Estimates of transition densities and their derivatives for jump \L evy processes

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Abstract

We give upper and lower estimates of densities of convolution semigroups of probability measures under explicit assumptions on the corresponding \L evy measure and the \L evy–Khinchin exponent. We obtain also estimates of derivatives of densities.

1 Introduction

Let $d \in \{1, 2, \ldots \}$, $b \in \mathbb{R}^d$, and $\nu$ be a \L evy measure on $\mathbb{R}^d$, i.e.,

$$\int_{\mathbb{R}^d} (1 \wedge |y|^2) \nu(dy) < \infty.$$ 

We always assume that $\nu(\mathbb{R}^d) = \infty$ and consider the convolution semigroup of probability measures $\{P_t, t \geq 0\}$ with the Fourier transform $\mathcal{F}(P_t)(\xi) = \int_{\mathbb{R}^d} e^{i\xi \cdot y} P_t(dy) = \ldots$

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\[ \exp(-t\Phi(\xi)), \]

where

\[ \Phi(\xi) = -\int (e^{i\xi y} - 1 - i\xi \cdot y) \nu(dy) - i\xi \cdot b, \quad \xi \in \mathbb{R}^d. \]

There exists a Lévy process \( \{X_t, t \geq 0\} \) corresponding to \( \{P_t, t \geq 0\} \), i.e., \( P_t \) is the transition function of \( X_t \). For the rotation invariant \( \alpha \)-stable Lévy processes we have \( \nu(dy) = c|y|^{-d-\alpha} \) and \( b = 0 \), where \( \alpha \in (0, 2) \). The asymptotic behaviour of its densities \( p_t \) is well known (see, e.g., [1]) and in this case we have \( p_t(x) \approx \min(t^{-d/\alpha}, t|x|^{-d-\alpha}) \).

Explicit estimates for the first derivative of the transition density in this case are given in [3, Lemma 5] and we have \( |\nabla_x p(1, x)| \leq c|x|(1 + |x|)^{-d-\alpha-2} \).

W.E. Pruitt and S.J. Taylor investigated in [25] stable densities in the general setting, i.e., \( \nu(dr d\theta) = r^{-1-\alpha} d\mu(d\theta) \), where \( \mu \) is a bounded measure on the unit sphere \( S \). They obtained the estimate \( p_t(x) \leq c(1 + |x|)^{-d/\alpha} \). Indeed the upper bound can be attained if the spectral measure \( \mu \) has an atom (see the estimates from below in [11] and [12]). P. Głowacki and W. Hebisch proved in [8] and [9] that if \( \mu \) has a bounded density, \( g_{\mu} \), with respect to the surface measure on \( S \) then \( p_t(x) \leq c(1 + |x|)^{-d/\alpha} \). When \( g_{\mu} \) is continuous on \( S \) we even have \( \lim_{r \to \infty} r^{d+\alpha} p_t(r \theta) = c g_{\mu}(\theta), \theta \in S \) and if \( g_{\mu}(\theta) = 0 \) then additionally \( \lim_{r \to \infty} r^{d+2\alpha} p_t(r \theta) = c_r > 0 \), which was proved by J. Dziubański in [7].

More recent asymptotic results for stable Lévy processes are given in papers [32] and [4]. In particular if for some \( \gamma \in [1, d] \) the measure \( \nu \) is a \( \gamma \)-measure on \( S \), i.e.,

\[ \nu(B(x, r)) \leq c r^\gamma \quad \text{for every} \quad x \in S, r \leq 1/2, \]

or equivalently

\[ \mu(B(\theta, r) \cap S) \leq c r^{\gamma-1}, \quad \theta \in S, r \leq 1/2, \]

then we have

\[ p_t(x) \leq c (1 + |x|)^{-\alpha-\gamma}, \quad x \in \mathbb{R}^d. \]

Here and below we denote \( B(x, r) = \{y \in \mathbb{R}^d : |y - x| < r\} \). By scaling \( p_t(x) \leq c t^{-d/\alpha}(1 + t^{-1/\alpha}|x|)^{-\alpha-\gamma} \) for every \( t > 0 \). It follows also from [32 Theorem 1.1] that if for some \( \theta_0 \in S \) we have

\[ \mu(B(\theta_0, r) \cap S) \geq c r^{\gamma-1}, \quad r \leq 1/2, \]

then

\[ p_t(r \theta_0) \geq c (1 + r)^{-\alpha-\gamma}, \quad r > 0. \]

The estimates for more general Lévy processes were next obtained in [30, 31, 16, 19]. A recent paper [2] contains some estimates of densities for isotropic unimodal Lévy processes with Lévy-Khintchine exponents having the weak local scaling at infinity. Bounds for the transition density of a class of Markov processes with jump intensities which are not necessarily translation invariant but dominated by the Lévy measure of the stable rotation invariant process were given in [5] [8] [15]. Estimates for processes
which are solutions of some stochastic differential equations driven by Lévy processes were given in [23].

The main goal of the present paper is to extend the estimates in [30, 31] to more general class of semigroups and processes. We want to emphasize that we consider an essentially wider class of Lévy processes with Lévy measures not necessarily absolutely continuous with respect to the underlying (e.g., Lebesgue) measure. We also include here processes with intensities of small jumps remarkably different than the stable one. The time-space asymptotics of the densities for this class of processes is still very little understood (see [20, 21]). The other novelty here are the estimates of the derivatives of the densities.

For a set $A \subset \mathbb{R}^d$ we denote $\delta(A) = \text{dist}(0, A) = \inf\{|y| : y \in A\}$ and $\text{diam}(A) = \sup\{|y-x| : x, y \in A\}$. By $\mathcal{B}(\mathbb{R}^d)$ we denote Borel sets in $\mathbb{R}^d$. We denote $\Psi(r) = \sup_{|\xi| \leq r} \text{Re}(\Phi(\xi))$, $r > 0$.

We note that $\Psi$ is continuous and nondecreasing and $\sup_{r>0} \Psi(r) = \infty$, since $\nu(\mathbb{R}^d) = \infty$ (see (17)). Let $\Psi^{-1}(s) = \sup\{r > 0 : \Psi(r) = s\}$ for $s \in (0, \infty)$ so that $\Psi(\Psi^{-1}(s)) = s$ for $s \in (0, \infty)$ and $\Psi^{-1}(\Psi(s)) \geq s$ for $s > 0$. Define

$$h(t) = \frac{1}{\Psi^{-1}(\frac{1}{t})}, \quad t > 0.$$ 

The function $h$ gives global estimates of densities of Lévy processes (see Lemma 6 and the metric defined in [14]) and it appears also in [28] where gradient estimates of semigroups were proved.

The main results of the present paper are the following theorems.

**Theorem 1.** Assume that $\nu$ is a Lévy measure such that $\nu(\mathbb{R}^d) = \infty$ and

$$\nu(A) \leq M_1 f(\delta(A))[\text{diam}(A)]^\gamma, \quad A \in \mathcal{B}(\mathbb{R}^d),$$

where $\gamma \in [0, d]$, and $f : [0, \infty) \to [0, \infty]$ is nonincreasing function satisfying

$$\int_{|y|>r} f\left(s \vee |y| - \frac{|y|}{2}\right) \nu(dy) \leq M_2 f(s) \Psi\left(\frac{1}{r}\right), \quad s > 0, r > 0,$$

for some constants $M_1, M_2 > 0$. We assume also that for a constant $M_3 > 0$ and a nonempty set $T \subseteq (0, \infty)$ we have

$$\int_{\mathbb{R}^d} e^{-t \text{Re}(\Phi(\xi))} |\xi| \, d\xi \leq M_3 (h(t))^{-d-1}, \quad t \in T.$$

Then the measures $P_t$ are absolutely continuous with respect to the Lebesgue measure and there exist constants $C_1, C_2, C_3$ such that their densities $p_t$ satisfy

$$p_t(x + tb_{h(t)}) \leq C_1 (h(t))^{-d} \min\left\{1, t [h(t)]^\gamma f(|x|/4) + e^{-C_2 \frac{h(t)}{h(t)}} \log\left(1 + \frac{C_3 |x|}{h(t)}\right)\right\},$$

$x \in \mathbb{R}^d$, $t \in T,$
where

\[ b_r = \begin{cases} 
  b - \int_{|y| < 1} y \nu(dy) & \text{if } r \leq 1, \\
  b + \int_{1 < |y| < r} y \nu(dy) & \text{if } r > 1.
\end{cases} \]

We note that \( T \) is an arbitrary subset of \((0, \infty)\) satisfying (3). In particular the Theorem 1 can be applied either for small or for large times \( t \). In Lemma 5 below we give conditions which yield (3) for \( T = (0, \infty) \). All the assumptions of Theorem 1 are satisfied by a wide class of semigroups and corresponding Lévy processes, including stable, tempered stable, layered, relativistic, Lamperti and truncated stable processes as well as geometric stable (for large times \( t \)) and some subordinated processes. Some specific examples will be discussed in Section 4.

The lower estimate for symmetric Lévy measures is given in the following theorem.

**Theorem 2.** Assume that the Lévy measure \( \nu \) is symmetric, i.e. \( \nu(D) = \nu(-D) \) for every \( D \in B(\mathbb{R}^d) \), \( \nu(\mathbb{R}^d) = \infty \) and (3) holds for a set \( T \subset (0, \infty) \), and there exists a constant \( M_4 > 0 \) such that

\[ \nu(B(x, r)) \geq M_4 r^\gamma f(|x| + r), \quad x \in A, r > 0, \]

for some \( A \in B(\mathbb{R}^d), \gamma \in [0, d] \) and a function \( f : (0, \infty) \to [0, \infty) \). Then there exist constants \( C_4, C_5 \) and \( C_6 > C_5 \) such that

\[ p_t(x + tb) \geq C_4 (h(t))^{-d} \quad \text{for } |x| < C_5 h(t), \quad t \in T, \]

\[ p_t(x + tb) \geq C_4 t [h(t)]^\gamma f(|x| + C_5 h(t)) \quad \text{for } |x| \geq C_5 h(t), \quad x \in A, \quad t \in T. \]

In particular

\[ p_t(x + tb) \geq C_4 (h(t))^{-d} \min \{1, t [h(t)]^\gamma f(\min\{|x| + C_5 h(t), 2|x|\})\}, \quad x \in A, \quad t \in T. \]

The following estimate of derivatives is an extension of the results obtained for stable processes in [29].

**Theorem 3.** If the Lévy measure \( \nu \) and a nonincreasing function \( f \) satisfy (1), (2), \( \nu(\mathbb{R}^d) = \infty \) and there exist a constant \( M_5 > 0 \) and a set \( T \subset (0, \infty) \) such that

\[ \int_{\mathbb{R}^d} e^{-t \Re(\Phi(\xi))} |\xi|^m d\xi \leq M_5 (h(t))^{-d-m}, \quad t \in T, \]

for some \( m \in \mathbb{N}_0, m > \gamma, \) then \( p_t \in C_b^m(\mathbb{R}^d) \) and for every \( n \in \mathbb{N}_0 \) such that \( m \geq n > \gamma \) and every \( \beta \in \mathbb{N}_0^d \) such that \( |\beta| \leq m - n \) there exists a constant \( C_7 = C_7(n, m) \) such that

\[ \left| \partial^\beta_x p_t(x + tb_{h(t)}) \right| \leq C_7 (h(t))^{-d-|\beta|} \min \left\{1, t [h(t)]^\gamma f(|x|/4) + \left(1 + \frac{|x|}{h(t)}\right)^{-n}\right\}, \quad x \in \mathbb{R}^d, \quad t \in T, \]

where \( b_{h(t)} \) is given by (4).
In Section 2 we give estimates of the real part of the characteristic exponent \( \text{Re} \Phi \) and the function \( \Psi \) in terms of the Lévy measure \( \nu \) and we consider sufficient conditions for assumptions (3) and (8). We prove also that an inequality opposite to (8) holds for every Lévy measure. In section 3 we prove all the main theorems. In Section 4 we discuss examples. We focus on the specific type of Lévy measures \( \nu \) such that \( \nu(\text{d}r\text{d}\theta) \approx r^{-1-\alpha}[\log(1+r^{-\kappa})]^{-\beta}d\mu(d\theta) \) for suitable constants \( \alpha, \kappa, \beta \) and a nondegenerate measure \( \mu \) on the unit sphere \( \mathbb{S} \).

We use \( c, C, M \) (with subscripts) to denote finite positive constants which depend only on \( \nu, b, \) and the dimension \( d \). Any additional dependence is explicitly indicated by writing, e.g., \( c = c(n) \). We write \( f(x) \approx g(x) \) to indicate that there is a constant \( c \) such that \( c^{-1}f(x) \leq g(x) \leq cf(x) \).

## 2 Estimates of characteristic exponent

The characteristic exponent (symbol) \( \Phi \) of the process is a continuous negative definite function and its basic properties are given, e.g., in [13]. In Proposition 1 we obtain both sides estimates for \( \text{Re} \Phi \) and \( \Psi(r) = \sup_{|\xi| \leq r} |\xi| \langle \Phi(\xi) \rangle \). We note that the estimates for \( \Psi \) follow also from combined results of [27], Remark 4.8 and Section 3. of [24] but we include here a short direct proof (see also Lemma 6 in [10]).

**Proposition 1.** Let

\[
H(r) = \int 1 \wedge \frac{|y|^2}{r^2} \nu(dy), \quad r > 0.
\]

We have

\[
(10) \quad (1 - \cos 1) \int_{|y| < 1/|\xi|} |\xi \cdot y|^2 \nu(dy) \leq \text{Re}(\Phi(\xi)) \leq 2H(1/|\xi|), \quad \xi \in \mathbb{R}^d \setminus \{0\},
\]

and there exists a constant \( C_8 \) such that

\[
(11) \quad C_8 H(1/r) \leq \Psi(r) \leq 2H(1/r), \quad r > 0.
\]

**Proof.** We have

\[
\text{Re}(\Phi(\xi)) = \int (1 - \cos(\xi \cdot y)) \nu(dy)
\leq \frac{1}{2} \int_{|y| \leq 1/|\xi|} |\xi \cdot y|^2 \nu(dy) + 2 \int_{|y| > 1/|\xi|} \nu(dy)
\leq \frac{1}{2} |\xi|^2 \int_{|y| \leq 1/|\xi|} |y|^2 \nu(dy) + 2 \int_{|y| > 1/|\xi|} \nu(dy)
\leq 2H(1/|\xi|), \quad \xi \in \mathbb{R}^d \setminus \{0\},
\]
and

\[ \Re(\Phi(\xi)) \geq (1 - \cos 1) \int_{|y| < \sqrt{1/|\xi|}} |\xi \cdot y|^2 \nu(dy), \]

since \(1 - \cos s \geq (1 - \cos 1)s^2\), for \(|s| \leq 1\), and (10) follows. The upper estimate in (11) follows directly from (10). For the lower estimate we use the obvious inequality

\[ \int_A g(x) \, dx \leq |A| \sup_{x \in A} g(x). \]

Let \(\delta \in (0, 1)\) and \(M_\delta = \bigcup_{k \in \mathbb{Z}} (\delta + k2\pi, 2\pi - \delta + k2\pi)\), \(c_1 = (1 - \cos \delta)/\delta^2\), \(\kappa = 3\delta/(2\pi - \delta)\) and let \(\omega_0 = 1\) and \(\omega_d = \pi^{d/2}/\Gamma(d/2 + 1)\) (the volume of the unit ball in \(\mathbb{R}^d\)). We get

\[ \Psi(r) = \sup_{|\xi| \leq r} \int (1 - \cos(\xi \cdot y)) \nu(dy) \]

\[ \geq \frac{1}{r^d\omega_d} \int_{|\xi| < r} \left( \int_{|\xi| < \delta} (1 - \cos(\xi \cdot y)) \nu(dy) + \int_{|\xi| \geq \delta} (1 - \cos(\xi \cdot y)) \nu(dy) \right) d\xi \]

\[ \geq \frac{1}{r^d\omega_d} \int_{|\xi| < r} \left( c_1 \int_{|\xi| < \delta} |\xi \cdot y|^2 \nu(dy) + c_1\delta^2 \int_{|\xi| \geq \delta} \nu(dy) \right) d\xi \]

\[ = \frac{c_1}{r^d\omega_d} \left( \int_{|\xi| < \delta} \int |\xi \cdot y|^2 d\xi \nu(dy) + \delta^2 \int_{|\xi| \geq \delta} \nu(dy) \right) \]

\[ = \frac{c_1}{r^d\omega_d} \left( \int_{|y| < \frac{\delta}{2\pi}} |y|^2 \int_{|\xi| < \delta} \xi^2 \nu(dy) + \delta^2 \int_{|y| \geq \frac{\delta}{2\pi}} \int_{|\xi| \geq \delta} \nu(dy) \right) \]

\[ \geq \frac{c_1}{r^d\omega_d} \left( \int_{|y| < \frac{\delta}{2\pi}} |y|^2 \int_{|\xi| < \delta} \xi^2 \nu(dy) + \delta^2 \int_{|y| \geq \frac{\delta}{2\pi}} \frac{\omega_d}{d+2} (\kappa r)^{d+2} \nu(dy) + \delta^2 \int_{|y| \geq \frac{\delta}{2\pi}} \int \frac{\omega_d}{d+2} \nu(dy) \right). \]

For \(r|y| > \delta/\kappa = (2\pi - \delta)/3\) we have

\[ \int_{|\xi| \in \frac{M_\delta}{|\xi| < r}} d\xi \leq \omega_d \int_{|y| < \frac{\delta}{2\pi}} \frac{2\delta}{|y|} \left( 2 \left| \frac{r|y| + \delta}{2\pi} \right| + 1 \right) \leq \omega_d \int_{|y| < \frac{\delta}{2\pi}} \frac{1}{2\omega_d r^d} \]

since if \(\left| \frac{r|y| + \delta}{2\pi} \right| \geq 1\) then also \(r|y| \geq 2\pi - \delta\). For \(\delta\) such that \(2\kappa \omega_d / \omega_d \leq 1/2\) this yields

\[ \int_{|\xi| \in \frac{M_\delta}{|\xi| < r}} d\xi \geq \frac{1}{2} \omega_d r^d, \]
and we obtain

\[
\Psi(r) \geq c_1 \left( \frac{\kappa^d}{d + 2} \int_{|y| < \frac{\delta}{\kappa r}} (\kappa r)^2 |y|^2 \nu(dy) + \frac{1}{2} \int_{|y| \geq \frac{\delta}{\kappa r}} \delta^2 \nu(dy) \right) \\
\geq \frac{c_1 \kappa^d}{d + 2} \int (|y| \kappa r \wedge \delta)^2 \nu(dy) \geq \frac{c_1 \kappa^d}{d + 2} H(1/r).
\]

Now we prove the following technical lemma.

**Lemma 1.** Assume that for a function \( f : (0, \infty) \rightarrow [0, \infty) \) exist a nonincreasing function \( g : (0, \infty) \rightarrow [0, \infty) \) and constants \( m > 0, \ a > \kappa \geq 0, \) and \( r_0 \geq 0 \) such that

\[
(12) \quad \int_0^r s^a f(s) \, ds \leq mr^\kappa g(r)
\]

for every \( r > r_0. \) Then we have

\[
\int_r^\infty f(s) \, ds \leq \frac{ma}{a - \kappa} r^{-a} g(r),
\]

for every \( r > r_0. \)

**Proof.** By (12) for every \( r > r_0 \) we have

\[
\int_r^\infty \frac{1}{t^{a+1}} \left( \int_0^t s^a f(s) \, ds \right) dt \leq m \int_r^\infty t^{\kappa-a-1} g(t) \, dt \leq mg(r) \int_r^\infty t^{\kappa-a-1} \, dt = \frac{m}{a - \kappa} r^{-a} g(r)
\]

Furthermore, changing the order of integration we obtain

\[
\int_r^\infty \frac{1}{t^{a+1}} \left( \int_0^t s^a f(s) \, ds \right) dt = \int_r^\infty s^a f(s) \int_s^\infty \frac{1}{t^{a+1}} \, dt \, ds + \int_r^\infty s^a f(s) \int_s^\infty \frac{1}{t^{a+1}} \, dt \, ds
\]

\[
= \frac{1}{ar^a} \int_0^r s^a f(s) \, ds + \frac{1}{a} \int_r^\infty f(s) \, ds
\]

\[
\geq \frac{1}{a} \int_r^\infty f(s) \, ds,
\]

and the lemma follows. \( \square \)

The following corollaries which give estimates of \( \text{Re} \Phi \) for the more specific case of Lévy measure follow directly from Proposition and Lemma.
Corollary 2. If \( \mu \) is nondegenerate, i.e., the support of \( \mu \) is not contained in any proper linear subspace of \( \mathbb{R}^d \), and
\[
\nu(A) \geq M_6 \int_S \int_0^\infty 1_A(s\theta)f(s)\, ds\mu(d\theta),
\]
where \( f : (0, \infty) \to [0, \infty) \), then
\[
\Re(\Phi(\xi)) \geq C_9|\xi|^2g_1(1/|\xi|),
\]
where \( g_1(r) = \int_0^r s^2f(s)\, ds \).

Corollary 3. Let \( f : (0, \infty) \to [0, \infty) \) be such that
\[
\nu(A) \leq M_7 \int_S \int_0^\infty 1_A(s\theta)f(s)\, ds\mu(d\theta),
\]
and
\[
\int_0^r s^2f(s)\, ds \leq M_8r^\kappa g_2(r), \quad r > 0,
\]
for constants \( M_7, M_8 > 0, 2 > \kappa \geq 0 \) and nonincreasing function \( g_2 : (0, \infty) \to [0, \infty) \). Then there exists a constant \( C_{10} \) such that
\[
\Re(\Phi(\xi)) \leq C_{10}|\xi|^{2-\kappa}g_2(1/|\xi|), \quad \xi \in \mathbb{R}^d \setminus \{0\}.
\]

In the following Lemma we will prove that the inequality opposite to (8) holds for every Lévy measure.

Lemma 4. If \( \nu(\mathbb{R}^d) = \infty \) then for every \( m \in \mathbb{N}_0 \) there exists a constant \( C_{11} = C_{11}(m) \) such that
\[
\int e^{-t\Re(\Phi(\xi))}|\xi|^m\, d\xi \geq C_{11}(h(t))^{-d-m},
\]
for every \( t > 0 \).

Proof. Using the fact that \( \Psi \) is increasing and \( \Psi(\Psi^{-1}(s)) = s \) for every \( s > 0 \), we get
\[
\int e^{-t\Re(\Phi(\xi))}|\xi|^m\, d\xi \geq \int e^{-t\Psi(|\xi|)}|\xi|^m\, d\xi
\]
\[
= c_1\int_0^\infty e^{-t\Psi(s)}s^{m+d-1}\, ds
\]
\[
\geq c_1\int_0^{\Psi^{-1}(1/t)} e^{-t\Psi(s)}s^{m+d-1}\, ds
\]
\[
\geq \frac{c_1e^{-1}}{m+d} [\Psi^{-1}(1/t)]^{m+d}
\]
\[
= \frac{c_1e^{-1}}{m+d} [h(t)]^{m-d}.
\]

\square
Now we give conditions which guarantee that the assumptions \( \mathfrak{a} \) and \( \mathfrak{b} \) hold.

**Lemma 5.** Assume that there is a strictly increasing function \( F : [0, \infty) \to [0, \infty) \) such that \( F(0) = 0, \lim_{s \to \infty} F(s) = \infty, \) which is differentiable and which satisfies
\[
F^{-1}(2s) \leq M_9 F^{-1}(s), \quad s > 0, \tag{13}
\]
and
\[
M_{10}^{-1} F(|\xi|) \leq \text{Re} \Phi(\xi) \leq M_10 F(|\xi|), \quad \xi \in \mathbb{R}^d,
\]
for some constants \( M_9, M_{10} \). Then there exists a constant \( C_{12} = C_{12}(m) \) such that
\[
C_{12}^{-1} (h(t))^{-d-m} \leq \int_{\mathbb{R}^d} e^{-t \text{Re} \Phi(\xi)} |\xi|^m d\xi \leq C_{12} (h(t))^{-d-m}, \quad t > 0.
\]
for every \( m \in \mathbb{N} \cup \{0\} \).

**Proof.** We follow the argumentation given in [28], proof of Theorem 1.3. We denote \( g(s) = F^{-1}(s) \). We have
\[
\Psi(s) = \sup_{|x| < s} \text{Re} \Phi(\xi) \approx \sup_{|x| < s} F(|\xi|) = F(s),
\]
and this yields
\[
\Psi^{-1}(c_1 s) \leq F^{-1}(s) \leq \Psi^{-1}(c_2 s), \quad s > 0.
\]
It follows from (13) that \( g(2^n s) \leq c_3^n g(s) \) for \( c_3 = M_9 \) and for \( c_4 = M_{10}^{-1} \) this yields
\[
\int e^{-t \text{Re} \Phi(\xi)} |\xi|^m d\xi \leq \int e^{-t c_4 F(|\xi|)} |\xi|^m d\xi
\leq c_5 \int_0^\infty e^{-t c_4 F(s)} s^{m+d-1} ds
\leq c_5 \int_0^1 + \int_1^\infty e^{-c_4 u / c_2} \left[ g \left( \frac{u}{c_2 t} \right) \right]^{m+d-1} g' \left( \frac{u}{c_2 t} \right) \frac{1}{c_2 t} du
\leq \frac{c_5}{m+d} \left[ g \left( \frac{1}{c_2 t} \right) \right]^{m+d}
\leq \frac{c_5}{m+d} \left[ g \left( \frac{1}{c_2 t} \right) \right]^{m+d} + \sum_{n=1}^{\infty} e^{-c_4 \frac{2^n}{c_2} / c_2} \left[ g \left( \frac{2^n}{c_2 t} \right) \right]^{m+d}
\leq \frac{c_5}{m+d} \left[ g \left( \frac{1}{c_2 t} \right) \right]^{m+d} + \left[ g \left( \frac{1}{c_2 t} \right) \right]^{m+d} \sum_{n=1}^{\infty} e^{-c_4 \frac{2^n}{c_2} / c_3} u^{(m+d)}
\leq c_6 \left[ g \left( \frac{1}{c_2 t} \right) \right]^{m+d} \leq c_6 \left[ \Psi^{-1}(1/t) \right]^{m+d} = c_6 [h(t)]^{-m-d}.
\]
The estimate from below in (14) follows from Lemma 4. \( \square \)
3 Proof of theorems

We will now prove the theorems. In the following we often assume that (3) is satisfied which gives the existence of densities $p_t \in C^1_b(\mathbb{R}^d)$ of $P_t$ for $t \in T$. We note that several necessary and sufficient conditions for the existence of (smooth) transition probability densities for Lévy processes and isotropic Lévy processes are are given in [18].

In the following two lemmas we obtain estimates of $p_t$ by constants depending on $t$.

**Lemma 6.** If $\nu(\mathbb{R}^d) = \infty$ and (3) holds then there exists a constant $C_{13}$ such that

$$p_t(x) \leq C_{13} (h(t))^{-d}, \quad t \in T.$$  

**Proof.** We have

$$p_t(x) = (2\pi)^{-d} \int e^{-ix \cdot \xi} e^{-t\Phi(\xi)} \, d\xi \leq (2\pi)^{-d} \int e^{-t \Re(\Phi(\xi))} \, d\xi$$

$$= (2\pi)^{-d} \left( \int_{|\xi| \leq (1/h(t))} e^{-t \Re(\Phi(\xi))} \, d\xi + \int_{|\xi| > (1/h(t))} e^{-t \Re(\Phi(\xi))} \, d\xi \right)$$

$$\leq (2\pi)^{-d} \left( c_1 (h(t))^{-d} + h(t) \int e^{-t \Re(\Phi(\xi))} \, d\xi \right)$$

$$\leq c_2 (h(t))^{-d},$$

for $t \in T$. Here we use (3) in the last inequality above. \hfill \Box

**Lemma 7.** If $\nu(\mathbb{R}^d) = \infty$ and (3) holds and $\nu$ is symmetric then there exist constants $C_{14}, C_6$ such that

$$p_t(x + tb) \geq C_{14} (h(t))^{-d}, \quad t \in T, \ |x| \leq C_6 h(t).$$

**Proof.** It follows from Lemma 4 and the symmetry of $\nu$ that

$$p_t(tb) = (2\pi)^{-d} \int e^{-t\Phi(\xi)} e^{-itb \cdot \xi} \, d\xi = (2\pi)^{-d} \int e^{-t \Re(\Phi(\xi))} \, d\xi$$

$$\geq c_1 (h(t))^{-d}.$$

For every $j \in \{1, ..., d\}$ and $t \in T$, by (3) we get

$$\left| \frac{\partial p_t}{\partial y_j} \right| = \left| (2\pi)^{-d} \int_{\mathbb{R}^d} (-i) \xi_j e^{-iy \cdot \xi} e^{-t\Phi(\xi)} \, d\xi \right|$$

$$\leq (2\pi)^{-d} \left( \int_{|\xi| \leq (1/h(t))} e^{-t \Re(\Phi(\xi))} \, d\xi + \int_{|\xi| > (1/h(t))} e^{-t \Re(\Phi(\xi))} \, d\xi \right)$$

$$\leq (2\pi)^{-d} \left( c_2 (h(t))^{-d-1} + \int e^{-t \Re(\Phi(\xi))} \, d\xi \right)$$

$$\leq c_3 (h(t))^{-d-1}, \quad y \in \mathbb{R}^d.$$
It follows that
\[ p_t(x + tb) \geq c_1 (h(t))^{-d} - dc_3 (h(t))^{-d-1} |x| \geq \frac{c_1}{2} (h(t))^{-d}, \]
provided that $|x| \leq \frac{c_1}{2 dc_3} h(t)$, which clearly yields (15). \hfill \Box

For $r > 0$ we denote $\tilde{\nu}_r(dy) = \mathbb{1}_{B(0,r)}(y) \nu(dy)$. We consider the semigroup of measures $\{\tilde{P}_t^r, t \geq 0\}$ such that
\[ \mathcal{F}(\tilde{P}_t^r)(\xi) = \exp \left( t \int (e^{i\xi \cdot y} - 1 - i\xi \cdot y) \tilde{\nu}_r(dy) \right), \quad \xi \in \mathbb{R}^d. \]
We have
\[
|\mathcal{F}(\tilde{P}_t^r)(\xi)| = \exp \left( -t \int_{|y|<r} (1 - \cos(y \cdot \xi)) \nu(dy) \right) 
= \exp \left( -t \left( \text{Re}(\Phi(\xi)) - \int_{|y| \geq r} (1 - \cos(y \cdot \xi)) \nu(dy) \right) \right) 
\leq \exp(-t \text{Re}(\Phi(\xi))) \exp(2t \nu(B(0,r)^c)), \quad \xi \in \mathbb{R}^d.
\]
(16)

It follows that if (3) holds then for every $r > 0$ and $t \in T$ the measure $\tilde{P}_t^r$ is absolutely continuous with respect to the Lebesgue measure with density, say, $\tilde{p}_t^r \in C^1_b(\mathbb{R}^d)$.

We will often use $\tilde{P}_t^r$ and $\tilde{p}_t^r$ with $r = h(t)$ and for simplification we will denote $\tilde{P}_t = \tilde{P}_t^{h(t)}$ and $\tilde{p}_t = \tilde{p}_t^{h(t)}$.

We note also that there exists a constant $M_0$ such that
\[ \nu(B(0,r)^c) \leq M_0 \Psi(1/r), \quad r > 0, \]
which follows from Proposition 11 (see also [28], the proof of Proposition 2.2, Step 3).

Using (16) and (17) we obtain
\[
|\mathcal{F}(\tilde{P}_t)(\xi)| \leq \exp(-t \text{Re}(\Phi(\xi))) \exp(2t \nu(B(0,h(t))^c))) 
\leq \exp(-t \text{Re}(\Phi(\xi))) \exp(2tM_0 \Psi(1/h(t))) 
= \exp(-t \text{Re}(\Phi(\xi))) \exp(2M_0), \quad \xi \in \mathbb{R}^d, t \in T,
\]
(18)
since $\Psi(1/h(t)) = 1/t$.

The Lévy measures with bounded support are discussed, e.g., in Section 26 of [26], where estimates of tails of corresponding distributions are included. We extended these results in [31] to estimates of densities and in the following lemma we use the results of [31] in our new more general context.

**Lemma 8.** If $\nu(\mathbb{R}^d) = \infty$ and (3) holds then there exist constant $C_{15}, C_{16}$ and $C_{17}$ such that
\[ \tilde{p}_t(x) \leq C_{15} [h(t)]^{-d} \exp \left( -\frac{C_{16}}{h(t)} |x| \log \left( 1 + \frac{C_{17}}{h(t)} |x| \right) \right), \quad x \in \mathbb{R}^d, t \in T. \]
(19)
Proof. Let \( g_t(y) = [h(t)]^d \tilde{p}_t(h(t)y) \). We consider the infinitely divisible distribution \( \pi_t(dy) = g_t(y)\, dy \). We note that

\[
\mathcal{F}(\pi_t)(\xi) = \exp \left( t \int \left( e^{i \xi(h(t)) - \frac{1}{2} \| \xi \|^2} - 1 - i \xi(h(t)) \cdot y \mathbb{1}_{B(0, h(t))}(y) \right) \tilde{v}_{h(t)}(dy) \right)
\]

\[
= \exp \left( \int \left( e^{i \xi y} - 1 - i \xi \cdot y \mathbb{1}_{B(0, 1)}(y) \right) \lambda_t(dy) \right), \quad \xi \in \mathbb{R}^d;
\]

where \( \lambda_t(A) = t \tilde{v}_{h(t)}(h(t)A) \) is the Lévy measure of \( \pi_t \).

From (18) and (3) for every \( j \in \{1, \ldots, d\} \) and \( t \in T \) we obtain

\[
\left| \frac{\partial g_t}{\partial y_j}(y) \right| = \left| h(t) \right|^{d+1} \left| (2\pi)^{-d} \int_{\mathbb{R}} (-i)\xi_j e^{-ib(t)\xi} \mathcal{F}(\tilde{p}_t)(\xi) d\xi \right|
\]

\[
\leq \left| h(t) \right|^{d+1} (2\pi)^{-d} \int |\xi| e^{2M_0} e^{-t \Re(\Phi(\xi))} d\xi
\]

\[
\leq c_1.
\]

Similarly we get

\[
g_t(y) = [h(t)]^d (2\pi)^{-d} \int e^{-ib(t)y \cdot \xi} \mathcal{F}(\tilde{p}_t)(\xi) d\xi
\]

\[
\leq \left| h(t) \right|^{d} (2\pi)^{-d} e^{2M_0} \left[ \int_{|\xi| < (1/h(t))} e^{-t \Re(\Phi(\xi))} d\xi + \int_{|\xi| > (1/h(t))} e^{-t \Re(\Phi(\xi))} d\xi \right]
\]

\[
\leq \left| h(t) \right|^{d} (2\pi)^{-d} e^{2M_0} \left[ c_2 |h(t)|^{-d} + h(t) \int_{|\xi| > (1/h(t))} |\xi| e^{-t \Re(\Phi(\xi))} d\xi \right]
\]

\[
\leq c_3.
\]

It follows from (2.16) in [28] that

\[
\int |y|^2 \lambda_t(dy) = t \int (|y|/h(t))^2 \tilde{v}_{h(t)}(dy) \leq c_4.
\]

We have also

\[
\int_{|y| > 1} y_j \lambda_t(dy) = t(h(t))^{-1} \int_{B(0, h(t))^c} y_j \tilde{v}_{h(t)}(dy) = 0.
\]

It follows from Lemma 2 in [31] and (20) that

\[
g_t(y) \leq c_5 \exp \left( -c_6 |y| \log (c_7 |y|) \right) \leq c_8 \exp \left( -c_9 |y| \log (1 + c_{10}|y|) \right),
\]

for \( y \in \mathbb{R}^d \), and this yields

\[
\tilde{p}_t(x) \leq c_8 (h(t))^{-d} \exp \left( -c_9 |x|/h(t) \log \left( 1 + c_{10}|x|/h(t) \right) \right),
\]

for \( x \in \mathbb{R}^d, t \in T \). \( \square \)
For $r > 0$ we denote $\bar{\nu}_r(dy) = 1_{B(0,r)}(y) \nu(dy)$ and consider the probability measures \{\bar{P}_t, t \geq 0\} such that
\begin{equation}
\mathcal{F}(\bar{P}_t)(\xi) = \exp \left( t \int (e^{i\xi y} - 1) \bar{\nu}_t(dy) \right), \quad \xi \in \mathbb{R}^d.
\end{equation}

Note that
\begin{equation}
\bar{P}_t = \exp(t(\bar{\nu}_r - |\bar{\nu}_r|\delta_0)) = \sum_{n=0}^{\infty} \frac{t^n (\bar{\nu}_r - |\bar{\nu}_r|\delta_0))^{n*}}{n!} = e^{-t|\bar{\nu}_r|} \sum_{n=0}^{\infty} \frac{t^n \bar{\nu}_r^{n*}}{n!}, \quad t \geq 0.
\end{equation}

**Lemma 9.** If $\nu$ is a Lévy measure and $f : [0, \infty) \to (0, \infty]$ is nonincreasing function satisfying (1) and if for some $r > 0$ we have
\begin{equation}
\int_{|y| > r} f \left( s \vee |y| - \frac{|y|}{2} \right) \nu(dy) \leq M_2 f(s) \Psi \left( \frac{1}{r} \right), \quad s > 0,
\end{equation}
with a constant $M_2$, then
\begin{equation}
\bar{\nu}_r^{n*}(A) \leq C_{18} m \left[ \Psi(1/r) \right]^{n-1} f(\delta(A)/2) \left[ \text{diam}(A) \right]^{\gamma},
\end{equation}
for $n \in \mathbb{N}$ and $A \in \mathcal{B}(\mathbb{R}^d)$ such that $\delta(A) > 0$, $\text{diam}(A) < \infty$, with a constant $C_{18} := \max\{M_0, M_1 + M_2\}$.

**Proof.** We use induction. For $n = 1$ the lemma follows from (1). Let (24) hold for some $n \in \mathbb{N}$ and constant $c_0 = C_{18}$ and $A$ be a set such that $\delta(A) > 0$. For $y \in \mathbb{R}^d$ we denote $D_y = \{z \in \mathbb{R}^d : |z| > \frac{1}{2}|z + y|\} = \left( B \left( \frac{1}{2}y, \frac{3}{2}|y| \right) \right)^c$. We have
\begin{align*}
\bar{\nu}_r^{(n+1)*}(A) &= \int \bar{\nu}_r(A - y) \bar{\nu}_r^{n*}(dy) \\
&= \int \bar{\nu}_r ((A - y) \cap D_y) \bar{\nu}_r^{n*}(dy) + \int \bar{\nu}_r ((A - y) \cap D_y^c) \bar{\nu}_r^{n*}(dy) \\
&= I + II.
\end{align*}

We note that for $z \in (A - y) \cap D_y$ we have $z + y \in A$ and $|z| > \frac{1}{2}|z + y|$, therefore $|z| > \frac{1}{2}\delta(A)$ and $\delta((A - y) \cap D_y) > \frac{1}{2}\delta(A)$. Furthermore, $\text{diam}((A - y) \cap D_y) \leq \text{diam}(A)$ and using (1) and (17) we obtain
\begin{align*}
I &\leq M_1 f(\delta(A)/2) (\text{diam}(A))^{\gamma} |\bar{\nu}_r^{n*}| \\
&\leq M_1 M_0^n (\Psi(1/r))^n f(\delta(A)/2) (\text{diam}(A))^{\gamma}.
\end{align*}
We have

\[ II = \int \int 1_{A-y}(z) 1_{D_y}(z) \tilde{\nu}_r(dz) \tilde{\nu}_r^n(dy) \]

\[ = \int \int 1_{A-z}(y) 1_{B(-z,2|z|)}(y) \tilde{\nu}_r^n(dy) \tilde{\nu}_r(dz) \]

\[ = \int \tilde{\nu}_r^n((A-z) \cap B(-z,2|z|)) \tilde{\nu}_r(dz), \]

Let \( y \in V_z := (A-z) \cap B(-z,2|z|) \). We then have \( y+z \in A \), so \( |y+z| \geq \delta(A) \), and \( |y| \geq |y+z|-|z| \) and this yields

\[ \delta(V_z) \geq \inf_{y \in V_z} |y+z|-|z| \geq \frac{1}{2} \delta(A), \]

and by (23) and induction we get

\[ II \leq c_0^n (\Psi(1/r))^{n-1} \int f \left( \left( \frac{(\delta(A) \vee 2|z|) - |z|}{2} \right) \tilde{\nu}_r(dz) \right) \]

\[ \leq c_0^n (\Psi(1/r))^{n-1} (\delta(A)) \gamma M_2 f (\delta(A)/2) \Psi(1/r) \]

\[ = M_2 c_0^n (\Psi(1/r))^{n} f (\delta(A)/2) (\delta(A)) \gamma. \]

Indeed, we see that the lemma follows by taking \( c_0 := \max\{ M_0, M_1 + M_2 \} \).

\[ \text{Corollary 10. If (7) and (2) hold then} \]

\[ \tilde{\nu}_r^n(B(x, \rho)) \leq C_{18}^n [\Psi(1/r)]^{n-1} f (|x|/4) (2 \rho)^\gamma, \]

for every \( x \in \mathbb{R}^d \setminus \{0\} \), \( \rho < |x|/2 \) and \( r > 0, n \in \mathbb{N} \).

\[ \text{Proof of Theorem 1} \]

We have

\[ P_t = \bar{P}_t^r * \bar{P}_t^r * \bar{\delta}_{b_r}, \quad t \geq 0, \]

where \( P_t^r \) is defined by (21) and \( b_r \) by (4). Of course

\[ p_t = \bar{p}_t^r * \bar{P}_t^r * \bar{\delta}_{b_r}, \quad t \in T. \]

We will denote

\[ P_t = P_t^{h(t)}. \]

We have \( \Psi(1/h(t)) = 1/t \) and it follows from Corollary 10 and (22) that

\[ P_t(B(x, \rho)) \leq c_1 t f (|x|/4) \rho^\gamma, \]

for \( \rho \leq \frac{1}{2} |x| \) and \( t > 0 \).

We denote

\[ g(s) = e^{-C_{16} s \log(1+C_{17}s)}, \quad s \geq 0, \]
where constants $C_{16}, C_{17}$ are given by (13). We note that $g$ is decreasing, continuous on $[0, \infty)$ and $g(s) \leq c_2 s^{-2\gamma}$, for some $c_2 > 0$, which yields that the inverse function $g^{-1} : (0, 1] \rightarrow [0, \infty)$ exists, is decreasing, and $g^{-1}(s) \leq (c_2/s)^{1/(2\gamma)}$. In particular

$$
\int_0^1 (g^{-1}(s))^{\gamma} \, ds < \infty.
$$

Using Lemma 8 and (25) we obtain

$$
\tilde{p}_t \ast \tilde{P}_t(x) = \int \tilde{p}_t(x-y) \tilde{P}_t(dy) \\
\leq \int C_{16}[h(t)]^{-d} g(|x-y|/h(t)) \tilde{P}_t(dy) \\
= C_{16}[h(t)]^{-d} \int \int_0^{g(|x-y|/h(t))} ds \tilde{P}_t(dy) \\
= C_{16}[h(t)]^{-d} \int_0^1 \int \mathbb{1}_{\{y \in \mathbb{R}^d, g(|x-y|/h(t)) > s\}} \tilde{P}_t(dy) ds \\
= C_{16}[h(t)]^{-d} \int_0^1 \tilde{P}_t(B(x, h(t)g^{-1}(s))) ds \\
\leq c_1C_{16}[h(t)]^{-d} \left( \int_0^1 t f(|x|/4) \left( h(t)g^{-1}(s) \right)^{\gamma} ds + \int_0^{g(|x|/2h(t))} ds \right) \\
\leq c_1C_{16}[h(t)]^{-d} \left( t[h(t)]^{\gamma} f \left( \frac{|x|}{4} \right) \int_0^1 \left( g^{-1}(s) \right)^{\gamma} ds + g \left( \frac{|x|}{2h(t)} \right) \right) \\
= c_3[h(t)]^{-d} \left( t[h(t)]^{\gamma} f \left( \frac{|x|}{4} \right) + \frac{|x|}{2h(t)} \right).
$$

This and Lemma 6 yield

$$
p_t(x + tb(h(t))) = \int \tilde{p}_t \ast \tilde{P}_t(x + tb(h(t)) - y) \delta_{tb(h(t))}(dy) \\
= \tilde{p}_t \ast \tilde{P}_t(x) \\
\leq c_4[h(t)]^{-d} \min \left\{ 1, t[h(t)]^{\gamma} f \left( \frac{|x|}{4} \right) + g \left( \frac{|x|}{2h(t)} \right) \right\},
$$

for $t \in T$. 

The following Lemma which will be used in the proof of Theorem 2 was communicated to us by Tomasz Grzywny.

**Lemma 11.** If $\nu(\mathbb{R}^d) = \infty$ then we have

$$
\lim_{a \to 0^+} \sup_{t > 0} \frac{h(at)}{h(t)} = 0.
$$

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Proof. Since $\nu(\mathbb{R}^d) = \infty$ function $H(r) = \int (1 \wedge (|y|^2/r^2)) \nu(dy)$ is strictly decreasing and $H(0, \infty) = (0, \infty)$. Moreover, we have $H(\lambda r) \geq \lambda^{-2} H(r)$, for $r > 0$, $\lambda > 1$, hence
\begin{equation}
\lambda H^{-1}(s) \leq H^{-1}(\lambda^{-2}s), \quad s > 0, \; \lambda > 1. \tag{26}
\end{equation}
It follows from Proposition II that
\[ C_8 H(1/r) \leq \Psi(r) \leq 2H(1/r), \quad r > 0, \]
which yields
\begin{equation}
\frac{1}{H^{-1}(s/2)} \leq \Psi^{-1}(s) \leq \frac{1}{H^{-1}(s/C_8)}, \quad s > 0. \tag{27}
\end{equation}
Using (27) and (26) we obtain
\[ \frac{h(at)}{h(t)} = \frac{\Psi^{-1}(1/t)}{\Psi^{-1}(1/at)} \leq \frac{H^{-1}(1/(2at))}{H^{-1}(1/(C_8 t))} \leq \sqrt{\frac{2a}{C_8}}, \]
for $a < C_8/2$, and the lemma follows. \hfill \Box

Proof of Theorem II. First we will prove that there exist constants $c_1$, $c_2$, $c_3$ such that for every $a \in (0, 1]$ we have
\begin{equation}
\hat{p}_t^{h(at)}(y) \geq c_1 (h(t))^{-d}, \tag{28}
\end{equation}
provided $|y| \leq c_2 e^{-c_3/a} h(t)$, $t \in T$.
By symmetry of $\nu$ we have
\[ \mathcal{F}(\hat{p}_t^{h(at)})(\xi) \geq |\mathcal{F}(p_t)(\xi)|, \quad \xi \in \mathbb{R}^d, \; t \in T, \]
and this and Lemma II yield
\[ \hat{p}_t^{h(at)}(0) \geq (2\pi)^{-d} \int e^{-t \Re(\Phi(\xi))} \, d\xi \geq c_4 (h(t))^{-d}, \quad t \in T. \]
By (16) and (17) we have
\[ |\mathcal{F}(\hat{p}_t^{h(at)})(\xi)| \leq |\mathcal{F}(p_t)(\xi)| e^{2 \nu(B(0,h(at)^c)} \leq e^{-t \Re(\Phi(\xi))} e^{2M_0 t \Psi(1/(h(at)))} = e^{-t \Re(\Phi(\xi))} e^{2M_0/a}, \]
and for every $j \in \{1, \ldots, d\}$ by (3) we get
\[ \left| \frac{\partial \hat{p}_t^{h(at)}}{\partial y_j}(y) \right| \geq |(2\pi)^{-d} \int (-i) \xi_j e^{-iy \xi} \mathcal{F}(\hat{p}_t^{h(at)})(\xi) d\xi| \]
\[ \leq c_5 e^{2M_0/a} \left( \int_{|\xi| \leq (1/h(t))} e^{-t \Re(\Phi(\xi))} |\xi| \, d\xi + \int_{|\xi| > (1/h(t))} e^{-t \Re(\Phi(\xi))} |\xi| \, d\xi \right) \]
\[ \leq c_5 e^{2M_0/a} \left( c_6 (h(t))^{-d-1} + \int e^{-t \Re(\Phi(\xi))} |\xi| \, d\xi \right) \]
\[ \leq c_7 e^{2M_0/a} (h(t))^{-d-1}. \]
It follows that
\[ p_t^{h(at)}(y) \geq c_1 (h(t))^{-d} - dc_7 e^{2M_0/a} (h(t))^{-d-1} |y| \geq \frac{1}{2} c_1 (h(t))^{-d}, \]
provided \(|y| \leq \frac{c_4}{2d} e^{-2M_0/a} h(t)\), which clearly yields (28).

Let \(a \in (0, 1)\) and \(t \in T\). For \(r > 0\), \(|x| > r + h(at)\) by (22) and (17) we get
\[ \tilde{P}^{h(at)}_t (B(x, r)) \geq e^{-M_0/a} t \nu_h (B(x, r)) = e^{-M_0/a} t \nu (B(x, r)). \]
This, (28) and (3) for \(x \in A\) yield
\[
\begin{align*}
p_t (x + tb) &= \tilde{P}^{h(at)}_t (x) = \int \tilde{P}^{h(at)}_t (x - z) \tilde{P}^{h(at)}_t (dz) \\
&\geq c_1 (h(t))^{-d} \tilde{P}^{h(at)}_t (B(x, c_2 e^{-c_3/a} h(t))) \\
&\geq c_8 t (h(t))^{-d+\gamma} f(|x| + c_2 e^{-c_3/a} h(t)),
\end{align*}
\]
for a constant \(c_8 = c_8(a)\), provided \(|x| > h(at) + c_2 e^{-c_3/a} h(t)\). By Lemma 7 we have \(p_t (x + tb) \geq C_{14} (h(t))^{-d}\) for \(|x| < C_6 h(t)\). Using Lemma 11 we choose \(a \in (0, 1)\) such that \(h(at)/h(t) + c_2 e^{-c_3/a} \leq C_6\) and we obtain (7) and (6) follows from Lemma 7.

**Lemma 12.** If \(\nu (\mathbb{R}^d) = \infty\) and (3) holds for some \(m \in \mathbb{N}_0\), then \(\tilde{p}_t \in C^m_b (\mathbb{R}^d)\) and for every \(n \in \mathbb{N}_0\) such that \(m \geq n\) and every \(\beta \in \mathbb{N}_0^d\) such that \(|\beta| \leq m - n\) there exists a constant \(C_{19} = C_{19}(m, n)\) such that
\[
|\partial_\beta^\gamma p_t (y)| \leq C_{19} [h(t)]^{-d-|\beta|} (1 + |y|/h(t))^{-n}, \quad y \in \mathbb{R}^d, t \in T.
\]

**Proof.** The existence of the density \(\tilde{p}_t \in C^m_b (\mathbb{R}^d)\) is a consequence of \(13\), 8 and [26 Proposition 28.1]. Similarly like in the proof of Lemma 8 we consider \(g_t (y) = \lfloor h(t) \rfloor \tilde{p}_t (h(t) y)\) and the infinitely divisible distribution \(\pi_t (dy) = g_t (y) dy\). It follows from (2.16) in [28] that there exists a constant \(c_1\) such that
\[
\int |y|^n \lambda_t (dy) \leq c_1, \quad t \in T,
\]
for every \(n \geq 2\), where \(\lambda_t\) is the Lévy measure of \(\pi_t\). Moreover using (8) and (18) we get
\[
\begin{align*}
\int |\mathcal{F}( \pi_t ) (\xi) | |\xi|^m d\xi &= \int |\mathcal{F} (\tilde{p}_t) (\xi/h(t)) | |\xi|^m d\xi \\
&= |h(t)|^{d+m} \int |\mathcal{F} (\tilde{P}_t) (\xi) | |\xi|^m d\xi \\
&\leq |h(t)|^{d+m} \int e^{2M_0} \exp [-t \Re (\Phi (\xi))] |\xi|^m d\xi \leq M_8 e^{2M_0}. 
\end{align*}
\]
Using [28, Proposition 2.1] we obtain
\[ |\partial^\beta_y g_t(y)| \leq c_2(1 + |y|)^{-n}, \quad y \in \mathbb{R}^d, \]
for \(|\beta| + n \leq m\), and \(c_2 = c_2(m, n)\), and the lemma follows.

\textbf{Proof of Theorem 3} The existence of the density \(p_t \in C^m_b(\mathbb{R}^d)\) is a consequence of (8) and [28, Proposition 28.1, or [22, Proposition 0.2. Using (8) we obtain

\begin{align*}
|\partial^\beta_x p_t(x)| &= (2\pi)^{-d} \left| \int (-i)^{|\beta|} \xi^\beta e^{-ix\xi} e^{-t\Phi(\xi)} d\xi \right| \\
&\leq (2\pi)^{-d} \left( \int_{|\xi| \leq (1/h(t))} e^{-t \text{Re}(\Phi(\xi))|\xi|^2} d\xi + \int_{|\xi| > (1/h(t))} e^{-t \text{Re}(\Phi(\xi))|\xi|^2} d\xi \right) \\
&\leq (2\pi)^{-d} \left( \int_{|\xi| \leq (1/h(t))} |\xi|^{|\beta|} d\xi + [h(t)]^{m-|\beta|} \int_{|\xi| > (1/h(t))} e^{-t \text{Re}(\Phi(\xi))|\xi|^m} d\xi \right) \\
&\leq c_1(h(t))^{-d-|\beta|}, \\
\end{align*}

for \(x \in \mathbb{R}^d\) and \(t \in T\). It follows from Lemma 12 Corollary 10 and (25) that

\begin{align*}
|\partial^\beta_x (\tilde{p}_t \ast \tilde{P}_t)(x)| &= \left| \int \partial^\beta_x \tilde{p}_t(x - y) \tilde{P}_t(dy) \right| \\
&\leq \int |\partial^\beta_x \tilde{p}_t(x - y)| P_t(dy) \\
&\leq C_{19} [h(t)]^{-d-|\beta|} \int (1 + |x - y|/h(t))^{-n} \tilde{P}_t(dy) \\
&= C_{19} [h(t)]^{-d-|\beta|} \int \int_0^1 (1 + |x - y|/h(t))^{-n} ds \tilde{P}_t(dy)ds \\
&= C_{19} [h(t)]^{-d-|\beta|} \int_0^1 \tilde{P}_t \left( B(x, h(t)(s^{-\gamma} - 1)) \right) ds \\
&\leq c_2[h(t)]^{-d-|\beta|} \left( \int_0^1 s^{-\gamma} \left( h(t)(s^{-\gamma} - 1) \right)^\gamma ds + \int_0^1 \left( 1 + \frac{|x|}{2h(t)} \right)^{-n} ds \right) \\
&\leq c_2[h(t)]^{-d-|\beta|} \left( t[h(t)]^\gamma f (|x|/4) \int_0^1 s^{-\gamma/n} ds + \left( 1 + \frac{|x|}{2h(t)} \right)^{-n} \right) \\
&= c_3[h(t)]^{-d-|\beta|} \left( t[h(t)]^\gamma f (|x|/4) + \left( 1 + \frac{|x|}{2h(t)} \right)^{-n} \right),
\end{align*}

for \(x \in \mathbb{R}^d\), \(t \in T\), and this and (29) yield (8).
4 Examples

In what follows we assume that
\[ \nu(A) \approx \int_S \int_0^\infty 1_A(s\theta)Q(s) \, ds \mu(d\theta), \]
for nondegenerate measure \( \mu \) and a nonincreasing function \( Q \). We assume also that \( \mu \) is a \( \gamma - 1 \)-measure on \( S \) for some \( \gamma \in [1, d] \), i.e.
\[ (30) \quad \mu(S \cap B(\theta, \rho)) \leq c \rho^{\gamma-1}, \quad \theta \in S, \rho > 0. \]

It is easy to check that
\[ \nu(A) \leq cQ(\delta(A))(\delta(A))^{1-\gamma}[\text{diam}(A)]^\gamma, \quad A \in \mathcal{B}(\mathbb{R}^d), \]
and so the assumption (1) is satisfied with \( f(s) = s^{1-\gamma}Q(s) \). Furthermore, it follows from (17) that (2) holds for every \( Q \) such that
\[ (31) \quad Q(s) \leq cQ(2s), \quad s > 0. \]

In the following theorem we obtain upper estimates for a specific class of jump processes. For simplification we include here only symmetric case and \( b = 0 \).

**Theorem 4.** Let \( \alpha \in (0, 2], \kappa > 0, \alpha > \kappa \beta > \alpha - 2, \) and \( \beta > 1 \) if \( \alpha = 2 \). If the Lévy measure \( \nu \) satisfies
\[ (32) \quad \nu(A) \approx \int_S \int_0^\infty 1_A(s\theta)s^{-\alpha-1}[\log (1 + s^{-\kappa})]^{-\beta} \, ds \mu(d\theta), \]
is symmetric, i.e. \( \nu(-A) = \nu(A), b = 0, \mu \) is nondegenerate and there exists a constant \( \gamma \in [1, d] \) such that (30) holds then the measures \( P_t \) are absolutely continuous with respect to the Lebesgue measure and their densities \( p_t \) satisfy the following estimates.

1. Short time estimates:

   (a) for \( \alpha \in (0, 2) \) there exists a constant \( C_{20} \) such that for every \( t \in (0, 1) \) and \( x \in \mathbb{R}^d \), we have
   \[ p_t(x) \leq C_{20}t^{-d/\alpha}(\log (1 + 1/t))^{d\beta/\alpha} \min \left\{ 1, \frac{t^{1+\gamma/\alpha}[\log (1 + |x|^{-\kappa})]^{-\beta}}{\log (1 + 1/t)} \frac{|x|^{\gamma+\alpha}}{|x|^{\gamma+\alpha}} \right\}. \]

   (b) for \( \alpha = 2 \) there exist constants \( C_{21}, C_{22}, C_{23} \) such that for every \( t \in (0, 1) \) and \( x \in \mathbb{R}^d \), we have
   \[ p_t(x) \leq C_{21}t^{-d/2}(\log (1 + 1/t))^{d(\beta-1)/2} \times \min \left\{ 1, \frac{t^{1+\gamma/2}[\log (1 + |x|^{-\kappa})]^{-\beta}}{(\log (1 + 1/t))^{\gamma(\beta-1)/2} |x|^{\gamma+2}} + e^{-C_{22}|x|} \log (1 + 1/t) \right\}. \]
2. Large time estimates: for $\alpha \in (0, 2]$ there exists a constant $C_{24}$ such that for every $t > 1$ and $x \in \mathbb{R}^d$, we have

$$p_t(x) \leq C_{24} t^{-d/(\alpha - \kappa \beta)} \min \left\{ 1, t^{1+\gamma/((\alpha - \kappa \beta))} |x|^{-\gamma - \alpha} \left[ \log \left( 1 + |x|^{-\kappa} \right) \right]^{-\beta} \right\}.$$ 

Proof. Let

$$Q(s) = s^{-1-\alpha} \left[ \log(1 + s^{-\kappa}) \right]^{-\beta}, \quad s \in (0, \infty).$$

The function $Q$ is decreasing, satisfies (31) and \( \int_0^\infty (1 \wedge s^2) Q(s) \, ds < \infty \). Furthermore for $r \in (0, 1)$ we have

$$\int_0^r s^2 Q(s) \, ds \approx \int_0^r s^{-1-\alpha} \left[ \log(2s^{-\kappa}) \right]^{-\beta} \, ds = \kappa^{-\beta} 2^{(2-\alpha)/\kappa} \int_0^\infty e^{-u(2-\alpha)u^{-\beta}} \, du \approx \begin{cases} r^{2-\alpha} \left[ \log(1 + \frac{1}{r}) \right]^{-\beta} & \text{for } \alpha \in (0, 2), \\ \left[ \log(1 + \frac{1}{r}) \right]^{-\beta+1} & \text{for } \alpha = 2, \end{cases}$$

and for $r > 1$ we get

$$\int_0^r s^2 Q(s) \, ds \approx \int_0^1 s^2 Q(s) \, ds + \int_1^r s^{-1-\alpha} s^{\kappa \beta} \, ds \approx r^{2-\alpha+\kappa \beta}.$$ 

Using Corollary 3 and 2 (with decreasing function $g(r) = r^{-(\kappa \beta/2)} \left[ \log \left( 1 + r^{-\kappa} \right) \right]^{-\beta}$ for $\alpha \in (0, 2)$ and $g(r) = r^{-\kappa \beta} \left[ \log \left( 1 + r^{-\kappa \beta/(\beta-1)} \right) \right]^{-1-\beta}$ for $\alpha = 2$) we obtain

$$\Phi(\xi) \approx |\xi|^\alpha \left[ \log \left( 1 + |\xi|^\kappa \right) \right]^{-\beta},$$

for $\alpha \in (0, 2)$, and

$$\Phi(\xi) \approx |\xi|^2 \left[ \log \left( 1 + |\xi|^{\kappa \beta/((\beta-1))} \right) \right]^{-\beta}$$

for $\alpha = 2$.

For $s > 0$, set $F_\alpha(s) = s^\alpha \left[ \log(1+s^\kappa) \right]^{-\beta}$ for $\alpha \in (0, 2)$ and $F_2(s) = s^2 \left[ \log \left( 1 + s^{\kappa \beta/(\beta-1)} \right) \right]^{-1-\beta}$. The functions $F_\alpha$ are increasing for every $\alpha$. We let $g_\alpha(r) = r^{(\kappa \beta/\alpha)} \left[ \log \left( 1 + r^{\kappa \beta/\alpha} \right) \right]^{-\beta}$ for $\alpha \in (0, 2)$ and $g_2(r) = \left( r \left[ \log(1+r) \right]^{-\beta/2} \right)^{1/2}$. Then there exists $r_0 = r_0(\alpha, \kappa, \beta)$ such that for $r > r_0$ and $\alpha \in (0, 2)$ we have

$$F_\alpha \left( g_\alpha(r) \right) = r \left( \log(r) \right)^\beta \left[ \log \left( 1 + \left( r \left( \log(1+r) \right)^\beta \right)^{\kappa/\alpha} \right) \right]^{-\beta} \approx r \left( \log(r) \right)^\beta \left( \log r + \beta \log \log r \right)^{-\beta} \approx r \left[ \frac{\log r + \beta \log \log r}{\log r} \right]^{-\beta} \approx r.$$
Similarly $F_2(g_2(r)) \approx r$ for sufficiently large $r$. This shows that $F_\alpha^{-1}(r) \approx g_\alpha(r)$ for $r > r_0$. For $r < r_1 = r_1(\alpha, \kappa, \beta)$ we have $F_\alpha(r) \approx r^{\alpha-\kappa\beta}$ and $F_\alpha^{-1}(r) \approx r^{1/(\alpha-\kappa\beta)}$. It yields
\[
h(t) = \frac{1}{\psi^{-1}(\frac{1}{t})} \approx t^{1/(\alpha-\kappa\beta)}, \quad t \geq 1,
\]
and
\[
h(t) \approx t^{1/\alpha} \left[ \log \left(1 + \frac{1}{t}\right) \right]^{1/\alpha}, \quad t \in (0, 1),
\]
for $\alpha \in (0, 2)$, and
\[
h(t) \approx t^{1/2} \left[ \log \left(1 + \frac{1}{t}\right) \right]^{1/2}, \quad t \in (0, 1),
\]
for $\alpha = 2$. Moreover the assumptions of Lemma 5 and Theorem 1 are satisfied with $T = (0, \infty)$. The estimate given in Theorem 1 holds, i.e.
\[
p_t(x) \lesssim C_1 (h(t))^{-d} \min \left\{ 1, t \left[ h(t) \right]^\gamma \frac{f(|x|/4)}{e^{-C_2 \frac{|x|}{h(t)}}} \right\},
\]
\[x \in \mathbb{R}^d, t \in (0, \infty),\]
for $f(s) = s^{1-\gamma}Q(s)$. We have
\[
t = \frac{1}{\psi(1/h(t))} \approx h(t)^\alpha \left[ \log(1 + h(t)^{-\kappa}) \right]^\beta,
\]
for $\alpha \in (0, 2)$ and
\[
t = \frac{1}{\psi(1/h(t))} \approx h(t)^2 \left[ \log(1 + h(t)^{\kappa\beta}) \right]^{\beta-1},
\]
for $\alpha = 2$. Let $g(t, |x|) = th(t)^\gamma f(|x|/4)$. For $\alpha \in (0, 2)$ we obtain
\[
g(t, |x|) \approx h(t)^{\alpha+\gamma} \left[ \log(1 + h(t)^{-\kappa}) \right]^{\beta} |x|^{-\alpha-\gamma} \left[ \log(1 + |x|^\kappa) \right]^{-\beta}
\]
\[= \left[ \frac{|x|}{h(t)} \right]^{-\alpha-\gamma} \left[ \log(1 + h(t)^{-\kappa}) \right]^\beta \left[ \log(1 + |x|^\kappa) \right]^{-\beta}.
\]
Using the fact that
\[
\frac{u}{v} \land 1 \leq \frac{\log(1+u)}{\log(1+v)} \leq \frac{u}{v} \lor 1, \quad u, v > 0,
\]
we get
\[
g(t, |x|) \geq C_1 e^{-C_2 \frac{|x|}{h(t)}} \log \left(1 + \frac{C_1|\lambda|}{\kappa}\right),
\]
(35)
for some constant \( c_1 \). Similarly for \( \alpha = 2 \) we have
\[
g(t, |x|) \approx \left[ \frac{|x|}{h(t)} \right]^{-2-\gamma} \left[ \frac{\log(1 + h(t)^{-\kappa})}{\log(1 + |x|^{-\kappa})} \right] \left[ \frac{\log(1 + h(t)^{\kappa\beta})}{\log(1 + h(t)^{-\kappa})} \right]^{\beta-1}.
\]

Let
\[
A(t) = \frac{\left[ \log(1 + h(t)^{\kappa\beta}) \right]^{\beta-1}}{\left[ \log(1 + h(t)^{-\kappa}) \right]^{\beta}}.
\]

We have \( c_2^{-1} \leq A(t) \leq c_2 \), for some constant \( c_2 \) and \( t \geq 1 \) (but \( A(t) \to 0 \) for \( t \to 0 \)). Therefore
\[
g(t, |x|) \geq c_3 e^{-C_2 \frac{|x|}{h(t)} \log(1 + \frac{C_2 |x|}{h(t)})}, \quad t \geq 1.
\]

The case of \( \nu \) satisfying locally \((32)\) with \( \alpha = 2, \beta \in (1, 2] \) and \( \gamma = d \) for \( t < 1 \) and \( |x| < 1 \) was investigated also in \([20]\). Theorem 1.1 in \([20]\) contains the estimate
\[
p_t(x) \leq c_1 \min \left\{ t^{-d/2} \left( \frac{\log 2}{t} \right)^{d(\beta-1)/2}, \frac{t}{|x|^{d+1} \left( \frac{2}{|x|} \right)^{\beta-1}} \right\}, \quad |x| < 1, t < 1,
\]
and here \((33)\) with \( \gamma = d \) yields
\[
p_t(x) \leq c_2 \min \left\{ t^{-d/2} \left( \frac{\log 2}{t} \right)^{d(\beta-1)/2}, \frac{t}{|x|^{d+1} \left( \frac{2}{|x|} \right)^{\beta}} + h(t)^{-d} e^{-C_2 \frac{|x|}{h(t)} \log(1 + \frac{C_2 |x|}{h(t)})} \right\},
\]
for \( |x| < 1 \) and \( t < 1 \). We note that there exists a constans \( c_3 \) such that
\[
\frac{t}{|x|^{d+2} \left( \frac{2}{|x|} \right)^{\beta}} + h(t)^{-d} e^{-C_2 \frac{|x|}{h(t)} \log(1 + \frac{C_2 |x|}{h(t)})} \leq c_3 \frac{t}{|x|^{d+2} \left( \frac{2}{|x|} \right)^{\beta-1}},
\]
for \( |x| < 1, t < 1 \), which can be shown similarly as \((36)\) in the proof of Theorem \([4]\) and so \((33)\) improves the results of \([20]\). Exact estimate in this case is still an open question.

Using Theorem \([2]\) we can obtain estimates from below. For example, if \( \nu \) satisfies the assumptions of Theorem \([4]\) and additionally for some finite set \( D_0 = \{ \theta_1, \theta_2, ..., \theta_n \} \subset S \) and a positive constant \( c_0 \) we have
\[
\mu(\{ \theta_k \}) \geq c_0, \quad \theta_k \in D_0, \quad k = 1, 2, ..., n,
\]
then
\[
p_t(x) \geq c_4 t^{-d/\alpha} \left( \log(1 + 1/t) \right)^{d\beta/\alpha} \min \left\{ 1, \left( \frac{t^{1+1/\alpha} \log(1 + |x|^{-\kappa})}{\log(1 + 1/t)} \right)^{\beta/\alpha} \right\}, \quad t \in (0, 1),
\]
for $\alpha \in (0, 2)$ and $x \in D = \{x \in \mathbb{R}^d : x = r\theta, r > 0, \theta \in D_0\}$, and
\[
    p_t(x) \geq c_5 t^{-d/2} (\log (1 + 1/t))^{d(\beta-1)/2} \min \left\{ 1, \frac{t^{3/2}[\log (1 + |x|^{-\kappa}]^{-\beta}}{(\log (1 + 1/t))^{(\beta-1)/2} |x|^\beta} \right\}, \ t \in (0, 1),
\]
for $\alpha = 2$ and $x \in D$, and
\[
    p_t(x) \geq c_6 t^{-d/(\alpha-\kappa\beta)} \min \left\{ 1, t^{1+1/(\alpha-\kappa\beta)}|x|^{-1-\alpha} [\log (1 + |x|^{-\kappa}]^{-\beta} \right\}, \ t \geq 1,
\]
for $\alpha \in (0, 2], x \in D$.

In the following corollary we consider the case of $\alpha \in (0, 2)$, $\gamma = d$ and give both side estimates of the densities and estimates of their derivatives. We omit the proof which is a verification of the assumption of Theorems 1, 2 and 3 analogous to the proof of Theorem 4.

**Corollary 13.** Let $\alpha \in (0, 2)$, $\kappa > 0$, $\alpha > \kappa\beta > \alpha - 2$. If the Lévy measure $\nu(dy) = g(y)dy$ satisfies
\[
g(y) \approx |y|^{-d-\alpha} [\log (1 + |y|^{-\kappa}]^{-\beta},
\]
and is symmetric, i.e. $g(-y) = g(y)$, then the measures $P_t$ are absolutely continuous with respect to the Lebesgue measure and their densities $p_t$ satisfy
\[
p_t(x) \approx \min \left\{ t^{-d/\alpha} (\log (1 + 1/t))^{d\beta/\alpha}, t|x|^{-d-\alpha} [\log (1 + |x|^{-\kappa}]^{-\beta} \right\}, \ t \in (0, 1), x \in \mathbb{R}^d,
\]
and
\[
p_t(x) \approx \min \left\{ t^{-d/(\alpha-\kappa\beta)}, t|x|^{-d-\alpha} [\log (1 + |x|^{-\kappa}]^{-\beta} \right\}, \ t \geq 1, x \in \mathbb{R}^d.
\]
Furthermore, for every $\eta \in \mathbb{N}_0^d$ there exist constants $C_{25}, C_{26}$ such that for $t \in (0, 1), x \in \mathbb{R}^d$, we have
\[
|\partial_x^n p_t(x)| \leq C_{25} t^{(-d-|\eta|)/\alpha} (\log (1 + 1/t))^{(d+|\eta|)/\beta} \min \left\{ 1, \frac{t^{1+d/\alpha} [\log (1 + |x|^{-\kappa}]^{-\beta}}{|x|^{d+\alpha} (\log (1 + 1/t))^{d\beta/\alpha}} \right\},
\]
and for $t \geq 1, x \in \mathbb{R}^d$, we have
\[
|\partial_x^n p_t(x)| \leq C_{26} t^{(-d-|\eta|)/(\alpha-\kappa\beta)} \min \left\{ 1, t^{1+d/(\alpha-\kappa\beta)}|x|^{-d-\alpha} [\log (1 + |x|^{-\kappa}]^{-\beta} \right\}.
\]
If $Q(s) \approx s^{-1-\alpha} q(s) \phi(s)$ where $\alpha \in (0, 2)$, and the functions $s^{-1-\alpha} q(s), \phi(s)$ are nonincreasing and positive on $[0, \infty)$, $q$ and $\phi$ are bounded and satisfy
\[
q(s) \leq cq(2s), \ \phi(s_1)\phi(s_2) \leq c\phi(s_1 + s_2),
\]
where $c > 0$ is independent of $s$. Then the measure $Q$ is absolutely continuous with respect to the Lebesgue measure and its density is bounded and satisfies
\[
\left| \partial_x^n Q(x) \right| \leq C_{27} t^{(-d-|\eta|)/\alpha} (\log (1 + 1/t))^{(d+|\eta|)/\beta} \min \left\{ 1, \frac{t^{1+d/\alpha} [\log (1 + |x|^{-\kappa}]^{-\beta}}{|x|^{d+\alpha} (\log (1 + 1/t))^{d\beta/\alpha}} \right\},
\]
and for $t \geq 1, x \in \mathbb{R}^d$, we have
\[
\left| \partial_x^n Q(x) \right| \leq C_{28} t^{(-d-|\eta|)/(\alpha-\kappa\beta)} \min \left\{ 1, t^{1+d/(\alpha-\kappa\beta)}|x|^{-d-\alpha} [\log (1 + |x|^{-\kappa}]^{-\beta} \right\}.
\]
for every $s, s_1, s_2 > 0$ and some constant $c$, and if

$$\int_0^\infty s^{1-\alpha} q(s) \frac{\phi(s)}{\phi(s/2)} ds < \infty,$$

then it can be checked that (2) holds. Such examples of Lévy measures were investigated in [31, 30, 15] so we do not repeat here detailed estimates of their densities but we give below some estimates of the derivatives which follow from Theorem 3.

**Corollary 14.** Let $m \geq 0$, $\beta \in (0, 1]$, $\alpha \in (0, 2)$, $\kappa \leq 1 + \alpha$, and $\kappa < \alpha$ if $m = 0$. If the Lévy measure $\nu$ satisfies

$$\nu(A) \approx \int_S \int_0^\infty 1_A(s\theta) s^{-1-\alpha}(1+s)^\kappa e^{-ms^\beta} ds \mu(d\theta),$$

is symmetric, i.e., $\nu(A) = \nu(-A)$, $b = 0$, $\mu$ is nondegenerate and fulfills (30) with $\gamma \in [1, d]$, then the measures $P_t$ are absolutely continuous with respect to the Lebesgue measure and their densities $p_t \in C_b^\infty(\mathbb{R}^d)$ satisfy

$$|\partial_\eta^2 p_t(x)| \leq C_{27} (h(t))^{-d-|\eta|} \min \left\{ 1, \frac{t |h(t)|^\gamma (1 + |x|)^\kappa}{|x|^{\gamma+\alpha}} e^{-m|x|/4} + \left( 1 + \frac{|x|}{h(t)} \right)^{-n} \right\},$$

for every $|\eta| \in \mathbb{N}_0^d$, every $n \in \mathbb{N}$, $n > \gamma$ and a constant $C_{27} = C_{27}(|\eta|, n)$, where

$$h(t) \approx t^{1/\alpha}, \quad \text{for} \quad t < 1,$$

and

$$h(t) \approx \begin{cases} 
  t^{1/2} & \text{for} \quad m > 0, \quad \kappa \leq 1 + \alpha, \\
  t^{1/2} & \text{for} \quad m = 0, \quad \kappa < \alpha - 2, \\
  (t \log(1+t))^{1/2} & \text{for} \quad m = 0, \quad \kappa = \alpha - 2, \\
  t^{1/(\alpha-\kappa)} & \text{for} \quad m = 0, \quad \alpha - 2 < \kappa < \alpha.
\end{cases}$$

for $t > 1$.

We consider in the last example the discrete Lévy measure

$$\nu(dy) = \sum_{i=1}^d \sum_{n=-\infty}^\infty 2^n \beta (\delta_{2^{-n}e_i}(dy) + \delta_{-2^{-n}e_i}(dy)),$$

where $0 < \beta < 2\kappa$ and $\{e_i\}_{i=1}^d$ is the standard basis in $\mathbb{R}^d$. Using Proposition 1 we easily get $\text{Re}(\Phi(\xi)) \approx |\xi|^{\beta/\kappa}$, $h(t) \approx t^{\kappa/\beta}$ in this case. It follows from Lemma 3 that (3) and (5) are satisfied with $T = (0, \infty)$ and it is also easy to check that (1) and (2) hold.
with $\gamma = 0$ and $f(s) = s^{-\beta/\kappa}$. Therefore for the corresponding semigroup (we let $b = 0$) we obtain the following estimate of the density

$$p_t(x) \leq c_1 t^{-d\kappa/\beta} \min \left\{ 1, t|x|^{-\beta/\kappa} \right\}, \quad t > 0, x \in \mathbb{R}^d,$$

and their derivatives

$$|\partial_\eta^np_t(x)| \leq c_2 t^{-(d+|\eta|)\kappa/\beta} \min \left\{ 1, t|x|^{-\beta/\kappa} \right\}, \quad t > 0, x \in \mathbb{R}^d,$$

for every $\eta \in \mathbb{N}_0^d$ with $c_2 = c_2(\eta)$. Using Theorem 2 with $A = \text{supp}\, \nu$ we obtain the lower estimate

$$p_t(x) \geq c_3 t^{-d\kappa/\beta} \min \left\{ 1, t|x|^{-\beta/\kappa} \right\}, \quad t > 0, x \in \{2^{-n}\kappa e_i, -2^{-n}\kappa e_i : n \in \mathbb{Z}, i = 1, \ldots, d\}.$$

The estimates in this case for $d = 1$ were obtained previously in [17] (see Example 4.2).

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