A NOTE ON A THEOREM OF H. CARTAN

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Abstract. We prove that if $D \subset \mathbb{C}^n$ is a bounded domain with real analytic boundary, and $D$ is either pseudoconvex or it satisfies condition R, then the compact open topology in $\text{Aut}(D)$, the group of holomorphic automorphisms of $D$ is the topology of uniform convergence on $D$.

0 Introduction

Let $D$ be a bounded domain in $\mathbb{C}^n$ with $n \geq 1$, and let $\text{Aut}(D)$ denote the group of holomorphic automorphisms of $D$. It was shown by H. Cartan [2, Chap. 9, th. 4] that $\text{Aut}(D)$, endowed with the compact open topology (that is, the topology of local uniform convergence on $D$), is a real Lie group whose Lie algebra $\text{aut}(D)$ consists of all complete holomorphic vector fields $X: D \to \mathbb{C}^n$, and that the natural action

$$\text{Aut}(D) \times D \to D, \quad (f, z) \mapsto f(z)$$

is real analytic in joint variables. This result is no longer valid if $\mathbb{C}^n$ is replaced by an arbitrary complex Banach space $E$; however, it is still valid if we assume that $D \subset E$ is a bounded symmetric domain ([4], [5]).

It is remarkable that no assumptions on the smoothness of the boundary $\partial D$ is needed either in the $\mathbb{C}^n$ setting or in the infinite dimensional case. This leads naturally to the question of whether every $f$ in $\text{Aut}(D)$ extends holomorphically beyond $\partial D$. Affirmative answers to this question have been found, in the $\mathbb{C}^n$ setting, by S. Chen S. Jang in [1, th.1.2] assuming that $D$ has real analytic boundary and that either $D$ is pseudoconvex or that condition (R) holds on $D$. By condition (R) we mean that the Bergmann projector (that is, the orthogonal projection from $L^2(D)$ onto the the closed subspace $O^2(D)$ of square integrable holomorphic functions) maps $C^\infty(D)$ continuously into itself. In the infinite dimensional context, an affirmative answer has been established in [6] for bounded circular domains with no restrictions on the boundary.

In turn, the possibility of extending every element $f$ in $\text{Aut}(D)$ to a neighbourhood of $\overline{D}$ gives new information about the topology of $\text{Aut}(D)$. Indeed, it has been proved in [6] that for bounded circular domains $D$ in a Banach space $E$, the topology on $\text{Aut}(D)$ of local uniform convergence over $D$ is actually the topology of uniform convergence on $D$. In this note we use the results in [1] to prove the following theorem:

**Theorem** Let $D \subset \mathbb{C}^n$, where $n \geq 1$, be a bounded domain with real analytic boundary, and suppose that either $D$ is pseudoconvex or it satisfies condition (R). Then the compact open topology on $\text{Aut}(D)$ is the same as the topology of uniform convergence over $D$.
For a domain $D \subset \mathbb{C}^n$, we let $\text{Hol}(D, \mathbb{C}^n)$ denote the vector space of all holomorphic mappings $h: D \to \mathbb{C}^n$, and the subset $\text{Hol}(D)$ consists of the mappings such that $f(D) \subset D$. A set $S \subset D$ is a vanishing set for $D$ if the relations $f \in \text{Hol}(D, \mathbb{C}^n)$ and $f|_S = 0$ entail $f = 0$. We say that $\partial D$ is an algebraically determining set for $\text{Aut}(D)$ if the relations $f, g \in \text{Aut}(D)$ and $f = g$ on $\partial D$ entail $f = g$. Similarly, the expression $\partial D$ is a topologically determining set for $\text{Aut}(D)$ means that whenever $(f_n)_{n \in \mathbb{N}}$ and $f$ are respectively a sequence and an element in $\text{Aut}(D)$ such that $f_n \to f$ uniformly over $\partial D$ we have $f_n \to f$ in $\text{Aut}(D)$. Obvious changes give us the meaning of the expressions $S \subset D$ is an algebraically or topologically determining set for $\text{aut}(D)$, see [3]

1 The main result.

1.1 Theorem. Let $D \subset \mathbb{C}^n$, where $n \geq 1$, be a bounded domain with real analytic boundary, and suppose that either $D$ is pseudoconvex or it satisfies condition (R). Then for every sequence $(f_n)_{n \in \mathbb{N}}$ in $\text{Aut}(D)$ and every $f$ in $\text{Hol}(D)$ the following conditions are equivalent:

1. $f \in \text{Aut}(D)$ and there is a neighbourhood $\Omega$ of $\overline{T}$ in $\mathbb{C}^n$ such that $(f_n)_{n \in \mathbb{N}}$ converges to $f$ uniformly on $\Omega$.

2. $f \in \text{Aut}(D)$ and $(f_n)_{n \in \mathbb{N}}$ converges to $f$ in the group $\text{Aut}(D)$.

3. There are a non void open subset $U \subset D$ and a point $a \in U$ such that $(f_n(z))_{n \in \mathbb{N}}$ converges to $f(z)$ pointwise on $U$ and $f(a) \notin \partial D$.

Proof. We only need to prove $3 \implies 1$. By the identity principle, $U$ is a vanishing set for $D$. Also the family $(f_n)_{n \in \mathbb{N}}$ is uniformly bounded in $D$. Therefore by Vitali’s theorem [2, Chap. 1 prop. 7] the sequence $(f_n)_{n \in \mathbb{N}}$ converges to $f$ uniformly on each compact subset $K \subset D$. Since $f(a) \notin \partial D$, we have $f \in \text{Aut}(D)$ by Cartan’s theorem [2, Chap. 5 th. 4]. On the other hand, since $D$ is a pseudocompact set or it satisfies condition (R), by [1, th. 1.2] there are a neighbourhood $\mathcal{V}_f$ of $f$ in $\text{Aut}(D)$ and a neighbourhood $\Omega_f$ of $\overline{T}$ in $\mathbb{C}^n$ such that every $g \in \mathcal{V}_f$ extends to some $\tilde{g} \in \text{Hol}(\Omega_f)$ and the action

$$\mathcal{V}_f \times \Omega_f \to \mathbb{C}^n, \quad (g, z) \mapsto \tilde{g}(z)$$

is real analytic on $\mathcal{V}_f \times \Omega_f$. By restricting ourselves to smaller neighbourhoods $W \subset \mathcal{V}_f$ and $\Omega \subset \Omega_f \subset \Omega_f$ we may assume that $\overline{W}$ and $\overline{\Omega}$ are compact and that the latter is connected. Since $f_n \to f$ in $\text{Aut}(D)$, we have $f_n \in W$ for $n$ large enough, say $n \geq n_0$ and so

$$\sup_{n \geq n_0, z \in \Omega} \|f_n(z)\| \leq M < \infty,$$

and a new application of Vitali’s theorem yields that $f_n \to f$ uniformly on each compact subset of $\Omega_1$, in particular on $\Omega$. \qed

1.2 Corollary. Let the domain $D \subset \mathbb{C}^n$ satisfy the assumptions in (1.1). Then

1. The compact open topology on $\text{Aut}(D)$ coincides with the topology of uniform convergence over $\overline{T}$ and with the topology of uniform convergence over $\partial D$.

2. $\partial D$ is a determining set for the topological group $\text{Aut}(D)$.

3. If $D$ is a bounded convex domain with real analytic boundary, then every element in $\text{Aut}(D)$ has a fixed point in $\overline{T}$.

Proof. (1): It is clear from the above proof that on $\text{Aut}(D)$ uniform convergence over the compact subsets of $D$ is the same as uniform convergence over $\overline{T}$, which in turn is the same as uniform convergence over $\partial D$ as a consequence of the maximum modulus theorem. (2) and (3): Extend to $\overline{T}$ the elements in $\text{Aut}(D)$ and apply respectively the maximum modulus theorem and Brower’s fixed point theorem. \qed

We now consider the similar problems at the Lie algebra level.
1.3 Theorem. Let $D \subset \mathbb{C}^n$, where $n \geq 1$, be a bounded domain with real analytic boundary, and suppose that either $D$ is pseudoconvex or it satisfies condition (R). Then there are a neighbourhood $\mathcal{M}_0$ of 0 in $\text{aut} (D)$ and a neighbourhood $\Omega_0$ of $\overline{D}$ in $\mathbb{C}^n$ such that every vector field $X \in \mathcal{M}_0$ extends to some $\tilde{X} \in \text{Hol} (\Omega_0, \mathbb{C}^n)$ that is the infinitesimal generator of a local one-parameter group of holomorphic transformations on $\Omega_0$.

Proof. Choose a neighbourhood $V_{\text{id}}$ of $\text{id}$ in the group $\text{Aut} (D)$ and a neighbourhood $\Omega_0$ of $\overline{D}$ in $\mathbb{C}^n$ such that every element $f \in V_{\text{id}}$ extends to a holomorphic mapping $\tilde{f}$ in $\Omega_0$ and the action

$$V_{\text{id}} \times \Omega_0 \rightarrow \mathbb{C}^n, \quad (f, z) \mapsto \tilde{f}(z)$$

is real analytic on $V_{\text{id}} \times \Omega_0$. The exponential mapping $\exp$ defines a homeomorphism of a neighbourhood $W_{\text{id}}$ of $\text{id}$ in $\text{Aut} (D)$ onto a neighbourhood of 0 in $\text{aut} (D)$. Let $U_0 := V_{\text{id}} \cap W_{\text{id}}$ and set $M_0 := \exp^{-1}U_0$. We may assume that $M_0$ is balanced hence connected. Let $X \in M_0$ and let $f_t := \exp tX \in \text{Aut} (D)$ denote the one-parameter group corresponding to $X$. For small values of $t$, say $|t| < \tau$, we have $f_t \in U_0$, hence $f_t$ extends to a holomorphic mapping $f_t \in \text{Hol} (\Omega_0, \mathbb{C}^n)$. By the identity principle we have $f_t \circ f_s = f_{t+s}$ whenever $|t+s| < \tau$. Thus $t \mapsto f_t$ is a local one-parameter group of holomorphic transformations on $\Omega_0$ and so its infinitesimal generator $\tilde{X} = \frac{d}{dt}|t=0}f_t$ is a holomorphic mapping on $\Omega_0$ that extends $X$. □

1.4 Corollary. Let the domain $D \subset \mathbb{C}^n$ satisfy the assumptions in (1.3). Then $\partial D$ is a determining set for $\text{aut} (D)$, and for every sequence $(X_n)_{n \in \mathbb{N}}$ in $\text{aut} (D)$ and every function $X \in \text{Hol} (D, \mathbb{C}^n)$ the following conditions are equivalent:

1. $X \in \text{aut} (D)$ and there is a neighbourhood $\Omega$ of $\overline{D}$ in $\mathbb{C}^n$ such that $X_n \rightarrow X$ uniformly on $\Omega$.
2. $X \in \text{aut} (D)$ and $(X_n)_{n \in \mathbb{N}}$ converges to $X$ in the Lie algebra $\text{aut} (D)$.
3. $(X_n(z))_{n \in \mathbb{N}}$ is uniformly bounded in $D$ and $X_n \rightarrow X$ pointwise on some open non void subset $U \subset D$.

Proof. Clearly 1 $\implies$ 2. Suppose that (2) holds; then a repetition of the arguments made in the proof of (1.3) show that $X$ and all $X_n$ extend holomorphically to a suitable neighbourhood $\Omega$ of $\overline{D}$ and $X_n \rightarrow X$ uniformly over $\Omega$, hence in particular $(X_n)_{n \in \mathbb{N}}$ is uniformly bounded on $D$. Now suppose (3) holds; by Vitali’s theorem we have $X_n \rightarrow X$ uniformly over the compact subset of $D$, hence $X \in \text{aut} (D)$, and by (1.3) all $X_n$ and $X$ extend holomorphically to some neighbourhood $\Omega$ of $\overline{D}$. A new application of Vitali’s theorem gives the result. The other claim is now obvious. □

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