Scalar field Brans-Dicke Universe in Lyra’s manifold

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Abstract In this paper, we investigate a scalar field Brans-Dicke cosmological model in Lyra’s manifold which is based on both the modifications in terms of geometry as well as energy terms of original Einstein’s field equations. We have examined the validity of proposed cosmological model on observational scale by performing statistical analysis from $H(z)$, SN Ia and joint $H(z)$ & BAO data sets. We find that estimated values of Hubble’s constant and energy density parameters nicely match with their corresponding values, obtained by recent observations of WMAP and Plank collaboration. We also derived deceleration parameter, age of universe and jerk parameter in terms of red-shift and computed its present values. The dynamics of deceleration parameter in derived universe shows a signature flipping from positive to negative value and also indicates that the present universe is in accelerating phase.

Keywords: Observational dataset; Modified theories of gravity; Large-scale Structure; Deceleration parameter;Jerk parameter
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1 Introduction

In 1915, Einstein had proposed General Theory of Relativity (GTR) and beautifully described the geometry of space and time in elegance with gravity. In this theory, it has been proposed that the energy-momentum tensor are due to curvature of space and time through famous Einstein’s field equation: $R_{ij} - \frac{1}{2}Rg_{ij} = \frac{8\pi G}{c^4}T_{ij}$, where $R_{ij}$, $T_{ij}$ are the Ricci curvature tensor and energy momentum tensor respectively. This equation specifies how the geometry of space and time is influenced by whatever matter and radiation are present, and form the core of Einstein’s general theory of relativity. Till now, various cosmological models have been studied in GTR and significantly described different phase of universe \cite{1,2,3,4,5,6}. Among these models, \Lambda CDM model is credited the most successful cosmological model to describe the current features of observed universe but it suffers cosmological constant problem or vacuum catastrophe. The cosmological constant problem gave clue to think about the modification in GTR. In the literature, various modification in Einstein’s theory have been proposed by cosmologist from time to time, since it’s inception. In this paper, we have applied modification in both the geometrical term as well as in energy-momentum term by taking into account Lyra’s geometry \cite{7} and Brans-Dicke theory of gravitation \cite{8} respectively.

Lyra’s geometry represents the modification of Riemannian geometry with aid of gauge function into structureless manifold. In this approach, gauge function naturally replace the cosmological constant and hence the cosmological constant problem. This means that time like constant gauge function or constant displacement field plays the role of cosmological constant - a physically accepted candidate of dark energy which is required to accelerate the universe in it’s present epoch.
It is worth to note that singularity free cosmological models have been developed in the framework of emergent universe scenario by assuming displacement field as time dependent while Ellis & Maartens have investigated a cosmological model with normal matter and a scalar field sources in general relativity. Recently there is an upsurge of interest in scalar fields in general relativity and alternative theories of gravitation in the context of inflationary cosmology. Therefore the study of cosmological models in Lyra’s geometry may be relevant for inflationary models. Some important applications of constant and time varying displacement field in Lyra’s geometry are found in Refs. Many experimental and theoretical tests of GTR confirm that the local motion of particle does not affect due to large scale matter distribution which is why Mach’s principle can be violated in GTR. So, Brans and Dicke had proposed a modified theory of gravity which was simply formulated to validate Mach’s principle. The proposed BD theory of gravitation not only validate the Mach’s principle but also describe the dynamics of universe from inflation era to present accelerating epoch. Note that BD theory would also modify the average expansion rate of universe due to appearance Brans-Dicke coupling constant ω. Recently, Akarsu et al. have studied dynamical behaviour of effective dark energy and the red-shift dependency of the expansion anisotropy in the framework of BD theory of gravity. This study reveals that high positive values of ω imply minimal deviation from the ΛCDM model while small values of ω produce huge deviation from standard ΛCDM model. In fact, BD theory of gravity is an interesting alternative to GTR and effectively introduces a scalar field φ, in addition to the metric tensor field gij. The scalar field φ play the role of GI and BD coupling parameter is constrained as ω ≃ 40000 for it’s consistency with solar system bounds. Several investigations have been made in BD cosmology with non-minimally coupling of scalar field and minimally interacting holographic dark energy models.

Motivated by the above discussion, we investigate, in this paper, a scalar field Brans-Dicke cosmological model in Lyra’s manifold. We choose ω = 40000 and displacement field β as time like constant vector. The outlines of this paper is as follows: Section I is introductory in nature. In section 2, we have developed the basic mathematical formalism of scalar field Brans-Dicke universe in Lyra’s manifold. Section 3 deals the statistical analysis of derived model with observational data sets and estimation of model parameters. In sections 4, we have computed the present values of deceleration parameter, age of universe and jerk parameter of the model under consideration. Finally, we summarized our findings in section 5.

The basic mathematical formalism of scalar field Brans-Dicke universe in Lyra’s manifold

The Einstein’s Brans-Dicke field equations in Lyra’s manifold is read as

\[ G_{ij} + \frac{3}{2} \psi_i \psi_j - \frac{3}{4} g_{ij} \psi^k \psi^k = \frac{8 \pi T_{ij}}{\phi c^4} - \frac{\omega}{\phi^2} \left( \dot{\phi} \right)^2 \left( \phi, \phi, \frac{1}{2} g_{ij} \dot{\phi} \dot{\phi} \right) - \frac{1}{\phi} \left( \phi, (\psi, \phi) - g_{ij} \Box \phi \right) \]  

\[ \Box \phi = \phi^i_i = \frac{8 \pi T}{(3 + 2 \omega)c^2} \]  

where \( G_{ij} \), \( \psi^i \), \( \omega \) and \( \phi \) are Einstein’s curvature tensor, displacement vector field of Lyra’s geometry, Brans-Dicke coupling constant and Brans-Dike scalar field respectively and the other symbols have their usual meaning in Riemannian geometry. We also suppose that \( \psi_i = (\beta(t), 0, 0, 0) \) where \( \beta(t) \) is time varying displacement vector.

The isotropic and homogeneous space-time is given by

\[ ds^2 = c^2 dt^2 - a^2(t)(dx^2 + dy^2 + dz^2) \]  

where \( a(t) \) is scale factor which define the rate of expansion.

The energy momentum tensor of perfect fluid is read as

\[ T_{ij} = (\rho + p)u_i u_j - p g_{ij} \]  

where co-moving co-ordinate \( u_i u^i = 1 \). For space-time, solving equations \( 1 \), \( 2 \) and \( 4 \) together, we obtain the following system of equations

\[ 3 \frac{\ddot{a}}{a^2} + \frac{3}{4} \beta^2 = \frac{8 \pi \rho}{\phi c^2} - \frac{\omega}{2} \left( \frac{\dot{\phi}}{\phi} \right)^2 + 3 \frac{\dot{\phi}}{a \phi} \]  

\[ -2 \frac{\ddot{a}}{a} - \frac{3}{4} \beta^2 = \frac{8 \pi p}{\phi c^2} + \frac{\omega \dot{\phi}^2}{2 \phi c^2} + 2 \frac{\ddot{\phi}}{a \phi} + \ddot{\phi} \]  

\[ \frac{\ddot{\phi}}{\phi} + 3 \frac{\dot{\phi}^2}{a \phi} = \frac{8 \pi (\rho - 3p)}{(2 \omega + 3) \phi c^2} \]
Here, over dot stands derivative with respect to time.
The Hubble’s parameter is defined as \( H = \frac{\dot{a}}{a} \Rightarrow \frac{\dot{a}}{a} = H + H^2 \)

Putting these values in equation (5), we obtain

\[-2\dot{H} - 3H^2 + \frac{3}{4} \beta^2 = \frac{8\pi p}{\phi c^2} + \frac{\omega \dot{\phi}^2}{2\dot{\phi}^2} + 2 \frac{\ddot{a}}{a} + \frac{\ddot{\phi}}{\phi} \quad (8)\]

The deceleration parameter in terms of \( H \) is read as
\( q = -\frac{\ddot{a}}{a H^2} = -1 - \frac{\dot{H}}{H^2} \)
\( \dot{H} = -1(1 + q)H^2 \)

From equation (5), we obtain

\[(2q - 1)H^2 + \frac{3}{4} \beta^2 = \frac{8\pi p}{\phi c^2} + \frac{\omega \dot{\phi}^2}{2\dot{\phi}^2} + 2 \frac{\ddot{a}}{a} + \frac{\ddot{\phi}}{\phi} \quad (9)\]

After some algebra, Equation (9) leads the following expression for \( q \)

\[2q = \frac{4H^2 - 3\beta^2}{H^2} + \frac{1}{H^2} \left[ \frac{8\pi p}{\phi c^2} + \frac{\omega \dot{\phi}^2}{2\dot{\phi}^2} + 2 \frac{\ddot{a}}{a} + \frac{\ddot{\phi}}{\phi} \right] \quad (10)\]

We have taken here the perfect fluid so isotropic pressure \( p \) is positive. Also, all the other quantities in second term of equation (10) on RHS are positive. For acceleration in the model \( q \) must be negative. Thus the acceleration is only possible when \( \frac{4H^2 - 3\beta^2}{H^2} < 0 \).

Hence for accelerating model, we have \( 4H^2 - 3\beta^2 < 0 \Rightarrow \frac{1}{3} < \frac{\pi}{\sqrt{H^2}} \). This is similar to cosmological constant and we can argue that that displacement vector \( \beta \) behaves like cosmological constant. Thus on theoretical ground, for acceleration \( \Omega_\beta = \frac{\beta^2}{3H^2} > 0.33 \)

Equation (5) in terms of \( H \) is given by

\[3H^2 - \frac{3}{4} \beta^2 = \frac{8\pi p}{\phi c^2} - \frac{\omega}{2} \left( \frac{\dot{\phi}}{\phi} \right)^2 + 3 \frac{\ddot{a}}{a} \phi \quad (11)\]

The continuity equation for perfect fluid is given by

\[\dot{\rho} + 3(\rho + p)H = 0 \quad (12)\]

where \( H = \frac{\dot{a}}{a} \) is the Hubble’s parameter. Also it is well known that \( p = \gamma \rho \) is the equation of state of perfect fluid having values of \( \gamma \) in the range \( 0 \leq \gamma \leq 1 \).

Integrating equation (12), we obtain

\[\rho = \rho_0 \left( \frac{a}{a_0} \right)^{-3(1+\gamma)} \quad (13)\]

where \( a_0 \) and \( \rho_0 \) are the constants of integration and is taken as the present value of scale factor and energy density.

Now, we define matter energy density parameter \( \Omega_m \) and \( \beta \) energy density parameter \( \Omega_\beta \) as following

\[\Omega_m = \frac{8\pi \rho}{3H^2 + 2\dot{\phi}} \quad \Omega_\beta = \frac{\beta^2}{4H^2} \quad (14)\]

Following Goswami [21], we have defined deceleration parameter in terms of \( a \) and \( \phi \) as

\[q = -\frac{\ddot{a}}{a H^2}, \quad q_\phi = -\frac{\ddot{\phi}}{\phi H^2} \quad (15)\]

Using equations (13), (14) and (30), the equations (5) - (7) takes the following form

\[\Omega_m + \Omega_\beta = 1 + \frac{\omega}{6} \Psi^2 - \Psi \quad (16)\]

\[\gamma \Omega_m + \Omega_\beta = \frac{2}{3} - \frac{1}{3}(q_\phi + 1) + \frac{\omega}{6} \Psi^2 + \frac{2}{3} \Psi \quad (17)\]

\[-q_\phi + 3\Psi = 3(3 - 3\gamma)\Omega_m \quad (18)\]

where \( \Psi = \frac{\dot{\phi}}{\phi} \).

Solving equations (16) - (18), we get

\[q = \frac{(\omega - \omega\gamma - 3\gamma + 2)}{1 - 3\gamma} q_\phi + \frac{3(\omega - 3\gamma - 4\gamma + 5)}{1 - 3\gamma} \Psi = 2 \quad (19)\]

Equation (19) leads the following relation

\[\phi = \phi_0 \left( \frac{a}{a_0} \right)^{\frac{1-3\gamma}{\omega - \omega\gamma - 3\gamma + 2}} \quad (20)\]

and

\[\Psi = \frac{1 - 3\gamma}{\omega - \omega\gamma - 3\gamma + 2} \quad (21)\]

where \( \phi_0 \) denotes the present values of scalar field.

Substituting the value of \( \Psi \) from equation (21) in equation (16), we obtain

\[\Omega_m + \Omega_\beta = 1 - \frac{(1 - 3\gamma)(5\omega - 3\omega\gamma - 18\gamma + 12)}{6(\omega - \omega\gamma - 3\gamma + 2)^2} \quad (22)\]
3 Likelihoods and Data

The scale factor in terms of red-shift is read as

\[ a = \frac{a_0}{1 + z} \quad (23) \]

Equations (21), (20), (22) and (23) leads the following expression for Hubble’s function in terms of red-shift as

\[ H(z) = \frac{H_0}{(1 - \frac{1}{\beta + 2})} \times \left(\frac{\Omega_m(1 + z)^{\frac{\beta}{3}} + (\Omega_\beta)^0}{(\Omega_m)_0(1 + z)\frac{\beta}{3}} + (\Omega_\beta)_0\right) \]

where \( H_0, (\Omega_m)_0 \) and \( (\Omega_\beta)_0 \) denote the values of Hubble constant, matter energy density parameter and \( \beta \) energy density parameter at present respectively.

We consider 46 observational \( H(z) \) data points tabulated in Table 1 to estimate the present values of \( H_0 \), \( (\Omega_m)_0 \) and \( (\Omega_\beta)_0 \) by well known statistical method of \( \chi^2 \) test. For this purpose, we defined \( \chi^2 \) as following (22):

\[ \chi^2_{46}(H(z), (\Omega_m)_0, (\Omega_\beta)_0) = \sum_{i=1}^{46} \left[ \frac{H(z_i, (\Omega_m)_0, (\Omega_\beta)_0) - H_{obs}(z_i)}{\sigma_i} \right]^2 \]

where \( H_{obs}(z_i) \) is the observed values of Hubble’s function given in Table 1. \( \sigma_i \) denotes the standard deviation.

Similarly, the joint \( \chi^2 \) text for combined \( H(z) \) and BAO data is read as

\[ \chi^2_{joint} = \chi^2_{H(z)} + \chi^2_{BAO} \quad (26) \]

The distance modulus is given by

\[ \mu(z) = m_b - M = 5\log_{10}D_L(z) + \mu_0 \quad (27) \]

where \( D_L \) is the luminosity distance and other parameters have their usual meaning. For model (3), \( D_L \) is obtained as

\[ D_L = \frac{c(1 + z)}{H_0} \int_0^z \frac{dz}{h(z)}, \quad h(z) = \frac{H(z)}{H_0} \]

In Table 1, \( H(z) \) is in unit of km s\(^{-1}\) Mpc\(^{-1}\).

Thus, \( \chi^2_{SN1a} \) is given by

\[ \chi^2_{SN1a}(H_0, (\Omega_m)_0, (\Omega_\beta)_0) = \sum_{i=1}^{581} \left[ \frac{\mu(z_i, (\Omega_m)_0, (\Omega_\beta)_0) - \mu_{obs}(z_i)}{\sigma_i} \right]^2 \]

where \( \mu_{obs}(z_i) \) is the observed values of distance modulus. The numerical results of statistical analysis by bounding the derived model with astrophysical observations are summarized in Table 2.

4 Cosmological parameters of the model

4.1 Deceleration parameter

The deceleration parameter of derived model in terms of red-shift is read as

\[ q = -1 + \frac{(1 + z)H'(z)}{H(z)} \quad (30) \]

where \( H'(z) \) is the first order derivative of \( H(z) \) with respect to \( z \).

Solving equations (3) and (30), we get

\[ q = -1 + \frac{(3\omega - 3\omega_\gamma^2 + 6\gamma + 7)}{(\omega - \omega_\gamma - 3\gamma + 2)} (\Omega_m)_0 (1 + z) \frac{3\omega - 3\omega_\gamma^2 + 6\gamma + 7}{(\omega - \omega_\gamma - 3\gamma + 2) + (\Omega_\beta)_0} \]

The graphical behaviour of deceleration parameter is depicted in Figure 3. It is now clear that one can easily obtain the present value of deceleration parameter as -0.574, -0.635 and -0.583 by bounding equation (31) with \( H(z) \), SN Ia and \( H(z) + BAO \) data sets respectively.

Figure 3 clearly shows the signature flipping behavior of deceleration parameter with decreasing value of redshift i.e. at beginning \( q \) was evolving with positive sign which indicates that early universe was in decelerating phase and it turns into accelerating mode at \( 0.7 \leq z \leq 0.9 \). This transitioning evolution of \( q \) from positive to negative value leads the concept of hybrid universe. In recent past, various cosmological models in different physical context with hybrid expansion law have been investigated [46,47,48,49,50,51,52,53,54].

4.2 Age of Universe

The age of scalar field Brans-Dike universe is obtained as

\[ H(z) = \frac{1}{1 + z} \frac{dz}{dt} \quad (32) \]
Table 1: Hubble parameter $H(z)$ with redshift and errors $\sigma_z$.

| S.N. | $z$  | $H(z)$ | $\sigma_z$ | References |
|------|------|--------|------------|------------|
| 1    | 0    | 67.77  | 1.30       | [30]       |
| 2    | 0.07 | 69     | 19.6       | [31]       |
| 3    | 0.09 | 69     | 12         | [32]       |
| 4    | 0.01 | 69     | 12         | [33]       |
| 5    | 0.12 | 68.6   | 26.2       | [31]       |
| 6    | 0.17 | 83     | 8          | [33]       |
| 7    | 0.179| 75     | 4          | [33]       |
| 8    | 0.1993| 75  | 5          | [33]       |
| 9    | 0.2  | 72.9   | 29.6       | [31]       |
| 10   | 0.24 | 79.7   | 2.7        | [32]       |
| 11   | 0.27 | 77     | 14         | [33]       |
| 12   | 0.28 | 88.8   | 36.6       | [31]       |
| 13   | 0.35 | 82.7   | 8.4        | [36]       |
| 14   | 0.352| 83     | 14         | [33]       |
| 15   | 0.38 | 81.5   | 1.9        | [37]       |
| 16   | 0.3802|83 | 13.5       | [38]       |
| 17   | 0.4  | 95     | 17         | [32]       |
| 18   | 0.4004|77 | 10.2       | [35]       |
| 19   | 0.4247|87.1 | 11.2       | [35]       |
| 20   | 0.43 | 86.5   | 3.7        | [35]       |
| 21   | 0.44 | 82.6   | 7.8        | [35]       |
| 22   | 0.44497|92.8 | 12.9       | [35]       |
| 23   | 0.47 | 89     | 49.6       | [40]       |
| 24   | 0.4783|80.9 | 9          | [38]       |
| 25   | 0.48 | 97     | 60         | [35]       |
| 26   | 0.51 | 90.4   | 1.9        | [37]       |
| 27   | 0.57 | 96.8   | 3.4        | [35]       |
| 28   | 0.593| 104    | 13         | [34]       |
| 29   | 0.6  | 87.9   | 6.1        | [35]       |
| 30   | 0.61 | 97.3   | 2.1        | [37]       |
| 31   | 0.68 | 92     | 8          | [34]       |
| 32   | 0.73 | 97.3   | 7          | [39]       |
| 33   | 0.781| 105    | 12         | [34]       |
| 34   | 0.875| 125    | 17         | [35]       |
| 35   | 0.88 | 90     | 40         | [35]       |
| 36   | 0.9  | 117    | 23         | [35]       |
| 37   | 1.037| 154    | 20         | [34]       |
| 38   | 1.3  | 168    | 17         | [35]       |
| 39   | 1.363| 160    | 33.6       | [42]       |
| 40   | 1.43 | 177    | 18         | [33]       |
| 41   | 1.53 | 140    | 14         | [35]       |
| 42   | 1.75 | 202    | 40         | [35]       |
| 43   | 1.965| 186.5  | 50.4       | [42]       |
| 44   | 2.3  | 224    | 8          | [45]       |
| 45   | 2.34 | 222    | 7          | [44]       |
| 46   | 2.36 | 226    | 8          | [45]       |

Table 2: Summary of the numerical result.

| Source/Data | Model parameters | Value at present |
|-------------|------------------|------------------|
| $H(z)$      | $(\Omega_m)_0$   | 0.284            |
| $H(z)$      | $(\Omega_b)_0$   | 0.7159           |
| $H(z)$      | $H_0$            | 0.0698 (Gyr$^{-1}$) |
| SN Ia       | $(\Omega_m)_0$   | 0.243            |
| SN Ia       | $(\Omega_b)_0$   | 0.7569           |
| SN Ia       | $H_0$            | 0.0725 (Gyr$^{-1}$) |
| $H(z)+BAO$  | $(\Omega_m)_0$   | 0.278            |
| $H(z)+BAO$  | $(\Omega_b)_0$   | 0.720            |
| $H(z)+BAO$  | $H_0$            | 0.0703 (Gyr$^{-1}$) |
Fig. 1 The likelihood contours at 1σ and 2σ confidence levels around the best fit values as $H_0 = 0.0698 \text{ Gyr}^{-1} \sim 68.297 \text{ km s}^{-1} \text{ Mpc}^{-1}$ & $(\Omega_m)_0 = 0.284$ (left panel), $H_0 = 0.0725 \text{ Gyr}^{-1} \sim 70.939 \text{ km s}^{-1} \text{ Mpc}^{-1}$ & $(\Omega_m)_0 = 0.243$ (middle panel) and $H_0 = 0.0703 \text{ Gyr}^{-1} \sim 68.786 \text{ km s}^{-1} \text{ Mpc}^{-1}$ & $(\Omega_m)_0 = 0.278$ (right panel) in $n - H_0$ plane obtained by bounding the derived model with $H(z)$, SN Ia and $H(z) + BAO$ observational data sets respectively.

Fig. 2 The best fit curve of derived model with 46 $H(z)$ (left panel), SN Ia (middle panel) and $H(z) + BAO$ (right panel) observational data sets are shown.

Fig. 3 Dynamical behavior of deceleration parameter with respect to $z$.

Solving equations (24) and (32), we get

$$t = \frac{2(\gamma(3 + \omega) - \omega - 2)}{\sqrt{(\Omega_\beta)_0 H_0(3\omega\gamma^2 - 3\omega - 6\gamma - 7)}} \times \arctanh \left( \frac{(\Omega_m)_0(1 + z)^{2 + 6\gamma + 3\omega - 3\omega z^2}}{(\Omega_\beta)_0} + (\Omega_\beta)_0 \right)$$

(33)

One can easily compute the present age of universe by putting $z = 0$ and estimated values of $H_0$, $(\Omega_m)_0$ and $(\Omega_\beta)_0$ given in table II. In doing so, we obtain the present age of derived universe as 14.0236 Gyr, 14.0902 Gyr and 13.98 Gyr in elegance with $H(z)$, SN Ia and $H(z) + BAO$ data sets. It is important to note that the present age of derived universe nicely match with the age of universe, predicted by recent WMAP observations [55] and Plank collaborations [56]. Therefore, the model under consideration have pretty consistency with recent astrophysical observations.
4.3 Jerk parameter

The jerk parameter ($j$) [19], in terms of red-shift and Hubble’s function is read as

$$j = 1 - (1 + z)^{-1} H'(z) + \frac{1}{2} (1 + z)^2 \frac{H''(z)}{H(z)}$$

(34)

Differentiating equation (24) with respect to $z$ and putting $H'(z)$ and $H''(z)$ in equation (34), we get

$$j = 1 - \left( \Omega_m(7 + 6\gamma + 3\omega - 3\gamma^2\omega)(1 + z) \frac{7 + 6\gamma + 3\omega - 3\omega\gamma^2}{2 + 2\omega - 3\omega\gamma} \right)$$

$$+ \frac{(\Omega_m)(1 + z)^{3\omega - 3\omega\gamma}}{\kappa_1(1 + z)^{3\omega - 3\omega\gamma}} + \frac{(\Omega_m)(1 + z)^{\gamma}}{\kappa_2(1 + z)^{\gamma}}$$

(35)

where

$$\kappa_1 = (7 + 6\gamma + 3\omega - 3\omega\gamma^2)^2(5 + 2\omega - 3\omega\gamma^2 + \gamma(9 + \omega))^2$$

and

$$\kappa_2 = 2(2 + \omega - \gamma(3 + \omega))^4.$$
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