ENTIRE SELF-EXPANDERS FOR POWER OF $\sigma_k$ CURVATURE FLOW IN MINKOWSKI SPACE

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ABSTRACT. In $[19]$, we prove that if an entire, spacelike, convex hypersurface $M_{u_0}$ has bounded principal curvatures, then the $\sigma_1^{1/\alpha}$ (power of $\sigma_k$) curvature flow starting from $M_{u_0}$ admits a smooth convex solution $u$ for $t > 0$. Moreover, after rescaling, the flow converges to a convex self-expander $\tilde{M} = \{(x, \tilde{u}(x)) \mid x \in \mathbb{R}^n\}$ that satisfies $\sigma_k(\kappa[\tilde{M}]) = (-\langle X_0, \nu_0 \rangle)^\alpha$. Unfortunately, the existence of self-expander for power of $\sigma_k$ curvature flow in Minkowski space has not been studied before. In this paper, we fill the gap.

1. INTRODUCTION

Let $\mathbb{R}^{n,1}$ be the Minkowski space with the Lorentzian metric

$$ds^2 = \sum_{i=1}^n dx_i^2 - dx_{n+1}^2.$$ 

In this paper, we will devote ourselves to the study of spacelike hypersurfaces $M$ with prescribed $\sigma_k$ curvature in Minkowski space $\mathbb{R}^{n,1}$. Here, $\sigma_k$ is the $k$-th elementary symmetric polynomial, i.e.,

$$\sigma_k(\kappa) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \kappa_{i_1} \cdots \kappa_{i_k}.$$ 

Any such hypersurface $M$ can be written locally as a graph of a function $x_{n+1} = u(x)$, $x \in \mathbb{R}^n$, satisfying the spacelike condition

$$|Du| < 1.$$ 

More specifically, we will study self-similar solutions of flow by powers of the $\sigma_k$ curvature. Namely, we are interested in entire, spacelike, convex hypersurfaces which move under $\sigma_k$ curvature flows by homothety.

Let $X(\cdot, t)$ be a spacelike, strictly convex solution of

$$\frac{\partial X}{\partial t}(p, t) = \sigma_k^2(p, t)\nu(p, t)$$

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for some $\beta \in (0, \infty)$. If the hypersurfaces $M(t)$ given by $X(\cdot, t)$ move homothetically, then $X(\cdot, t) = \phi(t)X_0$ for some positive function $\phi$. Since the normal vector field is unchanged by homotheties, by taking the inner product of (1.2) with $\nu_0 = \nu(\cdot, t)$ we obtain
$$\phi'(X_0, \nu_0) = -\sigma^\beta_k(\kappa[X_0])\phi^{-k\beta},$$
where $\kappa[X_0] = (\kappa_1, \ldots, \kappa_n)$ is the principal curvatures of $M_0$ at $X_0$. Therefore, we must have
$$\phi'(t)\phi^{k\beta}(t) = \lambda$$
and
$$\sigma^\beta_k = -\lambda \langle X_0, \nu_0 \rangle.$$

In this paper, we will consider the case when $\lambda > 0$, which we call expanding solutions. Through rescaling, we may also assume $\lambda = 1$.

Complete noncompact self similar solutions of curvature flows in Euclidean space have been studied intensively (for example [1, 6, 3, 9, 14, 15, 16]). However, in Minkowski space, there is no corresponding known result yet.

It is well-known that the hyperboloid is a self-expander. In [18], we have proved the rescaled convex curvature flows, including Gauss curvature flow, converge to the hyperboloid. Therefore, a natural question to ask is whether there exist self-expanders other than the hyperboloid? Moreover, if such self-expanders exist, can we construct some curvature flows such that their rescaled flows converge to these new self-expanders? In this paper and an upcoming paper [19], we give affirmative answers to both questions.

Now consider $M_{u_0} = \{(x, u_0(x)) \mid x \in \mathbb{R}^n\}$, an entire, spacelike, convex hypersurface satisfying $u_0(x) - |x| \to \varphi \left( \frac{x}{|x|} \right)$ as $|x| \to \infty$. By translating $M_{u_0}$ vertically we may also assume $\varphi \left( \frac{x}{|x|} \right) > 0$. In an upcoming paper [19], we prove that, if in addition $M_{u_0}$ also has bounded principal curvatures, then the equation
$$\begin{cases}
\frac{\partial X}{\partial t} = \sigma^{1/\alpha}_{k} \nu \\
X(x, 0) = M_{u_0},
\end{cases}$$
where $\alpha \in (0, 1]$, admits a smooth convex solution $u$ for $t > 0$. Moreover, after rescaling the flow converges to a convex self-expander $\tilde{M} = \{(x, \tilde{u}(x)) \mid x \in \mathbb{R}^n\}$ that satisfies
$$\sigma_k(\kappa[\tilde{M}]) = (-\langle X, \nu \rangle)^\alpha$$
and
$$\tilde{u} - |x| \to \varphi \left( \frac{x}{|x|} \right) \text{ as } |x| \to \infty.$$

Unfortunately, the existence of solutions of equations (1.3) and (1.4) have not been studied before. In this paper, we fill the gap and prove the following theorems.
Theorem 1. Suppose $\varphi$ is a positive $C^2$ function defined on $\mathbb{S}^{n-1}$, i.e., $\varphi \in C^2(\mathbb{S}^{n-1})$. Then there exists a unique, entire, strictly convex, spacelike hypersurface $\mathcal{M}_u = \{(x, u(x)) \mid x \in \mathbb{R}^n\}$ satisfying

$$\sigma_n(\kappa[\mathcal{M}_u]) = (-\langle X, \nu \rangle)^\alpha \text{ for any } \alpha \in (0, n],$$

and

$$u(x) - |x| \to \varphi \left(\frac{x}{|x|}\right) \text{ as } |x| \to \infty.$$

Remark 2. Note that, unlike previously known results on spacelike hypersurfaces with prescribed Gauss curvature (see [2, 8, 10, 13]), the right hand side of (1.3) is unbounded. Therefore, the proof of Theorem 1 is different from earlier works, and we need to develop new techniques to prove it.

Using the solution we obtained in Theorem 1 as a subsolution we can also prove

Theorem 3. Suppose $\varphi$ is a positive $C^2$ function defined on $\mathbb{S}^{n-1}$, i.e., $\varphi \in C^2(\mathbb{S}^{n-1})$. Then there exists a unique, entire, strictly convex, spacelike hypersurface $\mathcal{M}_u = \{(x, u(x)) \mid x \in \mathbb{R}^n\}$ satisfying

$$\sigma_k(\kappa[\mathcal{M}_u]) = (-\langle X, \nu \rangle)^\alpha \text{ for any } \alpha \in (0, k],$$

and

$$u(x) - |x| \to \varphi \left(\frac{x}{|x|}\right) \text{ as } |x| \to \infty,$$

where $k \leq n - 1$.

The paper is organized as follows. In Section 2 we prove Theorem 1. In particular, we develop new techniques to prove the local estimates. In Section 3 combining the result we obtained in Section 2 with our ideas developed in [13] and [17], we prove Theorem 3. The arguments in this section are modifications of our arguments in [13] and [17].

2. GAUSS CURVATURE SELF-EXPANDER

In this section, we want to show there exists an entire, strictly spacelike, convex solution to the following equation

$$\sigma_n(\kappa[\mathcal{M}_u]) = (-\langle X, \nu \rangle)^\alpha, \ 0 < \alpha \leq n,$$

and

$$u(x) - |x| \to \varphi \left(\frac{x}{|x|}\right) \text{ as } |x| \to \infty.$$
where \( \varphi \) is a positive function defined on \( S^{n-1} \). Let \( u^* \) be the Legendre transform of \( u \). By Section 3 of [10] and Lemma 14 in [17], we know when \( u \) is a solution of (2.1) and (2.2), then \( u^* \) satisfies the following PDE,

\[
\begin{align*}
\sigma_n(D^2 u^*) &= \frac{w^* \alpha - n - 2}{(-u^*)^\alpha} \quad \text{in } B_1 \\
u^* &= \varphi^*(\xi) \quad \text{on } \partial B_1,
\end{align*}
\]

where \( \varphi^* = -\varphi < 0 \) on \( \partial B_1 \) and \( w^* = \sqrt{1 - |\xi|^2} \). Since (2.3) is a degenerate equation, we will study the following approximate problem instead

\[
\begin{align*}
\sigma_n(D^2 u^*) &= \frac{(1 - s|\xi|^2)^{(\alpha - n - 2)/2}}{(-u^*)^\alpha} \quad \text{in } B_1 \\
u^* &= \varphi^*(\xi) \quad \text{on } \partial B_1,
\end{align*}
\]

where \( 0 < s < 1 \).

**Remark 4.** In this remark, we want to explain why \( \alpha \) needs to be less than or equal to \( n \). Note that here we want to construct an entire solution to equation (2.1). This requires \( |Du^*(\xi)| \to \infty \) as \( \xi \to \partial B_1 \). One can see that if \( u^* \) is a solution to (2.3) then we have

\[
\int_{B_1} \det D^2 u^* = \int_{Du^*(B_1)} 1 \sim \int_{B_1} (1 - |\xi|^2)^{\frac{\alpha - n - 2}{2}} \sim \int_{1/2}^1 r^{n-1}(1 - r^2)^{\frac{\alpha - n - 2}{2}}dr \sim \int_{1/2}^1 (1 - r^2)^{\frac{\alpha - n - 2}{2}}dr^2.
\]

Since \( Du^*(B_1) = \mathbb{R}^n \), we know \( \int_{1/2}^1 (1 - r^2)^{\frac{\alpha - n - 2}{2}}dr^2 \) blows up, which implies \( \alpha \leq n \).

### 2.1. Solvability of equation (2.4)

We will show there exists a solution \( u^{**} \) of (2.4) for \( 0 < s < 1 \). For our convenience, in the following, when there is no confusion, we will drop the superscript \( s \) and denote \( u^{**} \) by \( u^* \).

**Lemma 5.** (**C**\(^0\) estimate for \( u^* \)) Let \( u^* \) be the solution of (2.4), then

\[
|u^*| < C,
\]

where \( C = C(|\varphi^*|_{C^0}) \).

**Proof.** Let \( -C_0 = \max_{\xi \in \partial B_1} \varphi^* < 0 \), by the convexity of \( u^* \) we know \( -C_0 > u^* \) in \( B_1 \). On the other hand, [10] proves that there exists a solution \( \bar{u}^* \) satisfies

\[
\begin{align*}
\sigma_n(D^2 \bar{u}^*) &= \frac{1}{K} (1 - |\xi|^2)^{-\frac{n+2}{2}} \quad \text{in } B_1 \\
\bar{u}^* &= \varphi^*(\xi) \quad \text{on } \partial B_1,
\end{align*}
\]

for any \( K \in \mathbb{R}_+ \).
Now let \( K \leq C_0^0 \) we have \( \sigma_n(D^2u^*) > \sigma_n(D^2u)^* \) in \( B_1 \), and \( u^* = u \) on \( \partial B_1 \). By the maximum principle we obtain
\[
(2.7) \quad -C_0 > u^* > u^*.
\]

Following \[4\], we can obtain the \( C^1 \) and \( C^2 \) estimates for the solution of \((2.4)\). Applying the method of continuity, we get the solvability of \((2.4)\). Therefore, in the following, we will focus on establishing local estimates for \( u^* \).

2.2. \textbf{Local} \( C^1 \) \textit{estimates}. This subsection contains two parts. In the first part, we will prove \((1 - s|\xi|^2)|Du^*|^2 < C\), where \( C \) is independent of \( s \) and \( \xi \). This estimate will be useful for obtaining local \( C^2 \) estimates in the next subsection. In the second part, we will show \(|Du^*|^2(\xi) \to \infty \) as \( s, |\xi| \to 1 \). This is to illustrate that the Legendre transform of \( u^* \), denoted by \( u^s \), converges to an entire solution of \((1.3)\) as \( s \to 1 \).

2.2.1. \textit{Local} \( C^1 \) \textit{upper bound}. In this part we will show \((1 - s|\xi|^2)|Du^*|^2 < C\), where \( C \) is a constant independent of \( s \).

First, applying \[4\] we know that we can solve the following equation
\[
(2.8) \quad \left\{ \begin{array}{ll}
\sigma_n(D^2u^*) = \frac{1}{C_1} < \frac{1}{(-\min u^*)^2} & \text{in } B_1 \\
u^* = \phi^*(\xi) & \text{on } \partial B_1.
\end{array} \right.
\]

We will denote the solution to \((2.8)\) by \( u^* \). It’s clear that \( u^* > u^s > u^* \), where \( u^* \) is the solution to \((2.6)\).

Next, denote \( h^s := 1 - s|\xi|^2 \) and \( V^s = |Du^*| \), we prove

\textbf{Lemma 6}. For any \( s \in \left[\frac{1}{2}, 1\right) \), if \( M^s := \max_{\xi \in B_1} h^sV^se^{-u^2} \) is achieved in \( B_1 \), then \( M^s \leq C \), where \( C = C(|u^*|_{C^0}) \) is a constant independent of \( s \).

\textit{Proof}. For our convenience, in this proof, we drop the superscript \( s \) from \( u^s, h^s, \) and \( V^s \). Consider \( \phi = hV^se^{-u^2} \) and assume \( \phi \) achieves its maximum at an interior point \( \xi_0 \in B_1 \). We may rotate the coordinate such that at \( \xi_0 \), we have \( V = u^2_1 \). Differentiating \( \phi \) we get
\[
\frac{2s\xi_i}{h} = \frac{u^2_1u^*_{i1} - 2u^*u^*_i}{V}, \quad 1 \leq i \leq n.
\]

Therefore, when \( i = 1 \), we obtain
\[
\frac{2s\xi_1}{h} = u^*_{11} - 2u^*V.
\]

By the convexity of \( u^* \) we know that \( u^*_{11} > 0 \), which gives
\[
\frac{2s\xi_1}{h} \geq 2|u^*|V.
\]

This completes the proof of the Lemma. \[\square\]
Finally, we want to show $h^s V^s$ is bounded on $\partial B_1$.

**Lemma 7.** For any $s \in [\frac{1}{2}, 1)$, if $M^s := \max_{\xi \in \bar{B}_1} h^s V^s e^{-u^{s^*2}}$ is achieved in $\partial B_1$, then $M^s \leq C$, where $C > 0$ is a constant independent of $s$.

**Proof.** For our convenience, in this proof, we drop the superscript $s$ from $u^{s^*}, h^s$, and $V^s$. Let $\psi = -kh^{1/2} + k(1 - s)^{1/2} + u_0^*$. For any $\xi \in B_1$, WLOG, we may assume $\xi = (r, 0, \cdots, 0)$. A direct calculation yields at $\xi$ we have

$$
\psi_{11} = ksh^{-3/2} + (u_0^*)_{11},
$$
$$
\psi_{ii} = ksh^{-1/2} + (u_0^*)_{ii} \text{ for } i \geq 2,
$$

and

$$
\psi_{ij} = (u_0^*)_{ij} \text{ for all other cases}.
$$

Since $u_0^*$ is strictly convex we get

$$
\sigma_n(D^2 u^*) > \sigma_n(D^2 (-kh^{1/2})) = (ks)^n h^{-n-2}.\]

Choosing $k = \frac{3}{\min(-\varphi^s)^{n/2}}$, we can see that $\psi$ is a subsolution of (2.4). Thus, on $\partial B_1$ we have

$$
|Du^*| < |D\psi| < \frac{C}{\sqrt{1-s}},
$$

where $C > 0$ is a constant independent of $s$. It is easy to see that (2.9) implies the Lemma. □

Combining Lemma 6 and Lemma 7 we conclude

**Lemma 8.** Let $u^{s^*}$ be the solution of (2.4) for $s \in \left[\frac{1}{2}, 1\right)$. Then there exists a constant $C > 0$, such that $|Du^{s^*}(\xi)| (1 - s|\xi|^2) \leq C$. Here, $C$ is a constant independent of $s$.

2.2.2. **Local $C^1$ lower bound near $\partial B_1$.** In this part, we will show $|Du^{s^*}(\xi)| \to \infty$ as $s, |\xi| \to 1$.

In order to obtain local $C^1$ lower bounds, we will construct a supersolution $\tilde{u}^{s^*}$ to

$$
\sigma_n(D^2 u^*) = \frac{(1-s|\xi|^2)^{(\alpha-n-2)/2}}{(-u^*)^\alpha} \text{ in } B_1
$$

for $\alpha \leq n$, which satisfies

$$
|D\tilde{u}^{s^*}(\xi)| \to \infty \text{ as } s \to 1 \text{ and } |\xi| \to 1.
$$

In the following, we will restrict ourselves to the case when $s \in [1/2, 1)$. Denote $h^s = 1 - s|\xi|^2$, then $h^s_i = -2s\xi_i$, and $h^s_{ij} = -2s\delta_{ij}$. Consider $g_1(h^s) = -h^s \log |\log h^s|$. By a straightforward calculation we get

$$
g'_1 = -\log |\log h^s| + \frac{1}{|\log h^s|},
$$
and

\[(2.13) \quad g''_1 = \left(1 + \frac{1}{|\log h^s|}\right) \frac{1}{h^s |\log h^s|}.\]

Therefore, at any point \(\xi \in B_1\) with \(|\xi| = r\) we have

\[(2.14) \quad \det(D^2 g_1) = s^n \left(2 \log |\log h^s| - \frac{2}{|\log h^s|}\right)^{n-1} \times \left[ \left(2 \log |\log h^s| - \frac{2}{|\log h^s|}\right) + 4sr^2 \left(1 + \frac{1}{|\log h^s|}\right) \frac{1}{h^s |\log h^s|} \right].\]

When \(h^s < \delta_0\) for some fixed \(\delta_0 > 0\) small, we have \(\det(D^2 g_1) \leq \frac{C}{r}\) for some constant \(C > 0\).

Here, \(C = C(\delta_0)\) is independent of \(s\). On the other hand, when \(h^s \geq \delta_0\), it’s easy to see that

\[(2.15) \quad \begin{cases} 
\det(D^2 g_2) = 1 & \text{in } B_\sqrt{\frac{1 - \delta_0}{s}}, \\
g_2 = -\delta_0 \log |\log \delta_0| & \text{on } \partial B_\sqrt{\frac{1 - \delta_0}{s}}.
\end{cases}
\]

Define

\[g = \begin{cases} 
g_1 & \text{for } h^s < \delta_0, \\
g_2 & \text{for } \delta_0 \leq h^s \leq 1,
\end{cases}\]

then \(g\) is a continuous and convex function in \(B_1\). By standard smoothing procedure, we can find a convex, rotationally symmetric function \(\Phi \in C^2(B_1)\) such that

\[\Phi(g) = \begin{cases} 
g_1 & \text{for } h^s < \frac{\delta_0}{2}, \\
g_2 & \text{for } 2\delta_0 \leq h^s \leq 1.
\end{cases}\]

We can see that for some suitable choice of \(\rho > 0\), \(\rho \Phi\) is a supersolution of \((2.10)\) that satisfies \((2.11)\). Here, \(\rho > 0\) only depends on \(|u^{*s}|_{C_0}\). Below we will denote this supersolution by \(\bar{u}^{*s}\).

Following [10] we can prove following Lemmas

\[\text{Lemma 9. Let } u^{*s} \text{ be the solution of } (2.4), \text{ then } |\partial u^{*s}| \text{ is bounded above by a constant } C_1 = C_1(|\varphi^*|_{C^1}).\]

\[\text{Proof. For our convenience, in this proof, we drop the superscript } s \text{ from } u^{*s}. \text{ We take the logarithms of both sides of } (2.4) \text{ and differentiate it with respect to } \xi_k, \text{ then find}
\]

\[u^{*ij}u^{*}_{ki} = \frac{\partial}{\partial \xi_k} \left[ \log(1 - s|\xi|^2) \cdot \frac{\alpha - n - 2}{2} \right] - \alpha \frac{\partial \log(-u^*)}{\partial \xi_k}.
\]

This implies

\[(2.16) \quad \sum u^{*ij}(\partial u^*)_{ij} = -\alpha \frac{\partial (-u^*)}{-u^*} = \alpha \frac{\partial u^*}{-u^*}.
\]
If $\partial u^*$ achieves interior positive maximum, we would have $0 \geq \alpha \frac{\partial u^*}{-u^*} > 0$. This leads to a contradiction. Similarly, if $\partial u^*$ achieves interior negative minimum, we would have $0 \leq \alpha \frac{\partial u^*}{-u^*} < 0$. This also leads to a contradiction. Therefore we conclude

$$|\partial u^*| \leq \max_{\partial B_1} |\partial \varphi^*|.$$

$$\square$$

Lemma 10. Let $u^{**}$ be the solution of (2.4), then $\partial^2 u^{**}$ is bounded above by a constant $C_2 = C_2(|\varphi^*|_{C^2})$.

Proof. For our convenience, in this proof, we drop the superscript $s$ from $u^{**}$. We have shown

$$\sum u^{**}_{ij} (\partial u^*)_{ij} = -\alpha \frac{\partial (-u^*)}{-u^*} = \alpha \frac{\partial u^*}{-u^*}.$$

Differentiating this equation once again we obtain

$$\sum u^{**}_{ij} \partial((\partial u^*)_{ij}) + \partial(u^{**}_{ij})(\partial u^*)_{ij} = \frac{\alpha}{(-u^*)^2} (\partial u^*)^2 + \frac{\alpha}{(-u^*)} \partial^2 u^*.$$

Following the argument of Lemma 5.2 in [10], we get

$$\sum u^{**}_{ij} [(\partial^2 u^*)_{ij}] \geq \frac{\alpha}{(-u^*)} (\partial^2 u^*).$$

Therefore, $\partial^2 u^*$ does not achieve positive maximum at interior points and we conclude

$$\partial^2 u^* \leq \max_{\partial B_1} |\partial^2 \varphi^*|.$$

$$\square$$

Lemma 11. Let $s \in [1/2, 1)$, $\sqrt{\frac{2-s}{2s}} < r < 1$, and $S^{n-1} = \{ \xi \in \mathbb{R}^n \mid \sum \xi_i^2 = r^2 \}$. For any point $\hat{\xi} \in S^{n-1}(r)$ there exists a function

$$\tilde{u}_s = \tilde{u}^{**} + b_1 \xi_1 + \cdots + b_n \xi_n + d$$

such that $\tilde{u}_s(\hat{\xi}) = u^{**}(\hat{\xi})$ and $\tilde{u}_s(\xi) > u^{**}(\xi)$ for any $\xi \in S^{n-1} \setminus \{ \hat{\xi} \}$. Here, $u^{**}$ is a solution of (2.4), $\tilde{u}^{**}$ is the supersolution constructed before, $b_1, \cdots, b_n$ are constants depending on $\hat{\xi}$, and $d$ is a positive constant independent of $\xi$ and $r$.

Proof. By rotating the coordinate we may assume $\hat{\xi} = (r, 0, \cdots, 0)$. We choose $b_k = \frac{\partial u^{**}}{\partial \xi_k}(r, 0, \cdots, 0)$, $k = 2, 3, \cdots, n$, and choose $b_1$ such that $u^{**}(r, 0, \cdots, 0) = \tilde{u}^{**}(r, 0, \cdots, 0) + b_1 r + d$. To choose $d$ we consider an arbitrary great circle $c(t)$ on $S^{n-1}(r)$ passing through $\hat{\xi}$, for example the circle

$$\xi_1 = r \cos t, \xi_2 = r \sin t, -\pi < t < \pi, \xi_3 = \xi_4 = \cdots = \xi_n = 0.$$

Let

$$F(t) = (\tilde{u}_s - u^{**})|_{c(t)} = \tilde{u}^{**}|_{c(t)} + b_1 \xi_1 + \cdots + b_n \xi_n + d - u^{**}|_{c(t)}$$

$$= \tilde{u}^{**}|_{c(t)} + b_1 r \cos t + b_2 r \sin t + d - u^{**}|_{c(t)}.$$
Note that by our construction of $\bar{u}^{ss}$, when $\frac{2-\delta_0}{2s} < r < 1$ we have

$$\bar{u}^{ss}|_{\xi(t)} = -\rho h^s \log |\log h^s|_{\{|\xi|=r\}} = \bar{u}^{ss}(r).$$

Therefore, we get

$$F(t) = \bar{u}^{ss}(r) + [u^{ss}(r, 0, \ldots, 0) - \bar{u}^{ss}(r) - d] \cos t + b_2 r \sin t + d - u^{ss}(t).$$

It's clear that $F(0) = 0$ and $\frac{dF}{dt}(0) = 0$. We will look at the second derivative of $F$. Since

$$\frac{d^2F(t)}{dt^2} = [d + \bar{u}^{ss}(r) - u^{ss}(r, 0, \ldots, 0)] \cos t - b_2 r \sin t - \frac{d^2u^{ss}}{dt^2},$$

when $-\frac{\pi}{2} < t < \frac{\pi}{2}$ we choose $d > u^{ss}(r, 0, \ldots, 0) - \bar{u}^{ss}(r)$ then we get

$$\frac{d^2F(t)}{dt^2} \geq \frac{1}{\sqrt{2}}[d + \bar{u}^{ss}(r) - u^{ss}(r, 0, \ldots, 0)] - \left|\frac{d u^{ss}}{dt}(0)\right| - \frac{d^2u^{ss}}{dt^2}.$$

$$\geq \frac{1}{\sqrt{2}}[d + \bar{u}^{ss}(r) - u^{ss}(r, 0, \ldots, 0)] - C_3$$

for some $C_3 > 0$ determined by Lemma\[9\] and [10]. When $t \in [-\pi, -\frac{\pi}{2}] \cup [\frac{\pi}{2}, \pi]$ we have

$$F(t) = d(1 - \cos t) + [u^{ss}(r, 0, \ldots, 0) - \bar{u}^{ss}(r)] \cos t + b_2 r \sin t - u^{ss}(t) + \bar{u}^{ss}(r)$$

$$\geq d \left(1 - \frac{\sqrt{2}}{2}\right) - C_4$$

By choosing $d > 0$ sufficiently large we prove this lemma. \(\square\)

Finally, we can prove

**Lemma 12.** Let $u^{ss}$ be the solution of (2.4) for $s \in (1/2, 1)$. Then there exists $C = C(|\varphi^s|_{C^2}) > 0$, such that when $\sqrt{\frac{2-\delta_0}{2s}} \leq \sqrt{\frac{2-\delta_1}{2s}} < |\xi| < 1$, we have

$$\frac{|Du^{ss}(\xi)|}{\log |\log h^s|} \geq C.$$

Here, $\delta_1 > 0$ is a small constant.

**Proof.** When $\sqrt{\frac{2-\delta_0}{2s}} \leq \sqrt{\frac{2-\delta_1}{2s}} < r < 1$, for any $\hat{\xi} \in S^{n-1}(r)$ we assume $\hat{\xi} = (r, 0, \ldots, 0)$. By Lemma\[11\] there exists a supersolution of (2.4) $\bar{u}^{ss}_s$, such that $\bar{u}^{ss}_s(\hat{\xi}) = u^{ss}(\hat{\xi})$ and $\bar{u}^{ss}_s(\xi) > u^{ss}(\xi)$ for any $\xi \in S^{n-1} \setminus \{\hat{\xi}\}$. By the maximum principle we get $\bar{u}^{ss}_s(\xi) > u^{ss}(\xi)$ in $B_r$. Hence at $\hat{\xi}$ we obtain

$$\frac{\partial u^{ss}}{\partial \xi_1} > \frac{\partial \bar{u}^{ss}_s}{\partial \xi_1} = \frac{\partial \bar{u}^{ss}}{\partial \xi_1} + b_1.$$ 

Therefore, when $\delta_1 > 0$ is chosen to be small we complete the proof of this Lemma. \(\square\)
2.3. **Local $C^2$ estimates.** Lemma 8 gives us local $C^1$ estimates for $u^{s*}$. In the following we will establish local $C^2$ estimates for the solution $u^{s*}$ of equation (2.4). Comparing with usual local $C^2$ estimates, the complication here is as $s \to 1$ and $|\xi| \to 1$, by Lemma 12 we know that $|Du^{s*}(\xi)| \to \infty$. In other words, we don’t have uniform $C^1$ estimates. Therefore, we need to introduce some new techniques to overcome this difficulty.

Let $u_0^{s*}$ be the solution of (2.8), denote $\eta^s := u_0^{s*} - u^{s*}$ and $f^s = (-u^{s*})^{-\alpha}(1 - s|\xi|^2)^{(\alpha - n - 2)/2}$, we prove

**Lemma 13.** Let $u^{s*}$ be a solution of (2.4) for $s \in [1/2, 1)$. Then we have

\[(2.19)\]

$$\eta^s < C(h^s)^{m_{\alpha}},$$

where $m_{\alpha} := \frac{n - 2 + \alpha}{2n}$, $h^s = 1 - s|\xi|^2$, and $C = C(n, \alpha, |u^s|_{C^0}) > 0$ is a constant independent of $s$.

**Proof.** For our convenience, in this proof, we will drop the superscript $s$ on $\eta^s, h^s, f^s$, and $u^{s*}$. Let $\gamma = \frac{1}{m_{\alpha}}$, since $\alpha \in (0, n)$, it’s clear that $\gamma > 1$. Assume $\max_{\xi \in B_1} \gamma \eta h^{-1}$ is achieved at an interior point $\xi_0$. We may rotate the coordinate such that at this point $u_{ij}^{s*} = u_{ii}^{s*} \delta_{ij}$. Moreover, at $\xi_0$ we have

$$0 = \gamma \eta_i - \frac{h_i}{h},$$

and

$$0 \geq \gamma \sigma_n^{ii} \left( \frac{\eta_{ii}}{\eta} - \frac{\eta_i^2}{\eta^2} \right) - \frac{\sigma_n^{ii} h_{ii}}{h} + \frac{\sigma_n^{ii} h_i^2}{h^2} = \gamma \sigma_n^{ii} \frac{\eta_{ii}}{\eta} + (\gamma^2 - \gamma) \sigma_n^{ii} \frac{\eta_i^2}{\eta^2} + 2s \sum \sigma_n^{ii} \frac{h_i}{h}.$$

Since $u_0^{s*}$ is convex, we get $\sigma_n^{ii}(u_0^{s*})_{ii} > 0$, the above inequality becomes

$$0 \geq -n \gamma \sigma_n \frac{\eta}{\eta} + 2s \sum \sigma_n^{ii} \frac{h_i}{h}.$$

Recall that $\sum \sigma_n^{ii} = \sigma_n - 1 \geq c(n) \sigma_n^{n-1}$ and $s \in [1/2, 1)$, we conclude

$$n \gamma \geq \frac{c(n) \eta}{h \sigma_n^{1/n}} \geq c_0 \frac{\eta}{h^{1 - \frac{n+2-\alpha}{2n}}} = c_0 \frac{\eta}{h^{m_{\alpha}}},$$

where $c(n) > 0$ is a constant depending on $n$ and $c_0$ is a constant depending on $|u^s|_{C^0}$ and $\alpha$. Therefore, we conclude that at $\xi_0$

$$\eta \gamma h^{-1} \leq C,$$

where $C = C(n, \alpha, |u^s|_{C^0})$ is independent of $s$. \hfill $\Box$

**Lemma 14.** Let $u^{s*}$ be a solution of (2.4) for $s \in [1/2, 1)$. Then we have

$$\max_{\xi \in B_1, \xi \in S^n} \eta^\beta u_{\xi \xi}^{s*} \leq C.$$
Here, $\beta = \frac{8}{m_0}$ and $C$ only depends on the $C^0$ estimates of $u^s$ and the local $C^1$ estimates we obtained in Lemma 8.

Proof. In this proof, for our convenience, we will drop the superscript $s$. We denote $h = 1 - s|\xi|^2$, then $h_i = -2s\xi_i$ and $h_{ij} = -2s\delta_{ij}$. We also note, differentiating $f = (-u^s)^{-\alpha}h^\frac{\alpha - n - 2}{2}$ twice we get

$$f_i = f \left[ \frac{\alpha u_i^s}{-u^s} + \frac{(\alpha - n - 2)}{2}h^{-1}h_i \right]$$

and

$$f_{ii} = f \left[ \frac{\alpha u_i^s}{-u^s} + \frac{(\alpha - n - 2)}{2}h^{-1}h_i \right]^2 + f \left[ \frac{\alpha u_{ii}^s}{-u^s} + \frac{\alpha u_{i}^2}{u^s} - \frac{(\alpha - n - 2)}{2}h^{-2}h_i^2 + \frac{(\alpha - n - 2)}{2}h^{-1}h_{ii} \right].$$

Moreover, applying Lemma 8 we may assume

$$h^2|Du^s|^2 < m_0 \text{ and } h^4|Du^s|^2 < m_0,$$

for some positive constant $m_0 > 1$. Let $g = h^4|Du^s|^2$ and differentiate $g$ twice, we get

(2.20) $$g_i = 4h^3h_i|Du^s|^2 + 2h^4\sum_k u_k^i u_k^s,$$

and

$$g_{ii} = 12h^2h_i^2|Du^s|^2 + 4h^3|Du^s|^2h_{ii} + 16h^2\sum_k h_i u_k^i u_k^s$$

(2.21)

$$+ 2h^4\sum_k u_{ki}^s + 2h^4\sum_k u_k^i u_{ki}^s.$$ 

Now we consider $\phi = \frac{\eta^3 u_\zeta}{|1 - \eta^2|}$, where $\beta > 0$, $M > 2m_0$ are some constants to be determined, and $\zeta \in \mathbb{S}^n$ is some direction. Suppose

$$\hat{M} := \max_{\xi \in B_1, \zeta \in \mathbb{S}^n} \phi$$

is achieved at an interior point $\xi_0 \in B_1$ in the direction of $\zeta_0 \in \mathbb{S}^n$. We may choose a local orthonormal frame $\{e_1, \ldots, e_n\}$ at $\xi_0$, such that $u^s_{ij}(\xi_0)$ is diagonal and we also assume $\zeta_0 = e_1$.

Then at $\xi_0$ we have

$$\log \phi = \beta \log \eta - \log \left(1 - \frac{g}{M}\right) + \log u_{11}^s.$$

Differentiating $\log \phi$ twice we get

(2.22) $$0 = \frac{\phi_{ii}}{\phi} = \frac{\beta \eta_i}{\eta} + \frac{g_i}{M - g} + \frac{u_{11i}^s}{u_{11}^s}$$

and

(2.23) $$0 > \sigma_{ii}^n \left[ \frac{\beta \eta_{ii}}{\eta^2} + \frac{g_{ii}}{M - g} + \frac{g_i^2}{(M - g)^2} + \frac{u_{11ii}^s}{u_{11}^s} - \left( \frac{u_{11i}^s}{u_{11}^s} \right)^2 \right].$$
By (2.22) we can see that when $i = 1$ we have

\[(2.24)\]
\[
\left(\frac{u_{11}^*}{u_{11}^*}\right)^2 = \left(\frac{\beta \eta_i}{\eta} + \frac{g_1}{M - g}\right)^2 \leq \frac{2\beta^2 \eta_i^2}{\eta^2} + \frac{2g_1^2}{(M - g)^2}.
\]

When $i \geq 2$

\[(2.25)\]
\[
\beta \left(\frac{\eta_i}{\eta}\right)^2 = \frac{1}{\beta} \left(\frac{g_i}{M - g} + \frac{u_{11i}^*}{u_{11}^*}\right)^2 \leq \frac{2g_i^2}{\beta(M - g)^2} + \frac{2}{\beta} \left(\frac{u_{11i}^*}{u_{11}^*}\right)^2.
\]

Note also that

\[
\sum_k \sigma_{n}^{ii} u_k^* u_{kii}^* = \sum_k u_k^* f_k = f \left(\frac{\alpha|Du^*|^2}{u^*} + \frac{(\alpha - n - 2)}{2} h^{-1} \sum_k h_k u_k^*\right).
\]

Therefore,

\[(2.26)\]
\[
\sigma_{n}^{ii} g_{ii} \geq 12h^2 |Du^*|^2 \sigma_{n}^{ii} \xi_i^2 - 8h m_0 \sum \sigma_{n}^{ii} - 32h^2 \sqrt{m_0} \sigma_n
\]
\[+ 2h^4 \sigma_n \sigma_1 + 2h^4 \sigma_n \left(\frac{\alpha|Du^*|^2}{u^*} + \frac{(\alpha - n - 2)}{2} h^{-1} \sum_k h_k u_k^*\right),
\]

and

\[(2.27)\]
\[
\sigma_{n}^{11} g_{ii} \leq \sigma_{n}^{11} \left(32h^6 h_2^2 |Du^*|^4 + 8h^8 u_{11}^* u_{11}^*\right) < 128 m_0 h^4 |Du^*|^2 \sigma_n \xi_1^2 + 8h^6 m_0 \sigma_n u_{11}^*.
\]

Combining (2.26) and (2.27) we obtain

\[(2.28)\]
\[
\frac{\sigma_{n}^{ii} g_{ii}}{M - g} - \frac{\sigma_{n}^{11} g_{11}}{(M - g)^2}
\]
\[\geq \frac{1}{(M - g)^2} \left\{ (M - g) |12h^2 |Du^*|^2 \sigma_{n}^{ii} \xi_i^2 - 36h^2 \sqrt{m_0} \sigma_n - 8h m_0 \sum \sigma_{n}^{ii} + 2h^4 \sigma_n \sigma_1
\]
\[\right. - 128 m_0 h^4 |Du^*|^2 \sigma_n \xi_1^2 - 8h^6 m_0 \sigma_n u_{11}^*\right\}
\]

Choose $M = 13m_0 + N$ such that $M - g \geq 12m_0 + N$ then

\[(2.29)\]
\[
\frac{\sigma_{n}^{ii} g_{ii}}{M - g} - \frac{\sigma_{n}^{11} g_{11}}{(M - g)^2}
\]
\[\geq \frac{1}{(M - g)^2} \left[ -(M - g)36h^2 \sqrt{m_0} \sigma_n - 8(M - g) h m_0 \sum \sigma_{n}^{ii} + 2N h^4 \sigma_n \sigma_1\right],
\]

where we have used $\sigma_1 > u_{11}^*$. Differentiating $\sigma_n = f$ twice we get

\[
\sigma_{n}^{ii} u_{11i}^* + \sigma_{n}^{pq rs} u_{pq1}^* u_{rs1}^* = f_{11}.
\]

Thus,

\[(2.30)\]
\[
\sigma_{n}^{ii} u_{11i}^* = f_{11} + \sum_{p\neq q} \sigma_{n}^{pq} u_{pq1}^* u_{pq1}^* - \sum_{p\neq q} \sigma_{n}^{pq} u_{pq1}^* u_{pq1}^* f_{11}^2.
\]
Notice that
\[
\begin{align*}
    f_{11} - f^2_f &= f \left[ \frac{\alpha u^*_1}{-u^*} + \frac{\alpha u^*_1^2}{-u^*} + \frac{(n + 2 - \alpha)h^{-2}h_1^2 + (n + 2 - \alpha)sh^{-1}}{2} \right] \\
    &\geq C_5u^*_1f,
\end{align*}
\]
we conclude
\begin{equation}
    (2.31)
    \frac{\sigma^{ii}u^{11ii}}{u^*_1} \geq C_5f + 2 \sum_{p=2}^n \frac{\sigma^{pp}_n}{u^*_1}u^{11p}_1.
\end{equation}

By a straightforward calculation we can see
\[
    \sigma^{ii}_n \eta^{ii} = \sigma^{ii}_n((u^*_0)_{ii} - u^*_1) \geq C_6 \sum \sigma^{ii}_n - n\sigma_n.
\]

Combining (2.24), (2.25) with (2.23) we obtain
\begin{equation}
    (2.32)
    0 \geq \frac{(C_6\sigma^{ii}_n - n\sigma_n)}{\eta^2} - \frac{(\beta + 2\beta^2)\sigma^{11}_n\eta^2}{\eta^2} + \frac{\sigma^{ii}_ng^1}{(M - g)^2} - \frac{\sigma^{11}_ng^2}{(M - g)^2} \\
    - \left(1 + \frac{2}{\beta}\right) \sum_{i \geq 2} \sigma^{ii}_n\left(\frac{u^{11}_1}{u^*_1}\right)^2 + \frac{\sigma^{ii}_nu^{11ii}_1}{u^*_1} + \left(1 - \frac{2}{\beta}\right) \sum_{i \geq 2} \frac{\sigma^{ii}_n g^2_i}{(M - g)^2}.
\end{equation}

When \( \beta \geq 2 \), applying (2.29) and (2.31) we get
\begin{equation}
    (2.33)
    0 \geq \frac{n\beta}{\eta^2}\sigma_n - \frac{(\beta + 2\beta^2)\sigma_n(C_6 + |Du^*|)^2}{\eta^2} - C_7\sigma_n + \frac{2Nh^4\sigma_n u^*_1}{M^2} + C_5\sigma_n.
\end{equation}

By Lemma \( \ref{lem:13} \) we know \( \eta^{1/\alpha} < Ch \), which gives \( h > C\eta^{1/\alpha} \). Thus, we have
\[
    \eta^{1/\alpha}|Du^*| < C\eta^\alpha.
\]

Now, let \( \beta = \frac{8}{m_\alpha} > 8 \) and multiplying (2.33) by \( \eta^{\beta} \), we obtain
\[
    0 \geq -n\beta \eta^{\beta - 1} - (\beta + 2\beta^2)(C_6 + |Du^*|)^2\eta^{\beta - 2} - C_7\eta^\beta + C_8N\eta^\beta u^*_1.
\]

Therefore, we conclude that \( \eta^\beta u^*_1 < C_9 \) at its interior maximum point, which implies \( \phi < 2C_9 \). 
\[\square\]
Proof of Theorem 1} By subsection 2.1 we know there exists a solution \( u^* \) of (2.4) for any \( s \in (0, 1) \). Combining Lemma 5, 8, 14 with the classic regularity theorem, we know that there exists a subsequence of \( u^* \) denoted by \( \{ u^{n*} \}_{n=1}^{\infty} \), converging locally smoothly to a convex function \( u^* \), which satisfies (2.3). Here, \( s_j \to 1 \) as \( j \to \infty \).

Moreover, applying Lemma 12 and Lemma 14 of [17] we conclude, the Legendre transform of \( u^* \), denoted by \( u \), is the desired entire solution of (2.1) satisfying the asymptotic condition (2.2). This completes the proof of Theorem 1. \( \square \)

3. \( \sigma_k \) CURVATURE SELF-EXPANDER

In this section we will show that there exists an entire, strictly spacelike solution to the following equation

\[
\sigma_k(\kappa[\mathcal{M}_u]) = (-\langle X, \nu \rangle)^\alpha,
\]

and

\[
u(x) - |x| \to \varphi \left( \frac{x}{|x|} \right) \text{ as } |x| \to \infty,
\]

where \( 0 < \alpha \leq k \) are constants. If \( u \) is a strictly convex solution satisfying (3.1) and (3.2), then subsection 2.3 and Lemma 14 of [17] imply its Legendre transform \( u^* \) satisfies

\[
\begin{align*}
\sigma_n(w^* \gamma^*_i u_k^* \gamma^*_l) &= \left( \frac{\sqrt{1 - |\xi|^2}^\alpha}{-u^*} \right) \text{ in } B_1, \\
u^* &= \varphi^* \text{ on } \partial B_1,
\end{align*}
\]

here \( \varphi^*(\xi) = -\varphi(\xi) \). By Section 2 we know there exists \( u \) such that

\[
\sigma_n(\kappa[\mathcal{M}_u]) = \frac{1}{(n_k) \pi} \left( -\langle X, \nu \rangle \right)^{\alpha n_k},
\]

and \( u(x) - |x| \to \varphi \left( \frac{x}{|x|} \right) \text{ as } |x| \to \infty \). Applying Maclaurin’s inequality we obtain

\[
\sigma_k(\kappa[\mathcal{M}_u]) \geq (-\langle X, \nu \rangle)^\alpha.
\]

We will denote the Legendre transform of \( u \) by \( u^* \), then \( u^* \) satisfies

\[
\begin{align*}
\sigma_n(w^* \gamma^*_i u_k^* \gamma^*_l) &\leq \left( \frac{\sqrt{1 - |\xi|^2}^\alpha}{-u^*} \right) \text{ in } B_1, \\
u^* &= \varphi^* \text{ on } \partial B_1.
\end{align*}
\]

We will study the following approximate equation

\[
\begin{align*}
\sigma_n(w^* \gamma^*_i u_k^* \gamma^*_l) &= \left( \frac{\sqrt{1 - |\xi|^2}^\alpha}{-u^*} \right) \text{ in } B_r, \\
u^* &= \varphi^* \text{ on } \partial B_r.
\end{align*}
\]
where $0 < r < 1$. In the following we denote $\Psi^* := \left(\frac{\sqrt{1-|\xi|^2}}{-u^*}\right)^\alpha$, and we can see that as long as $-u^* > 0$ we have

$$\frac{\partial \Psi^*}{\partial u^*} = \alpha \left(\sqrt{1-|\xi|^2}\right)^\alpha (-u^*)^{-\alpha-1} > 0.$$ 

This guarantees that the maximum principle holds for (3.5). Now assume $\max_{\xi \in \partial B_1} \varphi^*(\xi) = -C_0 < 0$, let $\bar{u}$ be a constant $\sigma_k$ curvature hypersurface satisfying $\sigma_k(\kappa[M_{\bar{u}}]) = C_0^n$, $\bar{u}$ is strictly convex, and $\bar{u}(x) - |x| \to \varphi \left(\frac{x}{|x|}\right)$ as $|x| \to \infty$. We denote the Legendre transform of $\bar{u}$ by $\bar{u}^*$, then $\bar{u}^*$ satisfies

$$\sigma_n \left(\sigma_n - w^* \gamma^*_i u^*_k \gamma^*_l \right) = \frac{1}{C_0^n} \geq \left(\frac{\sqrt{1-|\xi|^2}}{-u^*}\right)^\alpha$$

in $B_1$

$$\bar{u}^* = \varphi^*$$

on $\partial B_1$.

By the maximal principle we know $\bar{u}^* < u^*$ in $B_1$. Moreover, for any solution $u^*$ of (3.5), it is easy to see that

$$\bar{u}^* < u^* < u^*$$

in $B_r$.

Therefore, we conclude

**Lemma 15.** Let $u^*$ be a solution of (3.5) and $\bar{u}^*$, $\bar{u}^*$ are constructed above. Then we have

$$\bar{u}^* < u^* < u^*$$

in $B_r$.

### 3.1. Global a priori estimates.

In the subsection, we will prove a priori estimates that needed for the solvability of (3.5).

**Lemma 16.** Let $u^*$ be a solution of (3.5), then there exists $C > 0$ such that

$$|Du^*| < C.$$ 

**Proof.** By Section 2 of [5], we know that for any $0 < r < 1$, we can construct a subsolution $u^*$ such that

$$\frac{\sigma_n}{\sigma_n - w^* \gamma^*_i u^*_k \gamma^*_l} \geq \frac{1}{C_0^n}$$

in $B_r$

$$u^* = u^*$$

on $\partial B_r$.

Then by the convexity of $u^*$ we have

$$|Du^*| \leq \max_{\partial B_r} |Du^*|.$$ 

□

Let $v = \langle X, \nu \rangle = \frac{x \cdot Du - u}{\sqrt{1-|Du|^2}} = \frac{u^*}{\sqrt{1-r^2}}$. We will consider the hyperbolic model of (3.5) (see [17] for detail).

$$F(v_{ij} - v \delta_{ij}) = (-v)^{-\alpha}$$

in $U_r$,

$$v = \frac{u^*}{\sqrt{1-r^2}}$$

on $\partial U_r$,

(3.7)
Proof. The proof of this Lemma is a modification of the proof of Lemma 18 in [17]. Set
\[ v_0 = \mathbf{v} - \mathbf{v}_0 \]
where \( v_0 \) denotes the covariant derivative with respect to the hyperbolic metric, \( U_r = P^{-1}(B_r) \subset \mathbb{H}^n(-1) \), \( F(v_i - v_i) = \frac{\pi_n}{\pi_{n-k}}(\lambda[v_i - v_i]) \), and \( \lambda[v_i - v_i] = (\lambda_1, \cdots, \lambda_n) \) denotes the eigenvalues of \( (v_i - v_i) \). Recall the following Lemma 27 from [12].

**Lemma 17.** There exist some uniformly positive constants \( B, \delta, \epsilon > 0 \) such that
\[
    h = (v - v) + B \left( \frac{1}{\sqrt{1 - r^2}} - x_{n+1} \right)
\]
satisfying \( \mathcal{L}h \leq -\alpha(1 + \sum_i F^{ii}) \) in \( U_{r\delta} \) and \( h \geq 0 \) on \( \partial U_{r\delta} \). Here \( \alpha > 0 \) is some positive constant, \( v = \frac{u^*}{\sqrt{1 - |\xi|^2}} \) is a subsolution, \( \mathcal{L}f := F^{ij}\nabla_{ij}f - f\sum_i F^{ii} \), and \( U_{r\delta} := \{ x \in U_r | \frac{1}{\sqrt{1 - r^2}} - x_{n+1} < \delta \} \).

Following the argument in [7], we obtain a \( C^2 \) boundary estimate for \( u^* \). So far, we have obtained the \( C^0, C^1 \), and \( C^2 \) boundary estimates for the solution of (3.5). To prove the solvability of (3.5), we only need to obtain the \( C^2 \) global estimates. We consider
\[
    \hat{F} = \left( \frac{\sigma_n}{\sigma_{n-k}} \right)^{\frac{1}{2}} (\Lambda_{ij}) = (-v)^{\frac{\alpha}{\beta}} := \tilde{\Psi},
\]
where \( \Lambda_{ij} = v_i - v_i \).

**Lemma 18.** Let \( v \) be the solution of (3.5) in a bounded domain \( U \subset \mathbb{H}^n \). Denote the eigenvalues of \( (v_i - v_i) \) by \( \lambda[v_i - v_i] = (\lambda_1, \cdots, \lambda_n) \). Then
\[
    \lambda_{\text{max}} \leq \max\{ C, \lambda_i \partial U \},
\]
and \( C \) is a positive constant only depending on \( U \) and \( \tilde{\Psi} \).

**Proof.** The proof of this Lemma is a modification of the proof of Lemma 18 in [17]. Set \( M = \max_{p \in U} \max_{|\xi| = 1, \xi \in T_p^0} \left( \log \Lambda_{\xi\xi} \right) + N x_{n+1} \), where \( x_{n+1} \) is the coordinate function. Without loss of generality, we may assume \( M \) is achieved at an interior point \( p_0 \in U \) for some direction \( \xi_0 \). Choose an orthonormal frame \( \{ e_1, \cdots, e_n \} \) around \( p_0 \) such that \( e_1(p_0) = \xi_0 \) and \( \Lambda_{ij}(p_0) = \lambda_i \delta_{ij} \). Now, let consider the test function
\[
    \phi = \log \Lambda_{11} + N x_{n+1}.
\]
At its maximum point \( p_0 \), we have
\[
    0 = \phi_i = \frac{\Lambda_{11i}}{\Lambda_{11}} + N(x_{n+1})_i,
\]
and
\[
    0 \geq \hat{F}^{ii} \phi_{ii} = \hat{F}^{ii} \frac{\Lambda_{11i}}{\Lambda_{11}} - \hat{F}^{ii} \left( \frac{\Lambda_{11i}}{\Lambda_{11}} \right)^2 + N(x_{n+1}) \sum_i \hat{F}^{ii}
\]
Since \( \Lambda_{11i} = \Lambda_{ii1} + \Lambda_{ij} - \Lambda_{i1} \) and
\[
    \hat{F}_{ii} = \hat{F}^{ii} \Lambda_{ii1} + \hat{F}^{pq,rs} \Lambda_{pq1} \Lambda_{rs1} = \tilde{\Psi}_{ii},
\]
we get

\[
\hat{F}^{ii} \Lambda_{11ii} = \hat{F}^{ii} \Lambda_{ii11} + \bar{\Psi} - \Lambda_{11} \sum_i \hat{F}^{ii}
\]

(3.11)

\[
= \bar{\Psi}_{11} - \hat{F}^{pp,qq} \Lambda_{pp1} \Lambda_{qq1} - \sum_{p \neq q} \frac{\hat{F}^{pp} - \hat{F}^{qq}}{\lambda_p - \lambda_q} \Lambda_{pq1}^2 + \bar{\Psi} - \Lambda_{11} \sum_i \hat{F}^{ii}.
\]

Since \( \hat{F} \) is concave, combining (3.11) and (3.10) we have

\[
0 \geq \frac{1}{\Lambda_{11}} \left\{ \bar{\Psi}_{11} + 2 \sum_{i \geq 2} \frac{\hat{F}^{ii} - \hat{F}^{11}}{\lambda_1 - \lambda_i} \Lambda_{11ii}^2 + \bar{\Psi} - \Lambda_{11} \sum_i \hat{F}^{ii} \right\}
\]

(3.12)

\[-\frac{\hat{F}^{ii} \Lambda_{11ii}^2}{\Lambda_{11}^2} + \lambda_{n+1} \sum_i \hat{F}^{ii}.\]

We need an explicit expression of \( \hat{F}^{ii} \). A straightforward calculation gives

\[
k \hat{F}^{k-1} \hat{F}^{ii} = \frac{\sigma_n}{\sigma_{n-k}} - \frac{\sigma_{n-1}}{\sigma_{n-k}^2} \sigma_{n-k-1}(\lambda|i).
\]

Since

\[
\sigma_{n-1}(\lambda|i)\sigma_{n-k} - \sigma_n \sigma_{n-k-1}(\lambda|i)
\]

\[
= \sigma_{n-1}(\lambda|i) \left[ \lambda_i \sigma_{n-k-1}(\lambda|i) + \sigma_{n-k}(\lambda|i) \right] - \sigma_n \sigma_{n-k-1}(\lambda|i)
\]

\[
= \sigma_{n-1}(\lambda|i) \sigma_{n-k}(\lambda|i),
\]

we get

\[
k \hat{F}^{k-1} \hat{F}^{ii} = \frac{\sigma_n}{\sigma_{n-k}^2} \sigma_{n-k}(\lambda|i).
\]

Therefore, we have

\[
k \hat{F}^{k-1} (\hat{F}^{ii} - \hat{F}^{11})
\]

\[
= \frac{1}{\sigma_{n-k}^2} \left[ \sigma_{n-2}(\lambda|i) \sigma_{n-k}(\lambda|i) - \sigma_{n-1}(\lambda|1) \sigma_{n-k}(\lambda|1) \right]
\]

\[
= \frac{1}{\sigma_{n-k}^2} \left[ \sigma_{n-2}(\lambda|1) \lambda_1 \sigma_{n-k}(\lambda|i) - \sigma_{n-2}(\lambda|1) \lambda_i \sigma_{n-k}(\lambda|i) \right]
\]

\[
= \frac{\sigma_{n-2}(\lambda|1)}{\sigma_{n-k}^2} \left[ \lambda_1 \sigma_{n-k}(\lambda|i) - \lambda_i \sigma_{n-k}(\lambda|1) \right]
\]

\[
= \frac{\sigma_{n-2}(\lambda|1) \lambda_1 - \lambda_i}{\sigma_{n-k}^2} \left[ (\lambda_1 + \lambda_i) \sigma_{n-k-1}(\lambda|1i) + \sigma_{n-k}(\lambda|1i) \right]
\]
When $i \geq 2$ we can see that
\[
\hat{F}^{k-1} \left( \frac{\hat{F}^{ii} - \hat{F}^{11}}{\lambda_1 - \lambda_i} - \frac{\hat{F}^{ii}}{\lambda_1} \right)
= \frac{\sigma_{n-2}(\lambda|1)}{\sigma_{n-k}^2} \left[ (\lambda_1 + \lambda_i)\sigma_{n-k-1}(\lambda|1i) + \sigma_{n-k}(\lambda|1i) - \sigma_{n-k}(\lambda|i) \right]
= \frac{\sigma_{n-1}(\lambda|1)}{\sigma_{n-k}^2} \sigma_{n-k-1}(\lambda|1i) > 0
\]

Thus, (3.12) can be reduced to

\[
0 \geq \frac{1}{\Lambda_{11}} \hat{\Psi}_{11} + (Nx_{n+1} - 1) \sum_i \hat{F}^{ii} - \frac{\hat{F}^{11} \Lambda_{11}^2}{\Lambda_{11}^2}
= \frac{\hat{\Psi}_{11}}{\Lambda_{11}} + (Nx_{n+1} - 1) \sum_i \hat{F}^{ii} - \hat{F}^{11} N^2 (x_{n+1})^2.
\]

Since $\hat{\Psi} = (-v)^{-\frac{\alpha}{k}}$ and $-v = \frac{|u^*|}{\sqrt{1 - |\xi|^2}} > \min_{\xi \in B_r} |u^*| > \max_{\xi \in \partial B_1} \varphi^* > 0$, a direct calculation yields
\[
\hat{\Psi}_{11} = \alpha \left( \frac{\alpha}{k} + 1 \right) (-v)^{-\frac{k-2}{k}v^2} + \frac{\alpha}{k} (-v)^{-\frac{k-1}{k}v_1}
\geq C_1 (\lambda_1 + \lambda)
\geq \frac{\alpha}{k} (-v)^{-\frac{k-1}{k}v_1}
\geq C_1 (\lambda_1 - C_2).
\]

Here, $C_1$ depends on $U$, since $-v \leq \frac{C}{\sqrt{1 - |\xi|^2}}$. Plugging the above inequality into (3.13) we obtain
\[
0 \geq C_1 - \frac{C_2}{\lambda_1} + (Nx_{n+1} - 1) \sum_i \hat{F}^{ii} - N^2 (x_{n+1})^2 \frac{C_3}{\lambda_1}.
\]

Here we have used
\[
k \hat{F}^{k-1} \hat{F}^{11} = \frac{\sigma_n \sigma_{n-k}(\lambda|1)}{\lambda_1 \sigma_{n-k}^2} \frac{1}{\lambda_1} \hat{F}^{-k} \leq \frac{C_3}{\lambda_1},
\]

where $C_3$ depends on $U$. Let $N = 2$ we can see that when $\lambda_1$ is large, we get an contradiction. This completes the proof of Lemma 18.$\square$

Therefore, we conclude that the approximate problem (3.5) is solvable.

3.2. **Local a priori estimates.** Let $u^{rs}$ be the solution of (3.5), $u_r$ be the Legendre transform of $u^{rs}$. In this section, we will study interior estimates of $u_r$, which will enable us to show there exists a subsequence of $\{u_r\}$ that converges to the desired entire solution $u$ of (3.1).

**Lemma 19.** (Lemma 5.1 of [2]) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $u, \bar{u}, \Psi : \Omega \to \mathbb{R}^n$ be strictly spacelike. Assume that $u$ is strictly convex and $u < \bar{u}$ in $\Omega$. Also assume that near $\partial \Omega$, we
have $\Psi > \bar{u}$. Consider the set where $u > \Psi$. For every $x$ in this set, we have the following gradient estimate for $u$:

$$\frac{1}{\sqrt{1 - |Du|^2}} \leq \frac{1}{u(x) - \Psi(x)} \sup_{\{u > \Psi\}} \frac{\bar{u} - \Psi}{\sqrt{1 - |D\Psi|^2}}.$$

### 3.2.1. Construction of $\Psi$. In order to obtain the local $C^1$ estimate, we introduce a new subsolution $u_1$ of (3.1), where $u_1$ satisfies

$$\sigma_n(\kappa[\mathcal{M}_{u_1}]) = 100 \left( - \langle X, \nu \rangle \right) \frac{\alpha_n}{k},$$

and

$$u_1(x) - |x| \to \varphi \left( \frac{x}{|x|} \right) \text{ as } |x| \to \infty.$$

**Lemma 20.** Let $u$ be a solution of

$$\sigma_n(\kappa[\mathcal{M}_u]) = \frac{1}{\binom{n}{k}} \left( - \langle X, \nu \rangle \right) \frac{\alpha_n}{k},$$

satisfying $u(x) - |x| \to \varphi \left( \frac{x}{|x|} \right) \text{ as } |x| \to \infty$, then $u_1 < u$.

**Proof.** We look at the Legendre transform of $u_1$, denoted by $u_1^*$. Then $u_1^*$ satisfies

$$\sigma_n(w^* \gamma_{ik}(u_1^*)_{kl} \gamma_{lj}) = \frac{1}{100} \left( \frac{\sqrt{1 - |\xi|^2}}{w^* - u_1^*} \right)^{\frac{\alpha_n}{k}};$$

while $u^*$ satisfies

$$\sigma_n(w^* \gamma_{ik}(u^*)_{kl} \gamma_{lj}) = \frac{n}{k} \left( \frac{\sqrt{1 - |\xi|^2}}{w^* - u^*} \right)^{\frac{\alpha_n}{k}}.$$

Moreover, $u_1^* = u^* = \varphi^*(\xi)$ on $\partial B_1$. Applying the maximal principle we conclude $u_1^* > u^*$ in $B_1$. Following the proof of Lemma 13 of [17] we get $u_1 < u$ in $\mathbb{R}^n$. \hfill $\square$

Now, for any compact domain $K \subset \mathbb{R}^n$, let $2\delta = \min\{u - u_1\}$. We define $\Psi = u_1 + \delta$. Denote $K' = \{x \in \mathbb{R}^n \mid \Psi \leq \bar{u}\}$, notice that as $|x| \to \infty$, we have $u_1 - \bar{u} \to 0$, this implies $K'$ is compact. Applying Lemma [19] for any $(\Omega_r, u^*)$, if $K' \subset \Omega_r$, we have

$$\sup_{K'} \frac{1}{\sqrt{1 - |Du^*|^2}} \leq \frac{1}{\delta} \sup_{K'} \frac{\bar{u} - \Psi}{\sqrt{1 - |D\Psi|^2}}.$$

### 3.2.2. Local $C^2$ estimates. We will follow the proof of Lemma 24 in [17].

**Lemma 21.** Let $u^*$ be the solution of (3.5), $u_r$ be the Legendre transform of $u^*$, and $\Omega_r = Du^*(B_r)$. For any giving $s > 1$, let $r_s > 0$ be a positive number such that when $r > r_s$, $u_r\mid_{\partial\Omega_r} > s$. Let $\kappa_{\max}(x)$ be the largest principal curvature of $\mathcal{M}_{u_r}$ at $x$, where $\mathcal{M}_{u_r} = \{(x, u_r(x)) \mid x \in \Omega_r\}$. Then, for $r > r_s$ we have

$$\max_{\{x \in \Omega_r \mid u_r(x) \leq s\}} (s - u_r) \kappa_{\max} \leq C.$$
Here, $C$ only depends on the $C^0$ and local $C^1$ estimates of $u_r$.

Proof. Consider the test function

$$
\phi = m \log(s-u) + \log P_m - m N \langle \nu, E \rangle,
$$

where $P_m = \sum_j \kappa_j^m$, $E = (0, \cdots, 0, 1)$, and $N, m > 0$ are some undetermined constants. Assume that $\phi$ achieves its maximum value on $M$ at some point $x_0$. We may choose a local orthonormal frame $\{\tau_1, \cdots, \tau_n\}$ such that at $x_0$, $h_{ij} = \kappa_i \delta_{ij}$ and $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n$. Differentiating $\phi$ twice at $x_0$ we have

$$
\sum_j \kappa_j^{m-1}h_{jjj} - N h_{ii} \langle \tau_i, E \rangle + \frac{\langle \tau_i, E \rangle}{s-u} = 0,
$$

and

$$
0 \geq \frac{1}{P_m} \left[ \sum_j \kappa_j^{m-1}h_{jjj} + (m-1) \sum_j \kappa_j^{m-2}h_{jjj}^2 + \sum_{p \neq q} \kappa_p^{m-1} - \kappa_q^{m-1} \frac{h_{pqi}^2}{\kappa_p - \kappa_q} \right]
$$

$$
- \frac{m}{P_m^2} \left( \sum_j \kappa_j^{m-1}h_{jjj} \right)^2 - N h_{ii} \langle \tau_i, E \rangle - N h_{ii}^2 \langle \nu, E \rangle + \frac{h_{ii} \langle \nu, E \rangle}{s-u} - \frac{u_i^2}{(s-u)^2}.
$$

Denote $\hat{v} = -\langle X, \nu \rangle$ then

$$
\hat{v}_j = -h_{jk} \langle X, \tau_k \rangle = -h_{jj} \langle X, \tau_j \rangle,
$$

and

$$
\hat{v}_{jj} = -h_{jjk} \langle X, \tau_k \rangle - h_{jk} \langle \tau_j, \tau_k \rangle - h_{jj}^2 \langle X, \nu \rangle
$$

$$
= -h_{jjk} \langle X, \tau_k \rangle - h_{jj} - h_{jj}^2 \langle X, \nu \rangle.
$$

Since $\sigma_k = \hat{v}^\alpha := G$, we can see that $\sigma_k^{ii}h_{iij} = G_j$ and $\sigma_k^{ii}h_{iijj} + \sigma_k^{pq,s} h_{pqj} h_{rsj} = G_{jj}$. Recall also that in Minkowski space we have

$$
h_{jjj} = h_{iij} + h_{ii}^2 h_{jj} - h_{ii} h_{jj}^2,
$$

thus (3.16) becomes

$$
0 \geq \frac{1}{P_m} \left[ \sum_j \kappa_j^{m-1} \sigma_k^{ii}(h_{iij} + h_{ii}^2 h_{jj} - h_{ii} h_{jj}^2) \right]
$$

$$
+ (m-1) \sigma_k^{ii} \sum_j \kappa_j^{m-2}h_{jjj}^2 + \sum_{p \neq q} \kappa_p^{m-1} - \kappa_q^{m-1} \frac{\sigma_k^{ii} h_{pqi}^2}{\kappa_p - \kappa_q} \right]
$$

$$
- \frac{m}{P_m^2} \sigma_k^{ii} \left( \sum_j \kappa_j^{m-1}h_{jjj} \right)^2 - N \langle \nabla G, E \rangle - N \sigma_k^{ii} \kappa_i^2 \langle \nu, E \rangle + \frac{kG \langle \nu, E \rangle}{s-u} - \frac{\sigma_k^{ii} u_i^2}{(s-u)^2}.
$$
This gives

\[
0 \geq \frac{1}{P_m} \left\{ \sum_j \kappa_j^{m-1} [G_{jj} - \sigma_k^{pq,rs} h_{pq} h_{rs} - kG h_{jj}^2] + (m-1)\sigma_k^{ii} \sum_j \kappa_j^{m-2} h_{ji}^2 + \sum_{p \neq q} \frac{\kappa_p^{m-1} - \kappa_q^{m-1}}{\kappa_p - \kappa_q} \sigma_k^{ii} h_{pq}^2 \right\}
\]

\[
(3.18)
\]

\[- \frac{m}{P_m} \sigma_k^{ii} \left( \sum_j \kappa_j^{m-1} h_{ji}^2 \right)^2 - N \langle \nabla G, E \rangle - N \sigma_k^{ii} \kappa_i^2 \langle \nu, E \rangle + \frac{kG \langle \nu, E \rangle}{s-u} - \frac{\sigma_k^{ii} u_i^2}{(s-u)^2}.
\]

We denote \( A_i = \frac{m-1}{P_m} \left[ K(\sigma_k)^2_i - \sum_{p \neq q} \sigma_k^{pp,qq} h_{pp} h_{qq} \right] \), \( B_i = 2 \frac{m-1}{P_m} \sum_j \sigma_k^{jj,ii} h_{ji}^2 \), \( C_i = \frac{m-1}{P_m} \sigma_k^{ii} \sum_j \kappa_j^{m-2} h_{ji}^2 \), \( D_i = \frac{2m}{P_m} \sum_j \frac{\kappa_j^{m-1} - \kappa_j^{m-1}}{\kappa_j - \kappa_i} h_{ji}^2 \), and \( E_i = \frac{m}{P_m} \sigma_k^{ii} \left( \sum_j \kappa_j^{m-1} h_{ji}^2 \right)^2 \).

Then (3.18) can be reduced to

\[
0 \geq \sum_i \left( A_i + B_i + C_i + D_i - E_i \right) - \sum_i \frac{K \kappa_i^{m-1} (G_i)^2}{P_m}
\]

\[
+ \frac{\sum_j \kappa_j^{m-1} G_{jj}}{P_m} - N \langle \nabla G, E \rangle - N \sigma_k^{ii} \kappa_i^2 \langle \nu, E \rangle
\]

\[
- \frac{\sum_j \kappa_j^{m+1}}{P_m} kG + \frac{kG \langle \nu, E \rangle}{s-u} - \frac{\sigma_k^{ii} u_i^2}{(s-u)^2}.
\]

A straightforward calculation shows

\[
\sum_j \frac{\kappa_j^{m-1} G_{jj}}{P_m} = \sum_j \frac{\kappa_j^{m-1} \left[ \alpha (\alpha - 1) \dot{\nu}^{\alpha-2} \dot{\nu}_j^2 + \alpha \dot{\nu}^{\alpha-1} \dot{\nu}_j \right]}{P_m}
\]

\[
= \alpha (\alpha - 1) \dot{\nu}^{\alpha-2} \sum_j \frac{\kappa_j^{m-1} \dot{\nu}_j^2}{P_m} + \alpha \dot{\nu}^{\alpha-1} \sum_j \frac{\kappa_j^{m-1} \left( h_{jj} \langle -X, \tau \rangle - \kappa_j + \kappa_j^2 \dot{\nu} \right)}{P_m}
\]

\[
(3.20)
\]

\[- \alpha \dot{\nu}^{\alpha-1} + \alpha \dot{\nu} \sum_j \frac{\kappa_j^{m+1}}{P_m}.
\]
Moreover, we have

\[
\frac{\alpha \hat{v}^{\alpha - 1} \sum \kappa_j^{m-1} h_{jji} \langle -X, \tau_i \rangle}{P_m} - N \langle \nabla G, E \rangle
\]

\[
= \frac{\alpha \hat{v}^{\alpha - 1} \sum \kappa_j^{m-1} h_{jji} \langle -X, \tau_i \rangle}{P_m} - N \alpha \hat{v}^{\alpha - 1} \hat{v}_l \langle \tau_l, E \rangle
\]

\[
= \alpha \hat{v}^{\alpha - 1} \left( \sum \frac{\kappa_j^{m-1} h_{jji} \langle -X, \tau_i \rangle}{P_m} - N \kappa_l \langle X, \tau_i \rangle u_l \right)
\]

\[
= \alpha \hat{v}^{\alpha - 1} \sum \langle X, \tau_i \rangle \left( N \kappa_l u_l - \frac{u_l}{s - u} \right)
\]

\[
= - \alpha \hat{v}^{\alpha - 1} \sum \langle X, \tau_i \rangle u_l
\]

where we have used (3.15). Combining (3.20), (3.21) with (3.19) we obtain

\[
0 \geq \sum_i (A_i + B_i + C_i + D_i - E_i) - \sum_i \frac{K \kappa_i^{m-1} (G_i)^2}{P_m}
\]

\[
+ \frac{\alpha \hat{v}^{\alpha - 2} \sum j \kappa_j^{m-1} \hat{v}_j^2}{P_m} - \alpha \hat{v}^{\alpha - 1} + \alpha \hat{v}^\alpha \sum j \kappa_j^{m+1}
\]

\[
- \frac{\alpha \hat{v}^{\alpha - 1} \sum \langle X, \tau_i \rangle u_l}{s - u} - N \sigma_k \kappa_i^2 \langle \nu, E \rangle
\]

\[
- kG \frac{\sum j \kappa_j^{m+1}}{P_m} + \frac{kG \langle \nu, E \rangle}{s - u} - \frac{\sigma_k^2 u_i^2}{(s - u)^2}
\]

Recall that

\[
\langle X, X \rangle + \langle \nu, X \rangle^2 = \sum_i \langle X, \tau_i \rangle^2,
\]

we know \(|\langle X, \tau_i \rangle|\) can be controlled by some constants depending on \(s\) and local \(C^1\) estimates. Therefore, applying Lemma 8 and 9 of [11] we may assume

\[
0 \geq -C \kappa_1 \sum_{i=2}^n \frac{\sigma_k^2}{P_m} \left( \sum_j \kappa_j^{m-1} h_{jji} \right)^2 - \frac{C}{s - u}
\]

\[
- \frac{N \sigma_k \kappa_i^2 \langle \nu, E \rangle}{s - u} + \frac{kG \langle \nu, E \rangle}{s - u} - \frac{\sigma_k^2 u_i^2}{(s - u)^2}.
\]
Now, for any fixed \( i \geq 2 \) by (3.15) we have

\[
\frac{\sigma^i_k u_i^2}{(s - u)^2} = \sigma^i_k \left[ \frac{\sum \kappa_j^{m-1} h_{ijji}}{P_m} + N \kappa_i u_i \right]^2
\]

\[
= \sigma^i_k \left( \frac{\sum \kappa_j^{m-1} h_{ijji}}{P_m} \right)^2 + 2N \sigma^i_k \kappa_i u_i \left( -N \kappa_i u_i + \frac{u_i}{s - u} \right) + N^2 \sigma^i_k \kappa_i^2 u_i^2
\]

(3.24)

\[
= \sigma^i_k \left( \frac{\sum \kappa_j^{m-1} h_{ijji}}{P_m} \right)^2 - N^2 \sigma^i_k \kappa_i^2 u_i^2 + 2N \sigma^i_k \kappa_i u_i^2 \frac{u_i}{s - u}
\]

Plugging (3.24) into (3.23) we get,

\[
0 \geq -C \kappa_1 - \frac{C}{s - u} - N \frac{\sigma^i_1 u_i^2}{(s - u)} + \frac{kG \langle \nu, E \rangle}{s - u} - \frac{\sigma^i_k u_i^2}{(s - u)^2} + \sum_{i=2}^{n} N^2 \sigma^i_k \kappa_i^2 u_i^2 - 2N \sum_{i=2}^{n} \frac{\sigma^i_k \kappa_i u_i^2}{s - u}
\]

Since there is some constant \( c_0 \) such that \( \sigma^i_1 \kappa_1 \geq c_0 > 0 \), we have

\[
0 \geq \left( -\frac{c_0 N \langle \nu, E \rangle}{2} - C \right) \kappa_1 - \frac{N}{2} \sigma^i_1 \kappa_1^2 \langle \nu, E \rangle - \sum_{i=2}^{n} 2N \sigma^i_k u_i^2 \frac{s - u}{s - u} + \frac{kG \langle \nu, E \rangle - C}{s - u} - \frac{\sigma^i_k u_i^2}{(s - u)^2},
\]

where we have used for any \( 1 \leq i \leq n \) (no summation), \( \sigma_k = \sigma^i_k \kappa_i + \sigma_k(\kappa|i) \geq \sigma^i_k \kappa_i \). Moreover, it’s clear that

\[
\sum_{i=2}^{n} u_i^2 = \sum_{i=2}^{n} \langle \tau_i, E \rangle^2 < \frac{1}{1 - |Du|^2} = \langle \nu, E \rangle^2.
\]

We conclude

\[
\left( \frac{2NC}{s - u} + \frac{\sigma^i_1 u_i^2}{(s - u)} \right) \langle \nu, E \rangle^2 \geq \frac{Nc_0 \kappa_1}{4} \langle -\nu, E \rangle + \frac{N}{2} \sigma^i_1 \kappa_1^2 \langle -\nu, E \rangle.
\]

This implies \( (s - u) \kappa_1 \leq C \), where \( C \) depends on \( s \) and local \( C^1 \) estimates. Therefore, we obtain the desired Pogorelov type \( C^2 \) local estimates.

Following the argument in subsection 6.4 of [17], we prove Theorem 3.

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