The microcanonical theory and pseudoextensive systems

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In the present paper are considered the self-similarity scaling postulates in order to extend the Thermodynamics to the study of one special class of nonextensive systems: the pseudoextensive, those with exponential behavior for the asymptotical states density of the microcanonical ensemble. It is shown that this kind of systems could be described with the usual Boltzmann-Gibbs’ Distribution with an appropriate selection of the representation of the movement integrals. It is shown that the pseudoextensive systems are the natural frame for the application of the microcanonical thermostatistics theory of D. H. E. Gross.

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I. INTRODUCTION

In our previous work we addressed the problem of generalizing the extensive postulates in order to extend the application of the Thermostatistics to study of the nonextensive systems [1]. Our proposition was that this derivation could be carried out taking into consideration the self-similarity scaling properties of the systems with the increasing of their degrees of freedom and analyzing the conditions for the equivalence of the microcanonical ensemble with some generalization of the canonical ensemble in the thermodynamic limit (ThL). The last argument has a most general character than the Gibbs’ one, since it is not sustained by the axiom of the separability of one subsystem from a whole system, invalid supposition for the nonextensive systems due to the long-range correlations that exist in them.

In order to apply the above considerations it must be taken into account the geometrical aspects of the probabilistic distribution functions (PDF) of the ensembles. In the ref. [1] we showed that the microcanonical ensemble possesses reparametrization invariance, $\text{Diff}(\mathcal{S}_N)$, for the movement integrals abstract space of the distribution, $\mathcal{S}_N$. This property is completely in contradiction with the ergodic chauvinism that characterizes the traditional Thermodynamics. Under the supposition that the generalized canonical ensemble could be derived from a Probabilistic Thermodynamic Formalism (PThF), those based on the probabilistic interpretation of the entropy concept, it is deduced that its PDF only admits a certain set of representations of $\mathcal{S}_N$ in agreement with the self-similarity scaling properties of the system, $\mathcal{M}_c = \{R_I\}$, which are related by the transformations of the Special Lineal Group, $\text{SL}(R^n)$, where $n = \dim(\mathcal{S}_N)$. This fact suggests that a spontaneous symmetry breaking takes place from $\text{Diff}(\mathcal{S}_N)$ to $\text{SL}(R^n)$ due to the scaling properties of the systems during the ThL, that is, the ThL could be identified with the establishment of the self-similarity scaling properties in the system.

The present paper will be devoted to the application of the above ideas to the analysis of those systems possessing a simplest scaling laws: the exponential. In this particular case, the self-similarity scaling laws of the systems in the ThL are defined by:

\[
\begin{align*}
N \to N(\alpha) &= \alpha N \\
I \to I(\alpha) &= \alpha^\pi I \\
a \to a(\alpha) &= \alpha^\kappa a
\end{align*}
\Rightarrow W_{\text{asym}}(\alpha) = \exp[\alpha^\pi \ln W_{\text{asym}}(1)] ,
\]

where $W_{\text{asym}}$ is the accessible volume of the macroscopic state in the system configurational space in the ThL, $\alpha$ is the scaling parameter, $\pi$, $\kappa$ and $\kappa$ are real constants characterizing the scaling transformations, where:

\[0 < \kappa \leq 1.\]

Finally, $W_{\text{asym}}(\alpha)$ is given by:

\[
W_{\text{asym}}(\alpha) = W_{\text{asym}}[I(\alpha), N(\alpha), a(\alpha)].
\]

The equality in the Eq. (2) can only take place when exist non-correlated states in the ThL. It will be referred as pseudoextensive those systems with scaling laws given by the above conditions. The reason for such denomination is easy to understand since the extensive properties of the systems described by the traditional Thermodynamic are one special case of the exponential scaling
laws. It is easy to see that a great variety of systems could be grouped under this denomination. In general, all the Hamiltonian systems of identical particles with an additive kinetic part are pseudoextensive independently of the interactions among them.

According to ref. [3], in this case it must be assumed the classical expression of the Boltzmann’s Principle [4]. The Boltzmann’s Principle constitutes the starting point of the microcanonical thermostatistics theory of D. H. E. Gross (see the refs. [2–4]). Following Boltzmann, all the statistical information about the general properties of the macroscopic state of any Hamiltonian system is contained in the accessible configurational space volume of the microcanonical ensemble, \( W(I, N, a) \):

\[
W(I, N, a) = \Omega(I, N, a) \delta I_o = \delta I_o \int \delta[I - I_N(X; a)] dX
\]

where \( \delta I_o \) is a suitable constant volume element, which makes \( W \) dimensionless. For the pseudoextensive systems the ordering information is contained by the function \( \gamma(I, N, a) \):

\[
\gamma(I, N, a) \equiv \frac{1}{N^\kappa} \ln W(I, N, a),
\]

which is scaling invariant when \( \mathcal{R}_I \in \mathcal{M}_c \) and the ThL is reached.

In this case, the corresponding information entropy is the very well known Shannon-Boltzmann-Gibbs’ extensive entropy (SBGE):

\[
S_{SBG} = - \sum_k p_k \ln p_k.
\]

In the thermodynamic equilibrium the SBGE leads to the ordinary canonical ensemble where its PDF are known as the Boltzmann-Gibbs’ Distributions:

\[
\omega(X; \beta, N, a) = \frac{1}{Z(\beta, N, a)} \exp[-\beta \cdot I_N(X; a)],
\]

where \( \beta \) are the canonical parameters and \( Z(\beta, a, N) \) is the partition function. The states density of the microcanonical ensemble and the partition function of the canonical ensemble are related by means of the Laplace Transformation:

\[
Z(\beta, N, a) = \int \exp[-\beta \cdot I] W(I, N, a) \frac{dI}{\delta I_o},
\]

which establishes the connection between the Boltzmann’s entropy with the fundamental thermodynamics potential of the canonical distribution, the Planck Potential:

\[
P(\beta, N, a) = -\ln Z(\beta, N, a)
\]

The Planck Potential is adopted since it does not introduce any preference with an specific integral of movement of the microcanonical distribution, in order to be consequent with the \( SL(R^n) \) invariance. The last integral can be rewritten as:

\[
\exp[-P(\beta, N, a)] = \int \exp[-\beta \cdot I + S_B(I, N, a)] \frac{dI}{\delta I_o}.
\]

Immediately, it is recognized in the exponential argument the Legendre’s Transformation between the thermodynamic potentials. If the integrals of movement and the Boltzmann’s entropy has the same scaling transformation in the ThL, that is:

\[
\chi = \kappa,
\]

this integral will have a very sharp peak around the maximum value, if this maximum exists, and therefore, the main contribution to this integral will come from it. Thus, for the equivalence between the microcanonical and the canonical ensembles only are possible those representations \( \mathcal{R}_I \) that satisfy the condition given by Eq. (12).

When the equivalence is held, the Planck potential could be approached by means of the Legendre’s formalism:

\[
P(\beta, N, a) \simeq \max_I [\beta \cdot I - S_B(I, N, a)],
\]

where \( \beta \) is obtained by:

\[
\beta_\mu = \left. \frac{\partial}{\partial I_\mu} S_B \right|_{I=I_M}.
\]

For the stability of the maximum is necessary that all the eigenvalues of the curvature tensor:

\[
(K_B)_{\mu \nu} = \left. \frac{\partial}{\partial I_\mu} \frac{\partial}{\partial I_\nu} S_B \right|_{I=I_M},
\]

be negatives, that is, the Boltzmann’s entropy must be locally concave around the maximum. In this case, in the canonical ensemble there will be small fluctuations of the integrals of movement around its mean values with a standard deviation:
\[
\sqrt{(I^\mu - I_M^\mu)(I^\nu - I_M^\nu)} \approx \frac{1}{a^2}. \tag{15}
\]

However, if the concavity condition for the entropy is not hold, there will be a catastrophe in the Laplace transformation and the canonical ensemble will not be able to describe the system. In the general sense, this situation is a sign of the occurrence of generalized phase transitions.

The scaling behavior of the system impose one restriction to the macroscopic observables of the systems, the generalized Duhem-Gibbs’ relation:

\[
\kappa (\beta \cdot I^*-S_B) + \mu N + \pi_a \beta \cdot f_a a = 0, \tag{16}
\]

where:

\[
\mu = \frac{\partial}{\partial N} S_B, \quad \beta \cdot f_a = \frac{\partial}{\partial a} S_a \text{ with } f_a = \left\langle -\frac{\partial}{\partial a} I_N^a (X;a) \right\rangle. \tag{17}
\]

Although the pseudoextensive systems are nonextensive in the usual sense, their study could be carried out with the same formalism used for the ordinary extensive systems with appropriate selection of the representation of the movement integrals space. The Eq.\([12]\) leads to condition of the **homogeneous scaling of the movement integrals and the entropy** in order to satisfy the requirement of the scaling invariance of the canonical parameters, the validity of the zero principle of the thermodynamics. This is the cause of the spontaneous symmetry breaking. Although in the microcanonical ensemble all the representations of the movement integrals are equivalent, in the generalized canonical ensemble only are possible those representations with an **homogeneous nondegenerated scaling**. To understand the term **nondegenerate**, let us show the following example. Let be \(A\) and \(B\) two integrals of movement with different scaling behavior:

\[
A \sim \alpha^a, \quad B \sim \alpha^b \quad \text{with} \quad a > b. \tag{18}
\]

In another representation, these integrals could be equivalently represented in the microcanonical ensemble by:

\[
I^\pm = A \pm B, \tag{19}
\]

but, in this case, their scaling behavior will be given by:

\[
I^\pm \approx A \sim \alpha^a. \tag{20}
\]

\(I^\pm\) have a homogeneous scaling in the ThL, but they are not independent. The canonical parameters derived from \(I^\pm, \beta^\pm = \frac{\partial S_B}{\partial I^\pm}\), will be identically in the ThL. This leads to the **trivial vanishing** of the curvature tensor determinant:

\[
\lim_{\alpha \to \infty} \alpha^{2a} \det (K_B)_{\mu\nu} = 0. \tag{21}
\]

This is an example of an homogeneous degenerate scaling representation. These representations are inadmissible for the generalized canonical ensemble. The above considerations give a criterium of an homogeneous nondegenerated scaling representation, the non trivial vanishing of the curvature tensor determinant:

\[
\lim_{\alpha \to \infty} \det (\alpha^\kappa \partial_\mu \partial_\nu S_\eta) \neq 0 \tag{22}
\]

A final remark: In general, each a representation \(R_I\) belonging to \(M_c\) will possess an specific region of \(\mathcal{Z}_N\) in which will be equivalent the microcanonical and the canonical ensemble. This fact reflects the inconsistency of the above analysis because the description of the system should not depend on the representation. It could be corrected introducing a local reparametrization invariant formalism, maybe, introducing the **covariant derivative**.

### III. CONCLUSIONS

We have showed that the pseudoextensive systems could be described by the ordinary Boltzmann-Gibbs’ Statistic with an appropriate selection of the representation of the movement integrals space of the microcanonical ensemble, \(\mathcal{Z}_N\). The key for this conclusion is the assuming of scaling postulates: The equivalence of the microcanonical description with a generalized canonical one during the realization of the Thermodynamic Limit by means of the self-similarity scaling properties of the fundamental physical observables. It is shown that the pseudoextensive systems are the natural frame for the application of the microcanonical thermostatistics theory of D. H. E. Gross.

The general lines of the present analysis is analogue to the method used by D. H. E. Gross in deriving the thermodynamic formalism of his formulation. However our derivation clarifies some incongruences present in this theory. The Gross’ theory does not take into account the reparametrization invariance of the microcanonical ensemble neither the scaling properties of the nonextensive systems in general way. In its present status, this theory could only be valid for those systems that in the ThL converge to an ordinary extensive systems, since its formalism is based on the consideration of the additive representations of the movement integrals (see for example in refs.\([3,4]\)).
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