APPROXIMATING THE LIKELIHOOD FUNCTION OF CMB EXPERIMENTS

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ABSTRACT We discuss the problem of constraining cosmological parameters with cosmic microwave background band–power estimates. Because these latter are variances, they do not have gaussian distribution functions and, hence, the standard $\chi^2$–approach is not strictly applicable. A general purpose approximation to experimental band–power likelihood functions is proposed, which requires only limited experimental details. Comparison with the full likelihood function calculated for several experiments shows that the approximation works well.

KEYWORDS:

1. INTRODUCTION

Current cosmic microwave background (CMB) observations are already capable of constraining cosmological parameters (Bond & Jaffe 1996; Lineweaver et al. 1997; Hancock et al. 1998; Bartlett et al. 1998a). The COBE data alone have for some time now been used to determine the amplitude and slope of the power spectrum of density perturbations, but newer data at smaller angular scales are now reaching the so–called “Doppler peaks”. This is the key region which will decide the fate of models such as inflation, cosmic defects or other contenters; this is the region which will, within the context a particular model, impose strong constraints on the fundamental cosmological parameters, such as the density parameter, $\Omega$.

The present observational situation is summarized in Figure 1 as a set of band–power estimates over multipole order $l$, the most common method of reporting CMB results. Our discussion here will focus on statistical methods of constraining parameters within the context of inflation–like models, which predict gaussian temperature fluctuations. It is important to remark at this stage that the construction of Figure 1 already supposes gaussianity, because the given band–power estimates assume gaussian fluctuations (note that this includes the noise). This will become more clear in the following. If the sky fluctuations where ever shown to be non–gaussian (or an experiment contained an important component of non–gaussian noise), then the band–power estimates would need to be recalculated, potentially changing Figure 1.

A common approach to parameter estimation applies the $\chi^2$–statistic to the band–power estimates of Figure 1. This technique, however, is not strictly applicable in this context because the points in Figure 1 are not gaussian distributed. This
is true even if the underlying sky fluctuations, the pixel values (including noise), are in fact gaussian random variables. Power estimates represent the variance of the temperature fluctuations, and an estimate of the variance of gaussian variables is not itself gaussian. The $\chi^2$–statistic, which assumes gaussianity, is therefore not the correct approach.

A rigorous analysis of the data in Figure 1 requires that the correct likelihood function be calculated for each experiment. Even with the present data set, this is a time consuming task, due to the variety of different experimental set–ups represented, and for the next generation experiments, with tens of thousands of pixels, it becomes computationally demanding in the extreme; for example, analysis of BOOMERANG’s North American flight (30,000 pixels) requires $\sim 10$ hours on a Cray T3E (Borril 1998). It is therefore important to find useful approximations to experimental likelihood functions (Bond et al. 1998; Wandelt et al. 1998). Here, we describe our efforts to derive a reasonable approximation. Some preliminary results from its application to current data can be found in Bartlett et al. (1998b).

2 APPROXIMATE LIKELIHOOD FUNCTION

For gaussian anisotropies (gaussian noise is assumed throughout), we may write the likelihood function for a set of parameters, represented by a vector $\Theta$, once
given the data, a set of pixel values arranged in a vector $\vec{d}$:
\[
L(\vec{\Theta}) \equiv \text{Prob}(\vec{d}|\vec{\Theta}) = \frac{1}{(2\pi)^{N\text{pix}/2}|C|^{1/2}}e^{-\frac{1}{2}\vec{d}\cdot C^{-1}\cdot \vec{d}}
\] (1)
where $C$ is the covariance matrix
\[
C_{ij} = \langle d_i d_j \rangle_{\text{ens}} = T_{ij} + N_{ij}
\] (2)
The average is understood to be over the theoretical ensemble of all possible anisotropy patterns realizable with the given parameter set $\vec{\Theta}$, of which the actual data set is but one realization. The covariance has two contributions, one intrinsic to the sky fluctuations, $T(\vec{\Theta})$, a function of the parameter vector, and the other due to the noise, $N$. For a data vector consisting of simple pixel values, from a map of the sky, we further have
\[
T_{ij} = \frac{1}{4\pi} \sum_l (2l + 1) C_l |B_l|^2 P_l(\cos \theta_{ij})
\] (3)
where, as usual, the power spectrum is the ensemble of $C_l$, $B_l$ describes the experimental beam (assumed spherically symmetric), $P_l$ is the Legendre polynomial of order $l$ and $\theta_{ij}$ is the angle separating pixels $i$ and $j$. The likelihood is a function of the parameters $\vec{\Theta}$, which may be either the $C_l$ or the world–model constants, such as $\Omega$, etc... In the latter situation, the parameter dependence enters the likelihood function through relations of the kind $C_l[\vec{\Theta}]$, specified by the adopted theory, e.g., inflation. In either case, the best estimates for the parameter values are found by maximizing the likelihood function; confidence intervals can be defined by treating the likelihood function as a probability distribution in $\vec{\Theta}$ (with non–uniform prior, if desired).

We have implicitly been working with simple pixel values, but it should be emphasized that all follows through for temperature differences, or any linear combination of sky temperatures, for these simply transform the covariance matrix $C$. Only Eq. (3) is altered; and it should be noted that in the more general case, $T_{ij}$ may not be expandable in Legendre polynomials because, e.g., a difference measurement breaks spherical symmetry (i.e., $T$ may depend on the orientation of the two difference pairs in a single difference scheme).

Experimental results are usually given in terms of band–powers, estimates of the variance of the temperature fluctuations over a finite range of $l$. These may either be defined by the differencing scheme employed during observation, or by applying a linear transformation to the pixel values of a map. For an ideal experiment with full sky coverage, the band–powers would simply be the individual $C_l$; however, limited sky coverage results in less resolution in $l$–space, permitting estimates only within finite bandwidths. A useful example is the single difference scheme, where one measures $\Delta \equiv d_1 - d_2$ with $d_1$ and $d_2$ separated by an angle $\theta$ on the sky. In this case, the diagonal elements of the covariance matrix may be written as
\[
\langle \Delta^2 \rangle_{\text{ens}} = \frac{1}{4\pi} \sum_l (2l + 1) C_l W_l
\] (4)
where the window function, \( W_l = 2|B_l|^2[1 - P_l(\cos \theta)] \), identifies the range of \( l \) to which the difference is sensitive. The common approach is to quote a flat band–power estimate, \( \delta T_f \), defined by \((\delta T_f)^2 = l(l+1)C_l/(2\pi)\), which leads to

\[
<\Delta^2>_{ens} = \frac{(\delta T_f)^2}{2} \sum_l \frac{(2l+1)}{l(l+1)} W_l
\]

The band–power, \( \delta T_f \), is then treated as the parameter to be estimated from the full likelihood function. It is this procedure which leads to the points and uncertainties shown in Figure 1. We see that it has indeed been constructed under the assumption of gaussian fluctuations, as mentioned in the Introduction. Notice also that because it contains all relevant information, the likelihood function includes the uncertainty on the power estimate due to sample, or so–called “cosmic”, variance.

It is convenient, and perhaps essential for future experiments, to use band–power estimates as the starting point for constraining cosmological parameters, instead of pixel vectors, as in Eq. (1). Besides being the principle result reported in the literature, and hence easy to find, band–powers represent a sort of data compression (Bond et al. 1998) – there are fewer band–powers than pixels for any given experiment, and hence fewer calculations required to explore a given parameter space. If the fluctuations are truly gaussian, then we have lost nothing in the compression. To work in this direction, we need to develop an easy–to–use approximation to the full likelihood function for each band–power estimate, \( \mathcal{L}(\delta T_f) \), one which hopefully requires little information about experimental details.

With this aim, note first that the band–powers shown in Figure 1 are proportional to the variance of measured temperatures (or differences), e.g., as in Eq. (5). To motivate an ansatz, consider a totally unrealistic case where the covariance matrix is strictly diagonal, including noise, and the noise is uniform with variance \( \sigma_N^2 \):

\[
C_{ij} = [(\delta T)^2 + \sigma_N^2] \delta_{ij}
\]

Here, \((\delta T)^2\) represents the true sky variance (proportional to the band–power) we are trying to estimate. In this case, we know that the maximum likelihood estimate of the variance of \( \tilde{d} \), including the noise, is \((1/N_{\text{pix}}) \sum_i \tilde{d}_i^2\), i.e., this is the best estimate of \((\delta T)^2 + \sigma_N^2\). We also know that the quantity \( \chi^2_{\text{pix}} = (1/N_{\text{pix}}) \sum_i \tilde{d}_i^2/[(\delta T)^2 + \sigma_N^2] \) is \( \chi^2 \)–distributed with \( N_{\text{pix}} \) degrees–of–freedom. We may therefore express the best estimate of the sky variance as

\[
(\delta T^{(c)})^2 = [(\delta T)^2 + \sigma_N^2]\chi^2_{\text{pix}} - \sigma_N^2
\]

and we know that its distribution, \( \text{Prob}(\delta T^{(c)}|\delta T) \), (in the frequentist sense) is related to a \( \chi^2 \)–distribution by the change of variable in Eq. (7). Given an actual best estimate of the sky variance from a particular experiment, \( \delta T^{(o)} \), we argue that the likelihood function for \( \delta T \) is \( \mathcal{L}(\delta T) = \text{Prob}(\delta T^{(o)}|\delta T) \).

These arguments apply only to this particular, simplistic case. For more general situations corresponding to actual experiments, we proceed by adopting the same
FIGURE 2. Comparison between the true likelihood function for the 1995 Saskatoon Q–band 4–point difference, solid line, with the approximation, shown as the dashed line (in green). The x–axis is in units of $\mu$K.

functional form, $L(\delta T; N_{\text{pix}}, \sigma_N^2)$, as an ansatz for the true likelihood function. Notice that we now explicitly write the dependence on both $N_{\text{pix}}$ and the noise. What are these quantities in real situations? In some sense, they are the number of independent pixels and an average noise level, respectively. Thus, using the number of pixels and the given band–power estimate, we fit our approximation by adjusting $\sigma_N$ so that 68% percent of the likelihood falls within the quoted (1$\sigma$) error bars.

How well does this work? The only way to answer that question is by comparing the approximate likelihood to the true, complete likelihood function for a number of experiments. We have performed such a comparison for the COBE, Saskatoon and MAX data sets. In all cases, the approximation works astoundingly well. As an example, Figure 2 shows the comparison for a particular combination of the Saskatoon data.

3. CONCLUSIONS

Band–power estimates are not gaussian distributed, and so use of the $\chi^2$–statistic to constrain models in the power spectrum plane is not justified. We have proposed an apparently good approximation to the likelihood function which may be used as the basis of a more correct statistical approach to the data in Figure 1. Some preliminary details of its application may be found in Bartlett et al. (1998b). A general result seems to be that the constraints imposed by a $\chi^2$–analysis tend to be too strong (small confidence regions) compared to the constraints from a more complete likelihood analysis. An example of this in the case of COBE is shown in Figure 3.
FIGURE 3. COBE likelihood contours on $Q$ and $n$ from the true likelihood function, dashed lines (green), compared to those from a $\chi^2$-analysis, dot-dashed lines (inner contour in red). The $Q$-axis is expressed in $\mu$K.

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REFERENCES

Bartlett, J.G., Blanchard, A., Le Dour, M., Douspis, M. & Barbosa, D. 1998a, in Fundamental Parameters in Cosmology, Moriond 1998 proceedings, in press [astro-ph/9804158]

Bartlett, J.G., Blanchard, A., Douspis, M. & Le Dour, M. 1998b, in Evolution of Large-scale Structure: from Recombination to Garching, MPA/ESO conference proceedings, in press

Bond, J.R. & Jaffe, A.H. 1996, in Microwave Background Anisotropies (Moriond proceedings), F.R. Bouchet & R. Gispert (eds.), Editions Frontieres, Gif-sur-Yvette, p.197

Bond, J.R., Jaffe, A.H. & Knox, L. 1998, astro-ph/980826

Borril, J. 1998, in 3K Cosmology, conference held at the Università di Roma “La Sapienza”, in press

Hancock, S., Rocha, G., Lasenby, A.N. & Gutierrez, C.M. 1998, MNRAS 296, L1

Lineweaver, C., Barbosa, D., Blanchard, A. & Bartlett, J.G. 1997, A&A 322, 365

Wandelt, B.D., Hivon, E. & Górski, K.M. 1998, astro-ph/9808292