Linear relations between writhe and minimal crossing number in Conway families of ideal knots and links

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Abstract. We showed earlier how to predict the writhe of any rational knot or link in its ideal geometric configuration, or equivalently the average of the 3D writhe over statistical ensembles of random configurations of a given knot or link (Cerf and Stasiak 2000 Proc. Natl Acad. Sci. USA 97 3795). There is no general relation between the minimal crossing number of a knot and the writhe of its ideal geometric configuration. However, within individual families of knots linear relations between minimal crossing number and writhe were observed (Katritch et al 1996 Nature 384 142). Here we present a method that allows us to express the writhe as a linear function of the minimal crossing number within Conway families of knots and links in their ideal configuration. The slope of the lines and the shift between any two lines with the same slope can be computed.

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1. Introduction

Quantization of writhe in knots and links is a puzzling phenomenon that was initially discussed only among a narrow group of specialists but has recently become quite popular [1]. Let us explain what this concept covers. Knots and links are one or several non-self-intersecting closed curves in 3D space. Writhe (or 3D writhe, or $\text{Wr}$) measures their chirality. It corresponds to the average signed number of crossings when the knot or link is observed from a random direction, where each right-handed crossing is scored as $+1$ and each left-handed crossing is scored as $-1$.

Writhe is usually calculated using a Gauss integral formula [2]:

$$\text{Wr} = \frac{1}{4\pi} \oint_c \oint_c \frac{(d\mathbf{r}_1 \times d\mathbf{r}_2) r_{12}}{|r_{12}|^3},$$

where $\mathbf{r}_1$ ($\mathbf{r}_2$) is a vector from origin to point 1 (point 2) and $r_{12}$ is a vector from point 1 to point 2, with points 1 and 2 running over the considered curve(s) $c$. Studies of random walks in a cubic lattice revealed that, while different realizations of random knots of a given type have stochastically distributed values of their writhe, the average of writhe over the statistical ensemble of knots of a given type, like right-handed trefoils for example, reaches a characteristic value that is independent of the length of the walk [3, 4]. Thus, for example, the average writhe of random right-handed trefoils in a cubic lattice does not change when the number of segments is increased from 34 to 250. The same studies pointed out that there is a constant increase (i.e. a quantization) of the average writhe between successive knots belonging to given families (e.g. torus knots 3$_1$, 5$_1$, 7$_1$, etc, or twist knots 4$_1$, 6$_1$, 8$_1$, etc), the specific difference between successive knots depending on the family considered. The writhe increases between sequential knots belonging to a given family, obtained by numerical simulations, were not exactly constant however, so we should use the term ‘quasi-quantization’ rigorously rather than ‘quantization’. It is not known whether the quantization is strict but hidden by numerical errors, or whether quasi-quantization is the actual phenomenon. We will include these two possibilities within the term ‘quantization’.

New Journal of Physics 5 (2003) 87.1–87.15 (http://www.njp.org/)
Studies of random knots that are not confined to a lattice also revealed the phenomenon of quantization. Actually, the average writhe over a statistical ensemble of random walks not confined to a lattice but forming a given knot type was practically the same as the average writhe of a statistical ensemble of random knots of the same type in a cubic lattice [2, 4]. The average writhe over a statistical ensemble of simulated configurations of a given knot also closely corresponds to the time-averaged writhe of a randomly fluctuating long polymeric chain forming the same knot type. Therefore a freely fluctuating long DNA molecule closed into a right-handed trefoil, for example, would show the same average writhe as a freely fluctuating long polyethylene molecule that is also closed into a right-handed trefoil. The time-averaged values of writhe seem to be independent of the size of the polymeric chain [2].

The quantization of writhe observed for statistical ensembles of different knots is reflected in the quantization of writhe for the so-called ideal knots (also called tight knots). Ideal knots are defined as the shortest possible paths of cylindrical tubes with uniform diameter that can still be closed into a given knot type [2]. Numerical simulations revealed that writhe of axial trajectories of unique realizations of ideal knots of a given type (denoted by \( W_{\text{ideal}} \)) corresponds to the time-averaged writhe of freely fluctuating knots of the same type [2]. Therefore unique representations of ideal knots of a given type ‘capture’ the essential statistical property of random knots of a given type [5]. To find the average writhe of fluctuating knots of a given type it becomes therefore much more practical to measure the writhe of one ideal configuration of this knot instead of simulating thousands of random configuration of this knot type.

The mathematical existence of an ideal configuration for each knot or link has been proved recently by two groups [6, 7]. Actually, some links have been shown to have several ideal configurations [7]. They, however, share the same writhe. The constancy of the writhe results from the fact that the writhe of a link is equal to the sum of ‘internal’ writhes of individual components plus twice the linking number between the constituting components (see, e.g., [8]). In all instances of links with multiple ideal configurations known up to now, the individual components are unknotted planar closed curves, therefore having internal writhes of zero. The linking number between the components is constant since it is a topological invariant, insensitive to the configuration. Thus, for those links the overall writhe of the different ideal configurations remains unchanged.

It has been noticed by Pieranski (see figure 8 of [9]) that the values of \( W_{\text{ideal}} \) of all prime knots up to 9 crossings are concentrated around nine well-defined levels, regardless of their families, thereby suggesting the existence of a general quantization pattern. Using this observation, we have demonstrated [10] that \( W_{\text{ideal}} \) of all oriented alternating knots and links can be predicted using a topological invariant, namely the predicted writhe \( PWR \), which is a linear combination of the nullification writhe \( w_x \) and the remaining writhe \( w_y \):

\[
PWR = \frac{10}{7}w_x + \frac{4}{7}w_y.
\]  

The nullification writhe \( w_x \) and the remaining writhe \( w_y \) are defined as follows [11]: transform a standard projection by nullifying (or smoothing) successive crossings until the unknot is reached, while at each step preventing the diagram from becoming disconnected. Then \( w_x \) is the sum of the signs of the nullified crossings and \( w_y \) is the sum of the signs of the remaining crossings. Examples of nullifications are shown in figures 3–9. A discussion about the coefficients \( 10/7 \) and \( 4/7 \) can be found in [5] and [10]. Since \( w_x \) and \( w_y \) are topological invariants, so is \( PWR \), i.e. it depends on the topological type of the knot or link but not on a particular configuration of it.
The matching between \( PWr \) and \( W_{\text{ideal}} \) is strikingly good [10, 12]. This supports the notion that the ideal configuration contains important information about knots and links. Moreover, the fact that the calculation of the 3D writhe of ideal knots can be performed using the minimal planar diagram of the knot greatly facilitates calculation of the time-averaged writhe of randomly fluctuating knotted polymers. Complex simulations of ideal configurations are not needed but just a simple scoring of crossings in any minimal diagram of the corresponding knot.

The method of writhe prediction that we proposed in [10] can be applied to any minimal diagram of alternating knots and links. If one considers all alternating knots and links, there is no simple function that can relate the writhe to the minimal crossing number. However, Huang and Lai [13] used formula (2) to calculate the writhe of ideal knots belonging to four families of knots: torus knots \((3_1, 5_1, 7_1, \ldots)\), even twist knots \((4_1, 6_1, 8_1, \ldots)\), odd twist knots \((5_2, 7_2, 9_2)\) and the family containing knots \(6_2, 8_2, 10_2, \ldots\). They observed that, within these families, the writhe varies linearly versus the minimal crossing number. We show here that, in the setting of formula (2), the writhe is naturally described as a linear function of the minimal crossing number in families of knots and links that we call Conway families.

This work had been first presented at the International Workshop on Topology in Condensed Matter Physics held from 13 May to 31 July 2002, at the Max-Planck-Institut für Physik Komplexer Systeme (Dresden, Germany), and it will appear in the proceedings of this workshop in a shorter version.

2. Rational tangles, rational links and Conway families

Rational tangles and rational links were introduced by Conway in 1970 [14]. A rational tangle is a region of a knot or link projection composed of a succession of vertical and horizontal rows of crossings, and is denoted by a sequence of numbers corresponding to the number of crossings in each row (see figure 1(a)). To avoid confusion, one always has to end with a horizontal row. If the latter contains no crossing, the sequence will end with (0). All crossings are done in order for the projection to be alternating (each strand alternately goes over and under other strands). Figure 1(a) shows positive rational tangles. If each crossing is inverted (i.e. the mirror image is considered), one gets negative rational tangles that will be denoted by a sequence of negative numbers. Conway proved that a rational tangle denoted by a mixed sequence of positive and negative numbers (i.e. a nonalternating projection) is always topologically equivalent to an alternating projection with all positive or all negative numbers. We will continue with positive rational tangles only. The extension to negative rational tangles is obvious.

The closure of a rational tangle is the operation of rejoining the two upper free ends and the two lower free ends on the projection (see figure 1(b)). The link so obtained is called a rational link (also called a two-bridge link or 4-plat). Rational links are completely classified. They all have either one or two components (let us remind the reader that a one-component link is a knot). Nearly all knots and links naturally occurring in closed polymer chains are rational links.

Let us now introduce a notation that will allow us to denote families of rational tangles. It will be easily understood using some examples. We denote by \((2)(1^{\Delta_2})(3)\) the family of tangles \((2)(1)(3), (2)(3)(3), (2)(5)(3), \ldots\) We denote by \((2^{\Delta_4})(1)(1)(3^{\Delta_2})\) the family of tangles \((2)(1)(1)(3), (6)(1)(1)(5), (10)(1)(1)(7), \ldots\) We call Conway families the families of links obtained by the closure of such tangles and denote them with a \(C\) followed by the considered sequence. With this notation, the family of torus knots is denoted by \(C(3^{\Delta_2})\), the family of even
The family of twist knots is denoted by $C(2^{\pm2})(2)$, the family of odd twist knots is denoted by $C(3^{\pm2})(2)$ and the family containing knots $6_2$, $8_2$, $10_2$, $\ldots$ is denoted by $C(3^{\pm2})(1)(2)$ (see figure 2).

3. Writhe of links belonging to Conway families

We will successively consider families of links obtained by the closure of tangles composed of one, two, three and $r$ rows.

3.1. Tangles with one row, denoted by $(n)$, $n$ positive integer

Two cases occur, depending on the parity of the number of crossings $n$.

3.1.1. $n$ odd. Let us consider a knot formed by the closure of tangle $(n)$ with $n$ odd and positive. Figure 3 shows the knot with $n = 5$ but the reader is asked to imagine any odd and positive $n$. The
same exercise will be done for figures 4–9. We now nullify [11] the knot of figure 3. Crossings are successively nullified (i.e. smoothed) until the unknot is reached, forbidding at each step the appearance of a disconnected component. The sum of the signs of the nullified crossings is \( w_x \) and the sum of the signs of the remaining crossings is \( w_y \). In this case, nullifying \( n - 1 \) positive crossings gives rise to the unknot. One cannot nullify the last crossing without disconnecting the link. So \( w_x = n - 1 \) and \( w_y = 1 \). Using formula (2) we get

\[
P_{Wr} = \frac{10}{7}(n - 1) + \frac{4}{7} = \frac{10}{7}n - \frac{6}{7}.
\]  

(3)

Incrementing \( n \) in steps of 2 (to keep it odd), we get a family \( C(3^{\Delta 2}) \). Notice that the family begins with three crossings because a knot with one crossing is an unknot. All members of this family have the same nullification scheme, so \( P_{Wr} \) is a linear function of the crossing number \( n \) with slope \( \frac{10}{7} \).

3.1.2. \( n \) even. Let us now close the tangle \( (n) \) with \( n \) even and positive. Figure 4(a) shows that we are now dealing with a two-component link. If we increment \( n \) in steps of 2, we get a family \( C(2^{\Delta 2}) \). Because there are two components, we must distinguish two possibilities of orientation. The family \( C(2^{\Delta 2}) \) is therefore split into two subfamilies, that we will call \( C(2^{\Delta 2}) \) type I and \( C(2^{\Delta 2}) \) type II.

(i) Conway family \( C(2^{\Delta 2}) \) type I contains links \( 2_1^2 \) ++, \( 4_1^2 \) ++, \( 6_1^2 \) ++, \( 8_1^2 \) ++ , etc. (The symbols + and − refer to orientation. The convention used can be found in [15].) Figure 4(b)
Figure 3. Knot formed by the closure of a rational tangle with one row of $n$ (odd) crossings. The nullification of this knot is shown.

Figure 4. (a) Two-component link formed by the closure of a rational tangle with one row of $n$ (even) crossings. Depending on the orientation chosen for the second component, the sign of the crossings changes, and the nullification scheme changes accordingly. The two cases are shown in (b) and (c).

shows the nullification process applied to those links. Here again we get $w_x = n - 1$ and $w_y = 1$. Therefore

$$PW_r = \frac{10}{7}(n - 1) + \frac{4}{7} = \frac{10}{7}n - \frac{6}{7}.$$  

$PW_r$ is a linear function of the crossing number $n$ with slope $10/7$.

(ii) Conway family $C(2^{\Delta^2})$ type II contains links $2_i^+ - -, 4_i^2 + - , 6_i^2 + + , 8_i^2 + +$, etc. Figure 4(c) shows that the $n$ crossings are now of negative sign and that only one crossing may be nullified. We already reached the unknot and any further nullification would create a disconnected component. Thus $w_x = -1$ and $w_y = -(n - 1)$. In words, we have nullified one negative crossing and there remain $n - 1$ negative crossings. Using formula (2) again, we get

$$PW_r = -\frac{10}{7} - \frac{4}{7}(n - 1) = -\frac{4}{7}n - \frac{6}{7}$$

which tells us that $PW_r$ is a linear function of the crossing number $n$ with slope $-4/7$. 

New Journal of Physics 5 (2003) 87.1–87.15 (http://www.njp.org/)
3.2. Tangles with two rows, denoted by \((a)(b)\), \(a\) and \(b\) positive integers.

Now the minimal crossing number \(n = a + b\). This time, four cases occur, depending on the parity of \(a\) and \(b\).

3.2.1. \(a\) odd, \(b\) even. The closure of this tangle gives rise to knots with \(n\) positive crossings: \(a\) in the vertical row and \(b\) in the horizontal row. Looking at figure 5, we see that we may nullify one positive crossing from the vertical row and \(b - 1\) positive crossings from the horizontal row. Thus \(w_x = 1 + (b - 1) = b\) and \(w_y = (a - 1) + 1 = a\). Using formula (2) we have

\[
PWr = \frac{10}{7}b + \frac{4}{7}a.
\]

Let us examine the Conway families containing such knots. The simplest ones involve an incrementation of the number of crossings in one row only. These are \(C(3^{\Delta 2})(2)\), which is the family of odd twist knots, and \(C(3)(2^{\Delta 2})\). Notice that we consider vertical rows with three crossings at least because a tangle \((1)(t)\) is equivalent to a simpler tangle \((t + 1)\). In \(C(3^{\Delta 2})(2)\), when \(a\) is increased by 2, \(n\) is increased by 2 and \(PWr\) is increased by 8/7 (since \(b\) remains constant). \(PWr\) is thus a linear function of \(n\) with slope 4/7. In \(C(3)(2^{\Delta 2})\), when \(b\) is increased...
Figure 6. Knot formed by the closure of a rational tangle with two rows, containing \( a \) (even) and \( b \) (even) crossings, respectively. The nullification of this knot is shown.

by 2, \( n \) is increased by 2 and \( PWr \) is increased by \( 20/7 \) (since \( a \) remains constant). \( PWr \) is thus a linear function of \( n \) with slope \( 10/7 \).

We may now combine incrementations of the numbers of crossings in both rows. Several combinations are possible. Let us begin with the family \( C(3^{\Delta_2})(2^{\Delta_2}) \). Successive members of this family have \( a \) increased by 2 and \( b \) increased by 2. Therefore \( n \) is increased by 4 and \( PWr \) is increased by 4 \( (20/7 + 8/7) \). \( PWr \) is thus a linear function of \( n \) with slope 1. It is obvious that in all Conway families \( (C(3 + 2i)^{\Delta_2})(2 + 2j)^{\Delta_2}) \) with \( i \) and \( j \) positive integers, \( PWr \) will be a linear function of \( n \) with slope 1. Only the intercept will be different. Let us now consider families \( (3 + 2i)^{\Delta_2}(2 + 2j)^{\Delta_4}) \). Successive members of one such family have \( a \) increased by 2 and \( b \) increased by 4. Therefore \( n \) is increased by 6 and \( PWr \) is increased by \( 48/7(40/7 + 8/7) \). \( PWr \) is thus a linear function of \( n \) with slope \( 48/42 \). The general statement is that in Conway families \( (3 + 2i)^{\Delta_2}(2 + 2j)^{\Delta_2}) \) with \( i, j, k, l \) positive integers, \( PWr \) is a linear function of
Figure 7. Knot formed by the closure of a rational tangle with two rows, containing \( a \) (even) and \( b \) (odd) crossings, respectively. The nullification of this knot is shown.

\[ n \text{ with} \]
\[ \text{slope} = \frac{10l + 4k}{7(l + k)} . \]  

(7)

In the whole section, \( i, j, k, l \) will be any positive integers.

3.2.2. \( a \) even, \( b \) even. The closure of this tangle gives rise to knots with \( a \) positive crossings in the vertical row and \( b \) negative crossings in the horizontal row. Figure 6 shows that we may nullify only one positive crossing from the vertical row and one negative crossing from the horizontal row. Thus \( w_x = 1 - 1 = 0 \) and \( w_y = (a - 1) - (b - 1) = a - b \). Formula (2) gives

\[ PWr = \frac{4}{7}(a - b) . \]  

(8)
Figure 8. (a) Two-component link formed by the closure of a rational tangle with two rows, containing \( a \) (odd) and \( b \) (odd) crossings, respectively. Depending on the orientation chosen for the second component, the sign of the crossings changes and the nullification scheme changes accordingly. The two cases are shown in (b) and (c).

As a result, in Conway families \( C(\{2 + 2 i\}^{\Delta 2k})(\{2 + 2 j\}^{\Delta 2l}) \) \( PWr \) is a linear function of \( n \) with
\[
slope = \frac{4(k - l)}{7(k + l)}.
\]

The family of even twist knots, \( C(2^{\Delta 2})(2) \), is an example where \( i = j = l = 0 \) and \( k = 1 \). The slope is therefore \( 4/7 \).

3.2.3. \( a \) even, \( b \) odd. The closure of this tangle gives rise to knots with \( n \) negative crossings: \( a \) in the vertical row and \( b \) in the horizontal row. Looking at figure 7, we see that we may nullify \( a - 1 \) negative crossings from the vertical row and one negative crossing from the horizontal row. Thus \( w_x = -(a - 1) - 1 = -a \) and \( w_y = -1 - (b - 1) = -b \). Using formula (2) we have
\[
PWr = -\frac{10}{7}a - \frac{4}{7}b.
\]

The same reasoning as before tells us that in Conway families \( C((2 + 2 i)^{\Delta 2k})(\{1 + 2 j\}^{\Delta 2l}) \) \( PWr \) is a linear function of \( n \) with
\[
slope = -\frac{10k + 4l}{7(k + l)}.
\]

3.2.4. \( a \) odd, \( b \) odd. Figure 8(a) shows that the closure of such a tangle gives rise to a two-component link. There are therefore two possible orientations for the second component. The Conway families \( C((3 + 2 i)^{\Delta 2k})(\{1 + 2 j\}^{\Delta 2l}) \) are split into two subfamilies: \( C((3 + 2 i)^{\Delta 2k})(\{1 + 2 j\}^{\Delta 2l}) \) type I and \( C((3 + 2 i)^{\Delta 2k})(\{1 + 2 j\}^{\Delta 2l}) \) type II.
Figure 9. Knot formed by the closure of a rational tangle with three rows, containing \(a\) (odd), \(b\) (odd) and \(c\) (even) crossings, respectively. The nullification of this knot is shown.

(i) Conway family \(C(\{3+2i\}^{\Delta 2k})(\{1+2j\}^{\Delta 2l})\) type I contains links with \(n\) positive crossings: \(a\) in the vertical row and \(b\) in the horizontal row. Looking at figure 8(b), we see that we may nullify one positive crossing from the vertical row and \(b - 1\) positive crossings from the horizontal row, as in the case where \(a\) is odd and \(b\) even (section 3.2.1). Thus \(w_x = 1 + (b - 1) = b\) and \(w_y = (a - 1) + 1 = a\). Using formula (2) we have, as in section 3.2.1,

\[
PWr = \frac{10}{7}b + \frac{4}{7}a,
\]

and in Conway families \(C(\{3+2i\}^{\Delta 2k})(\{1+2j\}^{\Delta 2l})\) type I, \(PWr\) is a linear function of \(n\) with slope

\[
\text{slope} = \frac{10l + 4k}{7(l + k)}
\]

In the diagram of \(PWr\) versus \(n\), each line corresponding to such a family of two-component links is thus parallel to a series of lines corresponding to Conway families \(C(\{3+2i\}^{\Delta 2k})(\{2+2j\}^{\Delta 2l})\) of knots having the same \(k\) and \(l\).

(ii) Conway family \(C(\{3+2i\}^{\Delta 2k})(\{1+2j\}^{\Delta 2l})\) type II contains links with \(n\) negative crossings: \(a\) in the vertical row and \(b\) in the horizontal row. Figure 8(c) shows that we may nullify \(a - 1\) negative crossings from the vertical row and one negative crossing from the horizontal...
row, as in the case where \( a \) is even and \( b \) odd (section 3.2.3). Thus \( w_x = -(a - 1) - 1 = -a \) and \( w_y = -1 - (b - 1) = -b \). Using formula (2) we have, as in section 3.2.3,

\[
P Wr = \frac{10}{7}a - \frac{4}{7}b,
\]

and in Conway families \( C((3 + 2i)^{\Delta 2k})((1 + 2j)^{\Delta 2l}) \) type II, \( P Wr \) is a linear function of \( n \) with slope \( \frac{-10k + 4l}{7(k + l)} \).

In the diagram of \( P Wr \) versus \( n \), each line corresponding to such a family of two-component links is thus parallel to a series of lines corresponding to Conway families \( C((2 + 2i)^{\Delta 2k})((1 + 2j)^{\Delta 2l}) \) of knots having the same \( k \) and \( l \).

3.3. Tangles with three rows, denoted by \((a)(b)(c)\), \( a, b \) and \( c \) positive integers

Now, \( n = a + b + c \). There are \( 2^3 = 8 \) cases to study, depending on the parity of \( a, b \) and \( c \). Let us illustrate the process with \( a \) odd, \( b \) odd and \( c \) even. There are \( a \) positive crossings in the first horizontal row, \( b \) positive crossings in the vertical row and \( c \) negative crossings in the last horizontal row (see figure 9). Looking at the figure, we see that we may nullify \( a - 1 \) positive crossings from the first row, one positive crossing from the second row and one negative crossing from the last row. Thus \( w_x = (a - 1) + 1 - 1 = a - 1 \) and \( w_y = 1 + (b - 1) - (c - 1) = 1 + b - c \). Formula (2) gives

\[
P Wr = \frac{10}{7}(a - 1) + \frac{4}{7}(1 + b - c).
\]

As before, we obtain Conway families by incrementing the number of crossings in one or several rows by steps in multiples of 2. As an example, \( C(3^{\Delta 2})(1)(2) \) is obtained by incrementing in steps of 2 the number of crossings in the first horizontal row only, and it contains knots 62, 82, 102, etc. Since only \( a \) varies in formula (16), we get a linear function of \( n \) with slope 10/7.

3.4. Tangles with \( r \) rows

We can generalize this approach to any Conway family of links formed by the closure of \( r \)-row tangles. Formula (2) will still hold, where each of \( w_x \) and \( w_y \) is a sum of the following form:

\[
w_{x/y} = \begin{cases} a - 1 & b - 1 & c - 1 \\ -(a - 1) & -(b - 1) & -(c - 1) \\ 1 & 1 & 1 \\ -1 & -1 & -1 \\ \end{cases} + \cdots.
\]

All derived Conway families obtained by incrementing the number of crossings in one or several rows by steps in multiples of 2 will have a linear relation between \( P Wr \) and \( n \).

4. Discussion

4.1. Shifts between Conway families with the same slope

In all Conway families, \( P Wr \) is a linear function of \( n \). As we have seen above, some families share the same slope. In that case, the shift between two such families can be computed.
As an example, let us consider two Conway families formed by the closure of a tangle with two rows: \((a)(b)\) with \(a\) odd and \(b\) even, and \((a)(b)\) with \(a\) and \(b\) even. Let us fix \(b\) at 2 in both families. The families we consider could be, for instance, \(C(3\Lambda^2)(2)\) and \(C(2\Lambda^2)(2)\). For the first family, equation (6) with \(b = 2\) becomes

\[
P Wr = \frac{4}{7}a + \frac{20}{7},
\]

with a slope of \(\frac{4}{7}\). For the second family, equation (8) with \(b = 2\) becomes

\[
P Wr = \frac{4}{7}a - \frac{8}{7},
\]

with the same slope of \(\frac{4}{7}\). The difference between equations (18) and (19) gives a shift of \(\frac{28}{7} = 4\) between both families.

As a second example, let us compute the shift between two Conway families having the same slope but formed by the closure of tangles presenting different numbers of rows. We will consider the family \(C(3\Lambda^2)\) of torus knots, and the family \(C(3\Lambda^2)(1)(2)\) containing knots 62, 82, 102, \ldots. We thus have to compare equations (3) and (16). Equation (16) with \(b = 1\) and \(c = 2\) becomes

\[
P Wr = \frac{10}{7}(a - 1),
\]

and we have the same slope for both families, namely \(\frac{10}{7}\). In order to compare with equation (3), we have to make \(n\) appear in the latter equation. Remembering that \(n = a + b + c\), \(b = 1\) and \(c = 2\), we have

\[
P Wr = \frac{10}{7}(n - 4) = \frac{10}{7}n - \frac{40}{7}.
\]

The difference between equations (3) and (21) gives a shift of \(\frac{34}{7}\) between both families.

### 4.2. Writhe of achiral knots

Since the 3D writhe is a measure of chirality of oriented closed curves in 3D space, it is a good test to see how achiral knots behave when seen as members of Conway families. Let us consider \(4_1\), which is the closure of tangle (2)(2). It belongs to the family \(C(2\Lambda^2)(2)\) of even twist knots \((4_1, 6_1, 8_1, \ldots)\). The \(P Wr\) of these knots is expressed by equation (8) with \(b = 2\), i.e.,

\[
P Wr = \frac{4}{7}(a - 2).
\]

For knot \(4_1, a = 2\), so \(P Wr = 0\) as it should. Now, let us consider knot \(4_1\) as a member of another Conway family: \(C(2\Lambda^2)(2\Lambda^2)\). This family contains achiral knots only. Their \(P Wr\) is expressed by equation (8) with \(a = b\), which indeed leads to \(P Wr = 0\).

### 4.3. Knots versus two-component links

As already mentioned, the closure of a rational tangle gives rise either to a knot or to a two-component link. Actually, this can be directly related to the parity of the nullification writhe \(w\) (see proposition 12 of [11]). For a family of knots \(w\) is even while for a family of two-component links \(w\) is odd.

*New Journal of Physics* 5 (2003) 87.1–87.15 (http://www.njp.org/)
5. Conclusion

Using the formula

\[ PWr = \frac{10}{7}w_s + \frac{4}{7}w_y \]

that was introduced in [10], we can predict the 3D writhe of any rational knot or link in its ideal configuration, or equivalently the ensemble average of the 3D writhe of random configurations of it. For every Conway family of knots or links, \( PWr \) presents a linear behaviour versus \( n \), the minimal crossing number of the knot or link. The slope of the lines can be computed, as can the shift between two lines having the same slope. We thus have at our disposal a formalism allowing us to predict a number of data usually obtained numerically, and which might help us shed some light on the ‘quantum mystery of knots’ [1].

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