Accelerated FRW Solutions in Chern-Simons Gravity

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(Dated: January 10, 2014)
Abstract

We consider a five-dimensional Einstein-Chern-Simons action which is composed of a gravitational sector and a sector of matter, where the gravitational sector is given by a Chern-Simons gravity action instead of the Einstein-Hilbert action and where the matter sector is given by the so called perfect fluid. It is shown that (i) the Einstein-Chern-Simons (EChS) field equations subject to suitable conditions can be written in a similar way to the Einstein-Maxwell field equations; (ii) these equations have solutions that describe accelerated expansion for the three possible cosmological models of the universe, namely, spherical expansion, flat expansion and hyperbolic expansion when $\alpha$, a parameter of theory, is greater than zero. This result allow us to conjecture that this solutions are compatible with the era of Dark Energy and that the energy-momentum tensor for the field $h^a$, a bosonic gauge field from the Chern-Simons gravity action, corresponds to a form of positive cosmological constant.

It is also shown that the EChS field equations have solutions compatible with the era of matter: (i) In the case of an open universe, the solutions correspond to an accelerated expansion ($\alpha > 0$) with a minimum scale factor at initial time that, when the time goes to infinity, the scale factor behaves as a hyperbolic sine function. (ii) In the case of a flat universe, the solutions describing an accelerated expansion whose scale factor behaves as a exponential function when time grows. (iii) In the case of a closed universe it is found only one solution for a universe in expansion, which behaves as a hyperbolic cosine function when time grows.

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I. INTRODUCTION

Some time ago was shown that the standard, five-dimensional General Relativity can be obtained from Chern-Simons gravity theory for a certain Lie algebra $B$, which was obtained from the anti de Sitter (AdS) algebra and a particular semigroup $S$ by means of the $S$-expansion procedure introduced in Refs. [2], [3].

The five dimensional Chern-Simons Lagrangian for the $B$ algebra is given by [1]

$$L_{ChS}^{(5)} = \alpha_1 l^2 \varepsilon_{abcd} R^{ab} R^{cd} e^e + \alpha_3 \varepsilon_{abcd} \left( \frac{2}{3} R^{ab} e^c e^d e^e + 2 l^2 k^{ab} R^{cd} T^e + l^2 R^{ab} R^{cd} h^e \right),$$

where $\alpha_1$, $\alpha_3$ are parameters of the theory [1], $l$ is a coupling constant, $R^{ab} = d\omega^{ab} + \omega^a_b$ corresponds to the curvature 2–form in the first-order formalism related to the 1–form spin connection [4], [5], [6], and $e^a$, $h^a$ and $k^{ab}$ are others gauge fields presents in the theory [1].

We can see that

i If one identifies the field $e^a$ with the vielbein, then, the covariant derivative of the field $e^a$ is the Torsion 2–form ($T^a = De^a = de^a + \omega^a_b e^b$). Therefore, the system consist of the Einstein-Hilbert action plus non-minimally coupled 1–form matter fields1 given by $h^a$ and $k^{ab}$.

ii It is possible to recover the odd-dimensional Einstein gravity theory from Chern-Simons gravity theory in the limit where the coupling constant $l$ equals to zero while keeping the effective Newton’s constant fixed [1].

Recently was found [7] that the standard five-dimensional FRW equations and some of their solutions can be obtained, in a certain limit, from the so-called Chern-Simons-FRW field equations, which are the cosmological field equations corresponding to a Chern-Simons gravity theory.

It is the purpose of this paper to show that the Einstein-Chern-Simons (EChS) field equations, subject to (i) the torsion-free condition ($T^a = 0$) and (ii) the variation of the matter Lagrangian with respect to ($w.r.t.$) the spin connection is zero ($\delta L_M / \delta \omega^{ab} = 0$) can be written in a similar way to the Einstein-Maxwell field equations. The interpretation of
the \( h^a \) field as a perfect fluid allow us to show that the Einstein-Chern-Simons field equations have an universe in accelerated expansion as a of their solutions.

This paper is organized as follows: In Section II we briefly review the Einstein-Chern-Simons field equations. In Section III we study the Einstein-Chern-Simons field equations in the range of validity of general relativity. In Section IV we consider accelerated solutions for Einstein-Chern-Simons field equations. We try to find solutions that describes accelerated expansion for cases of open universes, flat universes and closed universes. In Section V we consider the consistency of the solutions with the ”Era of Matter”. A summary and an appendix conclude this work.

II. EINSTEIN-CHERN-SIMONS FIELD EQUATIONS

In Ref. [7] was found that in the presence of matter the lagrangian is given by

\[
L = L^{(5)}_{\text{ChS}} + \kappa L_M
\]  

where \( L^{(5)}_{\text{ChS}} \) is the five-dimensional Chern-Simons lagrangian given by (1), \( L_M = L_M(e^a, h^a, \omega^{ab}) \) is the matter Lagrangian and \( \kappa \) is a coupling constant related to the effective Newton’s constant. The variation of the lagrangian (2) w.r.t. the dynamical fields vielbein \( e^a \), spin connection \( \omega^{ab}, h^a \) and \( k^{ab} \), leads to the following field equations

\[
\varepsilon_{abcde}\left(2\alpha_3 R^{ab} e^c e^d + \alpha_1 l^2 R^{ab} R^{cd} + 2\alpha_3 l^2 D_\omega k^{ab} R^{cd}\right) = \kappa \frac{\delta L_M}{\delta e^e},
\]  

\[
\alpha_3 l^2 \varepsilon_{abcde} R^{ab} R^{cd} = \frac{\delta L_M}{\delta h^e},
\]  

\[
2\alpha_3 l^2 \varepsilon_{abcde} R^{cd} T^e = \frac{\delta L_M}{\delta k^{ab}},
\]  

\[
2\varepsilon_{abcde}\left(\alpha_1 l^2 R^{cd} T^e + \alpha_3 l^2 D_\omega k^{ab} T^e + \alpha_3 e^c e^d T^e + \alpha_3 l^2 D_\omega h^e\right)
+ 2\alpha_3 \varepsilon_{abcde} l^2 R^{cd} k^{e} e^f = \frac{\delta L_M}{\delta \omega^{ab}}.
\]

For simplicity, we will assume that the torsion vanishes \( (T^a = 0) \) and \( k^{ab} = 0 \). In this case the Eqs.(3-6) takes the form
This field equations system can be written in the form

\[ \epsilon_{abcde} R^{ab} e^c e^d = 4\kappa_5 \left( \frac{\delta L_M}{\delta e^e} + \alpha \frac{\delta L_M}{\delta h^e} \right), \]  
(7)

\[ l^2 \epsilon_{abcde} R^{ab} R^{cd} = 8\kappa_5 \frac{\delta L_M}{\delta h^e}, \]  
(8)

\[ l^2 \epsilon_{abcde} R^{cd} D_\omega h^e = 4\kappa_5 \frac{\delta L_M}{\delta \omega^{ab}} \]  
(9)

where we introduce \(\kappa_5 = \kappa/8\alpha_3\) and \(\alpha = -\alpha_1/\alpha_3\). Imposing the condition \(\delta L_M/\delta \omega^{ab} = 0\) (for consistency with the condition \(T^a = 0\)) we find that the equations (7-9) can be written in the form

\[ \epsilon_{abcde} R^{ab} e^c e^d = 4\kappa_5 \left( \frac{\delta L_M}{\delta e^e} + \alpha \frac{\delta L_M}{\delta h^e} \right), \]  
(10)

\[ \frac{\delta L_M}{\delta h^e} = \frac{l^2}{8\kappa_5} \epsilon_{abcde} R^{ab} R^{cd}, \]  
(11)

\[ \epsilon_{abcde} R^{cd} D_\omega h^e = 0. \]  
(12)

This means that the Einstein-Chern-Simons field equations, subject to the conditions \(T^a = 0, \ k^{ab} = 0\) and \(\delta L_M/\delta \omega^{ab} = 0\), can be re-written in a way similar to the Einstein-Maxwell field equations. Equation (10) is analogous to Einstein’s equation, where \(\delta L_M/\delta h^a\) correspond to the energy-momentum tensor for the field \(h^a\).

From (10-12) we can see that if \(L_M = 0\), then in five dimensions there is no solution of Schwarzschild type [1], [8].

### III. EINSTEIN-CHERN-SIMONS EQUATIONS IN THE RANGE OF VALIDITY OF GENERAL RELATIVITY

From (10-11) we can see that general relativity is valid when (i) the curvature \(R^{ab}\) takes values not excessively large (ii) the parameter \(l\) takes small values \((l \to 0)\) [1]; (iii) the
constant $\alpha$ takes values not excessively large. In fact, in this case we have that (11) takes the form
\[ \frac{\delta L_M}{\delta h^e} \approx 0. \] (13)

Introducing (13) into (10) we obtain the Einstein’s field equation
\[ \varepsilon_{abcd} R^{ab} e^c e^d \approx 4 \kappa_5 \frac{\delta L_M}{\delta e^e}. \] (14)

If $R^{ab}$ is not large then $\delta L_M / \delta e^a$ is also not large. This means that General Relativity can be seen as a low energy limit of Einstein-Chern-Simons gravity. So that, in the range of validity of the General Relativity, the equations (10-12) are given by
\[ \varepsilon_{abcd} R^{ab} e^c e^d = 4 \kappa_5 \frac{\delta L_M}{\delta e^e}, \] (15)
\[ \varepsilon_{abcd} R^{cd} D_\omega h^e = 0. \] (16)

On the other hand, if $R^{ab}$ is large enough, so that when it is multiplied by $l^2$ (which is very small) will have a non-negligible results, then we will find that $\delta L_M / \delta h^a$ is not negligible. This means that, in this case, we must consider the entire system of equations (10-12).

IV. ACCELERATED SOLUTION FOR EINSTEIN-CHERN-SIMONS FIELD EQUATIONS

From Ref. [7] we know that the vielbein for five dimensional FRW metric is given by
\[ e^0 = dt, \]
\[ e^1 = \frac{a(t)}{\sqrt{1 - kr^2}} dr, \]
\[ e^2 = a(t) r d\theta_2, \]
\[ e^3 = a(t) r \sin \theta_2 d\theta_3, \]
\[ e^4 = a(t) r \sin \theta_2 \sin \theta_3 d\theta_4 \] (17)
where $a(t)$ is the scale factor of the universe and $k$ is the sign of the curvature of spacetime: (i) $+1$ for a closed space ($S^4$), (ii) $0$ for a flat space ($E^4$) and (iii) $-1$ for an open space (hyperbolic).
Following Ref. [7], we postulate that the bosonic field $h^a$ is given by

\begin{align*}
  h^0 &= h(0) dt = h(0)e^0, \\
  h^1 &= h(t) \frac{a(t)}{\sqrt{1 - kr^2}} dr = h(t)e^1, \\
  h^2 &= h(t)a(t)r d\theta_2 = h(t)e^2, \\
  h^3 &= h(t)a(t)r \sin \theta_2 d\theta_3 = h(t)e^3, \\
  h^4 &= h(t)a(t)r \sin \theta_2 \sin \theta_3 d\theta_4 = h(t)e^4
\end{align*}

(18)

where $h(0)$ is a constant and $h(t)$ is a function of time $t$ that must be determined.

In accordance with the equation (10), we will consider a fluid composed of two perfect fluids, the first one related to ordinary energy-momentum tensor and the second one related to field $h^a$. The energy-momentum tensors in the comoving frame, where the fluids are at rest, are given by

\[ T_{\mu\nu} = \text{diag}(\rho, p, p, p, p), \]

(19)

\[ T^{(h)}_{\mu\nu} = \text{diag}(\rho^{(h)}, p^{(h)}, p^{(h)}, p^{(h)}, p^{(h)}), \]

(20)

where $\rho$ is the matter density and $p$ is the pressure of fluid. Then, the energy-momentum tensor for the composed fluid is

\[ \tilde{T}_{\mu\nu} = T_{\mu\nu} + \alpha T^{(h)}_{\mu\nu} \]

(21)

\[ = \text{diag}\left( \rho + \alpha \rho^{(h)}, p + \alpha p^{(h)}, p + \alpha p^{(h)}, p + \alpha p^{(h)}, p + \alpha p^{(h)} \right) \]

(22)

\[ = \text{diag}(\tilde{\rho}, \tilde{p}, \tilde{p}, \tilde{p}, \tilde{p}). \]

(23)

Introducing (17-23) into eqs. (10-12) we find the following field equations (see Ref. [7]
and A)

\[
6 \left( \frac{\dot{a}^2 + k}{a^2} \right) = \kappa_5 \rho, \tag{24}
\]

\[
3 \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}^2 + k}{a^2} \right) \right] = -\kappa_5 \ddot{p}, \tag{25}
\]

\[
\frac{3l^2}{\kappa_5} \left( \frac{\dot{a}^2 + k}{a^2} \right)^2 = \rho^{(h)}, \tag{26}
\]

\[
\frac{3l^2}{\kappa_5} \frac{\ddot{a}}{a} \left( \frac{\dot{a}^2 + k}{a^2} \right) = -p^{(h)}, \tag{27}
\]

\[
\left( \frac{\dot{a}^2 + k}{a^2} \right) \left[ (h - h(0)) \frac{\dot{a}}{a} + \dot{h} \right] = 0. \tag{28}
\]

We should note that equation (24) was studied in Ref. [9] in the context of inflationary cosmology. In this work we consider the study of the present acceleration for the complete Einstein-Chern-Simons field equations.

The Equations (24) and (25) are very similar to the Friedmann equations in five dimensions. However now \( \rho \) and \( p \) are subject to restrictions imposed by the remaining equations.

We can consider the case where \( T_{\mu\nu} = 0 \), i.e., when the contribution from the ordinary matter is negligible compared to the contribution from the field \( h^a \). In this case, the energy-momentum tensor \( \tilde{T}_{\mu\nu} \) fluid is given by

\[
\tilde{T}_{\mu\nu} = \text{diag}(\tilde{\rho}, \tilde{p}, \tilde{p}, \tilde{p})
\]

\[= \alpha \tilde{T}^{(h)}_{\mu\nu} \]

\[= \text{diag}(\alpha \rho^{(h)}, \alpha p^{(h)}, \alpha p^{(h)}, \alpha p^{(h)}) \tag{29}
\]

and the equations (24 - 28) take the form

\[
6 \left( \frac{\dot{a}^2 + k}{a^2} \right) = \kappa_5 \alpha \rho^{(h)}, \tag{30}
\]

\[
3 \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}^2 + k}{a^2} \right) \right] = -\kappa_5 \alpha p^{(h)}, \tag{31}
\]

\[
\frac{3l^2}{\kappa_5} \left( \frac{\dot{a}^2 + k}{a^2} \right)^2 = \rho^{(h)}, \tag{32}
\]

\[
\frac{3l^2}{\kappa_5} \frac{\ddot{a}}{a} \left( \frac{\dot{a}^2 + k}{a^2} \right) = -p^{(h)}, \tag{33}
\]

\[
\left( \frac{\dot{a}^2 + k}{a^2} \right) \left[ (h - h(0)) \frac{\dot{a}}{a} + \dot{h} \right] = 0. \tag{34}
\]
A. Case $T_{\mu\nu} = 0$ and $k = -1$

Introducing (32) into (30) we obtain

$$6 \left( \frac{\dot{a}^2 + k}{a^2} \right) = 3l^2 \alpha \left( \frac{\dot{a}^2 + k}{a^2} \right)^2$$

(35)

which can be rewritten

$$\left( \frac{\dot{a}^2 + k}{a^2} \right) \left( \frac{2}{\alpha l^2} - \frac{\dot{a}^2 + k}{a^2} \right) = 0.$$  

(36)

1. Solution $\ddot{a} = 0$

Consider the solution $\ddot{a} = 0$, i.e., a solution without accelerated expansion. For the first term in left side of (36) we have

$$\frac{\dot{a}^2 + k}{a^2} = 0,$$

(37)

remembering $k = -1$, we have

$$\dot{a} = \sqrt{-k}.$$  

(38)

The solution is

$$a(t) = \sqrt{-k}(t - t_0) + a_0.$$  

(39)

In this case $a(t)$ is increase linearly, i.e., there is no accelerated expansion.

Replacing this solutions into equations (30 - 33) we find

$$\rho^{(h)} = p^{(h)} = 0$$

(40)

and equation (34) is satisfied for $h(t)$ arbitrary.

2. Solution $\ddot{a} \neq 0$

From (36) we obtain we obtain

$$\dot{a}^2 - \frac{2}{\alpha l^2}a^2 = -k.$$  

(41)

From (41) we can see two options (i) $\alpha > 0$ and (ii) $\alpha < 0$. 
a. Case $\alpha > 0$:

Consider the case where the constant $\alpha$ is positive. Using the following ansatz\(^1\)

$$a(t) = A \sinh \left( \sqrt{\frac{2}{\alpha l^2}} (t - t') \right)$$

\(^1\) This ansatz can be obtained from

$$\dot{a} = \sqrt{\frac{2}{\alpha l^2} a^2 - k}$$

whose solution is ($\alpha > 0$, $k = -1$)

$$\int_{t'}^{t} \frac{da}{\sqrt{\frac{2}{\alpha l^2} a^2 - k}} = t - t'$$

using an hyperbolic substitution

$$\sqrt{\frac{\alpha l^2}{2}} \text{arsinh} \left( \sqrt{\frac{2}{\alpha l^2 k}} a \right) = t - t'.$$
where \( t' \) is a constant of integration, we obtain

\[
A = \sqrt{-\frac{\alpha}{2}}
\]  

(43)

and therefore

\[
a(t) = \sqrt{-\frac{\alpha}{2}} \sinh \left( \sqrt{\frac{2}{\alpha l^2}} (t - t') \right),
\]  

(44)

the initial condition \( a_0 = a(t = t_0) \) leads

\[
a(t) = \sqrt{-\frac{\alpha}{2}} \times \sinh \left[ \sqrt{\frac{2}{\alpha l^2}} (t - t_0) + \text{arsinh} \left( \sqrt{\frac{2}{\alpha l^2}} a_0 \right) \right]
\]  

(45)

and

\[
\dot{a}(t) = \sqrt{-k} \times \cosh \left[ \sqrt{\frac{2}{\alpha l^2}} (t - t_0) + \text{arsinh} \left( \sqrt{\frac{2}{\alpha l^2}} a_0 \right) \right]
\]  

(46)

This results shows that if \( \alpha > 0 \), then there is an accelerated expansion (see Fig. 2).

On the other hand, from (45) and (46) we can see that

\[
\dot{a}(t) = \frac{2}{\alpha l^2} a(t),
\]  

(47)

replacing (45), (46) and (47) into (30 - 33) we obtain

\[
\rho^{(h)} = -p^{(h)} = \frac{12}{\kappa_5 \alpha l^2},
\]  

(48)

i.e., we have an accelerated expansion when the energy density is positive and pressure is negative (like a cosmological constant positive).

From equation (34) we find

\[
- \frac{\dot{h}}{h - h(0)} = \frac{\dot{a}}{a}.
\]  

(49)

Integrating, we find

\[
h(t) = \frac{C}{\sinh \left[ \sqrt{\frac{2}{\alpha l^2}} (t - t_0) + \text{arsinh} \left( \sqrt{\frac{2}{\alpha l^2}} a_0 \right) \right]} + h(0)
\]  

(50)
where \( C \) is a constant of integration. The initial condition \( h_0 = h(t_0) \) leads

\[
h(t) = \frac{\left(h_0 - h(0)\right) \sqrt{-\frac{2}{\alpha^2 k}} a_0}{\sinh \left[ \sqrt{-\frac{2}{\alpha^2 k}}(t - t_0) + \text{arsinh} \left( \sqrt{-\frac{2}{\alpha^2 k}} a_0 \right) \right]} + h(0)
\]

from where we can see that \( h(t) \to h(0) \) when \( t \to \infty \)

b. *Case \( \alpha < 0 \):

Consider now the case when the constant \( \alpha \) is negative. The ansatz
\[ a(t) = A \sin \left( \sqrt{-\frac{2}{\alpha l^2}} (t - t') \right) \]  
\[ \text{with } t' \text{ a constant of integration, leads} \]
\[ A = \sqrt{\frac{\alpha l^2 k}{2}}, \]  
\[ \text{therefore} \]
\[ a(t) = \sqrt{\frac{\alpha l^2 k}{2}} \sin \left( \sqrt{-\frac{2}{\alpha l^2}} (t - t') \right). \]

The initial condition \( a_0 = a(t = t_0) \), leads
\[ a(t) = \sqrt{\frac{\alpha l^2 k}{2}} \sin \left[ \sqrt{-\frac{2}{\alpha l^2}} (t - t_0) + \arcsin \left( \frac{2}{\sqrt{\alpha l^2 k}} a_0 \right) \right] \]  
and
\[ \dot{a}(t) = \sqrt{-k} \]
\[ \times \cos \left[ \sqrt{-\frac{2}{\alpha l^2}} (t - t_0) + \arcsin \left( \frac{2}{\sqrt{\alpha l^2 k}} a_0 \right) \right]. \]

Therefore if \( a(t) > 0 \) then \( \ddot{a}(t) < 0 \), which shows that if \( \alpha < 0 \), then there is a decelerated expansion (see Fig. 3).

On the another hand, replacing (54) and (55) into (30 - 33) we obtain
\[ \rho^{(h)} = -p^{(h)} = \frac{12}{\kappa_5 \alpha l^2}. \]  
Since the energy momentum tensor is given by
\[ \tilde{T}_{\mu\nu} = \alpha T_{\mu\nu}^{(h)} = \text{diag} \left( \alpha \rho^{(h)}, \alpha p^{(h)}, \alpha p^{(h)}, \alpha p^{(h)} \right) \]  
we have that the corresponding energy density and pressure are \((\alpha < 0)\)
\[ \tilde{\rho} = \alpha \rho^{(h)} = \frac{12}{\kappa_5 \alpha l^2} < 0, \]
\[ \tilde{p} = \alpha p^{(h)} = -\frac{12}{\kappa_5 \alpha l^2} > 0, \]
i.e., the energy density is negative and the pressure is positive (like a cosmological constant negative).
From equation (34) we find

$$- \frac{\dot{h}}{h - h(0)} = \frac{\dot{a}}{a}. \quad (60)$$

Integrating, we find

$$h(t) = \frac{C}{\sin \left[ \sqrt{-\frac{2}{\alpha^2}}(t - t_0) + \arcsin \left( \sqrt{\frac{2}{\alpha^2}}a_0 \right) \right]} + h(0) \quad (61)$$

where $C$ is a constant of integration. The initial condition $h_0 = h(t_0)$, leads

$$h(t) = \frac{(h_0 - h(0)) \sqrt{\frac{2}{\alpha^2}}a_0}{\sin \left[ \sqrt{-\frac{2}{\alpha^2}}(t - t_0) + \arcsin \left( \sqrt{\frac{2}{\alpha^2}}a_0 \right) \right]} + h(0). \quad (62)$$

**B. Case $T_{\mu\nu} = 0$ and $k = 0$**

Introducing (32) into (30) and considering $k = 0$, we obtain
\[ 6 \left( \frac{\dot{a}}{a} \right)^2 = 3 \ell^2 \alpha \left( \frac{\dot{a}}{a} \right)^4 \]  \hspace{1cm} (63)

which can be rewritten as

\[ \left( \frac{\dot{a}}{a} \right)^2 \left( \frac{2}{\alpha \ell^2} - \frac{\dot{a}}{a} \right) = 0. \]  \hspace{1cm} (64)

1. Static solution \( \dot{a} = 0 \)

The solution for an static universe is given by

\[ a(t) = a_0 \]  \hspace{1cm} (65)

which leads

\[ \rho^{(h)} = p^{(h)} = 0 \]  \hspace{1cm} (66)

and the equation (34) is satisfied for all \( h(t) \).

2. Non-static solution \( \dot{a} \neq 0 \)

From (64) we obtain

\[ \dot{a}^2 - \frac{2}{\alpha \ell^2} a^2 = 0. \]  \hspace{1cm} (67)

This equation have solution, only if \( \alpha > 0 \).

a. Case \( \alpha > 0 \):

In this case we have an expanding universe

\[ a(t) = A \exp \left( \sqrt{\frac{2}{\alpha \ell^2}} t \right). \]  \hspace{1cm} (68)

The initial condition \( a_0 = a(t_0) \) leads

\[ a(t) = a_0 \exp \left( \sqrt{\frac{2}{\alpha \ell^2}} (t - t_0) \right) \]  \hspace{1cm} (69)

and

\[ \rho^{(h)} = -p^{(h)} = \frac{12}{\kappa_5 \alpha \ell^2}. \]
Replacing (69) into equation (34), solving for $h(t)$ and using the initial condition $h_0 = h(t_0)$, we find

$$h(t) = \frac{h_0 - h(0)}{\exp \left( \sqrt{\frac{2}{\alpha l^2}} (t - t_0) \right) + h(0)}. \quad (70)$$

\textit{b. Case } \alpha < 0: \\
In this case it is not possible to find a solution.
C. Case $T_{\mu\nu} = 0 \text{ and } k = 1$

Introducing (32) into (30) we obtain

$$6 \left( \frac{\dot{a}^2 + k}{a^2} \right) = 3l^2 \alpha \left( \frac{\dot{a}^2 + k}{a^2} \right)^2$$

which can be rewritten as

$$\left( \frac{\dot{a}^2 + k}{a^2} \right) \left( \frac{2}{\alpha l^2} - \frac{\dot{a}^2 + k}{a^2} \right) = 0.$$  \hspace{1cm} (72)

1. Case $\ddot{a} = 0$

In this case it is not possible to find a solution.

2. Case $\ddot{a} \neq 0$

From equation (72) we obtain

$$\frac{2}{\alpha l^2} a^2 - \dot{a}^2 = k.$$  \hspace{1cm} (73)

From (73) we can see two cases:

a. Case $\alpha > 0$

If $\alpha > 0$ we can postulate a solution given by

$$a(t) = A \cosh \left( \sqrt{\frac{2}{\alpha l^2}} (t - t') \right)$$

where $t'$ is a constant of integration, which leads

$$A = \sqrt{\frac{\alpha l^2 k}{2}}.$$  \hspace{1cm} (75)

The initial condition $a_0 = a(t = t_0)$ leads

$$a(t) = \sqrt{\frac{\alpha l^2 k}{2}} \cosh \left[ \sqrt{\frac{2}{\alpha l^2}} (t - t') + \text{arcosh} \left( \sqrt{\frac{2}{\alpha l^2}} a_0 \right) \right]$$

and

$$\dot{a}(t) = \sqrt{k} \sinh \left[ \sqrt{\frac{2}{\alpha l^2}} (t - t_0) + \text{arcosh} \left( \sqrt{\frac{2}{\alpha l^2}} a_0 \right) \right]$$

which shows an accelerated expansion (see Fig. 5)
FIG. 5. Graph of \( a(t) \) with \( \alpha > 0 \) and \( k = 1 \). See equation (76).

Replacing (76) and (77) into (30 - 33) we obtain

\[
\rho^{(h)} = -p^{(h)} = \frac{12}{\kappa_5 \alpha l^2}
\]

i.e., we have an accelerated expansion when the energy density is positive and pressure is negative (like a cosmological constant positive).

From equation (34) we find

\[
-\frac{\dot{h}}{h-h(0)} = \frac{\ddot{a}}{a}, \tag{78}
\]

so that

\[
h(t) = \frac{C}{\cosh \left[ \sqrt{\frac{2}{\alpha \pi}}(t-t_0) + \text{arcosh} \left( \sqrt{\frac{2}{\alpha \pi k}} a_0 \right) \right]} + h(0) \tag{79}
\]
where $C$ is a constant of integration. The initial condition $h_0 = h(t_0)$ leads

$$h(t) = \frac{(h_0 - h(0))\sqrt{\frac{2}{\alpha l^2 k}} a_0}{\cosh \left[ \sqrt{\frac{2}{\alpha l^2 k}}(t - t_0) + \text{arcosh} \left( \sqrt{\frac{2}{\alpha l^2 k}} a_0 \right) \right]} + h(0)$$

from where we can see that $h(t) \to h(0)$ when $t \to \infty$.

\[ b. \ Case \ \alpha < 0 : \]

If $\alpha < 0$ the equation (73) have no solution.

**D. Era of Dark Energy from Einstein-Chern-Simons gravity**

The results in the previous section are summarized in Tables I, II and III.

**TABLE I. Solutions for scale factor of an open space $k = -1$ (hyperbolic).**

| Dynamics   | $\alpha$ | $\rho^{(h)}$ | $p^{(h)}$ | $\Lambda$ |
|------------|----------|--------------|-----------|-----------|
| $a(t)$     |          |              |           | compatible|
| Accelerated| $> 0$    | $> 0$        | $< 0$     | $> 0$     |
| Decelerated| $< 0$    | $< 0$        | $> 0$     | $< 0$     |
| No accelerated| any  | 0            | 0         | $-$       |
| (Vacuum)   |          |              |           |           |

**TABLE II. Solutions for scale factor of a flat space $k = 0$.**

| Dynamics   | $\alpha$ | $\rho^{(h)}$ | $p^{(h)}$ | $\Lambda$ |
|------------|----------|--------------|-----------|-----------|
| $a(t)$     |          |              |           | compatible|
| Accelerated| $> 0$    | $> 0$        | $< 0$     | $> 0$     |
| Stationary|   any    | 0            | 0         | $-$       |
| (Vacuum)   |          |              |           |           |

So that we have found solutions that describe accelerated expansion for the three possible cosmological models of the universe. Namely, spherical expansion ($k = 1$), flat expansion ($k = 0$) and hyperbolic expansion ($k = -1$) when the constant $\alpha$ is greater than zero. This means that the Einstein-Chern-Simons field equations have as a of their solutions a universe
TABLE III. Solutions for scale factor of a closed space $k = 1$.

| Dynamics | $\alpha$ | $\rho^{(h)}$ | $p^{(h)}$ | $\Lambda$ |
|----------|----------|---------------|-----------|-----------|
| $a(t)$   | $> 0$    | $> 0$         | $< 0$     | $> 0$     | compatible |

in accelerated expansion. This result allow us to conjure that this solutions are compatible with the era of Dark Energy and that the energy-momentum tensor for the field $h^a$ corresponds to a form of positive cosmological constant.

From this solutions we can see that as time passes, the $h(t)$ decreases rapidly to $h(0)$, a constant value, keeping constant matter density.

We have also shown that the EChS field equations have solutions that allows us to identify the energy-momentum tensor for the field $h^a$ with a negative cosmological constant.

V. CONSISTENCY OF THE SOLUTIONS WITH THE "ERA OF MATTER"

In the previous section, we find that the solutions of EChS field equations, with $T_{\mu \nu} = 0$, can be useful as models of the era of dark energy. In this section we review the consistency of these equations with the era of matter.

In the era of matter, ordinary matter is modeled as dust, i.e., pressure corresponding to the era of matter is zero. In this case the field equations (24 - 28) takes the form

$$6 \left( \frac{\ddot{a}^2 + k}{a^2} \right) = \kappa_5 (\rho + \alpha \rho^{(h)})$$  \hspace{1cm} (80)

$$3 \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}^2 + k}{a^2} \right) \right] = -\kappa_5 \alpha p^{(h)}$$  \hspace{1cm} (81)

$$\frac{3l^2}{\kappa_5} \left( \frac{\dot{a}^2 + k}{a^2} \right)^2 = \rho^{(h)}$$  \hspace{1cm} (82)

$$\frac{3l^2}{\kappa_5} \frac{\ddot{a}}{a} \left( \frac{\dot{a}^2 + k}{a^2} \right) = -p^{(h)}$$  \hspace{1cm} (83)

$$\left( \frac{\ddot{a}^2 + k}{a^2} \right) \left[ (h - h(0)) \frac{\dot{a}}{a} + \dot{h} \right] = 0$$  \hspace{1cm} (84)

and the conservation equations (divergence-free energy-momentum tensor) for each fluids are given by

$$\dot{\rho} + \frac{\dot{a}}{a} \rho = 0$$  \hspace{1cm} (85)
and
\[ \dot{\rho}^{(h)} + 4 \frac{\dot{a}}{a} \left( \rho^{(h)} + p^{(h)} \right) = 0. \] (86)

The equation (85) have as solution
\[ \rho(t) = \left( \frac{a_0}{a(t)} \right)^4 \rho_0 \] (87)

where the initial conditions \( a_0 = a(t_0) \) and \( \rho_0 = \rho(t_0) \) has been set.

Replacing (87) and (82) into equation (80) we have
\[ \left( \frac{\dot{a}^2 + k}{a^2} \right)^2 - 2A \left( \frac{\dot{a}^2 + k}{a^2} \right) + AB \frac{a_0^4}{a^4} = 0 \] (88)

where we defined
\[ A := \frac{1}{\alpha l^2}, \quad B := \frac{\kappa_5 \rho_0}{3}. \] (89)

A. Case \( k = -1 \)

In this case, the equation (88) can be rewritten
\[ \left( \frac{\dot{a}^2 - 1}{a^2} \right)^2 - 2A \left( \frac{\dot{a}^2 - 1}{a^2} \right) + AB \frac{a_0^4}{a^4} = 0 \] (90)

where we find
\[ \dot{a} = \pm \sqrt{Aa^2 \left( 1 \pm \text{sgn}(A) \sqrt{1 - \frac{B a_0^4}{A a^4}} \right) + 1}. \] (91)

1. Case \( \alpha > 0 \)

In this case
\[ A = \frac{1}{\alpha l^2} > 0. \] (92)

From (91) we can see that \( \dot{a} \) is well defined if
\[ a \geq a_{\text{min}} = \sqrt[4]{\frac{B}{A}} a_0 \] (93)

where
\[ a_{\text{min}} = \sqrt[4]{\frac{\kappa_5 \alpha l^2 \rho_0}{3}} a_0. \]
On the other hand $a_0$ must satisfy

$$a_0 \geq a_{\text{min}}$$  \hspace{1cm} (94)

so that

$$\frac{B}{A} \leq 1 \quad \text{i.e.,} \quad B \leq A$$  \hspace{1cm} (95)

and therefore

$$\rho_0 \leq \rho_{\text{max}} = \frac{3}{\kappa_5 \alpha l^2}$$  \hspace{1cm} (96)

These results allow us to analyze the radicand in (91)

$$AAa^2 \left(1 \pm \sqrt{1 - \frac{a_{\text{min}}^4}{a^4}}\right) + 1 \geq 0,$$  \hspace{1cm} (97)

i.e.,

$$-A^2a_{\text{min}}^4 \leq 1 + 2Aa^2$$  \hspace{1cm} (98)

which is satisfied for all $a$.

\textit{a. Plus or minus sign?}

The choice of the sign into the radicand has information about the allowed values of $\dot{a}$. Let us consider $\dot{a} > 0$ (the analysis of the case $\dot{a} < 0$ is very similar)

$$\dot{a} = \sqrt{AAa^2 \left(1 \pm \sqrt{1 - \frac{a_{\text{min}}^4}{a^4}}\right) - k.}$$  \hspace{1cm} (99)

The function $\dot{a}(a)$ is monotonically increasing (decreasing) if we consider the plus (minus) sign in front of the square root.

From (99) we can see that there exist $\dot{a}_{\text{cri}}$

$$\dot{a}_{\text{cri}} := \dot{a}(a_{\text{min}}) = \sqrt{\frac{\kappa_5 \rho_0}{3\alpha l^2} a_0^2 - k.}$$  \hspace{1cm} (100)

If we consider the plus (minus) sign in front of the square root, $\dot{a}_{\text{cri}}$ is the minimum (maximum) value of $\dot{a}$.

If there is a limit to $a \gg a_{\text{min}}$, then
FIG. 6. For every $a_0$ allowed there are two different values for $\dot{a} > 0$: evolution with $\dot{a}$ approximate constant and evolution accelerated(decelerated).

\[
\dot{a} = \pm \sqrt{Aa^2 \left(1 \pm \sqrt{1 - \frac{a_{\text{min}}^4}{a^4}}\right) - k}
\]

\[
\approx \pm \sqrt{2Aa^2 - k},
\]

where $k = -1$.

b. Case where the sign is “+”

In this case

\[
\dot{a} = \pm \sqrt{Aa^2 \left(2 - \frac{a_{\text{min}}^4}{2a^4}\right) - k} \approx \pm \sqrt{2Aa^2 - k},
\]
whose approximate solution is

\[ a(t) = \pm \sqrt{\frac{-\alpha l^2 k}{2}} \times \sinh \left[ \sqrt{\frac{2}{\alpha l^2}} (t - t_0) + \text{arsinh} \left( \sqrt{\frac{-2}{\alpha l^2}} a_0 \right) \right] \]  

(103)

where we use \( A = \frac{1}{\alpha^2} \) and \( k = -1 \).

\[ \dot{a} = \dot{a}_{\text{cri}} \]  

\[ c. \text{ Case where the sign is } - \]  

In this case

\[ \dot{a} = \pm \sqrt{A a_{\text{min}}^4 - 2a^2} \approx \pm \sqrt{-k} \]  

(104)
whose approximate solution is

\[
a(t) = \pm \sqrt{-k} (t - t_0) + a_0
\]

where we use \( k = -1 \).

FIG. 8. Numerical solution with \( A > 0, k = -1 \) and \( \dot{a}_0 < \dot{a}_{\text{cri}} \) of

\[
\dot{a} = \sqrt{Aa^2 \left( 1 + \sqrt{1 - \frac{a^4_{\text{min}}}{a^4}} \right)} - k
\]

2. Case \( \alpha < 0 \)

In this case

\[
A = \frac{1}{\alpha l^2} < 0.
\]
From (91) we can see that $\dot{a}$ is well defined if
\[
1 - \frac{B a_0^4}{A a^4} \geq 0,
\]
but this condition is satisfied for all $a$.

a. Case where the sign is ‘+’

In this case
\[
Aa^2 \left( 1 + \sqrt{1 - \frac{Ba_0^4}{Aa^4}} \right) - k \geq 0,
\]
so that
\[
\frac{k - Aa^2}{Aa^2} \geq \sqrt{1 - \frac{Ba_0^4}{Aa^4}}.
\]
The left side of the last equation must be positive, i.e.,
\[
k - Aa^2 \leq 0 \text{ or } a \leq \sqrt{\frac{k}{A}}.
\]
From (108) we obtain
\[
k^2 - 2Aka^2 \geq -ABa_0^4
\]
and again, the left side of the last equation must be positive, i.e.,
\[
k^2 - 2Aka^2 \geq 0 \iff a \leq \sqrt{\frac{k}{2A}}
\]
and from (110) we find
\[
a \leq \sqrt{\frac{k^2 + ABa_0^4}{2Ak}}.
\]
Since
\[
\sqrt{\frac{k}{A}} > \sqrt{\frac{k}{2A}} > \sqrt{\frac{k^2 + ABa_0^4}{2Ak}} = a_{\text{max}} \geq a,
\]
we have found a maximum value for $a$
\[
a_{\text{max}} = \sqrt{\frac{3\alpha l^2 k^2 + \kappa_5 \rho_0 a_0^4}{6k}}
\]
and therefore
\[
\dot{a}(a = a_{\text{max}}) = 0
\]
I. e., $a_{\text{max}}$ is a local maximum. It is direct to prove that $\dot{a} \neq 0$ for $a \neq a_{\text{max}}$. If $a$ has a maximum value $a_{\text{max}}$ then (see (87))
\[
\rho(t) = \left( \frac{a_0}{a(t)} \right)^4 \rho_0 \geq \left( \frac{a_0}{a_{\text{max}}} \right)^4 \rho_0 = \rho_{\text{min}}.
\]
This means that $\rho$ has a minimum value $\rho_{\text{min}}$ given by

$$
\rho_{\text{min}} = \left( \frac{6ka_0^2}{3\alpha^2k^2 + \kappa\rho_0a_0^4} \right)^2 \rho_0
$$

(117)

where $k = -1$.

---

Consider the case where $\dot{a} > 0$. We just consider $\dot{a} > 0$ because the analysis of the case $\dot{a} < 0$ looks very similar. In this case

$$
\dot{a} = \sqrt{Aa^2 \left( 1 - \sqrt{1 - \frac{Ba_0^4}{Aa^4}} \right) - k}
$$

(118)
is a decreasing function. We can see that the minimum value of $\dot{a}$ is given by

$$
\dot{a}_{\min} = \dot{a}(a_{\max}) = 0
$$

and the maximum value of $\dot{a}$ is given by

$$
\dot{a}_{\max} = \dot{a}(a = 0) = \sqrt{-\frac{\kappa_5 \rho_0}{3\alpha l^2} a_0^2 - k}.
$$

**FIG. 10. Phase space for** $A < 0$ **and** $k = -1$ **with** “+” **sign.**

**b. Case where the sign is** “−”

In this case we obtain the following condition

$$
Aa^2 \left( 1 - \sqrt{1 - \frac{Ba_0^4}{Aa^4}} \right) - k \geq 0
$$

(120)
where \( A = -\frac{1}{a^2} \) and \( k = -1 \). This condition is trivially satisfied for all \( a \).

This result implies that \( \dot{a} \neq 0 \). This means that \( a \) has no local maximums/minimums, so \( a \) is monotonically increasing or monotonically decreasing.

If there is a limit to \( a \gg \sqrt{-\frac{B}{A} a_0} \), then

\[
\dot{a} = \pm \sqrt{A a^2 \left( 1 - \sqrt{1 - \frac{Ba_0^4}{2Aa^4}} \right)} - k
\]

and

\[
\dot{a} = \pm \sqrt{\frac{Ba_0^4}{2a^2}} - k \approx \pm \sqrt{-k},
\]

whose approximate solution is

\[
a(t) = \pm \sqrt{-k}(t - t_0) + a_0
\]

where we use \( k = -1 \).

In this case

\[
\dot{a} = \sqrt{A a^2 \left( 1 - \sqrt{1 - \frac{Ba_0^4}{2Aa^4}} \right)} - k
\]

is a decreasing function. The maximum value of \( \dot{a} \) is given by

\[
\dot{a}_{\text{max}} = \dot{a}(a = 0) = \sqrt{-\frac{\kappa_5 \rho_0}{3\alpha l^2} a_0^2} - k
\]

and we can see that \( \dot{a} \) tends to a minimum value given by

\[
\dot{a}_{\text{min}} = \dot{a}(a \to \infty) = \sqrt{-k} = 1
\]

B. Case \( k = 0 \)

In this case, the equation (88) takes the form

\[
\left( \frac{\dot{a}}{a} \right)^4 - 2A \left( \frac{\dot{a}}{a} \right)^2 + AB \frac{a_0^4}{a^4} = 0
\]

from where

\[
\dot{a} = \pm \sqrt{Aa^2 \left( 1 \pm \text{sgn}(A) \sqrt{1 - \frac{B a_0^4}{A a^4}} \right)}.
\]
FIG. 11. Solution of $\dot{a} = \sqrt{Aa^2\left(1 - \sqrt{1 - \frac{B\dot{a}_0^4}{\lambda a^4}}\right)} - k$ with $A < 0$, $k = -1$ and $1 < \dot{a}_0 < \dot{a}_{\text{max}}$.

1. Case $\alpha > 0$

In this case

$$A = \frac{1}{\alpha l^2} > 0.$$ (129)

From (128) we can see that $\dot{a}$ is well defined if

$$a \geq \sqrt[4]{\frac{B}{A}} a_0$$ (130)

and therefore a minimum value for $a$ is given by

$$a_{\text{min}} = \sqrt[4]{\frac{\kappa_5\alpha l^2 \rho_0}{3}} a_0.$$ (131)
On the other hand $a_0 \geq a_{\text{min}}$, so that

$$B \leq A \quad \text{i.e.,} \quad \rho_0 \leq \rho_{\text{max}} = \frac{3}{\kappa_5 \alpha l^2}. \quad (132)$$

These results leads

$$Aa^2 \left( 1 \pm \text{sgn}(A) \sqrt{1 - \frac{B}{A} \frac{a_0^4}{a^4}} \right) \geq 0, \quad (133)$$

i.e., $a$ has no local maximums/minimums $^2$, so that $a$ is monotonically increasing or monotonically decreasing.

**Plus or minus sign?**

$^2$ Only it has a local maximum/minimum if we consider the minus sign into the radicand. In that case the local minimum is $a_{\text{min}}$. We can prove that there is no local maximum.
The choice of the sign into the radicand has information about the allowed values of $\dot{a}$. Let us consider $\dot{a} > 0$, the analysis of the case $\dot{a} < 0$ is very similar

$$\dot{a} = \sqrt{Aa^2 \left( 1 \pm \sqrt{1 - \frac{a_{\text{min}}^4}{a^4}} \right)}.$$ \hspace{1cm} (134)

The function $\dot{a}(a)$ is monotonically increasing(decreasing) if we consider the plus(minus) sign in front of the square root.

From (134) we can see that exist $\dot{a}_{\text{cri}}$

$$\dot{a}_{\text{cri}} := \dot{a}_{\text{min}} = \sqrt{A} a_{\text{min}} = \sqrt{\frac{\kappa_5 \rho_0}{3 \alpha l^2}} a_0.$$
If we consider the plus (minus) sign in front of the square root, $\dot{a}_{\text{cri}}$ is the minimum(maximum) value of $\dot{a}$.

If there is a limit to $a \gg a_{\text{min}}$ then

$$\dot{a} = \pm \sqrt{Aa^2 \left( 1 \pm \sqrt{1 - \frac{a_{\text{min}}^4}{a^4}} \right)}$$

$$\approx \pm a \sqrt{A \left( 1 \pm \left( 1 - \frac{a_{\text{min}}^4}{2a^4} \right) \right)}$$

(135)

a. Case where the sign is “+”
In this case

\[ \dot{a} = \pm a \sqrt{A \left( 2 - \frac{a_{\text{min}}^4}{2a^4} \right)} \approx \pm a \sqrt{2A} \]  

(136)

whose approximate solution is

\[ a(t) = a_0 \exp \left( \pm \sqrt{\frac{2}{\alpha l^2}} (t - t_0) \right) \]  

(137)

where \( A = \frac{1}{\alpha l^2} > 0 \).

![Graph](image)

**FIG. 15.** Solution of \( \dot{a} = \sqrt{Aa^2 \left( 1 + \sqrt{1 - \frac{a_{\text{min}}^4}{a^4}} \right)} \) with \( A > 0 \) and \( \dot{a}_0 > \dot{a}_{\text{cri}} \).

b. Case where the sign is “−”
In this case

\[ \dot{a} \approx \pm \sqrt{\frac{A}{2}} \frac{a_{\text{min}}^2}{a} \]  

(138)

whose approximate solution is

\[
a(t) = \pm \sqrt{a_0^2 \pm \sqrt{\frac{2}{\kappa l^2 a_{\text{min}}^2} (t - t_0)}}
\]

\[
= \pm a_0 \sqrt{1 \pm \sqrt{\frac{2\kappa_5 \rho_0}{3a_0^4} (t - t_0)}}
\]  

(139)

where we use \( A = \frac{1}{\kappa l^2} > 0 \) and \( a_{\text{min}} = \frac{4}{3} \sqrt{\frac{\kappa_5 \rho_0}{a_0^4}} a_0 \).

*FIG. 16.* Solution of \( \dot{a} = \sqrt{Aa^2 \left(1 - \sqrt{1 - \frac{a_{\text{min}}^4}{a^4}}\right)} \) with \( A > 0 \) and \( \dot{a}_0 < \dot{a}_{\text{cri}} \).
2. Case $\alpha < 0$

In this case

$$A = \frac{1}{\alpha l^2} < 0$$

(140)

From (128) we can see that $\dot{a}$ is well defined if

$$1 \mp \sqrt{1 - \frac{B}{A} \frac{a_0^4}{a^4}} \leq 0.$$  

(141)

This condition is only satisfied if we use the minus sign “−” for all $a$, i.e.,

$$1 - \sqrt{1 - \frac{B}{A} \frac{a_0^4}{a^4}} < 0$$

(142)

and therefore $a$ has no local maxima/minima, so $a$ is monotonically increasing or monotonically decreasing. So that $\dot{a}$ has a maximum value in $a = 0$, i.e.,

$$\dot{a}_{\text{max}} = \dot{a}(a = 0) = \sqrt{-\frac{\kappa_0 \rho_0}{3 \alpha l^2}} a_0$$

(143)

and $\dot{a}$ tends to a minimum value given by

$$\dot{a}_{\text{min}} = \dot{a}(a \to \infty) = 0.$$  

(144)

If exist a limit for $a \gg \sqrt[4]{\frac{B}{A}} a_0$ then

$$\dot{a} = \pm \sqrt{A a^2 \left(1 \mp \sqrt{1 - \frac{B}{A} \frac{a_0^4}{a^4}}\right)} \approx \pm \sqrt{\frac{B}{2}} \frac{a_0^2}{a}.$$  

(145)

whose approximate solution is

$$a(t) = a_0 \sqrt{1 \pm \sqrt{\frac{2 \kappa_0 \rho_0}{3 a_0^5}} (t - t_0)}$$

(146)

where we use $B = \frac{\kappa_0 \rho_0}{3}.$

C. Case $k = 1$

In this case, the equation (88) can be rewritten as

$$\left(\frac{\dot{a}^2 + 1}{a^2}\right)^2 - 2A \left(\frac{\dot{a}^2 + 1}{a^2}\right) + AB \frac{a_0^4}{a^4} = 0,$$

(147)

from where

$$\dot{a} = \pm \sqrt{A a^2 \left(1 \pm \text{sgn}(A) \sqrt{1 - \frac{B}{A} \frac{a_0^4}{a^4}}\right) - k}$$

(148)

with $k = 1$. 

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1. Case $\alpha > 0$

In this case

$$A = \frac{1}{\alpha l^2} > 0.$$  \hfill (149)

From (148) we can see that $\dot{a}$ is well defined if

$$a_{\min} = 4\sqrt{\frac{\kappa_5\alpha l^2}{3}} a_0,$$  \hfill (150)

so that

$$B \leq A \quad \text{i.e.,} \quad \rho_0 \leq \rho_{\max} = \frac{3}{\kappa_5\alpha l^2}.$$  \hfill (151)

With these considerations we can analyze if the radicand is positive in (148)

$$Aa^2 \left( 1 \pm \sqrt{1 - \frac{a_{\min}^4}{a^4}} \right) - k.$$  \hfill (152)
FIG. 18. Solution of $\dot{a} = \sqrt{Aa^2 \left(1 - \sqrt{1 - \frac{B_0^2}{Aa^4}}\right)}$ with $A < 0$, $k = 0$ and $\dot{a}_0 < \dot{a}_{\text{max}}$.

a. Plus or minus sign?

Let us consider $\dot{a} > 0$, the analysis of the case $\dot{a} < 0$ is very similar

$$\dot{a} = \sqrt{Aa^2 \left(1 \pm \sqrt{1 - \frac{a_{\text{min}}^4}{a^4}}\right)} - k. \quad (153)$$

The function $\dot{a}(a)$ is monotonically increasing (decreasing) if we consider the plus (minus) sign in front of the square root.
From (153) we can see that exist $\dot{a}_{\text{cri}}$

$$\dot{a}_{\text{cri}} := \dot{a}(a_{\text{min}}) = \sqrt{\frac{\kappa_5 \rho_0}{3 \alpha l^2} a_0^2 - k}.$$  \hfill (154)

If we consider the plus(minus) sign in front of the square root, $\dot{a}_{\text{cri}}$ is the minimum(maximum) value of $\dot{a}$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig19.png}
\caption{Phase space for $A > 0$ and $k = 1$.}
\end{figure}

b. Case where the sign is “+”

In this case

$$A a^2 \left(1 + \sqrt{1 - \frac{a_{\text{min}}^4}{a^4}} \right) - k \geq A a_{\text{min}}^2 - k \geq 0,$$  \hfill (155)

so that

$$a_{\text{min}} \geq \sqrt{\frac{k}{A}} \iff \rho_0 a_0^4 \geq 3 \frac{\alpha l^2 k^2}{\kappa_5},$$  \hfill (156)
but (see equation (87))

\[ \rho(t) = \left( \frac{a_0}{a(t)} \right)^4 \rho_0 \implies \rho a^4 = \rho_0 a_0^4, \]

then

\[ \rho a^4 \geq 3\frac{\alpha l^2 k^2}{\kappa_5}. \]  \hspace{1cm} (157)

It is direct to prove that \( \dot{a} \neq 0 \) for \( a > a_{\text{min}} \), then \( a \) has no local maximums/minimums, and therefore \( a \) is monotonically increasing or monotonically decreasing.

If there is a limit to \( a \gg a_{\text{min}} \), then

\[ \dot{a} = \pm \sqrt{A a^2 \left( 1 + \sqrt{1 - \frac{a_{\text{min}}^4}{a^4}} \right)} - k \approx \pm \sqrt{2 A a^2 - k} \]  \hspace{1cm} (158)

whose approximate solution is

\[ a(t) = \pm \sqrt{\frac{\alpha l^2 k}{2}} \times \cosh \left[ \sqrt{\frac{2}{\alpha l^2}} (t - t_0) + \text{arcosh} \left( \sqrt{\frac{2}{\alpha l^2}} a_0 \right) \right] \]

where we use \( A = \frac{1}{\alpha l^2} \) and \( k = 1 \).

\textit{c. Case where the sign is “-”}

In this case

\[ A a^2 \left( 1 - \sqrt{1 - \frac{a_{\text{min}}^4}{a^4}} \right) - k \geq 0, \]  \hspace{1cm} (159)

therefore

\[ \frac{A a^2 - k}{A a^2} \geq \sqrt{1 - \frac{a_{\text{min}}^4}{a^4}}. \]  \hspace{1cm} (160)

This condition must be also satisfied by \( a_{\text{min}} \)

\[ A a_{\text{min}}^2 - k \geq 0 \iff a_{\text{min}} \geq \sqrt{\frac{k}{A}}, \]  \hspace{1cm} (161)

so that,

\[ \rho_0 a_0^4 \geq 3\frac{\alpha l^2 k^2}{\kappa_5}, \]  \hspace{1cm} (162)

but (see equation (87))

\[ \rho a^4 = \rho_0 a_0^4 \]
FIG. 20. Solution of \( \dot{a} = \sqrt{Aa^2 \left( 1 - \sqrt{1 + \frac{a_{\text{min}}^4}{a^4}} \right)} - k \) with \( A > 0, k = 1 \) and \( \dot{a}_0 > \dot{a}_{\text{cri}} \).

and therefore

\[
\rho a^4 \geq 3 \frac{\alpha l^2 k^2}{\kappa_5}. \tag{163}
\]

From (160) we obtain

\[
a \leq a_{\text{max}} = \sqrt{\frac{k^2 + A^2 a_{\text{min}}^4}{2Ak}}, \tag{164}
\]

i.e.,

\[
a_{\text{max}} = \sqrt{\frac{3\alpha l^2 k^2 + \kappa_5 \rho_0 a_0^4}{6k}}. \tag{165}
\]

From (165) we have

\[
\rho = \frac{a_0^4}{a^4 \rho_0}, \tag{166}
\]
FIG. 21. Phase space for $A > 0$ and $k = 1$ with “−” sign.

from where

$$\rho_{\text{min}} = \frac{a_0^4}{a_{\text{max}}^4} \rho_0$$  \hspace{1cm} (167)

and therefore

$$\rho_{\text{min}} = \left( \frac{6k\alpha_0^2}{3\alpha l^2 k^2 + \kappa_5 \rho_0 a_0^4} \right)^2 \rho_0$$  \hspace{1cm} (168)

2. Case $\alpha < 0$

In this case

$$A = \frac{1}{\alpha l^2} < 0.$$  \hspace{1cm} (169)
FIG. 22. Solution of $\dot{a} = \sqrt{Aa^2 \left(1 - \sqrt{1 - \frac{a_{\text{min}}^4}{a^4}}\right) - k}$ with $A > 0$, $k = 1$ and $\dot{a}_0 < \dot{a}_{\text{cri}}$.

From (148) we can see that $\dot{a}$ is well defined if

$$Aa^2 \left(1 \pm \text{sgn}(A) \sqrt{1 - \frac{B a_0^4}{A a^4}}\right) - k \geq 0.$$  \hspace{1cm} (170)

this constrain exclude the case with plus sign “+” in front of square root. This condition leads

$$a \leq a_{\text{max}} = \sqrt{\frac{-ABa_0^4 - k^2}{-2Ak}}$$  \hspace{1cm} (171)

where $k = 1$ and $A = \frac{1}{a_0^2} < 0$. There is a maximum value for $a$

$$a_{\text{max}} = \sqrt{\frac{3\alpha l^2 k^2 + \kappa_5 \rho \rho_0 a_0^4}{6k}},$$  \hspace{1cm} (172)
this maximum leads
\[ \rho_0 a_0^4 \geq -3 \frac{\alpha l^2 k^2}{\kappa_5}, \]  
(173)
but (see equation (87))
\[ \rho a^4 = \rho_0 a_0^4, \]
so that,
\[ \rho a^4 \geq -3 \frac{\alpha l^2 k^2}{\kappa_5}. \]  
(174)
If there is a maximum \( a_{\text{max}} \) then, must exist a minimum for \( \rho \)
\[ \rho_{\text{min}} = \left( \frac{6 k a_0^2}{3 \alpha l^2 k^2 + \kappa_5 \rho_0 a_0^4} \right)^2 \rho_0. \]  
(175)

FIG. 23. Phase space for \( A < 0 \) and \( k = 1 \) with “−” sign.

There is no a limit to \( a \rightarrow \infty \) and therefore it is impossible find an approximate solution for
\[ \dot{a} = \pm \sqrt{A a^2 \left(1 - \sqrt{1 - \frac{B a_0^4}{A a^4}}\right)} - k. \quad (176) \]

FIG. 24. Solution of \( \dot{a} = \sqrt{A a^2 \left(1 - \sqrt{1 - \frac{B a_0^4}{A a^4}}\right)} - k \) with \( A < 0, k = 1 \) and \( \dot{a}_0 < \dot{a}_{\text{max}} \).

D. Solutions for era of matter

We have found a family of solutions for era of matter.

If we consider an open space \((k = -1)\), the solutions found include (i) an accelerated expansion \((\alpha > 0)\) with a minimum scale factor at initial time that, when the time goes to infinity, the scale factor behaves as a hyperbolic sine function (Fig. 7) (ii) a decelerated
expansion \((\alpha < 0)\), with a Big Crunch in a finite time \(t_{\text{max}}\) (Fig. 9) (iii) and a couple of solutions without accelerated expansion, whose scale factor tends to a constant value: \(\alpha > 0\) (Fig. 8) and \(\alpha < 0\) (Fig. 11). See Table IV and Table V.

**TABLE IV.** Expanding universe solutions for scale factor of an open space \(k = -1\) (hyperbolic) with \(\alpha > 0\), where \(a_{\text{min}} = \frac{4}{\sqrt{\kappa \alpha l^2 \rho_0}} a_0\), \(\omega = \sqrt{\frac{2}{\alpha l^2}}\), \(\phi = \text{arsinh}\left(\frac{\omega}{\sqrt{-k}} a_0\right)\), \(\rho_{\text{max}} = \frac{3}{\kappa \alpha l^2}\) and \(\dot{a}_{\text{cri}} = \sqrt{\frac{\kappa \rho_0}{3 \alpha l^2 a_0^2}} - k\).

| \(a\) | Accelerated | No accelerated |
|------|-------------|----------------|
| \(a_{\text{min}} \leq a\) | \(a_{\text{min}} \leq a\) |
| \(a(t \to \infty)\) | \(\sim \sinh\left(\omega(t - t_0) + \phi\right)\) | \(\sim (t - t_0)\) |
| \(\rho \sim \frac{1}{a^2}\) | \(0 < \rho \leq \rho_{\text{max}}\) | \(0 < \rho \leq \rho_{\text{max}}\) |
| \(\dot{a}\) | \(\dot{a}_{\text{cri}} < \dot{a}\) | \(\sqrt{-k} < \dot{a} < \dot{a}_{\text{cri}}\) |
| \(\dot{a}(t \to \infty)\) | \(\sim \cosh\left(\omega(t - t_0) + \phi\right)\) | \(\sim \sqrt{-k}\) |

**TABLE V.** Expanding universe solutions for scale factor of an open space \(k = -1\) (hyperbolic) with \(\alpha < 0\), where \(a_{\text{max}} = \sqrt{\frac{3 \alpha l^2 k^2 + \kappa \rho_0 a_0^2}{6k}}\), \(\rho_{\text{min}} = \left(\frac{\kappa a_0^2}{3 \alpha l^2 + \kappa \rho_0 a_0^2}\right)^2 \rho_0\) and \(\dot{a}_{\text{max}} = \pm \sqrt{\frac{\kappa \rho_0}{3 \alpha l^2 a_0^2}} - k\).

A decelerated solution describes *Big Crunch* in a finite time \(t_{\text{max}}\).

| \(a\) | Decelerated | No accelerated |
|------|-------------|----------------|
| \(0 \leq a \leq a_{\text{max}}\) | \(0 \leq a\) |
| \(a(t \to \infty)\) | \(-\) | \(\sim (t - t_0)\) |
| \(\rho \sim \frac{1}{a^2}\) | \(\rho_{\text{min}} \leq \rho\) | \(0 < \rho\) |
| \(\dot{a}\) | \(-\dot{a}_{\text{max}} \leq \dot{a} \leq \dot{a}_{\text{max}}\) | \(\sqrt{-k} < \dot{a} \leq \dot{a}_{\text{max}}\) |
| \(\dot{a}(t \to \infty)\) | \(-\) | \(\sim \sqrt{-k}\) |

From models found in Section V A we can see that there are solutions with \(\alpha > 0\) for accelerated contracting universe and no accelerated contracting universe (see Figure 6, \(\dot{a} < 0\)). These solutions were not studied.

Solutions found for a flat universe \((k = 0)\) in expansion are (i) an accelerated expansion whose scale factor behaves as an exponential function when time grows and starts from a minimum value (Fig. 15) (ii) and a couple of solutions with decelerated expansion whose scale factor tends to square root function: \(\alpha > 0\) (Fig. 16) and \(\alpha < 0\) (Fig. 18). See Table
VI and Table VII.

TABLE VI. Expanding universe solutions for scale factor of a flat space $k = 0$ with $\alpha > 0$, where

$$a_{\text{min}} = \sqrt[4]{\frac{\kappa \lambda^2 \rho_0}{3}} a_0, \quad \omega = \sqrt{\frac{2}{\alpha^2}}, \quad \rho_{\text{max}} = \frac{3}{\kappa \lambda^2} \quad \text{and} \quad \dot{a}_{\text{cri}} = \sqrt[4]{\frac{\kappa \rho_0}{3 \alpha^2}} a_0.$$

| Accelerated       | Decelerated                      |
|-------------------|----------------------------------|
| $a$               | $a_{\text{min}} \leq a$         |
| $a(t \to \infty)$ | $\sim \exp \left(\omega(t - t_0)\right)$ | $\sim \sqrt{1 + \omega(a_{\text{min}}/a_0)^2 (t - t_0)}$ |
| $\rho \sim \frac{1}{a^2}$ | $0 < \rho \leq \rho_{\text{max}}$ | $0 < \rho \leq \rho_{\text{max}}$ |
| $\dot{a}$         | $\dot{a}_{\text{cri}} < \dot{a}$ | $0 < \dot{a} < \dot{a}_{\text{cri}}$ |
| $\dot{a}(t \to \infty)$ | $\sim \exp \left(\omega(t - t_0)\right)$ | $\sim \frac{1}{\sqrt{1 + \omega(a_{\text{min}}/a_0)^2 (t - t_0)}}$ |

In this case there are also solutions of contraction universe ($\dot{a} < 0$) (i) one ends with a minimum value $a_{\text{min}}$ when $\alpha$ is positive (Fig. 14) (ii) and other ends with a Big Crunch when $\alpha$ is negative (Fig. 17).

TABLE VII. Expanding universe solutions for scale factor of a flat space $k = 0$ with $\alpha < 0$, where

$$a_{\text{ref}} = \sqrt[4]{\frac{-\kappa \lambda^2 \rho_0}{3}} a_0, \quad \omega = \sqrt{-\frac{2}{\alpha^2}}, \quad \text{and} \quad \dot{a}_{\text{max}} = \sqrt[4]{\frac{\kappa \rho_0}{3 \alpha^2}} a_0.$$

| Accelerated   | Decelerate                      |
|---------------|---------------------------------|
| $a$           | $0 \leq a$                      |
| $a(t \to \infty)$ | $\sim \sqrt{1 + \omega(a_{\text{ref}}/a_0)^2 (t - t_0)}$ |
| $\rho \sim \frac{1}{a^2}$ | $0 \leq \rho$ | $0 < \dot{a} \leq \dot{a}_{\text{max}}$ |
| $\dot{a}(t \to \infty)$ | $\sim \frac{1}{\sqrt{1 + \omega(a_{\text{cri}}/a_0)^2 (t - t_0)}}$ |

Finally, we only found one solution for a closed universe ($k = 1$) in expansion. This solution is found when $\alpha$ is greater than zero. It behaves as a hyperbolic cosine function when time grows and starts from a minimum value (Fig 20). See Table VIII.

Furthermore, there are two contracting universe solutions, both ends in a finite time (i) one ends with a minimum value $a_{\text{min}}$, when $\alpha$ is positive (Fig. 22) (ii) and other ends with a Big Crunch, when $\alpha$ is negative (See Table IX and Fig. 24).
TABLE VIII. Expanding universe solutions for scale factor of a closed space \( k = 1 \) with \( \alpha > 0 \), where \( a_{\text{min}} = \frac{4}{3} \sqrt{\frac{\kappa_5 \rho_0 a_0}{a_0^2}} \), \( a_{\text{max}} = \sqrt{\frac{3 a_0^2 k^2 + \kappa_5 \rho_0 a_0^2}{a_0^2}} \), \( \omega = \frac{2}{a_0^2} \), \( \phi = \text{arcosh} \left( \frac{a_0}{\sqrt{k} a_0} \right) \), \( \rho_{\text{min}} = \left( \frac{6 \kappa_5 \rho_0}{3 a_0^2 + a_0^2} \right)^2 \rho_0 \), \( \rho_{\text{max}} = \frac{3}{\kappa_5 a_0^2} \) and \( \dot{a}_{\text{cri}} = \sqrt{\frac{\kappa_5 \rho_0}{3 a_0^2} a_0^2} - k \). A decelerated solution describes an expanding universe, which then stops the expansion and then contracts until scale factor reaches a minimum \( a_{\text{min}} > 0 \), in a finite time \( t_{\text{max}} \).

| \( a \) | \( a_{\text{min}} \leq a \) | \( a_{\text{min}} \leq a \leq a_{\text{max}} \) | Decelerated |
|---|---|---|---|
| \( a(t \to \infty) \) | \( \sim \cosh (\omega (t - t_0) + \phi) \) | \( - \) | Accelerated |
| \( \rho \sim \frac{1}{a^2} \) | \( 0 < \rho \leq \rho_{\text{max}} \) | \( \rho_{\text{min}} \leq \rho \leq \rho_{\text{max}} \) | \( \rho_{\text{min}} \leq \rho \leq \rho_{\text{max}} \) |
| \( \dot{a} \) | \( \dot{a}_{\text{cri}} < \dot{a} \) | \( -\dot{a}_{\text{cri}} < \dot{a} < \dot{a}_{\text{cri}} \) | \( -\dot{a}_{\text{cri}} < \dot{a} < \dot{a}_{\text{cri}} \) |
| \( \dot{a}(t \to \infty) \) | \( \sim \sinh (\omega (t - t_0) + \phi) \) | \( - \) | \( \sim \sinh (\omega (t - t_0) + \phi) \) |

TABLE IX. Expanding universe solutions with Big Crunch for scale factor of a closed space \( k = 1 \) with \( \alpha < 0 \), where \( a_{\text{max}} = \sqrt{\frac{3 a_0^2 k^2 + \kappa_5 \rho_0 a_0^2}{a_0^2}} \), \( \rho_{\text{min}} = \left( \frac{6 \kappa_5 \rho_0}{3 a_0^2 + a_0^2} \right)^2 \rho_0 \) and \( \dot{a}_{\text{cri}} = \sqrt{\frac{\kappa_5 \rho_0}{3 a_0^2} a_0^2} - k \). This solution describes a expanding universe, which then stops the expansion and then contracts until a Big Crunch, in a finite time \( t_{\text{max}} \).

| \( a \) | \( a \leq a_{\text{max}} \) | \( a \leq a_{\text{max}} \) | Decelerate |
|---|---|---|---|
| \( a(t \to \infty) \) | \( - \) | \( - \) | \( - \) |
| \( \rho \sim \frac{1}{a^2} \) | \( \rho_{\text{min}} \leq \rho \) | \( \rho_{\text{min}} \leq \rho \) | \( \rho_{\text{min}} \leq \rho \) |
| \( \dot{a} \) | \( -\dot{a}_{\text{max}} < \dot{a} < \dot{a}_{\text{max}} \) | \( -\dot{a}_{\text{max}} < \dot{a} < \dot{a}_{\text{max}} \) | \( -\dot{a}_{\text{max}} < \dot{a} < \dot{a}_{\text{max}} \) |
| \( \dot{a}(t \to \infty) \) | \( - \) | \( - \) | \( - \) |

VI. SUMMARY

We have considered a five-dimensional Einstein-Chern-Simons action \( S = S_g + S_M \) which is composed of a gravitational sector and a sector of matter, where the gravitational sector is given by a Chern-Simons gravity action instead of the Einstein-Hilbert action and where the matter sector is given by the so called perfect fluid. We have shown that (i) the Einstein-Chern-Simons EChS field equations subject to the condition \( T^a = 0 \) and \( \delta L_M / \delta \omega^{ab} = 0 \), can
be written in a similar way to the Einstein-Maxwell field equations; (ii) the interpretation of the $h^a$ field as a perfect fluid allow us to find find solutions to the field equations which describe accelerated expansion universes.

In fact, in Section IV we have found solutions that describes accelerated expansion for the three possible cosmological models of the universe. Namely, spherical expansion ($k = 1$), flat expansion ($k = 0$) and hyperbolic expansion ($k = -1$) when the constant $\alpha$ is greater than zero. This mean that the Einstein-Chern-Simons field equations have as a of their solutions an universe in accelerated expansion. This result allow us to conjeture that this solutions are compatible with the era of Dark Energy and that the energy-momentum tensor for the field $h^a$ corresponds to a form of positive cosmological constant. We have also shown that the EChS field equations have solutions that allows us to identify the energy-momentum tensor for the field $h^a$ with a negative cosmological constant.

On the other hand, in Section V we have found a family of solutions for era of matter. In the case $k = -1$ (open universe), the solutions correspond to (i) an accelerated expansion ($\alpha > 0$) with a minimum scale factor at initial time that, when the time goes to infinity, the scale factor behaves as a hyperbolic sine function (ii) a decelerated expansion ($\alpha < 0$), with a Big Crunch in a finite time $t_{\text{max}}$ (iii) and a couple of solutions without accelerated expansion, whose scale factor tends to a constant value. In the case $k = 0$ (flat universe), the solutions describing (i) an accelerated expansion whose scale factor behaves as a exponencial function when time grows and starts from a minimum value (ii) and a couple of solutions with decelerated expansion whose scale factor tends to square root function. In the case $k = 1$ it is found only one solution for a closed universe in expansion, which behaves as a hyperbolic cosine function when time grows and starts from a minimum value. However there are two contracting universe solutions, both ends in a finite time. One ends with a minimum value $a_{\text{min}}$, when $\alpha$ is positive and other ends with a Big Crunch, when $\alpha$ is negative.

**ACKNOWLEDGMENTS**

This work was supported in part by FONDECYT Grants 1130653 and by Universidad de Concepción through DIUC Grant 212.011.056-1.0. Two of the authors (F.G., C.Q.) were supported by grants from the Comisión Nacional de Investigación Científica y Tec-
nológica CONICYT and from the Universidad de Concepción, Chile. M.C. was supported by Grant FONDECYT 1121030 and by Dirección de Investigación de la Universidad del Bío-Bío through Grants DIUBB 1210072/R and GI1221407/VBC. S.delC. was supported by Grant FONDECYT 1110230 and by Pontificia Universidad Católica de Valparaíso through Grants PUCV 123.710

Appendix A: Obtaining equations (24-28)

From equations (28-32) of Ref. [7] we know that

\[ 48\alpha_3 \left( \frac{\dot{a}^2 + k}{a^2} \right) + 24\alpha_1 l^2 \left( \frac{a^2 + k}{a^2} \right)^2 = \beta_1 T_{00}, \]  
(A1)

\[ -24\alpha_3 \left[ \frac{\dot{a}}{a} + \left( \frac{a^2 + k}{a^2} \right) \right] \] 
\[ -24\alpha_1 l^2 \frac{\dot{a}}{a} \left( \frac{a^2 + k}{a^2} \right) = \beta_1 T_{11}, \]  
(A2)

\[ 24\alpha_3 l^2 \left( \frac{a^2 + k}{a^2} \right)^2 = \beta_2 T_{00}^{(b)}, \]  
(A3)

\[ -24\alpha_3 l^2 \frac{\ddot{a}}{a} \left( \frac{a^2 + k}{a^2} \right) = \beta_2 T_{11}^{(b)}, \]  
(A4)

\[ 24\alpha_3 l^2 \left( \frac{a^2 + k}{a^2} \right) \left[ (g - f) \frac{\dot{a}}{a} + \dot{g} \right] = 0 \]  
(A5)

where

\[ h^0 = f(t) e^0 \]  
(A6)

\[ h^p = g(t) e^p, \quad p = 1, ..., 4. \]  
(A7)

In this article we have considered \( \beta_1 = \beta_2 = \kappa \). Making this replacement in (A1-A7) and
dividing it by $8\alpha_3$ we have

\[
6 \left( \frac{a^2 + k}{a^2} \right) + \left( \frac{\alpha_1}{\alpha_3} \right) \left[ 3l^2 \left( \frac{a^2 + k}{a^2} \right)^2 \right] = \left( \frac{\kappa}{8\alpha_3} \right) T_{00}, \tag{A8}
\]

\[-8 \left[ \frac{\ddot{a}}{a} + \left( \frac{a^2 + k}{a^2} \right) \right] - \left( \frac{\alpha_1}{\alpha_3} \right) \left[ 3l^2 \frac{\ddot{a}}{a} \left( \frac{a^2 + k}{a^2} \right) \right] = \left( \frac{\kappa}{8\alpha_3} \right) T_{11}, \tag{A9}
\]

\[3l^2 \left( \frac{a^2 + k}{a^2} \right)^2 = \left( \frac{\kappa}{8\alpha_3} \right) T^{(h)}_{00}, \tag{A10}\]

\[-3l^2 \frac{\ddot{a}}{a} \left( \frac{a^2 + k}{a^2} \right) = \left( \frac{\kappa}{8\alpha_3} \right) T^{(h)}_{11}, \tag{A11}\]

\[3l^2 \left( \frac{a^2 + k}{a^2} \right) \left[ (g - f) \frac{\ddot{a}}{a} + \dot{g} \right] = 0. \tag{A12}\]

Consider now the definition of the constants of Section II

\[\kappa_5 = \frac{\kappa}{8\alpha_3}, \quad \alpha = -\frac{\alpha_1}{\alpha_3}. \tag{A13}\]

With these constants, equations (A8 - A12) take the form

\[6 \left( \frac{a^2 + k}{a^2} \right) - \alpha \left[ 3l^2 \left( \frac{a^2 + k}{a^2} \right)^2 \right] = \kappa_5 T_{00}, \tag{A14}\]

\[-8 \left[ \frac{\ddot{a}}{a} + \left( \frac{a^2 + k}{a^2} \right) \right] + \alpha \left[ 3l^2 \frac{\ddot{a}}{a} \left( \frac{a^2 + k}{a^2} \right) \right] = \kappa_5 T_{11}, \tag{A15}\]

\[3l^2 \left( \frac{a^2 + k}{a^2} \right)^2 = \kappa_5 T^{(h)}_{00}, \tag{A16}\]

\[-3l^2 \frac{\ddot{a}}{a} \left( \frac{a^2 + k}{a^2} \right) = \kappa_5 T^{(h)}_{11}, \tag{A17}\]

\[3l^2 \left( \frac{a^2 + k}{a^2} \right) \left[ (g - f) \frac{\ddot{a}}{a} + \dot{g} \right] = 0. \tag{A18}\]

Replacing now (A16) in square brackets (A14), and (A17) in square brackets (A15), and passing those terms on the right side of the equations, we find
In Section IV was considered an energy-momentum tensor of the form
\[
\tilde{T}_{\mu\nu} = T_{\mu\nu} + \alpha T_{\mu\nu}^{(h)}
\]
(A29)
\[
= \text{diag}(\rho, p, p, p, p) + \alpha \text{diag}(\rho^{(h)}, p^{(h)}, p^{(h)}, p^{(h)}, p^{(h)})
\]
(A30)
\[
= \text{diag}(\rho + \alpha\rho^{(h)}, p + \alpha p^{(h)}, p + \alpha p^{(h)}, p + \alpha p^{(h)}, p + \alpha p^{(h)})
\]
(A31)
\[
= \text{diag}(\tilde{\rho}, \tilde{p}, \tilde{p}, \tilde{p}, \tilde{p})
\]
(A32)
where

\[ T_{\mu\nu} = \text{diag}(\rho, p, p, p, p) \]  \hspace{1cm} (A33)

\[ T^{(h)}_{\mu\nu} = \text{diag}\left(\rho^{(h)}, p^{(h)}, p^{(h)}, p^{(h)}, p^{(h)}\right) \]  \hspace{1cm} (A34)

Writing the functions \( f \) and \( g \) as

\[ f = h(0), \quad g = h, \]  \hspace{1cm} (A35)

we find that the equations (A24 - A28) take the form

\[ 6 \left( \frac{\ddot{a}}{a^2} + k \right) = \kappa_5 \bar{\rho}, \]  \hspace{1cm} (A36)

\[ 3 \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}^2 + k}{a^2} \right) \right] = -\kappa_5 \bar{p}, \]  \hspace{1cm} (A37)

\[ \frac{3l^2}{\kappa_5} \left( \frac{\ddot{a}}{a^2} + k \right)^2 = \rho^{(h)}, \]  \hspace{1cm} (A38)

\[ \frac{3l^2}{\kappa_5} \frac{\ddot{a}}{a} \left( \frac{\dot{a}^2 + k}{a^2} \right) = -p^{(h)}, \]  \hspace{1cm} (A39)

\[ \left( \frac{\ddot{a}}{a^2} + k \right) \left[ (h - h(0)) \frac{\ddot{a}}{a} + \dot{\varphi} \right] = 0. \]  \hspace{1cm} (A40)

which correspond to the equations (24 - 28).

[1] F. Izaurieta, P. Minning, A. Pérez, E. Rodríguez, P. Salgado, Phys. Lett. B 678 (2009) 213.
[2] F. Izaurieta, E. Rodríguez, P. Salgado, Jour. Math. Phys. 47 (2006) 123512.
[3] F. Izaurieta, A. Perez, E. Rodríguez, P. Salgado, Jour. Math. Phys. 50 (2009) 073511.
[4] A. H. Chamseddine, Phys. Lett. B 233 (1989) 291.
[5] A. H. Chamseddine, Nucl. Phys. B 346 (1990) 213.
[6] J. Zanelli, "Lecture notes on Chern-Simons (super)gravities. Second edition (February 2008)."
[7] F. Gomez, P. Minning, P. Salgado, Phys. Rev. D 84, 063506 (2011).
[8] C.A.C. Quinzacara and P. Salgado, Phys. Rev. D 85 (2012) 124026.
[9] S. del Campo, JCAP, 1212 (2012) 005.