INJECTIVE RESOLUTIONS OF $BG$ AND DERIVED MODULI SPACES OF LOCAL SYSTEMS

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Let $X$ be a finite connected CW-complex, $x_0 \in X$ be a point, and $G$ be an affine algebraic group over $\mathbb{C}$. A $G$-local system on $X$ is just a locally constant sheaf of $G$-torsors. Let $\mathcal{L}S_G(X, x_0)$ denote the set of isomorphism classes of $G$-local systems trivialized over $x_0$. For such a local system $E$ let $[E]$ denote the corresponding point of $\mathcal{L}S_G(X, x_0)$. This set is naturally an algebraic variety; it is just the variety of all homomorphisms $\pi_1(X, x_0) \to G$. It is acted upon by $G$, and the quotient $\mathcal{L}S_G(X) = \mathcal{L}S_G(X, x_0)/G$ is the set of isomorphism classes of local systems. Since the $G$-action may not be free, $\mathcal{L}S_G(X)$ may not exist as an algebraic variety, but is well defined as an algebraic stack. The first order deformation theory gives an identification $T_{[E]} \mathcal{L}S_G(X) = H^1(X, \text{ad}(E))$ for any local system. The stack $\mathcal{L}S_G(X)$ and the variety $\mathcal{L}S_G(X, x_0)$ may be not smooth: the jumping of the dimension of the tangent space is made possible by the corresponding jumping of the dimensions of the higher cohomology of $\text{ad}(E)$.

This situation is typical for many other problems of deformation theory (e.g., the moduli stack of algebraic vector bundles on a variety $M$ is smooth when $M$ is a curve, but not when $\dim(M) > 1$). It has been proposed by several people (among them Deligne, Drinfeld and Kontsevich, see [K]) that one could overcome this difficulty by systematically working in the derived category, i.e., constructing a kind of non-Abelian derived functor of the moduli space functor. The appropriate language for such derived functors (over a field of characteristic 0) is that of dg-schemes, i.e., schemes together with a sheaf of (negatively graded) differential graded algebras, cf. [Q], [Mun]. From this point of view the reason that the moduli space is singular is that we disregard the higher cohomology by artificially truncating the “true” derived moduli space, which should be the right object to consider in all geometric studies.

It is not, however, exactly straightforward to construct such derived moduli spaces, and the purpose of the present note is to do so in the simplest case, that of local systems. Our construction is based on the observation that the moduli space space can be represented as $[N, BG]$, the set of simplicial homotopy classes of simplicial maps from a Čech nerve $N$ of $X$ to the simplicial classifying space of $G$. To extend this into derived category, we construct an appropriate “injective” resolution $RBG$ of $BG$. It may seem surprising that the space such as $BG$ needs an additional injective resolution, since, regarded as a simplicial set, it is fibrant. The reason is that we are interested in a geometric and not topological concept of fibrations for (dg)-schemes.

To keep the paper short, we avoided going into foundational matters related to dg-
schemes; for instance, the derived moduli spaces should really be “dg-stacks”, but we consider the rigidified situation when the stack structure is not necessary. We also did not consider in any detail the (rudiments of a) closed model structure on the category of dg-schemes with the role of fibrations played by smooth maps etc.

I am grateful to M. Kontsevich who convinced me that the dg-point of view is more flexible than the more exotic approaches to circumventing the singularity of moduli spaces I was experimenting with. This research was supported by an NSF grant and an A.P.Sloan Fellowship.

§1. Injective resolutions of $BG$.

(1.1) Dg-schemes. We work everywhere over the field $\mathbb{C}$ of complex numbers. By a complex (or dg-vector space) we mean a cochain complex, i.e., a graded vector space with a differential of degree $+1$. By a dg-algebra we always mean a commutative $\mathbb{Z}_{\leq 0}$-graded differential algebra $A$. Note that in such an algebra the differential $d$ is $A^0$-linear, so its cohomology forms a graded $A^0$-module $H^\bullet(A)$. A quasi-isomorphism of dg-algebras is a morphism inducing an isomorphism in the cohomology.

It is also convenient to use the dual geometric language and to speak about dg-schemes. By definition, a dg-scheme $M$ is an ordinary scheme $M^0$ equipped with a quasi-coherent sheaf $\mathcal{O}_M^\bullet$ of dg-algebras on $M^0$. Thus affine dg-schemes (those with $M^0$ affine) are in anti-equivalence with dg-algebras. For a dg-scheme $M$ we have the scheme $\pi_0(M) = \text{Spec}(H^0(\mathcal{O}_M))$ which is a closed subscheme in $M^0$. It is possible to view dg-schemes as superschemes in the sense of [Man] equipped with additional structure as follows.

(1.1.1) Proposition. Let $Y$ be a scheme. Then the category of dg-schemes $M$ with $M^0 = Y$ is equivalent to the category of superschemes $\tilde{M} \to Y$ affine over $Y$ and equipped with the following additional structures:

1. A section $i : S \to \tilde{M}$:
2. An (algebraic) action of the multiplicative semigroup $(\mathbb{C}, \times)$ on $\tilde{M}$ whose fixed point subscheme is $i(Y)$;
3. An odd vector field $d$ on $\tilde{M}$ satisfying $\{d, d\} = 0$ and having degree 1, i.e., such that $[d, L] = L$ where $L$ is the vector field generating the action of $G_m \subset \mathbb{C}$.

The proof is obvious and left to the reader: the action of $G_m$ gives a $\mathbb{Z}$-grading, the fact that it is situated in degrees $\leq 0$ is encoded by saying that the action extends to an action of $\mathbb{C}$ etc.

Because of this proposition we can easily reduce several foundational questions re-
garding dg-schemes (e.g., the properties of the sheaves of differentials and derivations) to those about superschemes, which have been treated in [Man].

(1.2) Tangent spaces. We will say that a dg-scheme $M$ is smooth (or is a dg-manifold), if $M^0$ is a smooth manifold, and locally on the Zariski topology of $M^0$ the sheaf $\mathcal{O}_M$ is free as a sheaf of graded commutative algebras, with finitely many generators in each degree.

Given a dg-manifold $M$ and a $\mathbb{C}$-point $x \in \pi_0(M)$, we have the tangent dg-space (complex) $T^\bullet_x M$. It is defined, as usual, as the graded vector space of $\mathbb{C}$-valued derivations. This is a complex of vector spaces concentrated in degrees $\geq 0$.

Any morphism $f : M \rightarrow N$ of dg-manifolds gives rise to a morphism of complexes $d_x f : T^\bullet_x M \rightarrow T^\bullet_{f(x)} N$. It is suggestive to use the topological notation for the cohomology of the tangent complex:

(1.2.1) $\pi_{-i}(M,x) := H^i(T^\bullet_x M), \quad i \geq 0.$

This notation is justified by the following fact.

(1.2.2) Proposition. For any smooth dg-scheme $M$ and any its $\mathbb{C}$-point $x$ there are natural bilinear maps (“Whitehead products”)

$$\pi_i(M,x) \otimes \pi_j(M,x) \rightarrow \pi_{i+j-1}(M,x)$$

which makes $\pi_{*+1}(M,x)$ into a graded Lie algebra. For any morphism $f : M \rightarrow N$ of dg-manifolds the induced morphism $\pi_\bullet(M,x) \rightarrow \pi_\bullet(N,f(x))$ is a homomorphism of graded Lie algebras.

Proof: Recall the concept of a weak Lie algebra (or a shLie algebra [St]). This is a graded vector space $g$ together with a continuous differential $D$ in $\hat{S}(g^*[-1])$, the completed symmetric algebra of the shifted dual vector space. By restricting $D$ to the degree 1 part, namely $g^*$ and dualizing the graded components of this restriction, one gets antisymmetric $n$-linear brackets of degree $2 - n$

$$\lambda_n : g^* \otimes^n \rightarrow g, \quad x_1 \otimes \ldots \otimes x_n \mapsto [x_1, \ldots, x_n]_n, \quad n \geq 1,$$

In particular, $d = \lambda_1$ is a differential in $g$, while $\lambda_2$ satisfies the jacobi identity up to $d$-boundaries, so that $H^*_d(g)$ is a graded Lie algebra.

Now, $M$ being a dg-manifold, the completion $\hat{O}^\bullet_{M,x}$ is isomorphic, as a graded algebra, to $\hat{S}(W^*)$, the completion of a free graded algebra generated by some graded vector space $W^*$. Let $V^* = W^*$ be the dual graded space. Such an isomorphism $\phi$ is just a formal coordinate system in $M$ near $x$. Given such $\phi$, its differential identifies $V^*$ with $T^\bullet_x M$. So $\phi$ identifies the graded algebra $\hat{S}(T^\bullet_x M)$ with the dg-algebra $\hat{O}^\bullet_{M,x}$, and thus we get by pullback a differential $D$ on $\hat{S}(T^\bullet_x M)$ which of course satisfies $D^2 = 0$. This means that $T^\bullet_x M[-1]$ becomes equipped with the structure of a weak Lie algebra, so its cohomology is a graded Lie algebra. If we choose a different isomorphism $\phi' : \hat{S}(V^*) \rightarrow \hat{O}^\bullet_{M,x}$ but with the same differential at 0, then we get what is is known as a weakly isomorphic weak Lie algebra, so that the Lie algebra structure on the cohomology will be the same.

The following fact can be seen as an analog of the Whitehead theorem in topology.
(1.2.3) Proposition. Let \( f : M \to N \) be a morphism of dg-manifolds. Then the following conditions are equivalent:

(a) \( f \) is a quasiisomorphism.

(b) The morphism of schemes \( \pi_0(f) ; \pi_0(M) \to \pi_0(N) \) is an isomorphism, and for any \( \mathbb{C} \)-point \( x \) of \( M \) the differential \( d_x f \) induces an isomorphism \( \pi_i(M, x) \to \pi_i(N, f(x)) \) for all \( i \leq 0 \).

Proof: It is enough to prove that for any \( x \in \pi_0(M) \) the map \( f \) induces the completed local dg-algebras \( \hat{f}^* \mathcal{O}_{N,f(x)} \to \hat{\mathcal{O}}_{M,x} \), which is a quasiisomorphism. For that, notice that \( \mathcal{O}_{M,x} \) has a filtration whose quotients are the symmetric powers of the cotangent dg-space \( T^*_x M \). So if \( f \) gives a quasiisomorphism of tangent dg-spaces, we find that \( \hat{f}^* \) induces quasiisomorphisms on the quotients of the natural filtrations. So the proof is accomplished by invoking a spectral sequence argument, which is legitimate (i.e., the spectral sequences converge) because the dg-algebras in question are \( \mathbb{Z}_{\leq 0} \)-graded.

(1.3) Twisted tensor products and fibrations. A dg-algebra \( C \) is called a twisted tensor product of dg-algebras \( A \) and \( B \), if \( C \simeq A \otimes B \) as a graded algebra, and with respect to this identification,

\[
d_C = d_A \otimes 1 + 1 \otimes d_B + \sum_{i \geq 2} d_i, \quad \deg(d_i) = (1 - i, i).
\]

In this case the natural embedding

\[
A \hookrightarrow C, \quad a \mapsto a \otimes 1,
\]

is a morphism of dg-algebras. See [May]. Note, in particular, the concept of a quasi-free dg-algebra \( C \) over \( A \). This just means that \( C \) is a twisted tensor product of \( A \) and a free dg-algebra.

(1.3.3) Definition. Let \( p : M \to N \) be a morphism of affine dg-schemes, \( M = \text{Spec}(C), N = \text{Spec}(A) \). We will say that \( p \) is a fibration with fiber \( F = \text{Spec}(B) \), if \( C \) is isomorphic to a twisted tensor product of \( A \) and \( B \) in such a way that the homomorphism \( p^* : A \to C \) becomes the canonical embedding (1.3.2).

It is clear that for a fibration we have a spectral sequence

\[
E_2 = H^\bullet(A) \otimes H^\bullet(B) \Rightarrow H^\bullet(C).
\]

(1.4) Resolutions. Let \( A \to R \) be any morphism of dg-algebras. Then it is standard, see, e.g., [Mun] how to replace \( R \) by a quasi-isomorphic dg-algebra \( \mathcal{R} \) which is quasi-free over \( A \). We construct \( \mathcal{R} \) as the union of an increasing sequence \( \mathcal{R}_n, n \geq 0 \) of sub-dg-algebras. We first take a subspace of algebra generators in \( H^0(R) \), and lift this space to a subspace \( V^0 \subset R^0 \). We define \( \mathcal{R}_0 \) to be the free \( A \)-algebra generated by \( V^0 \) (where the
differential is set to vanish on $V^0$). Thus we have a morphism of dg-algebras $d_0 : \mathcal{R}_0 \to R$ surjective on $H^0$. Then we take a space of generators of the ideal $\text{Ker}(d_0)$ and denote by $V^{-1}$ the vector space freely spanned by these generators. Then we have a natural morphism $d_{-1} : V^{-1} \to \mathcal{R}_0$ which gives rise to a dg-algebra structure on the free $\mathcal{R}_0$-algebra generated by $V^{-1}$. Denote the dg-algebra thus obtained by $\mathcal{R}_1$. By construction, $H^0(\mathcal{R}_1) = R$. As the next step, we take a space $V^{-2}$ of generators of the $H^0(\mathcal{R}_1)$-module $H^{-1}(\mathcal{R}_1)$, lift $V^{-2}$ to a subspace of cocycles and define in this way a morphism of graded vector spaces $d_{-2} : V^{-2} \to \mathcal{R}_1$. This gives a quasi-free dg-algebra $\mathcal{R}_2$. Continuing in this way, we inductively construct $\mathcal{R}_n$ so as to kill the $(-n+1)$st cohomology of $\text{Ker}(\mathcal{R}_{n-1} \to R$ and not to affect the $j$th cohomology, $j > -n + 1$.

Let us summarize the well known properties of this construction.

(1.4.1) Proposition. (a) Any two quasi-free resolutions constructed in this way, are quasiisomorphic.
(b) If $A, R$ have only finitely many generators in each degree, then we can choose $\mathcal{R}$ with the same property.
(c) If, in the situation of (b), $H$ is a reductive algebraic group acting on $A, R$ so that the morphism $A \to R$ is equivariant, then it is possible to choose $\mathcal{R}$ so as to possess a $G$-action compatible with the maps and to have finitely many generators in each degree.

To see part (c), just notice that it is possible to take the spaces of generators on each step of construction to be $H$-invariant and finite-dimensional.

(1.5) Simplicial objects and classifying spaces. We will be using simplicial objects in the categories of sets, schemes and dg-schemes. See [F] [May] for general background. By $\Delta[n]$ we denote the standard $n$-simplex regarded as a simplicial set. If $I$ is any set (scheme), then the collection of Cartesian powers $(I^n)_{n \geq 0}$ forms a simplicial set (scheme) which we denote $\Delta(I)$ and call the unoriented $I$-simplex.

Let $C$ be any category. Its nerve (or classifying space) $B_\bullet C$ is the simplicial set whose $n$-simplices are “commutative $n$-simplices” in $C$, i.e., diagrams consisting of $n + 1$ objects $A_0, ..., A_n$ and morphisms $g_{ij} : A_i \to A_j$ satisfying the conditions

\[(1.5.1) \quad g_{jk}g_{ij} = g_{ik}, \quad i < j < k.\]

Let also $\tilde{B}_\bullet C$ be the set consisting of all, not necessarily commutative, simplex-shaped diagrams in $C$. In other words, an element of $\tilde{B}_n C$ is an arbitrary collection of objects $A_0, ..., A_n$ and morphisms $g_{ij} : A_i \to A_j$. It is clear that these sets unite into a simplicial set $\tilde{B}_\bullet C$ containing $B_\bullet C$.

A group $G$ over $C$ can be considered as a category with one object and the set of automorphisms $G$, so $B_\bullet G$ is defined. If $G$ is an affine algebraic group over $C$, then $B_\bullet G$ and $\tilde{B}_\bullet G$ are simplicial schemes. Moreover, $B_\bullet G \to \tilde{B}_\bullet G$ is a closed embedding, given by the equations (1.5.1). In other words, for every $n$ we have a surjection of algebras of
functions

\[(1.5.2) \quad C[\tilde{B}_nG] = \bigotimes_{0 \leq i < j \leq n} C[G] \rightarrow C[B_nG].\]

Notice also that the group \(G^{n+1}\) acts on \(\tilde{B}_nG\) by “gauge transformations”:

\[(1.5.2) \quad (g_0, \ldots, g_n) : (g_{ij}) \mapsto (g_j g_{ij} g_i^{-1}),\]

and this action preserved \(B_nG\). The simplicial scheme \(\Delta(G)\) formed by the \(G^n\) is actually a simplicial algebraic group, acting on the simplicial scheme \(\tilde{B}G\) and preserving \(BG\).

**1.6 Injective simplicial dg-schemes.** A simplicial dg-scheme \(X_\bullet\) is called affine if each \(X_i\) is affine. Such a scheme is the same as a cosimplicial dg-algebra.

If \(S\) is a simplicial set and \(X\) is a simplicial dg-scheme, then we have a dg-scheme \(\text{Hom}(S, X)\), which is affine when \(X\) is affine.

**1.6.1 Definition.** An affine dg-scheme \(X_\bullet\) is called injective, if for any cofibration (i.e., embedding) \(S' \subset S\) of simplicial sets the induced map of dg-schemes \(\text{Hom}(S, X) \rightarrow \text{Hom}(S', X)\) is smooth and a fibration with fiber \(\text{Hom}(S/S', X)\).

So this definition is a direct analog of the concept of an injective object in an Abelian category.

**1.6.2 Example.** The simplicial scheme \(\tilde{B}_\bullet G\) (with trivial dg-structure) is injective, but its simplicial subscheme \(B_\bullet G\) is not.

More generally, we have the following fact, whose proof is achieved by using induction over the simplices.

**1.6.3 Proposition.** A simplicial dg-scheme \(X\) is injective if and only if \(X_0\) is smooth and for any \(n \geq 1\) the morphism

\[X_n = \text{Hom}(\Delta[n], X) \rightarrow \text{Hom}(\partial \Delta[n], X)\]

is a fibration with smooth fiber.

**1.7 Theorem.** Let \(G\) be any reductive group. One can replace \(BG\) by a quasiisomorphic (dimension by dimension) injective simplicial dg-manifold \(R\)\(BG\) with an action of the simplicial group \(\Delta(G)\). This replacement is canonical up to an equivariant quasiiusomorphism of simplicial dg-manifolds.

Proof: We set \(RB_mG = B_mG\) for \(m = 0, 1\), then take for \(C[RB_2G]\) any free dg-resolution of \(C[B_2G]\) as an algebra over \(C[B_2G]\), and then continue inductively as follows. Suppose we already constructed dg-schemes \(RB_nG, n \leq m\) and face morphisms \(\partial_i\) satisfying the simplicial identities (i.e., suppose we constructed the \(m\)-th skeleton \(RB_{\leq m}G\)). Then the dg-scheme \(\text{Hom}(\partial \Delta[m+1], RB_{\leq m}G)\) is defined (because \(\partial \Delta[m+1]\) is \(m\)-dimensional). Let
A(m+1) be its dg-algebra of functions. We have a natural dg-algebra morphism $A(m+1) \to C[B_{m+1}G]$ (the latter algebra has, of course, trivial dg-structure). Further, there is a natural action of $G^{m+2}$ on $A(m+1)$ so that the morphism becomes equivariant. Define $C[RB_{m+1}G]$ to be an equivariant quasi-free dg-resolution of $C[B_{m+1}G]$ as an $A(m+1)$-algebra. Continuing in this way, we get a required dg-resolution of the entire $BG$.

(1.8) Explicit resolution for $G = GL(r)$. In the case $G = GL(r)$ it is possible to write down a canonical injective resolution by analyzing the syzygies among the equations (1.5.1).

We set $G = GL(r)$. The scheme $\tilde{B}_n G$ is just the product $\prod_{0 \leq i < j \leq n} G$ of $n(n+1)/2$ copies of $G$. We will denote the $(i,j)$th copy by $G_{ij}$. The ring $C[G]$ of functions on $G$ is generated by the matrix elements of one matrix-valued function $g$ which is required to satisfy $\det(g) \neq 0$. The ring of functions on $\tilde{B}_n G$ i.e., $\bigotimes_{0 \leq i < j \leq n} C[G_{ij}]$ is therefore generated by the matrix elements of $n(n+1)/2$ matrix-valued functions which we denote by $g_{ij}$, $0 \leq i < j \leq n$. The scheme $B_n G$ is described inside $\tilde{B}_n G$ by $\binom{n+1}{3}$ equations (1.5.1). We now define a dg-algebra $C[RB_n G]$ over $C[\tilde{B}_n G]$ to be generated by the matrix elements of the $(r \times r)$-matrix functions $g_{i_0 \ldots i_p}$, $0 \leq i_0 < \ldots < i_p \leq n$, with the degree of (each matrix element of) $g_{i_0 \ldots i_p}$ equal to $1 - p$, and the differential $d$ (of degree +1) defined on the generators by

\[
d(g_{i_0 \ldots i_p}) = \sum_{\nu=1}^{p-1} (-1)^\nu (g_{i_0 \ldots \hat{i_\nu} \ldots i_p} - g_{i_\nu \ldots i_\nu i_0 \ldots i_\nu}).
\]

Note in particular that for $p = 2$ we get

\[
d(g_{ijk}) = g_{jki} g_{ij} - g_{ijk},
\]

so the image of the last differential is the ideal in $C[\tilde{B}_n G]$ generated by the equations (1.5.1).

One verifies immediately that the condition $d^2 = 0$ is satisfied on the generators (and hence on the entire algebra). So $C[RB_n G]$ is a free dg-algebra over $C[\tilde{B}_n G]$. We denote by $RB_n G$ the affine dg-scheme whose ring of functions is $C[RB_n G]$.

(1.8.3) Theorem. (a) The dg-algebra $C[RB_n G]$ is quasiisomorphic to $C[B_n G]$, i.e., it provides a free resolution of the equations (1.5.1).

(b) The dg-schemes $RB_n G$, $n \geq 0$, arrange into a simplicial dg-scheme $RB_\bullet G$, containing $B_\bullet G$ as a closed simplicial subscheme (with trivial dg-structure).

(c) The simplicial dg-scheme $RB G$ is injective.

Proof: Part (b) is obvious, part (a) will be proved in the next subsection.

(1.9) $RB_n GL(r)$ and simplicial connections. It is easy to understand the meaning of the algebra $C[RB_n G]$. For any group $G$, points of the space $\tilde{B}_n G = \prod_{0 \leq i < j \leq n} G$ can be viewed as simplicial $G$-connections on the $n$-simplex $\Delta[n]$, while points of the subscheme
$B_n G$ can be viewed as flat connections. When $G = GL(r)$ (which assumption we will keep), this analogy can be made more precise as follows.

For any associative algebra $R$ let $C^\bullet(\Delta[n], R)$ be the (normalized) simplicial cochain complex of $\Delta[n]$ with coefficients in $R$. Non-degenerate $p$-faces of $\Delta[n]$ are labelled by sequences $(i_0, ..., i_p)$, $0 \leq i_0 < ... < i_p \leq n$, and thus an element of $C^p(\Delta[n], R)$ is a function $\phi$ associating to any such sequence an element $\phi(i_0, ..., i_p) \in R$. The Alexander-Whitney multiplication

$$ (\phi \cdot \psi)(i_0, ..., i_{p+q}) = \phi(i_q, ..., i_{p+q})\psi(i_0, ..., i_q), \quad \phi \in C^p(\Delta[n], R), \psi \in C^q(\Delta[n], R), $$

makes $C^\bullet(\Delta[n], R)$ into an associative dg-algebra.

Now let $\text{gl}(r)$ be the associative algebra of $r$ by $r$ matrices. A point $g \in \tilde{B}_n G$ is nothing but an element of $C^1(\Delta[n], \text{gl}(r))$ whose components are all invertible. The condition for $g$ to lie in the subscheme $B_n G$ can be expressed as

$$ dg + \gamma \cdot g = 0 \text{ in } C^2(\Delta[n], \text{gl}(r)). $$

Further, let $A^\bullet$ be any dg-algebra. An element $\gamma \in C^\bullet(\Delta[n], \text{gl}(r)) \otimes A^\bullet$ of degree 1 can be split into its components $\gamma_p \in C^p(\Delta[n], \text{gl}(r)) \otimes A^{1-p}$. Each $\gamma_p$ can be viewed as a collection of matrices $\gamma_{i_0, ..., i_p}$ whose matrix elements belong to $A^{1-p}$. By comparing the formula for the Alexander-Whitney map with the definition of the differential in the algebra $C[RB_n G]$ we get the following characterization of the latter.

**(1.9.3) Proposition.** Let $A^\bullet$ be any dg-algebra. A dg-homomorphism $C[RB_n G] \to A^\bullet$ is the same as a degree 1 element $\gamma \in C^\bullet(\Delta[n], \text{gl}(r)) \otimes A^\bullet$ satisfying $d\gamma + \gamma \cdot \gamma = 0$ and such that all the matrices $\gamma_{i_0, i_1}$ with entries in $A^0$ are invertible.

**(1.10) Proof of Theorem 1.8.3 (a).** By Proposition 1.2.3, it is enough to prove the following fact.

**(1.10.1) Lemma.** Let $g \in B_n G \subset RB_n G$. Then the tangent dg-space $T_g RB_n G$ has no cohomology in degrees other than 0.

To see the lemma, let us regard $g$ as a simplicial local system $V$ on $\Delta[n]$. Then, we identify the complex $T_g RB_n G$ (concentrated in non-negative degrees) with the shifted and truncated cochain complex

$$ C^1(\Delta[n], \text{End}(V)) \to C^2(\Delta[n], \text{End}(V)) \to ... $$

which is clearly exact outside the leftmost term.
2. The derived space of local systems.

(2.1) Ordinary moduli space. Let $S$ be a connected simplicial set and $G$ be a reductive algebraic group, as before. Consider the scheme $\underline{\text{Hom}}(S, BG)$. It is acted upon by the group $\prod_{x \in S_0} G$. Two points $f, g \in \underline{\text{Hom}}(S, BG)$ are equivalent with respect to this action if and only if the morphisms $f, g$ are elementary homotopic, i.e., can be obtained as restrictions of one morphism $F : S \times \Delta[1] \to BG$. A morphism $f : S \to BG$ is just a rule associating to any edge $\gamma \in S_1$ an element of $G$ so that for any 2-simplex $\sigma \in S_2$ we have the condition $g_{\partial_0 \sigma} g_{\partial_2 \sigma} = g_{\partial_1 \sigma}$. So, geometrically, it can be viewed as a flat simplicial $G$-connection on $S$ and the elements of $\prod_{x \in S_0} G$ are discrete analogs of gauge transformations.

Let $x_0 \in S_0$ be a vertex. Define

\[(2.1.1) \quad \mathcal{L}S(S, x_0) := \frac{\underline{\text{Hom}}(S, BG)}{\prod_{x \in S, x \neq x_0} G}.
\]

Notice that the action of the subgroup $\prod_{x \neq x_0} G$ is free, so the quotient exists as an algebraic variety. It is clear that

\[(2.1.2) \quad \mathcal{L}S(S, x_0) = \underline{\text{Hom}}(\pi_1(|S|, x_0), G),
\]

the moduli space of all homomorphisms from the fundamental group to $G$. This variety can be singular.

(2.2) Derived moduli space. We keep the notations and assumptions of the previous subsection. Choose a $\Delta(G)$-equivariant injective dg-resolution $RBG$ of $BG$, and consider the dg-scheme $\underline{\text{Hom}}(S, RBG)$. Because of the injectivity, this is a smooth dg-manifold. The group $\prod_{x \in S_0} G$ acts on this manifold.

(2.2.1) Proposition. The action of the subgroup $\prod_{x \neq x_0} G$ on $\underline{\text{Hom}}(S, RBG)$ is free.

Proof: The ordinary simplicial scheme underlying $RBG$ (i.e., the simplicial scheme formed by the spectra of the 0th graded components of the dg-algebras $C[R^b_nG]$ is just $\tilde{BG}$. So the ordinary scheme underlying $\underline{\text{Hom}}(S, RBG)$ is $\underline{\text{Hom}}(S, \tilde{BG})$, which is just the product of as many copies of $G$ as there are nondegenerate 1-simplices in $S$. In other words, it the space of all simplicial $G$-connections on $S$, flat or not. The action of $\prod_{x \in S_0, x \neq x_0} G$ on this space is clearly free. The inclusion of the 0th component into any $Z_{\leq 0}$-graded dg-algebra is a morphism of dg-algebras. This means that we have a morphism (in fact, a fibration) of dg-schemes $\underline{\text{Hom}}(S, RBG) \to \underline{\text{Hom}}(S, \tilde{BG})$. This morphism is equivariant and the action on the target is free. So the action on the source is free.

(2.2.2) Definition. The derived moduli space of local systems is defined as

\[R\mathcal{L}S(S, x_0) = \underline{\text{Hom}}(S, RBG) \bigg/ \prod_{x \in S_0, x \neq x_0} G.
\]
Clearly, this moduli space is well defined up to a quasiisomorphism. Further, a morphism \( f : (S, x_0) \to (T, y_0) \) of pointed simplicial sets induces a morphism \( f^* : \mathcal{R}\mathcal{L}\mathcal{S}_G (T, y_0) \to \mathcal{R}\mathcal{L}\mathcal{S}(S, x_0) \) of dg-manifolds.

(2.3) **Theorem.** (a) \( \mathcal{R}\mathcal{L}\mathcal{S}(S, x_0) \) is a smooth dg-manifold, with

\[
\pi_0(\mathcal{R}\mathcal{L}\mathcal{S}(X, x_0)) = \mathcal{L}\mathcal{S}(S, x_0).
\]

(b) For a \( \mathbf{C} \)-point \([E]\) of \( \mathcal{R}\mathcal{L}\mathcal{S}_G (S, x_0) \) represented by a \( G \)-local system \( E \) on \( S \), the cohomology of the tangent dg-space is found as follows:

\[
H^i(T^\bullet_{[\nabla]} \mathcal{R}\mathcal{L}\mathcal{S}(S, x_0)) = \begin{cases} 
H^{i+1}(S, \text{ad}(E)), & i \geq 1 \\
H^1_{\text{res}}(S, \text{ad}(E)), & i = 0
\end{cases}
\]

where \( H^1_{\text{res}}(S, \text{ad}(E)) = Z^1/dC^0_{\text{res}} \) with \( Z^1 \) being the space of 1-cocycles and \( C^0_{\text{res}} \) being the space of 0-cochains whose values at \( x_0 \) is zero.

**Proof:** (a) The smoothness follows from the smoothness of \( \underline{\text{Hom}}(S, \mathbf{R}BG) \) and the freeness of the action. The statement about \( \pi_0 \) follows because \( \pi_0 \) commutes with \( \underline{\text{Hom}}(S, -) \) (exactness of the cokernel with respect to colimits).

(b) Let \( \nabla \) be a morphism \( S \to BG \), i.e., a flat simplicial connection on \( S \), and \( E \) be the local system represented by \( \nabla \). We will prove that

\[
H^i(T^\bullet_{[\nabla]} \underline{\text{Hom}}(S, \mathbf{R}BG)) = \begin{cases} 
H^{i+1}(S, \text{ad}(E)), & i \geq 1 \\
Z^1(S, \text{ad}(E)), & i = 0
\end{cases}
\]

This will imply our statement since \( \mathcal{R}\mathcal{L}\mathcal{S}(S, x_0) \) is the quotient of \( \underline{\text{Hom}}(S, \mathbf{R}BG) \) by the group \( \prod_{x \neq x_0} G \).

To prove (2.3.1), we first consider the case when \( S = \Delta[n] \) is the \( n \)-simplex. Then, \( \underline{\text{Hom}}(\Delta[n], \mathbf{R}BG) \) is just \( \mathbf{R}B_n G \) which is a resolution of \( B_n G = \underline{\text{Hom}}(\Delta[n], BG) \). So the tangent dg-space at \([\nabla]\) to \( \underline{\text{Hom}}(\Delta[n], \mathbf{R}BG) \) is quasiisomorphic to the ordinary tangent space at \([\nabla]\) to \( \underline{\text{Hom}}(\Delta[n], BG) \) situated in degree 0.

Now, consider the case of general \( S \) and represent \( S \) as the union (direct limit) of its simplices and realize accordingly the dg-scheme \( \underline{\text{Hom}}(S, \mathbf{R}BG) \) as an inverse limit:

\[
S = \text{colim}_{s \in S, n \geq 0} \Delta[n], \quad \underline{\text{Hom}}(S, \mathbf{R}BG) = \text{lim}_{s \in S, n \geq 0} \underline{\text{Hom}}(\Delta[n], \mathbf{R}BG).
\]

On the level of tangent dg-spaces this implies:

\[
T^\bullet_{[\nabla]} \underline{\text{Hom}}(S, \mathbf{R}BG) = \text{lim}_{s \in S, n \geq 0} T^\bullet_{[\nabla]} \underline{\text{Hom}}(\Delta[n], \mathbf{R}BG).
\]

But because of the injectivity of \( \mathbf{R}BG \) all the maps in the diagram whose inverse limit is (2.3.3), are surjective morphisms of complexes. Therefore the limit is quasiisomorphic to the homotopy inverse limit:

\[
T^\bullet_{[\nabla]} \underline{\text{Hom}}(S, \mathbf{R}BG) \sim \text{holim}_{s \in S, n \geq 0} T^\bullet_{[\nabla]} \underline{\text{Hom}}(\Delta[n], \mathbf{R}BG) \sim \text{holim}_{s \in S, n \geq 0} Z^1(\Delta[n], \text{ad}(E)).
\]
But the last homotopy inverse limit, if we calculate it via the nerve, will have exactly the cohomology described in (2.3.1). Theorem is proved.

**2.4 Corollary.** A weak equivalence \((S, x_0) \to (T, y_0)\) of pointed simplicial sets induces a quasi-isomorphism of dg-manifolds \(R\mathcal{L}S(T, y_0) \to R\mathcal{L}S(S, x_0)\).

*Proof:* This follows from Theorem 2.3 and the “Whitehead theorem” 1.2.3.

**2.5 The case of simple local systems.** Suppose that \(E\) is a \(G\)-local system on \(S\) such that \(\text{Aut}(E) = \{1\}\). If \(\nabla \in \text{Hom}(S, BG)\) is any simplicial connection representing \(E\), then the action of the full group \(\prod_{x \in S_0} G\) is free on the neighborhood of \([\nabla]\) in \(\text{Hom}(S, BG)\). The corresponding formal germ of the quotient is denoted by \(\text{Def}(E)\) and is called the formal deformation space of \(E\). This is a formal scheme with one closed point \([E]\). In this case the action on a neighborhood of \([\nabla]\) in \(\text{Hom}(S, RBG)\) is also free and we get a smooth dg-thickening \(R\text{Def}(E)\) which is a formal dg-scheme with \(\pi_0 R\text{Def}(E) = \text{Def}(E)\). By factorizing the equality (2.3.1) by \(\prod_{x \in S_0} G\) we get the following fact.

**2.5.1 Proposition.** We have \(H^i T[E] \text{RDef}(E)) = H^{i+1}(S, \text{ad}(E))\) for all \(i \geq 0\). Thus the dg-algebra structure on the ring \(C[R\text{Def}(E)]\) makes \(T[E] \text{RDef}(E)[-1] \sim R\Gamma(S, \text{ad}(E))\) into a weak Lie algebra.

This result provides a “derived” generalization of the main theorem of Hinich and Schechtman [H] [HS] [Sch] (for the case of local systems). Note that the ring of functions on \(R\text{Def}\) serves as the cochain complex of the weak Lie algebra structure on \(R\Gamma(S, \text{ad}(E))\), so, in particular, the 0th cohomology of this weak Lie algebra is the algebra of functions on the ordinary formal moduli space.

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