Abstract

We investigate some properties of topological groups related to disconnectedness or Archimedeaness. We prove or disprove the preservation of those under operations as subgroups, quotients, products, etc. Characterizations of non-Archimedeaness are obtained by using embedding into universal groups. We also clarify the differences between the properties by constructing miscellaneous Polish groups.

1 Introduction

For topological groups, there are several notions of connectedness or disconnectedness; some are topological and others are topologico-algebraic. The topological groups treated here may be connected or zero-dimensional at the extremes as to the former, and Archimedean or non-Archimedean (in the terminology of $\mathbb{R}$, $\mathbb{R}$) as to the latter.

Our main concern is for topologico-algebraic properties, i.e., richness or poorness in open subgroups. We investigate the interrelation among each other and with some operations and topological properties.
In Section 2, we introduce the properties and obtain preservation results under subgroups, quotients, direct products, topological closures and group extensions. Section 3 is for the characterizations of non-Archimedeaness. We show universal groups in general and Abelian case, respectively. In Section 4, we mainly treat Abelian Polish groups. We exhibit peculiar examples with or without several properties.

All topological spaces we consider here are assumed to be Hausdorff unless otherwise stated. In particular, a quotient of a topological group is meant to be that by a closed subgroup.

We mainly follow the notation of standard texts such as [7]. The neutral element of a group may be denoted by 1 (or 0 in Abelian case). We also adopt some set-theoretic notations. Let $X^Y$ denote the set of functions from $X$ into $Y$ possibly with some induced structure. The set $\{0,1,2,...\}$ of natural numbers is identified with the ordinal number $\omega$ and the cardinal number $\aleph_0$.

2 Many or few open subgroups

We consider following properties for topological groups concerning the magnitude of open subgroups.

**Definition 2.1** (1) A topological group is *topologically non-Archimedean* (TNA) if every neighborhood of the neutral element contains an open subgroup.

(2) A topological group has *sufficiently many open subgroups* (SMOG) if the intersection of all open subgroups consists of only the neutral element.

(3) A topological group is *topologically Archimedean* (TA) if it has no proper open subgroups.

These have close relation to connectedness as open subgroups are clopen. We recall some previously known or easily observable facts.

**Remark 2.2** (1) The terminology of (non-)Archimedeaness for Abelian topological groups appear in [12] and [13]. As noted in [13], a topological group is TNA if and only if it is Archimedean with respect to
the right (or left) group uniformity. Being TA is equivalent to each of
the following properties: every non-empty open set (neighborhood of
1, respectively) generates the whole group.

(2) The following implications and incompatibility are straightforward:

\[
\begin{align*}
\text{TNA} & \Rightarrow \text{zero-dimensional} \\
\downarrow & \quad \downarrow \\
\text{SMOG} & \Rightarrow \text{totally-disconnected}
\end{align*}
\]

connected \(\Rightarrow\) TA;

A TA group with SMOG is trivial.

(3) If a topological group has SMOG, then the open subgroups induce
another group topology, which is TNA. So a topological group has
SMOG if and only if it has a weaker (Hausdorff) TNA group topology.

(4) For a locally compact group, each arrow above can be reversed (cf. [7],
Theorem 7.7, Corollary 7.9).

(5) As remarked in [3, p. 9], the Polish group \(F\) in [4, pp. 77–78] is zero-
dimensional but not TNA; it has SMOG. Ancel, Dobrowolski and
Grabowski [1] investigate topological groups with similar properties
in Banach spaces. Our \(\Gamma_0\) in Example 4.1 is typical one of them.

(6) A totally-disconnected TA Polish group is constructed by T.C. Stevens
[14] (see also Hartman, Mycielski, Rolewicz and Schinzel [3] and Schinzel
[11]), which is not zero-dimensional (Theorem 4.4).

We have preservation under some operations. Some are immediate by
definition.

**Proposition 2.3**  (1) Subgroups, quotient groups and products of TNA
groups are also TNA.

(2) Having SMOG is hereditary to subgroups and products (but not to
quotients, see Example 4.1).

(3) Quotient groups and products of TA groups are also TA; subgroups
need not.
Proof. We show (3) for a product. Suppose that \( \langle G_i : i \in I \rangle \) is a family of TA groups. Let \( J \) be a finite subset of \( I \) and \( U_i \) a neighborhood of 1 in \( G_i \) for \( i \in J \). We shall prove that the neighborhood \( U = \prod_{i \in I \setminus J} G_i \times \prod_{i \in J} U_i \) generates the whole product \( \prod_{i \in I} G_i \). For any \( g = \langle g_i : i \in I \rangle \in \prod_{i \in I} G_i \), set \( g' = \langle g_i : i \in I \setminus J \rangle \cup \langle 1_{G_i} : i \in J \rangle \), i.e., \( g' = \langle g'_i : i \in I \rangle \) with \( g'_i = g_i \) for \( i \in I \setminus J \) and \( g'_i = 1_{G_i} \) for \( i \in J \). For each \( i \in J \), there exists a finite sequence \( g_{i1}, \ldots, g_{im_i} \) in \( U_i \) such that \( g_i = g_{i1} \cdots g_{im_i} \). We may assume that \( U_i^{-1} = U_i \) and \( n_i = n \) for all \( i \in J \). Let \( h_k \) for \( 1 \leq k \leq n \) denote \( \langle 1_{G_i} : i \in I \setminus J \rangle \cup \langle g_{ik} : i \in J \rangle \). Then \( g', h_1, \ldots, h_n \in U \) and \( g' h_1 \cdots h_n = g \). So we are done. ■

Topological closure preserves two properties of the three.

**Proposition 2.4** (1) A topological group is TA if it has a dense TA subgroup.

(2) TNA-ness of a dense subgroup implies that of the whole group.

Proof. (2) Assuming that \( H \) is a dense TNA subgroup of \( G \), we shall show that \( G \) is also TNA. For a neighborhood \( U \) of 1 in \( G \), there exists another \( V \) such that \( \overline{V} \subseteq U \). By the assumption, we find an open subgroup \( K \) of \( H \) with \( K \subseteq \overline{V \cap H} \). Then the closure \( \overline{K} \) in \( G \) is an open subgroup contained in \( U \). ■

**Example 2.5** The additive group of \( \mathbb{R} \times \mathbb{Q}_p \) has a dense subgroup \( \{ (x, x) : x \in \mathbb{Q} \} \). While the latter has SMOG, the former does not.

Next we consider group extensions. Suppose that \( G \) is a topological group and \( N \) is a normal subgroup.

**Proposition 2.6** If \( N \) and \( G/N \) are TA, so is \( G \).

Proof. Let \( H \) be an open subgroup of \( G \). Then \( H \cap N \) is open in \( N \), and hence \( H \cap N = N \), i.e., \( N \subseteq H \). Since \( H/N \) is open in \( G/N \), we also have \( H/N = G/N \). Therefore \( H = G \). ■

**Theorem 2.7** If both \( N \) and \( G/N \) are TNA, then so is \( G \).
Proof. Let $U$ be a neighborhood of 1 in $G$. We shall find an open subgroup $H$ contained in $U$.

We choose neighborhoods $U_0$, $V$ and $W$ of 1 in $G$ as follows. First let $U_0$ be such that $U_0^2 \subseteq U$. By the assumption, there is an open subgroup $M$ of $N$ contained in $N \cap U_0$. Let $V \subseteq U_0$ be open with $V^{-1} = V$ and $V^3 \cap N \subseteq M$. We denote by $\pi$ the natural homomorphism $G \to G/N$. Since $\pi(V)$ is open in $G/N$, it contains an open subgroup $K$. We set $W = V \cap \pi^{-1}(K)$.

Now let $H$ be the subgroup of $G$ generated by $W$. Then $H$ is open and $H \subseteq WM \subseteq U_0^2 \subseteq U$ as desired. ■

The “three-group property” above does not hold for SMOG (Example 4.2).

3 Universal non-Archimedean groups

TNA groups are characterized by embeddings into certain topological groups. For a (usually infinite) set $\Omega$, $\text{Sym}(\Omega)$ denotes the symmetric group on $\Omega$. We topologize $\text{Sym}(\Omega)$ as a subspace of the product $\Omega^\Omega$ with $\Omega$ discrete, which makes $\text{Sym}(\Omega)$ a topological group.

We recall some cardinal invariants for topological spaces (cf. [8]). Suppose $X$ is a topological space. The weight, the cellularity and the character of $X$ are defined respectively as follows:

$$w(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ a base for } X\} + \aleph_0,$$

$$c(X) = \sup\{|\mathcal{V}| : \mathcal{V} \text{ a disjoint family of open sets in } X\} + \aleph_0,$$

$$\chi(X) = \sup\{\chi(p, X) : p \in X\} + \aleph_0,$$

where

$$\chi(p, X) = \min\{|\mathcal{V}| : \mathcal{V} \text{ a local base for } p\}.$$  

Note that $c(X), \chi(X) \leq w(X)$ and that in case $X$ is a topological group,

$$\chi(X) = \chi(1, X) + \aleph_0.$$
Theorem 3.1 (1) A topological group is TNA if and only if it is isomorphic to a subgroup of a symmetric group. Specifically, for a topological group $G$ and an infinite cardinal $\kappa$, the following are equivalent.

(a) $G$ is isomorphic to a subgroup of $\text{Sym}(\kappa)$.
(b) $G$ is TNA and $w(G) \leq \kappa$.
(c) $G$ is TNA and $c(G), \chi(G) \leq \kappa$.

(2) An Abelian topological group is TNA if and only if it is isomorphic to a subgroup of a direct product of discrete groups. Suppose that $A$ is an Abelian topological group and $\kappa$ is an infinite cardinal. Then the following are equivalent.

(a) $A$ is isomorphic to a subgroup of the direct product of $\kappa$ or less discrete groups each of size at most $\kappa$.
(b) $A$ is isomorphic to a subgroup of

$$
\prod_{\kappa} \left( \bigoplus_{\kappa} \mathbb{Q} \bigoplus_{p: \text{prime}} \bigoplus_{\kappa} \mathbb{Z}(p^{\infty}) \right)_d,
$$

where $\mathbb{Z}(p^{\infty})$ is the quasicyclic group and $\cdot_d$ denotes the topological group endowed with the discrete topology.
(c) $A$ is TNA and $w(A) \leq \kappa$.
(d) $A$ is TNA and $c(A), \chi(A) \leq \kappa$.

Proof. (1): We proceed as in the argument for the universality of $\text{Sym}(\omega)$ among the TNA Polish groups (cf. [2, Theorem 1.5.1]). Since $\text{Sym}(\kappa)$ is TNA and of weight $\kappa$, we have that (a) $\Rightarrow$ (b). The implication (b) $\Rightarrow$ (c) is straightforward.

We show that (c) $\Rightarrow$ (a). Suppose that $G$ is a TNA group. Let $\mathcal{V}$ be a local base for 1 consisting of open subgroups. We may assume that $|\mathcal{V}| \leq \chi(G)$. We denote by $\Omega$ the disjoint union of quotient spaces

$$
\bigcup \{G/H : H \in \mathcal{V}\}.
$$

The natural action of $G$ on $\Omega$ such that

$$
g \cdot (xH) = (gx)H
$$
induces an isomorphic embedding $G \to \text{Sym}(\Omega)$. Since $|G/H| \leq c(G)$ for each open subgroup $H$, we get $|\Omega| \leq c(G)\chi(G)$.

(2): The implication (a) $\Rightarrow$ (b) follows from the universality of

$$\bigoplus_{\kappa} \mathbb{Q} \oplus \bigoplus_{p: \text{prime}} \bigoplus_{\kappa} \mathbb{Z}(p^\infty)$$

among (discrete) Abelian groups of size $\kappa$ (see [4], Theorem 0.1).

Let $A$ be an Abelian TNA group. By virtue of [3, III, §7, No 3, Proposition 5], there exists an isomorphic embedding from $A$ into

$$\lim\left\{A/B : B \text{ is an open subgroup of } A\right\}.$$

The estimation of the cardinality is similar as in (1). ■

**Remark 3.2** A metric $d$ is said to be non-Archimedean if it satisfies the strong triangle inequality

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}.$$ 

It is easily seen that a topological group with a right (or left) invariant non-Archimedean metric is TNA. For metrizable groups, these two notions coincides: if $\{U_n : n < \omega\}$ is a neighborhood basis of 1 consisting of decreasing sequence of open subgroups with $U_0$ the whole group, then $d(x, y) = 2^{-\max\{n < \omega : xy^{-1} \in U_n\}}$ for $x \neq y$ determines a right invariant compatible non-Archimedean metric.

G. Rangan asserted in [10]: “Lemma 3.1: Suppose a topological group $G$ is such that its topology is given by a non-archimedean metric $d$ then there is an equivalent non-archimedean right (or left) invariant metric $\rho$ on $G.$” and “Theorem 3.2: Let $G$ be a separable totally disconnected ordered topological group. Then $G$ is non-archimedean metrizable.” (A topological group is said to be non-archimedean metrizable if there exists a non-archimedean right (or left) invariant metric on $G$ which induces the topology of $G.$)

The additive group of rational numbers $\mathbb{Q}$ is, however, a counterexample to both of the statements, which is separable, metrizable, zero-dimensional and TNA.
4 Disconnected Polish groups

Among the properties in Remark 2.2 (2), no implication other than indicated holds even for Abelian Polish groups. We construct witnessing examples from sequence spaces.

Example 4.1 As usual, let $c_0$ be the Banach space of the real sequences converging to 0 with the norm $||\cdot||_\infty$ and $l^1$ that of absolutely summable ones with $||\cdot||_1$. We set

$$R = \{a \in \omega \mathbb{R} : (\forall n < \omega)(a(n) \in (1/(n + 1)! \mathbb{Z}))\},$$

$$S = \left\{ a \in \bigoplus_{\omega} \mathbb{Z} : \sum_{n < \omega} a(n) = 0 \right\},$$

$$\Gamma_0 = c_0 \cap R,$$

$$\Gamma_1 = l^1 \cap R,$$

where $\bigoplus_{\omega} \mathbb{Z}$ is identified with $\{a \in \omega \mathbb{Z} : a(n) = 0 \text{ for all but finitely many } n\} \subset \omega \mathbb{R}$. Then $\Gamma_0, \Gamma_1, \Gamma_0/S$ and $\Gamma_1/S$ are Polish groups with the following properties.

| Property          | $\Gamma_0$ | $\Gamma_0/S$ | $\Gamma_1$ | $\Gamma_1/S$ |
|-------------------|-------------|--------------|-------------|--------------|
| totally-disconnected | Yes         | Yes          | Yes         | Yes          |
| zero-dimensional  | Yes         | Yes          | No          | No           |
| SMOG              | Yes         | No           | Yes         | No           |
| TNA               | No          | No           | No          | No           |
| TA                | No          | No           | No          | No           |

Proof. $\Gamma_0$ is zero-dimensional: Every open ball is clopen.

$\Gamma_0$ and $\Gamma_1$ have SMOG: Suppose that $a \neq 0$. Then there is $n$ with $a(n) \neq 0$. The open subgroup $\{b : b(n) = 0\}$ excludes $a$.

$\Gamma_0$ is not TNA: For each $a \neq 0$, we have that $\lim_{n \to \infty} ||na||_\infty = \infty$. Accordingly every nontrivial subgroup is unbounded.

$\Gamma_0/S$ is zero-dimensional: Since $S$ is discrete, zero-dimensionality of $\Gamma_0$ is preserved under the quotient.
Γ₀/S and Γ₁/S do not have SMOG: Let eₙ denote the n-th unit vector (0,...,0,1,0,...). We show that e₀ mod S(≠ 0) belongs to all open subgroups. Let A be an open subgroup. Then for sufficiently large n, we have eₙ/(n+1) mod S ∈ A. Accordingly e₀ mod S = (n+1)(eₙ/(n+1) mod S) ∈ A.

Γ₀/S and Γ₁/S are not TA: the subgroup \{b : b(1) ∈ Z\} mod S is open and proper.

Γ₁ is not zero-dimensional: Theorem 4.4 below.

Γ₁/S is totally-disconnected: Suppose that a mod S ≠ 0. Then for r > 0 sufficiently small, (B₀(0,r) ∩ Γ₁) mod S is clopen and does not include a, where B₀(a,r) is the open ball of c₀ with center a and radius r.

Γ₁/S is not zero-dimensional: Non-zero-dimensionality is also hereditary to quotients by a discrete subgroup. ■

These examples behave wildly as to other properties.

Example 4.2 While Γ₀/S does not have SMOG, it has a subgroup ⊔ω Z/S ∼= Z with quotient (Γ₀/S)/(⊔ω Z/S) ∼= Γ₀/⊔ω Z such that both have SMOG.

Remark 4.3 Since every totally-disconnected locally compact group is TNA, its quotient is always totally-disconnected as well. On the other hand, as seen from

Γ₁ \left/ \left\{ a ∈ Γ₁ : \sum_{n<ω} a(n) = 0 \right\} \right. \cong \mathbb{R},

a quotient of a totally-disconnected Polish group may be nontrivially connected.

Let c denote the space of convergent sequences with the norm \| \cdot \|_∞. Then we have (c ∩ R)/(c₀ ∩ R) ∼= \mathbb{R}. But c ∩ R, which has SMOG similarly as Γ₀, is not zero-dimensional by Theorem 4.4.

Problem 1 Concerning Polish groups, is zero-dimensionality inherited by quotients?

We do not know whether the examples in Remark 2.2 (4) (5) may be superseded by a “strong” one.
PROBLEM 2 Does there exist a zero-dimensional TA Polish group?

Some “naive constructions” toward answering the problems have failed due to the limitation below for zero-dimensional complete metric groups, which is obtained by modifying the argument by Erdős [5] for a subgroup of the Hilbert space $l^2$.

THEOREM 4.4 Suppose that $A$ is an Abelian topological group topologized with a complete invariant metric $d$ and $\langle a_n : n < \omega \rangle$ is a sequence in $A$ converging to $0$. We denote by $C$ and $\tilde{C}$ the sets of finite subsums and converging ones, respectively, $\sum_{n \in X} a_n$ for $X \subset \omega$. We set $\rho = \sup \{d(y,0) : y \in A\} \in \mathbb{R} \cup \{\infty\}$ and assume that there exists a function $\nu : C \to \mathbb{R}_{\geq 0}$ such that

1. $\sum_{n=0}^{\infty} \nu(a_n) = \infty$,

2. $\sum_{n \in X} \nu(a_n) = \nu \left( \sum_{n \in X} a_n \right)$ for each finite $X \subset \omega$.

3. $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in C)(\nu(x) < \delta \Rightarrow d(0,x) < \varepsilon)$,

4. $(\forall \varepsilon < \rho)(\exists \delta < \infty)(\forall x \in C)(\nu(x) > \delta \Rightarrow d(0,x) > \varepsilon)$,

Then $\tilde{C}$ is not zero-dimensional.

Proof. Assuming that $U$ is an open neighborhood of $0$ contained in some open ball $B(0,r)$ with center $0$ and radius $r < \rho$, we show that $U \cap \tilde{C}$ is not closed in $\tilde{C}$.

We define sequences $\langle c_m : m < \omega \rangle$ in $U$ and $\langle n_m : m < \omega \rangle$ and $\langle n'_m : 1 \leq m < \omega \rangle$ of integers by induction such that $c_m = \sum_{k=1}^{m} \sum_{n=n'_k}^{n_k} a_n$. We set $c_0 = 0$ and $n_0 = -1$. Since $U$ is open and $c_m + a_n \to c_m$ as $n \to \infty$, there is a natural number $n'_{m+1} > n_m$ with $c_m + a_{n'_{m+1}} \in U$. Due to (4), $\sum_{n=n'_{m+1}}^{\infty} \nu(a_n) = \infty$ as well, and hence for sufficiently large $i$, it occurs
that $c_m + \sum_{n=n_m+1}^i a_n \not\in B(0, r)$ by (3), where we use (3) freely. So we may find $n_{m+1}$ such that

$$c_m + \sum_{n=n_{m+1}}^{n_{m+1}+1} a_n \not\in U.$$ 

Then we have $c_m \in C \cap U$ and $c_m + a_{n_{m+1}} \in C \setminus U$ for each $m$. We show that the sequence $\langle c_m : m < \omega \rangle$ is convergent. Since it is in $B(0, r)$, the sequence of $\nu(c_m) = \sum_{k=1}^m \nu \left( \sum_{n=n_k}^{n_{k+1}} a_n \right)$ is bounded due to (4). The latter is increasing as well, and hence it is convergent. Therefore for any $\delta > 0$, sufficiently large $i$ and every $j > i$ satisfy $\nu(c_j - c_i) = \sum_{k=i+1}^j \nu \left( \sum_{n=n_k}^{n_{k+1}} a_n \right) < \delta$. Since $\delta$ is arbitrary, (3) yields that the sequence $\langle c_m : m < \omega \rangle$ is Cauchy, so convergent.

Since the sequences $\langle c_m : m < \omega \rangle$ in $C \cap U$ and $\langle c_m + a_{n_{m+1}} : m < \omega \rangle$ in $C \setminus U$ converge to the same point in $\tilde{C}$, we conclude that $U \cap C$ is not clopen in $\tilde{C}$. ■

We exhibit some examples to which the theorem is applicable.

Remark 4.5 The totally-disconnected TA Polish group in [14] is seen not to be zero-dimensional as follows. If we set $a_n = 2^{-2n}$ and $\nu = D$, then the premises of the theorem hold.

For $\Gamma_1$ in Example 4.1, $a_n = e_n/(n + 1)$ and $\nu = || \cdot ||_1$ will do.

As to $c \cap \Gamma_0$, if we set $a_n = (0, \ldots, 0, 1, 1, \ldots)/n$ and $\nu = || \cdot ||_\infty$, then it goes well. The last argument is based on the proof provided by Professor Katsuya Eda to the author that the set of rational points in $c$ is not zero-dimensional.

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