efficient.

Let us introduce some notations and definitions. For any polygon \( K \in T_h \) we denote by \( \mathcal{E}_K \) the set of edges of \( K \) and

\[ \mathcal{E} := \bigcup_{K \in T_h} \mathcal{E}_K. \]

We decompose \( \mathcal{E} \) as \( \mathcal{E} := \mathcal{E}_\Omega \cup \mathcal{E}_\Gamma \), where \( \mathcal{E}_\Gamma := \{ \ell \in \mathcal{E} : \ell \subset \Gamma \} \) and \( \mathcal{E}_\Omega := \mathcal{E} \setminus \mathcal{E}_\Gamma \). On the other hand, given \( \xi \in L^2(\Omega)^2 \), for each \( K \in T_h \) and \( \ell \in \mathcal{E}_\Omega \), we denote by \( [\xi] \cdot \ell \) the tangential jump of \( \xi \) across \( \ell \), that is \( [\xi] \cdot \ell := (\xi|_K - \xi|_{K'})|_{\ell} \cdot \ell \), where \( K \) and \( K' \) are elements of \( T_h \) having \( \ell \) as a common edge. Due to the regularity assumptions on the mesh, for each polygon \( K \in T_h \) there is a sub-triangulation \( T^K_h \) obtained by joining each vertex of \( K \) with the midpoint of the ball with respect to which \( K \) is star-shaped. We define

\[ \widehat{T}_h := \bigcup_{K \in T_h} T^K_h. \]
For each polygon $K$, we define the following computable and local terms

$$
R_K^2 := h_K^2 \left\| \text{rot}(\Pi_h^K w_h) \right\|_{0, K}^2, \quad \theta_K^2 := S^K (w_h - \Pi_h^K w_h, w_h - \Pi_h^K w_h),
$$

$$
J_\ell := \|\Pi_h^K w_h \cdot \ell\|,
$$

which allows us to define the local error indicator

$$
\eta_K^2 := R_K^2 + \theta_K^2 + \sum_{\ell \in \mathcal{E}_K} h_K \| J_\ell \|_{0, \ell}^2,
$$

and hence, the global error estimator

$$
\eta := \left\{ \sum_{K \in T_h} \eta_K^2 \right\}^{1/2}.
$$

In what follows we will prove that (24) is reliable and locally efficient. With this aim, we begin by decomposing the error $w - w_h$, using the classical Helmholtz decomposition as follows:

$$
w - w_h = \nabla \psi + \text{curl} \beta,
$$

with $\psi \in \tilde{H}^1(\Omega) := \{ v \in H^1(\Omega) : (v, 1)_{0, \Omega} = 0 \}$ and $\beta \in H_0^1(\Omega)$. Satisfying

$$
\| \psi \|_{1, \Omega} + \| \text{curl} \beta \|_{0, \Omega} \lesssim \| w - w_h \|_{0, \Omega}.
$$

Note that above inequality is a consequence of the orthogonality of $\nabla \psi$ and $\text{curl} \beta$ in $L^2(\Omega)^2$ and the Poincaré inequality. With this decomposition at hand, we split the $L^2(\Omega)$-norm of the error $w - w_h$ into two terms,

$$
\| w - w_h \|_{0, \Omega}^2 = (w - w_h, \nabla \psi)_{0, \Omega} + (w - w_h, \text{curl} \beta)_{0, \Omega}.
$$

To conclude the reliability, we begin by proving the following results.

**Lemma 4.1** There holds

$$
(w - w_h, \nabla \psi)_{0, \Omega} \lesssim h^{2\bar{r}} \| w - w_h \|_{0, \Omega},
$$

where $\bar{r} \in (\frac{1}{2}, 1]$ and the hidden constant are independent of $h$.

**Proof** An integration by parts reveals that

$$
(w - w_h, \nabla \psi)_{0, \Omega} = - (\text{div}(w - w_h), \psi)_{0, \Omega} + ((w - w_h) \cdot n, \psi)_{0, \Gamma}.
$$

Using that $(w - w_h) \cdot n = 0$ on $\Gamma$, the fact that $\text{div} w = -\lambda u$, $\text{div} w_h = -\lambda_h u_h$, and adding and subtracting $\lambda_h (u - P_h u, \psi)_{0, \Omega}$, we obtain

$$
(w - w_h, \nabla \psi)_{0, \Omega} = (\lambda - \lambda_h) u, \psi)_{0, \Omega} + \lambda_h (u - P_h u, \psi)_{0, \Omega} + \lambda_h (P_h u - u_h, \psi)_{0, \Omega}.
$$

For the term I, we use (12), the fact that $\| u \|_{0, \Omega} = 1$ and (25) in order to obtain

$$
(\lambda - \lambda_h) u, \psi)_{0, \Omega} \lesssim h^{2 \min\{k+1\}} \| \psi \|_{0, \Omega} \lesssim h^{2 \min\{k+1\}} \| w - w_h \|_{0, \Omega},
$$
with \( r > \frac{1}{2} \) as in Lemma 2.1. Applying the approximation properties (11) and (25) on \( \Pi \), Lemma 2.1 and the fact that \( \|u\|_{0,\Omega} = 1 \), we obtain
\[
(u - P_k u, \psi)_{0,\Omega} = (u - P_k u, \psi - P_k \psi)_{0,\Omega} \lesssim h\|u - P_k u\|_{0,\Omega}\|\psi\|_{1,\Omega} \lesssim h^2\|w - w_h\|_{0,\Omega}.
\]

Finally, for \( \Pi \), we use Lemma 3.7 and the bound for \( \|\psi\|_{0,\Omega} \) to write,
\[
\lambda_h (P_k u - u_h, \psi)_{0,\Omega} \lesssim h^{2r} \|w - w_h\|_{0,\Omega}.
\]
The proof follows by combining the above estimates.

Given \( k \in \mathbb{N} \cup \{0\} \), let us consider the following virtual discrete subspace of \( H^1(\Omega) \).
\[
V^1_h := \left\{ \xi \in H^1(\Omega) : \Delta \xi \in P_{k-1}(K) \quad \forall K \in T_h, \xi \in C(\partial K) : \xi|_\ell \in P_{k+1}(\ell), \quad \forall \text{edge } \ell \subseteq \partial K \right\}.
\]

Then, there exists \( \xi_I \in V^1_h \) that satisfies (see the proof of [35, Lemma 3.4])
\[
\|\xi - \xi_I\|_{0,\ell} \lesssim h^{1/2}\|\xi\|_{1,K} \quad \text{and} \quad \|\xi - \xi_I\|_{0,K} \lesssim h_K \|\xi\|_{1,K} \quad \forall \xi \in H^1(K).
\]

With this result at hand, now we prove the following result.

**Lemma 4.2** There holds
\[
(w - w_h, \text{curl } \beta)_{0,\Omega} \lesssim \eta \|\text{curl } \beta\|_{0,\Omega},
\]
where the hidden constant is independent of \( h \) and the discrete solution.

**Proof** Since \( \text{curl } \beta \in H_0(\text{div}^0; \Omega) \), we have that \( (w, \text{curl } \beta)_{0,\Omega} = 0 \). Thus,
\[
(w - w_h, \text{curl } \beta)_{0,\Omega} = (w_h - \Pi_h w_h, \text{curl } \beta)_{0,\Omega} + (\Pi_h w_h, \text{curl } \beta)_{0,\Omega}.
\]

For the first term on the right-hand side of the above equality we have
\[
(w_h - \Pi_h w_h, \text{curl } \beta)_{0,\Omega} \lesssim \eta \|\text{curl } \beta\|_{0,\Omega}.
\]

Next, we introduce \( \beta_I \in V^1_h \) as the virtual interpolant of \( \beta \). We have that \( \text{curl } \beta_I \in H_0(\text{div}^0; \Omega) \) and \( \text{curl } \beta_I \in H_h \). In fact, we have that \( \text{rot}(\text{curl } \beta_I)|_K = 0 \), \( \text{div}(\text{curl } \beta_I)|_K = 0 \) and \( \text{curl } \beta_I \cdot n \in P_k(\ell) \). Thus, by testing the first equation of Problem 3.2 with \( \text{curl } \beta_I \in H_h \) and using integration by parts we have.
\[
(\Pi_h w_h, \text{curl } \beta)_{0,\Omega} = \sum_{K \in T_h} \int_K \Pi^K_h w_h \cdot \text{curl}(\beta - \beta_I) + S^K(w_h - \Pi^K_h w_h, \text{curl } \beta_I - \Pi^K_h \text{curl } \beta_I)
\]
\[
= \sum_{K \in T_h} \left( \int_K \text{rot}(\Pi^K_h w_h)(\beta - \beta_I) + \int_{\partial K} (\Pi^K_h w_h \cdot n)(\beta - \beta_I) \right)
\]
\[
+ S^K(w_h - \Pi^K_h w_h, \text{curl } \beta_I - \Pi^K_h \text{curl } \beta_I).
\]

Hence, applying Cauchy-Schwarz inequality and property of approximation of \( \beta_I \) and (10), in the estimate above yields to
\[
(\Pi_h w_h, \text{curl } \beta)_{0,\Omega} \lesssim \sum_{K \in T_h} \eta_K \|\beta\|_{1,\omega_K} \leq C \eta \|\text{curl } \beta\|_{0,\Omega}.
\]
Now, combining (27), (28) and (29) we conclude the proof.

We now provide an upper bound for our error estimator.

**Lemma 4.3** The following bound holds
\[ \|w - w_h\|_{0, \Omega} \lesssim \eta + h^{2\gamma}, \]
where the hidden constants are independent of \( h \) and the discrete solution.

**Proof** The proof is a consequence of (26), Lemmas 4.1 and 4.2, together with (25).

Thanks to the previous lemmas, we have the following result

**Lemma 4.4** The following bound holds
\[ \|w - \Pi_h w_h\|_{0, \Omega} \lesssim \eta + h^{2\gamma}, \]
where the hidden constant is independent of \( h \).

**Proof** From the triangle inequality, together to (10), for the stability of the \( \Pi_h \)-projector and Lemma 4.3, we have
\[ \|w - \Pi_h w_h\|_{0, \Omega} \leq \|w - w_h\|_{0, \Omega} + \|w_h - \Pi_h w_h\|_{0, \Omega} \]
\[ \lesssim \eta + h^{2\gamma}. \]

Hence, we conclude the proof.

Now we are in position to establish the reliability of our estimator.

**Corollary 4.1** [Reliability] The following error estimate hold
\[ \|w - w_h\|_{0, \Omega} + \|w - \Pi_h w_h\|_{0, \Omega} \lesssim \eta + h^{2\gamma}, \]
where the hidden constants are independent of \( h \).

**Remark 4.1** From Corollary 4.1, we note that \( \mathcal{O}(h^{2\gamma}) \) can be considered a “higher order term” when lowest order VEM \((k = 0)\) is used. When \( k \geq 1 \), the term can be considered a “higher order term” when the eigenfunction is singular. This usually happens when the eigenproblem is solved in non-convex polygonal domains.

### 4.1 Efficiency

Now our aim is to prove that the local indicator \( \eta_K \) defined in (23) provides a lower bound of the error \( w - w_h \) in a vicinity of any polygon \( K \). To do this task, we proceed as is customary for the efficiency analysis, using suitable bubble functions for the polygons and their edges.

The bubble functions that we will consider are based on [16]. Let \( \chi_K \in H^1_0(\Omega) \) be an interior bubble function defined in a polygon \( K \). These bubble functions can
be constructed piecewise as the sum of the cubic bubble functions for each triangle of the sub-triangulation $\mathcal{T}_h^K$ that attain the value 1 at the barycenter of each triangle. Also, the edge bubble function $\chi_\ell \in \partial K$ is a piecewise quadratic function attaining the value of 1 at the barycenter of $\ell$ and vanishing on the triangles $K \in \hat{\mathcal{T}}_h$ that do not contain $\ell$ on its boundary.

The following technical results for the bubble functions are a key point to prove the efficiency bound.

**Lemma 4.5** For any $K \in \mathcal{T}_h$, let $\chi_K$ be the corresponding interior bubble function. Then, there hold

$$\|p\|^2_{0,K} \lesssim \int_K \chi_K p^2 \lesssim \|p\|^2_{0,K} \quad \forall p \in \mathbb{P}_k(K);$$

$$\|p\|_{0,K} \lesssim \|\chi_K p\|_{0,K} + h_K \|\nabla (\chi_K p)\|_{0,K} \lesssim \|p\|_{0,K} \quad \forall p \in \mathbb{P}_k(K);$$

where the hidden constants are independent of $h_K$.

**Lemma 4.6** For any $K \in \mathcal{T}_h$ and $\ell \in \mathcal{E}_K$, let $\chi_\ell$ be the corresponding edge bubble function. Then, there holds

$$\|p\|^2_{0,\ell} \lesssim \int_\ell \chi_\ell p^2 \lesssim \|p\|^2_{0,\ell} \quad \forall p \in \mathbb{P}_k(\ell).$$

Moreover, for all $p \in \mathbb{P}_k(\ell)$, there exists an extension of $p \in \mathbb{P}_k(K)$, which we denote simply by $p$, such that

$$h_K^{-1/2} \|\chi_\ell p\|_{0,K} + h_K^{1/2} \|\nabla (\chi_\ell p)\|_{0,K} \lesssim \|p\|_{0,\ell},$$

where the hidden constants are independent of $h_K$.

Now we are in position to establish the main result of this section.

**Theorem 4.1** For any $K \in \mathcal{T}_h$, there holds

$$\eta_K \lesssim \|w_h - w\|_{0,\omega_\ell} + \|w - \Pi^K_h w_h\|_{0,\omega_\ell},$$

where $\omega_\ell$ denotes the union of two polygons sharing an edge with $K$, and the hidden constant is independent of $h$ and the discrete solution.

**Proof** The aim is to estimate each term of the local indicator (23). The proof is divided in three steps:

- Step 1: We begin by estimating $R^2_K$ in (22). Invoking the properties of the bubble function $\chi_K$, Cauchy-Schwarz inequality, and Lemma 4.5, we have

$$R^2_K \lesssim \int_K \chi_K \text{rot}(\Pi^K_h w_h) \cdot \text{rot}(\Pi^K_h w_h) = \int_K \chi_K \text{rot}(\Pi^K_h w_h) \cdot \text{rot}(\Pi^K_h w_h - w_h)$$

$$= -\int_K (\Pi^K_h w_h - w_h) \text{curl}(\chi_K \text{rot}(\Pi^K_h w_h))$$

$$\lesssim \\|\Pi^K_h w_h - w_h\|_{0,K} h_K^{-1} \|\text{rot}(\Pi_h w_h)\|_{0,K},$$
which implies that
\[ R_K = h_K \| \text{rot}(\Pi_h^K \lesssim w_h) \|_{0,K} \| w - w_h \|_{0,K} + \| w - \Pi_h^K w_h \|_{0,K}. \]  
\[ (30) \]

- Step 2: Now we estimate \( J_h \). Following the proof of [17, Lemma 5.16], we obtain
\[
\| J_\ell \|^2_{0,\ell} \lesssim \| \chi_\ell J_\ell \|^2_{0,\ell} = \int_{\omega_\ell} (\chi_\ell J_\ell) \cdot J_\ell = \int_{\omega_\ell} (w - \Pi_h^K w_h) \cdot \text{curl}(\chi_\ell J_\ell) \\
+ \int_{\omega_\ell} \chi_\ell J_\ell \text{rot}(\Pi_h^K w_h).
\]
Hence, from Cauchy-Schwarz inequality, the bubble function properties and (30), we have
\[
\| J_\ell \|^2_{0,\ell} \lesssim \| \chi_\ell J_\ell \|_{1,\omega_\ell} \| w - \Pi_h^K w_h \|_{0,\omega_\ell} + \| \chi_\ell J_\ell \|_{0,\omega_\ell} \| \text{rot}(\Pi_h^K w_h) \|_{0,\omega_\ell},
\]
\[
\lesssim h_K^{-1/2} \left( \| w_h - w \|_{0,\omega_\ell} + \| w - \Pi_h^K w_h \|_{0,\omega_\ell} \right) \| J_\ell \|_{0,\ell}.
\]
Hence, we conclude that
\[
h_K^{1/2} \| J_\ell \|_{0,\ell} \lesssim \| w_h - w \|_{0,\omega_\ell} + \| w - \Pi_h^K w_h \|_{0,\omega_\ell}.
\]  
\[ (31) \]

- Step 3: The final step is to control the term \( \theta_K \). To do this task, we use the stability property (10), add and subtract \( w \) with the purpose of applying triangular inequality as follows
\[
\theta_K \leq c_1 \| w_h - \Pi_h^K w_h \|_{0,K} \lesssim \| w_h - w \|_{0,K} + \| \Pi_h^K w_h - w \|_{0,K}. \]  
\[ (32) \]
Hence, the proof is complete by gathering (30), (31) and (32).

As a direct consequence of theorem above, we have the following result that allows us to conclude the efficiency of the local and global error estimators for the acoustic problem, and hence, for its equivalent mixed problem.

**Corollary 4.2** [Efficiency] There holds
\[
\eta \lesssim \| w - w_h \|_{0,\Omega} + \| w - \Pi_h w_h \|_{0,\Omega},
\]
where the hidden constants are independent of \( h \).

### 5 Numerical results

In this section, we report numerical tests in order to assess the behavior of the a posteriori estimator defined in (24). With this aim, we have implemented in a MATLAB code a lowest order VEM scheme on arbitrary polygonal meshes.

We have tested the method by using different families of meshes (see Fig. 1):
- tria: triangular meshes;
- tria – quad: coupled triangular and squares meshes;
- hexa: non-structured hexagonal meshes made of convex hexagons;
- voro: non-structured Voronoi meshes;
• trap: trapezoidal meshes which consist of partitions of the domain into $N \times N$ congruent trapezoids, all similar to the trapezoid with vertices $(0, 0), \left(\frac{1}{2}, 0\right), (1, \frac{2}{3})$, and $(0, \frac{1}{3})$.

We have used the mesh refinement algorithm described in [7], which consists in splitting each element of the mesh into $n$ quadrilaterals ($n$ being the number of edges of the polygon) by connecting the barycenter of the element with the midpoint of each edge, which will be named as Adaptive VEM. Notice that although this process is initiated with a mesh of triangles, the successively created meshes will contain other kind of convex polygons, as it can be seen in Fig. 2. The scheme is based on the strategy of refining those elements $K \in \mathcal{T}_h$ that satisfy

$$\eta_K \geq 0.5 \max_{K' \in \mathcal{T}_h} \eta_{K'}.$$

5.1 Test 1: L-shaped domain

We will consider the non-convex domain $\Omega := (0, 1) \times (0, 1) \setminus [1/2, 1] \times [1/2, 1]$. It is clear that the first eigenfunction of the acoustic problem in this domain is not smooth enough, due the presence of a geometrical singularity at $\left(\frac{1}{2}, \frac{1}{2}\right)$. This leads to a lack of regularity due to the re-entrant angle $\omega = 3\pi/2$. Therefore, according to [9], using quasi-uniform meshes, the convergence rate for the eigenvalues should be $|\lambda - \lambda_h| = \mathcal{O}(h^{4/3}) \approx \mathcal{O}(N^{-2/3})$, where $N$ denotes the number of degrees of freedom. Then, the proposed a posteriori estimator (24) must be capable to recover
the optimal order $|\lambda - \lambda_h| = \mathcal{O}(N^{-1})$, when the adaptive refinement is performed near to the singularity point.

For the numerical tests, we have computed the smallest eigenvalue and its corresponding eigenfunction using the MATLAB command \texttt{eigs}.

Figures 3 and 4 show the adaptively refined meshes obtained with VEM procedures and different initial meshes. Figure 3 is initiated with a \textit{tria} mesh, while Fig. 4 is initiated with an \textit{hexa} mesh.

Figures 3 and 4 show that our estimator identifies the singularity of the domain, leading to a refinement on the region of the re-entrant angle. This refinement allows to achieve the optimal order of convergence for the eigenvalue.

In order to compute the errors $|\lambda_1 - \lambda_{h,1}|$, and since an exact eigenvalue is not known, we have used an approximation based on a least squares fitting of the computed values obtained with extremely refined meshes. Thus, we have obtained the value $\lambda_1 = 5.9017$, which has at least four accurate significant digits.

We report in Table 1 the lowest eigenvalue $\lambda_{h,1}$ on uniformly refined meshes, adaptively refined meshes with VEM schemes initiated with a \textit{tria} mesh and in the
last column we report adaptively refined meshes with VEM schemes initiated with an "hexa" mesh. Each table includes the estimated convergence rate.

In Fig. 5 we present error curves where we observe that the two refinement schemes lead to a correct convergence rate. It can be seen from Table 1 and Fig. 5, that the uniform refinement leads to a convergence rate close to that predicted by the theory, while the adaptive VEM schemes allow us to recover the optimal order of convergence $O(N^{-1})$.

We report in Table 2 the error $\epsilon_{\nu}(\lambda_1) := |\lambda_1 - \lambda_{h,1}|/\lambda_1$ and the estimators $\eta$ at each step of the adaptive VEM scheme initiated with a "tria" mesh. We include in the

Table 1  Test 1. Computed lowest eigenvalue $\lambda_{h,1}$ computed with different schemes

|                | Uniform VEM | Adaptive VEM tria | Adaptive VEM hexa |
|----------------|-------------|--------------------|--------------------|
| $N$            | $\lambda_{h,1}$ | $N$                | $\lambda_{h,1}$ | $N$     | $\lambda_{h,1}$ |
| 245            | 5.6831      | 245                | 5.6831           | 829     | 5.8283          |
| 940            | 5.8231      | 266                | 5.7356           | 872     | 5.8495          |
| 3680           | 5.8732      | 288                | 5.7554           | 945     | 5.8605          |
| 14560          | 5.8914      | 381                | 5.7805           | 1131    | 5.8688          |
| 57920          | 5.8982      | 889                | 5.8440           | 2010    | 5.8833          |
| 231040         | 5.9008      | 1206               | 5.8620           | 3296    | 5.8895          |
| 113600         | 5.9000      | 1731               | 5.8713           | 4932    | 5.8928          |
| 7653           | 5.8736      | 3639               | 5.8876           | 7287    | 5.8955          |
| 5206           | 5.8924      | 5206               | 5.8924           | 12003   | 5.8982          |
| 7653           | 5.8949      | 7653               | 5.8946           | 19349   | 5.8998          |
| 14560          | 5.8985      | 14560              | 5.8985           | 30751   | 5.9007          |
| 57920          | 5.8900      | 22982              | 5.9000           | 50421   | 5.9014          |
| 231040         | 5.9006      | 33844              | 5.9006           |         |                 |
| 61641          | 5.9015      | 61641              | 5.9015           |         |                 |
| Order          | $O(N^{-0.79})$ | Order             | $O(N^{-1.08})$   | Order   | $O(N^{-1.14})$  |
| $\lambda_1$   | 5.9017      | $\lambda_1$       | 5.9017           | $\lambda_1$ | 5.9017 |

Fig. 4  Test 1. Adaptively refined meshes obtained with VEM scheme at refinement steps 0, 1 and 8 initiated with an "hexa" mesh (Adaptive VEM "hexa")
table the terms $\theta^2 := \sum_{K \in T_h} \theta^2_K$, which appears from the inconsistency of the VEM, and $J_h := \sum_{K \in T_h} \left( \sum_{\ell \in \partial K} h_K \| J_{\ell} \|_{0, \ell}^2 \right)$, which arise from the edge residuals. We also report in the table the effectivity indexes $\text{eff}(\eta) := \frac{\text{err}(\lambda_1)}{\eta^2}$.

From Table 2 we observe that the effectivity indexes are bounded and far from zero. Also, the inconsistency and edge residual terms are, roughly speaking, of the same order. These results are similar to those obtained in [35]. We end this test presenting in Fig. 6 the displacement field and the pressure fluctuation of the fluid on the L-shaped domain, associated to the first eigenfunction.

5.2 Test 2: H-shaped domain

The aim of this test is to assess the performance of the adaptive scheme when solving a problem with a singular solution. In this test $\Omega$ consists of an H-shaped domain
Table 2  Components of the error estimator and effectivity indexes on the adaptively refined meshes with VEM initiated with a tria mesh

| $N$  | $\lambda_{h,1}$ | $\mbox{err}(\lambda_1)$ | $\theta^2$ | $J_h^2$ | $\eta^2$ | $\mbox{eff}(\eta)$ |
|------|-----------------|--------------------------|------------|---------|---------|------------------|
| 245  | 5.6831          | 3.7048e-02               | 9.4718e-03 | 1.3226e-01 | 1.4174e-01 | 2.6139e-01 |
| 266  | 5.7356          | 2.8145e-02               | 1.2881e-02 | 8.5612e-02 | 9.8493e-02 | 2.8576e-01 |
| 288  | 5.7554          | 2.4794e-02               | 1.2791e-02 | 7.5101e-02 | 8.7892e-02 | 2.8210e-01 |
| 381  | 5.7805          | 2.0529e-02               | 1.2944e-02 | 5.6072e-02 | 6.9016e-02 | 2.9745e-01 |
| 889  | 5.8440          | 9.7831e-03               | 9.4599e-03 | 1.5576e-02 | 2.5036e-02 | 3.9076e-01 |
| 1206 | 5.8620          | 6.7293e-03               | 6.8149e-03 | 1.0962e-02 | 1.7777e-02 | 3.7854e-01 |
| 1731 | 5.8713          | 5.1531e-03               | 5.1742e-03 | 8.1822e-03 | 1.3356e-02 | 3.8582e-01 |
| 3639 | 5.8876          | 2.3936e-03               | 2.7744e-03 | 3.2286e-03 | 6.0030e-03 | 3.9874e-01 |
| 5206 | 5.8924          | 1.5825e-03               | 1.8562e-03 | 2.3210e-03 | 4.1771e-03 | 3.7885e-01 |
| 7653 | 5.8949          | 1.1586e-03               | 1.3746e-03 | 1.6477e-03 | 3.0223e-03 | 3.8334e-01 |
| 14545| 5.8985          | 5.3529e-04               | 7.4205e-04 | 9.1237e-03 | 1.6544e-03 | 3.2355e-01 |
| 22982| 5.9000          | 2.9214e-04               | 4.6344e-04 | 5.9698e-04 | 1.0604e-03 | 2.7549e-01 |
| 33844| 5.9006          | 1.8454e-04               | 3.3863e-04 | 4.2666e-04 | 7.6529e-04 | 2.4114e-01 |

that represents the union of two pools. More precisely, the geometry of this domain is given by

$$
\Omega := \{(0, 3/2) \times (0, 3)\} \setminus \{(1/2, 1) \times [0, 5/4] \cup \{(1/2, 1) \times [15/8, 3]\}\}.
$$

According to the definition of this domain, four singularities are present, leading once again to a lack of regularity for the eigenfunctions of our acoustic problem. Hence, the proposed estimator $\eta$ defined in (24) must be capable of identifying these singularities of the geometry and performing adaptive refinement, with different polygonal meshes, in order to recover optimal order of convergence.

Figures 7, 8, and 9 show the adaptively refined meshes obtained with VEM procedures and different initial meshes. In Fig. 7 we start with a tria — quad mesh, while in Fig. 8 we begin with a trap mesh. Finally, in Fig. 9 we start with a voro mesh.

Similarly to Test 5.1, the computations of the errors $|\lambda_2 - \lambda_{h,2}|$ have been obtained with a least squares fitting of the calculated values obtained with extremely refined

Fig. 6 Test 1. Eigenfunctions of the acoustic problem corresponding to the first lowest eigenvalue: displacement field $w_{h,1}$ (left), pressure fluctuation $p_{h,1}$ (right)
Fig. 7 Adaptively refined meshes obtained with VEM scheme at refinement steps 0, 1 and 8 (Adaptive VEM tria – quad)

Fig. 8 Adaptively refined meshes obtained with VEM scheme at refinement steps 0, 1 and 8 (Adaptive VEM trap)

Fig. 9 Adaptively refined meshes obtained with VEM scheme at refinement steps 0, 1 and 8 (Adaptive VEM voro)
In Table 3 we report the second lowest eigenvalue $\lambda_{h,2}$ on uniformly refined meshes, adaptively refined meshes with different types of initial meshes. Each table includes the estimated convergence rate.

In Fig. 10 we present error curves where we observe that the three refinement schemes lead to a correct convergence rate. It can be seen from Table 3 and Fig. 10, that the uniform refinement leads to a convergence rate close to that predicted by the theory, while the adaptive VEM schemes allow us to recovering the optimal order of convergence $O(N^{-1})$.

In Fig. 11 we present plots of the computed eigenfunctions $w_{h,2}$ (displacement field) and $p_{h,2}$ (pressure fluctuation) corresponding to the second eigenvalue.

### 5.3 Test 3: Circular domain with obstacles

As a third test, we have considered a configuration closer to a real application: four square tubes immersed in a fluid occupying a circular cavity. Clearly in this test there are two relevant geometrical issues: on the one hand, we have a non polygonal domain for which we are making an approximation by means of polygonal meshes, and the four rigid squares that lie in the interior of the circle. These tubes lead to non smooth eigenfunctions when the solutions for the acoustic problem are approximated, due the singularities of the corner on each square.

To make matters precise, let us define the circular domain by $\Omega_C := \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1 \}$ and the squares $\Omega_1 := [1/5, 3/5] \times [1/5, 3/5]$, $\Omega_{II} := [-3/5, -1/5] \times [1/5, 3/5]$, $\Omega_{III} := [-3/5, -1/5] \times [-3/5, -1/5]$ and $\Omega_{IV} := [1/5, 3/5] \times [-3/5, -1/5]$. Hence, the computational domain is $\Omega := \Omega_C \setminus (\Omega_1 \cup \Omega_{II} \cup \Omega_{III} \cup \Omega_{IV})$.
Fig. 10  Error curves of $|\lambda_2 - \lambda_{h,2}|$ for uniformly refined meshes and adaptively refined meshes VEM with different initial meshes.

Fig. 11  Test 2. Eigenfunctions of the acoustic problem corresponding to the second lowest eigenvalue: displacement field $w_{h,2}$ (left), pressure fluctuation $p_{h,2}$ (right)
In the sequel, we consider the fourth eigenfunction. In Fig. 12 we present an adaptive refinement of our estimator with a voro mesh. On the left-hand side we present the initial mesh and, after 1 and 8 iterations of our numerical method, we observe that the estimator $\eta$ identifies the singularities on the geometry that cause the poor regularity of the eigenfunction, and starts the refinement around these corners in order to recover the optimal order of convergence.

![Fig. 12 Adaptively refined meshes obtained with VEM scheme at refinement steps 0, 1 and 8 (Adaptive VEM voro)](image)

![Fig. 13 Error curves of $|\lambda_d - \lambda_{k,d}|$ for uniformly refined meshes and adaptively refined meshes VEM](image)
Figure 13 shows a logarithmic plot of the errors between the calculated approximations of the fourth smallest positive eigenvalue and the “exact” one, versus the number of degrees of freedom $N$ of the meshes. As in the previous two tests, the exact value of the fourth eigenvalue is obtained by using a least squares fit. The figure shows the results obtained with “uniform” meshes and with adaptively refined meshes and shows how the optimal order of convergence is recovered. Finally, Fig. 14 shows the eigenfunctions of the acoustic problem corresponding to the fourth lowest eigenvalue at different levels of refinement.

6 Conclusions

In this work, we have derived and analyzed an a posteriori error estimate for the acoustic vibration problem by means of mixed virtual element discretization. The theoretical analysis developed in this work was strongly supported by superconvergence results for mixed spectral formulations. Several numerical tests that substantiate the theoretical results were presented, confirming that the proposed estimator is capable of recover the optimal order of convergence, as theory predicts. Moreover, we stress that the present analysis can be extended to the tridimensional case by using the VEM spaces introduced in [6] and the recent results for interpolation estimates derived in [8].

Declarations

Competing interests The authors declare no competing interests.

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