A NOTE ON ENDMORPHISMS AND C*-ALGEBRAS OF GRAPHS

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Abstract. It is a well-known fact, first noted by Arveson [1], that endomorphisms of $B(H)$ are intimately connected with families of mutually orthogonal isometries, i.e. with representations of the so-called Toeplitz C*-algebras. In this paper we consider a natural generalization of this connection between the representation theory of certain C*-algebras associated to graphs and endomorphisms of certain subalgebras of $B(H)$.

In [3], Laca determines that given a normal *-endomorphism $\alpha$ of $B(H)$ there exists an $n \leq \infty$ and *-representation $\pi : \mathcal{E}_n \to B(H)$, where $\mathcal{E}_n$ denotes the Toeplitz algebra for $n$ orthogonal isometries $v_1, \ldots, v_n$, such that

$$\alpha(T) = \sum_{i=1}^{n} \pi(v_i)T\pi(v_i)^*$$

for each $T \in B(H)$. The $n$ value is unique but the representation $\pi$ may differ by automorphisms of $\mathcal{E}_n$ which arise from unitary transformations of the Hilbert space $\ell^2(\{v_1, \ldots, v_n\}) \subseteq \mathcal{E}_n$ [3 Proposition 2.2].

Our goal is to extend the connection between endomorphisms and representations to a class of C*-algebras termed “Toeplitz algebras for C*-correspondences” which include the classical Toeplitz algebras. Why we are considering this class of C*-algebras and not the perhaps more natural “graph C*-algebras” will be made apparent in due time.

1. Preliminaries

First we will establish our terminology and notation.

Definition 1.1. A graph is a tuple $E = (E^0, E^1, r, s)$ consisting of a vertex set $E^0$, an edge set $E^1$, and range and source maps $r, s : E^1 \to E^0$.

We will only consider graphs where $E^0$ and $E^1$ are at most countable.

Definition 1.2. Let $A$ be a C*-algebra. A set $X$ is a C*-correspondence over $A$ provided that it is a right Hilbert $A$-module and there is a *-homomorphism $\phi : A \to L(X)$, where $L(X)$ denotes the space of adjointable $A$-module homomorphisms from $X$ to itself.

Given $X$ a C*-correspondence over $A$, we will denote the $A$-valued inner product on $X$ by $\langle x, y \rangle_A$ (perhaps omitting the $A$) and the right action will be written as “$x \cdot a$”. The map $\phi$ may sometimes be written as $\phi_X$ for clarity.

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Our primary objects of study will be certain $C^*$-correspondences which arise from graphs. The following construction is due originally to Fowler and Raeburn [2, Example 1.2].

**Definition 1.3.** Given a graph $E$, the graph correspondence $X(E)$ is the set of all functions $x : E^1 \to \mathbb{C}$ for which $\hat{x}(v) := \sum_{e \in s^{-1}(v)} |x(e)|^2$ extends to a function $\hat{x} \in C_0(E^0)$. We give $X(E)$ the structure of a $C^*$-correspondence over $C_0(E^0)$ as follows:

$$x \cdot a : e \mapsto x(e)a(s(e)),$$

$$\phi(a)x : e \mapsto a(r(e))x(e),$$

$$\langle x, y \rangle : v \mapsto \sum_{e \in s^{-1}(v)} \overline{x(e)}y(e).$$

which is to say that $a \in C_0(E^0)$ acts on the right of $X(E)$ as multiplication by $a \circ s$ and acts on the left as multiplication by $a \circ r$.

Note that this structure reverses the roles of $r$ and $s$ as in [2]. The sets $\{\delta_e : e \in E^1\}$ and $\{\delta_v : v \in E^0\}$ are dense in $X(E)$ and $C_0(E^0)$, respectively, in the appropriate senses. For $e \in E^1$ and $v \in E^0$ we have the following useful relations:

$$\langle \delta_e, \delta_e \rangle = \delta_{s(e)}, \delta_e \cdot \delta_v = \delta_e \text{ if } v = s(e) \text{ and is 0 otherwise, and } \phi(\delta_v)\delta_e = \delta_e \text{ if } v = r(e) \text{ and is 0 otherwise.}$$

**Definition 1.4.** [2, Example 1.2] Given a $C^*$-correspondence $X$ over $A$ and given another $C^*$-algebra $B$, a Toeplitz representation of $X$ in $B$ is a pair $(\sigma, \pi)$ consisting a linear map $\sigma : X \to B$ and a $*$-homomorphism $\pi : A \to B$ such that for all $x, y \in X$ and $a \in A$

1. $\sigma(x \cdot a) = \sigma(x)\pi(a),$
2. $\sigma(\phi(a)x) = \pi(a)\sigma(x)$, and
3. $\pi(\langle x, y \rangle) = \sigma(x)^* \sigma(y).$

For a graph correspondence $X(E)$ a Toeplitz representation $(\sigma, \pi)$ is determined entirely by the values $\{\sigma(\delta_e) : e \in E^1\}$ and $\{\pi(\delta_v) : v \in E^0\}$. Property (3) of a Toeplitz representation guarantees that $\sigma(\delta_e)$ is a partial isometry with source projection $\pi(\delta_{s(e)})$.

**Definition 1.5.** [2, Proposition 1.3] Given a $C^*$-correspondence $X$ over $A$, the Toeplitz algebra of $X$ is the $C^*$-algebra, denoted $T_X$, which is universal in the following sense: there exists a Toeplitz representation $(\sigma_u, \pi_u)$ of $X$ in $T_X$ such that if $(\sigma, \pi)$ is another Toeplitz representation of $X$ in a $C^*$-algebra $B$ then there exists a unique $*$-homomorphism $\rho_{\sigma, \pi} : T_X \to B$ such that $\sigma = \rho_{\sigma, \pi} \circ \sigma_u$ and $\pi = \rho_{\sigma, \pi} \circ \pi_u$.

That $T_X$ exists was proven by Pimnser in [5].

Given a graph $E$ we may consider the Toeplitz algebra of its graph correspondence, cumbersomely denoted $T_X(E)$. Unless there is danger of confusion, we will abuse notation and make no distinction between elements of $X(E)$ and $C_0(E^0)$ and their images in $T_X(E)$ under the universal maps $\sigma_u$ and $\pi_u$.

If $r : T_X(E) \to B(H)$ is a $*$-representation then, for each $v \in E^1$, $\tau(\delta_v)$ is a partial isometry with source projection $\tau(\delta_{s(v)})$ and range projection contained in $\tau(\delta_{r(v)})$. 
If \( E \) is the graph with but a single vertex and \( n \) edges then \( X(E) \) is a Hilbert space of dimension \( n \) and \( T_X(E) \) is isomorphic to the classical Toeplitz algebra \( \mathcal{E}_n \). In this case the elements \( \{ \delta_e : e \in E^1 \} \) are precisely the generating isometries of \( \mathcal{E}_n \). The space \( X(E) \) plays a significant role in Laca’s analysis of endomorphisms of \( B(H) \), and it is for this reason that we are considering the generalized Toeplitz algebras \( T_X(E) \) in our investigations.

2. Coherent Unitary Equivalence

Two graphs \( E \) and \( F \) are isomorphic if there are two bijections \( \psi^0 : E^0 \to F^0 \) and \( \psi^1 : E^1 \to F^1 \) for which \( r_F \circ \psi^1 = \psi^0 \circ r_E \) and \( s_F \circ \psi^1 = \psi^0 \circ s_E \). In order to encode such an isomorphism at the level of the graph correspondences \( X(E) \) and \( X(F) \), we offer the following definition.

**Definition 2.1.** Let \( X \) and \( Y \) be \( C^* \)-correspondences over \( A \) and \( B \), respectively. A coherent unitary equivalence between \( X \) and \( Y \) is a pair \((U, \alpha)\) consisting of a bijective linear map \( U : X \to Y \) and a \( * \)-isomorphism \( \alpha : A \to B \) for which

1. \( U(x \cdot a) = (Ux) \cdot \alpha(a) \) for all \( x \in X \) and \( a \in A \),
2. \( U(\phi_X(a)x) = \phi_Y(\alpha(a))Ux \) for all \( x \in X \) and \( a \in A \), and
3. \( \langle Ux, y \rangle_Y = \alpha(\langle x, U^{-1}y \rangle_x) \) for all \( x \in X \) and \( y \in Y \).

Routine calculations will verify that coherent unitary equivalence is an equivalence relation.

**Proposition 2.2.** If \( E \) and \( F \) are isomorphic graphs then \( X(E) \) and \( X(F) \) are coherently unitarily equivalent.

**Proof.** We’ll assume \((\psi^0, \psi^1)\) to be an isomorphism from \( F \) to \( E \).

For \( a \in C_0(E^0) \), \( \alpha(a) := a \circ \psi^0 \) clearly defines a \( * \)-isomorphism \( \alpha : C_0(E^0) \to C_0(F^0) \). For \( x \in X(E) \) define \( Ux := x \circ \psi^1 \). For \( v \in F^1 \) we have

\[
\sum_{e \in s_E^{-1}(v)} |Ux(e)|^2 = \sum_{e \in s_E^{-1}(v)} |x(\psi^1(e))|^2 = \sum_{f \in s_E^{-1}(\psi^0(v))} |x(f)|^2
\]

(using the fact that if \( s_E(v) = v \) then \( s_F(\psi^1(v)) = \psi^0(v) \)) and so \( \hat{U}x = \hat{x}(\psi^1(v)) \).

As \( \hat{x} \in C_0(E^0) \) it follows immediately that \( \hat{U}x \in C_0(F^0) \), i.e. \( Ux \in X(F) \). Identical arguments show that \( U^{-1}y := y \circ (\psi^1)^{-1} \) is a map from \( X(F) \) to \( X(E) \) which is a two-sided inverse for \( U \). Hence \( U : X(E) \to X(F) \) is a bijection which is naturally linear.

Given \( x \in X(E), \ a \in C_0(E^0), \) and \( e \in E^1 \) we have

\[
U(x \cdot a) = (x(a \circ s_E)) \circ \psi^1 = (x \circ \psi^1)(a \circ s_E \circ \psi^1) = (Ux)(a \circ \psi^0 \circ s_E) = Ux \cdot \alpha(a)
\]

\[
U(\phi_E(a)x) = ((a \circ r_E)x) \circ \psi^1 = (a \circ r_E \circ \psi^1)(x \circ \psi^1) = (a \circ \psi^0 \circ r_F)(Ux) = \phi_F(\alpha(a))Ux
\]

and, given \( v \in F^0 \),

\[
\langle Ux, y \rangle(v) = \sum_{e \in s_E^{-1}(v)} \overline{Ux(e)}y(e) = \sum_{e \in s_E^{-1}(v)} \overline{x(\psi^1(e))}y(e) = \sum_{f \in s_E^{-1}(\psi^0(v))} \overline{x(f)}y((\psi^1)^{-1}(f))
\]

\[
= \sum_{f \in s_E^{-1}(\psi^0(v))} \overline{x(f)}U^{-1}y(f) = \langle x, U^{-1}y \rangle(\psi^0(v)) = \alpha(\langle x, U^{-1}y \rangle)(v)
\]

(the first inner product is that of \( X(F) \) and the later two are that of \( X(E) \)). Thus the pair of \( U \) and \( \alpha \) satisfies the definition of a coherent unitary equivalence. \( \square \)
Not every coherent unitary equivalence is built from a graph isomorphism in the sense of the preceding Proposition. As a simple example, consider the graph $E$ with but a single vertex $v$ and two edges $e_1$ and $e_2$. In this case $C_0(E^0) = \mathbb{C}$ and $X(E) = \mathbb{C}^2$. Hence any unitary $U \in M_2(\mathbb{C})$ forms (with the identify on $C_0(E^0)$) a coherent unitary equivalence. However, the only such equivalences arising from graph isomorphisms would be those of the two permutation matrices in $M_2(\mathbb{C})$.

**Proposition 2.3.** If there is a coherent unitary equivalence between $X$ and $Y$ then $T_X$ and $T_Y$ are *-isomorphic.

**Proof.** Let $A$ and $B$ be the coefficient $C^*$-algebras for $X$ and $Y$, respectively. Suppose that $(U, \alpha)$ is a coherent unitary equivalence between $X$ and $Y$ and let $(\sigma, \pi)$ be a Toeplitz representation of $Y$. For $x \in X$ and $a \in A$ 
$$
\sigma(U(x \cdot a)) = \sigma(U x \alpha(a)) = \sigma(U x) \pi(\alpha(a)) \\
\sigma(U(\phi_X(a)x)) = \sigma(\alpha(a)U x) = \pi(\alpha(a))\sigma(U x)
$$
and for $x_1, x_2 \in X$
$$
\pi \circ \alpha(\langle x_1, x_2 \rangle_A) = \pi((U x_1, U x_2)_B) = \sigma(U x_1)^* \sigma(U x_2).
$$
Hence $(\sigma \circ U, \pi \circ \alpha)$ is a Toeplitz representation of $X$.

In particular, $(\sigma_Y \circ U, \pi_B \circ \alpha)$ is a Toeplitz representation of $X$ where $(\sigma_Y, \pi_B)$ is the universal Toeplitz representation of $Y$ in $T_Y$. By the universal property of $T_X$, there is a *-homomorphism $\theta : T_X \rightarrow T_Y$ such that $\theta \circ \sigma_X = \sigma_Y \circ U$ and $\theta \circ \pi_A = \pi_B \circ \alpha$, where $(\sigma_X, \pi_A)$ is the universal representation of $X$ in $T_X$.

Similarly $(\sigma_X \circ U^{-1}, \pi_A \circ \alpha^{-1})$ is a Toeplitz representation of $Y$ and induces a *-homomorphism $\theta' : T_Y \rightarrow T_X$ for which $\theta' \circ \sigma_Y = \sigma_X \circ U^{-1}$ and $\theta' \circ \pi_B = \pi_A \circ \alpha^{-1}$. Thus
$$
\sigma_Y = \sigma_Y \circ U \circ U^{-1} = \theta \circ \sigma_X \circ U^{-1} = \theta \circ \theta' \circ \sigma_Y
$$
and similarly $\pi_B = \theta \circ \theta' \circ \pi_B$. Since the identity $id$ on $T_Y$ also has the property that $\pi_B = id \circ \pi_B$ and $\sigma_Y = id \circ \sigma_Y$, it follows by the universal property of $T_Y$ that $\theta \circ \theta' = id$. Identical reasoning verifies that $\theta' \circ \theta$ is the identity on $T_X$. Thus $\theta$ is our desired *-isomorphism. $\square$

Going forward we will be exclusively interested in Toeplitz algebras associated to graph correspondences, and so offer the following corollary.

**Corollary 2.4.** Let $E$ and $F$ be graphs. If $(U, \alpha)$ is a coherent unitary equivalence between $X(E)$ and $X(F)$ then there is a *-isomorphism $\Gamma_{U,\alpha} : T_{X(E)} \rightarrow T_{X(F)}$ for which $\Gamma_{U,\alpha}(\delta_e) = U \delta_e$ and $\Gamma_{U,\alpha}(\delta_v) = \alpha(\delta_v)$ for all $e \in E^1$ and $v \in E^0$.

This is immediately seen from the proof of the previous proposition if we recall that we identify $X(E)$ and $X(F)$ with their images in $T_{X(E)}$ and $T_{X(F)}$, respectively, under the appropriate universal maps.

3. **Endomorphisms from Graphs**

Throughout this section we will let $E$ be a graph. All *-representations will be assumed non-degenerate.

**Proposition 3.1.** Given a *-representation $\tau : T_{X(E)} \rightarrow B(H)$, the assignments
$$
Ad_{\tau}(w) = \sum_{e \in E^1} \tau(\delta_e) w \tau(\delta_e)^*,
$$
(the sum is taken as a SOT limit) define a $\ast$-endomorphism $Ad_{\tau}$ of the von Neumann algebra $W = \{\tau(\delta_v) : v \in E^0\}$.

**Proof.** First, notice that for $e \in E^1$ and $w \in W$ the term $\tau(\delta_e)w\tau(\delta_e)^*$ has its support projection contained in $\tau(\delta_e^*\delta_e)$. Since the partial isometries $\tau(\delta_e)$ have mutually orthogonal ranges, it follows that for every $h \in H$, $\tau(\delta_e)w\tau(\delta_e)^*h$ is nonzero for at most one $e \in E^1$. Thus the sum converges in the SOT.

Certainly $Ad_{\tau}$ is linear and has $Ad_{\tau}(w^*) = Ad_{\tau}(w)^*$ for each $w \in W$. Given $w_1, w_2 \in W$ we find that

$$Ad_{\tau}(w_1)Ad_{\tau}(w_2) = \left( \sum_{e \in E^1} \tau(\delta_e)w_1\tau(\delta_e)^* \right) \left( \sum_{f \in E^1} \tau(\delta_f)w_2\tau(\delta_f)^* \right)$$

$$= \sum_{e, f \in E^1} \tau(\delta_e)w_1\tau(\delta_e)^*\tau(\delta_f)w_2\tau(\delta_f)^*$$

$$= \sum_{e \in E^1} \tau(\delta_e)w_1\tau(\delta_{s(e)})w_2\tau(\delta_e)^*$$

$$= \sum_{e \in E^1} \tau(\delta_e)\tau(\delta_{s(e)})w_1w_2\tau(\delta_e)^*$$

$$= \sum_{e \in E^1} \tau(\delta_e)w_1w_2\tau(\delta_e)^*$$

$$= Ad_{\tau}(w_1w_2)$$

and so $Ad_{\tau}$ is multiplicative. Note that any potential issues with SOT-convergence of the product are circumvented by $E^1$ being at most countable. All that remains is to verify that $Ad_{\tau}(w) \in W$ for each $w \in W$. To that end we first note that $\delta_v^*\delta_v = \delta_v^*$ if $v = r(e)$ and is zero otherwise. By taking adjoints, $\delta_v\delta_v = \delta_v$ if $v = r(e)$ and is zero otherwise. Thus, given $w \in W$ and $v \in E^0$ we find

$$Ad_{\tau}(w)\tau(\delta_v) = \sum_{e \in r^{-1}(v)} \tau(\delta_e)w\tau(\delta_e)^* = \tau(\delta_v)Ad_{\tau}(w)$$

and so $Ad_{\tau}(w)$ commutes with each $\tau(\delta_v)$. $\square$

The following is a construction which we believe to be folklore, but use of it is motivated by observations made by Muhly and Solel \[4]. Given a $\ast$-representation $\tau : T_X(E) \to B(H)$ let $W = \{\tau(\delta_v) : v \in E^0\}'$. The space

$$\mathcal{I}_\tau := \{T \in B(H) : Ad_{\tau}(w)T = Tw, \; w \in W\}$$

is a $C^*$-correspondence over $W'$. The left and right actions of $W'$ are simply multiplication within $B(H)$ and the $W'$-valued inner product is defined by $\langle T, S \rangle_{W'} := T^*S$.

Because our endomorphism is of the form $Ad_{\tau}$, we can say more: for $w \in W$ and $e \in E^1$

$$Ad_{\tau}(w)\tau(\delta_e) = \sum_{f \in E^1} \tau(\delta_f)w\tau(\delta_f)^*\tau(\delta_e) = \tau(\delta_e)w\tau(\delta_{s(e)}) = \tau(\delta_e)\tau(\delta_{s(e)})w = \tau(\delta_e)w$$

and so $\tau(\delta_e) \in \mathcal{I}_\tau$ for each $e \in E^1$. As $\tau(\delta_v) \in W'$ for each $v \in E^0$ we finally have $\tau(X(E)) \subseteq \mathcal{I}_\tau$. 

Theorem 3.2. Suppose that $\tau_1$ and $\tau_2$ are two faithful $*$-representations of $T_{X(E)}$. If $\text{Ad}_{\tau_1} = \text{Ad}_{\tau_2}$ on $W = \{\tau_1(\delta_v) : v \in E^0\}' = \{\tau_2(\delta_v) : v \in E^0\}'$ then there is a coherent unitary equivalence $(U, \alpha)$ between $X(E)$ and itself such that $\tau_2 = \tau_1 \circ \Gamma_{U, \alpha}$.

Here $\Gamma_{U, \alpha}$ is the $*$-automorphism of $T_{X(E)}$ as defined in Corollary 2.3.

Proof. Since $\{\tau_1(\delta_v) : v \in E^0\}$ and $\{\tau_2(\delta_v) : v \in E^0\}$ are sets of orthogonal projections with the same commutant they are in fact equal. To ease notation we’ll denote these projections by $P_v$, $v \in E^0$, (with no assumption that $P_v = \tau_1(\delta_v)$ or similar) hence

$$\{P_v : v \in E^0\} = \{\tau_1(\delta_v) : v \in E^0\} = \{\tau_2(\delta_v) : v \in E^0\}.$$  

As $\text{Ad}_{\tau_1} = \text{Ad}_{\tau_2}$ we have that $\mathcal{I}_{\tau_1} = \mathcal{I}_{\tau_2}$ and we’ll call this module simply $\mathcal{I}$.

As $\tau_1(\delta_c) \in \mathcal{I}$ for each $c \in E^1$ we have

$$\tau_1(\delta_c) = \tau_1(\delta_c)I = \text{Ad}_{\tau_2}(I)\tau_1(\delta_c) = \sum_{f \in E^1} \tau_2(\delta_f)\tau_2(\delta_f)^*\tau_1(\delta_c)$$

hence $\tau_1(\delta_c)$ is in the $W'$-submodule of $\mathcal{I}$ generated by $\tau_2(X(E))$. Similarly, for each $c \in E^1$, $\tau_2(\delta_c)$ is in the $W'$-submodule generated by $\tau_1(X(E))$. Thus they generate the same $W'$-submodule of $\mathcal{I}$.

Given $c, f \in E^1$ we have seen that

$$\tau_2(\delta_f)^* \tau_1(\delta_c) \in W' = \{P_v : v \in E^0\}'' = \ell^\infty(\{P_v : v \in E^0\}).$$

Notice however that $\tau_2(\delta_f)^* \tau_1(\delta_c)\tau_2(\delta_0) = 0$ unless $v = s(c)$ and hence $\tau_2(\delta_f)^* \tau_1(\delta_c)$ is a multiple of $\tau_1(\delta_{s(c)})$ only, i.e. is an element of $C_0(\{P_v : v \in E^0\})$. Since before we obtained $\tau_1(\delta_c) = \sum_{f \in E^1} \tau_2(\delta_f)\tau_2(\delta_f)^*\tau_1(\delta_c)$ for all $c \in E^1$, it now follows that $\tau_1(X(E))$ and $\tau_2(X(E))$ generate the same correspondence over $C_0(\{P_v : v \in E^0\})$. It is important to note that this correspondence has three different actions of $C_0(\{P_v : v \in E^0\})$: the ones inherited through $\tau_1$ and $\tau_2$ and simple operator multiplication in $B(H)$.

Finally we have that $\tau_1(C_0(E^0)) = \tau_2(C_0(E^0))$ and $\tau_1(X(E)) = \tau_2(X(E))$ as sets and, because both representations are faithful by hypothesis, so $\tau_2^{-1} \circ \tau_1$ is a well-defined bijection on both $X(E)$ and $C_0(E^0)$. Denote by $U$ and $\alpha$ the restrictions of $\tau_2^{-1} \circ \tau_1$ to $X(E)$ and to $C_0(E^0)$, respectively.

Given $x \in X(E)$ and $a \in C_0(E^0)$ we have

$$U(xa) = \tau_2^{-1} \circ \tau_1(xa) = \tau_2^{-1} \circ \tau_1(x) \tau_2^{-1} \circ \tau_1(a) = (Ux)\alpha(a),$$

$$U(\phi(a)x) = \tau_2^{-1} \circ \tau_1(\phi(a)x) = \tau_2^{-1} \circ \tau_1(a) \tau_2^{-1} \circ \tau_1(x) = \alpha(a)Ux,$$

$$\langle Ux, y \rangle = [\tau_2^{-1} \circ \tau_1(x)]^*y = \tau_2^{-1} \circ \tau_1(x^* \tau_1^{-1} \circ \tau_2(y)) = \alpha((x, \tau_1^{-1} \circ \tau_2(y))) = \alpha((x, U^{-1}y)).$$

and so $(U, \alpha)$ is a coherent unitary equivalence between $X(E)$ and itself.

It follows from Corollary 2.3 that $(U, \alpha)$ induces an automorphism $\Gamma_{U, \alpha}$ of $T_{X(E)}$ and, by construction, $\tau_2 \circ \Gamma_{U, \alpha} = \tau_1$, as desired. $\square$

Our result is a generalization of Laca’s [3, Proposition 2.2]. When $E$ is the graph with a single vertex and $n \leq \infty$ edges we have already seen that $T_{X(E)} = \mathcal{E}_n$. If $\tau_1$ and $\tau_2$ are faithful and nondegenerate then $W = B(H)$. The map $\alpha$ is the identity on $C_0(E^0) = \mathbb{C}$ and $U$ is a unitary operator on the Hilbert space $X(E)$.

We will conclude this section with a brief discussion of conjugacy conditions for endomorphisms of the type we’ve been examining. Recall that two endomorphisms $\alpha$ and $\beta$ are said to be conjugate if there is an automorphism $\gamma$ such that $\alpha \circ \gamma = \gamma \circ \beta$. 


Lemma 3.3. If $P_1, P_2, \ldots \in B(H)$ is an at most countable family of orthogonal projections and $\gamma$ is a $*$-automorphism of $W = \{P_1, P_2, \ldots\}'$ then there exists a unitary $U \in B(H)$ such that $\gamma(w) = UwU^*$ for all $w \in W$.

Proof. Note that for each $n$, $\gamma$ restricts to a $*$-isomorphism $\gamma_n$ between $P_n B(H) P_n = B(P_n H)$ and $\gamma(P_n) B(H) \gamma(P_n) = B(\gamma(P_n) H)$. Such isomorphisms are always spatial and so there are unitaries $U_n : B(P_n H) \to B(\gamma(P_n) H)$ such that $\gamma_n(w) = U_n w U_n^*$. It is then immediate that $U = \bigoplus U_n$ is a unitary in $B(H)$ and $U w U^* = \gamma(w)$ for each $w \in W$.

□

Theorem 3.4. Suppose that $\tau_1, \tau_2 : T_{X(E)} \to B(H)$ are two faithful $*$-representations such that $Ad_{\tau_1}$ and $Ad_{\tau_2}$ are conjugate $*$-endomorphisms of $W = \{\tau_1(\delta_v) : v \in E^0\}' = \{\tau_2(\delta_v) : v \in E^0\}'$. Then there is a coherent unitary equivalence $(U, \alpha)$ between $X(E)$ and itself such that $\tau_2$ and $\tau_1 \circ \Gamma_{U, \alpha}$ are unitarily equivalent $*$-representations.

Proof. Let $\gamma$ be an $*$-automorphism of $W$ such that $Ad_{\tau_1} \circ \gamma = \gamma \circ Ad_{\tau_2}$ and let $V \in B(H)$ be the unitary for which $\gamma(w) = V w V^*$ according the Lemma 3.3. Then $Ad_{\tau_2}(w) = V^* Ad_{\tau_1}(V w V^*) V$ for all $w \in W$. Define $\kappa(t) := V \tau_1(t) V^*$ and note that $\kappa$ is a $*$-representation of $T_{X(E)}$ such that

$$Ad_{\kappa}(w) = \sum_{e \in E^1} \kappa(\delta_{\varepsilon}) w \kappa(\delta_{\varepsilon})^* = \sum_{e \in E^1} V \tau_1(\delta_{\varepsilon}) W \tau_1(\delta_{\varepsilon})^* V^* = V Ad_{\tau_1}(V^* w V) V^*$$

and so $Ad_{\kappa} = Ad_{\tau_2}$ on $W$. Applying Theorem 3.2 we obtain a coherent unitary equivalence $(U, \alpha)$ inducing the $*$-automorphism $\Gamma_{U, \alpha}$ of $T_{X(E)}$ such that $\tau_2 = \kappa \circ \Gamma_{U, \alpha}$. As now $\tau_2(t) = V \tau_1(\gamma(t)) V^*$ for each $t \in T_{X(E)}$, we have that $\tau_2$ and $\tau_1 \circ \Gamma_{U, \alpha}$ are unitarily equivalent, as desired.

□

4. Graphs from Endomorphisms

Theorem 4.1. Let $P_1, P_2, \ldots \in B(H)$ be pairwise disjoint projections, $W = \{P_1, P_2, \ldots\}'$, and $\alpha$ a normal $*$-endomorphism of $W$. Then there exists a graph $E$ and $*$-representation $\tau : T_{X(E)} \to B(H)$ such that $\alpha = Ad_\tau$.

Proof. Without loss of generality we may assume that $\sum P_i = I$. If this were not the case then the same procedure outlined below would yield a degenerate representation of $T_{X(E)}$.

For $i > 0$ define $H_i = P_i H$. For $i, j > 0$ and $x \in W$ define $\alpha_{ij}(x) = P_j \alpha(P_i x)$. Then $\alpha_{ij}$ restricts to a $*$-homomorphism between $B(H_i) = P_i B(H) P_i$ and $B(H_j) = P_j B(H) P_j$ as seen by

$$P_j \alpha(P_i x) P_j \alpha(P_j y) = P_j (P_j \alpha(P_i x)) \alpha(P_j y) = P_j \alpha(P_i x P_j y) = P_j \alpha(P_j x y).$$

Thus by Proposition 2.1 if $\alpha_{ij}$ is nonzero there exists $n_{ij} \in \mathbb{N} \cup \{\infty\}$ and isometries $V_{k}^{(ij)} \in B(H_i, H_j)$, $k = 1, \ldots, n_{ij}$ such that $\|\alpha_{ij}|_{B(H_i)}(T) = \sum_{k=1}^{n_{ij}} V_{k}^{(ij)} T V_{k}^{(ij)*}$.

We will identify the $V_{k}^{(ij)}$ with their associated partial isometries in $B(H)$, so that $V_{k}^{(ij)*} V_{k}^{(ij)} = P_i$ and $V_{k}^{(ij)} V_{k}^{(ij)*} \leq P_j$.

Set $E^0 = \{P_1, P_2, \ldots\}$ and $E^1 = \bigcup_{i,j} \{V_{k}^{(ij)} : k = 1, \ldots, n_{ij}\}$. Define maps $r, s : E^1 \to E^0$ by $r(V_{k}^{(ij)}) = P_j$ and $s(V_{k}^{(ij)}) = P_i$. Then $E = (E^0, E^1, r, s)$ is a graph. It is trivial to see that the identity maps on $E^1$ and $E^0$ extend to a Toeplitz covariant representation of $X(E)$, $\tau$. 
Finally, we have that for each $x \in W$

$$\alpha(x) = \sum_{i,j>0} P_j x\alpha(P_i x) = \sum_{i,j>0} \alpha_{ij}(x) = \sum_{i,j>0} \sum_{k=1}^{n_{ij}} V_k^{(ij)} x V_k^{(ij)*} = \sum_{f \in E^1} \tau(\delta_f) x \tau(\delta_f)^*$$

as desired. □

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