Invariance of success in Grover’s search under local unital noise with memory

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We analyze the robustness of Grover’s quantum search algorithm performed by a quantum register under a possibly time-correlated noise acting locally on the qubits. We model the noise as originating from arbitrary but fixed unitary evolution by $U$ of some noisy qubits. The noise occurs with some probability in the interval between every consecutive noiseless Grover evolutions. This general noise reproduces some well-known decoherence models when the time-correlations are absent. We derive a set of $U$’s, called the ‘good noises,’ for which the success probability of the algorithm remains unchanged with respect to varying the non-trivial total number ($m$) of noisy qubits in the register. The result holds irrespective of the presence of any time-correlations in the noise. We show that only when $U$ is either of the Pauli matrices $\sigma_x$ and $\sigma_z$ (which give rise to $m$-qubit bit-flip and phase-damping channels respectively in the memory-less case), the algorithm’s success probability stays unchanged when increasing or decreasing $m$. In contrast, when $U$ is the third Pauli matrix $\sigma_y$ (giving rise to $m$-qubit bit-phase flip channel in the memory-less case), the success probability at all times stays unaltered as long as the parity (even or odd) of the total number $m$ remains the same. This asymmetry between the Pauli operators stems from the inherent symmetry-breaking existing within the Grover circuit. We further show that the positions of the noisy sites are irrelevant in the case of any of the Pauli matrices as noise. The results are illustrated in case of both types of noise: with and without time-correlations. We find that the former case leads to a better performance of the noisy algorithm. In this paper, the time-correlations have been restricted to be Markovian.

I. INTRODUCTION

The last few decades saw the advent and flourishing of the field of quantum information and computation. One of the most important classes of discoveries made in this field has to be that of quantum algorithms which provide or are believed to provide substantial computational advantages over their classical counterparts. The most significant ones include the Deutsch-Jozsa algorithm [1, 2], Shor’s factoring algorithm [3, 4], the quantum search algorithms [5–10] and the quantum simulation algorithms [11–15]. The advantages of these quantum algorithms are assumed to be derived from the efficient use of quantum coherence and entanglement.

After Grover’s seminal proposal [5, 6] of his eponymous quantum search algorithm, which has been shown to be a special case of the more general amplitude amplification algorithm [16], an extensive amount of research effort has been directed towards implementing and studying the effects of noise on the efficiency of the algorithm in an actual quantum device. The experimental implementation of the algorithm was first done using nuclear magnetic resonance techniques [17]. Later on, the efficiency of the Grover’s algorithm was studied in [18] and a generalization of the algorithm for an arbitrary amplitude distribution was done in [19]. For more works on the applications of the quantum search algorithm, see [20–28] and for some experimental implementations, see [29–37].

Even if a quantum algorithm theoretically provides a significantly better efficiency in comparison with its classical counterpart, the efficiency in an implementation of the same undoubtedly depends on the actual fabrication of the relevant quantum circuit. Due to possible impurities in circuit components and their erroneous implementations, there may arise fluctuations or drifts, which can affect the performance of the quantum algorithm considerably. Therefore, characterizing such deviations from the ideal situation, caused by decoherence and noise, is important to assess the usefulness of an algorithm. The disturbances may cause a unitary noise on the ideal system, i.e., a small perturbation can arise in the Hamiltonians describing the unitary gates, conserving the hermiticity of the Hamiltonian as well as the unitarity of the quantum gates. See e.g. [38–40]. Studies on the consequences of noisy scenarios in quantum algorithms has started sometime back. See e.g. [41]. The effect of noise on the Grover’s search algorithm was studied in [42], which investigated the effect of random Gaussian noise on the algorithm’s efficiency at each step. A perturbative method was used in [43] to study decoherence in a noisy Grover algorithm where each qubit suffers phase-flip error independently after each step. The effect of a noisy oracle was considered in [44, 45]. In [46], the effect of depolarizing channels on all qubits was examined and it was found that the number of iterations needed to obtain the maximal efficiency of the success probability decreases with increasing decoherence. The effect of the Grover unitary becoming noisy was considered in [47] using a noisy Hadamard gate, with unbiased and isotropic noise, uncorrelated in each iteration of the Grover operators. An upper bound on the strength of the noise parameters up to which the algorithm works efficiently was deduced. A comparison of the effects of several completely positive trace preserving maps on the efficiency and computational complexity of the algorithm was described in [48]. The performance of the algorithm under localized dephasing was studied in [49]. For more discussions and further ramifications of noise on the Grover’s algorithm, see [50–52]. Fault-ignorant quantum search was proposed in [53] where the searched element is reached eventually but with the runtime depending on the noise level. Steane’s [54] quantum error correction code was employed in presence of the depolarizing channel in [55].

Noise with correlations in time [56–60] and space [61–63] have been observed in realistic quantum computing devices.
Detrimental effects of such noise on quantum error correcting codes have also been reported [64–68]. As we introduce our model of time-correlated noise in the subsequent sections, it will become evident that such scenario could arise if the noisy qubits in the quantum register get coupled to an external degree of freedom acting as a physical memory state [69–71]. Incidentally, long coherence times have already been achieved in photonic [72–74] and trapped ion [75, 76] settings.

In this paper, we study the effects of a noise that originates from random and probabilistic unitary evolution of some noisy qubits in between any two Grover operation. In particular, we look for a set of noisy yet unitary qubit evolutions, for which the success probability of the algorithm remains unaffected by the strength of the noise, i.e., the number of noisy qubits. We refer to those special noise unitaries as “good noise”. We extend our investigation to a type of correlated noise with partial memory considered in [77–79].

We have organized the paper as follows. After reviewing the noiseless Grover algorithm in Sec. II A, we introduce our noise model in Sec. II B and II C. The time-correlation-less case is then described in Sec. II D. In Sec. III, we analytically find the unitaries representing “good noise”. A measure of the algorithm’s performance is introduced in Sec. IV A. The effect of a memory-less noise and that of a Markovian-correlated noise on the efficiency of the Grover’s algorithm are numerically studied in Sec. IV B. Sec. V concludes the paper.

II. GROVER SEARCH: NOISELESS CASE AND A NOISE MODEL

The Grover search algorithm aims to find a single marked element from a search space of finite size, and attains quadratic speed-up over the best classical search. In our paper, we consider Grover search under a time-correlated local noise. In the succeeding subsections contain we discuss the ideal Grover algorithm and then introduce our noise model.

A. The noiseless scenario

The algorithm begins with a search space consisting of $N = 2^n$ elements with $n$ being an integer. The elements of the space are denoted by $x = 1, 2, \ldots, N$ and a function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ is defined such that for the marked element $w$, $f(w) = 1$ and $\forall x \neq w, f(x) = 0$. To solve this problem, a classical computer evaluates $f$ for each of the elements until it gives the value 1, i.e., the marked element is found and therefore requires $O(N)$ operations. The advantage of Grover’s search algorithm over the classical treatment is that, by using a sequence of unitary operations, it can find the marked element by using only $O(\sqrt{N})$ queries to $f$. The steps of the algorithm are described as follows and a schematic demonstration is shown in Fig. 1.

It starts with all the qubits of an $n$-qubit register in the $|0\rangle$ state, where $|0\rangle$ is the eigenvector of Pauli-$\sigma_z$ operator corresponding to the eigenvalue 1. The next step is to act on each qubit by the Hadamard operator, $H = \frac{1}{\sqrt{2}} (\sigma_x + \sigma_z)$, where $\sigma_x$ and $\sigma_z$ are Pauli operators. Thus the total register comes to an uniform superposition state,

$$|s\rangle = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)^\otimes n = \frac{1}{\sqrt{N}} \left(\sum_{x=1}^{N} |x\rangle + |w\rangle\right),$$  \hfill (1)

where $|w\rangle$ is the marked state, i.e., the state corresponding to the element we are searching for in the database of $N = 2^n$ elements. The state $|s\rangle$ is then acted on by the Grover operator $G$, given by $G = D \ O$, where $D = (2|s\rangle\langle s| - I_N)$ is called the Diffuser and $O = (I_N - 2|w\rangle\langle w|)$ is the Oracle. For a detailed discussion about the construction of the Diffuser $D$, Oracle $O$ and the Grover operator $G$, see e.g. [40, 80]. The operator $G$ has the form,

$$G = -I_N + 2|s\rangle\langle s| - \frac{4}{\sqrt{N}} |s\rangle\langle w| + 2|w\rangle\langle w|. \quad \hfill (2)$$

It acts on successive states until the state of the register, $|\psi(t)\rangle = G^t |s\rangle$ reaches close enough to the marked state $|w\rangle$. Here $t$ stands for the number of times the Grover operator is employed after the first step, i.e., after the Hadamard operation. The success probability, i.e., the probability to find the marked state after $t^{th}$ operation, is given as $P(t) = |\langle w|\psi(t)\rangle|^2$. 

![FIG. 1. Grover’s search algorithm in the noiseless situation. The register containing a string of $n$ qubits, each in the state $|0\rangle$, is each subjected to a Hadamard operation in the first step. The second step is the operation of Grover operator, for $t$ times, which is followed by a measurement on the output state of the register in the computational basis. Time taken to reach the maximal success probability is $O(2^{n/2})$. Further discussions presented in text.]

![FIG. 2. Noiseless Grover algorithm for $n=5$ qubits in the register. The number, $t$, of Grover iterations is along the horizontal axis and the success probability, $P(t)$, for finding the marked state, is plotted on the vertical axis. The smallest $t$ for which $P(t)$ is maximal is given by $t=4$. All quantities plotted are dimensionless.]


It can be checked that the marked element is reached after around \( t = \frac{1}{4} \sqrt{N} \) Grover iterations. See Fig. 2 for the profile of the success probability with time, when the database has 32 entries.

### B. The noise model

For a large database, the number of iterations of the Grover operator will still be large - although quadratically more efficient than the classical algorithm - for reaching the maximal success probability. A high number of applications of Grover operator may result in some noise or fluctuations in the circuit parameters performing the computation, significantly affecting the efficiency of the algorithm.

In this paper, we consider that in the interval between any two consecutive Grover operations on the \( n \) qubits, any \( m \) qubits evolve under some unitary \( U \) that we call the ‘noise unitary’. The effect of this noisy evolution is expressed in the form of the total noise unitary \( \chi_m \). For example, it can be \( \chi_m = (U \otimes (\mathbb{1}_2) \otimes (n-m) \otimes I \otimes (m-1)) \), meaning \( m \) noisy qubits evolving under \( U \) and the \((n-m)\) noiseless qubits acted on by the identity operator \( \mathbb{1}_2 \). We will call the number of noisy qubits \( m \) as the ‘noise strength’. The positions of the \( m \) noise sites are allowed to be arbitrary. The noise \( \chi_m \) occurs with some well-defined probability after every Grover iteration and we can incorporate its effect on the algorithm by considering the unitary operator \( G' = \chi_m G \), which we will call the ‘noisy Grover operator’. Using Eq. (2),

\[
G' = -\chi_m + 2(\chi_m|s\rangle\langle s| + \chi_m|w\rangle\langle w|) - \frac{4}{\sqrt{N}} \chi_m|s\rangle\langle w|.
\]

The probabilistic occurrences of this noise could possibly even be correlated in time. So, we have considered that possibility by assuming the noise at each pair of consecutive time steps to be Markovian-correlated. In Section II.C we have introduced a ‘memory parameter’ that controls the strength of the time-correlation. We further show in Section II.D that when this parameter is zero, our noise model reduces to some common decoherence processes, depending on the local noise unitaries \( U \).

### C. Markovian-correlated noise

In the noise model given in the previous paragraph, a total unitary evolution by \( \chi_m \) (Eq. (II.B)) of locally evolving \( m \) noisy qubits is a probabilistic process, happening after each noiseless Grover evolution \( G \). This noisy evolution is ‘probabilistic’ in the sense that after a given Grover evolution \( G \), the state of the register is a convex mixture of two possible states: one corresponding to no noise after evolution by \( G \) and another corresponding to a noisy evolution by \( \chi_m \) after \( G \).

Now, it can happen that the probability of noise at a given time depends on the history of the register’s noisy evolution [81, 82]. In this paper, we will consider the simplest such situation where this noise is Markovian-correlated in time (see Appendix B), i.e., noise at each given instant is affected only by the noisy evolution at the immediately previous time step. This potentially important variety of noise with memory has not yet been studied before in case of the Grover algorithm. It is also to be noted here that the results shown in the paper are not exclusive to only this kind of noise, and validity in this case will serve as an indication to the generality of the results.

It is easy to understand the situation by considering that the \( m \) noisy qubits become connected to another degree of freedom which we denote as the ‘walker’. The walker has two orthogonal states \( |g\rangle \) and \( |g'\rangle \) and at each time step, it performs a transition between these states with some well-defined probability. A schematic diagram is shown in Fig. 3. When it transitions to \( |g'\rangle \), all the \( m \) qubits connected to it are rotated by a unitary \( U \) and the other \((n-m)\) left as they were. When it transitions to \( |g\rangle \), all the \( n \) qubits connected to it are left as they were. Thus, with each time step (iteration), application of an ideal unitary Grover operator \( G \) is followed by each of the following with some corresponding probabilities:

(I) any \( m \) out of \( n \) qubits are rotated by a unitary \( U \), i.e., walker is in state \( |g'\rangle \), or,

(II) all the \( n \) qubits are left untouched, i.e., walker is in state \( |g\rangle \).

We assume the transition probabilities for \( |g\rangle \leftrightarrow |g'\rangle \) to be determined by the dichotomous Markov chain considered in [77], and described by Eqs. (B1) and (B2) (in Appendix B). To make the situation clearer, let us say that after the noisy \((t-1)\)th Grover iteration, the \( n \) qubit register is in a state given by the density matrix \( \rho_{t-1} \). After the noisy \( t\)th iteration, the register will be a convex mixture of the following two possible states:

(I) \( \rho_t = G' \rho_{t-1} G'^\dagger \), with \( G' = \chi_m G \),

(II) \( \rho_t = G \rho_{t-1} G\dagger \).

At \( t = 1 \), i.e., on the first Grover iteration, the probabilities of (I) and (II) are determined by the initial probabilities of the walker to be in states \( |g'\rangle \) and \( |g\rangle \) respectively. These probabilities are called stationary probabilities and are taken to be \( p_{g'} = p \) and \( p_g = (1-p) \) respectively. Here \( p \) can be referred to as the noise probability. At any later time \( t \), the probabilities are determined by the \((t-1)\)th iteration and the memory parameter, \( \mu \). That is, for \( t \geq 2 \),

\[
p_{k|l} = \begin{cases} 
(1-\mu) p_l + \mu, & \text{for } k = l \\
(1-\mu) p_k, & \text{for } k \neq l.
\end{cases}
\]

where \( k, l \) can be \( G \) or \( G' \). The memory \( \mu \) is a measure of the time it takes for a walker state to transition to the other orthogonal state. When \( \mu = 0 \), noise at each time step is independent of what happened in the previous step, since \( p_{k|l} = p_k \) from Eqs. (4) and (5). On the other hand, \( \mu = 1 \) leads to \( p_{k|l} = 1 \), meaning that in the case of perfect memory, the walker state remains fixed throughout the evolution. See Appendix B for further discussions.

Before the application of the first Grover iteration, the \( n\)-qubit register is in the uniform superposition state \( |s\rangle \) and
thus the register’s state corresponds to the density matrix $\rho_0 = |s\rangle \langle s|$. So, the density matrix of the composite system containing the walker and the register before applying any Grover iteration is given as $R_0 = \frac{|s\rangle + |g\rangle}{\sqrt{2}} \frac{|g\rangle + |g\prime\rangle}{\sqrt{2}} \otimes |s\rangle \langle s|$. 

So, the state of the register after the first Grover iteration will be $\rho_1 = \text{Tr}_{\text{walker}} \{ R_1 \}$. The state of the register after the second Grover iteration will likewise be given as, $\rho_2 = \text{Tr}_{\text{walker}} \{ R_2 \}$. We can write $R_1 = S_0 R_0$ and $R_2 = S R_1$, where $S_0$ and $S$ are the transition matrices given by

$$S_0 = \left( p_{g|g} |g\rangle \otimes \Phi^0_{[\cdot]} + p_{g|g}^\prime |g\prime\rangle \otimes \Phi^0_{[\cdot]} + p_{g|g}^\prime |g\prime\rangle \otimes |g\prime\rangle \otimes \Phi^0_{[\cdot]} \right),$$

$$S = \left( p_{g|g} |g\rangle \otimes \Phi^0_{[\cdot]} + p_{g|g}^\prime |g\prime\rangle \otimes |g\prime\rangle \otimes \Phi^0_{[\cdot]} + p_{g|g}^\prime |g\prime\rangle \otimes |g\prime\rangle \otimes \Phi^0_{[\cdot]} \right),$$

where $\Phi^0_{[\cdot]} = G \rho G^\dagger$ and $\Phi^1_{[\cdot]} = G^\prime \rho G^\dagger$. Similarly, for $t \geq 2$, we have

$$R_t = S^{t-1} R_1,$$

$$\rho_t = \text{Tr}_{\text{walker}} \{ R_t \}.$$  

The success probability, i.e., the probability to find the marked state at time $t$, is given as

$$P(t) = |\langle w|\rho_t|w\rangle|.$$  

This kind of correlated noise with partial memory can potentially be found in real quantum devices, and it has been shown to provide an enhancement in the transmission of classical information as compared to transmission through noisy channels without memory [77]. In the next subsection we show that the memory-less case corresponds to some well-known decoherence processes.

**D. $\mu = 0$ and unital decoherence processes**

In the case when the noise in consecutive steps do not have any time-correlations, i.e., $\mu = 0$, Eqs. (4) and (5) have the form $p_{g|g} = p_g = p_{g|g}^\prime$ and $p_{g|g} = p_g = p_{g|g}^\prime$. Putting these in Eqs. (6) and (7) leads to $S = S_0$. Taking the initial register state $\rho_0$ and the composite walker and register state $R_0$ as given in previous section, we get the register’s state $\rho_1$ after the first noisy Grover iteration as

$$\rho_1 = \text{Tr}_{\text{walker}} \{ R_1 \} = \text{Tr}_{\text{walker}} \{ S_0 R_0 \} = (1 - p) \Phi^0_{[\rho]} + p \Phi^1_{[\rho]} = (1 - p)(G \rho G^\dagger) + p \chi_m (G \rho_0 G^\dagger) \chi_m^\dagger.$$  

Thus, the noisy evolution after the first noiseless Grover operation is a quantum operation $\mathcal{E}$ given by the Kraus operators $K_1 = \sqrt{1 - p} \mathbb{1}_N$ and $K_2 = \sqrt{p} \chi_m$ so that $\mathcal{E}[\rho] = K_1 \rho K_1^\dagger + K_2 \rho K_2^\dagger$ and $\rho_1 = \mathcal{E}[G \rho_0 G^\dagger]$. Since $S_0 = S$ in case of $\mu = 0$, we have for $t \geq 1$,

$$\rho_t = \mathcal{E}[G \rho_{t-1} G^\dagger].$$  

Note that for $\mu > 0$, an expression like Eq. (11) is not possible because of noise being conditioned on the application at the previous time step.

For $U = \sigma_x$, the noisy operation $\mathcal{E}$ thus becomes an $m$-qubit bit-flip channel. Similarly, $U = \sigma_y$ leads to phase-damping and $U = \sigma_z$ to bit-phase flip channels. A comparison of the effects of these channels on the Grover algorithm was done extensively in [48]. In fact, all these are examples of unital channels (that is, $\sum_i K_i^\dagger K_i = 1$) and $\mathcal{E}$ is a stochastic combination of unitary channels [83].

We have shown how the memory-less special case of the Markovian-correlated noise gives rise to some of the popular decoherence channels. It is a general feature that the success probability of an algorithm reduces with increase in strength of noise, as seen e.g. in [47]. Whereas, there is a possibility of identifying such noise for which the decrease in the success probability does not depend on the number of noisy qubits, $m$. For an example, see Appendix A. If it is possible to choose between different noise generating unitaries in an experimental setup, it will be helpful to have those noise unitaries which do not decrease the success probability with increase in the noise strength. We can christen such noise unitaries as “good noise”. In the succeeding section, we try to identify the form of such good noise.

**III. SEARCH FOR “GOOD NOISE”**

To find what the good noises are, we will start with the most general single-qubit unitary matrix (in the $\{|0\rangle, |1\rangle\}$ basis),

$$U = \begin{pmatrix} a & b \\ -\overline{b}e^{i\theta} & \overline{a}e^{i\theta} \end{pmatrix},$$

with $a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1, \theta \in [0, 2\pi]$, and $\tau$ denoting the complex conjugate of $z$. The good noise corresponds to the values of $a, b$ and $\theta$, for which the success probability $P(t)$ (Eq. (9)) remains unchanged on changing the value of $m$. If a noise acts on $m$ sites, we will denote $P(t)$ in that case as...
The good noises will be found through elimination of $U$’s for which $P_m(t)$ changes with $m$. We will start by finding the conditions for keeping $P_m(t) = (1 - p)|\langle w|G|s\rangle|^2 + p|\langle w|G'|s\rangle|^2$ constant with changing $m$. We have

$$|\langle w|G'|s\rangle|^2 = \left|1 - \frac{4}{N}\right|\langle w|\chi_m|s\rangle + \frac{2}{\sqrt{N}}\langle w|\chi_m|w\rangle|^2$$

$$= \frac{1}{N}\left(1 - \frac{4}{N}\sum_{j=1}^{N}(\chi_m)_{w,j} + 2(\chi_m)_{w,w}\right)^2 \tag{13}$$

where, $\chi_m|s\rangle = \frac{1}{\sqrt{N}}\left(\sum_{j=1}^{N}(\chi_m)_{1,j} \ldots (\chi_m)_{N,j}\right)$. It can be shown that

$$\sum_{j=1}^{N}(\chi_m)_{k,j} = e^{i\eta(a + b)N - q}(\bar{a} - \bar{b})^q := \psi_q \tag{14}$$

and $(\chi_m)_{k,k} = e^{i\eta\alpha^{m-q}}\pi^q$, where $q \in [0, m]$ and $q$ depends on $k$. Here each $\psi_q$ appears $(\frac{N}{m^q})$ times in $\chi_m|s\rangle$. Since $|\langle w|G|s\rangle|^2$ is independent of $m$, we can conclude from the expression of $|\langle w|G'|s\rangle|^2$ in Eq. (13) that for getting $P_{m+1}(1) = P_m(1)$, we need either $|a| = 0$ or $|b| = 0$. Thus, for satisfying the condition $P_m(t) = P_{m+1}(t)$, we cannot have both $a$ and $b$ non-zero, and therefore we get our first condition for constructing a good noise which gives the constraints, Eqs. (15) and (16). So, a good noise needs to obey

**Condition 1:** $|a| = 1$ or $|b| = 1$

Thus, $U = \left\{ \begin{array}{ll} \left( \begin{array}{cc} a & 0 \\ 0 & \bar{a}e^{i\theta} \end{array} \right), & \text{for } |a| = 1, \\ \left( \begin{array}{cc} 0 & b \\ -\bar{b}e^{i\theta} & 0 \end{array} \right), & \text{for } |b| = 1. \end{array} \right. \tag{15}$

Hence $\chi_m$ has to be a generalized permutation matrix with

$$w' = \begin{cases} w, & \text{for } |a| = 1, \\ N - \frac{N}{2m} - w + 2 \left( w \mod \frac{N}{2m} \right), & \text{for } |b| = 1. \end{cases} \tag{16}$$

Thus, $\chi_m|s\rangle = \begin{cases} c \sqrt{\frac{N-1}{N}}|s'_1\rangle + \frac{1}{\sqrt{N}}|w\rangle, & \text{for } |a| = 1, M = 1 \\ c \sqrt{\frac{N-2}{N}}|s'_1\rangle + \frac{1}{\sqrt{N}}|w\rangle + \frac{1}{\sqrt{N}}|w'\rangle, & \text{for } |b| = 1, M = 1 \end{cases} \tag{20}$

and, $\chi_m|s\rangle = \begin{cases} c_1 \sqrt{\frac{N-2}{2N}}|s'_1\rangle + \frac{1}{\sqrt{N}}|w\rangle + \frac{c_2}{\sqrt{2}}|s'_2\rangle, & \text{for } |a| = 1, M = 2 \\ c_1 \sqrt{\frac{N-4}{2N}}|s'_1\rangle + \frac{1}{\sqrt{N}}|w\rangle + \frac{1}{\sqrt{N}}|w'\rangle + \frac{c_2}{\sqrt{2}}|s'_2\rangle, & \text{for } |b| = 1, M = 2, \alpha = \beta \end{cases} \tag{22}$

and, $\chi_m|s\rangle = \begin{cases} c_1 \sqrt{\frac{N-2}{2N}}|s'_1\rangle + \frac{1}{\sqrt{N}}|w\rangle + \frac{c_2}{\sqrt{2}}|s'_2\rangle, & \text{for } |a| = 1, M = 2 \\ c_1 \sqrt{\frac{N-4}{2N}}|s'_1\rangle + \frac{1}{\sqrt{N}}|w\rangle + \frac{1}{\sqrt{N}}|w'\rangle + \frac{c_2}{\sqrt{2}}|s'_2\rangle, & \text{for } |b| = 1, M = 2, \alpha \neq \beta \end{cases} \tag{24}$

where $\Delta_i = \text{dim}(\text{Span}\{|d_i\rangle\})$, $ \langle s'_i|s'_j\rangle = \delta_{ij}$, $|c_i| = 1 = |\alpha| = |\beta|$, i.e., we get the basis $B = \{|s'_1\rangle, |s'_2\rangle, \ldots, |s'_{M}\rangle, |w\rangle\}$ of dimension $(M + 1)$ from Eq. (18) and $B = \{|s'_1\rangle, |s'_2\rangle, \ldots, |s'_{M}\rangle, |w\rangle, |w'\rangle\}$ of dimension $(M + 2)$ from Eq. (19) respectively.

Now, there are two possibilities for a matrix $U$ of the form in Eqs. (15) and (16): its two non-zero elements are either equal (Case (i)), or unequal (Case (ii)). Case (i) suggests $M = 1$ and directly leads to the constraints, given in Eqs. (20) and (21), which have to be satisfied by the unitary presenting the good noise. If we have a $U$ as in Case (ii), we need to put further restrictions for the success probability to stay conserved with $m$. Since $|\langle w|G'|s\rangle|$ in basis $B$ must not change with $m$, we do not want $\text{dim}(B)$ to change with the same. Thus, the number of distinct $c_i$’s in Eqs. (18) and (19) must remain constant with $m$. There are total $M$ of these coefficients. For $m = 1$, e.g., for $\chi_1 = U \otimes 1_{N/2}$ in Case (ii), there are only two distinct non-zero elements in $\chi_1$, because $U$ has two distinct non-zero elements. This implies $M = 2$. Since $M$ should remain constant with $m$, Case (ii) leads to the restrictions for the unitaries constructing good noise to be satisfied given in Eqs. (22), (23), (24). So, to summarise, we have another necessary (but not sufficient) condition:

**Condition 2:** $M = 1$ or 2
So, only the \( U \)'s that satisfy one of the Eqs. (20)-(24), are the unitaries corresponding to the good noise for which \( P(t) \) does not depend on the number of noise sites \( m \). It can be shown that \( \psi_q \) appears \((\frac{N}{2})^m \) times in the column vector \( \chi_m |s \rangle \).

We have the following observations:

(1) If \( U \) satisfies Eq. (20), then \( b = 0 \) and \( \psi_q = c, \forall q \).

Solving for \( a \) and \( \theta \) gives \( a = e^{i\phi} = \sqrt{c}, \theta = 2\phi \), i.e., \( U = \sqrt{c} |0 \rangle_1 = \sqrt{c} 1_2 \).

(2) If \( U \) satisfies Eq. (22), then it turns out that we need

(a) \( \psi_q = \psi_{q+2} = c_1, \forall q \) even and (b) \( \psi_q = \psi_{q+2} = c_2, \forall q \) odd. That is because, \((\frac{m}{1}) + (\frac{m}{3}) + (\frac{m}{5}) + \ldots = (\frac{m}{1}) + (\frac{m}{3}) + (\frac{m}{5}) + \ldots = 2^{m-1}, \) i.e., the sum of multiplicities of elements in \( \chi_m |s \rangle \) from the set \{\( \psi_q |q \rangle \) even\} is equal to that in case of elements from the set \{\( \psi_q |q \rangle \) odd\}. Since \( c_1 \neq c_2 \), solving \( (a) \) and \( (b) \) for \( a \) and \( \theta \) give the solution, \( a = \sqrt{c} 1_2, \theta = 2\phi - \pi \).

Putting these values in Eqs. (25), (26), (27), we can see that \( P_m(2) \) is independent of \( m \).

For \( U = \sigma_x \), we have \( b = 0, a = 1 \) and \( \theta = \pi \). So, \( \langle s | \chi_m |s \rangle = 1 \), \( \langle w | \chi_m |s \rangle = \frac{1}{\sqrt{N}} \) and \( \langle w | \chi_m |w \rangle = 0, \forall m \).

Putting these values in Eqs. (25), (26), (27), we can see that \( P_m(2) \) is independent of \( m \).

In case of \( U = \sigma_y \), \( a = 0, b = -i \) and \( \theta = \pi \). So we have \( \langle s | \chi_m |s \rangle = 0 \), \( \langle w | \chi_m |w \rangle = 0 \), \( \langle w | \chi_m |s \rangle = \frac{1}{\sqrt{N}} e^{i\pi} (-1)^m = (-1)^m \langle s | \chi_m |w \rangle \). So, from Eq. (26), \( |\langle w | G' |s \rangle|^2 \) is independent of \( m \). But, Eq. (25) leads to \( |\langle w | G' |s \rangle|^2 = \frac{1}{N} (1 - \frac{4}{N} + \frac{4}{N} (-1)^m)^2 \) and Eq. (27) leads to \( |\langle w | G' |s \rangle|^2 = \frac{1}{N} (1 - \frac{4}{N}) (-1)^m - \frac{4}{N} + 3^2 \).

From these two expressions, we can infer that for \( U = \sigma_y \), the success probabilities \( P_m(2) = P_{m+2}(2), \forall m \), i.e., although for any consecutive \( m \), the success probability is not constant, it does remain the same for \( m \) staying either odd or even. We have not analytically shown here if this is true for \( t > 2 \) in case of \( \sigma_y \). But in Fig. 6, it is shown that indeed \( P_m(t) = P_{m+2}(t), t \geq 1, \forall m \).

For any \( t \geq 2, P_m(t) = |\langle w | G' |t \rangle|^2 \) consists of the terms \( \langle s | \chi_m |s \rangle, \langle w | \chi_m |s \rangle, \langle w | \chi_m |w \rangle \) or combinations of these terms. Since it turns out that for \( \sigma_x \) and \( \sigma_z \), \( P_m(t) \) are independent of \( m \), we have \( P_m(t) = P_{m+2}(t), \forall m \).

So, for example, if we have total of \( n = 50 \) qubits in the register performing the search algorithm, it turns out that the evaluation of the success probability in case of \( m = 22 \) noise sites and that in case of \( m = 41 \) noise sites will be indistinguishable if the qubits in those sites evolve under the good noises, i.e., \( U \in \{\sigma_x, \sigma_z\} \). The success probabilities in the cases where \( m = 10, m = 40 \) or \( m = 50 \), will be exactly the same in case of \( U = \sigma_y \). Similarly, the cases of \( m = 9, m = 33 \) or \( m = 45 \) will be indistinguishable among themselves when \( U = \sigma_y \). It is to be noted that there may be some

\[
U = \sqrt{c} |\frac{1}{0} \rangle_1 = \sqrt{c} \sigma_z.
\]

(3) If \( U \) satisfies Eq. (21), then \( a = 0 \) and \( \psi_q = c, \forall q \).

Solving for \( b \) and \( \theta \) gives \( b = e^{i\phi} = \sqrt{c}, \theta = 2\phi - \pi \), i.e., \( U = \sqrt{c} |\frac{0}{1} \rangle_1 = \sqrt{c} \sigma_x \).

(4) If \( U \) satisfies Eqs. (23) or (24), a similar analysis as above can be performed and the solution is \( U = \sqrt{c} |\frac{1}{0} \rangle_1 = \sqrt{c} i \sigma_y \).

Here \( \sqrt{c} \) is only a constant phase factor. We can see from the above discussion that the restricted set of unitary qubit evolutions that are candidates for being good noise, are the matrices \( e^{i\phi} 1_2, e^{i\phi} \sigma_x, e^{i\phi} \sigma_y \) and \( e^{i\phi} \sigma_z \), for any \( \phi \in [0, 2\pi) \) for \( t = 1 \).

We will now check if this set of noise unitaries are ‘good noise’ for all times \( t \geq 1 \). From Eq. (9) we get \( P_m(2) = p_{g|g|g} |\langle w | G'G' |s \rangle|^2 + p_{g|g|g} |\langle w | G'G' |s \rangle|^2 + p_{g|g|g} |\langle w | G'G' |s \rangle|^2 + p_{g|g|g} |\langle w | G'G' |s \rangle|^2 \). Similar to Eq. (13) we can write for \( t = 2, \)

\[
|\langle w | G'G' |s \rangle|^2 = \left( 1 - \frac{4}{N} \right)^2 \langle w | \chi_m |s \rangle + \frac{2}{\sqrt{N}} \langle w | \chi_m |w \rangle + \frac{2}{\sqrt{N}} \langle w | \chi_m |w \rangle - \frac{4}{\sqrt{N}} \langle w | \chi_m |s \rangle + \frac{2}{\sqrt{N}} \langle w | \chi_m |w \rangle \right)^2 + \frac{4}{\sqrt{N}} \langle w | \chi_m |s \rangle + \frac{2}{\sqrt{N}} \langle w | \chi_m |w \rangle - \frac{4}{\sqrt{N}} \langle w | \chi_m |s \rangle + \frac{2}{\sqrt{N}} \langle w | \chi_m |w \rangle \right)^2 + \frac{4}{\sqrt{N}} \langle w | \chi_m |s \rangle + \frac{2}{\sqrt{N}} \langle w | \chi_m |w \rangle - \frac{4}{\sqrt{N}} \langle w | \chi_m |s \rangle + \frac{2}{\sqrt{N}} \langle w | \chi_m |w \rangle \right)^2.
\]
unitary $U$, other than these Pauli matrices, which makes the success probability independent of $m$ for some particular time $t$ and not at other times. The Pauli matrices $\sigma_x$ and $\sigma_z$ are special in the sense that when $U$ is one of these, the success probability becomes independent of $m$, for all $t$.

Another important observation is that none of the conditions used above put restrictions on what the positions of the $m$ unitaries are, out of the total $n$ positions. The coefficients $c_i$ (in Eqs. (18) and (19)) remain the same for any arrangement of the $m$ noisy qubits. So, the success probability does not depend on the positions of the qubits which evolve under the noise unitary $U \in \{\sigma_x, \sigma_y, \sigma_z\}$. This result is also depicted in Fig. 8. We investigate the effects of Markovian correlated noise on the Grover’s search algorithm numerically, and the results are gathered in the section below.

IV. EXAMPLES

Before numerically showing the invariance of success probabilities proved in the previous section in presence of the good noises, we will first identify the regime of our noise model which performs better than the classical search with at least some confidence. The noisy algorithm loses its quantum advantage if it can not provide any substantial speed-up over a classical search.

A. Performance of the noisy algorithm

The preservation of success probability upon increasing the number of noisy sites is a potentially important feature. Nevertheless, increasing the noise probability still has a detrimental effect on the performance of the algorithm, as will be evident in the analysis in next section. Since probability of finding the marked element in the Grover’s algorithm is given by the success probability which never reaches 1 in the noisy scenario, the algorithm needs to be re-run multiple times to find the element with some confidence [48].

Say a noisy register, searching for a marked state out of total $N$ states, reaches its first success probability maximum $P$ at time $T$. A classical search would find the element in $\frac{N}{2}$ time steps on average. Thus, assuming $T < \frac{N}{2}$, the quantum algorithm reaches its first maximum approximately $q = \lfloor \frac{N}{2T} \rfloor$ times faster than the classical one. But it being likely that $P$ is much less than 1 for a noisy algorithm, we can claim that the quantum algorithm is at least as good as the classical one, only if, after running the noisy algorithm $q$ times, the probability $P = (1 - (1 - P)^q)$ of finding the marked element at least once, is close to unity. Here we take a probability of 0.95 to be the lower bound of such confidence.

In Fig. 4, we have shown the values of $\mu$ and $p$ for which the register, under $U = \sigma_x$ noise, searching from a collection of $N$ elements is at least as good as the classical algorithm. We can see that a higher memory $\mu$ helps the algorithm to perform better than its classical counterpart up to higher noise probabilities. Another observation is that the quantum advantage becomes more prominent in case of higher database sizes $N$.

B. Patterns of success probability

In this subsection, we will first show that the invariance of the success probabilities in case of Pauli noise unitaries, persists irrespective of any time-correlation in the noise. Then, the independence from positions of the noise sites in case of the good noises and the effect of memory on the algorithm is shown numerically.

1. Noise without memory

The case of $\mu = 0$, discussed in Sec. II D, is a noise without any memory or time-correlation. So at each time step, the probability for the Grover operation to become noisy is $p_g = p$. In Fig. 5, we compare the behavior in case of two noise unitaries $U$. The noise sites are the first $m$ qubits in the register, i.e., $\chi_m = U^{\otimes m} \otimes 1^\otimes(10^m - m)$.

The case of $U = \sigma_x$ here corresponds to an $n$-qubit bit-flip channel, as was shown in Sec. II D. We see that the success probability’s evolution, $P(t)$, for a given noise probability $p$, is unchanged when the number of noisy qubits is increased from $m = 1$ to $m = 5$ for $U = \sigma_x$.

We contrast this with the evolution of $P(t)$ in case of $U = (\sigma_x + \sigma_z)/\sqrt{2}$, i.e., the Hadamard operator. This $U$ is a linear combination of two Pauli matrices and thus is not a good noise. $P(t)$ in presence of this noise changes when the number of noise sites is increased from $m = 1$ to $m = 5$, as expected.

We have also plotted in Fig. 6 the success probability’s evolution for $m = 1, 2, 4, 5$ in presence of noise unitary $U = \sigma_y$ and $\mu = 0$. As discussed in Sec. III, the behavior of $P(t)$ is exactly the same for odd number of total noise sites, i.e., for
for non-zero (positive) memory $m$ are for different $m$ used are dimensionless. Whereas for a non-Pauli unitary $U$ depicted in Fig. 7. Here we have used the form of noise as elements in classical list) for two different noise unitaries are in the plots for different values of the noise probability $p$. The plots are for different $U$ and $m$ as displayed below each plot. All quantities used are dimensionless.

$$m = 1 \text{ and } m = 5.$$ The same is true for even noise strengths $m = 2 \text{ and } m = 4$.

2. Noise with finite time-correlation

The success probability $P(t)$ of Grover’s search algorithm for non-zero (positive) memory $\mu$ and for $n = 8$ qubits (64 elements in classical list) for two different noise unitaries are depicted in Fig. 7. Here we have used the form of noise as

$$X_m = U \otimes m \otimes I^{(8-m)}$$

with $m = 1 \text{ and } m = 4$. We can observe that the success probability $P(t)$ depends on the noise probability $p$ and the memory parameter $\mu$. It is obvious that the success probability reduces with increasing noise probability, and we can see from all the four panels that for a very high noise probability, the oscillatory behaviour of $P(t)$ tends to vanish.

It can be seen from Fig. 7(a) and 7(b) that for the good noise $U = \sigma_x$, $P(t)$ for a given $p$ and $\mu$, remains unaffected when we change the number of noise sites $m$ on which $U$ is applied. Whereas for a non-Pauli unitary $U = (\sigma_y + \sigma_z)/\sqrt{2}$, which was shown not to be a good noise before, the success probability $P(t)$ changes with the noise strength $m$. Compare panels 7(c) and 7(d). We can see that in case of a noise with low memory $\mu = 0.2$ and a high noise probability $p = 0.4$, the success probability evolution of the algorithm almost disappears. The noisy Grover’s search algorithm achieves greater efficiency for lower values of $p$ and higher values of $\mu$. Moreover, for higher values of $p$ for which the oscillation of $P(t)$ completely vanishes, the correlated noise helps in achieving higher success probabilities. For example, compare the lines corresponding to $(p, \mu) = (0.4, 0.2)$ and $(0.4, 0.9)$ in the figure. The analysis for $U = \sigma_x$ in case of perfect memory ($\mu = 1$) is summarised in Appendix C.

The success probabilities for the good noises are plotted with respect to time in Fig. 8 for different locations and number of noise sites. As we have commented previously, the positions of the noisy qubits do not matter if $U$ is a good noise. But, the total number of noise sites is important in case of $U = \sigma_y$. In Fig. 8(a) and 8(c), the total number $m$ of noise sites is the same but the positions of these sites are different. $P(t)$ in both these cases are exactly the same, as expected. In Fig. 8(b), all the qubits in the register are noisy. So, $m = 10$ in this case and it is an even number. Since in the other two sub-figures $m = 3$, an odd number, the behavior of $P(t)$ in...
FIG. 7. Success probabilities of Grover’s search algorithm in presence of noise. The plots are for \( n = 8 \). Here we have plotted \( P(t) \) on the vertical axis and the number of Grover iterations along the horizontal axis. The inset table exhibits the symbols used in the plots for different pairs of values of the noise probability \( p \) and memory parameter \( \mu \). The different plots are for different \( U \) and \( m \), as displayed below each plot. All quantities used are dimensionless.

FIG. 8. Success probabilities of Grover algorithm for good noises. Here \( n = 10 \) and \( \mu = 0.9 \). Each of the colored curves are for different \( p \) values and noise unitaries \( U \), as shown in the inset. The noise matrices \( \chi \), are displayed above each plot. The noise unitary acts on different sites in each plot. The number \( m \) of the noisy qubits also changes. The behavior in case of \( U = \sigma_y \), is not exactly the same as those cases. Whereas, \( P(t) \) in case of \( U = \sigma_x \) and \( \sigma_z \) remains unaltered in all three sub-figures of Fig. 8. From the figure, we see that given a good noise \( U \), \( P(t) \) is independent of the positions of noise sites.

Fig. 9 gives an overview of the effects of memory, database size and noise probability on the algorithm’s success probability. We have presented here the values of \( P(t^*) \), with respect to the noise probability \( p \), for different \( \mu \)’s. The algorithm is performed on \( n = \log_2 N \) qubits. Here the noise unitary considered is \( U = \sigma_x \). All quantities used are dimensionless.

\[ N \]

- \( \diamondsuit \) 8
- \( \triangle \) 16
- \( \square \) 128
- \( \triangleleft \) 256
- \( \triangledown \) 1024

FIG. 9. Effects of memory, database size and noise probability on the algorithm’s success probability. We have presented here the values of \( P(t^*) \), with respect to the noise probability \( p \), for different \( \mu \)’s. The algorithm is performed on \( n = \log_2 N \) qubits. Here the noise unitary considered is \( U = \sigma_x \). All quantities used are dimensionless.

V. CONCLUSION

The Grover’s algorithm can be employed to achieve a quadratic speed-up over classical methods in an unstructured search. While this gives an advantage, a practical quantum circuit will undoubtedly be affected by different types of noise and several studies have already been pursued on the effects of such noises on the algorithm’s performance. In our study, we consider the quantum register performing the algorithm to be under a local, unital noise which can also be correlated in
time. In this setting, we find that the success probability of the algorithm at all times remains unchanged with respect to the number of noisy qubits, if and only if the local noisy evolutions are given by some special unitaries which we call the ‘good noises’. These noises are shown to reduce to multi-qubit bit-flip or phase-damping errors and in some cases bit-phase flip errors in the absence of time-correlations. Locations of the noisy qubits are also shown to be not relevant in case of the good noises. This can be a potentially useful information in an actual implementation of the search algorithm on a register. The result that two of the noises behave in a different way than the third, can be explained by the symmetry breaking due to the choice of the initial state of the algorithm’s register (which is a product of eigenvectors of the Pauli $\sigma_z$ operator) and the ensuing Hadamard rotation (which connects the $\sigma_x$ and $\sigma_z$ eigenbases). Numerically, we have shown that a noise with Markovian time-correlation can lead to higher success of the algorithm. It will be interesting to investigate if such enhancement persists even in presence of more general time-correlated noise [84–91] in the quantum circuit.

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Appendix A: Invariance of success probability with respect to noise strength: a special case

In case of $U = \sigma_x$ from Eq. (3), we get $\chi_{m}|s\rangle = |s\rangle$. So, we can express all the states in terms of the orthogonal basis vector set $\{|s\rangle, |w\rangle, |w'\rangle\}$, with

$$|s\rangle := \frac{1}{\sqrt{N-2}} \sum_{x=1}^{N} |x\rangle.$$  \hspace{1cm} (A1)

So, in this basis $\langle s | = (\frac{\sqrt{N-2}}{N-2}, \frac{1}{(N-2)^{1/2}})$. Hence,

$$G = \begin{pmatrix}
\frac{2N-2}{N} & -\frac{2N}{N} & \frac{2N}{N} \\
\frac{2N-2}{N} & -\frac{2}{N} & \frac{2}{N}
\end{pmatrix}.$$  \hspace{1cm} (A2)

$$G' = \begin{pmatrix}
\frac{2N-2}{N} & -\frac{2N}{N} & \frac{2N}{N} \\
\frac{2N-2}{N} & -\frac{2}{N} & \frac{2}{N}
\end{pmatrix}.$$  \hspace{1cm} (A3)

It is evident from the expressions above that, at least for the case $U = \sigma_x$, although changing $m$ does change the forms of the basis vectors $|w'\rangle$ and $|s\rangle$ in the computational basis, elements of all the states or operators like $|s\rangle$ or $G'$ remain the same in $\{|s\rangle, |w\rangle, |w'\rangle\}$ basis. Thus, increasing or decreasing the noise strength $m$ does not affect the success probability of the algorithm in case of $U = \sigma_x$ and $m \geq 1$.

Appendix B: Markovian correlated noise

An important example of noise with memory is the Markovian correlated Pauli channel investigated in [77–79]. In that paper they studied the classical capacity of channels with partial memory. More specifically, they considered a channel that applies $\pi$-rotations along random sets of axes $l_1, l_2, \ldots, l_n$ on a sequence of $n$ qubits, with joint probability $p_{1i2i3i\ldots}$, where $\sum_{l_1,l_2,\ldots} p_{1i2i3i\ldots} = 1$. They also assumed that the rotation axes $l_1, l_2, \ldots, l_n$ form a Markov chain so that

$$p_{1i\ldots} = p_{1i}p_{2i|1} \cdots p_{ni|1\ldots},$$  \hspace{1cm} (B1)

where $p_{ij}$ denotes the conditional probability of rotation along $i$-axis given that the previous one was along $j$-axis. The conditional probabilities are given as

$$p_{ij} = (1 - \mu) p_i + \mu \delta_{i,j}.$$  \hspace{1cm} (B2)

Here $\mu$ corresponds to the relaxation time or “memory”. For example, if $\mu = 1$, the same rotation axis $l_1$ is used at all subsequent rotations, i.e., $l_1l_1 \ldots l_1$ on the qubits.

Appendix C: Evolution of success probability for perfect memory for $U = \sigma_x$

Here, we consider the case when $\mu = 1$, i.e., perfect memory. On the first noisy iteration (i.e., $t = 1$), $G$ occurs with probability $(1 - p)$ and $G'$ with $p$. Let us assume at $t = 1$, $G$ is applied. Due to perfect memory, for all $t \geq 2$, the same operator $G$ will be applied. This scenario corresponds to an ideal noiseless Grover algorithm. The success probability in this case will be denoted as $P(t)$ and the marked state is reached at $t \approx \pi \sqrt{N}$ [40].

If $G'$ is applied at $t = 1$, for $t \geq 2$ the state of the whole $n$-qubit register would be $|\psi(t)\rangle = G'^t|s\rangle$. Using the form of $G'$ in Eq. (A3) for $U = \sigma_x$,

$$\langle w|G'^t|s\rangle = \frac{(-1)^{t+1}}{\sqrt{N}} \text{Im} \left[ \left( \tan \left( \frac{\theta}{2} \right) - i \right) e^{it0} \right]$$

where $\theta = \cos^{-1}(\frac{\pi}{2N})$ and $\text{Im}[:]$ denotes the imaginary part of a complex number.

Then the success probability at time $t$ in this case is

$$P'(t) = |\langle w|G'^t|s\rangle|^2 = \frac{\cos^2(\theta t)}{N} \left( \tan \left( \frac{\theta}{2} \right) \tan(\theta t) - 1 \right)^2.$$  \hspace{1cm} (C1)

The first maximum of $P(t)$ in Eq. (C1) is analytically found at $t = \left( \frac{\pi}{2N} - \frac{1}{2} \right) \approx \left( \frac{\pi}{2N} + \frac{\pi}{2N} \right)$.

Combining the above two cases, the success probability of a noisy algorithm at time $t$, with noise probability $p$ and perfect memory $\mu = 0$, then becomes $(1 - p) P(t) + p P'(t)$.
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