Divergencies in the Casimir energy for a medium with realistic ultraviolet behavior

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Abstract

We consider a dielectric medium with an ultraviolet behavior as it follows from the Drude model. Compared with dilute models, this has the advantage that, for large frequencies, two different media behave the same way. As a result one expects the Casimir energy to contain less divergencies than for the dilute media approximation. We show that the Casimir energy of a spherical dielectric ball contains just one divergent term, a volume one, which can be renormalized by introducing a contact term analogous to the volume energy counterterm needed in bag models.
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Due to recent progress in experimental techniques, it is now possible to measure Casimir forces between macroscopic testbodies very accurately [1, 2]. Even quantum mechanical actuation of microelectromechanical systems by the Casimir force is under discussion [3]. For these reasons a thorough theoretical
understanding of the Casimir effect with realistic media is desirable. There are essentially two ways to calculate the Casimir energy. One possibility is to sum up retarded van der Waals forces between individual molecules [4, 5]. The second way makes use of quantum field theory in the background of a dielectric medium (see e.g. [6, 7, 8, 9, 10, 11, 12, 13]). In this second approach one tries to recover retarded van der Waals forces by calculating the vacuum energy for the electro-magnetic field in a dielectric background. The relation between these two approaches is not well established. Only for a dilute ball, up to the second order of a perturbative expansion, the methods have been shown to yield the same answers [14, 15]. In all other cases of dielectric media the presence of divergencies forbids to give a physical interpretation to the finite parts obtained. Even for a dilute medium, once the full dependence of the pole on the dielectric constants is considered, the result is not even expressible in terms of known special functions [16]. So the counterterms needed to renormalize the divergent contributions is extremely complicated and the interpretation of the classical model associated with the counterterms is completely unclear.

One possible reason for this briefly described difficulty might be the fact that the models of dielectric media treated in the quantum field theory framework mostly do not fulfill a realistic frequency dependent dispersion relation. A direct consequence of an unreasonable ultraviolet behavior is, for example, the appearance of divergencies which make it impossible to extract, in a physically reasonable way, a finite value for the Casimir energy.

For those reasons, it seems natural to analyze the divergencies of the Casimir effect in a model with a dispersion relation that shows realistic features, at least in the ultraviolet range. As we will see, once this behavior is assumed, the pole structure resulting in the $\zeta$-function regularization scheme is in fact very simple. Namely just one pole exists and it is a term completely analogous to the one renormalized via the bag constant in the bag model. So, introducing only one counterterm, namely a volume energy, the Casimir energy can be rendered finite. Apparently there is no intrinsic way within the field theory framework, to fix the finite part of this counterterm. In principle, as is the case in bag models, it has to be fixed by experiments. However, given the simple structure of the renormalization needed, we feel this is an important step in order to understand the relation between different approaches employed and to extract finite Casimir energies in realistic media.

So, let us start with a description of the simple model we are going to analyze. We will consider a nonmagnetic ($\mu = \mu_0$) dielectric ball of radius $a$, with a frequency dependent permittivity given by $\epsilon(\omega) = \epsilon_0(1 - \Omega^2/\omega^2)$, immersed in another medium of permittivity $\epsilon_0$. This frequency dependent permittivity is the so-called plasma model, which follows from the Drude model in the high frequency approximation. The phenomenological parameter $\Omega$ is usually referred to as the effective plasma frequency. Given that we are going to concentrate only on the pole structure of the Casimir energy for the described setting, this high frequency model is sufficient for our purposes. Formally, the Casimir energy of this configuration is defined by summing over the vacuum energies of
each mode of the electromagnetic field,

\[ E_{\text{Cas}}(a) = \sum_n \frac{1}{2} \hbar \omega_n = \frac{\hbar c}{2a} \sum_n z_n, \quad (1) \]

where the dimensionless quantities \( z_n = \omega_n / c \) are the eigenvalues associated with a radius one ball. As it stands, the sum in Eq. (1) is divergent, and a regularization procedure must be adopted. To properly define the Casimir energy we will use the zeta function regularization. In this scheme, \( E_{\text{Cas}}(a) \) is defined as the analytic continuation of the function

\[ E_{\text{Cas}}(a) = \left. \frac{\hbar c}{2a} \zeta(s) \right|_{s \to -1}, \quad (2) \]

where \( \zeta(s) \) is defined through the eigenfrequencies of the electromagnetic field by means of the series

\[ \zeta(s) = \sum_n z_n^{-s}, \quad (3) \]

which is absolutely and uniformly convergent for \( \Re(s) \) sufficiently large. As is well known, usually the zeta function has a pole at \( s = -1 \), in which case a suitable renormalization procedure has to be employed.

In Eq. (3), the eigenfrequencies \( \omega_n \) must be determined by solving the field equation

\[ \triangle \vec{E} + \mu \epsilon \omega^2 \vec{E} = 0 \quad (4) \]

(and similarly for \( \vec{B} \)), subject to matching conditions appropriate to the interphase between two dielectric media,

\[ E_{\theta,\phi} \bigg|_{r=a^+} = E_{\theta,\phi} \bigg|_{r=a^-}, \quad \frac{1}{\mu_1} B_{\theta,\phi} \bigg|_{r=a^+} = \frac{1}{\mu_2} B_{\theta,\phi} \bigg|_{r=a^-}. \quad (5) \]

One can consider the transversal electric (TE) modes, for which the electric field has the form

\[ \vec{E}_{i,m} = f_i(r) \vec{L} Y_{i,m}(\theta, \phi), \quad (6) \]

and separately the transversal magnetic modes (TM), with the magnetic field given by

\[ \vec{B}_{i,m} = g_i(r) \vec{L} Y_{i,m}(\theta, \phi). \quad (7) \]

Here, as usual,

\[ \vec{L} = -i \vec{\rho} \times \vec{n}. \quad (8) \]

The imposition of the conditions in Eq. (5) leads to

\[ f_i(r = a^+) = f_i(r = a^-), \]
\[ \partial_r [r f_i(r)]|_{r=a^+} = \partial_r [r f_i(r)]|_{r=a^-}. \quad (9) \]
for the TE modes, and
g_l(r = a^+) = g_l(r = a^-),
\begin{equation}
\frac{1}{\epsilon(\omega)} \left. \frac{\partial_r [rg_l(r)]}{\partial_r [rg_l(r)]} \right|_{r=a^+} = \frac{1}{\epsilon(\omega)} \left. \frac{\partial_r [rg_l(r)]}{\partial_r [rg_l(r)]} \right|_{r=a^-},
\end{equation}

for the TM modes.

In order to have a discrete spectrum, one might enclose the system into a large conducting sphere of radius \(R\), sending \(R \to \infty\) at a suitable intermediate step. This implies also the following boundary condition at \(r = R\) for \(f_l(r)\) and \(g_l(r)\):
\begin{equation}
f_l(r)|_{r=R} = 0, \quad \partial_r (rg_l(r))|_{r=R} = 0.
\end{equation}

Alternatively it is possible to use a formulation in terms of the Jost function of the corresponding scattering problem, where a subtraction of the Minkowski space contribution is performed at the beginning and the infinite volume limit taken implicitly.

Similarly, for the TM modes the eigenfrequencies are determined by the zeroes of the function (we put \(\nu = l + 1/2\))
\begin{equation}
\Delta_{\nu}^{TM}(z) = \mathcal{J}_\nu(\bar{z}_1) \{ \mathcal{Y}_\nu(\bar{z}_0)\mathcal{J}'_\nu(\bar{z}_2) - \mathcal{J}_\nu(\bar{z}_0)\mathcal{Y}'_\nu(\bar{z}_2) \} -
-\xi \mathcal{J}'_\nu(\bar{z}_1) \{ \mathcal{Y}_\nu(\bar{z}_0)\mathcal{J}_\nu(\bar{z}_2) - \mathcal{J}_\nu(\bar{z}_0)\mathcal{Y}_\nu(\bar{z}_2) \},
\end{equation}

where
\begin{equation}
\mathcal{J}_\nu(w) = w \ j_\nu(w) = \sqrt{\pi/2} j_\nu(w)
\end{equation}
\begin{equation}
\mathcal{Y}_\nu(w) = w \ y_\nu(w) = \sqrt{\pi/2} y_\nu(w)
\end{equation}
are the Riccati-Bessel functions, being \(z = a (\omega/c)\), \(\bar{z}_1 = z n(z)\), \(\bar{z}_2 = z n_0\), \(\bar{z}_0 = z R n_0/a\), \(\xi = n(z)/n_0\), \(n_0 = \sqrt{\epsilon_0}\), and \(n(z) = \sqrt{\epsilon(\omega)}\).

Similarly, for the TM modes the eigenfrequencies are determined by the zeroes of
\begin{equation}
\Delta_{\nu}^{TM}(z) = \mathcal{J}_\nu(\bar{z}_1) \{ \mathcal{Y}_\nu(\bar{z}_0)\mathcal{J}'_\nu(\bar{z}_2) - \mathcal{J}_\nu(\bar{z}_0)\mathcal{Y}'_\nu(\bar{z}_2) \} -
-\frac{1}{\xi} \mathcal{J}'_\nu(\bar{z}_1) \{ \mathcal{Y}_\nu(\bar{z}_0)\mathcal{J}_\nu(\bar{z}_2) - \mathcal{J}_\nu(\bar{z}_0)\mathcal{Y}_\nu(\bar{z}_2) \}.
\end{equation}

We will suppose that these functions have only real and simple zeroes in the open right half plane.

In the following we will consider explicitly only the TE modes, since the treatment of the TM modes is entirely similar. Due to the spherical symmetry of the problem, the \(\zeta\)-function has the general appearance
\begin{equation}
\zeta(s) = \sum_{\nu = \lambda/2}^{\infty} 2\nu \sum_n z_{\nu,n}^{-s},
\end{equation}
where \( \nu \) labels the angular momentum and, for a given \( \nu \), the index \( n \) labels the zeroes of \( \Delta_{TE}^\nu(z) \) in Eq. (12). We need to construct the analytic continuation of this function to \( s \approx -1 \). As we will see, \( \zeta(s) \) has a simple pole at \( s = -1 \) and \( E_{TE}(a) \), defined as in Eq. (6), contains a divergent term which depends on the particular dispersion relation adopted for \( \epsilon(\omega) \). The method employed for the analytic continuation has been explained in great detail, e.g., in [17, 16, 18, 19, 20], and we will simply follow this procedure. However, let us mention that the idea of applying this method in the specific context of dielectric media goes back at least to [21].

For \( \Re(s) \) large enough, we can represent the \( \zeta \)-function as an integral in the complex \( z \)-plane employing the Cauchy’s theorem. For the TE modes we have

\[
\zeta_\nu(s) := \sum_{n=1}^{\infty} z_{\nu,n}^{-s} = \frac{1}{2\pi i} \oint_C z^{-s} \frac{\Delta_{TE}'^\nu(z)}{\Delta_{TE}^\nu(z)} \, dz, \tag{16}
\]

where the curve \( C \) encloses counterclockwise all the positive zeros of \( \Delta_{TE}^\nu(z) \). This curve can be deformed into a straight vertical line, crossing the horizontal axis at \( \Re(z) = x \), where \( x \) is any value satisfying \( 0 < x < z_{\nu,1} \), \( z_{\nu,1} \) being the first positive zero of \( \Delta_{TE}^\nu(z) \). Obviously the integral in eq. (16) does not depend on the particular value of \( x \) in this range.

Indeed, expressing the integrand in terms of the modified Bessel functions and taking into account their asymptotic behavior for large arguments, it is easily seen that the integral

\[
\zeta_\nu(s) = -\frac{1}{\pi} \Re \left\{ \int_{-\infty}^{\infty} z^{-s} \frac{\Delta_{TE}'^\nu(z)}{\Delta_{TE}^\nu(z)} \, dz \right\}, \tag{17}
\]

converges absolutely and uniformly to an analytic function in the open half-line \( s > 1 \), which can be meromorphically extended to the whole complex \( s \)-plane.

Expression (17) can also be written as

\[
\zeta_\nu(s) = -\frac{1}{\pi} \Re \left\{ \nu^{-s} e^{-\frac{\pi}{2} s} \int_0^\infty (y-i)^{-s} \frac{\Delta_{TE}'^\nu(\nu x(y-i))}{\Delta_{TE}^\nu(\nu x(y-i))} \, dy \right\}, \tag{18}
\]

where the prime means derivative with respect to the argument, and we have made use of the properties of the Bessel functions of complex argument.

Changing the integration variable to \( t \equiv z(y-i) \), with \( z = x/\nu > 0 \), we finally get

\[
\zeta_\nu(s) = -\frac{1}{\pi} \Re \left\{ \nu^{-s} e^{-\frac{\pi}{2} (s+1)} \int_{-iz}^{iz} t^{-s} \frac{d(\ln \Delta_{TE}^\nu(\nu t))}{dt} \, dt \right\}. \tag{19}
\]

Notice that the right hand side of Eq. (19) does not depend on \( z \) for \( z(>0) \) small enough.
In order to construct the analytic extension of the expression in Eq. (2) to 
$s \approx -1$, we subtract and add the first few terms of the asymptotic expansion of
the integrand in (19) (obtained from the uniform asymptotic Debye expansion
for the modified Bessel functions appearing in $\Delta^{TE}_\nu (\omega t)$). In fact, in order to
isolate the singularities of the Casimir energy it is sufficient to retain in this
expansion terms up to the order $\nu^{-3}$:

$$
\frac{d(\ln \Delta^{TE}_\nu (\omega t))}{dt} = D^{TE}_\nu (t) + \mathcal{O}(\nu^{-4}),
$$

with

$$
D^{TE}_\nu (t) = \nu D^{(1)}_{TE}(t) + D^{(0)}_{TE}(t) + \nu^{2} D^{(-1)}_{TE}(t) + \nu^{3} D^{(-2)}_{TE}(t) + \nu^{4} D^{(-3)}_{TE}(t),
$$

where $D^{(k)}_{TE}(t)$, $k = 1, ..., -3$, are algebraic functions of $t$ given in the Appendix.

Notice that we have also discarded terms which are exponentially vanishing for
$R \to \infty$.

So, we must consider the series

$$
\sum_{\nu=3/2}^{\infty} \nu \Re \left\{ -\frac{\nu^{-s}}{\pi} e^{-\frac{\pi}{2}(s+1)} \int_{-iz}^{iz} t^{-s} \frac{d(\ln \Delta^{TE}_\nu (\omega t))}{dt} \ dt \right\} =
$$

$$
- \sum_{\nu=3/2}^{\infty} \nu \Re \left\{ \frac{\nu^{-s}}{\pi} e^{-\frac{\pi}{2}(s+1)} \int_{-iz}^{iz} t^{-s} D^{TE}_\nu (t) \ dt \right\} -
$$

$$
\sum_{\nu=3/2}^{\infty} \nu \Re \left\{ \frac{\nu^{-s}}{\pi} e^{-\frac{\pi}{2}(s+1)} \int_{-iz}^{iz} t^{-s} \left\{ \frac{d(\ln \Delta^{TE}_\nu (\omega t))}{dt} - D^{TE}_\nu (t) \right\} \ dt \right\}.
$$

The second term in the right hand side of (22) converges for $s > -2$ by con-
struction. Therefore, we can put $s = -1$ inside the sum and the integral, and
numerically evaluate this contribution when necessary.

In order to investigate the ultraviolet divergencies in $E_{Cas}$ it is sufficient to
consider the first term in the right hand side of (22)

1 It should be stressed that the real part in the argument of the series in the first term in
the right hand side of Eq. (22) is in fact independent of $z$: Taking into account the analyticity
of the integrand, one can investigate the $z$-dependence by studying the integral

$$
\int_{-iz}^{iz} t^{-s} D^{TE}_\nu (t) \ dt,
$$

which can be exactly solved in terms of hypergeometric functions. It is straightforward to
verify that the $z$-dependent part is imaginary for all $s > 1$, and therefore is dropped out
when taking the real part in Eq. (22). This feature will be useful in what follows.
For $s > 1$ we can study each term in $D_{TE}^{(s)}(t)$ separately, and evaluate the expressions

$$
\sum_{\nu=3/2}^{\infty} \nu^{-(s-k-1)} \Im \left\{ \frac{-e^{-\frac{\pi}{2}(s+1)}}{\pi} \int_{-iz}^{iz} t^{-s} D_{TE}^{(k)}(t) \, dt \right\},
$$

(24)

with the index $k = 1, \ldots, -3$ corresponding to the order in the Debye expansion.

Since the real part of the expression between brackets in (24) is independent of $\nu$ (it is independent of $z = x/\nu$ - see footnote 1), the sum over $\nu$ can be performed by means of the Hurwitz zeta function,

$$
\sum_{\nu=3/2}^{\infty} \nu^{-(s-k-1)} = \zeta_H(s - k - 1, 1/2) - 2^{s-k-1}.
$$

(25)

Only for $k = -3$ one gets a singularity at $s = -1$ given by

$$
\zeta_H(s + 2, 1/2)_{|z=-1} = \frac{1}{s+1} + (\gamma + \ln(4)) + O(s + 1).
$$

(26)

For the other values of $k$ a regular extension to $s = -1$ is obtained.

Therefore, it is only for $k = -3$ that we need to calculate both finite and singular parts of the analytic extension of the integral in Eq. (24) around $s = -1$, while for the other values of $k$ only the singular terms are needed.

All these terms can be worked out exactly to give for the divergent part of the contribution of the TE modes to the Casimir energy (in the limit $R \to \infty$),

$$
E_{TE}(a) = \frac{\hbar c}{a} \left\{ -\frac{n_0^3 a^4 \Omega^4}{24 \pi c^4 (1 + s)} - \frac{n_0 a^2 \Omega^2}{4 \pi c^2 (1 + s)} + O(s + 1)^0 \right\}.
$$

(27)

Notice that the first term in the right hand side corresponds to a volume contribution, while the second one is a curvature contribution. In agreement with [22], no surface contribution appears.

For the TM modes one gets similarly

$$
E_{TM}(a) = \frac{\hbar c}{a} \left\{ -\frac{n_0^3 a^4 \Omega^4}{24 \pi c^4 (1 + s)} + \frac{n_0 a^2 \Omega^2}{4 \pi c^2 (1 + s)} + O(s + 1)^0 \right\}.
$$

(28)

Finally, adding up the contributions coming from the TE and TM modes, we get for the singular piece of the Casimir energy of the dielectric ball just a volume contribution,

$$
E(a) = \frac{\hbar c}{a} \left\{ -\frac{n_0^3 a^4 \Omega^4}{12 \pi c^4 (1 + s)} + O(s + 1)^0 \right\},
$$

(29)

since the divergent curvature terms have canceled out between TE and TM modes. Replacing the volume of the ball, $V = 4\pi a^3/3$, we find

$$
E(a) = -\frac{\hbar n_0^3 V \Omega^4}{16 \pi^2 c^3 (1 + s)} + O(s + 1)^0.
$$

(30)
Thus, we have found that the pole structure of the Casimir energy in the framework of the $\zeta$-function regularization scheme is very simple, once a realistic behavior of the dielectric medium at high frequencies is assumed. The pure volume divergence might be seen to represent a contribution to the mass density of the material and it is to be absorbed into a phenomenological counterterm, much in the way it is done in bag models. Neither a surface tension nor a curvature counterterm is needed in this model, since divergencies nicely cancel out between TE and TM modes.

Given this simple pole structure it makes sense to analyze further finite parts of the Casimir energy in a quantum field theory context for realistic media. As a first step one might assume the Drude model for the medium and analyze how the Casimir energy depends on e.g. the plasma and the relaxation frequency. More ambitiously, the dielectric constants might be obtained from tabulated refractive indices using the Kramer-Kronig relation. This is under consideration and will allow a detailed analysis of the Casimir energy for a spherically symmetric situation with realistic media.

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A Debye expansion for the TE and TM modes

The functions $D_{TE}^{(k)}(t)$, $k = 1, ..., -3$, coefficients of the expansion $D_{\nu}^{TE}(t)$ (see Eqs. (20) and (21)), which are obtained from the uniform asymptotic Debye expansion for the modified Bessel functions appearing in $\Delta_{\nu}^{TE}(u t)$, are given by

$$D_{TE}^{(3)}(t) = \frac{\sqrt{1 + \frac{n_0^2 R^2 t^2}{a^2}}}{t},$$

$$D_{TE}^{(0)}(t) = \left(2 t + \frac{2 n_0^2 R^2 t^3}{a^2}\right)^{-1},$$

$$D_{TE}^{(-1)}(t) = \frac{n_0^2 R^2 t \left(1 - \frac{n_0^2 R^2 t^2}{4 a^2}\right)}{2 a^2 \left(1 + \frac{n_0^2 R^2 t^2}{a^2}\right)^{3/2}} - \frac{\left(n_0^2 + \frac{2}{Z^2}\right) Z^2}{2 t \sqrt{1 + n_0^2 t^2}}.$$
\[ D^{(-2)}_{TE}(t) = \frac{a^6 n_0^2 R^2 t}{2(a^2 + n_0^2 R^2 t^2)^4} \left( \frac{5 n_0^2 t^2}{2} - \frac{n_0^4 R^4 t^4}{4 a^4} - \frac{6 n_0^2 R^2 Z^2}{a^2} \right) \]

\[ - \frac{a^2 Z^2}{n_0^2 R^2 t^4} \frac{4 Z^2}{t^2} - \frac{4 n_0^4 R^4 t^4 Z^2}{a^4} - \frac{n_0^6 R^6 t^4 Z^2}{a^6} - 1 \right) \]

\[ (34) \]

and

\[ D^{(-3)}_{TE}(t) = \frac{n_0^2 R^2 t \left(64 a^6 - 560 a^4 n_0^2 R^2 t^2 + 456 a^2 n_0^4 R^4 t^4 - 25 n_0^6 R^6 t^6\right) + }{128 \left(1 + \frac{n_0^2 R^2 t^2}{a^2}\right)^{1/2}} \]

\[ Z^2 \left(16 Z^2 + 56 n_0^2 t^2 Z^2 + 6 n_0^4 t^4 \left(t^2 + 12 Z^2\right) - 2 n_0^4 \left(\ell^2 - 35 t^4 Z^2\right)\right) \]

\[ 16 t^5 (1 + n_0^2 t^2)^2 \]

Similarly, for the TM modes the functions \( D^{(k)}_{TM}(t) \), \( k = 1, \ldots, -3 \), appearing in the expansion \( D^{(k)}_{TM}(t) \), are given by

\[ D^{(1)}_{TM}(t) = \frac{\sqrt{1 + \frac{n_0^2 R^2 t^2}{a^2}}}{t}, \]

\[ (36) \]

\[ D^{(0)}_{TM}(t) = \frac{-1}{2 \left(t + \frac{n_0^2 R^2 t^2}{a^2}\right)}, \]

\[ (37) \]

\[ D^{(-1)}_{TM}(t) = \frac{-\left(n_0^2 R^2 t \left(8 a^2 + n_0^2 R^2 t^2\right)\right)}{8 \left(a^2 + n_0^2 R^2 t^2\right)^2} \frac{2 + n_0^2 t^2}{t^3 \sqrt{1 + n_0^2 t^2}} Z^2 \]

\[ (38) \]

\[ D^{(-2)}_{TM}(t) = \frac{n_0^2 R^2 t \left(10 - \frac{10 n_0^2 R^2 t^2}{a^2} + \frac{n_0^4 R^4 t^4}{a^4}\right)}{8 a^2 \left(1 + \frac{n_0^2 R^2 t^2}{a^2}\right)^2} \frac{2 + n_0^2 t^2}{t^3} Z^2 \]

\[ (39) \]

and

\[ D^{(-3)}_{TM}(t) = \frac{-\left(n_0^2 R^2 t \left(176 a^6 - 784 a^4 n_0^2 R^2 t^2 + 480 a^2 n_0^4 R^4 t^4 - 23 n_0^6 R^6 t^6\right)\right) + }{128 a^6 \left(1 + \frac{2 a^2 R^2 t^2}{a^2}\right)^2} \]

\[ (40) \]

\[ \frac{c^2 \left(2 + n_0^2 t^2\right) \left(4 + 12 n_0^2 t^2 + 3 n_0^4 t^4\right) Z^2 + 2 \left(1 + n_0^2 t^2\right)^2 \left(8 + 12 n_0^2 t^2 + 3 n_0^4 t^4\right) Z^4}{16 t^5 (1 + n_0^2 t^2)^2} \]

\[ \right) \]

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