The Generating Functions Enumerating 12...d-Avoiding Words with r occurrences of each of 1, ..., n are D-finite for all d and all r

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Dedicated to Neil James Alexander Sloane (born October 10, 1939), on his A000027[75]-th birthday (alias A005408[38]-th, alias A002808[53]-th, alias A001477[74]-th, alias A014612[17]-th, and 13520 other aliases.)

Introduction

In a recent beautiful article, Nathaniel Shar and Doron Zeilberger ([ShZ]) proved that for any positive integer r, the generating function of the sequence enumerating 123-avoiding words with r occurrences of each of the letters 1, ..., n is always algebraic. In other words for each r, the generating function, let’s call it $f_r(x)$, satisfies an equation of the form

$$P_r(x, f_r(x)) = 0,$$

for some polynomial, $P_r$, of two variables. The actual polynomials, $P_r(x, y)$, were computed for $r \leq 4$.

This is no longer true for 12...d-avoiding words with $d \geq 4$, even for $r = 1$.

In 1990 Doron Zeilberger ([Z]) showed that for and any positive integer d, the generating function enumerating 1...d-avoiding permutations (i.e. words in {1, ..., n} where each letter occurs exactly 1 times) is the next-best-thing to being algebraic, which is being D-finite (aka as holonomic). Recall that a formal power series is D-finite if it satisfies a linear differential equation with polynomial coefficients, or equivalently, the enumerating sequence itself is P-recursive, i.e. satisfies a linear recurrence equation with polynomial coefficients. Ira Gessel ([G]) famously discovered (and proved) a beautiful determinant with Bessel functions, for the generating function, (of the sequence divided by $n!^2$) (that also implies the above result), and Amitai Regev ([R]) famously derived delicate and precise asymptotics.

In the present article, dedicated to guru Neil Sloane on his 75-th birthday, we observe that the analogous generating functions for multi-set permutations (alias words), where every letter appears the same number of times, say r, are still always D-finite, (for every d and every r), and we actually crank out the first few terms of quite a few of them, many of whom are not yet in the OEIS ([Sl]).

All this data, often with linear recurrences (that we know a priori exist, and hence it justifies their discovery by pure guessing), and very precise asymptotics, is collected in the front of this article

http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/sloane75.html
where links to two useful Maple packages, that were used to generate all that data, SLOANE75 and NEIL, can be found and downloaded, and readers who have Maple and computer time to spare are welcome to use in order to generate yet more data.

Last but not least, we pledge 100 dollars to the OEIS in honor of the first one to prove our conjectured asymptotic formula for the number of 1...d-avoiding words in \{1^r \ldots n^r \} that generalizes Regev’s ([R]) famous formula for \( r = 1 \). We pledge another 100 dollars for extending Gessel’s Bessel determinant, from the \( r = 1 \) case to general \( r \).

**Why is the Sequence Enumerating 1...d-avoiding words in \{1^r \ldots n^r \} P-recursive?**

By the Robinson-Schenstead-Knuth (RSK) famous correspondence, our quantity of interest, let’s call it \( A_{d,r}(n) \) is given by

\[
A_{d,r}(n) = \sum_{\lambda \vdash r n \atop \text{length}(\lambda) \leq d} f_\lambda g_\lambda^{(r)},
\]

where \( f_\lambda \) is the number of standard Young tableaux of shape \( \lambda = (\lambda_1, \ldots, \lambda_d) \), and \( g_\lambda^{(r)} \) is the number of column-strict Young tableaux with exactly \( r \) occurrences of each of 1, ..., \( n \). For \( \lambda = (\lambda_1, \ldots, \lambda_d) \) (where we pad it with zeroes if the length is less than \( d \)), \( f_\lambda \) is closed-form (thanks to Young-Frobenius, or the hook-length formula), and hence ipso facto, holonomic in its \( d \) discrete arguments.

Furthermore, for \( r > 1 \), \( g_\lambda^{(r)} \), while no longer closed-form, is easily seen to be holonomic in its \( d \) discrete arguments (one way to see this is to note that their redundant generating function (in the sense of MacMahon) is a rational formal power series in \( x_1, \ldots, x_d \)). It follows, by general holonomic nonsense ([Z]), that for any fixed integers \( r \) and \( d \) the sequence, in \( n \), \( \{ A_{d,r}(n) \} \), is \( P \)-recursive.

Computationally speaking, it is fairly easy to compute \( g_\lambda^{(r)} \), and hence crank-out the first few terms of the sequences \( \{ A_{d,r}(n) \} \) for quite a few \( d \) and \( r \), that for \( r \) and \( d \) not too large may be used to guess (in real time) the recurrences empirically, that we know must be the right ones.

**The 100 dollars conjecture generalizing Regev’s Asymptotics**

**Conjecture** (100 donation to the OEIS in honor of the first prover)

Let \( A_{d,r}(n) \) be the number of 1...d-avoiding words in \{1^r \ldots n^r \}, then there exists a constant \( C_{r,d} \) such that

\[
A_{d,r}(n) \sim C_{r,d} \cdot \left( \binom{d + r - 2}{d - 2} (d - 1)^r \right) n \frac{1}{n^{((d-1)^2 - 1)/2}}.
\]

**Extra Credit** (25 additional dollars): find an explicit expression for \( C_{r,d} \) in terms of \( r \) and \( d \) (involving \( \pi \), of course).

**The 100 dollars Challenge to generalize Gessel’s Spectacular Theorem**

This is more open-ended, but it would be nice to get a determinant expression, in the style of Ira Gessel’s ([G]) famous expression for the generating function of \( A_{d,1}(n)/n!^2 \), canonized in the bible
([W], p. 996, Eq. (5)). Here it is: Let \( u_k(n) := A_{k,1}(n) \), then

\[
\sum_{n \geq 0} \frac{u_k(n)}{n!^2} x^{2n} = \det(I_{|i-j|}(2x))_{i,j=1,\ldots,k},
\]

in which \( I_\nu(t) \) is (the modified Bessel function)

\[
I_\nu(t) = \sum_{j=0}^{\infty} \frac{(\frac{t}{2})^{2j+\nu}}{j!(j+\nu)!}.
\]

Guru Herb Wilf (ibid) goes on to wax eloquently:

“At any rate, it seems fairly “spectacular” to me that when you place various infinite series such as the above into a \( k \times k \) determinant, and then expand the determinant, you should find that the coefficient of \( x^{2n} \), when multiplied by \( n!^2 \), is exactly the number of permutations of \( n \) letters with no increasing subsequence longer than \( k \).”

It would be even more spectacular, if you, dear reader, would generalize this to \( r > 1 \!

References

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[R] A. Regev, Asymptotic values for degrees associated with strips of Young diagrams, Adv. Math. 41 (1981), 115-136.

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[Sl] N. J.A. Sloane, The Online-Encyclopedia of Integer Sequences (OEIS) https://oeis.org/

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