CYCLE CONNECTIVITY AND PSEUDOCONCAVITY
OF FLAG DOMAINS

TATSUKI HAYAMA

Abstract. We discuss cycle connectivity of non-classical flag domains. We
construct a certain subspace determined by the root spaces which is one-
connected to a cycle. As an application, we prove that a flag domain is
pseudoconcave if a certain positive compact root exists. Moreover, we inves-
tigate cycle connectivity of some Mumford–Tate domains in connection with
degenerating Hodge structures.

1. Introduction

A flag domain is an open real group orbit in a flag variety. For example Hermitian
symmetric domains are flag domains, which are classically studied. Except for such
classical cases, general flag domains have no non-constant global function. Such flag
domains are said to be non-classical. Period domains and Mumford–Tate domains,
which are important objects in Hodge theory, are non-classical flag domains in
general. By Huckleberry [Huc2] and Griffiths, Robles and Toledo [GRT], a flag
domain is non-classical if and only if it is cycle connected. We investigate the cycle
connectivity further in this paper.

A flag variety is a homogeneous space with a holomorphic action of a connected
complex semisimple Lie group $G$. Let $D$ be an open $G_{\mathbb{R}}$-orbit in this flag variety
with a real form $G_{\mathbb{R}}$ of $G$. We assume that $D$ is non-classical and the isotropy
subgroup at a point in $D$ is compact through this paper. For a reference point
$o \in D$, we have the maximal compact subgroup $K_{\mathbb{R}}$ of $G_{\mathbb{R}}$ containing the isotropy
subgroup at $o$. The $K_{\mathbb{R}}$-orbit $C_0$ at $o$ is a compact submanifold contained in $D$.
Here any two points of $D$ are connected by a chain $g_1C_0 \cup \cdots \cup g_\ell C_0$ contained in
$D$ with $g_1, \ldots, g_\ell \in G$, where $D$ is said to be cycle connected.

From the complex geometric aspect, Huckleberry studied cycle connectivity of
flag domains (cf. [Huc1], [Huc2] and [Huc3]). He proved that a flag domains is
not holomorphically convex if it is cycle connected. Moreover, a flag domain is
pseudoconcave if it is one-connected. Here one-connectivity is cycle connectivity in
a strong sense, which requires that any two points are connected by a one cycle.
He proved that some kind of flag domains is one-connected.

Cycle connectivity also has a relation with a property of holomorphic sections
of line bundles over $D$. Kollár [K] showed the vanishing order of a holomorphic
section of a line bundle $L$ depends on the length of a chain of cycles, which induces
finiteness of $\dim H^0(D, L)$ (see also [Huc3]). A uniform estimate for the length of
chains would reflects a geometric property of flag domains.

Date: January 9, 2015.
2000 Mathematics Subject Classification. 14D07, 32G20, 14M15, 58A14.
Key words and phrases. Flag domain, Mumford–Tate domain, degenerating Hodge structure.
In this paper, we discuss cycle connectivity of flag domains using the roots of semisimple Lie algebras. A flag variety is a homogeneous space $G/P$ with a parabolic subgroup $P$, where the structure of a parabolic subalgebra is determined by positive simple roots of the Lie algebra $\mathfrak{g}$. In Lemma 2.6, we show that a certain subset of $D$ is one-connected to $C_0$ if there exists a positive compact root satisfying some condition.

By using this lemma, the structure of the roots determines pseudoconcavity of $D$. Huckleberry [Huc] indicated that pseudoconcavity of flag domains is induced from existence of a neighborhood of $C_0$ which is filled out by cycles and where any points are one-connected to $C_0$. In Theorem 2.7, we prove that we can construct such a neighborhood if a certain positive compact root exists.

From the Hodge theoretic aspect, it is important when $G$ has a rational form $G_\mathbb{Q}$ which is a Mumford–Tate group. In this case, $D$ is called a Mumford–Tate domain. Degenerating Hodge structures, where the Mumford–Tate groups are contained in $G_\mathbb{Q}$, arises nilpotent orbits on $D$. The behavior of nilpotent orbits in the framework of representation theory has been studied by Green, Griffiths, Kerr, Pearlstein and Robles (cf., [KP], [KR] and [GGR]). We investigate cycle connectivity of some Mumford–Tate domains in connection with nilpotent orbits.

A nilpotent orbit is a orbit in the flag variety, i.e. the compact dual, $\check{D}$ defined by a rational nilpotent element in $\mathfrak{g}$. Given a one-variable nilpotent orbit, we have the horizontal representation of $SL_2(\mathbb{R})$ and the associated embedding $\mathbb{P}^1(\mathbb{C}) \to \check{D}$ by the $SL_2$-orbit theorem. We show that a generic point in the image of $\mathbb{P}^1(\mathbb{C})$ is one-connected to a point in $C_0$ in some cases.

We deal with two kinds of Mumford–Tate domains in this paper. The first one is Mumford–Tate–Chevalley domains introduced by [KR]. In this case, the rational structure is given by a Chevalley basis, which defines a Hodge structure on $\mathfrak{g}$. [KR] showed that a positive non-compact simple root defines a nilpotent orbit. We show the one-connected property for this nilpotent orbit on the assumption of the existence of a particular compact root in Theorem 3.2.

The second one is period domains. We consider minimal degenerations in period domains in the sense of [GGR]. In [H1], we have shown a cycle connected property for a specific case. Generalizing this result, in Proposition 4.2, we show the one-connected property for nilpotent orbits of minimal degeneration on a certain assumption on Hodge numbers. If the weight is odd, we have a stronger one-connectivity property in Proposition 4.4.

In the end of this introduction, we would like to point out that it could be interesting to consider a relationship between the cycle connectivity and the reduced limit in the sense of [GGR]. As is mentioned in the introduction of [GRT], it is important to study how the cycles $gC_0$’s approach the boundary of $D$. In the setting of Proposition 4.4, we have a sequence of $gC_0$’s in the cycle space which converges to a cycle containing the reduced limit.

2. Flag domains

We review some materials from the structure theory of semisimple Lie algebra, and then we discuss cycle connectivity and pseudoconcavity of flag domains.

2.1. Flag varieties. Let $G$ be a connected complex semisimple Lie group, and let $P$ be a parabolic subgroup. The homogeneous complex manifold $\check{D} := G/P$ admits
the structure of rational homogeneous variety, which is called a flag variety. We fix Cartan and Borel subgroups $H \subset B \subset P$. Let $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{p} \subset \mathfrak{g}$ be the associated Lie algebras. The Cartan algebra $\mathfrak{h}$ determines a set $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}^*$ of roots. A root $\alpha \in \Delta$ defines the root space $\mathfrak{g}_\alpha$, and we have the root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$. For a subalgebra $\mathfrak{s}$, we denote by $\Delta(\mathfrak{s})$ the subset of roots of which root spaces contained in $\mathfrak{s}$. The Borel subalgebra $\mathfrak{b}$ defines a positive root system by

$$\Delta^+ := \Delta(\mathfrak{b}) = \{ \alpha \in \Delta \mid \mathfrak{g}_\alpha \subset \mathfrak{b} \}.$$ 

Let $\mathcal{S} = \{ \sigma_1, \ldots, \sigma_r \}$ be the set of simple roots, and let $\{ \mathcal{S}^1, \ldots, \mathcal{S}^r \}$ be the dual basis to $\mathcal{S}$. An integral linear combination $E = \sum_j n_j \mathcal{S}^j$ is called a grading element.

The $E$-eigenspaces $g^\ell = \bigoplus_{\alpha \in \Delta(\mathfrak{E})} g^\ell_\alpha$, $g^0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{E})} g^0_\alpha$ determines a graded Lie algebra decomposition $g = g^{-k} \oplus \cdots \oplus g^k$ in the sense that $[g^\ell, g^m] \subset g^\ell+m$. The grading $E$ defines a parabolic subalgebra $\mathfrak{p}_E = \bigoplus_{\ell \geq 0} g^\ell$. On the other hand, setting $I(\mathfrak{p}) = \{ i \mid \mathfrak{g}^{-\sigma_i} \not\subset \mathfrak{p} \}$, the parabolic subalgebra $\mathfrak{p}_E$ defined by $E = \sum_{i \in I(\mathfrak{p})} \mathcal{S}^i$ coincides with $\mathfrak{p}$.

**Example 2.1** ($\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$). A basis of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ is given by

$$x^\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H^\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad x^{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (2.1)$$

Here $\mathfrak{h} = \mathbb{C} H^\alpha$ is a Cartan subalgebra, and $x^{\pm \alpha}$ is in the root space $\mathfrak{g}^{\pm \alpha}$ where $\alpha$ is the root given by $\alpha(H^\alpha) = 2$. This triple satisfies

$$[x^\alpha, x^{-\alpha}] = H^\alpha, \quad [H^\alpha, x^\alpha] = 2x^\alpha, \quad [H^\alpha, x^{-\alpha}] = -2x^{-\alpha}, \quad (2.2)$$

which is called a Chevalley basis discussed in the next section. In this case, $\Delta = \{ \pm \alpha \}$ and we set $\alpha$ as a positive root. We define a grading $E = \mathcal{S}$ where $\mathcal{S}$ is the dual of $\alpha$. Then

$$g^1 = \mathfrak{g}^\alpha, \quad g^0 = \mathfrak{h}, \quad g^{-1} = g^{-\alpha},$$

and $\mathfrak{p}_E = g^1 \oplus g^0$. The parabolic subgroup $P$ corresponding to $\mathfrak{p}_E$ is

$$\left\{ \begin{pmatrix} a & b \\ 0 & -1/a \end{pmatrix} \right\}, \quad a, b \in \mathbb{C}, a \neq 0 \quad (2.3)$$

and hence the flag variety associated with $P$ is

$$G/P \cong \{ \begin{pmatrix} 1 \\ z \end{pmatrix} \mid z \in \mathbb{C} \} \cup \{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \} \cong \mathbb{P}^1 \mathbb{C}.$$ 

Here $P$ is the isotropy subgroup at $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. In general, a triple in a semisimple Lie algebra satisfying $(2.2)$ is called a standard triple.
2.2. Chevalley basis. First, we review the Chevalley basis and their properties (cf. [Hum §25]). Since the Killing form defines a non-degenerating positive definite symmetric form on $\mathfrak{h}$, we have the form $(\cdot, \cdot)$ on $\mathfrak{h}^*$ induced by the one. Let $\alpha \in \Delta \cup \{0\}$ and let $\beta \in \Delta$. The set of all members of $\Delta \cup \{0\}$ of the form $\alpha + n\beta$ for $n \in \mathbb{Z}$ is called the $\beta$-string containing $\alpha$. Then the $\beta$-string containing $\alpha$ is given by

$\{\alpha + n\beta \mid -r \leq n \leq q\}$.  

(2.4)

If $\alpha$ and $\beta$ are linearly independent, we have

$$\langle \alpha, \beta \rangle := 2 \frac{(\alpha, \beta)}{(\beta, \beta)} = r - q.$$ 

We can choose $H^\alpha \in \mathfrak{h}$ and $x^\alpha \in \mathfrak{g}^\alpha$ for all $\alpha \in \Delta$ satisfying

$$[x^\alpha, x^{-\alpha}] = H^\alpha, \quad [H^\beta, x^\alpha] = (\alpha, \beta)x^\alpha,$$

$$[x^\alpha, x^\beta] = \begin{cases} c_{\alpha, \beta}x^{\alpha+\beta} & \text{if } \alpha + \beta \in \Delta \\ 0 & \text{if } \alpha + \beta \not\in \Delta \end{cases}$$

where $c_{\alpha, \beta} = -c_{-\alpha, -\beta} \in \mathbb{Z}$.

Here $c_{\beta, \alpha} = \pm (r + 1)$ if $\alpha$ and $\beta$ are linearly independent and $\alpha + \beta \in \Delta$. Now

$$\{x^\alpha \mid \alpha \in \Delta\} \cup \{H^\sigma \mid \sigma \in S\}$$

is a basis of $\mathfrak{g}$, which is called a Chevalley basis.

Lemma 2.2 ([Hum §25.2]). If $\alpha$ and $\beta$ are linearly independent, then

$$[x^{\beta}, [x^{\beta}, x^\alpha]] = q(r + 1)x^\alpha$$

where the $\beta$-string containing $\alpha$ is given by the form (2.4).

Next, we show the following proposition by using the above properties of the Chevalley basis.

Proposition 2.3. If $\alpha$ and $\beta$ are linearly independent, then

$$\text{Ad}(\exp \left( \frac{\pi}{2} (x^{\beta} - x^{-\beta}) \right))x^\alpha = \begin{cases} \pm x^{\alpha+\beta} & \text{if } (r, q) = (0, 1), \\ \pm x^{\alpha+2\beta} & \text{if } (r, q) = (0, 2). \end{cases}$$

Proof. First, we consider the case for $(r, q) = (0, 1)$. Since $\alpha - \beta \not\in \Delta$,

$$\text{ad}(x^{\beta} - x^{-\beta})x^\alpha = \text{ad}(x^{\beta})x^\alpha = c_{\beta, \alpha}x^{\alpha+\beta}.$$ 

By Lemma 2.2

$$[x^{-\beta}, [x^{\beta}, x^\alpha]] = x^\alpha.$$ 

Then

$$(\text{ad}(x^{\beta} - x^{-\beta}))^2 x^\alpha = -[x^{-\beta}, [x^{\beta}, x^\alpha]] = -x^\alpha.$$ 

Therefore

$$\text{Ad}(\exp \left( \frac{\pi}{2} (x^{\beta} - x^{-\beta}) \right))x^\alpha = \sum_{\ell=0}^{\infty} \frac{(-1)^{2\ell}}{(2\ell)!} \left( \frac{\pi}{2} \right)^{2\ell} x^\alpha + \sum_{k=0}^{\infty} \frac{(-1)^{k}}{(2k + 1)!} \left( \frac{\pi}{2} \right)^{2k+1} c_{\beta, \alpha}x^{\alpha+\beta}$$

$$= \cos \left( \frac{\pi}{2} \right) x^\alpha + \sin \left( \frac{\pi}{2} \right) c_{\beta, \alpha}x^{\alpha+\beta} = c_{\beta, \alpha}x^{\alpha+\beta}$$

Since $c_{\beta, \alpha} = \pm 1$, the equation holds.
Next, we consider the case for \((r, q) = (0, 2)\). As is the case for \((r, q) = (0, 1)\), we have
\[
\text{ad} (x^\beta - x^{-\beta}) x^\alpha = c_{\beta, \alpha} x^{\alpha + \beta}, \quad [x^{-\beta}, [x^\beta, x^\alpha]] = 2x^\beta.
\]
Then
\[
(\text{ad} (x^\beta - x^{-\beta}))^2 x^\alpha = c_{\beta, \alpha + \beta} c_{\beta, \alpha} x^{\alpha + 2\beta} - 2x^\alpha.
\]
By the Lemma 2.2,
\[
[x^\beta, [x^{-\beta}, x^{\alpha + 2\beta}]] = 2x^{\alpha + 2\beta}.
\]
On the other hand,
\[
[x^\beta, [x^{-\beta}, x^{\alpha + 2\beta}]] = c_{\beta, \alpha + \beta} c_{-\beta, \alpha + 2\beta} x^{\alpha + 2\beta}.
\]
Then \(c_{\beta, \alpha + \beta} c_{-\beta, \alpha + 2\beta} = 2\). Therefore,
\[
(\text{ad} (x^\beta - x^{-\beta}))^3 x^\alpha = -c_{-\beta, \alpha + 2\beta} c_{\beta, \alpha + \beta} c_{\beta, \alpha} x^{\alpha + \beta} - 2c_{\beta, \alpha} x^{\alpha + \beta} = -4c_{\beta, \alpha} x^{\alpha + \beta}.
\]
To summarize the above calculations, we have
\[
\text{Ad} \left( \exp \left( \frac{\pi}{2} (x^\beta - x^{-\beta}) \right) \right) x^\alpha = x^\alpha + \sum_{k=0}^{\infty} \frac{1}{(2k + 1)!} \left( \frac{\pi}{2} \right)^{2k+1} (-4)^k c_{\beta, \alpha} x^{\alpha + \beta} + \sum_{\ell=1}^{\infty} \frac{1}{(2\ell)!} \left( \frac{\pi}{2} \right)^{2\ell} (-4)^{\ell-1} (c_{\beta, \alpha + \beta} c_{\beta, \alpha} x^{\alpha + 2\beta} - 2x^\alpha),
\]
where
\[
\sum_{k=0}^{\infty} \frac{1}{(2k + 1)!} \left( \frac{\pi}{2} \right)^{2k+1} (-4)^k = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)!} \left( \frac{\pi}{2} \right)^{2k+1} 2^{2k+1} = \frac{1}{2} \sin \pi = 0,
\]
\[
\sum_{\ell=1}^{\infty} \frac{1}{(2\ell)!} \left( \frac{\pi}{2} \right)^{2\ell} (-4)^{\ell-1} = \frac{1}{4} \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell}}{(2\ell)!} \left( \frac{\pi}{2} \right)^{2\ell} 2^{2\ell} = \frac{1}{4} (\cos \pi - 1) = \frac{1}{2}.
\]
Hence
\[
\text{Ad} \left( \exp \left( \frac{\pi}{2} (x^\beta - x^{-\beta}) \right) \right) x^\alpha = \frac{1}{2} c_{\beta, \alpha + \beta} c_{\beta, \alpha} x^{\alpha + 2\beta}.
\]
Since \(c_{\beta, \alpha + \beta} = \pm 2\) and \(c_{\beta, \alpha} = \pm 1\), the equation holds. \(\square\)

In the case for \((r, q) = (1, 0)\) or \((r, q) = (2, 0)\), the same equation holds if \(\beta\) is replaced by \(-\beta\).

2.3. \(G_\mathbb{R}\)-orbits. Let \(\mathfrak{g}_\mathbb{R}\) be a real form of \(\mathfrak{g}\). We denote by \(G_\mathbb{R}\) the connected Lie group corresponding to \(\mathfrak{g}_\mathbb{R}\). There are finitely many \(G_\mathbb{R}\)-orbits in the flag varieties \(\mathcal{D}\). An open \(G_\mathbb{R}\)-orbit \(\mathcal{D}\) in \(\mathcal{D}\) with the compact isotropy subgroup is called a flag domain.

There exists a unique maximal compact subgroup \(K_\mathbb{R}\) containing the isotropy subgroup at a point in \(\mathcal{D}\). Let \(\mathfrak{k}_\mathbb{R}\) be the Lie algebra of \(K_\mathbb{R}\). We then have the Cartan decomposition \(\mathfrak{g}_\mathbb{R} = \mathfrak{k}_\mathbb{R} \oplus \mathfrak{p}_\mathbb{R}^\perp\). Let \(\theta\) be the Cartan involution associated with this decomposition. A point \(x \in \mathcal{D}\) is called a Matsuki point if the Lie algebra \(\mathfrak{p}_x\) of the stabilizer \(P_x\) of \(G_\mathbb{C}\) at \(x\) contains a Cartan subalgebra \(\mathfrak{h}_x\) that is stable under the complex conjugation and the Cartan involution \(\theta\). Every \(G_\mathbb{R}\)-orbit contains a Matsuki point. A Cayley transform maps Matsuki points to Matsuki points.
Example 2.4 \((g = \mathfrak{sl}_2(\mathbb{C}))\). This is continuation of Example 2.1. We set

\[ u^\alpha := i(x^\alpha - x^{-\alpha}) = i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad h^1 := i\mathfrak{h}^\alpha = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]

\[ v^\alpha := x^\alpha + x^{-\alpha} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]

where \(i = \sqrt{-1}\) and define a real form \(g_\mathbb{R} = \mathbb{R}u^\alpha + \mathbb{R}h^1 + \mathbb{R}v^\alpha\), which coincide with \(\mathfrak{su}(1,1)\). Here \(t_\mathbb{R} = \mathbb{R}h^1\) is a maximal torus and \(t_\mathbb{R}^\perp = \mathbb{R}u^\alpha + \mathbb{R}v^\alpha\), which define the Cartan involution \(\theta\) over \(\mathbb{R}\). Then

\[ o := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad o' := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad o_\alpha := \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

are Matsuki points, and these \(G_\mathbb{R}\)-orbits are

\[ \left\{ \begin{pmatrix} 1 \\ z \end{pmatrix} \mid |z| < 1 \right\}, \quad \left\{ \begin{pmatrix} z \\ 1 \end{pmatrix} \mid |z| < 1 \right\}, \quad \left\{ \begin{pmatrix} 1 \\ z \end{pmatrix} \mid |z| = 1 \right\}. \]

The Cayley transform is \(c_\alpha = \text{Ad}(\tilde{\varrho})\) with

\[ \tilde{\varrho} = \exp\left(\frac{\pi}{4}(x^{-\alpha} - x^\alpha)\right) = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \]

where \(\tilde{\varrho}o = o_\alpha\).

2.4. Cycle connectivity. Let \(D\) be a flag domain. We choose a base point \(o \in D\). Let \(K_\mathbb{R}\) be the maximal compact subgroup containing the isotropy subgroup of \(G_\mathbb{R}\) at \(o\). Here the \(K_\mathbb{R}\)-orbit \(C_0 := K_\mathbb{R}o\) is a compact submanifold contained in \(D\). In fact, the complexification \(K\) of \(K_\mathbb{R}\) acts transitively on \(C_0\), and then \(C_0\) is a compact complex manifold.

We define the set \(\mathcal{M}_D := \{gC_0 \mid g \in G_\mathbb{C}\}\). Then \(\mathcal{M}_D\) has a natural structure of \(G\)-invariant homogeneous structure [FHW, §5]. The topological component \(\mathcal{M}_D\) of \(C_0\) in \(\{C \in \mathcal{M}_D \mid C \subset D\}\) is called the cycle space. We recall the definition of cycle connectivity.

Definition 2.5. For two points \(p\) and \(q\) in \(D\), we say \(p\) and \(q\) are cycle connected if there exists finitely many cycles \(C_1, \ldots, C_\ell\) in \(\mathcal{M}_D\) so that \(p \in C_1, q \in C_\ell\) and the chain \(C_1 \cup \cdots \cup C_\ell\) is connected. Moreover, \(D\) is said to be cycle connected if any two points in \(D\) are cycle connected.

The flag domain \(D\) is called classical if \(G_\mathbb{R}/K_\mathbb{R}\) is Hermitian symmetric and the quotient map \(D \to G_\mathbb{R}/K_\mathbb{R}\) is holomorphic or anti-holomorphic. Otherwise, \(D\) is said to be non-classical. By Huckleberry [Huc1], [Huc2], [Huc3] and [GRT], the following are equivalent:

- \(D\) is non-classical;
- \(O(D) = \mathbb{C}\);
- \(D\) is not holomorphically convex;
- \(D\) is cycle connected;

Furthermore, Huckleberry [Huc1] showed that some flag domains are cycle connected in a strong sense called one-connectivity. That is, \(p\) and \(q\) in \(D\) are one-connected if these are connected by a one cycle. He proved that \(D\) is pseudoconcave if it is generically one-connected.

We show a relationship between the one-connectivity and the roots of \(g\). Let \(P\) be the parabolic subgroup which stabilizes \(o\), and let \(\mathfrak{p}\) be the Lie subalgebra of \(P\).
By [FHW] Theorem 4.2.2, the subalgebra \( g_R \cap p \) contains a Cartan subalgebra \( h_R \), and we have a Borel subalgebra \( b \) such that
\[
h_R \otimes \mathbb{C} =: h \subset b \subset p
\]
and \( p \) is defined by a set of simple roots in the positive root system \( \Delta(b) \) of \( \Delta = \Delta(g, h) \). Now \( h_R \subset \mathfrak{t}_R \), and then \( h \subset \mathfrak{t} \). Proposition 2.3 induces the following lemma:

**Lemma 2.6.** Let \( \alpha_1, \ldots, \alpha_\ell \in \Delta(t^+) \cap \Delta(g^{>0}) \), and let \( \beta \in \Delta(t) \). Suppose that the \( \beta \)-string containing \( \alpha_i \), which is given by the form (2.4), for \( 1 \leq i \leq \ell \) satisfies one of the following conditions:

- \( (r, q) = (0, 1) \) and \( \alpha_i + \beta \in \Delta(p) \);
- \( (r, q) = (0, 2) \) and \( \alpha_i + 2\beta \in \Delta(p) \).

We choose a sufficiently small \( \varepsilon > 0 \) so that
\[
U := \left\{ k, \prod_{i=1}^\ell \exp (\varepsilon_i x^{\alpha_i}), z \mid k \in K_R, z \in C_0, |\varepsilon_i| < \varepsilon \right\}
\]
is contained in \( D \). For \( z' \in U \), there exists a point in \( C_0 \) which is one-connected to \( z' \).

**Proof.** Let \( z' \in U \), and we write \( z' = k \xi \cdot z \) with \( k \in K_R \) and
\[
\xi = \prod_{i=1}^\ell \exp (\varepsilon_i x^{\alpha_i}).
\]
For \( k_0 = \exp (\frac{\pi}{2} (x^\beta - x^{-\beta})) \), we have
\[
\text{Ad} (k_0) \xi = \prod_{i=1}^\ell \exp (\varepsilon_i \text{Ad} (k_0)(x^{\alpha_i})),
\]
which is contained in \( P \) by Proposition 2.3 and the hypothesis. Then \( \text{Ad} (k_0) \xi \cdot o = o \), and therefore \( \xi_{k_0^{-1}} \cdot o = k_0^{-1} \cdot o \) and \( k_0^{-1} \cdot o \in C_0 \cap \xi \cdot C_0 \). Moreover, \( k_0^{-1} \cdot o \) is in \( C_0 \cap k \xi \cdot C_0 \). It is concluded that \( z' \) and \( k \cdot k_0^{-1} \cdot o \) are one-connected by \( k \xi \cdot C_0 \).

2.5. **Pseudoconcavity.** We show that \( D \) is pseudoconcave if it satisfies a certain condition on the roots. We recall the definition of pseudoconcavity. A flag domain is pseudoconcave if it contains a relatively compact open subset \( \text{Int}(K) \) such that for every point of its closure \( K \) there exists a 1-dimensional holomorphic disc \( V_p \) with \( p \) at its center such that the boundary of \( \text{bd}(V_p) \) is contained in \( \text{Int}(K) \). In a similar manner to [Huc1, Theorem 3.7], it is enough to show that there exists a relatively compact open set which is filled out by cycles.

**Theorem 2.7.** Suppose that there exist a compact root \( \beta \in \Delta(t) \) such that for any \( \alpha \in \Delta(t^+) \cap \Delta(g^{>0}) \) the \( \beta \)-string containing \( \alpha \) satisfies one of the conditions of Lemma 2.6. Then \( D \) is pseudoconcave.

**Proof.** Let
\[
\{\alpha_1, \ldots, \alpha_\ell, \beta_1, \ldots, \beta_r\} = \Delta(g^{>0}),
\]
\[
\alpha_1, \ldots, \alpha_\ell \in \Delta(t^+), \quad \beta_1, \ldots, \beta_r \in \Delta(t).
\]
Choose $\varepsilon > 0$ sufficiently small so that
\[
\mathcal{K} := \left\{ k, \prod_{i=0}^{\ell} \exp (\varepsilon_i x_{\alpha_i}^{\pm}) \prod_{j=1}^{r} \exp (\varepsilon_j x_{\beta_j}), o \left| k \in K_R, \quad |\varepsilon_i|, |\varepsilon_j| \leq \varepsilon \right. \right\}
\]
is contained in $D$. Since $T_o(D) \cong g^{<0}$, the interior $\text{Int}(\mathcal{K})$ is a relatively compact open subset in $D$ and $C_0$ is contained in $\text{Int}(\mathcal{K})$.

For $z \in \mathcal{K}$, we can write $z = k_1 \xi k_2 o$ with
\[
k_1 \in K_R, \quad \xi = \prod_{i=0}^{\ell} \exp (\varepsilon_i x_{\alpha_i}^{\pm}), \quad k_2 = \prod_{j=1}^{r} \exp (\varepsilon_j x_{\beta_j}).
\]
By Lemma 2.6, we have a point $\tilde{z} \in C_0$ which is one-connected to $z'$. In fact, $\tilde{z} = k_1 k_0^{-1} o$ and $k_1 \xi C_0$ connects $z$ and $\tilde{z}$. We define
\[
z_t := k_1 \xi \exp ((1 - t)\frac{\pi}{2} (x_\beta - x_\beta)), \prod_{j=1}^{r} \exp (t \varepsilon_j x_{\beta_j}). o.
\]
for $t \in \mathbb{C}$. We then have a projective line $Y = \{z_t\}_{t \in \mathbb{P}^1}$ contained in the cycle $k_1 q C_0$, where
\[
z_0 = k_1 \xi k_0^{-1} o = k_1 k_0^{-1} o = \tilde{z}, \quad z_1 = k_1 \xi k_2 o = z.
\]
Choose a sufficient small disc $V_{z_0}$ so that $\text{bd}(V_{z_0}) \subset \text{Int}(\mathcal{K})$ and define $V_{z}$ to be the complement in $Y$. Then $\text{bd}(V_z) \subset \text{Int}(\mathcal{K})$. □

**Remark 2.8.** The above proof does not require that $D$ is non-classical. The assumption of Theorem 2.7 induces that $D$ is non-classical.

Huckleberry [Huc1, §3.2] proved that every flag domain of $SL(n, \mathbb{R})$ is one-connected. We see three examples of non-classical flag domains with 2-dimensional Cartan subgroups which are also Mumford–Tate domains (cf. [GGK2] and [KP]).

**Example 2.9.** Let $D$ be the Carayol domain. That is, the Mumford–Tate domain with $h^{2,0} = h^{1,1} = 2$ and $G_R = SU(2,1)$. The root diagram is depicted in the Figure 1 where compact roots are those within a box and the shaded area is a Weyl Chamber. The flag variety $\tilde{D}$ is associated with the grading element $S^1 + S^2$.

![Root Diagram of \text{sl}(3, \mathbb{C})](image)

Here $\{-\sigma_1, -\sigma_2\} = \Delta(t^+) \cap \Delta(g^{<0})$, and $\sigma_1 + \sigma_2 \in \Delta(t)$ satisfies the condition of Theorem 2.7. Hence $D$ is pseudoconcave.
Example 2.10. Let $D$ be the period domain with $h^{2,0} = 2$ and $h^{1,1} = 1$. Then $G_{\mathbb{R}} = SO(2, 1)$, and the root diagram is depicted in the Figure 2. The flag variety $\hat{D}$ is associated with the grading element $S^1$. Here $\{-\sigma_1, -\sigma_1 - \sigma_2\} = \Delta(t^+) \cap \Delta(g^{<0})$, and $2\sigma_1 + \sigma_2 \in \Delta(\mathfrak{t})$ satisfies the condition. Hence $D$ is pseudoconcave.

Example 2.11. Let $D$ be the period domain with $h^{3,0} = h^{2,1} = 1$. Then $G_{\mathbb{R}} = Sp(4, \mathbb{R})$, and the root diagram is depicted in the Figure 3. The flag variety $\hat{D}$ is associated with the grading element $S^1 + S^2$. In this case, the $(\sigma_1 + \sigma_2)$-string containing $-\sigma_1$ does not satisfy the condition.

3. Mumford–Tate–Chevalley Domains

We discuss cycle connectivity in connection with nilpotent orbits on Mumford–Tate–Chevalley domains. We fix the subgroups $H \subset B \subset P \subset G$ as in the previous section.

3.1. Integral structures. Let $E$ be a grading element associated with $p$. As we saw in §2.2 we have the Chevalley basis

$$\{x^\alpha \mid \alpha \in \Delta\} \cup \{e^{\sigma_1}, \ldots, e^{\sigma_r} \mid \sigma_1, \ldots, \sigma_r \in S\}.$$
We set
\[ h^j := iH^j, \]
\[ u^\alpha := \begin{cases} x^\alpha - x^{-\alpha} & \text{if } \alpha(E) \text{ is even}, \\ 1(x^\alpha - x^{-\alpha}) & \text{if } \alpha(E) \text{ is odd}; \end{cases} \]
\[ v^\alpha := \begin{cases} i(x^\alpha - x^{-\alpha}) & \text{if } \alpha(E) \text{ is even}, \\ x^\alpha - x^{-\alpha} & \text{if } \alpha(E) \text{ is odd}. \end{cases} \]

We define \( g_\mathbb{Z} = \mathfrak{t}_\mathbb{Z} \oplus \mathfrak{t}^*_\mathbb{Z} \) by
\[
\mathfrak{t}_\mathbb{Z} := \text{span}_\mathbb{Z}\{h^j, u^\alpha, v^\alpha \mid 1 \leq j \leq r, \alpha(E) \text{ is even}\}
\]
\[
\mathfrak{t}^*_\mathbb{Z} := \text{span}_\mathbb{Z}\{u^\alpha, v^\alpha \mid \alpha(E) \text{ is odd}\},
\]
which induces a Cartan decomposition (cf. [KR, §6.3]). The \( \mathbb{Q} \)-algebraic group \( G_\mathbb{Q} \) of the Lie algebra \( g_\mathbb{Q} := g_\mathbb{Z} \otimes \mathbb{Q} \) is called a Mumford–Tate–Chevalley group.

### 3.2. Hodge structures

The \( \mathbb{E} \)-eigenspace decomposition \( g = \bigoplus_{\alpha} g^\alpha \) defines a polarized \( \mathbb{Q} \)-Hodge structure \((g_\mathbb{Q}, Q_g(\bullet, \bullet))\) of weight zero where \( Q_g \) is the Killing form (cf. [KR, §6.4]). Indeed, \( g^\alpha \) is the \((p, -p)\)-component of this Hodge decomposition. Let \( F^*_g \) be the Hodge filtration of \( g \) determined \( F^*_g = \bigoplus_{k \geq 0} g^p \). Then the Lie subgroup \( P \) of the parabolic subalgebra \( p_\mathbb{Z} \) is the isotropy subgroup of \( \text{Ad}(G) \) at \( F^*_g \), and the set \( \hat{D} \) of \( \text{Ad}(g)F^*_g \) for all \( g \in G \) is isomorphic to the flag variety \( \hat{D} = G/P \). We define a Cartan subalgebra over \( \mathbb{Z} \) by
\[ h_\mathbb{Z} := \text{span}_\mathbb{Z}\{H^1, \ldots, H^r\} \subset \mathfrak{t}_\mathbb{Z}. \]

Then \( g_\mathbb{R} \cap p = g_\mathbb{R} \cap g^0 \) is compact and contains the Cartan subalgebra \( h_\mathbb{R} \otimes \mathbb{R} = h_\mathbb{R} \). Let \( o \in \hat{D} \) be the point corresponding to \( F^*_g \). We then have the Mumford–Tate domain \( D := G_{\mathbb{R}o} \). A flag domain arising in this way is called a Mumford–Tate–Chevalley domain. Here \( D \) parametrizes weight zero Hodge structures on \( g_\mathbb{Q} \) that are polarized by the Killing form and have the Mumford-Tate groups contained in \( G_\mathbb{Q} \).

### 3.3. \( SL_2 \)-orbits

We have the Cartan decomposition \( g = \mathfrak{k} \oplus \mathfrak{k}^\perp \) induced by (3.1). For a non-compact positive root \( \alpha \), i.e. \( \alpha \in \Delta(\mathfrak{k}^\perp) \cap \Delta(\mathfrak{g}^{>0}) \), we have the standard triple \((x^\alpha, H^\alpha, x^{-\alpha})\) in \( \mathfrak{g} \). Let
\[ \mathfrak{s}^\alpha_2(\mathbb{C}) := \text{span}_\mathbb{C}\{x^\alpha, H^\alpha, x^{-\alpha}\}, \]
and let \( SL_2^\alpha(\mathbb{C}) \) be the connected Lie subgroup with the Lie algebra \( \mathfrak{s}^\alpha_2(\mathbb{C}) \). Then \( H^\alpha, x^\alpha \in p \) and we have the orbit
\[ \hat{D} \supset \mathcal{O}_\alpha := SL_2^\alpha(\mathbb{C})o \cong \mathbb{P}^1 \mathbb{C} \]
by Example 2.1.

Now \( \mathfrak{s}^\alpha_2(\mathbb{R}) := \mathfrak{s}^\alpha_2(\mathbb{C}) \cap g_\mathbb{R} \) is generated by the \( \mathbb{Z} \)-basis \( \{iH^\alpha, u^\alpha, v^\alpha\} \). Then
\[ \mathfrak{s}^\alpha_2(\mathbb{R}) = (\mathfrak{s}^\alpha_2(\mathbb{R}) \cap \mathfrak{t}_\mathbb{R}) \oplus (\mathfrak{s}^\alpha_2(\mathbb{R}) \cap \mathfrak{t}^*_\mathbb{R}) \]
is a Cartan decomposition. In the orbit \( \mathcal{O}_\alpha \), we have the Matsuki points with respect to the complex conjugation and the Cartan involution associated with (3.1). We

---

This is a Hodge domain in the sense of [GGK1].
choose the Matsuki point \( o' \) in the other open \( SL_2^0(\mathbb{R}) \)-orbit, which does not contain \( o \) in \( O_\alpha \) as in Example 2.4. Then
\[
O_\alpha' := O_\alpha - \{ o' \} = \exp (g^{-\alpha}) o.
\]
We choose \( C_0 = K_{\mathbb{R}, o} \) as a base cycle and define \( \mathcal{M}_D \) and \( \mathcal{M}_D \) as in the previous section.

**Lemma 3.1.** Let \( \beta \in \Delta(t) \). Suppose that the \( \beta \)-string \( \{-\alpha + n\beta \mid r \leq n \leq q\} \) containing \(-\alpha\) satisfies one of the following conditions:
\begin{itemize}
  \item \( (r, q) = (0, 1) \) and \(-\alpha + \beta \in \Delta(p)\);
  \item \( (r, q) = (0, 2) \) and \(-\alpha + 2\beta \in \Delta(p)\).
\end{itemize}
Then there exists a point \( z_0 \in C_0 \) which is one-connected to any point in \( O_\alpha' \) by a cycle in \( \mathcal{M}_D \). Moreover, \( z_0 \) and \( z \in O_\alpha' \) are one-connected by a cycle in \( \mathcal{M}_D \) if \( z \) is sufficiently close to \( o \).

**Proof.** For \( k = \exp \left( \frac{\pi}{2} (x^\beta - x^{-\beta}) \right) \), we have \( \text{Ad} (k) x^{-\alpha} = \pm x^{-\alpha + \beta} \) or \( \pm x^{-\alpha + 2\beta} \). Then \( \text{Ad}(k) \exp(g^{-\alpha}) \in P \), and therefore
\[
k^{-1} o = k^{-1} \text{Ad} (k) \exp(g^{-\alpha}) o = \exp(g^{-\alpha}) k^{-1} o.
\]
Now \( \exp(\lambda x^{-\alpha}) o \) with \( \lambda \in \mathbb{C} \) and \( k^{-1} o \) are one-connected by \( \exp(\lambda x^{-\alpha}) C_0 \in \mathcal{M}_D \). Then \( \exp(\lambda x^{-\alpha}) C_0 \in \mathcal{M}_D \) if \( \lambda \) is sufficiently small.

### 3.4. Nilpotent orbits

We may identifies a point in \( \hat{D} \) with a filtration of \( g \). A pair \((F^0, N)\) consisting of a nilpotent \( N \in g_{\mathbb{Q}} \), as an element of \( \text{End} (g) \), and \( F^0 \in \hat{D} \) is called a nilpotent orbit if it satisfies the following conditions:
\begin{itemize}
  \item \( \exp(zN) F^0 \in D \) if \( \text{Im} (z) \) is sufficiently large;
  \item \( NF^p \subset F^{p-1} \).
\end{itemize}

Let \( \alpha \in \Delta(g^1) \). We have the Cayley transform \( c_\alpha = \text{Ad}(\tilde{g}) \) as (2.5), and put
\[
o_\alpha = \tilde{g} o,
\]
\[
y^\alpha := -ic_\alpha(x^\alpha) = \frac{i}{2} (x^{-\alpha} - x^\alpha + H^\alpha) \in g_{\mathbb{Q}},
\]
\[
y^{-\alpha} := ic_\alpha(x^{-\alpha}) = \frac{i}{2} (x^{-\alpha} - x^\alpha - H^\alpha) \in g_{\mathbb{Q}},
\]
\[
H^\alpha := c_\alpha(H^\alpha) = x^\alpha + x^{-\alpha} \in t_\mathbb{C}^1.
\]
Then \((y^\alpha, H^\alpha, y^{-\alpha})\) is a standard triple in \( g_{\mathbb{Q}} \), and \((o_\alpha, y^{-\alpha})\) is a nilpotent orbit by [KR], Lemma 6.50. In fact, if we identify the triple \((x^\alpha, H^\alpha, x^{-\alpha})\) with the triple (2.1) in \( SL(2, \mathbb{C}) \) and identify \( o_\alpha \in O_\alpha \) with \( \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in \mathbb{P}^1 \mathbb{C} \), we have
\[
\text{iy}^{-\alpha} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix},
\]
\[
\exp (i \text{iy}^{-\alpha}) o_\alpha = \exp \left( \frac{1}{2} \begin{pmatrix} t & t \\ -t & -t \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1 + t \\ 1 - t \end{pmatrix}.
\]
Here \( \exp (i \text{iy}^{-\alpha}) o_\alpha \) is in \( D \) if and only if \( t > 0 \) (see Example 2.4). In addition, the reduced limit is
\[
F^\infty := \lim_{t \to \infty} \exp (i \text{iy}^{-\alpha}) o_\alpha = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\]
and
\[ o = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \exp (iy^{-\alpha}) o_\alpha, \quad o' = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \exp (-iy^{-\alpha}) o_\alpha \]
are the Matsuki points in \( O_\alpha \). Then Lemma 3.1 induces the following:

**Theorem 3.2.** Let \( \beta \in \Delta (\mathfrak{t}) \cap \Delta (\mathfrak{g}^{>0}) \). If the \( \beta \)-string containing \(-\alpha\) satisfies one of the conditions of Lemma 3.1, then there exists a point \( z_0 \) in \( C_0 \) which is one-connected to any point in
\[ O_\alpha^* = \{ \exp (\lambda y^{-\alpha}) o_\alpha \mid \mathbb{C} \ni \lambda \neq -i \} \cup \{ F^* \} \]
by a cycle in \( M_\beta \). Moreover, \( z_0 \) and \( \exp (\lambda y^{-\alpha}) o_\alpha \) are one-connected by a cycle in \( M_D \) if \( \lambda \) is sufficiently close to 1.

**Remark 3.3.** For strongly orthogonal roots in \( \Delta (\mathfrak{g}^1) \), we can have a several variable nilpotent orbit by induction (see [KR, Theorem 6.38]). We then have the several variable \( SL_2 \)-orbit, and the several variable version of Theorem 3.2 would follow.

**Example 3.4.** Let \( \mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C}) \), and we fix a Cartan subalgebra \( \mathfrak{h} \). The Dynkin diagram for \( \mathfrak{g} \) is
\[
\begin{array}{cccccccc}
\bullet & - & \cdots & - & \bullet \\
\sigma_1 & & \sigma_2 & & \cdots & & \sigma_{n-1} & & \sigma_n \\
\end{array}
\]
For a suitable base \( \{ e_1, \ldots, e_n \} \) of \( \mathfrak{h}^* \), the set \( \Delta \) of roots is
\[ \{ \pm 2e_i, \pm e_i \pm e_j \mid i \neq j, 1 \leq i, j \leq n \} \]
and the simple roots are
\[ \sigma_i = e_i - e_{i+1} \ (i < n), \quad \sigma_n = 2e_n. \]
We set \( \alpha = \sigma_n \) and \( \beta = \sigma_{n-1} + \sigma_n \in \Delta \). Then the \( \beta \)-string containing \(-\alpha\) is
\[ -\alpha, \ -\alpha + \beta = \sigma_{n-1}, \ -\alpha + 2\beta = 2\sigma_{n-1} + \sigma_n. \]
If a grading element is given by
\[ E = S^1 + \cdots + S^{n-1} + S^n \ (\ell \leq n - 1), \]
then \( \alpha (E) = 1 \) and \( \beta (E) = 2 \), i.e. \( \alpha \in \Delta (\mathfrak{g}^1) \) and \( \beta \in \Delta (\mathfrak{t}) \cap \Delta (\mathfrak{g}^{>0}) \). Therefore, the one-connectivity of Theorem 3.2 holds.

4. Period Domains

We consider cycle connectivity of a period domain in connection with minimal degenerations. In this section, \( D \) is a period domain parameterizing Hodge structures of weight \( n \) on a rational vector space \( V \) polarized by \( Q \). Here \( G = \text{Aut} (V \otimes \mathbb{C}, Q) \) is the complex semisimple Lie group endowed with the rational and real structure.

4.1. Limit mixed Hodge structures. Let \((N, F^*)\) be a nilpotent orbit. By [S], there exists the monodromy weight filtration \( W_\bullet (N) \), and \((W_\bullet (N)[-n], F^*)\) is a mixed Hodge structure, which is called a limit mixed Hodge structure. Let
\[
\mathbb{V}_C = \bigoplus I^{p,q} \]
be the Deligne decomposition with respect to \((W_\bullet (N)[-n], F^*)\). We assume the limit mixed Hodge structure is defined over \( \mathbb{R} \), i.e. the splitting associated with (1.1) is defined over \( \mathbb{R} \). Then there exists \( Y \in \mathfrak{g}_R \) which acts on \( I^{p,q} \) by the scalar \( p + q - n \). Moreover, we have a unique element \( N^+ \in \mathfrak{g}_R \) such that \((N^+, Y, N)\)
is a standard triple. By the $\mathrm{SL}_2$-orbit theorem [S], we have a homomorphism $v : \mathrm{SL}(2, \mathbb{R}) \to G_{\mathbb{R}}$ such that

$$v_* \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = N^+, \quad v_* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Y, \quad v_* \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = N,$$

and the $\mathrm{SL}_2(\mathbb{C})$-equivalent horizontal holomorphic map $\psi : \mathbb{P}^1 \mathbb{C} \to \tilde{D}$ given by

$$\left( \frac{1}{z} \right) \mapsto \exp(zN)F^*.$$

Here $\exp(zN)F^* \in D$ for $\text{Im}(z) > 0$, and $\psi$ defines a $\mathrm{SL}_2(\mathbb{R})$-equivalent map from the upper half plane to $D$. We define

$$\varphi := \exp \left( \frac{i\pi}{4} (N^+ + N) \right).$$

Then $\varphi : S^1 \cong \{ e^{2\pi i t} \in \mathbb{C} \mid 0 \leq t < 1 \} \to G_{\mathbb{R}}$ by $\varphi(z)v = z^{2p-n}v$ for $v$ in the $(p, n-p)$-component $V^{p, n-p}$ of the Hodge decomposition with respect to $\varphi$. The grading element is

$$E_x = \frac{1}{4\pi i} \varphi'(1).$$

The image $\varphi(S^1)$ is contained in a compact maximal torus $T$ ([GGKII Proposition IV.A.2]), and the complexification $\mathfrak{h} = \mathfrak{h}_{\mathbb{R}} \otimes \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}$.

Let

$$V_{\mathbb{C}} = \bigoplus V^\mu$$

be the weight space decomposition with respect to $\mathfrak{h}$. That is, $\mu \in \mathfrak{h}^*$ and $v \in V^\mu$ if and only if $\xi(v) = \mu(\xi)v$ for all $\xi \in \mathfrak{h}$. Then we have

$$V^{p, n-p} = \bigoplus_{\mu(\mathbb{E}_x) = (2p-n)/2} V^\mu.$$

Now $\varphi$ defines the Hodge structure on $\mathfrak{g}$. In fact, the $(p, -p)$-component $\mathfrak{g}^{p, -p}$ is the eigenspace $\mathfrak{g}^p$ of $\text{ad}(\mathbb{E}_x)$. Here the Lie algebra of the parabolic subgroup stabilizing $\varphi$ is

$$\mathfrak{p} = \bigoplus_{p \leq 0} \mathfrak{g}^p.$$

Then the infinitesimal period relation is bracket-generating in the sense of [R (3.12)]. Since $\varphi(1)$ is the Weil operator, $\text{Ad}(\varphi(1))$ is a Cartan involution defined over $\mathbb{R}$, which defines the Cartan decomposition

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{t}_{\mathbb{R}}^\perp$$

such that

$$\mathfrak{t}_{\mathbb{R}} \otimes \mathbb{C} = \bigoplus_{p \; \text{even}} \mathfrak{g}^p, \quad \mathfrak{t}_{\mathbb{R}}^\perp \otimes \mathbb{C} = \bigoplus_{p \; \text{odd}} \mathfrak{g}^p.$$

The maximal compact subalgebra $\mathfrak{k}_{\mathbb{R}}$ contains the maximal torus contained in $\varphi(S^1)$. 

We set a standard triple 
\[ (\mathcal{E}, \mathcal{Z}, \mathcal{E}) := \text{Ad}_g(N^+, Y, N) \]
in \( \mathfrak{g}_C \). Then
\[
\mathcal{E} = \frac{1}{2}(N + N^+ - iY), \quad \mathcal{Z} = i(N - N^+), \quad \mathcal{E} = \frac{1}{2}(N + N^+ + iY).
\]
The horizontality condition of nilpotent orbits induces
\[ \mathcal{E} \in \mathfrak{g}^1, \quad \mathcal{Z} \in \mathfrak{g}^0, \quad \mathcal{E} \in \mathfrak{g}^{-1}. \]

By [R, Theorem 5.3(d)] and [R, Theorem 5.8(b)], for the \((p, q)\)-component of \( I^{p,q} \) of (4.1), we have
\[
\varrho(I^{p,q}) = \bigoplus_{\mu(\mathcal{E}) = (2p-n)/2, \mu(\mathcal{Z}) = p+q-n} V^\mu
\]
where \( V^\mu \) is the weight space (4.2).

4.2. Minimal degenerations. Let \((N, F^\bullet)\) be a nilpotent orbit. Then we have the reduced limit
\[
F^\bullet_{\infty} = \lim_{\text{Im}(z) \to \infty} \exp(zN)F^\bullet,
\]
which is in a \( G_2 \)-orbit in \( \bar{D} \). If \( F^\bullet_{\infty} \) is in a \( G_2 \)-orbit of codimension 1, \((N, F^\bullet)\) is called a minimal degeneration. By [GGR, Theorem 1.7], a minimal degeneration in a period domain have either
Type I: \( N \neq 0, N^2 = 0 \) and \( \text{rank} \ N = 1, 2 \), or
Type II: \( N^2 \neq 0, N^3 = 0 \) and \( \text{rank} \ N = 2 \).

For a nilpotent orbit \((N, F^\bullet)\), we have the limit mixed Hodge structure \((W(N), F^\bullet)\). We assume \((W(N), F^\bullet)\) is split over \( \mathbb{R} \). The standard triple \((N^+, Y, N)\) defines \( g \in G_C \) and \( \varphi = \varrho(F^\bullet) \). We denote by \( i^{p,q} \) the dimension of the \((p, q)\)-component \( I^{p,q} \) of the Deligne decomposition with respect to \((W(N), F^\bullet)\), and we denote by \( h^{p,q} \) the dimension of the \((p, q)\)-component \( V^{p,q} \) of the Hodge decomposition with respect to \( \varphi \). The relationship between \( \{i^{p,q}\} \) and \( \{h^{p,q}\} \) is given as follows (see Figure 4). A nilpotent orbit \((N, F^\bullet)\) is a minimal degeneration of type I if and only if there exists \( p_0 \in \mathbb{Z} \) such that \( 2p_0 < n \) satisfying the following conditions:
\[
\begin{align*}
(\text{I-1}) & \quad i^{p_0+1,n-p_0} = i^{p_0,n-p_0-1} = 1; \\
(\text{I-2}) & \quad i^{p_0,n-p_0} = h^{p_0,n-p_0} = 1 \\
(\text{I-3}) & \quad \text{for all other } p \text{ such that } 2p < n, i^{p,n-p} = h^{p,n-p}.
\end{align*}
\]
Moreover, \((N, F^\bullet)\) is a minimal degeneration of type II if and only if \( n = 2m \) is even and it satisfies the following conditions:
\[
\begin{align*}
(\text{II-1}) & \quad i^{m-1,m-1} = i^{m+1,m+1} = 1; \\
(\text{II-2}) & \quad i^{m-1,m-1} = h^{m-1,m-1} \\
(\text{II-3}) & \quad \text{for all other } p \text{ such that } 2p < n, i^{p,n-p} = h^{p,n-p}.
\end{align*}
\]

**Lemma 4.1.** Let \((N, F^\bullet)\) be a minimal degeneration.

(1) Suppose that \((N, F^\bullet)\) is of type I. Let \( v \in I^{p_0+1,n-p_0} \). We then have
\[
\varrho(v) = \frac{1}{\sqrt{2}}(v + 1Nv), \quad \varrho(Nv) = \frac{i}{\sqrt{2}}(v - iNv), \quad \varrho(\bar{v}) = i \cdot \varrho(Nv), \quad \varrho(N\bar{v}) = 1 \cdot \varrho(v).
\]

\[2\text{The index of the Deligne splitting here is shifted from the one in [R, Theorem 5.8(b)].} \]
In particular, \( \rho(v) = i \cdot \rho(N^2v) \) if \( n = 2p_o + 1 \).

(2) Suppose that \((N,F^*)\) is of type II. Let \( v \in I^{m+1,m+1} \). We then have

\[
\rho(v) = \frac{1}{2}v + \frac{1}{2}iNv - \frac{1}{4}N^2v, \quad \rho(Nv) = i(v + \frac{1}{2}N^2v), \quad \rho(N^2v) = -2\rho(v)
\]

Proof. First, we consider (1). Since

\[
(N + N)v = Nv, \quad (N + N)^2v = N^2Nv = [N^+,N]v = Yv = v,
\]

we have

\[
\rho(v) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell)!}(\frac{\pi}{4})^{2\ell}v + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(\frac{\pi}{4})^{2k+1}iNv
\]

\[
= \cos\left(\frac{\pi}{4}\right)v + i\sin\left(\frac{\pi}{4}\right)Nv = \frac{1}{\sqrt{2}}(v + iNv),
\]

and

\[
\rho(Nv) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell)!}(\frac{\pi}{4})^{2\ell}Nv + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(\frac{\pi}{4})^{2k+1}iNv
\]

\[
= \cos\left(\frac{\pi}{4}\right)Nv + i\sin\left(\frac{\pi}{4}\right)v = \frac{i}{\sqrt{2}}(v - iNv).
\]

The equations for \( \rho(\bar{v}) \) and \( \rho(N\bar{v}) \) follows from similar calculations. Since \( v = \bar{v} \) if \( n = 2p_o + 1 \), then the equation for \( n = 2p_o + 1 \) follows.

Next, we consider (2). Since

\[
N^+Nv = [N^+,N]v = Yv = 2v, \quad N^+N^2v = [N^+,N]Nv + N^+N^2v = YNv + 2Nv = 2Nv,
\]

we have

\[(N^+ + N)v = Nv, \quad (N^+ + N)^2v = N^+Nv + N^2v = 2v + N^2v, \quad (N^+ + N)^3v = 2Nv + N^+N^2v = 4Nv.\]
Then
\[ \varrho(v) = v + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (\frac{\pi}{4})^{2k+1} 4^k iNv + \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{(2\ell)!} (\frac{\pi}{4})^{2\ell} 4^{\ell-1} (2v + N^2v). \]

Here
\begin{align*}
(4.7) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (\frac{\pi}{4})^{2k+1} 4^k &= \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (\frac{\pi}{4})^{2k+1} 2^{2k+1} = \frac{1}{2} \sin(\frac{\pi}{2}) = \frac{1}{2}, \\
\sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{(2\ell)!} (\frac{\pi}{4})^{2\ell} 4^{\ell-1} &= \frac{1}{4} \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{(2\ell)!} (\frac{\pi}{4})^{2\ell} 2^{2\ell} = \frac{1}{4} (\cos(\frac{\pi}{2}) - 1) = -\frac{1}{4}.
\end{align*}

Therefore,
\[ \varrho(v) = \frac{1}{2} v + \frac{1}{2} iNv - \frac{1}{4} N^2v. \]

By (4.6) and (4.7),
\[ \varrho(Nv) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell)!} (\frac{\pi}{4})^{2\ell} 4^{\ell} (Nv) + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (\frac{\pi}{4})^{2k+1} 4^k (2v + N^2v) \]
\[ = \cos(\frac{\pi}{2}) Nv + \frac{i}{2} (2v + N^2v) = i(v + \frac{1}{2} N^2v), \]

and, since \((N + N)N^2v = N^2N^2v = 2Nv,
\[ \varrho(N^2v) = N^2v + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (\frac{\pi}{4})^{2k+1} 4^k 2iNv + \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{(2\ell)!} (\frac{\pi}{4})^{2\ell} 4^{\ell-1} 2(2v + N^2v) \]
\[ = N^2v + iNv - \frac{1}{2} (2v + N^2v) = \frac{1}{2} N^2v + iNv - v = -2\varrho(v). \]

\[ \square \]

Let \( K_R \) be the Lie group corresponding to \( \frak{t}_R \) of (1.4). We then have \( \mathcal{M}_D \) and \( \mathcal{M}_D \) with the base cycle \( C_0 = K_R \varphi. \)

**Proposition 4.2.** There exists a point \( z_0 \in C_0 \) which is one-connected to any point in \( \operatorname{exp}(C \mathcal{E}) \varphi \) by a cycle in \( \mathcal{M}_D \) if there exists \( p \in \mathbb{Z} \) such that \( i^{p,n-p} \neq 0 \) satisfying
\[ p = p_o + 2\ell \text{ or } p = p_o - 2\ell + 1 \text{ with } \ell \geq 1 \text{ if } (N, F^*) \text{ is of type I,} \]
\[ p = m + 2\ell + 1 \text{ with } \ell \in \mathbb{Z} \text{ if } (N, F^*) \text{ is of type II.} \]

Moreover, \( \operatorname{exp}(\lambda \mathcal{E}) \varphi \) and \( z_0 \) are one-connected by a cycle in \( \mathcal{M}_D \) if \( \lambda \) is sufficiently small.

**Proof.** It is enough to show that there exists \( k \in K \) such that \( \operatorname{Ad}(k)(\mathcal{E}) \in p. \) We first consider the case for type I. Now \( V_C = \bigoplus I^{p,q} \) is the Deligne complex with respect to \( (W_*(N)[-n], F^*) \), and \( V_C = \bigoplus V^{p,n-p} \) is the Hodge complex with respect to \( \varphi. \) Let \( v \in I^{p,o+1,n-p_o}. \) Since \( N \) is a \((-1,-1)-\)morphism, \( Nv \in I^{p_o,n-p_o-1}. \) Then, by Lemma 4.3 and (4.3),
\begin{align*}
(4.8) \quad \varrho(v) &\in \varrho(I^{p_o+1,n-p_o}) \subset V^{p_o+1,n-p_o-1}, \\
\mathcal{E} \varrho(v) &= \varrho(Nv) \in \varrho(I^{p_o,n-p_o-1}) \subset V^{p_o,n-p_o}, \\
\overline{\mathcal{E} \varrho(v)} &= -i\varrho \bar{v}, \quad \mathcal{E} \cdot \overline{\varrho(v)} = -i\varrho(N\bar{v}) = \overline{\varrho(v)}.
\end{align*}
In particular, if $n = 2p_o + 1$, then

\[(4.9) \quad \varphi(v) = i \cdot \overline{E \varphi(v)}, \quad E \cdot \overline{E \varphi(v)} = -i E \varphi(v).\]

Suppose that there exists $p$ such that $i^{p,n-p} \neq 0$ satisfying $p = p_o + 2\ell$ with $\ell \geq 1$. We put $w := E \varphi(v)$. We may assume $\|w\| = 1$ for the Hodge norm $\| \cdot \| = Q(\varphi(i) \cdot, \cdot)$. Let $u \in V^{p,n-p}$ such that $\|u\| = 1$. We define $k \in \text{Aut}(V_C)$ by

\[(4.10) \quad kw = u, \quad ku = w, \quad k\bar{u} = \bar{w}, \quad k\bar{w} = \bar{u}, \quad k\bar{v}' = v' \quad \text{if} \quad v' \perp w, u, \bar{w} \quad \text{and} \quad \bar{u} \quad \text{with respect to} \quad Q.

Then $k \in G$, and $\text{Ad}(\varphi(i))k = k$, i.e. $k \in K$ (see Figure 5).

**Figure 5.** Hodge decompositions with respect to $\varphi$
By (4.18) and (4.19),
\[
\text{Ad} (k)(\mathcal{E}) \phi(v) = \begin{cases} 
\text{i Ad} (k)(\mathcal{E}) \bar{w} = \text{i} k \mathcal{E} \bar{u} = 0 & \text{if } n = 2p_o + 1, \\
kw = u & \text{otherwise};
\end{cases}
\]
\[
\text{Ad} (k)(\mathcal{E}) \bar{u} = k \mathcal{E} \bar{w} = \begin{cases} 
-\text{i} kw = -\text{i} u & \text{if } n = 2p_o + 1, \\
\mathcal{E} \bar{w} & \text{otherwise};
\end{cases}
\]
\[
\text{Ad} (k)(\mathcal{E}) v' = 0 & \text{if } v' \perp u \text{ and } \overline{\phi(v)}.
\]
Therefore,
\[
(4.11) \quad \text{Ad} (k)(\mathcal{E}) \in \begin{cases} 
g^{4\ell-1} \subset \mathfrak{p} & \text{if } n = 2p_o + 1, \\
g^{2\ell-1} \subset \mathfrak{p} & \text{otherwise}.
\end{cases}
\]
If there exists \( p \) such that \( i^{p,n-p} \neq 0 \) satisfying \( p = p_o - 2\ell + 1 \) with \( \ell \geq 1 \), we put \( w = \phi(v) \) instead and define \( k \in K \) as (4.10). Then we obtain (4.11).

Next, we consider the case for type II. Let \( v \in \mathcal{I}^{m+1,m+1} \) such that \( \|\phi(v)\| = 1 \) and put \( w = \phi(v) \). Then
\[
w \in g(I^{m+1,m+1}) \subset V^{m+1,m-1},
\]
\[
\overline{\mathcal{E}w} = -\mathcal{E}w \in g(I^{m,m}) \subset V^{m,m},
\]
\[
-2\bar{w} = \mathcal{E}^2 w.
\]
If there exists \( p \) such that \( i^{p,n-p} \neq 0 \) satisfying \( p = m + 2\ell + 1 \), we may assume \( \ell < 0 \), then we choose \( u \in V^{p,n-p} \) so that \( \|u\| = 1 \). We define \( k \in K \) as (4.10), and then
\[
\text{Ad} (k)(\mathcal{E}) u = k \mathcal{E} w = \mathcal{E} w, \quad \text{Ad} (k)(\mathcal{E}) \bar{u} = -2\bar{u},
\]
\[
\text{Ad} (k)(\mathcal{E}) v' = 0 & \text{if } v' \perp \bar{u} \text{ and } \overline{\mathcal{E}w}.
\]
Therefore, \( \text{Ad} (k)(\mathcal{E}) \in g^{-2\ell-1} \subset \mathfrak{p} \).

\[ \square \]

**Remark 4.3.** In the minimal degeneration, \( N \) is a root vector with respect to a Cartan subalgebra \( \mathfrak{h}' \) defined over \( \mathbb{R} \) contained in the stabilizer of \( F^* \) by [K.P]. Then \( \mathcal{E} \) is a root vector with respect to the Cartan subalgebra \( \text{Ad} (\mathfrak{h}) \mathfrak{h}' \).

If the weight \( n \) is odd, then \( \mathfrak{G}_\mathbb{R} \cong Sp(r,\mathbb{R}) \) and \( \mathfrak{K}_\mathbb{R} \cong U(r) \) where
\[
r = \sum_{p < (n+1)/2} h^{p,n-p}.
\]
Hence \( \mathfrak{G}_\mathbb{R}/\mathfrak{K}_\mathbb{R} \) is isomorphic to the Hermitian symmetric domain \( \mathcal{B} \) of type III, and \( \mathcal{D} \) is Hermitian case in the sense of [F.H.W. §5.2], i.e. \( \mathcal{M}_D \cong \mathcal{B} \times \mathcal{B} \). In this case, any minimal degeneration is type I. Moreover, any minimal degeneration is either even type or odd type in the sense of [H2]. By Lemma 4.1 and (4.18), \( \|\mathcal{E}\| = 1 \) for the operator norm. Therefore, \( \exp (z \mathcal{E}) C_o \subset D \), i.e. \( \exp (z \mathcal{E}) C_o \in \mathcal{M}_D \), if \( |z| < 1 \) (see [H2 §2.1–2.2] for more detail). It concludes the following proposition:

**Proposition 4.4.** Let us assume the weight \( n \) is odd. There exists a point \( z_0 \in C_0 \) which is one-connected to any two points in
\[
\{ \exp (\lambda \mathcal{E}) \varphi \in D \mid |\lambda| < 1 \}
\]
by a cycle in \( \mathcal{M}_D \) if the Hodge numbers satisfies the condition of Proposition 4.2.

Here \( z_0 \) and \( \exp (\lambda \mathcal{E}) \varphi \) are one-connected by \( \exp (\lambda \mathcal{E}) C_o \), and \( \exp (\mathcal{E}) \varphi \) is the reduced limit.
References

[FHW] G. Fels, A. Huckleberry and J. A. Wolf, Cycle Spaces of Flag Domains: A Complex Geometric Viewpoint, Progress in Mathematics 45. Birkhauser Boston, Inc., 2006.

[GGK1] M. Green, P. Griffiths and M. Kerr, Mumford-Tate groups and domains: their geometry and arithmetic, Annals of Math Studies 183. Princeton University Press, 2012.

[GGK2] M. Green, P. Griffiths and Matt Kerr, Hodge theory, complex geometry, and representation theory, CBMS Regional Conference Series in Mathematics 118. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2013.

[GGR] M. Green, P. Griffiths and C. Robles, Extremal degenerations of polarized Hodge structures, preprint [arXiv:1403.0640].

[GRT] P. Griffiths, C. Robles and D. Toledo, Quotients of non-classical flag domains are not algebraic, Algebr. Geom. 1 (2014), no. 1, 1–13.

[H1] T. Hayama, On the boundary of the moduli spaces of log Hodge structures, II: non-trivial torsors, Nagoya Math. J. 213, 1-20, (2014).

[H2] T. Hayama, Boundaries of cycle spaces and degenerating Hodge structures, Asian J. of Math. 18 (2014), 687–706.

[Hum] J. Humphreys, Introduction to Lie algebras and representation theory, Graduate Texts in Mathematics, 9. Springer-Verlag, New York-Berlin, 1972.

[Huc1] A. Huckleberry, Remarks on homogeneous manifolds satisfying Levi-conditions, Bollettino U.M.I. (9) III (2010) 1-23.

[Huc2] A. Huckleberry, Hyperbolicity of cycle spaces and automorphism groups of flag domains, Amer. J. Math. 135 (2013), no. 2, 291–310.

[Huc3] A. Huckleberry, Cycle Connectivity and Automorphism Groups of Flag Domains, in Developments and Retrospectives in Lie Theory, Developments in Mathematics 37, Springer International Publishing, 2014, pp 113–126.

[K] J. Kollár, Neighborhoods of subvarieties in homogeneous spaces, preprint (arXiv:1308.5603).

[KR] M. Kerr and C. Robles, Hodge theory and real orbits in flag varieties, preprint (arXiv:1407.3507).

[KP] M. Kerr and G. Pearlstein, Naive boundary strata and nilpotent orbits, to appear in Annales de l’Institut Fourier.

[R] C. Robles, Classification of horizontal SL(2)’s, preprint (arXiv:1405.3163).

[S] W. Schmid, Variation of Hodge structure: the singularities of the period mapping, Invent. Math. 22 (1973), 211–319.

Mathematical Sciences Center, Tsinghua University, Haidian District, Beijing 100084, China
E-mail address: tatsuki@math.tsinghua.edu.cn