SYMMETRY OF POSITIVE SOLUTIONS TO FRACTIONAL EQUATIONS IN BOUNDED DOMAINS AND UNBOUNDED CYLINDERS

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(Communicated by Wenxiong Chen)

Abstract. The aim of this paper is to study symmetry and monotonicity for positive solutions to fractional equations. We first consider the following problems in bounded domains in the sense of distributions

\[\begin{cases}
(-\Delta)^s u = \frac{g(u)}{|x|^{2s}} + f(x, u) & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}\]

where \(n > 2s, 0 < s < 1\). We prove that all positive solutions are radically symmetric about the origin. Compare to results in [1], we use a completely different method under the weaker conditions in \(f\). Next we consider a problem in infinite cylinders. We establish the symmetry and monotonicity of positive solutions by using the method of moving planes. This result can be seen as the nonlocal counterparts of [9].

1. Introduction. In this paper we study qualitative properties of positive solutions for nonlinear equations involving fractional Laplacian in bounded domains and unbounded cylinders. First, we obtain the symmetry and monotonicity for positive solutions to the following nonlocal problems

\[\begin{cases}
(-\Delta)^s u = \frac{g(u)}{|x|^{2s}} + f(x, u) & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}\] (1.1)

where \(\Omega\) is bounded domain, \(n > 2s, 0 < s < 1\). \((-\Delta)^s\) is the fractional Laplacian, which is defined as

\[\begin{align*}
(-\Delta)^s u &= C_{n,s} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x - y|^{n+2s}}\,dy = C_{n,s} \text{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}}\,dy, 
\end{align*}\] (1.2)

where \(C_{n,s}\) is a normalizing positive constant and \(\text{PV}\) stands for the Cauchy principal value. In order to ensure the integral (1.2) is well defined, we assume that

2010 Mathematics Subject Classification. Primary: 35R11; Secondary: 35B09.

Key words and phrases. Fractional Laplacian, the method of moving plane, radially symmetry, monotonicity of solutions, convex domain.
u ∈ C^{1,1}_{loc}(Ω) \cap L_{2s}(\mathbb{R}^n) with
\[ L_{2s} = \left\{ u \in L^1_{loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2s}} dx < \infty \right\}. \]

Then, for the case \( g(u) = 0 \) in (1.1), we establish the symmetry of positive solutions in unbounded cylinders \( \Omega \).

When exploiting the method of moving planes in nonlocal problems, many difficulties arise due to non-locality of the fractional Laplacian. To overcome these difficulties, instead of using the conventional extension method introduced by Caffarelli and Silvestre [6], Chen-Li-Li [10] introduced a direct method of moving planes for the fractional Laplacian, which allows us to study the monotonicity and symmetry of positive solutions of various nonlocal problems. For more articles concerning the moving plane methods involving the nonlocal problems, see [7, 9, 11, 12, 13] and the references therein.

Due to the presence of the Hardy Leray potential in the right hand side of (1.1), the solutions may be unbounded at the origin. To overcome this difficulty, Barrios, Montoro and Sciunzi [1] introduced a weak comparison principle in a small domain, which allows them to move the plane. Furthermore, the strong comparison principle they use in [1] is the other important tool from [15]. Later, in [13] the strong comparison principle was refined under the additional assumptions that \( f(x,t) \) and \( g(t) \) are nondecreasing with respect to \( t \).

In last decades, there have been a large number of results about the symmetry and monotonicity of positive solutions to semilinear elliptic equations in unbounded domains. For example, the authors in [2, 3, 4, 5] obtained symmetry and monotonicity for positive solutions \( u \) of equation
\[-\Delta u = f(u) \quad \text{in } \Omega.\]

Berestycki, Caffarelli and Nirenberg [2] proved that in the half upper space, a bounded solution is a function of \( x_n \) alone and is increasing with respect to \( x_n \). Later, in [4] they considered the domain
\[ \Omega = \{ x = (x_1, ..., x_n) \in \mathbb{R}^n; x_n > \varphi(x_1, ..., x_{n-1}) \}, \]
where \( \varphi : \mathbb{R}^{n-1} \to \mathbb{R} \) is a globally Lipschitz continuous function. Under certain assumptions on \( f \), the authors proved that all positive solutions \( u \) are strictly increasing with respect to \( x_n \) in \( \Omega \). In particular, the domain \( \Omega \) is the half upper space when \( \varphi(x_1, ..., x_{n-1}) = 0 \). In [3], the authors continued this program by considering another type of domains, in unbounded cylinders
\[ \Omega = \mathbb{R}^{n-j} \times H, \]
where \( H \) is a smooth bounded domain in \( \mathbb{R}^j \). In contrast, not much is known for fractional equations. In this context, we are interested in the symmetry and monotonicity of positive solutions of the problem
\[ \begin{cases} (-\Delta)^s u = f(u), & u > 0 \quad \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^n \setminus \Omega, \end{cases} \]
where \( \Omega = \mathbb{R}^{n-j} \times H \) and \( H \) is a smooth bounded domain in \( \mathbb{R}^j \). We denote by \( x = (x_1, ..., x_{n-j}) \) the coordinates in \( \mathbb{R}^{n-j} \), and by \( y = (y_1, ..., y_j) \) the coordinates in \( H \).

Now we state our main results.
Theorem 1.1. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), convex in the \( x_1 \) direction and symmetric with respect to the plane \( \Gamma_0 = \{ x \in \mathbb{R}^n | x_1 = 0 \} \). Let \( u \) be a solution of (1.1). Assume that

\begin{enumerate}
\item[(H1)] \( f(x, t) \) and \( g(t) \) are locally Lipschitz continuous in \( t \) with \( g(t) \geq 0 \);
\item[(H2)] \( f(x, t) \leq f(x, t) \) if \( \lambda < 0 \), \( x \in \Omega \cap \{ x_1 < \lambda \} \) and \( t \in [0, \infty) \), where \( x_\lambda = (2\lambda - x_1, x_2, ..., x_n) \);
\item[(H3)] \( f(x, t) = f(x_0, t) \) if \( x \in \Omega \cap \{ x_1 < 0 \} \) and \( t \in [0, \infty) \).
\end{enumerate}

Then \( u \) is symmetric in \( x_1 \) and \( u_{x_1} > 0 \) for \( x_1 < 0 \). Moreover, if \( \Omega \) is a ball, then \( u \) is radially symmetric and strictly decreasing in \( |x| \).

A similar result was obtained in [1], in which they require

\( f(x, t) \) and \( g(t) \) are nondecreasing with respect to \( t \). (*)

Here in this paper, we used a completely different approach to obtain the result without the assumption (*). An important tool we used is the maximum principle on a punctured ball for antisymmetric function in [14]. This allows us to consider the problems which solutions are not smooth.

Let us now turn to the symmetry and monotonicity of positive solutions of (1.3) in unbounded cylinders under some different assumptions on \( f \).

Theorem 1.2. Let \( u \) be a positive bounded solution of (1.3). Assume that \( H \) is convex in the \( y_1 \) direction and symmetric with respect to the hyperplane \( y_1 = 0 \). Assume that \( f(u) \) is Lipschitz continuous and \( f(0) \geq 0 \). Then \( u \) is symmetric in \( y_1 \) and strictly monotone decreasing with respect to \( y_1 \) for \( y_1 > 0 \).

Corollary 1 follows easily from Theorem 1.2.

Corollary 1. Assume that \( H \) is a ball with \( |y| < R \) in \( \mathbb{R}^l \). Assume also that \( f(u) \) is Lipschitz continuous and \( f(0) \geq 0 \). If \( u \) is a bounded solution of (1.3), then \( u \) is radially symmetric in \( y \) and strictly decreasing in \( \rho = |y| \) for \( 0 < \rho < R \).

This paper divides into two parts. Section 2 is devoted to studying symmetry and monotonicity of positive solutions to fractional equations with the Hardy potential in bounded domains. In section 3, we give the proof of Theorem 1.2.

Notation. In this paper, positive constants are denoted by \( c \) (with subscript in some cases) and are allowed to vary within a single line or formula. We denote by \( |A| \) the Lebesgue measure of a measurable set \( A \).

2. Symmetry of positive solutions in bounded domain. In this section, we prove Theorem 1.1. Here we first prove more general monotonicity results by making use of a version of the maximum principle on a punctured ball for antisymmetric functions. The maximum principle established in [14] plays an essential role in handling the issue that the solutions \( u \) blow up near the origin.

Theorem 2.1 ([14], Theorem 2). Let \( H = \{ x = (x_1, x') \in \mathbb{R}^n | x_1 < 0 \} \), and assume that \( w(-x_1, x') = -w(x_1, x') \), \( \forall x \in H, B_r(x^0) \subset H \),

\[
\begin{aligned}
(-\Delta)^s w(x) + a(x)w(x) &\geq 0 & \text{in} & B_r(x^0) \setminus \{ x^0 \}, \\
 w(x) &\geq m > 0 & \text{on} & B_r(x^0) \setminus B_2^R(x^0), \\
 w(x) &\geq 0 & \text{on} & H, \\
w(x) &\in L^2, & n &\geq 2
\end{aligned}
\]
and \( a(x) \leq D \) for some constant \( D \). Then there exists a positive constant \( c = c(n, s, D) < 1 \) depending on \( n, s \) and \( D \) only such that
\[
w(x) \geq cm, \ x \in B_r(x^0) \setminus \{ x^0 \}, \text{ for all } r \leq 1.
\]

In order to state our result, we introduce some well known notation. Let \( \nu \) be a direction in \( \mathbb{R}^n \), i.e., \( \nu \in \mathbb{R}^n, |\nu| = 1 \). For \( \lambda \in \mathbb{R} \), let
\[
T^\nu_\lambda = \{ x \in \Omega : x \cdot \nu = \lambda \}.
\]
be the moving plane. Moreover, we set
\[
\Sigma^\nu_\lambda = \{ x \in \Omega : x \cdot \nu < \lambda \}
\]
and let
\[
x^\nu_\lambda = R^\nu_\lambda(x) = x + 2(\lambda - x \cdot \nu)\nu,
\]
be the reflection of the point \( x \) about the plane \( T^\nu_\lambda \). In particular, the typical points in \( \mathbb{R}^n \) are \( x = (x_1, \ldots, x_{n-1}, x_n) = (x, x') \) and \( x_\lambda = (2\lambda - x_1, x') \). To compare the value of \( u^\nu_\lambda(x) \) and \( u(x) \), we denote
\[
u^\nu_\lambda(x) = u^\nu_\lambda(x) - u(x),
\]
where \( u^\nu_\lambda(x) = u(x^\nu_\lambda) \). Define
\[
\lambda(\nu) = \inf_{x \in \Omega} x \cdot \nu.
\]
When \( \lambda > \lambda(\nu) \), since \( \Sigma^\nu_\lambda \) is nonempty, we set
\[
(\Sigma^\nu_\lambda)' = R^\nu_\lambda(\Sigma^\nu_\lambda) \quad \text{and} \quad \lambda_1(\nu) = \sup \{ \lambda : (\Sigma^\nu_\lambda)' \subseteq \Omega \}.
\]

As a consequence of the maximum principle just recalled, we have the following result:

**Theorem 2.2.** Let \( u \) be a solution of (1.1). Let \( \Omega \) be convex domain with respect to the \( \nu \) direction and symmetric with respect to the region \( T^\nu_0 = \{ x \in \mathbb{R}^n : x \cdot \nu = 0 \} \). Assume that \( f(x, t) \) and \( g(t) \) satisfy the following conditions:

(h1) \( f(x, t) \) and \( g(t) \) are locally Lipschitz continuous in \( t \), and \( g(t) \) is nonnegative function;

(h2) \( f(x, t) \leq f(x^\nu_\lambda, t) \) if \( \lambda < 0, x \in \Sigma^\nu_\lambda \) and \( t \in (0, \infty) \);

(h3) \( f(x, t) = f(x^\nu_\lambda, t) \) if \( x \in \Sigma^\nu_\lambda \) and \( t \in (0, \infty) \).

Then \( u \) is symmetric with respect to \( T^\nu_0 \) and monotone increasing with respect to the \( \nu \)-direction in \( \Sigma^\nu_0 \). Moreover, if \( \Omega \) is a ball, then \( u \) is radially symmetric and monotone increasing.

We shall make use of a more general result of Theorem 2.2 which is an easy consequence. It states as follows:

**Proposition 1.** Let \( u \) be a solution of (1.1). Let \( \Omega \) be convex with respect to the \( \nu \) direction and symmetric with respect to the region \( T^\nu_{\lambda_1(\nu)} = \{ x \in \mathbb{R}^n : x \cdot \nu = \lambda_1(\nu) \} \). Assume that \( f(x, t) \) and \( g(t) \) are locally Lipschitz continuous in \( t \), and \( g(t) \) is nonnegative function. Set \( \lambda_0^\nu(\nu) = \min \{ 0, \lambda_1(\nu) \} \). Assume that \( f(x, t) \leq f(x^\nu_\lambda, t) \) if \( \lambda < \lambda_0^\nu(\nu), x \in \Sigma^\nu_\lambda \) and \( t \in [0, \infty) \). Then, for \( x \in \Omega \), \( u(x) \leq u^\nu_\lambda(x) \) if \( a(\nu) \leq \lambda \leq \lambda_0^\nu(\nu) \). Moreover, \( u \) is strictly monotone increasing with respect to the \( \nu \) direction in \( \Sigma^\nu_0 \).

**Proof.** Without loss of generality, we carry out the proof only for the direction \( \nu = e_1 = (1, 0, 0, \ldots 0) \). We introduce some standard notation. Denote
\[
T_\lambda = \{ x \in \Omega : x_1 = \lambda \}, \ \Sigma_\lambda = \{ x \in \Omega : x_1 < \lambda \}, \text{ and } w_\lambda(x) = u_\lambda(x) - u(x),
\]
where \( u_\lambda(x) = u(x^\lambda) \). Moreover, \( a = \inf_{x \in \Omega} x_1 \). When \( \lambda > a \), we set
\[
(S_\lambda)' = R_\lambda(S_\lambda).
\]
We also set
\[
\lambda_1 = \sup\{\lambda : (S_\lambda)' \subseteq \Omega \} \quad \text{and} \quad \lambda_1^0 = \min\{0, \lambda_1\}.
\]
By our assumptions on \( f(x, u) \) and \( g(u) \), we have
\[
(-\Delta)^s w_\lambda(x) = \frac{g(u_\lambda)}{|x^\lambda|^{2s}} - \frac{g(u)}{|x|^{2s}} + f(x_\lambda, u_\lambda) - f(x, u)
\]
\[
\geq \frac{g(u_\lambda) - g(u)}{|x^\lambda|^{2s}} + f(x_\lambda, u_\lambda) - f(x, u) \geq \frac{-c_1 w_\lambda(x)}{|x|^{2s}} - c_2 w_\lambda(x). \tag{2.2}
\]
We obtain the first inequality from the fact
\[
\frac{1}{|x|} < \frac{1}{|x^\lambda|} \quad \text{for} \quad \lambda < 0.
\]
From (2.2), we see that for \( \lambda < 0 \)
\[
(-\Delta)^s w_\lambda + c(x) w_\lambda \geq 0,
\]
where \( c(x) \) is bounded. Then we apply Theorem 2 of [10] to conclude that for \( \lambda - a \) small and \( \lambda < 0 \),
\[
w_\lambda(x) \geq 0 \quad \text{in} \quad S_\lambda. \tag{2.3}
\]
Now we move the plane to the right as long as (2.3) holds to the limiting position. Define
\[\overline{x} = \sup\{a < \lambda < \lambda_1^0 : u \leq u_\mu \quad \text{in} \quad \Sigma_\mu \setminus \{0\} \quad \text{for all} \quad a \leq \mu \leq \lambda\}.\]
The proof of this theorem is completed once we prove \( \overline{x} = \lambda_1^0 \). To prove this, we will argue by contradiction and assume \( \overline{x} < \lambda_1^0 \). Let us fix \( \varepsilon > 0 \) such that \( \overline{x} + \varepsilon < \lambda_1^0 \).
We want to prove there exists small \( \varepsilon > 0 \) such that for all \( \varepsilon < \varepsilon \) and \( \lambda \in \overline{x}, \overline{x} + \varepsilon \),
(2.3) still holds which contradicts the definition of \( \overline{x} \).
By continuity of \( w_\lambda \) with respect to \( \lambda \), we get \( w_{\overline{x}} \geq 0 \) in \( S_{\overline{x}} \setminus \{0\} \). Let us show that for \( \overline{x} < \lambda_1^0 \),
\[
w_{\overline{x}} > 0 \quad \text{in} \quad S_{\overline{x}} \setminus \{0\}. \tag{2.4}
\]
Suppose (2.4) is not true, then there exists a point \( \hat{x} \in S_{\overline{x}} \setminus \{0\} \) such that
\[
w_{\overline{x}}(\hat{x}) = \min_{S_{\overline{x}}} w_{\overline{x}}(x) = 0. \tag{2.5}
\]
By the definition of fractional Laplacian, we calculate
\[
(-\Delta)^s w_{\overline{x}}(\hat{x}) = C_{n,s} PV \int_{\mathbb{R}^n} \frac{w_{\overline{x}}(\hat{x}) - w_{\overline{x}}(y)}{|\hat{x} - y|^{n+2s}} dy
\]
\[= C_{n,s} PV \left\{ \int_{S_{\overline{x}}} \frac{-w_{\overline{x}}(y)}{|\hat{x} - y|^{n+2s}} dy + \int_{\mathbb{R}^n \setminus S_{\overline{x}}} \frac{-w_{\overline{x}}(y)}{|\hat{x} - y|^{n+2s}} dy \right\}
\]
\[= C_{n,s} PV \left\{ \int_{S_{\overline{x}}} \frac{-w_{\overline{x}}(y)}{|\hat{x} - y|^{n+2s}} dy + \int_{S_{\overline{x}}} \frac{-w_{\overline{x}}(y)}{|\hat{x} - y|^{n+2s}} dy \right\}
\]
\[= C_{n,s} PV \int_{S_{\overline{x}}} \left( \frac{1}{|\hat{x} - y|^{n+2s}} - \frac{1}{|\hat{x} - y|^{n+2s}} \right) w_{\overline{x}}(y) dy
\]
\[< 0. \tag{2.6}
\]
The above inequality follows from the fact \( |\hat{x} - y| > |\hat{x} - y| \).
On the other hand, by our assumptions on \( f(x, u) \) and \( g(u) \), we have
\[
(-\Delta)^{\ast} w_{\lambda}(\hat{x}) = \frac{g(\hat{u}_{\lambda})}{|\hat{x}|^{2\lambda}} + f(\hat{x}, u_{\lambda}) - \frac{g(u)}{|\hat{x}|^{2\lambda}} - f(\hat{x}, u) \\
\geq f(\hat{x}, u_{\lambda}) - f(\hat{x}, u) \geq 0,
\]
which contradicts the inequality (2.6). This verifies (2.4).

Then what left is to prove there exists a constant \( c_{0} \) such that for sufficiently small \( \delta \)
\[
w_{\lambda} \geq c_{0} > 0 \quad \text{in} \quad \Sigma_{\lambda-\delta} \setminus \{0\}_{\lambda}.
\]
We choose \( r > 0 \) small such that \( B_{2r}(0) \subseteq \Sigma_{\lambda-\delta} \). It follows from (2.4) and \( w_{\lambda} > 0 \)
in \( \Sigma_{\lambda-\delta} \setminus B_{r}(0) \) that there exist \( c_{1} \) and \( \rho \) such that
\[
w_{\lambda} \geq c_{1} > 0 \quad \text{in} \quad \Sigma_{\lambda-\delta} \setminus B_{r}(0) \tag{2.8}
\]
and
\[
w_{\lambda} \geq \rho > 0 \quad \text{in} \quad B_{2r}(0) \setminus B_{r}(0). \tag{2.9}
\]
By Theorem 2.1, there exists some constant \( c_{2} \) such that
\[
w_{\lambda} \geq c_{2}\rho > 0 \quad \text{in} \quad B_{2r}(0) \setminus \{0\}. \tag{2.9}
\]
We assume that \( \varepsilon < \frac{\rho}{2} \), thus obtaining \( \partial_{\lambda+\varepsilon} \subseteq B_{2r}(0) \). Since \( w_{\lambda} \) is continuous
with respect to \( \lambda \), by (2.8) and (2.9), there exists \( c_{0} > 0 \) such that for sufficiently small \( \delta \), \( w_{\lambda} \geq c_{0} > 0 \)
in \( \Sigma_{\lambda-\delta} \setminus \{0\}_{\lambda} \). Now Theorem 2 of [10] can be applied with \( \Sigma_{\lambda+\varepsilon} \setminus \Sigma_{\lambda-\delta} \), we see that for all \( \lambda \in [\overline{\lambda}, \overline{\lambda} + \varepsilon] \)
\[
w_{\lambda} \geq 0 \quad \text{in} \quad \Sigma_{\lambda} \setminus \{0\}_{\lambda}. \tag{2.9}
\]
This contradicts the definition of \( \overline{\lambda} \). Therefore, we must have
\[
\overline{\lambda} = \lambda_{1}^{0}.
\]
This implies that for \( a(\nu) \leq \lambda \leq \lambda_{1}^{0}(\nu) \),
\[
u(x) \leq u_{\lambda}^{\nu}(x).
\]
Now we will prove \( u \) is strictly monotone increasing in \( \Sigma_{\lambda}^{\nu} \). Let \( t_{0} \) be the midpoint
\( (t_{1}, x') \) and \( (t_{2}, x') \) in \( \Sigma_{\lambda_{1}^{0}}^{\nu} \) with \( t_{1} < t_{2} \). Let \( \lambda = \frac{t_{1} + t_{2}}{2} \). Then, as proved above, we have
\[
w_{\lambda} > 0 \quad \text{in} \quad \Sigma_{\lambda}.
\]
Therefore,
\[
0 < w_{\lambda}(t_{1}, x') = u(2\lambda - t_{1}, x') - u(t_{1}, x') = u(t_{2}, x') - u(t_{1}, x').
\]
That is, \( u(t_{1}, x') < u(t_{2}, x') \). \( \square \)

Proof of Theorem 2.2. Since \( \Omega \) is convex in \( \nu \)-direction and symmetric with respect
to \( T_{0}^{\nu} = \{ x \in \Omega : x \cdot \nu = 0 \} \), we have
\[
\lambda_{1}^{0} = \lambda_{1} = 0.
\]
Applying Proposition 1, we have
\[
u(x) \leq u_{\lambda}^{\nu}(x), \quad \text{for} \quad x \in \Sigma_{0}^{\nu}.
\]
In the same way, using the method of moving plane along the opposite direction \( -\nu \), we see that
\[
u(x) \geq u_{\lambda}^{\nu}(x), \quad \text{for} \quad x \in \Sigma_{0}^{\nu}.
\]
This implies that $u$ is symmetric with respect to $\nu$-direction and nondecreasing with respect to $\nu$-direction in $\Sigma_\nu$.

If $\Omega$ is a ball, using the same argument in any arbitrary direction, we conclude that $u$ is radially symmetric. Finally, we prove that $u(r)$ is strictly decreasing with respect to $r$.

**Proof of Theorem 1.1.** Theorem 1.1 is an easy consequence of Theorem 2.2 by considering the case $\nu = (1, 0, \ldots, 0)$.

3. **Monotonicity of positive solutions in unbounded cylinders.** In this section, we obtain the symmetry of positive solutions to fractional equations

$$\begin{cases}
(-\Delta)^s u = f(u), & u > 0 \text{ in } \Omega, \\
u = 0, & \text{on } \mathbb{R}^n \setminus \Omega,
\end{cases}$$

where $\Omega$ is a unbounded cylinder. For this purpose we use the method of moving planes.

**Proof of Theorem 1.2.** We fix $j = n - 1$. The proof can be carried on for other values of $j$ with trivial modifications. We therefore consider $\Omega = \mathbb{R} \times H$, where $H$ is convex in $y_1$ direction and symmetric with respect to the hyperplane $y_1 = 0$. We first introduce some standard notation. For $0 < \lambda < R$, denote

$$w_{\lambda}(x, y) = u(x, y) - u(x, y^\lambda), \quad H_{\lambda} = \{ y \in H : y_1 > \lambda \}$$

and

$$\Sigma_{\lambda} = \{(x, y) \in \mathbb{R}^n : y_1 > \lambda \},$$

where $y^\lambda = (2\lambda - y_1, y_2, \ldots, y_{n-1})$ is the reflection of $y$ about the plane $\{y_1 = \lambda\}$ and $u_{\lambda}(x, y) = u(x, y^\lambda)$. Set

$$\overline{R} = \sup_{H} y_1.$$

As usual, $w_{\lambda}(x, y)$ satisfies some linear equation

$$(-\Delta)^s w_{\lambda}(x, y) + c(x, y)w_{\lambda}(x, y) = 0 \text{ in } \Sigma_{\lambda}.$$

Since $f$ is Lipschitz continuous, there exists some constant $b$ such that

$$\|c(x, y)\|_{L^\infty(\Sigma_{\lambda})} \leq b \quad \forall \lambda \in (0, \overline{R}).$$

For $\overline{R} - \lambda$ small, we apply Theorem 1 of [7] to conclude that

$$w_{\lambda}(x, y) \leq 0 \quad \text{in } \Sigma_{\lambda}.$$

Now we start the moving plane procedure by setting

$$\lambda_0 = \inf \{ \lambda \in (0, \overline{R}) : w_{\lambda}(x, y) \leq 0 \text{ in } \Sigma_{\lambda} \}.$$

We shall show that $\lambda_0 = 0$. Proceeding by contradiction, we assume that $\lambda_0 > 0$. By continuity of $w_{\lambda}(x, y)$ with respect to $\lambda$, we have that $w_{\lambda_0}(x, y) \leq 0$ in $\Sigma_{\lambda_0}$. In fact, we have

$$w_{\lambda_0}(x, y) < 0 \quad \text{in } \Sigma_{\lambda_0}.$$

If not, there exists some point $(\overline{x}, \overline{y})$ such that

$$w_{\lambda_0}(\overline{x}, \overline{y}) = \max_{\Sigma_{\lambda_0}} w_{\lambda_0}(x, y) = 0.$$

It follows that

$$(-\Delta)^s w_{\lambda_0}(\overline{x}, \overline{y})$$
\[ C_{n,s} \text{PV} \int_{\mathbb{R}^n} \frac{w_{\lambda_j}(x,y) - w_{\lambda_j}(x,y)}{|(x,y) - (x,y)|^{n+2s}} \, dx \, dy \]
\[ = C_{n,s} \text{PV} \int_{\Sigma_{\lambda_j}} \frac{w_{\lambda_j}(x,y) - w_{\lambda_j}(x,y)}{|(x,y) - (x,y)|^{n+2s}} \, dx \, dy + \int_{\mathbb{R}^n \setminus \Sigma_{\lambda_j}} \frac{w_{\lambda_j}(x,y) - w_{\lambda_j}(x,y)}{|(x,y) - (x,y)|^{n+2s}} \, dx \, dy \]
\[ = C_{n,s} \text{PV} \int_{\Sigma_{\lambda_j}} \frac{1}{|(x,y) - (x,y)|^{n+2s}} - \frac{1}{|(x,y) - (x,y)|^{n+2s}} w_{\lambda_j}(x,y) \, dx \, dy > 0. \quad (3.2) \]

We obtain the above inequality from the fact
\[ |(x,y) - (x,y)| > |(x,y) - (x,y)|. \]

On the other hand,
\[ (-\Delta)^s w_{\lambda_j}(x,y) = 0, \]
which contradicts (3.2). This proves (3.1).

Next we show that for \( \varepsilon > 0 \) small, we can decrease \( \lambda \) continuously such that for \( \lambda \in (\lambda_0 - \varepsilon, \lambda_0) \),
\[ w_\lambda \leq 0 \quad \text{in} \quad \Sigma_\lambda, \]
which contradicts the definition of \( \lambda_0 \). Therefore, we must have \( \lambda_0 = 0 \).

Fix an open set \( \beta \subset \beta' \subset H_{\lambda_0} \) such that \( |H_{\lambda_0} \setminus \beta'| \) is sufficiently small. Set \( K = \mathbb{R} \times \beta' \).

We first prove that for any \( \lambda \in (\lambda_0 - \varepsilon, \lambda_0) \),
\[ w_\lambda \leq 0 \quad \text{in} \quad \Sigma_\lambda, \quad (3.3) \]
Suppose (3.3) is not true, then there exist sequences \( (x_j, y_j) \in K \) and \( \lambda_j \to \lambda_0 \) such that
\[ w_{\lambda_j}(x_j, y_j) = u(x_j, y_j) - u(x_j, y_j^{\lambda_j}) > 0. \quad (3.4) \]
Consider the translated function
\[ u^j(x, y) = u(x + x_j, y). \]

We observe that \( u^j(x, y) \) satisfies
\[ \begin{cases} (-\Delta)^s u^j = f(u^j) & \text{in} \quad \Omega, \\ u^j = 0 & \text{on} \quad \mathbb{R}^n \setminus \Omega. \end{cases} \]

For a subsequence, we may suppose that
\[ y_j \to \overline{y} \in \beta' \quad \text{and} \quad \alpha^j = u^j(0, y_j) \to \alpha. \]

Set
\[ z^j(x, y) = \frac{1}{\alpha^j} u^j(x, y). \]

Then \( z^j(0, y^j) = 1 \) and
\[ (-\Delta)^s z^j(x, y) = \frac{1}{\alpha^j} f(\alpha^j z^j(x, y)). \]
We observe that \( f(\alpha^j z^j(x, y)) \) is Lipschitz continuous in \( z^j(x, y) \). Since \( u(x, y) \) is bounded in \( \Omega \), we can find that for some constant \( a > 0 \),
\[
z^j(x, y) \leq a \quad \text{in } (-R, R) \times \overline{\Omega},
\]
where \( R > 1 \). For a subsequence, we may assume that
\[
\frac{1}{\alpha^j} f(\alpha^j z^j(x, y)) \to \overline{g} \quad \text{uniformly on } [0, a].
\]
We observe that \( \overline{g} \) is Lipschitz continuous with Lipschitz constant \( k \). Using the standard elliptic estimates, we may infer that \( z^j(x, y) \) converge uniformly in \([-1, 1] \times \overline{\Omega} \) to a function \( \overline{z}(x, y) \geq 0 \), and \( \overline{z}(x, y) \) satisfies
\[
\begin{cases}
(\Delta)^s \overline{z} = \overline{g}(\overline{z}) & \text{in } (-1, 1) \times H, \\
\overline{z} = 0 & \text{on } (-1, 1) \times (\mathbb{R}^{n-1} \setminus H), \\
\overline{z}(0, \overline{y}) = 1.
\end{cases}
\]
Set \( \overline{w}(x, y) = \overline{z}(x, y) - \overline{z}(x, y^{\lambda_0}) \) in \((-1, 1) \times H_{\lambda_0} \). Using (3.1) and (3.4), we see that \( \overline{w}(x, y) \) satisfies
\[
\begin{cases}
(\Delta)^s \overline{w}(x, y) + \overline{z}(x, y)\overline{w}(x, y) = 0 & \text{in } (-1, 1) \times H_{\lambda_0}, \\
\overline{w}(x, y) \leq 0 & \text{in } (-1, 1) \times H_{\lambda_0}, \\
\overline{w}(0, \overline{y}) = 0 & \text{for some } \overline{y} \in \overline{\Omega},
\end{cases}
\tag{3.5}
\]
where \( \overline{z}(x, y) \) is bounded (here we use the local Lipschitz continuity of \( f \) in \( u \)). Note that the equation (3.5) is also true in \((-R, R) \times H_{\lambda_0} \). We now prove
\[
\overline{w}(x, y) \equiv 0 \quad \text{in } (-1, 1) \times H_{\lambda_0}.
\tag{3.6}
\]
Suppose (3.6) does not hold, then there exists some point \((x^0, y^0)\) in \((-1, 1) \times H_{\lambda_0} \) such that
\[
\overline{w}(x^0, y^0) < 0.
\]
By the definition of fractional Laplacian, we compute
\[
(\Delta)^s \overline{w}(0, \overline{y})
= C_{n,s}PV \int_{\mathbb{R}^n} \frac{\overline{w}(0, \overline{y}) - \overline{w}(x, y)}{|(0, \overline{y}) - (x, y)|^{n+2s}} \, dx \, dy
= C_{n,s}PV \int_{\Sigma_{\lambda_0}} \frac{\overline{w}(0, \overline{y}) - \overline{w}(x, y)}{|(0, \overline{y}) - (x, y)|^{n+2s}} \, dx \, dy + \int_{\mathbb{R}^n \setminus \Sigma_{\lambda_0}} \frac{\overline{w}(0, \overline{y}) - \overline{w}(x, y)}{|(0, \overline{y}) - (x, y)|^{n+2s}} \, dx \, dy
= C_{n,s}PV \int_{\Sigma_{\lambda_0}} \frac{1}{|(0, \overline{y}) - (x, y^{\lambda_0})|^{n+2s}} - \frac{1}{|(0, \overline{y}) - (x, y)|^{n+2s}} \overline{w}(x, y) \, dx \, dy
+ \int_{D_1} \frac{1}{|(0, \overline{y}) - (x, y^{\lambda_0})|^{n+2s}} - \frac{1}{|(0, \overline{y}) - (x, y)|^{n+2s}} \overline{w}(x, y) \, dx \, dy
+ \int_{D_2} \frac{1}{|(0, \overline{y}) - (x, y^{\lambda_0})|^{n+2s}} - \frac{1}{|(0, \overline{y}) - (x, y)|^{n+2s}} \overline{w}(x, y) \, dx \, dy
= I_1 + I_2 + I_3,
\]
where $\Sigma_{\lambda_0} = \mathbb{R} \times H_{\lambda_0}$, $D_1 = (-1, 1) \times H_{\lambda_0}$, $D_2 = (1, R) \times H_{\lambda_0} \cup (-R, -1) \times H_{\lambda_0}$, 
$D_3 = \Sigma_{\lambda_0} \setminus (D_1 \cup D_2)$.

We first estimate $I_1$. Since $\overline{w}(x, y)$ is continuous and $\overline{w}(x^0, y^0) < 0$, we choose $r$ small such that
\[
\overline{w}(x, y) < -c \quad \text{for } (x, y) \in B_r(x^0, y^0) \subset D_1.
\] (3.7)

Then we see by (3.7) that
\[
I_1 \geq \int_{B_r(z_0)} \frac{1}{|(0, \overline{y}) - (x, y^0)|^{n+2\sigma}} - \frac{1}{|(0, \overline{y}) - (x, y)|^{n+2\sigma}} \overline{w}(x, y) dy > c.
\]

It is easy to verify that $I_2 \geq 0$ and $I_3 \to 0$ as $R \to \infty$. The above inequalities implies that
\[
(-\Delta)^s \overline{w}(0, \overline{y}) > \frac{c}{2}.
\] (3.8)

On the other hand, we have
\[
(-\Delta)^s \overline{w}(0, \overline{y}) + \overline{w}(x, y) \overline{w}(0, \overline{y}) = 0,
\]
which contradicts (3.8). Therefore, $\overline{w}(x, y) \equiv 0$. Since $(0, y) \in \partial \Omega$, we conclude that
\[
\overline{w}(0, y^0) = 0 \quad \text{for some } (0, y^0) \in \Omega.
\]

It implies that $\overline{w}$ attains its minimum at some point $(0, y^0) \in \Omega$. Similar to the above arguments, one can derive that
\[
(-\Delta)^s \overline{w}(0, y^0) < 0,
\]
which contradicts the fact $f(0) \geq 0$.

Then what left is to show that
\[
w_{\lambda} \leq 0 \quad \text{in } \mathbb{R} \times (H_{\lambda} \setminus \overline{\Omega}).
\] (3.9)

If (3.9) is not true, then there exists $(\tilde{x}, \tilde{y})$ such that $w_{\lambda}(\tilde{x}, \tilde{y}) > 0$ in $\mathbb{R} \times (H_{\lambda} \setminus \overline{\Omega})$. Let
\[
A = \sup_{\mathbb{R} \times (H_{\lambda} \setminus \overline{\Omega})} w_{\lambda}(x, y) > 0.
\]

Then there exists a sequence $\{(x^k, y^k)\}$ in $\mathbb{R} \times (H_{\lambda} \setminus \overline{\Omega})$ such that
\[
w_{\lambda}(x^k, y^k) \to A \quad \text{as } k \to \infty.
\]

Since $\mathbb{R} \times (H_{\lambda} \setminus \overline{\Omega})$ is an unbounded domain, there may not exist a point in $\mathbb{R} \times (H_{\lambda} \setminus \overline{\Omega})$ such that $w_{\lambda}$ attains its maximum. To overcome this difficulty, we introduce a smooth function
\[
\varphi(z) = \begin{cases} 
ce^{-|z|^{-1}} & \text{if } |z| < 1, \\
0 & \text{if } |z| \geq 1,
\end{cases}
\]
where $z = (x, y)$ and $c$ is a positive constant such that $\max_{\mathbb{R}^n} \varphi(z) = 1$. It is easy to verify that $\varphi(z)$ is radially decreasing with respect to $|z|$, and its support is $B_1(0)$.

Set
\[
\psi_k(z) = \varphi\left(\frac{z - z^k}{r_k}\right),
\]
where $z^k = (x^k, y^k)$ and $r_k = \frac{\text{dist}(z^k, \partial(H_{\lambda} \setminus \overline{\Omega}))}{2}$. Choose $\epsilon_k > 0$ and $\epsilon_k \to 0$ ($k \to \infty$) such that for $z \in B_{r_k}(z^k)$
\[
w_{\lambda}(z^k) + \epsilon_k \psi_k(z^k) \geq A \geq w_{\lambda}(z) + \epsilon_k \psi_k(z) \quad \text{for all } z \in (\mathbb{R} \times H_{\lambda}) \setminus B_{r_k}(z^k).
\]
Then there exists a point \( z^k ∈ B_{r_k}(z^k) \) such that

\[
w_λ(z^k) + \varepsilon_k \psi_k(z^k) = \max_{R × H_λ} (w_λ(z) + \varepsilon_k \psi_k(z)).
\]

By the definition of fractional Laplacian, we have

\[
(-Δ)^s(w_λ(z^k) + \varepsilon_k \psi_k(z^k)) = C_{n,s}\text{PV} \int_{R^n} \frac{w_λ(z^k) + \varepsilon_k \psi_k(z^k) - w_λ(z)}{|z^k - z|^{n+2s}}dz
\]

\[
= C_{n,s}\text{PV} \int_{\Sigma^H_λ} \frac{w_λ(z^k) + \varepsilon_k \psi_k(z^k) - w_λ(z)}{|z^k - z|^{n+2s}}dz
\]

\[
+ C_{n,s} \int_{\Sigma^H_λ} \frac{w_λ(z^k) + \varepsilon_k \psi_k(z^k) - w_λ(z)}{|z^k - z|^{n+2s}}dz
\]

\[
≥ C_{n,s} \int_{\Sigma^H_λ} \frac{2[w_λ(z^k) + \varepsilon_k \psi_k(z^k)] - \varepsilon_k \psi_k(z)}{|z^k - z|^{n+2s}}dz
\]

\[
≥ C_{n,s} \int_{\Sigma^H_λ} \frac{2[w_λ(z^k) + \varepsilon_k \psi_k(z^k)] - \varepsilon_k \psi_k(z)}{|z^k - z|^{n+2s}}dz
\]

\[
≥ C_{n,s} \int_{\Sigma^H_λ} \frac{2A - \varepsilon_k}{r_k^{2s}} \int_{\Sigma^H_λ \setminus \overline{B}} \frac{1}{|t|^{n+2s}} dt
\]

\[
= \frac{2A - \varepsilon_k}{r_k^{2s}}, \tag{3.10}
\]

where \( \Sigma^H_λ = R × H_λ \) and \( z^λ = (x, y^λ) \). We obtain the third inequality from the fact

\[
|z^k - z| ≤ |z^k - z^λ| + |z^λ - z^k| ≤ 2|z^k - z^λ|.
\]

On the other hand, we have that for some positive constants \( γ \)

\[
(-Δ)^s(w_λ(z^k) + \varepsilon_k \psi_k(z^k)) ≤ -c(z^k)w_λ(z^k) + \frac{ε_k}{r_k^{2s}}
\]

\[
≤ γ w_λ(z^k) + \frac{ε_k}{r_k^{2s}} ≤ γ A + \frac{ε_k}{r_k^{2s}}. \tag{3.11}
\]

Using the above two inequalities (3.10) and (3.11), we deduce that

\[
2A ≤ γ r_k^{2s} A + ε_k. \tag{3.12}
\]

Since the measure of \(|H_λ \setminus \overline{B}|\) is sufficiently small, \( r_k \) is small. Therefore,

\[
γ r_k^{2s} A < A. \tag{3.13}
\]

Combining (3.12), (3.13) with the fact \( ε_k → 0 (k → ∞) \), we have \( A = 0 \). This contradicts \( A > 0 \). Hence we must have \( w_λ(z) ≤ 0 \). This contradicts the definition
of \( \lambda_0 \), thus we must have \( \lambda_0 = 0 \). It implies that
\[
    u(x, y_1, y_2, \ldots, y_{n-1}) \leq u(x, -y_1, y_2, \ldots, y_{n-1}) \quad \text{for } y_1 \geq 0.
\]
Using the same argument from the other side, we conclude that
\[
    u(x, y_1, y_2, \ldots, y_{n-1}) \geq u(x, -y_1, y_2, \ldots, y_{n-1}) \quad \text{for } y_1 \geq 0.
\]
Hence we obtain that \( u \) is symmetric with respect to the plane \( y_1 = 0 \).

Finally, we prove \( u \) is strictly monotone decreasing with respect to \( y_1 \) for \( y_1 > 0 \).

Let us consider \( 0 < t_1 < t_2 < R \) and let \( \lambda = \frac{1-t_2}{2} \). Then, as proved above, we have
\[
    w_\lambda(x, y) < 0 \quad \text{in } \Sigma_\lambda.
\]

Then
\[
    0 > w_\lambda(x, t_1, y_2, \ldots, y_{n-1}) = u(x, t_1, y_2, \ldots, y_{n-1}) - u(x, 2\lambda - t_1, y_2, \ldots, y_{n-1}),
\]
\[
    = u(x, t_1, y_2, \ldots, y_{n-1}) - u(x, t_2, y_2, \ldots, y_{n-1}),
\]
that is, \( u(x, t_1, y_2, \ldots, y_{n-1}) > u(x, t_2, y_2, \ldots, y_{n-1}). \)

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Received September 2019; revised January 2020.

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