MINIMAL MODEL THEORY FOR SURFACES OVER AN IMPERFECT FIELD

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ABSTRACT. In this paper, we establish a minimal model theory for surfaces over a field of positive characteristic. More precisely, we show the minimal model program and the abundance theorem for log canonical surfaces.

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0. Introduction

In this paper, we establish the minimal model program and the abundance theorem for log canonical surfaces over an arbitrary field of positive characteristic. More precisely, we prove the following two theorems.

**Theorem 0.1** (Minimal model program). Let $k$ be a field of characteristic $p > 0$. Let $(X, \Delta)$ be a log canonical surface over $k$, where $\Delta$ is an effective $\mathbb{R}$-divisor. Let $f : X \rightarrow S$ be a projective $k$-morphism to a scheme $S$ which is separated and of finite type over $k$. Then, there exists a sequence of projective birational $S$-morphisms

$$(X, \Delta) =: (X_0, \Delta_0) \xrightarrow{\varphi_0} (X_1, \Delta_1) \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{s-1}} (X_s, \Delta_s) =: (X^\dagger, \Delta^\dagger)$$

where $(\varphi_{i-1})_* (\Delta_{i-1}) =: \Delta_i$ with the following properties.

1. Each $(X_i, \Delta_i)$ is a log canonical surface over $k$, which is projective over $S$.
2. Each $\text{Ex}(\varphi_i) =: C_i$ is an irreducible projective curve over $k$ such that $$(K_{X_i} + \Delta_i) \cdot C_i < 0.$$ 
3. $(X^\dagger, \Delta^\dagger)$ satisfies one of the following conditions.
   a. $K_{X^\dagger} + \Delta^\dagger$ is nef over $S$.
   b. There is a projective surjective $S$-morphism $\mu : X^\dagger \rightarrow Z$ to a variety $Z$ over $k$, which is projective over $S$, such that $\mu_* \mathcal{O}_{X^\dagger} = \mathcal{O}_Z$, $\dim X^\dagger > \dim Z$, $-(K_{X^\dagger} + \Delta^\dagger)$ is $\mu$-ample and $\rho(X^\dagger/S) - 1 = \rho(Z/S)$.

**Theorem 0.2** (Abundance theorem). Let $k$ be a field of characteristic $p > 0$. Let $(X, \Delta)$ be a log canonical surface over $k$, where $\Delta$ is an effective $\mathbb{R}$-divisor. Let $f : X \rightarrow S$ be a projective $k$-morphism to a scheme $S$ which is separated and of finite type over $k$. If $K_X + \Delta$ is $f$-nef, then $K_X + \Delta$ is $f$-semi-ample.

0.3 (Overview of related results). We summarize some results related to this paper.

*Surface theory.* The Italian school (Enriques, Castelnuovo, and many others) established the classification theory for smooth projective surfaces over $\mathbb{C}$, which was generalized by Kodaira, Shafarevich and Bombieri–Mumford. In particular, Shafarevich studies a minimal model theory (in the classical sense) for regular surfaces. Recently, Fujino and T1 establish a minimal model theory for log canonical and $\mathbb{Q}$-factorial surfaces by a view point of the higher dimensional minimal model theory. For related results, see also Fujita2, KK, T2.
**Minimal model theory.** For results on the 3-dimensional minimal model theory in positive characteristic, we refer to [Birkar2], [BW], [CTX], [HX], [Kawamata], [Keel], [Kollár1] and [Xu].

**Others.** Maddock constructs a regular del Pezzo surface $X$ over an imperfect field with $H^1(X, \mathcal{O}_X) \neq 0$ (cf. [Schröer]), although such surfaces do not exist over an algebraically closed field.

0.4 (Proofs). The proof of Theorem 0.1 is almost all the same as the case when $k$ is an algebraically closed field. Actually, one of the key results is a contractible criterion (Theorem 3.1), which is obtained by using Keel’s result (cf. Theorem 1.13). Moreover, the cone theorem is already obtained in [T3, Theorem 0.7].

Thus our main ingredient is the abundance theorem (Theorem 0.2). If $k$ is perfect, then we can show Theorem 0.2 just by taking the base change to the algebraic closure. However, if the base field is imperfect, the situation is subtler. First, the base changed scheme may be non-normal or non-reduced. Second, for the case when $k$ is algebraically closed, the proof of the abundance theorem for smooth surfaces depends on Noether’s formula and Albanese morphisms. These techniques cannot be used in general for varieties over non-closed fields. Therefore it is difficult to imitate the proof for the case over algebraically closed fields.

Let us overview the proof of Theorem 0.2 only for the case when $X$ is a regular surface $X$, $\Delta = 0$, and $S = \text{Spec} \, k$. This case is proved in Section 6 (Theorem 6.3). Let $X$ be a projective regular surface over a field of characteristic $p > 0$. Assume that $K_X$ is nef. We would like to show that $K_X$ is semi-ample. First we can assume that $k$ is separably closed by taking the base change to the separable closure. Such a base change is harmless because, for example, a finite separable extension is etale. Second, we take the normalization $Y$ of $(X \times_k \overline{k})_{\text{red}}$, where $(X \times_k \overline{k})_{\text{red}}$ is the reduced structure of $X \times_k \overline{k}$. Set $f : Y \to X$ to be the induced morphism:

$$f : Y \to (X \times_k \overline{k})_{\text{red}} \to X \times_k \overline{k} \to X.$$ 

We can check that $Y$ is $\mathbb{Q}$-factorial (Lemma 11.3)). By [T3, Theorem 0.1] (cf. Theorem 1.5), we can find an effective $\mathbb{Z}$-divisor $E$ such that

$$K_Y + E = f^*K_X.$$ 

By using adjunction formula and other known results, we can assume that $E \sim_\mathbb{Q} \Delta_Y$ for some $\mathbb{Q}$-divisor $\Delta_Y$ on $Y$ whose coefficients are contained in $[0, 1]$. Then, $K_X$ is semi-ample by using the following abundance theorem obtained in [T1].
Theorem 0.5 (Theorem 1.2 of [11]). Let $Y$ be a projective normal $\mathbb{Q}$-factorial surface over an algebraically closed field of characteristic $p > 0$. Let $\Delta_Y$ be a $\mathbb{Q}$-divisor on $Y$ whose coefficients are contained in $[0, 1]$. If $K_Y + \Delta_Y$ is nef, then $K_Y + \Delta_Y$ is semi-ample.

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1. Preliminaries

1.1. Notation. Let $k$ be a field. We say $X$ is a variety over $k$ (or $k$-variety) if $X$ is an integral scheme which is separated and of finite type over $k$. We say $X$ is a curve over $k$ (resp. a surface over $k$) if $X$ is a variety over $k$ with $\dim X = 1$ (resp. $\dim X = 2$). For a scheme $X$, set $X_{\text{red}}$ to be the reduced scheme whose underlying topological space is equal to $X$.

We will not distinguish the notations invertible sheaves and divisors. For example, we will write $L + M$ for invertible sheaves $L$ and $M$. For a proper scheme $X$ over a field $k$ and a coherent sheaf $F$ on $X$, we set

$$h^i(X, F) := \dim_k H^i(X, F), \quad \chi(X, F) := \sum_{i=0}^{\dim X} (-1)^i h^i(X, F).$$

Note that these numbers depend on $k$.

We will freely use the notation and terminology in [Kollár2]. In the definition in [Kollár2, Definition 2.8], for a pair $(X, \Delta)$, $\Delta$ is not necessarily effective. However, in this paper, we assume that $\Delta$ is an effective $\mathbb{R}$-divisor.

Let $\Delta$ be an $\mathbb{R}$-divisor on a normal variety over a field. We write $\Delta \leq a$ (resp. $\Delta \geq b$) if, for the prime decomposition $\Delta = \sum_{i \in I} \delta_i \Delta_i$, $\delta_i \leq a$ (resp. $\delta_i \geq b$) holds for every $i \in I$. For example, $0 \leq \Delta \leq 1$ means $0 \leq \delta_i \leq 1$ for every $i \in I$.

For a $\mathbb{Z}$-module $M$, we set $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$ and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$.

1.2. Dualizing sheaves and purely inseparable base changes. In this subsection, we summarize basic properties of dualizing sheaves and purely inseparable base changes (cf. [13, Subsections 1.2, 1.3]).
Lemma 1.1. Let $k$ be a field. Let $X$ be a normal variety over $k$. Then the following assertions hold.

1. The normalization morphism $Y \to (X \times_k k^{1/p^\infty})_{\text{red}}$ is a universal homeomorphism.
2. $\rho(X) = \rho(Y)$.
3. If $X$ is $\mathbb{Q}$-factorial, then so is $Y$.

Proof. See [T3, Lemma 1.2, Proposition 1.4, Lemma 1.5].

We recall the definition of dualizing sheaves and collect some basic properties.

Definition 1.2. Let $k$ be a field. Let $X$ be a $d$-dimensional separated scheme of finite type over $k$. We set

$$\omega_{X/k} := \mathcal{H}^{-d}(f^!k),$$

where $f : X \to \text{Spec } k$ is the structure morphism.

A dualizing sheaf does not change by enlarging a base field.

Lemma 1.3. Let $k/k_0$ be a finite field extension. Let $X$ be a separated scheme of finite type over $k$. Note that $X$ is also of finite type over $k_0$. Then, there exists an isomorphism

$$\omega_{X/k} \simeq \omega_{X/k_0}.$$

Proof. Set $\alpha : X \xrightarrow{\beta} \text{Spec } k \xrightarrow{\theta} \text{Spec } k_0$. By $\alpha^! = \beta^! \theta^!$, it suffices to show $\theta^!k_0 \simeq k$. This follows from $\text{Hom}_{k_0}(k, k_0) \simeq k$.

Thanks to Lemma 1.3 we can define canonical divisors $K_X$ independent of a base field.

Definition 1.4. Let $k$ be a field. If $X$ is a normal variety over $k$, then it is well-known that $\omega_{X/k}$ is a reflexive sheaf. Let $K_X$ be a divisor which satisfies

$$\mathcal{O}_X(K_X) \simeq \omega_{X/k}.$$ 

Such a divisor $K_X$ is called canonical divisor. Note that a canonical divisor is determined up to linear equivalence.

Theorem 1.5. Let $k$ be a field of characteristic $p > 0$. Let $X$ be a normal variety over $k$. Let $\nu : Y \to (X \times_k k^{1/p^\infty})_{\text{red}}$ be the normalization of $(X \times_k k^{1/p^\infty})_{\text{red}}$ and set $f : Y \to X$ to be the induced morphism:

$$f : Y \xrightarrow{\nu} (X \times_k \overline{k})_{\text{red}} \to X \times_k \overline{k} \to X,$$
where the second arrow is taking the reduced structure and the third arrow is the first projection. If $K_X$ is $\mathbb{Q}$-Cartier, then there exists an effective $\mathbb{Z}$-divisor $E$ on $Y$ such that

$$K_Y + E = f^*K_X.$$ 

Proof. See [T3, Theorem 3.2].

1.3. Intersection numbers and Riemann–Roch theorem. We recall the definition of intersection numbers (cf. [Kleiman, Ch I, Sections 1 and 2]).

Definition 1.6. Let $k$ be a field.

(1) Let $C$ be a proper curve over $k$. Let $M$ be an invertible sheaf on $C$. It is well-known that

$$\chi(C, mM) = \dim_k(H^0(C, mM)) - \dim_k(H^1(C, mM)) \in \mathbb{Z}[m]$$

and that its degree is at most one (cf. [Kleiman, Ch I, Section 1, Theorem in page 295]). We define the degree $\deg_C(L) = \deg(L)$ by the coefficient of $m$, that is, the top coefficient.

(2) Let $X$ be a separated scheme of finite type over $k$ and let $C \hookrightarrow X$ be a closed immersion such that $C$ is a proper $k$-curve. Let $L$ be an invertible sheaf on $X$. We define the intersection number by $L \cdot C := \deg_C(L|_C)$.

The following is the Riemann–Roch theorem for curves.

Theorem 1.7. Let $k$ be a field. Let $X$ be a proper curve over $k$.

(1) If $L$ is a Cartier divisor on $X$, then

$$\chi(X, L) = \deg(L) + \chi(X, \mathcal{O}_X).$$

(2) Assume that $X$ is regular. Let $D$ be a Weil divisor and let $D = \sum_i a_i P_i$ be the prime decomposition. Then

$$\deg(D) = \sum_i a_i \dim_k(k(P_i))$$

where $k(P_i)$ is the residue field at $P_i$.

Proof. The assertion (1) follows from Definition 1.6(1). The assertion (2) holds by the same argument as the case when $k$ is algebraically closed (cf. the proof of [Hartshorne, Ch IV, Theorem 1.3]).

As a corollary, we obtain a formula between $\deg(\omega_X)$ and $\chi(X, \mathcal{O}_X)$ for Gorenstein curves.
Corollary 1.8. Let $k$ be a field. Let $X$ be a proper Gorenstein curve over $k$. Then
\[ \deg(\omega_X) = 2(h^1(X, \mathcal{O}_X) - h^0(X, \mathcal{O}_X)) = -2\chi(X, \mathcal{O}_X). \]

Proof. By Theorem 1.7(1), we obtain $\chi(X, \omega_X) = \deg(\omega_X) + \chi(X, \mathcal{O}_X)$. By Serre duality, we obtain $\chi(X, \omega_X) = -\chi(X, \mathcal{O}_X)$, which implies the assertion. \qed

Corollary 1.9. Let $k$ be a field. Let $X$ be a projective regular $k$-surface and let $C$ be a curve in $X$. Then,
\[ (K_X + C) \cdot C = \deg K_C = 2(\dim_k H^1(C, \mathcal{O}_C) - \dim_k H^0(C, \mathcal{O}_C)). \]

Proof. The first equality follows from the adjunction formula (cf. [Kollár2 (4.1.3)]). The second equality holds by Corollary 1.8. \qed

The following is the Riemann–Roch theorem for surfaces.

Theorem 1.10. Let $k$ be a field. Let $X$ be a projective regular surface over $k$. Let $D$ be a $\mathbb{Z}$-divisor on $X$. Then
\[ \chi(X, D) = \chi(X, \mathcal{O}_X) + \frac{1}{2} D \cdot (D - K_X). \]

Proof. We can write $D = \sum_i A_i - \sum_j B_j$, where all $A_i$ and $B_j$ are prime divisors. Thus, it suffices to prove that if $D$ satisfies the required equation, so do $D + C$ and $D - C$ for a prime divisor $C$ on $X$. Assume
\[ \chi(X, D) = \chi(X, \mathcal{O}_X) + \frac{1}{2} D \cdot (D - K_X). \]

For a prime divisor $C$, we only show
\[ \chi(X, D + C) = \chi(X, \mathcal{O}_X) + \frac{1}{2} (D + C) \cdot (D + C - K_X). \]

By an exact sequence
\[ 0 \rightarrow \mathcal{O}_X(D) \rightarrow \mathcal{O}_X(D + C) \rightarrow \mathcal{O}_X(D + C)|_C \rightarrow 0, \]

we obtain
\[ \chi(X, D + C) - \chi(X, D) = \chi(C, \mathcal{O}_X(D + C)|_C) = (D + C) \cdot C + \chi(C, \mathcal{O}_C). \]

On the other hand, we see
\[ \frac{1}{2} (D + C) \cdot (D + C - K_X) - \frac{1}{2} D \cdot (D - K_X) = \frac{1}{2} (2D \cdot C + C^2 - K_X \cdot C). \]

Thus it is enough to prove
\[ (D + C) \cdot C + \chi(C, \mathcal{O}_C) = \frac{1}{2} (2D \cdot C + C^2 - K_X \cdot C), \]
which is equivalent to
\[(K_X + C) \cdot C = -2\chi(C, O_C).\]
This holds by Corollary 1.9.

1.4. Mumford’s intersection theory. Intersection theory gives us
intersection numbers \(D \cdot C\) for a Cartier divisor \(D\) and a curve \(C\). For
the surface case, we can generalize this to Weil divisors.

**Definition 1.11.** Let \(k\) be a field. Let \(X\) be a normal \(k\)-surface. Fix
a proper birational morphism \(f : X' \to X\) from a regular surface \(X'\)
and let \(E_1, \cdots, E_n\) be the \(f\)-exceptional curves. Take a curve \(C\) in \(X\)
proper over \(k\) and let \(D\) be an \(\mathbb{R}\)-divisor in \(X\). Let \(C'\) and \(D'\) be their
proper transforms, respectively.

1. We define \(f^*D := D' + \sum_{1 \leq i \leq n} e_i E_i\) where all \(e_i\) are real numbers
determined by the linear equations
\[(D' + \sum_{1 \leq i \leq n} e_i E_i) \cdot E_j = 0\]
for \(j = 1, \cdots, n\). Such real numbers \(e_i\) are uniquely determined
since the intersection matrix \((E_i \cdot E_j)\) is negative definite (cf. [Kollár2, Theorem 10.1]).

2. We define an intersection number by
\[C \cdot D := f^*C \cdot f^*D = C' \cdot f^*D.\]

3. If \(X\) is proper over \(k\), then we can naturally extend this inter-
tersection number to Weil divisors with \(\mathbb{Q}\) or \(\mathbb{R}\) coefficients by
linearity.

1.5. Keel’s result. We recall a result obtained by [Keel], which plays
a crucial role in this paper.

**Theorem 1.12.** Let \(k\) be a field of characteristic \(p > 0\). Let \(X\) be a
projective normal variety over \(k\) and let \(L\) be a nef \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor.
We define a reduced closed subscheme \(\mathcal{E}(L)\) of \(X\) by
\[\text{Supp}\mathcal{E}(L) = \bigcup_{L|_Y \text{ is not big}} Y\]
where \(Y\) runs over subvarieties of \(X\). If \(L|_{\mathcal{E}(L)}\) is semi-ample, then \(L\)
is semi-ample.

**Proof.** See [Keel Theorem 0.2].
Theorem 1.13. Let \( k \) be a field of positive characteristic. Let \( X \) be a projective normal surface over \( k \) and let \( L \) be a nef and big \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor. We define a reduced closed subscheme \( E(L) \) of \( X \) by

\[
\text{Supp} E(L) = \bigcup_{L \cdot C = 0} C
\]

where \( C \) runs over curves in \( X \). If \( L|_{E(L)} \) is semi-ample, then \( L \) is semi-ample.

Proof. The assertion follows from Theorem 1.12.

\( \square \)

1.6. Some criteria for semi-ampleness. We collect some known results (at least for algebraically closed fields) on semi-ampleness.

Lemma 1.14. Let \( k \) be a field. Let \( X \) be a projective normal surface over \( k \). Let \( D \) be an effective \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor and let \( D = \sum d_i D_i \) be the irreducible decomposition. If \( D \cdot D_i > 0 \) for every \( i \), then \( D \) is semi-ample.

Proof. We show that \( B(D) = \emptyset \), where \( B(D) \) is the stable base locus of \( D \). We obtain \( B(D) \subset \text{Supp}(D) \). On the other hand, by [Birkar3, Theorem 1.3], we obtain \( B(D) \subset B_+(D) = \mathcal{E}(D) \). Thus we see

\[
B(D) \subset \text{Supp}(D) \cap \mathcal{E}(D).
\]

By the assumption \( D \cdot D_i > 0 \), \( \text{Supp}(D) \cap \mathcal{E}(D) \) is empty or zero-dimensional. Therefore, [Fujita1, Corollary 1.14] implies that \( D \) is semi-ample.

\( \square \)

Lemma 1.15. Let \( k \) be a field. Let \( X \) be a projective normal surface over \( k \). Let \( L \) be a \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor. If \( L \) is nef and \( \kappa(X, L) = 1 \), then \( L \) is semi-ample.

Proof. If \( k \) is algebraically closed, then the assertion is well-known (cf. [Fujita2, Theorem 4.1]). We reduce the problem to this case. Replacing \( X \) with the base change to the separable closure of \( k \), we may assume \( k \) is separably closed. Note that algebraic separable base changes do not change the normality.

Let \( \beta : X \times_k \overline{k} \to X \) be the base change to the algebraic closure of \( k \) and take its reduced closed subscheme \( (X \times_k \overline{k})_{\text{red}} \) and its normalization \( (X \times_k \overline{k})_{\text{red}}^N \):

\[
(X \times_k \overline{k})_{\text{red}}^N \xrightarrow{\nu} (X \times_k \overline{k})_{\text{red}} \to X \times_k \overline{k} \xrightarrow{\beta} X.
\]

Then, we can find a finite purely inseparable extension \( \overline{k} \) of \( k \) and a finite purely inseparable morphism \( f : Y \to X \) where \( Y \) is a normal surface over \( \overline{k} \) with \( Y \times_{\overline{k}} \overline{k} \simeq (X \times_k \overline{k})_{\text{red}}^N \). Then, we can reduce the problem to the case where \( k \) is algebraically closed.

\( \square \)
Lemma 1.16. Let $k$ be a field. Let $X$ be a projective normal surface over $k$ and let $\Delta$ be an effective $\mathbb{R}$-divisor such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $L$ be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$. Assume the following conditions.

1. $L$ is nef.
2. $L^2 = 0$.
3. There exists a curve $C$ in $X$ with $L \cdot C > 0$.
4. $L \cdot (K_X + \Delta) < 0$.

Then, $L$ is semi-ample.

Proof. By replacing $X$ with the minimal resolution, we may assume that $X$ is regular and $\Delta = 0$. By (3), we see $H^2(X, mL) = 0$ for $m \gg 0$. Then, by the Riemann–Roch formula (Theorem 1.10), we obtain

$$
\chi(X, mL) = mL \cdot (mL - K_X) + \chi(X, O_X) = mL \cdot (-K_X) + \chi(X, O_X),
$$

which implies $\kappa(X, L) \geq 1$. By the condition (2), we obtain $\kappa(X, L) = 1$. Then, $L$ is semi-ample by Lemma 1.15. \hfill \Box

1.7. Cone theorem. We recall cone theorems for surfaces over a (non-closed) field.

Proposition 1.17. Let $k$ be a field. Let $X$ be a normal $k$-surface. Let $f : X \to S$ be a projective morphism to a separated scheme of finite type over $k$. If $\dim f(X) \geq 1$, then there exist finitely many proper $k$-curves $C_1, \ldots, C_m$ such that each $f(C_i)$ is one point and that

$$
\overline{NE}(X/S) = NE(X/S) = \sum_{i=1}^{m} \mathbb{R}_{\geq 0}[C_i].
$$

Proof. The same proof as [11, Proposition 6.1] can be applied. \hfill \Box

Theorem 1.18. Let $k$ be a field. Let $X$ be a normal $k$-surface and let $\Delta$ be an effective $\mathbb{R}$-divisor such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $f : X \to S$ be a projective morphism to a separated scheme of finite type over $k$. Let $A$ be an $f$-ample $\mathbb{R}$-divisor. Then there exist finitely many proper $k$-curves $C_1, \ldots, C_m$ in $X$ such that each $f(C_i)$ is one point and that

$$
\overline{NE}(X/S) = \overline{NE}(X/S)_{K_X + \Delta + A \geq 0} + \sum_{i=1}^{m} \mathbb{R}_{\geq 0}[C_i].
$$
Proof. If \( \dim f(X) \geq 1 \), then the assertion follows from Proposition 1.17. If \( \dim f(X) = 0 \), then the assertion holds by [T3] Theorem 0.7. □

1.8. **Semiample perturbation.** In this subsection, we show Lemma 1.20. We recall the following classical result of Bertini type.

**Theorem 1.19** (Seidenberg). Let \( k \) be an infinite field. Let \( X \) be a projective normal \( k \)-variety and let \( X \subset \mathbb{P}^N_k \) be an closed immersion. Then, general hyperplane sections of \( X \) are reduced, that is, there exists a non-empty open subset \( U \subset \mathbb{P}(H^0(\mathbb{P}^N_k, \mathcal{O}_{\mathbb{P}^N_k}(1))) \) such that, for every \( k \)-rational point \( u \in U(k) \), the scheme-theoretic intersection \( X \cap H_u \) is reduced, where \( H_u \) is the hyperplane corresponding to \( u \).

Proof. See [Seidenberg, Corollary 1]. □

**Lemma 1.20.** Let \( k \) be an infinite field. Let \( X \) be a projective normal \( k \)-variety and let \( D \) be a semi-ample \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor. If \( \epsilon \in \mathbb{Q}_{>0} \) and \( x_1, \ldots, x_\ell \in X \), then there exists an effective \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor \( D' \) such that \( D' \sim D \), \( 0 \leq D' \leq \epsilon \), and that \( x_i \notin \text{Supp} D' \) for every \( i \).

Proof. By replacing \( D \) with a large multiple, we may assume that \( \epsilon = 1 \) and that \( D \) is a base point free Cartier divisor whose linear system induces a morphism

\[
f : X \to W
\]

with \( f_* \mathcal{O}_X = \mathcal{O}_W \) and \( D = f^* \mathcal{O}_W(1) \), where \( \mathcal{O}_W(1) \) is very ample. Since \( f \) is projective, we can apply the noetherian normalization theorem and obtain the followings.

- \( W^0 \subset W \) is a non-empty open subset and set \( X^0 := f^{-1}(W^0) \).
- The induced morphism \( f|_{X^0} : X^0 \to W^0 \) can be written by

\[
f|_{X^0} : X^0 \xrightarrow{\pi} W_0 \times_k \mathbb{P}^r \xrightarrow{pr_1} W^0,
\]

where \( \pi \) is a finite surjective morphism.

Since \( D \) is base point free, it suffices to show that there exists \( \nu \in \mathbb{Z}_{>0} \) such that every general hyperplane section \( H \sim \mathcal{O}_W(1) \) satisfies

\[
0 \leq f^* H \leq \nu.
\]

Let

\[
\pi : X^0 \xrightarrow{\pi} V \xrightarrow{\pi} W^0
\]

be the factorization corresponding to the separable closure. We can find \( e \in \mathbb{Z}_{>0} \) such that the absolute Frobenius \( F_V^e \) factors through

\[
F_V^e : V \to X^0 \xrightarrow{\pi} V.
\]

Then, we can check that for every prime divisor \( P \) on \( W^0 \), we obtain

\[
\pi^*(P) \leq p^e \deg(\pi_*) =: \nu.
\]
Here, note that $\pi^*(P)$ is defined by the closure of the pull-back of the restriction of $P$ to the regular locus of $W^0$. For a general hyperplane section $H \subset W$, its pull-back $f^*H$ is equal to the closure of $(f|_{X^0})^*(H|_{W^0})$. By Theorem 1.19, a general hyperplane section $H \sim O_W(1)$ is reduced. Thus, also $\text{pr}_1^*(H|_{W^0})$ is reduced, that is, $\text{pr}_1^*(H|_{W^0}) \leq 1$. Therefore, we obtain

$$\pi^*\text{pr}_1^*(H|_{W^0}) \leq p^f \deg(\pi_s) = \nu.$$  

This implies $0 \leq f^*H \leq \nu$. We are done. \hfill \Box

2. Adjunction

In this section, we summarize results on adjunction formula.

**Proposition 2.1.** Let $k$ be a field. Let $X$ be a normal $k$-surface and let $C$ be a curve in $X$. Then, there exists an exact sequence

$$0 \to \mathcal{T} \to (\omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(C)|_C \to \omega_C \to 0$$

where $\mathcal{T}$ is a skyscraper sheaf.

*Proof.* See the proof of [Fujino, Lemma 4.4]. \hfill \Box

**Lemma 2.2.** Let $k$ be a field. Let $X$ be a normal $k$-surface and let $C$ be a proper $k$-curve in $X$. Let $L$ be a Cartier divisor on $X$. If $H^1(C, \mathcal{O}_X(L)|_C) \neq 0$, then

$$H^0(C, (\mathcal{O}_X(r(K_X + C - L))|_C) \neq 0$$

for every $r \in \mathbb{Z}_{\geq 0}$.

*Proof.* Set

$$\omega_X(C - L) := \omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(C - L) \simeq \omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(C) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-L).$$

By Proposition 2.1, we obtain an exact sequence:

$$0 \to \mathcal{T} \otimes_{\mathcal{O}_C} \mathcal{O}_X(-L)|_C \to (\omega_X(C - L)|_C \to \omega_C \otimes_{\mathcal{O}_C} (\mathcal{O}_X(-L)|_C \to 0.$$

Since $\mathcal{T}$ is a skyscraper sheaf, we have $H^1(C, \mathcal{T} \otimes_{\mathcal{O}_C} \mathcal{O}_X(-L)|_C) = 0$. Since

$$H^0(C, \omega_C \otimes_{\mathcal{O}_C} (\mathcal{O}_X(-L)|_C) \simeq H^1(C, \mathcal{O}_X(L)|_C)^* \neq 0,$$

we can find a non-zero element

$$\xi \in H^0(C, \omega_X(C - L)|_C) \neq 0$$

whose image $\xi' \in H^0(C, \omega_C \otimes_{\mathcal{O}_C} (\mathcal{O}_X(-L)|_C)$ is also non-zero. Thus there exists a map

$$\mathcal{O}_C \to \omega_X(C - L)|_C, \ 1 \mapsto \xi.$$
which is an injective \( \mathcal{O}_X \)-module homomorphism on some non-empty open set. Tensoring \((\omega_X(C - L))|_C\) one by one, we obtain a sequence of maps
\[
\mathcal{O}_C \to (\omega_X(C - L))|_C \to (\omega_X(C - L))^\otimes 2|_C \to \cdots \to (\omega_X(C - L))^\otimes r|_C
\]
which are injective on some non-empty open set. On the other hand, \(\omega\) is injective on some non-empty open set. Combining these maps, we have a map
\[
\mathcal{O}_C \to \mathcal{O}_X(r(K_X + C - L))|_C,
\]
which is injective on some non-empty open set. Thus, the kernel \(K\) of this map is a torsion subsheaf of \(\mathcal{O}_C\). Then, we have \(K = 0\). Therefore, we obtain an injection \(\mathcal{O}_C \hookrightarrow \mathcal{O}_X(r(K_X + C - L))|_C\). This implies \(H^0(C, \mathcal{O}_X(r(K_X + C - L))|_C) \neq 0\).

Using Lemma 2.2, we obtain the following theorem, which plays a crucial role in this paper.

**Theorem 2.3.** Let \(k\) be a field. Let \(X\) be a normal \(k\)-surface and let \(C\) be a proper \(k\)-curve in \(X\) such that \(r(K_X + C)\) is Cartier for some positive integer \(r\).

1. Assume \((K_X + C) \cdot C < 0\). If \(L\) is a Cartier divisor on \(X\) such that \(L \cdot C = 0\), then \(L|_C \simeq \mathcal{O}_C\).

2. If \((K_X + C) \cdot C = 0\), then there exists \(s \in r\mathbb{Z}_{>0}\) such that \(\mathcal{O}_X(s(K_X + C))|_C \simeq \mathcal{O}_C\).

**Proof.** (1) It is enough to show \(H^0(C, L|_C) \neq 0\). Since \((K_X + C) \cdot C < 0\) implies \(H^0(C, \mathcal{O}_X(r(K_X + C - L))|_C) = 0\), we obtain \(H^1(C, L|_C) = 0\) by Lemma 2.2. In particular, we see \(H^1(C, \mathcal{O}_C) = 0\). By the Riemann–Roch formula, we obtain
\[
h^0(C, L|_C) = \chi(C, L|_C) = \chi(C, \mathcal{O}_C) = h^0(C, \mathcal{O}_C) \neq 0.
\]

(2) By replacing \(r\) with \(2r\), we can assume \(r \geq 2\). Assume \(H^1(C, \mathcal{O}_C) \neq 0\). Then, we can apply Lemma 2.2 for \(L := 0\) and we obtain
\[
H^0(C, r(K_X + C)|_C) \neq 0.
\]
The, we may assume \(H^1(C, \mathcal{O}_C) = 0\).

Assume \(H^1(C, r(K_X + C)|_C) \neq 0\). Then, we can apply Lemma 2.2 for \(L := r(K_X + C)\), and we obtain
\[
H^0(C, -(r - 1)r(K_X + C)|_C) \neq 0.
\]
Thus, we may assume \(H^1(C, r(K_X + C)|_C) = 0\).
It is enough to show that $H^0(C, r(K_X + C)|_C) \neq 0$, assuming
$$H^1(C, \mathcal{O}_C) = H^1(C, r(K_X + C)|_C) = 0.$$  
This follows from the Riemann–Roch formula:
$$h^0(C, r(K_X + C)|_C) = \chi(C, r(K_X + C)|_C) = \chi(C, \mathcal{O}_C) = h^0(C, \mathcal{O}_C) \neq 0.$$  

Example 2.4. Let $k$ be a field and let $C$ be a proper $k$-curve. If $k$ is algebraically closed, then $H^1(C, \mathcal{O}_C) = 0$ implies that $C$ is regular. However, if $k$ is not algebraically closed, then $C$ may not be regular, as the following examples show.

1. Let $C := \{x^2 + y^2 = 0\} \subset \mathbb{P}^2_k$, where $[x : y : z]$ is a homogeneous coordinate. Then $C$ is a projective $\mathbb{R}$-curve with $H^1(C, \mathcal{O}_C) = 0$, however $C$ is not regular.

2. Let $k$ be a field of characteristic two. Assume that there exists $t \in k$ such that $t^{1/2} \notin k$. Then $C := \{x^2 + ty^2 = 0\} \subset \mathbb{P}^2_k$ is a projective $k$-curve, with $H^1(C, \mathcal{O}_C) = 0$, which is neither regular nor geometrically reduced.

3. **Contractible criterion**

In the classical minimal theory for surfaces, Castelnuovo’s criterion plays a crucial role. We establish a similar result for singular surfaces of positive characteristic.

**Theorem 3.1.** Let $k$ be a field of characteristic $p > 0$. Let $X$ be a quasi-projective normal surface over $k$. Let $C$ be a curve in $X$ proper over $k$. Assume that the following conditions hold.

- $K_X$ and $C$ are $\mathbb{Q}$-Cartier.
- $(K_X + C) \cdot C < 0$.
- $C^2 < 0$.

Then there exists a projective birational morphism $f : X \to Y$ which satisfies the following properties.

1. $Y$ is a quasi-projective normal surface with $f_* \mathcal{O}_X = \mathcal{O}_Y$.
2. $\text{Ex}(f) = C$.
3. Let $L$ be a Cartier divisor on $X$ such that $L \cdot C = 0$. Then, $mL = f^*L_Y$ for some integer $m > 0$ and a Cartier divisor $L_Y$ on $Y$.
4. If $X$ is $\mathbb{Q}$-factorial, then $Y$ is $\mathbb{Q}$-factorial.
5. If $X$ is projective, then $Y$ is projective and $\rho(Y) = \rho(X) - 1$. 

Proof. By taking a projective compactification $X \subset \overline{X}$ such that $\overline{X}$ is regular along $\overline{X} \setminus X$, we can assume that $X$ is projective. Let $A$ be an ample Cartier divisor on $X$. We define $D := A + qC$ and $q \in \mathbb{Q}_{>0}$ by $D \cdot C = (A + qC) \cdot C = 0$. In order to show that $D$ is semi-ample, it suffices to check that $D|_{\overline{X}}$ is semi-ample by Theorem 1.13. This follows from Theorem 2.3(1). Thus, for a sufficiently divisible integer $n \in \mathbb{Z}_{>0}$, the complete linear system $|nD|$ induces a birational morphism $f : X \to Y$ which satisfies (1) and (2). We show (3). Let $L$ be a Cartier divisor on $X$ such that $L \cdot C = 0$. Fix an ample Cartier divisor $A_Y$ on $Y$. Then, the divisor $L + nf^*A_Y$, with $n \gg 0$, is a nef and big Cartier divisor such that $\mathcal{E}(L) = C$. By the same argument as above, $L$ is semi-ample. By the Zariski main theorem, $|m(L + nf^*A_Y)|$ also induces the morphism $f : X \to Y$. Therefore, $mL = f^*L_Y$ for some Cartier divisor $L_Y$. We show (4). Let $D_Y$ be a prime divisor and let $D_X$ be the proper transform of $D_Y$. We can find $r \in \mathbb{Q}$ such that $(D_X + rC) \cdot C = 0$. By (3), $D_X + rC = f^*L_Y$ where $L_Y$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Taking $f^*$, we obtain $D_Y = L_Y$. We are done.

The equation $\rho(T) = \rho(X) - 1$ in (5) follows from (3).

4. Basic properties of log canonical singularities

In this section, we describe log canonical singularities in surfaces by using the contraction theorem (Theorem 3.1).

Definition 4.1. We say a pair $(X, \Delta)$ is a numerically log canonical surface if a normal surface $X$ and an $\mathbb{R}$-divisor $\Delta$ satisfy the following properties.

(1) For an arbitrary proper birational morphism $f : Y \to X$ and the divisor $\Delta_Y$ defined by

$$K_Y + \Delta_Y = f^*(K_X + \Delta),$$

the inequality $\Delta_Y \leq 1$ holds. Note that $f^*(K_X + \Delta)$ is defined by Definition 1.11.

(2) $\Delta$ is effective.

We say $(X, \Delta)$ is a log canonical surface if it is numerically log canonical and $K_X + \Delta$ is $\mathbb{R}$-Cartier.

The goal of this section is Proposition 4.4, which states that numerical log canonical surfaces are log canonical, that is, $K_X + \Delta$ is $\mathbb{R}$-Cartier. For the case where $k$ is algebraically closed, one of useful tools is the classification of surface log canonical singularities. To avoid such a classification result, we establish the following lemma.
Lemma 4.2. Let \((X, \Delta)\) be a numerically log canonical surface over a field of characteristic \(p > 0\). Assume that \(x \in X\) is a unique non-regular closed point of \(X\). Then one of the following assertions hold.

1. \(X\) is \(\mathbb{Q}\)-factorial.
2. \(x \not\in \text{Supp}\Delta\) and there exists a projective birational morphism 
   
   \[ f : Z \to X \]

   from a normal \(\mathbb{Q}\)-factorial surface \(Z\) such that \(E := \text{Ex}(f)\) is a projective irreducible curve and that \((K_Z + E) \cdot E = 0\).

Proof. Let \(g : Y \to X\) be the minimal resolution of \(X\). We assume that there exist \(j \in \mathbb{Z}_{\geq 0}\) and a sequence of projective birational morphisms

\[ g : Y =: Y_0 \xrightarrow{g_0} Y_1 \xrightarrow{g_1} \cdots \xrightarrow{g_{j-1}} Y_j \xrightarrow{F} X \]

\(\Delta_Y =: \Delta_0, (g_i)_*\Delta_i =: \Delta_{i+1}\)

such that each \((Y_i, \Delta_i)\) and \(g_i\) satisfy the following properties.

(a) Each \(Y_i\) is a normal \(\mathbb{Q}\)-factorial surface.
(b) Each \(g_i\) is a projective birational morphism and \(E_i := \text{Ex}(g_i)\) is irreducible.
(c) Each \(E_i\) satisfies \((K_{Y_i} + E_i) \cdot E_i < 0\).

It suffices to show that one of the following assertions hold.

- \(F\) is an isomorphism.
- \(Y_j =: Z\) and \(F =: f\) satisfies the same properties as (2).
- There exists a projective birational morphisms
  
  \[ Y_j \xrightarrow{g_j} Y_{j+1} \xrightarrow{F'} X \]

  which satisfy (a)–(c).

Thus, we can assume that \(F\) is not an isomorphism.

If we can find an \(F\)-exceptional proper curve \(E_j\) on \(Y_j\) such that \((K_{Y_j} + E_j) \cdot E_j < 0\), then we obtain a contraction of \(E_j\)

\[ g_j : Y_j \to Y_{j+1} \]

to a \(\mathbb{Q}\)-factorial surface \(Y_{j+1}\) by Theorem 3.1. Therefore, we can assume that every \(F\)-exceptional proper curve \(E_j\) satisfies \((K_{Y_j} + E_j) \cdot E_j \geq 0\).

Thus every \(F\)-exceptional curve \(E_j\) satisfies

\[ 0 \leq (K_{Y_j} + E_j) \cdot E_j \leq (K_{Y_j} + \Delta_j) \cdot E_j = F^*(K_X + \Delta) \cdot E_j = 0, \]

where \(\Delta_j\) is defined by \(K_{Y_j} + \Delta_j = F^*(K_X + \Delta)\). Therefore the coefficient of \(E_j\) in \(\Delta_j\) is one.

Assume that \(\text{Ex}(F)\) is reducible. Then there exists an \(F\)-exceptional curve \(E_j'\) such that \(E_j \cap E_j' \neq \emptyset\). We have

\[ (K_{Y_j} + E_j') \cdot E_j' < (K_{Y_j} + \Delta_j) \cdot E_j' = F^*(K_X + \Delta_j) \cdot E_j' = 0. \]
This case is excluded.

Thus we see that $E := \text{Ex}(F)$ is irreducible. We show that $Y_j =: Z$ and $F =: f$ satisfies the same properties as (2). We have $(K_{Y_j} + E) \cdot E = 0$. We show $x \not\in \text{Supp}\Delta$. If $x \in \text{Supp}\Delta$, then we obtain a contradiction:

$$(K_{Y_j} + E) \cdot E < (K_{Y_j} + \Delta_j) \cdot E = 0.$$ 

This implies (2). □

To show that $K_X + \Delta$ is $\mathbb{R}$-Cartier for a numerically log canonical surface, it suffices to consider only the case (2) in Lemma 4.2. In the following lemma, we prove that $K_X + \Delta$ is $\mathbb{R}$-Cartier in this case.

**Lemma 4.3.** Let $k$ be a field of characteristic $p > 0$. Let $f : Z \to X$ be a projective birational morphism of normal $k$-surfaces. Assume the following conditions.

1. $Z$ is $\mathbb{Q}$-factorial.
2. $E := \text{Ex}(f)$ is irreducible.
3. $(Z, E)$ is log canonical.
4. $(K_Z + E) \cdot E = 0$.

Then $K_X$ is $\mathbb{Q}$-Cartier.

**Proof.** Set $x := f(E)$. By shrinking $X$ around $x$, we may assume that $X$ is affine. Moreover, by taking a projective compactification $\overline{X}$ which is regular along $\overline{X} \setminus X$, we may assume that $X$ and $Z$ are projective.

Let $A_X$ be an ample Cartier divisor on $X$. Consider the following divisor $L := K_Z + E + mf^*A_X$ for $m \gg 0$. We can check that $L$ is nef and big and that, for a curve $C$ on $Z$, $L \cdot C = 0$ if and only if $C = E$.

For the time being, we assume that $L$ is semi-ample and let us show the assertion. We see that the induced morphism by $L$ is the same as $f$. Then, $nL = n(K_Z + E + mf^*A_X) = f^*L_X$ for some Cartier divisor $L_X$ on $X$. Taking the push-forward $f_*$, we obtain

$$n(K_X + mA_X) = L_X.$$ 

Thus, $K_X$ is $\mathbb{Q}$-Cartier.

It suffices to show that $L = K_Z + E + mf^*A_X$ is semi-ample. By Theorem 2.3(2), we obtain

$$L|_E = (K_Z + E + mf^*A_X)|_E = (K_Z + E)|_E \sim_{\mathbb{Q}} 0.$$ 

Therefore, Theorem 1.13 implies that $L$ is semi-ample. □

We show the main result in this section.

**Proposition 4.4.** Let $(X, \Delta)$ be a numerically log canonical surface over a field of characteristic $p > 0$. Then the following assertions hold.
(1) $K_X$ and all the irreducible components of $\Delta$ are $\mathbb{Q}$-Cartier. In particular, $(X, \Delta)$ is log canonical, that is, $K_X + \Delta$ is $\mathbb{R}$-Cartier.

(2) There exists an open subset $X^0 \subset X$ such that $\text{Supp} \Delta \subset X^0$ and that $X^0$ is $\mathbb{Q}$-factorial.

Proof. By Lemma 4.2 and Lemma 4.3, $K_X$ is $\mathbb{Q}$-Cartier. The assertion (2) follows from Lemma 4.2. In particular, all the irreducible components of $\Delta$ are $\mathbb{Q}$-Cartier. Thus (1) holds. □

We show the following lemma, which will be used in Section 5.

Lemma 4.5. Let $(X, \Delta)$ be a log canonical surface over a field $k$ of characteristic $p > 0$. Let $C$ be a proper $k$-curve $C$ in $X$ such that $(K_X + \Delta) \cdot C < 0$ and that $C^2 < 0$. Then $C$ is $\mathbb{Q}$-Cartier.

Proof.

Step 1. In this step, we show that we may assume that $X$ has a unique non-regular closed point $x$ and that $x \in C$.

If $C$ contains no non-regular closed points of $X$, then there is nothing to show. Fix a non-regular closed point $x$ of $X$ which lies on $C$. It suffices to show that $C|_{\text{Spec} \mathcal{O}_{X,x}}$ is $\mathbb{Q}$-Cartier. Thus, by applying the minimal resolution for the other non-regular points, we can assume that this point is a unique non-regular point of $X$.

Step 2. In this step, we show that we can assume that $\Delta = 0$.

If $C \subset \text{Supp} \Delta$, then the assertion follows from Proposition 4.4. Thus we may assume $C \not\subset \text{Supp} \Delta$. In particular, $\Delta \cdot C \geq 0$ and we obtain $K_X \cdot C < 0$.

Thus, by Proposition 4.4, we can replace $K_X + \Delta$ with $K_X$.

Step 3. In this step, we show that we may assume that there exists a projective birational morphism $g : Z \to X$

from a $\mathbb{Q}$-factorial surface $Z$ such that $E := \text{Exc}(g)$ is irreducible and that $K_Z + E = g^* K_X$.

We can apply Lemma 4.2 and $x \in X$ satisfies (1) or (2) in Lemma 4.2. If Lemma 4.2(1) holds, then $X$ is $\mathbb{Q}$-factorial, hence the assertion in the lemma holds. Thus, we may assume that Theorem 4.2(2) holds. This implies the assertion in this step.
**Step 4.** Let $C_Z \subset Z$ be the proper transform of $C$. In this step, we show that there exists a projective birational morphism

$$\text{cont}_{C_Z} = \varphi : Z \to Z'$$

to a $\mathbb{Q}$-factorial surface $Z'$ such that $\text{Ex}(\varphi) = C_Z$.

By Theorem 3.1 it suffices to show that $C_Z^2 < 0$ and that $(K_Z + C_Z) \cdot C_Z < 0$. These follows from

$$C_Z^2 \leq C_Z \cdot g^*(C) = C^2 < 0,$$

and

$$K_Z \cdot C_Z \leq (K_Z + E) \cdot C_Z = g^* (K_X \cdot C_Z) = K_X \cdot C < 0.$$

**Step 5.** Set $E' := \varphi_*(E)$. In this step, we show that there exists a projective birational morphism

$$\text{cont}_{E'} = g' : Z' \to X'$$

to a $\mathbb{Q}$-factorial surface $X'$ such that $\text{Ex}(g') = E'$.

By Theorem 3.1 it suffices to prove that $E'^2 < 0$ and that $(K_{Z'} + E') \cdot E' < 0$. First, we show $E'^2 < 0$. Since

$$E'^2 = (\varphi^*(E'))^2 = (E + cC_Z)^2$$

for some $c \in \mathbb{Q}$, it is enough to prove that $(aE + bC_Z)^2 < 0$ for every $(a, b) \in \mathbb{Q}^2 \setminus \{(0,0)\}$. This follows from

$$(aE + bC_Z)^2 = (a'E + bg^*C)^2 = a'^2 E^2 + b^2 C^2 < 0$$

for $(a', b) \in \mathbb{Q}^2 \setminus \{(0,0)\}$, because $(a, b) \neq (0,0)$ if and only if $(a', b) \neq (0,0)$. Thus we obtain $E'^2 < 0$. Second, we show $(K_{Z'} + E') \cdot E' < 0$. Let us consider the rational number $d$ defined by

$$K_Z + E = \varphi^*(K_{Z'} + E') + dC_Z.$$

Here, taking the intersection with $E$, we obtain

$$0 = (K_{Z'} + E') \cdot E' + dC_Z \cdot E$$

by $(K_Z + E) \cdot E = 0$. Since $x \in C$, we see $C_Z \cdot E > 0$. Thus it is sufficient to prove $d > 0$. This holds by the following inequality

$$0 > K_X \cdot C = g^* (K_X) \cdot C_Z = (K_Z + E) \cdot C_Z = dC_Z^2.$$

Thus we obtain $(K_{Z'} + E') \cdot E' < 0$.

**Step 6.** In this step, we show that $C$ is $\mathbb{Q}$-Cartier.

We have a commutative diagram

$$\begin{array}{ccc}
Z & \xrightarrow{\varphi} & Z' \\
\downarrow g & & \downarrow g' \\
X & \xrightarrow{\psi} & X',
\end{array}$$
where $\psi : X \to X'$ is the contraction of $C$. Since $X'$ is $\mathbb{Q}$-factorial,

$$K_X + \alpha C = \psi^* K_{X'},$$

is $\mathbb{Q}$-Cartier, where $\alpha \in \mathbb{Q}$. On the other hand, $K_X$ is $\mathbb{Q}$-Cartier because $(X, \Delta = 0)$ is log canonical (Proposition 4.4). Thus it suffices to show that $\alpha \neq 0$, which follows from $K_X \cdot C \neq 0$, $C^2 \neq 0$ and $\psi^* K_{X'} \cdot C = 0$.

\[ \square \]

5. Minimal model program

In this section, we establish the minimal model program for log canonical surfaces. Since we know the cone theorem (Subsection 1.7), let us prove the contraction theorem.

**Theorem 5.1** (Contraction theorem). Let $k$ be a field of characteristic $p > 0$. Let $(X, \Delta)$ be a log canonical $k$-surface and let $f : X \to S$ be a projective $k$-morphism to a separated scheme $S$ of finite type over $k$. Let $R = \mathbb{R}_{\geq 0}[C]$ be a $(K_X + \Delta)$-negative extremal ray of $\overline{NE}(X/S)$. Then there exists a projective $S$-morphism $\varphi_R : X \to Y$ to a $k$-variety $Y$, which is projective over $S$, with the following properties: (1)–(5).

1. Let $C'$ be a proper $k$-curve on $X$ such that $f(C')$ is one point. Then $\varphi_R(C')$ is one point if and only if $[C'] \in R$.
2. $(\varphi_R)_*(O_X) = O_Y$.
3. Assume that $\dim Y \neq 0$. If $L$ is a Cartier divisor on $X$ with $L \cdot C = 0$, then $nL = (\varphi_R)^* L_Y$ for some Cartier divisor $L_Y$ on $Y$ and for some positive integer $n$.
4. $\rho(Y/S) = \rho(X/S) - 1$.
5. If $\dim Y = 2$, then $(Y, (\varphi_R)_*(\Delta))$ is a log canonical surface.

**Remark 5.2.** After we prove the abundance theorem established in Section 7, the assumption $\dim Y \neq 0$ in (3) will be dropped.

**Remark 5.3.** Let $N^1(X/S)_\mathbb{Q}$ be the finite dimensional $\mathbb{Q}$-vector space of $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor modulo numerical equivalence over $S$. Note that $\rho(X/S) = \dim \mathbb{Q} N^1(X/S)_\mathbb{Q}$.

Assume that we obtain a morphism $\varphi_R : X \to Y$ which satisfies (1) and (2). Consider a sequence:

$$0 \to N^1(Y/S)_\mathbb{Q} \xrightarrow{(\varphi_R)^*} N^1(X/S)_\mathbb{Q} \xrightarrow{\cdot C} \mathbb{Q} \to 0.$$ 

Clearly the former map $(\varphi_R)^*$ is injective and the latter map $\cdot C$ is surjective. It is obvious that the composition map $\cdot C \circ (\varphi_R)^*$ is zero. Thus, if (3) holds, then the above sequence is exact. In particular, (3) implies (4).
5.4 (A reduction step). By taking the Stein factorization of \( f : X \to S \), we may assume that \( f_* \mathcal{O}_X = \mathcal{O}_S \). In particular, \( S \) is a normal \( k \)-variety with \( 0 \leq \dim S \leq \dim X = 2 \). We have the three cases: \( C^2 < 0 \), \( C^2 = 0 \), and \( C^2 > 0 \).

Note that we obtain the following implications.
- If \( \dim S = 2 \), then \( C^2 < 0 \).
- If \( \dim S = 1 \), then \( C^2 \leq 0 \).

**Proof of the case when \( C^2 < 0 \).** By Lemma 4.5, \( C \) is \( \mathbb{Q} \)-Cartier. If \( S \) is quasi-projective, then by Theorem 3.1, we obtain a \( S \)-morphism \( \varphi_R : X \to Y \) which satisfies (1)(2)(3)(4). For the general case, we take an affine open neighborhood \( f(C) \in S_1 \subset S \) and set \( S_2 := S \setminus \{ f(C) \} \).

We see \( S = S_1 \cup S_2 \). Since \( S_1 \) is quasi-projective, we can obtain a required contraction over \( S_1 \). We do nothing over \( S_2 \) and glue them together. Then, we obtain a projective \( S \)-morphism \( \varphi_R : X \to Y \) which satisfies (1)(2)(3)(4).

We show (5). We see that \( (Y, (\varphi_R)_* \Delta) \) is numerically log canonical. By Proposition 4.4, this is log canonical. □

**Proof of the case when \( C^2 > 0 \).** In this case, we see \( \dim S = 0 \) (cf. (5.4)). First, we prove that every curve \( D \) in \( X \) satisfies \( D \in \mathcal{R}_{\geq 0}[C] \).

Let \( f : X' \to X \) be a resolution of singularities. Since \( f^*(C)^2 > 0 \), \( f^*(C) \) is a nef and big \( \mathbb{Q} \)-divisor. By Kodaira’s lemma, we obtain

\[
nf^*(C) \sim f^*D + E
\]

for some \( n \in \mathbb{Z}_{>0} \) and effective divisor \( E \). By applying \( f_* \) to this equation, we obtain

\[
nC \equiv D + f_*(E),
\]

that is, \( L \cdot nC = L \cdot (D + f_*(E)) \) for every \( \mathbb{R} \)-Cartier \( \mathbb{R} \)-divisor \( L \) on \( X \). Since \( \mathbb{R}_{>0}[C] \) is extremal, we have \( D \in \mathbb{R}_{>0}[C] \).

We see that every curve \( D \) in \( X \) satisfies \( D \in \mathbb{R}_{>0}[C] \). In particular, we obtain \( \rho(X) = 1 \). Consider the Stein factorization of the structure morphism: \( X \to \text{Spec} k' \to \text{Spec} k \). Set \( Y := \text{Spec} k' \). Then \( \varphi_R : X \to Y \) satisfies (1), (2) and (4). In this case, (3) has no assertion and we are done. □

**Proof of the case when \( C^2 = 0 \).** There are two cases (a) \( \dim S = 0 \) and (b) \( \dim S = 1 \) (cf. (5.4)).

(a) Assume \( \dim S = 0 \) and we show the assertion. We prove that \( C \) is \( \mathbb{Q} \)-Cartier and that there exists a morphism \( \varphi : X \to Y \) to a projective regular curve with \( \varphi_* \mathcal{O}_X = \mathcal{O}_Y \) such that \( C = (\varphi^{-1}(y))_{\text{red}} \) for some closed point \( y \in Y \). Let \( f : X' \to X \) be the minimal resolution. By \( K_X \cdot C \leq (K_X + \Delta) \cdot C < 0 \), we see \( K_{X'} \cdot f^*C < 0 \). Then, by
$K_X \cdot f^*C < 0$ and $f^*(C)^2 = 0$, the $\mathbb{Q}$-divisor $f^*(C)$ is semi-ample by Lemma 1.16. We consider the projective surjective morphism $\rho : X' \to Y$ to a projective regular curve, with $\rho_* \mathcal{O}_{X'} = \mathcal{O}_Y$, obtained by the complete linear system $|nf^*(C)|$ for some sufficiently divisible $n \in \mathbb{Z}_{>0}$. For every $f$-exceptional curve $E \subset X'$, we have $f^*(C) \cdot E = 0$. This means that $\rho$ factors through $X$:

$$\rho : X' \xrightarrow{f} X \xrightarrow{\varphi} Y.$$ 

Since $\text{Supp}(f^*(C))$ is a union of fibers of $\rho$, $C$ is contained in a fiber $\varphi^{-1}(y)$. If $\varphi^{-1}(y)$ is reducible, then we obtain $C^2 < 0$, hence $C = (\varphi^{-1}(y))_{\text{red}}$. In particular, $C$ is $\mathbb{Q}$-Cartier.

Therefore, the morphism $\varphi : X \to Y$ satisfies the required properties (1) and (2). We show (3). By Lemma 5.5(2), it suffices to show that every fiber of $\varphi$ is irreducible. Otherwise, there exists a curve $C' \in \mathbb{R}_{\geq 0}[C]$ such that $C'^2 < 0$. By Lemma 4.5, $C'$ is $\mathbb{Q}$-Cartier. However $C \cdot C' = 0$ and $C'^2 < 0$ implies a contradiction $C' \notin \mathbb{R}_{\geq 0}[C]$. Thus (3) holds, hence also (4) holds. This completes the proof of the case (a).

(b) Assume $\dim S = 1$ and we show the assertion. In this case, $C$ is an irreducible fiber of $f : X \to S$ since $C^2 = 0$. By the same argument as the case (a), we can show that every fiber of $f$ is irreducible.

Since every fiber of $\varphi$ is irreducible, we obtain $\rho(X/S) = 1$. Set $Y := S$. Then, the properties (1) and (2) hold. We show (3). Take projective compactifications $\overline{\varphi} : \overline{X} \to \overline{Y}$ of $X$ and $Y$ such that $\overline{Y}$ is regular and $\overline{X}$ is regular along $\overline{X} \setminus X$. If every fiber of $\overline{\varphi}$ is irreducible, then Lemma 5.5(2) implies the required assertion (3) because a Cartier divisor on $X$ can be extended to one on $\overline{X}$. Thus we show that we can find a compactification $\overline{\varphi} : \overline{X} \to \overline{Y}$ which has irreducible fibers. We have a compactification $\overline{\varphi} : \overline{X} \to \overline{Y}$ such that $\overline{X}$ is regular along reducible fiber. Moreover, we know $K_X \cdot F < 0$ for a general fiber of $\overline{\varphi}$. Considering the intersection number $K_X \cdot (\sum G_i) < 0$ where $\sum G_i$ is a reducible fiber, we can find a curve $G_i$ such that $K_X \cdot G_i < 0$ and $G_i^2 < 0$. We can contract $G_i$ (Theorem 3.1) and we can repeat this procedure. Then, every fiber of the end result $\overline{X} \to \overline{Y}$ is irreducible. Thus we obtain (3), which implies (4). This completes the proof of the case (b).

The following lemma is used in the proof of the case $C^2 = 0$.

**Lemma 5.5.** Let $k$ be a field. Let $\varphi : X \to Y$ be a $k$-morphism from a projective normal $k$-surface to a projective regular $k$-curve with $\varphi_* \mathcal{O}_X = \mathcal{O}_Y$. Take the fiber $F$ of $\varphi$ of a closed point of $Y$. Assume that the following conditions hold.

...
• Every fiber of $\varphi$ is irreducible.
• There exists an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $\Delta$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier and that $(K_X + \Delta) \cdot F < 0$.

Then, the following assertions hold.

1. $\mathbb{R}_{\geq 0}[F]$ is an extremal ray of $\overline{NE}(X)$.
2. For every $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $L$ on $X$ with $L \cdot F = 0$, there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $L_Y$ on $Y$ such that $L = \varphi^*L_Y$.

Proof. (1) By the cone theorem (Theorem 1.18), $[F] \in \overline{NE}(X)_{K_X + \Delta < 0} \cap F^\perp$ implies that $F^\perp$ contains a $(K_X + \Delta)$-negative extremal ray $\mathbb{R}_{\geq 0}[C]$. However, $F \cdot C = 0$ implies that $C$ is contained in a fiber of $\varphi$. Since every fiber of $\varphi$ is irreducible, $C$ is equal to the reduced structure of a fiber of $g$. Therefore, $\mathbb{R}_{\geq 0}[C] = \mathbb{R}_{\geq 0}[F]$ is an extremal ray.

(2) First, we show that for every nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $M$ with $\overline{NE}(X) \cap M^\perp = \mathbb{R}_{\geq 0}[F]$, there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $M_Y$ such that $M = \varphi^*M_Y$. Take an integer $n \gg 0$ such that

$$(M + nF) \cdot (K_X + \Delta) < 0.$$  

Since $M \cdot F = F^2 = 0$, we obtain $M^2 = 0$, otherwise the nefness of $M$ implies $M^2 > 0$ which gives a contradiction $F^2 < 0$. Thus, we obtain $(M + nF)^2 = 0$ and see that $M + nF$ is semi-ample by Lemma 1.16. Thus, $M + nF$ induces a induces a morphism $\psi : X \to Z$ to a curve $Z$ with $\psi_*\mathcal{O}_X = \mathcal{O}_Z$ such that $M + nF$ is the pull-back of a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $Z$. Since $(M + nF) \cdot C = 0$, we obtain the following factorization:

$$\psi : X \xrightarrow{\varphi} Y \to Z.$$  

Then, the equation $\psi_*\mathcal{O}_X = \mathcal{O}_Z$ implies $Y \simeq Z$. Since $M + nC$ and $C$ are the pull-backs of $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on $Y$, so is $M$.

By the cone theorem (Theorem 1.18), we can find a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $M$ on $X$ such that $\overline{NE} \cap M^\perp = \mathbb{R}_{\geq 0}[F]$. Again, by the cone theorem (Theorem 1.18), the divisor $M' := L + \ell M$, with $\ell \gg 0$, is also a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor with $\overline{NE}(X) \cap (M')^\perp = \mathbb{R}_{\geq 0}[F]$. Then, we obtain $M = \varphi^*M_Y$ (resp. $M' = \varphi^*M'_Y$) for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $M_Y$ (resp. $M'_Y$) on $Y$. Therefore, $L = \varphi^*(M'_Y - \ell M)$. We are done.

We obtain the minimal model program which is one of the main result in this paper.

**Theorem 5.6** (Minimal model program). Let $k$ be a field of characteristic $p > 0$. Let $(X, \Delta)$ be a log canonical $k$-surface and let $f : X \to S$ be a projective $k$-morphism to a separated scheme $S$ of finite type over
Then, there exists a sequence of projective birational $S$-morphisms
\[(X, \Delta) =: (X_0, \Delta_0) \xrightarrow{\varphi_0} (X_1, \Delta_1) \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{s-1}} (X_s, \Delta_s) =: (X^\dagger, \Delta^\dagger)\]
where \((\varphi_{i-1})_* (\Delta_{i-1}) =: \Delta_i\)
with the following properties.

1. Each \((X_i, \Delta)\) is a log canonical $k$-surface which is projective over $S$.
2. Each $\text{Ex}(\varphi_i) =: C_i$ is a projective $k$-curve such that
   \[(K_{X_i} + \Delta_i) \cdot C_i < 0.\]
3. \((X^\dagger, \Delta^\dagger)\) satisfies one of the following conditions.
   a. $K_{X^\dagger} + \Delta^\dagger$ is nef over $S$.
   b. There is a projective surjective morphism $\mu : X^\dagger \to Z$ to a $k$-variety $Z$, which is projective over $S$, such that $\mu_* \mathcal{O}_{X^\dagger} = \mathcal{O}_Z$, $\dim X^\dagger > \dim Z$, $-(K_{X^\dagger} + \Delta^\dagger)$ is $\mu$-ample and $\rho(X^\dagger/S) - 1 = \rho(Z/S)$.

Proof. The assertion follows from Theorem 1.18 and Theorem 5.1. \(\square\)

6. ABUNDANCE THEOREM FOR KLT SURFACES

In this section, we prove a special case of the abundance theorem (Theorem 6.2). This theorem implies the abundance theorem for klt surfaces (Theorem 6.4). First we give a criterion for semi-ampleness.

Proposition 6.1. Let $k$ be a field of characteristic $p > 0$. Let $X$ be a projective normal surface over $k$. Let $E$ be an effective $\mathbb{Q}$-divisor on $X$ and let $E = \sum e_j E_j$ be the prime decomposition. Assume the following conditions.

1. $K_X$ and $E$ are $\mathbb{Q}$-Cartier.
2. $K_X \cdot E_j = E \cdot E_j = 0$ for every $j$.
3. There exists $s \in \mathbb{Z}_{>0}$ such that $sE$ is Cartier and that $\mathcal{O}_X(sE)|_E \simeq \mathcal{O}_E$.

Then, $E$ is semi-ample.

Proof.

Step 1. By taking the base change to the separable closure of $k$, we may assume that $k$ is separably closed. From now on, we assume that $k$ is separably closed.

Step 2. In this step we show that we may assume that $X$ is $\mathbb{Q}$-factorial. Take a resolution $h : X' \to X$ and let $E' := h^* E$. We can replace $X$ and $E$ with $X'$ and $E'$, respectively. Thus we may assume that $X$ is $\mathbb{Q}$-factorial.
Step 3. In this step we show that we may assume that $X$ is \(\mathbb{Q}\)-factorial and $E$ is irreducible.

By Step 2, we may assume that $X$ is \(\mathbb{Q}\)-factorial. Clearly we may assume that $E$ is connected. We reduce the proof to the case when $E$ is irreducible. Assume that $E$ is not irreducible. Fix an irreducible component $E_1$ of $E$. Since $E$ is connected and $E \cdot E_1 = 0$, we obtain $E_1^2 < 0$. This implies \((K_X + E_1) \cdot E_1 < 0\). By Theorem 3.1, we can contract $E_1$:

$$\text{cont}_{E_1} = \pi : X \to Z$$

to a projective $\mathbb{Q}$-factorial surface $Z$. For the time being, we assume that $Z$ and $E' := \pi_* E$ satisfies the same properties as (1), (2), and (3). If $E'$ is irreducible, then we are done. Otherwise, we can apply the same argument as above. Since the number of the irreducible components of $E'$ is strictly less than the one of $E$, this repeating procedure will terminate.

Therefore, it suffices to show that $Z$ and $E'$ satisfy (1), (2), and (3). Since $Z$ is \(\mathbb{Q}\)-factorial, the property (1) holds automatically. By $K_X \cdot E_1 = 0$, we can write

$$K_X = \pi^* K_Z.$$ 

By $E \cdot E_1 = 0$ and Theorem 3.1(3), we obtain $E = \pi^* \pi_* E = \pi^*(E')$, where $E' := \pi_* E$. Set $E_j' := \pi_* E_j$ for $j \neq 1$. Thus $K_Z$ and $E'$ satisfy (2) by

$$K_Z \cdot E_j' = K_Z \cdot \pi_* E_j = \pi^* K_Z \cdot E_j = K_X \cdot E_j = 0$$

and

$$E' \cdot E_j' = E' \cdot \pi_* E_j = \pi^* E' \cdot E_j = E \cdot E_j = 0.$$ 

We show that $E'$ satisfies (3), that is, $O_Z(sE')|_{E'} \simeq O_{E'}$ for some integer $s > 0$. Set

$$E \xrightarrow{\alpha} E'' \xrightarrow{\beta} E'$$

to be the Stein factorization. Since every fiber of $E \to E'$ is connected and $k$ is separably closed, $E'' \to E'$ is a universally homeomorphism. Thus it suffices to show that

$$\beta^*(O_Z(sE')|_{E'}) \simeq O_{E''}$$

for some $s > 0$. Fix $s > 0$ such that $sE$ and $sE'$ are Cartier and that $O_X(sE)|_E \simeq O_E$. Then, we obtain an isomorphism $\beta^*(O_Z(sE')|_{E'}) \simeq O_{E''}$ by

$$\alpha_*(O_X(sE)|_E) \simeq \alpha_* O_E \simeq O_{E''}$$

and

$$\alpha_*(O_X(sE)|_E) \simeq \alpha_*(\pi^* O_Z(sE')|_E) \simeq \alpha_*(\alpha^* \beta^*(O_Z(sE')|_{E'})) \simeq \beta^*(O_Z(sE')|_{E'}).$$
**Step 4.** In this step, we reduce the proof to the case when \( k \) is algebraically closed.

By Step 3, we can assume the following conditions.
- \( k \) is separably closed.
- \( X \) is \( \mathbb{Q} \)-factorial.
- \( E \) is irreducible.

Let \( Y \) be the normalization of \( (X \times_k \overline{k})_{\text{red}} \) and let
\[
f : Y \to X
\]
be the induced morphism. By Theorem 1.5, we obtain
\[
K_Y + D = f^*K_X
\]
for some effective \( \mathbb{Z} \)-divisor \( D \) on \( Y \). Note that \( Y \) is \( \mathbb{Q} \)-factorial by Lemma 1.1(3). Set \( E_Y := f^*E \). It suffices to show that \( E_Y \) is semi-ample. For this, we check that \( Y \) and \( E_Y \) satisfies the properties (1), (2), and (3). Since \( Y \) is \( \mathbb{Q} \)-factorial, (1) holds automatically. Since
\[
\mathcal{O}_E \simeq \mathcal{O}_X(sE)|_{E_Y}
\]
we can replace \( E \) with \( qE \) for \( q \in \mathbb{Q} > 0 \). Since \( k \) is algebraically closed, the assertion follows from [Mašek, Lemma in page 682].

Step 4 and Step 5 imply the assertion in the proposition. \( \Box \)

We prove the main theorem in this section.

**Theorem 6.2.** Let \( k \) be a separably closed field of characteristic \( p > 0 \). Let \( X \) be a projective normal \( \mathbb{Q} \)-factorial surface over \( k \) and let \( \Delta \) be a \( \mathbb{Q} \)-divisor with \( 0 \leq \Delta \leq 1 \). Assume the following condition \((\ast)\).
For a curve $C$ in $X$, if $(K_X + \Delta) \cdot C = 0$, then $C^2 \geq 0$.

If $K_X + \Delta$ is nef, then $K_X + \Delta$ is semi-ample.

Proof. Since $k$ is separably closed, $X$ is geometrically irreducible. Let $Y$ be the normalization of $(X \times_k \overline{k})_{\text{red}}$ and set
$$f : Y \to X$$
to be the induced morphism. Then, $Y$ is $\mathbb{Q}$-factorial (Lemma 1.1(3)) and we obtain
$$K_Y + E + f^*\Delta = f^*(K_X + \Delta)$$
for some effective $\mathbb{Z}$-divisor $E$ on $Y$ by Theorem 1.5.

Step 1. In this step, we show that we may assume that $(K_X + \Delta)^2 = 0$ and that, for every irreducible component $C$ of $\text{Supp}E \cup \text{Supp}f^*\Delta$, we obtain
$$f^*(K_X + \Delta) \cdot C = 0.$$
In particular, $C^2 \geq 0$.

For a rational number $0 < \epsilon < 1$, we consider the following equation:
$$K_Y + E + f^*\Delta = \epsilon K_Y + (\epsilon(E + f^*\Delta) + (1 - \epsilon)f^*(K_X + \Delta)).$$
Note that, for $0 < \epsilon \ll 1$, the latter divisor
$$\epsilon(E + f^*\Delta) + (1 - \epsilon)f^*(K_X + \Delta)$$
is nef by (*). We fix such a small rational number $0 < \epsilon \ll 1$. If this divisor is big, then we obtain
$$K_Y + E + f^*\Delta = \epsilon K_Y + (\epsilon(E + f^*\Delta) + (1 - \epsilon)f^*(K_X + \Delta)) \sim_{\mathbb{Q}} \epsilon(K_Y + \Delta_Y)$$
where $\Delta_Y$ is a $\mathbb{Q}$-divisor whose coefficients are contained in $[0, 1]$. Then, we can apply [11, Theorem 1.2]. Thus, we may assume $(K_X + \Delta)^2 = 0$. Moreover, we may assume that, for every irreducible component $C$ of $\text{Supp}E \cup \text{Supp}f^*\Delta$,
$$f^*(K_X + \Delta) \cdot C = 0.$$
The condition (*) implies $C^2 \geq 0$.

Step 2 (Notation). Let
$$E + g^*\Delta = \sum_i C_i$$
be the decomposition into the connected component and let
$$C_i = \sum_j c_{ij} C_{ij}$$
be the irreducible decomposition. Note that $C_{ij}^2 \geq 0$ for every $i$ and $j$ by Step [11].
Step 3. We fix an arbitrary index $i$ and we prove the following assertions.

(1) If $C_i$ is reducible, then $C_i$ is semi-ample.
(2) Assume that $C_i$ is irreducible. Let $C_i = c_{i0}C_{i0}$ where $c_{i0} \in \mathbb{Q}_{>0}$ and $C_{i0}$ is the prime divisor.

(2a) If $C_i \geq 0$, then $C_i$ is semi-ample.
(2b) If $C_i^2 = 0$, then $C_i$ is semi-ample or $0 \leq c_{i0} \leq 1$.

1. Assume $C_i$ is reducible. Then $C_{ij}^2 \geq 0$ implies $C_i \cdot C_{ij} > 0$ for every $j$. We see that $C_i$ is semi-ample by Lemma [1, 14].
(2a) The assertion follows from Lemma [1, 14].
(2b) Assume that $c_{i0} > 1$, and we show that $C_i$ is semi-ample. Set $C_{i0, X} := f(C_{i0})$ to be the curve. Since $(K_X + \Delta) \cdot C_{i0, X} = 0$ (cf. Step 1) and $C_{i0, X}^2 = 0$, we obtain $K_X \cdot C_{i0, X} \leq 0$. If $K_X \cdot C_{i0, X} < 0$, then $C_{i0, X}$ is semi-ample by Lemma [1, 16]. Thus, we may assume

$$K_X \cdot C_{i0, X} = \Delta \cdot C_{i0, X} = 0.$$

This implies $(K_Y + E + g^*\Delta) \cdot C_{i0} = 0$. Since $C_{i0}^2 = 0$, we obtain $K_Y \cdot C_{i0} \leq 0$. If $K_Y \cdot C_{i0} < 0$, then $C_{i0}$ is semi-ample by Lemma [1, 16]. Thus we can assume

$$K_Y \cdot C_{i0} = (E + g^*\Delta) \cdot C_{i0} = 0.$$

This implies that $C_{i0}$ (resp. $C_{i0, X}$) is a connected component of Supp($E + g^*\Delta$) (resp. Supp$\Delta$). Thus, for sufficiently divisible $s \in \mathbb{Z}_{>0}$, we obtain

$$\mathcal{O}_X(s(K_X + \Delta))|_{C_{i0, X}} \simeq \mathcal{O}_X(s(K_X + \delta C_{i0, X}))|_{C_{i0, X}}, \quad \text{and}$$
$$\mathcal{O}_Y(s(K_Y + E + g^*\Delta))|_{C_{i0}} \simeq \mathcal{O}_Y(s(K_Y + c_{i0} C_{i0}))|_{C_{i0}},$$

where $0 \leq \delta \leq 1$ is the coefficient of $C_{i0, X}$ in $\Delta$. Since $(K_X + C_{i0, X}) \cdot C_{i0, X} = 0$ and $(K_Y + C_{i0}) \cdot C_{i0} = 0$, we can apply Theorem [2, 3] and obtain

$$\mathcal{O}_X(s(K_X + C_{i0, X}))|_{C_{i0, X}} \simeq \mathcal{O}_{C_{i0, X}}, \quad \text{and}$$
$$\mathcal{O}_Y(s(K_Y + C_{i0}))|_{C_{i0}} \simeq \mathcal{O}_{C_{i0}}$$

for sufficiently divisible $s \in \mathbb{Z}_{>0}$. Summarizing above, we have

$$\mathcal{O}_X(s(K_X + \Delta))|_{C_{i0, X}} \simeq \mathcal{O}_X(s(K_X + \delta C_{i0, X}))|_{C_{i0, X}} \simeq \mathcal{O}_X(s(-(1 - \delta)C_{i0, X}))|_{C_{i0, X}},$$
$$\mathcal{O}_Y(s(K_Y + E + g^*\Delta))|_{C_{i0}} \simeq \mathcal{O}_Y(s(K_Y + c_{i0} C_{i0}))|_{C_{i0}} \simeq \mathcal{O}_Y(s((c_{i0} - 1)C_{i0}))|_{C_{i0}}.$$

Since $\mathcal{O}_Y(s(K_Y + E + g^*\Delta)) \simeq g^*\mathcal{O}_X(s(K_X + \Delta))$, we obtain

$$\mathcal{O}_Y(s((c_{i0} - 1)C_{i0} + (1 - \delta)g^* C_{i0, X}))|_{C_{i0}} \simeq \mathcal{O}_Y(s(K_Y + E + g^*\Delta - g^*(K_X + \Delta))|_{C_{i0}} \simeq \mathcal{O}_{C_{i0}}.$$
Thus, for some sufficiently divisible $s' \in \mathbb{Z}_{>0}$, we obtain

$$O_Y(s'C_{i0})|C_{i0} \simeq O_{C_{i0}}.$$  

Since $K_Y \cdot C_{i0} = C_{i0}^2 = 0$, we can apply Lemma 6.1 and $C_{i0}$ is semi-ample.

**Step 4.** By Step 3 and Lemma 1.20 we can write

$$E + f^*\Delta = \sum_i C_i \sim_{\mathbb{Q}} \Delta_Y$$

for some $\mathbb{Q}$-divisor $\Delta_Y$ whose coefficients are contained in $[0,1]$. Then the divisor $K_Y + E + f^*\Delta \sim_{\mathbb{Q}} K_Y + \Delta_Y$ is semi-ample by [11, Theorem 1.2].

As consequences of Theorem 6.2, we obtain the abundance theorem for the regular and klt cases.

**Theorem 6.3.** Let $k$ be a field of characteristic $p > 0$. Let $X$ be a projective regular surface over $k$ and let $\Delta$ be a $\mathbb{Q}$-divisor with $0 \leq \Delta < 1$. If $K_X + \Delta$ is nef, then $K_X + \Delta$ is semi-ample.

**Proof.** Take the base change to the separable closure. Every conditions are stable under this base change. Thus we may assume that $k$ is separably closed.

It is enough to show that we may assume that the condition $(\ast)$ in Theorem 6.2 holds. Assume that there is a curve $C$ such that $(K_X + \Delta) \cdot C = 0$ and that $C^2 < 0$. Since $0 \leq \Delta < 1$, we obtain

$$(K_X + C) \cdot C < (K_X + \Delta) \cdot C = 0.$$  

Then, by Theorem 3.1 we obtain a contraction $f : X \to X'$ of $C$ to a $\mathbb{Q}$-factorial surface. We repeat this procedure and this will terminate.

**Theorem 6.4.** Let $k$ be a field of characteristic $p > 0$. Let $(X, \Delta)$ be a projective klt surface over $k$, where $\Delta$ is a $\mathbb{Q}$-divisor. If $K_X + \Delta$ is nef, then $K_X + \Delta$ is semi-ample.

**Proof.** Take the minimal resolution and we may assume that $X$ is regular. Then the assertion follows from Theorem 6.3.

7. **Abundance theorem for log canonical surfaces**

In this section, we show the abundance theorem for log canonical surfaces (Theorem 7.10), that is, for a projective log canonical surface $(X, \Delta)$ with a $\mathbb{Q}$-divisor $\Delta$, if $K_X + \Delta$ is nef, then it is semi-ample. Subsection 7.1 is devoted to show that $\kappa(X, K_X + \Delta) \geq 0$.  


In Subsection 7.2, we prove that $K_X + \Delta$ is semi-ample for each case $\kappa(X, K_X + \Delta) = 0, 1, \text{and 2}$. In Subsection 7.3, we generalize our result to the relative settings.

7.1. **Non-vanishing theorem.** The goal of this subsection is to show Theorem 7.7. A rough overview of this subsection is as follows. To show the non-vanishing theorem (Theorem 7.7), we can assume that $\kappa(X, K_X) = -\infty$. Thanks to the abundance theorem for the klt case (Theorem 6.4), we may assume that $(X \times_k \overline{k})_{\text{red}}$ is a ruled surface. The rational case (resp. the irrational case) is treated in Proposition 7.1 (resp. Proposition 7.6).

**Proposition 7.1.** Let $k$ be a separably closed field of characteristic $p > 0$. Let $(X, \Delta)$ be a projective log canonical surface over $k$, where $\Delta$ is a $\mathbb{Q}$-divisor. Assume the following conditions.

- $X$ is regular.
- $(X \times_k \overline{k})_{\text{red}}$ is a rational surface.

If $K_X + \Delta$ is nef, then $\kappa(X, K_X + \Delta) \geq 0$.

**Proof.** Let $Y$ be the minimal resolution of the normalization $(X \times_k \overline{k})_{\text{red}}$ of $X \times_k \overline{k}$:

$$f: Y \to (X \times_k \overline{k})_{\text{red}} \to (X \times_k \overline{k})_{\text{red}} \to X \times_k \overline{k} \to X.$$  

By our assumption, $Y$ is a smooth rational surface.

If $K_X + \Delta \equiv 0$, then the pull-back to $Y$ is torsion, hence also $K_X + \Delta$ is torsion. Thus we may assume that $K_X + \Delta \not\equiv 0$. Then, by Theorem 1.5, we obtain

$$K_Y + \Delta_Y = f^*(K_X + \Delta)$$

for some effective $\mathbb{Q}$-divisor $\Delta_Y$. Fix $m_0 \in \mathbb{Z}_{>0}$ such that $m_0(K_Y + \Delta_Y)$ is Cartier. Then, we can check that $H^2(Y, mm_0(K_Y + \Delta_Y)) = 0$ for every large integer $m \gg 0$. Therefore, by the Riemann–Roch formula, we obtain

$$h^0(Y, mm_0(K_Y + \Delta_Y)) \geq \chi(Y, \mathcal{O}_Y) + \frac{1}{2} mm_0(K_Y + \Delta_Y) \cdot (mm_0(K_Y + \Delta_Y) - K_Y)$$

$$= 1 + \frac{1}{2} mm_0(K_Y + \Delta_Y) \cdot (\Delta_Y + (mm_0 - 1)(K_Y + \Delta_Y))$$

$$\geq 1.$$

The above equality follows since $Y$ is rational. We are done. \qed

To show Lemma 7.3, we establish two auxiliary results on Mori fiber spaces (Lemma 7.2 and Lemma 7.3).
Lemma 7.2. Let \( k \) be a separably closed field of characteristic \( p > 0 \). Let \( X \) be a projective log terminal \( k \)-surface such that \(-K_X\) is ample and that \( \rho(X) = 1 \). Then \((X \times_k \overline{k})_{\text{red}}\) is a rational surface.

Proof. If \( k = \overline{\mathbb{F}}_p \), then the assertion is well-known. Thus we can assume that \( k \neq \overline{\mathbb{F}}_p \). We see \( \overline{\mathbb{F}}_p \subsetneq k \). In particular, \( \overline{k} \neq \overline{\mathbb{F}}_p \).

Assume that \((X \times_k \overline{k})_{\text{red}}\) is not a rational surface and let us derive a contradiction. Set \( Y \) to be the normalization of \((X \times_k \overline{k})_{\text{red}}\). Then, by Lemma [11] \( Y \) is a projective normal \( \mathbb{Q} \)-factorial surface such that \( \rho(Y) = 1 \). Moreover, by Theorem [13] and \( \rho(Y) = 1 \), \(-K_Y\) is ample. Let \( f : Z \to Y \) be the minimal resolution. We obtain \( K_Z + E = f^*K_Y \) for some effective \( \mathbb{Q} \)-divisor \( E \) on \( Z \). Thus, \( Z \) has a ruled surface structure \( \pi : Z \to B \). Since \( \overline{k} \neq \overline{\mathbb{F}}_p \) and \( Z \) is not rational, we can apply [11] Theorem 3.20. Therefore, every \( f \)-exceptional curves are \( \pi \)-vertical. Thus, \( Z \to B \) factors through \( Y \). In particular, the induced morphism \( Y \to B \) is a surjection. However, this contradicts \( \rho(Y) = 1 \). We are done. \( \square \)

Lemma 7.3. Let \( k \) be a separably closed field of characteristic \( p > 0 \). Let \( g : Z \to B \) be a surjective morphism from a regular projective \( k \)-surface to a regular projective \( k \)-curve. Assume that \( g \) is a \( K_Z \)-Mori fiber space structure, that is, \( g_*\mathcal{O}_Z = \mathcal{O}_B \), \( \rho(Z/B) = 1 \), and \(-K_Z\) is \( g \)-ample. Consider the following commutative diagram:

\[
\begin{array}{cccc}
Z & \leftarrow & Z \times_k \overline{k} & \leftarrow (Z \times_k \overline{k})_{\text{red}} \leftarrow (Z \times_k \overline{k})^N_{\text{red}} =: W \\
g \downarrow & & (g \times_k \overline{k})_{\text{red}} \downarrow (g \times_k \overline{k})^N_{\text{red}} & \\
B & \leftarrow & B \times_k \overline{k} & \leftarrow (B \times_k \overline{k})_{\text{red}} \leftarrow (B \times_k \overline{k})^N_{\text{red}} =: \Gamma',
\end{array}
\]

where \( Y^N \) is the normalization of a variety \( Y \). Let

\[(g \times_k \overline{k})^N_{\text{red}} : W \twoheadrightarrow \Gamma \xrightarrow{\gamma} \Gamma'\]

be the Stein factorization of \((g \times_k \overline{k})^N_{\text{red}}\). Then the following assertions hold.

1. \( W \) is \( \mathbb{Q} \)-factorial.
2. \( \gamma : \Gamma \to \Gamma' \) is a universally homeomorphism.
3. A general fiber of \( h : W \to \Gamma \) is \( \mathbb{P}^1_k \).
4. If \((Z \times_k \overline{k})_{\text{red}}\) is not rational, then \( H^1(B, \mathcal{O}_B) \neq 0 \) and neither \( \Gamma \) nor \( \Gamma' \) is a rational curve.

Proof. (1) The assertion follows from Lemma [11].

(2) Since \( g_*\mathcal{O}_Z = \mathcal{O}_B \) and the base change \((-) \times_k \overline{k} \) is flat, we obtain \((g \times_k \overline{k})_*\mathcal{O}_{Z \times_k \overline{k}} = \mathcal{O}_{B \times_k \overline{k}} \). Moreover, horizontal arrows

\[(Z \times_k \overline{k})^N_{\text{red}} \to Z \times_k \overline{k}, \quad (B \times_k \overline{k})^N_{\text{red}} \to B \times_k \overline{k} \]

...
are universally homeomorphisms. Thus every fiber of \((g \times_k \mathbb{A})^N_{\text{red}}\) is connected. Therefore \(\gamma : \Gamma \to \Gamma' = (B \times_k \mathbb{A})^N_{\text{red}}\) is a finite morphism whose fibers are connected. In particular, \(\gamma\) is a universally homeomorphism.

(3) Let \(f : W \to Z\) be the induced morphism. By Theorem 1.5, we can find an effective \(Z\)-divisor \(D\) on \(W\) such that
\[K_W + D = f^*K_Z.\]
Let \(F_g\) (resp. \(F_h\)) be a general fiber of \(g\) (resp. \(h\)). By (2), \(K_Z : F_g < 0\) implies \(K_W : F_h < 0\). Thus, we see \((K_W + F_h) \cdot F_h < 0\) and \(F_h \simeq \mathbb{P}^1_{\overline{k}}\).

(4) First, we show that neither \(\Gamma\) nor \(\Gamma'\) is a rational curve. By (3), there exists a non-empty open subset \(\Gamma^0 \subset \Gamma\) such that \(h^{-1}(\Gamma^0) \simeq \Gamma^0 \times_{\overline{k}} \mathbb{P}^1_{\overline{k}}\). Since \((Z \times_k \mathbb{A})^N_{\text{red}}\) is not rational, \(\Gamma\) is not a rational curve. By (2), \(\gamma : \Gamma \to \Gamma'\) is a finite surjective purely inseparable morphism. Therefore also \(\Gamma'\) is not a rational curve.

We show \(H^1(B, \mathcal{O}_B) \neq 0\). Assume that \(H^1(B, \mathcal{O}_B) = 0\) and let us derive a contradiction. By Corollary 1.8, we obtain \(\deg(K_B) < 0\). By Theorem 1.5, \(\Gamma' := (B \times_k \mathbb{A})^N_{\text{red}}\) satisfies \(\deg(K_{\Gamma'}) < 0\). Thus \(\Gamma' \simeq \mathbb{P}^1_{\overline{k}}\). However, we have already shown \(\Gamma' \not\simeq \mathbb{P}^1_{\overline{k}}\), which is a contradiction. \(\square\)

**Remark 7.4.** Lemma 7.3(4) states that \(H^1(B, \mathcal{O}_B) \neq 0\) for a projective regular curve \(B\). This implies
\[\chi(B, \mathcal{O}_B) = \dim_k H^0(B, \mathcal{O}_B) - \dim_k H^1(B, \mathcal{O}_B) \leq 0\]
as follows.

Let \(B \to \text{Spec} k_B \to \text{Spec} k\) be the Stein factorization of the structure morphism. Then, we obtain \(H^0(B, \mathcal{O}_B) \simeq k_B\). On the other hand, \(H^1(B, \mathcal{O}_B)\) has a \(k_B\)-vector space structure and we obtain
\[\dim_{k_B} H^1(B, \mathcal{O}_B) = [k_B : k] \dim_k H^1(B, \mathcal{O}_B).\]
Therefore, \(H^1(B, \mathcal{O}_B) \neq 0\) implies
\[\dim_k H^0(B, \mathcal{O}_B) - \dim_k H^1(B, \mathcal{O}_B) = [k_B : k] \dim_k H^1(B, \mathcal{O}_B) \leq 0.\]

The following result is a key result to show the non-vanishing theorem for the irrational case (Proposition 7.6).

**Lemma 7.5.** Let \(k\) be a separably closed field of characteristic \(p > 0\). Let \(g : Z \to B\) be a surjective morphism from a regular projective surface \(Z\) to a regular projective curve \(B\). Let \(\Delta_Z\) be a \(\mathbb{Q}\)-divisor on \(Z\) with \(0 \leq \Delta_Z \leq 1\). Assume the following conditions.
- \(g\) is a \(K_Z\)-Mori fiber space structure, that is, \(g_* \mathcal{O}_Z = \mathcal{O}_B\), \(\rho(Z/B) = 1\), and \(-K_Z\) is \(g\)-ample.
• $K_Z + \Delta_Z$ is $g$-nef.
• $(Z \times_k \bar{k})_{\text{red}}$ is not rational

Then $\kappa(Z, K_Z + \Delta_Z) \geq 0$.

Proof. Fix a general fiber $F_g$ of $g : Z \to B$. Note that $F_g$ is irreducible. We may assume that $\Delta_Z$ has no $g$-vertical prime divisors. Moreover, by reducing the coefficients of $\Delta_Z$, we may assume that $(K_Z + \Delta_Z) \cdot F_g = 0$. Then, by Theorem 5.1, we obtain $K_Z + \Delta_Z = g^* L_B$ for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $L_B$ on $B$. If $L_B$ is ample, then there is nothing to show. Thus, we may assume that $\deg(L_B) \leq 0$.

We prove that $K_Z + \Delta_Z$ is semi-ample if there exists a $g$-horizontal curve $C$ with $C^2 < 0$. We obtain

$$0 \geq g^* L_B \cdot C = (K_Z + \Delta_Z) \cdot C$$

$$\geq (K_Z + C) \cdot C = 2(h^1(C, \mathcal{O}_C) - h^0(C, \mathcal{O}_C)) \geq 0,$$

where the second equality follows by Corollary 1.9 and the last inequality holds by Lemma 7.3 (4) and Remark 7.4. Thus all the above inequalities are equalities. In particular, we obtain $\deg(L_B) = 0$ and $g^*(sL_B)|_C \simeq \mathcal{O}_Z(s(K_Z + \Delta_Z))|_C \equiv \mathcal{O}_Z(s(K_Z + C))|_C$

for a sufficiently divisible integer $s \in \mathbb{Z}_{>0}$. By Theorem 2.3, this is torsion. Therefore, $L_B$ is also torsion, hence its pull-back $g^* L_B \simeq K_Z + \Delta_Z$ is semi-ample.

Thus, we may assume that every $g$-horizontal curve $C$ in $Z$ has non-negative self-intersection number. In particular, every curve $D$ on $Z$ satisfies $D^2 \geq 0$. Assume that there exists a $g$-horizontal curve $C$ with $C^2 = 0$. By the same calculation as above, we obtain $K_Z + \Delta_Z = g^* L_B \equiv 0$. Therefore, $K_Z + \Delta_Z$ is semi-ample by Theorem 6.2.

We can assume that every $g$-horizontal curve has positive self-intersection number. By Lemma 1.14, every $g$-horizontal curve is semi-ample. On the other hand, every $g$-vertical curve is semi-ample. Thus, every curve on $Z$ is semi-ample. Consider the following commutative diagram:

$$
\begin{array}{ccc}
Z & \hookrightarrow & Z \times_k \bar{k} \hookrightarrow (Z \times_k \bar{k})_{\text{red}} \hookrightarrow (Z \times_k \bar{k})_{\text{red}}^N \equiv: W \\
\downarrow g & & \downarrow (g \times_k \bar{k})_{\text{red}} & \downarrow (g \times_k \bar{k})_{\text{red}}^N \\
B & \hookrightarrow & B \times_k \bar{k} \hookrightarrow (B \times_k \bar{k})_{\text{red}} \hookrightarrow (B \times_k \bar{k})_{\text{red}}^N \equiv: \Gamma',
\end{array}
$$

where $Y^N$ is the normalization of a variety $Y$. Let $(g \times_k \bar{k})_{\text{red}}^N : W \xrightarrow{h} \Gamma \xrightarrow{g} \Gamma'$. 
be the Stein factorization of \((g \times_k \overline{k})^N_{\text{red}}\) and we obtain the following commutative diagram

\[
\begin{array}{ccc}
 Z & \xleftarrow{f} & W \\
 \downarrow{g} & & \downarrow{h} \\
 B & \xleftarrow{\mu} & \Gamma
\end{array}
\]

By Theorem \[\text{[1.5]},\] we can find an effective \(Q\)-divisor \(\Delta_W\) on \(W\) such that

\[
K_W + \Delta_W = f^*(K_Z + \Delta) = g^*f^*L_B = h^*\rho^*L_B.
\]

We see that \(\Delta_W\) is semi-ample. Let \(\mu : V \to W\) be the minimal resolution and set

\[
q : V \xrightarrow{\mu} W \xrightarrow{h} \Gamma.
\]

We can find an effective \(\mu\)-exceptional \(Q\)-divisor \(E_V\) on \(V\) such that

\[
K_V + E_V = \mu^*K_W,
\]

\[
K_V + E_V + \mu^*\Delta_W = \mu^*(K_W + \Delta_W).
\]

Clearly, every \(\mu\)-exceptional divisor is vertical over \(\Gamma\). Therefore, the divisor

\[
K_V + \mu^*\Delta_W = \mu^*(K_W + \Delta_W) - E_V = \mu^*h^*\rho^*L_B - E_V
\]

satisfies \((K_V + \mu^*\Delta_W) \cdot F_q = 0\) for any general fiber \(F_q\) of \(q : V \to \Gamma\). Therefore, \(\kappa(V, K_V + \mu^*\Delta_W) \geq 0\) by \([\text{Fujita2}, \text{Theorem 2.2}]\). Note that we can apply \([\text{Fujita2}, \text{Theorem 2.2}]\) because \(\mu^*\Delta_W\) is semi-ample and \(\Gamma\) is irrational by Lemma \[\text{[7.3]},(4)\]. In particular, we obtain

\[
\kappa(Z, K_Z + \Delta_Z) = \kappa(W, K_W + \Delta_W) = \kappa(V, K_V + \mu^*\Delta_W + E_V) \geq 0.
\]

We are done. \(\square\)

We prove the non-vanishing theorem for the irrational case.

**Proposition 7.6.** Let \(k\) be a separably closed field of characteristic \(p > 0\). Let \((X, \Delta)\) be a projective log canonical surface over \(k\), where \(\Delta\) is a \(Q\)-divisor. Assume the following conditions.

- \(X\) is regular.
- \((X \times_k \overline{k})_{\text{red}}\) is not a rational surface.
- \(\kappa(X, K_X) = -\infty\).

If \(K_X + \Delta\) is nef, then \(\kappa(X, K_X + \Delta) \geq 0\).

**Proof.** We run a \(K_X\)-MMP with scaling \(\Delta\). Since \(\kappa(X, K_X) = -\infty\), the end result \(Z\) is a Mori fiber space by Theorem \[\text{[6.3]}\]. Thus we obtain morphisms \(X \to Z \to B\) where \(\pi : Z \to B\) is a \(K_Z\)-Mori fiber space.

We see that \(K_Z + c\Delta_Z\) is \(\pi\)-nef for some \(0 \leq c \leq 1\) and that \(\kappa(X, K_X +\)
We show the main result in this subsection.

**Theorem 7.7.** Let \( k \) be a field of characteristic \( p > 0 \). Let \((X, \Delta)\) be a projective log canonical surface over \( k \), where \( \Delta \) is a \( \mathbb{Q} \)-divisor. If \( K_X + \Delta \) is pseudo-effective, then \( \kappa(X, K_X + \Delta) \geq 0 \).

**Proof.** By Theorem 5.6, we may assume that \( K_X + \Delta \) is nef. Taking the base change to the separable closure of \( k \), we may assume that \( k \) is separably closed. By replacing \( X \) with its minimal resolution, we may assume that \( X \) is regular.

If \( \kappa(X, K_X) \geq 0 \), then there is nothing to show. If \( \kappa(X, K_X) = -\infty \), then the assertion follows from Proposition 7.1 and Proposition 7.6. \( \square \)

### 7.2. Abundance theorem

In this subsection, we show the abundance theorem with \( \mathbb{Q} \)-coefficients (Theorem 7.10). In Proposition 7.8 (resp. Proposition 7.9), we treat the case \( \kappa(X, K_X + \Delta) = 0 \) (resp. \( \kappa(X, K_X + \Delta) = 2 \)).

#### Proposition 7.8.

Let \( k \) be a separably closed field of characteristic \( p > 0 \). Let \( X \) be a projective normal \( \mathbb{Q} \)-factorial surface over \( k \) and let \( \Delta \) be a \( \mathbb{Q} \)-divisor on \( X \) with \( 0 \leq \Delta \leq 1 \). If \( K_X + \Delta \) is nef and \( \kappa(X, K_X + \Delta) = 0 \), then \( K_X + \Delta \) is semi-ample.

**Proof.** Since \( \kappa(X, K_X) \geq 0 \), we obtain \( K_X + \Delta \sim_{\mathbb{Q}} D \) for some effective \( \mathbb{Q} \)-divisor \( D \). We assume \( D \neq 0 \) and let us derive a contradiction. Let

\[
D = \sum_{i \in I} d_i D_i
\]

be the irreducible decomposition with \( d_i \in \mathbb{Q}_{>0} \).

**Step 1.** In this step, we show that we may assume that if \( (K_X + \Delta) \cdot C = 0 \) for a curve \( C \) in \( X \), then one of the followings holds.

1. \( C^2 \geq 0 \).
2. \( C \subseteq \text{Supp}(\Delta) \) and \( C \) is a connected component of \( \text{Supp} \Delta \).

Let \( C \) be a curve in \( X \), with \( (K_X + \Delta) \cdot C = 0 \), which satisfies neither (1) nor (2). We obtain \( C^2 < 0 \). We consider the following inequality

\[
(K_X + C) \cdot C \leq (K_X + \Delta) \cdot C = 0.
\]

If the above inequality is an equality, then (2) holds. Thus we obtain \( (K_X + C) \cdot C < 0 \). By Theorem 6.1, we obtain a birational morphism \( h: X \to Y \) to a projective \( \mathbb{Q} \)-factorial surface \( Y \) such that \( \text{Ex}(h) = C \).
We can check that $Y$ and $\Delta_Y := h_*\Delta$ satisfies the same properties as $X$ and $\Delta$, that is, $0 \leq \Delta_Y \leq 1$ and $K_Y + \Delta_Y$ is nef. Moreover, if $K_Y + \Delta_Y$ is semi-ample, then $K_X + \Delta = h^*(K_Y + \Delta)$ is also semi-ample. Thus the problem is reduced to the one of $(Y, \Delta_Y)$. If there exists a curve $C'$ on $Y$, with $(K_Y + \Delta_Y) \cdot C' = 0$, which satisfies neither (1) nor (2), then we can repeat the same procedure as above. This procedure will terminate because the Picard number dropped: $\rho(Y) = \rho(X) - 1$. Therefore, we may assume that every curve $C$ on $X$, with $(K_X + \Delta) \cdot C = 0$, satisfies (1) or (2).

**Step 2.** In this step, we show that any connected component of $D$ is irreducible.

Assume that $D_{i_1} \cap D_{i_2} \neq \emptyset$ for some prime components $D_{i_1} \neq D_{i_2}$ of $\text{Supp} \, D$. Since $D$ is nef and

$$D^2 = (K_X + \Delta) \cdot D = 0,$$

we obtain $D \cdot D_{i_a} = (K_X + \Delta) \cdot D_{i_a} = 0$ for each $a = 1, 2$. Since $D_{i_1} \cdot D_{i_2} > 0$, we obtain $D_{i_a}^2 < 0$ for each $a = 1, 2$. By Step 1, each $D_{i_a}$ is a connected component of $\text{Supp} \, \Delta$. This is a contradiction.

**Step 3.** In this step, we show

$$K_X \cdot D_i = \Delta \cdot D_i = D \cdot D_i = D_i^2 = 0$$

for every $i \in I$.

Since $K_X + \Delta \sim_{Q} D$ is nef, we obtain

$$(K_X + \Delta) \cdot D_i = D \cdot D_i = D_i^2 = 0$$

for every $i \in I$. Now, we prove $K_X \cdot D_i \geq 0$ for every $i \in I$. If $K_X \cdot D_i < 0$, then we obtain $\kappa(X, D) \geq \kappa(X, D_i) \geq 1$ by Lemma 1.16. This contradicts $\kappa(X, K_X + \Delta) = \kappa(X, D) = 0$. We have $K_X \cdot D_i \geq 0$ for every $i \in I$. Since $D_i$ is nef, we obtain $\Delta \cdot D_i \geq 0$. Thus, $(K_X + \Delta) \cdot D_i = 0$ implies

$$K_X \cdot D_i = \Delta \cdot D_i = 0.$$

**Step 4.** In this step, we prove the assertion of the proposition. For this, it suffices to show that $D_i$ is semi-ample for some $i \in I$. Fix $i \in I$. By $D_i^2 = 0$ and $\Delta \cdot D_i = 0$, we can write

$$\Delta = \delta D_i + \Delta'$$

where $0 \leq \delta \leq 1$ and $D_i \cap \text{Supp} \, \Delta' = \emptyset$. Since

$$K_X \cdot D_i = D_i^2 = 0,$$
it suffices to show that $O_X(rD_i)|_{D_i} \simeq O_{D_i}$ for some $r \in \mathbb{Z}_{>0}$ by Proposition [6.1]. This follows from

\[ 0 \sim_Q O_X(K_X + D_i)|_{D_i} \]
\[ = O_X(K_X + D_i + (\Delta - \delta D_i))|_{D_i} \]
\[ = O_X(K_X + \Delta + (1 - \delta)D_i)|_{D_i} \]
\[ \sim_Q O_X(D + (1 - \delta)D_i)|_{D_i} \]
\[ = O_X((d_i + (1 - \delta))D_i)|_{D_i} \]

and $d_i + (1 - \delta) \geq d_i > 0$. The first $\sim_Q$ follows by Theorem [2.3(2)].

\[ \square \]

Proposition 7.9. Let $k$ be a field of characteristic $p > 0$. Let $X$ be a projective normal $\mathbb{Q}$-factorial surface and let $\Delta$ be a $\mathbb{Q}$-divisor with $0 \leq \Delta \leq 1$. If $K_X + \Delta$ is nef and big, then $K_X + \Delta$ is semi-ample.

Proof. Although the proof is the same as [11, Proposition 3.29], we give a proof for the sake of the completeness.

By Keel’s result (Theorem [1.13]), it is sufficient to prove that if $E := \bigcup_{C : (K_X + \Delta) = 0} C = C_1 \cup \cdots \cup C_r$, then $(K_X + \Delta)|_E$ is semi-ample. Let $C \subset E$. Then we have

\[ (K_X + C) \cdot C \leq (K_X + \Delta) \cdot C = 0. \]

Step 1. In this step, we reduce the proof to the case when if $C \subset E$, then $(K_X + C) \cdot C = 0$.

Assume $C \subset E$ and $(K_X + C) \cdot C < 0$. Then $C^2 < 0$. Thus, by Theorem [3.1] we can contract $C$. Let $f : X \to Y$ be the contraction and $\Delta_Y := f_*(\Delta)$. Then since $K_X + \Delta = f^*(K_Y + \Delta_Y)$ and $Y$ is $\mathbb{Q}$-factorial, if we can prove that $K_Y + \Delta_Y$ is semi-ample, then $K_X + \Delta$ is semi-ample. We can repeat this procedure and we obtain the desired reduction.

Step 2. In this step, we prove that $E$ is a disjoint union of irreducible curves and if $C \subset E$, then $O_X(s(K_X + \Delta))|_C = O_X(s(K_X + C))|_C$ for sufficiently divisible $s \in \mathbb{Z}_{>0}$.

Let $C \subset E$. By Step 1, we have $(K_X + C) \cdot C = 0$. Then, the inequality over Step 1 is an equality. Thus $C \subset \text{Supp}(\Delta)$ and $C$ is disjoint from any other component of $\Delta$.

By Step 2, it is sufficient to prove that, if $(K_X + C) \cdot C = 0$, then $O_X(s(K_X + C))|_C$ is semi-ample for sufficiently divisible $s \in \mathbb{Z}_{>0}$. This holds by Theorem [2.3(2)].

\[ \square \]
We prove the main theorem in this subsection.

**Theorem 7.10** (Abundance theorem). Let $k$ be a field of characteristic $p > 0$. Let $(X, \Delta)$ be a projective log canonical surface over $k$, where $\Delta$ is a $\mathbb{Q}$-divisor. If $K_X + \Delta$ is nef, then $K_X + \Delta$ is semi-ample.

**Proof.** By taking the base change to the separable closure of $k$, we may assume that $k$ is separably closed. Set $\kappa := \kappa(X, K_X + \Delta)$. By Theorem 7.7, we obtain $\kappa \geq 0$. If $\kappa = 0$ (resp. $\kappa = 1$, resp. $\kappa = 2$), then $K_X + \Delta$ is semi-ample by Proposition 7.8 (resp. Lemma 1.15, resp. Proposition 7.9). □

7.3. Relativization. The purpose of this section is to prove Theorem 7.12, which generalizes our abundance theorem (Theorem 7.10) to the relative case. First we establish a lemma to compare nef divisors with relative nef divisors.

**Lemma 7.11.** Let $k$ be a field of characteristic $p > 0$. Let $X$ be a projective normal surface over $k$ and let $\Delta$ be an effective $\mathbb{R}$-divisor such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $f : X \to S$ be a morphism to a projective $k$-scheme $S$. Let $A_S$ be an ample invertible sheaf on $S$. If $K_X + \Delta$ is $f$-nef, then $K_X + \Delta + mf^*A_S$ is nef for some $m \in \mathbb{Z}_{>0}$.

**Proof.** Taking the base change to the separable closure of $k$, we can assume that $k$ is separably closed. Taking the Stein factorization of $f : X \to S$, we may assume $f_*O_X = O_S$. Moreover, by taking the Stein factorization of the structure morphism $S \to \text{Spec} \ k$, we can assume that $\alpha_*O_X = O_{\text{Spec} \ k}$, where $\alpha : X \to \text{Spec} \ k$ is the structure morphism.

Let $Y$ be the normalization of $(X \times_k \overline{k})_{\text{red}}$ and let $g : Y \to X$ be the induced morphism. By Theorem 1.5, we obtain

$$K_Y + \Delta_Y = g^*(K_X + \Delta)$$

for some effective $\mathbb{R}$-divisor $\Delta_Y$ on $Y$. Note that $g_S^*A_S$ is ample, where $g_S : S \times_k \overline{k} \to S$. If $K_Y + \Delta_Y + mf^*g_S^*A_S$ is nef, then so is $K_X + \Delta + mf^*(A_S)$. Thus, we may assume that $k$ is algebraically closed. Then, the assertion follows from follows from [11] Theorem 3.13]. □

We prove the main result of this section.

**Theorem 7.12.** Let $k$ be a field of characteristic $p > 0$. Let $(X, \Delta)$ be a log canonical surface over $k$, where $\Delta$ is a $\mathbb{Q}$-divisor. Let $f : X \to S$ be a projective morphism to a separated scheme $S$ of finite type over $k$. If $K_X + \Delta$ is $f$-nef, then $K_X + \Delta$ is $f$-semi-ample.
Proof. In order to use \([T4, \text{Theorem 1}]\), we reduce the proof to the case 
k is an \(F\)-finite field containing \(\overline{\mathbb{F}}_p\). For this, first we take the base change to \(k\overline{\mathbb{F}}_p\), where \(k\overline{\mathbb{F}}_p\) is the minimum field in the algebraic closure \(\overline{k}\) containing \(k\) and \(\mathbb{F}_p\). By replacing \(k\) with \(k\overline{\mathbb{F}}_p\), we may assume that \(\overline{\mathbb{F}}_p \subset k\). Second, take an intermediate field \(\mathbb{F}_p \subset k_1 \subset k\) such that \(k_1\) is finitely generated over \(\mathbb{F}_p\) and that every scheme and divisor are defined over \(k_1\), i.e. there exists \(X_1\) of finite type over \(k_1\) with \(X_1 \times_{k_1} k \simeq X\) etc. By replacing \(k\) with \(k_1\), we can assume that \(k\) is an \(F\)-finite field containing \(\mathbb{F}_p\).

We reduce the proof to the case when \(X\) and \(S\) are projective. Since the problem is local on \(S\), we may assume that \(S\) is affine. We can find projective compactifications \(S \subset \overline{S}\) and \(X \subset \overline{X}\), that is, there exists a commutative diagram

\[
\begin{array}{ccc}
X & \longrightarrow & \overline{X} \\
\downarrow{f} & & \downarrow{f} \\
S & \longrightarrow & \overline{S}
\end{array}
\]

such that \(\overline{X}\) and \(\overline{S}\) are projective and each horizontal arrow is an open immersion. By replacing \(\overline{X}\) with a resolution along \(\overline{X} \setminus X\), we may assume that \(\overline{X}\) is regular along \(\overline{X} \setminus X\). Taking more blowing-ups, we may assume that the support of the closure \(\text{Supp}(\Delta)\) is regular at every point contained in \(\overline{X} \setminus X\). In particular, \((\overline{X}, \overline{\Delta})\) is log canonical. Note that \((\overline{X}, \overline{\Delta})\) may not be \(f\)-nef. However, by running a \((K_{\overline{X}} + \overline{\Delta})\)-MMP over \(\overline{S}\) (Theorem 5.6), the end result \((\overline{X}', \overline{\Delta}')\) is log canonical and nef over \(\overline{S}\). Therefore, by replacing \((X, \Delta) \to S\) with \((\overline{X}', \overline{\Delta}') \to \overline{S}\), we can assume that \(X\) and \(S\) are projective.

Fix an ample invertible sheaf \(A_S\) on \(S\). By Lemma 7.11, \(K_X + \Delta + mf^*A_S\) is nef for some \(m \in \mathbb{Z}_{>0}\). We can find \(\Delta' \sim_{\mathbb{Q}} \Delta + mf^*A_S\) such that \((X, \Delta')\) is log canonical by \([T4, \text{Theorem 1}]\). Thus, by Theorem 7.10, \(K_X + \Delta'\) is semi-ample. In particular, \(K_X + \Delta\) is \(f\)-semi-ample. \(\square\)

8. Abundance Theorem with \(\mathbb{R}\)-coefficients

In this section, we generalize our relative log canonical abundance theorem (Theorem 7.12) to the case of \(\mathbb{R}\)-coefficients (Theorem 8.9). The main strategy is to use Shokurov polytope, which is the same as the case when \(k\) is algebraically closed.

However, there are some differences as follows. Let \(X\) be a projective normal (geometrically connected) variety over a field \(k\) and let \(Y\) be the normalization of \((X \times_k \overline{k})_{\text{red}}\). Set \(f : Y \to X\) to be the induced
homomorphism. In the proof of our main result (Theorem 8.9), we consider the following $\mathbb{Z}$-bilinear homomorphism:

$$M \times N \rightarrow \mathbb{Z}, \ (D, C) \mapsto f^* D \cdot C.$$  

where $M$ is a free $\mathbb{Z}$-module generated by finitely many Cartier divisors on $X$ and $N$ is the free $\mathbb{Z}$-module generated by the curves in $Y$. Since $Y$ has more curves than $X$, we can establish an appropriate boundedness result (Lemma 8.7).

In Subsection 8.1 we establish a key result in a setting of convex geometry (Proposition 8.5). In Subsection 8.2 we prove the abundance theorem with $\mathbb{R}$-coefficients (Theorem 8.9) by applying Proposition 8.5 to our setting.

8.1. Shokurov polytope in a convex geometric setting. We fix the following notation. The goal of this subsection is to prove Proposition 8.5. The argument in this subsection is mainly extracted from [Birkar1, Proposition 3.2(1)(2)(3)].

**Notation 8.1.** Let $M$ and $N$ be torsion free $\mathbb{Z}$-modules and let

$$\varphi : M \times N \rightarrow \mathbb{Z}$$

be a $\mathbb{Z}$-bilinear homomorphism. For $D \in M$ and $C \in N$, we write

$$\varphi(D, C) = D \cdot C = C \cdot D$$

by abuse of notation. Assume that $\dim_{\mathbb{R}} M_{\mathbb{R}} < \infty$, i.e. $M$ is a free $\mathbb{Z}$-module of finite rank. Fix a rational polytope $L \subset M_{\mathbb{R}}$. Fix an $\mathbb{R}$-linear basis of $M_{\mathbb{R}}$ and we denote the sup norm with respect to this basis by $\| \cdot \|$.

**Example 8.2.** Let $k$ be a field. Let $X$ be a proper normal variety. Set $N := \bigoplus \mathbb{Z} C$ where $C$ runs over the curves on $X$. Fix Cartier divisors $D_1, \cdots, D_r$ and set $M := \bigoplus_{1 \leq i \leq r} \mathbb{Z} D_i$. Then the intersection theory induces a $\mathbb{Z}$-bilinear homomorphism $M \times N \rightarrow \mathbb{Z}$.

**Remark 8.3.** In the proof of our main theorem of this section (Theorem 8.9), we does not use the naive bilinear homomorphism as in Example 8.2, but as in Notation 8.6.

We establish an auxiliary lemma.

**Lemma 8.4.** We use the same notation as Notation 8.1. Fix $K \in M_{\mathbb{Q}}$, $\Delta \in L$, and $\rho \in \mathbb{R}_{>0}$. Then, there exist positive real numbers $\epsilon, \delta > 0$, depending on $K$, $\Delta$, and $\rho$, which satisfy the following properties.

1. For every $C \in N$ such that $-(K + B) \cdot C \leq \rho$ for all $B \in L$, if $(K + \Delta) \cdot C > 0$, then $(K + \Delta) \cdot C > \epsilon$.  

(2) If $C \in N$ and $B_0 \in \mathcal{L}$ satisfy $\|B_0 - \Delta\| < \delta$, $(K + B_0) \cdot C \leq 0$, and $-(K + B) \cdot C \leq \rho$ for all $B \in \mathcal{L}$, then $(K + \Delta) \cdot C \leq 0$.

Proof. Let $V_1, \cdots, V_s \in \mathcal{L}$ be the vertices of $\mathcal{L}$. Note that, for every $B \in \mathcal{L}$, we obtain the irreducible decomposition

$$B = \sum_{i=1}^{s} v_i V_i$$

for some real numbers $v_i \geq 0$ with $\sum_{i=1}^{s} v_i = 1$. Note that this expression is not unique in general.

(1) We can write $\Delta := \sum v_i V_i$ as above. Then we have

$$(K + \Delta) \cdot C = \sum_{i=1}^{s} v_i (K + V_i) \cdot C.$$ Suppose $(K + \Delta) \cdot C < 1$. Fix an index $1 \leq i_0 \leq s$. We obtain

$$v_{i_0} (K + V_{i_0}) \cdot C < 1 - \sum_{i \neq i_0} v_i (K + V_i) \cdot C \leq 1 + \sum_{i \neq i_0} v_i \rho \leq 1 + (s-1) \rho.$$ Thus, if $v_{i_0} \neq 0$, then we obtain

$$-\rho \leq (K + V_{i_0}) \cdot C < \frac{1}{v_{i_0}} (1 + (s-1) \rho).$$ Since $C \in N$, there are only finitely many possibilities for the number $(K + V_{i_0}) \cdot C$.

Thus, if $(K+\Delta) \cdot C < 1$, then there are only finitely many possibilities for the number $(K + \Delta) \cdot C = \sum_{i=1}^{s} v_i (K + V_i) \cdot C$. Therefore we can find the desired number $\epsilon > 0$.

(2) First, we reduce the proof to the case when $\mathcal{L}$ is a simplex. Thus we assume that our assertion holds for simplices. Fix a decomposition into simplices:

$$\mathcal{L} = \bigcup_{\lambda \in \Lambda} \mathcal{L}_\lambda,$$

where $\Lambda$ is a finite set. We divide the index set $\Lambda$ into two subsets: $\Lambda = \Lambda_1 \sqcup \Lambda_2$ where $\Delta \in \mathcal{L}_{\lambda_1}$ for every $\lambda_1 \in \Lambda_1$ and $\Delta \not\in \mathcal{L}_{\lambda_2}$ for every $\lambda_2 \in \Lambda_2$. For every $\lambda_1 \in \Lambda_1$, we can find $\delta_{\lambda_1} > 0$ such that

(2)$_{\lambda_1}$ If $C \in N$ and $B_0 \in \mathcal{L}_{\lambda_1}$ satisfy $\|B_0 - \Delta\| < \delta_{\lambda_1}$, $(K + B_0) \cdot C \leq 0$, and $-(K + B) \cdot C \leq \rho$ for all $B \in \mathcal{L}_{\lambda_1}$, then $(K + \Delta) \cdot C \leq 0$. 
Fix $\delta_{A_2} > 0$ such that
\[
\left( \bigcup_{\lambda_2 \in A_2} \mathcal{L}_{\lambda_2} \right) \cap \{ B \in M_{\mathbb{R}} \mid \| B - \Delta \| < \delta_{A_2} \} = \emptyset.
\]
Set
\[
\delta := \min\{ \min_{\lambda_1 \in A_1} \{ \delta_{\lambda_1} \}, \delta_{A_2} \}.
\]
Then we can check that this $\delta$ satisfies the required property.

Second, we reduce the proof to the case when $\mathcal{L}$ is a simplex which contains the origin as a vertex. Fix a vertex $A \in \mathcal{L}$ and we can write
\[
\mathcal{L} = A + \mathcal{L}'
\]
where $\mathcal{L}'$ is a simplex which contains the origin as a vertex. Set $K' := K + A$ and $\Delta' := \Delta - A$. We see $\Delta' \in \mathcal{L}'$. By our assumption, we can find $\delta > 0$ such that
\[
(2)' \text{ If } C \in N \text{ and } B_0' \in \mathcal{L}' \text{ satisfy } \| B_0' - \Delta' \| < \delta, (K' + B_0') \cdot C \leq 0, \text{ and } -(K' + B') \cdot C \leq \rho \text{ for all } B' \in \mathcal{L}', \text{ then } (K' + \Delta') \cdot C \leq 0.
\]
Take $C \in N$ and $B_0 \in \mathcal{L}$ such that $\| B_0 - \Delta \| < \delta$, $(K + B_0) \cdot C \leq 0$, and $-(K + B) \cdot C \leq \rho$ for all $B \in \mathcal{L}$. Set $B_0' := B_0 - A$ and we obtain $B_0' \in \mathcal{L}'$, $\| B_0' - \Delta' \| < \delta$, $(K' + B_0') \cdot C \leq 0$, and $-(K' + B') \cdot C \leq \rho$ for all $B' \in \mathcal{L}'$. Thus the above (2)' implies
\[
(K + \Delta) \cdot C = (K' + \Delta') \cdot C \leq 0.
\]
This implies (2).

Thus we can assume that $\mathcal{L}$ is a simplex which contains the origin as a vertex. Suppose that the statement is not true. Then, for an arbitrary $\delta \in \mathbb{R}_{>0}$, there exist $C \in N$ and $B_0 \in \mathcal{L}$ which satisfy $\| B_0 - \Delta \| < \delta$, $(K + B_0) \cdot C \leq 0$, $-(K + B) \cdot C \leq \rho$ for all $B \in \mathcal{L}$, and $(K + \Delta) \cdot C > 0$. Set $\delta := 1/m$ for any $m \in \mathbb{Z}_{>0}$. Then we obtain two infinite sequences $\{ C_m \}_{m \in \mathbb{Z}_{>0}} \subset N$ and $\{ B_m \}_{m \in \mathbb{Z}_{>0}} \subset \mathcal{L}$ which satisfy
\[
(K + B_m) \cdot C_m \leq 0,
\]
\[
-(K + B) \cdot C_m \leq \rho \text{ for all } B \in \mathcal{L}, \text{ and}
\]
\[
(K + \Delta) \cdot C_m > 0,
\]
and $\| B_m - \Delta \|$ converges to zero.

Let $\Delta = \sum v_i V_i$ and $B_m = \sum v_{i,m} V_i$ as above. We can assume that $V_1$ is the origin and that $v_1 = v_{1,m} = 0$ for every $m$. Moreover, since $\mathcal{L}$ is a simplex, these expressions are unique. Therefore we obtain $v_i = \lim_m v_{i,m}$.

Here, for each $j$, the set $\{(K + V_j) \cdot C_m \}_{m}$ has a lower bound $-\rho$. 
Let us show that, if \( v_j \neq 0 \), then the set \( \{(K + V_j) \cdot C_m\}_m \) has an upper bound. Since \( 0 < v_j = \lim v_{j,m} \), we may assume \( v_{j,m} > 0 \) for all \( m \) by replacing the sequence with a suitable sub-sequence. By the inequality

\[
0 \geq (K + B_m) \cdot C_m = \sum_{i=1}^{s} v_{i,m}(K + V_i) \cdot C_m,
\]

we have

\[
(K + V_j) \cdot C_m \leq \frac{1}{v_{j,m}} (-\sum_{i \neq j} v_{i,m}(K + V_i) \cdot C_m)
\]

\[
\leq \frac{1}{v_{j,m}} \left( \sum_{i \neq j} v_{i,m}\rho \right)
\]

\[
\leq \frac{1}{v_{j,m}} (s - 1)\rho.
\]

Since the set \( \{1/v_{j,m}\}_m \) has an upper bound, the set \( \{(K + V_j) \cdot C_m\}_m \) also has an upper bound.

Then, for \( m \gg 0 \), we have

\[
(K + B_m) \cdot C_m
\]

\[
= (K + \Delta) \cdot C_m + \sum_{i=1}^{s} (v_{i,m} - v_i) (K + V_i) \cdot C_m
\]

\[
> \epsilon + \sum_{i \neq 0} (v_{i,m} - v_i) (K + V_i) \cdot C_m
\]

\[
= \epsilon + \sum_{v_i \neq 0} (v_{i,m} - v_i) (K + V_i) \cdot C_m + \sum_{v_i = 0} v_{i,m}(K + V_i) \cdot C_m
\]

\[
\geq \epsilon + \sum_{v_i \neq 0} (v_{i,m} - v_i) (K + V_i) \cdot C_m + \sum_{v_i = 0} v_{i,m}(-\rho)
\]

\[
> 0.
\]

The first inequality follows from (1). The last inequality follows when \( m \gg 0 \). Note that, if \( v_i \neq 0 \), then the set \( \{(K + V_i) \cdot C_m\}_m \) is bounded from the both sides. This is a contradiction. \( \square \)

We show the main result of this subsection.

**Proposition 8.5.** We use the same notation as Notation 8.1. Let \( K \in M_Q \) and \( \rho \in \mathbb{R}_{>0} \). Fix a subset \( \{C_t\}_{t \in T} \subset N \) such that

\[
-(K + B) \cdot C_t \leq \rho
\]
for every \( t \in T \) and every \( B \in \mathcal{L} \). For any subset \( T' \subset T \), we define
\[
\mathcal{N}_{T'} := \{ B \in \mathcal{L} \mid (K + B) \cdot C_t \geq 0 \text{ for every } t \in T' \}.
\]
Then there exists a finite subset \( S \subset T \) such that
\[
\mathcal{N}_T = \mathcal{N}_S.
\]
In particular \( \mathcal{N}_T \) is a rational polytope.

Proof. We show the assertion by the induction on \( \dim \mathcal{L} \). If \( \dim \mathcal{L} = 0 \), then there is nothing to show. Thus, we assume \( \dim \mathcal{L} > 0 \). We may assume that, for each \( t \in T \), there exists \( B \in \mathcal{L} \) with \((K + B) \cdot C_t < 0\).

We see that \( \mathcal{N}_T \) is a compact set. Then, by Lemma 8.4(2) and by the compactness of \( \mathcal{N}_T \), there exist \( \Delta_1, \ldots, \Delta_n \in \mathcal{N}_T \) and positive real numbers \( \delta_1 > 0, \ldots, \delta_n > 0 \) such that \( \mathcal{N}_T \) is covered by
\[
B_i := \{ B \in \mathcal{L} \mid ||B - \Delta_i|| < \delta_i \}
\]
and that if \( B \in B_i \) with \((K + B) \cdot C_t < 0\) for some \( t \in T \), then \((K + \Delta_i) \cdot C_t = 0\). Set
\[
T_i := \{ t \in T \mid (K + B) \cdot C_t < 0 \text{ for some } B \in B_i \}.
\]
Then, for every \( t \in T_i \), we have \((K + \Delta_i) \cdot C_t = 0\).

Here, we prove
\[
\mathcal{N}_T = \bigcap_{i=1}^n \mathcal{N}_{T_i}.
\]
The inclusion \( \mathcal{N}_T \subset \bigcap \mathcal{N}_{T_i} \) is obvious. Thus we want to prove \( \mathcal{N}_T \supset \bigcap \mathcal{N}_{T_i} \). Let \( B \notin \mathcal{N}_T \). Since \( \mathcal{N}_T \) is compact, we can find an element \( B' \in \mathcal{N}_T \) with
\[
||B' - B|| = \min \{ ||B^* - B|| \mid B^* \in \mathcal{N}_T \}.
\]
Here we have \( B' \in B_i \) for some \( i \). Since \( B_i \cap \overline{BB'} \) is an open subset of \( \overline{BB'} \) where \( \overline{BB'} \) is the line segment, we have an element \( B'' \) such that \( B'' \in B_i \cap \overline{BB'} \), \( B'' \neq B \) and \( B'' \neq B' \). This means that there is a real number \( \beta \) with \( 0 < \beta < 1 \) such that
\[
\beta B + (1 - \beta)B' = B''.
\]
We obtain
\[
\beta(K + B) + (1 - \beta)(K + B') = K + B''.
\]
Moreover, we see that \( B'' \notin \mathcal{N}_T \). Here, since \( B'' \in B_i \setminus \mathcal{N}_T \), we have \((K + B'') \cdot C_t < 0\) for some \( t \in T_i \). Thus we obtain the following
inequality
\[ \beta(K + B) \cdot C_t = (K + B') \cdot C_t - (1 - \beta)(K + B') \cdot C_t < -(1 - \beta)(K + B') \cdot C_t \leq 0. \]
Therefore we have \((K + B) \cdot C_t < 0\). This means \(B \notin \mathcal{N}_T\).

Thus by replacing \(T\) with \(T_i\), we may assume that there exists \(\Delta_0 \in \mathcal{N}_T\) such that \((K + \Delta_0) \cdot C_t = 0\) for every \(t \in T\). Let \(\mathcal{L}^1, \cdots, \mathcal{L}^u\) be the proper faces of \(\mathcal{L}\) whose codimensions are one. For every \(1 \leq u' \leq u\), we can write
\[ \mathcal{N}_T \cap \mathcal{L}^{u'} = \{B \in \mathcal{L}^{u'} | (K + B) \cdot C_t \geq 0 \text{ for every } t \in T\}. \]
By the induction hypothesis, for every \(1 \leq u' \leq u\), we can find a finite subset \(S_{u'} \subset T\) such that
\[ \mathcal{N}_T \cap \mathcal{L}^{u'} = \mathcal{N}_{S_{u'}} \cap \mathcal{L}^{u'}. \]
Set
\[ S := \bigcup_{1 \leq u' \leq u} S_{u'}. \]
Clearly, the inclusion \(S \subset T\) holds. In particular, we obtain \((K + \Delta_0) \cdot C_s = 0\) for every \(s \in S\), and \(\mathcal{N}_T \subset \mathcal{N}_S\). Thus it suffices to show that \(\mathcal{N}_T \supset \mathcal{N}_S\).

Here, take an arbitrary element \(B \in \mathcal{N}_T\) (resp. \(B \in \mathcal{N}_S\)) with \(B \neq \Delta_0\). Then we can find \(B' \in \mathcal{L}^{u'}\) for some \(1 \leq u' \leq u\) such that \(B\) is on the line segment defined by \(\Delta_0\) and \(B'\). Since \((K + \Delta_0) \cdot C_t = 0\) for all \(t \in T\) (resp. for all \(t \in S\)), we have \(B' \in \mathcal{N}_T \cap \mathcal{L}^{u'}\) (resp. \(B' \in \mathcal{N}_S \cap \mathcal{L}^{u'}\)). Thus we see that \(\mathcal{N}_T\) (resp. \(\mathcal{N}_S\)) is the convex hull of \(\Delta_0\) and all the \(\mathcal{N}_T \cap \mathcal{L}^{u'}\) (resp. \(\Delta_0\) and all the \(\mathcal{N}_S \cap \mathcal{L}^{u'}\)). On the other hand, we have
\[ \mathcal{N}_T \cap \mathcal{L}^{u'} = \mathcal{N}_{S_{u'}} \cap \mathcal{L}^{u'} \supset \mathcal{N}_S \cap \mathcal{L}^{u'}. \]
Therefore, we obtain \(\mathcal{N}_T \supset \mathcal{N}_S\). We are done. \(\square\)

8.2. **Proof.** In this subsection, we often use the following notation.

**Notation 8.6.** Let \(k\) be a separably closed field of characteristic \(p > 0\). Let \(X\) be a projective \(\mathbb{Q}\)-factorial log canonical surface over \(k\). Let \(B_1, \cdots, B_\ell\) be prime divisors. Fix \(m \in \mathbb{Z}_{>0}\) such that \(mK_X\) and all \(mB_i\) are Cartier. Set
\[ M := \mathbb{Z}(mK_X) \bigoplus_{i=1}^\ell \mathbb{Z}(mB_i). \]
Let
\[ \mathcal{L} := \{ D = \sum_{i=1}^{\ell} b_i B_i \in M_{\mathbb{R}} \mid (X, D) \text{ is log canonical} \}. \]

Let \( Y \) be the normalization of \((X \times_k \overline{k})_{\text{red}}\) and let \( g : Y \to X \) be the induced morphism. Note that \( Y \) is \( \mathbb{Q} \)-factorial (Lemma 1.1(3)). Let
\[ N := \bigoplus \mathbb{Z}C_Y \]
where \( C_Y \) runs over all the curves on \( Y \). We obtain a \( \mathbb{Z} \)-bilinear homomorphism
\[ \varphi : M \times N \to \mathbb{Z}, \ (D, C) \mapsto g^* D \cdot C. \]

To apply Proposition 8.5, we need the following result on the boundedness of extremal rays.

**Lemma 8.7.** We use the same notation as Notation 8.6. Then, there exists \( \rho \in \mathbb{Z}_{>0} \) such that, for every extremal ray \( R \) of \( NE(Y) \) spanned by a curve, there exists a curve \( C \) on \( Y \) such that \( R = \mathbb{R}_{\geq 0}[C] \) and that
\[ -g^*(K_X + B) \cdot C \leq \rho \]
for every \( B \in \mathcal{L} \).

**Proof.** We obtain
\[ K_Y + \Delta_Y = g^* K_X \]
for some effective \( \mathbb{Z} \)-divisor \( \Delta_Y \) by Theorem 1.5.

We see \( \mathcal{L} \subset \{ \sum_{i=1}^{\ell} b_i B_i \in M_{\mathbb{R}} \mid 0 \leq b_i \leq 1 \} \). Let \( g^* B_i = \beta_i B'_i \), where \( B'_i \) is the prime divisor and \( \beta_i \in \mathbb{Z}_{>0} \). Set \( \beta := \max_{1 \leq i \leq \ell} \beta_i \) and
\[ \mathcal{L}' := \{ \sum_{i=1}^{\ell} b'_i B'_i \mid b'_i \in \mathbb{R}, 0 \leq b_i \leq \beta \}. \]

By [11] Lemma 3.37], we can find \( \rho \in \mathbb{Z}_{>0} \) such that, for every extremal ray \( R \) of \( NE(Y) \) spanned by a curve, there exists a curve \( C \) on \( Y \) such that
\[ -(K_Y + \Delta_Y + B') \cdot C \leq \rho \]
for every \( B' \in \mathcal{L}' \). Since \( B' := g^* B \in \mathcal{L}' \) for every \( B \in \mathcal{L} \), we obtain
\[ -g^*(K_X + B) \cdot C = -(K_Y + \Delta_Y + B') \cdot C \leq \rho. \]
This implies the assertion. \( \Box \)

By Proposition 8.5, we obtain the following result.
Theorem 8.8. We use the same notation as Notation 8.6. Let \( \{R_t\}_{t \in T} \) be the family of the extremal rays of \( \overline{NE}(Y) \) spanned by a curve. Then the set

\[
N_T := \{ B \in \mathcal{L} \mid g^*(K_X + B) \cdot R_t \geq 0 \text{ for every } t \in T \}
\]

is a rational polytope.

Proof. We fix \( \rho \in \mathbb{Z}_{>0} \) as in Lemma 8.7. By Lemma 8.7, for every \( t \in T \), there exists a curve \( C_t \) such that \( R_t = \mathbb{R}_{\geq 0}[C_t] \) and that \( -g^*(K_X + B) \cdot C_t \leq \rho \) for all \( B \in \mathcal{L} \). Thus, the assertion follows from Proposition 8.5.

Now, we prove the abundance theorem with \( \mathbb{R} \)-coefficients.

Theorem 8.9. Let \( k \) be a field of characteristic \( p > 0 \). Let \( (X, \Delta) \) be a log canonical \( k \)-surface, where \( \Delta \) is an \( \mathbb{R} \)-divisor. Let \( f : X \to S \) be a projective \( k \)-morphism to a separated scheme \( S \) of finite type over \( k \). If \( K_X + \Delta \) is \( f \)-nef, then \( K_X + \Delta \) is \( f \)-semi-ample.

Proof. We may assume that \( k \) is separably closed. By replacing \( X \) with its minimal resolution, we may assume that \( X \) is regular. In particular, \( X \) is \( \mathbb{Q} \)-factorial.

Let \( \Delta = \sum_{i=1}^{\ell} \delta_i B_i \) be the irreducible decomposition, where each \( B_i \) is a prime divisor. Set

\[
M := \mathbb{Z}K_X \oplus \bigoplus_{i=1}^{\ell} \mathbb{Z}B_i
\]

Note that, since \( X \) is regular, \( K_X \) and all \( B_i \) are Cartier. Let

\[
\mathcal{L} := \{ D = \sum_{i=1}^{\ell} b_i B_i \in M_{\mathbb{R}} \mid (X, D) \text{ is log canonical} \}.
\]

Let \( \{R_t\}_{t \in T} \) be the set of the extremal rays of \( \overline{NE}(X/S) \) spanned by proper \( k \)-curves \( C \) such that \( f(C) \) is one point. By Theorem 1.18 we obtain

\[
N_T := \{ B \in \mathcal{L} \mid (K_X + B) \cdot R_t \geq 0 \text{ for every } t \in T \}
= \{ B \in \mathcal{L} \mid K_X + B \text{ is nef} \}.
\]

If \( \dim f(X) = 0, N_T \) is a rational polytope by Theorem 8.8. If \( \dim f(X) > 0 \), then the \( f \)-nef cone is a rational polytope by Theorem 1.17, hence also \( N_T \) is a rational polytope. In any case, \( N_T \) is a rational polytope.

Since \( \Delta \in N_T \), we can find \( \mathbb{Q} \)-divisors \( \Delta_1, \ldots, \Delta_\ell \) such that \( \Delta_i \in N_T \) for all \( i \) and that \( \sum r_i \Delta_i = \Delta \) where positive real numbers \( r_i \) satisfy
\[ \sum r_i = 1. \] Thus we have
\[ K_X + \Delta = \sum r_i(K_X + \Delta_i) \]
and \( K_X + \Delta_i \) is \( f \)-nef. By Theorem 7.12, \( K_X + \Delta_i \) is \( f \)-semi-ample. \( \square \)

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