A NOTE ON EQUIVARIANT K-STABILITY

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Abstract. We define $G$-pseudovaluations on a variety with a group action $G$. By introducing $G$-pseudovaluations, we are able to give some criteria for $G$-equivariant K-stability of Fano varieties which are parallel to existing results for usual K-stability.

1. Introduction

We work over the complex number $\mathbb{C}$. A $\mathbb{Q}$-Fano variety is a normal projective variety with klt singularities such that the anti-canonical divisor is ample.

It is conjectured that in order to test K-polystability of a $\mathbb{Q}$-Fano variety it is enough to examine equivariant test configurations with respect to a finite or connected reductive subgroup $G$ of $\text{Aut}(X)$. For the case of Fano manifolds, an analytic proof is given in [DS16]. An algebraic proof is also provided in [LWX18] when $G$ is a torus group.

The purpose of this short note however is to provide another perspective on equivariant K-stability for $\mathbb{Q}$-Fano varieties with arbitrary group action. We give parallel results to some existing theorems on characterizing K-stability by replacing the space of valuations with a special collection of pseudovaluations in terms of the group action. Indeed, for any variety $X$, let $G \subset \text{Aut}(X)$ denote a group action on $X$. For any valuation $v$ on $X$, we define $G \cdot v := \inf_{g \in G} g \cdot v$, where $g$ acts on the valuation $v$ by $g \cdot v(f) = v(f \circ g)$ for any $f \in \mathbb{C}(X)$. We call $G \cdot v$ a $G$-pseudovaluation and denote all $G$-pseudovaluations on $X$ by $G\text{Val}_X$. Note that all $G$-invariant valuations, which we denote by $\text{Val}^G_X$, are contained in $G\text{Val}_X$. For any $G$-pseudovaluation $G \cdot v$, and a nonnegative real number $x$, we can define the ideal sheaf $a_x(G \cdot v)$ to be

$$a_x(G \cdot v) = \bigcap_{g \in G} a_x(g \cdot v),$$

where for any valuation $w$, $a_x(w)$ is the ideal sheaf of regular functions with vanishing order no less than $x$ with respect to $w$. Refer to Section 2 for details about the definition of $G$-pseudovaluations.

The first theorem is about valuative criteria of equivariant K-stability parallel to the main results in [Fuj16]. Let $X$ be a $\mathbb{Q}$-Fano variety and $G \subset \text{Aut}(X)$ a group action on $X$. We define the $G$-equivariant beta invariant of $F$ to be

$$\beta^G(F) := A_X(F)(-K_X)^n - \int_0^{+\infty} \text{vol}_X(\mathcal{O}_X(-K_X) \otimes a_x(G \cdot \text{ord}_F)) \, dx.$$
We say that $F$ is of finite orbit if the orbit of the valuation $\text{ord}_F$ under $G$-action is finite. We say that $F$ is $G$-dreamy if $F$ is of finite orbit and moreover the graded ring
\[ \bigoplus_{k,j \geq 0} H^0(X, \mathcal{O}_X(-kK_X) \otimes \mathfrak{a}_j(G \cdot \text{ord}_F)) \]
is finitely generated.

Define
\[ \tau^G(F) := \sup \{ t > 0 | \text{vol}_X(\mathcal{O}_X(-K_X) \otimes \mathfrak{a}_t(G \cdot \text{ord}_F)) > 0 \} \]
and
\[ j^G(F) = \int_0^{\tau^G(F)} (\text{vol}_X(-K_X) - \text{vol}_X(\mathcal{O}_X(-K_X) \otimes \mathfrak{a}_x(G \cdot \text{ord}_F))) \, dx. \]

Note that for $G$-invariant divisors over $X$, the above definitions coincide with the usual ones defined in [Fuj16].

The following theorem gives valuative criteria of K-stability in terms of $\beta^G(F)$:

**Theorem A.** Let $X$ be a $\mathbb{Q}$-Fano variety with $G \subset \text{Aut}(X)$ a group action on $X$.

1. The following are equivalent:
   - (i) $X$ is uniformly $G$-equivariantly K-stable;
   - (ii) there exists $0 < \delta < 1$, such that $\beta^G(F) \geq \delta j^G(F)$ for any finite-orbit prime divisor $F$ over $X$;
   - (iii) there exists $0 < \delta < 1$, such that $\beta^G(F) \geq \delta j^G(F)$ for any $G$-dreamy prime divisor $F$ over $X$.

2. The following are equivalent:
   - (i) $X$ is $G$-equivariantly K-semistable;
   - (ii) $\beta^G(F) \geq 0$ for any finite-orbit prime divisor $F$ over $X$;
   - (iii) $\beta^G(F) \geq 0$ for any $G$-dreamy prime divisor $F$ over $X$.

3. The following are equivalent:
   - (i) $X$ is $G$-equivariantly K-stable;
   - (ii) $\beta^G(F) > 0$ for any $G$-dreamy prime divisor $F$ over $X$.

**Remark 1.1.** When $G$ is finite, every prime divisor over $X$ is of finite orbit. Moreover, by an argument provided by Yuchen Liu, we can take the quotient of each $G$-equivariant test configuration and run the process in [LX14] to get a special test configuration. Then by [Fuj16], we know that it is enough to check $G$-invariant divisors for K-stability for finite $G$. When $G$ is connected, we know that every finite-orbit divisor is $G$-invariant. In general, when $G$ is not finite, all the prime divisors induced by $G$-special test configurations (see Section 2.3 for definition) are still of finite orbit. Therefore, we are not losing any information in terms of test configurations and K-stability by focusing only on divisors of finite orbit.

We can also characterize equivariant K-stability in terms of equivariant normalized volume of $G$-pseudovaluations. Normalized volume of $G$-pseudovaluations can be defined similarly as the normalized volume of usual valuations in [Li15] and we will use the same notation. See Section 2 for more details.

Let $X$ be a $\mathbb{Q}$-Fano variety with $G$-action, denote by $Y = C(X, -K_X)$ the cone over $X$ and $o \in Y$ the vertex of the cone. Suppose $\pi : Z = \text{Bl}_o Y \to Y$ is the blow-up of $Y$ at $o$. Let $E$ be the exceptional divisor of the blow-up. Denote the divisorial valuation $\text{ord}_E$ by $v_0$. Note that there is a natural $G$-action induced on the cone $Y$ and the blow-up $Z$. Since
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E is a $G$-invariant divisor, we know that $v_0 \in \Val^G_{Y,o} \subset \GVal_{Y,o}$, where $\Val^G_{Y,o}$ and $\GVal_{Y,o}$ refer to $G$-invariant valuations and $G$-pseudovaluations centered at $o$ respectively.

Under the above notation, we have the following characterization of $G$-equivariant K-semistability compared to the results in [Li17, LL16, LX16]:

**Theorem B.** $X$ is $G$-equivariantly K-semistable iff the normalized volume function $\hat{\vol}_{Y,o}$ is minimized at $v_0$ among all finite-orbit $G$-pseudovaluations on $Y$ centered at $o$.

**Remark 1.2.** If one can show that the minimizer of $\hat{\vol}_{Y,o}$ among all valuations on $Y$ centered at $o$ is unique, which is a long existing conjecture first proposed in [Li15], then it is necessarily $G$-invariant. As it is well known, this would immediately imply the equivalence between $G$-equivariant K-semistability and usual K-semistability by a similar argument as in the proof of Theorem E in [LX16]. In particular, it would follow that it is enough to consider only $G$-invariant divisors and $G$-invariant valuations to check K-semistability.

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2. PSEUDOVALUATIONS, NORMALIZED VOLUMES AND EQUIVARIANT K-STABILITY

We include in this section relevant equivariant version of notions about valuations and K-stability for reader’s convenience.

2.1. VALUATIONS AND PSEUDOVALUATIONS. For a variety $X$ with a group action $G$, we define $G$-pseudovaluations in the following way:

**Definition 2.1.** Let $G$ be a group action on $X$ and $v$ a valuation on $X$. Define

$$G \cdot v := \inf_{g \in G} g \cdot v,$$

where $g \cdot v$ is the valuation given by $g \cdot v(f) = v(f \circ g)$ for any $f \in \mathbb{C}(X)$. We call $G \cdot v$ a $G$-pseudovaluation and denote all $G$-pseudovaluations on $X$ by $\GVal_X$. The center of $G \cdot v$ is defined to be the union of the centers of $g \cdot v$ for all $g \in G$. We say $G \cdot v$ is of finite orbit if the orbit of $v$ under $G$-action is finite.

**Remark 2.1.** In general, $G$-pseudovaluations are not valuations because they do not satisfy the product property. Indeed, for any $f, g \in \mathbb{C}(X)$, we only have

$$G \cdot v(fg) \geq G \cdot v(f) + G \cdot v(g).$$

If $U \subset X$ is an affine open set containing all the centers of the valuations $g \cdot v$, then $G \cdot v$ induces a pseudovaluation on $O_X(U)$ in the sense of [dFM15]. When $G$ is finite, we can always find such $U$. Note that pseudovaluations on an affine variety do not extend to its function field due to the lack of product property. In general there is not a clear way to define pseudovaluations on a projective variety.

For a valuation $v$ on $X$ and a nonnegative real number $x$, the ideal sheaf $a_x(v) \subset O_X$ is defined as follows. For $U \subset X$ an open affine subset of $X$, if $U$ contains the center of $v$, then define

$$a_x(v)(U) = \{ f \in O_X(U) | v(f) \geq x \}.$$
If $U$ does not contain the center of $v$, we set $a_x(v)(U) = \mathcal{O}_X(U)$. For a $G$-pseudovaluation $G \cdot v$, and $x$ a nonnegative real number, we define the ideal sheaf $a_x(G \cdot v)$ to be
\[
a_x(G \cdot v) = \bigcap_{g \in v} a_x(g \cdot v).
\]

2.2. **Equivariant normalized volume.** Let $x$ be a $G$-invariant point on $X$. Denote by $\text{GVal}_{X,x}$ all $G$-pseudovaluations centered at $x$. We can define the normalized volume $\text{vol}^G$ on the $\text{GVal}_{X,x}$ almost the same way as normalized volume of usual valuactions. First of all, for any $G$-pseudovaluation $G \cdot v$, we define the volume
\[
\text{vol}(G \cdot v) = \lim_{\lambda \to \infty} \frac{\dim C_{X,x}/a_{\lambda}(G \cdot v)}{\lambda^n/n!}.
\]
Note that $A_X(g \cdot v) = A_X(v)$ for any $g \in G$, so we define the log discrepancy of $G \cdot v$ to be $A_X(v)$. Then the normalized volume of $G \cdot v$ is defined as
\[
\widehat{\text{vol}}(G \cdot v) = A_X(v)^n \text{vol}(G \cdot v).
\]

2.3. **Equivariant K-stability.** We first give the definition of equivariant test configuration.

**Definition 2.2.** Let $(X, L)$ be a polarized variety. A (semi-)test configuration $(\mathcal{X}, \mathcal{L})$ of $(X, L)$ with exponent $r$ consists of the following data:

1. a proper flat family $\pi : \mathcal{X} \to \mathbb{A}^1$,
2. an equivariant $\mathbb{C}^*$-action on $\pi : \mathcal{X} \to \mathbb{A}^1$, where $\mathbb{C}^*$ acts on $\mathbb{A}^1$ by multiplication in the standard way, and
3. a $\mathbb{C}^*$-equivariant line bundle $\mathcal{L}$ on $\mathcal{X}$ which is $\pi$-relatively (semi-)ample, such that $(\mathcal{X}, \mathcal{L})|_{\pi^{-1}(\mathbb{A}^1 \setminus \{0\})}$ is $\mathbb{C}^*$-equivariantly isomorphic to $(X \times (\mathbb{A}^1 \setminus \{0\}), L^{\otimes r}_{\mathbb{A}^1 \setminus \{0\}})$, where $L_{\mathbb{A}^1 \setminus \{0\}}$ is the pull back of $L$ from $X$ to $X \times (\mathbb{A}^1 \setminus \{0\})$. In addition, let $G$ be a group action on $(X, L)$. We say $(\mathcal{X}, \mathcal{L})$ is a $G$-equivariant test configuration if $G$ can be extended to an action on $(\mathcal{X}, \mathcal{L})$ such that it commutes with the $\mathbb{C}^*$ on $(\mathcal{X}, \mathcal{L})$, fixes fibers of $\mathcal{X}$ and restricts to the $G$-action on all fibers of $\mathcal{X}$ other than $\mathcal{X}_0$.

Next, we will focus on $\mathbb{Q}$-Fano varieties with the polarization to be $-K_X$. By replacing $-K_X$ with a sufficiently divisible multiple of itself, we may assume $-K_X$ is already Cartier.

The definition of Donaldson-Futaki invariant for an equivariant test configuration is the same as the usual one. We include a definition using intersection formula here which will come up in later computation.

**Definition 2.3.** Let $X$ be a $\mathbb{Q}$-Fano variety of dimension $n$. Pick a rational number $r$ such that $rK_X$ is Cartier. Let $(\mathcal{X}, \mathcal{L})$ be a normal semi-test configuration of $(X, -rK_X)$. We can compactify the test configuration into a flat family $(\mathcal{X}, \mathcal{L})$ over $\mathbb{P}^1$, such that over $\mathbb{P}^1 \setminus \{0\}$, the family $(\mathcal{X}, \mathcal{L})$ is $\mathbb{C}^*$-equivariantly isomorphic to $X \times \mathbb{P}^1 \setminus \{0\}$ with trivial $\mathbb{C}^*$-action on the fibers. Then we can define the Donaldson-Futaki invariant of $(\mathcal{X}, \mathcal{L})$ to be
\[
\text{DF}(\mathcal{X}, \mathcal{L}) := \frac{1}{(n + 1)(-K_X)^n} \left( \frac{n}{r^{n+1}} \bar{L}^{n+1} + \frac{n+1}{r^n} \bar{L}^n \cdot K_{\mathcal{X}/\mathbb{P}^1} \right) \tag{2.1}
\]

We also include the definition of $J^{\text{NA}}(\mathcal{X}, \mathcal{L})$ following [Fuj16], which can be viewed as the norm of $(\mathcal{X}, \mathcal{L})$. Let
be a common resolution of $X \times \mathbb{P}^1$ and $\tilde{X}$. We set
\[
\lambda_{\text{max}}(\mathcal{X}, \mathcal{L}) := \frac{p^*((-K_{X \times \mathbb{P}^1})^n \cdot q^* \mathcal{L}}{(-K_X)^n},
\]
and define
\[
J_{\text{NA}}(\mathcal{X}, \mathcal{L}) := \lambda_{\text{max}}(\mathcal{X}, \mathcal{L}) - \frac{\mathcal{L}^{n+1}}{(n+1)(-rK_X)^n}
\]

**Definition 2.4.** Let $X$ be a $\mathbb{Q}$-Fano variety with $G \subset \text{Aut}(X)$ a group action on $X$. We have the following three definitions of K-stability:

1. $(X, -K_X)$ is said to be $G$-equivariantly K-semistable if the Donaldson-Futaki invariant is nonnegative for all $G$-equivariant normal test configurations.
2. $(X, -K_X)$ is said to be $G$-equivariantly K-stable if the Donaldson-Futaki invariant is positive for all nontrivial $G$-equivariant normal test configurations.
3. $(X, -K_X)$ is said to be uniformly $G$-equivariantly K-stable if there exists $0 < \delta < 1$ such that $DF(\mathcal{X}, \mathcal{L}) \geq \delta J_{\text{NA}}(\mathcal{X}, \mathcal{L})$ for all $G$-equivariant normal test configurations.

Following the argument in [LX14], we can get a collection of equivariant test configurations that plays the same role as special test configurations for K-stability.

**Theorem 2.5.** For any $G$-equivariant normal test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ of $(X, -K_X)$, there exists a finite morphism $\phi : \mathbb{A}^1 \rightarrow \mathbb{A}^1$, a test configuration $(\mathcal{X}^s, \mathcal{L}^s)$ with the central fiber being reduced and $G$-irreducible and a both $\mathbb{C}^*$- and $G$-equivariant birational map $\mathcal{X}^s \rightarrow \mathcal{X} \times_\phi \mathbb{A}^1$ over $\mathbb{A}^1$, such that for any $0 \leq \delta \leq 1$, we have
\[
DF(\mathcal{X}^s, \mathcal{L}^s) - \delta J_{\text{NA}}(\mathcal{X}^s, \mathcal{L}^s) \leq \deg \phi (DF(\mathcal{X}, \mathcal{L}) - \delta J_{\text{NA}}(\mathcal{X}, \mathcal{L})).
\]
In addition, we can choose $\mathcal{L}^s = -K_{\mathcal{X}^s/\mathbb{A}^1}$.

**Proof.** By running the $G$-equivariant version of each steps in the proof of the main theorem in [LX14], we get the $G$-equivariant test configuration $(\mathcal{X}^s, \mathcal{L}^s)$ and the birational map $\mathcal{X}^s \rightarrow \mathcal{X} \times_\phi \mathbb{A}^1$. The computation in [LX14] and [Fuj16] gives us the inequality.

Note that both $\mathcal{L}^s$ and $K_{\mathcal{X}^s/\mathbb{A}^1}$ are $G$-invariant. Then since $\mathcal{L}^s + K_{\mathcal{X}^s/\mathbb{A}^1}$ supports on the central fiber $\mathcal{X}^s_0$, it can only be a multiple of the whole fiber $\mathcal{X}^s_0$. Then by definition we have $DF(\mathcal{X}^s, \mathcal{L}^s) = DF(\mathcal{X}^s, -K_{\mathcal{X}^s/\mathbb{A}^1})$ and $J_{\text{NA}}(\mathcal{X}^s, \mathcal{L}^s) = J_{\text{NA}}(\mathcal{X}^s, -K_{\mathcal{X}^s/\mathbb{A}^1})$. \hfill \Box

We call the resulting test configuration $(\mathcal{X}^s, -K_{\mathcal{X}^s/\mathbb{A}^1})$ in Theorem 2.5 a $G$-special test configuration. As in the usual K-stability case, we know from Theorem 2.5 that it is enough to check only $G$-special test configurations for $G$-equivariant K-stability.

A test configuration $(\mathcal{X}, \mathcal{L})$ of $(X, -K_X)$ induces a filtration $\mathcal{F}$ on $V_k = H^0(X, -kK_X)$ in the following way:
\[
\mathcal{F}^s V_k = \{ s \in V_k \mid t^{-|s|} \tilde{s} \in H^0(\mathcal{X}, k\mathcal{L}) \},
\]
where $\tilde{s}$ is the $\mathbb{C}^*$-invariant section of $k\mathcal{L}$ on $\mathcal{X} \setminus \mathcal{X}_0$ induced by $s$. Note that $\mathcal{F}$ is decreasing, left-continuous, multiplicative and linearly bounded. Filtrations in this paper will always be assumed to satisfy these four properties.
Conversely, let $\mathcal{F}$ be a filtration on $V_\bullet$ such that $\bigoplus_{k \in \mathbb{Z}_{\geq 0}, j \in \mathbb{Z}} \mathcal{F}^j V_k$ is finitely generated. We may assume it is generated in degree $k = 1$. Then we can define a test configuration

$$(\text{Proj}_\mathbb{A}^1 \bigoplus_{k \in \mathbb{Z}_{\geq 0}, j \in \mathbb{Z}} t^{-j} \mathcal{F}^j V_k, \mathcal{O}(1))$$

The following proposition gives the relation between filtrations and test configurations.

**Proposition 2.6** (Proposition 2.15, [BHJ15]). The above construction sets up a one-to-one correspondence between test configurations of $(X, -K_X)$ and finitely generated filtrations on $V_\bullet$.

For any prime divisor $F$ over $X$, we can construct a $G$-invariant filtration

$$\mathcal{F}^x V_r = \begin{cases} H^0(X, \mathcal{O}_X(-rK_X) \otimes a[x](G \cdot \text{ord}_F)), & x \geq 0, \\ V_r, & x < 0, \end{cases} \quad (2.2)$$

which induces a $G$-equivariant test configuration.

To conclude this section, we look at some basic examples that illustrate the difference between $G$-equivariant K-stability and usual K-stability.

**Example 2.7.** Consider the projective space $X = \mathbb{P}^n$ with $G = \text{PGL}(n + 1)$-action. Then the only $G$-equivariant test configuration of $(\mathbb{P}^n, -K_{\mathbb{P}^n})$ is the trivial test configuration $\mathbb{P}^n \times \mathbb{A}^1$. Therefore by definition we know that $\mathbb{P}^n$ is uniformly $G$-equivariantly K-stable. Note that for any $G$-pseudovaluation $G \cdot v$, we have that $a_x(G \cdot v) = (0)$ for any $x > 0$. Therefore for any prime divisor $F$ over $\mathbb{P}^n$, we know that the corresponding $G$-invariant filtration

$$\mathcal{F}^x V_r = \begin{cases} 0, & x > 0, \\ V_r, & x \leq 0, \end{cases}$$

which of course induces the trivial test configuration $\mathbb{P}^n \times \mathbb{A}^1$.

**Example 2.8.** Consider $X = \mathbb{P}^1 \times \mathbb{P}^1$ with $G = \text{PGL}(2)$ acting on the first component. Pick any point $p \in X$. Let $E$ be the exceptional divisor of the blow-up of $X$ at $p$. Let $H$ be the horizontal line through $p$, and we know that $H$ is the orbit of $p$ under $G$-action. Therefore $E$ and $H$ induce the same $G$-invariant filtration. Note that although $E$ is not of finite orbit, we know $H$ is $G$-invariant. The compactified test configuration corresponding to the $G$-invariant filtration is $\pi : \mathbb{P}^1 \times F_1 \to \mathbb{P}^1$, with $G$ acting on the first component and $\pi$ induced by the Hirzebruch surface $F_1 \to \mathbb{P}^1$.

Similar examples can also be constructed easily when $G$ is non-compact, e.g. a torus action $(\mathbb{C}^*)^r$ on $\mathbb{P}^n$.

## 3. Equivariant valuative criteria

We separate the proof of Theorem A into 3 parts. We first prove the following theorem which gives a necessary valuative condition of equivariant uniform K-stability in Theorem A.

**Theorem 3.1.** Let $X$ be a $\mathbb{Q}$-Fano variety with $G \subset \text{Aut}(X)$ a group aciton on $X$. If $X$ is uniformly $G$-equivariantly K-stable, then there exists $0 < \delta < 1$, such that $\beta^G(F) \geq \delta j^G(F)$ for any finite-orbit prime divisor $F$ over $X$. 

Proof. We may assume $-K_X$ is already Cartier. Given any divisor $F$ of finite orbit, let $\pi: Y \to X$ be a $G$-equivariant resolution such that $F$ is a smooth divisor on $Y$. Following the notation in (2.2), we consider the $G$-invariant filtration of $F^r V_r$ defined by $F$. Note that $F$ is saturated. Let $I_{(r,x)} := \text{Im}(F^r V_r \otimes \mathcal{O}_X(rK_X) \to \mathcal{O}_X)$ be the base ideal of $F^r V_r$. Suppose $F_1 = F, \ldots, F_N$ form the orbit of $F$ under the $G$-action. We have

$$I_{(r,x)} \cdot \mathcal{O}_Y \subset \mathcal{O}_Y \left( [-x] \sum_{i=1}^{N} F_i \right).$$

Now the same computation as in the proof of Theorem 4.1 in [Fuj16] will give us $\beta^G(F) \geq \delta j^G(F)$.

Remark 3.1. Note that when $F$ is not of finite orbit, it is not possible to find a $G$-equivariant resolution $Y \to X$ as in the above proof.

Next we study the relation between Donaldson-Futaki invariants of $G$-special test configurations and equivariant beta invariants.

**Theorem 3.2.** Let $(\mathcal{X}, \mathcal{L})$ be a normal $G$-equivariant test configuration of $(X, -K_X)$ with $\mathcal{X}_0$ reduced and $G$-irreducible. Suppose the central fiber of $\mathcal{X}$ can be decomposed into irreducible components $\mathcal{X}_0^1, \ldots, \mathcal{X}_0^N$. Let $v_i$ be the restriction on $X$ of the divisorial valuation $\text{ord}_{\mathcal{X}_0}$. Then $v_i$ is $G$-dreamy. We have $\text{DF}(\mathcal{X}, \mathcal{L}) = \beta^G(v_i)/(-K_X)^n$ and $J^{NA}(\mathcal{X}, \mathcal{L}) = j^G(v_i)/(-K_X)^n$ for any $i$.

Proof. First note that by Lemma 2.5, we may assume that $\mathcal{L} = -K_{\mathcal{X}/\mathbb{A}^1}$. Then we claim that each $v_i$ is a divisorial valuation on $X$ corresponding to distinct divisor $F_i$ over $X$. Indeed, Suppose $\text{ord}_{\mathcal{X}_0}$ restricts to the same valuation $v_i$ and $v_j$ on $X$. Then since $\text{ord}_{\mathcal{X}_0}(t) = \text{ord}_{\mathcal{X}_0}(t) = 1$, we know that the two valuations are the same on $X$. The $G$-action permutes all the irreducible components of the central fiber $\mathcal{X}_0$, so we have that $F_1, \ldots, F_N$ forms the orbit of $F_1$ under the $G$-action.

Now the same argument as in the proof of Theorem 5.1 in [Fuj16] gives us the conclusion.

The following theorem is an immediate consequence of Theorem 3.2.

**Theorem 3.3.** If there exists some $0 < \delta < 1$, such that $\beta^G(F) > 0(\geq \delta j^G(F))$ for any $G$-dreamy divisor $F$ over $X$, then $X$ is (uniformly) $G$-equivariantly $K$-stable.

Note that if we set $\delta = 0$ in Theorem 3.1 and Theorem 3.3, we get the corresponding valuative criterion for $K$-semistability.

By Proposition 2.6, we have a one-to-one correspondence between filtrations on $V_\bullet$ and test configurations. Then combining Theorem 3.2, we have the following theorem:

**Theorem 3.4.** Let $X$ be a $\mathbb{Q}$-Fano variety and $F$ a $G$-dreamy divisor over $X$. Define a filtration $\mathcal{F}$ on $V_\bullet$ as in (2.2). Then the test configuration

$$(\mathcal{X}, \mathcal{L}) = \left( \text{Proj}_{\mathbb{A}^1} \bigoplus_{k \in \mathbb{Z}_{\geq 0}, j \in \mathbb{Z}} t^{-j} \mathcal{F}^j V_k, \mathcal{O}(1) \right)$$

is a $G$-special test configuration such that $\text{DF}(\mathcal{X}, \mathcal{L}) = \beta^G(F)/(-K_X)^n$ and $J^{NA}(\mathcal{X}, \mathcal{L}) = j^G(F)/(-K_X)^n$. 

Proof. We only need to show that $\mathcal{X}_0$ is reduced and $G$-irreducible. Note that

$$\mathcal{X}_0 = \text{Proj} \bigoplus_{k,j \geq 0} \text{gr}^j_F V_k,$$

where

$$\text{gr}^j_F V_k = \frac{\mathcal{F}^j V_k}{\mathcal{F}^{j+1} V_k}$$

is the graded piece of the filtration $\mathcal{F}$. Apparently $\bigoplus_{k,j \geq 0} \text{gr}^j_F V_k$ is reduced. Now pick any $f_i \in \mathcal{F}^j V_{k_i} \setminus \mathcal{F}^{j+1} V_{k_i}$ for $i = 1, 2$. Let $F_1, \ldots, F_N$ form the orbit of $F$ under $G$-action. Since the orbit is finite, we can find $F_i$ such that $\text{ord}_{F_i}(f_i) = G \cdot \text{ord}_F(f_i) = j_i$ for $i = 1, 2$. Suppose $\text{ord}_{F_2} = g \cdot \text{ord}_{F_1}$ for some $g \in G$. Then we have

$$G \cdot \text{ord}_F(g(f_1)f_2) = \text{ord}_{F_2}(g(f_1)f_2) = j_1 + j_2,$$

and consequently $g(f_1)f_2 \in \mathcal{F}^{j_1 + j_2} V_{k_1 + k_2} \setminus \mathcal{F}^{j_1 + j_2 + 1} V_{k_1 + k_2}$. Therefore we know that $\mathcal{X}_0$ is $G$-irreducible. \hfill $\Box$

An immediate consequence is the following theorem:

**Theorem 3.5.** If $X$ is $G$-equivariant $K$-stable, then $\beta^G(F) > 0$ for any $G$-dreamy divisor $F$ over $X$.

Combining the above results we finish the proof of Theorem A.

4. **EQUIVARIANT NORMALIZED VOLUMES**

Let $X$ be an $n$-dimensional $\mathbb{Q}$-Fano variety with group action $G$ and $F$ a prime divisor over $X$. Denote by $Y = C(X, -K_X)$ the cone over $X$ with respect to the polarization $-K_X$ and $O \in Y$ the vertex of the cone. Suppose $\pi : Z = \text{Bl}_O Y \to Y$ is the blow-up of $Y$ at $O$. Let $E$ be the exceptional divisor, and $\mathcal{F}$ the pull back of $F$ to $Z$. Denote the divisorial valuation $\text{ord}_E$ by $v_0$ and $\text{ord}_F$ by $v_F$. Then for $t > 0$, we have $v_t := v_0 + tv_F$ to be a quasi-monomial valuation centered at $O$. Note that there is a natural $G$-action induced on the cone $Y$ and the blow-up $Z$. Now we consider the $G$-pseudovaluation $G \cdot v_t$. The following proposition gives a relation between the derivative of the normalized volume $\hat{\text{vol}}(G \cdot v_t)$ and $\beta^G(F)$.

**Proposition 4.1.** Under the above notations, we have

$$\frac{d}{dt} \hat{\text{vol}}(v_t) \biggr|_{t=0} = (n + 1) \beta^G(F).$$

**Proof.** First of all, we have $A_Y(\mathcal{F}) = A_X(F)$, and $A_Y(0) = 1$. Therefore $A_Y(v_t) = 1 + tA_X(F)$. Next we compute the volume of $G \cdot v_t$. Let $V_m = H^0(X, -mK_X)$ and $V = \oplus V_m$. Note that for $f \in V_m$, we have $v_0(f) = m$. Then

$$\dim V/a_{\lambda}(G \cdot v_t) = \sum_{m=0}^{\lfloor \lambda \rfloor} \dim V_m/a_{\lambda}(G \cdot v_t) = \sum_{m=0}^{\lfloor \lambda \rfloor} \dim V_m - \sum_{m=0}^{\lfloor \lambda \rfloor} \dim V_m \cap a_{\lambda}(G \cdot v_t).$$

By asymptotic Riemann-Roch, we know that

$$\sum_{m=0}^{\lfloor \lambda \rfloor} \dim V_m = \frac{(-K_X)^n \lambda^{n+1}}{(n+1)!} + O(\lambda^n).$$
On the other hand, for any \( f \in V_m \), we know that \( g \cdot v_t(f) \geq \lambda \) is equivalent as \( g \cdot v_F(f) \geq \frac{\lambda - m}{t} \). Then we know that
\[
V_m \cap a_{\lambda}(G \cdot v_t) = H^0 \left( \mathcal{O}_X(-mK_X) \otimes a_{\lambda-m}(G \cdot \text{ord}_F) \right).
\]
According to Lemma 4.5 in [Li17], we know that
\[
\sum_{m=0}^{[\lambda]} \dim V_m \cap a_{\lambda}(G \cdot v_t) = \lambda^{n+1} \int_0^{+\infty} \frac{\text{vol}_X(\mathcal{O}_X(-K_X) \otimes a_x(G \cdot \text{ord}_F))}{(1 + tx)^{n+2}} \, dx + O(\lambda^n).
\]
Putting the above expressions together, we have
\[
\text{vol}(G \cdot v_t) = (-K_X)^n - (n + 1) \int_0^{+\infty} \frac{\text{vol}_X(\mathcal{O}_X(-K_X) \otimes a_x(G \cdot \text{ord}_F))}{(1 + tx)^{n+2}} \, dx.
\]
Taking the derivative we get
\[
\frac{d}{dt} \text{vol}(G \cdot v_t) \bigg|_{t=0} = (n + 1) \beta^G(F).
\]

An immediate consequence of Proposition 4.1 gives one direction of Theorem B:

**Corollary 4.2.** If the normalized volume function \( \text{vol} \) is minimized at \( v_0 \) among all finite-orbit \( G \)-pseudovaluations on \( Y \) centered at \( o \), then \( X \) is \( G \)-equivariantly K-semistable.

Repeating a similar computation as in the proof of Theorem 4.5 in [LX16] also gives the other direction of Theorem B.

### 5. Other Related Results

In this section, we list some other results we can get by introducing \( G \)-pseudovaluations. Let \( X \) be a variety with \( G \subset \text{Aut}(X) \) a group action on \( X \). By replacing usual valuations with \( G \)-pseudovaluations, we can define the \( G \)-log canonical threshold of any effective divisor \( D \) to be
\[
\text{Glct}(D) := \inf_E \frac{A_X(E)}{G \cdot \text{ord}_E(D)}.
\]
Next assume in addition that \( X \) is \( \mathbb{Q} \)-Fano. We define the \( G \)-equivariant alpha invariant of \( X \) to be
\[
\alpha_G(X) = \inf \{ \text{Glct}(D) | 0 \leq D \sim_{\mathbb{Q}} -K_X \}.
\]

**Remark 5.1.** Note that Tian first defines \( \alpha_G(X) \) analytically in [Tia87]. It is then shown in [CS08] that the analytic definition of \( \alpha_G(X) \) is the same as the following algebraic one:
\[
\alpha_G(X) = \inf_m \left\{ \text{lct} \left( X, \frac{1}{m} \Sigma \right) \left| \Sigma \text{ is a } G\text{-invariant linear subsystem in } | -mK_X| \right. \right\}.
\]
The above two algebraic definitions of \( \alpha_G(X) \) are in fact the same. Indeed, for any \( G \)-invariant linear system \( \Sigma \sim -mK_X \), pick any divisor \( D \in \Sigma \). Then for any prime divisor \( E \) over \( X \) and \( g \in G \), we have \( \text{ord}_E(gD) \geq \text{ord}_E(\Sigma) \). Therefore we know that
\[
\text{Glct} \left( \frac{1}{m} D \right) \leq \text{lct} \left( \frac{1}{m} \Sigma \right).
\]
Conversely, for any effective divisor \( D \in |-mK_X| \), let \( \Sigma \) be the linear subsystem of \( |-mK_X| \) spanned by \( \{ gD | g \in G \} \). Then for any effective divisor \( D' \in \Sigma \), we know that \( \text{ord}_E(D') \geq G \cdot \text{ord}_E(D) \) and hence \( \text{ord}_E(\Sigma) \geq G \cdot \text{ord}_E(D) \). Therefore we know that
\[
\text{Glct} \left( \frac{1}{m} D \right) \geq \text{lct} \left( \frac{1}{m} \Sigma \right).
\]

Next we will give another proof of the following result in [OS12] which is the \( G \)-equivariant version of Tian’s criterion.

**Theorem 5.1** (Theorem 1.10, [OS12]). Let \( X \) be a \( \mathbb{Q} \)-Fano variety of dimension \( n \geq 2 \) and \( G \subset \text{Aut}(X) \) a group action on \( X \). If \( \alpha_G(X) > n/(n+1) \), then \( X \) is \( G \)-equivariantly K-stable.

**Proof.** The idea of the proof follows from [Fuj17]. Take any \( G \)-special test configuration \((X, L)\). Let \( \text{ord}_F \) be the divisorial valuation on \( X \) induced by one of the irreducible components of \( X_0 \). Then the orbit of \( F \) under the action \( G \) is induced by all irreducible components of \( X_0 \) and hence finite. Let \( \pi : Y \to X \) be a \( G \)-equivariant birational morphism such that \( F_1 = F \ldots , F_N \) are prime divisors on \( Y \) and form the orbit of \( F \) under \( G \)-action. Set
\[
S^G(F) = \frac{1}{(-K_X)^n} \int_0^{+\infty} \text{vol}_Y \left( - \pi^* K_X - x \sum F_i \right) \, dx.
\]
It suffices to show that \( A_X(F) > S^G(F) \).

Let
\[
\tau^G(F) = \sup \{ t > 0 | \text{vol}_Y \left( - \pi^* K_X - t \sum F_i \right) > 0 \}.
\]
Using integration by parts, we have
\[
\int_0^{\tau^G(F)} \left( x - S^G(F) \right) \frac{d}{dx} \text{vol}_Y \left( - \pi^* K_X - x \sum F_i \right) \, dx = 0.
\]
Note that by Theorem A and Theorem B of [BFJ09], we have
\[
-\frac{1}{n} \frac{d}{dx} \text{vol}_Y \left( - \pi^* K_X - x \sum F_i \right) = N \text{vol}_{Y|F} \left( - \pi^* K_X - x \sum F_i \right),
\]
where \( \text{vol}_{Y|F} \) denotes the restricted volume (see [ELM+09] for definition). For simplicity, we use \( V(x) \) to denote the restricted volume function \( \text{vol}_{Y|F} \left( - \pi^* K_X - x \sum F_i \right) \). Then we have
\[
\int_0^{\tau^G(F)} \left( x - S^G(F) \right) V(x) \, dx = 0.
\]
Using log concavity of restricted volume, we have
\[
\left( x - S^G(F) \right) V(x) \leq \left( x - S^G(F) \right) V \left( S^G(F) \right) \left( \frac{x}{S^G(F)} \right)^{n-1}.
\]
Therefore we get that
\[
S^G(F) \leq \frac{n}{n+1} \tau^G(F).
\]
Now suppose \( A_X(F) \leq S^G(F) \). Then we know that
\[
A_X(F) \leq \frac{n}{n+1} \tau^G(F).
\]
For arbitrarily small $\epsilon > 0$, pick $0 \leq D \sim_{\mathbb{Q}} -K_X$ such that $G \cdot \text{ord}_F(D) = \tau_G(F) - \epsilon$. Then we know that

$$\text{Glct}(D) \leq \frac{A_X(F)}{G \cdot \text{ord}_F(D)} \leq \frac{n}{n+1} \cdot \epsilon.$$

Contradicting to the assumption that $\alpha_G(X) > n/(n+1)$.

□

Using $G$-log canonical threshold, we can also define the $G$-delta invariant of $X$ to be

$$G\delta(X) := \lim_{m} \sup \text{Glct}_m(X),$$

where

$$G\delta_m(X) := \inf \{ \text{Glct}(D) \mid D \text{ is of } m \text{-basis type} \}. $$

For the definition of usual delta invariant, we refer to [BJ17] for details. We have the following theorem parallel to one of the results in [BJ17]:

**Theorem 5.2.** Let $X$ be a $\mathbb{Q}$-Fano variety with $G \subset \text{Aut}(X)$ a group action on $X$. We have that $G\delta(X) = \lim_{m} G\delta_m(X)$. Let

$$S^G(F) := \frac{1}{(-K_X)^n} \int_{0}^{+\infty} \text{vol}(\mathcal{O}_X(-K_X) \otimes a_x(G \cdot \text{ord}_F)) \, dx$$

Then we have

$$G\delta(X) = \inf_{F} \frac{A_X(F)}{S^G(F)}.$$ 

**Proof.** By definition, we have

$$G\delta_m(X) = \inf_{D} \inf_{F} \frac{A_X(F)}{G \cdot \text{ord}_F(D)}$$

$$= \inf_{F} \inf_{D} \frac{A_X(F)}{G \cdot \text{ord}_F(D)}$$

$$= \inf_{F} \sup_{D} \{ G \cdot \text{ord}_F(D) \},$$

where $D$ runs through all $m$-basis type divisors and $F$ runs through all prime divisors over $X$.

Let $V_m = H^0(X, -mK_X)$. For any prime divisor $F$ over $X$, we construct the following filtration

$$\mathcal{F}^x V_m = H^0(X, \mathcal{O}_X(-mK_X) \otimes a_x(G \cdot \text{ord}_F)), \, x \geq 0.$$ 

In order to make $G \cdot \text{ord}_F(D)$ as large as possible, we adapt a basis of $V_m$ to $\mathcal{F}^* V_m$ and get

$$\sup_{D} \{ G \cdot \text{ord}_F(D) \} = \frac{1}{mN_m} \left( \sum_{i=0}^{mT_m-1} i \left( \text{dim} \mathcal{F}^i V_m - \mathcal{F}^{i+1} V_m \right) + mT_m \cdot \mathcal{F}^m T_m V_m \right)$$

$$= \frac{1}{mN_m} \sum_{i=0}^{mT_m} \text{dim} \mathcal{F}^i V_m,$$

where $N_m = \text{dim} V_m$ and $T_m = \frac{1}{m} \max \{ G \cdot \text{ord}_F(s) \mid s \in V_m \}$. We follow the notation in [BJ17] and denote

$$S^G_m(F) = \frac{1}{mN_m} \sum_{i=0}^{mT_m} \text{dim} \mathcal{F}^i V_m.$$
By Corollary 2.12 of [BJ17] we know that \( \lim S^G_m (F) = S^G (F) \), and this finishes the proof. \( \square \)

Theorem A and Theorem 5.2 immediately gives the following corollary:

**Corollary 5.3.** Let \( X \) be a \( \mathbb{Q} \)-Fano variety with \( G \subset \text{Aut}(X) \) a finite group action on \( X \). Then

1. \( X \) is \( G \)-equivariantly K-semistable if and only if \( G \delta (X) \geq 1 \);
2. \( X \) is uniformly \( G \)-equivariantly K-stable if and only if \( G \delta (X) > 1 \).

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