Estimates on Green functions and Schrödinger-type equations for non-symmetric diffusions with measure-valued drifts

Panki Kim
Department of Mathematics
Seoul National University
Seoul 151-742, Republic of Korea
Email: pkim@snu.ac.kr
URL: www.math.snu.ac.kr/~pkim
Telephone number: 82-2-880-4077
Fax number: 82-2-887-4694

and

Renming Song*
Department of Mathematics
University of Illinois
Urbana, IL 61801, USA
Email: rsong@math.uiuc.edu
URL: www.math.uiuc.edu/~rsong
Telephone number: 1-217-244-6604
Fax number: 1-217-333-9576

March 29, 2022

Abstract

In this paper, we establish sharp two-sided estimates for the Green functions of non-symmetric diffusions with measure-valued drifts in bounded Lipschitz domains. As consequences of these estimates, we get a 3G type theorem and a conditional gauge theorem for these diffusions in bounded Lipschitz domains.

Informally the Schrödinger-type operators we consider are of the form $L + \mu \cdot \nabla + \nu$ where $L$ is uniformly elliptic, $\mu$ is a vector-valued signed measure belonging to $K_{d,1}$ and $\nu$ is a signed measure belonging to $K_{d,2}$. In this paper, we establish two-sided estimates for the heat kernels of Schrödinger-type operators in bounded $C^{1,1}$-domains and a scale invariant boundary Harnack principle for the positive harmonic functions with respect to Schrödinger-type operators in bounded Lipschitz domains.

AMS 2000 Mathematics Subject Classification: Primary: 58C60, 60J45; Secondary: 35P15,
Keywords and phrases: Brownian motion, diffusion, diffusion process, non-symmetric diffusion, Kato class, measure-valued drift, transition density, Green function, Lipschitz domain, 3G theorem, Schrödinger operator, heat kernel, boundary Harnack principle, harmonic function

1 Introduction

This paper is a natural continuation of [11, 12, 14], where diffusion (Brownian motion) with measure-valued drift was discussed. For a vector-valued signed measure \( \mu \) belonging to \( K_{d,1} \), a diffusion with measure-valued drift \( \mu \) is a diffusion process whose generator can be informally written as \( L + \mu \cdot \nabla \). In this paper we consider Schrödinger-type operators \( L + \mu \cdot \nabla + \nu \) (see below for the definition) and discuss their properties.

In this paper we always assume that \( d \geq 3 \). First we recall the definition of the Kato class \( K_{d,\alpha} \) for \( \alpha \in (0,2] \). For any function \( f \) on \( \mathbb{R}^d \) and \( r > 0 \), we define

\[
M_f^\alpha (r) = \sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq r} \frac{|f(y) dy}{|x-y|^{d-\alpha}}, \quad 0 < \alpha \leq 2.
\]

In this paper, we mean, by a signed measure, the difference of two nonnegative measures at most one of which can have infinite total mass. For any signed measure \( \nu \) on \( \mathbb{R}^d \), we use \( \nu^+ \) and \( \nu^- \) to denote its positive and negative parts, and \( |\nu| = \nu^+ + \nu^- \) its total variation. For any signed measure \( \nu \) on \( \mathbb{R}^d \) and any \( r > 0 \), we define

\[
M_\nu^\alpha (r) = \sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq r} \frac{|\nu(dy)|}{|x-y|^{d-\alpha}}, \quad 0 < \alpha \leq 2.
\]

Definition 1.1 Let \( 0 < \alpha \leq 2 \). We say that a function \( f \) on \( \mathbb{R}^d \) belongs to the Kato class \( K_{d,\alpha} \) if \( \lim_{r \downarrow 0} M_f^\alpha (r) = 0 \). We say that a signed Radon measure \( \nu \) on \( \mathbb{R}^d \) belongs to the Kato class \( K_{d,\alpha} \) if \( \lim_{r \downarrow 0} M_\nu^\alpha (r) = 0 \). We say that a \( d \)-dimensional vector valued function \( V = (V^1, \cdots, V^d) \) on \( \mathbb{R}^d \) belongs to the Kato class \( K_{d,\alpha} \) if each \( V^i \) belongs to the Kato class \( K_{d,\alpha} \). We say that a \( d \)-dimensional vector valued signed Radon measure \( \mu = (\mu^1, \cdots, \mu^d) \) on \( \mathbb{R}^d \) belongs to the Kato class \( K_{d,\alpha} \) if each \( \mu^i \) belongs to the Kato class \( K_{d,\alpha} \).

Rigorously speaking a function \( f \) in \( K_{d,\alpha} \) may not give rise to a signed measure \( \nu \) in \( K_{d,\alpha} \) since it may not give rise to a signed measure at all. However, for the sake of simplicity we use the convention that whenever we write that a signed measure \( \nu \) belongs to \( K_{d,\alpha} \) we are implicitly assuming that we are covering the case of all the functions in \( K_{d,\alpha} \) as well.

Throughout this paper we assume that \( \mu = (\mu^1, \cdots, \mu^d) \) is fixed with each \( \mu^i \) being a signed measure on \( \mathbb{R}^d \) belonging to \( K_{d,1} \). We also assume that the operator \( L \) is either \( L_1 \) or \( L_2 \) where

\[
L_1 := \frac{1}{2} \sum_{i,j=1}^d \partial_i (a_{ij} \partial_j) \quad \text{and} \quad L_2 := \frac{1}{2} \sum_{i,j=1}^d a_{ij} \partial_i \partial_j.
\]
with $A := (a_{ij})$ being $C^1$ and uniformly elliptic. We do not assume that $a_{ij}$ is symmetric.

Informally, when $a_{ij}$ is symmetric, a diffusion process $X$ in $\mathbb{R}^d$ with drift $\mu$ is a diffusion process in $\mathbb{R}^d$ with generator $L + \mu \cdot \nabla$. When each $\mu^i$ is given by $U^i(x)dx$ for some function $U^i$, $X$ is a diffusion in $\mathbb{R}^d$ with generator $L + U \cdot \nabla$ and it is a solution to the SDE $dX_t = dY_t + U(X_t) \cdot dt$ where $Y$ is a diffusion in $\mathbb{R}^d$ with generator $L$. For a precise definition of a (non-symmetric) diffusion $X$ with drift $\mu$ in $K_{d,1}$, we refer to section 6 in [12] and section 1 in [14]. The existence and uniqueness of $X$ were established in [1] (see Remark 6.1 in [1]). In this paper, we will always use $X$ to denote the diffusion process with drift $\mu$.

In [11, 12, 14], we have already studied some potential theoretical properties of the process $X$. More precisely, we have established two-sided estimates for the heat kernel of the killed diffusion process $X^D$ and sharp two-sided estimates on the Green function of $X^D$ when $D$ is a bounded $C^{1,1}$ domain; proved a scale invariant boundary Harnack principle for the positive harmonic functions of $X$ in bounded Lipschitz domains; and identified the Martin boundary $X_D$ in bounded Lipschitz domains.

In this paper, we will first establish sharp two-sided estimates for the Green function of $X^D$ when $D$ is a bounded Lipschitz domain. As consequences of these estimates, we get a 3G type theorem and a conditional gauge theorem for $X$ in bounded Lipschitz domains. We also establish two-sided estimates for the heat kernels of Schrödinger-type operators in bounded $C^{1,1}$-domains and a scale invariant boundary Harnack principle for the positive harmonic functions with respect to Schrödinger-type operators in bounded Lipschitz domains. The results of this paper will be used in proving the intrinsic ultracontractivity of the Schrödinger semigroup of $X^D$ in [15].

Throughout this paper, for two real numbers $a$ and $b$, we denote $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. The distance between $x$ and $\partial D$ is denote by $\rho_D(x)$. In this paper we will use the following convention: the values of the constants $r_i, i = 1 \cdots 6, C_0, C_1, M, M_i, i = 1 \cdots 5,$ and $\varepsilon_1$ will remain the same throughout this paper, while the values of the constants $c, c_1, c_2, \cdots$ may change from one appearance to another. In this paper, we use “:=” to denote a definition, which is read as “is defined to be”.

2 Green function estimates and 3G theorem

In this section we will establish sharp two-sided estimates for the Green function and a 3G theorem for $X$ in bounded Lipschitz domains. We will first establish some preliminary results for the Green function $G_D(x, y)$ of $X^D$. Once we have these results, the proof of the Green function estimates is similar to the ones in [3], [5] and [10]. The main difference is that the Green function $G_D(x, y)$ is not (quasi-) symmetric.

For any bounded domain $D$, we use $\tau_D$ to denote the first exit time of $D$, i.e., $\tau_D = \inf\{t > 0 : X_t \notin D\}$. Given a bounded domain $D \subset \mathbb{R}^d$, we define $X^D_t(\omega) = X_t(\omega)$ if $t < \tau_D(\omega)$ and $X^D_t(\omega) = \partial$ if $t \geq \tau_D(\omega)$, where $\partial$ is a cemetery state. The process $X^D$ is called a killed diffusion
with drift $\mu$ in $D$. Throughout this paper, we use the convention $f(\partial) = 0$.

It is shown in [12] that, for any bounded domain $D$, $X^D$ has a jointly continuous and strictly positive transition density function $q^D(t,x,y)$ (see Theorem 2.4 in [12]). In [12], we also showed that there exist positive constants $c_1$ and $c_2$ depending on $D$ via its diameter such that for any $(t,x,y) \in (0,\infty) \times D \times D$,

$$q^D(t,x,y) \leq c_1 t^{-\frac{d}{2}} e^{-\frac{c_2|x-y|^2}{2t}}$$

(see Lemma 2.5 in [12]). Let $G_D(x,y)$ be the Green function of $X^D$, i.e.,

$$G_D(x,y) := \int_0^{\infty} q^D(t,x,y) dt.$$  

By (2.1), $G_D(x,y)$ is finite for $x \neq y$ and

$$G_D(x,y) \leq \frac{c}{|x-y|^{d-2}}$$

for some $c = c(\text{diam}(D)) > 0$.

From Theorem 3.7 in [12], we see that there exist constants $r_1 = r_1(d,\mu) > 0$ and $c = c(d,\mu) > 1$ depending on $\mu$ only via the rate at which $\max_{1 \leq i \leq d} M_{\mu_i}(r) \to 0$ such that for $r \leq r_1$, $z \in \mathbb{R}^d$, $x,y \in B(z,r)$,

$$c^{-1} |x-y|^{-d+2} \leq G_{B(z,r)}(x,y) \leq c |x-y|^{-d+2}, \quad x,y \in \overline{B(z,2r/3)}.$$  

(2.3)

**Definition 2.1** Suppose $U$ is an open subset of $\mathbb{R}^d$.

(1) A Borel function $u$ defined on $U$ is said to be harmonic with respect to $X$ in $U$ if

$$u(x) = \mathbb{E}_x [u(X_{\tau_B})], \quad x \in B,$$

for every bounded open set $B$ with $\overline{B} \subset U$;

(2) A Borel function $u$ defined on $\overline{U}$ is said to be regular harmonic with respect to $X$ in $U$ if $u$ is harmonic with respect to $X$ in $U$ and (2.4) is true for $B = U$.

Every positive harmonic function in a bounded domain $D$ is continuous in $D$ (see Proposition 2.10 in [12]). Moreover, for every open subset $U$ of $D$, we have

$$\mathbb{E}_x[G_D(X_{T_U}, y)] = G_D(x,y), \quad (x,y) \in D \times U$$

(2.5)

where $T_U := \inf\{ t > 0 : X_t \in U \}$. In particular, for every $y \in D$ and $\varepsilon > 0$, $G_D(\cdot,y)$ is regular harmonic in $D \setminus B(y,\varepsilon)$ with respect to $X$ (see Theorem 2.9 (1) in [12]).

We recall here the scale invariant Harnack inequality from [11].
Theorem 2.2 (Corollary 5.8 in [11]) There exist $r_2 = r_2(d, \mu) > 0$ and $c = c(d, \mu) > 0$ depending on $\mu$ only via the rate at which $\max_{1 \leq i \leq d} M_{\mu_i}(r)$ goes to zero such that for every positive harmonic function $f$ for $X$ in $B(x_0, r)$ with $r \in (0, r_2)$, we have

$$\sup_{y \in B(x_0, r/2)} f(y) \leq c \inf_{y \in B(x_0, r/2)} f(y)$$

Recall that $r_1 > 0$ is the constant from (2.3).

Lemma 2.3 For any bounded domain $D$, there exists $c = c(D, \mu) > 0$ such that for every $r \in (0, r_1 \wedge r_2)$ and $B(z, r) \subset D$, we have for every $x \in D \setminus B(z, r)$

$$\sup_{y \in B(z, r/2)} G_D(y, x) \leq c \inf_{y \in B(z, r/2)} G_D(y, x) \quad (2.6)$$

and

$$\sup_{y \in B(z, r/2)} G_D(x, y) \leq c \inf_{y \in B(z, r/2)} G_D(x, y) \quad (2.7)$$

Proof. Fix $x \in D \setminus \overline{B(z, r)}$. Since $G_D(\cdot, x)$ is harmonic for $X$ in $B(z, r)$, (2.6) follows from Theorem 2.2. So we only need to show (2.7).

Since $r < r_1$, by (2.2) and (2.3), there exist $c_1 = c_1(D) > 1$ and $c_2 = c_2(d) > 1$ such that for every $y, w \in B(z, \frac{3r}{4})$

$$c_2^{-1} \frac{1}{|w-y|^{d-2}} \leq G_B(z, r)(w, y) \leq G_D(w, y) \leq c_1 \frac{1}{|w-y|^{d-2}}.$$ 

Thus for $w \in \partial B(z, \frac{3r}{4})$ and $y_1, y_2 \in B(z, \frac{r}{2})$, we have

$$G_D(w, y_1) \leq c_1 \left( \frac{|w-y_2|}{|w-y_1|} \right)^{d-2} \frac{1}{|w-y_2|^{d-2}} \leq 4^{d-2} c_2 c_1 G_D(w, y_2). \quad (2.8)$$

On the other hand, by (2.5), we have

$$G_D(x, y) = E_x \left[ G_D(X_{T_{B(z, \frac{r}{2})}}, y) \right], \quad y \in B(z, \frac{r}{2}) \quad (2.9)$$

Since $X_{T_{B(z, \frac{r}{2})}} \in \partial B(z, \frac{3r}{4})$, combining (2.8) and (2.9), we get

$$G_D(x, y_1) \leq 4^{d-2} c_2 c_1 E_x \left[ G_D(X_{T_{B(z, \frac{r}{2})}}, y_2) \right] = 4^{d-2} c_2 c_1 G_D(x, y_2), \quad y_1, y_2 \in B(z, \frac{r}{2})$$

In fact, (2.7) is true for every $x \in D$. \qed

Recall that a bounded domain $D$ is said to be Lipschitz if there is a localization radius $R_0 > 0$ and a constant $\Lambda_0 > 0$ such that for every $Q \in \partial D$, there is a Lipschitz function $\phi_Q : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$
satisfying $|\phi_Q(x) - \phi_Q(z)| \leq \Lambda_0 |x - z|$, and an orthonormal coordinate system $CS_Q$ with origin at $Q$ such that

$$B(Q, R_0) \cap D = B(Q, R_0) \cap \{ y = (y_1, \cdots, y_{d-1}, y_d) =: (\tilde{y}, y_d) \in CS_Q : y_d > \phi_Q(\tilde{y}) \}.$$ 

The pair $(R_0, \Lambda_0)$ is called the characteristics of the Lipschitz domain $D$.

Any bounded Lipschitz domain satisfies $\kappa$-fat property: there exists $\kappa_0 \in (0, 1/2]$ depending on $\Lambda_0$ such that for each $Q \in \partial D$ and $r \in (0, R_0)$ (by choosing $R_0$ smaller if necessary), $D \cap B(Q, r)$ contains a ball $B(A_r(Q), \kappa_0 r)$.

In this section, we fix a bounded Lipschitz domain $D$. We recall here the scale invariant boundary Harnack principle for $X^D$ in bounded Lipschitz domains from [12].

**Theorem 2.4** (Theorem 4.6 in [12]) Suppose $D$ is a bounded Lipschitz domain. Then there exist constants $M_1, c > 1$ and $r_3 > 0$, depending on $\mu$ only via the rate at which $\max_{1 \leq i \leq d} M_1^i(r)$ goes to zero such that for every $Q \in \partial D$, $r < r_3$ and any nonnegative functions $u$ and $v$ which are harmonic with respect to $X^D$ in $D \cap B(Q, M_1 r)$ and vanish continuously on $\partial D \cap B(Q, M_1 r)$, we have

$$\frac{u(x)}{v(x)} \leq c \frac{u(y)}{v(y)} \quad \text{for any } x, y \in D \cap B(Q, r). \quad (2.10)$$

For any $Q \in \partial D$, we define

$$\Delta_Q(r) := \{ y \in CS_Q : \phi_Q(\tilde{y}) + 2r > y_d > \phi_Q(\tilde{y}), \ |\tilde{y}| < 2(M_1 + 1)r \},$$

$$\partial_1 \Delta_Q(r) := \{ y \in CS_Q : \phi_Q(\tilde{y}) + 2r \geq y_d > \phi_Q(\tilde{y}), \ |\tilde{y}| = 2(M_1 + 1)r \},$$

$$\partial_2 \Delta_Q(r) := \{ y \in CS_Q : \phi_Q(\tilde{y}) + 2r = y_d, \ |\tilde{y}| \leq 2(M_1 + 1)r \},$$

where $CS_Q$ is the coordinate system with origin at $Q$ in the definition of Lipschitz domains and $\phi_Q$ is the Lipschitz function there. Let $M_2 := 2(1 + M_1) \sqrt{1 + \Lambda_0^2} + 2$ and $r_4 := M_2^{-1}(R_0 \wedge r_1 \wedge r_2 \wedge r_3)$. If $z \in \overline{\Delta_Q(r)}$ with $r \leq r_4$, then

$$|Q - z| \leq |(\tilde{z}, \phi_Q(\tilde{z})) - (\tilde{z}, 0)| + 2r \leq 2r(1 + M_1) \sqrt{1 + \Lambda_0^2} + 2r = M_2 r \leq M_2 r_4 \leq R_0.$$

So $\overline{\Delta_Q(r)} \subset B(Q, M_2 r) \cap D \subset B(Q, R_0) \cap D$.

**Lemma 2.5** There exists constant $c > 1$ such that for every $Q \in \partial D$, $r < r_4$, and any nonnegative functions $u$ and $v$ which are harmonic in $D \setminus B(Q, r)$ and vanish continuously on $\partial D \setminus B(Q, r)$, we have

$$\frac{u(x)}{u(y)} \leq c \frac{v(x)}{v(y)} \quad \text{for any } x, y \in D \setminus B(Q, M_2 r). \quad (2.11)$$
**Proof.** Throughout this proof, we fix a point $Q$ on $\partial D$, $r < r_4$, $\Delta Q (r) \cap \partial_1 \Delta Q (r)$ and $\partial_2 \Delta Q (r)$. Fix an $\tilde{y}_0 \in \mathbb{R}^{d-1}$ with $|\tilde{y}_0| = 2(M_1 + 1)r$. Since $|(\tilde{y}_0, \phi_Q (\tilde{y}_0))| > r$, $u$ and $v$ are harmonic with respect to $X$ in $D \cap B((\tilde{y}_0, \phi_Q (\tilde{y}_0)), 2M_1r)$ and vanish continuously on $\partial D \cap B((\tilde{y}_0, \phi_Q (\tilde{y}_0)), 2M_1r)$. Therefore by Theorem 2.4

$$\frac{u(x)}{u(y)} \leq c_1 \frac{v(x)}{v(y)} \quad \text{for any } x, y \in \partial_1 \Delta_Q (r) \text{ with } \tilde{x} = \tilde{y} = \tilde{y}_0,$$

for some constant $c_1 > 0$. Since $\text{dist}(D \cap B(Q, r), \partial_2 \Delta_Q (r)) > cr$ for some $c := c(\Lambda_0)$, the Harnack inequality (Theorem 2.2) and a Harnack chain argument imply that there exists a constant $c_2 > 1$ such that

$$c_2^{-1} < \frac{u(x)}{u(y)}, \frac{v(x)}{v(y)} < c_2, \quad \text{for any } x, y \in \partial_2 \Delta_Q (r).$$

In particular, (2.13) is true with $y := (\tilde{y}_0, \phi_Q (\tilde{y}_0) + 2r)$, which is also in $\partial_1 \Delta_Q (r)$. Thus (2.12) and (2.13) imply that

$$c_3^{-1} \frac{u(x)}{u(y)} \leq \frac{v(x)}{v(y)} \leq c_3 \frac{u(x)}{u(y)}, \quad x, y \in \partial_1 \Delta_Q (r) \cup \partial_2 \Delta_Q (r)$$

for some constant $c_3 > 0$. Now, by applying the maximum principle (Lemma 7.2 in [11]) twice, we get that (2.14) is true for every $x \in D \setminus \Delta_Q (r) \supset D \setminus B(Q, M_2r)$. \hfill \Box

Combining Theorem 2.4 and Lemma 2.5, we get a uniform boundary Harnack principle for $G_D (x, y)$ in both variables. Recall $\kappa_0$ is the $\kappa$-fat constant of $D$.

**Lemma 2.6** There exist constants $c > 1$, $M > 1/\kappa_0$ and $r_0 \leq r_4$ such that for every $Q \in \partial D$, $r < r_0$, we have for $x, y \in D \setminus B(Q, r)$ and $z_1, z_2 \in D \cap B(Q, r/M)$

$$\frac{G_D (x, z_1)}{G_D (y, z_1)} \leq c \frac{G_D (x, z_2)}{G_D (y, z_2)} \quad \text{and} \quad \frac{G_D (z_1, x)}{G_D (z_1, y)} \leq c \frac{G_D (z_2, x)}{G_D (z_2, y)}.$$  \hfill (2.15)

Fix $z_0 \in D$ with $r_0/M < \rho_D (z_0) < r_0$ and let $\varepsilon_1 := r_0/(12M)$. For $x, y \in D$, we let $r(x, y) := \rho_D (x) \vee \rho_D (y) \vee |x - y|$ and

$$B(x, y) := \{ A \in D : \rho_D (A) > \frac{1}{M} r(x, y), |x - A| \vee |y - A| < 5r(x, y) \}$$

if $r(x, y) < \varepsilon_1$, and $B(x, y) := \{ z_0 \}$ otherwise.

By a Harnack chain argument we get the following from (2.2) and (2.3).

**Lemma 2.7** There exists a positive constant $C_0$ such that $G_D (x, y) \leq C_0 |x - y|^{-d+2}$, for all $x, y \in D$, and $G_D (x, y) \geq C_0^{-1} |x - y|^{-d+2}$ if $2|x - y| \leq \rho_D (x) \vee \rho_D (y)$.
Let $C_1 := C_0 2^{d-2} \rho_D(z_0)^{2-d}$. The above lemma implies that $G_D(\cdot, z_0)$ and $G_D(z_0, \cdot)$ are bounded above by $C_1$ on $D \setminus B(z_0, \rho_D(z_0)/2)$. Now we define

$$g_1(x) := G_D(x, z_0) \wedge C_1 \quad \text{and} \quad g_2(y) := G_D(z_0, y) \wedge C_1.$$ 

Using Lemma 2.3 and a Harnack chain argument, we get the following.

**Lemma 2.8** For every $y \in D$ and $x_1, x_2 \in D \setminus B(y, \rho_D(y)/2)$ with $|x_1 - x_2| \leq k(\rho_D(x_1) \wedge \rho_D(x_2))$, there exists $c := c(D, k)$ independent of $y$ and $x_1, x_2$ such that

$$G_D(x_1, y) \leq c G_D(x_2, y) \quad \text{and} \quad G_D(y, x_1) \leq c G_D(y, x_2). \quad (2.16)$$

The next two lemmas follow easily from the result above.

**Lemma 2.9** There exists $c = c(D) > 0$ such that for every $x, y \in D$,

$$c^{-1} g_1(A_1) \leq g_1(A_2) \leq c g_1(A_1) \quad \text{and} \quad c^{-1} g_2(A_1) \leq g_2(A_2) \leq c g_2(A_1), \quad A_1, A_2 \in B(x, y).$$

**Lemma 2.10** There exists $c = c(D) > 0$ such that for every $x \in \{y \in D; \rho_D(y) \geq \varepsilon_1/(8M^3)\}$, $c^{-1} \leq g_i(x) \leq c$, $i = 1, 2$.

Using Lemma 2.3, the proof of the next lemma is routine (for example, see Lemma 6.7 in [8]). So we omit the proof.

**Lemma 2.11** For any given $c_1 > 0$, there exists $c_2 = c_2(D, c_1, \mu) > 0$ such that for every $|x - y| \leq c_1(\rho_D(x) \wedge \rho_D(y))$,

$$G_D(x, y) \geq c_2 |x - y|^{-d+2}.$$ 

In particular, there exists $c = c(D, \mu) > 0$ such that for every $|x - y| \leq (8M^3/\varepsilon_1)(\rho_D(x) \wedge \rho_D(y))$,

$$c^{-1} |x - y|^{-d+2} \leq G_D(x, y) \leq c |x - y|^{-d+2}.$$

With the preparations above, the following two-sided estimates for $G_D$ is a direct generalization of the estimates of the Green function for symmetric processes (see [8] for a symmetric jump process case).

**Theorem 2.12** There exists $c := c(D) > 0$ such that for every $x, y \in D$

$$c^{-1} \frac{g_1(x)g_2(y)}{g_1(A)g_2(A)} |x - y|^{-d+2} \leq G_D(x, y) \leq c \frac{g_1(x)g_2(y)}{g_1(A)g_2(A)} |x - y|^{-d+2} \quad (2.17)$$ 

for every $A \in B(x, y)$.  

8
Proof. Since the proof is an adaptation of the proofs of Proposition 6 in [13] and Theorem 2.4 in [10], we only give a sketch of the proof for the case $\rho_D(x) \leq \rho_D(y) \leq \frac{1}{13M|x-y|}$.

In this case, we have $r(x,y) = |x-y|$. Let $r := \frac{1}{2}(|x-y| \wedge \epsilon_1)$. Choose $Q_x, Q_y \in \partial D$ with $|Q_x - x| = \rho_D(x)$ and $|Q_y - y| = \rho_D(y)$. Pick points $x_1 = A_{r/M}(Q_x)$ and $y_1 = A_{r/M}(Q_y)$ so that $x, x_1 \in B(Q_x, r/M)$ and $y, y_1 \in B(Q_y, r/M)$. Then one can easily check that $|z_0 - Q_x| \geq r$ and $|y - Q_x| \geq r$. So by the first inequality in (2.15), we have

$$c_1^{-1} \frac{G_D(x_1, y)}{g_1(x_1)} \leq \frac{G_D(x, y)}{g_1(x)} \leq c_1 \frac{G_D(x_1, y)}{g_1(x_1)},$$

for some $c_1 > 1$. On the other hand, since $|z_0 - Q_y| \geq r$ and $|x_1 - Q_y| \geq r$, applying the second inequality in (2.15),

$$c_1^{-1} \frac{G_D(x_1, y_1)}{g_2(y_1)} \leq \frac{G_D(x, y)}{g_2(y)} \leq c_1 \frac{G_D(x_1, y_1)}{g_2(y_1)}.$$

Putting the four inequalities above together we get

$$c_1^{-2} \frac{G_D(x_1, y_1)}{g_1(x_1)g_2(y_1)} \leq \frac{G_D(x, y)}{g_1(x)g_2(y)} \leq c_1^2 \frac{G_D(x_1, y_1)}{g_1(x_1)g_2(y_1)}.$$

Moreover, $\frac{1}{3} |x - y| < |x_1 - y_1| < 2|x - y|$ and $|x_1 - y_1| \leq (8M^2/\epsilon_1)(\rho_D(x_1) \wedge \rho_D(y_1))$. Thus by Lemma 2.11 we have

$$\frac{1}{2^{d-2}c_2c_1^2} \frac{|x - y|^{-d+2}}{g_1(x_1)g_2(y_1)} \leq \frac{G_D(x, y)}{g_1(x)g_2(y)} \leq \frac{3^{d-2}c_2c_1^2}{g_1(x_1)g_2(y_1)} |x - y|^{-d+2},$$

for some $c_2 > 1$.

If $r = \epsilon_1/2$, then $r(x,y) = |x-y| \geq \epsilon_1$. Thus $g_1(A) = g_2(A) = g_1(z_0) = g_2(z_0) = C_1$ and $\rho_D(x_1), \rho_D(y_1) \geq r/M = \epsilon_1/(2M)$. So by Lemma 2.10

$$C_1^{-2}C_3^{-2} \leq \frac{g_1(A)g_2(A)}{g_1(x_1)g_2(y_1)} \leq C_1^2C_3^2,$$

for some $c_3 > 1$.

If $r < \epsilon_1/2$, then $r(x,y) = |x-y| < \epsilon_1$ and $r = \frac{1}{2}r(x,y)$. Hence $\rho_D(x_1), \rho_D(y_1) \geq r/M = r(x,y)/(2M)$. Moreover, $|x_1 - A|, |y_1 - A| \geq 6r(x,y)$. So by applying the first inequality in (2.16) to $g_1$, and the second inequality in (2.16) to $g_2$ (with $k = 12M$),

$$c_4^{-1} \leq \frac{g_1(A)}{g_1(x_1)} \leq c_4 \quad \text{and} \quad c_4^{-1} \leq \frac{g_2(A)}{g_2(y_1)} \leq c_4$$

for some constant $c_4 = c_4(D) > 0$.

\[\square\]

Lemma 2.13 (Carleson’s estimate) For any given $0 < N < 1$, there exists constant $c > 1$ such that for every $Q \in \partial D$, $r < r_0$, $x \in D \setminus B(Q, r)$ and $z_1, z_2 \in D \cap B(Q, r/M)$ with $B(z_2, Nr) \subset D \cap B(Q, r/M)$

$$G_D(x, z_1) \leq cG_D(x, z_2) \quad \text{and} \quad G_D(z_1, x) \leq cG_D(z_2, x) \quad \text{(2.18)}$$
Proof. Recall that $CS_Q$ is the coordinate system with origin at $Q$ in the definition of Lipschitz domains. Let $\overline{y} := (\overline{0}, r)$. Since $z_1, z_2 \in D \cap B(Q, r/M)$, by (2.14),

$$G_D(\overline{y}, z_1) \leq c_1 r^{-d+2} \quad \text{and} \quad G_D(z_1, \overline{y}) \leq c_1 r^{-d+2},$$

for some constant $c_1 > 0$. On the other hand, since $\rho_D(\overline{y}) \geq c_2 r$ for some constant $c_2 > 0$ and $\rho_D(z_2) \geq Nr$, by Lemma 2.11

$$G_D(\overline{y}, z_2) \geq c_3 |\overline{y} - z_2|^{-d+2} \geq c_4 r^{-d+2} \quad \text{and} \quad G_D(z_2, \overline{y}) \geq c_3 |\overline{y} - z_2|^{-d+2} \geq c_4 r^{-d+2},$$

for some constants $c_3, c_4 > 0$. Thus from (2.15) with $y = \overline{y}$, we get

$$G_D(x, z_1) \leq c_5 \left( \frac{c_1}{c_4} \right) G_D(x, z_2) \quad \text{and} \quad G_D(z_1, x) \leq c_5 \left( \frac{c_1}{c_4} \right) G_D(z_2, x)$$

for some constant $c_5 > 0$. □

Recall that, for $r \in (0, R_0)$, $A_r(Q)$ is a point in $D \cap B(Q, r)$ such that $B(A_r(Q), \kappa_0 r) \subset D \cap B(Q, r)$. For every $x, y \in D$, we denote $Q_x$, $Q_y$ by points on $\partial D$ such that $\rho_D(x) = |x - Q_x|$ and $\rho_D(y) = |y - Q_y|$ respectively. It is easy to check that if $r(x, y) < \varepsilon_1$

$$A_{r(x,y)}(Q_x), A_{r(x,y)}(Q_y) \in B(x, y).$$

(2.19)

In fact, by the definition of $A_{r(x,y)}(Q_x)$, $\rho_D(A_{r(x,y)}(Q_x)) \geq \kappa_0 r(x, y) > r(x, y)/M$. Moreover,

$$|x - A_{r(x,y)}(Q_x)| \leq |x - Q_x| + |Q_x - A_{r(x,y)}(Q_x)| \leq \rho_D(x) + r(x, y) \leq 2r(x, y)$$

and

$$|y - A_{r(x,y)}(Q_x)| \leq |x - y| + |x - A_{r(x,y)}(Q_x)| \leq 3r(x, y).$$

Lemma 2.14 There exists $c > 0$ such that the following holds:

(1) If $Q \in \partial D$, $0 < s \leq r < \varepsilon_1$ and $A = A_r(Q)$, then

$$g_i(x) \leq c g_i(A) \quad \text{for every} \quad x \in D \cap B(Q, Ms) \cap \{ y \in D \; : \; \rho_D(y) > \frac{s}{M} \}, \quad i = 1, 2.$$

(2) If $x, y, z \in D$ satisfy $|x - z| \leq |y - z|$, then

$$g_i(A) \leq c g_i(B) \quad \text{for every} \quad (A, B) \in B(x, y) \times B(y, z), \quad i = 1, 2.$$

Proof. This is an easy consequence of the Carleson’s estimates (Lemma 2.13), Lemma 2.11 and Lemmas 2.9, 2.11 (see page 467 in [10]). Since the proof is similar to the proof on page 467 in [10], we omit the details □

The next result is called a generalized triangle property.
Theorem 2.15 There exists a constant $c > 0$ such that for every $x, y, z \in D$,
\[
\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} \leq c \left( \frac{g_1(y)}{g_1(x)} G_D(x,y) \vee \frac{g_2(y)}{g_2(z)} G_D(y,z) \right) \tag{2.20}
\]

Proof. Let $A_{x,y} \in B(x,y)$, $A_{y,z} \in B(y,z)$ and $A_{z,x} \in B(z,x)$. If $|x - y| \leq |y - z|$ then $|x - z| \leq |x - y| + |y - z| \leq 2|y - z|$. So by (2.17) and Lemma 2.14 (2), we have
\[
\frac{G_D(y,z)}{G_D(x,z)} \leq c_1 \frac{g_1(A_{x,z}) g_2(A_{x,z})}{g_1(A_{y,z}) g_2(A_{y,z})} \frac{|x - z|^{d-2} g_1(y)}{|y - z|^{d-2} g_1(x)} \leq c_1 c_2 2^{d-2} \frac{g_1(y)}{g_1(x)},
\]
for some $c_1, c_2 > 0$. Similarly if $|x - y| \geq |y - z|$, then
\[
\frac{G_D(x,y)}{G_D(x,z)} \leq c_1 \frac{g_1(A_{x,z}) g_2(A_{x,z})}{g_1(A_{y,z}) g_2(A_{y,z})} \frac{|x - z|^{d-2} g_1(y)}{|x - y|^{d-2} g_1(x)} \leq c_2 c_2 2^{d-2} \frac{g_2(y)}{g_2(z)}.
\]
Thus
\[
\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} \leq c_1 c_2 2^{d-2} \left( \frac{g_1(y)}{g_1(x)} G_D(x,y) \vee \frac{g_2(y)}{g_2(z)} G_D(y,z) \right).
\]

Lemma 2.16 There exists $c > 0$ such that for every $x, y \in D$ and $A \in B(x,y)$,
\[
g_i(x) \vee g_i(y) \leq c g_i(A), \quad i = 1, 2.
\]

Proof. If $r(x, y) \geq \varepsilon_1$, the lemma is clear. If $r(x, y) < \varepsilon_1$, from Lemma 2.11 (1), it is easy to see that that
\[
g_i(x) \leq c g_i(A_{r(x,y)}(Q_x))
\]
for some $c > 0$, where $Q_x$ is a point on $\partial D$ such that $\rho_D(x) = |x - Q_x|$. Thus the lemma follows from Lemmas 2.9 and and (2.15) \(\square\).

Now we are ready to prove the 3G theorem.

Theorem 2.17 There exists a constant $c > 0$ such that for every $x, y, z \in D$,
\[
\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} \leq c \frac{|x - z|^{d-2}}{|x - y|^{d-2}|y - z|^{d-2}}. \tag{2.21}
\]

Proof. Let $A_{x,y} \in B(x,y)$, $A_{y,z} \in B(y,z)$ and $A_{z,x} \in B(z,x)$. By (2.17), the left-hand side of (2.21) is less than and equal to
\[
\left( \frac{g_1(y)}{g_1(A_{x,z})} \right) \left( \frac{g_2(y)}{g_2(A_{x,y})} \right) \frac{|x - z|^{d-2}}{|x - y|^{d-2}|y - z|^{d-2}}.
\]
If $|x - y| \leq |y - z|$, by Lemma 2.14 and Lemma 2.16 we have
\[
\frac{g_1(y)}{g_1(A_{x,y})} \leq c_1, \quad \frac{g_2(y)}{g_2(A_{x,y})} \leq c_1, \quad \frac{g_1(A_{x,z})}{g_1(A_{y,z})} \leq c_2 \quad \text{and} \quad \frac{g_2(A_{x,z})}{g_2(A_{y,z})} \leq c_2
\]
for some constants $c_1, c_2 > 0$. Similarly, if $|x - y| \geq |y - z|$, then
\[
\frac{g_1(y)}{g_1(A_{y,z})} \leq c_1, \quad \frac{g_2(y)}{g_2(A_{y,z})} \leq c_1, \quad \frac{g_1(A_{x,z})}{g_1(A_{x,y})} \leq c_2 \quad \text{and} \quad \frac{g_2(A_{x,z})}{g_2(A_{x,y})} \leq c_2.
\]

\[\square\]

Combining the main results of this section, we get the following inequality.

**Theorem 2.18** There exist constants $c_1, c_2 > 0$ such that for every $x, y, z \in D$,
\[
\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} \leq c_1 \left( \frac{g_1(y)}{g_1(x)} G_D(x,y) \vee \frac{g_2(y)}{g_2(z)} G_D(y,z) \right) \leq c_2 \left( |x - y|^{-d+2} \vee |y - z|^{-d+2} \right).
\]

(2.22)

**Proof.** We only need to prove the second inequality. Applying Theorem 2.12 we get that there exists $c_1 > 0$ such that
\[
\frac{g_1(y)}{g_1(x)} G_D(x,y) \leq c_1 \frac{g_1(y) g_2(y)}{g_1(A) g_2(A)} |x - y|^{-d+2}
\]
and
\[
\frac{g_2(y)}{g_2(z)} G_D(y,z) \leq c_1 \frac{g_1(y) g_2(y)}{g_1(B) g_2(B)} |x - y|^{-d+2}
\]
for every $(A, B) \in B(x, y) \times B(y, z)$. Applying Lemma 2.16 we arrive at the desired assertion.

3 **Schrödinger semigroups for $X^D$**

In this section, we will assume that $D$ is a bounded Lipschitz domain. We first recall some notions from [14]. A measure $\nu$ on $D$ is said to be a smooth measure of $X^D$ if there is a positive continuous additive functional (PCAF in abbreviation) $A$ of $X^D$ such that for any $x \in D$, $t > 0$ and bounded nonnegative function $f$ on $D$,
\[
\mathbb{E}_x \int_0^t f(X_s^D) dA_s = \int_0^t \int_D q^D(s, x, y)f(y)\nu(dy)ds.
\]
(3.1)

The additive functional $A$ is called the PCAF of $X^D$ with Revuz measure $\nu$.

For a signed measure $\nu$, we use $\nu^+$ and $\nu^-$ to denote its positive and negative parts of $\nu$ respectively. A signed measure $\nu$ is called smooth if both $\nu^+$ and $\nu^-$ are smooth. For a signed smooth measure $\nu$, if $A^+$ and $A^-$ are the PCAFs of $X^D$ with Revuz measures $\nu^+$ and $\nu^-$ respectively, the additive functional $A := A^+ - A^-$ of is called the CAF of $X^D$ with (signed) Revuz measure $\nu$.

When $\nu(dx) = c(x)dx$, $A_t$ is given by $A_t = \int_0^t c(X_s^D)ds$.

We recall now the definition of the Kato class.
Definition 3.1 A signed smooth measure $\nu$ is said to be in the class $S_\infty(X^D)$ if for any $\varepsilon > 0$ there is a Borel subset $K = K(\varepsilon)$ of finite $|\nu|$-measure and a constant $\delta = \delta(\varepsilon) > 0$ such that

$$\sup_{(x,z) \in (D \times D) \setminus K} \int_{D \setminus K} \frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} |\nu|(dy) \leq \varepsilon$$

and for all measurable set $B \subset K$ with $|\nu|(B) < \delta$,

$$\sup_{(x,z) \in (D \times D) \setminus B} \int_{B} \frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} |\nu|(dy) \leq \varepsilon.$$  

A function $q$ is said to be in the class $S_\infty(X^D)$, if $q(x) dx$ is in $S_\infty(X^D)$.

It follows from Proposition 7.1 of [14] and Theorem 2.17 above that $K_{d,2}$ is contained in $S_\infty(X^D)$. In fact, by Theorem 2.18 we have the following result. Recall that $g_1(x) = G_D(x,z_0) \land C_1$ and $g_2(y) = G_D(z_0,y) \land C_1$.

Proposition 3.2 If a signed smooth measure $\nu$ satisfies

$$\sup_{x \in D} \lim_{r \downarrow 0} \int_{D \setminus \{|x-y| \leq r\}} \frac{g_1(y)}{g_1(x)} G_D(x,y) |\nu|(dy) = 0$$

and

$$\sup_{x \in D} \lim_{r \downarrow 0} \int_{D \setminus \{|x-y| \leq r\}} \frac{g_2(y)}{g_2(x)} G_D(y,x) |\nu|(dy) = 0,$$

then $\nu \in S_\infty(X^D)$.

Proof. This is a direct consequence of Theorem 2.18. }

In the remainder of this section, we will fix a signed measure $\nu \in S_\infty(X^D)$ and we will use $A$ to denote the CAF of $X^D$ with Revuz measure $\nu$. For simplicity, we will use $e_A(t)$ to denote $\exp(A_t)$. The CAF $A$ gives rise to a Schrödinger semigroup:

$$Q^D_t f(x) := E_x [e_A(t) f(X^D_t)].$$

The function $x \mapsto E_x[e_A(\tau_D)]$ is called the gauge function of $\nu$. We say $\nu$ is gaugeable if $E_x[e_A(\tau_D)]$ is finite for some $x \in D$. In the remainder of this section we will assume that $\nu$ is gaugeable. It is shown in [14], by using the duality and the gauge theorems in [4] and [7], that the gauge function $x \mapsto E_x[e_A(\tau_D)]$ is bounded on $D$ (see section 7 in [14]).

For $y \in D$, let $X^D,y$ denote the $h$-conditioned process obtained from $X^D$ with $h(\cdot) = G_D(\cdot,y)$ and let $E^y_x$ denote the expectation for $X^D,y$ starting from $x \in D$. We will use $\tau^y_D$ to denote the
lifetime of $X_{D,y}$. We know from [14] that $E_x^y[e_A(\tau_D^y)]$ is continuous in $D \times D$ (also see Theorem 3.4 in [6]) and

$$\sup_{(x,y) \in (D \times D) \setminus d} E_x^y[|A|_{\tau_D^y}] < \infty$$

(3.4)

(also see [4] and [7]) and therefore by Jensen’s inequality

$$\inf_{(x,y) \in (D \times D) \setminus d} E_x^y[e_A(\tau_D^y)] > 0,$$

(3.5)

where $d$ is the diagonal of the set $D \times D$. We also know from section 7 in [14] that

$$V_D(x, y) := E_x^y[e_A(\tau_D^y)]G_D(x, y)$$

(3.6)

is the Green function of $\{Q_t^D\}$, that is, for any nonnegative function $f$ on $D$,

$$\int_D V_D(x, y)f(y) \, dy = \int_0^\infty Q_t^D f(x) \, dt$$

(3.3)

and the continuity of $E_x^y[e_A(\tau_D^y)]$ imply that $V_D(x, y)$ is comparable to $G_D(x, y)$ and $V_D(x, y)$ is continuous on $(D \times D) \setminus d$. Thus there exists a constant $c > 0$ such that for every $x, y, z \in D$,

$$\frac{V_D(x, y)V_D(y, z)}{V_D(x, z)} \leq c \frac{|x - z|^{d-2}}{|x - y|^{d-2}|y - z|^{d-2}}.$$  

(3.7)

4 Two-sided heat kernel estimates for $\{Q_t^D\}$

In this section, we will establish two-sided estimates for the heat kernel of $Q_t^D$ in bounded $C^{1,1}$ domains.

Recall that a bounded domain $D$ in $\mathbb{R}^d$ is said to be a $C^{1,1}$ domain if there is a localization radius $r_0 > 0$ and a constant $\Lambda > 0$ such that for every $Q \in \partial D$, there is a $C^{1,1}$ function $\phi = \phi_Q : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ satisfying $\phi(0) = \nabla \phi(0) = 0$, $\|\nabla \phi\|_\infty \leq \Lambda$, $|\nabla \phi(x) - \nabla \phi(z)| \leq \Lambda|x - z|$, and an orthonormal coordinate system $y = (y_1, \cdots, y_{d-1}, y_d) := (\tilde{y}, y_d)$ such that $B(Q, r_0) \cap D = B(\tilde{y}, r_0) \cap \{y : y_d = \phi(\tilde{y})\}$.

We will always assume in this section that $D$ is a bounded $C^{1,1}$ domain. Since we will follow the method in [11] (see also [19]), the proof of this section will be little sketchy.

First, we recall some results from [11]. For every bounded $C^{1,1}$ domain $D$ and any $T > 0$, there exist positive constants $c_i$, $i = 1, \ldots, 4$, such that

$$C_1 \psi_D(t, x, y)t^{-\frac{d}{2}}e^{-\frac{c_2|x-y|^2}{t}} \leq q^D(t, x, y) \leq C_3 \psi_D(t, x, y)t^{-\frac{d}{2}}e^{-\frac{c_4|x-y|^2}{t}}$$

(4.1)

for all $(t, x, y) \in (0, T] \times D \times D$, where

$$\psi_D(t, x, y) := (1 \wedge \frac{\rho_D(x)}{\sqrt{t}})(1 \wedge \frac{\rho_D(y)}{\sqrt{t}})$$
(see (4.27) in [11]).

For any \( z \in \mathbb{R}^d \) and \( 0 < r \leq 1 \), let

\[
D_r^z := z + rD, \quad \psi_{D_r^z}(t, x, y) := (1 \land \frac{\rho_{D_r^z}(x)}{\sqrt{t}})(1 \land \frac{\rho_{D_r^z}(y)}{\sqrt{t}}), \quad (t, x, y) \in (0, \infty) \times D_r^z \times D_r^z
\]

where \( \rho_{D_r^z}(x) \) is the distance between \( x \) and \( \partial D_r^z \). Then, for any \( T > 0 \), there exist positive constants \( t_0 \) and \( c_j, 5 \leq j \leq 8 \), independent of \( z \) and \( r \) such that

\[
c_5 t^{- \frac{d}{2}} \psi_{D_r^z}(t, x, y) e^{- \frac{c_8|x-y|^2}{2t}} \leq q_{D_r^z}^z(t, x, y) \leq c_7 t^{- \frac{d}{2}} \psi_{D_r^z}(t, x, y) e^{- \frac{c_8|x-y|^2}{2t}}
\]  \hspace{1cm} (4.2)

for all \( (t, x, y) \in (0, t_0 \land (r^2T)) \times D_r^z \times D_r^z \) (see (5.1) in [11]). We will sometimes suppress the indices from \( D_r^z \) when there is no possibility of confusion.

For the remainder of this paper, we will assume that \( \nu \) is in the Kato class \( K_{d,2} \). Using the estimates above and the joint continuity of the densities \( q^D(t, x, y) \) (Theorem 2.4 in [12]), it is routine (For example, see Theorem 3.17 [8], Theorem 3.1 [2] and page 4669 in [4]) to show that \( Q^D_t \) has a jointly continuous density \( r^D(t, \cdot, \cdot) \) (also see Theorem 2.4 in [12]). So we have

\[
E_x \left[ e_A(t)f(X^D_t) \right] = \int_D f(y)r^D(t, x, y)dy
\]  \hspace{1cm} (4.3)

where \( A \) is the CAF of \( X^D \) with Revuz measure \( \nu \) in \( D \).

**Theorem 4.1**  The density \( r^D(t, x, y) \) satisfies the following equation

\[
r^D(t, x, y) = q^D(t, x, y) + \int_0^t \int_D r^D(s, x, z)q^D(t-s, z, y)\nu(dz)ds
\]  \hspace{1cm} (4.4)

for all \( (t, x, y) \in (0, \infty) \times D \times D \).

**Proof.**  Recall that \( A \) is the CAF of \( X^D \) with Revuz measure \( \nu \) in \( D \) and Let \( \theta \) be the usual shift operator for Markov processes.

Since for any \( t > 0 \)

\[
e_A(t) = e^{A_t} = 1 + \int_0^t e^{A_{t-s}}dA_s = 1 + \int_0^t e^{A_{t-s}\theta_s}dA_s
\]

We have

\[
E_x \left[ e_A(t)f(X^D_t) \right] = E_x \left[ f(X^D_t) \right] + E_x \left[ f(X^D_t) \int_0^t e^{A_{t-s}\theta_s}dA_s \right]
\]  \hspace{1cm} (4.5)

for all \( (t, x) \in (0, \infty) \times D \) and all bounded Borel-measurable functions \( f \) in \( D \).

By the Markov Property and Fubini’s theorem, we have

\[
E_x \left[ f(X^D_t) \int_0^t e^{A_{t-s}\theta_s}dA_s \right] = \int_0^t E_x \left[ f(X^D_t)e^{A_{t-s}\theta_s}dA_s \right]
\]

\[
= \int_0^t E_x \left[ E_X^D \left[ f(X^D_{t-s})e_A(t-s) \right] dA_s \right].
\]
Thus by (3.1) and (4.3),
\[
\mathbb{E}_x \left[ f(X^D_t) \int_0^t e^{A_{t-s}q^s} dA_s \right] = \int_D f(y) \int_0^t \int_D r^D(s, x, z) q^D(t - s, z, y) \nu(dz) ds dy.
\]

(4.6)

Since \( r^D(s, \cdot, \cdot) \) and \( q^D(t - s, \cdot, \cdot) \) are jointly continuous, combining (4.5)-(4.6), we have proved the theorem.

\[\boxed{}\]

The proof of the next lemma is almost identical to that of Lemma 3.1 in [20]. We omit the proof.

**Lemma 4.2** For any \( a > 0 \), there exists a positive constants \( c \) depending only on \( a \) and \( d \) such that for any \((t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \),
\[
\int_0^t \int_{\mathbb{R}^d} s^{-d+1} e^{-\frac{a|x-z|^2}{2s}} (t-s)^{-\frac{d}{2}} e^{-\frac{a|x-y|^2}{4s}} |\nu|(dz)ds \\
\leq ct^{-\frac{d+1}{2}} e^{-\frac{a|x-z|^2}{2t}} (t-s)^{-\frac{d}{2}} e^{-\frac{a|x-y|^2}{4s}} |\nu|(dz)ds
\]

and
\[
\int_0^t \int_{\mathbb{R}^d} s^{-d+1} e^{-\frac{a|x-z|^2}{2s}} (t-s)^{-\frac{d}{2}} e^{-\frac{a|x-y|^2}{4s}} |\nu|(dz)ds \\
\leq ct^{-\frac{d+1}{2}} e^{-\frac{a|x-z|^2}{2t}} (t-s)^{-\frac{d}{2}} e^{-\frac{a|x-y|^2}{4s}} |\nu|(dz)ds
\]

**Lemma 4.3** For any \( a > 0 \), there exists a positive constant \( c \) depending only on \( a \) and \( d \) such that for any \((t, x, y) \in (0, \infty) \times D \times D \),
\[
\int_0^t \int_D (1 \wedge \frac{\rho(x)}{\sqrt{s}})(1 \wedge \frac{\rho(y)}{\sqrt{s}}) s^{-\frac{d}{2}} e^{-\frac{a|x-z|^2}{2s}} (1 \wedge \frac{\rho(y)}{\sqrt{t-s}})(1 \wedge \frac{\rho(y)}{\sqrt{t}})(t-s)^{-\frac{d}{2}} e^{-\frac{a|x-y|^2}{4s}} |\nu|(dz)ds \\
\leq c(1 \wedge \frac{\rho(x)}{\sqrt{t}})(1 \wedge \frac{\rho(y)}{\sqrt{t}})(t-s)^{-\frac{d}{2}} e^{-\frac{a|x-y|^2}{4s}} \sup_{u \in D} \int_0^t \int_{\mathbb{R}^d} s^{-\frac{d}{2}} e^{-\frac{a|x-z|^2}{2s}} |\nu|(dz)ds
\]

(4.7)

**Proof.** With Lemma 4.2 in hand, we can follow the proof of Theorem 2.1 (page 389-391) in [17] to get the next lemma. So we skip the details. \[\boxed{}\]

Recall that
\[
M^1_\mu(r) = \sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq r} \frac{|\mu^1(dy)|}{|x-y|^d-1} \quad \text{and} \quad M^2_\nu(r) = \sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq r} \frac{|\nu(dy)|}{|x-y|^d-2}, \quad r > 0, i = 1 \ldots d.
\]
Theorem 4.4  (1) For each $T > 0$, there exist positive constants $c_j, 1 \leq j \leq 4$, depending on $\mu$ and $\nu$ only via the rate at which $\max_{1 \leq i \leq d} M_{\mu i}^2(r)$ and $M_{\nu i}^2(r)$ go to zero such that

$$c_1 t^{-\frac{d}{2}} \psi_D(t, x, y) e^{-\frac{c_4|x-y|^2}{2t}} \leq r^D(t, x, y) \leq c_3 t^{-\frac{d}{2}} \psi_D(t, x, y) e^{-\frac{c_4|x-y|^2}{2t}}$$

(4.8)

(2) There exist $T_1 = T_1(D) > 0$ such that for any $T > 0$, there exist positive constants $t_1$ and $c_j, 5 \leq j \leq 8$, independent of $z$ and $r$ such that

$$c_5 t^{-\frac{d}{2}} \psi_{D^r}(t, x, y) e^{-\frac{c_4|x-y|^2}{2t}} \leq r^{D^r}(t, x, y) \leq c_7 t^{-\frac{d}{2}} \psi_{D^r}(t, x, y) e^{-\frac{c_4|x-y|^2}{2t}}$$

(4.9)

for all $r \in (0, 1]$ and $(t, x, y) \in (0, t_1 \wedge (r^2(T \wedge T_1))] \times D_r \times D_r^z$.

Proof. We only give the proof of (4.9). The proof of (4.8) is similar. Fix $T > 0$ and $z \in \mathbb{R}^d$. Let $D_r := D_r^z, \rho_D(x) := \rho_D(x)$ and $\psi_D(t, x, y) := \psi_D^r(t, x, y)$. We define $\tilde{I}_k(t, x, y)$ recursively for $k \geq 0$ and $(t, x, y) \in (0, \infty) \times D \times D$:

$$I_0^r(t, x, y) := q^{D^r}(t, x, y),$$

$$I_{k+1}^r(t, x, y) := \int_0^t \int_{D_r} I_k^r(s, x, z) q(z) q^{D^r}(t-s, z, y) dz ds.$$

Then iterating the above gives

$$r^{D^r}(t, x, y) = \sum_{k=0}^{\infty} I_k^r(t, x, y), \quad (t, x, y) \in (0, \infty) \times D_r \times D_r.$$  

(4.10)

Let

$$N^2_\nu(t) := \sup_{u \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} s^{-\frac{d}{2}} e^{-\frac{|u-x|^2}{2s}} |\nu|^2(dz) ds, \quad t > 0.$$  

It is well-known (See, for example, Proposition 2.1 in [11]) that for any $r > 0$, there exist $c_1 = c_1(d, r)$ and $c_2 = c_2(d)$ such that

$$N^2_\nu(t) \leq (c_1 t + c_2) M^2_\nu(r), \quad \text{for every } t \in (0, 1).$$  

(4.11)

We claim that there exist positive constants $c_3, c_4$ and $A$ depending only on the constants in (4.12) and (4.17) such that for $k = 0, 1, \cdots$ and $(t, x, y) \in (0, t_0 \wedge (r^2T)) \times D_r \times D_r$

$$|I_k^r(t, x, y)| \leq c_3 \psi_D^r(t, x, y) t^{-\frac{d}{2}} e^{-\frac{c_4|x-y|^2}{2t}} \left(c_4 N^2_\nu \left(\frac{2t}{A}\right)\right)^k, \quad 0 < r \leq 1.$$  

(4.12)

We will prove the above claim by induction. By (4.12), there exist constants $t_0, c_3$ and $A$ such that

$$|I_0^r(t, x, y)| = |q^{D^r}(t, x, y)| \leq c_3 \psi_D^r(t, x, y) t^{-\frac{d}{2}} e^{-\frac{A|x-y|^2}{2t}}$$  

(4.13)
for \((t, x, y) \in (0, t_0 \wedge (r^2 T)) \times D_r \times D_r\). On the other hand, by Lemma 4.3.3 there exists a positive constant \(c_5\) depending only on \(A\) and \(d\) such that

\[
\int_0^t \int_{D_r} \psi_r(s, x, z) s^{-\frac{d}{2}} e^{-\frac{|x-z|^2}{2t}} \left(1 \wedge \frac{\rho_r(y)}{\sqrt{t-s}}\right)(t-s)^{-\frac{d}{2}} e^{-\frac{|x-r|^2}{4t}} \omega((dz)ds
\]

\[
\leq c_5 \psi_r(t, x, y) t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{2t}} \sup_{u \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} s^{-\frac{d}{2}} e^{-\frac{|u-z|^2}{4t}} \omega((dz)ds.
\]

So there exists \(c_6 = c_6(d)\) such that

\[
|I_1(t, x, y)| \leq c_3 c_5 \psi_r(t, x, y) t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{2t}} \sup_{u \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} s^{-\frac{d}{2}} e^{-\frac{|u-z|^2}{4t}} \omega((dz)ds
\]

\[
\leq c_3 c_5 c_6 A^\frac{d}{2} \psi_r(t, x, y) t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{2t}} N_\nu(\frac{2t}{A})
\]

for \((t, x, y) \in (0, t_0 \wedge (r^2 T)) \times D_r \times D_r\). Therefore \(4.12\) is true for \(k = 0, 1\) with \(c_4 := c_3 c_5 c_6 A^\frac{d}{2}\).

Now we assume \(4.12\) is true up to \(k\). Then by \(4.13\)-\(4.14\), we have

\[
|I_k+1(t, x, y)| \leq \int_0^t \int_{D_r} |I_k(t, s, x, z)| q_{Dr} (t-s, z, y) |\omega((dz)ds
\]

\[
\leq c_3(1 \wedge \frac{\rho_r(y)}{\sqrt{t-s}})(t-s)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{2t}} \omega((dz)ds
\]

\[
\leq c_3 \left( c_4 N_\nu(\frac{2t}{A}) \right)^k \int_0^t \int_{D_r} \psi_r(s, x, z) s^{-\frac{d}{2}} e^{-\frac{|x-z|^2}{2t}} \left(1 \wedge \frac{\rho_r(y)}{\sqrt{t-s}}\right)
\]

\[
\times (t-s)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{2t}} \omega((dz)ds
\]

\[
\leq c_3 \left( c_4 N_\nu(\frac{2t}{A}) \right)^k \int_0^t \int_{D_r} \psi_r(t, x, y) t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{2t}} N_\nu(\frac{2t}{A})
\]

\[
\leq c_3 \psi_r(t, x, y) t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{2t}} \left( c_4 N_\nu(\frac{2t}{A}) \right)^{k+1}.
\]

So the claim is proved.

Choose \(t_1 < (1 \wedge t_0)\) small so that

\[
c_4 N_\nu(\frac{2t_1}{A}) \leq \frac{1}{2}.
\]

By \(4.11\), \(t_1\) depends on \(\nu\) only via the rate at which \(M_\nu^2(r)\) goes to zero. \(4.10\) and \(4.12\) imply that for \((t, x, y) \in (0, t_1 \wedge (r^2 T)) \times D_r \times D_r\)

\[
r^{D_r}(t, x, y) \leq \sum_{k=0}^{\infty} |I_k(t, x, y)| \leq 2c_3 \psi_r(t, x, y) t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{2t}}.
\]

\[
(4.16)
\]
Now we are going to prove the lower estimate of \( r^{D_r}(t, x, y) \). Combining (4.10), (4.12) and (4.15) we have for every \((t, x, y) \in (0, t_1 \wedge (r^2 T)] \times D_r \times D_r,\)

\[
| r^{D_r}(t, x, y) - q^{D_r}(t, x, y) | \leq \sum_{k=1}^{\infty} | I_k(t, x, y) | \leq c_3 c_4 N_\nu \left( \frac{2t_1}{A} \right) \psi_\nu(t, x, y) t^{-\frac{d}{2}} e^{-\frac{c_7 |x-y|^2}{2t}}.
\]

Since there exist \( c_7 \) and \( c_8 \leq 1 \) depending on \( T \) such that

\[
q^{D_r}(t, x, y) \geq 2c_8 \psi_\nu(t, x, y) t^{-\frac{d}{2}} e^{-\frac{c_7 |x-y|^2}{2t}},
\]

we have for \(|x-y| \leq \sqrt{t}\) and \((t, x, y) \in (0, t_1 \wedge (r^2 T)] \times D \times D,\)

\[
r^{D_r}(t, x, y) \geq \left( 2c_8 e^{-2c_7} - c_3 c_4 N_\nu \left( \frac{2t_1}{A} \right) \right) \psi(t, x, y) t^{-\frac{d}{2}}. \tag{4.17}
\]

Now we choose \( t_2 \leq t_1 \) small so that

\[
c_3 c_4 N_\nu \left( \frac{2t_2}{A} \right) < c_8 e^{-2c_7}. \tag{4.18}
\]

Note that \( t_2 \) depends on \( \nu \) only via the rate at which \( M_\nu^2(r) \) goes to zero. So for \((t, x, y) \in (0, t_2 \wedge (r^2 T)] \times D \times D \) and \(|x-y| \leq \sqrt{t}\), we have

\[
r^{D_r}(t, x, y) \geq c_8 e^{-2c_7} \psi_\nu(t, x, y) t^{-\frac{d}{2}}. \tag{4.19}
\]

It is easy to check (see pages 420–421 of [21]) that there exists a positive constant \( T_0 \) depending only on the characteristics of the bounded \( C^{1,1} \) domain \( D \) such that for any \( \hat{t} \leq T_0 \) and \( x, y \in D \) with \( \rho_D(x) \geq \sqrt{\hat{t}}, \rho_D(y) \geq \sqrt{\hat{t}}, \) one can find a arclength-parameterized curve \( l \subset D \) connecting \( x \) and \( y \) such that the length \(|l|\) of \( l \) is equal to \( \lambda_1 |x-y| \) with \( \lambda_1 \leq \lambda_0, \) a constant depending only on the characteristics of the bounded \( C^{1,1} \) domain \( D \). Moreover, \( l \) can be chosen so that

\[
\rho_D(l(s)) \geq \lambda_2 \sqrt{\hat{t}}, \quad s \in [0, |l|]
\]

for some positive constant \( \lambda_2 \) depending only on the characteristics of the bounded \( C^{1,1} \) domain \( D \). Thus for any \( t = r^2 \hat{t} \leq r^2 T_0 \) and \( x, y \in D_r \) with \( \rho_r(x) \geq \sqrt{\hat{t}}, \rho_r(y) \geq \sqrt{\hat{t}}, \) one can find a arclength-parameterized curve \( l \subset D_r \) connecting \( x \) and \( y \) such that the length \(|l|\) of \( l \) is equal to \( \lambda_1 |x-y| \) and

\[
\rho_r(l(s)) \geq \lambda_2 \sqrt{\hat{t}}, \quad s \in [0, |l|].
\]

Using this fact and (4.19), and following the proof of Theorem 2.7 in [9], we can show that there exists a positive constant \( c_9 \) depending only on \( \nu \) and the characteristics of the bounded \( C^{1,1} \) domain \( D \) such that

\[
r^{D_r}(t, x, y) \geq \frac{1}{2} c_8 e^{-2c_7} \psi_\nu(t, x, y) t^{-\frac{d}{2}} e^{-\frac{c_9 |x-y|^2}{\hat{t}}}. \tag{4.20}
\]

for all \( t \in (0, t_2 \wedge r^2(T \wedge T_0)] \) and \( x, y \in D_r \) with \( \rho_r(x) \geq \sqrt{\hat{t}}, \rho_r(y) \geq \sqrt{\hat{t}}. \)
It is easy to check that there exists a positive constant \( T_1 \leq T_0 \) depending only on the characteristics of the bounded \( C^{1,1} \) domain \( D \) such that for \( t \leq T_1 \) and arbitrary \( x, y \in D \), one can find \( x_1, y_1 \in D \) be such that \( \rho_D(x_1) \geq \sqrt{t}, \rho_D(y_1) \geq \sqrt{t} \) and \( |x - x_0| \leq \sqrt{t}, |y - y_0| \leq \sqrt{t} \). Thus for any \( t, x, y \in D \), one can find \( x_1, y_1 \in D \), be such that \( \rho_r(x_1) \geq \sqrt{t}, \rho_r(y_1) \geq \sqrt{t} \) and \( |x - x_0| \leq \sqrt{t}, |y - y_0| \leq \sqrt{t} \). Now Using (4.17) and (4.20) one can repeat the last paragraph of the proof of Theorem 2.1 in [17] to show that there exists a positive constant \( c_{10} \) depending only on \( d \) and the characteristics of the bounded \( C^{1,1} \) domain \( D \) such that

\[
r_D^D(t, x, y) \geq c_{10}c^{-2c_7}t^{-d/4}G^d_{10}e^{2c_9|x-y|^2/t} \tag{4.21}
\]

for all \( (t, x, y) \in (0, t_2 \wedge r^2(T \wedge T_1)] \times D_r \times D_r \).

Using (4.11) instead of (4.12) The proof of (4.8) up to \( t \leq t_3 \) for some \( t_3 \) depending on \( T \) and \( D \) is similar (and simpler) to the proof of (4.9). To prove (4.13) for a general \( T > 0 \), we can apply the Chapman-Kolmogorov equation and use the argument in the proof of Theorem 3.9 in [18]. We omit the details.

\[\square\]

**Remark 4.5** Theorem 4.4 (2) will be used in [15] to prove parabolic Harnack inequality, parabolic boundary Harnack inequality and the intrinsic ultracontractivity for the semigroup \( Q^D_t \).

## 5 Uniform 3G type estimates for small Lipschitz domains

Recall that \( r_1 > 0 \) is the constant from [2.3] and \( r_3 > 0 \) is the constant from Theorem 2.2. The next lemma is a scale invariant version of Lemma 2.3. The proof is similar to the proof of Lemma 2.3.

**Lemma 5.1** There exists \( c = c(d, \mu) > 0 \) such that for every \( r \in (0, r_1 \wedge r_3), Q \in \mathbb{R}^d \) and open subset \( U \) with \( B(z, l) \subset U \subset B(Q, r) \), we have for every \( x \in U \setminus B(z, l) \)

\[
\sup_{y \in B(z,l/2)} G_U(y, x) \leq c \inf_{y \in B(z,l/2)} G_U(y, x) \quad \tag{5.1}
\]

and

\[
\sup_{y \in B(z,l/2)} G_U(x, y) \leq c \inf_{y \in B(z,l/2)} G_U(x, y) \tag{5.2}
\]

**Proof.** (5.1) follows from Theorem 2.2. So we only need to show (5.2). Since \( r < r_1 \), by (2.3), there exists \( c = c(d) > 1 \) such that for every \( x, w \in B(z, \frac{3l}{4}) \)

\[
c^{-1} \frac{1}{|w - x|^{d-2}} \leq G_{B(z,l)}(w, x) \leq G_U(w, x) \leq G_{B(Q,r)}(w, x) \leq c \frac{1}{|w - x|^{d-2}}.
\]

20
Thus for \( w \in \partial B( z, \frac{3r}{4} ) \) and \( y_1, y_2 \in B( z, \frac{r}{2} ) \), we have
\[
G_U( w, y_1 ) \leq c \left( \frac{|w - y_2|}{|w - y_1|} \right)^{d-2} \frac{1}{|w - y_2|^{d-2}} \leq 4^{d-2} c^2 G_U( w, y_2 ). \tag{5.3}
\]

On the other hand, from (2.5), we have
\[
G_U(x, y) = E_x \left[ G_U( X_{T_{B(z, \frac{r}{4})}}(x), y ) \right], \quad y \in B( z, \frac{r}{2} ) \tag{5.4}
\]

Since \( X_{T_{B(z, \frac{r}{4})}} \in \partial B( z, \frac{3r}{4} ) \), combining (5.3)-(5.4), we get
\[
G_U( x, y_1 ) \leq 4^{d-2} c^2 E_x \left[ G_U( X_{T_{B(z, \frac{r}{4})}}(x), y_2 ) \right] = 4^{d-2} c^2 G_U( x, y_2 ), \quad y_1, y_2 \in B( z, \frac{r}{2} )
\]

\[\square\]

In the remainder of this section, we fix a bounded Lipschitz domain \( D \) with characteristics \( (R_0, \Lambda_0) \). For every \( Q \in \partial D \) we put
\[
\Delta_Q( r ) := \{ y \in CS_Q : \phi_Q( \tilde{y} ) + r > y_d > \phi_Q( \tilde{y} ), |\tilde{y}| < r \}
\]
where \( CS_Q \) is the coordinate system with origin at \( Q \) in the definition of Lipschitz domains and \( \phi_Q \) is the Lipschitz function there. Define
\[
r_5 := \frac{R_0}{\sqrt{1 + \Lambda_0^2} + 1} \wedge r_1 \wedge r_3. \tag{5.5}
\]

If \( z \in \overline{\Delta_Q}( r ) \) with \( r \leq r_5 \), we have
\[
|Q - z| \leq |(\tilde{z}, \phi_Q( \tilde{z} )) - (\tilde{Q}, 0)| + r \leq \left( \sqrt{1 + \Lambda_0^2} + 1 \right) r \leq R_0.
\]
So \( \overline{\Delta_Q}( r ) \subset B(Q, R_0) \cap D \).

For any Lipschitz function \( \psi : \mathbb{R}^{d-1} \to \mathbb{R} \) with Lipschitz constant \( \Lambda_0 \), let
\[
\Delta^\psi := \{ y : r_5 > y_d - \psi( \tilde{y} ) > 0, |\tilde{y}| < r_5 \},
\]
so that \( \Delta^\psi \subset B(0, R_0) \). We observe that, for any Lipschitz function \( \varphi : \mathbb{R}^{d-1} \to \mathbb{R} \) with the Lipschitz constant \( \Lambda \), its dilation \( \varphi_r(x) := r \varphi(x/r) \) is also Lipschitz with the same Lipschitz constant \( \Lambda_0 \). For any \( r > 0 \), put \( \eta = \frac{r}{r_5} \) and \( \psi = (\phi_Q)_\eta \). Then it is easy to see that for any \( Q \in \partial D \) and \( r \leq r_5 \),
\[
\Delta_Q(r) = \eta \Delta^\psi.
\]
Thus by choosing appropriate constants \( \Lambda_1 > 1, R_1 < 1 \) and \( d_1 > 0 \), we can say that for every \( Q \in \partial D \) and \( r \leq r_5 \), the \( \Delta_Q(r)'s \) are bounded Lipschitz domains with the characteristics \( (rR_1, \Lambda_1) \) and the diameters of \( \Delta_Q(r)'s \) are less than \( rd_1 \). Since \( r_5 \leq r_1 \wedge r_3 \), Lemma 5.1 works for \( G_{\Delta_Q(r)}(x, y) \) with \( Q \in \partial D \) and \( r \leq r_5 \). Moreover, we can restate the scale invariant boundary Harnack principle in the following way.
Theorem 5.2 There exist constants $M_3, c > 1$ and $s_1 > 0$, depending on $\mu$, $\nu$ and $D$ such that for every $Q \in \partial D$, $r < r_5$, $s < r s_1$, $w \in \partial \Delta_Q(r)$ and any nonnegative functions $u$ and $v$ which are harmonic with respect to $X^D$ in $\Delta_Q(r) \cap B(w, M_3 s)$ and vanish continuously on $\partial \Delta_Q(r) \cap B(w, M_3 s)$, we have
\[
\frac{u(x)}{v(x)} \leq c \frac{u(y)}{v(y)} \text{ for any } x, y \in \Delta_Q(r) \cap B(w, s). \tag{5.6}
\]

In the remainder of this section we will fix the above constants $r_5$, $M_3$, $s_1$, $\Lambda_1$, $R_1$ and $d_1 > 0$, and consider the Green functions of $X$ in $\Delta_Q(r)$ with $Q \in \partial D$ and $r > 0$. We will prove a scale invariant $3G$ type estimates for these Green functions for small $r$. The main difficulties of the scale invariant $3G$ type estimates for $X$ are the facts that $X$ does not have rescaling property and that the Green function $G_{\Delta_Q(r)}(x, \cdot)$ is not harmonic for $X$. To overcome these difficulties, we first establish some results for the Green functions of $X$ in $\Delta_Q(r)$ with $Q \in \partial D$ and $r$ small.

Let $\delta^Q_0(x) := \text{dist}(x, \partial \Delta_Q(r))$. Using Lemma 5.1 and a Harnack chain argument, the proof of the next lemma is almost identical to the proof of Lemma 6.7 in [3]. So we omit the proof.

Lemma 5.3 For any given $c_1 > 0$, there exists $c_2 = c_2(D, c_1, \mu) > 0$ such that for every $Q \in \partial D$, $r < r_5$, $|x - y| \leq c_1(\delta^Q_0(x) \wedge \delta^Q_0(y))$, we have
\[
G_{\Delta_Q(r)}(x, y) \geq c_2 |x - y|^{-d+2}.
\]

Recall that $M_3 > 0$ and $s_1 > 0$ are the constants from Theorem 5.2. Let $M_4 := 2(1 + M_3)\sqrt{1 + \Lambda^2} + 2$ and $R_4 := R_1/M_4$. The next lemma is a scale invariant version of Lemma 2.5. The proof is similar to the proof of Lemma 2.5. We spell out the details for the reader’s convenience.

Lemma 5.4 There exists constant $c > 1$ such that for every $Q \in \partial D$, $r < r_5$, $s < r R_4$, $w \in \partial \Delta_Q(r)$ and any nonnegative functions $u$ and $v$ which are harmonic in $\Delta_Q(r) \setminus B(w, s)$ and vanish continuously on $\partial \Delta_Q(r) \setminus B(w, s)$, we have
\[
\frac{u(x)}{u(y)} \leq c \frac{v(x)}{v(y)} \text{ for any } x, y \in \Delta_Q(r) \setminus B(w, M_4 s). \tag{5.7}
\]

Proof. We fix a point $Q$ on $\partial D$, $r < r_5$, $s < r R_4$ and $w \in \partial \Delta_Q(r)$ throughout this proof. Let
\[
\Delta^s := \{ y \in CS_w : \varphi_w(\tilde{y}) + 2s > y_d > \varphi_w(\tilde{y}), |\tilde{y}| < 2(M_3 + 1)s \},
\partial_1 \Delta^s := \{ y \in CS_w : \varphi_w(\tilde{y}) + 2s \geq y_d > \varphi_w(\tilde{y}), |\tilde{y}| = 2(M_3 + 1)s \},
\partial_2 \Delta^s := \{ y \in CS_w : \varphi_w(\tilde{y}) + 2s = y_d, |\tilde{y}| \leq 2(M_3 + 1)s \},
\]
where $CS_w$ is the coordinate system with origin at $w$ in the definition of the Lipschitz domain $\Delta_Q(r)$ and $\varphi_w$ is the Lipschitz function there. If $z \in \Delta^s$,
\[
|w - z| \leq |(\tilde{z}, \varphi_w(\tilde{z})) - (\tilde{z}, 0)| + 2s \leq 2s (1 + M_3)\sqrt{1 + \Lambda^2} + 2s = M_4 s \leq r R_1.
\]
So $\Delta^s \subset B(Q, M_4s) \cap D \subset B(Q, rR_1) \cap D$. For $|y| = 2(M_3 + 1)s$, we have $|(\bar{y}, \varphi_w(\bar{y}))| > s$. So $u$ and $v$ are harmonic with respect to $X$ in $\Delta_Q(r) \cap B((\bar{y}, \varphi_w(\bar{y})), 2M_3s)$ and vanish continuously on $\partial \Delta_Q(r) \cap B((\bar{y}, \varphi_w(\bar{y})), 2M_3s)$ where $|\bar{y}| = 2(M_3 + 1)s$. Therefore by Theorem 5.2

$$u(x) \leq c \frac{v(x)}{v(y)}$$

for any $x, y \in \partial_1 \Delta^s$ with $\bar{x} = \bar{y}$. (5.8)

Since $\text{dist}(\Delta_Q(r) \cap B(w, s), \partial_2 \Delta^s) > cs$ for some $c_1 = c_1(D)$, if $x \in \partial_2 \Delta^s$, the Harnack inequality (Theorem 2.2) and a Harnack chain argument give that there exists constant $c_2 > 1$ such that

$$c_2^{-1} \frac{u(x)}{u(y)} \leq c \frac{v(x)}{v(y)} < c_2.$$  

(5.9)

In particular, (5.9) is true with $x = x_s := (\bar{x}, \varphi_w(\bar{x}) + 2s)$, which is also in $\partial_1 \Delta^s$. Thus (5.8) and (5.9) imply that

$$c_3^{-1} \frac{u(x)}{u(y)} \leq \frac{v(x)}{v(y)} \leq c_3 \frac{u(x)}{u(y)}, \quad x, y \in \partial_1 \Delta^s \cup \partial_2 \Delta^s$$

(5.10)

for some $c_3 > 1$. Now, by applying the maximum principle (Lemma 7.2 in [11]) twice ($x$ and $y$), (5.10) is true for every $x \in \Delta_Q(r) \setminus \Delta^s$.

Combining Theorem 5.2 and Lemma 5.4, we get the following as a corollary.

**Corollary 5.5** There exists constant $c > 1$ such that for every $Q \in \partial D$, $r < r_5$, $w \in \partial(\Delta_Q(r))$, and $s < rR_4$, we have for $x, y \in \Delta_Q(r) \setminus B(w, M_4s)$ and $z_1, z_2 \in \Delta_Q(r) \cap B(w, s)$

$$\frac{G_{\Delta_Q(r)}(x, z_1)}{G_{\Delta_Q(r)}(y, z_1)} \leq c \frac{G_{\Delta_Q(r)}(x, z_2)}{G_{\Delta_Q(r)}(y, z_2)} \quad \text{and} \quad \frac{G_{\Delta_Q(r)}(z_1, x)}{G_{\Delta_Q(r)}(z_1, y)} \leq c \frac{G_{\Delta_Q(r)}(z_2, x)}{G_{\Delta_Q(r)}(z_2, y)}.$$  

(5.11)

**Corollary 5.6** For any given $N \in (0, 1)$, there exists constant $c = c(N, M_4, D) > 1$ such that for every $Q \in \partial D$, $r < r_5$, $w \in \partial(\Delta_Q(r))$ and $s < rR_4$, we have

$$G_{\Delta_Q(r)}(x, z_1) \leq c G_{\Delta_Q(r)}(x, z_2) \quad \text{and} \quad G_{\Delta_Q(r)}(z_1, x) \leq c G_{\Delta_Q(r)}(z_2, x)$$

(5.12)

for $x \in \Delta_Q(r) \setminus B(w, M_4s)$ and $z_1, z_2 \in \Delta_Q(r) \cap B(w, s)$ with $B(z_2, Ns) \subset \Delta_Q(r) \cap B(w, s)$.

**Proof.** Fix $Q \in \partial D$, $r < r_5$, $w \in \partial(\Delta_Q(r))$ and $s < rR_4$. Recall from the proof of Lemma 5.4 that $CS_w$ is the coordinate system with origin at $w$ in the definition of the Lipschitz domain $\Delta_Q(r)$. Let $\bar{y} := (0, M_4s)$. By (2.2),

$$G_{\Delta_Q(r)}(\bar{y}, z_1) \leq c_1 |y - z_2|^{-d+2} \leq c_2 s^{-d+2} \quad \text{and} \quad G_{\Delta_Q(r)}(z_1, \bar{y}) \leq c_1 |y - z_2|^{-d+2} \leq c_2 s^{-d+2},$$

for some constants $c_1, c_2 > 0$.
Note that, since $\Delta_Q(r)$'s are bounded Lipschitz domains with the characteristics $(rR_1, \Lambda_1)$ and $s < rR_4$, it is easy to see that there exists a positive constant $c_3$ such that $\rho^Q_\ell(\bar{y}) \geq c_3 M_4 s$ and $\rho^Q_\ell(z_2) \geq Ns$. Thus by Lemma 5.3,

$$G_{\Delta_Q(r)}(y, z_2) \geq c_4 |y - z_2|^{-d+2} \geq c_5 s^{-d+2} \quad \text{and} \quad G_{\Delta_Q(r)}(z_2, y) \geq c_4 |y - z_2|^{-d+2} \geq c_5 s^{-d+2}$$

for some constants $c_4, c_5 > 0$.

Now apply (5.11) with $y = \bar{y}$ and get

$$G_{\Delta_Q(r)}(x, z_1) \leq c_6 G_{\Delta_Q(r)}(x, z_2) \quad \text{and} \quad G_{\Delta_Q(r)}(z_1, x) \leq c_6 G_{\Delta_Q(r)}(z_2, x),$$

for some $c_6 > 1$.

With lemma 5.4, Corollary 5.5, and Corollary 5.6 in hand, one can follow either the argument in Section 2 of this paper or the argument on page 170-173 of [8]. So we skip the details.

**Theorem 5.7** There exists a constant $c > 0$ such that for every $Q \in \partial D$, $r < r_5$ and $x, y, z \in \Delta_Q(r)$,

$$\frac{G_{\Delta_Q(r)}(x, y)G_{\Delta_Q(r)}(y, z)}{G_{\Delta_Q(r)}(x, z)} \leq c \left( |x - y|^{-d+2} + |y - z|^{-d+2} \right).$$

(5.13)

### 6 Boundary Harnack principle for the Schrödinger operator of $X^D$ in bounded Lipschitz domains

Recall that $\nu$ belongs to the Kato class $K_{d,2}$ and $A$ is continuous additive functional associated with $\nu|_D$. We also recall $e_A(t) = \exp(A_t)$ and the Schrödinger semigroup

$$Q_t^D f(x) = E_x \left[ e_A(t) f(X_t^D) \right].$$

Using the Martin representation for Schrödinger operators (Theorem 7.5 in [6]) and the uniform 3G estimates (Theorem 5.7), we will prove the boundary Harnack principle for the Schrödinger operator of diffusions with measure-valued drifts in bounded Lipschitz domains. In the remainder of this section, we fix a bounded Lipschitz domain $D$ with its characteristics $(R_0, \Lambda_0)$. Recall

$$\Delta_Q(r) = \{ y \in CS_Q : \phi_Q(\bar{y}) + r > y_d > \phi_Q(\bar{y}), |\bar{y}| < r \},$$

where $CS_Q$ is the coordinate system with origin at $Q \in \partial D$ in the definition of Lipschitz domains and $\phi_Q$ is the Lipschitz function there. We also recall that $r_5$ is the constant from (5.5) and that the diameters of $\Delta_Q(r)$'s are less than $r d_1$.

For $Q \in \partial D$, $r < r_5$ and $y \in \Delta_Q(r)$, let $X^{Q, r, y}$ denote the $h$-conditioned process obtained from $X^{\Delta_Q(r)}$ with $h(\cdot) = G_{\Delta_Q(r)}(\cdot, y)$ and let $E_x^{Q, r, y}$ denote the expectation for $X^{Q, r, y}$ starting from $x \in \Delta_Q(r)$. Now define the conditional gauge function

$$u_r^Q(x, y) := E_x^{Q, r, y} \left[ e_{A^\nu} \left( \tau^{y}_{\Delta_Q(r)} \right) \right].$$
By Theorem 5.7
\[
E_x^{Q,r,y} \left[ A \left( \tau_{\Delta_Q(r)}^y \right) \right] \leq \int_{\Delta_Q(r)} G_{\Delta_Q(r)}(x,a) G_{\Delta_Q(r)}(a,y) \frac{|\nu(da)|}{G_{\Delta_Q(r)}(x,y)} \nu(da) \\
\leq c \int_{\Delta_Q(r)} \left( |x-a|^{-d+2} + |a-y|^{-d+2} \right) \nu(da), \quad r < r_5.
\]

Since the above constant is independent of \( r < r_5 \), we have
\[
\sup_{x,y \in \Delta_Q(r)} E_x^{Q,r,y} \left[ A \left( \tau_{\Delta_Q(r)}^y \right) \right] \leq c \sup_{x \in \mathbb{R}^d} \int_{|x-a| \leq rd} \frac{|\nu(da)|}{|x-a|^{d-2}} = cM_5^2(rd) < \infty, \quad r < r_5, Q \in \partial D.
\]

Thus \( \nu \in S_\infty(X^{\Delta_Q(r)}) \) for every \( r < r_5 \) and there exists \( r_6 \leq r_5 \) such that
\[
\sup_{x,y \in \Delta_Q(r)} E_x^{Q,r,y} \left[ A \left( \tau_{\Delta_Q(r)}^y \right) \right] \leq \frac{1}{2}, \quad r < r_6, Q \in \partial D.
\]

Hence by Khasminskii’s lemma,
\[
\sup_{x,z \in \Delta_Q(r)} u^Q(x,y) \leq 2, \quad r < r_6, Q \in \partial D.
\]

By Jensen’s inequality, we also have
\[
\inf_{x,z \in \Delta_Q(r)} u^Q(x,y) > 0, \quad r < r_6, Q \in \partial D.
\]

Therefore, we have proved the following lemma.

**Lemma 6.1** For \( r < r_6 \), \( \nu|_{\Delta_Q(r)} \in S_\infty(X^{\Delta_Q(r)}) \) and \( \nu|_{\Delta_Q(r)} \) is gaugeable. Moreover, there exists a constant \( c \) such that \( c^{-1} \leq u^Q(x,y) \leq c \) for \( x,y \in \Delta_Q(r) \) and \( r < r_6 \).

**Theorem 6.2** (Boundary Harnack principle) Suppose \( D \) be a bounded Lipschitz domain in \( \mathbb{R}^d \) with the Lipschitz characteristic \((R_0, \Lambda_0)\) and let \( M_5 := (\sqrt{1+\Lambda_0^2} + 1) \). Then there exists \( N > 1 \) such that for any \( r \in (0, r_6) \) and \( Q \in \partial D \), there exists a constant \( c > 1 \) such that for any nonnegative functions \( u,v \) which are \( \nu \)-harmonic in \( D \cap B(Q, rM_5) \) with respect to \( X^D \) and vanish continuously on \( \partial D \cap B(Q, rM_5) \), we have
\[
\frac{u(x)}{v(x)} \leq c \frac{u(y)}{v(y)} \quad \text{for any } x,y \in D \cap B(Q, \frac{r}{N}).
\]

**Proof.** Note that, with \( M_5 = (\sqrt{1+\Lambda_0^2} + 1), \Delta_Q(r) \subset D \cap B(Q, M_5r) \). So \( u,v \) are \( \nu \)-harmonic in \( \Delta_Q(r) \). For the remainder of the proof, we fix \( Q \in \partial D \), \( r \in (0, r_5) \) and a point \( x^Q_r \in \Delta_Q(r) \). Let
\[
M(x,z) := \lim_{U \ni y \rightarrow z} \frac{G_U(x,y)}{G_U(x^Q_r,y)}, \quad K(x,z) := \lim_{U \ni y \rightarrow z} \frac{V_U(x,y)}{V_U(x^Q_r,y)}.
\]
Since $u, v$ are $\nu$-harmonic with respect to $X^{\Delta_Q(r)}$, by Theorem 7.7 in \cite{6} and our Lemma 6.1, there exist finite measures $\mu_1$ and $\nu_1$ on $\partial U$ such that
\[
 u(x) = \int_{\partial \Delta_Q(r)} K(x, z) \mu_1(dz) \quad \text{and} \quad v(x) = \int_{\partial \Delta_Q(r)} K(x, z) \nu_1(dz), \quad x \in \Delta_Q(r).
\]
Let
\[
u_1(x) := \int_{\partial \Delta_Q(r)} M(x, z) \mu_1(dz) \quad \text{and} \quad v_1(x) := \int_{\partial \Delta_Q(r)} M(x, z) \nu_1(dz), \quad x \in \Delta_Q(r).
\]
By Theorem 7.3 (2) in \cite{6} and our Lemma 6.1, we have for every $x \in U$
\[
 \frac{u(x)}{v(x)} = \frac{\int_{\partial \Delta_Q(r)} K(x, z) \mu_1(dz)}{\int_{\partial \Delta_Q(r)} K(x, z) \nu_1(dz)} \leq c_1^2 \frac{\int_{\partial \Delta_Q(r)} M(x, z) \mu_1(dz)}{\int_{\partial \Delta_Q(r)} M(x, z) \nu_1(dz)} = c_1^2 \frac{u_1(x)}{v_1(x)} \leq c_1 \frac{u_1(x)}{v(x)}.
\]
Since $u_1, v_1$ are harmonic for $X^U$ and vanish continuously on $\partial \Delta_Q(r) \cap \partial D$, by the boundary Harnack principle (Theorem 4.6 in \cite{12}), there exist $N$ and $c_2$ such that
\[
 \frac{u_1(x)}{v_1(x)} \leq c_2 \frac{u_1(y)}{v_1(y)}, \quad x, y \in D \cap B(Q, \frac{r}{N}).
\]
Thus for every $x, y \in D \cap B(Q, \frac{r}{N})$
\[
 \frac{u(x)}{v(x)} \leq c_2 \frac{u_1(x)}{v_1(x)} \leq c_2 c_1^2 \frac{u_1(y)}{v_1(y)} \leq c_2 c_1^4 \frac{u(y)}{v(y)}.
\]

References

[1] R. F. Bass and Z.-Q. Chen, Brownian motion with singular drift. *Ann. Probab.* 31(2) (2003), 791–817.

[2] Ph. Blanchard and Z. M. Ma, Semigroup of Schrödinger operators with potentials given by Radon measures, In *Stochastic processes, physics and geometry*, 160–195, World Sci. Publishing, Teaneck, NJ, 1990.

[3] K. Bogdan, Sharp estimates for the Green function in Lipschitz domains. *J. Math. Anal. Appl.* 243 (2000), 326-337.

[4] Z.-Q. Chen, Gaugeability and conditional gaugeability. *Trans. Amer. Math. Soc.* 354 (2002), 4639–4679.

[5] Z.-Q. Chen and P. Kim, Green function estimate for censored stable processes. *Probab. Theory Relat. Fields* 124 (2002), 595-610.

[6] Z.-Q. Chen and P. Kim, Stability of Martin boundary under non-local Feynman-Kac perturbations. *Probab. Theory Relat. Fields* 128 (2004), 525-564.

[7] Z.-Q. Chen and R. Song, General gauge and conditional gauge theorems. *Ann. Probab.* 30 (2002), 1313-1339.
[8] K. L. Chung and Z. X. Zhao, *From Brownian motion to Schrödinger’s equation*, Springer-Verlag, Berlin, 1995.

[9] E. B. Fabes and D. W. Stroock, A new proof of Moser’s parabolic Harnack inequality using the old ideas of Nash. *Arch. Rational Mech. Anal.* **96**(4) (1986), 327–338.

[10] W. Hansen, Uniform boundary Harnack principle and generalized triangle property. *J. Funct. Anal.* **226**(2), (2005), 452–484.

[11] P. Kim and R. Song, Two-sided estimates on the density of Brownian motion with singular drift, to appear in the Doob-memorial volume of the Ill. J. of Math., 2006.

[12] P. Kim and R. Song, Boundary Harnack principle for Brownian motions with measure-valued drifts in bounded Lipschitz domains, Preprint, 2006.

[13] P. Kim and R. Song, Intrinsic ultracontractivity of non-symmetric diffusion semigroups in bounded domains, Preprint, 2006.

[14] P. Kim and R. Song, On dual processes of non-symmetric diffusions with measure-valued drifts, Preprint, 2006.

[15] P. Kim and R. Song, Intrinsic ultracontractivity of non-symmetric diffusions with measure-valued drifts and potentials, Preprint, 2006.

[16] D. Revuz, Mesures associées aux fonctionnelles additives de Markov. I, *Trans. Amer. Math. Soc.* **148** (1970), 501–531.

[17] L. Riahi, Comparison of Green functions and harmonic measure of parabolic operators, *Potential Anal.* **23**(4) (2005), 381–402.

[18] R. Song, Sharp bounds on the density, Green function and jumping function of subordinate killed BM, *Probab. Theory Related Fields* **128** (2004), 606–628.

[19] Q. S. Zhang, A Harnack inequality for the equation $\nabla(a\nabla u) + b\nabla u = 0$, when $|b| \in K_{n+1}$, *Manuscripta Math.* **89** (1996), 61–77.

[20] Qi S. Zhang, Gaussian bounds for the fundamental solutions of $\nabla(A\nabla u) + B\nabla u - u_t = 0$, *Manuscripta Math.* **93** (1997), 381–390.

[21] Q. S. Zhang, The boundary behavior of heat kernels of Dirichlet Laplacians, *J. Differential Equations* **182**(2002), 416–430.