On computational aspects
of two classical knot invariants

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Abstract
We look into computational aspects of two classical knot invariants. We look for ways of simplifying the computation of the coloring invariant and of the Alexander module. We support our ideas with explicit computations on pretzel knots.

1 Introduction
Knots ([1, 2, 4, 8, 11, 13, 16]) are embeddings or placements of the standard circle, $S^1$, into 3-space. Roughly speaking, the idea is to break $S^1$ at one point, make the line segment so obtained go over and under itself a number of times and finally to connect the two ends. We consider two such embeddings to represent the same knot if one of the embeddings can be deformed into the other. In the sequel, knot will stand for a class of these embeddings that are deformable into one another or for one individual embedding, the context will make the choice clear.

Although Knot Theory has varied and interesting applications and connections to other fields of study, we believe that one of its basic goals is the classification of the knots modulo deformations, or at least an attempt to do so. Ideally, an invariant of knots would yield different outcomes when applied to knots which are not deformable into each other. It is still an open question whether such an invariant exists or not. On the other hand, invariants should be tested for their efficiency. This can be accomplished by using classes of knots and testing the efficiency of a given invariant in telling apart the elements of the given class. The next step is to look into the possible simplifications in the calculation of the invariant.

In this article we will focus on these issues of testing and simplifying on two classical invariants of knots: the coloring invariant and the Alexander module. Moreover, we will be using the class of pretzel knots as our working example.

Knots are usually represented by drawing a diagram on a plane. This begins with a projection of the embedding on a plane, possibly after some deformation of the embedding so that each point of intersection in the projection has the following property. There is a neighborhood of each such point such that the intersection of this neighborhood with the projection of the knot is formed by exactly two arcs meeting transversally. These points of intersection are called crossings. At crossings, in order to make sense of what goes over and what goes under in the embedding, the line that goes under in the embedding is broken in the projection. The result so obtained is called a knot diagram, see Figure 1.

Knot diagrams are useful in characterizing and visualizing knots. Another important aspect of knot diagrams stems from the so-called Reidemeister moves. These are transformations on knot diagrams known as Reidemeister moves of type I, II, and III as shown in Figures 2, 3 and 4. Their interplay with knot diagrams is stated in Theorem 1.
**Theorem 1.1 (Reidemeister-Alexander)** If two knot diagrams are related by a finite number of Reidemeister moves, then the corresponding knots are deformable into each other. Conversely, if two knots are deformable into each other, then any diagram of one of these two knots is related to any diagram of the other knot by a finite sequence of Reidemeister moves.

See [15] or [6] for a proof. The Reidemeister moves are local i.e., the changes they make on the diagrams occur only inside a given neighbourhood of the diagram. Outside this neighbourhood the diagram remains the same.

A possible way of obtaining a knot invariant is to construct an assignment that uses data from the knot diagrams and whose outcome (essentially) does not change upon the performance of the Reidemeister moves. In Section 2 we will see examples of such assignments.

### 1.1 Organization and acknowledgements

This article is organized as follows. In Section 2 we develop the background material on colorings, filling in what we believe to be a gap in the literature. In Section 3 we calculate the coloring matrices for pretzel knots. In Section 4 we calculate the Alexander polynomial for two subclasses of pretzel knots partially recovering, but also extending, previous work of Parris ([14]).
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2 Background material on colorings

We begin this Section by defining Integer colorings of diagrams and Coloring matrix of a diagram.

Definition 2.1 (Integer coloring of a diagram) Given a knot $K$, consider a diagram $D_K$ of it. An integer coloring of $D_K$ is an assignment of integers to the arcs of this diagram such that, at each crossing, twice the integer assigned to the over-arc equals the sum of the integers assigned to the under-arcs (see Figure 5). There are always the so-called trivial colorings. In a trivial coloring of a diagram the same integer is assigned to each and every arc of the diagram.

$$2b = a + c$$

Figure 5: Coloring at a crossing
**Definition 2.2 (Coloring matrix of a diagram)** We keep the notation and the terminology of Definition 2.1. From each crossing of a knot diagram stems an arc. In this way, there are as many arcs as crossings. Suppose there are \( n \) arcs (resp., crossings). Enumerating the arcs (resp., the crossings) as we go along the knot diagram under study starting at a given reference point, we say that the \( i \)-th arc stems from the \( i \)-th crossing. We then set up a system of \( n \) linear equations with \( n \) variables as follows. Each arc stands for a variable and each crossing stands for an equation. Each of these equations states that the sum of the variables corresponding to the under-arcs minus twice the variable corresponding to the over-arc at the crossing at issue equals zero. The integer solutions of this system of equations are the \textbf{Integer colorings} of Definition 2.1. The matrix of integer coefficients of this system of equations is the \textbf{Coloring matrix of the diagram}.

We now define \textbf{Elementary transformations on Integer matrices} and prove that Reidemeister moves on diagrams give rise to \textbf{Coloring matrices of diagrams} which are related by a finite number of \textbf{Elementary transformations}. This being related by a finite number of Elementary transformations is an equivalence relation on the set of integer matrices. As a consequence, the equivalence class of a coloring matrix of a diagram of a given knot is a topological invariant of that knot.

**Definition 2.3 (Elementary Transformations on Integer Matrices [2, 5, 11])** Given an integer matrix \( M \), an \textbf{elementary transformation} on \( M \) is obtained by performing a finite sequence of the following operations (and/or their inverses)

1. Permutation of rows or columns;
2. Replacement of the matrix \( M \) by \[
\begin{pmatrix}
M & 0 \\
0 & 1
\end{pmatrix}
\]
3. Addition of a scalar multiple of a row (resp., column) to another row (resp., column)

**Proposition 2.1** Consider a knot diagram of a knot. Upon performance of Reidemeister moves on this diagram, the Coloring matrix of the resultant diagram is obtained by performing a finite sequence of elementary transformations on the Coloring matrix of the original diagram.

Proof: We split the proof into three parts, each one concerning one type of Reidemeister move.

- **Type I Reidemeister move**, see Figure 2
  Upon performance of this type of move (in one direction) the new diagram has an extra arc and crossing with respect to the original diagram, see Figure 2. Let this extra arc be denoted \( a' \) and let the remaining arcs keep the notation of the previous diagram. The equation at the new crossing reads
  \[
a + a' - 2a' = 0 \iff a' = a \iff a' + a - 2a = 0
\]
  which states that the new variable equals one of the old variables. Let us now see what happens with the Coloring matrices. We will start from the Coloring matrix corresponding to the diagram on the right of Figure 2. This matrix has an extra row and an extra column due to the extra arc \( a' \). We write down only the entries in the rows and columns that have directly to do with \( a' \). In this way, there is a row formed by 1 and 1 \(-2\) which corresponds to the equation read at the extra crossing \( a + a' - 2a' = 0 \). The row with a sole 1 corresponds to the contribution of the other end of arc \( a' \) to another equation where it enters as an under-arc (hence the 1). This portion of the Coloring matrix is the one down to the left. The other matrices were obtained by performing Elementary transformations which we hope are clear to the reader. In the matrix to the far right we will then perform a Transformation 2. to get rid of the row and column with the sole \(-1\). The matrix so obtained corresponds to the diagram on the left of Figure 2

\[
\begin{pmatrix}
1 & 1 - 2 & \ldots \\
0 & 1 & \ldots
\end{pmatrix} \leftrightarrow \begin{pmatrix}
0 & -1 & \ldots \\
1 & 1 & \ldots
\end{pmatrix} \leftrightarrow \begin{pmatrix}
0 & -1 & \ldots \\
1 & 0 & \ldots
\end{pmatrix}
\]
- Type II Reidemeister move, see Figure 3.
Upon performance of this type of Reidemeister move (in one direction) the new diagram has two extra arcs and two extra crossings, with respect to the original diagram, see Figure 3. Let these extra arcs be denoted $a'_1$ and $a'_2$. The equations at the new crossings read
\[
\begin{align*}
&\begin{cases}
a + a'_1 - 2b = 0 \\
a'_1 - a'_2 - 2b = 0 \\
\end{cases} \\
&\iff \begin{cases}
a'_1 = 2b - a \\
a'_2 = a \\
\end{cases}
\end{align*}
\]

We write down the progression of the portions of the matrices relevant to this move, under the Elementary Transformations. In the matrix to the left, the top row corresponds to the equation read off at the top crossing of the diagram on the right of Figure 3, the second row corresponds to the bottom crossing on the same Figure, and the last row corresponds to an equation to which the bottom left arc contributes also.

\[
\begin{pmatrix}
1 & 1 & 0 & -2 \\
0 & 1 & 1 & -2 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 1 & \ldots
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 1 & \ldots
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & -1 & 0 & \ldots
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
1 & 0 & 0 & \ldots
\end{pmatrix}
\]

- Type III Reidemeister move, see Figure 4.
Without further remarks, we write down the progression of the relevant portions of the matrices under the Elementary Transformations.

\[
\begin{pmatrix}
1 & 1 & 0 & \ldots & -2 & \ldots & \ldots \\
0 & 1 & 1 & \ldots & \ldots & \ldots & -2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & 1 & 1 & \ldots & -2
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
1 & 1 & 0 & \ldots & -2 & \ldots & \ldots \\
0 & 1 & 1 & \ldots & -1 & -1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & 1 & 1 & \ldots & -2
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
1 & 1 & 0 & \ldots & 0 & 1 & \ldots & -2 \\
0 & 1 & 1 & \ldots & -1 & -1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & 1 & 1 & \ldots & -2
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
1 & 1 & 0 & \ldots & 0 & 0 & \ldots & -2 \\
0 & 1 & 1 & \ldots & -2 & 0 & \ldots & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & 1 & 1 & \ldots & -2
\end{pmatrix}
\]

Corollary 2.1 Given a knot $K$ along with one of its diagrams, consider the coloring matrix of this diagram. The equivalence class of the coloring matrix of this diagram under elementary transformations as described in Definition 2.3 is a topological invariant of the knot $K$.

Proof: Omitted.

Definition 2.4 (Coloring matrix of a knot) Given a knot $K$ we call Coloring matrix of $K$ any matrix which is obtained from a Coloring matrix of a diagram of $K$ by performing a finite number of transformations as described in Definition 2.3.
In view of Corollary 2.1, given a Coloring matrix we would like to diagonalize it, using the elementary transformations. That this is possible is a result of Smith on integer matrices ([17]) which further states that the diagonalized matrix, say \( \text{diag}(d_1, d_2, \ldots, d_s, 0, \ldots, 0) \), can be obtained so that \( d_i | d_{i+1} \) (see [5, 12]). This is called the Smith normal form of the matrix under consideration. It is unique modulo multiplication of the \( d_i \)'s by units. The \( d_i \)'s are called the invariant factors.

Using elementary transformation 2., \( \pm 1 \)'s can be introduced in or removed from the diagonal thus increasing or decreasing the diagonal’s length. In this way, two Smith normal forms of the same matrix may differ on the length of the diagonal, and the difference of the lengths is the surplus of \( \pm 1 \)'s one has with respect to the other. We distinguish the Smith normal forms without \( \pm 1 \)'s, in our work.

**Definition 2.5 (Normal form of the coloring matrix)** Consider a knot \( K \). Consider the Smith normal form of any one of its Coloring matrices. Eliminate the \( \pm 1 \)'s from the diagonal of this matrix along with the corresponding rows and columns. We call the matrix so obtained Normal form of the Coloring matrix of \( K \).

In case the diagonal of the Smith normal form consists only of \( \pm 1 \)'s, we call Normal form of the Coloring matrix of \( K \) the \( 1 \times 1 \) matrix formed by the entry 1.

It is also a known result (see [3]) that the invariant factors can be calculated in the following way. Calculate the greatest common divisors (g.c.d.’s) of the \( i \)-rowed minors i.e., the g.c.d.’s of the determinants of all \( i \times i \) submatrices of the matrix under study, denoted \( \Delta_i \)’s. Then, modulo units,

\[
d_1 = \Delta_1, \quad d_2 = \Delta_2 \cdot \Delta_1^{-1}, \quad d_3 = \Delta_3 \cdot \Delta_2^{-1}, \quad \ldots
\]

**Corollary 2.2** Given a knot \( K \), the invariant factors (resp., the greatest common divisors of the \( i \)-rowed minors) of any Coloring matrix of \( K \) form a set of topological invariants of \( K \).

Proof: Omitted.

**Definition 2.6 (Torsion invariants)** The torsion invariants of a knot \( K \) are the invariant factors of any Coloring matrix of \( K \).

**Corollary 2.3** Given a knot \( K \), the determinant of any of its coloring matrices is zero.

Proof: Consider a coloring matrix of \( K \) obtained by associating a system of equations to a given diagram of \( K \) in the way indicated in 2.2. Since the equations read at each crossing have the form \( a_i + a_{i+1} - 2a_j = 0 \), then the entries of the Coloring matrix along each row add up to zero. If we add all columns but the last one to the last one we obtain a new matrix which is identical to the preceding one except at the last column which is now made of zero’s. Since the passage from the former to the latter matrices is obtained via a finite sequence of elementary transformations, their determinants are equal. Since the latter has a column made of zeros the determinant is zero.

**Definition 2.7 (The determinant of the knot)** Consider an \( n \times n \) coloring matrix of a given knot \( K \). The determinant of \( K \) is the greatest common divisor of the \( (n-1) \times (n-1) \)-rowed minors of this coloring matrix. It is clearly a topological invariant of \( K \).

**Corollary 2.4** Given a knot \( K \) and a coloring matrix of a diagram of \( K \), its determinant can be computed by eliminating any row and any column of this matrix and computing the determinant of the matrix so obtained.

Proof: We prove that certain linear combinations of the rows (respect., columns) yield zero. Furthermore the coefficients of these linear combinations are \( \pm 1 \). We thus prove that any row (respect., column) is a linear combination of the remaining rows (respect., columns) hence proving the statement in this Corollary.
In the proof of Corollary 2.3 it was shown how to obtain a column of 0’s by adding all columns but the one in stake to it. We now consider how to obtain a row of zero’s. Assume the diagram $D$ that is being used is an alternating diagram i.e., each arc goes over exactly one crossing. The contribution of each arc to the coloring matrix of $D$ is then as follows. It contributes a $2$ at the crossing of which it is the over-arc; it contributes a $-1$ at each of the two crossings of which it is an under-arc. In this way, each column contains one $2$ and two $-1$’s. Then, adding all but a given row to this row changes it to a new row made of 0’s.

Here is the algorithm for the general case. Consider a knot diagram $D$ and introduce a checkerboard shading on it. This is a shading of the faces of the diagram using two tones (grey and white) such that at each crossing the faces that share a common arc receive distinct tones, as (partially) illustrated in Figure 6.

Now, go along the diagram starting at a given point. As you go over a crossing you print a mark to the right of the crossing and just before it. Do this for all crossings. Note that, at each crossing the markings either fall on grey faces or on white faces. Also note that, at each crossing, this falling of the markings on grey or white faces of the diagram is independent of the orientation of the components of the diagram. As a matter of fact, should the orientation of any of these components be reversed, then at each crossing of this component, the marking would now fall on the opposite side of this crossing. But shadings on opposite sides of crossings are the same.

When you associate the $a_{i-1} + a_i - 2a_j = 0$ equation to the $i$-th crossing, you multiply this equation by $-1$ if the mark corresponding to this crossing is in a grey face, and you leave the equation as it is otherwise. It is now easy to see that adding all rows, a row of 0’s is obtained. In Figure 6 we show illustrative instances of this row cancellation. The left case is intended to illustrate the case when an $a_i$ is an over-arc for an even number of crossings and the right case for the odd number of crossings. Finally, note that the markings associated to the other arcs meeting at the crossing $a_i$ stems from, comply with the markings associated to $a_i$, so that the overall effect is to obtain a row of zeros. □

![Figure 6: Illustrative cases of the row cancellation](image_url)

We now know that the **Normal form of the Coloring matrix** of any knot has at least one 0 along its diagonal. We define:

**Definition 2.8 (Reduced coloring matrix)** A **Reduced coloring matrix** is a matrix obtained from...
a Coloring matrix by eliminating one row and one column - or any matrix equivalent to it. The determinant of the knot is the determinant of the reduced coloring matrix.

To some extent and from the topological point of view, the Reduced coloring matrix is the Coloring matrix with some redundant information removed. Note that the procedure explained above leading to the elimination of a row corresponds to changing all but the last variable to a new set of variables each of which equals the old one minus the variable associated with the given row. Suppose the determinant of the knot is \(d\). If \(d\) is a unit i.e., an element such that there is a \(\bar{d}\) with \(\bar{d}d = 1 = d\bar{d}\) then we can use Cramer’s rule (see [12]) to obtain a unique solution for this reduced system. According to what was previously said, this unique solution corresponds to trivial colorings. If \(d = 0\) then this knot would have more solutions then just the trivial ones. On the other hand since we are working with matrices over the integers, the units are \(\pm 1\). Also, there are always infinitely many solutions even if these are only the trivial ones. Note that if the ring we were using were finite then the number of solutions would constitute a topological invariant of the knot. It would also constitute a computable topological invariant of the knot.

This is the situation when we replace the ring of integers, \(\mathbb{Z}\), by the ring of residues modulo a given \(r\), \(\mathbb{Z}_r\). Clearly, these rings comply with the set up developed above.

**Definition 2.9 (r-colorings of a knot \(K\))** The \(r\)-colorings of a knot \(K\) are the solutions of the system of equations whose coefficient matrix is a Coloring matrix of \(K\) over the ring \(\mathbb{Z}_r\).

**Corollary 2.5** The number of \(r\)-colorings of a knot \(K\) is a topological invariant of \(K\).

Proof: Omitted.

We distinguish two types of situations, as far as \(r\) is concerned. If \(r\) is prime, then \(\mathbb{Z}_r\) is a field. In this case, after diagonalizing the Coloring Matrix we count the number of 0’s modulo \(r\) in the diagonal, say \(k(\geq 1)\). This \(k\) is then the dimension of the linear subspace of the solutions; the number of the solutions is \(r^k\). \(k\) is at least 1 on account of the trivial solutions. Moreover if \(r\) divides the determinant of the knot, then there are non-trivial solutions.

If \(r\) is not prime then \(\mathbb{Z}_r\) is a ring which is not a field. In this case, if the determinant of the knot is a unit (say \(d \in \mathbb{Z}_r\) such that there is \(\bar{d} \in \mathbb{Z}_r\) and \(\bar{d}d = 1 = d\bar{d}\)) then we can use Cramer’s rule to obtain a unique solution for the reduced matrix which means as discussed above, there are only trivial solutions. If the determinant of the knot is not a unit (which in these rings is equivalent to saying that \(r\) is a zero divisor) then there are more than just the trivial colorings.

The prime \(r\) case is relatively known since it comes down to linear algebra over fields, as \(\mathbb{Z}_r\) is a field. The case for non-prime \(r\) is still being studied (see [9]).

There is another aspect which makes this use of rings of residues worthwhile. As a matter of fact, given any diagram of a knot we can set up the Coloring matrix of this diagram and proceed to counting solutions in a given modulus \(r\) by brute force. Of course, once the diagonalization of the Coloring matrix of the diagram is achieved, the counting of solutions in any given modulus is very easy. But, given the known difficulties with diagonalization of integer matrices, the method above is particularly interesting from the practical point of view.

This point of view is explored in [8]. In this article the coloring equations are set up for each knot from their minimal diagrams. Then the candidates to solutions are tried out on this system of equations. The candidates that comply with the system of equations are the solutions and are thus counted. The knots considered are all prime knots up to ten crossings.

Our stand point in the current article is more on the theoretical side as mentioned before. In this way, we also diagonalize the coloring matrices of each knot of a class of knots, the pretzel knots. This is done in Section [8]

### 3 Coloring matrices of pretzel knots

#### 3.1 Preliminary formulas
In this subsection we set up formulas for the colorings of twists.

**Definition 3.1** An $n$-twist is obtained by producing $n$ half-twists on two line segments. Tassel on $n$ crossings, and twist on $n$ crossings are synonyms to $n$-twist. We refer to top arcs and bottom arcs and order crossings as shown for the $n = 3$ instance in Figure 7. We note that this is a different sort of ordering from that presented in Definition 2.2.

![Diagram of n-twist](image)

**Figure 7:** Aspect and terminology of an $n$-twist at the $n = 3$ instance

A coloring of an $n$-twist is the assignment of integers to the arcs of the twist so that at each crossing the color on the emergent under-arc is $2b - a$, where $b$ is the color on the over-arc and $a$ is the color on the incoming under-arc (see Figure 8). We will also use expressions as colored $n$-twist to refer to an $n$-twist endowed with a coloring.

![Diagram of coloring](image)

**Figure 8:** Coloring of an $n$-twist at the $n = 3$ instance

Typically, such a coloring is established once the $a$ and $b$ colors on the top arcs of the twist are specified, as shown in Proposition 3.1.

**Proposition 3.1** Consider an $n$-twist and assign integers $a$ and $b$ to its top arcs as shown in Figure 8 for $n = 3$. These integers induce a coloring of the twist such that the color on the arc emerging from the $i$-th crossing is $b + i(b - a)$ for each $1 \leq i \leq n$. 
Proof: By induction on $n$. The $n = 3$ instance is illustrated in Figure 8. Assume the statement is true for an $n > 2$. Upon juxtaposition of the $(n + 1)$-th crossing to a colored $n$-twist (see Figure 9) we obtain on the arc emerging from the $(n + 1)$-th crossing:

$$2\left(b + n(b - a)\right) - \left(b + (n - 1)(b - a)\right) = b + (n + 1)(b - a)$$

This concludes the proof.

\[
\begin{align*}
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Figure 10: Symbolic representation of an \( n \)-twist and a colored \( n \)-twist

\[
b + (n - 1)(b - a) \quad b + n(b - a)
\]

Figure 11: Knot diagram of a \( P(n_1, n_2, n_3, \ldots, n_N) \) pretzel knot

**Definition 3.3** Given a positive integer \( N \) and \( n_1, n_2, \ldots, n_N \) integers, a \( P(n_1, n_2, \ldots, n_N) \) pretzel knot is represented by the knot diagram in Figure 11 (disregard the \( a'_i \)'s and \( b'_i \)'s for now).

We will now consider the \( N = 3 \) situation with three positive integers, \( n_1, n_2, n_3 \), and write down its coloring system of equations. Consider Figure 12.

From Proposition 3.1 we know that once the \( a_i \) colors have been assigned to the tops of the twists that compose the pretzel knot, the colors on the arcs along each of these twists are uniquely expressed in terms of the \( a_i \)'s. Thus, we just have to equate the two different ways of writing each \( b_i \). In this way the coloring system of equations for this pretzel knot is

\[
\begin{align*}
b_2 : \quad & a_2 + n_1(a_2 - a_1) = a_2 + n_2(a_3 - a_2) \\
b_3 : \quad & a_3 + n_2(a_3 - a_2) = a_3 + n_3(a_1 - a_3) \\
b_1 : \quad & a_1 + n_3(a_1 - a_3) = a_1 + n_1(a_2 - a_1)
\end{align*}
\]
This is equivalent to

\[
\begin{align*}
-n_1a_1 + (n_1 + n_2)a_2 - n_2a_3 &= 0 \\
n_3a_1 - n_2a_2 + (n_2 + n_3)a_3 &= 0 \\
(n_3 + n_1)a_1 - n_1a_2 - n_3a_3 &= 0
\end{align*}
\]

The Coloring matrix of this diagram is then

\[
\begin{pmatrix}
-n_1 & n_1 + n_2 & -n_2 \\
-n_3 & -n_2 & n_2 + n_3 \\
n_3 + n_1 & -n_1 & -n_3
\end{pmatrix}
\]

As we saw before, along any row the coefficients add up to zero and analogously along any column.

Adding the first two columns to the third one, and adding the first two rows to the third one, we obtain

\[
\begin{pmatrix}
-n_1 & n_1 + n_2 & 0 \\
-n_3 & -n_2 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

In order to keep up with what we will do later in the general case, we add the first column to the second one to obtain

\[
\begin{pmatrix}
-n_1 & n_2 & 0 \\
-n_3 & -n_3 - n_2 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

The greatest common divisors of the \(i\)-rowed minors are now easy to calculate.

\[
\Delta_1 = (n_1, n_2, n_3) \\
\Delta_2 = n_1n_2 + n_1n_3 + n_2n_3 \\
\Delta_3 = 0
\]

where we use the following

**Definition 3.4** Given \(I\) integers, \(a_1, a_2, \ldots, a_I\), we denote their greatest common divisor (g.c.d.) by

\[
(a_1, a_2, \ldots, a_I) \quad \text{or} \quad (a_i)_{i=1,\ldots,I}
\]
With this notation, the diagonalized Coloring matrix is:

\[
\begin{pmatrix}
  (n_1, n_2, n_3) \\
  0 \\
  0 \\
  0 \\
\end{pmatrix}
\begin{pmatrix}
  0 & n_1 n_2 + n_3 n_4 + n_2 n_3 \\
  0 & n_1 n_2 + n_3 n_4 + n_2 n_3 \\
  0 & n_1 n_2 + n_3 n_4 + n_2 n_3 \\
  0 & n_1 n_2 + n_3 n_4 + n_2 n_3 \\
\end{pmatrix}
\begin{pmatrix}
  0 & 0 \\
  0 & 0 \\
  0 & 0 \\
  0 & 0 \\
\end{pmatrix}
\]

Now for the general case of a pretzel knot with \(N\) tassels. The result describing the diagonalized Coloring matrix is Theorem 3.1. It is the corollary to Propositions 3.2 and 3.3 and Lemma 3.1.

**Proposition 3.2** Given positive integers \(N > 2\) and \(n_1, \ldots, n_N\), consider the Pretzel knot \(P(n_1, \ldots, n_N)\) (see Figure 11). Its coloring matrix is equivalent to

\[
\begin{pmatrix}
  -n_1 & n_1 + n_2 & -n_2 & 0 & \cdots & 0 \\
  0 & -n_2 & n_2 + n_3 & -n_3 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  -n_N & -n_N & -n_N & \cdots & -n_N & n_N - n_N - 1 \\
  n_N + n_1 & -n_1 & 0 & \cdots & 0 & -n_N \\
\end{pmatrix}
\]

Proof: We remark that the \(N = 3\) instance is the calculation we worked out before Proposition 3.2. For any \(N \geq 3\) the coloring matrix is

\[
\begin{pmatrix}
  -n_1 & n_1 + n_2 & -n_2 & 0 & \cdots & 0 \\
  0 & -n_2 & n_2 + n_3 & -n_3 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  -n_N & -n_N & -n_N & \cdots & -n_N & n_N - n_N - 1 \\
  n_N + n_1 & -n_1 & 0 & \cdots & 0 & -n_N \\
\end{pmatrix}
\]

In each row (resp., column) the coefficients add up to zero. We then add all but the last column (resp., row) to the last column (resp., row). In this way we obtain a column (resp., row) of zero’s for the new last column (resp., row). We now add the first column to the second one, the resultant second column to the third one and so on and so forth to obtain

\[
\begin{pmatrix}
  -n_1 & n_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\
  0 & -n_2 & n_3 & 0 & \cdots & 0 & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & -n_{N-3} & n_{N-2} & 0 & 0 \\
  0 & 0 & 0 & \cdots & 0 & -n_{N-2} & n_{N-1} & 0 \\
  -n_N & -n_N & -n_N & \cdots & -n_N & n_N - n_N - 1 & 0 \\
  0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[\blacksquare\]

**Lemma 3.1** For an integer \(N > 2\) consider the \((N - 2) \times (N - 1)\) matrix

\[
\begin{pmatrix}
  -n_1 & n_2 & 0 & 0 & \cdots & 0 & 0 \\
  0 & -n_2 & n_3 & 0 & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & -n_{N-3} & n_{N-2} & 0 \\
  0 & 0 & 0 & \cdots & 0 & -n_{N-2} & n_{N-1} \\
\end{pmatrix}
\]

Its \(i\)-rowed minors are either zero or the product of the \(i\) distinct \(n_s\)’s from the corresponding square submatrix. Moreover, for each \(1 \leq i \leq N - 2\), and for each \(1 \leq s_1 < s_2 < s_3 < \cdots < s_i \leq N - 1\), there is at least one \(i\)-rowed minor of the indicated matrix equal to

\[n_{s_1} n_{s_2} n_{s_3} \cdots n_{s_i}\]
Proof: By induction on $i$. Let $i = 2$. Any $2 \times 2$ submatrix of the indicated matrix has at least one zero entry for otherwise there would be two consecutive rows with non-zero entries along two consecutive columns. Thus, any 2-rowed minor of the indicated matrix is either zero or the product of two $n_s$'s with distinct indices, for those with the same indices lie along the same column. Moreover, for each $1 \leq s < s' \leq N - 1$, each $n_sn_s'$ equals at least one of the 2-rowed minors, e.g.,

$$
\det \begin{pmatrix} -n_s & n_{s'} \\ 0 & -n_{s'} \end{pmatrix}
$$

where

$$
n_{s'} = \begin{cases} n_{s'}, & \text{if } |s - s'| = 1 \\ 0, & \text{if } |s - s'| > 1 \end{cases}
$$

Now suppose that for some $i$, for each $1 \leq j \leq i$, any $j$-rowed minor is either zero or the product of $j$ distinct $n_s$'s and, moreover, for any $1 \leq s_1 < s_2 < \cdots < s_j \leq N - 1$, the product $n_{s_1}n_{s_2} \cdots n_{s_j}$ is realized. Consider an $(i + 1) \times (i + 1)$ submatrix of the indicated matrix. If its determinant is not zero then it has at least one column with only one non-zero entry. Then this determinant equals this non-zero entry times an $i$-rowed minor. Further this $i$-rowed minor is the product of $i$ distinct $n_s$'s by the induction hypothesis and none of these $n_s$'s equals the indicated non-zero entry - for it was the only non-zero entry in its column.

Consider a sequence of $i + 1$ integers $1 \leq s_1 < s_2 < \cdots < s_i < s_{i+1} \leq N - 1$. Then

$$
n_{s_2} \cdots n_{s_i}n_{s_{i+1}}
$$

can be realized as a minor of some $i \times i$ submatrix say $M$ by the induction hypothesis. Then

$$
\begin{vmatrix} -n_{s_1} & * & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -n_{N-3} \\
0 & 0 & 0 & \cdots & 0 \\
-n_N & -n_N & -n_N & \cdots & -n_N \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0
\end{vmatrix} = n_{s_1}n_{s_2} \cdots n_{s_i}n_{s_{i+1}}
$$

modulo sign, where * is either 0 or $n_{s_1}$. This completes the proof.

\[\blacksquare\]

**Proposition 3.3** Given an integer $N > 2$ and integers $n_1,n_2,\ldots,n_N$ consider the $N \times N$ matrix

$$
\begin{pmatrix}
-n_1 & n_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & -n_2 & n_3 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -n_{N-3} & n_{N-2} & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & -n_{N-2} & n_{N-1} & 0 \\
-n_N & -n_N & -n_N & \cdots & -n_N & -n_N & -n_N & -n_{N-1} \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix}
$$

Then,

$$
\Delta_N = 0
$$

$$
\Delta_{N-1} = \sum_{1 \leq s_1 < s_2 < \cdots < s_{N-1} \leq N} n_{s_1}n_{s_2} \cdots n_{s_{N-1}}
$$

$$
\Delta_i = (n_{s_1}n_{s_2} \cdots n_{s_{i-1}})_{1 \leq s_1 < s_2 < \cdots < s_i \leq N}, \quad \text{for } 1 \leq i \leq N - 2
$$

Proof: $\Delta_N = 0$ since there is one row of zeroes (and one column of zeroes) in the indicated $N \times N$ matrix.

There is only one $(N - 1) \times (N - 1)$ submatrix, call it $M$, with non-zero determinant. We calculate the determinant of $M$ by Laplace expansion on the $(N - 1)$-th row ($|M|_{N-1,s}$ stands for the $N - 1, s$
concludes the proof. We see though from the symmetry of this knot that there is no preferred twist. This would indicate that there would be something special about the factors $n_1 \cdots n_{s-1} n_N$ that this product includes factors $n_i$ that are non-zero entries. Then

$$
\sum_{s=1}^{N-2} (-n_N)(-1)^{N-1+s}|M|_{N-1,s} + (-n_N - n_{N-1})(-1)^{N-1+N-1}|M|_{N-1,N-1} = 
$$

$$
= \sum_{s=1}^{N-2} n_N(-1)^{N+s}(-n_1)(-n_2)\cdots(-n_s) n_{s+1}\cdots n_{N-1} + (-1)^1(n_N + n_{N-1})(-1)^{N-2}n_1n_2\cdots n_{N-2} = 
$$

$$
= \sum_{s=1}^{N-2} (-1)^{N+s}(-1)^{s-1}n_1\cdots n_s n_{N-1}n_N + (-1)^{N-1} \left( n_1\cdots n_{N-2}n_N + n_1\cdots n_{N-2}n_{N-1} \right) = 
$$

$$
= (-1)^{N-1} \sum_{s=1}^{N} n_1n_2\cdots n_s n_{s+1}\cdots n_{N-1}n_N
$$

We will now investigate the possible contributions of the $i$-rowed minors to $\Delta_1$, for an otherwise arbitrary $1 \leq i \leq N - 2$. Note that, in the original matrix, $n_i$ is found only along column $s$, for each $1 \leq s \leq N - 1$, $n_s$, while $n_N$ is found only along row $N - 1$. In this way, $i$-rowed minors are either zero, the product of $i$'s no two of them with the same index, or sums of such products. Thus, these non-zero $i$-rowed minors are generated by products of $i$'s no two of them with the same index. We will now prove that either such a product equals a specific $i$-rowed minor or is a linear combination of some of these $i$-rowed minors, in this way concluding the proof. Note that products not involving $n_N$ are dealt with in Lemma 3.1. So here we will just consider products involving $n_N$.

We first consider products of $i$'s no two of them having the same index such that one of them has index $N$ and none of them has index $1$ i.e., consider the sequence of indices

$$2 \leq s_1 < s_2 < \cdots < s_{i-1} < s_i = N$$

Then

$$n_{s_1}n_{s_2}\cdots n_{s_{i-1}}n_N$$

is realized, modulo sign, by the determinant of the following $i \times i$ submatrix of the indicated matrix

$$
\begin{pmatrix}
0 & -n_{s_1} & * & \cdots & 0 \\
0 & 0 & -n_{s_2} & \cdots & 0 \\
0 & 0 & 0 & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -n_{s_{i-1}} \\
-n_N & -n_N & -n_N & \cdots & -n_N \\
\end{pmatrix}
$$

where * is a possible non-zero entry.

Assume now that each product of $i$'s no two of them with the same index, involve the factors $n_1$ and $n_N$ but do not involve the factor $n_{N-1}$. Then

$$n_1n_{s_2}\cdots n_{s_{i-1}}n_N$$

(where $1 < s_2 < \cdots < s_{i-1} < N - 1$) is realized modulo sign and modulo addition or subtraction of a product obtained in Lemma 3.1, by the determinant of the following $i \times i$ submatrix of the indicated matrix

$$
\begin{pmatrix}
-n_1 & * & 0 & \cdots & 0 \\
0 & -n_{s_2} & * & \cdots & 0 \\
0 & 0 & \ddots & \vdots \\
0 & 0 & \cdots & -n_{s_{i-1}} & 0 \\
-n_N & -n_N & \cdots & -n_{s_{i-1}} & -n_N - n_{N-1} \\
\end{pmatrix}
$$

This proof is now complete except for products of $i$'s no two of them with distinct indices, such that this product includes factors $n_{N-1}$ and $n_{N}$. If there were a formally distinct expression for these products, this would indicate that there would be something special about the $(N - 1)$-th twist in the pretzel knot. We see though from the symmetry of this knot that there is no preferred twist. This concludes the proof.
**Theorem 3.1** Given positive integers \( N > 2 \) and \( n_1, \ldots, n_N \), consider the Pretzel knot \( P(n_1, \ldots, n_N) \) - see Figure 11. Its coloring matrix is equivalent to an \( N \times N \) diagonal matrix whose \( i \)-th entry along the diagonal is

\[
\begin{cases}
\Delta_1, & i = 1 \\
\Delta_i/\Delta_{i-1}, & 2 \leq i \leq N
\end{cases}
\]

where

\[
\Delta_i = (n_{s_1}n_{s_2}\cdots n_{s_i})_{1 \leq s_1 < s_2 < \cdots < s_i \leq N},
\quad 1 \leq i \leq N - 2
\]

\[
\Delta_{N-1} = \sum_{1 \leq s_1 < s_2 < \cdots < s_{N-1} \leq N} n_{s_1}n_{s_2}\cdots n_{s_{N-1}}
\]

\[
\Delta_N = 0
\]

Proof: This is a straightforward consequence of Propositions 3.2 and 3.3 and Lemma 3.1.

Pretzel knots constitute an interesting class of knots when it comes to coloring matrices. Rational knots are poor in this sense. As a matter of fact, the coloring matrices of rational knots reduce to \( 2 \times 2 \) matrices i.e., their reduced coloring matrices have only one entry, the determinant of the knot (see [10]).

### 4 Seifert surfaces and the Alexander module

In this Section we address another invariant of knots, the so-called Alexander module of a knot. We outline the basic facts pertaining to the Alexander module below; we elaborate on them in the next subsection.

The Alexander module is presented by a(n equivalence class of) matrix(ies) of the form \( tS - ST \), where \( S \) is a Seifert matrix and \( t \) is an indeterminate. A Seifert matrix is obtained from a Seifert surface of the knot under study. A Seifert surface of a knot is any connected orientable surface whose boundary is this knot. Consider then the generators of the first homology group of such a surface. Consider also their translates by pushing off each generator slightly along the direction of the normal to the surface. The Seifert matrix, \( S \), is then the matrix whose elements are the linking numbers between generators and their translates. It is a theorem ([11]) that the matrix \( tS - ST \) is a presentation matrix for the Alexander module, which is, further, a knot invariant. In particular, the determinant of this matrix is a knot invariant, the so-called Alexander polynomial.

In this Section we concentrate on computational aspects of this presentation matrix. Our working examples will be subclasses of pretzel knots. We will partially recover work of Parris ([14]) on pretzel knots with an odd number of tassels, each tassel with an odd number of crossings. On the other hand we will also look into a subclass of pretzel knots not considered by Parris.

In [7], Kauffman outlined a way of simplifying the calculation of the Seifert matrix. We have the same goal in mind in the current Section although we believe we came up with a different approach. In order to achieve our goal, we look for a simpler form of the Seifert surface so that the generators of the first homology group are easily identified and the linking with their translates easily calculated.

#### 4.1 Definitions of Seifert surface and Seifert matrix

We start with the Definition of Seifert surface followed by the proof of its existence, for any knot.

**Definition 4.1 (Seifert surface of a knot)** A Seifert surface of a knot is a connected, orientable surface whose boundary is the given knot.
Proposition 4.1  There is at least one Seifert surface for each knot.

Proof: We consider a knot diagram of the given knot and orient it. We smooth each crossing as shown in Figure 13. Upon the performance of these smoothings, the knot diagram becomes a finite sequence of closed curves. We cap off each of these closed curves with discs and connect them at the crossings with half-twists. Each of these half-twists has to be consistent with the corresponding crossing so the final result is an orientable surface whose boundary is the given knot. If this surface is not connected, connect any two distinct components by cutting one small disc on each of them and gluing the boundaries of a thin tube to the boundaries of the removed discs. This completes the proof.

Figure 14 illustrates this procedure for the trefoil knot. The different shadings represent the different sides of the surface. As a matter of fact, note that the orientation of the knot induces an orientation of the normal to the Seifert surface by the right-hand rule. It is this orientation of the normal that we will consider in the Seifert surfaces, in the sequel.

Definition 4.2  Given a knot endowed with an orientation and one of its Seifer t surfaces, the orientation of the normal to this Seifert surface is induced by the orientation of the knot and the right-hand rule (see Figure 16).

We now proceed to define “Signs at crossings” and “Linking numbers” in order to define a Seifert matrix of a diagram.

Definition 4.3 (Signs at crossings) We define signs at crossings of an oriented knot diagram as shown in Figure 15.

Definition 4.4 (Linking Number) Suppose $l_1$ and $l_2$ are two knots and consider any diagram where both $l_1$ and $l_2$ are depicted. The linking number, $lk(l_1, l_2)$ of $l_1$ and $l_2$ is half the sum of the signs at the crossings where one strand is from $l_1$ and the other one is from $l_2$. The linking number is a topological invariant ([1, 6]).
Definition 4.5 (Seifert Matrix) Given a Seifert surface, $F$, consider a set of generators of the first homology group of $F$, say $\{l_1, \ldots, l_g\}$. For any $i \in \{1, \ldots, g\}$, let $l_i^+$ denote the closed curve obtained by pushing slightly $l_i$ in the direction of the normal to the Seifert surface. Finally, let

$$l_{ij} := \text{lk}(l_i, l_j^+)$$

The Seifert matrix of the given Seifert surface is the square $g \times g$ matrix whose $i, j$ entry is $l_{ij}$. We will usually denote a Seifert matrix by the letter $S$.

We calculate a Seifert matrix for the diagram of our previous example, the trefoil (see also Figures 16 and 17):

$$S = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$$

A Seifert matrix, per se, is not a topological invariant. On the other hand via the Seifert matrix one obtains a topological invariant. In fact
Theorem 4.1 Consider a knot $K$ along with a Seifert matrix of $K$, say $S$. Let $t$ be an indeterminate. Then the matrix $tS - S^T$ is a presentation matrix for the so-called Alexander module of $K$ which is a topological invariant of $K$ ($S^T$ denotes the transpose of $S$).

Proof: Omitted. See [11].

Given a presentation matrix of a module, its elementary ideals are defined as follows.

Definition 4.6 ([2, 11]) Consider an $m \times n$ presentation matrix, $A$, of a module $M$ over a commutative ring $R$. The $r$-th elementary ideal $I_r$ of $M$ is the ideal of $R$ generated by all the $(m - r + 1) \times (m - r + 1)$ minors of $A$.

Elementary ideals of a module and the generators (modulo multiplication by units) of the principal ideals contained in them are invariants of the module at issue ([2]). In this way the following are also topological invariants of knots.

Definition 4.7 ([2, 11]) The $r$-th Alexander ideal of an oriented link $L$ is the $r$-th elementary ideal of the $\mathbb{Z}[t, t^{-1}]$ Alexander module. The $r$-th Alexander polynomial of $L$ is a generator of the smallest principal ideal of the Alexander module that contains the $r$-th Alexander ideal. The first Alexander polynomial is called the Alexander polynomial. We remark that the Alexander polynomials are unique up to multiplication by units, $\pm t^n$.  

Figure 17: A pictorial representation of a Seifert matrix of the trefoil
Resuming the study of the trefoil, we recall that a Seifert matrix for a diagram of the trefoil read:

$$S = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$$

and so a presentation matrix for the Alexander module of the trefoil is

$$tS - ST = \begin{pmatrix} -t + 1 & -1 \\ t & -t + 1 \end{pmatrix}$$

Therefore, the Alexander polynomial of the trefoil is:

$$\det(tS - ST) = t^2 - t + 1$$

### 4.2 Simplifying Seifert surfaces

The purpose of this Subsection is to suggest a way of simplifying the calculation of the Alexander polynomial (and if possible the other Alexander polynomials) by simplifying the Seifert surface. Given a knot diagram of the knot under study we calculate the associated Seifert surface by the algorithm in Proposition 4.1. We then proceed to simplifying this Seifert surface by reducing it via deformations to the form “disc plus ribbons” (see Definition 4.8 below). We note that in this way each ribbon gives rise to a generator for the first homology group of the Seifert surface (see below). In the next Subsections we will use the pretzel knots to illustrate the benefits of these ideas.

**Definition 4.8 (Disc plus ribbons)** By **disc** we mean the standard disc on the plane or one of its isotopes in 3-space. By **ribbon** we mean a rectangle (or one of its isotopes in 3-space), although we will prefer rectangles such that two parallel sides are significantly shorter than the other sides. A **disc with ribbons** or **disc plus ribbons** is then a disc with a set of ribbons attached to it along the boundaries. Each ribbon has both short sides attached to the disc; the long sides of the ribbons are not attached to anything (see Figures 26 and 40). We remark that the ribbons may be twisted.

**Definition 4.9 (Seifert surface in standard form)** A **Seifert surface** is said in **standard form** if it is in a disc plus ribbons form. Figures 26 and 40 provide illustrative examples.

In general a Seifert surface consists of several discs with ribbons connecting them. Our strategy in transforming this Seifert surface into a Seifert surface in standard form will be to choose one disc and to merge the other discs into this one by shrinking some ribbons. The preferred disc will, in general, be one with more ribbons connecting to it.

Once a Seifert surface (or for that matter, any surface) is in standard form it is immediate to obtain a set of generators of the first homology group of the surface. Each ribbon will give rise to a generator in the following way. Consider this ribbon augmented by a second ribbon embedded in the disc of the standard form of the Seifert surface we are working with. This second ribbon connects the short sides of the first ribbon, these connecting short sides of both ribbons being of identical size. The ensemble of these two ribbons constitutes a closed ribbon which retracts to a circle embedded in three dimensions. This circle is the generator of the first homology group we associate with the ribbon we first considered. The retract mentioned above is realized by shrinking the ensemble of the two ribbons along the short side. See dotted lines in Figure 27.

**Definition 4.10 (Standard generators of first homology group of a standard Seifert surface)** Given a Seifert surface in standard form, we call the generators of the first homology group as described in the preceding paragraph, the **standard generators** of this standard Seifert surface.
4.3 Calculations: pretzel knots on three tassels, each with an odd number of crossings

As a working example for the ideas described in the preceding Subsection, we consider a pretzel knot on three tassels, each with an odd number of crossings, $P(2i_1 + 1, 2i_2 + 1, 2i_3 + 1)$, see Figure 18. In Figures 19 and 20 we use the algorithm described in Proposition 4.1 to obtain a Seifert surface in Figure 21. We move the third tassel up (Figure 22) and we rotate the disc in the center of Figure 23 to undo the twisting in the third tassel. As a consequence we merge this disc into the other one giving rise to a Seifert surface in standard form (Figure 26). This produces also a braiding of the first two tassels and an increase in their crossings of $2i_3 + 1$.

With the Seifert surface in standard position, we easily identify the standard generators of the first homology group of this Seifert surface (see Figure 27): we associate one generator of the first homology group of the Seifert surface to each ribbon (and vice-versa). We then calculate the linking numbers between generators and their translates as illustrated in Figures 28 and 29. Note that the relevant crossings between generator and translate of generator for this calculation are those shown in Figure 28. As for Figure 29 there is one relevant crossing missing. In fact, in this case the two dotted lines will go over one another one more time, after they leave the top right part of this Figure. For $lk(l_i, l_j^+)$, it is $l_i^-$ that goes over $l_1$; for $lk(l_i, l_j^+$) it is $l_i^+$ that goes over $l_2$.

The Seifert matrix is then:

$$S = \begin{pmatrix} i_1 + i_3 + 1 & i_3 + 1 \\ i_3 & i_2 + i_3 + 1 \end{pmatrix}$$

and the presentation matrix of the Alexander module for the pretzel knot $P(2i_1 + 1, 2i_2 + 1, 2i_3 + 1)$ is

$$tS - ST = \begin{pmatrix} (i_1 + i_3 + 1)(t - 1) & (i_3 + 1)t - i_3 \\ i_3t - (i_3 + 1) & (i_2 + i_3 + 1)(t - 1) \end{pmatrix}$$

which is equivalent to the following matrix:

$$\begin{pmatrix} i_1t - (i_1 + 1) & (i_3 + 1)t - i_3 \\ i_2 - (i_2 + 1)t & (i_2 + i_3 + 1)(t - 1) \end{pmatrix}$$

It then follows that the Alexander polynomial of $P(2i_1 + 1, 2i_2 + 1, 2i_3 + 1)$ is

$$\det(tS - ST) = i_1t - (i_1 + 1)(i_2 + i_3 + 1)(t - 1) - (i_2 - (i_2 + 1)t)((i_3 + 1)t - i_3)$$

We record here two results implicit in the preceding calculations which will be useful in subsequent calculations.

**Proposition 4.2 (The $lk(l_i, l_j^+)$ linking numbers)** Let $L_i$ be the ribbon which gives rise to the $l_i$ generator of the first homology group of the surface. The contribution to $lk(l_i, l_j^+)$ of the ribbon $L_i$ as depicted in Figure 28 is one half the number of the crossings in the Figure. If each of these crossings were reversed then it would be negative one half the number of these crossings.

Proof: Omitted.

**Proposition 4.3 (The $lk(l_i, l_j^+)$ linking numbers)** Let $L_i$ and $L_j$ be the ribbons which give rise to the $l_i$ and $l_j$ generators of the first homology group of the surface ($i \neq j$). The contribution to $lk(l_i, l_j^+)$ and to $lk(l_i, l_j^+)$ from the braiding of $L_i$ and $L_j$ as depicted in Figure 29 is one half the number of crossings of these two ribbons, $L_i$ and $L_j$. If each of these crossings were reversed then it would be negative one half the number of these crossings.

Proof: Omitted.
Figure 18: Pretzel knot on three tassels, each with an odd number of crossings, $P(2i_1 + 1, 2i_2 + 1, 2i_3 + 1)$
Figure 19: Towards constructing a Seifert surface of the $P(2i_1 + 1, 2i_2 + 1, 2i_3 + 1)$ (1)
Figure 20: Towards constructing a Seifert surface of the $P(2i_1 + 1, 2i_2 + 1, 2i_3 + 1)$ (2)
Figure 21: A Seifert surface of the $P(2i_1 + 1, 2i_2 + 1, 2i_3 + 1)$ (1)
Figure 22: A Seifert surface of $P(2i_1 + 1, 2i_2 + 1, 2i_3 + 1)$ (2)
Figure 23: A Seifert surface of $P(2i_1 + 1, 2i_2 + 1, 2i_3 + 1)$ (3)
Figure 24: A Seifert surface of \( P(2i_1 + 1, 2i_2 + 1, 2i_3 + 1) \)
Figure 25: A Seifert surface of $P(2i_1 + 1, 2i_2 + 1, 2i_3 + 1)$ (5)
Figure 26: A Seifert surface of $P(2i_1 + 1, 2i_2 + 1, 2i_3 + 1)$ - the standard form
Figure 27: The standard form with the standard generators for the homology
\[ \text{lk}(l_1, l_1^+) = (i_1 + i_3) + 1 \]

\[ \text{lk}(l_2, l_2^+) = (i_2 + i_3) + 1 \]

Figure 28: The \( \text{lk}(l_i, l_i^+) \) linking numbers
\[ \text{lk}(l_1, l_2^+) = i_3 + 1 \quad \text{lk}(l_2, l_3^+) = i_3 \]

Figure 29: The $\text{lk}(l_1, l_2^+)$ linking numbers
4.4 Calculations: pretzel knots on $N$ tassels, each with an odd number of crossings

We will now consider a pretzel knot on $N$ tassels, each with an odd number of crossings. The next Proposition will distinguish two types in this class of knots.

**Proposition 4.4** Consider a pretzel knot on $N$ tassels, each with an odd number of crossings. If $N$ is odd this is a 1-component knot. If $N$ is even this is a 2-component knot.

Proof: Since each tassel has an odd number of crossings, then starting from the top right (respect., left) side of the tassel and going along the knot down this tassel we end up at the bottom left (respect., right). Analogously when going along the knot and up each tassel. So, suppose the orientation of the knot takes us down from the top left of the $N$-th tassel to the bottom right of this tassel, which is the bottom left of the $(N-1)$-th tassel. Then we will be taken up along the $(N-1)$-th tassel to its top left. Iterating this procedure we will get to the first tassel after going through each one only once. If $N$ is odd then we will get to the bottom right of the first tassel which is the bottom left of the $N$-th tassel. A second trip along the knot will take us to the the top right of the first tassel which is the top left of the $N$-th tassel. In this way we went along the entire knot and came back to the starting point for odd $N$. If $N$ is even, then after going once along the knot, going through each tassel only once, we will get to the top right of the first tassel, which is the top left of the $N$-th tassel. Then we will have reached the starting point without having gone through the entire knot. If we now start at the top right of the $N$-th tassel, for even $N$, we will go through the second component of this knot. This completes the proof. ■

**Corollary 4.1** A pretzel knot with an even number of tassels, each with an odd number of crossings has four possible distinct orientations of its two components.

Proof: Omitted. ■

**Definition 4.11** We will always choose an orientation of the pretzel knot such that, in the first tassel, the over-arc of the first crossing is oriented downwards and the over-arc of the second crossing is oriented upwards.

Consider now a pretzel knot on $N$ tassels, each tassel with an odd number of crossings, $P(2i_1+1, 2i_2+1, 2i_3+1, \ldots, 2i_N+1)$. Starting from the defining diagram, we move the $N$-th tassel upwards as we did in the $N = 3$ instance. Then we undo the twists in this tassel by producing a braiding of the remaining tassels and increasing the number of crossings in each of the remaining tassels by the original number of crossings in the $N$-th tassel (see case $N = 2$). After this procedure we obtain a Seifert surface in standard form. Each of the $N-1$ braided ribbons represents a standard generators of the first homology group of this Seifert surface. Moreover, the linking numbers between standard generators and their translates are analogous to those calculated for the $N = 3$ case. As a matter of fact, in the braiding of these $(N-1)$ ribbons, the braiding of two distinct of them is like the braiding of ribbons (tassels) 1 and 2 in case $N = 3$ (just remove the other ribbons (tassels) in case $N > 3$). In this way the linking numbers are, for $j, k = 1, \ldots, N-1$,

$$\text{lk}(l_j, l_k^+) = \begin{cases} 
  i_N + 1, & \text{if } j < k \\
  i_j + i_N + 1, & \text{if } j = k \\
  i_N, & \text{if } j > k
\end{cases}$$

The Seifert matrix is then the following $(N-1) \times (N-1)$ matrix

$$S = \begin{pmatrix}
  i_1 + i_N + 1 & i_N + 1 & i_N + 1 & \ldots & i_N + 1 \\
  i_N & i_2 + i_N + 1 & i_N + 1 & \ldots & i_N + 1 \\
  i_N & i_N & i_3 + i_N + 1 & \ldots & i_N + 1 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  i_N & i_N & \ldots & i_N & i_{N-1} + i_N + 1
\end{pmatrix}$$

34
The presentation matrix of the Alexander module of \( P(2i_1 + 1, 2i_2 + 1, 2i_3 + 1, \ldots, 2i_N + 1) \) is then

\[ tS - ST = \]

\[
\begin{pmatrix}
(i_1 + i_N + 1)(t - 1) & (i_N + 1)t - i_N & (i_N + 1)t - i_N & \ldots & (i_N + 1)t - i_N \\
(i_N - (i_N + 1)t) & (i_2 + i_N + 1)(t - 1) & (i_N + 1)t - i_N & \ldots & (i_N + 1)t - i_N \\
i_N - (i_N + 1)t & i_N - (i_N + 1)t & (i_3 + i_N + 1)(t - 1) & \ldots & (i_N + 1)t - i_N \\
i_N - (i_N + 1)t & i_N - (i_N + 1)t & i_N - (i_N + 1)t & \ldots & i_N - (i_N + 1)t \\
i_N - (i_N + 1)t & i_N - (i_N + 1)t & i_N - (i_N + 1)t & \ldots & i_N - (i_N + 1)t + (i_N + 1)(t - 1)
\end{pmatrix}
\]

By subtracting the second column from the first column, the third from the second, \ldots, the \( N \)-th from the \((N-1)\)-th we obtain the equivalent matrix

\[
\begin{pmatrix}
i_1t - (i_1 + 1) & 0 & 0 & \ldots & 0 & (i_N + 1)t - i_N \\
i_2 - (i_2 + 1)t & i_2t - (i_2 + 1) & 0 & \ldots & 0 & (i_N + 1)t - i_N \\
i_3t - (i_3 + 1) & i_3t - (i_3 + 1) & \ldots & 0 & (i_N + 1)t - i_N \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & i_{N-2}t - (i_{N-2} + 1) & (i_N + 1)t - i_N \\
0 & 0 & 0 & \ldots & i_{N-1}t - (i_{N-1} + 1) & (i_N + 1)(t - 1)
\end{pmatrix}
\]

By doing Laplace expansion over the last column we obtain the following expression for \( \det(tS - ST) \), the Alexander Polynomial of \( P(2i_1 + 1, 2i_2 + 1, \ldots, 2i_N + 1) \):

\[
(i_{N} + 1)t - i_{N} \cdot \sum_{k=1}^{N-2} (-1)^k \cdot \prod_{j=1}^{k-1} (i_jt - (i_j + 1)) \cdot \prod_{j=k+1}^{N-2} (i_j - (i_j + 1)t) +
\]

\[+(-1)^{N-1}(i_{N-1} + i_{N} + 1)(t - 1) \cdot \prod_{k=1}^{N-2} (ikt - (ik + 1))\]

where a \((-1)^{N-1}\) factor has been omitted throughout.

4.5 Calculations: pretzel knots on four tassels, exactly one with an even number of crossings

In this Subsection we work out the calculations for a Pretzel knot on four tassels with exactly one tassel with an even number of crossings. This is intended to pave the way for the calculations for a pretzel knot with an arbitrary even number of tassels and with exactly one tassel with an even number of crossings which will done in the next Subsection.

We choose pretzel knot \( P(5, 3, 7, 4) \) to do our calculations (see Figure 33). A Seifert surface for this pretzel knot is obtained as before using the algorithm in Proposition 4.1 (see Figure 31). Only one side of the Seifert surface will be shaded in order not to blur the interpretation of the Figures later on. We remark that on the shaded side of the Seifert surface the normal points to the reader. Our strategy here will be first to transfer the twists in the ribbons from left to right, by deformations; the twists will ultimately accumulate in the ribbons to the far right. This will, however, produce a braiding of the ribbons. Then we will shrink some of these ribbons in order to merge the center discs into the larger disc. We will finally obtain a Seifert surface in standard form. We now specify each step of this procedure.
Figure 32 is obtained by stretching the central portion of Figure 31 so that the ribbons connecting the different discs are clearly identified. Then the twists in the ribbons to the left in Figure 32 are transferred to the right by a $\frac{\pi}{2}$ rotation of the portion of the diagram boxed by the dotted rectangle. The result of this rotation is shown in Figure 33. This Figure is further obtained from Figure 32 by rotation of the whole by a $\frac{\pi}{2}$ clockwise on the plane of the page. The upper twists in ribbons in Figure 33 are transferred down by $\frac{\pi}{2}$ rotation of the portion of the diagram boxed by the dotted rectangle. The result of this rotation is shown in Figure 34. The upper twists in ribbons in Figure 34 are transferred down using the same technique. The result is shown in Figure 35 where the twists in ribbons are all in the lower portion of this Figure. As of this Figure we proceed to merge the smaller discs into the larger one mainly by shrinking specific ribbons.

In Figure 35 we identify three vertices of the diagram by the letters $A$, $B$, and $C$. These are the vertices to the immediate left of each of these letters. The arrows to the right of these letters indicate that a deformation of the surface will be performed by, also, pushing on the corresponding vertices. The same letters in Figure 36 indicate the new location of the corresponding points after the deformation. Furthermore, in this Figure, there is a dotted and oriented arc inside a ribbon, in the upper part of the diagram. This indicates that the corresponding ribbon will be shrunk in the indicated direction thus merging the disc about point $A$ into the larger disc. The result of this operation is shown in Figure 37. In this Figure, the dotted and oriented arc inside a ribbon indicates the same operation on the corresponding ribbon. The result of this operation is shown in Figure 38. Without further remarks, in Figure 39 we show the result of the merging of the last small disc into the larger one. After some deforming we obtain a standard form in Figure 40. Here we enumerate the ribbons as indicated in the top of the diagram. We thus obtain an enumeration of the standard generators of the first homology group of this surface (see Definition 4.10). We orient these generators counterclockwise.

Consider again the Seifert surface in standard form depicted in Figure 40. Since each ribbon is twisted, each standard generator links with its translate. Thus, every diagonal element of the Seifert matrix will be non-null. Except for the four generators with numbers 13, 14, 15 and 16 which potentially link with every other generator, the remaining generators exhibit linking within given subsets of generators. It is further clear that these subsets of generators are directly related to definite tassels in $P(5, 3, 7, 4)$. Specifically, the four generators with numbers 13, 14, 15, 16, stem from the only tassel with the even number of crossings (four crossings). The subset of generators 7, 8, 9, 10, 11, 12 stems from the tassel with seven crossings; the subset of generators 5, 6 stems from the tassel with three crossings; and the subset of generators 1, 2, 3, 4 stems from the tassel with five crossings. In the general case of a pretzel knot with (even) $N$ tassels exactly one with an even number of crossings, there will be, after the procedure described before, a subset of standard generators for the first homology group with as many elements as there are crossings in the tassel these generators stem from (we call this tassel the reference tassel); the remaining tassels will generate subsets of standard generators whose cardinality is the number of crossings of the tassel minus one. The cause for this “minus one” difference has to do with the shrinking of one of the ribbons in the passages from Figure 36 to Figure 37, 37 to 38, and 38 to 39. It is clear that the reference tassel should be chosen to be the one with least number of crossings since it is the generators stemming from this tassel that link with each of the other generators.

Finally, invoking Propositions 4.2 and 4.3 we obtain the following Seifert matrix.
Figure 30: A diagram of $P(5, 3, 7, 4)$
Figure 31: A Seifert surface for $P(5, 3, 7, 4)$ (1)
Figure 32: A Seifert surface for $P(5, 3, 7, 4)$ (2)
Figure 33: A Seifert surface for \( P(5, 3, 7, 4) \) (3)
Figure 34: A Seifert surface for $P(5, 3, 7, 4)$ (4)
Figure 35: A Seifert surface for $P(5, 3, 7, 4)$ (5)
Figure 36: A Seifert surface for $P(5,3,7,4)$ (6)
Figure 37: A Seifert surface for $P(5, 3, 7, 4)$ (7)
Figure 38: A Seifert surface for $P(5, 3, 7, 4)$ (8)
Figure 39: A Seifert surface for $P(5, 3, 7, 4)$ (9)
Figure 40: A Seifert surface for $P(5, 3, 7, 4)$ - the standard form
We now calculate the presentation matrix for the Alexander module of $P(5, 3, 7, 4)$ from the Seifert matrix above. We will further use elementary transformations on matrices to pass on to a more convenient presentation matrix of this module, in order to calculate the Alexander polynomial.

$$
tS - S^T =
\begin{pmatrix}
1 - t & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & t & t & t \\
t & 1 - t & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & t & t & t \\
t & t & 1 - t & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & t & t & t \\
t & t & t & 1 - t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & t & t & t \\
0 & 0 & 0 & 0 & t & 1 - t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & t & t & t \\
0 & 0 & 0 & 0 & 0 & 0 & 1 - t & -1 & -1 & -1 & -1 & -1 & -1 & -1 & t & t & t & t \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & t & 1 - t & -1 & -1 & -1 & -1 & -1 & t & t & t & t \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & t & 1 - t & -1 & -1 & -1 & -1 & t & t & t & t \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & t & t & 1 - t & -1 & -1 & -1 & t & t & t & t \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & t & t & t & 1 - t & -1 & -1 & -1 & t & t & t & t \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 - 2 t & 1 - 2 t & 1 - 2 t & 1 - 2 t \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 - 2 t & 2 - 2 t & 2 - 2 t & 1 - 2 t \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 - 2 t & 2 - 2 t & 2 - 2 t & 2 - 2 t \\
\end{pmatrix}
$$

The next matrix is the result of subtracting, in the preceding matrix, the 14-th column from the 13-th column, the 15-th from the 14-th, and the 16-th from the 15-th.

$$
tS - S^T \rightarrow
\begin{pmatrix}
1 - t & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t \\
t & 1 - t & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t \\
t & t & 1 - t & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t \\
t & t & t & 1 - t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t \\
0 & 0 & 0 & 0 & 1 - t & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t \\
0 & 0 & 0 & 0 & 0 & t & 1 - t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t \\
0 & 0 & 0 & 0 & 0 & 0 & t & t & 1 - t & 0 & 0 & 0 & 0 & 0 & t \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & t & t & t & 1 - t & 0 & 0 & 0 & t \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & t & t & 1 - t & 0 & 0 & t \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & t & 1 - 2 t & 1 - 2 t \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & t & 1 - 2 t & 1 - 2 t \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & t & 2 - 2 t & 2 - 2 t \\
\end{pmatrix}
$$

The next matrix is the result of subtracting the 16-th row from the 13-th, the 14-th, and the 15-th rows of the preceding matrix.

$$
tS - S^T \rightarrow
\begin{pmatrix}
1 - t & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t \\
t & 1 - t & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t \\
t & t & 1 - t & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t \\
t & t & t & 1 - t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t \\
0 & 0 & 0 & 0 & 1 - t & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & t \\
0 & 0 & 0 & 0 & 0 & t & 1 - t & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & t \\
0 & 0 & 0 & 0 & 0 & 0 & t & t & 1 - t & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & t \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & t & t & t & 1 - t & -1 & -1 & -1 & 0 & 0 & 0 & 0 & t \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & t & t & 1 - t & -1 & -1 & 0 & 0 & 0 & 0 & t \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & t \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & t \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & t \\
\end{pmatrix}
$$

The next matrix is the result of subtracting the second row from the first, the third from the second, and the fourth from the third; the sixth from the fifth; the eighth from the seventh, the ninth from the eighth, the tenth from the ninth, the eleventh from the tenth, and the twelfth from the eleventh, in the preceding matrix.
The next matrix is the result of subtracting the second column from the first, the third from the second, and the fourth from the third; the sixth from the fifth; the eighth from the seventh, the ninth from the minors, we obtain

\[
\begin{vmatrix}
-2 & t & t & t & 1-t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & t -2 -2t
\end{vmatrix}
\]

Using Laplace’s expansion over the last row and conveniently relocating the last column to compute the minors, we obtain

\[
(-1)(-1)\det\begin{vmatrix}
3-3t & t-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & t -2 -2t
\end{vmatrix}
\]

\[
+ (-1)(-1)\det\begin{vmatrix}
3-3t & t-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & t -2 -2t
\end{vmatrix}
\]

\[
+ (-1)(-1)\det\begin{vmatrix}
3-3t & t-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & t -2 -2t
\end{vmatrix}
\]

\[
+ t(-1)\det\begin{vmatrix}
3-3t & t-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & t -2 -2t
\end{vmatrix}
\]

\[
+ (2 - 2t)\det\begin{vmatrix}
3-3t & t-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & t -2 -2t
\end{vmatrix}
\]

\[
= 2 + 192t + 972t^2 - 12289t^3 + 49274t^4 - 120582t^5 + 213765t^6 - 295426t^7 + 328185t^8 - 295426t^9 + 213765t^{10} - 120582t^{11} + 49274t^{12} - 12289t^{13} + 972t^{14} + 192t^{15} + 2t^{16}
\]
From this matrix it is now clear how to obtain the matrix for any given even number of tassels such that one of the tassels has an even number of crossings.

### 4.6 Calculations: pretzel knots on \( N \) tassels, exactly one with an even number of crossings

Let us assume that there are \( N \) tassels, for a given even positive integer \( N \), and that the \( N \)-th tassel has \( 2i_N \) crossings (\( i_N > 1 \)), and each of the remaining tassels have \( 2i_k + 1 \) tassels, for \( 1 \leq k \leq N-1 \) (\( i_k \)'s > 0). The presentation matrix of the Alexander module of the Pretzel knot \( P(2i_1+1, 2i_2+1, \ldots, 2i_{N-1}+1, 2i_N) \) is equivalent to the sum of the following two \( (\sum_{i=1}^{N} 2i_j) \times (\sum_{j=1}^{N} 2i_j) \) matrices, \( M_1 \) and \( M_2 \).

\( M_1 \) is a matrix whose last column and last row have non-zero entries, all other entries being zero. The only non-zero entries of the last row are \(-1\)'s at the \( \sum_{j=1}^{k} 2i_j \) positions, for \( 1 \leq k \leq N-1 \). The only non-zero entries of the last column are \( t \)'s at the \( \sum_{j=1}^{k} 2i_j \) positions, for \( 1 \leq k \leq N-1 \).

\( M_2 \) is a matrix in block-diagonal form. Each of the blocks is a \( 2i_k \times 2i_k \) matrix, for \( 1 \leq k \leq N \). For \( 1 \leq k \leq N-1 \), the \( k \)-th block is of the following type:

\[
B_k = \begin{pmatrix}
-3 + 3t & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & 0 \\
2t - 1 & 3 - 3t & t - 2 & \cdots & \cdots & 0 & 0 & 0 & 0 \\
0 & 2t - 1 & 3 - 3t & t - 2 & \cdots & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & 0 & 2t - 1 & 3 - 3t & t - 2 & 0 & 0 \\
0 & 0 & \cdots & \cdots & 0 & 2t - 1 & 3 - 3t & t - 2 & 0 \\
0 & 0 & 0 & \cdots & \cdots & 0 & 2t - 1 & 3 - 3t & t - 2 \\
0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 2t - 1 & 1 - t \\
\end{pmatrix}
\]

The \( N \)-th block is of the following type:

\[
B_N = \begin{pmatrix}
1 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & -t & -1 \\
t & 1 & 0 & \cdots & \cdots & 0 & 0 & -t & -1 \\
0 & t & 1 & 0 & \cdots & \cdots & 0 & -t & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & 0 & t & 1 & 0 & -t & -1 \\
0 & 0 & \cdots & \cdots & 0 & t & 1 & -t & -1 \\
0 & 0 & 0 & \cdots & \cdots & 0 & t & 1 - t & -1 \\
0 & 0 & 0 & 0 & \cdots & \cdots & 0 & t & I - It \\
\end{pmatrix}
\]

where \( I \) is such that \( N = 2I \).

The Alexander polynomial of \( P(2i_1+1, 2i_2+1, \ldots, 2i_{N-1}+1, 2i_N) \) is the determinant of \( M_1 + M_2 \). We compute it by doing Laplace’s expansion on the last row of \( M_1 + M_2 \). In order to do that the following matrices will be helpful. For \( 1 \leq k \leq N-1 \), \( B_k' \) is a \( 2i_k \times 2i_k \) matrix; it is obtained by replacing the last column of \( B_k \) by a column with a \( t \) in the last entry and otherwise zero. \( B_N' \) is a \( (2i_N - 1) \times (2i_N - 1) \) matrix; it is obtained by removing the last column and the last row from \( B_N \). \( B_N' \) is also a \( (2i_N - 1) \times (2i_N - 1) \) matrix; it is is obtained by removing the column before the last one and the last row from \( B_N \).
The Alexander polynomial is then:

\[
B'_k = \begin{pmatrix}
3 - 3t & t - 2 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 \\
2t - 1 & 3 - 3t & t - 2 & 0 & \cdots & \cdots & 0 & 0 & 0 \\
0 & 2t - 1 & 3 - 3t & t - 2 & 0 & \cdots & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \cdots & \ddots & \cdots & \cdots & \vdots \\
0 & \cdots & \cdots & 0 & 2t - 1 & 3 - 3t & t - 2 & 0 & 0 \\
0 & 0 & \cdots & \cdots & 0 & 2t - 1 & 3 - 3t & t - 2 & 0 \\
0 & 0 & 0 & \cdots & \cdots & 0 & 2t - 1 & 3 - 3t & 0 \\
0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 2t - 1 & t
\end{pmatrix}
\]

\[
B'_N = \begin{pmatrix}
1 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & -1 \\
t & 1 & 0 & 0 & \cdots & \cdots & 0 & 0 & -1 \\
0 & t & 1 & 0 & \cdots & \cdots & 0 & 0 & -1 \\
\vdots & \vdots & \vdots & \ddots & \cdots & \ddots & \cdots & \cdots & \vdots \\
0 & \cdots & \cdots & 0 & t & 1 & 0 & 0 & -1 \\
0 & 0 & \cdots & \cdots & 0 & t & 1 & 0 & -1 \\
0 & 0 & 0 & \cdots & \cdots & 0 & t & 1 & -1 \\
0 & 0 & 0 & 0 & \cdots & \cdots & 0 & t & -1
\end{pmatrix}
\]

\[
B''_N = \begin{pmatrix}
1 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & -t \\
t & 1 & 0 & 0 & \cdots & \cdots & 0 & 0 & -t \\
0 & t & 1 & 0 & \cdots & \cdots & 0 & 0 & -t \\
\vdots & \vdots & \vdots & \ddots & \cdots & \ddots & \cdots & \cdots & \vdots \\
0 & \cdots & \cdots & 0 & t & 1 & 0 & 0 & -t \\
0 & 0 & \cdots & \cdots & 0 & t & 1 & 0 & -t \\
0 & 0 & 0 & \cdots & \cdots & 0 & t & 1 & -t \\
0 & 0 & 0 & 0 & \cdots & \cdots & 0 & t & -1
\end{pmatrix}
\]

The Alexander polynomial is then:

\[
\det B''_N \sum_{j=1}^{N-1} \det B'_j \prod_{l=1, l \neq j}^{N-1} \det B_l - t \det B'_N \prod_{l=1}^{N-1} \det B_l + (I - It) \det B''_N \prod_{l=1}^{N-1} \det B_l
\]

We remark that it is easy to see, by Laplace expansion on the last column, that

\[
\det B'_N = \sum_{j=1}^{2N-1} (-1)^j (-1)^{j+1} t^{2N-1-j}
\]

and

\[
\det B''_N = \sum_{j=1}^{2N-2} (-t)(-1)^{j+1} t^{2N-1-j} + (1 - t)
\]
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