Lattice of intermediate subalgebras

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Dedicated to V. S. Sunder

Abstract

Analogous to subfactor theory, employing Watatani’s notions of index and $C^*$-basic construction of certain inclusions of $C^*$-algebras, (a) we develop a Fourier theory (consisting of Fourier transforms, rotation maps and shift operators) on the relative commutants of any inclusion of simple unital $C^*$-algebras with finite Watatani index, and (b) we introduce the notions of interior and exterior angles between intermediate $C^*$-subalgebras of any inclusion of unital $C^*$-algebras admitting a finite index conditional expectation. Then, on the lines of Bakshi et al. (Trans. Amer. Math. Soc. 371 (2019) 5973–5991), we apply these concepts to obtain a bound for the cardinality of the lattice of intermediate $C^*$-subalgebras of any irreducible inclusion as in (a), and improve Longo’s bound for the cardinality of intermediate subfactors of an irreducible inclusion of type $III$ factors with finite index. Moreover, we also show that for a fairly large class of inclusions of finite von Neumann algebras, the lattice of intermediate von Neumann subalgebras is always finite.

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1. Introduction

Among the various significant themes of operators algebras, the theory of subfactors has attracted a fair share of limelight during the last three and a half decades because of the deep relationship and implications it has exhibited to various other branches of mathematics as well as theoretical physics. The modern theory of subfactors was initiated by Vaughan Jones in 1983 in his seminal work [20], wherein, among various deep and original ideas, he formalized the notion of the index $[M : N]$ as the Murray–von Neumann’s coupling constant $\dim_N L^2(M)$, for any subfactor $N \subset M$ of type $II_1$, and introduced the notion of the basic construction for any unital inclusion of finite von Neumann algebras. Later, Kosaki [28] generalized the notion of index and basic construction in terms of suitable conditional expectations for subfactors of any type. And, in 1990, Watatani [48] generalized Jones’ and Kosaki’s indices to the index of a conditional expectation associated to any inclusion of $C^*$-algebras. In the same article, using the language of Hilbert $C^*$-modules, Watatani also provided an analog of their notions of basic
construction for any pair $B \subset A$ of unital $C^*$-algebras with respect to a finite index conditional expectation. Over the years, many authors have used Watatani’s notions of index and $C^*$-basic construction to prove significant results in the theories of $C^*$-algebras, von Neumann algebras and Hilbert $C^*$-modules — see \[10, 15–18, 24–27, 43\].

Since the basic flavor of the theory of subfactors revolves around the analysis of the relative position of a subfactor inside an ambient factor, it is a very natural and fundamental question to analyze the lattice consisting of all intermediate subfactors. Needless to mention, a substantial amount of work has been done in this direction too. For instance, Bisch \[5\] exhibited a dictionary between the intermediate subfactors of a subfactor $N \subset M$ of type $II_1$ and the so-called biprojections in the relative commutant space $N' \cap M_1$. See \[7\] for some interesting results in this direction. The crucial ingredient in Bisch’s biprojection theory is the Fourier theory on the relative commutants $N' \cap M_k$ formulated by Ocneanu and Jones — see \[5, 21, 37\]. Furthermore, subfactor theory has gained a lot from the structures of Popa’s $\lambda$-lattice \[41\] and Jones’ planar algebra \[21\] on the standard invariant of any subfactor of type $II_1$ with finite Jones index, both of which were formulated by exploiting the techniques of Fourier theory quite heavily (see \[44\] for details).

On the other hand, the study of $C^*$-subalgebras of a given $C^*$-algebra has also attracted good attention and that too from different perspectives — see \[8, 9, 15, 18, 46\] and the references therein. In Section 2, after a quick recollection of Watatani’s notions of index and $C^*$-basic construction, and the notion of minimal conditional expectations by Hiai, Kosaki and Longo \[13, 27, 29, 32, 33, 48\], given any inclusion $B \subset A$ of simple unital $C^*$-algebra with finite Watatani index, we single out a sequence of consistent tracial states on the tower of relative commutants, which then allows us to obtain a bound for the dimension of each relative commutant $B' \cap A_k$.

Then, in Section 3, we provide a $C^*$-version of the Fourier theory for any such pair of simple unital $C^*$-algebras. The subtle difference between our approach and that of Ocneanu and Jones lies in the fact that, unlike for finite factors, we neither have a tracial state on $A$ to begin with nor the ‘modular conjugation operator’ on the $L^2$-completion of $A$. As mentioned above, we found a way around using the notion of minimal conditional expectations. We provide a detailed theory of Fourier transforms, rotation maps and shift operators on the relative commutants of appropriate inclusions of $C^*$-algebras.

In Section 4, motivated by Bisch’s characterization of intermediate subfactors in terms of biprojections, we formulate the notions of biunitaries, bipartial isometries and biprojections and their behavior under Fourier transforms and rotations. As the first application of the $C^*$-Fourier theory, given any irreducible pair $B \subset A$ of simple unital $C^*$-algebras with finite Watatani index and a biprojection in $B' \cap A_1$, we provide a recipe to obtain an intermediate $C^*$-subalgebra of the dual pair $A \subset A_1$ in Theorem 4.24.

Recently, the first named author along with Das, Liu and Ren, in \[2\], introduced the notions of interior and exterior angles between intermediate subfactors of a subfactor of type $II_1$ to understand the relative position of two intermediate subfactors. Motivated by them, in Section 5, we begin with the introduction of the notions of interior and exterior angles between any two intermediate $C^*$-subalgebras of an inclusion $B \subset A$ of unital $C^*$-algebras with a finite index conditional expectation, and provide some useful expressions for the same. Then, very much like the minimal intermediate subfactors of a subfactor of type $II_1$ (as was exhibited in \[2\]), we show in Theorem 5.17 that, in terms of relative positions, there is a certain rigidity observed by the minimal intermediate $C^*$-subalgebras of an irreducible pair of simple $C^*$-algebras in the sense that the interior angle between any two such $C^*$-subalgebras is always greater than $\pi/3$. The proof is based on the $C^*$-Fourier theory that we develop.

On the other hand, Watatani (in \[49\], see also \[40\]) and then Teruya and Watatani (in \[47\]) showed that the lattice of intermediate subfactors of an irreducible subfactor of type $II_1$ and type $III$, respectively, is finite. Then, Longo (in \[35\]) proved that the number of intermediate
subfactors of an irreducible subfactor $N \subset M$ (of any type) with finite index is bounded by $([M : N])^2 [M : N]$ and had asked whether the bound could be improved to $[M : N]^{[M : N]}$. The authors of [2] exploited the notion of interior angle satisfactorily to answer this question and showed that for an irreducible subfactor $N \subset M$ of type $II\_1$ the bound can be improved significantly to $\min\{9[M : N], [M : N]^{[M : N]}\}$. However, the question for irreducible subfactors of type $III$ remained unanswered. For $C^\ast$-algebras, Ino and Watatani (in [15, Corollary 3.9]) had shown that every irreducible pair $B \subset A$ of simple unital $C^\ast$-algebras with a conditional expectation of finite index has only finitely many intermediate $C^\ast$-subalgebras. However, they did not provide any bound on the number of such intermediate $C^\ast$-subalgebras.

As another useful application of the $C^\ast$-Fourier theory and the notion of interior angle, on the lines of [2], we deduce (in Theorem 5.18) that the number of intermediate $C^\ast$-subalgebras of an irreducible pair $B \subset A$ of simple unital $C^\ast$-algebras with finite Watatani index is bounded by $\min\{9[A : B]_0^2, ([A : B]_0^2)^{A : B]_0^2}\}$, where $[A : B]_0$ denotes the Watatani index of the pair $B \subset A$. As was observed in [2], the essence of this proof lies in the above-mentioned rigidity phenomenon observed by the minimal intermediate $C^\ast$-subalgebras, which then allows one to deduce that the number of such intermediate subalgebras is bounded by the Kissing number $\tau_n$ of the $n$-dimensional sphere, where $n = \dim\mathbb{C}(B' \cap A_1)$. The same tools allow us to deduce (in Theorem 5.20) that the improved bound obtained in [2] holds even for the lattice of intermediate subfactors of an irreducible $\sigma$-finite subfactor of type $III$ of finite index, and thereby answers the question of Longo [35] for the type $III$ case as well.

Finally, in the last section, using Christensen’s perturbation technique from [8] and Watatani’s compactness argument from [49], we generalize the above-mentioned finiteness results of Watatani and Teruya by showing (in Theorem 6.4) that the lattice of intermediate von Neumann subalgebras of an unital inclusion $N \subset M$ is finite if $M$ is a finite von Neumann algebra with a normal tracial state $tr$ on $M$ such that the unique $tr$-preserving conditional $E^M_N : M \to N$ has finite Watatani index, $\mathcal{Z}(N)$ is finite dimensional and $N' \cap M$ equals either $\mathcal{Z}(N)$ or $\mathcal{Z}(M)$. We conclude the paper with some nice corollaries.

2. Inclusions of simple unital $C^\ast$-algebras

Generalizing the notions of indices and basic constructions by Jones [20] and Kosaki [28], Watatani, in [48], proposed the notion of a vector-valued index for conditional expectations of inclusions of $C^\ast$-algebras and the notion of basic construction of such inclusions. In this section, we briefly recall the two notions and present some consequences which will be used later and are of independent interest as well.

2.1. Watatani index of conditional expectations

Given a pair $B \subset A$ of unital $C^\ast$-algebras (with a common identity), a conditional expectation $E : A \to B$ is a positive projection with norm one such that $E(abx) = aE(x)b$ for all $a, b \in B$ and $x \in A$. A conditional expectation $E : A \to B$ is said to have finite index if there exists a finite set $\{\lambda_1, \ldots, \lambda_n\} \subset A$ such that $x = \sum_{i=1}^n E(x\lambda_i)\lambda_i^\ast = \sum_{i=1}^n \lambda_i E(\lambda_i^\ast x)$ for every $x \in A$. Such a set $\{\lambda_1, \ldots, \lambda_n\}$ is called a quasi-basis for $E$. This is a generalization of the notion of the Pimsner–Popa basis [39] for a pair of von Neumann algebras with a conditional expectation. The Watatani index of $E$ is given by

$$\text{Ind}(E) = \sum_{i=1}^n \lambda_i \lambda_i^\ast,$$

and is independent of the quasi-basis. Let $\mathcal{E}_0(A, B)$ denote the set of all conditional expectations from $A$ onto $B$ of finite index.
In general, Ind($E$) is not a scalar but it is an invertible positive element of $\mathcal{Z}(A)$. Motivated by the values of Jones index for subfactors [20], Watatani showed the following theorem.

**Theorem 2.1 [48].** Let $B \subset A$ be an inclusion of unital $C^*$-algebras with a finite index conditional expectation $E : A \to B$. If Ind($E$) is a scalar, then it takes values in the set

$$\left\{4\cos^2\left(\frac{\pi}{n}\right), n = 3, 4, 5, \ldots \right\} \cup \{4, \infty\}.$$

In particular, if $B \subset A$ is an inclusion of simple unital $C^*$-algebras with Ind($E$) < 4, then there are no intermediate $C^*$-subalgebras of $B \subset A$.

**Remark 2.2.** (1) The assumption that the inclusion has common identity is redundant, because, if $E : A \to B$ is a conditional expectation of finite index, and $B$ is also a unital $C^*$-algebra with unit $1_B$, then

$$1_A = \sum E(1_A \lambda_i) \lambda_i^* = \sum E(\lambda_i) \lambda_i^* = \sum 1_B E(\lambda_i) \lambda_i^* = \sum E(1_B \lambda_i) \lambda_i^* = 1_B.$$

(2) A conditional expectation of finite index is automatically faithful. (It follows from Proposition 2.3.)

We now recall a useful result which says that a conditional expectation with finite Watatani index also has finite probabilistic index (of Pimsner and Popa — see [39]).

**Proposition 2.3 [48].** Let $B \subset A$ be an inclusion of $C^*$-algebras with a conditional expectation $E$ of finite index. Then, there exists a $c > 0$ such that

$$E(x) \geq cx \text{ for all } x \in A_+.$$  \hspace{1cm} (2.1)

Izumi showed that the converse also holds for inclusions of simple unital $C^*$-inclusions.

**Theorem 2.4 [18].** Let $B \subset A$ be an inclusion of simple unital $C^*$-algebras with a conditional expectation $E : A \to B$. If $E$ satisfies the Pimsner–Popa inequality (2.1), then $E$ has finite Watatani index.

For more on Watatani index, we suggest the reader to see [48].

2.1.1. **Minimal conditional expectations.** Recall that if $B \subset A$ is an inclusion of unital $C^*$-algebras such that $\mathcal{Z}(A) = \mathbb{C}$, then every finite index conditional expectation has scalar index and a conditional expectation $E_0 \in \mathcal{E}_0(A,B)$ is said to be minimal if it satisfies Ind($E_0$) \leq Ind($E$) for all $E \in \mathcal{E}_0(A,B)$. See [13, 27, 32, 33, 48] for details.

Under some hypothesis, there exists only one conditional expectation.

**Theorem 2.5 [48, Corollary 1.4.3].** Let $B \subset A$ be an inclusion of unital $C^*$-algebras and $E \in \mathcal{E}_0(A,B)$. If $B' \cap A \subset B$, then $E$ is the unique conditional expectation from $A$ onto $B$.

In particular, $E$ is a minimal conditional expectation from $A$ onto $B$.

Interestingly, when the $C^*$-algebras are simple, then we have a unique minimal conditional expectation.

**Theorem 2.6 [48, Theorem 2.12.3].** Let $B \subset A$ be an inclusion of simple unital $C^*$-algebras such that $\mathcal{E}_0(A,B) \neq \emptyset$. Then, there exists a unique minimal conditional expectation from $A$ onto $B$ (which will be denoted by $E_A^B$).
We now briefly recall the theory of expectation $E$ by inclusion map. Recall that the space $\mathcal{E}_0(A, C)$ is a subalgebra of $B \subset A$, $F \in \mathcal{E}_0(A, C)$ and $E \in \mathcal{E}_0(C, B)$. Then, $E \circ F$ is minimal if and only if both $E$ and $F$ are minimal.

Moreover, the Watatani index is multiplicative, that is, $[A : B]_0 = [C : B]_0[A : C]_0$.

**Theorem 2.8** [27]. Let $B \subset A$ be as in Theorem 2.6, $C$ be an intermediate simple $C^*$-subalgebra of $B \subset A$, $F \in \mathcal{E}_0(A, C)$ and $E \in \mathcal{E}_0(C, B)$. Then, $E \circ F$ is minimal if and only if both $E$ and $F$ are minimal.

Moreover, the Watatani index is multiplicative, that is, $[A : B]_0 = [C : B]_0[A : C]_0$.

**Lemma 2.9** [15, 18, 48]. Let $B \subset A$ and $E_B^A$ be as in Theorem 2.6 and $C$ be an intermediate simple $C^*$-subalgebra of $B \subset A$. Then, there exist unique minimal conditional expectations $E_B^C : C \to B$ and $E_A^C : A \to C$, and they satisfy the relation $E_B^C \circ E_A^C = E_B^A$. Moreover, if $B \subset A$ is irreducible, that is, $B' \cap A = C$, then $E_B^A = E_B^A|_{C'}$.

**Example 2.10** [48]. Consider a unital simple $C^*$-algebra $B$ with a finite group $G$ acting outerly on $B$ and consider the $C^*$-crossed product $B \rtimes G$. Then, the canonical conditional expectation $E : B \rtimes G \to B$ given by $E(\sum_{g \in G} x_g u_g) = x_e$ is minimal, the proof of which can be read off [48] and we omit the necessary details.

### 2.2. Watatani’s $C^*$-basic construction

We now briefly recall the theory of $C^*$-basic construction introduced by Watatani in [48]. Let $A$ be a $C^*$-algebra and $H$ be a pre-Hilbert $A$-module. Recall that the map

$$\mathcal{H} \ni x \mapsto \|x\|_H := \|\langle x, x \rangle_A\|^{1/2} \in [0, \infty)$$

is a norm on $\mathcal{H}$; and that, $\mathcal{H}$ is called a Hilbert $A$-module if it is complete with respect to this norm. For details about the theory of Hilbert $C^*$-modules, we refer the reader to [30].

Now, suppose that $B \subset A$ is a unital inclusion of $C^*$-algebras with a faithful conditional expectation $E_B$ from $A$ onto $B$. Then, $A$ becomes a pre-Hilbert $B$-module with respect to the $B$-valued inner product given by

$$\langle x, y \rangle_B = E_B(x^* y) \text{ for all } x, y \in A. \quad (2.2)$$

Here, we follow the physicists’ convention of keeping conjugate linearity in the first coordinate. Let $\mathfrak{A}$ denote the Hilbert $B$-module completion of $A$ and $\iota : A \to \mathfrak{A}$ denote the isometric inclusion map. Recall that the space $\mathcal{L}_B(\mathfrak{A})$ consisting of adjointable $B$-linear maps on $\mathfrak{A}$ is a $C^*$-algebra.

For each $a \in A$, consider $\lambda(a) \in \mathcal{L}_B(\mathfrak{A})$ given by $\lambda(a)(\iota(x)) = \iota(ax)$ for $x \in A$. The map $\iota(A) \ni \iota(x) \mapsto \iota(E_B(x)) \in \iota(A)$ extends to an adjointable projection on $\mathfrak{A}$, and is denoted by $e_B \in \mathcal{L}_B(\mathfrak{A})$. The projection $e_B$ is called the Jones projection for the pair $B \subset A$; thus, $e_B(\iota(x)) = \iota(E_B(x))$ for all $x \in A$. The $C^*$-basic construction $C^*(A, e_B)$ is defined to be the
C*-subalgebra generated by \( \{ \lambda(A), e_B \} \) in \( \mathcal{L}_B(\mathfrak{A}) \). It turns out that \( C^*(A, e_B) \) equals the closure of the linear span of \( \{ \lambda(x)e_B \lambda(y) : x, y \in A \} \) in the \( C^* \)-algebra \( \mathcal{L}_B(\mathfrak{A}) \); \( \lambda \) is an injective \(*\)-homomorphism and thus we can consider \( A \) as a \( C^* \)-subalgebra of \( C^*(A, e_B) \). The following inequality, known as the Kadison–Schwarz inequality, holds:

\[
E_B(x)^* E_B(x) \leq E_B(x^* x) \quad \text{for all } x \in A.
\] (2.3)

Interestingly, when the conditional expectation has finite index, then \( A \) turns out to be complete with respect to the above norm as we show below.

**Lemma 2.11** [48]. Let \( B \subset A \) be a unital inclusion of \( C^* \)-algebras and \( E_B \in \mathcal{E}_0(A, B) \). Then, \( A \) is a Hilbert \( B \)-module with respect to the \( B \)-valued inner product given as in equation 2.2.

**Proof.** Since a conditional expectation with finite index is faithful (Remark 2.2), \( A \) is a pre-Hilbert \( B \)-module. By Proposition 2.3, we have \( E_B(x^* x) \geq L x^* x \) for every \( x \in A \) for some positive constant \( L \). Therefore, \( \| x \|_A \geq L \| x \| \) for all \( x \in A \). In particular, if \( \{ x_n \} \) is a Cauchy sequence in \( A \) with respect to \( \| . \|_A \), then so is it with respect to \( \| . \| \), and therefore, converges to some element \( x \in A \). On the other hand, \( \| y \|_A^2 = \| E_B(yy^*) \| \leq \| y \|_A^2 \) for all \( y \in A \). So, \( \{ x_n \} \) converges to \( x \) with respect to \( \| . \|_A \) as well. Thus, \( A \) is complete with respect to \( \| . \|_A \). \( \square \)

A simple algebraic calculation yields the following useful and standard equality, and is left to the reader.

**Proposition 2.12.** Let \( A, B \) and \( E_B \) be as in Lemma 2.11 and \( \{ \lambda_i : 1 \leq i \leq n \} \) be a quasi-basis for \( E_B \). Then,

\[
\sum_{i=1}^{n} \lambda_i e_B \lambda_i^* = 1.
\]

**Theorem 2.13** [26, 48]. Let \( A, B \) and \( E_B \) be as in Lemma 2.11 and let \( A_1 \) denote the \( C^* \)-basic construction of \( B \subset A \) with respect to \( E_B \). Then, we have the following.

1. There exists a unique finite index conditional expectation \( \tilde{E}_B : A_1 \rightarrow A \) satisfying

\[
\tilde{E}_B(\lambda(x)e_B \lambda(y)) = \lambda(x)\lambda(\text{Ind}(E_B)^{-1}) \lambda(y) = \lambda(\text{Ind}(E_B)^{-1} xy)
\]

for all \( x, y \in A \). (\( \tilde{E}_B \) is called the dual conditional expectation of \( E_B \).) [48, Proposition 1.6.1]

2. If \( A \) and \( B \) are both simple, then \( A_1 \) is also simple and if \( E_0 : A \rightarrow B \) denotes the unique minimal conditional expectation, then the dual conditional expectation \( \tilde{E}_0 : A_1 \rightarrow A \) is minimal as well and \( \text{Ind}(E_0) = \text{Ind}(E_0) \). [48, 2.2.14 and 2.3.4], [26]

**Remark 2.14.** For \( A, B \) and \( A_1 \) as in Theorem 2.13(2), we have \([A_1 : A]_0 = [A : B]_0 \).

The following lemma is an extremely useful observation and is a direct adaptation of the so called ‘Push-down Lemma’ from [39]. As is standard in subfactor theory, we write \( e_1 \) for the projection \( e_B \) in \( A_1 \), which implements the \( C^* \)-basic construction for the inclusion \( B \subset A \).

**Lemma 2.15.** Let \( A, B, E_B, A_1 \) and \( \tilde{E}_B \) be as in Theorem 2.13. If \( x_1 \in A_1 \), then there exists a unique \( x_0 \in A \) such that \( x_1 e_1 = x_0 e_1 \); this element is given by \([A : B]_0 \tilde{E}_B(x_1 e_1) \).

**Proof.** Uniqueness is trivial. Suppose that \( \{ \lambda_i : 1 \leq i \leq n \} \) is a quasi-basis for \( E_B \). Then, by [48, Proposition 1.6.6], \([A : B]_0^{1/2} \lambda_i e_1 : 1 \leq i \leq n \) is a quasi-basis for \( \tilde{E}_B \). Therefore,
$A_1 = A e_1 A := \text{span}\{a e_1 b : a, b \in A\}$; and, it is easy to see that $[A : B]_0 \widetilde{E}_B(a e_1 b e_1) e_1 = a e_1 b e_1$ for all $a, b \in A$. This completes the proof. \hfill \Box

### 2.3. Iterated $C^*$-basic constructions and the relative commutants

Throughout this subsection, $B \subset A$ will denote a fixed pair of simple unital $C^*$-algebras such that $\mathcal{E}_0(A, B) \neq \emptyset$; and $\tau := [A : B]_0^{-1}$.

From Theorem 2.13, we know that $A \subset A_1$ is also a pair of simple unital $C^*$-algebras (with common identity) and that $[A_1 : A]_0 = [A : B]_0$. Thus, like Jones’ tower of basic constructions of a finite index subfactor of type $II_1$, we can repeat the process of $C^*$-basic construction to obtain a tower of simple unital $C^*$-algebras

$$B \subset A \subset A_1 \subset A_2 \subset \cdots \subset A_k \subset \cdots$$ \hspace{1cm} (2.4)

with unique (dual) minimal conditional expectations $E_k : A_k \to A_{k-1}$, $k \geq 0$, with the convention that $A_{-1} := B$ and $A_0 := A$. We shall call this tower the tower of $C^*$-basic constructions of the inclusion $B \subset A$. For each $k \geq 1$, let $e_k$ denote the Jones projection in $A_k$ which implements the $C^*$-basic construction of the inclusion $A_{k-2} \subset A_{k-1}$ with respect to the (minimal) conditional expectation $E_{k-1} : A_{k-1} \to A_{k-2}$. For each $k \geq 0$, the relative commutants of $B$ in $A_k$ are given by

$$B' \cap A_k = \{ x \in A_k : xb = bx \text{ for all } b \in B \}. \hspace{1cm} (2.5)$$

**Proposition 2.16** [48]. $B' \cap A_k$ is finite dimensional for all $k \geq 0$.

*Proof*. Since $A_k$ is simple and the conditional expectation $E_0 \circ E_1 \circ \cdots \circ E_{k-1} \circ E_k : A_k \to B$ has finite index (by [48, Proposition 1.7.1]), it follows from [48, Proposition 2.7.3] that $B' \cap A_k$ is finite dimensional. \hfill \Box

We shall provide a bound for the dimension of $B' \cap A_k$ in terms of index of $B \subset A$ in the next subsection.

**$k$-step $C^*$-basic construction.** The multi-step basic construction holds exactly like in [39]. See [1] for an easier proof. Out here, we use the characterization of $C^*$-basic construction given by Watatani in [48, Proposition 2.2.11].

**Proposition 2.17** [39]. For each $n \geq 1$, the tower $B \subset A_n \subset A_{2n+1}$ is an instance of $C^*$-basic construction with the corresponding Jones projection given by

$$e_{[-1,2n+1]} := \tau^{-\frac{n(n+1)}{2}}(e_{n+1} e_n \cdots e_1)(e_{n+2} e_{n+1} \cdots e_2) \cdots (e_{2n+1} e_{2n} \cdots e_{n+1}).$$

**Proposition 2.18** [22, 48]. Let $\{\lambda_i : 1 \leq i \leq m\}$ be a quasi-basis for $E_0$. Then, for each $n \geq 1$, the collection

$$\left\{ \tau^{-\frac{n(n+1)}{2}} \lambda_{i_n}(e_1 e_2 \cdots e_{n-1} e_n) \lambda_{i_{n-1}}(e_1 e_2 \cdots e_{n-2} e_{n-1}) \cdots \lambda_{i_2}(e_1 e_2) \lambda_{i_1} : 1 \leq i_1, \ldots, i_n \leq m \right\}$$

is a quasi-basis for the minimal conditional expectation $E_0 \circ E_1 \circ \cdots \circ E_{n-1} \circ E_n : A_n \to B$.

### 2.4. Tracial states on the relative commutants

Like in the preceding subsection, $B \subset A$ will again denote a fixed pair of simple unital $C^*$-algebras such that $\mathcal{E}_0(A, B) \neq \emptyset$; and $\tau := [A : B]_0^{-1}$.

Being finite dimensional, the higher relative commutants $B' \cap A_k$ admit numerous tracial states. However, using the minimal conditional expectations, we can single out a consistent
Markov-type trace, which then allows one to talk about the ‘standard invariant’ and the ‘principal graph’ of such an inclusion, as is done for any finite index subfactor. Izumi has also mentioned about this aspect in [18]. But we are not aware of any literature in this direction. However, we will not delve into these topics in this paper.

First, we recall two auxiliary results from [48] that will be fundamental in obtaining the tracial states of our choice.

**Proposition 2.19** [48]. Let \( E \in \mathcal{E}_0(A, B) \) and \( \{ \lambda_i : 1 \leq i \leq n \} \) be a quasi-basis for \( E \). Consider the map \( H_E : B' \cap A \to A \) given by

\[
H_E(x) = \sum_i \lambda_i x \lambda_i^*, \quad x \in B' \cap A.
\]

Then, \( H_E \) maps \( B' \cap A \) onto \( Z(A) \) and does not depend on the choice of the quasi-basis.

Moreover, the map \( G_E : B' \cap A \to Z(A) \) given by

\[
G_E(x) = \frac{1}{\text{Ind}(E)} \sum_i \lambda_i x \lambda_i^*, \quad x \in B' \cap A
\]

is a conditional expectation.

**Theorem 2.20** [48]. Let \( E \in \mathcal{E}_0(A, B) \). Then, the following are equivalent.

1. \( E \) is minimal.
2. \( E|_{B' \cap A} \) (respectively, \( H_E \)) is a tracial state (respectively, tracial map) on \( B' \cap A \) and \( H_E = \text{Ind}(E) \) \( E|_{B' \cap A} \).
3. \( H_E = c E|_{B' \cap A} \) for some constant \( c \).

**Proposition 2.21.** For each \( k \geq 0 \), \( B' \cap A_k \) admits a faithful tracial state \( \text{tr}_k \) such that

\[
\text{tr}_k(xe_k) = \tau \text{tr}_k(x) \quad \text{for all} \quad x \in B' \cap A_{k-1},
\]

(2.6)

and \( \text{tr}_k|_{B' \cap A_{k-1}} = \text{tr}_{k-1} \) for all \( k \geq 1 \). (We will drop \( k \) and denote \( \text{tr}_k \) simply by \( \text{tr} \).)

**Proof.** Define \( \text{tr}_k : B' \cap A \to \mathbb{C} \) as \( \text{tr}_k = (E_0 \circ E_1 \circ \cdots \circ E_k)|_{B' \cap A_k} \). Then, by Theorem 2.8, Theorem 2.20 and Remark 2.2(2), \( \text{tr}_k \) is a faithful tracial state and, by definition, \( \text{tr}_k|_{B' \cap A_{k-1}} = \text{tr}_{k-1} \) for all \( k \geq 1 \).

We prove the Markov-type property only for \( k = 1 \). Other cases follow similarly. We have

\[
\text{tr}(xe_1) = E_0 \circ E_1 (xe_1) = E_0(xE_1(e_1)) = \tau E_0(x) = \tau \text{tr}(x)
\]

for all \( x \in B' \cap A \).

**Remark 2.22.** Denote the minimal conditional expectation \( E_0 \circ E_1 \circ \cdots \circ E_k \) simply by \( F_k \). Then, in view of Theorem 2.20, \( H_{F_k} \) is \( \tau^{-k} \text{tr}_k \) on \( B' \cap A_k \).

**Lemma 2.23.** Let \( \{ \lambda_i : 1 \leq i \leq n \} \subset A \) be a quasi-basis for the minimal conditional expectation \( E_0 \). Then, the tr-preserving conditional expectation from \( B' \cap A_k \) onto \( A' \cap A_k \) is given by

\[
E_{A' \cap A_k}^{B' \cap A_k}(x) = \frac{1}{[A : B]_0} \sum_i \lambda_i x \lambda_i^*, \quad x \in B' \cap A_k.
\]
Proof. Consider \( G_{E_0} : B' \cap A_k \to A_k \) given by
\[
G_{E_0}(x) = \tau \sum_i \lambda_i x \lambda_i^* , x \in B' \cap A_k.
\]
We assert that \( G_{E_0}(x) \in A' \cap A_k \). Indeed, for any \( a \in A \) and \( x \in B' \cap A_k \), as in [48, Proposition 1.2.9], we observe that
\[
G_{E_0}(x)a = \tau \sum_i \lambda_i E_0(x \lambda_i^* a) \lambda_i^* = \tau \sum_{i,j} \lambda_i E_0(\lambda_i^* a \lambda_j) x \lambda_j^*
\]
\[
= a \sum_j \lambda_j x \lambda_j^* = a G_{E_0}(x).
\]
Now, we show that \( G_{E_0} \) is the tr-preserving conditional expectation \( E_{B' \cap A_k} \). Let \( x \in B' \cap A_k \). Then, by the definition of tr, for any \( y \in A' \cap A_k \), we have
\[
\text{tr} \left( \sum_i \lambda_i x \lambda_i^* y \right) = E_k \left( \sum_i \lambda_i x \lambda_i^* y \right) = E_k \left( \sum_i \lambda_i x y \lambda_i^* \right).
\]
Clearly, \( E_1 \circ \cdots \circ E_k(x y) \in B' \cap A \). Hence, in view of Theorem 2.20, we observe that \( H_{E_0}(E_1 \circ \cdots \circ E_k(x y)) \in A' \cap A = \mathbb{C} \). Thus,
\[
F_k \left( \sum_i \lambda_i x y \lambda_i^* \right) = E_0 \left( \sum_i \lambda_i (E_1 \circ \cdots \circ E_k(x y)) \lambda_i^* \right) = H_{E_0}(E_1 \circ \cdots \circ E_k(x y)).
\]
Therefore, \( \text{tr} (\sum_i \lambda_i x \lambda_i^* y) = \tau^{-1} F_k(x y) = \tau^{-1} \text{tr}(x y) \), again by Theorem 2.20. This proves that \( G_{E_0} = E_{B' \cap A_k}^{A' \cap A_k} \). \( \Box \)

In view of Proposition 2.12, we have the following corollary.

**Corollary 2.24.** \( E_{B' \cap A_k}^{A' \cap A_k}(e_1) = \tau \).

We now provide a bound for the dimension of each relative commutant \( B' \cap A_k \), whose proof is motivated by that of [11, Lemma 3.6.2(b)] (see [32] for the type \( \text{III} \) case). We will need the following observation related to the local behavior of conditional expectation and index.

**Proposition 2.25.** For each non-zero projection \( p \) in \( B' \cap A \), consider the \( C^* \)-inclusion \( pBp \subset pAp \) and the map \( E_p : pAp \to pBp \) given by
\[
E_p(x) = \frac{E_0(x)p}{\text{tr}(p)} , x \in pAp.
\]
Then, the following hold.

1. The pair \( pBp \subset pAp \) is an inclusion of simple unital \( C^* \)-algebras with common identity \( p \).
(2) $E_p$ is a conditional expectation of finite index and, for any quasi-basis \{\lambda_i : 1 \leq i \leq n\} of $E_0$, \{\sqrt{\text{tr}(p)} \ p\lambda_i \ : \ 1 \leq i \leq n\} is a quasi-basis for $E_p$.

(3) $E_p$ is the unique minimal conditional expectation from $pAp$ onto $pBp$.

(4) $[pAp : pBp]_0 = \text{tr}(p)^2 [A : B]_0 p$.

(5) Suppose that \{\rho_i\}_{i=1}^n is a partition of identity consisting of projections in $B' \cap A$. Then, $[A : B]_0^{1/2} = \sum_{i=1}^n \|\rho_i Ap : Bp_i]\|^{1/2}$.

**Proof.** (1) It is well known that $pAp$ is a hereditary $C^*$-subalgebra of $A$. Since any hereditary $C^*$-subalgebra of a simple $C^*$-algebra is simple, $pAp$ is simple. On the other hand, since $p \in B' \cap A$, $B \sim Bp$; so, $pBp$ is also simple. Hence, $pBp \subset pAp$ is an inclusion of simple $C^*$-algebras with common identity $p$.

(2) Clearly, $E_p$ is a conditional expectation. And, for any $x \in pAp$, we observe that 
\[
\begin{align*}
\sum_i E_p(xp\lambda_i p) p\lambda_i^* p &= \frac{1}{\text{tr}(p)} \sum_i E_0(xp\lambda_i p) p\lambda_i^* p \\
&= \frac{1}{\text{tr}(p)} \sum_i E_0(x\lambda_i p) p\lambda_i^* p \\
&= \frac{1}{\text{tr}(p)} \sum_i E_0(px\lambda_i) p\lambda_i^* p \\
&= \frac{1}{\text{tr}(p)} x.
\end{align*}
\]
Hence, $E_p$ has finite index with a quasi-basis \{\sqrt{\text{tr}(p)} \ p\lambda_i : 1 \leq i \leq n\}.

(3) For each $x \in (pBp)' \cap pAp$, we have 
\[
\begin{align*}
H_{E_p}(x) &= \sum_i \text{tr}(p) \ p\lambda_i pxp\lambda_i^* p \\
&= \text{tr}(p) \ p \left( \sum_i \lambda_i pxp\lambda_i^* \right) p \\
&= \text{tr}(p) \ pH_E(x)p \\
&= c E_p(x). \quad \text{(for some constant $c$)}
\end{align*}
\]
Therefore, from Theorem 2.20, we conclude that $E_p$ is a minimal conditional expectation. Uniqueness follows from Theorem 2.6.

(4) We have 
\[
[pAp : pBp]_0 = \text{Ind}(E_p) \\
= \text{tr}(p) \sum_i p\lambda_i p\lambda_i^* p \\
= \text{tr}(p) \ p \left( \sum_i \lambda_i p\lambda_i^* \right) p \\
= \text{tr}(p) \ pH_{E_0}(p)p \\
= \text{tr}(p)^2 [A : B]_0 p,
\]
where the last equality follows from the fact that $H_{E_0}(p) = \text{Ind}(E_0) E_0(p) = \text{Ind}(E_0) \text{tr}(p)$, by Theorem 2.20.

(5) follows readily from Item (4).

\[ \square \]

**Proposition 2.26.** We have

\[ \dim_C(B' \cap A_k) \leq [A : B]_0^{k+1} \]  

(2.7)

for all $k \geq 0$.

**Proof.** Since Watatani index is multiplicative (Remark 2.14), it suffices to prove (2.7) for $k = 0$.

Let $\{p_i : 1 \leq i \leq m\}$ be a maximal family of mutually orthogonal minimal projections in $B' \cap A$ such that $\sum_{i=1}^m p_i = 1$.

Note that, for each projection $p$ in $B' \cap A$, by Proposition 2.25 and [48, Lemma 2.3.1], we have

\[ [A : B]_0 p = \frac{pAp : pBp}_0 \geq \frac{p}{\text{tr}(p)}; \]

so that $\text{tr}(p)[A : B]_0 \geq \frac{1}{\text{tr}(p)}$. Thus, $[A : B]_0 \geq \sum_{i=1}^m \frac{1}{\text{tr}(p_i)}$. Since $\sum_{i=1}^m \text{tr}(p_i) = 1$, it follows that $\sum_{i=1}^m \frac{1}{\text{tr}(p_i)} \geq m^2$. Hence,

\[ \dim_C(B' \cap A) \leq m^2 \leq \sum_{i=1}^m \frac{1}{\text{tr}(p_i)} \leq [A : B]_0. \]

This completes the proof. \[ \square \]

**Corollary 2.27.** If $[A : B]_0 < 4$, then $B \subset A$ is irreducible.

**Proof.** From the proof of Proposition 2.26, it follows that if $\{p_i : 1 \leq i \leq m\}$ is a maximal family of mutually orthogonal minimal projections in $B' \cap A$ such that $\sum_{i=1}^m p_i = 1$, then $m^2 \leq [A : B]_0$. Thus, if $[A : B]_0 < 4$, then we must have $m = 1$. This completes the proof. \[ \square \]

3. **Fourier theory for inclusions of simple unital $C^*$-algebras**

In the theory of subfactors, some of the crucial tools include certain naturally occurring operations on the higher relative commutants, namely, the so-called Fourier transforms, shift operators and rotation maps. These were introduced by Ocneanu (see [37]) and played a major role in the development of the subject. Details may be found in [5–7]. A significant application of the Fourier theory has been that the rotation maps on the higher relative commutants were highly instrumental in the formalism of the structure of Jones’ planar algebra on the standard invariant of any extremal subfactor (see [20]). According to Jones (see [21]), the rotation operator is ‘the most interesting algebraic ingredient of a subfactor seen from the planar point of view.’ The formulation of Fourier transforms and rotation maps for a subfactor $N \subset M$ depends heavily on the unique tracial state on the $II_1$ factor $M$ and the modular conjugation operator $J$ on $L^2(M)$. Needless to mention, both of these tools are absent for general inclusions of simple unital $C^*$-algebras. Still, given any pair $B \subset A$ of simple unital $C^*$-algebras, based on the consistent tracial states on the relative commutants that we obtained in the preceding section, we will show that an analogous Fourier theory can be developed.
As an application, we shall provide bounds for the cardinality of the lattice of intermediate subalgebras of such pairs of $C^*$-algebras as well as of subfactors of type $III$. We believe that, very much like $II_1$-factors and their subfactors, the $C^*$-Fourier theory will also have a significant say in the understanding of simple unital $C^*$-algebras and their $C^*$-subalgebras.

As in the preceding section, throughout this section too, $B \subset A$ will denote a fixed pair of simple unital $C^*$-algebras such that $E_0(A, B) \neq \emptyset$; and $\tau := [A : B]_0^{-1}$.

### 3.1. Fourier transforms

**Definition 3.1.** For each $k \geq 0$, the Fourier transform $F_k : B' \cap A_k \to A' \cap A_{k+1}$ is defined as

$$F_k(x) = \tau^{-\frac{k+2}{2}} E_{A' \cap A_{k+1}}^{B' \cap A_k}(xe_{k+1}e_k \ldots e_2e_1), \quad x \in B' \cap A_k.$$ 

And, the inverse Fourier transform $F_k^{-1} : A' \cap A_{k+1} \to B' \cap A_k$ is defined as

$$F_k^{-1}(x) = \tau^{-\frac{k+2}{2}} E_{k+1}(xe_1e_2 \ldots e_ke_{k+1}), \quad x \in A' \cap A_{k+1}.$$ 

The usage of the word ‘inverse’ in the preceding definition is justified by the following proposition.

**Proposition 3.2.** We have

$$F_k \circ F_k^{-1} = \text{Id}_{A' \cap A_{k+1}} \quad \text{and} \quad F_k^{-1} \circ F_k = \text{Id}_{B' \cap A_k}$$

for all $k \geq 0$. In particular, if $B \subset A$ is irreducible, then so is $A \subset A_1$.

**Proof.** First, observe that, for any $a \in A$, we have

$$(e_{k+1}e_k \ldots e_2e_1)a(e_1e_2 \ldots e_ke_{k+1}) = \tau^k E_0(a)e_{k+1}. \quad (3.1)$$

Similarly, it is easy to see that

$$(e_1e_2 \ldots e_ke_{k+1})(e_{k+1}e_k \ldots e_2e_1) = \tau^k e_1. \quad (3.2)$$

Also, notice that if $\{\lambda_i : 1 \leq i \leq n\}$ is a quasi-basis for $E_0$, then using Lemma 2.23, we readily obtain

$$E_{A' \cap A_{k+1}}^{B' \cap A_k}(xe_{k+1}e_k \ldots e_2e_1) = \tau \sum_i \lambda_i xe_{k+1}e_k \ldots e_2e_1 \lambda_i^* \quad (3.3)$$

for all $x \in B' \cap A_{k+1}$. Then, for any $x \in B' \cap A_k$, we have

$$F_k^{-1}F_k(x) = \tau^{-(k+2)} E_{k+1} \left( E_{A' \cap A_{k+1}}^{B' \cap A_k}(xe_{k+1}e_k \ldots e_2e_1)e_1e_2 \ldots e_ke_{k+1} \right)$$

$$= \tau^{-(k+1)} E_{k+1} \left( \sum_i \lambda_i xe_{k+1}e_k \ldots e_2e_1 \lambda_i^* e_1e_2 \ldots e_ke_{k+1} \right) \quad \text{(by equation (3.3))}$$

$$= \tau^{-(k+1)} \tau^k \sum_i \lambda_i x E_{k+1}(E_0(\lambda_i^*)e_{k+1}) \quad \text{(by equation (3.1))}$$

$$= \tau^{-(k+1)} \tau^k \tau \sum_i \lambda_i E_0(\lambda_i^*)x \quad \text{(since \quad x \in B' and \quad E_{k+1}(e_{k+1}) = \tau)}$$

$$= x.$$
On the other hand, for any \( y \in A' \cap A_{k+1} \), we see that
\[
F_k F_k^{-1}(y) = \tau^{-(k+2)} E_{A' \cap A_{k+1}}^B (E_{k+1}(ye_1e_2 \cdots e_ke_{k+1})e_{k+1}e_k \cdots e_{2}e_1)
= \tau^{-(k+1)} E_{A' \cap A_{k+1}}^B (y(e_1e_2 \cdots e_ke_{k+1})(e_{k+1}e_k \cdots e_{2}e_1))
= \tau^{-1} E_{A' \cap A_{k+1}}^B (ye_1)
= y.
\]
This completes the proof.

We now proceed to show that the Fourier transform \( F_1 \) and its inverse are both isometries. First, a lemma that will be required.

**Lemma 3.3.** Let \( \{ \lambda_i : 1 \leq i \leq n \} \) be a quasi-basis for \( E_0 \). Then, for any two elements \( x, y \in B' \cap A_1 \), we have

1. \( \sum_i E_0(\lambda_i) E_1(y^* \lambda_i^* x) = E_1(y^* x) \); and
2. \( \sum_i \lambda_i e_1 E_1(y^* \lambda_i^* x) \) is independent of the quasi-basis for \( E_0 \) and belongs to \( B' \cap A_1 \).

**Proof.** (1) We have
\[
\sum_i E_0(\lambda_i) E_1(y^* \lambda_i^* x) = \sum_i E_1(E_0(\lambda_i)y^* \lambda_i^* x)
= \sum_i E_1(y^* E_0(\lambda_i) \lambda_i^* x)
= E_1\left(y^* \left( \sum_i E_0(\lambda_i) \lambda_i^* \right) x \right)
= E_1(y^* x).
\]

(2) First, we show that the operator \( t := \sum_i \lambda_i e_1 E_1(y^* \lambda_i^* x) \) is independent of the quasi-basis for \( E_0 \). Suppose that \( \{ \mu_j : 1 \leq j \leq m \} \) is some other quasi-basis for \( E_0 \). Then,
\[
\sum_i \lambda_i e_1 E_1(y^* \lambda_i^* x) = \sum_j \mu_j E_0(\mu_j^* \lambda_i) e_1 E_1(y^* \lambda_i^* x)
= \sum_j \mu_j E_0(\mu_j^* \lambda_i) E_1(y^* \lambda_i^* x)
= \sum_j \mu_j E_1(\mu_j^* \lambda_i) y^* \lambda_i^* x
= \sum_j \mu_j E_1(y^* E_0(\mu_j^* \lambda_i) \lambda_i^* x)
= \sum_j \mu_j E_1\left(y^* \left( \sum_i E_0(\mu_j^* \lambda_i) \lambda_i^* \right) x \right)
= \sum_j \mu_j e_1 E_1(y^* \mu_j^* x).
\]
Now, fix an \( u \in U(B) \). Then, clearly, \( \{u\lambda_i\} \) is also a quasi-basis for \( E_0 \). Therefore,
\[
t = \sum_i u\lambda_i e_1 E_1(y^*\lambda_i^* u^*x) = u \sum_i \lambda_i e_1 E_1(y^*\lambda_i^* x) u^* = u t u^*.
\]
Since \( u \) was fixed arbitrarily, \( t \in B' \cap A_1 \). This completes the proof. \( \square \)

**Notation 3.4.** For simplicity, we denote the Fourier transform \( F_1 \) by \( F \).

**Theorem 3.5.** \( F \) and \( F^{-1} \) are isometries with respect to the norm given by \( \|x\|_2 = \text{tr}(x^*x) \).

**Proof.** Since \( F \) is a linear isomorphism, it suffices to show that one of them is an isometry. Let \( \{\lambda_i : 1 \leq i \leq n\} \) be a quasi-basis for \( E_0 \) and \( x, y \in B' \cap A_1 \). Then,
\[
\langle F(x), F(y) \rangle = \text{tr}((F(x))^* F(y))
\]
\[
= \tau^{-3} \text{tr}(E_{A' \cap A_2}^{B' \cap A_2}(e_1 e_2 x^*) E_{A' \cap A_2}^{B' \cap A_2}(ye_1 e_2))
\]
\[
= \tau^{-3} \text{tr}(E_{A' \cap A_2}^{B' \cap A_2}(E_{A' \cap A_2}^{B' \cap A_2}(e_1 e_2 x^*) ye_1 e_2))
\]
\[
= \tau^{-3} \text{tr}(E_{A' \cap A_2}^{B' \cap A_2}(e_1 e_2 x^*) ye_1 e_2)
\]
\[
= \tau^{-2} \text{tr}(\sum_i \lambda_i e_1 e_2 x^* \lambda_i^* ye_1 e_2)
\] (by Lemma 2.23)
\[
= \tau^{-2} \text{tr}(e_1 \sum_i \lambda_i e_1 E_1(x^* \lambda_i^* y) e_2)
\]
\[
= \tau^{-1} \text{tr}(e_1 \sum_i \lambda_i e_1 E_1(x^* \lambda_i^* y))
\] (by Lemmas 3.3 and 2.21)
\[
= \tau^{-1} \text{tr}(e_1 \sum_i E_0(\lambda_i) E_1(x^* \lambda_i^* y))
\]
\[
= \text{tr}(E_1(x^* y))
\] (by Lemma 3.3)
\[
= \text{tr}(x^* y)
\]
\[
= \langle x, y \rangle.
\]
This completes the proof. \( \square \)

**Remark 3.6.** An astute reader must have noted that the analogs of Lemma 3.3 and Theorem 3.5 are in fact true for all \( k \geq 0 \). Some amount of book keeping will do the job. Since we need it only for \( k = 1 \), we leave the details to the reader.

### 3.2. Rotation maps and shift operators

Following [5], we show that the relative commutants can be endowed with certain *rotation* maps. Throughout this subsection, \( \{\lambda_i : 1 \leq i \leq n\} \) will denote a fixed quasi-basis for the minimal conditional expectation \( E_0 : A \to B \).
**Definition 3.7.** For each $k \geq 0$, the rotation map $\rho_k^{BCA} : B' \cap A_k \to B' \cap A_k$ is defined as
\[
\rho_k^{BCA}(x) = \left( F_k^{-1}(F_k(x^*)) \right)^*, \quad x \in B' \cap A_k.
\] (3.4)

**Remark 3.8.** By Lemma 2.23, it is easily seen that
\[
\rho_k^{BCA}(x) = \tau^{-k} \sum_i E_k(e_k e_{k-1} \cdots e_2 e_1 \lambda_i x) e_k e_{k-1} \cdots e_2 e_1 \lambda_i^* \tag{3.5}
\]
for all $x \in B' \cap A_k$.

For simplicity, we will focus mainly on $k = 0$ and $k = 1$ only. The higher cases are straightforward generalizations and left to the reader. First, we show that a certain square root exists for $\rho_1^{BCA}$.

**Proposition 3.9.** $(\rho_3^{BCA})^2 = \rho_1^{BCA}$.

**Proof.** We have $\rho_3^{BCA}(x) = \tau^{-3} \sum_i E_3(e_3 e_2 e_1 \lambda_i x) e_3 e_2 e_1 \lambda_i^*$ for all $x \in B' \cap A_3$. On the other hand, by Proposition 2.17, we know that $B \subset A_1 \subset A_3$ is an instance of basic construction. So, from Remark 3.8 and Proposition 2.18, we have
\[
\rho_1^{BCA_1}(x) = \tau^{-5} \sum_{i,j} E_2 \circ E_3(e_2 e_1 e_3 e_2 e_1 \lambda_i \lambda_j x) e_2 e_1 e_3 e_2 \lambda_j^* e_1 \lambda_i^*
\]
for all $x \in B' \cap A_3$. Thus, for any $x \in B' \cap A_3$, we obtain
\[
(\rho_3^{BCA})^2(x) = \tau^{-3} \sum_i E_3(e_3 e_2 e_1 \lambda_i \rho_3^{BCA}(x)) e_3 e_2 e_1 \lambda_i^*
\]
\[= \tau^{-6} \sum_{i,j} E_3(e_3 e_2 e_1 \lambda_i E_3(e_3 e_2 e_1 \lambda_j x) e_3 e_2 e_1 \lambda_j^*) e_3 e_2 e_1 \lambda_i^*
\]
\[= \tau^{-6} \sum_{i,j} E_2(e_2 e_1 \lambda_i E_3(e_3 e_2 e_1 \lambda_j x)) e_3 e_2 e_1 \lambda_i^*
\]
\[= \tau^{-6} \sum_{i,j} E_2(e_2 e_1 \lambda_i E_3(e_3 e_2 e_1 \lambda_j x)) E_3(e_3 e_2 \lambda_j^* e_3 e_2 e_1 \lambda_i^*)
\]
\[= \tau^{-5} \sum_{i,j} E_2 \circ E_3(e_2 e_1 \lambda_i e_3 e_2 e_1 \lambda_j x) e_2 e_1 \lambda_j^* e_3 e_2 e_1 \lambda_i^*
\]
\[= \rho_1^{BCA_1}(x).
\]
This completes the proof. \(\square\)

**Notation 3.10.** Following [37] (see also [5]), we denote the rotations $\rho_1^{BCA}$ and $\rho_1^{BCA_1}$, respectively, by $\gamma_0$ and $\gamma_1$.

Ocneanu called them mirrorings (for the $II_1$-factor case). These will prove to be very important in what follows.

**Remark 3.11.** Following Remark 3.8 and the above notation, we see that
\[
\gamma_0(x) = \tau^{-1} \sum_i E_1(e_1 \lambda_i x) e_1 \lambda_i^*,
\] (3.6)
for every \( x \in B' \cap A_1 \). Similarly, using Propositions 2.17 and 2.18, we have
\[
\gamma_1(y) = \tau^{-5} \sum_{i,j} E_2 \circ E_3(e_2 e_1 e_2 \lambda_i e_1 \lambda_j x e_2 e_1 e_2 \lambda_j^* e_1 \lambda_i^*), \quad (3.7)
\]
for every \( y \in B' \cap A_3 \).

We next show that \( \gamma_0 \) and \( \gamma_1 \) are both \( * \)-preserving anti-automorphisms. This requires some work. We break the proof into various steps.

**Lemma 3.12.** \( \gamma_k \), for \( k \in \{0, 1\} \), is a \( * \)-preserving map.

**Proof.** We prove only for \( k = 0 \). The proof for \( \gamma_1 \) will follow once we apply the same technique for the inclusion \( B \subset A_1 \) with the minimal conditional expectation \( E_0 \circ E_1 \), since \( \gamma_1 = \rho^{B \subset A_1}_I \).

Let \( x \in B' \cap A_1 \). Then, for \( a, b \in A \), we have
\[
\langle \gamma_0(x)(a), b \rangle_B = \tau^{-1} \langle \sum_i E_1(e_1 \lambda_i x) E_0(\lambda_i^* a), b \rangle_B
\]
\[
= \tau^{-1} \langle \sum_i E_1(e_1 \lambda_i x E_0(\lambda_i^* a)), b \rangle_B
\]
\[
= \tau^{-1} \langle E_1(\sum_i \lambda_i E_0(\lambda_i^* a)) x, b \rangle_B \quad \text{(since } x \in B')
\]
\[
= \tau^{-1} \langle E_1(e_1 a x), b \rangle_B
\]
\[
= \tau^{-1} E_0(E_1(x^* a^* e_1) b)
\]
and, on similar lines,
\[
\langle a, \gamma_0(x^*)(b) \rangle_B = \tau^{-1} \langle a, \sum_i E_1(e_1 \lambda_i x^*) e_1 \lambda_i^* b \rangle_B
\]
\[
= \tau^{-1} \langle a, E_1(e_1 b x^*) \rangle_B
\]
\[
= \tau^{-1} E_0(a^* E_1(e_1 b x^*)).
\]

Since \( x \in B' \cap A_1 \) and \( E_0 \circ E_1 : A_1 \to B \) is the minimal conditional expectation, thanks to [26, Lemma 3.11], we have \( E_0 \circ E_1(x^* a^* e_1 b) = E_0 \circ E_1(a^* e_1 b x^*) \). Hence, \( \gamma_0(x) \) is adjointable with \( (\gamma_0(x))^* = \gamma_0(x^*) \).

**Remark 3.13.** From Lemma 3.12, it is obvious that \( (\mathcal{F}_k(x))^* = \mathcal{F}_k(\gamma_{k-1}(x^*)) \) for \( k \in \{1, 2\} \).

**Lemma 3.14.** \( \gamma_k^2 = \text{Id} \) for \( k \in \{0, 1\} \).

**Proof.** Observe that for \( x \in B' \cap A_1 \), we have
\[
\gamma_0^2(x) = \tau^{-1} \sum_i E_1(e_1 \lambda_i \gamma_0(x)) e_1 \lambda_i^*
\]
\[
= \tau^{-2} \sum_{i,j} E_1(e_1 \lambda_i E_1(e_1 \lambda_j x) e_1 \lambda_j^*) e_1 \lambda_i^*
\]
\[
= \tau^{-2} \sum_{i,j} E_1(E_0(\lambda_i E_1(e_1 \lambda_j x)) e_1 \lambda_j^*) e_1 \lambda_i^*
\]
\[= \tau^{-2} \sum_{i,j} E_1(E_0(\lambda_i E_1(e_1 \lambda_j x))e_1)\lambda_j^*e_1 \lambda_i^*\]

\[= \tau^{-2} \sum_{i,j} E_0(\lambda_i E_1(e_1 \lambda_j x)) E_1(e_1)\lambda_j^*e_1 \lambda_i^*\]

\[= \tau^{-1} \sum_{i,j} E_0 \circ E_1(\lambda_i e_1 \lambda_j x)\lambda_j^*e_1 \lambda_i^*\]

\[= \tau^{-1} \sum_{i,j} E_0 \circ E_1(x \lambda_i e_1 \lambda_j^*)\lambda_j^*e_1 \lambda_i^* \text{ (by [26, Lemma 3.11])}\]

\[= \tau - \frac{1}{2} \sum_{i,j} E_0 \circ E_1(x \lambda_i e_1 \lambda_j^*)\lambda_j^*e_1 \lambda_i^* \text{ (by Lemma 2.15)}\]

\[= \gamma_0(x).\]

In the last equality, we have used the fact (Proposition 2.18) that the collection \(\{\tau^{-1/2} \lambda_i e_1 \lambda_j : 1 \leq i, j \leq n\}\) is a quasi-basis for the minimal conditional expectation \(E_0 \circ E_1\).

That \(\gamma_1^2 = \text{Id}\) follows once we repeat the same procedure as above using Propositions 2.17 and 2.18. This completes the proof. \(\square\)

**Lemma 3.15.** \(\gamma_k\), for \(k \in \{0, 1\}\), is an anti-homomorphism.

**Proof.** For \(x, y \in B' \cap A_1\), we have

\[
\gamma_0(x)\gamma_0(y) = \tau^{-2} \sum_{i,j} E_1(e_1 \lambda_i x) e_1 \lambda_j^* E_1(e_1) e_1 \lambda_j^* \]

\[= \tau^{-2} \sum_{i,j} E_1(e_1 \lambda_i x) E_0(\lambda_j^* E_1(e_1) e_1) e_1 \lambda_j^* \]

\[= \tau^{-2} \sum_{i,j} E_1(e_1 \lambda_i x E_0(\lambda_j^* E_1(e_1) e_1)) e_1 \lambda_j^* \]

\[= \tau^{-2} \sum_{i,j} E_1(e_1 \lambda_i x E_0(\lambda_j^* E_1(e_1) e_1)) x e_1 \lambda_j^* \]

\[= \tau^{-2} \sum_{i,j} E_1(e_1 E_1(e_1) \lambda_j^* x e_1 \lambda_j^* \]

\[= \tau^{-1} \sum_{i,j} E_1(e_1 \lambda_i y x) e_1 \lambda_j^* \text{ (by Lemma 2.15)}\]

The proof for \(\gamma_1\) is similar and is left to the reader. \(\square\)

We have thus proved the following:

**Theorem 3.16.** The rotation map \(\gamma_k : B' \cap A_{2k+1} \rightarrow B' \cap A_{2k+1}\), for \(k \in \{0, 1\}\), is a *-preserving anti-automorphism.

When \(B \subset A\) is irreducible, then it turns out that \(\gamma_0\) preserves trace as well.

**Lemma 3.17.** If \(B \subset A\) is irreducible, then \(\gamma_0\) is a tr-preserving map on \(B' \cap A_1\).
Proof. For \( x \in B' \cap A_1 \), we have
\[
\text{tr}(\gamma_0(x)) = \tau^{-1}E_0 \circ E_1 \left( \sum_i E_1(e_1 \lambda_ix)e_1 \lambda_i^* \right) 
= \sum_i E_0(E_1(e_1 \lambda_ix)\lambda_i^*) 
= E_0 \circ E_1 \left( e_1 \left( \sum_i \lambda_ix\lambda_i^* \right) \right).
\]

Thanks to Proposition 2.19 we see that \( \sum_i \lambda_ix\lambda_i^* \in A' \cap A_1 \). Since \( B' \cap A = \mathbb{C} \), using Proposition 3.2 and Lemma 2.23, we see that \( \sum_i \lambda_ix\lambda_i^* = \tau^{-1}\text{tr}(x) \). Thus, \( \text{tr}(\gamma_0(x)) = \text{tr}(x) \). This completes the proof.

We can now talk about the shift operator \( \gamma_1 \gamma_0 \) from \( B' \cap A_1 \) onto \( A' \cap A_3 \).

**Theorem 3.18.** \( \gamma_1 \gamma_0 \) is a tr-preserving \(*\)-isomorphism from \( B' \cap A_1 \) onto \( A' \cap A_3 \) and its inverse is the map \( \gamma_0 \gamma_1 \).

Proof. We first prove that \( \gamma_1 \) maps \( B' \cap A_1 \) into \( A' \cap A_3 \). Suppose that \( \{\lambda_i : 1 \leq i \leq n\} \) is a quasi-basis for \( E_0 \). Put \( \lambda_{ij} = \tau^{-\frac{1}{2}}\lambda_i e_1 \lambda_j \). Then, by Proposition 2.18, it follows that \( \{\lambda_{ij} : 1 \leq i, j \leq n\} \) is a quasi-basis for \( E_0 \circ E_1 \). Therefore, for any \( y \in B' \cap A_3 \), by Remark 3.11, we have
\[
\gamma_1(y) = \tau^{-2} \sum_{i,j} E_2 \circ E_3(e_{[-1,1]}\lambda_{ij}y)e_{[-1,1]}\lambda_{ij}^*,
\]
where \( e_{[-1,1]} \) is as in Proposition 2.17. In particular, for any \( x \in B' \cap A_1 \), we obtain
\[
\gamma_1(x) = \tau^{-2} \sum_{i,j} E_2 \circ E_3(e_{[-1,1]}\lambda_{ij}x)e_{[-1,1]}\lambda_{ij}^* 
= \tau^{-2} \sum_{i,j} E_2 \circ E_3(e_{[-1,1]})(\lambda_{ij}xe_{[-1,1]})\lambda_{ij}^* 
= \sum_{ij} \lambda_{ij}xe_{[-1,1]}\lambda_{ij}^* 
= \tau^{-2}E_{A'_1 \cap A_3}(xe_{[-1,1]}).
\] (by Lemma 2.23) (3.8)

This proves that \( \gamma_1(x) \in A' \cap A_3 \) for all \( x \in B' \cap A_1 \). Thus, by Theorem 3.16, \( \gamma_1 \gamma_0 \) is an injective \(*\)-homomorphism from \( B' \cap A_1 \) into \( A' \cap A_3 \).

In order to show that \( \gamma_0 \gamma_1 \) is the inverse of \( \gamma_1 \gamma_0 \), we first show that \( \gamma_1 \) maps \( A'_1 \cap A_3 \) into \( B' \cap A_1 \). Let \( z \in A'_1 \cap A_3 \). Then
\[
\gamma_1(z) = \tau^{-5} \sum_{i,j} E_2 \circ E_3(e_2e_1e_3e_2\lambda_i e_1 \lambda_j z)e_2e_1e_3e_2\lambda_i^* e_1 \lambda_j^* 
= \tau^{-5} \sum_{i,j} E_2 \circ E_3(e_2e_1e_3e_2z\lambda_i e_1 \lambda_j)e_2e_1e_3e_2\lambda_j^* e_1 \lambda_i^* 
= \tau^{-5} \sum_{i,j} E_2 \circ E_3(e_2e_1e_3e_2z)\lambda_i e_1 \lambda_j e_2e_1e_3e_2\lambda_j^* e_1 \lambda_i^* 
\]

$= \tau^{-5} \sum_{i,j} E_2 \circ E_3(e_2e_1e_3e_2z)\lambda_i e_1 e_2 \lambda_j e_3 e_2 e_1 \lambda_i^*$

$= \tau^{-5} \sum_i E_2 \circ E_3(e_2e_1e_3e_2z)\lambda_i e_1 e_2 e_3 e_2 e_1 \lambda_i^*$  
(by Proposition 2.12)

$= \tau^{-3} \sum_i E_2 \circ E_3(e_2e_1e_3e_2z)\lambda_i e_1 \lambda_i^*$

$= \tau^{-5} E_2 \circ E_3(e_2e_1e_3e_2z) \in B' \cap A_1.$

Therefore, $\gamma_0^* \gamma_1 : A_1' \cap A_2 \to B' \cap A_1$ is a well-defined *-homomorphism and we have $(\gamma_1 \gamma_0^*)(\gamma_0^* \gamma_1) = \text{Id}$, by Lemma 3.14. In particular, this proves that $\gamma_1 \gamma_0$ and $\gamma_0^* \gamma_1$ are inverses of each other.

To see that $\gamma_1 \gamma_0$ is $\text{tr}$-preserving we use equation (3.8) to note that for any $x \in B' \cap A_1$, we have

$\text{tr}(\gamma_1 \gamma_0(x)) = \text{tr}\left(\tau^{-2} E_{A_1' \cap A_1}(xye_{[-1,1]})\right)$

$= \tau^{-2} \text{tr}(xe_{[-1,1]})$

$= \text{tr}(x).$  
(by Proposition 2.21)

This completes the proof of the theorem.

\[ \square \]

3.3. **Coproduct on the relative commutants**

Each higher relative commutant comes equipped with another product, the so-called coproduct (Ocneanu called it ‘convolution’), as defined below.

**Definition 3.19.** The coproduct of any two elements $x$ and $y$ of $B' \cap A_k$, denoted by $x \circ y$, is defined as

$x \circ y = F_k^{-1}(F_k(y)F_k(x)).$

**Lemma 3.20.** The coproduct $\circ$ is associative.

**Proof.** The proof is basically a simple book keeping exercise. Consider $x, y$ and $z$ in $B' \cap A_k$. Let us agree to denote $e_{k+1}e_k \cdots e_2 e_1$ by $v_k+1$. From the definition, it follows that $x \circ y$ equals

$\tau^{-\frac{3(k+2)}{2}} E_{A' \cap A_k+1}(x e_{k+1} e_k \cdots e_2 e_1 y e_{k+1} e_k \cdots e_2 e_1).$

Therefore,

$E_{A' \cap A_k+1}(x \circ y y_k+1)$

$= \tau^{-\frac{3(k+2)}{2}} E_{A' \cap A_k+1}(x e_{k+1} e_k \cdots e_2 e_1 y e_{k+1} e_k \cdots e_2 e_1).$

$= \tau^{-\frac{3(k+2)}{2}} E_{A' \cap A_k+1}(x e_{k+1} e_k \cdots e_2 e_1 e_1 e_2 \cdots e_k e_{k+1}).$

$= \tau^{-\frac{3(k+2)}{2}} E_{A' \cap A_k+1}(x e_{k+1} e_k \cdots e_2 e_1 e_1 e_2 \cdots e_k e_{k+1}).$

$= \tau^{-\frac{3(k+2)}{2}} E_{A' \cap A_k+1}(x e_{k+1} e_k \cdots e_2 e_1 e_1 e_2 \cdots e_k e_{k+1}).$

$= \tau^{-\frac{3(k+2)}{2}} E_{A' \cap A_k+1}(x e_{k+1} e_k \cdots e_2 e_1 e_1 e_2 \cdots e_k e_{k+1}).$

$= \tau^{-\frac{3(k+2)}{2}} E_{A' \cap A_k+1}(x e_{k+1} e_k \cdots e_2 e_1 e_1 e_2 \cdots e_k e_{k+1}).$

(by Lemma 2.15)

(by equation (3.2))
Thus,

\[(x \circ y) \circ z = \tau^{-\frac{(k+2)}{2}} E_{k+1} \left( E_{A'}^{B' \cap A_{k+1}} \left( (z v_{k+1}) E_{A'}^{B' \cap A_{k+1}} \left( (y v_{k+1}) E_{A'}^{B' \cap A_{k+1}} \left( (y v_{k+1}) E_{A'}^{B' \cap A_{k+1}} \left( (x v_{k+1}) v_{k+1} \right) \right) \right) \right) \right)\]

\[= \tau^{-2(k+2)} E_{k+1} \left( E_{A'}^{B' \cap A_{k+1}} \left( (y v_{k+1}) E_{A'}^{B' \cap A_{k+1}} \left( (x v_{k+1}) v_{k+1} \right) \right) \right)\]

\[= \tau^{-2(k+2)} E_{k+1} \left( (x v_{k+1}) \right) \]

\[= x \circ (y \circ z).\]

This completes the proof of associativity. \(\Box\)

Below we determine the identity element with respect to the coproduct.

**Proposition 3.21.** For every \(x \in B' \cap A_k\), we have

\[x \circ (\tau^{-k/2} e_1 e_2 \cdots e_k) = x = (\tau^{-k/2} e_1 e_2 \cdots e_k) \circ x.\]

**Proof.** By equation 3.2, we have \((e_1 e_2 \cdots e_{k-1} e_k)(e_{k+1} e_2 \cdots e_{k-1}) = \tau^k e_1\). So,

\[F_k(e_1 e_2 \cdots e_k) = \tau^{-k/2} \tau^k E_{A'}^{B' \cap A_{k+1}}(e_1) = \tau^{k/2}.\] (3.9)

Therefore, \(x \circ (e_1 e_2 \cdots e_k) = F_k^{-1}(F_k(e_1 e_2 \cdots e_k)(F_k(x)) = \tau^{k/2} x\) by Proposition 3.2. Similarly, \((e_1 e_2 \cdots e_k) \circ x = \tau^{k/2} x\). This completes the proof. \(\Box\)

4. Biprojections and intermediate \(C^\ast\)-subalgebras

In this section, we show how we can apply the results of the previous sections to understand the intermediate \(C^\ast\)-subalgebras of an irreducible inclusion of simple unital \(C^\ast\)-algebras. When the inclusion is a subfactor of \(II_1\), a result by Bisch [6] shows that the intermediate subfactors are in bijective correspondence with certain projections in the relative commutant, the so-called biprojections. Later, Bisch, Jones and Landau found a nice pictorial description of the biprojections in the planar algebraic language (see [31]). We obtain similar results for irreducible inclusions of simple unital \(C^\ast\)-algebras and we obtain an analogue of the Bisch’s characterization.

Throughout this section, \(B \subset A\) will denote a fixed irreducible pair (that is, \(B' \cap A = C\)) of simple unital \(C^\ast\)-algebras such that \(E_0(A, B) \neq \emptyset\).

4.1. Intermediate \(C^\ast\)-subalgebras

**Notation 4.1.** Let \(C\) be an intermediate \(C^\ast\)-subalgebra of \(B \subset A\). Clearly, \(C\) is also simple and there exists a unique (minimal) condition \(E_{C}^{A}\) from \(A\) onto \(C\).

(1) We denote the \(C^\ast\)-basic construction of the irreducible pair \(C \subset A\) by \(C_1\) with Jones projection \(e_C\) corresponding to the minimal conditional expectation \(E_{C}^{A}\), that is, \(C_1 = C^\ast(C, e_C)\).

(2) We denote \([A : C]_{0}^{-1}\) by \(\tau_C\); so that \([C : B]_{0}^{-1} = \tau / \tau_C\).

We first provide few useful observations which will be used ahead.
Lemma 4.2. Let $B \subset C \subset A$ be as in Notation 4.1. Then, we have the following.

1. $C_1$ is simple and unital.
2. $C_1 \subset A_1$ is an irreducible pair with common identity.
3. The unique (minimal) conditional expectations $E_{A_1}^{A_1}$ and $E_{C_1}^{C_1}$ satisfy $E_{A_1}^{A_1}|_{C_1} = E_{C_1}^{C_1}$.
4. $E_{C_1}^{A_1}|_{B' \cap A_1}$ is the unique tr-preserving conditional expectation from $B' \cap A_1$ onto $B' \cap C_1$.
5. The tracial state on $C' \cap C_1$ induced by the inclusion $C \subset A \subset C_1$ (as in Proposition 2.21) is same as the restriction of the tracial state of $B' \cap A_1$.

Proof. (1) follows from Theorem 2.13.

(2) First note that $e_C \circ e_B = e_B$. Indeed, for an arbitrary $a \in A$ we have $e_C \circ e_B(a) = e_C(e_B(e_B(a))) = E_B^B(e_B(a)) = E_B^B(a) = e_B(a)$.

Now, recall from [48, Proposition 1.6.6] that $C_1 = \text{span}\{x_1 e_C x_2 : x_1, x_2 \in A\}$ and $A_1 = \text{span}\{y_1 e_B y_2 : y_1, y_2 \in A\}$. Since $A_1$ is unital, it suffices to show that $C_1 A_1 \subset A_1$, which is rather trivial as $(x_1 e_C x_2)(y_1 e_B y_2) = x_1 e_C x_2 y_1 e_C e_B y_2 = x_1 e_C(x_2 y_1) e_B y_2 \in A_1$ for all $x_1, y_1 \in A$, $i = 1, 2$.

(3) is now immediate from Lemma 2.9 because $A \subset A_1$ is also an irreducible pair of simple unital $C^*$-algebras, by Proposition 3.2.4.

(4) Let $x \in B' \cap A_1$. Clearly, $E_{C_1}^{C_1}(x) \in B' \cap C_1$ and, by (3), we have $E_{A_1}^{A_1} \circ E_{C_1}^{A_1}(xy) = E_{A_1}^{A_1} \circ E_{C_1}^{A_1}(xy)$ for every $y \in B' \cap C_1$. Hence,

$$\text{tr}(E_{C_1}^{A_1}(xy)) = E_{A_1}^{A_1} \circ E_{C_1}^{A_1}(xy) = E_{A_1}^{A_1} \circ E_{C_1}^{A_1}(xy) = \text{tr}(xy)$$

for every $y \in B' \cap C_1$.

(5) It suffices to show that $E_B^B \circ E_{A_1}^{A_1}|_{C' \cap C_1} = E_C^C \circ E_{C_1}^{C_1}$. For any $z \in C' \cap C_1$, we see that $E_B^B \circ E_{A_1}^{A_1}(z) = E_B^B \circ E_C^C \circ E_{C_1}^{C_1}(z)$, by (3). But $E_{C_1}^{C_1}(z) \in C' \cap A$ and so $E_B^B \circ E_C^C \circ E_{C_1}^{C_1}(z) \in Z(C) = \mathbb{C}$. Therefore, $E_B^B \circ E_{A_1}^{A_1}(z) = E_B^B(E_C^C \circ E_{C_1}^{C_1}(z)) = E_C^C \circ E_{C_1}^{C_1}(z)$. This completes the proof. \qed

Notation 4.3. Let $A \subset C \subset B$ be as in Lemma 4.2.

1. Let $C_2$ denote the $C^*$-basic construction of $C_1 \subset A_1$ with Jones projection denoted by $e_{C_1}$, corresponding to the unique minimal conditional expectation $E_{C_1}^{C_1}$.
2. As in Lemma 4.2, $C_2 \subset A_2$ is a simple unital irreducible inclusion. Let $C_3$ denote its $C^*$-basic construction with Jones projection denoted by $e_{C_2}$.

Lemma 4.4. Let $B \subset C \subset A$ be as in Lemma 4.2. Then,

1. $\gamma_0(e_C) = e_C$. In particular, $\gamma_0(e_1) = e_1$.
2. $E_{C_1}^{C_1}(e_1) = \frac{1}{|C:B|_0} e_C$.

Proof. (1) Observe that $e_1 e_C = e_C e_1 = e_1$. Fix a quasi-basis $\{\lambda_i\}$ for $E_0$. Then,

$$\gamma_0(e_C) = \tau^{-1} \sum_{i} E_1(e_1 \lambda_i e_C) e_1 \lambda_i^*$$

$$= \tau^{-1} \sum_{i} E_1(e_1 e_C \lambda_i e_C) e_1 \lambda_i^*$$

$$= \tau^{-1} \sum_{i} E_1(e_1 E_0^A(\lambda_i) e_C) e_1 \lambda_i^*$$

$$= \tau^{-1} \sum_{i} E_1(e_1 e_C \lambda_i e_C) e_1 \lambda_i^*$$

$$= \tau^{-1} \sum_{i} E_1(e_1 \lambda_i e_C) e_1 \lambda_i^*$$

$$= \tau^{-1} \sum_{i} E_1(e_1 \lambda_i e_C) e_1 \lambda_i^*$$

$$= \frac{1}{|C:B|_0} e_C.$$
= ∑_{i} E_{C}^{A}(\lambda_{i}) e_{1} \lambda_{i}^{*} \\
= ∑_{i} \lambda_{i} e_{C} e_{1} \lambda_{i}^{*} \\
= e_{C} \left( ∑_{i} \lambda_{i} e_{1} \lambda_{i}^{*} \right) \\
= e_{C}. \quad \text{(by Proposition 2.12)}

(2) Let \( w = ∑_{i} x_{i} e_{C} y_{i} \in C' \cap C_{1} \) with \( x_{i}, y_{i} \in A \) for all \( i \). By Lemma 4.2, the tracial state on \( C' \cap C_{1} \) is the restriction of the tracial state on \( B' \cap A_{1} \). So, we have

\[
\text{tr}(e_{C}w) = \text{tr}\left( ∑_{i} e_{C} x_{i} e_{C} u_{i} \right) \\
= E_{B}^{A} \circ E_{A}^{A_{1}} \left( ∑_{i} e_{C} E_{C}^{A}(x_{i}) y_{i} \right) \\
= [A : C]_{0}^{-1} E_{B}^{A_{1}} \left( ∑_{i} E_{C}^{A}(x_{i}) y_{i} \right). \quad \text{(since } E_{A}^{A_{1}}(e_{C}) = [A : C]_{0}^{-1})
\]

And, on the other hand,

\[
\text{tr}(e_{1}w) = \text{tr}\left( ∑_{i} e_{1} x_{i} e_{C} y_{i} \right) \\
= E_{B}^{A_{1}} \left( ∑_{i} e_{1} E_{C}^{A}(x_{i}) y_{i} \right) \quad \text{(since } e_{1} e_{C} = e_{C} e_{1} = e_{1}) \\
= \tau E_{B}^{A_{1}} \left( ∑_{i} E_{C}^{A}(x_{i}) y_{i} \right).
\]

Thus, \( \text{tr}(e_{C}w) = [C : B]_{0} \text{tr}(e_{1}w) \) for all \( w \in B' \cap C_{1} \); so that \( E_{C' \cap C_{1}}^{B' \cap A_{1}}(e_{1}) = \frac{1}{[C : B]_{0}} e_{C} \). Then, by Lemma 4.2, we deduce that \( E_{C_{1}}^{A_{1}}(e_{1}) = \frac{1}{[C : B]_{0}} e_{C} \). \( \square \)

A more general version of the following lemma will be established in the next section.

**Lemma 4.5.** Let \( B \subset C \subset A \) be as in Lemma 4.2. Then,

\[
e_{C} e_{2} e_{C} = [A : C]_{0}^{-1} e_{C} e_{C_{1}} \text{ and } e_{C} e_{3} e_{C_{1}} = [C : B]_{0}^{-1} e_{C_{1}} e_{C_{2}}.
\]

**Proof.** This proof is essentially borrowed from [6, 7, 12]. We first show that

\[
\text{tr}(e_{C}) = \tau_{e} \quad \text{and} \quad \text{tr}(e_{C} e_{C_{1}}) = \tau. \quad (4.1)
\]
The first equality follows trivially from the definition. To see the second equality note that
\[
\text{tr}(e_Ce_{C_1}) = E_A^c \circ E_{C_1}^c \circ E_{A_1}^c(e_Ce_{C_1})
\]
\[
= E_C^c \circ E_{C_1}^c \left( e_C e_{A_1}^c \circ E_{A_1}^c(e_{C_1}) \right)
\]
\[
= E_C^c \circ E_{C_1}^c \left( e_C[A_1 : C_1]_0^{-1} \right)
\]
\[
= \tau,
\]
where the last equality follows from the derivation
\[
[A_1 : C_1]_0 = \frac{[A_1 : A]_0}{[C_1 : A]_0} = \frac{[A : B]_0}{[A : C]_0} = [C : B]_0. \quad \text{(by Theorem 2.8)}
\]

Now, \(e_2e_{C_2}e_2 = E_{A_1}^c(e_C)e_2 = \tau e_2\). So, \(v := \frac{1}{\sqrt{\tau}}e_2e_C\) is a partial isometry and hence \(v^*v = \frac{1}{\tau}e_2e_Ce_2\) is a projection, say, \(q\). Clearly, \(q\) majorizes the projection \(e_Ce_{C_1}\). And, on the other hand, by Proposition 2.21, we have \(\text{tr}(q) = \frac{1}{\tau}\text{tr}(e_Ce_2) = \frac{\tau}{\tau}\text{tr}(e_C) = \tau\). Therefore, by equation 4.1, \(q = e_Ce_{C_1}\). This proves that \(e_Ce_2e_C = [A : C]_0^{-1}e_Ce_{C_1}\). The other implication follows similarly and we omit the details.

**Proposition 4.6.** Let \(B \subset C \subset A\) be as in Lemma 4.2. Then,
\[
\mathcal{F}_1(e_C) = \frac{1}{\sqrt{[A : B]_0}}[A : C]_0 e_{C_1}. \quad \text{(4.2)}
\]

In particular, \(\mathcal{F}_1(e_1) = \frac{1}{\sqrt{[A : B]_0}}1_{A_1}\).

**Proof.** This follows immediately from the preceding lemma as follows:
\[
\mathcal{F}_1(e_C) = \tau^{-3/2}E_{A' \cap A_2}^c(e_Ce_2e_1)
\]
\[
= \tau^{-3/2}E_{A' \cap A_2}^c(e_Ce_2e_1)
\]
\[
= \tau^{-3/2}E_{A' \cap A_2}^c(e_Ce_1 e_1) \quad \text{(by Lemma 4.5)}
\]
\[
= \tau e_C^{-1/2}e_{C_1}. \quad \text{(by Lemma 2.23)}
\]

**Remark 4.7.** Similar to Proposition 4.6, we also have the following expression:
\[
\mathcal{F}_1^{A' \cap A_1}(e_{C_1}) = \frac{1}{\sqrt{[A : B]_0}}e_{C_2},
\]
where \(\mathcal{F}_1^{A' \cap A_1}\) is the Fourier transform from \(A' \cap A_2\) onto \(A'_1 \cap A_3\) defined by
\[
\mathcal{F}_1^{A' \cap A_1}(x) = \tau^{-3/2}E_{A'_1 \cap A_3}^c(x e_2).
\]

The map \(\gamma_1\gamma_0\) in Theorem 3.18 has the following special property.

**Proposition 4.8.** \(\gamma_1\gamma_0(e_C) = e_{C_2}\).

**Proof.** First, note that, \(\gamma_1\gamma_0(e_C) = \gamma_1(e_C)\), by Lemma 4.4. And then, we have
\[
\gamma_1(e_C) = \tau^{-2}E_{A'_1 \cap A_3}^c(e_Ce_{[-1,1]}).
\]
This finishes the proof. □

The following proposition will be used in the last section.

**Proposition 4.9.** Let \( B \subseteq C \subseteq A \) be as in Lemma 4.2. Then,

\[
\gamma_0(B' \cap C_1) = C' \cap A_1 \quad \text{and} \quad \gamma_0(C' \cap A_1) = B' \cap C_1.
\]

**Proof.** Consider \( x \in C' \cap A_1 \). Let \( \{ \gamma_j : j \in J \} \) be a quasi-basis for \( E_B^A \) and \( \{ \lambda_i : i \in I \} \) be a quasi-basis for \( E_B^A \). Then,

\[
\gamma_0(x) = \tau^{-1} \sum_{i,j} E_A^{A'_1} (e_1 \gamma_j^* e C_1^A (\gamma_j \lambda_i) x) e_1 \lambda_i^*
\]

\[
= \tau^{-1} \sum_{i,j} E_A^{A'_1} (e_1 \gamma_j^* x) E_C^A (\gamma_j \lambda_i) e_1 \lambda_i^*
\]

\[
= \tau^{-1} \sum_{j} E_A^{A'_1} (e_1 \gamma_j^* x) e C \gamma_j. \quad \text{(since } \sum_i \lambda_i e_1 \lambda_i^* = 1)\]

Therefore, \( \gamma_0(x) \in C_1 \). In other words,

\[
\gamma_0(C' \cap A_1) \subseteq B' \cap C_1 \quad \text{(**).}
\]

Next, consider \( y \in B' \cap C_1 \). We show that \( \gamma_0(y) \in C' \cap A_1 \). To see this note that since for any \( u \in U(C) \) (the set of all unitaries of \( C \)) we have \( \{ u \lambda_i : i \in I \} \) is a quasi-basis for \( E_B^A \) and hence

\[
\gamma_0(y) = \tau^{-1} \sum_i E_A^{A'_1} (e_1 u_1 \lambda_i y) e_1 \lambda_i^* u^*
\]

\[
= \tau^{-1} \sum_i E_{A'_{C_1}} \circ E_A^{A'_1} (e_1 u_1 \lambda_i y) e_1 \lambda_i^* u^*
\]

\[
= \tau^{-1} \sum_i E_A^{C_{C_1}} (E_{C_{C_1}}^A (e_1) u_1 \lambda_i y) e_1 \lambda_i^* u^*
\]

\[
= \tau^{-1} \frac{1}{[C : B]} \sum_i E_A^{C_{C_1}} (e_1 u_1 \lambda_i y) e_1 \lambda_i^* u^* \quad \text{(by Lemma 4.4)}
\]
\[
\tau^{-1} \frac{1}{[C:B]_0} u \left( \sum_i E_{A_i}^{C_i} (e_C \lambda_i y e_1 \lambda_i^*) \right) u^* = \tau^{-1} u \left( \sum_i E_{A_i}^{C_i} (E_{C_i}^{A_i} (e_1 \lambda_i y) e_1 \lambda_i^*) \right) u^* = u \left( \tau^{-1} \sum_i E_{A_i}^{A_1} (e_1 \lambda_i y) e_1 \lambda_i^* \right) u^* = u \gamma_0(y) u^*.
\]

Therefore, \( \gamma_0(y) \in C' \). In other words, we have proved that,

\[
\gamma_0(B' \cap C_1) \subseteq C' \cap A_1 \tag{**}.
\]

Combining (\( \ast \)) and (\( \ast \ast \)) together with Lemma 3.14, we establish the desideratum. \( \square \)

4.2. Biunitaries, biprojections and bipartial isometries

Motivated by [6, 19, 21, 37], we propose the following definitions. The notion of ‘bipartial isometry’ was introduced and studied effectively in [19] for subfactors.

**Definition 4.10.** (1) A unitary \( u \in B' \cap A_k \) will be called a biunitary for the inclusion \( B \subset A \) if \( \mathcal{F}_k(u) \) is again a unitary in \( A' \cap A_{k+1} \). We denote the collection of all biunitaries by \( \text{BU}_k(B, A) \).

(2) A projection \( e \in B' \cap A_k \) will be called biprojection for the inclusion \( B \subset A \) if \( \mathcal{F}_k(e) \) is a multiple of a projection. We denote the collection of all biprojections by \( \text{BP}_k(B, A) \).

(3) An element \( e \in B' \cap A_k \) will be called a bipartial isometry for the inclusion \( B \subset A \) if both \( e \) and \( \mathcal{F}_k(e) \) are multiples of partial isometries. We denote the collection of all bipartial isometries by \( \text{BPI}_k(B, A) \).

**Notation 4.11.** Suppose \( e \in \text{BP}_k(B, A) \). So, \( \mathcal{F}_k(e) = tf \) for some positive scalar \( t \) and a projection \( f \in A' \cap A_{k+1} \). We shall denote the projection \( f \) by the symbol \( \{ \mathcal{F}_k(e) \} \).

**Remark 4.12.** By Proposition 4.6 and Corollary 4.7, \( e_C \) and \( e_{C_1} \) are both biprojections. More precisely, \( e_C \in \text{BP}_1(B, A) \) and \( e_{C_1} \in \text{BP}_1(A, A_1) \).

**Proposition 4.13.** Fourier transform of a biunitary is a bipartial isometry. More precisely, for every \( k \geq 1 \),

\[
\mathcal{F}_k(u) \in \text{BPI}_{k+1}(B, A) \text{ for all } u \in \text{BU}_k(B, A).
\]

**Proof.** First, notice that, being a unitary, \( \mathcal{F}_k(u) \) is also a partial isometry in \( A' \cap A_{k+1} \subset B' \cap A_{k+1} \). Next, we show that \( \mathcal{F}_{k+1}(\mathcal{F}_k(u)) \) is a multiple of a partial isometry. We have

\[
\mathcal{F}_{k+1}(\mathcal{F}_k(u)) = \tau^{-\frac{k+3}{2}} E_{A' \cap A_{k+2}}^{B' \cap A_{k+2}}(\mathcal{F}_k(u)e_{k+2}e_{k+1} \cdots e_2 e_1) = \tau \tau^{-\frac{k+3}{2}} \mathcal{F}_k(u)e_{k+2}e_{k+1} \cdots e_3 e_2.
\]

Let \( w := \mathcal{F}_{k+1}(\mathcal{F}_k(u)) \). Then,

\[
ww^* = \tau^2 \tau^{-(k+3)} \mathcal{F}_k(u)(e_{k+2}e_{k+1} \cdots e_3 e_2)(e_2 e_3 \cdots e_{k+1} e_{k+2}) \mathcal{F}_k(u)^*.
\]
Clearly, \((e_{k+2}e_{k+1} \cdots e_3e_2)(e_2e_3 \cdots e_{k+1}e_{k+2}) = \tau^k e_{k+2}\). So, \(ww^* = \tau^{-1}F_k(u)e_{k+2}F_k(u)^*\) and since \(F_k(u)\) is a unitary it follows that \(w\) is a multiple of a partial isometry. This completes the proof. \(\Box\)

**Lemma 4.14.** \(e_1\) is a minimal as well as a central projection in \(B' \cap A_1\).

**Proof.** We first assert that \(e_1\) is a minimal projection in \(B' \cap A_1\).

Let \(u \in \mathcal{U}(B' \cap A_1)\). By Lemma 2.15, we have \(ue_1 = [A : B]'_0E_0(ue_1)e_N\). Since \(u, e_1 \in B' \cap A_1, E^{B_0}_0(ue_1) \in B' \cap A = \mathbb{C}\). Thus \(e_1ue_1 \in \mathbb{C}e_1\). So, \(e_1\) is minimal in \(B' \cap A_1\).

Next, we show that \(e_1\) is central as well. Let \(\lambda_0 := [A : B]'_0E_0(ue_1) \in \mathbb{C}\). We now show that \(|\lambda_0| = 1\). We have \(ue_1u^* = \lambda_0e_1\lambda_0\). Applying \(E_1\) on both sides, we get
\[
[A : B]'_0^{-1} = \text{tr}(e_1) = \text{tr}(ue_1u^*) = E_1(ue_1u^*) = |\lambda_0|^2E_1(e_1) = [A : B]'_0^{-1}|\lambda_0|^2.
\]

Hence, \(|\lambda_0| = 1\) and we get \(ue_1u^* = e_1\). Since \(u\) was an arbitrary unitary in \(B' \cap A_1\), we deduce that \(e_1 \in \mathcal{Z}(B' \cap A_1)\). \(\Box\)

**Lemma 4.15.** Let \(e \in \text{BP}_1(B, A)\) and \(F_1(e) = tf\) for some \(t > 0\) and projection \(f \in A' \cap A_2\).

Then,

1. \(ee_1 = e_1ee = e_1\) and \(fe_2 = e_2f = e_2\);
2. \(E_1(e) = \text{tr}(e) = t \tau^{1/2}\) and \(E_2(f) = \text{tr}(f) = t^{-1}\tau^{1/2}\);
3. \(\text{tr}(ef) = \tau\) and
4. \(fe_1e_2 = t^{-1}\tau^{1/2}ee_2e_2\).

**Proof.** (1) We first assert that \(ee_1 = e_1\). To see this, use Lemma 2.15 to obtain \(ee_1 = \tau^{-1}E_1(ee_1)e_1 = se_1\) for some scalar \(s > 0\). Indeed, \(E_1(ee_1) \in B' \cap A = \mathbb{C}\) and hence \(E_1(ee_1) = E_0 \circ E_1(ee_1) = \text{tr}(e_1)\). And, by Lemma 4.14, \(e_1\) is a projection. Therefore, \(s = 1\). The assertion about \(f\) follows similarly.

(2) We first show that \(E_1(e) = \text{tr}(e)\). Note that \(e \in B' \cap A_1\) and hence, for any \(b \in B, E_1(e)b = E_1(eb) = E_1(be) = beE_1(e)\). Therefore, \(E_1(e) \in B' \cap A = \mathbb{C}\). Thus, \(E_1(e) = E_0 \circ E_1(e) = \text{tr}(e)\). Similarly, we obtain \(E_1(f) = \text{tr}(f)\).

Further, from the definition of \(f\), we obtain \(e = \tau^{-\frac{3}{2}}E_2(efe_2)\). Thus, \(\text{tr}(e) = t \tau^{\frac{3}{2}}E_0 \circ E_1 \circ E_2(efe_2)\). Then, by [26, Lemma 3.11], we get
\[
\text{tr}(e) = t \tau^{\frac{3}{2}}E_0 \circ E_1 \circ E_2(efe_1) = t \tau^{\frac{3}{2}}\tau^2 = t \tau^{\frac{1}{2}}.
\]

For \(\text{tr}(f)\), observe that
\[
\text{tr}(f) = t^{-1}\tau^{-\frac{3}{2}}\text{tr}\left(E_{A' \cap A_2}(efe_1)\right)
= t^{-1}\tau^{-\frac{3}{2}}\text{tr}(efe_2e_1)
= t^{-1}\tau^{-\frac{3}{2}}\text{tr}(e_1e_2)
= t^{-1}\tau^{\frac{1}{2}}.
\]

(3)
\[
\text{tr}(ef) = E_0 \circ E_1 \circ E_2(ef)
= E_0 \circ E_1(eE_2(f))
= E_0 \circ E_1(e) \text{tr}(f)
= \text{tr}(e) \text{tr}(f)
\]
\[(t\tau^\frac{1}{2})(t^{-1} \tau^\frac{1}{2}) = \tau. \tag{4}\]

\[fe_1 e_2 = t^{-1} \tau^{-\frac{3}{2}} E_{A' \cap A_2}^{B' \cap A_2} (ee_2 e_1) e_1 e_2 = t^{-1} \tau^{-\frac{3}{2}} \sum_i \lambda_i ee_2 e_1 \lambda_i^* e_1 e_2 = t^{-1} \tau^{-\frac{1}{2}} \sum_i \lambda_i E_0 (\lambda_i^*) ee_2 \quad (\text{since } e \in B') = t^{-1} \tau^\frac{3}{2} ee_2. \]

Using above observations, we deduce the following exchange relation.

**Proposition 4.16.** Let \(e, t\) and \(f\) be as in Lemma 4.15. Then, \(ef = fe\).

**Proof.** We first show that \(\text{tr}(fee_2 e) = \tau \text{tr}(e)\).

By Lemma 4.15(4), we see that \(ee_2 e = t^2 f e_1 f\). Then, by other parts of the same lemma, we obtain

\[\text{tr}(fee_2 e) = t^2 \text{tr}(fe_1 f) = t^2 E_0 \circ E_1 (E_2 (f) e_1) = t^2 \text{tr}(f) E_0 \circ E_1 (e_1) = \tau t^2 \text{tr}(f) = \tau \text{tr}(e).\]

Now, note that \(e_2 ee_2 = E_1 (e) e_2\), and hence, by Lemma 4.15(2), we have \(e_2 ee_2 = \text{tr}(e) e_2\).

Further,

\[
\|ee_2 e - \text{tr}(e) ef\|_2^2 = \text{tr}((ee_2 e - \text{tr}(e) f e)(ee_2 e - \text{tr}(e) ef)) = \text{tr}(ee_2 ee_2) - 2 \text{tr}(e) \text{tr}(fee_2 e) + \text{tr}(e)^2 \text{tr}(ef) = \tau \text{tr}(e)^2 - 2 \tau \text{tr}(e)^2 + \tau \text{tr}(e)^2 (\text{by Lemma 4.15(3)}) = 0;
\]

so that, \(ee_2 e = \text{tr}(e) ef\), and after taking adjoint we obtain

\[ef = \frac{ee_2 e}{\text{tr}(e)} = fe. \tag{4.3}\]

**Theorem 4.17.** Let \(e, t\) and \(f\) be as in Lemma 4.15. Then, \(ef \in \text{BP}_2(B, A)\).

**Proof.** By Proposition 4.16, \(ef\) is a projection and

\[F_2(ef) = F_2(fe) = \tau^{-2} E_{A' \cap A_3}^{B' \cap A_3} (fee_3 e_2 e_1) = \tau^{-2} f E_{A' \cap A_3}^{B' \cap A_3} (ee_3 e_2 e_1) = \tau^{-2} f E_{A' \cap A_3}^{B' \cap A_3} (e_3 e e_2 e_1)\]
\[ = \tau^{-\frac{1}{2}}fe_3F_1(e) \]
\[ = t\tau^{-\frac{1}{2}}fe_3f \]
\[ = \frac{1}{\text{tr}(f)}fe_3f. \]

Then, by Lemma 4.15(2), we have \( e_3fe_3 = \text{tr}(f)e_3 \). Thus, \( v := \frac{1}{\sqrt{\text{tr}(f)}}e_3f \) is a partial isometry with \( vv^* = e_3 \). Hence, \( v^*v = \frac{1}{\text{tr}(f)}fe_3f \) is a projection too. In particular, \( ef \) and \( F_2(ef) \) are both projections. This completes the proof. \( \square \)

The following shows that, up to a scalar, a biprojection is also an idempotent with respect to the coproduct.

**Lemma 4.18.** If \( e \in \text{BP}_k(B, A) \), then \( e \circ e = te \) for some scalar \( t \).

**Proof.** By definition. \( \square \)

Now we give an analogue of Lemma 4.18 for biunitary element.

**Lemma 4.19.** If \( u \in \text{BU}_1(B, A) \), then \( u \circ (\gamma_0(u^*)) = (\gamma_0(u^*)) \circ u = \tau^{-1/2}e_1 \).

**Proof.** Apply Lemma 3.12 and Remark 3.13. \( \square \)

4.2.1. Behavior under inclusions, Fourier transforms and rotation maps. This short subsection discusses the relationships between biprojections, biunitaries and bipartial isometries. Although we do not use any result of this section, we include it to show how bipartial isometry comes naturally into the picture to describe higher dimensional biprojections, which generalizes Bisch’s biprojections which are elements of the ‘two-box space.’

An arbitrarily fixed \( u \in \text{BU}_k(B, A) \) may also be thought of as a unitary element in \( B' \cap A_{k+1} \) via the canonical inclusion map. A natural question is whether \( u \in \text{BU}_{k+1}(B, A) \) or not? This might not be the case always. However, \( u \in \text{BPI}_{k+1}(B, A) \) as the following result shows.

**Proposition 4.20.** (1) If \( u \in \text{BU}_k(B, A) \), then \( u \in \text{BPI}_{k+1}(B, A) \) for all \( k \geq 1 \).

(2) If \( e \in \text{BP}_1(B, A) \), then \( e \in \text{BPI}_{k+1}(B, A) \) for all \( k \geq 1 \).

**Proof.** (1) We just need to prove that \( F_{k+1}(u) \) is a multiple of a partial isometry. Indeed, since \( ue_{k+2} = e_{k+2}u \) it readily follows that
\[
F_{k+1}(u) = \tau^{-1}e_{k+2}F_k(u).
\]

Hence, \( F_{k+1}(u)^*F_{k+1}(u) = \tau^{-2}F_k(u)^*e_{k+2}F_k(u) \). Since \( F_k(u) \) is a unitary, this proves that \( F_{k+1}(u) \) is a multiple of a partial isometry. This finishes the proof.

(2) We need to show that \( F_k(e) \) is a multiple of a partial isometry for all \( k \geq 1 \). For \( k = 1 \) it follows from Definition 4.10 that \( F_k(e) = tf \) for some projection \( f \). Thus we need to prove only for \( k \geq 2 \). To see this observe that
\[
F_k(e) = \tau^{-\frac{k+2}{2}}P_{B' \cap A_{k+1}}^B(ee_{k+1}e_k \cdots e_2e_1) \]
\[
= \tau^{-\frac{k+2}{2}}P_{A' \cap A_{k+1}}^B(ee_{k+1}e_k \cdots e_3ee_2e_1) \]
\[
\frac{1}{2} \tau e_{k+1} e_k \cdots e_3 E_{A' \cap A_{k+1}} (ee_2 e_1) = \tau^{-\frac{1}{2}} te_{k+1} e_k \cdots e_3 f.
\]

Thus,
\[(F_k(e))^* F_k(e) = t \tau^{-\frac{1}{2}} \frac{fe_3 f}{\text{tr}(f)}.\]

And, we saw in the proof of Theorem 4.17 that \(\frac{fe_3 f}{\text{tr}(f)}\) is a projection. Thus, \(F_k(e)\) is a multiple of a partial isometry. This completes the proof.

\[\square\]

The next result shows that Fourier transform of a biprojection (in \(B' \cap A_1\)) is a bipartial isometry.

**Proposition 4.21.** If \(e \in BP_1(B, A)\), then \(F_1(e) \in BPI_2(B, A)\).

**Proof.** Write, as before, \(F_1(e) = tf\) for some scalar \(t > 0\) and some projection \(f \in A' \cap A_2 \subset B' \cap A_2\). Then,
\[
F_2(f) = \tau^{-2} E_{A' \cap A_3} (fe_3 e_2 e_1) = \tau^{-1} fe_3 e_2.
\]

Now, as before, \(\frac{fe_3 f}{\text{tr}(f)}\) being a projection, it is obvious that \(fe_3 e_2\), and hence \(F_2(f)\), is a multiple of a partial isometry. \[\square\]

The following result shows that the biunitaries and biprojections (in \(B' \cap A_1\)) are preserved under the rotation map.

**Proposition 4.22.**
1. If \(u \in BU_1(B, A)\), then \(\gamma_0(u), \gamma_0(u^*) \in BU_1(B, A)\).
2. If \(e \in BP_1(B, A)\), then \(\gamma_0(e) \in BP_1(B, A)\).

**Proof.** (1) follows from Remark 3.13 and Lemma 3.12.

(2) By Lemma 3.12, it follows immediately that \(\gamma_0(e)^* = \gamma_0(e)\). Also, \[
\gamma_0(e)^2 = \tau^{-2} \sum_{i,j} E_1(e_1 \lambda_i e_1 \lambda_j^* E_1(e_1 \lambda_j e_1 \lambda_j^*) e_1 \lambda_j^* = \tau^{-2} \sum_{i,j} E_1(e_1 \lambda_i E_0(\lambda_j^* E_1(e_1 \lambda_j e)) e_1 \lambda_j^* = \tau^{-2} \sum_{i,j} E_1(e_1 \lambda_i E_0(\lambda_j^* E_1(e_1 \lambda_j e)) e_1 \lambda_j^* = \tau^{-2} \sum_j E_1(e_1 \lambda_j e_1 \lambda_j^* e_1 \lambda_j^* = \tau^{-1} \sum_j E_1(e_1 \lambda_j e_1 \lambda_j^* = \gamma_0(e).
\]

Thus, \(\gamma_0(e)\) is a projection. On the other hand, by Remark 3.13, we have
\[
F_1(\gamma_0(e)) = (F_1(e))^* = tf.
\]

Hence, \(\gamma_0(e)\) is a biprojection. \[\square\]
In this paper, we discuss only about biprojections. The applications of biunitaries will be analyzed in a future paper.

4.3. From biprojections to intermediate $C^*$-subalgebras

As in the case of a type $II_1$-subfactor, we have the following lemma.

**Lemma 4.23.** $B = \{e_1\}' \cap A$.

**Proof.** Clearly, $B \subseteq \{e_1\}' \cap A$. To see the other direction, take $x \in \{e_1\}' \cap A$. Then, $e_1x = e_1xe_1 = e_1E_0(x)$. Then, by Lemma 2.15, we see that $x = E_0(x) \in B$. □

An abstract characterization of the intermediate subfactors of a type $II_1$ subfactor $N \subseteq M$ in terms of biprojections was established by Bisch in [6, Theorem 3.2]. His proof crucially uses a canonical conjugate linear unitary operator on the standard space $L^2(M, \text{tr}_M)$ and the notion of downward basic construction of a subfactor, both of which are not available for inclusions of general simple $C^*$-algebras. Still, based on the $C^*$-Fourier theory developed above, a suitable adaptation of Bisch’s proof yields the following result to obtain intermediate $C^*$-subalgebras of the dual pair. Recall that, given any intermediate $C^*$-algebra $C$ of $A \subseteq B$, the Jones projection $e_C$ will always be a biprojection. In this sense, the following can be thought of as a partial converse.

**Theorem 4.24.** Let $B \subset A$ be an irreducible inclusion of simple unital $C^*$-algebras with a conditional expectation $E$ of finite index and $e \in B' \cap A_1$ be a biprojection. Then, $[F(e)]$ implements a conditional expectation onto the intermediate $C^*$-subalgebra $P$ of $A \subset A_1$ given by $P = [F(e)]' \cap A_1$. Moreover,

$$P = AeA, \quad [F(e)] = e_P \quad \text{and} \quad [A_1 : P] = \text{tr}([F(e)])^{-1}.$$  

**Proof.** First, recall that, since $B \subset A$ is irreducible, $E$ is the unique minimal conditional expectation from $A$ onto $B$ (see Theorem 2.5). Now, let $f$ denote the projection $[F(e)]$ in $A' \cap A_2$ and $P := \{f\}' \cap A_1$.

**Step I:** For each $x_1 \in A_1$, we show that $fx_1f = y_1f$ for some $y_1 \in A_1$.

Since $A_1 = Ae_1A$, it suffices to consider $x_1$ of the form $x_1 = ae_1b$ for some $a, b \in A$. Since $f \in A' \cap A_2$, we get $fx_1f = fae_1bf = af e_1 f b$. But,

$$fe_1f = \tau^{-1}fe_1e_2e_1f = t^{-2}ee_2e,$$  

(4.4)

where the last equality follows from Lemma 4.15. Thus, applying equation (4.3) followed by Lemma 4.15, we obtain

$$fx_1f = t^{-2}a(ee_2b) = t^{-2}t_2\text{tr}(e)a(ef)b = t^{-2}t_2\text{tr}(e)abf = \text{tr}(f)abf.$$  

Putting $\text{tr}(f)abf = y_1 \in A_1$, we get $fx_1f = y_1f$.

**Step II:** We show that $y_1$ obtained in Step I is an element of $P$.

Without loss of generality we may assume that $x_1 = x_1^*$. Then, $y_1f = fx_1f = (fx_1f)^* = f y_1^*$; so that $E_2(y_1f) = E_2(fy_1^*)$, which implies that $y_1 = y_1^*$, by Lemma 4.15(2). In particular, $fy_1 = y_1f$ and, hence, $y_1 \in \{f\}' \cap A_1 = P$.

**Step III:** We show that $f$ implements a conditional expectation from $A_1$ onto $P$. Since $B' \cap A = \mathbb{C}$, it follows that $P$ is a simple $C^*$-subalgebra of $A \subset A_1$. Let us denote by $E_P$ the unique minimal conditional expectation from $A_1$ onto $P$. We shall show that $fx_1f = E_P(x_1)f$.

Thanks to above steps and Lemma 4.15(2), it is clear that for any $x_1 \in A_1$ there exists a unique $y_1 \in P$ such that $fx_1f = y_1f$. Define $\phi : A_1 \to P$ by $\phi(x_1) = y_1$. We show that $\phi = E_P$. 


To see that $\phi$ is positive, assume that $x_1$ is positive, that is, $x_1 = s^*s$ for some element $s = \sum_{i=1}^n a_i e_i b_i \in A_1$ for some $a_i, b_i \in A$. Then, as in Step I, we get $y_1 = \sum_{i,j} \tau^{1/2} b_i^* e E_0 (a_i^* a_j) e b_j$. Since a conditional expectation is always completely positive, we have $y_1 \geq 0$.

Next, we show that $\phi$ is a retraction on $P$. This is obvious. If $x_1 \in P$, then by the definition of $P$, we must have $fx_1f = x_1f$; so, by the uniqueness of $y_1$, we immediately get $y_1 = x_1 \in P$. In other words, $\phi(x_1) = x_1$ for all $x_1 \in P$. In particular, $\|\phi\| = \|\phi(1_{A_1})\| = \|1_P\| = 1$.

Finally, we show that $\phi$ is a $P$-bimodule map. This also follows trivially. Let $c_1, c_2 \in P$. By the definition of $P$, we have $f \in P' \cap A_2$; so, for each $x_1 \in A_1$, we see that

$$\phi(c_1 x_1 c_2)f = f c_1 x_1 c_2 f = c_1 f x_1 f c_2 = c_1 \phi(x_1) f c_2 = c_1 \phi(x_1) c_2 f.$$ 

Hence, by the definition of $\phi$, we obtain $\phi(c_1 x_1 c_2) = c_1 \phi(x_1) c_2$.

Thus, $\phi$ implements a conditional expectation from $A_1$ onto $P$. Since $B \subset A$ is irreducible, we must have $\phi = E_P$, by Theorem 2.5. Hence, $fx_1f = E_P(x_1)f$ as desired.

In order to show that $P = AeA$, first note that, $AeAf = AeA = (ee_2)eA$. Then, by equation (4.4), we obtain

$$A(ee_2)eA = A(fe_1f)A = fA_1f = E_P(A_1)f = Pf.$$ 

Thus, $(AeA)f = Pf$ and hence $AeA = P$.

Finally, we show that $[F(e)] = e_P$.

First note that,

$$F_1^{-1}(e_P) = \tau^{-3/2} E_2(\tau e_1 e_2)$$

$$= \tau^{-3/2} E_2(\tau e_1 e_2)$$

$$= \tau^{-3/2} E_2(E_P^{A_1}(e_1)e_2)$$

$$= \tau^{-1/2} E_P^{A_1}(e_1).$$

On the other hand,

$$F_1(E_P^{A_1}(e_1)) = \tau^{-3/2} E_{A_1}^{B' \cap A_2}(E_P^{A_1}(e_1)e_2e_1)$$

$$= \tau^{-3/2} E_{A_1}^{B' \cap A_2}(E_P^{A_1}(e_1)e_2e_1)$$

$$= \tau^{-3/2} E_{A_1}^{B' \cap A_2}(fe_1 e_2 e_1)$$

$$= \tau^{-1/2} E_{A_1}^{B' \cap A_2}(fe_1)$$

$$= \tau^{1/2} f.$$ 

Hence, $f = e_P$. This completes the proof. \qed

**Remark 4.25.** We do not know at present whether the biprojection ‘$e$’ corresponds to an intermediate $C^*$-subalgebra of $B \subset A$ or not. A natural candidate would be $C := \{e\}' \cap A$. But we cannot see how $e$ will implement the conditional expectation from $A$ onto $C$.

Further we would like to know whether $P$ is the basic construction of the pair $C \subset A$ or not? We feel that this may not be plausible because of possible $K$-theoretical obstruction. It will be interesting to analyze this in detail.

Applying Remark 4.12 and Theorem 4.24 we get the following.
COROLLARY 4.26. If \( e \in \text{BP}_1(B, A) \), then \( f(\{F(e)\}) \in \text{BP}_1(A, A_1) \), where \( e \) is as in Theorem 4.24.

5. An angle between intermediate \( C^* \)-subalgebras

In this section, motivated by [2], we introduce the notions of interior and exterior angles between any two intermediate \( C^* \)-subalgebras \( C \) and \( D \) of a given inclusion \( B \subset A \) of unital \( C^* \)-algebras with a finite index conditional expectation. As mentioned in the Introduction, a significant application of the notion of angle in [2] was to better Longo’s bound for the cardinality of the lattice of intermediate subfactors of type \( II_1 \), thereby answering a question of Longo. On similar lines, toward the end of this section, we exploit our notion of interior angle and some aspects of the \( C^* \)-Fourier theory that we developed to

(a) obtain a bound for the cardinality of the set of intermediate \( C^* \)-subalgebras of an irreducible inclusion of simple unital \( C^* \)-algebras and

(b) improve Longo’s upper bound for the cardinality of intermediate subfactors of an irreducible subfactor of type \( III \).

5.1. Interior and exterior angles between intermediate \( C^* \)-subalgebras

Let \( B \subset A \) be an inclusion of unital \( C^* \)-algebras with a conditional expectation \( E : A \to B \) of finite index. By Lemma 2.11, \( A_1 \) is a Hilbert \( A \)-module with respect to the \( A \)-valued inner product \( (a_1, b_1)_A := E(a_1^* b_1) \) for \( a_1, b_1 \in A_1 \).

As in [15], let \( \text{IMS}(B, A, E) \) denote the set of all intermediate \( C^* \)-subalgebras \( C \) of \( B \subset A \) with a conditional expectation \( F : A \to C \) such that \( E_{(a)} \circ F = E \). Then, as in Lemma 4.2, it is easily seen that \( C_1 \subset A_1 \). For any pair \( C, D \in \text{IMS}(B, A, E) \), let \( e_C \) and \( e_D \) denote the corresponding Jones projections in \( C_1 \) and \( D_1 \), respectively. Then, by Cauchy–Schwarz inequality (see [38]), we have

\[
\| (e_C - e_B, e_D - e_B)_A \| \leq \| e_C - e_B \|_A \| e_D - e_B \|_A. \tag{5.1}
\]

Based on this, we propose the following:

DEFINITION 5.1. Let \( (B, C, D, A) \) be a quadruple of \( C^* \)-algebras as above.

1. The interior angle between \( C \) and \( D \), denoted by \( \alpha^B_A(C, D) \), is defined by the expression

\[
\cos (\alpha^B_A(C, D)) = \frac{\| (e_C - e_B, e_D - e_B)_A \|}{\| e_C - e_B \|_A \| e_D - e_B \|_A}.
\]

2. The exterior angle (or dual angle) between \( C \) and \( D \), denoted by \( \beta^B_A(C, D) \), is defined as the interior angle between \( C_1 \) and \( D_1 \), that is,

\[
\beta^B_A(C, D) := \alpha^A_A(C_1, D_1).
\]

We take the value of \( \alpha \) (and, hence, of \( \beta \)) only in the interval \([0, \pi/2] \).

REMARK 5.2. (1) It is clear that the definition of \( \alpha(C, D) \) (and \( \beta(C, D) \)) depends on the conditional expectations from \( A \) onto \( C \) and \( D \). However, when \( B \subset A \) is an irreducible inclusion of simple unital \( C^* \)-algebras, then by Theorem 2.5, there will be unique (minimal) conditional expectations from \( A \) onto \( C \) and \( D \) and hence there would not be any ambiguity.

(2) If \( B \subset A \) is a pair of simple unital \( C^* \)-algebras and \( C \) and \( D \) are both simple, then one can work with the minimal conditional expectations.
(3) If $B \subset A$ is an irreducible inclusion of simple unital $C^*$-algebras such that $\mathcal{E}_0(A, B) \neq \emptyset$ and $E^B_0 : A \to B$ is the unique minimal conditional expectation. Then, by Lemma 2.9, IMS($B, A, E^B_0$) consists of all intermediate $C^*$-subalgebras of $B \subset A$.

(4) For a subfactor of type $II_1$ factor with finite Jones index, the interior angle defined here is different from that in [2]. Recall that the trace preserving conditional expectation need not be the minimal conditional expectation.

By a quadruple $\mathcal{G} = (B, C, D, A)$ of unital $C^*$-algebras, we shall mean that $A$ is a unital $C^*$-algebra with unital $C^*$-subalgebras $B, C$ and $D$ such that $B \subset C \cap D$. The quadruple $\mathcal{G}$ is said to be irreducible if $B \subset A$ is an irreducible inclusion.

We will be interested only in analyzing intermediate $C^*$-subalgebras of simple inclusions.

### 5.1.1. Angle between intermediate $C^*$-subalgebras of a simple unital inclusion

Throughout this subsection, $\mathcal{G} = (B, C, D, A)$ will denote a quadruple of simple unital $C^*$-algebras such that $\mathcal{E}_0(A, B) \neq \emptyset$. We first list some notations that will be used ahead.

**Notation 5.3.**

1. We denote the corresponding unique minimal conditional expectations by $E^A_C, E^B_D$ and $E^A_B$ (see Theorem 2.6).
2. We let $e_C$ and $e_D$ denote the Jones projections corresponding to $E^A_C$ and $E^A_B$, respectively.
3. As in Lemma 4.2, $C_1$ and $D_1$ are both simple and contained in $A_1$; the quadruple $\tilde{\mathcal{G}} := (A, C_1, D_1, A_1)$ will be called the dual quadruple.
4. Using multiplicativity of the Watatani index (Theorem 2.8), it is clear that $\frac{[C: B]_0}{[D: B]_0} = \frac{[D, B]_0}{[A: D]_0}$. We denote this common value by $r$. If $r = 1$, we call the quadruple a parallelogram.
5. The quadruple $\mathcal{G}$ will be called a commuting square if $E^A_C E^A_D = E^B_D E^B_C = E^A_B$. And $\bar{\mathcal{G}}$ will be called a co-commuting square if the dual quadruple $\tilde{\mathcal{G}}$ is a commuting square.

**Remark 5.4.** As in [2], it can be seen easily that $\alpha(C, D) = \pi/2$ if and only if $(B, C, D, A)$ is a commuting square. The dual statement holds similarly; thus, $\beta(C, D) = \pi/2$ if and only if $(B, C, D, A)$ is a co-commuting square.

We may also do a $C^*$-algebraic version of the theory developed in [2, Section 2]. The details are similar and left to the interested readers.

We now provide some useful expressions for the interior and exterior angles.

**Proposition 5.5.** We have

$$\cos \left( \alpha^B_A(C, D) \right) = \frac{r}{\sqrt{|A: C|_0 - 1} \sqrt{|A: D|_0 - 1}} \cos \left( \beta^B_A(C, D) \right) + \frac{r - 1}{\sqrt{|C: B|_0 - 1} \sqrt{|D: B|_0 - 1}}.$$

**Proof.** Let $v_C := \frac{e_C - e_1}{\|e_C - e_1\|_2}$, where $\| \cdot \|_2$ is defined with respect to the tracial state $tr$ on $B' \cap A_1$ as in Proposition 2.21. Then, applying Proposition 4.6, we get

$$\mathcal{F}(v_C) = \frac{\sqrt{|A: B|_0} e_C - \frac{r - 1}{\sqrt{|A: B|_0}} 1_{A_1}}{\|e_C - e_1\|_2}.$$

By definition, $\cos(\alpha^B_A(C, D)) = \langle v_C, v_D \rangle$. So, by Theorem 3.5, we obtain

$$\cos \left( \alpha^B_A(C, D) \right) = \langle \mathcal{F}(v_C), \mathcal{F}(v_D) \rangle.$$
Again apply Proposition 5.5 to obtain the desired result. Details are obvious.

\[
\begin{align*}
&= \frac{1}{\|e_C - e_1\|_2 \|e_D - e_1\|_2} \left( [A : B]_0 e_{C_1} - \frac{1}{\sqrt{|A : B|}_0} 1_{A_1}, [A : B]_0 e_{D_1} - \frac{1}{\sqrt{|A : B|}_0} 1_{A_1} \right) \\
&= \frac{1}{\|e_C - e_1\|_2 \|e_D - e_1\|_2} \left( r(e_{C_1}, e_{D_1}) - \frac{1}{|A : D|}_0 \text{tr}(e_{D_1}) - \frac{1}{|A : C|}_0 \text{tr}(e_{C_1}) + \frac{1}{|A : B|}_0 \right) \\
&= \frac{1}{\|e_C - e_1\|_2 \|e_D - e_1\|_2} \left( r \cos(\beta_A(C, D)) \|e_{C_1} - e_2\|_2 \|e_{D_1} - e_2\|_2 - \frac{1}{|A : B|}_0 + \frac{r}{|A : B|}_0 \right) \\
&= r \frac{\sqrt{|A : C|}_0 - 1/\sqrt{|A : D|}_0 - 1}{\sqrt{|C : B|}_0 - 1/\sqrt{|D : B|}_0 - 1} \cos(\beta_A(C, D)) + \frac{r - 1}{\sqrt{|C : B|}_0 - 1/\sqrt{|D : B|}_0 - 1}. \quad \Box
\end{align*}
\]

From the preceding proposition, the following three results follow easily. We leave it to the reader to check the details. These results have been mentioned for $I_1$ factors in [2] and [3].

The following result says that if a commuting square (respectively, co-commuting square) is a parallelogram, then it must be a co-commuting square (respectively, commuting square).

**Corollary 5.6.** If $\beta_A(C, D) = \pi/2$, then
\[
\cos(\alpha_A(C, D)) = \frac{r - 1}{\sqrt{|C : B|}_0 - 1/\sqrt{|D : B|}_0 - 1}.
\]

On the other hand, if $\alpha = \pi/2$, then
\[
\cos(\beta_A(C, D)) = \frac{1 - r}{\sqrt{|A : C|}_0 - 1/\sqrt{|A : D|}_0 - 1}.
\]

In particular, the preceding result says that if $(B, C, D, A)$ is a commuting square, then $|A : D| \geq |C : B|_0$ (and so, $|A : C|_0 \geq |D : B|_0$). On the other hand for a co-commuting square $|A : D|_0 \leq |C : B|_0$ (and so, $|A : C|_0 \leq |D : B|_0$).

**Corollary 5.7.** If $(B, C, D, A)$ is a parallelogram (that is, if $r = 1$), then $\alpha_A(C, D) = \beta_A(C, D)$.

In view of the above corollary we can interpret the interior and exterior angles as opposite angles of a parallelogram. By definition, $\cos(\beta_A(C_1, D_1)) = \cos(\alpha_A(C_2, D_2))$. However, we can say the following.

**Corollary 5.8.** $\cos(\alpha_A(C, D)) = \cos(\beta_A(C_1, D_1))$.

**Proof.** Apply Proposition 5.5 for the quadruple $(A, C_1, D_1, A_1)$ to get
\[
\cos(\beta_A(C, D)) = \frac{1}{r} \sqrt{|C : B|}_0 - 1/\sqrt{|D : B|}_0 - 1 \cos(\beta_A(C_1, D_1))
\]
\[
+ \frac{1 - r}{\sqrt{|A : C|}_0 - 1/\sqrt{|A : D|}_0 - 1}. \quad \Box
\]

Again apply Proposition 5.5 to obtain the desired result. Details are obvious.

The above result justifies the name ‘dual angle.’

**Remark 5.9.** Corollary 5.8 can also be independently proved using Theorem 3.18.
A priori, it not clear why \( \alpha(C, D) = 0 \) if and only if \( C = D \). However, when the pair \( B \subset A \) is irreducible, we have the following proposition.

**Proposition 5.10.** Let \((B, C, D, A)\) be an irreducible quadruple of simple unital \(C^*\)-algebras such that \( \mathcal{E}_0(A, B) \neq \emptyset \). Then,

1. \( \alpha(C, D) = 0 \) if and only if \( C = D \) and
2. the interior and exterior angles between \( C \) and \( D \) are given by

\[
\cos(\alpha^B_A(C, D)) = \frac{\text{tr}(e_Ce_D) - \tau}{\sqrt{\text{tr}(e_C) - \tau\text{tr}(e_D) - \tau}}, \quad \text{and} \tag{5.2}
\]
\[
\cos(\beta^B_A(C, D)) = \frac{\text{tr}(e_Ce_D) - \text{tr}(e_C)\text{tr}(e_D)}{\sqrt{\text{tr}(e_C) - \text{tr}(e_C)^2}\sqrt{\text{tr}(e_D) - \text{tr}(e_D)^2}}. \tag{5.3}
\]

**Proof.** (1) By Proposition 2.16, \( B' \cap A_1 \) is finite dimensional; so, it is a Hilbert space with respect to the inner product induced by the tracial state \( \text{tr} \). Further, if \( a_1, b_1 \in B' \cap A_1 \), then \( E_1(a_1^*b_1) \in B' \cap A = \mathbb{C} \) and, therefore, \( \langle a_1, b_1 \rangle_A = E_1(a_1^*b_1) = E_0 \circ E_1(a_1^*b_1) = \text{tr}(a_1^*b_1) = \langle a_1, b_1 \rangle_1 \). One then easily deduces that \( \alpha(C, D) = 0 \) if and only if \( C = D \). See, for instance, [2, Proposition 2.3].

(2) The formulas for angles follow easily (as in [2]) and we omit the details. For example, formula for \( \alpha \) follows from the fact that \( E_A(e_Ce_D - e_B) \in B' \cap A = \mathbb{C} \).

\[ \square \]

We now proceed to apply our notions of \( C^* \)-Fourier theory and interior angle to obtain a bound on the cardinality of intermediate \( C^* \)-subalgebras of a given inclusion of simple \( C^* \)-algebras. Note that if the pair \( B \subset A \) is not irreducible, then its intermediate \( C^* \)-algebras are not necessarily simple and conjugating by unitaries of \( B' \cap A \), we obtain infinitely many intermediate \( C^* \)-subalgebras from any given one. So, we will now restrict our analysis to irreducible pairs only.

First, we develop some auxiliary results (which are also of independent interest) that will be required ahead.

### 5.2. Two auxiliary operators associated to an irreducible quadruple of \( C^* \)-algebras

Throughout this subsection, \( \mathcal{G} = (B, C, D, A) \) will be assumed to be an irreducible quadruple of simple unital \( C^* \)-algebras such that \( \mathcal{E}_0(A, B) \neq \emptyset \).

Fix any two quasi-bases \( \{\gamma_j : 1 \leq j \leq m\} \) and \( \{\delta_k : 1 \leq k \leq l\} \) for \( E^C_B \) and \( E^D_B \), respectively. We wish to show that the two positive elements

\[
\sum_{j,k} \gamma_j \delta_k e_B \delta_k^* \gamma_j^* \quad \text{and} \quad \sum_{j,k} \delta_k \gamma_j^* e_B \gamma_j^* \delta_k^*
\]

remain same even when we vary the quasi-bases. Similar operators have also been studied and used in [2–4].

**Lemma 5.11.** We have

\[
\sum_k \delta_k e_B \delta_k^* = e_D \quad \text{and} \quad \sum_j \gamma_j e_B \gamma_j^* = e_C.
\]

In particular, \( \sum_{j,k} \gamma_j \delta_k e_B \delta_k^* \gamma_j^* \in D_1 \) and \( \sum_{j,k} \delta_k \gamma_j e_B \gamma_j^* \delta_k^* \in C_1 \).
Proof. For any $a \in A$, we have
\[
\sum_k \delta_k e_B \delta^*_k(a) = \sum_k \delta_k E_B^A(\delta^*_k a) \\
= \sum_k \delta_k E_B^D \circ E_D^A(\delta^*_k a) \\
= \sum_k \delta_k E_B^D(\delta^*_k E_A^D(a)) \\
= E_B^A(a) \quad \text{(since } \{\delta_k\} \text{ is a quasi-basis for } E_B^D) \\
= e_D(a).
\]

Similarly, for $e_C$. □

**Proposition 5.12.** We have

1. $E_{B \cap A_1}(e_D) = \left[ C : B \right]^{-1} \sum_{j,k} \gamma_j \delta_k e_B \delta^*_k \gamma_j^*$ and
2. $E_{D \cap A_1}(e_C) = \left[ D : C \right]^{-1} \sum_{j,k} \delta_k \gamma_j e_B \gamma^*_j \delta_k^*$.

In particular, $\sum_{j,k} \gamma_j \delta_k e_B \delta^*_k \gamma_j^* \in C' \cap D_1$, $\sum_{j,k} \delta_k \gamma_j e_B \gamma^*_j \delta_k^* \in D' \cap C_1$ and they are independent of the quasi-bases $\{\gamma_j\}$ and $\{\delta_k\}$.

Proof. (1) In view of Lemma 5.11, it suffices to generalize Lemma 2.23 as follows.

We show that
\[
E_{C \cap A_k}(x) = \left[ C : B \right]^{-1} \sum_j \gamma_j x \gamma^*_j \quad \text{for all } x \in B' \cap A_k.
\]

It is easy to see that $\sum_j \gamma_j x \gamma^*_j \in C' \cap A_k$. Indeed, for any $c \in C$, we have
\[
\sum_j \gamma_j x \gamma^*_j c = \sum_{j,j'} \gamma_j x E_B^C(\gamma_j^* c \gamma_{j'}) \gamma_{j'} \\
= \sum_{j,j'} \gamma_j E_B^C(\gamma_j^* c \gamma_{j'}) x \gamma_{j'} \\
= c \sum_j \gamma_j x \gamma_{j'}^*.
\]

So, it now suffices to show that $\text{tr}(\sum_j \gamma_j x \gamma^*_j y) = \left[ B : C \right]_{B_0} \text{tr}(x y)$ for all $y \in C' \cap A_k$. For any such $y$, we have
\[
\text{tr} \left( \sum_j \gamma_j x \gamma^*_j y \right) = E_0 \circ E_1 \circ \cdots \circ E_k \left( \sum_j \gamma_j x \gamma^*_j y \right) \\
= E_0 \circ E_1 \circ \cdots \circ E_k \left( \sum_j \gamma_j x y \gamma^*_j \right) \quad \text{(since } y \in C') \\
= E_0 \left( \sum_j \gamma_j (E_1 \circ \cdots \circ E_k(x y)) \gamma^*_j \right) \\
= \left[ B : C \right]_{B_0} \text{tr}(x y),
\]
where the last equality follows because $E_1 \circ \cdots \circ E_k(xy) \in B' \cap A = \mathbb{C}$ and is thus equal to $\text{tr}(xy)$. Thus, equation (5.4) holds.

(2) follows by symmetry.

**Definition 5.13.** We define two positive elements $p(C, D)$ and $q(C, D)$ in $C' \cap D_1$ and $D' \cap C_1$, respectively, by

$$p(C, D) = \sum_{i,j} \gamma_i \delta_j e_B \delta_j^* \gamma_i^* \quad \text{and} \quad q(C, D) = \sum_{i,j} \delta_i \gamma_i e_B \gamma_i^* \delta_j^*$$

for any two quasi-bases $\{\gamma_i : 1 \leq i \leq m\}$ and $\{\delta_j : 1 \leq j \leq n\}$ for $E_B^C$ and $E_B^D$, respectively.

Interestingly, these two auxiliary operators get mapped to each other under the rotation on $B' \cap A_1$.

**Proposition 5.14.** $\gamma_0(p(C, D)) = q(C, D)$ and $\gamma_0(q(C, D)) = p(C, D)$.

**Proof.** As before, suppose $\{\lambda_i : 1 \leq i \leq n\}$, $\{\gamma_j : 1 \leq j \leq m\}$ and $\{\delta_k : 1 \leq k \leq l\}$ are quasi-bases for $E_B^A$, $E_B^C$, $E_B^D$, respectively. We have

$$\gamma_0(p(C, D)) = \tau^{-1} \sum_i E_1 \left( e_1 \lambda_i \left( \sum_j \gamma_j e_D \gamma_j^* \right) \right) e_1 \lambda_i^*$$

$$= \tau^{-1} \sum_{i,j} E_1(e_1 e_D \lambda_i \gamma_j e_D \gamma_j^*) e_1 \lambda_i^*$$

$$= \tau^{-1} \sum_{i,j} E_1(e_1 E_D^A(\lambda_i \gamma_j) \gamma_j^*) e_1 \lambda_i^*$$

$$= \sum_{i,j} E_D^A(\lambda_i \gamma_j) \gamma_j^* e_1 \lambda_i^*$$

$$= \sum_{i,j, k} \delta_k E_D^B(\delta_k^* E_D^A(\lambda_i \gamma_j)) \gamma_j^* e_1 \lambda_i^*$$

$$= \sum_{i,j, k} \delta_k E_D^B \circ E_D^A(\delta_k^* \lambda_i \gamma_j) \gamma_j^* e_1 \lambda_i^*$$

$$= \sum_{i,j, k} \delta_k E_D^B \circ E_D^C(\delta_k^* \lambda_i \gamma_j) \gamma_j^* e_1 \lambda_i^*$$

$$= \sum_{i,j, k} \delta_k E_D^B(\delta_k^* \lambda_i) \gamma_j^* e_1 \lambda_i^* \quad \text{(since $\{\gamma_j\}$ is a quasi-basis for $E_B^C$)}$$

$$= \sum_{i,k} \delta_k E_D^A(\delta_k^* \lambda_i) e_1 \lambda_i^*$$

$$= \sum_{i,k} \delta_k E_D^A(\delta_k^* \lambda_i) e_C e_1 \lambda_i^*$$

$$= \sum_{i,k} \delta_k e_C \delta_k^* \lambda_i e_1 \lambda_i^*$$
\[\sum_k \delta_k e_C \delta_k^* = q(C, D).\]

Then, on the other hand, applying Lemma 3.14, we also obtain \(\gamma_0(q(C, D)) = p(C, D). \square\)

We deduce some consequences that will be used ahead.

**Proposition 5.15.** There exists a positive scalar \(t\) such that

1. \(p(C, D)e_D = te_D\) and \([p(C, D)] = \frac{1}{t}p(C, D),\)
2. \(q(C, D)e_C = te_C\) and \([q(C, D)] = \frac{1}{t}q(C, D)\) and
3. \(p(C, D)e_C = te_C\) and \(q(C, D)e_D = te_D,\)

where \([x]\) denotes the support projection of \(x\).

Moreover, \(e_C \vee e_D \leq [p(C, D)] \wedge [q(C, D)]\) and \(t = [A : B]_0 \text{tr}(e_C e_D).\) In particular, \(e_C\) and \(e_D\) are never orthogonal to each other.

**Proof.** (1) Since \(p(C, D) \in D_1\), using Lemma 2.15, we have

\[p(C, D)e_D = [A : D]_0 E_{D_1}^D(p(C, D)e_D)e_D.\]

Since \(p(C, D)e_D \in B' \cap D_1\), we must have \(E_{D_1}^D(p(C, D)e_D) \in B' \cap A = C\). In other words, \(p(C, D)e_D = te_D\) for some scalar \(t\). Now, applying Proposition 5.12, we obtain

\[p(C, D)^2 = [C : B]_0 p(C, D) E_{D_1}^C \cap D_1 (e_D)\]
\[= [C : B]_0 E_{D_1}^C \cap D_1 (p(C, D)e_D)\]
\[= t[C : B]_0 E_{D_1}^C \cap D_1 (e_D)\]
\[= t p(C, D).\]

Since \(p(C, D)\) and \(p(C, D)^2\) are both positive and non-zero, we must have \(t > 0\). We also deduce that \([p(C, D)] = \frac{1}{t}p(C, D) \geq e_D\).

(2) By symmetry, \(q(C, D)\) also satisfies above properties. To show that the scalars have the same value, observe that, using Theorem 3.16, we obtain \(\gamma_0(p(C, D))^2 = \gamma_0(p(C, D)^2)\). Then, by applying Proposition 5.14, we get \(q(C, D)^2 = t q(C, D)\).

(3) We have \(q(C, D)e_C = te_C\). Thus, by applying \(\gamma_0\), we obtain \(te_C = e_C p(C, D) = p(C, D)e_C\), by Proposition 5.14 and Lemma 4.4.

From Item (2), we have \([p(C, D)] = \frac{1}{t} p(C, D) \geq e_D\) and \([q(C, D)] = \frac{1}{t} q(C, D) \geq e_C\); and from Item (3), we obtain \([p(C, D)] \geq e_C\) and \([q(C, D)] \geq e_D\). Thus, \(e_C \vee e_D \leq [p(C, D)] \wedge [q(C, D)]\).

Finally, from Item (3) and Lemma 5.11, we obtain \(te_C = p(C, D)e_C = \sum_i \gamma_i e_D e_C \gamma_i^*\); so that

\[\frac{t}{[A : C]_0} = t E_A^{A_1}(e_C) = \sum_i \gamma_i E_A^{A_1}(e_D e_C) \gamma_i^* = [C : B]_0 E_A^{A_1}(e_D e_C),\]

where the last equality follows from the facts that \(E_A^{A_1}(e_D e_C) \in B' \cap A = C\) and that \(\sum_i \gamma_i \gamma_i^* = [C : B]_0\). Hence, \(t = [A : B]_0 \text{tr}(e_C e_D). \square\)

We conclude this subsection with some useful expressions for the above auxiliary operators.
Proposition 5.16. We have

\[ p(C, D) = |D : B|_0 E_{D_1}^{A_1}(e_C) \quad \text{and} \quad q(C, D) = |C : B|_0 E_{C_1}^{A_1}(e_D). \]

In particular, \( \text{tr}(p(C, D)) = r = \text{tr}(q(C, D)) \).

Proof. To prove this we need the following general statement:

\[ \gamma_0 \left( E_{D_1}^{B_1 \cap A_1}(x) \right) = E_{D_1}^{A_1}(\gamma_0(x)) \quad \text{for any} \quad x \in B' \cap A_1. \tag{5.6} \]

To see this, first note that, by Proposition 4.9, \( \gamma_0 \left( E_{D_1}^{B_1 \cap A_1}(x) \right) \in B' \cap D_1 \) for all \( x \in B' \cap A_1 \). Now, let \( x_1 \in B' \cap D_1 \). Then, by Proposition 4.9 again, there exists a \( y_1 \in D' \cap A_1 \) such that \( \gamma_0(y_1) = x_1 \), so

\[
\begin{align*}
\text{tr} \left( \gamma_0 \left( E_{D_1}^{B_1 \cap A_1}(x_1) \right) \right) &= \text{tr} \left( \gamma_0 \left( E_{D_1}^{B_1 \cap A_1}(x_1) \right) \gamma_0(y_1) \right) \\
&= \text{tr} \left( \gamma_0 \left( y_1 E_{D_1}^{B_1 \cap A_1}(x_1) \right) \right) \quad \text{(by Theorem 3.16)} \\
&= \text{tr} \left( y_1 E_{D_1}^{B_1 \cap A_1}(x_1) \right) \quad \text{(by Lemma 3.17)} \\
&= \text{tr} \left( E_{D_1}^{B_1 \cap A_1}(y_1 x_1) \right) \\
&= \text{tr}(y_1 x_1) \\
&= \text{tr}(\gamma_0(x_1)). \\
\end{align*}
\]

This proves equation 5.6.

Now, by Proposition 5.12, we have \( p(C, D) = |C : B|_0 E_{C_1}^{B_1 \cap A_1}(e_D) \) and by, Proposition 5.14, we know that \( \gamma_0(p(C, D)) = q(C, D) \). Thus, applying equation 5.6, we obtain

\[ q(C, D) = |C : B|_0 E_{C_1}^{A_1}(\gamma_0(e_D)) = |C : B|_0 E_{C_1}^{A_1}(e_D), \]

by Lemma 4.4. The expression for \( p(C, D) \) follows by symmetry.

\[ \square \]

5.3. A bound for the cardinality of intermediate subalgebras

For any unital pair \( N \subset M \) of von Neumann algebras, let \( \mathcal{I}(N \subset M) \) denote the set of its intermediate von Neumann subalgebras. Then, \( \mathcal{I}(N \subset M) \) forms a lattice under the following two natural operations:

\[ P \land Q := P \cap Q \quad \text{and} \quad P \lor Q := (P \cup Q)^\prime. \]

If we assume that \( N \subset M \) is an irreducible subfactor (of any type), then \( \mathcal{I}(N \subset M) \) becomes the lattice of its intermediate subfactors. Watatani, in [49] (implicitly in [40]), showed that if \( N \subset M \) is a finite index irreducible subfactor of type \( II_1 \), then \( \mathcal{I}(N \subset M) \) is a finite lattice. Subsequently, Teruya and Watatani (in [47]) showed that \( \mathcal{I}(N \subset M) \) is finite also if \( N \subset M \) is a finite index irreducible subfactor of type \( III \).

On the other hand, if we consider a unital inclusion of \( C^* \)-algebras \( B \subset A \), the set of intermediates \( C^* \)-subalgebras, denoted by \( \mathcal{L}(B, A) \) (to distinguish it from the \( W^* \)-version), also forms a lattice under the following two operations:

\[ A \land B := A \cap B \quad \text{and} \quad A \lor B := C^*(A, B). \]

Recently, Ino and Watatani (in [15]) proved that \( \mathcal{L}(B, A) \) is finite if \( A \) and \( B \) are simple unital \( C^* \)-algebras with \( B' \cap A = \mathbb{C} \) and \( [A : B]_0 < \infty \). They did not provide any bound for
the cardinality of $\mathcal{L}(A, B)$. Below, we provide an upper bound for the cardinality of $\mathcal{L}(B, A)$. We also improve Longo’s bound for the cardinality of $\mathcal{I}(N \subset M)$ for any finite index irreducible subfactor of type $III$.

5.3.1. Intermediate $C^*$-subalgebras of an irreducible pair of simple unital $C^*$-algebras. We first observe a certain rigidity phenomenon among the minimal intermediate $C^*$-subalgebras as was discovered for the minimal subfactors of an irreducible subfactor of type $II_1$ in [2].

**Theorem 5.17.** Let $B \subset A$ be an irreducible inclusion of simple unital $C^*$-algebras with a conditional expectation $E : A \to B$ of finite Watatani index. Then, the interior angle between any two distinct minimal intermediate $C^*$-subalgebras $C$ and $D$ of $B \subset A$ is greater than $\pi/3$.

**Proof.** First, note that, by Theorem 2.5, $E$ is unique and hence is same as the minimal conditional expectation $E_0$. As usual, let $E_1$ denote the dual (also minimal) conditional expectation of $E_0$. Let $C$ and $D$ be two distinct minimal intermediate $C^*$-subalgebras of $B \subset A$. Then, the expression (5.2) for interior angle yields

$$
\cos (\alpha(C, D)) = \frac{\text{tr}(e_C e_D) - ([A : B]_0)^{-1}}{\sqrt{([A : C]_0)^{-1} - ([A : B]_0)^{-1} \sqrt{([A : D]_0)^{-1} - ([A : B]_0)^{-1}}}
$$

$$
= \frac{[A : B]_0 \text{tr}(e_C e_D) - 1}{\sqrt{[C : B]_0 - 1 \sqrt{[D : B]_0 - 1}}}
$$

$$
= \frac{t - 1}{\sqrt{[C : B]_0 - 1 \sqrt{[D : B]_0 - 1}}}
$$

where the last equality follows from Proposition 5.15. Also, we have $\text{tr}(p(C, D)) = r := \frac{|C : [B]_0|}{|A : [D]_0|}$, by Proposition 5.16. Thus, from Proposition 5.15, we obtain $\text{tr}(p(C, D)) = \frac{r}{\text{tr}(e_C e_D)}$.

Next, recall that the projections $e_C \lor e_D - e_C$ and $e_D - e_C \land e_D$ are Murray von Neumann equivalent in the finite-dimensional von Neumann algebra $B' \cap A_1$. Therefore,

$$
\text{tr}(e_C \lor e_D) = \text{tr}(e_C) + \text{tr}(e_D) - \text{tr}(e_C \land e_D).
$$

Since $C$ and $D$ are distinct minimal intermediate $C^*$-subalgebras, it is clear that $e_C \land e_D = e_B$. So, we have

$$
\frac{1}{t} \geq \frac{1}{[C : B]_0} + \frac{1}{[D : B]_0} - \frac{1}{[C : B]_0 [D : B]_0}.
$$

Thus, as in [2], we obtain

$$
\frac{t - 1}{\sqrt{[C : B]_0 - 1 \sqrt{[D : B]_0 - 1}}} \leq \frac{\sqrt{[C : B]_0 - 1 \sqrt{[D : B]_0 - 1}}}{[C : B]_0 + [D : B]_0 - 1}
$$

$$
< \frac{\sqrt{[C : B]_0 - 1 \sqrt{[D : B]_0 - 1}}}{[C : B]_0 - 1 + [D : B]_0 - 1}
$$

$$
\leq \frac{1}{2}.
$$

Therefore, $\alpha(C, D) > \pi/3$. This completes the proof. \hfill \Box

For an irreducible subfactor (of any type), Longo (in [35]) gave an explicit bound for the number of intermediate subfactors by showing that the number is bounded by $\ell^\ell$, where $\ell = [M : N]^2$. He then asked whether the number of intermediate subfactors could be bounded
by $[M : N]^{[M : N]}$. This question was settled for type $II_1$ subfactors in [2] using the notion of interior angle between intermediate subfactors.

Now that all the necessarily tools are available to us, analogous to the bound obtained by Longo [35], we first obtain a bound for the cardinality of the lattice of intermediate $C^*$-subalgebras of an irreducible pair $B \subset A$ of simple unital $C^*$-algebras and then answer Longo's question for type $II_1$ case.

The procedure that we employ is exactly the same as was employed in [2]. We provide an outline for the reader's convenience.

**Theorem 5.18.** Let $B \subset A$ be an irreducible inclusion of simple unital $C^*$-algebras with a conditional expectation of finite Watatani index. Then, the number of intermediate $C^*$-subalgebras of $B \subset A$ is bounded by

$$\min \left\{ 9^{[A : B]^2}, \left( \frac{[A : B]^2}{2} \right)^{[A : B]^2} \right\}.$$ 

**Proof.** Let $\mathcal{L}(B, A)$ (respectively, $\mathcal{L}_m(B, A)$) denote the set of all intermediate (respectively, minimal intermediate) $C^*$-subalgebras of $B \subset A$. Then, in view of Theorem 5.17, imitating the proof of [2, Theorem 4.1], we deduce that

$$|\mathcal{L}_m(B, A)| \leq 3^{\dim_{c}(B' \cap A)}.$$ 

From Proposition 2.26, we know that $\dim_{c}(B' \cap A) \leq [A : B]_0^2$. Thus, $|\mathcal{L}_m(B, A)| \leq 3^{[A : B]^2}$. Now, for any $\delta^2 \geq 2$, consider (as in [2, Definition 4.3])

$$I(\delta^2) := \sup \{ |\mathcal{L}(Q, P)| : Q \subset P \text{ is an irreducible inclusion of simple unital } C^* - \text{algebras with } [P : Q]_0 \leq \delta^2 \};$$

$$m(\delta^2) := \sup \{ |\mathcal{L}_m(Q, P)| : Q \subset P \text{ is an irreducible inclusion of simple unital } C^* - \text{algebras with } [P : Q]_0 \leq \delta^2 \}.$$ 

So, for any $\delta^2 \geq 2$, we have $m(\delta^2) \leq 3^{\delta^4}$. Further, since every $II_1$ factor is a simple unital $C^*$-algebra, on the lines of [2, Lemma 4.5], we must have $I(\delta^2) \leq m(\delta^2)I(\delta^2)/2$.

Finally, in view of Theorem 2.1, proceeding as in [2, Theorems 4.6 and 4.7], we obtain the desired bound. $\square$

### 5.3.2. Intermediate subfactors of an irreducible subfactor of type $III$

Recall that every $\sigma$-finite (equivalently, countably decomposable) type $III$ factor is known to be simple as a $C^*$-algebra. Also, if $N \subset M$ is a $\sigma$-finite subfactor of type $III$ and $P$ is an intermediate subfactor of $N \subset M$, then $P$ is also $\sigma$-finite and of type $III$; hence, $P$ is also a simple unital $C^*$-algebra.

Now, suppose that $N \subset M$ is a $\sigma$-finite irreducible subfactor of type $III$ with finite Watatani index. Then, by Theorem 2.5, it admits a unique (and hence minimal) conditional expectation, say, $E_N^M : M \to N$; and also $[M : N]_0 = \text{Ind}(E_N^M)$. Clearly, $E_N^M$ is faithful and, since $E_N^M$ satisfies the Pimsner–Popa inequality (Proposition 2.3), $E_N^M$ is normal as well, by [42, Proposition 1.1]. So, by [48, Proposition 2.5.3], $[M : N]_0$ is equal to the Kosaki index of $E_N^M$ (see [28]).

**Proposition 5.19.** Let $N \subset M$ be an irreducible $\sigma$-finite subfactor of type $III$ with finite Watatani index. Then, $\dim(N' \cap M_1) \leq [M : N]_0$, where $M_1$, denotes the Watatani’s $C^*$-basic construction for $N \subset M$ with respect to $E_N^M$.

**Proof.** As observed above, $E_N^M : M \to N$ is a faithful normal conditional expectation with finite Kosaki index. So, by [28], given any faithful normal state $\varphi$ on $N$, there is a projection
f ∈ \mathcal{N} ∩ B(\mathcal{H}) such that faf = E_N^M(a)f for all a ∈ M, where \mathcal{H} is the Hilbert space L^2(M, \varphi \circ E_N^M). Then, \tilde{M}_1 := \psi\mathcal{N}(M, f) ⊆ B(\mathcal{H}) is called the von Neumann basic construction of N ⊂ M with respect to E_N^M and \varphi. Further, since N and M are of type III and the Kosaki index of E_N^M is finite, it is known that dim_\mathbb{C}(N' ∩ \tilde{M}_1) ≤ [A : B]_0 — see, for example, [45]. Also, by [28, Lemmas 3.2 & 3.3], the mapping M ⊃ a ↦ faf ∈ M_1 is injective. Thus, by [48, Proposition 2.2.11] (uniqueness of \mathcal{C}^*-basic construction), there exists an injective ***-homomorphism \varphi : M_1 → \tilde{M}_1 such that \varphi(e_N) = f and \varphi(a) = a for all a ∈ M. In particular, \varphi(M_1) = \text{span}\{xfy : x, y ∈ M\} and \varphi maps \mathcal{N}' ∩ M_1 injectively into \mathcal{N}' ∩ \tilde{M}_1. Hence,
\[
\text{dim}_\mathbb{C}(N' ∩ \tilde{M}_1) ≤ \text{dim}_\mathbb{C}(N' ∩ M_1) ≤ [M : N]_0.
\]
This completes the proof.

**Theorem 5.20.** Let N ⊂ M be an irreducible \(\sigma\)-finite subfactor of type III with finite Watatani index. Then, the number of intermediate subfactors of N ⊂ M is bounded by
\[
\min\left\{9^{[M : N]_0}, [M : N]_0^{[M : N]_0}\right\}.
\]

**Proof.** As observed above, every intermediate subfactor of N ⊂ M is a simple unital \mathcal{C}^*-subalgebra.

Let \mathcal{I}(N ⊂ M) (respectively, \mathcal{I}_m(N ⊂ M)) denote the set of all intermediate (respectively, minimal intermediate) subfactors of N ⊂ M. Then, in view of Theorem 5.17, imitating the proof of [2, Theorem 4.1], we deduce that
\[
|\mathcal{I}_m(N ⊂ M)| ≤ 3^{\text{dim}_\mathbb{C}(N' ∩ M_1)}.
\]
Thus, by Proposition 5.19, \(|\mathcal{I}_m(N ⊂ M)| ≤ 3^{[M : N]_0}\). Now, for any \(\delta^2 ≥ 2\), consider (as in [2, Definition 4.3])
\[
I(\delta^2) := \sup\{|\mathcal{I}(K ⊂ L)| : K ⊂ L \text{ is a } \sigma\text{-finite irreducible subfactor of type III with } [L : K]_0 ≤ \delta^2\}; \text{ and}
\]
\[
m(\delta^2) := \sup\{|\mathcal{I}_m(K ⊂ L)| : K ⊂ L \text{ is a } \sigma\text{-finite irreducible subfactor of type III with } [L : K]_0 ≤ \delta^2\}.
\]
So, for any \(\delta^2 ≥ 2\), we have \(m(\delta^2) ≤ 3^{\delta^2}\).

Furthermore, there always exists a \(\sigma\)-finite hyperfinite factor of type III which admits an outer action of every finite group; thus, imitating the proof of [2, Lemma 4.5], we obtain \(I(\delta^2) ≤ m(\delta^2)I(\delta^2/2)\).

Finally, in view of Theorem 2.1, proceeding as in [2, Theorems 4.6 and 4.7], we obtain the desired bound. □

6. Lattice of intermediate von Neumann subalgebras

Let \mathcal{N} ⊂ \mathcal{M} be a unital inclusion of von Neumann algebras. For any such pair, as above, let \mathcal{I}(\mathcal{N} ⊂ \mathcal{M}) denote the lattice of intermediate von Neumann subalgebras. The main theorem of this section will show that, for a fairly large class of such pairs, the lattice \mathcal{I}(\mathcal{N} ⊂ \mathcal{M}) is always finite.

In order to achieve this, we will use the notion of a metric between two subalgebras of a given \mathcal{C}^*-algebra introduced by Kadison and Kastler (in [23]) and Christensen’s theory of
perturbations of operator algebras based on this metric. Recall that if $B$ and $C$ are two $C^*$-subalgebras of a $C^*$-algebra $A$, then the (Kadison–Kastler) distance between $B$ and $C$ is defined as

$$d(B, C) = \max \left\{ \sup_{a \in \text{ball}(B)} \inf_{b \in \text{ball}(C)} \|a - b\|, \sup_{b \in \text{ball}(C)} \inf_{a \in \text{ball}(B)} \|a - b\| \right\}.$$ 

The following useful elementary observation is well known — see, for instance, [14].

**Lemma 6.1.** Let $B$ and $C$ be $C^*$-subalgebras of a $C^*$-algebra $A$. If $B \subset C$ and $d(B, C) < 1$, then $B = C$.

**Notation 6.2.** Let $N \subset M$ be a unital inclusion of finite von Neumann algebras with a (fixed) faithful normal tracial state $\text{tr}$ on $M$. Let $E^M_N : M \to N$ denote the unique tr-preserving faithful normal conditional expectation. Also, when we restrict $\text{tr}$ to $P$, we obtain another unique tr-preserving normal conditional expectation $E^P_N : P \to N$ and we have $E^P_N \circ E^M_P = E^M_N$.

**Proposition 6.3.** In the setup of Notation 6.2, suppose that $E^M_N$ has finite Watatani index. Then, the conditional expectations $E^P_N$ and $E^P_M$ also have finite Watatani index.

**Proof.** That $E^P_N$ has finite index follows from [48, Proposition 1.7.2]. And that $E^P_M$ has finite index follows from [36, Proposition 3.5]. \qed

We now prove the main result of this section, which generalizes [49, Theorem 2.2]. We will break the proof into two steps. First, combining Christensen’s perturbation technique from [8] and an improvement by Ino [14], we show that if the distance between two intermediate von Neumann subalgebras $P$ and $Q$ is sufficiently small, then they are unitarily equivalent. Then, following an idea of Watatani [49] (see also [15]), we use a compactness argument combined with the first step to conclude that there are only finitely many intermediate von Neumann subalgebras.

**Theorem 6.4.** Let $N \subset M$ be a unital inclusion of finite von Neumann algebras with a normal tracial state $\text{tr}$ on $M$ such that the unique tr-preserving conditional $E^M_N : M \to N$ has finite Watatani index. If $N$ has finite-dimensional center and $N' \cap M$ equals either $\mathcal{Z}(N)$ or $\mathcal{Z}(M)$, then the lattice $\mathcal{I}(N \subset M)$ is finite.

**Proof.** Step I: Following [14] and [8], we show that, for every pair $P, Q \in \mathcal{I}(N \subset M)$ with $d(P, Q) < 1/15$, there exists a unitary $u$ in $N' \cap M$ such that $uPu^* = Q$.

From Notation 6.2 and Proposition 6.3, we see that the conditional expectations $E^M_P : M \to P$ and $E^M_Q : M \to Q$ both have finite index. So, they satisfy the Pimsner–Popa inequality (Proposition 2.3). Thus, by [14, Proposition 3.1], there exists a $*$-isomorphism $\Phi : Q \to P$ such that $\Phi \big|_{N'} = \text{Id}_{N'}$ and

$$\sup_{x \in \text{ball}(Q)} \|\Phi(x) - x\| < 14d(P, Q) < 1. \quad (6.1)$$

Then, in view (6.1), there exists a unitary $u \in M$ such that $\Phi(x) = uxu^*$ for all $x \in Q$, by [8, Proposition 4.4]. And, since $\Phi \big|_{N'} = \text{Id}_{N'}$, it follows that $u \in N' \cap M$.

Step II: We show that $\mathcal{I}(N \subset M)$ is finite.

We will again use Watatani’s notion of $C^*$-basic construction. Let $N \subset M \subset C^*(M, e_N)$, $P \subset M \subset C^*(M, e_P)$ and $Q \subset M \subset C^*(M, e_Q)$ denote the respective $C^*$-basic constructions with the corresponding $C^*$-Jones projections $e_N$, $e_P$ and $e_Q$, respectively. Since $\text{Ind}(E^M_N)$ is
invertible [48, Lemma 2.3.1], the dual conditional expectation $E^{C^*\langle M, e_M \rangle}_{N} : C^*\langle M, e_M \rangle \to M$ of $E^M_{N}$ exists and has finite index, by [48, Propositions 1.6.1 & 1.6.6]; so that, $E^M_{N} \circ E^{C^*\langle M, e_M \rangle}_{N} : C^*\langle M, e_M \rangle \to N'$ also has finite index. Thus, since $Z(N)$ is finite dimensional, the relative commutant $N' \cap C^*\langle M, e_M \rangle$ is finite dimensional, by [48, Proposition 2.7.3]. Hence, the set

$$S := \{ p \in N' \cap C^*\langle M, e_M \rangle : p \text{ is a projection} \}$$

is a compact Hausdorff space with respect to the norm topology. So, for any $r > 0$, there exist finitely many open balls of diameter $r$ which cover $S$.

Fix any $0 < r < \frac{1}{15\|\text{Ind}(E^N_{M})\|}$. If $e_P$ and $e_Q$ both lie in same such ball, then $\|e_P - e_Q\| < r$; and, following Notation 6.2 and Proposition 6.3, we have $\text{IMS}(N, M, E^M_{N}) = \mathcal{I}(N \subset M)$; so, by [15, Lemma 3.3], we obtain $d(P, Q) < 1/15$. Thus, by Step I, there exists a unitary $u \in N' \cap M$ such that $uPQ = Q$. Then, either $N' \cap M \subseteq N' \subseteq Q$ or $N' \cap M = Z(M) \subset M'$ or $Q'$, in both cases, we get $P = Q$. Thus, there are only finitely many intermediate von Neumann subalgebras of the pair $N' \subset M$. This completes the proof of the theorem. \hfill \Box

Recall that, for any unital inclusion $N \subset M$ of finite von Neumann algebras, a representation $\pi$ of $M$ on a Hilbert space $H$ is said to be a finite representation of the pair $N \subset M$ if $\pi(N)' \subseteq B(H)$ is a finite von Neumann algebra. And the pair $N' \subset M$ is said to be of finite GHJ index if it admits a finite faithful representation — see [11, §3.5].

**Corollary 6.5.** Let $N \subset M$ be a unital inclusion of finite direct sums of finite factors with finite GHJ index. If $N$ has finite-dimensional center and either $N' \cap M = Z(N)$ or $N' \cap M = Z(M)$, then the lattice $\mathcal{I}(N' \subset M)$ is finite.

**Proof.** Fix a faithful normal tracial state $\text{tr}$ on $M$. Then, by [11, Theorem 3.6.4], the unique $\text{tr}$-preserving conditional expectation $E^M_{N} : M \to N$ has finite Watatani index. The rest follows from Theorem 6.4. \hfill \Box

**Corollary 6.6.** Let $N$ be a finite direct sum of II$_1$ factors with a finite group $G$ acting outerly on $N$. Then, the lattice $\mathcal{I}(N' \subset N \rtimes G)$ is finite.

**Proof.** Let $M := N \rtimes G$. We know that $\text{Ind}(E) = |G|$, $E$ is the canonical conditional expectation from $N \rtimes G$ onto $N$. Further, the outerness of the action implies that $N' \cap M = Z(N)$. Applying Theorem 6.4, we obtain the desired result. \hfill \Box

The following consequence can be thought of as an appropriate generalization of [49, Theorem 2.2] in the non-irreducible case.

**Corollary 6.7.** Let $N \subset M$ be a subfactor of type II$_1$ with finite Jones index. Then, $\mathcal{I}(R \subset M)$ is a finite lattice, where $R := N \vee (N' \cap M)$.

**Proof.** Since $R \cong N \otimes (N' \cap M)$, it is clear that $R$ is a direct sum of finitely many II$_1$ factors. Then, observe that $R' \cap M \subseteq N' \cap M \subset R$. Thus, $R' \cap M \subset R \cap R'$; so that $R' \cap M = Z(R)$. And, by [36], the $\text{tr}_M$-preserving conditional expectation $E^M_{R} : M \to R$ has finite Watatani index. The rest again follows from Theorem 6.4. \hfill \Box
Lemma 6.8. Let $\mathcal{N} \subset \mathcal{M}$ be a pair of von Neumann algebras with common identity. Let $\mathcal{R} := \mathcal{N} \cap (\mathcal{N} \cap \mathcal{M})$ and $\mathcal{R}_0 := \mathcal{N} \cap \mathcal{Z}(\mathcal{N} \cap \mathcal{M})$. Then, we have the following.

1. $\mathcal{Z}(\mathcal{R}_0) = \mathcal{Z}(\mathcal{R})$.
2. $\mathcal{Z}(\mathcal{R}_0) = \mathcal{Z}(\mathcal{N} \cap \mathcal{M}) = \mathcal{Z}(\mathcal{R}_0)$.

The following implications are obvious once we apply Theorem 6.4 and the preceding lemma.

**Corollary 6.9.** Let $\mathcal{N} \subset \mathcal{M}$ be a subfactor of type $II_1$ with $[\mathcal{M} : \mathcal{N}] < \infty$. Then, the lattice $\mathcal{I}(\mathcal{N} \subset \mathcal{R})$ is finite.

In particular, if $\mathcal{N} \cap \mathcal{M}$ is abelian, then the lattices $\mathcal{I}(\mathcal{N} \subset \mathcal{R})$ and $\mathcal{I}(\mathcal{R} \subset \mathcal{M})$ are both finite.

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