Optimal Sampled-Data Control of a Nonlinear System

Yasuaki Oishi† and Noboru Sakamoto†

Optimal sampled-data control of a nonlinear system is considered with the stable-manifold approach and extensive use of numerical techniques. The idea is to notice the Hamiltonian system associated with the considered optimal control problem and to compute trajectories on its stable manifold. Since the control input accompanied with those trajectories is proved to be optimal, the optimal control law can be obtained through interpolation. The stable-manifold approach was originally proposed for continuous-time optimal control and here it is adapted for sampled-data control based on the works of Navasca. In the case of sampled-data control, the approach requires the state transition of the controlled plant during one sampling period together with its derivatives with respect to the state and the input. Their computation is achieved by numerical techniques. Moreover, a shooting method is proposed for systematic generation of the trajectories and extension is considered for the intersample behavior to be taken into account. The proposed method is applied to tracking control of a wheeled mobile robot. It works successfully with a rather long sampling period.

Keywords: sampled-data control, optimal control, nonlinear control, stable-manifold approach, minimum principle, Hamilton–Jacobi–Bellman equation, shooting method, intersample behavior.

1. Introduction

A sampled-data control system is a system where a continuous-time plant is controlled by a discrete-time controller together with sampling and hold devices. Nowadays, most of control systems are sampled-data control systems because a controller is usually implemented with a digital technique. Sampled-data control of a linear plant has been thoroughly investigated and a methodology for the optimal control is established according to various performance indices possibly with intersample behavior taken into account [4].

In contrast, sampled-data control of a nonlinear system is far from matured and most of the existing control methods are based on approximation. One approach is to design a continuous-time controller first for a given continuous-time plant and then to approximate it by a discrete-time controller [11,12]. Another approach is to discretize a given continuous-time plant by the Euler approximation or something similar and then to design a discrete-time controller [14,8,13,7]. In both approaches, the sampling period has to be sufficiently small for the approximation to be valid. In [2,3], dynamic programming of a sampled-data control system is considered in a general framework without approximation. The authors do not know however an existing method that gives an optimal sampled-data control law in a practical state-feedback form without resorting to approximation. One of the difficulties should be that the state transition during one sampling period is hard to obtain analytically for a nonlinear plant.

In this paper, we consider optimal sampled-data control of a nonlinear plant with the stable-manifold approach and extensive use of numerical techniques. The stable-manifold approach is originally developed for

†Department of Mechanical Engineering and System Control, Nanzan University, Yamazatocho 18, Showa-ku, Nagoya 466-8673, Japan; email: oishi@nanzan-u.ac.jp, noboru.sakamoto@nanzan-u.ac.jp
continuous-time control and has been applied to various practical nonlinear control problems [21, 17, 16]. There, the Hamiltonian system is considered for a given optimal control problem and its trajectories on the stable manifold are computed. Since a trajectory on the stable manifold gives a correspondence between a state and the optimal input, one can have an optimal control law in a state-feedback form through interpolation. In order to apply this approach to sampled-data control, one can utilize the results of Navasca [9, 10], where the Hamiltonian system is considered for discrete-time control and its connection to the optimal control is discussed. Computation of the optimal sampled-data control law is however not straightforward. Since the Hamiltonian system in [9, 10] is implicit both in forward time and backward time, a numerical technique is necessary to compute its trajectory. The technique further requires the state transition of the controlled plant and its derivative with respect to the initial state and the given control input. With a numerical technique, again, they can be computed and this is the basis of the proposed method. Although the control performance is evaluated only at the sampling instants first, extension is possible to take care of the intersample performance. There, a numerical technique again plays a central role.

The rest of the paper is structured as follows. Section 2 presents the optimal control problem to be considered. After some preliminaries in Section 3, the Hamiltonian system is presented in Section 4. Section 5 provides a shooting method for efficient generation of a trajectory. Optimal control taking care of intersample performance is considered in Section 6. After a numerical example is given in Section 8, the paper is concluded in Section 9.

The following notation is used. The symbol $\mathbb{R}$ stands for the set of real numbers and $\mathbb{R}^n$ for the set of $n$-dimensional real column vectors. The transpose of a vector and a matrix is denoted by $^T$. The derivative of a scalar-valued function $H$ with respect to a vector variable $u = (u_1 \cdots u_m)^T$ is an $m$-dimensional row vector $\partial H/\partial u$ whose $i$th component is $\partial H/\partial u_i$. The derivative of an $n$-dimensional vector-valued function $\phi = (\phi_1 \cdots \phi_n)^T$ with respect to $u = (u_1 \cdots u_m)^T$ is an $n \times m$ matrix $\partial \phi/\partial u$ whose $(i, j)$-component is $\partial \phi_i/\partial u_j$. Moreover, the second-order derivative of a scalar-valued function $H$ with respect to $u = (u_1 \cdots u_m)^T$ and $x = (x_1 \cdots x_n)^T$ is an $m \times n$ matrix $\partial^2 H/\partial u \partial x$ whose $(i, j)$-component is $\partial^2 H/\partial u_i \partial x_j$.

2. Considered problem

A plant to be controlled is a continuous-time system

$$\dot{x}(t) = f(x(t), u(t))$$

(1)

with the state $x(t) \in \mathbb{R}^n$ and the input $u(t) \in \mathbb{R}^m$. Here, the function $f(x, u)$ is assumed to be sufficiently smooth and to satisfy $f(0, 0) = 0$. We consider to control this plant with a discrete-time controller that produces a discrete-time input $u[k]$ from a sampled state $x[k]$. See Figure 1. Here, $x[k]$ is the state $x(t)$ sampled at $t = kh$, $k = 0, 1, \ldots$, for some sampling period $h > 0$ and $u[k]$ is converted to the continuous-time input $u(t)$ by $u(t) = u[k]$ for $kh \leq t < (k + 1)h$ and $k = 0, 1, \ldots$.

Let us consider the portion enclosed by the dashed rectangle in the figure, which is regarded as a discrete-time system and is called the discretized plant. To have its expression, consider the initial-value problem

$$\dot{x}(t) = f(x(t), u_0), \quad x(0) = x_0.$$  

(2)
Figure 1. The sampled-data control system to be considered. The portion enclosed by the dashed rectangle can be regarded as a discrete-time system and is called the discretized plant.

We assume that this initial-value problem has a unique solution \( \phi(t; x_0, u_0) \), \( 0 \leq t \leq h \), for any \( x_0 \in X \) and any \( u_0 \in U \), where \( X \subset \mathbb{R}^n \) and \( U \subset \mathbb{R}^m \) are some bounded open sets containing the origin and \( U \) is also convex. It is known [18, Chapter 6] that \( \phi(t; x_0, u_0) \) is smooth in \((t, x_0, u_0)\). Now, the discrete-time system

\[
x[k + 1] = \phi(h; x[k], u[k])
\]

is the desired discretization of the plant (1).

Our problem to be considered is the following.

**Problem.** Let \( Q \) be some positive semidefinite matrix and \( R \) be some positive definite matrix. Obtain the optimal input \( u[k] \in U \) for each \( k = 0, 1, \ldots \) that minimizes the objective function

\[
\sum_{k=0}^{\infty} (hx[k]^TQx[k] + hu[k]^TRu[k]).
\]

(4)

**Remark 1.** To the objective function above, the sampling period \( h \) is multiplied. It is for consistency with the objective function in Section 7, where the intersample behavior of the system is considered.

\[\diamond\]

### 3. Preliminaries

Given a nonlinear function \( f(x, u) \), it is hard to obtain an analytic form of \( \phi(t; x, u) \) in the initial-value problem (2). For a specific value of \((t, x, u)\), however, the value of \( \phi(t; x, u) \) can be computed by a numerical technique. The same is true for the derivatives of \( \phi(t; x, u) \). The details are explained in this section as a preliminary for the rest of the paper.

Suppose that some \((x, u) \in X \times U\) is given. By definition, the function \( \phi(t; x, u) \) satisfies

\[
\frac{d}{dt} \phi(t; x, u) = f(\phi(t; x, u), u), \quad \phi(0; x, u) = x.
\]

(5)

We are to apply a numerical technique to compute \( \phi(t; x, u) \) for various \( t \) between 0 and \( h \). One example of such technique is the classical fourth-order Runge–Kutta method [19, Section 12.5]. There, with \( M \) being some positive integer and \( \hat{\phi}_0 := x \), we repeat the following procedure for \( j = 0, 1, \ldots, M - 1 \):

\[
g_{j+1} := f(\hat{\phi}_j, u),
\]

\[\diamond\]
Then, \( \hat{\phi}_j \) gives the value of \( \phi(jh/M; x, u) \) for \( j = 1, 2, \ldots, M \). For the moment, we need only the final value \( \phi(h; x, u) \) that appears in the expression (3) of the discretized plant. Later in Section 7, however, we also use the intermediate values \( \phi(jh/M; x, u) \) to assess the intersample behavior of the system.

For computation of the derivative of \( \phi(t; x, u) \) with respect to \( x \) and \( u \), differentiate both sides of (5) as

\[
\frac{d}{dt} \frac{\partial \phi(t; x, u)}{\partial x} = \frac{\partial f(\phi(t; x, u), u)}{\partial x} \quad \text{and} \quad \frac{d}{dt} \frac{\partial \phi(t; x, u)}{\partial u} = \frac{\partial f(\phi(t; x, u), u)}{\partial u} \cdot \frac{\partial \phi(t; x, u)}{\partial x}.
\]

Application of a numerical technique simultaneously to (5)–(7) gives the values of \( \phi(jh/M; x, u) \) and \( \partial \phi/\partial u(jh/M; x, u) \) for \( j = 1, 2, \ldots, M \). It is also possible to evaluate the second-order derivatives in a similar way. In later sections, we need \( \phi(t; x, u) \), \( \partial \phi/\partial x(t; x, u) \), \( \partial \phi/\partial u(t; x, u) \), \( \partial^2 \phi_i/\partial u \partial x(t; x, u) \), and \( \partial^2 \phi_i/\partial u^2(t; x, u) \) for \( i = 1, 2, \ldots, n \). This means that the total dimension of the differential equations to be considered is \( n + n^2 + mn + n^2m + mn^2 \). In Section 6, we additionally need \( \partial^2 \phi_i/\partial x^2(t; x, u) \) for \( i = 1, 2, \ldots, n \), which increases the dimension by \( n^3 \).

For the original continuous-time plant (1), let us write its linearization as

\[
\dot{x}(t) = Ax(t) + Bu(t)
\]

for \( A = (\partial f/\partial x)(0, 0) \) and \( B = (\partial f/\partial u)(0, 0) \). Similarly, the linearization of the discretized plant (3) is

\[
x[k + 1] = A_h x[k] + B_h u[k]
\]

for \( A_t = (\partial \phi/\partial x)(t; 0, 0) \) and \( B_t = (\partial \phi/\partial u)(t; 0, 0) \). The following relationships follow immediately.

**Proposition 1.** There hold \( A_t = e^{At} \) and \( B_t = \int_0^t e^{A\tau} \, d\tau B \).

**Proof.** Substitute \( x = 0 \) and \( u = 0 \) into (5) and note \( f(0, 0) = 0 \) to see \( \phi(t; 0, 0) = 0 \) for any \( t \geq 0 \). Substitute \( x = 0 \) and \( u = 0 \) into (6) and use \( \phi(t; 0, 0) = 0 \) and \( A = (\partial f/\partial x)(0, 0) \). Then, \( (\partial \phi/\partial x)(t; 0, 0) = e^{At} \) follows. A similar reasoning with (7) gives \( (\partial \phi/\partial u)(t; 0, 0) = \int_0^t e^{A\tau} \, d\tau B \).

We make the following assumption.

**Assumption 1.** The original plant (1) has the linearization such that \( (A, B) \) is stabilizable and \( (Q, A) \) is detectable.
Proposition 2. For a small enough sampling period \( h \), the discretized plant (3) has the linearization such that \((A_h, B_h)\) is stabilizable and \((Q, A_h)\) is detectable. Then the discrete-time Riccati equation

\[
A_h^T S A_h - S + hQ - A_h^T S B_h (B_h^T S B_h + hR)^{-1} B_h^T S A_h = O
\]

has a unique stabilizing solution \( S \), which is positive semidefinite.

Proof. See [4, Theorem 3.2.1] for the former statement. See [4, Theorem 6.3.2] or [1, Section 4.1] for the latter statement. □

4. Hamiltonian System

In order to solve our optimal control problem, we consider the corresponding Hamiltonian and the Hamiltonian system.

The Hamiltonian [1] for our problem is

\[
H(x[k], u[k], p[k+1]) = \phi(h; x[k], u[k])^T p[k+1] + hx[k]^T Qx[k] + hu[k]^T Ru[k]
\]

with the costate \( p[k] \in \mathbb{R}^n \). As a domain of the costate \( p[k] \), we fix some bounded open set \( P \subset \mathbb{R}^n \) containing the origin. To simplify the notation, let us write \( x^+[k] = x[k+1] \) and \( p^+[k] = p[k+1] \) and drop the dependence on \( k \). The following assumption is made.

Assumption 2. The Hamiltonian \( H(x, u, p^+) \) is convex in \( u \in U \) for any \((x, p^+) \in X \times P\). ◻

This assumption is satisfied for a small enough sampling period \( h \) in the following special case.

Proposition 3. Suppose that the original plant (1) is affine in the input \( u(t) \). Then Assumption 2 is satisfied for a small enough sampling period \( h \).

Proof. The Hessian of the Hamiltonian \( H(x, u, p^+) \) with respect to \( u \) is

\[
\frac{\partial^2}{\partial u^2} \phi(h; x, u)^T p^+ + 2hR.
\]

Here, the first term is the Hessian of the scalar-valued function \( \phi(h; x, u)^T p^+ \). Note that the Hessian \( (\partial^2 \phi_i/\partial u^2)(0; x, u) \) is equal to zero for each of \( i = 1, 2, \ldots, n \). Hence, as \( h \to 0 \),

\[
\frac{1}{h} \frac{\partial^2 \phi_i}{\partial u^2}(h; x, u) \to \frac{d}{dt} \frac{\partial^2 \phi_i}{\partial u^2}(0; x, u).
\]

The last quantity is in fact equal to \( (\partial^2 f_i/\partial u^2)(x, u) \) for each \( i = 1, 2, \ldots, n \). This can be seen by differentiating the first equation of (7) with respect to \( u \) and noting that the derivatives \( (\partial \phi/\partial u)(0; x, u) \) and \( (\partial^2 \phi_i/\partial u^2)(0; x, u) \) are all equal to zero. Moreover, the affinity of \( f(x, u) \) implies that the right-hand side quantity in (10) equals to zero. The convergence is uniform on \( X \times P \) due to the smoothness of \( \phi(t; x, u) \). Hence, the Hessian (9) is positive definite on \( X \times P \) for a small enough \( h \). □
Henceforth, we write \( \phi(h; x, u) \) as \( \phi_h(x, u) \) in short. The minimum principle for our problem is the following:

\[
\begin{align*}
  x^+ &= \frac{\partial H}{\partial p^+}(x, u, p^+)^T = \phi_h(x, u), \\
  0 &= \frac{\partial H}{\partial u}(x, u, p^+)^T = \frac{\partial \phi_h}{\partial u}(x, u)^T p^+ + 2hRu, \\
  p &= \frac{\partial H}{\partial x}(x, u, p^+)^T = \frac{\partial \phi_h}{\partial x}(x, u)^T p^+ + 2hQx.
\end{align*}
\]

The equations (11)–(13) are regarded to define some dynamics on the state \((x, p)\) around the origin. Indeed, we have the following property.

**Proposition 4.** The equations (11)–(13) uniquely determine a smooth function from \((x, p) \in X \times P\) to \((x^+, u, p^+) \in X \times U \times P\) around the origin. Conversely, the equations (11)–(13) uniquely determine a smooth function from \((x^+, p^+) \in X \times P\) to \((x, u, p) \in X \times U \times P\) around the origin.

**Proof.** When \(x, p, u, x^+, p^+\) are all equal to zero, the equations (11)–(13) are obviously satisfied. Suppose first that \((x, p) \in X \times P\) is given and look for \((x^+, u, p^+)\) satisfying (11)–(13). Differentiating the right-hand sides of (12) and (13) with respect to \((u, p^+)\) to have

\[
\begin{pmatrix}
  \frac{\partial^2}{\partial u^2} \phi_h(x, u)^T p^+ + 2hR \frac{\partial \phi_h}{\partial u}(x, u)^T \\
  \frac{\partial^2}{\partial x \partial u} \phi_h(x, u)^T p^+ + \frac{\partial \phi_h}{\partial x}(x, u)^T
\end{pmatrix}.
\]

Again, \((\partial^2 / \partial u^2) \phi_h(x, u)^T p^+\) is the second derivative of the scalar-valued function \(\phi_h(x, u)^T p^+\). When \(x, p, u, x^+, p^+\) are all equal to zero, this matrix is reduced to

\[
\begin{pmatrix}
  2hR & B_h^T \\
  0 & A_h^T
\end{pmatrix},
\]

which is nonsingular because \(A_h = e^{Ah}\) due to Proposition 1. Hence, the implicit function theorem implies the existence of a unique mapping from \((x, p)\) to \((u, p^+)\) around the origin and then to \(x^+\) by (11).

The converse case where \((x^+, p^+) \in X \times P\) is given can be similarly considered. \(\square\)

5. Proposed Method

Navasca [9, 10] considered optimal control of a discrete-time nonlinear system and showed in particular the existence of a stable manifold for the associated Hamiltonian system and its connection to the optimal control. Application of these results to the present framework gives the following. Sketch of the proof is presented in the appendix for completeness.

**Proposition 5.** The Hamiltonian system (11)–(13) has a stable manifold around the origin that can be expressed as \(p = s(x)\). Here, \(s(x)\) is a smooth function satisfying \(s(0) = 0\) and \((\partial s / \partial x)(0) = 2S\) for the stabilizing solution \(S\) of the Riccati equation (8).
**Proposition 6.** For the function $s(x)$ in the previous proposition, there exists a smooth scalar-valued function $V(x)$ around the origin such that $(\partial V/\partial x)(x)^T = s(x)$.

**Proposition 7.** Let $s(x)$ and $V(x)$ be the functions in the preceding two propositions. Define the function $u^*(x)$ by the value of $u$ corresponding to $(x, p = s(x))$ in the Hamiltonian system (11)–(13). Then, $V(x)$ and $u^*(x)$ satisfy the following equation around the origin:

$$V(x) = \min_{u \in U} \left[ V(\phi_h(x, u)) + h x^T Q x + h u^T R u \right] = V(\phi_h(x, u^*(x))) + h x^T Q x + h u^*(x)^T R u^*(x).$$

This is the Hamilton–Jacobi–Bellman equation of dynamic programming and tells that $V(x)$ is the optimal cost-to-go function for the state $x$ and $u^*(x)$ gives the optimal control.

Based on the last proposition, we propose a method to solve our problem. That is, we obtain a sufficient number of points $(x, p)$ as well as the corresponding $u$ on the stable manifold and approximate the relationship between $x$ and $u$ to have the function $u^*(x)$. In particular, we choose $x[N]$ sufficiently close to the origin for some positive number $N$ and consider the point $(x[N], p[N])$ for $p[N] = 2S x[N]$. Since this point can be regarded to be on the stable manifold, so are the points $(x[k], p[k])$, $k = N - 1, N - 2, \ldots, 0$, generated by the Hamiltonian system (11)–(13) backward in time. Thus we can collect desired pairs $(x[k], u[k])$. Repeat this procedure for various values of $x[N]$ and interpolate the obtained $u$ as a function of $x$.

There are some issues for execution of this procedure.

First, we need to consider how the initial value $x[N]$ is to be chosen. As will be discussed in the next section, it is desired to be chosen so that the corresponding $x[0]$ matches some prespecified value. This is accomplished by some shooting method, to be proposed there.

Next, backward simulation of the Hamiltonian system (11)–(13) is discussed. For a given $(x^+, p^+)$, the nonlinear equations (11) and (12) have to be solved to have $(x, u)$. Newton’s method is used for this purpose.

**Algorithm 1.**

Input: the pair $(x[N], p[N])$.

Output: a sequence of triplets $(x[k], u[k], p[k])$ for $k = N - 1, \ldots, 1, 0$.

Procedure:

1. Set $k := N - 1$.

2. With $(x^+, p^+) = (x[k + 1], p[k + 1])$ compute $(x, u, p)$ satisfying (11)–(13) by the following procedure.

   (1) Set the initial value $(x(0), u(0))$ in some way. One possibility is to set $x(0)$ to $x^+ = x[k + 1]$ and $u(0)$ to $u[k + 1]$. When $k = N - 1$, since $u[k + 1]$ is not available, set $u(0) = -(B_k^T S B_k + hR)^{-1} B_k^T S A_h x^+$ (linear optimal control) instead, for example.

   (2) For $j = 0, 1, \ldots$, repeat the following until convergence:

   $$\begin{pmatrix} x(j+1) \\ u(j+1) \end{pmatrix} = \begin{pmatrix} x(j) \\ u(j) \end{pmatrix} - \begin{pmatrix} \frac{\partial \phi_h}{\partial x} (x(j), u(j)) \\ \frac{\partial^2 \phi_h}{\partial x^2} (x(j), u(j))^T \end{pmatrix} p^+ - \begin{pmatrix} \frac{\partial \phi_h}{\partial u} (x(j), u(j))^T \\ \frac{\partial^2 \phi_h}{\partial u^2} (x(j), u(j))^T \end{pmatrix} p^+ + 2hR.$$

   The necessary derivatives of $\phi_h(x, u)$ can be computed as in Section 3.
(3) Substitute the obtained \((x, u)\) to (13) to have \(p\).

3. Define \((x[k], u[k], p[k])\) by the obtained \((x, u, p)\).

4. If \(k = 0\), output the obtained sequence and stop. Otherwise, decrease \(k\) by one and go back to Step 2.

Finally, interpolation of the obtained pairs \((x, u)\) is considered. When the number of the pairs is medium-sized and the dimensions of \(x\) and \(u\) are small, any method such as least-squares approximation with a polynomial can be used. When the problem size is large, a more sophisticated method such as moving least squares and a radial basis function is preferable [20].

6. Shooting Method

In this section, we present a shooting method to choose the value of \(x[N]\) so that the corresponding \(x[0]\) matches the prespecified value. As will be seen in Section 8, a practical plant can have some important domain of the state through which the plant moves under control. Suppose that we choose a sufficient number of points over the domain and let each of them be \(x[0]\) to compute the corresponding optimal input. Then, through interpolation, we can obtain the optimal control law \(u^*(x)\) for any \(x\) in the domain. This should be more efficient than computing \(u^*(x)\) blindly in a large domain. The idea of the shooting method has been presented in [15] for optimal control of a continuous-time system.

To present the method, assume that we have an original trajectory of the Hamiltonian system (11)–(13) \((x[k], u[k], p[k]), k = 0, 1, \ldots, N - 1,\) and \((x[N], p[N])\) on the stable manifold. Here, \(x[N]\) is assumed to be close enough to the origin and \(x[0]\) be not very distant from the prespecified value \(x^*\). We iteratively update the trajectory so that \(x[0]\) approaches the desired value \(x^*\). To this aim, we consider to add a small perturbation \(\Delta x[N]\) to \(x[N]\). The perturbation to \(p[N]\) is chosen as \(\Delta p[N] = 2S\Delta x[N]\) so that the point remains on the stable manifold. We compute the corresponding perturbation that arises at each point \((x[k], p[k])\) in the trajectory.

Suppose that we have the perturbation \((\Delta x[k + 1], \Delta p[k + 1])\) for the \((k + 1)\)st point \((x[k + 1], p[k + 1])\) and want to compute the perturbation for the next point \((x[k], p[k])\), where \(k = N - 1, \ldots, 1, 0\). Linearization of the Hamiltonian system (11)–(13) gives

\[
\Delta x[k + 1] = \frac{\partial \phi_h}{\partial x} \Delta x[k] + \frac{\partial \phi_h}{\partial u} \Delta u[k],
\]

\[
0 = \frac{\partial^2}{\partial u \partial x} \phi_h^T p^+ \Delta x[k] + \left( \frac{\partial^2}{\partial u^2} \phi_h^T p^+ + 2hR \right) \Delta u[k] + \left( \frac{\partial \phi_h}{\partial u} \right)^T \Delta p[k + 1],
\]

\[
\Delta p[k] = \left( \frac{\partial^2}{\partial x^2} \phi_h^T p^+ + 2hQ \right) \Delta x[k] + \frac{\partial^2}{\partial x \partial u} \phi_h^T p^+ \Delta u[k] + \left( \frac{\partial \phi_h}{\partial x} \right)^T \Delta p[k + 1],
\]

where the derivatives are all evaluated at \((x, u, p^+) = (x[k], u[k], p[k + 1])\). Delete \(\Delta u\) and solve them for \(\Delta x[k]\) and \(\Delta p[k]\). The result can be expressed as

\[
\begin{bmatrix}
\Delta x[k] \\
\Delta p[k]
\end{bmatrix} = H_k \begin{bmatrix}
\Delta x[k + 1] \\
\Delta p[k + 1]
\end{bmatrix}
\]
with the matrix $H_k$ defined as

$$H_k = \left( \begin{array}{c} I \\ \frac{\partial^2}{\partial x^2} \phi_h^T p^+ + 2hQ \\ \frac{\partial^2}{\partial u \partial x} \phi_h^T p^+ \\ \frac{\partial^2}{\partial u^2} \phi_h^T p^+ + 2hR \end{array} \right) \left( \begin{array}{c} \frac{\partial \phi_h}{\partial x} \\ \frac{\partial \phi_h}{\partial u} \end{array} \right) \left( \begin{array}{c} I \\ O \\ \frac{\partial \phi_h}{\partial u} \end{array} \right)^T + \left( \begin{array}{c} O \\ O \\ \frac{\partial \phi_h}{\partial u} \end{array} \right)^T. \right) (17)$$

The matrix $H_k$ contains the new derivative $(\partial^2/\partial x^2) \phi_h(x, u)^T p^+$, which is however computable as in Section 3. Successive multiplication of the matrix $H_k$ gives the perturbation for $(x[0], p[0])$, that is,

$$\begin{pmatrix} \Delta x[0] \\ \Delta p[0] \end{pmatrix} = H_0 H_1 \cdots H_{N-1} \left( \begin{array}{c} I \\ 2S \end{array} \right) \Delta x[N].$$

Since we want to make $x[0] + \Delta x[0]$ equal to $x^*$, an appropriate choice of $\Delta x[N]$ should be

$$\Delta x[N] = \left[ \left( I \ O \right) H_0 H_1 \cdots H_{N-1} \left( \begin{array}{c} I \\ 2S \end{array} \right) \right]^{-1} (x^* - x[0]).$$

Modify the present $x[N]$ and $p[N]$ by adding this $\Delta x[N]$ and $2S \Delta x[N]$, respectively, and update the trajectory with Algorithm 1 applied to this new initial point. If the resulting $x[0]$ is close enough to the desired value $x^*$, we are done. If this is not the case, we again consider to add a perturbation to $x[N]$ and update the trajectory in a similar way. Since this is basically Newton’s method, convergence can be expected if the original $x[0]$ is not very distant from the desired $x^*$.

This is the idea of the shooting method. To make it more practical, some numerical techniques are applied.

Since the matrix $H_k$ originates from the Hamiltonian system, it can have both stable and unstable eigenvalues. This means that it may have a large condition number and its successive multiplication may produce a large numerical error. In order to attenuate the error, the (thin) QR-factorization [5, Section 5.2] [19, Section 2.9] should be useful. First, factorize $(I \ 2S)^T$ into

$$\begin{pmatrix} Y_N \\ Z_N \end{pmatrix} R_N = \begin{pmatrix} I \\ 2S \end{pmatrix}, \right) (18)$$

so that the columns of $(Y_N^T \ Z_N^T)^T$ are orthonormal and the matrix $R_N$ is square and upper-triangular. Multiply $H_{N-1}$ only to $(Y_N^T \ Z_N^T)^T$ and factorize the resulting matrix as

$$\begin{pmatrix} Y_{N-1} \\ Z_{N-1} \end{pmatrix} R_{N-1} = H_{N-1} \begin{pmatrix} Y_N \\ Z_N \end{pmatrix}$$

again with the QR-factorization. Since each column of $(Y_N^T \ Z_N^T)^T$ is orthonormal, the multiplication of $H_{N-1}$ can be made in a more numerically stable manner than its direct multiplication to $(I \ 2S)^T$. Due to the final form

$$\begin{pmatrix} \Delta x[0] \\ \Delta p[0] \end{pmatrix} = \begin{pmatrix} Y_0 \\ Z_0 \end{pmatrix} R_0 R_1 \cdots R_N \Delta x[N],$$

$\Delta x[N]$ can be computed by $R_N^{-1} \cdots R_1^{-1} R_0^{-1} Y_0(x^* - x[0])$. Since each $R_k$ is upper-triangular, the multiplication of its inverse can be computed by simple sequential substitution.

It is possible that the added perturbation $\Delta x[N]$ is too large and the resulting $x[0]$ does not improve its distance to $x^*$. In that case, the damping technique can be useful. That is, we replace the perturbation by its
half \((1/2)\Delta x[N]\) and update the trajectory with Algorithm 1. If the resulting \(x[0]\) improves the distance to \(x^*\), we accept it. If this is not the case, we again halve the perturbation to \((1/2^2)\Delta x[N]\) and repeat the procedure.

Combination of these ideas gives the following algorithm.

**Algorithm 2.**

Input: a sequence \((x[k], u[k], p[k]), k = 0, 1, \ldots, N - 1\), and \((x[N], p[N])\) on the stable manifold such that \(x[N]\) is close enough to the origin and \(x(0)\) is not too distant from the desired value \(x^*\).

Output: a sequence \((x[k], u[k], p[k]), k = 0, 1, \ldots, N - 1\), and \((x[N], p[N])\) on the stable manifold such that \(x[0]\) is close enough to \(x^*\).

Procedure:

1. Evaluate and store the present distance \(d = \|x^* - x[0]\|\) in the Euclid norm.
2. Apply the QR-decomposition to \((I - L\tilde{S})^T\) to have \(Y_N, Z_N, \text{ and } R_N\) as in (18).
3. For \(k = N - 1, \ldots, 1, 0\), compute \(H_k\) through (17) and apply the QR-decomposition to the product of \(H_k\) and \((Y_{k+1}^T Z_{k+1}^T)^T\) as

\[
\begin{pmatrix}
Y_k \\
Z_k
\end{pmatrix}
R_k = H_k
\begin{pmatrix}
Y_{k+1} \\
Z_{k+1}
\end{pmatrix}.
\]

4. Determine the perturbation \(\Delta x[N]\) by

\[
R_N^{-1} \cdots R_1^{-1} R_0^{-1} Y_0^{-1}(x^* - x[0]) \quad \text{and} \quad \Delta p[N] \quad \text{by} \quad 2S \Delta x[N].
\]

With \((x[N] + \Delta x[N], p[N] + \Delta p[N])\) being the new initial point, produce a trajectory using Algorithm 1. If the resulting \(x[0]\) improves the distance to \(x^*\) as \(\|x^* - x[0]\| < d\), accept the produced trajectory and proceed to the next step. If this is not the case, replace \(\Delta x[N]\) and \(\Delta p[N]\) by their halves \((1/2)\Delta x[N]\) and \((1/2)\Delta p[N]\), respectively, and produce a trajectory again.

5. If the distance \(\|x^* - x[0]\|\) is small enough, output the present trajectory and stop. Otherwise, go back to Step 1.

7. Intersample Behavior

So far we have considered optimal control of a nonlinear system (1) with respect to the objective function (4), which is evaluated only at the sampling instants. Such objective function can be problematic, however, because the controlled system may behave badly between the sampling instants. For sampled-data control of a linear system, there is a well-established methodology to take account of the intersample behavior (See [4] and the references therein). The objective of this section is to extend the method developed so far in this direction.

Let us replace the objective function (4) by

\[
\sum_{k=0}^{\infty} \left[ \int_0^h \phi(t; x[k], u[k])^T Q \phi(t; x[k], u[k]) \, dt + hu[k]^T Ru[k] \right].
\]

Since this objective function reflects the system behavior not only at the sampling instants but also between them, the resulting control law is expected to give more acceptable control performance. We will see what
needs to be modified in the method for adaptation to the new objective function. Let us write the integral in \([19]\) as \(q(x[k], u[k])\) for simplicity.

First we need to modify the Riccati equation \([8]\), which gives the optimal control law around the origin. Due to Proposition \([1]\) the function \(\phi(t; x, u)\) can be approximated by \(A_t x + B_t u\) for small \(x\) and \(u\) with \(A_t = (\partial \phi / \partial x)(t; 0, 0) = e^{A t}\) and \(B_t = (\partial \phi / \partial u)(t; 0, 0) = \int_0^t e^{A t} \, dt B\). Hence, the function \(q(x, u)\) is approximated by

\[
(x^T u^T) \int_0^h \begin{bmatrix} A_t^T \\ B_t^T \end{bmatrix} Q(A_t, B_t) \, dt \left( \begin{array}{c} x \\ u \end{array} \right).
\]

Let us combine this with the input penalty \(hu^T Ru\) and write

\[
\begin{bmatrix} \bar{Q} \\ \bar{W}^T \\ \bar{R} \end{bmatrix} = \int_0^h \begin{bmatrix} A_t^T \\ B_t^T \end{bmatrix} Q(A_t, B_t) \, dt + \begin{bmatrix} O \\ O \\ hR \end{bmatrix}.
\]

This is the weight matrix on the state and the input in the approximate version of our control problem. Its value can be computed by numerical integration, which will be discussed soon. Now, with this matrix, the Riccati equation \([8]\) is modified as

\[
A_h^T S A_h - S + \bar{Q} - (A_h^T S B_h + \bar{W})(B_h^T S B_h + \bar{R})^{-1}(B_h^T S A_h + \bar{W}) = O.
\]  

(20) We assume the existence of its stabilizing solution \(\bar{S}\) and use it in place of \(S\).

With the new objective function, the Hamiltonian changes to

\[
\bar{H}(x, u, p^+) = \phi_h(x, u)^T p^+ + q(x, u) + hu^T Ru.
\]

The Hamiltonian system accordingly changes to

\[
x^+ = \frac{\partial \bar{H}}{\partial p^+}(x, u, p^+)^T = \phi_h(x, u),
\]

(21)

\[
0 = \frac{\partial \bar{H}}{\partial u}(x, u, p^+)^T = \frac{\partial \phi_h}{\partial u}(x, u)^T p^+ + \frac{\partial q}{\partial u}(x, u)^T + 2hRu,
\]

(22)

\[
p = \frac{\partial \bar{H}}{\partial x}(x, u, p^+)^T = \frac{\partial \phi_h}{\partial x}(x, u)^T p^+ + \frac{\partial q}{\partial x}(x, u)^T.
\]

(23)

These equations determine a unique dynamics in the space of \((x, p)\) around the origin for a small enough sampling period \(h\). Indeed, the right-hand side of \((22)\) and \((23)\) differentiated by \((u, p^+)\) is

\[
\begin{bmatrix} 2\bar{R} & B_h^T \\ 2\bar{W} & A_h^T \end{bmatrix} \]

for \(x, p, u, x^+, p^+\) all equal to zero. This matrix is invertible if and only if so are \(2\bar{R} \) and \(A_h^T - 2\bar{W} (2\bar{R})^{-1} B_h^T\). Because the latter is true for a small enough \(h\), the same reasoning as in the proof of Proposition \([4]\) implies that the mapping from \((x, p)\) to \((u, p^+)\) is uniquely determined and thus so is the mapping to \(x^+\). The situation is similar on the mapping from \((x^+, p^+)\) to \((x, u, p)\).

Propositions \([5,7]\) hold as well after appropriate changes. Hence it should be possible to obtain the optimal control law. In order to have a trajectory on the stable manifold, the initial point \((x[N], p[N])\) should be chosen
by \( p[N] = 2 \bar{S} x[N] \) with the stabilizing solution \( \bar{S} \) of the modified Riccati equation (20). Algorithm 1 needs to be modified so as to be consistent with the new Hamiltonian system (21)–(23). In particular, the linear optimal control used in Step 2 (1) is replaced by \( u^{(0)} = -(B_h^T \bar{S} B_h + \bar{R})^{-1}(B_h^T \bar{S} A_h + \bar{W})x^+ \). Newton’s method in Step 2 (2) should be replaced by

\[
\begin{pmatrix}
(x^{(i+1)})_1 \\
(x^{(i+1)})_2 \\
(x^{(i+1)})_3 \\
(x^{(i+1)})_4
\end{pmatrix} = 
\begin{pmatrix}
(x^{(i)})_1 \\
(x^{(i)})_2 \\
(x^{(i)})_3 \\
(x^{(i)})_4
\end{pmatrix} 
- h \begin{pmatrix}
\frac{\partial h}{\partial x} \phi h^T p^+ + \frac{\partial \phi}{\partial u} \phi h^T p^+ + 2 \phi h R \\
\frac{\partial \phi}{\partial x} p^+ + \phi^T \phi h^T p^+ + 2 h R p^+
\end{pmatrix}
\]

where the derivatives are all evaluated at \((x^{(i)}, u^{(i)})\). Here we need evaluation of the derivatives of \( q(x, u) = \int_0^h \phi(t; x, u)^T Q \phi(t; x, u) \, dt \), which is possible with numerical integration. For example, evaluation of the derivative \( (\partial q/\partial u)(x, u) \) can be made by the composite Simpson’s rule [19, Section 7.5]:

\[
\frac{\partial q}{\partial u}(x, u) = \int_0^h 2 \phi(0; x, u)^T Q \frac{\partial \phi}{\partial u}(t; x, u) \, dt 
= \frac{h}{5M} \left[ 2 \phi(0; x, u)^T Q \frac{\partial \phi}{\partial u}(0; x, u) + 4 \sum_{j=1}^{M/2-1} \phi(2jh; x, u)^T Q \frac{\partial \phi}{\partial u}(2jh; x, u) 
+ 8 \sum_{j=1}^{M/2} \phi((2j-1)h; x, u)^T Q \frac{\partial \phi}{\partial u}((2j-1)h; x, u) + 2 \phi(h; x, u)^T Q \frac{\partial \phi}{\partial u}(h; x, u) \right]
\]

for an even positive number \( M \). It requires the values of the derivative \( (\partial \phi/\partial u)(t; x, u) \) at \( t = i h/M \), \( i = 1, 2, \ldots, M \), which can be computed by the Runge–Kutta method as in Section 3.

**Remark 2.** The composite Simpson’s rule has the numerical error in the order of \( O(M^{-4}) \) and is suitable to use with the classical fourth-order Runge–Kutta method, whose numerical error is also in the order of \( O(M^{-4}) \). The composite trapezoidal rule, which is more common for numerical integration, produces a larger numerical error \( O(M^{-2}) \) and is not suitable.

In Step 3 of Algorithm 1, the computed \((x, u)\) should be substituted into (23) in place of (13) to give the value of \( p \). Here, we need \((\partial q/\partial x)(x, u)\), which again is computable with numerical integration.

The shooting method can be used, too, with the new objective function. To this end, evaluation of \((\partial^2 q/\partial x^2)(x, u)\) is necessary, which is possible similarly to that of other derivatives.

### 8. Example

For illustration of the proposed method, tracking control of a wheeled mobile robot is considered in the framework of [6]. The position of the controlled robot is \((x_c, y_c)\) in the \(xy\)-coordinates and its direction is \(\theta_c\) measured counterclockwise from the positive direction of the \(x\)-axis (Figure 2). The linear velocity \(v_c\) and the angular velocity \(\omega_c\) can be specified independently. The objective of control is to make the position \((x_c, y_c)\) and the direction \(\theta_c\) of the controlled robot equal to those of the reference robot \((x_r, y_r, \theta_r)\), which is driven by the known linear and angular velocities \(v_r\) and \(\omega_r\). The state of the dynamics is chosen as

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta}
\end{bmatrix} = 
\begin{bmatrix}
\cos \theta_c & \sin \theta_c & 0 \\
-\sin \theta_c & \cos \theta_c & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_r - x_c \\
y_r - y_c \\
\theta_r - \theta_c
\end{bmatrix},
\]

12
where \((a, b)\) stands for the relative position of the reference robot in the coordinate system fixed to the controlled robot and \(\theta\) is the direction of the reference robot relative to that of the controlled robot. With the relative velocities \(v = v_r - v_c\) and \(\omega = \omega_r - \omega_c\) for inputs, one can write the dynamics as

\[
\frac{d}{dt} \begin{bmatrix} a(t) \\ b(t) \\ \theta(t) \end{bmatrix} = \begin{bmatrix} v_r(t)(\cos \theta(t) - 1) + b(t)\omega_r(t) + v(t) - b(t)\omega(t) \\ v_r(t)\sin \theta(t) - a(t)\omega_r(t) + a(t)\omega(t) \\ \omega(t) \end{bmatrix}.
\]

Suppose that the reference robot has \((x_r(t), y_r(t), \theta_r(t)) = (0, 0, 0)\) at the initial time \(t = 0\) and moves in the positive direction of the \(x\)-axis with the constant speed \(v_r(t) \equiv 1\) [m/s] and \(\omega_r(t) \equiv 0\) [rad/s]. On the other hand, the controlled robot has \((x_c(t), y_c(t), \theta_c(t)) = (0, 0, -\pi)\) at \(t = 0\), that is, it is located at the same position as the reference robot but in the opposite direction. Hence the initial state is \((a(0), b(0), \theta(0))^T = (0, 0, \pi)^T\). We want to make it converge to the origin by appropriate choice of the inputs \(v(t)\) and \(\omega(t)\). A sampled-data control law is designed for the purpose with the sampling period \(h = 1\) [s] and the weight matrices \(Q\) and \(R\) being identities.

Note that, if the inputs \(v(t)\) and \(\omega(t)\) are generated by the sampled-data control and thus are piecewise constant, the inputs \(v_c(t)\) and \(\omega_c(t)\) actually given to the controlled robot also become piecewise constant.

As described in Section 5, by choosing some \(x[N]\) close to the origin and applying Algorithm 1, we can have a trajectory on the stable manifold. Because our initial state is \((a(0), b(0), \theta(0))^T = (0, 0, \pi)^T\), we want a trajectory whose \(x[0]\) coincides with this vector. For the purpose, we invoke the shooting method of Algorithm 2 with the desired state \(x^* = (0, 0, \pi)^T\). The result is shown in Figure 3. Here, the intersample behavior of the state is presented for reference. After three updates, the desirable trajectory that reaches \((0, 0, \pi)^T\) was obtained. In the figure, the last update is too small to recognize. Figure 4 shows the obtained trajectory in the original coordinate \((x_c, y_c)\), where the position at the sampling instants is expressed by the circles and the direction by the arrows. Note that the scale of the \(y\)-coordinate is presented 10 times larger. We can see that the robot first goes backward with changing the direction counterclockwise and then goes forward in the negative direction.
Figure 3. The trajectory updates made by the shooting method. The original trajectory is updated three times and gives a trajectory reaching the desired point $(0 \ 0 \ \pi)^T$. Here, the third and the fourth trajectories are too close to distinguish in this scale. The intersample behavior is presented for reference though not required by the method.

Figure 4. The computed trajectory of the controlled robot in the original coordinate $(x_c, y_c)$. The position and the direction at the sampling instants are shown by the circles and the arrows, respectively. The robot successfully catches up the reference robot with a long sampling period $h = 1 \, [s]$. 
Figure 5. Tracking control under the measurement noise. Three realizations are presented by the lines of different color and style. Tracking is successfully made though some fluctuation is found.

Figure 6. The trajectory of the controlled robot with the intersample behavior considered. Deviation from the $x$-axis is smaller than that of the trajectory in Figure 4.
of the $y$-axis. It gradually changes the direction and finally catches up the reference robot. Although the inputs are kept constant between the sampling instants, still the control objective is achieved successfully.

In order to have a control law in a feedback form, we applied the shooting method with setting $x^*$ to various values and obtained the corresponding input $u$. In particular, we added Gaussian noise of standard deviation $0.2$ to each component of $x[0], x[1], x[2], x[3]$ of the finally obtained trajectory in Figure 3 and also to the origin and produced 100 values of $x^*$. With $x^*$ set to each value of them, the corresponding $u$ was obtained. The obtained values of $u$ were approximated by a third-order polynomial of $x$, which is our control law. The obtained control law was applied to tracking control. In order to see its sensitivity to the noise, we added Gaussian noise of standard deviation $0.02$ to each component of the measured state and then passed it to the control law. The result is presented in Figure 5. Even with the measurement noise, the tracking was successfully made though some fluctuation is found.

Figure 6 shows the optimal trajectory of the robot when the intersample behavior is taken into account. Compared with the trajectory of Figure 4 the deviation from the $x$-axis is smaller both in the positive and negative directions of the $y$ axis. Indeed, in the objective function (19), the sum of the first term, i.e., the state penalty, is $5.52$ in this trajectory, which is smaller than the value $5.86$ in the trajectory of Figure 4. On the other hand, the sum of the second term of the objective function, i.e., the input penalty, is larger in the present trajectory and thus the overall value of the objective function is not very different. In particular, it is $13.90$ in the present trajectory while $13.93$ in the trajectory of Figure 4.

9. Conclusion

In this paper, optimal sampled-data control is considered for a nonlinear system with the stable-manifold approach. The approach can be adapted for sampled-data control thanks to the works of Navasca and can be carried out with extensive use of numerical techniques. A shooting method is proposed for systematic choice of an initial point of a trajectory. The extension is considered for the intersample behavior of the system to be taken into account. An example shows the efficacy of the proposed method.

Appendix

Outline of the proof is presented for Propositions 5-7. See [9, 10] for details.

(Outline of the Proof of Proposition 5) The existence of the stable manifold is shown by the contraction mapping theorem. Expansion in $x$ gives $s(0) = 0$ and $(\partial s/\partial x)(0) = 2S$.

(Outline of the Proof of Proposition 6) Due to the form of the Hamiltonian system (11)–(13), the symplectic form satisfies $\sum_{i=1}^n dp_i \wedge dx_i = \sum_{i=1}^n dp_i^+ \wedge dx_i^+$, which means $\sum_{i=1}^n dp_i \wedge dx_i = 0$ on the stable manifold because $(x[k], p[k]) \to 0$ as $k \to \infty$ there. Noting that $p$ is a function of $x$ on the stable manifold, substitute $dp_i = \sum_{j=1}^n (\partial p_i/\partial x_j)dx_j$ to have $(\partial p_i/\partial x_j) - (\partial p_j/\partial x_i) = 0$ for any $i$ and $j$. Poincaré’s lemma then implies the existence of the desired $V(x)$. 

16
Substitute $p = s(x) = (\partial V/\partial x)(x)^T$ and $u = u^*(x)$ into (12) to have

$$0 = \frac{\partial \phi_h}{\partial u}(x, u^*(x))^T \frac{\partial V}{\partial x}(\phi_h(x, u^*(x)))^T + 2hRu^*(x)$$

$$= \frac{\partial}{\partial u} \left[ V(\phi_h(x, u)) + hx^TQx + hu^TRu \right]_{u = u^*(x)}.$$

This shows that $u = u^*(x)$ minimizes the quantity in the bracket.

Similarly, substitution of $p = s(x) = (\partial V/\partial x)(x)^T$ and $u = u^*(x)$ into (13) gives

$$\frac{\partial V}{\partial x}(x)^T = \frac{\partial \phi_h}{\partial x}(x, u^*(x))^T \frac{\partial V}{\partial x}(\phi_h(x, u^*(x)))^T + 2hQx$$

$$= \frac{\partial}{\partial x} \left[ V(\phi_h(x, u)) + hx^TQx + hu^TRu \right]_{u = u^*(x)}$$

$$= \frac{\partial}{\partial x} \left[ V(\phi_h(x, u^*(x))) + hx^TQx + hu^*(x)^TRu^*(x) \right]^T.$$

See [1] Lemma 3.3.1 for the derivation of the last expression. This shows that $V(x) - V(\phi_h(x, u^*(x))) - hx^TQx - hu^*(x)^TRu^*(x)$ is a constant independent of $x$. This constant is actually zero because $u^*(x) = 0$ for $x = 0$. □

References

[1] D. P. Bertsekas, Dynamic Programming and Optimal Control, vol. 1, second ed. Athena Scientific, Belmont, USA, 2000.

[2] L. Bourdin and E. Trélat, “Optimal sampled-data control, and generalization on time scales,” [arXiv:1501.07361v2 [math.OC]], 2015.

[3] L. Bourdin and E. Trélat, “Linear-quadratic optimal sampled-data control problems: Convergence result and Riccati theory,” Automatica, vol. 79, pp. 273–281, 2017.

[4] T. Chen and B. Francis, Optimal Sampled-Data Control Systems. Springer, London, UK, 1995.

[5] G. H. Golub and C. F. Van Loan, Matrix Computations, third ed. Johns Hopkins University Press, Baltimore, USA, 1996.

[6] Y. Kanayama, Y. Kimura, F. Miyazaki, and T. Noguchi, “A stable tracking control method for an autonomous mobile robot,” in Proceedings of the 1990 IEEE International Conference on Robotics and Automation, Cincinnati, USA, May 1990, pp. 384–389.

[7] H. Katayama and H. Aoki, “Straight-line trajectory tracking control for sampled-data underactuated ships,” IEEE Transactions on Control Systems Technology, vol. 22, no. 4, pp. 1638–1645, 2014.

[8] D. S. Laila, D. Nešić, and A. Astolfi, “Sampled-data control of nonlinear systems,” in Advanced Topics in Control Systems Theory: Lecture Notes from FAP 2005, A. Loría, F. Lamnabhi-Lagarrigue, and E. Panteley, Eds. Springer, London, UK, 2006, pp. 91–137.

[9] C. L. Navasca, “Local solutions of the dynamic programming equations and the Hamilton Jacobi Bellman PDE,” PhD Dissertation, University of California at Davis, Davis, USA, 2002.
[10] C. Navasca, “Local stable manifold for the bidirectional discrete-time dynamics,” arXiv:math/0309026 [math.OC], 2003.

[11] D. Nešić and L. Grüne, “Lyapunov-based continuous-time nonlinear controller redesign for sampled-data implementation,” Automatica, vol. 41, no. 7, pp. 1143–1156, 2005.

[12] D. Nešić and L. Grüne, “A receding horizon control approach to sampled-data implementation of continuous-time controllers,” Systems & Control Letters, vol. 55, no. 8, pp. 660–672, 2006.

[13] D. Nešić and A. R. Teel, “Stabilization of sampled-data nonlinear systems via backstepping on their Euler approximate model,” Automatica, vol. 42, no. 10, pp. 1801–1808, 2006.

[14] D. Nešić, A. R. Teel, and P. V. Kokotović, “Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations,” Systems & Control Letters, vol. 38, nos. 4–5, pp. 259–270, 1999.

[15] Y. Oishi and N. Sakamoto, “Numerical computational improvement of the stable-manifold method for nonlinear optimal control,” IFAC PapersOnLine, vol. 50, no. 1, pp. 5103–5108, 2017.

[16] N. Sakamoto, “Case studies on the application of the stable manifold approach for nonlinear optimal control design,” Automatica, vol. 49, no. 2, pp. 568–576, 2013.

[17] N. Sakamoto and A. J. van der Schaft, “Analytical approximation methods for the stabilizing solution of the Hamilton–Jacobi equation,” IEEE Transactions on Automatic Control, vol. 53, no. 10, pp. 2335–2350, 2008.

[18] T. C. Sideris, Ordinary Differential Equations and Dynamical Systems. Atlantis Press, Paris, France, 2013.

[19] E. Süli and D. Mayers, An Introduction to Numerical Analysis. Cambridge University Press, Cambridge, UK, 2003.

[20] H. Wendland, Scattered Data Approximation. Cambridge University Press, Cambridge, UK, 2005.

[21] Y. Yamashita and M. Shima, “A new method of solving the Hamilton–Jacobi partial differential equation” (in Japanese), Transactions of the Society of Instrument and Control Engineers, vol. 34, no. 6, pp. 571–576, 1998.