Octonion and Split Octonion Representation of $SO(8)$

Symmetry

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Abstract

The $8 \times 8$ matrix representation of $SO(8)$ Symmetry has been defined by using the direct product of Pauli matrices and Gamma matrices. These $8 \times 8$ matrices are being used to describe the rotations in $SO(8)$ symmetry. The comparison of $8 \times 8$ matrices with octonions has also been shown. The transformations of $SO(8)$ symmetry are represented with the help of Octonions and split Octonions spinors.
1 Introduction

Octonions form a division algebra of the highest possible dimension 8. The group of rotations in eight dimensions has been described \[1, 2\] as the extensions for the symmetries of elementary particles. Günydin and Gürsey \[3\] discussed the Lie algebra of \(G_2\) group and its embedding \[4, 5, 6\] in \(SO(7)\) and \(SO(8)\) groups. The triality properties of the rotation group \(SO(8)\) which are closely related to octonions are described by Curtright \[7\]. A dynamical scheme of quark and lepton family unification based on non associative algebra has also been discussed \[8\]. Generators of \(SO(8)\) are constructed by using Octonion structure tensors \[9\], and the representations of these generators are given as products of Octonions. Lassig and Joshi \[10\] introduced the bi-modular representation of octonions and formulated the \(SO(8)\) gauge theory equivalent to the octonionic construction. Furthermore, some attention has been given to octonions \[11\] in theoretical physics in order to extend the 3+1 space-time to eight dimensional space-time as the consequence to accommodate the ever increasing quantum numbers and internal symmetries related to elementary particles and gauge fields. It is shown \[12\] that three dimensional vector space may be extended to seven dimensional space by means of octonions under certain permutations of combinations of structure constant associated with the octonion multiplication rules. Recently, we have used quaternions and octonions to defined the Quantum Chromo Dynamics \[13\], symmetry breaking \[13\], Flavor symmetry \[15\] and Casimir operator \[16\] in their algebraic form. Here, we have constructed the \(8 \times 8\) matrix by using Pauli matrices and Gamma matrices which represents the \(SO(8)\) symmetry. The eight dimensional space has been defined by considering octonions as a spinor.

2 Matrix representation of \(SO(8)\) Symmetry

\(SO(8)\) represents the special orthogonal group of eight-dimensional rotations. By using direct product of Pauli matrices and Gamma matrices, we have constructed the eight dimensional representation of \(SO(8)\) Symmetry.

As the Gamma matrices for Dirac Pauli representation are given by

\[
\gamma_j = \begin{bmatrix} 0 & -i\sigma_j \\ i\sigma_j & 0 \end{bmatrix}, \quad \gamma_4 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}
\] (1)

where \(\sigma_j\) are Pauli matrices and \(I\) is a \(2 \times 2\) unit matrix.

Now to construct the eight dimensional representation \[3\] of \(SO(8)\) Symmetry, we have chosen the following representations,
\[ \beta_1 = \sigma_1 \otimes \gamma_1 = i \left[ \Sigma_{36} + \Sigma_{45} + \Sigma_{72} + \Sigma_{81} - \Sigma_{18} - \Sigma_{27} - \Sigma_{54} - \Sigma_{63} \right] ; \\
\beta_2 = \sigma_3 \otimes \gamma_1 = i \left[ \Sigma_{32} + \Sigma_{41} + \Sigma_{58} + \Sigma_{67} - \Sigma_{14} - \Sigma_{23} - \Sigma_{76} - \Sigma_{85} \right] ; \\
\beta_3 = \sigma_2 \otimes \gamma_3 = \left[ \Sigma_{28} + \Sigma_{82} + \Sigma_{35} + \Sigma_{53} - \Sigma_{64} - \Sigma_{46} - \Sigma_{71} - \Sigma_{17} \right] ; \\
\beta_4 = \sigma_3 \otimes \gamma_2 = \left[ \Sigma_{23} + \Sigma_{32} + \Sigma_{58} + \Sigma_{65} - \Sigma_{14} - \Sigma_{41} - \Sigma_{67} - \Sigma_{76} \right] ; \\
\beta_5 = \sigma_1 \otimes \gamma_3 = i \left[ \Sigma_{28} + \Sigma_{35} + \Sigma_{64} + \Sigma_{71} - \Sigma_{17} - \Sigma_{46} - \Sigma_{63} - \Sigma_{82} \right] ; \\
\beta_6 = \sigma_3 \otimes \gamma_3 = i \left[ \Sigma_{24} + \Sigma_{31} + \Sigma_{57} + \Sigma_{86} - \Sigma_{13} - \Sigma_{42} - \Sigma_{68} - \Sigma_{75} \right] ; \\
\beta_7 = \sigma_1 \otimes \gamma_4 = \left[ \Sigma_{15} + \Sigma_{26} + \Sigma_{51} + \Sigma_{62} - \Sigma_{37} - \Sigma_{48} - \Sigma_{73} - \Sigma_{84} \right] ; \\
\beta_8 = \sigma_1 \otimes \gamma_1 = \left[ \Sigma_{11} + \Sigma_{22} + \Sigma_{77} + \Sigma_{88} - \Sigma_{33} - \Sigma_{44} - \Sigma_{55} - \Sigma_{66} \right] ; \tag{2} \]

where \( \otimes \) denotes the direct product of matrices. \( \sigma \)'s are Pauli matrices and \( \gamma \)'s are Dirac matrices. These 8 matrices \( \beta_1, \beta_2, \ldots, \beta_8 \) are \( 8 \times 8 \) Hermitian matrices and \( \Sigma_{mn} \) are \( 8 \times 8 \) matrices in which \( mn \)th matrix element is unity and rest elements are zero. \( \Sigma_{mn} \) are the \( 8 \times 8 \) matrix representation of the generators \( \text{SO}(8) \).

The eight dimensional space on which \( \text{SO}(8) \) acts can be given by the structure of a non associative algebra. Here such algebra is described by the octonions as a general spinor \( \psi \) in eight dimensions.

Under this symmetry, the spinor \( \psi \) transforms as

\[
\psi \rightarrow \psi' = \exp \left[ \sum_{A=1}^{8} f_A \beta_A \right] \psi
= e^{X \cdot \psi} \tag{3}
\]

where vector \( X \) is,

\[
X = \sum_{A=1}^{8} f_A \beta_A \tag{4}
\]

with \( f_1, f_2, f_3, \ldots, f_8 \) as the component of vector. Since the representation given in equation \( (2) \) will contain a vector space of dimension 8, for which we want to introduce an octonionic description. This description, should be invariant under the appropriate \( \text{SO}(8) \) symmetry group.
Therefore Spinor $\psi$ is defined as,

$$\psi = \begin{bmatrix} 1 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{bmatrix}$$ \hspace{1cm} (5)

where $e_1, e_2, \ldots, e_7$ are Octonion units. After expanding equation (4), $X$ becomes as,

$$X = \begin{bmatrix} f_8 & 0 & -if_6 & -f_4 - if_2 & f_7 & 0 & -f_3 - if_5 & -if_1 \\ 0 & f_8 & f_4 - if_2 & if_6 & 0 & f_7 & -if_1 & f_3 + if_5 \\ if_6 & f_4 + if_2 & -f_8 & 0 & f_3 + if_5 & if_1 & -f_7 & 0 \\ -f_4 + if_2 & -if_6 & 0 & -f_8 & if_1 & -f_3 - if_5 & 0 & -f_7 \\ f_7 & 0 & f_3 - if_5 & -if_1 & -f_8 & 0 & if_6 & f_4 + if_2 \\ 0 & f_7 & -if_1 & -f_3 + if_5 & 0 & -f_8 & -f_4 + if_2 & -if_6 \\ -f_3 + if_5 & if_1 & -f_7 & 0 & -if_6 & -f_4 - if_2 & f_8 & 0 \\ if_1 & f_3 - if_5 & 0 & -f_7 & f_4 - if_2 & if_6 & 0 & f_8 \end{bmatrix}$$ \hspace{1cm} (6)

which is a traceless Hermitian matrix. In compact form $X$ can be written as,

$$X = \begin{bmatrix} A & B^\dagger \\ B & -A \end{bmatrix}$$ \hspace{1cm} (7)
where $A$ and $B$ are given as,

$$A = \begin{bmatrix}
  f_8 & 0 & -i f_6 & -f_4 - i f_2 \\
  0 & f_8 & f_4 - i f_2 & i f_6 \\
  i f_6 & f_4 + i f_2 & -f_8 & 0 \\
  -f_4 + i f_2 & -i f_6 & 0 & -f_8 \\
\end{bmatrix}; \quad (8)$$

$$B = \begin{bmatrix}
  f_7 & 0 & f_3 - i f_5 & -i f_1 \\
  0 & f_7 & -i f_1 & -f_3 + i f_5 \\
  -f_3 + i f_5 & i f_1 & -f_7 & 0 \\
  i f_1 & f_3 - i f_5 & 0 & -f_7 \\
\end{bmatrix}. \quad (9)$$

Matrices $A$ and $B$ are independent of each other.

Furthermore, the constructed $8 \times 8$ matrices given in equation (2) are being used to describe the rotation in $SO(8)$ Symmetry.

### 3 Rotation in $SO(8)$ Representation

As an infinitesimal rotation by an angle $\theta$ in the plane $(k, l)$ is obtained by the following operator [1],

$$R_{kl} = 1 + \theta \beta_k \beta_l \quad (10)$$

which acts on a vector $X$ to form a rotated vector $X'$ as,

$$X' = R_{kl} X R_{kl}^{-1} \quad (11)$$

By using Equation (10), the rotation operator $R_{12}$ becomes,
The rotation $R_{12}$ gives the rotated vector $X'$ as follows,

$$X' = X + 2\theta \begin{bmatrix} 1 & 0 & 0 & 0 & -\theta & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\theta & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\theta & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\theta \\ \theta & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \theta & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \theta & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \theta & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since $\theta$ is an infinitesimal rotation, therefore neglecting $\theta^2$ terms. Rotation $R_{12}$ transforms the components of a vector $X$ as follows:

$$X' = X + 2\theta \begin{bmatrix} -f_7 & 0 & if_5 & if_1 & f_8 & 0 & -if_6 & -(f_4 + if_2) \\ 0 & -f_7 & if_1 & -if_5 & 0 & f_8 & (f_4 - if_2) & if_6 \\ -if_5 & -if_1 & f_7 & 0 & if_6 & f_4 + if_2 & -f_8 & 0 \\ -if_1 & if_5 & 0 & f_7 & -f_4 + if_2 & -if_6 & 0 & -f_8 \\ f_8 & 0 & -if_6 & -(f_4 + if_2) & f_7 & 0 & -if_5 & -if_1 \\ 0 & f_8 & (f_4 - if_2) & if_6 & 0 & f_7 & -if_1 & if_5 \\ if_6 & (f_4 + if_2) & -f_8 & 0 & if_5 & if_1 & -f_7 & 0 \\ -(f_4 - if_2) & -if_6 & 0 & -f_8 & if_1 & -if_5 & 0 & -f_7 \end{bmatrix}.$$ 

(13)
Table 1: $8 \times 8$ matrix multiplication

We have calculated all the possible rotations of $R_{kl}$, and observed that $R_{56}$ and $R_{78}$ give the same transformation as $R_{12}$.

4 Comparison of $8 \times 8$ matrix with Octonions

As the Octonions are described over the algebra of real numbers having the vector space of dimension 8. Here, we construct following matrices using $\beta$ matrices for showing similarity between eight dimensional matrices with Octonion basis elements.

Let

\begin{align*}
E_0 &= I_8 \\
E_1 &= \beta_1 \beta_5 = \beta_2 \beta_6; \\
E_2 &= \beta_1 \beta_7 = \beta_2 \beta_8; \\
E_3 &= \beta_7 \beta_5 = \beta_8 \beta_6; \\
E_4 &= \beta_7; \\
E_5 &= \beta_5; \\
E_6 &= \beta_1; \\
E_7 &= \beta_7 \beta_5 \beta_1 = \beta_8 \beta_6 \beta_1 = E_3 E_6. \quad (15)
\end{align*}

where $I_8$ is $8 \times 8$ unity matrix. Multiplication Table for the above matrices are given as in Table 1. Octonion multiplication \cite{8} are given in Table 2.

On comparing $8 \times 8$ matrix multiplication table (Table 1) with the octonion multiplication table (Table 2), we can see some similarities between these two. It means that multiplication of $8 \times 8$ matrices satisfied some of the multiplication of the 8 dimensional algebra of Octonions. Out of 64 combinations, 48 combinations are identical, however, rest 16 combinations have opposite signs for octonions and $8 \times 8$ matrix. This dissimilarity can be attributed to the non associativity of
Table 2: Octonion multiplication

| . | $e_0$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|---|
| $e_0$ | $e_0$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
| $e_1$ | $e_1$ | $-e_0$ | $e_3$ | $-e_2$ | $e_7$ | $-e_6$ | $e_5$ | $-e_4$ |
| $e_2$ | $e_2$ | $-e_3$ | $-e_0$ | $e_1$ | $e_6$ | $e_7$ | $-e_4$ | $-e_5$ |
| $e_3$ | $e_3$ | $e_2$ | $-e_1$ | $-e_0$ | $-e_5$ | $e_4$ | $e_7$ | $-e_6$ |
| $e_4$ | $e_4$ | $-e_7$ | $-e_6$ | $e_5$ | $-e_0$ | $-e_3$ | $e_2$ | $e_1$ |
| $e_5$ | $e_5$ | $e_6$ | $-e_7$ | $-e_4$ | $e_3$ | $-e_0$ | $-e_1$ | $e_2$ |
| $e_6$ | $e_6$ | $-e_5$ | $e_4$ | $-e_7$ | $-e_2$ | $e_1$ | $-e_0$ | $e_3$ |
| $e_7$ | $e_7$ | $e_4$ | $e_5$ | $e_6$ | $-e_1$ | $-e_2$ | $-e_3$ | $-e_0$ |

Octonions. Thus, we have made a comparative study between the multiplicative properties of the $8 \times 8$ matrix and the octonions.

5 Split Octonions

Split Octonion algebra [17] with its split base units is defined as

\[
\begin{align*}
    u_0 &= \frac{1}{2} (e_0 + ie_7), & u_0^* = \frac{1}{2} (e_0 - ie_7); \\
    u_m &= \frac{1}{2} (e_m + ie_{m+3}), & u_m^* = \frac{1}{2} (e_m - ie_{m+3}).
\end{align*}
\]

(16) (17)

where $m=1,2,3$.

These basis element satisfy the following algebra

\[
\begin{align*}
    u_i u_j &= -u_j u_i = \epsilon_{ijk} u_k^*, & u_i^* u_j^* &= -u_j^* u_i^* = \epsilon_{ijk} u_k; \\
    u_i u_j^* &= -\delta_{ij} u_0, & u_i^* u_j &= -\delta_{ij} u_0^*; \\
    u_0 u_i &= u_i u_0^* = u_i, & u_0^* u_i^* &= u_i^* u_0 = u_i^*; \\
    u_i u_0 &= u_0 u_i^* = 0, & u_i^* u_0^* &= u_0 u_i = 0; \\
    u_0 u_0^* &= u_0^* u_0 = 0 & u_0^2 = u_0, & u_0^2 = u_0^*.
\end{align*}
\]

(18)

These relations [18] are invariant [18] under $G_2$ group as a automorphism of octonions. Unlike octonions, the split octonion algebra contains zero divisors and is therefore not a division algebra.
6 Split Octonion representation of SO(8) Symmetry

Now, spinor $\psi$ in terms of split octonion $[3]$ i.e.

$$\psi = \left[ \begin{array}{c} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_0' \\ u_1' \\ u_2' \\ u_3' \end{array} \right] = \left[ \begin{array}{c} \frac{1}{2} (1 + ie_7) \\ \frac{1}{2} (e_1 + ie_4) \\ \frac{1}{2} (e_2 + ie_5) \\ \frac{1}{2} (e_3 + ie_6) \\ \frac{1}{2} (1 - ie_7) \\ \frac{1}{2} (e_1 - ie_4) \\ \frac{1}{2} (e_2 - ie_5) \\ \frac{1}{2} (e_3 - ie_6) \end{array} \right] = \left[ \begin{array}{c} u \\ u^* \end{array} \right].$$

(19)

The mapping due to split octonions can be given as

$$\phi \mapsto \phi' = e^Y \phi$$

(20)

where

$$Y = \left[ \begin{array}{cccccccc} f_8 & 0 & -i f_6 & -f_4 - i f_2 & f_8 & 0 & -i f_6 & -f_4 - i f_2 \\ 0 & f_8 & f_4 - i f_2 & i f_6 & 0 & f_8 & f_4 - i f_2 & i f_6 \\ i f_6 & f_4 + i f_2 & -f_8 & 0 & i f_6 & f_4 + i f_2 & -f_8 & 0 \\ -f_4 + i f_2 & -i f_6 & 0 & -f_8 & -f_4 + i f_2 & -i f_6 & 0 & -f_8 \\ f_7 & 0 & f_3 - i f_5 & -i f_1 & f_7 & 0 & f_3 - i f_5 & -i f_1 \\ 0 & f_7 & -i f_1 & -f_3 + i f_5 & 0 & f_7 & -i f_1 & -f_3 + i f_5 \\ -f_3 + i f_5 & i f_1 & -f_7 & 0 & -f_3 + i f_5 & i f_1 & -f_7 & 0 \\ i f_1 & f_3 - i f_5 & 0 & -f_7 & i f_1 & f_3 - i f_5 & 0 & -f_7 \end{array} \right]$$

$$+ \left[ \begin{array}{cccccccc} -f_1 & -i f_7 & 0 & -f_5 + i f_3 & f_1 & i f_7 & 0 & -i f_3 + f_5 \\ f_5 - i f_3 & 0 & -i f_7 & -f_1 & -f_5 + i f_3 & 0 & i f_7 & f_1 \\ 0 & f_5 - i f_3 & f_1 & i f_7 & 0 & i f_3 - f_5 & -f_1 & -i f_7 \\ i f_7 & f_1 & -f_5 + i f_3 & 0 & -i f_7 & -f_1 & f_5 - i f_3 & -f_8 \\ -i(f_4 + i f_2) & i f_8 & 0 & f_6 & i(f_4 + i f_2) & -i f_8 & 0 & -f_6 \\ -f_6 & 0 & i f_8 & i f_4 - f_2 & f_6 & 0 & -i f_8 & -i f_4 + f_2 \\ 0 & -f_6 & i f_4 - f_2 & f_8 & 0 & f_6 & -i f_4 + f_2 & -f_8 \\ -i f_8 & -i f_4 - f_2 & -i f_6 & 0 & i f_8 & i f_4 + f_2 & i f_6 & 0 \end{array} \right].$$

(21)
$$Y = \begin{bmatrix} A & A \\ B & B \end{bmatrix} + \begin{bmatrix} C & -C \\ D & -D \end{bmatrix} = \begin{bmatrix} A + B & A - C \\ B + D & B - D \end{bmatrix}$$

Equation (22)

Y is calculated by using equation (5), (6), (16) and equation (17). A and B are already defined in equations (8) and (9). C and D are 4×4 matrices given as,

$$C = \begin{bmatrix} -f_1 & -i f_7 & 0 & -f_5 + i f_3 \\ f_5 - i f_3 & 0 & -i f_7 & -f_1 \\ 0 & f_5 - i f_3 & f_1 & i f_7 \\ i f_7 & f_1 & -f_5 + i f_3 & 0 \end{bmatrix}$$

and

$$D = \begin{bmatrix} -i(f_4 + i f_2) & i f_8 & 0 & f_6 \\ -f_6 & 0 & i f_8 & i f_4 - f_2 \\ 0 & -f_6 & i f_4 - f_2 & f_8 \\ -i f_8 & -i f_4 - f_2 & -i f_6 & 0 \end{bmatrix}.$$  

Equation (22) represent how the SO(8) Symmetry split the Octonion representation given in equation (7). This is the split octonion representation in $SO(8)$ Symmetry.

7 Result and Discussion

The mathematical properties on the space of eight dimensions are presented for their possible applications for the study of symmetries of elementary particles. Here the eight dimensional space on which $SO(8)$ acts has been defined in terms $8 \times 8$ matrices. These $8 \times 8$ matrix transformations are used as a general spinor in eight dimensions. An infinitesimal rotation transformations in $8 \times 8$ matrices are defined. After using the different combinations of these matrices, we compare the similarity of octonions with these matrices. Split octonion representation of $SO(8)$ symmetry and their transformations are also defined. These representations of 8 dimensional orthogonal groups may be used to give octonionic descriptions of the Clifford groups. Since the generators of our fundamental representation are also generators of the $SO(8)$ symmetry group. Therefore, we have used the representation of the octonion algebra of the $SO(8)$ symmetry group. Octonionic symmetry has
been used to represents the $SO(8)$ symmetry. We have calculated an embedding of the octonion symmetry in $SO(8)$. A similar description for the exceptional Lie group $G_2$, which is the automorphism group of the octonionic algebra has also been found earlier [3]. The octonionic description of the vector representations of $SO(8)$ can give a unified picture of the triality automorphisms of $SO(8)$.

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