ENHANCED DISSIPATION FOR THE THIRD COMPONENT OF 3D ANISOTROPIC NAVIER-STOKES EQUATIONS

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Abstract. In this paper, we study the decay rates for the global small smooth solutions to 3D anisotropic incompressible Navier-Stokes equations. In particular, we prove that the horizontal components of the velocity field decay like the solutions of 2D classical Navier-Stokes equations. While the third component of the velocity field decays as the solutions of 3D Navier-Stokes equations. We remark that such enhanced decay rate for the third component is caused by the interplay between the divergence free condition of the velocity field and the horizontal Laplacian in the anisotropic Navier-Stokes equations.

Keywords: Anisotropic Navier-Stokes equations, Littlewood-Paley theory, large time behavior

AMS Subject Classification (2000): 35Q30, 76D03

1. Introduction

We investigate the large time behavior of the global small smooth solutions to the following 3D anisotropic Navier-Stokes equations

\[
\begin{align*}
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} - \Delta_h \mathbf{v} + \nabla p &= 0, \\
\text{div} \mathbf{v} &= 0, \\
\mathbf{v}|_{t=0} &= \mathbf{v}_0,
\end{align*}
\]

where $\Delta_h = \partial_{x_1}^2 + \partial_{x_2}^2$ denotes the horizontal Laplacian operator, $\mathbf{v} = (v^1, v^2, v^3)$ and $p$ designate the velocity and the pressure of the fluid respectively.

Systems of this type appear in geophysical fluid dynamics (see for instance [3, 14]). In fact, meteorologists often modelize turbulent diffusion by putting a viscosity of the form: $\mu_1 \Delta_h - \mu_3 \partial_{x_1}^2$, where $\mu_1$ and $\mu_3$ are empirical constants, and $\mu_3$ is usually much smaller than $\mu_1$. For simplicity, we shall take $\mu_1 = 1$ and $\mu_3 = 0$ in this context.

Just as the classical Navier-Stokes system where the horizontal Laplacian $\Delta_h$ in (1.1) is replaced by the full Laplacian $\Delta = \sum_{j=1}^3 \partial_j^2$, it is still big open question concerning whether or not singularity will be developed in finite time for (1.1) with general initial data. Considering system (1.1) has only horizontal dissipation, it is reasonable to use the following anisotropic Sobolev space to study the well-posedness of the system (1.1):

**Definition 1.1.** For any $(s, s') \in \mathbb{R}^2$, the anisotropic Sobolev space $\dot{H}^{s,s'}(\mathbb{R}^3)$ denotes the space of homogeneous tempered distribution $a$ such that

\[
||a||_{\dot{H}^{s,s'}}^{2} \overset{\text{def}}{=} \int_{\mathbb{R}^3} |\xi|^s |\xi'|^{s'} |\hat{a}(\xi)|^2 d\xi < \infty \quad \text{with} \quad \xi_3 = (\xi_1, \xi_2).
\]

Chemin et al. [3] and Iftimie [9] proved that (1.1) is locally well-posed with initial data in $L^2(\mathbb{R}^3) \cap H^{0,1/2+x}(\mathbb{R}^3)$ for some $\varepsilon > 0$, and is globally well-posed if in addition

\[
||v_0||_{L^2} ||v_0||_{H^{0,1/2+x}}^{1-\varepsilon} \leq c
\]

for some sufficiently small constant $c$.

Notice that just as the classical Navier-Stokes system, the system (1.1) has the following scaling invariant property:

\[
v_{\lambda}(t, x) \overset{\text{def}}{=} \lambda v(\lambda^2 t, \lambda x) \quad \text{and} \quad v_{0,\lambda}(x) \overset{\text{def}}{=} \lambda v_0(\lambda x),
\]

which means that if $\mathbf{v}$ is a solution of (1.1) with initial data $\mathbf{v}_0$ on $[0, T]$, $\mathbf{v}_\lambda$ determined by (1.3) is also a solution of (1.1) with initial data $\mathbf{v}_{0,\lambda}$ on $[0, T/\lambda^2]$.

It is easy to observe that the smallness condition (1.2) in [3] is scaling invariant under the scaling transformation (1.3), nevertheless, the norm of the space $H^{0,1/2+x}$ is not. To investigate the well-posedness
of (1.1) with initial data in the critical spaces, we recall the following anisotropic dyadic operators from [1]:

\[
\Delta^h \varphi \overset{\text{def}}{=} F^{-1}(\varphi(2^{-k}|\xi_h|)\hat{a}), \quad \Delta^v \varphi \overset{\text{def}}{=} F^{-1}(\varphi(2^{-\ell}|\xi_v|)\hat{a}),
\]

where \(\xi_h = (\xi_1, \xi_2)\), \(F a\) or \(\hat{a}\) denotes the Fourier transform of \(a\), while \(F^{-1} a\) designates the inverse Fourier transform of \(a\), \(\varphi(\tau)\) and \(\varphi(\tau)\) are smooth functions such that

\[
\text{Supp } \varphi \subset \left\{ \tau \in \mathbb{R} : \frac{3}{4} \leq |\tau| \leq \frac{8}{3} \right\} \text{ and } \forall \tau > 0, \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\tau) = 1;
\]

\[
\text{Supp } \chi \subset \left\{ \tau \in \mathbb{R} : |\tau| \leq \frac{4}{3} \right\} \text{ and } \forall \tau \in \mathbb{R}, \chi(\tau) + \sum_{j \geq 0} \varphi(2^{-j}\tau) = 1.
\]

**Definition 1.2.** We define \(B^{0, \frac{1}{2}}(\mathbb{R}^3)\) to be the set of homogeneous tempered distribution \(a\) so that

\[
\|a\|_{B^{0, \frac{1}{2}}} \overset{\text{def}}{=} \sum_{\ell \in \mathbb{Z}} 2^{\frac{\ell}{2}} \|\Delta^\ell a\|_{L^2(\mathbb{R}^3)} < \infty.
\]

Paicu [12] proved the local well-posedness of the system (1.1) with initial data in \(B^{0, \frac{1}{2}}\) and also global well-posedness of (1.1) provided that \(\|v_0\|_{B^{0, \frac{1}{2}}}\) is sufficiently small. Chemin and the second author [4] introduced a critical anisotropic Besov-Sobolev type space with negative index and proved the global well-posedness of (1.1) with small initial data in this space. In particular, this result implies the global well-posedness of (1.1) with highly oscillatory initial data in the horizontal variables. Those global well-posedness results in [4, 12] were improved in [13, 19] that the system has a unique global solution with two components of the initial data being small enough in the critical spaces. Very recently, Liu, Paicu and the second author [10] proved the global well-posedness of (1.1) with uni-directional derivative of the initial data being sufficiently small. More precisely, as long as \(\|D_0|^{-1}\partial_3 v_0\|_{B^{0, \frac{1}{2}}}\) is small enough, (1.1) has a unique global solution, which in particular improves the previous global well-posedness results in [12, 13, 19]. One may check [10] and the references therein for details concerning the well-posedness theory of (1.1). The authors of [11] studied the lower bound to the lifespan of the local smooth solutions to the system (1.1).

Lately, Ji, Wu and Yang [8] investigated the decay rate for the global smooth solution of (1.1). More precisely,

**Theorem 1.1** (Theorem 1.1 of [8]). Let \(\frac{4}{3} \leq \sigma < 1\). Assume that

\[
v_0 \in H^4(\mathbb{R}^3), \quad \nabla \cdot v_0 = 0, \quad v_0, \partial_3 v_0 \in \dot{H}^{-\sigma, 0}(\mathbb{R}^3).
\]

Then there is \(\varepsilon > 0\) such that if

\[
\|v_0\|_{H^4} + \|v_0, \partial_3 v_0\|_{\dot{H}^{-\sigma, 0}} \leq \varepsilon,
\]

then (1.1) has a unique solution \(v\) satisfying

\[
\|v\|_{L^\infty(\mathbb{R}^+; H^4)} + \|v\|_{L^\infty(\mathbb{R}^+; \dot{H}^{-\sigma, 0})} \leq C\varepsilon,
\]

\[
\|v(t)\|_{L^2} + \|\partial_3 v(t)\|_{L^2} \leq C\varepsilon(t)^{-\frac{\sigma}{2}},
\]

\[
\|\nabla v(t)\|_{L^2} \leq C\varepsilon(t)^{-\frac{\sigma + 1}{2}}.
\]

We remark that although the linear part of the system (1.1) is 2D Stokes system, yet the Fourier splitting method introduced by Wieger in [18] (see also the works [7, 15, 16]) to study the decay-in-time estimates for 2D Navier-Stokes equation can not be applied to investigate the decay rates for the global small solutions of (1.1). Instead the authors of [8] employed the integral formulation of the system (1.1) and used a continuous argument to prove Theorem 1.1.

Our first observation in this paper is that under the critical smallness condition (1.6), we can propagate the regularity of the solution to (1.1) in the vertical variable. We can also propagate the full high regularity of the solution to (1.1). Precisely, our first result states as follows:

**Theorem 1.2.** Let \(s \geq 1\) and \(v_0 \in (B^{0, \frac{1}{2}} \cap \dot{H}^s)(\mathbb{R}^3)\) be a solenoidal vector field which satisfies

\[
\|v_0\|_{B^{0, \frac{1}{2}}} \leq c_0,
\]

(1.6)
for some $c_0 > 0$ sufficiently small. Then the system (1.1) has a unique global solution $v$ so that

\[ v \in C((0, \infty); (B^{0, \frac{1}{2}} \cap \dot{H}^s)(\mathbb{R}^3)) \quad \text{and} \quad \nabla_h v \in L^2(\mathbb{R}_+; (B^{0, \frac{1}{2}} \cap \dot{H}^s)(\mathbb{R}^3)),\]

and

\[
\begin{align*}
\|v\|_{L^\infty(\mathbb{R}_+; \dot{H}^{0,s})}^2 + \|\nabla_h v\|_{L^2(\mathbb{R}_+; \dot{H}^{0,s})}^2 &\leq C\|v_0\|_{\dot{H}^{0,s}}^2, \\
\|v\|_{L^\infty(\mathbb{R}_+; \dot{H}^{s,0})}^2 + \|\nabla_h v\|_{L^2(\mathbb{R}_+; \dot{H}^{s,0})}^2 &\leq C\|v_0\|_{\dot{H}^{s,0}}^2.
\end{align*}
\]

(1.7a) Here and in what follows $C > 0$ denotes a universal constant that may vary from line to line.

Our second result is concerned with the large time behavior of the global small solutions to the system (1.1).

**Theorem 1.3.** Let $s_1 > 2$ and $\alpha \in \left(\frac{1+3\alpha}{10(s_1-1)}, 1\right)$. Let $v_0 \in \left(\dot{H}^{0,s_1} \cap \dot{H}^{-s,0} \cap \dot{H}^{-s,-\frac{2}{3}-\frac{4}{5}}\right)(\mathbb{R}^3)$ with $\partial_t v_0 \in \dot{H}^{-\frac{1}{2},0}(\mathbb{R}^3)$. We denote

\[ A_s \overset{\text{def}}{=} \|v_0\|^2_{\dot{H}^{-s,0}} + \|\nabla v_0\|^2_{\dot{H}^{-s,0}} \quad \text{and} \quad B_s \overset{\text{def}}{=} \|v_0\|^2_{\dot{H}^{-s,0}} + \|\nabla v_0\|^2_{\dot{H}^{-s,0}} + A_s^2. \]

(1.8) Then there exists $\varepsilon_0 > 0$ such that if besides (1.6), there hold in addition

\[ A_s B_s \leq \varepsilon_0 \quad \text{and} \quad \mathcal{E}_0 \overset{\text{def}}{=} \|\partial_t v_0\|^2_{\dot{H}^{-s,0}} + \|\nabla v_0\|^2_{\dot{H}^{-s,0}} + (A_s B_s)^\alpha \leq \varepsilon_0. \]

(1.9) Then (1.1) has a unique global solution so that

1. (Propagation of the regularity)

\[
\begin{align*}
\|v(t)\|^2_{L^\infty(\mathbb{R}_+; \dot{H}^{-s,0})} + \|\nabla_h v(t)\|^2_{L^2(\mathbb{R}_+; \dot{H}^{-s,0})} &\leq C\|v_0\|_{\dot{H}^{-s,0}}^2, \\
\|v(t)\|^2_{L^\infty(\mathbb{R}_+; \dot{H}^{-s,0})} + \|\nabla_h v(t)\|^2_{L^2(\mathbb{R}_+; \dot{H}^{-s,0})} &\leq C\|v_0\|_{\dot{H}^{-s,0}}^2.
\end{align*}
\]

(1.10a) (b) (decay estimates)

\[
\begin{align*}
\|v(t)\|^2_{L^\infty(\mathbb{R}_+; \dot{H}^{-s,0})} + \|\nabla_h v(t)\|^2_{L^2(\mathbb{R}_+; \dot{H}^{-s,0})} &\leq CA_s(t)^{-\alpha}, \\
\|\partial_t v(t)\|^2_{L^\infty(\mathbb{R}_+; \dot{H}^{-s,0})} + \|\nabla v(t)\|^2_{L^2(\mathbb{R}_+; \dot{H}^{-s,0})} &\leq C\mathcal{E}_0(t)^{-\frac{1}{2}},
\end{align*}
\]

(1.11a) (b) (enhanced dissipation for the third component)

\[
\|v(t)\|^2_{L^\infty(\mathbb{R}_+; \dot{H}^{-s,0})} + \|\nabla_h v(t)\|^2_{L^2(\mathbb{R}_+; \dot{H}^{-s,0})} \leq C B_s(t)^{-\frac{2}{5}s - \frac{4}{5}}.
\]

(1.12)

**Remark 1.1.**

1. The scaling invariant norm $\|v(t)\|_{L^2(\dot{H}^{-s,0})}$ is crucial to propagate the negative horizontal regularity of the solution to the system (1.1) (see (3.18) below). That is the reason why we need the smallness condition on $v_0 \in \dot{H}^{-\frac{1}{2},0}(\mathbb{R}^3)$.

2. $s_1 = 4$ in Theorem 1.3 corresponds to the assumption: $v_0 \in H^4(\mathbb{R}^3)$ in Theorem 1.1. Yet with $s_1 = 4$, we have $s \in \left(\frac{11}{10}, 1\right)$ in Theorem 1.3, which improves the assumption that $\sigma \in \left[\frac{2}{3}, 1\right]$ in Theorem 1.1. We do not know if the above result can be extended to any $s \in (0, 1)$ as for the decay rates of classical 2D Navier-Stokes equations (see for instance [18]).

3. We observe from (1.12) that the third component $v^3(t)$ of the $v(t)$ decays faster than the horizontal components $v^h(t)$. This is purely due to the special structure of the third equation of (1.1), precisely, (3.30), and the divergence free condition of $v$. If we replace the norm $\|v_0\|_{\dot{H}^{-s,-\frac{2}{3}+\frac{4}{5}}}$ in Theorem 1.3 by $\|v_0\|_{\dot{H}^{-s,-\frac{2}{3}}}$, then the solution $v(t)$ decays according to

\[
\|v^3(t)\|^2_{L^2} \leq CB_s(t)^{-\frac{1}{2}s} \quad \text{and} \quad \|\nabla_h v^3(t)\|^2_{L^2} \leq CB_s(t)^{-\frac{2}{5}s - 1},
\]

(1.13) which is exactly the decay rate for the global solutions of classical 3D Navier-Stokes system (see for instance [15]).

4. In general, the interplay between transport $u \cdot \nabla$ and the diffusion of the following equation may cause the solution $f$ decay more rapidly than the diffusion alone (see for instance [5, 6, 17])

\[
\partial_t f + u \cdot \nabla f - \nu \Delta f = 0.
\]

Here the enhanced decay rate of the third component $v^3$ is caused by interplay between the divergence free condition $\partial_1 v^1 + \partial_2 v^2 + \partial_3 v^3 = 0$, which transports the horizontal regularity of $v^h$ to the vertical regularity of $v^3$, and the horizontal Laplacian in the $v$ equations in (1.1). That is the reason why we borrow the word ”enhanced dissipation” from [5, 6, 17].
Let us end this section with some notations that will be used throughout this paper.

**Notations:** Let $A, B$ be two operators, we denote $[A; B] = AB - BA$, the commutator between $A$ and $B$. For $a \lesssim b$, we mean that there is a uniform constant $C$, which may be different in each occurrence, such that $a \leq Cb$. We shall denote by $(a|b)$ the $L^2(\mathbb{R}^3)$ inner product of $a$ and $b$. $(c_k)_{k \in \mathbb{Z}}$ designates a generic elements on the unit sphere of $l^2(\mathbb{Z})$, i.e., $\sum_{k \in \mathbb{Z}} c_k^2 = 1$ (resp. $\sum_{k \in \mathbb{Z}} c_k(t)^2 = 1$).

Finally, we denote $L^p_c(L^p_h)$ to be the space $L^p([0, T]; L^p(\mathbb{R}^2; L^q(\mathbb{R}^3)))$, and $\nabla_h \overset{\text{def}}{=} (\partial_{x_1}, \partial_{x_2})$, $\text{div}_h = \nabla_h \cdot$.

2. The Propagation of Regularities to Paicu’s Solution

In this section, we shall prove the propagation of regularities for the solution obtained by Paicu in [12] based on the estimate (2.1) below. Namely, we are going to present the proof of Theorem 1.2.

**Proof of Theorem 1.2.** For simplicity, we just present the *a priori* estimates for smooth enough solutions of (1.1). Indeed under the assumption of (1.6), we deduce from [12] that the system (1.1) has a unique global solution $v$ so that

$$\|v\|_{L^\infty([0, T]; Z_0^0)} + \|\nabla_h v\|_{L^2([0, T]; Z_0^0)} \leq C\|v_0\|_{Z_0^0}. \quad (2.1)$$

With the estimate (2.1), we can prove the propagation of regularity of $v$ in the vertical variable. We first get, by applying $\Delta_t^\gamma$ to the $v$ equation of (1.1) and taking $L^2$ inner product of the resulting equation with $\Delta_t^\gamma v$, that

$$\frac{1}{2} \frac{d}{dt}\|\Delta_t^\gamma v(t)\|_{L^2}^2 + \|\nabla_h \Delta_t^\gamma v\|_{L^2}^2 = -(\Delta_t^\gamma (v \cdot \nabla v) \cdot \Delta_t^\gamma v). \quad (2.2)$$

The estimate of term on the right-hand side (r.h.s.) of (2.2) relies on the following lemma:

**Lemma 2.1.** Let $s > 0$, one has

$$\left| (\Delta_t^\gamma (v \cdot \nabla v) \cdot \Delta_t^\gamma v) \right| \lesssim c_\ell(t)^2 2^{-2\ell s} \left( \|\nabla_h v\|_{Z_0^0} \|v\|_{Z_0^0} \|\nabla_h v\|_{H^s} \right) \quad (2.3)$$

Here and in all that follows, we shall always denote $(c_\ell(t))_{\ell \in \mathbb{Z}}$ to be a generic element of $l^2(\mathbb{Z})$ so that $\sum_{\ell \in \mathbb{Z}} c_\ell^2(t) = 1$.

We postpone the proof of this lemma till we finish the proof of Theorem 1.2.

By inserting the estimate (2.3) into (2.2), and then multiplying the inequality by $2^{-2\ell s}$, finally by summing up the resulting inequalities for $\ell$ in $\mathbb{Z}$, we achieve

$$\frac{1}{2} \frac{d}{dt}\|v(t)\|_{H^{2s}}^2 + \|\nabla_h v(t)\|_{H^{2s}}^2 \leq C \left( \|\nabla_h v\|_{Z_0^0} \|v\|_{Z_0^0} \|\nabla_h v\|_{H^s} \right) \quad (2.4)$$

Applying Young’s inequality gives

$$\text{r.h.s. of (2.4)} \leq C \left( 1 + \|v\|_{Z_0^0}^2 \|\nabla_h v\|_{Z_0^0}^2 \|v\|_{H^s}^2 + \frac{1}{2} \|\nabla_h v\|_{H^s}^2 \right),$$

so that there holds

$$d\|v(t)\|_{H^{2s}}^2 + \|\nabla_h v(t)\|_{H^{2s}}^2 \leq C \left( 1 + \|v\|_{Z_0^0}^2 \|\nabla_h v\|_{Z_0^0}^2 \|v\|_{H^s}^2 \right) \quad (2.5)$$

Applying Gronwall’s inequality yields for any $s > 0$ that

$$\|v\|_{L^\infty([0, T]; Z_0^0)} + \|\nabla_h v(t)\|_{L^2([0, T]; Z_0^0)} \leq \|v_0\|_{Z_0^0} \exp \left( C \left( 1 + \|v\|_{L^\infty([0, T]; Z_0^0)}^2 \right) \|\nabla_h v\|_{L^2([0, T]; Z_0^0)}^2 \right),$$

which together with (2.1) leads to (1.7a).

To handle the estimate of (1.7b) for case when $s \geq 1$, we get, by applying $\Delta_h^1$ to the $v$ equation of (1.1) and then taking $L^2$-inner product of the resulting equation with $\Delta_h^1 v$, that

$$\frac{1}{2} \frac{d}{dt}\|\Delta_h^1 v(t)\|_{L^2}^2 + \|\nabla_h \Delta_h^1 v\|_{L^2}^2 = -(\Delta_h^1 v \cdot \nabla v \cdot \Delta_h^1 v). \quad (2.6)$$

The estimate of the term on the r.h.s. of (2.6) relies on the following lemma, the proof of which will be postponed at the end of this section.
Lemma 2.2. Let \( s > -1 \), there hold
\[
\left| (\Delta_h^k (v^b \cdot \nabla_h v) - \Delta_h^k v) \right| \lesssim c_k^2(t) 2^{-2ks} \left( \| \nabla_h v \|_{E^0} \frac{1}{2} \| v \|_{H^s} \| \Delta_h v \|_{H^{s-2}} \right),
\]
(2.7)

and
\[
\left| (\Delta_h^k (v^3 \partial_3 v) - \Delta_h^k v) \right| \lesssim c_k^2(t) 2^{-2ks} \left( \| \nabla_h v \|_{E^0} \frac{1}{2} \| v \|_{H^s} + \| v \|_{E^0} \frac{1}{2} \| \nabla_h v \|_{E^0} \right).
\]
(2.8)

By inserting the estimates (2.7) and (2.8) into (2.6), and then multiplying the resulting inequalities by \( 2^{2ks} \) and summing up \( k \) over \( \mathbb{Z} \), we arrive at
\[
\frac{1}{2} \frac{d}{dt} \| v(t) \|_{H^{s-1},0}^2 + \| \nabla_h v(t) \|_{H^s,0}^2 \leq C \left( \| v \|_{E^0} \frac{1}{2} \| \nabla_h v \|_{E^0} \frac{1}{2} \| v \|_{H^s} \| \partial_3 v \|_{H^{s} - 1,0} \| \partial_3 v \|_{H^{s},0} \right.
\]
\[
+ \left. \| v \|_{E^0} \frac{1}{2} \| \nabla_h v \|_{E^0} \frac{1}{2} \| v \|_{H^s} \right). \tag{2.9}
\]

However, notice that for \( s \geq 1 \),
\[
\| \partial_3 v \|_{H^{s-1},0} = \int_{\mathbb{R}^3} |\xi_h|^{2(s-1)} |\xi_3|^2 |\widehat{\alpha}(\xi)|^2 d\xi
\]
\[
\leq \left( \int_{\mathbb{R}^3} |\xi_3|^2 |\widehat{\alpha}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |\xi_3|^2 |\widehat{\alpha}(\xi)|^2 d\xi \right)^{\frac{1}{2}} = \| a \|_{H^{s-1},0} \| a \|_{H^{s-1},0}^2,
\]

then we get, by applying Young’s inequality, that
\[
C \| v \|_{E^0} \frac{1}{2} \| \nabla_h v \|_{E^0} \frac{1}{2} \| \Delta_h v \|_{H^{s-1},0} \| \partial_3 v \|_{H^{s},0}^2 \leq C \| v \|_{E^0} \frac{1}{2} \| \nabla_h v \|_{E^0} \frac{1}{2} \| v \|_{H^s} \| \partial_3 v \|_{H^{s} - 1,0} \| \partial_3 v \|_{H^{s},0} \right.
\]
\[
+ \left. \| v \|_{E^0} \frac{1}{2} \| \nabla_h v \|_{E^0} \frac{1}{2} \| v \|_{H^s} \right), \tag{2.9}
\]

and
\[
C \| v \|_{E^0} \frac{1}{2} \| \nabla_h v \|_{E^0} \frac{1}{2} \| \Delta_h v \|_{H^{s-1},0} \| \partial_3 v \|_{H^{s},0}^2 \leq C \| v \|_{E^0} \frac{1}{2} \| \nabla_h v \|_{E^0} \frac{1}{2} \| v \|_{H^s} \| \partial_3 v \|_{H^{s} - 1,0} \| \partial_3 v \|_{H^{s},0} \right.
\]
\[
+ \left. \| v \|_{E^0} \frac{1}{2} \| \nabla_h v \|_{E^0} \frac{1}{2} \| v \|_{H^s} \right). \tag{2.9}
\]

By substituting the above estimates into (2.9), we achieve
\[
\frac{d}{dt} \| v(t) \|_{H^{s},0}^2 + \| \nabla_h v \|_{H^s,0}^2 \leq C \| v \|_{E^0} \frac{1}{2} \left( 1 + \| v \|_{E^0} \frac{1}{2} \right) \| v \|_{H^s,0}^2
\]
\[
+ C \| v \|_{E^0} \frac{1}{2} \| \nabla_h v \|_{E^0} \frac{1}{2} \| v \|_{H^s,0}^2 + \| \nabla_h v \|_{H^s,0}^2.
\]

Applying Gronwall’s inequality gives rise to
\[
\| v \|_{L^2_r(H^{s-1},0)}^2 + \| \nabla_h v \|_{L^2_r(H^s,0)}^2 \leq \left( \| v_0 \|_{H^{s},0}^2 + C \| v \|_{L^2_r(E^0,\frac{1}{2})} \| \nabla_h v \|_{L^2_r(E^0,\frac{1}{2})} \| v \|_{L^2_r(H^{s},0)}^2 \right.
\]
\[
+ \left. \| \nabla_h v \|_{L^2_r(H^{s},0)}^2 \right) \exp \left( C \| v \|_{L^2_r(E^0,\frac{1}{2})} \left( 1 + \| v \|_{L^2_r(E^0,\frac{1}{2})} \right) \right),
\]

which together with (2.1), (1.6) and (1.7a) ensures
\[
\| v \|_{L^2_r(H^{s-1},0)}^2 + \| \nabla_h v \|_{L^2_r(H^s,0)}^2 \leq C \| v_0 \|_{H^{s},0}^2. \tag{2.10}
\]

By combining (1.7a) with (2.10), we deduce (1.7b). This completes the proof of Theorem 1.2. \(\square\)

Theorem 1.2 has been proved provided that we provide the proof of Lemmas 2.1 and 2.2, which we present below.
Proof of Lemma 2.1. Observing that

\[ (\Delta \tilde{\gamma}(v \cdot \nabla v) \mid \Delta \tilde{\gamma}v) = (\Delta \tilde{\gamma}(v^h \cdot \nabla_h v) \mid \Delta \tilde{\gamma}v) + (\Delta \tilde{\gamma}(v^3 \partial_3 v) \mid \Delta \tilde{\gamma}v), \]

we split the proof of (2.3) into the following two steps:

**Step 1.** The estimate of \((\Delta \tilde{\gamma}(v^h \cdot \nabla_h v) \mid \Delta \tilde{\gamma}v)\).

By applying Bony’s decomposition (A.1) to \(v^h \cdot \nabla_h v\) in the vertical variables, we find

\[ v^h \cdot \nabla_h v = T^\nu_h \cdot \nabla_h v + \bar{T}^\nu_{v_h} v^h \quad \text{with} \quad \bar{T}^\nu_{v_h} v^h \overset{\text{def}}{=} T^\nu_{v_h} v^h + R^\nu(v^h, \nabla_h v), \]

due to the support properties to the Fourier transform of the terms in \(T^\nu_{v_h} \cdot \nabla_h v\), we infer

\[
|\langle \Delta \tilde{\gamma}(T^\nu_{v_h} \cdot \nabla_h v) \mid \Delta \tilde{\gamma}v \rangle| \lesssim \sum_{|\ell' - \ell| \leq 4} \|S_{\ell' - 1}^\nu v^h\|_{L^2(\mathbb{R}^2)} \|\Delta \tilde{\gamma}v\|_{L^2} \|\Delta \tilde{\gamma}v\|_{L^2(\mathbb{R}^2)} \\
\quad \lesssim \sum_{|\ell' - \ell| \leq 4} \|v^h\|_{L^2(\mathbb{R}^2)} \|\Delta \tilde{\gamma}v\|_{L^2} \|\Delta \tilde{\gamma}v\|_{L^2} \|\Delta \tilde{\gamma}v\|_{L^2},
\]

which together with the first inequality of Lemma A.3, i.e.,

\[
\|a\|_{L^6(\mathbb{R}^2)} \lesssim \|a\|_{g^{0, \frac{3}{2}}} \|a\|_{g^{0, rac{3}{2}}},
\]

ensures that

\[
|\langle \Delta \tilde{\gamma}(T^\nu_{v_h} \cdot \nabla_h v) \mid \Delta \tilde{\gamma}v \rangle| \lesssim \left( \sum_{|\ell' - \ell| \leq 4} c_{\ell}(t) 2^{-\ell' s} c_{\ell}(t) 2^{-\ell s} \|v^h\|_{g^{0, \frac{3}{2}}} \|\nabla_h v\|_{g^{0, \frac{3}{2}}} \|v\|_{H^0, s} \|\nabla_h v\|_{H^0, s} \right) \\
\lesssim c_2(t)^2 2^{-2s} \|v\|_{g^{0, \frac{3}{2}}} \|\nabla_h v\|_{g^{0, \frac{3}{2}}} \|v\|_{H^0, s} \|\nabla_h v\|_{H^0, s}.
\]

Along the same line, due to \(s > 0\), we get by using Lemma A.3 that

\[
|\langle \Delta \tilde{\gamma}(\bar{T}^\nu_{v_h} v^h) \mid \Delta \tilde{\gamma}v \rangle| \lesssim \sum_{\ell' \geq \ell - N_0} \|S_{\ell' - 1}^\nu v^h\|_{L^2(\mathbb{R}^2)} \|\Delta \tilde{\gamma}v\|_{L^2} \|\Delta \tilde{\gamma}v\|_{L^2(\mathbb{R}^2)} \\
\quad \lesssim \left( \sum_{\ell' \geq \ell - N_0} c_{\ell}(t) 2^{-\ell' s} c_{\ell}(t) 2^{-\ell s} \|v^h\|_{g^{0, \frac{3}{2}}} \|\nabla_h v\|_{g^{0, \frac{3}{2}}} \|v\|_{H^0, s} \|\nabla_h v\|_{H^0, s} \right) \\
\quad \lesssim c_2(t)^2 2^{-2s} \|v\|_{g^{0, \frac{3}{2}}} \|\nabla_h v\|_{g^{0, \frac{3}{2}}} \|v\|_{H^0, s} \|\nabla_h v\|_{H^0, s}.
\]

By summarizing the above estimates, we arrive at

\[
|\langle \Delta \tilde{\gamma}(v^h \cdot \nabla_h v) \mid \Delta \tilde{\gamma}v \rangle| \lesssim c_2(t)^2 2^{-2s} \left( \|\nabla_h v\|_{g^{0, \frac{3}{2}}} \|v\|_{H^0, s} \|\nabla_h v\|_{H^0, s} \\
+ \|v\|_{g^{0, \frac{3}{2}}} \|\nabla_h v\|_{g^{0, \frac{3}{2}}} \|v\|_{H^0, s} \|\nabla_h v\|_{H^0, s} \right). \tag{2.11}
\]

**Step 2.** The estimate of \((\Delta \tilde{\gamma}(v^3 \partial_3 v) \mid \Delta \tilde{\gamma}v)\).

We first get, by using a standard commutator’s argument as that in [3, 4, 12], that

\[
(\Delta \tilde{\gamma}(T^\nu_{v_h} \partial_3 v) \mid \Delta \tilde{\gamma}v) = \sum_{|\ell' - \ell| \leq 4} \left( \langle [\Delta \tilde{\gamma}v^3; S_{\ell' - 1}^\nu v^3] \partial_3 \Delta \tilde{\gamma}v \mid \Delta \tilde{\gamma}v \rangle \\
+ \langle (S_{\ell' - 1}^\nu v^3; \partial_3 \Delta \tilde{\gamma}v) \mid \Delta \tilde{\gamma}v \rangle \right) + (S_{\ell}^\nu v^3 \partial_3 \Delta \tilde{\gamma}v \mid \Delta \tilde{\gamma}v) \tag{2.12}
\]

By applying Lemmas A.2 and A.1 and using \(\text{div} v = 0\), we find

\[
\sum_{|\ell' - \ell| \leq 4} \left| \langle \Delta \tilde{\gamma}v^3; S_{\ell' - 1}^\nu v^3 \partial_3 \Delta \tilde{\gamma}v \mid \Delta \tilde{\gamma}v \rangle \right| \\
\lesssim \sum_{|\ell' - \ell| \leq 4} 2^{-\ell} \|S_{\ell' - 1}^\nu v^3\|_{L^2(\mathbb{R}^2)} \|\partial_3 \Delta \tilde{\gamma}v\|_{L^2} \|\Delta \tilde{\gamma}v\|_{L^2(\mathbb{R}^2)} \\
\lesssim \sum_{|\ell' - \ell| \leq 4} \|\text{div}_h v^h\|_{L^2(\mathbb{R}^2)} \|\Delta \tilde{\gamma}v\|_{L^2} \|\nabla_h \Delta \tilde{\gamma}v\|_{L^2} \|\Delta \tilde{\gamma}v\|_{L^2} \|\nabla_h \Delta \tilde{\gamma}v\|_{L^2} \\
\lesssim c_2(t)^2 2^{-2s} \|\nabla_h v\|_{g^{0, \frac{3}{2}}} \|v\|_{H^0, s} \|\nabla_h v\|_{H^0, s}.
\]

Similar estimate holds for the second term in (2.12).
While we get, by using integration by parts and \( \text{div} \, v = 0 \), that
\[
\left| \langle S_y^v \partial_3 \Delta_y^v \mid \Delta_y^v \rangle \right| = \frac{1}{2} \left| \int_{\mathbb{R}^3} S_y^v \partial_3 \Delta_y^v |^2 \, dx \right|
\leq \frac{1}{2} \| \text{div}_h \, v^h \|_{L^2_h(L^\infty)} \| \Delta_y^v \|^2_{L^2_h(L^s)}
\lesssim c^2(t) 2^{-2ks} \| \nabla_h \, v \|_{g^{0,0}} \| v \|_{H^{s,0}} \| \nabla_h \, v \|_{H^{s,0}}.
\]
This leads to
\[
\left| \langle \Delta_y^v (T^v_{\partial_3} \partial_3 v) \mid \Delta_y^v \rangle \right| \lesssim c^2(t) 2^{-2ks} \| \nabla_h \, v \|_{g^{0,0}} \| v \|_{H^{s,0}} \| \nabla_h \, v \|_{H^{s,0}}. \tag{2.13}
\]
On the other hand, again due to \( \text{div} \, v = 0 \) and \( s > 0 \), we deduce from Lemmas A.1 and A.3 that
\[
\left| \langle \Delta_y^v (\hat{T}^v_{\partial_3} v^3) \mid \Delta_y^v \rangle \right| \lesssim \sum_{k' \geq k - N_0} \| S_{k' - 1}^v \partial_3 v \|_{L^2_h(L^\infty)} \| \Delta_y^v \partial_3 \Delta_y^v \|_{L^2_h(L^s)}
\lesssim \sum_{k' \geq k - N_0} \| v^h \|_{L^2_h(L^\infty)} \| \Delta_y^v \partial_3 \Delta_y^v \|_{L^2_h(L^s)}
\lesssim \left( \sum_{k' \geq k - N_0} c_k(t) 2^{-k' \ell} \right) c_k(t) 2^{-ks} \| v \|_{g^{0,0}}^2 \| \nabla_h \, v \|_{g^{0,0}}^2 \| v \|_{H^{s,0}} \| \nabla_h \, v \|_{H^{s,0}}^2
\lesssim c^2(t) 2^{-2ks} \| v \|_{g^{0,0}}^2 \| \nabla_h \, v \|_{g^{0,0}}^2 \| v \|_{H^{s,0}} \| \nabla_h \, v \|_{H^{s,0}}^2.
\]
which together with (2.13) ensures that
\[
\left| \langle \Delta_y^v (v^3 \partial_3 v) \mid \Delta_y^v \rangle \right| \lesssim c^2(t) 2^{-2ks} \left( \| \nabla_h \, v \|_{g^{0,0}} \| v \|_{H^{s,0}} \| \nabla_h \, v \|_{H^{s,0}} + \| v \|_{g^{0,0}} \| \nabla_h \, v \|_{g^{0,0}} \| v \|_{H^{s,0}} \| \nabla_h \, v \|_{H^{s,0}}^2. \right) \tag{2.14}
\]
By combining (2.11) with (2.14), we conclude the proof of (2.3).

Proof of Lemma 2.2. We divide the proof into the following two steps:

**Step 1.** The estimate of \( \langle \Delta_k^h (v^h \cdot \nabla_h \, v) \mid \Delta_k^h \rangle \).

By applying Bony’s decomposition (A.1) to \( v^h \cdot \nabla_h \, v \) in the horizontal variables, we find
\[
v^h \cdot \nabla_h \, v = T^h_{v^h} \cdot \nabla_h \, v + T^h_{\nabla_h \, v^h} \cdot v^h + R^h(v^h, \nabla_h \, v).
\]
Due to the support properties to the Fourier transform of the terms in \( T^h_{v^h} \cdot \nabla_h \, v, T^h_{\nabla_h \, v^h} \cdot v^h, R^h(v^h, \nabla_h \, v) \), we infer by using Lemma A.3 that
\[
\left| \langle \Delta_k^h (T^h_{v^h} \cdot \nabla_h \, v) \mid \Delta_k^h \rangle \right| \lesssim \sum_{|k' - k| \leq 4} \| S_{k' - 1}^h \nabla_h \, v \|_{L^2_h(L^\infty)} \| \Delta_k^h \nabla_h \, v \|_{L^2_h(L^s)}
\lesssim \sum_{|k' - k| \leq 4} \| v^h \|_{L^2_h(L^\infty)} \| \Delta_k^h \nabla_h \, v \|_{L^2_h(L^s)}^2 \| \Delta_k^h \nabla_h \, v \|_{L^2_h(L^s)}
\lesssim c^2_k(t) 2^{-2ks} \| v \|_{g^{0,0}}^2 \| \nabla_h \, v \|_{g^{0,0}}^2 \| v \|_{H^{s,0}} \| \nabla_h \, v \|_{H^{s,0}}^2.
\]
Along the same line, we get
\[
\left| \langle \Delta_k^h (T^h_{\nabla_h \, v} \cdot v^h) \mid \Delta_k^h \rangle \right| \lesssim \sum_{|k' - k| \leq 4} \| S_{k' - 1}^h \nabla_h \, v \|_{L^2_h(L^\infty)} \| \Delta_k^h \nabla_h \, v \|_{L^2_h(L^s)} \| \Delta_k^h \nabla_h \, v \|_{L^2_h(L^s)}
\lesssim \left( \sum_{|k' - k| \leq 4} c_{k'}(t) 2^{k' s} \right) c_k(t) 2^{-ks} \| \nabla_h \, v \|_{g^{0,0}} \| v \|_{H^{s,0}} \| \nabla_h \, v \|_{H^{s,0}}
\lesssim c^2_k(t) 2^{-2ks} \| \nabla_h \, v \|_{g^{0,0}}^2 \| v \|_{H^{s,0}} \| \nabla_h \, v \|_{H^{s,0}}.
\]
While by applying Lemma A.1 and using the fact that \( s > -1 \), we find
\[
\left| \langle \Delta_k^h (R^h(v^h, \nabla_h \, v)) \mid \Delta_k^h \rangle \right| \lesssim 2^k \sum_{k' \geq k - 3} \| \Delta_k^h \nabla_h \, v \|_{L^2_h(L^\infty)} \| \Delta_k^h \nabla_h \, v \|_{L^2_h(L^\infty)} \| \Delta_k^h \nabla_h \, v \|_{L^2_h(L^\infty)}
\lesssim 2^k \left( \sum_{k' \geq k - 3} c_k(t) 2^{-k' (s + 1)} \right) c_k(t) 2^{-ks} \| \nabla_h \, v \|_{g^{0,0}}^2 \| v \|_{H^{s,0}} \| \nabla_h \, v \|_{H^{s,0}}
\lesssim c^2_k(t) 2^{-2ks} \| \nabla_h \, v \|_{g^{0,0}}^2 \| v \|_{H^{s,0}} \| \nabla_h \, v \|_{H^{s,0}}.
By summarizing the above estimates, we obtain (2.7).

**Step 2.** The estimate of \((\Delta_k^h (v^3, \partial_3 v)) \mid \Delta_k^h \partial_3 v)\).

Once again we get, by applying Bony’s decomposition (A.1) to \(v^3 \partial_3 v\) in the horizontal variables, that
\[
\begin{align*}
\partial_3 v = T^h_3 \partial_3 v + \mathcal{T}^h_3 v^3 + R^h (v^3, \partial_3 v).
\end{align*}
\]
(2.15)

By virtue of Lemma A.3, it is easy to observe that

\[
\begin{align*}
\| (\Delta_k^h (T^h_3 \partial_3 v) \mid \Delta_k^h v) \| & \leq \sum_{|k|, |k'| \leq 4} \| S^h_{k' - 1} v^3 \|_{L^2_k (L^\infty_v)} \| \Delta_k^h \partial_3 v \|_{L^2_k (L^2_v)} \| \Delta_k^h v \|_{L^2}
\leq \sum_{|k|, |k'| \leq 4} c_k (t) 2^{-k'(s - 1)} c_k (t) 2^{-k(s + 1)}
\times \| v^3 \|_{L^\infty_k (L^\infty_v)} \| \partial_3 v \|_{H^{s - 0}} \| v \|_{H^{s - 0}} \| \nabla v \|_{H^{s - 0}}
\leq c^2_k (t) 2^{-2k s} \| v^3 \|_{B^{s, 1}_0} \| \nabla v \|_{B^{s, 1}_0} \| v \|_{B^{s, 1}_0} \| \nabla v \|_{H^{s - 0}}.
\end{align*}
\]

While we get, by using integration by parts, that

\[
\begin{align*}
\Delta_k^h (T^h_3 \partial_3 v) \mid \Delta_k^h v) = - (\Delta_k^h (T^h_3 \partial_3 v^3) \mid \Delta_k^h v) - (\Delta_k^h (T^h_3 v^3) \mid \Delta_k^h \partial_3 v).
\end{align*}
\]

Using \(\text{div} v = 0\) and Lemma A.3, we find

\[
\begin{align*}
\| (\Delta_k^h (T^h_3 \partial_3 v^3) \mid \Delta_k^h v) \| & \leq \sum_{|k|, |k'| \leq 4} \| S^h_{k' - 1} v \|_{L^2_k (L^\infty_v)} \| \Delta_k^h \text{div} v \|_{L^2_k} \| \Delta_k^h v \|_{L^2_k (L^\infty_v)}
\leq \sum_{|k|, |k'| \leq 4} c_k (t) 2^{-k s} c_k (t) 2^{-k s} \| v \|_{L^2_k (L^\infty_v)}
\times \| \nabla v \|_{H^{s - 0}} \| \partial_3 v \|_{H^{s - 0}}
\leq c^2_k (t) 2^{-2k s} \| v \|_{B^{s, 1}_0} \| \nabla v \|_{B^{s, 1}_0} \| v \|_{B^{s, 1}_0} \| \nabla v \|_{H^{s - 0}}.
\end{align*}
\]

Similarly, one has

\[
\begin{align*}
\| (\Delta_k^h (T^h_3 v^3) \mid \Delta_k^h \partial_3 v) \| & \leq \sum_{|k|, |k'| \leq 4} \| S^h_{k' - 1} v \|_{L^2_k (L^\infty_v)} \| \Delta_k^h v^3 \|_{L^2} \| \Delta_k^h \partial_3 v \|_{L^2_k (L^\infty_v)}
\leq \sum_{|k|, |k'| \leq 4} c_k (t) 2^{-k'(s + 1)} c_k (t) 2^{-k(s - 1)} \| v \|_{L^2_k (L^\infty_v)}
\times \| \nabla v \|_{H^{s - 0}} \| \partial_3 v \|_{H^{s - 0}}
\leq c^2_k (t) 2^{-2k s} \| v \|_{B^{s, 1}_0} \| \nabla v \|_{B^{s, 1}_0} \| v \|_{B^{s, 1}_0} \| \nabla v \|_{H^{s - 0}}.
\end{align*}
\]

Along the same line, by using integration by parts, we find

\[
\begin{align*}
(\Delta_k^h (R^h (v^3, \partial_3 v)) \mid \Delta_k^h v) = -(\Delta_k^h (R^h (\partial_3 v^3, v)) \mid \Delta_k^h v) - (\Delta_k^h (R^h (v^3, v)) \mid \Delta_k^h \partial_3 v).
\end{align*}
\]

Notice that \(s > -1\), by using div \(v = 0\) and Lemma A.1, we obtain

\[
\begin{align*}
\| (\Delta_k^h (R^h (v^3, \partial_3 v)) \mid \Delta_k^h v) \| & \leq 2k \sum_{k \geq k - 3} \| \Delta_k^h v \|_{L^2_k (L^\infty_v)} \| \Delta_k^h \partial_3 v \|_{L^2_k (L^\infty_v)} \| \Delta_k^h v \|_{L^2}
\leq 2k \sum_{k \geq k - 3} c_k (t) 2^{-k'(s + 1)} c_k (t) 2^{-k s} \| v \|_{L^2_k (L^\infty_v)}
\times \| \nabla v \|_{H^{s - 0}} \| \partial_3 v \|_{H^{s - 0}}
\leq c^2_k (t) 2^{-2k s} \| v \|_{B^{s, 1}_0} \| \nabla v \|_{B^{s, 1}_0} \| v \|_{B^{s, 1}_0} \| \nabla v \|_{H^{s - 0}}.
\end{align*}
\]

Similarly, by virtue of Lemma A.3, one has

\[
\begin{align*}
\| (\Delta_k^h (R^h (v^3, v)) \mid \Delta_k^h \partial_3 v) \| & \leq \sum_{k \geq k - 3} \| \Delta_k^h v \|_{L^2_k (L^\infty_v)} \| \Delta_k^h v^3 \|_{L^2} \| \Delta_k^h \partial_3 v \|_{L^2_k (L^\infty_v)}
\leq \sum_{k \geq k - 3} c_k (t) 2^{-k'(s + 1)} c_k (t) 2^{-k(s - 1)} \| v \|_{L^2_k (L^\infty_v)}
\times \| \nabla v \|_{H^{s - 0}} \| \partial_3 v \|_{H^{s - 0}}
\leq c^2_k (t) 2^{-2k s} \| v \|_{B^{s, 1}_0} \| \nabla v \|_{B^{s, 1}_0} \| v \|_{B^{s, 1}_0} \| \nabla v \|_{H^{s - 0}}.
\end{align*}
\]
By summarizing the above estimates, we obtain (2.8). This finishes the proof of Lemma 2.2. □

3. LARGE TIME BEHAVIOR OF THE GLOBAL SMALL SOLUTION TO (1.1)

In this section, we shall present the proof of Theorem 1.3 concerning the large time behavior of the global small solution to (1.1), especially the enhanced dissipation for the third component of the solution. In order to do so, we need several auxiliary estimates which will be presented in the following subsections.

3.1. The estimate of $\|\nabla_h v(t)\|_{L^2}$. The goal of this subsection is to present the $L^2$ estimate to the horizontal derivatives of $v$.

Lemma 3.1. Let $v$ be a smooth enough solution of (1.1) on $[0,T]$. Then for $t \leq T$, one has

$$\frac{d}{dt} \|\nabla_h v(t)\|_{L^2}^2 + \|\Delta_h v\|_{L^2}^2 \leq C(\|\nabla_h v\|_{L^2}^2 + \|\partial_3 v\|_{H^4}^2)\|\nabla_h v\|_{L^2}^2. \quad (3.1)$$

Proof. By taking $L^2$-inner product of the $v$ equation of (1.1) with $-\Delta_h v$ and using integration by parts, we find

$$\frac{1}{2} \frac{d}{dt} \|\nabla_h v\|_{L^2}^2 + \|\Delta_h v\|_{L^2}^2 = (v \cdot \nabla v | \Delta_h v). \quad (3.2)$$

To handle the term $(v \cdot \nabla v | \Delta_h v)$, we get, by using integrating by parts and $\text{div } v = 0$, that

$$(v \cdot \nabla v | \Delta_h v) = -\sum_{j=1}^2 \left((\partial_j v \cdot \nabla v | \partial_j v) + (v \cdot \nabla \partial_j v | \partial_j v)\right)$$

Yet it follows from (A.3) that

$$\sum_{j=1}^2 |(\partial_j v^h \cdot \nabla_h v | \partial_j v)| \lesssim \|\nabla_h v^h\|_{L^2(L^\infty)} \|\nabla_h v\|_{L^2(L^2)}^2 \quad (3.3)$$

On the other hand, we observe that

$$\sum_{j=1}^2 |(\partial_j v^3 \partial_3 v | \partial_j v)| \lesssim \|\nabla_h v^3 \partial_3 v\|_{L^2} \|\nabla_h v\|_{L^2}. \quad (3.4)$$

Applying Bony’s decomposition (A.1) to $\nabla_h v^3 \partial_3 v$ in the horizontal variables yields

$$\nabla_h v^3 \partial_3 v = T_{h,v^3}^{h} \partial_3 v + \Delta_h v^3 \nabla_h \partial_3 v + R^h(\nabla_h v^3, \partial_3 v).$$

For the term $T_{h,v^3}^{h} \partial_3 v$, we have

$$\|\Delta_h (T_{h,v^3}^{h} \partial_3 v)\|_{L^2} \lesssim \sum_{|k-k'| \leq 4} \|\nabla_h v^3\|_{L^\infty} \|\Delta_h^k \partial_3 v\|_{L^2} \|\Delta_h^{k'} \partial_3 v\|_{L^2} \quad (3.5)$$

Yet it follows from Lemma A.1, (A.3) and $\text{div } v = 0$ that

$$\|\nabla_h v^3\|_{L^\infty} \lesssim \sum_{l \leq k' - 2} 2^{2l} \|\Delta_h^l v^3\|_{L^2(L^\infty)} \lesssim \sum_{l \leq k' - 2} 2^{2l} \|\Delta_h^l v^3\|_{L^2}^{1/2} \|\Delta_h^{l'} v^3\|_{L^2}^{1/2} \lesssim \sum_{l \leq k' - 2} 2^{2l} \|\Delta_h^l v^3\|_{L^2}^{1/2} \|\Delta_h^l (\nabla_h \cdot v^h)\|_{L^2}^{1/2} \lesssim 2^{k' - 2} \|\Delta_h v\|_{L^2},$$

from which, we infer

$$\|\Delta_h (T_{h,v^3}^{h} \partial_3 v)\|_{L^2} \lesssim \|\Delta_h v\|_{L^2} \sum_{|k-k'| \leq 4} 2^{k'} \|\Delta_h^k \partial_3 v\|_{L^2} \lesssim c_h(t) \|\Delta_h v\|_{L^2} \|\partial_3 v\|_{H^{4/6}}. \quad (3.6)$$
Similarly, one has
\[
\| \Delta_k^h (T_{\partial_3 v} \nabla_h v^3) \|_{L^2} \lesssim \sum_{|k-k'| \leq 4} \| S_{k-1}^h \partial_3 v \|_{L^2(L^\infty_h)} \| \Delta_k^h \nabla_h v^3 \|_{L^\infty_h(L^2_h)} \\
\lesssim \| \partial_3 v \|_{\dot{H}^{1/2}_h} \cdot \sum_{|k-k'| \leq 4} 2^k \| \Delta_k^h \nabla_h v^3 \|_{L^2_h} \| \Delta_k^h \nabla_h \partial_3 v \|_{L^2_h} \\
\lesssim c_k(t) \| \Delta_h v \|_{L^2} \| \partial_3 v \|_{\dot{H}^{1/2}_h}.
\]

Finally we deduce from Lemma A.1 and \( \text{div} v = 0 \) that
\[
\| \Delta_k^h (R^h (\nabla_h v^3, \partial_3 v)) \|_{L^2} \lesssim 2^k \sum_{k' \geq k-3} \| \Delta_{k'}^h \nabla_h v^3 \|_{L^\infty_h(L^2_h)} \| \Delta_{k'}^h \partial_3 v \|_{L^2_h} \\
\lesssim 2^k \sum_{k' \geq k-3} \| \Delta_{k'}^h \nabla_h v^3 \|_{L^2_h} \| \Delta_{k'}^h \nabla_h \partial_3 v \|_{L^2_h} \cdot \| \Delta_{k'}^h \partial_3 v \|_{L^2_h} \\
\lesssim 2^k \sum_{k' \geq k-3} c_{k'}(t) 2^{-k'} \| \Delta_k v \|_{L^2_h} \| \partial_3 v \|_{\dot{H}^{1/2}_h} \\
\lesssim c_k(t) \| \Delta_h v \|_{L^2} \| \partial_3 v \|_{\dot{H}^{1/2}_h}.
\]

As a result, it comes out
\[
\| \nabla_h v^3 \partial_3 v \|_{L^2} \lesssim \| \partial_3 v \|_{\dot{H}^{1/2}_h} \| \Delta_h v \|_{L^2}.
\]

By inserting the above estimate into (3.4), we achieve
\[
\sum_{j=1}^{2} \left| (\partial_j v^3 \partial_3 v \mid \partial_3 v) \right| \lesssim \| \partial_3 v \|_{\dot{H}^{1/2}_h} \| \Delta_h v \|_{L^2} \| \nabla_h v \|_{L^2}. \tag{3.5}
\]

Substituting (3.3) and (3.5) into (3.2) gives rise to
\[
1 \frac{d}{dt} \| \nabla_h v \|_{L^2}^2 + \| \Delta_h v \|_{L^2}^2 \lesssim (\| \nabla_h v \|_{\dot{H}^{1/2}_h} + \| \partial_3 v \|_{\dot{H}^{1/2}_h}) \| \nabla_h v \|_{L^2} \| \Delta_h v \|_{L^2},
\]
which implies (3.1). This finishes the proof of Lemma 3.1. \( \square \)

3.2. The estimate of \( \| \partial_3 v(t) \|_{\dot{H}^{1/2}_h} \). In order to close the estimate (3.1), we need to handle the estimate of \( \| \partial_3 v \|_{L^2_t(\dot{H}^{1/2}_h)} \), which is the purpose of this subsection.

**Lemma 3.2.** Let \( v \) be a smooth enough solution of (1.1) on \([0, T]\). Then for any \( t \leq T \), one has
\[
\frac{d}{dt} \| \partial_3 v(t) \|_{\dot{H}^{1/2}_h}^2 + \| \partial_3 v \|_{\dot{H}^{1/2}_h}^2 \leq C \left( \| \nabla_h v \|_{L^2}^2 \| \partial_3 v \|_{\dot{H}^{1/2}_h}^2 \right. \\
+ \| \nabla_h v^3 \|_{L^2}^2 \| \text{div}_h v^h \|_{L^2}^2 \| \partial_3 v \|_{\dot{H}^{1/2}_h} \right). \tag{3.6}
\]

**Proof.** We first get, by applying the operator \( \Delta_k^h \) to (1.1) and then taking \( L^2 \)-inner product of the resulting equations with \( -\partial_3^2 \Delta_k^h v \), that
\[
\frac{d}{dt} \| \partial_3 v \|_{L^2}^2 + \| \Delta_h \partial_3 v \|_{L^2}^2 = (\Delta_k^h (\partial_3 v \cdot \nabla v) \mid \Delta_k^h \partial_3 v) + (\Delta_k^h (v \cdot \nabla \partial_3 v) \mid \Delta_k^h \partial_3 v). \tag{3.7}
\]

- **Estimate of** \( (\Delta_k^h (\partial_3 v \cdot \nabla v) \mid \Delta_k^h \partial_3 v) \).

Due to \( \text{div} v = 0 \), we write
\[
(\Delta_k^h (\partial_3 v \cdot \nabla v) \mid \Delta_k^h \partial_3 v) = (\Delta_k^h (\partial_3 v^b \cdot \nabla_h v) \mid \Delta_k^h \partial_3 v) - (\Delta_k^h (\text{div}_h v^b) \partial_3 v \mid \Delta_k^h \partial_3 v). \tag{3.8}
\]

By applying Bony’s decomposition (A.1) to \( \partial_3 v^b \cdot \nabla_h v \) in the horizontal variables, we get
\[
\partial_3 v^b \cdot \nabla_h v = T_{\partial_3 v^b} \cdot \nabla_h v + T_{\nabla_h} \partial_3 v^b + R^h(\partial_3 v^b, \nabla_h v).
\]
For the term $T^h_{k_0} v, \nabla_h v$, we have

$$\left| \left( \Delta^h_k \left( T^h_{k_0} v, \nabla_h v \right) \right) \right| \lesssim \sum_{|k-k'| \leq 4} \| S^h_{k_0-1} \partial_3 v^h \|_{L^\infty_h (L^2_h)} \| \Delta^h_k \nabla_h v \|_{L^2_h (L^\infty_h)} \| \Delta^h_k \partial_3 v \|_{L^2}$$

$$\lesssim 2^{-k} \sum_{|k-k'| \leq 4} 2^{k'} \| S^h_{k'-1} \partial_3 v^h \|_{L^2} \| \Delta^h_{k'} \nabla_h v \|_{L^2} \| \Delta^h_k \nabla_h \partial_3 v \|_{L^2}$$

$$\lesssim c_2^2(t) 2^k \| \nabla_h v \|_{L^1_{g_0}} \| \partial_3 v \|_{H^{-1/2}_0} \| \nabla_h \partial_3 v \|_{H^{-1/2}_0}.$$

While for term $T^h_{k_0} v, \partial_3 v^h$, we get, by applying (A.3), that

$$\left| \left( \Delta^h_k \left( T^h_{k_0} v, \partial_3 v^h \right) \right) \right| \lesssim \sum_{|k-k'| \leq 4} \| S^h_{k_0-1} \nabla_h v \|_{L^2_h (L^\infty_h)} \| \Delta^h_k \partial_3 v^h \|_{L^2_h (L^\infty_h)} \| \Delta^h_k \partial_3 v \|_{L^2}$$

$$\lesssim c_2^2(t) 2^k \| \nabla_h v \|_{L^1_{g_0}} \| \partial_3 v \|_{H^{-1/2}_0} \| \nabla_h \partial_3 v \|_{H^{-1/2}_0}.$$

Finally by applying Lemma A.1, we obtain

$$\left| \left( \Delta^h_k (R^h (\partial_3 v^h, \nabla_h v)) \right) \right| \lesssim 2^k \sum_{k' \geq k-3} \| \Delta^h_k \partial_3 v^h \|_{L^2} \| \Delta^h_k \nabla_h v \|_{L^2} \| \Delta^h_k \partial_3 v \|_{L^2}$$

$$\lesssim 2^k \left( \sum_{k' \geq k-3} c_k(t) 2^{-k'} \right) \| \Delta^h_k \nabla_h \partial_3 v \|_{H^{-1/2}_0} \| \nabla_h v \|_{L^1_{g_0}} \| \partial_3 v \|_{H^{-1/2}_0}$$

$$\lesssim c_2^2(t) 2^k \| \nabla_h v \|_{L^1_{g_0}} \| \partial_3 v \|_{H^{-1/2}_0} \| \nabla_h \partial_3 v \|_{H^{-1/2}_0}.$$

As a result, it comes out

$$\left| \left( \Delta^h_k (\partial_3 v^h \cdot \nabla_h v) \right) \right| \lesssim c_2^2(t) 2^k \| \nabla_h v \|_{L^1_{g_0}} \| \partial_3 v \|_{H^{-1/2}_0} \| \nabla_h \partial_3 v \|_{H^{-1/2}_0}. \quad (3.9)$$

The estimate of the second term in (3.8) is the same. Therefore, we obtain

$$\left| \left( \Delta^h_k (\partial_3 v \cdot \nabla v) \right) \right| \lesssim c_2^2(t) 2^k \| \nabla_h v \|_{L^1_{g_0}} \| \partial_3 v \|_{H^{-1/2}_0} \| \nabla_h \partial_3 v \|_{H^{-1/2}_0}. \quad (3.10)$$

- **Estimate for term** $\left| \left( \Delta^h_k (v \cdot \nabla \partial_3 v) \right) \right|$.

By applying Bony’s decomposition (A.1) for $v \cdot \nabla \partial_3 v$ in the horizontal variable, we write

$$v \cdot \nabla \partial_3 v = T^h_v \cdot \nabla \partial_3 v + T^h_{\nabla \partial_3 v} \cdot v + R^h (v, \nabla \partial_3 v).$$

Let us first deal with the estimate of $\left( \Delta^h_k (T^h_v \nabla \partial_3 v) \right) \left( \Delta^h_k \partial_3 v \right)$. Indeed we get, by using a standard commutator’s argument, that

$$\left( \Delta^h_k (T^h_v \cdot \nabla \partial_3 v) \right) \left( \Delta^h_k \partial_3 v \right) = \sum_{|k-k'| \leq 4} \left( \Delta^h_k (S^h_{k_0-1} v \cdot \Delta^h_k, \nabla \partial_3 v) \right) \left( \Delta^h_k \partial_3 v \right) = I_k + II_k + III_k,$$

where

$$I_k \overset{\text{def}}{=} \sum_{|k-k'| \leq 4} \left( \Delta^h_k (S^h_{k_0-1} v - S^h_{k_0-1} v) \cdot \nabla \Delta^h_k \partial_3 v \right) \left( \Delta^h_k \partial_3 v \right),$$

$$II_k \overset{\text{def}}{=} \sum_{|k-k'| \leq 4} \left( \Delta^h_k \left( S^h_{k_0-1} v \cdot \nabla \Delta^h_k \partial_3 v \right) \right) \left( \Delta^h_k \partial_3 v \right),$$

$$III_k \overset{\text{def}}{=} \left( S^h_{k_0-1} v \cdot \nabla \Delta^h_k \partial_3 v \right) \left( \Delta^h_k \partial_3 v \right).$$

Due to $\text{div} \ v = 0$, we find

$$III_k = -\frac{1}{2} \left( \nabla \cdot \Delta^h_{k_0-1} v \Delta^h_k \partial_3 v \right) \left( \Delta^h_k \partial_3 v \right) = 0. \quad (3.11)$$

For $I_k$, we have

$$|I_k| \lesssim \sum_{|k-k'| \leq 4} \sum_{|k-j| \leq 4} \left( ||\Delta^h_k (\Delta^h_j v \cdot \nabla \Delta^h_{k-j} \partial_3 v) \right) \left( \Delta^h_j \partial_3 v \right)|$$

$$\lesssim \sum_{|k-k'| \leq 4} \sum_{|k-j| \leq 4} \left( ||\Delta^h_k v^h ||_{L^\infty} ||\nabla \Delta^h_{k-j} \partial_3 v ||_{L^2} + ||\Delta^h_j v^h ||_{L^2} ||\Delta^h_{k-j} \partial_3 v ||_{L^2} \right) ||\Delta^h_k \partial_3 v ||_{L^2}.$$
Yet it follows from Lemma A.1 that
\[
\|\Delta_k^h v^h\|_{L^\infty} \lesssim 2^k \|\Delta_k^h v^h\|_{L^\infty(L^\infty)} \lesssim \|\Delta_k^h v^h\|_{L^\infty} \lesssim \|\Delta_k^h v^h\|_{H^{-\frac{1}{2},0}},
\]
\[
\|\Delta_k^h v^3\|_{L^\infty} \lesssim 2^k \|\Delta_k^h v^3\|_{L^\infty} \lesssim \|\Delta_k^h v^3\|_{L^\infty} \lesssim 2^k \|\Delta_k^h v^3\|_{L^\infty} \|\Delta_k^h v^3\|_{L^2} \div v^h \|_{L^2}.
\]
so that we obtain
\[
|I_k| \lesssim \|\Delta_k^h v^h\|_{L^\infty} \|\Delta_k^h v^h\|_{L^2} \|\Delta_k^h v^h\|_{L^2} \sum_{|k-k'| \leq 4} \|\Delta_k^h \nabla v^h\|_{L^2} \|\Delta_k^h \partial_3 v^h\|_{L^2} \sum_{|k-k'| \leq 4} \|\Delta_k^h \partial_3^2 v^h\|_{L^2} \|\Delta_k^h \partial_3 v^h\|_{L^2},
\]
\[
\lesssim c_k^2(t) 2^k \|\Delta_k^h v^h\|_{L^\infty} \|\nabla v^h\|_{L^\infty} \|\Delta_k^h v^h\|_{H^{-\frac{1}{2},0}} \|\Delta_k^h \partial_3 v^h\|_{L^2} \|\nabla v^h\|_{L^2} \|\div v^h\|_{L^2} \|\Delta_k^h v^3\|_{L^2} \|\Delta_k^h \partial_3 v^3\|_{L^2} \|\partial_3 v^h\|_{L^2}.
\]
For $I_k$, we get by applying the commutator's estimate (A.2), that
\[
|I_k| \lesssim \|\nabla v^h\|_{L^\infty} \|\nabla v^h\|_{L^2} \|\Delta_k^h v^3\|_{L^2} \|\Delta_k^h \partial_3 v^3\|_{L^2} \|\Delta_k^h \partial_3 v^3\|_{L^2} \|\Delta_k^h \partial_3 v^h\|_{L^2}.
\]
Notice that
\[
\|\nabla v^h\|_{L^\infty} \|\nabla v^h\|_{L^2} \|\Delta_k^h v^3\|_{L^2} \|\Delta_k^h \partial_3 v^3\|_{L^2} \|\Delta_k^h \partial_3 v^3\|_{L^2} \|\Delta_k^h \partial_3 v^h\|_{L^2}.
\]
from which, we deduce from a similar derivation of (3.12), that
\[
|I_k| \lesssim c_k^2(t) 2^k \|\nabla v^h\|_{L^\infty} \|\nabla v^h\|_{L^2} \|\Delta_k^h v^3\|_{L^2} \|\Delta_k^h \partial_3 v^3\|_{L^2} \|\partial_3 v^h\|_{L^2} \|\div v^h\|_{L^2} \|\Delta_k^h \partial_3 v^3\|_{L^2} \|\partial_3 v^h\|_{L^2}.
\]
By summarizing the estimates (3.11), (3.12) and (3.13), we obtain
\[
|\Delta_k^h (T^h v \cdot \nabla \partial_3 v) | \lesssim c_k^2(t) 2^k \|\nabla v^h\|_{L^\infty} \|\nabla v^h\|_{L^2} \|\Delta_k^h \partial_3 v^h\|_{L^2} \|\div v^h\|_{L^2} \|\Delta_k^h \partial_3 v^3\|_{L^2} \|\partial_3 v^h\|_{L^2}.
\]
For the term $\Delta_k^h (T^h v \cdot \nabla \partial_3 v) | \Delta_k^h \partial_3 v$), we observe that
\[
|\Delta_k^h (T^h v \cdot \nabla \partial_3 v) | \lesssim \sum_{|k-k'| \leq 4} \left( \|\nabla v^h\|_{L^\infty} \|\nabla v^h\|_{L^2} \|\Delta_k^h v^h\|_{L^\infty} \|\Delta_k^h v^3\|_{L^\infty} \right) \|\Delta_k^h \partial_3 v^h\|_{L^2} \|\Delta_k^h \partial_3 v^3\|_{L^2} \|\partial_3 v^h\|_{L^2}.
\]
It follows from Lemma A.1 and $\div v = 0$, that
\[
\|\nabla v^h\|_{L^\infty} \|\nabla v^h\|_{L^2} \|\Delta_k^h v^h\|_{L^\infty} \|\nabla v^h\|_{L^\infty} \|\nabla v^h\|_{L^2} \|\Delta_k^h \partial_3 v^h\|_{L^2} \|\Delta_k^h \partial_3 v^3\|_{L^2} \|\partial_3 v^h\|_{L^2}.
\]
from which, we deduce that
\[
|\Delta_k^h (T^h v \cdot \nabla \partial_3 v) | \lesssim c_k^2(t) 2^k \|\nabla v^h\|_{L^\infty} \|\nabla v^h\|_{L^2} \|\Delta_k^h \partial_3 v^h\|_{L^2} \|\div v^h\|_{L^2} \|\Delta_k^h \partial_3 v^3\|_{L^2} \|\partial_3 v^h\|_{L^2}.
\]
Finally we deal with the estimate of $\Delta_k^h (R^h(v, \nabla \partial_3 v) | \Delta_k^h \partial_3 v)$. Indeed by applying Lemma A.1, we find
\[
|\Delta_k^h (R^h(v, \nabla \partial_3 v) | \Delta_k^h \partial_3 v) |
\lesssim 2^k \sum_{k' \geq k-3} \left( \|\Delta_k^h v^h\|_{L^\infty} \|\nabla \Delta_k^h \partial_3 v^h\|_{L^2} + \|\Delta_k^h v^3\|_{L^\infty} \|\Delta_k^h \partial_3 v^3\|_{L^2} \right) \|\Delta_k^h \partial_3 v^h\|_{L^2} \|\div v^h\|_{L^2} \|\Delta_k^h \partial_3 v^3\|_{L^2} \|\partial_3 v^h\|_{L^2}.
\]
which implies
\[
\left| (\Delta_k^h (R^h (v, \nabla \partial_3 v)) | \Delta_k^h \partial_3 v) \right| \lesssim c_k^2 (t) 2^k \left( \| \nabla_h v \|_{L^5} \| \nabla_h \partial_3 v \|_{H^{\frac{1}{2}, 0}} + \| \nabla_h v^{H^1 \| L^2_h} \| \| \partial_3^2 v \|_{L^2} \right) \| \partial_3 v \|_{H^{\frac{1}{2}, 0}}. \tag{3.16}
\]

Thanks to (3.14), (3.15) and (3.16), we obtain
\[
\left| (\Delta_k^h (v \cdot \nabla \partial_3 v) | \Delta_k^h \partial_3 v) \right| \lesssim c_k^2 (t) 2^k \left( \| \nabla_h v \|_{L^5} \| \nabla_h \partial_3 v \|_{H^{\frac{1}{2}, 0}} + \| \nabla_h v^{H^1 \| L^2_h} \| \| \partial_3^2 v \|_{L^2} \right) \| \partial_3 v \|_{H^{\frac{1}{2}, 0}}. \tag{3.17}
\]

By inserting the estimates (3.10) and (3.17) into (3.7) and multiplying the resulting inequality by \(2^{-k}\) and summing over \(Z\) with respect to \(k\), we achieve
\[
\frac{1}{2} \frac{d}{dt} \| \partial_3 v \|_{H^{\frac{1}{2}, 0}}^2 + \| \partial_3 v \|_{H^{\frac{1}{2}, 0}}^2 \leq C \left( \| \nabla_h v \|_{L^5} \| \nabla_h \partial_3 v \|_{H^{\frac{1}{2}, 0}} + \| \nabla_h v^{H^1 \| L^2_h} \| \| \partial_3^2 v \|_{L^2} \right) \| \partial_3 v \|_{H^{\frac{1}{2}, 0}},
\]

which leads to (3.6). This finishes the proof of Lemma 3.2. \(\square\)

3.3. The estimate of \(\| v(t) \|_{H^{\frac{1}{2}, 0}}\). In order to derive the decay in time estimate for the solutions of (1.1), we need the negative derivative estimate of \(v\) in the horizontal variables (see (3.25) below).

Lemma 3.3. Let \(s \in (0, 1)\) and \(v\) be a smooth enough solution of (1.1) on \([0, T]\). Then for \(t \leq T\), we have
\[
\frac{d}{dt} \| v(t) \|_{H^{-s, 0}} \leq C \left( \| \nabla_h v \|_{L^5} \| \nabla_h \partial_3 v \|_{H^{\frac{1}{2}, 0}} + \| \nabla_h v^{H^1 \| L^2_h} \| \| \partial_3^2 v \|_{L^2} \right) \| v \|_{H^{\frac{1}{2}, 0}}. \tag{3.18}
\]

Proof. In view of (2.6) and (2.7), it remains to handle the estimate of \( (\Delta_k^h (v^3 \partial_3 v) | \Delta_k^h v) \). Indeed due to \(\text{div} \ v = 0\) and \(s > 0\), one has
\[
\| S_{k-1}^h v^3 \|_{L^2_h (L^\infty)} \lesssim \sum_{j \leq k - 2} \| \Delta_j^h v^3 \|_{L^2} \| \Delta_j^h \partial_3 v^3 \|_{L^2} \lesssim c_k (t) 2^k \| v^3 \|_{H^{\frac{1}{2}, 0}} \| \text{div} \ v^h \|_{H^{\frac{1}{2}, 0}},
\]

so that we deduce
\[
\left| (\Delta_k^h (T_{k, v^3} \partial_3 v) | \Delta_k^h v) \right| \lesssim \sum_{|k - l| \leq 4} \| S_{k-1}^h v^3 \|_{L^2_h (L^\infty)} \| \Delta_k^h \partial_3 v \|_{L^2_h (L^\infty)} \| \Delta_k^h v \|_{L^2_h (L^\infty)} \lesssim \left( \sum_{|k - l| \leq 4} c_k (t) 2^{ks} \right) \| v^3 \|_{H^{\frac{1}{2}, 0}} \| \text{div} \ v^h \|_{H^{\frac{1}{2}, 0}} \| \partial_3 v \|_{H^{\frac{1}{2}, 0} \| v \|_{H^{\frac{1}{2}, 0}} \| \nabla_h v \|_{H^{\frac{1}{2}, 0}} \| \nabla_h v \|_{H^{\frac{1}{2}, 0}} \]
so that by applying Lemma A.1 once again, for \( s < 1 \), we infer

\[
| (\Delta^k_h (R^h (v^3, \partial_3 v)) | (\Delta^k_h v) | \lesssim 2^k \sum_{k' \geq k-3} \| \Delta^k_h v^3 \|_{L^2_k (L^\infty_x)} \| \Delta^k_h \partial_3 v \|_{L^2} \| \Delta^k_h v \|_{L^2} \\
\lesssim 2^k \left( \sum_{k' \geq k-3} c_k(t) 2^{k'(s-1)} \right) c_k(t) 2^{ks} \| \partial_3 v^h \|_{H^{s,0}_x} \| \nabla_h v \|_{H^{-s,0}_x} \| v \|_{H^{-r,0}_x} \\
\lesssim 2^k(t) 2^{2ks} \| \partial_3 v \|_{H^{s,0}_x} \| v \|_{H^{-s,0}_x} \| \nabla_h v \|_{H^{-r,0}_x}. 
\]

Then in view of (2.15), we deduce that

\[
| (\Delta^k_h (v^3 \partial_3 v) | (\Delta^k_h v) | \lesssim 2^k(t) 2^{-2ks} \| \partial_3 v \|_{H^{s,0}_x} \| v \|_{H^{-s,0}_x} \| \nabla_h v \|_{H^{-r,0}_x}. \tag{3.19}
\]

By inserting the estimates (2.7) and (3.19) into (2.6), and then multiplying the resulting inequality by \( 2^{2ks} \) and summing up \( k \) over \( \mathbb{Z} \), we arrive at

\[
\frac{1}{2} \frac{d}{dt} \| v(t) \|^2_{H^{-s,0}_x} + \| \nabla_h v(t) \|^2_{H^{-s,0}_x} \lesssim C \left( \| v \|_{L^2_t (H^{s,0}_x)} \| v \|_{L^2_k (H^{\frac{s}{2},0}_x)} \right) \left( \| v \|_{L^2_t (H^{s,0}_x)} \| v \|_{L^2_k (H^{\frac{s}{2},0}_x)} \right) \\
+ \left( \| \nabla_h v \|_{L^2_t (H^{\frac{s}{2},0}_x)} + \| \partial_3 v \|_{H^{s,0}_x} \right) \| v \|_{H^{-s,0}_x} \| \nabla_h v \|_{H^{-r,0}_x}, \tag{3.20}
\]

which leads to (3.18). This finishes the proof of Lemma 3.3. \( \square \)

3.4. **Proof of Theorem 1.3.** We are now in a position to complete the proof of Theorem 1.3 which relies on the continuity argument and Lemmas 3.1, 3.2 and 3.3.

**Proof of Theorem 1.3.** Under the assumption of (1.6), we deduce from Theorem 1.2 that the system (1.1) has a unique global solution which satisfies

\[
\| v \|_{L^\infty_t (H_0^{s,1})} + \| \nabla_h v \|_{L^2_t (H_0^{s,1})} \leq C \| v_0 \|_{H_0^{s,1}}. \tag{3.21}
\]

Next, we shall only present the *a priori* estimate for the smooth enough solution of the system (1.1).

In what follows, we divide the proof into the following steps:

**Step 1. Ansatz for the continuity argument.**

We denote

\[
T^* \equiv \sup \left\{ T > 0 : \| \partial_3 v \|_{L^2_t (H^{-\frac{s}{2},0})} + \| \nabla_h v \|_{L^2_t (H^{-s,0})} \leq \varepsilon \right\}, \tag{3.22}
\]

where \( \varepsilon \) is a small enough positive constant which will be determined later on.

Then for \( t \leq T^* \) and \( s \in (0,1) \), by using Gronwall’s inequality, we deduce from (2.1) and (3.18) that

\[
\| v \|_{L^2_t (H^{-s,0})} + \| \nabla_h v \|_{L^2_t (H^{-s,0})} \leq \| v_0 \|_{H^{-s,0}} \exp \left( C \left( \| v_0 \|_{L^2_t (H^{s,0})} + \| \partial_3 v \|_{L^2_t (H^{s,0})} \right) \right) \\
\leq \| v_0 \|_{H^{-s,0}} \exp \left( C (\varepsilon^2_0 + \varepsilon) \right) \leq e \| v_0 \|_{H^{-s,0}}, \tag{3.23}
\]

as long as \( c_0 \) in (1.6) and \( \varepsilon \) in (3.22) are so small that \( Cc_0^2 \leq \frac{1}{2} \) and \( CE \leq \frac{1}{2} \).

**Step 2. The decay estimate of \( \| v(t) \|_{L^2} \).**

Due to \( \text{div} \, v = 0 \), we get, by taking \( L^2 \)-inner product of the \( v \) equation of (1.1) with \( v \), that

\[
\frac{1}{2} \frac{d}{dt} \| v(t) \|^2_{L^2} + \| \nabla_h v \|^2_{L^2} = 0. \tag{3.24}
\]

While by applying Hölder’s inequality in the frequency space and using (3.23), we find

\[
\| v(t) \|_{L^2} \leq \| v(t) \|^2_{H^{-s,0}} \| \nabla_h v(t) \|^2_{L^2} \leq e^{\varepsilon^2} \| v_0 \|^2_{H^{-s,0}} \| \nabla_h v(t) \|^2_{L^2},
\]

which implies

\[
\| \nabla_h v(t) \|^2_{L^2} \geq \| v(t) \|^2_{H^{-s,0}} \left( C \| v_0 \|_{H^{-s,0}} \right)^2. \tag{3.25}
\]

Thanks to (3.24) and (3.25), we infer

\[
\frac{d}{dt} \| v(t) \|^2_{L^2} + \frac{2}{(C \| v_0 \|_{H^{-s,0}})^2} \left( \| v(t) \|^2_{L^2} \right)^{1+\frac{s}{2}} \leq 0,
\]
Then for any $t \leq T^*$, we obtain
\[
\|v(t)\|_{L^2}^2 \leq \left( \frac{1}{\|v_0\|_{L^2}^2 + 2s^{-1}(e\|v_0\|_{H^{-s,0}})^{-s}} \right)^s \leq CA_s(t)^{-s} \quad \text{with}
\]
\[
A_s \equiv \|v_0\|_{L^2}^2 + \|v_0\|_{H^{-s,0}}^2.
\]

Step 3. The decay estimate of $\|\nabla v(t)\|_{L^2}$.

Motivated by the study of the decay-in-time estimate for the derivatives of the global solutions to classical Navier-Stokes system (see [7] for instance), for any $0 \leq t_0 < t \leq T^*$, we get, by multiplying $t - t_0$ to (3.1), that
\[
\frac{d}{dt}[(t-t_0)\|\nabla v(t)\|_{L^2}^2] + [(t-t_0)\|\Delta v\|_{L^2}^2] \leq \|\nabla v\|_{L^2}^2 + C\left(\|\nabla v\|_{L^2}\|\nabla v\|_{L^2} + \|\partial_3 v\|_{H_{-s,0}}^2\right)(t-t_0)\|\nabla v\|_{L^2}^2.
\]

By applying Gronwall’s inequality and using (1.6) and (3.22) for sufficiently small $c_0$ and $\varepsilon$, we find
\[
(t-t_0)\|\nabla v(t)\|_{L^2}^2 + \int_{t_0}^t (t'-t_0)\|\Delta v(t')\|_{L^2}^2 dt' \\
\leq \int_{t_0}^t \|\nabla v(t')\|_{L^2}^2 dt' \exp\left(C\left(\|\nabla v\|_{L^2}^2\|\nabla v\|_{L^2} + \|\partial_3 v\|_{H_{-s,0}}^2\right)\right) \leq e \int_{t_0}^t \|\nabla v(t')\|_{L^2}^2 dt'.
\]

On the other hand, we get, by integrating (3.24) over $[\tau, t]$ for any $0 \leq \tau < t$, that
\[
\int_{\tau}^{t} \|\nabla v(t')\|_{L^2}^2 dt' \leq \frac{1}{2} \|v(\tau)\|_{L^2}^2,
\]
from which, (3.26) and (3.27), we deduce that
\[
\|\nabla v(t)\|_{L^2}^2 + \frac{2}{t} \int_{t/2}^t (t' - t/2)\|\Delta v(t')\|_{L^2}^2 dt' \\
\leq \frac{2e}{t} \int_{t/2}^t \|\nabla v(t')\|_{L^2}^2 dt' \leq \frac{e}{t} \|v(t/2)\|_{L^2}^2 \leq C_s A_s(t)^{-s} t^{-1} \quad \forall t \leq T^*.
\]

This together with (3.26) ensures that (1.11a) holds for $t \leq T^*$.

Step 4. The decay estimate of $\|v^3(t)\|_{L^2}$.

By taking space divergence to the $v$ equation of (1.1), we write
\[
-\Delta p = \nabla \cdot (v \cdot \nabla v) = \sum_{j,k=1}^3 \partial_j \partial_k (v^j v^k),
\]
which yields
\[
\nabla p = \nabla (-\Delta)^{-1} \sum_{j,k=1}^3 \partial_j \partial_k (v^j v^k).
\]

While by applying Fourier transform to the $v^3$ equation of (1.1), we find
\[
\partial_t \tilde{v}^3 + |\xi_h|^2 \tilde{v}^3 = -\mathcal{F}(v \cdot \nabla v^3 + \partial_3 p).
\]

Using $\text{div } v = 0$ and (3.29), we infer
\[
\tilde{v}^3(t, \xi) = e^{-t|\xi|^2} \tilde{v}^3_0(\xi) + \int_0^t e^{-(t-\tau)|\xi|^2} F_1(\tau, \xi) d\tau + \int_0^t e^{-(t-\tau)|\xi|^2} F_2(\tau, \xi) d\tau
\]
\[
\overset{\text{def}}{=} \tilde{v}^3_1(t, \xi) + \tilde{v}^3_{N_1}(t, \xi) + \tilde{v}^3_{N_2}(t, \xi) \quad \text{with}
\]
\[
F_1(t, \xi) \overset{\text{def}}{=} -i|\xi|^2 \tilde{v}^3_0(\xi) \sum_{j=1}^3 \xi_j \xi_k \tilde{v}^j(\xi, t, \xi) \quad \text{and} \quad F_2(t, \xi) \overset{\text{def}}{=} i|\xi|^2 \tilde{v}^3_0(\xi) \sum_{j=1}^3 \xi_j \xi_k \tilde{v}^j(\xi, t, \xi).
\]

By taking $L^2$-inner product of (3.30) with $\tilde{v}^3(t, \xi)$, we obtain
\[
\|\tilde{v}^3(t, \cdot)\|_{L^2}^2 \leq \|\tilde{v}^3_1(t, \cdot)\|_{L^2}^2 + \|\tilde{v}^3(t, \cdot)\|_{L^2}^2 + \left(\tilde{v}^3_{N_1}(t, \xi) + \tilde{v}^3_{N_2}(t, \xi) \right) \tilde{v}^3(t, \xi),
\]
from which, we infer
\[
\|\hat{v}^3(t, \cdot)\|_{L^2}^2 \leq \|\hat{v}^3_L(t, \cdot)\|_{L^2}^2 + 2\left|\left(\hat{u}_{N1}^3(t, \xi) + \hat{v}^3_{N2}(t, \xi)\right)\bar{v}^3(t, \xi)\right|.
\]  
(3.31)

(1) Decay estimate of \(\|v^3_L(t)\|_{L^2}\).

By virtue of the definition of \(\hat{v}^3_L(t, \xi)\) in (3.30), we write
\[
\|\hat{v}^3_L(t)\|_{L^2}^2 = \int_{\mathbb{R}^3} e^{-2t|\xi_h|^2}\hat{v}^3_0(\xi)^2 d\xi = \int_{|\xi_h| \leq |\xi_s|} e^{-2t|\xi_h|^2}\hat{v}^3_0(\xi)^2 d\xi + \int_{|\xi_h| \geq |\xi_s|} e^{-2t|\xi_h|^2}\hat{v}^3_0(\xi)^2 d\xi \overset{\text{def}}{=} A_1 + A_2.
\]  
(3.32)

Due to \(s \in (0,1)\), we get, by using \(\text{div} v_0 = 0\), that
\[
A_1 = \int_{|\xi_h| \leq |\xi_s|} e^{-2t|\xi_h|^2} \frac{1}{|\xi_s|^2} |i\xi_h\hat{v}^3_0(\xi)|^2 d\xi \\
\leq \int_{\mathbb{R}^3} \int_{|\xi_h| \geq |\xi_s|} \left|\frac{v^3_0(\xi_h, \xi_s)}{|\xi_s|^2}\right|^2 |\xi_h|^2 e^{-2t|\xi_h|^2}|d\xi_h| d\xi \\
\leq \int_{\mathbb{R}^3} e^{-2t|\xi_h|^2} |\xi_h|^3 + \frac{4}{3} \cdot |\xi_h|^{-2s} |\xi_h|^{-s + \frac{2}{3}} \hat{v}^3_0(\xi)^2 d\xi \\
\overset{\text{(3.34)}}{\leq} t^{-\left(\frac{4}{3} + \frac{2}{3}\right)} \int_{\mathbb{R}^3} |\hat{v}^3_0(\xi_h, \xi_s)|^2 |\xi_h|^{-2s} |\xi_s|^{-s + \frac{2}{3}} d\xi,
\]  
(3.33)

which implies
\[
A_1 \leq t^{-\left(\frac{4}{3} + \frac{2}{3}\right)} \|v^3_0\|_{\dot{H}^{-s,\frac{5}{4} - \frac{s}{2}}}^2.
\]  
(3.34)

While we observe that
\[
A_2 = \int_{|\xi_h| \geq |\xi_s|} e^{-2t|\xi_h|^2} |\xi_h|^3 + \frac{4}{3} \cdot |\xi_h|^{-2s} |\xi_h|^{-s + \frac{2}{3}} \hat{v}^3_0(\xi)^2 d\xi \\
\leq \int_{\mathbb{R}^3} e^{-2t|\xi_h|^2} |\xi_h|^3 + \frac{4}{3} \cdot |\xi_h|^{-2s} |\xi_h|^{-s + \frac{2}{3}} \hat{v}^3_0(\xi)^2 d\xi \\
\overset{\text{(3.34)}}{\leq} t^{-\left(\frac{4}{3} + \frac{2}{3}\right)} \int_{\mathbb{R}^3} |\hat{v}^3_0(\xi_h, \xi_s)|^2 |\xi_h|^{-2s} |\xi_s|^{-s + \frac{2}{3}} d\xi,
\]  
(3.35)

which gives rise to
\[
A_2 \leq t^{-\left(\frac{4}{3} + \frac{2}{3}\right)} \|v^3_0\|_{\dot{H}^{-s,\frac{5}{4} - \frac{s}{2}}}^2.
\]  
(3.36)

By inserting the estimates (3.33) and (3.34) into (3.32), we obtain
\[
\|v^3_L(t)\|_{L^2} \leq t^{-\left(\frac{4}{3} + \frac{2}{3}\right)} \|v_0\|_{\dot{H}^{-s,\frac{5}{4} - \frac{s}{2}}},
\]

which together with the fact that \(\|v^3_L(t)\|_{L^2} \leq \|v^3_0\|_{L^2}\) ensures that
\[
\|v^3_L(t)\|_{L^2} \leq C(\|v^3_0\|_{L^2} + \|v_0\|_{\dot{H}^{-s,\frac{5}{4} - \frac{s}{2}}})t^{-\left(\frac{4}{3} + \frac{2}{3}\right)}.
\]  
(3.37)

(2) Decay estimate for term involving \(v^3_{N1} + v^3_{N2}\).

By virtue of the definition of \(F_2(t, \xi)\) in (3.30), we get, by using \(\text{div} v = 0\), that
\[
\left(\hat{u}^3_{N2}(t, \xi) | \hat{v}^3(t, \xi)\right) = -\int_0^t e^{-t(\tau)} \langle |\xi_h|^{-2} \sum_{j=1}^3 \sum_{k=1}^2 \xi_j \xi_k v^j \bar{v}^k(\tau, \xi) d\tau \rangle \langle i\xi_h \hat{v}^3(t, \xi)\rangle \\
\overset{\text{(3.34)}}{=} \int_0^t e^{-t(\tau)} \langle |\xi_h|^{-2} \sum_{j=1}^3 \sum_{k=1}^2 \xi_j \xi_k v^j \bar{v}^k(\tau, \xi) d\tau \rangle \langle \bar{v}^3(t, \xi)\rangle,
\]

from which and the definition of \(F_1(t, \xi)\) in (3.30), we infer
\[
\left(\hat{u}^3_{N1}(t, \xi) + \hat{v}^3_{N2}(t, \xi) | \hat{v}^3(t, \xi)\right) \leq \int_0^t e^{-t(\tau)} \langle |\xi_h|^{-1} \bar{v} \otimes \bar{v}(\tau, \xi) d\tau \rangle |\nabla_v v(t)|_{L^2}.
\]  
(3.38)
Yet by applying Hölder’s inequalities, we have
\[
\left\| \int_0^t e^{-(t-\tau)\xi_h^2} |\xi_h||\xi|^{-1} \hat{v} \otimes \hat{v}(\tau, \xi) \, d\tau \right\|_{L^2_H} \\
\leq \left\| \|\xi|^{-1}\right\|_{L^2} \left\| \int_0^t e^{-(t-\tau)\xi_h^2} |\xi_h||\xi|^{-1} \hat{v} \otimes \hat{v}(\tau, \xi) \, d\tau \right\|_{L^\infty_\xi} \left\| \hat{v}(\tau, \xi) \right\|_{L^2_{\xi}} ,
\]
so that we get, by using Young’s inequality, that
\[
\left\| \int_0^t e^{-(t-\tau)\xi_h^2} |\xi|^{-1} \hat{v} \otimes \hat{v}(\tau, \xi) \right\|_{L^2_{\xi}} \lesssim \left\| \int_0^t (t-\tau)^{-\delta} \left| \int_0^1 \xi \hat{v} \otimes \hat{v}(\tau, \xi) \, d\xi \right| \, d\tau \right\|_{L^2_{\xi}} \lesssim \left\| \int_0^t (t-\tau)^{-\delta} \left| \int_0^1 \xi \hat{v} \otimes \hat{v}(\tau, \xi) \, d\xi \right| \, d\tau \right\|_{L^2_{\xi}} \leq CA \int_0^t (t-\tau)^{-\delta} (r)^{-s}(r)^{-\left(\frac{3}{4}\right)} d\tau.
\]
Notice that for \( \delta \in (s-\frac{3}{4}, \frac{3}{4}) \), one has
\[
\int_0^t (t-\tau)^{-\delta} (r)^{-s}(r)^{-\left(\frac{3}{4}\right)} d\tau \leq \int_0^t (t-\tau)^{-\delta} (r)^{-s}(r)^{-\left(\frac{3}{4}\right)} d\tau + \int_0^t (t-\tau)^{-\delta} (r)^{-s}(r)^{-\left(\frac{3}{4}\right)} d\tau \lesssim (t)^{-s} t^{\frac{3}{4}},
\]
we find
\[
\left\| \int_0^t e^{-(t-\tau)\xi_h^2} |\xi_h||\xi|^{-1} \hat{v} \otimes \hat{v}(\tau, \xi) \, d\tau \right\|_{L^2_H} \leq CA_s (t)^{-s} t^{\frac{3}{4}},
\]
which together with (3.28) and (3.36) ensures that
\[
\left| \left( v_{N1}^3(t, \xi) + v_{N2}^3(t, \xi) \right) \right| \leq CA_s^3 (t)^{-\frac{3}{4}} t^{-\frac{1}{4}}.
\]
By inserting the estimates (3.35) and (3.39) into (3.31), we achieve
\[
\left\| v(t) \right\|_{L^2} \leq CB_s (t)^{-\frac{3}{4}} t^{-\frac{1}{4}} \quad \text{with} \quad B_s \equiv \left\| v_0^3 \right\|_{L^2} + \left\| v_0 \right\|_{L^2}^2 + A_s^3 ,
\]
for any \( t \leq T^* \).

**Step 5. The decay estimate of \( \| \nabla_h v(t) \|_{L^2} \).**

We first get, by a similar derivation of (3.36), that
\[
\frac{1}{2} \frac{d}{dt} \left\| v(t) \right\|_{L^2}^2 + \left\| \nabla_h v(t) \right\|_{L^2}^2 \lesssim \left\| \int_0^t |\xi|^{-1} \hat{v} \otimes \hat{v}(\tau, \xi) \, d\tau \right\|_{L^2_{\xi}} \left\| \hat{v}(\tau, \xi) \right\|_{L^2_{\xi}}.
\]
Yet along the same line to the proof of (3.39), we find
\[
\left\| D_h |D|^{-1}(v \otimes v) \right\|_{L^2} = \left\| \xi_h \right\| |\xi|^{-1} \left\| \hat{v} \otimes \hat{v} \right\|_{L^2} \lesssim \left\| |\xi|^{-1} \left\| \xi_h \right\| \left\| \xi_h \right\| \left\| \hat{v} \otimes \hat{v} \right\|_{L^2} \lesssim \left\| \hat{v} \right\|_{L^2} \lesssim \left\| \hat{v} \right\|_{L^2} \lesssim \left\| \hat{v} \right\|_{L^2} \lesssim \left\| \hat{v} \right\|_{L^2}.
\]
By inserting the above estimate into (3.41) and integrating the resulting inequality over \([t/2, t]\), we achieve
\[
\frac{1}{2} \int_{t/2}^t \left\| \nabla_h v^3(t') \right\|_{L^2}^2 dt' \leq \frac{1}{2} \int_{t/2}^t \left\| v^3(t'/2) \right\|_{L^2}^2 + C \int_{t/2}^t \left\| v(t') \right\|_{L^2}^2 \left\| \nabla_h v(t') \right\|_{L^2}^2 dt',
\]
which together with (3.26), (3.28) and (3.40) implies
\[
\frac{1}{2} \int_{t/2}^t \left\| \nabla_h v^3(t') \right\|_{L^2}^2 dt' \leq B_s(t) - \frac{2}{3} s^{-1} t^{-\frac{2}{3}} \quad \forall t \leq T^*.
\] (3.42)
On the other hand, by taking \(L^2\)-inner product of the \(v^3\) equation of (1.1) with \(\Delta_h v^3\), we deduce from a similar derivation of (3.36) that
\[
\frac{1}{2} \frac{d}{dt} \left\| \nabla_h v^3(t) \right\|_{L^2}^2 + \left\| \Delta_h v^3 \right\|_{L^2}^2 \lesssim \left\| D_h |D|^{-1}(v \otimes v) \right\|_{L^2} \left\| \Delta_h v \right\|_{L^2}.
\] (3.43)
We observe that
\[
\left\| D_h |D|^{-1}(v \otimes v) \right\|_{L^2} = \left\| |\xi|^{-1} |\xi_h|^2 \hat{v} \otimes \hat{v} \right\|_{L^2} \lesssim \left\| |\xi|^{-1} \left\| \xi_h \right\| \left\| \xi_h \right\| \left\| \hat{v} \otimes \hat{v} \right\|_{L^2} \lesssim \left\| \hat{v} \right\|_{L^2} \lesssim \left\| \hat{v} \right\|_{L^2} \lesssim \left\| \hat{v} \right\|_{L^2} \lesssim \left\| \hat{v} \right\|_{L^2} \lesssim \left\| \hat{v} \right\|_{L^2}.
\]
By inserting the above estimate into (3.43), we find
\[
\frac{1}{2} \frac{d}{dt} \left\| \nabla_h v^3(t) \right\|_{L^2}^2 + \left\| \Delta_h v^3 \right\|_{L^2}^2 \leq C \left\| v \right\|_{L^2}^2 \left\| \nabla_h v \right\|_{L^2} \left\| \Delta_h v \right\|_{L^2}^2.
\]
For any \(0 \leq t_0 \leq t\), by multiplying the above inequality by \(t - t_0\) and then integrating the resulting inequality over \([t_0, t]\), we obtain
\[
(t - t_0) \left\| \nabla_h v^3(t) \right\|_{L^2}^2 \leq \int_{t_0}^t \left\| \nabla_h v^3(t') \right\|_{L^2}^2 dt' + C \int_{t_0}^t (t' - t_0) \left\| v(t') \right\|_{L^2}^2 \left\| \nabla_h v(t') \right\|_{L^2} \left\| \Delta_h v(t') \right\|_{L^2} dt'
\]
\[
\lesssim \int_{t_0}^t \left\| \nabla_h v^3(t') \right\|_{L^2}^2 dt' + \left( \int_{t_0}^t (t' - t_0) \left\| v(t') \right\|_{L^2}^2 \left\| \nabla_h v(t') \right\|_{L^2} \left\| \Delta_h v(t') \right\|_{L^2} dt' \right)^{\frac{1}{2}} \left( \int_{t_0}^t (t' - t_0) \left\| \Delta_h v(t') \right\|_{L^2}^2 dt' \right)^{\frac{1}{2}}.
\]
Taking \(t_0 = \frac{t}{2}\) in the above inequality and using (3.26), (3.28) and (3.42), we find
\[
\frac{1}{2} \left\| \nabla_h v^3(t) \right\|_{L^2}^2 \leq CB_s(t) - \frac{2}{3} s^{-1} t^{-\frac{2}{3}} \quad \forall t \leq T^*.
\] (3.44)
This together with (3.40) ensures that (1.12) holds for \(t \leq T^*\).

**Step 6. Closing of the continuity argument.**

To close the continuity argument, we shall use (3.6) in Lemma 3.2. We first observe that for any \(s_1 > 2\),
\[
\left\| \partial_s^2 v \right\|_{L^2} \lesssim \left\| \partial_s v \right\|_{L^2}^{\frac{3}{2}} \left\| v \right\|_{H^{s_1}} \lesssim \left\| \partial_s v \right\|_{L^2}^{\frac{3}{2}} \left\| \partial_s v \right\|_{H^{1+s_1}} \left\| v \right\|_{H^{s_1+1}},
\]
so that we get, by applying Young’s inequality, that
\[
\left\| \nabla_h v^3 \right\|_{L^2} \lesssim \left\| \nabla_h v^3 \right\|_{L^2} \left\| \partial_s v \right\|_{L^2} \left\| v \right\|_{H^{s_1}} \lesssim \left\| \nabla_h v^3 \right\|_{L^2} \left\| \partial_s v \right\|_{L^2} \left\| v \right\|_{H^{s_1}} \left\| \partial_s v \right\|_{L^2} \left\| v \right\|_{H^{s_1+1}}.
\]
Substituting the above inequality into (3.6) gives rise to
\[
\frac{d}{dt} \left\| \partial_s v \right\|_{H^{s_1}}^2 + \left\| \partial_s v \right\|_{H^{s_1}}^2 \leq C \left\| \nabla_h v^3 \right\|_{L^2} \left\| v \right\|_{H^{s_1}} \left\| \partial_s v \right\|_{L^2} \left\| v \right\|_{H^{s_1+1}}.
\]
Applying Gronwall’s inequality leads to
\[
\|\partial_3 v\|_{L_t^\infty(\dot{H}^{-s\over 4})}^2 + \|\partial_3 v\|_{L_t^2(\dot{H}^{s\over 4})}^2 \\
\leq C\left( \|\partial_3 v_0\|_{\dot{H}^{-s\over 4}}^2 + \int_0^t \left( \|\nabla_h v^3\|_{L^2} \|\text{div} h v^h\|_{L^2} \right)^{\frac{2(s_1+1)}{3s_1-2}} dt' \|v\|_{L_t^\infty(\dot{H}^{s_1})}^2 \right) \exp\left(C\|\nabla_h v^h\|_{L^2}^2 \|\text{div} h v^h\|_{L^2} \right)^{\frac{2(s_1+1)}{3s_1-2}} dt'.
\]
(3.45)

Yet for \( t \leq T^* \), it follows from (3.28) and (3.44) that
\[
\int_0^t \left( \|\nabla_h v^3\|_{L^2} \|\text{div} h v^h\|_{L^2} \right)^{\frac{2(s_1+1)}{3s_1-2}} dt' \leq C\left( A_s B_s \right)^{\frac{s_1}{3s_1-2}} \int_0^t \left( (\tau)^{-\frac{s}{2}} \tau^{-\frac{s}{2}} \right)^{\frac{2(s_1+1)}{3s_1-2}} d\tau \leq C\left( A_s B_s \right)^{\frac{s_1}{3s_1-2}},
\]
if
\[
\left( \frac{9}{8} + \frac{5}{4} s \right) \frac{2(s_1-1)}{3s_1-2} > 1 \Leftrightarrow s > \frac{1+3s_1}{10(s_1-1)}.
\]

Therefore thanks to (3.21), we deduce from (3.45) that for \( t \leq T^* \)
\[
\|\partial_3 v\|_{L_t^\infty(\dot{H}^{-s\over 4})}^2 + \|\partial_3 v\|_{L_t^2(\dot{H}^{s\over 4})}^2 \\
\leq C\left( \|\partial_3 v_0\|_{\dot{H}^{-s\over 4}}^2 + \|v_0\|_{\dot{H}^{s_1}}^2 \left( A_s B_s \right)^{\frac{s_1}{3s_1-2}} \right) \exp\left(C_0 + \left( C_0 + \left( C_0 \right)^{\frac{s_1}{3s_1-2}} \right) \right).
\]
(3.46)

In particular, under the assumption (1.9), we infer
\[
\|\partial_3 v\|_{L_t^\infty(\dot{H}^{-s\over 4})}^2 + \|\partial_3 v\|_{L_t^2(\dot{H}^{s\over 4})}^2 \leq \frac{\varepsilon}{2}
\]
for \( t \leq T^* \),
(3.47)
which contradicts with (3.22). This in turn shows that \( T^* = \infty \). Furthermore, (3.21), (3.23) and (3.46) ensures (1.10), and there hold (1.11a) and (1.12). To complete the proof of Theorem 1.3, it remains to prove (1.11b).

**Step 7. The decay estimate of \( \|\partial_3 v(t)\|_{L^2} \).**

We first deduce from (2.1) and (2.5) that
\[
\frac{d}{dt} \|\partial_3 v(t)\|_{L^2}^2 + \|\nabla_h \partial_3 v\|_{L^2}^2 \leq C\|\nabla_h v^h\|_{L^2}^2 \|\partial_3 v\|_{L^2}^2.
\]
(3.48)

Let us denote
\[
X(t) := e^{-C \int_0^t \|\nabla_h v^h(\tau)\|_{L^2}^2 \frac{d\tau}{s_0^{\frac{s}{4}}} d\tau} \|\partial_3 v(t)\|_{L^2}^2 
\quad \text{and} \quad D(t) := e^{-C \int_0^t \|\nabla_h v^h(\tau)\|_{L^2}^2 \frac{d\tau}{s_0^{\frac{s}{4}}} d\tau} \|\nabla_h \partial_3 v(t)\|_{L^2}^2.
\]

Then we deduce from (3.48) that
\[
\frac{d}{dt} X(t) + D(t) \leq 0.
\]
(3.49)

It follows from (2.1) that
\[
\int_0^t \|\nabla_h v^h(\tau)\|_{L^2}^2 \frac{d\tau}{s_0^{\frac{s}{4}}} d\tau \leq C\|v_0\|_{L^2}^2 \leq Cc_0^2,
\]
so that there holds
\[
e^{-C \int_0^t \|\nabla_h v^h(\tau)\|_{L^2}^2 \frac{d\tau}{s_0^{\frac{s}{4}}} d\tau} \geq e^{-Cc_0^2} \geq e^{-1} \quad \text{if} \quad Cc_0^2 \leq 1,
\]
which implies
\[
e^{-1}\|\partial_3 v(t)\|_{L^2} \leq X(t) \leq \|\partial_3 v(t)\|_{L^2}^2, \quad \text{and} \quad e^{-1}\|\nabla_h \partial_3 v(t)\|_{L^2} \leq D(t) \leq \|\nabla_h \partial_3 v(t)\|_{L^2}^2.
\]
(3.50)

On the other hand, we observe that
\[
\|\partial_3 v(t)\|_{L^2} \leq \|\partial_3 v(t)\|_{\dot{H}^{-s\over 4}}^{\frac{2}{s_1}} \|\nabla_h \partial_3 v(t)\|_{L^2}^{\frac{3}{s_1}},
\]
from which and (1.10c), (3.50), we infer
\[
D(t) \geq e^{-1}\|\nabla_h \partial_3 v(t)\|_{L^2} \geq e^{-1}\frac{\|\partial_3 v(t)\|_{L^2}^6 \|\nabla_h \partial_3 v(t)\|_{L^2}^4}{\|\partial_3 v(t)\|_{\dot{H}^{-s\over 4}}^{\frac{3}{s_1}}} \geq e^{-1} \frac{X^3(t)}{\|\partial_3 v(t)\|_{\dot{H}^{-s\over 4}}^{\frac{3}{s_1}}} \geq (cE_0^2)^{-1} X^3(t).
\]

Then we deduce from (3.49) that
\[
\frac{d}{dt} X(t) + (cE_0^2)^{-1} X^3(t) \leq 0,
\]
which implies
\[
X(t) \leq \left( \frac{1}{X(0)^{-2} + (eE_0^2)^{-1}} \right)^{\frac{1}{2}} \leq \left( \frac{1}{\| \partial_3 v_0 \|_{L^2}^4 + (eE_0^2)^{-1}} \right)^{\frac{1}{2}} \leq \left( \| \partial_3 v_0 \|_{L^2}^2 + \sqrt{eE_0^2} \right) t^{-\frac{1}{2}},
\]
This together with (3.50) ensures (1.11b).

As a consequence, we complete the proof of Theorem 1.3. \(\Box\)

APPENDIX A. TOOL BOX ON LITTLEWOOD–PALEY THEORY

For the convenience of readers, we collect some basic facts on anisotropic Littlewood–Paley theory in this section. We first observe from Definition 1.1 and (1.4) that for any \(n\),
\[
\text{Lemma A.2.}
\]

We recall the following anisotropic version of Bernstein type lemma from [1, 4, 12]:

\textbf{Lemma A.1.} Let \(B_h\) (resp. \(B_v\)) be a ball of \(\mathbb{R}^2\) (resp. \(\mathbb{R}\)), and \(C_h\) (resp. \(C_v\)) a ring of \(\mathbb{R}^2\) (resp. \(\mathbb{R}\)); let \(1 \leq p_2 \leq p_1 \leq \infty\) and \(1 \leq q_2 \leq q_1 \leq \infty\). Then there hold:

If the support of \(\tilde{a}\) is included in \(2^k B_h\), then
\[
\| \partial^a h \|_{L_{p_1}^k L_{q_1}^\infty} \lesssim 2^k \left( |a| + 2 \left( \frac{h}{r} - \frac{1}{p_1} \right) \right) \| a \|_{L_{p_2}^k L_{q_2}^\infty}, \quad \text{for} \quad h = (\partial_1, \partial_2).
\]

If the support of \(\tilde{a}\) is included in \(2^k B_v\), then
\[
\| \partial^a h \|_{L_{p_1}^k L_{q_1}^\infty} \lesssim 2^k \left( |a| + 2 \left( \frac{h}{r} - \frac{1}{p_1} \right) \right) \| a \|_{L_{p_2}^k L_{q_2}^\infty}.
\]

If the support of \(\tilde{a}\) is included in \(2^k C_h\), then
\[
\| a \|_{L_{p_1}^k L_{q_1}^\infty} \lesssim 2^{-kN} \| \partial^a h \|_{L_{p_2}^k L_{q_2}^\infty}.
\]

If the support of \(\tilde{a}\) is included in \(2^k C_v\), then
\[
\| a \|_{L_{p_1}^k L_{q_1}^\infty} \lesssim 2^{-kN} \| \partial^a h \|_{L_{p_2}^k L_{q_2}^\infty}.
\]

To deal with the estimate of product of two distributions, we constantly use the following parado- differential decomposition from [2] in the horizontal variables: for any functions \(f, g \in S' (\mathbb{R}^3)\),
\[
fg = T^h_f g + T^h_g f + R^h(f, g), \quad (A.1)
\]
where
\[
T^h_f g \overset{\text{def}}{=} \sum_{k \in \mathbb{Z}} S^h_{k-1} f \Delta^h_k g, \quad R^h(f, g) \overset{\text{def}}{=} \sum_{k \in \mathbb{Z}} \Delta^h_k f \Delta^h_k g, \quad \text{with} \quad \Delta^h_k g \overset{\text{def}}{=} \sum_{k' = k - 1}^{k+1} \Delta^h_{k'} g.
\]

We also employ Bony’s decomposition in the vertical variable.

The following technical lemmas are very useful in this context. The first one is concerned with the commutator’s estimates involving \(\Delta^h_k\) and \(\Delta^v_k\). Indeed it follows from the classic commutator’s estimate from [1] and Hölder’s inequality that

\textbf{Lemma A.2.} Let \(p, q, r, s \in [1, \infty]\) which satisfy \(\frac{1}{p} = \frac{1}{r} + \frac{1}{s}\). Then for any \(f, g \in \mathcal{S}(\mathbb{R}^3), j, k \in \mathbb{Z}\), there hold
\[
\| [\Delta^h_j; f] g \|_{L_{p}^k L_{q}^r} \lesssim 2^{-k} \| \nabla_h f \|_{L_{p}^k L_{q}^r} \| g \|_{L_{p}^k L_{q}^r},
\]
\[
\| [\Delta^v_k; f] g \|_{L_{p}^k L_{q}^r} \lesssim 2^{-k} \| \nabla_v f \|_{L_{p}^k L_{q}^r} \| g \|_{L_{p}^k L_{q}^r}. \quad (A.2)
\]

By virtue of Lemma A.1 and the following 2D interpolation inequality
\[
\| a \|_{L_{p}^z (\mathbb{R}^2)} \lesssim \| a \|_{L_{2}^z (\mathbb{R}^2)} \| \nabla_h a \|_{L_{2}^z (\mathbb{R}^2)}^{\frac{1}{2}},
\]
we deduce that
\[
\text{Therefore,}
\]
\[
\| a \|_{L_{p}^z (\mathbb{R}^2)} \lesssim \| a \|_{L_{2}^z (\mathbb{R}^2)} \| \nabla_h a \|_{L_{2}^z (\mathbb{R}^2)}^{\frac{1}{2}}.
\]
Lemma A.3. For any $u \in S(\mathbb{R}^3)$, there hold
\[
\|u\|_{L^4_h(L^\infty_v)} \lesssim \|u\|_{B^{\frac{1}{2},1}_{2,1}} \lesssim \|u\|_{B^{\frac{3}{4},1}_{2,1}} \|\nabla_{h} u\|_{L^\infty_v}^{\frac{1}{2}}
\]
\[
\|u\|_{L^4_h(L^2_v)} \lesssim \|u\|_{L^2_h(L^2_v)} \lesssim \|u\|_{L^2_h(H^1_v)} \lesssim \|u\|_{L^2_v} \|\nabla_{h} u\|_{L^2_v}^{\frac{1}{2}}
\]
\[
\|u\|_{L^4_v(L^2_h)} \lesssim \|u\|_{L^2_v(L^2_h)} \lesssim \|u\|_{B^{\frac{1}{4},1}_{2,1}} \lesssim \|u\|_{L^4_v} \|\nabla_{h} u\|_{L^2_v}^{\frac{1}{2}}.
\]

Acknowledgments. Li Xu is supported by NSF of China under grant 11671383. Ping Zhang is partially supported by K. C. Wong Education Foundation and NSF of China under Grants 11731007, 12031006 and 11688101.

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