Abstract. Carathéodory’s and Kobayashi’s infinitesimal metrics on Teichmüller spaces of dimension two or more are never equal in the direction of any tangent vector defined by a separating cylindrical differential.

Introduction

This paper deals with infinitesimal and global conformal metrics on a complex manifold $M$. In the case at hand $M$ will always be either a Teichmüller space or a Riemann surface. We will use the notation $d(\cdot, \cdot)$ for both global and infinitesimal metrics. If the first entry inside the parentheses is a point $p$ in $M$ and the second is a tangent vector $V$ at the point $p$ then $d(p, V)$ will be an infinitesimal form, namely, the assignment of a norm $\|V\|_p$ on the second entry $V$ to every such pair $(p, V)$ that depends in a Lipschitz manner on the first entry $p$. Metrics of this type are called Finsler metrics. If both entries of $d(\cdot, \cdot)$ are points $p_1$ and $p_2$ in $M$ then the meaning changes; $d(p_1, p_2)$ means the integrated form of $d(p, V)$. By this we mean $d(p_1, p_2)$ is the infimum of the arclength integrals

$$\int \gamma d(\gamma(t), \gamma'(t)) dt,$$

where the infimum is taken over all piecewise differentiable curves $\gamma(t)$ that join $p_1$ to $p_2$ in $M$.

The most important example is when $M$ is the open unit disc in the complex plane $\mathbb{C}$, that is,

$$M = \Delta = \{z \in \mathbb{C} : |z| < 1\}$$

and the metric is the Poincaré metric, which we denote by $\rho$. With this convention the infinitesimal form is

$$\rho(p, V) = \frac{|dp(V)|}{1 - |p|^2},$$

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where $V$ is a tangent vector at a point $p$ with $|p| < 1$ and the global form of $\rho$ is
\begin{equation}
\rho(p_1, p_2) = \frac{1}{2} \log \frac{1 + r}{1 - r},
\end{equation}
where $r = \frac{|p_1 - p_2|}{|1 - p_1 p_2|}$.

The infinitesimal form $\rho$ on $\Delta$ induces infinitesimal forms on any complex manifold $M$ in two ways. The first way uses the family $\mathcal{F}$ of all holomorphic functions $f$ from $\Delta$ to $M$. It is commonly called Kobayashi’s metric and we denote it by $K$. The second way uses the family $\mathcal{G}$ of all holomorphic functions $g$ from $M$ to $\Delta$. It is commonly called Carathéodory’s metric and we denote it by $C$.

In general if $f : M \rightarrow N$ is a mapping from a manifold $M$ to a manifold $N$ and $f(p) = q$, we denote by $(df)_p$ the induced linear map from the fiber of tangent space to $M$ at $p$ to the fiber of the tangent space to $N$ at $q$.

For a point $p$ in a complex manifold $M$ we let $\mathcal{F}(p)$ be the subset of $f \in \mathcal{F}$ for which $f(0) = p$ and we let $\mathcal{G}(p)$ consist of those functions $g \in \mathcal{G}$ for which $g(p) = 0$. Given a point $p \in M$ and a vector $V$ tangent to $M$ at $p$, the infinitesimal form $K$ is defined by the infimum
\begin{equation}
K_M(p, V) = \inf_{f \in \mathcal{F}(p)} \left\{ \frac{1}{|a|} : df_0(1) = aV \text{ and } f \in \mathcal{F}(p) \right\}
\end{equation}
and the infinitesimal form $C$ is defined by the supremum
\begin{equation}
C_M(p, V) = \sup_{g \in \mathcal{G}(p)} \left\{ |b| : dg_p(V) = b \text{ and } g \in \mathcal{G}(p) \right\}.
\end{equation}

When $M = \Delta$ it is an exercise in the use of Schwarz’s lemma to show that the metrics $K_\Delta$, $C_\Delta$ and $\rho$ coincide.

For general manifolds $M$, Schwarz’s lemma applied to $g \circ f : \Delta \rightarrow \Delta$ where $f \in \mathcal{F}(\tau)$ and $g \in \mathcal{G}(\tau)$ implies
\begin{equation}
|b| \leq \frac{1}{|a|}
\end{equation}
and so the formulas (4), (5) and (6) together imply for all complex manifolds $M$
\begin{equation}
C_M(p, V) \leq K_M(p, V).
\end{equation}

The two metrics $C$ and $K$ represent categorical extremes of a general property of so called Schwarz-Pick metrics. $C$ is the smallest conformal metric for which holomorphic mappings are contracting and $K$ is the largest. This idea is explained further in Appendix A.
In the case $M$ is equal to a Teichmüller space $Teich(R)$ one can use the Bers embedding $[2, 4]$ to obtain a reverse inequality so one ends up with the double inequality

$$(8) \quad \frac{1}{3} K_{Teich(R)}(\tau, V) \leq C_{Teich(R)}(\tau, V) \leq K_{Teich(R)}(\tau, V).$$

Since the arbitrary points are now points in a Teichmüller space, we have switched notations and have denoted those points by the letter $\tau$ instead of $p$. We refer to Appendix B for the proof of the left hand inequality in (8).

A global metric $d$ satisfying mild smoothness conditions has an infinitesimal form given by the limit:

$$d(\tau, V) = \lim_{t \to 0} d(\tau, \tau + tV).$$

(See $[8]$.) When this is the case the integral of the infinitesimal form $d(\tau, V)$ gives back a global metric $\bar{d}$. By definition the metric $\bar{d}(\tau_1, \tau_2)$ on pairs of points $\tau_1$ and $\tau_2$ is the infimum of the arc lengths of arcs in the manifold that join $\tau_1$ to $\tau_2$. In general $\bar{d}$ is symmetric, satisfies the triangle inequality, determines the same topology as $d$ and one always has the inequality $\bar{d} \geq d$. However, in many situations $\bar{d}$ is larger than $d$. It turns out that when $M$ is a Teichmüller space and $d$ is Teichmüller’s metric, then $d$ and $\bar{d}$ coincide, $[9, 20]$. Moreover, for all Teichmüller spaces Teichmüller’s and Kobayashi’s metrics coincide (see $[31]$ for finite dimensional cases and $[12, 13]$ for infinite dimensional cases).

The main result of this paper is the following theorem.

**Theorem 1.** Assume $Teich(R)$ has dimension more than 1 and $V$ is a tangent vector corresponding to a separating cylindrical differential. Then there is strict inequality;

$$(9) \quad C_{Teich(R)}(\tau, V) < Kob_{Teich(R)}(\tau, V).$$

In some ways this result is surprising partly because of the metric equivalence implied by $[8]$ explained in Appendix B and partly because of Kra’s equality theorem $[21]$ explained in Appendix C. That theorem shows there are many directions in which the two infinitesimal forms for $C$ and $K$ are equal.

To explain Theorem 1 we must define a *separating cylindrical differential*. Since any simple closed curve $\gamma$ embedded in a Riemann surface divides it into one or two components, we call $\gamma$ separating if this number is two. No matter whether $\gamma$ is separating or not we can try to maximize the modulus of a cylinder in $R$ with core curve homotopic to $\gamma$. If this modulus is
bounded it turns out that there is a unique embedded cylinder of maximal modulus. By a theorem of Jenkins (\cite{18}, \cite{19}) and Strebel \cite{34} it corresponds to a unique quadratic differential $q_\gamma$ which is holomorphic on $R$ with the following properties. Its noncritical horizontal trajectories all have length $2\pi$ in the metric $|q_\gamma|^{1/2}$, these trajectories fill the interior of the cylinder and no other embedded cylinder in the same homotopy class has larger modulus. If for a separating simple closed curve such a cylinder exists and if $\text{Teich}(R)$ has dimension more than 1 then we call $q_\gamma$ a separating cylindrical differential.

We use the notation $\mathcal{A}_r = \{z : 1/r < |z| < r\}$ for the standard annulus which has modulus $(1/\pi) \log r$ and $\mathcal{C}_r$ for the standard cylinder which is conformal to $\mathcal{A}_r$ and which has circumference $2\pi$ and height $2 \log r$. The conformal map $z \mapsto \zeta$ is realized by putting

$$\zeta = -i \log z = -i(\log |z| + i \arg z)$$

to get a point in the infinite strip

$$\{\zeta = \xi + i\eta : -\log r < \eta < \log r\}$$

and then factoring the strip by the translation $\zeta \mapsto \zeta + 2\pi$.

The rough idea for the proof of Theorem 1 is to move in the direction of a separating cylindrical differential towards points of $\text{Teich}(R)$ that represent surfaces containing tall cylinders with constant circumference. These directions point to where anomalous aspects of the extremal length boundary of Teichmüller space become interesting (see for example \cite{24}, \cite{35}) and \cite{27}. By deforming in the direction of a maximal annulus $\mathcal{A}_r$ in the given homotopy class of a separating closed curve $\alpha$ in $R$ we are able to construct a sequence of waist curves $\beta_n(t)$ for cylinders $\mathcal{C}_r^n$ lying in Riemann surfaces $R_n$ constructed from $R$ that have complexity roughly equal to $n$ times the complexity of $R$.

By estimating the length of each $\beta_n$ with respect to the Teichmüller density $\lambda_{\mathcal{A}_r^n}$, where

$$\text{(10)} \quad (1/2) \rho_{\mathcal{A}_r^n} \leq \lambda_{\mathcal{A}_r^n} \leq \rho_{\mathcal{A}_r^n}$$

and where in general $\rho_R$ is Poincaré’s density on any Riemann surface $R$ we arrive at a contradiction. In \cite{10} the Teichmüller density $\lambda_R$ is defined for any Riemann surface $R$ although we apply it only when $R$ is conformal to one of the annuli $\mathcal{A}_r^n$. The constant $(1/2)$ in inequality \text{(10)} is of course independent of $n$ and so it provides a bridge that enables comparison of lengths of curves in maximal annuli $\mathcal{A}_r^n$ to lengths of curves in Teichmüller spaces $\text{Teich}(R_n)$. 

The Teichmüller density $\lambda(p)dp$ at a point $p \in R$ is the restriction of the Teichmüller infinitesimal form of Teichmüller’s metric on $Teich(R - p)$ to the fiber of the forgetful map

$$\Psi : Teich(R - p) \to Teich(R).$$

More detail is given in section 2.

On the one hand the right hand side of inequality (10) and a computation shows that this length increases proportionately to $n$. On the other hand, the left hand inequality of (10) and the assumption that (11)

$$C_{Teich(R)}(\tau, V) = K_{Teich(R)}(\tau, V),$$

shows that this number is bounded independently of $n$.

This paper is divided into ten sections. Section 1 explains a consequence of the equality

$$C_M(\tau, V) = K_M(\tau, V)$$

when $M = Teich(R)$, the Teichmüller space of a Riemann surface $R$ and where $V$ is a tangent vector to $M$ at $\tau$. If $f(0) = [0]$ and $f(t) = [t|q|/q]$, then $df_0(1) = V$ where

$$\frac{\partial V}{\partial z} = |q|/q.$$

The square bracket denotes Teichmüller equivalence class and $q$ is an integrable holomorphic quadratic differential on $R$. Equality in (11) implies the existence of a holomorphic function $g$ defined on $Teich(R)$ mapping onto the unit disc $\Delta$ such that $g \circ f(t) = t$. Thus to prove Theorem 1 it suffices to show that under the given hypothesis such a function $g$ cannot exist.

Section 2 introduces the Bers fiber space $F(Teich(R))$ of $Teich(R)$ which is bianalytically equivalent to $Teich(R - p)$ for any point $p \in R$. The maps $f$ and $g$ lift to this fiber space giving maps $\hat{f}$ and $\hat{g}$ for which $\hat{g} \circ \hat{f}(t) = t$. See Figure 3 in section 6.

Section 3 briefly describes a natural conformal metric $\lambda_R$ on any Riemann surface which is induced by Bers’ fiber space over $Teich(R)$. If $\rho$ is the Poincaré metric on $R$ then one has the inequality (10). We call $\lambda_R(p)|dp|$ the Teichmüller density because it is induced by the infinitesimal form of Teichmüller’s metric restricted to the fiber over the identity in $Teich(R)$ for the forgetful map $\Psi : Teich(R - p) \to Teich(R)$. We only need inequality (10) in the case that $R$ is an annulus but understanding how it is constructed is key to understanding the proof of Theorem 1.

Section 4 estimates the decay of Carathéodory’s infinitesimal metric $C_{A_r}(1)$ where $A_r$ is the annulus $1/r < |z| < r$ as $r$ approaches infinity.
Section 5 focuses on a quasiconformal selfmapping of $A_r$ which we call $Spin_r$. It spins the point 1 on the core curve $|z| = 1$ of $A_r$ through a counterclockwise angle of length $t$ while holding fixed all of the boundary points of $A_r$ and while shearing points along concentric circles. This quasiconformal map induces a new conformal structure on $A_r$ and in this new conformal structure the annulus is conformal to an annulus of the form $A_{r(t)}$ for some annulus with outer radius $r(t)$ and inner radius $1/r(t)$. We give estimates for $r(t)$ from above and below in terms of $r$ and $t$.

Section 6 shows how to prove Theorem 1 in the very special case that the Riemann surface $R$ is just an annulus and the separating simple closed curve is the core curve which separates its two boundary components. It shows that if $V$ is the tangent vector pointing in the direction of enlarging the modulus of the annulus then $Car(\tau, V)$ is strictly less than $Kob(\tau, V)$. This result is the model for the more general result in Theorem 1. In Theorem 1 we apply the same idea to an embedded annulus $A_r$ which is maximal annulus in its homotopy class.

Section 7 develops the Jenkins-Streel theory of a maximal separating cylinder on an arbitrary Riemann surface and the associated cylindrical quadratic differential $q$.

Section 8 explains the spinning map as an element in the mapping class group of $R - p$

Section 9 explains how to create a countable a family $R_n$ of Riemann surfaces induced by $q$ and by unrolling the cylinder along an infinite strip.

Section 10 explains contradictory estimates of the length of waist curves that are core curves of annuli $A_{r,n}$ embedded in $R_n$. The estimates depend on inequality [10].

Since writing the first version of this paper in 2015 I have learned of a paper [25] by V. Markovic that proves a related result concerning the ratio of the global Carathéodory’s and Kobayashi’s metrics.

1. Teichmüller discs

Definition 1. A Teichmüller disc is the image $\mathbb{D}(q)$ of any map

$$f : \Delta \to Teich(R)$$

of the form

$$f(t) = [t|q|/q]$$

for $|t| < 1$ where $q$ is an integrable, holomorphic, non-zero quadratic differential on $R$. 

By Teichmüller’s theorem this map is injective and isometric with respect to Poincaré’s metric on $\Delta$ and Teichmüller’s metric on $Teich(R)$. It is interesting to note that any disc isometrically embedded in a finite dimensional Teichmüller space necessarily has the form $f(t)$ or $f(\bar{t})$ where $f$ has Teichmüller form, [3].

**Lemma 1.** Suppose $V$ is a vector tangent to $Teich(R)$ tangent to the embedding of a Teichmüller disc given by $f$ in (12). In particular, this means $V_\tau = \frac{|w|}{q}$. Then $C(\tau, V) = K(\tau, V)$ if and only if there exists a holomorphic function $g \in G(\tau, 0)$ for which $(g \circ f)(t) = t$ for all $|t| < 1$ and for which $dg_0(f(0))$ is equal to 1.

**Proof.** By Schwarz’s lemma and a normal families argument the supremum in (5) is realized by a holomorphic function $g$ defined on $Teich(R)$ for which $g \circ f(t) = t$ for all $t \in \Delta$. □

**2. Bers fiber space**

Over every Teichmüller space $Teich(S)$ of a hyperbolic quasiconformal surface $S$ there is a canonical fiber space [5] called the Bers fiber space.

$$\Psi : F(Teich(S)) \to Teich(S),$$

which has dimensions one more than the dimension of $Teich(S)$.

To describe $F(Teich(S))$ we begin by going over the definition of $Teich(S)$. Assume $S$ is a fixed quasiconformal surface in the sense that $S$ is a topological Hausdorff space equipped with a finite system of charts $z_j$ mapping an open subset $U_j \subset S$, $1 \leq j \leq n$, such that

$$\bigcup_{j=1}^{n} U_j = S,$$

and for which there is an $M > 0$ such that for each $j$ and $k$ the dilatation $K$ of $z_j \circ z_k^{-1}$ satisfies

$$K(z_j \circ z_k^{-1}) \leq M.$$

The Teichmüller equivalence relation is an equivalence relation on the set of quasiconformal maps $f$ from $S$ to variable Riemann surfaces $f(S)$. That $f(S)$ is a Riemann surface means that it has a system of coordinates $w_j$ on domains $U_k$ that cover $f(S)$ such that on each overlap $U_k \cap U_\ell$

$$K(w_k \circ (w_\ell)^{-1}) = 1.$$
This equivalence relation makes quasiconformal maps \( f_0 : S \to f_0(S) \) and \( f_1 : S \to f_1(S) \) equivalent if there is a conformal map \( c \) and an isotopy through quasiconformal maps \( h_t \) such that \( h_0 = f_0, h_1 = c \circ h_1 \) and \( h_t(p) = f_0(p) = c \circ f_1(p) \) for all points in the ideal boundary of \( S \) and all real numbers \( t \) with \( 0 \leq t \leq 1 \).

\[
\begin{array}{c}
\begin{array}{cc}
\downarrow c & \\
S & f_0(S)
\end{array} \\
\begin{array}{c}
\uparrow f_1(S) \\
f_0 \\
\end{array}
\end{array}
\]

**FIGURE 1.** Equivalence of \( f_0 \) and \( f_1 \)

To define the Bers fibration \( \Psi : F(\text{Teich}(S)) \to \text{Teich}(S) \) one picks an arbitrary point \( p \) in \( S \) and considers quasiconformal maps \( f \) to variable Riemann surfaces \( f(S) \). Since \( f \) is a quasiconformal homeomorphism that maps to a Riemann surface \( f(S) \), it restricts to a map \( f : S - p \to f(S) - f(p) \). The equivalence relation has a similar description: two maps \( f_0 \) and \( f_1 \) from \( S - p \) to \( f_0(S - p) \) and to \( f_1(S - p) \) are equivalent if there is an isotopy \( f_t \) connecting \( f_0 \) to \( c \circ f_1 \) as described above. But now since \( f(p) \) is a boundary point of \( f(S) - f(p) \) the isotopy \( f_t \) must also pin down the point \( f_t(p) \). Thus the composition \( f_1 \circ f_0^{-1} \) keeps track of the variable conformal structure of the marked Riemann surface \( f(S) \) as well as the movement of the point \( f(p) \) within that surface.

Since by definition a representative \( f \) of a class \( [[f]] \in \text{Teich}(R - p) \) is quasiconformal on \( S - p \), it extends uniquely to a quasiconformal map defined on \( S \). Moreover two such representatives that are isotopic through quasiconformal maps on \( S - p \) are also isotopic on \( S \). Therefore the covering

\[
\Psi : F(\text{Teich}(S)) \to \text{Teich}(S),
\]

which forgets that the isotopy must fix the point at \( f_t(p) \), is well-defined. Thus, the map \( \Psi \) which maps \( [[f]] \) to \( [f] \) is well-defined.

In general \( F(\text{Teich}(S)) \) is a complex manifold and for each point \( \tau \in \text{Teich}(S) \), and

\[
(13) \quad \mathbb{K}_\tau = \Psi^{-1}(\tau)
\]

is a one-dimensional properly embedded submanifold of \( F(\text{Teich}(S)) \) which is conformal to a disc. Bers explicitly describes \( F(\text{Teich}(S)) \) as a moving family of normalized quasicircles. One side of each of these quasicircles determines a point in \( \text{Teich}(S) \) while the other side realizes the fiber \( \mathbb{K}_\tau \).
as a plane domain conformal to a disc. This disc is properly embedded in $F(\text{Teich}(S))$.

Now assume $S$ is given an underlying complex structure which makes it into a Riemann surface $R$. Then denote by $\hat{R}$ the Riemann surface $R - p$ from which the point $p$ has been excised. Bers shows that the fibration $\Psi$ is isomorphic to the fibration

$$\Psi_p : \text{Teich}(\hat{R}) \to \text{Teich}(R),$$

where $\hat{R} = R - p$. Also the derivative $d\Psi$ of $\Psi$ is dual to the inclusion map of the integrable holomorphic quadratic differentials on $R$ into the integrable quadratic differentials holomorphic on $R - p$. Except in the case when $R$ is a complex torus, at any point $p$ on $R$ there exists an integrable holomorphic quadratic differential $R$ with a simple pole at $p$. Thus $\Psi$ has a surjective derivative at every point $\hat{\tau} \in F(\text{Teich}(R))$ except when $R$ is a torus.

One system of global holomorphic charts for $\text{Teich}(\hat{R})$ is given by the Bers embedding of a Fuchsian universal covering group $\hat{G}$ that covers $R - p$ at any point $\tau \in \text{Teich}(R)$ and any point $p \in R$. Bers main theorem gives a second global holomorphic chart for $\text{Teich}(\hat{R})$. Assume $\tau = \Psi(\hat{\tau})$ where $\hat{\tau} \in \text{Teich}(\hat{R})$. One takes a Fuchsian universal cover $\Pi : \Delta \to \Delta/G \cong R$ with covering group $G$ that maps $0$ to $p$ and then one forms the quasi-Fuchsian group $G_\tau = \psi^\tau \circ G \circ (\psi^\tau)^{-1}$. Here $\psi^\tau$ has Beltrami coefficient $\mu$ supported on the exterior of the unit disc and $\mu$ represents the point $\tau \in \text{Teich}(R)$. One uses the fact that two Beltrami coefficients $\mu_1$ and $\mu_2$ are Teichmüller equivalent if and only if $w^{\mu_1}(z) = w^{\mu_2}(z)$ for all $z \in \Delta$. Note that $\mu$ is identically equal to $0$ in $\Delta$ and $\mu$ represents the marked Riemann surface $\hat{R}(\tau)$ in the complement $\Delta^* = \overline{\mathbb{C}} \setminus \Delta$.

Let $QF(G)$ be the space of quasiconformal conjugacies $w^{\mu} \circ G \circ (w^{\mu})^{-1}$, where it is assumed that

$$\mu(\gamma(z)) \frac{\gamma'(z)}{\gamma'(z)} = \mu(z)$$

for all $\gamma \in G$. This implies that $w^{\mu} \circ G \circ (w^{\mu})^{-1}$ is a group of Möbius transformations. We say such a Beltrami coefficient $\mu$ is equivariant for $G$. If we take $||\mu||_\infty = 1$, the mappings $w^{t\mu}$ for $|t| < 1$ form a holomorphic motion of $\Delta$ parameterized by $\{t : |t| < 1\}$ and this motion is equal to the identity when $t = 0$. Bers shows that the map

$$(14) \quad \text{Teich}(\hat{R}) \ni \hat{\tau} \mapsto (w^{\Psi(\hat{\tau})}, w^{\Psi(\hat{\tau})}(0)) \in QF(G) \times \mathbb{C}$$

is a global holomorphic chart for $\text{Teich}(\hat{R})$. 
By Bers theorem for a fixed point \( \tau \in \text{Teich}(R) \) the range of values
\[
\{ w : w = \infty \text{ or } w = w^\tau(z) \text{ where } |z| > 1 \}
\]
is conformal to \( K_\tau \) defined in (13), the quasi-Fuchsian group \( G^\tau = w^\tau \circ G \circ (w^*)^{-1} \) acts discontinuously on \( K_\tau \) and the quotient space is conformal to \( R_\tau \).

In the following lemma we assume \( V \) is a tangent vector to \( \text{Teich}(R) \) of the form \( \partial V = |q|/q \), where \( q \) is a quadratic differential form supported in the complement of \( \Delta \) and where \( \partial V = 0 \) in \( \Delta \).

Note that if \( q \) is integrable and holomorphic on \( R \) then it is also integrable and holomorphic on \( \hat{R} \). Without changing the notation we view \( V \) as representing a vector tangent to \( \text{F}(\text{Teich}(R)) \) and at the same time a vector tangent to \( \text{Teich}(R) \).

**Lemma 2.** Suppose \( \tau \in \text{Teich}(R - p) \) and \( \partial V = |q|/q \) where \( q \) is a holomorphic quadratic differential on \( R \). In the setting just described suppose
\[
C_{\text{Teich}(R)}(\Psi(\tau), V) = K_{\text{Teich}(R)}(\Psi(\tau), V).
\]
Then \( C_{\text{F}(\text{Teich}(R))}(\tau, V) = K_{\text{F}(\text{Teich}(R))}(\tau, V) \).

**Proof.** First observe that since \( q \) is holomorphic and integrable on \( R \), it is also holomorphic and integrable on \( R - p \). This implies that \( V \) can be viewed both as a tangent vector to \( \text{Teich}(R) \) and to \( \text{Teich}(R - p) \). Since any holomorphic quadratic differential \( q \) integrable on \( R \) is also integrable on \( \hat{R} = R - p \), the Teichmüller disc \( f(\Delta) = \{ [t \frac{w}{q}] : |t| < 1 \} \) in \( \text{Teich}(R) \) lifts to a Teichmüller disc \( \{ [t \frac{w}{q}] : |t| < 1 \} \) in \( \text{Teich}(\hat{R}) \) and we denote the corresponding lifted mapping by \( \hat{f} \). Let \( \hat{g} = g \circ \Psi \) where \( \Psi \) is the projection from Bers fiber space \( \text{F}(\text{Teich}(R)) \) to \( \text{Teich}(R) \) and where \( g \) is defined in section 1 and referred to in Lemma 1. We obtain
\[
\hat{f} : \Delta \to \text{F}(\text{Teich}(R)) \text{ and } \hat{g} : \text{F}(\text{Teich}(R)) \to \Delta
\]
where both \( \hat{f} \) and \( \hat{g} \) are holomorphic and
\[
\hat{g} \circ \hat{f}(t) = t.
\]
Thus the conclusion of the lemma follows from Lemma 1.

\[\square\]

From Lemmas 1 and 2 each point \( \tau \in \text{Teich}(R) \) induces two kinds of conformal discs in \( \text{F}(\text{Teich}(R)) \). The first is the fiber \( \mathbb{K}_\tau \) of the forgetful map \( \Psi \) over the point \( \tau \). The second is a bundle of Teichmüller discs, \( \mathbb{D}_q \), defined in Definition 1, one for each projective class of non-zero quadratic differential \( q \) holomorphic on \( R_\tau \).
3. Comparing metrics on surfaces

For every \( \tau \in \text{Teich}(R) \) the fiber \( \mathbb{K}_\tau = \Psi^{-1}(\tau) \) of the forgetful map,

\[
\Psi : F(\text{Teich}(R)) \to \text{Teich}(R),
\]

carries two metrics that are induced by inherent geometry. The first is the Poincaré metric \( \rho \) defined in \( [3] \). It is natural because \( \mathbb{K}_\tau \) is conformal to a disc. When viewed as a metric on \( \mathbb{K}_\tau \) we denote it by \( \rho_\tau \) and when viewed as a metric on \( R \) we denote it by \( \rho_R \). The second metric is the restriction of the infinitesimal form of Teichmüller’s metric on \( F(\text{Teich}(R)) \) to \( \mathbb{K}_\tau \). Similarly, when viewed as a metric on \( \mathbb{K}_\tau \) we denote it by \( \lambda_\tau \), and when viewed as a metric on \( R = R_\tau \) we denote it by \( \lambda_R \). In \( [16] \) we have called \( \lambda_R \) the Teichmüller density.

**Definition 2.** \( \lambda_\tau(z)|dz| \) is the pull-back by a conformal map \( c : \Delta \to \mathbb{K}_\tau \) of the infinitesimal form of Teichmüller’s metric on \( F(\text{Teich}(R)) \) restricted to \( \mathbb{K}_\tau \).

From the previous section we know that \( F(\text{Teich}(R)) \) has a global coordinate given by \( \text{Teich}(R-p) \), where \( R \) is a marked Riemann surface and \( p \) is a point on \( R \). For any point \( [[g]] \in \text{Teich}(R-p) \) the evaluation map \( \text{Ev} \) is defined by \( \text{Ev}([[g]]) = g(p) \) is holomorphic and, since the domain of \( \text{Ev} \) is simply connected, \( \text{Ev} \) lifts to \( \Delta \) by the covering map \( \Pi_p \) to \( \tilde{\text{Ev}} \) mapping \( \text{Teich}(R-p) \) to \( \Delta \). Note that the target of the map \( \text{Ev} \) is the Riemann surface \( R \). For this reason we can interpret the inequality of the next theorem as an inequality of conformal metrics on \( R = R_\tau \).

The following theorem is proved in \( [16] \).

**Theorem 2.** For any surface \( R \) with marked conformal structure \( \tau \) whose universal covering is conformal to \( \Delta \), the conformal metrics \( \rho_\tau \) and \( \lambda_\tau \) satisfy

\[
(1/2)\rho_\tau(p)|dp| \leq \lambda_\tau(p)|dp| \leq \rho_\tau(p)|dp|.
\]

This theorem has a companion theorem that expresses \( \lambda_\tau \) in terms residues of quadratic quadratic differentials holomorphic on \( R-p \) that have integral norm less than or equal to 1. For the statement see Appendix D. In the proof of the main theorem of this paper, that is, Theorem \( [1] \) we use Theorem \( 2 \) only for the case \( R \) is conformal to an annular domain \( A_r = \{ z : 1/r < |z| < r \} \).
4. The $C$ and $K$ Metrics on an Annulus

Let $C_r$ and $K_r$ be the infinitesimal forms of Carathéodory’s and Kobayashi’s metrics on the annulus

$$\mathcal{A} = \mathcal{A}_r = \{ w : (1/r) < |w| < r \}.$$ 

We identify the tangent space of the plane domain $\mathcal{A}_r$ with direct product $\mathcal{A}_r \times \mathbb{C}$. Since $\mathcal{A}_r$ is acted on by the continuous group of rotations the value of either of these forms is invariant along the core curve $\{ w : |w| = 1 \}$ and we focus only on the values of $K_{\mathcal{A}_r}(1, 1)$ and $C_{\mathcal{A}_r}(1, 1)$, which are defined by the general formulas (4) and (5) in the introduction. Also, we shorten these notations to $C_r(1)$ and $K_r(1)$.

The logarithm provides a conformal map from $\mathcal{A}_r$ to a cylinder which we can think of as being vertical with the circumference equal to $2\pi$ and height equal to $2\log r$. With the same circumference it becomes taller as $r$ becomes larger. In the sequel the only results we need from this section are the following Lemma and formula (22).

**Lemma 3.** Assume $r > 1$ and $n$ is a positive integer. Then

$$C_{r^n}(1) \leq \frac{1}{n} \cdot \frac{r}{r - 1}. \tag{19}$$

**Proof.** We assume the supremum in (5) for the tangent vector 1 at the point 1 in $\mathcal{A}_{r^n}$ is realized by $|\tilde{g}'(1)|$ where $\tilde{g}(1) = 0$ and $\tilde{g}$ is a holomorphic function with domain $\mathcal{A}_{r^n}$ and range contained in the unit disc $\Delta$. If we put $g(z) = \tilde{g}(z^n)$, then $g$ is holomorphic and defined on $\mathcal{A}_r$ with $g(1) = 0$ and has the same range as $\tilde{g}$. Since $C_{\mathcal{A}_r}(1)$ is the supremum of the values of $|g'(1)|$ for all such functions $g$, we have

$$C_{\mathcal{A}_{r^n}}(1) = |\tilde{g}'(1)| = (1/n) \cdot |g'(1)| \leq (1/n) \cdot C_r(1).$$

Any holomorphic function $g$ mapping $\mathcal{A}_r$ into the unit disc with $g(1) = 0$ restricts to a map of radius $1 - r$ centered at the point 1. By Schwarz’s lemma $|g'(1)|$ satisfies

$$|g'(1)| < \frac{1}{1 - 1/r} = r/(r - 1).$$

This proves the lemma. \qed

Although it is unnecessary for the proof of Theorem 1 it is instructive to compare the rates of decay of $C_r$ and $K_r$. The following explicit formula is given by Simha [33]

$$C_r(w)(V)|dw(V)| = \frac{2}{r} \cdot \frac{\prod_1^\infty (1 + r^{-4n})^2(1 - r^{-4n})^2}{\prod_1^\infty (1 + r^{-4n+2})^2(1 - r^{-4n+2})^2}|dw(V)|. \tag{20}$$
That the first factor is \( \frac{2}{r} \) and that the second “product” factor is bounded as \( r \to \infty \) is what is important to us. The formula for \( K_r(1) \) is much more simple and can be derived by using the logarithmic coordinate \( \zeta = -i \log w \) for \( \mathcal{A}_r \). Then the core is covered by the real axis \( \eta = 0 \) and \( \mathcal{A}_r \) is conformal to the rectangle

\[
\{ \zeta = \xi + i \eta : \log(1/r) < \eta < \log r, \quad -\pi \leq \xi < \pi \},
\]

with the two sides at \( \xi = -\pi \) and \( \xi = \pi \) identified by the translation \( \zeta \mapsto \zeta + 2\pi \). Then the Kobayashi (Poincaré) metric is

\[
K_r(\zeta, V) = \frac{\pi |d\eta(V)|}{(2 \log r) \cos\left(\frac{\eta\pi}{2 \log r}\right)}
\]

In the \( \zeta \)-coordinate along the core curve \( \eta = 0 \) where the cosine is equal to 1 we have

\[
K_r(1) = \frac{\pi}{2 \log r}.
\]

Of course from inequality (7) for any tangent vector \( V \) and any point \( P \) in any complex manifold \( M \), the ratio \( C_M(P, V)/K_M(P, V) \leq 1 \). From formulas (22) and (20) when \( M \) is the annulus \( \mathcal{A}_r \) we have the more precise formula valid for any point \( P \) along the core curve, namely, the ratio

\[
C_r(|w| = 1, V)/K_r(|w| = 1, V) = \frac{4}{\pi} \cdot \frac{\log r}{r} \cdot \frac{\prod_{n=1}^{\infty} (1 + r^{-4n})^2 (1 - r^{-4n})^2}{\prod_{n=1}^{\infty} (1 + r^{-4n+2})^2 (1 - r^{-4n+2})^2}.
\]

In particular, the graph of this ratio is shown in Figure 2 and was provided to me by Patrick Hooper and Sean Cleary. It shows that the ratio is always less than 1 and monotonically decreases to 0 as \( r \) increases to \( \infty \).

**Figure 2.** metric ratio
5. Spinning an Annulus

We are interested in a particular selfmap $\text{Spin}_t$ of the annulus
$$A_r = \{ z : 1/r < |z| < r \}.$$ 
$\text{Spin}_t$ spins counterclockwise the point $p = 1$ on the unit circle
$$\{ z : |z| = 1 \} \subset A_r$$
to the point $p = e^{it}$ for real numbers $t$ while keeping points on the boundary of $A_r$ fixed. The formula for $\text{Spin}_t$ is easier to write in the rectangular coordinate $\zeta$ where we put
$$\zeta = \xi + i\eta = -i \log z.$$ 
We call $\zeta$ rectangular because the strip
$$(24) \quad \{ \zeta = \xi + i\eta : -\log r \leq \eta \leq +\log r \}$$
factored by the translation $\zeta \mapsto \zeta + 2\pi$ is a conformal realization $A_r$ with a fundamental domain that is the rectangle
$$\{ \zeta : 0 \leq \xi \leq 2\pi \text{ and } -\log r \leq \eta \leq +\log r \}.$$ 
The covering map is $\Pi(\zeta) = z = e^{-i\zeta}$ and the points $\zeta = 2\pi n$ for integers $n$ cover the point $z = 1$. In the $\zeta$ coordinate we make the following definition.

**Definition 3.** For $\zeta = \xi + i\eta$ in the strip $-\log r \leq \eta \leq \log r$, let
$$(25) \quad \text{Spin}_t(\zeta) = \xi + t \left( 1 - \frac{|\eta|}{\log r} \right) + i\eta.$$ 
Note that $\text{Spin}_t(\zeta + 2\pi) = \text{Spin}_t(\zeta) + 2\pi$ and $\text{Spin}_t$ shears points with real coordinate $\xi$ and with imaginary coordinate $\pm \eta$ along horizontal lines to points with real coordinate $\xi + t(1 - |\eta|/\log r)$ while fixing points on the boundary of the strip where $\eta = \pm \log r$.

We let $\mu_t$ denote the Beltrami coefficient of $\text{Spin}_t$ and let $A_r(\mu_t)$ be the annulus $A_r$ with conformal structure determined by $\mu_t$. By the uniformization theorem applied to the annulus $A_r(\mu_t)$ we know that there is a positive number $r(t)$ such that $A_r(\mu_t)$ is conformal to $A_{r(t)}$.

**Theorem 3.** Assume $t > 2\pi$. Then the unique real number $r(t)$ with the property that $A_r(\mu_t)$ is conformal to $A_{r(t)}$ satisfies
$$(26) \quad t \cdot \frac{(1 - 2\pi t)^2 + ((\log r)/t)^2}{\sqrt{1 + ((\log r)/t)^2}} \leq \log r(t) \leq t \cdot \left( \frac{1}{t^2} + \frac{1}{(\log r)^2} \right)^{1/2}.$$
Proof. We begin the proof with a basic inequality valid for any pair of measured foliations \(|du|\) and \(|dv|\) on any orientable surface \(S\) with a marked conformal structure \(\tau\). Actually, this inequality is true for any pair of weak measured foliations. In this paper we only need the result for measured foliations. See [14] for the definition of a weak measured foliation.

We denote by \(S(\tau)\) the surface \(S\) with the conformal structure \(\tau\).

For any smooth surface the integral \(\int \int_S |du\wedge dv|\) is well defined for any pair of weak measured foliations \(|du|\) and \(|dv|\) on defined \(S\). The Dirichlet integral of a measured foliation \(|du|\) defined by

\[
Dir_\tau(|du|) = \int \int_{S(\tau)} (u_x^2 + u_y^2)dx dy,
\]

is well defined only after a complex structure \(\tau\) has been assigned to \(S\). \(\tau\) gives invariant meaning to the form \((u_x^2 + u_y^2)dx \wedge dy\) where \(z = x + iy\) is any holomorphic local coordinate. The following is a version of the Cauchy-Schwarz inequality.

Lemma 4. With this notation

\[
\left( \int \int_S |du\wedge dv| \right)^2 \leq \int \int_{S(\tau)} (u_x^2 + u_y^2)dx dy \int \int_{S(\tau)} (v_x^2 + v_y^2)dx dy.
\]

Proof. See [14].

The measured foliations \(|d\eta|\) and \(|d\xi|\) on the strip (24) induce horizontal and vertical foliations whose corresponding trajectories are concentric circles and radial lines on \(A_r\). Correspondingly, \(|d\eta \circ (Spin_t)^{-1}|\) and \(|d\xi \circ (Spin_t)^{-1}|\) are horizontal and vertical foliations on \(A_r(\mu_t)\). We apply Lemma 4 to these foliations with \(S = A_r(\mu_t)\) and \(S(\tau) = A_r\) and we get

\[
(4\pi \log r(t))^2 \leq [4\pi \log r] \left(4\pi \log r(1 + \frac{t^2}{(\log r)^2})\right),
\]

which leads to the right hand side of (26).

To obtain the left hand side we work with the conformal structure on \(A_r(\mu_t)\) where the slanting level lines of \(\xi \circ (Spin_t)^{-1}\) are viewed as orthogonal to the vertical lines of \(\eta\). By definition the number \(r(t)\) is chosen so that \(A_r(\mu_t)\) is conformal \(A_r(t)\) and this modulus is equal to

\[
\frac{2 \log r(t)}{2\pi}.
\]

On the other hand this modulus is bounded below by any of the fractions

\[
\frac{(\inf_\beta \int_\beta \sigma)^2}{\text{area}(\sigma)},
\]
where $\beta$ is any arc that joins the two boundary contours of the annulus and $\sigma$ is any metric on $A_r(t)$.

From the definition of extremal length and using the metric $\sigma = |d\zeta|$ where $\zeta$ is the strip domain parameter one obtains

$$\frac{2(\sqrt{(t-2\pi)^2 + (\log r)^2})^2}{4 \left(\sqrt{t^2 + (\log r)^2}\right)} \leq \log r(t) \log r,$$

This lower bound leads to the left hand side of (31).

6. Comparing $C$ and $K$ on $Teich(\mathcal{A}_r)$

**Theorem 4.** For every point $\tau \in Teich(\mathcal{A}_r)$ and for the tangent vector $V$ with $\mathcal{D}V = |q|/q$ where $q = (\frac{dz}{z})^2$,

$$C_{Teich(\mathcal{A}_r)}(\tau, V) < K_{Teich(\mathcal{A}_r)}(\tau, V).$$

**Proof.** By Teichmüller’s theorem the embedding

$$\{s : |s| < 1\} \ni s \mapsto f(s) = [s|q]/q \in Teich(\mathcal{A}_r)$$

is isometric in Teichmüller’s metric and it lifts to an isometric embedding

$$\{s : |s| < 1\} \ni s \mapsto \hat{f}(s) = [[s|q]] = Teich(\mathcal{A}_r) \ni g \ni \{s : |s| < 1\}$$

![Figure 3. $\hat{f}, \hat{g}, f$ and $g$](image)

The map $f$ in Figure 3 is defined by $f(s) = [s|q]/q$ and $\hat{f}(s) = [[s|q]]$ where the single and double brackets denote equivalence classes of Beltrami coefficients representing elements of $Teich(\mathcal{A}_r)$ and $Teich(\mathcal{A}_r - \{1\})$, respectively. The map $g$ is any holomorphic map from $Teich(\mathcal{A}_r)$ into $\Delta$ and $\hat{g} = g \circ \Psi$ where $\Psi$ is the forgetful map. That is, $\Psi$ applied to an equivalence class $[[\mu]]$ forgets the requirement that the isotopy in the equivalence must pin down the point 1 in $\mathcal{A}_r$ while keeping the requirement that it must pin down all of the points on both the inner and outer boundaries of $\mathcal{A}_r$. 
Just as in (24) we use the coordinate $\zeta = -i \log z$ that realizes the annulus $A_r$ as the quotient space of

$$\text{Strip} = \{ \zeta = \xi + i \eta : |\eta| < \log r \}$$

factored by the translation $\zeta \mapsto \zeta + 2\pi$. In the coordinate $\zeta$ the quadratic differential $q$ and the tangent vector $V$ have simple expressions:

$$q = (d\zeta)^2 \quad \text{and} \quad \partial V = \frac{d\zeta}{d\zeta}.$$

For any positive integer $n$ we let $A_r^n$ be the same strip factored by the translation $\zeta \mapsto \zeta + 2\pi n$ and consider the core curves $\beta_n$ that are covered by the interval $\eta = 0$, $0 < \xi \leq 2\pi n$ in the strip. Denote by $\lambda_n, c_n$ and $\rho_n$ the Teichmüller, Carathéodory and Kobayashi densities on the surface $R = A_r^n$. Now use Theorem 2 and the assumption that there is equality in (28) to derive contradictory estimates for the $\lambda_n(\beta_n)$, that is, the $\lambda_n$-length of $\beta_n$ in the annulus $A_r^n$. Since $\lambda_n > (1/2)\rho_n$ from (21) we have

$$\lambda_n(\beta_n) > (1/2)\rho_n(\beta_n) = n \cdot \pi \cdot \frac{1}{4 \log r}. \quad (29)$$

On the other hand since we assume inequality (28) is actually an equality there exists a holomorphic function $g_n : \text{Teich}(A_r^n) \to \Delta$ such that $g_n \circ f_n(s) = s$ where $f_n(s) = [s|q_n]/q_n$ and $q_n = (d\zeta)^2$ in the same strip. Moreover $\hat{g}_n \circ \hat{f}_n(s) = s$ where $\hat{f}_n(s) = [[s|q_n]/q_n]$ and where $\hat{g}_n = g_n \circ \Psi$. $\hat{g}_n$ is holomorphic and automorphic for the translation $\zeta \mapsto \zeta + 2\pi n$ so its restriction the Teichmüller disc $[[s|q_n]/q_n]$ determines a function on the annulus for which $\hat{g}_n \circ f_n(s) = s$. This restriction must realize the extremal value for the extremal problem in (5) that defines $c_n$. From Theorem 2 and Lemma 3 the $\lambda_n$-lengths of the curves $\beta_n$ are bounded independently of $n$ because

$$\lambda_n(\beta_n) \leq \rho_n(\beta_n) = c_n(\beta_n) \leq n \cdot (1/n) \cdot \frac{r}{1 - r} = \frac{r}{1 - r},$$

which contradicts (29).

□

7. Maximal separating cyclinders

Let $R$ be a Riemann surface for which $\text{Teich}(R)$ has dimension at least 2 and assume $R$ is of finite analytic type, by which we mean that the fundamental group of $R$ is finitely generated. Consider a simple closed curve $\gamma$ that divides $R$ into two connected components $R_1$ and $R_2$ and assume that both $R_1$ and $R_2$ have non-trivial topology in the sense that
each of them has a fundamental group generated by two or more elements. From a theorem of Jenkins [18] and also of Strebel [34], there exists on \( R \) an integrable, holomorphic, quadratic differential \( q_\gamma(z)(dz)^2 \) that realizes \( R \) in cylindrical form, in the following sense. A cylinder realized as the factor space of a horizontal strip in the \( \zeta \)-plane

\[(30) \quad C = \{ \zeta = \xi + i\eta : -\log r \leq \eta \leq \log r \} / (\zeta \mapsto \zeta + 2\pi)\]

with \( r > 1 \) is embedded in \( R \) with the following properties:

i) \((d\zeta)^2 = -(d\log z)^2\) is the restriction of a global holomorphic quadratic differential \( q_\gamma(z)(dz)^2 \) on \( R \) with \( \int_R |q|dxdy = 2\pi \log r \),

ii) the horizontal line segments in the \( \zeta \)-plane that fill the rectangle \( \{ \zeta = \xi + i\eta : 0 \leq \xi < 2\pi, \xi < 2\pi, -\log r < \eta < \log r \} \) comprise all of the regular horizontal closed trajectories of \( q \) on \( R \),

iii) the core curve \( \gamma = \{ \zeta = \xi + i\eta : 0 \leq \xi \leq 2\pi \} \) separates \( R \) into two components \( R_1 \) and \( R_2 \),

iv) the remaining horizontal trajectories of \( q \) on \( R \), that is, those which run into singular points of \( q \), comprise two critical graphs, one lying in the subsurface \( R_1 \) and the other lying in the subsurface \( R_2 \),

v) the critical graphs \( G_1 \subset R_1 \) and \( G_2 \subset R_2 \) include as endpoints all the punctures of \( R \) as simple poles of \( q \), and

vi) each of the graphs \( G_1 \) and \( G_2 \) is connected,

vii) for every closed curve \( \tilde{\gamma} \) homotopic in \( R \) to the core curve \( \gamma \),

\[\int_{\tilde{\gamma}} |q_\gamma|^{1/2} \geq \int_{\gamma} |q_\gamma|^{1/2}.\]

**Theorem 5.** [Jenkins and Strebel] The condition that the embedded cylinder described above has maximal modulus in its homotopy class implies that \( q_\gamma \) is a global holomorphic quadratic differential on \( R \) whose restriction to the interior of the cylinder is equal to \((d\zeta)^2\). This \( q_\gamma \) is maximal in the sense that any quadratic differential \( q \) holomorphic on \( R \) satisfying \( ||q|| = ||q_\gamma|| \) and the inequality \( \int_{\gamma} |q|^{1/2} \geq \int_{\gamma} |q_\gamma|^{1/2} \) for every \( \tilde{\gamma} \) homotopic to the core curve \( \gamma \) is identically equal to \( q_\gamma \). Moreover, the critical graphs \( G_1 \) and \( G_2 \) are connected.

**Proof.** This theorem is proved in [34] except for the part concerning the components of the critical graph \( G_1 \) and \( G_2 \). To see that \( G_1 \) is connected take any two points in \( G_1 \). Since \( R \) is arcwise connected, by assumption they can be joined by an arc in \( R \). If that arc crosses the core curve \( \gamma \), cut off all the parts that lie in \( R_2 \) and join the endpoints by arcs appropriate subarcs of the core curve \( \gamma \) to obtain a closed curve \( \alpha \) in \( R_1 \cup \gamma \) that joins the two points. Then use the vertical trajectories of the cylinder to project
every point of $\alpha$ continuously to a curve that lies in the critical graph $G_1$ that joins the same two points. This shows that $G_1$ is arcwise connected. Of course, the same is true for $G_2$.

That the two conditions $|q| = |q_\gamma|$ and the inequality $\int_{\gamma} |q|^{1/2} \geq \int_{\gamma} |q_\gamma|^{1/2}$ imply that $q = q_\gamma$ follows from Grötzsch’s argument or from the minimum norm principle proved in [13]. Finally, if $(d\zeta)^2$ failed to be the restriction of a global holomorphic quadratic differential on $R$, it would be possible to deform the embedding of the cylinder inside $R$ so as to increase the modulus of the cylinder. □

**Definition 4.** We call a cyclindrical differential $q_\gamma$ with the properties of the previous theorem *separating* and any simple closed curve homotopic to $\gamma$ is also called a *separating curve*.

8. **Spinning on an arbitrary surface**

We extend the estimates for moduli of annuli given in Theorem 3 to moduli of a maximal embedded separating cylinder $C \subset R$. Just as in Theorem 4 we let $\mu_t$ be the Beltrami coefficient of $\text{Spin}_t$. However in this section we use the fact that since $\text{Spin}_t$ fixes every point on the two sides of the strip and since the conformal structure of $R$ is determined by the conformal structure of $A_r$ and side identifications of certain intervals on its boundary, our objective in this section is to estimate the modulus of the embedded annulus with this new conformal structure. We denote this new annulus by $A_r(\mu_t)$.

The measured foliations $|d\eta \circ \text{Spin}_t^{-1}|$ and $|d\xi \circ \text{Spin}_t^{-1}|$ on $R - p(t)$, where $p(t)$ is a point on the core curve of a maximal cylinder $C \subset R$ is represented by the point $t$ on the real axis in the rectangular coordinate that represents the cylinder defined in (24). In this way $\mu_t$ determines a conformal structure on $R$ and $R - p(t)$.

**Definition 5.** The real number $r(t)$ is the number for which the standard annulus $A_r(t)$ with inner and outer radii equal to $1/(r(t))$ and $r(t)$ is conformal to $A(\mu_t)$ where $\mu(t)$ is viewed as a Beltrami coefficient on $R$.

**Theorem 6.** The unique real number $r(t)$ with the property that $A_r(\mu_t)$ is conformal to $A_r(t)$ satisfies

$$t \cdot \frac{(1 - 2\pi/t)^2 + ((\log r)/t)^2}{(\sqrt{1 + ((\log r)/t)^2})} \leq \log r(t) \leq t \cdot \left(\frac{1}{t^2} + \frac{1}{(\log r)^2}\right)^{1/2}.$$  

**Proof.** Consider the the measured foliations $|d\eta \circ \text{Spin}_t^{-1}|$ and $|d\xi \circ \text{Spin}_t^{-1}|$, which are horizontal and vertical foliations on the annulus $A_r(\mu_t)$ which has
conformal structure given by $\mu_t$. Applying Lemma 4 to these foliations with $S = A_r(\mu_t)$ and $S(\tau) = A_r$ and we get

$$(4\pi \log r(t))^2 \leq [4\pi \log r](4\pi \log r(1 + \frac{t^2}{(\log r)^2})],$$

which leads to the right hand side of (31).

To obtain the left hand side we work with the conformal structure on $A_r(\mu_t)$ where the slanting level lines of $\xi \circ (Spin_t)^{-1}$ are viewed as orthogonal to the vertical lines of $\eta$. By definition the number $r(t)$ is chosen so that $A_r(\mu_t)$ is conformal $A_r(t)$ and so the modulus of this annulus is equal to

$$\frac{2 \log r(t)}{2\pi}.$$

This is bounded below by any of the fractions

$$\frac{1}{4} \cdot \frac{(\inf_{\beta} \int_{\beta} \sigma)^2}{\text{area}(\sigma)},$$

where $\beta$ is any homotopy class of closed curve on $R$ that joins the two boundary contours of the annulus and has intersection number 2 with the core curve of $A_r(t)$. We may take $\sigma$ to be the flat $\sigma(\zeta)|d\zeta|$ on $A_r(t)$ given by $\sigma(\zeta) \equiv 1$. Then

$$\int_{\beta} \sigma(\zeta)|d\zeta| = \int_{\beta} |d\zeta| \geq 4 \log r(t).$$

From the definition of extremal length one obtains

$$\frac{(2\sqrt{(t-2\pi)^2 + (\log r)^2})^2}{4 \left(\sqrt{t^2 + (\log r)^2}\right) \log r} \leq \frac{\log r(t)}{\log r},$$

This lower bound leads to the left hand side of (31). 

9. UNROLLING

Just as in the previous sections we let $C_\gamma$ be the maximal closed cylinder corresponding to a separating simple closed curve $\gamma$ contained in $R$. Then

$$\text{Strip}_\gamma/((\zeta \mapsto \zeta + 2\pi) \cong C_\gamma.$$ 

Moreover the interval $[0, 2\pi]$ along the top perimeter of $C_\gamma$ partitions into pairs of subintervals of equal length that are identified by translations or half translations of the form

$$\zeta \mapsto \pm\zeta + \text{constant}$$

that realize the conformal structure of $R$ along the top side of $C_\gamma$. 


Similarly the bottom perimeter of $C_\gamma$ partitions into pairs of intervals similarly identified that realize the local conformal structure of $R$ on the bottom side.

In this section there is a fixed value of $r$ which is determined by separating curve and the conformal structure of $R$. The value of $r$ for annulus $A_r$ is that value that makes that $A_r$ conformal to the annulus in $\mathbb{R}$ that maximizes this modulus. But we also consider annuli $A_{r_n}$ that are maximal annuli in $R_n$ where $R_n$ is obtained from the same strip factored by the translation $\zeta \mapsto \zeta + 2\pi n$. The same Beltrami coefficient $\mu$ determines a well-defined conformal structure on each surface $R_n$ and our task is to estimate the modulus of the embedded annulus in $R_n$ and in $R_n - p$. There is one case when our estimate fails. That is when $R$ is the Riemann sphere minus four points and the separating curve is one that separates two of these from the other two. Then the Teichmüller space $T(R)$ has dimension 1 and the comparison of the moduli for the annuli on $R_n$ and on $R_n - p$ breaks down.

As we have just explained we use the same strip to define a sequence of Riemann surfaces $R_n$ for any integer $n \geq 1$. By definition $R_1 = R$ and $R_n$ is the Riemann surface obtained by factoring $\text{Strip}_\gamma$ by the translation $\zeta \mapsto \zeta + 2\pi n$ and using the same identifications along the top and bottom horizontal sides of the strip generated by those that determine $R_1$. We let $C_n$ be the associated cylinder and $A_n$ the annulus which is conformal to $C_n$.

**Lemma 5.** Assume $\gamma$ is a separating closed curve in $R$ and let $q_n$ be the quadratic differential $(d\zeta)^2$ on $C_n$ and $\Delta = \{ t : |t| < 1 \}$. Then

$$
\Delta \ni t \mapsto [f^t|q_n|/q_n]_{R_n} \in \text{Teich}(R_n)
$$

is an injective isometry.

**Proof.** Since the domain $A_{r_n}$ has extremal modulus in $R_n$ the quadratic differential $q_n$ on $A_{r_n}$ is the restriction of an integrable holomorphic quadratic differential on $R_n$. Thus the statement follows from Teichmüller’s theorem. \qed

We let $p$ be a point on the core curve $\gamma$ of $C_\gamma$ corresponding to $\zeta = 0$ in the strip. Similarly, we let $p_n$ corresponding to the same point $\zeta = 0$ in the strip but in the factor space

$$
C_{\gamma_n} \cong \text{Strip}_\gamma/(\zeta \mapsto \zeta + 2\pi n).
$$

With these choices Bers’ fiber space theorem identifies the fiber spaces $F(\text{Teich}(R_n))$ and $F(\text{Teich}(R))$ with Teichmüller spaces $\text{Teich}(R_n - p_n)$.
and $Teich(R-p)$, that is, the Teichmüller spaces of the punctured Riemann surfaces $R-p$ and $R_n-p_n$.

**Lemma 6.** Assume $q_n$ is the holomorphic quadratic differential on $R$ whose restriction to the maximal annulus $A_{r_n}$ is equal to $(d\zeta)^2$. Also assume $V_n$ is a tangent vector to $Teich(R_n)$ for which $V_n = |q_n|/q_n$ and $R_n$ is Riemann surface with marked complex structure $\tau_n$. If $Car_{Teich(R)}(\tau,V) = Kob_{Teich(R)}(\tau,V)$ then for every integer $n \geq 1$

$$Car_{F(Teich(R_n))}(\tau_n,V_n) = Kob_{F(Teich(R_n))}(\tau_n,V_n).$$

**Proof.** Choose the point $p_n$ on $R_n$ which corresponds in the $\zeta$ coordinate to $\zeta = 0$. The integrable holomorphic quadratic differential $q_n$ on $R_n$ is also integrable and holomorphic on $R_n - p_n$ and the Teichmüller disc $[[t|q_n|/q_n]$ in $Teich(R_n)$ lifts to the Teichmüller disc $[[t|q_n|/q_n]]$ in $Teich(R_n - p_n)$. □

**Lemma 7.** Assume the hypotheses of the previous lemma and that $Teich(R)$ has dimension more than 1. Also assume $\tilde{p}$ is the point 0 on the horizontal line $\eta = 0$ in the strip and $p$ is equal to the point the core curve of the annulus $A_r$ covered by $\tilde{p}$. Also put $p_n$ equal to the point on the annulus $A_{r_n}$ covered by $\tilde{p}$. Then $Spin_{2\pi n}$ represents an element of the mapping class group of $Teich(R_n - p)$ that preserves the Teichmüller disc

$$D(q_n) = \{[[s|q_n|/q_n]] \in Teich(R_n - p) : |s| < 1\}$$

and $Spin_{2\pi n}$ preserves $D(q_n)$ restricted to this disc it is a hyperbolic transformation.

**Proof.** $Spin_{2\pi n}$ preserves $D(q_n)$ because on $R_n$ it is homotopic to the translation $\zeta \mapsto \zeta + 2\pi n$ and $q_n(\zeta)(d\zeta)^2 = (d\zeta)^2$ is automorphic for this translation. Since the dimension of $Teich(R)$ is more than 1 the translation length of this transformation is positive.

Note that $D(q_n)$ is a conformal disc lying in the Teichmüller space $Teich(R_n)$. On it the mapping class element $Spin_{2\pi n}$ acts as the translation $\zeta \mapsto \zeta + 2\pi n$. Because it represents this translation on the strip it preserves this strip. Because of the extremal length estimates given, the two points $+\infty$ and $-\infty$ on the boundary of the strip are realized as distinct points on the extremal length boundary of Teichmüller space, which means the $Spin_{2\pi n}$ realizes a hyperbolic transformation on $D(q_n)$. □

Several authors have used spin elements of the mapping class group in different contexts and different ways. But none of them in the manner used here where $Spin_{2\pi n}$ is viewed as spinning once around a simple curve on a
changing family surfaces \( R_n \) whose complexity increases proportionately to \( n \). See \([6]\) and \([22]\).

For readers who know the theory of pseudo-Anosov elements of the mapping class group and also know that \( \text{Spin}_{2\pi n} \) cannot be a pseudo-Anosov map, we can already make the conclusion of Theorem 1. We prefer to find a contradiction by using the same argument used to prove Theorem 4 in section 6 together with the estimates described in the next section.

10. The waist sequence

Just as in previous sections we fix the point \( p \) in a Riemann surface \( R \) lying on the core curve of a separating cylindrical differential \( q_\gamma \) and assume \( p \) corresponds to \( \zeta = 0 \) in the maximal cylindrical strip \( C_{\gamma} \) which is covered by the horizontal strip \( \text{Strip}(q_\gamma) \).

**Definition 6.** Let \( C_n \) be the cylinder conformal to

\[
\text{Strip}(q_n)/ (\zeta \mapsto \zeta + 2\pi n)
\]

and \([\mu_t]\) the equivalence class of Beltrami coefficient of \( \text{Spin}_t \) in \( F(\text{Teich}(R_n)) \). Then we call

\[
\alpha_n(t) = [\mu_t] \in \text{Teich}(R_n - p)
\]

for \( \mu_t, 0 \leq t \leq 2\pi n \) the \( n \)-th spin curve.

The spin curve \( \alpha_n(t) \) determines a waist curve \( \beta_n(t) \) in \( \mathbb{C} \). To obtain \( \beta_n \) we use the Bers isomorphism that realizes \( \text{Teich}(R_n - p) \) as a fiber space over \( \text{Teich}(R_n) \). In the normalization used by Bers \( R_n \) is covered by Fuchsian group acting on the upper half plane and one assumes the point \( p \) on the surface \( R_n \) is covered by the point \( i \) in the upper half plane. Then Bers isomorphism of \( \text{Teich}(R_n - p_n) \) with \( F(\text{Teich}(R_n)) \) is realized by the map

\[
\text{Teich}(R_n - p_n) \ni [\mu] \mapsto ([\mu], w^{[\mu]}(i)) \in \text{Teich}(R_n) \times \Delta_{\mu}.
\]

Here \( \Delta_{\mu} \) is by definition the unit disc \( \Delta \) with moving complex structures given by the Beltrami coefficients \( \mu \) that represent elements \([\mu]\) of \( \text{Teich}(R_n) \).

**Definition 7.** [the waist sequence] The \( n \)-th waist curve is the plane curve

\[
\beta_n(t) = w^{[\mu_t]}(i)
\]

given by the second entry in (32), where \([\mu_t] = \Psi(\alpha_n(t))\) and \( \alpha_n(t) = [\mu_t] \in \text{Teich}(R_n - p), 0 \leq t \leq 2\pi n \).
We will use the upper and lower bounds from Theorem 2 to estimate the length of \( \beta_n(t) \), \( 0 \leq t \leq 2\pi n \) with respect to the Teichmüller density \( \lambda_{A_{rn}} \) on the annulus \( A_{rn} \) in two ways. From Theorem 2 of section 3 we have for all Riemann surfaces \( R \)

\[
\frac{1}{2} \rho_R \leq \lambda_R \leq \rho_R.
\]

For each positive integer \( n \) we apply this inequality to the subannuli \( A_{rn} \) contained in \( R_n \).

We use notation \( \lambda_n(\beta_n) \), \( \rho_n(\beta_n) \) and \( c_n(\beta_n) \) to denote the lengths of the curve \( \beta_n \), respectively, with respect to \( \lambda_n \), \( \rho_n \) and \( c_n \), that is, the Teichmüller the Poincaré and the Carathéodory metrics, respectively.

Our assumption

\[
Car_{Teich(R)} = K_{ob_{Teich(R)}},
\]

implies the existence of a holomorphic function \( g_n \) defined on \( A_{rn} \) with image in \( \Delta \) for which \( g_n(1) = 0 \) and for which

\[
g_n \circ f_n(t) = t \quad \text{for all } t \in \Delta.
\]

Thus we have

\[
\lambda_n(\beta_n) \leq \rho_n(\beta_n) = c_n(\beta_n) \leq \frac{M}{n} \cdot n = M.
\]

The equality in the middle of (35) follows from (34) and the existence of the number \( M \) independent of \( n \) in the second inequality of (35) follows from the left hand inequality of (31) in Theorem 6.

Using the left hand side of (33) to estimate the lengths \( \lambda_{A_{rn}}(\beta_n) \) of \( \beta_n \) from below we have have

\[
\lambda_{A_{rn}}(\beta_n) \geq \frac{1}{2} \rho_{A_{rn}}(\beta_n) \geq \frac{n}{2 \log r}.
\]

Since \( \beta_n \) is comprised of \( n \) intervals of equal length in the interval \( [0, 2\pi n] \), this estimate implies \( \lambda_{A_{rn}}(\beta_n) \) is unbounded in its dependence on \( n \), which contradicts (35) and proves Theorem 1.

**Appendix A: Schwarz-Pick metrics**

If \( h \) is a holomorphic function from a complex manifold \( M \) to a complex manifold \( N \), the derivative \( dh_\tau \) of \( h \) at \( \tau \in M \) is a complex linear map from the tangent space to \( M \) at \( \tau \) to the tangent space to \( N \) at \( h(\tau) \). From Schwarz’s lemma it follows both \( C_M \) and \( K_M \) have the pull-back contracting property, namely, for any point \( \tau \in M \) and tangent vector \( V \) at \( \tau \), \( C \) and \( K \) satisfy

\[
K_N(h(\tau), dh_\tau(V)) \leq K_M(\tau, V)
\]
and
\[
C_N(h(\tau), dh_\tau(V)) \leq C_M(\tau, V). \tag{38}
\]

Any infinitesimal form on a complex manifold satisfying this pull-back contracting property for all holomorphic functions \(h\) is called a Schwarz-Pick metric by Harris in [17] and by Harris and Earle in [7]. They also observe that any infinitesimal form with this property must necessarily lie between \(C_M(\tau, V)\) and \(K_M(\tau, V)\).

APPENDIX B: EQUIVALENCE OF THE INFINITESIAL FORMS OF \(C\) AND \(K\).

The Bers embedding is a biholomorphic map \(\Phi_\tau\) from \(\text{Teich}(R)\) onto a bounded simply connected domain in the Banach space of bounded cusp forms on the Riemann surface \(R_\tau\) which is the differentiable surface together with a marked complex structure \(\tau\) indicated by the point \(\tau \in \text{Teich}(R)\). A function \(\varphi(z)\) holomorphic in the upper half plane is a bounded cusp form for \(R_\tau\) if it satisfies the inequality
\[
\sup_{\{z=x+iy: y>0\}} |\varphi(z)|y^2 < \infty
\]
and if
\[
\varphi(\gamma(z))\gamma'(z)^2 = \varphi(z)
\]
for every \(\gamma\) in a Fuchsian covering group \(\Gamma_\tau\) of \(R_\tau\).

We consider the set of quasiconformal self mappings \(z \mapsto w\) of the complex plane for which \(w \circ \Gamma_\tau \circ w^{-1}\) is a subgroup of the group of Moebius self-mappings of \(\mathbb{C}\). Any \(w\) in this set satisfies \(\{w, z\} = \varphi(z)\) is a bounded cusp form in the lower half plane where \(\{w, z\}\) is notation for the Schwarzian derivative, namely,
\[
\{w, z\} = N' - (1/2)N^2 \quad \text{and} \quad N = \frac{w''}{w'}. \tag{1.2}
\]

Given any Beltrami coefficient \(\mu\) for which \(||\mu(z)||_\infty < 1\) and for which
\[
\mu(\gamma(z))\overline{\gamma'(z)} = \mu(z)
\]
the quasiconformal selfmapping \(z \mapsto w\) of \(\mathbb{C}\) with Beltrami coefficient \(\mu\) in the upper half plane and identically equal to 0 in the lower half plane has the property that holomorphic mapping \(z \mapsto w\) is holomorphic and univalent in the lower half plane. We put \(\mu\) equal to the Beltrami coefficient of \(w\) and \(\Phi_\tau(\mu) = \{w, z\}\).

Then Bers shows that the image of \(\Phi_\tau : \text{Teich}(R_\tau) \to B_\tau\) in \(B_\tau\) contains the open ball of radius 2 and is contained in the ball of radius 6, [2]. Assume
$V$ is a tangent vector $||V||_{B_{r}} = 1$. Then the complex linear map $V \mapsto (1/6)V$ extends by the Hahn-Banach theorem to a complex linear map $L : B_{r} \rightarrow \mathbb{C}$ with $||L|| \leq 1/6$. Then because $\Phi_{r}(Teich(R))$ is contained in the ball of radius 6, $L(\Phi_{r}(Teich(R)))$ is contained in the unit disc and $L \in \mathcal{G}$. Thus from definition (5)

$$C_{Teich(R)}(\tau, V) \geq 1/6.$$ 

On the other hand, put $f(t) = 2tV$. Then since the image of $\Phi_{r}$ contains $f(\Delta)$ from definition (4)

$$K_{Teich(R)}(\tau, V) \leq 1/2.$$ 

Putting the preceding two inequalities together we find

$$(39) \quad (1/3)K_{Teich(R)}(\tau, V) \leq C_{Teich(R)}(\tau, V).$$

(39) and (7) together yield (8). Inequality (39) has been pointed out to me by Stergios Antonokoudis and the same argument is given by Miyachi in [28] to prove the parallel result for asymptotic Teichmüller space.

**Appendix C: Kra’s equality theorem**

Inequality (9) is a feature that distinguishes between different tangent vectors. In fact by a theorem of Kra [21] for some tangent vectors one can have equality

$$(40) \quad C(\tau, V) = K(\tau, V).$$

Kra’s theorem states (40) holds when $V$ has the form $\partial V = |q|/q$ where $q = \omega^{2}$ and $\omega$ is a multiple of an element of a canonical basis for the abelian differentials of the first kind, see Theorem 4.2 of McMullen’s paper [26].

Kra’s result is obtained by using the Riemann period relations and Ahlfors’ first variation formula for the motion of a point $w^{\mu}(z)$ in terms of the variation of the Beltrami coefficient $\mu$, see [2]. This formula is sometimes called Rauch’s variation formula [30] when it is applied to the variation of a diagonal entry $\int_{\alpha_{i}} \omega_{i}$ of the period matrix. When first understood Rauch’s observation was important because it showed that the complex structure coming from the unit ball in the complex Banach space of Beltrami coefficients had to coincide with the complex structure coming from the embedding of Teichmüller space in the Siegel upper half plane.

We call a disc that has the form $[t|q]/q$ for $|t| < 1$ where $q = \omega^{2}$ where $\omega_{i}$ is a diagonal entry in the period matrix an abelian Teichmüller disc. A consequence of this infinitesimal result is that for any two points $\tau_{1}$ and $\tau_{2}$ in the same abelian Teichmüller disc, $C(\tau_{1}, \tau_{2}) = K(\tau_{1}, \tau_{2})$. Note that
the usual simple closed curves \( \alpha_j \) and \( \beta_j \), \( 1 \leq j \leq g \), taken as a basis for the homology group of a surface of genus \( g \) are all non-separating and homologous to zero.

An instructive example is a genus 2 surface \( X \) with is a separating cylindrical quadratic differential \( q \) that separates the surface into two genus 1 components. It is possible for \( q \) to have one double zero in each component. In that case \( q = \omega^2 \) where \( \omega \) is a holomorphic differential 1-form on \( X \). By Theorem 1 it cannot be a differential occurring in the form \( \int \omega_i \) where \( \omega_i \) is part of a homology basis.

In contrast to abelian tangent vectors, any tangent vector \( V \) corresponding to a separating cylindrical differential will originate from what are sometimes called \( \mathcal{F} \)-structures \([10]\) and sometimes called half-translation structures \([11]\).

**Appendix D: Lakic’s dual formulas for the Teichmüller density**

If \( \psi \) is a holomorphic \( q \)-differential \( \psi \) on \( R \) except for possibly a simple pole at a point \( p \) in \( R \) we use the notation \( \text{res}_p(\psi) \) to mean the residue of \( \psi \) at \( p \). In general, if \( \psi \) is a \( q \)-differential then \( \text{res}_p(\psi) \) is a \((q-1)\)-differential. In Definition 2 we have defined the Teichmüller density at a point \( p \) on any Riemann surface.

**Theorem 7.** Let \( p \) be a point in a Riemann surface \( R \) and \( Q(R - p) \) be the space of quadratic differentials \( \psi \) which are holomorphic on \( R - p \) but which may have a simple pole at \( p \) and for which \( ||\psi|| = \int R |\psi| < \infty \). Then the Teichmüller density \( \lambda(t)dt \) on \( R \) satisfies

\[
\lambda_r(t)|dt| = \sup_{\psi} \{ \pi \cdot \text{res}_p(\psi) \} = \inf_V \{ ||V||_{\infty} \},
\]

where the supremum is taken over integrable holomorphic quadratic differentials \( \psi \) in \( Q(R - p) \) with \( ||\psi|| \leq 1 \) and where the infimum is taken over all continuous vector fields \( V \) on \( R \) that are equal to zero on the boundary of \( R \) and equal to 1 at \( p \).

**Proof.** See \([15]\) and \([16]\). \(\square\)

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