Internal wave propagation in a two-fluid rigid lid system over submerged bars

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Abstract. In this research, the wave propagation in a two-fluid system of different density is investigated. The lower fluid bounded underneath by a flat bottom with two bars over it. The upper fluid is bounded above by a rigid lid, which means that the surface waves have negligible amplitude. Shallow water equation is used to explain the internal wave’s profile. The effect of the upper fluid to the transmitted wave’s amplitude of the lower fluid after passing the two bars will be analyzed. The optimal dimension of the two bars and the distance between them are determined so that the transmitted wave’s amplitude of the lower fluid is minimum. Both the optimal dimension and the distance of the bars are depending on the value of the upper fluid’s density. As the upper fluid’s density increases, the optimal dimension, and distance are decreasing with the overall value of the optimal dimension is smaller than the optimal distance of the bars.

1. Introduction

Wave propagation in a two-fluid system of different density has been investigated by some researchers such as Cao, et. al [1] using KdV equation, Xu [2] using ILW equation, Kampf [3], Meng [4] using Benjamin-Ono equation, and Barros [5] using combination between KdV and ILW equation. The limitation using the KdV equation is we only can simulate one direction propagation wave. Then KDV will fail to simulate the scattering process of wave propagation over a rigid structure. In 1999, Tuck and Wiryanto [7] derived the model of steady internal wave based on a composite long wave equation. In case of the bottom topography is involved, the model is modified by adding an external force in the form of a new term in the equation, like what has been done by Choi, et. al. [6]. To simplify the problem, we consider a submerged bar as the bottom topography. In the case of one-layer fluid, Wiryanto [8] has calculated the transmitted and reflected the amplitude of the wave and compared them with respect to the incoming amplitude of the wave. The optimal dimension of the bar that gives minimal transmitted amplitude has also been determined. Jamhuri and Wiryanto [9] have extended this problem to the case of the bottom with two bars over it. Wiryanto and Mungkasi [10] have extended the problem to the case of two-layer fluid with the surface of the upper fluid is defined as rigid using Potential Theory. Using the potential theory, we end up with a system of governing equation that is more difficult to solve analytically and numerically. Therefore, to overcome that issue, we use shallow water type model that is relatively easy to solve numerically as written in Magdalena et.al [11, 12, 13].

In this paper, we propose a two-layer shallow water model to investigate the two-fluid system over two submerged bars with a rigid lid on the upper layer. The equations will be solved to determine the
dispersion relation that will be used to calculate the wave number. The transmitted and reflected amplitude of the wave is determined using wave elevation and flux continuity as the depth of the lower layer is changed. The wave elevation and flux continuity give a system of equations that can be solved analytically or numerically easier than the previous research. By observing the transmitted and reflected amplitude, the optimal dimension of the bars and the distance between them can be obtained. The results will be compared to the results presented in Wiryanto and Mungkasi [10].

2. Mathematical model

Consider a rigid lid system over two bars as described in Figure 1, consists of two layers of fluid with $\rho_1$ is the density of upper fluid and $\rho_2$ is the density of the lower fluid. In case of the rigid lid, we set the upper fluid’s surface as a solid flat boundary, so there will be no change in its elevation or $\eta_1(x, t) = 0$. On the contrary, the lower fluid will flow over the bars with elevation $\eta_2(x, t)$. The horizontal velocity of the upper fluid and lower fluid are $u_1(x, t)$ and $u_2(x, t)$, respectively. The two bars at the bottom have the same length and height which are $L$ and $d$ respectively, with the distance between them denoted as $S$. The water thickness is $h_1$ for the upper layer and $h_2$ for the lower layer. The depth of the lower layer over the bar is $h_3 = h_2 - d$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1}
\caption{Illustration of two-layer fluid with a rigid lid system.}
\end{figure}

Here, we use the shallow water equation for two layers fluid written as

\begin{align}
\eta_{1t} - \eta_{2t} &= -h_1 u_{1x}, \\
\eta_{2t} &= -h u_{2x}, \\
u_{2t} &= -\frac{\rho_1}{\rho_2} g \eta_{1x} - \frac{\rho_2 - \rho_1}{\rho_2} g \eta_{2x}, \\
u_{1t} &= -g \eta_{1x},
\end{align}

where $g$ is gravity acceleration and $h$ is water depth which following the function below.

$$h = \begin{cases} h_2 & \text{for domain } \Omega_1, \Omega_3, \text{and } \Omega_5 \\ h_3 & \text{for domain } \Omega_2 \text{ and } \Omega_4 \end{cases}$$

In this case, we set $\eta_1(x, t) = 0$, so that $\eta_{1t} = 0$ and $\eta_{1x} = 0$, and it will produce

\begin{align}
\eta_{2t} &= h_1 u_{1x}, \\
\eta_{2t} &= -h u_{2x}, \\
u_{2t} &= -\frac{\rho_2 - \rho_1}{\rho_2} g \eta_{2x}, \\
u_{1t} &= 0.
\end{align}

The equations will be solved analytically to investigate the lower fluid or internal wave propagation due to the upper fluid’s profile.
3. Result and Discussion

3.1. Dispersion relation
We assume the internal wave is a monochromatic wave that has an amplitude \( a_l \) and follows the function

\[
\eta_2 = a_l e^{i(k_l x - \omega t)},
\]

in which \( i = \sqrt{-1} \), \( k \) is wave number and \( \omega \) is angular frequency. We divided the domain of observation into five domains as illustrated in Figure 1, which are \( \Omega_j \) where \( j = 1,2,3,4,5 \). The corresponding wave number for each domain is denoted by \( k_j \). The dispersion relation that will be determined in this section is the relation between \( k_j \) and \( \omega \) for each domain of observation. Differentiate equation (2) with respect to \( t \) and (3) with respect to \( x \), we obtain \( u_{2xt} = -\frac{1}{h} \eta_{2t} \) and \( u_{2tx} = -\frac{\rho_2 - \rho_1}{\rho_2} \eta_{2x} \). Because the function \( u_2(x,t) \) is continue, then we can write that \( u_{2xt} = u_{2tx} \), so that we get

\[
\frac{1}{h} \eta_{2tt} = \frac{\rho_2 - \rho_1}{\rho_2} \eta_{2xx}.
\]

Substituting equation (5) to equation (6), we get dispersion relation written as

\[
c = \frac{\omega}{k_j} = \sqrt{g' h_j},
\]

where \( g' = \frac{\rho_2 - \rho_1}{\rho_2} g \). By solving the dispersion relation, we will obtain the value of \( k_j \), with \( k_1 = k_3 = k_5 \) is the corresponding wave number for a domain with depth \( h_2 \) and \( k_2 = k_4 \) is the corresponding wave number for a domain with depth \( h_3 \). To see the effect of the upper fluid to the internal wave, we can compare the dispersion relation in (7) with the dispersion relation for the one-fluid system. We set the value of depth without bar \( h_2 = 1 \), \( \alpha = 2 \) and \( g = 10 \), that will give the wave number for the one-fluid system is \( k_1 = 0.6325 \). Using \( \rho_2 = 1 \) and \( \rho_1 = 0.05 \), wave number for the internal wave in the two-fluid system given by \( k_1 = 0.6489 \), which is a little bit bigger than the wave number for the one-fluid system. By increasing density \( \rho_1 = 0.8 \), we obtain the wave number that also increases which is \( k_1 = 1.4142 \). Compared to the result in [10], using equation (7) give us a smaller wave number in case of a two-fluid system when \( \rho_1 = 0.05 \), but bigger when \( \rho_1 = 0.8 \).

3.2. The optimal length of one bar
In this section, we will investigate the optimal length of one bar, so that the transmitted amplitude of the internal wave that flows over the bar can be as minimum as possible. In order to obtain the optimal length of the bar, we need to determine the transmitted amplitude of the internal wave. Slightly different from what has been described in Figure 1, here, we divided the domain of observation into three domains only, which are \( \Omega_1 = (-\infty,0) \), \( \Omega_2 = (0,L) \), and \( \Omega_3 = (L,\infty) \), where \( L \) is the length of the bar. As mentioned in the previous section, the depth of the internal wave is \( h_2 \), and the depth of the internal wave above the bar is \( h_3 = h_2 - d \), with \( d \) is the height of the bar. Therefore, the corresponding wave number to fluid depth \( h_2 \) and \( h_3 \) is \( k_1 \) and \( k_2 \) respectively calculated using equation (7).

Since the internal wave propagates to the right direction, in sub-domain \( \Omega_1 \) there are incoming and reflected waves with amplitude \( a_l \) and \( a_r \) respectively. In the second sub-domain \( \Omega_2 \) there are waves with \( b_l \) as transmitted amplitude and \( b_r \) is reflected amplitude. For domain \( \Omega_3 \), there is only transmitted wave with amplitude \( c_l \). Consequently, we can express the surface elevation of the internal wave as

\[
(8)
\]
flux of the internal wave must be the same. By limiting of we obtain transmitted and no wave that is reflected. For those purposes, we need to make sure that function (8) and (9) is continuous. It means that at the point \( \frac{x}{g_{1859}} = 10 \), function of the flux that comes with the internal wave. By substituting function (8) to equation (1), (2), (3) and (4), then solve the equation, we will produce the function of flux written as

\[
\eta_2(x, t) = \begin{cases} 
 a_1 e^{i(k_1 x - \omega t)} + a_r e^{-i(k_1 x + \omega t)} & \text{if } x \in \Omega_1, \\
 b_t e^{i(k_2 x - \omega t)} + b_r e^{-i(k_2 x + \omega t)} & \text{if } x \in \Omega_2, \\
 c_t e^{i(k_1 x - \omega t)} & \text{if } x \in \Omega_3.
\end{cases}
\]

To determine the transmitted and reflected amplitude, we need to solve our model to obtain the function of the flux that comes with the internal wave. By substituting function (8) to equation (1), (2), (3) and (4), then solve the equation, we will produce the function of flux written as

\[
u_2(x, t) = \begin{cases} 
 \frac{\omega}{k_1 b_2} (a_t e^{i(k_1 x - \omega t)} - a_r e^{-i(k_1 x + \omega t)}) & \text{if } x \in \Omega_1, \\
 \frac{\omega}{k_2 b_3} (b_t e^{i(k_2 x - \omega t)} + b_r e^{-i(k_2 x + \omega t)}) & \text{if } x \in \Omega_2, \\
 \frac{\omega}{k_1 b_2} c_t e^{i(k_1 x - \omega t)} & \text{if } x \in \Omega_3.
\end{cases}
\]

To obtain the optimal length of the bar, we need to determine the value of \( \frac{|c_t|}{|a_t|} \) for the various value of \( L \) and find the minimum one. In case \( L \to 0 \), it should be \( \frac{|c_t|}{|a_t|} \to 1 \), meaning all the wave is transmitted and no wave that is reflected. For those purposes, we need to make sure that function (8) and (9) is continuous. It means that at the point \( x = 0 \) and \( x = L \), the value of surface elevation and flux of the internal wave must be the same. By limiting \( \eta_2 \) and \( u_2 \) from both side at \( x \to 0 \) and \( x \to L \) we obtain

\[
a_t + a_r = b_t + b_r, \\
b_t e^{ik_2 L} + b_r e^{-i k_2 L} = c_t e^{i k_1 L}, \\
k_2 (a_t - a_r) = k_1 (b_t - b_r), \\
k_3 (b_t e^{i k_2 L} - b_r e^{-i k_2 L}) = k_2 c_t e^{i k_1 L}.
\]

Solving equation (10), we will get the value of \( a_t, b_t, b_r \) and \( c_t \) that are scaled with respect to incoming amplitude \( a_t \). To simplify the calculation, we define \( a_t = 1 \). Then we plot \( |c_t| \) versus the various value of \( L \), so that we can obtain the optimal value of \( L \) that can cause \( |c_t| \) as minimum as possible.

To produce the results in Figure 2, we use the same parameters that have been used in [10], so that we can compare the results. The parameters we use are \( h_2 = 3.5, h_3 = 2.0, \rho_2 = 1 \), with \( \omega = 2 \) and \( g = 10 \). The results presented in Figure 2 obtained when \( \rho_1 = 0.1 \). For \( L = 3.04 \), Wiryanto in [10] obtained the transmitted amplitude \( |c_t| = 0.98563 \) and stated that the larger value of \( \rho_1 \), the smaller transmitted amplitude obtained. Here, using the same parameters, we obtained the minimum transmitted amplitude for \( L = 3.04 \) is \( |c_t| = 0.9628 \), which is smaller than the transmitted amplitude obtained in [10]. For the same \( L \) and \( \rho_1 = 0.5 \), we obtained \( |c_t| = 0.9664 \), which is slightly larger than the result for \( \rho_1 = 0.1 \). Yet, if we use \( \rho_1 = 0.2 \), we obtained \( |c_t| = 0.9622 \), which is slightly smaller than the result when \( \rho_1 = 0.1 \). It doesn’t confirm the statement claimed by Wiryanto in [10], because, from our results, we can’t exactly say whether it will be larger or smaller when \( \rho_1 \) is larger.

In Figure 2, we collected various value of \( L \) and present the value of \( |c_t| \) corresponding to \( L \). We define the optimal length of the bar \( L_{opt} \) is the smallest \( L \) giving the smallest \( |c_t| \). Using \( \rho_1 = 0.1 \), we obtained the minimum transmitted amplitude is \( |c_t| = 0.9621 \) given by \( L_{opt} = 3.33 \). Compared to the result given in [10], which is \( |c_t| = 0.98562 \) for \( L = 3.00 \), our \( L_{opt} \) is larger, yet, gives smaller \( |c_t| \). Similar to those results, for \( \rho_1 = 0.5 \), we obtained \( L_{opt} = 2.50 \) corresponding to \( |c_t| = 0.9621 \), which have larger \( L_{opt} \) and smaller \( |c_t| \) compared to the result in [10].
Following the previous section, we need to substituting function (11) to equation (1), (2), (3), and (4) and then solve it in order to get the function of flux that comes with the internal wave. We got this following function.

\[
\eta_2(x, t) = \begin{cases} 
\frac{\omega}{k_1 h_2} (a_t e^{ik_1 x - \omega t} - a_r e^{-ik_1 x + \omega t}) & \text{if } x \in \Omega_1, \\
\frac{\omega}{k_2 h_3} (b_t e^{ik_2 x - \omega t} - b_r e^{-ik_2 x + \omega t}) & \text{if } x \in \Omega_2, \\
\frac{\omega}{k_1 h_2} (c_t e^{ik_1 x - \omega t} - c_r e^{-ik_1 x + \omega t}) & \text{if } x \in \Omega_3, \\
\frac{\omega}{k_2 h_3} (d_t e^{ik_2 x - \omega t} - d_r e^{-ik_2 x + \omega t}) & \text{if } x \in \Omega_4, \\
\frac{\omega}{k_1 h_2} e^{ik_1 x - \omega t} & \text{if } x \in \Omega_5, 
\end{cases}
\]

(11)

Following the previous section, we found that after passes a bar, the amplitude of the transmitted internal wave will be reduced. The reduction also can be done after the wave passes through the next bar. In this section, our task is to determine the distance between the first bar and the second bar so that the amplitude of the transmitted wave is minimum. Similar to the previous section, we formulate the function for the elevation of the internal wave by combining transmitted and reflected wave, but in this section, we divided the domain into five domain, which are \( \Omega_1 = (-\infty, 0) \), \( \Omega_2 = (0, L) \), \( \Omega_3 = (L, L + S) \), \( \Omega_4 = (L + S, 2L + S) \) and \( \Omega_5 = (2L + S, \infty) \). The two bars have the same length which is \( L \) and the same height which is \( d \), while \( S \) is the distance between two bars. The fluid depth is \( h_2 \) and the fluid depth above the bars is \( h_3 = h_2 - d \), with the wave number corresponding to that depth, is \( k_1 \) and \( k_2 \). The formulation of internal wave elevation can be written as

\[
\eta_2(x, t) = \begin{cases} 
\frac{a_t e^{ik_1 x - \omega t} + a_r e^{-ik_1 x + \omega t}}{2} & \text{if } x \in \Omega_1, \\
\frac{b_t e^{ik_2 x - \omega t} + b_r e^{-ik_2 x + \omega t}}{2} & \text{if } x \in \Omega_2, \\
\frac{c_t e^{ik_1 x - \omega t} + c_r e^{-ik_1 x + \omega t}}{2} & \text{if } x \in \Omega_3, \\
\frac{d_t e^{ik_2 x - \omega t} + d_r e^{-ik_2 x + \omega t}}{2} & \text{if } x \in \Omega_4, \\
\frac{e_t e^{ik_1 x - \omega t}}{2} & \text{if } x \in \Omega_5. 
\end{cases}
\]

(13)

Similar to the previous section, here, the function of elevation and the flux of the internal wave have to continue. By limiting from both side at \( x \to 0, x \to L, x \to L + S, \) and \( x \to 2L + S, \) we obtain

**Figure 2.** Plot of \(|c_1|\) (vertical) versus \(L\) (horizontal) for \( \rho_1 = 0.1 \), obtained \( L_{opt} = 3.33 \) corresponding to \(|c_1| = 0.9621\).
\[ a_t + a_r = b_t + b_r \]
\[ b_t e^{ik_2L} + b_r e^{-ik_2L} = c_t e^{ik_1L} + c_r e^{-ik_1L} \]
\[ c_t e^{ik_1(L+S)} + c_r e^{-ik_1(L+S)} = d_t e^{ik_2(L+S)} + d_r e^{-ik_2(L+S)} \]
\[ d_t e^{ik_2(2L+S)} + d_r e^{-ik_2(2L+S)} = e_t e^{ik_1(2L+S)} \]

Solving equation (13), we will get the value of all the amplitudes in the form of the ratio between the amplitudes and incoming amplitude \( a_t \). Further, we will calculate the optimal distance between two bars denoted by \( S_{opt} \) which is a distance between two bars with a length \( l_{opt} \) from the previous section that gives the minimum value of \( |e_t|/a_t \). Similar to the previous section, we defined \( a_t = 1 \). In Figure 3 (Left), we take \( h_2 = 3.5, h_3 = 2.0, \rho_2 = 1, \rho_1 = 0.1, \) and \( L = 3.33 \) which is the optimal length from the previous section. We defined the optimal distance between two bars \( S_{opt} \) as the smallest value of \( S \) correspond to \( l_{opt} \) giving the smallest value of \( |e_t| \). Figure (3) shows that for \( S = 0 \), we obtain \( |e_t| = 1 \). This can be explained that for \( S = 0 \), we are back to the system for one bar with \( L \) is two times larger than \( l_{opt} \), or \( L = 6.66 \). The given transmitted amplitude is the same as our previous result in Figure (2). We plot the value of \( |e_t| \) for the various value of \( S \) to determine \( S_{opt} \). We obtain the optimal distance \( S_{opt} = 4.4 \) giving the value of the transmitted amplitude \( |e_t| = 0.8615 \).

\[ \text{Figure 3. (Left) Plot of } |e_t| \text{ (vertical) versus } S \text{ (horizontal) for } \rho_1 = 0.1, \text{ obtained } S_{opt} = 4.40 \text{ corresponding to } |e_t| = 0.8615. \] (Right) Plot of \( l_{opt} \) (lower curve, vertical) and \( S_{opt} \) (upper curve, vertical) versus \( \rho_1 \) (horizontal).

In the need for comparison, we then take \( L = 3.0 \) as the length of the bar. Wiryanto [10] obtain that for given \( L \), the optimal distance is \( S_{opt} = 3.55 \) giving \( |e_t| = 0.9445 \), while using our model, we obtain for given \( L \) that the optimal distance is \( S_{opt} = 4.8 \) which is larger than the result in [10] but gives smaller transmitted amplitude which is \( |e_t| = 0.8645 \). Instead, for \( |e_t| = 0.9445 \), the corresponding distance is \( S = 2.2 \) which is smaller than the result in [10]. Figure 3 (Right) shows the plot of \( l_{opt} \) (lower curve) and \( S_{opt} \) (upper curve) for the various value of \( \rho_1 \). The curve of \( l_{opt} \) and \( S_{opt} \) has the same pattern, which is decreasing by increasing \( \rho_1 \), with the overall value of \( l_{opt} \) is smaller than \( S_{opt} \). As \( \rho_1 \to 0 \) we obtain surface wave which means there will be no obstacle above the fluid, that is why the transmitted wave becomes higher than another case with bigger \( \rho_1 \). Consequently, longer bars are needed to minimize the transmitted amplitude. As \( \rho_1 \to 1 \) the fluids
become one, and since there is rigid lid above the fluids, there will be no transmitted wave propagates through the channel. It means only short bars will be needed to minimalize the transmitted wave.

4. Conclusion
We have formulated a model based on Shallow Water Equation to investigated the two-fluid rigid lid system over two submerged bars. Using the model, we have presented the effect of rigid lid system to the propagation of the internal wave. Using wave elevation and flux continuity, we have obtained the transmitted and reflected the amplitude of the wave, which is dependent on the wave number that obtained using dispersion relation. Then, this transmitted amplitude is compared to the incoming amplitude. By studied the comparison, we have determined the optimal length of one bar and the optimal distance between two bars which depend on the optimal length. We have presented the profile of optimal length and optimal distance as the density of the upper fluid is changed. Compared to the previous research, the pattern of our results for both the length of one bar and the distance between two bars are the same, but with a slightly different value of the optimal length and optimal distance.

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