Structural scale q-derivative and the LLG equation in a scenario with fractionality

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Abstract – In the present contribution, we study the Landau-Lifshitz-Gilbert equation with two versions of structural derivatives which were recently proposed: the scale q-derivative in the non-extensive statistical mechanics and the axiomatic metric derivative. The latter presents the Mittag-Leffler functions as eigenfunctions. The use of structural derivatives aims to take into account long-range forces, possible non-manifest or hidden interactions and the dimensionality of space. Having this purpose in mind, we build up an evolution operator and a deformed version of the LLG equation. Damping in the oscillations naturally shows up without an explicit Gilbert damping term.

Introduction. – In recent works, we have developed connections and a variational formalism to treat deformed or metric derivatives, considering the relevant space-time/phase space as fractal or multifractal [1]. We have also presented a variational formulation to approach dissipative systems, contemplating also cases of a time-dependent mass [2].

The use of deformed-operators was justified based on our proposition that there exists an intimate relationship between dissipation, coarse-grained media and a limit energy scale for the interactions. Concepts and connections like open systems, quasi-particles, energy scale and the change in the geometry of space-time at its topological level, non-conservative systems, non-integer dimensions of space-time connected to a coarse-grained medium, have been discussed. With this perspective, we argued that deformed or, we should say, metric or structural derivatives, similarly to the fractional calculus (FC), could allow us to describe and emulate certain dynamics without explicit many-body, dissipation or geometrical terms in the dynamical governing equations. Also, we emphasized that the paradigm we adopt was different from the standard approach in the generalized statistical mechanics context [3–5], where the modification in the definition of entropy leads to the modification of the algebra and, consequently, the concept of a derivative [1,2]. This was set up by a mapping into a continuous fractal space [6–8] which naturally yields the need of modifications in the derivatives, that we have named deformed or, better, metric derivatives [1,2]. The modifications of the derivatives, in connection with the metric, yield a change in the algebra involved, which, in turn, may lead to a generalized statistical mechanics with some adequate definition of entropy.

To provide the reader with a more complete picture of the background of structural derivatives, including experimental facts that reinforce the use of the aforementioned deformed operators, see refs. [9,10].

The Landau-Lifshitz-Gilbert (LLG) equation sets out as a fundamental approach to describe physics in the field of applied magnetism. It exhibits a wide spectrum of effects stemming from its non-linear structure, and its mathematical and physical consequences open up a rich field of study. An interesting overview about the subject can be found in ref. [11]. We pursue the investigation of the LLG equation in a scenario where complexity may play a role. The connection between LLG and fractionality, represented by an α-deformation parameter in the deformed differential equations, has not been exploited with due attention. Here, the use of metric derivatives aims to take
into account long-range forces, possible non-manifest or hidden interactions and/or the dimensionality of space.

To get some additional insight into the general interest in and more recent developments on the LLG studies and applications, we can cite, for example, a relativistic theory of spin relaxation mechanisms in ref. [12], non-linear and chaotic magnetization dynamics in ref. [13], anisotropic Landau-Lifshitz-Gilbert models of dissipation in qubits in ref. [14], experimental investigation of temperature-dependent Gilbert damping in permalloy thin films in ref. [15], effects of Landau-Lifshitz-Gilbert damping on domain growth in ref. [16]. In terms of Lagrangian formalism and LLG equations, we cast the interesting work in refs. [17,18].

In this contribution, considering intrinsically the presence of complexity and possible dissipative effects, and aiming to tackle these issues, we apply our approach to study the LLG equation with two metric or structural derivatives, the recently proposed scale derivative, the recently proposed scale

The key of our present work is the scale $q$-derivative (Sq-D) that we have recently defined as

$$ D_{(q),x}^\lambda f(\lambda x) := [1 + (1 - q)\lambda x] \frac{df(\lambda x)}{dx}. \quad (2) $$

An eigenvalue equation holds for this derivative operator, as the reader may check:

$$ D_{(q),x}^\lambda f(\lambda x) = \lambda f(\lambda x). \quad (3) $$

The solution to eq. (3) with this Sq-D is $f = c_\lambda(\lambda x)$. The $q$-deformed Heisenberg equation in the non-extensive statistics context. With the aim to obtain a scale $q$-deformed Heisenberg equation, we now consider the scale $q$-derivative [2]

$$ \frac{d^q}{dt^q} = (1 + (1 - q)\lambda x \frac{d}{dx} \quad (4) $$

and the scale $q$-deformed Schrödinger equation [2],

$$ i\hbar D_{q,t}^\lambda \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi - V \psi = H \psi, \quad (5) $$

that, as we have shown in [2], is related to the non-linear Schrödinger equation referred to in ref. [19] as NRT-like Schrödinger equation (with $q = q' - 2$ compared to the $q$-index of the reference) and can be thought of as resulting from a time scale $q$-deformed-derivative applied to the wave function $\psi$.

Considering in eq. (5), $\psi(\vec{r},t) = U_q(t,t_0)\psi(\vec{r},t_0)$, the $q$-evolution operator naturally emerges if we take into account a time scale $q$-deformed-derivative (do not confuse this with the formalism of the discrete scale time derivative):

$$ U_q(t,t_0) = e_q(-\frac{\hbar}{\hbar M_q} H t). \quad (6) $$

Here, $M_q$ is a constant for dimensional regularization reasons. Note that the $q$-deformed evolution operator is neither Hermitian nor unitary, and the possibility of a $q$-unitality as $U_q^\dagger(t,t_0) \otimes_q U_q(t,t_0) = 1$ could be thought to come over these facts. In this work, we assume the case in which the commutativity of $U_q$ and $H$ holds, but the $q$-unitality is also a possibility.

Now, we follow a similar reasoning to the one which can be found in ref. [20] by considering the Sq-D.

So, with these considerations, we are now ready to write a non-linear scale $q$-deformed Heisenberg equation as

$$ D_{(q),t}^{\lambda} \hat{A}(t) = -i\frac{\hbar}{\hbar M_q} [\hat{A}, H], \quad (7) $$

where we supposed that $U_q$ and $H$ commute and $M_q$ is some factor only for dimensional equilibrium.

$q$-deformed LLG equation. To build up the scale $q$-deformed Landau-Lifshitz-Gilbert equation, we consider eq. (7), with $\hat{A}(t) = \hat{S}_q$

$$ D_{(q),t}^{\lambda} \hat{S}_q(t) = -i\frac{\hbar}{\hbar M_q} [\hat{S}_q, H]. \quad (8) $$

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where we supposed that $U_q$ and $\mathcal{H}$ commute,

$$\mathcal{H} = -\frac{g_\alpha}{\hbar M_q} \dot{S}_q \circ \hat{H}_{\text{eff}}. \quad (9)$$

Here, $\hat{H}_{\text{eff}}$ is some effective Hamiltonian whose form we shall clearly write down in the following.

The scale $q$-deformed momentum operator is here defined as $\hat{p}^q_\alpha = -i\hbar M_q [1 + \lambda (1 - q')x] \partial x$.

Considering this operator, we obtain a deformed algebra, here in terms of the commutation relation between coordinates and their corresponding momenta:

$$[\hat{x}^q_i, \hat{p}^q_j] = i[1 + \lambda (1 - q')x] \hbar M_q \delta_{ij} I$$

and, for angular momentum components, we have

$$[\hat{L}^q_i, \hat{L}^q_j] = i[1 + \lambda (1 - q')x] \hbar M_q \hat{\delta}_{ij}. \quad (11)$$

The $q'$ factor in $\hat{p}^q_i, \hat{p}^q_j, \hat{L}^q_i, \hat{L}^q_j, M_q$ is only an index and $q$ is not necessarily equal to $q'$.

The resulting scale $q$-deformed LLG equation is now written as

$$D^\lambda_{t,q} \dot{S}_q(t) = -\frac{[1 + \lambda (1 - q')x] g_\alpha \mu B}{\hbar M_q} \dot{S}_q \times \hat{H}_{\text{eff}}. \quad (12)$$

Take $\dot{m}_q \equiv \gamma_q \ddot{S}_q, \gamma_q \equiv \frac{[1 + \lambda (1 - q')x] g_\alpha \mu B}{\hbar M_q}.

If we consider that the spin algebra is not affected by any emergent effects, we can take $q'=1$.

Considering eq. (7) with $\hat{A}(t) = \ddot{S}_q$ and $\dot{m}_q = |\gamma_q| \dddot{S}_q$ and $q'=1$, we obtain the $q$-time-deformed LLG dynamical equation for magnetization as

$$D^\lambda_{t,q} \dot{m}_q(t) = -|\gamma| \dot{m}_q \times \hat{H}_{\text{eff}}. \quad (13)$$

By taking $\hat{H}_{\text{eff}} = H_{00}\hat{k}$, we have the solution

$$m_{x,q} = \rho \cos_q(\theta_0) \cos_q(\gamma H_0 t) + \rho \sin_q(\theta_0) \sin_q(\gamma H_0 t). \quad (14)$$

In fig. 1, $\theta_0 = 0$.

Applying the axiomatic derivative and the $\alpha$-deformed Heisenberg equation. – Now, to compare the results with two different local operators, we apply the axiomatic metric derivative. Following the steps in [20] and considering the axiomatic MD [21], there holds the eigenvalue equation $D^\alpha_t E_\alpha(\lambda x^\alpha) = \lambda E_\alpha(\lambda x^\alpha)$, where $E_\alpha(\lambda x^\alpha)$ is the Mittag-Leffler function that is of crucial importance to describe the dynamics of complex systems. It involves a generalization of the exponential function and several trigonometric and hyperbolic functions. The eigenvalue equation above is only valid if we consider $\alpha$ very close to 1. This is what we call low-level fractionality [21]. Our proposal is to allow the use of the Leibniz rule, even if it would result in an approximation. So, we can build up an evolution operator:

$$U_\alpha(t, t_0) = E_\alpha(-\frac{i}{\hbar^\alpha} \mathcal{H}^\alpha), \quad (15)$$

and for the deformed Heisenberg equation

$$D^\alpha_t A^H_\alpha(t) = -\frac{i}{\hbar^\alpha} [A^H_\alpha, \mathcal{H}], \quad (16)$$

where we supposed that $U_\alpha$ and $\mathcal{H}$ commute.

To build up the deformed Landau-Lifshitz-Gilbert equation, we use eq. (16) and consider the spin operator $\dot{S}_\alpha(t)$ in such a way that we can write the deformed Heisenberg equation as

$$\dot{D}^\alpha_t \dot{S}_\alpha(t) = -\frac{i}{\hbar^\alpha} [\dot{S}_\alpha, \mathcal{H}], \quad (17)$$

with

$$\mathcal{H} = -g_\alpha \frac{\mu B}{\hbar^\alpha} \ddot{S}_\alpha \circ \hat{H}_{\text{eff}}. \quad (18)$$

Here, $\hat{H}_{\text{eff}}$ is some effective Hamiltonian whose form that we will turn out clear forward.

Consider now the deformed momentum operator as [20,22,23]

$$\tilde{\rho}^\alpha = -i (\hbar^\alpha M_q \partial x^\alpha). \quad (19)$$

Taking this operator, we obtain a deformed algebra, here in terms of commutation relation for coordinate and momentum,

$$[\hat{x}^\alpha_i, \hat{p}^\alpha_j] = i\Gamma(\alpha + 1) \hbar^\alpha M_q \delta_{ij} I$$

and for angular momentum components as

$$[\hat{L}^\alpha_i, \hat{L}^\alpha_j] = i\Gamma(\alpha + 1) \hbar^\alpha M_q \hat{\delta}_{ij}. \quad (21)$$

The resulting $\alpha$-deformed LLG equation is now written as

$$D^\alpha_t \dot{S}_\alpha(t) = -\frac{M_\alpha \Gamma(\alpha + 1) g_\alpha \mu B}{\hbar^\alpha} \ddot{S}_\alpha \times \hat{H}_{\text{eff}}. \quad (22)$$

If we take $\dot{m}_\alpha \equiv \gamma_\alpha \dddot{S}_\alpha, \gamma_\alpha \equiv \frac{M_\alpha \Gamma(\alpha + 1) g_\alpha \mu B}{\hbar^\alpha}$, we can rewrite the equation as the $\alpha$-deformed LLG equation,

$$D^\alpha_t \dot{m}_\alpha(t) = -|\gamma_\alpha| \dot{m}_\alpha \times \hat{H}_{\text{eff}}. \quad (23)$$
Fig. 2: (a) Damping of oscillations. \( \beta = 1 \). (b) Increase of oscillations.

with \( \vec{H}_{\text{eff}} = H_0 \hat{k} \). We have the solution of eq. (23):

\[
m_{\alpha z} = A \cos \theta_0 E_{2\alpha}(-\omega_0^2 t^{2\alpha}) + A \sin \theta_0 x. E_{2\alpha, 1+\alpha}(-\omega_0^2 t^{2\alpha}).
\]

In fig. 2, the reader may notice the behavior of the magnetization, considering \( \theta_0 = 0 \).

For \( \alpha = 1 \), the solution reduces to \( m_x = A \cos(\omega_0 t + \theta_0) \), the standard simple harmonic-oscillator solution for the precession of magnetization.

The presence of complex interactions and dissipative effects that are not explicitly included into the Hamiltonian can be seen with the use of deformed metric derivatives. Without explicitly adding up the Gilbert damping term, the damping in the oscillations could reproduce the damping described by the Gilbert term or could it disclose some new extra damping effect. Also, depending on the relevant parameter, the \( q \)-entropic parameter, or for \( \alpha \), the increasing oscillations can signal that it is sensible to expect fractionality to interfere on the effects of polarized currents as the Slonczewski term describes. We point out that there are qualitative similarities in both cases, as the damping or the increasing of the oscillations, depending on the relevant control parameters. Despite that, there are also some interesting differences, such as the change in phase for the axiomatic-derivative application case.

Here, we cast some comments about an apparent paradox: If we carry out, as usually done in the literature of the LLG equation, the scalar product in eq. (13) with \( \vec{m}_q \), we obtain the apparent paradox that the modulus of \( \vec{m}_q \) does not change. On the other hand, if instead of \( \vec{m}_q \), we proceed now with a scalar product with \( \vec{H}_{\text{eff}} \), we obtain thereby the indications that the angle between \( \vec{m}_q \) and \( \vec{H}_{\text{eff}} \) does not change. So, how to explain the damping in oscillations for \( \vec{m}_q \)? This question can be explained by the following arguments. Even the usual LLG equation, with the term of Gilbert, can be rewritten in a form similar to the LLG equation without term of Gilbert. See eq. (2.7) in the ref. [11]. The effective \( \vec{H}_{\text{eff}} \) field now stores information about the interactions that cause damping. In our case, when carrying out the simulations, we have taken \( \vec{H}_{\text{eff}} \) as a constant effective field. Here, we can argue that the damping term, eq. (2.8) in ref. [11], being small, this would cause the effective field \( \vec{H}_{\text{eff}} = \vec{H}(t) + \vec{k}(\vec{S} \times \vec{H}) \) to be approximately \( \vec{H}(t) \). In this way, the scalar product would dominate over the term of explicit dissipation. This could, therefore, explain the possible inconsistency.

Additional remarks on physical arguments and a glance on the comparison with fractional calculus. In order to reinforce the use of the metric derivatives, we present here a few comments on the physical aspects of our results. Following similar arguments as in ref. [24], we argue that, since we are dealing with open systems, as already commented, particles should in fact be seen as dressed entities or pseudoparticles that exchange energy with other particles and the environment. The system composed by particles and their surroundings may be considered non-conservative due to the possible energy exchange. This energy exchange may be responsible for the resulting non-integer dimension of space-time, giving rise then to an effective coarse-grained medium. Considering the existence of factors like dissipative effects, not explicitly included in the equation of integer-order models, we argue that the approach with the deformed operators is suitable to treat systems with dissipative forces or non-holonomic systems, since it also includes the scale in time, allowing to consider the effects of internal time scales of the systems. Based on these arguments, the effects of damping, or even amplifying oscillations, could be expected. Another question that might strengthen our results arises from the FC studies. In the standard FC, the mathematical rules for derivatives, as the Leibniz and the chain rules, are not the usual ones. This renders the way we have obtained equations like the Heisenberg equation natural, a more difficult task or even limited to more specific function classes. Even so, for comparative intention, results in the literature for fractional Bloch equations [25,26] indicate damping in the oscillations, qualitatively reinforcing our results, since the final structures of the Bloch equations are similar to those here obtained.

Conclusions and outlook. – In short: Here, we tackle the problem of LLG equations considering the presence of complexity and dissipation or other interactions that give rise to the term proposed by Gilbert and the one by Slonczewski.

With this aim, we have applied scale \( q \)-derivative and the axiomatic metric derivative to build up deformed
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Heisenberg equations. The evolution operator naturally emerges with the use of each case of the structural derivatives. The deformed LLG equations are solved by a simple case, with either structural or metric derivatives. Remarks on physics arguments and a brief comparative study with fractional calculus were also given.

Also, in connection with the LLG equation, we can cast some final considerations for future investigations:

Does fractionality simply reproduce the damping described by the Gilbert term or could it disclose some new effect in addition to damping?

Is it sensible to expect fractionality to interfere on the effects of polarized currents as the Slonczewski term describes?

These two points are relevant in connection with fractionality and the recent high-precision measurements in magnetic systems may open up a new venue to strengthen the relationship between the fractional properties of space-time and condensed-matter systems.

Also, as future work, we shall revisit the subject discussed here in a Lagrangian approach, using structural derivatives in a formulation similar to the one in refs. [17,18].

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