PECCULARITIES OF MATTER MOTION IN METRIC-AFFINE GRAVITATIONAL THEORY

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Abstract

On the basis of the Lie derivative method in a metric-affine space-time \((L_4, g)\) it is shown that in the metric-affine gravitational theory the energy-momentum conservation law and therefore the equations of the matter motion are the consequence (as in the GR) of the gravitational field equations. The possibility of the detection of the space-time non-metric properties is discussed.

1. Introduction

It is well known that in the General Relativity the equations of the matter motion are the consequence of the gravitational field equations. The reason of this fact consists in the identical vanishing (as the consequence of the Bianchi identities) of the covariant divergence of the left part of the Einstein equations, which leads to the covariant energy-momentum conservation law of matter

\[ (R) \nabla_\beta T^\beta_\alpha = 0 \]  \hspace{1cm} (1)

(here \((R) \nabla\) is the covariant differentiation with respect to the connection of a Riemann space-time). The equations of the matter motion are the consequence of (1).

In the Einstein-Cartan theory \([1, 2]\) the same situation takes place \([3]\). The more complicated case occurs in the generalized theories of gravity with torsion in a Riemann-Cartan space-time \(U_4\) based on non-linear Lagrangians, where the some new covariant identities appear. In the theories of a such type as it was shown in \([4, 5]\) the equations of the matter motion are also the consequence of the gravitational field equations.

In this paper we generalize the results of \([4, 5]\) to the theory of gravitation in the metric-affine space-time \((L_4, g)\) with the curvature \(R_{\mu\nu\sigma}^\lambda\), torsion \(T_{\mu\nu}^\lambda\) and nonmetricity \(Q_{\lambda\sigma}^\rho := \nabla_\lambda g^{\sigma\rho}\). The metric-affine theory of gravitation was proposed in \([6]\) and now is of a great interest in connection with the problem of the relation of the gravitation and the elementary particles physics \([7, 8]\).
2. The energy-momentum conservation law as the consequence of the field equations in the metric-affine gravitational theory

We shall use the Lie derivative method for deriving the differential identities in the metric-affine space \((L_4, g)\). In a Riemann-Cartan space this method was described in detail in [9]. Let us consider the Lie transport \(\mathcal{L}_\xi\) in the direction of an arbitrary vector field \(\xi^\sigma\). Then the transformation law of the gravitational field Lagrangian density \(L_0 = \sqrt{-g}L_0\) is

\[
\mathcal{L}_\xi L_0 = \xi^\sigma \nabla_\sigma L_0 + L_0 \hat{\nabla}_\sigma \xi^\sigma ,
\]

where \(\hat{\nabla}_\sigma := \nabla_\sigma + T_\sigma\), and \(T_\sigma := T_\sigma^\tau\). Here \(\nabla\) is the covariant differentiation with respect to the connection of \((L_4, g)\).

From the other side the result of this Lie differentiation can be calculated as the consequence of the explicit dependence of the Lagrangian density \(L_0 = \sqrt{-g}L_0(g^{\sigma\rho}, R^{\lambda\mu\nu\sigma}, T^{\mu\nu\lambda})\) from the metric, curvature and torsion tensors of \((L_4, g)\), the Lie differentiation of those being calculated by the corresponding rules that valid in \((L_4, g)\):

\[
\mathcal{L}_\xi g^{\sigma\rho} = \xi^\mu Q_\mu^{\sigma\rho} - 2\xi^{(\sigma\rho)}, \quad \xi^\mu := \nabla_\mu \xi^\rho - T^{\mu\rho}_{\sigma\nu} \xi^\nu ,
\]

\[
\mathcal{L}_\xi R^{\lambda\mu\nu\sigma} = \xi^\rho \nabla_\rho R^{\lambda\mu\nu\sigma} + R^{\rho\lambda\nu}_{\mu\rho\sigma} \xi^\mu + R^{\rho\mu\sigma}_{\nu\rho\lambda} \xi^\nu + R^{\rho\mu\lambda}_{\nu\rho\sigma} \xi^\nu + R^{\rho\nu\lambda}_{\mu\rho\sigma} \xi^\nu ,
\]

\[
\mathcal{L}_\xi T^{\mu\nu\lambda} = \xi^\rho \nabla_\rho T^{\mu\nu\lambda} + T^{\mu\nu\lambda}_{\sigma} \xi^\sigma + T^{\mu\nu\lambda}_{\rho} \xi^\rho - T^{\mu\rho\lambda}_{\nu\lambda\sigma} \xi^\sigma .
\]

Comparing the results of the both Lie differentiation methods, we get the covariant correlation, on the base of which, taking into account that \(\xi^\sigma\) and \(\nabla_\lambda \xi^\sigma\) are arbitrary, we obtain the two covariant identities. Because of the arbitrariness of \(\nabla_\lambda \xi^\sigma\) we get

\[
\sqrt{-gt^\sigma_\sigma}[L_0] = L_0 \delta^\lambda_\sigma - 2R_{\mu\rho\sigma\beta} \frac{\partial L_0}{\partial R^{\mu\rho\sigma\beta}} + T^{\mu\nu}_{\lambda} \frac{\partial L_0}{\partial T^{\mu\nu}_{\lambda}}
\]

\[
-2(\delta^\sigma_\rho \hat{\nabla}_\nu - T^{\sigma\nu}_{\tau\rho}) \frac{\partial L_0}{\partial T^{\nu\lambda\sigma}} ,
\]

where the following notations have been used:

\[
\sqrt{-gt^\lambda_\sigma}[L_0] := \sqrt{-g}T^{\lambda_\sigma}[L_0] + \hat{\nabla}_\nu (\sqrt{-g}J^{\lambda_\nu}[L_0]) ,
\]

\[
\sqrt{-g}T^{\sigma_\rho}[L_0] := -2\frac{\delta L_0}{\delta g^{\sigma_\rho}} , \quad \sqrt{-g}J^{\lambda_\nu}[L_0] := -\frac{\delta L_0}{\delta \Gamma^{\lambda_\nu}_{\mu\lambda\sigma}} .
\]

Because of the arbitrariness of \(\xi^\sigma\) we get the second identity:

\[
(\delta^\rho_\sigma \hat{\nabla}_\lambda - T^{\rho\lambda}_{\sigma\rho}) \sqrt{-gt^\lambda_\sigma}[L_0] - R^{\sigma\nu}_{\rho\lambda} \frac{\delta L_0}{\delta \Gamma^{\rho\nu}_{\lambda\sigma}} - Q^{\sigma_\beta}_{\rho\nu} \frac{\delta L_0}{\delta g^{\sigma_\beta}} = 0 .
\]
When obtaining (11) and (10) the following identities have played the essential role

\[ \frac{\delta L_0}{\delta \Gamma_{\sigma\nu}^{\lambda}} = 2 \nabla_\rho \frac{\partial L_0}{\partial R_{\sigma\rho\nu}^{\lambda}} + T_{\alpha\beta} \frac{\partial L_0}{\partial R_{\alpha\beta\nu}^{\lambda}} + 2 \frac{\partial L_0}{\partial T_{\sigma\rho}^{\lambda}}, \]

(11)

\[ \nabla_\sigma \frac{\delta L_0}{\delta \Gamma_{\sigma\nu}^{\lambda}} = (R_{\alpha\beta\nu}^{\lambda} \delta_\rho^\lambda - R_{\alpha\beta}^{\lambda} \delta_{\sigma}^\nu) \frac{\partial L_0}{\partial R_{\alpha\beta\nu}^{\lambda}} + 2 \nabla_\sigma \frac{\partial L_0}{\partial T_{\sigma\rho}^{\lambda}}, \]

(12)

which take place in \((L_4, g)\) for the the Lagrangian density \((3)\) \((4)\), for some details see Appendix).

Taking into account (8), (9), the field equations of the metric-affine gravitational theory can be introduced in the following form

\[ \delta g_{\sigma\rho} : T_{\sigma\rho} = -T_{\sigma\rho} := T_{\sigma\rho}[L_0], \]

(13)

\[ \delta \Gamma_{\nu\lambda}^{\sigma} : J_{\sigma\nu}^{\lambda} = -J_{\sigma\nu}^{\lambda} := J_{\sigma\nu}[L_0], \]

(14)

where \(L_m\) is the Lagrangian density of matter, generating the gravitational field. The system (13), (14) with the help of the identity

\[ R_{\sigma\rho} = \frac{1}{2} \nabla_\tau M_{\sigma\rho}^{\tau} - \frac{1}{2} V_{\sigma\rho} = T_{\sigma\rho} + 2 \delta_{[\sigma} T_{\rho]}^{\rho}, \]

(15)

where \(V_{\alpha\beta}\) is the tensor of homothetic curvature of \((L_4, g)\):

\[ V_{\alpha\beta} = R_{\alpha\beta}^{\tau} = \nabla_{[\alpha} Q_{\beta]} + \frac{1}{2} T_{\alpha\beta}^{\tau} Q_{\tau}, \quad Q_{\sigma} = Q_{\sigma}^{\tau}, \]

(16)

can be written in the equivalent form:

\[ t_{\sigma}^{\lambda}[L_0] = -t_{\sigma}^{\lambda}, \quad t_{\sigma}^{\lambda} := t_{\sigma}^{\lambda}[L_m]. \]

(17)

\[ J_{\sigma}^{\lambda\nu}[L_0] = -J_{\sigma}^{\lambda\nu}, \]

(18)

where in the right part of the gravitational field equations (17), (18) we have the canonical energy-momentum tensor and the hypermomentum tensor of all matter that generates the gravitational field. These quantities are calculated by means of the replacing the matter Lagrangian density \(L_m\) instead of \(L_0\) into (8), (3). The question of deriving the field equations (17) will be discussed in detail in the following paper \([12]\).

Substituting the field equations (17), (18) into (10), we obtain the quasi-conservation law for the canonical energy-momentum tensor of matter in the metric-affine gravitational theory:

\[ \nabla_\lambda (\sqrt{-g} t_{\sigma}^{\lambda}) - \sqrt{-g} T_{\sigma\lambda}^{\rho} t_{\rho}^{\lambda} + \sqrt{-g} R_{\sigma\nu\lambda}^{\rho} J_{\rho}^{\lambda\nu} + \frac{1}{2} \sqrt{-g} Q_{\alpha}^{\sigma\beta} T_{\alpha\beta} = 0. \]

(19)
The equation (19) was derived in [10] (see also [11]) as the consequence of the matter Lagrangian invariance with respect to the infinitesimal coordinate transformations.

3. The various cases of the matter motion in the metric-affine space-time \((L_4, g)\)

The hypermomentum tensor can be split up in the following way [6]

\[
J_{\sigma\rho}^\lambda = S_{\sigma\rho}^\lambda + \mathcal{J}_{\sigma\rho}^\lambda + \frac{1}{4} g_{\sigma\rho} J^\lambda, \quad S_{\sigma\rho}^\lambda := J_{[\sigma\rho]}^\lambda, \quad \mathcal{J}_{\sigma}^{\sigma\lambda} = 0 .
\] (20)

In \((L_4, g)\) the affine connection coefficients have the form [13]

\[
\Gamma_{\nu\rho}^\sigma = \frac{1}{2} g^{\sigma\rho} \Delta_{\nu\rho}^{\alpha\beta\gamma} (\partial_\alpha g_{\beta\gamma} - T_{\alpha\beta\gamma} + Q_{\alpha\beta\gamma}) ,
\] (21)

\[
\Delta_{\nu\rho}^{\alpha\beta\gamma} := \delta_\nu^{\alpha} \delta_\rho^{\beta} \delta_\gamma^{\gamma} + \delta_\nu^{\alpha} \delta_\gamma^{\beta} \delta_\rho^{\gamma} - \delta_\nu^{\alpha} \delta_\rho^{\beta} \delta_\gamma^{\gamma} .
\] (22)

The quasiconservation law (19) with the help of (20), (21) yields

\[
\sqrt{-g} \nabla_{\lambda} (t_{\sigma}^\lambda) + \sqrt{-g} R_{\sigma\nu\lambda}^\rho S_{\rho}^{\lambda\nu} + \frac{1}{4} \sqrt{-g} V_{\sigma\nu} J^\nu
\]

\[
- (T_{\sigma}^{[\lambda} + \frac{1}{2} T^\rho_\sigma - Q_\nu^{[\rho} \sigma + \frac{1}{2} Q_\sigma^{[\rho} \nu) \nabla_\nu (\sqrt{-g} J^{\lambda\nu})
\]

\[
+ \sqrt{-g} (\nabla_{[\sigma} Q_{\nu]\lambda\rho + \frac{1}{2} T_{\sigma\nu}^{\tau} Q_{\tau\lambda\rho}) J^{\lambda\rho\nu} = 0 .
\] (23)

Let us consider some particular cases of the matter motion.

I. The matter hypermomentum tensor is equal to the spin momentum tensor: \(J_{\sigma\rho}^\lambda = S_{\sigma\rho}^\lambda\).

In this case some terms with nonmetricity vanish in (23) except the terms \((Q_\nu^{[\rho} \sigma + \frac{1}{2} Q_\sigma^{[\rho} \nu) \nabla_\nu (\sqrt{-g} S^{\lambda\nu})\). Therefore in this case the matter motion depends on both torsion and nonmetricity.

II. The matter hypermomentum tensor vanishes: \(J_{\sigma\rho}^\lambda = 0\).

In this case the canonical energy-momentum tensor of matter (17) reduces to the metric one: \(t_{\sigma\lambda} = T_{\sigma\lambda}\) and the quasiconservation law (23) reduces to the matter energy-momentum conservation law in a Riemann space-time (1). Therefore we have proved the following Theorem [14, 15].

**Theorem:** In \((L_4, g)\) the motion of matter without hypermomentum coincides with the motion in the Riemann space-time, which the metric tensor coincides with the metric tensor of \((L_4, g)\).

Thus bodies and mediums without hypermomentum are not subjected to the influence of the possible nonmetricity of the space-time (in contrast to the generally accepted opinion) and therefore can not be the tools for the detection of the deviation of the real space-time properties from the Riemann space structure.
Therefore for the investigation of the different manifestations of the possible space-time nonmetricity one needs to use the bodies and mediums endowed with the hypermomentum, i.e. the spin particles, the perfect spinning fluid or the perfect fluid with the intrinsic hypermomentum [14, 17, 18].

Appendix

Let us consider the identities (11), (12) in more detail. The identity (11) is the direct consequence of (3) and the explicit forms of the curvature tensor \( R_{\mu\nu\sigma}^{\lambda} \) and the torsion tensor \( T_{\mu\nu}^{\lambda} \).

In order to derive the identity (12) let us evaluate the covariant derivative \( \hat{\nabla}_\sigma \) from the both sides of the (11):

\[
\hat{\nabla}_\sigma \frac{\delta L_0}{\delta \Gamma_{\sigma\nu\lambda}} = 2\nabla_\sigma \nabla_\alpha T_\beta^\alpha + 2\nabla_\sigma T_\alpha^\beta \frac{\partial L_0}{\partial R_{\alpha\beta\nu}^{\lambda}} + 2\nabla_\sigma T_\alpha^\beta \frac{\partial L_0}{\partial T_{\alpha\beta}^{\lambda}} + 2\nabla_\sigma \frac{\partial L_0}{\partial T_{\sigma\nu}^{\lambda}}.
\]  

(24)

The first term in the right part of the (24) may be calculated with the help of Ricci identity in \((L_4, g)\) [19]:

\[
2\nabla_{[\alpha} \nabla_{\beta]} W^{\lambda} = R_{\alpha\beta\sigma}^{\lambda} W^{\sigma} - T_{\alpha\beta}^{\sigma} \nabla_\sigma W^{\lambda} - V_{\alpha\beta} W^{\lambda}.
\]  

(25)

Here \( V_{\alpha\beta} \) is the tensor of homothetic curvature (16); \( W^{\lambda} \) is a contravariant vector density, the minus appearing in the first term in the right part of the (25) in case of the covariant vector density \( W_\lambda \). As a result one has

\[
\hat{\nabla}_\sigma \frac{\delta L_0}{\delta \Gamma_{\sigma\nu}^{\lambda}} = (2\nabla_\alpha T_\beta + \nabla_\sigma T_\alpha^\sigma + T_\sigma T_\alpha^\sigma - 3R_{[\alpha\beta\sigma]}^{\lambda}) \frac{\partial L_0}{\partial R_{\alpha\beta\nu}^{\lambda}}
\]

\[
+ R_{\alpha\beta\sigma}^{\nu} \frac{\partial L_0}{\partial R_{\alpha\beta\sigma}^{\lambda}} - R_{\alpha\beta\sigma}^{\lambda} \frac{\partial L_0}{\partial R_{\alpha\beta\nu}^{\sigma}} + 2\nabla_\sigma \frac{\partial L_0}{\partial T_{\sigma\nu}^{\lambda}}.
\]  

(26)

Then using in (26) the second identity for the curvature [19]:

\[
R_{[\alpha\beta\sigma]}^{\lambda} = \nabla_{[\alpha} T_{\beta\sigma]}^{\lambda} - T_{[\alpha\beta} T_{\sigma]}^{\lambda},
\]  

(27)

we convince oneself that the terms in the parentheses in the right part of the (26) vanishes. As a result we get the identity (12).

References

[1] T.W.B. Kibble, J. Math. Phys. 2 (1961) 212.

[2] A. Trautman, Symp. Math. 12 (1973) 139.
[3] F.W. Hehl, Phys Letters 36A (1971) 225.

[4] O.V. Babourova, V.N. Ponomariov, B.N. Frolov, in: Gravitation and electromagnatic field, Universitetskoe,Minsk,1988, p.6 [in Russian].

[5] O.V. Babourova, The variational theory of perfect fluid with intrinsic degrees of freedom in generalized spaces of the modern theory of gravitation, Dr. thesis, VNICPV, Moscow, 1989 [in Russian].

[6] F.W. Hehl, G.D. Kerlick, P. Heyde, Z. Naturforsch 31A (1976) 111, 524, 823.

[7] Y. Ne`eman, Dj. Šijački, Ann. Phys. 120 (1979) 292.

[8] F.W. Hehl, Dj. Šijački, Gen. Rel. Grav. 12 (1980) 83.

[9] W. Kopczyński, Ann. Phys. (NY) 203 (1990) 308.

[10] F.W. Hehl, J.D. McCrea, E.W. Mielke, Y. Ne´eman, Found. Phys. 19 (1989) 1075.

[11] R. Hecht et al, Phys. Lett. A172 (1992) 13.

[12] O.V.Baburova, B.N. Frolov, On the field equations in the metric-affine gravitational theory, preprint gr-qc 9502.

[13] F.W. Hehl, P. von der Heyde, G.D. Kerlick, J.M. Nester, Rev. Mod. Phys. 48 (1976) 393.

[14] O.V. Babourova, in: Abstracts of contrib. papers of the Cornelius Lanczos Int. conf. (NC State Univ., USA) 1993, p. 100.

[15] O.V.Babourova, M. Yu. Koroliov, B.N. Frolov, Izvestiya Vysshykh Uchebnykh Zavedenij. Fizika (Russian Physics Letters) N1 (1994) 76.

[16] O.V. Babourova, in: Gravitation and fundamental interaction (theses), UDN, Moscow, 1988, p. 119 [in Russian].

[17] O.V.Baburova, B.N. Frolov, M. Yu. Koroliov, in: 13th Int. Conf. gen. rel. grav. (Abstract of contr. papers), Cordoba (Argentina), 1992, eds. P.W. Lamberti and O.E. Ortiz, p. 131.

[18] O.V.Babourova, B.N. Frolov, M. Yu. Koroliov, in: Nauchnye trudy MPGU (Ser.: Est. nauki) ("Prometej", Moscow), 1994, Part 1, P. 89 [in Russian].

[19] J.A. Schouten, Ricci-Calculus, Springer, Berlin, 1954.