Sharp Estimates for Module of Continuity of Fractional Integrals and Derivatives.

E.Ostrovsky\textsuperscript{a}, L.Sirota\textsuperscript{b}

\textsuperscript{a} Corresponding Author. Department of Mathematics and computer science, Bar-Ilan University, 84105, Ramat Gan, Israel.
E-mail: eugostrovsky@list.ru

\textsuperscript{b} Department of Mathematics and computer science. Bar-Ilan University, 84105, Ramat Gan, Israel.
E-mail: sirota3@bezeqint.net

Abstract.

We derive the bilateral estimates for the module of continuity of the fractional integrals and derivatives for the functions from the classical Lebesgue-Riesz spaces.

Key words and phrases: Fractional derivatives and integrals of a Riemann-Liouville type, Riesz potential, test functions, examples and counterexamples, fundamental function for rearrangement invariant space, indicator function, upper and lower estimate, sharp estimate, Lebesgue-Riesz and Grand Lebesgue spaces (GLS), measurable set, measurable function.

Mathematics Subject Classification (2000): primary 60G17; secondary 60E07; 60G70.

1 Notations. Statement of problem.

"Fractional derivatives have been around for centuries but recently they have found new applications in physics, hydrology and finance", see [29].

Another applications: in the theory of Differential Equations are described in [30]; in statistics see in [5], [36]; see also [14], [8]; in the theory of integral equations etc. see in the classical monograph [51].

Let $\alpha = \text{const} \in (0,1)$; and let $g = g(x), \ x \in \mathbb{R}_+$ be measurable numerical function. The fractional derivative of a Riemann-Liouville type of order $\alpha$: $D^\alpha[g](x) = g^{(\alpha)}(x)$ [50], [27] is defined as follows:

$$
\Gamma(1-\alpha) \ D^\alpha[g](x) = \Gamma(1-\alpha) \ D_x^\alpha[g](x) \stackrel{\text{def}}{=} \frac{d}{dx} \int_0^x \frac{g(t) \ dt}{(x-t)^\alpha}.
$$

(1.1)

see, e.g. the classical monograph of S.G.Samko, A.A.Kilbas and O.I.Marichev [51], pp. 33-38; see also [30].

The case when $\alpha \in (k, k+1), \ k = 1,2,\ldots$ may be considered analogously, through the suitable derivatives of integer order.
Hereafter $\Gamma(\cdot)$ denotes the ordinary Gamma function.

We agree to take $D^\alpha[g](x_0) = 0$, if at the point $x_0$ the expression $D^\alpha[g](x_0)$ does not exist.

Notice that the operator of the fractional derivative is non-local, if $\alpha$ is not integer non-negative number.

Recall also that the fractional integral $I^{(\alpha)}[g](x) = I^\alpha[g](x)$ of a Riemann-Liouville type of an order $\alpha$, $0 < \alpha < 1$ is defined as follows:

$$I^{(\alpha)}[g](x) \overset{def}{=} \frac{1}{\Gamma(\alpha)} \int_0^x \frac{g(t) \, dt}{(x-t)^{1-\alpha}}, \quad x, t > 0. \quad (1.2)$$

It is known (theorem of Abel, see [51], chapter 2, section 2.1) that the operator $I^{(\alpha)[\cdot]}$ is inverse to the fractional derivative operator $D^{(\alpha)[\cdot]}$, at least in the class of absolutely continuous functions.

Another approach to the introducing of the fractional derivative, more exactly, the fractional Laplace operator leads us to the using of Fourier transform

$$F[f](t) = \int_{\mathbb{R}^d} e^{i(t,x)} \, f(x) \, dx$$

in the space $\mathbb{R}^d$, $d = 1, 2, \ldots$:

$$R_{\alpha,F}[f] := C_1(\alpha, d) F^{-1} \left[ |x|^\alpha F[f](x) \right], \quad 0 < \alpha < d,$$

which leads us in turn up to multiplicative constant to the well-known Riesz potential

$$R_{\alpha}[f] \overset{def}{=} \int_{\mathbb{R}^d} \frac{f(y) \, dy}{|x-y|^{d-\alpha}}, \quad 0 < \alpha < d. \quad (1.3)$$

Hereafter $(t,x) = \sum_{m=1}^d t_m x_m$, $|x| = \sqrt{(x,x)}$, $t, x \in \mathbb{R}^d$.

We intent to obtain in this short article the bilateral estimates for module of continuity of the fractional integrals and derivatives for the functions from the classical Lebesgue-Riesz spaces.

The classical Lebesgue-Riesz $L_p$ estimations for the fractional integrals and derivatives are investigated in many works, see, e.g. [1], chapter 2,3; [2], [12], [13], [16], [17], [18], [19], [11], [20], [11], [24], [28], [41], [42], [43], [44], [48], [51], chapters 2-3 etc. The module of continuous estimates ones (previous works) see in [15], [31] - [34], [51], pp, 66-71.

Note that in the articles [15], [32], [34] is considered the case of the so-called Lebesgue-Riesz spaces with variable exponent $p = p(x)$ for the function $f(\cdot)$.

Recall that the classical Lebesgue-Riesz $L(p)$ norm $|f|_p$ of a function $f$ is defined by a formula

$$|f|_p = \left[ \int_{\mathbb{R}^d} |f(x)|^p \, dx \right]^{1/p}, \quad 1 \leq p < \infty$$

or correspondingly

$$|f|_p = \left[ \int_{\mathbb{R}^+} |f(x)|^p \, dx \right]^{1/p}, \quad 1 \leq p < \infty.$$
2 Module of continuity of the fractional derivatives.

The module of continuity \( \omega(f, h) \), \( h \geq 0 \) of (uniformly continuous) function \( f : \mathbb{R}^d \to \mathbb{R}, \) or \( f : (0, b) \to \mathbb{R}, b = \text{const} \in (0, \infty] \) is defined as usually as follows

\[
\omega(f, h) = \sup_{(x,y):|x-y| \leq h} |f(x) - f(y)|. \tag{2.1}
\]

Let \( f : (0, b) \to \mathbb{R} \) be uniformly continuous function such that \( f(0) = 0 \) and let \( \alpha \in (0, 1) \). The inequality

\[
\omega(D^{\alpha}[f], h) \leq C_D(\alpha) \int_0^h \frac{\omega(f, t) \, dt}{t^{1+\alpha}} \tag{2.2}
\]

is proved, e.g. in [51], p. 250-253, theorem 3.16. Define for simplicity the following set of continuous functions

\[
S(\alpha) \overset{\text{def}}{=} \left\{ f : f(0) = 0, \int_0^h \frac{\omega(f, t) \, dt}{t^{1+\alpha}} < \infty \right\}. \tag{2.3}
\]

We define the minimal value os the ”constant” \( C_D(\alpha) \) as \( K_D(\alpha) \):

\[
K_D(\alpha) \overset{\text{def}}{=} \sup_{h \in (0, 1)} \sup_{f \neq \text{const}, f \in S(\alpha)} \left[ \omega(D^{\alpha}[f], h) : \int_0^h \frac{\omega(f, t) \, dt}{t^{1+\alpha}} \right]. \tag{2.4}
\]

It is proved in fact in [51], p. 250-253

\[
K_D(\alpha) \leq C \alpha^{-1} \Gamma(1 - \alpha), \quad \alpha \in (0, 1). \tag{2.5}
\]

where \( C \) is an absolute constant.

**Proposition 2.1.**

\[
K_D(\alpha) \geq \Gamma(1 - \alpha), \quad \alpha \in (0, 1). \tag{2.6}
\]

**Proof.** We set \( b = 1 \) and choose \( g(x) = x^{\beta}, \beta = \text{const} \in (\alpha, 1], \) (test function), then \( g(\cdot) \in S(\alpha) \) and \( \omega(g, h) = g(h) = h^{\beta}. \) Further,

\[
\int_0^h \frac{\omega(g, t)}{t^{1+\alpha}} \, dt = \frac{h^{\beta - \alpha}}{\beta - \alpha},
\]

\[
D^{\alpha}[g] = \frac{d}{dx} x^{1+\beta-\alpha} \int_0^1 z^\beta (1 - z)^{-\alpha} \, dz =
\]

\[
(1 + \beta - \alpha) x^{\beta - \alpha} B(\beta + 1, 1 - \alpha), \tag{2.7}
\]

where as ordinary \( B(\cdot, \cdot) \) denotes the usually Beta function;

\[
\omega(D^{\alpha}[g], h) = (1 + \beta - \alpha) h^{\beta - \alpha} B(\beta + 1, 1 - \alpha),
\]
\[ \omega(D^\alpha[g], h) : \int_0^h \frac{\omega(g, t)}{t^{1+\alpha}} \, dt = (1 + \beta - \alpha)(\beta - \alpha)B(\beta + 1, 1 - \alpha), \]

and we deduce using the well-known recursion for the Gamma function

\[ K_D(\alpha) \geq \frac{\Gamma^2(1 - \alpha) \Gamma(1 + \beta)}{\Gamma(\beta - \alpha)}, \tag{2.8} \]

therefore

\[ K_D(\alpha) \geq \sup_{\beta \in (\alpha, 1)} \left[ \frac{\Gamma^2(1 - \alpha) \Gamma(1 + \beta)}{\Gamma(\beta - \alpha)} \right] = \Gamma(1 - \alpha). \tag{2.9} \]

### 3 Module of continuity of the fractional integrals.

Let again \( \alpha \in (0, 1) \) and let now \( p > 1/\alpha, \ f \in L_p(0, \infty) \); then the function \( g(x) := I^\alpha[f](x) \) is continuous and moreover

\[ \omega(g, h) = o(h^{\alpha-1/p}), \ h \to 0+, \]

see [51], pp. 104-110.

Denote

\[ Z(\alpha, p) = \left[ \frac{p-1}{\alpha p - 1} \right]^{1-1/p}, \tag{3.1} \]

and for any function \( f \in L_p(0, \infty), \)

\[ \Delta_p(f, h) \overset{def}{=} \sup_{|\delta| < h} \sup_x \left[ \int_x^{x+\delta} |f(t)|^p \, dt \right]^{1/p}, \tag{3.2} \]

where the function \( f(t) \) is presumed to be continued as zero for negative values \( t \). Evidently,

\[ \lim_{h \to 0^+} \Delta_p(f, h) = 0, \ f \in L_p, \]

and \( \Delta_p(f, h) \leq |f|_p. \)

We conclude after some calculations following [51], pp, 66-71

\[ \Gamma(\alpha)|I^\alpha[f](x)| \leq Z(\alpha, p) \cdot x^{\alpha-1/p} \cdot \Delta_p(f, x), \ f \in L_p, \ x \in [0, 1]; \tag{3.3} \]

\[ \omega(\Gamma(\alpha)I^\alpha[f], h) \leq 4 \cdot Z(\alpha, p) \cdot h^{\alpha-1/p} \cdot \Delta_p(f, h), \ f \in L_p, \ h \in [0, 1]; \tag{3.4} \]

see also [49] for technical details.

In turn,
\[
\Gamma(\alpha)|I^\alpha[f](x)| \leq Z(\alpha, p) \cdot x^{\alpha - 1/p} \cdot |f|_p, \ f \in L_p, \ x \in [0, 1]; \quad (3.3a)
\]

\[
\omega(\Gamma(\alpha)I^\alpha[f], h) \leq 4 \cdot Z(\alpha, p) \cdot h^{\alpha - 1/p} \cdot |f|_p, \ f \in L_p, \ h \in [0, 1]. \quad (3.4a)
\]

Recall that here \( \alpha > 1/p \); the case \( \alpha = 1/p \) is considered in [51], p. 69. The case of greatest values \( \alpha \) can be reduced to the considered here by means of differentiating, see [51], p. 69-77.

We intend to show in this section that the exponent \( \alpha - 1/p \) is exact.

**A. Case** \( X = \{x\} = (0, 1) \).

**Proposition 3.A.** For all the values \( \epsilon \in (0, (\alpha - 1/p)/2) \) there exists a function \( f_0, f_0 \in L_p(0, 1) \), for which

\[
\omega(I^\alpha[f_0], h) \geq C(\alpha, p) \cdot h^{\alpha - 1/p + \epsilon}, \ C(\alpha, p) > 0, \ h \in (0, 1). \quad (3.5)
\]

**Proof.** Let us consider the following example (test function)

\[
f_0(x) = x^{-\beta} \cdot I(x \in (0, 1)), \ \beta < 1/p,
\]

and denote \( g_0(x) = I^\alpha[f_0](x)/\Gamma(1 - \alpha) \); then

\[
\forall \beta \in (0, 1/p) \Rightarrow f_0 \in L_p(0, 1) \subset L_p(0, \infty) :
\]

\[
|f_0|_p = (1 - \beta p)^{-1/p} < \infty.
\]

Further, let \( x \in (0, 1) \); we have consequently

\[
g_0(x) = \int_0^x \frac{y^{-\beta} \, dy}{(x - y)^{1-\alpha}} = x^{\alpha - \beta} \cdot B(1 - \beta, \alpha)
\]

and therefore

\[
\omega(g_0, h) = C(\alpha, \beta) \cdot h^{\alpha - \beta}, \ C(\alpha, \beta) = B(1 - \beta, \alpha) > 0, \ h \in (0, 1). \quad (3.6)
\]

Since the value \( \beta \) is arbitrary from the set \( (0, 1/p) \), the proposition is proved.

**B. Case** \( X = \{x\} = (0, \infty) \).

**Proposition 3.B.** Suppose the inequality

\[
\omega(I^\alpha([f], h)) \leq F_{\alpha,p}(f) \cdot h^{\mu(\alpha,p)}, \ p > 1/\alpha.
\]

there holds for any \( f \in L_p(R_+) \) Then

\[
\mu(\alpha, p) = \alpha - 1/p. \quad (3.8)
\]
Proof. We will use the well-known scaling, or dilation method, see [52], chapter 3, [51], chapter 3, [53]. Namely, let \( \rho(\cdot) \) be arbitrary non-zero function from the space \( L^p(R_+) \) such that

\[
\omega(I^\alpha[\rho], h) \leq K h^\gamma |\rho|_p, \quad h > 0.
\]

(3.9)

Let also \( \lambda \) be arbitrary positive number; the dilation (linear) operator \( T_{\lambda} \) is defined by an equality

\[
T_{\lambda}[\rho](x) \overset{\text{def}}{=} \rho(\lambda x).
\]

Evidently, \( T_{\lambda}[\rho](\cdot) \in L^p(R_+) \) and moreover

\[
|T_{\lambda}[\rho]|_p = \lambda^{-1/p} |\rho|_p
\]

and analogously

\[
I^\alpha T_{\lambda}[\rho] = \lambda^{-\alpha} T_{\lambda} I^\alpha[\rho].
\]

(3.10)

We deduce from the source inequality (3.8) applied to the function \( T_{\lambda}[\rho] \)

\[
\lambda^{-\alpha} \omega(I^\alpha \rho, \lambda h) \leq K \lambda^{-1/p} h^\gamma |\rho|_p,
\]

or equally after change of variables \( \lambda h = \delta, \delta \in (0, \infty) \)

\[
\omega(I^\alpha[\rho], \delta) \leq K \delta^\gamma \lambda^{\alpha-\gamma-1/p} |\rho|_p.
\]

(3.11)

Thus, \( \alpha - \gamma - 1/p = 0 \), Q.E.D.

4 Module of continuity of Riesz potential.

The necessary and sufficient condition for existence (a.e.) of Riesz potential (1.3) is the following

\[
\int_{R^d} (1 + |y|)^{\alpha-d} |f(y)| \, dy < \infty;
\]

we will suppose in what follows in this and in the next sections that this condition on the (measurable) function \( f : R^d \to R \) is satisfied.

Further, assume \( \alpha \in (0, d) \), \( f \in L_p(R^d) \) for some \( p > d/\alpha \):

\[
\int_{R^d} |f(y)|^p \, dy < \infty, \quad p = \text{const} > d/\alpha.
\]

(4.2)

We introduce therefore the following norm for a (measurable) function \( f : R^d \to R \)

\[
|f|_{\alpha,d,p} \overset{\text{def}}{=} \max \left\{ \int_{R^d} (1 + |y|)^{\alpha-d} |f(y)| \, dy, \ |f|_p \right\}
\]

(4.3)

and the correspondent Banach space consisting on all the measurable functions \( F : R^d \to R \) with finite norm.
\[ L_{\alpha,d,p} = \{ f, f : \mathbb{R}^d \to \mathbb{R}, \ |f|_{\alpha,d,p} < \infty \} \].

Y. Mizuta et al in [31]-[34] proved that under condition \(|f|_{\alpha,d,p} < \infty\) the Riesz potential
\[ r_\alpha[f](x) := R_\alpha[f](x) \quad (4.4) \]
is uniformly continuous and moreover
\[ \omega(r_\alpha[f], h) \leq C \cdot \frac{p - 1}{\alpha p - d} \cdot h^{\alpha - d/p} \cdot |f|_{\alpha,d,p}, \ h > 0. \quad (4.5) \]

It turns out that the exponent \(\alpha - d/p\) in the considered case is non-improvable still for the Riesz potential.

We denote by \(K_R(\alpha, p)\) the optimal, i.e. minimal value \(C\) in the last inequality:
\[ K_R(\alpha, p) \overset{\text{def}}{=} \sup_{h > 0} \sup_{0 \neq f \in L_{\alpha,d,p}} \left[ \frac{\omega(r_\alpha[f], h) \cdot (\alpha p - d)^{1-1/p}}{(p - 1)^{1-1/p} \cdot h^{\alpha - d/p} \cdot |f|_{\alpha,d,p}} \right], \quad (4.6) \]
then as before for some finite positive constants \(C_1(d), C_2(d)\) depending only on the dimension \(d\)
\[ \frac{C_1(d)}{\alpha} \leq K_R(\alpha, p) \leq \frac{C_2(d)}{\alpha(d - \alpha)}, \ 0 < \alpha < d. \]

**Proposition 4.C.** Suppose the inequality
\[ \omega(R_\alpha([f], h)) \leq F_{\alpha,p}(f) \cdot h^{\nu(\alpha,p)}, \ p > d/\alpha, \ h \geq 0 \]
there holds for arbitrary \(f \in L_{\alpha,d,p}(\mathbb{R}^d)\), i.e. and such that \(\int_{\mathbb{R}^d} (1 + |y|)^\alpha - d |f(y)| \, dy < \infty\) and \(f \in L_p(\mathbb{R}^d)\) Then
\[ \nu(\alpha, p) = \alpha - d/p. \quad (4.7) \]

**Proof** is at the same as in the proposition 3.B by means of scaling method an may be omitted.

Further, let us introduce the following Young-Orlicz function
\[ \Phi_{p,\gamma}(u) = |u|^p (\ln |u|)^\gamma, \ |u| > e, \quad (4.8a) \]
\[ \Phi_{p,\gamma}(u) = e^{p-2} u^2, \ |u| < e; \ p = \text{const} > 1, \ \gamma = \text{const} > 0; \quad (4.8b) \]
and denote the correspondent Orlicz space by \(L(\Phi_{p,\gamma})\) with a norm \(||f||L(\Phi_{p,\gamma})\). Since this function \(u \to \Phi_{p,\gamma}(u)\) satisfies the \(\Delta_2\) condition, the belonging of arbitrary (measurable) function \(f : \mathbb{R}^d \to \mathbb{R}\) to this Orlicz space: \(f \in L(\Phi_{p,\gamma})\) is completely equivalent to the convergence of the following integral
\[ \int_{\mathbb{R}^d} \Phi_{p,\gamma}(f(x)) \, dx < \infty. \quad (4.9) \]
Introduce also the following norm

\[ \|f\|_{L^{\alpha,d}(\Phi_p,\gamma)} \overset{def}{=} \|f\|_{L^{\Phi_p,\gamma}} + \int_{\mathbb{R}^d} (1 + |y|)^{\alpha - d} |f(y)| \, dy. \quad (4.10) \]

It follows immediately from the articles [31] - [34] that if \( \|f\|_{L^{\alpha,d}(\Phi_p,\gamma)} < \infty \), then

\[ \omega(R_{\alpha}([f],h)) \leq C(d) \cdot \left[ \frac{p - 1}{\alpha p - d} \right]^{1-1/p} \times \]

\[ h^{\alpha - d/p} |\ln h|^{\gamma/p} \cdot \|f\|_{L^{\alpha,d}(\Phi_p,\gamma)}, \ 0 < h < 1/e. \quad (4.11) \]

5 Module of continuity of fractional integrals for the functions from Grand Lebesgue Spaces.

Recently appear the so-called Grand Lebesgue Spaces \( GLS = G(\psi) = G(\psi; A, B), \ A, B = \text{const}, A \geq 1, A < B \leq \infty, \) spaces consisting on all the measurable functions \( f : \mathbb{R}^d \to \mathbb{R} \) with finite norms

\[ \|f\|_{G(\psi)} \overset{def}{=} \sup_{p \in (A,B)} \|f|_p/\psi(p)\|. \quad (5.1) \]

Here \( \psi(\cdot) \) is some continuous positive on the open interval \((A, B)\) function such that

\[ \inf_{p \in (A,B)} \psi(p) > 0, \ \psi(p) = \infty, \ p \notin (A, B). \]

We will denote

\[ \text{supp}(\psi) \overset{def}{=} (A, B) = \{p : \psi(p) < \infty,\} \quad (5.2) \]

The set of all \( \psi \) functions with support \( \text{supp}(\psi) = (A, B) \) will be denoted by \( \Psi(A, B) \).

This spaces are rearrangement invariant, see [6], and are used, for example, in the theory of probability [23], [37], [38]; theory of Partial Differential Equations [22]; functional analysis [22], [38]; theory of Fourier series, theory of martingales, mathematical statistics, theory of approximation etc.

Notice that in the case when \( \psi(\cdot) \in \Psi(A, \infty) \) and a function \( p \to p \cdot \log \psi(p) \) is convex, then the space \( G\psi \) coincides with some exponential Orlicz space.

Conversely, if \( B < \infty \), then the space \( G\psi(A, B) \) does not coincides with the classical rearrangement invariant spaces: Orlicz, Lorentz, Marcinkiewicz etc.

The fundamental function of these spaces \( \phi(G(\psi),\delta) = ||I_A||G(\psi),\mes(A) = \delta, \ \delta > 0, \) where \( I_A \) denotes as ordinary the indicator function of the measurable set \( A \), by the formulae

\[ \phi(G(\psi),\delta) = \sup_{p \in \text{supp}(\psi)} \left[ \frac{\delta^{1/p}}{\psi(p)} \right]. \quad (5.2) \]
The fundamental function of arbitrary rearrangement invariant spaces plays very important role in functional analysis, theory of Fourier series and transform [6] as well as in our further narration.

Many examples of fundamental functions for some $G\psi$ spaces are calculated in [37], [38].

**Remark 5.1.** If we introduce the discontinuous function

$$\psi_r(p) = 1, \quad p = r; \quad \psi_r(p) = \infty, \quad p \neq r, \quad p, r \in (A, B)$$

and define formally $C/\infty = 0$, $C = \text{const} \in R^1$, then the norm in the space $G(\psi_r)$ coincides with the $L_r$ norm:

$$||f||_{G(\psi_r)} = |f|_r.$$  

Thus, the Grand Lebesgue Spaces are direct generalization of the classical exponential Orlicz’s spaces and Lebesgue spaces $L_r$.

Suppose that the function $f : R^d \to R$ and parameters $\alpha, d, p$ satisfy all the conditions of the previous section. Suppose also that the function $p \to |f|_{\alpha,d,p}$ allows the following estimation

$$|f|_{\alpha,d,p} \leq \psi_{\alpha,d}(p), \quad (5.3)$$

where

$$\psi_{\alpha,d}(\cdot) \in \Psi(A, B), \quad \exists A = \text{const} > d/\alpha, \quad \exists B > A. \quad (5.3a)$$

One can choose, for instance,

$$\psi_{\alpha,d}(p) := |f|_{\alpha,d,p}, \quad (5.3b)$$

if the function $p \to \psi_{\alpha,d}(p)$ satisfies of the condition (5.3a).

Define a new $\psi$ – function $\nu_{\alpha,d}(p)$ as follows:

$$\nu_{\alpha,d}(p) := \psi_{\alpha,d}(p) \cdot K_R(\alpha, p) \cdot \left[\frac{p - 1}{\alpha p - d}\right]^{1-1/p}, \quad p \in (A, B). \quad (5.4)$$

**Theorem 5.1.** We propose under formulated above conditions

$$\omega((r_\alpha[f], \delta) \leq \frac{\delta^\alpha}{\phi(G\nu_{\alpha,d}(p)), \delta^d)}, \quad \delta > 0. \quad (5.5)$$

**Proof.** We use the Mizuta’s inequality and direct definition of the constant $K_R(\alpha, p)$

$$\omega(r_\alpha[f], h) \leq K_R(\alpha, p) \cdot \left[\frac{p - 1}{\alpha p - d}\right]^{1-1/p} \cdot h^{\alpha-d/p} \cdot |f|_{\alpha,d,p} \leq$$

$$K_R(\alpha, p) \psi_{\alpha,d,p}(p) h^{\alpha-d/p} \leq \nu_{\alpha,d,p}(p) h^{\alpha-d/p}, \quad p \in (A, B).$$

Therefore
\[
\frac{\omega(r_\alpha[f], h)}{h^\alpha} \leq \frac{h^{-d/p}}{1/\nu_{\alpha,d,p}(p)} = \left[ \frac{h^{d/p}}{\nu_{\alpha,d,p}(p)} \right]^{-1}.
\] (5.6)

Since the left-hand side of the last inequality does not depend on the variable \( p \), we can take the infimum from both the sides one:

\[
\frac{\omega(r_\alpha[f], h)}{h^\alpha} \leq \inf_{p \in (A,B)} \left\{ \frac{h^{-d/p}}{1/\nu_{\alpha,d,p}(p)} \right\} = \frac{1}{\phi(G\nu_{\alpha,d,p}; h^d)},
\] (5.7)

which is equivalent to the proposition of theorem.

Let us consider a slight generalization of theorem 5.1.

Suppose \( ||f||_{L_\alpha,d(\Phi_p,\gamma)} < \infty \). Suppose also that the function \( p \to ||f||_{L_\alpha,d(\Phi_p,\gamma)} \) allows the following estimation

\[
||f||_{L_\alpha,d(\Phi_p,\gamma)} \leq \theta_{\alpha,d}(p),
\] (5.8)

where

\[
\theta_{\alpha,\gamma,d}(\cdot) \in \Psi(A_1, B_1), \ \exists A_1 = \text{const} > d/\alpha, \ \exists B_1 > A_1. \tag{5.8a}
\]

One can choose, for instance,

\[
\theta_{\alpha,\gamma,d}(p) := ||f||_{L_\alpha,d(\Phi_p,\gamma)},
\]

if the function \( p \to ||f||_{L_\alpha,d(\Phi_p,\gamma)} \) satisfies of the condition (5.8a).

Define a new \( \psi - \) function \( \zeta_{\alpha,\gamma,d}(p) \) as follows:

\[
\zeta_{\alpha,\gamma,d}(p) := \theta_{\alpha,\gamma,d}(p) \cdot K_R(\alpha, p) \cdot \left[ \frac{p-1}{\alpha p-d} \right]^{1-1/p}, \ p \in (A_1, B_1). \tag{5.10}
\]

**Theorem 5.2.** We propose under formulated above conditions

\[
\omega(r_\alpha[f], h) \leq \frac{h^\alpha}{\phi(G\zeta_{\alpha,\gamma,d}; h^d \cdot |\ln h|^{-\gamma})}, \ h \in (0, 1/e).
\] (5.11)

Consider ultimately the case when \( \exists (A_2, B_2) = \text{const}, \ A_2 > d/\alpha, B_2 \in (A_2, \infty] \), such that

\[
\forall p \in (A_2, B_2) \Rightarrow \int_{R^d} |f(y)|^p \left[ \ln^+ |f(y)| \right]^\gamma_0 \gamma_1^p \ dy < \infty, \tag{5.12}
\]

where \( \ln^+ z = \max(1, \ln z), \ z \geq 0 \), for certain finite non-negative constants \( \gamma_0, \gamma_1 \).

We introduce a non-negative function \( \kappa_0 = \kappa_0(p) \) by an equality

\[
\kappa_0^p(p) := \int_{R^d} |f(y)|^p \left[ \ln^+ |f(y)| \right]^p \ dy = |f \cdot \left[ \ln^+ |f(y)| \right]_p|_p^p, \ p \in (A_2, B_2),
\]
and define the new finite in the interval $(A_2, B_2)$ function

$$
\kappa(p) := \max \left\{ \int_{R^d} (1 + |y|)^{\alpha-d} |f(y)| \, dy, \, \kappa_0(p) \right\}.
$$

(5.13)

**Theorem 5.3.** We assert under formulated above conditions, for instance, conditions (4.1) and (5.12)

$$
\omega(r_\alpha[f], h) \leq C(\alpha, \gamma_0, \gamma_1, d) \frac{h^\alpha |\ln h|^{-\gamma_1}}{\phi(G_K, h^d \cdot |\ln h|^{-\gamma_0})}, \quad h \in (0, 1/e).
$$

(5.14)

6 Concluding remarks.

It is interest perhaps to obtain the estimations of the module continuity for the weight Riesz potential as well as for the weight fractional integrals and derivatives. The $L_p$ estimates for ones are investigates in [25], [26], [11], [43], [44].

Also it is interest by our opinion to investigate the multidimensional fractional integrals and derivatives.

References

[1] D.R.Adams, L.I. Hedberg. *Function Spaces and Potential Theory*. Springer Verlag, Berlin, Heidelberg, New York, 1996.

[2] D.R.Adams, R.J.Bagby. *Translation-dilation invariant estimates for Riesz potential*. Indiana Univ. Math. Journal. 1974, V.23, N 1, 1051-1067.

[3] D.R. Adams. *Choquet integrals in potential theory*. Publ. Mat. 42 (1998), 3-66.

[4] D.R. Adams. *On the existence of capacitary strong type estimates in $R^n$*. Arkiv för Matematik, 14 (1976), 125-140.

[5] I. B. Bapna and Nisha Mathur. *Application of Fractional Calculus in Statistics*. Int. J. Contemp. Math. Sciences, Vol. 7, 2012, no. 18, 849-856.

[6] Bennett C. and Sharpley R. *Interpolation of operators*. Orlando, Academic Press Inc.,1988.

[7] Andrea Borla and Costen Protoapoeescu. *Nonparametric Estimation of the Fractional Derivative of a Function Distribution*. Internet publication, PDF, (2014).

[8] Farida Enikeeva. *Adaptive minimax estimation of a fractional derivative*. Statistics Probability Letters, 76, (2006), 1441-1448.
[9] A. Fiorenza. *Duality and reflexivity in grand Lebesgue spaces.* Collectanea Mathematica (electronic version), **51**, 2, (2000), 131-148.

[10] A. Fiorenza and G.E. Karadzhov. *Grand and small Lebesgue spaces and their analogs.* Consiglio Nationale Delle Ricerche, Instituto per le Applicazioni del Calcolo Mauro Picone, Sezione di Napoli, Rapporto tecnico n. 272/03, (2005).

[11] R.L. Frank and E.H. Lieb. *Inversion Positivity and the sharp Hardy-Littlewood-Sobolev Inequality.* Electronic Publications, arXiv:0904.4275v1 [math.FA] 27 Apr 2009.

[12] Fuglede B. *On the theory of potentials in locally compact spaces.* Acta. Math. 103, (1960), 139-215.

[13] A.E. Gatto, C. Segovia and S. V'agi. *On fractional differentiation and integration on spaces of homogeneous type.* Rev. Mat. Iberoamericana, **12**, (1996), 111-145.

[14] Golubev, G.K., Enikeeva, F. (2001.) *On the minimax estimation problem of a fractional derivative.* Theory Probab. Appl. 46, 619-635.

[15] Mubariz G. Hajibayov. *Continuity Properties of Potentials on Spaces of Homogeneous Type.* Int. Journal of Math. Analysis, Vol. 2, 2008, no. 7, 315-328.

[16] Hardy G.H. *On some properties of integrals of fractional order.* Messenger. Math. 1917, V. 47 N 10, 145-150.

[17] Hardy G.H., Littlewood J.E. *Some properties of fractional integrals.* Proc. London Math. Soc., Ser. 2, (1928), V.24, 77-141.

[18] Hardy G.H., Littlewood J.E. *Some properties of fractional integrals. I.* Math. Zeitschrift, (1928), V.27, N 4, 565-606.

[19] Hardy G.H., Littlewood J.E. *Some properties of fractional integrals. II.* Math. Zeitschrift, (1932), V.34, N 34, 403-439.

[20] Harboure E., Macias R.A., Segovia C. *Boundedness of fractional operators on L(p) spaces with different weight.* Trans. Amer. Soc., 1984, V.285 N 2, 629-647.

[21] T. Iwaniec and C. Sbordone. *On the integrability of the Jacobian under minimal hypotheses.* Arch. Rat.Mech. Anal., 119, (1992), 129143.

[22] T. Iwaniec, P. Koskela and J. Onninen. *Mapping of finite distortion: Monotonicity and Continuity.* Invent. Math. 144 (2001), 507-531.

[23] Kozachenko Yu. V., Ostrovsky E.I. (1985). *The Banach Spaces of random Variables of subgaussian type.* Theory of Probab. and Math. Stat., (in Russian). Kiev, KSU, **32**, 43-57.
[24] Leoni G. A first Course in Sobolev Spaces. Graduate Studies in Mathematics. v. 105, AMS, Providence, Rhode Island, (2009).

[25] Lieb E.H, Loss M. Analysis. Providence, Rhode Island, 1997.

[26] Lieb E.H. Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities. Ann. of Math., (2), 118 (1983), no 2, 349-374.

[27] Liouville J. Mémoire sur il’ integration des équations différentielles a indices fractionnaires. J. École Roy. Polytéchn., 1835, V.15 Sect. 24, 17-54.

[28] J. Maly and L. Pick. The sharp Riesz potential estimates in metric spaces, Indiana Univ. Math. J., 51 (2002), 251268.

[29] Mark Meerschaert, Jeff Mortensen, and Hans-Peter Scheffler. Vector Grünwald formula for fractional derivatives. Internet electronic publication, 2014.

[30] K. Miller and B. Ross. (1993) An Introduction to Fractional Calculus and Fractional Differential Equations. Wiley, New York.

[31] Yoshihiro Mizuta and Tetsu Shimomura. Continuity properties of Riesz potential of Orlicz functions. Tohoku Math. J., 61 (2009), 225-240.

[32] Y. Mizuta. Continuity properties of Riesz potentials and boundary limits of Beppo Levi functions. Math. Scand. 63 (1988), 238-260.

[33] Y. Mizuta. Continuity properties of potentials and Beppo-Levi-Deny functions. Hiroshima Math. J. 23 (1993), 79-153.

[34] Y. Mizuta. Potential theory in Euclidean spaces. Gakkōtosyo, Tokyo, 1996.

[35] Juno Nuutinen and Pilar Silvestre. The Riesz capacity in metric spaces. arXiv:1501.05746v1 [math.FA] 23 Jan 2015

[36] E. Liflyand, E. Ostrovsky and L. Sirota. Structural properties of Bilateral Grand Lebesque Spaces. Turk. Journal of Math., 34, (2010), 207-219. TUBITAK, doi:10.3906/mat-0812-8

[37] Ostrovsky E.I. Exponential estimates for the random fields and its applications. (1999), Moskow-Obninsk, OINPE, (in Russian).

[38] Buldygin V.V., Mushtary D.Ch., Ostrovsky E.I., Puchalskii A.W. New Trends in Probability Theory and Statistics. (1992), VSP (Utrecht, Tokyo, New York).

[39] E. Ostrovsky and L. Sirota. Moment Banach spaces: theory and applications. HAIT Journal of Science and Engineering, C, Volume 4, Issues 1-2, pp. 233-262, (2007).

[40] E. Ostrovsky and L. Sirota. Well Posedness of the Problem of Estimation Fractional Derivative for a Distribution Function. arXiv:1412.6829v1 [math.ST] 21 Dec 2014.
[41] E. Ostrovsky and L. Sirota. Cesaro-Hardy operators on bilateral Grand Lebesgue Spaces. arXiv:1307.5481v1 [math.FA] 20 Jul 2013

[42] E. Ostrovsky and L. Sirota. Riesz’s and Bessel’s operators in bilateral Grand Lebesgue Spaces. arXiv:0907.3321v1 [math.FA] 19 Jul 2009

[43] E. Ostrovsky and L. Sirota. Hardy-Littlewood inequalities for Riesz’s potential. Low bounds estimations for different powers. arXiv:0909.5663v1 [math.FA] 30 Sep 2009

[44] E. Ostrovsky and L. Sirota. Weight Hardy-Littletwood inequalities for different powers. arXiv:0910.5880v1 [math.FA] 30 Oct 2009

[45] E. Ostrovsky and L. Sirota. Moment Banach spaces: theory and applications. HAIT Journal of Science and Engineering, C, Volume 4, Issues 1-2, pp. 233-262, (2007).

[46] E. Ostrovsky and L. Sirota. Boundedness of operators in bilateral Grand Lebesgue Spaces, with exact and weakly exact constant calculation. arXiv:1104.2963v1 [math.FA] 15 Apr 2011

[47] E. Ostrovsky and L. Sirota. Multiple weight Riesz and Fourier transforms in bilateral anosotropic Grand Lebesgue Spaces. arXiv:1208.2392v1 [math.FA] 12 Aug 2012

[48] E. Ostrovsky and L. Sirota. Lebesgue Spaces Norm Estimates for Fractional Integrals and Derivatives. arXiv:1502.00696v1 [math.FA] 3 Feb 2015

[49] E. Ostrovsky, L. Sirota. Lebesgue Spaces Norm Estimates for Fractional Integrals and Derivatives. arXiv:1502.00696v1 [math.FA] 3 Feb 2015

[50] Riemann G. Versuch einer algemainen Auffasung der Integration und Differentiation. Gesamelte Math. Werke, Leipzig, Teubner Verlag, 1922-1923, 114-126.

[51] S. G. Samko, A. A. Kilbas and O. I. Marichev. Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach Science Publishers, Yverdon, 1993.

[52] E.M. Stein. Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton, (1992).

[53] G. Talenty. Best constant in Sobolev inequality. Instit. Mat. Univ. Firenze, 22, (1974-1975), p. 1-32.