Abstract. We collect, survey and develop methods of (one-dimensional) stochastic approximation in a framework that seems suitable to handle fairly broad generalizations of Pólya urns.

To show the applicability of the results we determine the limiting fraction of balls in an urn with balls of two colors. We consider two models generalizing the Pólya urn, in the first one ball is drawn and replaced with balls of (possibly) both colors according to which color was drawn. In the second, two balls are drawn simultaneously and replaced along with balls of (possibly) both colors according to what combination of colors were drawn.

Contents

1. Introduction 1
2. The method of stochastic approximation 5
3. Generalized Pólya urns 24
References 39

2000 Mathematics Subject Classification. 60G99, 62L20.
Key words and phrases. Stochastic approximation, unstable equilibrium, stable equilibrium, touchpoint, Generalized Pólya urns.
1. Introduction

1.1. Urns. The urn is a common tool in probability theory and statistics and no student thereof can avoid it. Imagine an urn with \( w \) white and \( b \) black balls. At a beginners level, urns provide examples of how to calculate probabilities, e.g. the probability of drawing a white ball is the number of white balls divided by the total number of balls, i.e. \( w/(w+b) \). If we sample more than one ball, say \( n \) balls, from the urn and count the number of white ball we get examples of the binomial distribution (with parameters \( n \) and \( w/(w+b) \)) and hypergeometric distributions (with parameters \( w+b \), \( n \) and \( w/(w+b) \)), depending on whether we sample with or without replacement. These distributions in turn are very important in statistical theory as they are the key to understanding properties of surveys, e.g. voter polls, such as margins of error.

More aspects of probability theory can be illustrated via urns. Suppose we draw two balls without replacement. The question “what is the probability that the second ball is white?” may introduce the concept of conditional probabilities, as the answer depends on the knowledge we have (or lack) regarding the outcome of the first draw. Urns are so useful that it is hard to imagine an introductory text on probability and statistics not ever mentioning urns of any kind. Any reader with a general interest in urns may consult [JK77].

In 1923 Eggenberger and Pólya introduced a new urn model in [EP23], now commonly referred to as a Pólya urn. An urn has one white and one black ball. We sample one ball and replace it along with one additional ball of the same color, and repeat this procedure. It was thought of as a simple model for a contagious disease. The first draw might correspond to a doctor examining the very first patient of the day. She then has a 50% risk of being infected. Now, the essence of a contagious disease is that the more people have it, the more likely you are to get it, and vice versa. This is now reflected in the model in the following way. Say white ball means “infected”. After we draw a white ball we replace at along with one additional white ball. Hence, the probability of drawing a white ball next time has risen to \( 2/3 \approx 67\% \). It basically means that the more infected patients the doctor gets, the more likely it is that there are yet more to come. Of course, the actual numbers in this example is by no means meant to be “realistic”, it is rather a qualitative model.

We can, however, play with the parameters of the model to better fit some specific situation if needed. First, the initial composition of the urn need not be 1 of each color. A rare disease might correspond to 10,000 black balls and only 1 white. Also, some diseases are more contagious than others. We could incorporate this by stating that we should not add one additional ball, but several, of the same color as the one drawn, corresponding to a faster spread of the disease.
Any reader interested in Pólya urns and generalizations thereof can start with [Mah08].

Our own interest in Pólya-like urn models comes from a similar situation as described above but rather than modelling infectious diseases, it can model how something is learned, e.g. a “brain” trying to learn what to do in a specific situation. Assume for simplicity that there are only two possible ways to act, act 1 and act 2, and that act 1 is the correct way to handle the situation and, as such, leads to a reward of some kind. Act 2 is wrong and has no benefit for our brain. However, at first it is not known to our brain which act is correct (if any). It must somehow learn this by trial and error. A very simple urn model describing how this brain could work is the following. To model an initial state of ignorance, there is one white ball (meaning “do act 1”) and one black ball (meaning “do act 2”) so that the first time it just picks one ball (act) randomly. Then, to model reinforcement learning, there is a rule that if an act is deemed successful, more balls of the color corresponding to the act just performed are added to the urn. In this case; if a white ball is drawn, add, say, one additional white ball and if a black ball is drawn replace it but add no more balls. Now, every time our brain performs the right act it becomes increasingly likely that it will do so again.

As with the previous model, the interest is mainly qualitative. One should not expect that any brain works exactly like an urn. However, it captures some of the dynamics of what one can think of as learning; one tends to be more likely to do things that have proved successful in the past.

Again, we can fine tune the parameters. More colors can mean more ways to act, different reinforcement rules between colors can specify how much benefit the brain gets from the different acts, and so on.

More specifically, it was questions relating to the so called “signaling problems” (communicated by Persi Diaconis and Brian Skyrms) that spawned the authors interest in these matters. These refer to the situation where two (or more) agents try to acquire a common language simultaneously via urns. Recently, one of these problems was solved in [APSV08] which also contains a more thorough description of the problem.

This is some of the motivation behind studying urns evolving along the lines of “draw one or several balls and add more balls according to some prescribed rule depending on the colors of the drawn balls”. It is also the motivation for only looking at the fraction of balls, as these dictate the probabilities of “acting correctly” in models of learning.

1.2. Stochastic approximation algorithms. A stochastic approximation algorithm \( \{X_n\} \) is usually defined as an \( \mathbb{R}^d \)-valued stochastic process adapted to a filtration \( \{\mathcal{F}_n\} \) such that

\[
X_{n+1} = X_n + \gamma_{n+1} [f(X_n) + \epsilon_{n+1}]
\]
holds, where the decreasing “steplengths” $\gamma_n > 0$ satisfy $\sum_n \gamma_n = \infty$ and $\sum_n \gamma_n^2 < \infty$. The random variables $\gamma_n$ can be considered stochastic or deterministic but in either case it is usually assumed that $\{\gamma_n \epsilon_n\}$ is a martingale difference sequence, i.e.

$$\mathbb{E}[\gamma_n \epsilon_n | \mathcal{F}_{n-1}] = 0.$$  

(1.2)

The origin of this subject is [RM51], in which Robbins and Monro considered the following one-dimensional problem; suppose that given an input $x$ to some system in which we get $M(x)$ as output, where $M$ is an unknown function and only observable through white noise. What we really observe is thus $M(x) + \epsilon$, for some random variable $\epsilon$ with $\mathbb{E}\epsilon = 0$. We want to find the input $\theta$ so that $M(\theta) = \alpha$ for some prescribed $\alpha$. For simplicity we might assume that $M$ is nondecreasing and that $M(x) = \alpha$ has a unique solution $\theta$.

A candidate algorithm for finding a sequence $\{X_n\}$ that converges (in some sense) to $\theta$ is to start with some initial input $X_0 = x_0$. Given a value $X_n$, with $n \geq 0$, create the next element by

$$X_{n+1} = X_n + \frac{1}{n+1}(\alpha - M(X_n) + \epsilon_{n+1}),$$

where $-\epsilon_n$ is the noise associated with the $n$th observation. The algorithm works on an intuitive level since whenever $X_n \neq \theta$ then, on average, $X_{n+1}$ takes a step in the direction of $\theta$.

This describes a stochastic approximation algorithm with drift function $f(x) = \alpha - M(x)$ and steplengths $\gamma_n = 1/n$. Of course, there is nothing in the formulation of the problem that demands us to set the steplengths to $1/n$. To demand $\sum_n \gamma_n = \infty$ is natural since this basically means that the algorithm can wander arbitrarily far, thus hopefully finding what it is looking for, and not converging in a trivial manner.

Next, since

$$X_n - x_0 = \sum_{k=1}^{n} \gamma_k (f(X_{k-1}) + \epsilon_k),$$

the requirement $\sum_n \gamma_n^2 < \infty$ makes $\text{Var}X_n$ bounded (under additional assumptions on the error terms and $f$).

In the multidimensional case the heuristics behind the algorithm (1.1) is that it constitutes a discrete time version of the ordinary differential equation

$$\frac{d}{dt}x_t = f(x_t),$$

subject to “noise”. If the noise vanishes for large $n$ it seems plausible that the interpolation of $X_n$ should estimate some trajectory of a solution $x_t$ of (1.3), an idea made precise in [Ben99], where more references may be found. An overview may also be found in [Pem07]. We are however only concerned with the one-dimensional case.

Any reader interested in other aspects of stochastic approximation and applications may find [Bor08] useful.
1.3. How they fit. Stochastic approximation is very well suited for urn models with reinforcements such as the classical Pólya urn and generalizations thereof. If a ball is drawn from an urn and (a bounded number of) balls are added according to some reinforcement scheme, the difference of the proportion of balls before and after is approximately some function of the proportion times 1/n.

As an example, consider the so called Friedman’s urn starting with one ball each of two colors where \( a > 0 \) balls of the same color and \( b > 0 \) balls of the other color are added along with the ball drawn. The proportion \( Z_n \) of either color then satisfies

\[
Z_{n+1} - Z_n = \frac{1}{2 + (n + 1)(a + b)} \left[ f(Z_n) + \text{“noise”} \right],
\]

with the drift function \( f(Z_n) = b(1 - 2Z_n) \) and where “noise” is a martingale difference sequence. This resembles the situation considered by Robbins and Monro and, as the drift always points towards \( 1/2 \), it seems intuitive that this is the point of convergence of \( Z_n \) (in some sense). That this is so will follow from Theorem 1 below. This is “easy” since \( 1/2 \) is the unique solution of \( f(x) = 0 \).

In other urn models \( f(x) = 0 \) may have several roots. There are known results that deal with multiple zeros, although often under the property (1.2). Urn schemes where the total number of balls added each time is not constant tend to lose this property. We will generalize existing results under an assumption slightly weaker than (1.2) and apply the results to generalized Pólya urns.

1.4. A generalized Pólya urn considered as a stochastic approximation algorithm. First, we will show more precisely how stochastic approximation algorithms fit urn schemes by presenting an application which will be studied in more detail below. Consider an urn with balls of two colors, white and black say. Let \( W_n \) and \( B_n \) denote the number of balls of each color, white and black respectively, after the \( n \)’th draw and consider the initial values \( W_0 = w_0 > 0 \) and \( B_0 = b_0 > 0 \) to be fixed. After each draw we notice the color and replace it along with additional balls according to the replacement matrix

\[
\begin{pmatrix}
W & B \\
\begin{pmatrix}
 a & b \\
 c & d \\
\end{pmatrix}
\end{pmatrix},
\]

where \( \min\{a, b, c, d\} \geq 0 \) and \( \max\{a, b, c, d\} > 0 \),

so that, e.g. a white ball is replaced along with \( a \) additional white and \( b \) additional black balls. We demand that \( a, b, c, d \) are nonnegative numbers.

This model is by no means new, chapter 3 of [Mah08] gives a historical overview. Setting \( a = d = 1, b = c = 0 \) and \( W_0 = B_0 = 1 \) gives the classical Pólya urn described in the introduction.

We let \( \mathbf{I}_{n+1}^w \) and \( \mathbf{I}_{n+1}^b \) denote the indicators of getting a white and black ball in draw \( n \), respectively. We set \( T_n = W_n + B_n \) and \( Z_n = W_n/T_n \).
Recursively, \( W_n \) and \( T_n \) evolve as
\[
W_{n+1} = W_n + a \mathbf{1}_{n+1}^W + c \mathbf{1}_{n+1}^B \quad \text{and} \quad T_{n+1} = T_n + (a + b) \mathbf{1}_{n+1}^W + (c + d) \mathbf{1}_{n+1}^B
\]
and hence, with \( \Delta Z_n = Z_{n+1} - Z_n \),
\[
\Delta Z_n = \frac{1}{T_{n+1}} \left[ W_n + a \mathbf{1}_{n+1}^W + c \mathbf{1}_{n+1}^B - Z_n \left( T_n + (a + b) \mathbf{1}_{n+1}^W + (c + d) \mathbf{1}_{n+1}^B \right) \right]
\]
\[
= \frac{1}{T_{n+1}} \left[ \mathbf{1}_{n+1}^W(a - (a + b)Z_n) + \mathbf{1}_{n+1}^B(c - (c + d)Z_n) \right] = \frac{Y_{n+1}}{T_{n+1}}.
\]
Let \( \mathcal{F}_n \) denote the history of the process up to time \( n \), i.e. the \( \sigma \)-algebra \( \sigma(X_1, \ldots, X_n) \). We will define
\[
f(Z_n) = \mathbb{E}[Y_{n+1} | \mathcal{F}_n] = Z_n(a - (a + b)Z_n) + (1 - Z_n)(c - (c + d)Z_n)
\]
\[
= \alpha Z_n^2 + \beta Z_n + c,
\]
where \( \alpha = c + d - a - b \) and \( \beta = a - 2c - d \).
In the form of a stochastic approximation algorithm we can write
\[
\Delta Z_n = \gamma_{n+1} \left[ f(Z_n) + U_{n+1} \right],
\]
where \( U_{n+1} = Y_{n+1} - f(Z_n) \) and \( \gamma_{n+1} = 1/T_{n+1} \).
Now, \( U_{n+1} \) is mean-zero “noise” but if \( a + b \neq c + d \) then in general \( \mathbb{E}_n \gamma_{n+1} U_{n+1} \neq 0 \). However, as will be shown later, \( |\mathbb{E}[\gamma_{n+1} U_{n+1} | \mathcal{F}_n]| = \mathcal{O}(T_n^{-1}) \), and \( T_n \) (usually) grows like \( n \), so this conditional expectation is vanishing fast.

2. The method of stochastic approximation

We will apply the stochastic approximation machinery to fractions and thus limit ourselves to processes in \([0,1]\). This naturally restricts the noise and the function to be bounded.

2.1. Definition. Stochastic variables are throughout assumed to be defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), although we will find no need to make any reference to the underlying measurable space \((\Omega, \mathcal{F})\). We will also consider a filtration \( \{\mathcal{F}_n, n \geq 0\} \) to be given.

To simplify notation, let \( \mathbb{E}_n(\cdot) = \mathbb{E}(\cdot | \mathcal{F}_n) \) and \( \mathbb{P}_n(\cdot) = \mathbb{P}(\cdot | \mathcal{F}_n) \) denote the conditional expectation and probability, respectively, with respect to \( \mathcal{F}_n \).

Definition 1.
A stochastic approximation algorithm \( \{X_n\} \) is a stochastic process taking values in \([0,1]\), adapted to the filtration \( \{\mathcal{F}_n\} \), that satisfies
\[
X_{n+1} - X_n = \gamma_{n+1}[f(X_n) + U_{n+1}],
\]
where \( \gamma_n, U_n \in \mathcal{F}_n, f : [0,1] \to \mathbb{R} \) and the following conditions hold a.s.
(\( i \)) \( c_l/n \leq \gamma_n \leq c_u/n \),
(\( ii \)) \( |U_n| \leq K_u \),
(iii) $|f(X_n)| \leq K_f$, and
(iv) $|\mathbb{E}_n(\gamma_{n+1}U_{n+1})| \leq K_e\gamma_n^2$,

where the constants $c_l, c_u, K_f, K_e$ are positive real numbers. For future reference, set $K_\Delta = c_u(K_f + K_u)$.

**Remark 1.** There is no consensus in the scientific literature as to exactly what constitutes a stochastic approximation algorithm. The main characteristic is that a relation of type (2.1) holds, although the range, measurability etc. of the ingredients $\gamma_n, U_n$ and $f$ may differ. In this section we state results concerning "the" process $\{X_n\}$ which throughout is understood to be a stochastic approximation algorithm according to our definition.

**Remark 2.** The condition (iv) could, in view of condition (i), equally well have been formulated as $|\mathbb{E}_n(\gamma_{n+1}U_{n+1})| \leq K'_e n^{-2}$, for some positive constant $K'_e$. The formulation above arises naturally for the applications toward the end of this paper.

Condition (iv) replaces the more common requirement (1.2), so that $\gamma_n U_n$ does not necessarily have conditional expectation 0, but this expectation is tending to zero quickly. In what follows, we verify that some results known to be true for condition (1.2) carry over to the present situation, as well as present some new results.

### 2.2. Limit points.

In this section we establish that the accumulation points of the process $\{X_n\}$ are a subset of the zeros of $f$, for continuous $f$. This property is well known and the ideas for the proofs of Lemma 2 and Lemma 4 are from [Pem07]. Moreover, Theorem 1 gives an existence result for the limit of the process $\{X_n\}$.

Let $W_n = \sum_1^n \gamma_k U_k$ so that we may write increments of the process $\{X_n\}$ as

$$X_{n+k} - X_n = \sum_{k=n+1}^{n+k} \gamma_k f(X_{k-1}) + W_{n+k} - W_n.$$  

**Lemma 1.** $\{W_n\}$ converges almost surely.

**Proof.** Set $Y_k = \gamma_k U_k$ and $\tilde{Y}_k = \mathbb{E}_{k-1}(\gamma_k U_k)$ and define the martingale $M_n = \sum_1^n (Y_k - \tilde{Y}_k)$. Then

$$\mathbb{E}M_n^2 = \mathbb{E}\left\{\sum_{k=1}^n (Y_k - \tilde{Y}_k)^2\right\} \leq \sum_{k=1}^n \mathbb{E}Y_k^2 \leq \sum_{k=1}^n \frac{c_u^2 K_u^2}{k^2} < \infty$$

so that $M_n$ is an $L^2$-martingale and thus convergent. Next, since

$$\sum_{k=1}^\infty |\tilde{Y}_k| \leq \sum_{k=1}^\infty \frac{c_u^2 K_e}{(k-1)^2} < \infty$$

we must also have that $\sum_{k=1}^{\infty} Y_k$ converges a.s. \qed
Definition 2. Let

\[ X_\infty = \bigcap_{n \geq 1} \{X_n, X_{n+1}, \ldots\} \]

be the set of accumulation points of \( \{X_n\} \).

Lemma 2. Suppose that \( f(x) < -\delta \) (or \( f(x) > \delta \)), for some \( \delta > 0 \), whenever \( x \in (a_0, b_0) \). Then

\[ X_\infty \cap (a_0, b_0) = \emptyset \quad \text{a.s.} \]

and either \( \limsup_n X_n \leq a_0 \) or \( \liminf_n X_n \geq b_0 \).

Proof. The proof follows that of Lemma 2.6 of [Pem07].

Let \([a, b] \subset (a_0, b_0)\) and let \( \Delta = \min\{a - a_0, b_0 - b\} \) be the smallest distance from \([a, b]\) to a point outside \((a_0, b_0)\). Let \( N > 4c_uK_f/\Delta \) be a (random) number large enough so that \( n, m \geq N \) implies

\[ |W_n - W_m| < \Delta/4, \]

which by Lemma 1 is possible a.s. due to the a.s. convergence of \( W_n \). Then we have for any \( n \geq N \)

\[ X_{n+1} - X_n = \gamma_{n+1} f(X_n) + W_{n+1} - W_n < \Delta/2, \]

so that the process after \( N \) cannot immediately go from a point to the left of \( a_0 \) to a point on the right of \( a \). Also, if \( n \geq N \), \( X_n \in (a_0, b] \) and \( X_{n+1}, \ldots, X_{n+k-1} \in (a_0, b_0) \) then

\[ X_{n+k} - X_n = \sum_{j=n+1}^{n+k} \gamma_j f(X_{j-1}) + W_{n+j} - W_n \]

\[ < -\delta \sum_{j=n+1}^{n+k} \gamma_j + \Delta/4. \]

The last step shows that after \( N \) the process cannot increase by more than \( \Delta/4 \) while inside \((a_0, b_0)\), hence cannot escape out to the right. Moreover, since \( \sum_{k>N} \gamma_k \to \infty \) a.s., we must have \( X_{N+k}^* < a_0 \) for some \( k^* > 0 \).

Now, once the process is below \( a_0 \) it will never reach above \( a_0 + \Delta/2 \) in one step. Once inside \((a_0, b_0)\) it will never increase by more than \( \Delta/4 \). Hence, it will never again reach above \( a_0 + 3\Delta/4 < a \). Obviously, we a.s. cannot have both \( \liminf_n X_n \leq a \) and \( \limsup_n X_n \geq b \).

The first results follows from choosing \([a_k, b_k] \subset (a_0, b_0), \) such that \( \bigcup_k [a_k, b_k] = (a_0, b_0) \), so that

\[ \mathbb{P}\{X_\infty \cap (a_0, b_0) \neq \emptyset\} \leq \sum_k \mathbb{P}\{X_n \in [a_k, b_k] \ \text{i.o.}\} = 0. \]

The second results follows by an analogous calculation, yielding

\[ \mathbb{P}\left(\{\liminf_{n \to \infty} X_n \leq a_0\} \cap \{\limsup_{n \to \infty} X_n \geq b_0\}\right) = 0, \]
and the observation that since we thus must have \( \liminf_{n} X_n > a_0 \) or \( \limsup_{n} X_n < b_0 \), we must in fact have either a.s.
\[
\liminf_{n \to \infty} X_n \geq b_0 \quad \text{or} \quad \limsup_{n \to \infty} X_n \leq a_0,
\]
since no accumulation points exist in \((a_0, b_0)\) by the first result.

The case where \( f(x) > \delta \) on \((a_0, b_0)\) is analogous. \( \square \)

Next, we introduce the concept of attainability that we need now and again to rule out trivialities.

**Definition 3.** Call a subset \( I \) **attainable** if for every fixed \( N \geq 0 \) there exists an \( n \geq N \) such that
\[
P(X_n \in I) > 0.
\]

Any "reasonable" stochastic approximation algorithm on \([0, 1]\) should have \( f(0) \geq 0 \) and \( f(1) \leq 0 \), otherwise it seems that the drift could push the processes out of \([0, 1]\). The notion of attainability gives a sufficient condition to ensure this.

**Lemma 3.** Assume that the drift function \( f \) is continuous at the boundary points 0 and 1. If all neighborhoods of the origin are attainable, then \( f(0) \geq 0 \). Similarly, if all neighborhoods of 1 are attainable, then \( f(1) \leq 0 \).

We postpone the proof of this as it will be included in the proof of Theorem 4 on page 19.

**Lemma 4.** Suppose \( f \) is continuous and let \( Q_f = \{x : f(x) = 0\} \) the zeros of \( f \). Then
\[
P\{X_\infty \subseteq Q_f\} = 1.
\]

**Proof.** The continuity of \( f \) makes the sets
\[
A_n = \{x \in (0, 1) : f(x) > 1/n \text{ or } f(x) < -1/n\}
\]
open. Hence, each \( A_n \) is a countable union of open intervals, each on which \( f \) is \( > 1/n \) or \( < -1/n \) and hence where no accumulation points may exist.

The only "loose end" here is the boundary. Suppose e.g. that \( f < 0 \) close to zero (but a priori not at zero). Then it seems that the process might be pushed down to zero (or beyond) even though \( 0 \notin Q_f \). This is however ruled out by Lemma 3, since if neighborhoods of the origin are attainable then \( f(0) \geq 0 \) and if they are not, then the process eventually is bounded away from the origin. Similarly we can not have \( f > 0 \) close to \( x = 1 \) and attainability of this boundary point simultaneously, unless \( f(1) = 0 \).

It is clear that if \( f > 0 \) close to the origin then the process will eventually be bounded away from there (and similarly if \( f < 0 \) close to \( x = 1 \) then the process will be bounded away from 1). \( \square \)

**Theorem 1.** If \( f \) is continuous then \( \lim_{n \to \infty} X_n \) exists a.s. and is in \( Q_f \).
Proof. If \( \lim_{n \to \infty} X_n \) does not exist, we can find two different rational numbers in the open interval \( (\lim\inf_{n \to \infty} X_n, \lim\sup_{n \to \infty} X_n) \).

Let \( p < q \) be two arbitrary different rational numbers. If we can show

\[
P \left( \lim_{n \to \infty} \inf X_n \leq p \cap \lim_{n \to \infty} \sup X_n \geq q \right) = 0,
\]

the existence of the limit will be established and the claim of the theorem
will follow from Lemma 4.

To do this we need to distinguish between whether or not \( p \) and \( q \) are in
the same connected component of \( Q_f \).

Case 1: \( p \) and \( q \) are in not in the same connected component of \( Q_f \).
Since \( Q_f \) is closed and \( f \) continuous, there must exist \( (a, b) \subseteq (p, q) \cap Q_f^c \)

such that \( f \) is non-zero and of the same sign for all \( x \in (a, b) \). By Lemma 2
it is impossible to have \( \lim\inf_{n \to \infty} X_n \leq a \) and \( \lim\sup_{n \to \infty} X_n \geq b \).

Case 2: \( p \) and \( q \) are in the same connected component of \( Q_f \).
Assume that \( \lim\inf_{n \to \infty} X_n \leq p \) and fix an arbitrary \( \epsilon \) in such a way that

\[ 0 < \epsilon < q - p. \]

Recall the notation \( W_n = \sum_{k=1}^{\tau_n} \gamma_k U_k \). We know from Lemma 1 that \( W_n \)
converges a.s., so for some stochastic \( N > 2K_\Delta/\epsilon \), we have that \( n, m \geq N \)
implies \( |W_n - W_m| < \epsilon/2 \). By assumption there is some stochastic \( n \geq N \)
such that \( X_n - p < \epsilon/2 \).

Set

\[ \tau_1 = \inf \{ k \geq n : X_k \geq p \} \quad \text{and} \quad \sigma_1 = \inf \{ k > \tau_1 : X_k < p \} \]

and define, for \( n \geq 1 \),

\[ \tau_{n+1} = \inf \{ k > \sigma_n : X_k \geq p \} \quad \text{and} \quad \sigma_{n+1} = \inf \{ k > \tau_n : X_k < p \}. \]

Now, for all \( k \),

\[
X_{\tau_k} = X_{\tau_k - 1} + \Delta X_{\tau_k - 1} \leq p + K_\Delta/\tau_k < p + \epsilon/2.
\]

Note that \( f(x) = 0 \) when \( x \in [p, q] \). Hence, if \( \tau_k + j - 1 \) is a time before the
exit time of the interval \( [p, q] \) then

\[
X_{\tau_k + j} = X_{\tau_k} + \sum_{\tau_k + 1}^{\tau_k + j} \gamma_i f(X_{i-1}) + W_{\tau_k + j} - W_{\tau_k} = X_{\tau_k} + W_{\tau_k + j} - W_{\tau_k}.
\]

As

\[
|W_{\tau_k + j} - W_{\tau_k}| < \epsilon/2
\]

the process will never be able to reach above \( p + \epsilon \) before \( \sigma_{k+1} \). Since (2.2)
and (2.3) is true for all \( k \), we must have \( \sup_{k \geq n} X_k \leq p + \epsilon \). \( \Box \)
2.3. Categorizing equilibrium points. Any point \( x \in Q_f = \{ x : f(x) = 0 \} \) is called an equilibrium point, or zero, of \( f \). In this paper we shall use the following terminology:

- A point \( p \in Q_f \) is called **unstable** if there exists a neighborhood \( \mathcal{N}_p \) of \( p \) such that \( f(x)(x - p) \geq 0 \) whenever \( x \in \mathcal{N}_p \).

  This means that \( f(x) \geq 0 \) when \( x \) is just above \( p \) and \( f(x) \leq 0 \) when \( x \) is just below \( p \), hence the drift is locally pushing the process away from \( p \) (or not pushing at all).

  If \( f(x)(x - p) > 0 \) whenever \( x \in \mathcal{N}_p \setminus \{ p \} \) we call \( p \) **strictly unstable**.

  If \( f \) is differentiable then \( f'(p) > 0 \) is sufficient to determine that \( p \in Q_f \) is strictly unstable.

- A point will be called **stable** if there exists a neighborhood \( \mathcal{N}_p \) of \( p \) such that \( f(x)(x - p) < 0 \) whenever \( x \in \mathcal{N}_p \setminus \{ p \} \). If \( f \) is differentiable then \( f'(p) < 0 \) is sufficient to determine that \( p \in Q_f \) is stable.

  Locally, the drift pushes the process towards \( p \) from both directions.

- A point \( p \in Q_f \cap (0,1) \) is called a **touchpoint** if there exists a neighborhood \( \mathcal{N}_p \) of \( p \) such that either \( f(x) > 0 \) for all \( x \in \mathcal{N}_p \setminus \{ p \} \) or \( f(x) < 0 \) for all \( x \in \mathcal{N}_p \setminus \{ p \} \). If \( f \) is twice differentiable then \( f(p) = f'(p) = 0 \) and \( f''(p) \neq 0 \) is sufficient to determine that \( p \in (0,1) \) is a touchpoint.

  A touchpoint may be thought of as having one stable and one strictly unstable side. Note that our definition does not allow touchpoints on the boundary.

2.4. Nonconvergence. In this section we narrow down the set of limit points of the process by excluding certain unstable points.

2.4.1. Unstable points with non-vanishing error terms. Here we exclude the unstable zeros of \( f \) as possible limit points, given that the error terms do not vanish at these points. For our applications below this is applicable to zeros of \( f \) in \((0,1)\) as the noise does vanish at the boundary \( \{0,1\} \), a problem addressed in the next section.

  Heuristically, the process \( \{X_n\} \) may arrive at an unstable point \( p \in (0,1) \) by “accident”. To ensure that it does not stay there, we need to know that there is enough noise to push the process out into the drift leading away from \( p \).

  The main result here, Theorem 2 below, is an adaptation of Theorem 3.5 of [Pem88], a sketch of which can be found in [Pem07] and a corresponding multidimensional result in [Pem90], whereby condition (1.2) is replaced by (iv). For results on nonconvergence to more general unstable sets in the multidimensional case the reader is referred to section 9 of [Ben99] and references there.

  To begin with we mention a result which will be used.
Lemma 5. Let \( A \in \mathcal{F}_\infty = \sigma(\cup_n \mathcal{F}_n) \) and suppose there is some integer \( N \) and a real number \( 0 < a \leq 1 \) such that \( n > N \implies \mathbb{P}(A|\mathcal{F}_n) \geq a \). Then \( \mathbb{P}(A) = 1 \).

Proof. The sequence \( \mathbb{P}(A|\mathcal{F}_n) = \mathbb{E}_n(\mathbb{I}_A) \) is an a.s. convergent martingale and \( \lim_n \mathbb{E}_n(A) = \mathbb{E}(\mathbb{I}_A|\mathcal{F}_\infty) = \mathbb{I}_A \) a.s., see Th. 35.6 of Bil95. If this variable is bounded away from zero it must be 1. \( \square \)

Also, the following will prove to be useful.

Lemma 6. Let \( N \geq 0 \) be an integer and \( \tau \) be a stopping time with respect to the filtration in Definition 1, such that \( \tau \geq N \) a.s. Let \( A \in \mathcal{F}_\infty, B = A^c \),

\[
Z_k = Z_k(N, \tau) = |\mathbb{E}_{k-1} \Delta X_{k-1} - \Delta X_{k-1}| \mathbb{I}_{\{N < k \leq \tau\}}, \quad \text{and}
\]

\[
W_m = \sum_{k=N+1}^{m} Z_k.
\]

Suppose that on \( A \) we have \( W_\tau > 0 \) or that we on \( A \) have \( W_\tau < 0 \), then

\[
(2.4) \quad \mathbb{E}_N^2[W_\tau|A] \frac{\mathbb{P}_N(A)}{\mathbb{P}_N(B)} \leq \mathbb{E}_N W_\tau^2 \leq \frac{K_A^2}{N}.
\]

Proof. First, we note that for any \( m > N \)

\[
\mathbb{E}_N W_m^2 \leq \sum_{k=N+1}^{\infty} \mathbb{E}_N [ (\Delta X_{k-1})^2] \leq \sum_{k=N+1}^{\infty} \frac{K_A^2}{k^2} \leq \frac{K_A^2}{N} < \infty,
\]

so that \( W_m \) is an \( L^2 \)-martingale and hence a.s. convergent. Due to the assumption that on \( A \) we have \( W_\tau \) strictly positive, or strictly negative, we must have \( \mathbb{P}_N(A) < 1 \), otherwise we would have \( 0 \neq \mathbb{E}_N W_\infty = \mathbb{E}_N W_\tau \). In particular, this assumption means that \( \mathbb{P}_N(B) > 0 \) so that we can make the following calculation

\[
0 = \mathbb{E}_N W_\infty = \mathbb{E}_N[W_\tau] = \mathbb{E}_N[W_\tau|A] \mathbb{P}_N(A) + \mathbb{E}_N[W_\tau|B] \mathbb{P}_N(B)
\]

\[
\iff -\mathbb{E}_N[W_\tau|B] = \mathbb{E}_N[W_\tau|A] \frac{\mathbb{P}_N(A)}{\mathbb{P}_N(B)}
\]

\[
\Rightarrow \mathbb{E}_N^2[W_\tau|B] = \mathbb{E}_N^2[W_\tau|A] \left[ \frac{\mathbb{P}_N(A)}{\mathbb{P}_N(B)} \right]^2.
\]

Next, since \( \mathbb{E} X^2 \geq \mathbb{E}^2 X \) is true for any random variable \( X \),

\[
\mathbb{E}_N[W_\tau^2] = \mathbb{E}_N[W_\tau^2|A] \mathbb{P}_N(A) + \mathbb{E}_N[W_\tau^2|B] \mathbb{P}_N(B)
\]

\[
\geq \mathbb{E}_N^2[W_\tau|A] \mathbb{P}_N(A) + \mathbb{E}_N^2[W_\tau|A] \frac{\mathbb{P}_N^2(A)}{\mathbb{P}_N(B)}
\]

\[
= \mathbb{E}_N^2[W_\tau|A] \mathbb{P}_N(A) \left( 1 + \frac{\mathbb{P}_N(A)}{\mathbb{P}_N(B)} \right)
\]

\[
= \mathbb{E}_N^2[W_\tau|A] \frac{\mathbb{P}_N(A)}{\mathbb{P}_N(B)}.
\]

\( \square \)
Theorem 2. Assume that there exist an unstable point \( p \) in \( Q_f \), i.e. such that \( f(x)(x - p) \geq 0 \) locally, and that
\[
E_n U_{n+1}^2 \geq K_L
\]
holds, for some \( K_L > 0 \), whenever \( X_n \) is close to \( p \). Then
\[
P\{X_n \to p \} = 0.
\]

Remark 3. The local assumptions \( f(x)(x - p) \geq 0 \) and (2.5) can without loss of generality be assumed, in the proof, to hold globally. Assume that the theorem is proved with global assumptions but that \( f(x)(x - p) \geq 0 \) and (2.5) are only satisfied when \( X_n \) is in a neighborhood \( \mathcal{N}_p \) of \( p \). Couple the process \( \{X_n\} \) after a late time \( N \) to another process \( \{Y_n\} \), such that
\[
Y_N = X_N \quad \text{and} \quad \Delta Y_n = \Delta X_n I_{\{n > N, X_n \in \mathcal{N}_p\}} + \Delta Y'_n I_{\{n > N, X_n \notin \mathcal{N}_p\}}.
\]
If \( \{Y'_n, n > N\} \) is constructed so as to satisfy the global assumptions of Theorem 2, then so does \( Y_n \). Now if \( P(X_n \to p) > 0 \), then the same would be true of \( \{Y_n\} \), contradicting the theorem.

Proof of Theorem 2. Following Pemantle’s proof there are two steps that need verification:

Step 1: Show that there is a \( \beta > 0 \) such that for all \( N \) large enough
\[
P_N \left[ \sup_{k \geq N} |X_k - p| > \beta / \sqrt{N} \right] \geq 1/2.
\]

Step 2: Let
\[
\tau = \inf\{k \geq N : |X_k - p| > \beta / \sqrt{N}\}.
\]
Conditional on \( \{\tau < \infty\} \) show that
\[
P_\tau \left[ \inf_{k \geq \tau} |X_k - p| \geq \beta / 2 \sqrt{N} \right] \geq a,
\]
for some \( a > 0 \) not depending on \( N \).

If (2.6) and (2.8) are true then
\[
P_N(p \notin X_\infty) \geq P_N(\tau < \infty) P_\tau \left( \sup_{k \geq \tau} |X_k - p| > \beta / 2 \sqrt{N} \mid \{\tau < \infty\} \right) \geq \frac{a}{2} > 0,
\]
and the result follows from Lemma 5.

Notation: Throughout the proof we will justify inequalities (as they appear in calculations) by stating that they hold if a parameter is sufficiently large. We will denote this by \( \ast \leq, \ast \geq \) or \( \leq, \geq \) if the inequality holds if \( n \) is sufficiently large, or if it is clear from the context which parameter is referred to, respectively. E.g. \( \frac{10}{n} + \frac{1}{\sqrt{n}} \leq \frac{2}{\sqrt{n}} \), since this is true if \( n \geq 100 \).

Verification of Step 1:
First, in view of Remark 3, we assume that \( f(x)(x - p) \geq 0 \) and \( E_n U_{n+1}^2 \geq K_L \) holds globally.
We aim to show that \( \mathbb{P}_N\{\tau = \infty\} \leq 1/2 \) where \( \tau \) is defined in (2.7).

Recall that \( K_\Delta = c_u(K_f + K_u) \), so that we have the bounds
\[
|\Delta X_n| \leq \frac{K_\Delta}{n + 1} \quad \text{and} \quad (\Delta X_n)^2 \leq \frac{K_\Delta^2}{(n + 1)^2}.
\]

We may assume that \( \tau > N \), otherwise there is nothing to prove. Examine the process \( |X_{\tau \wedge m} - p|^2 \) for \( m > N \). An upper bound on this quantity is given by
\[
\begin{align*}
|X_{\tau \wedge m} - p| &= |X_{\tau \wedge m-1} - p + \Delta X_{\tau \wedge m-1}| \\
&\leq \beta \sqrt{\frac{1}{\tau}} + \frac{K_\Delta}{\tau \wedge m} \leq \beta \sqrt{\frac{N}{N}} + \frac{K_\Delta}{N} \leq 2 \beta \sqrt{\frac{N}{N}},
\end{align*}
\]
and so
\[
G_n(m) = \mathbb{E}_N[(X_{\tau \wedge m} - p)^2] \leq \frac{4\beta^2}{N}.
\]

Next, we make use of the relation
\[
(X_{\tau \wedge m} - p)^2 = [X_{\tau \wedge (m-1)} - p + \Delta X_{m-1}I_{r \geq m}]^2 \\
= (X_{\tau \wedge (m-1)} - p)^2 + 2(X_{\tau \wedge (m-1)} - p)\Delta X_{m-1}I_{r \geq m} \\
+ (\Delta X_{m-1})^2I_{r \geq m}.
\]

Since \( m > N \) we have \( \mathcal{F}_N \subset \mathcal{F}_{m-1} \) so any conditional expectation \( \mathbb{E}_N(\cdot) \) can be calculated as \( \mathbb{E}_N\mathbb{E}_{m-1}(\cdot) \). Hence,
\[
G_N(m) = G_N(m-1) + 2\mathbb{E}_N\{I_{r \geq m}(X_{m-1} - p)\mathbb{E}_{m-1}[\Delta X_{m-1}]\} \\
+ \mathbb{E}_N\{I_{r \geq m}\mathbb{E}_{m-1}[(\Delta X_{m-1})^2]\}.
\]

Now, by the assumption \( \mathbb{E}_n U_{n+1}^2 \geq K_L \) we get
\[
\mathbb{E}_{m-1}[(\Delta X_{m-1})^2] = \mathbb{E}_{m-1}[\gamma_m^2(f(X_{m-1}) + U_m)^2] \\
\geq \frac{c_l}{m}\mathbb{E}_{m-1}[\gamma_m f^2(X_{m-1}) + \gamma_m U_m^2 + 2f(X_{m-1})\gamma_m U_m] \\
\geq \frac{c_l}{m}\left[ \frac{c_l}{m} f^2(X_{m-1}) + \frac{c_l}{m}\mathbb{E}_{m-1}U_m^2 - 2[f(X_{m-1})] \cdot |\mathbb{E}_{m-1}\gamma_m U_m| \right] \\
\geq \frac{c_l^2 {K_L}}{m^2} - \frac{c_l c_u^2 K_f K_e}{m(m-1)^2} \\
\geq \frac{s m c_l^2 K_L}{2m^2}.
\]

Also, by the assumption \( f(x)(x - p) \geq 0 \) we have that
\[
(X_{m-1} - p)\mathbb{E}_{m-1}[\Delta X_{m-1}] \\
= (X_{m-1} - p)f(X_{m-1})\mathbb{E}_{m-1}\gamma_m + (X_{m-1} - p)\mathbb{E}_{m-1}\gamma_m U_m \\
\geq 0 - \frac{|X_{m-1} - p| K_e c_u^2}{(m-1)^2}.
\]
We can now get a lower bound on $G_N(m)$. Continuing (2.10), using (2.11) and combining (2.12) with the fact that $|X_{m-1} - p| < \beta/\sqrt{N}$ when $N < m \leq \tau$, we see that

$$
G_N(m) \geq G_N(m-1) + \frac{c_l^2 K_l}{2m^2} E_N \{ I_{\tau \geq m} \} - 2 \frac{c_u^2 K_u \beta}{\sqrt{N} (m-1)^2} E_N \{ I_{\tau \geq m} \}
$$

$$
\geq G_N(m-1) + \frac{c_l^2 K_l}{4m^2} P_N \{ \tau \geq m \}
$$

(2.13) $$
\geq G_N(m-1) + \frac{c_l^2 K_l}{4m^2} P_N \{ \tau = \infty \},
$$

where the last inequality is true for any $m$ since $\{ \tau \geq m \} \supset \{ \tau = \infty \}$.

Expanding this recursion gives us

$$
G_N(m) \geq G_N(N) + \frac{1}{4} \frac{c_l^2 K_l}{N+1} \sum_{k=N+1}^{m} \frac{1}{k^2}
$$

$$
\geq G_N(N) + \frac{1}{4} \frac{c_l^2 K_l}{N+1} \left( \frac{1}{N+1} - \frac{1}{m+1} \right).
$$

Letting $m \to \infty$ and combining this with (2.9) we have

$$
P_N(\tau = \infty) \leq \frac{16 \beta^2}{c_l^2 K_l} \frac{N+1}{N} \leq \frac{32 \beta^2}{c_l^2 K_l}.
$$

Choosing $\beta \leq \sqrt{c_l^2 K_l/64}$ makes $P_N(\tau = \infty) \leq 1/2$.

**Verification of Step 2:**

Assume throughout that $\{ \tau < \infty \}$, $\tau$ defined by (2.7), is realized through the event $\{ X_\tau > p + \beta/\sqrt{N} \}$. The case when $\{ X_\tau < p - \beta/\sqrt{N} \}$ is similar.

Set

$$
\hat{\tau} = \inf \{ k \geq \tau : X_k < p + \beta/2\sqrt{N} \}.
$$

We aim to show that $P_\tau \{ \hat{\tau} = \infty \} \geq a$, with $a > 0$.

With notation as in Lemma 6 let $A = \{ \hat{\tau} < \infty \}$ and set $Z_k = Z_k(\tau, \hat{\tau})$.

Notice that by conditioning on $\tau$ we may consider it fixed (so that Lemma 5 is indeed applicable).

Observe that by the assumption $f(x)(x-p) \geq 0$ we must have $f(X_{k-1}) \geq 0$ when $\tau < k \leq \hat{\tau}$, since $X_{k-1} - p > 0$ in this case. This gives us

$$
E_{k-1} \Delta X_{k-1} = f(X_{k-1})E_{k-1} \gamma_{k} + E_{k-1} \gamma_{k} U_{k} \geq -\frac{c_u^2 K_u}{(k-1)^2}
$$
and hence on the event $A = \{ \hat{\tau} < \infty \}$,
\[
W_{\hat{\tau}} = \sum_{\tau+1}^{\hat{\tau}} Z_k = \sum_{\tau+1}^{\hat{\tau}} E_{k-1} \Delta X_{k-1} - \sum_{\tau+1}^{\hat{\tau}} \Delta X_{k-1} \\
\geq -\sum_{\tau+1}^{\hat{\tau}} \frac{c_u^2 K_e}{(k-1)^2} - (X_{\hat{\tau}} - X_\tau) \\
\geq -\frac{c_u^2 K_e}{\tau - 1} - p - \frac{\beta}{2\sqrt{N}} + p + \frac{\beta}{\sqrt{N}} \\
\geq \frac{\beta}{2\sqrt{N}} - \frac{c_u^2 K_e}{N - 1} \geq \frac{\beta}{4\sqrt{N}}.
\]
Lemma 6 now gives us
\[
\mathbb{P}_N(\hat{\tau} = \infty) = \mathbb{P}_N(\hat{\tau} < \infty) \geq \frac{\tau \beta^2}{K_\Delta \tau} \geq \frac{\beta^2}{16K_\Delta^2} = a' > 0,
\]
which implies $\mathbb{P}_\tau(\tau = \infty) \geq a'/(1 + a') = a > 0$.

2.4.2. Strictly unstable boundary points. In this section we deal with strictly unstable zeros on the boundary. This present a new problem as the error terms tend to vanish, making Theorem 2 inapplicable. This new result motivated a separate paper [Ren09].

Interestingly, the key ingredient here is an upper bound on how fast the error terms are vanishing when the process gets near the unstable point on the boundary. This is quite the opposite to the situation in Theorem 2, which required a lower bound on the error terms. This may at first seem odd. However, the heuristics is that if the process cannot arrive at the boundary in a finite number of steps, knowing that the error terms get small enough means an increasing tendency for the process to follow the drift.

Theorem 3. Suppose of the process $\{X_n\}$ from Definition 1 that $X_n \in (0,1)$ for all $n$. Assume that $p \in \{0,1\} \cap Q_f$ is such that $f(x)(x-p) > 0$ whenever $x \neq p$ is close to $p$ and that there are positive constants $K'_f, K'_u$ such that a.s.

(2.14) \[ \mathbb{E}_n u_{n+1}^2 \leq K'_u |X_n - p|, \]
(2.15) \[ |f(x)|^2 \leq K'_f |x - p|, \]
(2.16) \[ \lim k \cdot |X_k - p| = \infty, \quad \text{as } k \to \infty. \]

Then $\mathbb{P}\{X_n \to p\} = 0$.

Remark 4. Consider the case $p = 0$ in Theorem 3. In our applications, $X_n$ is the fraction of white balls in an urn. If $W_n$ and $T_n$ denote the number of white balls and the total number of balls in the urn at time $n$ respectively, then $X_n = W_n/T_n$. What is usually easy to verify is that $T_n = \mathcal{O}(n)$, say $T_n \leq Cn$, which implies $nX_n \geq \frac{1}{c}W_n$ so that assumption (2.16) just means that we need that $W_n \to \infty$. 
Proof of Theorem 3. We will, for ease of notation, assume in the proof that \( p = 0 \). Let \( \epsilon > 0 \) be a number such that \( f(x) > 0 \) if \( 0 < x \leq \epsilon \).

The idea of the proof is to show that should the process ever be close to the origin it is very likely that it doubles its value before it decreases to a fraction of its value. So likely in fact, that it will do this time and time again until it reaches above \( \epsilon \).

Consider the process \( \{X_n\} \) after time \( N \). Let \( \lambda > 0 \) be a small constant and let \( a \in (0, 1 - \lambda) \). Define

\[
\tau_1 = \inf\{k \geq N : X_k \geq (2X_N) \land \epsilon\} \quad \text{and} \quad \hat{\tau}_1 = \inf\{k \geq N : X_k \leq aX_N\}.
\]

(2.17)

Since we assume that \( X_n \in (0, 1) \) for all \( n \), we know that \( X_N > 0 \) and thus \( \hat{\tau}_1 > N \). Let \( \tau = \tau_1 \land \hat{\tau}_1 \) and define the two events \( A = \{\hat{\tau}_1 < \tau_1\} \) and \( B = \{\tau_1 < \hat{\tau}_1\} \). Anticipating an application of Lemma 6, we let \( Z_k = Z_k(N, \tau) \) and \( W_m \) as in that lemma.

On the event \( A \) we have for any \( N < k \leq \hat{\tau}_1 \) that \( X_{k-1} < \epsilon \) and hence

\[
\mathbb{E}_{k-1} \Delta X_{k-1} = f(X_{k-1})\mathbb{E}_{k-1} \gamma_k + \mathbb{E}_{k-1} \gamma_k U_k > \frac{\epsilon^2 \lambda}{(k-1)^2}.
\]

Using this estimate gives us, on the event \( A \),

\[
W_\tau = \sum_{N+1}^{\tau_1} \mathbb{E}_{k-1} \Delta X_{k-1} - \sum_{N+1}^{\hat{\tau}_1} \Delta X_{k-1}
\geq - \sum_{k=N+1}^{\infty} \frac{\epsilon^2 \lambda}{(k-1)^2} - (X_{\hat{\tau}_1} - X_N) \geq X_N(1-a) - \frac{\epsilon^2 \lambda}{N-1}
\geq X_N \left( 1-a - \frac{\epsilon^2 \lambda}{X_N(N-1)} \right) \geq X_N(1-\lambda-a),
\]

where the last step is justified by assumption (2.16) if \( X_N N \geq \frac{\epsilon^2 \lambda}{\epsilon} + 1 \).

Next, we use assumptions (2.14) and (2.15) to get

(2.18) \[ \mathbb{E}_{k-1} (\Delta X_k)^2 \leq 2\mathbb{E}_{k-1} \gamma_k^2 \left[f^2(X_{k-1}) + \epsilon^2 \right] \leq \frac{C_1 X_{k-1}}{k^2}, \]

where \( C_1 = 2\epsilon^2 (K'_f + K'_u) \). This in turn gives, since \( X_k < (2X_N) \land \epsilon \leq 2X_N \) whenever \( k < \tau \),

\[
\mathbb{E}_N [W_\tau^2] \leq \mathbb{E}_N \left[ \sum_{N+1}^{\tau} (\Delta X_{k-1})^2 \right] \leq C_1 2X_N \sum_{N+1}^{\infty} \frac{1}{k^2} \leq C_1 \frac{2X_N}{N}.
\]

Since \( f(x) > 0 \) on \( 0 < x < \epsilon \) we know from Lemma 2 that \( X_n \) eventually must leave \( (\zeta, \epsilon) \), for any \( 0 < \zeta < \epsilon \), and hence that \( B = A^c \). So, we can apply Lemma 6 to get

(2.19) \[ \frac{\mathbb{P}_N(B)}{\mathbb{P}_N(A)} \geq \frac{\mathbb{E}_N [W_\tau | A]}{\mathbb{E}_N [W_\tau^2]} \geq \frac{[X_N(1-\lambda-a)]^2 N}{C_1 2X_N} = \frac{[1-\lambda-a]^2}{2C_1} N X_N. \]
Exploiting that \( P(A) + P(B) = 1 \), we see that (2.19) is equivalent to, with \( c_a = (1 - \lambda - a)^2/2C_1 \),

\[
(2.20) \quad P_N(B) \geq \frac{c_aN X_N}{1 + c_aN X_N} = 1 - \frac{1}{1 + c_aN X_N} \geq 1 - \frac{1}{c_aN X_N}.
\]

Notice that this estimate decreases if \( a \) increases.

Next, define stopping times recursively from (2.17)

\[
\tau_{n+1} = \inf\{k \geq \tau_n : X_k \geq (2X_{\tau_n}) \wedge \epsilon\} \quad \text{and}
\]

\[
\hat{\tau}_{n+1} = \inf\{k \geq \tau_n : X_k \leq aX_N\} = \inf\{k \geq \tau_n : X_k \leq a_n X_{\tau_n}\},
\]

where \( a_n \) is some (stochastic) number s.t. \( a_n X_{\tau_n} = aX_N \) and thus \( a_n \leq a \) since either \( X_N \geq \epsilon \) (in which case \( a_n = a \)) or

\[
X_{\tau_n} \geq (2X_{\tau_n-1}) \wedge \epsilon \geq (2^n X_N) \wedge \epsilon > X_N.
\]

(in which case \( \tau_n > N \) and \( a_n < a \)). Define the events \( A_k = \{\tau_k < \hat{\tau}_k\} \) and stopping times \( T_k = \tau_k \wedge \hat{\tau}_k \). Then (2.20) yields, if \( X_{\tau_k} \geq 2^k X_N \),

\[
P(A_{k+1}|A_k, \mathcal{F}_{T_k}) \geq 1 - \frac{1}{c_a \tau_k X_{\tau_k}} \geq 1 - \frac{1}{c_a 2^k N X_N},
\]

since \( \tau_k \geq N \) and \( a_k \leq a \). If \( X_{\tau_k} \geq \epsilon \) then \( P(A_{k+1}|A_k, \mathcal{F}_{T_k}) = 1 \). In either case

\[
P(A_{k+1}|A_k, \mathcal{F}_{T_k}) \geq 1 - \frac{1}{c_a 2^k N X_N},
\]

holds.

Now, \( \bigcap_k A_k \) is a subset of the event that the process after \( N \) reaches above \( \epsilon \). Hence

\[
P_N\left(\sup_{j \geq N} X_j \geq \epsilon\right) \geq P_N\left(\bigcap_{k=1}^{\infty} A_k\right) \geq \prod_{k=1}^{\infty} \left(1 - \frac{1}{c_a 2^{k-1} N X_N}\right)
\]

\[
\geq 1 - \sum_{k=1}^{\infty} \frac{1}{c_a 2^k N X_N} = 1 - \frac{2}{c_a N X_N} \to 1, \quad \text{as} \quad N \to \infty.
\]

This contradicts the assumption that \( P\{X_n \to 0\} > 0 \) since this requires that there is a positive probability that for every prescribed \( \delta > 0 \) there is an \( N_\delta \) such that \( n \geq N_\delta \) implies \( X_n < \delta \). \( \square \)

2.5. Convergence. Now we know when we may exclude some unstable points from the set of limit points. Next, we need to check that stability of a point \( p \) is in fact enough to ensure positive probability of convergence to \( p \). After that we also need to know what happens at a touchpoint. A touchpoint \( p' \) may be thought of as having a “stable side” and an “unstable side”. Intuitively, one may think that convergence to \( p' \) might be possible from the stable side, which is indeed the case.

For the results of the sections to follow we need the notion of attainability, recall Definition 3. This is just to rule out trivialities, as there might exists a stable point in a neighborhood where the process is somehow forbidden to go. Consider e.g. the urn model studied in [HLS80]; an urn has balls of two
colors, white and black say, and at each timepoint \( n \) there is a proportion \( X_n \) of white balls and a ball is drawn and replaced along with one additional ball of the same color. The probability of drawing a white ball is not \( X_n \) but \( h(X_n) \) where \( h : [0, 1] \to [0, 1] \). This yields a drift function of \( f(x) = h(x) - x \). Consider e.g. \( h(x) = 0 \) if \( 0 \leq x \leq 1/2 \) and define \( h \) on \((1/2, 1]\) in such a way that \( h > 0 \) and a stable zero \( p \) of \( f \) exists there. Then the attainability of neighborhoods of this \( p \) depends on the initial condition \( X_0 \). If \( X_0 \leq 1/2 \) then \( p \) can not be reached as \( X_n \) (strictly) decreases to zero.

2.5.1. Stable points. That convergence to stable points is possible is known in related models, e.g. [HLS80] has a similar result as Theorem 4 below. For related multidimensional results, see section 7.1 of [Ben99].

**THEOREM 4.** Suppose \( p \in Q_f \) is stable, i.e. \( f(x)(x - p) < 0 \) whenever \( x \neq p \) is close to \( p \). If every neighborhood of \( p \) is attainable then \( P(X_n \to p) > 0 \).

**PROOF.** \textbf{Case 1:} \( p \in (0, 1) \).

We can find \( a \) and \( b \) such that \( a < p < b \) and \( f > 0 \) on \((a, p)\) and \( f < 0 \) on \((p, b)\). Let

\[
\delta = \min\{b - p, p - a\} \quad \text{and} \quad \epsilon = \delta / 2.
\]

Define \( A_j = \{p - \epsilon \leq X_j < p + \epsilon\} \) and let \( N \) be large enough so that

\[
\frac{C}{\delta^2(N - 1)} \leq \frac{1}{12}, \quad \text{where} \quad C = K^2 + 2K\epsilon^2.
\]

For \( k \geq n \), define \( Y_k = (X_k - p)^2 \). By attainability there exists an \( n \geq N \) such that \( P\{A_n\} > 0 \). Define

\[
\tau = \inf\{k > n : X_k \leq a \text{ or } X_k \geq b\}.
\]

We want to show that \( P(\tau = \infty | A_n) \) is non-zero.

Notice that on \( A_n \) we have \( Y_n \leq \epsilon^2 \) and if \( \tau < \infty \) then \( Y_\tau \geq \delta^2 \).

On \( A_n \), for any \( n < k \leq \tau \), we have \( f(X_{k-1})(X_{k-1} - p) < 0 \), so that

\[
E_n Y_k = E_n (X_{k-1} - p + \Delta X_{k-1})^2
\]

\[
= E_n (X_{k-1} - p)^2 + E_n (\Delta X_{k-1})^2
\]

\[
+ E_n [2\gamma_k(X_{k-1} - p)f(X_{k-1})] + 2E_n(X_{k-1} - p)E_{k-1}\gamma_k U_k
\]

\[
\leq E_n Y_{k-1} + \frac{K^2}{K^2} + \frac{2K\epsilon^2}{(k - 1)^2}
\]

\[
\leq E_n Y_{k-1} + C(k - 1)^{-2}.
\]

Expanding the above recursion gives a bound on the conditional expectation

\[
E(Y_\tau | A_n) \leq E(Y_n | A_n) + \sum_{k=n+1}^{\tau} \frac{C}{(k - 1)^2}
\]

\[
\leq E(Y_n | A_n) + \frac{C}{n - 1} \leq \epsilon^2 + \frac{C}{n - 1}.
\]

(2.21)

Now,

\[
E(Y_\tau | A_n) \geq E_n(Y_\tau I_{\tau < \infty} | A_n) \geq \delta^2 P(\tau < \infty | A_n),
\]
and this fact in combination with (2.21) yields
\[ \mathbb{P}(\tau < \infty | A_n) \leq \left( \frac{\epsilon}{\delta} \right)^2 + \frac{C}{\delta^2(n - 1)} \leq \frac{1}{4} + \frac{1}{12} = \frac{1}{3}. \]

Hence, \( \mathbb{P}(\tau = \infty | A_n) \geq 2/3 \) so there is a positive probability that \( \{X_{n+k}\} \) never leaves \((a, b)\). On the event \( \{\tau = \infty\} \) Lemma 2 implies that \( \lim X_n \in \{a, p, b\} \). Since we can repeat our argument with any \( a' \in (a, p) \) instead of \( a \), and any \( b' \in (p, b) \) instead of \( b \), this implies that \( \lim X_n = p \) (on the event \( \{\tau = \infty\} \)).

**Case 2:** \( p \in \{0, 1\} \)

We will prove the statement for \( p = 0 \), with \( p = 1 \) being analogous. Assume that \( f < 0 \) on \((0, \delta_s] \), for some \( \delta_s > 0 \). Set \( N \) so large that \( 4C/\delta^2(N - 1) \leq 1/12 \), \( \epsilon_s = \delta_s/4 \), \( \mathbb{E}_n(A_j = X_j \leq \epsilon_s) \) and \( \tau = \inf \{k > n : X_k \geq \delta_s/2\} \), where \( n \geq N \) is such that \( \mathbb{P}(A_n) > 0 \). Analogous to Case 1 we calculate \( \mathbb{E}_n(X^2|A_n) \leq \epsilon^2 + C/(n - 1) \) and \( \mathbb{E}_n(X^2|A_n) \geq \frac{\delta^2}{4} \mathbb{P}_n(\tau < \infty|A_n) \), so that \( \mathbb{P}_n(\tau < \infty|A_n) \geq 2/3 \). We know that \( \mathbb{P}(A_n) > 0 \) for some \( n \geq N \) by attainability.

So, there is a positive probability of the event \( B = \{X_{n+k} \leq \delta_s/2 \} \). By Lemma 2 it follows that on this event \( B \) we must have \( \lim X_n \in \{0, \delta_s/2\} \). But we may repeat the argument above, choosing any \( \delta'_s \in (0, \delta_s) \) in place of \( \delta_s \), concluding that \( \lim X_n \in \{0, \delta'_s/2\} \). This makes it clear that given the event \( B \) we must have \( \lim X_n = 0 \).

**Postponed proof of Lemma 3**

We will prove Lemma 3 in the case when the origin is attainable, the case of the other boundary point, \( x = 1 \), is analogous. We assume that \( f \) is continuous at \( x = 0 \) and we need to prove that \( f(0) \geq 0 \). Assume the contrary, i.e. that \( f(0) < 0 \) and hence, by continuity, that \( f < 0 \) on \([0, \delta_s] \), for some \( \delta_s > 0 \).

Recall the notation of Lemma 1 and 2; \( W_n = \sum_1^n \gamma_k U_k \). Lemma 1 ensures that \( \{W_n\} \) converges. Hence, for some large (stochastic) \( N_W \geq n \) we have that \( i, j \geq N_W \) implies \( |W_i - W_j| < \delta_s/2 \).

From identical calculations as in Case 2 above, we can conclude that there is a positive probability of \( \{X_{n+k} < \delta_s/2 \} \). Then
\[ X_{N_W + k} < X_{N_W} + \sum_{j=N_W+1}^{N_W+k} \gamma_j f(X_{j-1}) + \delta_s/2 \to -\infty, \quad \text{as} \; k \to \infty, \]
with positive probability, which is a contradiction. Hence, \( f(0) \geq 0 \). An analogous argument shows that \( f(1) \leq 0 \), and Lemma 3 follows. \( \square \)

2.5.2. **Touchpoints.** Theorem 5 below asserts that as long as the slope toward a touchpoint \( p \) (from the stable side) is not too steep, convergence is possible. \( p \) need in fact not be a touchpoint as the proof only shows that convergence to \( p \) may happen from the stable side. In our applications, the drift function is differentiable and thus the slope tends to zero, making the result applicable.
The method of proof is taken from a similar result of [Pem91], which deals with the same urn model as [HLS80]. The interested reader is advised to read this article for more, and stronger, results on touchpoints, albeit not in this more general setting of stochastic approximation.

**Theorem 5.** Suppose that $p$ is such that $K(p-x) < f(x) < 0$ for some $K < \frac{1}{2cu}$ whenever $x < p$ is close to $p$.

Also, assume the following technical condition:

$\star$ Suppose there exists some $p' > p$ such that for every $N \geq 0$ and every $y \in (p, p')$ there exists an $n \geq N$ such that $\mathbb{P}(X_n > y$ and $X_{n+1} < y) > 0$.

[Or similarly suppose that $0 < f(x) < K(p-x)$ for some $K < \frac{1}{2cu}$ whenever $x > p$ is close to $p$ and assume the existence of a $p' > p$ such that for every $N \geq 0$ and every $y \in (p, p')$ there exists some $n \geq N$ such that $\mathbb{P}(X_n < y$ and $X_{n+1} > y)$]

**Remark 5.** Condition $\star$ states that every point in some neighborhood to the right – the stable side – of $p$ can potentially be down-crossed at some “later” time.

**Proof.** First, without loss of generality we make the global assumption that $f(x) < 0$ for $x \in (0, 1] \setminus \{p\}$ (remember Remark 3). The reason that the origin is not included in the interval where $f$ is negative is Lemma 3. These global assumptions are somewhat superfluous, as we will only be concerned with the behavior of the process to the right of $p$. We will however assume that the inequality $K(p-x) < f(x) < 0$ holds for all $x > p$.

The idea here is to show that

(2.22) \[ \mathbb{P}\{\exists N : n \geq N \text{ implies } X_n > p\} > 0, \]

i.e. that it might happen that the process never again reaches below $p$. Given the event that the process stays above $p$, Lemma 2 implies that the process must converge to $p$ (from above).

The proof is rather technical and there are numerous constants that needs fine tuning in order for everything to work. First, we will use a sequence of times $0 < T_n < T_{n+1} \nearrow \infty$ and a sequence of points $1 > p_n > p_{n+1} \searrow p$ starting with an index $N$ large enough so that $p_N < p'$, where $p'$ is defined by condition $\star$.

We define

\[ \tau_N = \inf\{j > T_N : X_j < p_N < X_{j-1}\} \quad \text{and for } n \geq N \]

\[ \tau_{n+1} = \inf\{j \geq \tau_n : X_j < p_{n+1}\}. \]

Notice that by $\star$ we have $\mathbb{P}(\tau_N < \infty) > 0$, and if $X_l \leq p$ for some $l > \tau_N$ then all stopping times are bounded, namely $\tau_n \leq l$, for all $n \geq N$.

If we can show that $\mathbb{P}(\tau_n > T_n$, for all $n \geq N) > 0$, this will imply (2.22).
For reasons that only become apparent later we set

\[
T_n = \exp \left\{ \frac{n(1-r)}{\gamma K_1} \right\} \quad \text{and} \quad p_n = p + r^n
\]

where \( c_u K < K_1 < 1/2 \) and \( \gamma > 1 \) such that \( \gamma K_1 < 1/2 \) and \( r \in (0,1) \) is to be specified by the demand that

\[
T_n \cdot r^{2n} > 1, \quad \text{i.e.} \quad \left( r \exp \left\{ \frac{1-r}{2\gamma K_1} \right\} \right)^{2n} > 1.
\]

If we let \( g(r) = r e^{(1-r)/2\gamma K_1} \), then \( g(1) = 1 \) and \( g'(1) = 1 - 1/2\gamma K_1 < 0 \), so that we know that there exists an \( r \in (0,1) \) such that \( g(r) > 1 \). From now on we fix\(^1\) such an \( r \). Let

\[
A_n = \{ \tau_n > T_n \} \quad \text{and} \quad B_n = \left\{ \sup_{j > \tau_n} X_j \leq p_n + q_n \right\},
\]

where \( q_n = r^n(\gamma - 1) > 0 \).

Set

\[
Z_k = E_{k-1} \Delta X_{k-1} - \Delta X_{k-1} \quad \text{and for } m > n \quad W_{n,m} = \sum_{k=n+1}^{m} Z_k.
\]

We always have the estimate, due to assuming \( f \leq 0 \),

\[
E_{k-1} \Delta X_{k-1} \leq f(X_{k-1}) c_1 k + \frac{c_2 K e}{(k-1)^2} \leq \frac{c_1}{(k-1)^2},
\]

where \( c_1 = c_2 u K e \), and hence on \( A_n \), for \( j > \tau_n \),

\[
W_{\tau_n,j} = \sum_{k=\tau_n+1}^{\tau_n} E_{k-1} \Delta X_{k-1} - (X_j - X_{\tau_n}) \leq p_n - X_j + \frac{c_1}{\tau_n - 1} \leq p_n - X_j + \frac{c_1}{\lfloor T_n \rfloor}.
\]

We will begin by bounding

\[
P(B_n^c | A_n) = P \left\{ \sup_{j > \tau_n} X_j > p_n + q_n \mid A_n \right\}
\]

\[
\leq P \left\{ \sup_{j > \tau_n} \left( p_n + \frac{c_1}{\lfloor T_n \rfloor} \right) - W_{\tau_n,j} > p_n + q_n \mid A_n \right\}
\]

\[
= P \left\{ \inf_{j > \tau_n} W_{\tau_n,j} < -q_n + \frac{c_1}{\lfloor T_n \rfloor} \mid A_n \right\}.
\]

Since \( 1/ \lfloor T_n \rfloor \leq 2/T_n < 2r^{2n} \) and \( q_n = r^n(\gamma - 1) \) means that we can make \( n \) large enough to ensure that

\[
h_1(n) = -q_n + c_1/ \lfloor T_n \rfloor < 0.
\]

\(^1\)We may assume that \( r \notin \{ 1 - \gamma K_1 \ln m/n : m, n \in \mathbb{N} \} \) so that \( T_n \notin \mathbb{N} \), as it is easier to consistently think of \( T_n \) as a non-integer.
Set $\hat{\tau} = \inf\{k > \tau_n : W_{\tau_n,k} < h_1(n)\}$ and combine the facts that
\[
\mathbb{E}[W_{\tau_n,\infty}^2|A_n] \leq K_\Delta^2/T_n \quad \text{and} \\
\mathbb{E}[W_{\tau_n,\hat{\tau}}^2|A_n] \geq \mathbb{E}[W_{\tau_n,\hat{\tau}}^2] \mathbb{P}(\hat{\tau} < \infty|A_n)
\]
so that we can continue the estimates of (2.25)
\begin{equation}
\mathbb{P}(B_n^c|A_n) \leq \mathbb{P}(\hat{\tau} < \infty|A_n) \leq \frac{K_\Delta^2}{T_n h_1^2(n)}.
\end{equation}

Notice that on the event $B_n$, meaning that $j \geq \tau_n$ implies $X_j \leq p_n + q_n$, we have
\[
\sum_{j=\tau_n+1}^{\tau_n+1} \mathbb{E}_{j-1} \Delta X_{j-1} = \sum_{j=\tau_n+1}^{\tau_n+1} \mathbb{E}_{j-1} \gamma_j (f(X_{j-1}) + U_j)
\begin{align*}
&\geq -\sum_{j=\tau_n+1}^{\tau_n+1} \frac{c_u}{j} K(X_{j-1} - p) - \sum_{j=\tau_n+1}^{\tau_n+1} |\mathbb{E}_{j-1} \gamma_j U_j| \\
&\geq -c_u K(p_n + q_n - p) \sum_{j=\tau_n+1}^{\tau_n+1} \frac{1}{j} - K c_u^2 \sum_{j=\tau_n+1}^{\tau_n+1} \frac{1}{(j-1)^2}.
\end{align*}
\]

Introduce $\kappa_n = c_u K(p_n + q_n - p) = c_u K r^n$. By the previous bound, given $A_n = \{\tau_n > T_n\}$, the events $A_{n+1}^c = \{\tau_{n+1} \leq T_{n+1}\}$ and $B_n$ together imply, first
\[
\sum_{j=\tau_n+1}^{\tau_n+1} \mathbb{E}_{j-1} \Delta X_{j-1} \geq -c_u K r^n \sum_{T_{n+1} < j < T_{n+1}+1} \frac{1}{j} - \frac{K c_u^2}{[T_n]}
\begin{equation}
\begin{aligned}
&\geq -c_u K r^n (\ln T_{n+1} - \ln T_n) - \frac{\kappa_n}{T_n} - \frac{K c_u^2}{[T_n]} \\
&\geq -c_u r^n (1 - r) K/K_1 - \frac{2K c_u^2}{[T_n]},
\end{aligned}
\end{equation}
\]
where [1] is motivated by the fact that if $a,b \in \mathbb{R} \setminus \mathbb{N}$ are such that $0 < a$ and $b > a + 1$, then $\sum_{a < j < b} \frac{1}{j+1} \leq \ln b - \ln a + 1/a$. [2] is motivated by having $n$ large enough since $\kappa_n$ tends to 0 as $n$ grows. Secondly, by setting $\delta_n = p_n - X_{\tau_n} \leq X_{\tau_n-1} - X_{\tau_n} \leq K_\Delta/\tau_n$,
\[
W_{\tau_n,\tau_{n+1}} = \sum_{j=\tau_n+1}^{\tau_n+1} \mathbb{E}_{j-1} \Delta X_{j-1} - (X_{\tau_n+1} - X_{\tau_n})
\begin{align*}
&\geq -c_u r^n (1 - r) K/K_1 - \frac{2K c_u^2}{[T_n]} - p_{n+1} + p_n - \delta_n \\
&\geq r^n (1 - r) [1 - c_u^2 K/K_1] - c_2/ [T_n] = h_2(n),
\end{align*}
\]
where \( c_2 = 2c_u^2 K_e + K_\Delta \). Notice that
\[
h_2(n) = r^n c_3 - c_2 / \lfloor T_n \rfloor \geq r^n c_3 - c_2 2r^{2n},
\]
with \( c_3 = (1 - r)(1 - c_u K/K_1) > 0 \) and \( r < 1 \), so if \( n \) is large enough \( h_2(n) \) is positive.

We have just shown that on the event \( A_n \) we have
\[
B_n \cap A_{n+1}^c \subseteq \{ W_{\tau_n, \tau_{n+1}} \geq h_2(n) \}.
\]
If we let \( \varsigma = \inf \{ m \geq \tau_n : W_{\tau_n, m} \geq h_2(n) \} \) then we also have \( A_{n+1}^c \cap B \subseteq \{ \varsigma < \infty \} \) on \( A_n \). An upper bound on \( \mathbb{P}(\varsigma < \infty | A_n) \) can be calculated analogously to the bound on \( \mathbb{P}(\hat{\tau} < \infty | A_n) \) in (2.26). This yields an upper bound on \( \mathbb{P}(A_{n+1}^c | A_n) \) given by
\[
\mathbb{P}(A_{n+1}^c | A_n) = \mathbb{P}(A_{n+1}^c \cap B_n^c | A_n) + \mathbb{P}(A_{n+1}^c \cap B_n | A_n)
\leq \mathbb{P}(B_n^c | A_n) + \mathbb{P}(\varsigma_n < \infty | A_n)
\leq \frac{K_\Delta^2}{T_n h_2^2(n)} + \frac{K_\Delta^2}{T_n h_2^2(n)}
= \frac{K_\Delta^3}{T_n r^{2n}} \left( \frac{1}{i_1^2(n)} + \frac{1}{i_2^2(n)} \right),
\]
where
\[
i_1^2(n) = \left( \gamma - 1 - \frac{c_1}{r^n \lfloor T_n \rfloor} \right)^2 \to (\gamma - 1)^2,
i_2^2(n) = \left( c_3 - \frac{c_2}{r^n \lfloor T_n \rfloor} \right)^2 \to c_3^2,
\]
as \( n \to \infty \) since \( \frac{1}{r^n \lfloor T_n \rfloor} < r^n 2r^{2n} \to 0 \). Thus we can get the bound
\[
\mathbb{P}(A_{n+1}^c | A_n) \leq \frac{C}{[g(r)]^2n},
\]
for some constant \( 0 < C < \infty \). So,
\[
\mathbb{P}(\tau_n > T_n, \forall n \geq N) = \mathbb{P}(\tau_N < \infty) \prod_{n \geq N} [1 - \mathbb{P}(A_{n+1}^c | A_n)] > 0,
\]
since the product converges as
\[
\sum_{n \geq N} \mathbb{P}(A_{n+1}^c | A_n) \leq \frac{C [g(r)]^2}{[g(r)]^2N (g(r)^2 - 1)} < \infty.
\]
3. Generalized Pólya urns

Now, we will apply these results to determine the limiting fraction of balls in two related urn models. The stochastic approximation machinery makes this fairly easy albeit hard work since the calculations to verify the required properties can be rather lengthy.

3.1. Evolution by one draw. We now return to the model defined in Section 1.4. An urn has $W_n$ white and $B_n$ black balls after the $n$’th draw. Each draw consists of drawing one ball uniformly from the contents of the urn, noticing the color and replacing it along with additional balls according to the replacement matrix

$$
\begin{pmatrix}
W & B \\
B & W
\end{pmatrix}
$$

where $\min\{a, b, c, d\} \geq 0$ and $\max\{a, b, c, d\} > 0$,

so that, e.g. a black ball is replaced along with $c$ additional white and $d$ additional black balls. The initial values $W_0 = w_0 > 0$ and $B_0 = b_0 > 0$ are considered fixed, although this makes no difference to the distribution of the limiting fraction of white balls, except when $a = d$ and $b = c = 0$ as we will see later.

$I_{W_{n+1}}$ and $I_{B_{n+1}}$ denote the indicators of getting a white and black ball in draw $n$, respectively. We define $T_n = W_n + B_n$ and $Z_n = W_n/T_n$ and $Y_{n+1}$ implicitly by $\Delta Z_n = Y_{n+1}/T_{n+1}$, which, after rewriting, gives

$$Y_{n+1} = (c + d - a - b)Z_nI_{W_{n+1}} + [(a - c)I_{W_{n+1}} - (c + d)Z_n] + c.$$

With $Y_{n+1}$ written on this form it is easy to see that the drift function $f(Z_n) = \mathbb{E}_n Y_{n+1}$ is given by

$$f(x) = (c + d - a - b)x^2 + (a - 2c - d)x + c.$$

By defining $U_{n+1} = Y_{n+1} - f(Z_n)$ and $\gamma_n = 1/T_n$ we arrive at the stochastic approximation representation

$$\Delta Z_n = \gamma_{n+1}[f(Z_n) + U_{n+1}].$$

Clearly, $f$ and $U_n$ are bounded since $Z_n \in [0, 1]$, so that conditions (ii) and (iii) of Definition 1 are satisfied.

**Condition (i):**
Recall that $\gamma_n = 1/T_n$. Define

$$t_{\min} = \min\{a + b, c + d\} \quad \text{and} \quad t_{\max} = \max\{a + b, c + d\}.$$

Assume $t_{\min} > 0$, then

$$T_n \leq T_0 + nt_{\max} \implies n\gamma_n \geq \frac{1}{T_0/n + t_{\max}} \geq \frac{1}{T_0 + t_{\max}} > 0$$

$$T_n \geq T_0 + nt_{\min} \implies n\gamma_n \leq \frac{1}{T_0/n + t_{\min}} < \frac{1}{t_{\min}} < \infty.$$
Throughout we will assume that \( t_{\text{min}} > 0 \) and handle the case \( t_{\text{min}} = 0 \) separately.

**Condition (iv):**

To verify condition (iv) of Definition 1 we calculate the expected value of

\[
\frac{U_{n+1}}{T_{n+1}} = \frac{a - (a + b)Z_n - f(Z_n)}{T_n + a + b} \mathbf{1}_{n+1}^W + \frac{c - (c + d)Z_n - f(Z_n)}{T_n + c + d} \mathbf{1}_{n+1}^n.
\]

\[
\mathbb{E}_n \left[ \frac{U_{n+1}}{T_{n+1}} \right] = \frac{a - c + (2c + d - 2a - b)Z_n + (a + b - c - d)Z_n^2}{T_n + a + b} Z_n
\]

\[
+ \frac{(c - a)Z_n + (a + b - c - d)Z_n^2}{T_n + c + d} (1 - Z_n)
\]

\[
= \frac{(a - c)Z_n + (2c + d - 2a - b)Z_n^2 + (a + b - c - d)Z_n^3}{T_n + a + b}
\]

\[
+ \frac{(c - a)Z_n + (2a + b - 2c - d)Z_n^2 + (c + d - a - b)Z_n^3}{T_n + c + d}
\]

\[
= \left[ C_1 Z_n + C_2 Z_n^2 + C_3 Z_n^3 \right] \frac{c + d - a - b}{(T_n + a + b)(T_n + c + d)}
\]

for coefficients \( C_1 = a - c, C_2 = 2c + d - 2a - b \) and \( C_3 = a + b - c - d \). So, there is a constant \( K_e \) (depending on \( a, b, c, d \)) such that

\[
\left| \mathbb{E}_n \left[ \frac{U_{n+1}}{T_{n+1}} \right] \right| \leq \frac{K_e}{T_n^2}.
\]

**The error function**

In order to apply Theorems 2 and 3 via verification of condition (2.5) and (2.14) we need to calculate what we will call the *error function*

\[
(3.5) \quad \mathcal{E}(Z_n) = \mathbb{E}_n U_{n+1}^2.
\]

One sees from (3.2) and (3.3) that

\[
U_{n+1} = Y_{n+1} - f(Z_n) = (\mathbf{1}_{n+1}^W - Z_n) \Psi(Z_n),
\]

where \( \Psi(Z_n) = a - c + (c + d - a - b)Z_n \) so that

\[
\mathcal{E}(Z_n) = \mathbb{E}_n [U_{n+1}^2] = Z_n (1 - Z_n) [\Psi(Z_n)]^2.
\]

**3.1.1. Limit points.** To determine the limit points of the fraction of white balls in this urn model we know from Theorem 1 that we need to look at zeros of

\[
(3.6) \quad f(x) = \alpha x^2 + \beta x + c,
\]

where \( \alpha = c + d - a - b \) and \( \beta = a - 2c - d \). First notice that

\[
(3.7) \quad f(0) = c \geq 0 \quad \text{and} \quad f(1) = -b \leq 0
\]

so that, by the continuity and differentiability of \( f \), there must be a point \( x^* \in [0, 1] \) such that \( f(x^*) = 0 \) and \( f'(x^*) \leq 0 \), see Fig. 3.1.1. A unique zero
must be the convergence point of the process \( \{ Z_n \} \) and if more than one zero exists, we must check which one is stable (if any).

We will look at the possible difficult zeros \( x_u \), i.e. the ones that are unstable and where the error terms are vanishing, in the sense that \( \mathcal{E}(x_u) = 0 \), recall (3.5).

First, \( \mathcal{E}(x) \equiv 0 \) if and only if \( a = c \) and \( b = d \). This is not surprising since there will be no error terms when there is no randomness; this is the urn scheme where \( a \) white and \( b \) black balls are added whatever color is drawn. The drift function is then \( f(x) = -(a + b)x + a \) so that \( x^* = a/(a + b) \) is a unique (stable) zero.

It follows from (3.7) that if at most two zeros exist\(^2\) and one of these is in \((0, 1)\) then that one is stable. Hence, an unstable zero, if it exists, must be at the boundary. By symmetry between colors we need only consider unstable zeros at the origin. To that end set \( c = 0 \) so that \( f(x) = \alpha x^2 + \beta x \), with \( \alpha = d - a - b \) and \( \beta = a - d \), has the property \( f(0) = 0 \). In order for the origin to be unstable we need parameters to make \( f(x) \geq 0 \) when \( x \) is very small. We need to consider two cases:

(i) \( \beta = 0 \) but \( \alpha \geq 0 \). This can only happen if \( a = d \) and \( b = 0 \) i.e. \( f(x) \equiv 0 \). Having \( f \equiv 0 \) makes the sequence \( \{ Z_n \} \) a (bounded) martingale and hence a.s. convergent. This is in fact the classical Pólya-Eggenberger urn model where it is well known that \( Z_n \) converges a.s. to a random variable that has a beta distribution with parameters \( w_0/a \) and \( b_0/a \), see e.g. Theorem 2.2 of [Fre65] or Theorem 3.2 of [Mah08].

(ii) \( \beta > 0 \), i.e. \( a > d \) and \( b \geq 0 \) arbitrary. The origin is an unstable zero and not a convergence point. To see why, we need only notice that \( \mathcal{E}(x) \) certainly can be bounded as (constant)\( x \) and the same is true of \( f(x) \). Considering Remark 4, \( W_n \to \infty \) is clear since \( a > 0 \) does imply that white balls are reinforced infinitely often. Hence, Theorem 3 is applicable.

\(^2\)If there are more than two zeros then \( f \) is identically zero, since \( f \) is a polynomial of order at most 2.
Loose ends
It remains to check what happens if $t_{\text{min}} = \min\{a + b, c + d\} = 0$ (but $a + b + c + d > 0$). If $c + d = 0$ then it is clear that $Z_n \to a/(a + b)$ a.s. which is the unique zero of the drift function $f(x) = -(a + b)x^2 + ax$.

The case $a + b = 0$ is symmetric.

So we have proved the following.

**Theorem 6.** Consider the Pólya urn scheme with replacement matrix (3.1), starting with a positive number of balls of each color. Then, the limit of the fraction of balls exists a.s. Furthermore, apart from the case when $a = d$ and $b = c = 0$, in which the fraction of white balls tends a.s. to a beta distribution, the a.s. limiting random variable has a one point distribution at $x^*$. This point $x^*$ is a zero of (3.6) in $[0, 1]$ and if two such points exists it has the additional property that $f'(x^*) < 0$.

The author does not expect that Theorem 6 is new (although we have never seen it written down). In [Gou89] one finds a similar proposition with less generality as it is demanded that $a + b = c + d$ although it has the benefit that $a$ and $d$ could be negative.

It seems likely that Theorem 3 could be proved using only the embedding method of Athreya and Karlin into multi-type branching processes, see e.g. chapter V of [AK68], but we have not attempted it. However, the model in the next section does not fit this embedding method.

In [HMPS03] a very general extension of the Pólya urn is studied. The urn may have balls of several colors and balls are drawn with a probability according to a function $h$ of the urn content. At any stage a replacement policy is randomly selected from a number of different policies, which may include nonbounded random variables, depending on the colors drawn. However, their convergence result (Theorem 2.1) is inapplicable to several cases in our study due to their assumption of a unique zero of the resulting drift function.

Any reader interested in other types of limit theorems for this model is advised to consult [AK68], [Jan04] and [Jan06].

**3.2. Evolution by two draws.** Again, we will consider an urn with balls of two colors but now we turn our attention to an urn scheme where two balls are drawn simultaneously and reinforcement is done according to which of the three possible combinations of colors this results in. $W_n$ and $B_n$ keep their meaning from the previous section but we now assume that $w_0, b_0 \geq 2$ so that all 3 combinations of draws have positive probability from the start.
The replacement matrix becomes

\[
\begin{pmatrix}
  a & b \\
  c & d \\
  e & f
\end{pmatrix},
\]

where \( \min\{a, b, \ldots, f\} \geq 0 \) and \( \max\{a, b, \ldots, f\} > 0 \).

From this we see, e.g. that if we draw a white and a black ball these will be replaced along with additional \( c \) white and \( d \) black balls.

This model has been studied e.g. in Chapter 10 of [Mah08], where a central limit theorem for the number of white balls (under parameter constraints, see remark after Theorem 7) is presented as well as applications of the model.

Let \( I_{n+1}^{WW}, I_{n+1}^{WB} \) and \( I_{n+1}^{BB} \) be the indicators of the events that draw \( n \) results in two white, one black and one white or two black balls, respectively.

Since balls are drawn simultaneously we have

\[
\begin{align*}
\mathbb{E}_n I_{n+1}^{WW} &= \frac{W_n(W_n - 1)}{T_n(T_n - 1)} = Z_n^2 - \frac{Z_n(1 - Z_n)}{T_n - 1}, \\
\mathbb{E}_n I_{n+1}^{WB} &= \frac{2W_n B_n}{T_n(T_n - 1)} = 2Z_n(1 - Z_n) + \frac{2Z_n(1 - Z_n)}{T_n - 1}, \\
\mathbb{E}_n I_{n+1}^{BB} &= \frac{B_n(B_n - 1)}{T_n(T_n - 1)} = (1 - Z_n)^2 - \frac{Z_n(1 - Z_n)}{T_n - 1}.
\end{align*}
\]

**Remark 6.** If two balls were drawn with replacement we would have the simpler situation

\[
\begin{align*}
\mathbb{E}_n I_{n+1}^{WW} &= Z_n^2, \\
\mathbb{E}_n I_{n+1}^{WB} &= 2Z_n(1 - Z_n) \quad \text{and} \quad \mathbb{E}_n I_{n+1}^{BB} = (1 - Z_n)^2.
\end{align*}
\]

As \( T_n \to \infty \) the rightmost parts of (3.9) suggest that for large \( n \) there is little difference in sampling the two balls with or without replacement. Sampling without replacement will make the calculations messier but with the added benefit that is it easy to see that the result, Theorem 7 below, will remain valid in the simpler case. In the calculations, terms named “\( R_i \)” or “\( R_i(k) \)” are terms that would be zero if we drew with replacement.

The number of white balls \( W_n \) and the total number of balls \( T_n \) evolve recursively as

\[
W_{n+1} = W_n + aI_{n+1}^{WW} + cI_{n+1}^{WB} + eI_{n+1}^{BB} \quad \text{and} \quad T_{n+1} = T_n + (a + b)I_{n+1}^{WW} + (c + d)I_{n+1}^{WB} + (e + f)I_{n+1}^{BB}.
\]

Hence, the increments of the fraction of white balls \( Z_n \) can be calculated as

\[
\Delta Z_n = \frac{1}{T_{n+1}} ([a - (a + b)Z_n]I_{n+1}^{WW} + [c - (c + d)Z_n]I_{n+1}^{WB} + [e - (e + f)Z_n]I_{n+1}^{BB} - Z_n),
\]

which we again denote as \( \Delta Z_n = Y_{n+1}/T_{n+1} \). From the above and (3.9) we calculate

\[
\mathbb{E}_n Y_{n+1} = g(Z_n) + R_n.
\]
where
\[ g(Z_n) = \alpha Z_n^3 + \beta Z_n^2 + \gamma Z_n + e \quad \text{and} \quad R_n = -\frac{Z_n(1 - Z_n)}{T_n - 1}(a - 2c + e + \alpha Z_n) \]
with
\[ \alpha = -a - b + 2c + 2d - e - f, \]
\[ \beta = a - 4c - 2d + 3e + 2f, \quad \text{and} \]
\[ \gamma = 2c - 3e - f. \]

Setting \( U_{n+1} = Y_{n+1} - g(Z_n) \) gives us the stochastic approximation representation
\[ \Delta Z_n = \frac{1}{T_{n+1}}[g(Z_n) + U_{n+1}]. \]
It is clear that (ii) and (iii) of Definition 1 are satisfied.

**Condition (i):** Define
\[ t_{\min} = \min\{a + b, c + d, e + f\} \quad \text{and} \quad t_{\max} = \{a + b, c + d, e + f\}. \]
Assume \( t_{\min} > 0 \), then
\[ T_n \leq T_0 + nt_{\max} \implies n \gamma_n \geq \frac{1}{T_0 + t_{\max}} > 0 \]
\[ T_n \geq T_0 + nt_{\min} \implies n \gamma_n < \frac{1}{t_{\min}} < \infty. \]
Throughout we will assume that \( t_{\min} > 0 \) and handle the case \( t_{\min} = 0 \) separately.

**Condition (iv):**
We write the expectation of \( U_{n+1}/T_{n+1} \) as
\[ \mathbb{E}_n\left[ \frac{Y_{n+1} - g(Z_n)}{T_{n+1}} \right] = \frac{p_1(Z_n)}{T_n + a + b} + \frac{p_2(Z_n)}{T_n + c + d} + \frac{p_3(Z_n)}{T_n + e + f}, \]
where each \( p_j \) in (3.12) has the form
\[ p_j = \sum_{k=0}^{5} C^{(j)}_k Z_n^k + R_j(n) \]
with coefficients \( C^{(j)}_k \) given by Table 1. As an example, \( p_1 \) is calculated from \( \mathbb{E}_n[a - (a + b)Z_n - g(Z_n)]T_{n+1}^{WW} \). Each \( R_j(n) \) is a polynomial in \( Z_n \) divided by \( T_n - 1 \), more precisely
\[ R_1(n) = \frac{Z_n(1 - Z_n)}{T_n - 1}[((e - a) + (\gamma + a + b)Z_n + \beta Z_n^2 + \alpha Z_n^3], \]
\[ R_2(n) = \frac{Z_n(1 - Z_n)}{T_n - 1}[2((c - e) - (\gamma + c + d)Z_n - \beta Z_n^2 - \alpha Z_n^3]; \]
\[ R_3(n) = \frac{Z_n(1 - Z_n)}{T_n - 1}[(\gamma + e + f)Z_n + \beta Z_n^2 + \alpha Z_n^3]. \]
We want to show that $|E_nU_{n+1}/T_{n+1}| = O(T_n^{-2})$. The $R_j$ terms clearly satisfy $|R_j(n)/T_{n+1}| = O(T_n^{-2})$.

Recalling (3.10), and plugging these in, shows that for each $k = 0, 1, \ldots, 5$ we have $C_k^{(1)} + C_k^{(2)} + C_k^{(3)} = 0$. This gives us

$$
(3.12) = \sum_{k=0}^{5} \left[ \frac{C_k^{(1)}}{T_n + a + b} + \frac{C_k^{(2)}}{T_n + c + d} + \frac{C_k^{(3)}}{T_n + e + f} \right] Z_n^k + \mathcal{R}(n)
$$

$$
= \sum_{k=0}^{5} \left[ \frac{[C_k^{(1)} + C_k^{(2)} + C_k^{(3)}]T_n^2 + c_1^{(k)}T_n + c_2^{(k)}}{(T_n + a + b)(T_n + c + d)(T_n + e + f)} \right] Z_n^k + \mathcal{R}(n)
$$

$$
= \sum_{k=0}^{5} \left[ \frac{c_1^{(k)}T_n + c_2^{(k)}}{T_n^3 + c_3^{(k)}T_n^2 + c_4^{(k)}T_n + c_5^{(k)}} \right] Z_n^k + \mathcal{R}(n),
$$

where $\mathcal{R}(n) = \frac{R_1(n)}{T_n + a + b} + \frac{R_2(n)}{T_n + c + d} + \frac{R_3(n)}{T_n + e + f}$ and $c_1^{(k)}, \ldots, c_5^{(k)}$ are some constants whose exact value is of no importance. This makes it clear that

$$
|E_nU_{n+1}/T_{n+1}| = O(T_n^{-2}).
$$

**The error function**

In order to apply Theorems 2 and 3 via verification of condition (2.5) and (2.14) we need to calculate the second moment of

$$
U_{n+1} = Y_{n+1} - g(Z_n) = [a - (a + b)Z_n][I_n^{WW} - Z_n^2] + [c - (c + d)Z_n][I_n^{WB} - 2Z_n(1 - Z_n)] + [e - (e + f)Z_n][I_n^{BB} - (1 - Z_n)^2].
$$

Excruciating calculations show that the error function is given by

$$
\mathcal{E}(Z_n) = E_nU_{n+1}^2 = Z_n(1 - Z_n)\Psi(Z_n) + R_n,
$$
where $R_n$ is a polynomial in $Z_n$ divided by $T_n - 1$ (so this term tends to zero for large $n$) and $\Psi(x)$ is a polynomial of order 4 given by

$$\Psi(x) = (a + b - 2c - 2d + e + f)^2 \cdot x^4 +$$

$$[-2(a + b - 2c - 2d + e + f)(a - 2c + e) + (e + f - a - b)^2 - 4(e + f - c - d)^2] \cdot x^3 +$$

$$[(a - 2c + e)^2 + 2(a - e)(e + f - a - b) - 8(c - e)(e + f - c - d) + 2(e + f - c - d)^2] \cdot x^2 +$$

$$[(a - e)^2 - 4(c - e)^2 + 4(c - e)(e + f - c - d)] \cdot x +$$

$$2(c - e)^2,$$

which is too complicated a formula to work with. Working through the expression one can arrive at the form

$$\Psi(x) = 2x^2(A_x + C_x)^2 + x(1 - x)B_x^2 + 2(1 - x)^2C_x^2,$$

where

$$A_x = (-a - b + 2c + 2d - e - f)x + a - 2c + e,$$

$$B_x = (e + f - a - b)x + a - e \quad \text{and}$$

$$C_x = (e + f - c - d)x + c - e,$$

and the relation

$$A_x = B_x - 2C_x$$

holds.

3.2.1. Limit points. To determine the limiting fraction of white balls we need to examine the zeros of

$$g(x) = \alpha x^3 + \beta x^2 + \gamma x + e,$$

where $\alpha = -a - b + 2c + 2d - e - f$, $\beta = a - 4c - 2d + 3e + 2f$ and $\gamma = 2c - 3e - f$.

We see that

$$g(0) = e \geq 0 \quad \text{and} \quad g(1) = -b \leq 0.$$

By continuity and differentiability there will thus exists a point $x^\ast$ with $f(x^\ast) = 0$ and $f'(x^\ast) \leq 0$. A difference with the previous model, where the urn evolved by a single draw each time, is that we now have more types of equilibrium points. Previously, we only encountered unstable zeros on the boundary, which we resolved with Theorem 3. Now we will also make use of Theorems 2 and 5.

Remark 7. There will be urn schemes with a unique zero, e.g. the case $(a, b, \ldots, f) = (3, 2, 2, 3, 1, 4)$, where

$$g(x) = -3x + 1,$$

and $1/3$ is unique. Another example is given by $(a, b, \ldots, f) = (9, 1, 2, 3, 1, 7)$ where $g(x) = -8(x - 1/2)^3$. 
Remark 8. On attainability and condition ★ of Theorem 5
Consider the replacement matrix 3.8. Let \( \hat{w}_n, \hat{m}_n \) and \( \hat{b}_n \) denote the number of times that draws up to time \( n \) has resulted in the combinations WW, WB and BB, respectively.

Then
\[
Z_n = \frac{W_0 + a\hat{w}_n + c\hat{m}_n + e\hat{b}_n}{T_0 + (a + b)\hat{w}_n + (c + d)\hat{m}_n + (e + f)\hat{b}_n}.
\]

Since
\[
P(\hat{w}_n = i, \hat{m}_n = j, \hat{b}_n = k) > 0
\]
for any combination \( 0 \leq i, j, k \) such that \( i + j + k = n \), it follows that any open set in \([L, U]\) is attainable, where
\[
L = \min \left\{ \frac{a}{a+b}, \frac{c}{c+d}, \frac{e}{e+f} \right\} \quad \text{and} \quad U = \max \left\{ \frac{a}{a+b}, \frac{c}{c+d}, \frac{e}{e+f} \right\}.
\]

Hence, if \( p_s \) is a stable zero of \( f \) and \( p \in [L, U] \) then the conditions of Theorem 4 is fulfilled and \( P(Z_n \to p_s) > 0 \).

We can also see that condition ★ of Theorem 5 is satisfied if \( p_t \) is a touchpoint and \( p_t \in (L, U) \). Furthermore, since the drift is continuously differentiable, the slope will tend to zero close to \( p_t \), making Theorem 5 applicable.

We will not attempt to prove that it will always be the case that neighborhoods of stable points are attainable and that condition ★ is satisfied whenever there is a touchpoint. Our attempts to do so yields too messy calculations, but it seems reasonable that this is true.

In any specific situation there is no problem in verifying these conditions.

Remark 9. There will be urn schemes where the set of stable zeros contains exactly two points and with unstable zeros in \((0, 1)\). For example the case \((a, b, \ldots, f) = (15, 3, 4, 1, 3, 21)\), where
\[
g(x) = -32x^3 + 48x^2 - 22x + 3 = -32(x - 1/4)(x - 1/2)(x - 3/4),
\]
and 1/2 is unstable whereas 1/4 and 3/4 are stable. Notice that \( L = 3/24 < 1/4 \) and \( 3/4 < U = 15/18 \) so that both stable points are possible convergence points by Remark 8.

Remark 10. Touchpoints may arise. If \((a, \ldots, f) = (35, 9, 1, 1, 3, 21)\) then
\[
g(x) = -64x^3 + 80x^2 - 28x + 3 = -64(x - 1/4)^2(x - 3/4),
\]
where 1/4 is a touchpoint and 3/4 is stable. Notice that \( L = 3/24 < 1/4 < 3/4 < U = 35/44 \) so that both the stable point and the touchpoint are possible convergence points by Remark 8.

No unstable equilibrium in \((0, 1)\) with vanishing error terms.
Now we will examine whether there could exist an unstable zero \( x_u \) in \((0, 1)\) such that \( E(x_u) = 0 \), i.e. an unstable zero to which we can not apply Theorem
2. We recall that $\mathcal{E}(x) = x(1-x)\Psi(x) + R_n$ where $R_n = \mathcal{O}(T_n^{-1})$ and $\Psi$ is given in (3.13). Hence, we need to look at points $x \in (0,1)$ such that $\Psi(x) = 0$.

First, $\Psi(x) \equiv 0$ if $A_x \equiv B_x \equiv C_x \equiv 0$. It is easy to calculate and intuitive that this can only happen if $a = c = d$ and $b = d = f$, since this is the case when there is no randomness involved in the urn scheme, a white and black balls are added whatever is drawn. Then $g(x) = -(a + b)x + a$ so that $x^* = a/(a + b)$ is unique.

Next, we need to solve $\Psi(x) = 0$ for $0 < x < 1$. We will do this by going through the cases when exactly one of $C_x, B_x$ or $A_x$, or none, is zero. This suffices due to relation (3.14).

Note, since the drift function is a polynomial of order at most 3 with boundary condition (3.16), the only time when an unstable $x^* \in (0,1)$ has $g'(x^*) = 0$ is when $g(x) \equiv 0$. This case is special and will be treated below. Hence, we need only verify that if $x^* \in (0,1)$ is a point where $\mathcal{E}$ vanishes, then $x^*$ is not strictly unstable, i.e. that $g'(x^*) \leq 0$.

The case $C_x \equiv 0$

If $C_x \equiv 0$, i.e. $c = e$ and $f = d$, then from (3.14) we have that $A_x = B_x = (e + f - a - b)x + a - e$. Then

$$\Psi(x) = 2x^2A_x^2 + x(1-x)A_x^2 = x(1 + x)A_x^2.$$ 

We assume $B_x$ is not identically zero, so if $e + f = a + b$ then $B_x = a - e$ is never zero. If $e + f \neq a + b$ then $B_x = 0$ for $x^* = \frac{e-a}{e+f-a-b} = \frac{1}{D}$. The drift function $g(x)$ is now

$$g(x) = (-a - b + e + f)x^3 + (a - e)x^2 - (e + f)x + e = Dx^3 - Nx^2 - (e + f)x + e,$$

so that $g(x^*) = e - \frac{(e+f)(e-a)}{e+f-a-b}$ and thus $g(x^*) = 0$ if $af = eb$. If $e \neq 0$ then

$$g'(x^*) = \frac{(e-a)^2}{e+f-a-af/e} - (e+f) = (1-a/e)\frac{e^2}{e+f} - (e+f) \leq 0.$$ 

If $e = 0$, then $af = 0$ so that $x^* = 0$ or $x^* = a/(a + b)$. In the latter case we have $g'(\frac{a}{a+b}) = -a^2/(a + b) \leq 0$.

The case $B_x \equiv 0$

If $B_x \equiv 0$ then $a = e$ and $f = b$, $A_x = -2C_x$ and

$$\Psi(x) = 2x^2(-C_x)^2 + 2(1-x)^2C_x^2 = C_x^2(4x^2 - 4x + 2),$$

where $4x^2 - 4x + 2$ has no roots in $(0,1)$. We assume that $A_x = -2C_x$ is not identically zero so if $a + b = c + d$ then $A_x = 2(a - c)$ is never zero. If
We assume that $B = 0$ simultaneously. A common zero $x$ so that $g(x) = 0$ imposes that $A = 0$.

So that $g(x^*) = a - \frac{(a+b)(a-c)}{a+b-c-d} = 0$ if $cb = ad$. If $a \neq 0$ then

$$g'(x^*) = -2 \frac{(a-c)^2}{a+b-c-cb/a} + a - 2c - b$$

$$= -2(1 - c/a) \frac{a^2}{a+b} + a - b - 2c = -\frac{a^2 + b^2 + 2bc}{a+b} \leq 0.$$

If $a = 0$ then $cb = 0$ so $x^* = 0$ or $x^* = c/(c + d)$. In the latter case we have

$$g'(x^*) = -2cd/(c + d) \leq 0.$$

The case $A \equiv 0$

If $A \equiv 0$ then $2c = a + e$, $2d = b + f$, $B_x = 2C_x$ and

$$\Psi(x) = 2x^2C_x^2 + (1 - x)(2C_x)^2 + 2(1 - x)^2C_x = 2C_x^2.$$

We assume that $B_x = 2C_x$ is not identically zero so if $e + f - a - b = 0$ then $B_x = a - e$ is never zero. If $e + f \neq a + b$ then $B_x \neq 0$ for $x^* = \frac{e-a}{e+f-a-b}$ and

$$g(x) = (e + f - a - b)x^2 + (a - 2e - f)x + e$$

$$= Dx^2 - (N + e + f)x + e,$$

so that

$$g'(x^*) = N - e - f = -a - f \leq 0.$$

The case $A_x$, $B_x$, $C_x$ not $\equiv 0$

Suppose $a \neq e \neq c$, $a+b \neq e+f \neq c+d$, $2c \neq a+c$ and $a+b+c+d \neq 2c+2d$.

The only chance of having $\Psi(x) = 0$, for $x \in (0, 1)$, is for $A_x$, $B_x$ and $C_x$ to be zero simultaneously. A common zero $x^*$ of $A_x$, $B_x$ and $C_x$ when none of these is identically zero imposes

$$x^* = \frac{e-a}{e+f-a-b} = \frac{e-c}{e+f-c-d} = \frac{a-2c+e}{a+b-2c-2d+e+f}$$

which is the case whenever $e(b-d) + c(f-b) + a(d-f) = 0$.

Setting $x^* = (e-a)/(e+f-a-b)$ and $d = [b(e-c) + f(c-a)]/(e-a)$ and using Maple yields the simple expression

$$g(x^*) = \frac{af - eb}{e + f - a - b},$$

which is zero if $af = eb$. The derivative simplifies to

$$g'(x^*) = -a\frac{e - a}{e - a + f - b} - (f + 2c)\frac{f - b}{f - b + e - a}.$$
If \( a = 0 \) then \( be = 0 \) so \( g'(x^*) = -(f + 2c) \) or \( -(f + 2c) \frac{f}{f + e} \). If \( a \neq 0 \) we can write \( f = be/a \) and
\[
g'(x^*) = -a - \frac{a}{a + b} - (f + 2c) \frac{b}{a + b} \leq 0.
\]

So we can determine that there is no strictly unstable zero of \( g(x) \) in \((0, 1)\) such that the error terms are vanishing.

**Unstable boundary points**

Next, we check the boundary. By symmetry of colors we need only consider an unstable zero of \( g \) at the origin. To that end set \( e = 0 \) so that \( g(x) = \alpha x^3 + \beta x^2 + \gamma x \) has \( g(0) = 0 \). We check the cases where \( g \geq 0 \) close to the origin:

(a) \( \gamma = 0, \beta = 0 \) and \( \alpha \geq 0 \) is only possible if \( 2c = f, 2d = a \) and \( b = 0 \) so that \( \alpha = 0 \) and hence \( g(x) \equiv 0 \). It is then, in some sense, the 2-draw version of the classical Pólya urn. The special case of \( 2c = 2d = a = f \) has been studied in [CW05] and they show that the limiting variable has an absolutely continuous distribution. They also include a simulation study that indicates that the limiting distribution “resembles” the beta distribution.

By Theorem 2, we may only conclude that the limiting distribution has no point masses on \((0, 1)\).

(b) \( \gamma = 0 \) and \( \beta > 0 \), i.e. \( 2c = f \) and \( a > 2d \).

(c) \( \gamma > 0 \), i.e. \( 2c > f \).

In both (b) and (c) the conditions of Theorem 3 are satisfied. \( kZ_k \) can be made arbitrarily big since \( W_k \to \infty \). In the second case this is due to \( c > 0 \).

Since the combination \( WB \) will always be drawn infinitely often, it follows that white balls will tend to infinity. In the first case, either \( c > 0 \) and we are done, or \( c = f = e = 0 \) and \( a > 0 \) so that the combination \( WW \) will be drawn infinitely often.

Also, both \( g(x) \) and \( E(x) \) behave like \((\text{constant}) \cdot x\) when \( x \) is close to zero. Thus, convergence to a strictly unstable boundary point is impossible.

**Loose ends**

Here we examine what happens if \( t_{\text{min}} = \min\{a + b, c + d, e + f\} = 0 \).

1. \( c = d = e = f = 0 \) has drift \( g(x) = -(a + b)x^3 + ax^2 \). It is clear that \( Z_n \) does converge to \( a/(a + b) \) a.s. (i.e. the stable zero of \( g \)) since any draw that alters the urns composition does so by increasing the number of white balls by \( a \) and the numbers of black balls by \( b \).

2. \( a = b = c = d = 0 \) is symmetric to the above case \( c = d = e = f = 0 \); \( Z_n \) will converge to the stable zero of \( g \).

3. \( a = b = e = f = 0 \) has
\[g(x) = 2(c + d)x^3 - (4c + 2d)x^2 + 2cx = 2x(1 - x)[c - (c + d)x]
\]
with \( c/(c + d) \) being the only stable zero. It is clear that \( Z_n \) converges to this point a.s.
4. $a = b = 0$ and $\min\{c + d, e + f\} > 0$ so that nothing happens when two white balls are drawn (except that they are replaced in the urn). Let $\tau_1 = \inf\{k \geq 1 : T_k > T_{k-1}\}$ and for $n \geq 1$

$$\tau_{n+1} = \inf\{k > \tau_n : T_k > T_{\tau_n}\}.$$ 

By looking at the sequence $Z_{\tau_n}$ we ignore the times when two white balls are drawn. This makes no difference to the limit since $Z_{\tau_n+k} = Z_{\tau_n}$ for $0 \leq k < \tau_{n+1} - \tau_n$.

However, since

$$E_{\tau_n} \Gamma_{\tau_{n+1}}^{BB} = \frac{B_{\tau_n}(B_{\tau_n}-1)}{T_{\tau_n}(T_{\tau_n}-1)} = \frac{B_{\tau_n}-1}{B_{\tau_n} + 2W_{\tau_n} - 1}$$

and

$$E_{\tau_n} \Gamma_{\tau_{n+1}}^{WB} = \frac{2W_{\tau_n}}{W_{\tau_n} + 2B_{\tau_n} - 1},$$

it is more convenient to define $\hat{T}_n = B_{\tau_n} + 2W_{\tau_n} - 1$ and $\hat{Z}_n = 2W_{\tau_n}/\hat{T}_n$. It is straightforward to compute $\Delta \hat{Z}_n = \hat{Y}_{n+1}/\hat{T}_{n+1}$ where

$$\hat{Y}_{n+1} = \Gamma_{\tau_{n+1}}^{WB}(2c - (2c + d)\hat{Z}_n) + \Gamma_{\tau_{n+1}}^{BB}(2e - (2e + f)\hat{Z}_n),$$

so that

$$\hat{g}(\hat{Z}_n) = E_{\tau_n} \hat{Y}_{n+1} = (2e + f - 2c - d)\hat{Z}_n^2 + (2c - 4e - f)\hat{Z}_n + 2e.$$  

We also get the error function, with $\hat{U}_{n+1} = \hat{Y}_{n+1} - g(\hat{Z}_n)$, as

$$E(\hat{Z}_n) = E_{\tau_n}(\hat{U}_{n+1}^2) = [(2e + f - 2c - d)\hat{Z}_n + 2c - 2e] \hat{Z}_n(1 - \hat{Z}_n).$$

Since we have assumed $\min\{c + d, e + f\} > 0$ we know that (i) of Definition (1) is satisfied, and one easily verifies (ii)-(iv).

As $g(0) = 2e \geq 0$ and $g(1) = -d \leq 0$ any unstable zero of $\hat{g}$ must be on the boundary $\{0, 1\}$. We can apply Theorem 3 to conclude that $\hat{Z}_n$ will not converge to an unstable boundary point.

So, we have the original process $Z_n = W_n/(W_n + B_n)$ described by the drift function

$$g(x) = (2c + 2c - e - f)x^3 + (-4c - 2d + 3e + 2f)x^2 + (2c - 3e - f)x + e$$  

$$= (1 - x)[(-2c - 2d + e + f)x^2 + (2c - 2e - f)x + e]$$

and our "new" process $\hat{Z}_n = W_{\tau_n}/(2W_{\tau_n} + B_{\tau_n} - 1)$ described by

$$\hat{g}(x) = (-2c - d + 2e + f)x^2 + (2c - 4e - f)x + 2e.$$  

Now, as $T_n \to \infty$, the limit $x = \lim_n Z_n$ and $y = \lim_n \hat{Z}_n$ are related as

$$y = \frac{2x}{x + 1} \quad \text{and} \quad x = \frac{y}{2 - y}.$$  

In particular $x = 0$ is equivalent to $y = 0$ and $x = 1$ is equivalent to $y = 1.$
A straightforward calculation shows that

$$\frac{1}{2}(1 + x)^2 \hat{g} \left( \frac{2x}{x+1} \right) = \frac{g(x)}{1-x},$$

so that \( \hat{g} \) and \( g \) have the same zeros, with the possible exception of \( x = 1 \).

For \( x = 1 \) we examine two cases:

(i) If \( d > 0 \) then \( \hat{g}(1) = d > 0 \), whereas \( g(1) = 0 \) always, so that \( 1 \notin \hat{Z}_\infty \), which implies \( 1 \notin Z_\infty \). But \( g'(1) = d > 0 \) so that non-convergence to 1 is what we would expect.

(ii) If \( d = 0 \) then \( \hat{g}(1) = g(1) = 0 \). We examine the behavior of these close to 1 via the calculations

\[
\hat{g}(1 - \epsilon) = \epsilon [2\epsilon + (2c - f)(1 - \epsilon)] \quad \text{and} \quad g(1 - \epsilon) = \epsilon^2 [\epsilon + (2c - f)(1 - \epsilon)].
\]

Hence \( x = 1 \) is stable/unstable simultaneously for \( \hat{g} \) and \( g \).

Differentiating (3.17) for \( x \neq 1 \) yields

\[
(1 + x)\hat{g} \left( \frac{2x}{x+1} \right) + \hat{g}' \left( \frac{2x}{x+1} \right) = \frac{1}{(1-x)^2}g(x) + \frac{1}{1-x}g'(x),
\]

so that at any point \( x < 1 \) where \( \hat{g} \) and \( g \) vanishes we have

\[
\hat{g}' \left( \frac{2x}{x+1} \right) = \frac{1}{1-x}g'(x),
\]

i.e. \( \hat{g} \) and \( g \) have the same stable and strictly unstable points (if any). For any \( x < 1 \) such that \( \hat{g}' \) and \( g' \) vanishes we also have

\[
\hat{g}'' \left( \frac{2x}{x+1} \right) = \frac{1}{1-x}g''(x),
\]

so that \( \hat{g} \) and \( g \) have identical touchpoints (if any).

Thus the convergence of \( \hat{Z}_n \) to a stable zero of \( \hat{g} \) implies the convergence of \( Z_n \) to a stable zero of \( g \). Also, the non-convergence to an unstable zero of \( \hat{g} \) implies the non-convergence to an unstable zero of \( g \).

5. \( e = f = 0 \) and \( \min\{a + b, c + d\} > 0 \) is symmetric to 4.

6. \( c = d = 0 \) and \( \min\{a + b, e + f\} > 0 \). Define \( \tau_n \) as in the previous section and \( \hat{Z}_n = Z_{\tau_n} \). Now

$$E_{\tau_n} \mathbf{I}^\text{BB}_{\tau_{n+1}} = \frac{W_{\tau_n}(W_{\tau_n} - 1)}{W_{\tau_n}(W_{\tau_n} - 1) + B_{\tau_n}(B_{\tau_n} - 1)}$$

is a difficult expression to work with directly, so we rewrite it as

$$E_{\tau_n} \mathbf{I}^\text{BB}_{\tau_{n+1}} = \frac{\hat{Z}_n^2}{\hat{Z}_n^2 + (1 - \hat{Z}_n)^2} + \frac{R_n}{T_n},$$

where

$$R_n = -\frac{\hat{Z}_n(1/2 - \hat{Z}_n)(1 - \hat{Z}_n)}{[\hat{Z}_n^2 + (1 - \hat{Z}_n)^2][\hat{Z}_n^2 + (1 - \hat{Z}_n)^2 - 1/T_n]}.$$
Next, set \( \hat{Y}_{n+1} = \hat{T}_{n+1} \Delta \hat{Z}_n \). Then
\[
\hat{Y}_{n+1} = (e + f - a - b)\hat{Z}_n I_{\text{BB}}^{\text{BB}} + (a - e)I_{\text{BB}}^{\text{BB}} - (e + f)\hat{Z}_n + e,
\]
so that
\[
\mathbb{E}_{\tau_n} \hat{Y}_{n+1} = g(\hat{Z}_n) + \frac{R_n[\hat{\alpha} \hat{Z}_n + (a - e)]}{T_n}, \quad \text{where}
\]
\[
g(x) = \frac{x^3}{x^2 + (1 - x)^2} + (a - e)\frac{x^2}{x^2 + (1 - x)^2} + (-e - f)x + e, \quad \text{and}
\]
\[
\hat{\alpha} = e + f - a - b.
\]

Next,
\[
\hat{U}_{n+1} = \hat{Y}_{n+1} - g(\hat{Z}_n) = [\hat{\alpha} \hat{Z}_n + (a - e)] \left[ I_{\text{BB}}^{\text{BB}} - \frac{\hat{Z}_n^2}{\hat{Z}_n^2 + (1 - \hat{Z}_n)^2} \right],
\]
which yields the error function
\[
\mathcal{E}(\hat{Z}_n) = \mathbb{E}_{\tau_n} \hat{U}_{n+1}^2 = [\hat{\alpha} \hat{Z}_n + (a - e)]^2 \left[ \frac{\hat{Z}_n^2(1 - \hat{Z}_n)^2}{[\hat{Z}_n^2 + (1 - \hat{Z}_n)^2]^2} + \frac{R_n[1 - 2\hat{Z}_n^2/(\hat{Z}_n^2 + (1 - \hat{Z}_n)^2)]}{T_n} \right].
\]

One may also verify (iv) of Definition (1) by calculating:
\[
\frac{\mathbb{E}_{\tau_n} \hat{U}_{n+1}}{T_{n+1}} = - \frac{1}{(T_n + a + b)(T_n + e + f)} \frac{\hat{\alpha}[\hat{\alpha} \hat{Z}_n + (a - e)]}{[\hat{Z}_n^2 + (1 - \hat{Z}_n)^2]^2}
\]
which certainly is \( O(T_n^{-2}) \).

Next we compare \( \hat{Z}_n \) with the original process \( Z_n \), with drift
\[
g(x) = (-e - f - a - b)x^3 + (a + 3e + 2f)x^2 + (-3e - f)x + e.
\]
It is straightforward to verify that
\[
(3.18) \quad [x^2 + (1 - x)^2] \hat{g}(x) = g(x),
\]
so that \( \hat{g} \) and \( g \) have the same equilibrium points. Differentiating (3.18) yields
\[
(4x - 1)\hat{g}(x) + [x^2 + (1 - x)^2] \hat{g}'(x) = g'(x),
\]
so, at any point where \( \hat{g} = g = 0 \) we have
\[
\sgn[\hat{g}'(x)] = \sgn[g'(x)].
\]
Differentiating again at a point where \( \hat{g}' = g' = 0 \) yields
\[
[3x^2 + (1 - x)^2] \hat{g}''(x) = g''(x).
\]

In conclusion, \( \hat{g} \) and \( g \) have "similar" equilibrium points in that they are stable, strictly unstable, or touchpoints simultaneously.

We have proved the following.

**Theorem 7.** Suppose that the Pólya urn scheme of drawing two balls (with or without replacement) according to (3.8) has \( w_0, b_0 > 1 \) (or just \( w_0, b_0 > 0 \) if drawn with replacement). Then, the limit of the fraction of balls exists a.s. Furthermore, apart from the case \( a = 2d, f = 2c \) and \( b = e = 0 \), in which all we may conclude is that the limiting variable of the
fraction of white balls has no point masses in $(0,1)$, the limiting random variable has support only on the zeros of $g$, defined in (3.15), such that the derivative $g'$ is nonpositive there.

In Theorem 10.1 of [Mah08] one can find a central limit theorem for the number of white balls in the urn scheme 3.8 with the added constraints of a constant row sum larger than one (i.e. $a + b = c + d = e + f = K ≥ 1$) and $a - 2c + e = 0$, which together has the effect that the drift function becomes linear (i.e. $α = β = 0$ in 3.10). It is also noted there (Proposition 10.3) that the fraction of white balls converge (in probability) to $-e/γ$.

Acknowledgements
This paper, with minor modifications, has served as my licentiate thesis which was defended 2009-03-06 at Uppsala university. I am indebted to my PhD supervisors Sven Erick Alm and Svante Janson for their encouragement and for providing me with numerous ways to improve this paper, both with the mathematics and clarity of exposition.

The thesis was partly finished while attending the Mittag-Leffler institute. I am indebted to the Royal Swedish Academy of Sciences for financial support towards attending the institute during spring 2009.

In the midst of writing this thesis I became a father. A most loving acknowledgement to my wonderful wife Ida who has looked after both me and our amazing little girl, Emma, during this time.

References

[AK68] K. B. Athreya, S. Karlin: Embedding of urn schemes into continuous time Markov branching processes and related limit theorems, Ann. Math. Statist., 39 (1968), 1801–1817.

[APSVO8] R. Argiento, R. Pemantle, B. Skyrms, S. Volkov: Learning to signal: analysis of a micro-level reinforcement model. Preprint (2008), 19 pages. To appear in Stoch. Proc. Appl.

[Ben99] M. Benaim: Dynamics of stochastic approximation algorithms, Séminaire de Probabilités. Lectures Notes in Mathematics, 1709, Springer (1999), 1-68.

[Bil95] P. Billingsley: Probability and Measure, 3rd ed. Wiley Series in Probability and Mathematical Statistics (1995).

[Bor08] V. Borkar: Stochastic approximation. A dynamical systems viewpoint, Cambridge University Press, (2008).

[CW05] M. Chen, C. Wei: A new urn model, J. Appl. Probab. 42 (2005), 964–976.

[EP23] F. Eggenberger, G. Pólya: Über die Statistik verketteter Vorgänge, Zeit. Angew. Math. Mech. 3 (1923), 279–289.

[Fre65] D. Freedman: Bernard Friedman’s urn, Ann. Math. Statist., 36 (1965), 956–970.

[Gou89] R. Gouet: A martingale approach to strong convergence in a generalized Pólya-Eggenberger urn model, Statist. Probab. Lett., 8 (1989), 225–228.

[HMP03] I. Higueras, J. Moler, F. Plo, M. San Miguel: Urn models and differential algebraic equations. J. Appl. Probab., 40 (2003), 401–412.

[HLS80] B. Hill, D. Lane, W. Sudderth: A strong law for some generalized urn processes, Ann. Probab., 8 No. 2 (1980), 214–226.
[Jan04] S. Janson: Functional limit theorems for multitype branching processes and generalized Pólya urns, Stochastic Process. Appl., 110 No. 2 (2004), 177–245.

[Jan06] S. Janson: Limit theorems for triangular urn schemes, Probab. Theory Related Fields, 134 No. 3 (2006), 417–452.

[JK77] N. Johnsson, S. Kotz: Urn models and their application. An approach to modern discrete probability theory. Wiley (1977).

[Mah08] H. M. Mahmoud: Pólya urn models, CRC Press (2009).

[Pem88] R. Pemantle: Random processes with reinforcement, Doctoral Dissertation. M.I.T. (1988).

[Pem90] R. Pemantle. Nonconvergence to unstable points in urn models and stochastic approximations, Ann. Probab., 18 No. 2 (1990), 698–712.

[Pem91] R. Pemantle. When are touchpoints limits for generalized Pólya urns?, Proc. Amer. Math. Soc., 113 No. 1 (1991), 235–243.

[Pem07] R. Pemantle: A survey of random processes with reinforcement, Probab. Surv., 4 (2007), 1–79.

[Ren09] H. Renlund: Nonconvergence to strictly unstable boundary points in stochastic approximation. Preprint (2009), 9 pages.

[RM51] H. Robbins, S. Monro: A stochastic approximation method, Ann. Math. Statist. 22 (1951), 400–407.

Department of Mathematics, Uppsala University, PO Box 480, S-751 06 Uppsala, Sweden

E-mail address: henrik.renlund@math.uu.se

URL: http://www.math.uu.se/~renlund/