Quantum Properties and Gravitational Field of a System with Oscillations in Time

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Abstract. We study the quantum properties and gravitational field of a system that has oscillations in time. Treating time as a dynamical variable, we first construct a wave with 4-vector amplitude that has matters vibrating in space and time. By analyzing its Hamiltonian density equation, we find that such system shall be treated as a quantized field. This quantized real scalar field obeys the Klein-Gordon equation and has properties resemble a zero spin bosonic field. In addition, the particle observed has oscillation in proper time. By neglecting all quantum effects and assuming the particle as a classical object that can remain stationary in space, we show that the spacetime geometry around the proper time oscillation has properties similar to the Schwarzschild gravitational field of a point mass in relativity.

1. Introduction
The treatment of time is a key issue in physics theories. For instance, time is assumed the same for all reference frames in Newtonian mechanics. It is basically treated as a parameter. On the other hand, there is no major difference how time is treated in quantum theory. Although there are suggestions that time shall play a more dynamical role in many quantum systems (e.g. tunneling time [12], decay time of an unstable particle [8], etc.), time is generally treated as a parameter. However, this is fundamentally different from how relativity is formulated where there is no globally defined time. This difference in approach has created constellation of problems especially when one tries to reconcile the two basic theories from a single framework [4,5].

In relativity, spacetime is weaved as unity. The relativistic dynamics require time to be treated on the same footing as space. However, when we consider a system like the simple harmonic oscillator, the treatment is not fully symmetric. The oscillation considered is only in the spatial directions. Thus, if space and time are to be treated on the same footing, can there be oscillation in the temporal direction? In fact, if we take analogy to the classical harmonic oscillator, it is theoretically possible to define an amplitude that has oscillation in the temporal direction [6]. Although it is feasible to construct an oscillator that has vibrations in time, can its properties have something to do with our real physical world? More importantly, how does it affect the surrounding spacetime geometry?

In the first half of this paper, we investigate the quantum properties of a system with oscillations in time. Taking time as a dynamical variable, we construct a plane wave with 4-vector amplitude \( (T, X) \) that has vibrations in space and time as shown in Section 2. By studying the Hamiltonian density equation of this plans waves in Section 3, we find that a harmonic oscillating system with vibrations of matter in proper time is a quantized field. The particle observed has
oscillation in proper time. Quantum properties of the system (e.g. Schrödinger equation, Klein-Gordon equation, bosonic field, probability density, unobservable overall phase etc.) will be discussed in Sections 4 and 5. The system with vibrations of matter in space and time produce the familiar structures of a real quantum system.

In the second half of this paper, we study the spherically symmetric spacetime geometry around the particle with oscillation in proper time. In Section 6, we study the Fourier decomposition of the proper time oscillator as a classical wave. We find that the system has additional radial oscillations at the immediate neighborhood of the proper time oscillator which can curve the surrounding spacetime. Their properties will be discussed in Section 7. The external spacetime geometry generated by the radial oscillations are derived in Sections 8 and 9. Our results mimic the Schwarzschild gravitational field for a point mass in relativity.

2. Plane Wave with Vibrations in Time

Consider the background coordinates \((t, x)\) for the flat spacetime as observed in an inertial frame. Time in this background is the 'external time’ as measured by clocks stationary at spatial infinity that are not coupled to the system under investigation \[7–9\]. Taking time as a dynamical variable, let us construct a plane wave that has both vibrations in the temporal and spatial directions. To study the vibrations that take place inside a plane wave, we will adopt a convention similar to the Lagrangian formulation in wave mechanics. In the undisturbed state without vibration, the clock of an observer at spatial coordinate \(x\) is synchronized with the clock at spatial infinity. Furthermore, the observer will remain stationary at \(x\). We will define real functions \(\text{Re}[\xi(t, x)]\) and \(\text{Re}[\zeta(t, x)]\) as the differences of the time measurement and spatial location from the undisturbed state labeled \((t, x)\). The function \(\text{Re}[\zeta(t, x)]\) does not tell us the spatial displacement of an observer which at time \(t\) have coordinate \(x\), but rather the displacement of an observer which have coordinate \(x\) in the undisturbed condition. Similarly, the function \(\text{Re}[\xi(t, x)]\) is the difference in time measured by an observer relative to the background time \(t\) originally from \(x\), and not at coordinate \(x\). An observer originally at \(x\) will be displaced to \(x' = x + \text{Re}(\xi)\), and measure a time \(t' = t + \text{Re}(\zeta)\) instead of time \(t\) at spatial infinity.

Let us first study a plane wave with only temporal vibrations in an inertial frame \(O'\). We will define the wave’s temporal amplitude \(T_0\) as the maximum difference between time \(t'\) observed inside the wave and time \(t'\) observed outside the wave by an inertial observer. Therefore, if an observer is placed inside the plane wave, time \(t'\) measured by the clock of this observer will be different from time \(t\) measured at spatial infinity. Time measured by the observer’s clock is running at a varying rate relative to the inertial observer’s clock. Without the vibration, the observer’s clock shall synchronize with the clock at spatial infinity in this undisturbed state. In addition, the plane wave has vibrations in the temporal direction but with no vibration in the spatial direction. We may then write,

\[
t'_f = t' - T_0 \sin(\omega_0 t') = t' + \text{Re}(\xi'_f),
\]

\[
x'_f = x',
\]

where

\[
\zeta'_f = -iT_0 e^{-i\omega_0 t'}.
\]

Therefore, time in the plane wave passes at the rate,

\[
\frac{\partial t'_f}{\partial t'} = 1 - \omega_0 T_0 \cos(\omega_0 t'),
\]

with respect to the external time and has an average value of 1. It will appear to travel along a timelike geodesic when averaged over many cycles.
By an appropriate Lorentz transformation, the undisturbed coordinates \((t', x')\) of inertial frame \(O'\) can be related to the undisturbed coordinates \((t, x)\) for the flat spacetime observed in another frame of reference \(O\). We assume that frame \(O'\) travels with velocity \(v\) relative to frame \(O\). Similarly, the displaced coordinates \((t'_f, x'_f)\) can be Lorentz transformed to the displaced coordinates \((t_f, x_f)\) as observed in frame \(O\). We can thus relate the displaced coordinates \((t_f, x_f)\) to the undisturbed coordinates \((t, x)\):

\[ t_f = t + T \sin(k \cdot x - \omega t) = t + \text{Re}(\zeta_t), \]

\[ x_f = x + X \sin(k \cdot x - \omega t) = x + \text{Re}(\zeta_x), \]

where

\[ \zeta_t = -i Te^{i(k \cdot x - \omega t)}, \]

\[ \zeta_x = -i X e^{i(k \cdot x - \omega t)}. \]

Amplitude \(X = (k/\omega_0)T_0\) is the maximum displacement of the wave from its undisturbed coordinate \(x\), and amplitude \(T = (\omega/\omega_0)T_0\) is its maximum displacement from time \(t\). The proper time displacement \(T_0\) can be seen as a Lorentz transformation of a 4-displacement vector: \((T_0, 0, 0, 0) \rightarrow (T, X)\) where \(T^2 = T^2_0 + |X|^2\). The amplitude of the plane wave is a 4-vector.

Inside a plane wave, time measured by the clock of an observer and its spatial location are displaced from the undisturbed state. We can further summarize these vibrations with a single function,

\[ \zeta = \frac{T_0}{\omega_0} e^{i(k \cdot x - \omega t)}. \]

The vibrations \(\zeta_t\) and \(\zeta_x\) from Eqs.\((7)\) and \((8)\) can be written as:

\[ \zeta_t = \frac{\partial \zeta}{\partial t}, \]

\[ \zeta_x = -\nabla \zeta. \]

3. Quantization of Wave Vibrations

Consider the function \(\zeta\) and its complex conjugate \(\zeta^*\). Both functions satisfy the wave equation:

\[ \partial_t \partial^\mu \zeta + \omega_0^2 \zeta = 0, \]

\[ \partial_t \partial^\mu \zeta^* + \omega_0^2 \zeta^* = 0. \]

Eqs. \((12)\) and \((13)\) are similar to the Klein-Gordon equation, except that we have yet to understand how \(\zeta\) can be related to the zero spin bosonic field in quantum theory. The corresponding Lagrangian density for the equations of motion is \(L = K[(\partial^\mu \zeta^*)(\partial_t \zeta) - \omega_0^2 \zeta^* \zeta]\), and the Hamiltonian density is

\[ H = K[(\partial_0 \zeta^*)(\partial_0 \zeta) + (\nabla \zeta^*) \cdot (\nabla \zeta) + \omega_0^2 \zeta^* \zeta], \]

where \(K\) is a constant of the system.

Let us examine the properties of this Hamiltonian density equation which is bounded from below. Substitute,

\[ \zeta_0 = \frac{T_0}{\omega_0} e^{-i\omega_0 t}, \]

into Eq. \((14)\), the Hamiltonian density of this plane wave with vibrations in proper time only is \(H_0 = 2KT_0^2\). This result is similar to the Hamiltonian density of a harmonic oscillating system.
in classical mechanics, except the vibrations are in time and not in space. For a system that can have multiple number of particles with mass \( m \) in a cube with volume \( V \), we make the ansatz
\[
K = \frac{m\omega_0^2}{2V},
\]
in analogous to its classical counterpart.

Under Lorentz transformation, the plane wave has vibrations in both time and space as observed in another reference frame. Substitute \( \zeta \) from Eq. (9) into Eq. (14), the Hamiltonian density is,
\[
H = H_1 + H_2 + H_3,
\]
where
\[
H_1 = \left( \frac{m\omega_0^2}{2V} \right) T^* T,
\]
\[
H_2 = \left( \frac{m\omega_0^2}{2V} \right) X^* \cdot X,
\]
\[
H_3 = \left( \frac{m\omega_0^2}{2V} \right) T_0^* T_0,
\]
where \( T_0, X \) and \( T \) are taken as complex amplitudes. The first term, \( H_1 \), on the right hand side of Eq. (17) is a Hamiltonian density with oscillations in time. The second term, \( H_2 \), is related to fluctuations of matter in the spatial direction. In the non-relativistic limit \( (\omega/\omega_0 \approx 1) \), our choice of \( K \) is not arbitrary but gives us the Hamiltonian density of a classical harmonic system with oscillations in the spatial direction. The third term, \( H_3 \), is related to the oscillations in proper time. After combining \( H_2 \) and \( H_3 \), we have,
\[
H = \left( \frac{m\omega_0^2}{V} \right) T^* T.
\]

Let us return to the plane wave with proper time vibrations only. In this plane wave, matter is stationary in space. Its Hamiltonian density, \( H_0 \), shall correspond to certain internal energy of matter at rest. Since the vibrations in proper time do not involve any force field, \( H_0 \) does not necessary have energy from charges. On the other hand, we have only consider matter with mass \( m \) in this simple harmonic oscillating system. No other energy is present in this system except the energy of mass \( m \). Here, we will consider \( H_0 \) as the internal mass-energy density generated by the proper time vibrations of matter.

From Eqs. (14), (15) and (16), the Hamiltonian density of a plane wave with vibrations in proper time is,
\[
H_0 = \frac{m\omega_0^2 T_0^* T_0}{V}.
\]
The energy inside volume \( V \) is \( E = m\omega_0^2 T_0^* T_0 \) of a harmonic oscillator in proper time with mass \( m \). If the energy of this harmonic oscillator is the internal energy of matter, it can only be observed as the energy of mass \( m \) which is on shell, i.e.
\[
E = m = m\omega_0^2 T_0^* T_0,
\]
or
\[
\omega_0^2 T_0^* T_0 = 1.
\]
In addition to the classical concepts of mass [10], we suggest here a possibility that a point mass \( m \) can have oscillation in proper time with amplitude \( T_0 = 1/\omega_0 \). On the other hand, this
A plane wave with proper time vibrations has energy for \( n = H_0 V/m \) oscillators to be observed in a volume \( V \). However, the equation does not predict where an oscillator can be detected. Its appearance is random. In fact, we can define a probability density of finding an oscillator as

\[
\rho_{pr} = \frac{n}{V} = \frac{H_0}{m}.
\]

The system with vibrations in space and time is a quantized field. The oscillators in time are the field quanta.

The internal time \( \dot{t}_f \) of the point mass’s internal clock is:

\[
\dot{t}_f(t) = t - \frac{\sin(\omega_0 t)}{\omega_0}.
\]

The internal time rate relative to the external time for the oscillator is

\[
\frac{\partial \dot{t}_f}{\partial t} = 1 - \cos(\omega_0 t).
\]

Not only the average of this time rate is 1, its value is bounded between 0 and 2 which is positive. Therefore, the internal time of the oscillator moves only forward. It cannot go backward to its past. If we assume the oscillator in time as a particle, e.g. electron (\( \omega_0 = 7.6 \times 10^{20} \) s and \( T_0 = 1.32 \times 10^{-21} \) s), the vibration frequency is large and the amplitude is small. The particle will appear to travel along a smooth time-like geodesic if the inertial observer’s clock is not sensitive enough to detect the high frequency and small amplitude of the oscillation. In fact, as the angular frequency increases and approaches infinity (\( \omega_0 \to \infty \)), the amplitude of oscillation becomes negligible (\( T_0 \to 0 \)). Such particle will travel along a near time-like geodesic with no oscillation observed.

The internal clock of the particle with angular frequency \( \omega_0 \to \infty \) is a clock suitable for the observer at spatial infinity. Its near time-like geodesic nature is sensitive enough to detect the varying internal time rate of another particle with lower frequency. However, this clock’s mass is infinite (\( m = \omega_0 \to \infty \)). As pointed out by Salecker and Wigner \[11\], to obtain infinite accuracy in measuring a clock’s time means infinite uncertainty in the clock’s mass, and thus the clock’s mass needs to reach infinity. Some of the studies regarding quantum clocks in the context of time-energy uncertainty relation can be found in references \[7, 8, 12\].

Taking mass \( m \) as the de Broglie mass/energy, we have the final form for the constant \( K \) of the system,

\[
K = \frac{\omega_0^3}{2V}.
\]

Under Lorentz transformation, a particle in the plane wave with angular frequency \( \omega \) and wave vector \( \mathbf{k} \) will travel at a velocity \( \mathbf{v} = \mathbf{k}/\omega \). It also has vibrations in time and space with amplitudes \( T = \omega/\omega_0^2 \) and \( X = k/\omega_0^2 \) from the Lorentz transform of amplitude \( T_0 = 1/\omega_0 \).

We can calculate the amplitudes of oscillation for a particle. For example, we can estimate the amplitude of spatial oscillation for an electron:

\[
|\mathbf{v}| = 0.99999 \Rightarrow |X| = 8.6 \times 10^{-9} \text{ cm,}
\]

\[
|\mathbf{v}| = 0.001 \Rightarrow |X| = 3.9 \times 10^{-14} \text{ cm.}
\]

In the second, non-relativistic example, the amplitude of the spatial oscillation is approximately equal to the diameter of a nucleus. However, this oscillation also has a very short time scale (\( \approx 10^{-21} \) s for electron). A particle will therefore appear to travel along a smooth trajectory if the measurements are not sensitive enough to detect the small oscillations.
4. Probability Density and Wave Function

The location that a particle can be observed in a plane wave is random. A region with higher Hamiltonian density shall have a greater chance to detect a particle. By taking the approximations \( \frac{\omega_0^2}{\zeta^* \zeta} >> (\nabla \zeta^*) \cdot (\nabla \zeta) \) and \( (\partial_0 \zeta^*)/(\partial_0 \zeta) \approx \frac{\omega_0^2}{V} \zeta \) in the non-relativistic limit, the Hamiltonian density \( H \) from Eq.(14) becomes:

\[
H \approx 2K \omega_0^2 \zeta^* \zeta = \frac{\omega_0^5}{V} \zeta^* \zeta .
\] (31)

The probability density \( \rho_{pr} \) of finding a particle in a region with Hamiltonian density \( H \) can be approximately defined as:

\[
\rho_{pr} \approx \frac{H}{\omega_0} \approx \frac{\omega_0^4}{V} \zeta^* \zeta .
\] (32)

Base on this probability density, we can establish a relationship between \( \zeta \) from Eq.(9) and the quantum mechanical wave function \( \psi \) for a plane wave in a cube with volume \( V \):

\[
\psi = \left[ \frac{\omega_0}{\sqrt{V}} e^{i(\omega_0 t + \chi)} \right] \zeta = \frac{a}{\sqrt{V}} e^{i(k \cdot x - \tilde{\omega} t + \chi)} ,
\] (33)

where

\[
a = \omega_0 T_0 ,
\] (34)

\[
\tilde{\omega} = k \cdot k / (2\omega_0) \approx \omega - \omega_0 ,
\] (35)

and \( e^{ix} \) is an arbitrary phase factor. Eq.(32) can then be written as:

\[
\rho_{pr} \approx \psi^* \psi .
\] (36)

Using the superposition principle, we can write a more general wave function

\[
\psi(x, t) = e^{i\chi} \sum_k \frac{\omega_0 T_0 k}{\sqrt{V}} e^{i(k \cdot x - \tilde{\omega} t)} .
\] (37)

By substituting \( \zeta \) with \( \psi \) in Eq.(12) and taking the non-relativistic limit, we obtain the Schrödinger equation for a free particle in quantum mechanics,

\[
i \frac{\partial}{\partial t} \psi(x, t) = -\frac{1}{2m} \nabla^2 \psi(x, t) .
\] (38)

As we can see, the phase factor \( e^{ix} \) in Eqs. (33) and (37) does not change the probability density. In fact, as demonstrated in quantum mechanics, the theory developed with wave functions \( \psi \) is invariant under global phase transformation but the relative phase factors are physical. Here, the wave function \( \psi \) serves as a mathematical tool for describing the quantization of the Hamiltonian density generated by the vibrations in time and space. A system with wave function \( \psi \) from the superposed plane waves can have a global phase shift \( \chi \) without changing the results in quantum mechanics. The overall phase of \( \psi \) is unobservable. However, function \( \psi \) is not required to have the same phase as \( \zeta \) that describes the vibrations in space and time.
5. Bosonic Field

The above analysis is based on a single particle system in the non-relativistic limit where approximations are taken to obtain the Schrödinger equation. As it is well known in quantum theory, when the Klein-Gordon equation is treated as a single particle equation in a relativistic theory, one will encounter the difficulties of negative energy solutions. Since \( \zeta \) satisfies an equation similar to the Klein-Gordon equation, we expect the system with vibrations in space and time shall have the same properties of a zero spin matter field in quantum theory.

For a many-particle system, it can have \( n \) integer number of oscillators. We can generalize condition \( \text{Eq. (22)} \) as

\[
\omega_0^2 T_0^* T_0 = n,
\]

which is a Lorentz invariant. The number of particles observed in the system shall remain same under Lorentz transformations. The Hamiltonian density from Eq. \( \text{Eq. (22)} \) for a plane wave \( \zeta_0 \) can be written as,

\[
H_0 = \frac{n \omega_0}{V}. \tag{40}
\]

The energy in this plane wave with vibrations in proper time only is quantized with \( n = 0, 1, 2, \ldots \).

Under a Lorentz transformation, \( \zeta_0 \rightarrow \zeta \). Instead, let us consider a plane wave \( \zeta_n \) which is normalized in volume \( V \) when \( n = 1 \),

\[
\zeta_n = \gamma^{-1/2} \zeta, \tag{41}
\]

where \( \gamma = (1 - |v|^2)^{-1/2} = \omega/\omega_0 \). Replace \( \zeta \) with \( \zeta_n \) in Eq. \( \text{Eq. (41)} \), the Hamiltonian density for plane wave \( \zeta_n \) is

\[
H_n = \gamma H_0 = \frac{n \omega}{V}. \tag{42}
\]

The energy in this plane wave \( \zeta_n \) is quantized with \( n \) particles of angular frequency \( \omega \) in a volume \( V \).

The vibrations in space and time are real physical quantities. As shown in Eqs. \( \text{Eq. (5)} \) and \( \text{Eq. (6)} \), only the real component of \( \zeta \) is relevant for obtaining these physical quantities. We retained the complex component of \( \zeta \) in previous analysis to simplify the derivation of the complex wave function. Here, \( \zeta \) can be combined with its complex conjugate. We can obtain a real scalar field by superposition of plane waves,

\[
\zeta(\vec{x}) = \frac{1}{\sqrt{2}} \sum_k [\zeta_{nk}(\vec{x}) + \zeta_{nk}^*(\vec{x})]
\]

\[
= (2\omega \omega_0)^{-1/2} \sum_k [T_{0k} e^{-i\vec{k} \cdot \vec{x}} + T_0^* e^{i\vec{k} \cdot \vec{x}}],
\]

which satisfies the Klein-Gordon equation.

To adopt the same convention in quantum field theory, we will switch to the use of field \( \varphi \) for describing the vibrations, i.e.,

\[
\varphi(\vec{x}) = \zeta(\vec{x}) \sqrt{\frac{\omega_0^3}{V}} = (2\omega V)^{-1/2} \sum_k [\omega_0 T_{0k} e^{-i\vec{k} \cdot \vec{x}} + \omega_0^* T_{0k} e^{i\vec{k} \cdot \vec{x}}]. \tag{43}
\]

From Eqs. \( \text{Eq. (14)} \) and \( \text{Eq. (43)} \), the Hamiltonian density equation for \( \varphi \) is,

\[
H = \frac{1}{2} [ (\partial_0 \varphi)^2 + (\nabla \varphi)^2 + \omega_0^2 \varphi^2 ] . \tag{44}
\]
In quantum field theory, the transition to a quantum field can be done via canonical quantization. Similarly, we can treat \( \varphi(\vec{x}) \) and \( H \) as operators. Condition (39) can be extended to the quantized field with

\[
N_k = \omega_0^2 T^\dagger_{0k} T_{0k},
\]

as the particle number operator. Ordering between \( T_{0k} \) and \( T^\dagger_{0k} \) shall be taken into account. We can also define the annihilation operator \( a_k \) and creation operator \( a_k^\dagger \) as,

\[
a_k = \omega_0 T_{0k},
\]

and

\[
a_k^\dagger = \omega_0 T^\dagger_{0k},
\]

such that \( N_k = a_k^\dagger a_k \). Comparing these results with quantum field theory, the real scalar field with vibrations in space and time has the physical structure of a zero-spin bosonic field.

6. Fourier Decomposition of Proper Time Oscillator

In the previous sections, we show that the energy of a point mass is the result of its oscillation in time. If we look at this in a different way, we can say that time at the location of a point mass is driven by its energy to oscillate. This energy is the source of the proper time oscillation. Taking this proper time oscillation as a part of the spacetime geometry, its properties are different from those of the assumed flat spacetime at spatial infinity where there is no oscillation in time. The geometry of spacetime at these two distant locations are different. Therefore, if the spacetime manifold outside the proper time oscillator is smooth and continuous, its structures cannot be flat. In the rest of this paper, we will study the gravitational field around the proper time oscillator. To simplify our analysis, we will neglect all quantum effects. The point mass will be treated as a classical object that can stay indefinitely at a location in space.

The oscillation in time driven by the energy of a point mass is shown in Eq. (26). Here, we will assume the proper time oscillator can remain indefinitely at the origin of spatial coordinates, \( x_0 \). This temporal oscillation is a part of the spacetime geometry. Outside the point mass, the region is source free and is a vacuum spacetime. As we shall recall from Section 3, matter with energy has oscillation in time. Thus, there shall have no temporal vibration outside the proper time oscillator if the external spacetime is source free, i.e. \( \dot{t}_f(t, x) = t \) when \( x \neq x_0 \). The temporal vibrations at and around the particle can be expressed as

\[
\dot{t}_f(t, x) = t - \frac{\Pi(x) \sin(\omega_0 t)}{\omega_0} = t + \dot{\zeta}(t, x),
\]

where

\[
\dot{\zeta}(t, x) = -\frac{\Pi(x)}{\omega_0} \sin(\omega_0 t),
\]

and

\[
\Pi(x) = 0 \text{ if } |x| \geq \epsilon/2,
\]

\[
\Pi(x) = 1 \text{ if } |x| < \epsilon/2.
\]

\( \Pi(x) \) is a pulse with width \( \epsilon \to 0 \).

As a wave, the vibrations in time \( \dot{\zeta} \) from Eq. (49) can be decomposed into Fourier series of plane waves. This can be done by the superposition of plane waves \( \zeta_{0k} = -i T^\dagger_{0k} e^{i(k \cdot x - \omega t)} \) base on Eq. (47) with their complex conjugates \( \zeta_{0k}^\dagger \) in a cube with volume \( V \) and then let \( V \) tend to infinity. Instead of treating \( \zeta_{0k} \) as quantized plane waves for the development of a quantum field, we will utilize them for another purpose. Here, \( \zeta_{0k} \) will be considered as functions for the Fourier decomposition of \( \dot{\zeta} \) as a classical wave.
As shown in Section 2, the amplitude of a plane wave that describes vibrations in time and space is necessary a 4-vector. For example, a plane wave with only vibrations in proper time can be Lorentz transformed to another plane wave that has vibrations in both space and time. \( \zeta_k \) is only the 0-component of a plane wave with 4-vector amplitude. The other component with vibrations in space, \( \zeta_{xk} = -iX_k e^{i(k \cdot x - \omega t)} \), are defined in Eq. (8). In addition, \( \zeta_k \) and \( \zeta_{xk} \) can be described by a plane wave \( \zeta_k \) as shown in Eqs. (9), (10), (11). The vibrations in time \( \zeta_t \) from Eq. (49) can be expressed in terms of \( \zeta_k \) and \( \zeta_{xk}^* \).

Let us define,

\[
\dot{\zeta}(t, x) = \frac{\Pi(x)}{\omega_0^2} \cos(\omega_0 t),
\]

which can be decomposed into Fourier series of \( \zeta_k \) and \( \zeta_{xk}^* \). Since the superpositions are linear, Eqs. (10) and (11) can be extended to the superposed wave \( \dot{\zeta} \) to obtain its vibrations in time and space \( (\dot{\zeta}_t, \dot{\zeta}_x) \), i.e.

\[
\dot{\zeta}_t = \frac{\partial \dot{\zeta}}{\partial t},
\]

\[
\dot{\zeta}_x = -\nabla \dot{\zeta}.
\]

Hence, \( \dot{\zeta}_t \) from Eq. (49) can be obtained from Eqs. (52) and (53). However, Eq. (54) shows that there are additional oscillations in space other than the oscillation in time at \( x_0 \). These oscillations in space are resulted from the covariant properties of the 4-vector amplitudes. Since the system we are considering is spherically symmetric, we can switch to a spherical coordinate system with the oscillator stationary at \( r = 0 \). The oscillations in space are described by \( \dot{\zeta}_r \),

\[
\dot{\zeta}_r(t, r) = -\frac{\Pi'(r)}{\omega_0^2} \cos(\omega_0 t).
\]

\( \Pi'(r) \) denotes the derivative of \( \Pi(r) \) with respect to \( r \), such that

\[
\Pi'(r) = 0 \text{ if } r \neq \epsilon/2,
\]

\[
\Pi'(r) = -\infty \text{ if } r = \epsilon/2.
\]

Therefore, apart from the proper time oscillation at \( r = 0 \), there are oscillations in the radial direction about \( r = \epsilon/2 \).

The radial oscillations about \( r = \epsilon/2 \) are revealed only after we study the Fourier decomposition of the proper time oscillator. They are not part of our study when we consider the quantum properties of the system. In fact, they can be omitted in our formulation for a bosonic field and their properties are not relevant in a quantum theory. However, these radial oscillations can have some importance in a spacetime theory.

7. Rest Mass System

We can summarize our results from Section 6 as follow:

At \( r = 0 \),

\[
\hat{t}_f(t, 0) = t - \frac{\sin(\omega_0 t)}{\omega_0},
\]

\[
\hat{r}_f(t, 0) = 0.
\]

At \( r = \epsilon/2 \),

\[
\hat{t}_f(t, \epsilon/2) = t,
\]
\[ \dot{r}_f(t, \epsilon/2) = \epsilon/2 + R_\infty \cos(\omega_0 t) \text{ with } R_\infty \to \infty. \]  

The system is spherically symmetric with oscillations in the temporal and radial directions only. From Eq. (60), the proper time oscillator is stationary at the origin of the spatial coordinate \( r = 0 \). It has no vibration in space where the point mass is located. On the other hand, the region outside the proper time oscillator is a vacuum spacetime which is source free. There are no vibrations in this vacuum spacetime except about a shell with radius \( r = \epsilon/2 \) as shown in Eq. (62). The point mass energy is the driving force of the system.

Let us look at this rest mass system in more detail. As shown, the point mass at rest has two oscillating components: the proper time oscillator at \( r = 0 \) and the radial oscillations about \( r = \epsilon/2 \). They are simple harmonic oscillators. Base on our knowledge about simple harmonic oscillating systems, their total energies are conserved over time. The rest mass system as a whole, therefore, shall have a symmetry under time translation as demanded by the Noether’s theorem.

As a part of the rest mass system, the proper time oscillator has a time translation symmetry. In Section 3, we show that the internal energy of a point mass is generated from the oscillation of matter in time. As a simple harmonic oscillator, the internal energy \( E \) of the proper time oscillator from Eq. (23) can be rewritten in terms of \( \dot{t}_f \) and \( \partial \dot{t}_f / \partial t \) from Eqs. (26) and (27) respectively,

\[ E = m I_t, \]  

where

\[ I_t = \omega_0^2 (\dot{t}_f - t)^2 + \left( \frac{\partial \dot{t}_f}{\partial t} - 1 \right)^2 = 1. \]  

Therefore, the internal energy \( E \) is the summation of two parts. The first part is the energy related to the displacement of internal proper time from the external time, \( (\dot{t}_f - t) \). The second part is the energy resulted from the difference between the internal time rate and external time rate, \( (\partial \dot{t}_f / \partial t - 1) \). This is analogous to the ‘potential’ and ‘kinetic’ energy components of a classical harmonic oscillator except the oscillation is in time and not in space. The quantities \( E \) and \( I_t \) are constant over time.

Spacetime outside the proper time oscillator is a vacuum. As shown in Eq. (61), a radial oscillation about \( r = \epsilon/2 \) does not have oscillation in time. This is unlike a particle with momentum that has vibrations in both space and time. In the absence of temporal oscillation, there is no related energy. The radial oscillation about \( r = \epsilon/2 \), therefore, has properties that are quite different from the vibrations of matter discussed so far.

Matter cannot have simple harmonic motion in space that has an amplitude of infinite magnitude and a finite frequency. This will violate the principles of relativity by allowing superluminal transfer of energy. From Eq. (62), the instantaneous velocity of the radial oscillation is,

\[ \dot{\hat{v}}_f(t, \epsilon/2) = \frac{\partial}{\partial t} \dot{r}_f(t, \epsilon/2) = -R_\infty \omega_0 \sin(\omega_0 t), \]  

which can exceed the speed of light. Transportation of an observer by the radial oscillation through space is forbidden by the principles of relativity. The radial oscillation therefore cannot be interpreted as vibration that can carry an observer through space. Instead, we shall study the effects of these radial oscillation on an observer that is stationary at \( r = \epsilon/2 \).

Any effects generated by the radial oscillations are negligible at spatial infinity where the spacetime is considered flat. An observer \( O \) stationary at spatial infinity is an inertial observer which is used as reference for our study. In a Minkowski spacetime, the clock of a stationary observer at any location shall be synchronized with the clock of the inertial observer \( O \). However, this is not the case for an observer \( O_+ \) stationary at \( r = \epsilon/2 \). As shown in Eqs. (61) and (62), it is the clock of a fictitious observer \( \bar{O} \) oscillating about \( r = \epsilon/2 \) that synchronize with the clock of
8. Thin Shell with Fictitious Radial Oscillations

Instead of working directly with the radial oscillations about $r = \epsilon/2$, let us first consider an infinitesimally thin spherical shell $\Sigma$ with radius $\tilde{r}(>2m)$. Relative to this shell, there are radial oscillations, i.e.

\[
\dot{t}_f(t, \tilde{r}) = t,
\]

\[
\dot{r}_f(t, \tilde{r}) = \tilde{r} + \tilde{R}\cos(\omega_0 t),
\]

\[
\dot{v}_f(t, \tilde{r}) = \frac{\partial}{\partial t} \dot{r}_f(t, \tilde{r}) = -\tilde{R}\omega_0 \sin(\omega_0 t),
\]

where $\tilde{R}\omega_0 < 1$. The properties of these radial oscillations with amplitude $\tilde{R}$ are analogous to those about $r = \epsilon/2$, except the magnitude of the instantaneous velocity $|\dot{v}_f|$ is now less than 1, and the amplitude of oscillation $\tilde{R}$ is finite. Apart from the oscillations in the radial direction, there are no other oscillations. The spacetime outside the shell is a vacuum and the system is spherically symmetric. We will further assume the system has a time translation symmetry as expected for a simple harmonic oscillating system.

As shown in Eqs. (67) and (68), it is the clock of a fictitious observer $\bar{O}$ oscillating about $r = \tilde{r}$ that synchronizes with the clock of an observer $O$ at spatial infinity. In its fictitious frame, $\bar{O}$ is an inertial observer. From Eqs. (68) and (69), an observer $\bar{O}$ stationary at $r = \tilde{r}$ has a fictitious displacement $\bar{r}_f$ and instantaneous velocity $\bar{v}_f$ relative to $O$,

\[
\bar{r}_f(t, \tilde{r}) = -\dot{r}_f(t, \tilde{r}) + \tilde{r} = -\tilde{R}\cos(\omega_0 t),
\]

\[
\bar{v}_f(t, \tilde{r}) = -\dot{v}_f(t, \tilde{r}) = \tilde{R}\omega_0 \sin(\omega_0 t).
\]

Although $\bar{O}$ is stationary relative to $O$ at spatial infinity, it is under the effects as if $\bar{O}$ is oscillating in the fictitious frame of $O$. As discussed before, the fictitious oscillation is not a vibration that carry an observer through space. It is information that will reflect the geometrical structure of spacetime at $r = \tilde{r}$.

The relativistic properties of a moving observer ($|\mathbf{v}| < 1$) are well defined in relativity. These properties can be applied in the fictitious frame of $O$. However, apart from the instantaneous velocity $\bar{v}_f$, $\bar{O}$ also has a displacement relative to the fictitious observer $\bar{O}$. As a simple oscillating system analogous to the proper time oscillator, we expect both the fictitious displacement and its instantaneous velocity can have effects on $\bar{O}$. Their combined effects shall remain constant such that the total Hamiltonian of the system is invariant over time. Although the effects of the fictitious displacement are not yet defined, we can obtain the spacetime geometrical properties of the thin shell $\Sigma$ when there is only a fictitious velocity with $|\bar{v}_f| < 1$.

At $t = t_m = \pi/(2\omega_0)$, the fictitious displacement and instantaneous velocity from Eqs. (70) and (71) are:

\[
\bar{r}_f(t_m, \tilde{r}) = \bar{r}_{fm} = 0,
\]

\[
\bar{v}_f(t_m, \tilde{r}) = \bar{v}_{fm} = 0.
\]
and
\[ v_f(t_m, \bar{r}) = v_{fm} = \bar{R}\omega_0 < 1. \quad (73) \]

\( \bar{O} \) is traveling with a velocity \( v_{fm} \) in the fictitious frame with no displacement relative to \( O \). To obtain the geometrical properties of spacetime at \( r = \bar{r} \), we need to understand how the clocks and measuring rods carried by \( O \) and \( \bar{O} \) are related at the instant \( t = t_m \).

Let us consider two events in frame \( \bar{O} \). The infinitesimal \( d\bar{t} \) and \( d\bar{r} \) are the differences in the temporal and radial coordinates between the two events at the instant \( t = t_m \). They can be related to the coordinate increments \( dt \) and \( dr \) for the same two events observed in frame \( O \),

\[ \begin{bmatrix} \frac{dt}{dr} \end{bmatrix} = \begin{bmatrix} \Upsilon^t_t & \Upsilon^t_r \\
\Upsilon^r_t & \Upsilon^r_r \end{bmatrix} \begin{bmatrix} d\bar{t} \\
 d\bar{r} \end{bmatrix}. \quad (74) \]

In the local frames of \( O \) and \( \bar{O} \), the basis vectors in the temporal and radial directions are orthogonal, i.e. \( \vec{e}_t \cdot \vec{e}_r = 0 \) and \( \vec{e}_{\bar{t}} \cdot \vec{e}_{\bar{r}} = 0 \). On the other hand, \( \bar{O} \) is stationary relative to the inertial frame \( O \). The temporal and radial basis vectors in frame \( O \) are parallel to their counterparts in frame \( \bar{O} \), i.e. \( \vec{e}_{\bar{t}} \parallel \vec{e}_t \) and \( \vec{e}_{\bar{r}} \parallel \vec{e}_r \). Under these conditions, the transformation matrix \( \Upsilon \) is diagonal,

\[ \Upsilon^t_t = \Upsilon^r_r = 0. \quad (75) \]

When \( d\bar{r} = 0 \), \( d\bar{t} \) is a proper time measured by the clock carried by \( \bar{O} \). This timelike interval can be Lorentz transformed to the fictitious frame of \( O \),

\[ dt = \gamma d\bar{t}, \quad (76) \]
\[ dr = \gamma v_{fm}d\bar{t}, \quad (77) \]

where \( \gamma = \sqrt{1 - (v_{fm})^2} \). In the fictitious frame, \( \bar{O} \) travels a distance \( d\bar{r} \) over a time \( d\bar{t} \). On the other hand, the clocks of \( O \) and \( \bar{O} \) are synchronized. \( O \) shall measure the same time as \( \bar{O} \),

\[ dt = dt = \gamma d\bar{t}. \quad (78) \]

However, \( O \) is physically stationary relative to \( \bar{O} \),

\[ dr = 0. \quad (79) \]

The underlined quantity in Eq. (77) is a fictitious displacement that appears only in the fictitious frame of \( O \). Its effect slows down the clock of \( \bar{O} \) but without relative movement between \( O \) and \( \bar{O} \). From Eqs. (73) and (78),

\[ \Upsilon^t_t = \gamma = \sqrt{1 - (v_{fm})^2} = \sqrt{1 - (\bar{R}\omega_0)^2}. \quad (80) \]

Next, we will consider a measuring rod with length \( d\bar{r} \) carried by \( \bar{O} \). This spacelike interval can be expressed as two events measured at the endpoints of the rod simultaneously, \( d\bar{t} = 0 \). Again, we can Lorentz transform these two events to the fictitious frame of \( O \),

\[ dt = \gamma v_{fm}d\bar{r}, \quad (81) \]
\[ dr = \gamma d\bar{r}. \quad (82) \]

From the viewpoint of \( O \), the rod carried by \( \bar{O} \) is moving at a velocity \( v_{fm} \). To obtain the moving length \( dl \) of the rod, we shall subtract \( dr \) by the distance traveled by the rod during \( dt \),

\[ dl = dr - v_{fm}dt = \gamma^{-1} d\bar{r}. \quad (83) \]
As inertial observers with their clocks synchronized, $O$ measures the same length of the rod as $O$,

$$\text{dr} = \text{dl} = \gamma^{-1} d\tilde{r}. \quad (84)$$

However, a rod carried by $\tilde{O}$ is stationary relative to $O$. The underlined quantities in Eqs. (81) and (83) are fictitious displacements in time and space that only appear in the fictitious frame of $\tilde{O}$. Their effects shorten the rod observed in frame $O$ but there is no relative movement between $O$ and $\tilde{O}$. The spacelike interval representing the length of the rod in frame $O$ is measured simultaneously at the endpoints,

$$dt = 0. \quad (85)$$

From Eqs. (73) and (84),

$$\Upsilon_{t\tilde{r}} = \gamma^{-1} = [1 - (\bar{v}f)_{m}^{2}]^{1/2} = (1 - \tilde{R}^{2} \omega_{0}^{2})^{1/2}. \quad (86)$$

The above results relate the clocks and measuring rods of $O$ and $\tilde{O}$ at one particular instant. As discussed, this simple harmonic oscillating system has a symmetry under time translation. The effects of the fictitious oscillations on $\tilde{O}$ shall be constant over time. Under this condition, we can define a constant,

$$\tilde{I} = \omega_{0}^{2} (\bar{r}f)^{2} + (\bar{v}f)^{2} = \tilde{R}^{2} \omega_{0}^{2}, \quad (87)$$

such that Eq. (74) becomes,

$$\begin{bmatrix} \frac{dt}{dr} \end{bmatrix} = \begin{bmatrix} (1 - \tilde{I})^{-1/2} & 0 \\ 0 & (1 - \tilde{I})^{1/2} \end{bmatrix} \begin{bmatrix} d\tilde{t} \\ d\tilde{r} \end{bmatrix}, \quad (88)$$

based on the results from Eqs. (75), (80) and (86). Analogous to the constant $I$ from Eq. (64), $\tilde{I}$ is the summation of two parts. The first part is the effect related to the fictitious displacement $\bar{r}f$. The second part is the effect resulted from the fictitious velocity $\bar{v}f$. Unlike a classical simple harmonic oscillator, the fictitious displacement and velocity are spacetime geometrical effects on $\tilde{O}$. There is no external spring for storing the potential effect nor there is any physical spatial movement. As geometrical properties, both $\bar{r}f$ and $\bar{v}f$ from Eqs. (70) and (71) have effects on $\tilde{O}$. Their summation is a constant under time translation. From Eq. (88), we can relate the basis vectors in frame $O$ and $\tilde{O}$,

$$\tilde{e}_{t} = \bar{e}_{t} (1 - \tilde{I})^{1/2}, \quad (89)$$

$$\tilde{e}_{\tilde{r}} = \bar{e}_{r} (1 - \tilde{I})^{-1/2}. \quad (90)$$

The oscillations in the system are entirely radial and temporal. It is spherically symmetric and there is no rotational motion. Thus, the line element at $r = \bar{r}$ can be written as [13],

$$ds^{2} = g_{tt}(\bar{r}) dt^{2} + 2g_{t\tilde{r}}(\bar{r}) dt d\tilde{r} + g_{\tilde{r}\tilde{r}}(\bar{r}) d\tilde{r}^{2} - \tilde{r}^{2} d\Omega^{2}, \quad (91)$$

where $\Omega$ is the metric induced on each 2-sphere using the radial coordinates of our reference system. As we shall note, external time $t$ is measured by a stationary clock located infinitely far from the source adopted in Section 2. The radial coordinate $r$ can be defined as the circumference, divided by $2\pi$, of a sphere centered around center of the shell. They are the same coordinate system adopted for the conventional Schwarzschild field.

The measurements in the temporal and radial directions in frames $O$ and $\tilde{O}$ have different scales. As a result, the metrics at $O$ and $\tilde{O}$ are different. From Eqs. (89) and (90),

$$g_{tt}(\bar{r}) = \bar{e}_{t} \cdot \bar{e}_{t} = (1 - \tilde{I}) \bar{e}_{t} \cdot \bar{e}_{t} = 1 - \tilde{I}, \quad (92)$$
\[ g_{rr}(\vec{r}) = \vec{e}_r \cdot \vec{e}_r = (1 - \vec{I})^{-1} \vec{e}_r \cdot \vec{e}_r = -(1 - \vec{I})^{-1}, \quad (93) \]
\[ g_{tr}(\vec{r}) = g_{rt}(\vec{r}) = \vec{e}_t \cdot \vec{e}_r = \vec{e}_t \cdot \vec{e}_r = 0, \quad (94) \]

where \( \vec{e}_t \cdot \vec{e}_t = 1, \vec{e}_r \cdot \vec{e}_r = -1, \) and \( \vec{e}_t \cdot \vec{e}_r = 0. \) Therefore, the line element at \( r = \vec{r} \) is,
\[ ds^2 = [1 - \vec{I}] dt^2 - [1 - \vec{I}]^{-1} dr^2 - \vec{r}^2 d\Omega^2. \quad (95) \]

Apart from the time translation symmetry, the spacetime at \( r = \vec{r} \) is also invariant under time reflection symmetry \( (t \rightarrow -t). \)

9. Schwarzschild Field

Eq. (95) is the line element of Schwarzschild metric at \( r = \vec{r} \) on the surface for a thin spherical shell with total mass \( m \) if we set
\[ \vec{I} = \frac{2m}{\vec{r}}, \quad (96) \]

or
\[ m = \frac{\vec{r}\vec{R}^2\omega_0^2}{2}. \quad (97) \]

From relativity, the vacuum spacetime \( \nu^+ \) outside this spherical thin shell \( \Sigma \) (a time-like hypersurface) is the Schwarzschild spacetime, i.e.
\[ ds^2 = [1 - \frac{\vec{r}\vec{R}^2\omega_0^2}{r}] dt^2 - [1 - \frac{\vec{r}\vec{R}^2\omega_0^2}{r}]^{-1} dr^2 - r^2 d\Omega^2. \quad (98) \]

The spacetime structure around the fictitious oscillations is static with time translation and time reflection symmetries. Eq. (98) relates the fictitious oscillations and the spacetime geometry outside the spherical shell \( \Sigma \).

The Birkhoffs theorem [14,15] states that the gravitational field of any spherically symmetric vacuum region is necessarily static, and its metric is that of the Schwarzschild spacetime. This applies to the external field of any non-rotating, spherical, uniform thin shell whether the shell is static, fluctuating or collapsing. Applying the same principle, the time-like hypersurface \( \Sigma \) can be expanded (or contracted) by carrying the fictitious oscillations along geodesics orthogonal to the original surface to a new sphere \( \Sigma' \). As long as mass \( m \) given in Eq. (97) is remaining constant during this transformation, the metric and curvature of the external field will not be affected. Under this condition, the amplitude of the radial oscillation is,
\[ \vec{R} = \sqrt{\frac{2}{\vec{r}\omega_0}}, \quad (99) \]

where we have used the equivalence \( m = \omega_0. \)

The amplitude \( \vec{R} \) from Eq. (99), and the related spacetime curvature tensors (e.g. the coordinate independent Kretschmann invariant [16], \( R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} = 48m^2/r^6 = 12\vec{r}^2\vec{R}^4\omega_0^4/r^6, \) etc.) derived from the metric, are well defined as the shell is contracted until it reaches a radius \( \vec{r} = \epsilon/2. \) At this point, the shell is infinitely small but has infinitely large amplitude of oscillations, \( \vec{R} \rightarrow \infty. \) Substitute \( \vec{R} = \vec{R}_\infty \) into Eq. (70), we obtain the fictitious radial displacement \( \vec{r}_f(t, \epsilon/2) \) given in Eq. (80). This infinitely small shell of radius \( \vec{r} = \epsilon/2 \) is the same shell we have described in Section 7 that has oscillations with amplitude \( \vec{R}_\infty \) and angular frequency \( \omega_0. \) As predicted by Birkhoffs theorem, the metric around this infinitely small shell is the Schwarzschild spacetime. As a result, the spacetime structure generated by the proper time oscillator can mimic the gravitational field of a point mass in relativity.
We note that when the shell is contracted to a radius $\bar{r} = 2m$ (the event horizon), the metric still encounters a coordinate singularity. Although the fictitious instantaneous velocity on a shell inside event horizon can exceed the speed of light (i.e. $v_{fm} > 1$ from Eqs. (73) and (99) when $\bar{r} < 2m$), they are not physical vibrations of matter. As information about the geometrical properties of spacetime, there is no superluminal transfer of energy for the fictitious oscillations on a shell inside the event horizon. The metric on the surface of the shell is well defined until the radius is contracted to $\bar{r} = \epsilon/2$. The shell, therefore, can be contracted even beyond radius $\bar{r} = 2m$ as allowed by Birkhoffs theorem while maintaining the same Schwarzschild geometry.

10. Conclusions and Discussions

In the first part of this paper, we treat time as a dynamical variable and study the possibility that matter not only can have vibrations in space but also in time. We show that the harmonic oscillator in proper time can be the generator for the energy of mass. However, the energy of a mass is necessary on shell meaning only one unique amplitude for the proper time harmonic oscillator can be observed, $T_0 = 1/\omega_0$. This is unlike a classical harmonic oscillator with oscillation in space that can take on different values as its amplitude. (There is no condition analogous to mass on shell that restrict amplitude of oscillation in space to an unique value.) The Hamiltonian of the system is quantized which can only correspond to those generated by integer number of oscillators. The assumption that matter has oscillations in time can lead to the quantization of a bosonic field. It is, therefore, possible that time can have a more dynamical role than generally considered in quantum theory.

In the second part of this paper, we consider the proper time oscillation as part of the spacetime geometry. The point mass energy is the source of the oscillation. By neglecting all quantum effects and study the Fourier decomposition of the proper time oscillator, we find that there are fictitious oscillations on a shell with infinitely small radius center around the proper time oscillator. The external spacetime is a vacuum which is curved by the shell with fictitious oscillations. Analogous to introducing a fictitious force to describe gravity, we can use the fictitious oscillations to explain the gravitational effects outside a thin shell. The resulting spacetime geometry obtained is similar to the results for a point mass in general relativity.

In general relativity, the criterion for a true singularity is geodesic incompleteness [17,18]. A singularity is present when the world line of a freely falling test object cannot be extended past that point. In our model with the proper time oscillator, a singularity exists in the spacetime manifold outside the proper time oscillator. The spacetime structure is discontinuous at $r = \epsilon/2$ where any causal geodesic (time-like or null) cannot be extended further. Although the singularity acts as a boundary for incoming geodesics, the spacetime structure and mass-energy content inside this boundary remain well defined. The point mass at the center has time oscillation relative to an observer at spatial infinity. There is no "unusual behavior" at the origin ($r = 0$), and the mass travels along a time-like geodesic when its oscillation is averaged over time.

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