Free massless particle Dirac like wave equations of any spin in curved space-time

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Abstract. We derived Dirac like equations for massless particles of any spin in a global curved space-time. In an axially symmetric space-time the wave functions factorize in two parts. A spin-connection independent function which resembles the solution in a flat space-time, and a second spin connection dependent function which is calculated analytically for static Friedman-Robertson-Walker and Schwarzschild metrics.

1. Introduction

In previous contributions [1] it was demonstrated that in a Minkowski flat space-time, wave equations for massless particles of any spin can be written as a matrix equation,

\[
\left( \begin{array}{c}
\gamma^0 - \gamma_1 \hat{p}_1 - \gamma_2 \hat{p}_2 - \gamma_3 \hat{p}_3
\end{array} \right) \Phi^{(4s)} = 0,
\]

where is the spin, \( \hat{E} = i \hbar \partial_0 \), \( \hat{p}_i = -i \hbar \partial_i \) (i = 1, 2, 3) are energy and momentum operators, \( \gamma_i \) are \( (4s) \times (4s) \) hermitian matrices and \( \gamma_0 \) is the \( (4s) \times (4s) \) unit matrix. The wave functions \( \Phi^{(4s)} \) form bases for \( D^{(s-1/2,1/2)} \) representations of the Lorentz group, which we may write as a column matrix with \( 2s-1 \) components of spin \( s-1 \) and \( 2s+1 \) components of spin \( s \), altogether \( 4s \) components. Explicitly,

\[
\Phi^{(4s)} = \begin{pmatrix}
\psi^{s-1}_{s-1} \\
\vdots \\
\psi^{s+1}_{s-1} \\
\psi^s_{s+1} \\
\vdots \\
\psi^s_{s}
\end{pmatrix}.
\]

Here we note that the Hermitian conjugate of \( \Phi^{(4s)} \) is defined as \( [\Phi^{(4s)}]^H = [\Phi^{(4s)}]^T \) and that \( [\Phi^{(4s)}]^H \gamma^0 [\Phi^{(4s)}] \) is a scalar. Likewise, \( [\Phi^{(4s)}]^H \gamma^a [\Phi^{(4s)}] \) form a vector. We shall use these observations latter on to calculate the spin connection for the wave function \( \Phi^{(4s)} \). For free particles the spin \( (s-1) \) components of the wave function are set to be zero. By doing so, the subsidiary conditions which restrict helicity values to forward helicity and backward helicity only are satisfied.
For spin 1 particles (a photon) Eq. (1) is equivalent to the Maxwell equations. For spin 2 Eq. (1) describes a free graviton with no sources. Our main objective in the present note is to reformulate these equations in a curved space-time. To this aim we make use of the Principle of Equivalence. Namely, Curved space-time is locally a Minkowski flat space-time. Accordingly, Eq. (1) is valid in within sufficiently small region around any point in a curved space-time. With this in mind, we calculate in Section 2 the appropriate spin connection for the function \( \Phi \) and rewrite Eq. (1) for a general global space-time. In section 3 we consider the equations derived in the vicinity of spherically symmetric massive body such as a neutron star or a black hole. Following Refs. [2,3] we factorize the wave function into a spin connection dependent term which is rather easy to calculate in axisymmetric space-time and a spin connection independent term which can be obtained from solving an equation rather similar to Eq. (1). We conclude in section 4.

2. Covariant formulation

In what follows we reformulate Eq. (1) in a general global curved space-time. We first rewrite Eq. (1) using natural units with \( \hbar = 1 \) and a metric signature \((+,-,-,-)\). Greek indices refer to the general world index, whilst Latin indices refer to the flat Minkowski tangent space. We denote the metric of the curved space by \( g_{\mu \nu} \), where \( g_{\mu \nu} = diag(+1,-1,-1,-1) \) stands for the Minkowski metric. With these notations we write Eq. (1) as,

\[
\gamma^{a} \eta_{ab} \partial_{a} \phi = 0 . \quad (3)
\]

The Gamma matrices in Eqs. (1,3) form presentations of bi-quaternions, they factorize the d’Alembertian operator and have eigenvalues +1, and −1 only. This is a reflection of the fact that the subsidiary conditions are encoded in the gamma matrices [1]. The gamma matrices satisfy the following relations,

\[
(y^{a})^\dagger = (y^{a}) ; \quad (y^{a})^{2} = y^{0} , \quad (4.1)
\]

\[
\{y^{a}, y^{b}\} = 2i(0)\delta^{ab} ; \quad a, b = 0,1,2,3 , \quad (4.2)
\]

\[
[y^{a}, y^{b}] = i\epsilon^{abc} y_{c} ; \quad a, b, c = 0,1,2,3, \quad (4.3)
\]

\[
\text{trace}(y^{a}) = 0 ; \quad a=1,2,3 . \quad (4.4)
\]

The gamma matrices above carry tangent space indices so that they maintain a flat space-time form. As shown below, the spinor representations of the Lorentz generators are commutators of the gamma matrices, an observation that simplifies the calculation of the affine connection for the wave function \( \Phi \).

In order to rewrite Eq. (3) in a curved space-time one must replace the Minkowski metric \( \eta_{ab} \) by a curved space-time metric tensor \( g_{\mu \nu} \) and replace the partial local derivatives by covariant derivatives. In addition, one must transform the local gamma matrices by global gamma matrices by means of vierbein fields [4]. One obtains,

\[
E_{a}^{\mu} \gamma^{a} g_{\mu \nu}[\partial_{\nu} + \Omega_{\nu}(x)]\Phi(x) = 0 . \quad (5)
\]

In the above equation \( E_{a}^{\mu} \) are inverse vierbein fields, and \( \Omega_{\nu} \) is the connection coefficient for the wave function. A parallel transport of the wave function should then satisfy the relation,

\[
\Phi(x + dx) = \Phi(x) - \Omega_{\mu}(x)\Phi(x) dx^{\mu} . \quad (6)
\]

To evaluate the connection \( \Omega_{\mu}(x) \) we recall that \( S(x) = \phi^{H}(x) y^{0} \Phi(x) \) is a scalar and that \( V^{\mu}(x) = \phi^{H}(x) y^{a} \Phi(x) \) is a vector and therefore must obey the transformation rules for scalars and vectors,
respectively. Consider first $S(x)$ which must remain unchanged under parallel transport. Using Eq. (6) one obtains,

\[ S(x + dx) = S(x) - \Phi^H(x) [ \gamma^a \Omega^H_\mu(x) + \Omega_\mu(x) \gamma^a ] \Phi(x) dx^\mu, \]  

where terms of order $dx^\mu dx^\nu$ have been neglected. For Eq. (8) to satisfy scalar parallel transport rule the quantity in the square bracket must vanish, i.e.,

\[ \Omega^H_\mu(x) = -\Omega_\mu(x) \]  

Next we consider the vector $V(x)$ which transports as a local vector. Again using Eq. (7) one obtains,

\[ V^a(x + dx) = V^a(x) - \Phi^H(x) [ \gamma^a, \Omega_\mu(x) ] \Phi(x) dx^\mu. \]  

Then the second term must satisfy,

\[ \Phi^H(x) [ \gamma^a, \Omega_\mu(x) ] \Phi(x) = \omega^a_{\mu b} V^b, \]  

where $\omega^a_{\mu b}$ is the spin connection [4]. This last expression indicates that the commutator in the expression above must be proportional to $\gamma^b$ and that $\Omega_\mu$ is related to multiplications of gamma matrices. Let us assume that,

\[ \Omega_\mu = C \omega_{\mu bc} \gamma^b \gamma^c, \]  

where $C$ is a constant. It is straightforward to show using properties of the gamma matrices Eqs. (4) above that,

\[ [ \gamma^a, \Omega_\mu(x) ] = 4C \omega^a_{\mu b} \gamma^b. \]  

Comparing this last expression with Eq. (11) we choose $C = 1/4$ so that $\Omega_\mu = (1/4) \omega_{\mu bc} \gamma^b \gamma^c$. Then the spin connection term in Eq. (5) can be written explicitly as,

\[ E^\mu_a \gamma^a \Omega_\mu = \frac{1}{2} \gamma^b [ E^\sigma_b \partial_\sigma ln e + \partial_\sigma E^\sigma_b ], \]  

where we have substitute the well-known expression [4] for the spin connection, i.e.,

\[ \omega^a_{\mu b} = e^a_\mu E^\sigma_b \Gamma^\nu_{\sigma \mu} + e^a_\mu \partial_\mu E^\sigma_b. \]  

In the expression above $g = det (g_{\mu \nu})$ stands for the determinant of the metric tensor and $e = \sqrt{-g}$. Substituting Eq. (14) in Eq. (5) gives the covariant form of Eq. (1), i.e.,

\[ E^\mu_a \gamma^a \frac{1}{\sqrt{e}} \partial_\mu (\sqrt{e} \Phi) = -\frac{1}{2} \gamma^b \partial_\sigma E^\sigma_b \Phi. \]  

For a Minkowski space-time the terms on the right vanish so that in the limit of the correspondence principle, Eq. (1) that was derived in a flat space-time is recovered.
3. Covariant Massless Particle Equation In Axis-symmetric Space-time

It may not be easy to solve Eq. (15) as it is for general global metric tensor. In what follows we attempt to define a scheme based on variable separation similar to that of Refs. [2,3] that may ease the solution in axially symmetric space-time. Black holes are stationary axis-symmetric solutions of the Einstein equation [5] and such a scheme can be useful to study the behavior of free massless particles such as the photons and gravitons in the environment of massive black holes. 

Axial symmetry restricts the form of the metric tensor and consequently the corresponding vierbein and inverse vierbein fields. Following Ref. [2] we assume that

\[ g_{\mu\nu} =\begin{cases} g_{\alpha\beta} & \text{for } \alpha, \beta = 0, \ldots, 3 \end{cases} \]

and require that the function \( \psi \) satisfies an equation independent of the spin connection, i.e.,

\[ E^\mu_a \gamma^a \partial_\mu \psi = 0. \tag{16} \]

This expression does not depend on the spin connection. This is absorbed into the function \( f \) which as we shall see below can be calculated for axially symmetric space-time analytically. Define \( E^0_\alpha \partial_\mu = \partial_\mu \), \( E^0_\alpha \partial_\kappa = -\partial_\alpha \) the equation for \( \psi \) becomes \( \gamma^0 \partial_0 - \gamma^a \partial_a \psi = 0 \), rather similar to Eq. (3) and can be solved as shown in Ref. [1]. The function \( f(x^1, x^2) \) satisfies the equation,

\[ E^a_\alpha \gamma^a \partial_\mu f = -\frac{1}{2} \left( \gamma^b E^\alpha_b \partial_0 \ln e + \gamma^b \partial_\alpha \right) f. \tag{17} \]

Multiplying the expression by \( \gamma^a \) on the left and take the matrix trace on both sides of the above expression the equation for \( f \) resumes a compact form,

\[ \partial_\mu \ln(f \sqrt{e}) = -\frac{1}{2} \left( e^b_\mu \partial_\sigma E^\sigma_b \right). \tag{18} \]

Eq. (18) depends on the metric tensor. For the Minkowski, FRW and Schwarzschild metrics Eq. (18) is integrable. A list of the function \( f \) for these is given in Table 1.

| Metric      | \( e_{\mu\nu} \)                      | \( f(x^1, x^2) \)                      |
|-------------|--------------------------------------|--------------------------------------|
| Minkowski   | diag \( (1, 1, r, r \sin \theta) \)  | \( 1/\sin^{1/2} \theta \)            |
| FRW         | \( \begin{cases} diag \left( 1, a/F, ar, ar \sin \theta \right) \end{cases} \) | \( 1/a^{3/2} \sin^{1/2} \theta \) |
| Schwarzschild | \( \begin{cases} diag \left( S, 1/S, r, rsin \theta \right) \end{cases} \) | \( 1/\sqrt{S} \sin^{1/2} \theta \) |

\( ^* \) \( a \) is the size of the universe and has the dimension of length; \( r \) is dimensionless parameter; and \( k = \{-1, 0, 1\} \).  

\( ^** \) For \( 2M/r \ll 1 \), the calculated \( f \) reduces, as expected, to that of the Minkowski space-time.
4. Conclusions

Dirac like equations for free massless particles of any spin were derived in a curved space-time. These can serve to study the behaviour of free photons and gravitons in near compact massive objects such as neutron stars or black holes. All black holes are stationary axis-symmetric solutions of the Einstein's equation [5] and our results focus on this class of space-time. The free particle wave function was written in terms of a scaler function \( f(x^1, x^2) \) and a spinor \( \psi(x^0, x^1, x^2, x^3) \) which satisfies an equation similar to that derived originally in Minkowski flat space. Axial symmetry imposes restrictions which are encoded in the vierbein fields and spin connection and renders the evaluation of \( f(x^1, x^2) \) to be rather simple. We believe that the equations presented above can serve as a platform to study eigenvalues and degeneracies of photons and gravitons on the 3-sphere.

References

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