Abstract

Common intervals have been defined as a modelisation of gene clusters in genomes represented either as permutations or as sequences. Whereas optimal algorithms for finding common intervals in permutations exist even for an arbitrary number of permutations, in sequences no optimal algorithm has been proposed yet even for only two sequences. Surprisingly enough, when sequences are reduced to permutations, the existing algorithms perform far from the optimum, showing that their performances are not dependent, as they should be, on the structural complexity of the input sequences.

In this paper, we propose to characterize the structure of a sequence by the number $q$ of different dominating orders composing it (called the domination number), and to use a recent algorithm for permutations in order to devise a new algorithm for two sequences. Its running time is in $O(q_1 q_2 p + q_1 n_1 + q_2 n_2 + N)$, where $n_1, n_2$ are the sizes of the two sequences, $q_1, q_2$ are their respective domination numbers, $p$ is the alphabet size and $N$ is the number of solutions to output. This algorithm performs better as $q_1$ and/or $q_2$ reduce, and when the two sequences are reduced to permutations (i.e. when $q_1 = q_2 = 1$) it has the same running time as the best algorithms for permutations. It is also the first algorithm for sequences whose running time involves the parameter size of the solution. As a counterpart, when $q_1$ and $q_2$ are of $O(n_1)$ and $O(n_2)$ respectively, the algorithm is less efficient than other approaches.

1 Introduction

One of the main assumptions in comparative genomics is that a set of genes occurring in neighboring locations within several genomes represent functionally related genes [9, 14, 18]. Such clusters of genes are then characterized by a highly conserved gene content, but a possibly different order of genes within different genomes. Common intervals have been defined to model clusters [20], and have been used since to detect clusters of functionally related genes [15, 19], to compute similarity measures between genomes [6, 2] and to predict protein functions [11, 21].

Depending on the representation of genomes in such applications, allowing or not the presence of duplicated genes, comparative genomics requires for finding common intervals either in sequences or in permutations over a given alphabet. Whereas the most general - and thus useful in practice - case is the one involving sequences, the easiest to solve is the one involving permutations. This is why, in some approaches [11, 3], sequences are reduced to permutations by renumbering the copies of the same gene according to evolutionary based hypothesis. Another way to exploit the performances of algorithms for permutations in dealing with sequences is to see each sequence as a combination of several permutations, and to deal with these permutations rather than with the sequences. This is the approach we use here.

In permutations on $p$ elements, finding common intervals may be done in $O(Kp + N)$ time where $K$ is the number of permutations and $N$ the number of solutions, using several algorithms proposed in the literature [20, 5, 10, 16]. In sequences (see Table 1), even when only two sequences $T$ and $S$ of respective sizes $n_1$ and $n_2$ are considered, the best solutions take quadratic time. In a chronological order, the first algorithm is due to Didier [3] and performs in $O(n_1 n_2 \ \log n_2)$ time and $O(n_1 + n_2)$ space. Shortly later, Schmidt and Stoye [17] propose an $O(n_1 n_2)$ algorithm which needs $O(n_1 n_2)$ space, and note that Didier’s algorithm may benefit from an existing result to achieve $O(n_1 n_2)$ running
Running time for the existing algorithms when (1) the input sequences are as general as possible (lengths $n_1$ and $n_2$), when (2) one of them is a permutation (lengths $n_1 = p = |\Sigma|$ and $n_2 \geq p$), and when (3) both are permutations (lengths $n_1 = n_2 = p = |\Sigma|$). The running time of Didier’s algorithm is updated according to Schmidt and Stoye’s remark. On the last line, we add the memory space needed by each algorithm ($n_1 = p$ and $n_1 = n_2 = p$ respectively hold for the second and third case). Parameters $l_1, l_2, q_1, q_2$ are described in the text.

The running times of all the existing algorithms have at least two main drawbacks: first, they do not involve at all the number $N$ of output solutions; second, they insufficiently exploit the particularities of the two sequences and, in the particular case where the sequences are reduced to permutations, need quadratic time instead of the optimal $O(p+N)$ time for two permutations on $p$ elements. That means that their performances insufficiently depend both on the inherent complexity of the input sequences, and on the amount of results to output. Unlike the algorithms dealing with permutations, the algorithms for sequences lack of criteria allowing them to decide when the progressive generation of a candidate must be stopped, since it is useless. This is the reason why their running time is independent of the number of output solutions. This is also the reason why when sequences are reduced to permutations the running time is very unsatisfactory.

The most recent optimal algorithm for permutations [16] proposes a general framework for efficiently searching for common intervals and all of their known subclasses in $K$ permutations, and has a twofold advantage, not proposed by other algorithms. First, it permits an easy and efficient selection of the common intervals to output based on two types of parameters. Second, assuming one permutation has been renumbered to be the identity permutation, it outputs all common intervals with the same minimum value together and in increasing order of their maximum value. We use here these properties to propose a new algorithm for finding common intervals in two sequences. Our algorithm strongly takes into account the structure of the input sequences, expressed by the number $q$ of different dominating orders (which are permutations) composing the sequence ($q = 1$ for permutations). Consequently, it has a complexity depending both on this structure and on the number of output solutions. It runs in optimal $O(p+N)$ time for two permutations on $p$ elements, is better than the other algorithms for sequences composed of few dominating orders and, as a counterpart, it performs less well as the number of composing dominating orders grows.

The structure of the paper is as follows. In Section 2, we define the main notions, including that of a dominating order, and give the results allowing us a first simplification of the problem. In Section 3, we propose our approach for finding common intervals in two sequences based on this simplification, for which we describe the general lines. In Sections 4-5 and 6, we develop each of these general lines and prove correctness and complexity results. Section 7 is the conclusion.
2 Preliminaries

Let \( T \) be a sequence of length \( n \) over an alphabet \( \Sigma := \{1, 2, \ldots, p\} \). We denote the length of \( T \) by \(|T|\), the set of elements in \( T \) by \( \text{Set}(T) \), the element of \( T \) at position \( i \), \( 1 \leq i \leq n \), by \( t_i \) and the subsequence of \( T \) delimited by positions \( i, j \) (included), with \( 1 \leq i \leq j \leq n \), by \( T[i..j] \). An interval of \( T \) is any set \( I \) of integers from \( \Sigma \) such that there exist \( i, j \) with \( 1 \leq i \leq j \leq n \) and \( I = \text{Set}(T[i..j]) \). Then \( [i, j] \) is called a location of \( I \) on \( T \). A maximal location of \( I \) on \( T \) is any location \([i, j]\) such that neither \([i−1, j]\) nor \([i, j+1]\) is a location of \( I \).

When \( T \) is the identity permutation \( \text{Id}_p := (1 \ldots p) \), we denote \((i..j) := \{i, i + 1, \ldots, j\} \), which is also \( \text{Set}(\text{Id}_p[i..j]) \). Note that all intervals of \( \text{Id}_p \) are of this form, and that each interval has a unique location on \( \text{Id}_p \). When \( T \) is an arbitrary permutation on \( p \) elements (denoted \( P \) in this case), we denote by \( P^{-1} \) the function which associates with each element of \( P \) its position in \( P \). For a subsequence \( P[i..j] \) of \( P \), we also say that it is delimited by its elements \( p_i \) and \( p_j \) located at positions \( i \) and \( j \). These elements are the delimiters of \( P[i..j] \) (note the difference between delimiters, which are elements, and their positions).

We now define common intervals of two sequences \( T \) and \( S \) of respective sizes \( n_1 \) and \( n_2 \):

\[\text{(T, S)-Common Intervals Searching}\]

**Input:** Two sequences \( T \) and \( S \) of respective lengths \( n_1 \) and \( n_2 \) over an alphabet \( \Sigma := \{1, 2, \ldots, p\} \).

**Requires:** Find all \((T, S)\)-maximal locations of common intervals of \( T \) and \( S \), without redundancy.

To address this problem, assume we add a new element \( X \) (not in \( \Sigma \)) at positions 0 and \( n_1 + 1 \) of \( T \). Let \( \text{Succ} \) be the \((n_1 + 1)\)-size array defined for each position \( i \) with \( 0 \leq i \leq n_1 \) by \( \text{Succ}[i] = j \) if \( t_i = t_j \) and \( j > i \) is the smallest with this property (if \( j \) does not exist, then \( \text{Succ}[i] = n_1 + 1 \)). Call the area of the position \( i \) on \( T \) the sequence \( A_i := T[i..\text{Succ}[i] − 1] \).

**Example 2.** With \( T = X 1 2 5 2 1 4 3 1 2 6 5 \) and \( S = 5 6 4 2 3 4 1 5 \), we have \( \text{Succ} = [12, 5, 4, 11, 9, 8, 12, 12, 12, 12, 12] \). Thus the area of position \( 2 \) in \( T \) is \( A_2 = T[2..\text{Succ}[1] − 1] = T[2..(5 − 1)] = 2 5 2 \). Similarly, \( A_4 = T[4..\text{Succ}[3] − 1] = T[4..10] = 2 1 4 3 1 2 6 \).

**Definition 2.** [8] The order \( O_i \) associated with a position \( i \) of \( T \), \( 1 \leq i \leq n_1 \), is the sequence of all elements in \( \text{Set}(A_i) \) ordered according to their first occurrence in \( A_i \). We note \( k_i = |\text{Set}(A_i)| = ||O_i|| \).

**Remark 1.** Note that:
- \( O_i \) may be empty, and this holds iff \( t_{i−1} = t_i \).
- if \( O_i \) is not empty, then its first element is \( t_i \).
- if \( O_i \) is not empty, then \( O_i \) contains each element in \( \text{Set}(A_i) \) exactly once, and is thus a permutation on a subset of \( \Sigma \).

In the subsequent, we consider that a pre-treatment has been performed on \( T \), removing every element \( t_i \) which is equal to \( t_{i−1} \), \( 2 \leq i \leq n_1 \), such that to guarantee that no empty order exists. In this way, the maximal locations are slightly modified, but this is not essential.
Let respectively \( b_1 (= i), b_2, \ldots, b_k \) be the positions in \( T \) of the elements \( a_1 (= t_i), a_2, \ldots, a_k \), defining \( O_1 \), i.e. the position in \( T \) of their first occurrences in \( A_i \). Now, define \( B_i := b_1 b_2 \ldots b_k \), to be the ordered sequence of these positions.

Example 3. With \( T = 12521431265 \), we have \( A_4 = 2143126 \) and thus \( O_4 = 21436 \) with \( B_4 = 456710 \). Note that \( \mathcal{O}_4[1..4] = 2143 \) and \( \mathcal{B}_4[1..4] = 4567 \) meaning that \( \text{Set}(\mathcal{O}_4[1..4]) \), i.e. \( \{1, 2, 3, 4\} \), is an interval of \( T \) a location of which is given by \( B_4[1] \) and \( B_4[4] \), i.e. \( [4, 7] \). This location is not maximal, but it is the \textit{maxmin} location corresponding to \([4, 9]\) as defined below.

Definition 3. Given a sequence \( T \) and an interval \( I \) of it, a \textit{maxmin location} of \( I \) on \( T \) is any location \( [i, j] \) of \( I \) which is left maximal and right minimal, that is, such that neither \( [i-1, j] \) nor \( [i, j-1] \) is a location of \( I \) on \( T \). A \((T, S)\)-\textit{maxmin location} of \( I \) is any pair \(([i, j], [y, z])\) of maxmin locations of \( I \) on \( T \) (this is \([i, j]\)) and respectively on \( S \) (this is \([y, z]\)).

It is easy to see that that maxmin locations and maximal locations are in bijection. We make this more precise as follows.

Claim 1. \textit{The function associating with each maximal location} \([i, j']\) of an interval in \( T \) the \textit{maxmin location} \([i, j]\) in \( T \) such that \( j \) is maximum with the properties \( j \leq j' \) and \( j \in \text{Set}(\mathcal{B}_i) \) is a bijection. Moreover, if \( j = \mathcal{B}_i[h] \), then \( j' \) may be computed in \( O(1) \) when \( i, h, \mathcal{B}_i \) and \( \text{Succ} \) are known.

Proof. It is easy to see that by successively removing from \( T[i..j'] \) the rightmost element as long as it has a copy on its left, we obtain a unique interval \( T[i..j] \) such that \( [i, j] \) is a minmax location of \( I, j \in \text{Set}(\mathcal{B}_i) \) and \( j \) is maximum with this property. The inverse operation builds \([i, j']\) when \([i, j]\) is given.

Moreover, if \( j = \mathcal{B}_i[h] \), then \( \text{Set}(T[i..j']) = \text{Set}(T[i..j]) = \text{Set}(\mathcal{O}_i[1..h]) \). Then, assuming \( i, h, \mathcal{B}_i \) and \( \text{Succ} \) are known and we want to compute \( j' \), we have two cases. If \( h = k_i \), then \( j' \) is the position of the last element in \( A_i \) and thus \( j' \) is computed as \( j' = \text{Succ}[i-1] - 1 \). If \( h < k_i \), then \( j' \) is the position in \( T \) of the element preceding \( \mathcal{O}_i[h+1] \), that is, \( j' = \mathcal{B}_i[h+1] - 1 \).

In the subsequent, and due to the preceding Claim, we solve the \((T, S)\)-\textit{COMMON INTERVAL SEARCHING} problem by replacing maximal locations with maxmin locations. Using Claim 1 it is also easy to deduce that:

Claim 2. \textit{The intervals of} \( T \) are the sets \( \text{Set}(\mathcal{O}_i[1..h]) \) with \( 1 \leq h \leq k_i \). As a consequence, the common intervals of \( T \) and \( S \) are the sets \( \text{Set}(\mathcal{O}_i[1..h]) \) with \( 1 \leq h \leq k_i \), which are also intervals of \( S \).

With these precisions, Didier’s approach consists then in considering each order \( \mathcal{O}_i \) and, in total time \( O(n_2 \log n_2) \) (reducible to \( O(n_2) \) according to \([17]\)), verifying whether the intervals \( \text{Set}(\mathcal{O}_i[1..h]) \) with \( 1 \leq h \leq ||\mathcal{O}_i|| \) are also intervals of \( S \). Our approach avoids to consider each order \( \mathcal{O}_i \) by defining dominating orders which contain other orders, with the aim of focalising the search for common intervals on each dominating order rather than spreading it on each of the orders it dominates.

We introduce now the supplementary notions needed by our algorithm.

Definition 4. Let \( d, i \) be two integers such that \( 1 \leq d \leq i \leq n_1 \). We say that the order \( \mathcal{O}_d \) \textit{dominates} the order \( \mathcal{O}_i \) if \( \mathcal{B}_i \) is a contiguous subsequence of \( \mathcal{B}_d \). We also say that \( \mathcal{O}_i \) is \textit{dominated} by \( \mathcal{O}_d \).

Equivalently, \( \mathcal{O}_i \) is a contiguous subsequence of \( \mathcal{O}_d \) and the positions on \( T \) of their common elements are the same.

Definition 5. Let \( d \) be such that \( 1 \leq d \leq n_1 \). Order \( \mathcal{O}_d \) is \textit{dominating} if it is not dominated by any other order of \( T \). The number of dominating orders of \( T \) is the \textit{domination number} \( q(T) \) of \( T \).

The set of orders of \( T \) is provided with an order, defined as \( \mathcal{O}_i \prec \mathcal{O}_j \) if \( i < j \). For each dominating order \( \mathcal{O}_d \) of \( T \), its \textit{strictly dominated orders} are the orders \( \mathcal{O}_i \) with \( i \geq d \) such that \( \mathcal{O}_i \) is dominated by \( \mathcal{O}_d \) but is not dominated by any order preceding \( \mathcal{O}_d \) according to \( \prec \).
For the other values of $\text{Example 5.}$

For each dominating order $O_d$ (which is a permutation), we need to record the suborders which correspond to the strictly dominated orders. Only the left and right endpoints of each suborder are recorded, in order to limit the space and time requirements. Then, let the domination function of a dominating order $O_d$ be the partial function $F_d : \{1, \ldots, k_d\} \to \{1, \ldots, k_d\}$ defined as follows.

$$F_d(s) := f \text{ if there is some } i \text{ such that } O_i \text{ is strictly dominated by } O_d \text{ and } B_d[s..f] = B_i.$$ 

For the other values of $s \in \{1, 2, \ldots, k_d\}$, $F_d(s)$ is not defined. Note that $F_d(1) = k_d$, since by definition any dominating order strictly dominates itself. See Figure 2.

**Example 5.** For $T = 12521431265$ (see also Figure 1), the dominating order $O_4$ strictly dominates $O_1$, which correspond respectively to the suborders $O_4[1..6]$ and $O_4[2..4]$ of $O_4$. The dominating function of $O_4$ is then given by $F_4(1) = 6$ and $F_4(2) = 4$ ($F_4$ is not defined for the other values).

We know that, according to Claim 2, the common intervals of $T$ and $S$ must be searched among the intervals $Set(O_i[1..h])$ or, if we focus on one dominating order $O_d$ and its strictly dominated orders identified by $F_d$, among the intervals $Set(O_i[s..u])$ for which $F_d(s)$ is defined and $s \leq u \leq F_d(s)$. We formalize this search as follows.

**Definition 6.** Let $P$ be a permutation on $p$ elements, and $F : \{1, 2, \ldots, p\} \to \{1, 2, \ldots, p\}$ be a partial function such that $F(1) = p$ and $w \leq F(w)$ for all values $w$ for which $F(w)$ is defined. A location $[s, u]$ of an interval of $P$ is valid with respect to $F$ if $F$ is defined for $s$ and $s \leq u \leq F(s)$.

**Claim 3.** The $(T, S)$-maxmin locations $([i, j], [y, z])$ of common intervals of $T$ and $S$ are in bijection with the triples $(d, [s, u])$, $([y, z])$ such that:

(a) $O_d$ is a dominating order of $T$

(b) the location $[s, u]$ on $O_d$ of the interval $Set(O_d[\ldots u])$ is valid with respect to $F_d$

(c) $[y, z]$ is a maxmin location of $Set(O_d[\ldots u])$ on $S$.

Moreover, the triple associated with $([i, j], [y, z])$ satisfies: $O_d$ is the dominating order that strictly dominates $O_i$, $i \in B_d[s]$ and $j \in B_d[u]$.

**Proof.** See Figure 2. By Claim 2 the common intervals of $T$ and $S$ are the sets $Set(O_i[1..h])$ with $1 \leq h \leq k_i$, which are intervals of $S$. We note that the sets $Set(O_i[1..h])$ are not necessarily distinct, but their locations $i, j$ on $T$, given by $[i, j] = [B_i[1], B_i[h]]$, are distinct. Then, the $(T, S)$-maxmin locations $([i, j], [y, z])$ of common intervals $Set(O_i[1..h])$ are in bijection with the pairs $(B_i[1..h], [y, z])$. 

![Figure 1: The orders of $T = 12521431265$. For a given position $i$ of $T$, the order $O_i$ is represented by the horizontal line whose intersection with the vertical line going down from $t_i$ is marked with a square. The elements of $O_i$ are $t_i$ (marked with the abovementioned square) and all the elements on the line $O_i$ marked with a circle. When an order contains only one element, as $O_3$ and $O_{11}$, both the square and the circle represent the unique element of the order.](image-url)
ever, an improved running time of $O(n^2)$ steps 3 and 4 the dominating orders, with a more efficient algorithm. In such a way, we keep the space step 1 (and similarly in step 2), is too time consuming to be reused in steps 3 and 4 when dominating orders are needed. Instead, minimal information from steps 1 and 2 is stored, which allows to recover in steps 3 and 4 the dominating orders, with a more efficient algorithm. In such a way, we keep the space

![Figure 2: Correspondence of positions between T, a dominating order $O_d$ of T and an order $O_t$ of T which is dominated by $O_d$. Black circles in $O_d$ and $O_t$ not identified by a position are other elements of $O_d$ and $O_t$, not important here.](image)

such that $[y, z]$ is a maxmin location of the interval on $S$, which are themselves in bijection with the pairs $(B_d[s..u], [y, z])$ such that the dominating order $O_d$ strictly dominates $O_t$ and $[s, u]$ is valid with respect to $F_d$. More precisely, $u = s + h - 1 \leq F_d(c)$.]

**Corollary 1.** Each $(T, S)$-maxmin location $([i, j], [y, z])$ of a common interval of $T$ and $S$ is computable in $O(1)$ time if the corresponding triple $(d, [s, u], [y, z])$ and the sequence $B_d$ are known.

Looking for the $(T, S)$-maxmin locations of the common intervals of $T$ and $S$ thus reduces to finding the $(O_d, S)$-maxmin locations of common intervals for each dominating order $O_d$ and for $S$, whose locations on $O_d$ are valid with respect to the dominating function $F_d$ of $O_d$. The central problem to solve now is thus the following one (replace $O_d$ by $P$, $F_d$ by $F$ and $k_d$ by $p$):

**$(P, S)$-GUIDED COMMON INTERVALS SEARCHING**

**Input:** A permutation $P$ on $p$ elements, a sequence $S$ of length $n_2$ on the same set of $p$ elements, a partial function $F : \{1, 2, \ldots, p\} \rightarrow \{1, 2, \ldots, p\}$ such that $F(1) = p$ and $w \leq F(w)$ for all $w$ such that $F(w)$ is defined.

**Requires:** Find all $(P, S)$- maxmin locations of common intervals of $P$ and $S$ whose locations on $P$ are valid with respect to $F$, without redundancy.

As before, we assume w.l.o.g. that $S$ contains all the elements in $P$, so that $n_2 \geq p$. Also, we denote $q_2 := q(S)$. In this paper, we show (see Section 3, Theorem 1) that $(P, S)$-GUIDED COMMON INTERVALS SEARCHING may be solved in $O(q_2n_2 + N_{P, S})$ time and $O(n_2)$ space, where $N_{P, S}$ is its number of solutions for $P$ and $S$. This running time gives the running time of our general algorithm. However, an improved running time of $O(n_2 + N_{P, S})$ for solving $(P, S)$-GUIDED COMMON INTERVALS SEARCHING would lead to a $O(q_1n_1 + q_1n_2 + N)$ algorithm for the case of two sequences, improving the complexity of the existing $O(n_1n_2)$ algorithms.

### 3 The approach

The main steps for finding the maxmin locations of all common intervals in two sequences using the reduction to $(P, S)$-GUIDED COMMON INTERVALS SEARCHING are given in Algorithm 1. Recall that for $T$ and $S$ we respectively denote $n_1, n_2$ their sizes, and $q_1, q_2$ their dominating numbers. The algorithms for computing each step are provided in the next sections.

To make things clear, we note that the dominating orders (steps 1 and 2) are computed but never stored simultaneously, whereas dominated orders are only recorded as parts of their corresponding dominating orders, using the domination functions. The initial algorithm for computing this information, in step 1 (and similarly in step 2), is too time consuming to be reused in steps 3 and 4 when dominating orders are needed. Instead, minimal information from steps 1 and 2 is stored, which allows to recover in steps 3 and 4 the dominating orders, with a more efficient algorithm. In such a way, we keep the space
requirements in $O(n_1 + n_2)$, and we perform steps 3, 4, 5 in global time $O(q_1 q_2 p)$, which is the best we may hope.

In order to solve $(P, S)$-GUIDED COMMON INTERVALS SEARCHING, our algorithm cuts $S$ into dominating orders and then looks for common intervals in permutations. This is done in steps 2, 4 and 5, as proved in the next theorem.

**Theorem 1.** Steps 2, 4 and 5 in Algorithm 1 solve $(P, S)$-GUIDED COMMON INTERVALS SEARCHING with input $P = O_d$, $F = F_d$ and $S$. Moreover, these steps may be performed in global $O(q_2 n_2 + N_{P, S})$ time and $O(n_2)$ space.

**Proof.** Claim 3 and Corollary 1 insure that the $(S, O_d)$-maxmin locations of common intervals of $S$ and $O_d$, in this precise order, are in bijection with (and may be easily computed from) the triples $(\delta, [s, u], [y, z])$ such that $\Omega_2$ is a dominating order of $S$, $[s, u]$ is valid with respect to $\Phi_3$ and $[y, z]$ is a maxmin location of $Set(\Omega_3 [s..u])$ on $O_d$. Note that since $O_d$ is a permutation, each location is a maxmin location. Reducing these triples to those for which $[y, z]$ is valid w.r.t. $F_d$, as indicated in step 5, we obtain the solutions of $(P, S)$-GUIDED COMMON INTERVALS SEARCHING with input $P = O_d$, $F = F_d$ and $S$.

In order to give estimations of the running time and memory space, we refer to results proved in the remaining of this paper. Step 2 takes $O(q_2 n_2)$ time and $O(n_2)$ space assuming the orders are not stored (as proved in Section 4, Theorem 3), step 4 needs $O(q_2 p)$ time and $O(n_2)$ space to successively generate the orders $\Omega_4$ from information provided by step 2 (Section 5, Theorem 4), whereas step 5 takes $O(p + N_{O_d, \Omega_4})$ time and $O(p)$ space, where $N_{O_d, \Omega_4}$ is the number of solutions for $(O_d, \Omega_4)$-GUIDED COMMON INTERVALS SEARCHING (Section 6, Theorem 6).

**Example 6.** With $T = 1 2 5 2 1 4 3 1 2 6 5$ and $S = 5 6 4 2 3 4 1 5$ we have three dominating orders in $T$, that is $O_1, O_4$ and $O_7$, and three dominating orders in $S$, that is $\Omega_1, \Omega_3$ and $\Omega_5$. Consider step 5 for $O_4 = 2 1 4 3 6$ and $\Omega_5 = 3 4 1 5$. We have $F_4(1) = 6$ and $F_4(2) = 4$, as well as $\Phi_5(1) = 4$ and $\Phi_5(2) = 4$ (note that $\Phi_5(3)$ and $\Phi_5(4)$ are not defined as $\Omega_7$ and $\Omega_8$ are strictly dominated by $\Omega_5$ and not by $\Omega_5$). That means we only look for common intervals which start in positions 1 or 2 in $O_4$ and in positions 1, or 2 in $\Omega_5$. Moreover, an interval which starts in position $s$ must end not later than $F_4(s)$ in $O_4$ (and similarly for $\Omega_5$). Thus the common intervals the algorithm will find for those two permutations are $\{1, 4\}$ (with locations $[2, 3]$ in $O_4$, and $[2, 3]$ in $\Omega_5$) and $\{1, 3, 4\}$ (with locations $[2, 4]$ in $O_4$ and $[1, 3]$ in $\Omega_5$). Note that these locations are valid with respect to $F_4$ and $\Phi_5$. The common interval $\{3, 4\}$ of $O_4$ and $\Omega_5$ is not output in this step of the algorithm since its location in $O_4$ is not valid. However, this is not a loss since such an interval would be redundant with the one output when $O_1$ and $\Omega_5$ are compared. Also note that $O_1$ and $\Omega_5$ are not permutations on the same set, and thus the algorithm we give in Section 6 must be applied on two slightly modified permutations.

**Theorem 2.** Algorithm 1 solves the $(T, S)$-COMMON INTERVALS SEARCHING problem in $O(q_1 n_1 + q_2 n_2 + q_1 q_2 p + N)$ time, where $N$ is the size of the solution, and $O(n_1 + n_2)$ space.

**Proof.** The correctness of the algorithm is insured by Claim 3 and Theorem 1.
We now discuss the running time and memory space, once again referring to results proved in the remaining sections. As proved in Theorem 3 (Section 4), Step 1 (and similarly Step 2) takes $O(q_1 n_1)$ - time and $O(n_1)$ space, assuming that the dominating orders $O_d$ are identified by their position $d$ on $T$ and are not stored (each of them is computed, used to find its dominating function and then discarded). The positions $d$ corresponding to dominating orders are stored in decreasing order in a stack $D_T$. The values of the dominating functions are stored as $q_1$ lists, one for each dominating order $O_d$, whose elements are the pairs $(s, F_d(s))$, in decreasing order of the value $s$. This representation needs a global memory space of $O(n_1)$.

In step 3 the progressive computation of the $q_1$ dominating orders is done in $O(q_1 p)$ time and $O(n_1)$ space using the sequence $T$ and the list $D_T$ of positions $d$ of the dominating orders. The algorithm achieving this is presented in Section 5, Theorem 4. For each dominating order $O_d$ of $T$, the orders $\Omega_d$ of $S$ are successively computed in global $O(q_2 p)$ time and $O(n_2)$ space by the same algorithm, and are only temporarily stored. Step 5 is performed for $O_d$ and $\Omega_d$ in $O(p + N_{\Omega_d, \Omega_d})$ time and $O(p)$ space, where $N_{O_d, \Omega_d}$ is the number of output solutions for $(O_d, \Omega_d)$-GUIDED COMMON INTERVALS SEARCHING (Section 6, Theorem 6).

Then the abovementioned running time of our algorithm easily follows. 

To simplify the notations, in the next sections the size of $T$ is denoted by $n$ and its domination number is denoted $q$. The vector $\text{Succ}$, as well as the vectors $\text{Prec}$ and $\text{Prec_S}$ defined similarly later, are assumed to be computed once at the beginning of Algorithm 1.

4 Finding the dominating and dominated orders of $T$

This task is subdivided into two parts. First, the dominating orders $O_d$ are found as well as, for each of them, the set of positions $i$ such that $O_d$ strictly dominates $O_i$. Thus $O_i = O_d[s..F_d(s)]$, where $s$ is known but $F_d(s)$ is not known yet. In the second part of this section, we compute $F_d(s)$. Note that in this way we never store any dominated order, but only its position on $T$ and on the dominating order strictly dominating it. This is sufficient to retrieve it from $T$ when needed.

4.1 Find the positions $i$ such that $O_i$ is dominating/dominated

As before, let $T$ be the first sequence, with an additional element $X$ (new character) at positions 0 and $n + 1$. Recall that we assumed that neighboring elements in $T$ are not equal, and that we defined $\text{Succ}$ to be the $(n + 1)$-size array such that, for all $i$ with $0 \leq i \leq n$, $\text{Succ}[i] = j$ if $t_i = t_j$ and $j > i$ is the smallest with this property (if $j$ does not exist, then $\text{Succ}[i] = n + 1$).

Given a subsequence $A = T[i..j]$ of $T$, slicing it into singletons means adding the character $Y$ at the beginning and the end of $A$, as well as a so-called $h$-separator (denoted $|_h$) after each element of $A$ which is the letter $h$. And this, for each $h$. Call $A^{sep}$ the resulting sequence on $\Sigma \cup \{Y\} \cup \{|_h \mid h \in \Sigma \cup \{Y\}\}$.

Example 7. With $T = 1 2 5 2 1 4 3 1 2 6 5$, let for instance $A := T[4..10] = A_4 = 2 1 4 3 1 2 6$. Slicing $A$ into singleton yields $A^{sep} = Y |_2 |_1 |_4 |_3 |_1 |_2 |_6 |_6 |_Y$. Slicing $A$ into singleton yields $A^{sep} = Y |_2 |_1 |_4 |_3 |_1 |_2 |_6 |_6 |_Y$.

Once $A^{sep}$ is obtained from $A$, successive removals of the separators are performed, and the resulting sequence is still called $A^{sep}$. Let a slice of $A^{sep}$ be any maximal interval $\{r, r + 1, \ldots, s\}$ of positions in $\{i, \ldots, j\}$ (recall that $A = T[i..j]$) such that no separator exists in $A^{sep}$ between $t_l$ and $t_{l+1}$ with $r \leq l < s$. Note that in this case a $r_{l-1}$-separator exists after $t_{l-1}$ and a $s_{t+1}$-separator exists after $t_s$, because of the maximality of the interval $\{r, r + 1, \ldots, s\}$. With $A^{sep}$ as defined above, immediately after $A$ has been sliced, every position in $A$ forms a slice.

Example 8. With $A = T[4..10]$ and $A^{sep}$ obtained by slicing $A$ into singletons as in the preceding example, let now $A^{sep} = Y |_2 |_1 |_4 |_3 |_1 |_2 |_6 |_6 |_Y$ be obtained after the removal of all the 1-separators. The slices are now $\{4\}$ (corresponding to $t_4 = 2$), $\{5, 6\}$ (corresponding to $t_5 = 1$ and
**Algorithm 2 Resolve(T, d)**

1. if $d$ is not resolved then
2. $A_d \leftarrow T[d....Succ[d-1]-1]$; find $O_d$ and $B_d$ by successively considering all elements in $A_d$
3. label $d$ as resolved and $O_d$ as dominating
4. $A^{sep} \leftarrow$ slice into singletons the sequence $A$ defined as $A := T[d-1..Succ[d-1]]$
5. for each position $i \in \operatorname{Set}(B_d)$ in decreasing order do
6. remove all the $t_i$-separators from $A^{sep}$ using $\operatorname{Succ}$
7. if $i$ is not already resolved and $\operatorname{Find}(i) = \operatorname{Find}(\operatorname{Succ}[i-1])$ then
8. label $i$ as resolved and $O_i$ as dominated by $O_d$
9. end if
10. end for
11. RightEnd($d$)
12. end if

$t_6 = 4$, $\{7\}$ (corresponding to $t_7 = 3$), $\{8, 9\}$ (corresponding to $t_8 = 1$ and $t_9 = 2$), and $\{10\}$ (corresponding to $t_{10} = 6$).

Slices are disjoint sets which evolve from singletons to larger and larger disjoint intervals using separator removals. Two operations are needed, defining - as the reader will easily note - a Union-Find structure:

- Remove a $h$-separator, thus merging two neighboring slices into a new slice. This is set union, between sets representing neighboring intervals.
- Find the slice a position belongs to. In the algorithm we propose, this function is denoted by $\operatorname{Find}$.

In the following, a position $d$ is **resolved** if its order $O_d$ has already been identified, either as a dominating or as a dominated order. Now, by calling Resolve$(T, d)$ in Algorithm 2 successively for all $d = 1, 2, \ldots, n$ (initially non-resolved), we find the dominating orders $O_d$ of $T$ and, for each of them, the positions $i$ such that $O_i$ is strictly dominated by $O_d$. Note that the rightmost position of each $O_i$ dominated by $O_d$ is computed by the procedure RightEnd$(d)$, given in Section 4.2.

**Example 9.** With $T = 12521431265$ and $d = 1$, step 2 computes $O_1 = 125436$ and $B_1 = 1236710$ by searching into $A_1$. Then 1 is labeled as resolved, and $O_1$ as dominating. Further, $A^{sep} = Y \backslash \{1 \mid 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Starting the for loop in step 5, with $i = 10$ we remove the 0-separator and the condition in step 7 is not fulfilled. The same holds with $i = 7$ after the 3-separator is removed. With $i = 6$, the 4-separator is removed and the condition in step 7 is verified, so that $O_6$ is labeled as dominated by $O_1$. Similarly, $O_3$, $O_2$ are further labeled as dominated by $O_1$.

To prove the correctness of our algorithm, we first need two results.

**Claim 4.** Order $O_i$ with $i > d$ is dominated by order $O_d$ iff $i < \operatorname{Succ}[i-1] \leq \operatorname{Succ}[d-1]$ and $i \in \operatorname{Set}(B_d)$ and $\operatorname{Set}(T[d..i-1]) \cap \operatorname{Set}(A_i) = \emptyset$.

**Proof.** Notice that, by definition, the positions in $B_i$ belong to $\{i, i+1, \ldots, \operatorname{Succ}[i-1]\}$.

"$\Rightarrow$": Properties $i < \operatorname{Succ}[d-1]$ and $i \in B_d$ are deduced directly from the definitions of an order and of order domination. If the condition $\operatorname{Succ}[d-1] \geq \operatorname{Succ}[i-1]$ is not true, then $\operatorname{Succ}[d-1]$ belongs to $O_d$ but not to $O_i$ (again by the definition of an order), a contradiction. Moreover, if, by contradiction, there is some $r \in \operatorname{Set}(T[d..i-1]) \cap \operatorname{Set}(A_i)$, occurring respectively in positions $a$ and $b$ (choose each of them as small as possible with $a > d$ and $b > i$), then $a \in \operatorname{Set}(B_d)$ and $b \in \operatorname{Set}(B_i) - \operatorname{Set}(B_d)$, since only the first occurrence of $r$ is recorded in $O_d$. But then $\operatorname{Set}(B_i) \subseteq \operatorname{Set}(B_d)$ and thus $O_i$ is not dominated by $O_d$, a contradiction.
"⇐": Let \( j \in \text{Set}(B_i) \). Then the first occurrence of the element \( t_j \) in \( A_i \) is, by definition, at position \( j \). Moreover, \( t_j \not\in \text{Set}(T[d..i-1]) \) by hypothesis and since \( \text{Succ}[d-1] \geq \text{Succ}[i-1] \), we deduce that the first occurrence of the element \( t_j \) in \( A_d \) is at position \( j \). Thus \( j \in \text{Set}(B_d) \). It remains to show that \( B_i \) is contiguous inside \( B_d \). This is easy, since any position in \( B_d \), not in \( B_i \) but located between two elements of \( B_i \) would imply the existence of an element whose first occurrence in \( A_d \) belongs to \( A_i \); this element would then belong to \( O_i \), and its position to \( B_i \), a contradiction.

Claim 5. Let \( d < i < \text{Succ}[d-1] \), and assume \( O_d \) is dominating. Then \( O_i \) is labeled as "dominated by \( O_d \)" in Resolve\((T, d)\) iff \( O_i \) is strictly dominated by \( O_d \).

Proof. Note that \( O_i \) may get a label during Resolve\((T, d)\) iff \( d \) is not resolved at the beginning of the procedure, in which case steps 2-3 of Resolve\((T, d)\) insure that \( O_d \) is labeled as "dominating". By hypothesis, we assume this label is correct. Now, \( O_i \) is labeled as "dominated by \( O_d \)" iff

- \( i \in B_d \) (step 5), and
- in step 7 we have that \( i \) is not already resolved, and \( i, \text{Succ}[i-1] \) are in the same slice in the sequence \( A^{sep} \) where all the \( t_j \)-separators satisfying \( j \in \text{Set}(B_d) \) and \( j \geq i \) have been removed (step 6).

The latter of the two conditions is equivalent to saying that \( A_i \) contains only characters equal to \( t_j \), \( j \in \text{Set}(B_d) \) and \( j \geq i \), that is, only characters whose first occurrence in \( A_d \) belongs to \( A_i \). This is equivalent to \( \text{Set}(T[d..i-1]) \cap A_i = \emptyset \) (i.e. no character in \( A_i \) appears before \( i \)) and \( \text{Succ}[i-1] \leq \text{Succ}[d-1] \) (all characters in \( A_i \) have a first occurrence not later than \( \text{Succ}[d-1] - 1 \)). But then the three conditions on the right hand of Claim 5 are fulfilled, and this means \( O_i \) is dominated by \( O_d \). Given that step 8 is executed only once for a given position \( i \), that is, when \( i \) is labeled as resolved, the domination is strict.

Now, the correctness of our algorithm is given by the following claim.

Claim 6. Assume temporarily that the procedure RightEnd\((d)\) is empty. Then calling Resolve\((T, d)\) successively for \( d = 1, 2, \ldots, n \) correctly identifies the dominating orders \( O_d \) and, for each of them, the positions \( i \) such that \( O_d \) strictly dominates \( O_i \). This algorithm takes \( O(qn) \) time and \( O(n) \) space.

Proof. We prove by induction on \( d \) that, at the end of the execution of Resolve\((T, d)\), we have for all \( i \) with \( 1 \leq i < \text{Succ}[d-1] \):

(a) \( O_i \) is labeled as "dominating" iff \( i \leq d \) and \( O_i \) is dominating

(b) \( O_i \) is labeled as "dominated by \( O_{d'} \)" iff \( d' \leq d \) and \( O_{d'} \) is dominating and \( O_i \) is strictly dominated by \( O_{d'} \).

Say that a position \( d \) is used if \( d \) is unresolved when Resolve\((T, d)\) is called. We consider two cases.

Case \( d = 1 \). The position \( d \) is necessarily used (no position is resolved yet), thus \( O_d \) is labeled as "dominating" (step 3) and no other order will have this label during the execution of Resolve\((T, 1)\). Now, \( O_d \) is really dominating, as there is no \( d' < d \), and property (a) is proved. To prove (b), recalling that \( 1 < i < \text{Succ}[d-1] \) and \( d = 1 \), we apply Claim 5. Note that \( i \neq d \) since in step 7 \( d \) is already resolved.

Case \( d > 1 \). Assume by induction the affirmation we want to prove is true before the call of Resolve\((T, d)\). If \( d \) is not used, that means \( d \) is already resolved when Resolve\((T, d)\) is called, and nothing is done. Properties (a) – (b) are already satisfied due to the position \( d' \) such that \( O_{d'} \) dominates \( O_d \).

Assume now that \( d \) is used. Then \( O_d \) is labeled "dominating" and we have to show that \( O_d \) is really dominating. If this was not the case, then \( O_d \) would be strictly dominated by some \( O_{d'} \) with \( d' < d \), and by the inductive hypothesis it would have been labeled as so (property (b) for \( d' \)). But this contradicts the assumption that \( d \) is unresolved at the beginning of Resolve\((T, d)\). We deduce that (a) holds. To prove property (b), notice that it is necessarily true for \( d' < d \) and the corresponding dominated orders, by the inductive hypothesis and since Resolve\((T, d)\) does not relabel any labeled order. To finish the proof of (b), we apply Claim 5. When all the elements in \( T \) are resolved, the algorithm stops and affirmations (a)-(b) for the largest used \( d \) guarantee that the labels are correct.

The memory needed is obviously in \( O(n) \) since \( A_d, O_d, B_d \) obtained in step 2 are only stored during the call of Resolve\((T, d)\). Step 2 needs \( O(|A_d|) \) time by considering all elements \( t_j \) in \( A_d \) from left to
Algorithm 3 RightEnd(d)

1: \( St \leftarrow \emptyset \)
2: \( \text{for } g \leftarrow 2 \text{ to } k_d \text{ do} \)
3: \( \text{if } St \neq \emptyset \text{ then} \)
4: \( (h, \text{succ}) \leftarrow \text{top}(St) \) \hspace{1cm} // do not remove \((h, \text{succ})\) from \( St \)
5: \( \text{while } St \neq \emptyset \text{ and } b_g > \text{succ} \text{ do} \)
6: \( \text{last}[h] \leftarrow g - 1 \)
7: \( \text{pop}(St) \)
8: \( \text{if } St \neq \emptyset \text{ then } (h, \text{succ}) \leftarrow \text{top}(St) \) \hspace{1cm} // do not remove \((h, \text{succ})\) from \( St \)
9: \( \text{end while} \)
10: \( \text{end if} \)
11: \( \text{if } \bigcup_{b_g} \text{ is strictly dominated by } \bigcup_{d} \text{ then} \)
12: \( \text{push } (g, \text{Succ}[b_g - 1]) \text{ on } St \)
13: \( \text{end if} \)
14: \( \text{end for} \)
15: \( \text{while } St \neq \emptyset \text{ do} \)
16: \( (h, \text{succ}) \leftarrow \text{top}(St) \)
17: \( \text{last}[h] \leftarrow k_d \)
18: \( \text{pop}(St) \)
19: \( \text{end while} \)

right, adding their positions in \( B_d \) and using \( \text{Succ} \) to mark (so as to avoid considering them) all the other elements in \( A_d \) with the same value. The marks are discarded when step 2 is finished.

The running time of steps 4-10 is given by the implementation of the abovementioned Union-Find structure, in which the universe is the set of positions from \( A \), and the sets are the slices. As these sets always contain consecutive elements, and set unions are performed between neighboring slices, we are in the particular case of the Union-Find structure proposed in [12]. With this structure, a sequence of (intermixed) \( u \) unions and \( f \) finds on a universe of \( u \) elements is performed in \( O(u + f) \) time. Here, each \( t_j \)-separator in \( A^{sep} \) is considered, and removed, exactly once thus implying one union between slices for each \( j \). Step 7 requires two \( \text{Find} \) calls, for each element in \( B_d \). Overall, the running time is linear in the size of \( A \), which is in \( O(n) \).

Then the running time of \( \text{Resolve}(T, d) \) is in \( O(n) \), for each dominating order \( \bigcup_{d} \). When the algorithm stops, the \( q \) dominating orders and the start positions of the orders they strictly dominate are found in \( O(qn) \) time.

4.2 Find, for each \( i \) such that \( \bigcup_{i} \) is dominated, the rightmost element of \( \bigcup_{i} \)

In the preceding section, we found the dominating orders, which are of the type \( \bigcup_{d} = a_1 a_2 \ldots a_{k_d} \), with \( a_1 = t_d \), for some \( d \), and whose corresponding position sequence is \( B_d = b_1 b_2 \ldots b_{k_d} \) (these are positions from \( T \)). This was done in step 2 of the algorithm \( \text{Resolve}(T, d) \). For each such dominating order, the positions \( i \in B_d \) with \( i > d \) and such that \( \bigcup_{i} \) is strictly dominated by \( \bigcup_{d} \) have been identified in step 8 of \( \text{Resolve}(T, d) \). For each such position \( i \), assuming \( i = b_h \), we must find now the endpoint \( b_{f(h)} \), with \( 1 < h \leq f(h) < k_d \), such that \( B_i = b_h b_{h+1} \ldots b_{f(h)} = B_d[h..f(h)] \). Recall Figure 2.

Consider Algorithm RightEnd(d) in Algorithm [3] where \( St \) is a stack containing pairs of integers of the form \((h, \text{Succ}[b_h - 1])\), where \( \bigcup_{b_h} \) is strictly dominated by \( \bigcup_{d} \) and \( h > 1 \). This algorithm computes, for each such \( b_h \), a value \( \text{last}[h] \).

In order to show that \( B_i \) is indeed equal to \( b_h(= i) b_{h+1} \ldots b_{\text{last}[h]} \) (i.e. \( f(h) = \text{last}[h] \)), we note that \( \text{last}[h] \) has a value only if \( h \) is on \( St \) (steps 6 and 17), that is, only if \( \bigcup_{b_h} \), with \( b_h = i \), is strictly dominated by \( \bigcup_{d} \) (these are the only positions pushed on \( St \), by step 12). Therefore, we already know that \( B_{b_h} \) is a contiguous subsequence of \( B_d \) which starts at \( b_h \) and which finishes at the largest position of \( B_d \) which is smaller than \( \text{Succ}[b_h - 1] \) (by the definition of an order). Then we must show that \( \text{last}[b_h] \)
is this element.

**Example 10.** With \( T = 1 2 5 2 1 4 3 1 2 6 5 \) and \( d = 1 \), we have \( O_d = 1 2 5 4 3 6 \) and \( B_d = 1 2 3 6 7 1 0 \). With \( g = 2 \) and \( g = 3 \) respectively, the pairs (2, 5) and (3, 4) are pushed in this order on \( St \). With \( g = 4 \), we first have \((h, succ) = (3, 4)\) and \( 4 < b_3\) (which is 6), thus \( last[3] \leftarrow 3\). Then (3, 4) is discarded from \( St \) and we have \((h, succ) = (2, 5)\) and we deduce \( last[2] \leftarrow 3\). The stack \( St \) is now empty, and \((4, 8)\) is pushed on it. With \( g = 5 \) nothing happens, whereas with \( g = 6 \) the algorithm produces \( last[4] \leftarrow 5\). The stack is empty and RightEnd(1) is finished.

We first prove that:

**Claim 7.** Let \( b_h \) and \( b_g \), with \( 1 < h < g \leq k_d \), be two positions such that \( O_{b_h} \) and \( O_{b_g} \) are dominated by \( O_d \). Then exactly one of the following statements holds:

(a) \( O_{b_h} \) dominates \( O_{b_g} \)

(b) \( A_{b_h} \) and \( A_{b_g} \) are disjoint subsequences of \( T \) such that \( b_g > Succ[b_h - 1] \). Consequently, \( O_{b_h} \) and \( O_{b_g} \) are disjoint subsequences of \( O_d \).

**Proof.** Obviously, the two affirmations cannot hold simultaneously. Now, assume by contradiction that none of them is true. Then \( b_g \leq Succ[b_h - 1] \). The value \( t_{b_h - 1} \) will then occur in \( O_d \) before \( b_h \) (since \( d < b_h \)), and will also occur in \( O_{b_g} \), since \( T[Succ[b_h - 1]] = t_{b_h - 1} \). But then one cannot have \( Set(T[d..b_g - 1]) \cap A_{b_g} = \emptyset \) and Claim 4 is contradicted.

**Claim 8.** In Algorithm RightEnd(d), a pair \((g, Succ[b_g - 1])\) is on \( St \) above \((h, Succ[b_h - 1])\) iff \( h < g \) and \( Succ[b_g - 1] \leq Succ[b_h - 1] \).

**Proof.** We show this is true in the case where \((b_h, Succ[b_h - 1])\) is on the top of \( St \) when \((g, Succ[b_g - 1])\) is pushed on \( St \). The claim thus follows by induction.

Assume then that \((b_h, Succ[b_h - 1])\) is on the top of \( St \). The pair \((g, Succ[b_g - 1])\) is pushed on \( St \) iff \( O_{b_g} \) is strictly dominated by \( O_d \) (steps 11-12), and \( b_g \leq Succ[b_h - 1] \) (step 5). Now, since \( b_h < b_g \) according to step 2, we first deduce that property (b) in Claim 7 does not hold. Then property (a) in Claim 7 must be true. The claim follows by Claim 4.

Now, for each position \( b_h \) such that \( O_{b_h} \) is strictly dominated by \( O_d \), recall that \( f(h) \) is such that \( B_{b_h} = bh \ldots f(h) \). We prove that:

**Claim 9.** At the end of Algorithm RightEnd(d), we have \( last[h] = f(h) \) for each \( h > d \) such that \( O_{b_h} \) is strictly dominated by \( O_d \). To achieve this, the algorithm takes \( O(p) \) time and space.

**Proof.** It is easy to observe that we have \( b_f(h) < Succ[b_h - 1] < b_f(h) + 1 \) (when \( b_f(h) + 1 \) exists), by the definition of an order and since \( B_{b_h} = B_d[h..f(h)] \).

Let us follow the steps of Algorithm RightEnd(d). When \( g = h \) in step 2, \((h, Succ[b_h - 1])\) is pushed on \( St \) in step 12. Since all the values \( b_{h+1}, \ldots, b_{f(h)} \) are smaller than \( Succ[b_h - 1] \), by the definition of an order, when \( g \) considers each of these values (step 2) the condition in step 5 is never fulfilled with \( Succ = Succ[b_h - 1] \). Then \((h, Succ[b_h - 1])\) is still on the stack at the end of the execution of the for loop in step 2 for \( g = f(h) \). Then we have two cases:

- Either \( g \leftarrow f(h) + 1 \) is possible in step 2 (i.e. \( f(h) < k_d \)) and we deduce that \( b_g = b_f(h) + 1 \) \( Succ[b_h - 1] \) by the observation above, and because of Claim 8, we also have \( b_f(h) + 1 > Succ[b_g - 1] \) for all pairs \((g', Succ[g' - 1])\) which are before \((h, Succ[b_h - 1])\) on \( St \). The while loop in steps 5-9 for \( g = f(h) + 1 \) will thus discard all these pairs, including \((h, Succ[b_h - 1])\). In step 6, we will then have \( last[h] \leftarrow f(h) + 1 - 1 \), i.e. \( last[h] \leftarrow f(h) \).

- Or \( f(h) = k_d \), and then in step 17 we have \( last[h] \leftarrow k_d(= f(h)) \).
and allows to easily deduce useless, and this is done by verifying, for each position in $O$, the positions in $W$ stores the positions in which is initially built such that immediately following $O$ precisely, we show that the dominating orders $O$ of $T$, $D$ computes for each $O$ position $O$ of $T_D$ that belongs to at least one order remaining to be generated. Thus, the overall execution of the algorithm, for a fixed $d$, is in $O(k_d)$, and thus in $O(p)$. The space requirements are obviously in $O(p)$, assuming the vector $Succ$ is computed only once at the beginning of the main algorithm. 

**Theorem 3.** Given a sequence $T$ of length $n$ over an alphabet $\Sigma$, there is an algorithm running in $O(qn)$ time and using $O(n)$ space to compute, without storing, the $q$ dominating orders of $T$ and, for each such order $O_d$, its dominating function $F_d$.

**Proof.** By Claim 6, Algorithm Resolve($T, d$) successively called for $d = 1, 2, \ldots, n$ finds in $O(qn)$ time and $O(n)$ space the dominating orders $O_d$, and, for each order strictly dominated by $O_d$, the position $i$ on $T$ of its first element. Moreover, assuming that $B_d = b_1 \ldots b_{k_d}$, Algorithm RightEnd($d$) computes for each $i$ such that $d$ the endpoint $f(h)$ with $1 < h < f(h) \leq k_d$ such that $B_i = B_d[h..f(h)]$. But then the pairs $(h, f(h))$, to which we must add the trivial pair $(1, k_d)$, are exactly the pairs $(s, F_d(s))$ in the definition of the dominating function $F_d$ of $O_d$:

$$F_d(s) := f \text{ if there is some } i \text{ such that } O_i \text{ is strictly dominated by } O_d \text{ and } B_d[s..f] = B_i.$$ 

The running time of Resolve($T, d$) for all values of $d$ is in $O(qn)$, when the running time of the RightEnd($d$) call is left apart, as already proved in Claim 6. For all $q$ dominating orders, RightEnd() takes $O(qp)$ time by Claim 9 and this does not change the overall running time.

## 5 Retrieving the dominating orders of $T$

Once the step 1 in Algorithm 1 is performed, the list $D_T$ of positions $d$ such that $O_d$ is a dominating order of $T$ is available. We assume it is a stack in which the positions are ordered in decreasing order from top to bottom. Moreover, we assume a vector $Prec$ has been built for $T$ (augmented with character $X$ on both its endpoints) and is available, defined similarly to the vector $Succ$. Vector $Prec$ is the $(n+1)$-size array defined for each $i$ with $1 \leq i \leq n+1$ by $Prec[i] = j$ if $t_i = t_j$ and $j < i$ is the largest with this property (if $j$ does not exist, then $Prec[i] = 0$). As was the case for $Succ$, the array $Prec$ can be built in $O(n)$ time.

Then we may retrieve all the dominating orders in global time of $O(qp)$ and using $O(n)$ space. More precisely, we show that the dominating orders $O_d$, for all $d$, may be found in decreasing order of $d$ in $O(qp)$ time. Note that, according to Algorithm 1, we do not need to store all the orders, but only to progressively generate and use them. Once used, each order is discarded.

Consider Algorithm Retrieve($T, D_T, Prec$) in Algorithm 4. The algorithm works on a sequence $W$ which is initially built such that $W[i] = i$ and which reduces as the algorithm progresses. Intuitively, $W$ stores the positions in $T$ of the elements of $T$ that belong to at least one order remaining to be generated. Starting at the end of $W$, the algorithm finds in position $d$ of $W$ the value $d$ that is on the top of $D_T$, i.e. the maximum position of a dominating order. Then the part of $W$ between $d$ (which is the first element of $O_d$) and a previously computed value $e$ (which turns out to be the last value in $B_d$) is $B_d$ and allows to easily deduce $O_d$. Finally, it remains to remove from $W$ the positions in $T$ which became useless, and this is done by verifying, for each position in $B_d$, if it is used or not by the dominating order immediately following $O_d$ on its left, that is, $O_{d'}$ where $d'$ is the new top of $D_T$.

**Example 11.** With $T = 12521431265$ and $D_T = \{7, 4, 1\}$ (7 on top, 1 on bottom), we have $W = Id_{11}$ and in step 3 of Algorithm 4 we have $d \leftarrow 7$ and $D_T \leftarrow \{4, 1\}$, thus $B_2 \leftarrow W[7..11] = 7891011$ resulting into $O_2 = 31265$, which is indeed $O_7$. In steps 9 and 11, we mark 11 and respectively 8,9 for removal. Values 8,9 cannot belong to another order, whereas 11 marks the end of the area $A_4$ for
Algorithm 4 Retrieve($T$, $D_T$, $Prec$)

1: $w \leftarrow ||T||; \; W \leftarrow 12 \ldots w; \; e \leftarrow w$
2: while $D_T$ is not empty do
3: $d \leftarrow \text{top}(D_T); \; \text{pop}(D_T)$
4: $B_d \leftarrow W[d..e]; \; k_d \leftarrow e - d + 1; \; O_d \leftarrow t_{B_d[1]} t_{B_d[2]} \ldots t_{B_d[k_d]}
\text{//here, }O_d \text{ and } B_d \text{ may be used, and then discarded}$
5: if $D_T$ is not empty then
6: $j \leftarrow d; \; d' \leftarrow \text{top}(D_T)$
7: while $j \leq w$ do
8: if $Prec[W[j]] = d' - 1$ and $d' \neq 1$ then
9: mark $j$ for removal from $W; \; j \leftarrow w + 1$
10: else
11: if $Prec[W[j]] \geq d'$ then mark $j$ for removal from $W$ else $e \leftarrow j$ endif
12: $j \leftarrow j + 1$
13: end if
14: end while
15: remove all marked positions from $W; \; \text{update } e \text{ to point to the same value in } W$
16: $w \leftarrow ||W||$
17: end if
18: end while

d the top $d' = 4$ of the stack $D_T$. Therefore, $W[e]$ points to the position in $T$ of the last validated element of $A_4$. This position is 10. After the removals, $W$ becomes $W = 123456710$ and $e$ becomes 8 (so as $W[e] = 10$ as before the removals). During the new execution of the while loop, with $d = 4$, we have $B_4 = W[4..8] = 456710$ resulting into $O_4 = 21436$, which is $O_4$. And so on.

The proof makes use of the following easy result:

Claim 10. Let $i, d, d'$ be positions of $T$ such that $d' < d < i$ and $i \in \text{Set}(B_d)$. If $i \notin \text{Set}(B_{d'})$ then at least one of the following affirmations holds:
(a) $i > \text{Succ}[d' - 1]$
(b) for all $d'' \leq d'$, $i \notin \text{Set}(B_{d''})$.

Proof. Assume affirmation (a) does not hold. Then $i \notin \text{Set}(B_d)$ implies that either there is $k$ with $d' \leq k < d < i$ such that $t_k = t_i$, or $i = \text{Succ}[d' - 1]$, which may be written $t_k = t_i$ with $k = d' - 1$. Then, for all $d'' \leq d'$ we have $d'' \leq k$ and thus $i$ cannot belong to $B_{d''}$ since $t_i$ is preceded by $t_k$. ■

Claim 11. After the execution of the while loop in step 2 of Algorithm Retrieve($T$, $D_T$, $Prec$) for $d \in D_T$ we have:
(a) the values $B_d$, $O_d$ and $k_d$ are equal respectively to $B_d$, $O_d$ and $k_d$
(b) $B_{d'} = t_{W[e]} t_{W[e]+1} \ldots t_{W[e]}$
(c) $\bigcup_{d' \in D_T \text{ and } d' \leq d} \text{Set}(B_{d'}) \subseteq \text{Set}(W)$.

Proof. We use induction on the execution number $\alpha$ of the while loop.

Case $\alpha = 1$. Then $d$ is the first element in $D_T$, and thus $O_d$ is the dominating order of $T$ with largest $d$. Moreover, in $T[d..w]$ one cannot have $t_l = t_r$ for two positions $l, r$ (assume $l < r$) since otherwise $O_r$ would be dominated neither by $O_d$ nor by an order preceding $O_d$ on its left, implying that $d$ is not the top of $D_T$, a contradiction. Consequently, $O_d = T[d..w]$ and thus affirmation (a) holds, since $e = w$ by step 1. To see that affirmations (b) - (c) hold, notice that steps 8-9 identify and prepare for removal from $W$ the position $\text{Succ}[d' - 1]$ which marks the end of $A_{d'}$ in $T$, whereas step 11 prepares for removal from $W$ the positions $W[j]$ of $T$ which have a copy on their left in $A_{d'}$. Affirmation (b) follows by observing that $e$ keeps trace of the last element which has no copy on its left in $A_{d'}$. Also, by property (b) in Claim 10 we deduce affirmation (c).
Case $\alpha > 1$. Given that the new value of $d$ is the value of $d'$ during the $(\alpha - 1)$-th execution of the while loop, the induction hypothesis (affirmation (b)) directly implies affirmation (a). To prove affirmation (b), notice that the while loop in steps 7-14 considers all elements in $W$ between $d$ and the minimum between $\text{Succ}[d' - 1]$ and $w$ (the end of $W$). By the induction hypothesis (affirmation (c)) all positions in $B_d$ are in $W[d..w]$. As before, steps 8-9 and respectively 11 insure, also using Claim that the elements in $B_d$ are correctly selected (thus affirmation (b) holds) and that the removed elements are now useless (thus affirmation (c) holds). □

Theorem 4. Algorithm Retrieve($T, D_T, \text{Prec}$) successively computes, without storing, the dominating orders of $T$ in $O(pq)$ time and $O(p)$ space.

Proof. Affirmation (a) in Claim guarantees the correctness of the algorithm. The running time of the algorithm assumes $\text{Prec}$ has been computed once at the beginning of the main algorithm. Then the running time is given by the number of executions of the while loop in steps 7-14 and by step 15. For each $d$, the while loop in steps 7-14 considers all elements in $W$ between $d$ and $\min\{\text{Succ}[d' - 1], w\}$. Overall, each position $j$ in $T$ is considered at most once for each $d'$ such that $j \in \text{Set}(B_d)$ (step 11) and exactly once for the largest $d'$ such that $j \notin B_d$ (step 9 or 11). Thus the overall running time of the algorithm is in $O(pq)$ time, since the size of an order is at most $p$ and $pq \ge n$. Obviously, the only extra-space used is that for temporarily storing $B_d$ and $O_d$ for a fixed $d$. To finish the proof, note that step 15 may be performed in time proportional to the number of elements removed from $W$ by representing $W$ as a double chained list. Two pointers are then sufficient. One of them starts in $w$ (step 1) and goes back up to the element containing $d$, which is also the $d$-th element of $W$. The other one is $j$, which goes from $d$ to $w$ in steps 4 and 7, and also allows to remove the marked elements in step 15. □

Remark 2. Obviously, performing steps 3, 4 and 5 in the main algorithm (Algorithm requires to call Retrieve($T, D_T, \text{Prec}$), to insert a call of Retrieve($S, D_S, \text{Prec}_S$) between steps 4 and 5 of Retrieve($T, D_T, \text{Prec}$), and a call for the step 5 in the main algorithm between steps 4 and 5 of Retrieve($S, D_S, \text{Prec}_S$). Once a call from Retrieve() is finished, the information computed in step 4 of Retrieve() is discarded.

6 Finding the common intervals of $O_d$ and $\Omega_\delta$

Note here that $O_d$ and $\Omega_\delta$ are, by construction, permutations on no more than $p$ elements. Without loss of generality, we assume here that they are permutations on $\Sigma$ (otherwise the same algorithm must be applied on two permutations obtained from the initial ones by adding the missing elements at the end of each permutation, but without modifying the functions $F_d, \Phi$).

Consequently, make the following changes of notation. Renumber the elements of $O_d$ such that $O_d$ becomes the identity permutation $Id_p$ (simplified hereafter as $Id$), and renumber the elements of $\Omega_\delta$ accordingly so as to obtain a permutation $\pi$. Call $F$ the dominating function of $Id$, and $\Phi$ that of $\pi$.

Remark 3. Note that the common intervals of $Id$ and $\pi$ that are valid with respect to $F$ and $\Phi$ are exactly the common intervals $(s..u)$ of $Id$ and $\pi$ with location $[y, z]$ on $\pi$ such that $F(s) \le F(z)$ and $y \le \Phi(y)$.

Also notice that by Claim we have:

Remark 4. Assume the dominating function $F$ of the dominating order $O_d$ is defined for two values $s_0, s_1$ with $s_0 < s_1$. Let $f_0 = F_d(s_0)$ and $f_1 = F_d(s_1)$. Then the orders $O_d[s_0..f_0]$ and $O_d[s_1..f_1]$ either dominate each other or are disjoint subsequences of $O_d$. As a consequence, the similar affirmation holds for each of $F$ and $\Phi$.

In [10], the LR-Search algorithm in Algorithm is proposed for finding the common intervals of $K$ permutations, for an arbitrary $K \ge 2$. In order to show that this algorithm may be easily adapted to find common intervals of $Id$ and $\pi$ that are valid with respect to $F$ and $\Phi$, we need to present the details of the algorithm. Note that the algorithm looks for common intervals of size of least 2 (those of size 1 are easy to obtain).
Definition 7. Let $P$ be a set of permutations over $\Sigma$. Then the MinMax-profile of $P$ with respect to $l$ and $r$ is the set of pairs $[l_s, r_s], s \in \{1, 2, \ldots, p-1\}$. The MinMax-profile of $P$ is the information needed by the LR-Search algorithm (see Algorithm 5) to compute the common intervals $(a..c)$ of $P$ containing both $l_s$ and $r_s$ for all $s \in \{a; a+1; \ldots; c-1\}$.

Remark 5. We assume that each of the stacks $L, R$ admits the classical operations pop, push, and that their elements may be read without removing them. In particular, the function top() returns the first element of the stack, without removing it, and the function next() returns the element immediately following $u$ on the stack containing $u$, if such an element exists.

6.1 The LR-Search algorithm

The presentation in this section follows very closely that in [16].

Let $P \equiv \{P_1, P_2, \ldots, P_K\}$ be a set of $K$ permutations over $\Sigma = \{1, 2, \ldots, p\}$ such that $P_1 = \text{Id}$. Now, let $m^k_s$ (respectively $M^k_s$) be the minimum (respectively maximum) value in the interval of $P_k$ delimited by $s$ and $s+1$ (both included). Also define

$$m_s := \min\{m^k_s \mid 2 \leq k \leq K\}, M_s := \max\{M^k_s \mid 2 \leq k \leq K\}.$$  

Note that, for each $k \in \{1, 2, \ldots, K\}, m_s \leq m^k_s \leq s < s+1 \leq M^k_s \leq M_s$.

We call bounding functions $l, r : \Sigma \to \Sigma$ any two functions such that $l(s) \leq m_s$ and $r(s) \geq M_s$, for all $s \in \{1, 2, \ldots, p-1\}$. We denote $l_s := l(s)$ and $r_s := r(s)$.

Definition 8. An LR-stack for an ordered set $\Sigma$ is a 5-tuple $(L, R, SL, SR, R^\top)$ such that:

- $L, R$ are stacks, each of them containing distinct elements from $\Sigma$ in either increasing or decreasing order (from top to bottom). The first element of a stack is its top, the last one is its bottom.
- $SL, SR \subseteq \Sigma$ respectively represent the set of elements on $L$ and $R$.
- $R^\top : SL \to SR$ is an injective function that associates with each $a$ from $SL$ a pointer to an element on $R$ such that $R^\top(a)$ is before $R^\top(a')$ on $R$ if $a$ is before $a'$ on $L$.

According to the increasing (notation $+$) or decreasing (notation $-$) order of the elements on $L$ and $R$ from top to bottom, an LR-stack may be of one of the four types $L^+R^+, L^-R^-, L^+R^-, L^-R^+$. 

Algorithm 5 The LR-Search algorithm

Input: Set $P$ of $K$ permutations over $\Sigma$, bounding functions $l$ and $r$, Filter procedure

Output: All common intervals $(s..u)$ of $P$ with $u \in \text{Set}_R(s)$, filtered by Filter

1: Compute $m_s^k, m_s$ and $l_s$ with $2 \leq k \leq K$ and $s \in \{1, 2, \ldots, p\}$
2: Compute $M_s^k, M_s$ and $r_s$ with $2 \leq k \leq K$ and $s \in \{1, 2, \ldots, p\}$
3: Initialize an $L^-R^+$-stack with empty stacks $L, R$
4: for $s \leftarrow p - 1$ to 1 do
5: POP$_L(l_s)$ // discard from $L$ all candidates larger than $l_s$ and push $l_s$ instead
6: POP$_R(r_s)$ // discard from $R$ all candidates smaller than $r_s$
7: if $r_s = s + 1$ then
8: Push$_{LR}(l_s, s + 1)$ // $s + 1$ is a new right candidate, suitable for each $s$ on $L$
9: end if
10: Call Filter to choose a subset of intervals $(s..u)$ with $u \in \text{Set}_R(s)$
11: end for

The presentation in this section follows very closely that in [16].

Let $P = \{P_1, P_2, \ldots, P_K\}$ be a set of $K$ permutations over $\Sigma = \{1, 2, \ldots, p\}$ such that $P_1 = \text{Id}$. Now, let $m^k_s$ (respectively $M^k_s$) be the minimum (respectively maximum) value in the interval of $P_k$ delimited by $s$ and $s+1$ (both included). Also define

$$m_s := \min\{m^k_s \mid 2 \leq k \leq K\}, M_s := \max\{M^k_s \mid 2 \leq k \leq K\}.$$  

Note that, for each $k \in \{1, 2, \ldots, K\}, m_s \leq m^k_s \leq s < s+1 \leq M^k_s \leq M_s$.

We call bounding functions $l, r : \Sigma \to \Sigma$ any two functions such that $l(s) \leq m_s$ and $r(s) \geq M_s$, for all $s \in \{1, 2, \ldots, p-1\}$. We denote $l_s := l(s)$ and $r_s := r(s)$.

Definition 7. Let $P$ be a set of permutations over $\Sigma$. Then the MinMax-profile of $P$ with respect to $l$ and $r$ is the set of pairs $[l_s, r_s], s \in \{1, 2, \ldots, p-1\}$.

The MinMax-profile of $P$ is the information needed by the LR-Search algorithm (see Algorithm 5) to compute the common intervals $(a..c)$ of $P$ containing both $l_s$ and $r_s$ for all $s \in \{a; a+1; \ldots; c-1\}$. In this way, the functions $l$ and $r$ allow a first selection among all common intervals in $P$. The Filter procedure in the input, used in step 10, completes the selection tools of the algorithm. During the computation, the interval candidates are stored in an abstract data structure called an LR-Stack.

Definition 8. An LR-stack for an ordered set $\Sigma$ is a 5-tuple $(L, R, SL, SR, R^\top)$ such that:

- $L, R$ are stacks, each of them containing distinct elements from $\Sigma$ in either increasing or decreasing order (from top to bottom). The first element of a stack is its top, the last one is its bottom.
- $SL, SR \subseteq \Sigma$ respectively represent the set of elements on $L$ and $R$.
- $R^\top : SL \to SR$ is an injective function that associates with each $a$ from $SL$ a pointer to an element on $R$ such that $R^\top(a)$ is before $R^\top(a')$ on $R$ if $a$ is before $a'$ on $L$.

According to the increasing (notation $+$) or decreasing (notation $-$) order of the elements on $L$ and $R$ from top to bottom, an LR-stack may be of one of the four types $L^+R^+, L^-R^-, L^+R^-, L^-R^+$. 

Remark 5. We assume that each of the stacks $L, R$ admits the classical operations pop, push, and that their elements may be read without removing them. In particular, the function top() returns the first element of the stack, without removing it, and the function next() returns the element immediately following $u$ on the stack containing $u$, if such an element exists.
We further denote, for each \( a \in SL \) and with \( a' = next(a) \), assuming that \( next(a) \) exists:

\[
Set_R(a) = \{ c \in SR \mid c \text{ is located on } R \text{ between } R^T(a) \text{ included and } R^T(a') \text{ excluded} \}
\]

When \( next(a) \) does not exist, \( Set_R(a) \) contains all elements between \( R^T(a) \) included and the bottom of \( R \) included. Then \( R^T(a) \) is the first (i.e. closest to the top) element of \( Set_R(a) \) on \( R \).

We define the following operations on the LR-stack. Note that they do not affect the properties of an LR-stack. Sets \( Set_R() \) are assumed to be updated without further specification whenever the pointers \( R^T() \) change. Say that \( a' \) is \( L \)-blocking for \( a \), with \( a' \neq a \), if \( a \) cannot be pushed on \( L \) when \( a' \) is already on \( L \) (because of the increasing/decreasing order of elements on \( L \)), and similarly for \( R \).

- **PopL(a)**, for some \( a \in \Sigma \): pop successively from \( L \) all elements that are \( L \)-blocking for \( a \), push \( a \) on \( L \) iff at least one \( L \)-blocking element has been found and \( a \) is not already on \( L \), and define \( R^T(top(L)) \) as \( top(R) \). At the end, either \( a \) is not on \( L \) and no \( L \)-blocking element exists for \( a \), or \( a \) is on the top of \( L \) and \( R^T(a) \) is a pointer to the top of \( R \).

- **PopR(c)**, for some \( c \in \Sigma \): pop successively from \( R \) all elements that are \( R \)-blocking for \( c \), update all pointers \( R^T() \) (here, \( R^T(a) = nil \) is accepted temporarily if \( Set_R(a) = \emptyset \)) and successively pop from \( L \) all the elements \( a \) with \( R^T(a) = nil \). At the end, either \( c \) is not on \( R \) and no \( R \)-blocking element exists for \( c \), or \( c \) is on the top of \( R \).

- **PushL(a, c)**, for some \( a, c \in \Sigma \) (performed when no \( L \)-blocking element exists for \( a \) and no \( R \)-blocking element exists for \( c \)): push \( a \) on \( L \) iff \( a \) is not already on the top of \( L \), push \( c \) on \( R \) iff \( c \) is not already on the top of \( R \), and let \( R^T(top(L)) \) be defined as \( top(R) \).

- **FindL(c)**, for some \( c \in SR \): return the element \( a \) of \( SL \) such that \( c \in Set_R(a) \). (Note that this operation is not explicitly used in the LR-Search algorithm, but may be used in a separate algorithm to solve its steps 1 and 2.)

**Remark 6.** Note that operations \( PopL(a) \) and \( PopR(c) \) perform similar but not identical modifications on stacks \( L \) and \( R \) respectively. Indeed, \( PopL(a) \) pushes \( a \) on \( L \) if at least one element of \( L \) has been discarded and \( a \) is not already on \( L \), whereas \( PopR \) discards elements, but never pushes \( c \) on \( R \).

Algorithm LR-Search (see Algorithm 5) works intuitively as follows. For each pair \((s, s+1)\), the pair \([l_s, r_s]\) of bounding values means that each common interval \((a..c)\) containing \( s \) and \( s+1 \) must satisfy the bounding condition \( a \leq l_s < r_s \leq c \). The LR-stack, initially empty, stores on \( L \) (respectively on \( R \)) the candidates for the left endpoint \( a \) (respectively right endpoint \( c \)) of a common interval \((a..c)\), in such a way that, after the execution of the for loop (step 4) for a value \( s \) with \( a \leq s \leq c - 1 \), we have \( c \in Set_R(a) \) iff \((a..c)\) satisfies all the bounding conditions previously imposed with \( s' \) such that \( s \leq s' \leq c - 1 \). It is obvious (and understood) that if \((a..c)\) satisfies those conditions, all intervals \((a'..c)\) with \( a' < a \) do. When the execution of the for loop considers \( s = a \), the bounding conditions are satisfied for all \( s' \) with \( a \leq s' \leq c - 1 \).

More precisely, we have the following theorem. Denote by \( Set^*_R(a) \) the value of \( Set_R(a) \) at the end of step 9 in the execution of the for loop for \( s \), for each \( a \) on \( L \).

**Theorem 5.** (16) Assuming the Filter procedure does not change the state of the LR-stack, the set \( Z \) defined as

\[
Z : = \bigcup_{1 \leq s \leq p} \{ (s..u) \mid u \in Set^*_R(s) \}
\]

computed by LR-Search is the set of all common intervals \((s..u)\) of \( P \) satisfying

\[
s = l_s = \min \{ l_w \mid s \leq w \leq u - 1 \} \tag{1}
\]

\[
u = r_{u-1} = \max \{ r_w \mid s \leq w \leq u - 1 \} \tag{2}
\]
6.2 Setting the parameters

With the aim of computing the common intervals of \( Id \) and \( \pi \) that are valid with respect to \( F \) and \( \Phi \), we set the parameters \( l \) and \( r \) as follows.

For each \( s \in \{1, 2, \ldots, p - 1\} \), let \( v_s \) be as large as possible such that \( \Phi(v_s) \) is defined and

\[
\pi[v_s, \Phi(v_s)] \text{ contains both } s, s + 1
\]

Exactly one pair satisfies this condition, since \( \Phi(1) = p \) and \( \Phi \) is a (partial) function. Now, let

\[
x_s = \max\{\pi^{-1}(s), \pi^{-1}(s + 1)\}
\]

i.e. \( x_s \) is the position of the rightmost element between \( s \) and \( s + 1 \) on \( \pi \). Note that \( x_s \leq \Phi(v_s) \), and let for all \( s \in \{1, 2, \ldots, p - 1\} \):

\[
l_s = \min \{\pi[v_s, x_s]\} \quad r_s = \max \{\pi[v_s, x_s]\},
\]

where \( \min \{\pi[v_s, x_s]\} \) (respectively \( \max \{\pi[v_s, x_s]\} \)) denotes the minimum (respectively maximum) value in \( \pi[v_s, x_s]\).

**Claim 12.** Assuming the Filter procedure does not change the state of the LR-stack, the set \( Z \) defined as

\[
Z = \cup_{1 \leq s < p} \{ (s, u) \mid u \in \text{Set}_r(s) \}
\]

computed by LR-Search with the settings \( l, r \) above is the set of common intervals of \( Id \) and \( \pi \) that are valid with respect to \( \Phi \).

**Proof.** By Theorem 5 we have to show that the set \( C \) of common intervals satisfying conditions (1) and (2) is exactly the set \( V \) of common intervals that are valid with respect to \( \Phi \).

\( C \subseteq V \): Let \( (s, u) \in C \), and let \( [y, z] \) be the location of \( (s, u) \) in \( \pi \). Then by (1) and (2):

\[
s = l_s = \min\{l_w \mid s \leq w \leq u - 1\} = \min\{\min \{\pi[v_w, x_w]\} \mid s \leq w \leq u - 1\},
\]

\[
u = u_{u-1} = \max\{u_w \mid s \leq w \leq u - 1\} = \max\{\max \{\pi[v_w, x_w]\} \mid s \leq w \leq u - 1\}.\]

We first show that \( y = \min\{v_w \mid s \leq w \leq u - 1\} \). Assume a contrario that this is not the case. By equations (5) and (6) we deduce that \( \pi[v_w] \in (s, u) \), for all \( w \) with \( s \leq w \leq u - 1 \). Thus \( y < \min\{v_w \mid s \leq w \leq u - 1\} \), since \( y \) is the left endpoint of the location of \( (s, u) \) on \( \pi \). Now, with \( \pi[y] \in (s, u) \), we deduce that:

- either \( \pi[y] < u \) and thus \( \pi[y] \) is one of the values \( w \) with \( s \leq w \leq u - 1 \). By equations (3) and (4), \( \pi[y] \) belongs to \( \pi[v_{\pi[y]}..x_{\pi[y]}] \), therefore \( y \geq v_{\pi[y]} \), a contradiction.

- or \( \pi[y] = u \) and thus \( \pi[y] - 1 \) is one of the values \( w \) with \( s \leq w \leq u - 1 \). By equations (3) and (4), \( \pi[y] \) belongs to \( \pi[v_{\pi[y]-1..x_{\pi[y]-1]}] \), therefore \( y \geq v_{\pi[y]-1} \), a contradiction.

Thus \( y = \min\{v_w \mid s \leq w \leq u - 1\} \), i.e. \( y = v_{f} \) with \( s \leq f \leq u - 1 \) and thus \( \Phi(y) \) is defined. In a similar way, we are able to show that \( z = \max\{x_w \mid s \leq w \leq u - 1\} \), i.e. \( z = x_{g} \) with \( s \leq g \leq u - 1 \) and thus there exists \( v_{g} \) such that \( v_{g} \leq x_{g} \leq \Phi(v_{g}) \).

In the case where \( \Phi(y) \geq z \), we have that \( [y, z] \) is valid with respect to \( \Phi \) and the proof is finished. Let us show that one cannot have \( \Phi(y) < z \). Indeed, if this is true, let \( w \) with \( s \leq w \leq u - 1 \) be such that one of \( w, w + 1 \) is in \( \pi[y, \Phi(y)] \), and the other one is in \( \pi[\Phi(y) + 1..z] \). Such a value \( w \) must exist, since \( f, f + 1 \) are in \( \pi[y, \Phi(y)] \), whereas at least one of \( g, g + 1 \) is in \( \pi[\Phi(y) + 1..z] \) by the definition of \( x_{g} = \pi \). Thus at least one integer \( w \) with \( \min(f, g) \leq w \leq \max(f, g) \) satisfies the required condition. But then \( v_{w} \leq \Phi(y) \) and \( x_{w} > \Phi(y) \), implying that \( \Phi(v_{w}) \geq x_{w} > \Phi(y) \). But by Remark 3, the orders \( \pi[y, \Phi(y)] \) and \( \pi[v_{w}, \Phi(v_{w})] \) should dominate each other, and this is impossible, since \( v_{w} > v_{f} \) (the
equality is forbidden by the definition of a function since \( \Phi(v_w) \neq \Phi(v_f) \), and \( v_w < v_f \) means that \( v_w < y \) and this is impossible, since we proved \( y = \min \{ v_w \mid s \leq w \leq u - 1 \} \).

"\( V \subseteq C " \): Let \( (s..u) \) be a common interval of \( \text{Id} \) and \( \pi \) which is valid with respect to \( \Phi \). Then the location \( [y..z) \) of \( (s..u) \) on \( \pi \) satisfies \( z \leq \Phi(y) \). We have to show that equations (1) and (2) hold.

For each \( w \) with \( s \leq w \leq u - 1 \), the interval \( \pi[v_w..x_w] \) is the minimum interval which is valid with respect to \( \Phi \) that contains both \( w \) and \( w + 1 \). Since \( \pi[y..z] \) also contains \( w, w + 1 \), we deduce from (4) that \( y \leq v_w \leq x_w \leq z \), implying that

\[
\min \pi[y..z] \leq \min \{ \pi[v_w..x_w] \mid s \leq w \leq u - 1 \} = \min \{ l_w \mid s \leq w \leq u - 1 \}
\]

(7)

Now, over all \( w \) we deduce:

\[
\min \pi[y..z] \leq \min \{ \min \pi[v_w..x_w] \mid s \leq w \leq u - 1 \} = \min \{ l_w \mid s \leq w \leq u - 1 \}
\]

(8)

\[
\max \pi[y..z] \geq \max \{ \max \pi[v_w..x_w] \mid s \leq w \leq u - 1 \} = \max \{ r_w \mid s \leq w \leq u - 1 \}
\]

(9)

Furthermore, we have \( \min \pi[y..z] = s \) and \( \max \pi[y..z] = u \) since \( [y..z] \) is the location of \( (u..s) \) on \( \pi \). Moreover, we have \( s = l_s \), otherwise by the definition of \( l_s \), we only have the possibility \( l_s < s \), contradicting equation (8). Similarly, \( u = r_u \) and the proof is finished.

Consider now the filtering procedure in Algorithm 6, which chooses among all common intervals produced by LR-Search, and which are valid w.r.t. \( \Phi \), those that are also valid w.r.t. \( F \).

Theorem 6. Algorithm LR-Search with settings \( l_s, r_s \) and Filter in Algorithm 6 outputs the common intervals of \( \text{Id} \) and \( \pi \) which are valid w.r.t. \( F \) and \( \Phi \). The algorithm runs in \( O(p + N_{\text{Id},\pi}) \) time, where \( N_{\text{Id},\pi} \) is the number of such common intervals, and uses \( O(p) \) space.

Proof. Claim 12 guarantees that the intervals \( (s..u) \) with \( u \in \text{Set}_R(s) \) are exactly the common intervals of \( \text{Id} \) and \( \pi \) which are valid with respect to \( \Phi \). Recall that \( \text{Set}_R(s) \) is \( \text{Set}_R(s) \) at the end of step 9 during the execution of the for loop for \( s \) in the LR-Search algorithm, that is, exactly the set \( \text{Set}_R(s) \) considered by Filter. Moreover, it is easy to see that the Filter procedure selects between these intervals (s..u) those for which \( F(s) \) is defined (step 1) and \( u \leq F(s) \) (step 4). Only these intervals are output, so that the correctness of the algorithm is proved.

The running time and memory space requirements are both in \( O(p) \), when Filter is left apart and \( l_s, r_s \) are supposed already computed, as proved in 16 (case of \( K = 2 \) permutations). This is done by implementing the two stacks \( L \) and \( R \) of the LR-stack as lists, so as to ensure that \( \text{Pop}_L, \text{Pop}_R \) are performed in linear time with respect to the number of elements removed from \( L \) and \( R \) respectively. Furthermore, the running time of Filter is proportional with the number of output intervals, and there are no supplementary space requirements. Note that \( R^+, R^- \) are easily computed when \( \text{Pop}_L, \text{Pop}_R \) and \( \text{Push}_{L,R} \) are performed.
It remains to show that $l_s$ and $r_s$ may be computed in global $O(p)$ time and space, for all values of $s$. To this end, algorithm $\text{ComputeV}(\pi)$ in Algorithm 2 efficiently computes the values $v_s$ as shown below. The values $x_s$ are easy to compute. Once this is done, computing $l_s$ (and similarly $r_s$) for each $s$ is solving a problem known as the RANGE MINIMUM QUERY problem [7, 4] for the set of pairs $Q = \{(v_s, x_s) \mid s = 1, 2, \ldots, p − 1\}$. This takes $O(1)$ for each pair once a $O(p)$ preprocessing of $\pi$ is done [7, 4]. Alternatively, one may use the simplest algorithm in [16], based on LR-stacks and which also guarantees a $O(p)$ global time.

In algorithm $\text{ComputeV}(\pi)$, every pair $(w, \Phi(w))$ is recorded on $\rho$ as a pair of separators (steps 2-4), defining slices similarly to Section 4.1 (see Example 12). For each pair of consecutive separators in $\rho$, the positions in $\pi$ of the elements located between these separators form a slice. Its identification number is the smallest value it contains. Slices are sets of consecutive values, implemented again as a stack.

Example 12. With $\pi = 5, 3, 1, 4, 2, 6$ and $\Phi$ defined as $\Phi(1) = 6$ and $\Phi(3) = 5$, the for loop in steps 2-4 of algorithm $\text{ComputeV}(\pi)$ yields $\rho = [1, 3, 5, 3, 1, 4, 2, 6, 1]$, with slices $\{1, 2\}$ (the positions of 5 and 3 in $\pi$), $\{3, 4, 5\}$ (the positions of 1, 4 and 2 in $\pi$), and $\{6\}$ (the position of 6 in $\pi$). Each slice is a set identified by its smallest element. The for loop in steps 6-14 starts with $j = 1$. As $\rho[1]$ is a left separator, nothing happens. For $j = 2$, the condition in step 10 is true, as $\rho[2] = 5$. However, $\rho^{-1}(4) = 6 > 2$ in step 11 and $\rho^{-1}(6) = 9 > 2$ in step 12, thus no value is set in these steps. For $j = 3, 4$ and 5, similar situations occur. Now, for $j = 6$ we have $\rho[j] = 4$, and the conditions in steps 11 and 12 are both true since $\rho^{-1}(3) = 3 < 6$ and $\rho^{-1}(5) = 2 < 6$. Then $v_3 \leftarrow \text{FindSet}(\pi^{-1}(3))$, which is $\text{FindSet}(2)$ i.e. 1, since the slice $\{1, 2\}$ has identification number 1. Similarly, $v_4 \leftarrow \text{FindSet}(\pi^{-1}(5))$ which is also 1. Continuing with $j = 7$, we have $\rho[j] = 2$ which gives $v_1 \leftarrow \text{FindSet}(\pi^{-1}(1))$ (which is 3) and $v_2 \leftarrow \text{FindSet}(\pi^{-1}(3))$ (which is 1). When $j = 8$, the condition in step 7 is true with $w = 3$ so that the left and right 3-separators are removed, yielding $\rho = [1, 5, 3, 1, 4, 2, 6]$, with the unique slice $\{1, 2, 3, 4, 5, 6\}$ (the positions of 5, 3, 1, 4, 2, 6). With $j = 9$ we obtain $v_5 \leftarrow 1$.

Algorithm 2 is now complete.

7 Conclusion

In this paper, we proposed an alternative view of the common interval searching in sequences, obtained by reducind both sequences to orders (which are permutations), and factorizing the search for common intervals by grouping several orders into a unique order, called a dominating order. In order to insure low time and space requirements, we had to avoid storing the orders, but also rebuilding them with the algorithm from scratch in Section 4, which was too time consuming to be used in steps 3 and 4 of the main algorithm. Then, we used minimal informations computed by the algorithm from scratch in order to propose in Section 5 a much more efficient algorithm to generate the orders. Finally, we used the parameterizable algorithm in [16] to give an algorithm able to find common intervals with constrained endpoints, which completes our search for common intervals in sequences.

To solve the $(T, S)$. COMMON INTERVAL SEARCHING problem, we reduced it to a problem called $(P, S)$.GUIDED COMMON INTERVALS SEARCHING, where $P$ is a permutation on $p$ elements. We
Algorithm 7 ComputeV(π)

Input: An order π and its dominating function Φ

Output: For each s with 1 ≤ s < p, the value v_s.

1: ρ ← π
2: for each w such that Φ(w) is defined do
3: insert a left w-separator [w immediately before the value π[w] and a right w-separator ]w immediately after the value π[Φ(w)] in ρ
4: end for
5: let each slice in ρ be a set of positions in π, identified by its smallest element
6: for j = 1 to ||ρ|| do
7: if ρ[j] is a right separator ]w then
8: remove the separators [w and ]w from ρ //virtually; in the facts, update slices and sets
9: else
10: if ρ[j] is a value from π then
11: if ρ[j] − 1 ≥ 1 and ρ[ρ[j] − 1] < j then ρ[ρ[j] − 1] ← FindSet(π[ρ[j] − 1]) end if
12: if ρ[j] + 1 ≤ p and ρ[ρ[j] + 1] < j then ρ[ρ[j]] ← FindSet(π[ρ[j] + 1]) end if
13: end if
14: end if
15: end for

proposed a O(q2n2 + N_p,s)-time algorithm for solving this problem. However, an improved running time of O(n2 + N_p,s) for solving this problem would lead to a O(q1n1 + q2n2 + N) algorithm for the case of two sequences, improving the existing O(n1n2) algorithms.

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