Model-independent aspects of the reaction $\bar{K} + N \rightarrow K + \Xi$

Benjamin C. Jackson,¹ Yongseok Oh,²,³ H. Haberzettl,⁴ and K. Nakayama¹,⁵

¹Department of Physics and Astronomy, The University of Georgia, Athens, GA 30602, USA
²Department of Physics, Kyungpook National University, Daegu 702-701, Korea
³Asia Pacific Center for Theoretical Physics, Pohang, Gyeongbuk 790-784, Korea
⁴Institute for Nuclear Studies and Department of Physics, The George Washington University, Washington, DC 20052, USA
⁵Institut für Kernphysik and Center for Hadron Physics, Forschungszentrum Jülich, 52425 Jülich, Germany

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Various model-independent aspects of the $\bar{K}N \rightarrow K\Xi$ reaction are investigated, starting from the determination of the most general structure of the reaction amplitude for $\Xi$ baryons with $J^P = \frac{1}{2}^\pm$ and $\frac{3}{2}^\pm$ and the observables that allow a complete determination of these amplitudes. Polarization observables are constructed in terms of spin-density matrix elements. Reflection symmetry about the reaction plane is exploited, in particular, to determine the parity of the produced $\Xi$ in a model-independent way. In addition, extending the work of Biagi et al. [Z. Phys. C 34, 175 (1987)], a way is presented of determining simultaneously the spin and parity of the ground state of $\Xi$ baryon as well as those of the excited $\Xi$ states.

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I. INTRODUCTION

Multi-strangeness baryons have played an important role in the development of our understanding of strong interactions. For example, the prediction and discovery of the $\Omega$ baryon, with strangeness $S = -3$, has been a spectacular confirmation of how well the SU(3) flavor symmetry works in strong interactions. Nevertheless, more than a half century later, our knowledge of multi-strangeness baryons is still very limited as can be seen from the fact that the spin of the $\Omega^-$ (1672) was only recently confirmed [1, 2].

In this work, we concentrate on the $\Xi$ baryons with strangeness $S = -2$. So far, studies of $\Xi$ physics have been very scarce. The present situation can be summarized as follows. (i) The SU(3) flavor symmetry allows as many $\Xi$ states as there are $N^*$ and $\Delta^*$ resonances combined (∼44). However, until now, only 11 $\Xi$ resonances have been discovered [2]. (ii) Being $S = -2$ baryons, if there are no strange particles in the initial state, $\Xi$ are produced only indirectly and have relatively low production rates. In fact, the yield is only of the order of nb in the photoproduction reaction [8], whereas the yield is of the order of $\mu b$ [1] in the hadronic, $K$-induced reaction (where the $\Xi$ is produced directly because of the presence of an $S = -1$ $K$ meson in the initial state).

With the advent of new particle accelerators capable of reaching higher energies and advances in technologies and experimental techniques, we are now in a better position than ever to study multi-strangeness baryons. Indeed, the CLAS Collaboration at the Thomas Jefferson National Accelerator Facility (JLab) plans to initiate a $\Xi$ spectroscopy program using the upgraded 12-GeV machine. The Collaboration is also expected to measure exclusive $\Omega$ photoproduction for the first time [5]. Some data for the $\Xi$ ground state are already available [3] obtained from the 6-GeV machine. In addition, J-PARC proposes to study the $\Xi$ baryons via the $\bar{K}N \rightarrow K\Xi$ and $\pi N \rightarrow KK\Xi$ reactions as well as $\Omega$ production [6, 7], and at the Facility for Antiproton and Ion Research (FAIR) of GSI, the reaction $\bar{pp} \rightarrow \Xi\Xi$ will be studied [8]. For a more complete compilation of baryon spectra, the $\Xi$ baryons should be studied as an integral part of any baryon spectroscopy program.

Theoretical studies of the $\Xi$ baryons are hampered mainly by the scarcity of experimental data. The existing theoretical models cannot be well constrained and, as a consequence, there is strong model-dependence in predictions of the $\Xi$ spectrum. In particular, one of the current open issues in the $\Xi$ spectrum concerns the low mass of the $\Xi$(1690) and $\Xi$(1620), i.e., the nature of the third lowest $\Xi$ state [9]. Here, different approaches, such as the non-relativistic and relativistic quark models [10–12], one-boson-exchange model [13], large $N_c$ model [14–18], QCD sum rules [19, 20], and Skyrme model [9], yield contradictory predictions for the nature of these resonances. The planned new experimental studies as mentioned above are expected to play a key role in addressing such open problems. Quite recently, lattice QCD calculations of the baryon spectra, including those of $\Xi$ and $\Omega$ baryons, have been reported [21, 22].

To extract relevant information on $\Xi$ resonances from the experimental data, a reliable reaction model is required. To date, for photoproduction reactions, there exist currently only the work of some of the present authors [23, 24] analyzing the available CLAS data [3]. In $K$-induced reactions, recent calculations are reported by...
Sharov et al. [27] and by Shyam et al. [26]. Thus, further theoretical studies on this subject are timely for suggesting directions to experimental studies by providing predictions on the \( \Xi \) baryon production processes and for giving tools to analyze the forthcoming data. One feature of these production processes is that the \( t \)-channel processes of an intermediate meson production are suppressed since the exchanged intermediate meson should be exotic having two units of strangeness. As a consequence, the production of a \( \Xi \) state is dominated by intermediate \( S = -1 \) hyperons. Therefore, by analyzing the production mechanisms of the \( \Xi \), one also hopes to gain some insight into the spectrum and couplings of the \( S = -1 \) hyperons.

Our ongoing efforts to understand better the production process of \( \Xi \) baryons are pursued along two lines of inquiry. On the one hand, to build understanding of the dynamics of reactions involving strange particles, we are engaged in model-dependent analyses within an effective Lagrangian approach along the lines employed in the photoprocesses reported by some of us in Refs. [23, 24]. Further results of such model-dependent analyses will be reported elsewhere. In the present work, on the other hand, we report on studying model-independent aspects of production processes of \( \Xi \) baryons exploiting, in particular, some basic symmetries of the reactions in question to determine the spin and parity quantum numbers of the \( \Xi \) resonances.

The present paper is organized as follows. In Sec. II, following the method of Ref. [27], the most general structure of the reaction amplitude for \( \bar{K}N \to K\Xi \) is derived for a \( \Xi \) of spin-1/2 or of spin-3/2, and a set of observables is identified that determine the reaction amplitude completely. Here, the reflection symmetry about the reaction plane is exploited to determine the parity of the \( \Xi \) resonances in a model-independent manner. Furthermore, the coefficients that multiply each spin structure in the reaction amplitude are expressed in terms of the \( \Xi \) resonances in a model-independent manner. Furthermore, some basic symmetries of the reactions in question to determine the spin and parity quantum numbers of the \( \Xi \) resonances.

## II. STRUCTURE OF THE \( \bar{K}N \to K\Xi \) REACTION AMPLITUDE

In this section, we derive the most general structure of the amplitude for the reaction of

\[
\bar{K}(q) + N(p) \to K(q') + \Xi(p'),
\]

following the method used in Ref. [27]. In the present work, we consider the production of \( \Xi \) of spin-1/2 and -3/2 with both positive and negative parities. The method is quite general and, in principle, can be applied to extract the spin structure of any reaction amplitude. In the above equation, the arguments denote the four-momenta of the respective particles.

The reaction in Eq. (1) is described in its center-of-momentum (CM) frame, where \( q = -p \) and \( q' = -p' \). For further convenience, we define the three mutually orthogonal unit vectors \( \hat{n}_i \) \( (i = 1, 2, 3) \) in terms of the independent momenta available in the reaction, i.e.,

\[
\begin{align*}
\hat{n}_1 & = \frac{(p \times p') \times p}{|p \times p'|^2}, \\
\hat{n}_2 & = \frac{p \times p'}{|p \times p'|}, \\
\hat{n}_3 & = \frac{p}{|p|},
\end{align*}
\]

where \( p \) and \( p' \) denote the three-momenta of the nucleon and \( \Xi \), respectively. Note that \( p \) and \( p' \) define the reaction plane, such that \( \hat{n}_2 \) is perpendicular to the reaction plane. The coordinate-system setup is shown in Fig. 1. Throughout this paper, the hat notation for vectors is used to indicate unit vectors, i.e., \( \hat{a} \equiv a/|a| \) for an arbitrary vector \( a \). The quantization axis is chosen to be along \( \hat{n}_3 \). We also use the alternative Cartesian notation \( i = x, y, z \) for the indices of the unit vectors \( \hat{n}_i \).

### A. Production of \( \Xi \) with \( J^P = \frac{1}{2}^\pm \)

First, we consider spin-parity \( J^P = \frac{1}{2}^\pm \) for the \( \Xi \) produced in reaction (1). Following the method of Ref. [27], the most general spin structure of the reaction amplitude, consistent with basic symmetries, is

\[
\begin{align*}
\hat{M} & = M_0^+ + M_2^+ \sigma \cdot (\hat{p} \times \hat{p}') , & \text{for } J^P = \frac{1}{2}^+, \\
\hat{M} & = M_0^- + M_2^- \sigma \cdot \hat{p} , & \text{for } J^P = \frac{1}{2}^-, \tag{3a, b}
\end{align*}
\]

where \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \) stands for the vector built up of usual Pauli spin operators. Equations (3a) and (3b) are

\[1\] Note that the spin structure for the positive-parity \( \Xi \) in Eq. (3a) is identical to the familiar structure of the \( \pi N \) elastic scattering amplitude. However, obviously, the isospin structure is different.
FIG. 1. Coordinate systems used in describing the Ξ production reaction and its subsequent decay process. On the left, the production reaction $KN \rightarrow K\Xi$ is shown in its center-of-momentum (CM) frame. The corresponding reaction plane (indicated in dark gray) contains the nucleon and Ξ momenta $p$ and $p'$, respectively. The basis vectors $\{\hat{n}_1, \hat{n}_2, \hat{n}_3\}$ are defined in Eq. (2), with $\hat{n}_3$ aligned with the nucleon momentum $p$ and $\hat{n}_2$ perpendicular to the reaction plane; $\theta$ indicates the Ξ emission angle. The (primed) frame $\{\hat{n}'_1, \hat{n}'_2, \hat{n}'_3\}$ is obtained from $\{\hat{n}_1, \hat{n}_2, \hat{n}_3\}$ by rotating the latter about the $\hat{n}_3$ axis by $\theta$, which aligns $\hat{n}'_3$ with $p'$ and leaves $\hat{n}'_2 \equiv \hat{n}_2$. The (light gray) plane tilted by the angle $\phi_\Lambda$ about the $\hat{n}'_3$ axis is spanned by the momenta of the decay products $\Lambda$ and $K$. The polar and azimuthal angles of the decay product $\Lambda$ in the rotated (primed) CM frame are indicated by $\theta_\Lambda$ and $\phi_\Lambda$, respectively. In the boosted frame on the right, the decay process of the produced Ξ at rest is described in the $\{\hat{n}'_1, \hat{n}'_2, \hat{n}'_3\}$ coordinate system. The polar and azimuthal angles of the decay product $\Lambda$ are indicated here by $\theta_\Lambda$ and $\phi_\Lambda$, respectively. For the latter angle, one has $\phi_\Lambda \equiv \phi'_\Lambda$ since the boost happens along the corresponding tilt axis.

direct consequences of the amplitude’s reflection symmetry about the reaction plane $\Xi$ (Ref. [22,28]), which is further exploited in our analysis presented below. Note that the coefficients $M'_1$ and $M'_2$ do not contain $S$-wave in the final state.

For further convenience, we rewrite Eq. (3) as

$$M = M_0 + M_2 \sigma \cdot \hat{n}_2$$

for $J^P = \frac{1}{2}^+$, (4a)

$$M = M_1 \sigma \cdot \hat{n}_1 + M_3 \sigma \cdot \hat{n}_3$$

for $J^P = \frac{1}{2}^−$, (4b)

using $p' = \cos \theta \hat{n}_3 + \sin \theta \hat{n}_2$ and $\hat{n}_3 = \hat{p}$. The respective coefficients in Eqs. (4a) and (4b) are related by

$$M'_0 = M_0$$

$$M'_2 = \frac{1}{\sin \theta} M_2$$

$$M'_1 = \frac{1}{\sin \theta} M_1$$

$$M'_3 = M_3 - \frac{\cos \theta}{\sin \theta} M_1.$$  

(5a)

(5b)

Following Ref. [27], one may also express these coefficients in terms of partial-wave matrix elements. The corresponding results are given in Appendix A which show, in particular, that the coefficients $M_1$ and $M_3$ vanish identically for $\Xi$ scattering angles $\theta = 0$ and $\pi$, as can be seen in Eq. (A3). The partial-wave expansions will become particularly relevant once sufficient experimental data become available to permit their full-fledged partial-wave analysis. The isospin structure of the amplitudes in Eq. (4) [or in Eq. (3)] is contained in the coefficients $M_i$ as given explicitly by Eq. (A3) in Appendix A.

Once the spin structure of the reaction amplitude is determined, all the observables can be readily expressed in terms of the amplitudes $M_i$ multiplying each spin structure. For the reaction under consideration, apart from the cross section $(d\sigma/d\Omega)$, a complete set of observables includes the target asymmetry ($T$), recoil Ξ polarization ($P$), and the spin-transfer coefficient ($K$). For arbitrary spin orientations along directions $\hat{a}$ and $\hat{b}$, their coordinate-independent expressions are

$$\frac{d\sigma}{d\Omega} \equiv \frac{1}{2} \text{Tr}[\hat{M}\hat{M}^\dagger],$$

(6a)

$$\frac{d\sigma}{d\Omega}T_a \equiv \frac{1}{2} \text{Tr}[\hat{M}\sigma \cdot \hat{a}\hat{M}^\dagger],$$

(6b)

$$\frac{d\sigma}{d\Omega}P_a \equiv \frac{1}{2} \text{Tr}[\hat{M}\hat{M}^\dagger \sigma \cdot \hat{a}],$$

(6c)

$$\frac{d\sigma}{d\Omega}K_{ab} \equiv \frac{1}{2} \text{Tr}[\hat{M} \sigma \cdot \hat{b}\hat{M}^\dagger \sigma \cdot \hat{a}].$$

(6d)

For Cartesian directions $\hat{a}_i$, enumerated by $i = 1,2,3$ (= $x,y,z$), in particular, one obtains

$$\frac{d\sigma}{d\Omega}T_i \equiv \frac{1}{2} \text{Tr}[\hat{M}\sigma_i\hat{M}^\dagger],$$

(7a)

$$\frac{d\sigma}{d\Omega}P_i \equiv \frac{1}{2} \text{Tr}[\hat{M}\hat{M}^\dagger \sigma_i],$$

(7b)

$$\frac{d\sigma}{d\Omega}K_{ij} \equiv \frac{1}{2} \text{Tr}[\hat{M}\sigma_i\hat{M}^\dagger \sigma_j].$$

(7c)

Of course, the $T$, $P$, and $K$ observables for arbitrary directions in Eq. (6) can be expressed as linear combinations of the specific Cartesian expressions given in Eq. (7).

Due to symmetries of the reaction, eight observables vanish identically, i.e., $T_i = P_i = K_{iy} = K_{yi} = 0$ for $i = x, z$, and of the remaining eight, only four are independent for a given parity, which completely determine

...
the amplitudes \( M_i \) in Eq. (4). Indeed, for a positive-parity \( \Xi \), we have
\[
\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega} K_{yy} = |M_0|^2 + |M_2|^2 , \quad (8a)
\]
\[
\frac{d\sigma}{d\Omega} K_{zz} = \frac{d\sigma}{d\Omega} K_{zz} = |M_0|^2 - |M_2|^2 , \quad (8b)
\]
\[
\frac{d\sigma}{d\Omega} T_y = \frac{d\sigma}{d\Omega} P_y = 2 \text{Re} [M_2 M_3^*] , \quad (8c)
\]
\[
\frac{d\sigma}{d\Omega} K_{xx} = -\frac{d\sigma}{d\Omega} K_{zz} = 2 \text{Im} [M_2 M_3^*] , \quad (8d)
\]

and for a negative-parity \( \Xi \), we obtain
\[
\frac{d\sigma}{d\Omega} = -\frac{d\sigma}{d\Omega} K_{yy} = |M_1|^2 + |M_3|^2 , \quad (9a)
\]
\[
\frac{d\sigma}{d\Omega} K_{xx} = -\frac{d\sigma}{d\Omega} K_{zz} = |M_1|^2 - |M_3|^2 , \quad (9b)
\]
\[
\frac{d\sigma}{d\Omega} T_y = -\frac{d\sigma}{d\Omega} P_y = 2 \text{Im} [M_3 M_1^*] , \quad (9c)
\]
\[
\frac{d\sigma}{d\Omega} K_{xx} = -\frac{d\sigma}{d\Omega} K_{zz} = 2 \text{Re} [M_3 M_1^*] . \quad (9d)
\]

The respective first two relations in the two equation sets determine the magnitudes of the amplitudes \( M_0 \), \( M_2 \) and \( M_1 \), \( M_3 \), respectively, whereas the respective last two relations determine their phase differences. Therefore, apart from an irrelevant overall phase, the observables in Eqs. (8) and (9) determine the amplitudes \( M_i \), \( i = 0, \ldots, 3 \), unambiguously. These results reveal that it is experimentally demanding to determine the reaction amplitude completely, for it requires measuring both the single- and double-polarization observables.

Comparing Eqs. (8a) and (9a), one obtains
\[
K_{yy} = \pi_{\Xi} , \quad (10)
\]
where \( \pi_{\Xi} \) stands for the parity of the produced \( \Xi \). This result is actually a direct consequence of reflection symmetry, as exploited in Bohr’s theorem \( \Xi \) and applied in Ref. [35]. It, therefore, provides a model-independent way of determining the parity of the \( \Xi \) resonance. Alternative expressions extracted from Eqs. (8) and (9) are [35]
\[
T_y = \pi_{\Xi} P_y , \quad (11)
\]
which involves only single polarization observables and
\[
K_{xx} = \pi_{\Xi} K_{zz} \quad \text{and} \quad K_{zz} = -\pi_{\Xi} K_{xx} . \quad (12)
\]

These results are all consequences of the reflection symmetry about the reaction plane.

In Sec. II, we will perform the analysis in terms of the SDM elements, which are equivalent to the observables discussed here. The SDM elements are convenient quantities when dealing with spin observables, especially when higher-spin particles are produced in the reaction. They can be extracted from the information on the subsequent decay processes of the produced particles, in conjunction with the self-analyzing property of the decaying particles via a weak decay, without the explicit measurement of the spin polarizations of these produced particles.

B. Production of \( \Xi \) with \( J^P = \frac{3}{2}^\pm \)

We now turn to the spin-parity \( J^P = \frac{3}{2}^\pm \). Again, following Ref. [27], the most general spin structure of the reaction amplitude is given by
\[
\hat{M} = F_1 \hat{T}^1 \cdot \hat{n}_2 + F_2 \hat{T}^1 \cdot \hat{n}_1 \hat{\sigma} \cdot \hat{n}_1 + F_3 \hat{T}^1 \cdot \hat{n}_3 \hat{\sigma} \cdot \hat{n}_3 + F_4 \hat{T}^1 \cdot \hat{\sigma} \cdot \hat{n}_3 , \quad (12a)
\]
\[
\hat{M} = G_1 \hat{T}^1 \cdot \hat{n}_3 \hat{\sigma} \cdot \hat{n}_3 + G_2 \hat{T}^1 \cdot \hat{\sigma} \cdot \hat{n}_3 + G_3 \hat{T}^1 \cdot \hat{n}_3 + G_4 \hat{T}^1 \cdot \hat{n}_3 , \quad (12b)
\]

for \( J^P = \frac{3}{2}^+ \) and
\[
\hat{M} = F_1 \hat{T}^1 \cdot \hat{n}_2 + F_2 \hat{T}^1 \cdot \hat{n}_2 \hat{\sigma} \cdot \hat{n}_2 + F_3 \hat{T}^1 \cdot \hat{n}_1 \hat{\sigma} \cdot \hat{n}_1 + F_4 \hat{T}^1 \cdot \hat{\sigma} \cdot \hat{n}_1 , \quad (13a)
\]
\[
\hat{M} = G_1 \hat{T}^1 \cdot \hat{n}_3 \hat{\sigma} \cdot \hat{n}_3 + G_2 \hat{T}^1 \cdot \hat{\sigma} \cdot \hat{n}_3 + G_3 \hat{T}^1 \cdot \hat{n}_3 + G_4 \hat{T}^1 \cdot \hat{n}_3 , \quad (13b)
\]

for \( J^P = \frac{3}{2}^- \). Here, \( \hat{T}^1 \) stands for the (spin-1/2 → spin-3/2) transition operator. Its explicit representation may be found elsewhere [36]. In contrast to the spin-1/2 case, each parity of the spin-3/2 case has four independent amplitudes, \( F'_i \) and \( G'_i \) \( (i = 1, \ldots, 4) \), respectively, and one needs at least eight independent observables to determine them completely apart from an irrelevant overall phase. From the above equations, it is obvious that only \( F'_i \) and \( G'_i \) contain an S-wave in the final state. Also, \( F'_2 \) and \( G'_2 \) contain only D- and higher-waves in the final state.

The amplitudes in Eq. (12) can be also rewritten as
\[
\hat{M} = F_1 \hat{T}^1 \cdot \hat{n}_2 + F_2 \hat{T}^1 \cdot \hat{n}_1 \hat{\sigma} \cdot \hat{n}_1 + F_3 \hat{T}^1 \cdot \hat{n}_3 \hat{\sigma} \cdot \hat{n}_3 + F_4 \hat{T}^1 \cdot \hat{\sigma} \cdot \hat{n}_3 , \quad (14a)
\]
\[
\hat{M} = G_1 \hat{T}^1 \cdot \hat{n}_3 \hat{\sigma} \cdot \hat{n}_3 + G_2 \hat{T}^1 \cdot \hat{\sigma} \cdot \hat{n}_3 + G_3 \hat{T}^1 \cdot \hat{n}_3 + G_4 \hat{T}^1 \cdot \hat{n}_3 , \quad (14b)
\]

for \( J^P = \frac{3}{2}^+ \). The coefficients \( F_i \) and \( G_i \) are expressed in terms of the partial-wave matrix elements as given in Appendix A. They are also related to the corresponding coefficients \( F'_i \) and \( G'_i \) in Eq. (13) by
\[
F'_1 = \frac{1}{\sin \theta} F_1 , \quad (15a)
\]
\[
F'_2 = \frac{1}{\sin \theta} F_2 , \quad (15b)
\]
\[
F'_3 = \frac{1}{\sin \theta} F_3 - \frac{\cos \theta}{\sin^2 \theta} F_2 , \quad (15c)
\]
\[
F'_4 = F_4 + \frac{\cos^2 \theta}{\sin^2 \theta} F_2 - 2 \frac{\cos \theta}{\sin \theta} F_3 , \quad (15d)
\]
and
\[
G'_1 = \frac{1}{\sin \theta} G_1 - \frac{\cos \theta}{\sin^2 \theta} G_2 , \quad (15e)
\]
The polarization observables for this case will be discussed in the next section in terms of the SDM elements.

### III. SPIN DENSITY MATRIX APPROACH

As mentioned before, when dealing with higher-spin Ξ (i.e., spins higher than 1/2) in particular, it is more convenient to continue the analysis of the $K N \to K \Xi$ reaction in terms of spin-density matrix (SDM) elements. A similar (but not identical) analysis to the present one based on the SDM formalism was performed in Ref. [37] for a general two-body reaction with unpolarized initial state. Also, the reaction $K N \to \omega \Lambda$ was analyzed within the SDM approach in Ref. [38].

In Sec. II we have exploited the mirror (or reflection) symmetry about the reaction plane in our analysis, in particular, for the parity determination of the Ξ resonances. In fact, as long as the production process conserves total parity, the reaction amplitude should have this symmetry [32, 33]. This mirror operation is equivalent to doing a parity transformation followed by a subsequent rotation by 180° about the $n_2$-axis: $\vec{P}_y = \hat{R}_y(180°)\hat{P}$. The resulting symmetry, in terms of the spin matrix element, is

$$\langle S_f m_f | \hat{M} | S_i m_i \rangle = \langle S_f m_f | \hat{P}_y \hat{M} \hat{P}_y | S_i m_i \rangle$$

$$= \pi_i \pi_f (-1)^{(S_f - m_f) - (S_i - m_i)} \times \langle S_f - m_f | \hat{M} | S_i - m_i \rangle \, , \quad (16)$$

and holds as long as the quantization axis is in the production plane. Here, $\pi_i(f)$ is the intrinsic parity of the initial (final) state.

Based on this symmetry, the $J^P = \frac{3}{2}^+$ Ξ production amplitude, $\hat{M}$ given by Eq. (14), is completely described by two complex helicity amplitudes, $H_1$ and $H_2$, given by the spin matrix elements,

$$H_1 \equiv \langle \lambda_\Xi = \frac{1}{2} | \hat{M} | \lambda_N = \frac{1}{2} \rangle = \pi_\Xi \langle \lambda_\Xi = -\frac{1}{2} | \hat{M} | \lambda_N = -\frac{1}{2} \rangle \, , \quad (17a)$$

$$H_2 \equiv \langle \lambda_\Xi = \frac{1}{2} | \hat{M} | \lambda_N = -\frac{1}{2} \rangle = -\pi_\Xi \langle \lambda_\Xi = -\frac{1}{2} | \hat{M} | \lambda_N = \frac{1}{2} \rangle \, , \quad (17b)$$

where $\lambda_N$ and $\lambda_\Xi$ denote the helicity of the initial nucleon and final Ξ, respectively. Here, reference to the spin quantum numbers $S_\Xi = S_N = 1/2$ has been suppressed. The helicity amplitudes are related to the coefficient amplitudes in Eq. (14) by

$$H_1 = M_0 \cos \frac{\theta}{2} + iM_2 \sin \frac{\theta}{2} \, , \quad (18a)$$

for a positive parity $\Xi$, and by

$$H_1 = M_3 \cos \frac{\theta}{2} + M_1 \sin \frac{\theta}{2} \, , \quad (19a)$$

$$H_2 = M_1 \cos \frac{\theta}{2} - M_3 \sin \frac{\theta}{2} \, , \quad (19b)$$

likewise, the production amplitude $\hat{M}$ for a $\Xi$ with $J^P = \frac{3}{2}^-$ determined by Eq. (14) is completely described by four complex amplitudes given as

$$H_1 \equiv \langle \lambda_\Xi = \frac{1}{2} | \hat{M} | \lambda_N = \frac{1}{2} \rangle = \pi_\Xi \langle \lambda_\Xi = -\frac{1}{2} | \hat{M} | \lambda_N = -\frac{1}{2} \rangle \, , \quad (20a)$$

$$H_2 \equiv \langle \lambda_\Xi = \frac{1}{2} | \hat{M} | \lambda_N = -\frac{1}{2} \rangle = -\pi_\Xi \langle \lambda_\Xi = -\frac{1}{2} | \hat{M} | \lambda_N = \frac{1}{2} \rangle \, , \quad (20b)$$

$$H_3 \equiv \langle \lambda_\Xi = \frac{1}{2} | \hat{M} | \lambda_N = \frac{1}{2} \rangle = -\pi_\Xi \langle \lambda_\Xi = -\frac{1}{2} | \hat{M} | \lambda_N = -\frac{1}{2} \rangle \, , \quad (20c)$$

$$H_4 \equiv \langle \lambda_\Xi = \frac{1}{2} | \hat{M} | \lambda_N = -\frac{1}{2} \rangle = \pi_\Xi \langle \lambda_\Xi = -\frac{1}{2} | \hat{M} | \lambda_N = \frac{1}{2} \rangle \, . \quad (20d)$$

These helicity amplitudes are related to the coefficient functions in Eq. (14) by

$$H_1 = \frac{1}{\sqrt{2}} \left[ i \cos \frac{\theta}{2} F_1 - \cos \theta \sin \frac{\theta}{2} F_2 - \cos \frac{3\theta}{2} F_3 + \sin \theta \cos \frac{\theta}{2} F_4 \right] \, , \quad (21a)$$

$$H_2 = \frac{1}{\sqrt{2}} \left[ i \sin \frac{\theta}{2} F_1 - \cos \theta \cos \frac{\theta}{2} F_2 + \sin \frac{3\theta}{2} F_3 - \sin \theta \sin \frac{\theta}{2} F_4 \right] \, , \quad (21b)$$

$$H_3 = \frac{1}{\sqrt{6}} \left[ -i \sin \frac{\theta}{2} F_1 + (2 + 3 \cos \theta) \cos \frac{\theta}{2} F_2 + 3 \sin \frac{3\theta}{2} F_3 - (1 + 3 \cos \theta) \cos \frac{\theta}{2} F_4 \right] \, , \quad (21c)$$

$$H_4 = \frac{1}{\sqrt{6}} \left[ i \cos \frac{\theta}{2} F_1 + (2 + 3 \cos \theta) \sin \frac{\theta}{2} F_2 + 3 \cos \frac{3\theta}{2} F_3 - (1 + 3 \cos \theta) \sin \frac{\theta}{2} F_4 \right] \, , \quad (21d)$$

for a positive-parity $\Xi$ and by

$$H_1 = \frac{1}{\sqrt{2}} \left[ (2 - \cos \theta) \cos \frac{\theta}{2} G_1 + 2i \sin^3 \frac{\theta}{2} G_2 - \cos \theta \cos \frac{\theta}{2} G_3 + \sin \theta \cos \frac{\theta}{2} G_4 \right] \, , \quad (22a)$$

$$H_2 = \frac{1}{\sqrt{2}} \left[ -(2 + \cos \theta) \sin \frac{\theta}{2} G_1 + 2i \cos^3 \frac{\theta}{2} G_2 \right] \, . \quad (22b)$$
for a negative-parity $\Xi$.

The SDM elements are defined by

$$\rho^{\Xi,i}_{\lambda,\lambda'} = \langle \lambda | \rho^{N,i} | \lambda' \rangle = \frac{1}{2} \langle \lambda | M_{\sigma_1 M_1^*} | \lambda' \rangle,$$

(23)

for $i = 0, \ldots, 3$, where $\lambda$ and $\lambda'$ stand for the helicity of the produced $\Xi$ baryon and $\sigma_0 = \pm$ is the $2 \times 2$ unit matrix. For completeness, a relevant part of the SDM formalism for the present work is presented in Appendix B.

The SDM elements are related by

$$\rho^{\Xi,i}_{\lambda,\lambda'} = (-1)^{i+\lambda-\lambda'} \rho^{\Xi,-i}_{-\lambda,-\lambda'},$$

(24a)

$$\rho^{\Xi,i}_{\lambda,\lambda'} = \rho^{\Xi,1*}_{\lambda',\lambda},$$

(24b)

due to the symmetry of the spin matrix element and the hermiticity of Eq. (23).

We now relate the SDM elements, $\rho^{\Xi,i}_{\lambda,\lambda'}$, to the helicity amplitudes $\mathcal{H}_j$ given by Eqs. (17) and (20) which determine the reaction amplitudes. The purpose is to find a set of SDM elements that fixes those helicity amplitudes completely.

### A. $\Xi$ of $J^P = \frac{1}{2}^\pm$

Starting with $J = \frac{1}{2}$, there are sixteen possible SDM elements $\rho^{\Xi,i}_{\lambda,\lambda'}$. However, only four of them are independent for a given parity and they determine the amplitudes $\mathcal{H}_1$ and $\mathcal{H}_2$ apart from an irrelevant overall phase. Inserting Eq. (17) into Eq. (23), a set of four independent SDM elements can be determined as

$$2 \rho^0_{\frac{1}{2},\frac{1}{2}} = 2i \pi_{\Xi} \rho^2_{\frac{1}{2},-\frac{1}{2}} = |\mathcal{H}_1|^2 + |\mathcal{H}_2|^2,$$

(25a)

$$2 \rho^2_{\frac{1}{2},\frac{1}{2}} = 2 \pi_{\Xi} \rho^0_{\frac{1}{2},-\frac{1}{2}} = |\mathcal{H}_1|^2 - |\mathcal{H}_2|^2,$$

(25b)

$$\rho^2_{\frac{1}{2},-\frac{1}{2}} = i \pi_{\Xi} \rho^0_{\frac{1}{2},-\frac{1}{2}} = \text{Im} |\mathcal{H}_1 \mathcal{H}_2^*|,$$

(25c)

$$\rho^0_{\frac{1}{2},-\frac{1}{2}} = -i \pi_{\Xi} \rho^2_{\frac{1}{2},-\frac{1}{2}} = \text{Re} |\mathcal{H}_1 \mathcal{H}_2^*|,$$

(25d)

where the superindex $\Xi$ in $\rho^{\Xi,i}_{\lambda,\lambda'}$ was dropped for simplicity. A complete list of SDM elements $\rho^{\Xi,i}_{\lambda,\lambda'}$ in terms of helicity amplitudes $\mathcal{H}_i$ is given in Appendix C.

The SDM elements are directly related to the observables defined by Eq. (10). For example, from Eqs. (24) and (B10d), we have

$$\frac{d\sigma}{d\Omega} = 2 \rho^0_{\frac{1}{2},\frac{1}{2}}, \quad \frac{d\sigma}{d\Omega} K_{yy'} = 2i \rho^2_{\frac{1}{2},-\frac{1}{2}} - \frac{1}{2},$$

(26a)

$$\frac{d\sigma}{d\Omega} T_y = 2 \rho^2_{\frac{1}{2},\frac{1}{2}}, \quad \frac{d\sigma}{d\Omega} P_{yy'} = 2i \rho^0_{\frac{1}{2},-\frac{1}{2}} - \frac{1}{2},$$

(26b)

$$\frac{d\sigma}{d\Omega} K_{xx'} = 2 \rho^0_{\frac{1}{2},\frac{1}{2}}, \quad \frac{d\sigma}{d\Omega} K_{zz'} = 2 \rho^2_{\frac{1}{2},\frac{1}{2}},$$

(26c)

$$\frac{d\sigma}{d\Omega} K_{zz'} = 2 \rho^0_{\frac{1}{2},\frac{1}{2}} - \frac{1}{2}, \quad \frac{d\sigma}{d\Omega} K_{xx'} = 2 \rho^2_{\frac{1}{2},\frac{1}{2}} - \frac{1}{2},$$

(26d)

where the primed Cartesian components correspond to the rotated frame (see Fig. 1 note that $y' \equiv y$). From Eqs. (10) and (20), we see, in particular, that

$$K_{yy'} = \frac{i \rho^2_{\frac{1}{2},-\frac{1}{2}}}{\rho^0_{\frac{1}{2},\frac{1}{2}}} = \pi_{\Xi}.$$

(27)

More generally, in terms of the SDM elements, one obtains

$$(-1)^{\frac{1}{2}-\lambda'-\lambda} \frac{i \rho^2_{\lambda,\lambda'} \rho^0_{\lambda',\lambda}}{\rho^0_{\lambda,\lambda'}} = (-1)^{\frac{1}{2}-\lambda'} \frac{\rho^1_{\lambda,-\lambda'}}{\rho^0_{\lambda,\lambda'}} = \pi_{\Xi}.$$

(28)

This result reveals that one needs to measure two SDM elements to determine the parity of the $\Xi$ baryon: either $\rho^0_{\lambda,\lambda'}$ with unpolarized target nucleon and $\rho^2_{\lambda,-\lambda'}$ with polarized target nucleon along the direction $n_2 \equiv \hat{n}_2$ perpendicular to the reaction plane, or $\rho^1_{\lambda',\lambda}$ with transversally polarized target along $\hat{n}_1$, and $\rho^3_{\lambda',\lambda'}$ with longitudinally polarized target along $\hat{n}_3 \equiv \hat{p}$. Note that $\rho^0_{\lambda,\lambda'}$ is directly related to the cross section $d\sigma/d\Omega$ when $\lambda = \lambda'$.

### B. $\Xi$ of $J^P = \frac{3}{2}^\pm$

For $J = 3/2$, analogously to the $J = 1/2$ case, inserting Eq. (20) into Eq. (23), the SDM elements are related to the four amplitudes $\mathcal{H}_i$ ($i = 1, \ldots, 4$) by

$$2 \rho^0_{\frac{3}{2},\frac{3}{2}} = |\mathcal{H}_1|^2 + |\mathcal{H}_2|^2,$$

(29a)

$$2 \rho^1_{\frac{1}{2},-\frac{3}{2}} = \pi_{\Xi} (|\mathcal{H}_1|^2 - |\mathcal{H}_2|^2),$$

(29b)

$$-i \rho^0_{\frac{3}{2},-\frac{3}{2}} = \pi_{\Xi} \text{Im} |\mathcal{H}_2 \mathcal{H}_1^*|,$$

(29c)

$$\rho^0_{\frac{1}{2},-\frac{1}{2}} = \text{Re} |\mathcal{H}_2 \mathcal{H}_1^*|,$$

(29d)

$$2 \rho^0_{\frac{3}{2},\frac{3}{2}} = |\mathcal{H}_3|^2 + |\mathcal{H}_4|^2,$$

(30a)

$$2 \rho^1_{\frac{1}{2},-\frac{3}{2}} = \pi_{\Xi} (|\mathcal{H}_4|^2 - |\mathcal{H}_3|^2),$$

(30b)

$$-i \rho^0_{\frac{3}{2},-\frac{3}{2}} = \pi_{\Xi} \text{Im} |\mathcal{H}_3 \mathcal{H}_4^*|,$$

(30c)

$$\rho^0_{\frac{1}{2},-\frac{1}{2}} = \text{Re} |\mathcal{H}_3 \mathcal{H}_4^*|,$$

(30d)

$$2 \rho^1_{\frac{1}{2},-\frac{3}{2}} = \pi_{\Xi} (\mathcal{H}_2 \mathcal{H}_4^* - \mathcal{H}_3 \mathcal{H}_3^*),$$

(31a)

$$2 \rho^0_{\frac{3}{2},\frac{3}{2}} = \mathcal{H}_2 \mathcal{H}_4^* + \mathcal{H}_3 \mathcal{H}_3^*,$$

(31b)
where $\zeta_L = 1$ for even $L$ and $\zeta_L = \alpha_\Xi$ for odd $L$, with $\alpha_\Xi$ denoting the \Xi decay-asymmetry parameter.

Note that since all moments vanish identically for $L > 2J$, Eq. (38) offers a way of determining the spin of the \Xi undergoing a single (weak) decay by measuring the moments as a function of $L$. In other words, the nonvanishing $H^i(L, M)$ with the largest $L$ value for some $i$ and $M$ determines $J$ as $J = L/2$. Experimentally, of course, this may be challenging since it is not a priori clear how small the measured values of the next higher moment $H^i(L+1, M)$ would need to be for being compatible with zero. And, moreover, one would need to confirm that the smallness of this moment is not accidental.

Similar results are obtained for excited \Xi resonances, \Xi*, undergoing a double decay process, as discussed in Appendix D. In this case, we have,

\[
\left( \frac{d\sigma}{d\Omega} \right) H^i(0,0,L,M) = t^i_{LM} \langle J \frac{1}{2} L 0 | J \frac{1}{2} \rangle \tag{39a}
\]

for even $L$ and

\[
\left( \frac{d\sigma}{d\Omega} \right) H^i(1,0,L,M) = \frac{\alpha_\Lambda}{3} t^i_{LM} \langle J \frac{1}{2} L 0 | J \frac{1}{2} \rangle \tag{39b}
\]

for odd $L$. Here, $\alpha_\Lambda$ denotes the \Lambda decay-asymmetry parameter for the decay chain $\Xi^* \to \Lambda + \bar{K}$ followed by $\Lambda \to N + \pi$. In the case of $\Xi^* \to \Xi + \pi$ instead, $\alpha_\Lambda$ needs to be replaced by $\alpha_\Xi$.

For \Xi resonances decaying along the double decay chain specified in Eq. (38), the result of the corresponding moments given by Eq. (32) leads to

\[
\frac{H^0(1,\pm 1,L,M)}{H^0(1,0,L,M)} = \pi_\Xi (-1)^{J+\frac{1}{2}} \frac{2J + 1}{\sqrt{2L(L+1)}} \tag{40}
\]

for an unpolarized target and for odd values of $L(\leq 2J)$. This offers a way of determining the spin and parity of the excited \Xi resonance simultaneously.

\section{IV. SUMMARY}

A model-independent analysis of the $\bar{K} + N \to K^* + \Xi$ reaction has been performed. Following the method of Ref. [27], we derived the most general spin structure of the reaction amplitude, consistent with basic symmetries, for \Xi baryons of $J^P = \frac{1}{2}^+$ and $\frac{3}{2}^+$. The coefficients multiplying each spin structure have been presented in partial-wave-decomposed form, thus permitting partial-wave analyses, once sufficient data become available for these reactions. The method of Ref. [27] is general, and can be applied, in principle, to derive the structure of the reaction amplitude involving higher-spin \Xi production.

Furthermore, a minimal set of independent observables required to determine completely the reaction amplitude has been identified. In addition to the unpolarized cross sections, one also needs single- and double-spin observables, which poses a formidable experimental challenge.
in particular, since one needs to measure the polarization of the outgoing \( \Xi \). Note that, for the \( \Xi \) of spin-1/2, there are two complex amplitudes to be determined, whereas for the \( \Xi \) of spin-3/2, there are four complex amplitudes. We then formulated the problem using the SDM approach and expressed the spin observables in terms of the SDM elements. Following Ref. [30, 31], it was shown that the latter can be extracted from the moments associated with the \( \Xi \) decay processes in conjunction with the self-analyzing nature of the hyperon (\( \Lambda \) or \( \Xi \)) resulting from the subsequent decay of the \( \Xi \) produced in the primary reaction. The moments, in turn, can be extracted from the measurement of the angular distribution of the decay products.

Since the determination of the spin and parity quantum numbers is a fundamental part of any spectroscopy study, reflection symmetry about the reaction plane has been exploited, in particular, to show that, apart from the spin-transfer coefficient \( K_{yy} \), the ratio of the SDM elements given by Eq. (36) determines the parity of a \( \Xi \) resonance with an arbitrary spin. Furthermore, the spin-transfer coefficient \( M_{t} \) determines the parity of the \( \Xi \) resonance simultaneously [31].

We also mention that the present analysis applies as given only to \( \Xi \) resonances that are sufficiently narrow to permit them being treated like on-shell particles. For broad resonances, a partial-wave analysis would be required to extract them from experimental data.

In summary, the present analysis provides the model-independent framework for developing reliable reaction theories of \( \Xi \) production to help in the planning of future experimental efforts in \( \Xi \) baryon spectroscopy. This will also help in analyzing the data to understand the production mechanisms of \( \Xi \) baryons.

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**Appendix A: Partial-Wave Decomposition**

In this Appendix, we give the partial wave decomposition of the coefficients multiplying each spin structure of the reaction amplitudes in Eqs. (4) and (11) for \( \Xi \) with spin-parity \( J^P = 1^\pm \) and \( 3^\pm \). The partial-wave expansion of the (plane-wave) matrix element \( \hat{M} \) in Eqs. (4) and (11) is

\[
\langle S'M_S'|\hat{M}(p',p)|S'M_S\rangle = \sum_{L,L'} \langle S'M_S L M_L|J M_J \rangle \langle S'M_S' L' M_L'|J M_J \rangle \times M^L_{J} (p', p) Y^L_{M_L}(\hat{p}) Y^L_{M_L'}(\hat{p'}),
\]

where \( S, L, J, T \) stand for the total spin, orbital angular momentum, total angular momentum, and the total isospin, respectively, of the initial \( KN \) state. The corresponding projection quantum numbers are denoted by \( M_S, M_L, \) and \( M_J \). The primed quantities represent the corresponding quantum numbers of the final \( K\Xi \) state. The summation runs over all quantum numbers not specified in the left-hand side of Eq. (A1). The relative momenta of the initial \( KN \) and final \( K\Xi \) states are denoted by \( \bm{p} \) and \( \bm{p'} \), respectively, and \( p = |\bm{p}|, \ p' = |\bm{p'}| \). In the following, without loss of generality, we choose \( \hat{n}_3 \) along the momentum \( \bm{p} \) of the nucleon in the CM system, i.e., \( \hat{n}_3 \equiv \hat{p} \) as specified in Fig. 1. In Eq. (A1), \( \hat{P}_T \) stands for the total isospin projection operator onto the isospin singlet \( (T = 0) \) and isospin triplet \( (T = 1) \) states,

\[
\hat{P}_0 = \frac{1}{4}(1 - \tau_1 \cdot \tau_2) \quad \text{and} \quad \hat{P}_1 = \frac{1}{4}(3 + \tau_1 \cdot \tau_2),
\]

where the \( \tau_i \) \( (i = 1, 2) \) are the usual vectors made out of isospin Pauli matrices.

For a \( \Xi \) of \( J^P = 1^\pm \), following Ref. [27], the coefficients \( M_t \) in Eq. (4) are given by

\[
M_0 = \frac{i}{4\pi} \sum_{L', T} \left[ (L' + 1) M^T_{L,L'}(p', p) + L' M^T_{L',L'}(p', p) \right] P_{L'}(\hat{p} \cdot \hat{p'}) \hat{P}_T,
\]

\[
M_2 = \frac{i}{4\pi} \sum_{L', T} \left[ M^T_{L,L'}(p', p) - M^T_{L',L'}(p', p) \right] P^1_{L'}(\hat{p} \cdot \hat{p'}) \hat{P}_T,
\]

\[
M_1 = \frac{i}{4\pi} \sum_{L', T} \left[ M^T_{L,L'-1}(p', p) + M^T_{L',L'+1}(p', p) \right] P^1_{L'}(\hat{p} \cdot \hat{p'}) \hat{P}_T,
\]

\[
M_3 = \frac{i}{4\pi} \sum_{L', T} \left[ (L' + 1) M^T_{L,L'+1}(p', p) - L' M^T_{L',L'-1}(p', p) \right] P_{L'}(\hat{p} \cdot \hat{p'}) \hat{P}_T,
\]
where \( J_\pm \equiv L' \pm \frac{1}{2} \), and \( P_{L'}(x) \) and \( P^{1}_{L'}(x) \) denote the Legendre and associated Legendre functions, respectively.

The amplitudes \( M_i \) here are operators in isospin space whose actions are specified by the projectors \( \hat{P}_T \) defined in Eq. (A2).

Likewise, for a \( \Xi \) of \( J^P = \frac{3}{2}^{+} \), following Ref. [27], the coefficients \( F_i \) and \( G_i \) in Eq. (14) are given by

\[
F_1 = \frac{3}{8\pi} \sum_{J,L',T} (-1)^{L'+\frac{3}{2}} [J]^2 \frac{[L']}{\sqrt{L'(L'+1)}} \left\{ \frac{1}{2} L'^{T}_{\frac{3}{2}} J^T \right\} M_{L,L'}^{T}(p', p) P_{L'}^{1}(\hat{p}' \cdot \hat{p}) \hat{P}_T, \tag{A4a}
\]

\[
F_2 = \frac{1}{\sqrt{2}} \sum_{J,L',L,T} i^{L'-L'_1} (-1)^{J+\frac{3}{2}} [J]^2 \left\{ \frac{1}{2} L'^{T}_{\frac{3}{2}} J^T \right\} M_{L,L'}^{T}(p', p) a_{L,L'} \hat{P}_T, \tag{A4b}
\]

\[
F_3 = \frac{1}{2\sqrt{2}} \sum_{J,L',L,T} i^{L'-L'_1} (-1)^{J+\frac{3}{2}} [J]^2 \left\{ \frac{1}{2} L'^{T}_{\frac{3}{2}} J^T \right\} M_{L,L'}^{T}(p', p) b_{L,L'} \hat{P}_T, \tag{A4c}
\]

\[
F_4 = \frac{1}{\sqrt{2}} \sum_{J,L',L,T} i^{L'-L'_1} (-1)^{J+\frac{3}{2}} [J]^2 \left\{ \frac{1}{2} L'^{T}_{\frac{3}{2}} J^T \right\} M_{L,L'}^{T}(p', p) c_{L,L'} \hat{P}_T, \tag{A4d}
\]

and

\[
G_1 = \frac{1}{2\sqrt{2}} \sum_{J,L',L,T} i^{L'-L'_1} (-1)^{J+\frac{3}{2}} [J]^2 \left\{ \frac{1}{2} L'^{T}_{\frac{3}{2}} J^T \right\} M_{L,L'}^{T}(p', p) a_{L,L'} \hat{P}_T, \tag{A5a}
\]

\[
G_2 = \frac{1}{2\sqrt{2}} \sum_{J,L',L,T} i^{L'-L'_1} (-1)^{J+\frac{3}{2}} [J]^2 \left\{ \frac{1}{2} L'^{T}_{\frac{3}{2}} J^T \right\} M_{L,L'}^{T}(p', p) b_{L,L'} \hat{P}_T, \tag{A5b}
\]

\[
G_3 = \frac{\sqrt{2}}{4\pi} \sum_{J,L',L,T} i^{L'-L'_1} (-1)^{J+\frac{3}{2}} [J]^2 \left\{ \frac{[LL']}{\sqrt{L'(L'+1)}} \right\} \langle L_0 L' 0 1 | 11 \rangle \left\{ \frac{1}{2} L'^{T}_{\frac{3}{2}} J^T \right\} M_{L,L'}^{T}(p', p) P_{L'}^{1}(\hat{p}' \cdot \hat{p}) \hat{P}_T, \tag{A5c}
\]

\[
G_4 = \frac{\sqrt{2}}{8\pi} \sum_{J,L',L,T} i^{L'-L'_1} (-1)^{J+\frac{3}{2}} [J]^2 [LL'] \langle L_0 L' 0 10 \rangle \left\{ \frac{1}{2} L'^{T}_{\frac{3}{2}} J^T \right\} M_{L,L'}^{T}(p', p) P_{L'}^{1}(\hat{p}' \cdot \hat{p}) \hat{P}_T, \tag{A5d}
\]

where we introduced the notation \( [J] = \sqrt{2J+1} \) and \( [j_1,j_2] = [j_1][j_2] \). The summations extend over all the quantum numbers \( J, L', L \) and \( T \). Note that total parity conservation imposes the condition \( (-1)^{L'+\frac{3}{2}} = \pm 1 \) as the parity of the \( \Xi \) baryon is \( \pi_\Xi = \pm 1 \). The coefficients \( a_{L,L'}, b_{L,L'}, c_{L,L'}, a'_{L,L'}, b'_{L,L'}, \) etc., are given by

\[
a_{L,L'} = 2\frac{[LL']}{4\pi} \langle L_0 L' 2 | 22 \rangle \sqrt{\frac{(L'-2)!}{(L'+2)!}} P_{L'}^{2}(\hat{p}' \cdot \hat{p}), \tag{A6a}
\]

\[
b_{L,L'} = 2\frac{[LL']}{4\pi} \langle L_0 L' 1 | 21 \rangle \sqrt{\frac{(L'-1)!}{(L'+1)!}} P_{L'}^{1}(\hat{p}' \cdot \hat{p}), \tag{A6b}
\]

\[
c_{L,L'} = \frac{[LL']}{4\pi} \left[ \langle L_0 L' 2 | 22 \rangle \sqrt{\frac{(L'-2)!}{(L'+2)!}} P_{L'}^{2}(\hat{p}' \cdot \hat{p}) + \sqrt{\frac{3}{2}} \langle L_0 L' 0 | 20 \rangle P_{L'}^{1}(\hat{p}' \cdot \hat{p}) \right], \tag{A6c}
\]

\[
a'_{L,L'} = ib_{L,L'}, \tag{A6d}
\]

\[
b'_{L,L'} = -ia_{L,L'}. \tag{A6e}
\]

---

**Appendix B: SDM formalism**

A density operator can be used to describe an ensemble of quantum states. It is defined as

\[
\hat{\rho} \equiv \sum_{\psi} I_{\psi} |\psi\rangle \langle \psi|, \tag{B1}
\]
where \( I_{\psi} \) denotes the probability of finding an element of the ensemble in the state \( \psi \), subject to the condition \( \sum_{\psi} I_{\psi} = 1 \). For the present application, the states \( |\psi\rangle \) are the spin states of the initial state, \( N \), or the final state, \( \Xi \). For the initial nucleon state, the spin-density operator reads

\[
\hat{\rho} \rightarrow \hat{\rho}^{N} \equiv \sum_{\psi_{N}} I_{\psi_{N}} |\psi_{N}\rangle \langle \psi_{N}| \equiv \frac{1}{2} (1 + \hat{P} \cdot \sigma) , \tag{B2}
\]

where \( \sigma = (\sigma_{1}, \sigma_{2}, \sigma_{3}) \) denotes the vector formed of Pauli spin matrices and \( \hat{P} \) is the polarization vector of the nucleon which is the difference between the probability of finding the nucleon in the \( m_{N} = +\frac{1}{2} \) spin state and the probability of finding the nucleon in the \( m_{N} = -\frac{1}{2} \) state (\( m_{N} \) is the spin projection quantum number along the \( P \) direction) or symbolically \( |I_{\psi_{+}} - I_{\psi_{-}}| = |\hat{P}| \).

An unpolarized ensemble has \( \hat{P} = 0 \). The trace of this spin-density matrix is normalized to 1. By introducing the notation \( \sigma_{i} \equiv \hat{1} \) and \( P_{0} \equiv 1 \), the nucleon SDM in Eq. (B2) can be rewritten as

\[
\hat{\rho}^{N} = \frac{1 + \hat{P} \cdot \sigma}{2} = \sum_{i=0}^{3} P_{i} \hat{\rho}^{N,i} \tag{B3}
\]

with

\[
\hat{\rho}^{N,i} = \frac{1}{2} \sigma_{i} \tag{B4}
\]

for \( i = 0, \ldots, 3 \).

The spin-density operator for a produced \( \Xi \) particle, \( \hat{\rho}^{\Xi} \), can be expressed in terms of the production amplitude \( \hat{M} \) which is an operator that maps the initial nucleon spin state \( \psi_{N} \) into a spin-state of the \( \Xi \). In the helicity basis for the produced \( \Xi \), the corresponding spin-density matrix elements read

\[
\rho_{\lambda \lambda'}^{\Xi}(|\psi_{N}\rangle \equiv \langle \lambda | \hat{M} | \psi_{N}\rangle \langle \psi_{N}| \hat{M}^\dagger | \lambda' \rangle , \tag{B5}
\]

where \( \lambda \) and \( \lambda' \) enumerate the \( \Xi \)'s helicities.

When the beam of anti-Kaons scatters off an ensemble of nucleons, one needs to average over all nucleon spin states with their appropriate probability weights, i.e.,

\[
\hat{\rho}^{\Xi} \equiv \sum_{\psi_{N}} I_{\psi_{N}} \rho_{\lambda \lambda'}^{\Xi}(|\psi_{N}\rangle \langle \psi_{N}| \hat{M} \cdot \sigma \hat{M}^\dagger | \lambda' \rangle
\]

\[
= \sum_{\psi_{N}} I_{\psi_{N}} \langle \lambda | \hat{M} | \psi_{N}\rangle \langle \psi_{N}| \hat{M}^\dagger | \lambda' \rangle
\]

\[
= \langle \lambda | \hat{M} \hat{\rho}^{N} \hat{M}^\dagger | \lambda' \rangle
\]

\[
= \langle \lambda | \hat{\rho}^{\Xi} | \lambda' \rangle , \tag{B6}
\]

where Eq. (B11) was used to show that the \( \Xi \) spin-density operator is given by

\[
\hat{\rho}^{\Xi} = \hat{M} \hat{\rho}^{N} \hat{M}^\dagger . \tag{B7}
\]

Using Eq. (B13) we may write

\[
\hat{\rho}^{\Xi} = \sum_{i=0}^{3} P_{i} \hat{\rho}^{\Xi,i} , \tag{B8}
\]

where

\[
\hat{\rho}^{\Xi,i} \equiv \frac{1}{2} \hat{M} \sigma_{i} \hat{M}^\dagger , \tag{B9}
\]

for \( i = 0, \ldots, 3 \). Here, \( \hat{\rho}^{\Xi,0} \) and \( \hat{\rho}^{\Xi,j} \) (\( j = 1, 2, 3 \)) provide the respective contributions for unpolarized and polarized initial nucleons.

For a \( \Xi \) baryon of spin-1/2, comparing Eqs. (7) and (B9) gives

\[
\frac{d\sigma}{d\Omega} = \frac{1}{2} \text{Tr} [\hat{M} \hat{M}^\dagger] = \text{Tr} [\hat{\rho}^{\Xi,0}] , \tag{B10a}
\]

\[
\frac{d\sigma}{dT_{i}} = \frac{1}{2} \text{Tr} [\hat{M} \sigma_{i} \hat{M}^\dagger] = \text{Tr} [\hat{\rho}^{\Xi,i}] , \tag{B10b}
\]

\[
\frac{d\sigma}{dP_{i}} = \frac{1}{2} \text{Tr} [\hat{M} \hat{M}^\dagger \sigma_{i}] = \text{Tr} [\hat{\rho}^{\Xi,0} \sigma_{i}] , \tag{B10c}
\]

\[
\frac{d\sigma}{dK_{ij}} = \frac{1}{2} \text{Tr} [\hat{M} \hat{M}^\dagger \sigma_{j}] = \text{Tr} [\hat{\rho}^{\Xi,i} \sigma_{j}] . \tag{B10d}
\]

When a \( \Xi \) baryon of spin-3/2 (or higher-spin) is involved, there will be many more possible degrees of polarization than the spin-1/2 case. For the particular case of the spin-transfer coefficient, \( K_{ij} \), discussed in connection to the parity of \( \Xi \), its definition given in Eq. (13) has to be generalized. For this purpose, we first introduce the operator \( \Omega(J \cdot \hat{n}) \) as

\[
\Omega(J \cdot \hat{n}) \equiv \sum_{M=-J}^{+J} (-1)^{\frac{J}{2} - M} P_{\hat{n}}^{J,M} , \tag{B11}
\]

where \( P_{\hat{n}}^{J,M} \) denotes the spin-projection operator onto an arbitrary direction \( \hat{n} \) for an arbitrary half-integer spin \( J \). It can be explicitly calculated as

\[
P_{\hat{n}}^{J,M} = \prod_{m=-J}^{+J} \frac{m - J \cdot \hat{n}}{m - M} , \tag{B12}
\]

where the prime indicates that the factor with \( m = M \) is omitted. Here, \( J \equiv (J_{1}, J_{2}, J_{3}) \) stands for the generator of spin-\( J \) rotation. This expression provides a rotationally invariant polynomial of order \( 2J \) in \( J \cdot \hat{n} \) that is a generalization to arbitrary spin of the usual \( (1 \pm \sigma \cdot \hat{n})/2 \) projectors for spin-1/2.

With the spin-projection operator defined above, the spin-transfer coefficient involving a \( \Xi \) baryon with an arbitrary spin \( J \) is now generalized to

\[
\frac{d\sigma}{d\Omega} K_{ba} = \frac{1}{2} \text{Tr} \left[ \hat{M} \sigma \cdot \hat{b} \hat{M}^\dagger \Omega(J \cdot \hat{a}) \right] , \tag{B13}
\]

where \( \hat{b} \) and \( \hat{a} \) are the spin directions. For \( J = 1/2 \), in view of \( \Omega(J \cdot \hat{a}) \rightarrow \sigma \cdot \hat{a} \), this reduces to the familiar
expression \( \Omega^J \), of course. For Cartesian directions \( \hat{b} = \hat{n}_i \) and \( \hat{a} = \hat{n}_j \), in particular, Eq. (B13) may be written as

\[
\frac{d\sigma}{d\Omega} K_{ij'} = \text{Tr} \left[ \rho^{\pm} \Omega^J \right], \tag{B14}
\]

where \( \Omega^J_{ij'} \equiv \Omega(J \cdot \hat{n}_j) \).

For the Cartesian frame \( \{\hat{n}_1', \hat{n}_2', \hat{n}_3'\} \equiv \{\hat{n}_2, \hat{n}_3, \hat{p}'\} \), aligned with the momentum \( \hat{p}' \) of the outgoing \( \Xi \) (see Fig. 1), explicit expressions for \( \Omega^J_{ij'} \) are found as

\[
\Omega^J_{z'} = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \tag{B15a}
\]

\[
\Omega^J_{y'} = \begin{pmatrix}
0 & 0 & 0 & -i \\
0 & 0 & i & 0 \\
0 & -i & 0 & 0 \\
i & 0 & 0 & 0
\end{pmatrix}, \tag{B15b}
\]

\[
\Omega^J_{x'} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \tag{B15c}
\]

which were derived with the help of the spin-3/2 generators in their spinor representation.

For arbitrary spin of \( \Xi \), \( K_{ba} \) of Eq. (B13) becomes

\[
K_{ba} = \frac{\Sigma_{ba}^{\text{even}} - \Sigma_{ba}^{\text{odd}}}{\Sigma_{ba}^{\text{even}} + \Sigma_{ba}^{\text{odd}}}, \tag{B17}
\]

where

\[
\Sigma_{ba}^{\text{even/odd}} = \sum_{m_a, m_b} \frac{d\sigma_{m_a, m_b}}{d\Omega}, \quad m_a - m_b = \text{even/odd}, \tag{B18}
\]

denotes the sum of all polarized differential cross sections such that the differences of all possible combinations of initial and final spin projections \( m_a \) and \( m_b \) along \( \hat{a} \) and \( \hat{b} \), respectively, are an even or odd number.

Appendix C: Explicit form of the SDM's

In this section, we list the SDM elements of each \( \rho^J (i = 0, \ldots, 3) \) in terms of the helicity amplitudes, \( H_i \).

The Hermitian \( \rho^J \) matrices are arranged according to

\[
\rho^J = \begin{pmatrix}
\rho^J_{1,1} & \rho^J_{1,1,-1} & \cdots & \rho^J_{1,-J} \\
\rho^J_{1,-1,1} & \rho^J_{1,-1,1,-1} & \cdots & \rho^J_{1,-1,-J} \\
\vdots & \vdots & \ddots & \vdots \\
\rho^J_{-J,1} & \rho^J_{-J,1,-1} & \cdots & \rho^J_{-J,-J}
\end{pmatrix}. \tag{C1}
\]

For \( J = \frac{1}{2} \), the matrices read explicitly

\[
\rho^0 = \frac{1}{2} \begin{pmatrix}
|H_1|^2 + |H_2|^2 & 2i\pi_{\Xi} \text{Im} \left[ H_2 H_1^* \right] \\
-2i\pi_{\Xi} \text{Im} \left[ H_2 H_1^* \right] & |H_1|^2 + |H_2|^2
\end{pmatrix}, \tag{C2a}
\]

\[
\rho^1 = \frac{1}{2} \begin{pmatrix}
2 \text{Re} \left[ H_2 H_1^* \right] & \pi_{\Xi} \left[ |H_1|^2 - |H_2|^2 \right] \\
\pi_{\Xi} \left[ |H_1|^2 - |H_2|^2 \right] & -2 \text{Re} \left[ H_2 H_1^* \right]
\end{pmatrix}, \tag{C2b}
\]

\[
\rho^2 = \frac{1}{2} \begin{pmatrix}
-2 \text{Im} \left[ H_2 H_1^* \right] & -i\pi_{\Xi} \left[ |H_1|^2 + |H_2|^2 \right] \\
i\pi_{\Xi} \left[ |H_1|^2 + |H_2|^2 \right] & -2 \text{Im} \left[ H_2 H_1^* \right]
\end{pmatrix}, \tag{C2c}
\]

\[
\rho^3 = \frac{1}{2} \begin{pmatrix}
|H_1|^2 - |H_2|^2 & -2\pi_{\Xi} \text{Re} \left[ H_2 H_1^* \right] \\
-2\pi_{\Xi} \text{Re} \left[ H_2 H_1^* \right] & -|H_1|^2 + |H_2|^2
\end{pmatrix}, \tag{C2d}
\]

where \( \pi_{\Xi} \) is the parity of the \( \Xi \).
For a $J = \frac{3}{2}$ resonance,

$$\hat{\rho}^0 = \frac{1}{2} \begin{pmatrix}
|\mathcal{H}_0|^2 + |\mathcal{H}_1|^2 & |\mathcal{H}_0| \mathcal{H}_1^* + |\mathcal{H}_1| \mathcal{H}_0^* & \pi_\Xi (|\mathcal{H}_0|^2 - |\mathcal{H}_1|^2) & 2i\pi_\Xi \text{Im}[\mathcal{H}_0 \mathcal{H}_1^*] \\
|\mathcal{H}_0| \mathcal{H}_1^* + |\mathcal{H}_1| \mathcal{H}_0^* & |\mathcal{H}_0|^2 + |\mathcal{H}_1|^2 & |\mathcal{H}_0| \mathcal{H}_1^* + |\mathcal{H}_1| \mathcal{H}_0^* & 2i\pi_\Xi \text{Im}[\mathcal{H}_3 \mathcal{H}_1^*] \\
-2i\pi_\Xi \text{Im}[\mathcal{H}_3 \mathcal{H}_1^*] & -2i\pi_\Xi \text{Im}[\mathcal{H}_3 \mathcal{H}_4^*] & |\mathcal{H}_0|^2 + |\mathcal{H}_1|^2 & -4\pi_\Xi \mathcal{H}_3 \mathcal{H}_4^* + \mathcal{H}_4 \mathcal{H}_3^* \\
-2i\pi_\Xi \text{Im}[\mathcal{H}_2 \mathcal{H}_1^*] & \pi_\Xi (-|\mathcal{H}_2|^2 + |\mathcal{H}_1|^2) & -\pi_\Xi |\mathcal{H}_2|^2 - |\mathcal{H}_1|^2 & |\mathcal{H}_2|^2 + |\mathcal{H}_1|^2
\end{pmatrix},$$

(C3a)

$$\hat{\rho}^1 = \frac{1}{2} \begin{pmatrix}
2 \text{Re}[\mathcal{H}_0 \mathcal{H}_1^*] & \mathcal{H}_1 \mathcal{H}_4^* + \mathcal{H}_3 \mathcal{H}_5^* & \pi_\Xi (\mathcal{H}_2 \mathcal{H}_4^* + \mathcal{H}_3 \mathcal{H}_5^*) & \pi_\Xi (-|\mathcal{H}_2|^2 + |\mathcal{H}_1|^2) \\
\mathcal{H}_1 \mathcal{H}_4^* + \mathcal{H}_3 \mathcal{H}_5^* & 2 \text{Re}[\mathcal{H}_3 \mathcal{H}_4^*] & \pi_\Xi (|\mathcal{H}_4|^2 - |\mathcal{H}_3|^2) & \pi_\Xi (-\mathcal{H}_3 \mathcal{H}_2^* + \mathcal{H}_5 \mathcal{H}_4^*) \\
\pi_\Xi (|\mathcal{H}_4|^2 - |\mathcal{H}_3|^2) & -2 \text{Re}[\mathcal{H}_4 \mathcal{H}_3^*] & |\mathcal{H}_3|^2 + |\mathcal{H}_4|^2 & -2 \text{Re}[\mathcal{H}_2 \mathcal{H}_1^*] \\
\pi_\Xi (-|\mathcal{H}_4|^2 + |\mathcal{H}_3|^2) & \pi_\Xi (-|\mathcal{H}_3|^2 + |\mathcal{H}_4|^2) & |\mathcal{H}_3|^2 + |\mathcal{H}_4|^2 & |\mathcal{H}_2|^2 + |\mathcal{H}_1|^2
\end{pmatrix},$$

(C3b)

$$\hat{\rho}^2 = \frac{1}{2} \begin{pmatrix}
2 \text{Im}[\mathcal{H}_0 \mathcal{H}_4^*] & i (|\mathcal{H}_4|^2 - |\mathcal{H}_3|^2) & i \pi_\Xi (\mathcal{H}_2 \mathcal{H}_4^* + \mathcal{H}_3 \mathcal{H}_5^*) & -i \pi_\Xi (|\mathcal{H}_2|^2 + |\mathcal{H}_1|^2) \\
i \pi_\Xi (|\mathcal{H}_4|^2 - |\mathcal{H}_3|^2) & 2 \text{Re}[\mathcal{H}_3 \mathcal{H}_4^*] & i \pi_\Xi (|\mathcal{H}_4|^2 + |\mathcal{H}_3|^2) & -i \pi_\Xi (\mathcal{H}_3 \mathcal{H}_2^* + \mathcal{H}_5 \mathcal{H}_4^*) \\
-i \pi_\Xi (|\mathcal{H}_4|^2 + |\mathcal{H}_3|^2) & i \pi_\Xi (|\mathcal{H}_4|^2 + |\mathcal{H}_3|^2) & 2 \text{Re}[\mathcal{H}_4 \mathcal{H}_3^*] & i (|\mathcal{H}_3|^2 - |\mathcal{H}_4|^2) \\
i \pi_\Xi (\mathcal{H}_2^2 + |\mathcal{H}_1|^2) & i \pi_\Xi (\mathcal{H}_2^2 + |\mathcal{H}_1|^2) & i (|\mathcal{H}_3|^2 - |\mathcal{H}_4|^2) & 2 \text{Re}[\mathcal{H}_2 \mathcal{H}_1^*]
\end{pmatrix},$$

(C3c)

$$\hat{\rho}^3 = \frac{1}{2} \begin{pmatrix}
-|\mathcal{H}_1|^2 & -|\mathcal{H}_1|^2 & \pi_\Xi (|\mathcal{H}_4|^2 + |\mathcal{H}_3|^2) & -2 \pi_\Xi \text{Re}[\mathcal{H}_2 \mathcal{H}_1^*] \\
-|\mathcal{H}_1|^2 & -|\mathcal{H}_1|^2 & \pi_\Xi (-|\mathcal{H}_3|^2 + |\mathcal{H}_4|^2) & 2 \pi_\Xi \text{Re}[\mathcal{H}_3 \mathcal{H}_4^*] \\
\pi_\Xi (|\mathcal{H}_4|^2 + |\mathcal{H}_3|^2) & \pi_\Xi (-|\mathcal{H}_3|^2 + |\mathcal{H}_4|^2) & |\mathcal{H}_3|^2 + |\mathcal{H}_4|^2 & -4\pi_\Xi \mathcal{H}_3 \mathcal{H}_4^* + \mathcal{H}_4 \mathcal{H}_3^* \\
-2 \pi_\Xi \text{Re}[\mathcal{H}_3 \mathcal{H}_1^*] & -2 \pi_\Xi \text{Re}[\mathcal{H}_3 \mathcal{H}_1^*] & -4\pi_\Xi \mathcal{H}_3 \mathcal{H}_4^* + \mathcal{H}_4 \mathcal{H}_3^* & |\mathcal{H}_2|^2 - |\mathcal{H}_1|^2
\end{pmatrix}.$$

(C3d)

Appendix D: Measuring the SDM Elements

In Sec. [H1] we identified a set of SDM elements that determines the reaction amplitude completely. A standard way of measuring the SDM elements is via the subsequent decay of the produced particle in a primary reaction by exploiting the self-analyzing property of the decay-product particle.

In the present work, the reaction in Eq. (14) is the primary (or production) reaction, where the Ξ baryon is produced. If the produced Ξ is a ground state Ξ, then, it decays via a single weak decay process into

$$\Xi \rightarrow \Lambda + \pi,$$

(D1)

whose associated Ξ decay-asymmetry parameters, $\alpha_\Xi$, are known to be [2]

$$\alpha_{\Xi^0} = -0.406 \pm 0.013,$$  
$$\alpha_{\Xi^-} = -0.458 \pm 0.012.$$  

(D2a)  
(D2b)

An excited Ξ resonance, $\Xi^*$, on the other hand, may undergo a double-decay process

$$\Xi^* \rightarrow \Xi + \pi \quad \downarrow \Lambda + \pi,$$  
(D3a)

or

$$\Xi^* \rightarrow \Xi + \bar{K} \quad \downarrow \Lambda + \pi.$$  
(D3b)

The associated $\Lambda$ decay-asymmetry parameter for the second-step process $\Lambda \rightarrow N + \pi$ is [2]

$$\alpha_{\Lambda^-} = +0.642 \pm 0.013 \quad (\Lambda^0 \rightarrow p + \pi^-),$$  
$$\alpha_{\Lambda^0} = +0.650 \pm 0.015 \quad (\Lambda^0 \rightarrow n + \pi^0).$$  

(D4)

The Ξ production process [11] is described in the CM frame of the reaction. The Ξ decay processes of Eqs. (D1) and (D3), on the other hand, are described in the rest frame of the produced Ξ, whose right-handed Cartesian
coordinate system \( \{\hat{n}_1', \hat{n}_2', \hat{n}_3' \} \) is fully specified in Fig. 1.

In the double-decay processes (D3), the subsequent \( \Lambda \) decay process is described in the rest frame of the decaying \( \Lambda \) denoted by \( \{\hat{n}_1'', \hat{n}_2'', \hat{n}_3''\} \), with \( \hat{n}_3'' \equiv \hat{p}_\Lambda \), where \( \hat{p}_\Lambda \) describes the direction of the \( \Lambda \)'s momentum in the \( \{\hat{n}_1', \hat{n}_2', \hat{n}_3'\} \) frame (see Fig. 1); the other two axes are given by \( \hat{n}_2'' = (\hat{n}_1' \times \hat{p}_\Lambda)/|\hat{n}_1' \times \hat{p}_\Lambda| \) and \( \hat{n}_1'' = \hat{n}_2'' \times \hat{n}_3'' \).

1. Single-decay process: Ground state \( \Xi \)

The ground-state \( \Xi \) decays weakly almost entirely into \( \Lambda + \pi \). We define the amplitude describing the \( \Xi \) production process \( K + N \rightarrow K + \Xi \), followed by the subsequent weak decay of the produced \( \Xi \), \( \Xi \rightarrow \Lambda + \pi \), as [30, 31]

\[
A = A(\Omega_\Xi, \Omega_\Lambda, \lambda_\Xi, \lambda_\Lambda, \lambda_N)
= \langle \Omega_\Lambda, \lambda_\Lambda | \hat{M}_D | \Omega_\Xi, \lambda_\Xi \rangle \langle \Omega_\Xi, \lambda_\Xi | \hat{M} | \lambda_N \rangle \quad (D5)
\]

with \( \langle \Omega_\Xi, \lambda_\Xi | \hat{M} | \lambda_N \rangle \) denoting the production reaction amplitude (in the corresponding CM frame) and

\[
\langle \Omega_\Lambda, \lambda_\Lambda | \hat{M}_D | \lambda_\Xi \rangle \equiv \sqrt{\frac{2J+1}{4\pi}} F_{\lambda_\Lambda}^{\Xi} D_{\lambda_\Xi, \lambda_\Lambda}^J (\Omega_\Lambda), \quad (D6)
\]

Here, \( \rho_{\lambda_N, \lambda_\Lambda}^{N,i} = \langle \lambda_N | \hat{N}_i | \lambda_\Lambda \rangle \) denotes the target nucleon SDM element with \( \hat{N}_i \) given by Eq. (D3), and Eq. (D7) was used in the last step. Also, we note that the explicit reference to the \( \Omega_\Xi \) dependence of the angular distribution \( I^i \) in Eqs. (D7) and (D8) has been suppressed for the sake of simplicity of notation. The same holds for the angular distribution in Eqs. (D19) and (D20) in the next subsection.

We now define the moments, \( H^i(L, M) \), of this distribution as

\[
H^i(L, M) = \int d\Omega_\Lambda I^i(\Omega_\Lambda) D_{LM,0}^J (\Omega_\Lambda)
= t_{LM}^{J,i} \sum_{\lambda_\Lambda} F_{\lambda_\Lambda}^{\Xi} F_{\lambda_\Lambda}^{\Xi} \langle J \lambda_\Lambda L 0 | J \lambda_\Xi \rangle, \quad (D9)
\]

where \( d\Omega_\Lambda = \sin \theta_\Lambda d\theta_\Lambda d\phi_\Lambda \). The quantity \( t_{LM}^{J,i} \) here is denoting the subsequent \( \Xi \) decay amplitude (in the rest frame of \( \Xi \)). Here, \( \Omega_\Xi \) and \( \Omega_\Lambda \) are short-hand notations, respectively, for the polar and azimuthal angles of the produced \( \Xi \) in the CM frame of the production, \( \Omega_\Xi = (\theta_\Xi, \phi_\Xi) \), and for the polar and azimuthal angles of the \( \Lambda \) in the rest frame of the produced \( \Xi \), \( \Omega_\Lambda = (\theta_\Lambda, \phi_\Lambda) \).

\( \lambda_\Xi(\lambda_\Lambda) \) is the helicity of the \( \Xi(\Lambda) \) in the respective frame, while \( J \) denotes the spin of the decaying \( \Xi \). \( F_{\lambda_\Lambda}^{\Xi} \) stands for the helicity \( \Xi \)-decay amplitude and \( D_{\lambda_\Xi, \lambda_\Lambda}^J (\Omega_\Lambda) \) is the usual Wigner rotation matrix. Here, the argument \( \Omega_\Lambda \) in \( D_{\lambda_\Xi, \lambda_\Lambda}^J (\Omega_\Lambda) \) is to be understood as the set of Euler angles \( \{\alpha, \beta, \gamma\} \), such that, \( D_{\lambda_\Xi, \lambda_\Lambda}^J (\Omega_\Lambda) = D_{\alpha, \beta, \gamma}^J (\Omega_\Lambda) = 0 \) in the conventions defined in Ref. [41].

The angular distribution of the \( \Lambda \) hyperon, \( I(\Omega_\Lambda) \), in the \( \Xi \rightarrow \Lambda + \pi \) decay (for fixed \( \Xi \) angle \( \Omega_\Xi \)) is given by

\[
I(\Omega_\Lambda) = \sum_{i=0}^{3} P_i I^i(\Omega_\Lambda), \quad (D7)
\]

where

\[
I^i(\Omega_\Lambda) = \sum_{\lambda_N} A(\Omega_\Xi, \Omega_\Lambda, \lambda_\Xi, \lambda_\Lambda, \lambda_N) \rho_{\lambda_N, \lambda_\Lambda}^{N,i} A^\dagger(\Omega_\Xi, \Omega_\Lambda, \lambda_\Xi, \lambda_\Lambda, \lambda_N)
= \frac{(2J+1)}{4\pi} \sum_{\lambda_N} F_{\lambda_N}^{\Xi} F_{\lambda_\Lambda}^{\Xi} M_{\lambda_\Xi, \lambda_\Lambda} \rho_{\lambda_N, \lambda_\Lambda}^{N,i} M_{\lambda_\Xi, \lambda_\Lambda}^* \rho_{\lambda_\Xi, \lambda_\Lambda}^{\Xi,i} D_{\lambda_\Xi, \lambda_\Lambda}^J (\Omega_\Lambda) D_{\lambda_\Xi, \lambda_\Lambda}^J (\Omega_\Lambda), \quad (D8)
\]

related to the SDM elements of the \( \Xi \) by [30]

\[
t_{LM}^{J,i} = \sum_{\lambda_N, \lambda_\Xi} \rho_{\lambda_\Xi, \lambda_\Lambda}^{\Xi,i} \langle J \lambda_\Xi L M | J \lambda_\Xi \rangle, \quad (D10)
\]

whose inversion produces

\[
\rho_{\lambda_\Xi, \lambda_\Lambda}^{\Xi,i} = \sum_{L} \frac{2L+1}{2J+1} \langle J \lambda_\Xi L M | J \lambda_\Xi \rangle t_{LM}^{J,i}, \quad (D11)
\]

where \( M = \lambda_\Xi - \lambda_\Lambda \).

Introducing further the quantities

\[
g_{\Xi}^\pm = F_{\pm}^{\Xi} F_{\pm}^{\Xi} \quad (D12)
\]

we can re-express the moments in Eq. (D9) as

\[
H^i(L, M) = t_{LM}^{J,i} \langle J \frac{1}{2} L 0 | J \frac{1}{2} \rangle [g_{\Xi}^+ + (-1)^L g_{\Xi}^-], \quad (D13)
\]
We note that \( g_\Xi^\Xi \) introduced in Eq. (D12) are related to the \( \Xi \) decay-asymmetry parameter, \( \alpha_\Xi \), given in Eq. (D2) by

\[
\frac{g_\Xi^\Xi - g_\Xi^0}{g_\Xi^\Xi + g_\Xi^0} = \alpha_\Xi . \tag{D14}
\]

Then, taking the ratio of the moments \( H^i(L, M) \) \((i = 1, 2, 3)\) and \( H^0(0, 0) \), we obtain

\[
\frac{H^i(L, M)}{H^0(0, 0)} = \zeta_L t_{LM}^{L,i} (J \frac{1}{2}, L | 0, J \frac{1}{2}) , \tag{D15}
\]

where \( \zeta_L = 1 \) for even \( L \) and \( \zeta_L = \alpha_\Xi \) for odd \( L \).

Now, from Eq. (D10) and the definition of \( \tilde{\rho}_0^\Xi \) in Eq. (B3), we get

\[
t_{00}^i = \frac{g_\Xi^\Xi - g_\Xi^0}{g_\Xi^\Xi + g_\Xi^0} = \frac{d\sigma}{d\Omega} . \tag{D16}
\]

One can now use Eq. (D15) to extract \( t^{L,i}_{LM} \). Once \( t^{L,i}_{LM} \) is known, the SDM elements \( \rho_{\Lambda_\Xi, \Lambda_\Xi}^{L,i} \) are obtained by making use of Eq. (D11). Note that the non-vanishing moments \( H^i(LM) \) are restricted to \( L \leq 2J \) and \( |M| \leq L \).

2. Double-decay process: Excited \( \Xi \) resonance

The double-decay processes shown in Eq. (D3) are treated analogously to the single-decay process of the previous subsection. Here, we discuss the decay chain with the subsequent decay of the \( \Lambda \) hyperon, \( \Lambda \rightarrow p + \pi^- \), but the results apply to any decay chain that is a strong decay followed by a weak decay and containing a single pseudoscalar meson at each step of the decay.

As for the single-decay process case discussed in the previous subsection, we begin by defining the amplitude describing the \( \Xi^* \) production process \( K + N \rightarrow K + \Xi^* \), followed by the strong decay of the produced \( \Xi^* \), \( \Xi^* \rightarrow \Lambda + K \), and the subsequent weak decay of \( \Lambda \), \( \Lambda \rightarrow N + \pi \), as

\[
A = A(\Omega_\Xi, \Omega_\Lambda, \Omega_p, \lambda_\Lambda, \lambda_\Xi, \lambda_\Lambda, \lambda_p)
\]

\[
= \langle \Omega_p, \lambda_p | \tilde{M}_D | \lambda_\Lambda \rangle \langle \Omega_\Lambda, \lambda_\Lambda | \tilde{M}_D | \lambda_\Xi \rangle \langle \Omega_\Xi, \lambda_\Xi | \tilde{M}_D | \lambda_N \rangle , \tag{D17}
\]

where \( \langle \Omega_\Xi, \lambda_\Xi | \tilde{M}_D | \lambda_N \rangle \) stands for the \( \Xi^* \) production amplitude and

\[
\langle \Omega_\Lambda, \lambda_\Lambda | \tilde{M}_D | \lambda_\Xi \rangle \equiv \sqrt{\frac{2J + 1}{4\pi}} F_{\lambda_\Lambda} D_{\lambda_\Xi, \lambda_\Xi}^J (\Omega_\Xi) , \tag{D18a}
\]

\[
\langle \Omega_p, \lambda_p | \tilde{M}_D | \lambda_\Lambda \rangle \equiv \sqrt{\frac{2}{4\pi}} F_{\lambda_p} D_{\lambda_\Xi, \lambda_p}^{\Xi*} (\Omega_p) , \tag{D18b}
\]

denote the subsequent \( \Xi^* \) strong-decay and \( \Lambda \) weak-decay amplitudes, respectively. We note that the \( \Xi \) production and decay amplitudes are calculated in the CM frame of the production reaction and the rest frame of the produced \( \Xi \), respectively, exactly in the same way as for the single-decay case discussed in the previous subsection. The subsequent \( \Lambda \)-decay amplitude, \( \langle \Omega_\Lambda, \lambda_\Lambda | \tilde{M}_D' | \lambda_\Lambda \rangle \), is calculated in the rest frame of the decaying \( \Lambda \) denoted by \( \{ \hat{n}_p^{\prime}, \hat{n}_\Xi^{\prime}, \hat{n}_\Lambda^{\prime} \} \) [cf. the second paragraph just below Eq. (D4)], where \( \Omega_\Lambda = (\theta_p, \phi_p) \) is a short-hand notation for the polar and azimuthal angles \( \theta_p \) and \( \phi_p \), respectively, of the decay-product proton measured in the \( \Lambda \) rest frame.

The angular distribution of the entire double decay process (for fixed \( \Xi \) production angle \( \Omega_\Xi \)) is given as

\[
I(\Omega_\Lambda, \Omega_p) = \sum_{i=0}^{3} P_i I(\Omega_\Lambda, \Omega_p) , \tag{D19}
\]

where

\[
I^i(\Omega_\Lambda, \Omega_p) \equiv \sum_{\lambda_\lambda} A(\Omega_\Xi, \Omega_\Lambda, \Omega_p, \lambda_\Lambda, \lambda_\Xi, \lambda_\Lambda, \lambda_p) \rho_{\lambda_\lambda, \lambda_\lambda}^{N,i} A^*(\Omega_\Xi, \Omega_\Lambda, \Omega_p, \lambda'_N, \lambda'_\Xi, \lambda'_\Lambda, \lambda'_p) \]

\[
= \frac{2(2J + 1)}{16\pi^2} \sum_{\lambda_\lambda} \rho_{\lambda_\Xi, \lambda_\Xi} g_{\lambda_\lambda} A_{\lambda_\lambda} A_{\lambda_\lambda}^* D_{\lambda_\lambda, \lambda_\lambda}^{\Xi*} (\Omega_p) D_{\lambda_\lambda, \lambda_\lambda}^{\Xi*} (\Omega_\Lambda) D_{\lambda_\lambda, \lambda_\lambda}^{\Xi*} (\Omega_\Xi) , \tag{D20}
\]

with

\[
g_{\lambda_\lambda} = F_{\lambda_\lambda}^{\lambda_\lambda} F_{\lambda_\lambda}^{\lambda_\lambda*} \text{ and } \quad g_{\lambda_\lambda}^{\Xi*} = F_{\lambda_\lambda}^{\Xi*} F_{\lambda_\lambda}^{\Xi*}. \tag{D21}
\]

To arrive at the last equality in Eq. (D20), we have made use of Eq. (D7).

We now define the moments \( H^i(l, m, L, M) \) as

\[
H^i(l, m, L, M) \equiv \int d\Omega_\Lambda d\Omega_p I^i(\Omega_\Lambda, \Omega_p) D^{L,i}_{LM,m}(\Omega_\Lambda) D^{L,i}_{m,0}(\Omega_p)
\]

\[
= t_{LM}^{L,i} \sum_{\lambda_\lambda} \rho_{\lambda_\lambda, \lambda_\lambda} D_{\lambda_\lambda, \lambda_\lambda}^{L,i} (J \lambda_\lambda | m \rangle (J \lambda_\lambda | m \rangle \langle \frac{1}{2} \lambda_\lambda | l \rangle \langle \frac{1}{2} \lambda_\lambda | m \rangle) \sum_{\lambda_p} g_{\lambda_p} \langle \frac{1}{2} \lambda_p | l \rangle | \frac{1}{2} \lambda_p \rangle , \tag{D22}
\]
with $t^{j,i}_{LM}$ given by Eq. \ref{eq:td}.\footnote{Babar Collaboration, A. Aubert et al., Phys. Rev. Lett. 97, 112001 (2006).}

The different $g_{\lambda \Lambda, \lambda' \Lambda'}$ are related to each other by

\begin{align}
&g_{\lambda \Lambda}^+ = g_{\lambda \Lambda}^- , \quad (D23a) \\
&g_{\lambda \Lambda}^+ = g_{\lambda \Lambda}^- = \pi \langle J + \frac{1}{2} \rangle g_{++}^\Xi . \quad (D23b)
\end{align}

The $g_{\lambda \Lambda}$ terms can be related to the $\Lambda$ decay asymmetry parameter, $\alpha_{\Lambda}$, by \cite{42, 43}

\begin{equation}
\frac{g_{\lambda \Lambda}^+ - g_{\lambda \Lambda}^-}{g_{++}^\Xi} = \alpha_{\Lambda} . \quad (D24)
\end{equation}

Note that the non-vanishing moments $H^l(l, m, L, M)$ are restricted to $|m| \leq L, \ l \leq 1, \ |M| \leq L$ and $L \leq 2J$, as can be read off from Eq. \ref{eq:td}. The moments $H^l(0, 0, L, M)$ and $H^l(1, m, L, M)$ vanish identically for odd and even $L$, respectively, due to Eqs. \ref{eq:td} and \ref{eq:td2}. Analogously to the single-decay case, the ratios of the moments

\begin{equation}
\frac{H^l(0, 0, L, M)}{H^0(0, 0, 0, 0)} = \frac{t^{j,i}_{LM}}{t_{00}^{j,i}} \langle J + \frac{1}{2} | J - \frac{1}{2} \rangle \quad (D25a)
\end{equation}

for even $L$ and

\begin{equation}
\frac{H^l(1, 0, L, M)}{H^0(0, 0, 0, 0)} = \frac{\alpha_{\Lambda}}{3} \frac{t^{j,i}_{LM}}{t_{00}^{j,i}} \langle J + \frac{1}{2} | J - \frac{1}{2} \rangle \quad (D25b)
\end{equation}

for odd $L$ allow us to determine $t^{j,i}_{LM}$. Since $t_{00}^{j,i} = d\sigma/d\Omega$, once $t^{j,i}_{LM}$ is extracted, the SDM elements $\rho_{\lambda \Lambda, \lambda' \Lambda'}$ can be determined via Eq. \ref{eq:rho}.
Wesley, Reading, MA, 1964), Ch. 17.

[43] S. Gasiorowicz, *Elementary Particle Physics* (John Wiley & Sons, New York, 1966), Ch. 32.