TRACE INEQUALITIES ON A GENERALIZED WIGNER-YANASE SKew INFORMATION

S. FURUICHI, K.YANAGI, AND K.KURIYAMA

Abstract. We introduce a generalized Wigner-Yanase skew information and then derive the trace inequality related to the uncertainty relation. This inequality is a non-trivial generalization of the uncertainty relation derived by S.Luo for the quantum uncertainty quantity excluding the classical mixture. In addition, several trace inequalities on our generalized Wigner-Yanase skew information are argued.

1. Introduction

Wigner-Yanase skew information

\[ I_{\rho}(H) = \frac{1}{2} Tr \left( \left( i \left[ \rho^{1/2}, H \right] \right)^2 \right) \]

was defined in [8]. This quantity can be considered as a kind of the degree for non-commutativity between a quantum state \( \rho \) and an observable \( H \). Here we denote the commutator by \( [X, Y] \equiv XY - YX \). This quantity was generalized by Dyson

\[ I_{\rho,\alpha}(H) = \frac{1}{2} Tr \left( \left( i \left[ \rho^\alpha, H \right] \right) \left( i \left[ \rho^{1-\alpha}, H \right] \right) \right) \]

which is known as the Wigner-Yanase-Dyson skew information. It is famous that the convexity of \( I_{\rho,\alpha}(H) \) with respect to \( \rho \) was successfully proven by E.H.Lieb in [5]. From the physical point of view, an observable \( H \) is generally considered to be an unbounded operator, however in the present paper, unless otherwise stated, we consider \( H \in B(\mathcal{H}) \), where \( B(\mathcal{H}) \) represents the set of all bounded linear operators on the Hilbert space \( \mathcal{H} \), as a mathematical interest. We also denote the set of all self-adjoint operators (observables) by \( L_h(\mathcal{H}) \) and the set of all density operators (quantum states) by \( \mathcal{S}(\mathcal{H}) \) on the Hilbert space \( \mathcal{H} \). The relation between the Wigner-Yanase skew information and the uncertainty relation was studied in [7]. Moreover the relation between the Wigner-Yanase-Dyson skew information and the uncertainty relation was studied in [4, 9]. In our previous paper [9], we defined a
generalized skew information and then derived a kind of an uncertainty relation. In the section 2, we introduce a new generalized Wigner-Yanase skew information. On a generalization of the original Wigner-Yanase skew information, our generalization is different from the Wigner-Yanase-Dyson skew information and a generalized skew information defined in our previous paper [9]. Moreover we define a new quantity by our generalized Wigner-Yanase skew information and then we derive the trace inequality expressing a kind of the uncertainty relation.

2. Trace inequalities on a generalized Wigner-Yanase skew information

Firstly we review the relation between the Wigner-Yanase skew information and the uncertainty relation. In quantum mechanical system, the expectation value of an observable $H$ in a quantum state $\rho$ is expressed by $\text{Tr} [\rho H]$. It is natural that the variance for a quantum state $\rho$ and an observable $H$ is defined by $V_\rho(H) \equiv \text{Tr}[\rho (H - \text{Tr}[\rho H])^2] = \text{Tr}[\rho H^2] - \text{Tr}[\rho H]^2$. It is famous that we have the Heisenberg’s uncertainty relation:

$$V_\rho(A)V_\rho(B) \geq \frac{1}{4} |\text{Tr}[\rho [A, B]]|^2,$$

for a quantum state $\rho$ and two observables $A$ and $B$. The further strong result was given by Schrödinger

$$V_\rho(A)V_\rho(B) - |\text{Cov}_\rho(A, B)|^2 \geq \frac{1}{4} |\text{Tr}[\rho [A, B]]|^2,$$

where the covariance is defined by $\text{Cov}_\rho(A, B) \equiv \text{Tr}[\rho (A - \text{Tr}[\rho A] I) (B - \text{Tr}[\rho B] I)]$. However, the uncertainty relation for the Wigner-Yanase skew information failed. (See [7, 4, 9].)

Recently, S. Luo introduced the quantity $U_\rho(H)$ representing a quantum uncertainty excluding the classical mixture:

$$U_\rho(H) \equiv \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_\rho(H))^2},$$

then he derived the uncertainty relation on $U_\rho(H)$ in [6]:

$$U_\rho(A)U_\rho(B) \geq \frac{1}{4} |\text{Tr}[\rho [A, B]]|^2.$$

Note that we have the following relation

$$0 \leq I_\rho(H) \leq U_\rho(H) \leq V_\rho(H).$$

The inequality (2.3) is a refinement of the inequality (2.1) in the sense of (2.4).

In this section, we study one-parameter extended inequality for the inequality (2.3).

**Definition 2.1.** For $0 \leq \alpha \leq 1$, a quantum state $\rho$ and an observable $H$, we define the Wigner-Yanase-Dyson skew information

$$I_{\rho, \alpha}(H) \equiv \frac{1}{2} \text{Tr} [(i [\rho^\alpha, H_0]) (i [\rho^{1-\alpha}, H_0])]$$

and we also define

$$J_{\rho, \alpha}(H) \equiv \frac{1}{2} \text{Tr} [(\rho^\alpha, H_0) (\rho^{1-\alpha}, H_0)],$$

for $0 \leq \alpha \leq 1$. The uncertainty relation for the Wigner-Yanase skew information is expressed by

$$I_{\rho, \alpha}(A)V_{\rho, \alpha}(B) \geq \frac{1}{4} |\text{Tr}[\rho [A, B]]|^2,$$
where \( H_0 \equiv H - Tr[\rho H]I \) and we denote the anti-commutator by \( \{X, Y\} = XY + YX \).

Note that we have
\[
\frac{1}{2} Tr \left[ (i [\rho^\alpha, H_0]) (i [\rho^{1-\alpha}, H_0]) \right] = \frac{1}{2} Tr \left[ (i [\rho^\alpha, H]) (i [\rho^{1-\alpha}, H]) \right]
\]
but we have
\[
\frac{1}{2} Tr \left[ \left\{ \rho^\alpha, H_0 \right\} \left\{ \rho^{1-\alpha}, H_0 \right\} \right] \neq \frac{1}{2} Tr \left[ \left\{ \rho^\alpha, H \right\} \left\{ \rho^{1-\alpha}, H \right\} \right].
\]
Then we have the following inequalities:
\[
(I_{\rho,\alpha}(H) \leq J_{\rho}(H) \leq J_{\rho,\alpha}(H),
\]
since we have \( Tr[\rho^{1/2}H\rho^{1/2}] \leq Tr[\rho^\alpha H\rho^{1-\alpha}H] \). (See [1, 2] for example.) If we define
\[
U_{\rho,\alpha}(H) \equiv \sqrt{V_{\rho}(H)^2 - (V_{\rho}(H) - I_{\rho,\alpha}(H))^2},
\]
as a direct generalization of Eq. (2.2), then we have
\[
0 \leq I_{\rho,\alpha}(H) \leq U_{\rho,\alpha}(H) \leq U_{\rho}(H)
\]
due to the first inequality of (2.6). We also have
\[
U_{\rho,\alpha}(H) = \sqrt{I_{\rho,\alpha}(H)J_{\rho,\alpha}(H)}.
\]

Remark 2.2. From the inequalities (2.4), (2.6) and (2.8), our situation is that we have
\[
0 \leq I_{\rho,\alpha}(H) \leq U_{\rho,\alpha}(H) \leq U_{\rho}(H)
\]
and
\[
0 \leq I_{\rho,\alpha}(H) \leq U_{\rho,\alpha}(H) \leq U_{\rho}(H).
\]
Therefore our first concern is the ordering between \( I_{\rho}(H) \) and \( U_{\rho,\alpha}(H) \). However we have no ordering between them. Because we have the following examples. We set the density matrix \( \rho \) and the observable \( H \) such as
\[
\rho = \begin{pmatrix} 0.6 & 0.48 \\ 0.48 & 0.4 \end{pmatrix}, \quad H = \begin{pmatrix} 1.0 & 0.5 \\ 0.5 & 5.0 \end{pmatrix}.
\]
If \( \alpha = 0.1 \), then \( U_{\rho,\alpha}(H) - I_{\rho}(H) \) approximately takes \(-0.14736\). If \( \alpha = 0.2 \), then \( U_{\rho,\alpha}(H) - I_{\rho}(H) \) approximately takes \(0.4451\).

Conjecture 2.3. Our second concern is to show an uncertainty relation with respect to \( U_{\rho,\alpha}(H) \) as a direct generalization of the inequality (2.3) such that
\[
(U_{\rho,\alpha}(X))U_{\rho,\alpha}(Y) \geq \frac{1}{4} Tr[\rho[X, Y]]^2
\]
However we have not found the proof of the above inequality (2.10). In addition, we have not found any counter-examples of the inequality (2.10) yet.

In the present paper, we introduce a generalized Wigner-Yanase skew information which is a generalization of the Wigner-Yanase skew information defined in Eq. (1.1), but different from the Wigner-Yanase-Dyson skew information defined in Eq. (2.5).
Definition 2.4. For $0 \leq \alpha \leq 1$, a quantum state $\rho$ and an observable $H$, we define a generalized Wigner-Yanase skew information by

$$K_{\rho,\alpha}(H) \equiv \frac{1}{2} Tr \left( \left( \frac{\rho^\alpha + \rho^{1-\alpha}}{2}, H_0 \right) \right)^2$$

and we also define

$$L_{\rho,\alpha}(H) \equiv \frac{1}{2} Tr \left( \left( \frac{\rho^\alpha + \rho^{1-\alpha}}{2}, H_0 \right) \right)^2.$$ 

Remark 2.5. For two generalized Wigner-Yanase skew informations $I_{\rho,\alpha}(H)$ and $K_{\rho,\alpha}(H)$, we have the relation:

$$I_{\rho,\alpha}(H) \leq K_{\rho,\alpha}(H).$$

Indeed, for a spectral decomposition of $\rho$ such as $\rho = \sum_k \lambda_k |\phi_k\rangle\langle\phi_k|$, we have the following expressions:

$$I_{\rho,\alpha}(H) = \frac{1}{2} \sum_{m,n} \left( \lambda_m^\alpha - \lambda_n^\alpha \right) \left( \lambda_m^{1-\alpha} - \lambda_n^{1-\alpha} \right) |\langle \phi_m | H | \phi_n \rangle|^2$$

and

$$K_{\rho,\alpha}(H) = \frac{1}{2} \sum_{m,n} \left( \frac{\lambda_m^\alpha - \lambda_n^\alpha + \lambda_m^{1-\alpha} - \lambda_n^{1-\alpha}}{2} \right)^2 |\langle \phi_m | H | \phi_n \rangle|^2.$$ 

By simple calculations, we see

$$\left( \frac{\lambda_m^\alpha - \lambda_n^\alpha + \lambda_m^{1-\alpha} - \lambda_n^{1-\alpha}}{2} \right)^2 - \left( \lambda_m^\alpha - \lambda_n^\alpha \right) \left( \lambda_m^{1-\alpha} - \lambda_n^{1-\alpha} \right) \geq 0.$$ 

Throughout this section, we put $X_0 \equiv X - Tr[\rho X]I$ and $Y_0 \equiv Y - Tr[\rho Y]I$. Then we show the following trace inequality.

Theorem 2.6. For a quantum state $\rho$ and observables $X, Y$ and $\alpha \in [0, 1]$, we have

$$W_{\rho,\alpha}(X) W_{\rho,\alpha}(Y) \geq \frac{1}{4} \left| Tr \left( \left( \frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right)^2 [X,Y] \right) \right|^2$$

where

$$W_{\rho,\alpha}(X) \equiv \sqrt{K_{\rho,\alpha}(X)L_{\rho,\alpha}(X)}.$$ 

Proof: Putting

$$M \equiv i \left( \frac{\rho^\alpha + \rho^{1-\alpha}}{2}, X_0 \right) x + \left( \frac{\rho^\alpha + \rho^{1-\alpha}}{2}, Y_0 \right)$$

for any $x \in \mathbb{R}$, then we have

$$0 \leq Tr[M^* M]$$

$$= \left( \frac{1}{4} Tr \left( (i[\rho^\alpha, X_0])^2 + (i[\rho^{1-\alpha}, X_0])^2 + I_{\rho,\alpha}(X) \right) \right) x^2$$

$$+ \frac{1}{2} Tr \left( i[\rho^\alpha, X_0] + i[\rho^{1-\alpha}, X_0] \right) \left( \{\rho^\alpha, Y_0\} + \{\rho^{1-\alpha}, Y_0\} \right) x$$

$$+ \left( \frac{1}{4} Tr \left( \{\rho^\alpha, Y_0\}^2 + \{\rho^{1-\alpha}, Y_0\}^2 \right) + J_{\rho,\alpha}(Y) \right).$$
Therefore we have
\[
\frac{1}{4} \left| \text{Tr} \left[ (\rho^\alpha + \rho^{1-\alpha})^2 (i [X, Y]) \right] \right|^2 \\
\leq 4 \left( \frac{1}{4} \text{Tr} \left[ (i [\rho^\alpha, X_0])^2 + (i [\rho^{1-\alpha}, X_0])^2 + I_{\rho,\alpha}(X) \right] \right) \\
\times \left( \frac{1}{4} \text{Tr} \left[ (\rho^\alpha, Y_0)^2 + (\rho^{1-\alpha}, Y_0)^2 \right] + J_{\rho,\alpha}(Y) \right),
\]

since we have
\[
\text{Tr} \left[ (i [\rho^\alpha, X_0])^2 + (i [\rho^{1-\alpha}, X_0])^2 \right] = \text{Tr} \left[ (\rho^\alpha + \rho^{1-\alpha})^2 (i [X, Y]) \right].
\]

As similar as we have
\[
\frac{1}{4} \left| \text{Tr} \left[ (\rho^\alpha + \rho^{1-\alpha})^2 (i [X, Y]) \right] \right|^2 \\
\leq 4 \left( \frac{1}{4} \text{Tr} \left[ (i [\rho^\alpha, Y_0])^2 + (i [\rho^{1-\alpha}, Y_0])^2 + I_{\rho,\alpha}(Y) \right] \right) \\
\times \left( \frac{1}{4} \text{Tr} \left[ (\rho^\alpha, X_0)^2 + (\rho^{1-\alpha}, X_0)^2 \right] + J_{\rho,\alpha}(X) \right).
\]

By the above two inequalities, we have
\[
W_{\rho,\alpha}(X) W_{\rho,\alpha}(Y) \geq \frac{1}{4} \left| \text{Tr} \left[ \frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right]^2 [X, Y] \right|^2.
\]

\[\square\]

**Corollary 2.7.** For a quantum state \( \rho \) and observables (possibly unbounded operators) \( X, Y \) and \( \alpha \in [0, 1] \), if we have the relation \([X, Y] = \frac{1}{2\pi i} I \) on \( \text{dom}(XY) \cap \text{dom}(YX) \) and \( \rho \) is expressed by \( \rho = \sum_k \lambda_k |\phi_k\rangle \langle \phi_k| \), \( |\phi_k\rangle \in \text{dom}(XY) \cap \text{dom}(YX) \), then
\[
W_{\rho,\alpha}(X) W_{\rho,\alpha}(Y) \geq \frac{1}{4} \left| \text{Tr} \left[ \rho [X, Y] \right] \right|^2.
\]

**Proof:** It follows from Theorem 2.6 and the following inequality:
\[
\frac{1}{4} \left| \text{Tr} \left[ \frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right]^2 [X, Y] \right|^2 \geq \frac{1}{4} |\text{Tr} \left[ \rho [X, Y] \right]|^2,
\]
whenever we have the canonical commutation relation such as \([X, Y] = \frac{1}{2\pi i} I \).

\[\square\]

**Remark 2.8.** Theorem 2.6 is not trivial one in the sense of the following (i) and (ii).

(i) Since the arithmetic mean is greater than the geometric mean, \( \text{Tr} \left[ (i [\rho^\alpha, X_0])^2 \right] \geq 0 \) and \( \text{Tr} \left[ (i [\rho^{1-\alpha}, X_0])^2 \right] \geq 0 \) imply \( K_{\rho,\alpha}(X) \geq I_{\rho,\alpha}(X) \), by the use of

Schwarz’s inequality. Similarly, \( \text{Tr} \left[ (\rho^\alpha, Y_0)^2 \right] \geq 0 \) and \( \text{Tr} \left[ (\rho^{1-\alpha}, Y_0)^2 \right] \geq 0 \) imply \( L_{\rho,\alpha}(Y) \geq J_{\rho,\alpha}(Y) \). We then have \( W_{\rho,\alpha}(X) \geq U_{\rho,\alpha}(X) \).

From the inequality 2.3 and the above, our situation is that we have
\[
U_{\rho,\alpha}(H) \leq U_{\rho}(H)
\]
and

\[ U_{\rho,\alpha}(H) \leq W_{\rho,\alpha}(H). \]

Our third concern is the ordering between \( U_{\rho}(H) \) and \( W_{\rho,\alpha}(H) \). However, we have no ordering between them. Because we have the following examples. We set

\[
\rho = \begin{pmatrix} 0.8 & 0.0 \\ 0.0 & 0.2 \end{pmatrix}, \quad H = \begin{pmatrix} 2.0 & 3.0 \\ 3.0 & 1.0 \end{pmatrix}.
\]

If we take \( \alpha = 0.8 \), then \( U_{\rho}(H) - W_{\rho,\alpha}(H) \) approximately takes \(-0.241367\). If we take \( \alpha = 0.9 \), then \( U_{\rho}(H) - W_{\rho,\alpha}(H) \) approximately takes \(0.404141\).

This example actually shows that there exists a triplet of \( \alpha, \rho \) and \( H \) such that \( W_{\rho,\alpha}(H) < V_{\rho}(H) \), since we have \( U_{\rho}(H) \leq V_{\rho}(H) \) in general.

(ii) We have no ordering between \( TR \left[ \left( \frac{\rho^{\alpha} + \rho^{1-\alpha}}{2} \right)^2 [X,Y] \right] \) and \( TR[\rho[X,Y]] \), by the following examples. If we take

\[
\rho = \frac{1}{2} \begin{pmatrix} 2 & 2i & 1 \\ -2i & 3 & -2i \\ 1 & 2i & 2 \end{pmatrix}, \quad X = \begin{pmatrix} 3 & 3 & -i \\ 3 & 1 & 0 \\ i & 0 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & -i & 1 & -i \\ i & 1 & i \\ 1+i & -i & 3 \end{pmatrix},
\]

then we have

\[
TR \left[ \left( \frac{\rho^{\alpha} + \rho^{1-\alpha}}{2} \right)^2 [X,Y] \right] \approx 0.348097, \quad TR[\rho[X,Y]] \approx 0.326531.
\]

If we take

\[
\rho = \frac{1}{2} \begin{pmatrix} 2 & 2i & 1 \\ -2i & 3 & -2i \\ 1 & 2i & 2 \end{pmatrix}, \quad X = \begin{pmatrix} 3 & 3 & -i \\ 1 & 3 & 0 \\ i & 0 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & -i & 0 \\ i & 1 & i \\ 0 & -i & 3 \end{pmatrix},
\]

then we have

\[
TR \left[ \left( \frac{\rho^{\alpha} + \rho^{1-\alpha}}{2} \right)^2 [X,Y] \right] \approx 0.304377, \quad TR[\rho[X,Y]] \approx 0.326531.
\]

**Remark 2.9.**

(i) If we take \( M = \rho^{1/2}X_0x + \rho^{1/2}Y_0 \) for any \( x \in \mathbb{R} \) presented in Eq. (2.12), we recover the Heisenberg uncertainty relation Eq. (2.1) shown in [3].

(ii) If we take \( \alpha = \frac{1}{2} \), then we recover the inequality (2.9) presented in [6].

(iii) We have another inequalities which are different from the inequality (2.11), by taking different self-adjoint operators \( M \) appeared in the proof of Theorem 2.6.

**Conjecture 2.10.** Our fourth concern is whether the following inequality:

\[
(2.13) \quad U_{\rho,\alpha}(X)U_{\rho,\alpha}(Y) \geq \frac{1}{4} \left[ TR \left[ \left( \frac{\rho^{\alpha} + \rho^{1-\alpha}}{2} \right)^2 [X,Y] \right] \right]^2
\]

holds or not. However we have not found its proof and any counter-examples yet.

\( K_{\rho,\alpha}(H) \) and \( L_{\rho,\alpha}(H) \) are respectively rewritten by

\[
K_{\rho,\alpha}(H) = TR \left[ \left( \frac{\rho^{\alpha} + \rho^{1-\alpha}}{2} \right)^2 H_0^2 - \left( \frac{\rho^{\alpha} + \rho^{1-\alpha}}{2} \right) H_0 \left( \frac{\rho^{\alpha} + \rho^{1-\alpha}}{2} \right) H_0 \right]
\]
and

\[ L_{\rho,\alpha}(H) = \text{Tr} \left[ \left( \frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right)^2 H_0^2 + \left( \frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right) H_0 \left( \frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right) H_0 \right]. \]

Thus we have

\[ \frac{1}{2} \text{Tr} \left[ \left( i \left( \frac{\rho^\alpha + \rho^{1-\alpha}}{2}, H_0 \right) \right)^2 \right] = \frac{1}{2} \text{Tr} \left[ \left( \frac{\rho^\alpha + \rho^{1-\alpha}}{2}, H \right) \right]^2 \]

but we have

\[ \frac{1}{2} \text{Tr} \left[ \left( \left\{ \frac{\rho^\alpha + \rho^{1-\alpha}}{2}, H_0 \right\} \right)^2 \right] \neq \frac{1}{2} \text{Tr} \left[ \left( \frac{\rho^\alpha + \rho^{1-\alpha}}{2}, H \right) \right]^2. \]

In addition, we have \( L_{\rho,\alpha}(H) \geq K_{\rho,\alpha}(H) \) which implies

\[ W_{\rho,\alpha}(H) \equiv \sqrt{K_{\rho,\alpha}(H)L_{\rho,\alpha}(H)} \geq \sqrt{K_{\rho,\alpha}(H)K_{\rho,\alpha}(H)} \geq K_{\rho,\alpha}(H). \]

Therefore our fifth concern is whether the following inequality for \( \alpha \in [0,1] \) holds or not:

\[ (2.14) \quad K_{\rho,\alpha}(X)K_{\rho,\alpha}(Y) \geq \frac{1}{4} \left| \text{Tr} \left[ \left( \frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right)^2 [X,Y] \right] \right|^2. \]

However this inequality fails, because we have a counter-example. If we set \( \alpha = \frac{1}{2} \) and

\[ \rho = \frac{1}{4} \left( \begin{array}{cc} 3 & 0 \\ 0 & 1 \end{array} \right), \quad X = \left( \begin{array}{cc} 0 & i \\ -i & 0 \end{array} \right), \quad Y = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right). \]

Then we have,

\[ K_{\rho,\alpha}(X)K_{\rho,\alpha}(Y) = I_{\rho}(X)I_{\rho}(Y) = \left( \frac{1 - \sqrt{3}}{2} \right)^2 \]

and

\[ \frac{1}{4} \left| \text{Tr} \left[ \left( \frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right)^2 [X,Y] \right] \right|^2 = \frac{1}{4} \left| \text{Tr} [\rho [X,Y]] \right|^2 = \frac{1}{4}. \]

Thus the inequality (2.14) does not hold in general.

Before closing this section, we reconsider the ordering \( W_{\rho,\alpha}(H) \) and \( V_{\rho}(H) \), although we have already stated an example of the triplet \( \alpha, \rho \) and \( H \) satisfying \( W_{\rho,\alpha}(H) < V_{\rho}(H) \) in the last line of (i) of Remark 2.8. If we set \( \alpha = \frac{1}{3} \) and

\[ \rho = \left( \begin{array}{cc} 0.3 & 0.45 \\ 0.45 & 0.7 \end{array} \right), \quad H = \left( \begin{array}{cc} 1 & 3 \\ 3 & 1 \end{array} \right). \]

Then \( V_{\rho}(H) - W_{\rho,\alpha}(H) \) approximately takes \(-0.3072\). If we set \( \alpha = \frac{1}{5} \) and

\[ \rho = \left( \begin{array}{cc} 0.3 & 0.4 \\ 0.4 & 0.7 \end{array} \right), \quad H = \left( \begin{array}{cc} 1 & 3 \\ 3 & 1 \end{array} \right). \]

Then \( V_{\rho}(H) - W_{\rho,\alpha}(H) \) approximately takes \(0.682011\). Therefore we have no ordering between \( W_{\rho,\alpha}(H) \) and \( V_{\rho}(H) \). Thus it is natural for us to have an interest in the following conjecture, since we have \( K_{\rho,\alpha}(H) \leq W_{\rho,\alpha}(H) \) in general.
Conjecture 2.11. Our final concern is whether the following inequality:

\[ K_{\rho,\alpha}(H) \leq V_{\rho}(H), \quad \alpha \in [0,1] \]

holds or not. However we have not found its proof and any counter-examples yet.

3. Concluding remarks

As we have seen, we introduced a generalized Wigner-Yanase skew information \( K_{\rho,\alpha}(H) \) and then defined a new quantity \( W_{\rho,\alpha}(H) \). We note that our generalized Wigner-Yanase skew information \( K_{\rho,\alpha}(H) \) is different type of the Wigner-Yanase-Dyson skew information \( I_{\rho,\alpha}(H) \). For the quantity \( K_{\rho,\alpha}(H) \), we do not have a trace inequality related to an uncertainty relation. However, we showed that we have a trace inequality related to an uncertainty relation for the quantity \( W_{\rho,\alpha}(H) \). This inequality is a non-trivial one-parameter extension of the uncertainty relation Eq.(2.23) shown by S.Luo in [6]. In addition, we studied several trace inequaities on informational quantities.

Finally, we give another generalized trace inequality of the inequality (2.3). For a quantum state \( \rho \) an observable \( H \) and \( \alpha \in [0,1] \), we define

\[
Z_{\rho,\alpha}(H) \equiv \frac{1}{4} \sqrt{\text{Tr} \left[ (i[\rho^\alpha, H_0])^2 \right] \text{Tr} \left[ (i[\rho^{1-\alpha}, H_0])^2 \right] \text{Tr} \left[ (\rho^\alpha, H_0)^2 \right] \text{Tr} \left[ (\rho^{1-\alpha}, H_0)^2 \right]},
\]

with \( H_0 \equiv H - \text{Tr}[\rho H]I \). Then we have the following inequality

\[
\sqrt{Z_{\rho,\alpha}(X)Z_{\rho,\alpha}(Y)} \geq \frac{1}{4} \left| \text{Tr} \left[ \rho^{2\alpha}[X,Y] \right] \text{Tr} \left[ \rho^{2(1-\alpha)}[X,Y] \right] \right|,
\]

for a quantum state \( \rho \), two observables \( X,Y \) and \( \alpha \in [0,1] \). We note that the inequality (3.1) recovers the inequality (2.3) by taking \( \alpha = 1/2 \) and we do not have any weak-strong relation between the inequality (2.11) and the inequality (3.1).

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