Spaces of Tilings, Finite Telescopic Approximations
and Gap-Labeling

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Abstract: The continuous Hull of a repetitive tiling $T$ in $\mathbb{R}^d$ with the Finite Pattern Condition (FPC) inherits a minimal $\mathbb{R}^d$-lamination structure with flat leaves and a transversal $\Gamma_T$ which is a Cantor set. This class of tiling includes the Penrose & the Amman Benkker ones in 2D, as well as the icosahedral tilings in 3D. We show that the continuous Hull, with its canonical $\mathbb{R}^d$-action, can be seen as the projective limit of a suitable sequence of branched, oriented and flat compact $d$-manifolds. As a consequence, the longitudinal cohomology and the $K$-theory of the corresponding $C^*$-algebra $A_T$ are obtained as direct limits of cohomology and $K$-theory of ordinary manifolds. Moreover, the space of invariant finite positive measures can be identified with a cone in the $d$th homology group canonically associated with the orientation of $\mathbb{R}^d$. At last, the gap labeling theorem holds: given an invariant ergodic probability measure $\mu$ on the Hull the corresponding Integrated Density of States (IDS) of any selfadjoint operators affiliated to $A_T$ takes on values on spectral gaps in the $\mathbb{Z}$-module generated by the occurrence probabilities of finite patches in the tiling.

1. Introduction

Let $\mathcal{L}$ be a discrete subset of $\mathbb{R}^d$. Following the ideas developed in [32] for $r > 0$, $\mathcal{L}$ is $r$-uniformly discrete whenever every open ball of radius $r$ meets $\mathcal{L}$ on one point at most. For $R > 0$, $\mathcal{L}$ is $R$-relatively dense whenever every open ball of radius $R$ meets $\mathcal{L}$ on one point at least. $\mathcal{L}$ is a Delone set [32] if it is both uniformly discrete and relatively dense. $\mathcal{L}$ is repetitive whenever given any finite subset $p \subseteq \mathcal{L}$, and any $\epsilon > 0$ there is $R > 0$ such that in any ball of radius $R$ there is a subset $p'$ of $\mathcal{L}$ which is a distance at most $\epsilon$ of some translated of $p$. $\mathcal{L}$ has finite type whenever $\mathcal{L} - \mathcal{L}$ is discrete.

With each discrete set $\mathcal{L}$ is associated the Radon measure $\nu^\mathcal{L} \in \mathcal{M}(\mathbb{R}^d)$ supported by $\mathcal{L}$ and giving mass one to each point of $\mathcal{L}$ [8]. The weak* topology on $\mathcal{M}(\mathbb{R}^d)$ (seen as the dual space of the space $C_c(\mathbb{R}^d)$ of continuous functions with compact support),
endows the set of discrete subsets of $\mathbb{R}^d$ with a metrizable topology. For such a topology the subset of an $r$-uniformly discrete set in $\mathbb{R}^d$ is compact [8]. If $\mathcal{L}$ is $r$-uniformly discrete let $\Omega$ be the closure of the set $\{t^a \mathcal{L} = \mathcal{L} + a; a \in \mathbb{R}^d\}$ of its translated. $\Omega$ is compact. Then $(\Omega, \mathbb{R}^d, \tau)$ becomes a topological dynamical system called the Hull of $\mathcal{L}$. If $\omega \in \Omega$, let $\mathcal{L}_\omega$ denote the uniformly discrete subset of $\mathbb{R}^d$ corresponding to $\omega$. The subset $\Gamma = \{\omega \in \Omega; 0 \in \mathcal{L}_\omega\}$ is called the canonical transversal.

The Hull of an $r$-discrete set $\mathcal{L} \subset \mathbb{R}^d$ is minimal if and only if $\mathcal{L}$ is repetitive [32]. In such a case $\mathcal{L}$ is necessarily Delone. If, in addition, $\mathcal{L}$ has finite type, then its canonical transversal $\Gamma$ is a Cantor set.

The Hull can also be seen as a lamination [20] or a foliated space [34], namely a foliation with non smooth transverse structure. On the other hand [41], the construction of the Voronoi cells from the point set of atomic positions, leads to a tiling of $\mathbb{R}^d$ by polyhedra touching face to face, from which the point set can be recovered by a dual construction. Hence, the construction of the Hull can equivalently be performed from three equivalent complementary point of view: (i) as a dynamical system, (ii) as a lamination or foliated space, (iii) as a tiling. This latter point of view permits to select constraints using the tiling language more easily than using the language of point sets. The tilings that have mostly attracted attention are those with a finite number of bounded patches modulo translations, the so-called finite pattern condition (FPC) and satisfying repetitivity. Such tilings are equivalent to repetitive finite type Delone sets. The Penrose tiling in $2D$, the Amman Benkker (or octagonal) one or the variousicosahedral tilings used to describe quasicrystals [28] belong to this class. However the pinwheel tiling [39] is excluded of the present study but is the subject of a future publication [12]. In this work we prove the following results whenever $\mathcal{L}$ is a repetitive, finite type Delone set in $\mathbb{R}^d$:

**Theorem 1.1 (Main results).** Let $\mathcal{L}$ be a repetitive Delone set of finite type with Hull $\Omega$.

1. There is a projective family $\cdots \to B_n+1 \xrightarrow{f_n} B_n \to \cdots$ of compact, branched, oriented, flat manifolds (BOF) of dimension $d$, with the $f_n$'s being BOF-submersions (in particular, $Df_n = \text{id}$), such that the Hull of $\mathcal{L}$ is conjugate by a homeomorphism to the inverse limit $\lim_{\to} (B_n, f_n)$. The $\mathbb{R}^d$-action is induced by the infinitesimal parallel transport by constant vector fields in each of the $B_n$’s.

2. Let $H_d(\Omega, \mathbb{R})$ be the $d$-longitudinal homology group defined as the inverse limit $\lim_{\to} (H_d(B_n, \mathbb{R}), f_n^*)$. Then $H_d(\Omega, \mathbb{R})$ has a canonical positive cone induced by the orientation of the $B_n$’s, which is in one-to-one correspondence with the space of $\mathbb{R}^d$-invariant positive finite measures on $\Omega$.

3. If the $\|f_n^*\|$'s are uniformly bounded in $n$, the Hull is uniquely ergodic. If, in addition, $\dim H_d(B_n, \mathbb{R}) = N < \infty$ there is no more than $N$ invariant ergodic probability measures on $\Omega$.

4. Let $A = C^*(\Omega) \times \mathbb{R}^d$ be the crossed product $C^*$-algebra associated with the Hull. Then through the Thom-Connes isomorphism, $K_0(A) = \lim_{\to} (K_{*+d}(B_n), f_n^*)$.

5. Any $\mathbb{R}^d$-invariant ergodic probability measure $\mu$ defines canonically a trace $T_\mu$ on $A$ together with an induced measure $\mu_{tr}$ on the transversal $\Gamma$ [16]. Then the image by $T_\mu$ of the group $K_0(A)$ coincides with $\int_{\Sigma} d\mu_{tr} C(\Sigma, \mathbb{Z})$, namely the $\mathbb{Z}$-module generated by the occurrence numbers (w.r.t. $\mu$) of patches of finite size of $\mathcal{L}$ (gap labeling theorem).