We present a generalization of previous results regarding the stability under gravitational perturbations of nakedly singular super extreme Kerr spacetime and Kerr black hole interior beyond the Cauchy horizon. To do so we study solutions to the radial and angular Teukolsky’s equations with different spin weights, particularly $s = \pm 1$ representing electromagnetic perturbations, $s = \pm 1/2$ representing a perturbation by a Dirac field and $s = 0$ representing perturbations by a scalar field. By analyzing the properties of radial and angular eigenvalues we prove the existence of an infinite family of unstable modes.

1. Introduction

The most general stationary black hole in electrovacuum is the Kerr-Newman one. In Boyer-Lindquist coordinates its line element reads:

$$ds^2 = -\frac{(\Delta - a^2 \sin^2 \theta)}{\Sigma} dt^2 - 2a \sin^2 \theta \frac{(r^2 + a^2 - \Delta)}{\Sigma} dt d\phi$$

$$+ \left[ \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma} \right] \sin^2 \theta d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2,$$

where $\Sigma = r^2 + a^2 \cos^2 \theta$ and $\Delta = r^2 - 2Mr + a^2 + Q^2$.

The parameters $M$, $a$ and $Q$ correspond to the mass, the angular momentum per unit mass and the electric charge respectively.

Depending on the nature of the roots of the function $\Delta$, the above solution represents a black hole or a naked singularity.

The stability against gravitational perturbations of spacetimes presenting naked singularities were studied: for negative mass Schwarzschild, for super charged Reissner-Nordström and for super extreme Kerr. The results obtained...
in these works shed some light into Penrose’s Cosmic Censorship Conjecture (CCC) showing systematically the existence of unstable modes. Both the Reissner-Nordström and the Kerr black holes have a two horizon structure through which spacetime can be extended, in this process new regions isometric to

\[ I: r > r_o, \]

\[ II: r_i < r < r_o \]

and

\[ III: r < r_i \]

arise ad infinitum. The inner and outer horizons are located at

\[ r_i = M - \sqrt{M^2 - a^2 - Q^2} \]

and

\[ r_o = M + \sqrt{M^2 - a^2 - Q^2}. \]

In this work we will focus in Kerr’s spacetime because it is the most relevant from an astrophysical point of view.

The main result of this work is the generalization of the results on gravitational perturbations in to scalar, spinor and Maxwell fields, shown to be unstable in the Kerr naked singularity and black hole interior.

2. Teukolsky Equations

Teukolsky’s equation uses a master variable \( \Phi_s \) to describe scalar, spinor, Maxwell and linear gravitational fields, for which \(|s| = 0, 1/2, 1 \) and \( 2 \) respectively. Solutions are separable,

\[ \Phi_s = R_{\omega,m,s}(r) S_{\omega,s}(\theta) \exp(i m \phi) \exp(-i \omega t), \]

and the equations reduce to a coupled system for \( S \) and \( R \):

\[ \Delta \frac{d^2 R}{dr^2} + (s + 1) \frac{d}{dr} R = F_r(r, a, \omega, m, s) R = 0. \]

This formalism was used to establish the modal stability under gravitational perturbation of region I of a Kerr black hole. In what follows we will prove that region III of an \( a < M \) Kerr spacetime, as well as the nakedly singular \( a > M \) Kerr spacetimes, are unstable under gravitational perturbations and all kinds of linear fields, as described above for different \( s \) values, generalizing our previous work.

2.1. Spin Weighted Spheroidal Harmonics (SWSH’s)

In this section we gather important aspects of the SWSH’s eigenvalues, these hold for every \( a \) and \( M \). In the axial case \( m = 0 \) the \( E \) eigenvalue for large purely imaginary \( \omega \) behaves as \((\ell = 0, 1, 2...)\):

\[ E_{\ell,s}(a \omega)|_{a \omega = ik} = (2\ell + 1) k + \mathcal{O}(k^0), \quad \text{as } k \to \infty \]

and in the \( a \omega = 0 \) limit, independently of the value of \( s \) as:

\[ E_{\ell,s}(a \omega)|_{a \omega = 0} = (\ell + |s|)(\ell + |s| + 1). \]
2.2. Teukolsky Radial Equation

As was shown for the gravitational case, equation (4) can generally be cast into a Schrödinger-like form, so (4) can be written as:

\[ H \psi := -\psi'' + V \psi = -E \psi. \] (7)

with primes denoting derivatives with respect to the radial variable \( r^* \) defined in 5.

The form of the \( E \) independent potential \( V \) for axial perturbations (\( m = 0 \)) with purely imaginary frequencies (\( \omega = ik/a \)), those relevant to unstable modes, is:

\[ V(r, k) := k^2 V_2(r, k) + k V_1(r, k) + V_0(r, k) \] (8)

Understanding the behaviour of the potential for different values of \( k \) is important when describing the nature of the eigenvalues of \( H \).

\( V_2(r, k) \) is independent of the value of the spin weight \( s \), so for large values of \( k \) the potential behaves like \( V_2(r, k) \) and so would the eigenvalues of \( H \).

Particularly different is the situation for small values of \( k \): for \( k \geq 0 \), \( V \) is bounded from below, but its minimum is not a continuous function of \( k \) at \( k = 0 \). This fact is a consequence of the appearance of a second minimum (for \( |s| = \pm 1, \pm 2 \)) in the function \( V(r, k \neq 0) \) that moves towards \( \infty \) as \( k \to 0^+ \) that is not present in the simpler \( V(r, k = 0) := V_0(r) \) function. So we have that:

\[ \min \{V(r, k = 0), r \in \mathbb{R}\} = 1/2 - s^2, \quad \lim_{k \to 0^+} \min \{V(r, k), r \in \mathbb{R}\} = 1/4 - s^2, \] (9)

3. Results

3.1. Unstable Modes of the Kerr Naked Singularity

The operator \( H \) in (7) is self-adjoint in the Hilbert space of square integrable functions of \( r^* \) with hermitian product \( \langle \alpha | \beta \rangle := \int \alpha \beta^* dr^* \). Given that \( V \) is smooth, bounded from below, and \( V \sim \left( \frac{Mkr^*}{a} \right)^2, |r^*| \to \infty \), the spectrum of the self-adjoint operator \( H \) is fully discrete and has a lower bound.

A careful analysis indicates that the square integrable eigenfunctions of \( H \) behave as

\[ \psi \sim \begin{cases} e^{-\frac{k}{2k_c} \left( \frac{M}{r} \right)^{\frac{3}{2}+s+2\frac{k}{k_c}} \left( 1 + O(M/r) \right) } & , r \to \infty \\ e^{\frac{k}{2} \left( \frac{M}{r} \right)^{\frac{3}{2}+s-2\frac{k}{k_c}} \left( 1 + O(M/r) \right) } & , r \to -\infty. \end{cases} \] (11)

To obtain information about the fundamental energy of the radial Hamiltonian (7), we need to analyze its potential (8). One can show that there is an interval \( r_1(M) < r < r_2(M) \) where \( V_2 \) is negative. Using a smooth test function supported in this interval we can then show that if \( -\epsilon_o(k) \) is the lowest eigenvalue of \( H \) then

\[ \epsilon_o(k = 0^+) < -\frac{1}{4} + s^2, \quad \epsilon_o(k) > \frac{1}{2} |\langle \psi | V_2 \psi \rangle| k^2, \quad k > k_c. \] (12)
From \([5, 6]\) and \([12]\) it is clear that, for any \(\ell > \ell_o\) (\(\ell_o\) a function of \(s\)), the curves \(\epsilon_o(k)\) and \(E_\ell(a\omega)\) intersect at some \(k_\ell > 0\).

This implies that there is a common eigenvalue \(E = E_\ell(a\omega)|_{a\omega = ik_\ell} = \epsilon_o(k_\ell)\) for the radial and angular Teukolsky equations and (infinitely many!) unstable perturbations \(\Phi_s(t,r,\theta) = S_{(\ell,m=0,k_o)}(\theta) \Delta^{-\frac{d-2}{2}} \psi_o(k_\ell)(r) \exp(\frac{k_\ell t}{a})\) with \(\psi_o(k_\ell)(r)\) behaving as in \([11]\).

### 3.2. Unstable Modes for Region III of a Kerr Black Hole.

The calculations above can be adapted to deal with perturbations in the interior region \(r < r_i\) of a Kerr black hole. The extreme and sub-extreme cases require separate treatments, we present the results for the extreme case (the subtleties in the subextreme case can be seen in \([5]\)). Using a similar approach to that used in the superextreme case we get that:

\[
V \sim \begin{cases} 
4k^2 \exp(2r^*) & , r^* \to \infty \\
 k^2 \exp(-2r^*) & , r^* \to -\infty,
\end{cases}
\]

then the spectrum of the self-adjoint operator \(H\) is again fully discrete and has a lower bound. The eigenfunctions behave as

\[
\psi \sim \begin{cases} 
\left(\frac{M}{M-r}\right)^{2k-s-\frac{d}{2}} \exp \left[-2k \left(\frac{M}{M-r}\right)\right] (1 + \mathcal{O}(\frac{M-r}{r})) & , r \to M^- \\
\left(\frac{M}{M-r}\right)^{k-2k-s} \exp \left[\frac{rk}{M}\right] (1 + \mathcal{O}(\frac{M}{r})) & , r \to -\infty
\end{cases}
\]

The argument of instability in the super extreme case goes through without modifications and thus can be used again to obtain the bound \([12]\).

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