A bijection between self-conjugate and ordinary partitions
and counting simultaneous cores as its application

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Abstract

We give a bijection between the set of self-conjugate partitions and that of ordinary partitions. Also, we show the relation between hook lengths of self conjugate partition and corresponding partition via the bijection. As a corollary, we give new combinatorial interpretations for the Catalan number and the Motzkin number in terms of self-conjugate simultaneous core partitions.

Keywords: partition, self-conjugate partition, hook length, simultaneous core partition

1. Introduction

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition of $n$. The Young diagram of $\lambda$ is a collection of $n$ boxes in $\ell$ rows with $\lambda_i$ boxes in row $i$. We label the columns of the diagram from left to right starting with column 1. The box in row $i$ and column $j$ is said to be in position $(i,j)$. For example, the Young diagram for $\lambda = (5,4,2,1)$ is below.

For the Young diagram of $\lambda$, the partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_\ell)$ is called the conjugate of $\lambda$, where $\lambda'_j$ denotes the number of boxes in column $j$. For each box in its Young diagram, we define its hook length by counting the number of boxes directly to its right or below, including the box itself. Equivalently, for the box in position $(i,j)$, the hook length of $\lambda$ is defined by

$$h_\lambda(i,j) = \lambda_i + \lambda'_j - i - j + 1.$$  

For example, hook lengths in the first row above are 8, 6, 4, 3, and 1, respectively. We denote $h_\lambda(i,j)$ by $h(i,j)$ when $\lambda$ is clear.

For a positive integer $t$, a partition $\lambda$ is called a $t$-core if none of its hook lengths are multiples of $t$. The number of $t$-core partitions of $n$ is denoted by $c_t(n)$. The study of core partitions arises from the representation theory of the symmetric group $S_n$. (See [13] for details)

In [13], we can find the generating function of $c_t(n)$:

$$\sum_{n=0}^{\infty} c_t(n)q^n = \prod_{n=1}^{\infty} \frac{(1 - q^n)^t}{1 - q^n}.$$  

Moreover, the generating function of $c_t(n)$ can be represented as the product of the Dedekind eta function. Therefore, many researches on core partitions are being made through various ways, such as representation theory and

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A number of properties of self-conjugate core partitions have been found and proved. (See, [5, 6]) Similarly, the generating function of \( sc(n) \) for \( n \geq 4 \) and \( a(n) \) for \( n \geq 1 \) has been suggested in [11] and asymptotics are provided in [1].

Theorem 1.3. Let \( c_{t_1,\ldots,t_p}(n) \) be the number of \( (t_1,\ldots,t_p) \)-core partitions of \( n \) and \( sc_{t_1,\ldots,t_p}(n) \) be the number of self-conjugate \( (t_1,\ldots,t_p) \)-core partitions of \( n \). For \( |q| < 1 \),

\[
\sum_{n=0}^{\infty} sc_{t_1,\ldots,t_p}(n)q^n = \left( \sum_{n=0}^{\infty} c_{t_1,\ldots,t_p}(n)q^n \right) \left( \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} \right).
\]

A special case of \( p = 1 \) is the following corollary.

Corollary 1.4. For \( |q| < 1 \),

\[
\sum_{n=0}^{\infty} sc_{2t-1}(n)q^n = \left( \sum_{n=0}^{\infty} a(n)q^n \right) \left( \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} \right).
\]

We remark that Corollary 1.2 and Corollary 1.4 can also be proved by analytic method using equations (1), (2), and Gauss’s well-known identity

\[
\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \prod_{n=1}^{\infty} \frac{1}{1 - q^{2n-1}}.
\]

However, it might be difficult to prove Theorem 1.3 by analytic approach.

At the end of this paper, a new expression of the Catalan number and the Motzkin number in terms of self-conjugate simultaneous core partitions is given (See Corollary 1.11 as a corollary of Theorem 1.3).

This paper is organized as follows. In Section 2 we introduce new classification of the set of self-conjugate partitions. In Section 3 we give a bijection between the set of self-conjugate partitions and that of ordinary partitions. In Section 4 we explain the relation between even hook lengths in a self-conjugate partition and hook lengths in the corresponding partition via the bijection. Furthermore, we give some new results of the number of self-conjugate simultaneous core partitions.
Lemma 2.2. Let \( \beta \) be a partition. We often use the notation \( \delta_i \) for the hook length \( h(i, i) \) of the \( i \)th box on the main diagonal. The set \( D(\lambda) = \{ \delta_i : i = 1, 2, \ldots \} \) is called the set of main diagonal hook lengths of \( \lambda \). It is clear that if \( \lambda \) is self-conjugate, then \( D(\lambda) \) determines \( \lambda \), and elements of \( D(\lambda) \) are all distinct and odd. Hence, for a self-conjugate partition \( \lambda \), \( D(\lambda) \) can be divided into the following two subsets:

\[
D_1(\lambda) = \{ \delta_i \in D(\lambda) : \delta_i \equiv 1 \pmod{4} \}, \\
D_3(\lambda) = \{ \delta_i \in D(\lambda) : \delta_i \equiv 3 \pmod{4} \}.
\]

Example 2.1. Let \( \lambda = (4, 4, 4, 3) \) be a self-conjugate partition of 15. Figure 1 shows the Young diagram and hook lengths of \( \lambda \). The set \( D(\lambda) = \{7, 5, 3\} \) of main diagonal hook lengths is divided into \( D_1(\lambda) = \{5\} \) and \( D_3(\lambda) = \{7, 3\} \).

![Figure 1: The Young diagram of a self-conjugate partition and its hook lengths](attachment:image.png)

The set of hook lengths of boxes in the first column of the Young diagram of \( \lambda \) is called the beta-set of \( \lambda \) and denoted by \( \beta(\lambda) \). If \( \lambda' \) is the conjugate partition of \( \lambda \), then the \( \beta(\lambda') \) is the set of hook lengths of boxes in the first row of the Young diagram of \( \lambda \). One may notice that if \( \lambda \) is self-conjugate, then \( \beta(\lambda) = \beta(\lambda') \). The following lemma plays an important role when we deal with hook lengths.

Lemma 2.2. Let \( \lambda = (\lambda_1, \lambda_2, \ldots) \) be a self-conjugate partition with \( D(\lambda) = \{\delta_1, \delta_2, \ldots, \delta_d\} \). If we let

\[
\beta_{\leq d}(\lambda) = \{h(i, 1) : i = 1, 2, \ldots, d\}, \\
\beta_{> d}(\lambda) = \{h(i, 1) : i = d + 1, d + 2, \ldots, \lambda_1\},
\]

so that \( \beta(\lambda) = \beta_{\leq d}(\lambda) \cup \beta_{> d}(\lambda) \), then

\[
\beta_{\leq d}(\lambda) = \left\{ \frac{\delta_1 + \delta_1}{2}, \frac{\delta_1 + \delta_2}{2}, \ldots, \frac{\delta_1 + \delta_d}{2} \right\}, \\
\beta_{> d}(\lambda) = \left\{ \frac{\delta_1 - 1}{2}, \frac{\delta_1 - 3}{2}, \ldots, 1 \right\} - \left\{ \frac{\delta_1 - \delta_d}{2}, \frac{\delta_1 - \delta_{d-1}}{2}, \ldots, \frac{\delta_1 - \delta_2}{2} \right\}.
\]

Proof. We consider the set \( \beta_{\leq d}(\lambda) \) first. Since \( \lambda \) is self-conjugate, \( \delta_1 = 2\lambda_1 - 2i + 1 \) for \( i \leq d \). Hence,

\[
h(i, 1) = \lambda_1 - i = \frac{\delta_1 + \delta_i}{2} \quad \text{for} \quad 1 \leq i \leq d.
\]

Now, we consider the set \( \beta_{> d}(\lambda) \). If \( \lambda \) has exactly \( d \) parts, then one can notice that \( \beta_{> d}(\lambda) \) must be an empty set. Since \( \lambda \) is the partition of \( d^2 \) with \( \lambda_i = d \) for all \( i = 1, 2, \ldots, d \), we have \( \delta_d = 1, \delta_{d-1} = 3, \ldots, \delta_2 = \delta_1 - 2 \), and therefore,

\[
\beta_{> d}(\lambda) = \left\{ \frac{\delta_1 - 1}{2}, \frac{\delta_1 - 3}{2}, \ldots, 1 \right\} - \left\{ \frac{\delta_1 - \delta_d}{2}, \frac{\delta_1 - \delta_{d-1}}{2}, \ldots, \frac{\delta_1 - \delta_2}{2} \right\} = \emptyset.
\]
If $\lambda$ has more than $d$ elements, then
\[ h(d + 1, 1) = \lambda_{d+1} + \lambda_1 - (d + 1) \leq d + \lambda_1 - (d + 1) = \lambda_1 - 1 = \frac{\delta_1 - 1}{2}. \]

Since $h(d + 1, 1)$ is the largest element of $\beta_{d, \ell}(\lambda)$, the set $\beta_{d, \ell}(\lambda)$ is a subset of $\{\frac{\delta_1 - 1}{2}, \frac{\delta_1 - 3}{2}, \ldots, 1\}$. If we suppose that $\frac{\delta_1 - 3}{2}$ is an element of $\beta_{d, \ell}(\lambda)$ for some $j \leq d$, then for some $i \in \{d + 1, d + 2, \ldots, \lambda_1\}$,
\[ h(i, 1) - \frac{\delta_1 - \delta_j}{2} = (\lambda_1 + \lambda_1 - i) - (\lambda_1 - \lambda_j + j - 1) = \lambda_i + \lambda_j - i - j + 1 = 0. \]

But $\lambda_i + \lambda_j - i - j + 1$ is the hook length $h(i, j)$, so it should be nonzero even if the box $(i, j)$ is in the Young diagram of $\lambda$. Hence, we have come to the conclusion that $\frac{\delta_1 - \delta_j}{2} \notin \beta_{d, \ell}(\lambda)$ for all $j \leq d$ and
\[ \beta_{d, \ell}(\lambda) \subset \left\{\frac{\delta_1 - 1}{2}, \frac{\delta_1 - 3}{2}, \ldots, 1\right\} - \left\{\frac{\delta_1 - \delta_1}{2}, \frac{\delta_1 - \delta_2}{2}, \ldots, \frac{\delta_1 - \delta_j}{2}\right\}. \]

In fact, $\beta_{d, \ell}(\lambda) = \left\{\frac{\delta_1 - 1}{2}, \frac{\delta_1 - 3}{2}, \ldots, 1\right\} - \left\{\frac{\delta_1 - \delta_1}{2}, \frac{\delta_1 - \delta_2}{2}, \ldots, \frac{\delta_1 - \delta_1}{2}\right\}$ since $|\beta_{d, \ell}(\lambda)| = \lambda_1 - d$. \hfill $\square$

### 2.2. Self-conjugate partitions with same disparity

We denote the set of self-conjugate partitions of $n$ by $\mathcal{SC}(n)$. Using the value $|D_1(\lambda)| - |D_3(\lambda)|$ of $\lambda \in \mathcal{SC}(n)$, we split $\mathcal{SC}(n)$ as follows: For nonnegative integers $m$ and $n$, we define a set $\mathcal{SC}^{(m)}(n)$ by
\[ \mathcal{SC}^{(m)}(n) = \{\lambda \in \mathcal{SC}(n) : |D_1(\lambda)| - |D_3(\lambda)| = (-1)^{m+1} \left\lceil \frac{m}{2} \right\rceil\}. \]

where $\lceil x \rceil$ is the least integer greater than or equal to $x$. We note that for a self-conjugate partition $\lambda$, if $|D_1(\lambda)| - |D_3(\lambda)| = k$ for $k \geq 1$, then $\lambda \in \mathcal{SC}^{(2k-1)}(n)$. Otherwise, if $|D_1(\lambda)| - |D_3(\lambda)| = -k$ for $k \geq 0$, then $\lambda \in \mathcal{SC}^{(2k)}(n)$. Therefore, $\mathcal{SC}(n) = \bigcup_{m=0}^{\infty} \mathcal{SC}^{(m)}(n)$.

We use the notation $\mathcal{sc}^{(m)}(n)$ for $|\mathcal{SC}^{(m)}(n)|$. If we let $\mathcal{SC}^{(m)} = \bigcup_{n=0}^{\infty} \mathcal{SC}^{(m)}(n)$, then $\mathcal{SC} = \bigcup_{n=0}^{\infty} \mathcal{SC}^{(m)}$.

For a partition $\lambda$, we define the disparity of $\lambda$ by
\[ dp(\lambda) = |\{(i, j) \in \lambda : h(i, j) \text{ is odd}\}| - |\{(i, j) \in \lambda : h(i, j) \text{ is even}\}|. \]

For example, for the partition $\lambda = (4, 4, 4, 3)$ given in Example 2.1, $|D_1(\lambda)| - |D_3(\lambda)| = -1$. Therefore, $\lambda$ is an element of $\mathcal{SC}^{(15)}$ with $dp(\lambda) = 9 - 6 = 3$.

In the following proposition, we show that each element of the set $\mathcal{SC}^{(m)}(n)$ has the same disparity.

**Proposition 2.3.** For a nonnegative integer $m$, if $\lambda$ is in the set $\mathcal{SC}^{(m)}(n)$, then its disparity $dp(\lambda)$ is $\frac{m(m+1)}{2}$.

**Proof.** Let $\lambda \in \mathcal{SC}^{(m)}(n)$ be a self-conjugate partition with $D(\lambda) = (\delta_1, \delta_2, \ldots, \delta_d)$. First, we focus on $d^2$ hook lengths $h(i, j)$ of $\lambda$ for $1 \leq i, j \leq d$. Since $\delta_i = 2\lambda_i - 2i + 1$ for $i \leq d$,
\[ h(i, j) = \lambda_i + \lambda_j - i - j + 1 = \frac{\delta_i + \delta_j}{2} \quad \text{for} \quad 1 \leq i, j \leq d. \]

Thus, among these $d^2$ hook lengths, there are $2|D_1(\lambda)||D_3(\lambda)|$ even numbers, and then
\[ |h(i, j) : \text{odd} \quad : \quad 1 \leq i, j \leq d| - |h(i, j) : \text{even} \quad : \quad 1 \leq i, j \leq d| = d^2 - 4|D_1(\lambda)||D_3(\lambda)|. \]

Now, we consider the multiset $H(\lambda)$ of hook lengths $h(i, j)$ of $\lambda$ for $i > d$ and $j \leq d$. If we let
\[ H_j(\lambda) = \{h(i, j) : d + 1 \leq i \leq \lambda_j\} \quad \text{for} \quad 1 \leq j \leq d, \]
\[ H_j(\lambda) = \{h(i, j) : d + 1 \leq i \leq \lambda_j\} \quad \text{for} \quad 1 \leq j \leq d, \]
then $H(\lambda) = \bigcup_{j=1}^{d} H_j(\lambda)$. We note that $H_1(\lambda)$ is the set $\beta_{-d}(\lambda)$, $H_2(\lambda)$ is the set $\beta_{-d-1}(\lambda)$, where $\lambda$ is the self-conjugate partition with $D(\lambda) = [\delta_2, \delta_3, \ldots, \delta_d]$, and $H_j(\lambda), \ldots, H_d(\lambda)$ are defined similarly. Hence, by Lemma 2.2, we have

\[
H_1(\lambda) = \left\{ \frac{\delta_1 - 1}{2}, \frac{\delta_1 - 3}{2}, \ldots, 1 \right\} - \left\{ \frac{\delta_1 - \delta_d}{2}, \frac{\delta_1 - \delta_d - 1}{2}, \ldots, \frac{\delta_1 - \delta_2}{2} \right\},
\]

\[
H_2(\lambda) = \left\{ \frac{\delta_2 - 1}{2}, \frac{\delta_2 - 3}{2}, \ldots, 1 \right\} - \left\{ \frac{\delta_2 - \delta_d}{2}, \frac{\delta_2 - \delta_d - 1}{2}, \ldots, \frac{\delta_2 - \delta_3}{2} \right\},
\]

\[\vdots\]

\[
H_d(\lambda) = \left\{ \frac{\delta_d - 1}{2}, \frac{\delta_d - 3}{2}, \ldots, 1 \right\}.
\]

Thus, we can rewrite the multiset $H(\lambda)$ as follows.

\[
H(\lambda) = \bigcup_{k=1}^{d} \left\{ \frac{\delta_k - 1}{2}, \frac{\delta_k - 3}{2}, \ldots, 1 \right\} - \left\{ \frac{\delta_i - \delta_j}{2} : 1 \leq i < j \leq d \right\}.
\]

We note that depend on whether $\delta_k \in D_1(\lambda)$ or $\delta_k \in D_3(\lambda)$, the number of odd elements is same as the number of even elements, or we have one more odd element in the set $\left\{ \frac{\delta_1}{2}, \frac{\delta_2}{2}, \ldots, 1 \right\}$.

Since there are $|D_1(\lambda)||D_3(\lambda)|$ odd elements in the multiset $\left\{ \frac{\delta_i}{2} : 1 \leq i < j \leq d \right\}$, we can conclude that among the hook lengths $h(i, j)$ with $i > d$ and $j \leq d$,

\[
|h(i, j) : \text{odd}| - |h(i, j) : \text{even}| = |D_3(\lambda)| - \left| D_1(\lambda)D_3(\lambda) - \left( \begin{pmatrix} d \\ 2 \end{pmatrix} - |D_1(\lambda)||D_3(\lambda)| \right) \right|.
\]

Since $\lambda$ is self-conjugate, the multiset of hook lengths $h(i, j)$ with $i \leq d$ and $j > d$ is equal to the set $H(\lambda)$. Hence, the value $|h(i, j) : \text{odd}| - |h(i, j) : \text{even}|$ is equal to that one in the case when $i > d$ and $j \leq d$. By combining all of these, we have

\[
dp(\lambda) = (d^2 - 4|D_1(\lambda)||D_3(\lambda)|) + 2\left| D_3(\lambda) + \begin{pmatrix} d \\ 2 \end{pmatrix} - 2|D_1(\lambda)||D_3(\lambda)| \right|
\]

\[
= 2d^2 - d + 2|D_3(\lambda)| - 8|D_1(\lambda)||D_3(\lambda)|.
\]

Now, we consider two cases:

- Let $m = 2k - 1$ for a positive integer $k$. By the definition of $\mathcal{SC}^{(m)}$, $|D_1(\lambda)| - |D_3(\lambda)| = k$. Since $|D_1(\lambda)| + |D_3(\lambda)| = d$, we have $|D_1(\lambda)| = \frac{d+k}{2}$, $|D_3(\lambda)| = \frac{d-k}{2}$, and

\[
dp(\lambda) = 2d^2 - d + (d-k) - 2(d^2 - k^2) = 2k^2 - k = \frac{2k(2k - 1)}{2} = \frac{m+1}{2}m.
\]

- Let $m = 2k$ for a nonnegative integer $k$. In this case, $|D_1(\lambda)| - |D_3(\lambda)| = -k$. Hence, we have $|D_1(\lambda)| = \frac{d+k}{2}$, $|D_3(\lambda)| = \frac{d-k}{2}$, and

\[
dp(\lambda) = 2d^2 - d + (d+k) - 2(d^2 - k^2) = 2k^2 + k = \frac{2k(2k + 1)}{2} = \frac{m(m+1)}{2}.
\]

This completes the proof.

By Proposition 2.3, one may notice that the disparity of a self-conjugate partition is a triangular number $\frac{m(m+1)}{2}$ for some integer $m \geq 0$, and the set of self-conjugate partitions with the disparity $\frac{m(m+1)}{2}$ is $\mathcal{SC}^{(m)}$. 

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3. Proof of Theorem 1.1

The set of partitions of $n$ is denoted by $\mathcal{P}(n)$, and the set of partitions is denoted by $\mathcal{P}$. In this section, we construct a bijection between two sets $\mathcal{SC}^m(4n + m(m + 1)/2)$ and $\mathcal{P}(n)$ which play a key role throughout the paper.

Before constructing a bijection, we give some notations. For a self-conjugate partition $\lambda$, if

$$D_1(\lambda) = \{4a_1 + 1, 4a_2 + 1, \ldots, 4a_r + 1\},$$

$$D_2(\lambda) = \{4b_1 - 1, 4b_2 - 1, \ldots, 4b_s - 1\},$$

we say that $\lambda$ has the diagonal sequence pair $((a_1, a_2, \ldots, a_r), (b_1, b_2, \ldots, b_s))$, where $a_1 > a_2 > \cdots > a_r \geq 0$ and $b_1 > b_2 > \cdots > b_s \geq 1$. For convenience, we allow empty sequence if $r$ or $s$ is equal to 0.

For $\lambda = (4, 4, 4, 3) \in \mathcal{SC}^2(15)$ considered in Example 2.1, its diagonal sequence pair is $((1), (2, 1))$.

We note that if $\lambda \in \mathcal{SC}^m(4n + m(m + 1)/2)$ has the diagonal sequence pair $((a_1, \ldots, a_r), (b_1, \ldots, b_s))$, then

$$r - s + (-1)^m \left\lceil \frac{m}{2} \right\rceil = 0$$

and

$$4 \left( \sum_{i=1}^{r} a_i + \sum_{j=1}^{s} b_j \right) + r - s = 4n + \frac{m(m + 1)}{2}.$$

Now, we are ready to construct our mapping.

**Mapping $\phi_n(m) : \mathcal{SC}^m(4n + m(m + 1)/2) \to \mathcal{P}(n)$**

Let $\lambda \in \mathcal{SC}^m(4n + m(m + 1)/2)$ with the diagonal sequence pair $((a_1, \ldots, a_r), (b_1, \ldots, b_s))$. We define $\phi_n(m)(\lambda)$ by the partition $\mu = (\mu_1, \mu_2, \ldots, \mu_t)$ such that

$$\mu_i = a_i + i + s - r \quad \text{for} \quad i \leq r,$$

and $(\mu_{r+1}, \mu_{r+2}, \ldots, \mu_t)$ is the conjugate of the partition $\gamma = (b_1 - s, b_2 - s + 1, \ldots, b_s - 1)$.

(We allow that $\gamma$ has some zero parts.)

In the figure below, the diagram after deleting the gray portion is the Young diagram of $\mu$.

Now, we show that the mapping $\phi_n(m)$ is well-defined:

Since $\mu_i = a_i + i + s - r$ for $i \leq r$, $\mu_r = a_r + r + s - r = a_r + s$. From $a_1 > a_2 > \cdots > a_r \geq 0$, we have $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r \geq s$. Since $\gamma$ has at most $s$ parts, $\mu_{r+1} \leq s$. Therefore, $\mu$ is a partition of

$$|\mu| = \sum_{i=1}^{r} (a_i + i + s - r) + \sum_{j=1}^{s} (b_j - s + j - 1).$$

- If $m = 2k - 1$, then $r - s = k$ and

$$|\mu| = \left( \sum_{i=1}^{r} a_i + \sum_{j=1}^{s} b_j \right) - \frac{k(k - 1)}{2} = \frac{4n + 2k^2 - 2k}{4} - \frac{k(k - 1)}{2} = n.$$

- If $m = 2k$, then $r - s = -k$ and

$$|\mu| = \left( \sum_{i=1}^{r} a_i + \sum_{j=1}^{s} b_j \right) - \frac{k(k + 1)}{2} = \frac{4n + 2k^2 + 2k}{4} - \frac{k(k + 1)}{2} = n.$$
Theorem 3.1. For nonnegative integers m and n, the mapping $\phi_n^{(m)}$ is bijective.

Proof. We prove the theorem by constructing the inverse mapping $\psi_n^{(m)}$. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell) \in \mathcal{P}(n)$.

- For $m = 2k - 1$, there exist a unique $s \geq 0$ such that $\mu_{s+k} \geq s$ and $\mu_{s+k+1} \leq s$. Let $r = s + k$.
- For $m = 2k$, there exist a unique $r \geq 0$ such that $\mu_r \geq r + k$ and $\mu_{r+1} \leq r + k$. Let $s = r + k$.

For any $m$, let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_s)$ be the conjugate of the partition $(\mu_{s+1}, \ldots, \mu_{\ell})$. Now, we define $\psi_n^{(m)}(\mu)$ by the partition $\lambda$ with the diagonal sequence pair $((a_1, \ldots, a_r), (b_1, \ldots, b_s))$, where

$$ a_i = \mu_i - i + s, \quad b_j = \gamma_j + j - s \quad \text{for} \quad i \leq r, \quad j \leq s. $$

Then $\psi_n^{(m)}$ is well-defined. Moreover, one can see that $\phi_n^{(m)}(\psi_n^{(m)}(\mu)) = \mu$ and $\psi_n^{(m)}(\phi_n^{(m)}(\lambda)) = \lambda$. Therefore, $\phi_n^{(m)}$ is the inverse of the mapping $\phi_n^{(m)}$. \hfill $\square$

We define the bijection $\phi_n^{(m)} : SC_n^{(m)} \rightarrow \mathcal{P}$ by $\phi_n^{(m)}(\lambda)$, for a partition $\lambda \in SC_n^{(m)}$ of $4n + \frac{m(m+1)}{2}$.

We say that $\mu$ is the corresponding partition of $\lambda$ when $\phi_n^{(m)}(\lambda) = \mu$.

We give two examples of the bijection $\phi_n^{(m)}$.

Example 3.2. We consider two self-conjugate partitions $\lambda$ and $\tilde{\lambda}$ with the set of main diagonal hook lengths $D(\lambda) = \{21, 15, 13, 9, 3, 1\}$ and $D(\tilde{\lambda}) = \{31, 19, 11, 5\}$, respectively.

- Since $D_1(\lambda) = \{21, 13, 9, 1\}$ and $D_3(\lambda) = \{15, 3\}$, $\lambda$ is in the set $SC^{(3)}$ and $((5, 3, 2, 0), (4, 1))$ is the diagonal sequence pair of $\lambda$. If we let $\mu$ be the partition $\phi_3^{(3)}(\lambda)$, then

$$ \mu_1 = 5 + 1 - 2 = 4, \quad \mu_2 = 3 + 2 - 2 = 3, \quad \mu_3 = 2 + 3 - 2 = 3, \quad \mu_4 = 0 + 4 - 2 = 2 $$

and $(\mu_5, \mu_6, \ldots)$ is the conjugate of the partition $(4 - 2, 1 - 2 + 1)$. Therefore, $\mu = (4, 3, 3, 2, 1, 1)$.

- Since $D_1(\tilde{\lambda}) = \{5\}$ and $D_3(\tilde{\lambda}) = \{31, 19, 11\}$, $\tilde{\lambda} \in SC^{(3)}$ and $((1), (8, 5, 3))$ is the diagonal sequence pair of 1. If we let $\tilde{\mu}$ be the partition $\phi_3^{(4)}(\tilde{\lambda})$, then $\mu_1 = 1 + 1 + 2 = 4$ and $(\mu_2, \mu_3, \ldots)$ is the conjugate of the partition $(8 - 3, 5 - 3 + 1, 3 - 3 + 2)$. Therefore, $\tilde{\mu} = (4, 3, 3, 2, 1, 1)$.  

\hfill $\blacksquare$
For given $\mu \in \mathcal{P}$ and $m \geq 0$, we consider the following diagram to find the correspondence of $\mu$. For convenience, even if $i \leq 0$, we set the $i$th column is the column on the left side of the $(i+1)$st column and the $i$th row is on the above of the $(i+1)$st row.

- For $m = 2k - 1$, we consider the diagram $\nu$ obtained from the Young diagram of $\mu$ by attaching $\frac{k(k-1)}{2}$ boxes on the left side such that $\nu$ has $\mu_i + k - i$ boxes in row $i$ for $i < k$ and $\mu_i$ boxes in row $i$ for $i \geq k$. Then, the number of (white) boxes $(i, j)$ in row $i$ such that $i - j < k$ is equal to $a_i$ and the number of (gray) boxes $(i, j)$ in column $j$ such that $i - j \geq k$ is equal to $b_j$. See the first diagram in Figure 3 for $\mu = (4, 3, 3, 2, 1, 1)$ and $m = 3$.

- For $m = 2k$, we consider the diagram $\nu$ obtained from the Young diagram of $\mu$ by attaching $\frac{k(k+1)}{2}$ boxes on the above such that $\nu$ has $k - i$ boxes in row $i$ for $i = 0, 1, \ldots, k-1$ and $\mu_i$ boxes in row $i$ for $i > 0$. Then, the number of (white) boxes $(i, j)$ in row $i$ such that $i - j < -k$ is equal to $a_i$ and the number of (gray) boxes $(i, j)$ in column $j$ such that $i - j \geq -k$ is equal to $b_j$. See the second diagram in Figure 3 for $\mu = (4, 3, 3, 2, 1, 1)$ and $m = 4$.

$\Downarrow$

$((5, 3, 2, 0), (4, 1))$

$\Downarrow$

$(((1), (8, 5, 3))$

Figure 3: Graphical interpretations for odd $m$ and even $m$ of the bijection $\phi^{(m)}$

**Proposition 3.3.** For integer $m \geq 0$, the number of self-conjugate partitions of $n$ with the disparity $\frac{m(m+1)}{2}$ is

$$sc^{(m)}(n) = \begin{cases} p(k) & \text{if } n = 4k + \frac{m(m+1)}{2}, \\ 0 & \text{otherwise}. \end{cases}$$

**Proof.** For a self-conjugate partition $\lambda$ with the disparity $\frac{m(m+1)}{2}$, suppose there are $\ell$ boxes $(i, j)$ in the Young diagram of $\lambda$ such that $h(i, j)$ is even. By Proposition 2.3, there are $\ell + \frac{m(m+1)}{2}$ boxes with odd hook lengths. Since $\lambda$ is self-conjugate, $\ell$ must be even. If we let $\ell = 2k$, then $n = 4k + \frac{m(m+1)}{2}$. Bijection $\phi^{(m)}_n$ gives that $sc^{(m)}\left(4k + \frac{m(m+1)}{2}\right) = p(k)$.

By Theorem 3.1 and Proposition 3.3, we have the following corollary and as a consequence of Corollary 3.4, we have Corollary 1.2.

**Corollary 3.4.** For a nonnegative integer $m$, we have

$$\sum_{\lambda \in sc^{(m)}} q^{||\lambda||} = q^{\frac{m(m+1)}{2}} \sum_{\mu \in \mathcal{P}} q^{||\mu||}.$$
4.1. Hook lengths of the first row or column

For the partitions \( \lambda \in \mathcal{SC}^{(m)} \) and \( \mu = \phi^{(m)}(\lambda) \), we give a relation between their hook lengths in the first row or in the first column.

**Lemma 4.1.** For a partition \( \mu = (\mu_1, \mu_2, \ldots, \mu_\ell) \), if we let \( \bar{\beta}(\mu) = (h(1,1) - h(i,1) : 2 \leq i \leq \ell) \), then
\[
\bar{\beta}(\mu) = \{1, 2, \ldots, h(1,1)\} - \beta(\mu'),
\]
where \( \mu' \) is the conjugate of \( \mu \).

**Proof.** For some \( i \) and \( j \), if \( h(1,1) - h(i,1) = h(1,j) \), then \( \mu_i + \mu'_j - i - j + 1 = 0 \). Since any hook length \( h(i,j) \) can not be zero, \( h(1,1) - h(i,1) \neq h(1,j) \) for all \( i \) and \( j \). This implies that \( \bar{\beta}(\mu) \subset \{1, 2, \ldots, h(1,1)\} - \beta(\mu') \). Since \( |\bar{\beta}(\mu)| = h(1,1) - \mu_1 \), we are done. \( \square \)

For a self-conjugate partition \( \lambda \), we define the half-even beta set of \( \lambda \) by
\[
\beta_{\ell/2}(\lambda) = \{h(i,1)/2 : h(i,1) \text{ is even}, 1 \leq i \leq \lambda_1\}.
\]

**Proposition 4.2.** Let \( \lambda \in \mathcal{SC}^{(m)} \) be a partition with \( D(\lambda) = [\delta_1, \delta_2, \ldots, \delta_\ell] \). If \( \mu = (\mu_1, \mu_2, \ldots, \mu_\ell) \) is the corresponding partition of \( \lambda \), then the half-even beta set of \( \lambda \) is
\[
\beta_{\ell/2}(\lambda) = \begin{cases} 
\beta(\mu') & \text{if } \delta_1 \in D_1(\lambda), \\
\beta(\mu) & \text{if } \delta_1 \in D_3(\lambda).
\end{cases}
\]

**Proof.** We let \((a_1, \ldots, a_\ell), (b_1, \ldots, b_\ell)\) be the diagonal sequence pair of \( \lambda \) so that
\[
D(\lambda) = [4a_1 + 1, 4a_2 + 1, \ldots, 4a_\ell + 1] \cup [4b_1 - 1, 4b_2 - 1, \ldots, 4b_\ell - 1].
\]

First we will find two sets \( \beta(\mu) \) and \( \beta(\mu') \). The partition \( \phi^{(m)}(\lambda) = \mu \) is defined by
\[
\mu_i = a_i + i + s - r \quad \text{for } i \leq r,
\]
and \( (\mu_{i+1}, \ldots, \mu_\ell) \) is the conjugate of the partition \( \gamma = (b_1 - s, b_2 - s + 1, \ldots, b_\ell - 1) \). We suppose that \( \gamma \) has \( \mu \) nonzero parts so that \( b_i - s + i - 1 = 0 \) for \( i > u \).

To find \( \beta(\mu) \), we consider hook lengths of boxes \((i,1)\) in the Young diagram of \( \mu \). For \( i \leq r \),
\[
h_i(i,1) = \mu_i + \mu'_i - i = (a_i + i + s - r) + (r + b_1 - s) - i = a_i + b_1.
\]
We note that the set \( \{h_i(i,1) : i > r\} \) is equal to the set \( \beta(\gamma') \), where \( \gamma' \) is the conjugate of \( \gamma \). Since
\[
h_i(i,1) = \gamma_i + \gamma'_i - i = (b_i - s + i - 1) + u - i = b_i - s + u - 1,
\]
we have
\[
\beta(\gamma') = \{h_i(1,1) - h_i(i,1) : 2 \leq i \leq u\} = \{b_1 - b_i : 2 \leq i \leq u\}.
\]
By Lemma 4.1, we have
\[
\beta(\gamma') = \{1, 2, \ldots, h_i(1,1)\} - \bar{\beta}(\gamma) = \{1, 2, \ldots, b_1 - s + u - 1\} - \{b_1 - b_i : 2 \leq i \leq u\}.
\]
Since \( \{b_1 - b_i : u < i \leq s\} = \{b_1 - s + i - 1 : u < i \leq s\} \), we can write as
\[
\beta(\gamma') = \{1, 2, \ldots, b_1 - 1\} - \{b_1 - b_i : 2 \leq i \leq s\},
\]
and therefore, the beta set of \( \mu \) is
\[
\beta(\mu) = \{a_i + b_1 : 1 \leq i \leq r\} \cup \{1, 2, \ldots, b_1 - 1\} - \{b_1 - b_i : 2 \leq i \leq s\}.
\]
To find the set $\beta(\mu')$, we consider hook lengths of boxes $(1, j)$ in the Young diagram of $\mu$. For $j \leq s$, since $\mu'_j = r + \gamma_j$,

$$h_{\nu}(1, j) = \nu_1 + \nu'_j - j = (a_1 + 1 + s - r) + (r + b_j - s + j - 1) - j = a_1 + b_j.$$  

We note that if we set $\nu = (\mu_1 - s, \mu_2 - s, \ldots, \mu_r - s)$, then the set \{h_{\nu}(1, j) : j > s\} is equal to the set $\beta(\nu')$, where $\nu'$ is the conjugate of $\nu$. We suppose that $\nu$ has $\nu$ nonzero parts so that $a_i + i - r = 0$ for $i > \nu$. Since

$$h_{\nu}(i, 1) = \nu_i + \nu'_1 - i = (\mu_i - s) + v - i = (a_i + i + s - r - s) + v - i = a_i - r + v,$$

we have

$$\beta(\nu) = [h_{\nu}(1, 1) - h_{\nu}(i, 1) : 2 \leq i \leq \nu] = [a_1 - a_i : 2 \leq i \leq \nu].$$

By Lemma 4.1

$$\beta(\nu') = [1, 2, \ldots, h_{\nu}(1, 1)] - \beta(\nu) = [1, 2, \ldots, a_1 - r + v] - [a_1 - a_i : 2 \leq i \leq \nu].$$

Since $[a_1 - a_i : v < i \leq r] = [a_1 - r + i : v < i \leq r]$, we can write as

$$\beta(\nu') = [1, 2, \ldots, a_1] - [a_1 - a_i : 2 \leq i \leq r],$$

and therefore, we have the beta set of $\mu'$,

$$\beta(\mu') = [a_1 + b_j : 1 \leq j \leq s] \cup [1, 2, \ldots, a_1] - [a_1 - a_i : 2 \leq i \leq r].$$

Now, we consider the half-even beta set $\beta_{r/2}(\lambda)$. By Lemma 4.2, we have

$$\beta(\lambda) = \left\{ \frac{\delta_1 + \delta_1}{2}, \frac{\delta_1 + \delta_2}{2}, \ldots, \frac{\delta_1 + \delta_d}{2} \right\} \cup \left\{ \frac{\delta_1 - 1}{2}, \frac{\delta_1 - 3}{2}, \ldots, 1 \right\} - \left\{ \frac{\delta_1 - 3}{2}, \frac{\delta_1 - \delta_{d-1}}{2}, \ldots, \frac{\delta_1 - \delta_2}{2} \right\}.$$

We consider two cases:

**Case 1** Let $\delta_1 \in D_1(\lambda)$ so that $\delta_1 = 4a_1 + 1$. Then,

$$\beta_{r/2}(\lambda) = [a_1 + b_j : 1 \leq j \leq s] \cup [1, 2, \ldots, a_1] - [a_1 - a_i : 2 \leq i \leq r] = \beta(\mu').$$

**Case 2** Let $\delta_1 \in D_3(\lambda)$ so that $\delta_1 = 4b_1 - 1$. Then,

$$\beta_{r/2}(\lambda) = [b_1 + a_i : 1 \leq i \leq r] \cup [1, 2, \ldots, b_1 - 1] - [b_1 - b_j : 2 \leq i \leq s] = \beta(\mu).$$

**Example 4.3.** Let $\lambda, \lambda'$ be self-conjugate partitions we considered in Example 3.3. We remind that $\phi(3)(\lambda) = \phi(3)(\lambda') = \mu = (4, 3, 3, 2, 1, 1)$. We note that $h_3(1, 1) = 21 \in D_3(\lambda)$ and $h_3(1, 1) = 31 \in D_3(\lambda)$. As in Proposition $4.2$, $\beta_{r/2}(\lambda) = \beta(\mu') = [9, 6, 4, 1]$ and $\beta_{r/2}(\lambda) = \beta(\mu) = [9, 7, 6, 4, 2, 1]$. See Figure 2 for the Young diagrams of $\mu, \lambda, \lambda'$, and their hook lengths.

4.2. Relations between hook lengths of $SC^{(m)}$ and $P$

One may notice that there are more relations between hook lengths of corresponding partitions from Figure 3. Now, we give our main theorem.

**Theorem 4.4.** Let $\lambda \in SC^{(m)}$ be a self-conjugate partition and let $\mu$ be its corresponding partition. For each positive integer $k$, the number of boxes $(i, j)$ with $h_{\lambda}(i, j) = 2k$ is equal to twice the number of boxes $(i, j)$ with $h_{\mu}(i, j) = k$.

The following proposition is necessary to prove Theorem 4.4.
Proposition 4.5. For $\lambda \in SC^{(m)}$ with $D(\lambda) = \{\delta_1, \delta_2, \ldots, \delta_d\}$, let $\bar{\lambda}$ be the self-conjugate partition with $D(\bar{\lambda}) = \{\delta_i \in D(\lambda) : 2 \leq i \leq d\}$, and let $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ and $\bar{\mu}$ be the corresponding partitions of $\lambda$ and $\bar{\lambda}$, respectively. If $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$, then

$$
\bar{\mu} = \begin{cases} 
(\mu_2, \mu_3, \ldots, \mu_\ell) & \text{if } \delta_1 \in D_1(\lambda) \\
(\mu_1 - 1, \mu_2 - 1, \ldots, \mu_\ell - 1) & \text{if } \delta_1 \in D_2(\lambda)
\end{cases}
$$

Proof. We let $((a_1, \ldots, a_r), (b_1, \ldots, b_s))$ be the diagonal sequence pair of $\lambda$ so that $\phi^{(m)}(\lambda) = \mu$ is constructed by

$$
\mu_i = a_i + i + s - r \quad \text{for } i \leq r,
$$

and $(\mu_{r+1}, \ldots, \mu_\ell)$ is the conjugate of the partition $\gamma = (b_1 - s, b_2 - s + 1, \ldots, b_s - 1)$.

We denote $\tilde{\mu} = (\tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_\ell)$ and consider the following two cases.

1. **Case 1** If $\delta_1 \in D_1(\lambda)$, the diagonal sequence pair of $\bar{\lambda}$ is $((a_2, \ldots, a_r), (b_1, \ldots, b_s))$ for $r \geq 2$ or $(\emptyset, (b_1, \ldots, b_s))$ for $r = 1$. Then the corresponding partition $\bar{\mu}$ of $\bar{\lambda}$ is defined by

$$
\bar{\mu}_i = a_{i+1} + i + s - (r-1) = a_{i+1} + (i+1) + s - r \quad \text{for } i \leq r-1,
$$

$$
\bar{\mu}_i = a_{i+1} + i + s - (r-1) = a_{i+1} + (i+1) + s - r \quad \text{for } i \leq r-1,
$$
and \((\bar{\mu}_i, \bar{\mu}_{i+1}, \ldots)\) is the conjugate of \((b_1 - s, b_2 - s + 1, \ldots, b_s - 1)\). Hence, \(\bar{\mu}_i = \mu_{i+1}\) for all \(i\).

**Case 2** If \(\delta_1 \in D_1(\lambda)\), the diagonal sequence pair of \(\lambda\) is \((a_1, \ldots, a_t, b_2, \ldots, b_s)\) for \(s \geq 2\) or \((a_1, \ldots, a_t, 0)\) for \(s = 1\). In this case, the corresponding partition \(\bar{\mu}\) of \(\lambda\) is defined by

\[
\bar{\mu}_i = a_i + i + (s - 1) - r = a_i + i + s - r - 1 \quad \text{for } i \leq r,
\]

and \((\bar{\mu}_r, \bar{\mu}_{r+1}, \ldots)\) is the conjugate of \(\lambda = (b_2 - (s - 1), \ldots, b_s - 1) = (b_2 - s + 1, \ldots, b_t - 1)\). Therefore, \(\bar{\mu}_i = \mu_i - 1\) for all \(i\).

\[
\square
\]

**Proof of Theorem 4.4.** We apply strong induction on \(|\lambda|\). For \(|\lambda| = 0, \lambda = \emptyset \in SC_0^{(0)}\). Since \(\phi^{(0)}(\lambda) = \emptyset\), the assertion holds.

Now, we assume that the assertion holds for \(|\lambda| < n\). Let \(\lambda\) be a self-conjugate partition of \(n \geq 1\) with \(D(\lambda) = \{\delta_1, \delta_2, \ldots, \delta_d\}\) and let \(\mu = (\mu_1, \mu_2, \ldots, \mu_t)\) be the corresponding partition of \(\lambda\). If \(\lambda\) is the self-conjugate partition of \(n - \delta_1\) with \(D(\lambda) = \{\delta_1 : 2 \leq i \leq d\}\) and \(\bar{\mu}\) is its corresponding partition, then by our assumption, the number of boxes \((i, j)\) with \(h_1(i, j) = 2k\) is equal to twice the number of boxes \((\bar{i}, \bar{j})\) with \(h_2(\bar{i}, \bar{j}) = k\) for each \(k \geq 1\). We consider the following two cases.

**Case 1** Let \(\delta_1 \in D_1(\lambda)\). By Proposition 4.3, \(\bar{\mu} = (\mu_2, \mu_3, \ldots, \mu_t)\). Therefore, it is enough to show that the number of boxes \((i, j)\) either in the first row or in the first column of \(\bar{\mu}\) with \(h_2(i, j) = k\) is equal to twice the number of boxes \((\bar{i}, \bar{j})\) in the first row in \(\mu\) with \(h_1(\bar{i}, \bar{j}) = k\) for each \(k \geq 1\). By Proposition 4.2, \(\beta_{c/2}(\lambda) = \beta(\bar{\mu}')\). Moreover, \(\beta_{c/2}(\bar{\lambda}') = \beta(\bar{\mu}')\) since \(\lambda\) is self-conjugate.

**Case 2** Let \(\delta_1 \in D_2(\lambda)\). By Proposition 4.3, \(\bar{\mu} = (\mu_1 - 1, \mu_2 - 1, \ldots, \mu_t - 1)\). Therefore, it is enough to show that the number of boxes \((i, j)\) either in the first row or in the first column of \(\bar{\mu}\) with \(h_2(\bar{i}, \bar{j}) = k\) is equal to twice the number of boxes \((\bar{i}, \bar{j})\) in the first column in \(\mu\) with \(h_1(\bar{i}, \bar{j}) = k\) for each \(k \geq 1\). By Proposition 4.2, \(\beta_{c/2}(\lambda) = \beta_{c/2}(\mu') = \beta(\mu)\), as desired.

\[
\square
\]

The following corollary is obtained directly from Theorem 4.4.

**Corollary 4.6.** For a self-conjugate partition \(\lambda \in SC^{(m)}\), let \(\mu\) be the corresponding partition of \(\lambda\). Then \(\lambda\) is a \((2t_1, 2t_2, \ldots, 2t_p)\)-core partition if and only if \(\mu\) is a \((t_1, t_2, \ldots, t_p)\)-core partition.

We denote the set of self-conjugate \((t_1, t_2, \ldots, t_p)\)-core partitions \(\lambda \in SC^{(m)}\) of \(n\) by \(SC_{(t_1,\ldots,t_p)}^{(m)}(n)\), and use notation \(SC_{(t_1,\ldots,t_p)}^{(m)}(n)\) for \(|SC_{(t_1,\ldots,t_p)}^{(m)}(n)|\).

By using Theorem 3.1 and Theorem 4.4, we obtain the cardinality of \(SC_{(2t_1,\ldots,2t_p)}^{(m)}(n)\).

**Proposition 4.7.** For \(m \geq 0\), the number of self-conjugate \((2t_1, 2t_2, \ldots, 2t_p)\)-core partitions of \(n\) with the disparity \(\frac{m(m+1)}{2}\) is

\[
SC_{(2t_1,\ldots,2t_p)}^{(m)}(n) = \begin{cases} 
 c_{(t_1,\ldots,t_p)}^{(m)}(k) & \text{if } n = 4k + \frac{m(m+1)}{2}, \\
 0 & \text{otherwise.}
\end{cases}
\]

From the previous proposition, we have Theorem 1.3.

4.3. Counting self-conjugate \((2t_1, \ldots, 2t_p)\)-core partitions with the disparity \(\frac{m(m+1)}{2}\) is equal to the number of \((t_1, \ldots, t_p)\)-core partitions.

In this subsection, we give some sets of self-conjugate partitions each of them is counted by known special numbers. It is well-known that there are finitely many \((t_1, \ldots, t_p)\)-core partitions when \(t_1, \ldots, t_p\) are relatively prime numbers. From Proposition 4.7, we have the following result.

**Corollary 4.8.** For relatively prime numbers \(t_1, \ldots, t_p\), the number of self-conjugate \((2t_1, \ldots, 2t_p)\)-core partitions with the disparity \(\frac{m(m+1)}{2}\) is equal to the number of \((t_1, \ldots, t_p)\)-core partitions.
Anderson [3] gives an expression for the Catalan number in terms of simultaneous core partitions, and Amdeberhan and Leven [2], Yang, Zhong, and Zhou [19], Wang [18], respectively, gives an identity for the Motzkin number.

**Theorem 4.9.** [3] For relatively prime integers \( t_1, t_2 \geq 1 \), the number of \((t_1, t_2)\)-core partitions is

\[
C(t_1, t_2) = \frac{1}{t_1 + t_2} \binom{t_1 + t_2}{t_1}.
\]

In particular, \( C(n, n+1) = C_n \), where \( C_n = \frac{1}{n+1} \binom{2n}{n} \) is the \( n \)th Catalan number.

**Theorem 4.10.** [18] For relatively prime integers \( n, d \geq 1 \), the number of \((n, n + d, n + 2d)\)-core partitions is

\[
C(n, n + d, n + 2d) = \frac{1}{n + d} \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{i, i + d, n - 2i}.
\]

In particular, \( C(n, n + 1, n + 2) \) is the \( n \)th Motzkin number \( M_n = \sum_{i \geq 0} \frac{1}{i+1} \binom{2i}{i} \).

By using Corollary 4.3 and the above known results, we have the following corollary.

**Corollary 4.11.** Let \( m \geq 0 \) be an integer.

(a) For relatively prime integers \( t_1, t_2 \geq 1 \), the number of self-conjugate \((2t_1, 2t_2)\)-core partitions with the disparity \( \frac{m}{2} \) is

\[
s_{\text{sc}}^{(m)}(2t_1, 2t_2) = \frac{1}{t_1 + t_2} \binom{t_1 + t_2}{t_1}.
\]

In particular, \( s_{\text{sc}}^{(m)}(2n, 2n+2) = C_n \), where \( C_n \) is the \( n \)th Catalan number.

(b) For relatively prime integers \( n, d \geq 1 \), the number of self-conjugate \((2n, 2n + 2d, 2n + 4d)\)-core partitions with the disparity \( \frac{m}{2} \) is

\[
s_{\text{sc}}^{(m)}(2n, 2n + 2d, 2n + 4d) = \frac{1}{n + d} \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{i, i + d, n - 2i}.
\]

In particular, \( s_{\text{sc}}^{(m)}(2n, 2n + 2, 2n + 4) \) is the \( n \)th Motzkin number.

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