The solution of the heat conduction problem for a rectangular area with mixed boundary conditions and internal source in general by using fast expansions method

A D Chernyshov
Voronezh State University of Engineering Technologies, Revolution av., 19, Voronezh, Russia
E-mail: chernyshovad@mail.ru

Abstract. The fundamentals of the fast expansions method are outlined, provided that the mixed Dirichlet-Neumann boundary conditions are specified. Then, using the fast expansions method, the solution of the problem in an explicit analytical form with an accuracy of quadrature of Fourier coefficients is obtained. Such a solution allows to explore temperature fields in a rectangle, depending on the given input data of the task, which include the size of the rectangle, the boundary conditions and the internal source. The resulting solution for a rectangle can also be used to construct a solution to a problem in a curvilinear domain, for a two-phase Stefan problem with a curvilinear boundary, and for other applied problems.

1. Introduction

Метод The method of fast expansions is based on the ideas of work [1], where fast expansions are given for Dirichlet + Dirichlet boundary conditions, or Neumann + Neumann. In this article, boundary conditions of the mixed type of Dirichlet + Neumann are set. This leads to fundamental differences from the first case in the construction of the used boundary functions, the compatibility conditions of the boundary conditions of the problem and the direct organization of fast expansions. The main advantages of the fast expansions method are in obtaining a solution in an analytical form, the possibility of term-wise multiple differentiation of the used Fourier series, which are considered on a finite segment, and their fast convergence. Conditions for the term-wise differentiation of the Fourier series of periodic functions defined on an infinite interval are considered in classical literature on a finite interval in [2]. The issues of convergence of spectral expansions are given in [3,4] and others.

In fast expansions, a sufficiently smooth function \( f(x) \) is represented as the sum of a certain boundary function \( M_q(x) \) and a Fourier series for the difference \( f(x) - M_q(x) \). Due to the specially designed \( M_q(x) \), the rate of convergence of the Fourier series is quite high. For a given \( f(x) \) the coefficients, that determine the form of, \( M_q(x) \) and the coefficients of the Fourier series are determined in advance. The function of \( M_q(x) \) is determined using the special polynomials \( P_s(x) \), called fast polynomials and defined by double integrals by recurrence formulas. This is a significant difference between the method proposed here and the method used in works [5-8]. In articles [5-8], to increase the rate of convergence of a Fourier series, the authors form a special system, from which each time for given \( f(x) \) they again find the coefficients for constructing an improvement function.
In addition, the approach of the authors of the articles [5-8] is not generalized to the possibility of its application to solving multidimensional nonlinear boundary integro-differential problems for curvilinear domains. The fast expansions method allows to solve similar boundary value problems of heightened difficulty.

Solutions of multidimensional problems in partial derivatives with mixed boundary conditions are of definite applied and scientific interest. Such solutions for classical fields are considered in numerous literature, in [9] for the heat equation, in [10] for problems on the deformation of plates and shells, in [11,12] for elastic problems. The fast expansions method has proved to be especially effective in solving complex problems [13-16] and others, where numerical example are also given.

The objective of the work is to show the great possibilities of using the fast expansions method considering multidimensional boundary value problems when the input conditions are given in general form, which is impossible to do in cases of using other known analytical methods with an increased convergence rate of Fourier series.

2. The organization of fast expansions when specifying the boundary conditions of the mixed type of Dirichlet + Neumann

Let \( f(x) \in C^{q+1} \) (0 \( \leq x \leq a \)), (where \( q \) – some given integer), satisfy homogeneous boundary conditions without first setting any applied problem

\[
 f(0) = f'(a) = 0
\]

In (2.1), a function is given at the left end of the segment \( [0,a] \), and its derivative is given at the right end. In the future, to solve various problems, the use of Fourier series is assumed. There fore, we first consider the simplest trigonometric expression

\[
 f(x) = A \cos \lambda x + B \sin \lambda x, \quad x \in [0,a]
\]

Constants \( A, B, \lambda \) are defined so that the boundary conditions (2.1) are satisfied, i.e.

\[
 A = 0, \quad \lambda a = n\pi + \frac{\pi}{2}, \quad n = 0,1,2,...
\]

Using (2.1)-(2.3) on the segment \( [0,a] \) we can construct the full spectrum of eigenfunctions and eigenvalues:

\[
 \left\{ \sin \frac{\lambda_n x}{a}, \quad \lambda_n = \frac{1}{a} \left( n\pi + \frac{\pi}{2} \right), \quad x \in [0,a], \quad n = 0,1,2,... \right\}
\]

Here, the process of building the spectrum of eigenfunctions and eigenvalues is not tied to the Laplace operator, and there is also no need to consider the Sturm-Liouville problem. The obtained spectra will be used in the development of the fast expansions method. The fast expansions method proposed below is intended for solving multidimensional nonlinear integro-differential problems of an applied nature. The method uses Fourier series with a high rate of convergence. Usually in such cases, when deriving formulas for the Fourier coefficients, the integration formulas are reused in parts. When applied directly, a necessary condition is the continuity and smoothness of functions. If the functions are discontinuous, then in such formulas they take into account the magnitude of the discontinuity, which creates certain problems when considering applied multidimensional problems and significantly complicates the problem of their use when considering differential problems. In this connection, we will further assume that the functions under consideration are continuous and sufficiently smooth.

When considering applied multidimensional problems, in which it is necessary to take into account discontinuities, the discontinuity surfaces are curvilinear, and it creates great difficulties for obtaining a solution. Usually, in such cases, the region \( \Omega \), in which the solution is sought and inside which the
curvilinear surface of discontinuity $S$ is located, is divided into several subregions $\Omega = \Omega_1 \cup \Omega_2$, separated by the surface of discontinuity $S$. In each of the subregions $\Omega_1$, $\Omega_2$ the functions under consideration are continuous and smooth. To such functions in each of the subregions $\Omega_1$, $\Omega_2$ separately the application of the fast expansions method will be very effective, since the Fourier series will converge quickly and in order to maintain the required accuracy in their partial sums it will be sufficient to retain a small number of terms.

When dividing the $\Omega$ into $\Omega_1$, $\Omega_2$ part it is necessary to fulfill on the discontinuity surface $S$ some conditions for conjugating solutions in $\Omega_1$ and $\Omega_2$, that play the role of boundary conditions. These conditions depend on the formulation of the original problem. Therefore, in the future we will consider only continuous and sufficiently smooth functions.

A certain function $f(x) \in C^{(q+1)}$, $x \in [0,a]$ can be represented by a Fourier series in terms of eigenfunctions (2.4)

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{x}{a} \left( n \pi + \frac{\pi}{2} \right), \quad f(x) \in C^{(q+1)}, \ x \in [0,a],$$

$$B_n = \frac{2}{a} \int_0^a f(x) \sin \frac{x}{a} \left( n \pi + \frac{\pi}{2} \right) dx, \ n = 0,1,2,...$$

In (2.5) we have the classical Fourier series, which generally converges slowly and therefore it is impractical to use it. To find the conditions under which the series (2.5) will quickly converge, the expression for $B_n$ will integrate in parts

$$B_n = \frac{2}{a} \int_0^a f(x) \sin \frac{x}{a} \left( n \pi + \frac{\pi}{2} \right) dx = -\frac{2}{(n \pi + \pi/2)} \int_0^a f(x) d \cos \frac{x}{a} \left( n \pi + \frac{\pi}{2} \right) =$$

$$= \frac{2}{(n \pi + \pi/2)} \left[ -f(x) \cos \frac{x}{a} \left( n \pi + \frac{\pi}{2} \right) \right]_0^a + \int_0^a f'(x) \cos \frac{x}{a} \left( n \pi + \frac{\pi}{2} \right) dx =$$

$$= \frac{2}{(n \pi + \pi/2)} f(0) + \frac{2}{(n \pi + \pi/2)} \int_0^a f'(x) \cos \frac{x}{a} \left( n \pi + \frac{\pi}{2} \right) dx.$$  

From (2.6) it can be seen if, in addition to the smoothness condition written in (2.5), we additionally require the equality

$$f'(0) = 0,$$  

then the expression for $B_n$ will take the form

$$B_n = \frac{2}{(n \pi + \pi/2)} \int_0^a f'(x) \cos \frac{x}{a} \left( n \pi + \frac{\pi}{2} \right) dx.$$  

From analysis (2.7) and (2.8), we conclude that, if the additional condition (2.7) is fulfilled, the coefficients $B_n$, calculated by formula (2.8) will decrease much faster than the classical Fourier coefficients calculated by formulas (2.5), when condition (2.7) is not satisfied.

The rate of convergence of such a Fourier series can be further increased. In this connection, in order to find additional conditions for further increasing the rate of convergence of the Fourier series, we apply the reintegration by parts of formula (2.8):
\[ B_n = \frac{2}{(n\pi + \pi/2)^3} \int_0^a f'(x) \cos \frac{x}{a} \left( n\pi + \frac{\pi}{2} \right) dx = \frac{2a}{(n\pi + \pi/2)^3} \int_0^a f'(x) \sin \frac{x}{a} \left( n\pi + \frac{\pi}{2} \right) dx = \]
\[ = \frac{2a}{(n\pi + \pi/2)^3} \left( f'(x) \sin \frac{x}{a} \left( n\pi + \frac{\pi}{2} \right) \right)_0^a - \int_0^a f''(x) \sin \frac{x}{a} \left( n\pi + \frac{\pi}{2} \right) dx = \]
\[ = \frac{2a}{(n\pi + \pi/2)^3} \left( f'(a)(-1)^n - f''(x) \sin \frac{x}{a} \left( n\pi + \frac{\pi}{2} \right) \right) \].

(2.9)

Hence we have, if, in addition to (2.7), we also require the fulfillment of the second condition 
\[ f'(a) = 0, \quad f''(a) = 0, \] 
then the Fourier coefficients can be calculated by the formula
\[ B_n = -2a \int_0^a f''(x) \cos \frac{x}{a} \left( n\pi + \frac{\pi}{2} \right) dx. \] 

(11.1)

After the third application of integration in parts to the formula (11.1) we get
\[ B_n = -2a \int_0^a f''(x) \cos \frac{x}{a} \left( n\pi + \frac{\pi}{2} \right) dx = -2a^2 \int_0^a f''(x) \cos \frac{x}{a} \left( n\pi + \frac{\pi}{2} \right) dx = \]
\[ = -2a^2 \int_0^a f''(x) \cos \frac{x}{a} \left( n\pi + \frac{\pi}{2} \right) dx = -2a^2 \int_0^a f''(x) \cos \frac{x}{a} \left( n\pi + \frac{\pi}{2} \right) dx \]
\[ = -2a^2 \left( f''(0) + f''(x) \cos \frac{x}{a} \left( n\pi + \frac{\pi}{2} \right) dx \right). \]

The condition \( f''(0) = 0 \), together with (2.10), i.e. now three additional conditions must be met
\[ f(0) = 0, \quad f'(a) = 0, \quad f''(0) = 0 \] 
the formula for calculating the Fourier coefficients (12.1) is simplified and takes the
\[ B_n = -2a^2 \int_0^a f''(x) \cos \frac{x}{a} \left( n\pi + \frac{\pi}{2} \right) dx. \]

(12.1)

For understanding the regularity of writing additional conditions like (2.13), under which the rate of convergence of the Fourier series continues to increase, the integration in parts is applicable to formula (12.1) for the fourth time
\[ B_n = -2a^3 \int_0^a f''(x) \cos \frac{x}{a} \left( n\pi + \frac{\pi}{2} \right) dx = -2a^3 \int_0^a f''(x) \cos \frac{x}{a} \left( n\pi + \frac{\pi}{2} \right) dx = \]
\[ = -2a^3 \left( f''(0) + f''(x) \cos \frac{x}{a} \left( n\pi + \frac{\pi}{2} \right) dx \right) \]
\[ = -2a^3 \left( f''(0) + f''(x) \cos \frac{x}{a} \left( n\pi + \frac{\pi}{2} \right) dx \right). \]
This shows that in order to increase the rate of convergence of the Fourier series, conditions (2.13) should be supplemented by the equality \( f''(a) = 0 \) and the total set of additional conditions is expressed by

\[
f(0) = 0, \quad f'(a) = 0, \quad f''(0) = 0, \quad f''(a) = 0.
\]

(2.16)

Let us denote by \((q + 1)\) the number of additional conditions of the type (2.16), under which the rate of convergence of the Fourier series increases, the value of \( q \) will be equal to the order of the highest derivative under these conditions. Then for formulas of type (2.15) and conditions (2.16) we can make a generalization:

if \((q + 1) = 2p - 1\) is even, then at the left end of the segment at \( x = 0 \), where Dirichlet conditions were originally set in (2.1), now \( p \) of zero conditions for even derivatives should be met: at the same time, at the right end of the segment at \( x = a \), where initially (1.1) Neumann conditions were set, \( p \) of zero conditions for odd derivatives should be met:

\[
f(0) = f''(0) = \ldots = f^{(2p-2)}(0) = 0, \quad f'(a) = f''(a) = \ldots = f^{(2p-1)}(a) = 0
\]

(2.17)

If additional conditions (2.17) are fulfilled, the Fourier coefficients except (2.5) can be calculated through the sines by the formula

\[
B_n = \frac{(-1)^p 2a^{2p-1}}{(n\pi + \pi/2)^{2p}} \int_0^a f^{(2p)}(x) \sin\left(\frac{x}{n\pi + \pi/2}\right) dx, \quad p = 0, 1, 2, \ldots
\]

(2.18)

If \((q + 1) = 2p + 1\) is an odd number of integrations by parts of expressions (2.5) for \( B_n \), then for \( x = 0 \) \( p \) zero conditions are set for even derivatives and \( p - 1 \) zero conditions for odd derivatives at \( x = a \)

\[
f(0) = f''(0) = \ldots = f^{(2p)}(0) = 0, \quad f'(a) = f''(a) = \ldots = f^{(2p+1)}(a) = 0
\]

(2.19)

Fulfillment of additional conditions (2.19) allows one to calculate the Fourier coefficients except (2.5) via cosines using the formula

\[
B_n = \frac{(-1)^p 2a^{2p}}{(n\pi + \pi/2)^{2p+1}} \int_0^a f^{(2p+1)}(x) \cos\left(\frac{x}{n\pi + \pi/2}\right) dx, \quad p = 0, 1, 2, \ldots
\]

(2.20)

Fulfilling conditions (2.17), or (2.19), the rate of convergence of Fourier series greatly increases, and the possibility of multiple term-by-term differentiation of these series appears. Then their use becomes promising, since in the partial sum of the series it is enough to use only a few first terms. These properties can be formulated by the following theorem.

**Theorem.** Theorem consists of three positions.

Let the given \( f(x) \in C^{2p+1}(x \in [0, a], x \in [0, a]) \) at the boundaries of the segment \([0, a]\) satisfy to the additional conditions (2.17), or \( 2p + 1 \) to the additional conditions (2.19) and is represented by the Fourier series in sine

\[
f(x) = \sum_{m=0}^{\infty} f_m \sin\left(\frac{m\pi + \pi/2}{2}\right) \frac{x}{a}, \quad f_m = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{m\pi + \pi/2}{2}\right) \frac{x}{a} dx, \quad x \in [0, a]
\]

(2.21)

Then the following three statements hold:

1) if \( 2p \) conditions (2.17) or \( 2p + 1 \) conditions (2.19) are fulfilled, the order of decreasing \( f_m \).
coefficients with increasing number \( m \) will be equal to \( (m\pi + \pi/2)^{-2p} \) in the first case, or \( (m\pi + \pi/2)^{-2p-1} \) in the second, the coefficients themselves can be calculated using formulas (2.18); or (2.20) respectively;

2) the Fourier series (2.21) allows term-by-term differentiation \( 2p \) times equal to the number of conditions in (2.17) in the first case, or, \( 2p + 1 \) times equal to the number of conditions in (2.19);

3) if the first condition is not fulfilled in (2.17) or (2.19), i.e. when inequality \( f(0) \neq 0 \) holds, the Fourier series in sine (2.21) can not be differentiated term by term.

The proof of the first proposition of the theorem essentially follows from the procedure for obtaining formulas (2.18), (2.20):

\[
f'(x) - \sum_{m=0}^{\infty} f_m' \frac{1}{a} \left( m\pi + \frac{\pi}{2} \right) \cos \left( m\pi + \frac{\pi}{2} \right) \frac{x}{a}
\]

(2.22)

In (2.22), the equal sign is replaced by the match sign \( \sim \), since this equality has not yet been proved. We write the Fourier series in cosines and for the derivative

\[
f'(x) = \sum_{m=0}^{\infty} f_m' \cos \left( m\pi + \frac{\pi}{2} \right) \frac{x}{a}, \quad f_m' = \frac{2}{a^2} \int_0^a f'(x) \cos \left( m\pi + \frac{\pi}{2} \right) \frac{x}{a} dx.
\]

(2.23)

Here, the expression \( f_m' \) is used in accordance with the classical definition of the Fourier coefficients for the derivative \( f' \). This expression is transformed by integration by parts

\[
f_m' = \frac{2}{a^2} \int_0^a f'(x) \cos \left( m\pi + \frac{\pi}{2} \right) \frac{x}{a} dx = \frac{2}{a^2} \int_0^a f(x) \cos \left( m\pi + \frac{\pi}{2} \right) \frac{x}{a} dx
\]

\[= \frac{2}{a^2} \int_0^a f(x) \cos \left( m\pi + \frac{\pi}{2} \right) \frac{x}{a} dx = \frac{2}{a^2} \int_0^a f(x) \sin \left( m\pi + \frac{\pi}{2} \right) \frac{x}{a} dx = \frac{1}{a^2} \int_0^a f(x) \sin \left( m\pi + \frac{\pi}{2} \right) \frac{x}{a} dx
\]

(2.24)

Upon receipt (2.24), the first condition from (2.17) was used. After comparing the coefficients \( f_m' \) before the cosine in (2.22) with the last equation (2.24), we have the proof of the possibility of calculating the first derivative by term differentiation of the Fourier series (2.21). Now the match sign \( \sim \) in (2.22) can be replaced by the equal sign. The validity of the multiple series (2.21) differentiation is proved in a similar way.

The proof of the third proposition of the theorem follows from the fact that if the inequality \( f(0) \neq 0 \) holds, then the last equality in (2.24) will not hold. The theorem is completely proved.

3. Solution of the heat equation for a rectangular region with arbitrary mixed boundary conditions and an arbitrary internal source

We write the heat equation for a rectangular region

\[U_{xx} + U_{yy} + F(x,y) = 0, \quad (x,y) \in \Omega_0 = ([0,a] \times [0,b])
\]

(3.1)

Here \( F(x,y) \) is a well-known internal source. The boundary conditions on two sides of the rectangle \( \Omega_0 \) at \( x = 0 \) and at \( y = 0 \) are given in the Dirichlet form, on its opposite sides at \( x = a \) and at \( y = b \) in the Neumann form:
\[ U(0, y) = g_1(y), \quad U(x, 0) = g_2(x), \quad \partial U / \partial x \bigg|_{x=a} = g_3(y), \quad \partial U / \partial y \bigg|_{y=b} = g_4(x). \]

\[ \{ U(x, y), \quad F(x, y), \quad g_1(y), \quad g_2(x) \} \in C^1(\Omega_0), \quad \{ g_3(y), \quad g_4(x) \} \in C^1(\Omega_0). \]  

(3.2)

The orders of smoothness conditions in (3.2) are dictated by the order of the derivatives used in the process of obtaining a solution to the problem to ensure high accuracy. Values \( a, b, g_1(y) - g_4(x), \quad F(x, y) \) are called input data. The functions \( g_1(y) - g_4(x), \quad F(x, y) \) in (3.2) are assumed to be smooth, integrable in \( L_2([0, a]) \) and satisfying the matching conditions in the corners \( \Omega_0 \) between themselves. The type of conditions for matching the input data of the problem will be obtained later.

The solution to problem (3.1), (3.2) is represented by fast expansions, where for definiteness we put \( q = 2p - 1 = 3 \):

\[ U(x, y) = \sum_{s=1}^{4} A_s(y) P_s(x) + \sum_{m=1}^{N} u_m(y) \sin \left( m\pi + \frac{\pi}{2} \right) \frac{x}{a}. \]  

(3.3)

Here \( N \) is the number of members counted in the Fourier series. Unknown in (3.3) are \( (4 + N) \) coefficients of this fast expansions, depending only on one variable \( y \):

\[ A_s(y), \quad s = 1\ldots4, \quad u_m(y), \quad m = 1\ldots N. \]  

(3.4)

To find these coefficients, we first substitute the expansion (3.3) into the boundary conditions (3.2) with \( x = 0 \), \( x = a \), and then we find \( A_s(y), \quad A_2(y) \):

\[ \sum_{s=1}^{4} A_s(y) P_s(0) = g_1(y) \Rightarrow A_1(y) = g_1(y), \quad \sum_{s=1}^{4} A_s(y) P_s(a) = g_4(y) \Rightarrow A_2(y) = g_4(y). \]  

(3.5)

Now we expand the decomposition (3.3) into the heat equation (3.1)

\[ \sum_{s=1}^{4} A_s P_{s-2}(x) + \sum_{s=1}^{4} A_s' P_s(x) + \sum_{m=1}^{N} u_m \left( \sin \left( m\pi + \frac{\pi}{2} \right) \frac{x}{a} \right) \]  

\[ = \sum_{m=1}^{N} \left( m\pi + \frac{\pi}{2} \right) \frac{x}{a} + F(x, y) = 0. \]  

(3.6)

The substitution of the expansion (3.3) into (3.1) is legitimate, since the Fourier series in (3.3) allows term-by-term differentiation more than two times. The left part (3.6) will be considered as a function of \( (x, y) \). Since the differential equation (3.1) of the second order, in which the Fourier series, taken from (3.3), is differentiated twice, so the first order operator \( Ch_{y,2} = Ch_{y,2} = Ch_1 \) is applicable to (3.6). Representation by fast expansions of the left part (3.6) with the operator \( Ch_1 \) assumes the following operations:

- in equation (3.6) we first set \( x = 0 \)

\[ A_1 + A_1' + F(0, y) = 0 \]  

(3.7)

- then we differentiate (3.6) by \( x \) and then we put in it \( x = a \)

\[ A_1 + A_1' + F_s(a, y) = 0 \]  

(3.8)

From (3.7) and (3.8) we obtain the expressions for \( A_s(y), A_s(y) \):

\[ A_1(y) = -A_1' - F(0, y) = -g_1(y) - F(0, y), \quad A_1(y) = -A_1' - F_s(a, y) = -g_1(y) - F_s(a, y). \]  

(3.9)
To calculate the Fourier coefficients corresponding to the fast decomposition of the left-hand side of (3.6) with the \( C_3 \) operator, we multiply (3.6) by \( \sin(n\pi + \pi/2)x/a \) and integrate over \( x \) on the segment \([0,a]\), where the expressions of the polynomials \( P_s(x) \), \( s = 1 + 4 \) have the form

\[ P_1(x) = 1, \quad P_2(x) = x, \quad P_3(x) = \frac{1}{2}x^2 - ax, \quad P_4(x) = \frac{1}{6}x^3 - \frac{1}{2}ax^2 \]  \hspace{1cm} (3.10)

After multiplying (3.6) by \( \sin(n\pi + \pi/2)x/a \) and integrating over \( x \in [0,a] \) for convenience of writing the resulting equation, we will replace the notation for the \( n \rightarrow m \) indices. As a result, we will have a differential equation for \( u_m(y) \):

\[ u_m'' - u_m\left(m\pi + \frac{\pi}{2}\right)^2 \frac{1}{a^2} = R_m(y), \]

\[ R_m(y) = -\frac{2a(-1)^m}{(m\pi + \pi/2)^2} A_4 + A_4' + \frac{2a^2 A_m''}{(m\pi + \pi/2)^2} A_m - 2\left(\frac{A_3 + A_3'}{m\pi + \pi/2}\right) F_m(y). \]  \hspace{1cm} (3.11)

To solve the system (3.11), we need the boundary conditions for \( u_m(0) \) and \( u_m'(b) \), which we find by substituting \( U(x,y) \) from (3.3) into the unused boundary conditions (3.2) with \( y = 0, y = b \):

\[ \sum_{i=1}^{4} A_i(0) P_i(x) + \sum_{m=1}^{N} u_m(0) \sin\left(m\pi + \frac{\pi}{2}\right) \frac{x}{a} = g_2(x), \]  \hspace{1cm} (3.12)

\[ \sum_{i=1}^{4} A'_i(b) P_i(x) + \sum_{m=1}^{N} u'_m(b) \sin\left(m\pi + \frac{\pi}{2}\right) \frac{x}{a} = g_4(x). \]

The left and right sides of system (3.12) can be represented by a fast decomposition with the third-order \( C_3 \) operator, since in (3.12) the Fourier series are not differentiated. The \( C_3 \) operator assumes the following actions should be performed on the system (3.12):
- put in both equations (3.12) \( x = 0 \),
- calculate the derivatives of the first and third orders with \( x = a \),
- calculate the second derivative at \( x = 0 \).

After these four actions from two equations (3.12) we will have 8 equalities:

\[ A_i(0) = g_2(0), \quad A'_i(b) = g_4(0), \quad A_i(0) = g'_2(0), \quad A'_i(b) = g'_4(0), \quad A_i(0) = g''_2(0), \quad A'_i(b) = g''_4(0). \]  \hspace{1cm} (3.13)

Substituting \( A_i(y) \div A_i(y) \) from (3.5) and (3.9) into (3.13), we will have the above conditions for matching the input data of problem (3.2) and differential equation (3.1) written by the following 8 equalities:

\[ g_1(0) = g_2(0), \quad g'_1(b) = g'_4(0), \quad g_3(0) = g'_2(0), \quad g'_3(b) = g'_4(0), \]

\[ -g''_1(0) - F(0,0) = g''_2(0), \quad -g''_3(0) - F(0,b) = g''_4(0), \]  \hspace{1cm} (3.14)

\[ -g''_1(0) - F(a,0) = g''_2(a), \quad -g''_3(0) - F(a,b) = g''_4(a). \]

Equalities (3.14) mean that the boundary conditions (3.2) and the internal source \( F(x,y) \) in
equation (3.1) can not be given by arbitrary functions to ensure sufficient smoothness of the solution of the problem \( U(x,y) \). They must satisfy the additional conditions of approvals (3.14). Otherwise, the intended solution \( U(x,y) \) will tolerate gaps in the corners of the rectangle \( \Omega \), and then \( U(x,y) \) will not belong to the class \( C^{(2p+1)} \) in the area \( \Omega \). For credibility, we will show this in the following examples.

Consider the boundary conditions at the boundaries \( x=0 \) and \( y=0 \). When moving along the border \( x=0 \) to the corner \( (x=0,y=0) \), then along the border \( y=0 \) to the same angle, we obtain the first matching condition \( g_1(0) = g_2(0) \). To obtain the second condition, we will move along the boundary \( x=0 \) to the corner \( (x=0,y=b) \). In this case, you can calculate the derivative \( \partial U/\partial y \big|_{x=0,y=b} = g'_1(b) \). This is the second condition of approvals. If we move along the side \( y=b \) to the same angle, then we will have \( g'_1(b) = g_4(0) \) the third condition of approvals. In the same way, one can obtain all the conditions of approvals (3.14). It should be noted that when applying fast expansions, these conditions are obtained automatically. We assume that the input data of the problem satisfy conditions (3.14). Perform the last action on equation (3.12). We multiply (3.12) by \( \sin(n\pi x a/2) \), after multiplication, we integrate it by \( x \in [0,a] \), we take the expressions of polynomials \( R_s(x) \) from (3.10), the coefficients \( A_s(0) \) and \( A_s'(b) \), \( s = 1 \div 4 \) from (3.13), replace the index designation \( n \to m \), and as a result we will have:

\[
\begin{align*}
  u_m(0) &= g_{2,m} - \frac{2}{(m\pi + \pi/2)} g_1(0) - \frac{2a(-1)^m/2 - (g'_1(0) - F(0,0))}{(m\pi + \pi/2)} - \frac{2a^2}{(m\pi + \pi/2)^3}, \\
  u'_m(b) &= g_{4,m} - \frac{2}{(m\pi + \pi/2)} g_1(b) - \frac{2a(-1)^m/2 - (g'_1(b) + F_1(0,b))}{(m\pi + \pi/2)^3} - \frac{2a^2}{(m\pi + \pi/2)^3}, \\
  g_{2,m} &= \frac{2}{a} \int g_2(x) \sin\left(m\pi + \frac{\pi}{2}\right) dx, \\
  g_{4,m} &= \frac{2}{a} \int g_1(x) \sin\left(m\pi + \frac{\pi}{2}\right) dx.
\end{align*}
\]  

Equivalences (3.15) serve as boundary conditions for the differential system (3.11). Its solution can be represented as a sum of the particular solution of the inhomogeneous system \( u'_m(y) \) and the general solution of the homogeneous system (3.11) with zero right-hand side, formally assuming \( R_m(y) = 0 \), i.e.

\[
u_m(y) = E_1 \cos\left(m\pi + \frac{\pi}{2}\right)a + E_2 \sin\left(m\pi + \frac{\pi}{2}\right) a + u'_m(y), \quad m = 1 \div N.
\]  

Constants \( E_1, E_2 \) are found using the boundary conditions (3.15) and have the form:
$$E_1 = u_m'(0) - u_m'(b), \quad E_2 = \frac{a(u_m''(b) - u_m''(b)) + (u_m(0) - u_m'(0)) \left( m\pi + \frac{\pi}{2} \right) \sin \left( m\pi + \frac{\pi}{2} \right) b}{m\pi + \frac{\pi}{2} \cos \left( m\pi + \frac{\pi}{2} \right) b \ a}.$$ (3.17)

4. Conclusion

Thus, by using the fast expansions method, the solution of the heat conduction problem (3.1), (3.2) with mixed boundary conditions and an internal source, specified in a general form, up to quadratures when calculating the Fourier coefficients $u_m(y)$ in an analytical form, is obtained rather simply. In this case, the input data of the problem must be consistent with each other and satisfy equations (3.14). The obtained solution can be used not only for engineering purposes, but also when considering a similar problem for a curvilinear region, for regions with moving boundaries, as it was done in [15,16], and solving many other problems of an applied and theoretical nature.

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