The Description of Neutrino and Muon Oscillations by Interfering Amplitudes of Classical Space-Time Paths

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Abstract

Particle oscillations following neutrino production processes are analysed within Feynman’s path amplitude formulation of quantum mechanics. Consideration of the temporal sequence of production and detection events reveals an important contribution to the oscillation phase from the space-time propagator of the decaying source particle. Formulae are derived for spatial neutrino oscillations following pion decay (at rest and in flight), muon decay and nuclear $\beta$-decay at rest, as well as for similar muon oscillations after pion decay at rest and in flight. In all cases studied, the neutrino oscillation phases found differ from the conventionally used one. A concise critical review is made of previous treatments of the quantum mechanics of neutrino and muon oscillations.

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1 Introduction

The quantum mechanical description of neutrino oscillations\(^1\)\(^2\) has been the subject of much discussion and debate in the recent literature. The ‘standard’ oscillation formula\(^3\), yielding an oscillation phase\(^4\) at distance \(L\) from the neutrino source, between neutrinos, of mass \(m_1\) and \(m_2\) and momentum \(P\), of:

\[
\phi_{12} = \frac{(m_1^2 - m_2^2)L}{2P},
\]

is derived on the assumption of equal momentum at production of the two neutrino mass eigenstates. Other authors have proposed, instead, equal energies\(^4\) or velocities\(^5\) at production, confirming, in both cases, the result of the standard formula. The latter reference claims, however, that the standard expression for \(\phi_{12}\) should be multiplied by a factor of two in the case of the equal energy or equal momentum hypotheses. However, the equal momentum, energy or velocity assumptions are all incompatible with energy-momentum conservation in the neutrino production process\(^6\). In two recent calculations\(^7\),\(^8\) a covariant formalism was used in which exact energy-momentum conservation was imposed. Using the invariant Feynman propagator\(^9\) to describe the space-time evolution of the neutrino mass eigenstates, the author of Ref.\(^7\) also found an oscillation phase a factor of two larger than given by the the standard formula. However, on convoluting the transition amplitude with a Gaussian wavefunction describing the spatial-temporal position of the source pion, the standard result was recovered. In Ref.\(^8\), a completely different formula was obtained for the neutrino oscillation phase following pion decay and it was predicted that only correlated oscillations of neutrinos and the recoiling decay muons could be observed. However, the author of Ref.\(^7\) as well as others\(^9\),\(^10\) claimed that muon oscillations would either be completely suppressed, or essentially impossible to observe.

The present paper calculates the probabilities of oscillation of neutrinos and muons produced by pions decaying both at rest and in flight, as well as the probabilities of neutrino oscillation following muon decay or \(\beta\)-decay of a nucleus at rest. The calculations, which are fully covariant, are based on Feynman’s reformulation of quantum mechanics\(^11\) in terms of interfering amplitudes associated with classical space-time particle trajectories. The essential interpretational formula of this approach\(^12\) though motivated by the seminal paper of Dirac on the Lagrangian formulation of quantum mechanics\(^13\), and much developed later in the work of Feynman and other authors\(^14\), was actually already given by Heisenberg in 1930\(^15\). The application of the path amplitude formalism to neutrino or muon oscillations is particularly straightforward, since, in the covariant formulation of quantum mechanics, energy and momentum are exactly conserved at all vertices and due to the macroscopic propagation distances of the observed neutrinos and muons all these particles follow essentially classical trajectories\((i.e.\) corresponding to the minima of the classical action\)) which are rectilinear paths with constant velocities.

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\(^1\) The interference term is proportional to \(\cos \phi_{12}\) or \(\sin^2 \frac{\phi_{12}}{2}\).

\(^2\) Units with \(\hbar/2\pi = c = 1\) are used throughout.

\(^3\) Postulate 1 and Eqn. (7) of Ref.\(^12\).

\(^4\) Heisenberg remarked that the fundamental formula (1.2) must be distinguished from that where the summation over intermediate states is made at the level of probabilities, rather than amplitudes, and that the distinction between the two formulae is ‘the centre of the whole quantum theory’.
The essential formula of Feynman’s version of quantum mechanics, to be employed in the calculations presented below, is \[ P_{fi} = \left| \sum_{k_1} \sum_{k_2} \cdots \sum_{k_n} \langle f|k_1\rangle\langle k_2|\cdots\langle k_n|i \rangle \right|^2 \] (1.2)

where \( P_{fi} \) is the probability to observe a final state \( f \), given an initial state \( i \), and \( k_j, j = 1, n \) are (unobserved) intermediate quantum states. In the applications to be described in this paper, Eqn(1.2) specialises to \[ P_{\bar{e}^{-}\pi^{+}} = \left| \sum_{k=1,2} \langle e^{-}|\nu_k\rangle\langle \nu_k, x_D|\nu_k, x_k \rangle\langle \nu_k|\pi^{+}\rangle\langle \pi^{+}, x_k|\pi^{+}, x_0 \rangle\langle \pi^{+}, x_0|S_{\pi} \rangle \right|^2 \] (1.3)

for the case of neutrino oscillations and

\[ P_{e^{+}\pi^{+}} = \left| \sum_{k=1,2} \langle e^{+}|\mu_k^{+}\rangle\langle \mu_k^{+}, x_D|\mu_k^{+}, x_k \rangle\langle \mu_k^{+}|\pi^{+}\rangle\langle \pi^{+}, x_k|\pi^{+}, x_0 \rangle\langle \pi^{+}, x_0|S_{\pi} \rangle \right|^2 \] (1.4)

for the case of muon oscillations. \( P_{\bar{e}^{-}\pi^{+}} \) is the probability to observe the charged current neutrino interaction: \( \nu_e n \rightarrow e^{-}p \) following the decay: \( \pi^{+} \rightarrow \mu^{+}\nu_{\mu} \), while \( P_{e^{+}\pi^{+}} \) is the probability to observe the decay \( \mu^{+} \rightarrow e^{+}\nu_{e}\overline{\nu}_{\mu} \), after the same decay process. In Eqns.(1.3),(1.4) \( \langle \nu_k \rangle, k = 1, 2 \) are neutrino mass eigenstates while \( |\mu_k^{+}\rangle, k = 1, 2 \) are the corresponding recoil muon states from pion decay. \( \langle \nu_k|\pi^{+}\rangle \), \( \langle \mu_k^{+}|\pi^{+}\rangle \) and \( \langle e^{+}|\mu_k^{+}\rangle \) denote invariant decay amplitudes, \( \langle e^{-}|\nu_k \rangle \) is the invariant amplitude of the charged current neutrino interaction, \( \langle p, x_2|p, x_1 \rangle \) is the invariant space-time propagator of particle \( p = \nu, \mu, \pi \) between the space-time points \( x_1 \) and \( x_2 \) and \( \langle \pi^{+}, x_0|S_{\pi} \rangle \) is an invariant amplitude describing the production of the \( \pi^{+} \) by the source \( S_{\pi} \) and its space-time propagation to the space-time point \( x_0 \).

The difference of the approach used in the present paper to previous calculations presented in the literature can be seen immediately on inspection of Eqns(1.3) and (1.4). The initial state is a pion at space time-point \( x_0 \), the final state an \( e^{-} \) or \( e^{+} \) produced at space-time point \( x_D \). These are unique points, for any given event and do not depend in any way on the masses of the unobserved neutrino eigenstates propagating from \( x_k \) to \( x_D \) in Eqn(1.3). On the other hand the (unobserved) space-time points \( x_k \) at which the neutrinos and muons are produced do depend on \( k \). Indeed, because of the different velocities of the propagating neutrino eigenstates, only in this case can both neutrinos and muons (representing alternative classical histories of the decaying pion) both arrive simultaneously at the unique point \( x_D \) where the neutrino interaction occurs (Eqn(1.3)) or the muon decays (Eqn(1.4)).

The crucial point in the above discussion is that the decaying pion, via the different path amplitudes in Eqns(1.3) and (1.4), interferes with itself. To modify very slightly...
Dirac’s famous statement\textsuperscript{[7]}: ‘Each pion then interferes only with itself. Interference between two different pions never occurs’. As will be seen in the discussion in Section 5 below, essentially all recent treatments of neutrino oscillations in the literature have attempted to contradict the second sentence in Dirac’s statement!

Because of the different possible decay times of the pion in the two interfering path amplitudes, the pion propagators \(\langle \pi^+, x_k | \pi^+, x_0 \rangle\) in Eqns(1.3) and (1.4) above give important contributions to the interference phase. To the author’s best knowledge, this effect has not been taken into account in any previously published calculation of neutrino oscillations.

The results found for the oscillation phase are, for pion decays at rest:

\[
\phi^{\nu, \pi}_{12} = \phi^{\mu, \pi}_{12} = \frac{2m_\pi m_\mu^2 \Delta m^2 L}{(m_\pi^2 - m_\mu^2)^2} \tag{1.5}
\]

and for pion decays in flight:

\[
\phi^{\nu, \pi}_{12} = \frac{m_\mu^2 \Delta m^2 L}{(m_\pi^2 - m_\mu^2) E_\nu \cos \theta_\nu} \tag{1.6}
\]
\[
\phi^{\mu, \pi}_{12} = \frac{2m_\mu^2 \Delta m^2 (m_\mu^2 E_\pi - m_\mu^2 E_\mu)L}{(m_\pi^2 - m_\mu^2)^2 E_\mu^2 \cos \theta_\mu} \tag{1.7}
\]

where

\[
\Delta m^2 \equiv m_1^2 - m_2^2
\]

The superscripts indicate the particles whose propagators contribute to the interference phase. Also \(E_\pi, E_\nu\) and \(E_\mu\) are the energies of the parent \(\pi\) and the decay \(\nu\) and \(\mu\) and \(\theta_\nu, \theta_\mu\) the angles between the pion and the neutrino, muon flight directions. In Eqns.(1.5) to (1.7) terms of order \(m_1^4, m_2^4\), and higher, are neglected, and in Eqns.(1.6) and (1.7) ultrarelativistic kinematics with \(E_\pi, \mu \gg m_\pi, \mu\) is assumed. Formulae for the oscillation phase of neutrino oscillations following muon decays or nuclear \(\beta\)-decays at rest, calculated in a similar manner to Eqn(1.5), are given in Section 3 below.

A brief comment is now made on the generality and the covariant nature of the calculations presented in this paper. Although the fundamental formula (1.2) is valid in both relativistic and non-relativistic quantum mechanics, it was developed in detail by Feynman \([12, 14]\) only for the non-relativistic case. For the conditions of the calculations performed in the present paper (propagation of particles in free space) the invariant space-time propagator can either be derived (for fermions) from the Dirac equation, as originally done by Feynman \([9]\) or, more generally, from the covariant Feynman path integral for an arbitrary massive particle, as recently done in Ref. \([7]\). In the latter case, the invariant propagator for any stable particle with pole mass \(m\), between space-time points \(x_i\) and \(x_f\) in free space is given by the path integral \([7]\):

\[
K(x_f, x_i; m) = \int \mathcal{D}[x(\tau)] \exp \left\{ -\frac{im}{2} \int_{x_i}^{x_f} \left( \frac{dx}{d\tau} \cdot \frac{dx}{d\tau} + 1 \right) d\tau \right\} \tag{1.8}
\]

\textsuperscript{7}Each photon then interferes only with itself. Interference between two different photons never occurs’ \([16]\).
where $\tau$ is the proper time of the particle. By splitting the integral over $x(\tau)$ on the right side of Eqn(1.8) into the product of a series of infinitesimal amplitudes corresponding to small segments, $\Delta \tau$, Gaussian integration may be performed over the intermediate space-time points. Finally, integrating over the proper time $\tau$, the analytical form of the propagator is found to be \[ K(x_f, x_i; m) \simeq \left( \frac{m^2}{4\pi is} \right) H_1^{(2)}(ims) \] (1.9)

where \[ s = \sqrt{(x_f - x_i)^2} \]

and $H_1^{(2)}$ is a first order Hankel function of the second kind, in agreement with Ref. 9.

In the asymptotic region where $s \gg m^{-1}$, or for the propagation of on-shell particles \[ 7], the Hankel function reduces to an exponential and yields the configuration space propagator $\simeq \exp(-ims)$ of Eqn(2.11) below. It is also shown in Ref. 7 that energy and momentum is exactly conserved in the interactions and decays of all such ‘asymptotically propagating’ particles. The use of quasi-classical particle trajectories and the requirement of exact energy-momentum conservation are crucial ingredients of the calculations presented below.

The structure of the paper is as follows. In the following Section the case of neutrino or muon oscillations following pion decay at rest is treated. Full account is taken of the momentum wave-packets of the propagating neutrinos and muons resulting from the Breit-Wigner amplitudes describing the distributions of the physical masses of the decaying pion and daughter muon. The corresponding oscillation damping corrections and phase shifts are found to be very small, indicating that the quasi-classical (constant velocity) approximation used to describe the neutrino and muon trajectories is a very good one. The incoherent effects, of random thermal motion of the source pion, and of finite source and detector sizes, on the oscillation probabilities and the oscillation phases, are also calculated. These corrections are found to be small in typical experiments, but much larger than those generated by the coherent momentum wave packets. In Section 3, formulae are derived to describe neutrino oscillations following muon decay at rest or the $\beta$-decay of radioactive nuclei. These are written down by direct analogy with those derived in the previous Section for pion decay at rest. In Section 4, the case of neutrino and muon oscillations following pion decay in flight is treated. In this case the two-dimensional spatial geometry of the particle trajectories must be related to the decay kinematics of the production process. Due to the non-applicability of the ultrarelativistic approximation to the kinematics of the muon in the pion rest frame, the calculation, although straightforward, is rather tedious and lengthy for the case of muon oscillations, so the details are relegated to an appendix. Finally, in Section 5, the positive aspects and shortcomings of previous treatments in the literature of the quantum mechanics of neutrino and muon oscillations are discussed in comparison with the method and results of the present paper.

\[ 8 \] The metric for four-vector products is time-like.
2 Neutrino and muon oscillations following pion decay at rest

To understand clearly the different physical hypotheses and approximations underlying the calculation of the particle oscillation effects it is convenient to analyse a precise experiment. This ideal experiment is, however, very similar to LNSD [17] and KARMEN [18] except that neutrinos are produced from pion, rather than muon, decay at rest.

The different space-time events that must be considered in order to construct the probability amplitudes for the case of neutrino oscillations following pion decay at rest are shown in Fig.1. A $\pi^+$ passes through the counter $C_A$, where the time $t_0$ is recorded, and comes to rest in a thin stopping target $T$ (Fig.1a)). For simplicity, the case of only two neutrino mass eigenstates $\nu_1$ and $\nu_2$ of masses $m_1$ and $m_2$ ($m_1 > m_2$) is considered. The pion at rest constitutes the initial state of the quantum mechanical probability amplitudes. The final state is an $e^-p$ system produced, at time $t_D$, via the process $\nu_e n \rightarrow e^-p$ at a distance $L$ from the decaying $\pi^+$ (Fig.1d)). Two different physical processes may produce the observed $e^-p$ final state, as shown in Fig.1b) and 1c), where the pion decays either at time $t_1$ into $\nu_1$ or at time $t_2$ into $\nu_2$. The probability amplitudes for these processes are, up to an arbitrary normalisation constant:

$$A_i = \int <e^-p|T|\nu_e><\nu_e|\nu_i>D(x_f-x_i,t_D-t_i,m_i)BW(W_{\mu(i)},m_\mu,\Gamma_\mu)<\nu_i|\nu_\mu>$$

$$<\nu_\mu\mu^+|T|\pi^+> e^{-\frac{E_\mu}{2}(t_i-t_0)}D(0,t_i-t_0,m_\pi)BW(W_\pi,m_\pi,\Gamma_\pi)dW_{\mu(i)} \quad i = 1,2 \quad (2.1)$$

Note that following the conventional ‘fi’ (final,initial) ordering of the indices of matrix elements in quantum mechanics, the path amplitude is written from right to left in order of increasing time. This ensures also correct matching of ‘bra’ and ‘ket’ symbols in the amplitudes. In Eqn(2.1), $<e^-p|T|\nu_e>$, $<\nu_e\mu^+|T|\pi^+>$ are the invariant amplitudes of the $\nu_e$ charged current scattering and pion decay processes, $BW(W_{\mu(i)},m_\mu,\Gamma_\mu)$ and $BW(W_\pi,m_\pi,\Gamma_\pi)$ are relativistic Breit-Wigner amplitudes, $<\nu_e|\nu_i>$ and $<\nu_i|\nu_\mu>$ are amplitudes describing the mixture of the flavour and mass neutrino eigenstates, and $D$ is the Lorentz-invariant configuration space propagator of the pion or neutrino. The pole masses and total decay widths of the pion and muon are denoted by $m_\pi$, $\Gamma_\pi$ and $m_\mu$, $\Gamma_\mu$ respectively. For simplicity, phase space factors accounting for different observed final states are omitted in Eqn(2.1) and subsequent formulae.

Because the amplitudes and propagators in Eqn(2.1) are calculated using relativistic quantum field theory, and the neutrinos propagate over macroscopic distances, it is a good approximation, as already discussed in the previous section, to assume exact energy-momentum conservation in the pion decay process, and that the neutrinos are on their mass shells, i.e. $p_i^2 = m_i^2$, where $p_i$ is the neutrino energy-momentum four-vector. In these circumstances the neutrino propagators correspond to classical, rectilinear, particle trajectories.

The pion and muon are unstable particles whose physical masses $W_\pi$ and $W_{\mu(i)}$ differ from the pole masses $m_\pi$ and $m_\mu$ appearing in the Breit-Wigner amplitudes and covariant space-time propagators in Eqn(2.1). The neutrino momentum $P_i$ will depend on these
physical masses according to the relation:

\[ P_i = \frac{\left[ W_\pi^2 - (m_i + W_{\mu(i)})^2 \right] \left[ W_\pi^2 - (m_i - W_{\mu(i)})^2 \right]}{2W_\pi} \]  

(2.2)

Note that, because the initial state pion is the same in the two path amplitudes in Eqn(2.1) \( W_\pi \) does not depend on the neutrino mass index \( i \). However, since the pion decays resulting in the production of \( \nu_1 \) and \( \nu_2 \) are independent physical processes, the physical masses of the unobserved muons, \( W_{\mu(i)} \), \( i = 1, 2 \), recoiling against the two neutrino mass eigenstates are not, in general, the same. In the following kinematical calculations sufficient accuracy is achieved by retaining only quadratic terms in the neutrino masses, and terms linear in the small quantities: \( \delta_\pi = W_\pi - m_\pi \), \( \delta_i = W_{\mu(i)} - m_\mu \). This allows simplification of the relativistic Breit-Wigner amplitudes:

\[ BW(W, m, \Gamma) \equiv \frac{\Gamma m}{W^2 - m^2 + im\Gamma} = \frac{\Gamma}{\delta(2m + \delta) + im\Gamma} = \frac{\Gamma}{2(\delta + \frac{\Gamma}{2})} + O(\delta^2) \equiv BW(\delta, \Gamma) + O(\delta^2) \]  

(2.3)

Developing Eqn(2.2) up to first order in \( m_i^2 \), \( \delta_i \) and \( \delta_\pi \) yields the relation:

\[ P_i = P_0 \left[ 1 - \frac{m_i^2(m_\pi^2 + m_\mu^2)}{(m_\pi^2 - m_\mu^2)^2} + \frac{\delta_\pi (m_\pi^2 + m_\mu^2)}{m_\pi (m_\pi^2 - m_\mu^2)} \right. \\
- \left. \frac{2\delta_i m_\mu}{m_\pi^2 - m_\mu^2} + \frac{\delta_\pi m_\pi^2 (m_\pi^2 + m_\mu^2)}{m_\pi (m_\pi^2 - m_\mu^2)^2} \right] \]  

(2.4)

where

\[ P_0 = \frac{m_\pi^2 - m_\mu^2}{2m_\pi} = 29.8 \text{MeV} \]  

(2.5)

The term \( \simeq \delta_\pi m_i^2 \) which is also included in Eqn(2.4) gives a negligible \( O(m_i^4) \) contribution to the neutrino oscillation formula. For muon oscillations, however, it gives a term of \( O(m_i^2) \) in the interference term, as discussed below. Similarly, the exact formula for the neutrino energy:

\[ E_i = \frac{W_\pi^2 - W_{\mu(i)}^2 + m_i^2}{2W_\pi} \]  

(2.6)

in combination with Eqn(2.4) gives, for the neutrino velocity:

\[ v_i = \frac{P_i}{E_i} = 1 - \frac{m_i^2}{2P_0^2} \left[ 1 - \frac{2\delta_\pi (m_\pi^2 + m_\mu^2)}{m_\pi (m_\pi^2 - m_\mu^2)} + \frac{4\delta_i m_\mu}{m_\pi^2 - m_\mu^2} \right] + O(m_i^4, \delta_\pi^2, \delta_i^2) \]  

(2.7)

This formula will be used below to calculate the neutrino times-of-flight: \( t^{fi}_i \) \( i = 1, 2 \).

The unitary transformation specifying the mixing of the flavour and mass neutrino eigenstates is defined by a single real mixing angle \( \theta \):

\[ |\nu_e > = \cos \theta |\nu_1 > + \sin \theta |\nu_2 > \]  

(2.8)

\[ |\nu_\mu > = - \sin \theta |\nu_1 > + \cos \theta |\nu_2 > \]  

(2.9)
where the flavour eigenstates $|\alpha >$, $\alpha = \nu_e, \nu_\mu$ and the mass eigenstates $|a >$, $a = \nu_1, \nu_2$ respect the orthonormality conditions:

$$< \alpha | \beta > = \delta_{\alpha \beta}, \quad < a | b > = \delta_{ab} \tag{2.10}$$

The parts of the amplitudes requiring the most careful discussion are the invariant space-time propagators $D$, as it is mainly their treatment that leads to the different result for the neutrino oscillation phase found in the present paper, as compared to those having previously appeared in the literature. In the limit of large time-like separations, the propagator may be written as $[7, 9]$:

$$D(\Delta x, \Delta t, m) = \left( \frac{m^2}{2\pi i \sqrt{(\Delta t)^2 - (\Delta x)^2}} \right)^{\frac{3}{2}} \exp[-im\sqrt{(\Delta t)^2 - (\Delta x)^2}] \tag{2.11}$$

$D$ is the amplitude for a particle, originally at a space-time point $(\vec{x}_i, t_i)$, to be found at $(\vec{x}_f, t_f)$ and $\Delta \vec{x} \equiv \vec{x}_f - \vec{x}_i$, $\Delta t \equiv t_f - t_i$. In the following, according to the geometry of the experiment shown in Fig.1, only one spatial coordinate will be considered ($\Delta x \equiv x_f - x_i$) and only the exponential factor in Eqn(2.11), containing the essential phase information for particle oscillations will be retained in the amplitudes. Solid angle correction factors, taken into account by the factor in large brackets in Eqn(2.11), are here neglected, but are easily included in the final oscillation formulae. Writing then

$$D(\Delta x, \Delta t, m) \simeq \exp[-im\sqrt{(\Delta t)^2 - (\Delta x)^2}] = \exp[-im\Delta \tau] \equiv \exp[-i\Delta \phi] \tag{2.12}$$

it can be seen that the increment in phase of the propagator, $\Delta \phi$, when the particle undergoes the space-time displacement $(\Delta x, \Delta t)$ is a Lorentz invariant quantity equal to the product of the particle mass and the increment, $\Delta \tau$, of proper time. Using the relativistic time dilatation formula:

$$\Delta t = \gamma \Delta \tau = \frac{E}{m} \Delta \tau \tag{2.13}$$

and also the relation, corresponding to a classical, rectilinear, particle trajectory:

$$\Delta t = \frac{L}{v} = \frac{E}{p} \tag{2.14}$$

gives, for the phase increments corresponding to the paths of the neutrinos and the pion in Fig.1:

$$\Delta \phi_\nu^i = m_i \Delta \tau_i = \frac{m_i^2}{E_i} \Delta t_i = \frac{m_i^2}{P_i} L$$

$$= \frac{m_i^2 L}{P_0^2} \left[ 1 - \delta_{\pi \nu} \left( \frac{m_\pi^2 + m_\mu^2}{m_\pi^2 - m_\mu^2} \right) + \frac{\delta_{\pi \mu} m_\mu}{m_\pi^2 - m_\mu^2} \right] \tag{2.15}$$

$$\Delta \phi_\pi^i = m_\pi (t_i - t_0) = m_\pi (t_D - t_0) - \frac{m_\pi L}{v_i}$$

$$= m_\pi (t_D - t_0) - m_\pi L \left\{ 1 + \frac{m_i^2}{2P_0^2} \left[ 1 - \frac{2\delta_{\pi \nu} \left( m_\pi^2 + m_\mu^2 \right)}{m_\pi^2 \left( m_\pi^2 - m_\mu^2 \right)} + \frac{4\delta_{\pi \mu} m_\mu}{m_\pi^2 - m_\mu^2} \right] \right\} \tag{2.16}$$
Making the substitution $t_i - t_0 \rightarrow t_D - t_0 - L/v_i$ in the exponential damping factor due to the pion lifetime in Eqn(2.1) and using Eqns(2.3),(2.12),(2.15) and (2.16) Eqns(2.1) may be written as:

$$A_i = \int \langle e^{-p|T|} \nu_e | \nu_e > \langle \nu_e | \nu_i > \langle \nu_i | \nu_\mu > \langle \nu_\mu | T | \pi^- > \frac{\Gamma_\mu}{2} e^{i \alpha_i \delta_i} \frac{\Gamma_\pi e^{i \alpha(i) \delta_\pi}}{2 (\delta_\pi + i \frac{F_\pi}{2})} e^{i \phi_0 - \frac{\Gamma_\pi}{2} (t_D - t_0 - t_i^f)} \exp i \left[ \frac{m_i^2}{P_0} \left( \frac{m_\pi}{2P_0} - 1 \right) L \right] d\delta_i \quad i = 1,2 \quad (2.17)$$

where

$$\phi_0 \equiv m_\pi (L - t_D + t_0) \quad (2.18)$$

$$\alpha_i \equiv \frac{4m_i^2 m_\pi (m_\pi^2 + m_\mu^2)}{(m_\pi^2 - m_\mu^2)^2} \left[ 1 - \frac{i \Gamma_\pi m_\pi}{m_\pi^2 + m_\mu^2} \right] \quad (2.19)$$

$$\alpha_\pi(i) \equiv -\frac{2m_i^2 (m_\pi^2 + m_\mu^2)}{(m_\pi^2 - m_\mu^2)^2} \left[ 1 - \frac{i \Gamma_\pi m_\pi}{m_\pi^2 + m_\mu^2} \right] \quad (2.20)$$

and

$$t_i^f = L(1 + \frac{m_i}{2P_0^2}) + O(m_i^4) \quad (2.21)$$

To perform the integral over $\delta_i$ in Eqn(2.17) it is convenient to approximate the modulus squared of the Breit-Wigner amplitude by a Gaussian, via the substitution:

$$\frac{\Gamma}{\delta + i \frac{F_\pi}{2}} = \frac{\Gamma}{2} \left( \frac{\delta - i \frac{\Gamma}{2}}{\delta^2 + \frac{\Gamma^2}{4}} \right) \rightarrow \frac{\Gamma}{2} (\delta - i \frac{\Gamma}{2}) \exp \left( -\frac{3 \delta^2}{\Gamma^2} \right) \quad (2.22)$$

where the width of the Gaussian is chosen so that it has approximately the same full width at half maximum, $\Gamma$, as the Breit-Wigner function. After the substitution (2.22), the integral over $\delta_i$ in Eqn(2.17) is easily evaluated by a change of variable to ‘complete the square’ in the argument of the exponential, with the result:

$$I_i = \frac{2}{\Gamma_\mu} \int_{-\infty}^{\infty} (\delta_i - i \frac{\Gamma_\mu}{2}) \exp \left( -\frac{3 \delta_i^2}{\Gamma_\mu^2} + i \alpha_i \delta_i \right) d\delta_i = i \sqrt{\frac{\pi}{3}} \Gamma_\mu \exp \left( -\frac{\alpha_i^2 \Gamma_\mu^2}{12} \right) \left[ \frac{\alpha_i \Gamma_\mu}{3} - 1 \right] \quad (2.23)$$

Since $\Gamma_\pi = 2.53 \times 10^{-14}$ MeV, the term dependent on $\Gamma_\pi$ in Eqn(2.19) may be neglected, so that $\alpha_i$ may be taken as a real number with the numerical value:

$$\alpha_i = 3.1 \times 10^{14} \left( \frac{m_i}{m_\pi} \right)^2 L(m) \quad \text{MeV}^{-1} \quad (2.24)$$

For typical physically interesting values (see below) of $m_i = 1$ eV and $L = 30$ m, $\alpha_i$ takes the value 0.48 MeV$^{-1}$, so that

$$\alpha_i \Gamma_\mu = 0.48 \times 3.00 \times 10^{-16} = 1.44 \times 10^{-16}$$

Then, to very good accuracy, $I_1 = I_2 = -i \sqrt{\pi/3 \Gamma_\mu}$, independently of the neutrino mass. It follows that for neutrino oscillations, the muon mass dependence of the amplitudes may be neglected for any physically interesting values of $m_i$ and $L$. 

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From Eqns (2.17) and (2.23) the probability to observe the reaction $\nu_e n \rightarrow e^- p$ at distance $L$ from the pion decay point and at time $t_D$ is:

$$P(e^- p|L, t_D) = |A_1 + A_2|^2$$

$$= \frac{\pi \Gamma_\pi^2}{3} |< e^- p|T|\nu_e >|^2 |< \nu_\mu \mu^+|T|\pi^+ >|^2$$

$$\times \sin^2 \theta \cos^2 \theta \exp(-\Gamma_\pi (t_D - \tau_0)) \frac{\Gamma_\pi^2}{4(\delta^2 + \Gamma_\pi^2/4)}$$

$$\times \left\{ e^{\Gamma_\pi t_1^{fl}} + e^{\Gamma_\pi t_2^{fl}} - 2e^{\Gamma_\pi \frac{(t_1^{fl} + t_2^{fl})}{2}} \Re \exp \left[ \frac{\Delta m^2}{P_0} \left( \frac{m_\pi}{2P_0} - 1 \right) L + [\alpha_\pi(1) - \alpha_\pi(2)]\delta_\pi \right] \right\}$$

(2.25)

The time dependent exponential factors in the curly brackets of Eqn(2.25) are easily understood. If $m_1 > m_2$ then $t_1^{fl} > t_2^{fl}$. This implies that the neutrino of mass $m_1$ results from an earlier decay than the neutrino of mass $m_2$, in order to be detected at the same time. Because of the exponential decrease with time of the pion decay amplitude, the contribution to the probability of the squared amplitude for the neutrino of mass $m_1$ is larger. The interference term resulting from the product of the decay amplitudes of the two neutrinos of different mass, has an exponential factor that is the harmonic mean of those of the squared amplitudes for each neutrino mass eigenstate, and so is also suppressed relative to the squared amplitude for the neutrino of mass $m_1$. The integral over the physical pion mass is readily performed by replacing the Breit-Wigner function by a Gaussian as in Eqn(2.22). This leads to an overall multiplicative constant $\sqrt{\pi/3}\Gamma_\pi$ and a factor:

$$\exp[-(\alpha_\pi(1) - \alpha_\pi(2))^2\Gamma_\pi^2/12]$$

multiplying the interference term. For $\Delta m^2 = (1eV)^2$ and $L = 30m$ the numerical value of this factor is $\exp(-1.3 \times 10^{-20})$. This tiny correction is neglected in the following equations.

Integrating over $t_D$ gives the average probability to observe the $e^- p$ final state at distance $L$:

$$P(e^- p|L) = \frac{\pi \Gamma_\pi^2}{3\sqrt{3}} |< e^- p|T|\nu_e >|^2 |< \nu_\mu \mu^+|T|\pi^+ >|^2 \sin^2 \theta \cos^2 \theta$$

$$\times \left\{ 1 - \exp[-\frac{\Gamma_\pi m_\pi^2 \Delta m^2}{(m_\pi^2 - m_\mu^2)^2}L] \cos \frac{2m_\pi m_\mu^2 \Delta m^2}{(m_\pi^2 - m_\mu^2)^2}L \right\}$$

(2.26)

Where all kinematical quantities are expressed in terms of $\Delta m^2$, $m_\pi$ and $m_\mu$. Note that the minimum value of $t_D$ is $t_0 + t_1^{fl}$, $t_0 + t_2^{fl}$ and $t_0 + t_1^{fl}$ for the squared amplitude terms of neutrinos of mass $m_1$, $m_2$ and the interference term, respectively. On integrating over $t_D$, the squared amplitude terms give equal contributions, the larger amplitude for mass $m_1$ being exactly compensated by a smaller range of integration. The exponential damping factor in the interference term in Eqn(2.26) is derived using the relations:

$$t_1^{fl} - t_2^{fl} = \left( \frac{1}{v_1(\nu)} - \frac{1}{v_2(\nu)} \right) L \simeq (v_2(\nu) - v_1(\nu))L$$

(2.27)
and

\[ v_i(\nu) = 1 - \frac{m_i^2}{2P_0^2} + O(m_i^4) \quad i = 1, 2 \]  

(2.28)

to obtain

\[ t_1^f = t_2^f = \frac{(m_1^2 - m_2^2) L}{2P_0^2} = \frac{\Delta m^2 L}{2P_0^2} \]  

(2.29)

The damping factor arises because the difference in the times-of-flight of the two neutrino paths is limited by the pion lifetime. It will be seen below, however, that for distances \( L \) of practical interest for the observation of neutrino oscillations, the damping effect is tiny.

The part of the oscillation phase in Eqn(2.25) originating from the neutrino propagators (the term associated with the ‘1’ within the large curved brackets) differs by a factor two from the corresponding expression in the standard formula. The contribution to the oscillation phase of the propagator of the decaying pion (the term associated with \( m_\pi/(2P_0) \) in the large curved brackets of Eqn(2.24)) has not been taken into account in any published calculation known to the author of the present paper. The oscillation phase in Eqn(2.26) is \( 2m_\mu^2/(m_\pi^2 - m_\mu^2) = 2.685 \) times larger than that given by the standard formula (1.1). For \( L = 30m \), as in the LNSD experiment, the first oscillation maximum occurs for \( \Delta m^2 = 0.46(eV)^2 \). Denoting by \( \phi_{12}^\nu,\pi \) the phase of the cosine interference term in Eqn(2.26), the pion lifetime damping factor can be written as:

\[ F^\nu(\Gamma_\pi) = \exp\left(-\Gamma_\pi \frac{m_\pi}{2m_\mu^2} \phi_{12}^\nu,\pi \right) = \exp(-1.58 \times 10^{-16} \phi_{12}^\nu,\pi) \]  

(2.30)

so the damping effect is vanishingly small when \( \phi_{12}^\nu,\pi \approx 1 \).

The oscillation formula (2.26) is calculated on the assumption that the decaying pion is at rest at the precisely defined position \( x_i \). In fact, the positive pion does not bind with the atoms of the target, but will rather undergo random thermal motion. This has three effects: an uncertainty in the value of \( x_i \), a Doppler shift of the neutrino energy and a time dilatation correction correction factor of \( 1/\gamma_\pi \) in the equation (2.16) for the pion phase increment. Assuming that the target is at room temperature (\( T= 270^\circ \text{K} \)), the mean kinetic energy of \( 3kT/2 \) corresponds to a mean pion momentum of \( 2.6 \times 10^{-3} \) MeV and a mean velocity of \( \approx 5.6 \) km/sec. The pion will move, in one mean lifetime \( (2.6 \times 10^{-8} \text{sec}) \), a distance of \( 146 \mu m \). This is negligible as compared to \( L \) (typically \( \geq 30m \)) and so Eqn(2.26) requires no modification to account for this effect.

The correction factor due to the Doppler effect and time dilatation is readily calculated on the assumption of a Maxwell-Boltzmann distribution of the pion momentum:

\[ \frac{dN}{dp_\pi} \approx p_\pi^2 \exp\left(-\frac{p_\pi^2}{\overline{p}_\pi^2}\right) \]  

(2.31)

Here \( \overline{p}_\pi = \sqrt{2kTm_\pi} = 2.64 \times 10^{-3} \) MeV. Details of the calculation are given in Appendix A. The interference term in Eqn(2.26) is modified by a damping factor:

\[ F^\nu(Dop) = \exp\left\{-\left(\frac{\overline{p}_\pi^2 \Delta m^2}{2m_\pi^2 P_0} \left[\frac{m_\pi}{P_0} - 1\right] L\right)^2\right\} \]  

(2.32)
while the argument of the cosine term acquires an additional phase factor:

$$\phi^\nu(Dop) = \frac{3}{4} \left( \frac{\overline{\nu}_\pi}{m_\pi} \right)^2 \frac{\Delta m^2}{P_0} \left[ \frac{3m_\pi}{2P_0} - 1 \right] L$$

(2.33)

For $\phi^\mu_{12} = 1$, $F^\nu(Dop) = 1 - 6.7 \times 10^{-10}$ and $\phi^\nu(Dop) = 1.2 \times 10^{-9}$.

If the target in which the pion stops is of thickness $\ell_T$, then the effect of different stopping points of the $\pi$ (assumed uniformly distributed) is to multiply the interference term in (2.26) by the factor:

$$F^\nu_{Targ} = \frac{(m_\pi^2 - m_\mu^2)^2}{m_\pi m_\mu \Delta m^2 \ell_T} \sin\left( \frac{m_\pi m_\mu \Delta m^2 \ell_T}{(m_\pi^2 - m_\mu^2)^2} \right)$$

(2.34)

If the position of the neutrino interaction point within the target has an uncertainty of $\pm \ell_D/2$ a similar correction factor is found, with the replacement $\ell_T \rightarrow \ell_D$ in Eqn(2.34). The calculation of this correction factor is also described in Appendix A.

If the target T in which the pion comes to rest (Fig.1a)) is chosen to be sufficiently thin, the pion decay process may be detected by observing the recoil muon in the counter $C_A$ at times $t_1$ or $t_2$ (Fig.1b) or 1c)). A sufficiently accurate measurement of the times $t_1$, $t_2$ and $t_D$ would, in principle, enable separation of the different processes in Figs.1b) and 1c) by the observation of separated peaks in the time-of-flight distribution at $t_1' = t_D - t_1$ and $t_2' = t_D - t_2$. In this case the interference term in Eqn(2.26) vanishes as the two alternative space-time paths leading to the neutrino interaction are distinguishable. However, for $L = 30m$ and $\Delta m^2 = 1(eV)^2$ the difference in the times of flight is only $5.6 \times 10^{-23}$ sec, more than ten orders of magnitude smaller than can be measured with existing techniques. As discussed in Section 5 below, the momentum smearing due to the Doppler effect at room temperature is some eleven orders of magnitude larger than the shift due to a neutrino mass difference with $\Delta m^2 = 1(eV)^2$. Thus, even with infinitely good time resolution, separation of such neutrino mass eigenstates by time-of-flight is not possible.

The ideal experiment, described above to study neutrino oscillations, is easily adapted to the case of oscillations in the decay probability of muons produced by charged pion decay at rest. As previously pointed out in Ref. [1], such oscillations will occur if neutrinos with different masses exist. As before, the pion stops in the target $T$ at time $t_0$ (see Fig.2a)). At time $t_1$ the pion decays into $\nu_1$ and the corresponding recoil muon ($\mu_1$), whose passage is recorded in the counter $C_B$ (Fig.2b)). Similarly, a decay into $\nu_2$ and $\mu_2$ may occur at time $t_2$ (Fig.2c)). With a suitable choice of the times $t_1$ and $t_2$, such that muons following the alternative paths both arrive at the same time $t_D$ at the point $x_f$, interference occurs between the path amplitudes when muon decay occurs at the space-time point $(x_f, t_D)$ in the detector D (Fig.2d)). The path amplitudes corresponding to muons recoiling against neutrinos of mass $m_1$ and $m_2$ are:

$$A_i = \int <e^+ \nu_i \overline{\nu}_\mu | T | \mu^+ > \frac{\Gamma_{\mu} \nu_\mu}{2m_\mu} D(x_f - x_i, t_D - t_i, m_\mu)BW(W_{\mu(i)}, m_\mu, \Gamma_\mu) < \nu_i | \nu_\mu >$$

$$< \nu_\mu \mu^+ | T | \pi^+ > e^{-\frac{\Gamma_{\pi}}{2}(t_i - t_0)} D(0, t_i - t_0, m_\pi)BW(W_\pi, m_\pi, \Gamma_\pi) dW_{\mu(i)} \quad i = 1, 2$$

(2.35)

The various factors in these equations are defined, mutatis mutandis, as in Eqn(2.1).
With the same approximations, concerning the neutrino masses and the physical pion and muon masses, as those made above, the velocity of the muon recoiling against the neutrino mass eigenstate $\nu_i$ is:

$$v_i^\mu = v_0^\mu \left[ 1 - \frac{4m_\pi^2 m_\mu^2}{(m_\pi^2 - m_\mu^2)^2(m_\pi^2 + m_\mu^2)} + \frac{4\delta_\pi m_\pi^2 m_\mu^2}{m_\pi^4 - m_\mu^4} - 4\delta_\pi m_\pi^2 m_\mu^2 \right]$$

where

$$v_0^\mu = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2}$$

Comparing with Eqn(2.7), it can be seen that for the muon case, unlike that where neutrino interactions are observed, there are pion and muon mass dependent correction terms that are independent of the neutrino masses, implying a velocity smearing effect due to the physical pion and muon masses that is $\simeq m_\pi^2/m_\mu^2$ larger than for the case of neutrino oscillations.

The phase increments corresponding to the paths of the muons and the pion in Fig. 2 are, using (2.4)\(^9\) and (12.12)-(12.14):

$$\Delta \phi_i^\mu = \frac{m_\pi^2 L}{p_i^\mu} = \frac{m_\pi^2 L}{p_0^\mu} \left[ 1 + \frac{m_\pi^2 E_0^\mu}{2m_\pi p_0^\mu} - \frac{\delta_\pi (m_\pi^2 + m_\mu^2)}{m_\pi (m_\pi^2 - m_\mu^2)} + \frac{2\delta_\pi m_\mu}{m_\pi^2 - m_\mu^2} \right]$$

$$\Delta \phi_i^{\pi(\mu)} = m_\pi(t_i - t_0) = m_\pi(t_D - t_0) = m_\pi L \frac{E_0^\mu}{v_i^\mu}$$

where

$$E_0^\mu = \frac{m_\pi^2 + m_\mu^2}{2m_\pi}$$

Using Eqns(2.12),(2.38) and (2.39) to re-write the space-time propagators in Eqn(2.35), as well as Eqn(2.3) for the Breit-Wigner amplitudes gives:

$$A_i^\mu = \int <e^+ \nu_e \bar{\nu}_\mu | T | \mu^+ > e^{-\frac{r_{\nu e}(L)}{2\mu}} - <\nu_\mu | T | \mu^+ > \frac{\Gamma_\mu}{2} \frac{e^{i\alpha_i^\mu \delta_i}}{\left( \frac{\delta_i}{L} + \frac{2m_\pi}{3} \right)} \Gamma_\pi \exp i \left[ \frac{m_\pi^2 m_i^2}{2P_0^3} \left( 1 - \frac{E_0^\mu}{m_\pi^2} \right) \right] d\delta_i \quad i = 1, 2$$

\(^9\)Note that, in the pion rest frame $P_i = P_i^\mu$. 

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\[
\phi_0^\mu \equiv m_\pi \left( \frac{L}{v_0^\mu} - t_D + t_0 \right) - \frac{m_\pi^2 L}{P_0^\mu} \tag{2.42}
\]

\[
\alpha^\mu \equiv \frac{4m_\mu m_\pi L}{m_\pi^2 - m_\mu^2} \left[ 1 - \frac{i\Gamma_\pi m_\pi}{2(m_\pi^2 - m_\mu^2)} \right] \tag{2.43}
\]

\[
\alpha^\mu_{\pi}(i) \equiv -\frac{2m_\mu^2 L}{m_\pi^2 - m_\mu^2} \left[ 1 + \frac{m_\pi^2(5m_\pi^6 - 10m_\pi^4m_\mu^2 - m_\pi^2m_\mu^4 - m_\mu^6)}{(m_\pi^4 - m_\mu^4)(m_\pi^2 - m_\mu^2)^2} - \frac{i\Gamma_\pi m_\pi}{m_\pi^2 - m_\mu^2} \right] \tag{2.44}
\]

Making the substitution (2.22) and performing the integral over \( \delta_i \) according to Eqn(2.23), the following formula is found for the probability for muon decay at distance \( L \) and time \( t_D \).

\[
P(e^+\nu_e\overline{\nu}_\mu|L, t_D) = |A_1^\mu + A_2^\mu|^2
\]

\[
= \frac{\pi\Gamma_\mu^2}{3} e^{-\frac{(\alpha^\mu\Gamma_\mu)^2}{3}} \left[ 1 - \frac{\alpha^\mu\Gamma_\mu}{3} \right]^2 |<e^+\nu_e\overline{\nu}_\mu|T|\mu^+>|^2 e^{-\frac{\nu_\mu v_0^\mu}{m_\pi^2} L}
\]

\[
\mathcal{O} \left\{ \sin^2 \theta e^{\Gamma_\pi t^\mu_{\pi(1)}} + \cos^2 \theta e^{\Gamma_\pi t^\mu_{\pi(2)}} - 2 \sin \theta \cos \theta e^{\Gamma_\pi t^\mu_{\pi(1)} + t^\mu_{\pi(2)}} \right\} \tag{2.46}
\]

The muon path difference yields the term associated with \( E_0^\mu/m_\pi \) in the interference phase in Eqn(2.46) while the pion path is associated with ‘1’ in the large round brackets. The numerical value of the damping factor:

\[
\exp \left[ -\frac{(\alpha^\mu\Gamma_\mu)^2}{6} \right] \left[ 1 - \frac{\alpha^\mu\Gamma_\mu}{3} \right]^2
\]

resulting from the integral over the physical muon mass is, for \( L = 30m, 0.774 \), so, unlike for the case of neutrino oscillations, the correction is by no means negligible. This is because, in the muon oscillation case, the leading term of \( \alpha^\mu \) is not proportional to the neutrino mass squared. The non-leading terms proportional to \( m_\pi^2 \) have been neglected in Eqn(2.43). This correction however effects only the overall normalisation of the oscillation formula, not the functional dependence on \( L \) arising from the interference term. Integrating over \( \delta_\pi \) using Eqns(2.22) and (2.23), as well as over \( t_D \), gives the probability of muon decay at distance \( L \) from the production point, where all kinematical quantities are expressed in terms of \( \Delta m^2, m_\pi \) and \( m_\mu \):

\[
P(e^+\nu_e\overline{\nu}_\mu|L) = \frac{\pi^2\Gamma_\mu^2}{3\sqrt{3}} \exp \left[ -\frac{8}{3} \left( \frac{\Gamma_\mu m_\mu m_\pi}{m_\pi^2 - m_\mu^2} L \right)^2 \right] \left[ 1 - \frac{4 \Gamma_\mu m_\mu m_\pi}{3 m_\pi^2 - m_\mu^2} L \right]^2
\]

\[
|<e^+\nu_e\overline{\nu}_\mu|T|\mu^+>|^2 \exp \left[ -\frac{2\Gamma_\mu m_\mu m_\mu(m_\pi^2 - m_\mu^2)}{(m_\pi^2 + m_\mu^2)^3} L \right] |<\nu_\mu \mu^+|T|\pi^+>|^2
\]

\[
\left\{ 1 - \sin 2\theta \exp \left[ -\frac{2\Gamma_\pi m_\mu^2 m_\pi^2\Delta m^2}{(m_\pi^2 - m_\mu^2)^3} L \right] \cos \frac{2m_\pi m_\mu^2\Delta m^2}{(m_\pi^2 - m_\mu^2)^2} L \right\} \tag{2.47}
\]
In this expression the correction due to the damping factor of the interference term:

\[ \exp[-(\alpha_\mu(1) - \alpha_\mu(2))^2 \Gamma_\pi^2 / 12] \]
arising from the integral over the physical pion mass has been neglected. For \( \Delta m^2 = (1 \text{eV})^2 \) and \( L = 30 \text{m} \) the numerical value of this factor is \( \exp(-3.9 \times 10^{-31}) \). Denoting by \( \phi_{12}^{\mu,\pi} \) the argument of the cosine in Eqn(2.47), the exponential damping factor due to the pion lifetime may be written as:

\[
F^\mu(\Gamma_\pi) = \exp \left( -\frac{\Gamma_\pi m_\pi}{(m_\pi^2 - m_\pi^2)} \phi_{12}^{\mu,\pi} \right) \quad (2.48)
\]

For \( \phi_{12}^{\mu,\pi} = 1 \), \( F^\mu(\Gamma_\pi) - 1 = 4.4 \times 10^{-16} \) so, as in the neutrino oscillation case, the pion lifetime damping of the interference term is very small.

Corrections due to time dilatation and the Doppler effect are calculated in a similar way to the neutrino oscillation case with the results (see Appendix A):

\[
F^\mu(Dop) = \exp \left\{ -\left( \frac{\Gamma_\pi m_\pi}{2m_\pi P_0^3} \right)^2 \left[ \frac{3}{2} - \frac{E_\mu}{m_\pi} \right] L \right\} \quad (2.49)
\]

and

\[
\phi^\mu(Dop) = \frac{3}{2} \left( \frac{\Gamma_\pi m_\pi}{m_\pi} \right)^2 \frac{m_\mu^2 \Delta m^2}{P_0^3} \left[ 1 - \frac{E_\mu}{2m_\pi} \right] L \quad (2.50)
\]

As for neutrino oscillations, the corresponding corrections are very small for oscillation phases of order unity.

The phase of the cosine in the interference term is the same in neutrino and muon oscillations, as can be seen by comparing Eqns(2.26) and (2.47). It follows that the target or detector size correction (Eqn(2.34)) is the same in both cases.

Neutrino and muon oscillations from pion decay at rest have an identical oscillation phase for given values of \( \Delta m^2 \) and \( L \). In view of the much larger event rate that is possible, it is clearly very advantageous in this case to observe muons rather than neutrinos. In fact, it is not necessary to observe muon decay, as in the example discussed above. The oscillation formula applies equally well if the muons are observed at the distance \( L \) using any high efficiency detector such as, for example, a scintillation counter. In contrast, the rate of neutrino oscillation events is severely limited by the very small neutrino interaction cross section.

### 3 Neutrino oscillations following muon decay at rest or beta-decays in nuclear reactors

The formula describing neutrino oscillations \( \bar{\nu}_\mu \rightarrow \bar{\nu}_e \) following the decay at rest of a \( \mu^+ \), \( \mu^+ \rightarrow e^+\nu_e\bar{\nu}_\mu \) is easily derived from the similar formula where a \( \nu_\mu \) is produced by \( \pi^+ \) decay at rest (2.25). Because the neutrino momentum spectrum is continuous, smearing effects due to the finite muon lifetime may be neglected from the outset. The
phase increment associated with the neutrino path is then given by Eqn(2.15) with the replacements \( P_0 \rightarrow P_{\nu} \) and \( \delta_\pi, \delta_i \rightarrow 0 \), where \( P_{\nu} \) is the antineutrino momentum. The phase increment of the decaying muon is given by the same replacements in Eqn(2.16) and, in addition, \( m_\pi \rightarrow m_\mu \) and \( \Gamma_\pi \rightarrow \Gamma_\mu \). The formula, analogous to Eqn(2.26), for the time-averaged probability to detect the process \( \bar{\nu}_e p \rightarrow e^+ n \) at a distance \( L \) from the muon decay point is then:

\[
P(e^+ n, \mu|L) = \frac{|<e^+ n|T|p\bar{\nu}_e>|^2 |<\nu_e \bar{\nu}_e e^+|T|\mu^+>|^2 2 \sin^2 \theta \cos^2 \theta}{\Gamma_\mu} \\
\times \left\{ 1 - \exp\left[ -\frac{\Gamma_\mu \Delta m^2}{4 P_{\nu}^2} L \right] \cos \left[ \frac{\Delta m^2}{P_{\nu}} \left( \frac{m_\mu}{2 P_{\nu}} - 1 \right) L \right] \right\}
\] (3.1)

The standard neutrino oscillation formula, hitherto used in the analysis of all experiments, has instead the expression \( \Delta m^2 L/(2 P_{\nu}) \) for the argument of the cosine term in Eqn(3.1). Denoting \( \Delta m^2_S \) the value of \( \Delta m^2 \) obtained using the standard formula, and \( \Delta m^2_{FP} \) that obtained using the Feynman Path (FP) formula (3.1) then:

\[
\Delta m^2_{FP} = \frac{\Delta m^2_S}{\frac{m_\mu}{P_{\nu}} - 2}
\] (3.2)

For a typical value of \( P_{\nu} \) of 45 MeV, Eqn(3.2) implies that \( \Delta m^2_{FP} \approx 2.9 \Delta m^2_S \). Thus the \( \bar{\nu}_\mu \) oscillation signal from \( \mu^+ \) decays at rest reported by the LNSD Collaboration [17] corresponding to \( \Delta m^2_S \approx 0.5 \) (eV)\(^2\) for \( \sin^2 \theta \approx 0.02 \) implies \( \Delta m^2_{FP} \approx 1.5 \) (eV)\(^2\) for a similar mixing angle.

For the case of \( \beta \)-decay:

\[
N(A, Z) \rightarrow N(A, Z + 1)e^- \bar{\nu}_e
\]

\( m_\pi \) in the first line of Eqn(2.16) is replaced by \( E_\beta \), the total energy release in the \( \beta \)-decay process:

\[
E_\beta = M_N(A, Z) - M_N(A, Z + 1)
\] (3.3)

where \( M_N(A, Z) \) and \( M_N(A, Z + 1) \) are the masses of the parent and daughter nuclei. That the phase advance of an unstable state, over a time, \( \Delta t \), is given by \( \exp(-iE^* \Delta t) \) where \( E^* \) is the excitation energy of the state, is readily shown by the application of time-dependent perturbation theory to the Schrödinger equation [13]. A more intuitive derivation of this result has been given in Ref. [20]. In the present case, \( E^* = E_\beta \). Omitting the lifetime damping correction, which is about eight orders of magnitude smaller than for pion decay, given a typical \( \beta \)-decay lifetime of a few seconds, the time-averaged probability to detect the \( \bar{\nu}_e \) via the process \( \bar{\nu}_e p \rightarrow e^+ n \), at distance \( L \) from the decay point is given by the formula, derived in a similar way to Eqns(2.26) and (3.1):

\[
P(e^+ n, \beta|L) = \frac{|<e^+ n|T|p\bar{\nu}_e>|^2 |<\bar{\nu}_e N(A, Z + 1)|T|N(A, Z)>|^2}{\Gamma_\beta} \\
\times \left\{ \sin^4 \theta + \cos^4 \theta + 2 \sin^2 \theta \cos^2 \theta \cos \left[ \frac{\Delta m^2}{P_{\nu}} \left( \frac{E_\beta}{2 P_{\nu}} - 1 \right) L \right] \right\}
\] (3.4)
where $\Gamma_\beta^{-1} = \tau_\beta$ the lifetime of the unstable nucleus $N(A,Z)$. Until now, all experiments have used the standard expression $\Delta m^2 L/(2P_\nu)$ for the neutrino oscillation phase. The values of $\Delta m^2$ found should be scaled by the factor $(E_\beta/P_\nu - 2)^{-1}$, suitably averaged over $P_\nu$, to obtain the $\Delta m^2$ given by the Feynman Path formula (3.4).

4 Neutrino and muon oscillations following pion decay in flight

In this Section, the decays in flight of a $\pi^+$ beam with energy $E_\pi \gg m_\pi$ into $\mu^+\nu_\mu$ are considered. As the analysis of the effects of the physical pion and muon masses have been shown above to give negligible corrections to the $L$ dependence of the oscillation formulae, for the case of decays at rest, such effects will be neglected in this discussion of in-flight decays. The overall structure of the path amplitudes for neutrinos and muons is the same as for decays at rest (see Eqns(2.1) and (2.35)). However, for in-flight decays, in order to calculate the interfering paths originating at different and terminating at common space-time points, the two-dimensional spatial geometry of the problem must be properly taken into account.

In Fig. 3 a pion decays at A into the 1 mass eigenstate, the neutrino being emitted at an angle $\theta_1$ in the lab system relative to the pion flight direction. If $m_1 > m_2$ a later pion decay into the 2 mass eigenstate at the angle $\theta_1 + \delta$ may give a path such that both eigenstates arrive at the point B at the same time. A neutrino interaction $\nu_e n \rightarrow e^- p$, or $\nu_\mu n \rightarrow e^- p$ occurring at this space-time point will than be sensitive to interference between amplitudes corresponding to the paths AB and ACB. The geometry of the triangle ABC and the condition that the 1 and 2 neutrino mass eigenstates arrive at B at the same time gives the following condition on their velocities:

$$\frac{v_1(\nu)}{v_2(\nu)} = \frac{\sin \theta_2 - \sin \theta_1}{\sin \theta_1} \frac{v_1(\nu) \sin(\theta_2 - \theta_1)}{v_\pi \sin \theta_1}$$

(4.1)

Expanding to first order in the small quantity $\delta = \theta_2 - \theta_1$, rearranging and neglecting terms of $O(m^4)$ gives:

$$v_2(\nu) - v_1(\nu) = \frac{\Delta m^2}{2E_\nu^2} \frac{\delta}{\sin \theta_1} \left[ 1 - v_\pi \cos \theta_1 \right]$$

(4.2)

where the relation:

$$v_i(\nu) = 1 - \frac{m_i^2}{2E_\nu^2} + O(m^4)$$

(4.3)

has been used. Rearranging Eqn(4.2):

$$\delta = \frac{\Delta m^2}{2E_\nu^2} \left[ \frac{v_\pi \sin \theta_1}{1 - v_\pi \cos \theta_1} \right]$$

(4.4)

The difference in phase of the neutrino paths AB and CB is (see Eqn(2.15)):

$$\phi_{\nu_2} = \frac{m_1^2 AB}{P_1} - \frac{m_2^2 CB}{P_2} + O(m_1^4, m_2^4)$$

(4.5)
Since the angle $\delta$ is $\simeq \Delta m^2$, the difference between $AB$ and $CD$ is of the same order, and so:

$$\phi_{12} = \frac{\Delta m^2 L}{\cos \theta_1 E_\nu} + O(m_1^4, m_2^4)$$

(4.6)

where $P_1 \simeq P_2 \simeq E_\nu$, the measured neutrino energy. From the geometry of the triangle ABC:

$$\frac{AC}{\sin \delta} = \frac{AB}{\sin \theta_2} = \frac{L}{\cos \theta_1 \sin(\theta_1 + \delta)}$$

(4.7)

So, to first order in $\delta$, and using Eqn(4.4):

$$AC \equiv \Delta x_\pi = \frac{L \delta}{\cos \theta_1 \sin \theta_1} = \frac{\Delta m^2 L}{2E_\nu^2 \cos \theta_1 (1 - v_\pi \cos \theta_1)}$$

(4.8)

and

$$\Delta t_\pi = \frac{\Delta x_\pi}{v_\pi} = \frac{\Delta m^2 L}{2E_\nu^2 \cos \theta_1 (1 - v_\pi \cos \theta_1)}$$

(4.9)

Eqns(4.8) and (4.9) give, for the phase increment of the pion path:

$$\Delta \phi_\pi = m_\pi (\tau_2 - \tau_1) = m_\pi \Delta \tau = E_\pi \Delta t_\pi - p_\pi \Delta x_\pi = \frac{\Delta m^2 E_\pi L}{2E_\nu^2 \cos \theta_1 (1 - v_\pi \cos \theta_1)} (1 - \frac{v_\pi^2}{E_\nu})$$

(4.10)

In Eqn(4.10), the Lorentz invariant character of the propagator phase is used. Setting $\cos \theta_1 = 1$ and $v_\pi = 0$, the pion phase increment of Eqn(2.25) is recovered. Eqns(4.6) and (4.10) give, for the total phase difference of the paths AB, ACB:

$$\phi_{12}^{\nu, \pi} = \Delta \phi^{AB} - \Delta \phi^{ACB} = \phi_{12}^{\nu} - \Delta \phi_\pi = \frac{\Delta m^2 L}{\cos \theta_1 E_\nu} \left[ 1 - \frac{E_\pi}{2E_\nu} \frac{(1 - v_\pi^2)}{(1 - v_\pi \cos \theta_1)} \right]$$

(4.11)

Using the expressions, valid in the ultra-relativistic (UR) limit where $v_\pi \simeq 1$:

$$1 - v_\pi \cos \theta_1 = \frac{m_\pi^2}{E_\pi^2} \frac{1}{(1 + \cos \theta_\nu^*)}$$

(4.12)

and

$$E_\nu = \frac{E_\pi (m_\pi^2 - m_\mu^2)}{2m_\pi^2 \cos \theta_\nu^*} (1 + \cos \theta_\nu^*)$$

(4.13)

where $\theta_\nu^*$ is the angle between the directions of the pion and neutrino momentum vectors in the pion rest frame, Eqn(4.11) may be rewritten as:

$$\phi_{12}^{\nu, \pi} = -\frac{\Delta m^2}{\cos \theta_1 E_\nu} \frac{m_\mu^2}{(m_\pi^2 - m_\mu^2) L}$$

(4.14)

For $\theta_1 = 0$ the oscillation phase is the same as for pion decay at rest (see Eqn(2.26)) since in the latter case, $E_\nu \simeq P_0 = (m_\pi^2 - m_\mu^2)/(2m_\pi)$. Using Eqn(4.14), the probability to observe a neutrino interaction, at point B, produced by the decay product of a pion decay occurring within a region of length, $l_{dec} \ll L$ centered at the point A, in a beam of energy $E_{\pi}$, is given by a formula analogous to Eqn(2.26):

$$P(e^{-} p | L, \theta_1) = \frac{l_{dec} m_\pi \Gamma_\pi}{E_\pi} \left| < e^{-} p | T | n\nu_\pi > \right| ^2 \left| < \nu_\mu \mu^+ | T | \pi^+ > \right| ^2 \sin^2 \theta \cos^2 \theta \times \left\{ 1 - \cos \left( \frac{m_\mu^2 \Delta m^2 L}{(m_\pi^2 - m_\mu^2) E_\nu \cos \theta_1} \right) \right\}$$

(4.15)
As in the case of pion decay at rest, Eqn(2.26), the oscillation phase differs by the factor 
\( \frac{2 \mu}{m^2_{\pi} - m^2_{\mu}} \) = 2.685 from that given by the standard formula.

The derivation of the formula describing muon oscillations following pion decays in flight is very similar to that just given for neutrino oscillations. The condition on the velocities so that the muons recoiling against the different neutrino mass eigenstates arrive at the point B (see Fig.3) at the same time, is given by a formula analogous to (4.2):

\[ \Delta v(\mu) = v_2(\mu) - v_1(\mu) = \frac{v_1(\mu)[v_1(\mu) - v_\pi \cos \theta_1]}{v_\pi \sin \theta_1} \delta \]  
(4.16)

The formula relating the neutrino masses to the muon velocities is, however, more difficult to derive than the corresponding relation for neutrinos, (4.3), as the decay muons are not ultra-relativistic in the pion rest frame. The details of this calculation are given in Appendix B. The result is:

\[ \Delta v(\mu) = \frac{m^2_\mu(m^2_\pi + m^2_\mu)\Delta m^2}{E^2_\mu(m^2_\pi - m^2_\mu)^2} \left( 1 - \frac{2m^2_\mu E_\pi}{(m^2_\pi + m^2_\mu)E_\mu} \right) + O(m^4_1, m^4_2) \]  
(4.17)

Using Eqn(4.16) and the relation, valid to first order in \( \delta \):

\[ \Delta t = \frac{L\delta}{v_\pi \cos \theta_1 \sin \theta_1} \]  
(4.18)

where \( \Delta t \) is the flight time of the pion from A to C in Fig.3 (and also the difference in the times-of-flight of the muons recoiling against the two neutrino eigenstates), the angle \( \delta \) may be eliminated to yield:

\[ \Delta t = \frac{\Delta v(\mu)L}{v_1(\mu) \cos \theta_1 [v_1(\mu) - v_\pi \cos \theta_1]} \]  
(4.19)

Using now the kinematical relation (see Appendix B):

\[ v_1(\mu) - v_\pi \cos \theta_1 = \frac{(m^2_\pi + m^2_\mu)}{2E_\pi E_\mu} \left( 1 - \frac{2m^2_\mu E_\pi}{(m^2_\pi + m^2_\mu)E_\mu} \right) \]  
(4.20)

and the expression for the phase difference of the paths AB and ACB:

\[ \phi^{\mu, \pi}_{12} = \Delta \phi^{AB} - \Delta \phi^{ACB} = \Delta t \left( \frac{m^2_\mu}{E_\mu} - \frac{m^2_\pi}{E_\pi} \right) \]  
(4.21)

together with Eqn(4.19), it is found, taking the UR limit, where \( v_1(\mu), v_\pi \approx 1 \), that

\[ \phi^{\mu, \pi}_{12} = \frac{2m^2_\mu \Delta m^2}{E^2_\mu(m^2_\pi - m^2_\mu)^2} \left[ \frac{(m^2_\mu E_\pi - m^2_\pi E_\mu)L}{\cos \theta_1} \right] \]  
(4.22)

The probability of detecting a muon decay at B is then:

\[ P(e^+ \nu_e \bar{\nu}_\mu|L, \theta_1) = \frac{l_{Dec} m_\pi \Gamma_\pi}{E_\pi} |< e^+ \nu_e \bar{\nu}_\mu|T|\mu^+ >|^2 \]

\[ \times \exp \left[ -\frac{\Gamma_{\mu \mu} m_\mu}{E_\mu \cos \theta_1} L \right] |< \nu_\mu^+ |T|\pi^+ >|^2 \]

\[ \times \left\{ 1 - \sin 2\theta \cos \frac{2m^2_\mu \Delta m^2}{E^2_\mu(m^2_\pi - m^2_\mu)^2} \left[ \frac{(m^2_\mu E_\pi - m^2_\pi E_\mu)L}{\cos \theta_1} \right] \right\} \]  
(4.23)
where \( l_{\text{Dec}} \) is defined in the same way as in Eqn(4.13).

## 5 Discussion

The quantum mechanics of neutrino oscillations has been surveyed in two recent review articles [21, 22], where further extensive lists of references may be found.

The standard derivation of the neutrino oscillation phase will first be considered, following the treatment of Ref. [3], but using the notation of the present paper. The calculation is performed in terms of the ‘flavour eigenstates of the neutrino’, \(|\alpha>\), which in the case of pion decay is, at \( t = 0 \), essentially the pure state \( \alpha = \nu_\mu \) corresponding to the superposition of the mass eigenstates \(|\nu_1>, |\nu_2>\) given in Eqn(2.9) above. This flavour eigenstate is assumed to evolve with laboratory time, \( t \), according to fixed energy solutions of the non-relativistic Schrödinger equation:

\[
|\alpha, t> = -\sin \theta e^{-iE_1t}|\nu_1, p> + \cos \theta e^{-iE_2t}|\nu_2, p>
\]

(5.1)

where \(|\nu_i, p>\) are mass eigenstates of fixed momentum \( p \), and \( E_1, E_2 \) are the laboratory energies of the neutrino mass eigenstates. The amplitude for transition into the state \(|\nu_e, p>\) at time \( t \) is then, using Eqns(2.8), (2.9):

\[
<\nu_e, p|\alpha, t> = \sin \theta \cos \theta \left( e^{-iE_1t} + e^{-iE_2t} \right)
\]

(5.2)

Assuming now that the neutrinos have the same momentum but different energies :

\[
E_i = \sqrt{p^2 + m_i^2} = p + \frac{m_i^2}{2p} + O(m_i^4)
\]

(5.3)

and using (5.2) and (5.3), the probability of the flavour state \( \nu_e \) at time \( t \) is found to be:

\[
P(\nu_e, t) = |<\nu_e, p|\alpha, t>|^2 = 2 \cos^2 \theta \sin^2 \theta \left( 1 + \cos \left[ \frac{(m_1^2 - m_2^2)}{2p} t \right] \right)
\]

(5.4)

Finally, since the velocity difference of the neutrino mass eigenstates is \( O(\Delta m^2) \), then, to the same order in the oscillation phase, the replacement \( t \rightarrow L \) can be made in Eqn(5.4) to yield the standard oscillation phase of Eqn(1.1).

The following comments may be made on this derivation:

(i) The time evolution of the neutrino mass eigenstates in Eqn(5.1) according to the Schrödinger equation yields a non-Lorentz-invariant phase \( \simeq Et \), to be compared with the Lorentz-invariant phase \( \simeq m^2t/E \) given in Eqn(2.12) above. Although the two expressions agree in the non-relativistic limit \( E \simeq m \) it is clearly inappropriate to use this limit for the description of neutrino oscillation experiments. It may be noted that the Lorentz-invariant phase is robust relative to different kinematical approximations. The same result is obtained to order \( m^2 \) for the phase of spatial oscillations independent of whether the neutrinos are assumed to have equal momenta or energies. This is not true in the non-relativistic limit. Assuming equal
momenta gives the standard result of Eqn(1.1), whereas the equal energy hypothesis results in a vanishing oscillation phase. A contrast may be noted here with the standard treatment of neutral kaon oscillations, which follows closely the derivation in Eqns(5.1) to (5.4) above, except that the particle phases are assumed to evolve with time according to \( \exp[-im\tau] \) where \( m \) is the particle mass and \( \tau \) is its proper time, in agreement with Eqn(2.12).

(ii) As pointed out in Ref. [6], the different neutrino mass eigenstates do not have equal momenta as assumed in Eqns(5.1) and (5.3). The approximation of assuming equal momenta might be justified if the fractional change in the momentum of the neutrino due to a non-vanishing mass were much less than that of the energy. However, in the case of pion decay as is readily shown from Eqns(2.4) and (2.6) above, the ratio of the fractional shift in momentum to that in energy is actually \( (m_\pi^2 + m_\mu^2)/(m_\pi^2 - m_\mu^2) = 3.67 \); so, in fact, the opposite is the case.

(iii) To derive (5.2) from (5.1), it is necessary to introduce the concept of a ‘flavour momentum eigenstate’ which cannot be defined in a theoretically consistent manner [23].

(iv) What are the physical meanings of \( t, p \) in Eqn(5.1)? In Eqn(5.2), it is assumed that the neutrino mass eigenstates are both produced, and both detected, at the same times. Thus both have the same time-of-flight \( t \). The momentum \( p \) cannot be the same for both eigenstates, as assumed in Eqn(5.3), if both energy and momentum are conserved in the decay process. For any given value of the laboratory time \( t \) the different neutrino mass eigenstates must be at different space-time positions because they have different velocities\(^{10}\), if it is assumed that both mass eigenstates are produced at the same time. It then follows that the \( \nu_e \) part of the different mass eigenstates cannot be probed at, some space-time point, by a neutrino interaction, whereas the latter must clearly occur at some unique and specific point in every event.

(v) The historical development of the calculation of the neutrino oscillation phase is of some interest. The first published prediction [24] actually obtained a phase a factor two larger than Eqn(1.1) i.e. in agreement with the covariant calculation of Ref. [4] (before the introduction of ‘wave packets’) and with the contribution from neutrino propagation found in the present paper. This prediction was later used, for example, in Ref. [25]. The derivation sketched above, leading to the standard result of Eqn(1.1) was later given in Ref. [26]. A subsequent paper [27] by the authors of Ref. [25], published shortly afterwards, cited both Ref. [24] and Ref. [26], but used now the prediction of the latter paper. No comment was made on the factor of two difference in the two calculations. In a later review article [28] by the authors of Ref. [24] a calculation similar to that of Ref. [26] was presented in detail. Subsequently, all neutrino oscillation experiments have been analysed on the assumption of the standard oscillation phase of Eqn(1.1).

It may be thought that the kinematical and geometrical inconsistencies mentioned in points (ii) and (iv) above result from a too classical approach to the problem. After

\(^{10}\)This is true not only in the case of energy-momentum conservation, but also if it is assumed, as in the derivation of the standard formula, that the neutrinos have the same momentum but different energies.
all, what does it mean, in quantum mechanics, to talk about the ‘position’ and ‘velocity’ of a particle, in view of the Heisenberg Uncertainty Relations [29]? This point will become clear later in the present discussion, but first, following the original suggestion of Ref. [30], and, as done in almost all subsequent work on the quantum mechanics of neutrino oscillations, the ‘wave packet’ description of the neutrino mass eigenstates will be considered. In this approach, both the ‘source’ and also possibly the ‘detector’ in the neutrino oscillation experiment are described by coherent spatial wave packets. Here the ‘source’ wavepacket treatment in the covariant approach of Ref. [7] will be briefly sketched. After discussing the results obtained, and comparing them with those of the present paper, the general consistency of the wave packet approach with the fundamental quantum mechanical formula (1.2) will be examined.

The basic idea of the wave packet approach of Ref. [7] is to replace the neutrino propagator (2.11) in the path amplitude by a four-dimensional convolution of the propagator with a ‘source wavefunction’ which, presumably, describes the space-time position of the decaying pion. For mathematical convenience this wavefunction is taken to have a Gaussian form with spatial and temporal widths $\sigma_x$ and $\sigma_t$ respectively. Thus, the neutrino propagator is replaced by $\tilde{D}$ where:

$$\tilde{D}(x_f - x_i, m_j) = \int d^4x D(x_f - x_i, m_j)\psi_{in}(x - x_i, j)$$ (5.5)

where $x_i$ and $x_f$ are 4-vectors and

$$\psi_{in}(x - x_i, j) = N_0 \exp \left[-ip_j \cdot (x - x_i) - \frac{(\vec{x} - \vec{x_i})^2}{4\sigma_x^2} - \frac{(t - t_i)^2}{4\sigma_t^2}\right]$$ (5.6)

The integral in (5.5) was performed by the stationary phase method, yielding the result (up to multiplicative and particle flux factors), and here assuming, for simplicity, one dimensional spatial geometry:

$$\tilde{D}(\Delta x, \Delta t, m_j) = \exp \left[-i\left(\frac{m_j^2}{E_j} \Delta t - P_j(\Delta x - v_j \Delta t)\right) - \frac{(\Delta x - v_j \Delta t)^2}{4(\sigma_x^2 + v_j^2 \sigma_t^2)}\right]$$ (5.7)

For the case of $\nu_\mu \to \nu_\ell$ oscillations, following $\pi^+$ decay at rest, the probability to observe flavour $\nu_\ell$ at time $t$ and distance $x$ is given by:

$$P(\nu_\ell, \Delta x, \Delta t) = \cos^2\theta \sin^2\theta \left| -\tilde{D}(\Delta x, \Delta t, m_1) + \tilde{D}(\Delta x, \Delta t, m_2) \right|^2$$ (5.8)

Performing the integral over $\Delta t$ and making the ultra-relativistic approximation $v_1, v_2 \simeq 1$ yields finally, with $\Delta x = L$:

$$P(\nu_\ell, L) = 2 \cos^2\theta \sin^2\theta \left(1 + \exp \left[-\frac{\Delta m^4(\sigma_x^2 + \sigma_t^2)}{2m_\nu^2} - \frac{\Delta m^4(m_\mu^2 + m_{\mu'}^2)^2 L^2}{4(m_\pi^2 - m_{\mu'}^2)^4(\sigma_x^2 + \sigma_t^2)}\right] \times \cos \frac{\Delta m^2 L}{2P_0}\right)$$ (5.9)

It can be seen that the oscillation phase in Eqn(5.9) is the same as standard one of Eqn(1.1). The exponential damping factors in Eqn(5.9) are the same as those originally
found in Ref. [31] for spatial wave packets ($\sigma_t = 0$). Considering now only spatial wave packets and using the property $\sigma_x \sigma_p = 1/2$ derived from the Fourier transform of a Gaussian, the two terms in the exponential damping factor may be written as:

$$F_p = \exp \left[ -\frac{\Delta m^4}{8m^2 \sigma_p^2} \right]$$  \hspace{1cm} (5.10)

and

$$F_x = \exp \left[ -\frac{\Delta m^4 (m^2 + m^2_\mu)^2 L^2}{4(m^2 - m^2_\mu)^4 \sigma_x^2} \right]$$  \hspace{1cm} (5.11)

The spatial damping factor $F_x$ is usually interpreted in terms of a ‘coherence length’ [32]. If

$$L \gg \frac{2(m^2 - m^2_\mu)^2 \sigma_x}{\Delta m^2 (m^2 + m^2_\mu)}$$  \hspace{1cm} (5.12)

then $F_x \ll 1$ and the neutrino oscillation term is strongly suppressed. Eqn(5.12) expresses the condition that oscillations are only observed provided that the wave packets overlap. Since $\Delta \nu \simeq \Delta m^2$ the separation of the wave packets is $\simeq \Delta \nu L \simeq \Delta m^2 L$, so that Eqn(5.12) is equivalent to $\Delta \nu L \gg \sigma_x$ (no wave packet overlap).

The damping factor $F_p$ is typically interpreted [33] in terms of the ‘Heisenberg Uncertainty Principle’. This factor is small, unless the difference in mass of the eigenstates is much less than $2\pi \sqrt{m \sigma_p}$, so it is argued that only for wide momentum wave packets can neutrino oscillations be observed, whereas in the contrary case, when the mass eigenstates are distinguishable, the interference effect vanishes. In the case of pion decay the difference in momentum of the two interfering mass eigenstates comes only from the $\delta_\iota$ term in Eqn(2.4), as $\delta_x$, being a property of the common initial state, is the same for both eigenstates. The neutrino momentum smearing in pion decay is then estimated from Eqn(2.4) as:

$$\sigma_p = |P_1 - P_0| = \frac{2P_0 \delta_\iota m_\mu}{m^2 - m^2_\mu} \simeq \frac{\Gamma_\mu m_\mu}{2m_\pi} = 1.14 \times 10^{-10} \text{eV}$$

For $\Delta m^2 = 1(\text{eV})^2$ the value of $F_p$ is found to be $\exp(-500) \simeq 7.1 \times 10^{-218}$ giving complete suppression of neutrino oscillations. This prediction is in clear contradiction with the tiny damping corrections found in the path amplitude analysis in of Section 2 above.

The physical relevance of the damping factors $F_x$ and $F_p$ will be reviewed after the examination of the physical meaning of the wave packet convolution formula (5.5) which now follows.

The convolution formula (5.5) may be re-written so as to isolate the contribution from the region:

$$x'_1 - \frac{\Delta x'_1}{2} < x_1 < x'_1 + \frac{\Delta x'_1}{2}$$

where $x'_1$ is an arbitrary fixed value of the coordinate $x_1$ (say, the spatial position of source pion parallel to the direction of neutrino propagation):
\[\tilde{D}(x_f - x_i, m_1) = \int D\psi_{in}\left[\int_{-\infty}^{x'_1 - \Delta x'_1} dx_1 + \int_{x'_1 + \Delta x'_1}^{\infty} dx_1\right] d^3x + \Delta x'_1 \int [D\psi_{in}]_{x_1 = x'_1} d^3x \]

\[\equiv \int(-\Delta x_i) D\psi_{in}d^4x + \Delta x'_1 \overline{D\psi_{in}(x'_1)} \]

(5.13)

Here the \(x_2, x_3, x_4\) averaged quantity \(\overline{D\psi_{in}(x'_1)}\) is a function of \((x_f)_k - (x_i)_k, k = 2, 3, 4\) as well as \((x_f)_1 - x'_1\) and \(x'_1 - (x_i)_1\). Similarly:

\[\tilde{D}(x_f - x_i, m_2) = \int(-\Delta x''_i) D\psi_{in}d^4x + \Delta x''_1 \overline{D\psi_{in}(x''_1)} \]

(5.14)

Now in calculating the interference term from the terms proportional to \(\Delta x'_1, \Delta x''_1\) of:

\[I(\Delta x'_1, \Delta x''_1) = \Delta x'_1 \Delta x''_1 2Re \overline{D\psi_{in}(x'_1)} \overline{D\psi_{in}(x''_1)} \]

(5.15)

This contribution corresponds to interference between a pion decaying at an arbitrary point \(x_1 = x'_1\) with another decaying at another arbitrary point \(x_1 = x''_1\). This is in contradiction with Eqn(1.2), according to which, the initial quantum state must be the same for all interfering path amplitudes. Referring to Fig 1., it can be seen that the different path amplitudes correspond to two (classically alternative) decay histories of the same pion which must clearly be at some unique spatial position at time \(t_0\). Each pion interferes only with itself. Interference between two different pions never occurs' [16]. Thus the convolution, at amplitude level, of the neutrino propagator with ‘source’ and/or ‘detector’ wavefunctions does not respect Heisenberg and Feynman’s fundamental law of quantum mechanics (1.2).

The preceding discussion of the derivation of the standard formula for the oscillation phase (1.1) in terms of ‘flavour eigenstates’ revealed contradictions and inconsistencies if the neutrinos are assumed to follow classical space-time trajectories. The ‘source wave packet’ treatment gives the standard result for the oscillation phase and predicts that the interference term will be more or less damped depending on the widths in configuration and momentum space of the wave packets. However the convolution formula (5.5) is evidently at variance with Eqn(1.2) and the requirement that particles can interfere only ‘with themselves’. So do wave packets actually play a role in the correct quantum mechanical description of neutrino oscillations, as suggested in Ref.[30]? Are the packets actually constrained by the Heisenberg Uncertainty Relations? How do the properties of the wave packet affect the possibility to observe neutrino oscillations? The answers to all these questions are contained in the results of the calculations presented in Section 2 above. They are now reviewed, with special emphasis on the basic assumptions made and the physical interpretation of the equations.

Referring again to Fig.1, in a) a single pion comes to rest in the stopping target \(T\). The time of its passage is recorded by the counter \(C_A\), which thus defines the initial state as a \(\pi^+\) at rest at time \(t_0\). This pion, being an unstable particle, has a mass \(W_\pi\) that is, in general, different from its most likely value which is the pole mass \(m_\pi\). What are shown in Fig.1b) and Fig.1c) are two different classical histories of this very same pion. In b) it decays into the neutrino mass eigenstate \(\nu_1\) at time \(t_1\), and in c) it decays into the
neutrino mass eigenstate $\nu_2$ at time $t_2$. Because these are independent classical histories, the physical masses $W_{\mu(1)}$, $W_{\mu(2)}$ of the muons recoiling against the mass eigenstates $\nu_1$ and $\nu_2$ respectively, are, in general, not equal. Taking into account now exact energy-momentum conservation in the decay processes (appropriate because of the covariant formulation used throughout) the eigenstates $\nu_1$ and $\nu_2$ will have momenta which depend on $W_\pi$ and $W_{\mu(1)}$ and $W_\pi$ and $W_{\mu(2)}$ respectively. These momenta are calculated in Eqn(2.4). The distributions of $W_\pi$ and $W_{\mu(i)}$ are determined by Breit-Wigner amplitudes, that are the Fourier transforms of the corresponding exponential decay laws, in terms of the decay widths $\Gamma_\pi$ and $\Gamma_\mu$ respectively. In accordance with Eqn(1.2), only the Breit-Wigner amplitudes corresponding to the physical muon masses are (coherently) integrated over at the amplitude level. The integral over $W_\pi$ (a property of the initial state) is performed (incoherently) at the level of the transition probability. Because of the long lifetimes of the $\pi$ and $\mu$, the corresponding momentum wave packets are very narrow, so that all corrections resulting from integration over the resulting momentum spectra are found to give vanishingly small corrections. Indeed, as is evident from the discussion of Eqn(5.10) above, the width of the momentum wave packet is much smaller than the difference in the momenta of the eigenstates expected for experimentally interesting values of the neutrino mass difference (say, $\Delta m^2 = 1(eV)^2$). However, contrary to the prediction of Eqn(5.10), this does not at all prevent the observation of neutrino oscillations. This is because the oscillations result from interference between amplitudes corresponding to different propagation times, not momenta, of the neutrino mass eigenstates. Table 1 shows contributions to the fractional smearing of the neutrino momentum from different sources:

(1) neutrino mass difference $\Delta m^2 = 1(eV)^2$ (Eqn(2.4))

(2) coherent effect due to the physical muon mass ($\delta_i = \Gamma_\mu/2$ in Eqn(2.4))

(3) incoherent effect due to the physical pion mass ($\delta_\pi = \Gamma_\pi/2$ in Eqn(2.4))

(4) incoherent Doppler effect assuming $p_\pi = 2.6 \times 10^{-3}$MeV

The pion mass effect is of the same order of magnitude as the neutrino mass shift. The muon mass effect is two orders of magnitude smaller, while the Doppler effect at room temperature gives a shift eleven orders of magnitude larger than a $(1eV)^2$ neutrino mass difference squared.

According to the usual interpretation of the Heisenberg Uncertainty Principle, the neutrinos from pion decay, which, as has been shown above, correspond to very narrow momentum wave packets, would be expected to have a very large spatial uncertainty. Indeed, interpreting the width of the coherent momentum wave packet generated by the spread in $W_\mu$ according to the the momentum-space Uncertainty Relation $\Delta p \Delta x = 1$

| Source | $\Delta m = 1eV$ | $W_\mu$ | $W_\pi$ | Doppler Effect |
|--------|----------------|----------|---------|----------------|
| $\Delta P_\nu/P_\nu$ | $4.4 \times 10^{-16}$ | $3.8 \times 10^{-18}$ | $3.3 \times 10^{-16}$ | $1.9 \times 10^{-5}$ |

Table 1: Different contributions to neutrino momentum smearing in pion decay at rest (see text).
gives $\Delta x = 1.27$ km. Does this accurately represent the knowledge of the position of a decay neutrino obtainable in the experiment shown in Fig.1? Without any experimental difficulty, the decay time of the pion can be measured with a precision of $10^{-10}$ sec, by detecting the decay muon. Thus, at any later time the distance $\Delta x$ of the neutrino from the decay point is known with a precision of $c \times 10^{-10}$ cm = 3 cm. This is a factor $4 \times 10^4$ more precise than the ‘uncertainty’ given by the Heisenberg relation. It is clear that the experimental knowledge obtainable on the position of the neutrino is essentially classical, in agreement with the theoretical description in terms of classical particle trajectories in space-time. In this case the momentum-position Uncertainty Relation evidently does not reflect the possible experimental knowledge of the position and momentum of the neutrino. This is because it does not take into account the prior knowledge that the mass of the neutrino is much less than that of the pion or muon, so that its velocity is, with negligible uncertainty, $c$. In spite of this, a Heisenberg Uncertainty Relation is indeed respected in the pion decay process. The Breit-Wigner amplitude that determines the coherent spread of neutrino momentum is just the Fourier transform of the exponential decay law of the muon. The width parameter of the Breit-Wigner amplitude and the muon mean lifetime do indeed respect the energy-time Uncertainty Relation $\Gamma \mu \tau \mu = 1$.

It is then clear, from this careful analysis of neutrino oscillations following pion decay at rest, that, in contradiction to what has been almost universally assumed until now, the neutrinos are not described by a coherent spatial wave packet. There is a coherent momentum wave packet, but it is only a kinematical consequence of a Breit Wigner amplitude. If for mathematical convenience, the momentum wave packet is represented by a Gaussian, a conjugate (and spurious) Gaussian spatial wave packet will be generated by Fourier transformation. Indeed, in the majority of wave packet treatments that have appeared in the literature, Gaussian momentum and spatial wave packets related by a Fourier transform with widths satisfying the ‘Uncertainty Relation’ $\sigma_p \sigma_x = 1/2$ have been used. The above discussion shows that such a treatment does not correspond to the actual knowledge of the space-time events that constitute realistic neutrino oscillation experiments.

The limitation on the detection distance $L$, for observation of neutrino oscillations, given by the damping factor $F^\nu(\Gamma_\pi)$ of Eqn(2.30) is easily understood in terms of the classical particle trajectories shown in Fig.1. For a given velocity difference, the time difference $t_2 - t_1$ becomes very large when both neutrinos, in the alternative classical histories, are required to arrive simultaneously at a far distant detector. Because of the finite pion lifetime, however, the amplitude for pion decay at time $t_2$ is smaller than that at $t_1$ by the factor $\exp[-(t_2 - t_1)\Gamma_\pi/2]$. Integrating over all decay times results in the exponential damping factor $F^\nu(\Gamma_\pi)$ of Eqn(2.30). It is clear that, contrary to the damping factor $F_x$ of Eqn(5.11), the physical origin of the $L$ dependent damping factor is quite unrelated to ‘wavepacket overlap’.

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11 Here, ‘distance’ is defined without specifying the flight direction of the neutrino. Precise, simultaneous, measurement of the flight direction of the recoil muon also determines, with a similar precision, the neutrino direction. Evidently the thickness of the stopping target may be chosen sufficiently small that its contribution to the neutrino position uncertainty is negligible.

12 It follows from this that discussions of quantum mechanical coherence in neutrino oscillations based on the properties of Gaussian wavepackets, as in Ref. [34], do not address the actual physical basis of neutrino oscillation experiments. A similar remark applies to the more recent and very extensive study of Ref. [35].

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For a given value of $t_2 - t_1$ the coherent neutrino momentum spread originating in the Breit Wigner amplitude for $W_\mu$ produces a corresponding velocity smearing that reduces the number of possible classical trajectories arriving at the detection event. This effect is taken into account in the integral shown in Eqn(2.23). The effect is shown, in Section 2 above, to be much smaller than the (already tiny) pion lifetime damping described above, and it is neglected in Eqn(2.26).

Finally, as described after Eqn(2.25), the (incoherent) integration over the Breit Wigner function containing $W_\pi$ gives an additional damping correction to the interference term that is also very tiny compared to that due to the pion lifetime.

As promised above, a final remark is now made on the physical interpretation of the damping factors (5.10) and (5.11) that have often been derived and discussed in the literature. Although as pointed out above, and to be further discussed below, the derivation of these factors is incorrect, as it results from coherent integration over different spatial positions of the neutrino source, similar factors do appear when performing the integrations over the (unobserved) neutrino momentum distributions. $F_x$ is replaced by $F_\nu(\Gamma_\pi)$ and $F_p$ by the factor resulting from the coherent integration over the physical muon mass $W_\mu$. The reason for the huge suppression factor predicted by Eqn(5.10), and the tiny one found in the path amplitude calculation, is that, in deriving (5.10) and (5.11), it is assumed that the neutrino eigenstates are both produced and detected at equal times. This will only be possible if both the hypothetical ‘wave packet overlap’ is appreciable (Eqn(5.11)) and also the momentum smearing is sufficiently large that the time-of-flight differences due to the different neutrino masses are washed out (Eqn(5.10). In the path amplitude calculation interference and hence oscillations are made possible by different decay times of the source pion, and the damping factors analogous to $F_x$ and $F_p$ turn out to give vanishingly small corrections to the oscillation term.

It may be remarked that the physical interpretation of ‘neutrino oscillations’ provided by the path amplitude description is different from the conventional one in terms of ‘flavour eigenstates’. In the latter the amplitudes of different flavours in the neutrino are supposed to vary harmonically as a function of time. In the amplitudes for the different physical processes in the path amplitude treatment there is, instead, no variation of lepton flavour in the propagating neutrinos. If the mass eigenstates are represented as superpositions of flavour eigenstates using the inverses of Eqns(2.8) and (2.9), there is evidently no temporal variation of the lepton flavour composition. Only in the detection process itself are the flavour eigenstates projected out, and the interference effect occurs that is described as ‘neutrino oscillations’. In the case of the observation of the recoil muons no such projection takes place, but exactly similar interference effects are predicted to occur. As previously emphasised [8], the ‘flavour oscillations’ of neutrinos, neutral kaons and b-mesons are just special examples of the universal phenomenon of quantum mechanical superposition that is the basis of Eqn(1.2).

A more detailed critical review will now be made of previous treatments of the quantum mechanics of both neutrino and muon oscillations in the literature.

By far the most widespread difference from the path amplitude treatment of the present paper is the non-respect of the basic quantum mechanical formula, (1.2), by the introduction of wave packets to describe ‘source’ and/or ‘detector’ particles[7,8,10,11,30-
Since in any practical neutrino oscillation experiment a single initial or final quantum state, as specified by Eqn(1.2), is not defined, but rather sets of initial and final states \( I = \sum_l i_l \) and \( F = \sum_m f_m \) determined by experimental conditions, Eqn(1.2) may be generalised to:

\[
P_{FI} = \sum_m \sum_l \left| \sum_{k_1} \sum_{k_2} ... \sum_{k_n} \langle f_m | k_1 \rangle \langle k_1 | k_2 \rangle ... \langle k_n | i_l \rangle \right|^2
\]

(5.16)

to be contrasted with the formula used in the references cited above:

\[
P_{fi} = \left| \sum_m \sum_l \sum_{k_1} \sum_{k_2} ... \sum_{k_n} \langle \psi_f | f_m \rangle \langle f_m | k_1 \rangle \langle k_1 | k_2 \rangle ... \langle k_n | i_l \rangle \langle i_l | \psi_i \rangle \right|^2
\]

(5.17)

Here, \( \psi_i \) and \( \psi_f \) are ‘source’ and ‘detector’ wave packets respectively. In Eqn(5.17) the initial states \( i_l \) and final states \( f_m \) of Eqn(5.16), that correspond to different spatial positions, and also, possibly, different kinematic properties, of the source particle or detection event, are convoluted with spatial and/or temporal ‘wave packets’, that, as discussed above, are devoid of physical significance. Use of Eqn(5.17) leads to the absurd prediction (see, for example, Eqn(5.15) above) that different initial or final state particles interfere with each other. A possible reason for the widespread use of Eqn(5.17) instead of (5.16) may be understood following a remark of the author of Ref. [39] concerning a ‘paradox’ of the complete quantum field theory calculation that takes into account, by a single invariant amplitude, production, propagation and detection of the neutrinos. It was noticed that, if the amplitude for the complete chain of processes is considered to correspond to one big Feynman diagram, then integration over the space-time coordinates of the initial and final states will reduce the exponential factors, containing the essential information on the interference phase, to energy-momentum conserving delta functions, and so no oscillations will be possible. A related remark was made by the authors of Ref. [11] who stated that, as they were assuming exact energy-momentum conservation, the integration over the space-time coordinates could be omitted. They still, however, (quite inconsistently, in view of the previous remark) retained the exponential factors containing the interference phase information. These considerations indicate a general confusion between momentum space Feynman diagram calculations, where it is indeed legitimate to integrate, at the amplitude level, over the unobserved space-time positions of the initial and final state particles, and the case of neutrino oscillations, where the amplitude must be defined in configuration space. In the latter case, it is the unobserved momenta of the propagating particles that should be integrated over, as is done in Eqns(2.1) and (2.35) above, and not the space-time positions of the ‘source’ or ‘detector’ particles, as in Eqn(5.17).

It is clear that exact energy-momentum conservation plays a crucial role in the path amplitude calculation. This is valid only in a fully covariant theory. Still, several authors, in spite of the ultra-relativistic nature of neutrinos, used a non-relativistic theory to describe the production, propagation and detection of neutrinos [37, 40, 44, 45]. As is well known, in such ‘Old Fashioned Perturbation Theory’ [17] energy is not conserved at the level of propagators and so no precise analysis of the kinematics and the space-time configurations of the production and detection events, essential in the covariant path amplitude analysis, is possible.

Even when, in some cases, the complete production, propagation and detection process of the neutrinos were described [35-37,39,40], equal neutrino energies [37,39,40], equal neu-
trino momenta [36] or either [35] were assumed, in contradiction with energy-momentum conservation and a consistent space-time description of the production and detection events. As follows directly from Eqn(5.3) (or the similar formula, for the neutrino momentum, obtained by assuming equal neutrino energies), in all the above cited references, the standard formula (1.1) for the oscillation phase was obtained.

An interesting discussion of the interplay between different kinematical assumptions (not respecting energy-momentum conservation) and the space-time description of the production and detection events was provided in Ref. [5]. This treatment was based on the Lorentz-invariant propagator phase of Eqn(2.12). By assuming either equal momentum or equal energy for the propagating neutrinos, but allowing different times of propagation for the two mass eigenstates, values of $\phi^\nu_{12}$ agreeing with Eqns(2.15) and (2.25) above were found, i.e. differing by a factor two from the standard formula. Alternatively, assuming equal velocities, (and hence equal propagation times) the standard result (1.1) was obtained. In this latter case, however, the masses, momenta and energies of the neutrinos must be related, up to corrections of $O(m_i^2)$, according to:

$$\frac{m_1}{m_2} = \frac{P_1}{P_2} = \frac{E_1}{E_2}$$

(5.18)

Since the ratio of the neutrino masses may take, in general, any value, so must then the ratio of their momenta. For the case of neutrino production from pion decay at rest with $m_1, m_2 \ll m_\mu, m_\pi$ the relation (5.18) is clearly incompatible with Eqn(2.4) which gives:

$$\frac{P_1}{P_2} = 1 - \frac{\Delta m^2(m_\pi^2 + m_\mu^2)}{(m_\pi^2 - m_\mu^2)^2} + O(m_i^4, m_j^4)$$

(5.19)

On the other hand, Eqn(5.19) is clearly compatible (up to corrections of $O(\Delta m^2)$) with the ‘equal momentum’ hypothesis. There is similar compatibility with the ‘equal energy’ hypothesis. Even so, the authors of Ref. [3] recommended the use of the equal velocity hypothesis. With the hindsight provided by the path amplitude analysis, in which the two neutrino mass eigenstates do indeed have different propagation times, it can be seen that the kinematically consistent ‘equal momentum’ and ‘equal energy’ choices are good approximations and the neutrino oscillation phase, resulting from the propagation of the neutrinos alone, is indeed a factor two larger than the prediction of the standard formula.

Only the author of Ref. [39] included the propagator of the decaying pion in the complete production-propagation-detection amplitude; compare Eqn(8) of Ref. [32] with Eqn(2.1) above. However, no detailed space-time analysis of production and detection events was performed. Square spatial wave packets for the ‘source’ and ‘detector’ were convoluted at amplitude level as in Eqn(5.17). As the neutrinos were assumed to have equal energies the standard result for the oscillation phase was obtained.

Although, following Ref. [37], most recent studies of the quantum mechanics of neutrino oscillations have considered the complete production-propagation-detection process, some authors still use, in spite of the criticisms of Ref. [23], the ‘flavour eigenstate’ description [4, 48, 49]. In the last two of these references a ‘quantum field theory’ approach is adopted, leading to an ‘exact’ oscillation formula [49] that does not make use of the usual ultra-relativistic approximation. This work did not include either exact kinematics, or an analysis of the space-time structure of production and detection events. Also the
effect of the propagator of the source particle was not considered. In Ref. [4] the equal energy hypothesis was used and in Ref. [18, 19] the equal momentum hypothesis. As a consequence, in all three cases the standard result was found for the oscillation phase in the ultra-relativistic limit.

Correlated production and detection of neutrinos and muons produced in pion decay were considered in Ref. [8]. The introduction to this paper contains a valuable discussion of the universality of the ‘particle oscillation’ phenomenon. It is pointed out that this is a consequence of the general principle of amplitude superposition in quantum mechanics, and so is not a special property of the $K^0 - \overline{K^0}$, $B^0 - \overline{B^0}$ and neutrino systems which are usually discussed in this context. This paper used a covariant formalism that employed the ‘energy representation’ of the space-time propagator. In the introduction, the important difference between Eqns(5.16) and (5.17) was also touched upon:

‘The reader will agree that one should not integrate over space if one is interested in spatial interference (or oscillation).’

Even so, in the amplitude for the correlated detection of the muon and neutrino (Eqn(2.10) of Ref. [8]) not only are the space-time positions of the production points of the neutrino and muon integrated over, but they are assumed to be at different space-time points. The propagator of the decaying pion is not included in the amplitude, and although exact energy-momentum conservation is imposed, no space-time analysis of the production and decay points is performed. Correlated spatial oscillations of neutrinos and muons are predicted, though with interference phases different from the results of both the present paper and the standard formula. Integrating over the spatial position of either the detected neutrino or muon destroys the interference term, so it is predicted that only correlated neutrino-muon oscillations may be observed, following pion decay, in contradiction with the results of the present paper, and, (in the case of neutrino oscillations) with all previous studies. Pion and muon lifetime effects were mentioned in Ref. [8], but neither the role of the pion lifetime in enabling different propagation times for the neutrinos nor the momentum smearing, induced by the Fourier-transform-related Breit Wigner amplitudes, were discussed.

The claim of Ref. [8] that correlated neutrino-muon oscillations should be observable in pion decay was questioned in Ref. [14]. The authors of the latter paper attempted to draw conclusions on the possibility, or otherwise, of particle oscillations by using ‘plane waves’, i.e. energy-momentum eigenfunctions. As is well known, such wavefunctions are not square integrable, and so can yield no spatial information. The probability to find a particle described by such a wave function in any finite spatial volume is zero. Due to the omission of the (infinite) normalisation constants of the wavefunctions many of the equations in Ref. [14] are, as previously pointed out [50], dimensionally incorrect. Momentum wavepackets for the decaying pion convoluted at amplitude level as in Eqn(5.17) were also discussed in Ref. [11]. Although exact energy-momentum conservation constraints were used, it was assumed, as in Ref. [8], that the muons and the different neutrino mass eigenstates are both produced and detected at common points (Eqn(35) of Ref. [11]). The latter assumption implies equal velocities, yielding the standard neutrino oscillation phase as well as the inconsistent kinematical relation (5.18). The authors of Ref. [11] concluded

\footnote{Although this conclusion follows from Eqn(3.22) of Ref. [8], it was not pointed out in the paper.}
that:

(a) correlated $\mu - \nu$ oscillations of the type discussed in Ref. [8] could be observed, though with different oscillation phases.

(b) oscillations would not be observed if only the muon is detected.

(c) neutrino oscillations can be observed even if the muon is not detected.

Conclusion (b) is a correct consequence of the (incorrect) assumption that the muons recoiling against the different neutrino mass eigenstates have the same velocity. As both muons have the same mass they will have equal proper time increments. So according to Eqn(2.12) the phase increments will also be equal and the interference term will vanish. The conclusion (c) is in contradiction to the prediction of Eqn(3.22) of Ref. [8], according to which, no neutrino oscillations can be observed when the decay position of muon is integrated over. The path amplitude calculation of the present paper shows that conclusion (b) is no longer valid when the different possible times of propagation of the recoiling muons are taken into account.

Observation of neutrino oscillations following pion decay, using a covariant formalism (Schwinger’s parametric integral representation of the space-time propagator) was considered in Ref. [41]. Exact energy-momentum conservation was imposed, but integration over the pion spatial position at amplitude level, as in Eqn(5.17), was done and no account was taken of the contribution of the pion space-time propagator to the oscillation phase. As the different eigenstates were assumed to be be produced or detected at identical space-time points equal propagation velocities were implicitly assumed, so that just as in Refs [5, 8, 11], where the same assumption was made, the standard neutrino oscillation phase was obtained. In the conclusion of this paper the almost classical nature of the space-time trajectories followed by the neutrinos was stressed, although this was not taken to its logical conclusion in the previous discussion, e.g. the kinematical inconsistency of the equal velocity hypothesis that requires the evidently impossible condition (5.18) to be satisfied.

In a recent paper [51], the standard neutrino oscillation formula with oscillation phase given by Eqn(1.1) was compared with a neutrino decoherence model. In order to take into account uncertainties in the position of the source and the neutrino energy, an average was made over the quantity $L/4E_{\nu}$, assuming that it is distributed according to a Gaussian with mean value $\ell$ and width $\sigma$. The average was performed in an incoherent manner. Thus the calculation is closely analogous to those for the effects of target or detector length or of thermal motion of the neutrino source, presented in the Appendix A of the present paper. Perhaps uniquely then, in the published literature, in Ref. [51] the effects of source position and motion are taken into account correctly, according to Eqn(5.16) instead of Eqn(5.17). However, the source of the neutrino energy uncertainty is not specified. In as far as it is generated from source motion the calculation is, in principle, correct. There is however also the (typically much smaller, see Table 1 above, for the case of pion decay at rest) coherent contribution originating from the variation in the physical masses of the unstable particles produced in association with the neutrino, as discussed in detail above. It was concluded in Ref. [51] that the Gaussian averaging procedure used gave equivalent results to the decoherence model for a suitable choice of parameters.
It is clearly of great interest to apply the calculational method developed in the present paper to the case of neutral kaon and b-meson oscillations. Indeed the use of the invariant path amplitude formalism has previously been recommended [52] for experiments involving correlated pairs of neutral kaons. Here, just a few remarks will be made on the main differences to be expected from the case of neutrino or muon oscillations. A more detailed treatment will be presented elsewhere [53].

In the case of neutrino and muon oscillations, the interference effect is possible as the different neutrino eigenstates can be produced at different times. This is because the decay lifetimes of all interesting sources (pions, muons, \(\beta\)-decaying nuclei) are much longer than the time difference between the paths corresponding to the interfering amplitudes. To see if a similar situation holds in the case of \(\bar{K}_S - K_L\) oscillations, three specific examples will be considered with widely differing momenta of the neutral kaons:

(I) \(\phi \to K_S K_L\)

(II) \(\pi^- p \to \Lambda K^0\) at \(\sqrt{s} = 2\) GeV

(III) \(\pi^- p \to \Lambda K^0\) at \(\sqrt{s} = 10\) GeV.

These correspond to neutral kaon momenta of 108 MeV, 750 MeV and 5 GeV respectively. In each case the time difference (\(\Delta t_K\)) of production of \(K_S\) and \(K_L\) mesons, in order that they arrive at the same time at a point distant \(c\gamma K\tau_S\) (where \(\gamma_K\) is the usual relativistic parameter) from the source is calculated. Exact relativistic kinematics is assumed and only leading terms in the mass difference \(\Delta m_K = m_S - m_L\) are retained. Taking the value of \(\Delta m_K\) and the various particle masses from Ref. [54] the following results are found for \(\Delta t_K\) in the three cases: (I) \(3.4 \times 10^{-23}\) sec, (II) \(8.3 \times 10^{-25}\) sec and (III) \(6.4 \times 10^{-26}\) sec. For comparison, for neutrino oscillations following pion decay at rest, with \(\Delta m^2 = (1\text{eV})^2\) and \(L = 30\) m, Eqn(2.29) gives \(\Delta t_\nu = 5.6 \times 10^{-23}\) sec. The result (I) may be compared with the mean life of the \(\phi\) meson of \(1.5 \times 10^{-22}\) sec [54]. Thus the \(\phi\) lifetime is a factor of about five larger than \(\Delta t_K\) indicating that \(K_S - K_L\) interference should be possible by a similar mechanism to neutrino oscillations following pion decays, \textit{i.e.} without invoking velocity smearing of the neutral kaon mass eigenstates. In cases (I) and (II) the interference effects observed will depend on the ‘characteristic time’ of the (non resonant) strong interaction process, a quantity that has hitherto not been susceptible to experimental investigation [14]. If this time is much less than, or comparable to, \(\Delta t_K\), essentially equal velocities (and therefore appreciable velocity smearing) of the eigenstates will be necessary for interference to occur. Since \(\Delta m_K\) and \(\Gamma_S\) are comparable in size, velocity smearing effects are expected to be, in any case, much larger than for neutrino oscillations following pion decay. These effects are readily calculated using the Gaussian approximation (2.22) of the present paper. The main contribution to the velocity smearing is due to the variation of the physical mass of the \(K_S\) rather than those of the \(K_L\) or \(\Lambda\).

For the \(B_1 - B_2\) oscillation case, analogous to (I) above, \(\Upsilon(4S) \to B_1 B_2\) \((p_B = 335\) MeV) the value of \(\Delta t_B\) is found to be \(1.8 \times 10^{-22}\) sec, to be compared with \(\tau(\Upsilon(4S)) = 4.7 \times 10^{-23}\) sec [54], which is a factor 3.8 smaller. Thus, velocity smearing effects are

\[\text{A similar physical quantity has been considered in Ref. [55], where the possibility of observable modifications to the exponential decay law and the Breit-Wigner line shape distribution is suggested.}\]
expected to play an important role in $B_1 - B_2$ oscillations. This is possible, since the neutral b-meson decay width ($4.3 \times 10^{-10}$ MeV), and mass difference ($3.1 \times 10^{-10}$ MeV), have similar sizes.

In closing, it is interesting to mention two types of atomic physics experiments where interference effects similar to the conjectured (and perhaps observed [17, 56, 57]) neutrino oscillations have already been clearly seen.

The first is quantum beat spectroscopy [58]. This type of experiment, which has previously been discussed in connection with neutrino oscillations [50], corresponds closely to the gedanken experiment used by Heisenberg [15] to exemplify the fundamental law of quantum mechanics Eqn(1.2). The atoms of an atomic beam are excited by passage through a thin foil or a laser beam. The quantum phase of an atom with excitation energy $E^*$ evolves with time according to: $\exp(-iE^*\Delta t)$ (see the discussion after Eqn(3.3) above). If decay photons from two nearby states with excitation energies $E_{\alpha}^*$ and $E_{\beta}^*$ are detected after a time interval $\Delta t$ (for example by placing a photon detector beside the beam at a variable distance $d$ from the excitation foil) a cosine interference term with phase:

$$\phi_{\text{beat}} = \frac{(E_{\alpha}^* - E_{\beta}^*)d}{\tau_{\text{atom}}}$$

(5.20)

where $\tau_{\text{atom}}$ is the average velocity of the atoms in the beam, is observed [58]. An atom in the beam, before excitation, corresponds to the neutrino source pion. The excitation process corresponds to the decay of the pion. The propagation of the two different excited states, alternative histories of the initial atom, correspond to the alternative propagation of the two neutrino mass eigenstates. Finally the deexcitation of the atoms and the detection of a single photon corresponds to the neutrino detection process. The particular importance of this experiment for the path amplitude calculations presented in the present paper, is that it demonstrates, experimentally, the important contribution to the interference phase of the space-time propagators of excited atoms, in direct analogy to the similar contributions of unstable pions, muons and nuclei discussed above.

An even closer analogy to neutrino oscillations following pion decay is provided by the recently observed process of photodetachment of an electron by laser excitation: the ‘Photodetachment Microscope’ [59]. A laser photon ejects the electron from, for example, an $^{16}\text{O}^{-}$ ion in a beam. The photodetached electron is emitted in an S-wave (isotropically) and with a fixed initial energy. It then moves in a constant, vertical, electric field that is perpendicular to the direction of the ion beam and almost parallel to the laser beam. An upward moving electron that is decelerated by the field eventually undergoes ‘reflection’ before being accelerated towards a planar position-sensitive electron detector situated below the beam and perpendicular to the electric field direction (see Fig.1 of Ref. [59]). In these circumstances, it can be shown [60] that, just two classical electron trajectories link the production point to any point in the kinematically allowed region of the detection plane. Typical parameters for $^{16}\text{O}^{-}$ are [61]: initial electron kinetic energy, 102 keV; detector distance, 51.4 cm; average time-of-flight, 117 ns; difference in emission times to arrive in spatial-temporal coincidence at the detector plane, 160 ps. An interference pattern is generated by the phase difference between the amplitudes corresponding to the two allowed trajectories. The phase difference, derived by performing the Feynman path integral of the classical action along the classical trajectories [61], gives a very good
description of the observed interference pattern. The extremely close analogy between this experiment and the neutrino oscillation experiments described in Sections 2 and 3 above is evident. Notice that the neutrinos, like the electrons in the photodetachment experiment, must be emitted at different times, in the alternative paths, for interference to be possible. This is the crucial point that was not understood in previous treatments of the quantum mechanics of neutrino oscillations.

Actually, Ref. [61] contains, in Section IV, a path amplitude calculation for electrons in free space that is geometrically identical to the discussion of pion decays in flight presented Section 4 above (compare Fig.3 of the present paper with Fig.3 of Ref. [61]). The conclusion of Ref. [61] is that, in this case, no interference effects are possible for electrons that are mononenergetic in the source rest frame. As is shown in Section 4 above, if these electrons are replaced either by neutrinos of different masses from pion decay, or muons recoiling against such neutrinos, observable interference effects are indeed to be expected.
Appendix A

Random thermal motion of the decaying pion in the target has two distinct physical effects on the phase of neutrino oscillations,

\[ \phi_{12}^{\nu,\pi} (0) = - \frac{\Delta m^2 L}{P_0} + \frac{m_\pi \Delta m^2 L}{2 P_0^2} \]  

(A1)

(where the first and second terms in Eqn(A1) give the contributions of the neutrino and pion paths respectively):

(1) The observed neutrino momentum, \( P_\nu \), is no longer equal to \( P_0 \), due to the boost from the pion rest frame to the laboratory system. (Doppler effect or Lorentz boost)

(2) The time increment of the pion path \( t_D - t_0 \) (see Eqn(2.16)) no longer corresponds to the pion proper time. (Relativistic time dilatation)

Taking into account (1) and (2) gives, for the neutrino oscillation phase:

\[ \phi_{12}^{\nu,\pi} (corr) = \frac{\Delta m^2 L}{P_\nu} + \frac{m_\pi \Delta m^2 L}{2 \gamma_\pi P_\nu^2} \]  

(A2)

where:

\[ P_\nu = \gamma_\pi P_0 (1 + v_\pi \cos \theta^*_\nu) \quad \text{and} \quad \gamma_\pi = \frac{E_\pi}{m_\pi} \]

Here, \( \theta^*_\nu \) is the angle between the neutrino momentum vector and the pion flight direction in the pion rest frame. Developing \( \gamma_\pi \) and \( v_\pi \) in terms of the small quantity \( p_\pi / m_\pi \), Eqn(A2) may be written as:

\[ \phi_{12}^{\nu,\pi} (corr) = \phi_{12}^{\nu,\pi} (0) + \frac{p_\pi}{m_\pi} \frac{\Delta m^2 L}{P_0} \left[ 1 - \frac{m_\pi}{P_0} \right] \cos \theta^*_\nu + \left( \frac{p_\pi}{m_\pi} \right)^2 \frac{\Delta m^2 L}{2 P_0} \left[ 1 - \frac{3 m_\pi}{2 P_0} \right]\]  

(A3)

Performing now the average of the interference term over the isotropic distribution in \( \cos \theta^*_\nu \):

\[ \langle \cos \phi_{12}^{\nu,\pi} (corr) \rangle_{\theta^*_\nu} = \frac{1}{4} Re \int_{-1}^{1} \exp [i \phi_{12}^{\nu,\pi} (corr)] d \cos \theta^*_\nu \]

\[ = \frac{1}{2} Re \exp \left\{ i \phi_{12}^{\nu,\pi} (0) + \left( \frac{p_\pi}{m_\pi} \right)^2 \left( \frac{\Delta m^2 L}{2 P_0} \left[ 1 - \frac{3 m_\pi}{2 P_0} \right] \right) \right\} \]

(A4)

In deriving Eqn(A4) the following approximate formula is used:

\[ \frac{1}{2} \int_{-1}^{1} e^{i \alpha c} dc = \frac{1}{2i \alpha} \left[ e^{i \alpha} - e^{-i \alpha} \right] = \frac{\sin \alpha}{\alpha} \approx 1 - \frac{\alpha^2}{6} \]  

(A5)

where

\[ \alpha \equiv \frac{p_\pi}{m_\pi} \frac{\Delta m^2 L}{P_0} \left[ 1 - \frac{m_\pi}{P_0} \right] \ll 1 \]
The average over the Maxwell-Boltzmann distribution (2.31) is readily performed by ’completing the square’ in the exponential, with the result:

\[
\langle \cos \phi_{12}^{\nu,\pi} \rangle_{\theta^\nu,p_\pi} = \frac{1}{2} \Re \exp \left\{ - \left( \frac{p_\pi \Delta m^2 L}{2m_\pi P_0} \left[ 1 - \frac{m_\pi}{P_0} \right] \right)^2 \right. \\
+ i \left[ \phi_{12}^{\nu,\pi}(0) + \frac{3}{4} \left( \frac{P_\pi}{m_\pi} \right)^2 \left( \frac{\Delta m^2 L}{P_0} \left[ \frac{3m_\pi}{2P_0} - 1 \right] \right) \right] \right\}
\equiv F^{\nu}(Dop) \cos[\phi_{12}^{\nu,\pi}(0) + \phi^{\nu}(Dop)]
\]  

(A6)

leading to Eqns(2.32) and (2.33) for the Doppler damping factor \( F^{\nu}(Dop) \) and phase shift \( \phi^{\nu}(Dop) \), respectively.

The correction for the effect of thermal motion in the case of muon oscillations \( \phi^{\nu}(Dop) \), respectively, is performed in a similar way. The oscillation phase:

\[
\phi_{12}^{\mu,\pi}(0) = - \frac{m_\mu^2 E_0^\mu \Delta m^2 L}{2m_\pi P_0^3} + \frac{m_\mu \Delta m^2 L}{2P_0^3}
\]  

(A7)

is modified by the Lorentz boost of the muon momentum and energy, and the relativistic time dilatation of the phase increment of the pion path, to:

\[
\phi_{12}^{\mu,\pi}(corr) = \phi_{12}^{\mu,\pi}(0) + \frac{p_\pi}{m_\pi} \left[ \frac{E_0^\mu}{m_\pi} - \frac{3}{2} \right] \cos \theta^*_\mu + \left( \frac{p_\pi}{m_\pi} \right)^2 \left[ \frac{E_0^\mu}{2m_\pi} - 1 \right]
\]  

(A9)

Performing the averages over \( \theta^*_\mu \) and \( p_\pi \) then leads to Eqns(2.49) and (2.50) for the damping factor \( F^{\mu}(Dop) \) and phase shift \( \phi^{\mu}(Dop) \), respectively.

The effect of the finite longitudinal dimensions of the target or detector is calculated by an appropriate weighting of the interference term according to the value of the distance \( X = x_f - x_i \) between the decay and detection points (see Fig.1). Writing the interference phase as \( \phi_{12} = \beta X \), and assuming a uniform distribution of decay points within the target of thickness \( \ell_T \):

\[
\langle \cos \phi_{12} \rangle = \frac{1}{\ell_T} \int_{L - \ell_T/2}^{L + \ell_T/2} \cos \beta X dX = \frac{1}{\beta \ell_T} \sin \frac{\beta \ell_T}{2} \cos \beta L = F_{\text{Target}} \cos \beta L
\]  

(A10)

Substituting the value of \( \beta \) appropriate to neutrino oscillations yields Eqn(2.34). Since the value of \( \beta \) is the same for neutrino and muon oscillations, the same formula is also valid in the latter case. The same correction factor, with the replacement \( \ell_T \rightarrow \ell_D \) describes the effect of a finite detection region of length \( \ell_D \):

\[
L - \frac{\ell_D}{2} + x_i < x_f < L + \frac{\ell_D}{2} + x_i
\]
Appendix B

The first step in the derivation of Eqn(4.17) relating $\Delta v(\mu)$ to $\Delta m^2$ is to calculate the angle $\delta^*$, in the centre-of-mass (CM) system of the decaying pion, corresponding to $\delta$ in the laboratory (LAB) system (see Fig.3). It is assumed, throughout, that the pion and muon are ultra-relativistic in the latter system, so that: $v_\pi, v_\mu(\mu) \simeq 1$. The Lorentz transformation relating the CM and LAB systems gives the relation:

$$
\sin \theta_i = \frac{v^*_i(\mu) \sin \theta^*_i}{\gamma_\pi(1 + v^*_i(\mu) \cos \theta^*_i)} \quad i = 1, 2 \quad (B1)
$$

The starred quantities refer to the pion CM system. Making the substitutions: $\theta_2 = \theta_1 + \delta$, $\theta^*_2 = \theta^*_1 + \delta^*$, Eqns(B1) may be solved to obtain, up to first order in $\delta$, $\delta^*$ and $\Delta m^2$:

$$
\Delta v^*(\mu) = v^*_2(\mu) - v^*_1(\mu) = \frac{\gamma_\pi(1 + v^*_0(\mu) \cos \theta^*_1)^2 \delta - v^*_0(\mu)(\cos \theta^*_1 + v^*_0(\mu))\delta^* \sin \theta^*_1}{\sin \theta^*_1} \quad (B2)
$$

where, (c.f. Eqn(2.37)):

$$
v^*_0(\mu) = \frac{m^2_\pi - m^2_\mu}{m^2_\pi + m^2_\mu} \quad (B3)
$$

Using Eqn(2.36) $\Delta v^*(\mu)$ may be expressed in terms of the neutrino mass difference:

$$
\Delta v^*(\mu) = \frac{4m^2_\mu m^2_\pi \Delta m^2}{(m^2_\pi - m^2_\mu)(m^2_\pi + m^2_\mu)^2} \quad (B4)
$$

Eliminating now $\Delta v^*(\mu)$ between (B2) and (B4) gives a relation between $\delta$, $\delta^*$ and $\Delta m^2$:

$$
\delta^* = \frac{\gamma_\pi(1 + v^*_0(\mu) \cos \theta^*_1)^2 \delta}{v^*_0(\mu)(\cos \theta^*_1 + v^*_0(\mu))} - \frac{4m^2_\mu m^2_\pi \Delta m^2 \sin \theta^*_1}{(m^2_\pi - m^2_\mu)^2(m^2_\pi + m^2_\mu)(\cos \theta^*_1 + v^*_0(\mu))} \quad (B5)
$$

In the LAB system, and in the UR limit, the difference of the velocities of the muons recoiling against the two neutrino mass eigenstates is:

$$
\Delta v(\mu) = v_2(\mu) - v_1(\mu) = \frac{P_2(\mu)}{E_2(\mu)} - \frac{P_1(\mu)}{E_1(\mu)} \simeq \frac{m^2_\mu}{E_\mu}[E_2(\mu) - E_1(\mu)] \quad (B6)
$$

where $E_\mu$ is the muon energy in the LAB system for vanishing neutrino masses. Making the Lorentz transformation of the muon energy from the pion CM to the LAB frames, and using Eqns(2.4) and (2.36) to retain only terms linear in $\Delta m^2$ and $\delta^*$, enables Eqn(B6) to be re-written as:

$$
\Delta v(\mu) = \frac{E_\pi}{2E^0_\mu} \left( \frac{m_\pi}{m_\mu} \right)^2 \left[ \frac{\Delta m^2(\cos \theta^*_1 + v^*_0(\mu))}{v^*_0(\mu)} - \delta^*(m^2_\pi - m^2_\mu) \sin \theta^*_1 \right] \quad (B7)
$$

where $E_\pi$ is the energy of the pion beam. By combining the geometrical constraint equation for the muon velocities, (4.16) with (B5) and (B7) the angles $\delta$ and $\delta^*$ may be eliminated to yield the equation for LAB frame velocity difference:

$$
\Delta v(\mu) = \frac{E_\pi \Delta m^2}{2m^2_\pi(m^2_\pi - m^2_\mu)} \frac{A}{B} \quad (B8)
$$
where
\[
A = (v_1(\mu) - v_\pi \cos \theta_1) \left\{ (\cos \theta_1^* + v_0^*(\mu))^2 + \frac{4m_\mu^2m_\pi^2 \sin^2 \theta_1^*}{(m_\mu^2 + m_\pi^2)^2} \right\} \quad (B9)
\]

\[
B = \frac{E_\pi (m_\pi^4 - m_\mu^4)(1 + v_0^*(\mu) \cos \theta_1^*)}{8m_\pi^4 m_\mu^2} \times \left\{ \frac{E_\pi^2 (m_\pi^2 + m_\mu^2)(1 + v_0^*(\mu) \cos \theta_1^*)^2 (\cos \theta_1^* + v_0^*(\mu))(v_1(\mu) - v_\pi \cos \theta_1)}{m_\pi^2 (m_\pi^2 - m_\mu^2)} + \frac{4m_\mu^2m_\pi^2 \sin^2 \theta_1^*}{(m_\pi^2 + m_\mu^2)^2} \right\} \quad (B10)
\]

To simplify (B8), the quantity \((v_1(\mu) - v_\pi \cos \theta_1)\) is now expressed in terms of kinematic quantities in the pion CM system. Within the UR approximation used,
\[
\theta_1, m_\pi/E_\pi, m_\mu/E_\mu \ll 1
\]
so that
\[
v_1(\mu) - v_\pi \cos \theta_1 = \frac{1}{2} \left( \frac{m_\pi^2}{E_\pi^2} - \frac{m_\mu^2}{E_\mu^2} + \theta_1^2 \right) + O\left( \left( \frac{m_\pi}{E_\pi} \right)^4, \left( \frac{m_\mu}{E_\mu} \right)^4, \theta_1^4 \right) \quad (B11)
\]

Writing Eqn(B1) to first order in \(\theta_1\), and neglecting terms of \(O(\theta_1 m_\pi^2)\):
\[
\theta_1 = \frac{m_\pi v_0^* \sin \theta_1^*}{E_\pi (1 + v_0^*(\mu) \cos \theta_1^*)} \quad (B12)
\]

Using Eqn(B12), and expressing \(E_\mu\) in terms of pion CM quantities, Eqn(B11) may be written as:
\[
v_1(\mu) - v_\pi \cos \theta_1 = \frac{m_\pi^2 (m_\pi^2 - m_\mu^2)(\cos \theta_1^* + v_0^*(\mu))}{E_\pi^2 (m_\pi^2 + m_\mu^2)(1 + v_0^*(\mu) \cos \theta_1^*)^2} \quad (B13)
\]

Expressing the RHS of (B13) in terms of \(E_\pi\) and \(E_\mu\), using the relation:
\[
\cos \theta_1^* = \frac{m_\pi^2 (2E_\mu - E_\pi) - m_\mu^2 E_\pi}{E_\pi (m_\pi^2 - m_\mu^2)} \quad (B14)
\]
gives Eqn(4.20) of the text.

On substituting (B13) into the RHS of (B10), it can be seen that the factor in the large curly brackets is the same in (B9) and (B10), and so cancels in the ratio \(A/B\) in Eqn(B8). It follows that:
\[
\Delta v(\mu) = \frac{m_\mu^2 \Delta m_\mu^2}{E_\mu^2 (m_\pi^2 - m_\mu^2)} \left( \frac{\cos \theta_1^* + v_0^*(\mu)}{1 + v_0^*(\mu) \cos \theta_1^*} \right) \quad (B15)
\]

Finally, using (B3) and (B14) to express the factor in large brackets in Eqn(B15) in terms of \(E_\mu\) and \(E_\pi\), Eqn(4.17) of the text is obtained.
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Figure 1: The space-time description of $\nu_\mu \rightarrow \nu_e$ oscillations following $\pi^+$ decay at rest, in Feynman’s formulation of quantum mechanics. In a) a $\pi^+$ comes to rest in the stopping target T at time $t_0$. The pion, at rest at time $t_0$, constitutes the initial state for the path amplitudes. In b) and c) are shown two alternative classical histories for the $\pi^+$; in b), c) the pion decays into the mass eigenstate $|\nu_1 \rangle$, $|\nu_2 \rangle$ at times $t_1$, $t_2$. If $m_1 > m_2$, and for suitable values of $t_1$ and $t_2$ ($t_2 > t_1$), the two classical histories may correspond to a common final state, shown in d) where the neutrino interaction $\nu_e n \rightarrow e^- p$ occurs at time $t_D$. As the initial and final states of the two classical histories are the same, the corresponding path amplitudes must be added coherently, as in Eqn(1.2), to calculate the probability of the whole process.
Figure 2: The space-time description of ‘muon oscillations’ induced by neutrino mass differences, following $\pi^+$ decay at rest, in Feynman’s formulation of quantum mechanics. As in Fig.1, b) and c) show alternative classical histories of the stopped $\pi^+$. If $m_1 > m_2$, the velocity of $\mu_1$ is less than that of $\mu_2$, and provided that $t_2 > t_1$, the muons may arrive at the same spatial point at the same time $t_D$ in both classical histories. If the muons are detected at this space-time point in any way (not necessarily by the observation of muon decay as shown in c)) interference between the corresponding path amplitudes occurs, according to Eqn(1.2), just as in the case of neutrino detection.
Figure 3: Two dimensional spatial geometry for the observation of neutrino or muon oscillations following pion decay in flight. Four possible classical histories of a pion, originally at the point A, are shown. In the first two, the pion decays either into the mass eigenstate |ν₁⟩, at point A or into |ν₂⟩ at point C. If m₁ > m₂, and for suitable values of the angles θ₁ and θ₂, the neutrinos may arrive at the point B at the same time. If a neutrino detection event, such as νₑν → e⁻p, then occurs at B at this time, the amplitudes corresponding to the paths AB and ACB will be indistinguishable so that they must be superposed, as in Eqn(1.2), to calculate the probability of the overall decay-propagation-detection process. The third and fourth classical histories are similar, except that the neutrino mass eigenstates are replaced by the corresponding recoil muons. The muons in the different histories may arrive at point B, at the same time, leading to interference and ‘muon oscillations’ if they are detected there.