Bent BPS domain walls of D=5 N=2 gauged supergravity coupled to hypermultiplets

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ABSTRACT

Within D=5 N=2 gauged supergravity coupled to hypermultiplets we derive consistency conditions for BPS domain walls with constant negative curvature on the wall. For such wall solutions to exist, the covariant derivative of the projector, governing the constraint on the Killing spinor, has to be non-zero and proportional to the cosmological constant on the domain walls. We also prove that in this case solutions of the Killing spinor equations are indeed solutions of the equations of motion. We present explicit, analytically solved examples of such domain walls, employing the universal hypermultiplet fields. These examples involve the running of two scalar fields and the space-time in the transverse direction that is cut off at a critical distance, governed by the magnitude of the negative cosmological constant on the wall.

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1 Introduction

The study of BPS domain wall solutions in supergravity theory was initiated in [1] where the first examples of such solutions within D=4 N=1 minimal supergravity coupled to chiral superfields were found. These examples correspond to static, flat domain walls where the neutral scalar component of a chiral superfield interpolates between isolated supersymmetric extrema of the matter potential with non-positive cosmological constant. Such solutions were further generalized [2] to examples of BPS domain walls which couple to the linear supermultiplet, leading to walls interpolating between isolated supersymmetric vacua with the varying dilaton field, corresponding to a scalar component of the linear supermultiplet. (For a review and generalizations to non-BPS domain wall configurations, see [3].)

The study of BPS domain wall configurations found new implications within AdS/CFT correspondence. There remains an on-going effort to elucidate D=4 N=1 super Yang-Mills theories as a dual description in terms of gauged supergravity in five dimensions or its decompactification to 10 or 11 dimensions. From the five-dimensional perspective this corresponds to finding domain wall solutions of the Killing spinor equations. One direction was to consider (deformed) sphere compactification of type IIB string theory, which can be seen as gauging of N=8 supergravity. (For the study of consistent non-linear Kaluza-Klein sphere compactifications see, e.g., [4] and references therein.) In this approach the domain wall solutions parameterize the renormalization group (RG) flows that preserves only four super charges of the N=1 dual field theory. By now there is a plethora of RG flow examples, with the best known example discussed in [5]. In a complementary approach one can address these questions directly within N=2 gauged supergravity, which contains as a subclass the models that can be understood as from N=8 supergravity. For the above flow this was done in [6].

Another important role played by BPS domain wall is that gravity could be trapped on the wall, providing the warp factor, describing the asymptotic AdS space-times, falls-off on both sides of the wall [7]. Within N=1 D=4 minimal supergravity such example have been established [1, 8] when the corresponding vacua of the matter potential are minima of the matter potential and the superpotential changes sign on either side of the wall. (The thin wall limit of such walls was analyzed in [9].)
the other hand within D=5 gauged supergravities, (such supergravities are believed to arise as consistent compactifications of M- or string theory), the existence of such gravity trapping, smooth configurations, remains elusive. While by now there is a rich plethora of domain wall solutions describing RG flows in dual field theories, on the other hand within D=5 N=2 gauged supergravities coupled to vector and tensor supermultiplets, so-called no-go theorems for the existence of smooth BPS domain walls that could trap gravity were established [10, 11]. If one includes non-trivial hypermultiplets the flow towards the infra-red can be regular, but there are no truly infra-red (IR) critical points – only saddle points with some IR directions are possible (at least as long as the scalar manifold is homogeneous) [12]. See however, recent work that establishes the existence of the smooth gravity trapping solution realized by using non-homogeneous scalar manifolds [13].

Another interesting direction constitutes a construction of BPS domain walls that are not flat, but have a constant negative curvature, i.e. the space-time on the wall is AdS. [The thin AdS vacuum domain walls, with the negative (as well as positive) curvature were first studied in D=4 in [14]. For generalizations to D-dimensions, see, e.g. [15, 16].] It turns out that such configurations may provide a background where gravity is “locally localized” [17], providing the warp factors fall-off in the sufficient vicinity on either side of the wall. However, the explicit realization of such configurations within D=5 gauged supergravity may prove to be a difficult task. On the other hand, explicit examples of bent BPS domain walls, even though their warp factors may not provide for the localization of gravity, remain to be of interest from the AdS/CFT correspondence. Establishing explicitly the existence of such configurations within D=5 N=2 gauged supergravity is the main purpose of this paper.

The main difficulty in establishing a more general classification of BPS domain wall configurations within D=5 N=2 gauged supergravity is due to the fact N=2 gauged supergravity with general matter couplings has been worked only recently [18, 19]. The hypermultiplet sector is very much similar to the D=4 N=2 gauged supergravity case which was known for some time (see [20] for a review). So far only very few models could be discussed explicitly [21, 22, 23], but unfortunately most the explicit examples are singular (note, however, regular flows can be constructed using the model of [13]).
As far as the existence of bent BPS domain walls in gauged supergravity goes, within D=4 N=2 gauged supergravity, their existence and explicit examples were studied in [23]. Within D=5, the discussion of bent AdS walls is given in [24, 25, 26], and examples with trivial scalars was given in [17].

In this paper we advance the study of bent BPS domain walls in several ways. In D=5 N=2 gauged supergravity theory coupled to a general set of hypermultiplets we derive first order Killing spinor equations for the metric (eq. (11)), scalar fields (eq. (13)) and the SU(2) valued projector Θ ≡ Θ^x σ^x (where σ^x-Pauli matrices, and Θ^x-“phase factors” (x = 1, 2, 3)) (eq. (16)). This projector governs the generalized constraint on the Killing spinor. In particular, we find that the generalization to the curved BPS walls involves a nontrivial projector whose covariant derivative (in the transverse direction of the wall) is non-zero and proportional to the cosmological constant on the domain wall (see eqs. (24), (25)). A detailed derivation of the Killing spinor equations and constraint for these bent BPS walls is given in Section 2.

We also prove that the first order system of the Killing spinor equations indeed satisfy the equations of motion, and thus genuinely represents the BPS solutions of the theory. The key ingredient role in the proof is played by the nontrivial projector. The proof is given in Section 3.

In Section 4 we present two explicit examples of such bent walls, employing the fields of the universal hypermultiplets. The result can be represented in the analytic form, and involves two scalar fields. The generic property of these space-times is that at a critical value of the transverse coordinate, governed by the cosmological constant on the wall, the space-time is cut off.

2 Solving the Killing spinor equations

N=2 supergravity in five dimensions has eight supercharges and matter fields enter vector, tensor or hypermultiplets. We consider the case where all vector and tensor multiplets are trivial and all scalars are part of hypermultiplets. In addition, the gravity multiplet has besides the graviton one graviphoton as bosonic fields. Since this model contains only one Abelian vector (the graviphoton), we can gauge only an Abelian symmetry and the SU(2) R-symmetry of N=2 supergravity will be broken to an Abelian subgroup. Due to supersymmetry the hyper scalars, which we will
denote by \( q^u \), have to parameterize a quaternionic space \( \mathcal{M} \) [27] with the metric \( h_{uv} \) and without making to much restrictions we can assume the existence of a number of isometries of \( \mathcal{M} \) parameterized by Killing vectors. Since \( \mathcal{M} \) is quaternionic, the holonomy group is contained in \( SU(2) \times Sp(n) \) and there is a triplet of covariantly constant Kähler 2-forms \( \Omega^x \), where \( x = 1, 2, 3 \) is the \( SU(2) \) index; for a recent nice summary of quaternionic geometry we refer to the appendix of [28]. Using these 2-forms one introduces for a given Killing vector \( k \) a triplet of Killing prepotentials \( P^x \) as follows

\[
P^x = \Omega^x{}^{uv}(\partial_u k_v) \equiv \Omega^x_{rs} h^{ru} h^{sv} (\partial_u k_v) ,
\]

which solve the following equation:

\[
\Omega^x_{uv} k^v = - (\nabla_u P)^x \equiv - (\partial_u P^x + \epsilon^{xyz} \omega_u^z P^x) ,
\]

where \( \omega_u^x \) is the \( SU(2) \) part of the spin connection, \( (\omega_u)_i^j \equiv i \omega_u^x (\sigma^x)_i^j \).

Before we can start to investigate bent domain wall solutions we have to discuss the gauge fields. Domain walls are codimension one objects and should not be charged with respect to vector fields. Nevertheless the decoupling of the vector fields is a subtle point: as a consequence of the gauging, the scalar fields correspond to charged matter and basically represent sources for gauge fields. In fact, gauging the isometry \( q^u \simeq q^u + k^u \) yields the covariant derivative \( Dq^u = dq^u + k^u A \) and the source current becomes \( J \sim k_u dq^u \). As we will see in eq. (20) the scalar flow becomes perpendicular to the Killing direction and therefore the charged scalars can remain constant. Hence, the corresponding current will vanish and the gauge fields can be consistently decoupled.

For a BPS configuration the fermionic supersymmetry variations, which have been derived for general couplings in [18], have to vanish. In our notation they are given by

\[
\delta \psi_{\mu i} = D_{\mu} \epsilon_i - \frac{i}{2} \Gamma_{\mu s} S_{ij} \epsilon^j ,
\]

\[
\delta \zeta_\alpha = V_\alpha^i \left[ \frac{i}{2} \Gamma^\mu (\partial_\mu q^u + i \frac{3}{2} k^u) \right] \epsilon_i ,
\]

where

\[
D_{\mu} \epsilon_i = (\partial_\mu + \frac{i}{2} \omega_{\mu ab} \Gamma_{ab}) \epsilon_i - i (Q_\mu)_i^j \epsilon_j ,
\]

\[
(Q_\mu)_i^j = i \frac{1}{2} \partial_\mu q^a \omega_u^a (\sigma^x)_i^j ,
\]

\[
S_{ij} = i P^x (\sigma^x)_i^k \epsilon_{jk} ,
\]

where \( \sigma^x \) are the Pauli matrices.
For a domain wall solution we make the metric Ansatz
\[ ds^2 = e^{2A(z)} \hat{e}^m \hat{e}^m + dz^2 , \]
where \( \hat{e}^m \hat{e}^m = ds^2 \) is the line element of the wall of constant negative curvature \( (\hat{R}_m^m = -3\lambda^2 \delta_m^m) \). We should take a four-dimensional AdS space, which ensures that the wall curvature does not give rise to further breaking of supersymmetry and the Killing spinor satisfies the equation
\[ \hat{D}_m \epsilon_i = \frac{\lambda}{2} Q^x \sigma^x \Gamma_m \epsilon_i \quad \text{with} \quad |Q^x|^2 = 1 , \]
where we introduced a general SU(2) phase (with \( \hat{D}_m Q^x = 0 \)) to solve the equations below. For the supersymmetry projector we make the following general Ansatz (we have summarized our convention in an Appendix):
\[ \epsilon_i = -\Gamma_5 \Theta^j_i \epsilon_j , \]
with \( \Gamma_5 = \Gamma_z \). This is a consistent projector if \( \Theta^k_i \Theta^j_k = \delta^j_i \) and therefore one writes \( \Theta^j_i \) as
\[ \Theta = \Theta^x \sigma^x \quad \text{with} \quad |\Theta^x|^2 = 1 . \]
Assuming that the warp factor as well as the scalars depend only on the radial coordinate \( (A = A(z), q^u = q^u(z) \) with \( \dot{A} \equiv \partial_z A \) the gravitino variation \( \delta \psi_{mi} \) \( (m = 1 \ldots 4) \) gives:
\[ 0 = \delta \psi_{mi} = \frac{1}{2} e^A \Gamma_n \Gamma_5 \hat{e}^n \left[ e^{-A} \lambda \Theta \cdot Q + \dot{A} \mathbb{I} + P \cdot \Theta \right]^j_i \epsilon_j , \]
\( (P \equiv P^x \sigma^x) \) which vanishes if
\[ -(P^x + \lambda e^{-A} Q^x) \Theta^x = \dot{A} , \quad \epsilon^{xyz} (P^x + \lambda e^{-A} Q^x) \Theta^y = 0 . \]
An obvious solution for \( \Theta^x \) is
\[ \Theta^x = \pm \frac{P^x + \lambda e^{-A} Q^x}{|P^x + \lambda e^{-A} Q^x|} , \quad \dot{A} = \mp |P^x + \lambda e^{-A} Q^x| . \]
The second equation is the modified flow equation for the warp factor in the metric and we infer that for flat walls \( (\lambda = 0) \) the phase in the projector coincides with the phase of the gravitino mass matrix, i.e. \( \Theta^x = P^x / |P^x| \). Non-zero curvature is related to a deviation of the two phases.
Before we address the transversal (radial) component of the gravitino variation, let us discuss the hyperino variation, which yields the flow equation for the scalar fields. In order to obtain the flow equation we use the projector and write

\[ 0 = V^i_{\alpha} \Gamma^5 \left[ q^u 1 + 3 k^u \Theta \right] \epsilon_j. \]  

(12)

Multiplying this equation with the quaternionic vielbein \( V^k_\alpha \) and using eq. (64) yields after some steps the new flow equation for the scalar fields

\[ h_u q^v = 3 \Theta^x (\nabla_u P)^x. \]  

(13)

Finally, from the radial component of the gravitino variation we derive a constraint that fixes the phase \( Q^x \). Again by employing the projector this variation becomes

\[ \delta \psi_z \sim \left[ \partial_z + Q_z + \frac{1}{2} P \cdot \Theta \right] \epsilon_j. \]  

(14)

Since we are dealing here with systems of ordinary differential equations the integration of these equations should at least in principle always be possible yielding an expression for the Killing spinor. But this solution has to be consistent with the projector (7), which is not obvious. Namely, we can also write \( \nabla_z \epsilon \equiv (\partial_z + Q_z) \epsilon = - (\partial_z + Q_z) \Gamma^5 \Theta \epsilon \) and find an alternative expression

\[ \delta \psi_z \sim \Gamma^5 \left[ - \nabla_z \Theta - \frac{1}{2} P \right] \epsilon_j. \]  

(15)

Multiplying this equation by \( \Theta (\Theta^2 = 1) \), dropping \( \Gamma^5 \) and adding this equation to (14) gives:

\[ \Theta \nabla_z \Theta = \frac{1}{2} [\Theta, \nabla_z \Theta] = -\frac{1}{2} [\Theta, P]. \]  

(16)

For flat walls with \( \Theta = P/|P| \) one obtains the constraint \( dq^u [P, \nabla_u P] = 0 \), which was mentioned in the paper [29], but which is fulfilled if the relation (21) holds. Using the expression for \( \Theta \) in (11) and since \( \nabla_z P^x \sim \Theta^x \), see (21), combined with the flow equation \( \nabla_z A = \mp |P^x + \lambda e^{-A} Q^x| \) this constraint can be written as

\[ \lambda e^{-A} [\Theta, \nabla_z A] = 0. \]  

(17)

For \( \lambda \neq 0 \) the solution becomes \( \nabla_z A \sim \Theta \) or in components:

\[ \nabla_z Q^x = B \Theta^x. \]  

(18)

These are three differential equations which determine the two non-trivial components of \( Q^x \) (recall \( Q^x Q^x = 1 \)) and the scalar function \( B \). In the next section we will investigate this relation further.
3 Solving the equations of motion

Solutions of the Killing spinor equations do not automatically satisfy also the equations of motion. For black holes e.g., one has to impose the Bianchi identities; see also the discussion in [24, 26]. So let us verify that our flow equations (11), (13) combined with constraint (18) solve the equations of motion. Before we start with the explicit equations we will derive some useful identities.

First, using the definition (2) for the Killing prepotentials we can write
\[ \dot{q}^u = 3 h^{uv}\Theta^v \nabla_u P^x = -3 h^{uv} \Omega^x_{uv} k^x \Theta^x. \] (19)

One can contract this relation with the Killing vector \( k^u \) and obtains
\[ 0 = k^u \dot{q}^u = 3 \left( k^u \nabla_u P^x \right) \Theta^x, \] (20)
which ensures that it is consistent to decouple all gauge fields (see discussion after equation (2)) and it implies that the flow becomes perpendicular to the Killing direction. Moreover, using the expression for \( \dot{q}^u \) and (63) and we derive
\[ \nabla_z P^x = \nabla_u P^x \dot{q}^u = k^u \Omega^x_{uv} \dot{q}^v = 3 |k|^2 \Theta^x, \] (21)
After a multiplication with \( \Theta^x \) (21) we find:
\[ |k|^2 = \frac{1}{3} \Theta^x (\nabla_z P)^x = \frac{1}{3} \Theta^x \nabla_u P^x \dot{q}^u = (\Theta^x \nabla_u P^x)(\Theta^y \nabla_u P^y) = \frac{1}{9} |\dot{q}|^2. \] (22)

After having an expression for \( \nabla_z P^x \) we can also derive analogous expression for \( \nabla_z Q^x \) and \( \nabla_z \Theta^x \). For this we explore the constraint (18) in more detail. There are two possibilities, \( B \) can vanish or not. In the first case one can show that the equations of motion are not solved, see also [26]. If however \( B \neq 0 \) we infer from \( Q^x \nabla_z Q^x = 0 \) that \( \Theta^x Q^x = 0 \) or \( P^x Q^x = -\lambda e^{-A} \) and therefore the flow equation for the warp factor can be simplified to
\[ -\dot{A} = \pm \sqrt{P^2 - \lambda^2 e^{-2A}} = \Theta^x P^x \] (23)
Next, the solution of eq. (10) reads
\[ \nabla_z \Theta^x = -P^x + C \Theta^x = \lambda e^{-A} Q^x \] (24)
where we used in the second step that $\Theta^x P^x = C$. Similarly we find for the constraint

$$\nabla_z Q^z = -\lambda e^{-A} \Theta^x.$$  \hfill (25)

Having all covariant derivatives of the SU(2)-valued quantities we can now start to discuss the equations of motion. The relevant Lagrangian has the form:

$$S = \int \left[ \frac{R}{2} - \frac{1}{2} h_{uv} \partial_q^u \partial_q^v - V(q) \right],$$  \hfill (26)

with the potential given by

$$V(q) = \frac{3}{2} \left( 3|k|^2 - 4 P^x P^x \right)$$  \hfill (27)

(see [18] for the complete expressions). With our metric Ansatz the Einstein equations become

$$R^m_n = - \left[ \ddot{A} + 4 \dot{A}^2 + 3 e^{-2A} \lambda^2 \right] \delta^m_n = \frac{2}{3} V \delta^m_n,$$

$$R^z_z = - 4 \left( \ddot{A} + \dot{A}^2 \right) = h_{uv} \dot{q}^u \dot{q}^v + \frac{2}{3} V.$$  \hfill (28)

Using our expressions from above we find for $\ddot{A}$:

$$\ddot{A} = \mp \frac{P^z \nabla_z P^z + \lambda^2 e^{-2A}}{\sqrt{P^2 - \lambda^2 e^{-2A}}}$$

$$= - 3|k|^2 + \lambda^2 e^{-2A}$$  \hfill (29)

and it becomes straightforward to verify the Einstein equations (after using the flow equations and eq. (22)). The scalar equation of motion is given by

$$\frac{1}{\sqrt{g}} \partial_z \left( \sqrt{g} h_{uv} \dot{q}^v \right) - \frac{1}{2} \left( \partial_u h_{rs} \right) \dot{q}^r \dot{q}^s \partial_u V$$  \hfill (30)

and we obtain for the different components

$$\partial_u V = 3 \dot{q}^x \Theta^x \nabla_u \nabla_v P^x - 12 P^x \nabla_u P^x - \frac{1}{2} \dot{q}^r (\partial_u h_{rs}) \dot{q}^s$$

$$\frac{\partial_u \sqrt{g}}{\sqrt{g}} h_{uv} \dot{q}^v = - 12 \left( \dot{q}^u \nabla_u P^x + \lambda e^{-A} Q^x \nabla_u P^x \right)$$  \hfill (31)

$$\partial_z (h_{uv} \dot{q}^v) = 3 \nabla_z (\Theta^x \nabla_u P^x) = 3 \left( 4 \lambda e^{-A} Q^x \nabla_u P^x + \Theta^x \dot{q}^v \nabla_u \nabla_v P^x \right)$$

where in the last step we used the relation $[\nabla_v, \nabla_u] P^x = -e^{xyz} \Omega^y_{uv} P^z$ from [28]. Thus all terms cancel and we have shown that the flow equations solve the equations of motion.
Note, we can integrate the equation for the warp factor by introducing a new radial coordinate \( y = y(z) \) defined by
\[
\frac{\partial y}{\partial z} = \sqrt{|P|^2 - \lambda^2 e^{-2A}},
\]
so that the equation for the warp factor becomes \( A' = \partial_y A = \mp 1 \) and the new metric reads:
\[
ds^2 = e^{\mp 2(y-y_0)} e^m e^m + \frac{dy^2}{|P|^2 - \lambda^2 e^{\pm 2(y-y_0)}},
\]
where one has to insert into \( |P|^2 = P^x P_x \) a solution of the scalar flow equations, which in the new radial coordinate takes the following form:
\[
h_{uv} \frac{dq^v}{dy} = \pm \frac{\Theta^x \nabla_u P_x}{\Theta^x P_x}.
\]
From the metric we can conclude that this curved wall solution can never asymptote to an IR critical point, where \( e^{\mp 2y} \to 0 \) while \( |P|^2 \) stays finite. Also a flat space vacuum with \( |P|^2 \to 0 \) while the warp factor stays finite is cut-off. On the other hand, near an UV critical point where the warp factor diverges \( (e^{\mp 2y} \to \infty) \) while \( |P|^2 \neq 0 \), the effect of the wall curvature drops out. We cannot make general statements about singular infra-red flows where \( |P|^2 \) and \( e^{\pm 2y} \) diverges.

4 Example: Bent walls with the universal hypermultiplet

In this Section we will discuss two concrete examples. In both cases we consider the coset \( SU(2,1)/U(2) \) related to the universal hypermultiplet, but in a different parameterization. In the model I the metric is explicitly spherical symmetric so that some scalars are compact, whereas in model II all scalars are non-compact.

4.1 Model I

We will use the notation used in \([23]\) and parameterize this coset space in terms of two complex scalars \( z_1 \) and \( z_2 \) with the Kähler potential:
\[
K = -\log(1 - |z_1|^2 - |z_2|^2),
\]
and the Kähler metric and the Kähler two-form take the form:
\[
\begin{align*}
\partial_A \partial_B K \ dz^A dz^B &= e^K \delta_{AB} \ dz^A dz^B + e^{2K} (z_A dz^A)(z_B dz^B), \\
\partial_A \partial_B K \ dz^A \wedge dz^B &= e^K \delta_{AB} \ dz^A \wedge dz^B + e^{2K} (z_A dz^A) \wedge (z_B dz^B).
\end{align*}
\]
Following essentially the parameterization employed in [30], it turns out to be more convenient to introduce polar coordinates in the following way:

\[ z_1 = r \left( \cos \left( \frac{\theta}{2} \right) e^{i(\psi + \varphi)/2} \right), \quad z_2 = r \left( \sin \left( \frac{\theta}{2} \right) e^{i(\psi - \varphi)/2} \right), \]  

(37)

with \( r \in [0, 1) \), \( \theta \in [0, \pi) \), \( \varphi \in [0, 2\pi) \) and \( \psi \in [0, 4\pi) \). The Kähler metric becomes:

\[ \partial_A \partial_B K dz^A dz^B = \frac{dr^2}{(1-r^2)^2} + \frac{r^2}{4(1-r^2)} (\sigma_1^2 + \sigma_2^2) + \frac{r^2}{4(1-r^2)^2} \sigma_3^2, \]  

(38)

where the \( \text{SU}(2) \) one-forms \( (d\sigma_i + \frac{1}{2} \epsilon_{ijk} \sigma_j \wedge \sigma_k = 0) \) are given by:

\[ \begin{align*}
\sigma_1 &= \cos \psi d\theta + \sin \psi \sin \theta d\varphi, \\
\sigma_2 &= -\sin \psi d\theta + \cos \psi \sin \theta d\varphi, \\
\sigma_3 &= d\psi + \cos \theta d\varphi.
\end{align*} \]  

(39)

For this model one obtains for the \( \text{SU}(2) \) connection is given [29]

\[ \begin{align*}
\omega_1 &= -\frac{\sigma_1}{\sqrt{1-r^2}}, \\
\omega_2 &= \frac{\sigma_2}{\sqrt{1-r^2}}, \\
\omega_3 &= -\frac{1}{2} (1 + \frac{1}{1-r^2}) \sigma_3.
\end{align*} \]  

(40)

In order to keep the Killing prepotential as simple as possible we gauge the Killing vector \( k = \partial_\psi \) and the corresponding Killing prepotential becomes

\[ P^x = \frac{r^2}{2(1-r^2)} \delta^{x3}. \]  

(41)

The general form of all Killing prepotentials can found in [29]. Using (23) and the fact that the \( \text{SU}(2) \) connection (40) have no radial component, the radial flow equation for the upper sign becomes

\[ g_{rr}(y) = 3 \frac{\Theta^3 \partial_r P^3}{\sqrt{|P|^2 - \lambda^2 e^{-2A}}} = 3 \partial_r \log P^3 \]  

(42)

which is solved by

\[ r^2 = 1 - e^{-12y}, \]  

(43)

\( y \in (0, +\infty) \). Therefore the metric reads

\[ ds^2 = e^{-2y} \hat{e}^m \hat{e}^m + \frac{4dy^2}{(e^{12y} - 1)^2 - 4\lambda^2 e^{2y}}. \]  

(44)

But this is not the complete solution, there is a second scalar running and we have to investigate the remaining equations

\[ h_{ij}(q^i)' = 3 \frac{\Theta^x \nabla_i P^x}{\Theta^3 P^3} = 3\lambda e^{-A} (Q^1 \omega_i^2 - Q^2 \omega_i^1) \frac{P^3}{|P|^2 - \lambda^2 e^{-2A}} \]  

(45)
Since $\omega^{1/2}$ have no $\psi$-component we find that $h_{\psi j}(q^j)' = 0$, which can be solved by $\psi' = \varphi' = 0$. This implies that $h_{\varphi j}(q^j)' = 0$, which is the case if: $Q^1 \cos \psi + Q^2 \sin \psi = 0$. Together with constraint $|Q| = 1$ this implies

$$Q^2 = -\cot \psi Q^1 = \frac{\sqrt{|P|^2 - \lambda^2 e^{-2A}}}{P^3} \cos \psi$$  \hspace{1cm} (46)

(recall $\psi$ is constant). Using this relation in the equation $h_{\theta j}(q^j)' = 0$ gives

$$h_{\theta \theta \theta} = \frac{3 \lambda e^{-A}}{\sqrt{1 - r^2} \sqrt{|P|^2 - \lambda^2 e^{-2A}}}$$  \hspace{1cm} (47)

and after inserting $P^3$ and the solution for $r^2$ in (43) we get the differential whose solution is of the form:

$$\theta(y) = \int^{y} \frac{24 \lambda e^7 y'}{(e^{12y'} - 1) \sqrt{(e^{12y'} - 1)^2 - 4 \lambda^2 e^{2y'}}} dy'.$$  \hspace{1cm} (48)

Even though the above integral cannot be expressed in terms of known functions, one can infer that at the critical value of the $y$ component the field $\theta$ remains finite.

Note that we have determined the phase $Q$ in (46) from the flow equation, but this phase has to solve the differential equation (25). We have checked explicitly that this nontrivial constraint is indeed satisfied.

In summary, we considered here a curved wall generalization of a flow to flat spacetime (with $|P|^2 = 0$). The solution shows that if one turns on the wall curvature ($\lambda \neq 0$), the flat space vacuum at $r = 0$ is cutoff and the flows stop at some finite value of $r$ where the transversal metric component has a pole, however the space-time remains regular. Our calculations also show, that in contrast to the flat walls the curved wall requires an additional nontrivial scalar, whereas the remaining two are arbitrary constants.

### 4.2 Model II

In the second model we parameterize the same coset space in a different way by using the Kähler potential [31]

$$K = -\frac{1}{2} \log(S + \bar{S} - 2C\bar{C})$$  \hspace{1cm} (49)

and write: $S = V + \theta^2 + \tau^2 + i\sigma$ and $C = \theta - i\tau$ which yield the metric

$$ds^2 = \frac{dV^2}{2V^2} + \frac{1}{2V^2} \left[ d\sigma + 2(\theta d\tau - \tau d\theta) \right]^2 + \frac{2}{V} \left[ d\tau^2 + d\theta^2 \right]$$  \hspace{1cm} (50)
and the SU(2) connection becomes
\[
\omega^1 = -\frac{d\tau}{\sqrt{V}}, \quad \omega^2 = \frac{d\theta}{\sqrt{V}}, \quad \omega^3 = -\frac{1}{4V} \left[ d\sigma + 2(\theta d\tau - \tau d\theta) \right].
\] (51)

We took these expressions from [3] (see also [32]), where a different convention is used, which basically implies an additional factor of "2" in the covariant derivatives.

As before we choose a gauging where the Killing prepotential becomes simple, which is the case for \( k = \partial_\sigma \) so that
\[
P^x = -\frac{1}{4V} \delta^{x3}. \quad (52)
\]
The computation goes analogous to the Model I. The connection \( \omega^x \) has no \( V \)-component and hence
\[
g_{VV}V'(y) = 3 \frac{\Theta^3 \partial_V P^3}{\sqrt{|P|^2 - \lambda^2 e^{-2A}}} = 3 \partial_V \log P^3 \quad (53)
\]
which is solved by
\[
V = e^{-6y} \quad (54)
\]
where we again dropped the integration constant. Thus, the metric becomes
\[
ds^2 = e^{-2y} \hat{e}^m \hat{e}^m + \frac{16 dy^2}{e^{12y} - 16 \lambda^2 e^{2y}} \quad (55)
\]
To solve the remaining equations note that \( \omega^{1/2} \) have no \( \sigma \)-component and hence
\[
\Theta^x \nabla_\sigma P^x \sim (\Theta^1 \omega^2_\sigma - \Theta^2 \omega^1_\sigma) = 0 \] yielding \( h_{\sigma i} \hat{q}^i = 0 \). This equations is solved if
\[
\theta = c \tau \quad , \quad \sigma = \text{constant} \quad (56)
\]
for some constant \( c \). From this relation follows that \( h_{\theta i} \hat{q}^i - c h_{\tau i} \hat{q}^i = 0 \) which yields
\[
Q^1 = c Q^2 = \frac{c \sqrt{|P|^2 - \lambda^2 e^{-2A}}}{\sqrt{1 + c^2} P^3} \quad (57)
\]
and we find as differential equation for \( \tau \)
\[
\tau' = -\frac{3c \lambda}{\sqrt{1 + c^2}} \frac{\sqrt{V} e^{-A}}{\sqrt{16 y^2 - \lambda^2 e^{-2A}}} = -\frac{12c \lambda}{\sqrt{1 + c^2}} \frac{e^{-2y}}{\sqrt{e^{12y} - 16 \lambda^2 e^{2y}}} \quad (58)
\]
which can be solved explicitly
\[
\theta = c \tau = 3 \sqrt{\frac{A}{1 + c^2}} \left( \frac{1}{2} \frac{x}{A(x^2 + A)^{(1/5)}} - \frac{3}{10} x \text{ hypergeom} \left[ \frac{1}{2}, \frac{1}{5}, \frac{3}{2}, \frac{-x^2}{A} \right] A^{(6/5)} \right) \quad (59)
\]
where $A = 16\lambda^2$ and $x = \sqrt{e^{10y} - A}$. Again at the critical value of $y$, $\theta$ remains finite.

As before, we have to check that the phase $Q$ in (57) satisfies the equation (25), which is again the case. Also in this example there are two scalars that flow, whereas the other two remain constant. In the flat wall limit ($\lambda = 0$) this model has a known M-theory embedding and corresponds to the intersection of three 5-branes over a common 3-brane, which becomes the domain wall upon compactification. This is a well-known supergravity solution that has been discussed in more detail in [32, 33] (and refs. therein). In this setup the scalar $V$ is basically the volume of internal space, whereas the scalars $\tau$ and $\theta$ are related to the radii of the (3,0) and (0,3) cycles. Therefore, a non-trivial wall curvature gives a cutoff for the volume scalar $V$ where the transversal metric component develops a pole. At this point the scalars $\tau$ and $\theta$ are driven to zero and we expect a singularity in the internal space.

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Appendix

A. Conventions

The SU(2) indices are raised with the $\epsilon$-tensor

$$\epsilon_i = \epsilon_{ij} \epsilon^j \quad , \quad \epsilon^i = \epsilon^{ij} \epsilon_j$$

(60)

with $\epsilon^{12} = \epsilon_{12} = 1$ and similarly the $Sp(n)$ indices with

$$V_\alpha = C_{\alpha \beta} V^\beta \quad , \quad V^\alpha = C^{\beta \alpha} V_\beta$$

(61)
with $C_{\alpha\beta} = -C_{\beta\alpha}$ and $C^2 = -1$. The three complex structures fulfill the algebra

$$J^x \cdot J^y = -\delta^{xy} \mathbb{1} + \epsilon^{xyz} J^z$$

and because the SU(2) curvatures are given by $\Omega^x = e^m J^x_{mn} \wedge e^n$ this relation becomes

$$h^{vt} \Omega^x_{uv} \Omega^y_{tw} = -\delta^{xy} h_{uw} + \epsilon^{xyz} \Omega^z_{uw}.$$  \hfill (63)

For the quaternionic vielbeine holds the relation

$$2 V^i_u V^j_v \mathbb{1} = h_{uv} \epsilon^{ij} + i (\Omega_{uv})^{ij}$$

with $(\Omega_{uv})^{ij} = i \Omega^x_{uv} (\sigma^x)^j_k \epsilon^{ki}$ and since the Pauli matrices are traceless one find $(\Omega_{uv})^{ij} \epsilon_{ij} = 0.$

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