Singular solutions for coercive quasilinear elliptic inequalities with nonlocal terms

Roberta Filippucci\textsuperscript{*} and Marius Ghergu\textsuperscript{† ‡}

January 28, 2022

Abstract

We study the inequality
\[
\text{div}(|x|^{-\alpha} \nabla u |m-2| \nabla u) \geq (I_{\beta} * u^p) u^q \quad \text{in } B_1 \setminus \{0\} \subset \mathbb{R}^N,
\]
where \(\alpha > 0\), \(N \geq 1\), \(m > 1\), \(p, q > m - 1\) and \(I_{\beta}\) denotes the Riesz potential of order \(\beta \in (0, N)\). We obtain sharp conditions in terms of these parameters for which positive singular solutions exist. We further establish the asymptotic profile of singular solutions to the double inequality
\[
a(I_{\beta} * u^p) u^q \geq \text{div}(|x|^{-\alpha} \nabla u |m-2| \nabla u) \geq b(I_{\beta} * u^p) u^q \quad \text{in } B_1 \setminus \{0\} \subset \mathbb{R}^N,
\]
where \(a \geq b > 0\) are constants.

Keywords: Quasilinear elliptic inequalities; weighted \(m\)-Laplace operator; singular solutions.

2010 AMS MSC: 35J62, 35A23, 35B09, 35B53

1 Introduction and the main results

In this paper we are concerned with the following quasilinear elliptic inequality
\[
\text{div}(|x|^{-\alpha} |\nabla u|^{m-2} \nabla u) \geq (I_{\beta} * u^p) u^q \quad \text{in } B_1 \setminus \{0\} \subset \mathbb{R}^N,
\]
and with the double inequality
\[
a(I_{\beta} * u^p) u^q \geq \text{div}(|x|^{-\alpha} |\nabla u|^{m-2} \nabla u) \geq b(I_{\beta} * u^p) u^q \quad \text{in } B_1 \setminus \{0\},
\]
where \(\alpha > 0\), \(\beta \in (0, N)\), \(m > 1\), \(N \geq 1\), \(p > 0\), \(q > m - 1\) and \(a \geq b > 0\).

Throughout this paper, \(B_R(z)\) denotes the open ball in \(\mathbb{R}^N\), \(N \geq 1\), with center at \(z \in \mathbb{R}^N\) and having radius \(R > 0\). When \(z = 0\), we simply use \(B_R\) instead of \(B_R(0)\).

\textsuperscript{*}Dipartimento di Matematica e Informatica, Università degli Studi di Perugia, Via Vanvitelli 1, 06123 Perugia, Italy; roberta.filippucci@unipg.it

\textsuperscript{†}School of Mathematics and Statistics, University College Dublin, Belfield, Dublin 4, Ireland; marius.ghergu@ucd.ie

\textsuperscript{‡}Institute of Mathematics Simion Stoilow of the Romanian Academy, 21 Calea Grivitei St., 010702 Bucharest, Romania
The quantity \( I_\beta \ast u^p \) represents the convolution operation

\[
(I_\beta \ast u^p)(x) = \int_{B_1} I_\beta(x - y)u^p(y)dy,
\]

where \( I_\beta : \mathbb{R}^N \to \mathbb{R} \) is the Riesz potential of order \( \beta \in (0, N) \) given by

\[
I_\beta(x) = \frac{A_\beta}{|x|^{N-\beta}}, \quad \text{with} \quad A_\beta = \frac{\Gamma\left(\frac{N-\beta}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)\pi^{N/2}} = C(N, \beta) > 0.
\]

By a positive solution of (1.1) we understand a function \( u \in W^{1,m}_{\text{loc}}(B_1 \setminus \{0\}) \cap C(B_1 \setminus \{0\}) \) which satisfies:

- \( u > 0, \quad u \in L^p(B_1), \quad \text{div}(|x|^{-\alpha}\nabla u|m-2\nabla u), \quad (I_\beta \ast u^p)u^q \in L^1_{\text{loc}}(B_1 \setminus \{0\}) \);
- for any \( \phi \in C_\infty_c(\Omega), \phi \geq 0 \) we have

\[
\int_{B_1} |x|^{-\alpha}\nabla u|m-2 \nabla u \cdot \nabla \phi + \int_{B_1} (I_\beta \ast u^p)u^q \phi \leq 0.
\]

Solutions of (1.1) are called singular if

\[
\limsup_{x \to 0} u(x) = \infty.
\]

**Remark.** Let us point out that the condition \( u \in L^p(B_1) \) is needed to ensure \( I_\beta \ast u^p \) is finite almost everywhere. In fact, these two conditions are equivalent since for \( x \in B_1 \setminus \{0\} \) we have

\[
\infty > (I_\beta \ast u^p)(x) = C \int_{B_1} \frac{u^p(y)}{|x - y|^{N-\beta}}dy \geq C \int_{B_1} \frac{u^p(y)}{2^{N-\beta}}dy,
\]

so \( u \in L^p(B_1) \). Conversely, if \( u \in L^p(B_1) \) then, by standard properties of convolution (see, e.g., [20, Chapter 2]) one has \( I_\beta \ast u^p \in L^1(B_1) \).

The study of quasilinear elliptic inequalities has received constant attention in the last decades, one general example is the inequality

\[
-L_Au = -\text{div}[A(x, u, \nabla u)] \geq f(x, u) \quad \text{in} \ \Omega,
\]

which has appeared in many research papers under various structural hypotheses on \( A \). The work by Mitidieri and Pohozaev [21] contains many results in this direction and provides the reader with a range of methods to investigate the nonexistence of a solution. The equality case in (1.1) naturally leads to a proper differential equation and has even a longer history. We only mention here the seminal work of Gidas and Spruck [19] for the semilinear case with power type nonlinearity but also some more recent results [10], [15], [23] dealing with other different situations.

A systematic study of the inequality

\[
L_Au = -\text{div}[A(x, u, \nabla u)] \geq |x|^\sigma u^q \quad \text{in} \ \Omega,
\]
along with the corresponding system
\[
\begin{aligned}
L_A u &= -\text{div}[A(x, u, \nabla u)] \geq |x|^a u^{p_1} v^{q_1} & \text{ in } \Omega, \\
L_B v &= -\text{div}[B(x, v, \nabla v)] \geq |x|^b u^{p_2} v^{q_2},
\end{aligned}
\]
is carried out in [2] for various domains $\Omega \subset \mathbb{R}^N$, such as open balls and their complements, half balls and half spaces (see also [6] for the case of general nonlinearities). More recently, quasilinear elliptic inequalities and systems integrate the gradient term in the nonlinearity: the authors in [9] and [11] discuss coercive quasilinear inequalities in the form
\[
\text{div}(g(x)|\nabla u|^{p-2}\nabla u) \geq h(x)f(u)\ell(|\nabla u|) \quad \text{in } \mathbb{R}^N,
\]
and respectively
\[
\text{div}(h(x)g(u)A(|\nabla u|)\nabla u) \geq f(x, u, \nabla u) \quad \text{in } \mathbb{R}^N.
\]

Systems of quasilinear elliptic inequalities of type
\[
\begin{aligned}
\begin{cases}
-\text{div}(h_1(x)A(|\nabla u|)\nabla u) & \geq f(x, u, v, \nabla u, \nabla v) \\
-\text{div}(h_2(x)B(|\nabla v|)\nabla v) & \geq g(x, u, v, \nabla u, \nabla v)
\end{cases}
\end{aligned}
\]
and
\[
\begin{aligned}
\begin{cases}
-\text{div}[A(x, u, \nabla u)] & \geq a(x)u^{p_1}v^{q_1}|\nabla u|^{\theta_1} \\
-\text{div}[B(x, v, \nabla v)] & \geq b(x)u^{p_2}v^{q_2}|\nabla u|^{\theta_2}\end{cases}
\end{aligned}
\]
are studied in [7] and [8] respectively.

To the best of our knowledge, the first results dealing with quasilinear elliptic inequalities in the presence of nonlocal terms appear in [3]. The authors in [3] obtain local estimates and Liouville type results for
\[
-\text{div}[A(x, u, \nabla u)] \geq K \ast u^q \quad \text{in } \mathbb{R}^N,
\]
where $K \in L^1_{\text{loc}}(\mathbb{R}^N)$, $K \geq 0$ and $q > 0$. Extensions to these results were recently obtained in [14] in the case $K(x) = |x|^{-\beta}$, $\beta \in (0, N)$. The related equation
\[
-\Delta u + V(x)u = (|x|^{-\beta} \ast u^p)u^q
\]
is known in the literature under the name of Choquard (or Choquard-Pekar) equation and arises in various fields ranging from quantum physics to one-component plasma and Newtonian relativity. A survey on the mathematical results on the Choquard equation is presented in [22]. Solutions to the Choquard equation featuring isolated singularities are studied in [4] and [5]. In [17] and [13] it is investigated the behaviour around the origin of singular solutions to
\[
0 \leq -\Delta u \leq (I_{\alpha} \ast u^p)u^q \quad \text{in } B_1 \setminus \{0\}
\]
and
\[
0 \leq -\Delta u \leq (I_{\alpha} \ast u^p)(I_{\beta} \ast u^p) \quad \text{in } B_1 \setminus \{0\},
\]
respectively. Returning to inequality (1.1), we are now ready to state our first main result.
Theorem 1.1. Assume $m > 1$, $N \geq 1$, $q > m - 1$, $\alpha > 0$ and $\beta \in (0, N)$.

(i) If $N \leq m + \alpha$ then (1.1) has always singular solutions.

(ii) If $N > m + \alpha$ and $p > m - 1$ then (1.1) has singular solutions if and only if

\[
\max\{p, q\} < \frac{N(m - 1)}{N - m - \alpha}, \quad p + q < \frac{(N + \beta)(m - 1)}{N - m - \alpha} \quad \text{and} \quad N - 2m < 2\alpha + \beta. \tag{1.5}
\]

We next proceed to the study of the double inequality (1.2). To formulate our main result on (1.2) we introduce the exponent

\[
\sigma = \frac{m + \alpha + \beta}{p + q - m + 1} > 0. \tag{1.6}
\]

Let also

\[
\Phi_{m, \alpha}(x) = \begin{cases} 
|x|^{-\frac{N-m-\alpha}{m-1}} & \text{if } N \neq m + \alpha, \\
\log \frac{5}{|x|} & \text{if } N = m + \alpha,
\end{cases} \tag{1.7}
\]

be the fundamental solution of the weighted $m$-Laplace operator for $m > 1$. Note that $\Phi_{m, \alpha}$ satisfies the distributional equality

\[-\text{div}(|x|^{-\alpha}|\nabla \Phi_{m, \alpha}|^{m-2}\nabla \Phi_{m, \alpha}) = c\delta_0 \quad \text{in } D'(\mathbb{R}^N),
\]

for some positive constant $c$.

Given two positive functions $f, g$ defined on $\overline{B_1 \setminus \{0\}}$, by $f \asymp g$ we understand that the quotient $f/g$ is bounded on $\overline{B_1 \setminus \{0\}}$ between two positive constants.

In case $\sigma p < N$ we have the following result on (1.2).

Theorem 1.2. Assume $m > 1$, $p, q > m - 1$, $\alpha > 0$, $\beta \in (0, N)$, $N \geq 1$, and $\sigma p < N$.

(i) (Existence) Assume $N \geq m + \alpha$

(ii) (Asymptotic behavior) Assume $N \geq m + \alpha$ and

\[
\begin{cases} 
m - 1 < q < \frac{N - (\sigma p - \beta)^+}{N - m - \alpha}(m - 1) & \text{if } N > m + \alpha, \\
m - 1 < q < \infty & \text{if } N = m + \alpha.
\end{cases} \tag{1.8}
\]

(iii1) If $\sigma p > \beta$ then any singular solution of (1.2) satisfies

\[
either \quad u(x) \asymp \Phi_{m, \alpha}(x) \quad \text{or} \quad u(x) \asymp |x|^{-\sigma}. \tag{1.9}
\]

(iii2) If $\sigma p < \beta$ then any singular solution of (1.2) satisfies

\[
either \quad u(x) \asymp \Phi_{m, \alpha}(x) \quad \text{or} \quad u(x) \asymp |x|^{-\frac{m+\alpha}{p-m+1}}. \tag{1.10}
\]
Theorem 1.2(ii) above states that any singular solution $u$ of (1.2) either behaves like the fundamental solution $\Phi_{m,\alpha}(x)$ in a neighborhood of the origin or has a stronger singularity precisely given by (1.9)–(1.10). In particular, the asymptotic behaviour in Theorem 1.2(ii) applies to singular solutions of the equation \[ \text{div}(|x|^{-\alpha}|\nabla u|^{m-2}\nabla u) = (I_{\beta} \ast u^p)u^q \text{ in } B_1 \setminus \{0\}. \]

Our asymptotic behaviour (1.9)-(1.10) is in line with [24, Theorem 1.1] (see also [12, Theorem 2.1]) where the authors considered the equation \[ \text{div}(|x|^{-\alpha}|\nabla u|^{m-2}\nabla u) = |x|^{-\theta}u^q \text{ in } B_1 \setminus \{0\}, \tag{1.11} \]

for $\theta < m + \alpha$, $m - 1 < q < (N - \theta)(m - 1)/(N - m - \alpha)$ (if $N > m + \alpha$) and $m - 1 < q < \infty$ (if $N = m + \alpha$). It is obtained in [24, Theorem 1.1] that any singular solution $u$ of (1.11) satisfies the following behaviour at the origin:

- either $|x|^{\frac{m + \alpha - \theta}{q - m + 1}}u(x) \to A$ as $x \to 0$, for some $A = A(N, m, q, \alpha, \theta) > 0$;
- or $\frac{u(x)}{\Phi_{m,\alpha}(x)} \to B$ as $x \to 0$, for some $B = B(N, m, q, \alpha, \theta) > 0$ and

\[ -\text{div}(|x|^{-\alpha}|\nabla u|^{m-2}\nabla u) + |x|^{-\theta}u^q = B\delta_0 \text{ in } \mathcal{D}'(B_1), \]

where $\delta_0$ denotes the Dirac delta mass concentrated at the origin.

In the case of (1.12) such exact behaviour seems difficult to capture due to the presence of the nonlocal term $I_{\beta} \ast u^p$.

Our approach relies on establishing several a priori estimates for the behavior of the singular solutions to (1.1). These combine the Keller-Osserman type estimates (Proposition 2.5), the Harnack inequality (Propositions 2.2 and 2.3) and various estimates for the convolution term $I_{\beta} \ast u^p$. We collect all these results in the next section. Sections 3 and 4 contain the proofs of our main results.

Throughout this paper by $c, C, C_1, C_2, \ldots$ we denote positive generic constants whose values may vary on each occasion. Also, all integrals are computed in the Riemann sense even if we omit the $dx$ or $dy$ symbol.

## 2 Preliminary Results

A key tool in our approach is the use of a priori estimates for solutions $u \in W_{loc}^{1,m}(\Omega) \cap C(\Omega)$ of the inequality

\[ \text{div}(|x|^{-\alpha}|\nabla u|^{m-2}\nabla u) \geq f(x) \quad \text{in } \Omega, \tag{2.1} \]

where $\Omega \subset \mathbb{R}^N$ is an open set and $f \in L^1_{loc}(\Omega)$, $f \geq 0$. Solutions $u$ of (2.1) are understood in the weak sense, that is, \[ \text{div}(|x|^{-\alpha}|\nabla u|^{m-2}\nabla u) \in L^1_{loc}(\Omega) \text{ and } \]

\[ \int_{\Omega} |x|^{-\alpha}|\nabla u|^{m-2}\nabla u \cdot \nabla \phi + \int_{\Omega} f(x)\phi \leq 0 \quad \text{for any } \phi \in C_c^\infty(\Omega), \phi \geq 0. \tag{2.2} \]

In [11, Proposition 2.1] the authors obtain a priori estimates for solutions to the general quasilinear inequality \[ \text{div}[A(x, u, \nabla u)] \geq f(x) \quad \text{in } \Omega, \]
where $A$ is weakly-$m$-coercive. A careful analysis of the proof of [1, Proposition 2.1] reveals that the same arguments can be employed for

$$\text{div}[|x|^{-\alpha}A(x,u,\nabla u)] \geq f(x) \quad \text{in } \Omega,$$

which contains as a particular case the inequality (2.1). The result below is a reformulation of [1, Proposition 2.1].

**Proposition 2.1.** Let $\Omega \subset \mathbb{R}^N$ be an open set such that $B_{4R} \setminus B_{R/2} \subset \Omega$ for some $R > 0$. Let $u \in C(\Omega) \cap W^{1,1}_{\text{loc}}(\Omega)$ be a positive solution of (2.1).

Take $\phi \in C^\infty_c(\Omega)$ such that

- $0 \leq \phi \leq 1$ and $\text{supp } \phi \subset B_{4R} \setminus B_{R/2}$;
- $\phi = 1$ in $B_{2R} \setminus B_R$;
- $|\nabla \phi| \leq \frac{C}{R}$ in $\Omega$.

Then, for any $\ell > m - 1$ there exists $\Lambda = \Lambda(m, \ell)$ such that for any $\lambda > \Lambda$ there exists $C > 0$ independent of $R$ with

$$\int_{\Omega} f(x) \phi^\lambda \leq CR^{N-m-\alpha} \frac{m-1}{\ell} \left( \int_{\Omega} a\phi^\lambda \right)^{\frac{m-1}{\ell}}. \quad (2.3)$$

In the next results we recall the strong and the weak Harnack inequality for the weighted $m$-Laplace operator.

**Proposition 2.2.** (Strong Harnack inequality)

Let $u \in W^{1,m}_{\text{loc}}(\Omega) \cap C(\Omega)$, $u \geq 0$ satisfy

$$\text{div}(\sqrt[m]{a}|\nabla u|^{m-2}\nabla u) + a(x)u^{m-1} = 0 \quad \text{in } \Omega,$$

where $|a(x)| \leq c|x|^{-m-\alpha}$ for some constant $c > 0$. Assume $x \in \Omega$ and $r > 0$ are such that $B_{3r}(x) \subset \Omega$. Then, there exists a constant $C > 0$ independent of $u$ such that

$$\max_{\overline{B}_r(x)} u \leq C \min_{\overline{B}_r(x)} u. \quad (2.4)$$

**Proof.** Note that $u$ satisfies the equation

$$\text{div}(|\nabla u|^{m-2}\nabla u) - \frac{\alpha}{|x|^2}|\nabla u|^{m-2}\nabla u \cdot x + b(x)u^{m-1} = 0 \quad \text{in } \Omega,$$

where $b(x) = a(x)|x|^{\alpha}$ and $|b(x)| \leq c|x|^{-m}$. The above equation fulfills the structural assumptions in [25, Theorem 1.1]. According to this result, $u$ satisfies (2.4).

**Proposition 2.3.** (Weak Harnack inequality)

Let $R > 0$ and $a,b,c$ be real numbers such that $a > b > 3c > 0$. Assume $\Omega \subset \mathbb{R}^N$ is an open set such that

$$\overline{B}_{(a+3c)R} \setminus B_{(b-3c)R} \subset \Omega.$$
Suppose \( u \in W^{1,m}_{\text{loc}}(\Omega) \cap C(\Omega) \) satisfies \( u \geq 0 \) and
\[
\text{div}(|x|^{-\alpha}|\nabla u|^{m-2}\nabla u) \geq 0 \quad \text{in } \Omega.
\]
(2.5)

Then, for any \( \ell > m - 1 \), there exists a constant \( C > 0 \) independent of \( R \) such that
\[
R^{N/\ell} \sup_{B_{aR}\setminus B_{bR}} u \leq C \left( \int_{B_{(a+2c)R}\setminus B_{(b-2c)R}} u^\ell \right)^{1/\ell}.
\]
(2.6)

Proof. Observe first that (2.5) is equivalent to
\[
\text{div}(|\nabla u|^{m-2}\nabla u) - \frac{\alpha}{|x|^2} |\nabla u|^{m-2}\nabla u \cdot x \geq 0 \quad \text{in } \Omega,
\]
which satisfies the structural assumptions in [25].

Let \( z_1, z_2, \ldots, z_k \in \Omega \) be such that \( \{B_{cR}(z_i)\}_{i=1}^k \) is a finite cover with open balls of the compact set \( \overline{B_{aR} \setminus B_{bR}} \). By the standard Harnack inequality (see Trudinger [25, Theorem 1.3]) we find
\[
R^{N/\ell} \sup_{B_{cR}(z_i)} u \leq C \left( \int_{B_{2cR}(z_i)} u^\ell \right)^{1/\ell} \leq C \left( \int_{B_{(a+2c)R}\setminus B_{(b-2c)R}} u^\ell \right)^{1/\ell}.
\]
Thus,
\[
R^{N/\ell} \sup_{B_{aR}\setminus B_{bR}} u \leq R^{N/\ell} \sup_{\cup_{i=1}^k B_{cR}(z_i)} u \leq C \left( \int_{B_{(a+2c)R}\setminus B_{(b-2c)R}} u^\ell \right)^{1/\ell}.
\]
\[\square\]

Proposition 2.4. Assume \( u \in W^{1,m}_{\text{loc}}(B_1 \setminus \{0\}) \cap C(B_1 \setminus \{0\}) \) satisfies \( u \geq 0 \) and
\[
\text{div}(|x|^{-\alpha}|\nabla u|^{m-2}\nabla u) \geq 0 \quad \text{in } B_1 \setminus \{0\}.
\]
Then, either \( u \) is bounded near the origin, or there exist \( C > 0 \) and \( r_0 \in (0, 1/2) \) such that
\[
\sup_{|x|=r} \frac{u(x)}{\Phi_{m,\alpha}(x)} \geq C \quad \text{for all } r \in (0, r_0),
\]
(2.7)
where \( \Phi_{m,\alpha} \) is defined in (1.7).

Proof. Assume that (2.7) does not hold. Hence, \[
\liminf_{r \to 0} \left( \sup_{|x|=r} \frac{u(x)}{\Phi_{m,\alpha}(x)} \right) = 0.
\]
Then, for any \( k \geq 1 \) there exists \( r_k \in (0, 1/2) \), with \( r_k \to 0 \) as \( k \to \infty \), such that
\[
\sup_{|x|=r_k} \frac{u(x)}{\Phi_{m,\alpha}(x)} \leq \frac{1}{k} \quad \text{for all } k \geq 1.
\]
A comparison principle in the annular region \( B_{1/2} \setminus B_{r_k} \) shows that for all \( k \geq 1 \) we have
\[
u(x) \leq \frac{1}{k} \Phi_{m,\alpha}(x) + \max_{|x|=1/2} u(x) \quad \text{in } B_{1/2} \setminus B_{r_k},
\]
Letting \( k \to \infty \) in the above estimate we deduce that \( u \) is bounded in the ball \( B_{1/2} \). \[\square\]
The result below provides a first important estimate for solutions to (1.1).

**Proposition 2.5.** (Keller-Osserman type estimates)

Assume \( p + q > 2(m - 1) \) and let \( u \in W^{1,m}_{loc}(B_1 \setminus \{0\}) \cap C(B_1 \setminus \{0\}) \) be a positive solution of (1.1). Then, there exist \( C > 0 \) such that

\[
u(x) \leq C|x|^{-\sigma} \quad \text{in} \quad B_1 \setminus \{0\},
\]

where \( \sigma > 0 \) is given by (1.6).

**Proof.** We use Proposition 2.1 with \( \Omega = B_1 \), \( R \in (0, 1/4) \), \( f(x) = (I_\beta \ast u^p)u^q \) and \( \ell = (p + q)/2 > m - 1 \). From (2.3) we find

\[CR^{N-m-\alpha-m-1/\ell} \left( \int_{B_1} u^\ell \phi^\lambda \right)^{m-1/\ell} \geq \int_{B_1} (I_\beta \ast u^p)u^q \phi^\lambda, \tag{2.9}\]

where \( \phi \in C_\infty^c(B_1 \setminus \{0\}) \) and \( \lambda > m \) are chosen as in Proposition 2.1. If \( x, y \in B_{2R} \) \( \subset \supp \phi \), then \( |x - y| \leq |x| + |y| \leq 4R \) so

\[
(I_\beta \ast u^p)(x) \geq C \int_{B_{4R}} \frac{u^p(y)}{|x - y|^{N-\beta}} dy
\]

\[
\geq C \int_{B_{2R}} \frac{u^p(y)}{(4R)^{N-\beta}} dy
\]

\[
\geq CR^{3-N} \int_{B_1} u^p(y)\phi^\lambda(y) dy,
\]

since \( 0 \leq \phi \leq 1 \). Using this fact in (2.9) together with Hölder’s inequality, for \( \ell = (p + q)/2 \) we find

\[CR^\tau \left( \int_{B_1} u^\ell \phi^\lambda \right)^{m-1/\ell} \geq \left( \int_{B_1} u^p \phi^\lambda \right) \left( \int_{B_1} u^q \phi^\lambda \right) \geq \left( \int_{B_1} u^\ell \phi^\lambda \right)^2,
\]

where

\[
\tau = 2N - m - \alpha - \beta - \frac{m-1}{\ell}N. \tag{2.10}\]

Now, using the fact that \( \phi = 1 \) in \( B_{2R} \setminus B_R \) and the weak Harnack inequality (2.6) with \( a = 7/4, b = 5/4 \) and \( c = 1/8 \) we deduce

\[CR^\tau \geq \left( \int_{B_1} u^\ell \phi^\lambda \right)^2 \geq \left( \int_{B_{2R}\setminus B_R} u^\ell \right)^2 \geq \left( \int_{B_{2R}\setminus B_R} u^\ell \right)^2 \geq \left( \int_{B_{2R}\setminus B_R} u^p \right)^2 \geq R^{2N-m-1} \left( \sup_{\frac{5R}{4} < |x| < \frac{7R}{4}} u^{p+q-m-1} \right).
\]

From here and (2.10) we derive (2.8). \( \square \)

Similar to Proposition 2.5 we have:
Proposition 2.6. Let $\theta \geq 0$ and $q > \max\{m-1, \theta\}$.
If $u \in W_{loc}^{1,m}(B_1 \setminus \{0\}) \cap C(\overline{B}_1 \setminus \{0\})$ is positive and satisfies
\[
\text{div}(|x|^{-\alpha}|\nabla u|^{m-2}\nabla u) \geq |x|^{-\theta} u^q \quad \text{in } B_1 \setminus \{0\},
\] (2.11)
then
\[
w(x) \leq C|x|^{-\theta q \left(\frac{m}{\alpha-q}+\frac{1}{q-m+1}\right)} \quad \text{for all } x \in B_1 \setminus \{0\},
\] (2.12)
for some constant $C > 0$.

Proof. According to (2.3) with $\ell = q > m - 1$ we have
\[
CR^{N-m-\alpha-\frac{m-1}{q}} \left( \int_{B_1} u^q \phi^\lambda \right)^{\frac{m-1}{q}} \geq \int_{B_1} |x|^{-\theta} u^q \phi^\lambda.
\]

Since $\text{supp } \phi \subset B_{4R} \setminus B_{R/2}$, from the above estimate and the weak Harnack inequality (2.6) with $a = 7/4$, $b = 5/4$ and $c = 1/8$ it follows that
\[
CR^{N-m-\alpha-\frac{m-1}{q}} \geq R^{-\theta} \left( \int_{B_{2R} \setminus B_R} u^q \right)^{\frac{m-1}{q}} \geq R^{-\theta} \left( R^N \sup_{\frac{R}{2} < |x| < \frac{3R}{2}} u^q \right)^{\frac{m-1}{q}} \geq R^{N-\theta-\frac{m-1}{q}} \left( \sup_{\frac{R}{2} < |x| < \frac{3R}{2}} u^{-m+1} \right).
\]
From here, we easily deduce (2.12). \qed

Lemma 2.7. (See [24, Theorem 1.1]) Let $m > 1$, $N \geq m + \alpha > \theta$ and
\[
\begin{aligned}
  m-1 < q < \frac{(N-\theta)(m-1)}{N-m-\alpha} & \quad \text{if } N > m + \alpha, \\
  m-1 < q < \infty & \quad \text{if } N = m + \alpha.
\end{aligned}
\] (2.13)

Let $u \in W^{1,m}(B_1 \setminus \{0\}) \cap C(B_1 \setminus \{0\})$, $u \geq 0$, be a singular solution of
\[
\text{div}(|x|^{-\alpha}|\nabla w|^{m-2}\nabla w) = |x|^{-\theta} w^q \quad \text{in } B_1 \setminus \{0\}.
\] (2.14)
Then,
\[
either w \asymp \Phi_{m,\alpha}(x) \text{ or } w \asymp |x|^{-\frac{m+\alpha-\theta}{q-m+1}}.
\]

Proposition 2.8. Assume $N > m + \alpha$ and $q \geq \frac{N(m-1)}{N-m-\alpha}$. Then, any solution of (1.1) is bounded around the origin.

Proof. We use some tools from [26, Proposition 1.2]. Let
\[
\nu = \frac{N(m-1)}{N-m-\alpha}
\]
and let $u$ be a positive solution of (1.1). We note that since $u \in L^p(B_1)$, $u$ satisfies
\[
\text{div}(|x|^{-\alpha}|\nabla u|^{m-2}\nabla u) \geq cu^q \quad \text{in } B_1 \setminus \{0\},
\] (2.15)
where $c = 2^{\alpha-N} \int_{B_1} u^p > 0$, by (1.3). Using Proposition 2.6 (with $\theta = 0$ and being $q \geq \nu > m - 1$) we deduce
\[
u > m \quad \text{and let } u \in L^p(B_1)
\]
In particular, again by $q \geq \nu$, it follows that
\[
u > m \quad \text{and let } u \in L^p(B_1)
\]
Also, from (2.15) we deduce
\[
u > m \quad \text{and let } u \in L^p(B_1)
\]
for some $C > 0$.

In order to proceed to the proof of Proposition 2.8 we need the following result.

**Lemma 2.9.** Assume $u$ satisfies (2.17). Then, for any $\phi \in C^1_c(B_1 \setminus \{0\})$, $\phi \geq 0$ and any number $M \geq (C/c)^\nu$ we have
\[
\left\| |x|^{-\alpha/m} \phi \nabla (u - M)^+ \right\|_{L^m(B_1)} \leq m \left\| |x|^{-\alpha/m} (u - M)^+ |\nabla \phi| \right\|_{L^m(B_1)}.
\] (2.18)

**Proof of Lemma 2.9.** Let $\{\eta_k\} \subset C^1(\mathbb{R})$ be such that $\eta_k \geq 0$,
\[
\eta_k' = 0 \quad \text{on } (-\infty,0), \quad \eta_k' > 0 \quad \text{on } (0,\infty),
\]
\[
\eta_k(t) \to \text{sign}^+(t), \quad \eta_k(t) \to t^+ \quad \text{as } k \to \infty,
\]
where $\text{sign}^+(t) = 1$ if $t > 0$ and $\text{sign}^+(t) = 0$ if $t < 0$. Take $\eta_k(u - M)\phi^m$ as a test function in (2.17). We find
\[
\int_{B_1} (cu^\nu - C)\eta_k(u - M)\phi^m dx + \int_{B_1} |x|^{-\alpha/|\nabla u|^2}\nabla u \cdot \nabla (\eta_k(u - M)\phi^m) dx \leq 0.
\]
Since $(cu^\nu - C)\eta_k(u - M)\phi^m \geq 0$, by the choice of $M$, it follows that
\[
\int_{B_1} |x|^{-\alpha/|\nabla u|^2}\nabla u \cdot \nabla (u - M)^+ dx + m \int_{B_1} |x|^{-\alpha/|\nabla u|^2}\nabla u \cdot \nabla \phi dx \leq 0.
\]
Letting $k \to \infty$, by Fatou’s lemma we find
\[
\int_{B_1} |x|^{-\alpha/|\nabla u|^2}\nabla (u - M)^+ m dx + m \int_{B_1} |x|^{-\alpha/|\nabla u|^2}(u - M)^+ \nabla u \cdot \nabla \phi dx \leq 0,
\]
so
\[
\int_{B_1} |x|^{-\alpha/|\nabla u|^2}\nabla (u - M)^+ m dx \leq m \int_{B_1} |x|^{-\alpha/|\nabla u|^2}(u - M)^+ \nabla \phi dx. \quad (2.19)
\]
By Hölder’s inequality and since \((u - M)^+|\nabla u| = (u - M)^+|\nabla(u - M)^+|\), we estimate the right hand-side of (2.19) as
\[
\int_{B_1} |x|^{-\alpha} \phi^m (u - M)^+ |\nabla \phi| dx \leq \left( \int_{B_1} |x|^{-\alpha} \phi^m (u - M)^+ |\nabla \phi| dx \right)^{1/m'} \times \left( \int_{B_1} |x|^{-\alpha} |\nabla \phi|'' (u - M)^+ |\nabla \phi|'' dx \right)^{1/m},
\]
where \(m'\) is the Hölder conjugate of \(m\). Using (2.21) into (2.19) we deduce (2.18).

We are now ready to proceed to the proof of Proposition 2.8 whose arguments will be divided into two steps.

**Step 1:** \(u \in L^\nu_{loc}(B_1)\). Let \(\eta \in C^1(\mathbb{R})\) be such that \(\eta \geq 0\), \(\eta\) is bounded, \(\eta = 0\) on \((-\infty, 0)\) and \(\eta' > 0\) on \((0, \infty)\). Let also \(\{\zeta_k\} \subset C^1_c(\mathbb{R}^N)\) be such that
\[
\zeta_k(x) = \begin{cases} 
0 & \text{if } |x| < \frac{1}{2k} \text{ or } |x| > \frac{2}{3}, \\
1 & \text{if } \frac{1}{k} < |x| < \frac{1}{2},
\end{cases}
\]
and \(|\nabla \zeta_k| \leq Ck\).

Define \(A_k = B_{1/k} \setminus B_{1/(2k)}\) and
\[
M \geq \max \left\{ (C/c)^\eta, \max_{1/2 \leq |x| \leq 2/3} u(x) \right\}.
\]

We next test (2.17) with \(\zeta_k \eta(u - M)\). We find
\[
\int_{B_1} (cu' - C) \zeta_k \eta(u - M) dx + \int_{B_1} |x|^{-\alpha} |\nabla u|'' \nabla u \cdot \nabla (\zeta_k \eta(u - M)) dx \leq 0.
\]
Since \(\eta' > 0\) and \(|\nabla u|'' \nabla u \cdot \nabla (u - M) = |\nabla u|^m \geq 0\), it follows that
\[
\int_{B_1} (cu' - C) \zeta_k \eta(u - M) dx \leq \Gamma_k := \int_{B_1} |x|^{-\alpha} \eta(u - M) |\nabla u|'' |\nabla \zeta_k| dx. \tag{2.21}
\]
Observe that \(\eta(u - M) \nabla \zeta_k = 0\) outside of \(A_k\), being \(M \geq \max_{1/2 \leq |x| \leq 2/3} u(x)\). Using the fact that \(\eta\) is bounded together with Hölder’s inequality we find
\[
\Gamma_k \leq \|\eta\|_\infty \int_{A_k} |x|^{-\alpha} |\nabla (u - M)^+ |\nabla \zeta_k| dx
\leq C \left\| |x|^{-\alpha/m} |\nabla (u - M)^+ | \right\|_{L^m(A_k)} \left\| |x|^{-\alpha/m} |\nabla \zeta_k| \right\|_{L^m(A_k)}. \tag{2.22}
\]
By the definition of \(\zeta_k\) and the fact that \(|\nabla \zeta_k| \leq ck\) we have
\[
\left\| |x|^{-\alpha/m} |\nabla \zeta_k| \right\|_{L^m(A_k)} \leq Ck^1 - \frac{N - \alpha}{m}.
\]
Using this fact in (2.22) together with \( \zeta_{2k} = 1 \) in \( A_k \) and \( \zeta_k \geq 0 \) in \( A_{2k} \), we further estimate

\[
\Gamma_k \leq Ck^{1 - \frac{N - \alpha}{m}} \left\| x^{-\alpha/m} |(u - M)^+| \right\|_{L^m(A_{2k})}^{m-1} \\
\leq Ck^{1 - \frac{N - \alpha}{m}} \left\| x^{-\alpha/m} \zeta_{2k} \nabla (u - M)^+ \right\|_{L^m(A_{2k} \cup A_k)}^{m-1} \\
\leq Ck^{1 - \frac{N - \alpha}{m}} \left\| x^{-\alpha/m} (u - M)^+ |\nabla \zeta_{2k}| \right\|_{L^m(A_{2k})}^{m-1},
\]

(2.23)

where in the last inequality we have used (2.18) with \( \phi = \zeta_{2k} \) and the fact that \( \nabla \zeta_{2k} = 0 \) in \( A_k \). From (2.16) we have

\[
\int_{A_{2k}} |x|^{-\alpha/2} |(u - M)^+|^m |\nabla \zeta_{2k}| dx \leq Ck^{\alpha + \nu} \int_{A_{2k}} |(u - M)^+|^m \leq Ck^{\alpha + \nu - N + \frac{m(m + \alpha)}{m + \nu}}.
\]

Hence, from (2.23) we deduce

\[
\Gamma_k \leq Ck^{1 - \frac{N - \alpha}{m} + (\alpha + \nu - N + \frac{m(m + \alpha)}{m + \nu})} = C.
\]

We now replace \( \eta \) in (2.21) by a sequence \( \{\eta_n\} \) such that \( \eta_n(t) \to \text{sign}^+(t) \) as \( n \to \infty \). Letting \( n \to \infty \) and then \( k \to \infty \) in (2.21), since \( \text{supp} \zeta_k = B_{2/3} \) and \( \zeta_k \to 1 \) in \( B_{1/2} \), we find

\[
\int_{B_1} (cu^\nu - C) \text{sign}^+(u - M) dx \leq C,
\]

so \( u \in L^\infty_{loc}(B_1) \).

**Step 2:** \( u \in L^\infty_{loc}(B_1) \). We return to the estimate (2.23) and split our analysis into two cases.

- **Case 2.1:** \( \nu \geq m \). By Hölder’s inequality we find

\[
\left\| x^{-\alpha/m} (u - M)^+ |\nabla \zeta_{2k}| \right\|_{L^m(A_{2k})} \leq Ck^{\nu + 1} \left\| (u - M)^+ \right\|_{L^m(A_{2k})}^{\nu + 1} \\
\leq Ck^{\frac{\alpha}{\nu} + 1} \left\| (u - M)^+ \right\|_{L^\nu(A_{2k})}^{\frac{1}{\nu}} |A_{2k}|^{\frac{1}{\nu}} \\
= Ck^{\frac{\alpha}{\nu} + 1 - N} \frac{1}{\nu} o(1) \text{ as } k \to \infty.
\]

Using this estimate in (2.23) we deduce \( \Gamma_k \leq Ck^{\frac{N(m - 1)}{\nu} - N + m + \alpha} = o(1) \) as \( k \to \infty \), thanks to the value of \( \nu \).

- **Case 2.2:** \( \nu < m \). From (2.16) we have

\[
\left\| x^{-\alpha/m} (u - M)^+ |\nabla \zeta_{2k}| \right\|_{L^m(A_{2k})} \leq Ck^{\frac{\alpha}{\nu} + 1} \left\| (u - M)^+ \right\|_{L^m(A_{2k})}^{\frac{1}{\nu}} \\
\leq Ck^{\frac{\alpha}{\nu} + 1} \sup_{A_{2k}} \left\| (u - M)^+ \right\|_{L^\nu(A_{2k})}^{\frac{1}{\nu}} \\
= Ck^{\frac{\alpha}{\nu} + 1 - \frac{m + \alpha}{\nu}} o(1) \\
\leq Ck^{\frac{\alpha}{\nu} + 1 - \frac{\nu}{\nu + m + 1}} (1 - \frac{\nu}{\nu + m + 1}) o(1) \text{ as } k \to \infty,
\]

and from (2.23) we again derive \( \Gamma_k \leq Ck^{N(\nu - m - \alpha)/m} o(1) \) as \( k \to \infty \), thanks to the value of \( \nu \).
We now return to (2.21) and let \( k \to \infty \) to deduce
\[
\int_{B_{1/2}} (cu' - C)\eta(u - M)dx = 0.
\]

Since \( \eta \geq 0 \), it follows that \( u \leq M \) in \( B_{1/2} \), so \( u \in L^\infty_{\text{loc}}(B_1) \) which completes our proof. \( \square \)

**Lemma 2.10.** Let \( a, b \in (0, N) \) and \( \theta \geq 0 \). Then, there exists \( C > c > 0 \) such that:

(i) If \( a + b > N \) one has
\[
c \left( \frac{\log \frac{5}{|x|}}{|x|^{a+b-N}} \right)^{-\theta} \leq \int_{|y|<1} \left( \frac{\log \frac{5}{|y|}}{|x-y|^{a}|y|^{b}} \right)^{-\theta} \leq \frac{C \left( \log \frac{5}{|x|} \right)^{-\theta}}{|x|^{a+b-N}} \text{ for all } x \in B_1 \setminus \{0\}. \tag{2.24}
\]

(ii) If \( a + b = N, \theta \neq 1 \), one has
\[
c \left( \frac{\log \frac{5}{|x|}}{|x|^{a+b-N}} \right)^{(1-\theta)^+} \leq \int_{|y|<1} \left( \frac{\log \frac{5}{|y|}}{|x-y|^{a}|y|^{b}} \right)^{(1-\theta)^+} \leq C \left( \log \frac{5}{|x|} \right)^{(1-\theta)^+} \text{ for all } x \in B_1 \setminus \{0\}. \tag{2.25}
\]

(iii) If \( a + b < N \) one has
\[
c \leq \int_{|y|<1} \left( \frac{\log \frac{5}{|y|}}{|x-y|^{a}|y|^{b}} \right)^{-\theta} \leq C \text{ for all } x \in B_1 \setminus \{0\}. \tag{2.26}
\]

The proof of the above lemma will be given in the Appendix.

**Remark 2.11.** A direct and useful calculation shows that if
\[
u(x) = \kappa |x|^{-\gamma} \left( \frac{5}{|x|} \right)^{-\tau}, \gamma > 0,
\]
then
\[
\text{div} \left( |x|^{-\alpha} \nabla u \right) = \kappa^{m-1} |x|^{-\gamma(m-1)-m-\alpha} \left( \frac{5}{|x|} \right)^{-\tau(m-1)-m} \times \\
\times \left\{ -\gamma \log \frac{5}{|x|} + \tau \left[ A \left( \frac{5}{|x|} \right)^2 + B \log \frac{5}{|x|} + C \right] \right\},
\]
where
\[
A = \gamma \left[ \gamma(m-1) - (N - m - \alpha) \right], \\
B = \tau \left[ -2 \gamma(m-1) + (N - m - \alpha) \right], \\
C = (m-1)\tau(\tau + 1), \tag{2.27}
\]

13
3 Proof of Theorem 1.1

(i) Let
\[ 0 < \gamma < \min \left\{ \frac{\beta}{p}, \frac{m + \alpha}{q - m + 1} \right\}. \] (3.1)

We show that \( u(x) = \kappa |x|^{-\gamma} \) is a singular radially symmetric solution of (1.1) for suitable \( \kappa \in (0, 1) \). Since from (3.1) we have \( p\gamma < \beta < N \) it follows that \( u \in L^p(B_1) \). By Remark 2.11 (in which we take \( \tau = 0 \)) one has
\[ \text{div}\left(|x|^{-\alpha} |\nabla u|^{m-2} \nabla u\right) = \kappa^{m-1} \gamma^{m-2} A |x|^{-(m-1)\gamma - m - \alpha} \quad \text{in } B_1 \setminus \{0\}, \] (3.2)
where \( A \) is defined in (2.27). From \( N \leq m + \alpha \) and \( \gamma > 0 \) we have \( A > 0 \). Further, since \( p\gamma < \beta \), by Lemma 2.10 (iii) with \( \theta = 0 \), \( a = N - \beta \), \( b = p\gamma \) and \( a + b < N \), we estimate
\[ (I_\beta * u^q)(x) \asymp \kappa^p |x|^{-\gamma q} \quad \text{in } B_1 \setminus \{0\}. \] (3.3)
Comparing (3.2) and (3.3) we see that for \( \kappa \in (0, 1) \) small enough, thanks to (3.1) and \( q > m - 1 \), one has that \( u(x) = \kappa |x|^{-\gamma} \) is a singular positive solution of (1.1).

(ii) Let \( u \) be a positive singular solution of (1.1). Using Propositions 2.4 and 2.5 there exists \( C > 0 \) such that for small \( R > 0 \) we find
\[ CR^{-\sigma} \geq \sup_{|x|=R} u \geq cR^{-\frac{N - m - \alpha}{m - 1}}, \] (3.4)
where \( \sigma > 0 \) is defined in (1.6). The above estimate implies \( \sigma \geq \frac{N - m - \alpha}{m - 1} \) which is equivalent to \( p + q \leq (N + \beta)(m - 1)/(N - m - \alpha) \).

We claim that both inequalities are strict. Assume by contradiction that \( \sigma = \frac{N - m - \alpha}{m - 1} \) and let \( x \in B_{1/2} \setminus \{0\} \). Combining the estimate (3.4) with the weak Harnack inequality (2.6) with \( a = 1, b = 1/2, c = 1/8 \) and \( \ell = p > m - 1 \), we find
\[ (I_\beta * u^q)(x) \geq \int_{B_{|x|/4} \setminus B_{|x|/16}} \frac{u^q(y)}{|x - y|^{N-\beta}} \, dy \]
\[ \geq \left( \frac{9|x|}{4} \right)^{\beta - N} \int_{B_{|x|/4} \setminus B_{|x|/16}} u^q(y) \, dy \]
\[ \geq C |x|^{\beta - N} \left( |x|^{N/p} \sup_{\partial B_{|x|}} u \right)^p \quad \text{(by Harnack’s inequality (2.6))} \]
\[ \geq C |x|^{\beta - \sigma p} \quad \text{(by estimate (3.4)).} \]

Hence, \( u \) satisfies
\[ \text{div}\left(|x|^{-\alpha} |\nabla u|^{m-2} \nabla u\right) \geq C |x|^{\beta - \sigma p} u^q \quad \text{in } B_{1/2} \setminus \{0\}. \]

For any \( k \geq 3 \) let \( v_k \in C^1(B_{1/2} \setminus B_{1/k}) \) be a radial function such that
\[
\begin{cases}
\text{div}\left(|x|^{-\alpha} |\nabla v_k|^{m-2} \nabla v_k\right) = C |x|^{\beta - \sigma p} v_k^q \quad \text{in } B_{1/2} \setminus B_{1/k}, \\
v_k = \sup_{|x|=1/k} u \quad \text{on } \partial B_{1/k}, \\
v_k = \sup_{|x|=1/2} u \quad \text{on } \partial B_{1/2}.
\end{cases}
\]
Observe that $u$ is a subsolution while $cΦ_{m,α}$ is a supersolution of the above problem for suitable $c > 0$. By the maximum principle we find that $k \mapsto v_k$ is increasing and
\[ cΦ_{m,α} \geq v_k \geq u \quad \text{in } B_{1/2} \setminus B_{1/k}, \tag{3.5} \]
for some constant $c > 0$. Thus, there exists $v(x) = \lim_{k \to \infty} v_k(x)$ for all $x \in \overline{B_{1/2}} \setminus \{0\}$ and $v \in C^1(B_{1/2} \setminus \{0\})$ satisfies
\[ \nabla(|x|^{-α}|∇v|^{m-2}∇v) = C|x|^{β-σp}v^q \quad \text{in } B_1 \setminus \{0\}. \tag{3.6} \]
Also $v$ is radial (since $v_k$ is radial) and from (3.5) we find
\[ cΦ_{m,α} \geq v \geq u \quad \text{in } B_{1/2} \setminus \{0\}. \tag{3.7} \]
Using this inequality it is easy to see that $v$ satisfies the conditions of Proposition 2.2 with
\[ a(x) = |x|^{β-σp}v(x)^q \leq c|x|^{β-σ(p+q-m+1)} = c|x|^{-m-α}. \]
Thus, by (2.4), (3.4) and (3.7) we find
\[ v(x) \geq c \sup_{|y|=|x|} v(y) \geq C|x|^{-σ} \quad \text{for all } x \in B_{1/4} \setminus \{0\}. \]
From (3.6) and the above estimate we find
\[ \left( r^{N-1-α}|v'(r)|^{m-2}v'(r) \right)' = Cr^{N-1+β-σp}v^q \geq Cr^{N-1+β-σ(p+q)} \quad \text{for all } 0 < r < 1/4. \]
Since
\[ σ = \frac{N-m-α}{m-1} = \frac{m+α+β}{p+q-m+1}, \]
the above estimate reads
\[ \left( r^{N-1-α}|v'(r)|^{m-2}v'(r) \right)' \geq Cr^{-1} \quad \text{for all } 0 < r < 1/4. \]
We now fix $\bar{r} \in (0, 1/4)$ and integrate in the above inequality over $[r, \bar{r}]$. We obtain
\[ -r^{N-1-α}|v'(r)|^{m-2}v'(r) \geq -\bar{r}^{N-1-α}|v'(\bar{r})|^{m-2}v'(\bar{r}) + C \ln \frac{\bar{r}}{r} \quad \text{for all } 0 < r < \bar{r} < 1/4. \]
From here we have
\[ \lim_{r \to 0^+} r^{N-1-α}|v'(r)|^{m-2}v'(r) = -∞, \]
so that
\[ \lim_{r \to 0^+} r^{-N-1-α} = -∞. \]

By l’Hôpital’s rule it follows that
\[ \lim_{r \to 0^+} \frac{v(r)}{Φ_{m,α}(r)} = 0. \]
which contradicts (3.7) and proves our claim. Hence \( \sigma > \frac{N-m-\alpha}{m-1} \) which yields (1.5)2. Also, (1.5)3 follows from (1.5)2 and the fact that \( p, q > m - 1 \).

To derive the first inequality in (1.5) we combine the weak Harnack inequality and Proposition 2.4 with the regularity condition \( u \in L^p(B_1) \). We find

\[
\infty > \int_{B_{1/2}} u^p \geq \sum_{k=1}^{\infty} \int_{2^{-1-3k} < |y| < 2^{-3k}} u^p(y)\,dy
\geq \frac{C}{2N} \sum_{k=1}^{\infty} 2^{-3kN} \left( \sup_{\frac{1}{2} 2^{-3k} < |y| < \frac{3}{2} 2^{-3k}} u(y) \right)^p
\geq \frac{C}{2N} \sum_{k=1}^{\infty} 2^{-3kN} \left( \sup_{|y|=3^{-2-3k}} u(y) \right)^p
\geq \frac{C}{2N} \sum_{k=1}^{\infty} \left( \frac{N-m-\alpha}{m-1} \right)^p k.
\]

This implies \( N - \frac{N-m-\alpha}{m-1} p > 0 \) which establishes the first inequality in (1.5) for \( p \).

If \( q \geq \frac{N(m-1)}{N-m-\alpha} \), by Proposition 2.8 we deduce \( u \in L^\infty(B_1) \), which is not possible since \( u \) is singular. Hence, \( q < \frac{N(m-1)}{N-m-\alpha} \).

Conversely, assume that (1.5) holds. We construct a singular radially symmetric solution \( u \) of (1.1) in the form \( u(x) = \kappa |x|^{-\gamma} \), with \( \kappa, \gamma > 0 \) to be determined. **Case 1:** \( \sigma p > \beta \).

Let

\[
\max \left\{ \frac{N-m-\alpha}{m-1} \cdot \frac{\beta}{p} \right\} < \gamma < \min \left\{ \sigma, \frac{N}{p} \right\}.
\]

Note that this choice of \( \gamma \) is possible thanks to (1.5)1 and to our assumption \( \sigma > \frac{N-m-\alpha}{m-1} \).

Also, \( u(x) = \kappa |x|^{-\gamma} \) satisfies (3.2), where now the positivity of \( A \) follows from the lower bound of \( \gamma \).

By Lemma 2.10(i) with \( \theta = 0 \), \( a = N - \beta \), \( b = p \gamma \) so that \( a + b > N \) being \( p \gamma > \beta \), we find

\[
(I_\beta * u^p)u^q(x) \leq C_{p,q} \kappa^p |x|^{-p} u^q(x) \leq C_{\kappa^p+q} |x|^{\beta-(p+q)\gamma} \quad \text{in } B_1 \setminus \{0\}.
\]

Using (3.2), (3.4) and the fact that \( p + q > m - 1 \) together with \( \gamma < \sigma \), we may take \( \kappa \in (0, 1) \) small enough such that

\[
\text{div}(|x|^{-\alpha} \nabla u)^{m-2} \nabla u) = C_1 \kappa^m-1 |x|^{-(m-1)\gamma-m-\alpha}
\geq C_2 \kappa^{p+q} |x|^\beta-(p+q)\gamma
\geq (I_\beta * u^p)u^q(x) \quad \text{in } B_1 \setminus \{0\}.
\]

This shows that \( u(x) = \kappa |x|^{-\gamma} \) is a positive singular solution of (1.1) in \( B_1 \setminus \{0\} \).

**Case 2:** \( \sigma p \leq \beta \).

Let us observe first that this condition is equivalent to

\[
\frac{m + \alpha}{q - m + 1} \leq \sigma \leq \frac{\beta}{p}.
\]
Indeed, by replacing in $\sigma p \leq \beta$ the value of $\sigma$ given in (1.6), we get

$$(m + \alpha)p \leq \beta(q - m + 1).$$

Adding $(m + \alpha)(q - m + 1)$ on both sides of the above inequality we find

$$(m + \alpha)(p + q - m + 1) \leq (m + \alpha + \beta)(q - m + 1),$$

namely

$$\frac{m + \alpha}{q - m + 1} \leq \frac{m + \alpha + \beta}{p + q - m + 1} = \sigma,$$

thus the required lower bound for $\sigma$ follows, as the upper bound trivially holds since we are in Case 1b.

Let

$$\frac{N - m - \alpha}{m - 1} < \gamma < \frac{m + \alpha}{q - m + 1} \leq \sigma \leq \frac{\beta}{p}.$$

(Note that from (1.5) we have $N - m - \alpha < \frac{m + \alpha}{q - m + 1}$). Letting $u(x) = \kappa|x|^{-\gamma}$, we have that $u$ satisfies (3.3), where here $A > 0$ by the upper bound of $\gamma$. Also, by Lemma 2.10(iii) with $\theta = 0$, $a = N - \beta$, $b = p\gamma$ so that $a + b < N$ being $\beta > p\gamma$, we have

$$(I_\beta \ast u^p)u^q(x) \leq C\kappa^p u^q(x) \leq C\kappa^{p+q}|x|^{-q\gamma} \text{ in } B_1 \setminus \{0\}. \quad (3.12)$$

Combining (3.2) and (3.12) in the same way as we did in Case 1 we derive that $u(x) = \kappa|x|^{-\gamma}$ is a singular solution of (1.1).

\[ \Box \]

4 Proof of Theorem 1.2

(i) Any singular solution $u$ of (1.2) fulfills in particular (1.1). Thus, by Theorem 1.1 conditions (1.5) must hold if $N > m + \alpha$.

Conversely, assume now that either $N \leq m + \alpha$ or $N > m + \alpha$ and (1.5) holds. Let $\sigma$ be defined by (1.6) and $\tau = \frac{1}{p+q-m+1} > 0$.

We claim that

$$u(x) = \begin{cases} |x|^{-\sigma} & \text{if } \sigma p > \beta, \\ |x|^{-\sigma}\left(\log \frac{5}{|x|}\right)^{-\tau} & \text{if } \sigma p = \beta, \\ |x|^{-\frac{m+\alpha}{q-m+1}} & \text{if } \sigma p < \beta, \end{cases}$$

is a solution of (1.2). A straightforward calculation using Remark 2.11 yields

$$\text{div}(|x|^{-\sigma}|\nabla u|^{m-2}\nabla u) \propto \begin{cases} |x|^{-\sigma(m-1)-m-\alpha} & \text{if } \sigma p > \beta, \\ |x|^{-\sigma(m-1)-m-\alpha}\left(\log \frac{5}{|x|}\right)^{-(m-1)\tau} & \text{if } \sigma p = \beta, \\ |x|^{-\frac{\sigma(m+\alpha)}{q-m+1}} & \text{if } \sigma p < \beta. \end{cases}$$
To see this we first note that (1.5) implies

\[ \sigma > \frac{N - m - \alpha}{m - 1}. \]  

Thus, the coefficient \( A \) defined in (2.27) (in which \( \gamma = \sigma \)) satisfies \( A > 0 \).

Also, by Lemma 2.10(i)-(iii) (we use \( \theta = \tau p \in (0, 1) \) if \( \sigma p = \beta \)) we have

\[ (I \beta * u^p) u^q(x) \lesssim \begin{cases} 
|x|^{\beta - \sigma(p+q)} & \text{if } \sigma p > \beta, \\
|x|^{-q\sigma} \left( \log \frac{5}{|x|} \right)^{1-\tau(p+q)} & \text{if } \sigma p = \beta, \\
|x|^{-q(m+\alpha) + 1} & \text{if } \sigma p < \beta, 
\end{cases} \]

where in the latter case \( \sigma p < \beta \), from Case 2 in the proof of Theorem 1.1(ii), we have that (3.10) holds with the strict sign so that we fall in Case(iii) of Lemma 2.10.

From the above estimates we have

\[ \text{div} \left( |x|^{-\alpha} \nabla u \right) \lesssim (I_\alpha * u^p) u^q \]

and thus, for suitable constants \( a \geq b > 0 \) we have that \( u \) satisfies (1.2).

(ii) Let \( u \) be a singular solution of (1.2). We divide our argument into two steps.

**Step 1: \( u \) satisfies the strong Harnack inequality (2.4).**

Note first that \( u \) satisfies the inequality

\[ \text{div} \left( |x|^{-\alpha} \nabla u \right) \geq cu^q \quad \text{in } \Omega \]

where \( c = 2^{-N} \int_{B_1} u^p dx > 0 \). Applying Proposition 2.6 with \( \theta = 0 \) we find

\[ u(x) \leq C|x|^{-\frac{m+\alpha}{q-m+1}} \quad \text{in } \Omega. \]

Using the above estimate (if \( \sigma p < \beta \) and (2.8) (if \( \sigma p > \beta \)), from Lemma 2.10(i),(iii) we obtain

\[ (I_\beta * u^p)(x) \leq C \varphi(x) \quad \text{in } \Omega, \]

where

\[ \varphi(x) = |x|^{-(\sigma p - \beta)^+}. \]

(we take \( (\sigma p - \beta)^+ = 0 \) if \( \sigma p - \beta \leq 0 \). Now, (4.3) together with (1.2) (if \( \sigma p < \beta \) and (2.8) (if \( \sigma p > \beta \)) imply

\[ (I_\beta * u^p) u^{q-m+1} \leq C|x|^{-m-\alpha} \quad \text{in } \Omega. \]

We are exactly in the frame of Proposition 2.2 which yields (2.4).

**Step 2: Proof of (1.9)-(1.10).**

Our analysis is split into two cases.

**Case 1:** Suppose

\[ \limsup_{x \to 0} \frac{u(x)}{\mathcal{F}_{m,\alpha}(x)} < \infty. \]
Let $c > 0$ be such that $u(x) \leq c\Phi_{m,\alpha}(x)$ in $B_1 \setminus \{0\}$. By Lemma 2.10 we have
\[
I_\beta \ast u^p \leq I_\beta \ast (c\Phi_{m,\alpha})^p \leq C|x|^{-\theta} \quad \text{in } B_1 \setminus \{0\},
\] where
\[
\theta = \begin{cases} 
\frac{N - m - \alpha}{m - 1} - \beta & \text{if } \frac{N - m - \alpha}{m - 1} > \beta, \\
\tau & \text{if } \frac{N - m - \alpha}{m - 1} = \beta, \\
0 & \text{if } \frac{N - m - \alpha}{m - 1} < \beta,
\end{cases}
\] and $\tau > 0$ is chosen small enough such that
\[
qu < \frac{N - (\sigma p - \beta)^+ - \tau}{N - m - \alpha}(m - 1).
\]

Also, by the definition (4.7) of $\theta$ and (1.5) we have $0 \leq \theta < m + \alpha$, this latter condition is required in the statement of Lemma 2.7. Indeed, this is easy to check if $pN - m - \alpha = \beta$. If $pN - m - \alpha > \beta$ then we observe that from (1.5) $q > m - 1$ we find
\[
p < \frac{m + \alpha + \beta}{N - m - \alpha}(m - 1),
\] and then
\[
\theta = \frac{N - m - \alpha}{m - 1} - \beta < m + \alpha.
\]

Since $u$ is a singular solution of (1.2), there exists a decreasing sequence $\{r_k\} \subset (0, 1)$, $r_k \to 0$ (as $k \to \infty$) such that
\[
\sup_{|x|=r_k} u(x) \to \infty \quad \text{as } k \to \infty.
\]
Using the strong Harnack inequality (2.4) we also have
\[
\inf_{|x|=r_k} u(x) \to \infty \quad \text{as } k \to \infty. \tag{4.8}
\]
For any $k \geq 1$ let $w_k \in C^1(B_1 \setminus B_{r_k})$ be a radial function such that
\[
\begin{cases} 
\text{div}(|x|^{-\alpha}|\nabla w_k|^{m-2}\nabla w_k) = C|x|^{-\theta} w_k^q & \text{in } B_1 \setminus \overline{B}_{r_k}, \\
w_k = \inf_{|x|=r_k} u & \text{on } \partial B_{r_k}, \\
w_k = \inf_{|x|=1} u & \text{on } \partial B_1.
\end{cases}
\]
Since $u$ satisfies (4.6), by the maximum principle we find that $k \mapsto w_k$ is increasing and $u \geq w_k$ in $B_1 \setminus B_{r_k}$. Thus, there exists $w(x) = \lim_{k \to \infty} w_k(x)$ for all $x \in \overline{B}_1 \setminus \{0\}$ and $w \in C^1(B_1 \setminus \{0\})$ satisfies
\[
\text{div}(|x|^{-\alpha}|\nabla w|^{m-2}\nabla w) = C|x|^{-\theta} w^q \quad \text{in } B_1 \setminus \{0\}.
\]
\footnote{We choose $\tau$ with the above conditions just to avoid the log terms that appear in estimating the convolution integrals}
Also $w$ is singular since by (4.8) we have
\[ \sup_{|x|=r_k} w \geq \sup_{|x|=r_k} w_k = \inf_{|x|=r_k} u \to \infty \quad \text{as} \quad k \to \infty. \]

Thus, by (4.5) and Lemma 2.7 (which can be applied since in the first and in the second case of (4.7), condition (1.8) implies (2.13) being $\sigma > (N - m - \alpha)/(m - 1)$ by virtue of (1.5)_2, the third case of (4.7) condition (1.8) is exactly (2.13)) we deduce $u \asymp \Phi_{m,\alpha}$.

Case 2: Suppose
\[ \limsup_{x \to 0} \frac{u(x)}{\Phi_{m,\alpha}(x)} = \infty. \]

Hence, one may find a decreasing sequence $\{r_k\} \subset (0, 1)$, $r_k \to 0$ (as $k \to \infty$) such that
\[ \sup_{|x|=r_k} \frac{u(x)}{\Phi_{m,\alpha}(x)} \to \infty \quad \text{as} \quad k \to \infty. \]

Using the strong Harnack inequality (2.4) for $u$ and the fact that $\Phi_{m,\alpha}(x) = \Phi_{m,\alpha}(|x|)$, one has
\[ \inf_{|x|=r_k} \frac{u(x)}{\Phi_{m,\alpha}(x)} \to \infty \quad \text{as} \quad k \to \infty. \]

Recall that $u$ satisfies (4.3)-(4.4). For any $k \geq 1$ let $w_k \in C^1(B_1 \setminus B_{r_k})$ be a radial function such that
\[ \begin{aligned}
\text{div}(|x|^{-\alpha}|\nabla w_k|^{m-2}\nabla w_k) &= C\varphi(x)w_k^q \quad \text{in} \quad B_1 \setminus \overline{B}_{r_k}, \\
w_k &= \inf_{|x|=r_k} u \quad \text{on} \quad \partial B_{r_k}, \\
w_k &= \inf_{|x|=1} u \quad \text{on} \quad \partial B_1,
\end{aligned} \]

where $\varphi$ is defined in (4.3). By the maximum principle we find that $k \mapsto w_k$ is increasing and $u \geq w_k$ in $B_1 \setminus B_{r_k}$. Thus, there exists $w(x) = \lim_{k \to \infty} w_k(x)$ for all $x \in \overline{B}_1 \setminus \{0\}$ and
\[ \text{div}(|x|^{-\alpha}|\nabla w|^{m-2}\nabla w) = C\varphi(x)w^q \quad \text{in} \quad B_1 \setminus \{0\}. \]

In particular, $w$ satisfies (2.14) with $\theta = (\sigma p - \beta)^+ < m + \alpha$ since $q > m - 1$, and
\[ u \geq w \geq w_k \quad \text{in} \quad B_1 \setminus B_{r_k}. \]

Using the above estimates and (4.9) it follows that
\[ \limsup_{x \to 0} \frac{u(x)}{\Phi_{m,\alpha}(x)} \geq \lim_{k \to 0} \sup_{|x|=r_k} \frac{w(x)}{\Phi_{m,\alpha}(x)} \geq \lim_{k \to 0} \sup_{|x|=r_k} \frac{w_k(x)}{\Phi_{m,\alpha}(x)} = \infty. \]

By Lemma 2.7 it follows that
\[ w \asymp |x|^{-\frac{m+\alpha-(\sigma p - \beta)^+}{q-m+1}}. \]

This fact combined with (4.2) (if $\sigma p < \beta$) and (2.8) (if $\sigma p > \beta$) implies the estimates in Theorem 1.2 ii).

Acknowledgements. RF was partly supported by the Italian MIUR project Variational methods, with applications to problems in mathematical physics and geometry (2015KB9WPT_009) and by Fondo Ricerca di Base di Ateneo Esercizio 2017-19 of the University of Perugia, named Problemi con non linearità dipendenti dal gradiente. RF is a member of the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).
Appendix: Proof of Lemma 2.10

In this section we present the proof of Lemma 2.10 which is rather technical. For reader’s convenience we include all details. We first establish the lower bound in the estimates (2.24)-(2.26), that is,

\[
\int_{|y|<1} \frac{\left( \frac{\log 5}{|y|} \right)^{-\theta}}{|x-y|^a|y|^b} \, dy \geq c \begin{cases} 
\frac{\left( \frac{\log 5}{|x|} \right)^{-\theta}}{|x|^{a+b-N}} & \text{if } a+b > N, \\
\left( \frac{\log 5}{|x|} \right)^{1-\theta} & \text{if } a+b = N \text{ and } \theta \neq 1, \\
1 & \text{if } a+b < N.
\end{cases}
\]

It is enough to establish the above inequality for all \( x \in B_{1/2} \setminus \{0\} \). Then, since all involved functions are continuous on \( \overline{B_1 \setminus B_{1/2}} \) we may take a smaller constant \( c > 0 \) such that the above estimate still holds for all \( x \in B_1 \setminus \{0\} \).

Indeed, if we denote by \( \phi(x) \) the function on RHS of the above estimate, then, for \( 1/2 \leq |x| < 1 \) we have (since \( |x-y| \leq |x| + |y| < 2 \))

\[
\int_{|y|<1} \frac{\left( \frac{\log 5}{|y|} \right)^{-\theta}}{|x-y|^a|y|^b} \, dy \geq \int_{|y|<1} \frac{\left( \frac{\log 5}{|y|} \right)^{-\theta}}{2^a|y|^b} \, dy = C_1 (= \text{const.}) \geq \frac{C_1}{C_2} \phi(x),
\]

where

\[ C_2 = \max_{1/2 \leq |x| \leq 1} \phi(x). \]

This shows that the inequality holds true on \( B_1 \setminus B_{1/2} \) so we need only to prove it on \( B_{1/2} \setminus \{0\} \).

Observe that

\[
\int_{|y|<1} \frac{\left( \frac{\log 5}{|y|} \right)^{-\theta}}{|x-y|^a|y|^b} \, dy \geq \int_{|x|<|y|<1} \frac{\left( \frac{\log 5}{|y|} \right)^{-\theta}}{|x-y|^a|y|^b} \, dy \\
\geq \int_{|x|<|y|<1} \frac{\left( \frac{\log 5}{|y|} \right)^{-\theta}}{(2|y|)^a|y|^b} \, dy \\
= \omega_N 2^{-a} \int_{|x|}^{1} t^{N-a-b} \left( \frac{\log 5}{t} \right)^{-\theta} \frac{dt}{t},
\]

where \( \omega_N \) is the surface area of the unit ball in \( \mathbb{R}^N \). From here we estimate as follows:

(i) If \( a + b > N \) then

\[
\int_{|x|}^{1} t^{N-a-b} \left( \frac{\log 5}{t} \right)^{-\theta} \frac{dt}{t} \geq \left( \frac{\log 5}{|x|} \right)^{-\theta} \int_{|x|}^{1} t^{N-a-b} \frac{dt}{t} \geq c \left( \frac{\log 5}{|x|^{a+b-N}} \right)^{-\theta},
\]
if \(0 < |x| < 1/2\), with \(c = \frac{1 - 2^{N-a-b}}{a+b-N} \).

(i2) If \(a + b = N\) then, for any \(0 < |x| < 1/2\) we have

\[
\int_{|x|}^{1} t^{N-a-b} \left( \log \frac{5}{t} \right)^{-\theta} \frac{dt}{t} = \int_{|x|}^{1} \left( \log \frac{5}{t} \right)^{-\theta} \frac{dt}{t} \\
= \begin{cases} 
\frac{1}{1-\theta} \left[ \left( \log \frac{5}{|x|} \right)^{1-\theta} - \left( \log 5 \right)^{1-\theta} \right] & \text{if } 0 \leq \theta \neq 1 \\
\log \left( \frac{1}{\log 5} \cdot \frac{5}{|x|} \right) & \text{if } \theta = 1 \\
\geq c \left( \log \frac{5}{|x|} \right)^{1-\theta} & \text{if } 0 \leq \theta < 1, \\
1 & \text{if } \theta \geq 1.
\end{cases}
\]

Indeed, if \(\theta = 1\) then

\[
\int_{|x|}^{1} \left( \log \frac{5}{t} \right)^{-\theta} \frac{dt}{t} = \log \left( \frac{1}{\log 5} \cdot \frac{5}{|x|} \right) \geq \log \left( \frac{\log 10}{\log 5} \right), \quad 0 < |x| < \frac{1}{2},
\]

while for \(\theta > 1\) we have

\[
\frac{1}{1-\theta} \left[ \left( \log \frac{5}{|x|} \right)^{1-\theta} - \left( \log 5 \right)^{1-\theta} \right] \geq \frac{(\log 5)^{1-\theta} - (\log 10)^{1-\theta}}{\theta - 1} > 0;
\]

finally if \(0 \leq \theta < 1\) we have

\[
\frac{\left( \log \frac{5}{|x|} \right)^{1-\theta}}{1-\theta} \left[ 1 - \left( \log \frac{5}{\log 5/|x|} \right)^{1-\theta} \right] \geq \frac{(\log 10)^{1-\theta}}{1-\theta} \left[ 1 - \left( \log \frac{5}{\log 10} \right)^{1-\theta} \right] > 0.
\]

(i3) If \(a + b < N\), and \(0 < |x| < 1/2\) we have

\[
\int_{|x|}^{1} t^{N-a-b} \left( \log \frac{5}{t} \right)^{-\theta} \frac{dt}{t} \geq \int_{1/2}^{1} t^{N-a-b} \left( \log \frac{5}{t} \right)^{-\theta} \frac{dt}{t} \geq \frac{1 - 2^{a+b-N}}{N - a - b} (\log 10)^{-\theta} > 0.
\]

In order to establish the upper bounds in the estimates (2.24)-(2.26) we proceed as in [16, Lemma 3.6] (see also [18, Lemma 10.4]). Let \(r = |x| \in (0,1)\) and use the change of variables \(x = r\zeta, y = r\eta\). In particular, we have \(|\zeta| = 1\). Thus
\[
\int_{|y|<1} \left( \frac{\log \frac{5}{|y|}}{|x-y|^a|y|^b} \right)^{-\theta} dy \leq \int_{|y|<2} \left( \frac{\log \frac{5}{|y|}}{|x-y|^a|y|^b} \right)^{-\theta} dy
\]
\[
= r^{N-a-b} \int_{|\eta|<2/r} \left( \frac{\log \frac{5}{r|\eta|}}{|\zeta-\eta|^a|\eta|^b} \right)^{-\theta} d\eta
\]
\[
\leq r^{N-a-b} \left[ A \left( \log \frac{5}{2r} \right)^{-\theta} + \int_{2|\eta|<2/r} \left( \frac{\log \frac{5}{r|\eta|}}{|\eta/(2r)|^a|\eta|^b} \right)^{-\theta} d\eta \right]
\]
where
\[
A = \max_{|\zeta|=1} \int_{0<|\eta|<2} \frac{d\eta}{|\zeta-\eta|^a|\eta|^b} \in (0, \infty),
\]
and where we have used the trivial property \(|\zeta-\eta|^a \geq ||\zeta|-|\eta||^a = (|\eta|-1)^a \geq (|\eta|/2)^a\), when \(|\eta| > 2\).

By virtue of
\[
\frac{\log 5/(2r)}{\log 5/r} \geq 1 - \frac{\log 2}{\log 5}, \quad 0 < r < 1,
\]
we immediately derive
\[
\left( \frac{\log 5}{2r} \right)^{-\theta} \leq c \left( \frac{\log 5}{r} \right)^{-\theta}, \quad 0 < r < 1,
\]
so that
\[
\int_{|y|<1} \left( \frac{\log \frac{5}{|y|}}{|x-y|^a|y|^b} \right)^{-\theta} dy \leq C r^{N-a-b} \left[ A \left( \log \frac{5}{r} \right)^{-\theta} + \int_{2/r}^{2/r} t^{N-a-b} \left( \frac{\log \frac{5}{rt}}{t} \right)^{-\theta} dt \right]
\]
\[
:= C r^{N-a-b} I(r, \theta).
\]

Next, a straightforward calculation leads to the desired estimates in the upper bounds of (i)-(iii). Indeed, we proceed as follows.

(iii) If \(a+b > N\), the upper bound in (2.24) is in force, because choosing \(\varepsilon \in (0, a+b-N)\) we obtain
\[
\int_{2/r}^{2/r} t^{N-a-b} \left( \frac{\log \frac{5}{rt}}{t} \right)^{-\theta} dt \leq c \frac{2^{N-a-b+\varepsilon}}{a+b-\varepsilon-N} \left( 1 - r^{a+b-\varepsilon-N} \right) \left( \log \frac{5}{r} \right)^{-\theta} \leq c \left( \log \frac{5}{r} \right)^{-\theta},
\]
where we have used that there exists \(c > 0\) such that
\[
t^{-\varepsilon} \left( \log \frac{5}{rt} \right)^{-\theta} \leq c \left( \log \frac{5}{r} \right)^{-\theta}, \quad \text{for all } t \in \left(2, \frac{2}{r}\right), \quad r \in (0,1).
\]
(ii2) If \( a + b = N \) then
\[
\int_2^{2/r} t^{N-a-b} \left( \frac{\log 5}{rt} \right)^{-\theta} \frac{dt}{t} = \begin{cases} \\
\frac{1}{1-\theta} \left[ (\log \frac{5}{rt})^{1-\theta} - (\log \frac{5}{r})^{1-\theta} \right] & \text{if } 0 \leq \theta \neq 1 \\
\log \log \frac{5}{r} - \log \log \frac{5}{2} & \text{if } \theta = 1 \\
\end{cases}
\]
Consequently, also the upper bound in (2.25) holds, since, if \( \theta > 1 \), using that
\[
\log \frac{5}{r} > \log 5, \quad \log \frac{5}{2r} > \log \frac{5}{2}, \quad 0 < r < 1,
\]
we have
\[
I(r, \theta) \leq (\log 5)^{-\theta} + \frac{1}{\theta - 1} \left[ (\log \frac{5}{2})^{1-\theta} - (\log 5)^{1-\theta} \right],
\]
while if \( 0 \leq \theta < 1 \), we arrive to
\[
I(r, \theta) \leq \left[ \left( \log \frac{5}{r} \right)^{-1} + \frac{1}{1-\theta} \right] \left( \log \frac{5}{r} \right)^{1-\theta} - \frac{1}{1-\theta} \left( \log \frac{5}{2} \right)^{1-\theta} \leq \left[ \frac{1}{\log 5} + \frac{1}{1-\theta} \right] \left( \log \frac{5}{r} \right)^{1-\theta}.
\]

(ii3) When \( a + b < N \), the upper bound in (2.26) follows immediately since
\[
r^{N-a-b} I(r, \theta) \leq r^{N-a-b}(\log 5)^{-\theta} + \left( \log \frac{5}{2} \right)^{-\theta} 2^{N-a-b} \frac{2^{N-a-b}}{N-a-b} (1 - r^{N-a-b}) \leq C
\]
for all \( 0 < r < 1 \).

\[ \square \]

References

[1] M.F. Bidaut-Véron, M. García-Huidobro and C. Yarur, Keller-Osserman estimates for some quasilinear elliptic systems, Commun. Pure Appl. Anal. 12 (2013) 1547–1568.

[2] M.F. Bidaut-Véron and S. Pohozaev, Nonexistence results and estimates for some nonlinear elliptic problems, J. Analyse Math. 84 (2001), 1–49.

[3] G. Caristi, E. Mitidieri and S. Pohozaev, Local estimates and Liouville theorems for a class of quasilinear inequalities. (Russian) Dokl. Akad. Nauk 418 (2008), 453–457; translation in Dokl. Math. 77 (2008), 85–89.

[4] H. Chen and F. Zhou, Classification of isolated singularities of positive solutions for Choquard equations, J. Differential Equations 261 (2016), 6668–6698.

[5] H. Chen and F. Zhou, Isolated singularities of positive solutions for Choquard equations in sublinear case, Commun. Contemp. Math. 20 (2018), 1750040.

[6] L. D’Ambrosio and E. Mitidieri, Quasilinear elliptic systems in divergence form associated to general nonlinearities, Adv. Nonlinear Analysis 7 (2018), 425–447.

[7] R. Filippucci, Nonexistence of nonnegative solutions of elliptic systems of divergence type, J. Differential Equations 250 (2011), 572–595.
[8] R. Filippucci, Quasilinear elliptic systems in $\mathbb{R}^N$ with multipower forcing terms depending on the gradient, *J. Differential Equations* **255** (2013), 1839–1866.

[9] R. Filippucci, Nonexistence of positive weak solutions of elliptic inequalities, *Nonlinear Anal.* **70** (2009), 2903–2916.

[10] R. Filippucci, P. Pucci and V. Rădulescu, Existence and non-existence results for quasilinear elliptic exterior problems with nonlinear boundary conditions, *Commun. Partial Differential Equations* **33** (2008), 706–717.

[11] R. Filippucci, P. Pucci and M. Rigoli, Nonlinear weighted $p$-Laplacian elliptic inequalities with gradient terms, *Commun. Contemp. Math.* **12** (2010), 501–535.

[12] A. Friedman and L. Véron, Singular solutions of some quasilinear elliptic equations, *Arch. Rational Mechanics and Analysis* **96** (1986), 359–387.

[13] M. Ghergu, Behavior around isolated singularity for integral inequalities with multiple Riesz potentials, *Complex Var. Elliptic Equations*, in press, https://doi.org/10.1080/17476933.2018.1551889

[14] M. Ghergu, P. Karageorgis and G. Singh, Positive solutions for quasilinear elliptic inequalities and systems with nonlocal terms, *J. Differential Equations*, in press, https://doi.org/10.1016/j.jde.2019.11.013

[15] M. Ghergu, S. Kim and H. Shahgholian, Exact behavior around isolated singularity for semilinear elliptic equations with a log-type nonlinearity, *Adv. Nonlinear Analysis* **8** (2019), 995–1003.

[16] M. Ghergu and S. Taliaferro, Asymptotic behavior at isolated singularities for solutions of nonlocal semilinear elliptic systems of inequalities, *Calc. Var. Partial Differential Equations* **54** (2015), 1243–1273.

[17] M. Ghergu and S. Taliaferro, Pointwise bounds and blow-up for Choquard-Pekar inequalities at an isolated singularity, *J. Differential Equations* **261** (2016), 189–217.

[18] M. Ghergu and S. Taliaferro, Isolated Singularities in Partial Differential Inequalities. Encyclopedia of Mathematics and its Applications, vol. 161, Cambridge University Press, 2016.

[19] B. Gidas and J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, *Comm. Pure Appl. Math.* **34** (1981), 525–598.

[20] E. Lieb and M. Loss, Analysis, Second Edition, Graduate Studies in Mathematics, vol. 14, Amer. Math. Soc., 2001.

[21] E. Mitidieri and S.I. Pohozaev, A priori estimates and blow up of solutions to nonlinear partial differential equations, *Proc. Steklov Inst. Math.* **234** (2001), 1–367.

[22] V. Moroz and J. Van Schaftingen, A guide to the Choquard equation, *J. Fixed Point Theory Appl.* **19** (2017), 773–813.
[23] S. Rolando, Multiple nonradial solutions for a nonlinear elliptic problem with singular and decaying radial potential, *Adv. Nonlinear Analysis* 8 (2019), 885–901.

[24] H. Song, J. Yin and Z. Wang, Isolated singularities of positive solutions to the weighted $p$-Laplacian, *Calc. Var. Partial Differential Equations* 55 (2016), 55–28.

[25] N. Trudinger, On Harnack type inequality and their applications to quasilinear elliptic equations, *Commun. Pure Appl. Math.* 12 (1967), 721–747.

[26] J.L. Vázquez and L. Véron, Removable singularities of some strongly nonlinear elliptic equations, *Manuscripta Math.* 33 (1980), 129–144.