Mutual Exclusion Statistics in the Exactly Solvable Model of the Mott Metal-Insulator Transition

Yasuhiro Hatsugai\textsuperscript{1}, Mahito Kohmoto\textsuperscript{2}, Tohru Koma\textsuperscript{3}, and Yong-Shi Wu\textsuperscript{4}

\textsuperscript{1} Department of Applied Physics, University of Tokyo, Hongo, Bunkyo-ku, Tokyo 113, JAPAN
\textsuperscript{2} Institute for Solid State Physics, University of Tokyo, Roppongi, Minato-ku, Tokyo 106, JAPAN
\textsuperscript{3} Department of Physics, Gakushuin University, Mejiro, Toshima-ku, Tokyo 171, JAPAN
\textsuperscript{4} Department of Physics, University of Utah, Salt Lake City, Utah 84112, U.S.A.

We study statistical characterization of the many-body states in the exactly solvable model with internal degree of freedom in more than one dimension. The model exhibits the Mott metal-insulator transition. It is shown that the ground state is described by that of a generalized ideal gas of particles (called exclusons) which have mutual exclusion statistics between different species. In addition to giving a perspective view of spin-charge separation, the model constitutes an explicit example of mutual exclusion statistics in more than two dimensions.

1 Introduction

Elementary particles or excitations are usually classified either as boson or as fermion. In recent years, however, it has been recognized that particles with “fractional statistics” intermediate between boson and fermion can exist in two-dimensional or in one-dimensional systems. In two dimensions, a type of fractional statistics can be defined on the basis of the phase factor, \( \exp(i\theta) \) with \( \theta \) allowed to be arbitrary, associated with an exchange of identical particles. One has \( \theta = 0 \) for bosons and \( \theta = \pi \) for fermions. A particle obeying such fractional statistics (with \( \theta \neq 0 \) or \( \pi \)) is called as “anyon”. It is believed that the quasiparticles and the quasiholes in the fractional quantum Hall liquids are anyons. Anyons can exist only in two spatial dimensions due to the braid group structure associated with them.

Another aspect of quantum statistics involves state counting, or the exclusive nature of the particles. Any number of bosons can be in a single-particle quantum state. Therefore there is no exclusion between bosons. On the other hand, the exclusion is perfect for fermions in a sense that a single particle state can accommodate at most one fermion. This aspect of quantum statistics can be generalized, as noticed by Haldane, who proposed a definite generalization of Pauli principle such that one can consider particles with non-perfect exclusion. He pointed out that a spinon in one-dimensional long-range interacting quantum spin chain can exclude, on average, half of other spinon in
occupying a single particle state. We shall call such a generalization as “exclusion statistics” and a particle obeying it as an “exclusion”. In contrast to usual bosons and fermions, the general concept of exclusons allows mutual statistics. Namely, there may exist statistical interactions or mutual exclusion between different species of particles. Haldane has recognized that quasiparticles in the fractional quantum Hall fluids are exclusons with mutual statistics between quasi-electrons and quasi-holes. More detailed discussions that show the differences and relationship between the anyon description and the exclusion description of quasiparticles in the fractional quantum Hall fluids are given in Ref. 7. This gives a concrete example of an exclusion system in two dimensions.

Thus the concepts of fractional anyon statistics and exclusion statistics constitute generalizations of two different aspects (exchange phase and exclusion) of usual quantum statistics. An essential difference between the two concepts is that anyons can exist only in two spatial dimensions, while in principle exclusons may exist in any dimensions.

Recently, one of us introduced the concept of generalized ideal gas of exclusons (see Sec. 2 for definition), and showed that its thermodynamic properties can be easily understood through a statistical distribution that interpolates between bosons and fermions. Later Bernard and Wu have shown that the Bethe ansatz solvable models in one dimension can be described as an ideal (or non-interacting) exclusion gas. This exemplifies that in certain circumstances particle-particle interactions can be totally absorbed by the statistical interactions. (A possible relation to a conformal field theory was discussed by Fukui and Kawakami.) Thus the concept of exclusion statistics may become a powerful tool in understanding certain interacting many-body problems. For example, recently it has been shown that the essential features of low-temperature physics of Luttinger liquids in one dimension can be approximately described by a system of noninteracting exclusons. This may provide a new approach to interacting many-body systems.

The examples considered by Bernard and Wu are the repulsive δ-function boson gas and the Calogero-Sutherland model. Both of them contain only single species. The present paper studies statistical interactions or mutual statistics in exactly solvable models with internal quantum numbers. We shall consider an exactly solvable model in arbitrary dimensions proposed by two of us, which has the Mott metal-insulator transition. An interesting feature is that there exists mutual exclusion between different species. The question we are going to address deals with the exclusion description for the physically interesting phenomenon of charge-spin separation. The model under consideration exhibits this phenomenon under certain conditions. We shall show that when
this happens, the model can be described by two species (spin and charge) of exclusonic excitations with nontrivial mutual statistics. This provides an example of (mutual) exclusion statistics in more than two dimensions.

2 Excluson Description

We consider a system with a total number \( N = \sum_{j, \mu} N^\mu_j \) of particles or quasi-particles, where \( N^\mu_j \) is the number of particles of species \( \mu \) with a set of good quantum numbers, collectively denoted by \( j \), specifying the states. Following Ref. 8, we assume that the total number of states with \( \{ N^\mu_j \} \) is

\[
W = \prod_{i, \mu} \frac{[D^\mu_i(\{ N^\nu_j \}) + N^\mu_i - 1]!}{N^\mu_i! [D^\mu_i(\{ N^\nu_j \}) - 1]!}, (1)
\]

where \( D^\mu_i(\{ N^\nu_j \}) \) is the number of available single particle states (counted as bosons), which by definition is given by

\[
D^\mu_i(\{ N^\nu_j \}) + \sum_{j, \nu} g^{\mu\nu}_{ij} N^\nu_j = G^\mu_i, (2)
\]

with statistical interactions \( g^{\mu\nu}_{ij} \). Here \( G^\mu_i \) is the number of available single particle states when there is no particle in the system. Namely, \( G^\mu_i = D^\mu_i(\{0\}) \).

The derivative of (2) is

\[
\frac{\partial D^\mu_i(\{ N^\nu_j \})}{\partial N^\nu_j} = -g^{\mu\nu}_{ij}, (3)
\]

agreeing with the original definition for “statistical interactions” proposed by Haldane. When the pair \((i, \mu)\) differs from \((j, \nu)\), we call \( g^{\mu\nu}_{ij} \) mutual statistics between particles labelled by \((i, \mu)\) and those by \((j, \nu)\).

We further assume that the total energy of the system with \( \{ N^\mu_j \} \) particles is always simply given by

\[
E = \sum_{j, \mu} N^\mu_j \epsilon^\mu_j \quad (4)
\]

with constant \( \epsilon^\mu_j \). Equations (1), (2) and (4) define the generalized ideal gas of exclusons. It is known that (4) is not satisfied for free anyons. One of us has derived statistical distribution for generalized ideal gas and the thermodynamics following from it. The equilibrium statistical distribution for \( \{ N^\mu_i \} \) is determined by

\[
w^{\mu}_{i} N^\mu_{i} + \sum_{j, \nu} g^{\mu\nu}_{ij} N^\nu_{j} = G^\mu_{i}, \quad (5)
\]
where $w_\mu^i$ satisfy the equations

$$(1 + w_\mu^i) \prod_{j, \nu} \left( \frac{w_\nu^j}{1 + w_\nu^j} \right)^{g_{\mu \nu}^{ji}} = \exp \left[ \frac{\epsilon_\mu^i - a_\mu^i}{T} \right],$$

(6)

where $a_\mu^i$ is the chemical potential for particles of species $\mu$. The thermodynamic potential is given by

$$\Omega \equiv -T \log Z$$

$$= -T \sum_{\mu, i} G_\mu^i \log \left[ \frac{G_\mu^i + N_\mu^i - \sum_{j, \nu} g_{\mu \nu}^{ji} N_\nu^j}{G_\mu^i - \sum_{j, \nu} g_{\mu \nu}^{ji} N_\nu^j} \right],$$

(8)

where $Z$ is the grand partition function.

3 Exactly Solvable Model in Higher Dimensions

In two dimensions, it is known that the quasiparticles of the fractional quantum Hall liquid are anyons. They can also be considered to be exclusons \cite{7}.

In this section we present an example of mutual exclusion between different species in an exactly solvable model in higher dimensions, that exhibits charge-spin separation under certain circumstances. This clearly shows that exclusion statistics is conceptually different from anyon statistics whose existence requires two (spatial) dimensions.

Recently two of us \cite{14} and Baskaran \cite{16} have proposed a model of interacting electrons that can be solved exactly in any dimensions. The Hamiltonian is

$$H = -\sum_{\langle i,j \rangle} c_{i,\sigma}^\dagger c_{j,\sigma} + h.c. + \frac{U}{L^d} \sum_{i,j,\ell,m} \delta_{i+j,\ell+m} c_{i,\uparrow}^\dagger c_{j,\uparrow} c_{\ell,\downarrow} c_{m,\downarrow} - \sum_{i,\sigma} (\mu + \sigma \mu_0 h) c_{i,\sigma}^\dagger c_{i,\sigma},$$

(9)

where $\langle i, j \rangle$ represents nearest neighbors in $d$ dimensions, and $L^d$ is the total number of lattice sites, $\mu$ the chemical potential, $\mu_0$ the magnetic moment and $h$ the external magnetic field.

This model is unrealistic in the sense that the interaction term with coefficient $U$ is of infinite-ranged in real space and of strength independent of distance. It, however, has an attractive feature of being exactly solvable. In fact, it can be easily diagonalized for each $k$ in momentum space. All the properties including the thermodynamic quantities were obtained in \cite{14}. Furthermore it is remarkable that this model exhibits a number of important features of the correlated electron problems in spite of its simplicity and unrealistic nature.
Table 1: Electronic states and spin charge labels

| Electronic State Label | $N_c$: Charge | $N_s$: Magnon |
|------------------------|----------------|---------------|
| 0                      | 0              | 0             |
| $\uparrow$             | 1              | 0             |
| $\downarrow$           | 2              | 1             |

It exhibits, for example, the Mott metal-insulator transition that was stressed by two of us \cite{14} and Continentino and Coutinho-Filho \cite{17}. The zero temperature phase diagram of this model in any dimensions is shown in Fig. ???. It has both the fixed-density and density-driven Mott transitions. These transitions in general may be in different universal classes \cite{18,19}. The critical exponents of the two types of transitions are, however, the same in the present model and they seem to be in the same universality class \cite{17}. In the zero temperature phase diagram Fig.1, the region OBC is a Mott insulator phase with half filled band. The rest is metallic phases. In the region OABC, a double occupancy is prohibited and, as we will show, the system can be described by an exclusion picture. There is a Fermi surface of the exclusion gas in the region OAB, and on the phase transition line OB is a quantum phase transition of the exclusion gas. On the other hand, the exclusion description breaks down on the phase transition line BC. The region outside OABC is a metallic phase which is described by the two species of fermions (spin up and spin down electrons) as it should be.

Let us concentrate on the region OAB. Assume that $U$ is large and $T$ is low, so that $U$ is much larger than both $T$ and the band width. Under these conditions, there is no activation of doubly occupied states, therefore there are only three states (0,1 and 2) for each momentum $k$. In the state 0 there is no electron, the state 1 an electron with spin up, and the state 2 an electron with spin down. Let us denote the number of charges as $N_c$ and the number of magnons (number of spin-down) as $N_s$. We regard $N_c$ and $N_s$ as independent variables (spin-charge separation). By definition, the state 0 has $N_c = 0$ and $N_s = 0$, the state 1 $N_c = 1$ and $N_s = 0$ and the state 2 $N_c = 1$ and $N_s = 1$ (See Table 1). Then from (3), we easily derive

$$G_c = 1, \ G_s = 0,$$

and

$$
\begin{bmatrix}
    g_{cc} & g_{cs} \\
    g_{sc} & g_{ss}
\end{bmatrix} = \begin{bmatrix}
    1 & 0 \\
    -1 & 1
\end{bmatrix}.
$$

It is easy to verify that the condition for the ideal exclusion gas \cite{11} is satisfied, so that the system with doubly occupied states suppressed can be
described as a generalized ideal gas with two (charge and magnon) species with the statistical interaction given by (11). Note the nontrivial value $-1$ for mutual statistics $g_{sc}$; i.e. the presence of a charge can create an available state for magnon, though there is no bare available single magnon state ($G_s = 0$) when there is no charge. It is straightforward to check that the thermodynamics of the generalized ideal exclusion gas obtained from Eqs. (6) and (8) is identical to the result of Ref. 14 in the low temperature limit.

Indeed, in the present case, the species index $\mu = c, s$, and the state index $j$ is the momentum $k$ in $d$-dimensional space. Equation (8) now takes the form

$$\Omega = -T \sum_\mu \int \frac{d^d k}{(2\pi)^d} \log \left[ \frac{1 - n_\mu(k) - \sum_\nu g_{\mu\nu} n_\nu(k)}{1 - \sum_\nu g_{\mu\nu} n_\nu(k)} \right],$$

where $n_\mu(k)$ is the occupation number distribution function of the charge ($\mu = c$) or spin ($\mu = s$) excitations in $k$-space. From Eqs. (5) and (6), we have

$$n_c(k)(1 + w_c(k)) = 1, \quad n_s(k)(1 + w_s(k)) = n_c(k),$$

where the statistics matrix (11) has been used, and $w_c(k), w_s(k)$ satisfy

$$w_c(k) \frac{1 + w_s(k)}{w_s(k)} = e^{(\epsilon_c(k) - \mu_c)/T},$$

$$w_s(k) = e^{(\epsilon_s(k) - \mu_s)/T},$$

where $\epsilon_c(k) = -2 \sum_{\alpha=1}^d \cos(k_\alpha), \epsilon_s = 2\mu_0 h$ (the energy of spin excitation, which is actually measured relative to the energy of spin-up electrons), and

$$\mu_c = \mu + \mu_0 h, \quad \mu_s = 0.$$  

Thus

$$w_c(k) = \frac{e^{(\epsilon_c(k) - \mu_c)/T}}{1 + e^{-(\epsilon_s(k) - \mu_s)/T}}.$$  

Substituting Eqs. (17), (15), and (13) into (12), we obtain

$$\Omega = -T \int \frac{d^d k}{(2\pi)^d} \log \left[ 1 + (e^{\mu_0 h/T} + e^{-\mu_0 h/T}) e^{(\mu_c - \epsilon_c)/T} \right]$$

$$= -T \int \frac{d^d k}{(2\pi)^d} \log \left[ 1 + e^{-(\epsilon_1(k) - \mu)/T} + e^{-(\epsilon_2(k) - \mu)/T} \right],$$

where $\epsilon_1(k) = \epsilon_c(k) - \mu_0 h$, and $\epsilon_2(k) = \epsilon_c(k) + \mu_0 h$ are the energy of spin-up and spin-down electrons respectively. Equation (18) is nothing but the result
of Ref.\[3\] in the low temperature limit. (Remember that here we consider the case with large $U$ and low $T$, so that doubly occupied states are suppressed.)

Here we emphasize that the concept of spin-charge separation is crucial. The effects that spin and charge excitations are not actually independent of each other have been taken care of by the statistical interaction or mutual statistics between them in the present formulation. It seems to us that similar situations may happen in other strongly correlated systems that exhibit charge-spin separation in higher dimensions.

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