Second Order Perturbation Theory for Improved Gluon and Staggered Quark Actions

Matthew A. Nobes\textsuperscript{a}, Howard D. Trottier\textsuperscript{a}, G. Peter Lepage\textsuperscript{b}, Quentin Mason\textsuperscript{b}

\textsuperscript{a}Physics Department, Simon Fraser University Burnaby, B.C., Canada V5A 1S6
\textsuperscript{b}Newman Laboratory of Nuclear Studies, Cornell University, Ithaca, NY, 14853

We present the results of our perturbative calculations of the static quark potential, small Wilson loops, the static quark self energy, and the mean link in Landau gauge. These calculations are done for the one loop Symanzik improved gluon action, and the improved staggered quark action.

It has long been known that perturbative calculations are necessary for precision measurements of many quantities. Unfortunately perturbation theory using a lattice cutoff is difficult, due to extra diagrams and vastly more complicated Feynman rules. The complexity of these calculations is an impediment to doing the higher–order perturbation theory required for many important applications of improved actions. The present work attempts to automate as much of the perturbation theory as possible in order to make these types of computations more straightforward.

For any given action some of the most basic things we would like to compute are various operators that can be built out of products of links. These include the static quark potential, small Wilson loops, the static quark self energy, and the mean link in Landau gauge.

Perturbative determinations of these quantities are central to many other applications. For example, the static quark potential can be used to determine a physical, renormalized coupling, and the perturbative expansions of small Wilson loops can be used to extract the strong coupling from simulations (see [1]). We have the techniques in place to calculate all of these, for an arbitrary gluon action, along with the contributions from quark loops for highly improved quark actions. This paper reports preliminary results for the above quantities, for the one-loop Symanzik improved gluon action, and the improved staggered quark action.

These calculations are similar to traditional continuum perturbative QCD. The major difference is that the ultraviolet regulator is the lattice cutoff, which leads to the computational difficulties mentioned above. A further problem is how to regulate infrared divergences beyond one–loop, in a gauge covariant manner. This can be done using twisted periodic boundary conditions [2]. Feynman rules and factors from the operators are generated using the Lüscher and Weisz vertex generation algorithm [2]. PYTHON or C++ scripts implement this program for a given set of links. An advantage to this method is the ease with which new actions and operators can be treated. We use FORTRAN programs to do the momentum sums. These are generally done on large but finite lattices (typically 100^4 volume).

The action we consider here is the one–loop Symanzik improved action for isotropic lattices,

\begin{equation}
S_G = \sum_{x; \mu < \nu} (1 - P_{\mu\nu}) + \sum_{x; \mu \neq \nu} (1 - R_{\mu\nu}) + \sum_{x; \mu < \nu < \sigma} (1 - C_{\mu\nu\sigma}),
\end{equation}

where

\begin{align}
\beta_{pl} &= \frac{10}{g^2}, \\
\beta_{rt} &= -\frac{\beta_{pl}^2}{20u_0^4}(1 + 0.4805\alpha_s) \\
\beta_{pg} &= -\frac{\beta_{pl}}{u_0^4}0.03325\alpha_s.
\end{align}

We present our results for an expansion in the bare lattice coupling $\alpha_{\text{latt}} = g^2/(4\pi)$. We include
tadpole counterterms for \( (\Box) \), which are generated by the first order expansion of the mean-field in the rectangle terms. In this paper we use the mean-field, \( u_0 \), defined by the average plaquette \( (\Box) \), which, for the action of \( (\Box) \), is given to first order by \( u_0 \approx 1 - 0.7671 \alpha_{\text{latt}} \). Our second order results also included the counterterm generated by the 1x1x1 paths in \( (\Box) \).

Next we consider the static quark potential which is the central quantity that needs to be computed. We can use it to define a renormalized coupling \( \beta \), which can be used as the expansion parameter for other quantities.

To compute the static quark potential we take the correlator of two Polyakov lines of length \( L_T \), separated by a distance \( R \), \( < L(R, L_T) > \). The static quark potential is then given by

\[
V(R) + 2E_0 = \lim_{L_T \to \infty} \frac{1}{L_T} \ln < L(R, L_T) > ,
\]

where \( E_0 \) is the self energy of an isolated quark (see below for our calculation of this quantity). Expanded in the bare coupling the static quark potential should have the following form \( (\Box) \)

\[
V(R) = -\frac{4}{3} \frac{\alpha_{\text{latt}}}{R} \times \{ 1 + \alpha_{\text{latt}} (2\beta_0 \ln(\pi R) + C(R)) \},
\]

where \( \beta_0 = (11 - 2/3n_f)/(4\pi) \).

As mentioned above, the static quark potential is used to define a renormalized coupling. This is done by demanding that the Fourier transform of \( (\Box) \) have the form

\[
V(q) = -\frac{4}{3} \frac{4\pi}{q^2} \alpha_V(q).
\]

Using this definition, we obtain the expansion for the bare coupling in terms of the physical one,

\[
\alpha_{\text{latt}} = \alpha_V \left\{ 1 - \alpha_V \left( 2\beta_0 \ln \left( \frac{\pi}{q} \right) + \tilde{C} \right) + \mathcal{O}(\alpha_V^2) \right\}.
\]

As a test of our calculations we reproduced the known result for the Wilson gluon action, \( \tilde{C} = 4.702 \). We have also determined \( \tilde{C} \) for the Symanzik improved gluon action. We find

\[
\tilde{C}_I = 3.23(13).
\]

We could easily recompute \( \tilde{C} \) for other gluon (and quark) actions, and with other definitions of \( u_0 \).

In addition to the static quark potential, we have computed a number of other quantities. Tables \( (\Box) \) and \( (\Box) \) give results for the logarithms of small Wilson loops, whose perturbative expansion is defined by,

\[
-\frac{1}{2(R + T)} \ln W(R, T) = \sum_n w_n(R, T) \alpha_{\text{latt}}^n.
\]

The results for the Wilson action agree with those of \( (\Box) \).

We have also computed the static quark self energy \( E_0 \) through second order. We define the self energy \( E_0(L) \) on a finite lattice according to

\[
E_0(L) = -\frac{1}{L} \ln [P_{\ell}(L)] = \sum_n c_n(L) \alpha_{\text{latt}}^n.
\]

Here \( P_{\ell}(L) \) is the Polyakov line on a lattice of size \( L^4 \).

Figures \( (\Box) \) and \( (\Box) \) show our results for the first and second order coefficients on a series of volumes, for three sets of boundary conditions (Wilson glue, PBC stands for periodic boundary conditions, Txy and Txyz stand for twisted periodic boundary conditions along two and three spatial planes, respectively). The infinite volume extrapolation agree with earlier estimates, \( E_0 = 2.1172 \alpha_{\text{latt}} + 11.152 \alpha_{\text{latt}}^2 \). These finite volume results were used in a determination of the third order self energy, using Monte-Carlo methods \( (\Box) \). We have also calculated the self energy for the improved gluon action \( (\Box) \) with the result: \( E_0 = 1.8347(5) \alpha_{\text{latt}} + 8.01(2) \alpha_{\text{latt}}^2 \).

Finally, we have computed the mean link in Landau gauge. In agreement with earlier determinations we have for the Wilson action, \( u_0 = 1 - 0.9738(2) \alpha_{\text{latt}} - 3.33(1) \alpha_{\text{latt}}^2 \). For the one loop Symanzik improved action we report, \( u_0 = 1 - 0.7501(1) \alpha_{\text{latt}} - 2.06(1) \alpha_{\text{latt}}^2 \).

The methods for automatic vertex generation can also be readily applied to complicated fermionic actions. For example, we computed the \( n_f \) part of the average plaquette at second order for improved staggered fermions \( (\Box) \). We find: \( u_2(1, 1) = 1.958(2) - 0.06969(4) n_f \). Calculations of the second order \( n_f \) pieces for the other quantities considered in this paper are in progress.
Figure 1. The $\mathcal{O}(\alpha_{\text{lat}})$ contribution to the static quark self energy.

Figure 2. The $\mathcal{O}(\alpha_{\text{lat}}^2)$ contribution to the static quark self energy.

We turn now to some conclusions. Our current results demonstrate the versatility of our approach. High precision results, necessary for accurate determinations of many quantities, are currently being generated. For improved staggered quarks this includes one loop improvement of the action, and the matching of the quark currents and four quark operators [8] as well as quark mass renormalization [9].

Table 1
Perturbative Wilson loops evaluated using Wilson glue, errors are from the VEGAS integrations.

| R | T | $w_1$       | $w_2$       |
|---|---|-------------|-------------|
| 1 | 1 | 1.0471(4)   | 3.548(7)    |
| 1 | 2 | 1.2041(2)   | 4.460(5)    |
| 1 | 3 | 1.2589(2)   | 4.816(6)    |
| 2 | 2 | 1.4342(3)   | 5.841(7)    |
| 2 | 3 | 1.5177(3)   | 6.41(1)     |
| 3 | 3 | 1.610(1)    | 7.09(4)     |

Table 2
Perturbative Wilson loops evaluated using Symanzik improved glue.

| R | T | $w_1$       | $w_2$       |
|---|---|-------------|-------------|
| 1 | 1 | 0.7673(2)   | 1.958(2)    |
| 1 | 2 | 0.9255(2)   | 2.661(3)    |
| 1 | 3 | 0.9849(2)   | 2.954(4)    |
| 2 | 2 | 1.1503(3)   | 3.735(6)    |
| 2 | 3 | 1.2342(3)   | 4.172(8)    |
| 3 | 3 | 1.3231(4)   | 4.666(13)   |

REFERENCES
1. C. Davies, et.al., Phys. Rev. D56 (1995) 2755
2. M. Lüscher and P. Weisz, Nucl. Phys. B266 (1986) 309
3. C. Bernard, et.al., Phys. Rev. D58 (1998) 014503
4. G.P. Lepage and P. Mackenzie, Phys. Rev. D48 (1993) 2250
5. U. Heller and F. Karsch, Nucl. Phys. B251 (1985) 254
6. H. Trottier, et.al., in preparation
7. G. P. Lepage, Phys. Rev. D59 (1999) 074502
8. P. Mackenzie, et.al., poster presented at this conference
9. J. Hein, talk presented at this conference