ON THE TORSION SUBGROUPS OF THE MODULAR JACOBIANS

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Abstract. For any positive integer \( N \), we prove that the rational torsion subgroup of \( J_0(N) \) agrees with its rational cuspidal subgroups up to a factor of \( 6N \prod_{p\mid N}(p^2-1) \). Moreover, for modular Jacobians of the form \( J_0(DC) \) with \( D \) a positive square-free integer and \( C \) any positive divisor of \( D \), we prove that the \( \psi \)-part of the torsion subgroup of \( J_0(DC) \) agrees with the \( \psi \)-part of its cuspidal subgroup up to a factor of \( 6D \prod_{p\mid D}(p^2-1) \), where \( \psi \) is any quadratic character of conductor dividing \( C \).

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1. Introduction

For any positive integer \( N \), let \( X_0(N) \) be the canonical model over \( \mathbb{Q} \) of the modular curve of level \( \Gamma_0(N) \) and let \( J_0(N) \) be the Jacobian variety of \( X_0(N) \) over \( \mathbb{Q} \). When \( N = p \) is a prime, Ogg proved

\[
C_0(p) \simeq \mathbb{Z}/\frac{p - 1}{(p - 1, 12)}\mathbb{Z},
\]

where \( C_0(p) \) is the cuspidal subgroup of \( J_0(p) \) generated by the class of the divisor \([0] - [\infty]\) with \([0]\) and \([\infty]\) the two cusps of \( X_0(p) \). In fact, for any positive integer \( N \), the set of cusps of \( X_0(N) \) is stable under the action of \( G_\mathbb{Z} \), and each positive divisor \( d \mid N \) corresponds to a unique \( G_\mathbb{Z} \)-orbit consisting of those cusps defined precisely over \( \mathbb{Q}(\mu_{(d,N/d)}) \) (see §1.3 of [11]).

The above conjecture of Ogg has been proved by Mazur in his celebrated work [4], where the unique normalized weight-two Eisenstein series \( E \) of level \( \Gamma_0(p) \) plays a fundamental role. In fact, \( C_0(p) \) is exactly the cuspidal subgroup associated to \( E \) (see Definition 2.1). Moreover, let \( \mathbb{T}_0(p) \) be the full Hecke algebra of level \( \Gamma_0(p) \) generated over \( \mathbb{Z} \) by the Hecke operators \( T_\ell \) for all the primes \( \ell \). Then the action of \( \mathbb{T}_0(p) \) on \( J_0(p) \) preserves \( C_0(p) \) and induces an isomorphism

\[
\mathbb{T}_0(p)/I_{\Gamma_0(p)}(E) \simeq C_0(p),
\]

where \( I_{\Gamma_0(p)}(E) \) is the Eisenstein ideal of \( E \) (see also Definition 2.1). This isomorphism, which gives us the structure of \( C_0(p) \) as a \( \mathbb{T}_0(p) \)-module, is one of the key ingredients in the proof of Ogg’s conjecture by Mazur. Here, we should remark that Mazur actually defined \( \mathbb{T}_0(p) \) to be the \( \mathbb{Z} \)-algebra generated by all \( T_\ell \)’s with \( \ell \neq p \) and the Atkin-Lehner operator \( w_p \). But since \( w_p = -T_p \) in this situation, these two definitions are in fact the same. After their pioneering work, one is naturally led to the following

Conjecture 1.1. (Generalized Ogg’s conjecture) For any positive integer \( N \), we have that

\[
J_0(N)(\mathbb{Q})_{tor} = C_0(N)(\mathbb{Q}),
\]

where \( C_0(N) \) is the subgroup of \( J_0(N)(\overline{\mathbb{Q}}) \) generated by degree zero divisor classes supported at the cusps of \( X_0(N) \), and \( C_0(N)(\mathbb{Q}) = C_0(N)(\mathbb{Q})^{G_\mathbb{Z}} \) is the \( \mathbb{Q} \)-rational subgroup of \( C_0(N) \).
It is clear that the above conjecture is equivalent to \( J_0(N)(\mathbb{Q})_{tor} \subseteq C_0(N) \) for any positive integer \( N \).

To this date, it has been proved that:

1. If \( p \geq 5 \) is a prime and \( \tau \in \mathbb{Z}_{\geq 1} \), then \( J_0(p^\tau)(\mathbb{Q})[q^\infty] \subseteq C_0(p^\tau)[q^\infty] \) for any prime \( q \nmid 6p \). See [4].

2. Let \( N \) be a square-free positive integer, then we have \( J_0(N)(\mathbb{Q})[q^\infty] = C_0(N)[q^\infty] \) for any prime \( q \nmid 6 \) (See [7]). Note that when \( N \) is square free, all the cusps of \( X_0(N) \) are in fact \( \mathbb{Q} \)-rational and hence \( C_0(N) = C_0(N)(\mathbb{Q}) \).

The first main result of this article is the following

**Theorem 1.2.** For any positive integer \( N \), we have that

\[
J_0(N)(\mathbb{Q})[q^\infty] = C_0(N)(\mathbb{Q})[q^\infty]
\]

for any prime \( q \nmid 6 \cdot N \cdot \varpi(N) \), where \( \varpi(N) := \prod_{p \mid N} (p^2 - 1) \).

Our proof of this theorem is based on a careful study of modular Jacobian varieties of the form \( J_0(DC) \), where \( D \) is a positive square-free integer and \( C \) is a positive divisor of \( D \). In fact, in this situation, we can prove that the torsion points of \( J_0(DC) \) over some quadratic fields also come from the cusps of \( X_0(DC) \). Note that since the cusps of \( X_0(DC) \) are all defined over \( \mathbb{Q}(\mu_C) \) as remarked before, so is the cuspidal subgroup \( C_0(DC) \) of \( J_0(DC)(\overline{\mathbb{Q}}) \). For any quadratic Dirichlet character \( \psi \) of conductor \( f_\psi \mid C \), we define

\[
C_0(DC)(\psi) := \{ P \in C_0(DC) : \sigma(P) = \psi(\sigma) \cdot P \text{ for any } \sigma \in G_\mathbb{Q} \},
\]

and define similarly

\[
J_0(DC)(\psi)[q^\infty] = C_0(DC)(\psi)[q^\infty]
\]

for any prime \( q \nmid 6 \cdot D \cdot \varpi(D) \).

In our investigation, the relation between the weight two Eisenstein series and the cuspidal subgroup plays a very important role, so we will give a brief review of this relation in the second section. Then, in the third section, we construct a Hecke eigenbasis \( \{ E_{M,L,\psi} \} \) for the space \( \mathcal{E}_2(\Gamma_0(DC), \mathbb{C}) \) of Eisenstein series of weight two and level \( \Gamma_0(DC) \) (see Definition [3.3] and Proposition [3.7]). While all these Eisenstein series are interesting, we will in this article focus on the study of those \( E_{M,L,\psi} \) with \( \psi \) a quadratic character. The associated group \( G_{\Gamma_0(DC)}(E_{M,L,\psi}) \) will be called as **cuspidal subgroup** of \( J_0(DC) \). The order and the Hecke module structure of these quadratic cuspidal subgroups are determined up to a factor of \( 6D \) (see Theorem 4.1.10 and Theorem 4.2.2) in the fourth section. This will enable us to prove our main results in the final section.

Notations: For any positive integer \( N = \prod_{p \mid N} p^{\nu_p(N)} \), we denote by \( \varpi(N) = \prod_{p \mid N} (p^2 - 1) \), \( \nu(N) = \sum_{p \mid N} \nu_p(N) \) and \( \mu(N) = \prod_{p \mid N} (p + 1) \).

Let \( q \) to be the function \( z \mapsto e^{2\pi i z} \) on the upper half plane. For any function \( g \) on the upper half plane and any \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2^+(\mathbb{R}) \), we denote by \( g(\gamma) \) to be the function \( z \mapsto \text{det}(\gamma) \cdot g(\gamma) \cdot (cz + d)^{-2} \).

2. **Background materials**

In this section, we are going to recall the relation between weight two Eisenstein series and cuspidal subgroups. For more details and proof, the reader is referred to [11] and [12].

2.1. In the following, we fix a positive integer \( N \) and denote by \( \Gamma \) to be either \( \Gamma_0(N) \) or \( \Gamma_1(N) \). Let \( M_2(\Gamma, \mathbb{C}) \) be the space of weight two modular forms of level \( \Gamma \), then

\[
M_2(\Gamma, \mathbb{C}) = S_2(\Gamma, \mathbb{C}) \oplus \mathcal{E}_2(\Gamma, \mathbb{C}),
\]

where \( S_2(\Gamma, \mathbb{C}) \) is the subspace of cusp forms and \( \mathcal{E}_2(\Gamma, \mathbb{C}) \) is the subspace of Eisenstein series. For any positive integer \( n \), there is a Hecke operator \( T_{n}^\Gamma \) acting on \( M_2(\Gamma, \mathbb{C}) \) with respect to the above decomposition. We denote the restriction of \( T_{n}^\Gamma \) to \( S_2(\Gamma, \mathbb{C}) \) by \( T_{n}^{\Gamma} \). Let \( \mathcal{T}_{\Gamma} \) be the \( \mathbb{Z} \)-algebra generated by \( \{ T_{n}^{\Gamma} \}_{n \geq 1} \). Then the **full** Hecke algebra \( \mathcal{T}_{\Gamma} \) of level \( \Gamma \) is defined to be the restriction of \( \mathcal{T}_{\Gamma} \) to \( S_2(\Gamma, \mathbb{C}) \), which is the \( \mathbb{Z} \)-algebra generated by all the \( T_{n}^{\Gamma} \)'s. When \( \Gamma = \Gamma_0(N) \), we will also denote \( \mathcal{T}_{\Gamma_0(N)} \) as \( \mathcal{T}_N(\mathbb{Q}) \), which is in fact generated by the \( T_{n}^{\Gamma_0(N)} \) for all the primes \( \ell \).
2.2. Let $X_\Gamma$ be the modular curve over $\mathbb{Q}$ of level $\Gamma$. We denote by $cusp(\Gamma)$ to be the set of cusps of $X_\Gamma$, and by $Y_\Gamma$ be the complement of $cusp(\Gamma)$ in $X_\Gamma$. Let $J_\Gamma$ be the Jacobian variety of $X_\Gamma$ over $\mathbb{Q}$. For any $g \in M_2(\Gamma, \mathbb{C})$, let $\omega_g$ be the meromorphic differential on $X_\Gamma(\mathbb{C})$ whose pullback to the Poincaré upper half-plane $\mathcal{H}$ equals $g(z)dz$. The differential $\omega_g$ has all its poles supported at the cusps of $X_\Gamma$. Moreover, $g$ is a cusp form if and only if $\omega_g$ is holomorphic, or, $Res_x(\omega_g) = 0$ for any $x \in cusp(\Gamma)$. Denote by $Div^0(cusp(\Gamma); \mathbb{C})$ to be $Div^0(cusp(\Gamma); \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$, then we define the following homomorphism of $\mathbb{C}$-vector spaces
\[
\delta_\Gamma : E_2(\Gamma, \mathbb{C}) \to Div^0(cusp(\Gamma); \mathbb{C}),
\]
such that
\[
E \mapsto 2\pi i \sum_{x \in cusp(\Gamma)} Res_x(\omega_E) \cdot [x],
\]
with $2\pi i \cdot Res_x(\omega_E) = e_x \cdot a_0(E; [x])$, where $e_x$ is the ramification index of $X_\Gamma$ at $x$ and $a_0(g; [x])$ is the constant term of the Fourier expansion of $g$ at the cusp $x$. The homomorphism $\delta_\Gamma$ is actually an isomorphism by the theorem of Manin-Drinfeld. Because the restriction of $\omega_E$ to $Y_\Gamma$ is holomorphic, this differential induces the following periods integral homomorphism
\[
\xi_E : H_1(Y_\Gamma(\mathbb{C}), \mathbb{Z}) \to \mathbb{C}, \quad [c] \mapsto \int_c \omega_E
\]
where $[c]$ is the homology class represented by a 1-cycle $c$ on $Y_\Gamma$. Note that, for any cusp $x$, we have
\[
\int_{c_x} \omega = 2\pi i \cdot Res_x(\omega_E),
\]
where $c_x$ is a small circle around $x$.

**Definition 2.1.** Let $E \in E_2(\Gamma, \mathbb{C})$ be a weight-two Eisenstein series of level $\Gamma$. We denote by $\mathcal{R}_\Gamma(E)$ to be the sub-$\mathbb{Z}$-module of $\mathbb{C}$ generated by the coefficients of $\delta_\Gamma(E)$, and by $\mathcal{R}(E)^{\vee}$ to be the dual $\mathbb{Z}$-module of $\mathcal{R}(E)$. Then :

1. The cuspidal subgroup $C_\Gamma(E)$ associated with $E$ is defined to be the subgroup of $J_\Gamma(\mathbb{Q})$ which is generated by $\{w_\Gamma(\phi \circ \delta_\Gamma(E))\}_{\phi \in \mathcal{R}(E)^{\vee}}$, where $w_\Gamma$ is the Atkin-Lehner involution;
2. The periods $\mathcal{P}_\Gamma(E)$ of $E$ is defined to be the image of $\mathcal{P}(E)$. Since $\mathcal{P}_\Gamma(E)$ contains $\mathcal{R}_\Gamma(E)$ by the above remark, we can define $A_\Gamma(E)$ to be the quotient $\mathcal{P}_\Gamma(E)/\mathcal{R}_\Gamma(E)$;
3. The Eisenstein ideal $I_\Gamma(E)$ of $E$ is defined to be the image of $Ann_\mathcal{R}_\Gamma(E)$ in $\mathcal{T}_\Gamma$.

**Remark 2.2.** The above definition of $C_\Gamma(E)$ is slightly different from that given in [11], as we have added an action of the Atkin-Lehner operator $w_\Gamma$. Since $w_\Gamma$ is an isomorphism, this modification does not change the order of the associated cuspidal subgroups. However, $C_\Gamma(E)$ is now annihilated by $I_\ell(E)$ under the usual action of the Hecke algebra, because $T_\ell^i \circ \delta_\Gamma = \delta_\Gamma \circ T_\ell$ and $T_\ell^i \circ w_\Gamma = w_\Gamma \circ T_\ell$ for any prime $\ell$.

2.3. By Proposition 1.1 and Theorem 1.2 of [12], $A_\Gamma(E)$ is finite and there is a perfect pairing $C_\Gamma(E) \times A_\Gamma(E) \to \mathbb{Q}/\mathbb{Z}$, thus, the determination of the order of $C_\Gamma(E)$ is reduced to that of $\mathcal{P}_\Gamma(E)$. In the following, we will recall a method due to Stevens for the computation of the periods. The reader is referred to [12] for details.

We first consider the case when $\Gamma = \Gamma_1(N)$. Denote by $S_N$ to be the set of all primes $p$ satisfying $p \equiv 1 \pmod{4N}$. Let $X_N$ be the set of all non-quadratic Dirichlet character $\chi$ whose conductor is a prime in $S_N$, and $X_N^\infty$ be the set of all non-quadratic Dirichlet character $\chi$ whose conductor is of the form $p_N^M$ with $p_N \in S_N$ and $M$ some positive integer.

For any $E = \sum_{n=0}^\infty a_n(E; [\infty]) \cdot q^n \in E_2(\Gamma_1(N), \mathbb{C})$ and any Dirichlet character $\chi$, the $L$-function associated to the pair $(E, \chi)$ is defined as
\[
L(E, \chi, s) := \sum_{n=1}^\infty \frac{a_n(E; [\infty]) \cdot \chi(n)}{n^s}.
\]
If $\chi \in X_N^\infty$ is of conductor $p_N^M$, then we define
\[
\Lambda(E, \chi, 1) := \frac{\tau(\chi) \cdot L(E, \chi, 1)}{2\pi i},
\]
\[
\Lambda_{\pm}(E, \chi, 1) := \frac{1}{2}(\Lambda(E, \chi, 1) \pm \Lambda(E, \chi \cdot (p_N^M), 1)),
\]
where \( \left( \frac{z}{p} \right) \) is the Legendre symbol associated to \( p \). It is proved in Theorem 1.3 of [12] that, if \( M \) is a finitely generated sub-\( \mathbb{Z} \)-module of \( \mathcal{C} \), then the following are equivalent:

(S1) \( \mathcal{P}_{\Gamma_1(N)}(E) \subseteq M \);  
(S2) \( \mathcal{R}_{\Gamma_1(N)}(E) \subseteq M \) and \( \Lambda_\pm(E, \chi, 1) \in M[\chi, \frac{1}{\chi}] \) for any \( \chi \in \mathcal{X}_0 \);  
(S3) \( \mathcal{R}_{\Gamma_1(N)}(E) \subseteq M \) and \( \Lambda_\pm(E, \chi, 1) \in M[\chi, \frac{1}{\chi}] \) for any \( \chi \in \mathcal{X}_\infty^\chi \).

Because \( \Lambda_\pm(E, \chi, 1) \) is essentially the Bernoulli numbers whose integrality and divisibility are well known (see Theorem 4.2 of [12]), we can then use the above result to determine the periods \( \mathcal{P}_{\Gamma_1(N)}(E) \) of \( E \) and hence the order of \( C_{\Gamma_1(N)}(E) \).

On the other hand, if \( \Gamma = \Gamma_0(N) \), then Stevens’ method can only determine \( C_{\Gamma_0(N)}(E) \) up to its intersection with the Shimura subgroup. Recall that, if we denote by \( \pi_\gamma \) to be the natural projection of \( X_1(N) \) to \( X_0(N) \), then the Shimura subgroup of \( J_0(N) \) is defined to be

\[
\Sigma_N := \ker (\pi_0^\gamma : J_0(N) \to J_1(N)),
\]

which is a finite abelian group and is of multiplicative type as a \( G_\mathbb{Q} \)-module. For any \( E \in \mathcal{E}_2(\Gamma_0(N), \mathbb{C}) \), we define

\[
A_{\Sigma_N}(E) := \left( \mathcal{P}_{\Gamma_1(N)}(E) + \mathcal{R}_{\Gamma_0(N)}(E) \right) / \mathcal{R}_{\Sigma_N}(E),
\]

then there is an exact sequence

\[
0 \to \Sigma_N \cap C_{\Gamma_0(N)}(E) \to C_{\Gamma_0(N)}(E) \to A_{\Sigma_N}(E) \to 0,
\]

which enables us to determine the order of \( C_{\Gamma_0(N)}(E) / \Sigma_N \cap C_{\Gamma_0(N)}(E) \).

2.4 Finally, we recall some basic properties of the collection of functions \( \{ \phi_\mathbf{z} \}_{\mathbf{z} \in (\mathbb{Q}/\mathbb{Z})^{\oplus 2}} \) due to Hecke (see [11], Chapter 2, §2.4) which we will need later. For any \( \mathbf{z} = (x_1, x_2) \in (\mathbb{Q}/\mathbb{Z})^{\oplus 2} \), the Fourier expansion of \( \phi_\mathbf{z} \) at infinity is

\[
\phi_\mathbf{z}(z) + \delta(\mathbf{z}) \cdot \frac{i}{2\pi(z-\mathbf{z})} = \frac{1}{2} B_2(x_1) - P_\mathbf{z}(z) - P_\mathbf{z}^\ast(z)
\]

for any \( \mathbf{z} \in \mathcal{H} \), where \( B_2(t) = (t)^2 - (t) + \frac{1}{6} \) is the second Bernoulli polynomial and

\[
P_\mathbf{z}(z) = \sum_{k \in \mathbb{Q}_{>0}, \mathbf{z} \equiv \mathbf{1}(1)} k \sum_{m=1}^{\infty} e^{2\pi i m k z + x z}
\]

and \( \delta(\mathbf{z}) \) is defined to be 1 or 0 according to \( \mathbf{z} = 0 \) or not. If \( \mathbf{z} \neq 0 \), then \( \phi_\mathbf{z} \) is a (holomorphic) Eisenstein series. Moreover, for any \( \mathbf{z} \in (\mathbb{Q}/\mathbb{Z})^{\oplus 2} \) and \( \gamma \in SL_2(\mathbb{Z}) \), we have

\[
\phi_\mathbf{z} \gamma = \phi_{\mathbf{z} \gamma}
\]

where \( \mathbf{z} \cdot \gamma \) is the natural right action of \( \gamma \) on the row vector of length two. The whole collection of functions satisfy the following important distribution law

\[
\phi_{\mathbf{z}} = \sum_{\mathbf{z} \cdot \mathbf{a} = \mathbf{z}} \phi_{\mathbf{a}}|\mathbf{a}|
\]

where \( \mathbf{a} \) is any matrix in \( M_2(\mathbb{Z}) \) with positive determinant.

3. AN EIGEN-BASIS FOR \( \mathcal{E}_2(\Gamma_0(DC), \mathbb{C}) \)

In this section, we will construct a basis for \( \mathcal{E}_2(\Gamma_0(DC), \mathbb{C}) \) which plays a fundamental role in our later investigations. We will also show that the Eisenstein series in this basis are all eigenforms.

3.1. We first introduce some operators on the \( \mathbb{C} \)-vector space \( \mathcal{M}_2 \) of weight-two holomorphic modular forms of all levels. For any prime \( p \), we define an operator \( \gamma_p \) on \( \mathcal{M}_2 \) as following

\[
\gamma_p : \mathcal{M}_2 \to \mathcal{M}_2, \ g \mapsto g \left( \begin{array}{cc} p & 0 \\ 0 & 1 \end{array} \right).
\]

If \( \psi \) be a Dirichlet character of conductor \( f_\psi \) and \( p \nmid f_\psi \) is a prime, then we define the following two operators \( [p]_{\psi}^\pm \) on \( \mathcal{M}_2 \) as

\[
[p]_{\psi}^+ := 1 - \psi(p) \cdot \gamma_p
\]

\[
[p]_{\psi}^- := 1 - p^{-1} \cdot \psi^{-1}(p) \cdot \gamma_p
\]
More precisely, for any $g \in M_2$ and any $z$ in the Poincaré upper half-plane $\mathcal{H}$, we have that

$$[p]_\psi^+(g)(z) = g(z) - p \cdot \psi(p) \cdot g(pz),$$

$$[p]_\psi^-(g)(z) = g(z) - \psi^{-1}(p) \cdot g(pz).$$

It is clear that if $p_1$ and $p_2$ are two primes not dividing $f_\psi$, then the four operators $[p_1]_\psi^+, [p_2]_\psi^+, [p_2]_\psi^-$ and $[p_1]_\psi^-$ are commutative with each other. Thus we can define, for any positive square-free integer $M$ prime to $f_\psi$, two operators $[M]_\psi^\pm$ on $M_2$ as

$$[M]_\psi^\pm := [p_1]_\psi^+ \circ [p_2]_\psi^+ \circ \ldots \circ [p_k]_\psi^+,$$

with $M = p_1 \cdot p_2 \cdots p_k$ in any order. When $\psi = 1$ is the trivial Dirichlet character, we will write $[M]_\psi^+$ simply as $[M]^\pm$ for any positive square-free integer $M$.

**Remark 3.1.** It is easy to see that the above operators $[M]_\psi^\pm$ can also be applied to any function on $\mathcal{H}$ in the same manner. In particular, we have that

$$[p]_\psi^+(\frac{1}{z - \overline{z}}) = \frac{1}{z - \overline{z}} - \frac{p}{pz - \overline{pz}} = 0,$$

for any prime $p$. It follows that $[M]_\psi^+(\frac{1}{z - \overline{z}}) = 0$ for any square-free integer $M > 1$.

**Lemma 3.2.** Let $\psi$ be a Dirichlet character of conductor $f_\psi$, $p \nmid f_\psi$ be a prime and $N$ be a positive integer, then $[p]_\psi^\pm$ maps $M_2(\Gamma_0(N), \mathbb{C})$ to $M_2(\Gamma_0(Np), \mathbb{C})$ and satisfies the following properties

1. For any prime $\ell \neq p$, we have that $\mathcal{T}_\ell^{\Gamma_0(Np)} \circ [p]_\psi^+ = [p]_\psi^+ \circ \mathcal{T}_\ell^{\Gamma_0(N)}$;
2. If $p \nmid N$, then $\mathcal{T}_p^{\Gamma_0(Np)} \circ [p]_\psi^+ = \mathcal{T}_p^{\Gamma_0(N)} - \gamma_p - p \cdot \psi(p)$ and $\mathcal{T}_p^{\Gamma_0(Np)} \circ [p]_\psi^- = \mathcal{T}_p^{\Gamma_0(N)} - \gamma_p - \psi^{-1}(p)$;
3. If $p | N$, then $\mathcal{T}_p^{\Gamma_0(Np)} \circ [p]_\psi^+ = \mathcal{T}_p^{\Gamma_0(N)} - p \cdot \psi(p)$ and $\mathcal{T}_p^{\Gamma_0(Np)} \circ [p]_\psi^- = \mathcal{T}_p^{\Gamma_0(N)} - \psi^{-1}(p)$.

**Proof.** Since $\gamma_0$ maps $M_2(\Gamma_0(N), \mathbb{C})$ to $M_2(\Gamma_0(Np), \mathbb{C})$ and $[p]_\psi^\pm$ is defined to be a linear combination of the identity map and $\gamma_p$, we find that $[p]_\psi^\pm$ also maps $M_2(\Gamma_0(N), \mathbb{C})$ to $M_2(\Gamma_0(Np), \mathbb{C})$. Moreover, if $\ell$ is a prime and $\ell \neq p$, then $\gamma_\ell$ commutes with $\mathcal{T}_\ell = \sum_{k=0}^{\ell-1} \begin{pmatrix} k & \ell \ 0 & \ell \end{pmatrix} + \begin{pmatrix} \ell & 0 \ 0 & 1 \end{pmatrix}$ (or $\sum_{k=0}^{\ell-1} \begin{pmatrix} k & 0 \ 0 & \ell \end{pmatrix}$) if $\ell \nmid N$ (or respectively $\ell | N$) as operators on corresponding space of modular forms, so the first assertion follows.

If $p \nmid N$, then we have by definition that

$$\mathcal{T}_p^{\Gamma_0(Np)} \circ [p]_\psi^+(g) = g\left[1 - \psi(p) \cdot \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right] \sum_{k=0}^{p-1} \begin{pmatrix} k & \ell \ 0 & p \end{pmatrix},$$

$$= g\sum_{k=0}^{p-1} \begin{pmatrix} 1 & k \\ 0 & p \end{pmatrix} - \psi(p) \cdot g\sum_{k=0}^{p-1} \begin{pmatrix} p & pk \\ 0 & p \end{pmatrix},$$

$$= \mathcal{T}_p^{\Gamma_0(N)}(g) - f|\gamma_p - p \cdot \psi(p) \cdot g,$$

for any $g \in M_2(\Gamma_0(N), \mathbb{C})$; similarly, we have by definition that

$$\mathcal{T}_p^{\Gamma_0(Np)} \circ [p]_\psi^-(g) = g\left[1 - p^{-1} \cdot \psi^{-1}(p) \cdot \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right] \sum_{k=0}^{p-1} \begin{pmatrix} k & \ell \ 0 & p \end{pmatrix},$$

$$= g\sum_{k=0}^{p-1} \begin{pmatrix} k & 0 \\ 0 & p \end{pmatrix} - p^{-1} \cdot \psi^{-1}(p) \cdot g\sum_{k=0}^{p-1} \begin{pmatrix} p & pk \\ 0 & p \end{pmatrix},$$

$$= \mathcal{T}_p^{\Gamma_0(N)}(g) - f|\gamma_p - \psi^{-1}(p) \cdot g,$$

so the second assertion follows. The proof of the third assertion is similar and we leave it to the reader. $\square$

3.2. It is well known that the number of cusps of $X_0(DC)$ is equal to $\sum_{1 \leq d | DC} \varphi(d, DC/d)$, so we find that $\dim_\mathbb{C} E_2(\Gamma_0(DC), \mathbb{C}) = \sum_{1 \leq d | DC} \varphi(d, DC/d)$. Here $\varphi(d, DC/d)$ means applying Euler’s $\varphi$-function to the greatest common divisor of $d$ and $DC/d$. We define $\mathcal{H}(DC)$ to be the set of all triples $(M, L, \psi)$ where $1 \leq M, L | D$ with $M \neq 1$, $D | ML | DC$ and $\psi$ is a Dirichlet character modulo $(M, L)$. Note that the condition "$M \neq 1$" is automatically satisfied if $\psi \neq 1$.

**Lemma 3.3.** $\# \mathcal{H}(DC) = \dim_\mathbb{C} E_2(\Gamma_0(DC), \mathbb{C})$
Proof. By the above remark, we only need to prove that \( \# \mathcal{H}(DC) = \sum_{1 < d | DC} \varphi(d, \frac{DC}{d}) \). We will first prove this when \( C = D \). For any positive divisor \( d \) of \( D^2 \), we can associate the following two positive integers

\[
M := \sqrt{d \cdot (d, \frac{D^2}{d})}, \quad L := \sqrt{\frac{D^2}{d} \cdot (d, \frac{D^2}{d})}
\]

such that \( 1 \leq M, L \mid D \) and \( D \mid ML \mid D^2 \). Conversely, to any pair of integers \( M \) and \( L \) with \( 1 \leq M, L \mid D \) and \( D \mid ML \mid D^2 \), we can associate a positive divisor \( d \) of \( D \) as

\[
d := \left[ \frac{M}{(M, L)} \right]^2 \cdot (M, L)
\]

It is easy to see that the above establishes a bijection between \( \{d : 1 \leq d \mid D^2\} \) and the set of all pair of integers \( M \) and \( L \) with \( 1 \leq M, L \mid D \) and \( D \mid ML \mid D^2 \). Moreover, under this bijection, the divisor 1 of \( D^2 \) corresponds to the pair \( M = 1 \) and \( L = D \), and we have \( (d, D^2/d) = (M, L) \) if \( d \) corresponds to \( M \) and \( L \). It follows that there is a bijection between \( \{(d, \psi) : 1 < d \mid D^2, \psi : (\mathbb{Z}/(d, D^2/d) \cdot \mathbb{Z})^\times \to \mathbb{C}^\times\} \) and \( \mathcal{H}(D^2) \) which proves the lemma in this situation.

In general, since \( DC = D^2 \cdot C^2 \), any positive divisor \( d \) of \( DC \) can be uniquely decomposed as \( d = d_0 \cdot d' \) with \( 1 \leq d_0 \mid \frac{D^2}{d} \) and \( 1 \leq d' \mid C^2 \). If such a positive divisor \( d' \) of \( C^2 \) corresponds to a pair of integer \( m \) and \( \ell \) with \( 1 \leq m, \ell \mid C \) and \( \ell \mid m \mid C^2 \) as above, then we can associate with \( d' \) the pair of integers \( M = d_0 \cdot m \) and \( DC \cdot d_0 \cdot \ell \) which satisfies \( 1 \leq M, L \mid D \) and \( D \mid ML \mid DC \). This establishes a bijection between \( \{d : 1 \leq d \mid DC\} \) and the set of all pair of integers \( M \) and \( L \) with \( 1 \leq M, L \mid D \) and \( D \mid ML \mid DC \). Moreover, we have \( 1 \mid D^2 \) corresponds to the pair \( M = 1 \) and \( L = D \), and \( (d, \frac{D^2}{d}) = (M, L) \) if \( d \) corresponds to \( M \) and \( L \). It follows that there is a bijection between \( \{(d, \psi) : 1 < d \mid D^2, \psi : (\mathbb{Z}/(d, DC/d) \cdot \mathbb{Z})^\times \to \mathbb{C}^\times\} \) and \( \mathcal{H}(DC) \) which completes the proof the lemma. \( \square \)

**Definition 3.4.** For any Dirichlet character \( \psi \) of conductor \( f = f_\psi \), let

\[
E_\psi := -\frac{1}{2g(\psi)} \sum_{a \in (\mathbb{Z}/f \mathbb{Z})^\times} \sum_{b \in (\mathbb{Z}/f^2 \mathbb{Z})^\times} \psi(a) \cdot \psi(b) \cdot \phi(\frac{a}{f}, \frac{b}{f}).
\]

Then we define

\[
E_{M, L, \psi} := \left[ f \right]_0 \circ \left[ \frac{M}{f} \right]_0 (E_\psi),
\]

for any \( (M, L, \psi) \in \mathcal{H}(DC) \), where \( g(\psi) \) is the Gauss sum of \( \psi \).

From Eq.\((2.1)\), it is easy to see that

\[
E_\psi = -\frac{\delta_\psi}{4\pi i (z - \overline{z})} - \frac{1}{4g(\psi)} \sum_{x \in (\mathbb{Z}/f \mathbb{Z})^\times} \sum_{y \in (\mathbb{Z}/f^2 \mathbb{Z})^\times} \psi(x) \cdot \psi(y) \cdot B_2(\frac{x}{f})
\]

\[
+ \frac{1}{g(\psi)} \sum_{x \in (\mathbb{Z}/f \mathbb{Z})^\times} \sum_{y \in (\mathbb{Z}/f^2 \mathbb{Z})^\times} \psi(x) \cdot \psi(y) \cdot P(\frac{x}{f}, \frac{y}{f}),
\]

where \( \delta_\psi \) is equal to 1 or 0 according to \( \psi \) is trivial or not. Since we have by Eq.\((2.2)\) that

\[
\sum_{x \in (\mathbb{Z}/f \mathbb{Z})^\times} \sum_{y \in (\mathbb{Z}/f^2 \mathbb{Z})^\times} \psi(x) \cdot \psi(y) \cdot P(\frac{x}{f}, \frac{y}{f}) = \sum_{k, m = 1}^{\infty} \frac{k \psi(k)}{f} \left( \sum_{y \in (\mathbb{Z}/f^2 \mathbb{Z})^\times} \psi(y) e^{2\pi i \frac{m}{f^2}} \right) e^{2\pi i \frac{km}{f} z}
\]

\[
= \sum_{k, m = 1}^{\infty} \frac{k \psi(k)}{f} \left( \sum_{y \in (\mathbb{Z}/f^2 \mathbb{Z})^\times} \psi(y) e^{2\pi i \frac{m}{f^2}} \right) e^{2\pi imkz}
\]

\[
= g(\psi) \sum_{k, m = 1}^{\infty} k \cdot \psi(k) \cdot \psi^{-1}(m) \cdot e^{2\pi imkz},
\]

with \( \psi(n) \) defined to be 0 when \( (n, f) \neq 1 \) as usual, we find thus

\[
E_\psi = -\frac{\delta_\psi}{4\pi i (z - \overline{z})} + a_0(E_\psi; [\infty]) + \sum_{n = 1}^{\infty} \sigma_0(n) \cdot q^n,
\]

(3.1)
Proposition 3.7. Notations are as above, then we have that

$\sigma_\psi(n) := \sum_{d \mid n} d \cdot \psi(d) \cdot \psi^{-1}(n/d)$

In particular, we find that $a_1(E_\psi; [\infty]) = 1$, which means $E_\psi$ is normalized. Because $|M|^+(\frac{1}{\psi}) = 0$ for any $M > 1$ as we have see in Remark 3.3, it follows from the definition and Eq. (3.1) that $E_{M,L,\psi}$ is always holomorphic and hence belongs to $E_2(\Gamma_0(DC), \mathbb{C})$.

Lemma 3.5. $E_{M,L,\psi}$ is normalized for any $(M, L, \psi) \in \mathcal{H}(DC)$.

Proof. Because $g \gamma_p = \sum_{n=0}^{\infty} (pa_n) \cdot q^n$ for any prime $p$ and function $g$ of the form $\sum_{n=0}^{\infty} a_n \cdot q^n$, we find that $a_1(g; [\infty]) = 0$ and hence $a_1([p]_\psi^\dagger(g; [\infty]) = a_1(g; [\infty])$. By the above discussion, $E_\psi$ is normalized, so the assertion follows.

Lemma 3.6. For any non-trivial Dirichlet character $\psi$ of conductor $f_\psi = f$, we have that

$T_{\ell}^{\Gamma_0(f^2)}(E_\psi) = \begin{cases} (\psi^{-1}(\ell) + \ell \cdot \psi(\ell)) \cdot E_\psi & , if \ell \nmid f \\ 0 & , if \ell \mid f. \end{cases}$

Proof. By Proposition 2.4.7 of [11], we have that

$T_{\ell}^{\Gamma(f^2)}(\phi(\overline{\mu}, \overline{\beta})) = \phi(\overline{\mu}, \overline{\beta}) + \ell \cdot \phi(\overline{\mu}, \overline{\beta})$

for any $\ell \nmid f$, where $\ell'$ is an integer such that $\ell' \equiv 1 \pmod{f}$ and $T_{\ell}^{\Gamma(f^2)}$ is the $\ell$-th Hecke operator of level $\Gamma(f^2)$. It follows that

$T_{\ell}^{\Gamma_0(f^2)}(E_\psi) = (\psi^{-1}(\ell) + \ell \cdot \psi(\ell)) \cdot E_\psi,$

for any prime $\ell \nmid f$. On the other hand, since

$E_\psi = -\frac{1}{2g(\psi)} \sum_{x, y \in (\mathbb{Z}/f\mathbb{Z})^2} \psi(x) \cdot \psi(y) \cdot \phi(\overline{x}, \overline{y}) \left( \begin{array}{cc} f & 0 \\ 0 & 1 \end{array} \right)$

by the distribution law, we find that

$T_{\ell}^{\Gamma_0(f^2)}(E_\psi) = -\frac{1}{2g(\psi)} \sum_{x, y \in (\mathbb{Z}/f\mathbb{Z})^2} \psi(x) \cdot \psi(y) \cdot \phi(\overline{x}, \overline{y}) \left( \begin{array}{cc} f & 0 \\ 0 & 1 \end{array} \right) \sum_{k=0}^{\ell-1} \binom{\ell - 1}{k} \binom{\ell - 1 - k}{\ell - 1} \left( \begin{array}{cc} 1 & k \\ 0 & \ell \end{array} \right) \left( \begin{array}{cc} f & 0 \\ 0 & \ell \end{array} \right)$

for any prime $\ell \nmid f$, with the last equality holds because $\psi$ is primitive of conductor $f$, and hence complete the proof of the lemma.

Proposition 3.7. Notations are as above, then we have that

1. $E_{M,L,\psi}$ is normalized for any $(M, L, \psi) \in \mathcal{H}(DC)$, that is to say, $a_1(E_{M,L,\psi}; [\infty]) = 1$ for any $(M, L, \psi) \in \mathcal{H}(DC)$. In particular, all these Eisenstein series are non-zero;

2. For any $(M, L, \psi) \in \mathcal{H}(DC)$, the Hecke operators act on $E_{M,L,\psi}$ as

$T_{\ell}^{\Gamma_0(DC)}(E_{M,L,\psi}) = \begin{cases} \left( \psi^{-1}(\ell) + \ell \cdot \psi(\ell) \right) \cdot E_{M,L,\psi} & , if \ell \nmid D \\ \psi^{-1}(\ell) \cdot E_{M,L,\psi} & , if \ell \mid \frac{M}{(M, L)} \\ \ell \cdot \psi(\ell) \cdot E_{M,L,\psi} & , if \ell \mid \frac{M}{(M, L)} \\ 0 & , if \ell \mid (M, L) \end{cases}$

3. $\mathcal{E}_2(\Gamma_0(DC), \mathbb{C}) = \bigoplus_{(M, L, \psi) \in \mathcal{H}(DC)} \mathbb{C} \cdot E_{M,L,\psi}$. 

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Proof. We have already proved the first assertion in Proposition 3.5. Lemma 3.3 implies that the number of the Eisenstein series that we introduced equals the dimension of the $C$-vector space $E_2(\Gamma_0(DC), C)$. Thus, to prove the third assertion, it is enough to show that all these Eisenstein series are linearly independent over $C$. So we only need to prove the second assertion, which implies that the Eisenstein series have different eigenvalues and hence are linearly independent.

If $\ell$ is a prime not dividing $D$, then we find by (1) of Lemma 3.2 and Lemma 3.6 that

$$\mathcal{T}_\ell \Gamma_0(DC) = \left[ \frac{L}{f} \right]_{\psi} \circ \left[ \frac{M}{f} \right]_{\psi} \circ \mathcal{T}_\ell \Gamma_0(f^2) \circ (E_\psi)$$

and hence complete the proof.

If $\ell$ is a prime divisor of $\frac{M}{(M, L)}$, then we have by (2) of Lemma 3.2 that

$$\mathcal{T}_\ell \Gamma_0(DC)(E_{M, L, \psi}) = \left[ \frac{L}{f} \right]_{\psi} \circ \left[ \frac{M}{f} \right]_{\psi} \circ \mathcal{T}_\ell \Gamma_0(f^2) \circ [\ell]_{\psi} (E_\psi)$$

$$= \left[ \frac{L}{f} \right]_{\psi} \circ \left[ \frac{M}{f} \right]_{\psi} \circ (\psi^{-1}(\ell) \cdot E_{M, L, \psi})$$

The proofs for those primes $\ell \mid \frac{L}{(M, L)}$ and $\ell \mid \frac{M}{(M, L)}$ are similar to the above, so we omit it here.

Finally, if $\ell \mid f$, then we have that

$$\mathcal{T}_\ell \Gamma_0(DC)(E_{M, L, \psi}) = \left[ \frac{L}{f} \right]_{\psi} \circ \left[ \frac{M}{f} \right]_{\psi} \circ \mathcal{T}_\ell \Gamma_0(f^2) \circ (\psi^{-1}(\ell) \cdot E_{M, L, \psi}) = 0$$

and hence complete the proof. □

4. The quadratic subgroups of $C_0(DC)$

4.1. In this section, we study the cuspidal subgroups associated to those $E_{M, L, \psi}$ with $\psi$ a quadratic character. We begin with some preliminaries.

Lemma 4.1. If we take $r$ to be a positive divisor of $\frac{D}{f}$, and let $s, t$ two positive divisors of $C$ satisfying $(s, t) = 1$ and let $x$ runs over a set of representatives of $(\mathbb{Z}/t\mathbb{Z})^\times$ which are prime to $D$, then $\{[\frac{r^2 s^2 x^2}{D}]\}$ is a full set of representatives for the cusps of $X_0(DC)$.

Proof. It is clear that any divisor of $DC = \frac{D}{f} \cdot C^2$ is of the form $r s^2 t$ with some $r, s, t$ as above. Since $(r s^2 t, \frac{DC}{r s^2 t}) = t$ for any such a divisor, we find that the above set has at most $\sum_{1 \leq d \mid DC} \varphi(d, \frac{DC}{d})$ elements. Thus, it is enough to prove that the above are all different cusps as the number of cusps of $X_0(DC)$ is also $\sum_{1 \leq d \mid DC} \varphi(d, \frac{DC}{d})$.

Suppose $\frac{x_1 s_1 t_1 x_1}{DC} = \frac{r x_2 s_2 t_2 x_2}{DC}$, then there exists some $\gamma = \left( \begin{array}{cc} \alpha & \beta \\ DC\delta & \omega \end{array} \right) \in \Gamma_0(DC)$ such that

$$\gamma(\frac{x_1 s_1 t_1 x_1}{DC}) = \frac{r x_2 s_2 t_2 x_2}{DC}. \quad \text{It follows that}$$

$$r_2 s_2^2 t_2 x_2 = r_1 s_1^2 t_1 \cdot \frac{\alpha r_1 x_1 + \beta r_1 x_1 + \omega}{r_1 s_1^2 t_1 x_1 + \omega}.$$ 

But since $\delta r_1 s_1^2 t_1 x_1 + \omega$ is a unit at every prime dividing $r_1 s_1 t_1$, we find that $r_1, s_1, t_1$ divides $r_2, s_2, t_2$ respectively, and hence $r_1 = r_2, s_1 = s_2$ and $t_1 = t_2$ by symmetry. If we choose some $u_1, v_1 (i = 1, 2)$ such that

$$\left( \begin{array}{c} x_1 \\ DC \end{array} \right) \in SL_2(\mathbb{Z}),$$

then

$$\gamma(\left( \begin{array}{c} x_1 \\ DC \end{array} \right)) = \left( \begin{array}{c} x_2 \\ DC \end{array} \right) \left( \begin{array}{c} u_2 \\ v_2 \end{array} \right),$$

so that there exists some integer $n$ such that

$$\pm \gamma(\left( \begin{array}{c} x_1 \\ DC \end{array} \right)) = \left( \begin{array}{c} x_2 \\ DC \end{array} \right) \left( \begin{array}{c} u_2 \\ v_2 \end{array} \right),$$

which implies, after a straight forward calculation, that

$$\frac{DC}{r s^2 t} (v_1) = \frac{DC}{r s^2 t} v_2 \equiv \frac{n DC}{r s^2 t} \mod DC.$$
Because $t^2 \mid DC$, it follows that $v_1 \equiv v_2 \pmod{t}$. We find thus $x_1 \equiv x_2 \pmod{t}$ which completes the proof of the lemma. 

We will always use the above kind of representatives for cusps in the following investigation.

**Lemma 4.2.** Let $p$ be a prime divisor of $D$ and $[\frac{rs^2tx}{DC}]$ be a cusp of $X_0(\frac{DC}{p})$, then we have that:

1. If $p \mid r$, then $[\frac{rs^2tx}{DC}] = [\frac{(r/p)rs^tx}{DC/p}]$ in $X_0(\frac{DC}{p})$;
2. If $p \mid s$, then $[\frac{rs^2tx}{DC}] = [\frac{rs^tx}{DC/p}]$ in $X_0(\frac{DC}{p^2})$;
3. If $p \mid t$, then $[\frac{rs^2tx}{DC}] = [\frac{(r/p)tx}{DC/p}]$ in $X_0(\frac{DC}{p^3})$;
4. If $p \mid \frac{DC}{t}$, then $[\frac{rs^2tx}{DC}] = [\frac{rs^tx}{DC/p}]$ in $X_0(\frac{DC}{p})$;
5. If $p \mid \frac{DC}{rs}$, then $[\frac{rs^2tx}{DC}] = [\frac{rs^tx}{DC/p}]$ in $X_0(\frac{DC}{p^2})$.

**Proof.** The first two assertions are obvious. Since the proofs of last three assertions are similar, we will in the following only give that of (3). If $[\frac{rs^2tx}{DC}] = [\frac{rs^tx}{DC/p}]$ in $X_0(\frac{DC}{p^3})$, then there exists some

$$\gamma = \left( \frac{\alpha}{DC}, \frac{\beta}{p}, \frac{\gamma}{\omega} \right) \in \Gamma_0(\frac{DC}{p^3})$$

sending the former point to the latter one, and we find thus

$$r's^2t'x' = rs^2(t/p) \cdot \frac{x + \frac{\beta}{rs^2(t/p)x + \omega p}}{\frac{\delta}{rs^2(t/p)x + \omega p}}.$$

Since $\delta rs^2(t/p)x + \omega p$ is a unit for any prime dividing $rs^2(t/p)$, it follows that $r, s, t/p$ divides $r', s', t'$ respectively. We find thus

$$\frac{r'}{r} \cdot \frac{s'^2}{s^2} \cdot \frac{t'}{t/p} \cdot x' = \frac{x + \frac{\beta}{rs^2(t/p)x + \omega p}}{\frac{\delta}{rs^2(t/p)x + \omega p}}.$$

If there is some prime $q \mid r's't'$ (so that $q \neq p$ if $p \mid t'$) but not dividing $rst$, then $x + \frac{\beta}{rs^2(t/p)x + \omega p}$ will be a $q$-adic unit. But this contradicts to the above equation, so we have proved the assertion. 

Let $K$ be a positive divisor of $D$ and $1 \leq \alpha \mid K$. It is not difficult to deduce from the above lemma that: if $(K, rs) = 1$, then

$$[\frac{rs^2tx}{DC}] = [\frac{rs^2t(K/K, C)x}{DC/K(K, C)}] \in X_0(\frac{DC}{K(K, C)});$$

and if $K \mid t$, then

$$[\frac{rs^2tx}{DC}] = [\frac{rs^2t(K/K, C)x}{DC/K^2}] \in X_0(\frac{DC}{K^2}).$$

We leave the verifications to the reader. Finally, we give some general observation about how the constant terms of modular forms behave under the operators $[p]_{\psi}^\beta$. Let $N$ be a positive integer and $g \in M_2(\Gamma_0(N), \mathbb{C})$. Let $[\mathfrak{a}]$ be a cusp represented by two co-prime integers $a, c$, and let $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$ be a matrix in $\text{SL}_2(\mathbb{Z})$ such that $\gamma([\mathfrak{a}]) = [\mathfrak{a}]$. For any prime $p$, we may and will always assume $p \mid d$ when $p \nmid c$. If $p$ is prime to the conductor of $\psi$, then since

$$\gamma_p \cdot \gamma = \left\{ \begin{array}{ll} \left( a \quad pb \\ c/p \quad d \end{array} \right) \left( \begin{array}{cc} p & 0 \\ 0 & 1 \end{array} \right), & \text{if } p \mid c \\
\left( ap \quad b \\ c \quad d/p \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right), & \text{if } p \nmid c, \right.$$

it follows that

$$a_0([p]_{\psi}^\beta(g); [\mathfrak{a}]) = \left\{ \begin{array}{ll} a_0(g; [\mathfrak{a}]) - p \cdot \psi(p) \cdot a_0(g; [\mathfrak{a}]), & \text{if } p \mid c \\
a_0(g; [\mathfrak{a}]) - p^{-1} \cdot \psi(p) \cdot a_0(g; [\mathfrak{a}]), & \text{if } p \nmid c, \right.$$

and

$$a_0([p]_{\psi}^\beta(g); [\mathfrak{a}]) = \left\{ \begin{array}{ll} a_0(g; [\mathfrak{a}]) - \psi^{-1}(p) \cdot a_0(g; [\mathfrak{a}]), & \text{if } p \mid c \\
a_0(g; [\mathfrak{a}]) - p^{-2} \cdot \psi^{-1}(p) \cdot a_0(g; [\mathfrak{a}]), & \text{if } p \nmid c. \right.$$


Thus, for any positive square-free integer $K$ prime to the conductor of $\psi$, we find by induction that
\begin{equation}
(4.3) \quad a_0([K]\psi(g); \frac{a}{c}) = \begin{cases} 
\sum_{1 \le \alpha \le K} (-1)^{\nu(\alpha)} \cdot \alpha \cdot \psi(\alpha) \cdot a_0(g; \frac{\alpha a}{c}) & \text{if } K \mid c \\
\sum_{1 \le \alpha \le K} (-1)^{\nu(\alpha)} \cdot \alpha^{-1} \cdot \psi(\alpha) \cdot a_0(g; \frac{\alpha a}{c}) & \text{if } (K, c) = 1,
\end{cases}
\end{equation}
and
\begin{equation}
(4.4) \quad a_0([K]\psi^{-1}(g); \frac{a}{c}) = \begin{cases} 
\sum_{1 \le \alpha \le K} (-1)^{\nu(\alpha)} \cdot \psi^{-1}(\alpha) \cdot a_0(g; \frac{\alpha a}{c}) & \text{if } K \mid c \\
\sum_{1 \le \alpha \le K} (-1)^{\nu(\alpha)} \cdot \alpha^{-2} \cdot \psi^{-1}(\alpha) \cdot a_0(g; \frac{\alpha a}{c}) & \text{if } (K, c) = 1.
\end{cases}
\end{equation}

4.2. The constant terms of $E_{M,L,\psi}$. Let $\psi$ be a Dirichlet character of conductor $\psi = f$. We extend $\psi$ to a function on $\mathbb{Z}$ so that $\psi(n) = 0$ if $(n, f) \neq 1$. For any cusp $\frac{s^2tx}{f^2}$ in $X_0(f^2)$ with $s, t \mid f$ and $(s, t) = 1$ as in Lemma 4.1, we can choose a matrix $\begin{pmatrix} \frac{x}{f} & u \\ \frac{y}{f} & v \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ which maps $[\infty]$ to $\frac{s^2tx}{f^2}$.

Then it follows from Eqs. (2.1) and (2.4) that
\begin{align*}
a_0(E\psi; \frac{s^2tx}{f^2}) &= -\frac{1}{4g(\psi)} \sum_{a \in (\mathbb{Z}/f^2) \times} \sum_{b \in (\mathbb{Z}/f^2)} \psi(a) \cdot \psi(b) \cdot B_2 \left( \frac{xa}{f} + \frac{b}{s^2t} \right) \\
&= -\frac{1}{4g(\psi)} \sum_{b \in (\mathbb{Z}/f^2)} \psi(b) \left( \sum_{a \in (\mathbb{Z}/f^2)} \psi(a) \cdot B_2 \left( \frac{xa}{f} + \frac{b}{s^2t} \right) \right).
\end{align*}

Since the function in the above bracket depends only on $b \pmod{s^2t}$ and $\psi$ is primitive of conductor $f$, we find that $a_0(E\psi; \frac{s^2tx}{f^2})$ must be zero unless $st = f$. However, if $st = f$, then
\begin{align*}
a_0(E\psi; \frac{s^2tx}{f^2}) &= -\frac{1}{4g(\psi)} \sum_{a \in (\mathbb{Z}/f^2) \times} \sum_{b \in (\mathbb{Z}/f^2)} \psi(a) \cdot \psi(b) \cdot B_2 \left( \frac{xa}{f} + \frac{b}{s} \right) \\
&= -\psi^{-1}(x) \sum_{a \in (\mathbb{Z}/f^2)} \psi(a) \left( \sum_{b, k \in (\mathbb{Z}/f^2)} \psi(b) \cdot B_2 \left( \frac{as + b + kf}{fs} \right) \right),
\end{align*}

with the function in the bracket depends only on $a \pmod{\frac{f}{s}}$ and hence is zero unless $s = 1$. It follows that
\begin{equation}
(4.5) \quad a_0(E\psi; \frac{s^2tx}{f^2}) = \begin{cases} \psi^{-1}(x) \cdot n_\psi & \text{if } s = 1 \text{ and } t = f \\
0 & \text{otherwise},
\end{cases}
\end{equation}
where
\begin{equation}
n_\psi := -\frac{f}{4g(\psi)} \sum_{a, b \in \mathbb{Z}/f^2} \psi(a) \cdot \psi(b) \cdot B_2 \left( \frac{a + b}{f} \right).
\end{equation}

In particular, we find that
\begin{equation}
(4.6) \quad a_0(E\psi; \frac{s^2t(\alpha x)}{f^2}) = \psi^{-1}(\alpha) \cdot a_0(E\psi; \frac{s^2tx}{f^2}),
\end{equation}
where $\alpha$ is any integer prime to $f$. While the above is valid for any $\psi$ (not necessarily quadratic), we will assume $\psi$ is quadratic in the rest of this paper.

**Lemma 4.3.** For any quadratic character $\psi$ of conductor $f \mid C$, the constant terms of $E_{D,f,\psi}$ are given as
\begin{equation}
a_0(E_{D,f,\psi}; \frac{s^2tx}{DC^2}) = \begin{cases} 
\varphi(D) \cdot n_\psi \cdot (-1)^{\nu_\psi} \cdot \psi(\frac{DC}{f^2}) & \text{if } (s, f) = 1 \text{ and } t = D \\
0 & \text{otherwise},
\end{cases}
\end{equation}
In particular, $a_0(E_{D,f,\psi}; \frac{s^2t(\alpha x)}{DC^2}) = \psi(a) \cdot a_0(E_{D,f,\psi}; \frac{s^2tx}{DC^2})$ for any integer $\alpha$ prime to $D$.

**Proof.** Recall that $E_{D,f,\psi}$ is defined as $[\frac{D}{\psi}]^+(E\psi)$. For any cusp $\frac{s^2tx}{DC^2}$ of $X_0(DC)$, we decompose $\frac{D}{f} = K_r \cdot K_s \cdot K_t \cdot K$ with $K_r := (\frac{D}{f}, r)$, $K_s := (\frac{D}{f}, s)$, and $K_t := (\frac{D}{f}, t)$. By Eqs. (4.1), (4.2) and the
first formula of Eq. (4.3), we find that
\[
a_0(E_{D,f,\psi}; \frac{rs^2l_x}{DC})
\]
\[
= \sum_{1 \leq |a|} (-1)^{\nu(a)} \cdot \psi(a) \cdot \alpha \cdot a_0(E_{K,f,\psi}; \frac{rs^2l_x(K(K,C) \alpha, \alpha)}{DC/K(K,C)})
\]
\[
= \sum_{1 \leq |a|, 1 \leq |a|} (-1)^{\nu(a)} \cdot \psi(a) \cdot \alpha \cdot a_0(E_{\frac{D}{rs^{\alpha}},f,\psi}; \frac{rs^2l_x(K(K,C) \alpha, \alpha)}{DC/K^2(K,C)})
\]
It then follows from the second formula of Eq. (4.3) together with (1) and (2) of Lemma 4.2 that
\[
a_0(E_{D,f,\psi}; \frac{rs^2l_x}{DC})
\]
\[
= \sum_{1 \leq |a|} (-1)^{\nu(a)} \cdot \psi(a) \cdot \alpha \cdot a_0(E_{\frac{D}{rs^{|a|}},f,\psi}; \frac{rs^2l_x(K(K,C) \alpha, \alpha)}{DC/K^2(K,C)})
\]
\[
= \psi(K,K(C)) \cdot \sum_{1 \leq |a|} (-1)^{\nu(a)} \cdot \psi(a) \cdot \alpha \cdot a_0(E_{\frac{D}{rs^{|a|}},f,\psi}; \frac{rs^2l_x(K(K,C) \alpha, \alpha)}{DC/K^2(K,C)})
\]
where \( \alpha, \alpha, \alpha, \alpha \) and \( \alpha \) run through all the positive divisors of \( K, K, K, K \) and \( K, K \) respectively. It follows from (4.5) and (4.6) that the above constant term equals
\[
\psi(K,K(C)) \cdot \prod_{p \mid K} \left( 1 - \frac{1}{p} \right) \cdot \prod_{p \mid K} \left( 1 - \frac{1}{p} \right) \cdot a_0(E_{\frac{D}{rs^{|a|}},f,\psi}; \frac{rs^2l_x(K(K,C) \alpha, \alpha)}{DC/K^2(K,C)})
\]
which is zero unless \( s = K, f, K, t \), or equivalently, \( s, f = 1 \) and \( f, t \). Moreover, if these conditions are satisfied, then \( K, K, K, K = \frac{rs}{t} \) and \( K, K = \frac{rs}{t} \), which completes the proof.

Lemma 4.4. For any quadratic character \( \psi \), the constant terms of \( E_{M,f},f,\psi \) are given as
\[
a_0(E_{M,f,\psi}; \frac{rs^2l_x}{DC}) = \begin{cases} \psi(D) \cdot \mu(D) \cdot n_{\psi} \cdot \frac{(-1)^{|\psi|} \psi(D)}{rs \cdot \frac{D}{rs^{|a|}}} , & \text{if } D \mid rs, (s, f) = 1 \text{ and } f \mid t \\ 0 , & \text{otherwise} \end{cases}
\]
where \( \mu(n) = \prod_{p \mid n} (1 + p) \) for any positive integer \( n \). In particular, \( a_0(E_{M,f,\psi}; \frac{rs^2l_x(|a|)}{DC}) = \psi(a) \cdot a_0(E_{M,f,\psi}; \frac{rs^2l_x(|a|)}{DC}) \) for any integer \( \alpha \) prime to \( D \).

Proof. Recall that, for any \( M \) divided by \( f \), \( E_{M,f,\psi} \) is defined as \( \frac{D}{rs^{|a|}},(E_{M,f,\psi}) \). For any cusp \( \frac{D}{rs^{|a|}} \) of \( X_0(DC) \), we decompose \( \frac{D}{rs^{|a|}} = H_t \cdot H_s \cdot H \) with \( H_t := \frac{D}{rs^{|a|}}, r), H_s := (\frac{D}{rs^{|a|}}, s) \) and \( H_t := (\frac{D}{rs^{|a|}}, t) \). By Eqs. (4.1), (4.2) and the first formula of Eq. (4.4), we find that
\[
a_0(E_{M,f,\psi}; \frac{rs^2l_x}{DC})
\]
\[
= \sum_{1 \leq |a|} (-1)^{\nu(a)} \cdot \psi^{-1}(a) \cdot a_0(E_{M,f,\psi}; \frac{rs^2l_x(|a|)}{DC/H(H,C)})
\]
\[
= \sum_{1 \leq |a|, 1 \leq |a|} (-1)^{\nu(a)} \cdot \psi^{-1}(a) \cdot a_0(E_{M,f,\psi}; \frac{rs^2l_x(|a|)}{DC/H^2(H,H,C)})
\]
It then follows from the second formula of Eq. (4.4), (1) and (2) of Lemma 4.2 and the last assertion of Lemma 4.3 that
\[
a_0(E_{M,f,\psi}; \frac{rs^2l_x}{DC})
\]
\[
= \sum_{1 \leq |a|, 1 \leq |a|} (-1)^{\nu(a)} \cdot \psi^{-1}(a) \cdot \alpha \cdot a_0(E_{M,f,\psi}; \frac{rs^2l_x(|a|)}{DC/M\cdot(M,C)})
\]
\[
= \psi(HH_t(H,C)) \sum_{1 \leq |a|} (-1)^{\nu(a)} \cdot \alpha \cdot a_0(E_{M,f,\psi}; \frac{rs^2l_x(|a|)}{DC/M\cdot(M,C)})
\]
where \( \alpha_r, \alpha_s, \alpha_t \) and \( \alpha \) runs through all the positive divisors of \( H_r, H_s, H_t \) and \( H \) respectively. It is easy to see that the above sum is zero unless \( H_t = H = 1, (s, f) = 1 \) and \( f \mid t \), or equivalently, \( \frac{D}{rs} \mid rs, (s, f) = 1 \) and \( f \mid t \). When these conditions are satisfied, then the assertion follows from the previous Lemma. \( \Box \)

**Proposition 4.5.** For any \((M, L, \psi) \in \mathcal{H}(DC)\) with \( \psi \) a quadratic character of conductor \( f_\psi = f \), the constant terms of \( E_{M,L,\psi} \) are given as

\[
a_0(E_{M,L,\psi}; \frac{rs^2tx}{DC}) = \begin{cases} n_\psi \cdot \frac{\varphi_N(\psi)}{L/f} \cdot c_{rstx}, & \text{if } (s, f) = 1, (M, L) \mid st \text{ and } \frac{D}{rs} \mid rs \\ 0, & \text{otherwise,} \end{cases}
\]

where \( c_{rstx} := \frac{(-1)^{(\frac{D}{rs})} \varphi_N(\psi)}{rs} \prod_{p \mid (s, M)} (1 - \frac{1}{p}). \)

**Proof.** We have already proved the assertion when \((M, L) = f\), so it remains to consider the case when \((M, L) \neq f\). Since \((M, L) | C\), \((\frac{M}{L})\) can be decomposed as \((\frac{M}{L}) = W_s \cdot W_t \cdot W\) for any cusp \([\frac{rs}{rs}]\) of \(X_0(DC)\), where \(W_s := (\frac{M}{L}, s)\) and \(W_t := (\frac{M}{L}, t)\). It then follows from Eq. (4.4) that

\[
a_0(E_{M,L,\psi}; \frac{rs^2tx}{DC}) = \sum (-1)^{\nu} \cdot \varphi(\alpha) \cdot a_0(E_{M,f,\psi}; W_s, W_t, \psi; \frac{rs^2tx}{DC}),
\]

where \(\alpha_r, \alpha_s, \alpha_t\) and \(\alpha\) runs over all positive divisors of \(W_s, W_t\) and \(W\) respectively. As a cusp of \(X_0(DC)\), we have

\[
\frac{\frac{\text{rs}^2t\alpha t\alpha x}{DC}}{} = \frac{\frac{s\alpha t^2(\frac{\text{rs}}{\text{rs}})(\alpha s + \frac{\text{rs}}{\alpha s})}{DC}}{}
\]

with \((\alpha_s + \frac{\text{rs}}{\alpha_s}, D) = 1\), and \(\alpha_s + \frac{\text{rs}}{\alpha_s} \equiv \alpha_s (mod f)\) because \((\alpha_s, f) = 1\). So we find by Lemma \([4.4]\) that

\[
a_0(E_{M,f,\psi}; \frac{rs^2tx}{DC}) = (-1)^{\nu} \cdot \varphi(\alpha) \cdot \alpha_s \cdot a_0(E_{M,f,\psi}; \frac{rs^2tx}{DC}),
\]

and hence

\[
a_0(E_{M,L,\psi}; \frac{rs^2tx}{DC}) = \sum (-1)^{\nu} \cdot \alpha_s \cdot a_0(E_{M,f,\psi}; \frac{rs^2tx}{DC}).
\]

Thus, the constant term is zero unless \(\frac{D}{rs} \mid rs, (s, f) = 1, f \mid t \) and \(W = 1\), or equivalently, \(\frac{D}{rs} \mid rs, (s, f) = 1\) and \((M, L) \mid st\). If these conditions are satisfied, then it is easy to derive the desired result from the previous lemma. \( \Box \)

**Corollary 4.6.** For any quadratic character \(\psi\) of conductor \(f\), we have that

\[
R_{\Gamma_0(DC)}(E_{M,L,\psi}) = n_\psi \cdot \left( \frac{\varphi_N(\psi)}{L/f} \cdot \left( \frac{D}{rs} \right) \cdot C \right) \mathcal{Z}
\]

and

\[
R_{\Gamma_1(DC)}(E_{M,L,\psi}) = n_\psi \cdot \left( \frac{\varphi_N(\psi)}{L/f} \cdot \left( \frac{D}{rs} \right) \cdot f \right) \mathcal{Z}.
\]

**Proof.** This follows immediately from the above result about constant terms, since the ramification index of \(X_0(DC)\) at the cusp \([\frac{rs^2tx}{rs}]\) equals to \(rs^2\), and the ramification index of \(X_1(DC)\) at a cusp over \([\frac{rs^2tx}{rs}]\) equals to \(rs^2t\). \( \Box \)

**4.3. The periods of \(E_{M,L,\psi}\).** Now we turn to the determination of the periods of the Eisenstein series \(E_{M,L,\psi}\) with \(\psi\) being a quadratic character.

**Lemma 4.7.** For any quadratic character \(\psi\) of conductor \(f\), the Fourier expansion of \(E_{D,f,\psi}\) at \([\infty]\) is given as

\[
E_{D,f,\psi} = a_0(E_{D,f,\psi}; [\infty]) + \sum_{n=1}^{\infty} \sigma_{D,\psi}(n) \cdot \psi(n) \cdot q^n,
\]

with \(\sigma_{D,\psi}(n) := \sum_{d|n, (d, D) = 1} d\) for any positive integer \(n\).
Proof. We prove the statement by induction on \( \nu(D) \). Because \( \psi \) is quadratic, it follows from Eqs. (3.1) and (3.3) that \( a_n(E_\psi; |\infty|) = (\sum_{1 \leq d|n} d) \cdot \psi(n) \) for any \( n \geq 1 \), which verifies the assertion if \( D = f \).

Suppose \( \frac{D}{f} \neq 1 \) and let \( p \) be an arbitrary prime divisor of it. Because the non-holomorphic terms is annihilated by \( [p]_\psi^+ \) (see Remark 3.1), it follows from the induction hypothesis that

\[
E_{D,f,\psi} = [p]_\psi^+(E_{D,f,\psi})
\]

\[
= a_0(E_{D,f,\psi}) + \sum_{n=1}^\infty \sigma_{L,k}(n) \cdot \psi(n) \cdot q^n
\]

Writing \( n = m \cdot p^{v_p(n)} \) with \( (m,p) = 1 \), then we find that

\[
\sigma_{M,f,\psi}(n) - \sigma_{M,f,\psi}(n/p)
\]

\[
= (v_p(n) + \ldots + 1) \cdot \sigma_{M,f,\psi}(m) - (v_p(n) - 1 + \ldots + 1) \cdot \sigma_{M,f,\psi}(m)
\]

\[
= p^{v_p(n)} \cdot \sigma_{M,f,\psi}(n),
\]

which proves the assertion in this case. In general, if \( (M,L) \neq 1 \), then we choose an arbitrary prime divisor \( p \mid (M,L) \mid C \) and find that

\[
E_{M,L,\psi} = [p]_\psi^-(E_{M,L,\psi})
\]

\[
= a_0(E_{M,L,\psi}; |\infty|) + \sum_{n=1}^\infty \left( \sigma_{M,L,\psi}(n) - \sigma_{M,L,\psi}(n/p) \right) \cdot \psi(p) \cdot e^{2\pi inz}
\]

We have thus complete the proof of the lemma since it is easy to see that \( \sigma_{M,L,\psi}(n) - \sigma_{M,L,\psi}(n/p) = 0 \) if \( p \mid n \).

\[
\Box
\]

Lemma 4.8. For any quadratic character \( \psi \) of conductor \( f \), the Fourier expansion of \( E_{M,L,\psi} \) at \( |\infty| \) is given as

\[
E_{M,L,\psi} = a_0(E_{M,L,\psi}) + \sum_{n=1}^\infty \sigma_{M,L}(n) \cdot \psi(n) \cdot q^n,
\]

where \( \sigma_{M,L}(n) \) is defined to be \( (\sum_{1 \leq d|n} d) \cdot \psi(n) \) or zero according to \( n \) is prime to \( (M,L) \) or not.

Proof. We first consider the case when \( (M,L) = f \) so that \( E_{M,L,\psi} = E_{M,f,\psi} \). We will prove the lemma in this situation by induction on \( \nu(D) \). If \( \frac{D}{f} = 1 \), then the assertion have already been verified in the previous lemma. If \( \frac{D}{f} > 1 \) and let \( p \) be an arbitrary prime divisor of it. Then it follows form the induction hypothesis that

\[
E_{M,f,\psi} = [p]_\psi^+(E_{M,f,\psi})
\]

\[
= a_0(E_{M,f,\psi}) + \sum_{n=1}^\infty \left( \sigma_{M,f,\psi}(n) - \sigma_{M,f,\psi}(n/p) \right) \cdot \psi(n) \cdot q^n
\]

We have thus complete the proof of the lemma since it is easy to see that \( \sigma_{M,L,\psi}(n) - \sigma_{M,L,\psi}(n/p) = 0 \) if \( p \mid n \).

\[
\Box
\]

Proposition 4.9. For any quadratic character \( \psi \) of conductor \( f \), we have \( \mathcal{P}_{\Gamma_1(DC)}(E_{M,L,\psi}) = \frac{g(\psi)}{L} \mathbb{Z} + \mathcal{R}_{\Gamma_1(DC)}(E_{M,L,\psi}) \).

Proof. Straight manipulation with the Fourier expansion of \( E_{M,L,\psi} \) given by Lemma 4.8 yields that

\[
L(E_{M,L,\psi}; \chi, s) = \prod_{p|\text{factors of } M} (1 - \chi(p) \cdot p^{1-s}) \cdot \prod_{p|\text{factors of } L} (1 - \chi(p) \cdot p^{-s}) \cdot L(\chi \psi, s-1) \cdot L(\chi, s),
\]
for any Dirichlet character $\chi$ of conductor prime to $D$. It follows that $\Lambda(E_{M,L},\chi,1) = 0$ if $\chi(p) = 1$, and

$$\Lambda(E_{M,L},\chi,1) = -\frac{\chi(-f)}{2f} \psi(f,\chi) \cdot \prod_{p \mid M/f} (1 - \chi(p)) \cdot \prod_{p \mid L/f} (1 - \frac{\chi(p)}{p}) \cdot B_{1,\chi} \cdot B_{1,\chi}$$

if $\chi(-1) = -1$. By 4.2 (b) of [12], this implies that $\frac{2\psi(f)}{L}Z + R_{\Gamma_1(DC)}(E_{M,L})$ satisfies the condition (St3), and hence $\mathcal{P}_{\Gamma_1(DC)}(E_{M,L}) \subseteq \frac{2\psi(f)}{L}Z + R_{\Gamma_1(DC)}(E_{M,L})$. Thus, it remains to prove $\mathcal{P}_{\Gamma_1(DC)}(E_{M,L}) \supseteq \frac{2\psi(f)}{L}Z$.

Let $q$ be an arbitrary prime. For any prime $p' \in S_{DC}$ not equal to $q$, both $\prod_{p \mid q} (\psi(p) - \chi(p))$ and $\prod_{p \mid q} (\psi(p) \cdot p - \chi(p))$ are $q$-adic units for all but finitely many $\chi \in X_{DC}$ whose conductor is a power of $p'$. It then follows from the above $L$-value formula and Theorem 4.2 (c) of [12] that $\frac{1}{\phi(q)} \Lambda(E_{M,L},\chi,1)$ is a $q$-adic unit for infinitely many $\chi \in X_{DC}$ and hence completes the proof.

**Theorem 4.10.** Let $\psi$ be a quadratic character of conductor $f$, then

$$C(E_{M,L},\psi) \otimes \mathbb{Z}_p \left[ \frac{1}{2M-1} \right] \cong \frac{2\psi(f)}{L}Z \cdot \frac{\varphi(\frac{f}{p}) \cdot \mu(\frac{f}{p}) \cdot (\frac{p}{\psi,f})_CZ}{\varphi(\frac{f}{p}) \cdot \mu(\frac{f}{p}) \cdot (\frac{p}{\psi,f})_CZ} \otimes \mathbb{Z}_p \left[ \frac{1}{2M-1} \right]$$

where $\delta_{M,L}$ equals 1 or 0 according to $(M,L) = 1$ or not.

**Proof.** It follows from Corollary 4.8 and Proposition 4.9 that

$$A^{(\psi)}(E_{M,L}) = \frac{\mathcal{P}_{\Gamma_1(DC)}E_{M,L,\psi} + R_{\Gamma_0(DC)}E_{M,L,\psi}}{R_{\Gamma_1(DC)}E_{M,L,\psi}} \cong \frac{2\psi(f)}{L}Z + \varphi(\frac{f}{p}) \cdot \mu(\frac{f}{p}) \cdot (\frac{p}{\psi,f})_CZ \otimes \mathbb{Z}_p \left[ \frac{1}{2M-1} \right]$$

Since the intersection $C(E_{M,L},\psi)$ is annihilated by $T_p$ for any $p \mid (M,L)$ and such $T_p$ acts on $\sum_{DC}$ as multiplication by $p$ by [3], it follows that $\sum_{DC} C(E_{M,L},\psi)$ is annihilated by $(M,L)$ and hence finishes the proof when $(M,L) \neq 1$.

However, if $(M,L) = 1$ and hence $\psi = 1$, then the cyclic group $\sum_{DC} C(E_{M,L},\psi)$ is both of multiplicative type and $\mathbb{Q}$-rational, so it must be contained in $\mu_2$. In particular, $\sum_{DC} C(E_{M,L})$ is annihilated by 2, and the result follows.

5. **Proof of Theorems 1.2 and 1.3**

5.1. **The new part of $J_0(N)$**. Let $N$ be a positive integer. For any positive divisors $n \mid N$ and $m \mid \frac{N}{n}$, we have the following homomorphism

$$S_2(\Gamma_0(n),\mathbb{C}) \rightarrow S_2(\Gamma_0(N),\mathbb{C}),$$

which maps $f(z)$ to $f(mz)$, and hence the following

$$\prod_{n \mid N, n \neq N, m \mid \frac{N}{n}} S_2(\Gamma_0(n),\mathbb{C}) \rightarrow S_2(\Gamma_0(N),\mathbb{C}),$$

whose cokernel is isomorphic to the subspace of new forms of level $\Gamma_0(N)$. The above homomorphism induces the following morphism between abelian varieties over $\mathbb{Q}$

$$\iota_N : J_0(N) \rightarrow \prod_{n \mid N, n \neq N, m \mid \frac{N}{n}} J_0(n).$$

The new part $J_0^{\text{new}}(N)$ of $J_0(N)$ is then defined to be the kernel of the above morphism, so we have the following cartesian diagram

$$\begin{array}{ccc}
J_0^{\text{new}}(N) & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
J_0(N) & \longrightarrow & \prod_{n \mid N, n \neq N, m \mid \frac{N}{n}} J_0(n)
\end{array}$$

5.2. **Proof of Theorem 1.2**. In fact, we claim that $J_0(N)(\mathbb{Q})[q^\infty] = 0$ for any prime $q \mid 6 \cdot N \cdot \varpi(N)$ which clearly implies Theorem 1.2. We prove this claim by induction on $\nu(N)$. When $\nu(N) = 1$ so that $N$ is a prime, the claim follows from the theorems of Ogg and Mazur. In general, if $q$ is a prime such that $q \mid 6 \cdot N \cdot \varpi(N)$, then we also have $q \mid 6 \cdot n \cdot \varpi(n)$ for any $n \mid N$. Thus, by the induction hypothesis, a point $P \in J_0(N)(\mathbb{Q})[q^\infty]$ must be mapped to zero by $\iota_N$ as $\nu(n) < \nu(N)$ for any $n \mid N$ and $n \neq N$. It
follows that $P \in J_0^{new}(\mathbb{Q})[g^{\infty}]$ and we are reduced to prove that $J_0^{new}(\mathbb{Q})[g^{\infty}] = 0$ for any prime $q | 6 \cdot N \cdot \omega(N)$.

We can write $N$ as $D \cdot C \cdot C_1 \cdots C_k$, where $D, C, C_1, \ldots, C_k$ are positive square-free integers such that $C_k | C_{k-1} | \ldots | C | D$. By the Eichler-Shimura theory, we have $T_{\ell}^{\Gamma_0(N)}(P) = (1 + \ell) \cdot P$ for any prime $\ell | D$. Moreover, by the newform theory, we have $T_{\ell}^{\Gamma_0(N)}$ acts on $J_0^{new}(N)$ as multiplication by $\epsilon_{\ell}$, where $\epsilon_{\ell} = \pm 1$ if $\ell | (D/C)$ and $\epsilon_{\ell} = 0$ if $\ell | C$.

Thus, if $0 \neq P \in J_0^{new}(N)(\mathbb{Q})[g^{\infty}]$, then we have

$$S_2(\Gamma_0(N), \mathbb{F}_q) \left[ \{ T_{\ell}^{\Gamma_0(N)} - (1 + \ell) \} \chi_{\ell D}, \{ T_{\ell}^{\Gamma_0(N)} - \epsilon_{\ell} \} \chi_{\ell D} \right] \neq 0$$

and is generated by a unique normalized $\Theta$. However, simple manipulation shows that

- If $\epsilon_{\ell} = 1$, then $[\ell^{-1}(\Theta)]$ belongs to $S_2(\Gamma_0(N\ell), \mathbb{F}_q)$ and is annihilated by $T_{\ell}^{\Gamma_0(ND/C)}$;
- If $\epsilon_{\ell} = -1$, then $\Theta + \frac{1}{\ell} \Theta |_{\gamma_{\ell}}$ belongs to $S_2(\Gamma_0(N\ell), \mathbb{F}_q)$ and is annihilated by $T_{\ell}^{\Gamma_0(ND/C)}$.

Thus, by raising the levels in such a way, we will finally get some normalized form which spans the following one-dimensional $\mathbb{F}_q$-vector space

$$S_2(\Gamma_0(ND/C), \mathbb{F}_q) / \{ T_{\ell}^{\Gamma_0(ND/C)} - (1 + \ell) \} \chi_{\ell D}, \{ T_{\ell}^{\Gamma_0(ND/C)} - \epsilon_{\ell} \} \chi_{\ell D},$$

with $ND/C = D^2 \cdot C_1 \cdot C_k$ being a multiple of $D^2$. By the $q$-expansion principle and Proposition 5.7 this normalized form is exactly $E_{D,D}$ modulo $q$. In particular, we find that $E_{D,D}$ must be a modulo $q$ cusp form, so that all its constant terms should be zero modulo $q$. But by Proposition 1.5 the non-zero constant terms of $E_{D,D}$ are all units in $\mathbb{Z}/(\frac{1}{\ell} - 1)\mathbb{Z}$, so we get a contradiction and hence complete the proof of our claim.

5.3. The indexes of the quadratic Eisenstein ideals. In the following, we will denote by $T$ to be the full Hecke algebra $T_0(DC)$ of level $\Gamma_0(DC)$ generated over $\mathbb{Z}$ by all the $T_\ell = T_{\ell}^{\Gamma_0(DC)}$ for all the primes $\ell$.

Lemma 5.1. For any quadratic character $\psi$ of conductor $f$, there is a natural isomorphism

$$T/I_{\Gamma_0(DC)}(E_{M,L,\psi}) \simeq \mathbb{Z}/m\mathbb{Z},$$

for some non-zero integer $m$.

Proof. It is obvious that the natural homomorphism $\mathbb{Z} \to T/I_{\Gamma_0(DC)}(E_{M,L,\psi})$ is surjective, so we only need to prove that the kernel of this homomorphism is non-zero. However, suppose the kernel is zero so that $\mathbb{Z} \simeq T/I_{\Gamma_0(DC)}(E_{M,L,\psi})$, then the ring homomorphism $T \to T/I_{\Gamma_0(DC)}(E_{M,L,\psi}) \simeq \mathbb{Z} \to \mathbb{C}$ gives rise to a normalized cusp form whose eigenvalue is $\psi(\ell) + \ell \cdot \psi(\ell)$ for any $\ell \nmid D$, which contradicts the Ramanujan bound. Thus the kernel must be of the form $(m)$ for some non-zero integer $m$ and we have hence proved the lemma.

Proposition 5.2. For any quadratic character $\psi$, there is a natural isomorphism

$$T/I_{\Gamma_0(DC)}(E_{M,L,\psi}) \otimes \mathbb{Z}[\frac{1}{6D}] \simeq C_{\Gamma_0(DC)}(E_{M,L,\psi}) \otimes \mathbb{Z}[\frac{1}{6D}],$$

which is induced from the action of $T$ on the cuspidal group $C_{\Gamma_0(DC)}(E_{M,L,\psi})$.

Proof. Recall that there is a perfect pairing of $\mathbb{Z}$-modules (see [9])

$$T \times S_2(\Gamma_0(DC), \mathbb{Z}) \to \mathbb{Z},$$

which maps any $(T, f)$ to $a_1(f; T; [\infty])$. Tensor with $\mathbb{Z}/m\mathbb{Z}$ over $\mathbb{Z}$, we get another perfect pairing

$$T/mT \times S_2(\Gamma_0(DC), \mathbb{Z}/m\mathbb{Z}) \to \mathbb{Z}/m\mathbb{Z},$$

where $m$ is the non-zero integer in Lemma 5.1. Because $T/I_{\Gamma_0(DC)}(E_{M,L,\psi})$ is a quotient of $T/mT$, it follows that there is a perfect pairing

$$T/I_{\Gamma_0(DC)}(E_{M,L,\psi}) \times S_2(\Gamma_0(DC), \mathbb{Z}/m\mathbb{Z})[I_{\Gamma_0(DC)}(E_{M,L,\psi})] \to \mathbb{Z}/m\mathbb{Z}$$

of $\mathbb{Z}/m\mathbb{Z}$-modules, and hence we get a canonical isomorphism

$$S_2(\Gamma_0(DC), \mathbb{Z}/m\mathbb{Z})[I_E] \simeq \mathbb{Z}/m\mathbb{Z},$$
which gives us a unique normalized cusp form $F \in S_2(\Gamma_0(DC), \mathbb{Z})$ such that $F \equiv E_{M,L,\psi} \pmod{m}$ in other words, there exists some $G \in M_2(\Gamma_0(DC), \mathbb{Z})$ such that $F = E_{M,L,\psi} + m \cdot G$. However, by Theorem 1.6.2 of [1], the constant terms of $G$ at the cusps are all in $\mathbb{Z}[\frac{1}{6D}, \mu_D]$, so we find that

$$\varphi(\frac{D}{f}) \cdot \mu(\frac{L}{f}) \in m \cdot \mathbb{Z}[\frac{1}{6D}, \mu_D] \cap \mathbb{Q} = m \cdot \mathbb{Z}[\frac{1}{6D}]$$

by Proposition 4.15 which gives the explicit values of the constant terms of $E_{M,L,\psi}$. On the other hand, since $C_{\Gamma_0(DC)}(E_{M,L,\psi})$ is cyclic, it follows that $\frac{z}{m_z} \simeq \frac{1}{(E_{M,L})}$ acts transitively on it, so that

$$m \in \varphi(\frac{D}{f}) \cdot \mu(\frac{L}{f}) \cdot \mathbb{Z}[\frac{1}{6D}]$$

by Corollary 4.10 about the explicit value of the order of $C_{\Gamma_0(DC)}(E_{M,L,\psi})$. We have thus completed the proof of the theorem.

**Remark 5.3.** When combined with Corollary 4.10 about the order of the quadratic cuspidal groups, the above theorem also give the index of the quadratic Eisenstein ideals in $\mathbb{T}$ up to a factor of $6D$.

**5.4. Proof of Theorem 1.3** For any $f \mid C$, let $\psi$ be the unique quadratic character of conductor $f$. Recall that

$$J_0(DC)(\psi) := \{ P \in J_0(DC) \mid \psi(P) = \psi(\sigma) \cdot P \text{ for any } \sigma \in G_\mathbb{Q} \}$$

We claim that, for any prime $q$ not dividing $6 \cdot D \cdot \varpi(D)$,

$$J_0(DC)(\psi)[q^\infty] = 0,$$

which of course implies Theorem 1.3. Since any positive divisor of $DC$ is of the form $dc$ with $1 \leq c \mid d \mid D$ and $c \mid C$, the commutative diagram defining the new part of $J_0(DC)$ can be written as

$$\begin{array}{ccc}
J_0^{\text{new}}(DC) & \rightarrow & 0 \\
\downarrow & & \downarrow \\
J_0(DC) & \rightarrow & \prod_{1 < \alpha \mid \frac{dc}{c}} J_0(dc)
\end{array}$$

**Lemma 5.4.** If $f \mid c$, then $J_0(d)(\psi)[q^\infty] = 0$.

**Proof.** Firstly, if $f \mid d$, then $J_0(d)$ has good reduction at any prime divisor $p$ of $f$ not dividing $d$. It follows that $\psi(d)(\psi)(q^\infty)$ is unramified at $p$. But $p \mid f$ implies that $\psi$ is ramified at $p$, so that $J_0(d)(\psi)(q^\infty)$ must be zero.

On the other hand, if $f \mid d$ but $f \nmid c$. Let $p$ be a prime divisor of $f$ not dividing $c$. Then $J_0(d)$ has semi-stable reduction at $p$, so the inertia group $I_p$ acts unipotently on $T_q(J_0(d))$. If $P \neq J_0(d)(\psi)(q^\infty)$, then $(1 - \sigma)^k(P) = 0$ for any $\sigma \in I_p$ with $k$ some positive integer. But there is some $\sigma \in I_p$ such that $\sigma(P) = \psi(\sigma) \cdot P = -P$ as $p \mid f$, so that $2^k \cdot P = 0$ for some $k$ which contradicts the assumption that $q \neq 2$. We have thus finished the proof of the lemma.

**Lemma 5.5.** $J_0^{\text{new}}(DC)(\psi)(q^\infty) = 0$.

**Proof.** By Eichler-Shimura theory, for any prime $\ell \mid D$, $T_\ell$ acts as multiplication by $\psi(\ell) + \ell \cdot \psi(\ell)$ on $J_0^{\text{new}}(DC)(\psi)(q^\infty)$. On the other hand, the new form theory tells us that $T_\ell$ acts as $\pm 1$ if $\ell \mid \frac{D}{f}$, and $T_{\ell}$ acts as $0$ if $\ell \mid C$. Thus, if $J_0^{\text{new}}(DC)(\psi)(q^\infty) \neq 0$, then

$$S_2(\Gamma_0(DC), \mathbb{F}_q) \left[ \{ T_\ell - (\psi(\ell) + \ell \cdot \psi(\ell)) \} \right]_{\ell \mid D}, \{ T_\ell \right)_{\ell \mid C}, \{ T_\ell - \delta_\ell \} \right)_{\ell \mid D} \neq 0$$

and is generated by a unique normalized $\theta$. Here, for any $\ell \mid \frac{D}{f}$, $\delta_\ell = \pm 1$ according to how $T_\ell$ acts. However, simple manipulation on Fourier expansions shows that

- If $\delta_\ell = 1$, then $\ell \cdot \theta$ belongs to $S_2(\Gamma_0(DC), \mathbb{F}_q)$ and is annihilated by $T_\ell$;
- If $\delta_\ell = -1$, then $\theta + \frac{1}{\ell \cdot \gamma_\ell}$ belongs to $S_2(\Gamma_0(DC), \mathbb{F}_q)$ and is also annihilated by $T_\ell$.

It follows that, by raising the levels in such a way, we will finally get some normalized form which spans the one-dimensional $\mathbb{F}_q$-vector space

$$S_2(\Gamma_0(D^2), \mathbb{F}_q) \left[ \{ T_\ell - (\psi(\ell) + \ell \cdot \psi(\ell)) \} \right]_{\ell \mid D}, \{ T_\ell \right)_{\ell \mid D}$$

Since the ideal $\left( \{ T_\ell - (\psi(\ell) + \ell \cdot \psi(\ell)) \} \right)_{\ell \mid D}$ is exactly the Eisenstein ideal $I_{\Gamma_0(D^2)}(E_{D^2,\psi})$, we find that $q$ divides the index of $I_{\Gamma_0(D^2)}(E_{D^2,\psi})$ in $\mathbb{T}_0(D^2)$. By Proposition 5.2 it follows that $q$ divides the order of $C_{\Gamma_0(D^2)}(E_{D^2,\psi})$ as we have assumed that $q \nmid 6D$. But because $q \mid \varphi(D) \cdot \mu(D)$, it is clear
from Theorem 4.10 that \( C_{\Gamma_0(D^2)}(E_{D,D,\psi})[q^{\infty}] = 0 \), so we get a contradiction and hence completes the proof. \( \square \)

**Proof of the claim:** Firstly, we prove that \( J_0(f^2)(\psi)[q^{\infty}] = 0 \). By Lemma 5.5, \( J_0(mn)(\psi)[q^{\infty}] \) is zero for any \( 1 \leq n \mid m \mid f \) with \( mn \neq f^2 \). Moreover, by applying Lemma 5.3 to the situation when \( DC = f^2 \), we find that \( J_0(q^{new}(f^2)(\psi))[q^{\infty}] \) is also zero. It follows that \( J_0(f^2)(\psi)[q^{\infty}] = 0 \). In general, by induction hypothesis, we have \( J_0(D\psi)[q^{\infty}] = 0 \) for any \( 1 \leq c \mid d \mid D \) with \( dc \neq DC \). Then, it follows that \( J_0(D\psi)[q^{\infty}] = J_0(q^{new}(DC)(\psi))[q^{\infty}] \), which is zero by Lemma 5.5. We have thus complete the proof of the claim and hence that of Theorem 1.3.

**Remark 5.6.** To have a complete understanding of these Hecke module structures, it seems that a deeper study of the arithmetic-geometric properties of \( X_0(D\psi) \) is required. Moreover, from the previous results, it is curious to ask whether there is also an intrinsic characterization of the whole cuspidal subgroup \( \text{new}(\psi) \mathfrak{C} \mathfrak{C} \).

**Appendix**

In this appendix, we complete the computations of the \( 2 \)-part of \( C_{\Gamma_0(D\psi)}(E_{M,L}) \) when \( D \) is odd. We will need some basic properties of Dedekind sums which we will now briefly recall. The reader is recommended to \[8\] for the details. For any two integers \( h, k \) with \( k \geq 1 \) and \( (h, k) = 1 \), the associated **Dedekind sum** is defined to be

\[
s(h, k) := \sum_{\mu=1}^{k} \left( \frac{h\mu}{k} \right) \left( \frac{k\mu}{h} \right)
\]

where \( \left( \langle x \rangle \right) \) is defined to be

\[
\langle x \rangle = \begin{cases} 
0 & \text{if } x \in \mathbb{Z} \\
 x - \lfloor x \rfloor - \frac{1}{2} & \text{otherwise}
\end{cases}
\]

for any real number \( x \). The famous **reciprocity formulas** for these Dedekind sums says that

\[
s(h, k) + s(k, h) = \frac{1}{4} + \frac{1}{12} \left( \frac{h}{k} + \frac{1}{hk} + \frac{k}{h} \right)
\]

for any two positive integers \( h, k \) with \( (h, k) = 1 \). More over, for any \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}) \), we have that (see \[8\], P48)

\[
\log \eta(\gamma z) - \log \eta(z) = \frac{1}{2} \cdot \text{sgn}(c)^2 \cdot \log \left( \frac{cz + d}{i \cdot \text{sgn}(c)} \right) + \pi i \cdot \frac{a + d}{12c} - \pi i \cdot \text{sgn}(c) \cdot s(d, |c|)
\]

where \( \eta \) is the Dedekind \( \eta \)-function, \( \text{sgn}(c) \) equals the sign of \( c \) if \( c \neq 0 \) and is defined to be zero if \( c = 0 \). If we define a function \( \Phi \) on \( SL_2(\mathbb{Z}) \) as

\[
\Phi(\gamma) := \begin{cases} 
b/d & \text{if } c = 0 \\
\frac{ad}{cz + d} & \text{if } c \neq 0
\end{cases}
\]

for any \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}) \), then we can also write the above transformation formulas as

\[
\log \eta(\gamma z) - \log \eta(z) = \frac{1}{2} \cdot \text{sgn}(c)^2 \cdot \log \left( \frac{cz + d}{i \cdot \text{sgn}(c)} \right) + \frac{\pi i}{12} \cdot \Phi(\gamma)
\]

Finally, if \( k \) is an **odd positive** integer, then we have the following congruence equation (\[8\], P37)

\[
12 \cdot k \cdot s(h, k) \equiv k + 1 - 2 \left( \frac{h}{k} \right) \pmod{8}
\]

which is useful in studying the periods of some Eisenstein series in \( E_2(\Gamma_0(N), \mathbb{Z}) \) as we will see in later sections.
Lemma 6.1. For any $1 \neq M \mid D$, we have that

$$\int_{\gamma}^{\gamma} E_{M, D/M}(\tau) d\tau = \frac{1}{24} \sum_{1 \leq r \mid D} (-1)^{\nu(r)-1} \frac{1}{(r, D/M)} \Phi \left( \frac{a}{r}, \frac{rb}{d} \right)$$

with any $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(DC)$ and $z \in \mathcal{H}$.

Proof. We will firstly consider the Eisenstein series $E_{D, 1}$. When $\nu(D) = 1$, so that $D = p$ for some prime and $C = 1$, then

$$E_{p, 1}(z) = \frac{1}{2} \left[ p \cdot \phi(0, 0)(pz) - \phi(0, 0)(z) \right] = \frac{1}{d} \left( \log \eta(pz) - \log \eta(z) \right)$$

because $(2\pi i) \cdot \phi(0, 0) = \frac{1}{2\pi} + \frac{2d}{dz} \log \eta$ by [11], Remark 2.4.3. It follows that

$$\int_{\gamma}^{\gamma} E_{p, 1}(\tau) d\tau = \frac{1}{2\pi i} \left[ \frac{d}{dz} (\log \eta(p\gamma z) - \log \eta(\gamma z)) - \frac{d}{dz} (\log \eta(pz) - \log \eta(z)) \right]$$

$$= \frac{1}{2\pi i} \left( \frac{d}{dz} (\log \eta(p\gamma p^{-1}(pz)) - \log \eta(pz)) - \frac{d}{dz} (\log \eta(\gamma z) - \log \eta(z)) \right)$$

$$= \frac{1}{24} \left[ \Phi(\gamma p \gamma p^{-1}) - \Phi(\gamma) \right]$$

which is the desired in this special situation. However, if $\nu(D) > 1$, then we choose an arbitrary prime divisor $p$ of $D$ and find inductively that

$$\int_{\gamma}^{\gamma} E_{D, 1}(\tau) d\tau = \int_{\gamma}^{\gamma} E_{D/p, 1}(\tau) d\tau - \int_{\gamma}^{\gamma} (E_{D/p, 1}(\gamma p)(\tau) d\tau$$

$$= \int_{\gamma}^{\gamma} E_{D/p, 1}(\tau) d\tau - \int_{\gamma}^{\gamma} E_{D/p, 1}(\tau) d\tau$$

$$= \frac{1}{24} \sum_{1 \leq r \mid D/p} (-1)^{\nu(r)-1} \Phi \left( \frac{a}{r}, \frac{rb}{d} \right) - \frac{1}{24} \sum_{1 \leq r \mid D/p} (-1)^{\nu(r)-1} \Phi \left( \frac{a}{r}, \frac{spb}{d} \right)$$

$$= \frac{1}{24} \sum_{1 \leq r \mid D} (-1)^{\nu(r)-1} \Phi \left( \frac{a}{r}, \frac{rb}{d} \right)$$

for any $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(DC)$. This completes the proof for the Eisenstein series $E_{D, 1}$. The proof for more general $E_{M, D/M}$ is similar, in which one precede inductively on $\nu(\frac{D}{M})$ as following

$$\int_{\gamma}^{\gamma} E_{M, D/M}(\tau) d\tau$$

$$= \int_{\gamma}^{\gamma} E_{M, D/Mp}(\tau) d\tau - \frac{1}{p} \int_{\gamma}^{\gamma} E_{M, D/Mp}(\tau) d\tau$$

$$= \frac{1}{24} \sum_{1 \leq r \mid D/p} (-1)^{\nu(r)-1} \frac{1}{(r, \frac{D}{Mp})} \Phi \left( \frac{a}{r}, \frac{rb}{d} \right) - \frac{1}{24} \sum_{1 \leq r \mid D/p} (-1)^{\nu(r)-1} \frac{1}{p(r, \frac{D}{Mp})} \Phi \left( \frac{a}{r}, \frac{prb}{d} \right)$$

which completes the proof.

In the following, we denote $\xi_{M, D/M}(\gamma)$ to be $\sum_{1 \leq r \mid D} (-1)^{\nu(r)-1} \frac{1}{(r, \frac{M}{D})} \Phi \left( \frac{a}{r}, \frac{rb}{d} \right)$ for any $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(DC)$. Now we can finally prove the first part of Theorem [13].
Theorem 6.2. Notations are as above, then \( C(E_{M,L}) \) is a finite cyclic abelian group. More over, the order \( N_{M,L} \) is given by the following

\[
N_{M,L} := \begin{cases} 
\frac{p - 1}{\varphi(D) \mu(L)} \left( \frac{D}{C} \right), & \text{if } DC = p \text{ for some prime } p \\
(24, \varphi(D) \mu(L), \left( \frac{D}{C} \right)), & \text{if otherwise}
\end{cases}
\]

Proof. We only need to prove the assertion about its order, as the acyclicity of \( C(E_{m,L}) \) follows immediately from the definition.

When \( D = p \) is a prime and \( C \) equals 1 (or, respectively, \( p \)), the corresponding assertions about the order of \( C_{\Gamma_0(p)}(E_{p,1}) \) (respectively, \( C_{\Gamma_1(p)}(E_{p,1}) \)) has been verified in \([6]\) (respectively, \([2]\)), we are thus reduced to consider those \( D \) with at least two prime divisors. Since now \( N_{M,L} \) is nothing but \( n_{M,L} \), it follows from Corollary 2 that we only need to verify the 2-part.

Firstly, if \((M, L) \neq 1 \) and \( p \) is a prime divisor of it, then \( T_p(E_{M,L}) = 0 \) by Theorem 3 and so that \( C(E_{M,L}) \) is also annihilated by \( T_p \). But \([3]\) has proved that \( T_p \) acts as multiplication by \( p \) on the Shimura subgroup \( \sum_{DC} \), and hence \( \sum_{DC} C(E_{M,L}) \subseteq \mu_2 \) must be annihilated by multiplication by \( p \). Because \( p \mid D \) is odd by our assumption, we find the intersection must be zero and hence prove the assertion when \((M, L) \neq 1 \).

It remains to prove the assertion for those \( E_{M,D/M} \)'s. We will distinguish into two situations in the following discussion.

(I) Firstly, we consider the Eisenstein series \( E_{D,1} \). For any \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(DC) \), we have that

\[
\xi_{D,1}(\gamma) = \sum_{1 \leq r \mid D} (-1)^{\nu(r) - 1} \left( \frac{a + d}{(c/r)} - 12 \cdot s(d, \frac{c}{r}) \right)
\]

\[
\equiv (-1)^{\nu(D) - 1} \cdot \frac{a + d - 1}{c} \cdot \varphi(D) - \frac{2}{c} \sum_{p \mid D} (1 - \frac{d}{p}) (\text{mod } 8)
\]

\[
\equiv (-1)^{\nu(D) - 1} \cdot \frac{a + d - 1}{c} \cdot \varphi(D) \quad (\text{mod } 8)
\]

with the last equality holds because \( D \) is odd and \( \nu(D) > 1 \). We have thus prove that \( \int_{-1}^{2} E_{D,1}(\tau) d\tau \in \mathbb{Z}_2 + \varphi(D)/24\mathbb{Z}_2 \) for any such \( \gamma \).

(II) If \( c \neq 0 \) is even, then \( d \) is odd and we may assume \( d > 0 \), so that

\[
\xi_{D,1}(\gamma) = \sum_{1 \leq r \mid D} (-1)^{\nu(r) - 1} \left( \frac{a + d}{(c/r)} - 12 \cdot sgn(c) \cdot s(d, \frac{c}{r}) \right)
\]

By the reciprocity law, we have

\[
s(d, \frac{c}{r}) + s(\frac{c}{r}, d) = -\frac{1}{4} + \frac{1}{12} d \cdot \frac{1}{|c/r| + \frac{r}{|c|d} + \frac{|c|}{dr}}
\]

It follow that

\[
\xi_{D,1}(\gamma) = \left( \sum_{1 \leq r \mid D} (-1)^{\nu(r) - 1} \cdot sgn(c) \cdot s(\frac{c}{r}, d) \right) - \left( \sum_{1 \leq r \mid D} (-1)^{\nu(r) - 1} \cdot \frac{c}{dr} \right) \quad (\text{mod } 8)
\]

\[
\equiv \frac{2}{d} \cdot \frac{|c|}{dr} \cdot sgn(c) \cdot \prod_{p \mid D} (1 - \frac{p}{d}) + \frac{c}{dr} \cdot \varphi(D) \quad (\text{mod } 8)
\]

\[
\equiv \frac{c}{dr} \cdot \varphi(D) \quad (\text{mod } 8)
\]

with the last equality holds because \( \nu(D) > 1 \). We have thus prove that \( \int_{-1}^{2} E_{D,1}(\tau) d\tau \in \mathbb{Z}_2 + \varphi(D)/24\mathbb{Z}_2 \) for any such \( \gamma \).

It follows that \( \int_{-1}^{2} E_{D,1}(\tau) d\tau \in \mathbb{Z}_2 + \varphi(D)/24\mathbb{Z}_2 \) for any such \( \gamma \). But as

\[
\mathcal{P}(E_{D,1}) = \mathcal{P}_{\Gamma_0(DC)}(E_{D,1}) = \mathbb{Z} + \frac{\varphi(D)}{24}\mathbb{Z}
\]
we find that 
\[ \mathcal{P}(E_{D,1}) \otimes \mathbb{Z}_2 = \mathbb{Z}_2 + \frac{\varphi(D)}{24} \mathbb{Z}_2 \]
and hence complete the proof for \( E_{D,1} \)

(II) Now we consider those \( E_{M,D/M} \) with \( \frac{D}{M} \neq 1 \). The proof is similar as above. For any \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(DC) \), we have that

(II.1) If \( c = 0 \), then

\[
\xi_{M,D/M}(\gamma) = \pm \sum_{1 \leq s | D} (-1)^{\nu(s)} \sum_{1 \leq t | M} (-1)^{\nu(t)-1} tsb 
\]

\[
= (\pm b) \sum_{1 \leq s | D} (-1)^{\nu(s)} \sum_{1 \leq t | M} (-1)^{\nu(t)-1} t = 0
\]

(II.2) If \( c \) is odd, then we find by definition (note that we may assume \( c > 0 \))

\[
\xi_{M,D/M}(\gamma) = \sum_{1 \leq s | D} (-1)^{\nu(s)} \sum_{1 \leq t | M} (-1)^{\nu(t)-1} \left( \frac{a + d}{c(t s)} - 12 \cdot s(d, \frac{c}{ts}) \right)
\]

\[
\equiv - \sum_{1 \leq s | D} (-1)^{\nu(s)} \sum_{1 \leq t | M} (-1)^{\nu(t)-1} ts \left( \frac{e}{ts} + 1 - 2 \left( \frac{d}{c} \right) \left( \frac{d}{ts} \right) \right) \quad (\text{mod } 8)
\]

\[
\equiv \left( \frac{d}{c} \right) \prod_{p | D} \left( 1 - \left( \frac{d}{p} \right) \right) \prod_{p | M} \left( 1 - \left( \frac{d}{p} \right) \right) \equiv 0 \quad (\text{mod } 8)
\]

with the last equality holds because \( D \) is odd and \( \nu(D) > 1 \).

(II.3) If \( c \neq 0 \) is even, then \( d \) is odd and we may assume \( d > 0 \). Similarly as before, a straight forward calculation by using the reciprocity law show that

\[
\xi_{M,D/M}(\gamma) \equiv \pm \frac{2}{d} \left( \frac{c}{d} \right) \prod_{p | D} \left( 1 - \left( \frac{p}{d} \right) \right) \prod_{p | M} \left( 1 - \left( \frac{p}{d} \right) \right) \equiv 0 \quad (\text{mod } 8)
\]

with the last equality holds because \( \nu(D) > 1 \). We have thus prove that \( \int_{\mathbb{Z}_2} E_{M,D/M}(\tau)d\tau \in \mathbb{Z}_2 \) for any \( \gamma \in \Gamma_0(DC) \) and hence completes the proof of the theorem. \( \square \)

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