Design finite-time output feedback controller for nonlinear discrete-time systems with time-delay and exogenous disturbances

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ABSTRACT
This paper considers the finite-time output feedback controller design for nonlinear discrete-time systems with time-delay and time-varying exogenous disturbances. The exogenous disturbances are uncertain bounded signals. In this regard, a theorem is given and the sufficient conditions are extracted which guarantee the finite-time boundedness of the time-delay closed-loop system via selecting the appropriate Lyapunov-Krasovski functional. Furthermore, the gain of output feedback is achieved through the feasibility testing of the derived linear matrix inequalities (LMIs). Finally, a numerical example is given to verify the effectiveness of the developed technique.

1. Introduction
The Lyapunov asymptotic stability is a largely known concept in control theory. However, a dynamical system may be asymptotically stable in theory but practically useless (Amato, Ariola, & Dorato, 2001). In order to deal with such situations, the concepts of Finite-Time Stability (FTS) (Binazadeh & Shafiei, 2013, 2014; Liu, Shi, Karimi, & Chadli, 2016; Zhang, Zhang, & Zhang, 2015) and Finite-Time boundedness (FTB) (He & Liu, 2011; Lin, Li, Li, & Zou, 2016; Song, Niu, & Zou, 2017; Wu, Cao, Alofi, Abdullah, & Elaiw, 2015) have been proposed in literature with many practical applications. Moreover, the finite-time stability has improved the robustness and disturbance rejection properties (Binazadeh, 2016).

On the other hand, time-delay and exogenous disturbances may be appeared in many of real systems which are source of instability and poor performance and add some complexity in the procedure of controller design. Cheng, Chen, Gao, Zhang, & Li, 2014; Binazadeh & Yousefi, 2017a, 2017b. Finite-time stabilization of dynamical systems in the presence of time-delay is one of the important fields of research in recent years (Lin, Liang, Li, Jiao, & Nie, 2017; Moulay, Dammare, Yeganefar, & Perruquetti, 2008; Song & He, 2013, 2015a; Trang, Pham, & Samir, 2016; Wang, Chen, & Sheng, 2016; Yang & Wang, 2013).

Authors of (Xiang & Xiao, 2011; Zhang, Shi, & Shi, 2017; Zong, Wang, Zheng, & Hou, 2015) have studied the finite-time stability of discrete-time systems with time-delay. Shen, Yu, and Jian (2008) designed a state feedback controller for finite-time stabilization of linear discrete-time systems with time-delay. Also, Stojanovic (2015) and Stojanovic, Debeljko, and Dimitrijevi (2012) have been addressed the finite-time stability analysis for discrete-time systems with time-delay. However, to the best knowledge of authors, the problem of designing finite-time output feedback controller for nonlinear discrete-time systems with time-delay and exogenous disturbances has not been studied, yet. This subject is studied in this paper.

This paper considers nonlinear discrete-time systems in the presence of time-delay and exogenous disturbances. The exogenous disturbances are uncertain time-varying signals with known upper bound. The main contribution of this paper is extracting the sufficient conditions which guarantees the FTB property of the considered nonlinear discrete-time systems. Considering the dynamical equations of system, an appropriate Lyapunov-Krasovski functional is chosen which is an appropriate functional for the considered time-delay systems. The static output feedback is designed and the sufficient LMI conditions are derived to ensure the robust FTB property of the closed-loop system. In this regard, a theorem is given and proved based on the Lyapunov approach. Additionally, in order to display the efficiency of the proposed controller, it is applied on a numerical example. Computer simulations are also done to verify the theoretical results and demonstrate the effective performance of the proposed method in achieving the FTB characteristic in the closed-loop system.
The rest of this paper is organized as follow: In the next section, the considered system is introduced and some definitions are presented. Section 3 consists of the main results of this paper. The sufficient conditions for achieving the FTB characteristic of the closed-loop system are given in this section. Design example and numerical simulations are presented in section 4. Finally, conclusions are made in section 5.

2. Problem statement

Consider the following nonlinear discrete-time system with time-delay and exogenous disturbances:

\[ x[k + 1] = Ax[k] + A_r x[k - \tau] + Bu[k] + Gw[k] + f(x[k]) \]
\[ y[k] = Cx[k], \quad k = 0, 1, 2, \ldots \]
\[ x[k_0] = \sigma[k_0], \quad k_0 = -\tau, -\tau + 1, \ldots, 0 \]  \hspace{1cm} (1)

where \( x[k] \in \mathbb{R}^n \) is state vector, \( y[k] \in \mathbb{R}^l \) is the output vector, \( u[k] \in \mathbb{R}^r \) is vector of control inputs, \( \sigma[k] \) is the vector of initial functions, \( A, B, G, C \) and \( A_r \) are constant matrices with appropriate dimensions, \( \tau \) is the constant time-delay which is an integer number and \( w[k] \) is vector of unknown exogenous disturbances which satisfies:

\[ w^T[k]w[k] \leq \delta^2, \quad k = 0, 1, 2, \ldots \]  \hspace{1cm} (2)

where \( \delta \) is known positive constant. Moreover \( f(x[k]) \) is a nonlinear vector function which is locally Lipschitz with respect to \( x \).

The goal is design of the static output feedback controller (i.e. \( u[k] = Ky[k] \)) such that closed-loop system (3) be FTB in the presence of exogenous disturbance vector \( \{w[k]\neq0\} \).

\[ x[k + 1] = \bar{A}x[k] + A_r x[k - \tau] + Gw[k] + f(x[k]) \]
\[ x[k] = \sigma[k], \quad \forall k \in [1, \ldots, N] \]  \hspace{1cm} (3)

where

\[ \bar{A} = A + BKC \]  \hspace{1cm} (4)

**Definition 1**: (Xiang & Xiao, 2011; Zhang et al., 2017): The system (3) is FTB with respect to \((c_1, c_2, N, \delta, \alpha \geq 1)\) if

\[ \max_{k_0 = -\tau, -\tau + 1, \ldots, 0} T[k_0]x[k_0] \leq c_1^2 \Rightarrow T[k]x[k] \leq c_2^2 \quad \forall k = 0, 1, \ldots, N \]  \hspace{1cm} (5)

where \( c_2(\geq c_1) \).

**Definition 2**: (Khalil, 2002): The vector function \( f(x[k]) \) is locally Lipschitz with respect to \( x \) in a region \( \Lambda \subset \mathbb{R}^n \) containing the origin if

\[
(f(x_1) - f(x_2))^T (f(x_1) - f(x_2)) \\
\leq \gamma^2 (x_1 - x_2)^T (x_1 - x_2); \quad \forall x_1, x_2 \in \Lambda
\]  \hspace{1cm} (6)

where \( \gamma > 0 \) is called the Lipschitz constant.

**Lemma 1**: (Song & He, 2015a): For any \( x \& y \in \mathbb{R}^n \) and any positive-definite matrix \( \Gamma \in \mathbb{R}^{n \times n} \), one has

\[
2x^T y \leq x^T \Gamma x + y^T \Gamma^{-1} y
\]  \hspace{1cm} (7)

3. Main results

In this section, the sufficient conditions are derived and the appropriate feedback gain (i.e. \( K \)) is designed to guarantee the FTB property for the closed-loop system (3). For this purpose, the following theorem is given and proved.

**Theorem 1**: For given positive constants \( c_1, c_2, N, \delta, \alpha \geq 1 \), the closed-loop systems (3) is FTB with respect to \((c_1, c_2, N, \delta)\), if there exist the symmetric positive definite matrices \( W_1, W_2, W_3, \) and real matrix \( L \), and positive real numbers \( \mu_1, \mu_2, \mu_3, \mu_4, \epsilon_1 \) such that the following LMIs be feasible:

\[
\begin{bmatrix}
3\epsilon_1 \gamma^2 I - \alpha W_1 + W_2 & 0 & 0 & A^T W_1 + C^T L^T \\
0 & -W_2 & 0 & A^T W_1 \\
0 & 0 & -\alpha W_3 & G^T W_1 \\
W_1 A + LC & W_1 A_r & W_1 G & -W_1 \\
0 & W_1 A_r & 0 & 0 \\
0 & 0 & W_1 G & 0 \\
W_1 A^T + LC & 0 & 0 & 0 \\
0 & 0 & A^T W_1 + C^T L^T & 0 \\
A^T W_1 & 0 & 0 & 0 \\
0 & G^T W_1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-(\epsilon_1 I - 2W_1) & 0 & 0 & 0 \\
0 & -(\epsilon_1 I - 2W_1) & 0 & 0 \\
0 & 0 & -(\epsilon_1 I - 2W_1) & 0
\end{bmatrix} < 0
\]  \hspace{1cm} (8)

\[
\begin{bmatrix}
\mu_1 I - W_1 & 0 & 0 & 0 & 0 \\
0 & \mu_2 I - W_1 & 0 & 0 & 0 \\
0 & 0 & \mu_3 I - W_2 & 0 & 0 \\
0 & 0 & 0 & W_3 - \mu_4 I & 0 \\
0 & 0 & 0 & 0 & 2W_1 - \epsilon_1 I
\end{bmatrix} < 0
\]  \hspace{1cm} (9)

\[ \mu_2 c_1^2 + \mu_3 c_2^2 \tau + \mu_4 \delta^2 N < \mu_1 c_1^2 \alpha^{-N} \]  \hspace{1cm} (10)

In these situations, the suitable gain of output-feedback controller can be obtained by

\[ K = (B^T B)^{-1} B^T W_1^{-1} L \]  \hspace{1cm} (11)
**Proof:** Consider the following Lyapunov-Krasovskii functional for system (3)

\[ V(x[k]) = x^T[k]W_1x[k] + \sum_{i=k-1}^{k-1} x^T[i]W_2x[i] \]  

(12)

Thus, one has

\[ V(x[k + 1]) = x^T[k + 1]W_1x[k + 1] + \sum_{i=k+1}^{k} x^T[i]W_2x[i] \]

\[ = (x^T[k]\bar{A}^T + x^T[k - \tau]\bar{A}_r^T + w^T[k]\bar{G}^T + f^T[k])W_1 (\bar{A}x[k] + A_1x[k - \tau] + Gw[k] + f[k]) \]

\[ + \sum_{i=k-1}^{k-1} x^T[i]W_2x[i] + x^T[k]W_2x[k] \]

\[ - x^T[k - \tau]W_2x[k - \tau] \]

\[ \leq (x^T[k]\bar{A}^T + x^T[k - \tau]\bar{A}_r^T + w^T[k]\bar{G}^T + f^T[k])W_1 (\bar{A}x[k] + A_1x[k - \tau] + Gw[k] + f[k]) \]

\[ + \alpha \sum_{i=k-1}^{k-1} x^T[i]W_2x[i] + x^T[k]W_2x[k] \]

\[ - x^T[k - \tau]W_2x[k - \tau] \]

\[ + \alpha x^T[k]W_1x[k] - \alpha \sum_{i=k-1}^{k-1} x^T[i]W_2x[i] \]

\[ - \alpha w^T[k]W_3w[k] \]

(13)

where \( \alpha \geq 1 \). Define the following function \( J \) and replace \( V(x[k + 1]) \) and \( V(x[k]) \) therein, results in

\[ J = V(x[k + 1]) - \alpha V(x[k]) - \alpha w^T[k]W_3w[k] \]

\[ \leq (x^T[k]\bar{A}^T + x^T[k - \tau]\bar{A}_r^T + w^T[k]\bar{G}^T + f^T[k])W_1 (\bar{A}x[k] + A_1x[k - \tau] + Gw[k] + f[k]) \]

\[ + \alpha \sum_{i=k-1}^{k-1} x^T[i]W_2x[i] + x^T[k]W_2x[k] \]

\[ - x^T[k - \tau]W_2x[k - \tau] \]

\[ - \alpha x^T[k]W_1x[k] - \alpha \sum_{i=k-1}^{k-1} x^T[i]W_2x[i] \]

\[ - \alpha w^T[k]W_3w[k] \]

(14)

The above expression can be rewritten as follows

\[ J \leq x^T[k]\bar{A}^T W_1\bar{A}x[k] + x^T[k]\bar{A}^T W_1A_1x[k - \tau] + x^T[k]\bar{A}^T W_1Gw[k] + 2x^T[k]\bar{A}^T W_1f[k] \]

\[ + x^T[k - \tau]\bar{A}_r^T W_1\bar{A}x[k] + x^T[k - \tau]\bar{A}_r^T W_1A_1x[k - \tau] + x^T[k - \tau]\bar{A}_r^T W_1Gw[k] + 2x^T[k - \tau]\bar{A}_r^T W_1f[k] \]

\[ + w^T[k]G^T W_1\bar{A}x[k] + w^T[k]G^T W_1A_1x[k - \tau] + \alpha w^T[k]W_3w[k] \]

(15)

On the other hand,

\[ 2x^T[k]\bar{A}^T W_1f[k] \]

\[ = 2x^T[k]\bar{A}^T W_1f[k] + 2f^T[k]W_1f[k] \]

\[ - \varepsilon_1 f^T[k]f[k] + \varepsilon_1 f^T[k]f[k] \]

\[ = 2x^T[k]\bar{A}^T W_1f[k] - f^T[k] (\varepsilon_1 I - 2W_1) f[k] \]

(16)

Using Lemma 1 and inequality (6), results in

\[ 2x^T[k]\bar{A}^T W_1f[k] - f^T[k]Qf[k] \leq x^T[k]\bar{A}^T W_1 Q^{-1} W_1 \bar{A}x[k] \]

\[ \varepsilon_1 f^T[k]f[k] \leq \varepsilon_1 \gamma^2 x^T[k]x[k] \]

(17)

Similar to (16) and (17), one can write

\[ 2x^T[k - \tau]\bar{A}_r^T W_1f[k] + 2f^T[k]W_1f[k] \]

\[ = 2x^T[k - \tau]\bar{A}_r^T W_1f[k] + 2f^T[k]W_1f[k] - \varepsilon_1 f^T[k]f[k] + \varepsilon_1 f^T[k]f[k] \]

\[ = 2x^T[k - \tau]\bar{A}_r^T W_1f[k] - f^T[k] (\varepsilon_1 I - 2W_1) f[k] \]

(18)

\[ + \varepsilon_1 f^T[k]f[k] \]

\[ \leq x^T[k - \tau]\bar{A}_r^T W_1 Q^{-1} W_1 \bar{A}_r x[k - \tau] \]

\[ + \varepsilon_1 \gamma^2 x^T[k]x[k] \]

(19)

From (15), (16), (18) and (19), one can get

\[ J \leq x^T[k]\bar{A}^T W_1\bar{A}x[k] + x^T[k]\bar{A}^T W_1A_1x[k - \tau] + \alpha w^T[k]W_3w[k] \]

\[ + x^T[k - \tau]\bar{A}_r^T W_1\bar{A}x[k] + x^T[k - \tau]\bar{A}_r^T W_1A_1x[k - \tau] + \alpha w^T[k]W_3w[k] \]

\[ + x^T[k - \tau]\bar{A}_r^T W_1Gw[k] + 2x^T[k - \tau]\bar{A}_r^T W_1f[k] \]

\[ + x^T[k - \tau]\bar{A}_r^T W_1Gw[k] + 2x^T[k - \tau]\bar{A}_r^T W_1f[k] \]

\[ + x^T[k - \tau]\bar{A}_r^T W_1A_1x[k - \tau] + \alpha w^T[k]W_3w[k] \]

\[ + x^T[k - \tau]\bar{A}_r^T W_1Gw[k] + \varepsilon_1 \gamma^2 x^T[k]x[k] \]

(20)
which are equivalent to

\[
J \leq \begin{bmatrix}
    x[k] \\
    x[k-\tau] \\
    w[k]
\end{bmatrix}^T \begin{bmatrix}
    \Psi_{11} + \tilde{A}_W A_t & \tilde{A}_W G \\
    A_t^T \tilde{A}_W A_t & A_t^T \tilde{A}_W G \\
    G_t W_1 A_t & G_t W_1 G \\
\end{bmatrix} \Omega \begin{bmatrix}
    x[k] \\
    x[k-\tau] \\
    w[k]
\end{bmatrix}
\]

(21)

where

\[
\begin{align*}
\Psi_{11} &= \tilde{A}_W A_t + \tilde{A}_W W_1 Q^{-1} W_1 A_t + 3\varepsilon_1 \gamma^2 I - \alpha W_1 + W_2, \\
\Psi_{22} &= A_t^T \tilde{A}_W Q^{-1} W_1 A_t - W_2 + A_t^T W_1 A_t \\
\Psi_{33} &= G_t W_1 G + G_t W_1 Q^{-1} W_1 G - \alpha W_3
\end{align*}
\]

The condition inequality \( \Omega < 0 \) implies \( J < 0 \) for all \( \xi[k] \neq 0 \). Then one has

\[
J < 0 \Rightarrow V(x[k+1]) \leq \alpha V(x[k]) + \alpha w^T[k] W_3 w[k]
\]

(22)

By developing inequality above we get

\[
\begin{align*}
\text{If } k &= 0 \Rightarrow V(x[1]) \leq \alpha V(x[0]) + \alpha w^T[0] W_3 w[0] \\
\text{If } k &= 1 \Rightarrow V(x[2]) \leq \alpha V(x[1]) + \alpha w^T[1] W_3 w[1] \\
& \leq \alpha^2 V(x[0]) + \alpha^2 w^T[0] W_3 w[0] \\
& + \alpha w^T[1] W_3 w[1] \\
& \vdots \\
V(x[k]) \leq \alpha^k V(x[0]) \\
& + \sum_{i=1}^{k} \alpha^i w^T[k-i] W_3 w[k-i]
\end{align*}
\]

(23)

On the other hand, considering (12), then

\[
V(x[0]) = x^T[0] W_1 x[0] + \sum_{i=-\tau}^{-1} x^T[i] W_2 x[i]
\]

(24)

Replacing (24) into (23), and considering (2), results in

\[
V(x[k]) \leq \alpha^k x^T[0] W_1 x[0] + \alpha^k \sum_{i=-\tau}^{-1} x^T[i] W_2 x[i] \\
& + \sum_{i=1}^{k} \alpha^i w^T[k-i] W_3 w[k-i] \\
& \leq \alpha^k \lambda_{\max}(W_1) x^T[0] x[0] \\
& + \alpha^k \lambda_{\max}(W_2) \sum_{i=-\tau}^{-1} x^T[i] x[i] \\
& \leq \alpha^k \lambda_{\max}(W_1) x^T[0] x[0] \\
& + \alpha^k \lambda_{\max}(W_2) \sum_{i=-\tau}^{-1} x^T[i] x[i]
\]

(25)

On the other hand

\[
V(x[k]) \geq x^T[k] W_1 x[k] \\
\geq \lambda_{\min}(W_1) x^T[k] x[k]
\]

(26)

By attention to (25) and (26), one can write

\[
\lambda_{\min}(W_1) x^T[k] x[k] \leq V(x[k]) \leq \alpha^k \lambda_{\max}(W_1) c_1^2 \\
& + \alpha^k \lambda_{\max}(W_2) c_1^2 \tau \\
& + \lambda_{\max}(W_3) \delta^2 \sum_{i=1}^{N} \alpha^i
\]

(27)

The above inequality is equivalent to

\[
\frac{\lambda_{\min}(W_1)}{\lambda_{\max}(W_1) c_1^2 (1 + \frac{2}{N})} \left( \frac{\lambda_{\max}(W_1) c_1^2 + \lambda_{\max}(W_2) c_1^2 \tau}{\lambda_{\min}(W_1) \alpha^{-N}} \right) \\
\leq \sum_{i=1}^{N} \alpha^i
\]

(28)

Let

\[
\mu_1 I \leq W_1 \leq \mu_2 I, \quad 0 < W_2 \leq \mu_3 I, \quad 0 < W_3 \leq \mu_4 I
\]

(29)

Now, from (28) and (29), one has

\[
\mu_2 c_1^2 + \mu_3 c_1^2 \tau + \mu_4 \delta^2 N < \mu_1 c_2^2 \alpha^{-N}
\]

(30)
Condition (9) obtained simply by (29) and the condition $Q = \varepsilon_1 I - 2W_1 > 0$. Therefore the FTB property for the closed-loop system (3) is achieved if $\Omega < 0$. The matrix $\Omega$ can be rewritten as follows:

$$\Omega = \begin{bmatrix} \tilde{A}^T W_1 \tilde{A} + 3\varepsilon_1 \gamma^2 I - \alpha W_1 + W_2 & \tilde{A}^T W_1 A_r & \tilde{A}^T W_1 G \\ A_r^T W_1 \tilde{A} & -W_2 + A_r^T W_1 A_r & A_r^T W_1 G \\ G^T W_1 \tilde{A} & G^T W_1 A_r & -\alpha W_3 \end{bmatrix} + \begin{bmatrix} \tilde{A}^T W_1 \\ 0 \\ 0 \end{bmatrix} Q^{-1} \begin{bmatrix} W_1 \tilde{A} & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ A_r^T W_1 \\ 0 \end{bmatrix} Q^{-1} \begin{bmatrix} 0 & A_r^T W_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ G^TW_1 \end{bmatrix} Q^{-1} \begin{bmatrix} 0 & 0 & G^TW_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ G^TW_1 \end{bmatrix} W_1^{-1} \begin{bmatrix} 0 & 0 & G^TW_1 \end{bmatrix}$$

(31)

According to above matrices inequality and applying Schur complement lemma (Song & He, 2015b), $\Omega < 0$ is equivalent with the following condition:

$$\begin{bmatrix} \varphi_{11} & \tilde{A}^T W_1 A_r & \tilde{A}^T W_1 G & 0 & 0 & 0 & \tilde{A}^T W_1 \\ A_r^T W_1 \tilde{A} & \varphi_{22} & A_r^T W_1 G & 0 & A_r^T W_1 & 0 & 0 \\ G^T W_1 \tilde{A} & G^T W_1 A_r & -\alpha W_3 & G^T W_1 & 0 & G^T W_1 & 0 \\ 0 & 0 & W_1 G & -W_1 & 0 & 0 & 0 \\ 0 & W_1 A_r & 0 & 0 & -Q & 0 & 0 \\ 0 & 0 & W_1 G & 0 & 0 & -Q & 0 \\ W_1 \tilde{A} & 0 & 0 & 0 & 0 & 0 & -Q \end{bmatrix} < 0$$

(32)

where

$$\varphi_{11} = \tilde{A}^T W_1 \tilde{A} + 3\varepsilon_1 \gamma^2 I - \alpha W_1 + W_2,$$

$$\varphi_{22} = -W_2 + A_r^T W_1 A_r.$$

Pre multiplying (32) by

$$\begin{bmatrix} 1 & 0 & 0 & -\tilde{A}^T & 0 & 0 & 0 \\ 0 & 1 & 0 & -A_r^T & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} > 0$$

(33)
and pos-multiplying it by the transpose of (33); results in:

$$\Omega_1 = \begin{bmatrix}
3\varepsilon_1 \gamma^2 l - \alpha W_1 + W_2 & 0 & 0 & \tilde{A}^T W_1 \\
0 & -W_2 & 0 & A_2^T W_1 \\
0 & 0 & -\alpha W_3 & G^T W_1 \\
W_1 \tilde{A} & W_1 A_1 & W_1 G & -W_1 \\
0 & W_1 A_1 & 0 & 0 \\
0 & 0 & W_1 G & 0 \\
W_1 \tilde{A} & 0 & 0 & 0
\end{bmatrix} < 0 \quad (34)$$

By means of variable $W_1 B K = L$ and replace $Q = \varepsilon_1 l - 2W_1$, LMI-based condition (8) is obtained which results in selecting the gain of output feedback as $K = (B^T B)^{-1} B^T W_1^{-1} L$ and the proof is completed.

### 4. Computer simulation

In this section an example is considered to show the efficiency of the proposed method. Consider system (1) with

$$A = \begin{bmatrix}
0.5 & 2 \\
-1 & -1
\end{bmatrix}, \quad B = \begin{bmatrix}
-1 \\
1
\end{bmatrix}, \quad G = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix},$$

$$A_1 = \begin{bmatrix}
-0.1 & 0.1 \\
-0.2 & -0.1
\end{bmatrix}, \quad C = \begin{bmatrix}
0.5 & 0.5
\end{bmatrix}$$

$$w[k] = \frac{1}{1 + k^2}, \quad \tau = 1, \quad \sigma(k_0) = \begin{bmatrix}
-0.4 & 0.4
\end{bmatrix}^T,$$

$$f(x[k]) = \begin{bmatrix}
0.3 \sin(e^{-x_2[k]}) \\
0.2 \cos(x_1[k]) + 0.15 \sin(e^{-x_1[k]})
\end{bmatrix} \quad (35)$$

The function $f(x[k])$ is Lipschitz with $\gamma = 0.41$. The goal is design of controller such that the closed-loop system be FTB with $(c_1, c_2, N, \delta) = (0.5, 2, 10, 0.5)$. Since $\alpha$ should be selected such that conditions (8)–(10) be feasible. A simple computer programme is written and $\alpha = 2.91$ is
selected. By solving the LMI conditions trough LMI toolbox of MATLAB, the following results are achieved:

\[
W_1 = \begin{bmatrix} 0.6577 & -0.1099 \\ -0.1099 & 1.2136 \end{bmatrix}, \\
W_2 = \begin{bmatrix} 221.6534 & 102.1203 \\ 102.1203 & 264.9688 \end{bmatrix}, \\
W_3 = \begin{bmatrix} 1.5704 & -0.1027 \\ -0.1027 & 1.5908 \end{bmatrix}, \\
L = \begin{bmatrix} -1.2876 \\ 2.6053 \end{bmatrix}^T, \quad K = 1.8117
\]

(36)

Time-history of norm of state vector is shown in Figure 1 for open-loop and closed-loop system. As seen the closed-loop system has FTB property with \((c_1, c_2, N, \delta) = (0.5, 2, 10, 0.5)\) while the open-loop system has not this property. Also, the time-responses of the state variables of the closed-loop system are shown in Figure 2. The time-response of the control signal is presented in Figure 3.

5. Conclusion
This paper presented the finite-time output feedback controller design for nonlinear time-delay discrete-time systems. Using appropriate Lyapunov-Krasovskii functional, the sufficient conditions were derived which guarantee the FTB characteristic of the closed-loop system via output feedback. The obtained conditions were expressed through the feasibility testing of the derived LMIs. At end, a numerical example was given to show effective performance of the proposed technique.

Disclosure statement
No potential conflict of interest was reported by the authors.

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