Information ratchets exploiting spatially structured information reservoirs

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Information ratchets extract work from heat baths using low entropy information reservoirs. We find that the structure of the reservoirs affects both their performance and phase diagram. By deriving exact probabilities of recurrence in d-dimensional reservoirs we calculate the net extraction of work. Performance is characterised by two critical dimensions (i) For $d < 3$ driven exploration of the reservoir is essential to extract work (ii) For $d > 4$ purely diffusive exploration is optimal. This has important consequences for computation where sequential operations constitute a 1D path.

I. INTRODUCTION.

There has been significant recent progress in understanding the connection between computation, information theory and thermodynamics [1–8]. Through frameworks such as stochastic thermodynamics [9, 10], order can be treated as a resource [8, 11]. Information ratchets are mechanised Maxwell’s demons [12, 13] that extract work from a single heat reservoir while randomising reservoirs of low entropy, or information reservoirs. Conceptually such ratchets are important because they elucidate the conceptual relationship between information, computation and thermodynamics and highlight the importance of information as a fundamental thermodynamic quantity [14].

A pioneering model of an information ratchet is due to Mandal and Jarzynski (MJ) [13]. Their demon extracts work from a heat reservoir while randomising a binary 1D information reservoir or “tape.” Essential for the MJ model is a mechanism whereby the tape slides past the demon (or equivalently the demon moves along the tape) at a constant speed exploiting some frictionless mechanism. Barato and Seifert [8, 16] (BS) later adapted the MJ model to a fully stochastic 2-state model. The BS model is based on three assumptions about the tape. (i) The BS model to a fully stochastic 2-state model. The BS model is based on three assumptions about the tape. (i) As with the MJ model the tape is of infinite size. Yet, in contrast to MJ they also assume (ii) that the tape has no spatial structure (in that sense it is not a conventional tape but rather an infinitely dimensional information reservoir and the BS demon is thus bathing in a perfectly mixed symbol-gas). (iii) Finally and again like the MJ model, BS assume that the low entropy state of the reservoir is due to an over-representation of one of the symbols in the reservoir.

It has been pointed out [14, 15] that this third assumption is not necessary and that demons could exploit correlations rather than symbol imbalances. Explicit models of such demons have been proposed [16, 20]. Yet, dropping this third assumption also requires relaxing the second assumption. Demons that exploit correlations between symbols necessarily require information reservoirs that are spatially structured. We will henceforth refer to such information reservoirs synonymously as tapes, irrespective of their dimension and spatial structure. The impact of the spatial structure of the tape on the performance of the demon has not yet been explored.

In this letter, we present a generalised expression for the recurrence properties of biased random walkers in d-dimensional discrete spaces. We show that the properties of such random walks impact significantly on the ability of the demon to extract work using a tape. Particularly in low dimensional spaces a random walker will frequently re-sample the same sites. Introducing a bias overcomes this, but requires work and thus reduces the scope for network extraction. The effect is particularly striking in the case of 1D tapes, where even in the best case scenario of an initial zero entropy tape, independently of the demon parameters, net-gain of work can only be achieved when the bias of the random walk is within a narrow range.

II. MODEL & WORK CALCULATION.

We consider a demon characterised by the possible states $\{H, L\}$ associated with energies $\Delta E$ and 0 respectively. At any one time the demon interacts with a symbol of a d-dimensional reservoir where the symbol 1 occurs with probability $\epsilon$. The states of the demon are entirely correlated with the tape symbols, so that states always come as the pairs $\{H, 1\}$ and $\{L, 0\}$. Transitions between these compound states can occur due to thermal fluctuation with rate $k^+$ from $\{L, 0\} \to \{H, 1\}$ and reverse rate $k^-$, where $k^- / k^+ = \exp(\Delta E / T)$ at temperature $T$ of the heat bath. Additionally the tape can transition to an adjacent symbol on the tape with rate $\gamma$ according to some general mechanism that could physically include the demon transitioning along the tape.

Consistent with the BS model, when the demon is in state $\{L, 0\}$ and the tape transitions to a symbol 1, then the demon is excited to state $\{H, 1\}$ with $\Delta E$ work expended by a work reservoir. Conversely if the system is in state $\{H, 1\}$ and the tape transitions to a 0 symbol then

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The transition diagram is illustrated in Fig. (1). The tape induced transition rates are indicated on the transition diagram. Additionally, there are vertical thermal transitions happening with rates $k^-$ and $k^+$; for simplicity these are indicated separately.

$\Delta E$ work is extracted from the environment by dropping to the state $\{L, 1\}$. Tape symbols can transition to the value currently associated with the demon at the expense of no work. As with the BS model, we may assume that the rate $\gamma \ll k_+, k_-$ such that the demon always reaches a steady state $p_s \equiv p_s(\{H, 1\})$. This implies the probability of a 1 symbol on any visited tape site is also equal to $p_s$.

In spatially structured reservoirs, such as the 1D tape, the demon can only move to neighbouring symbols. Even for tapes containing an infinite number of symbols this entails that the probability of encountering new and hence unexploited tape elements reduces over time. To quantify this dimensionality constraint, we construct a mean field model. First, we extend the model by constructing 4 pseudo-states $\{\{H, 1, n\}, \{H, 1, o\}, \{L, 0, n\}, \{L, 0, o\}\}$ where $n$ indicates that the currently bound symbol has not been visited before and $o$ indicates that it has. Transitions $\{H, 1, o\} \leftrightarrow \{H, 1, n\}$ and $\{L, 0, o\} \leftrightarrow \{L, 0, n\}$ incur no exchange of heat in the demon nor expenditure of work. Transitions $\{H, 1, o\} \leftrightarrow \{L, 0, n\}$ and $\{H, 1, n\} \leftrightarrow \{L, 0, n\}$ can occur via the thermal mechanism or via exchange of work due to the transitions of the tape. Finally transitions $\{H, 1, o\} \leftrightarrow \{L, 0, n\}$ and $\{L, 0, o\}$ coincide with exchange of work. Transitions to ‘new’ unseen sites are controlled by the probability $P_u$ which is derived from the random walk statistics and depends on the bias of the walk and the dimension of the tape. Consequently, when a site is sampled for the first time, it contains a 1 symbol with probability $c$; otherwise the probability of a 1 is $p_s$ for previously seen sites. A transition diagram is illustrated in Fig. (1). As in the MJ model, the device can operate in three phases characterised by (i) positive work extraction, (ii) a reduction in entropy of the tape (an information eraser) and (iii) no extraction or erasure, a ‘dud’. In this letter we discuss the balance between the extraction and dud phases, but note similar arguments for erasing phases follow.

This model implicitly assumes that the demon samples from visited and unvisited sites according to the average statistics of the random walk revealing a mean field assumption. The model thus captures the mean behaviour of the demon correctly, but cannot predict correctly fluctuations. Finally we note that like the BS model, a key assumption in the dynamics relies on the separation of timescales between tape and demon. Relaxing this constraint introduces unaccounted for correlations between the demon and specific visited & unvisited sites which will effect the performance of the demon, however we include the general result as a crude approximation for insight.

The transitions indicated in Fig. (1) are sufficient to formulate a master equation with the steady state solution

$$p_s = P(\{H, 1, o\}) + P(\{H, 1, n\}) = \frac{k^+ + \epsilon \gamma P_u}{k^+ + k^- + \gamma P_u}. \quad (1)$$

With this solution the work the demon can extract when the tape transitions to new elements can be calculated, if $P_u$ is known,

$$\langle \dot{W}_u \rangle = \langle [P(\{H, 1, o\}) + P(\{H, 1, n\})] w(\rightarrow \{L, 0, n\}) - [P(\{L, 0, o\}) + P(\{L, 0, n\})] w(\rightarrow \{H, 1, n\}) \rangle \Delta E = (p_s \gamma P_u (1 - \epsilon) - (1 - p_s) \gamma \epsilon P_u \Delta E. \quad (2)$$

Here, $w(\rightarrow \{L, 0, n\})$ is the transition rate to the state $\{L, 0, n\}$ from the states $P(\{H, 1, o\})$ and $P(\{H, 1, n\})$ and can be read off from Fig. (1). The work extracted from the visited tape elements equals zero since the distributions characterising the demon and tape are statistically identical. Indeed, we find $\langle W_e \rangle = \Delta E (p_s, w(\rightarrow \{L, 0, o\}) + (1 - p_s) w(\rightarrow \{H, 1, o\})) = (p_s \gamma (1 - P_u) (1 - p_s) - (1 - p_s) \gamma (1 - P_u) p_s) \Delta E = 0$. The mean work extracted per tape transition then reads

$$\gamma^{-1} \langle \dot{W} \rangle = \frac{(k^- + \epsilon (k^+ + k^-)) P_u (p) T}{k^+ + k^- + \gamma P_u} \ln \left( \frac{k^-}{k^+} \right)$$

$$= \frac{(1 - \epsilon (1 + \epsilon \Delta E))(k^- + k^+ + akP_u(p))}{(1 + \epsilon \Delta E)(k^+ + k^- + akP_u(p))} \quad (3)$$

where $k = k^+ + k^-$ is the mean escape rate characterising the timescale of the demon, $a = \gamma / k$ characterises the relative timescales of the tape and the demon, and $p$ quantifies the directional bias in tape transitions. An optimal demon maximises work extraction, which is achieved by choosing $a \rightarrow 0$ ($k \gg k$), $\epsilon = 0$ and $\Delta E = T(1 + W(e^{-1}))$ such that $\gamma^{-1} \langle \dot{W}_{\text{max}} \rangle = W(e^{-1}) P_u T \approx 0.27 P_u T$ where $W$ is the Lambert W function.

At this point we consider the cost of driving the tape. Unlike in the BS model, this is a necessary inclusion because the performance of the demon, when the physical features of the tape are incorporated, is explicitly dependent on $P_u(p)$ and thus the bias $p$ which incurs cost. This in turns defines the random access assumption $P_u(p) = P_u(1/2) = 1$.

To assess the cost of driving the tape we naturally require the timescale of the tape fluctuations to be $\gamma$ for
consistency, but allow biased transitions in either direction through the use of rates $r^+$ and $r^-$ such that in characteristic time $1/\gamma$ we can expect a transition to the right with probability $p = r^+/ (r^+ + r^-)$. Consequently we appreciate that at temperature $T$ the work rate required to drive the tape (per tape transition) is given by

$$
\gamma^{-1}\langle \dot{W}_{\text{tape}} \rangle = T \frac{r^+}{r^+ + r^-} \ln \left( \frac{r^+}{r^-} \right) + T \frac{r^-}{r^+ + r^-} \ln \left( \frac{r^-}{r^+} \right)
$$

$$
= T(2p - 1) \ln \frac{p}{1 - p}
$$

(4)

arising from the assumption that all tape positions are of equal internal energy owing to the intervention of the work reservoir. Crucially, both the maximum possible work extraction performance and cost of driving the tape are linear in temperature such that the upper bound on the total performance is determined entirely by $P_u(p)$.

III. EXACT TRANSIENT SOLUTION FOR A DEMON AND 1D TAPE.

In Appendix A we calculate that the probability that $n$-th transition of a random walk in 1D to an unvisited site is given by

$$
P_a(p, n) = |1 - 2p| + C(m, p) F_1 \left( \frac{1}{2} + m; 2 + m; 4p(1 - p) \right)
$$

(5)

$$
C(m, p) = \frac{2^{1 + 2m} (1 - p)^m + m \Gamma \left( \frac{1}{2} + m \right)}{\sqrt{\pi \Gamma \left( \frac{1}{2} + m \right)}}
$$

where $F_1$ is the hypergeometric function and $m = \lfloor n/2 \rfloor$. This gives the long term limit $\lim_{n \to \infty} P_a(p, n) = P_u(p) = |1 - 2p|$ whereupon the distinction between discrete and continuous time becomes unimportant. Choosing $p = r^+/ (r^+ + r^-)$ gives the probability that the $n$-th transition of the tape will be to a site that has never been visited before. Particularly, for $p = 0, 1$ this probability is always 1, whilst for $p = 1/2$ in the $n \to \infty$ limit the probability vanishes. A particular solution exists for $p = 1/2$, $P_u(1/2, n) = \Gamma[n + 1/2] / (\sqrt{\pi} \Gamma[n + 1])$.

Conversion to a continuous time process gives

$$
P_a(p, t) = \sum_{n=0}^{\infty} P_a(p, n) e^{\gamma t} / n!
$$

which may be used to calculate an exact form for the work extraction rate by substituting into Eq. (4), valid in the $\gamma \ll k, a \to 0$, regime. In particular an asymptotically exact solution exists for the unbiased walker $P_u(1/2, t) \simeq (2/\pi \gamma t)^{1/2}$. Convergence to the $t \to \infty$, steady state, behaviour consequently follows a power law for $p = 1/2$, but exhibits increasingly fast exponential tails as $p$ deviates to 0, 1. However, for $p = 1/2$, within 1000 tape transitions, or $\sim 1000 \gamma^{-1}$ seconds, the probability of discovering a new site has dropped to around 0.025, and to 0.008 within 10000 transitions with commensurate loss of work extraction (see Appendix A for details).

At this point we explicitly deal with the steady state, long term limit, behaviour of the demon for the remainder of the letter. By considering the cost of a biased/driver tape, the central object of inquiry is now the net work extracted, given by the difference $\gamma^{-1}\langle \dot{W}_{\text{net}} \rangle = \gamma^{-1}\langle \dot{W} \rangle - \gamma^{-1}\langle \dot{W}_{\text{tape}} \rangle$. We observe that in the undriven regime ($p = 1/2$) both the work extracted and the work required are 0 - a diffusive tape requires no work to drive it, but no work can be extracted either in the long run.

Next, by expanding the work extracted and expended about $p = 1/2$ we observe a linear dependence in $(p - 1/2)$ in the extracted work, but quadratic dependence in the work spent driving the tape

$$
\gamma^{-1}\langle \dot{W}_{\text{tape}} \rangle \simeq 8T (p - 1/2)^2 + O ((p - 1/2)^3),
$$

(6)

revealing that it is always possible to extract net positive work from the 1D tape in the limit of small driving, provided the random access model is capable of work extraction, since the cost in this regime scales more slowly than the extracted work. This net positive work may be very small compared to the possible work that can be extracted using the tape with random access.

FIG. 2. Phase diagram. Black areas indicate an information erasing phase, white areas indicate work extracting phase whilst grey areas indicate ‘dud’ phases that either utilise work from the work reservoir, but do not erase or that extract less work than is required to drive the tape. The top left indicates the phase diagram for the BS model which here corresponds to the limit $p \to 1/2$ in our machine, the top right to a bias of 0.51, the bottom left to a bias of $p = 0.05 \simeq 0.5348$ and finally on the bottom right a bias of $p > p_c \simeq 0.57$ at which point no net work extraction is possible for any parameters. The $x$ axis indicates the probability distribution of the unseen tape $\epsilon$, whilst the $y$ axis is the quantity $\gamma k = (\epsilon \Delta E / T - 1) / (\epsilon \Delta E / T + 1)$. Dashed red lines indicate contours of optimal $\Delta E$ given $\epsilon$ whilst blue dots indicate phase points corresponding to the maximum possible performance of a such a demon. We set $\gamma = 10^{-4}, T = 1$ and $k = k^+ + k^- = 1$. 
However, as the tape bias is increased, the work required to drive the tape rises faster than the resultant increase in the incoming rate of unseen sites. Hence, the net-extracted work tends to zero at large biases. Consequently we observe that net work extraction is only possible in a defined range $[1 - p_c, p_c]$ (excluding $p = 1/2$), where $p_c = (1 + eW(e^{-1}))^{-1} \approx 0.569$, and find two (symmetrical) optimal biases given by $p_{\text{opt}} = (1 - p_{\text{opt}}) \exp[W(e^{-1}) + \frac{1}{2}(p_{\text{opt}} - (1 - p_{\text{opt}}))]$ which has an approximate solution (derived in Appendix E) $p_{\text{opt}} \simeq 1/2 \pm W(e^{-1})/8 \approx \{0.535, 1 - 0.535\}$. Further, we define $\eta$ to be the ratio of the net work extraction at optimal bias to the total work that can be extracted from the random access model. In 1D we find $\eta = (2p_{\text{opt}} - 1)\left(1 + W(e^{-1})\ln\frac{p_{\text{opt}}}{1-p_{\text{opt}}}\right) \approx 0.035$ quantifying the dramatic reduction in performance of such a machine. Moreover the required finite bias limits the net work that can be extracted from higher entropy tapes enlarging the ‘dud’ regions in the 1D phase diagram (see Fig. 2).

IV. GENERALISATION TO $d > 1$ RESERVOIRS.

We now generalise from tapes to reservoirs with $1 < d < \infty$, thus bridging the gap between the MJ model that assumed conventional 1D tapes and BS model that assumed tapes/reservoirs of infinite dimension. We can characterise such situations by considering a demon/tape that performs a random walk on arbitrary $d$-dimensional lattices. If the rates in each positive and negative spatial dimension are $r^+/d$ and $r^-/d$ then, conveniently, the work rate associated with driving such a random walk is independent of the dimensionality.

Our next main result is an expression for $P_u(d, p)$ for arbitrary dimensions $d$ and bias $p$, obtained by adapting results by Montroll [21]. An ansatz solution allows us to obtain the generating function for the conditional probability of the walker at a particular time, having started at the origin, which in turn allows for a general expression (see Appendix B for a detailed derivation)

$$P_u(d, p) = \left[\int_0^\infty e^{-z} z^d I_0(2z \sqrt{p(1-p)}) dz\right]^{-1}, \quad (7)$$

where $I_0$ is the modified Bessel function of the first kind. The extracted work for arbitrary bias for the first few dimensions is shown in Fig. 3. We note that this expression has continued the domain to $d \in \mathbb{R}_+$, where it remains analytic for $p \neq 1/2$. We discuss the behaviour of such a function with $d \in \mathbb{R}_+$ in order to better understand and illustrate the case $d \in \mathbb{Z}_+$, but remain agnostic on its physical significance. We note that at complete bias, $p = 0, 1$, the expression reduces to $\int_0^\infty e^{-z} dz = 1$ reflecting that the essentially ballistic motion is guaranteed to always discover new sites with each transition. For $d = 1$ and $d = 2$ this integral can be evaluated exactly, thus recovering the 1-D result $P_u(1, p) = |1 - 2p|$, and the 2-D result $P_u(2, p) = \pi/(2K[4p(1-p)])$ where $K$ is the complete elliptic integral of the first kind. Immediately we find our next main result. Since the integrals above converge, for $p = 1/2$, only for $d > 2$ we identify the first critical dimension, $d = 3$, which mirrors the critical dimension separating recurrence and transience in random walks due to Polyá, above which work can be extracted in the limit $t \to \infty$ at 0 bias and below which no work can be extracted. This behaviour, continued into $d \in \mathbb{R}_+$, is shown in Fig. 4. We note the recovery of the random access limit $\lim_{d \to \infty} P_u(d, 1/2) = 1$.

Our next main result concerns the qualitatively distinct behaviour observed in how a demon set up would maximise the net work extracted given that there is an increasing cost associated with biasing the reservoir (random walk). The distinct behaviours are whether the optimal bias is finite or 0. We investigate this by considering whether small perturbations away from $p = 1/2$ increase or decrease the net work extracted. Since the above integral is not analytic at $p = 1/2$ this then requires a more careful asymptotic analysis which is carried out in Appendix C. To summarise, around $P_u(d, 1/2)$, we find that for $d > 4$ there is an $\mathcal{O}(p-1/2)^2$ contribution which exists alongside a contribution in $\mathcal{O}(p-1/2)^{d-2}$ for all $d$. Additionally a contribution in $\mathcal{O}(p-1/2)^{2} \ln(\sqrt{2d}(p-1/2))$ arises when $d$ is an even integer. Meanwhile the cost of driving the reservoir, per transition, has an asymptotic expansion cap-
FIG. 4. Behaviour of $P_0(d, 1/2)$, $p_{opt}$ and $\eta$ varying with $d$. Distinct behaviour in $P_0(d, 1/2)$ is seen either side of $2^\ast$, leading to the first (integer) critical dimension, $d = 3$. The second critical dimension is determined by the the lowest value $d$ for which $p_{opt} = 1/2$, $\eta = P_0(d, 1/2)$ which occurs at $d \sim 4.0369$. The approach, $p_{opt} \rightarrow 1/2$ is shown in detail, inset. Integer dimension are marked with circles. Note only $p_{opt} \geq 0.5$ shown.

tured by a Taylor series with leading term $8T(p-1/2)^2$ as per Eq. (6). Therefore, since the work extracted must scale faster than the work to drive the reservoir for $d \leq 4$ and $P_0(d, p)$ is monotonically increasing about $p = 1/2$ we conclude that some finite bias is always optimal for $d \leq 4$ so long as the demon is capable of work extraction in the random access limit (i.e. is not a ‘dud’ or an information eraser). For $d > 4$ we observe that both the work extracted and work expended driving the reservoir scale quadratically to leading order. To investigate the behaviour here requires the quadratic coefficient in work extracted per reservoir transition, which, for $d > 4$ is given by

$$
\frac{(1 - \epsilon(1 + e^{\frac{dx}{\Delta E}})) \Delta E (2d^2M[f_d; 2] - 2dM[f_d; 1])}{(1 + e^{\frac{dx}{\Delta E}})(a + dM[f_d; 1])^2}
$$

where $M[f, z] = \int_0^\infty x^{z-1}f(x)dx$ is the Mellin transform and $f_d(x) = e^{-dx}[I_0(x)]^d$. The cross-over in behaviour in $d \in \mathbb{R}_+$, separating the regions where finite or zero bias is optimal, occurs when this quadratic coefficient equals $8T$, since above this point all deviations from $p = 1/2$ reduce the net extracted work. The real value, $d$, at which this occurs is bounded from above by the behaviour of the optimal demon corresponding to $d \approx 4.0369$ such that the cross-over must lie in $(4, 4.0369)$. Consequently we conclude the second (integer) critical dimension is $d = 5$. This mirrors the dimension that separates random walks that are strongly transient [22] and those which are not at zero bias, discussed further in Appendix D. Solving leading order terms allows illustration of the behaviour of the optimal bias, $p_{opt}$, in this region and is illustrated in Fig. 4 for an optimal demon. Finally, we remark briefly on the $d = 4$ case that the optimal bias is exceptionally small (see Appendix C.), indeed numerically undetectable but mathematically significant. For all practical purposes, however, all bias is detrimental to work extraction on hyper-cubic structures.

V. DISCUSSION AND CONCLUSION.

Demons operating on low dimensional reservoirs are limited significantly in their ability to extract work from a heat reservoir. As the dimension of the reservoir increases, the scope for work extraction gradually improves. The BS model emerges as a limiting case of an infinite dimensional reservoir. Throughout this contribution, we assumed that the demon exploits biases in the occurrence of symbols in the reservoir. If the low entropy is instead due to spatial correlations in the reservoir, as suggested by Boyd and co-workers [19], then the maximal possible work that can be extracted from the reservoir is lower than that of the symbol bias. Furthermore, the ability to exploit spatial correlations would depend on the ability of the demon to move into a given direction reliably, which in turn requires work input. Given that work extraction is precarious already in the case of symbol biases, it is difficult to see how demons could exploit spatial correlations to extract positive net work. Further, one may conceive of the exploitation of symbol bias in a structured medium as a prototypical example of an information processing task. In this regard, whilst our results indicate increasing the dimensionality of the medium reduces the necessity to bias the medium and thus the incurred cost, this is only viable if all unvisited sites can be treated indistinguishably. For instance, this would improve the prospects of erasure processes, but not more complicated processes which are sensitive to the order of its operations in the sense of a traditional deterministic computation which forms a one dimensional structure of operations leading to a halting state.

Appendix A: Transient solution for $P_u(p, n)$

As described in the main text, $P_u(p, n)$ is the probability that a random walker transitions to an unvisited site on the $n$th transition. This is simply related to the expected number of unique sites, $S(n)$, during a random walk of length $n$ and the first passage probability $F_i(j)$ at site $j$ and time $i$, $S(n) = 1 + \sum_{i=-n}^n \sum_{j=1}^n F_i(j)$. Given the
we thus have

\[ S(z) = (P_0(z)(1 - z)^2)^{-1} \]  

we may write

\[ P_1(z) = (1 - 4p(1 - p)z^2)^{-1/2} \left( \frac{1 - 4p(1 - p)z^2}{2z(1 - p)} \right)^{|i|} \]

with \( P_0(z) = (1 - 4p(1 - p)z^2)^{-1/2} \). By substitution one can then evaluate

\[ S(n) = \frac{1}{n!} \frac{\partial^n}{\partial z^n} S(z) \bigg|_{z=0} \]

which after repeated differentiation yields a geometric expression in terms of Pochhammer numbers which may be expressed

\[ S(n) = \sum_{i=1}^{[(n+1)/2]} (2i - n - 3)g(i)p^{i-1}(1 - p)^{i-1} \]

\[ g(i) = -\frac{2^{2i-3}\Gamma[i - 3/2]}{\sqrt{\pi\Gamma[i]}}. \]

In order to calculate the discrete derivative \( dS(n)/dn = S(n + 1) - S(n) \) we find

\[ \frac{dS(n)}{dn} = \sum_{i=1}^{[(n+2)/2]} (2i - n - 4)g(i)p^{i-1}(1 - p)^{i-1} - \sum_{i=1}^{[(n+1)/2]} (2i - n - 3)g(i)p^{i-1}(1 - p)^{i-1} \]

\[ = \sum_{i=1}^{[(n+1)/2]} -\frac{2^{2i-3}\Gamma[i - 3/2]}{\sqrt{\pi\Gamma[i]}} + \begin{cases} 0 & \text{if } n \text{ odd} \\ ((2\Gamma[n+2]) - n - 4)g(i)p^{(n+2)/2-1}(1 - p)^{(n+2)/2-1} & \text{if } n \text{ even} \end{cases} \]

\[ = \sum_{i=1}^{[(n+1)/2]} -\frac{2^{2i-3}\Gamma[i - 3/2]}{\sqrt{\pi\Gamma[i]}} + \pi^{2m}((1 - 2p)(p(1 - p))^{1+m}\Gamma[1/2 + m]_2F_1(1, 1/2 + m; 2 + m; 4p(1 - p)) \]

where \( _2F_1 \) is the hypergeometric function and \( m = [n/2] \).

A particular solution follows for \( p = 1/2 \). Since for \( p = 1/2 \) we have the form \( _2F_1(1, 1/2 + m; 2 + m; 1) = 2(1 + m) \) we may simplify

\[ P_n(1/2, n) = \frac{\Gamma[1/2 + [n/2]]}{\sqrt{\pi\Gamma[1 + [n/2]]}}. \]

Furthermore, since we may represent the Gamma function with large argument \( \Gamma[z] \simeq \sqrt{2\pi}z^{-1/2}e^{-z} \) it follows that for large \( n \) we may write

\[ P_n(1/2, n) \simeq (\pi[n/2])^{-1/2} \]

\[ \simeq \sqrt{\frac{2}{n\pi}}. \]

The above result is valid for a discrete time random walker. To convert to continuous time we write

\[ P_u(p, t) = \sum_{n=0}^{\infty} e^{-\gamma t}(\gamma t)^n/n!P_u(p, n) \]
FIG. 5. Asymptotic approximation Eq. (E2) contrasted with explicit expression Eq. (A10) with $\gamma = 1$. 

![Graph](image)

FIG. 6. Behaviour of $P_u(p, t) - |1 - 2p|$ ranging from $p = 0.55$ (top) to $p = 0.9$ (bottom) in steps of 0.05 in $p$ with $\gamma = 1$. 

![Graph](image)

where $\gamma$ is the escape rate of the continuous time random walker and $p = r^+/ (r^+ + r^-)$. An appeal to the central limit theorem for large $n$ yields $P_u(p, \gamma t) \simeq P_u(p, n)|_{n=\gamma t}$, yielding, in particular, the asymptotically exact result for $p = 1/2$,

$$P_u(p, t) \simeq \sqrt{\frac{2}{\pi \gamma t}}.$$  \hspace{1cm} (A11)

which demonstrably exhibits power law decay with $t$. The accuracy of such an approximation against Eq. (A10) is shown in Fig. 5. The behaviour for $p \neq 1/2$ is shown in Fig. 6 demonstrating exponential decay with increasing speed as $p \rightarrow 0, 1$.

**Appendix B: $P_u(d, p)$ for $d > 1$**

Following Montroll [21] we consider the discrete time random walk in dimension $d$ on a (hyper) cubic lattice. We consider the lattice point $x = (x_1, x_2, \ldots, x_d)$ and thus the probability of being at that lattice point at time $t$ to be $P(x; t)$. Our random walk, characterised by a uniform bias in all dimensions, is then given by the difference equation

$$P(x_1, x_2, \ldots, x_d, t + 1) = \sum_{i=1}^d \frac{p}{d} P(x_1, \ldots, x_i - 1, \ldots, x_d, t) + \frac{(1-p)}{d} P(x_1, \ldots, x_i + 1, \ldots, x_d, t).$$  \hspace{1cm} (B1)

This is then solved, subject to periodic boundary conditions $P(x_1 + m_1 N, x_2 + m_2 N, \ldots, x_d + m_d N, t) = P(x_1, x_2, \ldots, x_d, t)$ where $m_i \in \mathbb{N}$ such that we have $N^d$ lattice points. These equations can be solved in a linear basis.
of terms

\[ b'(l) N^{-\frac{d}{2}} \exp \left[ \frac{2\pi i}{d} (1 \cdot \mathbf{x}) \right] \]  
(B2)

where \( I = (l_1, l_2, \ldots, l_d) \) where \( l_i \in \{1, 2, \ldots, N\} \). Directly substituting into Eq. (B1) gives

\[
P(x_1, x_2, \ldots, x_d, t + 1) \sim b^{t+1}(l) N^{-\frac{d}{2}} \exp \left[ \frac{2\pi i}{d} (1 \cdot \mathbf{x}) \right]
\]

\[
= \sum_{i=1}^{d} d^{-1} p b'(l_i) N^{-\frac{d}{2}} \exp \left[ \frac{2\pi i}{d} (l_1 x_1, \ldots, l_i(x_i - 1), \ldots, l_d x_d) \right] \\
+ d^{-1} (1 - p) b'(l) N^{-\frac{d}{2}} \exp \left[ \frac{2\pi i}{d} (l_1 x_1, \ldots, l_i(x_i + 1), \ldots, l_d x_d) \right] \]  
(B3)

giving

\[
b(l) = \sum_{i=1}^{d} d^{-1} p \exp \left[ \frac{2\pi i}{d} l_i \right] + (1 - p) \exp \left[ \frac{2\pi i}{d} l_i \right]
\]

\[
= \sum_{i=1}^{d} d^{-1} \cos \left( \frac{2\pi l_i}{d} \right) + id^{-1}(1 - 2p) \sin \left( \frac{2\pi l_i}{d} \right) \]  
(B4)

such that

\[
P(x, t) = \sum_{l} a(l) b'(l) N^{-\frac{d}{2}} \exp \left[ \frac{2\pi i}{d} (1 \cdot \mathbf{x}) \right] \]  
(B5)

where the sum over \( l \) indicates \( d \) sums over \( l_1, l_2, \ldots, l_d \). \( a(l) \) is then determined by considering an initial condition \( P(x, 0) \). Writing

\[
P(x, 0) = \sum_{y} P(y, 0) \delta(x - y)
\]

\[
= N^{-d} \sum_{y} P(y, 0) \sum_{l} \exp \left[ \frac{2\pi i}{d} l \cdot (x - y) \right]
\]

\[
= N^{-d} \sum_{l} \left\{ \sum_{y} P(y, 0) \exp \left[ -\frac{2\pi i}{d} l \cdot y \right] \right\} \exp \left[ \frac{2\pi i}{d} l \cdot x \right]
\]

\[
= \sum_{l} a(l) N^{-\frac{d}{2}} \exp \left[ \frac{2\pi i}{d} (1 \cdot \mathbf{x}) \right] \]  
(B6)

shows

\[
a(l) = N^{-\frac{d}{2}} \sum_{y} P(y, 0) \exp \left[ -\frac{2\pi i}{d} l \cdot y \right]. \]  
(B7)

This then allows us to write the probability of being at a lattice site \( x \) having started at the origin, \( o = (x_1 = 0, x_2, = 0, \ldots, x_d = 0) \), at time \( t = 0 \) as

\[
P(x, t|o, 0) = N^{-d} \sum_{l} \left[ \sum_{i=1}^{d} d^{-1} \cos \left( \frac{2\pi l_i}{d} \right) + id^{-1}(1 - 2p) \sin \left( \frac{2\pi l_i}{d} \right) \right]^{t} \exp \left[ \frac{2\pi i}{d} l \cdot x \right]. \]  
(B8)

Letting \( N \to \infty \) allows expression as a \( d \) dimensional integral

\[
P(x, t|o, 0) = (2\pi)^{-d} \int_{\theta}^{2\pi} \cdots \int_{\theta}^{2\pi} \left[ d^{-1} \sum_{i=1}^{d} \cos (\theta_i) + i(1 - 2p) \sin (\theta_i) \right]^{t} \exp [i \theta \cdot x] d\theta_1, \ldots, d\theta_d \]  
(B9)
where \( \theta = (\theta_1, \theta_2, \ldots, \theta_d) \).

Using the above we can express the generating function

\[
G(x, s) = \sum_{t=0}^{\infty} P(x, t|0, 0) s^{-t}
\]

as

\[
G(x, s) = \frac{s}{(2\pi)^d} \int_0^{2\pi} \cdots \int_0^{2\pi} \exp[i\theta \cdot x] \frac{1}{s - d^{-1} \sum_{i=1}^{d} \cos(\theta_i) + i(1 - 2p) \sin(\theta_i)} d\theta_1, \ldots, d\theta_d
\]

or

\[
G(x, s) = s \int_0^{\infty} \prod_{i=1}^{d} \left[ e^{-z \theta_i} \frac{1}{2\pi} \int_0^{2\pi} e^{ix \theta_i} e^{z \cos(\theta_i) + i(1 - 2p) \sin(\theta_i)} dz \right] dz.
\]

The generating function with \( x = 0 \), \( G(0, s) \), then concerns the probability of being at the origin at time \( t \), having started at the origin at time \( t = 0 \), \( P_t(d, p, t) = P(0, t|0, 0) \). One can then consider the probability of returning to the origin for the first time at time \( t \), \( P_f(d, p, t) \), with corresponding generating function \( H(0, s) = \sum_{t=0}^{\infty} P_f(d, p, t) s^{-t} \).

Crucially one can show that these generating functions are related as \( G(0, s) - 1 = G(0, s)H(0, s) \) [21]. By then considering the probability of ever returning to the origin being \( \sum_{t=0}^{\infty} P_f(d, p, t) = H(0, 1) \), one finds the probability of ever returning to the origin to be given by \( P_r(d, p, t) = 1 - (G(0, 1))^{-1} \). Since \( 1 - P_r(d, p, t) = \lim_{t \to \infty} dS(t)/dt \) due to a lemma by Dvoretsky and Erdös [24, 25] and \( dS(t)/dt \) is also the mean field probability of discovering a new site, \( P_u(d, p) \), we finally have \( P_u(d, p) = (G(0, 1))^{-1} \) where we now introduce the explicit quantity \( u(d, p) = G(0, 1) \).

Consequently, this then leads to the expression for \( P_u(d, p) = u^{-1}(d, p) \)

\[
P_u^{-1}(d, p) = u(d, p) = \frac{1}{(2\pi)^d} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{d\theta_1 \cdots d\theta_d}{1 - d^{-1} \sum_{i=1}^{d} \cos(\theta_i) + i(1 - 2p) \sin(\theta_i)}
\]

\[
= \int_0^{\infty} e^{-z} \left[ \frac{1}{2\pi} \int_0^{2\pi} e^{z \cos(\theta_i) + i(1 - 2p) \sin(\theta_i)} \right] dz.
\]

Since \( (1 - 2p) \) must lie in \([-1, 1]\) we may define \( \cos(i\delta) = (1 - (1 - 2p)^2)^{-1/2}, \sin(i\delta) = i\delta(1 - (1 - 2p)^2)^{-1/2} \). Since \( \cos^{-1}(b) \cos(a - b) = \cos(a) - \sin(a) \tan(b) \) we have, letting \( a \to \theta, b \to i\delta, \cos(\theta_i) + i(1 - 2p) \sin(\theta_i) = 2(p(1 - p))^{1/2} \cos(\theta - i\delta) \). Considering then that the rectangular contour integral \( z = \theta + i\delta = -i\delta + 2\pi - i\delta \to 2\pi \to 0 \to -i\delta \) contains no poles, \( P_u^{-1}(d, p) \), corresponding to the first edge must be independent of \( \delta \) (since \( \cos(-i\delta) = \cos(2\pi - i\delta) \)) allowing us to set \( \delta = 0 \) and thus recognise

\[
P_u^{-1}(d, p) = u(d, p) = \int_0^{\infty} e^{-z} \left[ I_0 \left( \frac{2z}{d} \sqrt{p(1 - p)} \right) \right] dz
\]

where \( I_0 \) is the zero-th modified Bessel function of the first kind, which follows from the integral representation \( I_0(x) = \pi^{-1} \int_0^{\pi} e^{\cos(\theta)} d\theta \). We note that for all \( d \), if \( p = 0 \) or \( p = 1 \) then the integral reduces to \( \int_0^{\infty} e^{-z} dz = 1 \) indicating an escape probability, and thus \( P_u(d, p) \), of 1 at total bias (complete irreversibility).

The above integral has known solutions for \( d = 1 \) and \( d = 2 \)

\[
P_u^{-1}(d, p) = \frac{1}{|1 - 2p|}
\]

\[
P_u^{-1}(d, p) = \frac{2K[4p(1 - p)]}{\pi}
\]

where \( K \) is the complete elliptic integral of the first kind and where the 1D result agrees with our result in the manuscript.
Appendix C: Asymptotics and identification of critical dimensions \(d = 3\) and \(d = 5\)

The modified Bessel functions of the first kind, \(I_n(x)\), can be asymptotically approximated for large \(x\) to be \(I_n(x) \sim e^x/\sqrt{2\pi x}\). Consequently we may test convergence of \(u(d,p)\) with a simple \(p\)-test. Noting that we can write the integral involving Bessel functions as

\[
I(x) = \int_0^\infty e^{-xI_0(x)} \frac{dx}{2\sqrt{p(1-p)}}
\]

we have

\[
\lim_{c \to \infty} \frac{d}{2\sqrt{p(1-p)}} \int_0^\infty e^{-x\frac{d}{2\sqrt{p(1-p)}}} \left[ e^{-xI_0(x)} \right]^d dx = \lim_{c \to \infty} \frac{d}{2\sqrt{p(1-p)}} \int_0^\infty e^{-x\frac{d}{2\sqrt{p(1-p)}}} \left[ e^{-xI_0(x)} \right]^d dx
\]

which demonstrably diverges for \(d = 1\) and \(d = 2\) when \(p = 1/2\) with convergence otherwise indicating the first critical dimension, \(d = 3\), where work extraction can occur at zero bias.

Looking towards the critical behaviour in whether zero bias is locally optimal, we consider the local behaviour in \(u(d,p)\) and thus the work extracted in the region \(p = 1/2\). We might naively attempt to approximate by a truncated Taylor series where we would note that the Taylor series where we would note that

\[
\frac{d}{dp} u(d,p) \bigg|_{p=1/2} = 0 \quad \text{for} \quad d \geq 5 \quad \text{and undefined elsewhere since the limit}
\]

\[
\lim_{p \to 1/2} \frac{du(d,p)}{dp} \quad \text{diverges and that the second derivative at this point is similarly given by}
\]

\[
\frac{d^2 u(d,p)}{dp^2} \bigg|_{p=1/2} = -4 \int_0^\infty z e^{-z} \left[ I_0 \left( \frac{z}{d} \right) \right]^{d-1} I_1 \left( \frac{z}{d} \right) dz
\]

which by similar logic to the above can be shown to converge for \(d \geq 5\) also. Attempting to expand the work extracted around this point gives

\[
\gamma^{-1} \langle W \rangle = \langle W \rangle_{p=1/2} - \frac{1}{2} \frac{(1 - e(1 + e^{\Delta E}))\Delta E}{(1 + e^{\Delta E})(a + u(d,1/2))^2} \left| \frac{d^2 u(d,p)}{dp^2} \right|_{p=1/2} \left( p - \frac{1}{2} \right)^2 + O \left( \left( p - \frac{1}{2} \right)^3 \right). \quad (C4)
\]

The speed of growth in \(p\) maximises for \(a = 0, \epsilon = 0, \Delta E = T(1 + W(e^{-1}))\), where \(W\) is the Lambert \(W\)-function, such that

\[
\gamma^{-1} \langle W \rangle = \gamma^{-1} \langle W \rangle_{p=1/2} - \frac{W(e^{-1})T}{2(u(d,1/2))^2} \left| \frac{d^2 u(d,p)}{dp^2} \right|_{p=1/2} \left( p - \frac{1}{2} \right)^2 + O \left( \left( p - \frac{1}{2} \right)^3 \right). \quad (C5)
\]

For \(d = 5\) the quadratic term \(\sim 0.324T\), \(d = 6\) it is \(\sim 0.177T\) and decreases monotonically as \(d\) increases. This may be sufficient for \(d \geq 5\), but fails for \(d < 5\) and moreover requires confirmation for \(d \geq 5\) as faster terms may contribute from a proper asymptotic analysis.

To achieve we this note that Eq.(C1) is in the form \(\int_0^\infty h(\lambda x) f(x) dx\) where \(h(t) = e^{-t}\) and \(\lambda = \frac{d((4p(1-p))^{1/2} - 1)}{2}\) reducing to a Laplace transform, i.e.

\[
u(d,p) = (\lambda + d) \int_0^\infty e^{-\lambda x} \left[ e^{-xI_0(x)} \right]^d dx
\]

for which standard asymptotic integral techniques can be used where the limit \(p \searrow 1/2\) corresponds to \(\lambda \to 0^+\)\cite{20,23}. To proceed we require the following asymptotic forms

\[
f_d(t) = \left[ e^{-tI_0(t)} \right]^d \sim \sum_{s=0}^\infty c_s t^{-k_s} = (2\pi t)^{-\frac{d}{2}} \quad t \to \infty
\]

\[
h(t) = e^{-t} \sim \sum_{i=0}^\infty p_i t^{a_i} = \sum_{i=0}^\infty \frac{(-1)^i}{i!} t^i \quad t \to 0^+
\]

\[\text{(C7)}\]

\[\text{We recall the } p\text{-test identifies convergence in integrals of the form } \int_c^\infty f(x) dx, \text{ if } 0 \leq f(x) \leq x^{-p} \forall x \in [c, \infty) \text{ when } c > 0 \text{ if } p > 1 \text{ resulting from the convergence of } \int_c^\infty x^{-p} dx \text{ for such values}\]
and follow [26] in constructing the $h$-transform (here the Laplace transform) as the integral across the real line

\[ 
\int_0^\infty h(\lambda t)f(t)dt = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} M[h(\lambda t); 1 - z]M[f_d(t); z]dz 
\]

where $r$ lies in the strip of analyticity of $M[h; 1 - z]M[f_d; z]$ and where $M[g; s]$, is the Mellin transform

\[ 
M[g; s] = \int_0^\infty x^{s-1}g(x)dx. 
\]

By constructing a contour that extends the line integral to an infinite rectangle around the analytic continuation of the integrand where it is no longer holomorphic, but meromorphic, along with the ability to disregard the other line integrals either generally (at $z = x \pm i\infty$) or in the $\lambda \to 0$ limit, it follows that the asymptotic terms are related to the residues of any poles of the above integrand contained in that contour [26]. Owing to the $(2\pi i)^{-1}$ factor the expansion is then expressed as those residues contained in the contour multiplied by $-1$. Since $h = e^{-t}$ it follows that $M[h; 1 - s] = \Gamma[1 - s]$ such that it is non-analytic at $s = 1, 2, 3, \ldots$ (i.e. at $s = a_i + 1$), whilst the $p$-test of $f(t)$ indicates a pole in $M[f_d; s]$ at $s = d/2$. Since we have the asymptotic expressions in Eq. (C7) which describes the divergent part of the integrals, we have the Laurent series for these poles which can be expressed

\[ 
M[f_d; s]_{s=\frac{d}{2}} = \frac{(2\pi)^{-\frac{d}{2}}}{s^{\frac{d}{2}}}, 
\]

\[ 
M[h; 1 - s]_{s=a_i+1} = -\frac{(-1)^i}{i!(s - a_i - 1)}. 
\]

If the poles in $M[f_d; s]$ and $M[h; 1 - s]$ do not coincide then the residues are of order 1 and can be expressed

\[ 
\text{res} \left\{ \lambda^{z-1}M[h; 1 - z]M[f_d; z] \right\} = -M[f_d; s]_{s=\frac{d}{2}} \sum_{i=0}^\infty \lambda^{a_i} \frac{(-1)^i}{i!} M[f_d; a_i + 1] 
\]

\[ 
= \lambda^{\frac{d}{2}-1}(2\pi)^{-\frac{d}{2}} \Gamma \left[ 1 - \frac{d}{2} \right] + \sum_{i=0}^\infty \lambda^{a_i} \frac{(-1)^i}{i!} M[f_d; a_i + 1] 
\]

\[ 
= \lambda^{\frac{d}{2}-1}(2\pi)^{-\frac{d}{2}} \Gamma \left[ 1 - \frac{d}{2} \right] + \sum_{i=0}^\infty \lambda^i \frac{(-1)^i}{i!} M[f_d; i + 1] 
\]

which is always the case for odd $n$. For even $n$ there are values of $a_i$ for which $M[h; 1 - z]M[f_d; z]$ is a double pole, namely when $z = a_i + 1 = d/2$. For this case the residue is written

\[ 
\text{res} \left\{ \lambda^{z-1}M[h; 1 - z]M[f_d; z] \right\} = \lambda^{\frac{d}{2}-1} \text{res} \left\{ \left( 1 + \left( z - \frac{d}{2} \right) \ln(\lambda) + \ldots \right) M[h; 1 - z]M[f_d; z] \right\} 
\]

which is a second order pole and a first order pole respectively (followed by higher order terms in the expansion which
do not lead to poles), such that we then write

\[- \lambda^{\frac{d}{2} - 1} \text{res}_{z = \frac{d}{2}} \left\{ \left( 1 + \left( z - \frac{d}{2} \right) \ln(\lambda) + \ldots \right) M[h; 1 - z] M[f_d; z] \right\} \]

\[= - \lambda^{\frac{d}{2} - 1} \ln(\lambda) \lim_{z \to \frac{d}{2}} \left[ \left( z - \frac{d}{2} \right)^2 M[h; 1 - z] M[f_d; z] \right] - \lambda^{\frac{d}{2} - 1} \lim_{z \to \frac{d}{2}} \frac{d}{dz} \left[ \left( z - \frac{d}{2} \right)^2 M[h; 1 - z] M[f_d; z] \right] \]

\[= - \lambda^{\frac{d}{2} - 1} \ln(\lambda)(2\pi)^{-\frac{d}{2}} \frac{(1 - 1)^{\frac{d}{2} - 1}}{(\frac{d}{2} - 1)!} - \lambda^{\frac{d}{2} - 1} \lim_{z \to \frac{d}{2}} \frac{d}{dz} \left[ \left( z - \frac{d}{2} \right)^2 M[h; 1 - z] M[f_d; z] \right] \]

\[= - \lambda^{\frac{d}{2} - 1} \ln(\lambda)(2\pi)^{-\frac{d}{2}} \frac{(1 - 1)^{\frac{d}{2} - 1}}{(\frac{d}{2} - 1)!} + \lambda^{\frac{d}{2} - 1} \lim_{z \to \frac{d}{2}} \left[ \left( -1 \right)^{\frac{d}{2} - 1} \frac{d}{2(\frac{d}{2} - 1)!} dz \left( \left( z - \frac{d}{2} \right)^2 M[f_d; z] \right) + (2\pi)^{-\frac{d}{2}} \frac{d}{dz} \left( \left( z - \frac{d}{2} \right)^2 M[h; 1 - z] \right) \right] \]

\[= - \lambda^{\frac{d}{2} - 1} \ln(\lambda)(2\pi)^{-\frac{d}{2}} \frac{(1 - 1)^{\frac{d}{2} - 1}}{(\frac{d}{2} - 1)!} + \lambda^{\frac{d}{2} - 1} \frac{d}{2(\frac{d}{2} - 1)!} dz \left( \left( z - \frac{d}{2} \right)^2 M[f_d; z] \right) \]

\[+ \lambda^{\frac{d}{2} - 1} \lim_{z \to \frac{d}{2}} \frac{d}{dz} \left( \left( z - \frac{d}{2} \right)^2 M[f_d; z] \right) \]

\[(C13) \]

where \( \gamma \approx 0.577 \) is the Euler Gamma constant and \( H_n = \sum_{i=1}^{n} i^{-1} \), \( (H_0 = 0) \), is the \( n \)-th harmonic number. Manipulations come from directly substituting the relevant terms from the Laurent series, differentiating by parts, applying \( \lim a \cdot b = \lim a \cdot \lim b \) and then recognising \( M[h; 1 - z] = \Gamma[1 - z] \). These terms then replace the corresponding omitted terms corresponding to \( z = a_i + 1 = d/2 \) in Eq. \( (C11) \). Importantly, when paired with the pre-factor from Eq. \( (C6) \), for even \( d \geq 4 \), all terms from this contribution are slower than \( (p - 1/2)^2 \).

Having established the asymptotic forms for even and odd \( d \), we wish to consider the leading order term for each \( d \). To so we first consider the leading term in Eq. \( (C11) \) which only contributes for odd \( d \). In these cases we find (assuming \( p \geq 1/2 \) for brevity)

\[(\lambda + d)(2\pi)^{-d/2}\Gamma[1 - d/2] \lambda^{\frac{d}{2} - 1} = \frac{d}{2\sqrt{p(1-p)}} (2\pi)^{-d/2} \Gamma[1 - d/2] \left( \frac{d}{2\sqrt{p(1-p)}} - d \right)^{\frac{d}{2} - 1} \]

\[= \left( \frac{d}{\pi} \right)^{\frac{d}{2}} \frac{\Gamma[1 - d/2]}{2} \left( p - \frac{1}{2} \right)^{d - 2} + \mathcal{O} \left( p - \frac{1}{2} \right)^d \]

\[(C14) \]

allowing us to identify contributing terms in \( u(1,p) \sim (1/2)(p - 1/2)^{-1} \), \( u(3,p) \sim -3\sqrt{3}\pi^{-1}(p - 1/2) \), \( u(5,p) \sim 50\sqrt{5}(3\pi^2)^{-1}(p - 1/2)^3 \), \ldots

Next, for even \( d \) we must consider the contribution from the double pole at \( a_i + 1 = d/2 \). The leading order expression is that which contains the \( \ln(\lambda) \) term which is finally

\[\frac{-1/(2\pi)^{\frac{d}{2}}}{(\frac{d}{2} - 1)!} (\lambda + d) \lambda^{\frac{d}{2} - 1} \ln(\lambda) = \frac{(d/\pi)^{d/2}}{2(\frac{d}{2} - 1)!} \left( p - \frac{1}{2} \right)^{d - 2} \ln \left( 2d \left( p - \frac{1}{2} \right)^2 \right) \]

\[(C15) \]

where we note a term of order \( (p - 1/2)^{d - 2} \) also contributes.

Next, we address the sum in Eq. \( (C11) \) and note that

\[M[f, 1 + i] = \int_{0}^{\infty} x^i \left[ e^{-x} I_0(x) \right]^d \, dx \]

\[(C16) \]

which by the above asymptotic arguments only converges for \( i < (d/2) - 1 \) which is to be interpreted as its valid domain, which by comparison with the double pole condition \( z = a_i + 1 = d/2 \) ensures that it contributes, where it exists, for all \( d \), odd and even. Considering the leading order contributions from this sum we understand that we
have, again including the relevant pre-factor,
\[
(\lambda + d)\lambda^i = \frac{d}{2\sqrt{p(1-p)}}d^i\left(\frac{1}{2\sqrt{p(1-p)}} - 1\right)^i
\]
\[
= \begin{cases} 
\mathcal{O}(1) + \mathcal{O}\left((p - \frac{1}{2})^2\right), & \text{if } i = 0, n > 2 \\
\mathcal{O}\left((p - \frac{1}{2})^2\right), & \text{if } i = 1, n > 4 \\
\mathcal{O}\left((p - \frac{1}{2})^4\right), & \text{if } i = 2, n > 6 \\
\mathcal{O}\left((p - \frac{1}{2})^6\right), & \text{if } i = 3, n > 8 \\
\ldots
\end{cases}
\]
\tag{C17}

revealing that the sum of such terms only contribute at most quadratically in \((p - 1/2)\), less a constant that appears as expected for \(n \geq 3\) when \(u(d, 1/2) \neq \infty\). One can use this to then confirm the form of \(u(d, 1/2)\) and the correctness of the quadratic approximation for \(d \geq 5\) from the above. The \(\mathcal{O}(1)\) term in \((p - 1/2)\) here, contributing only for \(d \geq 3\) is given by
\[
u(d, 1/2) = dM[f_d, 1] = d \int_0^\infty e^{-x} I_0(x) dx
\]
\[
= \int_0^\infty e^{-x} \left[I_0\left(\frac{x}{d}\right)\right]^d dx
\tag{C18}
\]
which is the Montroll extension of the Polyá result as expected \cite{21}. In turn the quadratic coefficient in \(u(d, p)\) from the above analysis and is found to be
\[
2dM[f_d, 1] - 2d^2M[f_d, 2] = 2 \int_0^\infty e^{-x} \left[I_0\left(\frac{x}{d}\right)\right]^d dx - 2 \int_0^\infty xe^{-x} \left[I_0\left(\frac{x}{d}\right)\right]^d dx
\]
\[
= 2 \int_0^\infty e^{-x} \left[I_0\left(\frac{x}{d}\right)\right]^d dx - \left[2e^{-x}(-1 - x) \left[I_0\left(\frac{x}{d}\right)\right]^d\right]_0^\infty
\]
\[
- 2 \int_0^\infty (1 + x)e^{-x} \left[I_0\left(\frac{x}{d}\right)\right]^d I_1\left(\frac{x}{d}\right) dx
\]
\[
= -2 \int_0^\infty xe^{-x} \left[I_0\left(\frac{x}{d}\right)\right]^{d-1} I_1\left(\frac{x}{d}\right) dx
\tag{C19}
\]
since \((d/dx)I_0(x) = I_1(x)\) which then matches the naive Taylor expansion result in Eq. \cite{53}, confirming its suitability for \(d \geq 5\) (though it does not confirm the validity of the Taylor series expansion), with the quadratic coefficient in work extracted for the optimal demon in turn being given by
\[
\frac{T W(e^{-1}) \left(2d^2M[f_d, 2] - 2dM[f_d, 1]\right)}{(dM[f_d, 1])^2}
\tag{C20}
\]

Finally, we can use these asymptotics to consider the leading terms in \(p - 1/2\) in the work extracted (less \(\mathcal{O}(1)\) terms). In summary, for both odd and even \(d\) there is a contribution of order \((p - 1/2)^{d-2}\), whilst for even \(d\) there is an additional contribution of order \((p - 1/2)^2\ln(\sqrt{2d}(p - 1/2))\). In addition to these contributions there is an \(\mathcal{O}(1)\) and \((p - 1/2)^2\) contribution for \(d > 2\), with an additional \((p - 1/2)^2\) contribution appearing for \(d > 5\).

Since \(P_u(d, p) = w^{-1}(d, p)\), for \(d = 1\) and \(d = 2\) where there are no \(\mathcal{O}(1)\) terms the leading order term in \(P_u(d, p)\) goes as the inverse leading order term in \(u(d, p)\), whereas for \(d \geq 3\) the leading order terms go as \(\mathcal{O}(1)\) followed by a term proportional to the (negative) next leading term in \(u(d, p)\). To summarise we have the following asymptotic behaviour in \(u(d, p)\) and \(P_u(d, p)\), abbreviating \(\Delta p = (p - 1/2)\)

| \(d\) | \(u(d, 1/2 + \Delta p)\) | \(P_u(d, 1/2 + \Delta p)\) |
|---|---|---|
| 1 | \(\mathcal{O}(\Delta p^{-1})\) | \(\mathcal{O}(\Delta p)\) |
| 2 | \(\mathcal{O}(\ln(2\Delta p))\) | \(\mathcal{O}(1/\ln(2\Delta p))\) |
| 3 | \(\mathcal{O}(1) + \mathcal{O}(\Delta p)\) | \(\mathcal{O}(1) + \mathcal{O}(\Delta p)\) |
| 4 | \(\mathcal{O}(1) + \mathcal{O}((\Delta p)^2 \ln(2\sqrt{2}\Delta p))\) | \(\mathcal{O}(1) + \mathcal{O}((\Delta p)^2 \ln(2\sqrt{2}\Delta p))\) |
| \(\geq 5\) | \(\mathcal{O}(1) + \mathcal{O}(\Delta p^2)\) | \(\mathcal{O}(1) + \mathcal{O}(\Delta p^2)\) |
This finally allows us to conclude that, because near $p = 1/2$ the cost of biasing the exploration of the storage medium can be approximated as $8T(p - 1/2)^2$ and that from Eq. (C4) we have established that for the quadratic term for $d \geq 5$ is well below this value, deviations in $p$ around $p = 1/2$ are always sub-optimal for $d \geq 5$ and that, conversely, for $d \leq 4$ there is always a deviation away from $p = 1/2$ which is beneficial so long as the demon has been properly set up (such that the random access model is capable of work extraction) demonstrating the claim of a second critical dimension $d = 5$.

Finally, we remark on the case $d = 4$. Numerically computing $u(4, p)$ indicates an optimal $p$ of 0.5 down to machine precision, however the asymptotic analysis points towards a maximum away from $p = 0.5$. The mathematical, if not numerically significant, maximum can be illustrated through explicit expansion of terms. For $d = 4$ we have

$$u(4, p) \sim 4M[f_4, 1] + \frac{8}{\pi^2} \ln \left[8(p - 1/2)^2\right] (p - 1/2)^2 + \left[8M[f_4, 1] + \frac{8}{\pi^2} (\gamma - 1) - 32 \lim_{z \to 2} \frac{d}{dz} (z - 2)M[f_4, z]\right] (p - 1/2)^2$$

(C21)

where $\gamma \approx 0.577$ is the Euler Gamma constant. The last limit term, $c$, is not obtainable analytically and difficult to numerically estimate with much precision, but simulation points towards a bound $c < b \approx 0.1$. By differentiating the maximum net work, $W(e^{-1})T/u(d, p) - (2p - 1)T \ln(p/(1 - p))$, and expanding to $O((p - 1/2)^2)$ we can solve for its stationary point, the optimal value of $p = p_{opt}^4$, which gives

$$p_{opt}^4 \approx \frac{1}{2} + 2^{-3/2} \exp\left[-\frac{1}{2} (\gamma + \pi^2 [M[f_4, 1] - 4c + 16(W(e^{-1})^{-1}(M[f_4, 1]^2)])\right].$$

Choosing $c = 0.1$ (which corresponds to a maximum estimate of gives a value of $p_{opt}^4$) gives $p_{opt}^4 \approx 0.5 + 6 \times 10^{-13}$ which is far below a value which can be distinguished from the $d \geq 5$ result computationally.

Appendix D: Relation of the critical dimensions to transience of random walkers

Here we make explicit the connection between the transience of the random walker to the critical dimensions. Initially we consider the first critical dimension which concerns work extraction at zero bias. Since the long term ability of a random walk to discover new sites is related to the transience of a random walk the first dimension directly follows the well known result that at zero bias, for $d < 2$ random walks are recurrent and for $d \geq 3$ random walks are transient [22].

Next we consider the second critical dimension. Here we consider whether, at zero bias, a random walker is strongly transient, which is to be understood the mean return time to the origin is finite [22]. This occurs when the following limit

$$\lim_{s^{-1} \to 1-} \frac{d}{ds} G(o, s^{-1})\big|_{p=1/2}$$

converges. We note that we may write

$$G(o, s^{-1})\big|_{p=1/2} = \int_0^\infty e^{-z} \left[I_0 \left(\frac{z}{2d}\right)\right]^d dz$$

(D2)

from Eq. (B12). The first derivative then has the exact same convergence properties as Eq. (C3), i.e. the random walker is strongly transient when $p_{opt}(d, p)$ is analytic in the limit $p \to 1/2$ which occurs when $d \geq 5$. Since the leading order terms, when it is analytic, and the work spent driving the tape are equal (quadratic) it then follows that any random walker that is not strongly transient has an optimal bias $p_{opt} \neq 1/2$. Having then demonstrating that the quadratic coefficient in the work spent driving the tape is greater than that extracting using it for $d \geq 5$ the general result follows: dimensions where unbiased walkers are strongly transient have optimal bias $p = 1/2$ whereas dimensions where unbiased random walkers are not strongly transient have optimal bias $p \neq 1/2$.

Appendix E: Generalisation to $d \in \mathbb{R}_+$ and approximate expressions for optimal $p$

By recasting Eq. (B13) in the form found in Eq. (B14) we note that we have extended the $d$-domain of $P_u(d, p)$ to the entire positive real line, perhaps opening up the possibility of exploration of tape symbols on fractal structures.
FIG. 7. Illustration of first critical dimension through variation of, $P_u(d, 1/2)$, as a function of dimensionality $d$, $d \in \mathbb{R}_+$, with no work extraction possible at zero bias for $d \leq 2$, but work extraction possible for $d > 2$.

In this general picture, work cannot be extracted at zero bias for $d \in [0, 2]$ whilst it can for $d \in (2, \infty)$ such that we recognise the first critical dimension to be $d_{\text{crit}}^1 = \sup [0, 2] = 2$. Similarly, if the random access demon can extract work at zero bias, the interval in $d$ for which some bias is optimal is $d \in (0, 4 + \delta)$ where $0 < \delta < 1$ such that $d_{\text{crit}}^2 = \sup (0, 4 + \delta) = 4 + \delta$. For an optimal demon $\delta$ takes its maximum value when the quadratic coefficient in Eq. (C20) equals the quadratic coefficient in the cost of driving the tape, i.e. it is the solution to

$$W(e^{-1}) \frac{(2(4 + \delta_{\text{max}})^2 M[f_{4+\delta_{\text{max}}}; 2] - 2(4 + \delta_{\text{max}}) M[f_{4+\delta_{\text{max}}}; 1])^2}{((4 + \delta_{\text{max}}) M[f_{4+\delta_{\text{max}}}; 1])^2} = 8$$

(E1)

from which we estimate $\delta_{\text{max}} \approx 0.036949$ and observe that as a demon is made less optimal (but still capable of work extraction), $\delta$ approaches 0, but never passes it. This goes some way to explaining the asymptotic behaviour at $d = 4$ as it is demonstrably the infimum in $d_{\text{crit}}^2$ that divides the two qualitative behaviours, i.e. $d_{\text{crit}}^2 \in (4, 4 + \delta_{\text{max}}]$. These two critical behaviours are illustrated in Figs. 7 and 8.

Moreover, generalising to $d \in \mathbb{R}_+$, we can approximate closed form analytical solutions for $p_{\text{opt}}^d$, as opposed to the ad hoc (albeit more precise) approximation utilised in Eq. (C22), in different domains of validity in order to then

\[2\] We note here we have a right open interval since $\delta$ is associated with equal quadratic behaviour in work extracted and expended. At this point higher order terms in work extracted are faster in work extracted and negative such that at $d = 4 + \delta$, $p = 1/2$ is optimal.
describe them for given (integer) $d$. For instance, by taking leading order terms in the work extracted less the work spent on driving the tape, differentiating and solving for the maxima we find the following approximations (noting that logarithmic terms at $d = 2, 4$ would require further individual treatment)

\[
\begin{align*}
\rho_{\text{opt}}^d &\approx \begin{cases} 
\frac{1}{2} + \frac{W(e^{-1})}{8} & 0 < d < 2 \\
\frac{1}{2} + \left(\frac{d}{2} \frac{7}{8}\frac{(d-2)}{11-\frac{d}{2}} \right)^{d-1} & 2 < d < 4 \\
\frac{1}{2} + \left(\frac{d}{2} \frac{7}{8}\frac{(d-2)}{11-\frac{d}{2}} \right)^{d-1} & 4 < d < 4 + \delta_{\text{max}} \\
\frac{1}{2} + \left(\frac{d}{2} \frac{7}{8}\frac{(d-2)}{11-\frac{d}{2}} \right)^{d-1} & d \geq 4 + \delta_{\text{max}}.
\end{cases}
\end{align*}
\]

(E2)

The first approximation performs well until $d \simeq 1.1$ where the optimal $p$ becomes too large to be described by leading order terms. Similarly the second and third approximations perform well from $d \sim 2.9$ to $d \sim 3.9$, below which it fails in the same manner as the first expression and above which second leading order terms in $(p - 1/2)$ in the extracted work cannot be readily neglected since leading order terms become very similar in magnitude. This behaviour is illustrated in Fig. 9. Fortunately they well describe the regions around $d = 1$ and $d = 3$ allowing us to express the following approximations

\[
\begin{align*}
\rho_{\text{opt}}^{1} &\approx \frac{1}{2} + \frac{W(e^{-1})}{8} \approx 0.5348 \\
\rho_{\text{opt}}^{3} &\approx \frac{1}{2} + \frac{3\sqrt{3}W(e^{-1})}{\pi (144(M[f_{\beta};1])^2 + 12W(e^{-1})M[f_{\beta};1])} \approx 0.5120.
\end{align*}
\]

(E3)
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