On weak separation property for self-affine Jordan arcs.

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Abstract

We consider self-affine arcs in $\mathbb{R}^2$ and prove that violation of "inner" weak separation property for such arcs implies that the arc is a parabolic segment. Therefore, if a self-affine Jordan arc is not a parabolic segment, then it is the attractor of some multizipper.

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1 Introduction

The idea of associated family of similarities for a system $S = \{S_1, ..., S_m\}$ of similarities in $\mathbb{R}^d$ was initially proposed by C. Bandt and S. Graf [2] to analyse the measure and dimension properties of the attractor $K$ of the system $S$. This approach was developed in [10] to result in Weak Separation Condition [6, 9, 5, 15]. Violation of WSC results not only in the measure drop for $K$ [8, 14] in its dimension, but it also implies some special geometric properties of $K$ and rigidity phenomena for the deformations of self-similar structure on $K$ [4, 12, 13].

Though this scope of ideas and methods initially had self-similar sets as its target, there always was an attractive idea to extend it to more general classes of self-similar sets.

We consider how Weak Separation Condition (or its violation) applies to self-affine Jordan arcs in plane and show that structure and rigidity theorems for self-similar Jordan arcs [1, 11] have their self-affine analogues.

The main result of the current paper is the following
Theorem 1 Let \( \gamma \) be a self-affine Jordan arc in \( \mathbb{R}^2 \) which is not a parabolic segment. Then \( \gamma \) is a component of the attractor of some self-affine multi-zipper \( Z \).

As a main step for this result we prove the following rigidity theorem for a very general class of self-affine arcs, which need not be finitely generated:

Theorem 2 Let \( \gamma = \gamma(a_0, a_1) \) be a Jordan arc with endpoints \( a_0, a_1 \) in \( \mathbb{R}^2 \) such that

(i) For any \( \varepsilon > 0 \) and for any non-degenerate subarc \( \gamma' \subset \gamma \) there is an affine map \( S \) such that \( S(\gamma) \subset \gamma' \) and \( \text{Lip } S < \varepsilon \)

(ii) There is a sequence of affine maps \( f_k \) converging to \( \text{Id} \) such that \( f_k(\gamma) \cap \gamma = \gamma(f(a_0), a_1) \) and \( \text{fix}(f_k) \cap \gamma = \emptyset \).

Then \( \gamma \) is a parabolic segment.

In finitely generated case this theorem becomes

Theorem 3 Let a Jordan arc \( \gamma \subset \mathbb{R}^2 \) with endpoints \( a_0, a_1 \) be the attractor of a system \( S = \{S_1, \ldots, S_m\} \) of contracting affine maps. Let \( \mathcal{F}(S) \) be the associated family for the system \( S \). If there is a sequence \( f_n \in \mathcal{F}(S) \setminus \{\text{Id}\} \) such that \( f_n \to \text{Id} \), and \( f_n(\gamma) \cap \gamma \neq \emptyset \) then \( \gamma \) is a parabolic segment.

The proof of Theorems 2 and 3 uses the result of C. Bandt and A. S. Kravchenko [3] that except for parabolic arcs and segments, there are no twice continuously differentiable self-affine curves in the plane.

1. Definitions and notation.

Let \( S = \{S_1, \ldots, S_m\} \) be a system of contracting affine maps in \( \mathbb{R}^d \). The unique nonempty compact set \( K = K(S) \) such that \( K = \bigcup_{i=1}^{m} S_i(K) \), is called the attractor of the system \( S \), or a self-affine set generated by the system \( S \).

A system \( S \) is irreducible if, for every proper subsystem \( S' \subset S \), the attractor of \( S' \) is different from the attractor of the system \( S \).

By \( I = \{1, 2, \ldots, m\} \) we denote the set of indices, \( I^* = \bigcup_{n=1}^{\infty} I^n \) is the set of all multiindices \( i = i_1i_2...i_n \), and we denote \( S_1 = S_{i_1}, S_{i_2}, \ldots, S_{i_n} \). The set of all infinite sequences \( I^\infty = \{\alpha = \alpha_1\alpha_2\ldots, \alpha_i \in I\} \) is the index space.
and $\pi : I^\infty \to K$ is the index map, which maps a sequence $\alpha$ to the point $\bigcap_{n=1}^{\infty} K_{\alpha_1...\alpha_n}$.

The set $\mathcal{F}$ of all compositions $S_j^{-1}S_i$, where $i, j \in I^*$ and $i_1 \neq j_1$ is called the associated family of affine mappings for the system $S$. The system $S$ has the weak separation property (WSP) if and only if $\text{Id} \notin \mathcal{F} \setminus \text{Id}$.

If $\gamma$ is a Jordan arc with endpoints $a_0, a_1$, we denote its subarc $\gamma'$ with endpoints $x, y \in \gamma$ by $\gamma(x, y)$. We order the points in $\gamma$ putting $a_0 < a_1$ and write $x < y$ if $y \in \gamma(x, a_1)$. We denote the diameter of a set $A$ by $|A|$.

2. Representing $\gamma$ as a limit of $\varepsilon$-nets $P(k, x)$.

Applying if necessary a coordinate change, we may suppose that the arc $\gamma$ lies in the unit disc $D = \{x^2 + y^2 \leq 1\}$.

It follows from the condition (ii) that the subarcs $\sigma_{k,0} = \gamma \setminus f_k(\gamma)$ and $\sigma_{k,1} = f_k(\gamma) \setminus f_k^2(\gamma)$ are disjoint. Proceeding by induction we get a sequence of subarcs

$$\sigma_{k,n} = f_k^n(\sigma_{k,0}) = f_k^n(\gamma) \setminus f_k^{n+1}(\gamma)$$

which have endpoints $f_k^n(a_0), f_k^{n+1}(a_0)$ and have disjoint interiors as long as respective subarcs lie in $\gamma$. Since $f_k$ has no fixed points in $\gamma$, there is a maximal number $N_k$ for which

$$\bigcup_{n=0}^{N_k-1} \sigma_{k,n} = \gamma(a_0, f_k^{N_k}(a_0)) \subset \gamma.$$ Let $\sigma_{k,N_k} = f_k^{N_k}(\sigma_{k,0}) \cap \gamma = \gamma(f_k^{N_k}(a_0), a_1)$.

By the compactness of the arc $\gamma$ for any $\varepsilon > 0$ there is such $\delta$, that if $x_1, x_2 \in \gamma$ and $d(x_1, x_2) < \delta$, then the diameter of the subarc $\gamma(x_1, x_2)$ is less than $\varepsilon$.

Therefore for any $\varepsilon > 0$ there is such $N$, that if $k < N$ then $\|f_k(x) - x\| < \delta$ for any $x \in \gamma$, therefore the diameters of the subarcs $\sigma_{k,n}$ are not greater than $\varepsilon$.

For any $k$ and for any $x \in \gamma$ the point $x$ lies in one of subarcs $f_k^{n_k}(\sigma_{x,0})$. Denote $P(k, x) = \{f_k^n(x), -n_k \leq n \leq N_k - n_0\}$. Then Hausdorff distance between $P(k, x)$ and $\gamma$ is not greater than $\max\{\|\sigma_{k,n}\|, 0 \leq n \leq N_k\}$. 

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Therefore for any choice of the sequence $x_k \in \gamma$ the sequence of sets $P(k, x_k)$ converges to $\gamma$ in Hausdorff metrics.

3. Five types of affine maps and their associated vector fields.

Since the sequence $f_k$ converges to $\text{Id}$, we suppose that all $f_k$ are sufficiently close to $\text{Id}$ so that for any $f_k$ we can correctly define its power $f_k^t, t \in \mathbb{R}$, satisfying the conditions:

1. For any $t_1, t_2 \in \mathbb{R}$, $f_k^{t_1} \circ f_k^{t_2} = f_k^{t_1 + t_2}$;
2. $f_k^0 = \text{Id}$ and $f_k^1 = f_k$.

For that reason we divide the set of non-degenerate affine maps $f(x) = Ax + b$ on $\mathbb{R}^2$, where A is a non-degenerate matrix and $b$ is a vector to five following types, depending on the eigenvalues $\lambda_1$ and $\lambda_2$ of the matrix $A$ and on the translation vector $b$:

Type 1. If both eigenvalues $\lambda_1$ and $\lambda_2$ are not equal to 1, then the map $f(x)$ has unique fixed point $x_0 = (E - A)^{-1}b$. By our assumptions, $\|A - E\| < 1$, therefore $A = e^B$, where $B = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(A - E)^n}{n}$ is the matrix logarithm of $A$. Since $f(x) = A(x - x_0) + x_0$, we put

$$f^t(x) = e^{Bt}(x - x_0) + x_0 \quad \text{(2)}$$

In this case for any $x \neq x_0$, $\{f^t(x), t \in \mathbb{R}\}$ is an integral curve of autonomic system $\dot{x} = B(x - x_0)$.

Types 2 and 3. If $\lambda_1 \neq 1$ and $\lambda_2 = 1$ and $e_1, e_2$ are respective eigenvectors, then the map $f$ can be represented by $f(x) = Ax + ae_1 + be_2$.

In this case the matrix logarithm $B$ has eigenvalues $\log \lambda_1$ and 0 and the equation

$$f^t(x) = e^{Bt}x + a \frac{\lambda_1^t - 1}{\lambda_1 - 1}e_1 + bte_2 \quad \text{(3) ?}$$

defines some integral curve of the autonomic system

$$\dot{x} = Bx + a \frac{\log \lambda_1}{\lambda_1 - 1}e_1 + be_2 \quad \text{(4) dyn2}$$

We refer $f$ to the Type 2 if $b = 0$. In this case the right side in (4) is a multiple of $e_1$, and integral curves are straight lines parallel to $e_1$. If
\[ x = \frac{a}{1 - \lambda_1} e_1 + t e_2, \] then \[ Bx = -\frac{a \log \lambda_1}{\lambda_1 - 1} e_1, \] so the right side in (4) vanishes, and \[ L = \left\{ \frac{ae_1}{1 - \lambda_1} + e_2 t, t \in \mathbb{R} \right\} \] is the line consisting of fixed points of \( f \).

\( S \) is referred to \textbf{Type 3} if \( b \neq 0 \). The system (4) has no fixed points in this case. The right side of (4) on the line \( L \) is equal to \( be \), so \( L \) is the invariant straight line. The vector field is invariant under translations by \( te_2, t \in \mathbb{R} \), and there is the minimal value for \( \| \dot{x} \| \) which is equal to \( |b| |e_1| \sin \alpha_{12} \), where \( \alpha_{12} \) is the angle between \( e_1 \) and \( e_2 \).

\textbf{Type 4.} It is the case when the eigenvalues of \( A \) are \( \lambda_1 = \lambda_2 = 1 \), and \( A \neq \text{Id} \), while \( f(x) = Ax + ae_1 \), where \( e_1 \) is the eigenvector for \( A \). In this case the matrix logarithm \( B \) is similar to degenerate Jordan cell. The lines \( f^t(x), t \in \mathbb{R} \) are the integral curves for the autonomous system \( \dot{x} = Bx + be_1 \). Since \( Bx \) is a real multiple of \( e_1 \), the right side of the equation (4) is the multiple of \( e_1 \), so these curves are straight lines parallel to \( e_1 \). The line \( L = \{-be_2 + te_1, t \in \mathbb{R}\} \) is the set of fixed points for \( f \).

\textbf{Type 5.} This is the case when \( \lambda_1 = \lambda_2 = 1 \), \( A \neq \text{Id} \), and \( f(x) = Ax + ue_1 + ve_2 \), where \( e_2 \) is the root vector for \( A \) and \( v \neq 0 \). In this case \( f \) has no fixed points. One can see that the integral curves corresponding to \( f \) are parabolas obtained from each other by parallel translations:

Notice that matrix logarithm of \( A \) is equal to \( B = A - E \) and \( B^2 = 0 \).

Therefore the system \( \dot{x} = Bx + \beta \) with initial value \( x(0) = x_0 \), has the solution

\[
x(t) = x_0 + (Bx_0 + \beta)t + B \frac{t^2}{2} \quad \text{ (5) } \]

Denoting \( ue_1 + ve_2 = b \), we get \( \beta = (I - \frac{B}{2}) \cdot b \) and \( x(t) = Bb \frac{t^2}{2} + (b - \frac{1}{2} Bb + Bx_0)t + x_0 \), while the vector field for \( f \) is

\[
\dot{x} = Bx + b - \frac{1}{2} Bb \quad \text{or} \quad \dot{x} = (A - I)x + \left( \frac{3}{2} E - \frac{1}{2} A \right) b. \quad \text{ (6) } \]

Taking into account that for \( x = \xi e_1 + \eta e_2 \) \( Bx = \eta e_1 \), we see, that the right side in (6) \( \eta e_1 + (u - v/2) e_1 + ve_2 \) does not depend on \( \xi \) and vanishes 0 if \( v = 0 \) and \( \eta = -u \), which corresponds to Type 4.
Therefore if \( f \) belongs to the Type 5 the vector field has no stationary points and is preserved by translations by \( te_1 \), so the minimal value of \( \|\dot{x}\| \) is \( |v| \cdot \|e_2\| \cdot |\sin \alpha_{12}| \), where \( \alpha_{12} \) is the angle between \( e_1 \) and \( e_2 \).

Lemma 4 Suppose that under the conditions of Theorem 2 all the maps \( f_n \) belong to the Type 1. Then there is such sequence of non-degenerate affine maps \( h_n \) satisfying the conditions of Theorem 2 that their fixed points \( y_n = \text{fix}(h_n) \notin \overline{D} \).

If \( \gamma \) is not a straight line segment, there is such a ball \( B_1 \subset \mathbb{D} \), that \( \gamma \cap B_1 \neq 0 \) and the set \( \{ n : \text{fix} f_n \subset CB_1 \} \) is infinite.

By the condition (i) of the Theorem 2 there is such affine map \( g \), that \( g(\gamma) \subset \gamma' \) and \( g(B_1) \subset \mathbb{D} \). Then, if \( \text{fix} f_n = x_n \in CB \), then \( \text{fix}(g^{-1} \cdot f \cdot g) = g^{-1}(x_n) \in Cg^{-1}(B_1) \subset CD \).

Thus all the fixed points of the sequence of maps \( f'_n = g^{-1} \cdot f_n \cdot g \) lie in the complement of \( D \).

If \( y = Tx + C \), then the fixed points \( y_n \) of the map \( f'_n(x) \) are given by the equation \( y_n = T^{-1}(x_n - C) \) and the map \( f'_n \) is given by the equation \( f'_n(x) = T^{-1}A_nT(x - y_n) + y_n \).

At the same time the eigenvalues of the matrix \( A'_n \) are the same as the ones of \( A_n \), and the sequence \( f'_n \rightarrow \text{Id} \).

Notice that for sufficiently large \( n \) \( f_n(g(a)) \subset g(\gamma) \). Since \( f_n \) has no fixed points in \( \gamma \), \( f_n(g(\gamma)) \cap g(\gamma) = \gamma(f_n(g(a_0), g(a_1))) \).

Therefore \( f'_n(\gamma) \cap \gamma = \gamma(f'_n(a_0), a_1) \) and the sequence \( f'_n \) satisfies the conditions of Theorem 2.

Proof of Theorem 2

Let \( f_n \) be the sequence of maps satisfying the conditions (i),(ii) of the Theorem 2.

Without loss of generality we may assume that all \( f_k \) belong to one and the same of the Types 1-5.

If all \( f_k \) belong to the Type 2 or 4 then the set \( P(x, a_0) \) lies on the segment \( l_k = [a_0, f_k^{N_k}(a_0)] \), and the sequence \( l_k \) converges to the segment \( [a_0, a_1] \), therefore \( \gamma = [a_0, a_1] \).

Thus we need to prove the statement of the Theorem 2 for the case when \( f_n \) belong to Type 1,3 or 5.
If \( f_n \) belong to the Type 3 or 5, then the maps \( f_n \) as well as their associated vector fields have no fixed points.

If all \( f_n \) belong to the Type 1, Lemma 4 allows us to assume that fixed points of the maps \( f_n \) lie outside of \( D \).

Let \( L_k \) denote the set \( \{ f_k^t(a_0), 0 \leq t \leq N_k \} \). Since \( P(k, a_0) \subset L_k \) and \( \lim_{k \to \infty} P(k, a_0) = \gamma \), we have \( \gamma \subset \lim_{k \to \infty} L_k \).

The sets \( L_k \) are the subarcs of integral curves of linear dynamical systems \( \dot{x} = B_k x + b_k \), and the endpoints of \( L_k \) are \( a_0 \) and \( f_k^{N_k}(a_0) \).

Let \( m_k = \max \{ \| B_k x + b_k \|, x \in D \} \). If we replace the right sides \( B_k x + b_k \) of respective equations by \( B'_k x + b'_k \), where \( B'_k = B_k \setminus m_k \) and \( b'_k = b_k \setminus m_k \), we obtain a sequence of linear dynamical systems in \( D \), which have no stationary points in \( D \), and whose integral curves are the same as the ones for the systems \( \dot{x} = B_k x + b_k \). At the same time \( \max \{ \| B'_k(x) + b'_k \|, x \in D \} \) is equal to 1 and by convexity of the function \( \| B'_k(x) + b'_k \| \), is assumed at some point \( x \in \partial D \).

Denote \( g_k(x) = B'_k x + b'_k \). The affine map \( g_k \) sends \( D \) to some ellipse \( g_k(D) \subset D \) which is tangent to \( \partial D \) at some point and which does not contain \( 0 \). The sequence of maps \( g_k \) satisfies the conditions of Arcela’s theorem and one can find a subsequence \( g_{n_k} \) which converges uniformly on \( D \) to some affine function \( g_0(x) \).

By continuous dependence of solutions of differential equations on their right sides, the solutions of the differential equations \( \dot{x}(t) = g_n(x), x(0) = a_0 \) converge uniformly with all their derivatives to the solution of the equation \( \dot{x}(t) = g_0(x), x(0) = a_0 \), and the integral curves \( L_{n_k} \) converge to the curve \( L_0 \). The curve \( L_0 \) belongs to the class \( C^2 \) if \( \| g_0(x) \| \neq 0 \) so we need to control zero points of \( g_0(x) \).

For that reason we consider the limit \( g_0(D) \) of the sequence of ellipses \( g_n(D) \).

If \( g_0(D) \) is a non-degenerate ellipse, then since \( g_0(D) = \lim g_{n_k}(D) \), and \( g_{n_k}(D) \notin 0, g_0(D) \) can contain \( 0 \) only on its boundary. Since \( \gamma \subset \bar{D}, g_0(\gamma) \notin 0 \) in this case.

If \( g_0(D) \) is a line segment, for which \( 0 \) is its inner point, then \( g_0^{-1}(0) \) is a chord \( \Lambda \) in the disc \( D \). If \( \gamma \subset \Lambda \) then \( \gamma \) is a line segment. Otherwise \( \gamma \) contains a subarc \( \gamma' \), which is disjoint from \( \Lambda \). By the condition (i) we may assume that \( \gamma' = S(\gamma) \) for some affine mapping \( S \). The arc \( \gamma' \) is contained in the integral curve of the equation \( \dot{x} = g_0(x) \), which starts at the point
Since $\|g_0(x)\| \neq 0$ on $\gamma'$, it belongs to the class $C^2$. Therefore $\gamma$ is twice differentiable.

By Theorem of C.Bandt and A.S.Kravchenko [3, Theorem 3], $\gamma$ is a segment of a parabola or straight line.

**Proof of Theorem 3.**

Let $f_n = S_{i_n}^{-1} S_{j_n}$ be the sequence converging to Id for which $f_n(\gamma) \cap \gamma \neq \emptyset$. Since $f_n$ is close to Id, the maps $f_n$ and $f_n^{-1}$ preserve the orientation on $\gamma$. Notice that for self-affine arcs the condition (i) of Theorem 2 holds automatically. Therefore, following the argument of Lemma 4, the sequence $f_n$ can be chosen in such a way that for any $n$, $\text{fix}(f_n) \cap \gamma = \emptyset$. Then up to permutation of $i$ and $j$ we may suppose that for any $n$, $S_{i_n}(\gamma) \cap S_{j_n}(\gamma) = \gamma(S_{j_n}(a_0), S_{i_n}(a_1))$. Therefore $f_n(\gamma) \cap \gamma = \gamma(f_n(a_0), a_1)$ and we can apply Theorem 2 to complete the proof.

**Definition 5.** Let $\gamma_1, \gamma_2$ be Jordan arcs in $\mathbb{R}^d$. We say that $\gamma_1$ and $\gamma_2$ have proper intersection if the set $\gamma_1 \cap \gamma_2$ is a non-degenerate subarc in $\gamma_1$ and $\gamma_2$ and one of its endpoints is an endpoint of $\gamma_1$ and the other is an endpoint of $\gamma_2$.

**Corollary 6.** Let $S$ be a system of non-degenerate contracting affine mappings with a Jordan attractor $\gamma$. Let $A_\delta(\gamma)$ be the set of subarcs $\alpha = h(\gamma) \cap \gamma$ such that $|\alpha| \geq \delta$, $h$ is an affine map, and the arcs $h(\gamma)$ and $\gamma$ have regular intersection. If the set $A_\delta(\gamma)$ is infinite, then $\gamma$ is a segment of parabola.

## 2 The partition to elementary subarcs.

**Theorem 7.** Let $S = \{S_1, ..., S_m\}$ be a system of contractive affine maps in $\mathbb{R}^2$ with Jordan attractor $\gamma$. If $\gamma$ is different from a segment of a parabola or straight line, there is a multizipper $Z$ such that the arc $\gamma$ is one of the components of the attractor of $Z$.

**Proof.** We suppose the system $S$ is irreducible. Let us order the maps $S_1, ..., S_m$ so that $\gamma_i \cap \gamma_j \neq \emptyset$ if and only if $|i - j| = 1$, while $a_0 \in \gamma_1$ and $a_1 \in \gamma_m$. For two points $x, y \in \gamma$ we write, that $x < y$, if $y \in \gamma(x, a_1)$.

First we construct such finite set $\mathcal{P} \subset \gamma$, whose points $a_0 = p_0 < p_1 < ... < p_{N-1} < p_N = a_1$ define a partition of $\gamma$ to subarcs $\delta_i, i = 1, ..., N$, satisfying the conditions
1. For any $\delta_i$ and any $k = 1, ..., m$ there is $\delta_j$ such that $S_k(\delta_i) \subseteq \delta_j$;
2. For any $k_1, k_2 = 1, ..., m$ and for any $\delta_{i_1}, \delta_{i_2}$, $S_{k_1}(\delta_{i_1})$ and $S_{k_2}(\delta_{i_2})$ are either equal or disjoint.

Let $\mathcal{G}$ be the set of all affine mappings $g$ such that the set $\gamma \cap g(\gamma)$ contains a connected component which is a subarc $\gamma_g \subset \gamma$, whose endpoints are the points $g(a_i)$ and $a_j$, $i, j \in \{0, 1\}$. Let $\mathcal{P}$ be the set consisting of $a_0, a_1$ and of points $g(a_i)$, where $g \in \mathcal{G}$, $i = 0, 1$, and $g(a_i) \in \gamma \cap \dot{\gamma}$. Let $\mathcal{P}_i$ be the set of those $p \in \mathcal{P} \cap \gamma$, which are the endpoints of subarcs $\gamma_g$, that do not contain $a_{1-i}$. Thus, $\mathcal{P} = \{a_0, a_1\} \cup \mathcal{P}_0 \cup \mathcal{P}_1$.

Notice two properties of $\mathcal{P}$, which directly follow from its definition:

- **b1.** Let $g$ be affine map of $\mathbb{R}^2$ for which $g(\gamma) \subset \gamma$. Then $\mathcal{P} \cap \dot{g}(\gamma) \subset g(\mathcal{P})$.
- **b2.** Let $g_1, g_2$ be two affine maps such that $g_1(\gamma), g_2(\gamma)$ are subarcs of $\gamma$, having proper intersection. Then the endpoint of the subarc $g_1(\gamma)$, contained in $g_2(\gamma)$, lies in $g_2(\mathcal{P})$, and vice versa.

In the case when a Jordan arc $\gamma$ is the attractor of a system of contracting affine maps $\mathcal{S}$, the conditions **b1** and **b2** imply the properties:

- **c1.** For any $j \in I$, $\mathcal{P} \cap \dot{\gamma}_j \subset S_j(\mathcal{P})$;
- **c2.** For any $1 \leq j \leq m-1$, $S_j(\{a_0, a_1\} \cap \dot{\gamma}_{j+1} \subset \gamma_{j+1}(\mathcal{P})$ and $S_{j+1}(\{a_0, a_1\} \cap \dot{\gamma}_{j+1} \subset \gamma_{j+1}(\mathcal{P})$.

**Lemma 8** Let a Jordan arc $\gamma \subset \mathbb{R}^2$ with endpoints $a_0, a_1$ be the attractor of irreducible system $\mathcal{S} = \{S_1, ..., S_m\}$ of contracting affine maps, and $\gamma$ is not a segment of a parabola or a straight line. Then:

- **d1.** The set of limit points of $\mathcal{P}$ is contained in $\{a_0, a_1\}$.
- **d2.** There are such neighbourhoods $U_i$ of the points $a_i$, where $i = 0, 1$, that $P_{1-i} \cap U_i = \emptyset$, and
- **d3.** If for some $k \in \{1, m\}$ and some $i, j \in \{0, 1\}$, $S_k(a_i) = a_j$, then $S_k$ is a bijection of $U_i \cap P_i$ to $S_k(U_i) \cap P_j$.

**Proof.** First we show that the set $\mathcal{P}$ has no limit points in $\dot{\gamma}$. Suppose there is a $c \in \dot{\gamma} \cap \mathcal{P}$. Then for one of the endpoints of $\gamma$, say, for $a_0$, there is a sequence $g_n \in \mathcal{G}$, such that $g_n(a_0) \to c$. It follows from Corollary 6, that $\gamma$ is a segment of a parabola, which contradicts the assumptions of the
Lemma, so \( d1 \) is true. The same argument shows that \( a_1 \) cannot be a limit point of \( P_0 \) and \( a_0 \) cannot be a limit point of \( P_1 \). Therefore there are such neighbourhood \( U_i \) of the points \( a_i \), that \( P_{1-i} \cap U_i = \emptyset \). Moreover, we choose \( U_0, U_1 \) in such a way that \( \gamma \cap U_0 \subset \gamma_1 \) and \( \gamma \cap U_1 \subset \gamma_m \).

To check \( d3 \), consider first the case when \( S_1(a_1) = a_0 \). If \( p \in P_0 \cap U_0 \) and \( p = g(a_i) \), then \( S^{-1}_1 \circ g \in S \) and \( S^{-1}_1(p) \in P_1 \cap S^{-1}_1(U_0) \). Conversely, if \( p \in P_1 \cap U_1 \), and \( p = g(a_i) \), then \( S_1 \circ g \in S \) and \( S_1(p) \in P_0 \cap S(U_1) \). Therefore \( S_1 \) defines a bijection \( P \cap U_0 \cap S_1(U_1) \) to \( P \cap U_1 \cap S^{-1}_1(U_0) \). Enumerating all possibilities:

1. \( S_1(a_0) = a_0, S_m(a_1) = a_1 \);
2. \( S_1(a_0) = a_0, S_m(a_1) = a_0 \);
3. \( S_1(a_0) = a_1, S_m(a_1) = a_1 \);
4. \( S_1(a_0) = a_1, S_m(a_1) = a_0 \);

we find the desired pairs of neighborhoods for each of the cases. ■

\( \text{Lemma 9} \) The set \( P \) contains a finite subset \( P' \), which also satisfies \( c1 \) and \( c2 \).

\textbf{Proof.} For each of the points \( S_k(a_i) \in \gamma \), where \( k \in I \) and \( i = 0, 1 \) we denote by \( w(k, i) \) the connected component of the set \( \gamma_k \setminus P \), which has \( S_k(a_i) \) as its endpoint, whereas for \( S_k(a_i) = a_j \) we put \( w(k, i) = U_j \). Let \( W_i = \bigcap_{k \in I} S^{-1}_k(w(k, i)) \cap U_i \).

Let \( P' = \{a_0, a_1\} \cup P \setminus (W_0 \cup W_1) \).

The set \( P' \) is finite, so we denote its elements by \( a_0 = p_0 < p_1 < ... < p_M = a_1 \), and the subarcs \( \gamma(p_{k-1}, p_k) \) by \( \delta_k \).

For any \( j \in I \), \( S_j(P) \subset S_j(W_0 \cup W_1) \cup S_j(P') \). At the same time the definition of \( P' \) implies that \( S_j(W_0 \cup W_1) \cup S_j(P') = S_j(\{a_0, a_1\}) \). Therefore \( P' \cap \gamma_j \subset S_j(P') \). Thus the set \( P' \) satisfies the condition \( c1 \). The condition \( c2 \) directly follows from the definition of \( P' \). ■

\( \text{Lemma 10} \) Each of the subarcs \( \delta_i, i = 1, ..., M \) and \( \gamma_i, i \in I \) is an union of subarcs \( S_j(\delta_k) \) for some \( j \in I \) and some \( k \in \{1, ..., M\} \) whose interiors are disjoint.

\textbf{Proof.} The system \( S \) is irreducible, therefore each subarc \( \gamma_j \), \( 1 < j < m \) intersects two adjacent subarcs \( \gamma_{j-1}, \gamma_{j+1} \), so that \( \gamma_j \setminus (\gamma_{j-1} \cup \gamma_{j+1}) \neq \emptyset \). For
any subarc $\bar{\gamma}_j = \gamma_j \setminus (\hat{\gamma}_{j-1} \cup \hat{\gamma}_{j+1})$ its endpoints by $c_2$ are contained in $S_j(P')$; let them be the points $S_j(p_{k_j}), S_j(p_{K_j})$. The arc $\bar{\gamma}_j$ has unique representation $\bigcup_{i=0}^{K_j-1} S_j(\delta_i)$. For each of the subarcs $\gamma_j \cap \gamma_{j+1}$ there are exactly two partitions: first, to the subarcs $S_j(\delta_i)$ and second, to the subarcs $S_{j+1}(\delta_i)$; choose one of them. Taking the union over all subarcs and renumbering all the points, we get the desired partition for the whole $\gamma$. By the property $c_1$, the partition we obtained is at the same time a partition for each of the subarcs $\delta_k$. ■

Proof of the Theorem 7 Now we can construct a Jordan multizipper, for which the components of the attractor will be the subarcs $\delta_j$. Each of the subarcs $\delta_j, j = 1, \ldots M$ is a finite union of consequent subarcs $S_i(\delta_k)$, which form a partition of $\delta_j$. Therefore we can create a graph $\tilde{G}$ whose vertices are $u_j = \delta_j$ and an edge $e_{ij}$ is directed from $u_i$ to $u_j$ if there is such $S_k$, that $S_k(U_j) \subset \delta_i$. ■

References

[ATK] [1] V. V. Aseev, A. V. Tetenov, A. S. Kravchenko, Self-similar Jordan curves on the plane// Sibirsk. Mat. Zh., 44(2003), pp. 481-492.

[SSS7] [2] C. Bandt, S. Graf, Self-similar sets 7. A characterization of self-similar fractals with positive Hausdorff measure.// Proc.Amer.Math.Soc., 114(1992), No.4, pp.995-1001.

[BK] [3] C. Bandt, A. S. Kravchenko, Differentiability of fractal curves //Nonlinearity 24 (2011) 2717

[BR] [4] C. Bandt and H. Rao, Topology and separation of self-similar fractals in the plane// Nonlinearity 20 (2007), pp. 1463 - 1474.

[DE] [5] M. Das, G. A. Edgar, Finite type, open set conditions and weak separation conditions // Nonlinearity 24 (2011), 2489

[Edgdas] [6] G. A. Edgar, M. Das , Separation properties for graph-directed self-similar fractals// Top.appl.,152(2005), 138-156.

[Fal] [7] K. J. Falconer , Fractal Geometry: Mathematical Foundations and Applications, John Wiley and Sons, 1990.
[KT2F][8] K. G. Kamalutdinov, A. V. Tetenov, Twofold Cantor sets in R// Siberian Electr. Math. Rep., 15 (2018), pp. 801-814, DOI 10.17377/semi.2018.15.066.

[Lau][9] K. S. Lau and S. M. Ngai, Multifractal measures and a weak separation condition, //Adv. Math. 141 (1999), 45–96. MR1667146

[Schief][10] A. Schief, Separation properties for self-similar sets// Proc. Amer. Math. Soc., 124:2 (1996), pp. 481–490.

[Atet1][11] A. V. Tetenov, Self-similar Jordan arcs and graph-directed systems of similarities //Sibirsk. Mat. Zh., 47 (2006), pp. 11471159.

[Trg][12] A. V. Tetenov, On the rigidity of one-dimensional systems of contraction similitudes // Siberian Electr. Math. Rep., 3 (2006), 342–345.

[TCh][13] A. V. Tetenov, A. K. B. Chand, On weak separation property for affine fractal functions// Siberian Electr. Math. Rep., 12 (2015), 967972.

[TKV][14] A. V. Tetenov, K. G. Kamalutdinov, D. A. Vaulin, Self-similar Jordan arcs which do not satisfy OSC, arXiv:1512.00290

[Zer][15] M. P. W. Zerner, Weak separation properties for self-similar sets./// Proc.Amer.Math.Soc. 1996, 124, No. 11, pp.3529–3539.