Analytical solution of the proca equation for the modified posch teller potential

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Abstract. We study the solution of the Proca equation for the modified Posch Teller potential. Proca equation is a relativistic wave equation for a massive spin-1 particle. Proca equation also describes the imaginary mass of photons. Vector fields are used to describe spin-1 mesons. The hypergeometric method was used to solve the Proca equation to obtain the relativistic energy and wave function of the particle. This work is limited to radial part of the spherical coordinate system. Using the improved approximation scheme to deal with the centrifugal term, we solve approximately the Proca equation for the modified Posch Teller potential. The relativistic energy eigenvalue and radial wave function equations are obtained.

1. Introduction
The quantum field theory (QFT) is a combination of special relativity and quantum mechanics. The QFT have great generality and flexibility of the methods to the domain of particle physics [1]. Quantum field theory describe about creation and annihilation of quanta or particles [2]. For quite a long time it has been known that the effects of a nonzero photon rest mass can be incorporated into electromagnetism through the Proca equation. Proca equation formulated the theory of imaginary mass of the particles [3]. Proca equation describes the continuity equation of particle charge [4]. From 1936 to 1941 Proca developed the theory of the spin 1 particles [5]. Quantum mechanics describes the energy and wave function of the particle [6]. With solve the Schrödinger Equation, we can obtain the energy and wave function of the particle [7]. QFT describes the particles which have relatively and massless [8]. Proca and Maxwell equation tells the massive photon [9], spin 1 particle [10,11], Bose-Einstein gravity condensate [12], the electric and magnetic fields [13]. Proca equation also used for supersymmetric non-Abelian [14], Dirac-Proca equation [15], invariance test [16], Tachyonic Cherenkov and [17] synchrotron radiation [18], thermal electrons radiations [19], Crab Nebula [20], Tachyonic gamma ray [21], Graviton-electron-magnetic equation [22], London-Proca-Hirsch equation [23] and superconducting theory [24].

1.1 Proca Equation
The phenomena of electromagnetic are characterized by the electric and magnetic fields which describe of a photons. Proca extended the Maxwell equation in the quantum field theory. Proca worked out a new equation which would allow for positive and negative both of the energy and the charge, and a finite spin and massive photon [10]

$$\Box A^\theta - \partial^\theta (\partial_\mu A^\mu) + \mu^2 A^\theta = j^\theta$$  

(1)
The Proca equation describes a massive gauge boson. But, we assuming $m \neq 0$ one has

$$\partial_a A^\mu = \left(\frac{1}{m^2}\right) \partial_a j^\theta$$

(2)

If the source current is conserved ($\partial_a j^\theta = 0$) or if there are no sources ($j^\theta = 0$) it follows that

$$\partial_a A^\theta = 0$$

(3)

The field equation gets simplified for free particles, leading to four Klein Gordon equation.

$$\Box A^\theta + \mu_\gamma^2 A^\theta = 0$$

(4)

The parameter $\mu_\gamma$ could be interpreted as the photon rest mass $m_\gamma$ with

$$m_\gamma = \frac{\mu_\gamma \hbar}{c}$$

(5)

The photon imaginary mass is expressed by the following equation:

$$m_\gamma = \frac{2}{\sqrt{3}} \left(\frac{\hbar}{c^2}\right)i$$

(6)

Equation (6) means that the photon has null real mass. Then parameter $\mu_\gamma$ must be also an imaginary variable. Thus, $\mu_\gamma^2$ is a negative real number similarly to $m_\gamma^2$. Consequently, the negative real number can write that

$$\mu_\gamma^2 = \frac{m_\gamma^2 c^2}{\hbar^2} = 4 \left(\frac{2\pi^2}{3}\right)^2 = \frac{4}{3} k_r^2$$

(7)

Whence FD Aquino recognize $k_r = \frac{2\pi}{\lambda}$ as the real part of the propagation vector $k$,

$$k = |k| = |k_r + ik_i| = \sqrt{k_r^2 + k_i^2}$$

(8)

Proca equation doesn’t contradict to the theory of electrodynamics in Maxwellian. If particle was massless ($m_\gamma = 0$), so that the Proca equation will be return to common form of Maxwell’s equation. Generally, the 4-dimensional form of Proca equation can be expressed in the form:

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{4}{3} k^2\right) A_\mu = -\mu_0 j_\mu$$

(9)

Where $A_\mu$ and $J_\mu$ is 4-vector potential ($A, i\frac{\phi}{c}$) and current density ($J, i\frac{\phi}{c}$). It is helpful to express the Proca equation in generality for free particles has to form

$$\left[\Box + \frac{m_\gamma^2 c^2}{\hbar^2}\right] \psi(x_\mu) = 0$$

(10)

2. Hypergeometric Method

Generally, the Schrödinger equation can be solved by reducing the system of particles to the 2nd order differential equation. Second-order differential equations are one of the most widely studied in mathematics, physical science, and engineering. The Newton’s second law that describe about motion of objects is expressed as a law that involves acceleration of a particle, which is the second derivative of position of the particle. In this paper, we used the hypergeometric method, which is the second order differential equation. The principles of the hypergeometric method are substituted with new variable and parameter to obtain a second-order differential equation of hypergeometric function which is expressed [5].

$$z(1 - z) \frac{d^2 \phi}{dz^2} + (c - (a + b + 1)z) \frac{d \phi}{dz} - ab \phi = 0$$

(11)
With the solution of the wave function is given as

$$\phi_\lambda (r) = \frac{m^2 c^2}{\hbar^2} + \left( \epsilon - V(r) \right)^2 \psi(r, t) = 0$$

The energy eigenvalue is obtained from the condition of the hypergeometric equation $a = -n$ or $b = -n$.

Where $n = 0, 1, 2, 3, \ldots$

3. **Results and Discussion**

Proca equation with energy eigenvalue and potential can be express as

$$\left[ \frac{\partial^2 \psi(t)}{c^2 \partial t^2} - d \right] \psi(r) = 0$$

Where $d$ is constant. The three-dimensional of the Proca equation express by

$$- \left[ \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \psi(r) = \left[ \frac{m^2 c^2}{\hbar^2} + \left( \epsilon - V(r) \right)^2 \right] \psi(r)$$

For the wave function $\psi(r)$, make the following separation ansatz

$$\psi(r) = R(r) \Theta(\theta) \Phi(\phi)$$

We get for radial part

$$\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial R}{\partial r} + \left[ \frac{m^2 c^2}{\hbar^2} + \left( \epsilon - V(r) \right)^2 - \frac{l(l+1)}{r^2} \right] R = 0$$

For angular part

$$\frac{1}{r^2} \frac{\partial}{\partial \theta} r^2 \frac{\partial \Theta}{\partial \theta} + \frac{1}{\sin \theta} \frac{\partial \Theta}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2} = -\frac{\lambda}{\hbar^2 c^2}$$

Where $\lambda$ is the separation constant, with $\lambda = \ell(l + 1), \ell = 0, 1, 2, \ldots$ and $m = 0, \pm 1, \pm 2, \pm 3, \ldots$

The Solution for radial differential equation follow as

$$\left[ \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{l(l+1)}{r^2} \right] R = 0$$

Hence, we use the common ansatz

$$R(r) = \frac{u(r)}{r}$$

We get

$$\left[ \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + \frac{m^2 c^4 + h^2 c^2 (\epsilon - V(r))^2}{\hbar^2 c^2} \right] u(r) = 0$$

In contrast to this, we now couple a potential to a square of the mass in the equation of motion. We perform the substitution $m^2 c^4 = m^2 c^4 + V^2$. A direct coupling to the mass would yield mixing terms of the mass and potential. Thus, the radial Proca equation with arbitrary potential reads

$$\left[ \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + \frac{m^2 c^4 + V^2}{\hbar^2 c^2} \right] u(r) = 0$$

Now, we set the potential by modified Posch Teller potential

$$V = \frac{\sigma(\sigma+1)}{\sinh^2 ar \cosh^2 ar}$$
Thus the radial Proca equation with modified Posch Teller potential reads
\[
\left[ \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + \frac{m_r^2 c^4}{\hbar^2 c^2} + \varepsilon^2 + \left( \frac{\nu(v-1)}{\sinh^2 ar} - \frac{\sigma(s+1)}{\cosh^2 ar} \right) \right] u(r) = 0
\]
(24)
Thus we get equation (23) by set \( E = \frac{m_r^2 c^4}{\hbar^2 c^2} + \varepsilon^2 \)
and applying approximation of centrifugal terms
\[
\frac{l(l+1)}{r^2} = \frac{\alpha^2 l(l+1)}{\sinh^2 ar}
\]
(25)
Substituting again with \( z = \cosh^2 ar \) and \( l(l+1) = \lambda \), we have
\[
z(1-z) \frac{d^2 u}{dz^2} + \frac{1}{2} (1-2az) \frac{du}{dz} + \left( \frac{1}{4a^2} \right) \left( \frac{\nu(v-1) - a^2 \lambda}{1-z} + \frac{\sigma(s+1)}{z} - E \right) = 0
\]
(26)
General solution for wave function describe as
\[
u = z^\rho (1-z)^\beta f(z)
\]
(27)
This equation can be simplified by setting with
\[
\frac{(\nu(v-1) - a^2 \lambda}{4a^2} = \nu'(v' - 1)
\]
(28)
\[
\frac{\sigma(s+1)}{4a^2} = \sigma'(s' + 1)
\]
(29)
Where the parameters set as \( \nu' = 2\beta \) and \( \sigma' = -2\rho \). Thus the radial Proca equation with modified Posch Teller potential can written
\[
z(1-z) \frac{d^2 f}{dz^2} + \left( \frac{2\rho + 1}{2} \right) - (2\rho + 2\beta + 1)z \frac{df}{dz} + \{k^2 - (\rho + \beta)^2\}f = 0
\]
(30)
This is second order differential equation of hypergeometric function, with the solution
\[
f = _2F_1(a, b, c; z)
\]
(31)
Where
\[
a = \rho + \beta + k
\]
\[
b = \rho + \beta - k
\]
Thus the solution of energet reads
\[
a = -n_r
\]
(32)
With \( n_r \) is the quantum number of radial part, where \( n_r = 0, 1, 2, \ldots \)
The definition in the principal quantum number
\[
n = l + 1 + n_r
\]
(33)
And we get the wave function for radial part as
\[
u = (\cosh^2 ar)^{-\sigma'}(\sinh^2 ar)^{-\nu'}_2F_1(a, b, c; z)
\]
(34)
The wave function of the radial Proca equation is also obtained in hypergeometric function. Thus, we get the energy eigen-value \( \varepsilon \) as
\[
\varepsilon = \frac{\hbar c k}{2a^2} \left( \frac{m_r^2 c^4}{\hbar^2 c^2} + \frac{1}{2} \right)^{1/2}
\]
(35)
The energy eigen-value is symmetric for the photon and antiphoton states. The energy eigenvalue $\epsilon$ does not depend on the orbital angular.

4. Conclusion
In this study, we have obtained of the radial Proca equation for the modified Posch Teller potential by the hypergeometric method. The solution of the energy eigenvalue of the radial Proca equation is obtained. The energy eigen-value is symmetric for the photon and antiphoton states. The energy eigenvalue $\epsilon$ does not depend on the orbital angular. The wave function of the radial Proca equation is also obtained in hypergeometric function.

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