On finite element approximation of system of parabolic quasi-variational inequalities related to stochastic control problems

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Abstract: In this paper, an optimal error estimate for system of parabolic quasi-variational inequalities related to stochastic control problems is studied. Existence and uniqueness of the solution is provided by the introduction of a constructive algorithm. An optimally $L^\infty$-asymptotic behavior in maximum norm is proved using the semi-implicit time scheme combined with the finite element spatial approximation. The approach is based on the concept of subsolution and discrete regularity.

1. Introduction

We consider the following Parabolic Quasi-Variational Inequalities (PQVI):

\[
\begin{align*}
\frac{\partial u^i}{\partial t} + \mathbb{A}u^i - f^i &\leq 0, \quad u^i \leq Mu^i, \\
\left(\frac{\partial u^i}{\partial t} + \mathbb{A}u^i - f^i\right)&= 0, \quad \text{in } \Omega_T = \Omega \times [0, T]; \\
u^i\big|_{t=0} &= u_0^i, \quad \forall i = 1, 2, \ldots, J, \text{ in } \Omega; \\
u^i &= 0, \quad \text{on } \sum_T = \Gamma, \end{align*}
\]

(1.1)

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PUBLIC INTEREST STATEMENT

The stationary and evolutionary free boundary problems are accomplished in some applications; for example, in stochastic control, their solution characterize the in mum of the cost function associated to an optimally controlled stochastic switching process without costs for switching and for the calculus of quasi-stationary state for the simulation of petroleum or gaseous deposit.
where $\Omega$ is a bounded convex domain in $\mathbb{R}^d$, $d \geq 1$ with smooth boundary $\Gamma$, and $\Omega_T$ set in $\mathbb{R}^d \times \mathbb{R}$, $\Omega_T = \Omega \times [0, T]$ with $T < +\infty$. Let $\mathcal{A}$ are second order, uniformly elliptic operators of the form

$$\mathcal{A}^i = -\sum_{j,k=1}^d a_{jk}^i(x) \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j=1}^d b_j^i(x) \frac{\partial}{\partial x_j} + c^i(x),$$

where $\forall i = 1, ..., I, a_{jk}^i, b_j^i, c^i \in C^2(\bar{\Omega})$, $x \in \bar{\Omega}$, $1 \leq j \leq d$ are sufficiently smooth coefficients and satisfy the following conditions:

$$a_{jk}^i(x) = \alpha_{jk}^i(x); \quad a_{ij}^i(x) \geq \beta > 0, \quad \beta \text{ is a constant}$$

and

$$\sum_{j,k=1}^n a_{jk}^i(x) \xi_j \xi_k \geq \gamma |\xi|^2; \quad \xi \in \mathbb{R}^d, \quad \gamma > 0, \quad x \in \Omega.$$ (1.3)

$f^i$ are given functions satisfying the following condition

$$f^i \in L^2(0, T, L^\infty(\Omega)) \cap C^1(0, T, H^{-1}(\Omega)), \quad \text{and} \quad f^i \geq 0.$$ (1.5)

$\mathcal{M}u^i$ represents the obstacle of stochastic control defined by:

$$\mathcal{M}u^i = k + u^{i+1}$$ (1.6)

where $k$ is a strictly positive constant.

This problem arises in stochastic control problems. It also plays a fundamental role in solving the Hamilton–Jacobi–Bellman equation (Evans & Friedman, 1979; Lions & Menaldi, 1979).

In this paper, we are concerned with the numerical approximation in the $L^\infty$ norm for the problem (1.1). From Lions and Menaldi (1979), we know that (1.1) can be approximated by a solution of the following system of parabolic quasi-variational inequalities (PQVI): find a vector $U = (u^1, u^2, ..., u^I) \in \left( L^2(0, T, H^1_0(\Omega)) \right)^I$ such that

$$\begin{aligned}
\frac{\partial}{\partial t} (u^i, v - u^i) + \mathcal{A}^i (u^i, v - u^i) &\geq (f^i, v - u^i), \quad \forall v \in H_0^1(\Omega); \\
u^i &\leq k + u^{i+1}, \quad v \leq k + u^{i+1}, \quad i = 1, 2, ..., J; \\
u^{i+1} &\equiv u^i; \\
u^i(0) &\equiv u_0^i,
\end{aligned}$$

where $\mathcal{A}(u, v)$ is a continuous and noncoercive bilinear form associated with elliptic operator $\mathcal{A}$ defined as: for any $u, v \in H^1(\Omega)$

$$a(u, v) = \int_{\Omega} \left( \sum_{j,k=1}^d a_{jk}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_k} + \sum_{k=1}^d b_k(x) \frac{\partial u}{\partial x_k} v + c(x)uv \right) dx$$

and $(\cdot, \cdot)$ is the inner product in $L^2(\Omega)$.

Next we give consideration to a discrete version of (1.1): let $r_h$ be a regular and quasi-uniform triangulation of $\Omega$; $h > 0$ is the mesh size. Let also $\mathcal{V}_h$ be the finite element space consisting of continuous piecewise linear functions vanishing on $\Gamma$, $\{ \phi_i \}, i = 1, ..., m(h)$ be the basis functions of $\mathcal{V}_h$, and $r_h$ the usual restriction operator. We consider the fully discretized problem: find

$$U_h^n = (u_h^n, v_h^n, ..., u_h^{N,n}) \in (\mathcal{V}_h)^M$$

such that for all $n = 1, 2, ..., N$
\[
\begin{align*}
&\left\{ \frac{u^n_h - u^{n+1}_h}{\Delta t}, \ v_h - u^n_h \right\} + g'(u^n_h, v_h - u^n_h) \geq (f^{i,n}, v_h - u^n_h); \quad \forall v_h \in \mathcal{V}_h; \\
&u^n_h \leq \tau_h(k + u^{i+1,n}_h), v_h \leq \tau_h(k + u^{i+1,n}_h); \\
&u^{n+1}_h = u^n_h; \\
&u^n_h(0) = u^n_{h,0}.
\end{align*}
\] (1.9)

with \(\Delta t = \frac{T}{N}\), \(t_n = n \Delta t\) the time step, \(f^{i,n} = f^i(t_n)\) and \(u^n_{h,0}\) an appropriate approximation of \(u^i_0\).

Error estimates for piecewise linear finite element approximations of parabolic variational and quasi-variational inequalities have been established in various papers (cf. e.g. Achdou, Hecht, & Pommier, 2008; Alfredo, 1987; Bencheikh Le Hocine, Boulaaras, & Haiour, 2016; Bensoussan & Lions, 1973; Berger & Falk, 1977; Boulaaras, Bencheikh Le Hocine, & Haiour, 2014; Diaz & Defonso, 1985; Scarpini & Vivaldi, 1977). More recently, Bencheikh Le Hocine and Haiour (2013) exploited the above arguments for system of parabolic quasi-variational inequalities, where they analyzed the semi-implicit Euler scheme with combined with a finite element spatial approximation and gave (for \(d \geq 1\)) the following \(L^\infty\)-asymptotic behavior:

\[
\|U^i_h(T, .) - U^\infty(.)\|_\infty \leq C(h^2|\log h|^4 + \left(\frac{1}{1 + \beta \Delta t}\right)^n)
\] (1.10)

The quasi-optimal \(L^\infty\)-asymptotic behavior (\(d \geq 1\)): for \(\theta \geq \frac{1}{2}\)

\[
\|U^i_h(T, .) - U^\infty(.)\|_\infty \leq C(h^3|\log h|^2 + \left(\frac{1}{1 + \beta \theta \Delta t}\right)^n)
\] (1.11)

and for \(\theta \in [0, \frac{1}{2}]\)

\[
\|U^i_h(T, .) - U^\infty(.)\|_\infty \leq C(h^3|\log h|^2 + \left(\frac{2Ch^2}{2Ch^2 + \beta \theta (1 - 2\theta) \rho(A)}\right)^n),
\] (1.12)

where \(\rho(A) = \min_{1 \leq i \leq n}\rho(A^i)\) is the spectral radius of the elliptic operator \(A^i\), has been obtained in Boulaaras and Haiour (2014).

In the current paper, we shall employ the concepts of subsolutions and discrete regularity (Bencheikh Le Hocine et al., 2016; Boubrachene, 2014, 2015a, 2015b; Cortey-Dumont, 1987, 1985). More precisely, we use the characterization the continuous solution (resp. the discrete solution) as the maximum elements of the set of continuous subsolutions (resp. the maximum elements of the set of discrete subsolutions), in order to yield the following optimal \(L^\infty\)-asymptotic behavior (for \(d \geq 1\)):

\[
\|U^i_h(T, .) - U^\infty(.)\|_\infty \leq C(h^2|\log h|^2 + \left(\frac{1}{1 + \beta \Delta t}\right)^n).
\] (1.13)

The paper is organized as follows. In Section 2, we present the continuous problem and study some qualitative properties. The discrete problem is proposed in Section 3. In Section 4, we derive an \(L^\infty\)-error estimate of the approximation. The main result of the paper is presented in Section 5.
2. Statement of the continuous system

2.1. Existence and uniqueness

2.1.1. The time discretization

We discretize the problem (1.1) or (1.7) with respect to time by using the semi-implicit scheme. Therefore, we search a sequence of elements $u_i^{jn} \in H_0^1(\Omega), \ 1 \leq i \leq J$, which approaches $u'(t_n), t_n = k \Delta t$, with initial data $u_0^{jn} = u_j(0)$.

Thus, we have for $n = 1, ..., N$

\[
\begin{cases}
    \left( u_i^{jn} - u_i^{j-1,n}, v - u_i^{jn} \right) + \alpha \left( u_i^{jn}, v - u_i^{jn} \right) \geq \left( f_i^{jn}, v - u_i^{jn} \right), \ \forall v \in H_0^1(\Omega); \\
    u_i^{jn} \leq k + u_i^{j+1,n}, v \leq k + u_i^{j+1,n}; \\
    u_i^{j+1,n} = u_i^{j+1,n}; \\
    u_i^{j,0} = u_i^{j,0},
\end{cases}
\]

(2.1)

where

\[
\Delta t = \frac{T}{N},
\]

(2.2)

By adding $\left( u_i^{j+1,n}, v - u_i^{jn} \right)$ to both parties of the inequalities (2.1), we get

\[
\begin{cases}
    \alpha \left( u_i^{jn}, v - u_i^{jn} \right) + \frac{1}{\Delta t} \left( u_i^{jn} - u_i^{jn-1}, v - u_i^{jn} \right) \geq \left( f_i^{jn} + \frac{u_i^{jn-1} - u_i^{jn}}{\Delta t}, v - u_i^{jn} \right); \\
    u_i^{jn} \leq k + u_i^{j+1,n}, v \leq k + u_i^{j+1,n}; \\
    u_i^{j+1,n} = u_i^{j+1,n}; \\
    u_i^{j,0} = u_i^{j,0},
\end{cases}
\]

(2.3)

The bilinear form $a(\cdot, \cdot)$, being noncoercive in $H_0^1(\Omega)$, there exist two constants $\alpha > 0$ and $\lambda > 0$ such that:

\[
a(\varphi, \varphi) + \lambda \| \varphi \|_{L^2(\Omega)}^2 \geq \alpha \| \varphi \|_{H_0^1(\Omega)}^2 \quad \text{for all} \quad \varphi \in H_0^1(\Omega).
\]

(2.4)

Set

\[
b(u, v) = a(u, v) + \lambda (u, v).
\]

(2.5)

Then the bilinear form $b(\cdot, \cdot)$ is strongly coercive and therefore, the continuous problem (2.3) reads as follows: find $U^n = (u_1^{jn}, ..., u_N^{jn}) \in (H_0^1(\Omega))^J$ such that for all $n = 1, ..., N$

\[
\begin{cases}
    b\left( u_i^{jn}, v - u_i^{jn} \right) \geq \left( f_i^{jn} + \lambda u_i^{jn-1}, v - u_i^{jn} \right), \ \forall v \in H_0^1(\Omega); \\
    u_i^{jn} \leq k + u_i^{j+1,n}, v \leq k + u_i^{j+1,n}; \\
    u_i^{j+1,n} = u_i^{j+1,n};
\end{cases}
\]

(2.6)

where

\[
b\left( u_i^{jn}, v - u_i^{jn} \right) = a^j \left( u_i^{jn}, v - u_i^{jn} \right) + \lambda \left( u_i^{jn}, v - u_i^{jn} \right),
\]

\[
\lambda = \frac{1}{\Delta t} > 0.
\]

(2.7)
Remark 1  The problem (2.6) is called the coercive continuous problem of elliptic quasi-variational inequalities (QVI).

Notation 1  We denote by \( u^n = \sigma \left( f^n; k + u^{i+1,n} \right) \) the solution of problem (2.6).

Let \( U^0 = (u^0_0, ..., u^0_J) \) be the solution of the following continuous equation:

\[
 b'(u^0_0, v) = \left( f' + \lambda u^0_0, v \right), \quad \forall v \in H^1_0(\Omega). \tag{2.8}
\]

The existence and uniqueness of a continuous solution is obtained by means of Banach’s fixed point theorem.

2.1.2. A fixed point mapping associated with continuous system (2.6)

Let \( \mathbb{H}^+ = \prod_{i=1}^J L^\infty_+ (\Omega) \), where \( L^\infty_+ (\Omega) \) is the positive cone of \( L^\infty (\Omega) \). We introduce the following mapping:

\[
 T : \mathbb{H}^+ \longrightarrow \mathbb{H}^+,
 W \rightarrow TW = \xi = (\xi^1, ..., \xi^J),
\]

where \( \xi^i = \sigma \left( f^i + \lambda w^i; k + \xi^{i+1} \right) \in H^1_0(\Omega) \) solves the following coercive system of QVI:

\[
 \begin{align*}
 b'(\xi^i, v - \xi^i) & \geq \left( f' + \lambda w^i, v - \xi^i \right); \\
 \xi^{i+1} & \leq k + \xi^{i+1}, \quad v \leq k + \xi^{i+1}; \\
 \xi^{J+1} & = \xi^1. 
\end{align*}
\tag{2.10}
\]

**Theorem 1**  Under the preceding hypotheses and notations, the mapping \( T \) is a contraction in \( \mathbb{H}^+ \) with a contraction constant \( \rho = \frac{1}{\Delta t + 1} \). Therefore, \( T \) admits a unique fixed point which coincides with the solution of problem (2.6).

**Proof**  Boulbrachene, Haiour, and Chentouf (2002), taking \( \lambda = \frac{1}{\Delta t} \), we have:

\[
 \left\| T \bar{W} - \bar{W} \right\|_\infty \leq \frac{1}{\Delta t + 1} \left\| W - \bar{W} \right\|_\infty,
\]

which completes the proof. \( \square \)

The mapping \( T \) clearly generates the following iterative scheme.

2.2. A continuous iterative scheme

Starting from \( U^0 = U_0 \) the solution of Equation (2.8), we define the sequence:

\[
 U^n = T \ U^{n-1}, \tag{2.11}
\]

where \( U^n \) is a solution of the problem (2.6).

2.2.1. Geometrical convergence

In what follows, we shall establish the geometrical convergence of the proposed iterative scheme.

**Proposition 1**  Under conditions of Theorem 1, we have:

\[
 \max_{1 \leq i \leq J} \left\| u^n_i - u^m_i \right\|_\infty \leq \left( \frac{1}{\rho \Delta t + 1} \right)^n \max_{1 \leq i \leq J} \left\| u^0_i - u^m_i \right\|_\infty, \tag{2.12}
\]
where \( U^\infty \) is the asymptotic solution of the problem of quasi-variational inequalities: find \( U^\infty = (u_1^\infty, ..., u_J^\infty) \in (H_0^1(\Omega))^J \) such that

\[
\begin{aligned}
&b'(u^\infty, v - u^\infty) \geq (f^i + \lambda_u u^\infty, v - u^\infty), \quad v \in H_0^1(\Omega); \\
u^\infty \leq k + u^{i+1,\infty}, \quad v \leq k + u^{i+1,\infty}; \\
u^{i+1,\infty} = u^{i,\infty}.
\end{aligned}
\]  

(2.13)

Proof Under Theorem 1, we have for \( n = 1 \)

\[
\|u^1 - u^\infty\|_\infty = \|T u^1 - T u^\infty\|_\infty \\
\leq \left( \frac{1}{\beta \Delta t + 1} \right) \|u^0 - u^\infty\|_\infty.
\]

Now, we assume that

\[
\|u^n - u^\infty\|_\infty \leq \left( \frac{1}{\beta \Delta t + 1} \right) \|u^0 - u^\infty\|_\infty,
\]

then

\[
\|u^{n+1} - u^{i,\infty}\|_\infty = \|T u^{n+1} - T u^{i,\infty}\|_\infty \\
\leq \left( \frac{1}{\beta \Delta t + 1} \right) \|u^n - u^{i,\infty}\|_\infty.
\]

Thus,

\[
\|u^{n+1} - u^{i,\infty}\|_\infty \leq \left( \frac{1}{\beta \Delta t + 1} \right) \cdot \left( \frac{1}{\beta \Delta t + 1} \right)^n \|u^0 - u^\infty\|_\infty \\
\leq \left( \frac{1}{\beta \Delta t + 1} \right)^{n+1} \|u^0 - u^\infty\|_\infty,
\]

which completes the proof. \( \square \)

In what follows, we shall give monotonicity and Lipschitz dependence with respect to the right-hand sides and parameter \( k \) for the solution of system (2.6). These properties together with the notion of subsolution will play a fundamental role in the study the error estimate between the \( n \)th iterates of the continuous system (2.6) and its discrete counterpart.

2.3. A monotonicity property
Let \( k \) and \( \bar{k} \) be two parameters, \((f^{1,n}, ..., f^{J,n})\) and \((\bar{f}^{1,n}, ..., \bar{f}^{J,n})\) be two families of right-hand sides.

We denote \((u^{1,n}, ..., u^{J,n})\) (resp. \((\bar{u}^{1,n}, ..., \bar{u}^{J,n})\)) the corresponding solution to system of quasi-variational inequalities (2.6) defined with \((f^{1,n}, ..., f^{J,n}; k)\) (resp. \((\bar{f}^{1,n}, ..., \bar{f}^{J,n}; \bar{k})\)). Then, we have the following

**Lemma 1** (cf. Boulbrachene et al., 2002) If \( f^{i,n} \geq \bar{f}^{i,n} \) and \( k \geq \bar{k} \), then

\[
u^{i,n} \geq \bar{u}^{i,n}.
\]  

(2.14)

2.4. Lipschitz dependence with respect to the right-hand sides and the parameter \( k \)

**Proposition 2** Under conditions of Lemma 1, we have
\[ \max_{x \in \Omega} \| u^n - \bar{u}^n \| \leq C \max_{x \in \Omega} \left( |k - \bar{k}| + \| f^n - \bar{f}^n \| \right), \]  
\hspace{1cm} (2.15)  
where \( C \) is a constant such that  
\[ \alpha_0 C \geq 1. \]  
\hspace{1cm} (2.16)  

**Proof** Let  
\[ \phi' = C \left( |k - \bar{k}| + \| f^n - \bar{f}^n \| \right). \]
Then, from (2.16) it is easy to see that  
\[ \tilde{f}^n \leq f^n + |k - \bar{k}| + \| f^n - \bar{f}^n \| \]  
\[ \leq f^n + \alpha_0 C \left( |k - \bar{k}| + \| f^n - \bar{f}^n \| \right) \]  
\[ \leq f^n + \alpha_0 \phi', \]  
and  
\[ \bar{k} \leq k + C \left( |k - \bar{k}| + \| f^n - \bar{f}^n \| \right) \]  
\[ \leq k + \phi'. \]  

So, due to Lemma 1 it follows that  
\[ \sigma \left( \tilde{f}^n; \bar{k} + \bar{u}^{i+1,n} \right) \leq \sigma \left( f^n + \alpha_0 \phi'; k + \phi' + \bar{u}^{i+1,n} \right) \]  
\[ \leq \sigma \left( f^n; k + \bar{u}^{i+1,n} \right) + \phi', \]  
and  
\[ \sigma \left( \tilde{f}^n; \bar{k} + \bar{u}^{i+1,n} \right) - \sigma \left( f^n; k + \bar{u}^{i+1,n} \right) \leq \phi'. \]  

Interchanging the role of \( f^n \) and \( \tilde{f}^n \), \( k \) and \( \bar{k} \) we also get  
\[ \sigma \left( f^n; k + \bar{u}^{i+1,n} \right) - \sigma \left( \tilde{f}^n; \bar{k} + \bar{u}^{i+1,n} \right) \leq \phi'. \]  
Then  
\[ \left\| \sigma \left( f^n; k + \bar{u}^{i+1,n} \right) - \sigma \left( \tilde{f}^n; \bar{k} + \bar{u}^{i+1,n} \right) \right\| \leq C \left( |k - \bar{k}| + \| f^n - \bar{f}^n \| \right), \]  
which completes the proof. \[ \square \]

### 2.5. Characterization of the solution of system (2.6) as the envelope of continuous subsolutions

**Definition 1** \( Z = (z^1, \ldots, z^l) \in (H^1_0(\Omega))^l \) is said to be a continuous subsolution for the system of quasi-variational inequalities (2.6) if  
\[ \begin{cases}  
b'(z', v) \leq (f' + \lambda z', v), & v \in H^1_0(\Omega);  
z' \leq k + z'^{i+1}, v \geq 0;  
z'^{i+1} = z^1. \end{cases} \]  
\hspace{1cm} (2.17)
**Notation 2** Let $\mathcal{X}$ denote the set of such subsolutions.

**Theorem 2** *(cf. Bensoussan & Lions, 1978)* The solution of (2.6) is the least upper bound of the set $\mathcal{X}$.

3. Statement of the discrete system

In this section we shall see that the discrete system below inherits all the qualitative properties of the continuous system, provided the discrete maximum principle assumption is satisfied. Their respective proofs shall be omitted, as they are very similar to their continuous analogues.

3.1. Spatial discretization

Let $M_l, 1 \leq s \leq m(h)$ denote the vertex of the triangulation $\tau_h$, and let $\phi_l, 1 \leq l \leq m(h)$, denote the functions of $V_h$ which satisfies:

$$\phi_l(M_l) = \delta_{ls}, \quad 1 \leq l, s \leq m(h).$$

(3.1)

So that the function $\phi_l$ from a basis of $V_h$, $\forall v_h \in L^2(0, T; H^1_0(\Omega)) \cap C(0, T; H^2_0(\Omega))$

$$r_h v = \sum_{l=1}^{m(h)} v(M_l) \phi_l(x),$$

(3.2)

represents the interpolate of $v$ over $\tau_h$.

3.1.1. The discrete maximum principle (dmp)

Denote by $\mathcal{B}$ is the matrix with generic entries $\forall i = 1, \ldots, J$

$$\mathcal{B}_{i,s} = b'(\phi_i, \phi_s) = a'\phi_i, \phi_s + \lambda \int_{\Omega} \phi_i \phi_s \ dx, \quad 1 \leq l, s \leq m(h).$$

(3.3)

**Lemma 2** *(cf. Cortey-Dumont, 1983)* The matrix $\mathcal{B}$ is an $M$-matrix.

3.2. Existence and uniqueness

The discrete problem of PQVI consists of seeking $u_h = (u_h^1, \ldots, u_h^m) \in (V_h)^M$ such that

$$\begin{cases}
\frac{d}{dt}(u_h^i, v_h - u_h^i) + a'(u_h^i, v_h - u_h^i) \geq (f^i, v_h - u_h^i), \quad \forall v_h \in V_h; \\
u_h^i \leq r_h(k + u_h^{i+1}); \\
u_h^i = u_h^i; \\
u_h^i(0) = u_{oh},
\end{cases}$$

(3.4)

or equivalently,

$$\begin{cases}
b'(u_h^i, v_h - u_h^i) \geq (f^i + \lambda u_h^i, v_h - u_h^i); \\
u_h^i \leq r_h(k + u_h^{i+1}); \\
u_h^i = u_h^i; \\
u_h^{i+1} = u_h^{i+1}.
\end{cases}$$

(3.5)

**Notation 3** We denote by $u_h^{i,n} = \sigma_h(f^i, r_h(k + u_h^{i+1,n}))$ the solution of system (3.5).

Let $U_h^0 = U_{oh} = (u_{oh}^1, \ldots, u_{oh}^m)$ be the solution of the following discrete equation:

$$b'(u_{oh}^i, v_h) = (f^i + \lambda u_{oh}^i, v_h), \quad \forall v_h \in V_h.$$
3.2.1. A fixed point mapping associated with discrete problem (3.5)

We consider the following mapping:

\[ T_h : \mathbb{R}^+ \rightarrow (\mathbb{V}_h)^l, \]
\[ W \mapsto T_h W = \xi_h = (\xi_h^1, \ldots, \xi_h^l), \]

where \( \xi_h^i \in \mathbb{V}_h \) is a solution of the following coercive system of QVI:

\[
\begin{cases}
 b^i(\xi_h^i, v_h - \xi_h^i) \geq (f^i + \lambda \cdot w^i, v_h - \xi_h^i), & v_h \in \mathbb{V}_h; \\
 \xi_h^i < r_h(k + \xi_h^{i+1}), & v \leq r_h(k + \xi_h^{i+1}); \\
 \xi_h^{i+1} = \xi_h^i.
\end{cases}
\]

(3.8)

**Theorem 3** Under the dmp and the preceding hypotheses and notations, the mapping \( T_h \) is a contraction in \( \mathbb{V}_h \) with a rate of contraction \( \rho = \frac{1}{\beta \Delta t + 1} \). Therefore, \( T_h \) admits a unique fixed point which coincides with the solution of system (3.5).

**Proof** It is very similar to that of the continuous case. \( \square \)

3.3. A discrete iterative scheme

Starting from \( U_0^h = U_{0,0} \) the solution of Equation (3.6), we define the sequence:

\[ U_n^h = T U_{n-1}^h, \quad n = 1, \ldots, N, \]

(3.9)

where \( U_n^h \) is a solution of the problem (3.5).

3.3.1. Geometrical convergence

**Proposition 3** Under the dmp and Theorem 3, we have:

\[
\| U_n^h - U^\infty_h \|_\infty \leq \left( \frac{1}{\beta \Delta t + 1} \right)^n \| U_0^h - U^\infty_h \|_\infty.
\]

(3.10)

where \( U^\infty_h \) is the asymptotic solution of the problem of quasi-variational inequalities: find \( U^\infty_h = (u_1^\infty, \ldots, u_l^\infty) \in (\mathbb{V}_h)^l \) such that

\[
\begin{cases}
 b^i(u^\infty_h, v_h - u^\infty_h) \geq (f^i + \lambda \cdot w^i, v_h - u^\infty_h), & v_h \in \mathbb{V}_h; \\
 u^\infty_h \leq r_h(k + u^{i+1,\infty}), & v \leq r_h(k + u^{i+1,\infty}); \\
 u_{i+1,\infty} = u_{i,\infty}.
\end{cases}
\]

(3.11)

**Proof** It is very similar to that of the continuous case. \( \square \)

3.4. A monotonicity property

Let \( u_n^h = \sigma_h(\bar{f}_n^h, k) \) (resp. \( \bar{u}_n^h = \sigma_h(\bar{f}_n^h, \bar{k}) \)) the solution to (3.5).

**Lemma 3** If \( f_n^h \geq \bar{f}_n^h \) and \( k \leq \bar{k} \) then

\[
u_n^h \geq \bar{u}_n^h.
\]

(3.12)

3.5. Lipschitz dependence with respect to the right-hand sides and parameter \( k \)

**Proposition 4** Under dmp and conditions of Lemma 3, we have

\[
\max_{1 \leq i \leq l} \| u_n^h - \bar{u}_n^h \|_\infty \leq C \max_{1 \leq i \leq l} \left( \| k - \bar{k} \| + \| f_n^h - \bar{f}_n^h \|_\infty \right)
\]

(3.13)
where $C$ is a constant such that
\begin{equation}
\alpha_k C \geq 1.
\end{equation}

**Proof** It is very similar to that of the continuous case. \hfill \Box

### 3.6. Characterization of the solution of problem (3.5) as the envelope of discrete subsolutions

**Definition 2** $Z_h = (z_h^1, \ldots, z_h^m) \in (V_h)^m$ is said to be a discrete subsolution for the system of quasi-variational inequalities (3.5) if
\begin{equation}
\begin{cases}
\begin{align*}
b'(z_h^s, \varphi_s) &\leq \left(f' + \lambda z_h^s, \varphi_s\right), \quad \forall s, s = 1, \ldots, m(h); \\
z_h^s &\leq r_h\left(k + z_h^{s+1}\right), \quad \varphi_s \geq 0; \\
z_h^{s+1} &\equiv z_h^s.
\end{align*}
\end{cases}
\end{equation}

**Notation 4** Let $\mathcal{X}_h$ be the set of such subsolutions.

**Theorem 4** Under the dmp, the solution of (3.5) is the least upper bound of the set $\mathcal{X}_h$.

### 3.7. The discrete regularity

A discrete solution $U_h^0$ of a system of quasi-variational inequalities is regular in the discrete sense if it satisfies:

**Theorem 5** There exists a constant $C$ independent of $k$ and $h$ such that
\begin{equation}
\left|b'(u_h^0, \varphi_s)\right| \leq C \|\varphi_s\|_{1(\Omega)}, \quad s = 1, \ldots, m(h).
\end{equation}

Moreover, there exists a family of right-hand sides $\left\{g_{h,0}^n\right\}_{n=0}^\infty$ bounded in $(L^\infty(\Omega))'$ such that
\begin{equation}
\|g_{h,0}^n\|_{L^\infty(\Omega)} \leq C
\end{equation}

and
\begin{equation}
b'(u_{h,0}^n, \varphi_s) = g_{h,0}^n(\varphi_s), \quad \varphi_s \in V_h.
\end{equation}

Let $U_{l_0}^0$ be the corresponding continuous counterpart of (3.18), that is
\begin{equation}
b'(U_{l_0}^0(\varphi), v) = g_{h,0}^n(\varphi), \quad v \in H_0^1(\Omega),
\end{equation}

then, there exists a constant $C$ independent of $k$ and $h$ such that
\begin{equation}
\|U_{l_0}^0\|_{W^{1,2}(\Omega)} \leq C,
\end{equation}

and
\begin{equation}
\|U_{l_0}^0 - U_h^0\|_{L^\infty(\Omega)} \leq Ch^2 \log h.
\end{equation}

**Proof** We adapt [\.]. \hfill \Box
Remark 2  This new concept of “discrete regularity”, introduced in Berger and Falk (1977), Cortey-Dumont (1985) (see also Boulbrachene and Cortey-Dumont, 2009; Boulbrachene, 2015b), can be regarded as the discrete counterpart of the Lewy-Stampacchia regularity estimate \( \| u' \| \leq C \) extended to the variational form through the \( L^\infty - L^1 \) duality. It plays a major role in deriving the optimal error estimate as it permits to regularize the discrete obstacle “\( k + u^{i,n} \)” with \( W^{2,p}(\Omega) \) regular ones.

4. Finite element error analysis
This section is devoted to demonstrate that the proposed method is optimally accurate in \( L^\infty \). We first introduce the following two auxiliary systems:

### 4.1. Definition of two auxiliary sequences of elliptic variational inequalities

#### 4.1.1. A discrete sequence of variational inequalities

We define the sequence \( \{ U^n_h \}_{n \geq 1} \) such that \( U^n_h = \left( \hat{u}^n_h, \ldots, \hat{u}^n_h \right) \) solves the discrete system of variational inequalities (VI):

\[
\begin{aligned}
&b' \left( \hat{u}^n_h, \nu - \hat{u}^n_h \right) \geq \left( f^{i,n} + \lambda, \nu - \hat{u}^n_h \right), \quad \nu \in \mathcal{V}_h; \\
&\hat{u}^n_h \leq r_h \left( k + u^{i+1,n-1} \right), \quad \nu \leq r_h \left( k + u^{i+1,n-1} \right),
\end{aligned}
\]

where \( U^n = \left( u^{1,n}, \ldots, u^{J,n} \right) \) is the solution of the continuous problem (2.6).

**Proposition 5**  There exists a constant \( C \) independent of \( h, \Delta t \) and \( k \) such that

\[
\max_{1 \leq i \leq J} \| \hat{u}^n_h - u^{i,n} \|_\infty \leq C h^2 \log h^2.
\]  

**Proof**  Since \( \hat{u}^n_h = \sigma_h \left( f^{i,n}; r_h \left( k + u^{i+1,n-1} \right) \right) \) is the approximation of \( u^{i,n} = \sigma \left( f^{i,n}; k + u^{i+1,n-1} \right) \). So, making use of Cortey-Dumont (1985), we get the desired result. \( \square \)

#### 4.1.2. A continuous sequence of variational inequalities

We define the sequence \( \{ U^n \}_{n \geq 1} \) such that \( U^n = \left( \hat{u}^n, \ldots, \hat{u}^n \right) \) solves the continuous system of variational inequalities (VI):

\[
\begin{aligned}
&b' \left( \hat{u}^n, \nu - \hat{u}^n \right) \geq \left( f^{i,n} + \lambda, \nu - \hat{u}^n \right), \quad \nu \in H^1_0(\Omega); \\
&\hat{u}^n \leq k + u^{i+1,n-1}, \quad \nu \leq k + u^{i+1,n-1},
\end{aligned}
\]

where \( U^n_h = \left( u^{1,n}_h, \ldots, u^{J,n}_h \right) \) is the solution of the discrete problem (3.5), and \( U^n_h = \left( u^{i,n}_h, \ldots, u^{M,n}_h \right) \) is the solution of Equation (3.19).

**Proposition 6**  There exists a constant \( C \) independent of \( h, k, \Delta t \) such that

\[
\max_{1 \leq i \leq J} \| \hat{u}^n_h - u^{i,n} \|_\infty \leq C h^2 \log h^2.
\]  

**Proof**  We adapt Boulbrachene (2015b). \( \square \)

**Lemma 4**  (cf. Nochetto & Sharp, 1988) There exists a constant \( C \) independent of \( h, k, \Delta t \) such that

\[
\max_{1 \leq i \leq J} \| \hat{u}_0 - u^{i,n} \| \leq C h^2 \log h.
\]  

4.1.3. Optimal \( L^\infty \)-error estimates
Here, we shall estimate the error in the \( L^\infty \)-norm between the \( n \)th iterates \( U^n \) and \( U^n_h \) defined in (2.11) and (3.9), respectively.
THEOREM 6  Under the previous hypotheses, there exists a constant $C$ independent of $h$, $k$ and $\Delta t$ such that

$$
\|U^n - U^n_i\| \leq C h^2 \log h^2.
$$

(4.6)

The proof is based on two Lemmas:

LEMMA 5  There exists a sequence of discrete subsolutions $a_i^n = (a_i^n, ..., a_i^{M_i})$ such that

$$
\begin{align*}
& a_i^n \leq u_i^n, \quad i = 1, ..., J;
\text{and} & \\
& \max_{1 \leq i \leq J} \| a_i^n - u_i^n \| \leq C h^2 \log h^2,
\end{align*}
$$

(4.7)

where the constant $C$ is independent of $h$, $k$ and $\Delta t$.

Proof  For $n = 1$, we consider the discrete system of variational inequalities

$$
\begin{align*}
b' \left( \tilde{u}_1^n, \varphi_i \right) & \leq \left( f_1^{i+1} + \lambda u^{i+1}_0, \varphi_i \right), \quad \forall \varphi_i; \\
\tilde{u}_1^n & \leq r_n \left( k + u^{i+1}_0 \right), \quad \forall \varphi_i.
\end{align*}
$$

Then, as $\tilde{u}_1^n$ is solution to a discrete variational inequalities, it is also a subsolution, i.e.

$$
\begin{align*}
b' \left( \tilde{u}_1^n, \varphi_i \right) & \leq \left( f_1^{i+1} + \lambda u^{i+1}_0 - \lambda u^{i+1}_0, \varphi_i \right); \\
\tilde{u}_1^n & \leq r_n \left( k + u^{i+1}_0 \right).
\end{align*}
$$

or

$$
\begin{align*}
b' \left( \tilde{u}_1^n, \varphi_i \right) & \leq \left( f_1^{i+1} + \lambda u^{i+1}_0 - \lambda u^{i+1}_0, \varphi_i \right); \\
\tilde{u}_1^n & \leq r_n \left( k + u^{i+1}_0 \right).
\end{align*}
$$

Then

$$
\begin{align*}
b' \left( \tilde{u}_1^n, \varphi_i \right) & \leq \left( f_1^{i+1} + \lambda \| u^{i+1}_0 - u^{i+1}_0 \| + \lambda u^{i+1}_0, \varphi_i \right); \\
\tilde{u}_1^n & \leq r_n \left( k + u^{i+1}_0 \right) + r_n \left( k + u^{i+1}_0 \right) - r_n \left( k + u^{i+1}_0 \right).
\end{align*}
$$

It follows

$$
\begin{align*}
b' \left( \tilde{u}_1^n, \varphi_i \right) & \leq \left( f_1^{i+1} + \lambda \| u^{i+1}_0 - u^{i+1}_0 \| + \lambda u^{i+1}_0, \varphi_i \right); \\
\tilde{u}_1^n & \leq k + \| u^{i+1}_0 - u^{i+1}_0 \| + \tilde{u}^{i+1}_1.
\end{align*}
$$

Using (4.5), we have

$$
\begin{align*}
b' \left( \tilde{u}_1^n, \varphi_i \right) & \leq \left( f_1^{i+1} + C h^2 \log h + \lambda u^{i+1}_0, \varphi_i \right); \\
\tilde{u}_1^n & \leq k + C h^2 \log h + \tilde{u}^{i+1}_1.
\end{align*}
$$
So, $\tilde{u}_i^1$ is a discrete subsolution for the quasi-variational inequalities whose solution is $\tilde{U}_h^1 = \sigma_h\left(f^{i,1} + C h^2 \log h; k + C h^2 \log h + \tilde{U}_h^{i,1}\right)$. Then, as $u_i^1 = \sigma_h\left(f^{i,1}; k + \tilde{U}_h^{i,1}\right)$ making use of Proposition 4, we have

$$\|\tilde{U}_h^1 - u_h^1\|_\infty \leq C\left(\|k + C h^2 \log h - k\| + \|f^{i,1} + C h^2 \log h - f^{i,1}\|_\infty\right) \leq C h^2 \log h + C h^2 \log h \\ \leq C h^2 \log h.$$  

Hence, making use of Theorem 4, we have

$$\tilde{u}_i^1 \leq \tilde{U}_h^1 \leq u_h^1 + C h^2 \log h.$$ 

Putting

$$a_i^{i,1} = \tilde{u}_i^1 - C h^2 \log h,$$

we get

$$a_i^{i,1} \leq u_i^1,$$

and

$$\|a_h^{i,1} - u_h^{i,1}\|_\infty = \|\tilde{U}_h^{i,1} - C h^2 \log h - u_h^{i,1}\|_\infty \leq \|\tilde{U}_h^{i,1} - u_h^{i,1}\|_\infty + C h^2 \log h.$$  

Using Proposition 5, we get

$$\|a_h^{i,1} - u_h^{i,1}\|_\infty \leq C h^2 \log h |2 + C h^2 \log h \\ \leq C h^2 \log h |^2.$$  

For $n + 1$, let us now assume that

$$\begin{cases} a_i^{n,1} \leq u_h^{n,1}, \\
\|a_h^{n,1} - u_h^{n,1}\|_\infty \leq C h^2 \log h |^2,
\end{cases}$$

and we consider the system

$$\begin{cases} b'\left(\tilde{U}_h^{n,1}, v_h - \tilde{U}_h^{n,1}\right) \geq \left(f^{i,n} + C h^2 \log h, v_h - \tilde{U}_h^{i,n+1}\right); \\
\tilde{U}_h^{n,1} \leq r_h \left(k + U^{i+1,n}\right), v_h \leq r_h \left(k + U^{i+1,n}\right).
\end{cases}$$

Then

$$\begin{cases} b'\left(\tilde{U}_h^{i,n+1}, \varphi_i\right) \leq \left(f^{i,n} + C h^2 \log h, \varphi_i\right); \\
\tilde{U}_h^{i,n+1} \leq r_h \left(k + U^{i+1,n}\right),
\end{cases}$$

or

$$\begin{cases} b'\left(\tilde{U}_h^{i,n+1}, \varphi_i\right) \leq \left(f^{i,n} + C h^2 \log h - C h^2 \log h, \varphi_i\right); \\
\tilde{U}_h^{i,n+1} \leq r_h \left(k + U^{i+1,n}\right) + r_h \left(k + U^{i+1,n}\right) - r_h \left(k + U^{i+1,n}\right).
\end{cases}$$
Then

\[
\begin{cases}
    b'(\bar{u}_{h}^{i,n+1}, \varphi, \lambda) \leq f^{i,n} + \lambda \|u^{i,n} - \bar{u}_{h}^{i,n}\|_{\infty} + \lambda \bar{u}_{h}^{i,n}, \\
    \bar{u}_{h}^{i,n+1} \leq k + \|u^{i+1,n} - \bar{u}_{h}^{i+1,n}\|_{\infty} + \bar{u}_{h}^{i+1,n}.
\end{cases}
\]

Using (4.2), we have

\[
\begin{cases}
    b'(\bar{u}_{h}^{i,n+1}, \varphi, \lambda) \leq f^{i,n} + C h^2 |\log h|^2 + \lambda \bar{u}_{h}^{i,n}, \\
    \bar{u}_{h}^{i,n+1} \leq k + C h^2 |\log h|^2 + \bar{u}_{h}^{i+1,n}.
\end{cases}
\]

So, \( \bar{u}_{h}^{i,n+1} \) is a discrete subsolution for the quasi-variational inequalities whose solution is \( \bar{u}_{h}^{i,n+1} = \sigma_{h}(f^{i,n} + C h^2 |\log h|^2; k + C h^2 |\log h|^2 + \bar{u}_{h}^{i+1,n}) \). Then, as \( u_{h}^{i,n+1} = \sigma_{h}(f^{i,n}; k + u_{h}^{i+1,n}) \) making use of Proposition 4, we have

\[
\|u_{h}^{i,n+1} - u_{h}^{i,n+1}\|_{\infty} \leq C \left( |k + C h^2 |\log h|^2 - k| + \|f^{i,n} + C h^2 |\log h|^2 - f^{i,n}\|_{\infty}\right) 
\]

\[
\leq C h^2 |\log h|^2.
\]

Hence, applying Theorem 4, we get

\[
u_{h}^{i,n+1} \leq \bar{u}_{h}^{i,n+1} \leq u_{h}^{i,n+1} + C h^2 |\log h|^2.
\]

Putting

\[a_{h}^{i,n+1} = \bar{u}_{h}^{i,n+1} - C h^2 |\log h|^2,
\]

we get

\[a_{h}^{i,n+1} \leq u_{h}^{i,n+1},
\]

and

\[
\|u_{h}^{i,n+1} - u^{i,n+1}\|_{\infty} = \|u_{h}^{i,n+1} - C h^2 |\log h|^2 - u^{i,n+1}\|_{\infty} 
\]

\[
\leq \|u_{h}^{i,n+1} - u^{i,n+1}\|_{\infty} + C h^2 |\log h|^2.
\]

Using Proposition 5, we get

\[
\|u_{h}^{i,n+1} - u^{i,n+1}\|_{\infty} \leq C h^2 |\log h|^2,
\]

which completes the proof. \( \square \)

**LEMMA 6**  There exists a sequence of continuous subsolutions \( \rho_{(h)}^{i,n} = \left( \rho_{(h)}^{i,n}, \ldots, \rho_{(h)}^{M,n}\right) \) such that

\[
\begin{cases}
    \rho_{(h)}^{i,n} \leq u^{i,n}, \\
    \max_{1 \leq j \leq l} \|\rho_{(h)}^{i,n} - u^{i,n}\|_{\infty} \leq C h^2 |\log h|^2,
\end{cases}
\]

(4.8)

where the constant \( C \) is independent of \( h, k \) and \( \Delta t \).

**Proof**  For \( n = 1 \), we consider the system of variational inequalities
\[
\left\{ \begin{array}{l}
b'\left( \bar{u}^{i,1}_{(b)} \right) \geq \left( f^{i,1} + \lambda u_{th}^{1} \right), \quad \forall \in H^1_0(\Omega); \\
\bar{u}^{i,1}_{(b)} \leq k + u^{i,1}_{(b)}, \quad \forall \in H^1_0(\Omega), \\
\end{array} \right.
\]

Then, as \( \bar{u}^{i,1}_{(b)} \) is solution to a continuous variational inequalities, it is also a subsolution, i.e.,

\[
\left\{ \begin{array}{l}
b'\left( \bar{u}^{i,1}_{(b)} \right) \leq \left( f^{i,1} + \lambda u_{th}^{1} \right); \\
\bar{u}^{i,1}_{(b)} \leq k + u^{i,1}_{(b)}, \\
\end{array} \right.
\]

or

\[
\left\{ \begin{array}{l}
b'\left( \bar{u}^{i,1}_{(b)} \right) \leq \left( f^{i,1} + \lambda u_{th}^{1} - \lambda u_{th}^{1} + \lambda u_{th}^{1} \right); \\
\bar{u}^{i,1}_{(b)} \leq k + u^{i,1}_{(b)} + \left( k + u^{i,1}_{(b)} \right) - \left( k + u^{i,1}_{(b)} \right) \\
+ \left( k + \bar{u}^{i,1}_{(b)} \right) - \left( k + \bar{u}^{i,1}_{(b)} \right).
\end{array} \right.
\]

Then

\[
\left\{ \begin{array}{l}
b'\left( \bar{u}^{i,1}_{(b)} \right) \leq \left( f^{i,1} + C h^2 |\log h| + \lambda u_{th}^{1} \right); \\
\bar{u}^{i,1}_{(b)} \leq k + C h^2 |\log h| + \bar{u}^{i,1}_{(b)}. \\
\end{array} \right.
\]

Using (4.5), we have

\[
\left\{ \begin{array}{l}
b'\left( \bar{u}^{i,1}_{(b)} \right) \leq \left( f^{i,1} + C h^2 |\log h| + \lambda u_{th}^{1} \right); \\
\bar{u}^{i,1}_{(b)} \leq k + C h^2 |\log h| + \bar{u}^{i,1}_{(b)}. \\
\end{array} \right.
\]

So, \( \bar{u}^{i,1}_{(b)} \) is a continuous subsolution for the variational inequalities whose solution is \( \bar{u}^{i,1}_{(b)} = \sigma \left( f^{i,1} + C h^2 |\log h| + k + C h^2 |\log h| + \bar{u}^{i,1}_{(b)} \right) \). Then, as \( \bar{u}^{i,1} = \sigma \left( f^{i,1} + k + \bar{u}^{i,1}_{(b)} \right) \); making use of Proposition 2, we have

\[
\left\| \bar{u}^{i,1}_{(b)} - u^{i,1} \right\| \leq C \left( \left| k + C h^2 |\log h| - k \right| + \left\| f^{i,1} + C h^2 |\log h| - f^{i,1} \right\| \right)
\]

\[
\leq C h^2 |\log h| + C h^2 |\log h|
\]

\[
\leq C h^2 |\log h|.
\]

Hence, making use of Theorem 2, we have

\[
\bar{u}^{i,1}_{(b)} \leq \bar{u}^{i,1}_{(b)} \leq u^{i,1} + C h^2 |\log h|.
\]

Putting

\[
\beta^{i,1}_{(b)} = \bar{u}^{i,1}_{(b)} - C h^2 |\log h|,
\]

we get

\[
\beta^{i,1}_{(b)} \leq u^{i,1}.
\]
and

\[ \| \dot{u}^{n+1}_n - u^{n+1}_h \|_\infty = \| \dot{u}^{n+1}_n - C h^2 |\log h| - \dot{u}^{n+1}_h \|_\infty \leq \| \dot{u}^{n+1}_n - u^{n+1}_h \|_\infty + C h^2 |\log h|. \]

Using Proposition 6, we get

\[ \| \dot{u}^{n+1}_n - u^{n+1}_h \|_\infty \leq C h^2 |\log h|^2 + C h^2 |\log h| \leq C h^2 |\log h|^2. \]

For \( n + 1 \), let us now assume that

\[
\left\{
\begin{array}{l}
\psi^{n+1} \leq u^{n+1}; \\
\text{and} \\
\| \psi^{n+1} - u^{n+1}_h \|_\infty \leq C h^2 |\log h|^2 ,
\end{array}
\right.
\]

and consider the system

\[
\left\{
\begin{array}{l}
b' \left( \ddot{u}^{n+1}_n, v - \ddot{u}^{n+1}_h \right) \geq \left( f^{n+1} + \lambda u^{n+1}_h, v - \ddot{u}^{n+1}_h \right); \\
\ddot{u}^{n+1}_n \leq k + u^{n+1}_h, v \leq k + u^{n+1}_h.
\end{array}
\right.
\]

Then

\[
\left\{
\begin{array}{l}
b' \left( \ddot{u}^{n+1}_n, v \right) \leq \left( f^{n+1} + \lambda u^{n+1}_h, v \right); \\
\ddot{u}^{n+1}_n \leq k + u^{n+1}_h,
\end{array}
\right.
\]

or

\[
\left\{
\begin{array}{l}
b' \left( \ddot{u}^{n+1}_n, v \right) \leq \left( f^{n+1} + \lambda u^{n+1}_h - \lambda \ddot{u}^{n+1}_h + \lambda \ddot{u}^{n+1}_h, v \right); \\
\ddot{u}^{n+1}_n \leq \left( k + u^{n+1}_h \right) - \left( k + u^{n+1}_h \right) + \left( k + u^{n+1}_h \right) + \left( k + u^{n+1}_h \right).
\end{array}
\right.
\]

Then

\[
\left\{
\begin{array}{l}
b' \left( \ddot{u}^{n+1}_n, v \right) \leq \left( f^{n+1} + \lambda \ddot{u}^{n+1}_h - \lambda \ddot{u}^{n+1}_h, v \right); \\
\ddot{u}^{n+1}_n \leq \ddot{u}^{n+1}_h - \left( k + u^{n+1}_h \right) + \left( k + u^{n+1}_h \right).
\end{array}
\right.
\]

Using (4.4), we have

\[
\left\{
\begin{array}{l}
b' \left( \ddot{u}^{n+1}_n, v \right) \leq \left( f^{n+1} + C h^2 |\log h|^2 + \lambda \ddot{u}^{n+1}_h, v \right); \\
\ddot{u}^{n+1}_n \leq \left( C h^2 |\log h|^2 + k + \ddot{u}^{n+1}_h \right).
\end{array}
\right.
\]

So, \( \ddot{u}^{n+1}_h \) is a continuous subsolution for the quasi-variational inequalities whose solution is

\( \ddot{u}^{n+1}_h = \sigma \left( f^{n+1} + C h^2 |\log h|^2 + k + C h^2 |\log h|^2 + \ddot{u}^{n+1}_h \right) \). Then, as \( u^{n+1}_h = \sigma \left( f^{n+1} + k + u^{n+1}_h \right) \), making use of Proposition 2, we have


\[ \| \hat{u}_h^{n+1} - u_h^{n+1} \|_\infty \leq C \left( \| k + C h^2 \| \log h \|^2 - k \right) + \| f^{u^n} + C h^2 \| \log h \|^2 - f^{u^n} \| \infty \) 

\leq C h^2 \| \log h \|^2. 

Hence, applying Theorem 2, we get

\[ \hat{u}_{(b)}^{n} \leq \hat{u}_{(b)}^{n+1} \leq u_{(b)}^{n+1} + C h^2 \| \log h \|^2. \]

Putting

\[ \beta_{(b)}^{n+1} = \hat{u}_{(b)}^{n+1} - C h^2 \| \log h \|^2, \]

we get

\[ \beta_{(b)}^{n+1} \leq u_{(b)}^{n+1} \]

and

\[ \| u_{(b)}^{n+1} - u_h^{n+1} \|_\infty = \| \hat{u}_{(b)}^{n+1} - C h^2 \| \log h \|^2 - u_h^{n+1} \| \infty \] 

\leq \| \hat{u}_{(b)}^{n+1} - u_h^{n+1} \|_\infty + C h^2 \| \log h \|^2. \]

Using Proposition 6, we get

\[ \| u_{(b)}^{n+1} - u_h^{n+1} \|_\infty \leq C h^2 \| \log h \|^2, \]

which completes the proof. \( \square \)

We are now in a position to prove the Theorem 6.

Proof Using (4.7), we have

\[ u^n \leq a^n + C h^2 \| \log h \|^2 \]

\leq u^n + C h^2 \| \log h \|^2 

thus

\[ u^n - u^n \leq C h^2 \| \log h \|^2 \]

and using (4.8), we have

\[ u^n \leq \beta_{(b)}^{n} + C h^2 \| \log h \|^2 \]

\leq u^n + C h^2 \| \log h \|^2 ,

thus, we get

\[ u^n - u^n \leq C h^2 \| \log h \|^2. \]

Therefore,

\[ \| u^n - u^n \|_\infty \leq C h^2 \| \log h \|^2, \]

which completes the proof.
5. $L^\infty$-Asymptotic behavior for a finite element approximation

This section is devoted to the proof of main result of the present paper, where we prove the theorem of the asymptotic behavior in $L^\infty$-norm for parabolic quasi-variational inequalities.

Now, we evaluate the variation in $L^\infty$-norm between $u_h(T, \cdot)$ the discrete solution calculated at the moment $T = n \Delta t$ and $u^\infty$ the solution of system (2.13).

**Theorem 7 (The main result)** Under Propositions 1 and 3, and Theorem 6, the following inequality holds:

$$\max_{1 \leq i \leq N} \left\| u_i^h(T, \cdot) - u^\infty(\cdot) \right\|_\infty \leq C \left( h^2 \log h + \left( \frac{1}{\beta \Delta t + 1} \right)^N \right), \quad (5.1)$$

**Proof** We have

$$u_h = u_h(t, \cdot) \text{ for all } t \in [n-1, n] \Delta t,$$

thus

$$\left\| u_i^h(T, \cdot) - u^\infty(\cdot) \right\|_\infty = \left\| u_i^N - u_i^\infty \right\|_\infty \leq \left\| u_i^N - u_i^0 \right\|_\infty + \left\| u_i^0 - u^\infty \right\|_\infty.$$

Indeed, combining estimates (2.12), (3.10), and (4.6), we get

$$\begin{align*}
\left\| u_i^h(T, \cdot) - u^\infty \right\|_\infty &\leq \left\| u_i^N - u_i^\infty \right\|_\infty + \left\| u_i^\infty - u^\infty \right\|_\infty \\
&\leq \left\| u_i^N - u_i^0 \right\|_\infty + \left\| u_i^0 - u_i^\infty \right\|_\infty + \left\| u_i^\infty - u^\infty \right\|_\infty \\
&\leq 2 \left\| u_i^N - u_i^0 \right\|_\infty + \left\| u^\infty - u^N \right\|_\infty + \left\| u^N - u^h \right\|_\infty.
\end{align*}$$

Using Propositions 1 and 3, we have

$$\left\| u^\infty - u^N \right\|_\infty \leq \left( \frac{1}{\beta \Delta t + 1} \right)^N \left\| u^\infty - u_0 \right\|_\infty,$$

and for the discrete case

$$\left\| u^\infty - u^N \right\|_\infty \leq \left( \frac{1}{\beta \Delta t + 1} \right)^N \left\| u^\infty - u^0 \right\|_\infty.$$

Applying the previous results of Propositions 1, 3 and Theorem 6 we get

$$\begin{align*}
\left\| u_i^h(T, \cdot) - u^\infty \right\|_\infty &\leq 2 \left( \frac{1}{\beta \Delta t + 1} \right)^N \left\| u_i^N - u_i^0 \right\|_\infty + \left( \frac{1}{\beta \Delta t + 1} \right)^N \left\| u^\infty - u^0 \right\|_\infty \\
&+ Ch^2 \left\| \log h \right\|^2.
\end{align*}$$

Then, the following result can be deduced:

$$\left\| u_i^h(T, \cdot) - u^\infty \right\|_\infty \leq C \left( h^2 \log h + \left( \frac{1}{\beta \Delta t + 1} \right)^N \right),$$

which completes the proof.

**Corollary 1** It can be seen that in the previous estimate (5.1), $\left( \frac{1}{\beta \Delta t + 1} \right)^N$ tends to 0 when $N$ approaches to $+\infty$. Therefore, the convergence order for the noncoercive elliptic system of quasi-variational inequalities related to stochastic control problems is

$$\max_{1 \leq i \leq N} \left\| u_i^h - u^\infty \right\|_\infty \leq Ch^2 \left\| \log h \right\|^2. \quad (5.2)$$
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