Vanishing Diffusion Limits and Long Time Behaviour of a Class of Forced Active Scalar Equations

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Abstract

We investigate the properties of an abstract family of advection diffusion equations in the context of the fractional Laplacian. Two independent diffusion parameters enter the system, one via the constitutive law for the drift velocity and one as the prefactor of the fractional Laplacian. We obtain existence and convergence results in certain parameter regimes and limits. We study the long time behaviour of solutions to the general problem and prove the existence of a unique global attractor. We apply the results to two particular active scalar equations arising in geophysical fluid dynamics, namely the surface quasigeostrophic equation and the magnetogreostrophic equation.

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1. Introduction

Active scalar equations have been a topic of considerable study in recent years, in part because they arise in many physical models and in part because they present challenging nonlinear PDEs. In particular, such equations are prevalent in mathematical fluid dynamics. One such equation is the surface quasi-geostrophic equation (SQG) which was introduced by Constantin, Majda and Tabak as a 2 dimensional analogue for the three dimensional Euler equations \[ \text{[8,25,32]} \]. Another model with related, but distinct, features, is the magnetogeostrophic equation (MG) which was proposed by Moffatt and Loper as a model for magnetogeostrophic turbulence \[ \text{[19,23,30,31]} \]. The physics of an active scalar equation is encoded in the constitutive law that relates the transport velocity vector \( u \) with a scalar field \( \theta \). This law produces a differential operator that when applied to the scalar field determines the velocity. The singular or smoothing properties of this operator are closely connected with the mathematics of the nonlinear advection equation for \( \theta \). In this present paper we study an abstract class of active scalar equations in \( \mathbb{T}^d \times (0, \infty) = [0, 2\pi]^d \times (0, \infty) \) with \( d \in \{2, 3\} \) of the form

\[
\begin{aligned}
\partial_t \theta + u \cdot \nabla \theta &= -\kappa \Delta^\gamma \theta + S, \\
u_j[\theta] &= \partial_{x_i} T_{ij}^\nu[\theta], \theta(x, 0) = \theta_0(x)
\end{aligned}
\]  

(1.1)

where \( \nu \geq 0, \kappa \geq 0, \gamma \in (0, 2] \) and \( \Delta := \sqrt{-\Delta} \). Here \( \theta_0 \) is the initial datum and \( S = S(x) \) is a given function that represents the forcing of the system. We assume that

\[
\int_{\mathbb{T}^d} \theta_0(x) dx = \int_{\mathbb{T}^d} S(x) = 0,
\]

and throughout this paper, we consider mean-zero (zero average) solutions.\(^2\) \( \{T_{ij}^\nu\}_{\nu \geq 0} \) is a sequence of operators which satisfy that

\begin{align*}
\text{A1} & \quad \partial_{x_i} \partial_{x_j} T_{ij}^\nu f = 0 \quad \text{for any smooth functions } f \quad \text{for all } \nu \geq 0. \\
\text{A2} & \quad T_{ij}^\nu : L^\infty(\mathbb{T}^d) \to BMO(\mathbb{T}^d) \quad \text{are bounded uniformly in } \nu \quad \text{for all } \nu \geq 0. \\
\text{A3} & \quad \text{For each } \nu > 0, \text{ there exists a constant } C_\nu > 0 \text{ such that for all } 1 \leq i, j \leq d, \\
& \quad \quad \quad |\hat{T}_{ij}^\nu(k)| \leq C_\nu |k|^{-3}, \quad \forall k \in \mathbb{Z}^d \setminus \{|k| = 0\}. \\
\text{A4} & \quad \text{For each } 1 \leq i, j \leq d \text{ and } \nu \geq 0, \hat{T}_{ij}^\nu(k) = 0 \text{ for } |k| = 0.
\end{align*}

\(^1\) We point out that most of the results given in our work hold for \( d \geq 2 \).

\(^2\) Such a mean zero assumption is common in many physical models which include SQG equation and MG equation; see \[ \text{[9] and [17]} \], for example.
A5 There exists a constant $C_0 > 0$ independent of $\nu$, such that for all $1 \leq i, j \leq d$,

$$\sup_{\nu \in [0, 1]} \sup_{(k \in \mathbb{Z}^d)} \sup_{\nu \in [0, 1]} |\hat{r}_{ij}^\nu(k)| \leq C_0.$$  

Remark 1.1. There are several remarks for the Assumptions A1–A5 given above; these are as follows:

- A1 implies that $u = u[\theta]$ is divergence-free for all $\nu \geq 0$. Hence together with (1.2), it immediately implies that $\theta$ obeys

$$\int_{\mathbb{R}^d} \theta(x, t) \, dx = 0, \quad \forall t \geq 0. \quad (1.3)$$

- A2 implies that the drift velocity $u$ lies in the space $L^\infty_t BMO_x^{-1}$ for all $\nu \geq 0$.

- A3 implies that $u_j[\cdot] = \partial_{x_i} T_{ij}^\nu[\cdot]$ are operators of smoothing order 2 for $\nu > 0$, in the sense that for any $s \geq 0$ and $f \in L^p$ with $p > 1$,

$$\|\Lambda^s u[f]\|_{L^p} \leq C_v \|\Lambda^{s'} f\|_{L^p}, \quad (1.4)$$

where $s' = \max\{s - 2, 0\}$. Here $C_v$ is a positive constant which depends on $\nu$, $p$ and $d$ only, and $C_v$ may blow up as $\nu \to 0$.

- A4 implies that $u$ has zero mean, which is consistent with $\theta$ also having zero mean.

- A5 implies that $u_j[\cdot] = \partial_{x_i} T_{ij}^0[\cdot]$ is a singular operator of order 1, in the sense that for any $f \in L^p$ with $p > 1$,

$$\|u[f]\|_{L^p} \leq C_0 \|\nabla f\|_{L^p}, \quad (1.5)$$

for some positive constant $C_0$ which depends on $p$ and $d$ only.

The abstract family of active scalar equation (1.1) satisfying the properties A1–A5 include as special cases the SQG equation and the MG equation, both of which model phenomena in rotating fluids. The critical SQG equation is an example where the dimension $d = 2$, the diffusive parameter $\nu = 0$, the thermal diffusion $\kappa > 0$, the fractional power $\gamma = 1$, and the relation between the velocity $u$ and the scalar field $\theta$ is given by the perpendicular Riesz transform. We note that this is a singular integral operator of degree zero. Global well-posedness for the critical SQG equation was first proved in Kiselev, Nazarov and Volberg [26] and Caffarelli and Vasseur [3]. More recently there has been a considerable literature on the long time dynamics of the forced SQG equation including [5, 9, 10] and references therein.

The MG equation is an example where the dimension $d = 3$, the diffusive parameter $\nu \geq 0$, the thermal diffusion $\kappa \geq 0$, and the fractional power $\gamma = 2$. The derivation of the MG equation via the postulates in [31] reduces the MHD system to an active scalar equation

$$\partial_t \theta + u \cdot \nabla \theta = \kappa \Delta \theta + S, \quad (1.6)$$
where the constitutive law is obtained from the linear system

\[ e_3 \times u = -\nabla P + e_2 \cdot \nabla b + \theta e_3 + \nu \Delta u, \]  
(1.7)

\[ 0 = e_2 \cdot \nabla u + \Delta b, \]  
(1.8)

\[ \nabla \cdot u = 0, \; \nabla \cdot b = 0. \]  
(1.9)

This system encodes the vestiges of the physics in the problem, namely the Coriolis force, the Lorentz force and gravity. Vector manipulations of (1.7)–(1.9) give the expression

\[
[(\nu/Delta) - (e_2 \cdot \nabla)^2]u = -(\nu \Delta^2 - (e_2 \cdot \nabla)^2)\nabla \times (e_3 \times \nabla \theta) \\
+ (e_3 \cdot \nabla) \Delta (e_3 \times \nabla \theta).
\]  
(1.10)

Here \((e_1, e_2, e_3)\) denote Cartesian unit vectors. The explicit expression for the components of the Fourier multiplier symbol \(\hat{M}^\nu\) as functions of the Fourier variable \(k = (k_1, k_2, k_3) \in \mathbb{Z}^3\) with \(k_3 \neq 0\) are obtained from the constitutive law (1.10) to give

\[
\hat{M}^\nu_1(k) = [k_2 k_3|k|^2 - k_1 k_3(k_2^2 + \nu|k|^4)]D(k)^{-1},
\]  
(1.11)

\[
\hat{M}^\nu_2(k) = [-k_1 k_3|k|^2 - k_2 k_3(k_2^2 + \nu|k|^4)]D(k)^{-1},
\]  
(1.12)

\[
\hat{M}^\nu_3(k) = [(k_1^2 + k_2^2)(k_2^2 + \nu|k|^4)]D(k)^{-1},
\]  
(1.13)

where

\[
D(k) = |k|^2 k_3^2 + (k_2^2 + \nu|k|^4)^2.
\]  
(1.14)

In the magnetostrophic turbulence model the parameters \(\nu\), the nondimensional viscosity, and \(\kappa\), the nondimensional thermal diffusivity, are extremely small. The behaviour of the MG equation is dramatically different when the parameters \(\nu\) and \(\kappa\) are present (that is positive) or absent (that is zero). The limit as either or both parameters vanish is highly singular. Since both parameters multiply a Laplacian term, their presence is smoothing. However \(\kappa\) enters (1.6) in a parabolic heat equation role whereas \(\nu\) enters via the constitutive law (1.10). The mathematical properties of the MG equation have been determined in various settings of the parameters via an analysis of the Fourier multiplier symbol \(\hat{M}^\nu\) given by (1.11)–(1.14). When \(\nu = 0\) the relation between \(u\) and \(\theta\) is given by a singular operator of order 1. The implications of this fact for the inviscid MG\(^0\) equation are summarized in the survey article by FRIEDLANDER, RUSIN and VICOL [23]. In particular, when \(\kappa > 0\) the inviscid but thermally dissipative MG\(^0\) equation is globally well-posed [19].\(^3\) In contrast, when \(\nu = 0\) and \(\kappa = 0\), the singular inviscid MG\(^0\) equation is ill-posed in the sense of Hadamard in any Sobolev space [20]. In a recent paper FRIEDLANDER and SUEN [17] examine the limit of vanishing viscosity in the case when \(\kappa > 0\). They prove global existence of classical solutions to the forced MG\(^\nu\) equations and obtain strong convergence of solutions as the viscosity \(\nu\) vanishes.

\(^3\) \(L^\infty\) is the critical Lebesgue space with respect to the natural scaling for both the critically diffusive SQG and MG\(^0\) equations.
The purpose of our current paper is to investigate properties of the abstract system (1.1) under assumptions A1–A5, with emphasis on convergence results in the context of the fractional Laplacian. In Sect. 4 the parameter \( \nu \) is taken to be positive and the ensuing smoothing properties of \( T_{ij}^\nu \) permits existence and convergence in Sobolev space \( H^s \) as \( \kappa \) goes to zero. In contrast, when the parameter \( \nu \) is set to zero, A5 implies that \( \partial_{x_i} T_{ij}^\nu \) is a singular operator. In this case the existence and convergence results proved in Sect. 5 are restricted to analytic and Gevrey-class solutions. We note that the results in Sect. 5 are valid in the example of the critical SQG equation. In Sect. 6 we study the long time behaviour of solutions to the abstract system when both \( \nu > 0 \) and \( \kappa > 0 \). We prove that the solution map associated with (1.1) possesses a unique global attractor \( G^\nu \) in \( H^1 \) for all \( \nu > 0 \). With the restriction that \( \gamma \) lies in \([1, \, 2]\), we prove that \( G^\nu \) has finite fractal dimension.

In Sect. 7 we apply the results proved in Sect. 6 to the specific example of the MG equation. We prove convergence of \( G^\nu \) as \( \nu \) goes to zero to the global attractor \( A \) in \( L^2 \) for the MG\(^0\) equation whose existence was demonstrated in [17]. It is of interest to note that although we prove that \( G^\nu \) has finite fractal dimension for all \( \nu > 0 \), it is unknown whether or not \( A \) has finite fractal dimension.

2. Main Results

The main results that we prove for the forced problem (1.1) are stated in the following theorems, and they will be proved in Sects. 4, 5 and 6. These results will then be applied to the magnetogeostrophic (MG) active scalar equation and surface quasigeostrophic (SQG) equation which will be further discussed in Sects. 5 and 7.

**Theorem 2.1.** \((H^s\)-convergence as \( \kappa \to 0 \) when \( \nu > 0 \)) Let \( \nu > 0 \) and \( \gamma \in (0, \, 2] \) be given in (1.1), and let \( \theta_0, \, S \in C^\infty \) be the initial datum and forcing term respectively which satisfy (1.2). If \( \theta^\kappa \) and \( \theta^0 \) are smooth solutions to (1.1) for \( \kappa > 0 \) and \( \kappa = 0 \) respectively, then

\[
\lim_{\kappa \to 0} \| (\theta^\kappa - \theta^0)(\cdot, \, t)\|_{H^s} = 0
\]  

for all \( s \geq 0 \) and \( t \geq 0 \).

**Theorem 2.2.** \((Analytic\ convergence as \( \kappa \to 0 \) when \( \nu = 0 \))\) Let \( \nu = 0 \) and \( \gamma \in (0, \, 2] \) be given in (1.1), and let \( \theta_0, \, S \) be the initial datum and forcing term respectively which satisfy (1.2). Suppose that \( \theta_0 \) and \( S \) are both analytic functions. Then if \( \theta^\kappa, \, \theta^0 \) are analytic solutions to (1.1) for \( \kappa > 0 \) and \( \kappa = 0 \) respectively with initial datum \( \theta_0 \) on \( \mathbb{T}^d \times [0, \, \bar{T}] \) with radius of convergence at least \( \bar{\tau} \), then there exists \( T \leq \bar{T} \) and \( \tau = \tau(t) < \bar{\tau} \) such that, for \( t \in [0, \, T] \), we have

\[
\lim_{\kappa \to 0} \| (\Lambda^r e^{\tau \Lambda} \theta^\kappa - \Lambda^r e^{\tau \Lambda} \theta^0)(\cdot, \, t)\|_{L^2} = 0,
\]  

where \( \Lambda := (-\Delta)^{\frac{1}{2}} \) and \( r > \frac{d}{2} + \frac{5}{2} \) is the Sobolev exponent.
Theorem 2.3. (Existence of global attractors) Let $S \in L^{\infty} \cap H^1$ be the forcing term. For $\nu, \kappa > 0$ and $\gamma \in (0, 2]$, let $\pi^{\nu}(t)$ be the solution operator for the initial value problem (1.1) via

$$\pi^{\nu}(t) : H^1 \rightarrow H^1, \quad \pi^{\nu}(t)\theta_0 = \theta(\cdot, t), \quad t \geq 0.$$  

Then the solution map $\pi^{\nu}(t) : H^1 \rightarrow H^1$ associated to (1.1) possesses a unique global attractor $G^{\nu}$ for all $\nu > 0$. In particular, if we assume that $\gamma \in [1, 2]$, then for all $\nu > 0$, the global attractor $G^{\nu}$ of $\pi^{\nu}(t)$ further enjoys the following properties:

- $G^{\nu}$ is fully invariant, namely
  $$\pi^{\nu}(t)G^{\nu} = G^{\nu}, \quad \forall t \geq 0.$$

- $G^{\nu}$ is maximal in the class of $H^1$-bounded invariant sets.
- $G^{\nu}$ has finite fractal dimension.

Remark 2.4. There are several remarks for the main results as given above:

- The existence of global-in-time $H^s$-solutions to (1.1) when $\nu > 0$ will be given by Theorem 4.1.
- The local-in-time existence of analytic solutions to (1.1) when $\nu = 0$ will be given by Theorem 5.1. In particular, under a smallness assumption on the initial data in terms of Sobolev norm, the analytic solutions as claimed by Theorem 5.1 can be extended globally in time; refer to Theorem 5.2 for more details.
- Under a stronger assumption on the Fourier symbols $\hat{T}^{\nu}_{ij}(k)$, one can extend Theorem 2.2 to Gevrey-class solutions when the initial datum $\theta_0$ and forcing term $S$ are both Gevrey-class functions; refer to Theorem 5.6 for more details.

3. Preliminaries

We introduce the following notations and conventions:

- We write $u := u[\theta]$, where $u$ is given by $u_j := \partial_{x_i} T^{\nu}_{ij}[\theta]$. We also refer $u[\cdot]$ to an operator in the sense that $u_j[f] = \partial_{x_i} T^{\nu}_{ij}[f]$ for appropriate functions $f$.
- To emphasise the dependence of solutions on $\nu$ and $\kappa$, we sometimes write $\theta = \theta^{\kappa}$ and $u = u^{\kappa} := u[\theta^{\kappa}]$ for varying $\kappa$, while we write $\theta = \theta^{(\nu)}$ and $u^{(\nu)}[\cdot] := \partial_{x_i} T^{\nu}_{ij}[\cdot]$ for varying $\nu$.
- $W^{s,p}$ is the usual inhomogeneous Sobolev space with norm $\| \cdot \|_{W^{s,p}(\mathbb{T}^d)}$, and we write $H^s = W^{s,2}$. For simplicity, we write $\| \cdot \|_{L^p} = \| \cdot \|_{L^p(\mathbb{T}^d)}$, $\| \cdot \|_{W^{s,p}} = \| \cdot \|_{W^{s,p}(\mathbb{T}^d)}$, etc. unless otherwise specified.
- We define $\langle \cdot, \cdot \rangle$ to be the $L^2$-inner product on $\mathbb{T}^d$, that is,

$$\langle f, g \rangle := \int_{\mathbb{T}^d} f g \, dx$$

for any $f, g \in L^2$. 
Regarding the constants used in this work, we have the following conventions:
– $C$ shall denote a positive and sufficiently large constant, whose value may change from line to line;
– $C$ is allowed to depend on the size of $T_d$ and other universal constants which are fixed throughout this work;
– In order to emphasise the dependence of $C$ on a certain quantity $Q$ we usually write $C_Q$ or $C(Q)$.

We recall the following Sobolev embedding inequalities from the literature (see for example Bahouri–Chemin–Danchin [2] and Ziemer [38]):

• Let $d \geq 2$ be the dimension. There exists $C = C(d) > 0$ such that
  \[
  \|f\|_{L^\infty} \leq C \|f\|_{W^{2,d}}. \tag{3.1}
  \]

• For $q > d$, there exists $C = C(q) > 0$ such that
  \[
  \|f\|_{L^\infty} \leq C \|f\|_{W^{1,q}}. \tag{3.2}
  \]

• If $k > l$ and $k - \frac{d}{p} > l - \frac{d}{q}$, then there exists $C = C(k, l, d, p, q) > 0$ such that
  \[
  \|f\|_{W^{l,q}} \leq C \|f\|_{W^{k,p}}. \tag{3.3}
  \]

• For $q > 1$, $q' \in [q, \infty)$ and \( \frac{1}{q'} = \frac{1}{q} - \frac{s}{d} \), if $\Lambda^s h \in L^q$, there exists $C = C(q, q', d, s) > 0$ such that
  \[
  \|h\|_{L^{q'}} \leq C \|\Lambda^s h\|_{L^q}. \tag{3.4}
  \]

We also recall the following product and commutator estimates: if $s > 0$ and $p > 1$, then for all $f, g \in H^s \cap L^\infty$, we have

\[
\|\Lambda^s (fg)\|_{L^p} \leq C \left( \|f\|_{L^{p_1}} \|\Lambda^s g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}} \|g\|_{L^{p_4}} \right), \tag{3.5}
\]

\[
\|\Lambda^s (fg) - f \Lambda^s (g)\|_{L^p} \leq C \left( \|\nabla f\|_{L^{p_1}} \|\Lambda^{s-1} g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}} \|g\|_{L^{p_4}} \right), \tag{3.6}
\]

where \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} \) and $p, p_2, p_3 \in (1, \infty)$.

As in [15,28] and [33], for $s \geq 1$, the Gevrey-class $s$ is defined by

\[
\bigcup_{\tau > 0} D(\Lambda^r e^{\tau \Lambda^s}),
\]

where for any $r \geq 0$,

\[
D(\Lambda^r e^{\tau \Lambda^s}) := \{ f \in H^r : \|\Lambda^r e^{\tau \Lambda^s} f\|_{L^2} < \infty \},
\]
and we use the Gevrey-class real-analytic norm given by
\[
\|\Lambda^r e^{r\Lambda^\frac{1}{2}} f\|_{L^2}^2 := \sum_{k\in\mathbb{Z}^d} |k|^2 e^{2r|k|^\frac{1}{2}} |\hat{f}(k)|^2,
\]
where \(\tau = \tau(t) > 0\) denotes the radius of convergence. When \(s = 1\), we recover the class of real-analytic functions with radius of analyticity \(\tau\). When \(s > 1\), the Gevrey-classes consist of \(C^\infty\)-functions which are in general not analytic. In view of the mean-zero assumption (1.2), it makes sense to take \(\mathbb{Z}^d_* = \mathbb{Z}^d \backslash \{|k| = 0\}\) in the definition of Gevrey-class norm.

4. Existence and Convergence of \(H^s\)-Solutions When \(\nu > 0\)

In this section, we first obtain the global-in-time existence of \(H^s\)-solutions to (1.1) when \(\nu > 0\) by applying the De Giorgi iteration method. We then prove \(H^s\)-convergence of the solutions \(\theta\) to the active scalar equation (1.1) as \(\kappa \to 0\).

More precisely, given \(\nu > 0\) and \(\gamma \in (0, 2]\), we prove that if \(\theta^\kappa\) and \(\theta^0\) are smooth solutions to (1.1) for \(\kappa > 0\) and \(\kappa = 0\) respectively, then \(\|((\theta^\kappa - \theta^0)(\cdot, t))\|_{H^s} \to 0\) as \(\kappa \to 0\) for all \(t > 0\) and \(s \geq 0\).

4.1. Existence of \(H^s\)-Solutions

In this subsection, we prove the following theorem which gives the desired global-in-time wellposedness for (1.1) in Sobolev space \(H^s\) when \(\kappa \geq 0\) and \(\nu > 0\):

**Theorem 4.1.** (Global-in-time wellposedness in Sobolev space) Fix \(\nu > 0\), \(\gamma \in (0, 2]\) and \(s \geq 0\), and let \(\theta_0 \in H^s\) and \(S \in H^s \cap L^\infty\) be given.

- For any \(\kappa > 0\), there exists a global-in-time solution to (1.1) such that
  \[
  \theta^\kappa \in C([0, \infty); H^s) \cap L^2([0, \infty); H^{s+\frac{\gamma}{2}}). \tag{4.1}
  \]

- For \(\kappa = 0\), if we further assume that \(\theta_0 \in L^\infty\), then there exists a global-in-time solution to (1.1) such that \(\theta^0(\cdot, t) \in H^s\) for all \(t \geq 0\).

In view of the case when \(\kappa > 0\), the most subtle part for proving Theorem 4.1 is to estimate the \(L^\infty\)-norm of \(\theta^\kappa(\cdot, t)\) when \(\theta_0\) is not necessarily in \(L^\infty\). In achieving our goal, we apply De Giorgi iteration method which will be illustrated in Lemma 4.5.

We first recall the following energy inequalities which were established in [9] for the case \(\gamma = 1\). The general cases for \(\gamma \in (0, 2]\) follows similarly and we omit the details here.

**Proposition 4.2.** Assume that \(S \in L^2 \cap L^\infty\), and let \(\theta^\kappa\) be a smooth solution to (1.1). For \(\kappa > 0\), we have
\[
\|\theta^\kappa(\cdot, t)\|_{L^2}^2 + \kappa \int_0^t \|\Lambda^\frac{\gamma}{2} \theta^\kappa(\cdot, \tau)\|_{L^2}^2 d\tau \leq \|\theta_0\|_{L^2}^2 + \frac{t}{c_0\kappa} \|S\|_{L^2}^2, \quad \forall t \geq 0. \tag{4.2}
\]
If we further assume that \( \theta_0 \in L^\infty \), for all \( \kappa > 0 \), we get
\[
\| \theta^\kappa (\cdot, t) \|_{L^\infty} \leq \| \theta_0 \|_{L^\infty} e^{-c_0 \kappa t} + \frac{\| S \|_{L^\infty}}{c_0 \kappa}, \quad \forall t \geq 0,
\]
where \( c_0 > 0 \) is a universal constant which depends only on the dimension \( d \).

**Remark 4.3.** Instead of the decay estimate given in (4.3), we also have the following bound on \( \| \theta^\kappa (\cdot, t) \|_{L^\infty} \) for all \( \kappa \geq 0 \) provided that \( \theta_0 \in L^\infty \):
\[
\| \theta^\kappa (\cdot, t) \|_{L^\infty} \leq \| \theta_0 \|_{L^\infty} + \| S \|_{L^\infty}, \quad \forall t \geq 0.
\]

Next we state and prove the following lemma which gives local-in-time existence results for the equation (1.1):

**Lemma 4.4.** Let \( \nu > 0 \), \( \kappa > 0 \), \( \gamma \in (0, 2] \) and fix \( s \geq 0 \). Assume that \( \theta_0 \in H^s \) and \( S \in H^s \cap L^\infty \), then there exists \( T_* = T_*(\theta_0, S) > 0 \) and a unique solution \( \theta^\kappa \) of (1.1) such that
\[
\theta^\kappa \in C([0, T_*); H^s) \cap L^2([0, T_*); H^{s+\gamma/2}).
\]
When \( \kappa = 0 \) and \( s \geq 0 \), if \( \theta_0, S \in H^s \cap L^\infty \), then there exists \( T_* = T_*(\theta_0, S) > 0 \) and a unique solution \( \theta^0 \) of (1.1) such that \( \theta^0 (\cdot, t) \in H^s \) for all \( t \in [0, T_*) \).

**Proof.** We consider only the case for \( \kappa > 0 \). We note that the case for \( \kappa = 0 \) was addressed in [18]. Following the argument given in the proof of [22, Theorem 3.1], applying (1.4) for \( \nu > 0 \) and using standard energy method, there exists \( T_* = T_*(\theta_0, S) > 0 \) and a unique solution \( \theta^\kappa \) of (1.1) such that
\[
\theta^\kappa \in L^\infty([0, T_*); H^s) \cap L^2([0, T_*); H^{s+\gamma/2}),
\]
and in particular, it holds for \( t \in [0, T_*) \) that
\[
\| \theta^\kappa (\cdot, t) \|_{H^s}^2 \leq \| \theta_0 \|_{H^s}^2 + C t \| S \|_{H^s}^2 + C \int_0^t \| \theta^\kappa (\cdot, \tau) \|_{H^{s+\gamma/2}}^2 d\tau,
\]
where \( C \) is a positive constant which is independent of \( t \). It then remains to prove the continuity of \( \theta^\kappa \) in time. By the standard bootstrap argument, it suffices to show that \( \theta^\kappa (\cdot, t) \to \theta_0 \) in \( H^s \) as \( t \to 0^+ \). Using the fact that \( \theta^\kappa \in L^\infty([0, T_*); H^s) \), we apply Lemma 1.4 in [36, Chapter 3] to show that \( \theta^\kappa \in C_w([0, T_*); H^s) \). Therefore, using (4.5), we obtain
\[
\| \theta^\kappa (\cdot, t) - \theta_0 \|_{H^s}^2 \leq 2\| \theta_0 \|_{H^s}^2 - 2\langle \theta^\kappa (\cdot, t), \theta_0 \rangle_{H^s} + C t \| S \|_{H^s}^2 + C \int_0^t \| \theta^\kappa (\cdot, \tau) \|_{H^{s+\gamma/2}}^2 d\tau,
\]
where \( \langle \cdot, \cdot \rangle_{H^s} \) is the inner product on \( H^s \). As \( t \to 0^+ \), one can show that \( \| \theta^\kappa (\cdot, t) - \theta_0 \|_{H^s} \to 0 \) which follows by the argument given in [29] and we omit the details here. \( \square \)

The lemma below gives the desired bound on \( \| \theta^\kappa (\cdot, t) \|_{L^\infty} \) for the case when the time \( t \) is small, which will be used for proving Theorem 4.1. The idea follows from the work given in [3] and [10].
Lemma 4.5. (From $L^2$ to $L^\infty$) Fix $\nu > 0$, $\kappa > 0$, $s \geq 0$ and $\gamma \in (0, 2]$. Let $\theta(t)$ be the solution to (1.1) with initial datum $\theta_0 \in H^s$. Then for all $t \in (0, 1]$, we have

$$\|\theta(t)\|_{L^\infty} \leq C \left[ \left( \frac{2}{t} + 1 \right) \frac{d+1-\gamma}{2\nu} \left( \|\theta_0\|_{L^2} + \frac{\|S\|^2_{L^2}}{c_0^2 \kappa^2} \right) + \|S\|_{L^\infty} \right],$$

(4.6)

where $C = C(d) > 0$ is a constant which only depends on the dimension $d$.

Remark 4.6. By exploiting the bounds (4.3) and (4.6) on $\|\theta\|_{L^\infty}$, for $\theta_0 \in H^s$ with $s \geq 0$, we have, for $t \geq 1$ (see also [10, Theorem 3.2]),

$$\|\theta(t)\|_{L^\infty} \leq \frac{C}{\kappa} \left[ \|\theta_0\|_{L^2} + \frac{\|S\|^2_{L^2}}{\kappa^2} \right] e^{-c_0\kappa t} + \frac{1}{c_0\kappa} \|S\|_{L^\infty},$$

(4.7)

for some constant $C > 0$.

Proof of Lemma 4.5. Fix $\kappa > 0$, we drop $\kappa$ from $\theta^\kappa$ and write $\theta = \theta^\kappa$ for simplicity. Suppose that $M \geq 2\|S\|_{L^\infty}$, where $M > 0$ to be fixed later, we define

$$\lambda_n := M(1 - 2^{-n}), \quad n \in \mathbb{N} \cup \{0\},$$

and $\theta_n$ to be the truncated function $\theta_n = \max(\theta(t) - \lambda_n, 0)$. Fix $t \in (0, 1]$, we define the time cutoffs by

$$T_n := t(1 - 2^{-n}),$$

and we denote the level set of energy by

$$Q_n := \sup_{T_n \leq t \leq 1} \|\theta_n(\cdot, \tau)\|_{L^2}^2 + 2\kappa \int_{T_n}^1 \|\Lambda^\frac{\gamma}{2} \theta_n(\cdot, \tau)\|_{L^2}^2 d\tau.$$ 

Using the point-wise inequality given in [12, Proposition 2.3], for all $s \in (T_{n-1}, T_n)$, we have the following level set inequality:

$$\sup_{T_n \leq t \leq 1} \|\theta_n(\cdot, \tau)\|_{L^2}^2 + 2\kappa \int_{T_n}^1 \|\Lambda^\frac{\gamma}{2} \theta_n(\cdot, \tau)\|_{L^2}^2 d\tau \leq \|\theta_n(\cdot, s)\|_{L^2}^2 + 2\|S\|_{L^\infty} \int_{T_{n-1}}^1 \|\theta_n(\cdot, \tau)\|_{L^1} d\tau.$$ 

We take the mean value in $s$ on $(T_{n-1}, T_n)$ and multiply by $\frac{2^n}{t}$ to obtain

$$Q_n \leq \frac{2^n}{t} \int_{T_{n-1}}^1 \|\theta_n(\cdot, \tau)\|_{L^2}^2 d\tau + 2\|S\|_{L^\infty} \int_{T_{n-1}}^1 \|\theta_n(\cdot, \tau)\|_{L^1} d\tau,$$

(4.8)

and using (4.2), we also have

$$Q_0 \leq \|\theta_0\|_{L^2}^2 + \frac{1}{c_0\kappa} \|S\|_{L^2}^2.$$ 

(4.9)

We aim at bounding the right side of (4.8) by a power of $Q_{n-1}$. Using Hölder inequality and Sobolev embedding, there exists $C_d > 0$, which depends on $d$ such that

$$Q_{n-1} \geq C_d \|\theta_{n-1}\|_{L^\frac{2(d+1)}{d+\gamma}(\mathbb{T}^d \times [T_{n-1}, 1])}^2, \quad \forall n \in \mathbb{N}.$$ 

(4.10)
Since \( \theta_{n-1} \geq 2^{-n} M \) on the set \( \{ (x, \tau) : \theta_n(x, \tau) > 0 \} \), together with (4.10), we have
\[
\frac{2^n}{t} \int_{T_n}^1 \| \theta_n(\cdot, \tau) \|^2_{L^2} d\tau \leq \frac{2^{n+1}}{t} \int_{T_n}^1 \int_{\mathbb{T}^d} \theta_n^2 d\tau \cdot 1_{\{ \theta_n > 0 \}} \\
\leq \frac{2^{n+1}}{t} \int_{T_n}^1 \int_{\mathbb{T}^d} \theta_n^2 d\tau \cdot \left( \frac{2^n \theta_{n-1}}{M} \right)^{\frac{2^y}{\gamma+1}} \\
\leq 2 \left( \frac{2^n \left( \frac{d+1}{\gamma} \right)}{t M} \right)^{\frac{2^y}{\gamma+1}} \int_{T_n}^1 \theta_n^2 d\tau \\
\leq 2C_d \left( \frac{2^n \left( \frac{d+1}{\gamma} \right)}{t M} \right)^{\frac{d+1}{\gamma+1} - \frac{1}{\gamma}} Q_{n-1}^{\frac{d+1}{\gamma+1}}.
\]

Similarly, we have
\[
2 \| S \|_{L^\infty} \int_{T_{n-1}}^1 \| \theta_n(\cdot, \tau) \|_{L^1} d\tau \leq 2C_d \| S \|_{L^\infty} \left( \frac{2^n \left( \frac{d+1}{\gamma} \right)}{M} \right)^{\frac{d+1}{\gamma+1} - \frac{1}{\gamma}} Q_{n-1}^{\frac{d+1}{\gamma+1}}.
\]

Since \( M \geq 2 \| S \|_{L^\infty} \), we further obtain that
\[
Q_n \leq \left( \frac{2C_d}{t} + C_d \right) \left( \frac{2^n \left( \frac{d+1}{\gamma} \right)}{M} \right)^{\frac{d+1}{\gamma+1} - \frac{1}{\gamma}} Q_{n-1}^{\frac{d+1}{\gamma+1}}.
\]

If we assume that
\[
M \geq \left( \frac{2C_d}{t} + C_d \right)^{\frac{d+1}{\gamma} - \frac{1}{\gamma}} Q_0^{\frac{1}{\gamma}} \tag{4.11}
\]
then \( Q_n \to 0 \) as \( n \to \infty \). Hence using (4.9), if we choose \( M > 0 \) such that
\[
M \geq \left( \frac{2C_d}{t} + C_d \right)^{\frac{d+1}{\gamma} - \frac{1}{\gamma}} \left( \| \theta_0 \|_{L^2} + \frac{\| S \|_{L^2}}{\epsilon_0^2 \kappa^2} \right) + 2 \| S \|_{L^\infty},
\]
then it implies that \( \theta \) is bounded above by \( M \). Applying the same argument to \( -\theta \), we conclude that the bound (4.6) holds for all \( \tau \in (0, 1] \). \( \square \)

With the help of the bound (4.6) on \( \| \theta(\cdot, \tau) \|_{L^\infty} \), we are ready to give the proof of Theorem 4.1.

**Proof of Theorem 4.1.** It is enough to establish an *a priori* estimate on \( \theta^\kappa \) with initial data \( \theta_0 \in H^s \) for \( s \geq 0 \). We multiply (1.1) by \( \Lambda^{2s} \theta^\kappa \) and integrate over \( \mathbb{T}^d \) to obtain
\[
\frac{1}{2} \frac{d}{dt} \| \Lambda^s \theta^\kappa \|_{L^2}^2 + \kappa \| \Lambda^{s+\frac{2}{\gamma}} \theta^\kappa \|_{L^2}^2 \leq \left| \int_{\mathbb{T}^d} S \Lambda^{2s} \theta^\kappa \right| + \left| \int_{\mathbb{T}^d} u^\kappa \cdot \nabla \theta^\kappa \Lambda^{2s} \theta^\kappa \right|, \tag{4.12}
\]
The term $\int_{\mathbb{T}^d} \Sigma \Lambda^2 \theta^k$ is readily bounded by $\| \Lambda^s S \|_{L^2} \| \Lambda^r \theta^k \|_{L^2}$, and for $\int_{\mathbb{T}^d} u^k \cdot \nabla \theta^k \Lambda^2 \theta^k$, it can be estimated as follows:

$$\left| \int_{\mathbb{T}^d} u^k \cdot \nabla \theta^k \Lambda^2 \theta^k \right| = \left| \int_{\mathbb{T}^d} (u^k \theta^k) \cdot \nabla \Lambda^2 \theta^k \right| \leq \left| \int_{\mathbb{T}^d} \left[ \Lambda^s(u^k \theta^k) - u^k \Lambda^s \theta^k \right] \cdot \nabla \Lambda^s \theta^k \right| + \left| \int_{\mathbb{T}^d} u^k \Lambda^s \theta^k \cdot \nabla \Lambda^s \theta^k \right|. \quad (4.13)$$

Since $\text{div}(u^k) = 0$, the second term on the right side of (4.13) can be rewritten as follows:

$$\left| \int_{\mathbb{T}^d} u^k \Lambda^s \theta^k \cdot \nabla \Lambda^s \theta^k \right| = \left| \int_{\mathbb{T}^d} \Lambda^s \theta^k (\Lambda (u^k \cdot \Lambda^s \theta^k) - u^k \cdot \Lambda^s \theta^k) \right|. \quad (4.13)$$

Hence using (3.6), we have

$$\left| \int_{\mathbb{T}^d} u^k \Lambda^s \theta^k \cdot \nabla \Lambda^s \theta^k \right| \leq C \| \Lambda^s \theta^k \|_{L^2} \| \nabla u^k \|_{L^\infty} \| \nabla \Lambda^s \theta^k \|_{L^2} \leq C \| \Lambda u^k \|_{L^\infty} \| \Lambda^s \theta^k \|_{L^2}^2. \quad (4.13)$$

For the leading term on the right side of (4.13), we apply (3.6) again to obtain

$$\left| \int_{\mathbb{T}^d} \left[ \Lambda^s(u^k \theta^k) - u^k \Lambda^s \theta^k \right] \cdot \nabla \Lambda^s \theta^k \right| \leq \| \Lambda^s(u^k \theta^k) - u^k \Lambda^s \theta^k \|_{L^2} \| \nabla \Lambda^s \theta^k \|_{L^2} \leq \left( \| \Lambda u^k \|_{L^\infty} \| \Lambda^s \theta^k \|_{L^2} + \| \Lambda^s u^k \|_{L^2} \| \theta^k \|_{L^\infty} \right) \| \nabla \Lambda^s \theta^k \|_{L^2}. \quad (4.13)$$

Hence we have, from (4.13), that

$$\left| \int_{\mathbb{T}^d} u^k \cdot \nabla \theta^k \Lambda^2 \theta^k \right| \leq C \left( \| \Lambda u^k \|_{L^\infty} \| \Lambda^s \theta^k \|_{L^2} + \| \Lambda^s u^k \|_{L^2} \| \theta^k \|_{L^\infty} \right) \| \Lambda \theta^k \|_{L^2}. \quad (4.14)$$

By (1.4), the term $\| \Lambda^s u^k \|_{L^2}$ can be bounded by $C \| \Lambda^s \theta^k \|_{L^2}$, and for the term $\| \Lambda u^k \|_{L^\infty}$, using (3.2) for $q = d + 1$ and together with (1.4), we have

$$\| \Lambda u^k \|_{L^\infty} \leq C \| \Lambda u^k \|_{W^{1,d+1}} \leq C \| \theta^k \|_{L^{d+1}} \leq C \| \theta^k \|_{L^\infty}. \quad (4.15)$$

Therefore, we obtain from (4.14) that

$$\left| \int_{\mathbb{T}^d} u^k \cdot \nabla \theta^k \Lambda^2 \theta^k \right| \leq C \| \theta^k \|_{L^\infty} \| \Lambda^2 \theta^k \|_{L^2}^2. \quad (4.16)$$
We apply (4.16) on (4.12) and deduce that
\[
\frac{d}{dt} \| \Lambda^s \theta^\kappa \|_{L^2}^2 + 2\kappa \| \Lambda^{s+\frac{\nu}{2}} \theta^\kappa \|_{L^2}^2 \leq C \| \theta^\kappa \|_{L^\infty}^2 \| \Lambda^s \theta^\kappa \|_{L^2}^2 + 2 \| \Lambda^s S \|_{L^2} \| \Lambda^s \theta^\kappa \|_{L^2}.
\]  
(4.17)

For the case when \( \kappa > 0 \), using the bounds (4.6) and (4.7) on \( \theta^\kappa \), for each \( \tau > 0 \), there exists \( C_\tau > 0 \) which depends on \( \kappa, \tau, c_0, \| \theta_0 \|_{L^2}, \| S \|_{L^\infty} \) but independent of \( t \) such that
\[
\| \theta^\kappa(\cdot,t) \|_{L^\infty} \leq C_\tau, \quad \forall t \geq \tau.
\]  
(4.18)

Furthermore, by the continuity in time as proved in Lemma 4.4, we choose \( \tau > 0 \) sufficiently small such that
\[
\sup_{0 \leq t \leq \tau} \| \Lambda^s \theta^\kappa(\cdot,t) \|_{L^2} \leq 2 \| \Lambda^s \theta_0 \|_{L^2},
\]
Hence with the help of Grönwall’s inequality, we conclude from (4.17) that for \( \kappa > 0 \),
\[
\| \Lambda^s \theta^\kappa(\cdot,t) \|_{L^2} \leq (2 \| \Lambda^s \theta_0 \|_{L^2} + C \| \Lambda^s S \|_{L^2})(e^{CC_\tau t} + 1), \quad \forall t \geq 0.
\]  
(4.19)

Therefore, the above inequality implies that \( \| \Lambda^s \theta^\kappa(\cdot,t) \|_{L^2} \) remains finite for all positive time when \( \kappa > 0 \).

For the case when \( \kappa = 0 \), under the assumption that \( \theta_0 \in L^\infty \), we use the bound (4.4) to deduce from (4.17) that
\[
\| \Lambda^s \theta^0(\cdot,t) \|_{L^2} \leq \exp \left( Ct(\| \theta_0 \|_{L^\infty} + \| S \|_{L^\infty}) \right)(\| \Lambda^s \theta_0 \|_{L^2} + C \| \Lambda^s S \|_{L^2}), \quad \forall t > 0,
\]
and hence \( \| \Lambda^s \theta^0(\cdot,t) \|_{L^2} \) remains finite for all \( t > 0 \) as well.  

4.2. Convergence of \( H^s \)-Solutions as \( \kappa \to 0 \)

We are now ready to give the proof of Theorem 2.1.

**Proof of Theorem 2.1.** We let \( \theta_0, S \in C^\infty \). The proof is divided into three parts.

**Uniform \( H^s \)-bound:** For fixed \( v > 0 \), we first obtain a uniform (independent of \( \kappa \)) \( H^s \)-bound for all \( s \geq 0 \) on \( \theta^\kappa \). Apply the estimate (4.17) on \( \theta^\kappa \), we have
\[
\frac{d}{dt} \| \Lambda^s \theta^\kappa \|_{L^2}^2 + 2\kappa \| \Lambda^{s+\frac{\nu}{2}} \theta^\kappa \|_{L^2}^2 \leq C \| \theta^\kappa \|_{L^\infty} \| \Lambda^s \theta^\kappa \|_{L^2}^2 + 2 \| \Lambda^s S \|_{L^2} \| \Lambda^s \theta^\kappa \|_{L^2}.
\]  
(4.20)

Since \( \theta_0 \in C^\infty \subset L^\infty \), we can apply the uniform bound (4.4) to deduce from (4.20) that
\[
\frac{d}{dt} \| \Lambda^s \theta^\kappa \|_{L^2}^2 + 2\kappa \| \Lambda^{s+\frac{\nu}{2}} \theta^\kappa \|_{L^2}^2 \leq C(\| \theta_0 \|_{L^\infty} + \| S \|_{L^\infty}) \| \Lambda^s \theta^\kappa \|_{L^2}^2 + 2 \| \Lambda^s S \|_{L^2} \| \Lambda^s \theta^\kappa \|_{L^2}.
\]
Upon integrating the above inequality from 0 to \( t \), we obtain, for all \( \kappa \geq 0 \) that

\[
\| \Lambda^t \theta^\kappa (\cdot, t) \|_{L^2} \leq C (\| \Lambda^t \theta_0 \|_{L^2} + \| \Lambda^t S \|_{L^2}) \exp \left( Ct (\| \theta_0 \|_{L^\infty} + \| S \|_{L^\infty}) \right), \quad \forall t > 0.
\] (4.21)

**\( L^2 \)-convergence:** Fix \( \nu > 0 \) and \( \gamma \in (0, 2] \). We let \( \theta^\kappa, \theta^0 \) be the smooth solution to (1.1) for \( \kappa > 0 \) and \( \kappa = 0 \) respectively. Define \( \varphi = \theta^\kappa - \theta^0 \), then \( \varphi \) satisfies

\[
\partial_t \varphi + (u^\kappa - u^0) \cdot \nabla \theta^0 + u^\kappa \cdot \nabla \varphi = -\kappa \Lambda^\gamma \theta^\kappa,
\] (4.22)

where \( u^\kappa \) and \( u^0 \) are given by

\[
u^i := \partial_i T^\nu_i [\theta^\kappa], \quad u^0_j := \partial_j T^\nu_j [\theta^0]
\]

for \( 1 \leq i, j \leq d \). Multiply (4.22) by \( \varphi \) and integrate,

\[
\frac{1}{2} \frac{d}{dt} \| \varphi (\cdot, t) \|_{L^2}^2 = - \int_{\mathbb{T}^d} (u^\kappa - u^0) \cdot \nabla \theta^0 \cdot \varphi - \int_{\mathbb{T}^d} u^\kappa \cdot \nabla \varphi \cdot \varphi - \kappa \int_{\mathbb{T}^d} \Lambda^\gamma \theta^\kappa \cdot \varphi.
\] (4.23)

Using the divergence-free assumption on \( u^\kappa \), the term \( - \int_{\mathbb{T}^d} u^\kappa \cdot \nabla \varphi \cdot \varphi \) is zero. On the other hand, using (1.4), we have

\[
\| (u^\kappa - u^0)(\cdot, t) \|_{L^2} \leq C_\nu \| (\theta^\kappa - \theta^0)(\cdot, t) \|_{L^2}.
\]

Hence the first term on the right side of (4.23) can be bounded as follows:

\[
- \int_{\mathbb{T}^d} (u^\kappa - u^0) \cdot \nabla \theta^0 \cdot \varphi \leq \| \nabla \theta^0 (\cdot, t) \|_{L^\infty} \| (u^\kappa - u^0)(\cdot, t) \|_{L^2} \| \varphi (\cdot, t) \|_{L^2} \leq C_\nu \| \nabla \theta^0 (\cdot, t) \|_{L^\infty} \| \varphi (\cdot, t) \|_{L^2}^2.
\]

For the third term on the right side of (4.23), we can write

\[
-\kappa \int_{\mathbb{T}^d} \Lambda^\gamma \theta^\kappa \cdot \varphi \leq 4\kappa \| \Lambda^\gamma \theta^\kappa (\cdot, t) \|_{L^2}^2 + \frac{\kappa}{4} \| \varphi (\cdot, t) \|_{L^2}^2.
\]

Applying the above estimates on (4.23), we obtain

\[
\frac{1}{2} \frac{d}{dt} \| \varphi (\cdot, t) \|_{L^2}^2 \leq 4\kappa \| \Lambda^\gamma \theta^\kappa \|_{L^2}^2 + \frac{\kappa}{4} \| \varphi (\cdot, t) \|_{L^2}^2 + C_\nu \| \nabla \theta^0 (\cdot, t) \|_{L^\infty} \| \varphi (\cdot, t) \|_{L^2}^2 \leq \left[ C_\nu \| \nabla \theta^0 (\cdot, t) \|_{L^\infty} + \frac{\kappa}{4} \right] \| \varphi (\cdot, t) \|_{L^2}^2 + 4\kappa \| \Lambda^\gamma \theta^\kappa \|_{L^2}^2.
\] (4.24)

Using the bound (4.21), for sufficiently large \( s > 1 \), we have

\[
\| \nabla \theta^0 (\cdot, t) \|_{L^\infty} \leq C \| \Lambda^s \theta^\kappa (\cdot, t) \|_{L^2} \leq C (\| \Lambda^s \theta_0 \|_{L^2} + \| \Lambda^s S \|_{L^2}) \exp \left( Ct (\| \theta_0 \|_{L^\infty} + \| S \|_{L^\infty}) \right),
\]
as well as
\[ \| \Delta^\gamma \theta^\kappa \|_{L^2} \leq C \| \Delta^s \theta^\kappa (\cdot, t) \|_{L^2} \leq C (\| \Delta^s \theta_0 \|_{L^2} + \| \Delta^s S \|_{L^2}) \exp \left( Ct (\| \theta_0 \|_{L^\infty} + \| S \|_{L^\infty}) \right). \]

Hence by integrating (4.24) over \( t \) and using Grönwall’s inequality, for \( t > 0 \) there exist positive functions \( C_1(t), C_2(t) \) depending on \( t, \theta_0 \) and \( S \) such that
\[ \| \varphi (\cdot, t) \|_{L^2}^2 \leq C_1(t) e^{C_2(t)}, \]
where we recall that \( \varphi (\cdot, 0) = 0 \). Therefore, as \( \kappa \to 0 \), we have
\[ \lim_{\kappa \to 0} \int_\Omega |\theta^\kappa - \theta^0|^2(x, t) dx = \lim_{\kappa \to 0} \| \varphi (\cdot, t) \|_{L^2}^2 = 0. \tag{4.25} \]

**5. Existence and Convergence of Analytic and Gevrey-Class Solutions When \( \nu = 0 \)**

In this section, we address the existence and convergence of the analytic and Gevrey-class solutions to (1.1) for \( \nu = 0 \) as \( \kappa \to 0 \). More precisely, we consider the following system:
\[ \begin{align*}
\partial_t \theta^\kappa + u^\kappa \cdot \nabla \theta^\kappa &= -\kappa \Delta^\gamma \theta^\kappa + S, \\
u^\kappa_j &= \partial_{x_i} T^0_{ij} [\theta^\kappa], \theta^\kappa (x, 0) = \theta_0(x),
\end{align*} \tag{5.1} \]

where \( T^0_{ij} \) satisfies the bounds
\[ \sup_{k \in \mathbb{Z}_d^d} |\widehat{T}^0_{ij}(k)| \leq C_0, \]
and \( C_0 \) is given in assumption A5. The results will be proved in the following subsections. In Sect. 5.1, we prove that the equation (5.1) possesses local-in-time analytic solutions \( \theta^\kappa \) for all \( \kappa \geq 0 \) and \( \gamma \in (0, 2] \), and we prove that \( \theta^\kappa \) converges to \( \theta^0 \) in terms of analytic norm as \( \kappa \to 0 \). In Sect. 5.2, we proceed to prove that the local-in-time analytic solutions obtained in Sect. 5.1 can be extended globally in time under a smallness assumption on the initial data in terms of a certain Sobolev norm. Finally, in Sect. 5.3, we prove the existence and convergence of the Gevrey-class \( s \) solutions to (5.1) for \( s \geq 1 \) when the Fourier symbols \( \partial_{x_i} T^0_{ij}(k) \) are bounded functions in \( k \).
5.1. Local-in-time Existence and Convergence of Analytic Solutions

In this subsection, we prove the convergence of analytic solutions as stated in Theorem 2.2. We first obtain the local-in-time existence of analytic solutions to the equation (5.1) when \( \kappa \geq 0 \) and \( \gamma \in (0, 2) \), which is illustrated by Theorem 5.1 below.

**Theorem 5.1.** (Local-in-time existence of analytic solutions) Let \( \kappa \geq 0 \) and \( \gamma \in (0, 2) \) be fixed, and let \( \theta_0 \) and \( S \) be the initial datum and forcing term respectively. Fix \( K_0 > 0 \). Suppose \( \theta_0 \) and \( S \) are analytic functions with radius of convergence \( \tau_0 > 0 \) and

\[
\| \Lambda^r e^{\tau_0 \Lambda} \theta_0 \|_{L^2} \leq K_0, \quad \| \Lambda^r e^{\tau_0 \Lambda} S \|_{L^2} \leq K_0, \tag{5.2}
\]

where \( \Lambda := (-\Delta)^{\frac{1}{2}} \) and \( r > \frac{d}{2} + \frac{\gamma}{2} \). Then there exists \( T_* = T_*(\tau_0, K_0) > 0 \) and a unique analytic solution on \([0, T_*)\) to the initial value problem associated to (5.1).

**Proof of Theorem 5.1.** A proof can be found in [20] and we include here for the sake of completeness. We fix \( r \) such that \( r > \frac{d}{2} + \frac{\gamma}{2} \). We denote \( \tau = \tau(t) \) and then multiply (5.1) by \( \Lambda^{2r} e^{2\tau \Lambda} \theta^\kappa \) and integrate to obtain

\[
\frac{1}{2} \frac{d}{dt} \| \Lambda^r e^{\tau \Lambda} \theta^\kappa \|_{L^2}^2 - \dot{\tau} \| \Lambda^{r+\frac{1}{2}} e^{\tau \Lambda} \theta^\kappa \|_{L^2}^2 + \kappa \| \Lambda^{r+\frac{\gamma}{2}} e^{\tau \Lambda} \theta^\kappa \|_{L^2}^2 = \mathcal{R} + \langle S, \Lambda^{2r} e^{2\tau \Lambda} \theta^\kappa \rangle, \tag{5.3}
\]

where \( \mathcal{R} \) is given by \( \mathcal{R} = \langle u^\kappa \cdot \nabla \theta^\kappa, \Lambda^{2r} e^{2\tau \Lambda} \theta^\kappa \rangle \). The term \( \langle S, \Lambda^{2r} e^{2\tau \Lambda} \theta^\kappa \rangle \) can be readily bounded by \( \| \Lambda^r e^{\tau \Lambda} S \|_{L^2} \| \Lambda^{r+\frac{\gamma}{2}} e^{\tau \Lambda} \theta^\kappa \|_{L^2} \) and thus we focus on \( \mathcal{R} \). Using the bound (1.5) that \( \| \hat{u}^\kappa(j) \| \leq C \| j \| \| \hat{\theta}^\kappa(j) \| \) and the fact \( |j|^\frac{1}{2} \leq 2|l|^\frac{1}{2} |k|^\frac{1}{2} \) for \( s \geq 1 \) and \( |j|, |k|, |l| \geq 1 \), we can bound \( \mathcal{R} \) by

\[
\mathcal{R} \leq C \sum_{j+k=l, j,k \in \mathbb{Z}^d} \| j \| \| \hat{\theta}^\kappa(j) \| |k| \| \hat{\theta}^\kappa(k) \| |l|^{2r+2|l|} \| \hat{\theta}^\kappa(l) \|
\leq C \sum_{j+k=l, j,k \in \mathbb{Z}^d} |\hat{\theta}^\kappa(j)| |\hat{\theta}^\kappa(k)| |\hat{\theta}^\kappa(k)| |\hat{\theta}^\kappa(l)| |\hat{\theta}^\kappa(l)| |j|^{1+r+\frac{\gamma}{2}} |k|^{1+r+\frac{\gamma}{2}} |l|^{1+r+\frac{\gamma}{2}}
\leq C \sum_{j+k=l, j,k \in \mathbb{Z}^d} \left[ |\hat{\theta}^\kappa(j)| |\hat{\theta}^\kappa(k)| |\hat{\theta}^\kappa(k)| |\hat{\theta}^\kappa(l)| |\hat{\theta}^\kappa(l)| \right]
+ C \sum_{j+k=l, j,k \in \mathbb{Z}^d} \left[ |\hat{\theta}^\kappa(j)| |\hat{\theta}^\kappa(k)| |\hat{\theta}^\kappa(k)| |\hat{\theta}^\kappa(l)| |\hat{\theta}^\kappa(l)| \right]
\leq C \| \Lambda^{r+\frac{1}{2}} e^{\tau \Lambda} \theta^\kappa \|_{L^2} \| \Lambda^{r+\frac{\gamma}{2}} e^{\tau \Lambda} \theta^\kappa \|_{L^2}.
\]
where the last inequality follows since \( r > \frac{d}{2} + \frac{5}{2} \). Hence we obtain from (5.3) that
\[
\frac{1}{2} \frac{d}{dt} \| \Lambda^r e^{\tau \Lambda} \theta^k \|_{L^2}^2 \leq (\dot{\tau}(t) + C \| \Lambda^r e^{\tau \Lambda} \theta^k \|_{L^2}) \| \Lambda^r e^{\tau + \frac{1}{2} \Lambda} \theta^k \|_{L^2}^2 + \| \Lambda^r e^{\tau \Lambda} S_1 \|_{L^2} \| \Lambda^r e^{\tau \Lambda} \theta^k \|_{L^2}.
\]
(5.4)

In view of (5.4), we can apply the same argument given in [18]. Specifically, we let \( \tau(t) \) be decreasing and satisfies
\[
\dot{\tau}(t) + 4CK_0 = 0,
\]
with initial condition \( \tau(0) = \tau_0 \), then we have \( \dot{\tau}(t) + C \| \Lambda^r e^{\tau \Lambda} \theta^k \|_{L^2} < 0 \), and from (5.4) that
\[
\| \Lambda^r e^{\tau \Lambda} \theta^k \|_{L^2} \leq \| \Lambda^r e^{\tau \Lambda} \theta_0 \|_{L^2} + 2t \| \Lambda^r e^{\tau \Lambda} S_1 \|_{L^2} = 3K_0
\]
as long as \( \tau(t) > 0 \) and \( t \leq 1 \). Hence it implies the existence of analytic solution \( \theta^k \) on \([0, T_*)\), where the maximal time of existence of the analytic solution is given by \( T_* = \min\{\frac{\tau_0}{4CK_0}, 1\} \). \( \square \)

Once we obtain the existence of analytic solutions to (5.1), we are ready to prove the convergence of analytic solutions as \( \kappa \to 0 \), thereby proving Theorem 2.2.

**Proof of Theorem 2.2.** Let \( \varphi = \theta^k - \theta^0 \), then we have
\[
\partial_t \varphi + (u^k - u^0) \cdot \nabla \theta^0 + u^k \nabla \varphi = -\kappa \Lambda^\gamma \theta^0.
\]
(5.6)

For \( r > \frac{d}{2} + \frac{5}{2} \), we multiply (5.6) by \( \Lambda^{2r} e^{2\tau \Lambda} \theta^0 \) and obtain
\[
\frac{1}{2} \frac{d}{dt} \| \Lambda^r e^{\tau \Lambda} \varphi(\cdot, t) \|_{L^2}^2 = \dot{\tau}(t) \| \Lambda^{r + \frac{1}{2}} e^{\tau \Lambda} \varphi(\cdot, t) \|_{L^2}^2 + \langle (u^k - u^0) \cdot \nabla \theta^0, \Lambda^{2r} e^{2\tau \Lambda^\frac{1}{2}} \varphi \rangle
\]
\[
+ \langle u^k \cdot \nabla \varphi, \Lambda^{2r} e^{2\tau \Lambda^\frac{1}{2}} \varphi \rangle - \langle \kappa \Lambda^\gamma \theta^0, \Lambda^{2r} e^{2\tau \Lambda^\frac{1}{2}} \varphi \rangle
\]
\[
= \dot{\tau}(t) \| \Lambda^{r + \frac{1}{2}} e^{\tau \Lambda} \varphi(\cdot, t) \|_{L^2}^2 + \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3,
\]
(5.7)

where
\[
\mathcal{R}_1 = \langle (u^k - u^0) \cdot \nabla \theta^0, \Lambda^{2r} e^{2\tau \Lambda^\frac{1}{2}} \varphi \rangle,
\]
\[
\mathcal{R}_2 = \langle u^k \cdot \nabla \varphi, \Lambda^{2r} e^{2\tau \Lambda^\frac{1}{2}} \varphi \rangle,
\]
\[
\mathcal{R}_3 = -\langle \kappa \Lambda^\gamma \theta^0, \Lambda^{2r} e^{2\tau \Lambda^\frac{1}{2}} \varphi \rangle.
\]
To estimate $|R_1|$, following the method of bounding $R$ as in the proof of Theorem 5.1, we have
\[
|R_1| \leq C \sum_{j+k=l, j, k \in \mathbb{Z}^2} (u^{(k)} - u^0)(j) \cdot k \tilde{\theta}(k) \|l\|e^{2\tau ||l||} \phi(-l)
\]
\[
\leq C \sum_{j+k=l, j, k \in \mathbb{Z}^2} |j||k| \left[ |j| \bar{r} + |k| \bar{r} \right] \bar{\phi}(j) \bar{\phi}(j) e^{\tau |j|} \tilde{\theta}(k) e^{\tau |k|} \|l\|e^{\tau ||l||} \phi(l)
\]
\[
\leq C \|\Lambda^r e^{\tau \Lambda} \theta^0\|_{L^2} \|\Lambda^{r+\frac{1}{2}} e^{\tau \Lambda} \varphi\|_{L^2}^2 + C \|\Lambda^{r+\frac{1}{2}} e^{\tau \Lambda} \theta^0\|_{L^2} \|\Lambda^{r+\frac{1}{2}} e^{\tau \Lambda} \varphi\|_{L^2} \|\Lambda^r e^{\tau \Lambda} \varphi\|_{L^2}
\]
\[
\leq C \|\Lambda^{r+\frac{1}{2}} e^{\tau \Lambda} \theta^0\|_{L^2} \|\Lambda^r e^{\tau \Lambda} \varphi\|_{L^2}.\]

Similarly, we can bound $|R_2|$ and $|R_3|$, respectively, by
\[
|R_2| \leq C \|\Lambda^{r+\frac{1}{2}} e^{\tau \Lambda} \theta^0\|_{L^2} \|\Lambda^r e^{\tau \Lambda} \varphi\|_{L^2}^2,
\]
and
\[
|R_3| \leq \kappa \|\Lambda^{r+\frac{1}{2}} e^{\tau \Lambda} \theta^0\|_{L^2} \|\Lambda^r e^{\tau \Lambda} \varphi\|_{L^2}.
\]

Hence we conclude from (5.7) that
\[
\frac{1}{2} \frac{d}{dt} \|\Lambda^r e^{\tau \Lambda} \varphi\|_{L^2} \leq \|\Lambda^{r+\frac{1}{2}} e^{\tau \Lambda} \varphi\|_{L^2}^2 \left[ \dot{\varphi} + C \|\Lambda^{r+\frac{1}{2}} e^{\tau \Lambda} \theta^0\|_{L^2} + C \|\Lambda^{r+\frac{1}{2}} e^{\tau \Lambda} \theta^\kappa\|_{L^2} \right]
\]
\[
+ \kappa \|\Lambda^{r+\frac{1}{2}} e^{\tau \Lambda} \theta^0\|_{L^2} \|\Lambda^r e^{\tau \Lambda} \varphi\|_{L^2}.
\]

Choosing $\tau > 0$ such that
\[
\dot{\varphi} + C \|\Lambda^{r+\frac{1}{2}} e^{\tau \Lambda} \theta^0\|_{L^2} + C \|\Lambda^{r+\frac{1}{2}} e^{\tau \Lambda} \theta^\kappa\|_{L^2} < 0,
\]
we get
\[
\frac{d}{dt} \|\Lambda^r e^{\tau \Lambda} \varphi\|_{L^2} \leq \kappa \|\Lambda^{r+\frac{1}{2}} e^{\tau \Lambda} \theta^0\|_{L^2},
\]
and the result follows by taking $\kappa \to 0$. \qed

5.2. Global-in-time Existence of Analytic Solutions with Small Sobolev Initial Data

In this subsection, we prove that for the case when $\kappa > 0$ and $\gamma \in [1, 2]$, under a smallness assumption on the initial data, the analytic solutions to (5.1) obtained in Theorem 5.1 exist for all time.

**Theorem 5.2.** (Global-in-time existence of analytic solutions) Let $\kappa > 0$ and $\gamma \in [1, 2]$, and suppose that both $\theta_0$ and $S$ are both analytic functions. There exists $\epsilon > 0$ depending on $\kappa$ such that, if $\theta_0$ and $S$ satisfy
\[
\|\theta_0\|_{L^2}^\beta \|\theta_0\|_{L^2}^{1-\beta} + \|\theta_0\|_{L^2}^\beta \|S\|_{L^2}^{1-\beta \gamma} \leq \epsilon,
\]

(5.8)
and
\[ \| \Lambda^\alpha \theta_0 \|^2_{L^2} + \frac{2}{\kappa^2} \| S \|_{H^{\alpha - \frac{\gamma}{2}}}^2 \leq \varepsilon, \tag{5.9} \]
where \( \alpha > \frac{d+2}{2} + (1 - \gamma) \) and \( \beta = 1 - \frac{1}{\alpha} \left[ \frac{d+2}{2} + (1 - \gamma) \right] \), then the local-in-time analytic solution \( \theta^\kappa \) as claimed by Theorem 5.1 can be extended to all time.

Before we give the proof of Theorem 5.2, we state and prove the following global-in-time existence theorem of \( H^\alpha \)-solution to (5.1) under the smallness assumption (5.8).

**Theorem 5.3.** (Global-in-time existence of Sobolev solutions) Let \( \kappa > 0 \), \( \gamma \in [1, 2] \) and \( S \in H^{\alpha - \frac{\gamma}{2}}(\mathbb{T}^d) \), where \( \alpha > \frac{d+2}{2} + (1 - \gamma) \). There exists a small enough constant \( \varepsilon > 0 \) depending on \( \kappa \), such that if \( \theta_0 \) satisfies (5.8), then there exists a unique global-in-time \( H^\alpha \)-solution to (5.1). In particular, for all \( t > 0 \), we have the following bound on \( \theta^\kappa \):
\[ \| \Lambda^\alpha \theta^\kappa (\cdot, t) \|_{L^2}^2 \leq \| \Lambda^\alpha \theta_0 \|_{L^2}^2 + \frac{2}{\kappa^2} \| S \|_{H^{\alpha - \frac{\gamma}{2}}}^2. \tag{5.10} \]

**Proof of Theorem 5.3.** The proof is similar to the one given in [22, Theorem 3.2]. We multiply (5.1) by \( \Lambda^{2\alpha} \theta^\kappa \), integrate by parts to obtain
\[ \frac{1}{2} \frac{d}{dt} \| \Lambda^\alpha \theta^\kappa \|^2_{L^2} + \kappa \| \Lambda^{\frac{\gamma}{2} + \alpha} \theta^\kappa \|^2_{L^2} \leq \left| \int_{\mathbb{T}^d} S \Lambda^{2\alpha} \theta^\kappa \right| + \left| \int_{\mathbb{T}^d} \kappa^\kappa \cdot \nabla \theta^\kappa \Lambda^{2\alpha} \theta^\kappa \right|. \tag{5.11} \]

The first term on the right side of (5.11) is bounded by
\[ \left| \int_{\mathbb{T}^d} S \Lambda^{2\alpha} \theta^\kappa \right| \leq \frac{1}{2\kappa} \| \Lambda^{\alpha - \frac{\gamma}{2}} S \|_{L^2}^2 + \frac{\kappa}{2} \| \Lambda^{\alpha + \frac{\gamma}{2}} \theta^\kappa \|^2_{L^2}. \tag{5.12} \]

The second term on the right side of (5.11) can be estimated as follows:
\[ \left| \int_{\mathbb{T}^d} \kappa^\kappa \cdot \nabla \theta^\kappa \Lambda^{2\alpha} \theta^\kappa \right| \leq \| \Lambda^{\alpha - \frac{\gamma}{2}} (\kappa^\kappa \cdot \nabla \theta^\kappa) \|_{L^2} \| \Lambda^{\alpha + \frac{\gamma}{2}} \theta^\kappa \|_{L^2}. \tag{5.13} \]

To estimate the product term \( \| \Lambda^{\alpha - \frac{\gamma}{2}} (\kappa^\kappa \cdot \nabla \theta^\kappa) \|_{L^2} \), using (3.5), we readily have
\[ \| \Lambda^{\alpha - \frac{\gamma}{2}} (\kappa^\kappa \cdot \nabla \theta^\kappa) \|_{L^2} \leq C \left( \| \Lambda^{\alpha - \frac{\gamma}{2}} \kappa^\kappa \|_{L^p} \| \nabla \theta^\kappa \|_{L^{2p}} + \| \Lambda^{\alpha - \frac{\gamma}{2}} \nabla \theta^\kappa \|_{L^p} \| \kappa^\kappa \|_{L^{2p}} \right), \tag{5.14} \]
where \( p = \frac{2d}{2(1-\gamma)+d} \), and notice that \( p > 2 \) as \( \gamma \in [1, 2] \). By (1.5), we have
\[ \| \kappa^\kappa \|_{L^p} \leq C \| \Lambda \theta^\kappa \|_{L^p}, \]
and together with (3.4), we further get
\[
\| \Lambda^{\alpha - \frac{\gamma}{2}} u^k \|_{L^p} \leq C |\Lambda^{\alpha - \frac{\gamma}{2} + 1} \theta^k |_{L^p} \\
\leq C \| \Lambda^{\alpha - \frac{\gamma}{2} + 1 + s} \theta^k |_{L^2},
\]
where \( s \) satisfies \( \frac{1}{p} = \frac{1}{2} - \frac{s}{d} \) and gives \( s = \gamma - 1 \). Therefore we obtain from (5.14) that
\[
\| \Lambda^{\alpha - \frac{\gamma}{2}} (u^k \cdot \nabla \theta^k) \|_{L^2} \\
\leq C \left( \| \Lambda^{\alpha - \frac{\gamma}{2} + 1 + \gamma - 1} \theta^k \|_{L^2} \| \nabla \theta^k \|_{L^\frac{2p}{2}} + \| \Lambda^{\alpha - \frac{\gamma}{2} + 1 + \gamma - 1} \theta^k \|_{L^p} \| \Lambda \theta^k \|_{L^\frac{2p}{2}} \right) \\
\leq C \| \Lambda^{\alpha + \frac{\gamma}{2}} \theta^k \|_{L^2} \| \Lambda \theta^k \|_{L^\frac{2p}{2}}.
\]
(5.15)
We apply (5.15) on (5.13) to get
\[
\left| \int_{\mathbb{T}^d} u^k \cdot \nabla \theta^k \Lambda^{2\alpha} \theta^k \right| \leq C \| \Lambda^{\alpha + \frac{\gamma}{2}} \theta^k \|_{L^2} \| \Lambda \theta^k \|_{L^\frac{2p}{2}}.
\]
(5.16)
Using the estimates (5.12) and (5.16) on (5.11) and applying Cauchy inequality,
\[
\frac{1}{2} \frac{d}{dt} \| \Lambda^{\alpha} \theta^k \|_{L^2}^2 + \frac{\kappa}{2} \| \Lambda^{\gamma + \alpha} \theta^k \|_{L^2}^2 \leq \frac{1}{2\kappa} \| \Lambda^{\alpha - \frac{\gamma}{2}} S \|_{L^2}^2 + C \| \Lambda^{\alpha + \frac{\gamma}{2}} \theta^k \|_{L^2}^2 \| \Lambda \theta^k \|_{L^\frac{2p}{2}}.
\]
(5.17)
For the term \( \| \Lambda \theta^k \|_{L^\frac{2p}{2}} \), we can bound it as follows. Take \( \beta = 1 - \frac{1}{\alpha} \left( \frac{d+2}{2} + (1-\gamma) \right) \), then \( \beta \in (0, 1) \) and we have
\[
\| \Lambda \theta^k \|_{L^\frac{2p}{2}} \leq C \| \theta^k \|_{L^2}^\beta \| \Lambda^{\alpha} \theta^k \|_{L^2}^{1-\beta}.
\]
If \( \theta^k \) satisfies
\[
\| \theta^k \|_{L^2}^\beta \| \Lambda^{\alpha} \theta^k \|_{L^2}^{1-\beta} \leq \frac{\kappa}{4C},
\]
(5.18)
then we obtain from equation (5.17) that
\[
\frac{1}{2} \frac{d}{dt} \| \Lambda^{\alpha} \theta^k \|_{L^2}^2 + \frac{\kappa}{2} \| \Lambda^{\gamma + \alpha} \theta^k \|_{L^2}^2 \leq \frac{1}{2\kappa} \| \Lambda^{\alpha - \frac{\gamma}{2}} S \|_{L^2}^2 + \frac{\kappa}{4} \| \Lambda^{\alpha + \frac{\gamma}{2}} \theta^k \|_{L^2}^2,
\]
which implies
\[
\frac{d}{dt} \| \Lambda^{\alpha} \theta^k \|_{L^2}^2 + \frac{\kappa}{2} \| \Lambda^{\gamma + \alpha} \theta^k \|_{L^2}^2 \leq \frac{1}{\kappa} \| S \|_{H^{\alpha - \frac{\gamma}{2}}}^2.
\]
(5.19)
We conclude from (5.19) that for all \( t > 0 \),
\[
\| \Lambda^{\alpha} \theta^k (\cdot, t) \|_{L^2}^2 \leq \| \Lambda^{\alpha} \theta_0 \|_{L^2}^2 + \frac{2}{\kappa} \| S \|_{H^{\alpha - \frac{\gamma}{2}}}^2,
\]
(5.20)
\[
\| \theta^k (\cdot, t) \|_{L^2}^2 \leq 2\kappa \int_0^t \| \Lambda^{\gamma} \theta^k (\cdot, \tilde{t}) \|_{L^2}^2 \, d\tilde{t} \leq \| \theta_0 \|_{L^2}^2.
\]
In view of (5.20), we can see that condition (5.18) is satisfied for all \( t > 0 \) if (5.8) holds for \( \epsilon \) being sufficiently small. Hence we complete the proof of Theorem 5.3. □
We are now ready to give the proof of Theorem 5.2.

**Proof of Theorem 5.2.** We fix $r$ such that $r > \frac{d}{2} + \frac{5}{2}$. We multiply (5.1)_1 by $\lambda^2 e^{\tau \lambda} \theta^\kappa$ and integrate to obtain

\[
\frac{1}{2} \frac{d}{dr} \| \Lambda^r e^{\tau \Lambda} \theta^\kappa \|_{L^2}^2 - \dot{\tau} \| \Lambda^{r+\frac{\gamma}{2}} e^{\tau \Lambda} \theta^\kappa \|_{L^2}^2 + \kappa \| \Lambda^{r+\frac{\gamma}{2}} e^{\tau \Lambda} \theta^\kappa \|_{L^2}^2 \leq \mathcal{R} + \| \Lambda^r e^{\tau \Lambda} S \|_{L^2} \| \Lambda^r e^{\tau \Lambda} \theta^\kappa \|_{L^2},
\]

(5.21)

where $\mathcal{R}$ is defined in the proof of Theorem 5.1. We modify the way for estimating the term $\mathcal{R}$ as follows. Using the bound

\[
e^{x} \leq 1 + xe^{x}, \quad \text{for all } x \geq 0,
\]

we can bound $\mathcal{R}$ by

\[
\mathcal{R} \leq C \sum_{j+k=l, j,k \in \mathbb{Z}^d} \left[ |\hat{\theta}(j)|||j||^{r+\frac{1}{2}} e^{||j||} \left[ |\hat{\theta}(k)|||k||^{r+\frac{1}{2}} e^{||k||} \right] \right] ||\hat{\theta}(l)|| ||l||^{r+\frac{1}{2}} e^{||l||} \]

\[
+ C \sum_{j+k=l, j,k \in \mathbb{Z}^d} \left[ |\hat{\theta}(j)|||j||^{r+\frac{1}{2}} e^{||j||} \left[ |\hat{\theta}(k)|||k||^{r+\frac{1}{2}} e^{||k||} \right] \right] ||\hat{\theta}(l)|| ||l||^{r+\frac{1}{2}} e^{||l||} \]

\[
\leq C \left[ \sum_{m \in \mathbb{Z}^d} |\hat{\theta}(m)||m||^{r+\frac{1}{2}} \right] ||\Lambda^{r+\frac{1}{2}} e^{\tau \Lambda} \theta^\kappa ||_{L^2}^2 \left[ a(t) + \tau \lambda \| \Lambda^r e^{\tau \Lambda} \theta^\kappa \|_{L^2} \right],
\]

(5.22)

where $a(t) := \sum_{m \in \mathbb{Z}^d} |m|^r |\hat{\theta}(m)|(t)$ and we used the fact that

\[
\sum_{m \in \mathbb{Z}^d} |m|^{r+\frac{1}{2}} |\hat{\theta}(m)| e^{\tau |m|} \leq \left( \sum_{m \in \mathbb{Z}^d} |m|^{5-2r} \right)^{\frac{1}{2}} \left( \sum_{m \in \mathbb{Z}^d} |m|^{2r} |\hat{\theta}(m)|^2 e^{2\tau |m|} \right)^{\frac{1}{2}} \leq C_r \| \Lambda^r e^{\tau \Lambda} \theta^\kappa \|_{L^2},
\]

for $C_r$ being a large enough positive constant depending only on $r$ and $r > \frac{d}{2} + \frac{5}{2}$. We apply (5.22) to (5.21) and obtain

\[
\frac{1}{2} \frac{d}{dr} \| \Lambda^r e^{\tau \Lambda} \theta^\kappa \|_{L^2}^2 + \kappa \| \Lambda^{r+\frac{\gamma}{2}} e^{\tau \Lambda} \theta^\kappa \|_{L^2}^2 \leq \left[ \dot{\tau} + C_r a(t) + \tau C_r \| \Lambda^r e^{\tau \Lambda} \theta^\kappa \|_{L^2} \right] \| \Lambda^{r+\frac{1}{2}} e^{\tau \Lambda} \theta^\kappa \|_{L^2}^2 + \| \Lambda^r e^{\tau \Lambda} S \|_{L^2} \| \Lambda^r e^{\tau \Lambda} \theta^\kappa \|_{L^2}.\]

(5.23)

For $\gamma \geq 1$, we have $\frac{\gamma}{2} \geq \frac{1}{2}$, hence

\[
\| \Lambda^{r+\frac{\gamma}{2}} e^{\tau \Lambda} \theta^\kappa \|_{L^2} \geq \| \Lambda^{r+\frac{1}{2}} e^{\tau \Lambda} \theta^\kappa \|_{L^2}.
\]
On the other hand, for $\alpha > \frac{d+2}{2} + \frac{1}{2}$, we have

$$a(t) \leq \left( \sum_{m \in \mathbb{Z}^d} |m|^{3-2\alpha} \right)^{\frac{1}{2}} \left( \sum_{m \in \mathbb{Z}^d} |m|^{2\alpha} |\hat{\theta}(m)|^2 \right)^{\frac{1}{2}} \leq C_\alpha \| \Lambda^\alpha \theta \|_{L^2}$$ (5.24)

for some large enough positive constant depending only on $\alpha$. If we assume that

$$\| \Lambda^\alpha \theta_0 \|_{L^2}^2 + \frac{2}{\kappa^2} \| S \|_{H^{\alpha-\frac{\gamma}{2}}}^2 \leq \frac{\kappa^2}{4C_r C_\alpha},$$

then using the bound (5.10), (5.24) implies that

$$a(t) \leq \frac{\kappa^2}{4C_r}.$$ (5.25)

Applying the bound (5.25) to (5.23), we obtain

$$\frac{1}{2} \frac{d}{dt} \| \Lambda^\alpha e^{\tau \Lambda} \theta \|_{L^2}^2 + \frac{\kappa}{2} \| \Lambda^\alpha e^{\tau \Lambda} e^{\tau \Lambda} \theta \|_{L^2}^2 \leq \left[ \dot{\tau} + \tau C_r \| \Lambda^\alpha e^{\tau \Lambda} \theta \|_{L^2} \right] \| \Lambda^\alpha e^{\tau \Lambda} \theta \|_{L^2}^2 + \| \Lambda^\alpha e^{\tau \Lambda} S \|_{L^2} \| \Lambda^\alpha e^{\tau \Lambda} \theta \|_{L^2}^2.$$ (5.26)

Choose $\dot{\tau} + \tau C_r \| \Lambda^\alpha e^{\tau \Lambda} \theta \|_{L^2} = 0$ in (5.26), then it gives

$$\frac{1}{2} \frac{d}{dt} \| \Lambda^\alpha e^{\tau \Lambda} \theta \|_{L^2}^2 + \frac{\kappa}{2} \| \Lambda^\alpha e^{\tau \Lambda} e^{\tau \Lambda} \theta \|_{L^2}^2 \leq \| \Lambda^\alpha e^{\tau \Lambda} S \|_{L^2} \| \Lambda^\alpha e^{\tau \Lambda} \theta \|_{L^2},$$

hence

$$\| \Lambda^\alpha e^{\tau \Lambda} \theta \|_{L^2}^2 \leq \| \Lambda^\alpha e^{\tau \Lambda} \theta_0 \|_{L^2} + t \| \Lambda^\alpha e^{\tau \Lambda} S \|_{L^2},$$

and for all $t > 0$, $\tau = \tau(t)$ satisfies the bound

$$\tau(t) = \tau_0 \exp \left( - C_r \int_0^t \| \Lambda^\alpha e^{\tau \Lambda} \theta \|_{L^2} d\tilde{t} \right) \geq \tau_0 \exp \left( - C_r \int_0^t (\| \Lambda^\alpha e^{\tau \Lambda} \theta_0 \|_{L^2} + \tilde{t} \| \Lambda^\alpha e^{\tau \Lambda} S \|_{L^2} ) d\tilde{t} \right) > 0.$$

Therefore the local-in-time analytic solution as claimed by Theorem 5.1 can be extended to all time, thereby proving Theorem 5.2. \qed

**Remark 5.4.** For the case when $\kappa > 0$ and $\gamma = 2$, we point out that the smallness assumption (5.8) on $\theta_0$ can be removed. The reason is that, we can obtain the bound (5.25) on $a(t)$ without the smallness assumption (5.8) (also refer to [21] for more detailed discussions). The critical MG$^0$ equation is an example of equation (5.1) with $\gamma = 2$.

**Remark 5.5.** It is also worth mentioning that all the abstract results obtained in Theorem 5.1, Theorem 5.2 and Theorem 5.3 can be applied to the critical SQG equation, which is a special example of (5.1) with $\gamma = 1$. 


5.3. Existence and Convergence of Gevrey-Class Solutions with Bounded Symbols $\partial_{x_i} T_{ij}^0$

In this subsection, we address the existence and convergence of Gevrey-class solutions to the equation (5.1) under a stronger assumption on the operators $T_{ij}^0$. More precisely, for all $1 \leq i, j \leq d$, we further assume that

$$\sup_{|k| \leq D_d}|\partial_{x_i} T_{ij}^0(k)| \leq C_*$$ (5.27)

for some positive constant $C_*$. The condition (5.27) implies that

$$|u^\kappa (j)| \leq C(C_*)|\hat{\theta}^\kappa (j)|, \quad \text{for all } j \in \mathbb{Z}_d^d.$$

The results are summarised in the following theorem:

**Theorem 5.6.** (Existence and convergence of Gevrey-class solutions) Let $\kappa \geq 0$ and $\gamma \in (0, 2]$ be fixed, and let $\theta_0$ and $S$ be the initial datum and forcing term respectively. Fix $s \geq 1$ and $K_0 > 0$. Suppose $\theta_0$ and $S$ both belong to Gevrey-class $s$ with radius of convergence $\tau_0 > 0$ and

$$\|\Lambda^r e^{\tau_0 \Lambda^\gamma} \theta^\kappa (\cdot, 0)\|_{L^2} \leq K_0, \quad \|\Lambda^r e^{\tau_0 \Lambda^\gamma} S\|_{L^2} \leq K_0,$$

where $r > \frac{d}{2} + \frac{\gamma}{2}$. Assume further that the condition (5.27) holds for $1 \leq i, j \leq d$. Then there exists $T_* = T_*(\tau_0, K_0) > 0$ and a unique Gevrey-class $s$ solution on $[0, T_*)$ to the initial value problem associated to (5.1). Moreover, there exists $T \leq T_*$ and $\tau = \tau (t) < \tau_0$ such that, for $t \in [0, T]$, we have

$$\lim_{\kappa \to 0} \|\Lambda^r e^{\tau \Lambda^\gamma} \theta^\kappa (\cdot, t)\|_{L^2} = 0.$$

**Proof of Theorem 5.6.** We fix $r$ such that $r > \frac{d}{2} + \frac{\gamma}{2}$ and multiply (5.1)$_1$ by $\Lambda^r e^{2\tau \Lambda^\gamma} \theta^\kappa$ and integrate to obtain

$$\frac{1}{2} \frac{d}{dt} \|\Lambda^r e^{\tau \Lambda^\gamma} \theta^\kappa \|^2_{L^2} - \hat{T} \|\Lambda^r e^{\tau \Lambda^\gamma} \theta^\kappa \|^2_{L^2} + \kappa \|\Lambda^r e^{\tau \Lambda^\gamma} \theta^\kappa \|^2_{L^2} = T + \langle S, \Lambda^r e^{2\tau \Lambda^\gamma} \theta^\kappa \rangle,$$

where $T$ is given by $T := \langle u^\kappa \cdot \nabla \theta^\kappa, \Lambda^2 r e^{2\tau \Lambda^\gamma} \theta^\kappa \rangle$. The term $\langle S, \Lambda^2 r e^{2\tau \Lambda^\gamma} \theta^\kappa \rangle$ can be readily bounded by $\|\Lambda^r e^{\tau \Lambda^\gamma} S\|_{L^2} \|\Lambda^r e^{\tau \Lambda^\gamma} \theta^\kappa \|_{L^2}$. For $T$, since $\text{div}(u^\kappa) = 0$, we have $\langle u^\kappa \cdot \nabla \Lambda^\gamma e^{\tau \Lambda^\gamma} \theta^\kappa, \Lambda^r e^{\tau \Lambda^\gamma} \theta^\kappa \rangle = 0$ and hence

$$|T| = |\langle u^\kappa \cdot \nabla \theta^\kappa, \Lambda^2 r e^{2\tau \Lambda^\gamma} \theta^\kappa \rangle - \langle u^\kappa \cdot \nabla \Lambda^\gamma e^{\tau \Lambda^\gamma} \theta^\kappa, \Lambda^r e^{\tau \Lambda^\gamma} \theta^\kappa \rangle| \leq T_1 + T_2,$$

where $T_1$ and $T_2$ are given by

$$T_1 := C \sum_{j+k=\ell; j,k \in \mathbb{Z}_d^d} \|l^r - k^r\| |u^\kappa (j)||\hat{\theta}^\kappa (k)||k||l^r|\hat{\theta}^\kappa (l)|e^{\tau |l|^\gamma} e^{\tau |k|^\gamma},$$

$$T_2 := C \sum_{j+k=\ell; j,k \in \mathbb{Z}_d^d} \|l^r - k^r\| |u^\kappa (j)||\hat{\theta}^\kappa (k)||k||l^r|\hat{\theta}^\kappa (l)|e^{\tau |l|^\gamma} e^{\tau |k|^\gamma}.$$
\[ T_2 := C \sum_{j+k+l: j,k,l \in \mathbb{Z}_+^d} |l^r| |e^{\tau |l|^\frac{1}{2} - \tau |k|^\frac{1}{2}} - 1||u^\kappa (j)||\hat{\theta}^\kappa (k)||k||l^r| e^{\tau |l|^\frac{1}{2} - \tau |k|^\frac{1}{2}}. \]

To estimate \( T_1 \), we apply the similar method given in [27] and [33]. Using mean value theorem, there exists \( \xi_{k,l} \in (0, 1) \) such that
\[ |l^r - |k|^r| = (|\xi_{k,l}||l| + (1 - \xi_{k,l})|k|^r) - |k|^r + (|l| - |k|)|k|^r. \]

Since \( j + k = l \), we have \(|(|l| - |k|)|k|^r - |k|^r| \leq |j||k|^r - |k|^r| \), as well as
\[ |(\xi_{k,l}||l| + (1 - \xi_{k,l})|k|^r) - |k|^r| \leq C|j|^2(|j|^r - 2 + |k|^r - 2). \]

Together with (5.28), we have
\[
T_1 \leq C \sum_{j+k=l: j,k,l \in \mathbb{Z}_+^d} |j|^2(|j|^r - 2 + |k|^r - 2)\|\hat{\theta}^\kappa (j)||\hat{\theta}^\kappa (k)||k||l^r| e^{\tau |l|^\frac{1}{2} e^{\tau |k|^\frac{1}{2}}
\]

\[ + C \sum_{j+k=l: j,k,l \in \mathbb{Z}_+^d} |j||k|^r - 1\|\hat{\theta}^\kappa (j)||\hat{\theta}^\kappa (k)||k||l^r| e^{\tau |l|^\frac{1}{2} e^{\tau |k|^\frac{1}{2}}
\]

\[ \leq C\|\Lambda^r \theta^\kappa\|_{L^2} \|\Lambda^r e^{\tau \Lambda^r \theta^\kappa}\|_{L^2}^2,
\]

where the last inequality holds for \( r > \frac{d}{2} + \frac{5}{2} \). To estimate \( T_2 \), we use (5.28) and the inequality \(|e^{\tau |l|^\frac{1}{2} - \tau |k|^\frac{1}{2}} - 1| \leq |\tau |l|^\frac{1}{2} - \tau |k|^\frac{1}{2} e^{\tau |l|^\frac{1}{2} - \tau |k|^\frac{1}{2}}| \) to obtain
\[
T_2 \leq C \tau \sum_{j+k=l: j,k,l \in \mathbb{Z}_+^d} |l^r||l|^\frac{1}{2} - |k|^\frac{1}{2} e^{\tau |l|^\frac{1}{2} - \tau |k|^\frac{1}{2} \|\hat{\theta}^\kappa (j)||\hat{\theta}^\kappa (k)||k||l^r| e^{\tau |l|^\frac{1}{2} e^{\tau |k|^\frac{1}{2}}
\]

\[ \leq C \tau \sum_{j+k=l: j,k,l \in \mathbb{Z}_+^d} |l^r||l|^\frac{1}{2} - |k|^\frac{1}{2} e^{\tau |j|^\frac{1}{2} \|\hat{\theta}^\kappa (j)||\hat{\theta}^\kappa (k)||k||l^r| e^{\tau |l|^\frac{1}{2} e^{\tau |k|^\frac{1}{2}}
\]

where the last inequality follows since \(|\tau |l|^\frac{1}{2} - \tau |k|^\frac{1}{2} \leq |j|^\frac{1}{2} \). Depending on the values of \(|j| \) and \(|k| \), we have the following estimates:

- If \(|j| \leq |k| \), using the inequality \(|l^r||l|^\frac{1}{2} - |k|^\frac{1}{2} \leq \frac{|j|}{|l|^\frac{1}{2} + |k|^\frac{1}{2}} \) for \( s \geq 1 \), we have
\[
|l^r||l|^\frac{1}{2} - |k|^\frac{1}{2} e^{\tau |j|^\frac{1}{2} \|\hat{\theta}^\kappa (j)||\hat{\theta}^\kappa (k)||k||l^r| e^{\tau |l|^\frac{1}{2} e^{\tau |k|^\frac{1}{2}}
\]

\[ \leq |k|^r e^{\tau |k|^\frac{1}{2}} \frac{|j|}{|l|^\frac{1}{2} + |k|^\frac{1}{2} \|\hat{\theta}^\kappa (j)||\hat{\theta}^\kappa (k)||k||l^r| e^{\tau |l|^\frac{1}{2} e^{\tau |k|^\frac{1}{2}}
\]

\[ \leq \left(|k|^r e^{\tau |k|^\frac{1}{2}} \|\hat{\theta}^\kappa (k)||k||l^r| e^{\tau |l|^\frac{1}{2} e^{\tau |k|^\frac{1}{2}} \|\hat{\theta}^\kappa (j)|\right). \]

- If \(|j| \geq |k| \), using the inequality \(|l|^\frac{1}{2} \leq C |j|^\frac{1}{2} \) for \( s \geq 1 \), we have
\[
|l^r||l|^\frac{1}{2} - |k|^\frac{1}{2} e^{\tau |j|^\frac{1}{2} \|\hat{\theta}^\kappa (j)||\hat{\theta}^\kappa (k)||k||l^r| e^{\tau |l|^\frac{1}{2} e^{\tau |k|^\frac{1}{2}}
\]

\[ \leq C |j|^\frac{1}{2} e^{\tau |j|^\frac{1}{2} \|\hat{\theta}^\kappa (j)||\hat{\theta}^\kappa (k)||k||l^r| e^{\tau |l|^\frac{1}{2} e^{\tau |k|^\frac{1}{2}}
\]

\[ \leq C\|\Lambda^r \theta^\kappa\|_{L^2} \|\Lambda^r e^{\tau \Lambda^r \theta^\kappa}\|_{L^2}^2,
\]

where the last inequality holds for \( r > \frac{d}{2} + \frac{5}{2} \). To estimate \( T_2 \), we use (5.28) and the inequality \(|e^{\tau |l|^\frac{1}{2} - \tau |k|^\frac{1}{2}} - 1| \leq |\tau |l|^\frac{1}{2} - \tau |k|^\frac{1}{2} e^{\tau |l|^\frac{1}{2} - \tau |k|^\frac{1}{2}}| \) to obtain
\[
T_2 \leq C \tau \sum_{j+k=l: j,k,l \in \mathbb{Z}_+^d} |l^r||l|^\frac{1}{2} - |k|^\frac{1}{2} e^{\tau |l|^\frac{1}{2} - \tau |k|^\frac{1}{2} \|\hat{\theta}^\kappa (j)||\hat{\theta}^\kappa (k)||k||l^r| e^{\tau |l|^\frac{1}{2} e^{\tau |k|^\frac{1}{2}}
\]

\[ \leq C \tau \sum_{j+k=l: j,k,l \in \mathbb{Z}_+^d} |l^r||l|^\frac{1}{2} - |k|^\frac{1}{2} e^{\tau |j|^\frac{1}{2} \|\hat{\theta}^\kappa (j)||\hat{\theta}^\kappa (k)||k||l^r| e^{\tau |l|^\frac{1}{2} e^{\tau |k|^\frac{1}{2}}
\]

where the last inequality follows since \(|\tau |l|^\frac{1}{2} - \tau |k|^\frac{1}{2} \leq |j|^\frac{1}{2} \). Depending on the values of \(|j| \) and \(|k| \), we have the following estimates:

- If \(|j| \leq |k| \), using the inequality \(|l^r||l|^\frac{1}{2} - |k|^\frac{1}{2} \leq \frac{|j|}{|l|^\frac{1}{2} + |k|^\frac{1}{2}} \) for \( s \geq 1 \), we have
\[
|l^r||l|^\frac{1}{2} - |k|^\frac{1}{2} e^{\tau |j|^\frac{1}{2} \|\hat{\theta}^\kappa (j)||\hat{\theta}^\kappa (k)||k||l^r| e^{\tau |l|^\frac{1}{2} e^{\tau |k|^\frac{1}{2}}
\]

\[ \leq |k|^r e^{\tau |k|^\frac{1}{2}} \frac{|j|}{|l|^\frac{1}{2} + |k|^\frac{1}{2} \|\hat{\theta}^\kappa (j)||\hat{\theta}^\kappa (k)||k||l^r| e^{\tau |l|^\frac{1}{2} e^{\tau |k|^\frac{1}{2}}
\]

\[ \leq \left(|k|^r e^{\tau |k|^\frac{1}{2}} \|\hat{\theta}^\kappa (k)||k||l^r| e^{\tau |l|^\frac{1}{2} e^{\tau |k|^\frac{1}{2}} \|\hat{\theta}^\kappa (j)|\right). \]
\[
\leq C |j|^{r + \frac{1}{2}} e^{\tau |j|^{\frac{1}{2}}} (|l|^{\frac{1}{2}} + |k|^{\frac{1}{2}}) |\dot{\theta}^k(j)||\dot{\theta}^k(k)||l|^{r - \frac{1}{2}} e^{\tau |l|^{\frac{1}{2}}} e^{\tau |k|^{\frac{1}{2}}}
\]
\[
\leq C \left( |k| e^{\tau |k|^{\frac{1}{2}}} |\dot{\theta}^k(k)| \right) \left( |l|^{r + \frac{1}{2}} e^{\tau |l|^{\frac{1}{2}}} |\dot{\theta}^k(l)| \right) \left( |j|^{r + \frac{1}{2}} e^{\tau |j|^{\frac{1}{2}} |\dot{\theta}^k(j)|} \right)
\]
\[+ C \left( |k|^{1 + \frac{1}{2}} e^{\tau |k|^{\frac{1}{2}}} |\dot{\theta}^k(k)| \right) \left( |l|^{r - \frac{1}{2}} e^{\tau |l|^{\frac{1}{2}}} |\dot{\theta}^k(l)| \right) \left( |j|^{r + \frac{1}{2}} e^{\tau |j|^{\frac{1}{2}} |\dot{\theta}^k(j)|} \right).
\]

Hence for \( r > \frac{d}{2} + \frac{5}{2} \), we have
\[
T_2 \leq C \tau \| \Lambda^r e^{\tau \Lambda^{\frac{1}{2}} \theta^k} \|_{L^2} \| \Lambda^{r + \frac{1}{2}} e^{\tau \Lambda^{\frac{1}{2}} \theta^k} \|_{L^2}^2.
\]

Combining the estimates on \( T_1 \) and \( T_2 \), we conclude that
\[
|T| \leq C \left( \| \Lambda^r \theta^k \|_{L^2} + \tau \| \Lambda^r e^{\tau \Lambda^{\frac{1}{2}} \theta^k} \|_{L^2} \right) \| \Lambda^{r + \frac{1}{2}} e^{\tau \Lambda^{\frac{1}{2}} \theta^k} \|_{L^2}^2.
\]  \( (5.32) \)

We apply \( (5.32) \) on \( (5.31) \) to obtain that
\[
\frac{1}{2} \frac{d}{dt} \| \Lambda^r e^{\tau \Lambda^{\frac{1}{2}} \theta^k} \|_{L^2}^2 \leq \left( \dot{\tau} + C \left( \| \Lambda^r \theta^k \|_{L^2} + \tau \| \Lambda^r e^{\tau \Lambda^{\frac{1}{2}} \theta^k} \|_{L^2} \right) \right) \| \Lambda^{r + \frac{1}{2}} e^{\tau \Lambda^{\frac{1}{2}} \theta^k} \|_{L^2}^2
\]
\[+ \| \Lambda^r e^{\tau \Lambda^{\frac{1}{2}} S} \|_{L^2} \| \Lambda^r e^{\tau \Lambda^{\frac{1}{2}} \theta^k} \|_{L^2} \]
\[
\leq \left( \dot{\tau} + C \left( \| \Lambda^r e^{\tau \Lambda^{\frac{1}{2}} \theta^k} \|_{L^2} \right) \right) \| \Lambda^{r + \frac{1}{2}} e^{\tau \Lambda^{\frac{1}{2}} \theta^k} \|_{L^2}^2
\]
\[+ \| \Lambda^r e^{\tau \Lambda^{\frac{1}{2}} S} \|_{L^2} \| \Lambda^r e^{\tau \Lambda^{\frac{1}{2}} \theta^k} \|_{L^2},
\]

where the last inequality follows provided that \( \tau \leq 1 \). We can then apply the same argument given in the proof of Theorem \( 5.1 \), which implies the existence of a Gevrey-class \( s \) solution \( \theta^k \) on \( [0, T_s) \), where the maximal time of existence of the Gevrey-class \( s \) solution is given by \( T_s = \frac{T_0}{4Ck_0} \).

To prove the convergence result \( (5.30) \), we let \( \varphi = \theta^k - \theta^0 \). Then we multiply \( (5.6) \) by \( \Lambda^{2r} e^{2\tau \Lambda^{\frac{1}{2}} \theta^0} \) to give
\[
\frac{1}{2} \frac{d}{dt} \| \Lambda^r e^{\tau \Lambda^{\frac{1}{2}} \varphi(\cdot, t)} \|_{L^2}^2 = \dot{\tau} \| \Lambda^{r + \frac{1}{2}} e^{\tau \Lambda^{\frac{1}{2}} \varphi(\cdot, t)} \|_{L^2}^2 + T_1 + T_2 + T_3,
\]  \( (5.33) \)

where \( T_1, T_2 \) and \( T_3 \) are given by
\[
T_1 = \left( \langle u^k - u^0 \rangle \cdot \nabla \theta^0, \Lambda^{2r} e^{2\tau \Lambda^{\frac{1}{2}} \varphi} \right),
\]
\[
T_2 = \left( u^k \cdot \nabla \varphi, \Lambda^{2r} e^{2\tau \Lambda^{\frac{1}{2}} \varphi} \right),
\]
\[
T_3 = -\langle \kappa \Lambda^r \theta^0, \Lambda^{2r} e^{2\tau \Lambda^{\frac{1}{2}} \varphi} \rangle.
\]
Using the method for bounding the term $T$, we have

$$|T_1| + |T_2| \leq C \||r + \frac{1}{2} e^{r \Lambda} \phi \|_{L^2} \|r + \frac{1}{2} e^{r \Lambda} \phi \|_{L^2}^2 + C \||r + \frac{1}{2} e^{r \Lambda} \phi \|_{L^2} \|r + \frac{1}{2} e^{r \Lambda} \phi \|_{L^2}^2,$$

and $T_3$ can be readily bounded by $\kappa \||r + \frac{1}{2} e^{r \Lambda} \phi \|_{L^2} \|r e^{r \Lambda} \phi \|_{L^2}$. We conclude from (5.33) that

$$\frac{1}{2} \frac{d}{dt} \||r + \frac{1}{2} e^{r \Lambda} \phi \|_{L^2}^2 \leq \||r + \frac{1}{2} e^{r \Lambda} \phi \|_{L^2} \hat{t} + C \||r + \frac{1}{2} e^{r \Lambda} \phi \|_{L^2} + C \||r + \frac{1}{2} e^{r \Lambda} \phi \|_{L^2}^2,$$

so we get

$$\frac{d}{dt} \||r + \frac{1}{2} e^{r \Lambda} \phi \|_{L^2} \leq \kappa \||r + \frac{1}{2} e^{r \Lambda} \phi \|_{L^2},$$

and (5.30) follows by taking $\kappa \to 0$. □

**Remark 5.7.** Theorem 5.6 can be applied on critical SQG equation as well as incompressible porous media (IPM) equation; see [13, 14, 18] for more discussions on IPM equation.

**Remark 5.8.** In the same spirit as Theorem 5.2, one can show that the Gevrey-class $s$ solutions obtained in Theorem 5.6 can be extended globally in time provided that

$$\|\theta_0\|_{L^2}^2 \leq C \|\theta_0\|_{H^r}^{1-\beta} + \|\theta_0\|_{L^2}^2 \|S\|_{H^{1-r}} \leq \epsilon$$

and

$$\||r \|_{L^2}^2 + \frac{2}{\kappa} \|S\|_{H^{1-r}} \leq \epsilon$$

for some sufficiently small number $\epsilon$. Here $r > \frac{d+1}{2}$ is the one given in Theorem 5.6 and $\beta = 1 - \frac{1}{r} \left[\frac{d+2}{2} + (1 - r)\right]$. The key of the proof is to apply the estimate (5.32) on $T$, which gives

$$\frac{1}{2} \frac{d}{dt} \||r \|_{L^2}^2 + \kappa \||r \|_{L^2}^2 \leq \left[\hat{t} + C \||r \|_{L^2} + \tau \||r \|_{L^2}^2 \right] \||r \|_{L^2}^2 + \||r \|_{L^2}^2 \|S\|_{L^2} \|\theta_0\|_{L^2}.$$
6. Long Time Behaviour for Solutions When $\nu > 0$ and $\kappa > 0$

In this section, we study the long time behaviour for solutions to the active scalar equation (1.1) when $\nu > 0$ and $\kappa > 0$. Based on the global-in-time existence results established in Theorem 4.1, for fixed $\nu > 0$ and $\kappa > 0$, we can define a solution operator $\pi^\nu(t)$ for the initial value problem (1.1) via

$$
\pi^\nu(t) : H^1 \to H^1, \quad \pi^\nu(t)\theta_0 = \theta(\cdot, t), \quad t \geq 0. \tag{6.1}
$$

We study the long-time dynamics of $\pi^\nu(t)$ on the phase space $H^1$. Specifically, we establish the existence of global attractors for $\pi^\nu(t)$ in $H^1$, which will be given in Sect. 6.1. Once we obtain the existence of global attractors, we further address some properties for the attractors which will be explained in Sect. 6.2. The results obtained in Sects. 6.1 and 6.2 will be sufficient for proving Theorem 2.3.

6.1. Existence of Global Attractors in $H^1$-Space

The following theorem gives the main results for this subsection:

**Theorem 6.1.** (Existence of $H^1$-global attractor) Let $S \in L^\infty \cap H^1$. For $\nu, \kappa > 0$ and $\gamma \in (0, 2]$, the solution map $\pi^\nu(t) : H^1 \to H^1$ associated to (1.1) possesses a unique global attractor $G^\nu$. Moreover, there exists $M_{G^\nu}$ which depends only on $\nu, \kappa, \gamma, \|S\|_{L^\infty \cap H^1}$ and universal constants, such that if $\theta_0 \in G^\nu$, we have that

$$
\|\theta(\cdot, t)\|_{H^1+^{\gamma/2}} \leq M_{G^\nu}, \quad \forall t \geq 0, \tag{6.2}
$$

and

$$
\frac{1}{T} \int_t^{t+T} \|\theta(\cdot, \tau)\|_{H^1+\gamma} d\tau \leq M_{G^\nu}, \quad \forall t \geq 0 \text{ and } T > 0, \tag{6.3}
$$

where $\theta(\cdot, t) = \pi^\nu(t)\theta_0$.

Theorem 6.1 will be proved in a sequence of lemmas. In the following lemma, we first show the existence of an $L^\infty$-absorbing set by using the bound (4.7) on $\|\theta(t)\|_{L^\infty}$ for $t \geq 1$.

**Lemma 6.2.** (Existence of an $L^\infty$-absorbing set) Let $c_0 > 0$ be defined in Proposition 4.2. Then the set

$$
B_\infty = \left\{ \phi \in L^\infty \cap H^1 : \|\phi\|_{L^\infty} \leq \frac{2}{c_0\kappa} \|S\|_{L^\infty} \right\}
$$

is an absorbing set for $\pi^\nu(t)$. Moreover, using (4.3), we have

$$
\sup_{t \geq 0} \sup_{\theta_0 \in B_\infty} \|\pi^\nu(t)\theta_0\|_{L^\infty} \leq \frac{3\|S\|_{L^\infty}}{c_0\kappa}. \tag{6.4}
$$
**Proof.** The proof follows by the same argument given in [10, Theorem 3.1]. For a fixed bounded set $B \subset H^1$, we let

$$R = \sup_{\phi \in B} \|\phi\|_{H^1}.$$ 

Using the bound (4.7) and the Poincaré inequality, we conclude that if $\theta_0 \in B$, then

$$\|\pi^\nu(t)\theta_0\|_{L^\infty} \leq \frac{C}{\kappa} \left[ R + \frac{\|S\|_{L^\infty}}{\kappa^{\frac{1}{2}}} \right] e^{-c_0 \kappa t} + \frac{\|S\|_{L^\infty}}{c_0 \kappa}, \quad \forall t \geq 1. \quad (6.5)$$

Choose $t_B = t_B(R, \|S\|_{L^2 \cap L^\infty}) \geq 1$ such that

$$\frac{C}{\kappa} \left[ R + \frac{\|S\|_{L^\infty}}{\kappa^{\frac{1}{2}}} \right] e^{-c_0 \kappa t_B} \leq \frac{\|S\|_{L^\infty}}{c_0 \kappa}.$$

Then we have $\pi^\nu(t)\theta_0 \in B_\infty$, and hence, $B_\infty$ is absorbing. □

Next we prove the following lemma which gives the necessary *a priori* estimate in $C^\alpha$-space with some appropriate exponent $\alpha \in (0, 1)$. As pointed out in [10], in view of Lemma 6.2, we can see that the solutions to (1.1) emerging from data in a bounded subset of $H^1$ are absorbed in finite time by $B_\infty$. Hence in proving Lemma 6.3, we can assume that $\theta_0 \in L^\infty$ and derive *a priori* bounds in terms of $\|\theta_0\|_{L^\infty}$.

**Lemma 6.3.** (Estimates in $C^\alpha$-space) Assume that $\theta_0 \in H^1 \cap L^\infty$ and fix $\nu, \kappa > 0$. There exists $\alpha \in (0, \frac{\gamma}{3+\gamma})$ which depends on $\|\theta_0\|_{L^\infty}, \|S\|_{L^\infty}, \nu, \kappa, \gamma$ such that

$$\|\theta(\cdot,t)\|_{C^\alpha} \leq C(K_\infty + \tilde{K}_\infty), \quad \forall t \geq t_\alpha := \frac{3}{2\gamma(1-\alpha)}, \quad (6.6)$$

where $C > 0$ is a positive constant, $K_\infty$ and $\tilde{K}_\infty$ are given respectively by

$$K_\infty := \|\theta_0\|_{L^\infty} + \frac{1}{c_0 \kappa} \|S\|_{L^\infty},$$

$$\tilde{K}_\infty := \kappa^{-\frac{1}{2}} K_\infty + \|S\|_{L^\infty}^{\frac{2+\gamma}{2(1+\gamma)}} \frac{1}{\kappa^{-\frac{1}{2(1+\gamma)}}} K_\infty^{\frac{2+\gamma}{2(1+\gamma)}} + \kappa^{-\frac{1}{2}} K_\infty^{\frac{6+\gamma}{2}}. \quad (6.7)$$

**Remark 6.4.** Using the bound (4.3), under the assumption that $\theta_0 \in L^\infty$, for $\kappa > 0$, we have

$$\|\theta(\cdot,t)\|_{L^\infty} \leq K_\infty, \quad \forall t \geq 0. \quad (6.8)$$

**Proof of Lemma 6.3.** The idea of the proof follows from [10, Theorem 4.1]. As suggested in [9, 10], we introduce the finite difference

$$\delta_h \theta(x, t) = \theta(x + h) - \theta(x, t),$$

where $x, h \in T^d$. Then $\delta_h \theta$ satisfies

$$L_h(\delta_h \theta)^2 + D[\delta_h \theta] = 2(\delta_h S)(\delta_h \theta), \quad (6.9)$$
where $\mathcal{L}_h$ is the operator given by $\mathcal{L}_h := \partial_t + u \cdot \nabla + (\delta_h u) \cdot \nabla_h + \Lambda^Y$, and $D[\delta_h \theta](x)$ is the function

$$D[\delta_h \theta](x) = \int_{\mathbb{R}^d} \frac{|\delta_h \theta(x) - \delta_h \theta(x + y)|^2}{|y|^{d+\gamma}} \, dy.$$ 

Let $\xi(t) = \xi : [0, \infty) \to [0, \infty)$ be a bounded decreasing differentiable function which will be defined later, and for $\alpha \in (0, \frac{\gamma}{\gamma + \gamma})$, we define $v$ by

$$v(x, t; h) = \frac{|\delta_h \theta(x, t)|}{(\xi(t)^2 + |h|^2) \frac{\gamma}{2}}.$$ 

Using (6.9), we can see that $v$ satisfies the following inequality:

$$\mathcal{L}_h v^2 + \kappa \frac{D[\delta_h \theta(x, t)]}{(\xi^2 + |h|^2) \alpha} \leq 2\alpha \frac{\xi}{\xi^2 + |h|^2} v^2 + 2\alpha \frac{|h|}{\xi^2 + |h|^2} |\delta_h u| v^2 + \frac{2\|S\|_{L^\infty}}{(\xi^2 + |h|^2) \frac{\gamma}{2}} v.$$

(6.10)

We estimate the term in (6.10) as follows. First, using the similar argument given in [9], it can be shown that for $r \geq 4|h|$, there exists some constant $C \geq 1$ such that

$$D[\delta_h \theta](x) \geq \frac{1}{2r\gamma} |\delta_h \theta(x)|^2 - C |\delta_h \theta(x)| \|\theta\|_{L^\infty} \frac{|h|}{r^{1+\gamma}},$$

Choose $r \geq 4(\xi^2 + |h|^2)^{\frac{1}{2}} \geq 4|h|$ such that

$$r = \frac{4C \|\theta\|_{L^\infty}}{|\delta_h \theta(x)|} (\xi^2 + |h|^2)^{\frac{(1-a)(2+\gamma)}{6} + \frac{\gamma}{2}},$$

so we have

$$\frac{D[\delta_h \theta](x, t)}{(\xi^2 + |h|^2) \alpha} \geq \frac{|v(x, t; h)|^{2+\gamma}}{c_2 \|\theta\|_{L^\infty}^{\gamma} (\xi^2 + |h|^2)^{\frac{1-a)(2+\gamma)}{6}}}$$

(6.11)

for some positive constant $c_2$. For the term $2\alpha \frac{\xi}{\xi^2 + |h|^2} v^2$, we take $\xi$ such that $\xi$ solves the ordinary differential equation

$$\dot{\xi} = -\xi^a, \quad \xi(0) = 1,$$

(6.12)

where $a = 1 - \frac{2}{3} \gamma (1 - \alpha)$. More explicitly, $\xi$ can be given by

$$\xi(t) = \begin{cases} 
1 - t(1-a)^{\frac{1}{1-a}}, & t \in [0, t_\alpha] \\
0, & t \in (t_\alpha, \infty),
\end{cases}$$

(6.13)

where $t_\alpha$ is given by

$$t_\alpha = \frac{3}{2\gamma (1 - \alpha)}.$$
Hence, for \( \alpha \leq \frac{\gamma}{3+\gamma} \), we have

\[
2\alpha |\xi|^2 \frac{v^2}{\xi^2 + |h|^2} \leq \frac{2\gamma}{3+\gamma} \frac{v^2}{(\xi^2 + |h|^2)^{\gamma(1-\alpha)/(3+\gamma)}}.
\]

Using Young’s inequality, we can further obtain that

\[
2\alpha |\xi|^2 \frac{v^2}{\xi^2 + |h|^2} \leq \frac{\kappa |v|^{2+\gamma}}{8c_2 \|\theta\|_{L^\infty}^\gamma (\xi^2 + |h|^2)^{\gamma(1-\alpha)/(3+\gamma)}} + C \kappa^{-\frac{2}{\gamma}} \|\theta\|_{L^\infty}^2 \tag{6.15}
\]

for some positive constant \( C \). For the term \( \frac{2\|S\|_{L^\infty}}{(\xi^2 + |h|^2)^{\gamma}} v \), using Young’s inequality, we have

\[
\frac{2\|S\|_{L^\infty}}{(\xi^2 + |h|^2)^{\gamma}} v \leq \frac{\kappa |v|^{2+\gamma}}{8c_2 \|\theta\|_{L^\infty}^\gamma (\xi^2 + |h|^2)^{\gamma(1-\alpha)/(3+\gamma)}} + C (\xi^2 + |h|^2)^b \|S\|_{L^\infty}^{\frac{2+\gamma}{\gamma - \frac{1}{1+\gamma}}} \kappa^{-\frac{1}{1+\gamma}} \|\theta\|_{L^\infty}^{\frac{\gamma}{\gamma - \frac{1}{1+\gamma}}}.
\]

where \( b = \left( \frac{\gamma(1-\alpha)}{6} - \frac{\alpha}{2} \right) \left( \frac{2+\gamma}{1+\gamma} \right) \) and note that \( b \geq 0 \) since \( \alpha \leq \frac{\gamma}{3+\gamma} \) and \( \gamma \in (0, 2] \).

Since \( \xi, |h| \leq 1 \), we infer that

\[
\frac{2\|S\|_{L^\infty}}{(\xi^2 + |h|^2)^{\gamma}} v \leq \frac{\kappa |v|^{2+\gamma}}{8c_2 \|\theta\|_{L^\infty}^\gamma (\xi^2 + |h|^2)^{\gamma(1-\alpha)/(3+\gamma)}} + C \|S\|_{L^\infty}^{\frac{2+\gamma}{\gamma - \frac{1}{1+\gamma}}} \kappa^{-\frac{1}{1+\gamma}} \|\theta\|_{L^\infty}^{\frac{\gamma}{\gamma - \frac{1}{1+\gamma}}} \tag{6.16}
\]

Similarly, for the term \( 2\alpha \frac{|h|}{\xi^2 + |h|^2} |\delta h u| v^2 \), we apply Young’s inequality again to obtain

\[
2\alpha \frac{|h|}{\xi^2 + |h|^2} |\delta h u| v^2 \leq 2\alpha \frac{|h|^2}{\xi^2 + |h|^2} \|\nabla u\|_{L^\infty} v^2 \leq \frac{\kappa |v|^{2+\gamma}}{8c_2 \|\theta\|_{L^\infty}^\gamma (\xi^2 + |h|^2)^{\gamma(1-\alpha)/(3+\gamma)}} + C \|\nabla v\|_{L^\infty}^{\frac{2+\gamma}{\gamma - \frac{1}{1+\gamma}}} \kappa^{-\frac{2}{\gamma}} \|\theta\|_{L^\infty}^{\frac{\gamma}{\gamma - \frac{1}{1+\gamma}}} \tag{6.17}
\]

Applying the bounds (6.11), (6.15), (6.16) and (6.17) on (6.10), we obtain

\[
\mathcal{L} v^2 + \kappa \cdot \frac{D[\delta h \theta(x, t)]}{(\xi^2 + |h|^2)^{\alpha}} + \frac{\kappa |v|^{2+\gamma}}{8c_2 \|\theta\|_{L^\infty}^\gamma (\xi^2 + |h|^2)^{\gamma(1-\alpha)/(3+\gamma)}} \leq C \left[ \kappa^{-\frac{2}{\gamma}} \|\theta\|_{L^\infty}^2 + \|S\|_{L^\infty}^{\frac{2+\gamma}{\gamma - \frac{1}{1+\gamma}}} \kappa^{-\frac{1}{1+\gamma}} \|\theta\|_{L^\infty}^{\frac{\gamma}{\gamma - \frac{1}{1+\gamma}}} + \|\nabla v\|_{L^\infty}^{\frac{2+\gamma}{\gamma - \frac{1}{1+\gamma}}} \kappa^{-\frac{2}{\gamma}} \|\theta\|_{L^\infty}^{\frac{\gamma}{\gamma - \frac{1}{1+\gamma}}} \right].
\]

Applying the bound (6.8) on \( \|\theta\|_{L^\infty} \), we have

\[
\|\theta\|_{L^\infty} \leq K, \tag{6.18}
\]

and following the estimate (4.15) on \( \Lambda u \), we further obtain that

\[
\|\nabla u\|_{L^\infty} \leq C \|\theta\|_{L^\infty} \leq C K_\infty.
\]
Together with the fact that \( \| S \|_{L^\infty} \leq c_0 \kappa K_\infty \), this implies that

\[
\mathcal{L}_h v^2 + \frac{\kappa |v|^{2+\gamma}}{8c_2 \| \theta \|_{L^\infty}^\gamma (\xi^2 + |h|^2)^{\frac{\gamma(1-\alpha)(2+\gamma)}{6}}} \leq C \tilde{K}_\infty^2,
\]

where \( \tilde{K}_\infty \) is defined in (6.7). Since \( (\xi^2 + |h|^2)^{\frac{\gamma(1-\alpha)(2+\gamma)}{6}} \leq 2^{\frac{\gamma(1-\alpha)(2+\gamma)}{6}} \leq 4 \), we conclude that

\[
\mathcal{L}_h v^2 + \frac{\kappa |v|^{2+\gamma}}{32c_2 K_\infty^\gamma} \leq C \tilde{K}_\infty^2.
\] (6.19)

Take \( \psi(t) = \| v(t) \|_{L^\infty_{x,h}}^2 \), then because of (6.19), \( \psi \) satisfies the following differential inequality:

\[
\frac{d}{dt} \psi + \frac{\kappa \psi^{1+\gamma}}{32c_2 K_\infty^\gamma} \leq C \tilde{K}_\infty^2.
\] (6.20)

Moreover, \( \psi(0) \) satisfies the bound given by

\[
\psi(0) \leq \frac{4\| \theta_0 \|_{L^\infty}^2}{\bar{\xi}(0)^{2\alpha}} \leq 4K_\infty^2,
\]

and hence it follows from (6.20) that

\[
\psi(t) \leq C(K_\infty + \tilde{K}_\infty), \quad \forall t \geq 0
\] (6.21)

for some sufficiently large constant \( C > 0 \). In view of (6.21), we prove that

\[
[\theta(t)]_{C^\alpha}^2 \leq C(K_\infty + \tilde{K}_\infty), \quad \forall t \geq t_\alpha.
\] (6.22)

where \( t_\alpha \) is given by (6.14). Finally, together with the bound (6.8), we obtain

\[
\| \theta(t) \|_{C^\alpha} = \| \theta(t) \|_{L^\infty} + [\theta(t)]_{C^\alpha}^2 \leq C(K_\infty + \tilde{K}_\infty), \quad \forall t \geq t_\alpha,
\]

which finishes the proof of Lemma 6.3. \( \Box \)

**Remark 6.5.** Given \( \alpha \in (0, \frac{\gamma}{3+\gamma}] \), if we further assume that \( \theta_0 \in C^\alpha \), then following the similar argument given in the proof of Lemma 6.3 as shown above (also see [9] for reference), we have

\[
\| \theta(t) \|_{C^\alpha} \leq \| \theta_0 \|_{C^\alpha} + C(K_\infty + \tilde{K}_\infty), \quad \forall t \geq 0.
\] (6.24)

With the help of Lemma 6.3, we obtain the following result which can be regarded as an improvement of the regularity of the absorbing set \( B_\infty \) defined in Lemma 6.2:
**Lemma 6.6.** (Existence of an \( C^\alpha \)-absorbing set) There exists \( \alpha \in (0, \frac{\nu}{3+\nu}] \) and a constant \( C_\alpha = C_\alpha(\|S\|_{L^\infty}, \alpha, \nu, \kappa, \gamma, K_\infty, \bar{K}_\infty) \geq 1 \) such that the set
\[
B_\alpha = \left\{ \phi \in C^\alpha \cap H^1 : \|\phi\|_{C^\alpha} \leq C_\alpha \right\}
\]
is an absorbing set for \( \pi^\nu(t) \). Moreover, we have
\[
\sup_{t \geq 0} \sup_{0 \leq t \leq \alpha} \|\pi^\nu(t)\|_{\alpha} \leq 2C_\alpha. \tag{6.25}
\]

**Proof.** It is enough to prove that the \( L^\infty \)-absorbing set \( B_\infty \) given in Lemma 6.2 is itself absorbed by \( B_\alpha \). Take \( \theta_0 \in B_\infty \), then by using (6.4), we have
\[
\|\pi^\nu(t)\theta_0\|_{L^\infty} \leq \frac{3\|S\|_{L^\infty}}{c_0\kappa} \quad \forall t \geq 0.
\]
Therefore, if we define \( C_\alpha \) by
\[
C_\alpha := \max \left\{ 1, \frac{3\|S\|_{L^\infty}}{c_0\kappa} + \kappa - \frac{1}{\gamma} \left( \frac{3\|S\|_{L^\infty}}{c_0\kappa} \right)^{\frac{\nu}{2(3+\nu)}} \right\},
\]
then \( \pi^\nu(t)\theta_0 \in B_\alpha \) for all \( t \geq t_\alpha \). Since \( t_\alpha \) only depends on \( \kappa, \gamma \) and \( \|S\|_{L^\infty} \), we conclude that \( \pi^\nu(t)B_\infty \subset B_\alpha \) for all \( t \geq t_\alpha \). Finally, the bound (6.25) follows immediately from (6.24) and our choice of \( C_\alpha \), and we finish the proof. \( \square \)

We now proceed to prove the existence of a bounded absorbing set in \( H^1 \).

**Lemma 6.7.** (Existence of an \( H^1 \)-absorbing set) There exists \( \alpha \in (0, \frac{\nu}{3+\nu}] \) and a constant \( R_1 = R_1(\|S\|_{L^\infty \cap H^1}, \alpha, \nu, \kappa, \gamma) \geq 1 \) such that the set
\[
B_1 = \{ \phi \in C^\alpha \cap H^1 : \|\phi\|_{H^1} + \|\phi\|_{C^\alpha}^2 \leq R_1^2 \}
\]
is an absorbing set for \( \pi^\nu(t) \). Moreover, we have
\[
\sup_{t \geq 0} \sup_{0 \leq t \leq \alpha} \left[ \|\pi^\nu(t)\theta_0\|_{H^1}^2 + \|\pi^\nu(t)\theta_0\|_{C^\alpha}^2 + \int_{t}^{t+1} \|\pi^\nu(\tau)\theta_0\|_{H^1}^2 \right] \leq 2R_1^2. \tag{6.26}
\]

**Proof.** As pointed out in [10], it is enough to establish an a priori estimate for initial data in \( H^1 \cap C^\alpha \). Suppose that \( \theta_0 \in H^1 \cap C^\alpha \). We apply \( \nabla \) to (1.1) and take the inner product with \( \nabla \theta \) to obtain
\[
(\partial_t + u \cdot \nabla + \kappa A\gamma) |\nabla \theta|^2 + \kappa D[\nabla \theta] = -2\partial_j u_j \partial_j \theta \partial_j \theta + 2\nabla S \cdot \nabla \theta, \tag{6.27}
\]
where \( D[\nabla \theta] \) is given by
\[
D[\nabla \theta](x) = \int_{\mathbb{R}^d} \frac{|\nabla \theta(x) - \nabla \theta(x+y)|^2}{|y|^{d+\gamma}} dy.
\]
Since by (6.25), we have \( \| \theta(t) \|_{C^\alpha} \leq 2C_\alpha \) where \( C_\alpha \) is defined in Lemma 6.6, hence we can apply [11, Theorem 2.2] to obtain
\[
D[\nabla \theta] \geq \frac{|\nabla \theta|^{2 + \frac{r}{1 - \alpha}}}{C[\theta]^{\frac{r}{1 - \alpha}}} \geq \frac{|\nabla \theta|^{2 + \frac{r}{1 - \alpha}}}{C(2C_\alpha)^{\frac{r}{1 - \alpha}}}.
\] (6.28)

Also, using Young’s inequality, we have
\[
| - 2 \partial_{x_l} u_j \partial_{x_j} \theta \partial_{x_l} \theta | \leq \frac{k}{4} \frac{|\nabla \theta|^{2 + \frac{r}{1 - \alpha}}}{C(2C_\alpha)^{\frac{r}{1 - \alpha}}} + \left( \frac{4C}{\kappa} \right)^{\frac{2 - 2\alpha}{r}} (2C_\alpha)^2 |\nabla u|^{1 + \frac{2(1 - \alpha)}{r}} + 2|\nabla S| |\nabla \theta|. \] (6.29)

We apply (6.28) and (6.29) on (6.27) to get
\[
(\partial_t + u \cdot \nabla + \kappa \Lambda^\gamma) |\nabla \theta|^2 + \frac{k}{2} D[\nabla \theta] \leq \left( \frac{4C}{\kappa} \right)^{\frac{2 - 2\alpha}{r}} (2C_\alpha)^2 |\nabla u|^{1 + \frac{2(1 - \alpha)}{r}} + 2|\nabla S| |\nabla \theta|. \] (6.30)

Integrate (6.30) over \( \mathbb{T}^d \), and using the identity \( \frac{1}{2} \int_{\mathbb{T}^d} D[\nabla \theta](x) \, dx = \| \theta \|^2_{H^{1 + \frac{r}{2}}} \), we obtain
\[
\frac{d}{dt} \| \theta \|_{H^1}^2 + \frac{k}{2} \| \theta \|_{H^{1 + \frac{r}{2}}}^2 \leq \left( \frac{4C}{\kappa} \right)^{\frac{2 - 2\alpha}{r}} (2C_\alpha)^2 \int_{\mathbb{T}^d} |\nabla u|^{1 + \frac{2(1 - \alpha)}{r}} + 2\| \nabla S \|_{H^1} \| \nabla \theta \|_{H^1}
\leq \left( \frac{4C}{\kappa} \right)^{\frac{2 - 2\alpha}{r}} (2C_\alpha)^2 \| \nabla u \|_{L^{\infty}}^{1 + \frac{2(1 - \alpha)}{r}} + 2\| \nabla S \|_{H^1} \| \nabla \theta \|_{H^1}
\leq \left( \frac{4C}{\kappa} \right)^{\frac{2 - 2\alpha}{r}} (2C_\alpha)^2 K_\infty^{1 + \frac{2(1 - \alpha)}{r}} + \frac{8}{c_{\gamma,d,K}} \| S \|_{H^1}^2 + \frac{k}{4} \| \theta \|_{H^{1 + \frac{r}{2}}}^2,
\] (6.31)

where the last inequality follows by the bound (6.18) and Young’s inequality, and \( c_{\gamma,d} > 0 \) is the dimensional constant for which it satisfies the bound
\[
\| \theta \|_{H^1}^2 \leq (c_{\gamma,d})^{-1} \| \theta \|_{H^{1 + \frac{r}{2}}}^2. \] (6.32)

If we choose \( K_1 \geq 1 \) such that
\[
K_1 = \frac{8}{c_{\gamma,d,K}} \left[ \left( \frac{4C}{\kappa} \right)^{\frac{2 - 2\alpha}{r}} (2C_\alpha)^2 K_\infty^{1 + \frac{2(1 - \alpha)}{r}} + \frac{8}{c_{\gamma,d,K}} \| S \|_{H^1}^2 \right],
\]
then by applying Grönwall’s inequality on (6.31), we conclude that
\[
\| \theta(t) \|_{H^1}^2 \leq \| \theta_0 \|_{H^1}^2 e^{-\frac{c_{\gamma,d,K}}{K_1}} + K_1. \] (6.33)

Now we define \( R_1 = 2K_1 \), then it is straightforward to see that the set \( B_1 \) is an absorbing set for \( \pi^\gamma(t) \). Moreover, upon integrating (6.31) on the time interval \( (t, t + 1) \) and applying (6.33), we further obtain (6.26) and the proof is complete.

Once Lemma 6.7 is established, we can further improve the regularity of the absorbing set \( B_1 \) to \( H^{1 + \frac{r}{2}} \), which is illustrated in the next lemma.

Lemma 6.8. (Existence of an $H^{1+\frac{\gamma}{2}}$-absorbing set) There exists a constant $R_{1+\frac{\gamma}{2}} \geq 1$ which depends on $\|S\|_{L_\infty \cap H^1}$, $\nu$, $\kappa$, $\gamma$ such that the set

$$B_{1+\frac{\gamma}{2}} = \left\{ \phi \in H^{1+\frac{\gamma}{2}} : \|\phi\|_{H^{1+\frac{\gamma}{2}}} \leq R_{1+\frac{\gamma}{2}} \right\}$$

is an absorbing set for $\pi^\nu(t)$. Moreover,

$$\sup_{t \geq 0} \sup_{\theta_0 \in B_{1+\frac{\gamma}{2}}} \|\pi^\nu(t)\theta_0\|_{H^{1+\frac{\gamma}{2}}} \leq 2R_{1+\frac{\gamma}{2}}. \quad (6.34)$$

Proof. Similar to the previous cases, it is enough to show that $B_{1+\frac{\gamma}{2}}$ absorbs the $H^1$-absorbing set $B_1$ obtained in Lemma 6.7. Suppose $\theta_0 \in B_1$, then from (6.26), we have

$$\sup_{t \geq 0} \int_{t}^{t+1} \|\pi^\nu(\tau)\theta_0\|^2_{H^{1+\frac{\gamma}{2}}} d\tau \leq 2R_1^2. \quad (6.35)$$

Following the argument given in the proof of Theorem 4.1 and using the bound (6.8) on $\|\theta^\kappa\|_{L_\infty}$, for $t \geq 0$, we arrive at

$$\frac{d}{dt} \|\Lambda^{1+\frac{\gamma}{2}} \theta^\kappa\|^2_{L_2} + \kappa \|\Lambda^{1+\gamma} \theta^\kappa\|^2_{L_2} \leq C \|\theta^\kappa\|_{L_\infty} \|\Lambda^{1+\frac{\gamma}{2}} \theta^\kappa\|^2_{L_2} + 2\|\Lambda^{1+\frac{\gamma}{2}} S\|_{L_2} \|\Lambda^{1+\frac{\gamma}{2}} \theta^\kappa\|_{L_2} \leq CK_{\infty} \|\Lambda^{1+\frac{\gamma}{2}} \theta^\kappa\|^2_{L_2} + \frac{4}{c_{y,d}\kappa} \|S\|^2_{H^{1+\frac{\gamma}{2}}} + \frac{\kappa}{2} \|\theta\|^2_{H^{1+\gamma}}, \quad (6.36)$$

where $c_{y,d}$ is defined in (6.32). Hence using the local integrability (6.35) and the uniform Grönwall’s lemma (see [9, Lemma C.1] for example), we obtain

$$\|\pi^\nu(t)\theta_0\|^2_{H^{1+\frac{\gamma}{2}}} \leq \left[2CK_{\infty}R_1^2 + \frac{4}{c_{y,d}\kappa} \|S\|^2_{H^{1+\frac{\gamma}{2}}} \right] e^{CK_{\infty}R_1^2}, \quad \forall t \geq 1. \quad (6.37)$$

By setting

$$R_{1+\frac{\gamma}{2}} := \left[2CK_{\infty}R_1^2 + \frac{4}{c_{y,d}\kappa} \|S\|^2_{H^{1+\frac{\gamma}{2}}} \right]^\frac{1}{2} e^{CK_{\infty}R_1^2},$$

we conclude that $\pi^\nu(t)B_1 \subset B_{1+\frac{\gamma}{2}}$ for all $t \geq 1$. Since $B_{1+\frac{\gamma}{2}}$ absorbs $B_1$, the bound (6.34) then follows by (6.26) and (6.37), and we finish the proof. \hfill \Box

Remark 6.9. By combining (6.36) with (6.37), if $\theta_0 \in B_1$, then for $t \geq 1$ and $T > 0$, we further obtain

$$\frac{1}{T} \int_{t}^{t+T} \|\pi^\nu(\tau)\theta_0\|^2_{H^{1+\gamma}} d\tau \leq R_{1+\gamma}^2, \quad (6.38)$$

where

$$R_{1+\gamma} := \left[CK_{\infty}R_1^2 + \frac{4}{c_{y,d}\kappa} \|S\|^2_{H^{1+\frac{\gamma}{2}}} \right]^\frac{1}{2}.$$
The existence and regularity of the global attractor claimed by Theorem 6.1 now follows from Lemma 6.8 by applying the argument given in [4, Proposition 8], and the bounds (6.2)–(6.3) are immediate consequences of (6.34) and (6.38) by taking \( M_G := \max\{2R_1, R_1 + \gamma\} \). We summarise the properties of the global attractor \( G^\nu \) as claimed by Theorem 6.1.

Corollary 6.10. The solution map \( \pi^\nu(t) : H^1 \to H^1 \) associated to (1.1) possesses a unique global attractor \( G^\nu \) with the following properties:

- \( G^\nu \subset H^{1+\gamma} \) and is the \( \omega \)-limit set of \( B_1 \), namely
  \[
  G^\nu = \omega(B_1) = \bigcap_{t \geq 0} \bigcup_{\tau \geq t} \pi^\nu(\tau)B_1,
  \]

- For every bounded set \( B \subset H^1 \), we have
  \[
  \lim_{t \to \infty} \text{dist}(\pi^\nu(t)B, G^\nu) = 0,
  \]
  where \( \text{dist} \) stands for the usual Hausdorff semi-distance between sets given by the \( H^1 \)-norm.

- \( G^\nu \) is minimal in the class of \( H^1 \)-closed attracting set.

6.2. Further Properties for the Global Attractors

In this subsection, we prove some additional properties for the global attractors obtained in Theorem 6.1. Recall that we have \( d = 2 \) or \( d = 3 \), and throughout this subsection, we impose an extra condition on the exponent \( \gamma \), namely

\[
\gamma \in [1, 2]. \tag{6.39}
\]

Under the assumption (6.39), our goal is to prove the following theorem:

Theorem 6.11. Let \( S \in L^\infty \cap H^1 \). For \( \nu, \kappa > 0 \), assume that the exponent \( \gamma \) satisfies (6.39). Then the global attractor \( G^\nu \) of \( \pi^\nu(t) \) further enjoys the following properties:

- \( G^\nu \) is fully invariant, namely
  \[
  \pi^\nu(t)G^\nu = G^\nu, \quad \forall t \geq 0.
  \]

- \( G^\nu \) is maximal in the class of \( H^1 \)-bounded invariant sets.

- \( G^\nu \) has finite fractal dimension.

We first give the following auxiliary estimates on \( u \) and \( \theta \) under the assumption (6.39). These estimates will be useful for the later analysis.
**Lemma 6.12.** Assume that $\gamma$ satisfies the assumption \eqref{6.39}. Then if $f \in H^{1+\frac{\gamma}{2}}$ and $g, \theta \in H^1$, we have

\[
\|\Lambda^{2-\frac{\gamma}{2}} f\|_{L^2} \leq C \|\Lambda^{1+\frac{\gamma}{2}} f\|_{L^2}, \tag{6.40}
\]
\[
\|g\|_{L^4} \leq C \|\Lambda g\|_{L^2}, \tag{6.41}
\]
\[
\|u\|_{L^\infty} \leq C \|\Lambda^{\frac{\gamma}{2}} \theta\|_{L^2} \leq C_v \|\Lambda \theta\|_{L^2}, \tag{6.42}
\]
\[
\|\nabla u\|_{L^\infty} \leq C_v \|\Lambda^{1+\frac{\gamma}{2}} \theta\|_{L^2}, \tag{6.43}
\]

where $u = u[\theta]$, $C$ is a positive constant which depends on $d$ only, and $C_v$ is a positive constant which depends on $d$ and $\nu$.

**Proof.** The bound \eqref{6.40} follows immediately by observing that $2 - \frac{\gamma}{2} \leq 1 + \frac{\gamma}{2}$ when $\gamma \geq 1$. To prove \eqref{6.41}, using \eqref{3.4} and the assumption $d \in \{2, 3\}$, we have

\[
\|g\|_{L^4} \leq C \|\Lambda^\frac{d}{2} g\|_{L^2} \leq C \|\Lambda g\|_{L^2}.
\]

To prove \eqref{6.42}, by using the bound \eqref{3.2} for $q = d + 1$ and together with \eqref{1.5}, we have

\[
\|u\|_{L^\infty} = \|u[\theta]\|_{L^\infty} \leq C \|u\|_{W^{1,d+1}}
\]
\[
\leq C \|\Lambda^\frac{d}{2} - \frac{d}{d+1} \|_{L^{d+1}}
\]
\[
\leq C_v \|\Lambda^\frac{d}{2} - \frac{d}{d+1} \theta\|_{L^{d+1}}
\]
\[
\leq C_v \|\Lambda^\frac{\gamma}{2} \theta\|_{L^2} \leq C_v \|\Lambda \theta\|_{L^2},
\]

where the last inequality follows since $\frac{d}{2} - \frac{d}{d+1} \leq 1 + \frac{\gamma}{2} \leq 2$ for $\gamma \leq 2$ and $d \in \{2, 3\}$. To prove \eqref{6.43} we have

\[
\|\nabla u\|_{L^\infty} \leq C \|\nabla u\|_{W^{1,d+1}}
\]
\[
\leq C \|\Lambda^\frac{d}{2} - \frac{d}{d+1} u\|_{L^{d+1}}
\]
\[
\leq C_v \|\Lambda^\frac{d}{2} - \frac{d}{d+1} \theta\|_{L^{d+1}} \leq C_v \|\Lambda^{1+\frac{\gamma}{2}} \theta\|_{L^2},
\]

provided that $\frac{d}{2} - \frac{d}{d+1} \leq 1 + \frac{\gamma}{2}$ holds. \qed

We now show that the solution map $\pi^\nu(t)$ is indeed continuous in the $H^1$-topology. More precisely, we have the following lemma:

**Lemma 6.13.** (Continuity of $\pi^\nu(t)$) Let $B_{1+\frac{\gamma}{2}}$ be the absorbing set for $\pi^\nu$ defined in Lemma 6.8. For every $t > 0$, the solution map $\pi^\nu(t) : B_{1+\frac{\gamma}{2}} \to H^1$ is continuous in the topology of $H^1$.

**Proof.** We fix $t > 0$. Let $\theta_0, \tilde{\theta}_0 \in B_{1+\frac{\gamma}{2}}$ be arbitrary such that $\theta = \pi^\nu(t)\theta_0$ and $\tilde{\theta} = \pi^\nu(t)\tilde{\theta}_0$ with

\[
\|\theta(t)\|_{H^{1+\frac{\gamma}{2}}} \leq 2R_{1+\frac{\gamma}{2}}, \quad \|\tilde{\theta}(t)\|_{H^{1+\frac{\gamma}{2}}} \leq 2R_{1+\frac{\gamma}{2}}, \quad \forall t \geq 0. \tag{6.44}
\]
Denote the difference by $\tilde{\theta} = \theta - \tilde{\theta}$ with $\tilde{\theta}_0 = \theta_0 - \tilde{\theta}_0$, then $\tilde{\theta}$ satisfies the following equation

$$\partial_t \tilde{\theta} + \kappa \Lambda^\gamma \tilde{\theta} - \tilde{u} \cdot \nabla \tilde{\theta} + \tilde{u} \cdot \nabla \tilde{\theta} + \tilde{u} \cdot \nabla \tilde{\theta} = 0,$$

(6.45)

where $\tilde{u} = u[\tilde{\theta}]$ and $\tilde{u} = u[\theta]$. Multiply (6.45) by $-\Delta \tilde{\theta}$ and integrate,

$$\int_{\Omega} (\tilde{u} \cdot \nabla \tilde{\theta}) (-\Delta \tilde{\theta}) = \int_{\Omega} \frac{1}{2} \frac{d}{dt} \| \tilde{\theta} \|^2_{H^1} + \kappa \| \tilde{\theta} \|^2_{H^{1+\varphi}^2} \leq \int_{\Omega} \left( \nabla \tilde{u} \cdot \nabla \tilde{\theta} \right) \cdot \nabla \tilde{\theta} + \int_{\Omega} (\tilde{u} \cdot \nabla \tilde{\theta}) (-\Delta \tilde{\theta}) \right| \right. \leq \left( \| \nabla \tilde{u} \|_{L^\infty} + \| \nabla \tilde{u} \|_{L^\infty} \right) \| \nabla \tilde{\theta} \|^2_{L^2} + \int_{\Omega} (\tilde{u} \cdot \nabla \tilde{\theta}) (-\Delta \tilde{\theta}) \right| \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. 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Hence we conclude from (6.49) that
\[ \| \hat{\theta}(t) \|_{H^1}^2 \leq K_t \| \hat{\theta}_0 \|_{H^1}^2, \]
where \( K_t = \exp \left( C t \left( R_{1+\frac{\gamma}{2}} + \frac{k}{\kappa} R_{1+\frac{\gamma}{2}}^2 \right) \right) \), which implies that \( \pi^v(t) \) is continuous in the \( H^1 \)-topology. \( \square \)

Following the argument given in [9, Proposition 5.5] and using the log-convexity method introduced by [1], we can also prove that the solution map \( \pi^v \) is injective on the absorbing set \( B_{1+\frac{\gamma}{2}} \), which is illustrated in the following lemma:

**Lemma 6.14.** (Backwards uniqueness) Fix \( v, \kappa > 0 \) and assume that \( \gamma \) satisfies (6.39). Let \( \tilde{\varphi}_0, \bar{\varphi}_0 \in H^1 \) be two initial data, and let
\[ \tilde{\varphi}, \bar{\varphi} \in C([0, \infty); H^1) \cap L^2([0, \infty); H^{1+\frac{\gamma}{2}}) \]
be the corresponding solutions of the initial value problem (1.1) for \( \tilde{\varphi}_0 \) and \( \bar{\varphi}_0 \) respectively. If there exists \( T > 0 \) such that \( \tilde{\varphi}(\cdot, T) = \bar{\varphi}(\cdot, T) \), then \( \tilde{\varphi}_0 = \bar{\varphi}_0 \) holds.

**Proof.** Define \( \theta = \tilde{\varphi} - \bar{\varphi} \) and \( \bar{\theta} = \frac{1}{2}(\tilde{\varphi} + \bar{\varphi}) \). Then \( \theta \) and \( \bar{\theta} \) satisfy
\[ \partial_t \theta + \kappa \Lambda^\frac{\gamma}{2} \theta + u \cdot \nabla \theta + u \cdot \nabla \bar{\theta} = 0, \quad \theta(\cdot, 0) = 0 := \tilde{\varphi}_0 - \bar{\varphi}_0, \]
where \( \bar{u} = u[\bar{\theta}] \) and \( u = u[\theta] \). We argue by contradiction: suppose that \( \theta_0 \neq 0 \), then by continuity in time, we have \( \| \theta(\cdot, t) \|_{L^2} > 0 \) for a sufficiently small \( t > 0 \), and we define \( \tau > 0 \) to be the minimal time such that
\[ \lim_{t \to \tau^-} \| \theta(\cdot, t) \|_{L^2} = 0. \] (6.50)
Let \( m = \max_{t \in [0, \tau]} \| \theta(\cdot, t) \|_{L^2} \), and we write \( v(t) = \log \left( \frac{2m}{\| \theta(\cdot, t) \|_{L^2}} \right) \), then \( v(t) \) is well-defined and positive on \( [0, \tau) \) with \( v(0) < \infty \), and \( v(t) \to \infty \) as \( t \to \tau^- \) by (6.50). Notice that \( \theta \) satisfies
\[ \frac{d}{dt} v = -\frac{1}{\| \theta \|_{L^2}^2} \int_{\mathbb{T}^d} \theta \partial_t \theta \leq -\frac{\kappa}{\| \theta \|_{L^2}^2} \| \Lambda^{\frac{\gamma}{2}} \theta \|_{L^2}^2 + \frac{1}{\| \theta \|_{L^2}^2} \left( \int_{\mathbb{T}^d} u \cdot \nabla \bar{\theta} \right). \] (6.51)
Using (6.42), we can bound \( \int_{\mathbb{T}^d} u \cdot \nabla \bar{\theta} \) as follows:
\[ \left| \int_{\mathbb{T}^d} u \cdot \nabla \bar{\theta} \right| \leq \| u \|_{L^\infty} \| \nabla \bar{\theta} \|_{L^2} \| \theta \|_{L^2} \leq C \| \Lambda^{\frac{\gamma}{2}} \theta \|_{L^2} \| \nabla \bar{\theta} \|_{L^2} \| \theta \|_{L^2}. \]
Hence using Cauchy–Schwartz inequality and integrating (6.51) in time, we obtain
\[ v(t) \leq v(0) + C \int_0^\tau \| \bar{\theta}(\cdot, s) \|_{H^1}^2 ds < \infty, \quad \forall t \in [0, \tau), \]
Hence \( \lim_{t \to \tau^-} v(t) < \infty \), which leads to a contradiction. Therefore we must have \( \theta_0 = 0 \) and \( \tilde{\varphi}_0 = \bar{\varphi}_0 \) as desired. \( \square \)
Remark 6.15. In view of the results obtained from Lemma 6.14, the solution map \( \pi^v(t) \) is injective on \( G^v \). The dynamics, when restricted to \( G^v \), actually defines a dynamical system. Hence \( \pi^v(t) \big|_{G^v} \) makes sense for all \( t \in \mathbb{R} \), not just for \( t \geq 0 \).

Applying Lemma 6.13–6.14 and the argument given in [10, Proposition 6.4], we can obtain the invariance and the maximality of the attractor \( G^v \) stated in Theorem 6.1. We summarise the results in the following corollary:

**Corollary 6.16.** The global attractor \( G^v \) of \( \pi^v(t) \) is fully invariant, namely
\[
\pi^v(t) G^v = G^v, \quad \forall t \geq 0.
\]
In particular, \( G^v \) is maximal among the class of bounded invariant sets in \( H^1 \).

Finally, we address the fractal dimensions for the global attractors \( G^v \) obtained in Theorem 6.1. Given a compact set \( X \), we give the following definition for fractal dimension \( \dim_f(X) \), which is based on counting the number of closed balls of a fixed radius \( \varepsilon \) needed to cover \( X \); see [34] for further explanation.

**Definition 6.17.** Given a compact set \( X \), let \( N(X, \varepsilon) \) be the minimum number of balls of radius \( \varepsilon \) that cover \( X \). The fractal dimension \( \dim_f(X) \) is given by
\[
\dim_f(X) := \limsup_{\varepsilon \to 0} \frac{\log N(X, \varepsilon)}{-\log \varepsilon}.
\]

In order to prove that \( \dim_f(G^v) \) is finite, we need to show that the solution map \( \pi^v \) given in (6.1) is uniform differentiable. More precisely, we have the following definition:

**Definition 6.18.** We say that \( \pi^v(t) \) is uniform differentiable on \( G^v \) if for every \( \theta_0 \in G^v \), there exists a linear operator \( D \pi^v(t, \theta_0) \) such that
\[
\sup_{\theta_0, \varphi_0 \in G^v; 0 < \| \theta_0 - \varphi_0 \|_{H^1} \leq \varepsilon} \frac{\| \pi^v(t)(\varphi_0) - \pi^v(t)(\theta_0) - D\pi^v(t)(\varphi_0 - \theta_0) \|_{H^1}}{\| \theta_0 - \varphi_0 \|_{H^1}} \to 0 \quad \text{as} \ \varepsilon \to 0,
\]
and
\[
\sup_{\theta_0 \in G^v} \sup_{\psi_0 \in H^1} \frac{\| D\pi^v(t, \theta_0)(\psi_0) \|_{H^1}}{\| \psi_0 \|_{H^1}} < \infty, \quad \text{for all} \ t \geq 0.
\]

The next lemma proves that \( \pi^v(t) \) is indeed uniform differentiable, and the associated linear operator \( D\pi^v(t, \theta_0) \) is given by
\[
D\pi^v(t, \theta_0)[\psi_0] := \psi(t),
\]
where \( \psi \) is the solution of the linearised problem with respect to (1.1)
\[
\begin{cases}
\frac{d\psi}{dt} = A_\theta[\psi], \\
\psi(x, 0) = \psi_0(x),
\end{cases}
\]
with \( \theta = \pi^v(t) \theta_0 \) and \( A_\theta \) is the elliptic operator given by
\[
A_\theta[\psi] = A_{\theta_0}(t)[\psi] := -\kappa \Lambda^v \psi - u[\theta] \cdot \nabla \psi - u[\psi] \cdot \nabla \theta.
\]
Lemma 6.19. For fixed $\nu > 0$, if $\gamma$ satisfies the condition (6.39), then the solution map $\pi^\nu(t)$ is uniform differentiable on $\mathcal{G}^\nu$ in the sense of Definition 6.18, and the associated linear operator $D\pi^\nu(t, \theta_0)$ is given by (6.54). Furthermore, the linear operator $D\pi^\nu(t, \theta_0)$ is compact.

Proof. For $\theta_0, \varphi_0 \in \mathcal{G}^\nu$, we let $\theta(t) = \pi^\nu(t)\theta_0$ and $\varphi(t) = \pi^\nu(t)\varphi_0$. We denote $\psi_0 = \varphi_0 - \theta_0$ and define $\psi(t) = D\pi^\nu(t, \theta_0)[\psi_0]$, where $\psi(t)$ satisfies (6.54). We also define

$$
\eta(t) = \varphi(t) - \theta(t) - \psi(t) = \pi^\nu(t)\varphi_0 - \pi^\nu(t)\theta_0 - D\pi^\nu(t, \theta_0)[\psi_0].$

Then $\eta(t)$ satisfies

$$
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\eta\|^2_{H^1} + \kappa \|\eta\|^2_{H^{1+\frac{\nu}{2}}} = \\
g &= \int_{\mathbb{T}^d} u[\eta] \cdot \nabla \theta \Delta \eta - \int_{\mathbb{T}^d} \partial_{x_k} u[\theta] \cdot \nabla \eta \partial_{x_k} \eta + \int_{\mathbb{T}^d} u[w] \cdot \nabla w \Delta \eta. 
\end{aligned}

(6.57)

Using (6.43) on $\nabla u$, the second integrand of (6.57) can be bounded by

$$
\left| \int_{\mathbb{T}^d} \partial_{x_k} u[\theta] \cdot \nabla \eta \partial_{x_k} \eta \right| \leq C \|\nabla u\|_{L^\infty} \|\nabla \eta\|^2_{L^2} \leq C \|\Lambda^{1+\frac{\nu}{2}} \theta\|_{L^2} \|\nabla \eta\|^2_{L^2}.

To estimate the first integrand appeared in (6.57), using the fact that $\text{div}(u[\eta]) = 0$ and integrating by parts, we have

$$
\left| \int_{\mathbb{T}^d} u[\eta] \cdot \nabla \theta \Delta \eta \right| = \left| \int_{\mathbb{T}^d} u[\eta] \theta \cdot \nabla \Delta \eta \right| = \left| \int_{\mathbb{T}^d} \Lambda^{2-\frac{\nu}{2}} (u[\eta]\theta) \cdot \Lambda^{\frac{\nu}{2}} \nabla \eta \right| \leq \|\Lambda^{2-\frac{\nu}{2}} (u[\eta]\theta)\|_{L^2} \|\Lambda^{1+\frac{\nu}{2}} \eta\|_{L^2}.

Using (6.40), we have

$$
\|\Lambda^{2-\frac{\nu}{2}} (u[\eta]\theta)\|_{L^2} \leq C \|\Lambda^{1+\frac{\nu}{2}} (u[\eta]\theta)\|_{L^2},

and using the product estimate (3.5),

$$
\|\Lambda^{1+\frac{\nu}{2}} (u[\eta]\theta)\|_{L^2} \leq C \left( \|\Lambda^{1+\frac{\nu}{2}} u[\eta]\|_{L^4} \|\theta\|_{L^4} + \|u[\eta]\|_{L^\infty} \|\Lambda^{1+\frac{\nu}{2}} \theta\|_{L^2} \right).

Using (1.4) and (6.41), the term $\|\Lambda^{1+\frac{\nu}{2}} u[\eta]\|_{L^4} \|\theta\|_{L^4}$ can be bounded by

$$
\|\Lambda^{1+\frac{\nu}{2}} u[\eta]\|_{L^4} \|\theta\|_{L^4} \leq C \|\Lambda^{2-\frac{\nu}{2}} u[\eta]\|_{L^2} \|\Lambda \theta\|_{L^2} \leq C \|\Lambda^{\frac{\nu}{2}} \eta\|_{L^2} \|\Lambda \theta\|_{L^2} \leq C \|\Lambda \eta\|_{L^2} \|\Lambda \theta\|_{L^2}.
\]
for $\gamma \leq 2$, while the term $\|u(\eta)\|_{L^\infty} \|\Lambda^{1+\frac{2}{\gamma}} \theta\|_{L^2}$ can be bounded by $C \|\Lambda \eta\|_{L^2} \|\Lambda^{1+\frac{2}{\gamma}} \theta\|_{L^2}$ with the help of (6.42). Therefore, we have

$$\left| \int_{\mathbb{T}^d} u[\eta] \cdot \nabla \theta \Delta \eta \right| \leq C \left( \|\Lambda \eta\|_{L^2} \|\Lambda \theta\|_{L^2} + \|\Lambda \eta\|_{L^2} \|\Lambda^{1+\frac{2}{\gamma}} \theta\|_{L^2} \right) \|\Lambda^{1+\frac{2}{\gamma}} \eta\|_{L^2}.$$  

Similarly, we can obtain the following estimates on $\int_{\mathbb{T}^d} u[w] \cdot \nabla w \Delta \eta$:

$$\left| \int_{\mathbb{T}^d} u[w] \cdot \nabla w \Delta \eta \right| \leq \|\Lambda^{1+\frac{2}{\gamma}} (u[w]w)\|_{L^2} \|\Lambda^{1+\frac{2}{\gamma}} \eta\|_{L^2}.$$  

Using (3.5) again, we have

$$\|\Lambda^{1+\frac{2}{\gamma}} (u[w]w)\|_{L^2} \leq C \left( \|\Lambda^{1+\frac{2}{\gamma}} u[w]\|_{L^4} \|w\|_{L^4} + \|u[w]\|_{L^\infty} \|\Lambda^{1+\frac{2}{\gamma}} w\|_{L^2} \right),$$  

and we apply (6.41) and (6.42) to obtain

$$\|\Lambda^{1+\frac{2}{\gamma}} u[w]\|_{L^4} \|w\|_{L^4} + \|u[w]\|_{L^\infty} \|\Lambda^{1+\frac{2}{\gamma}} w\|_{L^2} \leq C \left( \|\Lambda^{2+\frac{2}{\gamma}} u[w]\|_{L^2} \|w\|_{L^2} + \|\Lambda w\|_{L^2} \|\Lambda^{1+\frac{2}{\gamma}} w\|_{L^2} \right),$$  

$$\leq C \|\Lambda w\|_{L^2} \|\Lambda^{1+\frac{2}{\gamma}} w\|_{L^2},$$  

where the last inequality follows by (1.4) and the assumption that $\gamma \leq 2$. Hence we have

$$\left| \int_{\mathbb{T}^d} u[w] \cdot \nabla w \Delta \eta \right| \leq C \|\Lambda w\|_{L^2} \|\Lambda^{1+\frac{2}{\gamma}} w\|_{L^2} \|\Lambda^{1+\frac{2}{\gamma}} \eta\|_{L^2},$$  

and we infer from (6.57) that

$$\frac{1}{2} \frac{d}{dt} \|\eta\|_{H^1}^2 + \kappa \|\eta\|_{H^{1+\frac{2}{\gamma}}}^2 \leq C \|\Lambda^{1+\frac{2}{\gamma}} \theta\|_{L^2} \|\nabla \eta\|_{L^2}^2 + C \left( \|\Lambda \theta\|_{L^2} + \|\Lambda^{1+\frac{2}{\gamma}} \theta\|_{L^2} \right) \|\Lambda \eta\|_{L^2} \|\Lambda^{1+\frac{2}{\gamma}} \eta\|_{L^2} + C \|\Lambda w\|_{L^2} \|\Lambda^{1+\frac{2}{\gamma}} w\|_{L^2} \|\Lambda^{1+\frac{2}{\gamma}} \eta\|_{L^2}. \quad (6.58)$$  

Next we focus on the function $w = w(t)$ which satisfies the following equations:

$$\begin{cases} 
\partial_t w + \kappa \Lambda^\gamma w + u[\varphi] \cdot \nabla w + u[w] \cdot \nabla \theta = 0, \\
w(x, 0) = \psi_0.
\end{cases} \quad (6.59)$$  

We take $L^2$-inner product of (6.59) with $-\Delta w$ to give

$$\frac{1}{2} \frac{d}{dt} \|w\|_{H^1}^2 + \kappa \|w\|_{H^{1+\frac{2}{\gamma}}}^2 = \int_{\mathbb{T}^d} u[\varphi] \cdot \nabla w \Delta w + \int_{\mathbb{T}^d} u[w] \cdot \nabla \theta \Delta w. \quad (6.60)$$  

Similar to the previous case for $\eta$, we have

$$\left| \int_{\mathbb{T}^d} u[\varphi] \cdot \nabla w \Delta w \right| \leq C \|\Lambda^{1+\frac{2}{\gamma}} \varphi\|_{L^2} \|\nabla w\|_{L^2}^2,$$  

$$\left| \int_{\mathbb{T}^d} u[w] \cdot \nabla \theta \Delta w \right| \leq C \|\Lambda^{1+\frac{2}{\gamma}} \theta\|_{L^2} \|\nabla \eta\|_{L^2}.$$
and
\[ \left| \int_{\mathcal{T}^d} u[w] \cdot \nabla \theta \Delta w \right| \leq C \left( \| \Lambda w \|_{L^2} \| \Lambda \theta \|_{L^2} + \| \Lambda w \|_{L^2} \| \Lambda^{1+\frac{\gamma}{2}} \theta \|_{L^2} \right) \| \Lambda^{1+\frac{\gamma}{2}} w \|_{L^2}. \]

Hence (6.60) implies
\[
\frac{1}{2} \frac{d}{dt} \| w \|_{H^1}^2 + \kappa \| w \|_{H^{1+\frac{\gamma}{2}}}^2
\leq C \| \Lambda^{1+\frac{\gamma}{2}} \varphi \|_{L^2} \| \nabla w \|_{L^2}^2 + C \left( \| \Lambda w \|_{L^2} \| \Lambda \theta \|_{L^2} + \| \Lambda w \|_{L^2} \| \Lambda^{1+\frac{\gamma}{2}} \theta \|_{L^2} \right) \| \Lambda^{1+\frac{\gamma}{2}} w \|_{L^2}
\leq \frac{\kappa}{2} \| w \|_{H^{1+\frac{\gamma}{2}}}^2 + C \left[ \| \Lambda^{1+\frac{\gamma}{2}} \varphi \|_{L^2} + \frac{1}{\kappa} (\| \Lambda \theta \|_{L^2} + \| \Lambda^{1+\frac{\gamma}{2}} \varphi \|_{L^2}) \right] \| w \|_{H^1}. 
\tag{6.61}
\]

Since \( \theta_0, \varphi_0 \in \mathcal{G}^v \), \( \theta \) and \( \varphi \) both satisfy the bounds (6.2)–(6.3), namely,
\[
\| \theta \|_{H^{1+\frac{\gamma}{2}}} \leq M_{\mathcal{G}^v}, \quad \| \varphi \|_{H^{1+\frac{\gamma}{2}}} \leq M_{\mathcal{G}^v}, \tag{6.62}
\]
and
\[
\frac{1}{T} \int_0^T \| \theta(\cdot, \tau) \|_{H^{1+\gamma}} d\tau \leq M_{\mathcal{G}^v}, \quad \frac{1}{T} \int_0^T \| \varphi(\cdot, \tau) \|_{H^{1+\gamma}} d\tau \leq M_{\mathcal{G}^v}, \quad \forall T > 0. \tag{6.63}
\]

We apply the bounds (6.62) on (6.61) and use Grönwall’s inequality to obtain
\[
\| w(\cdot, t) \|_{H^1}^2 \leq \| \psi_0 \|_{H^1}^2 K(t, M_{\mathcal{G}^v}), \quad \forall t \geq 0, \tag{6.64}
\]
as well as
\[
\int_0^t \| w(\cdot, \tau) \|_{H^{1+\frac{\gamma}{2}}}^2 d\tau \leq \| \psi_0 \|_{H^1}^2 K(t, M_{\mathcal{G}^v}), \quad \forall t \geq 0. \tag{6.65}
\]
where \( K(t, M_{\mathcal{G}^v}) \) is a positive function in \( t \). We combine (6.58) with (6.62) and (6.64) to get
\[
\frac{d}{dt} \| \eta \|_{H^1}^2 + \kappa \| \eta \|_{H^{1+\frac{\gamma}{2}}}^2 \leq CM_{\mathcal{G}^v} \| \eta \|_{H^1}^2 + \frac{C M^2_{\mathcal{G}^v}}{\kappa} \| \eta \|_{H^1}^2
+ \frac{C}{\kappa} K(t, M_{\mathcal{G}^v}) \| \psi_0 \|_{H^1}^2 \| w \|_{H^{1+\frac{\gamma}{2}}}^2. \tag{6.66}
\]

Using Grönwall’s inequality on (6.66) and recalling the fact that \( \eta(0) = 0 \), we conclude that
\[
\| \eta(\cdot, t) \|_{H^1}^2 \leq \exp \left( (CM_{\mathcal{G}^v} + \frac{C M^2_{\mathcal{G}^v}}{\kappa}) \frac{C}{\kappa} K(t, M_{\mathcal{G}^v}) \| \psi_0 \|_{H^1}^2 \right) \int_0^t \| w(\cdot, \tau) \|_{H^{1+\frac{\gamma}{2}}}^2 d\tau
\leq \tilde{K}(t, M_{\mathcal{G}^v}) \| \psi_0 \|_{H^1}^4, \tag{6.67}
\]
where \( \tilde{K}(t, M_{G^\nu}) := \exp \left( (CM_{G^\nu} + \frac{CM_{G^\nu}^2}{\kappa})t \right) \frac{C}{\kappa} K^2(t, M_{G^\nu}). \) Hence we prove that

\[
\lim_{\varepsilon \to 0} \left( \sup_{\theta_0, \varphi_0 \in G^\nu: 0 < \|\varphi_0\|_{H^1} \leq \varepsilon} \frac{\|\eta(\cdot, t)\|_{H^1}}{\|\psi_0\|_{H^1}} \right) \leq \lim_{\varepsilon \to 0} \tilde{K}(t, M_{G^\nu}) \varepsilon^2 = 0,
\]

and (6.52) follows.

To prove (6.53), it suffices to consider \( \psi_0 \) to be normalised so that \( \|\psi_0\|_{H^1} = 1 \). Let \( \theta_0 \in G^\nu \) be arbitrary, then using the similar estimates as given above, we have

\[
\frac{1}{2} \frac{d}{dt} \|\psi\|_{H^{1+\frac{\kappa}{2}}}^2 + \kappa \|\psi\|^2_{H^{1+\frac{\kappa}{2}}}
\leq \left| \int_{\mathbb{T}^d} u[\theta] \cdot \nabla \psi \Delta \psi \right| + \left| \int_{\mathbb{T}^d} u[\theta] \cdot \nabla \theta \Delta \psi \right|
\leq C \|\Lambda^{1+\frac{\kappa}{2}} \theta\|_{L^2} \|\nabla \psi\|_{L^2}^2 + C \left( \|\Lambda \psi\|_{L^2} \|\Lambda \theta\|_{L^2} + \|\Lambda \psi\|_{L^2} \|\Lambda^{1+\frac{\kappa}{2}} \theta\|_{L^2} \right) \|\Lambda^{1+\frac{\kappa}{2}} \psi\|_{L^2}.
\]

(6.68)

which gives

\[
\|\psi(\cdot, t)\|_{H^1}^2 \leq K(t, M_{G^\nu}), \quad \forall t \geq 0,
\]

and (6.53) holds as well.

Finally, we show that for any \( t > 0 \) and \( \theta_0 \in G^\nu \), the linear operator \( D\pi^\nu(t, \theta_0) \) is compact. Following the argument given in [9], it suffices to show that if \( U_1 \) is the unit ball in \( H^1 \), then \( D\pi^\nu(t, \theta_0)U_1 \subset H^{1+\frac{\kappa}{2}} \). In view of (6.68) and (6.69), we obtain

\[
\int_0^t \|\psi(\cdot, \tau)\|_{H^{1+\frac{\kappa}{2}}}^2 d\tau \leq K(t, M_{G^\nu}), \quad \forall t \geq 0.
\]

(6.70)

Hence using the mean value theorem, for \( t > 0 \), there exists \( \tau \in (0, t) \) such that

\[
\|\psi(\cdot, \tau)\|_{H^{1+\frac{\kappa}{2}}}^2 \leq \frac{1}{t} K(t, M_{G^\nu}).
\]

(6.71)

We take the \( L^2 \)-inner product of (6.54) with \( \Lambda^{2+\gamma} \psi \) and obtain

\[
\frac{1}{2} \frac{d}{dt} \|\psi\|_{H^{1+\frac{\kappa}{2}}}^2 + \kappa \|\psi\|^2_{H^{1+\frac{\kappa}{2}}} = \int_{\mathbb{T}^d} u[\theta] \cdot \nabla \psi \Lambda^{2+\gamma} \psi + \int_{\mathbb{T}^d} u[\theta] \cdot \nabla \theta \Lambda^{2+\gamma} \psi.
\]

Using the commutator estimate (3.6) and the bound (6.41), the term \( \left| \int_{\mathbb{T}^d} u[\theta] \cdot \nabla \psi \Lambda^{2+\gamma} \psi \right| \) can be bounded by

\[
\left| \int_{\mathbb{T}^d} u[\theta] \cdot \nabla \psi \Lambda^{2+\gamma} \psi \right|
\leq \left| \int_{\mathbb{T}^d} (\Lambda^{1+\frac{\kappa}{2}} (u[\theta] \nabla \psi) - u[\theta] \cdot \Lambda^{1+\frac{\kappa}{2}} \nabla \psi) \Lambda^{1+\frac{\kappa}{2}} \psi \right|
\leq \|\Lambda^{1+\frac{\kappa}{2}} (u[\theta] \nabla \psi) - u[\theta] \cdot \Lambda^{1+\frac{\kappa}{2}} \nabla \psi\|_{L^2} \|\Lambda^{1+\frac{\kappa}{2}} \psi\|_{L^2}
\]
\[ \leq C \left( \| \nabla (u[\theta]) \|_{L^\infty} \| \Lambda^{1+\frac{\gamma}{2}} \psi \|_{L^2} + \| \Lambda^{1+\frac{\gamma}{2}} u[\theta] \|_{L^4} \| \nabla \psi \|_{L^4} \right) \| \Lambda^{1+\frac{\gamma}{2}} \psi \|_{L^2} \]

\[ \leq C \left( \| \Lambda \theta \|_{L^2} \| \Lambda^{1+\frac{\gamma}{2}} \psi \|_{L^2} + \| \Lambda^{\frac{\gamma}{2}} \theta \|_{L^2} \| \Lambda^{1+\gamma} \psi \|_{L^2} \right) \| \Lambda^{1+\frac{\gamma}{2}} \psi \|_{L^2}, \]

and similarly, we also have

\[ \int_{\mathbb{T}^d} u[\psi] \cdot \nabla \theta \Lambda^{2+\gamma} \psi = \int_{\mathbb{T}^d} \Lambda^2 (u[\psi][\theta]) \Lambda^\gamma \nabla \psi \]

\[ \leq \| \Lambda^2 (u[\psi][\theta]) \|_{L^2} \| \Lambda^{1+\gamma} \psi \|_{L^2} \]

\[ \leq C \left( \| \Lambda^2 (u[\psi]) \|_{L^4} \| \theta \|_{L^4} + \| u[\psi] \|_{L^\infty} \| \Lambda^2 \psi \|_{L^2} \right) \| \Lambda^{1+\gamma} \psi \|_{L^2} \]

\[ \leq C \left( \| \Lambda \psi \|_{L^2} \| \Lambda \theta \|_{L^2} + \| \Lambda \psi \|_{L^2} \| \Lambda^{1+\gamma} \psi \|_{L^2} \right) \| \Lambda^{1+\gamma} \psi \|_{L^2}. \]

Therefore, using Young’s inequality, we obtain

\[ \frac{1}{2} \frac{d}{dt} \| \psi \|_{H^{1+\frac{\gamma}{2}}}^2 + \frac{\kappa}{2} \| \psi \|_{H^{1+\gamma}}^2 \]

\[ \leq C \left( \| \Lambda \theta \|_{L^2} + \| \Lambda^{\frac{\gamma}{2}} \theta \|_{L^2}^2 + \| \Lambda \theta \|_{L^2}^2 + \| \Lambda^{1+\gamma} \theta \|_{L^2}^2 \right) \| \psi \|_{H^{1+\frac{\gamma}{2}}}^2. \]  

(6.72)

Integrating (6.72) from \( \tau \) to \( t \), applying Grönwall’s inequality and using the bounds (6.63) and (6.71), we arrive at

\[ \| \psi(\cdot, t) \|_{H^{1+\frac{\gamma}{2}}}^2 \leq \| \psi(\cdot, \tau) \|_{H^{1+\frac{\gamma}{2}}}^2 \exp \left( \frac{C}{\kappa} \int_{\tau}^t \| \theta(\cdot, s) \|_{H^{1+\gamma}}^2 ds \right) \]

\[ \leq \frac{1}{t} K(t, M_{\mathcal{G}^v}) \exp \left( \frac{C}{\kappa} M_{\mathcal{G}^v} t \right), \]  

(6.73)

hence \( \psi(t) \in H^{1+\frac{\gamma}{2}} \), and we finish the proof of lemma 6.19. \( \square \)

Next we show that there is an \( N \) such that volume elements which are carried by the flow of \( \pi^v(t) \theta_0 \), with \( \theta_0 \in \mathcal{G}^v \), decay exponentially for dimensions larger than \( N \). We recall the following proposition for which the proof can be found in [6, 7]:

**Proposition 6.20.** Consider \( \theta_0 \in \mathcal{G}^v \), and an initial orthogonal set of infinitesimal displacements \( \{\psi_{1,0}, \ldots, \psi_{n,0}\} \) for some \( n \geq 1 \). Suppose that \( \psi_i \) obey

\[ \partial_t \psi_i = A_{\theta_0}(t) [\psi_i], \quad \psi_i(0) = \psi_{i,0} \]  

(6.74)

for all \( i \in \{1, \ldots, n\} \) and \( t \geq 0 \), where \( A_{\theta_0}(t) \) is given by (6.55). Then the volume elements

\[ V_n(t) = \| \psi_1(t) \wedge \cdots \wedge \psi_n(t) \|_{H^1} \]

satisfy

\[ V_n(t) = V_n(0) \exp \left( \int_0^t \text{Tr}(P_n(s) A_{\theta_0}(s)) ds \right), \]
where the orthogonal projection \( P_n(s) \) is onto the linear span of \( \{\psi_1(s), \ldots, \psi_n(s)\} \) in the Hilbert space \( H^1 \), and \( \text{Tr}(P_n(s)A_\theta) \) is defined by

\[
\text{Tr}(P_n(s)A_\theta) = \sum_{j=1}^{n} \int_{\mathbb{T}^d} (-\Delta \psi_j(s))A_\theta[\psi_j(s)]dx
\]

for \( n \geq 1 \), with \( \{\psi_1(s), \ldots, \psi_n(s)\} \) an orthonormal set spanning the linear span of \( \{\psi_1(s), \ldots, \psi_n(s)\} \). If we define

\[
\langle P_n A_{\theta_0} \rangle := \limsup_{T \to \infty} \frac{1}{T} \int_0^T \text{Tr}(P_n(t)A_{\theta_0}(t))dt,
\]

then we further obtain

\[
V_n(t) \leq V_n(0) \left( t \sup_{\theta_0 \in G^\nu} \sup_{P_n(0)} \langle P_n A_{\theta_0} \rangle \right), \quad \forall t \geq 0, \quad (6.75)
\]

where the supremum over \( P_n(0) \) is a supremum over all choices of initial \( n \) orthogonal set of infinitesimal displacements that we take around \( \theta_0 \).

Using Proposition 6.20, we show that the \( n \)-dimensional volume elements actually decay exponentially in time for \( n \) is sufficiently large, which is based on the following lemma:

**Lemma 6.21.** (Contractivity of large dimensional volume elements) There exists \( N \) such that for any \( \theta_0 \in G^\nu \) and any set of initial orthogonal displacements \( \{\psi_i, 0\}_{i=1}^n \), we have

\[
\langle P_n A_{\theta_0} \rangle < 0, \quad (6.76)
\]

whenever \( n \geq N \). Here \( N \) can be chosen explicitly from (6.79) below.

**Proof.** Let \( \xi \in H^1 \) be arbitrary. For \( \theta_0 \in G^\nu \), using the definition of \( A_\theta \) in (6.55) and the fact that \( \text{div} u[\theta] = 0 \), we have

\[
\int_{\mathbb{T}^d} \Lambda^2 \xi A_\theta[\xi]dx \leq -\kappa \|\xi\|^2_{H^1} + \left| \int_{\mathbb{T}^d} \partial_{x_k} u[\theta] \cdot \nabla \xi \partial_{x_k} \xi dx \right| + \left| \int_{\mathbb{T}^d} (u[\xi] \cdot \nabla \theta) \Lambda^2 \xi dx \right|.
\]

Using the bound (6.42), we readily have

\[
\left| \int_{\mathbb{T}^d} \partial_{x_k} u[\theta] \cdot \nabla \xi \partial_{x_k} \xi dx \right| \leq C \|\nabla u[\theta]\|_{L^\infty} \|\nabla \xi\|^2_{L^2} \leq C \|\Lambda \theta\|_{L^2} \|\nabla \xi\|^2_{L^2}.
\]

Using the product estimate (3.5) and the bounds (6.41)–(6.42), we have

\[
\left| \int_{\mathbb{T}^d} (u[\xi] \cdot \nabla \theta) \Lambda^2 \xi dx \right| = \left| \int_{\mathbb{T}^d} (u[\xi] \theta) \cdot \nabla \Lambda^2 \xi dx \right| \leq \Lambda^{2-\frac{p}{2}} (u[\xi] \theta)_{L^2} \|\Lambda^{1+\frac{p}{2}} \xi\|_{L^2} \leq \Lambda^{2-\frac{p}{2}} (u[\xi] \theta)_{L^2} \|\Lambda^{1+\frac{p}{2}} \xi\|_{L^2} \leq C \left( \Lambda^{2-\frac{p}{2}} [\theta]_{L^4} + \|u[\xi]\|_{L^\infty} \|\Lambda^{2-\frac{p}{2}} \theta\|_{L^2} \right) \|\Lambda^{1+\frac{p}{2}} \xi\|_{L^2}.
\]
\begin{align*}
\int_{\mathbb{T}^d} \Lambda^2 \xi A_\theta [\xi] \, dx \leq -\frac{\kappa}{2} \|\xi\|^2_{H^1 + \gamma} + C \left( \|\Lambda \theta\|_{L^2} + \frac{1}{\kappa} \|\theta\|^2_{H^1 + \gamma} \right) \|\xi\|^2_{H^1}. \quad (6.77)
\end{align*}

On the other hand, for \( T > 0 \), we have
\begin{align*}
\frac{1}{T} \int_0^T \text{Tr}(P_n(t)A_{\theta_0}(t)) \, dt = \frac{1}{T} \int_0^T \sum_{j=1}^n \int_{\mathbb{T}^d} (\Lambda^2 \varphi_j(t)) A_\theta [\varphi_j(t)] \, dx \, dt. \quad (6.78)
\end{align*}

Hence we apply (6.77) on (6.78), and together with the bound (6.2) on \( \theta \),
\begin{align*}
\frac{1}{T} \int_0^T \text{Tr}(P_n(t)A_{\theta_0}(t)) \, dt \\
\leq -\frac{\kappa}{2T} \int_0^T \sum_{j=1}^n \|\varphi_j(t)\|^2_{H^1 + \gamma} + C \int_0^T \left( \|\Lambda \theta\|_{L^2} + \frac{1}{\kappa} \|\theta\|^2_{H^1 + \gamma} \right) \sum_{j=1}^n \|\varphi_j\|^2_{H^1} \\
\leq -\frac{\kappa}{2T} \int_0^T \text{Tr}(P_n(t)\Lambda^\gamma) + C (M G^\nu + M_0^2) n \leq -\frac{\kappa}{C} n^{1+\gamma} + C (M G^\nu + M_0^2) n,
\end{align*}

where the last inequality follows from the fact that the eigenvalues \( \{\lambda_j^{(\gamma)}\} \) of \( \Lambda^\gamma \) obey the estimate (see [37, Theorem 1.1])
\begin{align*}
\sum_{j=1}^n \lambda_j^{(\gamma)} \geq \frac{1}{C} n^{1+\gamma}
\end{align*}
for some universal constant \( C > 0 \) which depends only on \( d \) and \( \gamma \). We choose \( N > 0 \) such that
\begin{align*}
-\frac{\kappa}{C} N^{1+\gamma} + C (M G^\nu + M_0^2) N < 0, \quad (6.79)
\end{align*}
then (6.76) holds whenever \( n \geq N \). \( \Box \)

From the results obtained in Lemma 6.19 and Lemma 6.21, together with Proposition 6.20, we obtain that
\begin{itemize}
  \item the solution map \( \pi^\nu(t) \) is uniform differentiable on \( G^\nu \);
  \item the linearisation \( D\pi^\nu(t, \theta_0) \) of \( \pi^\nu(t) \) is compact;
  \item the large-dimensional volume elements which are carried by the flow of \( \pi^\nu(t)\theta_0 \), with \( \theta_0 \in G^\nu \), have exponential decay in time.
\end{itemize}
Therefore, following the lines of the argument in [7, pp. 115–130, and Chapter 14], we can finally conclude that \( \dim_f(G^\nu) \) is finite. The results can be summarised in the following corollary:

**Corollary 6.22.** (Finite dimensionality of the attractor) Let \( N \) be as defined in Lemma 6.21. Then the fractal dimension of \( G^\nu \) is finite, and we have \( \dim_f(G^\nu) \leq N \).

**Proof of Theorem 2.3.** For \( \nu, \kappa > 0 \) and \( \gamma \in (0, 2] \), the existence and regularity of unique global attractor \( G^\nu \) follows by Theorem 6.1 and Corollary 6.10. For the case \( \gamma \in [1, 2] \), the finite dimensionality of the attractor follows by Corollary 6.22. \( \Box \)
7. Applications to Magneto-Geostrophic Equations

7.1. The MG Equations in the Class of Drift-Diffusion Equations

We now apply our results claimed by Sect. 2 to the magnetogeostrophic (MG) active scalar equation. Specifically, we are interested in the following active scalar equation in the domain $\mathbb{R}^3 \times (0, \infty) = [0, 2\pi]^3 \times (0, \infty)$ (with periodic boundary conditions):

\[
\begin{align*}
\partial_t \theta + u \cdot \nabla \theta &= \kappa \Delta \theta + S, \\
u &= M^\nu[\theta], \quad \theta(x, 0) = \theta_0(x),
\end{align*}
\]

(7.1)

via a Fourier multiplier operator $M^\nu$ which relates $u$ and $\theta$. More precisely,

\[
u_j = M^\nu_j[\theta] = (\hat{M}^\nu_j \hat{\theta})^\vee
\]

(7.2)

for $j \in \{1, 2, 3\}$. The explicit expression for the components of $\hat{M}^\nu$ as functions of the Fourier variable $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$ with $k_3 \neq 0$ are given by (1.11)–(1.14), as discussed in the introduction, in particular $\hat{M}^\nu$ is not defined on the set $\{k_3 = 0\}$.

Since for self-consistency of the model, we assume that $\theta$ and $u$ have zero vertical mean, and we take $\hat{M}^\nu_j(k) = 0$ on $\{k_3 = 0\}$ for all $j = 1, 2, 3$ and $\nu \geq 0$. We write $M^\nu_j = \partial_{x_i} T^\nu_{ij}$ for convenience. We also refer to (7.1) as the MG$^\nu$ equation when $\nu > 0$, and to the case when $\nu = 0$ as the MG$^0$ equation (see [16–18] for related discussions).

To apply the results from Sect. 2, it suffices to show that the sequence of operators $\{T^\nu_{ij}\}_{\nu \geq 0}$ satisfy the Assumptions A1–A5 given in Sect. 1.

**Proposition 7.1.** We define $T^\nu_{ij}$ by $M^\nu_j = \partial_{x_i} T^\nu_{ij}$. Then $T^\nu_{ij}$ satisfy the Assumptions A1–A5 given in Sect. 1.

**Proof.** The details for the proof can be found in [17, Lemma 5.1–5.2] and from the discussion in [19, Section 4]. We omit the details here. $\Box$

In view of Proposition 7.1, the abstract Theorem 2.1, Theorem 2.2 and Theorem 2.3 can then be applied to the MG equation (7.1). More precisely, we have

**Theorem 7.2.** ($H^s$-convergence as $\kappa \to 0$ for MG equations) Let $\nu > 0$ be given as in (7.1), and let $\theta_0, S \in C^\infty$ be the initial datum and forcing term respectively with zero mean. If $\theta^\kappa$ and $\theta^0$ are smooth solutions to (7.1) for $\kappa > 0$ and $\kappa = 0$ respectively, then

\[
\lim_{\kappa \to 0} \| (\theta^\kappa - \theta^0)(\cdot, t) \|_{H^s} = 0
\]

for all $s \geq 0$ and $t \geq 0$. 
Theorem 7.3. (Analytic convergence as $\kappa \to 0$ for MG equations) Let $\nu = 0$ be given as in (7.1), and let $\theta_0, S$ the initial datum and forcing term respectively. Suppose that $\theta_0$ and $S$ are both analytic functions with zero mean. Then if $\theta^\kappa, \theta^0$ are analytic solutions to (7.1) for $\kappa > 0$ and $\kappa = 0$ respectively with initial datum $\theta_0$ and with radius of convergence at least $\bar{\tau}$, then there exists $T \leq \bar{T}$ and $\tau = \tau(t) < \bar{\tau}$ such that, for $t \in [0, T]$, we have

$$\lim_{\kappa \to 0} \| (\Lambda^T \text{e}^{\tau A} \theta^\kappa - \Lambda^T \text{e}^{\tau A} \theta^0)(\cdot, t) \|_{L^2} = 0.$$ 

Theorem 7.4. (Existence of global attractors for MG equations) Let $S \in L^\infty \cap H^1$. For $\kappa > 0$, let $\pi^\nu(t)$ be solution operator for the initial value problem (7.1) via (6.1). Then the solution map $\pi^\nu(t) : H^1 \to H^1$ associated to (1.1) possesses a unique global attractor $\mathcal{G}^\nu$ for all $\nu > 0$. In particular, for each $\nu > 0$, the global attractor $\mathcal{G}^\nu$ of $\pi^\nu(t)$ enjoys the following properties:

- $\mathcal{G}^\nu$ is fully invariant, namely
  $$\pi^\nu(t)\mathcal{G}^\nu = \mathcal{G}^\nu, \quad \forall t \geq 0.$$ 

- $\mathcal{G}^\nu$ is maximal in the class of $H^1$-bounded invariant sets.

- $\mathcal{G}^\nu$ has finite fractal dimension.

In the coming subsection, we will further address the limiting properties of $\mathcal{G}^\nu$ which are related to the critical MG$^0$ equation.

7.2. Behaviour of Global Attractors for Varying $\nu \geq 0$

In the work [17], the authors proved the existence of a compact global attractor $\mathcal{A}$ in $L^2(\mathbb{T}^3)$ for the MG$^0$ equations, namely the equation (7.1) when $\kappa > 0$, $\nu = 0$ and $S \in L^\infty \cap H^1$. More precisely, $\mathcal{A}$ is the global attractor generated by the solution map $\tilde{\pi}^0$ via

$$\tilde{\pi}^0(t) : L^2 \to L^2, \quad \tilde{\pi}^0(t)\theta_0 = \theta(\cdot, t), \quad t \geq 0, \quad (7.3)$$

where $\theta$ is the solution to the MG$^0$ equation with $\theta(\cdot, 0) = \theta_0$. In this subsection, we obtain results when $\nu$ is varying, which can be summarised in the following theorem:

Theorem 7.5. Let $\kappa > 0$ be fixed in (7.1). Then we have

1. If $\mathcal{G}^\nu$ are the global attractors for the MG$^\nu$ equation (7.1) as obtained by Theorem 7.4, then $\mathcal{G}^\nu$ and $\mathcal{A}$ satisfy

$$\sup_{\phi \in \mathcal{G}^\nu} \inf_{\psi \in \mathcal{A}} \| \phi - \psi \|_{L^2} \to 0 \text{ as } \nu \to 0. \quad (7.4)$$

2. Let $\nu_* > \nu_0 > 0$ be arbitrary. For each $\nu_0 \in [\nu_*, \nu^*]$, the collection $\{\mathcal{G}^\nu\}_{\nu \in [\nu_*, \nu^*]}$ is upper semicontinuous at $\nu_0$ in the following sense:

$$\sup_{\phi \in \mathcal{G}^\nu} \inf_{\psi \in \mathcal{G}^{\nu_0}} \| \phi - \psi \|_{H^1} \to 0 \text{ as } \nu \to \nu_0. \quad (7.5)$$
Remark 7.6. Here are some relevant remarks regarding Theorem 7.5.
- Since \( A \subset L^2 \) and \( G^\nu \subset H^1 \subset L^2 \) for all \( \nu > 0 \), it makes sense to address the \( L^2 \)-difference \( \| \phi - \psi \|_{L^2} \) for \( \phi \in G^\nu \) and \( \psi \in A \).
- Although \( G^\nu \) has finite fractal dimension for all \( \nu > 0 \), it is unknown whether \( A \) has finite fractal dimension.

7.2.1. Convergence of Attractors as \( \nu \to 0 \)

We first prove the convergence result as claimed by (7.4). We recall from [17, Theorem 6.3] that for \( \kappa > 0 \), \( \nu \in [0, 1] \) and \( S \in L^\infty \cap H^2 \), there exists global attractor \( A^\nu \) in \( L^2 \) generated by the solution map \( \tilde{\pi}^\nu \) via

\[
\tilde{\pi}^\nu(t) : L^2 \to L^2, \quad \tilde{\pi}^\nu(t) \theta_0 = \theta(\cdot, t), \quad t \geq 0,
\]

where \( \theta \) is the solution to (7.1) with \( \theta(\cdot, 0) = \theta_0 \), and in particular \( A^\nu \big|_{\nu=0} = A \).

Furthermore, \( A^\nu \) is upper semicontinuous at \( \nu = 0 \) in the following sense (a proof can be found in [17]):

Proposition 7.7. For \( \nu \in [0, 1] \), the global attractors \( A^\nu \subset L^2 \) as mentioned above are upper semicontinuous with respect to \( \nu \) at \( \nu = 0 \), which means that

\[
\sup_{\phi \in A^\nu} \inf_{\psi \in A^\nu} \| \phi - \psi \|_{L^2} \to 0 \quad \text{as} \quad \nu \to 0.
\]

In view of Proposition 7.7, the result (7.4) now follows easily from (7.7) by observing that \( G^\nu \subset A^\nu \) for all \( \nu \in (0, 1] \), which gives

\[
\sup_{\phi \in G^\nu} \inf_{\psi \in A^\nu} \| \phi - \psi \|_{L^2} \leq \sup_{\phi \in A^\nu} \inf_{\psi \in A^\nu} \| \phi - \psi \|_{L^2}, \quad \forall \nu \in (0, 1].
\]

Remark 7.8. Concerning \( \tilde{\pi}^\nu \) and \( A^\nu \),
- We readily have \( \tilde{\pi}^\nu \big|_{H^1} = \pi^\nu \), where \( \pi^\nu \) is the solution map defined in (6.1) for the initial value problem (7.1) with \( \theta(\cdot, 0) = \theta_0 \).
- For each \( \nu > 0 \), it is not clear whether the two attractors \( G^\nu \) and \( A^\nu \) coincide. In particular, it is unknown whether \( A^\nu \) has finite fractal dimension.

7.2.2. Upper Semicontinuity of Global Attractors at \( \nu > 0 \)

For fixed \( \nu^* > \nu_* > 0 \), we define

\[
I^* = [\nu_*, \nu^*].
\]

In order to obtain the convergence result claimed by (7.5), we need to prove that
- \( \text{L1} \) there is a compact subset \( \mathcal{U} \) of \( H^1 \) such that \( G^\nu \subset \mathcal{U} \) for every \( \nu \in I^* \); and
- \( \text{L2} \) for \( t > 0 \), \( \pi^\nu \theta_0 \) is continuous in \( I^* \), uniformly for \( \theta_0 \) in compact subsets of \( H^1 \).

Once the conditions L1 and L2 are fulfilled, we can apply the result from [24] to conclude that (7.5) holds as well.

Conditions L1 and L2 will be proved in the subsequent lemmas. We first give the following bound on \( u^{(\nu)} \) in terms of \( \theta^{(\nu)} \) for \( \nu \in I^* \):

\[
\|
\]
Lemma 7.9. We fix $\nu^* > \nu_* > 0$ and define $I^*$ by (7.8). Then for any $\nu \in I^*$, $s \in [0, 2]$ and $f \in L^p$ with $p > 1$, we have
\[
\|\Lambda^s u^{(\nu)}[f]\|_{L^p} \leq C_* \|f\|_{L^p},
\] (7.9)
where $u^{(\nu)}$ is given by (7.2) and (1.11)--(1.14) for $\nu \in I^*$, and $C_*$ is a positive constant which depends only on $p$, $\nu_*$ and $\nu^*$.

Proof. By examining the explicit expression for the components of $\hat{M}^\nu$ given by (1.11)--(1.14), it is not hard to see that for $s \in [0, 2]$, the Fourier symbols of $\Lambda^s u^{(\nu)}[\cdot]$ can be bounded in terms of $\nu_*$ and $\nu^*$ but independent of $s$. The rest follows by the standard Fourier multiplier theorem (see [35] for example) and we omit the details here. ⊓⊔

Remark 7.10. Using the estimate (7.9) on $u^{(\nu)}[\cdot]$ and following lines by lines the arguments and proofs given in Sect. 6.1, we can obtain parallel results given in Sect. 6.1 for all $\nu \in I^*$. In particular, as claimed by Lemma 6.8, there exists a constant $R_2 \geq 1$ which depends only on $\nu_*$, $\nu^*$, $\kappa$, $\|S\|_{L^\infty \cap H^1}$ such that the set
\[
B_2 = \{ \phi \in H^2 : \|\phi\|_{H^2} \leq R_2 \}
\]
enjoys the following properties:

- $B_2$ is a compact set in $H^1$ which depends only on $\nu_*$, $\nu^*$, $\kappa$, $\|S\|_{L^\infty \cap H^1}$;
- $G^\nu \subset B_2$ for all $\nu \in I^*$.

The next lemma gives the necessary $H^1$-estimates which will be useful to our analysis.

Lemma 7.11. We fix $\nu^* > \nu_* > 0$ and define $I^*$ by (7.8). Define $U = \{ \phi \in H^1 : \|\phi\|_{H^1} \leq R_U \}$ where $R_U > 0$. For any $\theta_0 \in U$ and $\nu \in I^*$, if $\theta^{(\nu)}(t) = \pi^\nu(t)\theta_0$, then $\theta^{(\nu)}(t)$ satisfies
\[
\sup_{0 \leq \tau \leq t} \|\theta^{(\nu)}(\cdot, \tau)\|_{H^1}^2 + \int_0^t \|\theta^{(\nu)}(\cdot, \tau)\|_{H^2}^2 d\tau \leq M_+(t), \quad \forall t > 0,
\] (7.10)
where $M_+(t)$ is a positive function in $t$ which depends only on $t$, $\kappa$, $\nu_*$, $\nu^*$, $\|S\|_{H^1}$ and $R_U$.

Proof. First of all, we recall the the following $L^2$-estimate on $\theta^{(\nu)}$ from (4.2), namely,
\[
\|\theta^{(\nu)}(\cdot, t)\|_{L^2}^2 + \kappa \int_0^t \|\Delta \theta^{(\nu)}(\cdot, \tau)\|_{L^2}^2 d\tau \leq \|\theta_0\|_{L^2}^2 + \frac{t}{c_0 \kappa} \|\Delta S\|_{L^2}^2, \quad \forall t \geq 0.
\] (7.11)
Following the argument given in the proof of Theorem 4.1, for all $\nu \in I^*$, we readily have
\[
\frac{1}{2} \frac{d}{dt} \|\nabla \theta^{(\nu)}\|_{L^2}^2 + \kappa \|\Delta \theta^{(\nu)}\|_{L^2}^2 \leq \|\Delta S\|_{L^2} \|\nabla \theta^{(\nu)}\|_{L^2}^2 + \int_{\mathbb{T}^d} u^{(\nu)} \cdot \nabla \theta^{(\nu)} \Delta \theta^{(\nu)}.
\] (7.12)
Using the fact that $\text{div}(u^{(v)}) = 0$ and the commutator estimate given in (3.6), we obtain
\[
\left| \int_{\mathbb{T}^d} u^{(v)} \cdot \nabla \theta^{(v)} \Delta \theta^{(v)} \right| \leq C \| \nabla u^{(v)} \|_{L^4} \| \nabla \theta^{(v)} \|_{L^4} \| \Delta \theta^{(v)} \|_{L^2} \\
\leq C \| \Delta^2 u^{(v)} \|_{L^2} \| \Delta^2 \theta^{(v)} \|_{L^2} \| \Delta \theta^{(v)} \|_{L^2},
\]
where the last inequality follows from (6.41). With the help of (7.9), we can further bound $\| \Delta^2 u^{(v)} \|_{L^2}$ by $C_{*} \| \theta^{(v)} \|_{L^2}$. Hence we conclude from (7.12) that
\[
\frac{1}{2} \frac{d}{dt} \| \nabla \theta^{(v)} \|_{L^2}^2 + \frac{\kappa}{2} \| \Delta \theta^{(v)} \|_{L^2}^2 \leq \| \Delta S \|_{L^2} \| \nabla \theta^{(v)} \|_{L^2} + \frac{C_{*}}{\kappa} \| \theta^{(v)} \|_{L^2}^2 \| \nabla \theta^{(v)} \|_{L^2},
\]
and the result (7.10) follows from (7.11), (7.13) and Grönwall’s inequality.

Next we estimate the difference between the Fourier symbols of $u^{(v_1)}$ and $u^{(v_2)}$ for $v_1, v_2 \in I^*$.

**Lemma 7.12.** We fix $v^* > v_{*} > 0$ and define $I^*$ by (7.8). There exists $C_{*} > 0$ which depends on $v^*$ and $v_{*}$ only such that for any $v_1, v_2 \in I^*$ and $k = (k_1, k_2, k_3) \in \mathbb{Z}^3 \setminus \{(0, 0, 0)\}$, we have
\[
|k|^2 |\hat{M}_{j}^{v_1}(k) - \hat{M}_{j}^{v_2}(k)| \leq C_{*}|v_1 - v_2|, \quad \forall j \in \{1, 2, 3\}. \quad (7.14)
\]

**Proof.** By direct computation, for each $k = (k_1, k_2, k_3) \in \mathbb{Z}^3 \setminus \{(0, 0, 0)\}$ and $v_1, v_2 \in I^*$,
\[
|k|^2 |\hat{M}_{1}^{v_1}(k) - \hat{M}_{1}^{v_2}(k)| = |v_1 - v_2||k|^2 \frac{|k_1 k_2^3 k_3^6 + k_1 k_2 |k|^4 (k_2^2 + v_1 |k|^4) (k_3^2 + v_2 |k|^4) - k_2 k_3 |k|^6 (2 k_3^2 + v_1 |k|^4 + v_2 |k|^4)|}{[|k|^2 k_3^2 + (k_2^2 + v_1 |k|^4)^2][|k|^2 k_2^2 + (k_2^2 + v_2 |k|^4)^2]}
\leq |v_1 - v_2||k|^2 |k|^{10} + |k|^{10}(1 + v^* |k|^2)^2 + |k|^{10}(2 + 2v^* |k|^4)
\leq \frac{C_{*}}{v_*^4} |v_1 - v_2|,
\]
where $C_{*} \geq \frac{1 + (1 + v^*)^2 + 2(1 + v^*)}{v_*^4}$. The cases for $j = 2$ and $j = 3$ are just similar and we omit the details.

We are now ready to state and prove the following lemma which gives the continuity of $\pi^v$ in $v \in I^*$.

**Lemma 7.13.** We fix $v^* > v_{*} > 0$ and define $I^*$ by (7.8). Then for each $t > 0$, $\pi^v(t)\theta_0$ is continuous in $I^*$, uniformly for $\theta_0$ in compact subsets of $H^1$. 
Proof. Given a compact set $\mathcal{U}$ in $H^1$, we choose $R_\mathcal{U} > 0$ such that $\mathcal{U} \subset \{ \phi \in H^1 : \| \phi \|^2_{H^1} \leq R_\mathcal{U} \}$. For each $\theta_0 \in \mathcal{U}$ and $\nu_1, \nu_2 \in I^*$, we define

$$\theta^{(\nu_i)}(t) = \pi^{\nu_i}(t)\theta_0, \quad i \in \{1, 2\}.$$ 

We write $\phi = \theta^{(\nu_1)} - \theta^{(\nu_2)}$, then $\phi$ satisfies $\phi(\cdot, 0) = 0$ and

$$\partial_t \phi + (u^{(\nu_1)} - u^{(\nu_2)}) \cdot \nabla \theta^{(\nu_2)} + u^{(\nu_1)} \cdot \nabla \phi = \kappa \Delta \phi,$$

(7.15)

where $u^{(\nu_i)} = u^{(\nu_i)}[\theta^{(\nu_i)}]$ for $i = 1, 2$. Multiply (7.15) by $-\Delta \phi$ and integrate to get

$$\frac{1}{2} \frac{d}{dt} \| \nabla \phi \|^2_{L^2} + \kappa \| \Delta \phi \|^2_{L^2} \leq \left| \int_{\mathbb{T}^d} (u^{(\nu_1)} - u^{(\nu_2)}) \cdot \nabla \theta^{(\nu_2)} \Delta \phi \right| + \left| \int_{\mathbb{T}^d} u^{(\nu_1)} \cdot \nabla \phi \Delta \phi \right|.$$

(7.16)

Upon integrating by part, and exploiting the fact that $\text{div}(u^{(\nu_1)}) = 0$, we readily obtain

$$\left| \int_{\mathbb{T}^d} u^{(\nu_1)} \cdot \nabla \phi \Delta \phi \right| \leq \| \nabla u^{(\nu_1)} \|_{L^\infty} \| \nabla \phi \|^2_{L^2}.$$

Using the estimates (3.2) and (7.9), we can bound $\| \nabla u^{(\nu_1)} \|_{L^\infty}$ by $C_* \| \Delta \theta^{(\nu_1)} \|_{L^2}$ for some positive constant $C_*$ which only depends on $\nu^*$ and $\nu_*$, and $C_*$ may change from line to line. On the other hand, using Hölder’s inequality and (6.41), we have

$$\left| \int_{\mathbb{T}^d} (u^{(\nu_1)} - u^{(\nu_2)}) \cdot \nabla \theta^{(\nu_2)} \Delta \phi \right| \leq C \| \nabla (u^{(\nu_1)} - u^{(\nu_2)}) \|_{L^2} \| \Delta \theta^{(\nu_2)} \|_{L^2} \| \Delta \phi \|_{L^2}.$$

To bound the term $\| \nabla (u^{(\nu_1)} - u^{(\nu_2)}) \|_{L^2}$, we apply the estimate (7.14) to obtain

$$\| \nabla (u^{(\nu_1)} - u^{(\nu_2)}) \|^2_{L^2} = \sum_{k \in \mathbb{Z}^3 \setminus \{(0,0,0)\}} |k|^2 |\hat{(u^{(\nu_1)} - u^{(\nu_2)})}(k)|^2 \leq \sum_{k \in \mathbb{Z}^3 \setminus \{(0,0,0)\}} |k|^2 |\hat{M}^{(\nu_1)}|^2 |\hat{\theta^{(\nu_1)}} - \hat{\theta^{(\nu_2)}}(k)|^2.$$

Hence we deduce from (7.16) that

$$\frac{1}{2} \frac{d}{dt} \| \nabla \phi \|^2_{L^2} + \frac{\kappa}{2} \| \Delta \phi \|^2_{L^2} \leq \frac{C_*}{\kappa} \| \Delta \theta^{(\nu_2)} \|^2_{L^2} \left( \| \phi \|^2_{L^2} + |\nu_1 - \nu_2|^2 \| \phi \|^2_{L^2} \right) + C_* \| \Delta \theta^{(\nu_1)} \|_{L^2} \| \nabla \phi \|^2_{L^2}. \quad (7.17)$$
By integrating (7.17) from 0 to $t$, using Grönwall’s inequality and applying the bound (7.10) on $\theta^{(\nu_1)}$ and $\theta^{(\nu_2)}$, there exists a positive function $C_*(t)$ which depends only on $t$, $\kappa$, $\nu_*$, $\|S\|_{H^1}$ and $R_U$ such that

$$\|\theta^{(\nu_1)}(\cdot, t) - \theta^{(\nu_2)}(\cdot, t)\|_{H^1}^2 \leq C_*(t)|\nu_1 - \nu_2|^2, \quad \forall t > 0,$$ (7.18)

and (7.18) implies $\pi^*(t)\theta_0$ is continuous in $I^*$ uniformly for $\theta_0$ in $U$. □

In view of Remark 7.10 and Lemma 7.13, the conditions L1 and L2 as stated at the beginning of this subsection follow immediately and we conclude that (7.5) holds. We finish the proof of Theorem 7.5.

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