QUANTIZATION OF KÄHLER MANIFOLDS ADMITTING $H$-PROJECTIVE MAPPINGS

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Abstract

We discuss the quantization of mechanical systems for which the Hamiltonian vector fields of observables form the deformation of $n$-dimensional oscillator algebra. Because of this fact these systems can be considered as "deformations" of the harmonic oscillator. The set of above-mentioned mechanical systems are realized at the classical level in the form of Kähler manifolds of constant holomorphic curvature. Such mechanical systems are quantized later with the help of the geometric quantization approach. We also discuss the quantization of more general Kähler manifolds (not necessarily of constant holomorphic curvature) admitting $H$-projective mappings.

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1 Introduction

Let us call the \( n \)-dimensional oscillator algebra \( \text{osc}(n) \) Lie algebra with \((2n + 1)\) basic generators \( T^\alpha, T^{\bar{\beta}} \) and \( T \) which obey the following commutation relations

\[
[T^\alpha, T^{\bar{\beta}}] = [T^\alpha, T^{\bar{\beta}}] = [T^{\bar{\beta}}, T^{\bar{\beta}}] = 0,
\]

\[
[T^\alpha, T] = -iT^\alpha, \quad [T^{\bar{\beta}}, T] = iT^{\bar{\beta}}, \quad \alpha = 1, \ldots, n.
\]

(1)

Let \( A_m \) be the manifold of Lie algebra structures in an \( m \)-dimensional vector space \( V_m \). The curve \( \ell(t) \) in \( A_m \) passing through the algebra \( \ell = \ell(0) \) is called the deformation of the Lie algebra \( \ell \).

It is well known that deformation of one-dimensional oscillator algebra play an important role in various considerations. They are closely related with many interesting physical and mathematical objects, for example, anti-de Sitter quantum mechanics \([4,6]\), symplectic geometry of one-dimensional complex disk \([5,7,14]\) and quantization of spinning particle \([12]\).

This paper is devoted to quantization of the mechanical systems for which the Hamiltonian vector fields of observables form the deformation of \( n \)-dimensional oscillator algebra. Because of this fact these systems can be considered as ”deformations” of the harmonic oscillator. The paper consists of three sections. In the first section we give short review of the geometric quantization procedure. In the second section is constructed the set of above-mentioned mechanical systems which are realized at the classical level in the form of Kähler symplectic manifolds of constant holomorphic curvature \([9]\). Such mechanical systems are quantized later with the help of the geometric quantization approach \([8,12]\).

As it is known the Kähler manifold of constant holomorphic curvature is \( H \)-projectively flat, i.e. it admits \( H \)-projective mappings on the flat space. In the third section of the paper we discuss the quantization of more general Kähler manifolds (not necessarily of constant holomorphic curvature) admitting \( H \)-projective holomorphic mappings.

2 Preliminaries

To start with, we recall some relevant facts about geometric quantization procedure \([8,12]\). Let \((M, \omega)\) be a symplectic manifold. According to Dirac quantization is the linear map \( Q : f \mapsto \hat{f} \) of Poisson (sub)algebra \( C^\infty(M) \) into the set of operators in some (pre)Hilbert space \( \mathcal{H} \) possessing the properties:

1. \( \hat{1} = 1 \);

2. \( \{\hat{f}, \hat{g}\}_h = \frac{i}{\hbar}(\hat{f}\hat{g} - \hat{g}\hat{f}) \);

3. \( \hat{f} = (\hat{f})^* \);
4. for some complete set of functions $f_1, ..., f_n$ the operators $\hat{f}_1, ..., \hat{f}_n$ also form the complete set, where bar is the complex conjugation, star denotes the conjugation of the operator and $\hbar = 2\pi \hbar$ is the Planck constant. The linear map $P : f \to \hat{f}$ possessing the first three properties is called prequantization. For the case $M = T^*M$, $\omega = d\alpha$ prequantization was constructed by Koopman, Van Hove and Segal. It has the form
\begin{equation}
\label{prequantization}
P f = \hat{f} = f - i\hbar V(f) - \alpha(V(f)),
\end{equation}
where vector field $V(f)$ is the Hamiltonian vector field of the function $f \in C^\infty(M)$ defined by the condition
\begin{equation}
\label{hamiltonian_vector_field}
\iota_X\omega = -df,
\end{equation}
where $\iota$ is the internal product. In local coordinates $x^i, i = 1, ..., 2n$ from here we have
\begin{equation}
\label{hamiltonian_vector_field_coordinates}
V(f) = \omega^{ij} \partial_j f \partial_i.
\end{equation}

Let $\mathcal{L}$ be the Hermitian line bundle with connection $D$ and $D$-invariant Hermitian structure $<,>$. Recall that $D$-invariance means that for each pair of sections $\lambda$ and $\mu$ of $\mathcal{L}$ and each real vector field $X$ on $M$ holds
\begin{equation}
\label{D_invariance}
X <\lambda, \mu> = <D_X\lambda, \mu> + <\lambda, D_X\mu>.
\end{equation}

Let $(x, U)$ be local coordinate system on $M$. If $\mu_0$ is nonvanishing section of $\mathcal{L}$ over $U$, then we can identify the space $\Gamma(\mathcal{L}, U)$ of section with $C^\infty(M)$ by the formula $C^\infty(U) \ni \varphi \leftrightarrow \varphi\mu_0 \in \Gamma(\mathcal{L}, U)$. The operator $D_X$ in this case takes the next form
\begin{equation}
\label{D_X_coordinates}
D_X\varphi = X\varphi - i\hbar^{-1}\alpha(X)\varphi,
\end{equation}
where 1-form $\alpha$ is given by the relation
\begin{equation}
\label{alpha}
D_X\mu_0 = -i\hbar^{-1}\alpha(X)\mu_0.
\end{equation}

By comparing (8) and (6) we have the prequantization formula of Souriau-Kostant
\begin{equation}
\label{souriau-kostant_formula}
\hat{f} = f - i\hbar D_V(f).
\end{equation}

The curvature form $\Omega$ of $D$ is defined by the identity
\begin{equation}
\label{curvature_form}
\Omega(X, Y) = \frac{1}{2\pi i} ([D_X, D_Y] - D_{[X,Y]}),
\end{equation}
and locally we have
\begin{equation}
\label{locally_curvature_form}
\Omega = h^{-1}d\alpha.
\end{equation}

**Theorem 1** [8]. *The Souriau-Kostant formula (6) defines the prequantization if and only if the curvature form $\Omega$ coincides with $h^{-1}\omega$. 


The construction of Hilbert space $\mathcal{H}$ in geometric quantization procedure essentially involves the choice of the polarization that is the involutive Lagrange distribution $F$ in $TM \otimes_R \mathbb{C}$. The polarization $F$ is called Kähler if $F \cap \overline{F} = \emptyset$, and the Hermitian form $b(X, Y) = i\omega(X, Y)$ is positively defined for $X \in F$.

If the polarization $F$ on $TM \otimes \mathbb{C}$ is chosen then the Hilbert space $\mathcal{H}$ consists of the sections $\lambda$ of $L$ which are covariantly constant along $F$.

\[ D_X \lambda = 0, \quad \lambda \in \mathcal{L}, \quad X \in F. \] (9)

It is said that the function $f \in C^\infty(M)$ preserves the polarization $F$ if the flow $V(f)$ of $f$ obeys the condition

\[ [V(f), X^\alpha] = a[f]^\alpha_\beta X^\beta, \] (10)

where $a[f]^\alpha_\beta$ are some smooth functions on $M$ and vector fields $X^\alpha$, $\alpha = 1, \ldots, n$ span $F$. Let us consider the particular case when the polarization $F$ is spanned by the complex Hamiltonian vector fields. In this case the functions preserving polarization can be quantized with the help of the next formula [12]:

\[ Qf = -\frac{i\hbar}{2} a[f], \] (11)

where $a[f] = \Sigma_{\alpha=1}^n a[f]^\alpha_\alpha$.

In general situation $f$ it does not preserve the polarization and for quantizing $f$ one have to use the Blattner-Kostant-Sternberg (BKS) kernel which connects representations for different polarizations [8,12]. It is easy to see that even in the case of an $n$-dimensional oscillator the flow of the Hamiltonian

\[ H = \frac{1}{2} \sum_{\alpha=1}^n ((p^\alpha)^2 + (q^\alpha)^2) \] (12)

does not preserve the polarization spanned by the Hamiltonian vector fields of both position $q^\alpha$ and momentum $p^\alpha$ variables. Therefore we must use BKS kernel to quantize $H$. However, when we introduce the complex coordinates

\[ z^\alpha = \frac{1}{\sqrt{2}}(p^\alpha + iq^\alpha), \quad \overline{z^\alpha} = \frac{1}{\sqrt{2}}(p^\alpha - iq^\alpha) \]

and Kähler polarization spanned by the vector fields $V(z^\alpha)$ we can see that $H = \Sigma z^\alpha z^{\overline{\alpha}}$ preserves the polarization and one can use (11) to quantize $H$. As the result they obtain the differential operator $\hat{H}$

\[ \hat{H} = Q_{FB} H = \hbar(z^\alpha \frac{\partial}{\partial z^\alpha} + \frac{n}{2}) \] (13)

on the space $\mathcal{O}(U)$ of holomorphic functions on $U \in \mathbb{C}^n$ which we denote $Q_{FB}$ because the corresponding representation is called Fock-Bargmann representation. In the considered case the representation space $\mathcal{H}$ consists of the sections of $\mathcal{L}$ which have the form $\psi(z)\mu_0$, where $\mu_0$ is a nonvanishing section of $\mathcal{L}$ and $\psi(z) \in \mathcal{O}(U)$. 

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Now we consider the mechanical system whose quantization is connected with generalization of Fock-Bargmann representation. Let \((M, \omega)\) be a \(2n\)-dimensional Kähler manifold with fundamental form \(-\omega\) and positively definite Kähler metric \(g\) to be given in local complex coordinates \((z^\alpha, \bar{z}^{\overline{\alpha}})\) by the formula
\[
\omega = -\omega^\alpha_{\overline{\beta}} dz^\alpha \wedge d\bar{z}^{\overline{\beta}} = -i \partial_{\alpha\overline{\beta}} \Phi \ dz^\alpha \wedge d\bar{z}^{\overline{\beta}},
\]
(14)
\[
g = g_{\alpha\overline{\beta}} dz^\alpha d\bar{z}^{\overline{\beta}} = \partial_{\alpha\overline{\beta}} \Phi \ dz^\alpha d\bar{z}^{\overline{\beta}},
\]
(15)
where \(\Phi\) is the Kähler potential. As the 2-form \(\omega\) is closed and nondegenerate, it defines the symplectic structure on \(M\) and we can consider \((M, \omega)\) as symplectic manifold and the phase space of some mechanical system. The classical observables \([8]\) of such system form Lie algebra \(C^\infty(M)\) with respect to Poisson brackets.

Let us define the Kähler polarization \(F\) on \(TM \otimes \mathbb{C}\) in the form
\[
F = \{ X \in TM \otimes \mathbb{C} | X = \xi_{\alpha} V(z^\alpha), \xi_{\alpha} \in C^\infty(M) \},
\]
(16)
where \(V(z^\alpha) = \omega^\alpha_{\overline{\beta}} \partial_{\overline{\beta}}\) is given by (3). By (10) the function \(f\) preserves the polarization \(F\) if and only if
\[
[V(f), V(z^\alpha)] = a[f]^{\alpha}_{\mu} V(z^\mu)
\]
(17)
where according to (3) and (14) \(V(f) = \omega^{\mu\nu} \partial_{\mu} f \partial_{\nu} - \partial_{\mu} f \partial_{\nu} \), whence
\[
-\omega^{\mu\nu} \partial_{\mu} f \partial_{\nu} - \omega^{\mu\nu} \partial_{\nu} f \partial_{\mu} + \omega^{\mu\nu} \partial_{\mu} \omega^{\rho\sigma} \partial_{\nu} f \partial_{\sigma} + \omega^{\nu\rho} \partial_{\nu} \omega^{\sigma\mu} \partial_{\rho} f \partial_{\sigma} + \omega^{\nu\rho} \partial_{\nu} \partial_{\rho} f \omega^{\sigma\mu} \partial_{\sigma} - \omega^{\mu\nu} \partial_{\nu} \omega^{\sigma\rho} \partial_{\mu} f \partial_{\sigma} - \omega^{\mu\nu} \partial_{\nu} \partial_{\mu} f \omega^{\rho\sigma} \partial_{\sigma} = -a[f]^{\alpha}_{\mu} \omega^{\nu\rho} \partial_{\nu} f \partial_{\rho}.
\]
Equating the corresponding components of vector fields in left and right parts of the last relation we find
\[
\partial_{\tau} \omega^{\nu\rho} \partial_{\tau} f + \omega^{\nu\rho} \partial_{\nu} f \partial_{\rho} = 0.
\]
(18)
Differentiating the equation \(\omega^{\nu\rho} \omega_{\nu\rho} = \delta_{\nu}^{\nu}\) we obtain the identity \(\omega^{\nu\rho} \partial_{\nu} \omega_{\nu\rho} = -\partial_{\nu} \omega^{\nu\rho} \omega_{\nu\rho}\). After this (18) takes the form
\[
\nabla_X \nabla_Y f = 0, \quad X, Y \in F
\]
(19)
where \(\nabla\) denotes the covariant derivation with respect to Kähler metric \(g\). By Theorem 1 we find from (8) \(\omega = d\alpha\) and from (14)
\[
\alpha = -i \partial_{\alpha} \Phi \ dz^\alpha
\]
(20)
modulo to the exact 1-form \(d\beta\).
If we choose both $\mu$ and $\lambda$ in (3) equal to nonvanishing section $\mu_0$, then using (6) we obtain

$$X <\mu_0, \mu_0> = \bar{\hbar}^{-1}(\alpha(X) - \overline{\alpha(X)}) <\mu_0, \mu_0>.$$  

Evaluating this formula on the vector fields $\partial_\alpha$, $\partial_{\overline{\alpha}}$, $\alpha = 1, ..., n$ we find with the help of (20)

$$<\mu_0, \mu_0> = \exp(\bar{\hbar}^{-1}\Phi)$$

up to constant multiplier which we omitted.

Now we determine the sections of Hermitian line bundle $L$ which form the representation space $H$. Being covariantly constant with respect to $D_{X \in F}$ these sections must obey the equation (9):

$$D_{V(z)} \mu = 0, \quad \mu \in \Gamma(L).$$

From here we find with the help of (16), (20) and (21) that $\mu = \psi(z)\mu_0$, where $\psi(z)$ is holomorphic function on $U \subset \mathbb{C}^n$.

If $M$ is contractible then using the Hermitian structure $<,>$ in $L$ we can define the scalar product in $H$ by the formula [8,12]

$$<\mu_1, \mu_2> = \int \psi_1(z)\overline{\psi}_2(z) <\mu_0, \mu_0> \omega^n,$$

where $\mu_1 = \psi_1(z)\mu_0$, $\mu_2 = \psi_2(z)\mu_0$ and $\omega^n$ is n-th external degree of $\omega$. Using (21) we find

$$<\mu_1, \mu_2> = \int \psi_1(z)\overline{\psi}_2(z)\exp(-\bar{\hbar}^{-1}\Phi) \omega^n.$$

In this case the representation Hilbert space associated with polarization $F$ given by (16) can be identified with Fock space $L^2_{hol}(U, dm)$ of holomorphic functions on $U \subset \mathbb{C}^n$ quadratically integrable with the measure $dm = \exp(-\bar{\hbar}^{-1}\Phi)\omega^n$.

Let us consider the Kähler space $K_{2n}$ of constant holomorphic curvature $k$ (see for example [9]). As it is known the space $K_{2n}$ is isometric to the projective space $\mathbb{CP}^n$ for $k > 0$, to the disk $D^R_n = \{z \in \mathbb{C}^n|\|z\| < R\}$ for $k < 0$ and to $\mathbb{C}^n$ for $k = 0$. The metric of the space $K_{2n}$ in the local complex coordinates is

$$ds^2 = 2g_{\alpha\overline{\beta}}dz^\alpha d\overline{z}^\beta, \quad g_{\alpha\overline{\beta}} = \partial_{\alpha\overline{\beta}}\Phi = (A\delta_{\alpha\beta} - k\overline{z}^\alpha z^\beta)A^{-2},$$

$$\Phi = \frac{4}{k}\ln A, \quad A = 1 + \frac{k}{4}\sum_{\alpha}z^\alpha \overline{z}^\alpha.$$

The curve $x(t)$ on Kähler manifold $M$ is called $H$-planar (or holomorphical planar) [11] if it obeys the following equation

$$\nabla_x \chi = a(t)\chi + b(t)J(\chi), \quad \chi \equiv \dot{x}$$

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where $a(t)$, $b(t)$ are some real-valued functions and $J$ is complex structure operator in $TM$.

Let $M$ and $M'$ be two Kähler manifolds. The mapping $f : M \to M'$ is called $H$-projective (see for example [13]) if it transforms $H$-planar curves of $M$ into $H$-planar curves of $M'$.

The contravariant components and nonvanishing Christoffel symbols of the metric (24) in local complex coordinates are given by the formula

$$g^{\alpha\beta} = (\delta^{\alpha\beta} + \frac{k}{4} \bar{z}^{\alpha} \bar{z}^{\beta}) A^2 = -i \omega^{\alpha\beta}, \quad \Gamma^{\alpha}_{\beta\gamma} = -\frac{k}{4} A^{-1} (\bar{z}^\gamma \delta^\beta_\gamma + \bar{z}^\beta \delta^\alpha_\gamma) = \bar{\Gamma}^{\alpha}_{\beta\gamma}$$

From here it is follows that considered metrics are $H$-projectively flat. Now we find from (19)

$$\partial_{\mu} f + 2 A^{-1} \partial_{(\mu} A \partial_{\nu)} f = 0$$

(25)

After substitution $f = WA^{-1}$ in this equation it takes the form

$$\partial_{\mu} W = 0$$

whence

$$W = u_{\mu}(z) \bar{z}^{\mu} + v(z),$$

where $u_{\mu}$ and $v$ are arbitrary holomorphic functions.

In [7] the system of observables

$$\tilde{H} = \frac{1 + \bar{z}}{1 - z} = 1 + \bar{z}, \quad N = \frac{z}{1 - \bar{z}}, \quad \bar{N} = \frac{\bar{z}}{1 - z}$$

(26)

was considered when quantizing 1-dimensional harmonic oscillator. The Hamiltonian vector fields $V(\tilde{H})$, $V(N)$ and $V(\bar{N})$ form the basis of holomorphic isometries Lie algebra in the space $\mathcal{K}_2$ of holomorphic curvature $k = -4$ (see for example [14]). We use the next system of observables

$$H = \frac{\Sigma z^{\alpha} \bar{z}^{\alpha}}{A}, \quad (u_{\alpha} = z^{\alpha}, \quad v = 0),$$

(27)

$$N^{\beta} = \frac{z^{\beta}}{A}, \quad (u_{\alpha} = 0, \quad v = z^{\beta}),$$

(28)

$$\bar{N}^{\beta} = \frac{\bar{z}^{\beta}}{A}, \quad (u_{\alpha} = \delta^{\beta}_{\alpha}, \quad v = 0).$$

(29)

One can easily check that $H$, $N^\alpha$ and $\bar{N}^{\alpha}$ are the solutions of equation (25). The Hamiltonian vector fields of this functions define infinitesimal isometries in $\mathcal{K}_{2n}$ and $H$-projective transformations in the flat Kähler manifold $C^n$. Note that these isometries do not form Lie algebra. In 1-dimensional case $N^1$, $N^\bar{1}$ coincide with $N, \bar{N}$ from (24) and $H$ can be obtained from $\tilde{H}$ by the linear substitution. The using of $H$ is more preferable from the point of view of
the limit transition to the flat space ($k = 0$). In the limit $k \to 0$ we obtain $H = \Sigma z^\nu \bar{z}^\mu$, i.e. the Hamiltonian of harmonic oscillator (12) written in complex coordinates.

The Hamiltonian vector fields of the functions $H, N^\alpha$ and $N^{\overline{\alpha}}$ have the form

$$T \equiv V(H) = \omega^\mu_\nu (\partial_\nu H \partial_\mu - \partial_\nu H \partial_\overline{\mu}) = i(z^\nu \partial_\nu - \bar{z}^\mu \partial_\mu),$$

(30)

$$T^\alpha \equiv V(N^\alpha) = -i\left(\frac{k}{4} z^\alpha z \nu \partial_\nu + \partial_\overline{\nu}\right),$$

$$T^{\overline{\alpha}} \equiv V(N^{\overline{\alpha}}) = i\left(\frac{k}{4} \bar{z}^{\overline{\alpha}} \bar{z} \overline{\nu} \partial_\overline{\nu} + \partial_{\alpha}\right),$$

Using this formulae we can calculate the commutators of the vector fields $T$, $T^\alpha$ and $T^{\overline{\alpha}}$

$$[T^\alpha, T^\beta] = 0, \quad [T^\alpha, T^{\overline{\beta}}] = i\left(\frac{k}{4} \delta^\alpha_\beta T + \overline{T}^{\alpha\overline{\beta}}\right),$$

(31)

$$[T^\alpha, T] = -iT^\alpha, \quad [T^{\overline{\alpha}}, T] = -iT^{\overline{\alpha}},$$

where $T^{\alpha\overline{\beta}} = V(z^\alpha \bar{z}^\beta A^{-1}) = i(z^\alpha \partial_\beta - \bar{z}^\beta \partial_\alpha)$.

The generators $T$, $T^\alpha$ and $T^{\overline{\alpha}}$ do not form a Lie algebra but if we join to them the generator $T^{\alpha\overline{\beta}}$ then we obtain

$$[T^\alpha, T^{\beta\overline{\gamma}}] = i\delta^\alpha_\beta T^{\beta\overline{\gamma}}, \quad [T^{\overline{\alpha}}, T^{\beta\overline{\gamma}}] = -i\delta^{\overline{\alpha}}_{\overline{\beta}} T^{\beta\overline{\gamma}},$$

$$[T^{\alpha\overline{\beta}}, T^{\gamma\delta}] = i(\delta^{\alpha\overline{\beta}}_{\gamma\delta} T^{\alpha\overline{\beta}} - \delta^{\alpha\gamma}_{\overline{\beta}\delta} T^{\overline{\alpha}\gamma}).$$

Because $T^\alpha$, $T^{\overline{\alpha}}$, $T^{\alpha\overline{\beta}}$ are linearly independent and $T = \Sigma T^{\alpha\overline{\alpha}}$ from (29), (30) it follows that $T^\alpha$, $T^{\overline{\alpha}}$ and $T^{\alpha\overline{\beta}}$ form the basis of $n(n + 4)$-dimensional (over $\mathbb{R}$) Lie algebra $\ell(k)$ which is the Lie algebra of infinitesimal isometries of the space $\mathcal{K}_{2n}$ preserving the complex structure.

The Poisson brackets $\{f, g\} = \omega^\alpha_\beta (\partial_\alpha f \partial_\beta g - \partial_\beta f \partial_\alpha g)$ of the functions $H^{\alpha\overline{\beta}} = z^\alpha \bar{z}^\beta A^{-1}$, $N^\alpha = z^\alpha A^{-1}$ and $N^{\overline{\beta}} = \bar{z}^\beta A^{-1}$ are

$$\{N^\alpha, N^{\overline{\beta}}\} = 0, \quad \{N^\alpha, N^\beta\} = i\left(\delta^\alpha_\beta H + N^{\overline{\alpha}\overline{\beta}}\right) - i\delta^\beta_\alpha,$$

$$\{N^\alpha, N^{\overline{\beta}}\} = i\delta^\alpha_\beta N^\beta, \quad \{N^\beta, N^{\overline{\alpha}}\} = -i\delta^\beta_\alpha N^{\overline{\alpha}},$$

$$\{N^{\overline{\alpha}}, N^{\overline{\beta}}\} = i(\delta_{\overline{\alpha}}^\overline{\beta} N^{\overline{\alpha}} - \delta_{\overline{\beta}}^\overline{\alpha} N^{\overline{\beta}}).$$

Note that if we take the limit $k \to 0$ [1], [2] turns to

$$[T^\alpha, T^\beta] = 0, \quad [T^\alpha, T^{\overline{\beta}}] = 0,$$

(33)

$$[T^\alpha, T^{\alpha\overline{\beta}}] = i\delta^\alpha_\beta T^{\alpha\overline{\beta}}, \quad [T^{\overline{\alpha}}, T^{\beta\overline{\gamma}}] = -i\delta^{\overline{\alpha}}_{\overline{\beta}} T^{\beta\overline{\gamma}},$$

$$[T^{\alpha\overline{\beta}}, T^{\gamma\delta}] = i(\delta^{\alpha\overline{\beta}}_{\gamma\delta} T^{\alpha\overline{\beta}} - \delta^{\alpha\gamma}_{\overline{\beta}\delta} T^{\overline{\alpha}\gamma}).$$
and define the $n(n+4)$-dimensional Lie algebra $\ell(0)$. The curve $\ell(k)$ in the manifold of the $n(n+4)$-dimensional Lie algebra structures is the deformation of algebra $\ell(0)$ defined by the commutation relations (33) and containing the $n$-dimensional harmonic oscillator algebra $\text{osc}(n)$ as the Lie subalgebra. That is why we can consider the mechanical system with the phase space $K_{2n}$, symplectic form $\omega$ and the observables $H$, $N^\alpha$ and $N^\overline{\alpha}$ as the "deformation" of classical $n$-dimensional harmonic oscillator.

Now we quantize the classical mechanical systems obtained in the preceding sections using the polarization $F$ defined by (16). We calculate now $a[f] = \Sigma a[f]_\nu^\nu$ (see (11)) for $f = H$, $N^\alpha$ and $N^\overline{\alpha}$. Substituting in (10) $V(z^\alpha) = \omega^\nu_\alpha \partial_\nu$ instead of $X^\alpha$ and $T$, $T^\alpha, T^\overline{\alpha}$ instead of $V(f)$ we find using formulæ (23) and (30)

$$[T, V(z^\beta)] = iV(z^\alpha),$$

$$[T^\alpha, V(z^\beta)] = -i \frac{k}{4} (z^\alpha \delta_\beta^\nu + z^\beta \delta_\nu^\alpha) V(z^\nu),$$

$$[T^\overline{\alpha}, V(z^\beta)] = 0$$

whence

$$a[N^\alpha] = -i \frac{k}{4} z^\alpha (n+1), \quad a[N] = 0, \quad a[H] = in.$$

After this from (27)-(29) using (3), (20) and (22) we obtain the following expressions for differential operators in $L_{2n}^\text{hol}(U, dm)$ which are the quantizations of the observables $H$, $N^\alpha$ and $N^\overline{\alpha}$

$$QH \equiv \hat{H} = \hbar (z^\nu \partial_\nu \psi + \frac{n}{2} \psi),$$

$$QN^\alpha \equiv \hat{N}^\alpha = -\hbar \frac{k}{4} z^\alpha z^\nu \partial_\nu \psi + z^\alpha (1 - \hbar \frac{k}{8} (n+1)) \psi,$$

$$QN^\overline{\alpha} \equiv \hat{N}^\overline{\alpha} = \hbar \partial_\alpha \psi.$$

Let $B : \mathcal{H} \to \mathcal{H}$ be the selfadjoint operator in Hilbert space $\mathcal{H}$. The set $\sigma(B) = \{ \rho \in \mathbb{R} | \exists \mu_\rho \in \mathcal{H} : B \mu_\rho = \rho \mu_\rho \}$ is called the spectrum of the operator $B$. The number $\rho \in \sigma(B) \subset \mathbb{R}$ and section $\mu_\rho \in \mathcal{H}$ are called the eigenvalue of $B$ and eigenstate with eigenvalue $\rho$.

Let us consider the eigenstate $\psi_E$ of the operator $QH$ with the eigenvalue $E$. Equation (34) yields

$$\hbar (z^\nu \partial_\nu \psi_E + \frac{n}{2} \psi_E) = E \psi_E$$

which is equivalent to

$$z^\nu \partial_\nu \psi_E = (E \hbar^{-1} - \frac{n}{2}) \psi_E$$
whence $\psi_E$ is a homogeneous function of $z$ of degree $l = (E\hbar^{-1} - \frac{n}{2})$. Since $\psi_E$ is holomorphic it follows that $l$ is non-negative integer, so that the spectrum of $QH$ is given by

$$E_l = (l + \frac{n}{2})\hbar, \quad l \in \{0\} \cup \mathbb{N}$$

and coincides with the spectrum of the $n$-dimensional harmonic oscillator Hamiltonian ([12]) (see for example [12]).

4 Quantization of Kähler manifolds admitting $H$-projective mappings

In this section we consider the quantization of Kähler spaces admitting $H$-projective mappings onto another Kähler spaces. Let $(M,\omega)$ and $(M',\omega')$ be two Kähler manifolds with fundamental forms $-\omega$ and $-\omega'$. Let $\varrho : M \to M'$ be $H$-projective mapping, it is well known that $H$-projective mapping preserves the complex structure. Therefore we can choose the local complex chart $(z^\alpha, z^\alpha, U)$ in $M$ such that for each point $p \in U$ with coordinates $(z^\alpha, z^\alpha)$ its image $\varrho(p) \in \varrho(U)$ has the same coordinates. The necessary and sufficient condition for the mapping $\varrho : M \to M'$ to be $H$-projective is expressed with the following equation [13]

$$b_{\alpha\beta;\gamma} = 2\varrho'_{\alpha}g_{\beta\gamma}, \quad (35)$$

where

$$b_{\alpha\beta} = e^{2\varphi}g^{\mu\nu}g_{\alpha\mu}g_{\beta\nu}, \quad b_{\alpha\beta} = b_{\beta\alpha} = 0,$$

$$\varrho'_{\alpha} = \partial_{\alpha}\varrho' = \partial_{\mu}\varphi e^{2\varphi}g^{\mu\nu}g_{\alpha\nu},$$

$$J^k_l = J^k_l', \quad J^\nu_l = -J^\nu_l', \quad J^\nu_l = J^\nu_l = 0,$$

$\phi$ is some function on $U$ and semicolon denotes the covariant derivation with respect to $g$.

Because of the positive definiteness of $g$ we can define the complex frame $\{Z_A, Z^A\}$ which is adapted for the Hermitian structure of $M$ [10]. Then for the frame components of $g$ we have

$$g_{AB} = \delta^A_B.$$

Transformations, preserving this form of $g$, belong to the unitary group $U(n)$ for each point $p \in M$. With the help of such transformations we can choose the frame $\{Y_A, Y^A\}$, so that

$$g_{AB} = \delta^A_B, \quad b_{AB} = \lambda_A\delta^A_B, \quad (36)$$
where $\lambda_A = \bar{\lambda_A}$ are the roots of the $\lambda$-matrix $(b - \lambda g)$. Written in the frame (see for example [1,2]) (19) and (35) have the form

$$Y_A^\dagger Y_B^f - \sum_S \gamma_{BSA}^\dagger Y_S f = 0,$$

(37)

$$\delta_{AB} Y_A^\dagger \lambda_A + \sum_S (\gamma_{SAC}^\dagger \lambda_S \delta_{SB} + \gamma_{SBC}^\dagger \lambda_S \delta_{SA}) = 2 \delta_{CB} Y_A^\dagger \phi',$$

(38)

where $\gamma_{SAC}^\dagger (\gamma_{SAC}^\dagger = \gamma_{SAC}) = 0$ are Ricci rotation coefficients of the frame.

Let \{\theta^A, \theta^A\} be the coframe dual to the frame \{Y^A, Y^A\}. Then the connection form $\alpha$ in the Hermitian line bundle $L$ (see §1) can be written in the following form

$$\alpha = -i Y_A^\dagger \Phi \theta^A$$

as in §2. From (14) and (30) it follows $\omega_A^\dagger = \omega^A = i \delta^A_B$. Then for $Y_A = \xi^\mu_A \theta_\mu$ we obtain

$$V(z^\alpha) = -i \sum B \xi_B^\mu Y_B^\dagger,$$

If the function $F \in C^\infty(U)$ preserves polarization $F$ it obeys the condition (17) which we can write as the form

$$[V(f), V(z^\alpha)] = a[f]^\mu_V(z^\mu) = \sum_{A,B} (Y_A^\dagger f Y_A^\dagger \xi_B^\alpha - Y_A^\dagger f Y_B^\dagger \xi_B^\alpha + \xi_A^\alpha (Y_A^\dagger Y_B^\dagger f))Y_B^\dagger,$$

and here we find

$$a[f] = i \xi^A_A \sum_B (Y_B^\dagger f Y_B^\dagger \xi_A^\alpha - Y_B^\dagger f Y_B^\dagger \xi_A^\alpha + \xi_B^\alpha Y_B^\dagger Y_A^\dagger f),$$

where $\xi_B^\alpha$ are components of the inverse to $(\xi^\alpha_B)$ matrix: $\xi^\mu_B \xi_A^\nu = \delta^\mu_\nu$.

At last we can evaluate the differential operator $Qf$ in $L^2_{hol}(U, dm)$ for function $f \in C^\infty(U)$ obeying (27)

$$(Q)f \psi \equiv \hat{f} \psi = \hbar \sum_A Y_A^\dagger f Y_A^\dagger \psi - Y_A^\dagger f Y_A^\dagger \Phi f \psi + f \psi - \frac{i\hbar}{2} a[f] \psi.$$
of the metric. The second case is more interesting. If $\lambda_1 \neq \lambda_2$ then from \cite{3} it follows

$$Y_1 \lambda_1 = Y_1 \lambda_2 = Y_2 \lambda_1 = Y_2 \lambda_2 = 0, \quad \gamma_{\alpha \beta} = Y_1 \ln |\lambda_1 - \lambda_2|,$$

$$\gamma_{\alpha \beta} = -Y_2 \ln |\lambda_1 - \lambda_2|, \quad \gamma_{\alpha \beta} = \gamma_{\alpha \beta}.$$

This work was partially supported by grant 1749 of International Science Foundation and grant RFFI-94-01-01118-a of Russian Foundation for Fundamental Investigations.
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