Analytical method for projecting the buckling form of composite palates with a cut-out

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Abstract. The work focused on the original concept of a thin-walled plate element with a cut-out, for use as a spring or a load-bearing element. The subject of the study were rectangular plates with a cut-out with variable geometrical parameters and with a variable angle of fibre arrangement, made of a carbon-epoxy composite with high strength properties in an asymmetrical lay-up, subjected to uniform compression. The paper presents a method of predicting the buckling form of composite plates with a cut-out ensuring the stable nature of the structure's work. The influence of geometrical parameters of the cut-out and the angle of fibre arrangement on the value of the critical load of the structure and buckling form was investigated. The commercial ABAQUS programme using the finite element method was used to develop the discrete model and perform numerical calculations.

1. Introduction
Thin-walled structural components are load-carrying structures that can work after the loss of stability, provided that they remain in the elastic state [1-9]. This is a significant limitation regarding components made of conventional structural materials such as metals, as their work in the post-critical state is determined by the yield strength of the material. The group of materials which maintain their elastic characteristics in the post-critical state, virtually until failure, are composite materials, particularly glass fibre reinforced plastic (GFRP) and carbon fibre reinforced plastic (CFRP). Due to their favourable strength properties in relation to density, these materials are now widely used in modern thin-walled structures. Another characteristic of structural elements made of fibrous composites is their wide margin of structural load capacity, i.e. the ability to work after the loss of stability, sometimes exceeding the critical load values two or three times [10-13]. There are numerous studies that investigate the behaviour of thin-walled laminated elements in the pre-critical and post-critical states [14-16]. The wide margin of load capacity of composite elements increases the safety of work of the structure, as even after the first symptoms of failure, the composite structure maintains its high stiffness until achieving the value of failure load [14]. These characteristics of fibrous composites combined with a low density of material offer numerous applications for these materials in many fields, such as aviation, construction industry [17], automotive or aerospace industries [18], where design solutions based on the use of plate and thin shell elements are dominant.

Uniform thin plates are relatively cheap to manufacture, however, due to their low bending stiffness, they can carry relatively small loads [19]. The slightest form of plate stability loss, which is the bending form, leads to its rapid destruction due to the increase in high deflection, with a slight increase in compressive load – low rigidity of the structure. However, the stiffness characteristics of
the structure change significantly when the work of the plate is forcing according to the higher flexural-torsional form of buckling. The plate is able to carry a much higher compressive load while maintaining much stiffer work characteristics in the post-critical range (figure 1).

![Figure 1](image1.png)

**Figure 1.** The possibility of using thin-walled plate elements as the elastic elements (own study).

The calculations carried out in former studies [20,21] show that plates with a forced higher form of buckling are characterised by stable, progressive paths of post-critical equilibrium, enabling their use as elastic elements. The characteristics of such elements can be designed in a wide range by changing the geometrical parameters of the cut-out, i.e. height and width [22], as well as by changing the angle of fibre arrangement [21]. The paper presents an uncomplicated method for the estimation of buckling forms of tested plates for different variables. The calculations presented in the work, were performed by means of Abaqus software, which is at present widely applicable in many fields of science [12,23]. The obtained results are of significant practical importance in the design of structures with elastic elements, allowing engineers and designers to achieve the required maintenance characteristics of the device.

2. **Research subject and methodology**

2.1. **Object of the study**

The subject of the study was a rectangular plate of fixed dimensions of 160x80 [mm] with a central cut-out of variable geometrical parameters and the corner rounding radius R=5 [mm] (figure 2).

The plate was made of carbon-epoxy composite with the following properties: Young’s modulus, E1 = 130.71 GPa and E2 = 6.36 GPa, Poisson’s ratio, ν12 = 0.32, shear modulus, G12 = 4.18 GPa and ply thickness, t= 0.131 [mm] in an asymmetrical lay-up of layers [03/θ/-θ/θ/-θ/θ/-θ/θ/θ/-θ/θ/θ/θ/θ/θ/θ/θ/θ/θ90/θ90]T, where θ – means angle (figure 3).

![Figure 2](image2.png)

**Figure 2.** The tested thin-walled composite plate with a central cut-out (own study).
2.2. Research methodology

In order to determine the form of plate buckling, linear calculations were carried out. A more detailed description of the numerical analysis along with boundary conditions is provided in other works [21,24], whereas a similar modelling method was used in the study by Rozylo [25].

Table 1 summarises the results of the critical state numerical analysis for various geometrical parameters of the cut-out and for various fibre angles. The height of the cut-out was changed in the range of 80\(\div\)120 [mm], the width of the cut-out in the range of 20 \(\div\) 40 [mm], while the angle of fibre arrangement was changed at every 150. The penultimate column contains the critical force values, while the last column contains the buckling forms for given cases.

| a [mm] | b [mm] | \(\theta^\circ\) | Critical Force [N] | 1st buckling mode |
|--------|--------|-----------------|-------------------|------------------|
| 100    | 20     | 15              | 1909.4            | Flexural-torsional |
| 100    | 20     | 30              | 1660.5            | Flexural-torsional |
| 100    | 20     | 45              | 1378.1            | Flexural         |
| 100    | 20     | 60              | 1129.3            | Flexural         |
| 100    | 20     | 75              | 1021              | Flexural         |
| 100    | 20     | 90              | 999.6             | Flexural         |
| 100    | 30     | 15              | 1477.2            | Flexural-torsional |
| 100    | 30     | 30              | 1244.4            | Flexural-torsional |
| 100    | 30     | 45              | 949.87            | Flexural-torsional |
| 100    | 30     | 60              | 770.6             | Flexural-torsional |
| 100    | 30     | 75              | 721.98            | Flexural-torsional |
| 100    | 30     | 90              | 711               | Flexural-torsional |
| 100    | 40     | 15              | 1151.1            | Flexural-torsional |
| 100    | 40     | 30              | 961.78            | Flexural-torsional |
| 100    | 40     | 45              | 711.85            | Flexural-torsional |
| 100    | 40     | 60              | 543.07            | Flexural-torsional |
| 100    | 40     | 75              | 488.87            | Flexural-torsional |
| 100    | 40     | 90              | 479               | Flexural-torsional |
| 80  | 30  | 15  | 1524.4 | Flexural-torsional |
|-----|-----|-----|--------|-------------------|
| 80  | 30  | 30  | 1412.6 | Flexural-torsional |
| 80  | 30  | 45  | 1107.1 | Flexural-torsional |
| 80  | 30  | 60  | 913.31 | Flexural           |
| 80  | 30  | 75  | 827    | Flexural           |
| 80  | 30  | 90  | 812    | Flexural           |
| 120 | 30  | 30  | 1185.91| Flexural-torsional|
| 120 | 30  | 45  | 841.42 | Flexural-torsional|
| 120 | 30  | 60  | 598.56 | Flexural-torsional|
| 120 | 30  | 75  | 526.46 | Flexural-torsional|
| 120 | 30  | 90  | 523.86 | Flexural-torsional|

The diagram below (Figure 4) shows the distribution of the buckling form which the test object assumes in a three-dimensional spatial system, depending on parameters such as the angle of the fibres $\theta$, the height of the cut-out $a$ and the width of the cut-out $b$.

![Diagram showing distribution of buckling form](image)

**Figure 4.** Dispersion of buckling form depending on the angle of fibre lay-up $\Theta$ and the height $a$ and width $b$ of the cut-out (own study).

From figure 4, it can be seen that bending forms occur in plates with a 100x20 [mm] and 80x30 [mm] cut-out. In the case of 100x20 [mm] cut-out, the bending form occurs for four variants of the fibre angle, i.e. 45°, 60°, 75° and 90°, while in the case of a 80x30 [mm] cut-out for three variants of the angle: 60°, 75°, 90°. In all cases, the bending form occurred for a fibre angle equal to or greater than 45°. Based on the values given in Table 1 and the dispersion obtained, it can be stated that the lowest buckling form depends on the above-mentioned parameters. Presenting the obtained results in two-dimensional space, depending on the height of the cut-out a/width of the cut-out b to the angle $\Theta$ (figure 5 and figure 6), it emerges that in some cases the points corresponding to different buckling forms are projected at the same points, which makes them difficult to use for further predictions. Based on the charts and the limited number of data, it is difficult to clearly determine the form that the element will take for a different set of parameter values.

As already mentioned, the buckling form depends on the height and width of the cut-out and the angle of the fibre. By creating a new variable, whose value depends on two selected parameters, we can present the distribution of buckling form in a more intelligible way, e.g. by calculating the ratio of the height to the width of the cut-out (figure 7).
**Figure 5.** Graph of buckling form depending on the height of the cut-out $a$ and the angle of the fibres $\Theta$ (own study).

**Figure 6.** Graph of buckling form depending on the width of the cut-out $b$ and the angle of the fibres $\Theta$ (own study).

**Figure 7.** Graph of buckling form depending on the ratio $a/b$ and the angle of fibre arrangement $\Theta$ (own study).
In this way, the points representing the different buckling forms do not overlap. However, despite the fact that the obtained graph is clearer, it is somewhat difficult to see the dependencies and describe them. It is difficult to predict with certainty when the appropriate form of buckling will occur. A much better result was obtained by multiplying the height and width parameters of the cut-out, calculating the area of the cut-out (figure 8).

\[ P = f(\theta, a, b) \]  

(1)

where:
- \( P \) – form of buckling
- \( \theta \) – laying angle of laminate layers
- \( a \) – height of cut-out
- \( b \) – width of cut-out

By dividing the obtained surface area values by the fibre angle, a new parameter \( p \) (equation 2) was obtained, which takes into account all factors affecting the buckling form of the element. On the buckling dispersion chart, depending on the area of the cut-out and the new variable \( p \) (figure 9), the points representing the respective buckling forms are grouped and form sets. These sets can be separated by a straight line drawn between the extreme points on the borders of the set. The separation of sets can be made using any line or function and this could be done already on the dispersion plot of the cut-out surface area and the layer arrangement, however, the obtained relationship would only take into account the surface area of the cut-out and forms of buckling would be constant for all lay-ups. In

**Figure 8.** Graph of buckling form depending on the area of the cut-out and the angle of fibre arrangement \( \theta \) (own study).
addition, the use of functions other than rectilinear function would require the use of additional transformations and mathematical operations. The advantage of this method is to prevent the random determination of the coefficients’ value of straight, thus preventing the discrepancy in later results. Nonetheless, the straight line created by this method does not separate the sets throughout the entire range. The value of parameter \( p \) corresponding to the cut-out 100x30 [mm] and \( \Theta \) equal to 90° is below the straight line and equals 33.3 (3), while the value of the parameter corresponding to the point on the dividing line equals 33.375.

\[
P = \frac{a + b}{a}
\]  

(2)

![Graph of buckling form depending on the area of the cut-out and parameter \( p \) (own study).](image)

Figure 9. Graph of buckling form depending on the area of the cut-out and parameter \( p \) (own study).

In order to compensate for this situation, the parameter \( p \) was subjected to a mathematical procedure consisting in the numerator rooting in the formula. In this way, a new parameter \( s \) was obtained, which is calculated from:

\[
s = \frac{\sqrt{a \cdot b}}{a}
\]  

(3)

Proceeding analogously as in the previous case, a separating line was drawn, which separates the sets in the whole range (figure 10).
Figure 10. Graph of buckling form depending on the area of the cut-out and parameter $s$ (own study).

The resulting straight line is described by the equation:

$$s = -0.0007a * b + 2.69$$  \hspace{1cm} (4)

By modifying the equation of a straight line, we obtain the formula for the next parameter $f$, on the basis of which we can determine (estimate) what buckling form the composite element will have:

$$f = \frac{s - 2.69}{a * b} = \frac{\sqrt{a * b} - 2.69}{a * b}$$  \hspace{1cm} (5)

The value of the parameter $f$ for flexural-torsional forms is greater than the value of the directional coefficient of the straight line used to separate sets is equal to 0.000724202 (figure 11), while for bending forms, it is smaller. The value of the directional coefficient of the straight line, with using the approximation method of separating sets, is the limit (reference) value, on the basis of which is possible to estimate the buckling form. The advantage of this division is that we include all variable parameters in the predictions.

Taking into account the results obtained and the methodology adopted, we arrive at the function $P$ for the buckling form:

$$P = \begin{cases} 
\text{flexural-torsional form} & f > -0.0007 \\
\text{flexural form} & f < -0.0007
\end{cases}$$  \hspace{1cm} (6)

The above function does not account for the scenario in which the value of the parameter $f$ is equal to the directional coefficient of the straight line. This was done on purpose because the presented method is only an approximate method, the accuracy of which may be boosted by increasing the amount of available data. When we use this method, it should be borne in mind that with values close to the limit value, the results may be questionable.
Figure 11. Graph of buckling form depending on the area of the cut-out and parameter f (own study).

3. Conclusions
The paper presents a method for simple estimation of buckling forms of tested plates for different variables. The obtained results are of significant practical importance in the design of structures with elastic elements, as they lead to obtaining the required maintenance characteristics of the device.

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