UNIQUENESS PROPERTY FOR SPHERICAL HOMOGENEOUS SPACES

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Abstract. Let $G$ be a connected reductive group. Recall that a homogeneous $G$-space $X$ is called spherical if a Borel subgroup $B \subset G$ has an open orbit on $X$. To $X$ one assigns certain combinatorial invariants: the weight lattice, the valuation cone and the set of $B$-stable prime divisors. We prove that two spherical homogeneous spaces with the same combinatorial invariants are equivariantly isomorphic. Further, we recover the group of $G$-equivariant automorphisms of $X$ from these invariants.

1. Introduction

At first, let us fix some notation and introduce some terminology.

Throughout the paper the base field $\mathbb{K}$ is algebraically closed and of characteristic zero. Let $G$ be a connected reductive group. Fix a Borel subgroup $B \subset G$. An irreducible $G$-variety $X$ is said to be spherical if $X$ is normal and $B$ has an open orbit on $X$. The last condition is equivalent to $\mathbb{K}(X)^B = \mathbb{K}$. Note that a spherical $G$-variety contains an open $G$-orbit. An algebraic subgroup $H \subset G$ is called spherical if $G/H$ is spherical.

The theory of spherical varieties and related developments seem to be the most important topic in the study of algebraic transformation groups for the last twenty five years (see [Bri4], [Kn1], [T] for a review of this theory). The first problem arising in the study of spherical varieties is their classification. The basic result here is the Luna-Vust theory, [LV], describing all spherical $G$-varieties with a given open $G$-orbit. The description is carried out in terms of certain combinatorial invariants. So the classification of all spherical varieties is reduced to the description of all spherical subgroups and the computation of the corresponding combinatorial invariants. However, there are many spherical subgroups and their explicit classification (i.e., as subsets of $G$) seems to be possible only in some very special cases. For example, connected reductive spherical subgroups were classified (in fact, partially) in [Kr], [Bri3],[M]. In the general case one should perform the classification using a different language. A reasonable language was proposed by Luna, [Lu2]. Essentially, his idea is to describe spherical homogeneous spaces in terms of combinatorial invariants established in [LV]. One should prove that these invariants determine a spherical homogeneous space uniquely and then check the existence of the spherical homogenous space corresponding to any set of invariants satisfying some combinatorial conditions. Luna completed his program in [Lu2] for groups of type $A$ (that is, when any simple normal subgroup is locally isomorphic to a special linear group). Later on, the full classification for groups of type $A-D$ was carried out by Bravi, [Bra1]. A partial classification for type $A-C$ was obtained in [Pe1]. Finally, recently Bravi obtained the classification in types $A$-$D$-$E$, [Bra2]. The main result of the

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present paper is the proof of the uniqueness part of Luna’s program. An advantage of our approach over Luna’s is that we do not use long case-by-case considerations.

Let us describe combinatorial invariants in interest. We will use standard notation recalled in Section 2 below. Fix a maximal torus \( T \subset B \).

Let \( X \) denote a spherical \( G \)-variety. The set \( \mathfrak{X}_{G,X} := \{ \mu \in \mathfrak{X}(T) \mid \mathbb{K}(X)_{\mu}^{(B)} \neq \{0\} \} \) is called the \textit{weight lattice} of \( X \). This is a sublattice in \( \mathfrak{X}(T) \). By the \textit{Cartan space} of \( X \) we mean \( \mathfrak{a}_{G,X} := \mathfrak{X}_{G,X} \otimes \mathbb{Q} \). This is a subspace in \( t(\mathbb{Q})^* \).

Next we define the valuation cone of \( X \). Let \( v \) be a \( \mathbb{Q} \)-valued discrete \( G \)-invariant valuation of \( \mathbb{K}(X) \). Since \( X \) is spherical, we have \( \dim \mathbb{K}(X)_{\mu}^{(B)} = 1 \) for any \( \mu \in \mathfrak{X}_{G,X} \). So one defines the element \( \varphi_v \in \mathfrak{a}_{G,X}^* \) by the formula
\[
(\varphi_v, \mu) = v(f_\mu), \forall \mu \in \mathfrak{X}_{G,X}, f_\mu \in \mathbb{K}(X)_{\mu}^{(B)} \setminus \{0\}.
\]

It is known, see [Kn1], that the map \( v \mapsto \varphi_v \) is injective. Its image is a finitely generated convex cone in \( \mathfrak{a}_{G,X}^* \). We denote this cone by \( V_{G,X} \) and call it the \textit{valuation cone} of \( X \).

Let \( \mathcal{D}_{G,X} \) denote the set of all prime \( B \)-stable divisors of \( X \). This is a finite set. To \( D \in \mathcal{D}_{G,X} \) we assign \( \varphi_D \in \mathfrak{a}_{G,X}^* \) by \( (\varphi_D, \mu) = \text{ord}_D(f_\mu), \mu \in \mathfrak{X}_{G,X}, f_\mu \in \mathbb{K}(X)_{\mu}^{(B)} \setminus \{0\} \). Further, for \( D \in \mathcal{D}_{G,X} \) set \( G_D := \{ g \in G \mid gD = D \} \). Clearly, \( G_D \) is a parabolic subgroup of \( G \) containing \( B \). Choose \( \alpha \in \Pi(g) \). Set \( \mathcal{D}_{G,X}(\alpha) := \{ D \in G_D \mid P_\alpha \not\subset G_D \} \). Here and below by \( P_\alpha \) we denote the parabolic subgroup of \( G \) generated by \( B \) and the one-dimensional unipotent subgroup of \( G \) corresponding to the root subspace of weight \(-\alpha\).

Below we regard \( \mathcal{D}_{G,X} \) as an abstract set equipped with two maps \( D \mapsto \varphi_D, D \mapsto G_D \). For instance, if \( X_1, X_2 \) are spherical \( G \)-varieties, then, when writing \( \mathcal{D}_{G,X_1} = \mathcal{D}_{G,X_2} \), we mean that there exists a bijection \( \iota : \mathcal{D}_{G,X_1} \to \mathcal{D}_{G,X_2} \) such that \( G_D = G_{\iota(D)}, \varphi_D = \varphi_{\iota(D)} \).

**Theorem 1.** Let \( H_1, H_2 \) be spherical subgroups of \( G \). If \( \mathfrak{X}_{G,G/H_1} = \mathfrak{X}_{G,G/H_2}, V_{G,G/H_1} = V_{G,G/H_2}, \mathcal{D}_{G,G/H_1} = \mathcal{D}_{G,G/H_2} \), then \( H_1, H_2 \) are \( G \)-conjugate.

Thanks to Theorem 1, one may hope to describe all invariants of a spherical homogeneous space \( X \) in terms of \( \mathfrak{X}_{G,X}, V_{G,X}, \mathcal{D}_{G,X} \).

We want to describe the group \( \text{Aut}^G(X) \) of \( G \)-equivariant automorphisms of \( X \). For any \( \varphi \in \text{Aut}^G(X), \lambda \in \mathfrak{X}_{G,X} \) there is \( a_{\varphi,\lambda} \in \mathbb{K}^* \) such that \( \varphi|_{\mathbb{K}(X)^*} = a_{\varphi,\lambda} \text{id} \). The map \( (\lambda, \varphi) \mapsto a_{\varphi,\lambda} \) gives rise to the homomorphism \( \text{Aut}^G(X) \to A_{G,X} := \text{Hom}_{\mathbb{Z}}(\mathfrak{X}_{G,X}, \mathbb{K}^*) \). This homomorphism is injective and its image \( \mathfrak{A}_{G,X} \) is closed, see, for example, [Kn4], Theorem 5.5. Therefore \( \mathfrak{A}_{G,X} \) is recovered from \( \Lambda_{G,X} := \{ \lambda \in \mathfrak{X}_{G,X} \mid (\lambda, \mathfrak{A}_{G,X}) = 1 \} \). We call \( \Lambda_{G,X} \) the \textit{root lattice} of \( X \). By the root lattice of an arbitrary spherical \( G \)-variety \( X \) (also denoted by \( \Lambda_{G,X} \)) we mean the root lattice of the open \( G \)-orbit in \( X \).

To describe \( \Lambda_{G,X} \) in terms of \( \mathfrak{X}_{G,X}, V_{G,X}, \mathcal{D}_{G,X} \) we need to recall some further facts about the structure of \( V_{G,X} \). Namely, fix an \( N_G(T) \)-invariant scalar product on \( t(\mathbb{Q}) \). It induces the scalar product on \( \mathfrak{a}_{G,X} \). With respect to this scalar product \( V_{G,X} \) becomes a Weyl chamber for a (uniquely determined) linear group \( W_{G,X} \) generated by reflections, see [Bri3]. It turns out, see, for example, [Kn4], Sections 4.6, that \( \Lambda_{G,X}, \mathfrak{X}_{G,X} \) are \( W_{G,X} \)-stable. Denote by \( \Psi_{G,X} \) (resp., \( \overline{\Psi}_{G,X} \)) the set of primitive elements \( \alpha \in \mathfrak{X}_{G,X} \) (resp., \( \alpha \in \Lambda_{G,X} \)) such that \( \ker \alpha \subset \mathfrak{a}_{G,X} \) is a wall of \( V_{G,X} \) and \( (\alpha, V_{G,X}) \leq 0 \). Clearly, \( \Psi_{G,X}, \overline{\Psi}_{G,X} \) are systems of simple roots with Weyl group \( W_{G,X} \). An element of \( \Psi_{G,X} \) is called a \textit{spherical root} of \( X \). It was proved in [Kn4] that \( \overline{\Psi}_{G,X} \) generates \( \Lambda_{G,X} \). On the other hand, in general, \( \Psi_{G,X} \) does not generate \( \mathfrak{X}_{G,X} \).

We recover \( \overline{\Psi}_{G,X} \) from \( \Psi_{G,X}, \mathcal{D}_{G,X} \). To do this we define a certain subset \( \Psi_{G,X}^+ \subset \Psi_{G,X} \) of \textit{distinguished} roots, see Definition 4.1.1.
Theorem 2. \( \Psi_{G,X} = (\Psi_{G,X} \cap \Lambda(g) \setminus \Psi_{G,X}^+) \cup \{2\alpha|\alpha \in \Psi_{G,X}^+ \cup (\Psi_{G,X} \setminus \Lambda(g))\} \).

There are other invariants of a spherical homogeneous space \( G/H \) that can be described in terms of \( \mathcal{X}_{G,G/H}, \mathcal{V}_{G,G/H}, \mathcal{D}_{G,G/H} \). For example, Knop, [Kn1], described the set \( \mathcal{H}_H \) of all algebraic subgroups \( \tilde{H} \subset G \) such that \( H \subset \tilde{H} \) and \( \tilde{H}/H \) is connected, see Subsection 3.4 for details. This description plays a crucial role in our proofs.

Theorems 1,2 seem to be quite independent from each other. The reason why they are brought together in a single paper is twofold. First, the key ideas of their proofs are much alike. Second, we use Theorem 2 in the proof of Theorem 1.

Now let us briefly describe the content of this paper. In Section 2 we present conventions and the list of notation we use. In Section 3 we gather some preliminary results. Most of them are standard. Section 4 is devoted to the proofs of Theorems 1,2. In the beginning of Sections 3,4 their content is described in more detail. All propositions, definitions, etc. are numbered within a subsection.

Finally, we would like to discuss applications of our results. Theorem 1 significantly simplifies the proof of the Knop conjecture, [Lo2], allowing to get rid of many ugly technical considerations.

As for Theorem 2, we use it in [Lo4] to prove Brion’s conjecture on the smoothness of Demazure embeddings, see [Bri3]. In fact, it is that application that motivated us to state Theorem 2. Let us recall the definition of the Demazure embedding. Let \( h \) be a subalgebra of \( g \). Suppose \( h = n(g) \) and the subgroup \( H := N_G(h) \subset G \) is spherical. Consider \( h \) as a point of the Grassmanian \( Gr_d(g) \). The \( G \)-orbit of \( h \) is isomorphic to the homogeneous space \( G/H \). By the Demazure embedding of \( G/H \) we mean the closure \( \overline{Gh} \) of \( Gh \) in \( Gr_d(g) \). Brion conjectured that \( \overline{Gh} \) is smooth. Again, for groups of type A this conjecture was proved by Luna in [Lu3].

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2. Notation, Terminology and Conventions

If an algebraic group is denoted by a capital Latin letter, then we denote its Lie algebra by the corresponding small German letter.

We fix a Borel subgroup \( B \subset G \) and a maximal torus \( T \subset B \). This allows us to define the root system \( \Delta(g) \), the Weyl group \( W(g) \), the root lattice \( \Lambda(g) \) and the system of simple roots \( \Pi(g) \) of \( g \). For simple roots and fundamental weights we use the notation of [OV]. Note that the character groups \( \mathcal{X}(T), \mathcal{X}(B) \) are naturally identified. By \( B^- \) we denote the Borel subgroup of \( G \) opposite to \( B \) and containing \( T \).

By a spherical subalgebra of \( g \) we mean the Lie algebra of a spherical subgroup of \( G \).

If \( X_1, X_2 \) are \( G \)-varieties, then we write \( X_1 \cong G X_2 \) when \( X_1, X_2 \) are \( G \)-equivariantly isomorphic. Note that if \( V_1, V_2 \) are \( G \)-modules, then \( V_1 \cong G V_2 \) iff \( V_1, V_2 \) are isomorphic as \( G \)-modules.

Let \( X_1, X_2 \) be spherical varieties. We write \( X_1 \cong G X_2 \) if \( \mathcal{X}_{G,X_1} = \mathcal{X}_{G,X_2}, \mathcal{V}_{G,X_1} = \mathcal{V}_{G,X_2}, \mathcal{D}_{G,X_1} = \mathcal{D}_{G,X_2} \). When \( X_1 = G/H_1, X_2 = G/H_2 \) and \( X_1 \cong G X_2 \) we write \( H_1 \equiv G H_2 \).

Let \( Q \) be a parabolic subgroup of \( G \) containing either \( B \) or \( B^- \). There is a unique Levi subgroup of \( Q \) containing \( T \), we call it the standard Levi subgroup of \( Q \).
Now let $S$ be a reductive group and $\Gamma$ be a group with a fixed homomorphism to the group of outer automorphisms of $S$. There is a natural right action of $\Gamma$ on the set of (isomorphism classes of) $S$-modules. Namely, let $V$ be an $S$-module and $\rho : G \to \text{GL}(V)$ be the corresponding representation. For $V^\gamma$, $\gamma \in \Gamma$, we take the $S$-module corresponding to the representation $\rho \circ \overline{\gamma}$, where $\overline{\gamma}$ is a representative of the image of $\gamma$.

$\sim_\Gamma$ the equivalence relation induced by an action of a group $\Gamma$

$A^{(B)}_\lambda = \{ a \in A | b.a = \lambda(b)a, \forall b \in B \}$.

$A^{(B)}_\lambda = \cup_{\lambda \in (\chi(B))} A^{(B)}_\lambda$.

$A^\chi(X) = \text{Hom}(\chi_{G,X}, \mathbb{K}^\times)$.

$a_{G,X}$ the Cartan space of a spherical $G$-variety $X$.

$\mathfrak{X}_{G,X}$ the image of $\text{Aut}^G(X^0)$ in $A_{G,X}$, where $X^0$ is the open $G$-orbit of a spherical variety $X$.

$\mathfrak{X}_{G,X}(\alpha) = \{ \varphi \in \mathfrak{X}_{G,X} | \langle \alpha, \varphi \rangle = -1 \}$.

$(a^\dddot{H}, D^\dddot{H})$ the colored subspace associated with $\dddot{H} \in \mathcal{H}_H$.

$\text{Aut}^G(X)$ the group of all $G$-equivariant automorphisms of $X$.

$\mathcal{C}S_{G,X}$ the set of all colored subspaces of $(a_{G,X}^*, D_{G,X})$.

$D_{G,X}$ the set of all prime $B$-stable divisors of $X$.

$D_{G,X}(\alpha) = \{ D \in D_{G,X} | P_\alpha \not\subset G_D \}$.

$f_x$ a nonzero element in $\mathbb{K}(X)_{\chi}^{(B)}$.

$(f)$ the divisor of a rational function $f$.

$(G, G)$ the derived subgroup of a group $G$.

$[\mathfrak{g}, \mathfrak{g}]$ the derived subalgebra of a Lie algebra $\mathfrak{g}$.

$G^0$ the unit component of an algebraic group $G$.

$G^\circ V$ the homogeneous bundle over $G/H$ with a fiber $V$.

$G_x$ the stabilizer of $x$ under an action of $G$.

$\text{Gr}_d(V)$ the Grassman variety consisting of all $d$-dimensional subspaces of a vector space $V$.

$\mathcal{H}_H$ the set of all algebraic subgroups $\dddot{H}$ of $G$ such that $H \subset \dddot{H}$ and $\dddot{H}/H$ is connected.

$\mathcal{H}^{\dddot{H}} = \{ \dddot{H} \in \mathcal{H}_H | R_u(H) \subset R_u(\dddot{H}), \dddot{H}/R_u(\dddot{H}) = H/R_u(H), R_u(\dddot{h})/R_u(h) \text{ is an irreducible } H\text{-module} \}$.

$\mathcal{H}^{\dddot{H}}$ the set of all algebraic subgroups $H \subset \dddot{H}$ such that $\dddot{H}/H$ is connected.

$L_{G,X}$ the standard Levi subgroup of $P_{G,X}$.

$N_{G}(H) = \{ g \in G | ghg^{-1} = H \}$.

$N_{G}(h) = \{ g \in G | \text{Ad}(g)h = h \}$.

$n_{\mathfrak{g}}(\mathfrak{h}) = \{ \xi \in \mathfrak{g} | [\xi, \mathfrak{h}] \subset \mathfrak{h} \}$.

$P_{G,X} = \cap_{D \in D_{G,X}} G_D$.

$P_\alpha$ the minimal parabolic subgroup of $G$ containing $B$ associated with $\alpha \in \Pi(\mathfrak{g})$.

Pic($X$) the Picard group of a variety $X$.

Pic$_G(X)$ the equivariant Picard group of a $G$-variety $X$.

$R_u(G)$ the unipotent radical of an algebraic group $G$.

$rk_{G}(X) = \text{rk} \chi_{G,X}$.
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\[ \text{Span}_A(M) = \{a_1m_1 + \ldots + a_km_k, a_i \in A, m_i \in M\}. \]

\[ \text{Supp}(\gamma) \text{ the support of } \gamma \in \text{Span}_Q(\Pi(g)), \text{ that is, the set } \{\alpha \in \Pi(g)|n_\alpha \neq 0\}, \text{ where } \gamma = \sum_{\alpha \in \Pi(g)} n_\alpha \alpha. \]

\[ X(G) \text{ the character group of an algebraic group } G. \]

\[ X_{G,X} \text{ the weight lattice of a spherical } G\text{-variety } X. \]

\[ X^+_G,X \text{ the weight monoid of a spherical } G\text{-variety } X. \]

\[ X/\Gamma \text{ the geometric quotient for an action } \Gamma : X. \]

\[ \#X \text{ the cardinality of a set } X. \]

\[ \Psi_{G,X} \text{ the system of spherical roots of a spherical } G\text{-variety } X. \]

\[ \Psi^+_{G,X} \text{ the set of all distinguished elements of } \Psi_{G,X}. \]

\[ \Psi_{G,X}^i \text{ the subset of } \Psi^+_{G,X} \text{ of all roots of type } i, i = 1, 2, 3. \]

\[ \varphi_D \text{ the vector in } a^*_G,X \text{ associated with } D \in D_{G,X}. \]

3. Preliminaries

This section does not contain new results. Its goal is to recall some basic facts about spherical varieties used in the proofs of the main theorems. We often present short proofs of some results when we do not know an appropriate reference.

\( X \) denotes a spherical \( G \)-variety. Throughout the section we set \( X := X_{G,X}, D := D_{G,X}, \Psi := \Psi_{G,X}, P := P_{G,X}, \text{ etc.} \)

In Subsection 3.1 we recall some basic facts about the group \( \text{Aut}^G(X) \). Subsection 3.2 is devoted to spherical homogeneous spaces admitting a so called wonderful embedding. We also present there results concerning systems of spherical roots. In Subsection 3.3 we state a finiteness result for spherical subalgebras coinciding with their normalizers. It plays an important role in the proofs of Theorems 1,2.

Subsection 3.4 deals with the structure of the set \( H_H \). This description is due to Knop, [Kn1], and is of major importance in our proofs. We describe the structure of the ordered set \( H_H \) in terms of so called colored subspaces, see Definition 3.4.1. Then we describe the combinatorial invariants of \( G/\tilde{H}, \tilde{H} \in H_H \).

Subsection 3.5 is devoted to different results related to the local structure theorem. This theorem is used to reduce the study of the action \( G : X \) to the study of an action of a certain Levi subgroup \( M \subset G \) on a certain subvariety \( X' \subset X \). Mostly, we use the theorem in the situation described in Example 3.5.2. Originally, the local structure theorem was proved independently in [BLV],[G], but we use its variant due to Knop, [Kn3].
Finally, in Subsection 3.6 we gather different simple properties of affine spherical varieties.

3.1. **Equivariant automorphisms.** Throughout this subsection \(X = G/H\).

One can identify \(\text{Aut}^G(X)\) with \(N_G(H)/H\). The group \(\text{Aut}^G(X)\) acts naturally on \(\mathcal{D}\) and the maps \(D \mapsto G_D, D \mapsto \varphi_D\) are \(\text{Aut}^G(X)\)-invariant.

As it was explained in Introduction, there is a natural monomorphism \(\text{Aut}^G(X) \to A\). So \(\text{Aut}^G(X)\) is commutative.

**Lemma 3.1.1.** \(N_G(H) = N_G(H^\circ)\).

*Proof.* Since \(H^\circ\) is also spherical, the group \(N_G(H^\circ)/H^\circ\) is commutative. Thus \(H\) is a normal subgroup of \(N_G(H^\circ)\). \(\square\)

The next lemma seems to be well-known, see, for example, [Lo3], Lemma 7.17.

**Lemma 3.1.2.** Let \(\overline{X}\) be a quasiaffine spherical variety, whose open \(G\)-orbit is isomorphic to \(X\). Then \(\text{Aut}^G(\overline{X}) = \text{Aut}^G(X)\).

**Lemma 3.1.3.** Let \(X_1, X_2\) be quasiaffine spherical \(G\)-varieties and \(\psi : X_1 \to X_2\) a dominant \(G\)-equivariant morphism. For any \(\varphi \in \text{Aut}^G(X_1)\) there exists a unique element \(\varphi \in \text{Aut}^G(X_2)\) such that \(\psi \circ \varphi = \varphi \circ \psi\).

*Proof.* Note that \(\psi^*(\mathbb{K}[X_2]) \subset \mathbb{K}[X_1]\) is \(\varphi\)-stable. For \(\varphi\) we take a unique element of \(\text{Aut}^G(X_2)\) coinciding with \(\varphi\) on \(\mathbb{K}[X_2]\). \(\square\)

**Lemma 3.1.4** ([Kn4], Corollary 6.5). The Lie algebra of \(\mathfrak{A}\) coincides with \(\mathbb{K} \otimes_{\mathbb{Q}} (\mathcal{V} \cap -\mathcal{V})\).

**Lemma 3.1.5.** Let \(\Gamma\) be an algebraic subgroup of \(\mathfrak{A} \cong \text{Aut}^G(X)\). Denote by \(\pi\) the quotient morphism \(X \to X/\Gamma\).

1. \(\mathfrak{X}_{G,X/\Gamma} = \{\chi \in \mathfrak{X}|(\chi, \Gamma) = 1\}\) and \(V_{G,X/\Gamma} = \pi_*(\mathcal{V})\), where \(\pi_*\) denotes the restriction projection \(\mathfrak{a}^* \to \mathfrak{a}^*_{G,X/\Gamma}\).
2. The map \(\pi_* : \mathcal{D} \to \mathcal{D}_{G,X/\Gamma}, D \mapsto \pi(D)\), is the quotient map for the action \(\Gamma : \mathcal{D}\). Further, \(G_D = G_{\pi_*(D)}, \varphi_{\pi_*(D)} = \pi_*(\varphi_D)\).
3. If \(\Gamma\) is finite, then \(\Lambda_{G,X/\Gamma} = \Lambda\).

*Proof.* The equality of the weight lattices and assertion 2 are easy. The equality of the valuation cones follows from Corollary 1.5 in [Kn1]. Assertion 3 is a special case of Theorem 6.3 from [Kn4]. \(\square\)

**Corollary 3.1.6.** Let \(H_1, H_2\) be spherical subgroups of \(G\). If \(\mathfrak{X}_{G,G/H_1} = \mathfrak{X}_{G,G/H_2}\) and \(\mathfrak{h}_1 \sim_G \mathfrak{h}_2\), then \(H_1 \sim_G H_2\).

*Proof.* We may assume that \(H_1^\circ = H_2^\circ\). By Lemma 3.1.5, \(\mathfrak{X}_{G,G/H_1} = \{\mu \in \mathfrak{X}_{G,G/H_1}|(\mu, H_1/H_1^\circ) = 1\}\). It follows that \(H_1/H_1^\circ = H_2/H_2^\circ\) (as subgroups in \(N_G(H_1)/H_1^\circ\)). \(\square\)

3.2. **Wonderful embeddings and spherical roots.** In this subsection \(X = G/H\).

We recall that a \(G\)-variety \(\overline{X}\) is called a **wonderful embedding** of \(X\) if

1. \(\overline{X}\) is smooth and projective.
2. There is an open \(G\)-equivariant embedding \(X \hookrightarrow \overline{X}\).
3. \(\overline{X} \setminus X\) is a divisor with normal crossings (in particular, this means that all irreducible components of \(\overline{X} \setminus X\) are smooth).
4. Let \(D_1, \ldots, D_r\) be irreducible components of \(\overline{X} \setminus X\). Then for any \(I \subset \{1, \ldots, r\}\) the subvariety \(\bigcap_{i \in I} D_i \setminus \bigcup_{j \notin I} D_j\) is a single \(G\)-orbit.
It is known that a wonderful embedding of \( X \) is unique if it exists, see, for example, [Lu2] or [T], Section 30.

**Proposition 3.2.1.** \( X \) has a wonderful embedding iff \( \mathfrak{x} = \text{Span}_\mathbb{Z}(\Psi) \).

**Proof.** In fact, this is proved in [Kn], Corollary 7.2. \( \square \)

**Corollary 3.2.2.** Let \( \Gamma \) be the annihilator of \( \Psi \) in \( \mathfrak{a} \) and \( \tilde{H} \) denote the inverse image of \( \Gamma \) in \( N_G(H) \). Then \( G/\tilde{H} \) admits a wonderful embedding. In particular, if \( H = N_G(H) \), then \( X \) admits a wonderful embedding.

**Proof.** Clearly, \( \mathfrak{x}_{G,G/\tilde{H}} = \text{Span}_\mathbb{Z}(\Psi) \). By Lemma 3.1.5, \( \Psi_{G,G/\tilde{H}} = \Psi \). \( \square \)

Thanks to Corollary 3.2.2, all pairs \( (\alpha, P) \), \( \alpha \in \Psi \), correspond to wonderful varieties of rank 1 and so are obtained from those listed in [Wa], Table 1 by a so called *parabolic induction*.

The following proposition describes a relation between \( \Psi, \mathcal{D} \). It was proved by Luna using wonderful varieties.

**Proposition 3.2.3** ([Lu1], Proposition 3.4). For \( \alpha \in \Pi(\mathfrak{g}) \) exactly one of the following possibilities holds:

(a) \( \mathcal{D}(\alpha) = \emptyset \).
(b) \( \alpha \in \Psi \). Here \( \mathcal{D}(\alpha) = \{D^+, D^-\} \) and \( \varphi_{D^+} + \varphi_{D^-} = \alpha^\vee|_a \).
(c) \( 2\alpha \in \Psi \). In this case \( \mathcal{D}(\alpha) = \{D\} \) and \( \varphi_D = \frac{1}{2}\alpha^\vee|_a \).
(d) \( \mathcal{Q} \alpha \cap \Psi = \emptyset, \mathcal{D}(\alpha) \neq \emptyset \). Here \( \mathcal{D}(\alpha) = \{D\} \) and \( \varphi_D = \alpha^\vee|_a \).

We say that \( \alpha \in \Pi(\mathfrak{g}) \) is of type a) (or b),c),d)) for \( X \) if the corresponding possibility of Proposition 3.2.3 holds for \( \alpha \).

**Proposition 3.2.4.** Let \( \alpha \in \Psi, \beta \in \Pi(\mathfrak{g}) \cap \Psi, D \in \mathcal{D}(\beta) \). Then \( \langle \varphi_D, \alpha \rangle \leq 1 \) and the equality holds iff \( \alpha \in \Pi(\mathfrak{g}), D \in \mathcal{D}(\alpha) \).

**Proof.** Using the localization procedure established in [Lu1], Subsection 3.5, (see also [Lu2], Subsection 3.2) we reduce the proof to the case \( \Psi = \{\alpha, \beta\} \). In this case everything follows from the classification in [Wa]. \( \square \)

### 3.3. A finiteness result.

**Proposition 3.3.1.** Let \( Y \) be a locally closed irreducible subvariety of \( \text{Gr}_d(\mathfrak{g}) \) such that \( \mathfrak{h}_0 \) is spherical and \( \mathfrak{n}_\mathfrak{g}(\mathfrak{h}_0) = \mathfrak{h}_0 \) for any \( \mathfrak{h}_0 \in Y \). Then any two elements of \( Y \) are \( G \)-conjugate.

**Proof.** By [AB], Corollary 3.2, there is a decomposition \( Y = Y_1 \sqcup \ldots \sqcup Y_k \) such that \( Y_i \) is an intersection of \( Y \) with a class of \( G \)-conjugacy of subalgebras. Clearly, \( Y_i \) is locally closed in \( Y \). Let \( Y_i \) be open in \( Y \) and \( \mathfrak{h}_0 \in Y_i \). Then \( \mathfrak{h}_1 \in G\mathfrak{h}_0 \) for any \( \mathfrak{h}_1 \in Y_i \). Since \( \mathfrak{n}_\mathfrak{g}(\mathfrak{h}_i) = \mathfrak{h}_i, i = 0, 1 \), we get \( \dim G\mathfrak{h}_1 = \dim G\mathfrak{h}_0 \). So \( \mathfrak{h}_1 \sim_G \mathfrak{h}_0 \). \( \square \)

### 3.4. Inclusions of spherical subgroups.

Throughout the subsection \( X = G/H \).

The goal of this subsection is to describe the ordered (by inclusion) set \( \mathcal{H}_H \) in terms of \( \mathfrak{a}, \mathcal{V}, \mathcal{D} \). All results are proved in [Kn1], Section 4. We remark that any \( \tilde{H} \in \mathcal{H}_H \) is spherical. Further, \( \mathfrak{a} \) naturally acts on \( \mathcal{H}_H \) preserving the partial order.

**Definition 3.4.1.** A pair \( (\mathfrak{a}^1, \mathcal{D}^1) \) consisting of a subspace \( \mathfrak{a}^1 \subset \mathfrak{a}^* \) and \( \mathcal{D}^1 \subset \mathcal{D} \) is called a *colored subspace* of \( (\mathfrak{a}^*, \mathcal{D}) \) if \( \mathfrak{a}^1 \) coincides with the cone spanned by \( \mathcal{V} \cap \mathfrak{a}^1 \) and \( \varphi_D \) for all \( D \in \mathcal{D}^1 \). The set of all colored subspaces of \( (\mathfrak{a}^*, \mathcal{D}) \) is denoted by \( \mathcal{CS}(=\mathcal{CS}_{G,X}) \).
In particular, \((a^*, D) \in CS\). Further, \((a^1, \emptyset) \in CS\) iff \(a^1 \subset \mathcal{V} \cap -\mathcal{V}\).

The set \(CS\) has a natural partial order: \((a^1, D^1) \preceq (a^2, D^2)\) if \(a^1 \subset a^2, D^1 \subset D^2\). Further, \(\mathfrak{A}\) acts on \(CS\): \(a.(a^1, D^1) = (a^1.a, D^1), a \in \mathfrak{A}\).

Choose \(\tilde{H} \in \mathcal{H}_H\). There is the natural morphism \(\pi : X \rightarrow G/\tilde{H}, gH \mapsto g\tilde{H}\). It induces the inclusion \(\pi^* : \mathfrak{X}_{G,G/\tilde{H}} \hookrightarrow \mathfrak{X}\), the projection \(\pi_* : a^* \mapsto a^*_{G,G/\tilde{H}}\) and the embedding \(\pi^* : \mathcal{D}_{G,G/\tilde{H}} \hookrightarrow \mathcal{D}, D \mapsto \pi^{-1}(D)\). Put \(a_{\tilde{H}}^\natural := \ker \pi_*^\natural, D_{\tilde{H}}^\natural := \{D \in \mathcal{D} | \pi(D) = G/\tilde{H}\} = \mathcal{D} \setminus \text{im} \pi^*\). Note that \(a_{\tilde{H}}^\natural\) depends only on \(a_{G,G/\tilde{H}}^\natural, a_{G,G/\tilde{H}}\).

**Proposition 3.4.2.** \((a_{\tilde{H}}^\natural, D_{\tilde{H}}^\natural) \in CS_{G,G/\tilde{H}}\) and the map \(\tilde{H} \mapsto (a_{\tilde{H}}^\natural, D_{\tilde{H}}^\natural)\) is an order-preserving \(\mathfrak{A}\)-equivariant bijection between \(\mathcal{H}_H\) and \(CS\).

Now we recover \(\mathfrak{X}_{G,G/\tilde{H}}, V_{G,G/\tilde{H}}, D_{G,G/\tilde{H}}\) from \((a_{\tilde{H}}^\natural, D_{\tilde{H}}^\natural)\).

**Proposition 3.4.3.** \(\mathfrak{X}_{G,G/\tilde{H}} = \{\lambda \in \mathfrak{X} | \langle \lambda, a_{\tilde{H}}^\natural \rangle = 0\}, V_{G,G/\tilde{H}} = \pi_*^\natural(\mathcal{V}), G_{\pi^*}(D) = G_D, \varphi_D = \pi_*^\natural(\varphi_{\pi^*}(D))\) for any \(D \in \mathcal{D}_{G,G/\tilde{H}}\).

For example, the subgroup \(\tilde{H} \in \mathcal{H}_H\) is parabolic iff \(a_{G,G/\tilde{H}}^\natural = \{0\}\) iff \(a_{\tilde{H}}^\natural = a^\natural\). In this case \(\tilde{H} \sim_G Q^-\), where \(Q^-\) denotes the parabolic subgroup of \(G\) opposite to \(Q := \cap_{D \in \mathcal{D} \setminus D_{\tilde{H}}} G_D\) (i.e., generated by \(B^-\) and the standard Levi subgroup of \(Q\)).

### 3.5. Around the local structure theorem.

Let \(Q\) be an arbitrary algebraic group. Choose a Levi decomposition \(Q = M \ltimes R_u(Q)\). Then one can regard \(R_u(Q)\) as a \(Q\)-variety by setting \(m.q = mqm^{-1}, q, l \in Q, m \in M, q, l \in R_u(Q)\). If \(X'\) is an \(M\)-variety, then one regards \(X^*\) as a \(Q\)-variety assuming that \(Q\) acts on \(X^*\) via the projection \(Q \rightarrow Q/R_u(Q) \cong M\).

**Proposition 3.5.1** (The local structure theorem, [Kn3], Theorem 2.3). Let \(X\) be an irreducible smooth (not necessarily spherical) \(G\)-variety and \(D'\) be a finite set of \(B\)-stable prime divisors on \(X\). Put \(X^0 := X \setminus (\cup_{D \in D'} D), Q := \cap_{D \in D'} G_D\) and let \(M\) be the standard Levi subgroup of \(Q\). Then there is a closed \(M\)-stable subvariety (a section) \(X' \subset X^0\) such that the map \(R_u(Q) \times X' \rightarrow X^0, (q, x) \mapsto qx\), is a \(Q\)-equivariant isomorphism.

**Example 3.5.2.** Let \(Q^- \in \mathcal{H}_{B^-}\) and \(M\) be the standard Levi subgroup of \(Q^-\). Put \(Q = BM\). Let \(H\) be an algebraic subgroup of \(Q^-\) and \(\pi\) denote the natural projection \(G/H \rightarrow G/Q^-\). Take \(\{\pi^{-1}(D), D \in \mathcal{D}_{G,G/Q^-}\}\) for \(D'\). Then one can take \(\pi^{-1}(eQ^-) = Q^-/H\) for \(X'\).

Below \(D', X^0, Q, M, X'\) have the same meaning as in Proposition 3.5.1. Note that \(X' \cong^M X^0/R_u(Q)\).

**Remark 3.5.3.** Let \(\tilde{X}\) be another smooth \(G\)-variety and \(\pi : \tilde{X} \rightarrow X\) a smooth surjective \(G\)-equivariant morphism with irreducible fibers. Put \(\tilde{D}' := \{\pi^{-1}(D), D \in D'\}\). Then one can take \(\pi^{-1}(X')\) for a section of the action \(Q : \pi^{-1}(X^0)\).

**Lemma 3.5.4.** If \(X\) is affine, then so is \(X'\).

**Proof.** Since \(X\) is smooth, any divisor is Cartier. It is known that the complement to a Cartier divisor in an affine variety is again affine. So \(X^0\) is affine. Being closed in \(X^0\), the variety \(X'\) is affine too.

Note that \(X\) is \(G\)-spherical iff \(X'\) is \(M\)-spherical.

Until further notices \(X\) is a smooth quasiprojective spherical \(G\)-variety.
Lemma 3.5.5. \quad (1) \( X_{M,X'} = X \).

(2) The map \( i : \mathcal{D}_{M,X'} \to \mathcal{D} \setminus D', D' \mapsto R_u(Q) \times D \) is bijective and satisfies \( \varphi_D = \varphi_{i(D)}, M_D = M \cap G_{i(D)} \). In particular, \( L_{M,X'} = L \).

(3) \( \Psi_{M,X'} = \Psi \cap \operatorname{Span}_Q(\Delta(m)) \).

(4) \( A \subset A_{M,X'} \).

Proof. Assertions 1,4 follow from the natural isomorphism between \( \mathbb{K}(X') \) and \( \mathbb{K}(X)^{R_u(Q)} \). Assertion 2 is straightforward.

We proceed to assertion 3. The claim for quasiaffine \( X \) follows from [Lo3], Proposition 8.2. In the general case, since \( X \) is quasiprojective, there is a quasiaffine \( G \)-variety \( \tilde{X} \) and a \( G \)-equivariant principal \( \mathbb{K}^\times \)-bundle \( \pi : \tilde{X} \to X \). Put \( \tilde{G} := G \times \mathbb{K}^\times, \tilde{M} := M \times \mathbb{K}^\times \). The group \( \tilde{G} \) acts naturally on \( \tilde{X} \). Set \( \tilde{D}' = \{ \pi^{-1}(D)|D \in D' \}, \tilde{X}' := \pi^{-1}(X') \). Then \( \Psi_{\tilde{M},\tilde{X}'} = \Psi_{G,\tilde{X}} \cap \operatorname{Span}_Q(\Delta(m)) \). By [Kn3], Theorem 5.1, \( \Psi_{M,X'} = \Psi_{\tilde{M},\tilde{X}'} \).

Lemma 3.5.6. \quad (1) Let \( G_1 \) be a semisimple normal subgroup of \( G \). If \( \Pi(\mathfrak{g}_1) \) consists of roots of type \( a \), then \( G_1 \) acts trivially on \( X \).

(2) If \( \mathcal{D} = \emptyset \), then \( (G,G) \) acts trivially on \( X \) and \( X \) is a single \( G/(G,G) \)-orbit.

Proof. Applying Proposition 3.5.1 to \( \mathcal{D}' = \mathcal{D} \), we reduce assertion 1 to assertion 2. The latter stems, say, from [Kn3], Proposition 2.4.

Applying Proposition 3.5.1 to \( \mathcal{D}' = \mathcal{D} \) and using Lemmas 3.5.5, 3.5.6, we get the following two well-known lemmas.

Lemma 3.5.7. \( \langle \mathfrak{a}_{G,X}, \alpha^\vee \rangle = 0 \) for any simple root \( \alpha \) of type \( a \).

Lemma 3.5.8. \( \dim X = \dim \mathfrak{x} + \dim G - \dim P \).

Till the end of the subsection \( X = G/H \) is a spherical homogeneous space and \( Q^- \in \mathcal{H}_{B'} \cap \mathcal{H}_H \). Put \( X' := Q^-/H \subset G/H \). Let \( M \) denote the standard Levi subgroup of \( Q^- \).

Proposition 3.5.9. Let \( Q^- \), \( H, M, X' \) be such as above. If \( G/H \) admits a wonderful embedding, then the following conditions are equivalent:

(a) \( R_u(Q^-) \subset H \).

(b) \( \Psi \subset \operatorname{Span}_Q(\Delta(m)) \).

Under these equivalent conditions, \( \Psi_{M,X'} = \Psi, \alpha \) is of type \( d \) for \( X \) and \( \mathcal{D}^Q_H \cap \mathcal{D}(\alpha) = \emptyset \) for any \( \alpha \in \Pi(\mathfrak{g}) \setminus \Delta(m) \), and \( \underline{\Psi} = \underline{\Psi}_{M,X'} \).

Proof. Everything except the equality \( \underline{\Psi}_{M,X'} = \underline{\Psi} \) follows from results of [Lu2], Subsection 3.4. Hence \( \mathcal{D}^Q_H \) is \( \mathfrak{A} \)-stable. By Proposition 3.4.2, \( N_G(H) \subset N_G(Q) = Q \). It follows that \( N_G(H)/H = N_M(H \cap M)/(H \cap M) \) whence \( \Lambda = \Lambda_{M,X'} \) or, equivalently, \( \underline{\Psi} = \underline{\Psi}_{M,X'} \).

There is another distinguished class of inclusions \( H \subset Q^- \). It is well known, see, for example, [We], that for any algebraic subgroup \( F \subset G \) there is a parabolic subgroup \( Q \subset G \) such that \( F \in \mathcal{F}_Q \). The following lemma seems to be standard.

Lemma 3.5.10. Let \( X, Q^-, M, X' \) be such as above. Assume, in addition, that \( H \in \mathcal{F}_Q \) and \( M \cap H \) is a maximal reductive subgroup of \( H \). Then \( X' \cong_{M \cap H} (R_u(q^-)/R_u(h)) \).
3.6. Some properties of affine spherical varieties. In this subsection $X$ is a spherical $G$-variety.

**Lemma 3.6.1.** If $X$ is quasi-affine, then $\varphi_D \neq 0$ for any $D \in D$.

**Proof.** There is a regular function vanishing on $D$. By the Lie-Kolchin theorem there is even a $B$-semiinvariant such function. \hfill \Box

**Definition 3.6.2.** By the weight monoid of $X$ we mean the set $\mathfrak{x}^{+}_{G,X} := \{\lambda \in \mathfrak{x}(T) | \mathbb{K}[X]^{(B)}_{\lambda} \neq \{0\}\}.$

**Lemma 3.6.3.**

\begin{equation}
\mathfrak{x}^{+} = \{\lambda \in \mathfrak{x}(\lambda, \varphi_D) \geq 0, \forall D \in D\}.
\end{equation}

If $X$ is quasiaffine, then

\begin{equation}
\mathfrak{x} = \text{Span}_{\mathbb{Z}}(\mathfrak{x}^{+}).
\end{equation}

**Proof.** (3.1) is clear. (3.2) stems from [PV], Theorem 3.3. \hfill \Box

**Proposition 3.6.4.** Let $X_1, X_2$ be affine spherical $G$-varieties. Let $X^0_i$ denote the open $G$-orbit in $X_i$, $i = 1, 2$. If $X_1 \cong^G X_2$, $X^0_1 \cong^G X^0_2$, then $X_1 \cong^G X_2$.

**Proof.** We may consider $\mathbb{K}[X_1], \mathbb{K}[X_2]$ as subalgebras in $\mathbb{K}[X^0_1] \cong \mathbb{K}[X^0_2]$. By Lemma 3.6.3, $\mathfrak{x}^{+}_{G,X_1} = \mathfrak{x}^{+}_{G,X_2}$. The highest weight theory implies $\mathbb{K}[X_1] = \mathbb{K}[X_2]$. \hfill \Box

**Lemma 3.6.5.** Let $X_1, X_2$ be affine spherical $G$-varieties with $X_1 \cong^G X_2$. Suppose there is a decomposition $G = G^1G^2$ into the locally direct product such that $X_1 \cong^{G^1 \times G^2} X^1_1 \times X^2_1$, where $X^1_1$ is a $G^1$-variety. Then $X^1_1$ is a spherical $G^1$-variety and there is a spherical $G^1$-variety $X^2_2, i = 1, 2$, such that $X_2 = X^1_2 \times X^2_2$ and $X^1_1 \cong^G X^1_2$.

**Proof.** We may assume that $G = G^1 \times G^2$. Everything will follow if we check the existence of a $G^1$-variety $X^1_2$ such that $X_2 \cong^{G^1 \times G^2} X^1_2 \times X^2_2$. Put $X^1_2 := X_2//G^2, X^2_2 := X_2//G^1$.

By Lemma 3.6.3, $\mathfrak{x}^{+}_{G,X_1} = \mathfrak{x}^{+}_{G,X_2}$. From the highest weight theory one easily deduces that $\mathfrak{x}^{+}_{G,X^1_1} = \mathfrak{x}^{+}_{G,X^1_2} \cap \mathfrak{x}(T \cap G^1), i = 1, 2$. Thus $\mathfrak{x}^{+}_{G,X^1_1} = \mathfrak{x}^{+}_{G,X^1_2}, i = 1, 2$, and $\mathfrak{x}^{+}_{G,X_2} = \mathfrak{x}^{+}_{G,X^1_2} + \mathfrak{x}^{+}_{G^2,X^1_2}$. In other words, $\mathbb{K}[X^1_2]^{(B)} = \mathbb{K}[X^1_2]^{(B)\cap G^1} \otimes \mathbb{K}[X^2_2]^{(B)\cap G^2}$. It follows from the highest weight theory that $\mathbb{K}[X_2] = \mathbb{K}[X^1_2] \otimes \mathbb{K}[X^2_2]$. \hfill \Box

**Proposition 3.6.6 ([KVS], Corollary 2.2).** If $X$ is smooth and affine, then $X \cong^G G *_{H} V$, where $H$ is reductive and $V$ is an $H$-module.

The following lemma seems to be well known.

**Lemma 3.6.7.**

1. Let $H_i, i = 1, 2$, be a reductive subgroup of $G$ and $V_i$ be an $H_i$-module. Then $G *_{H_i} V_i \cong^G G *_{H_2} V_2$ iff there are $g \in G$ and a linear isomorphism $\iota : V_1 \rightarrow V_2$ such that $gh_1g^{-1} = h_2$ and $\iota(v) = (gh^{-1}_i)\iota(v)$ for all $h \in H_1, v \in V_1$.

2. An element $g \in N_G(H)/H$ lies in the image of the natural homomorphism $\text{Aut}^G(X) \rightarrow \text{Aut}^G(G/H)$ iff $V \cong^H V^g$.

**Proof.** Let $\pi_i : G *_{H_i} V_i \rightarrow G/H_i, i = 1, 2$, denote the natural projection. Let $\varphi : G *_{H_1} V_1 \rightarrow G *_{H_2} V_2$ be a $G$-equivariant isomorphism and $x = [e, 0] \in G *_{H_1} V_1$. Clearly, $G_{\varphi(x)} = H_1$ is conjugate to a subgroup of $H_2$. Analogously, $H_2$ is conjugate to a subgroup of $H_1$. So we may assume that $H_1 = H_2$. Let $V'_2$ denote the slice module $T_{\varphi(x)}(G *_{H_1} V_2)/g_*\varphi(x)$. Note that $\varphi$ gives rise to the linear $H_1$-equivariant isomorphism $d_{x,\varphi} : V_1 \rightarrow V'_2$. Since $V'_2 \cong^{H_1} \pi_2^{-1}(\pi_2(\varphi(x)))$, it follows that $V'_2 \sim^{N_G(H_1)} V_2$ whence the first assertion. Assertion 2 stems easily from the previous argument. \hfill \Box
4. Proofs of Theorems 1,2

Subsection 4.1 contains plenty of quite simple auxiliary results used in the proof of Theorem 2. We introduce the notions of a distinguished spherical root (Definition 4.1.1) and of an automorphism doubling it (Definition 4.1.6) and show that Theorem 2 is equivalent to the claim that any distinguished spherical root has an automorphism doubling it (Lemma 4.1.8). Other lemmas study different properties of distinguished roots and automorphisms doubling them.

To prove our main theorems we use "induction". The "base", roughly, consists of all homogeneous spaces of the form $G/H$, where $N_G(H)$ does not lie in a proper parabolic subgroup of $G$. The main goal of Subsection 4.2 is to prove "base of induction": assertion 1 of Proposition 4.2.1 for Theorem 2 and Proposition 4.2.3 for Theorem 1. The other two results, assertion 2 of Proposition 4.2.1 and Proposition 4.2.4, are of more technical nature. Their purposes are described in the beginning of the subsection. A common feature of all these results is that they concern smooth affine spherical varieties.

Subsections 4.3,4.4 contain "induction steps" of the proofs of Theorems 2 and 1, respectively. Very roughly, we need to replace a homogeneous space $G/H$ with $G/	ilde{H}$ for an appropriate subgroup $\tilde{H} \in \mathcal{H}_H$. The proofs are described in more details in each subsection.

4.1. Distinguished roots and automorphisms doubling them. Throughout this subsection $X$ is a smooth spherical $G$-variety.

Definition 4.1.1. An element $\alpha \in \Psi_{G,X}$ is said to be distinguished if one of the following conditions holds:

1. $\alpha \in \Pi(g)$ and $\varphi_D = \frac{1}{2} \alpha^\vee |_{\mathfrak{a}_{G,X}}$ for any $D \in D_{G,X}(\alpha)$.
2. There is a subset $\Sigma \subseteq \Pi(g)$ of type $B_k$, $k \geq 2$, such that $\alpha = \alpha_1 + \ldots + \alpha_k$ and $D_{G,X}(\alpha_i) = \emptyset$ for any $i > 1$.
3. There is a subset $\Sigma \subseteq \Pi(g)$ of type $G_2$ such that $\alpha = 2\alpha_2 + \alpha_1$.

By the type of a distinguished root we mean its number in the previous list. By $\Psi_{G,X}, i = 1,2,3$, we denote the subset of $\Psi_{G,X}$ consisting of all distinguished roots of type $i$. We put $\tilde{\alpha} := \alpha$ (resp., $\tilde{\alpha} = \alpha_1$, $\tilde{\alpha} = \alpha_2$) for $\alpha \in \Psi_{G,X}, i = 1$ (resp., $i = 2,3$). Set $\Psi_{G,X}^+ := \sqcup_{i=1}^3 \Psi_{G,X}^i$.

Recall that, by our conventions, $\alpha_k$, $\alpha_2$ denote the short simple roots in $B_k$, $G_2$. Note that $\text{Supp}(\alpha) \subseteq \{\tilde{\alpha}\} \cup \Pi(l_{G,X})$ for any $\alpha \in \Psi_{G,X}^+$.

Distinguished roots were previously considered in Pezzini’s paper [Pe2].

First of all, we study functorial properties of distinguished roots (Lemma 4.1.3). For this we need the following auxiliary lemma that is somewhat similar to Proposition 3.3.2 in [Lu2].

Lemma 4.1.2. Let $H$ be a spherical subgroup of $G$ and $\tilde{H} \in \mathcal{H}_H$. Let $\Psi_0 \subset \Psi$ be such that $\langle \varphi_D, \Psi_0 \rangle = 0$ for any $D \in D_{\tilde{H}}$. Then $\Psi_0 \subset \Psi_{G,G/\tilde{H}}$ and any $\beta \in \Psi_{G,G/\tilde{H}} \setminus \Psi_0$ is of the form $\sum_{\alpha \in \Psi \setminus \Psi_0} n_\alpha \alpha, n_\alpha \geq 0$.

Proof. At first, consider the situation when $\tilde{H}$ is generated by $\varphi_D, D \in D_{\tilde{H}}$. Note that ker $\alpha$ is a wall of $\mathcal{V} + \tilde{a}_H$ for any $\alpha \in \Psi_0$. Thus $\Psi_0 \subset \Psi_{G,G/\tilde{H}}$. Since $\langle \beta, \mathcal{V} \rangle \leq 0$, we obtain $\beta = \sum_{\alpha \in \Psi} n_\alpha \alpha$ for some $n_\alpha \geq 0$. Put $\beta' := \sum_{\alpha \in \Psi \setminus \Psi_0} n_\alpha \alpha$. Then $\langle \beta', \mathcal{V} \rangle \leq 0$ and $\langle \beta - \beta', a_H \rangle = 0$. Since ker $\beta$ is a wall of $\mathcal{V} + a_H$, we get $\beta = \beta'$.

We proceed to the general case. We may replace $H$ with the element of $\mathcal{H}_H$ corresponding to $\langle \text{Span}_{\mathbb{Q}}(\varphi_D | D \in D_{\tilde{H}}), D_{\tilde{H}} \rangle$. In this case $\tilde{H}/H$ is a torus and we use Lemma 3.1.5. □
Assertion 1 is clear. Let us check assertion 2. It follows from Lemma 4.1.2 that

\[ (1) \quad \text{Lemma 4.1.3.} \]

\[ (2) \quad \text{Lemma 4.1.4.} \]

\[ (3) \quad \text{Lemma 4.1.5.} \]

Proof. Assertion 1 is clear. Let us check assertion 2. It follows from Lemma 4.1.2 that \( \alpha \in \Psi_{G,X_2} \). By assertion 1, \( \alpha \in \Psi_{G,X_2} \), \( i = j \). Assertion 3 follows easily from the observation (see [Wa], Table 1) that for any \( \alpha \in \Psi_{G,X_1} \) either \( \alpha \) or \( 2 \alpha \) lies in \( \Psi_{G,X_2} \).

Now let us study the behavior of \( \Psi_{i,G,X} \) under some modifications of the pair \((G, X)\).

Lemma 4.1.4. Let \( \pi : \tilde{X} \to X \) be a \( G \)-equivariant principal \( \mathbb{K}^\times \)-bundle, \( \tilde{G} := G \times \mathbb{K}^\times \). Then \( \Psi_{i,G,\tilde{X}} \subset \Psi_{G,X} \). \( \Psi_{i,G,\tilde{X}} \) is denoted by \( \Psi_{i,G,X} \), \( i = 2, 3 \). Let \( \alpha \in \Psi_{i,G,X} \). Then \( \alpha \in \Psi_{i,G,\tilde{X}} \) iff \( \text{ord}_{D_i}(\sigma) \), where \( D_{G,X}(\alpha) = \{D_1, D_2\} \) and \( \sigma \) is a rational \( B \)-semiinvariant section of \( \pi \).

Proof. In the proof we may assume that \( X \) is homogeneous. In this case \( \tilde{X} \) is a homogeneous \( \tilde{G} \)-space. Note that \( \mathfrak{X}_{G,\tilde{X}} = \mathbb{Z}\chi \oplus \mathfrak{X}_{G,X} \), where \( \chi \) is the sum of the canonical generator of \( \mathfrak{X}(\mathbb{K}^\times) \) and the weight of \( \sigma \). Now everything follows from Lemma 3.1.5 applied to \( \mathbb{K}^\times \subset \mathfrak{A}_{G,\tilde{X}} \) and Lemma 4.1.3.

Lemma 4.1.5. Let \( D', Q, M, X' \) be such as in Proposition 3.5.1. Then \( \Psi_{i,M,X'} = \{\alpha \in \Psi_{G,X}|\tilde{\alpha} \in \Delta(m)\}, i = 1, 2, 3 \).

Proof. By Lemma 3.5.5, \( D_{M,X'} \) is naturally identified with \( D_{G,X} \setminus D', L_{M,X'} = L_{G,X} \), and \( \Psi_{M,X'} = \Psi_{G,X} \cap \text{Span}_Q(\Delta(m)) \). Since \( \text{Supp}(\alpha) \subset \tilde{\alpha} \cup \Delta(L_{G,X}) \), we see that the inclusions \( \alpha \in \text{Span}_Q(\Delta(m)) \) and \( \tilde{\alpha} \in \Delta(m) \) are equivalent. Now everything stems from Definition 4.1.1.

Definition 4.1.6. We say that \( \varphi \in \mathfrak{A}_{G,X} \) doubles \( \alpha \in \Psi_{G,X}^+ \) if \( \langle \alpha, \varphi \rangle = -1 \). The set of all \( \varphi \in \mathfrak{A}_{G,X} \) doubling \( \alpha \) is denoted by \( \mathfrak{A}_{G,X}(\alpha) \).

Note that \( \langle \alpha, \mathfrak{A}_{G,X} \rangle = 1 \). So \( \mathfrak{A}_{G,X}(\alpha) \) is a union of \( \mathfrak{A}_{G,X}^\circ \)-cosets.

Remark 4.1.7. Let \( \alpha \in \Psi_{G,X} \). The set \( \mathfrak{A}_{G,X}(\alpha) \) consists precisely of those \( \varphi \in \mathfrak{A}_{G,X} \) that transpose elements of \( D_{G,X}(\alpha) \). This stems from Proposition 3.2.3 applied to \( X \) and \( X/\Gamma \), where \( \Gamma \) denotes the algebraic subgroup of \( \mathfrak{A}_{G,X} \) generated by \( \varphi \). If \( \alpha \in \Pi(g) \) is such that \( D_{G,X}(\alpha) \) contains a \( \mathfrak{A} \)-unstable divisor, then \( \alpha \in \Psi_{G,X} \). Conversely, let \( \iota : D_{G,X} \to D_{G,X} \) be a bijection satisfying \( G_D = G_{l(D)}, \varphi_D = \varphi_{l(D)} \). If \( X \) is homogeneous and Theorem 2 holds for \( X \), then \( \iota \) is induced by some element of \( \mathfrak{A}_{G,X} \).

Lemma 4.1.8. Theorem 2 is equivalent to the following assertion:

\[ (*) \quad \mathfrak{A}_{G,X}(\alpha) \neq \emptyset \; \text{for any} \; \alpha \in \Psi_{G,X}^+ \]

Proof. We may assume that \( X = G/H \). Since \( \overline{\mathfrak{T}_{G,X}} = \Psi_{G,G/NG(H),1} \), Theorem 2 implies \( (*) \). From [Wa], Table 1, it follows that \( \overline{\mathfrak{T}_{G,X}} = \Psi_1 \sqcup 2\Psi_2 \) for some partition \( \Psi_{G,X} = \Psi_1 \sqcup \Psi_2 \) with \( \Psi_{G,X} \cap \Lambda(g) \setminus \Psi_{G,X}^+ \subset \Psi_1 \). If \( (*) \) holds, then \( \Psi_{G,X}^+ \subset \Psi_2 \). Since the image of \( Z(G) \) in \( \text{Aut}(X) \) lies in \( \text{Aut}^G(X) \), we see that \( \Psi_{G,X} \cap \Lambda(g) \subset \Psi_2 \).
Lemma 4.1.9.  

(1) Let $\alpha \in \Psi_{G,G/H}^+$. Then $\mathfrak{A}_{G,G/H^0}^+(\alpha) \neq \emptyset$ is equivalent to $\mathfrak{A}_{G,G/H}(\alpha) \neq \emptyset$.

(2) (*) holds for $G/H$ whenever it holds for $G/N_G(H)$.

Proof. Assertion 1 stems from Lemma 3.1.1.

Proceed to assertion 2. To prove it we need an auxiliary claim. Namely, let $X = G/H$ be such that $H$ is connected and $\widetilde{G}, \widetilde{X}$ be such as in Lemma 4.1.4. We want to show that if Theorem 2 holds for $X$, then it holds for $\widetilde{X}$.

There is a natural exact sequence

$$1 \to \mathbb{K}^\times \to \mathfrak{A}_{\widetilde{G},\widetilde{X}} \to \mathfrak{A}_{G,X}.$$

Let $\mathcal{L}$ be the line $G$-bundle on $X$ corresponding to $\widetilde{X} \to X$. The image of the homomorphism $\mathfrak{A}_{\widetilde{G},\widetilde{X}} \to \mathfrak{A}_{G,X}$ coincides with the stabilizer of $\mathcal{L}$ under the action $\mathfrak{A}_{G,X} : \text{Pic}(G)(X)$. Let $\pi$ denote the forgetful homomorphism $\text{Pic}_G(X) \to \text{Pic}(X)$. Since $H$ is connected, we see that $\mathfrak{A}(H)$ is torsion-free. Using the argument of the proof of [Kn4], Lemma 7.3, we get $(\mathfrak{A}_{G,X})_\mathcal{L} = (\mathfrak{A}_{G,X})_{\pi(\mathcal{L})}$. By Lemma 4.1.4 and Remark 4.1.7, $\varphi \in (\mathfrak{A}_{G,X})_{\pi(\mathcal{L})}$ iff $(\alpha, \varphi) = 1$ for all $\alpha \in \Psi_{G,X}^+ \setminus \Psi_{G,X}^\pm$. Thus Theorem 2 holds for $\widetilde{X}$.

Complete the proof of assertion 2. By assertion 1, (*) holds for $G/N_G(H)^0$. Repeatedly applying the auxiliary claim above we obtain that (*) holds for $X := G/H^0$ considered as a $G := G \times N_G(H)^0/H^0$-variety. Any $D \in \mathcal{D}_{G,X}$, any $f \in \mathbb{K}(X)^H$ and any $v \in \mathcal{V}_{G,X}$ are $N_G(H)^0/H^0$-stable (the last two claims follow, for example, from [Kn2], Satz 8.1). Moreover, $\text{Aut}^G(X) = \text{Aut}^\widetilde{G}(X)$. From these observations and Lemma 3.1.4 it follows that $\Psi_{G,X} = \Psi_{G,X}^0, \mathfrak{V}_{G,X} = \mathfrak{V}_{G,X}^0, \Psi_{G,X}^+ = \Psi_{G,X}^+$. Therefore (*) holds for $X$ considered as a $G$-variety. Applying assertion 1 one more time, we complete the proof.

Essentially, the proof of assertion 2 is based on Luna’s augmentation construction, see [Lu2], Section 6.

Lemma 4.1.10. Let $X = G/H$ and $\alpha \in \Psi_{G,X}^3$. Then $\mathfrak{A}_{G,X}(\alpha) \neq \emptyset$.

Proof. Thanks to Lemma 4.1.9, we may assume that $G/H$ admits a wonderful embedding. Note that $G = G' \times G_2$ for some reductive group $G'$. In the notation of Definition 4.1.1, $\langle \alpha \rangle, \Psi_{G,X} = 0$. Hence $\Psi_{G,X} = \{\alpha\} \sqcup (\Psi_{G,X} \cap \text{Span}_\mathbb{Q}(\Delta(g')))$. It follows from Propositions 3.2.3.3.2.4 that $\langle \varphi_D, \alpha \rangle = 0$ for any $D \in \mathcal{D}_{G,X} \setminus \mathcal{D}_{G,X}(\widehat{\alpha})$. It follows from [Lu2], Proposition 3.5, that $H = H_0 \times H'$, where $H_0 \subset G_2, H' \subset G'$. So we may assume that $G = G_2$. In this case $H = A_2$ and the claim follows.

4.2. Case of smooth affine $X$. This subsection studies some properties of smooth affine spherical varieties. Proposition 4.2.1 deals with some special (in fact, almost all) spherical affine homogeneous spaces. Assertion 1 proves Theorem 2 for these homogeneous spaces, while assertion 2 is an auxiliary result used in the proof of Theorem 1. Two assertions of Proposition 4.2.1 are brought together because their proofs use similar ideas. Proposition 4.2.3 proves Theorem 1 for some special reductive subgroups of $G$. Proposition 4.2.4 establishes a relation between the sets of distinguished roots for an affine homogeneous vector bundle and its closed orbit. This result is very important in the proof of Proposition 4.3.1 in Subsection 4.3.

Proposition 4.2.1. Let $H$ be a reductive subgroup of $G$ such that $N_G(H)$ is not contained in a proper parabolic subgroup of $G$. Then
(1) The condition (*) of Lemma 4.1.8 holds for $X = G/H$.
(2) $g^2 \in N_G(H)^0 Z(G)$ for any $g \in N_G(H)$.

**Lemma 4.2.2.** Let $X$ be an arbitrary smooth spherical $G$-variety. Then there is an epimorphism $\text{Pic}(X) \twoheadrightarrow \mathbb{Z}^{\Psi_G(X)}$.

**Proof.** This stems easily from the observation that $\text{Pic}(X) \cong \mathbb{Z}^{D_G(X)/(f_\lambda), \lambda \in \mathfrak{X}_{G,X}}$, see [Bri2] or Section 17 of [T].

**Proof of Proposition 4.2.1.** It is enough to check both assertions only when $H = N_G(H)^0$. This is clear for assertion 2 and follows from Lemma 4.1.9 for assertion 1. One may assume, in addition, that $h$ is indecomposable, i.e., there are no proper complementary ideals $g_1, g_2 \subset g$ such that $h = (h \cap g_1) \oplus (h \cap g_2)$. In particular, $G$ is semisimple. If $\text{rk}_G(X) = 1$, then the pair $(g,h)$ is presented in Table 1 of [Wa] and we use a case by case argument. Below we assume that $\text{rk}_G(X) > 1$.

Note that if $H \neq (H,H)$, then $(H,H)$ is not spherical and $N_G(H)$ acts on $\mathfrak{z}(h)$ without nonzero fixed vectors. The latter follows from the assumption that $N_G(H)$ is not contained in a parabolic. If $(H,H)$ is spherical, then the group $N_G((H,H))/(H,H)$ is abelian whence $N_G(H)$ and $Z(H)^0$ commute.

Connected reductive spherical subgroups $H \subset G$ with $N_G(H)^0 - H$ were classified in [Kr] (G is simple) and [Bri1],[M] (G is not simple). In the case of simple $G$ the monoid $\mathfrak{X}_{G,G/H}^\mathfrak{z}$ is presented in Table 3 of [Kr]. Recall that $\mathfrak{X}_{G,X} = \text{Span}_\mathbb{Z}(\mathfrak{X}_{G,X})$, see Lemma 3.6.3.

Let us prove assertion 1. We may assume that $G$ is of adjoint type. Since $N_G(H)$ acts on $\mathfrak{z}(h)$ without nonzero fixed vectors, we get $\# \text{Pic}(G/N_G(H)) < \infty$. By Lemma 4.2.2, $\Psi_{G,G/N_G(H)} = \emptyset$. In other words, $\mathfrak{A}_{G,X}(\alpha) \neq \emptyset$ for any $\alpha \in \Psi_{G,X}^1$.

Now let $\alpha \in \Psi_{G,X}^2$ and $g_1$ be a unique simple ideal of $g$ with $\alpha \in \Pi(g_1)$. Then $g_1 \cong B_n, C_n, n > 1, F_4$, and $(a_{G,X}, \beta') = 0$ for some short simple root $\beta \in \Pi(g_1)$. It follows from [Kr],[Bri1],[M] that $g$ does not have an ideal of type $F_4$. Suppose $g_1 \cong B_n, n > 2$. Using the computations of $a_{G,X}$ ([Lo1]), we see that $G = SO_{2n+1}, H = SO_{n \times SO_{2n+1-n_1}}, 1 < n_1 < [\frac{n}{2}]$. But here $D_{G,X}(\alpha_i) = \emptyset, i > n_1$, and $\alpha_{n_1} + \ldots + \alpha_n \notin \mathfrak{X}_{G,X}$. So $g_1 \cong \mathfrak{sp}_{2k}$ and $D_{G,X}(\alpha_{k-1}) = \emptyset, \alpha_k + \alpha_{k-1} \in \Psi_{G,X}$, where $\alpha_{k-1}, \alpha_k$ are simple roots in $C_k$. Since $H$ is semisimple or $(H,H)$ is not spherical, we get from [Bri1],[M] that all summands of $g$ are of type $C$. Using computations of $a_{G,X}$ from [Lo1], we obtain $g = \mathfrak{sp}_{4n}, h = \mathfrak{sp}_{2n} \times \mathfrak{sp}_{2n}$. In this case indeed $\#\Psi_{G,X}^2 = 1$ and a nontrivial element of $N_G(H)/H$ doubles the element from $\Psi_{G,X}^2$.

Proceed to assertion 2. At first, suppose $G/H$ is a symmetric space. We may assume that $G$ is simply connected. Then $H = G^\sigma$ for some involutory automorphism $\sigma$ of $G$, see [OV]. It follows from [V], Lemma 1 in 2.2, that for any $g \in N_G(H)$ there is $z \in Z(G)$ such that $\sigma(g) = g^2 z$. Assertion 2 will follow if we check $\sigma(g^2 z^{-1}) = g^2 z^{-1}$. Since $\sigma^2 = id$, we get $g = \sigma(gz) = g \sigma(z) z$ whence $\sigma(z) = z^{-1}$. So $\sigma(g^2 z^{-1}) = g \sigma(z)^2 \sigma(z^{-1}) = g^2 \sigma(z) = g^2 z^{-1}$.

Consider the situation when $G/H$ is not symmetric. Here we assume that $G$ is of adjoint type. One checks that if $G$ is not simple, then $N_G(H) = H$. If $H$ remains to consider the following pairs $(g,h)$: $(\mathfrak{so}_{2n+1}, \mathfrak{gl}_n), (\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2} \times \mathfrak{so}_2), (\mathfrak{so}_{10}, \mathfrak{spin}_7 \times \mathfrak{so}_2), (\mathfrak{so}_9, \mathfrak{spin}_7), (\mathfrak{so}_8, G_2)$. One checks case by case that $N_G(H)/H \cong \mathbb{Z}_2$ or 1.

**Proposition 4.2.3.** Let $H_1, H_2$ be reductive algebraic subgroups of $G$. Suppose $H_i, i = 1,2$, is not contained in any proper parabolic subgroup of $G$ and $H_i = N_G(H_i)$. If $H_1 \cong G, H_2$, then $H_1 \sim_G H_2$. 

\[\square\]
Proof. At first, consider the case of simple $G$. Since $a_{G,G/H_1} = a_{G,G/H_2}$, inspecting Tabelle in [Kr], we see that $g = so_{2n+1}, h_1 = gl_n, h_2 = so_n \times so_{n+1}$. But

\[ X_{so_{2n+1},so_{2n+1}}/GL_m = \text{Span}_\mathbb{Z}(\pi_1, \ldots, \pi_{n-1}, 2\pi_n), \]

\[ X_{so_{2n+1},so_{2n+1}}/so_n \times so_{n+1} = \text{Span}_\mathbb{Z}(2\pi_1, \ldots, 2\pi_n), \]

and $X_{G,G/H_1} = X_{G,G/H_2}$ is a subgroup of index 2 in both these lattices, which is absurd.

Let us consider the case when $G$ is not simple. Thanks to Lemma 3.6.5, we may assume that both $h_1, h_2$ are indecomposable.

Suppose $G/H_1$ is a symmetric space. Then we may assume that $G \cong H_1 \times H_1$ and $H_1$ is embedded diagonally into $G$. It is clear that $X_{G,G/H_1} \cap \text{Span}_\mathbb{Q}(\Pi(g)) = \{0\}$ for any simple ideal $g \subset g$. It follows that the projection of $h_2$ to both ideals of $g$ is surjective. Thence $H_1 \cong H_2$. From $X_{G,G/H_1} = X_{G,G/H_2}$ we deduce that $H_1 \sim_G H_2$.

It remains to consider the case when neither $G/H_1$ nor $G/H_2$ is symmetric. Using the classification of [Bri1], [M] and the equality $\dim H_1 = \dim H_2$, we see that $G = Sp_4 \times Sp_{2m} \times Sp_{2n}, H_1, H_2 \cong SL_2 \times SL_2 \times SL_{2m-2} \times SL_{2n-2}$, where one of the normal subgroups $SL_2 \subset H_1$ is embedded into $Sp_4$ while the normal subgroups $SL_2 \subset H_2$ are embedded diagonally into $Sp_4 \times Sp_{2n}, Sp_4 \times Sp_{2m}$. By the Frobenius reciprocity, $\pi_1' + \pi_1'' \in X_{G,G/H_1,SO_{2n+1}} \cap X_{G,G/H_2}$ (here $\pi_1', \pi_1''$ denote the highest weights of tautological representations of $Sp_{2m}, Sp_{2n}$, respectively). Contradiction with (3.1).

Proposition 4.2.4. Let $X$ be a spherical $G$-variety of the form $G \ast_H V$, where $H$ is a reductive subgroup of $G$ and $V$ is an $H$-module. Then $\Psi_{i,X}^i \subset \Psi_{i,G,G/H}^i$ for $i = 1, 2$.

Proof. Let $\pi$ denote the natural projection $G \ast_H V \to G/H$ and $\pi^*: D_{G,G/H} \hookrightarrow D_{G,G \ast H V}$ be the corresponding embedding.

Step 1. Let us show that if $H = G$, then $\Psi_{i,X}^i = \emptyset$. For $i = 1$ this follows from Lemma 4.2.2, since $X$ is factorial. Now suppose $\alpha \in \Psi_{2,X}^2$. In the notation of Definition 4.1.1, $D_{G,X}(\alpha) = \emptyset, i = 2, k, D_{G,X}(\alpha) \neq \emptyset$. In other words, any element in $K[X]^{(B)}$ is $P_{\alpha}$-semi-invariant for $i = 2, \ldots, k$, but there is a prime element in $K[X]^{(B)}$ that is not $P_{\alpha}$-semi-invariant. From the classification of spherical $G$-modules (see, for example, [Le], Theorems 1.4.2.5 and Tables 1, 2, or [Kn5], Section 5) one gets $G = SO(2k + 1) \otimes K^x$ as a linear group. However, here $\Psi_{G,X} = \{2(\alpha_1 + \ldots + \alpha_k)\}$.

Step 2. Let us show that $\Psi_{G,X} = \emptyset, i = 1, 2$, whenever $(G, G) \subset H$. Indeed, replacing $G$ with a covering, we may assume that $G = Z \times H^o$, where $Z \cong (K^x)^m$. Now there is an étale $G$-equivariant morphism $Z \times V \to X$, where $G$ acts on $V$ via the projection $G \to H^o$. By assertion 3 of Lemma 4.1.3, $\Psi_{i,G,X} \subset \Psi_{i,G,Z \times V}$. Obviously, $\Psi_{i,G,Z \times V} = \Psi_{i,H^o,V}$. By the previous step, $\Psi_{i,G,Z \times V} = \emptyset$.

Step 3. Let us check that $D_{G,X}(\alpha) = \pi^*(D_{G,G/H}(\alpha))$. Assume the contrary: $\pi(D) = G/H$ for some $D \in D_{G,X}(\alpha)$. Using Lemma 3.6.1, we see that $\pi(D) = G/H$ for any $D \in D_{G,X}(\alpha)$. Thus $D_{G,G/H}(\alpha) = \emptyset$. Applying Proposition 3.5.1 to $(X, \pi^*(D_{G,G/H}))$, $(G/H, D_{G,G/H})$, we get the Levi subgroup $M = L_{G,G/H}$ and sections $X_0 \subset G/H, X' := \pi^{-1}(X_0)$. By Lemma 4.1.5, $\alpha \in \Psi_{M,X}$. Note that $D_{M,X_0} = \emptyset$. From Lemma 3.5.6 it follows that $X' \cong M \ast S V$, where $(M, M) \subset S$. By step 2, $\Psi_{M,X} = \emptyset$. Contradiction.

Step 4. If $\alpha \in \Psi_{G,X}^1$, then everything follows from step 3. So assume that $\alpha \in \Psi_{G,X}^2$. Let $X_0 = G/H_0$ denote the open $G$-orbit in $X$. Since fibers of the natural morphism $G/H_0 \to G/H$ are irreducible, we see that $H \in H_{H^0}$. Since $\#D_{G,G/H}(\alpha) = \#D_{G,G/H}(\alpha) = 1$, it follows from Propositions 3.2.3, 3.3.2.4 that $\langle \phi_D, \alpha \rangle \leq 0$ for any $D \in D^o_H$. Therefore
Lemma 4.3.4. Let \( \langle \varphi_D, \alpha \rangle = 0 \) for all such \( D \). It remains to use Lemma 4.1.2 and the first assertion of Lemma 4.1.3. \( \square \)

4.3. Completing the proof of Theorem 2. The scheme of the proof is as follows. First, using results of Subsections 4.1, 4.2, we explain how to reduce the proof to the case when \( N_G(H) = H \) and \( H \) is contained in a proper parabolic. Here we need to show that there are no distinguished spherical roots. Propositions 4.3.1, 4.3.2 together contradict the existence of distinguished roots. More precisely, if there is a distinguished root \( \alpha \in \Psi_{G,G/H} \) then we show that there exists \( \tilde{H} \in \mathcal{H}_H \) with \( \alpha \in \Psi_{G,G/\tilde{H}} \) (Proposition 4.3.1). But, according to Proposition 4.3.2, existence of such \( \tilde{H} \) contradicts the inductive assumption in the next paragraph.

So let \( X = G/H \) be a spherical homogeneous space. In the proof we may assume that Theorem 2 is proved for all groups \( G \) with \( \dim G_0 < \dim G \) and for all spherical homogeneous spaces \( G/H_0 \) with \( \dim G/H_0 < \dim G/H \). By Lemma 4.1.9 and Proposition 4.2.1, for any \( \alpha \in \Delta(m) \), if \( \tilde{H} \) is contained in some proper parabolic subgroup \( Q \subset G \).

By assertion (1) of Proposition 4.2.1, \( H \) is contained in some proper parabolic subgroup of \( G \). Thanks to Lemma 4.1.8, Theorem 2 stems from the following two propositions.

**Proposition 4.3.1.** Let \( \alpha \in \Psi_{i} \) for some \( i = 1, 2 \). Suppose \( H \) is contained in some proper parabolic subgroup of \( G \). Then there is \( \tilde{H} \in \mathcal{H}_H \) such that \( \alpha \in \Psi_{i} \).

**Proposition 4.3.2.** Let \( \alpha \in \Psi_{i} \), \( i = 1, 2 \). There is no \( \tilde{H} \in \mathcal{H}_H \) such that \( \alpha \in \Psi_{i} \).

**Lemma 4.3.3.** Let \( \tilde{H} \) be a spherical subgroup of \( G \), \( \alpha \in \Psi_{i} \), \( i = 1, 2 \), \( Q^- \in \mathcal{H}_{B^-} \), and \( M \) be the standard Levi subgroup of \( Q^- \).

1. If \( \tilde{H} \subset Q^- \) and \( i = 2 \), then \( \tilde{\alpha} \in \Delta(m) \).
2. If \( i = 1, N_G(\tilde{H}) \subset Q^- \), and \( \mathfrak{A}_{G,G/\tilde{H}}(\alpha) \neq \emptyset \), then \( \alpha \in \Delta(m) \).

**Proof.** The inclusions \( \tilde{\alpha} \in \Delta(m) \) and \( \mathcal{D}_{G,G/\tilde{H}}(\tilde{\alpha}) \subset \mathcal{D}_{\tilde{H}}^{Q^-} \) are equivalent. Note that, thanks to Propositions 3.2, 3.2.3.2.4, \( \langle \varphi_D, \alpha \rangle \leq 0 \) whenever \( D \notin \mathcal{D}_{G,G/\tilde{H}}(\tilde{\alpha}) \). Therefore \( \mathcal{D}_{\tilde{H}}^{Q^-} \cap \mathcal{D}_{G,G/\tilde{H}}(\tilde{\alpha}) \neq \emptyset \). This observation proves assertion 1. To prove assertion 2 we note that \( \mathcal{D}_{\tilde{H}}^{Q^-} \) is \( \mathfrak{A}_{G,G/\tilde{H}} \)-stable whence \( \mathcal{D}_{\tilde{H}}^{Q^-} \supset \mathcal{D}_{G,G/\tilde{H}}(\tilde{\alpha}) \). \( \square \)

**Lemma 4.3.4.**

1. If \( \tilde{H} \in \mathcal{H}_H \), then \( [R_u(\tilde{h}), R_u(\tilde{h})] \subset R_u(h) \).
2. If \( H \) is contained in a parabolic subgroup \( Q \subset G \) and \( R_u(Q) \not\subset H \), then \( \mathcal{H}_H \cap \mathcal{H}^{R_u(Q)} \neq \emptyset \).

**Proof.** Note that \( R_u(q) \) is a nilpotent Lie algebra.

Put \( n := R_u(h), \tilde{n} := R_u(\tilde{h}) \). The projection of \( n \) to \( \tilde{n}/[\tilde{n}, \tilde{n}] \) is not surjective. Otherwise \( n \) generates \( \tilde{n} \) whence coincides with \( \tilde{n} \). Since \( \tilde{n}/n \) is an irreducible \( H \)-module, we see that \( \tilde{n}/\tilde{n} \subset n \).

Proceed to assertion 2. Let \( q_i, i = 0, 1, 2, \ldots \) be the sequence of ideals of \( q \) defined inductively by \( q_0 = R_u(q), q_i = [q_0, q_{i-1}] \). Choose the minimal number \( i \) such that \( q_i \subset h \). Let \( v \) be an irreducible \( H \)-submodule in \( q_{i-1}/(q_{i-1} \cap h) \), \( \tilde{v} \) be the inverse image of \( v \) in \( q_{i-1} \), and \( \tilde{N} \) the connected subgroup of \( R_u(Q) \) corresponding to \( \tilde{v} \). Then \( H \tilde{N} \in \mathcal{H}_H \cap \mathcal{H}^{R_u(Q)} \). \( \square \)
Proof of Proposition 4.3.1. Assume the contrary.

Step 1. Let us show that there exists \( Q^- \in \mathcal{H}_B^- \) such that:

1. \( \tilde{\alpha} \in \Delta(m) \), where \( M \) is the standard Levi subgroup of \( Q^- \).
2. There is \( g \in G \) such that \( gHg^{-1} \in \overline{\mathcal{F}}^{Q^-} \).

If \( \alpha \in \Psi^i_{G,X} \), then the claim follows from Lemma 4.3.3. So we may assume that \( \alpha \in \Psi^1_{G,X} \). Let us check that there is a \( G \)-equivariant principal \( \mathbb{K}^\times \)-bundle \( X_1 \rightarrow X \) such that \( X_1 \) is quasiaffine and \( \alpha \in \Psi^1_{G_1,X_1} \). For \( X_1 \) we take the principal \( \mathbb{K}^\times \)-bundle over \( X \) corresponding to a divisor of the form \( m \sum_{D \in \Delta_G,X} D \). For sufficiently large \( m \) the divisor \( m \sum_{D \in \Delta_G,X} D \) is very ample on the wonderful embedding \( \overline{X} \) of \( X \) (see [Bri2], Section 2).

Being an open subset in the affine cone over \( \overline{X} \), the variety \( X_1 \) is quasiaffine. By Lemma 4.1.4, \( \alpha \in \Psi_1^{G_1,X_1} \). Clearly, \( X_1 \) is a homogeneous \( G_1 \)-space. Let \( H_1 \) denote the stabilizer of a point in \( X_1 \) lying over \( eH \). So the projection of \( H_1 \) to \( G \) coincides with \( H \). By Sukhanov’s theorem, there is a parabolic subgroup \( Q^- \subset G \), a \( G_1 \)-module \( V \), and a nonzero \( Q^- \)-semiinvariant vector \( v \in V \) such that \( H_1 \in \overline{\mathcal{F}}^{Q^-} \), where \( Q^- := Q^- \times \mathbb{K}^\times \). We may assume that \( B^- \subset Q^- \). By assertion 2 of Lemma 4.3.4, there is \( \tilde{H}_1 \in \mathcal{H}_{H_1} \cap \overline{\mathcal{H}}^{Q^-} \). Automatically, \( \tilde{H}_1 \in \overline{\mathcal{H}}^{Q^-} \). From Sukhanov’s theorem it follows that \( G_1/\tilde{H}_1 \) is quasiaffine. By Lemma 3.6.1, \( \varphi_D \neq 0 \) for all \( D \in \Delta_{G_1,G_1/\tilde{H}_1} \). Therefore \( \# \Delta_{G_1,X_1}(\alpha) \cap \Delta_{H_1}(\alpha) = 0 \) or \( 2 \).

Let \( \tilde{H} \) denote the projection of \( \tilde{H}_1 \) to \( G \). Obviously, \( G_1/\tilde{H}_1 \rightarrow G/\tilde{H} \) is a \( G \)-equivariant principal \( \mathbb{K}^\times \)-bundle. It follows that \( \# \Delta_{G_1,G_1/\tilde{H}_1}(\alpha) = \# \Delta_{G,G/\tilde{H}}(\alpha) \). By our assumptions, \( \Delta_{G,G/\tilde{H}}(\alpha) = \emptyset \). Since \( \tilde{H} \subset Q^- \), we get \( \Delta_{G,G/\tilde{H}}(\alpha) = \emptyset \), q.e.d.

Step 2. Let \( Q^- \), \( M \) satisfy (1),(2) with \( g = 1 \). We may assume that \( S := H \cap M \) is a maximal reductive subgroup of \( H \). Note that, by our assumptions, \( H \subseteq \tilde{H} := H R_u(Q) \). Put \( \overline{X} := G/\tilde{H} \), \( X' := Q^-/H \cong M/S \times (R_u(q^-)/R_u(h)), X'' := Q^-/\tilde{H} \cong M/S \). By Lemma 4.1.5, \( \alpha \in \Psi^i_{M,X''} \). Thanks to Proposition 4.2.4, \( \alpha \in \Psi^i_{M,X'} \). Applying Lemma 4.1.5 again, we see that \( \alpha \in \Psi^i_{G,\overline{X}} \). By Lemma 4.3.4, there is \( \tilde{H} \in \mathcal{H}_H \cap \overline{\mathcal{H}} \). Assertion 2 of Lemma 4.1.3 implies \( \alpha \in \Psi^i_{G,G/\tilde{H}} \). □

Proof of Proposition 4.3.2. Assume the contrary: let such \( \tilde{H} \) exist. We may assume that there is \( Q^- \in \mathcal{H}_B^- \) such that \( N_G(\tilde{H}) \in \overline{\mathcal{F}}^{Q^-} \) and \( M \cap N_G(\tilde{H}) \) is a maximal reductive subgroup of \( N_G(\tilde{H}) \), where, as usual, \( M \) is the standard Levi subgroup of \( Q^- \). Put \( S := M \cap H \). It is a maximal reductive subgroup of \( \tilde{H} \) or, equivalently, of \( H \). Set \( X := G/\tilde{H} \) and let \( \rho \) denote the natural projection \( X \rightarrow \overline{X} \). Put \( X' := Q^-/H, X'' := Q^-/\tilde{H} \).

Let \( \pi_a \) be an element of \( a^\vee_{G,X} \) such that \( \langle \pi_a, \beta \rangle = \delta_{\alpha \beta} \) for all \( \beta \in \Psi_{G,X} \). Define \( \varphi_\alpha \in A_{G,X} \) by \( \langle \varphi_\alpha, \lambda \rangle = \exp(\pi(i \langle \pi_a, \lambda \rangle)) \) (actually, \( \varphi_\alpha \in \text{Hom}_Z(\mathfrak{z}_{G,X}, \overline{\mathbb{K}}^\times) \), where \( \overline{\mathbb{K}} \) denotes the algebraic closure of \( \mathbb{K} \)).

Step 1. By our assumptions, Theorem 2 holds for \( X', \overline{X} \). Applying Lemma 4.3.3 to \( \overline{X} \), we get \( \tilde{\alpha} \in \Delta(m) \). By Lemma 4.1.5, \( \alpha \in \Psi^i_{M,X'} \).

Let us check that \( \varphi_\alpha \in \Psi^i_{M,X'} \). Since Theorem 2 holds for \( X' \), we get \( \overline{\Psi}_{M,X'} = \{2\alpha\} \cup \Psi^i_1 \), where \( \Psi^i_1 \subset \text{Span}_Q(\Psi_{G,X} \setminus \{\alpha\}) \). Thus \( \langle \varphi_\alpha, \overline{\Psi}_{M,X'} \rangle = 1 \), q.e.d.
Let us check that \( \rho_\ast(\varphi_\alpha) \in \mathfrak{A}_{G,X} \), where \( \rho_\ast \) denotes the homomorphism \( A_{G,X} \to A_{G,X} \) induced by \( \rho \). By Lemma 4.1.2, \( \overline{\mathfrak{A}}_{G,X} = \{2\alpha\} \cup \Psi_2 \), where \( \Psi_2 \subset \text{Span}_Q(\Psi_{G,X} \setminus \{\alpha\}) \). Again, \( \langle \rho_\ast(\varphi_\alpha), \overline{\mathfrak{A}}_{G,X} \rangle = 1 \).

Step 2. Let \( \gamma \) denote the image of \( \varphi_\alpha \) in \( \text{Aut}^M(M/S) \) (see Lemma 3.1.3). Put \( \tilde{n} := R_u(b), n := R_u(\theta), v := h/b \). Clearly, \( v \) is an irreducible \( S \)-module. Let us check that \( v \cong S \nu \). Thanks to step 1, \( \varphi_\alpha \in \text{Aut}^M(X') \). By Lemma 3.1.2 and assertion 4 of Lemma 3.5.5, \( \rho_\ast(\varphi_\alpha) \in \text{Aut}^M(X') \). Lemma 3.6.7 implies that \( R_u(q^-)/n \cong (R_u(q^-)/\nu \cong (R_u(q^-)/\tilde{n})^{\gamma} \) whence \( v \cong S \nu \). 

Step 3. Put \( \Gamma := (M \cap N_G(\tilde{H}))/S \). From the assumptions on \( Q, M \) it follows that the natural homomorphism \( \Gamma \to N_G(\tilde{H})/\tilde{H} \) is an isomorphism. The inverse of this isomorphism is induced by the projection \( Q^- \to Q^-/R_u(Q^-) \cong M \). Under the identification \( \Gamma \cong \mathfrak{A}_{G,X} \), we have \( \gamma = \rho_\ast(\varphi_\alpha) \). Let \( Y \) denote the subvariety in \( \text{Gr}_{\dim \tilde{n}}(\tilde{n}) \) consisting of all \( S \)-stable subspaces \( n_0 \) such that \( [\tilde{n}, \tilde{n}] \subset n_0 \) and \( \tilde{n}/n_0 \) is an irreducible \( S \)-module. Clearly, \( Y \) is a disjoint union of projective spaces. Let \( Y_0 \) denote the component of \( Y \) consisting of all \( n_0 \in Y \) such that \( \tilde{n}/n_0 \cong S \nu \). By assertion 1 of Lemma 4.3.4, \( n \in Y_0 \). There is a natural action \( \Gamma : Y \). The subgroup \( \Gamma_v \subset \Gamma \) (the stabilizer of \( v \)) coincides with the stabilizer of \( Y_0 \) under this action. The subset \( Y_0^0 := \{n_0 \in Y_0 | n_0(s + n_0) = s + n_0 \} \) is open and \( \Gamma_v \)-stable. By Proposition 3.3.1, \( s + n_0 \sim_G h \) for any \( n_0 \in Y_0^0 \).

Let \( n_1, n_2 \in Y \) be such that \( g(s + n_1) = (s + n_2) \) for some \( g \in \tilde{H} \). It is easily seen that \( n_1 = \tilde{n} \cap n 2 \cap (g(s + n_1)) \) whence \( n_1 = n_2 \). So \( n_1, n_2 \in Y \) are \( \Gamma \)-conjugate iff \( s + n_1, s + n_2 \) are \( N_G(\tilde{H}) \)-conjugate.

4.4. **Proof of Theorem 1.** Assume the contrary, let \( G, H_1, H_2 \) be such that \( H_1 \cong H_2 \) but \( H_1 \not\cong G \).

In the proof we may assume that Theorem 1 is proved for all reductive groups \( G \) such that \( \dim G < \dim G \) and all subgroups \( \tilde{H}_1, \tilde{H}_2 \subset G \) such that \( \tilde{H}_1 \equiv G \tilde{H}_2 \) and \( \dim \tilde{H}_1 > \dim \tilde{H}_2 \). Luna in [Lu2], Section 6, proved that it is enough to prove Theorem 1 only when both \( G/H_1, G/H_2 \) have wonderful embeddings. His proof works for an arbitrary reductive group \( G \). In fact, he proved that, once Theorem 1 is proved for \( G/H_1, G/H_2 \) having wonderful embeddings (equivalently, \( x_{G,G/H_1} \) is spanned by \( \Psi_{G,G/H_1} \), see Proposition 3.2.1), homogeneous spaces \( G/H_1, G/H_2 \) with \( D_{G,G/H_1} = D_{G,G/H_2}, \Psi_{G,G/H_1} = \Psi_{G,G/H_2} \) are isomorphic provided \( x_{G,G/H_1} = x_{G,G/H_2} \).

So it is enough to assume that both \( G/H_1, G/H_2 \) admit wonderful embeddings. Fix some identification of \( D_{G,G/H_1} \) and \( D_{G,G/H_2} \).

The following lemma allows to carry out an "induction step".

**Lemma 4.4.1.** Let \( \tilde{H} \in \mathcal{H}_{H_1} \). Then, possibly after conjugating \( H_2 \) in \( G \), we get \( \tilde{H} \in \mathcal{H}_{H_2} \), \( \mathcal{D}_{H_1} = \mathcal{D}_{H_2} \) and \( \iota_1(D) = \iota_2(D) \), where \( \iota_i \) is the bijection \( D_{G,G/\tilde{H}} \to D_{G,G/H_i} \setminus D_{H_i}, i = 1, 2 \), induced by taking the preimage.
Proof. Let \( \tilde{H}_2 \in \mathcal{H}_{H_2} \) correspond to the colored subspace \( (\alpha_{\tilde{H}_1}, \mathcal{D}_{\tilde{H}_1}) \). By Proposition 3.4.3, \( \tilde{H} \equiv G \tilde{H}_2 \). Since \( \dim G/\tilde{H} < \dim G/H_1 \), it follows that there is \( g \in G \) such that \( \tilde{H} = g\tilde{H}_2g^{-1} \). Replace \( H_2 \) with \( gH_2g^{-1} \). By Remark 4.1.7, the bijection \( \iota_2^{-1}_1 : \mathcal{D}_{G,G/\tilde{H}} \to \mathcal{D}_{G,G/\tilde{H}} \) is induced by some element \( g \in N_G(\tilde{H}) \). To obtain \( \iota_1(D) = \iota_2(D) \) it remains to replace \( H_2 \) with \( g^{-1}H_2g \). \( \square \)

The proof of Theorem 1 is in six steps. At first (step 1), we reduce the proof to the case \( N_G(H_i) = H_i, i = 1, 2 \). Then (steps 2, 3) we show that it is enough to consider the situation when there is \( \tilde{H} \in \mathcal{H}_{H_1} \cap \mathcal{H}_{H_2} \) and \( H_1, H_2, \tilde{H} \) have a common maximal reductive subgroup, say \( S \). On steps 4, 5 we show that one, in addition, can assume that \( R_u(h_1) \cong^S R_u(h_2) \). Finally (step 6), we use Proposition 3.3.1 to show that \( H_1 \sim_G H_2 \).

Step 1. It follows from Theorem 2 that \( N_G(H_1) \equiv_G N_G(H_2). \) By our assumptions, \( G/H_i \) admits a wonderful embedding. So \( N_G(H_i)^0 \subset H_i, i = 1, 2 \) (this can be deduced, for example, from Proposition 3.2.1 and Lemma 3.1.4). Assume that we have checked \( N_G(H_1) \sim_G N_G(H_2). \) Then Corollary 3.1.6 implies \( H_1 \sim_G H_2. \) Proposition 4.2.3 implies that at least one of subgroups \( H_1, H_2 \) is contained in a proper parabolic subgroup of \( G \).

Step 2. Let \( Q^- \) be a proper parabolic subgroup of \( G \) such that \( H_i \subset Q^- \) for some \( i = 1, 2 \). Let us show that \( R_u(Q^-) \not\subset H_i. \) Assume the contrary, let, say, \( R_u(Q^-) \subset H \subset Q^- \). Set \( M = Q^-/R_u(Q^-) \). Thanks to Lemma 4.4.1, we may assume that \( H_2 \subset Q^- \) and \( \mathcal{D}^Q_{H_1} = \mathcal{D}^Q_{H_2}. \) Then, by Proposition 3.5.9, \( R_u(Q^-) \subset H_2, H_1/R_u(Q^-) \equiv^M H_2/R_u(Q^-). \) Since \( \dim M < \dim G, \) we see that \( H_1 \sim_Q H_2. \)

Step 3. Thanks to step 1, we may assume that there is \( i \in \{1, 2\} \) and a proper parabolic subgroup \( Q^- \in \mathcal{H}_H \) such that \( H_i \in Q^- \), and \( S := M \cap H_i \) is a maximal reductive subgroup of \( H_i \), where \( M \) is the standard Levi subgroup of \( Q^- \). To be definite, assume \( i = 1 \). By step 2 and assertion 2 of Lemma 4.3.4, there exists \( \tilde{H} \in \mathcal{H}_{H_1} \cap \mathcal{H}_{Q^-} \). We may replace \( Q^- \) with a minimal parabolic subgroup of \( G \) containing \( N_G(\tilde{H}) \). In particular, the subgroup \( S \subset M \) satisfies the conditions of Proposition 4.2.1. Let \( \tilde{H}, H_1, H_2 \) satisfy the conditions of Lemma 4.4.1. In this case, automatically, \( \mathcal{D}^Q_{H_1} = \mathcal{D}^Q_{H_2}. \)

Clearly, \( \dim H_2/R_u(H_2) \leq \dim S. \) From symmetry between \( H_1 \) and \( H_2 \) we get \( \dim H_2/R_u(H_2) = \dim S \) whence \( H_2 \in \mathcal{H}_{H_1}. \) Conjugating \( H_2 \) in \( \tilde{H} \), we obtain that \( S \subset H_2. \)

Step 4. Let \( Q^-, M, S \) be such as on the previous step. To simplify the notation, put \( n_i := R_u(h_i), n := R_u(h), \tilde{n} := R_u(q^-)/\tilde{n}, v_i := \tilde{n}/n_i, i = 1, 2, X := G/\tilde{H}, X' := M \ast_S \tilde{n}. \) Let us check that \( (\tilde{v} \oplus v_1) \sim_{N_M(S)} (\tilde{v} \oplus v_2). \)

Put \( X'_i := M \ast \tilde{v}\tilde{n} i, i = 1, 2 \). By Lemma 3.6.7, it is enough to check that \( X'_1 \cong^M X'_2 \). From Lemmas 3.5.5, 3.5.10 it follows that \( X'_1 \equiv^M X'_2 \). Thanks to Proposition 3.6.4 and the assumption that Theorem 1 holds for \( M \), we get \( X'_1 \equiv^M X'_2. \)

Step 5. Step 4 implies \( \tilde{H} \in \mathcal{H}_{H_1}. \)

Let us check that there is \( g \in N_G(\tilde{H}) \cap M \) such that \( v_1 \cong^S v_2. \) Assume that \( v_1 \not\cong^S v_2. \)

Recall that \( \Psi_{M,X'} = \Psi_{G,X} \cap \text{Span}_Q(\Delta(m)) \) and \( \mathfrak{A}_{G,X} \subset \mathfrak{A}_{M,X'} \) (Lemma 3.5.5). It follows from Theorem 2 and Lemma 4.1.5 that \( \Psi_{G,X} \cap \text{Span}_Q(\Delta(m)) = \Psi_{M,X'}. \) So \( \Lambda_{M,X'} \) is a direct summand of \( \Lambda_{G,X} \). Equivalently, \( \mathfrak{A}_{M,X'}/\mathfrak{A}_{G,X} \) is connected. Therefore it is enough to check that there is \( g \in \mathfrak{A}_{M,X'} \) such that \( v_1 \cong^S v_2. \) (here we consider the natural homomorphism \( \iota : \mathfrak{A}_{M,X'} \to N_M(S)/S \). By step 4, there is \( g \in N_M(S) \) such that

\[
(4.1) \quad \tilde{v} \oplus v_1 \cong^S \tilde{v} \oplus v_2.
\]
By assertion 2 of Proposition 4.2.1, \(g^2 \in N_M(S) \circ Z(M)^\circ\). Therefore

\[(4.2) \quad \tilde{v}_g + v_1^g \cong \mathcal{S} \tilde{v} + v_2.\]

Adding (4.1) and (4.2), one easily gets \(\tilde{v} \cong \mathcal{S} \tilde{v}_g\) (in other words, \(g\) is lifted to \(\mathfrak{A}_{M,X^\vee}\)) and \(v_1 \cong \mathcal{S} v_2^g\).

Replacing \(H_2\) with \(gH_2g^{-1}\), we get \(v_1 \cong \mathcal{S} v_2\).

**Step 6.** Complete the proof. By Lemma 4.3.4, [\(\bar{n}, \bar{n}\) \(\subset n_1 \cap n_2\) Let \(Y\) be the subvariety of \(\text{Gr}_{\dim n_1}(\bar{n})\) consisting of all \(S\)-stable subspaces \(n_0 \subset \bar{n}\) such that \(n_1 \cap n_2 \subset n_0\) and \(n/n_0 \cong \mathcal{S} n/n_1\). Note that \(Y \cong \mathbb{P}^1\). Put \(Y^0 := \{n_0 \in Y| n_0(s + n_0) = s + n_0\}\). Since \(n_1, n_2 \in Y^0\), we get \(Y^0 \neq \emptyset\). By Proposition 3.3.1, \(h_1 \sim_G h_2\).

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