On the existence and linear approximation of the power flow solution in power distribution networks

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Abstract—We consider the problem of deriving an explicit approximate solution of the nonlinear power equations that describe a power distribution network. We give sufficient conditions for the existence of a practical solution to the power flow equations, and we propose an approximation that is linear in the active and reactive power demands and generations. For this approximation, which is valid for generic power line impedances and grid topology, we derive a bound on the approximation error as an explicit function of the grid parameters.

Index Terms—Power systems, power distribution networks, modeling, power flow analysis, fixed point theorem.

I. INTRODUCTION

The problem of solving the power flow equations that describe a power system, i.e. computing the steady state of the grid (typically the bus voltages) given the state of power generators and loads, belongs to the most classical tasks in circuit and power system theory. An analytical solution of the power flow equations is typically not available, given their nonlinear nature. For this reason, notable effort has been devoted to the design of numerical methods to solve systems of power flow equations, to be used both in offline analysis of a grid and in real time monitoring and control of the system (see for example the review in [1]).

Some tools have been specifically derived for the approximate solution of such equations, based on some assumptions on the grid parameters. In particular, if the power lines are mostly inductive, equations relating active power flows and bus voltage angles result to be approximatively linear, and decoupled from the reactive power flow equations, resulting in the DC power flow model (see the review in [2], and the more recent discussion in [3]).

We focus here on a specific scenario, which is the power distribution grid. More specifically, we are considering a medium voltage grid which is connected to the power transmission grid in one point (the distribution substation, or PCC, point of common coupling), and which hosts loads and possibly also microgenerators.

The power distribution grid has recently been the object of an unprecedented interest. Its operation has become more challenging since the deployment of distributed microgeneration and the appearance of larger constant-power loads (electric vehicles in particular). These challenges motivated the deployment of ICT in the power distribution grid, in the form of sensing, communication, and control devices, in order to operate the grid more efficiently, safely, reliably, and within its voltage and power constraints. These applications have been reviewed in [4], and include real-time feedback control [5]–[7], automatic reconfiguration [8], [9], and load scheduling [10], [11]. In order to design the control and optimization algorithms for these application, an analytical (rather than numerical) solution of the power flow equation would be extremely convenient. Unfortunately, because in the medium voltage grid the power lines are not purely inductive, power flow equations include both the active and reactive power injection/demands, and both the voltage angles and magnitudes, in an entangled way. The explicit DC power flow model therefore does not apply well.

In this paper we derive a tractable approximate solution to the power flow problem, linear in the complex power injections. We give sufficient conditions for the existence of a practical solution of the original nonlinear power flow equations, and we derive a bound on the approximation error introduced by the linear model.

In the remainder of this section, we review relevant related works. In Section II, we present the nonlinear equations that define the power flow problem. In Section III we present our main existence result, together with the linear approximate solution. We illustrate such approximation via simulations in Section IV, and we discuss these results in Section V.

A. Related works

Conditions have been derived in order to guarantee the existence of a solution to the power flow equations in the scenario of a grid of nonlinear loads.

Many results are based on the degree theory [12]–[14]. In [15], for example, exponential model is adopted for the loads, and sufficient conditions for the existence of a solution are derived. These conditions are however quite restrictive, and do not include constant power (PQ) buses. In [16], on the other hand, the existence of a solution is proved by exploiting the radial structure of the grid, via an iterative procedure which is closely related to a class of iterative numerical methods specialized for the power distribution networks [17], [18].

The existence of solutions to the power flow equations has been also studied in order to characterize the security region of a grid, i.e. the set of power injections and demands that yield acceptable voltages across the network. These results include [19], and others where however the decoupling between active
and reactive power flows is assumed [20]. Other works in which the DC power flow assumption plays a key role are [21] and [22], both focused on active power flows across the grid. On the other hand, the results in [23] focus on the reactive power flows and on the voltage magnitudes at the buses.

In [24] the implicit function theorem is used in order to advocate the existence of a solution to the power flow equations, without providing an approximate expression for that.

It is worth noticing that the linear approximate model that we are presenting in this paper shares some similarities with the method of power distribution factors [25], which allow to express variations in the state (voltage angles) as a linear function of active power perturbations. This method is also typically based on the DC power flow assumptions, even if a formulation in rectangular coordinates (therefore modeling reactive power flows) has been proposed in [26]. Notice that, except for the seminal works on this method [27], [28], and the more recent results in [29], most of the related results consists in algorithms that allow to compute this factors numerically, from the Jacobian of the power flow equations.

The approximate power flow solution proposed in this paper has been presented in a preliminary form in [6] (and extended to include multiple voltage regulated buses in [30]), where however no guarantees on the existence of such solution and on the quality of the approximation were given.

II. POWER FLOW EQUATIONS

We are considering a portion of a symmetric and balanced power distribution network, connected to the grid at one point, delivering power to a number of buses, each one hosting loads and possibly also microgenerators.

We denote by \( \mathcal{V} = \{0, 1, \ldots, n\} \) the set of buses, where the index 0 refers to the PCC.

We limit our study to the steady state behavior of the system, when all voltages and currents are sinusoidal signals at the same frequency. Each signal can therefore be represented via a complex number \( y = |y|e^{j\phi} \) whose absolute value \( |y| \) corresponds to the signal root-mean-square value, and whose phase \( \phi \) corresponds to the phase of the signal with respect to an arbitrary global reference.

In this notation, the steady state of the network is described by the voltage \( v_h \in \mathbb{C} \) and by the injected current \( i_h \in \mathbb{C} \) at each node \( h \in \mathcal{V} \). We define the vectors \( v, i \in \mathbb{C}^n \), with entries \( v_h \) and \( i_h \), respectively.

Each bus \( h \) of the network is then characterized by a law relating its injected current \( i_h \) with its voltage \( v_h \). We model bus 0 as a slack node, in which a voltage is imposed

\[
v_0 = V_0 e^{i\theta}.
\]

where \( V_0, \theta \in \mathbb{R} \) are such that \( V_0 \geq 0 \) and \( -\pi < \theta \leq \pi \). We model all the other nodes as PQ buses, in which the injected complex power (active and reactive powers) is imposed and does not depend on the bus voltage. This model describes the steady state of most loads, and also the behavior of microgenerators, that are typically connected to the grid via power inverters [31]. According to the PQ model, we have that, at every bus,

\[
v_h i_h = s_h \quad \forall h \in \mathcal{L} := \{1, \ldots, n\},
\]

where \( s_h \) is the imposed complex power.

A more compact way to write these nonlinear power flow equations is the following. Let the vectors \( i_{\mathcal{L}}, v_{\mathcal{L}}, s_{\mathcal{L}} \) be vectors having \( i_h, v_h, s_h, h \in \mathcal{L} \) as entries. Then we have

\[
\begin{cases}
  v_0 = V_0 e^{i\theta} \\
  s_{\mathcal{L}} = \text{diag}(i_{\mathcal{L}}) v_{\mathcal{L}}
\end{cases}
\]

where \( i_{\mathcal{L}} \) is the vector whose entries are the complex conjugate of the entries of \( i_{\mathcal{L}} \) and where diag(\cdot) denotes a diagonal matrix having the entries of the vector as diagonal elements.

We model the grid power lines as series impedances, neglecting their shunt admittance. While power distribution networks are typically radial, that is not a necessary assumption for the results in this paper. We therefore have a linear relation between bus voltages and currents, in the form

\[
i = Y v
\]

where \( Y \in \mathbb{C}^{n \times n} \) is the admittance matrix of the grid which satisfies

\[
Y 1 = 0,
\]

where 1 is the vector of all ones. The matrix \( Y \) results to be the weighted Laplacian of the graph describing the grid, with edge weights equal to the admittance of the corresponding power lines.

Considering the same partitioning of the vectors \( i, v \) as before, we can partition the admittance matrix \( Y \) accordingly and rewrite (4) as

\[
\begin{bmatrix}
  i_0 \\
  i_{\mathcal{L}}
\end{bmatrix} =
\begin{bmatrix}
  Y_{00} & Y_{0\mathcal{L}} \\
  Y_{\mathcal{L}0} & Y_{\mathcal{L}\mathcal{L}}
\end{bmatrix}
\begin{bmatrix}
  v_0 \\
  v_{\mathcal{L}}
\end{bmatrix}.
\]

Using (5) we then obtain

\[
v_{\mathcal{L}} = 1 v_0 + Z i_{\mathcal{L}}
\]

where we introduce the impedance matrix \( Z := Y_{\mathcal{L}\mathcal{L}}^{-1} \in \mathbb{C}^{n \times \mathcal{L}} \).

Objective of the power flow analysis is to determine from these equations the voltages \( v_h \) and the currents \( i_h \) as functions of \( V_0, \theta \) and \( s_1, \ldots, s_n \), namely

\[
\begin{aligned}
v_h &= v_0(V_0, \theta, s_1, \ldots, s_n) \\
i_h &= i_0(V_0, \theta, s_1, \ldots, s_n)
\end{aligned}
\]

In general, because of the nonlinear nature of the loads, we may have no solution or more than one solution for fixed \( V_0, \theta \) and \( s_h \), as the following simple example shows.

Example (Two-bus case). Consider the simplest grid made by two nodes, node 0 being the slack bus (where we let \( \theta = 0 \)), and node 1 being a PQ bus. In this case we have that the following equations have to be satisfied

\[
\begin{cases}
  v_1 i_1 = s_1 \\
  v_1 = V_0 + Z_{11} i_1
\end{cases}
\]

Assume that \( Z_{11} = 1 \), and that \( s_1 \) is real. The system of equations (7) can then be solved analytically. In fact it can be
found that if \( V_0^2 + 4s_1 < 0 \) there are no solutions. When on the contrary \( V_0^2 + 4s_1 > 0 \), there are two solutions
\[
i_1 = -\frac{V_0 \pm \sqrt{V_0^2 + 4s_1}}{2}.
\]

Notice that, if \( V_0 \) is large, then the solutions exist and, since \( \sqrt{V_0^2 + 4s_1} = V_0\sqrt{1 + 4s_1/V_0^2} \approx V_0(1 + 2s_1/V_0^2) = V_0 + 2s_1/V_0 \), the current \( i_1 \) take the two values
\[
i_1^+ \approx s/V_0, \quad i_1^- \approx -V_0 - \frac{s}{V_0}.
\]

Therefore, when \( V_0 \) is large, one solution consists in small currents (thus small power losses and voltage close to the nominal voltage across all the network), while the other consists in larger currents, larger power losses, and larger voltage drops. Of course, the system should be controlled so that it works at the first working point.


The intuition from this simple example is developed in the next section, where the existence and uniqueness of a practical solution to the power flow equations is studied, and an approximate power flow solution (linear in the power terms) is proposed.

## III. MAIN RESULT

Define
\[
f := v_0\ddot{v} - s = V_0 e^{j\theta} \ddot{v} - s,
\]
so that we have
\[
i = \frac{1}{i_0}(\ddot{f} + \ddot{s}) = \frac{e^{j\theta}}{V_0}(\ddot{f} + \ddot{s}).
\]

By putting together (8) with (3) and (6), we get
\[
s = \text{diag}(i) v,
\]
and therefore
\[
f = -\frac{1}{V_0^2} \text{diag}(f + s) Z(\ddot{f} + \ddot{s}).
\]

We can determine a ball where there exists a unique solution \( f \) to this equation by applying the Banach fixed point theorem. In order to do so, define the function
\[
G(f) := -\frac{1}{V_0^2} \text{diag}(f + s) Z(\ddot{f} + \ddot{s}).
\]

Consider the standard 2-norm \( \| \cdot \| \) on \( \mathbb{C}^L \) defined as
\[
\|x\| := \sqrt{\sum_k |x_k|^2}.
\]

Let us then define the following matrix norm\(^1\) on \( \mathbb{C}^{L \times L} \)
\[
\|A\|^* := \max_h \|A_{h*}\| = \max_h \sqrt{\sum_k |A_{hk}|^2} \quad (10)
\]

where the notation \( A_{hk} \) stands for the \( h \)-th row of \( A \).

The following result holds.

**Theorem 1.** Consider the vector 2-norm \( \| \cdot \| \) on \( \mathbb{C}^L \), and the matrix norm \( \| \cdot \|^* \) defined in (10). If
\[
V_0^2 > 4\|Z\|^* s
\]
then there exists a unique solution \( v \) of the equations (3) and (6) in the form
\[
v = V_0 e^{j\theta} \left( 1 + \frac{1}{V_0^2} Z \ddot{s} + \frac{1}{V_0^2} Z \lambda \right)
\]
where \( \lambda \) is such that
\[
\|\lambda\| \leq 4\|Z\|^* s^2.
\]

**Proof:** Let
\[
\delta := \frac{4\|Z\|^*}{V_0^2} s^2
\]
and \( B = \{ f \in \mathbb{C}^L \ | \ \|f\| \leq \delta \} \). We want to show first that, under the hypotheses of the theorem,
\[
G(f) \in B \text{ for all } f \in B \quad (14)
\]
and
\[
\|G(f') - G(f'')\| \leq k\|f' - f''\| \text{ for all } f', f'' \in B \quad (15)
\]
for a suitable constant \( 0 < k < 1 \). We prove first (14). Observe that, by using Lemma A.1, we have
\[
\|G(f)\| \leq \frac{1}{V_0^2} \|Z\|^* \|f + s\|^2 \\
\leq \frac{1}{V_0^2 N} \|Z\|^* (\|f\| + s|^2)^2 \\
\leq \frac{1}{V_0^2} \|Z\|^* (\|f\| + s|^2)^2,
\]
where we used the fact that \( \|f\| \leq \delta \). Now, using the definition of \( \delta \) and Lemma A.5 in the Appendix (with \( a = \|Z\|^* / V_0^2 \) and \( b = s / \|s\| \)) we can argue that, if (11) is true, then
\[
\|G(f)\| \leq \frac{1}{V_0^2} \|Z\|^* (\|f\| + s|^2)^2 \leq \delta.
\]

We prove now (15). It is enough to notice that, by applying Lemma A.4 in the Appendix (with \( A = -Z/V_0^2, x = f \) and \( a = s / \|s\| \)) we obtain that
\[
\|G(f') - G(f'')\| \\
\leq \frac{1}{V_0^2} \|Z\|^* (\|f' + f''\| + 2sL)|f' - f''| \\
\leq \frac{2}{V_0^2} \|Z\|^* (\|f' + f''\| + 2sL) |f' - f''| \\
= \frac{2}{V_0^2} \|Z\|^* \left( \frac{4\|Z\|^* sL^2 + 2sL} {V_0^2} \right) |f' - f''| \]
\[
= k |f' - f''|
\]
\(^1\)The * sign indicates that we are not referring to the norm induced by the vector norm.
where
\[ k := \frac{2}{V_0^2} \|Z\| \left( \frac{4\|Z\| \|s_L\|^2 + \|s_L\|}{V_0^2} \right). \]
Finally, notice that by using (11) we obtain
\[ k < \frac{2}{V_0^2} \|Z\|^* \frac{V_0^2}{4\|Z\|^*} \left( \frac{\|Z\|^*}{V_0^2} \frac{V_0^2}{4\|Z\|^*} + 1 \right) = 1. \]
Now from (14) and (15), by applying the Banach fixed point theorem, we can argue that there exists unique solution \( f \in B \) of equation (9). Then by using (8) we have that
\[ v_L = 1 V_0 e^{j\theta} + Z_1 \]
\[ = 1 V_0 e^{j\theta} + Z L (s_L + \bar{f}) \]
\[ = V_0 e^{j\theta} \left( 1 + \frac{1}{V_0^2} Z s_L + \frac{1}{V_0^2} Z \bar{f} \right). \]
In order to prove (13) it is enough to define \( \lambda := V_0^2 \bar{f}. \)

**Remark.** The norm \( \|Z\|^* \) can be put in direct relation with the induced matrix 2-norm \( \|Z\| \), and with structural properties of the graph that describes the power grid. Indeed
\[ \|Z\|^* = \max_h \|e_h^T Z\| \leq \max_h \|e_h^T v\| = \|Z\|, \]
where \( e_h \) is the \( h \)-th vector of the canonical base. It can be shown that \( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \) is one possible pseudoinverse of the weighted Laplacian \( Y \) of the grid. Therefore we have that
\[ \|Z\|^* \leq \frac{1}{\sigma_2(Y)}, \]
where \( \sigma_2(Y) \) is the second smallest singular value of \( Y \) (the smallest one being zero).

In the special case in which all the power lines have the same \( X/R \) ratio (i.e., their impedances have the same angle but different magnitudes), then it is easy to show that \( \sigma_2(Y) \) corresponds also to the second smallest eigenvalue of the Laplacian, which is a well known metric for the graph connectivity.

Given this relation between \( \|Z\|^* \) and \( \|Z\| \), the assumption (11) in Theorem 1 is satisfied if
\[ V_0^2 > 4 \|s_L\| \frac{\|s_L\|}{\sigma_2(Y)}. \]
and the bound (13) on the error can be replaced by
\[ \|\lambda\| \leq 4 \|s_L\|^2 \frac{\|s_L\|}{\sigma_2(Y)}. \]

Theorem 1 provides a sufficient condition ensuring that there exists a neighborhood in the state space where a unique solution to the power flow nonlinear equations exists. This allows to derive an approximate solution for the power flow equations that is conveniently linear in the power injection terms, as the following corollary shows.

**Corollary.** Consider the vector 2-norm \( \| \cdot \| \) on \( C^L \), and the matrix norm \( \| \cdot \|^* \) defined in (10). Let the assumption of Theorem 1 be satisfied. Then the solution \( v_L \) of the power flow nonlinear equations is approximated by
\[ \hat{v}_L := V_0 e^{j\theta} \left( 1 + \frac{1}{V_0^2} Z s_L \right), \]
and the approximation error satisfies
\[ |v_L - \hat{v}_L| \leq \frac{4}{V_0^4} \|Z_{h\bullet}\| \|Z\|^* \|s_L\|^2, \]
where, as before, \( Z_{h\bullet} \) is the \( h \)-th row of \( Z \).

Proof: We have, from (12), for any bus \( h \in L \),
\[ |v_L - \hat{v}_L| = \frac{1}{V_0^2} |Z_{h\bullet}| \|Z\|^* \|s_L\|^2, \]
By using Cauchy-Schwarz inequality and the bound (13) we obtain (17).

The result of Theorem 1 holds also for other vector norms, different from the vector 2-norm, as the following remarks show.

**Remark (1-norm).** Consider the vector 1-norm
\[ \|x\|_1 := \sum_h |x_h|. \]
We have that Theorem 1 holds with the matrix norm
\[ \|Z\|_1 := \max_{h,k} |Z_{hk}|, \]
and this can be proved by using Lemma A.2 instead of Lemma A.1.

The sufficient condition (11) becomes
\[ V_0^2 > 4 \|Z\|_1 \|s_L\|_1 \]
and has in this case a physical interpretation. It can be inferred from (6) that, as power line impedances have positive resistance and reactance,
\[ |X_{hk}| \geq |X_{kk}| \quad \text{for all } h, k \in L. \]
Therefore \( \|Z\|_1^* = \max_h |Z_{hk}| \) corresponds to the breadth of the grid, defined as the length of the longest path (in terms of the magnitude of the path impedance) connecting a bus \( h \) to bus 0. On the other hand, \( \|s_L\|_1 \) corresponds to a metric for the total load of the grid, defined as the sum of the apparent powers \( |s_h| \) at every node \( h \in \mathcal{L} \).

As before, the statement of Theorem 1 allows to derive a bound on the approximation error (this time using Hölder's inequality), obtaining in this case
\[ |v_L - \hat{v}_L| \leq \frac{4}{V_0^4} |Z_{h\bullet}| \|Z\|_1^* \|s_L\|^2, \]

**Remark (∞-norm).** Similarly, if the \( \infty \)-norm
\[ \|x\|_\infty := \max_h |x_h| \]
is adopted, Theorem 1 holds with the matrix norm
\[ \|Z\|_\infty := \max_{h,k} |Z_{hk}|_1, \]
and this can be proved by using Lemma A.3 instead of Lemma A.1. The bound (17) becomes
\[ |v_L - \hat{v}_L| \leq \frac{4}{V_0^4} |Z_{h\bullet}|_1 \|Z\|_\infty \|s_L\|^2. \]
In general, Theorem 1 gives only sufficient conditions for the existence and uniqueness of a solution of the nonlinear power flow equations, and a bound on the error of the approximation (16). By revisiting the example proposed in Section II we show that both the sufficient condition (11) and the bound (13) can be tight.

**Example** (Two-bus case). Consider the same example considered in Section II. Notice that, regardless of the vector norm that is chosen,

\[ ||Z||_1 = |Z_{11}| = 1 \quad and \quad ||s|| = |s_1|. \]

The sufficient condition (11) in Theorem 1 corresponds then to

\[ V_0^2 \geq 4|s_1| \]

which is a necessary condition for the existence of a solution, in the case that we were considering in the example, with \( s_1 \) real and negative (corresponding to an active power load). In the special case in which the condition is marginally satisfied (\( s_1 = -V_0^2/4 \)), it is possible to compute both the exact solution

\[ v_1 = V_0 + i_1 = \frac{1}{2} V_0, \]

and the approximate solution according to (16),

\[ \hat{v}_1 = V_0 \left( 1 + \frac{1}{V_0^2} s_1 \right) = \frac{3}{2} V_0. \]

The approximation error is clearly \( V_0/4 \), corresponding to

\[ \lambda_1 = V_0^3 (v_1 - \hat{v}_1) = -\frac{1}{4} V_0^4, \]

which marginally satisfies the approximation error bound (16), i.e.

\[ ||\lambda|| \leq 4|s_1|^2 = \frac{1}{4} V_0^4. \]

**IV. Simulations**

In order to illustrate the results presented in the previous section, we considered a testbed inspired to the IEEE 123 test feeder [32], in which we assumed that all power lines are symmetric and all loads are balanced.

In a per-unit notation, where \( V_0 \) is taken as the reference nominal voltage, and \( S_N = 1 \)MW is taken as the reference nominal power, we have that

\[ ||s|| = 0.4058 \quad ||Z||_1 = 0.2264 \]
\[ ||s||_1 = 3.3937 \quad ||Z||_1^* = 0.0669, \]

and

\[ ||s||_\infty = 0.1320 \quad ||Z||_\infty^* = 2.0158. \]

For all the considered norms, the sufficient condition of Theorem 1 is verified, and therefore existence and uniqueness of the solution of the power flow equations is guaranteed.

In Figure 1 we reported both the true solution of the original nonlinear power flow equations (obtained numerically) and the approximate linear solution (16). We also reported the bound (17) on the approximation error in case of the 2-norm. Clearly, in this testbed, the approximation error is much smaller than this bound. Further simulative analysis showed that this effect is mainly due to the fact that the load is well distributed across the network in this testbed. It is interesting to notice that Theorem 1 also guarantees the uniqueness of the solution to the power flow equations in this neighborhood (while other uninteresting solutions could exists, with unacceptable voltage levels).

We then considered a different scenario (plotted in Figure 2), where the total grid load has been concentrated at bus 114, which is the furthest bus (in terms of power line impedance) from the PCC. In this case the network is not being operated properly, as a severe undervoltage condition happens around bus 114. Under these conditions, the approximate model (16) introduces a larger error, even if it still describes the voltage profile across the grid in a meaningful way. In Figure 2 we also plotted the bound (19) on the approximation error, using this time the vector 1-norm.

**V. Conclusion**

We considered a power distribution network, that we modeled as a grid of constant power (PQ) nodes and one slack node (the PCC). For this model, we derived sufficient conditions for the existence of a practical solution to the nonlinear power flow equations, and we proposed a linear approximated solution whose approximation error can be effectively bounded.

As discussed in the Introduction, the proposed linear approximated solution has the potential of serving as a linear model for the design of control, monitoring, and estimation strategies for the power distribution grid.

The applicability of the results presented in this paper can potentially be extended to more general models for the grid, as the following remarks show.

The fact that we modeled the PCC as an ideal voltage generator is a minor assumption. In the case in which the voltage at the PCC is not regulated, and thus it depends on the total demand of the power distribution grid, one can just add an auxiliary slack node and edge to represent a Thevenin equivalent of the transmission grid, and model the PCC as a PQ bus with zero power injection. The same results would follow seamlessly.

It has been shown in [6] how the same approximate power flow solution proposed in this paper (but not the existence result, in the form of Theorem 1) can be also used in the case of exponential loads, where power demands depends exponentially on the bus voltage magnitude (therefore including constant power, constant impedance, and constant current loads).

Finally, the case of unbalanced loads on a symmetrical power distribution grid can also be cast into the same framework proposed in this paper, via a change of coordinates, namely via the tool of symmetrical components [33, ch. 11].

**Appendix**

Given a vector norm \( || \cdot || \), the proof of Theorem 1 relies on the existence of a constant \( \rho \), that is a function of \( A \), such that

\[ ||\text{diag}(x)Ay|| \leq \rho||x||||y||. \]
In the following lemmas we consider three possible vector norms (the 2-norm, the 1-norm, and the \( \infty \)-norm), and for each of them we define a matricial norm \( \| \cdot \| \) for which the inequality is satisfied. In all three cases, the inequality is indeed tight, i.e. there exists two vectors \( x \) and \( y \) for which \( \| \text{diag}(x)Ay \| = \| A \|^* \| x \| \| y \| \).

**Lemma A.1** (2-norm). Let \( \| x \| \) be the 2-norm of a complex vector, and \( \| A \|^* \) be the matrix norm defined in (10). We have that

\[
\max_{\| x \| = 1} \| \text{diag}(x)Ay \| = \| A \|^*.
\]

**Proof:** We first prove one direction. We have that, if \( z := \text{diag}(x)Ay \), then

\[
|z_h|^2 = |x_h| \sum_k A_{hk}y_k^2 \leq |x_h|^2 \left( \sum_k |A_{hk}|^2 \right) \left( \sum_k |y_k|^2 \right) \leq |x_h|^2 \| A \|^* \sum_k |y_k|^2.
\]

Hence

\[
\| z \|^2 = \sum_h |z_h|^2 \leq \sum_h |x_h|^2 \| A \|^2 \sum_k |y_k|^2 = \| A \|^* \| x_h \| \sum_k |y_k|^2.
\]

On the other hand, we have that

\[
\max_{\| x \| = 1} \| \text{diag}(x)Ay \| \geq \max_{\| y \| = 1} \| \text{diag}(e_h)Ay \| = \max_h \| A_{h*} \| = \max_h \sqrt{\sum_k |A_{hk}|^2}.
\]

**Lemma A.2** (1-norm). Let \( \| x \|_1 \) be the 1-norm of a complex vector, and \( \| A \|^* \) be the matrix norm defined in (18). We have that

\[
\max_{\| x \|_1 = 1} \| \text{diag}(x)Ay \|_1 = \| A \|^*.
\]

**Proof:** We first prove one direction. We have that, if \( z := \text{diag}(x)Ay \), then

\[
|z_h| = |x_h| \sum_k A_{hk}y_k \leq |x_h| \sum_k |A_{hk}| |y_k| \leq |x_h| \| A \|^*_1 \sum_k |y_k| = |x_h| \| A \|^*_1 \| y \|_1.
\]

Hence

\[
\| z \|_1 = \sum_h |z_h| \leq \sum_h |x_h| \| A \|^*_1 \| y \|_1 = \| A \|^*_1 \| x_h \| \| y \|_1.
\]
On the other hand, we have that

\[
\max_{\|x\|_1 = 1, \|y\|_1 = 1} \| \text{diag}(x)Ay \|_1 \geq \max_h \max_{\|y\| \leq 1} \| \text{diag}(e_h)Ay \|_1
\]

\[
= \max_h \| e_h^T A \|_\infty = \max_{hk} |A_{hk}|.
\]

\[\square\]

**Lemma A.3** (\(\infty\)-norm). Let \(\|x\|_\infty\) be the \(\infty\)-norm of a complex vector, and \(\|A\|_\infty\) be the matrix norm defined in (20). We have that

\[
\max_{\|x\|_\infty = 1, \|y\|_\infty = 1} \| \text{diag}(x)Ay \|_\infty = \|A\|_\infty^*.
\]

**Proof:** We first prove one direction. We have that, if \(z := \text{diag}(x)Ay\), then

\[
|zh| = |zh| \sum_k A_{hk} y_k \leq \|x\|_\infty \|y\|_\infty \sum_k |A_{hk}|
\]

\[
\leq \|x\|_\infty \|y\|_\infty \|A\|_\infty^*
\]

which implies that \(\| \text{diag}(x)Ay \|_\infty \leq \|x\|_\infty \|y\|_\infty \|A\|_\infty^*\). On the other hand, let \(h\) such that \(\sum_k |A_{hk}| = \|A\|_\infty^*\) and let \(x = 1\) and \(y_k = \text{sign}(A_{hk})\) for all \(k\). Then

\[
\max_{\|x\|_\infty = 1, \|y\|_\infty = 1} \| \text{diag}(x)Ay \|_\infty \geq \| \text{diag}(x')Ay' \|_\infty = \|Ay\|_\infty
\]

\[\square\]

**Lemma A.4.** Assume that we have a vector norm \(\| \cdot \|\) and a matrix norm \(\| \cdot \|^*\) such that \(\| \text{diag}(x)Ay \| \leq \|A\|^* \|x\| \|y\|\) for any \(x, y\). If we define the function

\[
F(x) := \text{diag}(x + a)A(x + a)
\]

then

\[
\|F(x_1) - F(x_2)\| \leq \|A\|^* (\|x_1 + x_2\| + 2\|a\|) \|x_1 - x_2\|.
\]

**Proof:** First observe that

\[
\|F(x_1) - F(x_2)\| = \| \text{diag}(x_1)A\overline{x}_1 + \text{diag}(x_2)A\overline{x}_2 + \text{diag}(a)A\overline{x}_1 - \text{diag}(a)A\overline{x}_2 \|
\]

\[
\leq \| \text{diag}(x_1)A\overline{x}_1 - \text{diag}(x_2)A\overline{x}_2 \| + \| \text{diag}(x_1 - x_2)A\overline{x}_1 \| + \| \text{diag}(a)A(\overline{x}_1 - \overline{x}_2) \|.
\]

Notice that

\[
\text{diag}(x_1)A\overline{x}_1 - \text{diag}(x_2)A\overline{x}_2
\]

\[
= \frac{1}{2} (\text{diag}(x_1 - x_2)A(\overline{x}_1 + \overline{x}_2) + \text{diag}(x_1 + x_2)A(\overline{x}_1 - \overline{x}_2))
\]

and so

\[
\| \text{diag}(x_1)A\overline{x}_1 - \text{diag}(x_2)A\overline{x}_2 \| \leq \|A\|^* \|x_1 - x_2\| \|x_1 + x_2\|.
\]

From this we can argue that

\[
\|F(x_1) - F(x_2)\| \leq \|A\|^* \|x_1 - x_2\| (\|x_1 + x_2\| + 2\|a\|) \|x_1 - x_2\|
\]

\[
= \|A\|^* (\|x_1 + x_2\| + 2\|a\|) \|x_1 - x_2\|.
\]

\[\square\]

**Lemma A.5.** Let \(x, a, b \geq 0\) such that \(ab \leq 1/4\). Then \(x = 4ab^2\) satisfies

\[
a(x + b)^2 \leq x
\]

(22)

**Proof:** To prove the lemma it is enough to substitute \(x = 4ab^2\) in (22) and to verify that the inequality holds. Doing this we need to verify that

\[
4ab^2 \geq a(4ab^2 + b^2) = ab^2(4ab + b)^2
\]

This inequality holds if and only if \(4 \geq (4ab + 1)^2\), which is true since by hypothesis we have that \(0 \leq a \leq 1\).

\[\square\]

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