On geometry of gonometric family of cycles

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Abstract

An examples solutions of the equation for curvature of congruence of cycles are considered. Their properties are discussed.

1 Gonometric family of cycles

Two parametrical family of cycles on the plane is determined by the equation

$$(\xi - x)^2 + (\eta - y)^2 - \phi(x, y)^2 = 0.$$  

The angle metric in a given family of cycles is defined by the expression [1]

$$ds^2 = \left(1 - \left(\frac{\partial}{\partial x} \phi(x, y)\right)^2\right) dx^2 - 2 \frac{\partial}{\partial x} \phi(x, y) \frac{\partial}{\partial y} \phi(x, y) dx dy + \left(1 - \left(\frac{\partial}{\partial y} \phi(x, y)\right)^2\right) dy^2 \quad (\phi(x, y))^2. \quad (1)$$

The expression for the curvature of the metric (1) has the form

$$K(x, y) = \frac{\phi \left(1 - \left(\frac{\partial}{\partial y} \phi\right)^2\right) \frac{\partial^2}{\partial x^2} \phi + 2 \left(\frac{\partial}{\partial y} \phi\right) \left(\frac{\partial}{\partial x} \phi\right) \frac{\partial}{\partial x \partial y} \phi + \left(1 - \left(\frac{\partial}{\partial x} \phi\right)^2\right) \frac{\partial}{\partial y^2} \phi}{\left(1 - \left(\frac{\partial}{\partial x} \phi\right)^2 - \left(\frac{\partial}{\partial y} \phi\right)^2\right)^2} - \frac{\phi^2 \left(\frac{\partial^2}{\partial x^2} \phi - \left(\frac{\partial^2}{\partial x \partial y} \phi\right)^2\right)}{\left(1 - \left(\frac{\partial}{\partial x} \phi\right)^2 - \left(\frac{\partial}{\partial y} \phi\right)^2\right)^2} - \left(1 - \left(\frac{\partial}{\partial x} \phi\right)^2 - \left(\frac{\partial}{\partial y} \phi\right)^2\right)^{-1} + 1. \quad (2)$$

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The congruence of cycles of constant curvature

The congruence of cycles of constant curvature $K(x, y) = K$ are defined by the Monge-Ampere equation [1]

$$K = \phi \left( \left( 1 - \frac{\partial \phi}{\partial y} \right)^2 \frac{\partial^2 \phi}{\partial x^2} + 2 \frac{\partial \phi}{\partial y} \frac{\partial}{\partial x} \phi \frac{\partial^2 \phi}{\partial x \partial y} + \left( 1 - \frac{\partial \phi}{\partial x} \right)^2 \frac{\partial^2 \phi}{\partial y^2} \right) - \phi^2 \left( \left( \frac{\partial^2 \phi}{\partial x^2} \right)^2 - \left( \frac{\partial^2 \phi}{\partial y^2} \right)^2 \right) - \left( \left( \frac{\partial \phi}{\partial x} \right)^2 - \left( \frac{\partial \phi}{\partial y} \right)^2 \right)^{-1} + 1. \quad (3)$$

2.1 Method of solution

For solutions of the Monge-Ampere equation

$$F(x, y, f_x, f_y, f_{xx}, f_{xy}, f_{yy}) = 0$$

we use the method of solution of the p.d.e.’s described first in [2] and developed later in [3].

This method allow us to construct particular solutions of the partial nonlinear differential equation

$$F(x, y, z, f_x, f_y, f_z, f_{xx}, f_{xy}, f_{yy}, f_{yz}, f_{xxx}, f_{xyy}, f_{xxy}, \ldots) = 0 \quad (4)$$

with the help of transformation of the function and corresponding variables.

Essence of the method consists in a following presentation of the functions and variables

$$f(x, y, z) \rightarrow u(x, t, z), \quad y \rightarrow v(x, t, z), \quad f_x \rightarrow u_x - \frac{u_t}{v_t} v_x,$$

$$f_z \rightarrow u_z - \frac{u_t}{v_t} v_z, \quad f_y \rightarrow u_t, \quad f_{yy} \rightarrow \frac{(u/v_t)_t}{v_t}, \quad f_{xy} \rightarrow \frac{(u/v_t)_t}{v_t}, \quad \ldots \quad (5)$$

where variable $t$ is considered as parameter.

Remark that conditions of the type

$$f_{xy} = f_{yx}, \quad f_{xz} = f_{zx} \ldots$$

are fulfilled at the such type of presentation.

In result instead of equation (4) one get the relation between the new variables $u(x, t, z), \ v(x, t, z)$ and their partial derivatives

$$\Psi(u, v, u_x, u_z, u_t, v_x, v_z, v_t, \ldots) = 0. \quad (6)$$

This relation coincides with initial p.d.e at the condition $v(x, t, z) = t$ and lead to the new p.d.e

$$\Phi(\omega, \omega_x, \omega_t, \omega_{xx}, \omega_{xt}, \omega_{tt}, \ldots) = 0 \quad (7)$$

when the functions $u(x, t, s) = F(\omega(x, t, z), \omega_t, \ldots)$ and $v(x, t, s) = \Phi(\omega(x, t, z), \omega_t, \ldots)$ are expressed through the auxiliary function $\omega(x, t, s)$.

Remark that there are a various means to reduce the relation (6) into the partial differential equation.

In a some cases the solution of equation (7) is a more simple problem than solution of equation (4).
Remark 1. As the example we consider the Monge-Ampere equation

\[
\left( \frac{\partial^2}{\partial x^2} f(x,y) \right) \frac{\partial^2}{\partial y^2} f(x,y) - \left( \frac{\partial^2}{\partial x \partial y} f(x,y) \right)^2 + 1 = 0.
\]

After the \((u, v)\)-transformation

\[
v(x, t) = \left( \frac{\partial}{\partial t} \omega(x, t) \right) t - \omega(x, t),
\]

\[
u(x, t) = \left( \frac{\partial}{\partial t} \omega(x, t) \right)
\]

it takes the form of linear equation

\[
t^4 \frac{\partial^2}{\partial t^2} \omega(x, t) - \frac{\partial^2}{\partial x^2} \omega(x, t) = 0.
\]

with general solution

\[
\omega(x, t) = t \left( \mathcal{F}_1 \left( - \frac{tx - 1}{t} \right) + \mathcal{F}_2 \left( \frac{tx + 1}{t} \right) \right),
\]

depending from two arbitrary functions.

Choice of the functions \(\mathcal{F}_i\) and elimination of the parameter \(t\) from corresponding relations lead to the function \(f(x,y)\) satisfying the Monge-Ampere equation.

2.2 Congruence of cycles of zero curvature

In the case \(K = 0\) we get the equation

\[
\phi(x, y) \frac{\partial^2}{\partial x^2} \phi(x, y) - \phi(x, y) \left( \frac{\partial^2}{\partial x^2} \phi(x, y) \right) \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \\
2 \phi(x, y) \left( \frac{\partial}{\partial y} \phi(x, y) \right) \left( \frac{\partial}{\partial x} \phi(x, y) \right) \frac{\partial^2}{\partial x \partial y} \phi(x, y) + \phi(x, y) \frac{\partial^2}{\partial y^2} \phi(x, y) - \\
- \phi(x, y) \left( \frac{\partial^2}{\partial y^2} \phi(x, y) \right) \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 - \left( \phi(x, y) \right)^2 \left( \frac{\partial^2}{\partial x^2} \phi(x, y) \right) \frac{\partial^2}{\partial y^2} \phi(x, y) + \\
+ \left( \phi(x, y) \right)^2 \left( \frac{\partial^2}{\partial x \partial y} \phi(x, y) \right)^2 - \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 - \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \\
+ \left( \frac{\partial}{\partial x} \phi(x, y) \right)^4 + 2 \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^4 = 0
\]

After applying of the \((u,v)\)-transformation at this equation is reduced to the relation

\[
\left( \frac{\partial}{\partial t} u(x,t) \right)^4 - 4 \left( \frac{\partial}{\partial t} u(x,t) \right) \left( \frac{\partial}{\partial t} v(x,t) \right) \left( \frac{\partial}{\partial t} u(x,t) \right)^3 \left( \frac{\partial}{\partial x} v(x,t) \right)^3 - \\
-4 \left( \frac{\partial}{\partial x} u(x,t) \right)^3 \left( \frac{\partial}{\partial t} v(x,t) \right)^3 \left( \frac{\partial}{\partial t} u(x,t) \right) \frac{\partial}{\partial x} v(x,t) + 
\]
$$+ 6 \left( \frac{\partial}{\partial x} u(x,t) \right)^2 \left( \frac{\partial}{\partial t} v(x,t) \right)^2 \left( \frac{\partial}{\partial x} u(x,t) \right)^2 \left( \frac{\partial}{\partial x} v(x,t) \right)^2 -$$

$$- u(x,t) \left( \frac{\partial}{\partial t} u(x,t) \right)^2 \left( \frac{\partial}{\partial t} v(x,t) \right)^2 \left( \frac{\partial}{\partial x} u(x,t) \right)^2 + u(x,t) \left( \frac{\partial}{\partial t} v(x,t) \right)^2 \left( \frac{\partial}{\partial t} u(x,t) \right)^2 \left( \frac{\partial}{\partial t} u(x,t) \right)^2$$

$$- (u(x,t))^2 \left( \frac{\partial}{\partial t} u(x,t) \right)^2 \left( \frac{\partial}{\partial x} v(x,t) \right)^2 \frac{\partial^2}{\partial t^2} v(x,t) + 2 \left( \frac{\partial}{\partial t} u(x,t) \right)^4 \left( \frac{\partial}{\partial x} v(x,t) \right)^2 -$$

$$- \left( \frac{\partial}{\partial t} u(x,t) \right)^2 \left( \frac{\partial}{\partial t} v(x,t) \right)^2 - 2 \left( \frac{\partial}{\partial t} u(x,t) \right)^2 \left( \frac{\partial}{\partial t} v(x,t) \right)^2 \left( \frac{\partial}{\partial t} u(x,t) \right) \frac{\partial^2}{\partial t \partial x} v(x,t) +$$

$$+ (u(x,t))^2 \left( \frac{\partial}{\partial t} u(x,t) \right) \left( \frac{\partial^2}{\partial x^2} v(x,t) \right) \left( \frac{\partial}{\partial t} v(x,t) \right) \frac{\partial^2}{\partial t^2} u(x,t) +$$

$$+(u(x,t))^2 \left( \frac{\partial^2}{\partial x^2} u(x,t) \right) \left( \frac{\partial}{\partial t} v(x,t) \right) \left( \frac{\partial}{\partial t} u(x,t) \right) \frac{\partial^2}{\partial t^2} v(x,t) + (u(x,t))^2 \left( \frac{\partial}{\partial t} u(x,t) \right)^2 \left( \frac{\partial^2}{\partial t \partial x} v(x,t) \right)^2 -$$

$$- u(x,t) \left( \frac{\partial}{\partial t} v(x,t) \right)^2 \left( \frac{\partial}{\partial t} u(x,t) \right) \frac{\partial^2}{\partial x^2} u(x,t) \left( \frac{\partial}{\partial t} v(x,t) \right) \left( \frac{\partial}{\partial t} u(x,t) \right)^3 \left( \frac{\partial}{\partial t} u(x,t) \right) \frac{\partial^2}{\partial t \partial x} v(x,t) -$$

$$- 2 \left( u(x,t) \right)^3 \left( \frac{\partial}{\partial t} v(x,t) \right) \frac{\partial^2}{\partial t \partial x} u(x,t) \left( \frac{\partial}{\partial t} v(x,t) \right) - 4 \left( \frac{\partial}{\partial t} u(x,t) \right)^3 \left( \frac{\partial}{\partial t} v(x,t) \right) \left( \frac{\partial}{\partial t} u(x,t) \right) \frac{\partial^2}{\partial t \partial x} v(x,t) -$$

$$- u(x,t) \left( \frac{\partial}{\partial t} v(x,t) \right) \left( \frac{\partial}{\partial t} u(x,t) \right) \frac{\partial^2}{\partial t^2} v(x,t) - u(x,t) \left( \frac{\partial}{\partial t} v(x,t) \right) \left( \frac{\partial}{\partial t} u(x,t) \right) \left( \frac{\partial}{\partial t} v(x,t) \right)^2 \frac{\partial^2}{\partial t^2} v(x,t) +$$

$$+ 2 \left( u(x,t) \right) \left( \frac{\partial}{\partial t} v(x,t) \right)^2 \left( \frac{\partial}{\partial t} u(x,t) \right) \frac{\partial^2}{\partial t \partial x} u(x,t) +$$

$$+ 2 \left( u(x,t) \right) \left( \frac{\partial}{\partial t} v(x,t) \right)^2 \left( \frac{\partial}{\partial t} u(x,t) \right) \left( \frac{\partial}{\partial t} v(x,t) \right) \frac{\partial^2}{\partial t \partial x} v(x,t) -$$

$$- 2 \left( u(x,t) \right)^2 \left( \frac{\partial}{\partial t} v(x,t) \right)^2 \left( \frac{\partial}{\partial t} u(x,t) \right) \frac{\partial^2}{\partial x^2} v(x,t) +$$

$$+ u(x,t) \left( \frac{\partial}{\partial t} v(x,t) \right) \left( \frac{\partial}{\partial t} u(x,t) \right) \frac{\partial^2}{\partial t^2} v(x,t) -$$

$$- (u(x,t))^2 \left( \frac{\partial^2}{\partial x^2} u(x,t) \right) \left( \frac{\partial}{\partial t} v(x,t) \right)^2 \left( \frac{\partial}{\partial t} u(x,t) \right)^2 + 2 \left( \frac{\partial}{\partial t} v(x,t) \right)^3 \left( \frac{\partial}{\partial t} u(x,t) \right) \left( \frac{\partial}{\partial t} v(x,t) \right) \left( \frac{\partial}{\partial t} u(x,t) \right) \frac{\partial}{\partial x} v(x,t) +$$

$$+ u(x,t) \left( \frac{\partial}{\partial t} u(x,t) \right)^3 \left( \frac{\partial}{\partial t} v(x,t) \right) \frac{\partial^2}{\partial x^2} v(x,t) + \left( \frac{\partial}{\partial t} u(x,t) \right)^4 \left( \frac{\partial}{\partial t} v(x,t) \right)^4 +$$

$$+ \left( \frac{\partial}{\partial t} u(x,t) \right)^4 \left( \frac{\partial}{\partial t} v(x,t) \right)^4 - \left( \frac{\partial}{\partial t} v(x,t) \right)^4 \left( \frac{\partial}{\partial t} u(x,t) \right)^2 -$$

$$- \left( \frac{\partial}{\partial t} v(x,t) \right)^2 \left( \frac{\partial}{\partial t} u(x,t) \right)^2 \left( \frac{\partial}{\partial t} v(x,t) \right)^2 + 2 \left( \frac{\partial}{\partial t} u(x,t) \right)^2 \left( \frac{\partial}{\partial t} v(x,t) \right)^2 \left( \frac{\partial}{\partial t} u(x,t) \right)^2 +$$
\[ \begin{align*}
+ u(x, t) \left( \frac{\partial}{\partial t} v(x, t) \right)^2 \frac{\partial^2}{\partial t^2} u(x, t) + u(x, t) \left( \frac{\partial}{\partial t} v(x, t) \right)^4 \frac{\partial^2}{\partial x^2} u(x, t) + \\
+ (u(x, t))^2 \left( \frac{\partial^2}{\partial t \partial x} u(x, t) \right)^2 \left( \frac{\partial}{\partial t} v(x, t) \right)^2 = 0.
\end{align*} \]

From here after the choice of the functions \( u \) and \( v \) in the form

\[ v(x, t) = t \frac{\partial}{\partial t} \omega(x, t) - \omega(x, t), \quad u(x, t) = \frac{\partial}{\partial t} \omega(x, t) \]

we find the equation

\[ - \left( \frac{\partial^2}{\partial t^2} \omega(x, t) \right) t^2 \left( \frac{\partial}{\partial x} \omega(x, t) \right)^2 - t \frac{\partial}{\partial t} \omega(x, t) - \left( \frac{\partial^2}{\partial t^2} \omega(x, t) \right) \left( \frac{\partial}{\partial t} \omega(x, t) \right) t \frac{\partial^2}{\partial x^2} \omega(x, t) + \\
+ \left( \frac{\partial^2}{\partial t^2} \omega(x, t) \right) \left( \frac{\partial}{\partial t} \omega(x, t) \right) t^3 \frac{\partial^2}{\partial x^2} \omega(x, t) + \left( \frac{\partial}{\partial t} \omega(x, t) \right) t \left( \frac{\partial^2}{\partial t \partial x} \omega(x, t) \right)^2 - \\
- \left( \frac{\partial}{\partial t} \omega(x, t) \right) t^3 \left( \frac{\partial^2}{\partial x^2} \omega(x, t) \right)^2 - \left( \frac{\partial}{\partial t} \omega(x, t) \right) t \left( \frac{\partial}{\partial x} \omega(x, t) \right)^2 + \\
+ \frac{\partial^2}{\partial t^2} \omega(x, t) - \left( \frac{\partial^2}{\partial t^2} \omega(x, t) \right) t^2 + \left( \frac{\partial^2}{\partial t^2} \omega(x, t) \right) \left( \frac{\partial}{\partial x} \omega(x, t) \right)^4 + \\
+ 2 \left( \frac{\partial}{\partial x} \omega(x, t) \right)^2 \left( \frac{\partial^2}{\partial t^2} \omega(x, t) \right) + 2 \left( \frac{\partial}{\partial t} \omega(x, t) \right) t^2 \left( \frac{\partial^2}{\partial t \partial x} \omega(x, t) \right) \frac{\partial}{\partial x} \omega(x, t) + \\
+ \left( \frac{\partial}{\partial t} \omega(x, t) \right)^2 \frac{\partial^2}{\partial x^2} \omega(x, t) = 0
\]

having the particular solution

\[ \omega(x, t) = A(t) + x, \]

where

\[ -2 \left( \frac{d^2}{dt^2} A(t) \right) t^2 - 2 t \frac{d}{dt} A(t) + 4 \frac{d^2}{dt^2} A(t) = 0. \]

General solution of this equation is

\[ A(t) = \text{C1} + \text{C2} \ln(t + \sqrt{t^2 - 2}) \]

Now elimination of the variable \( t \) from the relations

\[ y \sqrt{t^2 - 2} - t \text{C2} + \text{C1} \sqrt{t^2 - 2} + \text{C2} \ln(t + \sqrt{t^2 - 2}) \sqrt{t^2 - 2} - 2 + x \sqrt{t^2 - 2} = 0, \]

and

\[ \phi(x, y) \sqrt{t^2 - 2} - \text{C2} = 0 \]

give us the function \( \phi(x, y) \) defined from the equation

\[ y - \sqrt{2 \left( \phi(x, y) \right)^2 + 1} + \text{C1} \ln\left( \frac{\sqrt{2 \left( \phi(x, y) \right)^2 + 1} + 1}{\phi(x, y)} \right) + x = 0 \]

satisfying the equation \( (8) \).
2.3 Congruence of positive constant curvature

In the case $K = 1$ from (3) we get the equation

$$
\phi(x, y) \frac{\partial^2}{\partial x^2} \phi(x, y) - \phi(x, y) \left( \frac{\partial^2}{\partial x^2} \phi(x, y) \right) \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + 
$$

$$
+ 2 \phi(x, y) \left( \frac{\partial}{\partial y} \phi(x, y) \right) \left( \frac{\partial}{\partial x} \phi(x, y) \right) \frac{\partial^2}{\partial x \partial y} \phi(x, y) + 
$$

$$
+ \phi(x, y) \frac{\partial^2}{\partial y^2} \phi(x, y) - \phi(x, y) \left( \frac{\partial^2}{\partial y^2} \phi(x, y) \right) \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 - 
$$

$$
- \left( \phi(x, y) \right)^2 \left( \frac{\partial^2}{\partial x^2} \phi(x, y) \right) \frac{\partial^2}{\partial y^2} \phi(x, y) + \left( \phi(x, y) \right)^2 \left( \frac{\partial^2}{\partial x \partial y} \phi(x, y) \right)^2 - 1 + 
$$

$$
+ \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 = 0. \quad (9)
$$

The $(u, v)$-transformation with condition

$$
v(x, t) = t \frac{\partial}{\partial t} \omega(x, t) - \omega(x, t), \quad u(x, t) = \frac{\partial}{\partial t} \omega(x, t)
$$

lead to the equation

$$
\left( \frac{\partial}{\partial t} \omega(x, t) \right)^2 \frac{\partial^2}{\partial x^2} \omega(x, t) + \left( \frac{\partial^2}{\partial t^2} \omega(x, t) \right) t^2 - \left( \frac{\partial^2}{\partial t^2} \omega(x, t) \right) t^4 + 
$$

$$
+ 2 \left( \frac{\partial}{\partial t} \omega(x, t) \right) t^2 \left( \frac{\partial^2}{\partial t \partial x} \omega(x, t) \right) \frac{\partial}{\partial x} \omega(x, t) + \left( \frac{\partial^2}{\partial t^2} \omega(x, t) \right) \left( \frac{\partial}{\partial t} \omega(x, t) \right) t^3 \frac{\partial^2}{\partial x^2} \omega(x, t) - 
$$

$$
- \left( \frac{\partial^2}{\partial t^2} \omega(x, t) \right) \left( \frac{\partial}{\partial t} \omega(x, t) \right) t \frac{\partial^2}{\partial x^2} \omega(x, t) - \left( \frac{\partial}{\partial t} \omega(x, t) \right) t^3 \left( \frac{\partial^2}{\partial t \partial x} \omega(x, t) \right)^2 + 
$$

$$
+ \left( \frac{\partial}{\partial t} \omega(x, t) \right) t \left( \frac{\partial^2}{\partial t \partial x} \omega(x, t) \right)^2 + \left( \frac{\partial^2}{\partial t^2} \omega(x, t) \right) t^2 \left( \frac{\partial}{\partial x} \omega(x, t) \right)^2 - t \frac{\partial}{\partial t} \omega(x, t) - 
$$

$$
- \left( \frac{\partial}{\partial t} \omega(x, t) \right) t \left( \frac{\partial}{\partial x} \omega(x, t) \right)^2 = 0. \quad (10)
$$

This equation admits particular solution in the form

$$
\omega(x, t) = A(t) + x
$$

where the function $A(t)$ is solution of equation

$$
2 \left( \frac{d^2}{dt^2} A(t) \right) t^2 - \left( \frac{d^2}{dt^2} A(t) \right) t^4 - 2 t \frac{d}{dt} A(t) = 0.
$$

So we get

$$
A(t) = -C1 + -C2 \sqrt{-2 + t^2}
$$
Now after elimination of the parameter $t$ from the relations
\[ y\sqrt{-2 + t^2} + C1 \sqrt{-2 + t^2} - 2 C2 + x\sqrt{-2 + t^2} = 0 \]
and
\[ \phi(x, y)\sqrt{t^2 - 2} - C2 = 0 \]
we obtain the simplest solution of the equation (9).

\[ \phi(x, y) = 1/2 \sqrt{2 y^2 + 4 y C1 + 4 y x + 2 C1^2 + 4 C1 x + 2 x^2 + 4 C2^2}. \]

Moore complicated solutions of the equation (10) in the form
\[ \omega(x, t) = A(t) + B(t)x \]
lead to the conditions
\[ B(t) = -\sqrt{t^2 - 1}, \]
and
\[ A(t) \]
is arbitrary function.

In result elimination of the parameter $t$ from the relations
\[ y\sqrt{t^2 - 1} - t \left( \frac{d}{dt} A(t) \right) \sqrt{t^2 - 1} + A(t) \sqrt{t^2 - 1} + x = 0, \]
and
\[ \phi(x, y)\sqrt{t^2 - 1} - \left( \frac{d}{dt} A(t) \right) \sqrt{t^2 - 1} + tx = 0 \]
with a given function $A(t)$ we get the solution of the equation (9) dependent from choice of arbitrary function.

As example in the case
\[ A(t) = \frac{1}{t} \]
we find the solution of the equation (9) in the form
\[ 16 (\phi(x, y))^4 + \left( 8 y^2 - 8 x^2 - 32 \right) (\phi(x, y))^3 + \left( -32 y^2 + 16 - 8 x^2 + x^4 + y^4 + 2 y^2 x^2 \right) (\phi(x, y))^2 + \]
\[ + \left( 8 y^2 - 10 y^4 + 8 x^4 - 2 y^2 x^2 + 32 x^2 \right) \phi(x, y) - y^6 - x^6 + 20 y^2 x^2 - 8 x^4 - 3 y^2 x^4 - 16 x^2 - 3 y^4 x^2 + y^4 = 0. \]

### 2.4 Congruence of negative constant curvature

In the case $K = -1$ from (3) we get the equation
\[
\phi(x, y) \frac{\partial^2}{\partial x^2} \phi(x, y) - \phi(x, y) \left( \frac{\partial^2}{\partial x^4} \phi(x, y) \right) \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \\
+ 2 \phi(x, y) \left( \frac{\partial}{\partial y} \phi(x, y) \right) \left( \frac{\partial}{\partial x} \phi(x, y) \right) \frac{\partial^2}{\partial x \partial y} \phi(x, y) + \phi(x, y) \frac{\partial^2}{\partial y^2} \phi(x, y) -
\]
\[-\phi(x, y) \left(\frac{\partial^2}{\partial y^2} \phi(x, y)\right) \left(\frac{\partial}{\partial x} \phi(x, y)\right)^2 - (\phi(x, y))^2 \left(\frac{\partial^2}{\partial x^2} \phi(x, y)\right) \frac{\partial^2}{\partial y^2} \phi(x, y) +
+ (\phi(x, y))^2 \left(\frac{\partial^2}{\partial x \partial y} \phi(x, y)\right)^2 + 1 - 3 \left(\frac{\partial}{\partial x} \phi(x, y)\right)^2 - 3 \left(\frac{\partial}{\partial y} \phi(x, y)\right)^2 +
+ 2 \left(\frac{\partial}{\partial x} \phi(x, y)\right)^4 + 4 \left(\frac{\partial}{\partial x} \phi(x, y)\right)^2 \left(\frac{\partial}{\partial y} \phi(x, y)\right)^2 + 2 \left(\frac{\partial}{\partial y} \phi(x, y)\right)^4 = 0. \] (11)

The \((u, v)\)-transformation with condition
\[u(x, t) = t \frac{\partial}{\partial t} \omega(x, t) - \omega(x, t), \quad v(x, t) = \frac{\partial}{\partial t} \omega(x, t)\]
lead to the equation
\[
\left(\frac{\partial}{\partial t} \omega(x, t)\right)^2 \frac{\partial^2}{\partial x^2} \omega(x, t) + \left(\frac{\partial^2}{\partial t^2} \omega(x, t)\right) t^2 - \left(\frac{\partial^2}{\partial t^2} \omega(x, t)\right) t^4 +
+ 2 \left(\frac{\partial}{\partial t} \omega(x, t)\right) t^2 \left(\frac{\partial^2}{\partial t \partial x} \omega(x, t)\right) \frac{\partial}{\partial x} \omega(x, t) + \left(\frac{\partial^2}{\partial t^2} \omega(x, t)\right) \left(\frac{\partial}{\partial t} \omega(x, t)\right) t^3 \frac{\partial^2}{\partial x^2} \omega(x, t) -
- \left(\frac{\partial^2}{\partial t^2} \omega(x, t)\right) \left(\frac{\partial}{\partial t} \omega(x, t)\right) t \frac{\partial^2}{\partial x^2} \omega(x, t) - \left(\frac{\partial}{\partial t} \omega(x, t)\right) t^3 \left(\frac{\partial^2}{\partial t \partial x} \omega(x, t)\right)^2 +
+ \left(\frac{\partial}{\partial t} \omega(x, t)\right) t \left(\frac{\partial^2}{\partial t \partial x} \omega(x, t)\right)^2 + \left(\frac{\partial^2}{\partial t^2} \omega(x, t)\right) t^2 \left(\frac{\partial}{\partial x} \omega(x, t)\right)^2 - t \frac{\partial}{\partial t} \omega(x, t) -
- \left(\frac{\partial}{\partial t} \omega(x, t)\right) t \left(\frac{\partial}{\partial x} \omega(x, t)\right)^2 = 0. \] (12)

This equation admits the particular solution
\[\omega(x, t) = \sqrt{-t^2 + 1} x + A(t)\]
with arbitrary function \(A(t)\).

In particular case after elimination of the parameter \(t\) from the relations
\[y \sqrt{-t^2 + 1} + x t - 2 t \sqrt{-t^2 + 1} = 0\]
and
\[\phi(x, y) \sqrt{-t^2 + 1} - t^2 \sqrt{-t^2 + 1} + x = 0\]
we find the solution of the equation (11) in the form
\[-16 \left(\phi(x, y)\right)^4 + \left(-32 + 8 y^2 - 8 x^2\right) \left(\phi(x, y)\right)^3 +
+ \left(-2 y^2 x^2 + 32 y^2 - 16 - y^4 + 8 x^2 - x^4\right) \left(\phi(x, y)\right)^2 +
+ \left(-2 y^2 x^2 - 10 y^4 + 32 x^2 + 8 x^4 + 8 y^2\right) \phi(x, y) -
- y^4 + 16 x^2 + 3 x^4 y^2 + 8 x^4 + 3 y^4 x^2 + x^6 + y^6 - 20 y^2 x^2 = 0\]
3 Geodesic equations

The geodesic of the metric (1) are equivalent to the equation

$$\frac{d^2 y(x)}{d x^2} + a_{-1}(x, y) \left( \frac{d}{dx} y(x) \right)^3 + 3 a_{-2}(x, y) \left( \frac{d}{dx} y(x) \right)^2 + 3 a_{-3}(x, y) \frac{d}{dx} y(x) + a_{-4}(x, y) = 0,$$

where

$$a_{-1}(x, y) = \frac{\left( \frac{\partial}{\partial x} \phi(x, y) \right) \left( \left( \frac{\partial}{\partial y} \phi(x, y) \right) \phi(x, y) - 1 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 \right)}{\phi(x, y) \left( -1 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 \right)},$$

$$3 a_{-2}(x, y) = \frac{\left( \frac{\partial}{\partial y} \phi(x, y) \right) \frac{\partial^2}{\partial x^2} \phi(x, y)}{-1 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2} + \frac{2 \frac{\partial}{\partial y} \phi(x, y)}{1 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2} + \frac{3 \frac{\partial}{\partial y} \phi(x, y)}{1 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2} + \frac{-3 \left( \frac{\partial}{\partial y} \phi(x, y) \right)^3 - 2 \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 \frac{\partial}{\partial y} \phi(x, y) + 3 \frac{\partial}{\partial x} \phi(x, y)}{1 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2},$$

$$3 a_{-3}(x, y) = \frac{\left( \frac{\partial}{\partial x} \phi(x, y) \right) \frac{\partial^2}{\partial x^2} \phi(x, y)}{-1 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2} + \frac{2 \frac{\partial}{\partial x} \phi(x, y)}{1 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2} + \frac{-2 \left( \frac{\partial}{\partial x} \phi(x, y) \right) \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 + 3 \frac{\partial}{\partial x} \phi(x, y) - 3 \left( \frac{\partial}{\partial x} \phi(x, y) \right)^3}{1 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2},$$

$$a_{-4}(x, y) = \frac{\left( \frac{\partial}{\partial y} \phi(x, y) \right) \left( \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 \phi(x, y) - 1 \right)}{\phi(x, y) \left( -1 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 \right)}.$$

4 The four-dimensional Riemann extension

The metric (1) has a following coefficients of connection

$$\Gamma_{11} = \frac{\left( \frac{\partial}{\partial x} \phi(x, y) \right) \left( 1 - \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 \right) \phi(x, y) - 2 \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2}{\phi(x, y) \left( -1 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 \right)},$$

$$\Gamma_{11}^2 = \frac{\frac{\partial}{\partial y} \phi(x, y) \left( \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 \phi(x, y) - 1 \right)}{\phi(x, y) \left( -1 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 \right)},$$

$$\Gamma_{12}^1 = \frac{\frac{\partial}{\partial y} \phi(x, y) \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 \phi(x, y) - \left( \frac{\partial}{\partial y} \phi(x, y) \right)^3}{\phi(x, y) \left( -1 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 \right)},$$
where $g$ space equipped with connection $\Gamma$. In explicit form the non zero components of the metric (13) looks as
\[
\phi(x, y) \left( -1 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 \right)^3
\]
\[
\phi(x, y) \left( -1 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 \right)
\]
\[
\phi(x, y) \left( -1 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 \right)^2
\]

Using given expressions for the connection coefficients we introduce a 4-dimensional Riemann space with the metric
\[
4ds^2 = (-2\Gamma_{ij}^z - 2\Gamma_{ij}^t) \, dx^i dx^j + 2dx dz + 2dy dt
\]
where $z$ and $t$ are an additional coordinates.

The Riemann space constructed on such a way is called the Riemann extension of the base space equipped with connection $\Gamma$.

In explicit form the non zero components of the metric (13) looks as
\[
g_{xx} = -2 \frac{\left( \frac{\partial}{\partial x} \phi(x, y) \right) z}{\phi(x, y) \left( -1 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 \right)} + 2 \frac{\left( \frac{\partial}{\partial x} \phi(x, y) \right)^3 z}{\phi(x, y) \left( -1 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 \right)}
\]
\[
-2 \frac{\left( \frac{\partial}{\partial x} \phi(x, y) \right) z \phi^2 \phi(x, y)}{-1 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2} + 4 \frac{\left( \frac{\partial}{\partial x} \phi(x, y) \right) z \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2}{\phi(x, y) \left( -1 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 \right)}
\]
\[
-2 \frac{\left( \frac{\partial}{\partial y} \phi(x, y) \right) t \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2}{\phi(x, y) \left( -1 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 \right)} - 2 \frac{\left( \frac{\partial}{\partial y} \phi(x, y) \right) t \frac{\partial^2}{\partial x \partial t} \phi(x, y)}{-1 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2}
\]
\[
+ 2 \frac{\left( \frac{\partial}{\partial y} \phi(x, y) \right) t}{\phi(x, y) \left( -1 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 \right)}
\]
\[
g_{xy} = -2 \frac{z \frac{\partial}{\partial y} \phi(x, y)}{\phi(x, y) \left( -1 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 \right)} - 2 \frac{z \left( \frac{\partial}{\partial y} \phi(x, y) \right)^3}{\phi(x, y) \left( -1 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 \right)}
\]
\[
+ 2 \frac{t \left( \frac{\partial}{\partial x} \phi(x, y) \right)}{\phi(x, y) \left( -1 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 \right)} - 2 \frac{t \frac{\partial^2}{\partial x^2} \phi(x, y)}{-1 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2}
\]
\[
- 2 \frac{t \left( \frac{\partial^2}{\partial x \partial y} \phi(x, y) \right) \frac{\partial}{\partial y} \phi(x, y)}{-1 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2} + 2 \frac{t \left( \frac{\partial^3}{\partial x^3} \phi(x, y) \right)}{\phi(x, y) \left( -1 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 \right)}.
\]
\[ g_{yy} = -2 \frac{\left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 - \left( \frac{\partial}{\partial x} \phi(x, y) \right)^4 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2}{1 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2} \] 

\[ g_{yt} = 1, \quad g_{xz} = 1. \]

**Proposition 1** Riemann space with the metric (13) is a Ricci-flat 

\[ R_{ij} = 0 \]

at the condition

\[ -2 \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 - \left( \frac{\partial}{\partial x} \phi(x, y) \right)^4 + \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 - \\
-2 \left( \frac{\partial}{\partial x} \phi(x, y) \right) \left( \frac{\partial}{\partial y} \phi(x, y) \right) \left( \frac{\partial^2}{\partial x \partial y} \phi(x, y) \right) + \left( \frac{\partial^2}{\partial x^2} \phi(x, y) \right) \phi(x, y) \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \\
+ \left( \frac{\partial^2}{\partial y^2} \phi(x, y) \right) \left( \frac{\partial^2}{\partial x^2} \phi(x, y) \right) \left( \frac{\partial}{\partial y} \phi(x, y) \right)^2 + \left( \frac{\partial^2}{\partial y^2} \phi(x, y) \right) \left( \frac{\partial}{\partial x} \phi(x, y) \right)^2 \phi(x, y) - \\
- \left( \frac{\partial^2}{\partial y^2} \phi(x, y) \right) \phi(x, y) - \left( \frac{\partial^2}{\partial x^2} \phi(x, y) \right) \phi(x, y) - \\
- \left( \frac{\partial^2}{\partial x \partial y} \phi(x, y) \right)^2 \phi(x, y)^2 - \left( \frac{\partial}{\partial y} \phi(x, y) \right)^4 = 0. \quad (14) \]

Remark that two dimensional metric (1) at this condition is a flat.

It is important to note that the space with the metric (13) with condition (14) does not a flat, the component \( R_{1212} \) of its Riemann tensor \( R_{1212} \neq 0 \).

So in result of the Riemann extension of the metric (1) we have got the Einstein space.

Finally we demonstrate some additional solutions of the equation (14).

The substitution

\[ \phi(x, y) = H(x + y) \]

into the equation (14) lead to the condition on the function \( H(x + y) = H(z) \)

\[ 2 \left( D(H)(z) \right)^4 + \left( D^{(2)}(H)(z) \right) H(z) - (D(H)(z))^2 = 0. \]
From here we find the function $H(z)$ in non explicit form

$$\sqrt{2} \left( H(z) \right)^2 - C1 + C1 \ln \left( \frac{-2 - C1 + 2 \sqrt{-C1} \sqrt{2 \left( H(z) \right)^2 - C1}}{H(z)} \right) \frac{1}{\sqrt{-C1}} - z - C2 = 0.$$ 

The substitution

$$\phi(x, y) = H\left( \frac{y}{x} \right) x$$

lead to the complex solutions.

In additive to the part (2.2) we show the solutions of the the equation (14) (or) which is connected with the function $\omega(x, t)$ in form

$$\omega(x, t) = A(t) + x\sqrt{t^2 - 1},$$

where $A(t)$ is arbitrary function.

In particular case $A(t) = t^2$ from here is followed that the function $\phi(x, y)$ defined from the equation

$$-(\phi(x, y))^6 + \left( 3 x^2 + 1 + 10 y + y^2 \right) (\phi(x, y))^4 +$$

$$+ \left( -2 y x^2 - 2 x^2 y^2 - 20 x^2 - 8 y - 8 y^3 - 3 x^4 - 32 y^2 \right) (\phi(x, y))^2 +$$

$$+ y^2 x^4 - 8 x^4 y + 16 y^4 + x^6 + 16 y^2 + 32 y^3 + 8 x^2 y^2 + 32 y x^2 + 16 x^2 - 8 x^4 - 8 y^3 x^2 = 0$$

is the solution of the equation (14).

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