1 Introduction

We usually define Floer homology for Lagrangian submanifolds in symplectic manifolds, but there are many important non-smooth guys such as algebraic varieties and Lagrangian subvarieties. Our first plan to construct Floer theory for such non-smooth objects is to do for something like open strata of them. The open strata are non-compact, and we will start with concave ends.

Floer’s chain complexes for Lagrangian submanifolds in closed symplectic manifolds are generated by intersection points of Lagrangian submanifolds and whose differentials count pseudo-holomorphic strips with Lagrangian boundary conditions. In this paper we will propose Floer’s chain complexes for Lagrangian submanifolds in symplectic manifolds with concave ends.

A symplectic form $\omega$ on a smooth manifold $X$ is a non-degenerate closed 2-form. The non-degeneracy induces the existence of almost complex structures $J$ such that $\omega(\cdot, J\cdot)$ is a metric on $X$. In particular, we consider time-dependent almost complex structures $J_t, t \in [0, 1]$. 
A Lagrangian submanifold $L$ is an $n$-dimensional submanifold in $X^{2n}$ such that $\omega |_{TL} = 0$. Here we assume the following conditions for Lagrangian submanifolds $L_0$ and $L_1$.

**Assumption 1.1 (Nondegeneracy of intersections)** $L_0$ and $L_1$ intersect transversally.

**Assumption 1.2 (Admissibility)** If $u : S^1 \times [0, 1] \to X$ is a map such that $u(S^1, 0) \subset L_0$ and $u(S^1, 1) \subset L_1$, then $\int_{S^1 \times [0,1]} u^* \omega = 0$.

Let $(X, J)$ be an almost complex manifold and $(\Sigma, j)$ a Riemann surface with a complex structure $j$. A pseudo-holomorphic curve is a map $u : \Sigma \to X$ such that

$$\bar{\partial}_Ju := \frac{1}{2}(du + J \circ du \circ j) = 0.$$ 

Our Riemann surface is $\mathbb{R} \times [0, 1]$ with the natural complex structure $i$, and our almost complex structures on $X$ are time-dependent, then we consider the following elliptic partial differential equation

$$\bar{\partial}_Ju(\tau, t) := \frac{\partial u(\tau, t)}{\partial \tau} + J_j(u(\tau, t)) \frac{\partial u(\tau, t)}{\partial t} = 0,$$

where $(\tau, t) \in \mathbb{R} \times [0, 1]$. Define $\mathcal{M}(p, q)$, for $p$ and $q \in L_0 \cap L_1$, to be the set of maps $u : \mathbb{R} \times [0, 1] \to X$ such that

- $u(\mathbb{R}, 0) \subset L_0$ and $u(\mathbb{R}, 1) \subset L_1$
- $\lim_{\tau \to -\infty} u(\tau, [0, 1]) = p$ and $\lim_{\tau \to \infty} u(\tau, [0, 1]) = q$
- $\bar{\partial}_Ju(\tau, t) = 0$.

We call a map satisfying the above conditions a pseudo-holomorphic strip and $\mathcal{M}(p, q)$ the moduli space of pseudo-holomorphic strips. Note that $\mathbb{R}$ acts on $\mathcal{M}(p, q)$ by $(a \ast u)(\tau, t) := u(\tau - a, t)$ with $a \in \mathbb{R}$. We denote the quotient by $\hat{\mathcal{M}}(p, q)$. Then Floer proved the following theorem \cite{5}.

**Theorem 1.3** If $J_t$ is generic, then $\mathcal{M}(p, q)$ is a smooth finite dimensional manifold.
The generic means that $J_t$ is an element of a Baire category set in a certain Banach space of time-dependent almost complex structures.

For the compactification of the moduli spaces in Floer’s context, we need the following condition.

**Assumption 1.4 (π₂-condition)** If $u : D^2 \to X$ is a map such that $u(\partial D^2) \subset L$, then $\int_{D^2} u^*\omega = 0$.

Assume that $L_0$ and $L_1$ satisfy the π₂-condition.

**Theorem 1.5** Let $X$ be a closed symplectic manifold. Then $\hat{\mathcal{M}}(p, q)$ can be compactified (with respect to the topology of uniform convergence with all derivatives on compact set).

In fact, if the dimension of $\hat{\mathcal{M}}(p, q)$ is equal to 0, then $\hat{\mathcal{M}}(p, q)$ is compact, and if the dimension of $\hat{\mathcal{M}}(p, q)$ is equal to 1, then it can be compactified so that the boundary is

$$\bigcup_{\dim \mathcal{M}(p,r) = \dim \hat{\mathcal{M}}(r,q) = 0} \mathcal{M}(p, r) \times \hat{\mathcal{M}}(r, q).$$

(To show (1) we need also the gluing theorem [5].)

Let $C$ be the free $\mathbb{Z}_2$-vector space over the elements of $L_0 \cap L_1$. We define a linear map $\partial : C \to C$ in terms of the canonical bases

$$\partial p := \sum_{q \in L_0 \cap L_1} \sharp \hat{\mathcal{M}}(p, q) q,$$

where the sum ranges over all $q \in L_0 \cap L_1$ such that $\dim \hat{\mathcal{M}}(p, q) = 0$ and where $\sharp \mathcal{M}(p, q)$ is the modulo 2 number of the elements of $\mathcal{M}(p, q)$. Then Floer proved the following theorem [5].

**Theorem 1.6** $\partial \circ \partial = 0$.

The idea of Theorem [1.6] is very important for us, hence we adopt the proof. For $p \in L_0 \cap L_1$

$$\partial \partial p = \partial \left( \sum_{q \in L_0 \cap L_1} \sharp \hat{\mathcal{M}}(p, q) q \right)$$

$$= \sum_{r \in L_0 \cap L_1} \sum_{q \in L_0 \cap L_1} \sharp \hat{\mathcal{M}}(p, q) \sharp \hat{\mathcal{M}}(q, r).$$
\[ \sum_{q \in L_0 \cap L_1} \sharp \hat{\mathcal{M}}(p,q) \sharp \hat{\mathcal{M}}(q,r) \] is nothing but the number of the boundary components of the compactification of a 1-dimensional smooth manifold \( \mathcal{M}(p,r) \), and then even.

We call the chain complex \((C, \partial)\) the Floer’s chain complex for \(L_0\) and \(L_1\) in \(X\).

By using the universal Novikov ring as in [9], we can remove the admissibility. On the other hand, many persons made effort to weaken the \(\pi_2\)-condition [15] and [16], and it grows into an obstruction theory of the boundary operators [9].

Next we will consider (1) from another angle. Let \(\{u_i\}_{i=1,2,...}\) be a sequence of elements in \(\hat{\mathcal{M}}(p,q)\) which converges to an element \((v,w) \in \hat{\mathcal{M}}(p,r) \times \hat{\mathcal{M}}(r,q)\), see Figure 1.

![Figure 1](image-url)

This phenomenon implies that, at the limit of the sequence, a generator \(r\) of the Floer’s chain complex appears at the intersection point \(r\). For simplicity, we assume that \(X\), around \(r\), is locally isomorphic to \(\mathbf{C}^n\), where \(r\) corresponds to the origin, and \(L_0\) and \(L_1\) are locally isomorphic to \(\mathbf{R}^n\) and \((\sqrt{-1}\mathbf{R})^n\), respectively. Consider \(\mathbf{C}^n \setminus \{0\}\) to be \((0, \infty) \times S^{2n-1}\) through the polar coordinate, and moreover transform it into \(\mathbf{R} \times S^{2n-1}\)
by
\[(0, \infty) \times S^{2n-1} \to \mathbb{R} \times S^{2n-1}, \ (\rho, x) \mapsto (\alpha, x) := (\log \rho, x),\]

so that \(L_0\) and \(L_1\) are locally diffeomorphic to \(\mathbb{R} \times \Lambda_0\) and \(\mathbb{R} \times \Lambda_1\), respectively, where \(\Lambda_0\) and \(\Lambda_1\) are (Legendrian) submanifolds in \(S^{2n-1}\).
(with the standard contact form). Then the limit of \( \{u_i\}_{i=1,2,...} \) looks like the following: \( u_i \)'s grow toward \(-\infty\), and at the limit there appear three pseudo-holomorphic strips \( v, w \) and \( z \) as in Figure 2. (The segments between \( \Lambda_0 \) and \( \Lambda_1 \) are Reeb chords.)

In comparison with the intersection point \( r \), this phenomenon implies that, at the limit of the sequence, a generator \( z \) of the Floer's chain complex for \( L_0 \setminus \{r\} \) and \( L_1 \setminus \{r\} \) in \( X \setminus \{r\} \) appears at \(-\infty\) of the (concave) end.

The content of our paper is as follows: Section 2 defines concave/convex ends of non-compact symplectic manifolds and pseudo-holomorphic strips. Section 3 proposes Floer’s chain complexes for Lagrangian submanifolds in symplectic manifolds with concave/convex ends. Section 4 proves some gluing arguments for pseudo-holomorphic strips and Section 5 observes the bubbling off phenomenon for pseudo-holomorphic curves.

Sign convention; through this paper, \( L^p \) is the space of functions/sections \( f \) such that \( f \left| f \right|^p + f \left| Df \right|^p < \infty \) for \( p > 2 \).

2 Pseudo-holomorphic strips in symplectic manifolds with concave ends

Let \( M \) be a smooth oriented manifold of dimension \( 2n + 1 \). A contact form on \( M \) is a 1-form \( \lambda \) such that \( \lambda \wedge (d\lambda)^n \) is a volume form on \( M \). Then the 2-form \( d(e^\alpha \lambda) \) is a symplectic structure in \( \mathbb{R} \times M \), where \( \alpha \in \mathbb{R} \). We call \((\mathbb{R} \times M, d(e^\alpha \lambda))\) the symplectization of \((M, \lambda)\), and there is a natural projection \( \pi_M : \mathbb{R} \times M \to M \).

Let \( X \) be a non-compact symplectic manifold which, out side a compact set, is diffeomorphic to cylinders. If a cylinder is symplectically isomorphic to \( ((-\infty, R_-) \times M_-, d(e^\alpha \lambda_-)) \) for \((M_-, \lambda_-)\) and \( R_- \in \mathbb{R} \), then we call the cylinder a concave end. Similarly, if a cylinder is symplectically isomorphic to \( ((R_+, \infty) \times M_+, d(e^\alpha \lambda_+)) \) for \((M_+, \lambda_+)\) and \( R_+ \in \mathbb{R} \), then we call the cylinder a convex end.

A Legendrian submanifold \( \Lambda \) is an \( n \)-dimensional submanifold in \((M^{2n+1}, \lambda)\)
which satisfies $T\Lambda \subset \ker \lambda$. Then $\mathbb{R} \times \Lambda$ is a Lagrangian submanifold in the symplectization.

Let $L$ be a Lagrangian submanifold in a non-compact symplectic manifold whose ends are concave or convex. If $L$ is non-compact, then we assume that $L$ satisfies the following.

Assumption 2.1 (Cone-condition) $L|(-\infty,R_-)\times M_- = (-\infty,R_-)\times \Lambda_-$ and $L|(R_+,\infty)\times M_+ = (R_+,\infty)\times \Lambda_+$, where $\Lambda_-$ and $\Lambda_+$ are Legendrian submanifolds.

Associated to $\lambda$ there are two important structures. First of all the so-called Reeb vector field $X = X_\lambda$ defined by $i_{X_\lambda} = 1$ and $i_{X_\lambda}d\lambda = 0$, and secondly the contact structure $\xi = \xi_\lambda$ given by $\xi := \ker \lambda \subset T M$. $T M$ splits into $\mathbb{R} X_\lambda \oplus \xi$, and we have a natural projection $\pi_\xi : T M \to \xi$. (We shall use the same notation $\pi_\xi$ to denote $\pi_\xi^* M$. ) Moreover, on the symplectization, $T(\mathbb{R} \times M)$ splits into $E \oplus \xi$, where $E := \mathbb{R} \frac{\partial}{\partial \theta} \oplus \pi_\xi^* \mathbb{R} X_\lambda$, and we will use a natural projection $\pi_E : T(\mathbb{R} \times M) \to E$.

Let $\{\phi_t\}_{0 \leq t \leq T}$ be the isotopy on $M$ such that

$$\frac{d}{dt}(\phi_t^* \lambda) = 0 \quad \text{and} \quad \frac{d}{dt}(\phi_t^* d\lambda) = 0,$$

and then $d\phi_T(\xi_p) = \xi_{\phi_T(p)}$ and $d\phi_T X_\lambda(p) = X_\lambda(\phi_T(p))$ for $p \in M$.

We call a map $\gamma : \mathbb{R}/T\mathbb{Z} \to M$ such that $\dot{\gamma} = X_\lambda(\gamma)$ a closed characteristic of period $T$. Similarly, we call a map $\gamma : [0,T] \to M$ such that $\dot{\gamma} = X_\lambda(\gamma)$ with $\gamma(0) \in \Lambda_0$ and $\gamma(T) \in \Lambda_1$, where $\Lambda_0$ and $\Lambda_1$ are Legendrian submanifolds, a Reeb chord from $\Lambda_0$ to $\Lambda_1$ of length $T$.

The restriction of $d\lambda$ on $\xi$ is non-degenerate, hence $d\lambda|_\xi$ induces complex structures $I$ on $\xi$ such that the bilinear form

$$d\lambda(x)(h,I(x)k), \ h,k \in \xi_x$$

is a positive definite inner product, and then

$$g_M(h,k) := \lambda(h)\lambda(k) + d\lambda(\pi_\xi(h), I\pi_\xi(k)), \ h,k \in TM,$$

gives a metric on $M$. For $a,b \in \mathbb{R}$ and $k \in \xi$,

$$T(\partial \frac{\partial}{\partial \theta} + bX_\lambda + k) := -b \frac{\partial}{\partial \theta} + aX_\lambda + Ik.$$
is an almost complex structure on $\mathbb{R} \times M$, and the equation of pseudo-holomorphic curve for $\pi := (\alpha, u) : \Sigma \to \mathbb{R} \times M$ turns out to be

$$\begin{cases}
\pi_{\xi} \circ du + I\pi_{\xi} \circ du \circ j = 0, \\
(u^*\lambda) \circ j = d\alpha.
\end{cases}$$

For pseudo-holomorphic strips we consider time-dependent complex structures $I_t$ on $\xi$, and if $X$ is a symplectic manifold with concave or convex ends, then we suppose that time-dependent almost complex structures $J_t$ on $X$, outside a compact set, are of the form $I_t$.

Let $L_0$ and $L_1$ be Lagrangian submanifolds in $X$ such that $L_0|_{(-\infty, R_-) \times M_-} = (-\infty, R_-] \times \Lambda_0^-$ and $L_0|_{[R_+, \infty) \times M_+} = [R_+, \infty) \times \Lambda_0^+$, and also $L_1|_{(-\infty, R_-) \times M_-} = (-\infty, R_-] \times \Lambda_1^-$ and $L_1|_{[R_+, \infty) \times M_+} = [R_+, \infty) \times \Lambda_1^+$. We assume that $L_0$ and $L_1$ satisfy Assumption 1.1 (and hence $\Lambda_0^- \cap \Lambda_1^- = \emptyset$ and $\Lambda_0^+ \cap \Lambda_1^+ = \emptyset$). Moreover we assume the following condition for $\Lambda_0^-$ and $\Lambda_1^-$.

**Assumption 2.2 (Nondegeneracy of Reeb chords)** Let $\gamma : [0, T] \to M$ be a Reeb chord from $\Lambda_i^-$ to $\Lambda_j^-$, $i \neq j$. Then $d\varphi_T(T_{\gamma(0)} \Lambda_i^-)$ and $T_{\gamma(T)} \Lambda_j^-$ intersect transversally in $\xi_{\gamma(T)}$, where $\{\varphi_t\}_{0 \leq t \leq T}$ is the isotopy generated by the Reeb vector field.

From the above assumption we can conclude that Reeb chords are isolated.

Now we will consider non-constant pseudo-holomorphic strips, i.e., maps $\pi : \Sigma = \mathbb{R} \times [0, 1] \to X$ such that

$$\bar{\partial}_J \pi(\tau, t) = 0$$

$$\pi(\mathbb{R}, 0) \subset L_0 \text{ and } \pi(\mathbb{R}, 1) \subset L_1,$$

with the following asymptotic conditions. (We are interested in concave ends, hence for simplicity we shall use notation $M$ instead of $M_-$.) In the following we denote $\pi$ on concave ends $(-\infty, R_-] \times M$ by $(\alpha, u)$:

**I** $\lim_{\tau \to -\infty} \pi(\tau, [0, 1]) = p_-$ and $\lim_{\tau \to -\infty} \pi(\tau, [0, 1]) = p_+$ for $p_-$ and $p_+ \in L_0 \cap L_1$.

**II** $\lim_{\tau \to -\infty} u(\tau, [0, 1]) = p_+ \in L_0 \cap L_1$, and there is a Reeb chord
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\( \gamma_- : [0, T_-] \to M \) from \( \Lambda_0^- \) to \( \Lambda_1^- \) such that \( \lim_{\tau \to -\infty} \alpha(t, \tau) = -\infty \) and \( \lim_{\tau \to -\infty} u(t, \tau) = \gamma_-(T_- t) \).

(II') \( \lim_{\tau \to -\infty} \bar{\omega}(\tau, [0, 1]) = p_- \in L_0 \cap L_1 \), and there is a Reeb chord \( \gamma_+ : [0, T_+] \to M \) from \( \Lambda_1^- \) to \( \Lambda_0^- \) such that \( \lim_{\tau \to \infty} \alpha(t, \tau) = -\infty \) and \( \lim_{\tau \to \infty} u(t, \tau) = \gamma_+(T_+ t) \).

(III) There is a Reeb chord \( \gamma_- : [0, T_-] \to M \) from \( \Lambda_1^- \) to \( \Lambda_0^- \) such that \( \lim_{\tau \to -\infty} \alpha(t, \tau) = -\infty \) and \( \lim_{\tau \to -\infty} u(t, \tau) = \gamma_-(T_- t) \), and a Reeb chord \( \gamma_+ : [0, T_+] \to M \) from \( \Lambda_0^- \) to \( \Lambda_1^- \) such that \( \lim_{\tau \to \infty} \alpha(t, \tau) = -\infty \) and \( \lim_{\tau \to \infty} u(t, \tau) = \gamma_+(T_+ t) \).

Moreover we will consider the following extra pseudo-holomorphic strips in the symplectization of \( M \), i.e., \( \bar{\omega} := (\alpha, u) : \Sigma = \mathbb{R} \times [0, 1] \to \mathbb{R} \times M \) such that

\[
\bar{\partial} \bar{\omega}(\tau, t) = 0
\]

\[\bar{\omega}(\mathbb{R}, 0) \subset \mathbb{R} \times \Lambda_0^- \text{ and } \bar{\omega}(\mathbb{R}, 1) \subset \mathbb{R} \times \Lambda_1^-\]

with the asymptotic conditions:

(IV) There is a Reeb chord \( \gamma_- : [0, T_-] \to M \) from \( \Lambda_1^- \) to \( \Lambda_0^- \) such that \( \lim_{\tau \to -\infty} \alpha(t, \tau) = \infty \) and \( \lim_{\tau \to -\infty} u(t, \tau) = \gamma_-(T_- (1 - t)) \), and a Reeb chord \( \gamma_+ : [0, T_+] \to M \) from \( \Lambda_0^- \) to \( \Lambda_1^- \) such that \( \lim_{\tau \to \infty} \alpha(t, \tau) = \infty \) and \( \lim_{\tau \to \infty} u(t, \tau) = \gamma_+(T_+ t) \).

(V) There is a Reeb chord \( \gamma_- : [0, T_-] \to M \) from \( \Lambda_0^- \) to \( \Lambda_1^- \) such that \( \lim_{\tau \to -\infty} \alpha(t, \tau) = -\infty \) and \( \lim_{\tau \to -\infty} u(t, \tau) = \gamma_-(T_- t) \), and a Reeb chord \( \gamma_+ : [0, T_+] \to M \) from \( \Lambda_0^- \) to \( \Lambda_1^- \) such that \( \lim_{\tau \to \infty} \alpha(t, \tau) = \infty \) and \( \lim_{\tau \to \infty} u(t, \tau) = \gamma_+(T_+ t) \).

(V') There is a Reeb chord \( \gamma_- : [0, T_-] \to M \) from \( \Lambda_1^- \) to \( \Lambda_0^- \) such that \( \lim_{\tau \to -\infty} \alpha(t, \tau) = \infty \) and \( \lim_{\tau \to -\infty} u(t, \tau) = \gamma_-(T_- (1 - t)) \), and a Reeb chord \( \gamma_+ : [0, T_+] \to M \) from \( \Lambda_1^- \) to \( \Lambda_0^- \) such that \( \lim_{\tau \to \infty} \alpha(t, \tau) = -\infty \) and \( \lim_{\tau \to \infty} u(t, \tau) = \gamma_+(T_+ (1 - t)) \).

If \( \gamma_- = \gamma_+ = \gamma \) of length \( T \), then there is a trivial pseudo-holomorphic strip \( \bar{\omega} := (T \tau, \gamma(T t)) \) of the type (V), and also \( \bar{\omega} := (-T \tau, \gamma(T(1 - t))) \)
of the type \((V')\).

For the each asymptotic, we put the following exponential decay condition:

\((1)\) If \(\lim_{\tau \to -\infty} \pi(\tau, t) = p_- \in L_0 \cap L_1\), then there are some constants \(\rho_-, C^-_\beta\) and \(d_- > 0\), and a function \(\xi_-(\tau, t) \in C^\infty(\mathbb{R} \times [0, 1], T_{p_-} X)\) such that

- \(\pi(\tau, t) = \exp_{p_-} \xi_-(\tau, t)\) for \(\tau < \rho_-\),
- \(|\partial^\beta \xi_-(\tau, t)| < C^-_\beta e^{d_- \tau}\).

\((1')\) If \(\lim_{\tau \to \infty} \pi(\tau, t) = p_+ \in L_0 \cap L_1\), then there are some constants \(\rho_+, C^+_\beta\) and \(d_+ > 0\), and a function \(\xi_+(\tau, t) \in C^\infty(\mathbb{R} \times [0, 1], T_{p_+} X)\) such that

- \(\pi(\tau, t) = \exp_{p_+} \xi_+(\tau, t)\) for \(\rho_+ < \tau\),
- \(|\partial^\beta \xi_+(\tau, t)| < C^+_\beta e^{-d_+ \tau}\).

Let \(\iota : [0, 1] \times \{z \in \mathbb{R}^{2n}, |z| < \varepsilon\} \to M\) be an immersion such that \(\iota(t, 0) = \gamma(Tt)\), where \(\gamma\) is a Reeb chord of length \(T\):

\((2)\) If \(\lim_{\tau \to -\infty} \alpha(\tau, t) = -\infty\) and \(\lim_{\tau \to -\infty} u(\tau, t) = \gamma_-(T_- t)\), then there are some \(\tau_-\) such that \(u(\tau, t) \in \text{Im} \iota_-\) for \(\tau < \tau_-\), where \(\iota_-\) is the immersion with respect to \(\gamma_-\). If we denote the \(\iota_-\) pull-back of \((\alpha, u)\) by

\((\alpha, \theta_-, z_-) : (-\infty, \tau_-] \times [0, 1] \to \mathbb{R} \times [0, 1] \times \{z \in \mathbb{R}^{2n}, |z| < \varepsilon\},\)

then there are some constants \(c_-\), \(C^-_\beta\) and \(d_- > 0\) such that

\(|\partial^\beta [\alpha(\tau, t) - (T_- \tau + c_-)]| \leq C^-_\beta e^{d_- \tau},\)
\(|\partial^\beta [\theta_-(\tau, t) - t]| \leq C^-_\beta e^{d_- \tau},\)
\(|\partial^\beta z_-(\tau, t)| \leq C^-_\beta e^{d_- \tau}.

\((2')\) If \(\lim_{\tau \to \infty} \alpha(\tau, t) = -\infty\) and \(\lim_{\tau \to \infty} u(\tau, t) = \gamma_+(T_+ (1 - t))\), then there are some \(\tau_+\) such that \(u(\tau, t) \in \text{Im} \iota_+\) for \(\tau_+ < \tau\), where \(\iota_+\) is the immersion with respect to \(\gamma_+\). If we denote the \(\iota_+\) pull-back of \((\alpha, u)\) by

\((\alpha, \theta_+, z_+) : [\tau_+, \infty) \times [0, 1] \to \mathbb{R} \times [0, 1] \times \{z \in \mathbb{R}^{2n}, |z| < \varepsilon\},\)
then there are some constants \( c_+, C^+_\beta \) and \( d_+ > 0 \) such that

\[
|\partial^\beta [\alpha(\tau, t) - (-T_+ \tau + c_+)]| \leq C^+_\beta e^{-d_+ \tau},
\]

\[
|\partial^\beta [\theta_+(\tau, t) - (1 - t)]| \leq C^+_\beta e^{-d_+ \tau},
\]

\[
|\partial^\beta z_+(\tau, t)| \leq C^+_\beta e^{-d_+ \tau}.
\]

(3) If \( \lim_{\tau \to -\infty} \alpha(\tau, t) = \infty \) and \( \lim_{\tau \to -\infty} u(\tau, t) = \gamma_-(T_- (1 - t)) \), then there are some \( \tau_- \) such that \( u(\tau, t) \in \text{Im} \), for \( \tau < \tau_- \), where \( \tau_- \) is the immersion with respect to \( \gamma_- \). If we denote the \( \tau_- \) pull-back of \((\alpha, u)\) by

\[
(\alpha, \theta_-, z_-) : (-\infty, \tau_-] \times [0, 1] \to \mathbb{R} \times [0, 1] \times \{ z \in \mathbb{R}^{2n}, |z| < \varepsilon \},
\]

then there are some constants \( c_-, C^-_\beta \) and \( d_- > 0 \) such that

\[
|\partial^\beta [\alpha(\tau, t) - (-T_- \tau + c_-)]| \leq C^-_\beta e^{d_- \tau},
\]

\[
|\partial^\beta [\theta_-(\tau, t) - (1 - t)]| \leq C^-_\beta e^{d_- \tau},
\]

\[
|\partial^\beta z_-(\tau, t)| \leq C^-_\beta e^{d_- \tau}.
\]

(3') If \( \lim_{\tau \to \infty} \alpha(\tau, t) = \infty \) and \( \lim_{\tau \to \infty} u(\tau, t) = \gamma_+(T_+ t) \), then there are some \( \tau_+ \) such that \( u(\tau, t) \in \text{Im} \), for \( \tau_+ < \tau \), where \( \tau_+ \) is the immersion with respect to \( \gamma_+ \). If we denote the \( \tau_+ \) pull-back of \((\alpha, u)\) by

\[
(\alpha, \theta_+, z_+) : [\tau_+, \infty] \times [0, 1] \to \mathbb{R} \times [0, 1] \times \{ z \in \mathbb{R}^{2n}, |z| < \varepsilon \},
\]

then there are some constants \( c_+, C^+_\beta \) and \( d_+ > 0 \) such that

\[
|\partial^\beta [\alpha(\tau, t) - (T_+ \tau + c_+)]| \leq C^+_\beta e^{-d_+ \tau},
\]

\[
|\partial^\beta [\theta_+(\tau, t) - t]| \leq C^+_\beta e^{-d_+ \tau},
\]

\[
|\partial^\beta z_+(\tau, t)| \leq C^+_\beta e^{-d_+ \tau}.
\]

Define \( \mathcal{M}_I(p, q) \) to be the set of pseudo-holomorphic strips of the form (I) with (1) and (1'), also \( \mathcal{M}_{II}(\gamma_-, p_+) \) of the form (II) with (2) and (1'), \( \mathcal{M}_{II'}(p_-, \gamma_+) \) of the form (II') with (1) and (2'), \( \mathcal{M}_{III}(\gamma_-, \gamma_+) \) of the form (III) with (2) and (2'), \( \mathcal{M}_{IV}(\gamma_-, \gamma_+) \) of the form (IV) with (3) and (3'), \( \mathcal{M}_{IV'}(\gamma_-, \gamma_+) \) of the form (V) with (2) and (3') and finally \( \mathcal{M}_{V'}(\gamma_-, \gamma_+) \) of the form (V') with (3) and (2'). Note that \( \mathbb{R} \) acts on the moduli spaces of the type (I), (II), (II') and (III) by \( \frac{d}{da} (a \ast \overline{a}), a \in \mathbb{R} \).
On the other hand, for the moduli spaces of the type (IV), (V) and (V'), $\mathbb{R}^2$ acts on them by $\frac{d}{dt}(a \ast \varpi)$ for $(a, 0) \in \mathbb{R}^2$ and $\frac{d}{db}(b \ast \varpi)$ for $(0, b) \in \mathbb{R}^2$, where $b \ast \varpi := (\alpha - b, u)$ for $\varpi := (\alpha, u)$. We shall denote these quotients by $\mathcal{M}_s(\ast, \ast)$.

First we recall the index for strips of the type (I) with (1) and (1'). (This part is so standard, the reader may skip to the next content.) Choose a trivialization $(\xi_1, \xi_2, \ldots, \xi_{2n+1}, \xi_{2n+2})$ of $T_p X$ such that $\omega_{p-}(e_{2i-1}, e_{2i}) = \delta_{ij}$, and similarly $(\xi_1^+, \xi_2^+, \ldots, \xi_{2n+1}^+, \xi_{2n+2}^+)$ of $T_{p^+} X$ such that $\omega_{p+}(e_{2i-1}^+, e_{2i}^+) = \delta_{ij}$. We assume that our time-dependent almost complex structures $J_t$ satisfy the following condition.

**Assumption 2.3** A time-dependent almost complex structure $J_t$, $t \in [0, 1]$, satisfies that $J_t$ on $T_{p^+} X$ is standard with respect to $(\xi_1^+, \xi_2^+, \ldots, \xi_{2n+1}^+, \xi_{2n+2}^+)$, i.e., $J_t e_{2i-1}^+ = e_{2i}^+$ and $J_t e_{2i}^+ = -e_{2i-1}^+$, and similarly $J_t$ on $T_{p^+} X$ is standard with respect to $(\xi_1, \xi_2, \ldots, \xi_{2n+1}, \xi_{2n+2})$.

(We can always choose such almost complex structures.) Let $g(t)$, $t \in [0, 1]$, be a time-dependent metric on $X$ such that $L_0$ is totally geodesic with respect to $g(0)$ and similarly $L_1$ is totally geodesic with respect to $g(1)$. We denote by $\exp := \exp(\cdot) : TX \to X$ the exponential map. Let $\varpi \in L^1(\mathbb{R} \times [0, 1]; X, L_0, L_1)$ be a map satisfying the Lagrangian boundary conditions and (I) with the decay conditions (1) and (1'). ($\varpi$ need not be pseudo-holomorphic.) For $\eta \in L^1(\varpi TX, \varpi TL_0, \varpi TL_1)$, a section of $\varpi$ TX with $\eta(\tau, 0) \in T_{\varpi(\tau, 0)}L_0$ and $\eta(\tau, 1) \in T_{\varpi(\tau, 1)}L_1$, we define a map $f_{\varpi} : L^1(\varpi TX, \varpi TL_0, \varpi TL_1) \to L^p(\varpi^* TX \otimes \wedge^{0, 1} T^* (\mathbb{R} \times [0, 1]))$ by

$$f_{\varpi}(\eta) := \Phi_{\varpi}(\eta)^{-1} \nabla_{\eta}(\exp_{\varpi}(\eta)), \quad (2)$$

where $\Phi_{\varpi}(\eta) : T\varpi X \to T_{\exp_{\varpi}} X$ denotes parallel transport of a connection along the geodesic $t \to \exp_{\varpi}(t \eta)$. The differential $Df_{\varpi}(0)$ is

$$Df_{\varpi}(0) \eta = \nabla_{\partial/\partial \tau} \eta + J_t(\varpi(\tau, t)) \nabla_{\partial/\partial \tau} \eta + (\nabla_{\eta} J_t) \partial_t \varpi.$$

As $\tau \to -\infty$, the right hand side is

$$\nabla_{\partial/\partial \tau} \eta + J_t(p_{-}) \nabla_{\partial/\partial \tau} \eta, \quad (3)$$
For simplicity we shall use $E_\pi$ to denote $Df_\pi(0)$ and put $E_\pi = \frac{d}{dr} - Q_t$. Then we conclude that $Q_{-\infty}$ has no eigenvectors of eigenvalue 0 from Assumption 2.4 Similarly, as $\tau \to \infty$, $Q_{\infty}$ also has no eigenvectors of eigenvalue 0. Since $Q_{-\infty}$ and $Q_{\infty}$ are invertible, $E_\pi: L^0(\pi^*TX, \pi^*TL_0, \pi^*TL_1) \to L^0(\pi^*TX \otimes \Lambda^{0,1}T^*(\mathbb{R} \times [0, 1]))$ is Fredholm. Ind$E_\pi$ denotes the index of $E_\pi$.

Similarly we will introduce an index for strips of the type (II) with (2) and (1'). For Reeb chords $\gamma : [0, T] \to M$ we define $\overline{\gamma}(t) := \gamma(Tt)$. Consider the pull-back $\overline{\gamma}^*\xi$ over $I := [0, 1]$ and choose a trivialization $\{e_1, e_2, \ldots, e_{2n}\}$ such that $e_i(t) = d\varphi_{Tt}e_i(0)$, $t \in [0, 1]$, and $\overline{\gamma}^*d\lambda(e_{2i-1}(0), e_{2j}(0)) = \delta_{ij}$. Take a time-dependent connection $\nabla^\lambda := \nabla^\lambda(t)$, $t \in [0, 1]$, on $\xi$ so that the holonomy of $\overline{\gamma}^*\nabla^\lambda(t)$ agrees with $d\varphi_{Tt}$ along the Reeb chords, i.e., $\overline{\gamma}^*\nabla^\lambda(t)e_i(t) = 0$. Let $\{f_1, f_2, \ldots, f_{2n}\}$ be another trivialization such that $\overline{\gamma}^*I(t)$ is the standard complex structure $J_0$ with respect to $\{f_1(t), f_2(t), \ldots, f_{2n}(t)\}$, i.e., $\overline{\gamma}^*I(t)f_{2i-1}(t) = f_{2i}(t)$ and $\overline{\gamma}^*I(t)f_{2i}(t) = -f_{2i-1}(t)$. If $e_i(t) = \sum_{j=1}^{2n} a_{ij}(t)f_j(t)$ and $A := [a_{ij}]$, then

$$\overline{\gamma}^*I(t)\overline{\gamma}^*\nabla^\lambda \sum_{i=1}^{2n} \eta_i(t)f_i(t) = [f_1f_2 \cdots f_{2n}] \left\{ J_0 \frac{\partial}{\partial t} \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_{2n} \end{bmatrix} - J_0 \frac{\partial A}{\partial t} A^{-1} \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_{2n} \end{bmatrix} \right\}.$$  

We conclude that $-J_0 \frac{\partial A}{\partial t} A^{-1}$ is symmetric if the complex structures satisfy the following condition.

**Assumption 2.4** A time-dependent complex structure $I_t$, $t \in [0, 1]$, on $\xi$ satisfies that $\overline{\gamma}^*d\lambda$ is standard with respect to $\{f_1(t), f_2(t), \ldots, f_{2n}(t)\}$, i.e., $\overline{\gamma}^*d\lambda(f_{2i-1}(t), f_{2j}(t)) = \delta_{ij}$.

(We can always choose $\nabla^\lambda$ and $I_t$ as above. In fact, if we choose them so that $e_i(t) = f_i(t)$, then $a_{ij}(t) = \delta_{ij}$.) The double $I'$ of $I$ is the circle $I' = I \cup_{\overline{\mathcal{I}}} \overline{I}$ where $\overline{I}$ is the mirror image of $I$ and corresponding points...
on the boundary are identified. We denote $\kappa$ a natural involution. The doubling of $\gamma^*\xi$ is completely similar. Let $\gamma^*\xi$ denote the vector bundle over $\mathcal{I}$ whose fiber $(\gamma^*\xi)_{\kappa(t)}$ is $(\gamma^*\xi)_{t}$ with the complex structure $-\gamma^*I_t(t)$.

Then the double $\gamma^*\xi'$ over $\mathcal{I}'$ is obtained by gluing $\gamma^*\xi$ and $\gamma^*\xi$ along $\partial\mathcal{I}$. The identification is realized by the automorphism $s$ of $\gamma^*\xi|_{\partial\mathcal{I}}$ which is the anti-complex reflection through $T_{\gamma(0)}\Lambda_{0}$ over $\gamma(0)$ and through $T_{\gamma(T)}\Lambda_{1}$ over $\gamma(T)$. We denote $\gamma^*I'_t$ the double of the complex structure on $\gamma^*\xi'$ which is $\gamma^*I_t$ on $\gamma^*\xi$ and $-\gamma^*I_t$ on $\gamma^*\xi$. An element $\eta' \in L^1_{\mathcal{I}}(\gamma^*\xi')$ is defined by $\eta'|_{\mathcal{I}} = \eta_1 \in L^1_{\mathcal{I}}(\gamma^*\xi)$ and $\bar{\kappa} \circ (\eta'|_{\mathcal{I}}) \circ \kappa = \eta_2 \in L^1_{\mathcal{I}}(\gamma^*\xi)$, where $\bar{\kappa}$ is the natural involution lifting $\kappa$. These satisfy $\eta_2|_{\partial\mathcal{I}} = s \circ (\eta_1|_{\partial\mathcal{I}})$ which is equivalent to

$$\eta_1 + \eta_2 \in L^1_{\mathcal{I}}(\gamma^*\xi; T_{\gamma(0)}\Lambda_0, T_{\gamma(T)}\Lambda_1),$$

$$\gamma^*I_t(\eta_1 - \eta_2) \in L^1_{\mathcal{I}}(\gamma^*\xi; T_{\gamma(0)}\Lambda_0, T_{\gamma(T)}\Lambda_1).$$

Conversely, any couple $(\eta_1, \eta_2) \in L^1_{\mathcal{I}}(\gamma^*\xi) \times L^1_{\mathcal{I}}(\gamma^*\xi)$ satisfying the above conditions defines $\eta' \in L^1_{\mathcal{I}}(\gamma^*\xi')$. We can now define the double operator $\gamma^*\nabla^\lambda(t)'$ by $\gamma^*\nabla^\lambda(t)$ on $\gamma^*\xi$ and $\bar{\kappa} \circ \gamma^*\nabla^\lambda(t) \circ \kappa$ on $\gamma^*\xi$. We conclude that the holonomy around the circle $\mathcal{I}'$ is equal to $-\text{id}$ from Assumption 2.2 and then the equation for sections $\eta'(t) \in L^1_{\mathcal{I}}(\gamma^*\xi')$

$$\gamma^*I'_t \gamma^*\nabla^\lambda(t)'\eta'(t) = 0$$

has no eigen functions of eigen value 0.

Let $\nabla := \nabla(t)$ be a time-dependent connection on $TX$ which, outside a compact set, is of the form: the restriction on $E$ is trivial, i.e., $\nabla(a\frac{\partial}{\partial t} + bX_\lambda) = da\otimes\frac{\partial}{\partial t} + db\otimes X_\lambda$, and the restriction on $\xi$ is the pull-back of $\nabla_\lambda$. Similarly, a time-dependent almost complex structure $J_t$ on $X$, outside a compact set, is of the form $\bar{t}_t$. Let $\bar{\pi}$ be a map which satisfies the Lagrangian boundary conditions and (II) and the decay condition (2). ($\bar{\pi}$ need not be pseudo-holomorphic.) Then, the differential $Df_{\bar{\pi}}(0)$ is

$$Df_{\bar{\pi}}(0)\eta = \nabla_{\partial/\partial \tau}\eta + J(\bar{\pi}(\tau, t))\nabla_{\partial/\partial \tau}\eta + (\nabla_\eta J)\partial_t \bar{\pi}.$$  

As $\tau \to -\infty$, the right hand side is

$$\nabla_{\partial/\partial \tau}\eta + \bar{t}_t(\gamma_-(T-t))\nabla_{\partial/\partial \tau}\eta,$$

(4)
and if we denote \( \eta = \eta_1 \frac{\partial}{\partial t} + \eta_2 X_\lambda + \sum_{i=3}^{2n+2} \eta_i f_i \), then an equation (1) splits into
\[
\begin{aligned}
\frac{\partial}{\partial \tau} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{2n+2} \end{bmatrix} + J_0 \frac{\partial}{\partial t} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{2n+2} \end{bmatrix} - J_0 \frac{\partial A}{\partial t} A^{-1} \begin{bmatrix} \eta_3 \\ \vdots \\ \eta_{2n+2} \end{bmatrix} = 0.
\end{aligned}
\]

If we consider the double operator of \( E_\tau \) on \( \mathbb{R} \times S^1 := (\mathbb{R} \times [0, 1]) \cup_{\partial(\mathbb{R} \times [0, 1])} (\mathbb{R} \times [0, 1]) \) (the double is exactly similar to that in the last paragraph), then \( Q_{-\infty} \) has eigenvectors \( (\eta_1, \eta_2, \eta_3, \ldots, \eta_{2n+2}) = (1, 0, 0, \ldots, 0) \) and \( (0, 1, 0, \ldots, 0) \) of eigenvalue 0. (If we simply consider \( E_\tau \) on \( \mathbb{R} \times [0, 1] \), then \( Q_{-\infty} \) on \( \mathcal{I} \) has an eigenvector \( (1, 0, \ldots, 0) \) of eigenvalue 0 under the Lagrangian boundary conditions.)

Now we introduce weighted Sobolev spaces for the Fredholm theory of \( E_\tau \). For \( 0 < \delta < 2\pi \) and \( \tau_- \) as in the decay condition (2), we define a smooth decreasing function \( \sigma \) by
\[
\sigma(\tau) := \begin{cases} 
-\delta(\tau - \tau_- + 1), & \tau \leq \tau_- - 2, \\
0, & \tau_- \leq \tau,
\end{cases}
\]
and a cut function \( \beta : \mathbb{R} \rightarrow [0, 1] \) by
\[
\beta(\tau) := \begin{cases} 
1, & \tau \leq \tau_- - 1, \\
0, & \tau_- \leq \tau.
\end{cases}
\]

For a section \( \eta \) of \( \pi^* TX|_{(-\infty, \tau_-] \times M} \), we denote \( \eta = \eta_E + \eta_\xi \), where \( \eta_E \) is the \( E \) component and \( \eta_\xi \) is the \( \xi \) component. Then we define weighted Sobolev norms by
\[
\| \eta \|_{L^p_{\delta, \sigma}} := \left( \int_{[\tau_- - \delta, \tau] \times [0, 1]} (1 - \beta(\tau))(|\eta|^p + |\nabla \eta|^p) d\tau dt \right)^{1/p} 
+ \int_{(-\infty, \tau_-] \times [0, 1]} e^{\sigma(\tau)} \beta(\tau) (|\eta_E|^p + |\nabla \eta_E|^p) d\tau dt + \beta(\tau) (|\eta_\xi|^p + |\nabla \eta_\xi|^p) d\tau dt \right)^{1/p},
\]
and similarly
\[
\| \eta \|_{L^p_0, \sigma} := \left( \int_{[\tau_- - \delta, \tau] \times [0, 1]} (1 - \beta(\tau))|\eta|^p d\tau dt + \int_{(-\infty, \tau_-] \times [0, 1]} e^{\sigma(\tau)} \beta(\tau)|\eta_E|^p + \beta(\tau)|\eta_\xi|^p d\tau dt \right)^{1/p}.
\]
We define \( L^p_{0,\sigma} \) to be the set of sections \( f \) such that \( \|f\|_{L^p_{0,\sigma}} < \infty \), and also \( L^p_{0,\sigma} \). Let \( I : L^2 \rightarrow L^2_{0,\sigma} \) be the following isometric transformation.

\[
I(\eta)(\tau, t) := \begin{cases} 
  e^{-\sigma(\tau)/2} \eta_{\xi}, & \tau \leq \tau_-
  \eta, & \tau_\leq < \tau
\end{cases}
\]

Then we obtain

\[
E_{\pi,\sigma}(\eta)(\tau, t) := I^{-1} E_{\pi}(\eta)(\tau, t) = (E_{\pi}\eta)(\tau, t) - \frac{1}{2} \frac{d\sigma}{dt} \eta_{E}.
\]

As \( \tau \rightarrow -\infty \), \( E_{\pi,\sigma} \) is equal to

\[
\frac{\partial}{\partial \tau} - Q_{-\infty} + \frac{\delta}{2} \pi_E.
\]

Since \( -Q_{-\infty} + \frac{\delta}{2} \pi_E \) is invertible (and \( Q_\infty \) is also invertible, see the last paragraph), \( E_{\pi,\sigma} : L^2_{1,\sigma} \rightarrow L^2_{1,\sigma} \) is a Fredholm operator, which implies that \( E_{\pi} : L^2_{0,\sigma} \rightarrow L^2_{0,\sigma} \) is Fredholm. \( \text{Ind} E_{\pi,\sigma} \) denotes the index of \( E_{\pi,\sigma} \).

Note that, if \( \pi \) is pseudo-holomorphic, then \( E_{\pi} \frac{d}{da}(a * \pi)\mid_{a=0} = 0 \). But \( \frac{d}{da}(a * \pi)\mid_{a=0} \) is not an element of \( L^1_{1,\sigma} \). Similarly, we can introduce weighted Sobolev spaces and indexes for strips of the type (IV) and (III).

For strips of the type (IV) we can also define weighted Sobolev spaces and indexes \( \text{Ind} E_{\pi,\sigma} \) for \( E_{\pi,\sigma} : L^p_{1} \rightarrow L^p_{0,\sigma} \). Note that, if \( \pi := (\alpha, \nu) \) is pseudo-holomorphic strip of the type (IV), then \( E_{\pi} \frac{d}{da}(a * \pi)\mid_{a=0} = 0 \). But \( \frac{d}{da}(a * \pi)\mid_{a=0} \) is not an element of \( L^1_{1,\sigma} \). Also \( \frac{d}{db}(b * \pi)\mid_{b=0} = 0 \). But \( \frac{d}{db}(b * \pi)\mid_{b=0} \) is also not an element of \( L^1_{1,\sigma} \).
Assumption 3.1. For pseudo-holomorphic strips \( T \) of the type (I), the linear operators \( E_{0}^{T} : L_{0}^{0}(\pi, TX, TL) \rightarrow L_{0}^{0}(\pi, TX, TL) \) are surjective, and pseudo-holomorphic strips \( T \) of the type (II), (III), and (IV) the linear operators \( E_{0}^{T} : L_{0}^{0}(\pi, TX, TL) \rightarrow L_{0}^{0}(\pi, TX, TL) \) are surjective. (The surjectivity or transversality problem will be observed in a forthcoming paper.) From the above assumption we can conclude that the moduli spaces coming paper [2] are smooth manifolds whose dimension at \( \pi \) is equal to \( \text{Ind}(\pi) \) and \( M_{\star}(\pi, \star) \) consists of the pseudo-holomorphic strips whose dimension is \( d \) by \( M_{\star}(\pi, \star) \), respectively.

For appropriate compactifications of moduli spaces we need Assumption 3.1 and the following.

Let \( Y \) be a symplectic manifold with finitely many concave or convex ends. We shall assume that the contact manifolds are compact without boundaries. Let \( L_{0} \) and \( L_{1} \) be the Lagrangian submanifolds in \( Y \) which satisfy Assumption 3.1. We propose Floer's chain complexes for Lagrangian submanifolds in concave/convex ends.

Floer's chain complexes in symplectic manifolds with concave ends.
**Assumption 3.2** There are no contractible closed characteristics and contractible Reeb chords from a Legendrian submanifold to itself in the contact manifolds of concave ends.

A closed characteristic in $M$ is contractible iff it represents 0 in $\pi_1(M)$, and similarly a Reeb chord from $\Lambda$ to $\Lambda$ is contractible iff it represents 0 of $\pi_1(M, \Lambda)$.

Moreover, for very technical reasons for the exponential decay conditions, we need the following assumption.

**Assumption 3.3** There is an open neighborhood $U \subset [0, 1] \times \mathbb{R}^{2n}$ of $[0, 1] \times \{0\}$ and an open neighborhood $V \subset M$ of a Reeb chord $x(Tt)$ of length $T$ and an immersion $\varphi : U \to V$ mapping $[0, 1] \times \{0\}$ onto $x(Tt)$ such that $\varphi^* \lambda = f \lambda_0$ with $\lambda_0 := dt + \sum_{i=1}^n x_i dy_i$ and a positive smooth function $f : U \to \mathbb{R}$ satisfying $f(t, 0, 0) = T$ and $df(t, 0, 0) = 0$ for all $t \in [0, 1]$.

From these assumptions we can conclude that.

**Theorem 3.4** $\hat{\mathcal{M}}_0^0(\ast, \ast)$ is compact, and $\hat{\mathcal{M}}_1(\ast, \ast)$ can be compactified whose boundaries are:

(a) $\partial\hat{\mathcal{M}}_1^0(p_-, p_+) = \bigcup_{q \in L_0 \cap L_1} \hat{\mathcal{M}}_1^0(p_-, q) \times \hat{\mathcal{M}}_1^0(q, p_+)$

$\cup \bigcup_{(\gamma_-, \gamma_+)} \hat{\mathcal{M}}_1^0(p_-, \gamma_-) \times \hat{\mathcal{M}}_1^0(\gamma_-, \gamma_+) \times \hat{\mathcal{M}}_1^0(\gamma_+, p_+),$

(b) $\partial\hat{\mathcal{M}}_1^0(\gamma_-, p_+) = \bigcup_{q \in L_0 \cap L_1} \hat{\mathcal{M}}_1^0(\gamma_-, q) \times \hat{\mathcal{M}}_1^0(q, p_+)$

$\cup \bigcup_{(\gamma_-, \gamma_+)} \hat{\mathcal{M}}_1^0(\gamma_-, \gamma_-') \times \hat{\mathcal{M}}_1^0(\gamma_-', \gamma_+) \times \hat{\mathcal{M}}_1^0(\gamma_+, p_+)$

$\cup \bigcup_{(\gamma_-, \gamma_+)} \hat{\mathcal{M}}_1^0(\gamma_-, \gamma) \times \hat{\mathcal{M}}_1^0(\gamma, p_+),$

(b') $\partial\hat{\mathcal{M}}_1^0(p_-, \gamma_+) = \bigcup_{q \in L_0 \cap L_1} \hat{\mathcal{M}}_1^0(p_-, q) \times \hat{\mathcal{M}}_1^0(q, \gamma_+)$

$\cup \bigcup_{(\gamma_-, \gamma_+)} \hat{\mathcal{M}}_1^0(p_-, \gamma_-') \times \hat{\mathcal{M}}_1^0(\gamma_-', \gamma_+) \times \hat{\mathcal{M}}_1^0(\gamma_+, \gamma_+)$

$\cup \bigcup_{(\gamma_-, \gamma_+)} \hat{\mathcal{M}}_1^0(p_-, \gamma) \times \hat{\mathcal{M}}_1^0(\gamma, \gamma_+),$
Floer's chain complexes in symplectic manifolds with concave ends

\( \partial\tilde{\mathcal{M}}_1^1(\gamma_-, \gamma_+) = \bigcup_{q \in L_0 \cap L_1} \tilde{\mathcal{M}}_I^0(\gamma_-, q) \times \tilde{\mathcal{M}}_I^0(q, \gamma_+) \\
\quad \cup \bigcup_{(\gamma', \gamma''')} \tilde{\mathcal{M}}_{III}^0(\gamma_-, \gamma') \times \tilde{\mathcal{M}}_{IV}^0(\gamma', \gamma''') \times \tilde{\mathcal{M}}_{II}^0(\gamma''', \gamma_+) \\
\quad \cup \bigcup_{\gamma} \tilde{\mathcal{M}}_{IV}^0(\gamma_-, \gamma) \times \tilde{\mathcal{M}}_{III}^0(\gamma, \gamma_+) \cup \bigcup_{\gamma} \tilde{\mathcal{M}}_{III}^0(\gamma_-, \gamma) \times \tilde{\mathcal{M}}_{IV}^0(\gamma, \gamma_+), \\
\)

where we used the notation \( \partial\tilde{\mathcal{M}}_1^1(\ast, \ast) \) to denote the boundary of the compactification of \( \tilde{\mathcal{M}}_1^1(\ast, \ast) \).

Note that, from the maximum principle, families of pseudo-holomorphic curves can not grow toward \( +\infty \) of convex ends, see Section 5.

Let \( C \) be the free \( \mathbb{Z}_2 \)-vector space over \( L_0 \cap L_1 \) and \( \bigcup_{(\gamma_-, \gamma_+)} \tilde{\mathcal{M}}_{IV}^0(\gamma_-, \gamma_+) \). We define a linear map \( \partial : C \to C \) in terms of the canonical bases: for \( p_- \in L_0 \cap L_1 \)

\[ \partial p_- := \sum_{p_+ \in L_0 \cap L_1} \sharp\tilde{\mathcal{M}}_I^0(p_-, p_+)p_+ + \sum_{(\gamma_-, \gamma_+), u \in \tilde{\mathcal{M}}_{IV}^0(\gamma_-, \gamma_+)} \sharp\tilde{\mathcal{M}}_{IV}^0(p_-, \gamma_-)u, \]

where \( \sharp\tilde{\mathcal{M}}_I^0(p_-, p_+) \) is the modulo 2 number of the elements of \( \tilde{\mathcal{M}}_I^0(p_-, p_+) \) and similarly where \( \sharp\tilde{\mathcal{M}}_{IV}^0(p_-, \gamma_-) \) is the modulo 2 number of the elements of \( \tilde{\mathcal{M}}_{IV}^0(p_-, \gamma_-) \) and the second sum ranges over all pairs of Reeb chords \( (\gamma_-, \gamma_+) \) with respect to the concave ends. and for \( u \in \tilde{\mathcal{M}}_{IV}^0(\gamma_-, \gamma_+) \)

\[ \partial u := \sum_{p_+ \in L_0 \cap L_1} \sharp\tilde{\mathcal{M}}_{II}^0(\gamma_+, p_+)p_+ + \sum_{(\gamma'_-, \gamma'_+), v \in \tilde{\mathcal{M}}_{IV}^0(\gamma'_-, \gamma'_+)} \sharp\tilde{\mathcal{M}}_{IV}^0(\gamma_+, \gamma_-)v, \]

where \( \sharp\tilde{\mathcal{M}}_{II}^0(\gamma_+, p_+) \) is the modulo 2 number of the elements of \( \tilde{\mathcal{M}}_{II}^0(\gamma_+, p_+) \) and similarly where \( \sharp\tilde{\mathcal{M}}_{IV}^0(\gamma_+, \gamma_-) \) is the modulo 2 number of the elements of \( \tilde{\mathcal{M}}_{IV}^0(\gamma_+, \gamma_-) \) and the second sum ranges over all pairs of Reeb chords \( (\gamma'_-, \gamma'_+) \) with respect to the concave ends.

**Assumption 3.5** There are no non-trivial pseudo-holomorphic strips of the type \((V)\) and \((V')\).

(It seems that the existence of non-trivial pseudo-holomorphic strips of the type \((V)\) and \((V')\) is an obstruction to \( \partial^2 = 0 \).) Then, we can prove
Theorem 3.6 $\partial \circ \partial = 0$.

Proof. For $p \in L_0 \cap L_1$

$$\partial \partial p = \partial \left( \sum_{q \in L_0 \cap L_1} \hat{\mathcal{M}}_I^0(p, q) \right) + \partial \left( \sum_{u \in \hat{\mathcal{M}}_I^0(\gamma_-) \cap \hat{\mathcal{M}}_I^0(\gamma_+)} \hat{\mathcal{M}}_I^0(p, \gamma_-) u \right)$$

$$= \sum_{q \in L_0 \cap L_1} \partial \left( \hat{\mathcal{M}}_I^0(p, q) \right) + \sum_{r \in L_0 \cap L_1} \hat{\mathcal{M}}_I^0(q, r) + \sum_{u \in \hat{\mathcal{M}}_I^0(\gamma_-) \cap \hat{\mathcal{M}}_I^0(\gamma_+)} \hat{\mathcal{M}}_I^0(p, \gamma_-) u$$

$$+ \sum_{u \in \hat{\mathcal{M}}_I^0(\gamma_-) \cap \hat{\mathcal{M}}_I^0(\gamma_+)} \left( \sum_{r \in L_0 \cap L_1} \hat{\mathcal{M}}_I^0(q, r) + \sum_{u \in \hat{\mathcal{M}}_I^0(\gamma_-) \cap \hat{\mathcal{M}}_I^0(\gamma_+)} \hat{\mathcal{M}}_I^0(p, \gamma_-) \hat{\mathcal{M}}_I^0(p, \gamma_+) r \right)$$

$$+ \sum_{u \in \hat{\mathcal{M}}_I^0(\gamma_-) \cap \hat{\mathcal{M}}_I^0(\gamma_+)} \left( \sum_{q \in L_0 \cap L_1} \hat{\mathcal{M}}_I^0(p, q) \hat{\mathcal{M}}_I^0(q, \gamma_-) + \sum_{u \in \hat{\mathcal{M}}_I^0(\gamma_-) \cap \hat{\mathcal{M}}_I^0(\gamma_+)} \hat{\mathcal{M}}_I^0(p, \gamma_-) \hat{\mathcal{M}}_I^0(p, \gamma_+) \hat{\mathcal{M}}_I^0(p, \gamma_-) \hat{\mathcal{M}}_I^0(p, \gamma_+) \right) v.$$

The number

$$\sum_{q \in L_0 \cap L_1} \hat{\mathcal{M}}_I^0(p, q) \hat{\mathcal{M}}_I^0(q, r) + \sum_{u \in \hat{\mathcal{M}}_I^0(\gamma_-) \cap \hat{\mathcal{M}}_I^0(\gamma_+)} \hat{\mathcal{M}}_I^0(p, \gamma_-) \hat{\mathcal{M}}_I^0(p, \gamma_+) r$$

is nothing but the one of the boundary components of the compactification of $\hat{\mathcal{M}}_I^0(p, r)$, and similarly the number

$$\sum_{q \in L_0 \cap L_1} \hat{\mathcal{M}}_I^0(p, q) \hat{\mathcal{M}}_I^0(q, \gamma_-) + \sum_{u \in \hat{\mathcal{M}}_I^0(\gamma_-) \cap \hat{\mathcal{M}}_I^0(\gamma_+)} \hat{\mathcal{M}}_I^0(p, \gamma_-) \hat{\mathcal{M}}_I^0(p, \gamma_+) \hat{\mathcal{M}}_I^0(p, \gamma_-) \hat{\mathcal{M}}_I^0(p, \gamma_+)$$
Floer’s chain complexes in symplectic manifolds with concave ends

is the one of the boundary components of the compactification of $\hat{M}_{1I}(p, \gamma_-')$, and hence $\partial \partial p = 0$. For $u \in \hat{M}_{1I}(\gamma_-', \gamma_+)$

$$\partial \partial u = \partial \sum_{p \in L_0 \cap L_1} \hat{\#} \hat{M}_{1I}(\gamma_+, p) + \partial \sum_{v \in \hat{M}_{1I}(\gamma_-', \gamma_+')} \hat{\#} \hat{M}_{III}(\gamma_-', \gamma_+') v$$

$$= \sum_{p \in L_0 \cap L_1} \hat{\#} \hat{M}_{1I}(\gamma_+, p) \left( \sum_{q \in L_0 \cap L_1} \hat{\#} \hat{M}_{1I}(\gamma_+, q) + \sum_{w \in \hat{M}_{1I}(\gamma_-', \gamma_+')} \hat{\#} \hat{M}_{III}(\gamma_-', \gamma_+') w \right)$$

$$+ \sum_{v \in \hat{M}_{1I}(\gamma_-', \gamma_+')} \hat{\#} \hat{M}_{III}(\gamma_-', \gamma_+) \left( \sum_{q \in L_0 \cap L_1} \hat{\#} \hat{M}_{1I}(\gamma_+, q) + \sum_{w \in \hat{M}_{1I}(\gamma_-', \gamma_+')} \hat{\#} \hat{M}_{III}(\gamma_-', \gamma_+') w \right)$$

$$= \sum_{q \in L_0 \cap L_1} \left( \sum_{p \in L_0 \cap L_1} \hat{\#} \hat{M}_{1I}(\gamma_+, p) \hat{\#} \hat{M}_{1I}(\gamma_+, q) + \sum_{v \in \hat{M}_{1I}(\gamma_-', \gamma_+')} \hat{\#} \hat{M}_{III}(\gamma_-', \gamma_+) \hat{\#} \hat{M}_{III}(\gamma_-', \gamma_+) \right) q$$

$$+ \sum_{w \in \hat{M}_{1I}(\gamma_-', \gamma_+')} \left( \sum_{p \in L_0 \cap L_1} \hat{\#} \hat{M}_{1I}(\gamma_+, p) \hat{\#} \hat{M}_{1I}(\gamma_+, q) + \sum_{v \in \hat{M}_{1I}(\gamma_-', \gamma_+')} \hat{\#} \hat{M}_{III}(\gamma_-', \gamma_+) \hat{\#} \hat{M}_{III}(\gamma_-', \gamma_+) \right) w.$$ The number

$$\sum_{p \in L_0 \cap L_1} \hat{\#} \hat{M}_{1I}(\gamma_+, p) \hat{\#} \hat{M}_{1I}(p, q) + \sum_{v \in \hat{M}_{1I}(\gamma_-', \gamma_+')} \hat{\#} \hat{M}_{III}(\gamma_-', \gamma_+) \hat{\#} \hat{M}_{III}(\gamma_-', \gamma_+')$$

is nothing but the one of the boundary components of the compactification of $\hat{M}_{1I}(\gamma_+, q)$, and similarly the number

$$\sum_{p \in L_0 \cap L_1} \hat{\#} \hat{M}_{1I}(\gamma_+, p) \hat{\#} \hat{M}_{1I}(\gamma_-', \gamma_') + \sum_{v \in \hat{M}_{1I}(\gamma_-', \gamma_+')} \hat{\#} \hat{M}_{III}(\gamma_-', \gamma_+) \hat{\#} \hat{M}_{III}(\gamma_-', \gamma_')$$
is the one of the boundary components of the compactification of $\mathcal{M}_{III}(\gamma_+\gamma_-)$, and hence $\partial \partial u = 0$.

We obtain the chain complex $(C, \partial)$ for $L_0$ and $L_1$ in $X$. If complex structures on contact structures vary, then the set $\bigcup(\gamma_-,\gamma_+)$ may change. This implies that the generators of the type $\bigcup(\gamma_-,\gamma_+)$ may appear and disappear, hence, at the time of this writing, the author does not know whether the homology is invariant under the variation of complex structures on contact structures. Concerning Assumption 3.2 and 3.5, he hopes that there are some relations between symplectic field theory and our chain complexes.

Similarly we can construct Floer’s chain complexes for periodic orbits of Hamiltonian flows on symplectic manifolds with concave ends, which will appear in a forthcoming paper.

4 Gluing arguments for pseudo-holomorphic strips

For our purpose we need the following gluing arguments. (We will define the notation soon later.)

**Theorem 4.1** For the compactification of the type (a) as in Theorem 3.4 we need the following (i) and (ii):

(i) Let $\hat{K} \subset \hat{M}^0_1(p_-,q)$ and $\hat{K}' \subset \hat{M}^0_1(q,p_+)$ be compact sets. Then there exist constants $\rho_0$ and a smooth map

$$\hat{\sharp} : \hat{K} \times \hat{K}' \times [\rho_0, \infty) \to \hat{M}^1_1(p_-,p_+).$$

Moreover, for $\bar{u}$ and $\bar{v}$ in the interior of $\hat{K}$ and $\hat{K}'$, there exist $\varepsilon > 0$ and $\rho$ so that $\hat{M}^1_1(p_-,p_+ \cap \bar{U}(\varepsilon,\rho)(\bar{u},\bar{v})$ is contained in the image of $\hat{\sharp}$.

(ii) Let $\hat{K} \subset \hat{M}^0_{II}(p_-,\gamma_-)$, $\hat{K}' \subset \hat{M}^0_{IV}(\gamma_-,\gamma_+)$ and $\hat{K}'' \subset \hat{M}^0_{II}(\gamma_+,p_+)$ be compact subsets. Then there exist constants $\rho_0$ and a smooth map

$$\hat{\sharp} : \hat{K} \times \hat{K}' \times \hat{K}'' \times [\rho_0, \infty) \to \hat{M}^1_1(p_-,p_+).$$
Moreover, for $\overline{u}$, $\overline{w}$ and $\overline{v}$ in the interior of $\hat{K}$, $\hat{K}'$ and $\hat{K}''$, there exist $\varepsilon > 0$ and $\rho$ so that $\mathcal{M}_{I'}^1(p_-, p_+) \cap \hat{U}_{(\varepsilon, \rho)}(\overline{u}, \overline{w}, \overline{v})$ is contained in the image of $\hat{\sharp}$.

For the type (b) we need the following (iii), (iv) and (v):

(iii) Let $\hat{K} \subset \mathcal{M}_{II}^0(\gamma_-, q)$ and $\hat{K}' \subset \mathcal{M}_{I}^0(q, p_+)$ be compact sets. Then there exist constants $\rho_0$ and a smooth map

$$\hat{\sharp} : \hat{K} \times \hat{K}' \mapsto [\rho_0, \infty) \rightarrow \mathcal{M}_{II}^1(\gamma_-, p_+).$$

Moreover, for $\overline{u}$ and $\overline{v}$ in the interior of $\hat{K}$ and $\hat{K}'$, there exist $\varepsilon > 0$ and $\rho$ so that $\mathcal{M}_{I}^1(\gamma_-, p_+ \cap \hat{U}_{(\varepsilon, \rho)}(\overline{u}, \overline{v}))$ is contained in the image of $\hat{\sharp}$.

(iv) Let $\hat{K} \subset \mathcal{M}_{III}^0(\gamma_-, \gamma')$, $\hat{K}' \subset \mathcal{M}_{II}^0(\gamma', \gamma'_+)$ and $\hat{K}'' \subset \mathcal{M}_{II}^0(\gamma'_+, p_+)$ be compact subsets. Then there exist constants $\rho_0$ and a smooth map

$$\hat{\sharp} : \hat{K} \times \hat{K}' \times \hat{K}'' \mapsto [\rho_0, \infty) \rightarrow \mathcal{M}_{II}^1(\gamma_-, p_+).$$

Moreover, for $\overline{u}$, $\overline{w}$ and $\overline{v}$ in the interior of $\hat{K}$, $\hat{K}'$ and $\hat{K}''$, there exist $\varepsilon > 0$ and $\rho$ so that $\mathcal{M}_{II}^1(\gamma_-, p_+ \cap \hat{U}_{(\varepsilon, \rho)}(\overline{u}, \overline{w}, \overline{v}))$ is contained in the image of $\hat{\sharp}$.

(v) Let $\hat{K} \subset \mathcal{M}_{IV}^0(\gamma_-, \gamma)$ and $\hat{K}' \subset \mathcal{M}_{II}^0(\gamma, p_+)$ be compact sets. Then there exist constants $\rho_0$ and a smooth map

$$\hat{\sharp} : \hat{K} \times \hat{K}' \mapsto [\rho_0, \infty) \rightarrow \mathcal{M}_{II}^1(\gamma_-, p_+).$$

Moreover, for $\overline{u}$ and $\overline{v}$ in the interior of $\hat{K}$ and $\hat{K}'$, there exists $\varepsilon > 0$ and $\rho$ so that $\mathcal{M}_{II}^1(\gamma_-, p_+ \cap \hat{U}_{(\varepsilon, \rho)}(\overline{u}, \overline{v}))$ is contained in the image of $\hat{\sharp}$.

For the type (b') we need the following (iii'), (iv') and (v'):

(iii') Let $\hat{K} \subset \mathcal{M}_{II}^0(p_-, q)$ and $\hat{K}' \subset \mathcal{M}_{II}^0(q, \gamma_+)$ be compact sets. Then there exist constants $\rho_0$ and a smooth map

$$\hat{\sharp} : \hat{K} \times \hat{K}' \mapsto [\rho_0, \infty) \rightarrow \mathcal{M}_{II}^1(p_-, \gamma_+).$$

Moreover, for $\overline{u}$ and $\overline{v}$ in the interior of $\hat{K}$ and $\hat{K}'$, there exists $\varepsilon > 0$ and $\rho$ so that $\mathcal{M}_{II}^1(p_-, \gamma_+ \cap \hat{U}_{(\varepsilon, \rho)}(\overline{u}, \overline{v}))$ is contained in the image of $\hat{\sharp}$.
(iv') Let $\hat{K} \subset \hat{M}_{IIV}^0(p_-, \gamma_-)$, $\hat{K}' \subset \hat{M}_{IIV}^0(\gamma_-, \gamma_+)$ and $\hat{K}'' \subset \hat{M}_{IIV}^0(\gamma'_+, \gamma_+)$ be compact subsets. Then there exist constants $\rho_0$ and a smooth map

$$\hat{\sharp} : \hat{K} \times \hat{K}' \times \hat{K}'' \times [\rho_0, \infty) \to \hat{M}_{IV}^1(p_-, \gamma_+).$$

Moreover, for $\bar{u}$, $\bar{w}$ and $\bar{v}$ in the interior of $\hat{K}$, $\hat{K}'$ and $\hat{K}''$, there exist $\varepsilon > 0$ and $\rho$ so that $\hat{M}_{IV}^1(p_-, \gamma_+) \cap \hat{U}(\varepsilon, \rho)(\bar{u}, \bar{w}, \bar{v})$ is contained in the image of $\hat{\sharp}$.

(v') Let $\hat{K} \subset \hat{M}_{IIV}^0(p_-, \gamma)$ and $\hat{K}' \subset \hat{M}_{IIV}^0(\gamma, \gamma_+)$ be compact sets. Then there exist constants $\rho_0$ and a smooth map

$$\hat{\sharp} : \hat{K} \times \hat{K}' \times [\rho_0, \infty) \to \hat{M}_{IIV}^1(p_-, \gamma_+).$$

Moreover, for $\bar{u}$ and $\bar{v}$ in the interior of $\hat{K}$ and $\hat{K}'$, there exists $\varepsilon > 0$ and $\rho$ so that $\hat{M}_{IIV}^1(p_-, \gamma_+) \cap \hat{U}(\varepsilon, \rho)(\bar{u}, \bar{v})$ is contained in the image of $\hat{\sharp}$.

For the type (c) we need the following (vi), (vii) and (vii'):

(vi) Let $\hat{K} \subset \hat{M}_{IIV}^0(\gamma_-, q)$ and $\hat{K}' \subset \hat{M}_{IIV}^0(q, \gamma_+)$ be compact sets. Then there exist constants $\rho_0$ and a smooth map

$$\hat{\sharp} : \hat{K} \times \hat{K}' \times [\rho_0, \infty) \to \hat{M}_{IIV}^1(\gamma_-, \gamma_+).$$

Moreover, for $\bar{u}$ and $\bar{v}$ in the interior of $\hat{K}$ and $\hat{K}'$, there exists $\varepsilon > 0$ and $\rho$ so that $\hat{M}_{IIV}^1(\gamma_-, \gamma_+) \cap \hat{U}(\varepsilon, \rho)(\bar{u}, \bar{v})$ is contained in the image of $\hat{\sharp}$.

(vii) Let $\hat{K} \subset \hat{M}_{IIV}^0(\gamma_-, \gamma_-')$, $\hat{K}' \subset \hat{M}_{IIV}^0(\gamma_-', \gamma_+)$ and $\hat{K}'' \subset \hat{M}_{IIV}^0(\gamma_+, \gamma_+)$ be compact subsets. Then there exist constants $\rho_0$ and a smooth map

$$\hat{\sharp} : \hat{K} \times \hat{K}' \times \hat{K}'' \times [\rho_0, \infty) \to \hat{M}_{IIV}^1(\gamma_-, \gamma_+).$$

Moreover, for $\bar{u}$, $\bar{w}$ and $\bar{v}$ in the interior of $\hat{K}$, $\hat{K}'$ and $\hat{K}''$, there exist $\varepsilon > 0$ and $\rho$ so that $\hat{M}_{IIV}^1(\gamma_-, \gamma_+) \cap \hat{U}(\varepsilon, \rho)(\bar{u}, \bar{w}, \bar{v})$ is contained in the image of $\hat{\sharp}$.

(viii) Let $\hat{K} \subset \hat{M}_{IIV}^0(\gamma_-, \gamma)$ and $\hat{K}' \subset \hat{M}_{IIV}^0(\gamma, \gamma_+)$ be compact sets. Then there exist constants $\rho_0$ and a smooth map

$$\check{\sharp} : \hat{K} \times \hat{K}' \times [\rho_0, \infty) \to \hat{M}_{IIV}^1(\gamma_-, \gamma_+).$$
Moreover, for $\overline{u}$ and $\overline{v}$ in the interior of $\tilde{K}$ and $\tilde{K}'$, there exists $\varepsilon > 0$ and $\rho$ so that $\tilde{M}^{1}_{III}(\gamma_{-}, \gamma_{+}) \cap \tilde{U}_{(\varepsilon, \rho)}(\overline{u}, \overline{v})$ is contained in the image of $\hat{z}$.

(viii') Let $\tilde{K} \subset \tilde{M}^{0}_{III}(\gamma_{-}, \gamma)$ and $\tilde{K}' \subset \tilde{M}^{0}_{r'}(\gamma, \gamma_{+})$ be compact sets. Then there exist constants $\rho_{0}$ and a smooth map

$$\hat{z} : \tilde{K} \times \tilde{K}' \times [\rho_{0}, \infty) \to \tilde{M}^{1}_{III}(\gamma_{-}, \gamma_{+}).$$

Moreover, for $\overline{u}$ and $\overline{v}$ in the interior of $\tilde{K}$ and $\tilde{K}'$, there exists $\varepsilon > 0$ and $\rho$ so that $\tilde{M}^{1}_{III}(\gamma_{-}, \gamma_{+}) \cap \tilde{U}_{(\varepsilon, \rho)}(\overline{u}, \overline{v})$ is contained in the image of $\hat{z}$.

In the following we denote by $\beta : \mathbb{R} \to [0, 1]$ a smooth function

$$\beta(\tau) \equiv \begin{cases} 
0, & \tau \leq 0, \\
1, & 1 \leq \tau,
\end{cases}$$

and by $\|f\|_{[r_{1}, r_{2}]; \ell_{p}^{1}}$ the restriction of the $\ell_{p}^{1}$ norm on $[r_{1}, r_{2}] \times [0, 1]$, i.e.,

$$\|f\|_{[r_{1}, r_{2}]; \ell_{p}^{1}} \equiv \left\{ \int_{[r_{1}, r_{2}] \times [0, 1]} |f|^p + |Df|^p d\tau dt \right\}^{1/p},$$

and also $\|f\|_{[r_{1}, r_{2}]; \ell_{p}} \equiv \left\{ \int_{[r_{1}, r_{2}] \times [0, 1]} |f|^p + |Df|^p d\tau dt \right\}^{1/p}$.

First we recall the proof of the gluing argument (i). (This part is so standard, the reader may skip to the next content.) For compact sets $K \subset M^{1}_{1}(p_{-}, q)$ and $K' \subset M^{1}_{1}(q, p_{+})$, we define $w_{\chi}(\tau, t) \in \ell_{p}^{1}(\mathbb{R} \times [0, 1]; X, L_{0}, L_{1})$ for $\chi := (\overline{u}, \overline{v}, \rho) \in K \times K' \times [\rho_{0}, \infty)$ by

$$w_{\chi}(\tau, t) \equiv \begin{cases} 
\overline{u}(\tau + \rho, t), & \tau \leq -1, \\
\exp_{q}(\beta(-\tau)\eta(\tau + \rho, t) + \beta(\tau)\zeta(\tau - \rho, t)), & -1 \leq \tau \leq 1, \\
\overline{v}(\tau - \rho, t), & 1 \leq \tau,
\end{cases}$$

where $\overline{u} = \exp_{q} \eta$ for $\rho - 1 \leq \tau$ and $\overline{v} = \exp_{q} \zeta$ for $\tau \leq -\rho + 1$. From $\overline{\partial}_{j}\overline{u} = \overline{\partial}_{j}\overline{v} = 0$ we conclude that

$$\|\overline{\partial}_{j}w_{\chi}\|_{\ell_{p}} \leq C \left( e^{-\rho_{1}} \|e^{\rho_{1} \tau} \eta\|_{[\rho_{-1}, \rho]; \ell_{p}^{1}} + e^{-\rho_{1}} \|e^{\rho_{1} \tau} \zeta\|_{[-\rho_{1}, -\rho]; \ell_{p}^{1}} \right),$$

where $C$ is a constant depending only on $K$, $K'$ and $\rho_{0}$. We denote the Taylor expansion of $f_{w_{\chi}}$ by

$$f_{w_{\chi}}(\xi) \equiv f_{w_{\chi}}(0) + Df_{w_{\chi}}(0)\xi + N_{w_{\chi}}(\xi).$$
Lemma 4.2 For $\|\eta\|_{L^p_t} \leq c$ and $\|\zeta\|_{L^p_t} \leq c$, the nonlinear part $N_{w_x}$ satisfies the estimate

$$\|N_{w_x}(\eta) - N_{w_x}(\zeta)\|_{L^p} \leq C(\|\eta\|_{L^p_t} + \|\zeta\|_{L^p_t})\|\eta - \zeta\|_{L^p_t},$$

(6)

where $C$ is a constant depending only on $\|
abla w_x\|_{L^p}$ and $c$.

Proof. Basically it is done by the Taylor expansion. (In the following we shall use $f$ to denote $f_{w_x}$.)

$$N_{w_x}(\eta) - N_{w_x}(\zeta)$$

$$= \int_0^1 (1-t)\{(d^2f)_{\eta}(\eta, \eta) - (d^2f)_{\zeta}(\zeta, \zeta)\}dt$$

$$= \int_0^1 (1-t)\{(d^2f)_{\eta}(\eta, \eta - \zeta) + (d^2f)_{\eta}(\eta, \zeta) - (d^2f)_{\zeta}(\eta, \zeta) + (d^2f)_{\zeta}(\eta - \zeta, \zeta)\}dt$$

$$= \int_0^1 (1-t)\{(d^2f)_{\eta}(\eta, \eta - \zeta) + \int_0^1 (d^3f)(1-s)\eta + st\zeta(t\eta - t\zeta, \eta, \zeta)ds + (d^2f)_{\zeta}(\eta - \zeta, \zeta)\}dt.$$

Then we can conclude

$$\|N_{w_x}(\eta) - N_{w_x}(\zeta)\|_{L^p} \leq C(\|\eta\|_{L^p_t} + \|\eta\|_{L^p_t}\|\zeta\|_{L^p_t} + \|\zeta\|_{L^p_t})\|\eta - \zeta\|_{L^p_t},$$

where the constant $C$ depends only on $\|
abla w_x\|_{L^p}$ and $c$. The reason of the independence of $C$ from $\|w_x\|_{L^\infty}$ is the compactness of the contact manifolds of the cylinders and boundedness of metrics and connections and so on. 

Let $\chi_i := (\overline{\nu}_i, \overline{\nu}_i, \rho_i), i = 1, 2, \ldots$, be a sequence of $K \times K' \times [\rho_0, \infty)$ such that $\rho_i \to \infty$. We assume that $w_{x_i}((-\rho_i, \rho_i), [0, 1])$ is contained in the Gaussian coordinate of $q$. For $\xi_i \in L^p_t(w_{x_i}^{*} TX, w_{x_i}^{*} TL_0, w_{x_i}^{*} TL_1)$, define $\xi_{0i} \in L^p_t([-\rho_i + \sqrt{\rho_i} - 1, \rho_i - \sqrt{\rho_i} + 1] \times [0, 1], T_qX, T_qL_0, T_qL_1)$ such that

$$D \exp_q(\xi_{0i}(\tau, t)) = \xi_i(\tau, t).$$

Note that, if we put $\beta_i(\tau) := \beta(-\tau + \rho_i + \sqrt{\rho_i} - 1)\beta(\tau + \rho_i - \sqrt{\rho_i} + 1)$, then $\beta_i\xi_{0i}$ is an element of $L^p_t(T_qX, T_qL_0, T_qL_1)$. Similarly define differential operators $E_{0i}$ on $[-\rho_i + \sqrt{\rho_i} - 1, \rho_i - \sqrt{\rho_i} + 1] \times [0, 1]$ by

$$D \exp_q(E_{0i}\xi_{0i}) = E_i\xi_i,$$
where $E_i$ denotes the differential $Df_{w_{k,i}}(0)$. Note that the sequence of
\(\{E_{0i}\}_{i=1,2,...}\) converges to the standard Cauchy-Riemann operator $\bar{\partial}_0$ on $[R_1, R_2] \times [0,1]$. (The convergence means that, if we denote $E_{0i} = a_i \frac{\partial}{\partial r} + b_i \frac{\partial}{\partial s} + c_i$, then $a_i \to 1$, $b_i \to J_0$ and $c_i \to 0$ in the $C^\infty$ topology.)

**Proposition 4.3** If $\|\xi_i\|_{L^p} = 1$ and $\|E_i \xi_i\|_{L^p} \to 0$, then there exists a subsequence $\{\xi_{i_0}\}$ such that

$$\|\xi_{i_0}\|_{L^p} \to 0.$$  

**Proof.** From the assumption $\|\xi_i\|_{L^p} = 1$, there is a constant $C$ such that $\|\beta_i \xi_{0i}\|_{L^p} \leq C$. By the Rellich’s theorem, there exists $\xi_{(R_1, R_2)} \in L^p([R_1, R_2] \times [0,1]; T_q X, T_q L_0, T_q L_1)$ and a subsequence $\{\beta_i \xi_{0i}\}$ such that

$$\|\xi_{(R_1, R_2)} - \beta_i \xi_{0i}\|_{L^p} \to 0.$$  

For simplicity, we use $\xi_{0i}$ to denote $\beta_i \xi_{0i}$ and assume $\{\beta_i \xi_{0i}\}$ satisfies

$$\|\beta_i \xi_{0i} - \xi_{(R_1, R_2)}\|_{L^p} \to 0$$

and $\|\xi_{(R_1', R_2')}\|_{L^p} \to 0$ for $[R_1, R_2] \subset [R_1', R_2']$. By the Gårding’s inequality

$$\|\xi_{0i} - \xi_{0j}\|_{L^p} \leq C(\|E_{0i} (\xi_{0i} - \xi_{0j})\|_{L^p} + \|\xi_{0i} - \xi_{0j}\|_{L^p}),$$

where $C$ is a constant depending on $R_1, R_2$ and $\delta$. We already know $\|\xi_{0i} - \xi_{0j}\|_{L^p} \to 0$, and from

$$\|E_{0i} (\xi_{0i} - \xi_{0j})\|_{L^p} \leq \|E_{0i} \xi_{0i}\|_{L^p} + \|E_{0i} - E_{0j}\|_{L^p} + \|E_{0j} \xi_{0j}\|_{L^p} \to 0,$$

we can conclude $\|E_{0i} (\xi_{0i} - \xi_{0j})\|_{L^p} \to 0$. Then $\{\xi_{0i}\}$ has a subsequence which converges to $\xi_{(R_1, R_2)}$ in the norm $\|\cdot\|_{L^p}$. Moreover, from $\|\beta_i \xi_{0i}\|_{L^p} \leq C$, we can conclude $\|\xi_{\infty}\|_{L^p} \leq C$, where $\xi_{\infty}\|_{\{[R_1, R_2] \times [0,1]\} := \xi_{(R_1, R_2)}$. On the other hand, from

$$\|\bar{\partial}_0 \xi_{\infty}\|_{L^p} \leq \|E_{0i} \xi_{0i}\|_{L^p} + \|E_{0i} - \bar{\partial}_0\|_{L^p} \|\xi_{0i}\|_{L^p} + \|\bar{\partial}_0 (\xi_{0i} - \xi_{\infty})\|_{L^p} \to 0,$$
\[ \overline{d}_\xi \infty = 0 \text{ and hence } \xi \infty = 0. \]

Finally, from \( \| \xi_i \|_{[R_1, R_2]; \ell_p^R} \leq C \| \xi_0 \|_{[R_1, R_2]; \ell_p^R} \), there exists a subsequence \( \{ \xi_{i_n} \} \) such that \( \| \xi_{i_n} \|_{[R_1, R_2]; \ell_p^R} \to 0. \)  

For \( \eta \in \text{Ker}\, E_{\pi} \) and \( \zeta \in \text{Ker}\, E_{\pi} \), we define \( \eta^\#_{\pi, \rho} \zeta \in L_p^p(w_*^* TX, w_*^* TL_0, w_*^* TL_1) \) by

\[
(\eta^\#_{\pi, \rho} \zeta)(\tau, t) := \begin{cases} 
\beta(-\tau - 2)\eta(\tau + \rho_i), & \tau \leq 2, \\
0, & -2 \leq \tau \leq 2, \\
\beta(\tau - 2)\zeta(\tau - \rho_i), & \tau \geq 2.
\end{cases}
\]

Let \( W^\perp_{w_*} \) be the \( L^2 \)-orthogonal complement of \( W_{w_*} := \{ \eta^\#_{\pi, \rho} \eta | \eta \in \text{Ker}\, E_{\pi}, \zeta \in \text{Ker}\, E_{\pi} \} \) in \( L_p^p(w_*^* TX, w_*^* TL_0, w_*^* TL_1) \). (Note that the dimension of \( W_{w_*} \) is equal to \( \dim\, \text{Ker}\, E_{\pi} + \dim\, \text{Ker}\, E_{\pi} \).)

**Proposition 4.4** There exist constants \( \rho_0 \) and \( C \) such that for \( \chi \in K \times K' \times [\rho_0, \infty) \) and \( \xi \in W^\perp_{w_*} \)

\[
\| \xi \|_{L_p^p} \leq C \| E_{w_*} \xi \|_{L^p}.
\]

**Proof.** Let \( \chi_i := (\bar{\pi}_i, \bar{\pi}_i, \rho_i), i = 1, 2, \ldots, \) be a sequence such that \( \rho_i \to \infty \) and there exist \( \xi_i \in W_{w_*}^\perp \) satisfying \( \| \xi_i \|_{L_p^p} = 1 \) and \( \| E_{w_*} \xi_i \|_{L^p} \to 0 \). Then we can conclude from Proposition 4.3 there exists a sequence \( \{ \xi_i \} \) such that \( \| \xi_i \|_{[-3,3]; L_p^p} \to 0 \). For simplicity we denote this subsequence by \( \{ \xi_i \} \) and define

\[
\eta_i(\tau, t) := \beta(-\tau + \rho_i - 1)\xi_i(\tau - \rho_i, t) \in L_p^p(\bar{\pi}_i^* TX, \bar{\pi}_i^* TL_0, \bar{\pi}_i^* TL_1),
\]

\[
\zeta_i(\tau, t) := \beta(\tau + \rho_i - 1)\xi_i(\tau + \rho_i, t) \in L_p^p(\bar{\pi}_i^* TX, \bar{\pi}_i^* TL_0, \bar{\pi}_i^* TL_1).
\]

Split \( \eta_i \) into \( k_i + n_i \), where \( k_i \in \text{Ker}\, E_{\pi_i} \) and \( n_i \in (\text{Ker}\, E_{\pi_i})^\perp \), then

\[
\| \eta_i \|_{L_p^p} \leq \| k_i \|_{L_p^p} + \| n_i \|_{L_p^p} \\
\leq \| k_i \|_{L_p^p} + C_{\pi_i} \| E_{\pi_i} n_i \|_{L^p} \\
\leq \| k_i \|_{L_p^p} + C_{\pi_i} \| E_{\pi_i} \eta_i \|_{L^p} \\
= \| k_i \|_{L_p^p} + C_{\pi_i} \| E_{w_*} (\beta(-\tau + \rho_i - 1)\xi_i(\tau - \rho_i, t)) \|_{L^p} \\
\leq \| k_i \|_{L_p^p} + C_{\pi_i} \| \xi_i \|_{[-2,1]; L^p} + C_{\pi_i}^w \| E_{w_*} \xi_i \|_{L^p}.
\]

(Note that, from the compactness of \( K \), \( C_{\pi_i}^w \) and \( C_{\pi_i}^w \) are bounded.) We already know \( \| \xi_i \|_{[-2,1]; L^p} \to 0 \) and \( \| E_{w_*} \xi_i \|_{L^p} \to 0 \). Let \( \{ e_i, \ldots, e_{i_r} \} \) be
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orthogonal bases of \( \text{Ker} E_u \),

\[
|\langle k_i, e^j_i \rangle | = |\langle \eta_i, e^j_i \rangle |
= |\langle \eta_i, (1 - \beta (-\tau - \rho_i + 2)) e^j_i \rangle |_{[\rho_i-2,\rho_i-1] \times [0,1]}
\leq \| \eta_i \|_{[\rho_i-2,\rho_i-1];L^p} \| e^j_i \|_{[\rho_i-2,\rho_i-1];L^{p/(p-1)}}
\leq C \| \xi_i \|_{[-2,-1];L^p} \| e^j_i \|_{[\rho_i-2,\rho_i-1];L^{p/(p-1)}}.
\]

Hence, from \( \| k_i \|_{L^p} \leq \sum_j |\langle k_i, e^j_i \rangle | \| e^j_i \|_{L^p} \), we conclude \( \| k_i \|_{L^p} \to 0 \). (Note that the compactness of \( K \) induces the boundedness of norms of \( e^j_i \).) Then we obtain \( \| \eta_i \|_{L^p} \to 0 \). Similarly we can prove also \( \| \zeta_i \|_{L^p} \to 0 \).

Put together them with

\[
\| \xi_i \|_{L^p} \leq \| \eta_i \|_{L^p} + \| \xi_i \|_{[-3,3];L^p} + \| \zeta_i \|_{L^p},
\]

we obtain \( \| \xi_i \|_{L^p} \to 0 \) which is a contradiction to the assumption \( \| \xi_i \|_{L^p} = 1 \). We finish proving the proposition.

**Proposition 4.5** There exist constants \( \rho_0 \) and \( C \) such that for \( \chi \in K \times K' \times [\rho_0, \infty) \) there exists a map \( G_{w_\chi} : L^p(w_\chi^* TX) \to W^\perp_{w_\chi} \subset L^p_1(w_\chi^* TX, w_\chi^* TL_0, w_\chi^* TL_1) \) such that

\[
E_{w_\chi} G_{w_\chi} = \text{id},
\]

\[
\| G_{w_\chi} \xi \|_{L^p} \leq C \| \xi \|_{L^p}.
\]

**Proof.** From Proposition 4.3 if \( \xi \in \text{Ker} E_{w_\chi} \cap W^\perp_{w_\chi} \), then \( \xi = 0 \) and

\[
\dim \text{Ker} E_{w_\chi} \leq \dim \text{Ker} E_{w_\chi} + \dim \text{Ker} E_{\overline{w_\chi}}.
\]

On the other hand, from Assumption 3.1

\[
\dim \text{Ker} E_{w_\chi} \geq \text{Ind} E_{w_\chi}
= \text{Ind} E_{w_\chi} + \text{Ind} E_{\overline{w_\chi}}
= \dim \text{Ker} E_{w_\chi} + \dim \text{Ker} E_{\overline{w_\chi}}.
\]

Hence we obtain

\[
\text{Ker} E_{w_\chi} \oplus W^\perp_{w_\chi} = L^p_1(w_\chi^* TX, w_\chi^* TL_0, w_\chi^* TL_1),
\]
and $E_{w\chi}$ is surjective. From the estimate of Proposition 4.3 we can obtain $G_{w\chi}$ as in the proposition.

So far we obtain the following: Let $K \subset \mathcal{M}^1_1(p-, q)$ and $K' \subset \mathcal{M}^1_1(q, p+)$ be compact sets. Then there are constants $\rho_0$ such that, for $\chi := (\pi, \eta, \rho) \in K \times K' \times [\rho_0, \infty)$, a map $f_{w\chi} : L^p_1(w^*_\chi TX, w^*_\chi TL_0, w^*_\chi TL_1) \to L^p_\infty(w^*_\chi TX)$ satisfies (5) and (6) and $Df_{w\chi}(0)$ possesses a right inverse $G_{w\chi} : L^p_\infty(w^*_\chi TX) \to L^p_1(w^*_\chi TX, w^*_\chi TL_0, w^*_\chi TL_1)$ satisfying (7). Then, from the Newton’s method in Appendix A, we can conclude that there are constants $\rho_0$ and $C$ and a smooth map

$$\hat{\gamma} : K \times K' \times [\rho_0, \infty) \to \mathcal{M}^1_1(p-, p+), \; \chi \mapsto \exp_{w\chi}(\xi_\chi)$$

with $\|\xi_\chi\|_{L^p_1} \leq C\|\partial_\tau w\chi\|_{L^p}$. Divide them by the $\mathbf{R}$ actions, and we obtain a gluing map $\hat{\gamma} : K \times K' \times [\rho_0, \infty) \to \mathcal{M}^1_1(p-, p+)$. 

The next step is to show the surjectivity of $\hat{\gamma} : K \times K' \times [\rho_0, \infty) \to \mathcal{M}^1_1(p-, p+) \cap U_{(\epsilon, \rho)}(\pi, \eta)$. Let $w$ be a map which satisfies the Lagrangian boundary conditions and (1) and the decay conditions (1) and (1'), and $w(\tau, t) = \exp_q \eta(\tau, t)$ for $-1 \leq \tau \leq 1$, where $q \in L_0 \cap L_1$. Then we define

$$x_\rho(\tau, t) := \begin{cases} w(\tau - \rho, t), & \tau \leq \rho - 1, \\ \exp_q \beta(-\tau + \rho)\eta(\tau - \rho, t), & \rho - 1 \leq \tau, \end{cases}$$

$$y_\rho(\tau, t) := \begin{cases} \exp_q \beta(\tau + \rho)\eta(\tau + \rho, t), & \tau \leq -\rho + 1, \\ w(\tau + \rho, t), & -\rho + 1 \leq \tau. \end{cases}$$

Moreover we define $U_{(\epsilon, \rho_0)}(\pi, \eta)$, for $(\pi, \eta) \in K \times K'$, to be the set of $w$ such that, for $\rho > \rho_0$, $\|\pi - x_\rho\|_{L^p_1} < \epsilon$ and $\|\eta - y_\rho\|_{L^p_1} < \epsilon$. (For simplicity, we shall use a letter $x$ to denote $x_\rho$, and also $y := y_\rho$.) If $w \in \mathcal{M}^1_1(p-, p+) \cap U_{(\epsilon, \rho_0)}(\pi, \eta)$, then for a smooth map $f_x$ there are constants $C$ and $C'$ such that

$$\|f_x(0)\|_{L^p} \leq C\|\eta\|_{[-1, 0]; L^p_1},$$

$$\|N_x(\xi) - N_x(\xi')\|_{L^p} \leq C'(\|\xi\|_{L^p_1} + \|\xi'\|_{L^p_1})\|\xi - \xi'\|_{L^p_1},$$

where $\|\xi\|_{L^p_1} \leq c$ and $\|\xi'\|_{L^p_1} \leq c$ and $C'$ depends on $c$, and also for $f_y$.

(The proofs of the above estimates are similar to those of (5) and (6).)
For \( \xi \in \text{Ker}E_w \), we define \( \hat{\xi} := (\xi_x, \xi_y) \in L^p_1(x^*TX, x^*TL_0, x^*TL_1) \oplus L^p_1(y^*TX, y^*TL_0, y^*TL_1) \) by
\begin{align*}
\xi_x &:= \beta(-\tau + \rho + 1) \xi(\tau - \rho, t), \\
\xi_y &:= \beta(\tau + \rho + 1) \xi(\tau + \rho, t).
\end{align*}

Let \( W_{(w, \rho)}^\perp \) the \( L^2 \)-orthogonal compliment of \( W_{(w, \rho)} := \{ \hat{\xi} | \xi \in \text{Ker}E_w \} \) in
\( L^p_1(x^*TX, x^*TL_0, x^*TL_1) \oplus L^p_1(y^*TX, y^*TL_0, y^*TL_1) \).

**Proposition 4.6** There are constants \( \varepsilon_0 > 0 \) and \( \rho_0 \) and \( C \) such that for \( w \in \mathcal{M}^2_I(p_-, p_+) \cap U(\varepsilon_0, \rho_0)(\tau, \overline{\tau}) \) and \( \xi := (\xi_x, \xi_y) \in W_{(w, \rho)}^\perp \)
\( \| \xi \|_{L^p_1} \leq C \| (E_x \xi_x, E_y \xi_y) \|_{L^p} \).

**Proof.** Let \( \{ \varepsilon_i \}_{i=1, 2, \ldots} \) and \( \{ \rho_i \}_{i=1, 2, \ldots} \) and \( w_i \in \mathcal{M}^2_I(p_-, p_+) \cap U(\varepsilon_i, \rho_i)(\tau, \overline{\tau}) \) be sequences such that \( \varepsilon_i \to 0 \) and \( \rho_i \to \infty \) and there exist \( \xi_i = (\xi_{x_i}, \xi_{y_i}) \in W_{(w_i, \rho_i)}^\perp \) satisfying \( \| \xi_i \|_{L^p_1} = 1 \) and \( \| (E_{x_i} \xi_{x_i}, E_{y_i} \xi_{y_i}) \|_{L^p} \to 0 \). Then, in a similar way to the proof of Proposition 4.4, we can prove that there exists a subsequence \( \{ \xi_{i_l} \} \) such that \( \| \xi_{i_l} \|_{L^p_1} \to 0 \), which contradicts the assumption \( \| \xi_i \|_{L^p_1} = 1 \).

**Proposition 4.7** There exist constants \( \varepsilon_0 > 0 \) and \( \rho_0 \) and \( C \) such that for \( w \in \mathcal{M}^2_I(p_-, p_+) \cap U(\varepsilon_0, \rho_0)(\tau, \overline{\tau}) \) there exists a map \( G_{(w, \rho)} : L^p(x^*TX) \oplus L^p(y^*TX) \to W_{(w, \rho)}^\perp \) such that
\( (E_x \oplus E_y)G_{(w, \rho)} = \text{id}, \)
\( \| G_{(w, \rho)} \xi \|_{L^p_1} \leq C \| \xi \|_{L^p}. \) (10)

**Proof.** From Proposition 4.6, if \( \xi \in \text{Ker}(E_x \oplus E_y) \cap W_{(w, \rho)}^\perp \), then \( \xi = 0 \) and
\( \dim \text{Ker}(E_x \oplus E_y) \leq \dim W_{(w, \rho)}. \)

On the other hand, from Assumption 3.1
\( \dim \text{Ker}(E_x \oplus E_y) \geq \text{Ind}E_x + \text{Ind}E_y \)
\( = \dim W_{(w, \rho)}. \)

Hence we obtain
\( \text{Ker}(E_x \oplus E_y) \oplus W_{(w, \rho)}^\perp = L^p_1(x^*TX, x^*TL_0, x^*TL_1) \oplus L^p_1(y^*TX, y^*TL_0, y^*TL_1), \)
and $E_x \oplus E_y$ is surjective. From the estimate of Proposition 4.5 we can obtain $G_{(u, \rho)}$ as in the proposition.

So far we obtain the following: There are constants $\varepsilon_0 > 0$ and $\rho_0$ such that, for $w \in \mathcal{M}_I^0(p_-, p_+ \cap U_{(\varepsilon_0, \rho_0)}(\bar{u}, \bar{v})$, a map $f(x, y) := (f_x, f_y) : L_1^p(x^*TX, x^*TL_0, x^*TL_1) \oplus L_1^p(y^*TX, y^*TL_0, y^*TL_1) \to L^p(x^*TX) \oplus L^p(y^*TX)$ satisfies (3) and (4), and $Df(x, y)(0)$ possesses a right inverse $G_{(w, \rho)} : L^p(x^*TX) \oplus L^p(y^*TX) \to L^p(x^*TX, x^*TL_0, x^*TL_1) \oplus L^p(y^*TX, y^*TL_0, y^*TL_1)$ satisfying (10). Then, from the Newton’s method in Appendix A, we can conclude that there are constants $\varepsilon > 0$ and $\rho$ and $C$ and a smooth map $\hat{\gamma}' : \mathcal{M}_I^0(p_-, p_+) \cap U_{(\varepsilon, \rho)}(\bar{u}, \bar{v}) \to K \times K'$, $w \mapsto (\exp_x(\xi_w, x), \exp_y(\xi_w, y))$

with $\|\xi_w\|_{L^p} \leq C\|\bar{\partial}, x\|_{L^p}$ and $\|\xi_w\|_{L^p} \leq C\|\bar{\partial}, y\|_{L^p}$. Divide them by the $\mathbb{R}$ actions, and we obtain a map $\hat{\gamma} : \mathcal{M}_I^0(p_-, p_+) \cap \hat{U}_{(\varepsilon, \rho)}(\bar{u}, \bar{v}) \to \hat{K} \times \hat{K}'$.

From the construction of $\hat{\gamma}$ and $\hat{\gamma}'$, if $\rho_0$ is large and $\varepsilon$ is small enough, then $\hat{\gamma} \circ \hat{\gamma}'$ and $\hat{\gamma}' \circ \hat{\gamma}$ are diffeomorphisms. We finish proving the gluing argument (i).

Next we will prove the gluing argument (ii). Take a lift of $\mathcal{M}_I^0(p_-, \gamma_-) \times \mathcal{M}_I^0(\gamma_-, p_+) \times \mathcal{M}_I^0(\gamma_-, \gamma_+) \times \mathcal{M}_I^0(\gamma_+, p_+)$ in $\mathcal{M}_I^0(p_-, \gamma_-) \times \mathcal{M}_I^0(\gamma_-, \gamma_+) \times \mathcal{M}_I^0(\gamma_+, p_+)$ and consider the orbit of the lift by the following $\mathbb{R}^2$-action: for $(a, 0) \in \mathbb{R}^2$

$$(a, 0) \cdot (\bar{u}, \bar{v}) := (\bar{u}, a \ast \bar{v}, \bar{v}),$$

and for $(0, b) \in \mathbb{R}^2$

$$(0, b) \cdot (\bar{u}, \bar{v}) := (\bar{u}, b \ast \bar{u}, \bar{v}).$$

Note that the orbit is diffeomorphic to $\mathcal{M}_I^0(p_-, \gamma_-) \times \mathcal{M}_I^0(\gamma_-, \gamma_+) \times \mathcal{M}_I^0(\gamma_+, p_+) \times \mathbb{R}^2$. We choose a compact set $S$ in the orbit, and we will construct a gluing map $\hat{\gamma} : S \times [\rho_0, \infty) \to \mathcal{M}_I^0(p_-, p_+)$. We consider a concave end which is isomorphic to $(-\infty, R] \times M$. Fix $(\bar{u}, \bar{v}, \bar{v}) \in S$. For simplicity, for $\bar{u}$ we assume $c_+ = 0$ and $\tau_+ = \tau_0$, where $c_+$ and $\tau_0$ are constants as in the decay condition (2'), and we denote $(\alpha, \theta_+, z_+)$ as in (2') by $(\alpha_u, \theta_+^u, z_+^u)$. Also for $\bar{v}$ we assume $c_- = 0$ and $\tau_- = \tau_0$, as in (2), and we denote $(\alpha, \theta_-, z_-)$ as in (2) by $(\alpha_v, \theta_-^v, z_-^v)$. Moreover, for $\bar{w}$
we assume $c_+ = c_- = 0$ and $\tau_+ = -\tau_- = \tau_0$, and we denote $(\alpha, \theta_-, z_-)$ by $(\alpha_w, \theta_w^-, z_w^-)$ and $(\alpha, \theta_+, z_+)$ by $(\alpha_w, \theta_w^+, z_w^+)$. We define $w_\chi(\tau, t) \in L^p_1(\mathbb{R} \times [0, 1]; X, L_0, L_1)$ for $\chi := (\underline{\nu}, \underline{\nu}, \underline{\nu}, \rho) \in S \times [\rho_0, \infty)$ by $w_\chi(\tau, t) := w_\chi(\alpha, \theta, z)$.
\begin{itemize}
  \item \(\overline{\pi}(\tau + 2T_+\rho, t), \text{ for } \tau \leq -T_+\rho - 1/T_-,\)
  \item \((\beta(T_+\tau - T_-T_+\rho)\alpha_u(\tau + 2T_+\rho, t) + \{1 - \beta(T_+\tau - T_-T_+\rho)\} - T_+\rho - 2T_-T_+\rho, \beta(T_+\tau - T_-T_+\rho)\theta_v^w(\tau + 2T_+\rho, 1 - t) + \{1 - \beta(T_+\tau - T_-T_+\rho)\} (1 - t), \beta(T_+\tau - T_-T_+\rho)z^u_+(\tau + 2T_+\rho, t)), \text{ for } -T_+\rho - 1/T_+ \leq \tau \leq -T_+\rho,\)
  \item \((\beta(T_+\tau + T_-T_+\rho)\alpha_w(\tau, t) - 2T_-T_+\rho) + \{1 - \beta(T_+\tau + T_-T_+\rho)\} (T_+\tau - 2T_-T_+\rho), \beta(T_+\tau + T_-T_+\rho)\theta_v^w(\tau, t) + \{1 - \beta(T_+\tau + T_-T_+\rho)\} t, \omega(T_+\tau + T_-T_+\rho)z^u_+(\tau, t)), \text{ for } -T_+\rho + 1/T_- \leq \tau \leq -T_+\rho,\)
  \item \((\alpha_w(\tau, t) - 2T_-T_+\rho, w(\tau, t)), \text{ for } -T_+\rho + 1/T_- \leq \tau \leq -T_+\rho - 1/T_+,\)
  \item \((\alpha_w(\tau, t) - 2T_-T_+\rho, w(\tau, t)), \text{ for } -T_+\rho + 1/T_- \leq \tau \leq -T_+\rho - 1/T_+,\)
  \item \(\overline{\pi}(\tau - 2T_-\rho, t) \text{ for } T_-\rho + 1/T_+ \leq \tau,\)
\end{itemize}

see Figure 3. Then \(\overline{\partial}_Lw_x = 0 \text{ for } \tau \leq -T_+\rho - 1/T_-, -T_+\rho + 1/T_- \leq \tau \leq -T_-\rho - 1/T_+ \text{ and } -T_-\rho + 1/T_+ \leq \tau, \text{ and there are constants } C_1 > 0 \text{ and } d > 0 \text{ such that}\)

\[
\|\overline{\partial}_Lw_x\|_{L^p} \leq C_1 e^{-dp}, \quad (11)
\]

where \(C_1\) depends only on \(S\) and \(\rho_0\), and \(N_{w_x}\) satisfies

\[
\|N_{w_x}(\xi) - N_{w_x}(\xi')\|_{L^p} \leq C_2(\|\xi\|_{L^p} + \|\xi'\|_{L^p})\|\xi - \xi'\|_{L^p}, \quad (12)
\]

where \(\|\xi\|_{L^p} \leq c\) and \(\|\xi'\|_{L^p} \leq c\) and \(C_2\) is a constant depending only on \(\|\nabla w_x\|_{L^p}\) and \(c\). (The proof is similar to that of \(\Theta\).)

From a spectral flow we can conclude

\[
\text{Ind}E_{w_x} = \text{Ind}E_{\pi,\sigma} + \dim \text{Ker}Q_\infty + \text{Ind}E_{\pi,\sigma} + \dim \text{Ker}Q'_\infty + \text{Ind}E_{\pi,\sigma},
\]

where \(E_{\pi} = \frac{\partial}{\partial \tau} - Q_\tau\) and \(E_{\pi} = \frac{\partial}{\partial \tau} - Q'_\tau\). In Section 2 we know \(\dim \text{Ker}Q_\infty = \dim \text{Ker}Q'_\infty = 1\), then \(\text{Ind}E_{w_x} = \text{Ind}E_{\pi,\sigma} + \text{Ind}E_{\pi,\sigma} + \text{Ind}E_{\pi,\sigma} + 2\). For simplicity, in the following, we assume that \(\text{Ind}E_{\pi,\sigma} = \text{Ind}E_{\pi,\sigma} = \text{Ind}E_{\pi,\sigma} = 0\) (and hence \(\text{Ind}E_{w_x} = 2\)). We will introduce the following two sections \(e^0_\rho\) and \(e^1_\rho\) of \(w_x^X TX\):
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Note that \( \frac{d}{d\alpha}(a \ast \mathbf{w})|_{a=0} \) is a section of \( \pi^* TX \) which, as \( \tau \to \infty \), is close to \( T_-(\frac{\partial}{\partial \alpha}) \), and \( \frac{d}{d\alpha}(a \ast \mathbf{w})|_{a=0} - T_+(\frac{\partial}{\partial \alpha}) \) is a section of \( \pi^* TX \) which, as \( \tau \to -\infty \), is close to \( (T_- + T_+)(\frac{\partial}{\partial \alpha}) \) and, as \( \tau \to \infty \), is close to 0. In a similar way to construct \( W_\alpha \), we glue the zero-section of \( \pi^* TX \) to construct \( e_\rho^0 \). Then \( E_{w, \gamma} e_\rho^0 = 0 \) for \( \tau \leq -2T_+\rho - \tau_0 - 1 - 2T_+\rho + \tau_0 \leq \tau \leq -T_\rho - 1/T_- \), \( -T_\rho - 1/T_+ \leq \tau \leq T_\rho - 1/T_+ \) and \( T_\rho - 1/T_+ \leq \tau \).

From the construction, we can conclude that \( e_\rho^0 \) is close to \( \frac{\partial}{\partial \alpha} \) on \( H_- := [\tau_0 - 2T_\rho, -\tau_0] \) and 0 on \( H_+ := [\tau_0, -\tau_0 + 2T_-\rho] \), and \( e_\rho^1 \) is close to 0 on \( H_- \) and \( \frac{\partial}{\partial \alpha} \) on \( H_+ \). Hence

\[
\lim_{\rho \to \infty} \frac{\|e_\rho^0\|_{H^1 L^2}}{\|e_\rho^0\|_{L^2}} = 1, \quad \lim_{\rho \to \infty} \frac{\|e_\rho^1\|_{H^1 L^2}}{\|e_\rho^1\|_{L^2}} = 1, \\
\lim_{\rho \to \infty} \int_{\mathbb{R} \times [0,1]} \left( \frac{e_\rho^0}{\|e_\rho^0\|_{L^2}}, \frac{e_\rho^1}{\|e_\rho^1\|_{L^2}} \right) d\tau dt = 0.
\]

**Proposition 4.8** Let \( \chi_i := (\mathbf{u}_i, \mathbf{w}_i, \mathbf{v}_i, \rho_i) \), \( i = 1, 2, \ldots \), be a sequence of \( S \times [\rho_0, \infty) \) such that \( \rho_i \to \infty \). If \( \|\xi_i\|_{L^p} = 1 \) and \( \|E_{w, \gamma_i} \xi_i\|_{L^p} \to 0 \), then there exists a subsequence \( \{\xi_{i_k}\} \) such that

\[
\|\xi_{i_k}\|_{[-T_+\rho + R_1, -T_+\rho + R_2]:L^p_1} \to 0,
\]

where \( R_1 \leq 0 \leq R_2 \).

**Proof.** The proof is similar to that of Proposition 4.3. Define \( \xi'_i(\tau, t) := \xi_i(\tau - T_+\rho, t) \). By Rellich’s theorem, there exists \( \xi'_{(R_1, R_2)} \in L^p([-R_1, R_2] \times [0,1]; \mathcal{E} \oplus \gamma^* \xi, \mathcal{R}(\frac{\partial}{\partial \alpha}) \oplus T_{\gamma_0(0)} \Lambda_{\gamma_0}, \mathcal{R}(\frac{\partial}{\partial \alpha}) \oplus T_{\gamma_0(1)} \Lambda_{\gamma_0}) \) and a subsequence
\{\xi'_i\} which converges \(\xi'_{(R_1,R_2)}\) in the norm \(\cdot\|_{[R_1,R_2];L^p}\). For simplicity, we assume \(\cdot\|_{[R_1,R_2]} - \xi'_i\|_{L^p} \rightarrow 0\) and \(\xi'_{(R_1,R_2)}[[R_1,R_2] \times [0,1] = \xi'_{(R_1,R_2)}\) for \([R_1,R_2] \subset [R_1',R_2']\). By the Gårding's inequality, we can conclude that \(\{\xi'_i\}\) has a subsequence which converges to \(\xi'_{(R_1,R_2)}\) in the norm \(\cdot\|_{[R_1,R_2];L^p}\). Moreover, from \(\cdot\|_{L^p}_1 \leq C\), we can conclude \(\cdot\|_{L^p}_{1,2} \leq C\), where \(\xi'\|_{[R_1,R_2] \times [0,1]} : = \xi'_{(R_1,R_2)}\). On the other hand, \(\xi'\|_{\infty}\) splits into E component \(\xi'\|_{E}\) and E component \(\xi'\|_{E}\), and

\[
\frac{\partial}{\partial \tau} (\xi'_{(\infty)})_E + \sqrt{1} \frac{\partial}{\partial \tau} (\xi'_{(\infty)})_E = 0,
\]

\[
\frac{\partial}{\partial \tau} (\xi'_{(\infty)})_E + J_{w_0} \frac{\partial}{\partial \tau} (\xi'_{(\infty)})_E - J_0 \frac{\partial}{\partial \tau} A^{-1}(\xi'_{(\infty)})_E = 0,
\]

where \(-J_0 \frac{\partial A}{\partial \tau} A^{-1}\) is the symmetric. Hence \(\xi'_{(\infty)} = 0\) (see \(\mathbb{I}^\infty\)), and then \(\cdot\|_{[R_1,R_2];L^p}_1 \rightarrow 0\).

Let \(W_{w_\chi}^p\) be the \(L^2\)-orthogonal compliment of \(W_{w_\chi} : = \langle e_\rho^0, e_\rho^1 \rangle\) in \(L^p(w_\chi^* TX, w_\chi^* TL_0, w_\chi^* TL_1)\).

**Proposition 4.9** There exist constants \(\rho_0\) and \(C_3\) such that for \(\chi \in S \times [\rho_0, \infty)\) and \(\xi \in W_{w_\chi}^p\)

\[
\|\xi\|_{L^p_1} \leq C_3 \rho^{\frac{1}{2} - \frac{1}{q}} E_{w_\chi} \xi \|_{L^p}.
\]

**Proof.** Let \(\chi_i : = (\tilde{w}, \tilde{w}_1, \tilde{w}_i, \rho_i), i = 1, 2, \ldots\), be a sequence such that \(\rho_i \rightarrow \infty\) and there exist \(\xi_i \in W_{w_\chi}^p\) satisfying \(\|\xi_i\|_{L^p} = 1\) and \(\rho_i^{\frac{1}{2} - \frac{1}{q}} E_{w_{\chi_i}} \xi_i \|_{L^p} \rightarrow 0\). We define the following smooth functions:

- \(\beta_i(\tau) : = \beta(-T_{-\tau} + T_{-\rho_i})\),
- \(\beta'_i(\tau) : = \beta(T_{-\tau} + T_{-\rho_i})\),
- \(\beta''_i(\tau) : = \beta(-T_{+\tau} - T_{-\rho_i})\),
- \(\beta'''_i(\tau) : = \beta(T_{+\tau} + T_{-\rho_i})\).

Then

\[
\|\xi_i\|_{L^p_1} \leq \|\beta_i \xi_i\|_{L^p_1} + \|\xi_i\|_{-T_{\rho_i-1/T_{-\tau} - T_{+\rho_i-1/T_{+\tau}}};L^p_1} + \|\beta'_i \xi_i\|_{L^p_1} + \|\beta''_i \xi_i\|_{L^p_1} + \|\beta'''_i \xi_i\|_{L^p_1}.
\]
From Proposition 4.9, we can conclude \( \|\xi_i\|_{[-T+\rho-1/T,-T,\rho+1/T]}L^p \rightarrow 0 \) and \( \|\xi_i\|_{[-T+\rho-1/T,-T,\rho+1/T]}L^p \rightarrow 0 \). Since we assume Ker\( E_{\pi \sigma} \neq 0 \), there exist constants \( C \) such that \( \|\beta_i \xi_i\|_{L^p} \leq C \|\pi E_{\pi \sigma}(\beta_i \xi_i)\|_{L^p} \), and then

\[
\|\beta_i \xi_i\|_{L^p} \leq C \|E_{\pi \sigma}(\beta_i \xi_i)\|_{L^p} \\
\leq C \|E_{\pi}(\beta_i \xi_i) - \frac{d\sigma}{d\tau} \pi E(\beta_i \xi_i)\|_{L^p} \\
= C \|E_{w_{\chi}}(\beta_i \xi_i) - \frac{d\sigma}{d\tau} \pi E(\beta_i \xi_i)\|_{L^p} \\
\leq C \|\beta_i E_{w_{\chi}} \xi_i\|_{L^p} + C \|\pi E \xi_i\|_{[\tau_0-2T+\rho-\tau_0]:L^p} + C'' \|\pi E \xi_i\|_{[\tau_0-2T+\rho-\tau_0]:L^p}.
\]

From the assumption \( \rho_{\frac{2}{2}-\frac{1}{p}} \|E_{w_{\chi}} \xi_i\|_{L^p} \rightarrow 0 \) we know \( \|\beta_i E_{w_{\chi}} \xi_i\|_{L^p} \rightarrow 0 \), and from Proposition 1.8 we conclude \( \|\beta_i \xi_i\|_{L^p} \rightarrow 0 \). Regard \( \pi E_{\xi_i} \) on \( [\tau_0-2T+\rho, -\tau_0] \times [0, 1] \) as a function on \( T_i := \mathbb{R}/2(T+\rho-\tau_0) \times [0, 1] \), and split \( \pi E_{\xi_i} \) into \( n_i \in \text{Ker} E_{w_{\chi}} \) and \( k_i \in (\text{Ker} E_{w_{\chi}})^\perp \). Since \( \xi_i \in W_{w_{\chi}}^\perp \), we obtain \( \lim_{i \rightarrow 0} \langle \pi E_{\xi_i} \varepsilon_{\rho_i}^0 / \|\varepsilon_{\rho_i}^0\|_{L^2} \rangle = 0 \), and then \( \lim_{i \rightarrow \infty} \|n_i\|_{L^2(T_i)} = 0 \). Moreover, since \( E_{w_{\chi}} E = \frac{\partial}{\partial T} + \sqrt{-T} \frac{\partial}{\partial T} \), then \( \|k_i\|_{L^2(T_i)} \leq \frac{T+\rho_0-\tau_0}{\pi} \|E_{w_{\chi}} k_i\|_{L^2(T_i)} \).

By the H"older's inequality \( \|y\|_{L^2(T_i)} \leq (2T+\rho_i - \tau_0)^{\frac{2}{2} - \frac{1}{p}} \|y\|_{L^p(T_i)} \) (we need \( p > 2 \)) and the assumption \( \rho_{\frac{2}{2}-\frac{1}{p}} \|E_{w_{\chi}} \xi_i\|_{L^p} \rightarrow 0 \), we can conclude \( \lim_{i \rightarrow \infty} \|k_i\|_{L^2(T_i)} = 0 \), and then \( \lim_{i \rightarrow \infty} \|\pi E_{\xi_i}\|_{L^2(T_i)} = 0 \). From the assumption \( \|\xi_i\|_{L^p} = 1 \) and the Sobolev's embedding theorem (we need \( p > 2 \)), we can conclude \( |\xi_i| < c \). Then \( \|\pi E_{\xi_i}\|_{L^p(T_i)} \leq c^{1-2/p} \|\pi E_{\xi_i}\|_{L^2(T_i)}^{2/p} \) (also we need \( p > 2 \)), and \( \lim_{i \rightarrow \infty} \|\pi E_{\xi_i}\|_{L^p(T_i)} = 0 \). Finally, we finish proving \( \lim_{i \rightarrow \infty} \|\beta_i ^{\prime \prime \prime} \xi_i\|_{L^p} = 0 \), which is a contradiction to the assumption \( \|\xi_i\|_{L^p} = 1 \).

**Proposition 4.10** There exist constants \( \rho_0 \) and \( C_3 \) such that for \( \chi \in S \times [\rho_0, \infty) \) there exists a map \( G_{w_{\chi}} : L^p(w_{\chi} TX) \rightarrow W_{w_{\chi}}^\perp \) such that

\[
E_{w_{\chi}} G_{w_{\chi}} = \text{id}, \\
\|G_{w_{\chi}} \xi\|_{L^p} \leq C_3 \rho_{\frac{2}{2}-\frac{1}{p}} \|\xi\|_{L^p}.
\]

**Proof.** From Proposition 4.9 if \( \xi \in \text{Ker} E_{w_{\chi}} \cap W_{w_{\chi}}^\perp \), then \( \xi = 0 \) and

\[
\dim \text{Ker} E_{w_{\chi}} \leq 2.
\]
On the other hand, from Assumption 3.1,
\[
\dim \ker E_w \geq \text{Ind} E_w = 2.
\]
Hence we obtain
\[
\ker E_w \oplus W^\perp_w \chi = L^p(w^*_{\chi} TX, w^*_{\chi} TL_0, w^*_{\chi} TL_1),
\]
and \(E_w\) is surjective. From the estimate of Proposition 4.9 we can obtain \(G_w\) as in the proposition.

So far we obtain the following: Let \(S\) be a compact set in the orbit. Then there are constants \(\rho_0\) such that, for \(\chi := (\pi, \overline{w}, \overline{v}, \rho) \in S \times [\rho_0, \infty)\), a map \(f_w : L^p(w^*_{\chi} TX, w^*_{\chi} TL_0, w^*_{\chi} TL_1) \to L^p(w^*_{\chi} TX)\) satisfies (11) and (12) and \(Df_w(0)\) possesses a right inverse \(G_w : L^p(w^*_{\chi} TX) \to L^p(w^*_{\chi} TX, w^*_{\chi} TL_0, w^*_{\chi} TL_1)\) satisfying (13). From (11) and (13)
\[
\|G_w f_w(0)\|_{L^p} \leq C_1 C_2 \rho^2 \frac{1}{p} e^{-d \rho},
\]
and from (12) and (13)
\[
\|G_w N_w(\xi) - G_w N_w(\xi')\|_{L^p} \leq C_2 C_3 \rho^2 \frac{1}{p} (\|\xi\|_{L^p} + \|\xi'\|_{L^p}) \|\xi - \xi'\|_{L^p}.
\]
Denote \(C := C_2 C_3 \rho^2 \frac{1}{p}\) and choose \(\rho_0\) large enough, then
\[
\|G_w f_w(0)\|_{L^p} \leq \frac{1}{8C}.
\]
Then, from the Newton’s method in Appendix A, we can conclude that there are constants \(\rho_0\) and \(C'\) and a smooth map
\[
\#: S \times [\rho_0, \infty) \to \mathcal{M}_1^2(p_-, p_+)\), \(\chi \mapsto \exp w(\xi_\chi)\)
\]
with \(\|\xi_\chi\|_{L^p} \leq C' \|
\]
\[
\partial \eta w(\xi)\|_{L^p}. \quad \text{Divide them by the } \mathbb{R} \text{ actions, then we obtain a gluing map } \tilde{\#: S \times [\rho_0, \infty) \to \mathcal{M}_1^2(p_-, p_+).}
\]

The next step is to show the surjectivity of \(\#: S \times [\rho_0, \infty) \to \mathcal{M}_1^2(p_-, p_+)\) and \(U(\varepsilon, \rho)(\pi, \overline{w}, \overline{v}). \quad \text{Let } \overline{h} \text{ be a map which satisfies the Lagrangian boundary conditions and (I) and the decay conditions (I') and (I'), and } \overline{h}(\tau, t) = (\alpha_h, \theta_h, z_h) \in \text{Im}_- \text{ for } -\tau_0 \leq \tau \leq \tau_0 - 2T_\rho \text{ and } \overline{h}(\tau, t) = (\alpha_h, \theta_h, z_h) \in \text{Im}_+ \text{ for } \tau_0 \leq \tau \leq -\tau_0 + 2T_\rho. \quad \text{Then we define } x_{\rho}(\tau + 2T_\rho, t) by
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- $\overline{h}(\tau, t)$ for $\tau \leq -T_+ \rho - 1/T_-$,

- $(\beta(-T_+ \tau - T_+ T_+ \rho) \alpha_h(\tau, t) + \{1 - \beta(-T_+ \tau - T_+ T_+ \rho)\}(\rho - T_+ \tau), \beta(-T_+ \tau - T_+ T_+ \rho) \theta^h(\tau, t) + \{1 - \beta(-T_+ \tau - T_+ T_+ \rho)\}(1 - t), \beta(-T_+ \tau - T_+ T_+ \rho) z^h(\tau, t))$

  for $-T_+ \rho - 1/T_- \leq \tau$,

\begin{align*}
& w_\rho(\tau, t) \text{ by} \\
& (\beta(T_+ \tau + T_+ T_+ \rho) \alpha_h(\tau, t) + 2T_+ T_+ \rho), + \{1 - \beta(T_+ \tau + T_+ T_+ \rho)\}(T_+ \tau), \beta(T_+ \tau + T_+ T_+ \rho) \theta^h(\tau, t) + \{1 - \beta(T_+ \tau + T_+ T_+ \rho)\}(1 - t), \beta(T_+ \tau + T_+ T_+ \rho) z^h(\tau, t)), \\
& \text{for } \tau \leq -T_+ \rho + 1/T_-, \\
& (\alpha_h(\tau, t) + 2T_+ T_+ \rho, h(\tau, t)), \text{ for } -T_+ \rho + 1/T_- \leq \tau \leq T_+ \rho - 1/T_+,
\end{align*}

and $y_\rho(\tau - 2T_+ \rho, t)$ by

\begin{align*}
& (\beta(T_+ \tau - T_+ T_+ \rho) \alpha_h(\tau, t) + \{1 - \beta(T_+ \tau - T_+ T_+ \rho)\}(T_+ \tau - 2T_+ T_+ \rho), \beta(T_+ \tau - T_+ T_+ \rho) \theta^h(\tau, t) + \{1 - \beta(T_+ \tau - T_+ T_+ \rho)\}(1 - t), \beta(T_+ \tau - T_+ T_+ \rho) z^h(\tau, t)), \\
& \text{for } \tau \leq T_+ \rho + 1/T_+,
\end{align*}

- $\overline{h}(\tau, t)$ for $T_- \rho + 1/T_+ \leq \tau$.

Moreover we define $U(\rho_0)(\overline{\pi}, \overline{\pi}, \overline{\sigma})$, for $(\overline{\pi}, \overline{\pi}, \overline{\sigma}) \in S$, to be the set of $\overline{h}$ such that, for $\rho > \rho_0$, $\overline{\pi} - x_\rho$ satisfies the $e^{-\frac{1}{T_+} \rho}$-exponential decay condition (see (14)), and also $\overline{\pi} - y_\rho$ and $\overline{\sigma} - y_\rho$. (For simplicity, we shall use a letter $x$ to denote $x_\rho$, and also $w := w_\rho$ and $y := y_\rho$.) If $\overline{h} \in M^2_\rho(p_-, p_+ \cap U(\rho_0)(\overline{\pi}, \overline{\pi}, \overline{\sigma})$, then for a smooth map $f_x$ there are constants $C$ and $C'$ such that

\begin{align}
\|f_x(0)\|_{L^0_{0, \sigma}} & \leq Ce^{-\frac{1}{T_+} \rho}, \\
\|N_x(\xi) - N_x(\xi')\|_{L^0_{0, \sigma}} & \leq C'(\|\xi\|_{L^1_{1, \sigma}} + \|\xi'\|_{L^1_{1, \sigma}})\|\xi - \xi'\|_{L^1_{1, \sigma}},
\end{align}

where $\|\xi\|_{L^1_{1, \sigma}} \leq c$ and $\|\xi\|_{L^0_{0, \sigma}} \leq c$ and $C'$ depends on $c$, and also for $f_w$ and $f_y$. (The proofs of the above estimates are similar to those of (5) and (6). We define two sections $e_\rho^0 := (e_x^0, e_z^0, e_y^0)$ and $e_\rho^1 := (e_x^1, e_z^1, e_y^1)$)
Define $W_{(\bar{n}, \rho)} := \langle e_0^0, \epsilon_0^1 \rangle$ and $H_{(\bar{n}, \rho)} := W_{(\bar{n}, \rho) \ominus L^p_{1, \sigma}} (x^*TX, x^*TL_0, x^*TL_1) \oplus L^p_{1, \sigma} (w^*TX, w^*TL_0, w^*TL_1) \oplus L^p_{1, \sigma} (y^*TX, y^*TL_0, y^*TL_1)$ and the $L^2$-inner product on $H_{(\bar{n}, \rho)}$ by

$$
\langle \xi, \xi' \rangle_{H_{(\bar{n}, \rho)}} := \langle \xi, \xi' \rangle_{L^2},
$$

$$
\langle e_0^0, \xi \rangle_{H_{(\bar{n}, \rho)}} = \langle e_0^1, \xi \rangle_{H_{(\bar{n}, \rho)}} := 0,
$$

$$
\langle e_0^0, e_0^1 \rangle_{H_{(\bar{n}, \rho)}} := 0,
$$

where $\xi, \xi' \in L^p_{1, \sigma} (x^*TX, x^*TL_0, x^*TL_1) \oplus L^p_{1, \sigma} (w^*TX, w^*TL_0, w^*TL_1) \oplus L^p_{1, \sigma} (y^*TX, y^*TL_0, y^*TL_1)$. Let $W^\perp_{(\bar{n}, \rho)}$ be the $L^2$-orthogonal compliment of $W_{(\bar{n}, \rho)}$ in $H_{(\bar{n}, \rho)}$.

**Proposition 4.11** There are constants $\epsilon_0 > 0$ and $\rho_0$ and $C$ such that for $\bar{n} \in M^2_L (p_-, p_+) \cap U(e_0, \rho_0) (\bar{u}, \bar{w}, \bar{v})$ and $\xi := (\xi_x, \xi_w, \xi_y) \in W^\perp_{(\bar{n}, \rho)}$

$$
\| \xi \|_{L^p_{1, \sigma}} \leq C \| (E_x \xi_x, E_w \xi_w, E_y \xi_y) \|_{L^p_{0, \sigma}}.
$$

**Proof.** Let $\{\epsilon_i\}_{i=1, 2, \ldots}$ and $\{\rho_i\}_{i=1, 2, \ldots}$ and $\bar{u}_i \in M^2_L (p_-, p_+) \cap U(\epsilon_i, \rho_i) (\bar{u}, \bar{w}, \bar{v})$ be sequences such that $\epsilon_i \to 0$ and $\rho_i \to \infty$ and there exist $\xi_i = (\xi_{x_i}, \xi_{w_i}, \xi_{y_i}) \in W^\perp_{(\bar{n}, \rho_i)}$ satisfying $\| \xi_i \|_{L^p_{1, \sigma}} = 1$ and $\| (E_x \xi_{x_i}, E_w \xi_{w_i}, E_y \xi_{y_i}) \|_{L^p_{0, \sigma}} \to 0$. Then, in a similar way to the proof of Proposition 4.10 we can prove that there exists a subsequence $\{\xi_{i_k}\}$ such that $\| \xi_{i_k} \|_{L^p_{1, \sigma}} \to 0$, which contradicts the assumption $\| \xi_i \|_{L^p_{1, \sigma}} = 1$.

**Proposition 4.12** There exist constants $\epsilon_0 > 0$ and $\rho_0$ and $C$ such that for $\bar{n} \in M^2_L (p_-, p_+) \cap U(e_0, \rho_0) (\bar{u}, \bar{w}, \bar{v})$ there exists a map $G_{(\bar{n}, \rho)} : L^p_{0, \sigma} (x^*TX) \oplus L^p_{0, \sigma} (w^*TX) \oplus L^p_{0, \sigma} (y^*TX) \to W^\perp_{(\bar{n}, \rho)}$ such that

$$
(E_x \oplus E_w \oplus E_y) G_{(\bar{n}, \rho)} = \text{id},
$$

$$
\| G_{(\bar{n}, \rho)} \xi \|_{L^p_{1, \sigma}} \leq C \| \xi \|_{L^p_{0, \sigma}}.
$$

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Proof. From Proposition 4.11 if $\xi \in \text{Ker}(E_x \oplus E_w \oplus E_y) \cap W_{(\tilde{h},\rho)}^\perp$, then $\xi = 0$ and

$$\dim \text{Ker}(E_x \oplus E_w \oplus E_y) \leq \dim W_{(\tilde{h},\rho)}.$$

On the other hand, from Assumption 3.1,

$$\dim \text{Ker}(E_x \oplus E_w \oplus E_y) \geq \text{Ind}_{E_x;\sigma} + \text{Ind}_{E_w;\sigma} + \text{Ind}_{E_y;\sigma} + 2 = \dim W_{(\tilde{h},\rho)}.$$

Hence we obtain

$$\text{Ker}(E_x \oplus E_w \oplus E_y) \geq W_{(\tilde{h},\rho)}^\perp,$$

and $E_x \oplus E_w \oplus E_y$ is surjective. From the estimate of Proposition 4.11 we can obtain $G_{(\tilde{h},\rho)}$ as in the proposition.

So far we obtain the following: There are constants $\varepsilon_0 > 0$ and $\rho_0$ such that, for $\tilde{h} \in \mathcal{M}^2_1(p_-, p_+) \cap U_{(\varepsilon_0, \rho_0)}(\mathfrak{h}, \mathfrak{p}, \mathfrak{v})$, a map $f_{(x,w,y)} := (f_x, f_w, f_y) : H_{(\tilde{h},\varepsilon)} \to L^p_{0,\sigma}(x^*TX) \oplus L^p_{0,\sigma}(w^*TX) \oplus L^p_{0,\sigma}(y^*TX)$ satisfies (14) and (15), and $Df_{(x,w,y)}(0)$ possesses a right inverse $G_{(\tilde{h},\rho)} : L^p_{0,\sigma}(x^*TX) \oplus L^p_{0,\sigma}(w^*TX) \oplus L^p_{0,\sigma}(y^*TX) \to H_{(\tilde{h},\rho)}$ satisfying (16). If we choose $\rho_0$ large enough, then

$$\|G_{(\tilde{h},\rho)}f_{(x,w,y)}(0)\|_{L^p_{1,\sigma}} \leq \frac{1}{8C}.$$

From the Newton’s method in Appendix A, we can conclude that there are constants $\varepsilon > 0$ and $\rho$ and $C'$ and a smooth map

$$\# : \mathcal{M}^2_1(p_-, p_+) \cap U_{(\varepsilon, \rho)}(\mathfrak{h}, \mathfrak{p}, \mathfrak{v}) \to S, \quad \tilde{h} \mapsto (\exp_x(\xi_{\tilde{h},x}), \exp_w(\xi_{\tilde{h},w}), \exp_y(\xi_{\tilde{h},y}))$$

with $\|\xi_{\tilde{h},x}\|_{L^p_{1,\sigma}} \leq C'\|\nabla f_{x}(x)\|_{L^p_{0,\sigma}}$, and also $w$ and $y$. Divide them by the $\mathbf{R}$ actions, then we obtain a map $\#' : \mathcal{M}^1_1(p_-, p_+) \cap \tilde{U}_{(\varepsilon, \rho)}(\mathfrak{h}, \mathfrak{p}, \mathfrak{v}) \to \tilde{S}$.

From the construction of $\#$ and $\#'$, if $\rho_0$ is large and $\varepsilon$ is small enough, then $\# \circ \#'$ and $\#' \circ \#$ are diffeomorphisms. We finish proving the gluing argument (ii).

Next we will prove the gluing argument (iii). (Most of the proof is similar to that of (i), we will show a sketch.) In a similar way of the
Moreover, for \( E \) conditions (2) and (I) and the decay conditions (2) and (I)

\[
\| \overrightarrow{J}_L w_\chi \|_{L^p_{0,\sigma}} \leq Ce^{-d\rho},
\]

where \( C \) and \( d > 0 \) are constants depending only on \( K, K' \) and \( \rho_0 \).

Proposition 4.13 There exist constants \( \rho_0 \) and \( C \) such that for \( \chi \in K \times K' \times [\rho_0, \infty) \) and \( \xi \in W^\perp_{w_\chi} \)

\[
\| \xi \|_{L^p_{1,\sigma}} \leq C \| E_{w_\chi} \xi \|_{L^p_{0,\sigma}}.
\]

Proposition 4.14 There exist constants \( \rho_0 \) and \( C \) such that for \( \chi \in K \times K' \times [\rho_0, \infty) \) there exists a map \( G_{w_\chi} : L^p_{0,\sigma} \to W^\perp_{w_\chi} \) such that 

\[
E_{w_\chi} G_{w_\chi} = \text{id},
\]

\[
\| G_{w_\chi} \xi \|_{L^p_{1,\sigma}} \leq C \| \xi \|_{L^p_{0,\sigma}}.
\]

From these propositions and the Newton’s method, we can conclude that there are constants \( \rho_0 \) and \( C \) and a smooth map

\[
\tilde{*} : K \times K' \times [\rho_0, \infty) \to M^2_{II}(\gamma, p_+), \chi \mapsto \exp_{w_\chi}(\xi_\chi)
\]

with \( \| \xi_\chi \|_{L^p_{1,\sigma}} \leq C \| \overrightarrow{J}_L w_\chi \|_{L^p_{0,\sigma}} \). Divide them by the \( R \) actions, we obtain a gluing map \( \hat{*} : K \times K' \times [\rho_0, \infty) \to \tilde{M}^1_{II}(\gamma, p_+) \).
There are constants $\varepsilon_0 > 0$ and $\rho_0$ and $C$ such that for $w \in \mathcal{M}_H^2(\gamma, p) \cap U(\varepsilon, \rho)(\pi, \nu)$ and $\xi := (\xi, \xi) \in W_{(w, \rho)}^\perp$,

$$\|\xi\|_{L_1^\perp} \leq C\|E_x \xi_x, E_y \xi_y\|_{L_0^\perp}.$$

**Proposition 4.16** There exist constants $\varepsilon_0 > 0$ and $\rho_0$ and $C$ such that for $w \in \mathcal{M}_H^2(\gamma, p) \cap U(\varepsilon, \rho)(\pi, \nu)$ there exists a map $G_{(w, \rho)} : L_0^\perp (x^*TX) \oplus L^p(y^*TX) \to W_{(w, \rho)}^\perp$ such that

$$(E_x \oplus E_y) G_{(w, \rho)} = \text{id},$$

$$\|G_{(w, \rho)} \xi\|_{L_1^\perp} \leq C\|\xi\|_{L_0^\perp}.$$

From these propositions and the Newton’s method, we can conclude that there are constants $\varepsilon > 0$ and $\rho$ and $C$ and a smooth map

$$
\tilde{\varphi} : \mathcal{M}_H^2(\gamma, p) \cap U(\varepsilon, \rho)(\pi, \nu) \to K \times K', \ w \mapsto (\exp_x(\xi_{w_0}), \exp_y(\xi_{w_1})),
$$

with $\|\xi_{w_0}\|_{L_1^\perp} \leq C\|\xi\|_{L_0^\perp}$ and $\|\xi_{w_1}\|_{L_1^\perp} \leq C\|\xi\|_{L_0^\perp}$. Divide them by the $R$ actions, and we obtain a map $\hat{\varphi} : \hat{\mathcal{M}}_H^1(\gamma, p) \cap \hat{U}(\varepsilon, \rho)(\hat{\pi}, \hat{\nu}) \to \hat{K} \times \hat{K}'$.
From the construction of $\bar{z}$ and $\bar{z}'$, if $\rho_0$ is large and $\varepsilon$ is small enough, then $\bar{z} \circ \bar{z}'$ and $\bar{z}' \circ \bar{z}$ are diffeomorphisms. We finish proving the gluing argument (iii).

Next we will prove the gluing argument (iv). (Most of the proof is similar to that of (ii), we will show a sketch.) Take a lift of $\hat{M}_{1II}(\gamma_-, \gamma'_-) \times \hat{M}_{2IV}(\gamma'_-, \gamma'_+) \times \hat{M}_{1II}(\gamma_-, \gamma'_-)$ and consider the orbit of the lift by $R_\rho(\hat{M}_{1II}(\gamma_-, \gamma'_-) \times \hat{M}_{2IV}(\gamma'_-, \gamma'_+) \times \hat{M}_{1II}(\gamma_-, \gamma'_-))$ and the orbit of the lift by $R^2$-action: for $(a, 0) \in R^2$, $(0, b) \cdot (\hat{z}, \hat{w}, \hat{v}) := (\hat{z}, a \cdot \hat{w}, \hat{v})$, and for $(0, b) \in R^2$, $(0, b) \cdot (\hat{z}, \hat{w}, \hat{v}) := (\hat{z}, b \cdot \hat{w}, \hat{v})$. Note that the orbit is diffeomorphic to $\hat{M}_{1II}(\gamma_-, \gamma'_-) \times \hat{M}_{2IV}(\gamma'_-, \gamma'_+) \times \hat{M}_{1II}(\gamma_-) \times R^2$. We choose a compact set $S$ in the orbit, and we will construct a gluing map $\# : S \times [\rho_0, \infty) \to \hat{M}_{1II}(\gamma_-, \gamma'_+)$. In a similar way of the proof for (ii) we construct a strip $w_\chi(\tau, t)$ for $\chi := (\hat{w}, \hat{v}, \hat{w}, \rho) \in S \times [\rho_0, \infty)$ which satisfies the Lagrangian boundary conditions and (II) and decay conditions (2) and (1') and

$$\|\bar{\partial}_k w_\chi\|_{L^\perp_{0,\sigma}} \leq C e^{-d\rho},$$

where $C$ and $d > 0$ are constants depending only on $S$ and $\rho_0$. Moreover, for $\|\xi\|_{L^\perp_{1,\sigma}} \leq c$ and $\|\xi'\|_{L^\perp_{1,\sigma}} \leq c$,

$$\|N_{w_\chi}(\xi) - N_{w_\chi}(\xi')\|_{L^\perp_{0,\sigma}} \leq C(\|\xi\|_{L^\perp_{1,\sigma}} + \|\xi'\|_{L^\perp_{1,\sigma}})\|\xi - \xi'\|_{L^\perp_{1,\sigma}},$$

where $C$ is a constant depending only on $\|\nabla w_\chi\|_{L^p_{0,\sigma}}$ and $c$. We define similar $e^0_\rho$ and $e^1_\rho$ as in the proof of (ii). Let $W^\perp_{w_\chi}$ be the $L^2$- orthogonal compliment of $W_{w_\chi} := \langle -\beta(-\tau - 2\tau_0 - 2T_-\rho) \frac{da}{da}(a \ast \hat{w}) |_{a=0}(\tau + 2T_-\rho, t) + e^0_\rho, e^1_\rho \rangle$ in $\langle \frac{da}{da}(a \ast w_\chi) |_{a=0} \rangle \oplus L^\perp_{1,\sigma}(w^*_\chi TX, w^*_\chi TL_0, w^*_\chi TL_1)$.

**Proposition 4.17** There exist constants $\rho_0$ and $C$ such that for $\chi \in S \times [\rho_0, \infty)$ and $W^\perp_{w_\chi}$

$$\|\xi\|_{L^\perp_{1,\sigma}} \leq C \rho^{\frac{3}{2} - \frac{\gamma}{2}} \|E_{w_\chi} \xi\|_{L^\perp_{0,\sigma}}.$$

**Proposition 4.18** There exist constants $\rho_0$ and $C$ such that for $\chi \in S \times [\rho_0, \infty)$ there exists a map $G_{w_\chi} : L^\perp_{0,\sigma} \to W^\perp_{w_\chi}$ such that

$$E_{w_\chi} G_{w_\chi} = \text{id},$$

$$\|G_{w_\chi} \xi\|_{L^\perp_{1,\sigma}} \leq C \rho^{\frac{3}{2} - \frac{\gamma}{2}} \|\xi\|_{L^\perp_{0,\sigma}}.$$
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From these propositions and the Newton’s method, we can conclude that there are constants $\rho_0$ and $C$ and a smooth map

$$\hat{\varepsilon} : S \times [\rho_0, \infty) \to \mathcal{M}_{II}^2(\gamma_-, p_+), \xi \mapsto \exp_{w_{\xi}}(\xi)$$

with $\|\xi\|_{L_{1,\sigma}^p} \leq C\|\overline{\mathcal{D}}_w w_{\xi}\|_{L_{0,\sigma}^p}$. Divide them by the $R$ actions, we obtain a gluing map $\hat{\varepsilon} : \tilde{S} \times [\rho_0, \infty) \to \tilde{\mathcal{M}}_{II}^2(\gamma_-, p_+)$. The next step is to show the surjectivity of $\hat{\varepsilon} : S \times [\rho_0, \infty) \to \mathcal{M}_{II}^2(\gamma_-, p_+)$. In a similar way of the proof for (ii), for a map $\hat{\h}$ which satisfies the Lagrangian boundary conditions and (II) and the decay conditions (2) and (1'), we define $x := x_{\rho}, w := w_{\rho}$ and $y := y_{\rho}$ and $U_{(\varepsilon, \rho_0)}(\overline{u}, \overline{w}, \overline{v})$. If $\hat{\h} \in \mathcal{M}_{II}^2(\gamma_-, p_+) \cap U_{(\varepsilon, \rho_0)}(\overline{u}, \overline{w}, \overline{v})$, then for a smooth map $f_x$ there are constants $C$ and $C'$

$$\|f_x(0)\|_{L_{1,\sigma}^p} \leq Ce^{-\frac{1}{\rho}},$$

$$\|N_x(\xi) - N_x(\xi')\|_{L_{0,\sigma}^p} \leq C'(\|\xi\|_{L_{1,\sigma}^p} + \|\xi'\|_{L_{0,\sigma}^p})\|\xi - \xi'\|_{L_{1,\sigma}^p},$$

where $\|\xi\|_{L_{1,\sigma}^p} \leq c$ and $\|\xi'\|_{L_{1,\sigma}^p} \leq c$ and $C'$ depend on $c$, and also for $f_w$ and $f_y$. We define similar $e^0_\rho$ and $e^1_\rho$ as in the proof of (ii). Define $W_{(\h, \rho)} := \left\{-(\beta(\tau - \tau_0))\frac{d}{da}(a \ast x)|_{a=0} + e^0_\rho, e^1_\rho\right\}$ and $H_{(\h, \rho)} := W_{(\h, \rho)} \oplus L_{1,\sigma}^p(x^*TX, x^*TL_0, x^*TL_1) \oplus L_{1,\sigma}^p(w^*TX, w^*TL_0, w^*TL_1) \oplus L_{1,\sigma}^p(y^*TX, y^*TL_0, y^*TL_1)$ and the $L^2$-inner product on $H_{(\h, \rho)}$ by

$$\langle \xi, \xi' \rangle_{H_{(\h, \rho)}} := \langle \xi, \xi' \rangle_{L^2},$$

$$\left\langle -\beta(\tau - \tau_0)\frac{d}{da}(a \ast x)|_{a=0} + e^0_\rho, \xi \right\rangle_{H_{(\h, \rho)}} = \langle e^1_\rho, \xi \rangle_{H_{(\h, \rho)}} := 0,$$

$$\left\langle -\beta(\tau + \tau_0)\frac{d}{da}(a \ast x)|_{a=0} + e^0_\rho, e^1_\rho \right\rangle_{H_{(\h, \rho)}} := 0,$$

where $\xi, \xi' \in L_{1,\sigma}^p(x^*TX, x^*TL_0, x^*TL_1) \oplus L_{1,\sigma}^p(w^*TX, w^*TL_0, w^*TL_1) \oplus L_{1,\sigma}^p(y^*TX, y^*TL_0, y^*TL_1)$. Let $W_{(\h, \rho)}^\perp$ be the $L^2$-orthogonal compliment of $W_{(\h, \rho)}$ in $H_{(\h, \rho)}$.

**Proposition 4.19** There are constants $\varepsilon_0 > 0$ and $\rho_0$ and $C$ such that for $\h \in \mathcal{M}_{II}^2(\gamma_-, p_+) \cap U_{(\varepsilon_0, \rho_0)}(\overline{u}, \overline{w}, \overline{v})$ and $\xi := (\xi_x, \xi_w, \xi_y) \in W_{(\h, \rho)}^\perp$

$$\|\xi\|_{L_{1,\sigma}^p} \leq C\|(E_x\xi_x, E_w\xi_w, E_y\xi_y)\|_{L_{0,\sigma}^p}.$$
Proposition 4.20  There exist constants $\varepsilon_0 > 0$ and $\rho_0$ and $C$ such that for $\mathbf{h} \in \mathcal{M}_I^2(\gamma_-, p_+) \cap U(\varepsilon_0, \rho_0)$ there exists a map $G(\mathbf{h}, \rho) : L_{0,\sigma}^p(x^*TX) \oplus L_{0,\sigma}^p(w^*TX) \oplus L_{0,\sigma}^p(y^*TX) \to W_{\mathbf{h}, \rho}$ such that

$$(E_x \oplus E_w \oplus E_y)G(\mathbf{h}, \rho) = \text{id},$$

$$\|G(\mathbf{h}, \rho)\|_{L_{1,\sigma}^p} \leq C\|\xi\|_{L_{0,\sigma}^p}.$$  

From these propositions and the Newton’s method, we can conclude that there are constants $\varepsilon > 0$ and $\rho$ and $C$ and a smooth map

$$\sharp' : \mathcal{M}_I^2(\gamma_-, p_+) \cap U(\varepsilon, \rho) \to S, \mathbf{h} \mapsto (\exp_{\mathbf{h}}(\zeta_{\mathbf{h}, x}), \exp_{\mathbf{h}}(\zeta_{\mathbf{h}, w}), \exp_{\mathbf{h}}(\zeta_{\mathbf{h}, y}))$$

with $\|\zeta_{\mathbf{h}, x}\|_{L_{1,\sigma}^p} \leq C\|\mathbf{h}\|_{L_{1,\sigma}^p}$, and also $w$ and $y$. Divide them by the $\mathbf{R}$ actions, then we obtain a map $\sharp' : \mathcal{M}_I^2(\gamma_-, p_+) \cap \hat{U}(\varepsilon, \rho) \to \hat{S}$.

From the construction of $\sharp$ and $\sharp'$, if $\rho_0$ is large and $\varepsilon$ is small enough, then $\sharp \circ \sharp'$ and $\sharp' \circ \sharp$ are diffeomorphisms. We finish proving the gluing argument (iv).

Next we will prove the gluing argument (v). (Most of the proof is similar to that of (ii), we will show a sketch.) Take a lift of $\hat{\mathcal{M}}_V^0(\gamma_-, \gamma) \times \hat{\mathcal{M}}_V^0(\gamma, p_+) \in \mathcal{M}_V^0(\gamma_-, \gamma) \times \mathcal{M}_V^1(\gamma, p_+)$ in $\mathcal{M}_V^2(\gamma_-, \gamma) \times \mathcal{M}_V^1(\gamma, p_+) \times \mathbf{R}^2$. We choose a compact set $S$ in the orbit, and we will construct a gluing map $\sharp : S \times [\rho_0, \infty) \to \mathcal{M}_I^2(\gamma_-, p_+)$. In a similar way of the proof for (ii) we construct a strip $w_{\chi}(\tau, t)$ for $\chi := (\mathbf{v}, \mathbf{v}, \mathbf{r}) \in S \times [\rho_0, \infty)$ which satisfies the Lagrangian boundary conditions and (ii) and decay conditions (ii) and (i') and

$$\|\mathcal{J}_t w_{\chi}\|_{L_{0,\sigma}^p} \leq Ce^{-d\rho},$$

where $C$ and $d > 0$ are constants depending only on $S$ and $\rho_0$. Moreover, for $\|\xi\|_{L_{1,\sigma}^p} \leq c$ and $\|\xi'\|_{L_{1,\sigma}^p} \leq c$,

$$\|N w_{\chi}(\xi) - N w_{\chi}(\xi')\|_{L_{0,\sigma}^p} \leq C(\|\xi\|_{L_{1,\sigma}^p} + \|\xi'\|_{L_{1,\sigma}^p})\|\xi - \xi'\|_{L_{1,\sigma}^p},$$

where $C$ is a constant depending only on $\|\nabla w_{\chi}\|_{L_{0,\sigma}^p}$ and $c$. We define similar $e_0^0$ and $e_1^0$ as in the proof of (ii). Let $W_{w_{\chi}}$ be the $L^2$- orthogonal complement of $W_{w_{\chi}} := \langle e_0^0, e_1^0 \rangle$ in $\langle \frac{d}{dt} a(a* w_{\chi})|_{a=0} \rangle \oplus L_{1,\sigma}^p(w_{\chi}^*TX, w_{\chi}^*TL_0, w_{\chi}^*TL_1)$. 


Proposition 4.21 There exist constants \( \rho_0 \) and \( C \) such that for \( \chi \in S \times [\rho_0, \infty) \) and \( W_{w, \chi}^\perp \)
\[
\| \xi \|_{L_{1, \sigma}^p} \leq C \rho^{\frac{3}{2} - \frac{1}{p}} \| E_{w, \chi} \xi \|_{L_{0, \sigma}^p}.
\]

Proposition 4.22 There exist constants \( \rho_0 \) and \( C \) such that for \( \chi \in S \times [\rho_0, \infty) \) there exists a map \( G_{w, \chi} : L_0^\sigma \to W_{w, \chi}^\perp \) such that
\[
E_{w, \chi} G_{w, \chi} = \text{id},
\]
\[
\| G_{w, \chi} \xi \|_{L_{1, \sigma}^p} \leq C \rho^{\frac{3}{2} - \frac{1}{p}} \| \xi \|_{L_{0, \sigma}^p}.
\]

From these propositions and the Newton’s method, we can conclude that there are constants \( \rho_0 \) and \( C \) and a smooth map
\[
\sharp : S \times [\rho_0, \infty) \to M_2^\perp (\gamma_-, p_+), \quad \xi \mapsto \exp_{w, \chi}(\xi)
\]
with \( \| \xi \|_{L_{1, \sigma}^p} \leq C \| \tilde{\partial}_L w, \chi \|_{L_{0, \sigma}^p} \). Divide them by the \( R \) actions, we obtain a gluing map \( \tilde{\sharp} : \tilde{S} \times [\rho_0, \infty) \to \tilde{M}_2^\perp (\gamma_-, p_+) \).

The next step is to show the surjectivity of \( \tilde{\sharp} : S \times [\rho_0, \infty) \to \tilde{M}_2^\perp (\gamma_-, p_+) \). In a similar way of the proof for (ii), for a map \( \bar{h} \) which satisfies the Lagrangian boundary conditions and (II) and the decay conditions (2) and (1'), we define \( x := x_\rho \) and \( y := y_\rho \) and \( U_{(\varepsilon, \rho_0)}(\bar{\rho}, \bar{\tau}) \). If \( \bar{h} \in M_2^\perp (\gamma_-, p_+) \cap U_{(\varepsilon, \rho_0)}(\bar{\rho}, \bar{\tau}) \), then for a smooth map \( f_x \) there are constants \( C \) and \( C' \)
\[
\| f_x(0) \|_{L_{0, \sigma}^p} \leq C e^{-\frac{1}{p}},
\]
\[
\| N_x(\xi) - N_x(\xi') \|_{L_{0, \sigma}^p} \leq C' (\| \xi \|_{L_{1, \sigma}^p} + \| \xi' \|_{L_{1, \sigma}^p}) \| \xi - \xi' \|_{L_{1, \sigma}^p},
\]
where \( \| \xi \|_{L_{1, \sigma}^p} \leq c \) and \( \| \xi' \|_{L_{1, \sigma}^p} \leq c \) and \( C' \) depend on \( c \), and also for \( f_y \).
Define \( W_{\bar{h}, \rho, \bar{\rho}} := \mathcal{E} \rho, \rho_0 \) and \( H_{\bar{h}, \rho, \bar{\rho}} := W_{\bar{h}, \rho, \bar{\rho}} \oplus L_{1, \sigma}^p (x^* TX, x^* TL_0, x^* TL_1) \oplus L_{1, \sigma}^p (y^* TX, y^* TL_0, y^* TL_1) \), and the \( L^2 \)-inner product on \( H_{\bar{h}, \rho, \bar{\rho}} \) in a similar way to that of (ii). Let \( W_{\bar{h}, \rho, \bar{\rho}}^\perp \) be the \( L^2 \)-orthogonal compliment of \( W_{\bar{h}, \rho, \bar{\rho}} \) in \( H_{\bar{h}, \rho, \bar{\rho}} \).

Proposition 4.23 There are constants \( \varepsilon_0 > 0 \) and \( \rho_0 \) and \( C \) such that for \( \bar{h} \in M_2^\perp (\gamma_-, p_+) \cap U_{(\varepsilon_0, \rho_0)}(\bar{\rho}, \bar{\tau}) \) and \( \xi := (\xi_x, \xi_y) \in W_{\bar{h}, \rho, \bar{\rho}}^\perp \)
\[
\| \xi \|_{L_{1, \sigma}^p} \leq C \| (E_x \xi_x, E_y \xi_y) \|_{L_{0, \sigma}^p}.
\]
**Proposition 4.24** There exist constants $\varepsilon_0 > 0$ and $\rho_0$ and $C$ such that for $\overline{\mathcal{T}} \in \mathcal{M}_I^2(\gamma_-, p_+) \cap U_{(\varepsilon_0, \rho_0)}(\overline{\pi}, \overline{\nu})$ there exists a map $G(\overline{\pi}, \rho) : L^p_{0,\sigma}(x^*TX) \oplus L^p_{0,\sigma}(y^*TX) \to W_{(\overline{\pi}, \rho)}^\perp$ such that

$$(E_x \oplus E_y)G(\overline{\pi}, \rho) = \text{id},$$

$$\|G(\overline{\pi}, \rho)\|_{L^p_{1,\sigma}} \leq C\|\xi\|_{L^p_{0,\sigma}}.$$  

From these propositions and the Newton’s method, we can conclude that there are constants $\varepsilon > 0$ and $\rho$ and $C$ and a smooth map

$$\eta' : \mathcal{M}_{II}^2(\gamma_- p_) \cap U_{(\varepsilon, \rho)}(\overline{\pi}, \overline{\nu}) \to \mathcal{S}, \overline{\pi} \mapsto (\exp_x(\xi_{\overline{\pi}, \overline{\nu}}), \exp_y(\xi_{\overline{\pi}, \overline{\nu}}))$$

with $\|\xi_{\overline{\pi}, \overline{\nu}}\|_{L^p_{1,\sigma}} \leq C\|\overline{\pi}_{x, \overline{\nu}}\|_{L^p_{0,\sigma}}$, and also $y$. Divide them by the $\mathbb{R}$ actions, then we obtain a map $\hat{\eta}' : \mathcal{M}_{II}^2(\gamma_- p_) \cap \hat{U}_{(\varepsilon, \rho)}(\overline{\pi}, \overline{\nu}) \to \hat{\mathcal{S}}$.

From the construction of $\hat{\eta}$ and $\hat{\eta}'$, if $\rho_0$ is large and $\varepsilon$ is small enough, then $\hat{\eta} \circ \hat{\eta}'$ and $\hat{\eta}' \circ \hat{\eta}$ are diffeomorphisms. We finish proving the gluing argument (v).

Next we will prove the gluing argument (vi). (Most of the proof is similar to that of (i), we will show a sketch.) In a similar way to the proof for (i), for compact sets $K \subset \mathcal{M}_{II}^1(\gamma_-, q)$ and $K' \subset \mathcal{M}_{II}^1(\gamma_+, q)$ we construct a strip $w_\chi(\tau, t)$ for $\chi := (\overline{\pi}, \overline{\nu}, \rho) \in K \times K' \times [\rho_0, \infty)$ which satisfies the Lagrangian boundary conditions and (III) and the decay conditions (2) and (2') and

$$\|\overline{\partial}_J w_\chi\|_{L^p_{0,\sigma}} \leq Ce^{-d\rho},$$

where $C$ and $d > 0$ are constants depending only on $K$, $K'$ and $\rho_0$. Moreover, for $\|\xi\|_{L^p_{1,\sigma}} \leq c$ and $\|\xi'\|_{L^p_{0,\sigma}} \leq c$,

$$\|N_{w_\chi}(\xi) - N_{w_\chi}(\xi')\|_{L^p_{0,\sigma}} \leq C(\|\xi\|_{L^p_{1,\sigma}} + \|\xi'\|_{L^p_{1,\sigma}})\|\xi - \xi'\|_{L^p_{1,\sigma}},$$

where $C$ is a constant depending only on $\|\nabla w_\chi\|_{L^p_{0,\sigma}}$ and $c$. We use maps $E_{\pi} : \langle \frac{d}{da}(a * \pi) \rangle_{a=0} \oplus L^p_{1,\sigma}(\varpi^*TX, \varpi^*TL_0, \varpi L_1) \to L^p_{0,\sigma}(\varpi^*TX)$ and $E_{\overline{\pi}} : \langle \frac{d}{da}(a * \overline{\pi}) \rangle_{a=0} \oplus L^p_{1,\sigma}(\overline{\varpi}^*TX, \overline{\varpi}^*TL_0, \overline{\varpi} L_1) \to L^\perp(\overline{-\varpi}^*TX)$. For $\eta \in \text{Ker}E_{\pi}$ and $\zeta \in \text{Ker}E_{\overline{\pi}}$, we define similar $\eta_{\rho, \pi}^\perp \zeta$ as in the proof of (i). Let $W_{w_\chi}^\perp$ be the $L^2$- orthogonal compliment of $W_{w_\chi} := \{\eta_{\rho, \pi}^\perp \zeta | \eta \in \text{Ker}E_{\pi}, \zeta \in \text{Ker}E_{\overline{\pi}}\}$ in $\langle \frac{d}{da}(a * w_\chi) \rangle_{a=0} \oplus L^p_{1,\sigma}(w_\chi^*TX, w_\chi^*TL_0, w_\chi^*TL_1)$.
Proposition 4.25 There exist constants \( \rho_0 \) and \( C \) such that for \( \chi \in K \times K' \times [\rho_0, \infty) \) and \( \xi \in W_{w,\chi}^\perp \)

\[
\| \xi \|_{L_{1,\sigma}^p} \leq C \| E_{w,\chi} \xi \|_{L_{0,\sigma}^p}.
\]

Proposition 4.26 There exist constants \( \rho_0 \) and \( C \) such that for \( \chi \in K \times K' \times [\rho_0, \infty) \) there exists a map \( G_{w,\chi} : L_{0,\sigma}^p \to W_{w,\chi}^\perp \) such that

\[
E_{w,\chi} G_{w,\chi} = \text{id},
\]

\[
\| G_{w,\chi} \xi \|_{L_{0,\sigma}^p} \leq C \| \xi \|_{L_{0,\sigma}^p}.
\]

From these propositions and the Newton’s method, we can conclude that there are constants \( \rho_0 \) and \( C \) and a smooth map

\[ 
\hat{\#:} K \times K' \times [\rho_0, \infty) \to M_{I\!I\!I}^2(\gamma_-, \gamma_+), \ \chi \mapsto \exp_{w,\chi}(\xi_\chi)
\]

with \( \| \xi_\chi \|_{L_{1,\sigma}^p} \leq C \| \partial_\chi w \|_{L_{0,\sigma}^p} \). Divide them by the \( \mathbb{R} \) actions, we obtain a gluing map \( \hat{\#:} \tilde{K} \times \tilde{K}' \times [\rho_0, \infty) \to \tilde{M}_{I\!I\!I}^1(\gamma_-, \gamma_+) \).

The next step is to show the surjectivity of \( \hat{\#:} K \times K' \times [\rho_0, \infty) \to M_{I\!I\!I}^2(\gamma_-, \gamma_+) \cap U_{(\varepsilon, \rho)}(\overline{\pi}, \overline{\tau}) \). In a similar way to the proof for (i), for a map \( w \) which satisfies the Lagrangian boundary conditions and (III) and the decay conditions (2) and (2'), we define \( x := x_\rho \) and \( y := y_\rho \) and \( U_{(\varepsilon, \rho)}(\overline{\pi}, \overline{\tau}) \). If \( w \in M_{I\!I\!I}^2(\gamma_-, \gamma_+) \cap U_{(\varepsilon, \rho)}(\overline{\pi}, \overline{\tau}) \), then for a smooth map \( f_x \) there are constants \( C \) and \( C' \) and \( d > 0 \) such that

\[
\| f_x(0) \|_{L_{0,\sigma}^p} \leq C e^{-d\rho},
\]

\[
\| N_x(\xi) - N_x(\xi') \|_{L_{0,\sigma}^p} \leq C' (\| \xi \|_{L_{1,\sigma}^p} + \| \xi' \|_{L_{1,\sigma}^p}) \| \xi - \xi' \|_{L_{1,\sigma}^p},
\]

where \( \| \xi \|_{L_{1,\sigma}^p} \leq c \) and \( \| \xi' \|_{L_{1,\sigma}^p} \leq c \) and \( C' \) depends on \( c \), and also for \( f_y \).

For \( \xi \in \text{Ker} \{ E_w : \langle \frac{\partial}{\partial a}(a \ast w) \rangle_{a=0} \oplus L_{1,\sigma}^p \to L_{0,\sigma}^p \} \), we define similar \( \hat{\xi} := (\xi_x, \xi_y) \in \langle \langle \frac{\partial}{\partial a} (a \ast x) \rangle_{a=0} \oplus L_{1,\sigma}^p (x^* T\! X, x^* T\! L_0, x^* T\! L_1) \rangle \oplus \langle \langle \frac{\partial}{\partial a} (a \ast y) \rangle_{a=0} \oplus L_{0,\sigma}^p (y^* T\! X, y^* T\! L_0, y^* T\! L_1) \rangle \) as in the proof of (i). Let \( W_{(w, \rho)}^\perp \) be the \( L^2 \)-orthogonal complement of \( W_{(w, \rho)}^\perp := \{ \hat{\xi} \in \text{Ker} E_w \} \) in \( \langle \langle \frac{\partial}{\partial a} (a \ast x) \rangle_{a=0} \oplus L_{1,\sigma}^p (x^* T\! X, x^* T\! L_0, x^* T\! L_1) \rangle \oplus \langle \langle \frac{\partial}{\partial a} (a \ast y) \rangle_{a=0} \oplus L_{0,\sigma}^p (y^* T\! X, y^* T\! L_0, y^* T\! L_1) \rangle \).
Proposition 4.27 There are constants $\varepsilon_0 > 0$ and $\rho_0$ and $C$ such that for $w \in M_{III}^i(\gamma_-, \gamma_+) \cap U(\varepsilon_0, \rho_0)(\pi, \nu)$ and $\xi := (\xi_x, \xi_y) \in W_{(w, \rho)}^1$

$$||\xi||_{L^p_{1, \sigma}} \leq C|| (E_x \xi_x, E_y \xi_y)||_{L^p_{0, \sigma}}.$$  

Proposition 4.28 There exist constants $\varepsilon_0 > 0$ and $\rho_0$ and $C$ such that for $w \in M_{III}^i(\gamma_-, \gamma_+) \cap U(\varepsilon_0, \rho_0)(\pi, \nu)$ there exists a map $G_{(w, \rho)} : L^p_{0, \sigma}(x^*TX) \oplus L^p_{0, \sigma}(y^*TX) \to W_{(w, \rho)}^1$ such that

$$(E_x \oplus E_y)G_{(w, \rho)} = \text{id},$$

$$||G_{(w, \rho)}\xi||_{L^p_{1, \sigma}} \leq C||\xi||_{L^p_{0, \sigma}}.$$  

From these propositions and the Newton’s method, we can conclude that there are constants $\varepsilon > 0$ and $\rho$ and $C$ and a smooth map

$$\sharp' : M_{III}^2(\gamma_-, \gamma_+) \cap U(\varepsilon, \rho)(\pi, \nu) \to K \times K', \ w \mapsto (\exp_x(\xi_{w; x}), \exp_y(\xi_{w; y}))$$

with $||\xi_{w; x}||_{L^p_{1, \sigma}} \leq C||\tilde{\mathcal{J}}_x||_{L^p_{0, \sigma}}$ and $||\xi_{w; y}||_{L^p_{1, \sigma}} \leq C||\tilde{\mathcal{J}}_y||_{L^p_{0, \sigma}}$. Divide by the $R$ actions, and we obtain a map $\sharp' : M_{III}^1(\gamma_-, \gamma_+) \cap U(\varepsilon, \rho)(\pi, \nu) \to \hat{K} \times \hat{K}'$.

From the construction of $\sharp$ and $\sharp'$, if $\rho_0$ is large and $\varepsilon$ is small enough, then $\sharp \circ \sharp'$ and $\sharp' \circ \sharp$ are diffeomorphisms. We finish proving the gluing argument (vi).

Next we will prove the gluing argument (vii). (Most of the proof is similar to that of (ii), we will show a sketch.) Take a lift of $\hat{M}_{III}^0(\gamma_-, \gamma') \times \hat{M}_{IV}^0(\gamma_+, \gamma_+) \times \hat{M}_{III}^0(\gamma', \gamma_+) \times M_{III}^1(\gamma_-, \gamma') \times M_{IV}^2(\gamma', \gamma_+) \times M_{III}^1(\gamma'_+, \gamma_+)$ and consider the orbit of the lift by the similar $R^2$-action to that of (ii). Note that the orbit is diffeomorphic to $\hat{M}_{III}^0(\gamma_-, \gamma') \times \hat{M}_{IV}^0(\gamma_-, \gamma'_+) \times \hat{M}_{III}^0(\gamma'_+, \gamma_+) \times \hat{M}_{III}^0(\gamma'_+, \gamma_+) \times R^2$. We choose a compact set $S$ in the orbit, and we will construct a gluing map $\sharp : S \times [\rho_0, \infty) \to M_{III}^2(\gamma_-, \gamma_+)$. In a similar way of the proof for (ii) we construct a strip $w_\chi(\tau, t)$ for $\chi := (\pi, \nu, \pi, \rho) \in S \times [\rho_0, \infty)$ which satisfies the Lagrangian boundary conditions and (III) and decay conditions (2) and (2') and

$$||\tilde{\mathcal{J}}_t w_\chi||_{L^p_{0, \sigma}} \leq Ce^{-d\rho},$$
where $C$ and $d > 0$ are constants depending only on $S$ and $\rho_0$. Moreover, for $\|\xi\|_{L^p_{0,\sigma}} \leq c$ and $\|\xi\|_{L^p_{1,\sigma}} \leq c$,
\[
\|N_{w_\chi}(\xi) - N_{w_\chi}(\xi')\|_{L^p_{0,\sigma}} \leq C(\|\xi\|_{L^p_{0,\sigma}} + \|\xi\|_{L^p_{1,\sigma}})\|\xi - \xi\|_{L^p_{1,\sigma}},
\]
where $\xi$ is a constant depending only on $\|\nabla w_\chi\|_{L^p_{0,\sigma}}$ and $c$. We define similar $e^0_\rho$ and $e^1_\rho$ as in the proof of (ii). Let $W_{w_\chi}^\perp$ be the $L^2$-orthogonal compliment of $W_{w_\chi} := \langle -\beta(\tau - 2\tau_0 - 2T_\rho)\partial_{da}(a * \overline{\pi})|_{a=0}(\tau + 2T_\rho, t) + e^0_\rho, \beta(\tau - 2\tau_0 - 2T_\rho)\partial_{da}(a * \overline{\pi})|_{a=0}(\tau - 2T_\rho, t) + e^1_\rho\rangle$ in $\langle \partial\overline{\omega}(a * \overline{w_\chi})|_{a=0}\rangle \oplus L^p_{0,\sigma}(w^*_\chi TX, w^*_\chi TL_0, w^*_\chi TL_1)$.

**Proposition 4.29** There exist constants $\rho_0$ and $C$ such that for $\chi \in S \times [\rho_0, \infty)$ and $W_{w_\chi}^\perp$
\[
\|\xi\|_{L^p_{1,\sigma}} \leq C\rho^{\frac{2}{2} - \frac{1}{p}}\|E_{w_\chi, \chi}\|_{L^p_{0,\sigma}}.
\]

**Proposition 4.30** There exist constants $\rho_0$ and $C$ such that for $\chi \in S \times [\rho_0, \infty)$ there exists a map $G_{w_\chi} : L^p_{0,\sigma} \to W_{w_\chi}^\perp$ such that
\[
E_{w_\chi, G_{w_\chi}} = \text{id},
\]
\[
\|G_{w_\chi, \chi}\|_{L^p_{1,\sigma}} \leq C\rho^{\frac{2}{2} - \frac{1}{p}}\|\xi\|_{L^p_{1,\sigma}}.
\]

From these propositions and the Newton’s method, we can conclude that there are constants $\rho_0$ and $C$ and a smooth map
\[
\sharp : S \times [\rho_0, \infty) \to \mathcal{M}_I(\gamma_-, \gamma_+), \ \xi \mapsto \exp_{w_\chi}(\xi_\chi)
\]
with $\|\xi_\chi\|_{L^p_{1,\sigma}} \leq C\|\overline{\partial}_J w_\chi\|_{L^p_{0,\sigma}}$. Divide them by the $R$ actions, we obtain a gluing map $\sharp : \tilde{S} \times [\rho_0, \infty) \to \tilde{\mathcal{M}}_{II}(\gamma_-, \gamma_+)$.

The next step is to show the surjectivity of $\sharp : S \times [\rho_0, \infty) \to \mathcal{M}_I(\gamma_-, \gamma_+)$, and with the Newtonian boundary conditions (III) and the decay conditions (2) and (2'), we define $x := x_\rho$, $w := w_\rho$ and $y := y_\rho$ and $U(\rho_0)(\pi, \overline{\pi}, \overline{\pi})$. If $\tilde{\mathcal{M}}_I(\gamma_-, \gamma_+) \cap U(\rho_0)(\pi, \overline{\pi}, \overline{\pi})$, then for a smooth map $f_x$ there are constants $C$ and $C'$
\[
\|f_x(0)\|_{L^p_{0,\sigma}} \leq Ce^{-\frac{1}{2}\rho},
\]
\[
\|N_x(\xi) - N_x(\xi')\|_{L^p_{0,\sigma}} \leq C'(\|\xi\|_{L^p_{0,\sigma}} + \|\xi\|_{L^p_{0,\sigma}})\|\xi - \xi\|_{L^p_{0,\sigma}}.
\]
where \(|\xi|_{L^p_{\sigma}} \leq c\) and \(|\xi'|_{L^p_{\sigma}} \leq c\) and \(C'\) depend on \(c\), and also for \(f_w\) and \(f_y\). We define similar \(e^0_\rho\) and \(e^1_\rho\) as in the proof (ii). Define \(W(\xi_\rho, \rho) := \langle -\beta(-\tau - \tau_0) \frac{d}{da}(a \ast x) \rangle_{|a=0} + e^0_{\rho, \xi} + e^1_{\rho, \xi} \rangle \) and \(H(\xi_\rho, \rho) := W(\xi_\rho, \rho) + L^p_{1,\rho}(x^*TX, x^*TL_0, x^*TL_1) + L^p_{2,\rho}(w^*TX, w^*TL_0, w^*TL_1) + L^p_{3,\rho}(y^*TX, y^*TL_0, y^*TL_1)\) and the \(L^2\)-inner product on \(H(\xi_\rho, \rho)\) by

\[
\langle \xi, \xi' \rangle_{H(\xi_\rho, \rho)} := \langle \xi, \xi' \rangle_{L^2},
\]

\[
\left\langle -\beta(-\tau - \tau_0) \frac{d}{da}(a \ast x) \right|_{a=0} + e^0_{\rho, \xi} + e^1_{\rho, \xi} \rangle_{H(\xi_\rho, \rho)} := 0,
\]

\[
\left\langle -\beta(-\tau - \tau_0) \frac{d}{da}(a \ast x) \right|_{a=0} + e^0_{\rho, \xi} + \beta(-\tau - \tau_0) \frac{d}{da}(a \ast y) \right|_{a=0} + e^1_{\rho, \xi} \rangle_{H(\xi_\rho, \rho)} := 0,
\]

where \(\xi, \xi' \in L^p_{1,\rho}(x^*TX, x^*TL_0, x^*TL_1) + L^p_{2,\rho}(w^*TX, w^*TL_0, w^*TL_1) + L^p_{3,\rho}(y^*TX, y^*TL_0, y^*TL_1)\). Let \(W(\xi_\rho, \rho)\) be the \(L^2\)-orthogonal complement of \(W(\xi_\rho, \rho)\) in \(H(\xi_\rho, \rho)\).

**Proposition 4.31** There are constants \(\varepsilon > 0\) and \(\rho_0\) and \(C\) such that for \(\overline{\gamma} \in \mathcal{M}_{II}^2(\gamma_-, \gamma_+) \cap U(\varepsilon, \rho_0)(\overline{\gamma}, \overline{\varepsilon}, \overline{\gamma})\) and \(\xi := (\xi_x, \xi_w, \xi_y) \in W(\xi_\rho, \rho)\)

\[
\|\xi\|_{L^p_{\sigma}} \leq C\|\langle E_x, E_w, E_y\rangle\|_{L^p_{\sigma}}.
\]

**Proposition 4.32** There exist constants \(\varepsilon_0 > 0\) and \(\rho_0\) and \(C\) such that for \(\overline{\gamma} \in \mathcal{M}_{II}^2(\gamma_-, \gamma_+) \cap U(\varepsilon_0, \rho_0)(\overline{\gamma}, \overline{\varepsilon}, \overline{\gamma})\) there exists a map \(G(\overline{\gamma}, \rho) : L^p_{0,\sigma}(x^*TX) + L^p_{2,\sigma}(w^*TX) + L^p_{3,\sigma}(y^*TX) \to W(\xi_\rho, \rho)\) such that

\[
(E_x \oplus E_w \oplus E_y)G(\overline{\gamma}, \rho) = \text{id},
\]

\[
\|G(\overline{\gamma}, \rho)\|_{L^p_{\sigma}} \leq C\|\xi\|_{L^p_{\sigma}}.
\]

From these propositions and the Newton’s method, we can conclude that there are constants \(\varepsilon > 0\) and \(\rho\) and \(C\) and a smooth map

\[
\gamma' : \mathcal{M}_{II}^2(\gamma_-, \gamma_+) \cap U(\varepsilon, \rho)(\overline{\gamma}, \overline{\varepsilon}, \overline{\gamma}) \to S, \quad \overline{\gamma} \mapsto (\exp_x(\xi_{\overline{\gamma}, x}), \exp_w(\xi_{\overline{\gamma}, w}), \exp_y(\xi_{\overline{\gamma}, y})),
\]

with \(\|\xi_{\overline{\gamma}, x}\|_{L^p_{\sigma}} \leq C\|\overline{\gamma}_t, x\|_{L^p_{\sigma}}\), and also \(w\) and \(y\). Divide them by the \(R\) actions, then we obtain a map \(\overline{\gamma}' = \mathcal{M}_{II}^2(\gamma_-, \gamma_+) \cap U(\varepsilon, \rho)(\overline{\gamma}, \overline{\varepsilon}, \overline{\gamma}) \to \overline{S}$. 

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From the construction of $\sharp$ and $\sharp'$, if $\rho_0$ is large and $\varepsilon$ is small enough, then $\sharp \circ \sharp'$ and $\sharp' \circ \sharp$ are diffeomorphisms. We finish proving the gluing argument (vii).

Finally we will prove the gluing argument (viii). (Most of the proof is similar to that of (ii), we will show a sketch.) Take a lift of $\mathcal{M}^0_v(\gamma_-, \gamma) \times \mathcal{M}^0_{III}(\gamma, \gamma_+)$ in $\mathcal{M}^2_v(\gamma_-, \gamma) \times \mathcal{M}^2_{III}(\gamma, \gamma_+)$ and consider the orbit of the lift by the similar $\mathbb{R}^2$-action to that of (v). Note that the orbit is diffeomorphic to $\mathcal{M}^0_v(\gamma_-, \gamma) \times \mathcal{M}^0_{III}(\gamma, \gamma_+)$ $\times \mathbb{R}^2$. We choose a compact set $S$ in the orbit, and we will construct a gluing map $\sharp : S \times [\rho_0, \infty) \to \mathcal{M}^2_{III}(\gamma_-, \gamma_+)$. In a similar way to the proof for (ii) we construct a strip $w_\chi(\tau, t)$ for $\chi := (\bar{w}, \bar{\gamma}, \rho) \in S \times [\rho_0, \infty)$ which satisfies the Lagrangian boundary conditions and (III) and decay conditions (2) and (2') and

$$\|\overline{\partial}_t w_\chi\|_{L^p_{0, \sigma}} \leq Ce^{-\rho},$$

where $C$ and $d > 0$ are constants depending only on $S$ and $\rho_0$. Moreover, for $\|\xi\|_{L^1_{1, \sigma}} \leq c$ and $\|\xi'\|_{L^1_{1, \sigma}} \leq c$,

$$\|N_{w_\chi}(\xi) - N_{w_\chi}(\xi')\|_{L^p_{0, \sigma}} \leq C(\|\xi\|_{L^1_{1, \sigma}} + \|\xi'\|_{L^1_{1, \sigma}})\|\xi - \xi'\|_{L^p_{1, \sigma}},$$

where $C$ is a constant depending only on $\|\nabla w_\chi\|_{L^p_{0, \sigma}}$ and $c$. We define similar $e^0_\rho$ and $e^1_\rho$ as in the proof of (ii). Let $W^\perp_{w_\chi}$ be the $L^2$-orthogonal compliment of $W_{w_\chi} := \{e^0_\rho, \beta(\tau - 2\tau_0 - 2T\rho)\frac{d}{da}(a*\bar{\gamma})|_{a=0}(\tau - 2T\rho, t) + e^1_\rho\}$ in $\langle \frac{d}{da}(a*w_\chi)|_{a=0} \rangle \oplus L^p_{1, \sigma}(w^*_X TX, w^*_\chi TL_0, w^*_\chi TL_1)$.

**Proposition 4.33** There exist constants $\rho_0$ and $C$ such that for $\chi \in S \times [\rho_0, \infty)$ and $W^\perp_{w_\chi}$

$$\|\xi\|_{L^1_{1, \sigma}} \leq C\rho^{\frac{1}{2} - \frac{1}{p}}\|E_{w_\chi}\xi\|_{L^p_{0, \sigma}}.$$

**Proposition 4.34** There exist constants $\rho_0$ and $C$ such that for $\chi \in S \times [\rho_0, \infty)$ there exists a map $G_{w_\chi} : L^p_{0, \sigma} \to W^\perp_{w_\chi}$ such that

$$E_{w_\chi}G_{w_\chi} = \text{id},$$

$$\|G_{w_\chi}\xi\|_{L^p_{1, \sigma}} \leq C\rho^{\frac{1}{2} - \frac{1}{p}}\|\xi\|_{L^p_{0, \sigma}}.$$
From these propositions and the Newton’s method, we can conclude that there are constants \( \rho_0 \) and \( C \) and a smooth map

\[
\hat{\sharp} : S \times [\rho_0, \infty) \to \mathcal{M}_{II}^2(\gamma_-, \gamma_+), \; \xi \mapsto \exp_{w_\chi}(\xi_{\chi})
\]

with \( \|\xi_{\chi}\|_{L^p_{1,\sigma}} \leq C\|\bar{\partial}_x w_\chi\|_{L^p_{0,\sigma}} \). Divide them by the \( \mathbf{R} \) actions, we obtain a gluing map \( \hat{\sharp} : \hat{S} \times [\rho_0, \infty) \to \hat{\mathcal{M}}_{II}(\gamma_-, \gamma_+) \).

The next step is to show the surjectivity of \( \hat{\sharp} : S \times [\rho_0, \infty) \to \mathcal{M}_{II}^2(\gamma_-, \gamma_+). \) In a similar way of the proof for (ii), for a map \( \overline{h} \) which satisfies the Lagrangian boundary conditions and (III) and the decay conditions (2) and (2'), we define \( x := x_\rho \) and \( y := y_\rho \) and \( U(\varepsilon, \rho_0)(\overline{u}, \overline{v}) \).

If \( \overline{h} \in \mathcal{M}_{II}^2(\gamma_-, \gamma_+) \cap U(\varepsilon, \rho_0)(\overline{u}, \overline{v}) \), then for a smooth map \( f_x \) there are constants \( C \) and \( C' \)

\[
\|f_x(0)\|_{L^p_{0,\sigma}} \leq Ce^{-\frac{1}{2}\rho},
\]

\[
\|N_x(\xi) - N_x(\xi')\|_{L^p_{0,\sigma}} \leq C'\|\xi\|_{L^p_{1,\sigma}} + \|\xi'\|_{L^p_{1,\sigma}}\|\xi - \xi'\|_{L^p_{1,\sigma}},
\]

where \( \|\xi\|_{L^p_{1,\sigma}} \leq c \) and \( \|\xi'\|_{L^p_{1,\sigma}} \leq c \) and \( C' \) depend on \( c \), and also for \( f_y \). Define \( W_{(\overline{h}, \rho)} := (e^0_\rho, e^1_\rho) \) and \( H_{(\overline{h}, \rho)} := W_{(\overline{h}, \rho)} \oplus L^p_{1,\sigma}(x^*TX, x^*TL_0, x^*TL_1) \oplus L^p_{0,\sigma}(y^*TX, y^*TL_0, y^*TL_1) \), and the \( L^2 \)-inner product on \( H_{(\overline{h}, \rho)} \) in a similar way to that of (v). Let \( W_{(\overline{h}, \rho)} \) be the \( L^2 \)-orthogonal compliment of \( W_{(\overline{h}, \rho)} \) in \( H_{(\overline{h}, \rho)} \).

**Proposition 4.35** There are constants \( \varepsilon_0 > 0 \) and \( \rho_0 \) and \( C \) such that for \( \overline{h} \in \mathcal{M}_{II}^2(\gamma_-, \gamma_+) \cap U(\varepsilon_0, \rho_0)(\overline{u}, \overline{v}) \) and \( \xi := (\xi_x, \xi_y) \in W_{(\overline{h}, \rho)} \)

\[
\|\xi\|_{L^p_{1,\sigma}} \leq C\|E_x \xi_x, E_y \xi_y\|_{L^p_{0,\sigma}}.
\]

**Proposition 4.36** There exist constants \( \varepsilon_0 > 0 \) and \( \rho_0 \) and \( C \) such that for \( \overline{h} \in \mathcal{M}_{II}^2(\gamma_-, \gamma_+) \cap U(\varepsilon_0, \rho_0)(\overline{u}, \overline{v}) \) there exists a map \( G_{(\overline{h}, \rho)} : L^p_{0,\sigma}(x^*TX) \oplus L^p_{0,\sigma}(y^*TX) \to W_{(\overline{h}, \rho)} \) such that

\[
(E_x \oplus E_y)G_{(\overline{h}, \rho)} = \text{id},
\]

\[
\|G_{(\overline{h}, \rho)}\|_{L^p_{1,\sigma}} \leq C\|\xi\|_{L^p_{1,\sigma}}.
\]
From these propositions and the Newton’s method, we can conclude that there are constants $\varepsilon > 0$ and $C$ and a smooth map

$$\hat{\#} : \mathcal{M}^2_{III}(\gamma^-, \gamma^+) \cap U_{(\varepsilon, \rho)}(x, y) \to S, \quad \hat{\#} \mapsto (\exp_x(\xi_{\hat{\#}, x}^y), \exp_y(\xi_{\hat{\#}, y}))$$

with $\|\xi_{\hat{\#}, x}\|_{L^p_{\rho, \sigma}} \leq C\|\partial_{\lambda} x\|_{L^p_{\rho, \sigma}}$ and also $y$. Divide them by the $R$ actions, then we obtain a map $\hat{\#}' : \mathcal{M}^1_{III}(\gamma^-, \gamma^+) \cap \hat{U}_{(\varepsilon, \rho)}(x, y) \to \hat{S}$.

From the construction of $\hat{\#}$ and $\hat{\#}'$, if $\rho_0$ is large and $\varepsilon$ is small enough, then $\hat{\#} \circ \hat{\#}'$ and $\hat{\#}' \circ \hat{\#}$ are diffeomorphisms. We finish proving the gluing argument (viii).

We observe the dimensions of the moduli spaces in gluing arguments. For example, we consider the following case. Take a lift of

$$\mathcal{M}^e_{IIV}(p_-, \gamma^1_-) \times \mathcal{M}^{e_1}_{IIV}(\gamma^1_-, \gamma^1_+) \times \mathcal{M}^{e_2}_{IIV}(\gamma^2_+ \gamma^2_+) \times \cdots$$

in

$$\mathcal{M}^{e_{k+1}}_{IIV}(p_-, \gamma^1_-) \times \mathcal{M}^{e_2}_{IIV}(\gamma^1_- \gamma^1_+) \times \mathcal{M}^{e_3}_{IIV}(\gamma^1_+ \gamma^2_+) \times \cdots$$

and consider the orbit of the lift by the following $\mathbb{R}^{2k}$-action:

for $((0, 0), \ldots, (a_l, 0), \ldots, (0, 0)) \in \mathbb{R}^{2k}$

$$((0, 0) \ldots, (a_l, 0), \ldots, (0, 0)) \cdot (w_0, w_1, \ldots, w_k, u)$$

$$:= (w_0, w_1, \ldots, a \ast w_l, \ldots, w_k, u),$$

and for $((0, 0), \ldots, (0, b_l), \ldots, (0, 0)) \in \mathbb{R}^{2k}$

$$((0, 0), \ldots, (0, b_l), \ldots, (0, 0)) \cdot (w_0, w_1, \ldots, w_k, u)$$

$$:= (w_0, w_1, \ldots, b_l \ast w_l, \ldots, w_k, u).$$

Then the orbit is diffeomorphic to

$$\mathcal{M}^{e_{k+1}}_{IIV}(p_-, \gamma^1_-) \times \mathcal{M}^{e_2}_{IIV}(\gamma^1_- \gamma^1_+) \times \mathcal{M}^{e_3}_{IIV}(\gamma^1_+ \gamma^2_+) \times \cdots$$

$$\times \mathcal{M}^{e_{k+1}}_{IIV}(\gamma^1_+ \gamma^1_+ \gamma^2_+) \times \mathcal{M}^{e_2}_{IIV}(\gamma^1_- \gamma^2_+) \times \mathcal{M}^{e_3}_{IIV}(\gamma^1_+ \gamma^2_+) \times \mathbb{R}^{2k},$$
and we can construct a smooth map
\[ \hat{\gamma} : S \times [\rho_0, \infty) \to M_{k_0 + \cdots + d_k + e_k + 2k}(p_-, p_+), \]
where \( S \) is a compact set in the orbit and \( \rho_0 \) is a constant depending on \( S \). Divide them by the \( \mathbb{R} \)-actions, then we obtain a gluing map
\[ \hat{\gamma} : \hat{S} \to \hat{M}_{k_0 + \cdots + d_k + e_k + 2k}(p_-, p_+). \]
This implies that each element of \( \hat{M}_{k_0 + \cdots + d_k + e_k + 2k}(p_-, p_+) \) contribute dimension 2 to
\[ \sum_{i=0}^{k_i} e_i + \sum_{j=1}^{k_j} d_i + 2k, \]
the dimension of \( M_{k_0 + \cdots + d_k + e_k + 2k}(p_-, p_+) \).

5 Bubbling off phenomena for pseudo-holomorphic curves

We owe most of this section to [11]. Let \( (R, \infty) \times M_+ \) be a convex end of \( X \) such that \( L_0|_{(R, \infty) \times M_+} \) and \( L_1|_{(R, \infty) \times M_+} \) are isomorphic to the products of \( (R, \infty) \) and Legendrian submanifolds. Assume that almost complex structures on the end are of the form \( \overline{T}_t \).

Lemma 5.1 If \( \overline{\pi} \) is a pseudo-holomorphic strip, then the image of \( \overline{\pi} \) is contained in \( X \setminus (R, \infty) \times M_+ \).

Proof. We denote \( \overline{\pi} \) on \( (R, \infty) \times M_+ \) by \( \overline{\pi} := (\alpha, u) \). We can compute
\[ u^*d\lambda_+ = \frac{1}{2}[g_{M_+}(\pi_{\xi+}u_\tau, \pi_{\xi+}u_\tau)^{1/2} + g_{M_+}(\pi_{\xi+}u_t, \pi_{\xi+}u_t)^{1/2}]d\tau dt, \]
where \( u_\tau := u_*(\frac{\partial}{\partial \tau}) \) and \( u_t := u_*(\frac{\partial}{\partial t}) \), and
\[
(-\Delta \alpha)d\tau \wedge dt = d(d\alpha \circ i)
= d(-u^*\lambda_+)
= -u^*d\lambda_+
= -\frac{1}{2}[g_{M_+}(\pi_{\xi+}u_\tau, \pi_{\xi+}u_\tau) + g_{M_+}(\pi_{\xi+}u_t, \pi_{\xi+}u_t)]d\tau \wedge dt,
\]
where \( \Delta = \partial^2/\partial \tau^2 + \partial^2/\partial t^2 \). Then, by the maximum principle, the maximum of \( \alpha \) has to be achieved at a boundary point \( p_0 \in \mathbb{R} \times \partial[0, 1] \). Since \( \overline{\pi}_*(\frac{\partial}{\partial \tau})|_{p_0} \) is tangent to \( L_0 \) or \( L_1 \), \( \overline{\pi}_*(\frac{\partial}{\partial \tau})|_{p_0} \in \xi_+ \). We assume that almost complex structures are of the form \( \overline{T}_t \), hence \( \overline{\alpha}_*(\frac{\partial}{\partial \tau})|_{p_0} \in \xi_+ \). Therefore
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the image of \( \overline{u} \) is tangent to \( \alpha(p_0) \times M_+ \) at the boundary point \( p_0 \), which
contradicts to the strong maximum principle.

From this lemma, it is enough for us to consider only concave ends.

In the following, we assume that our almost complex structures on
the symplectization of \((M, \lambda)\) are of the form \( I_t \). We recall the following
important matter \[11\] and \[12\]. Let \( \Phi := \{ \varphi : \mathbb{R} \to [0,1], \varphi' \geq 0 \} \). To \( \varphi \in \Phi \) we associate
\( \omega_\varphi := d(\varphi \lambda) \) and define
\[
E_\Phi(\mathbf{\pi}) := \sup_{\varphi \in \Phi} \int_{\Sigma} \mathbf{\pi}^* \omega_\varphi,
\]
for \( \mathbf{\pi} : \Sigma \to \mathbb{R} \times M \). Note that, if \( \mathbf{\pi} \) is pseudo-holomorphic, then
\[
\mathbf{\pi}^* \omega_\varphi = \mathbf{\pi}^*(d\varphi \wedge \lambda + \varphi d\lambda) = \frac{1}{2}[(\varphi'(\alpha)\alpha_t^2 + \alpha_i^2 + \lambda(\alpha_t)^2 + \lambda(\alpha_i)^2)\]
\[
+ \varphi(\alpha)(g_M(\pi_{\xi}u_{\tau}, \pi_{\xi}u_{\tau}) + g_M(\pi_{\xi}u_{t}, \pi_{\xi}u_{t})))d\tau \wedge dt.
\]
Hence \( \mathbf{\pi}^* \omega_\varphi \geq 0 \).

Hofer proved the following Lemma 5.2, Proposition 5.3, Theorem 5.4
and Theorem 5.5. Let \( \Sigma \) be \( \mathbb{C} \) or \( \mathbb{R} \times S^1 \) and \( \mathbf{\pi} = (\alpha, u) : \Sigma \to \mathbb{R} \times M \) a
pseudo-holomorphic map.

**Lemma 5.2** If \( E_\Phi(\mathbf{\pi}) < \infty \) and \( \int_{\Sigma} u^* \lambda = 0 \), then \( \mathbf{\pi} \) is a constant map.

**Proposition 5.3** If \( E_\Phi(\mathbf{\pi}) < \infty \), then \( \sup_{\mathbf{z} \in \Sigma} |\nabla \mathbf{\pi}| < \infty \).

**Theorem 5.4** If there is a constant \( c \) such that \( \sup_{\mathbf{z} \in \Sigma} |\nabla \mathbf{\pi}| < c \), then
for \( \beta := (\beta_1, \beta_2) \) there exist constants \( c_\beta \) such that \( |D^\beta \mathbf{\pi}| \leq c_\beta \).

Let \( \phi : \mathbb{R} \times S^1 \to \mathbb{C} \setminus \{0\} \) be a map \( \phi(\tau, t) := e^{2\pi(\tau + it)} \). We use \( \mathbf{\overline{v}} = (\alpha, v) \)
to denote \( \mathbf{\overline{v}} \circ \phi : \mathbb{R} \times S^1 \to \mathbb{R} \times M \).

**Theorem 5.5** Let \( \mathbf{\overline{u}} : \mathbb{C} \to \mathbb{R} \times M \) be a non-constant pseudo-holomorphic
map such that \( E_\Phi(\mathbf{\overline{u}}) < \infty \). Then there is a closed characteristic \( x : \mathbb{R}/TZ \to M \)
and a sequence \( s_k \to \infty \) such that \( x_k(t) := v(s_k, t/T) \) converges to \( x(t) \) in the \( C^\infty \) topology.
Note that the above closed characteristic $x$ is contractible. We can prove completely parallel arguments for $\Sigma = \{z \in C|\mathrm{Im}z \geq 0\}$ to the above results. Let $\Lambda$ be a Legendrian submanifold and $\bar{\pi} = (\alpha, u) : \Sigma \to R \times M$ be a non-constant pseudo-holomorphic map such that $\bar{\pi}(|\{z \in C|\mathrm{Im}z = 0\}|) \subset R \times \Lambda$.

**Lemma 5.6** If $E_u(\bar{\pi}) < \infty$ and $\int_{\Sigma} u^* \lambda = 0$, then $\bar{\pi}$ is a constant map.

**Proposition 5.7** If $E_\phi(\bar{\pi}) < \infty$, then $\sup_{z \in \Sigma} |\nabla \bar{\pi}| < \infty$.

**Theorem 5.8** If there is a constant $c$ such that $\sup_{z \in \Sigma} |\nabla \bar{\pi}| < c$, then for $\beta := (\beta_1, \beta_2)$ there exist constants $c_\beta$ such that $|D^\beta \bar{\pi}| \leq c_\beta$.

Let $\phi : R \times [0, 1] \to \{z \in R|\mathrm{Im}z \geq 0\}$ be a map $\phi(\tau, t) := e^{\pi(\tau + it)}$. We use $\pi = (\alpha, v)$ to denote $\pi := \pi \circ \phi : R \times [0, 1] \to R \times M$.

**Theorem 5.9** Let $\pi : \{z \in C|\mathrm{Im}z \geq 0\} \to R \times M$ be a non-constant pseudo-holomorphic map such that $E_u(\pi) < \infty$. Then there is a Reeb chord $x : [0, T] \to M$ from $\Lambda$ to itself and a sequence $s_k \to \infty$ such that $x_k(t) := v(s_k, t/T)$ converges to $x(t)$ in the $C^\infty$ topology.

Note that the above Reeb chord $x$ is contractible. Moreover, we can similarly prove the following theorem. Let $\Lambda_0$ and $\Lambda_1$ be Legendrian submanifolds such that $\Lambda_0 \cap \Lambda_1 = \emptyset$.

**Theorem 5.10** Let $\pi : \{z \in C|\mathrm{Im}z \geq 0\} \setminus \{0\} \to R \times M$ be a non-constant pseudo-holomorphic map such that $E_u(\pi) < \infty$ and $\pi(|\{z \in C|\mathrm{Im}z = 0, \mathrm{Re}z > 0\}|) \subset R \times \Lambda_0$ and $\pi(|\{z \in C|\mathrm{Im}z = 0, \mathrm{Re}z < 0\}|) \subset R \times \Lambda_1$. Then $\pi$ satisfies one of the following:

- There are Reeb chords $x : [0, T] \to M$ from $\Lambda_0$ to $\Lambda_1$ and $x' : [0, T'] \to M$ from $\Lambda_0$ to $\Lambda_1$ and sequences $s_k \to \infty$ and $s_k' \to -\infty$ such that $x_k(t) := v(s_k, t/T)$ converges to $x(t)$ and $x_k'(t) := v(s_k', t/T')$ converges to $x'(t)$ in the $C^\infty$ topology. ($\lim_{k \to \infty} \alpha(s_k, [0, 1]) = \infty$ and $\lim_{k \to \infty} \alpha(s_k', [0, 1]) = -\infty$.)

- There are Reeb chords $x : [0, T] \to M$ from $\Lambda_1$ to $\Lambda_0$ and $x' : [0, T'] \to M$ from $\Lambda_1$ to $\Lambda_0$ and sequences $s_k \to -\infty$ and $s_k' \to \infty$ such that $x_k(t) := v(s_k, t/T)$ converges to $x(1 - t)$ and $x_k'(t) := v(s_k', t/T')$ converges to $x'(1 - t)$ in the $C^\infty$ topology. ($\lim_{k \to \infty} \alpha(s_k, [0, 1]) = \infty$ and $\lim_{k \to \infty} \alpha(s_k', [0, 1]) = -\infty$.)
• There are Reeb chords \(x: [0, T] \rightarrow M\) from \(\Lambda_1\) to \(\Lambda_0\) and \(x': [0, T'] \rightarrow M\) from \(\Lambda_0\) to \(\Lambda_1\) and sequences \(s_k \rightarrow -\infty\) and \(s'_k \rightarrow \infty\) such that \(x_k(t) := v(s_k, t/T)\) converges to \(x(1 - t)\) and \(x'_k(t) := v(s'_k, t/T')\) converges to \(x'(t)\) in the \(C^\infty\) topology. (\(\lim_{k \rightarrow \infty} \alpha(s_k, [0, 1]) = \infty\) and \(\lim_{k \rightarrow \infty} \alpha(s'_k, [0, 1]) = \infty\).)

(Note that the first case is (V), the second is (V') and the third is (IV).)

From here \((-\infty, R] \times M\) denotes a concave end of \(X\). Let \(\overline{\pi}_i : R \times [0, 1] \rightarrow X\) be a sequence of pseudo-holomorphic strips with the Lagrangian boundary conditions and \(\{p_i\}\) a sequence of \(R \times [0, 1]\) such that \(|\nabla \overline{\pi}_i(p_i)| \rightarrow \infty\). Then there is a sequence \(\varepsilon_i \rightarrow 0\) such that

\[
\varepsilon_i |\nabla \overline{\pi}_i(p_i)| \rightarrow \infty,
\]

\[
|\nabla \overline{\pi}_i(z)| \leq 2|\nabla \overline{\pi}_i(p_i)| \text{ for } |z - p_i| \leq \varepsilon_i,
\]

see [11] and [13]. If \(\bigcup_i \{\overline{\pi}_i(z), |z - p_i| \leq \varepsilon_i\}\) is contained in a compact set, then we can adopt the usual bubbling off phenomena in closed symplectic manifolds. Put \(\overline{\pi}_i := (\alpha_i, u_i)\) on the concave end. Let \(q_i\) be a sequence such that \(|p_i - q_i| \leq \varepsilon_i\) and \(\alpha_i(q_i) \rightarrow -\infty\). Take a sequence \(\delta_i \rightarrow 0\) such that \(\delta_i \leq \varepsilon_i\) and \(\alpha_i(z) < R\) for \(|z - q_i| < \delta_i\) and \(\overline{\pi}_i(\{|z - q_i| = \delta_i\}) \cap \{R\} \neq \emptyset\). Then, from the mean value theorem, there is a point \(q'_i \in \{|z - q_i| \leq \delta_i\}\) such that \(\delta_i|\nabla \overline{\pi}_i(q'_i)| \rightarrow \infty\). By slightly modifying \(\{q_i\}\) we may assume \(\delta_i|\nabla \overline{\pi}_i(q_i)| \rightarrow \infty\). First we consider the case when we can choose a subsequence \(\{q_{i_k}\}\) such that \(\{|z - q_{i_k}| < \delta_{i_k}\} \subset R \times (0, 1)\). (In the following we shall use \(\{q_i\}\) to denote the subsequence.) Put

\[
\overline{\pi}_i(z) = (b_i(z), v_i(z)) := (\alpha_i(q_i + z/|\nabla \overline{\pi}_i(q_i)|) - \alpha_i(q_i), u_i(q_i + z/|\nabla \overline{\pi}_i(q_i)|)),
\]

then

\[
b_i(0) = 0, \ |\nabla \overline{\pi}_i(0)| = 1, \ |\nabla \overline{\pi}_i(z)| \leq 2 \text{ for } |z| \leq \delta_i |\nabla \overline{\pi}_i(q_i)|.
\]

Then we can conclude that there is a pseudo-holomorphic map \(\overline{\nu} = (b, v) : C \rightarrow R \times M\) and a subsequence of \(\{\overline{\pi}_i\}\) which converges to \(\overline{\nu}\) in the \(C^\infty_{loc}\) topology. Second we consider the case of \(\Im q_i \rightarrow 0\) and \(\{|z - q_i| < \delta_i\} \cap \{R \times \{0\}\} \neq \emptyset\), or \(\Im q_i \rightarrow 1\) and \(\{|z - q_i| < \delta_i\} \cap \{R \times \{1\}\} \neq \emptyset\). Then we can similarly obtain a pseudo-holomorphic map \(\overline{\nu} : \{z \in C | \Im z \geq 0\}

\]
Let $\pi_i : \mathbb{R} \times [0,1] \to X$ be a sequence of pseudo-holomorphic strips with the Lagrangian boundary conditions and \{q_i\} a sequence of $\mathbb{R} \times [0,1]$ such that $\alpha_i(q_i) \to -\infty$, where $\pi_i := (\alpha, u_i)$ on a concave end. Take a sequence $\delta_i$ such that $|\alpha_i(z)| < R$ for $|z - q_i| < \delta_i$ and $\pi_i(\{ |z - q_i| = \delta_i \}) \cap \{ R \times M \} \neq \emptyset$. If $\delta_i$ are bounded, then, from the mean value theorem, there are points $p_i$ such that $|q_i - p_i| < \delta_i$ and $|\nabla \pi_i(p_i)| \to \infty$. In this case we can return to the previous one. If there is a subsequence $\delta_{i_k} \to \infty$, then we put

$$w_k := (\alpha_{i_k}(z - \text{Re}q_{i_k}) - \alpha(q_k), u_{i_k}(z - \text{Re}q_{i_k})).$$

If there are points $r_k$ such that $|\nabla w_k(r_k)| \to \infty$, then we can also return to the previous bubbling phenomena. Hence we assume that the differential of $w_k$ are bounded. Then there is a pseudo-holomorphic strip $\pi : \mathbb{R} \times [0,1] \to \mathbb{R} \times M$ and a subsequence of $\{w_k\}$ which converges to $\pi$ in the $C^\infty_{loc}$ topology.

**Proposition 5.11** If $\pi_i$ are of the type (I) or (II) or (II') or (III) or (III'), and $\int_{\mathbb{R} \times [0,1]} \pi_i^* \omega \leq C$, then $E_{\varphi}(\pi) \leq e^{-RC}$.

**Proof.** Notation: $K_{[a,b]} := [a,b] \times M$, $M_d := d \times M$ and $C_d := \pi^{-1}((-\infty, d] \times M)$. Let $d_1 \leq d_2 \leq R$ and $\varphi = \varphi(\alpha) : \mathbb{R} \to [0,1]$ a function such that $\varphi' \geq 0$. Since $\partial^* \pi^{-1}(K_{[d_1,d_2]}) = \pi^{-1}(M_{d_1}) \cup \pi^{-1}(M_{d_2}) \cup \pi^{-1}([d_1,d_2] \times \Lambda)$, where $\Lambda$ is Legendrian,

\[
\int_{C_{d_2}} \overline{\pi}^* d(\varphi \lambda) - \int_{C_{d_1}} \overline{\pi}^* d(\varphi \lambda) = \int_{\pi^{-1}(K_{[d_1,d_2]})} \overline{\pi}^* d(\varphi \lambda) = \int_{\partial^* \pi^{-1}(K_{[d_1,d_2]})} \overline{\pi}^* (\varphi \lambda) = \int_{\pi^{-1}(M_{d_2})} \overline{\pi}^* (\varphi \lambda) - \int_{\pi^{-1}(M_{d_1})} \overline{\pi}^* (\varphi \lambda) = \varphi(d_2) e^{-d_2} \int_{\pi^{-1}(M_{d_2})} \overline{\pi}^* (e^{\alpha} \lambda) - \varphi(d_1) e^{-d_1} \int_{\pi^{-1}(M_{d_1})} \overline{\pi}^* (e^{\alpha} \lambda) = \varphi(d_2) e^{-d_2} \int_{C_{d_2}} \pi^* \omega - \varphi(d_1) e^{-d_1} \int_{C_{d_1}} \pi^* \omega.
\]
If \( \bar{u} \) satisfies the exponential decay conditions, then
\[
\lim_{d_1 \to -\infty} \int_{C_{d_1}} \bar{u}^* d(\varphi \lambda) = \lim_{d_1 \to -\infty} T(\varphi(d_1) - \varphi(-\infty)) = 0.
\]
Hence
\[
\int_{C_{d_2}} \bar{u}^* d(\varphi \lambda) = \varphi(d_2) e^{-d_2} \int_{C_{d_2}} \bar{u}^* \omega - \lim_{d_1 \to -\infty} \varphi(d_1) e^{-d_1} \int_{C_{d_1}} \bar{u}^* \omega \leq e^{-R} C.
\]
For any \( \varepsilon > 0 \), there are \( \varphi \in \Phi \) such that
\[
E_\Phi(\bar{v}) - \varepsilon \leq \int_{\Sigma} \bar{u}^* d(\varphi \lambda).
\]
For any compact set \( K \subset \Sigma \), there are \( \bar{u}_i \) such that
\[
\left| \int_K \bar{u}_i^* d(\varphi \lambda) - \int_K (\alpha_i \circ \phi_i - \alpha_i(q_i), u_i \circ \phi_i)^* d(\varphi \lambda) \right| \leq \varepsilon,
\]
where \( \phi_i : K \to C_R \) is a suitable map. Since \( \bar{u}_i^* d(\varphi \lambda) \geq 0 \),
\[
\int_K (\alpha_i \circ \phi_i - \alpha_i(q_i), u_i \circ \phi_i)^* d(\varphi \lambda) = \int_{\phi_i(K)} \bar{u}_i^* d(\varphi_i \lambda) \leq \int_{C_R} \bar{u}_i^* d(\varphi_i \lambda),
\]
where \( \varphi_i(\alpha) := \varphi(\alpha - \alpha(q_i)) \). Hence
\[
E_\Phi(\bar{v}) \leq e^{-R} C.
\]

From this proposition we can apply Theorem 5.5 to \( \bar{v} \) which is the limit of the sequence \( \{\bar{v}_i\} \) and also Theorem 5.10 to \( \bar{v} \), the limit of the sequence \( \{\bar{v}_k\} \). Moreover, we can conclude the exponential decay conditions for \( \bar{v} \).

**Proposition 5.12** If there is an open neighborhood \( U \subset S^1 \times \mathbb{R}^{2n} \) of \( S^1 \times \{0\} \) and an open neighborhood \( V \subset M \) of a closed characteristic \( x \) with the minimal period \( \tau_0 \) and a diffeomorphism \( \varphi : U \to V \) mapping \( S^1 \times \{0\} \) to \( x \) such that \( \varphi^* \lambda = f \lambda_0 \) with \( \lambda_0 := d\theta + \sum_{n=1}^n x_n dy_n \) and a positive smooth function \( f : U \to \mathbb{R} \) satisfying \( f(\theta, 0, 0) = \tau_0 \) and \( df(\theta, 0, 0) = 0 \) for all \( \theta \in S^1 \), then \( x_k \) as in Theorem 5.10 converges to \( x \) with the exponential decay conditions.

Similarly we can prove
Proposition 5.13 Under Assumption 3.3, $x_k$ and $x'_k$ as in Theorem 5.9 and 5.10 converge to $x$ and $x'$ with the exponential decay conditions.

Under Assumption 3.2 there are no point $q_i$ such that $|\nabla u_i(q_i)| \to \infty$ and $\alpha_i(q_i) \to -\infty$, and under Assumption 3.5 $v$, the limit of $w_k$, is of the type (IV) or the trivial ones of the type (V) and (V'). Let $\{q_i\} \subset \mathbb{R} \times [0,1] \to X$ be a sequence of pseudo-holomorphic strips of $\mathcal{M}^2_I(p_-, p_+)$ and let $\{q_i^s\} \subset \mathbb{R} \times [0,1]$ such that $\alpha_i(q_i^s) \to -\infty$. Take a sequence $\delta_i^s$ such that $|\alpha_i(z)| < R$ for $|z - q_i^s| < \delta_i^s$ and $\{|z - q_i^s| < \delta_i^s\} \cap \{|z - q_i^{s_2}| < \delta_i^{s_2}\} = \emptyset$ for $s_1 \neq s_2$. We assume $\delta_i^s \to \infty$, and put

$$\bar{w}_i^s := (\alpha_i(z - \text{Re} q_i^s) - \alpha(q_i^s), u_i(z - \text{Re} q_i^s)).$$

Then there are pseudo-holomorphic strips $\bar{\varphi} : \mathbb{R} \times [0,1] \to \mathbb{R} \times M$ of $\mathcal{M}^2_{IV}(\gamma_-, \gamma_+^s)$ or the trivial ones of (V) or (V') and subsequences of $\{\bar{w}_i^s\}_{i=1,2,...}$ which converges to $\bar{\varphi}$ in the $C^\infty_{\text{loc}}$ topology. Assume that $\bar{\varphi}^1$ and $\bar{\varphi}^2$ are not the trivial ones, i.e., of the type (IV). Then we obtain $\bar{x} \in \bar{\mathcal{M}}_{IV}^{d_x}(p_-, \gamma_-^s)$, $\bar{y} \in \bar{\mathcal{M}}_{IV}^{d_y}(\gamma_+^1, \gamma_-^2)$ and $\bar{w} \in \bar{\mathcal{M}}_{IV}^{d_w}(\gamma_+^2, p_+)$ such that we can glue them with $\bar{\varphi}^1$ and $\bar{\varphi}^2$ to reconstruct $\bar{w}_i$. From Assumption 3.2 we can calculate $d_x + d_1 + d_y + d_2 + d_w = 2$, which contradicts to $d_x, d_y, d_w \geq 0$ and $d_1, d_2 \geq 2$. Then there is at most one $\bar{\varphi}$ of the type (IV) which appears at the limit of the sequence $\{\bar{w}_i\}$. Similarly also for sequences of pseudo-holomorphic strips of $\mathcal{M}^2_{II}(\gamma_-, p_+)$ or $\mathcal{M}^2_{II}(p_-, \gamma_+)$ or $\mathcal{M}^2_{III}(\gamma_-, \gamma_+)$.

Finally we completely finish proving Theorem 3.4.

A Newton’s method

In this appendix we adopt Newton’s method with proof [5] and [7].

Proposition A.1 (Newton’s method) Let $(E, \| \cdot \|_E)$ and $(F, \| \cdot \|_F)$ be Banach spaces and $f : E \to F$ a smooth map. We denote the Taylor expansion of $f$ by

$$f(\xi) = f(0) + Df(0)\xi + N(\xi),$$

where $Df(0) : E \to F$ is the derivative of $f$ at $0$ and $N(\xi)$ is the remainder term.

Newton’s method aims to find the root of $f(x) = 0$ by iteratively applying the following update rule:

$$x_{n+1} = x_n - Df(x_n)^{-1} f(x_n).$$

The method is known to converge quadratically if the initial guess is sufficiently close to the root and $f$ is continuously differentiable and $Df(x_n)$ is invertible in a neighborhood of the root.
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and assume that $Df(0)$ has a right inverse $G : F \to E$, $Df(0)G = \text{id}_F$, such that
\begin{equation}
\|GN(\xi) - GN(\zeta)\|_E \leq C_N(\|\xi\|_E + \|\zeta\|_E)\|\xi - \zeta\|_E,
\end{equation}
for some constant $C_N$. Then the zero-set of $f$ in $B := \{\xi \in E|\|\xi\|_E < (4C_N)^{-1}\}$ is a smooth manifold, whose dimension is equal to that of $\text{Ker}Df(0)$. In fact, if we put
\begin{align*}
K &= \{\xi \in \text{Ker}Df(0)|\|\xi\|_E < (4C_N)^{-1}\}, \\
K^\perp &= \{\xi \in GF|\|\xi\|_E < (4C_N)^{-1}\},
\end{align*}
then there is a smooth map $\phi : K \to K^\perp$ such that $f(\xi + \phi(\xi)) = 0$, and all zeroes of $f$ in $B$ are of the form $\xi + \phi(\xi)$. Moreover, we have the estimate
\begin{equation}
\|\phi(\xi)\|_E \leq 2\|Gf(0)\|_E.
\end{equation}

On the other hand, if we have the inequality
\begin{equation}
\|Gf(0)\|_E \leq (8C_N)^{-1},
\end{equation}
then there must exist the zeros of $f$ in $B$.

Proof. If we put $v := Df(0)u$ for $u \in E$, then $Df(0)Gv = v = Df(0)u$, and $Df(0)(Gv - u) = 0$. Moreover, if we have $Gb \in \text{Ker}Df(0) \cap GF$, then $0 = Df(0)Gb = b$, and $\text{Ker}Df(0) \cap GF = \{0\}$. Hence we obtain the direct decomposition $E = \text{Ker}Df(0) \oplus GF$. Denote $E_1 := \text{Ker}Df(0)$, $E_2 := GF$ and $F := G \circ f : E_1 \oplus E_2 \to E_2$. For the natural projection $\pi : E_1 \oplus E_2 \to E_2$, we obtain $DF(0) = GDf(0) = \pi$, and the Taylor expansion of $F$ is
\begin{equation*}
F(\xi) = F(0) + \pi(\xi) + N(\xi),
\end{equation*}
where $N := GN$. The inequalities (17) (19) become
\begin{align}
\|N(\xi) - N(\zeta)\|_E &\leq C_N(\|\xi\|_E + \|\zeta\|_E)\|\xi - \zeta\|_E, \\
\|F(0)\|_E &\leq (8C_N)^{-1}.
\end{align}

Because $G : F \to GF$ is an isomorphism, we will prove the proposition for $F : E \to E$ instead of $f : E \to F$. 

For simplicity we denote the norm $\| \cdot \|_E$ by $\| \cdot \|$. From (20) we have

$$\| D\mathcal{N}(\xi)\zeta \| \leq \| \mathcal{N}(\xi + \zeta) - \mathcal{N}(\xi) \| + o(\zeta)\|\zeta\|$$

$$\leq C_N(\|\xi + \zeta\| + \|\zeta\|)\|\zeta\| + o(\zeta)\|\zeta\|,$$

where $o(\zeta)$ is a function such that $\lim_{\zeta \to 0} o(\zeta) = 0$, and then

$$\| D\mathcal{N}(\xi)(\varepsilon \zeta/\|\varepsilon \zeta\|) \| \leq C_N(\|\xi\| + \|\xi + \varepsilon \zeta\|) + o(\varepsilon \zeta).$$

The limit of the above estimate as $\varepsilon \to 0$ is

$$\| D\mathcal{N}(\xi)(\zeta/\|\zeta\|) \| \leq 2C_N\|\xi\|,$$

and hence the operator norm of $D\mathcal{N}(\xi)$ satisfies

$$\| D\mathcal{N}(\xi) \| < 1/2$$

for $\xi \in B$. Moreover, from the differential of the Taylor expansion of $\mathcal{F}$ we obtain

$$D\mathcal{F}(\xi) = \pi + D\mathcal{N}(\xi),$$

and then

$$D\mathcal{F}(\xi)|_{E_2} = \text{id}_{E_2} + D\mathcal{N}(\xi)|_{E_2}.$$ 

Hence, the restriction $D\mathcal{F}(\xi)|_{E_2} : E_2 \to E_2$ is an isomorphism for $\xi \in B$.

Now we use:

**Theorem A.2 (Implicit function theorem)** Let $\mathcal{F} : E_1 \oplus E_2 \to E_2$ be a smooth map, where $E_1$ and $E_2$ are Banach spaces, such that the differential $D\mathcal{F}(\xi_1, \xi_2)|_{E_2} : E_2 \to E_2$ is an isomorphism at zeros of $\mathcal{F}$, i.e., $\mathcal{F}(\xi_1, \xi_2) = 0$. Then there is a neighborhood $W_{\xi_1} \subset E_1$ of $\xi_1$ and a smooth map $\phi_{\xi_1} : W_{\xi_1} \to E_2$ such that, for any $\zeta \in W_{\xi_1}$, $\mathcal{F}(\zeta, \phi_{\xi_1}(\zeta)) = 0$.

Let $\xi := (\xi_1, \xi_2)$ and $\xi' := (\xi_1, \xi'_2)$ be in $B$ such that $\mathcal{F}(\xi) = \mathcal{F}(\xi') = 0$. From the Taylor expansion of $\mathcal{F}$

$$\mathcal{F}(0) + \xi_2 + \mathcal{N}(\xi) = \mathcal{F}(0) + \xi'_2 + \mathcal{N}(\xi') = 0,$$

and then

$$\|\xi_2 - \xi'_2\| = \|\mathcal{N}(\xi) - \mathcal{N}(\xi')\|$$

$$\leq C_N(\|\xi\| + \|\xi'\|)\|\xi - \xi'\|$$

$$\leq \frac{1}{2}\|\xi_2 - \xi'_2\|.$$
Hence we can write the zeros of $F$ in $B$ of the form $(\zeta, \phi(\zeta)), \zeta \in E_1$, and then $\{\xi \in B|F(\xi) = 0\}$ is a smooth manifold whose tangent spaces are isomorphic to $E_1$.

From the equations

$$F(\xi_1 + \phi(\xi_1)) = F(0) + \phi(\xi_1) + \mathcal{N}(\xi_1 + \phi(\xi_1)) = 0$$

$$F(\xi_1) = F(0) + \mathcal{N}(\xi_1)$$

for $\xi_1 \in E_1$, we obtain

$$\phi(\xi_1) = -F(\xi_1) - \mathcal{N}(\xi_1 + \phi(\xi_1)) + \mathcal{N}(\xi_1),$$

and then

$$\|\phi(\xi_1)\| \leq \|F(\xi_1)\| + \|\mathcal{N}(\xi_1 + \phi(\xi_1)) - \mathcal{N}(\xi_1)\|$$

$$\leq \|F(\xi_1)\| + C_N(\|\xi_1 + \phi(\xi_1)\| + \|\xi_1\|)\|\phi(\xi_1)\|$$

$$\leq \|F(\xi_1)\| + \frac{1}{2}\|\phi(\xi_1)\|.$$
Theorem A.3 (Fixed points theorem) Let \((X, d)\) be a complete metric space, \(g : X \to X\) a map such that
\[d(g(x), g(y)) \leq Cd(x, y), \ x, y \in X\]
for some constants \(C < 1\). Then there uniquely exists the point \(y_0\) such that \(g(y_0) = y_0\).

We can find the fixed point \(y_0 \in H\) with respect to a map \(g|_H : H \to H\). Then, from the Taylor expansion of \(\mathcal{F}\) and the definition of \(g\),
\[\mathcal{F}(y_0) = \mathcal{F}(0) + \pi(y_0) + \mathcal{N}(y_0) = 0,\]
i.e., \(y_0\) is a zeros of \(\mathcal{F}\). Note that \(y_0 = \phi(0)\).

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