Exceptional points in composite structures consisting of two
dielectric diffraction gratings with Lorentzian line shape

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Abstract. Using scattering matrix formalism we derive analytical expressions for the
eigenmodes of a composite structure consisting of two dielectric diffraction gratings with
Lorentzian profile in reflection. Analyzing these expressions we prove formation of two
distinct pairs of exceptional points, provide analytical approximations for their coordinates and
by rigorous simulation demonstrate eigenmodes interchange as a result of encircling said
exceptional points.

1. Introduction
Exceptional points (EPs) are degeneracies in non-Hermitian systems, which appear when several
eigenmodes coalesce [1]. The properties of such system dramatically change in the vicinity of EPs and
lead to such phenomena as enhanced optical sensing [2], loss-induced transparency [3], unidirectional
transmission or reflection [4], and lasers with reversed pump dependence [5] or single-mode operation
[6]. One promising feature of an EP is that adiabatically encircling an EP can result in an exchange of
the eigenstate. Such behavior is expected to have applications in asymmetric mode switching [7] and
on-chip non-reciprocal transmission [8] and light stopping [9].

As a rule, exceptional points are studied through analyzing the eigenvalues and eigenvectors of the
proper Hamiltonian [10] or by analyzing the eigenmodes dispersion relation [11]. In this work we
demonstrate the formation of EPs using \( \omega-k_x \) resonant approximation of Lorentzian line shape. By
obtaining analytical expressions for the eigenmodes of a composite structure consisting of two
dielectric diffraction gratings (DGs) with Lorentzian line shape we show the formation of two distinct
EPs which can be achieved by varying the distance between said stacked gratings \( l \). This theoretical
conclusion is supported by rigorous calculation results that show eigenmodes swapping as a result of
encircling said EPs in the \( l-k_x \) parameter space.

2. \( \omega-k_x \) Lorentzian line shape in composite structures
A scattering matrix \( S \) relates the complex amplitudes of the plane waves incident on the diffraction
structure from the superstrate \( I_s \) and the substrate \( I_d \) regions with the amplitudes of the transmitted \( T \)
and reflected \( R \) diffraction orders [12]. For a horizontally symmetrical subwavelength DG, which
allows only the 0th reflected and transmitted diffraction orders to propagate, the \( S \) matrix takes the
form:
where \( R_i(k_x, \omega) \) and \( T_i(k_x, \omega) \) are the complex reflection and transmission coefficients of the DG for a unit-amplitude incident wave. It is worth noting that the scattering matrix in (1) does not describe the near-field effects associated with the evanescent diffraction orders of the DG.

In this paper we consider the elements of the scattering matrix (1) to be functions of the angular frequency \( \omega \) and the in-plane wave component \( k_x \) of the incident light. Let the DG have Lorentzian line shape profile. In this case the appropriate reflection and transmission coefficients can be approximated as follows [13]:

\[
R_i(k_x, \omega) = e^{i \psi} \frac{i \text{Im} \omega_{p,1} (\omega - \omega_{p,1})}{v_x^2 k_x^2 - (\omega - \omega_{p,1})(\omega - \omega_{p,2})}, \quad T_i(k_x, \omega) = e^{i \psi} \frac{v_x^2 k_x^2 - (\omega - \text{Re} \omega_{p,1})(\omega - \omega_{p,2})}{v_x^2 k_x^2 - (\omega - \omega_{p,1})(\omega - \omega_{p,2})},
\]

(2)

As a DG under consideration we propose a dielectric structure shown in figure 1a. The agreement between its reflection spectra calculated using rigorous coupled-wave analysis and the one calculated using the approximations (2) confirms that said DG has Lorentzian reflection profile (figure 1b). See the caption of figure 1 for the parameters of the DG as well as the parameters for the approximation.

\[
|R(\omega)|^2 \text{ and } |T(\omega)|^2 \text{ spectra of the DG (TE-polarization).}
\]

Let us consider a composite DG consisting of two subwavelength DGs described by scattering matrices (1) and separated by a homogeneous dielectric layer with the thickness \( l \) and the refractive index \( n_{\text{env}} \) (figure 2a). In this case, the scattering matrix of the composite DG can be expressed through the matrix \( S_i(\omega) \) in the form [14]

\[
S_i(k_x, \omega) = S_i(k_x, \omega) \ast L(l, k_x, \omega) \ast S_i(k_x, \omega)
\]

(3)

where the symbol \( \ast \) denotes the Redheffer star product, \( L(l, k_x, \omega) = \exp(i \psi(l, k_x, \omega)) \) is the scattering matrix of the intermediary layer and \( \psi(l, k_x, \omega) \) is the phase shift the plane waves acquire by propagation through this layer:

\[
\psi(l, k_x, \omega) = ik_x \sqrt{n_{\text{env}} \omega/c - k_x^2(\omega)},
\]

(4)

where \( k_x(\omega) = (\omega/c)n_{\text{env}} \sin \theta \), \( \theta \) being the angle of incidence.
By substituting (2) into (3), we obtain the scattering matrix $S_z$ of the composite DG with the following reflection and transmission coefficients:

$$R_z(k_x, \omega) = \frac{\text{i} e^{\text{i} \pi} (\omega - \omega_{p,2}) \text{Im} \omega_{p,1} \left(k_x^2 v_g^2 - (\omega - \omega_{p,1}) (\omega - \omega_{p,2}) + e^{2 \text{i} \pi \phi} \left(k^2 v_g^2 - (\omega - \omega_{p,1}) (\omega - \omega_{p,2})\right)\right)}{\left(k^2 v_g^2 - (\omega - \omega_{p,1}) (\omega - \omega_{p,2})\right) \left(k_x^2 v_g^2 - (\omega - \omega_{p,1}) (\omega - \omega_{p,2})\right)},$$

$$T_z(k_x, \omega) = \frac{e^{2 \text{i} \pi \phi} \left(k^2 v_g^2 - (\omega - \omega_{p,1}) (\omega - \omega_{p,2})\right) (k^2 v_g^2 - (\omega - \omega_{p,2}) (\omega - \omega_{p,1}))}{\left(k^2 v_g^2 - (\omega - \omega_{p,1}) (\omega - \omega_{p,2})\right)^2},$$

where $\omega_{\text{mod},1,2} = \text{Re} \omega_{p,1} + \text{i} \text{Im} \omega_{p,1} \left(1 \pm e^{\text{i} \phi}\right)$.

3. Exceptional points

According to (5) the eigenmodes of the composite structure (reflection and transmission coefficients complex poles) have the following form:

$$\omega_{\text{polar}} = \begin{cases} 
\frac{1}{2} \text{Re} \omega_{p,1} + \omega_{p,2} + \text{i} \text{Im} \omega_{p,1} \left(1 + e^{\text{i} \phi} \omega\right) \pm \sqrt{4k^2 v_g^2 + \left[\text{Re} \omega_{p,1} - \omega_{p,2} + \text{i} \text{Im} \omega_{p,1} \left(1 + e^{\text{i} \phi} \omega\right)\right]^2}, \\
\frac{1}{2} \text{Re} \omega_{p,1} + \omega_{p,2} + \text{i} \text{Im} \omega_{p,1} \left(1 - e^{\text{i} \phi} \omega\right) \pm \sqrt{4k^2 v_g^2 + \left[\text{Re} \omega_{p,1} - \omega_{p,2} + \text{i} \text{Im} \omega_{p,1} \left(1 - e^{\text{i} \phi} \omega\right)\right]^2}. 
\end{cases}$$

(6)

Note that expressions (6) contain square roots, which gives rise to branch points when the expressions under the radicals equal zero:

$$4k^2 v_g^2 + \left[\text{Re} \omega_{p,1} - \omega_{p,2} + \text{i} \text{Im} \omega_{p,1} \left(1 \pm e^{\text{i} \phi} \omega\right)\right]^2 = 0.$$

(7)

Let us consider the vicinity of normal incidence ($k_x \ll k_0 = \frac{\omega}{c} \text{Re} \omega_{p,1}$) so that the phase shift $\psi$ is defined only by intermediary layer thickness $l$: $\psi(l, k_x, \omega) \approx \psi(l) = l k_x$. This way solving (7) with respect to $\psi$, we arrive at two pairs of EPs in the $\psi - k_x$ parameter space:

$$\psi = \begin{cases} 
\pm \arcsin \frac{\text{Re} \omega_{p,1} - \omega_{p,2}}{\text{Im} \omega_{p,1}} - \phi + 2 \pi m, & m \in \mathbb{Z} \\
\pi \mp \arcsin \frac{\text{Re} \omega_{p,1} - \omega_{p,2}}{\text{Im} \omega_{p,1}} - \phi + 2 \pi m, & m \in \mathbb{Z}
\end{cases} \Rightarrow k_x = \begin{cases} 
\frac{1}{2} \text{Im} \omega_{p,1} \left(1 + \cos \psi\right) v_g & \text{for } \psi \leq l \text{Im} \omega_{p,1} \left(1 - \cos \psi\right) v_g \\
\frac{1}{2} \text{Im} \omega_{p,1} \left(1 - \cos \psi\right) v_g & \text{for } \psi \leq l \text{Im} \omega_{p,1} \left(1 - \cos \psi\right) v_g
\end{cases}$$

(8)

The composite structure will possess exceptional points if and only if the value for $\psi$ in (8) is real, i.e. if $\left|\text{Re} \omega_{p,1} - \omega_{p,2}\right|/\text{Im} \omega_{p,1} \leq 1$. This condition demands that the structure possesses low Q-factor resonance (high $\text{Im} \omega_{p,1}$) as well as narrow band gap (small $\left|\text{Re} \omega_{p,1} - \omega_{p,2}\right|$), and is hard to satisfy since generally speaking increasing the diffraction grating’s contrast (straying away from slab waveguide) widens the band gap while simultaneously lowering the quality factor of the resonance. The necessity to meet this requirement explains the bulkiness of the structure under consideration (figure 1a).

Using the approximations (8) we derive the following estimates for the exceptional point coordinates $\psi_1 = 0.883$ ($l_1 = 0.742 \mu m$, $k_{s,1} = 4.6 \cdot 10^{-3} \mu m^{-1}$) and $\psi_2 = 2.26$ ($l_1 = 0.885 \mu m$, $k_{s,2} = 2.06 \cdot 10^{-2} \mu m^{-1}$). Let us choose a contour in the $l$-$k_x$ parameter space centering at the analytically estimated EPs coordinates (figures 2b and 2d). After encircling the EPs counterclockwise following said contours one can notice that the two rigorously calculated using RCWA complex poles corresponding to the same square root in (6) swap places (figures 2c and 2e). This interchange of eigenmodes is an intrinsic feature of EPs and proves the existence of EPs. More accurate estimate of the EPs location can be acquired by solving (7) accounting for $\psi$ being the function of $\omega$ and $\theta$. 3
Figure 2. (a) Geometry of a composite structure containing two identical DGs in figure 1. Analytically estimated EPs positions (8) (red asterisks) and trajectories of the encircling in the $l$-$k_x$ parameter space (b, d). Trajectories of the eigenmodes’ complex frequencies when encircling the corresponding EPs (c, e)

4. Conclusion
By means of scattering matrix formalism we derived analytical expressions for the eigenmodes of a composite structure consisting of two dielectric diffraction gratings with Lorentzian profile in reflection. Using said approximations we formulate a criterion for the grating eigenmodes that if satisfied allows formation on two distinct pairs of exceptional points. Rigorous calculation results show eigenmodes interchange upon encircling said EPs.

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