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On representations of the set of supermartingale measures
and applications in discrete time

Abstract We investigate some new results concerning the m-stability property. We show in particular under the martingale representation property with respect to a bounded martingale \( S \) that an m-stable set of probability measures is the set of supermartingale measures for a family of discrete integral processes with respect to \( S \).

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1 Introduction

The m-stability property plays a primordial role within the theory of dynamic risk measures and in Financial Mathematics in general. In dynamic setting it is crucial that a dynamic risk measure satisfies the recursiveness property when evaluating risk for a financial position, by taking into account the new incoming information in a consistent way. It should also be the case for a pricing mechanism when pricing a financial claim in incomplete markets, avoiding so the creation of arbitrage opportunities in a given time axis. In decision making and in econometrics the m-stability property known by rectangularity is an essential assumption in modeling preferences and utility functions and in constructing set of priors. We refer to \([5,6]\) for more details.

Let us consider a probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\ldots,T}, \mathbb{P})\) and a set \( Q \) of probability measures in \( \mathbb{P} \) containing at least one equivalent to \( \mathbb{P} \). We denote the set of \( \mathbb{P} \)-absolutely continuous (resp., \( \mathbb{P} \)-equivalent) probability measures. In \([2]\) it was shown that the m-stability assumption on \( Q \) is a cornerstone in generalizing a number of results from one-period case to multi-period case. We recall in particular that for an m-stable set \( Q \), we get the following:

(i) Suppose \( Q = \{ Q \in \mathbb{P} : \mathbb{E}_Q(X) = 0 \} =: \mathcal{M}(0, X) \) for a vector-valued random variable \( X = (X^1, \ldots, X^k) \).

Then \( Q \) is the set of martingale measures for the \( \mathbb{R}^k \)-valued adapted process \( M \) defined by \( M^i_t = \mathbb{E}_Q(X^i) := \mathbb{E}_Q(X^i|\mathcal{F}_t) \) for some (or any) \( Q \in Q^c \) and \( i = 1, \ldots, k \), where \( Q^c \) is given by \( Q^c = Q \cap \mathbb{P}^c \). In this case we can write a \( Q \)-supermartingale as the sum of a local martingale and a decreasing process, by applying the well-known theorem of Föllmer and Kabanov \([7]\).

(ii) The set \( Q' \), to be redefined later, is the set of martingale measures for a family \( \mathcal{Y} \) of adapted processes. Such family \( \mathcal{Y} \) can be replaced by a finite one if we suppose further that \( Q \) is optionally m-stable with respect to a vector-valued bounded random variable \( V \).
Our goal in this paper is to prove the $\mathcal{Q}$-supermartingale decomposition under minimal assumptions on $\mathcal{Q}$. To do that, we will start by proving in Sect. 3 some interesting properties of the set $\mathcal{Q}^{\text{st}}$ (See Definition 2.2 below). We will state in particular two fundamental assertions: (1) $\mathcal{Q}$-supermartingales are $\mathcal{Q}^{\text{st}}$-supermartingales, and (2) $\mathcal{Q}^{\text{st}}$ is the set of supermartingale measures for a family $\mathcal{Y}$ of bounded adapted processes.

In Sect. 4 we suppose the martingale representation property of the filtration with respect to an adapted process $\mathcal{S}$ with values in $\mathbb{R}^d$. We shall state the existence of a convex cone $\mathcal{C}$ of vector-valued adapted processes such that $\mathcal{Q}^{\text{st}}$ is the set of supermartingale measures for a family of processes of the form $\alpha \cdot \mathcal{S} := \sum_{s \in \mathcal{S}} \alpha_s, \Delta_t \mathcal{S}$ with $\alpha \in \mathcal{C}$ and $\Delta_t \mathcal{S} = \mathcal{S}_t - \mathcal{S}_s$. We apply such result and deduce that any positive (or bounded) $\mathcal{Q}$-supermartingale $\mathcal{X}$ can be written as $X_0 + \alpha \cdot \mathcal{S} - \mathcal{B}$ with $\mathcal{B}$ an increasing process and $\alpha \in \mathcal{C}$.

In Sect. 5 and further under the assumption $\mathcal{Q} = \mathcal{Q}'$, we shall prove that $\mathcal{Q}^{\text{st}}$ is the set of martingale measures for a family $\mathcal{Y}$ of bounded adapted processes and the process $\alpha \cdot \mathcal{S}$ appearing in the previous decomposition of $\mathcal{X}$ is a local $\mathcal{Q}$-martingale, generalizing the theorem of Föllmer and Kabanov.

We prove also under the martingale representation property that $(\mathcal{Q}^{\text{st}})' = (\mathcal{Q}')^{\text{st}}$, which is the commutativity property of the two operators $\mathcal{Q} \rightarrow \mathcal{Q}^{\text{st}}$ and $\mathcal{Q} \rightarrow \mathcal{Q}'$.

## 2 Notation and review

In this section we recall the definition of the main properties which will be used along this paper, and also some established characterizations of these properties.

### 2.1 Notation

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a discrete time filtration $(\mathcal{F}_t)_{t \in \mathcal{T}}$ with $\mathcal{T} = \{0, \ldots, T\}$ and $\mathcal{T}^* = \mathcal{T} \setminus \{T\}$. We denote by $\Pi$ (resp., $\Pi^*$) the set of all (resp., $L^1$-closed convex) subsets in $\mathbb{P}$, with $\Pi^e = \{Q \in \Pi : \mathcal{Q} \cap \mathbb{P}^e \neq \emptyset\}$ and $\Pi^{e,e} = \Pi^e \cap \Pi^e$. For a set $\mathcal{Q}$ of probability measures in $\mathbb{P}$, we define the dynamic expectation operator $\mathcal{E}$ by $\mathcal{E}^i(X) = \text{esssup}_{\mathcal{Q} \in \mathcal{Q}} \mathcal{E}^i_{\mathcal{Q}}(X)$ for $t \in \mathcal{T}$ and $X \in L^\infty$ with $\mathcal{E}^i_{\mathcal{Q}}(X) = \mathcal{E}^i_{\mathcal{Q}}(X | \mathcal{F}_t)$, the family of acceptance sets $\mathcal{A}_{s,u} = \{X \in L^\infty(\mathcal{F}_u) : \mathcal{E}^i_{\mathcal{Q}}(X) \leq u \ a.s.\}$ for $t < u$ with $\mathcal{A}_{u} := \mathcal{A}_{0,u}$, and the set of Radon–Nikodym densities $\mathcal{Z} = \{\mathcal{Z}^\mathcal{Q} := d\mathcal{Q} / d\mathcal{P} : \mathcal{Q} \in \mathcal{Q}\}$, $\mathcal{Z}^e = \{\mathcal{Z} \in \mathcal{Z} : \mathcal{Z} > 0 \ a.s.\}$ with $\mathcal{Z}_t := \mathcal{E}^i_t(\mathcal{Z})$ for $\mathcal{Z} \in \mathcal{Z}$ and a stopping time $\tau$.

### 2.2 On the m-stability property

**Definition 2.1** We say that a set of probability measures $\mathcal{Q}$ is m-stable with respect to the filtration $(\mathcal{F}_t)_{t \in \mathcal{T}}$ if for any $\mathcal{Z}^1, \mathcal{Z}^2 \in \mathcal{Z}$ with $\mathcal{Z}^1 > 0 \ a.s.$ and a stopping time $\tau$ we have $\mathcal{Z} := \mathcal{Z}^1 / \mathcal{Z}^2 \in \mathcal{Z}$.

**Definition 2.2** For any $\mathcal{Q} \in \Pi^{e,e}$, we define $\mathcal{Q}^{\text{st}}$ to be the intersection of all m-stable closed convex subsets in $\mathcal{P}$ containing $\mathcal{Q}$, and denote by $\mathcal{E}^{\text{st}}$ and $\mathcal{A}^{\text{st}}$, respectively, the dynamic expectation operator and the acceptance set associated to $\mathcal{Q}^{\text{st}}$.

We recall some interesting characterizations of the m-stability property, stated in Delbaen [5].

**Proposition 2.3** Let $\mathcal{Q} \in \Pi^{e,e}$. Then the following assertions are equivalent:

1. $\mathcal{Q}$ is m-stable,
2. For any $t \in \mathcal{T}^*$, $F \in \mathcal{F}_t$ and $\mathcal{Z}, \mathcal{Z}^1, \mathcal{Z}^2 \in \mathcal{Z}$ with $\mathcal{Z}^1 > 0$ and $\mathcal{Z}^2 > 0 \ a.s.$, we have $\mathcal{Z}_t \left( \frac{\mathcal{Z}^1}{\mathcal{Z}^2} + \frac{\mathcal{F}_t}{\mathcal{Z}^2} \right) \in \mathcal{Z}$,
3. $\mathcal{E}$ satisfies the recursiveness property, i.e., for any $X \in L^\infty$ and for any stopping times $\tau \leq \sigma$ we have $\mathcal{E}_\tau(\mathcal{E}_\sigma(X)) = \mathcal{E}_\tau(X)$,
4. $\mathcal{E}$ is time consistent, i.e., for any $X, Y \in L^\infty$ and for stopping times $\tau \leq \sigma$ we have $\mathcal{E}_\tau(X) \geq \mathcal{E}_\sigma(Y)$ implies $\mathcal{E}_\tau(X) \geq \mathcal{E}_\sigma(Y)$,
5. for any $X \in L^\infty$, the process $\mathcal{E}(X)$ is a $\mathcal{Q}$-supermartingale,
6. for any stopping time $\tau > 0$ and $\mathcal{A} := \mathcal{A}_{0,\tau}$ we have $\mathcal{A} = \mathcal{A}_{0,\tau} + \mathcal{A}_{\tau,\tau}$.
2.3 On the optional m-stability property

This subsection is devoted to recalling the definition of the optional m-stability. This concept was first introduced by Jacka et al. in [8]. We shall say that an $\mathbb{R}^d$-valued random variable $V$ is viable if for each component $v$ of $V$ satisfies $v \in L^\infty$ and $1/v \in L^\infty$. For a random variable $Y \in L^1$, with $\mathbb{E}(Y) = 1$, we define the probability measure $Q^Y$ by $Q^Y(F) = \mathbb{E}(Y 1_F)$ for $F \in \mathcal{F}$.

**Definition 2.4** (See Jacka and Berkaoui [8]) We say that $Q$ is optionally m-stable with respect to a viable random vector $V$ if for all $t = 0 \ldots T - 1$, whenever $Q^1, Q^2 \in Q$ are such that there exists $Q \in Q$, $F \in \mathcal{F}_t$, $\alpha' \in \mathbb{L}^0(\mathcal{F}_{t+1})$ with each $\alpha' Z^i \in L^1$ where $Z^i$ and $Z$ are respective densities of $Q^i$ and $Q$ for $i = 1, 2$ and $Y = 1_F \alpha' Z^1 + 1_F \alpha^2 Z^2$ satisfies $Q_t(V) = Q^Y_t(V)$, then we have $Q^Y \in Q$.

Characterizations of this property can be found in [1,8].

2.4 On the smallest set of martingale measures

Here we recall the definition of the optional m-stability. This concept was first introduced by Jacka et al. in [8]. We shall say that an $\mathbb{R}^d$-valued random variable $V$ is viable if for each component $v$ of $V$ satisfies $v \in L^\infty$ and $1/v \in L^\infty$. For a random variable $Y \in L^1$, with $\mathbb{E}(Y) = 1$, we define the probability measure $Q^Y$ by $Q^Y(F) = \mathbb{E}(Y 1_F)$ for $F \in \mathcal{F}$.

**Definition 2.4** (See Jacka and Berkaoui [8]) We say that $Q$ is optionally m-stable with respect to a viable random vector $V$ if for all $t = 0 \ldots T - 1$, whenever $Q^1, Q^2 \in Q$ are such that there exists $Q \in Q$, $F \in \mathcal{F}_t$, $\alpha' \in \mathbb{L}^0(\mathcal{F}_{t+1})$ with each $\alpha' Z^i \in L^1$ where $Z^i$ and $Z$ are respective densities of $Q^i$ and $Q$ for $i = 1, 2$ and $Y = 1_F \alpha' Z^1 + 1_F \alpha^2 Z^2$ satisfies $Q_t(V) = Q^Y_t(V)$, then we have $Q^Y \in Q$.

Characterizations of this property can be found in [1,8].

3 Intermediate results

Now we investigate some properties of the mapping $Q \rightarrow Q^t$.

3.1 Properties

First we express the triplet $(Q^{st}, A^{st}, \mathcal{E}^{st})$ in terms of the triplet $(Q, A, \mathcal{E})$.

**Proposition 3.1** Let $Q \in \Pi^{c,e}$. Then

1. $Q^{st}$ is the closed convex hull in $L^1$ of the set $B := \bigcap_{t \in I^*} \{Q \in \mathbb{P}^c : Z^Q_{t+1}/Z^Q_t = 1_F Z^1_{t+1}/Z^1_t + 1_F Z^2_{t+1}/Z^2_t \}$ for some $Z^1, Z^2 \in \mathbb{E}^c, F \in \mathcal{F}_t$.

2. $A^{st}_{s,t} = A_{s,s+1} + A^{st}_{s+1,t} = \oplus_{u=s+2} \ldots t A_{u,u+1}$ for all $s, t \in I^*$ with $s < t$. In particular, we have $A^{st}_{t,t-1} = A_{t,t-1}$ for $t \in I^*$.

3. $E^{st}_t = E_t \circ E^{st}_{t+1} = \ldots \circ E^{st}_{T-1}$ for $t \in I^*$.

**Proof** (1) For the reverse inclusion, let $Q \in B$ and $Z := Z^Q$ which means that for each $t \in I^*$ we have $Z^Q_{t+1}/Z^Q_t = 1_F Z^1_{t+1}/Z^1_t + 1_F Z^2_{t+1}/Z^2_t$ for some $Z^1, Z^2 \in \mathbb{E}^c$ and $F_t \in \mathcal{F}_t$. We define $R^0 = 1_F Z^0, 1 + 1_F Z^0, 2$ and for $t = 1 \ldots T - 1$, we define $R^t = R^{t-1}_t \left[ 1_F Z^t, 1/Z^t, 1 + 1_F Z^t, 2/Z^t, 2 \right]$. Thanks to Assertion (2) in Proposition 2.3 we deduce by induction on $t = 0 \ldots T - 1$ that all $R^t \in \mathcal{Z}^{st}$ and since $Z = R^{T-1}$ we deduce that $B \subseteq Q^{st}$ and, therefore, $\mathcal{V}_{Q}(B) \subseteq Q^{st}$. For the direct inclusion we remark that $B$ is m-stable and $Q \subseteq B$, so $Q^{st} \subseteq \mathcal{V}_{Q}(B)$.
(2) Fix $s < t$ and take $X \in A^t_{s,t}$ with $E^t_s(X) = 0$. We have $A^t_{s,t} \subseteq A^t_s$, then there exists some $X_u \in A_{s,u+1}$ for $u \in T^*$ such that $X = X_0 + \cdots + X_{T-1}$. We apply $E^t_s$ on both parts of the equality and obtain that $0 = E^t_s(X) = X_{0,s-1} + E^t_s(X_{s,T-1})$ with $X_{r,u} = X_r + \cdots + X_u$. We get $X_{0,s-1} = -E^s_s(X_{s,T-1}) \geq 0$ and, therefore, $X_{0,s-1} = 0$. Now we apply $E^t_{t+1}$ and obtain that $X = E^t_{t+1}(X) = X_{s,t} + E^t_{t+1}(X_{t+1,T-1})$, which means that $X_{t+1,T-1} = E^t_{t+1}(X_{t+1,T-1}) =: Y_{t+1} \in L^\infty(F_{t+1})$. So $X = X_{s,t} + Y_{t+1} \in A_{s,s+1} + \cdots + A_{s,t}$, Inversely we have $A_{s,s+1} + \cdots + A_{s,t} \subseteq A^t_{s,t}$ and then $A_{s,s+1} + \cdots + A_{s,t} \subseteq A^t_{s,t}$.

(3) We have $E^t_{t+1} = E^t_s \circ E^t_{t+1}$, thanks to Assertion (3) in Proposition 2.3. It suffices to prove that $E^t_s(X) = E^t_s(X)$ for all $X \in L^\infty(F_{t+1})$. Indeed we have $E^t_s(X) \leq E^t_s(X)$ since $Q \subseteq Q^t$ and for $X \in L^\infty(F_{t+1})$ we have $X - E^t_s(X) \in A_{s,t+1}$ and $A^t_{s,t+1} \subseteq A_{s,t+1}$. So $E^t_s(X - E^t_s(X)) \leq 0$ which means that $E^t_s(X) \leq E^t_s(X)$.

Therefore, $E^t_{t+1} = E^t_s \circ E^t_{t+1} = E^t_s \circ E^t_{t+1}$.

Theorem 3.2
Let $Q$, $Q_1$, $Q_2 \in \Pi^{c,e}$. Then we have the following:

1. $(Q^t_{s,t})^{st} = Q^{st}$.
2. $Q^{st}_{s,t} \subseteq Q^{st}$ if $Q_1 \subseteq Q_2$.
3. $(Q_1 \cap Q_2)^{st} = (Q_1)^{st} \cap (Q_2)^{st}$.
4. For an increasing sequence $Q^n \in \Pi^{c,e}$ with $Q := \bigcup_{n \geq 1} Q^n$, we have
   - (i) $Q$ is m-stable if each $Q^n$ is.
   - (ii) $Q^{st} = \bigcup_{n \geq 1} (Q^n)^{st}$, with the closure taken in $L^1$.

Proof. Assertions (1) and (2) are trivial. (3) We shall show that $(A^1 + A^2)_{s,t+1} = A^1_{s,t+1} + A^2_{s,t+1}$ for all $t \in T^*$, with the closure taken in weak star sense in $L^\infty$. The inverse inclusion is trivial. Let us prove the direct one. We know that the dynamic expectation operator $E$ of $Q_1 \cap Q_2$ is given by $E(X) = \text{essinf}_{s \leq t} E(X)$. So for $X \in (A^1 + A^2)_{s,t+1}$ we have $X = E(X) = E(X) = E(X) + 1_{F}(X - E(X)) = E(X) + X^1 + X^2 \leq X^1 + X^2$ with $F = E(X) \leq E(X)$.

(4,i) Let $Z^1, Z^2 \in Z$ with $Z^2 > 0$ a.s. and a stopping time $t$. Then there exists an integer $n$ such that $Z^1, Z^2 \in Z^n$ and since $Q^n$ is m-stable we deduce that $Z := Z^1 / Z^2 \in Z^n \subseteq Z$.

(4,ii) We have from one side $Q^n \subseteq (Q^t_{s,t})^{st}$. Hence, $\bigcup_{n \geq 1} Q^n \subseteq \bigcup_{n \geq 1} (Q^n)^{st} =: \hat{Q}$ and since $\hat{Q}$ is an m-stable closed convex set we deduce that $Q^{st} \subseteq \hat{Q}$. For the other inclusion we have $Q^n \subseteq Q$ which implies $\hat{Q} \subseteq Q^{st}$.

3.2 Link with the concept of supermartingale

We investigate the relationship of the m-stability property with the concept of supermartingale. We start by giving the definition of a supermartingale measure.

Definition 3.3
We say that a probability measure $Q \in \Pi$ is a supermartingale measure for a family $Y$ of adapted processes if each element $Y \in Y$ is a $Q$-supermartingale. We denote by $M_{sp}(Y)$ the set of all supermartingale measures for the family $Y$, and we will say that $Q$ is a set of supermartingale measures if it is the set of supermartingale measures for a family $Y$ of bounded adapted processes.

Now we state results related to that.

Theorem 3.4
Let $Q \in \Pi^{c,e}$. Then we have the following:

1. $Q^{st}$ is the set of supermartingale measures for the family $Y = \{E^t_s(X) : X \in A^t_s\}$.
2. $Q^{st}$ is the smallest set of supermartingale measures containing $Q$.

Proof (1) we know from Assertion (4) in Proposition 2.3 that the process $E^t_s(X)$ is a $Q^{st}$-supermartingale for all $X \in L^\infty$, so in particular $Q^{st} \subseteq M_{sp}(Y)$. Inversely let $Q \in M_{sp}(Y)$ then $E^Q(X) = E^Q(E^t_s(X)) \leq E^t_s(X) \leq 0$ for all $X \in A^t_s$. So $Q \subseteq Q^{st}$.

(2) Let $Q = M_{sp}(Y)$ for a family $Y$ of bounded adapted processes with $Q \subseteq \hat{Q}$. Since $\hat{Q}$ is an m-stable closed convex set, we deduce that $Q^{st} \subseteq \hat{Q}$.

For $Q \in \Pi$ we denote by $m(Q)$ (resp., $spm(Q)$) the set of all bounded $Q$-martingales (resp., $Q$-supermartingales).
Theorem 3.5 Let $Q \in \Pi^{c,e}$. Then we have the following:

1. $\text{spm}(Q) = \{Y = (Y_t)_{t \in \mathcal{T}} : Y_{t+1} - Y_t \in A_{t,t+1} \text{ for all } t \in \mathcal{T}^*\}$.
2. $A_{t,t+1} = \{Y_{t+1} - Y_t : Y \in \text{spm}(Q)\} = B_t$ for all $t \in \mathcal{T}^*$.
3. $\text{spm}(Q) = \text{spm}(Q^{it})$.
4. $Q^{it}$ is the greatest subset in $P$ that satisfies $\text{spm}(Q) = \text{spm}(Q^{it})$.
5. $m(Q) = m(Q^{it})$.

Proof (1) It is straightforward from the definitions of $\text{spm}(Q)$ and $A_{t,t+1}$ for $t \in \mathcal{T}^*$.

(2) The inclusion $B_t \subseteq A_{t,t+1}$ is straightforward. For the direct inclusion let $X \in A_{t,t+1}$ and define the process $Y$ by $Y_s = X$ for $s > t$ and $Y_s = 0$ for $s \leq t$. Then $Y \in \text{spm}(Q)$ by (1) and $X = Y_{t+1} - Y_t$.

(3) We apply Assertion (1) and the fact that $A_{t,t+1} = A_{t,t+1}^{it}$ for all $t \in \mathcal{T}^*$.

(4) Let $Q' \subseteq P$ satisfying $\text{spm}(Q) = \text{spm}(Q')$. Then from Assertion (2) we get $A_{t,t+1} = A_{t,t+1}^{it}$ for all $t \in \mathcal{T}^*$ and, therefore, $A^{it} = A^{it}$. We conclude that $Q' \subseteq Q^{it} = Q^{it}$.

(5) We remark that $X \in m(Q)$ if and only if $\pm X \in \text{spm}(Q)$. We use Assertion (3) and deduce the result. $\square$

An immediate consequence of Theorem 3.5 is as follows:

Corollary 3.6 Let $Q^1, Q^2 \in \Pi^{c,e}$. Then the following three assertions are equivalent:

1. $\text{spm}(Q^1) \subseteq \text{spm}(Q^2)$.
2. $A^{1}_{t,t+1} \subseteq A^{2}_{t,t+1}$ for all $t \in \mathcal{T}^*$.
3. $(Q^{i1})^{it} \subseteq (Q^{i2})^{it}$.

For a set $B \subseteq \mathbb{L}^0$ we denote $\text{lin}(B) = B \cap -B$ and for $Q \subseteq P$ we denote $Q' = \{Q \in P : \mathbb{E}Q(X) = 0 \text{ for all } X \in \text{lin}(A)\}$. The set $Q'$ was first introduced by Berkaoui in [2] and defined properly in [3]. We refer to Theorem 2.1 in [3] for more details on this set.

Theorem 3.7 Let $Q \in \Pi^{c,e}$. Then we have the following:

1. $m(Q) = \{Y = (Y_t)_{t \in \mathcal{T}} : Y_{t+1} - Y_t \in \text{lin}(A_{t,t+1}) \text{ for all } t \in \mathcal{T}^*\}$.
2. $\text{lin}(A_{t+1}) = \{Y_{t+1} - Y_t : Y \in m(Q)\} =: I_t$ for all $t \in \mathcal{T}^*$.
3. $m(Q) = m((Q^{it})')$.
4. $(Q^{it})'$ is the greatest subset in $P$ that satisfies $m(Q) = m((Q^{it})')$.

Proof (1) It is straightforward from the definitions of $m(Q)$ and $\text{lin}(A_{t,t+1})$ for $t \in \mathcal{T}^*$.

(2) The inclusion $I_t \subseteq A_{t,t+1}$ is straightforward. For the direct inclusion let $X \in \text{lin}(A_{t,t+1})$ and define the process $Y$ by $Y_s = X$ for $s > t$ and $Y_s = 0$ for $s \leq t$. Then $Y \in m(Q)$ and $X = Y_{t+1} - Y_t$.

(3) Since $m(Q) = m(Q^{it})$ it suffices to show that $m(Q) = m(Q')$. We apply Theorem 2.5 to deduce that $\text{lin}(A_{t,t+1}) = \text{lin}(A^{it}_{t,t+1})$ for all $t \in \mathcal{T}^*$. The result is concluded thanks to Assertion (1).

(4) Let $Q \in \Pi$ such that $m(Q) = m(Q')$, then from Assertion (2) we get $\text{lin}(A_{t,t+1}) = \text{lin}(A^{it}_{t,t+1})$ for all $t \in \mathcal{T}^*$ and, therefore, $(Q^{it})' = (A^{it})'$ thanks to Theorem 2.5. We conclude that $Q \subseteq (Q^{it})'$.$\square$

An immediate consequence of Theorem 3.7 is as follows:

Corollary 3.8 Let $Q^1, Q^2 \in \Pi^{c,e}$. Then the following three assertions are equivalent:

1. $m(Q^1) \subseteq m(Q^2)$.
2. $\text{lin}(A^1_{t,t+1}) \subseteq \text{lin}(A^2_{t,t+1})$ for all $t \in \mathcal{T}^*$.
3. $((Q^{i1})^{it})' \subseteq ((Q^{i2})^{it})'$.

4 Main results

In this section we precise further the results of Theorem 3.4. In what follows, we suppose the assumption MRP(S): the filtration $(\mathcal{F}_t)_{t \in \mathcal{T}}$ satisfies the martingale representation property with respect to a bounded martingale $S$ with values in $\mathbb{R}^d$, which means that any square integrable martingale $X$ can be written as the sum $\mathbb{E}(X) + \alpha \cdot S$ for some vector-valued adapted process $\alpha$. 
Theorem 4.1 Let $Q \in \Pi^{c,e}$. Then

1. $Q^{it} = M_{sp}(C \bullet S)$ with $C \bullet S = \Theta_{t \in T} C_t, \Delta_t S$ and each $C_t$ is a convex cone in $L^{\infty}(F_t; \mathbb{R}^d)$ given by $C_t = \{ \alpha_t \in L^{\infty}(F_t; \mathbb{R}^d) : \alpha_t, \Delta_t S \in A_{t+1} \}$.

2. Suppose $Q$ has a finite number of extreme points, then $Q^{it} = M_{sp}(g \bullet S)$ for a matrix-valued adapted process $g$.

We state first in the next theorem a similar version of Theorem 4.1 in the one-period model. We say that an element $Q \in Q$ is an $F_{t+1}$-extreme point of $Q$ if there are no elements $Q^1, Q^2 \in Q$ and $\alpha_t \in L^{\infty}(F_t; \{0, 1\})$ such that $\mathbb{E}_t^Q(X) = \alpha_t \mathbb{E}_t^{Q^1}(X) + (1 - \alpha_t) \mathbb{E}_t^{Q^2}(X)$ for all $X \in L^{\infty}(F_{t+1})$.

Theorem 4.2 Let $Q \in \Pi^{c,e}$ and suppose $I = \{0, 1\}$ and a non-necessarily trivial $F_0$. Then

1. $Q^{it} = M_{sp}(0; C_0(S_1 - S_0))$ where $C_0$ is a convex cone in $L^{\infty}(F_0; \mathbb{R}^d)$ given by $C_0 = \{ \alpha_0 \in L^{\infty}(F_0; \mathbb{R}^d) : \alpha_0(S_1 - S_0) \in A \}$.

2. Suppose $Q$ has a finite number of $F_0$-extreme points, then $Q^{it} = M_{sp}(0; g_0(S_1 - S_0))$ for a matrix-valued $F_0$-measurable random variable $g_0$.

Proof (1) We will show that $Q$ is optionally $\mu$-stable with respect to $\Delta S := S_1 - S_0$. Let $Q \in P$ such that $\mathbb{E}_0^Q(\Delta S) = \mathbb{E}_0^{Q^1}(\Delta S)$ for some $Q^1 \in Q$. Then for all $X \in A$ and thanks to the assumption $MRP(S)$, there exists some vector-valued $F_0$-measurable random variable $\alpha_0$ such that $X = \mathbb{E}_0(X) + \alpha_0 \Delta S$ and, therefore, $\mathbb{E}_0^Q(X) = \mathbb{E}_0(X) + \alpha_0 \mathbb{E}_0^Q(\Delta S) = \mathbb{E}_0(X) + \alpha_0 \mathbb{E}_0^Q(\Delta S) = \mathbb{E}_0^Q(\Delta S) \leq 0$, so $Q \in Q$. By applying Theorem 2.17 in [8] we deduce that $Q = M_{sp}(0; C_0, \Delta S)$ with $C_0 = \{ \alpha_0 \in L^{\infty}(F_0; \mathbb{R}^d) : \alpha_0 \Delta S \in A \}$.

(2) We define the set $K$ to be the closure of $C_0$ in $L^{\infty}(\mathbb{R}^d)$ with respect to the topology of convergence in measure and verify that $K = \{ \alpha_0 \in L^{0}(F_0; \mathbb{R}^d) : \alpha_0 Y^i \leq 0 \text{ for all } i = 1\ldots k \}$ where $Y^i = \mathbb{E}_0^Q(\Delta S)$ and $(Q^1, \ldots, Q^k)$ are the $F_0$-extreme points of $Q$. The set $K$ is an $F_0$-stable closed convex cone in $L^{0}(F_0; \mathbb{R}^d)$. Thanks to Theorem 4.6 in [9] there exists a random closed convex cone $W$ in $\mathbb{R}^d$ such that $\alpha_0 \in K$ if and only if $\alpha_0 \in L^{0}(F_0; \mathbb{R}^d)$ and $\alpha_0 \in W$ a.s. which means that $\alpha_0 Y^i \leq 0$ a.s. for $i = 1\ldots k$. We denote by $g_0 = (g_0^1, \ldots, g_0^d)$ the generating family of $W$, i.e., for all $\alpha_0 \in W$, there exists positive $F_0$-measurable scalar random variables $\lambda^1, \ldots, \lambda^d$ such that $\alpha_0 = \lambda^i g_0^i$. We shall prove that $Q = M_{sp}(0; g_0, \Delta S)$. For the direct inclusion we have $\mathbb{E}_0^Q(g_0, \Delta S) \leq 0$ for all $Q \in Q$ since $g_0 \in K$. For the inverse inclusion let $Q \in M_{sp}(0; g_0, \Delta S)$. Then there exists a sequence $\alpha_n^0 \in C_0$ such that $X = \lim_{n \to \infty} \alpha_n^0 \Delta S$ with the limit taken in weak star topology. There exists then a sequence $\lambda^0_n \geq 0$ such that $\alpha_n^0 = \lambda^0_n g_0$; therefore, $\alpha_n^0 \Delta S = \lambda^0_n g_0 \Delta S$ and then $\mathbb{E}_n^Q(X) = \lim_{n \to \infty} \mathbb{E}_n^Q(\lambda^0_n g_0 \Delta S) = \lim_{n \to \infty} \lambda^0_n \mathbb{E}_0^Q(\Delta S) \leq 0$. So $Q \in Q$.

Now we prove Theorem 4.1.

Proof We shall show first that for each $t \in T$, the assumption $MRP(S_t, S_{t+1})$ is satisfied on the one-period model $(t, t+1) + \Delta t S := S_{t+1} - S_t$. Indeed for a process $X(t, X_{t+1})$ with $E_t(X_{t+1}) = X_t$ we define the process $Y$ by $Y_s = E_s(X_{t+1})$ for $s \in T$ and remark that $Y$ is a martingale. So there exists a process $\alpha$ such that $Y = Y_0 + \alpha \bullet S$. In particular, we get $X_{t+1} - X_t = Y_{t+1} - Y_t = \alpha_t \Delta_t S$.

We denote by $Q^t$ for $t \in T^*$, the set of probability measures $Q \in P$, defined on $(\Omega, F_{t+1})$ such that $E_t^Q(X) \leq 0$ for all $X \in A_{t+1}$. For Assertion (1) we apply Assertion (1) in Theorem 4.2 and obtain that $Q^t = M_{sp}(0; C_t, \Delta t S)$. Now to prove that $Q^{it} = M_{sp}(C \bullet S)$, we remark that the direct inclusion is trivial from the definition of $C$. For the inverse inclusion let $Q \in M_{sp}(C \bullet S)$ and $X \in A^{it}$. So $X = X_0 + \cdots + X_{T-1}$ with each $X_t \in A_{t+1}$, then $E_t^Q(X_t) = E_t^Q(X_t)$ where $Q^t$ is the restriction of $Q$ on $F_{t+1}$. So $Q^t \in Q^t$ and, therefore, $E_t^Q(X_t) = E_t^Q(X_t) \leq 0$.

For Assertion (2) we shall show that $Q^t$ has a finite number of $F_t$-extreme points. Let $(Q^1, \ldots, Q^k)$ be the extreme points of  $Q$ with their respective densities $(Z^1, \ldots, Z^k)$. Then the probability measures $(Q^{1}, \ldots, Q^{k})$, defined on $F_{t+1}$ by their respective densities $(Z^{1}, \ldots, Z^{k}) := (Z_{t+1}^{1}, \ldots, Z_{t+1}^{k})$ are the $F_t$-extreme points of $Q^t$. Indeed we have $A_{t+1}$ as the dual cone of $Q^t$ in $L^{\infty}(F_{t+1})$ and $A_{t+1} := \{ X \in L^{\infty}(F_{t+1}) : E_t^Q(X) \leq 0 \text{ for all } Q \in Q \} = \{ X \in L^{\infty}(F_{t+1}) : E_t^{Q^i}(X) \leq 0 \text{ for all } i = 1\ldots k \}$. 

Remark 4.3 For the finite sample space case, all assertions of Theorem 4.1 are satisfied since the singleton $\{ P \}$ is the unique martingale measure for a vector-valued adapted process $S$, and then the assumption $MRP(S)$ is satisfied.
We prove easily that $\mathcal{Q} := [(1/2, 1/2), (1/3, 2/3)] = \mathcal{M}_{sp}(0; X)$ since $\mathcal{Q}$ has only two extreme points with $X = (X^1, X^2), X^1 = (1, -1)$ and $X^2 = (-2, 1)$.

5 Applications

We suppose again that the assumption $\text{MRP}(S)$ is satisfied along this section and investigate the decomposition of $\mathcal{Q}$-supermartingales.

**Theorem 5.1** Let $\mathcal{Q} \in \Pi^{c.e}$. Then any bounded $\mathcal{Q}$-supermartingale $X$ can be decomposed as follows: $X = X_0 + \alpha \ast S + C$ with $C$ an increasing process and the process $\alpha \ast S$ is a $\mathcal{Q}$-supermartingale.

**Proof** For $t \in I^*$ we denote by $\mathcal{Q}'$ for $t \in I^*$, the set of probability measures $\mathcal{Q} \in \mathcal{P}$, defined on $(\Omega, \mathcal{F}_{t+1})$ such that $\mathbb{E}_{\mathcal{Q}}^0(X) < 0$ for all $X \in \mathcal{A}_{t,t+1}$. It has been proved in the proofs of Theorems 4.1 and 4.2 that the assumption $\text{MRP}(S, S_{t+1})$ is satisfied on the one-period model $\{t, t + 1\}$ with $\Delta S := S_{t+1} - S_t$ and that $\mathcal{Q}'$ is optionally m-stable with respect to $\Delta S$.

We apply then Proposition 3.3 in [2] and obtain that $\mathcal{A}_{t,t+1} \subseteq K_t, \Delta S + \mathbb{L}_t^0(\mathcal{F}_{t+1})$ where $K_t := \{\alpha_t \in \mathbb{L}_t^0(\mathcal{F}_{t+1}; \mathbb{R}^d) \mid \alpha_t \Delta S \in \mathcal{A}_{t,t+1}\}$. So for a bounded $\mathcal{Q}$-supermartingale $X$ we have $X_{t+1} - X_t \in \mathcal{A}_{t,t+1}$ for each $t \in I^*$, and then there exists some $\alpha_t \in K_t$ and $B_t \in \mathbb{L}_t^0(\mathcal{F}_{t+1})$ such that $X_{t+1} - X_t = \alpha_t \Delta S - B_t$. We deduce that $X = X_0 + \alpha \ast S + C$ with $C_0 = 0$ and $C_t = B_0 + \cdots + B_{t-1}$ for $t \in \{1, \ldots, T\}$. \hfill \Box

**Corollary 5.2** Let $\mathcal{Q} \in \Pi^{c.e}$. Then any bounded random variable $Y$ can be written as follows: $Y = \mathbb{E}_0^q(Y) + (\alpha \ast S)_T - B$ with $B$ a positive random variable and the process $\alpha \ast S$ is a $\mathcal{Q}$-supermartingale.

**Proof** We apply Theorem 5.1 for the $\mathcal{Q}$-supermartingale $\mathbb{E}_0^q(Y)$. \hfill \Box

An immediate consequence of Theorem 5.1 is as follows:

**Corollary 5.3** Let $\mathcal{Q} \in \Pi^{c.e}$. Then any bounded $\mathcal{Q}$-martingale $X$ can be decomposed as follows: $X = X_0 + \alpha \ast S$.

Next we investigate the case where the process $\alpha \ast S$ appearing in Theorem 5.1 is a local $\mathcal{Q}$-martingale. We generalize then the result of Föllmer and Kabanov [7].

**Theorem 5.4** Let $\mathcal{Q} \in \Pi^{c.e}$ such that $\mathcal{Q} = \mathcal{Q}'$. Then any bounded $\mathcal{Q}$-supermartingale $X$ can be decomposed as follows: $X = X_0 + \alpha \ast S - C$ with $C$ an increasing process and the process $\alpha \ast S$ is a local $\mathcal{Q}$-martingale.

To prove Theorem 5.4, we state first some preliminary Lemmas.

**Lemma 5.5** Let $\mathcal{Q} \in \Pi^{c.e}$. Then $\mathcal{A}' \subseteq K' + L_0$ where $K'$ is the closure in $L_0$ of $\text{lin}(A)$.

**Proof** We shall prove that the set $K' + L_0$ is closed in $L_0$. We prove it first for the one-period model. We suppose then that $T = 1$ and $\mathcal{F}_0$ is not necessarily trivial. Let us define $D := \{\alpha \in L_0(\mathcal{F}_0; \mathbb{R}^d) \mid \alpha \Delta S \in K'\}$ and verify easily that $D$ is an $\mathcal{F}_0$-stable closed vector space in $L_0(\mathcal{F}_0; \mathbb{R}^d)$. So thanks to Lemma A.4 in [11] and Lemma 2.5 in [4] there exists a generating family $g = (g_1, \ldots, g_d)$ of the set $D$. We deduce that $K' = \{\alpha Y : \alpha \in L_0(\mathcal{F}_0; \mathbb{R}^d), Y = g \Delta S\}$. We apply Lemma 2.1 in [10] to conclude that $K' + L_0$ is closed in $L_0$. Now for the multi-period case we proceed as follows: for all $t \in I^*$ we define $D_t := \{\alpha_t \in L_0(\mathcal{F}_t; \mathbb{R}^d) \mid \alpha_t \Delta S \in K'\}$, $g_t = (g_1, \ldots, g_d)$ the generating family of $D_t$. $Y_t = g_t \Delta S$ and $B_t = (\alpha_t, Y_t) : \alpha \in L_0(\mathcal{F}_t; \mathbb{R}^d)$. We shall prove by induction on $t = T - 1, \ldots, 0$ that the set $B'_t := B_t + \cdots + B_{t+1} + L_0$ is closed in $L_0$. For $t = T - 1$ we apply the one-period case. Now we suppose that the sets $B^{t+1}, \ldots, B^{T-1}$ are closed, we shall prove that $B'_t$ is closed. We remark that $B'_t = B_t + B^{t+1}$ and follow the proof of Proposition 3.3 in [2]. Define the set $N := \{\alpha_t \in L_0(\mathcal{F}_t; \mathbb{R}^d) : -\alpha_t, Y_t \in B^{t+1} \text{ a.s.}\}$.

We prove easily that $N$ is a closed $\mathcal{F}_t$-stable convex cone in $L_0(\mathcal{F}_t; \mathbb{R}^d)$. We prove now that it is a vector space. Let $\alpha_t \in N$ so there exists $\alpha_{s, t} \in L_0(\mathcal{F}_s; \mathbb{R}^d)$ for $s = t + 1, \ldots, T - 1$ and $z \in L_0^+$ such that $\alpha_t = \alpha_s + \cdots + \alpha_{T-1} = z \geq 0$; therefore, $z = 0$ and $\alpha_t = -\alpha_{s+1} - \cdots - \alpha_{T-1} = z^{t+1}$ which means that $-\alpha_t \in N$. We denote by $N^{\perp}$ the orthogonal vector space of $N$. The decomposition of $B'$ becomes

\[\mathcal{Q} = \mathcal{Q}' + N^{\perp}\]
\(B^t = N_{t-1}^t Y_t + B_t^{t+1}\). Let \(X^n \in B^t\) be a sequence converging in measure to some \(X\). Then, there exists a subsequence, denoted also by \(X^n\), converging a.s. to \(X\). So there exists some \(\alpha_t^0 \in N_{t-1}^t\) and \(Y^n \in B_t^{t+1}\) such that \(X^n = \alpha_t^0 Y_t + Y^n\). We claim that \(\liminf_{n \to \infty} |\alpha_t^n| < \infty\) a.s. In fact, if we suppose the opposite, then there exists an integer-valued random sequence \(n_k\) such that \(\limk_{k \to \infty} |\alpha_t^{n_k}| = \infty\). Therefore,

\[
\frac{X_{n_k}}{|\alpha_t^{n_k}|} = \frac{\alpha_t^{n_k}}{|\alpha_t^{n_k}|} Y_t + \frac{Y_{n_k}}{|\alpha_t^{n_k}|} =: \lambda_t^{n_k} Y_t + J^{n_k},
\]

where \(\lambda_t^{n_k} \in N_{t-1}^t\) is bounded and \(J^{n_k} \in B_t^{t+1}\). Then, there exists a subsequence \(\lambda_t^{n_k}\), converging a.s. to some \(\lambda_t \in N_{t-1}^t\). Therefore, the sequence \(J^{n_k}\) converges a.s. to some \(J \in B_t^{t+1}\) and then \(\lambda_t, Y_t + J = 0\), which means that \(\lambda_t \in N\) and since \(\lambda_t \in N_{t-1}^t\) then \(\lambda_t = 0\) which contradicts the fact that \(|\lambda_t| = \limk_{k \to \infty} |\lambda_t^{n_k}| = 1\). Now, since \(\liminf_{n \to \infty} |\alpha_t^n| < \infty\) a.s., there exists an integer-valued random sequence \(n_k\) such that \(\alpha_t^{n_k} \in L_0^0(F_t; \mathbb{R}^d)\) converges a.s. to some \(\alpha_t \in L_0^0(F_t; \mathbb{R}^d)\) and then \(Y_{n_k} \in B_t^{t+1}\) converges a.s. to some \(Y \in B_t^{t+1}\) since \(B_t^{t+1}\) is closed by induction hypothesis. We conclude that \(B^t\) is closed and then \(B_0^0 = K^t + L_0^0\) is closed.

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