THE NUMBER OF ITERATES OF THE CARMICHAEL LAMBDA FUNCTION REQUIRED TO REACH 1

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Abstract. The Carmichael lambda function \( \lambda(n) \) is defined to be the smallest positive integer \( m \) such that \( a^m \equiv 1 \pmod{n} \) for all \((a, n) = 1 \). \( \lambda_k(n) \) is defined to be the \( k \)th iterate of \( \lambda(n) \). Let \( L(n) \) be the smallest \( k \) for which \( \lambda_k(n) = 1 \). It’s easy to show that \( L(n) \ll \log n \). It’s conjectured that \( L(n) \approx \log \log n \), but previously it was not known to be \( o(\log n) \) for almost all \( n \). We will show that \( L(n) \ll (\log n)^\delta \) for almost all \( n \), for some \( \delta < 1 \). We will also show \( L(n) \gg \log \log n \) for almost all \( n \) and conjecture a normal order for \( L(n) \).

1. Introduction

The Carmichael lambda function \( \lambda(n) \) is defined to be the exponent of the multiplicative group \((\mathbb{Z}/n\mathbb{Z})^\times\). It can be computed using the identity \( \lambda(\text{lcm}\{a, b\}) = \text{lcm}\{\lambda(a), \lambda(b)\} \) and its values at prime powers. Those values are \( \lambda(p^k) = \phi(p^k) = p^k - p^{k-1} \) for odd primes \( p \), and \( \lambda(2) = 1, \lambda(4) = 2 \), and \( \lambda(2^k) = \phi(2^k)/2 = 2^{k-2} \) for \( k \geq 3 \). The \( k \)-fold iterated Carmichael lambda function is defined recursively as follows.

\[
\lambda_0(n) = n, \quad \lambda_k(n) = \lambda(\lambda_{k-1}(n)), \text{ for } k \geq 1.
\]

This paper is about some analytical properties of a related function.

Definition 1. Let \( L(n) \) be the smallest non-negative integer \( k \) such that \( \lambda_k(n) = 1 \).

Since \( \lambda(n) \) is either even or 1, and \( \lambda(n) \leq n/2 \) for even \( n \), we easily see that \( L(n) \leq \lfloor \log n / \log 2 + 1 \rfloor \). By considering when \( n \) is a power of 3 we can note that \( L(n) \geq 1 + (1/\log 3) \log n \) for infinitely many values of \( n \). As for upper bounds, Martin and Pomerance [3] gave a construction for which \( L(n) < (1/\log 2 + o(1)) \log \log n \) for infinitely many \( n \). Probabilistically, these examples have asymptotic density 0. It is conjectured that for a set of positive integers with asymptotic density 1, that \( L(n) \approx \log \log n \), however no previous results have shown \( L(n) = o(\log n) \) for almost all \( n \).

The Pratt tree for a prime \( p \) is defined as follows. Let the root node be \( p \). Below \( p \) are nodes labelled with the primes \( q \) such that \( q \mid p - 1 \). The nodes below \( q \) are primes dividing \( q - 1 \) and so on until we are left with just 2. For example, if we want to take the prime 3691, the primes dividing 3690 are 2, 3, 5 and 41. The primes dividing 3 − 1 is 2, dividing 5 − 1 is 2 and dividing 41 − 1 are 2 and 5. Continuing we obtain the tree

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In a recent paper by Ford, Konyagin and Luca [2], they found bounds on the height of the Pratt tree $H(p)$. The height is closely related to $L(p)$ for a prime $p$. It’s easy to see that $H(p) \leq L(p)$, so any lower bound on $H$ acts as a lower bound on $L$. The Bombieri–Vinogradov Theorem implies

\[ \sum_{n \leq Q} \max_{y \leq x} \left| \pi(y; n, 1) - \frac{\ln(y)}{\phi(m)} \right| \ll x(\log x)^{-A} \]

with $Q = x^{1/2}(\log x)^{-B}$ for any $A > 0$ and $B = B(A)$. The Elliot–Halberstam conjecture says that (1) holds for $Q = x^\theta$ for any $\theta < 1$. Let $\theta'$ be such that (1) holds for $Q = x^{\theta'}$. In [2] they showed for any $c < 1/(e^{-1} - \log \theta')$,

\[ H(p) > c \log \log p \]

for all but $O\left(x/(\log x)^K\right)$ primes $p$, for some $K > 1$. Bombieri–Vinogradov allows us to take any $c < 1/(e^{-1} + \log 2)$, and under Elliot–Halberstam, we can take any $c < e$.

It’s easy to see if $n = \prod p_i^{\alpha_i}$, then

\[ L(n) = \max_i \{L(p_i^{\alpha_i})\} \]

and

\[ L(p^{\alpha}) = \alpha - 1 + L(p) \geq L(p). \]

These two equations imply $L(n) \geq L(p)$ for any $p \mid n$, motivating the following theorem.

**Theorem 2.** There exists some $c > 0$ such that

\[ L(n) \geq c \log \log n \]

for all $n$ as $n \to \infty$.

For an upper bound, from [2] we have

\[ H(p) \leq (\log p)^{0.95022} \]

for all $p \leq x$ outside a set of size $O\left(x \exp\left(-((\log x)^\delta\right)\right)$ for some $\delta > 0$. We extend this to a result about $L(n)$.

**Theorem 3.** If $H(p) \leq (\log p)^\gamma$ for almost all $p \leq x$ outside a set of size $O\left(x \exp\left(-((\log x)^\delta\right)\right)$ for some $\delta > 0$, then for some function $\psi$ and any $\epsilon > 0$,

\[ L(n) \ll (\log n)^\gamma \psi(n) \]

for almost all $n$ as $n \to \infty$.

The function $\psi(x)$ can be taken to be as small as $O(\log \log \log x)$. Using Theorem 3 along with equation (5) yields the following corollary.
Corollary 4. For any $\epsilon > 0$, for almost all $n$,

$$L(n) \ll (\log n)^{0.9503}.$$  

In [2], the authors described a probabilistic model which suggested a conjecture on the normal order for $H(p)$ is $e \log \log p$. Assuming this conjecture, we give some evidence to suggest a related conjecture for $L(n)$.

**Conjecture 5.** The normal order of $L(n)$ is $e \log \log n$.

Throughout the paper, $p$ and $q$ will always denote primes, $\log_k(n)$ will denote the $k$th iterate of the logarithm function, and $y = y(x) = \log_2(x) = \log \log x$. Also the notation $q \preceq q'$ is defined to mean $q \mid q' - 1$.

2. Lower bound for $L(n)$

For any $p \mid n$, we know that $L(n) \geq L(p)$, which implies that $L(n) > c \log_2(p)$. However, if all the primes $p$ dividing $n$ are small relative to $n$, this will not imply that $L(n) > c \log_2(n)$. The proof of Theorem 2 therefore relies on showing that not many $n$ are composed entirely of small primes as well as dealing with the exceptional set for which (2) doesn’t hold.

**Proof of Theorem 2** Let $Y = Y(x) \leq x$. Let $c$ from Equation (2) which can be shown to be any constant $c < 1/(e^{-1} + \log 2)$. Define a set $S(x) = S(x,Y) = \{p : p \geq Y, H(p) < c \log_2(p)\}$. We have that $\#S(x) \ll x/(\log x)^K$ for some $K > 1$, so if $p \mid n$ for some $p \notin S(x)$,

$$L(n) \geq L(p) \geq c \log_2(p).$$

If $n$ is only composed of $p \in S(x)$, then either there exists $p \geq Y, p \in S(x)$ such that $p \mid n$ or $n$ is composed entirely of primes less than or equal to $Y$. The number of $n \leq x$ where there exists $p \mid n$ with $p \in S(x)$ is bounded by

$$\sum_{n \leq x} \sum_{p \mid n \atop p \in S(x)} 1 = \sum_{p \leq x \atop p \in S(x)} \sum_{n \leq x \atop n \equiv 0 \pmod{p}} 1 \leq \sum_{p \leq x \atop p \in S(x)} \frac{x}{p} = x \left( \frac{|S(x)|}{x} + \int_Y^x \frac{S(t)dt}{t^2} \right) \ll \frac{x}{\log^K x} + \int_Y^x \frac{dt}{t \log^K t} \ll \frac{x}{\log^{K-1} Y}$$

using partial summation. Let $\Psi(x, z)$ be the number of $n \leq x$ composed of primes $p \leq z$ and let $z = x^{1/u}$. By [4, Theorem 7.2],

$$\Psi(x, z) \ll x p(u)$$
where $\rho(u)$ is the Dickman function. It’s known that $\rho(u) \to 0$ as $u \to \infty$. Given $\epsilon > 0$, choose $Y$ such that $\log Y = (\log x)^{1-\epsilon}$. Since $Y < x^\gamma$ for all $\gamma > 0$, this choice yields $L(n) \geq c(1-\epsilon) \log_2(x)$ for all but $O\left(x/(\log Y)^{K-1} + \Psi(x,Y)\right) = o(x)$ such $n$. This completes the theorem.

\[\square\]

It’s worth noting that under the Elliot–Halberstam conjecture, that constant can be replaced by any $c < e$.

3. Upper Bound for $L(n)$

The Pratt tree for a prime $p$ describes the primes $q$ where $q < \cdots < p$. This is useful in calculating $L(p)$, however $L(p)$ is also increased by prime powers for which the Pratt tree does not describe. The proof of Theorem 3 hinges on bounding the contribution of these large prime powers. We begin with the following lemma.

**Lemma 6.** Fix a prime $q$ and positive integers $k, \alpha$. The number of $n \leq x$ such that there exists $q^\alpha \mid q_{k-1} - 1, q_{k-1} \mid q_{k-2} - 1, \ldots, q_1 \mid p - 1$ and $p \mid n$ is at most

\[
\frac{x(cy)^k}{q^\alpha}
\]

for some absolute constant $c$.

**Proof.** We’ll make use out of the Brun–Titchmarsh inequality

\[
\pi(t; m, a) \leq \frac{2t}{\phi(m) \log(t/m)}.
\]

Partial summation yields

\[
\sum_{p \leq x \atop p \equiv 1 \pmod{m}} \frac{1}{p} \ll \frac{\log x}{\phi(m)}.
\]

Noting that $\phi(m) \gg m$ if $m$ is a prime or prime power implies

\[
\sum_{p \leq x \atop p \equiv 1 \pmod{m}} \frac{1}{p} \leq \frac{c \log_2 x}{m} = \frac{cy}{m}
\]

4
if $m$ is a prime or prime power. Repeated uses of (7) gives us the number of such $n$ is bounded by

$$
\sum_{n \leq x} \sum_{p | n} \sum_{q_1 | p - 1} \cdots \sum_{q_{k-1} | q_{k-2} - 1} \sum_{q_k | q_{k-1} - 1} 1
$$

$$
= \sum_{q_{k-1} \equiv 1 \pmod{q^\alpha}} \sum_{q_{k-2} \equiv 1 \pmod{q_{k-1}}} \cdots \sum_{p \equiv 1 \pmod{q_1}} \sum_{n \leq x} \sum_{n \equiv 0 \pmod{p}} 1
$$

$$
\leq \sum_{q_{k-1} \equiv 1 \pmod{q^\alpha}} \sum_{q_{k-2} \equiv 1 \pmod{q_{k-1}}} \cdots \sum_{p \equiv 1 \pmod{q_1}} \frac{x}{p}
$$

$$
\leq \sum_{q_{k-1} \equiv 1 \pmod{q^\alpha}} \sum_{q_{k-2} \equiv 1 \pmod{q_{k-1}}} \cdots \sum_{q_1 \equiv 1 \pmod{q_2}} \frac{x cy}{q_1}
$$

$$
\leq \frac{x (cy)^k}{q^\alpha}.
$$

\[ \square \]

We will show Theorem 3 is a corollary to the main proposition, that the difference between $H(p)$ and $L(p)$ cannot be too great.

**Proposition 7.** Let $b > 0$ and $c$ be the constant from (6). Suppose $H(p) \leq (\log p)^\gamma$ for all $p \leq x$ outside a set of size $O(x \exp(-\log x)^\delta)$ and let $\psi(x)$ be a function such that

$$
(8) \quad \frac{x (cy)^{\log x}^{\gamma + 1}}{2^b (log x)^{\gamma} \psi(x) - 2} = o(x).
$$

Then

$$
L(n) \ll (\log x)^\gamma \psi(x)
$$

for almost all $n \leq x$, for which the excluded $n$ are divisible by at least one prime $p$ in the above excluded set.

Note that if $\psi'(x)$ is some function such that $b \psi'(\log x)^\gamma - \log (cy) \rightarrow \infty$ and $\psi(x) > \frac{1}{3} \log (cy) + \psi'(x)$, then

$$
\frac{x (cy)^{\log x}^{\gamma + 1}}{2^b (log x)^{\gamma} \psi(x) - 2} = \frac{x \exp \left( (\log x)^\gamma + 1 \log (cy) \right)}{\exp \left( (b \psi'(x) \log x)^\gamma - 2 \log 2 \right)} \ll x \exp \left( \log (cy) - b \psi'(x) (\log x)^\gamma \right) = o(x).
$$

Specifically we can choose $\psi(x) \ll_b \log_2(x)$. The proof of Proposition 4 begins by analyzing the ways that $L(p)$ can be much larger than $H(p)$ and then showing in those cases that it cannot happen for many $p$.

**Proof of Proposition 7.** Let $n = \prod p_i^{\alpha_i}$ be the prime factorization of $n$ where $p | n \rightarrow H(p) \leq (log p)^\gamma$. By equations (3) and (4), $L(n) = \max \{ \alpha_i - 1 + L(p) \}$. Our first goal is to show that the number of $n$ for which there exists a large $\alpha$ with $p^\alpha | n$ is small. Fixing a prime $p,$
the number $n \leq x$ such that $p^n \mid n$ is at most $x/p^n$. Hence the number of bad $n$ is bounded by

$$
\sum_{p \leq x} \frac{x}{p^n} \leq x \sum_{m=1}^{x} \frac{1}{p^n} \lesssim \frac{x}{\alpha}.
$$

Applying this with any $\alpha = \xi(x)$ with $\xi(x) \to \infty$ makes the number of such $n$ be $o(x)$. Therefore for almost all $n \leq x$ we can assume

$$
L(n) \leq \max_{p \mid n}(L(p) + \xi(x)) = \max_{p \mid n}(L(p) + o((\log x)^\gamma))
$$

by taking $\xi(x) = o((\log x)^\gamma)$.

Let $\psi(x)$ be a function satisfying the hypothesis of the proposition. We must determine how $L(p)$ can be larger than $H(p)$ and by how much. First note that for any prime in the Pratt tree, the difference between the factors of $q - 1$ and the primes in the Pratt tree are just the powers of that prime which divide $q - 1$. Therefore, if we have a branch of the Pratt tree, $2 = q_k < q_{k-1} < \cdots < q_1 < q_0 = p$, then $L(p) \leq \max\{H(p) + \sum_{i=1}^{k}(\alpha_i - 1)\}$ where $q_i^{\alpha_i} \mid q_i - 1$ and the max is taken over all the branches of the Pratt tree. The inequality $q_i^{\alpha_i} < q_i - 1$ holds for all $i$ which implies

$$
2 \prod_{i=1}^{k} \alpha_i < p.
$$

Therefore we need to maximize the sum $\sum_{i=1}^{k}(\alpha_i - 1)$ subject to $\prod_{i=1}^{k} \alpha_i < \log x / \log 2$.

Suppose we have $rs = tu$, where $2 \leq r, s, t, u \leq M$. The larger of $r + s$ and $t + u$ will be where the two terms are further apart. Consequently if we wish to maximize a sum subject a fixed product and number of terms, we want some terms to be the lowest possible value, in this case $2$, and the rest to be the largest value, in this case $M$. Suppose $\sum_{i=1}^{k}(\alpha_i - 1) \succ \psi(x)(\log x)^\gamma$, where $2 \leq \alpha \leq M$ and $M = o(\psi(x)(\log x)^\gamma)$. By the above reasoning we know the sum is bounded by $2(k - l) + lM$ for some $l \leq k$. However, $M^l \leq \log x / \log 2$ implying $l \leq (\log_2 x - \log_2 2) / \log M$. Since $k \ll \log_2(x)$, $2(k - l) + lM$ is bounded above by

$$
O\left(\log_2(x) + M(\log_2 x - \log_2 2) / \log M\right) = o(\psi(x)(\log x)^\gamma),
$$

contradicting the bound on $M$. As a result, we know there exists some $\alpha_i \geq b\psi(x)(\log x)^\gamma$ for some $b > 0$.

It remains to show that the number of $n \leq x$ such that there exists $q^\alpha \mid q_k - 1, q_{k-1} \mid q_{k-2} - 1, \ldots, q_1 \mid p - 1, p \mid n$, with $\alpha \geq b\psi(x)(\log x)^\gamma$ is $o(x)$. Note that $k \leq H(p) \leq (\log x)^\gamma$.

By Lemma 6, the number of $n$ is bounded by

$$
\sum_{\alpha \geq b\psi(x)(\log x)^\gamma} \sum_{k \leq (\log x)^\gamma} \sum_{q} x(cy)^k q^\alpha.
$$

Summing $q$ over all integers at least 2 instead of primes and using $\alpha \geq 2$ makes this

$$
\ll \sum_{\alpha \geq b\psi(x)(\log x)^\gamma} \sum_{k \leq (\log x)^\gamma} x(cy)^k \frac{2^\alpha - 1}{2^\alpha - 1}.
$$
Summing the geometric series under both α and k yields
\[ x(cy)\left(\log x\right)^{\gamma+1} \leq \frac{x}{2b\psi(x)(\log x)^{\gamma-2}}. \]
By the choice of ψ this is o(x) and hence for almost all \( n \leq x, \)
\[ L(n) \leq o((\log x)^{\gamma}) + \max_{p/n} \left\{ H(p) + \sum_{i=1}^{k} (\alpha_i - 1) \right\} \leq (\log n)^{\gamma} + \psi(x)(\log x)^{\gamma} \ll \psi(x)(\log x)^{\gamma}. \]

We are now in a position to prove Theorem 3. Proposition 7 yields the theorem provided \( n \) wasn’t divisible by any primes for which (5) fails to hold, so it remains to consider when \( n \) is divisible by a prime.

Proof of Theorem 3. Let \( Y = Y(x) \to \infty \) such that \( \log Y \ll (\log x)^{\gamma} \). As in the proof of Theorem 2 we know that the set of \( n \leq x \) which are composed entirely of primes less than or equal to \( Y \) has density 0. Therefore we only need to consider values of \( n \) for which there exists a prime greater than \( Y \) where \( H(p) > (\log p)^{\gamma} \). Let \( S(x) \) be the set \( \{ Y < p \leq x \mid L(p) > (\log p)^{\gamma} \} \). Since \( L(p) > H(p), \) by [3] we know that \( \#S(x) \ll x \exp\left( -(\log t)^{\delta} \right). \) The number of \( n \leq x \) where \( n \) is divisible by a prime in \( S(x) \) is bounded by
\[
\sum_{n \leq x} \sum_{p \in S(x)} \frac{1}{p} \leq \sum_{p \in S(x)} \frac{x}{p} = \frac{x|S(x)|}{x} + x \int_{Y}^{x} \frac{|S(t)|dt}{t^2} \ll x \exp\left( -(\log x)^{\delta} \right) + x \int_{Y}^{x} \frac{\exp\left( -(\log t)^{\delta} \right)dt}{t} \ll x \exp\left( -(\log x)^{\delta} \right) + \frac{x}{\log x} + \frac{x}{\log Y} \]
using partial summation. In the last line we used \( \exp\left( -(\log t)^{\delta} \right) \ll (\log t)^{-2}. \) By our choice of \( Y \) the number of \( n \) is \( o(x) \) completing the theorem. \( \Box \)

4. Conjecture for the normal order of \( L(n). \)

The purpose of this section is to justify Conjecture 4 assuming the conjecture in 2 which implies \( H(p) \leq e \log p \) for almost all \( p. \) To do this, we wish to analyze the difference \( L(p) - H(p) \) to show that it is not too large. As we saw in the previous section, this difference is created when a branch of the Pratt tree has \( p_i^n | p_{i-1} - 1 \) where \( a > 1. \) Let \( Y = Y(x) \leq x. \) Also let a branch of the Pratt tree be \( p_1 > p_2 > \cdots > p_l > p_{l+1} > \cdots > p_k = 2 \) where \( p_i^n || p_{i-1} - 1 \) and let \( l \) be the largest index such that \( p_l > Y. \) We will separate our arguments into the cases where \( i < l + 1, i > l + 1, \) and finally \( i = l + 1. \)

By the trivial estimate \( L(n) \ll \log n \) we know \( L(p_{l+1}) \ll \log Y. \) By a suitable choice of \( Y \) this will be made to be \( o(\log_2 x). \)

For \( i \leq l, \) we wish to know the probability that \( n \) has a factor \( p^a, \) where \( p > Y. \) We use the following lemma.

Lemma 8. The number of \( n \leq x \) for which there exists \( p > Y \) where \( p^a \parallel n \) is \( O(x/Y^{a-1}). \)
Proof. The number of \( n \) is bounded by

\[
\sum_{n \leq x} \sum_{p > Y \mid p \mid n} 1 \leq \sum_{p > Y} \frac{x}{p^2} \ll \frac{x}{Y^{a-1}}.
\]

By Lemma 8 we should expect a proportion of at most \( c/Y^a \). This implies that the probability of \( p_i^a \mid p_{i-1} - 1 \) where \((a_2 - 1) + (a_3 - 1) + \cdots + (a_l - 1) = \psi(x)\) is bounded by \( c/Y^{\psi(x)} \). Since the number of possible branches of the Pratt tree is trivially bounded by \( \log x \), the probability of there existing such a string of \( a_i \) is bounded by

\[
1 - \left( 1 - \frac{c}{Y^{\psi(x)}} \right)^{\log x}.
\]

This bound will approach 0 provided \( \log x = o(Y^{\psi(x)}/c) \). Under the assumption that \( H(p) \leq e \log_2(p) \), we have \( l \leq H(p) \leq e \log_2(p) \). Therefore a choice of \( Y = \exp((\log_2(x))^{3/4}) \) and \( \psi(x) = (\log_2(x))^{3/4} \) makes the contribution to \( L(p) - H(p) \) be \( o(\log_2(x)) \) for \( i \neq l + 1 \).

For \( i = l + 1 \), we have \( p_{l+1} \mid p_l - 1 \). The remaining contribution to \( L(p) - H(p) \) is \( a_{l+1} - 1 \), if \( p_{l+1} > 2 \) and \( [(a_{l+1} + 1)/2] \) if \( p_{l+1} = 2 \). For the \( a_{l+1} \) to contribute a lot to \( L(p) \), it must be at the end of a long prime chain, i.e. \( l \gg \log_2(p) \), otherwise the conjectured value of \( H(p) \) being \( e \log_2(p) \) would nullify the contribution. To show this is unlikely, we use a result from [1] which implies that the number of primes at a fixed level \( n \) of the Pratt tree is \( \sim (\log_2(p))^n/n! \). If we allow some dependence and use \( n = c \log_2(p) \), for \( 0 < c < \log_2(p) \) we get roughly \( (e/c)^{c \log_2(p)} = (\log_2(p))^{c \log(e/c)} \) primes at level \( n \). We show that the probability of none of these primes being congruent to 1 modulo \( p_{l+1}^a \) goes to 1 provided \( p_{l+1}^a \) is large enough.

Suppose we have \( N \) primes. The probability that any one of them is congruent to 1 modulo \( r^a \) for a prime \( r \) and positive integer \( a \), is \( 1/\phi(r^a) \). Assuming some independence, the probability that none of the \( N \) primes are congruent to 1 modulo \( r^a \) is

\[
\left( 1 - \frac{1}{\phi(r^a)} \right)^N.
\]

Let \( \psi \) be a function going to infinity. Furthermore, let \( r^a > N\psi(N) \), be a prime power. Since \( r \) is prime, we know \( \phi(r^a) \geq r^a/2 \). This bound implies the probability is bounded below by

\[
\left( 1 - \frac{2}{r^a} \right)^N.
\]

Using our lower bound on \( r^a \) we get

\[
\left( 1 - \frac{2}{r^a} \right)^N \geq 1 - \left( 1 - \frac{2}{N\psi(N)} \right)^N \rightarrow 1.
\]

We know wish to use the lower bound on \( r^a \) to bound \( a_{l+1} \) and therefore our contribution to \( L(p) - H(p) \). Suppose \( q_{l+1} \neq 2 \). If the level \( l \approx c \log_2(p) \), for almost all \( p \), we expect

\[
a_{l+1} \leq \frac{\log(N \log N)}{\log q_{l+1}} = \frac{c \log(e/c)}{\log q_{l+1}} \log_2(p) + O(\log_3(p)).
\]
Combining all the contributions along any particular branch, we get

\( L(p) \leq \left( c + \frac{c \log(e/c)}{\log q_{i+1}} \right) \log_2 p + o(\log_2 p). \) (9)

If \( q_{i+1} = 2 \), since, \( \lambda(2^a) = 2^{a-2} \) we get

\[ \left( c + \frac{c \log(e/c)}{2 \log 2} \right) \log_2 p + o(\log_2 p) = \left( c + \frac{c \log(e/c)}{\log 4} \right) \log_2 p + o(\log_2 p). \]

Consequently, 3 is the value of \( q_{i+1} \) which yields the largest coefficient of \( \log_2 p \) in (9). Since \( c + c \log(e/c)/\log 3 \leq e \) for \( 0 < c < e \), we conclude that for almost all \( p \leq x \), \( L(p) \sim e \log_2 p \).

The reason that we can interchange \( p \) and \( n \) is the same reason as in Theorem 2. It may seem obvious to conclude \( L(p) \sim e \log_2 p \), since \( H(p) \sim e \log_2 p \). However, note that the function \( \left( c + \frac{c \log(e/c)}{\log 2} \right) \) does not yield a maximum value of \( e \), but instead has its maximum of \( 2/\log(2) \) at \( c = 2 \). This may suggest if we had a function \( L'(n) \) similar to \( L(n) \) except that \( \lambda'(2^a) = 2^{a-1} \) for all positive integers \( a \), that we may get a different normal order, perhaps even \( 2 \log_2 n/\log(2) \).

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