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Characterising Sobolev inequalities by controlled coarse homology and applications for hyperbolic spaces

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Abstract. We give a Sobolev inequality characterisation for the vanishing of a fundamental class in the controlled coarse homology of Nowak and Špakula for quasiconvex uniform spaces that support a local weak \((1,1)\)-Poincaré inequality. As applications, we consider visual Gromov hyperbolic spaces and Carnot groups.

Mathematics subject classification (2000): Primary 53C23, 30L99; Secondary 58J32

Key words: Controlled coarse homology, Sobolev inequalities.

1 Introduction

In this article, a metric measure space \((X,d,\mu)\) is a metric space \((X,d)\) with Borel regular outer measure \(\mu\) such that \(\mu(X) > 0\) and \(\mu(B(x,r)) < \infty\) for every \(x \in X\) and \(r > 0\). In what follows, we call a function \(\varrho: [0,\infty) \rightarrow [0,\infty)\) a control function if it is non-decreasing, \(\varrho(0) = 1\), and satisfies the conditions

\[
\varrho(\varepsilon + t) \leq L(\varepsilon) \varrho(t) \quad (\varrho_1)
\]

and

\[
\varrho(\varepsilon t) \leq M(\varepsilon) \varrho(t), \quad (\varrho_2)
\]

for some functions \(L, M : (0,\infty) \rightarrow (0,\infty)\) whenever \(t, \varepsilon > 0\). The space \((X,d,\mu)\) satisfies the global \(\varrho\)-weighted \((1,1)\)-Sobolev inequality \((S^{\varrho}_{1,1})\) if for some control function \(\varrho\) there exists \(C > 0\) and \(o \in X\) such that

\[
\int_X |u|d\mu \leq C \int_X |\nabla u| \varrho(d(o,\cdot))d\mu
\]

for every \(u \in N^{1,1}(X,d,\mu)\) with bounded support. Here, \(N^{1,1}(X,d,\mu)\) is the Newton-Sobolev space of equivalence classes of integrable functions \(u: X \rightarrow [-\infty,\infty]\) with integrable upper gradient, and \(|\nabla u|: X \rightarrow [0,\infty]\) the 1-weak minimal upper gradient of \(u\); see [10] Section 7.1. If \((S^{\varrho}_{1,1})\) holds for \(\varrho \equiv 1\) we say that \((X,d,\mu)\) satisfies \((S_{1,1})\).

Given \((X,d,\mu)\) what is the relationship between the \(\varrho\)-isoperimetry of \((X,d)\) and \((S^{\varrho}_{1,1})\)? Previously, \(\varrho\)-isoperimetry has been studied in [7] [10] [13], and Sobolev inequalities including \((S_{1,1})\) for Riemannian manifolds in [4] [5] [6] and...
for metric measure spaces in [8][17]. Our main result is a metric measure version of a result of Nowak and Špakula [16] Theorem 4.2, in part saying the following.

**Theorem A.** Let \((X, d, \mu)\) be a quasiconvex uniform space supporting a local weak \((1, 1)\)-Poincaré inequality. Then \((X, d, \mu)\) satisfies \(\left(S^0_{1,1}\right)\) for a given control function \(g\) if and only if \(0 = [\Gamma] \in H^0_0(\Gamma)\) for any quasi-lattice \(\Gamma \subseteq X\).

For the proof of Theorem A as well as several other equivalent statements; see Theorem 12. For the terminology related to controlled coarse homology \(H^0_0\) and its relationship with \(\varrho\)-isoperimetry, we refer to Section 2. A quasi-lattice in \((X, d)\) is any \((C-)\)cobounded set \(\Gamma \subseteq X\), that is \(N_C(\Gamma) := \{x \in X : d(x, \Gamma) < C\} = X\) for some \(C > 0\), which is uniformly locally finite in the sense that there exists a function \(N : (0, \infty) \rightarrow \mathbb{N}\) for which the cardinality \(#(\Gamma \cap B(x, r))\) \(\leq N(r)\) for every open ball \(B(x, r) \subseteq X\). The space \((X, d)\) is \((Q-)\)quasiconvex if there exists \(Q \geq 1\) such that for any \(x, y \in X\) there is a rectifiable path \(\gamma\) from \(x\) to \(y\) of length \(\ell(\gamma) \leq Qd(x, y)\). A Borel regular outer measure \(\mu\) is uniform if there exist non-decreasing functions \(f, g : (0, \infty) \rightarrow (0, \infty)\) such that \(f(r) \leq \mu(B(x, r)) \leq g(r)\) for all \(0 < r < \infty\) and \(B(x, r) \subseteq X\). We say that \((X, d, \mu)\) is uniform if \(\mu\) is uniform. Examples of uniform spaces are locally Ahlfors regular spaces and second-countable locally compact compactly generated groups with respect to a left-invariant metric and Haar measure; see [3] Proposition 4.B.9.

A space \((X, d, \mu)\) supports a local weak \((1, p)\)-Poincaré inequality (up to scale \(R_p\)) for \(1 \leq p < \infty\) if there exist \(C_p, R_p > 0\) and \(\tau \geq 1\) such that for all \(B(x, r) \subseteq X\) with \(0 < r \leq R_p, 0 < \mu(B(x, \tau r)) < \infty\) and

\[
\frac{1}{r} \int_{B(x, r)} |u - u_{B(x, r)}|d\mu \leq C_p r \left( \int_{B(x, \tau r)} g_u^p d\mu \right)^{1/p}
\]

for \(u : X \rightarrow \mathbb{R}\) such that \(u \in L^1(B(x, \tau r), d, \mu)\) and its minimal \(p\)-weak upper gradient of \(g_u : X \rightarrow [0, \infty]\); this is the local version of [10] Proposition 8.1.3. Here as usual,

\[
f_A = \int_A f d\mu = \frac{1}{\mu(A)} \int_A f d\mu
\]

assuming that \(A \subseteq X\) is a \(\mu\)-measurable set \(0 < \mu(A) < \infty\) and \(f : A \rightarrow [-\infty, \infty]\) is integrable over \(A\).

Contained in the proof of Theorem 12 is also the following partial result that does not rely on a local weak \((1, 1)\)-Poincaré inequality.

**Theorem B.** Let \((X, d, \mu)\) be quasiconvex uniform metric measure space satisfying \(\left(S^0_{1,1}\right)\). Then \([\Gamma] = 0\) in \(H^0_0(\Gamma)\) for any quasi-lattice \(\Gamma \subseteq X\).

We now list some immediate applications motivating Theorem A.

**Corollary C.** Let \((X, d, \mu)\) and \((X', d', \mu')\) be quasiconvex uniform spaces that support a local weak \((1, 1)\)-Poincaré inequality. If \((X, d)\) and \((X', d')\) are quasi-isometric, then \((X, d, \mu)\) satisfies \(\left(S^0_{1,1}\right)\) if and only if \((X', d', \mu')\) satisfies \(\left(S^0_{1,1}\right)\).

Recall that \((X, d)\) and \((X', d')\) are quasi-isometric if there exists \(f : X \rightarrow X'\) and constants \(\lambda \geq 1\) and \(\mu \geq 0\) such that

\[
\lambda^{-1}d(x, x') - \mu \leq d'(f(x), f(x')) \leq \lambda d(x, x') + \mu
\]

2
for all \(x, x' \in X\), and \(f(X) \subseteq X'\) is \(\mu\)-cobounded. Corollary C now follows from Theorem A as controlled coarse homology is a quasi-isometry invariant; see [16, Corollary 2.3]. A metric space \((X,d)\) with a quasi-lattice \(\Gamma \subseteq X\) is amenable if for any \(\varepsilon, r > 0\) there exists a non-empty finite \(F \subseteq \Gamma\) such that

\[
\frac{\# \partial_r F}{\# F} < \varepsilon,
\]

where \(\partial_r F = \{x \in \Gamma: d(x, \Gamma) < r \text{ and } d(x, \Gamma \setminus F) < r\}\). If \((X,d)\) is not amenable, we say that it is non-amenable. As observed by Block and Weinberger, a space \((X,d)\) with a quasi-lattice \(\Gamma \subseteq X\) is non-amenable if and only if \(0 = [\Gamma] \in H^0_1(\Gamma)\) where \(H^0_1(\Gamma)\) denotes 0-dimensional controlled coarse homology group for \(\varrho \equiv 1\); see [1, Proposition 2.3, Theorem 3.1] as well as [16]. With this in mind, we give the following characterisation.

**Corollary D.** Let \((X,d,\mu)\) be a quasiconvex uniform space that supports a local weak \((1,1)\)-Poincaré inequality. Then \((X,d)\) is non-amenable if and only if \((X,d,\mu)\) satisfies \((S_{1,1})\).

Corollary D follows directly from Theorem A and the characterisation [1, Theorem 3.1]. Note the similarity between Corollary D and [6, Theorem 7.1]; see also [17, Example 5.8].

**Theorem E.** Let \((X,d,\mu)\) be a quasiconvex uniform visual Gromov hyperbolic space defined using the Gromov product. If \((X,d,\mu)\) supports a local weak \((1,1)\)-Poincaré inequality and its Gromov boundary \(\partial X\) is connected and contains at least two points, \((X,d,\mu)\) satisfies \((S_{1,1})\).

**Proof.** Since \((X,d,\mu)\) is uniform visual and Gromov hyperbolic with connected boundary containing at least two points, \(0 = [\Gamma] \in H^0_1(\Gamma)\) for any quasi-lattice \(\Gamma \subseteq X\); see [15]. The claim now follows from Theorem A.

We give a further application of Corollary E to the Dirichlet problem at infinity that generalises a result of Cao [2, Corollary 1.1]; see also [12].

**Theorem F.** Suppose \((X,d,\mu)\) is a locally compact quasiconvex visual Gromov hyperbolic metric measure space defined using the Gromov product having uniform measure that supports a local weak \((1,1)\)-Poincaré inequality. Suppose its Gromov boundary \(\partial X\) is connected and contains at least two points. Then, if \(f: \partial X \to \mathbb{R}\) is a bounded continuous function, there exists a continuous function \(u: X^* \to \mathbb{R}\) on the Gromov closure \(X^*\) of \(X\) that is \(p\)-harmonic for \(p > 1\) in \(X\) and \(u|\partial X = f\).

**Proof.** By Theorem E, the space \((X,d,\mu)\) satisfies \((S_{1,1})\) and hence the corresponding \((p,p)\)-Sobolev inequality for \(1 \leq p < \infty\); see [12, Example 8]. By Hölder’s inequality, \((X,d,\mu)\) supports a local weak \((1,p)\)-inequality for \(1 \leq p < \infty\) as well. Thus, \((X,d,\mu)\) satisfies all the assumptions of [12, Theorem 1.1] (see Lemma [11] and the claim follows.

We finish with an example illustrating the case when \(\varrho \not\equiv 1\). Write \(f \preceq g\) for non-decreasing functions \(f, g: [0,\infty) \to [0,\infty)\) for which there exist constants \(\lambda, \mu > 0\) and \(c \geq 0\) such that \(f(r) \leq \lambda g(\mu r + c)\) for all \(r \geq 0\). Also, write \(f \asymp g\) if \(f \preceq g\) but \(g \not\preceq f\).
Example G. The first real Heisenberg group $(H_1(\mathbb{R}), d_H, \mu)$ with Heisenberg metric satisfies $(S^q_{1,1})$ for $\varrho(t) = t + 1$ but not $(S^q_{1,1})$ for any other control function $\xi(t) \sim t + 1$.

Proof. As the first integer Heisenberg group $H_1(\mathbb{Z}) \leq H_1(\mathbb{R})$ is a uniform lattice, there exists a quasi-isometry

$$f : (H_1(\mathbb{Z}), d_S) \to (H_1(\mathbb{R}), d_H)$$

where $d_S$ is the word metric; see [1] Definition 4.B.1 and [3] Proposition 5.C.3. In particular, $H^q_0(H_1(\mathbb{Z})) \cong H^q_0(H_1(\mathbb{R}))$ are isomorphic. As the group $H_1(\mathbb{Z})$ is infinite polycyclic, $0 = [H_1(\mathbb{Z})] \in H^q_0(H_1(\mathbb{Z}))$ if and only if $\varrho(t) = t + 1$; see [16] Corollary 5.5. In particular, $0 \neq [H_1(\mathbb{Z})] \in H^q_0(H_1(\mathbb{Z}))$ for $\xi(t) \sim t + 1$. The claim now follows from Theorem A.

Similar arguments hold for Carnot groups; again Theorem A gives a homological way to deduce $(S^q_{1,1})$ from algebraic growth data.

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2 Tools of controlled coarse homology

We first recall some terminology; see [16] for details. A metric space $(X, d)$ is uniformly coarsely proper if it has a quasi-lattice $\Gamma \subseteq X$.

Remark 1. A metric space $(X, d)$ is uniformly coarsely proper if and only if there exists $r_0 > 0$ and $N : (r_0, \infty) \times (r_0, \infty) \to \mathbb{N}$ such that, for all $R > r > r_0$, any open ball of radius $R$ in $X$ can be covered by $N(R, r)$ open balls of radius $r$ in $X$; see [3] Section 3.

A pointed uniformly coarsely proper space $(X, d, o)$ always has a quasi-lattice $\Gamma \ni o$. For $q \in \mathbb{N}$, we denote by $(X^{q+1}, d, o)$ the corresponding pointed $(q + 1)$-Cartesian product with basepoint $\bar{o} = (o, \ldots, o)$ and metric

$$d(\bar{x}, \bar{y}) = \max_{0 \leq i \leq q} d(x_i, y_i)$$

where $\bar{x} = (x_0, \ldots, x_q) \in X^{q+1}$ and $\bar{y} = (y_0, \ldots, y_q) \in X^{q+1}$. For a quasi-lattice $\Gamma \ni o$ and a control function $\varrho$, we denote by $C^\varrho_q(\Gamma)$ the space of functions $c : \Gamma^{q+1} \to \mathbb{R}$ for which

(a) there exists a constant $K(c) \geq 0$, which may depend on $c$, such that $|c(\bar{x})| \leq K(c)\varrho(d(\bar{x}, \bar{o}))$ for all $\bar{x} \in \Gamma^{q+1}$;
(b) \( c \) is alternating, that is \( c(x_{\sigma(0)}, \ldots, x_{\sigma(q)}) = \text{sign}(\sigma)c(x_0, \ldots, x_q) \) for all \( (x_0, \ldots, x_q) \in \Gamma^{q+1} \) and all permutations \( \sigma: \{0, \ldots, q\} \to \{0, \ldots, q\} \);

(c) there exists a constant \( P(c) \geq 0 \), which may depend on \( c \), such that \( c(x_0, \ldots, x_q) = 0 \) if \( \max_{i \neq j} d(x_i, x_j) > P(c) \).

Note that \( C_0^q(\Gamma) \) is an \( \mathbb{R} \)-module that does not depend on the choice of basepoint by \((21)\). A function \( c \in C_0^q(\Gamma) \) is called a **controlled coarse \( q \)-chain** and we write

\[
c = \sum_{(x_0, \ldots, x_q) \in \Gamma^{q+1}} c(x_0, \ldots, x_q) [x_0, \ldots, x_q]
\]

where the abstract \( q \)-cell \([x_0, \ldots, x_q] \in C_0^q(\Gamma)\) is the characteristic function \( \chi_{(x_0, \ldots, x_q)}: \Gamma^{q+1} \to \mathbb{R} \) of the point \((x_0, \ldots, x_q)\). The **controlled coarse homology** \( H_0^q(\Gamma) \) is the homology of the chain complex

\[
\cdots \xrightarrow{\partial_3} C_3^q(\Gamma) \xrightarrow{\partial_2} C_2^q(\Gamma) \xrightarrow{\partial_1} C_1^q(\Gamma) \xrightarrow{\partial_0} 0
\]

where the boundary homomorphism \( \partial_q: C_q^q(\Gamma) \to C_{q-1}^q(\Gamma) \) is given by

\[
\partial_q([x_0, \ldots, x_q]) = \sum_{i=0}^q (-1)^i[x_0, \ldots, \hat{x}_i, \ldots, x_q]
\]

for each abstract \( q \)-cell \([x_0, \ldots, x_q]\) and extended linearly to \( C^q(\Gamma) \) for \( q \in \mathbb{N} \setminus \{0\}\); as usual, \( [x_0, \ldots, \hat{x}_i, \ldots, x_q] \) denotes the abstract \( q \)-cell obtained from \([x_0, \ldots, x_q]\) by omitting its \( i \)th coordinate. In particular, \( \partial_{q-1} \circ \partial_q = 0 \) and \( \partial_q c \in C_{q-1}^q(\Gamma) \) by \((21)\). The \( q \)-dimensional controlled coarse homology group is explicitly

\[
H_q^q(\Gamma) = \ker \partial_q / \text{im} \partial_{q+1}.
\]

A special role is played by the homology class \([\Gamma] \in H_0^0(\Gamma)\) of the characteristic function

\[
\chi_\Gamma = \sum_{x \in \Gamma} [x] \in C_0^0(\Gamma),
\]

called the **fundamental class**. Its vanishing characterises the \( q \)-isoperimetry of the space. In what follows we use the notation \([x, y] := d(\bar{o}, (x, y))\).

**Theorem [16, Lemma 4.1, Theorem 4.2]**. For a quasi-lattice \( \Gamma \ni o \), assume that there exists \( C \in (0, 1) \) such that \( d(x, y) \geq C \) whenever \( x, y \in \Gamma \) are distinct, and that for all \( x, y \in \Gamma \) there is a sequence \((x = x_0, \ldots, x_n = y)\) in \( \Gamma \) such that \( n \leq d(x, y) \) and \( d(x_i, x_{i+1}) \leq 1 \) for every \( 0 \leq i \leq n-1 \). Then, the following are equivalent:

1. \( 0 = [\Gamma] \in H_0^0(\Gamma) \),
2. there exists \( C' > 0 \) such that for every finitely supported \( \eta: \Gamma \to \mathbb{R} \)

\[
\sum_{x \in \Gamma} |\eta(x)| \leq C' \left( \sum_{x \in \Gamma} \sum_{y \in B(x,1)} |\eta(x) - \eta(y)| \varphi(\| (x, y) \|) \right)
\]

where \( B(x, 1) = \{ y \in \Gamma: d(x, y) \leq 1 \} \),
(3) there exists $C'' > 0$ such that for all finite $F \subseteq \Gamma$

$$
\# F \leq C'' \sum_{x \in \partial F} \varrho(d(o, x)),
$$

where $\partial F = \{x \in \Gamma: d(x, F) = 1 \text{ or } d(x, \Gamma \setminus F) = 1\}$.

**Lemma 16.** Assume $\Gamma \subseteq X$ is a quasi-lattice for which there exists $c = \sum_{x \in F} c(x)[x] \in C^0_0(\Gamma)$ such that $\inf_{x \in \Gamma} c(x) > 0$ and $[c] = 0$ in $H^0_0(\Gamma)$. Then $0 = [\Gamma] \in H^0_0(\Gamma)$.

This leads us to the following observation which shows that if $[\Gamma] = 0$ for some quasi-lattice $\Gamma \subseteq X$ then $[\Gamma'] = 0$ for every quasi-lattice $\Gamma' \subseteq X$.

**Lemma 2.** Let $f: \Gamma \to \Gamma'$ be a quasi-isometry between quasi-lattices. Then, $[\Gamma] = 0$ in $H^0_0(\Gamma)$ if and only if $[\Gamma'] = 0$ in $H^0_0(\Gamma')$.

**Proof.** The quasi-isometry $f: \Gamma \to \Gamma'$ induces a chain map $f_q: C^0_q(\Gamma) \to C^0_q(\Gamma')$ extending the map $[x_0, \ldots, x_q] \mapsto [f(x_0), \ldots, f(x_q)]$ linearly to $C^0_q(\Gamma)$. By (3) and (2), $f_q$ is well-defined. In particular

$$f_0 \left( \sum_{x \in \Gamma} [x] \right) = \sum_{x \in \Gamma} [f(x)] = \sum_{y \in f(\Gamma)} c(y)[y] = c' \in C^0_0(\Gamma')$$

where $c(y) = \# f^{-1}(y) \geq 1$ for $y \in f(\Gamma)$. Since $f(\Gamma) \subseteq \Gamma'$ is a quasi-lattice and $0 = [\Gamma]$ implies that $0 = [c'] \in H^0_0(\Gamma')$ there exists for every $y \in f(\Gamma)$ a controlled coarse 1-chain

$$t_y = \sum_{i=0}^{\infty} [x_i, x_{i+1}] \in C^1_f(f(\Gamma))$$

where $x_0 = y$ so that

$$t = \sum_{y \in f(\Gamma)} t_y \in C^1_f(f(\Gamma))$$

by the proof of (3); see also [1] Lemma 2.4. By coboundedness, fix $C > 0$ such that $N_C(f(\Gamma)) = \Gamma'$. To begin, let $y_1 \in f(\Gamma)$ and let

$$t_{w, y_1} = [w, y_1] + t_y \in C^1_f(\Gamma')$$

for each $w \in B(y_1, C) \setminus \{y_1\}$. Since $\Gamma'$ is uniformly locally finite, there is at most $\# (B(y_1, C) \cap \Gamma') \leq N(C)$ chains $t_{w, y_1}$. Next, let $y_2 \in f(\Gamma) \setminus \{y_1\}$ and let

$$t_{w, y_2} = [s, y_2] + t_{y_2} \in C^1_f(\Gamma')$$

for each $w \in (B(y_2, C) \setminus \{y_2\}) \setminus B(y_1, C)$. Again, there is at most $N(C)$ chains $t_{w, y_2}$. Continuing in the obvious way, we obtain a controlled coarse 1-chain

$$t' = \sum_{i=1}^{\infty} t_{w, y_i} + \sum_{y \in f(\Gamma)} t_y \in C^1_f(\Gamma')$$

whose boundary is $\partial t' = \sum_{y \in f(\Gamma)} [y]$. In other words, $0 = [\Gamma'] \in H^0_0(\Gamma')$ as claimed.

\[ \square \]
3 Uniform metric measure spaces, discretisation, and smoothing

A metric measure space \((X, d, \mu)\) is a \((DV)_{\text{loc}}\) space if it has the \((DV)_{\text{loc}}\) property saying that there exists a function \(C: (0, \infty) \to (0, \infty)\) such that

\[
0 < \mu(B(x, 2r)) \leq C(r) \mu(B(x, r)) < \infty
\]

for all \(B(x, r) \subseteq X\); see \[6\]. This implies that the space is separable; see \[10\] Lemma 3.3.30. Examples of \((DV)_{\text{loc}}\) spaces are locally compact groups acting by measure preserving isometries on metric measure spaces \[17\] Example 5.4, and uniform spaces with \(C\) saying that there exists a function \(\kappa\) in \((DV)_{\text{loc}}\) for all \(u \in N\).

### 3.1 Discretisation and smoothing: from discrete to smooth

A maximal \(\varepsilon\)-net in \((X, d)\) is a \(\varepsilon\)-cobounded subset \(N(X, \varepsilon) \subseteq X\) such that \(d(x, y) \geq \varepsilon\) whenever \(x, y \in N(X, \varepsilon)\) are distinct. We also write \(q \sim p\) saying that \(q\) is a neighbour of \(q\) if \(p, q \in N(X, \varepsilon)\) and \(0 < d(p, q) \leq 3\varepsilon\). By Zorn’s lemma, for any \(\varepsilon > 0\) and \(o \in X \neq \emptyset\) there exists a maximal \(\varepsilon\) net \(N(X, \varepsilon) \ni o\).

Adapting the argument for doubling spaces in \[10\] Section 4.1, we record the following fact.

**Remark 3.** A \((DV)_{\text{loc}}\) space \((X, d, \mu)\) is uniformly coarsely proper as a metric space. In particular any \(N(X, \varepsilon)\) is a quasi-lattice.

**Lemma 4.** Let \((X, d, \mu)\) be an unbounded quasiconvex \((DV)_{\text{loc}}\) space that supports a local weak \((1, 1)\)-Poincaré inequality up to scale \(R_{\mu}\) Then, given \(0 < \varepsilon \leq R_{\mu}/4\), a quasi-lattice \(N(X, \varepsilon) \ni o\), where \(\mu\{\{o\}\} = 0\), and a control function \(\varrho: [0, \infty) \to [0, \infty)\), there exists \(C > 0\) for which

\[
\sum_{p \in N(X, \varepsilon)} \int_{q \sim p} |u_{B(p, 4\varepsilon)} - u_{B(q, 4\varepsilon)}|q(d(o, p))\mu(B(p, \varepsilon)) \leq C \int_X |\nabla u(x)|q(d(o, x))d\mu(x)
\]

for every \(u \in N^{1,1}(X, d, \mu)\).

This lemma is well-known for complete Riemannian manifolds of bounded geometry when \(\varrho \equiv 1\) \[11\] Lemma 33; see also \[13\]. Here the point to note is that using inequality \[23\] the classic result can additionally be weighted by the control function \(\varrho\) which connects it to controlled coarse homology.

**Proof of Lemma 4.** Let \(p \in N(X, \varepsilon)\) and \(x \in B(p, 8\varepsilon)\) where \(\tau \geq 1\). Now \(d(o, p) \leq d(o, x) + d(x, p) \leq d(o, x) + 8\varepsilon\), and since \(\varrho\) is non-decreasing

\[
(1) \quad \varrho(d(o, p)) \int_{B(p, 8\varepsilon)} |\nabla u(x)|d\mu(x) \leq \int_{B(p, 8\varepsilon)} |\nabla u(x)|\varrho(d(o, x) + 8\varepsilon)d\mu(x)
\]

\[
= \int_{B(p, 8\varepsilon)} \{\nabla u(x) |\varrho(d(o, x) + 8\varepsilon)d\mu(x)
\]

\[
\leq L(8\varepsilon) \int_{B(p, 8\varepsilon)} |\nabla u(x)|\varrho(d(o, x))d\mu(x),
\]

by \[23\]. The proposition follows from estimating (1) from below using the local weak \((1, 1)\)-Poincaré inequality. First, choose a neighbour \(q \sim p\) noting that the
space is quasiconvex and unbounded. Now $B(p, 4\tau \varepsilon) \cup B(q, 4\tau \varepsilon) \subseteq B(p, 8\tau \varepsilon)$ and

$$\int_{B(p, 8\tau \varepsilon)} |\nabla u(x)|d\mu(x) \geq \frac{1}{2} \int_{B(p, 4\tau \varepsilon)} |\nabla u(x)|d\mu(x) + \frac{1}{2} \int_{B(q, 4\tau \varepsilon)} |\nabla u(x)|d\mu(x).$$

By the local weak $(1, 1)$-Poincaré inequality

$$\int_{B(p, 4\tau \varepsilon)} |\nabla u(x)|d\mu(x) \geq \frac{1}{4C} \int_{B(p, 4\varepsilon)} |u(x) - u_{B(p, 4\varepsilon)}|d\mu(x),$$

and since $\mu(B(p, 4\tau \varepsilon)) \geq \mu(B(p, 4\varepsilon))$,

$$\int_{B(p, 4\tau \varepsilon)} |\nabla u(x)|d\mu(x) \geq C \int_{B(p, 4\varepsilon)} |u(x) - u_{B(p, 4\varepsilon)}|d\mu(x)$$

for some $C > 0$. Hence

$$\int_{B(p, 8\tau \varepsilon)} |\nabla u(x)|d\mu(x)$$

$$\geq \frac{1}{2} \int_{B(p, 4\tau \varepsilon)} |\nabla u(x)|d\mu(x) + \frac{1}{2} \int_{B(q, 4\tau \varepsilon)} |\nabla u(x)|d\mu(x)$$

$$\geq \frac{C}{2} \int_{B(p, 4\varepsilon)} |u(x) - u_{B(p, 4\varepsilon)}|d\mu(x) + \frac{C}{2} \int_{B(q, 4\varepsilon)} |u(x) - u_{B(q, 4\varepsilon)}|d\mu(x)$$

$$\geq \frac{C}{2} \int_{B(p, 4\varepsilon) \cap B(q, 4\varepsilon)} (|u(x) - u_{B(p, 4\varepsilon)}| + |u(x) - u_{B(q, 4\varepsilon)}|)d\mu(x)$$

$$\geq \frac{C}{2} |u_{B(p, 4\varepsilon)} - u_{B(q, 4\varepsilon)}| \int_{B(p, \varepsilon)} d\mu(x)$$

$$= \frac{C}{2} |u_{B(p, 4\varepsilon)} - u_{B(q, 4\varepsilon)}| \mu(B(p, \varepsilon)),$$

since $B(p, \varepsilon) \subseteq B(p, 4\varepsilon) \cap B(q, 4\varepsilon)$. Using this to estimate (1) gives

$$\int_{B(p, 8\tau \varepsilon)} |\nabla u(x)|g(d(o, x))d\mu(x) \geq \frac{\rho(d(o, p))}{L(8\tau \varepsilon)} \int_{B(p, 8\tau \varepsilon)} |\nabla u(x)|d\mu(x)$$

$$\geq \frac{C\rho(d(o, p))}{2L(8\tau \varepsilon)} |u_{B(p, 4\varepsilon)} - u_{B(q, 4\varepsilon)}| \mu(B(p, \varepsilon)).$$

Since $N(X, \varepsilon)$ is uniformly locally finite, the number of neighbours $q \sim p$ is uniformly bounded and hence

$$\int_{B(p, 8\tau \varepsilon)} |\nabla u(x)|g(d(o, x))d\mu(x)$$

$$\geq C' \rho(d(o, p)) \sum_{q \sim p} |u_{B(p, 4\varepsilon)} - u_{B(q, 4\varepsilon)}| \mu(B(p, \varepsilon)).$$

for some $C' > 0$ independent of $u$. Similarly, every $x \in X$ belongs to a uniformly bounded number of open balls of radius $8\tau \varepsilon$ having a center in $N(X, \varepsilon)$, and
altogether
\[ \sum_{p \in N(X, \varepsilon)} \sum_{q \sim p} |u_{B(p, 4\varepsilon)} - u_{B(q, 4\varepsilon)}|g(d(o, p))\mu(B(p, \varepsilon)) \]
\[ \leq C' \sum_{p \in N(X, \varepsilon)} \int_{B(p, 7\varepsilon)} |\nabla u(x)|g(d(o, x))d\mu(x) \]
\[ \leq C'' \int_X |\nabla u(x)|g(|x|)d\mu(x) \]
for some \( C'' > 0 \) independent of \( u \), which proves the claim.

We now show that the inequality obtained in Lemma 4 implies \((S^q_{1, 1})\). This time we need both \([71]\) and \([72]\).

**Proposition 5.** Let \((X, d, \mu)\) be a quasiconvex \((DV)_{\text{loc}}\) space that supports a local weak \((1, 1)\)-Poincaré inequality up to scale \(R_p\). Let \(N(X, \varepsilon) \ni o\) be a quasi-lattice, where \(\mu(\{o\}) = 0\) and \(0 < \varepsilon \leq R_p/4\). Suppose there exists a control function \(g: [0, \infty) \rightarrow [0, \infty)\) and a constant \(C > 0\) such that
\[ \sum_{p \in N(X, \varepsilon)} |v(p)|\mu(B(p, \varepsilon)) \leq C \sum_{p \in N(X, \varepsilon)} \sum_{q \sim p} |v(p) - v(q)|g(|(p, q)|)\mu(B(p, \varepsilon)) \]
for every \(v: N(X, \varepsilon) \rightarrow \mathbb{R}\) having finite support. Then \((X, d, \mu)\) satisfies \((S^q_{1, 1})\).

**Proof.** Let \(u: X \rightarrow [0, \infty)\) be a function in \(N^{1, 1}(X, d, \mu)\) having bounded support. Now,
\[ u_{B(.4\varepsilon)}: N(X, \varepsilon) \rightarrow [0, \infty) \]
is finitely supported, and since \(|(p, q)| = d(o, (p, q)) \leq 2d(o, p) + 3\varepsilon\), we have
\[ \sum_{p \in N(X, \varepsilon)} u_{B(p, 4\varepsilon)}\mu(B(p, \varepsilon)) \]
\[ \leq C \sum_{p \in N(X, \varepsilon)} \sum_{q \sim p} |u_{B(p, 4\varepsilon)} - u_{B(q, 4\varepsilon)}|g(|(p, q)|)\mu(B(p, \varepsilon)) \]
\[ \leq C \sum_{p \in N(X, \varepsilon)} \sum_{q \sim p} |u_{B(p, 4\varepsilon)} - u_{B(q, 4\varepsilon)}|g(2d(o, p) + 3\varepsilon)\mu(B(p, \varepsilon)) \]
\[ + C \sum_{q \sim o} |u_{B(o, 4\varepsilon)} - u_{B(q, 4\varepsilon)}|g(3\varepsilon)\mu(B(o, \varepsilon)). \]
In this inequality, the first sum on the right-hand side contains every neighbour of \(o\). To estimate the second sum observe that \(g(3\varepsilon) \leq g(2d(o, p) + 3\varepsilon)\) for every \(p \in N(X, \varepsilon)\), and when \(p \sim o\) we have \(B(o, \varepsilon) \subseteq B(o, 4\varepsilon) \subseteq B(p, 8\varepsilon)\) which gives \(\mu(B(o, \varepsilon)) \leq C(4\varepsilon)C(2\varepsilon)C(\varepsilon)\mu(B(p, \varepsilon))\) using the \((DV)_{\text{loc}}\) property. Put together, this gives the estimate
\[ \sum_{p \in N(X, \varepsilon)} u_{B(p, 4\varepsilon)}\mu(B(p, \varepsilon)) \]
\[ \leq 2CC(4\varepsilon)C(2\varepsilon)C(\varepsilon) \sum_{p \in N(X, \varepsilon) \setminus \{o\}} \sum_{q \sim p} |u_{B(p, 4\varepsilon)} - u_{B(q, 4\varepsilon)}|g(2d(o, p) + 3\varepsilon)\mu(B(p, \varepsilon)). \]
Now, using both (P1) and (P2) this gives
\[
\sum_{p \in \mathcal{N}(X, \varepsilon)} u_{B(p, 4\varepsilon)} \mu(B(p, \varepsilon)) \\
\leq C' \sum_{p \in \mathcal{N}(X, \varepsilon) \setminus \{o\}} \sum_{q \prec p} |u_{B(p, 4\varepsilon)} - u_{B(q, 4\varepsilon)}| \delta(d(o, p)) \mu(B(p, \varepsilon)) \\
\leq C' \sum_{p \in \mathcal{N}(X, \varepsilon) \setminus \{o\}} \sum_{q \prec p} |u_{B(p, 4\varepsilon)} - u_{B(q, 4\varepsilon)}| \delta(d(o, p)) \mu(B(p, \varepsilon)).
\]
for some $C' > 0$ independent of $u$. By Lemma [H]
\[
\sum_{p \in \mathcal{N}(X, \varepsilon) \setminus \{o\}} \sum_{q \prec p} |u_{B(p, 4\varepsilon)} - u_{B(q, 4\varepsilon)}| \delta(d(o, p)) \mu(B(p, \varepsilon)) \leq C' \int_X |\nabla u(x)| \delta(d(o, x)) d\mu(x),
\]
so
\[
\sum_{p \in \mathcal{N}(X, \varepsilon)} u_{B(p, 4\varepsilon)} \mu(B(p, \varepsilon)) \leq C' \int_X |\nabla u(x)| \delta(d(o, x)) d\mu(x)
\]
for some $C'' > 0$ independent of $u$. On the other hand, by the $(DV)_{\text{loc}}$ property
\[
\int_X u(x) d\mu(x) \leq \sum_{p \in \mathcal{N}(X, \varepsilon)} \int_{B(p, 4\varepsilon)} u(x) d\mu(x) = \sum_{p \in \mathcal{N}(X, \varepsilon)} u_{4B(p, 4\varepsilon)} \mu(B(p, 4\varepsilon)) \\
\leq C(2\varepsilon) C(\varepsilon) \sum_{p \in \mathcal{N}(X, \varepsilon)} u_{4B(p, \varepsilon)} \mu(B(p, \varepsilon))
\]
from which the claim follows for $u : X \rightarrow [0, \infty)$ in $N(X, d, \mu)$ having bounded support. The claim for any $u \in N^{1,1}(X, d, \mu)$ having bounded support follows by replacing $u$ with $|u|$ and noticing that $|\nabla|u|| \leq |\nabla u|$. \qed

### 3.2 From smooth to discrete

To begin, recall the notion of Lipschitz partition of unity associated to $N(X, \varepsilon)$ and Lipschitz extensions.

**Definition 6.** [A Section 1.12] A Lipschitz partition of unity associated to $N(X, \varepsilon)$ of a metric space $(X, d)$ is a locally finite family $\{\varphi_p : p \in \mathcal{N}(X, \varepsilon)\}$ of $L$-Lipschitz functions $\varphi_p : X \rightarrow [0, 1]$ such that
\[
\sum_{p \in \mathcal{N}(X, \varepsilon)} \varphi_p(x) = 1
\]
for every $x \in X$ and $\varphi_p|[X \setminus B(p, 2\varepsilon)] \equiv 0$.

The following lemma is a modification of [H] Section 1.12; the proofs are essentially identical.

**Lemma 7.** Let $(X, d)$ be a quasiconvex and uniformly coarse properly proper space and $N(X, \varepsilon)$ a quasi-lattice where $0 < \varepsilon \leq 2$. Then, the family $\{\varphi_p : p \in \mathcal{N}(X, \varepsilon)\}$ where
\[
\varphi_p(x) = \frac{\psi_p(x)}{\psi(x)}
\]
\[ \psi(x) = \min \left\{ 1, \frac{2}{\varepsilon} \text{dist} (x, X \setminus B(p, 3\varepsilon/2)) \right\}, \] and \( \psi(x) = \sum_{p \in N(X, \varepsilon)} \psi_p(x) \), is a Lipschitz partition of unity associated to \( N(X, \varepsilon) \).

**Definition 8.** Let \((X, d)\) be a quasiconvex uniformly coarsely proper space and \(N(X, \varepsilon)\) a quasi-lattice where \(0 < \varepsilon \leq 2\). Given any function \(v: N(X, \varepsilon) \to \mathbb{R}\), its locally Lipschitz extension \(\overline{v}: X \to \mathbb{R}\) associated to \(\{\varphi_p: p \in N(X, \varepsilon)\}\) is defined by

\[ \overline{v}(x) = \sum_{p \in N(X, \varepsilon)} v(p) \varphi_p(x), \]

where \(\{\varphi_p: p \in N(X, \varepsilon)\}\) is the Lipschitz partition of unity associated to \(N(X, \varepsilon)\).

The pointwise upper Lipschitz constant at \(x \in X\) of a function \(v: X \to \mathbb{R}\) from a metric space \((X, d)\) is

\[ \text{Lip} v(x) = \limsup_{r \to 0} \sup_{y \in B(x, r)} \frac{|v(x) - v(y)|}{r}. \]

Note that \(\text{Lip} \overline{v}: X \to [0, \infty]\) is an upper gradient of the locally Lipschitz extension \(\overline{v}: X \to \mathbb{R}\) of \(v: N(X, \varepsilon) \to \mathbb{R}\); see \([10, \text{Lemma 6.2.6}]\). We are now ready to prove the following lemma.

**Lemma 9.** Let \((X, d, \mu)\) be a quasiconvex \((DV)_{\text{loc}}\) space, \(N(X, \varepsilon) \ni o\) a quasi-lattice where \(0 < \varepsilon \leq 2\), \(\mu(o) = 0\), and \(g: [0, \infty) \to [0, \infty)\) a control function. Then there exists \(C > 0\) such that

\[ \int_X \text{Lip} \overline{v}(x) g(d(o, x)) d\mu(x) \leq C \sum_{p \in N(X, \varepsilon)} \sum_{q \sim p} |v(p) - v(q)| g(d(o, p)) \mu(B(p, \varepsilon)) \]

for any \(v: N(X, \varepsilon) \to \mathbb{R}\).

**Proof.** Let \(v: N(X, \varepsilon) \to \mathbb{R}\) be any function and \(\overline{v}: X \to \mathbb{R}\) its locally Lipschitz extension as in Lemma 7. Arguing as in \([12, \text{Lemma 3.2}]\), there exists a constant \(C > 0\) such that, for any \(p \in N(X, \varepsilon)\) and \(x, y \in B(p, \varepsilon)\),

\[ \frac{|\overline{v}(x) - \overline{v}(y)|}{d(x, y)} \leq C \sum_{q \in B(p, 3\varepsilon) \cap N(X, \varepsilon)} |v(q) - v(p)|. \]

In particular,

\[ \text{Lip} \overline{v}(x) = \limsup_{r \to 0} \sup_{y \in B(x, r)} \frac{|\overline{v}(x) - \overline{v}(y)|}{r} \leq C \sum_{q \in B(p, 3\varepsilon) \cap N(X, \varepsilon)} |v(q) - v(p)|. \]

Thus,

\[ \int_X \text{Lip} \overline{v}(x) g(d(o, x)) d\mu(x) \leq \sum_{p \in N(X, \varepsilon)} \int_{B(p, \varepsilon)} \text{Lip} \overline{v}(x) g(d(o, x)) d\mu(x) \]

\[ \leq C \sum_{p \in N(X, \varepsilon)} \sum_{q \in B(p, 3\varepsilon) \cap N(X, \varepsilon)} |v(q) - v(p)| \int_{B(p, \varepsilon)} g(d(o, x)) d\mu(x). \]
The claim now follows by an application of inequality \([21]\). Indeed, if \(x \in B(p, \varepsilon)\), then \(d(o, x) \leq d(x, p) + d(p, o) \leq \varepsilon + d(o, p)\), and we have \(\varrho(d(o, x)) \leq L(\varepsilon)\varrho(d(o, p))\) whenever \(p \neq o\). Hence,

\[
\int_X \text{Lip} \, \overline{\varpi}(x) \varrho(d(o, x)) d\mu(x) \leq CL(\varepsilon) \sum_{p \in N(X, \varepsilon)} \sum_{q \sim p} |v(q) - v(p)| \varrho(d(o, p)) \mu(B(p, \varepsilon))
\]

as claimed. \(\square\)

At this point, we have the following intermediate version of [16, Theorem 4.2] for quasiconvex \((DV)_{\text{loc}}\) spaces.

**Theorem 10.** If \((X, d, \mu)\) is a quasiconvex \((DV)_{\text{loc}}\) space that supports a local weak \((1, 1)\)-Poincaré inequality up to scale \(R_p\). Then the following are equivalent:

1. \((X, d, \mu)\) satisfies \((S^q_{1,1})\);
2. For any \(0 < \varepsilon \leq \min\{2, R_p/4\}\) and \(N(X, \varepsilon) \ni o\) such that \(\mu(\{o\}) = 0\), there exists \(C > 0\) such that

\[
\sum_{p \in N(X, \varepsilon)} |v(p)| \mu(B(p, \varepsilon)) \leq C \sum_{p \in N(X, \varepsilon)} \sum_{q \sim p} |v(p) - v(q)| \varrho((p, q)) \mu(B(p, \varepsilon))
\]

for every \(v: N(X, \varepsilon) \to \mathbb{R}\) with finite support.

**Proof.** By Proposition \([14]\) it follows that (2) implies (1). To prove that that (1) implies (2) let \(v: N(X, \varepsilon) \to [0, \infty)\) be finitely supported and let \(\overline{\varpi}: X \to [0, \infty)\) be its locally Lipschitz extension

\[
\overline{\varpi}(x) = \sum_{p \in N(X, \varepsilon)} v(p) \varphi_p(x) = \sum_{p \in N(X, \varepsilon)} v(p) \frac{\varphi_p(x)}{\varphi(x)},
\]

now with bounded support. Since \(\overline{\varpi}\) is locally Lipschitz, \(\text{Lip} \, \overline{\varpi}\) is an upper gradient of \(\overline{\varpi}\). In particular, \(\overline{\varpi}\) and has a minimal 1-weak upper gradient |\(\nabla \overline{\varpi}\)|; see [10, Theorem 6.3.20]. Thus, by \((S^q_{1,1})\)

\[
\int_X \overline{\varpi}(x) d\mu(x) \leq C \int_X |\nabla \overline{\varpi}| \varrho(d(o, x)) d\mu(x) \leq C \int_X \text{Lip} \, \overline{\varpi}(x) \varrho(d(o, x)) d\mu(x).
\]

By Lemma \([9]\)

\[
\int_X \text{Lip} \overline{\varpi}(x) \varrho(d(o, x)) d\mu(x) \leq C' \sum_{p \in N(X, \varepsilon)} \sum_{q \sim p} |v(p) - v(q)| \varrho((p, q)) \mu(B(p, \varepsilon)).
\]

Since \(\psi\) appearing in the Lipschitz partition of unity is uniformly bounded, there exists \(C'' > 0\) for which \(\psi(x) \leq C''\) for all \(x \in X\) and

\[
\int_X \overline{\varpi}(x) d\mu(x) = \int_X \sum_{p \in N(X, \varepsilon)} v(p) \varphi_p(x) d\mu(x) = \int_X \sum_{p \in N(X, \varepsilon)} v(p) \frac{\varphi_p(x)}{\varphi(x)} d\mu(x)
\]

\[
\geq \frac{1}{C''} \int_X \sum_{p \in N(X, \varepsilon)} v(p) \varphi_p(x) d\mu(x)
\]

\[
\geq \frac{1}{C''} \sum_{p \in N(X, \varepsilon)} v(p) \mu(B(p, \varepsilon)),
\]

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Then, the following are equivalent:

By Proposition 10, (1) and (2) are equivalent. By uniformity 0

Proof. for

for every $\mu$ equipping $N$ for every $X,d,\mu$ that (3) implies (4). First, we approximate $(\Psi)$

$\therefore$, it remains to prove that (3) and (4) are equivalent and we first show

fundamental class in $\H$ Combining the previous results, we are ready to prove that the vanishing of a

local weak $(1, \epsilon)$ Let

Theorem 12.

Lemma 11.

uniform. We begin with the following fact.

as $\psi_p|B(p, \epsilon) \equiv 1$; and altogether for some $C'' > 0$ independent of $v$

for every $v: N(X, \epsilon) \to [0, \infty)$ with finite support. The general claim for any $v: N(X; \epsilon) \to \mathbb{R}$ with finite support now follows observing that the claim holds

for $|v|$ by the previous, and by the triangle inequality for $v$.

3.3 Connecting $H_0^g$ to $(S^g_{1,1})$

Combining the previous results, we are ready to prove that the vanishing of a fundamental class in $H_0^g$ of a quasiconvex $(DV)_{loc}$ space that supports a local weak $(1, 1)$-Poincaré inequality is characterised by $(S^g_{1,1})$ whenever the space is uniform. We begin with the following fact.

Lemma 11. A quasiconvex uniform space $(X,d,\mu)$ has at most exponential volume growth.

Proof. Fix a quasi-lattice $N(X, \epsilon)$ and let $k \in \mathbb{N}\setminus\{0\}$. Since $N(X, \epsilon)$ is uniformly locally finite any open ball $B(x, 2k\epsilon) \subseteq X$ can be covered by $N(3\epsilon)^k$ balls of radius $\epsilon$. Since $(X,d,\mu)$ is uniform,

for every $k \in \mathbb{N}\setminus\{0\}$.

Theorem 12. Let $(X,d,\mu)$ be a quasiconvex uniform space that supports a local weak $(1, 1)$-Poincaré inequality up to scale $R_p$. Let $0 < \epsilon \leq \min\{2, R_p/4\}$, $N(X, \epsilon) \supseteq o$, where $\mu(\{o\}) = 0$, and $\varphi: [0, \infty) \to [0, \infty)$ a control function. Then, the following are equivalent:

(1) $(X,d,\mu)$ satisfies $(S^g_{1,1})$;

(2) there exists $C_1 > 0$ such that for every $v: N(X, \epsilon) \to \mathbb{R}$ with finite support

$(\sum p \in N(X, \epsilon)) |v(p)|\mu(B(p, \epsilon)) \leq C_1 \sum p \in N(X, \epsilon) v(p)|v(p)|\mu(B(p, \epsilon))$;

(3) there exists $C_2 > 0$ such that for every $v: N(X, \epsilon) \to \mathbb{R}$ with finite support

$(\sum p \in N(X, \epsilon)) |v(p)| \leq C_2 \sum p \in N(X, \epsilon) v(p)|v(p)|\mu(B(p, \epsilon))$;

(4) $0 = [\Gamma] \in H_0^g(\Gamma)$ for any quasi-lattice $\Gamma \subseteq X$.

Proof. By Proposition 10 (1) and (2) are equivalent. By uniformity $0 < f(\epsilon) \leq \mu(B(p, \epsilon)) \leq g(\epsilon) < \infty$ for all $p \in N(X, \epsilon)$, and so (2) and (3) are equivalent. Hence, it remains to prove that (3) and (4) are equivalent and we first show that (3) implies (4). First, we approximate $(X,d)$ by the space obtained from equippping $N(X, \epsilon)$ with the edge path length $\delta: N(X, \epsilon) \times N(X, \epsilon) \to \mathbb{N}\cup\{\infty\}$ given by

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δ(x, y) = 0 if x = y, \\
δ(x, y) = k if the shortest 3ε-path from x to y is of length k, \\
δ(x, y) = ∞ if there is no 3ε-path from x to y,

where a 3ε-path from x to y of length k is any sequence of points x = x₀, . . . , xₖ = y in N(X, ε) where 0 < d(xᵢ, xᵢ₊₁) ≤ 3ε. Since (X, d) is uniformly coarsely proper and (Q)-quasiconvex, δ is a metric on N(X, ε) and (N(X, ε), δ) is quasi-isometric to (X, d); see [13, Proposition 3.D.16], and

\[
\frac{1}{3\varepsilon} d(p, q) \leq \delta(p, q) \leq \frac{Q}{\varepsilon} d(p, q) + 1 \quad \text{(QI)}
\]

for all p, q ∈ N(X, ε) adapting [13, Lemma 2.5] for geodesic spaces to quasiconvex spaces. Thus \(\varrho(d(\bar{o}, (p, q))) \leq 3 \varepsilon \delta(\bar{o}, (p, q))\) by (QI), and using (22) we see that (N(X, ε), δ) satisfies

\[
\sum_{x \in N(X, \varepsilon)} |\eta(x)| \leq C_2 M(3\varepsilon) \left( \sum_{x \in \Gamma \setminus \{y : \delta(y, x) = 1\}} \sum_{y} |\eta(x) - \eta(y)| \varrho(\{(x, y)\}) \right)
\]

for every finitely supported \(\eta : N(X, \varepsilon) \to \mathbb{R}\). Equivalently, 0 = [N(X, ε)] ∈ \(H^0_0(N(X, \varepsilon))\) where \(H^0_0(N(X, \varepsilon))\) is defined using the metric δ; see [16, Lemma 4.1, Theorem 4.2]. Since id : (N(X, ε), δ) → (N(X, ε), d) is a quasi-isometry, we conclude that 0 = [(N(X, ε))] ∈ \(H^0_0(N(X, \varepsilon))\), where \(H^0_0(N(X, \varepsilon))\) is defined using the metric d, and hence 0 = \([\Gamma]\) ∈ \(H^0_0(\Gamma)\) for any quasi-lattice \(\Gamma \subseteq X\) by Lemma 2. It remains to prove that (4) implies (3). By assumption, 0 = \([\Gamma]\) ∈ \(H^0_0(\Gamma)\) for any quasi-lattice \(\Gamma \subseteq X\); in particular for N(X, ε) ⊆ X. Since id : (N(X, ε), d) → (N(X, ε), δ) is a quasi-isometry, 0 = [(N(X, ε))] ∈ \(H^0_0(N(X, \varepsilon))\) defined using the metric δ, equivalently, for some \(D > 0\)

\[
\sum_{x \in N(X, \varepsilon)} |\eta(x)| \leq D \left( \sum_{x \in N(X, \varepsilon) \setminus \{y : \delta(y, x) = 1\}} \sum_{y} |\eta(x) - \eta(y)| \varrho(\{(x, y)\}) \right)
\]

for every finitely supported \(\eta : N(X, \varepsilon) \to \mathbb{R}\). Applying (QI), (21) and (22), respectively, \(\varrho(\delta(\bar{o}, (p, q))) \leq L(1)M(Q/\varepsilon)\varrho(d(\bar{o}, (p, q)))\). Using this to estimating the above inequality from above gives (3).

Theorem A summarises this by stating the equivalence between (1) and (4) above. Theorem B follows from the observation that the local weak (1, 1)-Poincaré inequality is not needed to prove that (1) implies (2) in Theorem 10.

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