Certain basic inequalities for submanifolds in a $(\kappa, \mu)$-contact space form

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Abstract. Certain basic inequalities between intrinsic and extrinsic invariants for a submanifold in a $(\kappa, \mu)$-contact space form are obtained. As applications we get some results for invariant submanifolds in a $(\kappa, \mu)$-contact space form.

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1 Introduction

In [3], B.-Y. Chen established a sharp inequality for a submanifold in a real space form involving intrinsic invariants, namely the sectional curvatures and the scalar curvature of the submanifold; and the main extrinsic invariant, namely the squared mean curvature. Similar inequalities were established in case of submanifolds of Sasakian space forms also ([2],[6]).

Recently, T. Koufogiorgos introduced the notion of $(\kappa, \mu)$-contact space form ([7]), which contains the well known class of Sasakian space forms for $\kappa = 1$. Thus it is worthwhile to study relationships between intrinsic and extrinsic invariants of submanifolds in a $(\kappa, \mu)$-contact space form. The paper is organized as follows. Section 2 contains necessary details about $(\kappa, \mu)$-contact space form and its submanifolds. In section 3, we state a Lemma relating scalar curvature, squared mean curvature and squared second fundamental form for a submanifold tangential to the structure vector field in a $(\kappa, \mu)$-contact space form, then we obtain two basic inequalities involving the scalar curvature and the sectional curvatures of the submanifold on left hand side and the squared mean curvature on the right hand side. In the last section, we study invariant submanifolds of...
(κ, µ)-contact space forms. We also obtain a B-Y. Chen inequality for Chen like δ-invariant for invariant submanifolds in a (κ, µ)-contact space form.

2 Preliminaries

A (2m+1)-dimensional differentiable manifold \( \tilde{M} \) is called an almost contact manifold if its structural group can be reduced to \( U(m) \times 1 \). Equivalently, there is an almost contact structure \((\varphi, \xi, \eta)\) consisting of a \((1,1)\) tensor field \( \varphi \), a vector field \( \xi \), and a 1-form \( \eta \) satisfying \( \varphi^2 = -I + \eta \otimes \xi \) and \( \eta(\xi) = 1 \), \( \varphi \xi = 0 \), \( \eta \circ \varphi = 0 \). The almost contact structure is said to be normal if the induced almost complex structure \( J \) on the product manifold \( \tilde{M} \times \mathbb{R} \) defined by \( J(X, \lambda d/dt) = (\varphi X - \lambda \xi, \eta(X) d/dt) \) is integrable, where \( X \) is tangent to \( \tilde{M} \), \( t \) the coordinate of \( \mathbb{R} \) and \( \lambda \) a smooth function on \( \tilde{M} \times \mathbb{R} \). The condition for being normal is equivalent to vanishing of the torsion tensor \([\varphi, \varphi] + 2d\eta \otimes \xi \), where \([\varphi, \varphi]\) is the Nijenhuis tensor of \( \varphi \). Let \( \langle , \rangle \) be a compatible Riemannian metric with \( \langle \varphi, \xi, \eta, (, ) \rangle \), that is, \( \langle X, Y \rangle = \langle \varphi X, \varphi Y \rangle + \eta(Y)X \) or equivalently, \( \Phi(X, Y) \equiv \langle X, \varphi Y \rangle = -\langle \varphi X, Y \rangle \) along with \( \langle X, \xi \rangle = \eta(X) \) for all \( X, Y \in T\tilde{M} \). Then, \( \tilde{M} \) becomes an almost contact metric manifold equipped with an almost contact metric structure \((\varphi, \xi, \eta, (, ))\). An almost contact metric structure becomes a contact metric structure if \( \Phi = d\eta \). A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if \( \langle \nabla_X \varphi Y - \langle X, Y \rangle \xi - \eta(Y)X \rangle \) for all \( X, Y \in T\tilde{M} \), where \( \nabla \) is Levi-Civita connection. Also, a contact metric manifold \( \tilde{M} \) is Sasakian if and only if the curvature tensor \( \tilde{R} \) satisfies \( \tilde{R}(X, Y)\xi = \eta(Y)X - \eta(X)Y \) for all \( X, Y \in T\tilde{M} \).

In a contact metric manifold \( \tilde{M} \), the \((1,1)\)-tensor field \( h \) defined by \( 2h = 2\xi \varphi \) is symmetric and satisfies

\[ h\xi = 0, \quad h\varphi + \varphi h = 0, \quad \nabla_X\xi = -\varphi X - \varphi h X, \quad \text{trace } (h) = \text{trace } (\varphi h) = 0. \quad (1) \]

The \((\kappa, \mu)\)-nullity distribution of a contact metric manifold \( \tilde{M} \) is a distribution

\[ N(\kappa, \mu) : p \to N_p(\kappa, \mu) = \left\{ Z \in T_p\tilde{M} \mid \tilde{R}(X, Y) Z = \kappa (\langle Y, Z \rangle X - \langle X, Z \rangle Y) + \mu (\langle Y, Z \rangle hX - \langle X, Z \rangle hY) \right\}, \]

where \( \kappa \) and \( \mu \) are constants. If \( \xi \in N(\kappa, \mu) \), \( \tilde{M} \) is called a \((\kappa, \mu)\)-contact metric manifold. Since in a \((\kappa, \mu)\)-contact metric manifold one has \( h^2 = (\kappa - 1) \varphi^2 \), therefore \( \kappa \leq 1 \) and if \( \kappa = 1 \) then the structure is Sasakian. Characteristic examples of non-Sasakian \((\kappa, \mu)\)-contact metric manifolds are the tangent sphere bundles of Riemannian manifolds of constant sectional curvature not equal to one. For more details we refer to [1] and [3].

The sectional curvature \( \tilde{K}(X, \varphi X) \) of a plane section spanned by a unit vector \( X \) orthogonal to \( \xi \) is called a \( \varphi \)-sectional curvature. If the \((\kappa, \mu)\)-contact metric manifold \( \tilde{M} \) has constant \( \varphi \)-sectional curvature \( c \) then it is called a \((\kappa, \mu)\)-contact space form and is denoted by \( \tilde{M}(c) \). The curvature tensor of \( \tilde{M}(c) \) is
We recall the following Chen’s Lemma for later use.

Lemma 3.1

Let M be a totally geodesic n-dimensional submanifold in a manifold \( \tilde{M} \) equipped with a Riemannian metric \( \langle \cdot, \cdot \rangle \). The Gauss and Weingarten formulae are given respectively by \( \nabla_X Y = \nabla_X Y - \sigma (X, Y) \) and \( \nabla_X N = -A_N X + \nabla^\perp_X N \) for all \( X, Y \in T M \) and \( N \in T^1 M \), where \( \nabla, \nabla^\perp \) are Riemannian, induced Riemannian and induced normal connections in \( \tilde{M} \), M and the normal bundle \( T^1 M \) of M respectively, and \( \sigma \) is the second fundamental form related to the shape operator \( A_N \) in the direction of N by \( \langle \sigma (X, Y), N \rangle = \langle A_N X, Y \rangle \). Then, the Gauss equation is given by

\[
\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) - \langle \sigma (X, Y), \sigma (Z, W) \rangle + \langle \sigma (X, Z), \sigma (Y, W) \rangle
\]

(3)

for all \( X, Y, Z, W \in T M \), where \( \tilde{R} \) and \( R \) are the curvature tensors of \( \tilde{M} \) and M respectively. The mean curvature vector H is expressed by \( n H = \text{trace} \, (\sigma) \).

The submanifold M is totally geodesic in \( \tilde{M} \) if \( \sigma = 0 \), and minimal if \( H = 0 \). If \( \sigma (X, Y) = \langle X, Y \rangle \, H \) for all \( X, Y \in T M \), then M is totally umbilical.

3 Certain basic inequalities

We recall the following Chen’s Lemma for later use.

Lemma 3.1 (\( \mathbb{R} \)) If \( a_1, \ldots, a_n, a_{n+1} \) are \( n + 1 \) \( (n > 1) \) real numbers such that

\[
\frac{1}{n - 1} \left( \sum_{i=1}^{n} a_i \right)^2 = \sum_{i=1}^{n} a_i^2 + a_{n+1},
\]

(\( \mathbb{R} \))
then $2a_1a_2 \geq a_{n+1}$, with equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_n$.

For a vector field $X$ on a submanifold $M$ of an almost contact metric manifold $\tilde{M}$, let $PX$ be the tangential part of $\varphi X$. Thus, $P$ is an endomorphism of the tangent bundle of $M$ and satisfies $\langle X, PY \rangle = -\langle PX, Y \rangle$. Let $\pi \subset T_pM$ be a plane section spanned by an orthonormal basis $\{e_1, e_2\}$. Then, $\alpha(\pi)$ given by

$$\alpha(\pi) = \langle e_1, Pe_2 \rangle^2$$

is a real number in the closed unit interval $[0, 1]$, which is independent of the choice of the orthonormal basis $\{e_1, e_2\}$. Let $\xi \in TM$ and put

$$\beta(\pi) = (\eta(e_1))^2 + (\eta(e_2))^2,$$

$$\gamma(\pi) = \eta(e_1)^2 \langle h^T e_2, e_2 \rangle + \eta(e_2)^2 \langle h^T e_1, e_1 \rangle - 2\eta(e_1)\eta(e_2) \langle h^T e_1, e_2 \rangle.$$

Then, $\beta(\pi)$ and $\gamma(\pi)$ are also real numbers and do not depend on the choice of the orthonormal basis $\{e_1, e_2\}$. Of course, $\beta(\pi) \in [0, 1]$.

In view of (2) and (3) we state the following Lemma.

**Lemma 3.2** In an $n$-dimensional submanifold $M$ in a $(\kappa, \mu)$-contact space form $\tilde{M}(c)$ such that $\xi \in TM$, the scalar curvature and the squared mean curvature satisfy

$$2\tau = \frac{1}{4} \left\{ n(n-1)(c+3) + 3(c-1) \|P\|^2 - 2(n-1)(c+3-4\kappa) \right\}
+ \frac{1}{2} \left\{ \| (\varphi h)^T \|^2 - \| h^T \|^2 - \left( \text{trace} \left( (\varphi h)^T \right) \right)^2 + \left( \text{trace} (h^T) \right)^2 \right\}
+ 2(\mu + n-2) \text{trace} (h^T) + n^2 \|H\|^2 - \|\sigma\|^2,$$

(4)

where

$$\|\sigma\|^2 = \sum_{i,j=1}^{n} \langle \sigma(e_i, e_j), \sigma(e_i, e_j) \rangle, \quad \|Q\|^2 = \sum_{i,j=1}^{n} \langle e_i, Qe_j \rangle^2, \quad Q \in \left\{ P, (\varphi h)^T, h^T \right\}$$

and $(\varphi h)^TX$ and $h^TX$ are the tangential parts of $\varphi hX$ and $hX$ respectively for $X \in TM$.

The equation (4) is of fundamental importance and will play main role to establish several inequalities.

Now, we prove the following contact version of Theorem 3 of [3].

**Theorem 3.3** Let $M$ be an $n$-dimensional $(n \geq 3)$ submanifold isometrically immersed in a $(2m+1)$-dimensional $(\kappa, \mu)$-contact space form $\tilde{M}(c)$ such that...
\[\xi \in TM.\] Then, for each point \(p \in M\) and each plane section \(\pi \subset T_pM,\) we have
\[
\tau - K(\pi) \leq \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + \frac{1}{8} n(n-3)(c+3) + (n-1) \kappa
\]
\[+ \frac{3(c-1)}{8} \{\|P\|^2 - 2\alpha(\pi)\} + \frac{1}{4} (c+3-4\kappa) \beta(\pi) - (\mu - 1) \gamma(\pi)
\]
\[T \frac{1}{2} \{2\text{trace}(h|\pi) + \text{det}(h|\pi) - \text{det}((\varphi h)|\pi)) + (\mu + n - 2) \text{trace}(h^T)
\]
\[+ \frac{1}{4} \left\{\|\varphi h^T\|^2 - \|h^T\|^2 - \left(\text{trace}((\varphi h)^T)\right)^2 + \left(\text{trace}(h^T)\right)^2\right\}.\] (5)

The equality in (5) holds at \(p \in M\) if and only if there exist an orthonormal basis \(\{e_1, \ldots, e_n\}\) of \(T_pM\) and an orthonormal basis \(\{e_{n+1}, \ldots, e_{2m+1}\}\) of \(T_p^\perp M\) such that (a) \(\pi = \text{Span}\{e_1, e_2\}\) and (b) the forms of shape operators \(A_r \equiv A_{e_r},\)
\[r = n + 1, \ldots, 2m + 1,\] become
\[
A_{n+1} = \begin{pmatrix}
a & 0 & 0 \\
0 & b & 0 \\
0 & (a + b) I_{n-2}
\end{pmatrix}, \quad (6)
\]
\[
A_r = \begin{pmatrix}
c_r & d_r & 0 \\
d_r & -c_r & 0 \\
0 & 0 & 0_{n-2}
\end{pmatrix}, \quad r = n + 2, \ldots, 2m + 1. \quad (7)
\]

**Proof.** Let \(\pi \subset T_pM\) be a plane section. Choose an orthonormal basis \(\{e_1, e_2, \ldots, e_n\}\) for \(T_pM\) and \(\{e_{n+1}, \ldots, e_{2m+1}\}\) for the normal space \(T_p^\perp M\) at \(p\) such that \(\pi = \text{Span}\{e_1, e_2\}\) and the mean curvature vector \(H\) is in the direction of the normal vector to \(e_{n+1}.\) We rewrite (4) as
\[
\frac{1}{n-1} \left(\sum_{i=1}^{n} \sigma_{ii}^n\right)^2 = \sum_{i=1}^{n} (\sigma_{ii}^{n+1})^2 + \sum_{i \neq j} (\sigma_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n} \sigma_{ij}^r = \rho, \quad (8)
\]
where
\[
\rho = 2\tau - \frac{n^2(n-2)}{n-1} \|H\|^2 - 2(\mu + n - 2) \text{trace}(h^T)
\]
\[T \frac{1}{2} \left\{2(\mu + n - 2) \text{trace}(h^T) + \frac{3}{8} \{\|P\|^2 - 2\alpha(\pi)\} + \frac{1}{4} (c+3-4\kappa) \beta(\pi) - (\mu - 1) \gamma(\pi)
\]
\[+ \frac{1}{4} \left\{\|\varphi h^T\|^2 - \|h^T\|^2 - \left(\text{trace}((\varphi h)^T)\right)^2 + \left(\text{trace}(h^T)\right)^2\right\}.\] (9)

and \(\sigma_{ij}^r = \langle \sigma(e_i, e_j), e_r\rangle,\) \(i, j \in \{1, \ldots, n\}; \ r \in \{n + 1, \ldots, 2m + 1\}.\) Now, applying Lemma 3.1 to (8), we obtain
\[
2\sigma_{11}^{n+1} \sigma_{22}^{n+1} \geq \rho + \sum_{i \neq j} (\sigma_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n} (\sigma_{ij}^r)^2. \quad (10)
\]
From equation (2) and (3) it also follows that

\[
K(\pi) = \frac{1}{4} \left\{ 3 + c + 3 (c - 1) \alpha(\pi) - (c + 3 - 4\kappa) \beta(\pi) + 4 (\mu - 1) \gamma(\pi) \right\}
\]
\[
+ \frac{1}{2} \left\{ 2 \text{trace} (h|_\pi) + \det (h|_\pi) - \det ((\varphi h)|_\pi) \right\}
\]
\[
+ \sigma_{11}^{n+1} \sigma_{22}^{n+1} - \left( \sigma_{12}^{n+1} \right)^2 + \sum_{r=n+2}^{2m+1} \left( \sigma_{11}^r \sigma_{22}^r - \left( \sigma_{12}^r \right)^2 \right).
\]

(11)

Thus, from (10) and (11) we have

\[
K(\pi) \geq \frac{1}{4} \left\{ 3 + c + 3 (c - 1) \alpha(\pi) - (c + 3 - 4\kappa) \beta(\pi) + 4 (\mu - 1) \gamma(\pi) \right\}
\]
\[
+ \frac{1}{2} \left\{ 2 \text{trace} (h|_\pi) + \det (h|_\pi) - \det ((\varphi h)|_\pi) \right\} + \frac{1}{2} \rho
\]
\[
+ \frac{1}{2} \sum_{r=n+1}^{2m+1} \sum_{i,j>2} \left((\sigma_{ij}^r)^2 + (\sigma_{2j}^r)^2\right) + \frac{1}{2} \sum_{i,j>2} \left(\sigma_{ij}^{n+1}\right)^2
\]
\[
+ \frac{1}{2} \sum_{r=n+2}^{2m+1} \sum_{i,j>2} (\sigma_{ij}^r)^2 + \frac{1}{2} \sum_{r=n+2}^{2m+1} (\sigma_{11}^r + \sigma_{22}^r)^2.
\]

(12)

In view of (8) and (12), we get (5).

If the equality in (5) holds, then the inequalities given by (10) and (12) become equalities. In this case, we have

\[
\sigma_{ij}^{n+1} = 0, \quad \sigma_{2j}^{n+1} = 0, \quad \sigma_{ij}^{n+1} = 0, \quad i \neq j > 2;
\]
\[
\sigma_{ij}^r = \sigma_{2j}^r = \sigma_{ij}^r = 0, \quad r = n + 2, \ldots, 2m + 1; \quad i, j = 3, \ldots, n;
\]
\[
\sigma_{11}^{n+2} + \sigma_{22}^{n+2} = \cdots = \sigma_{11}^{2m+1} + \sigma_{22}^{2m+1} = 0.
\]

(13)

Now, we choose $e_1$ and $e_2$ so that $\sigma_{12}^{n+1} = 0$. Applying Lemma 3.1 we also have

\[
\sigma_{11}^{n+1} + \sigma_{22}^{n+1} = \sigma_{33}^{n+1} = \cdots = \sigma_{nn}^{n+1}.
\]

(14)

Thus, choosing a suitable orthonormal basis $\{e_1, \ldots, e_{2m+1}\}$, the shape operator of $M$ becomes of the form given by (11) and (12). The converse is easy to follow.

\[
\nabla_\xi \xi = 0 \quad \text{and} \quad \sigma(\xi, \xi) = 0.
\]

(15)

Next, we prove the following theorem.
Theorem 3.4 Let $M$ be an $n$-dimensional ($n \geq 3$) submanifold isometrically immersed in a $(2m + 1)$-dimensional $(\kappa, \mu)$-contact space form $\tilde{M}(c)$ such that $\xi \in TM$. Then, for each point $p \in M$ and each plane section $\pi \subset D_p$, we have

$$
\tau - K(\pi) \leq \frac{n^2(n-2)}{2(n-1)} ||H||^2 + \frac{1}{8} n(c+3)(n-3) + (n-1)\kappa \\
+ \frac{3(c-1)}{8} \left\{ ||P||^2 - 2\alpha(\pi) \right\} + (\mu + n-2) \text{trace}(h^T) \\
- \frac{1}{2} \left\{ 2\text{trace}(h|_\pi) + \det(h|_\pi) - \det((\varphi h)|_\pi) \right\} \\
+ \frac{1}{4} \left\{ \| (\varphi h)^T \|^2 - \| h^T \|^2 - \left( \text{trace}((\varphi h)^T) \right)^2 + \left( \text{trace}(h^T) \right)^2 \right\} .
$$

(16)

The equality in (16) holds at $p \in M$ if and only if there exist an orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_pM$ and an orthonormal basis $\{e_{n+1}, \ldots, e_{2m+1}\}$ of $T_p^\perp M$ such that (a) $e_n = \xi$, (b) $\pi = \text{Span}\{e_1, e_2\}$ and (c) the forms of shape operators $A_r = A_{\varphi r}$, $r = n + 1, \ldots, 2m + 1$, become $[\tilde{F}]$ and

$$
A_{n+1} = \begin{pmatrix} a & 0 & 0 \\
0 & -a & 0 \\
0 & 0 & 0 
\end{pmatrix} .
$$

(17)

Proof. Let $\pi \subset D_p$ be a plane section at $p \in M$. We choose an orthonormal basis $\{e_1, e_2, \ldots, e_n = \xi\}$ for $T_pM$ and $\{e_{n+1}, \ldots, e_{2m+1}\}$ for the normal space $T_p^\perp M$ at $p$ such that $\pi = \text{Span}\{e_1, e_2\}$ and the mean curvature vector $H(p)$ is parallel to $e_{n+1}$. Using $\eta(e_1) = 0 = \eta(e_2)$, we get $\beta(\pi) = 0 = \gamma(\pi)$. Thus, proof of (16) is similar to that of (10). In equality case, using (15), (14) becomes

$$
\sigma_1^{n+1} + \sigma_2^{n+1} = \sigma_3^{n+1} = \cdots = \sigma_{nn}^{n+1} = 0 ,
$$

(18)

and thus (10) is modified to (17). □

Since in case of non-Sasakian $(\kappa, \mu)$-contact space form, we have $\kappa < 1$, and thus $c = -2\kappa - 1$ and $\mu = \kappa + 1$. Putting these values in (10) and (14), we can have direct corollaries to Theorems 3.3 and 3.4. For example, Corollary to Theorem 3.3 is as follows.

Corollary 3.5 Let $M$ be an $n$-dimensional ($n \geq 3$) submanifold isometrically immersed in a non-Sasakian $(\kappa, \mu)$-contact space form $\tilde{M}(c)$ such that $\xi \in TM$. Then, for each point $p \in M$ and each plane section $\pi \subset T_pM$, we have

$$
\tau - K(\pi) \leq \frac{n^2(n-2)}{2(n-1)} ||H||^2 - \frac{1}{4} n(n-3)(\kappa-1) + (n-1)\kappa \\
- \frac{3}{4} (\kappa + 1) ||P||^2 + \frac{1}{2} \left\{ 3(\kappa + 1) \alpha(\pi) - (3\kappa - 1) \beta(\pi) - 2\kappa\gamma(\pi) \right\} .
$$

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\[-\frac{1}{2} \{2 \text{trace}(h) + \det(h) - \det((\phi h))\} + (\kappa + n - 1) \text{trace}(h^T) + \frac{1}{4} \left\{ \left\| (\phi h)^T \right\|^2 - \left\| h^T \right\|^2 - \left( \text{trace}\left((\phi h)^T\right) \right)^2 + \left( \text{trace}(h^T) \right)^2 \right\}. \quad (19)\]

The equality in (19) holds at \( p \in M \) if and only if there exist an orthonormal basis \( \{e_1, \ldots, e_n\} \) of \( T_p M \) and an orthonormal basis \( \{e_{n+1}, \ldots, e_{2m+1}\} \) of \( T_p^\perp M \) such that (a) \( \pi = \text{Span}\{e_1, e_2\} \) and (b) the shape operators \( A_r \equiv A_{e_r}, r = n + 1, \ldots, 2m + 1 \) are of forms given by (6) and (7).

If \( \kappa = 1 \), the \( (\kappa, \mu) \)-contact space form reduces to Sasakian space form \( \tilde{M}(c) \); thus \( h = 0 \) and (2) becomes the equation in Theorem 7.14 of [1]. Therefore, Theorem 3.3 and Theorem 3.4 provide Sasakian versions as Theorem 3.2 of [6] and [2] respectively. Now, we recall Chen’s \( \delta \)-invariant given by

\[\delta_M(p) = \tau(p) - (\inf K)(p) = \tau(p) - \inf\{K(\pi) \mid \pi \text{ is a plane section } \subset T_p M\},\]

which is certainly an intrinsic character of \( M \). Improving Theorem 4.1 of [6], we have the following [8]

Theorem 3.6 Let \( M \) be an \( n \)-dimensional Riemannian manifold isometrically immersed in a Sasakian space form \( \tilde{M}(c) \) of constant \( \varphi \)-sectional curvature \( c < 1 \) with the structure vector field \( \xi \) tangent to \( M \). \( M \) satisfies Chen’s basic equality

\[\delta_M = \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + \frac{1}{8} \left\{ n(n-3)c + 3n^2 - n - 8 \right\}, \quad (20)\]

if and only if \( M \) is a 3-dimensional minimal invariant submanifold. Hence, Chen’s invariant becomes \( \delta_M = 2 \).

4 Invariant submanifolds

A submanifold \( M \) of an almost contact metric manifold \( \tilde{M} \) with the structure \( (\varphi, \xi, \eta, (\cdot, \cdot)) \) is called an invariant submanifold if \( \varphi T_p M \subset T_p M \). If \( \tilde{M} \) is contact also, then \( \xi \in TM, \sigma(X, \xi) = 0 \) and \( M \) is minimal ([3]). On the other hand, we have the following

Proposition 4.1 Every totally umbilical submanifold \( M \) of a contact metric manifold such that \( \xi \in TM \) is minimal and consequently totally geodesic.

The proof follows from \( H = (\xi, \xi)H = \sigma(\xi, \xi) = 0 \), where (15) is used. Choosing an orthonormal basis \( \{e_i, \varphi e_i, \xi\}, i = 1, \ldots, \frac{n-1}{2} \), we also can prove

Proposition 4.2 In an \( n \)-dimensional invariant submanifold of a contact metric manifold, we have

\[\|P\|^2 = n - 1, \text{trace}(h^T) = \text{trace}\left((\varphi h)^T\right) = 0, \left\| (\varphi h)^T \right\|^2 = \left\| h^T \right\|^2. \quad (21)\]
Thus, for an $n$-dimensional invariant submanifold in a $(\kappa, \mu)$-contact space form $\tilde{M}(c)$, the scalar curvature and the second fundamental form satisfy

$$2\tau = \frac{1}{4} (n-1) \{(n+1) c + 8\kappa + 3n - 9\} - \|\sigma\|^2.$$  \hfill (22)

In view of the above equation, we can state the following theorem.

**Theorem 4.3** For an $n$-dimensional invariant submanifold isometrically immersed in a $(\kappa, \mu)$-contact space form $\tilde{M}(c)$, we get

$$\tau \leq \frac{1}{8} (n-1) \{(n+1) c + 8\kappa + 3n - 9\}$$ \hfill (23)

with equality if and only if the invariant submanifold is totally umbilical, where $c = -2\kappa - 1$ if $\kappa < 1$.

For each point $p \in M$, we put

$$\delta^p_M(p) = \tau(p) - \inf \{K(\pi) : \text{plane sections } \pi \subset D_p\}.$$ \hfill (24)

Now, we conclude the paper with the following theorem.

**Theorem 4.4** For an $n$-dimensional invariant submanifold isometrically immersed in a $(\kappa, \mu)$-contact space form $M(c)$, we get

$$\delta^p_M \leq \frac{n-3}{8} \{(n+3) c + 3(n-1)\} + (n-1) \kappa.$$ \hfill (25)

**Proof.** Let $\pi \subset D_p$ be a plane section at $p \in M$. We can choose unit vectors $e$ and $Pe$ such that $\pi = \text{Span} \{e, Pe\}$. Thus, we get $\alpha(\pi) = 1$, $\text{trace}(h|_\pi) = 0$ and $\det(h|_\pi) = \det((\varphi h)|_\pi)$. Using these facts along with (21) in (16), we get (25). \hfill $\square$

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