MULTIPLICATIVE FORMS AT THE INFINITESIMAL LEVEL

HENRIQUE BURSZTYN AND ALEJANDRO CABRERA

Abstract. We describe arbitrary multiplicative differential forms on Lie groupoids infinitesimally, i.e., in terms of Lie algebroid data. This description is based on the study of linear differential forms on Lie algebroids and encompasses many known integration results related to Poisson geometry. We also revisit multiplicative multivector fields and their infinitesimal counterparts, drawing a parallel between the two theories.

Contents

1. Introduction 1
Acknowledgments 4
Notation, conventions and identities 4
2. Linear forms on vector bundles 5
2.1. Tangent and cotangent bundles of vector bundles 5
2.2. The structure of linear forms on vector bundles 6
2.3. Tangent lifts 8
3. Linear forms on Lie algebroids 9
3.1. Core and linear sections 9
3.2. Tangent Lie algebroids 10
3.3. IM-forms 11
4. Infinitesimal description of multiplicative forms 15
4.1. From multiplicative to IM forms 16
4.2. Integration of IM forms 18
5. Relation with the Weil algebra and Van Est isomorphism 22
6. The dual picture: multiplicative multivector fields 25
6.1. Linear multivector fields and derivations 26
6.2. Infinitesimal description of multiplicative multivector fields 29
6.3. The universal lifting theorem revisited 30
References 35

1. Introduction

This paper is devoted to the study of multiplicative differential forms on Lie groupoids, with focus on their infinitesimal counterparts. Given a Lie groupoid $\mathcal{G}$ over a manifold $M$, recall that a $k$-form $\omega \in \Omega^k(\mathcal{G})$ is called multiplicative if $m^*\omega = \text{pr}_1^*\omega + \text{pr}_2^*\omega$, where $m: \mathcal{G}^{(2)} = \mathcal{G} \times_M \mathcal{G} \to \mathcal{G}$ is the groupoid multiplication, and $\text{pr}_i: \mathcal{G}^{(2)} \to \mathcal{G}$, $i = 1, 2$, are the natural projections. Our goal is to characterize multiplicative forms on $\mathcal{G}$ solely in terms of information from its Lie algebroid. We will also discuss the analogous problem for multiplicative multivector fields.
Multiplicative differential forms and multivector fields on Lie groupoids have been studied for over 20 years in a variety of contexts. On Lie groups, multiplicative bivector fields came to notice in the late 1980s when the concept of Poisson Lie group (see e.g. [20] and references therein) was systematized. Multiplicative 2-forms on Lie groupoids appeared around the same time as the symplectic forms on symplectic groupoids [9, 29], originally introduced as part of a quantization scheme for Poisson manifolds. Poisson Lie groups and symplectic groupoids naturally led to Poisson groupoids [30], which gave further impetus to the study of multiplicative structures. In particular, multiplicative vector fields on Lie groupoids turn out to encompass classical lifting processes of general interest in differential geometry, see [23]. There are further connections between multiplicative 2-forms and bivector fields and the theory of moment maps, found e.g. in [4, 5, 18, 25, 31].

A central issue when considering multiplicative geometrical structures on Lie groupoids concerns their infinitesimal description, i.e., their description in terms of Lie-algebroid data. As usual in Lie theory, relating global and infinitesimal objects involves suitable differentiation and integration procedures. Important classes of examples of multiplicative structures and their infinitesimal counterparts include Poisson Lie groups and Lie bialgebras (see e.g. [20]), symplectic groupoids and Poisson manifolds (see [7, 9]), and, more generally, the correspondence between Poisson groupoids and Lie bialgebroids [22, 24]. Dirac structures [10, 28] fit into a similar picture, being the infinitesimal versions of certain multiplicative 2-forms, see [5]. As observed in [11], generalized complex structures [16, 17] provide another class of geometrical structures encoding infinitesimal information that can be integrated to multiplicative structures on Lie groupoids. In establishing these infinitesimal-global correspondences, different methods have been employed, some based on the integrability of Lie-algebroid morphisms, e.g. [3, 22, 24], others on infinite-dimensional arguments, e.g. [5, 7, 8, 18]. Although identifying the infinitesimal versions of multiplicative forms is key in some of the aforementioned works, only special cases of this problem have been considered. In this paper, we treated it in full generality.

From yet another perspective, multiplicative differential forms arise as constituents of the Bott-Schulman double complex of Lie groupoids [2] (see also [1] and references therein), which computes the cohomology of their classifying spaces. So the problem of understanding multiplicative forms infinitesimally may be seen as part of the problem of finding infinitesimal models for the cohomology of classifying spaces. This broader viewpoint is explored in the recent work [1], leading to results closely related to ours; a comparison between them is also discussed in this paper.

Our approach to describe multiplicative forms infinitesimally starts with the study of linear differential forms on vector bundles $A \to M$. We observe (Theorem 2.5) that any linear $k$-form on $A$ is equivalent to a pair $(\mu, \nu)$ of vector-bundle maps $\mu : A \to \wedge^{k-1}T^*M$, $\nu : A \to \wedge^kT^*M$, covering the identity on $M$. If $A$ carries a Lie algebroid structure, with bracket $[\cdot, \cdot]$ and anchor $\rho$, we say that the pair $(\mu, \nu)$ is an $IM$ $k$-form ($IM$ standing for infinitesimally multiplicative) if the following compatibility conditions are satisfied: for all $u, v \in \Gamma(A)$,

\begin{align*}
(1) \quad & i_{\rho(u)}\mu(v) = -i_{\rho(v)}\mu(u), \\
(2) \quad & \mu([u, v]) = \mathcal{L}_{\rho(u)}\mu(v) - i_{\rho(v)}d\mu(u) - i_{\rho(v)}\nu(u), \\
(3) \quad & \nu([u, v]) = \mathcal{L}_{\rho(u)}\nu(v) - i_{\rho(v)}d\nu(u).
\end{align*}
We prove in Theorem 4.6 that multiplicative $k$-forms on a source-simply-connected Lie groupoid $\mathcal{G}$ over $M$ are in one-to-one correspondence with IM $k$-forms on its Lie algebroid $A \to M$. Concretely, the IM $k$-form $(\mu, \nu)$ associated with a multiplicative $k$-form $\omega \in \Omega^k(\mathcal{G})$ is defined by
\[
\langle \mu(u), X_1 \wedge \ldots \wedge X_{k-1} \rangle = \omega(u, X_1, \ldots, X_{k-1}),
\]
\[
\langle \nu(u), X_1 \wedge \ldots \wedge X_k \rangle = d\omega(u, X_1, \ldots, X_k),
\]
where $X_i \in TM$, $i = 1, \ldots, k$, and we view $M \subseteq \mathcal{G}$ and $A \subseteq T\mathcal{G}|_M$.

A special class of IM-forms is obtained as follows. Any closed form $\phi \in \Omega^{k+1}(M)$ determines a map $\nu : A \to \wedge^k T^*M$, $\nu(u) = -i_{\rho(u)} \phi$, satisfying condition (3) above. The IM $k$-forms $(\mu, \nu)$ with $\nu$ of this type are referred to as IM $k$-forms relative to $\phi$; they are the infinitesimal versions of multiplicative $k$-forms satisfying
\[
d\omega = s^* \phi - t^* \phi,
\]
where $s$ and $t$ denote the groupoid source and target maps. For $k = 2$, IM forms relative to $\phi$ include $\phi$-twisted Poisson and Dirac structures [28], and our Theorem 4.6 recovers their known integrations [5, 8]. For arbitrary $k$, IM forms relative to $\phi$ were studied in [1] in connection with the Weil algebra of a Lie algebroid. These and other examples are discussed in this paper.

The method we use to integrate IM forms on Lie algebroids to multiplicative forms on Lie groupoids relies entirely on the known correspondence between Lie-algebroid and Lie-groupoid morphisms (Lie’s second theorem for Lie algebroids); in particular, we do not resort to the path spaces of [7, 12, 27], hence avoiding infinite dimensional constructions. Although our method is inspired by [3, 22, 24], it brings a technical difference in that we represent differential $k$-forms on a manifold $N$ by functions $\oplus^k TN \to \mathbb{R}$ (as opposed to maps $\oplus^{k-1} TN \to T^*N$); this small variation greatly simplifies computations, so even when restricted to known situations, our general proof seems more direct than existing ones. The integration of IM forms is carried out in two steps: first, we show that an IM $k$-form on a Lie algebroid $A \to M$ defines an element in $\Omega^k(A)$ whose associated function $\oplus^k TA \to \mathbb{R}$ is a Lie-algebroid morphism; second, upon integration, one obtains a groupoid morphism $\oplus^k T\mathcal{G} \to \mathbb{R}$ which defines a multiplicative $k$-form$^1$.

In the last part of the paper, we revisit multiplicative multivector fields on Lie groupoids, as in [18]. We show how the very same techniques used to study multiplicative forms apply to the dual situation of multivector fields, leading to an alternative proof of the universal lifting theorem of [18] (not involving path spaces) and drawing a clear parallel between the two theories.

As a final remark, we note that the results in this paper admit a natural formulation in terms of graded geometry. Multiplicative forms and multivector fields on a given Lie groupoid $\mathcal{G}$ may be seen as multiplicative functions on the associated graded Lie groupoids $T[1]G$ and $T^*[1]G$, respectively. On ordinary Lie groupoids, the infinitesimal counterpart of a multiplicative function is a Lie-algebroid cocycle. The same holds at the graded level and, from this perspective, our results consist

$^1$Forms on $\mathcal{G}$ are commonly viewed as sections $\mathcal{G} \to \wedge^k T^*\mathcal{G}$. The multiplicative ones, for $k = 1$, are such that $\mathcal{G} \to T^*\mathcal{G}$ is a groupoid morphism. However, this viewpoint does not extend to $k \geq 2$, as in this case $\wedge^k T^*\mathcal{G}$ inherits no canonical groupoid structure in general; in contrast, $\oplus^k T\mathcal{G}$ is always a groupoid and multiplicative forms are conveniently described by groupoids morphisms $\oplus^k T\mathcal{G} \to \mathbb{R}$. An analogous discussion holds for multivector fields.
in using the geometry of $T[1]G$ and $T^*[1]G$ to obtain concrete descriptions of their Lie-algebroid cocycles. For example, using the the natural multiplicative vector field on $T[1]G$ (the de Rham differential on $G$), one identifies its Lie-algebroid cocycles with IM forms (see Theorem 3.1); for an analogous description of the Lie-algebroid cocycles of the graded groupoid $T^*[1]G$ (see Theorem 6.1), ones uses its canonical multiplicative symplectic structure (defined by the Schouten bracket on $G$). We will not elaborate on the supergeometric viewpoint in this paper, though it makes our results more intuitive.

The paper is organized as follows. In Section 2, we consider linear differential forms on vector bundles $A \to M$ and establish their correspondence with pairs of vector-bundle maps $(\mu, \nu)$, where $\mu : A \to \wedge^{k-1}T^*M$ and $\nu : A \to \wedge^k T^*M$. In Section 3, we define IM $k$-forms on Lie algebroids and prove a compatibility result with tangent Lie algebroid structures (Theorem 3.1). Section 4 is devoted to Theorem 4.6, which is the correspondence between IM forms on Lie algebroids and multiplicative forms on Lie groupoids; we also discuss several special cases of this result. Section 5 explains the relationship between Theorem 4.6 and the Van Est isomorphism of [1]. In Section 6, we revisit the theory of multivector fields from [18].

**Acknowledgments.** We thank D. Iglesias Ponte and C. Ortiz for helpful discussions related to this project. Cabrera thanks CNPq for financial support and IMPA for its hospitality and stimulating environment during the development of this work. Bursztyn’s research has been supported by CNPq and Faperj. The authors acknowledge the anonymous referees for their many useful suggestions, especially regarding improvements in the introduction.

**Notation, conventions and identities.** For vector bundles $A \to M$ and $B \to M$ over the same base $M$, a vector-bundle map $\Psi : A \to B$ is always assumed to cover the identity map on $M$, unless stated otherwise. We denote its transpose, or dual, by $\Psi^* : B^* \to A^*$. We denote the $k$-fold direct sum of a vector bundle $q_A : A \to M$ by $\oplus^k_M A$, or simply $\oplus^k A$ if there is no risk of confusion. We may also use the notation $\prod^k_M A$ if we want to be explicit about the projection map $q_A$ (this is relevant when dealing with double vector bundles).

For a Lie groupoid $G$ over $M$, we usually denote its source and target maps by $s$ and $t$. The set $G^{(2)} \subset G \times G$ of composable pairs is defined by the condition $s(g) = t(h)$, and the multiplication is denoted by $m : G^{(2)} \to G$, $m(g, h) = gh$. The unit map $e : M \to G$ is often used to identify $M$ with its image in $G$. The Lie algebroid of $G$ is $AG = \ker(Ts)|_M$, with anchor $Tt|_A : A \to M$ and bracket induced by right-invariant vector fields. For a Lie algebroid $A \to M$, we denote its anchor by $\rho_A$ and bracket by $[,]_A$ (or simply $\rho$ and $[,]$, if there is no risk of confusion).

We introduce some notation and collect some identities that will be useful for later computations. If $U_1, \ldots, U_m$ are vector fields on a manifold $M$, we set

\begin{equation}
I^U_{m,r} := i_{U_m} \ldots i_{U_r}, \quad r \leq m,
\end{equation}

where $i_U$ is the usual contraction. An inductive application of Cartan’s formula gives

\begin{equation}
I^U_{m,1} = \sum_{i=1}^{m} (-1)^{i+1} I^U_{m,i+1} \mathcal{L}_{U_i} I^U_{i,1} + (-1)^n d^U_{m,1},
\end{equation}
where \( d \) denotes the de Rham differential and \( \mathcal{L}_U \) is the Lie derivative. Given another vector field \( X \) and recalling the commutator formula \( i_{[X,U]} = \mathcal{L}_X i_U - i_U \mathcal{L}_X \), we obtain
\[
(1.3) \quad \mathcal{L}_X I^U_{m,1} = \sum_{l=1}^m I^U_{m,l+1} i_{[X,U]} I^U_{l-1,1} + I^U_{m,1} \mathcal{L}_X.
\]

Given a differential form \( \alpha \), we also have
\[
(1.4) \quad I^U_{m,1}(df \wedge \alpha) = \sum_{l=1}^m (-1)^{l+1} df(U_l) I^U_{m,l+1} I^U_{l-1,1} \alpha + (-1)^m df \wedge I^U_{m,1} \alpha.
\]

We often use Einstein’s summation convention when there is no risk of confusion.

2. Linear forms on vector bundles

In order to define linear forms, we recall a few facts about tangent and cotangent bundles of vector bundles.

2.1. Tangent and cotangent bundles of vector bundles. Let \( q_A : A \rightarrow M \) be a vector bundle, and let \( TA \) be the tangent bundle of the total space \( A \). Besides its natural vector bundle structure over \( A \), with projection map denoted by \( p_A : TA \rightarrow A \), it is also a vector bundle over \( TM \), with respect to the map \( Tq_A : TA \rightarrow TM \).

It is useful to consider a coordinate description of these bundles. Let \( (x^j) \) be coordinates on \( M \), \( j = 1, \ldots, \dim(M) \), and let \( \{e_d\} \) be a basis of local sections of \( A_d = 1, \ldots, \rank(A) \). The corresponding coordinates on \( A \) are denoted by \( (x^j, u^d) \), and tangent coordinates on \( TA \) by \( (x^j, u^d, \dot{x}^j, \dot{u}^d) \). In this notation, given \( x = (x^j) \), the coordinates \( (u^d) \) specify a point in \( A_x \), \( (\dot{x}^j) \) a point in \( T_x M \), whereas \( (\dot{u}^d) \) determines a point on a second copy of \( A_x \), tangent to the fibres of \( A \rightarrow M \). Note that \( p_A(x^j, u^d, \dot{x}^j, \dot{u}^d) = (x^j, u^d) \), and \( Tq_A(x^j, u^d, \dot{x}^j, \dot{u}^d) = (x^j, \dot{x}^j) \).

Similarly, consider the cotangent bundle \( T^*A \), with local coordinates \( (x^j, u^d, p_j, \xi_d) \), where \( (p_j) \) determines a point in \( T^*_x M \), and \( (\xi_d) \) a point in \( A^*_x \), dual to the direction tangent to the fibres of \( A \rightarrow M \). In this case, besides the natural vector bundle structure \( c_A : T^*A \rightarrow A \), \( c_A(x^j, u^d, p_j, \xi_d) = (x^j, u^d) \), \( T^*A \) is also a vector bundle over \( A^* \) [23], with respect to the projection map given in coordinates by
\[
(2.1) \quad r : T^*A \rightarrow A^*, \quad r(x^j, u^d, p_j, \xi_d) = (x^j, \xi_d).
\]

The total spaces \( TA \) and \( T^*A \) are examples of double vector bundles, see [21, 26]. They fit into the following commutative diagrams:

\[
\begin{array}{cccc}
TA & Tq_A & TM \\
p_A & & & p_M \\
A & q_A & M
\end{array}
\]

\[
\begin{array}{cccc}
T^*A & c_A & A^* \\
r & & & q_{A^*} \\
A & q_A & M
\end{array}
\]

where
\[
(2.2) \quad p_M : TM \rightarrow M, \quad p_M(x^j, \dot{x}^j) = (x^j), \quad q_{A^*} : A^* \rightarrow M, \quad q_{A^*}(x^j, \xi_d) = (x^j),
\]

are the natural projections. Recall, see e.g. [21], that the intersection of the kernels of the top and left arrows on each diagram defines a vector bundle over \( M \), known
as the core. In the case of $TA$, the core is identified with $A \to M$, with coordinates $(x^i, u^d)$; for $T^*A$, the core is $T^*M$, with coordinates $(x^i, p_j)$.

2.2. The structure of linear forms on vector bundles. Let $A \to M$ be a vector bundle, with local coordinates $(x^j, u^d)$, and let us consider the $k$-fold direct sum of $TA$ over $A$,

$$\oplus_kTA := TA \times_A \cdots \times_A TA,$$

locally described by coordinates $(x^j, u^d, \ldots, \hat{x}^j_k, \hat{u}^d_k, \ldots, \hat{u}^d_k)$. It is a vector bundle over $A$, with projection map

$$(x^j, u^d, \ldots, \hat{x}^j_k, \hat{u}^d_k, \ldots, \hat{u}^d_k) \mapsto (x^j, u^d),$$

and also a vector bundle over $\oplus^kTM = TM \times_M \cdots \times_M TM$, with projection map

$$(x^j, u^d, \ldots, \hat{x}^j_k, \hat{u}^d_k, \ldots, \hat{u}^d_k) \mapsto (x^j, \hat{x}^j_k, \ldots, \hat{x}^j_k).$$

Given a $k$-form $\Lambda \in \Omega^k(A)$ on the total space of $A \to M$, let us consider the induced maps

$$(2.3) \quad \Lambda^A : \oplus_k^{-1}TA \to T^*A, \quad \Lambda^A(U_1, \ldots, U_{k-1}) = i_{U_{k-1}} \cdots i_{U_1} \Lambda,$$

$$(2.4) \quad \Lambda : \oplus_k^A TA \to \mathbb{R}, \quad \Lambda(U_1, \ldots, U_k) = i_{U_k} \cdots i_{U_1} \Lambda,$$

which are alternating and linear in each of their entries\(^2\).

**Definition 2.1.** A $k$-form $\Lambda$ is called linear if the induced map $\Lambda^A$ (2.3) is a morphism of vector bundles with respect to the vector bundle structures $\oplus_k^{-1}TA \to \oplus^k^{-1}TM$ and $T^*A \to A^*$. The space of linear $k$-forms on $A$ is denoted by $\Omega^k_{\text{lin}}(A)$.

In particular, $\Lambda^A$ covers a base map $\lambda : \oplus_k^{-1}TM \to A^*$,

$$\begin{array}{ccc}
\oplus_k^{-1}TA & \xrightarrow{\Lambda^A} & T^*A \\
\downarrow & & \downarrow \lambda \\
\oplus_k^{-1}TM & \xrightarrow{\lambda} & A^*.
\end{array}$$

The map $\lambda$ is skew symmetric on its entries, so it can be viewed as a vector-bundle map $\Lambda^kTM \to A^*$. Its transpose is the vector-bundle map

$$(2.6) \quad \lambda^t : A \to \Lambda^kTM.$$

A simple computation in coordinates shows the following.

**Lemma 2.2.** Given a $k$-form $\Lambda \in \Omega^k(A)$, the following are equivalent:

1. $\Lambda$ is linear.
2. In local coordinates $(x^j, u^d)$ on $A$, $\Lambda$ has the form

$$(2.7) \quad \Lambda = \frac{1}{k!} \Lambda_{i_1 \ldots i_k} dx^i_1 \wedge \cdots \wedge dx^i_k + \frac{1}{(k-1)!} \lambda_{i_1 \ldots i_k} dx^i_1 \wedge \cdots \wedge dx^i_{k-1} \wedge du^d,$$

where $\lambda_{i_1 \ldots i_k} = \langle \lambda(\partial_{x^i_1}, \ldots, \partial_{x^i_{k-1}}), e_d \rangle$, and $\partial_{x^i} = \frac{\partial}{\partial x^i}$.

\(^2\)Notice that, since $\Lambda^t$ (resp. $\Lambda$) is multilinear in its entries, it is not a vector-bundle morphism from the direct sum $\oplus_k^{-1}TA \to A$ (resp. $\oplus^kTA \to A$) to $T^*A \to A^*$, unless $k = 2$ (resp. $k = 1$).
(3) The map $\overline{\Lambda}: \bigoplus_A^k TA \rightarrow \mathbb{R}$ defines a vector-bundle map

\[(2.8) \quad \bigoplus_A^k TA \xrightarrow{\overline{\Lambda}} \mathbb{R}, \quad \bigoplus^k TM \xrightarrow{\{\ast\}}.\]

Given a vector-bundle map $\mu: A \rightarrow \wedge^k T^*M$, let us consider the linear $k$-form $\Lambda_\mu$ on $A$ given at a point $u \in A$ by

\[(2.9) \quad (\Lambda_\mu)_u := Tq_A^t_u \mu(u).\]

In local coordinates $(x^i, u^d)$ on $A$, $\Lambda_\mu$ is written as

\[(2.10) \quad (\Lambda_\mu)_u = \frac{1}{k!} \mu_{i_1 \ldots i_k, d}(x) u^d dx^{i_1} \wedge \ldots \wedge dx^{i_k},\]

where $\mu_{i_1 \ldots i_k, d}$ is defined by

$$\mu_{i_1 \ldots i_k, d} = \left\langle \mu(e_d), \frac{\partial}{\partial x^{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial x^{i_k}} \right\rangle.$$

**Example 2.3.** When $k = 1$, a direct computation in coordinates shows that the linear 1-form $\Lambda_\mu$, defined by the vector-bundle map $\mu: A \rightarrow T^*M$, satisfies

\[(2.11) \quad \Lambda_\mu = \mu^* \theta_{\text{can}},\]

where $\theta_{\text{can}} = p^i dx^i$ is the canonical 1-form on $T^*M$. (When $A = M \rightarrow M$ is the vector bundle with zero fibres, (2.11) recovers the well-known “tautological” property $\mu^* \theta_{\text{can}} = \mu_\star$.)

**Lemma 2.4.** A linear $k$-form $\Lambda$ covers the fibrewise zero map in (2.5) if and only if it is of the form $\Lambda_\mu$ (as in (2.9)) for a vector bundle map $\mu: A \rightarrow \wedge^k T^*M$.

**Proof.** We can use Lemma 2.2, or argue more globally as follows. Consider the projection $r: T^* A \rightarrow A^*$, as in (2.1). One can directly check that

$$\ker(r)_u = (\ker(Tq_A^t_u))^0 = \text{im}(Tq_A^t_u),$$

where $^0$ stands for the annihilator. It follows from (2.9) that $r \circ \Lambda_\mu^* = 0$, which means that $\Lambda_\mu$ covers the fibrewise zero map in (2.5). Conversely, if $\Lambda$ covers the fibrewise zero map, then $r \circ \Lambda^* = 0$; so, given $U_1, \ldots, U_k \in T_u A$, $\Lambda^t(U_1, \ldots, U_k-1) = Tq_A^t_u \alpha$ for some $\alpha \in T^*_{q_A(u)} M$. Since $\Lambda$ is skew symmetric, we conclude that $\Lambda(U_1, \ldots, U_k)$ only depends on $Tq_A^t(u_j), j = 1, \ldots, k$. Hence, for each $u \in A$, there exists $\mu(u) \in \wedge^k T^*_{q_A(u)} M$ such that $(\Lambda)_u = Tq_A^t_u \mu(u)$. The linear dependence of $\mu(u)$ on $u$ follows from the linear dependence of $(\Lambda)_u$ on $u$, see (2.7); the resulting vector bundle map $\mu: A \rightarrow \wedge^k T^* M$ is smooth by the local expression (2.10). \qed

**Proposition 2.5.** There is a one-to-one correspondence between linear $k$-forms $\Lambda$ on $A$, covering a map $\lambda: \bigoplus_{A}^{k-1} TM \rightarrow A^*$, and pairs $(\mu, \nu)$, where $\mu: A \rightarrow \wedge^{k-1} T^* M$ and $\nu: A \rightarrow \wedge^k T^* M$ are vector bundle morphisms. The correspondence is given by

\[(2.12) \quad \Lambda = d\Lambda_\mu + \Lambda_\nu,\]

where $\mu = (-1)^{k-1} \lambda^t$. 
Proof. Let $\Lambda$ be a linear $k$-form on $A$, and set $\mu = (-1)^{k-1}\lambda^t$. A direct computation using the local expression (2.10) and Lemma 2.2 shows that the $k$-form $d\Lambda_\mu$ is linear and covers the same map $\lambda$, hence the linear $k$-form $\Lambda - d\Lambda_\mu$ covers the fibrewise zero map. By Lemma 2.4, there is a unique $\nu : A \rightarrow \wedge^k T^*M$ such that $\Lambda - d\Lambda_\mu = \Lambda_\nu$. \hfill $\square$

A direct consequence of (2.12) is that if $\Lambda$ is a linear form, then so is $d\Lambda$.

**Example 2.6.** Let $\Lambda \in \Omega^2(A)$ be a linear 2-form with $d\Lambda = 0$. According to the previous proposition, we can write it as $\Lambda = d\Lambda_\mu + \Lambda_\nu$, and $\Lambda$ being closed amounts to $d\Lambda_\nu = 0$; this condition immediately implies that $\nu = 0$, so $\Lambda = d\Lambda_\mu$. Using (2.11), it follows that

$$\Lambda = (\lambda^t)^*\omega_{\text{can}},$$

where $\omega_{\text{can}} = -d\theta_{\text{can}} = dx^i \wedge dp_i$ the canonical symplectic form on $T^*M$ (see [19, Sec. 7.3], and also [3, Prop. 4.3]).

**2.3. Tangent lifts.** We now briefly discuss linear forms obtained via the tangent lift operation [13, 32] (see also [3] and [23]), that assigns to any $k$-form on a manifold $M$ a linear $k$-form on the total space of its tangent bundle $p_M : TM \rightarrow M$.

Let us consider the operation

$$\tau : \Omega^l(M) \rightarrow \Omega^{l-1}(TM), \quad \tau(\beta)|_X := (Tp_M|_X)^t(i_X\beta),$$

where $X \in TM$ and $l \geq 1$; i.e., for $U_1, \ldots, U_{l-1} \in T_X(TM)$,

$$i_{U_{l-1}} \ldots i_{U_1} \tau(\beta)|_X = \beta(X, Tp_M(U_1), \ldots, Tp_M(U_{l-1})).$$

In the notation of Section 2.2, $\tau(\beta)$ is a linear $(l-1)$-form on the vector bundle $A = TM$ of type $\Lambda_\nu$, where

$$\nu : TM \rightarrow \wedge^{l-1}T^*M, \quad \nu(X) = i_X\beta.$$

It directly follows from Example 2.3 that, if $\omega \in \Omega^2(M)$, then $\tau(\omega) = (\omega^*)^*\theta_{\text{can}}$, where $\theta_{\text{can}} = p_t dx^i$ is the canonical 1-form on $T^*M$.

The **tangent lift** operation,

$$\Omega^k(M) \rightarrow \Omega^k(TM), \quad \alpha \mapsto \alpha_T,$$

assigns to $\alpha \in \Omega^k(M)$ the form $\alpha_T \in \Omega^k(TM)$ defined by the Cartan-like formula

$$\alpha_T = d\tau(\alpha) + \tau(d\alpha).$$

It follows directly from (2.15) that $\alpha_T$ is linear and that the operation (2.14) is compatible with exterior derivatives, in the sense that $(d\alpha)_T = d\alpha_T$.

We will also need an equivalent characterization of the tangent lift, see e.g. [13]. Given $\alpha \in \Omega^k(M)$, consider the associated map

$$\overline{\alpha} : \oplus^k TM \rightarrow \mathbb{R}, \quad (X_1, \ldots, X_k) \mapsto \alpha(X_1, \ldots, X_k).$$

Let $\prod_{p_M}^k(T(TM))$ denote the fibred product with respect to the vector bundle

$$Tp_M : T(TM) \rightarrow TM, \quad Tp_M(x^i, \dot{x}^j, \delta x^j, \delta \dot{x}^j) = (x^i, \delta x^j),$$

where $(x^i, \dot{x}^j, \delta x^j, \delta \dot{x}^j)$ are the local coordinates on $T(TM)$ induced by the tangent coordinates $(x^i, \dot{x}^j)$ on $TM$. We have a natural identification $T(\oplus^k TM) \cong \mathbb{R}$.
\[ \prod_{T_{PM}}^k T(TM), \text{ so we can view the differential of the function } \overline{\alpha} \text{ in } C^\infty(\oplus^k TM) \text{ as a map} \]
\[
\overline{\alpha} : \prod_{T_{PM}}^k T(TM) \rightarrow \mathbb{R}.
\]

Note that the canonical involution
\[
J_M : T(TM) \rightarrow T(TM), \quad J_M(x^j, \dot{x}^j) = (x^j, \dot{x}^j),
\]
induces an identification
\[
J^{(k)}_M : \prod_{T_{PM}}^k T(TM) \rightarrow \prod_{T_{PM}}^k T(TM).
\]

One can prove (see e.g. [13]) that, given \( \alpha \in \Omega^k(M) \), its tangent lift \( \alpha_T \in \Omega^k(TM) \) is uniquely determined by the condition
\[
\alpha_T = \overline{\alpha} \circ J^{(k)}_M : \prod_{T_{PM}}^k T(TM) \rightarrow \mathbb{R}.
\]

3. Linear forms on Lie algebroids

3.1. Core and linear sections. Any vector bundle that fits into a double vector bundle admits two distinguished types of sections, known as linear and core sections; a detailed discussion can be found e.g. in [15, Sec. 2.3] and [21]. For our purposes, we are mostly interested in the particular (double) vector bundles \( \oplus^k TA \) (and, later in Section 4, also in \( \oplus^k T^*A \)), so we restrict ourselves to these cases.

Each section \( u \) of a vector bundle \( A \rightarrow M \) defines a core section \( \hat{u} \) and a linear section \( Tu \) of \( TA \rightarrow TM \) as follows. The tangent bundle of \( A \) along its zero section \( M \leftarrow A \) naturally splits as \( TA|_M = TM \oplus A \), and we can define, for each \( X \in T_x M \), \( \hat{u}(X) := X(x) + u(x) \), where the sum is with respect to the previous decomposition of \( TA|_M \). The linear section \( Tu : TM \rightarrow TA \) is obtained by applying the tangent functor to \( u : M \rightarrow A \).

Let us consider local coordinates \( (x^j) \) on \( M \), a basis of local sections \( \{e_a\} \) of \( A \), and dual basis \( \{e^d\} \) of \( A^* \). As in Section 2, we denote the corresponding coordinates on \( A \) by \( (x^j, u^d) \), and on \( A^* \) by \( (x^j, \xi_d) \), while coordinates on \( TA \) are denoted by \( (x^j, u^d, \dot{x}^j, \dot{u}^d) \), and on \( T^*A \) by \( (x^j, u^d, p_j, \xi_d) \).

The core and linear sections of \( TA \rightarrow TM \) defined by a local section \( e_a \) of \( A \) are explicitly given by
\[
\bar{e}_a(x^j, \dot{x}^j) = (x^j, 0, \dot{x}^j, \delta^d_a), \quad Te_a(x^j, \dot{x}^j) = (x^j, \delta^d_a, \dot{x}^j, 0),
\]
where \( \delta^d_a \) is the \( d \)-th component of \( e_a \), i.e., 1 if \( d = a \) or zero otherwise.

More generally, \( e_a \) locally defines two types of local sections of \( \oplus^k TA \rightarrow \oplus^k TM \) as follows: the first type is given, for each \( n \in \{1, \ldots, k\} \), by
\[
\bar{e}_{a,n}(\dot{x}_{1} \oplus \cdots \oplus \dot{x}_n) := \tilde{0} (\dot{x}_1) \oplus \cdots \oplus \tilde{0} (\dot{x}_{n-1}) \oplus \bar{e}_a (\dot{x}_n) \oplus \tilde{0} (\dot{x}_{n+1}) \oplus \cdots \oplus \tilde{0} (\dot{x}_k),
\]
where \( \dot{x}_l = (x^j, \dot{x}^j_l) \) belongs to the \( l \)-th component of \( \oplus^k TM \) and \( \tilde{0}(\dot{x}_l) = (x^j, 0, \dot{x}^j_l, 0) \); the second type is
\[
(Te_a)^{(k)}(\dot{x}_1 \oplus \cdots \oplus \dot{x}_k) := Te_a(\dot{x}_1) \oplus \cdots \oplus Te_a(\dot{x}_k).
\]
The sections \( \hat{e}_{a,n} \) and \((Te_a)^k\) are the core and linear sections on \( \bigoplus_A^k TA \), respectively. A key property is that they generate the module of local sections of \( \bigoplus_A^k TA \to \bigoplus^k TM \). Note also that, under the natural projection \( \bigoplus_A^k TA \to A \), core sections \( \hat{e}_{a,n} \) are sent to the zero section of \( A \to M \), while linear sections \((Te_a)^k\) map to the section \( e_a \).

### 3.2. Tangent Lie algebroids

Suppose that \( A \to M \) carries a Lie algebroid structure (see e.g. [6, 21]), with Lie bracket \([\cdot,\cdot]_A\) on \( \Gamma(A) \) and anchor map \( \rho_A : A \to TM \). Then the vector bundle \( TA \to TM \) inherits a natural Lie algebroid structure, known as the tangent Lie algebroid, see e.g. [22]. We will need local expressions for the tangent Lie algebroid in terms of the coordinates introduced in Section 3.1.

The Lie algebroid \( A \to M \) is locally determined by structure functions \( \rho_a^j \) and \( C_{ab}^c \) defined by

\[
(3.4) \quad \rho_A(e_a) = \rho_a^j \frac{\partial}{\partial x^j}, \quad [e_a, e_b]_A = C_{ab}^c e_c.
\]

The tangent Lie algebroid structure on \( TA \to TM \) is defined in terms of core and linear sections (3.1) by

\[
(3.5) \quad [\hat{e}_a, \hat{e}_b]_{TA} = 0, \quad [Te_a, \hat{e}_b]_{TA} = C_{ab}^c \hat{e}_c, \quad [Te_a, Te_b]_{TA} = C_{ab}^c Te_c + \dot{x}^i \frac{\partial C_{ab}^c}{\partial x^i} \hat{e}_c,
\]

\[
(3.6) \quad \rho_{TA}(Te_a) = \rho_a^j \frac{\partial}{\partial x^j} + \dot{x}^i \frac{\partial \rho_a^j}{\partial x^i} \frac{\partial}{\partial \dot{x}^j}, \quad \rho_{TA}(\hat{e}_a) = \rho_a^j \frac{\partial}{\partial \dot{x}^j}.
\]

In (3.6), we have identified points in \( T(TM) \), written in coordinates as \((x^j, \dot{x}^j, \delta x^j, \delta \dot{x}^j)\), with tangent vectors

\[
\frac{\delta x^j}{\partial x^j} \frac{\partial}{\partial x^j} + \frac{\delta \dot{x}^j}{\partial \dot{x}^j} \frac{\partial}{\partial \dot{x}^j} \bigg|_{(x^j, \dot{x}^j)}.
\]

We notice that the tangent Lie algebroid induces a Lie algebroid structure on the direct sum \( \bigoplus_A^k TA \to \bigoplus^k TM \). This is a general property of \( VB\)-algebroids [15, Sec. 2.1], which we directly verify in this example. A simple consequence of (3.5) and (3.6) is that if \( U \) and \( V \) are local sections of \( TA \to TM \), each of type \( \hat{e}_a \) or \( Te_a \), then

\[
(3.7) \quad Tp_M(\rho_{TA}(U)) = \rho_A(p_A(U)), \quad p_A([U,V]_{TA}) = [p_A(U), p_A(V)]_A,
\]

where \( p_M : TM \to M \) and \( p_A : TA \to A \) are the natural projections. It follows from the first equation in (3.7) that if \( U_1 \oplus \ldots \oplus U_k \in \bigoplus_A^k TA \) is of type (3.2) or (3.3), then

\[
T p_M(\rho_{TA}(U_l)) = T p_M(\rho_{TA}(U_m)), \quad \forall l, m \in \{1, \ldots, k\}.
\]

As a result, \((\rho_{TA}(U_1), \ldots, \rho_{TA}(U_k))\) defines an element in \( \prod_{TPM}^k T(TM) \). Using the natural identification \( \prod_{TPM}^k T(TM) = T(\oplus^k TM) \), we obtain a vector bundle map \( \rho_k : \bigoplus_A^k TA \to T(\oplus^k TM) \),

\[
(3.8) \quad \rho_k(U_1 \oplus \ldots \oplus U_k) := \rho_{TA}(U_1) \oplus \ldots \oplus \rho_{TA}(U_k).
\]
Writing $\bigoplus^k TM$ in local coordinates $(x^j, \dot{x}^j_1, \ldots, \dot{x}^j_k)$, we have the following explicit formulas:

\begin{align}
\rho_k(\hat{e}_{a,n}) &= p^j_a \frac{\partial}{\partial x^j_n}, \\
\rho_k((Te_a)^k) &= \rho^j_a \frac{\partial}{\partial x^j} + \sum_{n=1}^k W^j_{a,n} \frac{\partial}{\partial \dot{x}^j_n},
\end{align}

where $W^j_{a,n} = \dot{x}^j_n \frac{\partial \rho^j_a}{\partial x^j_n} \in C^\infty(\bigoplus^k TM)$.

The second equation in (3.7) implies that if $U_1 \oplus \ldots \oplus U_k$ and $V_1 \oplus \ldots \oplus V_k$ are local sections of $\bigoplus_A^k TA \to \bigoplus^k TM$ of type (3.2) or (3.3), then

\begin{align}
[U_1 \oplus \ldots \oplus U_k, V_1 \oplus \ldots \oplus V_k] &= [U_1, V_1]_{TA} \oplus \ldots \oplus [U_k, V_k]_{TA}
\end{align}
is a well-defined local section of $\bigoplus^k_A TA \to \bigoplus^k TM$. Explicitly, we have:

\begin{align}
[c_{a,n}, c_{b,m}] &= 0, \\
[(Te_a)^k, c_{b,m}] &= C^d_{ab} c_{d,m}
\end{align}

\begin{align}
[(Te_a)^k, (Te_b)^k] &= C^d_{ab} (Te_d)^k + \sum_{n=1}^k \dot{x}^j_n \frac{\partial C^d_{ab}}{\partial x^j} c_{d,n}.
\end{align}

The induced Lie algebroid structure on $\bigoplus^k_A TA \to \bigoplus^k TM$ is defined by $\rho_k$ and the extension of $[,]_k$ to all sections via the Leibniz rule$^3$.

3.3. IM-forms. Let $\Lambda \in \Lambda^k(A)$ be a linear $k$-form on a Lie algebroid $A \to M$, $k \geq 1$. Following Prop. 2.5, let $\mu : A \to \Lambda^{k-1}T^*M$ and $\nu : A \to \Lambda^k T^*M$ be the vector-bundle maps such that $\Lambda = d\Lambda_{\mu} + \Lambda_{\nu}$. Let us consider the bundle map

\begin{align}
\bigoplus^k A T A &\xrightarrow{\pi} \mathbb{R} \\
\bigoplus^k T M &\to \{\ast\}.
\end{align}

The following is the main result of this section.

**Theorem 3.1.** The map (3.15) is a Lie algebroid morphism if and only if the following holds for all $u, v \in \Gamma(A)$:

\begin{align}
i_{\rho(u)} \mu(v) &= -i_{\rho(v)} \mu(u) \\
\mu([u, v]) &= L_{\rho(u)} \mu(v) - i_{\rho(v)} d\mu(u) - i_{\rho(v)} \nu(u) \\
\nu([u, v]) &= L_{\rho(u)} \nu(v) - i_{\rho(v)} d\nu(u).
\end{align}

For a Lie algebroid $A \to M$ and vector-bundle maps

\begin{align}
\mu : A \to \Lambda^{k-1}T^*M, & \quad \nu : A \to \Lambda^k T^*M, \quad k \geq 1,
\end{align}

we say that the pair $(\mu, \nu)$ is an **IM** $k$-**form** on $A$ if conditions (3.16), (3.17) and (3.18) are satisfied. The terminology IM stands for *infinitesimally multiplicative*, and it will be clarified in Section 4. The space of IM $k$-forms on $A$ is denoted by $\Omega_{IM}(A)$.

$^3$We adopt the simplified notation $\rho_k$, $[,]_k$, instead of $\rho_{\bigoplus^k A T A}$ and $[,]_{\bigoplus^k A T A}$; in particular, $\rho_1 = \rho_{TA}$ and $[,]_1 = [,]_{TA}$.
We note that Theorem 3.1 can be alternatively phrased in terms of the map \( \Lambda^2 \) (2.5), as this map is a Lie algebroid morphism if and only if so is \( \overline{\Lambda} \).

**Remark 3.2.** Given an IM-form \((\mu, \nu)\), it follows from (3.17), using the skew-symmetry and Jacobi identity for the Lie algebroid bracket \([,]\) on \(\Gamma(A)\), that \(\nu\) automatically satisfies

\[
(3.19) \quad i_{\rho(u)}\nu(v) = -i_{\rho(v)}\nu(u),
\]

\[
(3.20) \quad i_{\rho(u)}(\mathcal{L}_{\rho(v)}\nu(u) - \mathcal{L}_{\rho(u)}\nu(v)) + \text{c.p.} = 0
\]

for all \(u, v, w \in \Gamma(A)\), where c.p. stands for cyclic permutations in \(u, v, w\).

**Example 3.3.** Consider a Lie algebroid \(A \to M\) and a \(k\)-form \(\eta \in \Omega^k(M)\). Then the pair \((\mu, \nu)\) of vector-bundle maps

\[
\mu : A \to \wedge^{k-1}T^*M, \quad \mu(u) = -i_{\rho(u)}\eta, \quad \text{and} \quad \nu : A \to \wedge^kT^*M, \quad \nu(u) = -i_{\rho(u)}d\eta,
\]

defines an IM \(k\)-form on \(A\).

**Example 3.4.** Let \(A \to M\) be a Lie algebroid, and let \(\phi \in \Omega^{k+1}(M)\) be such that \(i_{\rho(u)}d\phi = 0, \forall u \in \Gamma(A)\). One directly checks that the vector-bundle map \(\nu : A \to \wedge^kT^*M\) given by

\[
(3.21) \quad \nu(u) := -i_{\rho(u)}\phi
\]

verifies (3.18). The particular IM \(k\)-forms \((\mu, \nu)\) on \(A\) for which \(\nu\) is given as in (3.21) for a closed form \(\phi \in \Omega^{k+1}(M)\) are called IM \(k\)-forms relative to \(\phi\). These special types of IM forms have first appeared in [5] (for \(k = 2\)), and more recently in [1] (for arbitrary \(k\)), in the study of multiplicative forms (see Section 4).

**Remark 3.5.** Let \(C : C \hookrightarrow M\) be an orbit of the Lie algebroid \(A \to M\), i.e., an integral leaf of the distribution \(\rho(A) \subset TM\). If \((\mu, \nu)\) is an IM \(k\)-form on \(A\), then we have induced forms \(\mu_C \in \Omega^k(C)\) and \(\nu_C \in \Omega^{k+1}(C)\) defined by

\[
i_{\rho(u)}\mu_C = i^*_C\mu(u), \quad i_{\rho(u)}\nu_C = i^*_C\nu(u).
\]

It follows from (3.16) and (3.19) that the formulas above do define differential forms on \(C\); moreover, (3.17) implies that \(d\mu_C = \nu_C\). In particular, we see that any IM \(k\)-form on a transitive Lie algebroid is like the one in Example 3.3.

In order to prove Thm. 3.1, we need some lemmas. We work in local coordinates \((x^j, u^a)\) on \(A\), induced by coordinates \((x^j)\) on \(M\) and the choice of a basis of local sections \(\{e_a\}\) of \(A\) (see Section 3.1).

**Lemma 3.6.** Let \(\dot{x} = (\dot{x}_1, \ldots, \dot{x}_k) \in \oplus^kTM\), where \(\dot{x}_l = (\dot{x}^j, \dot{x}^j_l)\) belongs to the \(l\)-th copy of \(TM\). Then:

\[
(3.22) \quad \overline{\Lambda}(\overline{e}_{a,n}(\dot{x}_1, \ldots, \dot{x}_k)) = (-1)^{n-1}I^x_{k,n+1}I^x_{n-1,1}\mu(e_a),
\]

\[
(3.23) \quad \overline{\Lambda}(T\overline{e}_a)^k(\dot{x}_1, \ldots, \dot{x}_k)) = I^x_{k,1}(d\mu(e_a) + \nu(e_a)),
\]

seen as functions in \(C^\infty(\oplus^kTM)\) (see (1.1) for notation).
Proof. Writing \( \Lambda = d\Lambda_\mu + \Lambda_\nu \) and recalling the local expressions of \( \Lambda_\mu \) and \( \Lambda_\nu \) (see (2.10)), we have

\[
\Lambda_{(x^j,u^d)} = \frac{1}{(k-1)!} u^d d\mu_{i_1 \ldots i_{k-1},d}(x) \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_{k-1}} + \frac{1}{k} \nu_{i_1 \ldots i_{k},d}(x) u^d dx^{i_1} \wedge \ldots \wedge dx^{i_k}.
\]

We write points in \( TA \) with coordinates \((x^j,u^d,\dot{x}^j,\dot{u}^d)\) in terms of horizontal tangent vectors \( \frac{\partial}{\partial x^j} \) and vertical tangent vectors \( \frac{\partial}{\partial u^d} \) as

\[
\dot{x}^j \frac{\partial}{\partial x^j} + \dot{u}^d \frac{\partial}{\partial u^d} |_{(x^j,u^d)}.
\]

In particular, recalling the local sections \( \hat{e}_a,\hat{0} \) and \( Te_a \) of \( TA \rightarrow TM \) from Section 3.1, we have

\[
\hat{0}(\dot{x}) = \dot{x}^j \frac{\partial}{\partial x^j} |_{(x^j,0)}, \quad \hat{e}_a(\dot{x}) = \dot{x}^j \frac{\partial}{\partial x^j} + \frac{\partial}{\partial u^a} |_{(x^j,0)}, \quad Te_a(\dot{x}) = \dot{x}^j \frac{\partial}{\partial x^j} |_{(x^j,\delta^j)},
\]

where \( \dot{x} = (x^j,\dot{x}^j) \in TM \). Using (3.2) and (3.3), formulas (3.22) and (3.23) follow from a direct calculation. \( \square \)

Let \((x^j,\dot{x}^j_1,\ldots,\dot{x}^j_k)\) be local coordinates on \( \oplus^k TM \), and fix \( n \in \{1,\ldots,k\} \).

Lemma 3.7. Let \( \alpha \in \Omega^l(\oplus^k TM) \) be such that \( \mathcal{L}_{\omega^j} \alpha = 0 \ \forall j \), and consider on \( \oplus^k TM \) the local vector fields \( \dot{x}_n = \dot{x}^j_n \frac{\partial}{\partial x^j} \), \( V^v = v^j(x) \frac{\partial}{\partial u^a} \), and \( V^k = v^j(x) \frac{\partial}{\partial x^j} \). Then \( \mathcal{L}_{V^v} i_{\dot{x}_n} \alpha = i_{V^k} \alpha \).

Proof. The proof follows from the identity \( i_{[X,Y]} = \mathcal{L}_X i_Y - i_Y \mathcal{L}_X \) and the fact that \( [v^j(x) \frac{\partial}{\partial x^j}, \dot{x}^j_n \frac{\partial}{\partial x^j}] = v^j \frac{\partial}{\partial x^j} - \dot{x}^j_n \frac{\partial}{\partial x^j} \frac{\partial v^j}{\partial x^j} \). \( \square \)

We now proceed to the proof of the main result.

Proof. (of Theorem 3.1)

To show that the map \( \overline{\Lambda} \) in (3.15) is a Lie algebroid morphism (see e.g. [21]), the only condition to be verified is

\[
\overline{\Lambda}([U,V]_k) = \mathcal{L}_{\rho_k(U)} \overline{\Lambda}(V) - \mathcal{L}_{\rho_k(V)} \overline{\Lambda}(U)
\]

for all \( U,V \) sections of \( \oplus^k TA \rightarrow \oplus^k TM \). Since sections of type \( \hat{e}_{a,n} \) (core) and \( (Te_a)^k \) (linear) locally generate the space of sections of \( \oplus^k TA \rightarrow \oplus^k TM \), it suffices to verify (3.25) taking \( U \) and \( V \) to be of these types.

Core-Core: Let us consider two core sections \( \hat{e}_{a,n} \) and \( \hat{e}_{b,m} \). Since \( [\hat{e}_{a,n}, \hat{e}_{b,m}]_k = 0 \) (3.12), condition (3.25) in this case becomes

\[
\mathcal{L}_{\rho_k(\hat{e}_{a,n})} \overline{\Lambda}(\hat{e}_{b,m}) - \mathcal{L}_{\rho_k(\hat{e}_{b,m})} \overline{\Lambda}(\hat{e}_{a,n}) = 0.
\]

Using (3.9) and (3.22), we see that

\[
\mathcal{L}_{\rho_k(\hat{e}_{a,n})} \overline{\Lambda}(\hat{e}_{b,m}) = (-1)^{n-1} \mathcal{L}_{\rho_k(\hat{e}_{a,n})} \overline{\Lambda}(\hat{e}_{b,m}) = \mu_\gamma(\hat{e}_{a,n}) \rho_k(\hat{e}_{b,m}).
\]

\( \square \)
This condition is trivially satisfied when \( n = m \), so we may assume that \( n > m \) (the case \( n < m \) leads to the same). Using Lemma 3.7, we see that the right-hand side of the last equation agrees with
\[
(-1)^{n-1} I_{k,n+1}^x \frac{\partial}{\partial x} \mu(e_b) = (-1)^{n-1} \frac{\partial}{\partial x} \mu(e_b).
\]
Hence we obtain
\[
\mathcal{L}_{\rho_k(\bar{e}_{a,n})} \bar{\Lambda}(\bar{e}_{b,m}) = -I_{k,n+1}^x \frac{\partial}{\partial x} \mu(e_b).
\]
An analogous computation leads to
\[
\mathcal{L}_{\rho_k(\bar{e}_{b,m})} \bar{\Lambda}(\bar{e}_{a,n}) = I_{k,n+1}^x \frac{\partial}{\partial x} \mu(e_b).
\]
It follows that (3.26) is equivalent to
\[
i_\rho(e_a) \mu(e_b) = -i_\rho(e_b) \mu(e_a).
\]

**Core-Linear:** We now consider sections \( \bar{e}_{b,m} \) and \( (Te_a)^k \), so that (3.25) reads
\[
(3.27)
\]
\[
\bar{\Lambda}([Te_a]^k, \bar{e}_{b,m}) = \mathcal{L}_{\rho_k((Te_a)^k)} \bar{\Lambda}(\bar{e}_{b,m}) - \mathcal{L}_{\rho_k(\bar{e}_{b,m})} \bar{\Lambda}([Te_a]^k).
\]
Using the linearity of \( \Lambda \), (3.13) and (3.22), we have
\[
(3.28)
\]
\[
\bar{\Lambda}([Te_a]^k, \bar{e}_{b,m}) = \bar{\Lambda}(C_{ab} e_{b,m}) = C_{ab}(-1)^{m-1} I_{k,m+1}^x \frac{\partial}{\partial x} \mu(e_a) = (-1)^{m-1} I_{k,m+1}^x \frac{\partial}{\partial x} \mu(e_a) + I_{m-1} \frac{\partial}{\partial x} \mu(e_b).
\]
For each fixed \( n \), consider the functions \( W_{2,n}^j = \frac{\partial}{\partial x} \bar{x}^j_n \) defined in (3.10), noticing the following identity (of local vector fields on \( \oplus^k TM \)):
\[
(3.29)
\]
\[
W_{2,n}^j \frac{\partial}{\partial x} = -[\rho(e_a), \bar{x}^j_n],
\]
where \( \bar{x}_n = \bar{x}^j_n \frac{\partial}{\partial x} \). Using (3.29) and Lemma 3.7, we see that
\[
\mathcal{L}_{\rho_k((Te_a)^k)} \bar{\Lambda}(\bar{e}_{b,m}) = \left( \mathcal{L}_{\rho(e_a)} + \sum_{l=1}^k \mathcal{L}_{W_{2,l}^j} \right) (-1)^{m-1} I_{k,m+1}^x \frac{\partial}{\partial x} \mu(e_b) - \sum_{l=1}^k I_{l+1}^x [\rho(e_a), U_l] \frac{\partial}{\partial x} \mu(e_b)
\]
where \( U = (U_1, \ldots, U_{k-1}) = (\bar{x}_1, \ldots, \bar{x}_{m-1}, \bar{x}_{m+1}, \ldots, \bar{x}_k) \). It follows from (1.3) that
\[
(3.30)
\]
\[
\mathcal{L}_{\rho_k((Te_a)^k)} \bar{\Lambda}(\bar{e}_{b,m}) = (-1)^{m-1} I_{k,m+1}^x \frac{\partial}{\partial x} \mu(e_b).
\]
Using (3.23) and Lemma 3.7, we obtain
\[
\mathcal{L}_{\rho_k(\bar{e}_{b,m})} \bar{\Lambda}((Te_a)^k) = \mathcal{L}_{\rho_k(\bar{e}_{b,m})} \left( I_{k,1}^x (\text{d}\mu(e_a) + \nu(e_a)) \right) = I_{k,m+1}^x \frac{\partial}{\partial x} \mu(e_a) = (-1)^{m-1} I_{k,m+1}^x I_{m-1}^y \frac{\partial}{\partial x} \mu(e_a) + \nu(e_a)).
\]
Combining this last equation with (3.28) and (3.30), we see that (3.27) is equivalent to
\[
(3.31)
\]
\[
\mu([e_a, e_b]) = \mathcal{L}_{\rho(e_a)} \mu(e_b) - i_\rho(e_a) \text{d}\mu(e_a) - i_\rho(e_b) \nu(e_a).
\]
Linear-Linear: We finally consider condition (3.25) for two linear sections:

(3.32) $\overline{\mathcal{L}}((Te_a)^k, (Te_b)^k) = \mathcal{L}_{\rho_a((Te_a)^k)}(Te_b)^k - \mathcal{L}_{\rho_b((Te_b)^k)}(Te_a)^k$.

Using (3.14) and the linearity of $\Lambda$, we have

$\overline{\mathcal{L}}((Te_a)^k, (Te_b)^k) = C_{ab}^d \overline{\mathcal{L}}((Te_d)^k) + \sum_{n=1}^{k} \lambda_{x_n} dC_{ab}(\hat{x}_n) \overline{\mathcal{L}}((Te_d)^k)$

(3.33)

It follows from (1.4) (also using that $I_{k,1}^x(\mu) = 0$, since $\mu(e_d)$ is a $(k-1)$-form) that

$I_{k,1}^x C_{ab}^d \mu(e_d) = I_{k,1}^x (d(\mu(C_{ab}\nu)) - I_{k,1}^x (dC_{ab}^d \wedge \mu(e_d))$

Comparing with (3.33), we conclude that

(3.34) $\overline{\mathcal{L}}((Te_a)^k, (Te_b)^k) = I_{k,1}^x (d(\mu(C_{ab}\nu)) + \nu(C_{ab}\nu))$

Using Lemma 3.7, (3.29) and (1.3), we directly obtain

(3.35) $\mathcal{L}_{\rho(Te_a)^k} \overline{\mathcal{L}}((Te_b)^k) = \left( \mathcal{L}_{\rho(e_a)} + \sum_{n=1}^{k} \mathcal{L}_{W_{a,n}} \frac{a}{a} \right) I_{k,1}^x (d(\mu(e_b) + \nu(e_b))$

Similarly

(3.36) $\mathcal{L}_{\rho((Te_b)^k)} \overline{\mathcal{L}}((Te_a)^k) = I_{k,1}^x \mathcal{L}_{\rho(e_a)}(d(\mu(e_a) + \nu(e_a))$

Combining (3.34), (3.35) and (3.36), we see that (3.32) is equivalent to

$d(\mu([e_a, e_b]) + \nu([e_a, e_b])) = \mathcal{L}_{\rho(e_a)}(d(\mu(e_b) + \nu(e_b)) - \mathcal{L}_{\rho(e_b)}(d(\mu(e_a) + \nu(e_a))$

We may assume that (3.31) holds, in which case one can directly check that the last equation is equivalent to

$\nu([e_a, e_b]) = \mathcal{L}_{\rho(e_a)}\nu(e_b) - i_{\rho(e_b)} d\nu(e_a)$.

4. Infinitesimal description of multiplicative forms

In this section, we relate IM-forms on Lie algebroids with multiplicative forms on Lie groupoids. Let $\mathcal{G}$ be a Lie groupoid over $M$, with source and target maps denoted by $s, t : \mathcal{G} \to M$, respectively; multiplication $m : \mathcal{G}^{(2)} \to \mathcal{G}$, and unit map $e : M \to \mathcal{G}$ (that we often use to view $M$ as a submanifold of $\mathcal{G}$). The Lie algebroid of $\mathcal{G}$ is denoted by $A(\mathcal{G})$, or simply $A$ if there is no risk of confusion; see Section 1.

A $k$-form $\alpha \in \Omega^k(\mathcal{G})$ is called multiplicative if

(4.1) $m^* \alpha = pr_1^* \alpha + pr_2^* \alpha$, 
where \( pr_1, pr_2 : G^{(2)} \to G \) are the natural projections. Alternatively, one may define multiplicative forms in terms of a natural groupoid structure on \( T G \) over \( T M \), known as the tangent groupoid, see e.g. [21]; it has source (resp. target) map \( Ts : T G \to T M \) (resp. \( T t : T G \to T M \)), multiplication \( Tm : (T G)^{(2)} = T G(2) \to T G \), and unit map \( T \epsilon : T M \to T G \). This groupoid structure can be naturally extended to the direct sum \( \bigoplus_k T G, k \geq 1 \), making it a Lie groupoid over \( \bigoplus^k T M \), with source (resp. target) map \( \bigoplus^k Ts \) (resp. \( \bigoplus^k T t \)), multiplication map \( \bigoplus^k Tm \), etc.

Let \( \alpha \in \Omega^k(G) \), and let us consider the associated map

\[
\alpha : \bigoplus^k_{\alpha} T G \to \mathbb{R}, \quad \alpha(U_1, \ldots, U_k) = i_{U_k} \cdots i_{U_1} \alpha.
\]

The following observation is immediate from (4.1).

**Lemma 4.1.** \( \alpha \) is multiplicative if and only if \( \alpha \) is a groupoid morphism. (Here \( \mathbb{R} \) is viewed as an additive group.)

We denote the space of multiplicative \( k \)-forms on \( G \) by \( \Omega^k_{\text{mult}}(G) \).

**4.1. From multiplicative to IM forms.** Let \( G \) be a Lie groupoid over \( M \), and consider the tangent lift operation \( \Omega^k(G) \to \Omega^k(T G) \), \( \alpha \mapsto \alpha_T \), recalled in Section 2.3. Using the natural inclusion \( t_A : A = AG \to T G \), we define a map

\[
\text{Lie} : \Omega^k(G) \to \Omega^k(A), \quad \alpha \mapsto \text{Lie}(\alpha) = \iota_A^* \alpha_T.
\]

Given \( \alpha \in \Omega^k(G) \), let us consider the associated bundle maps \( \mu : A \to \wedge^{k-1} T^*M \) and \( \nu : A \to \wedge^k T^*M \),

\[
\langle \mu(u), X_1 \land \ldots \land X_{k-1} \rangle = \alpha(u, X_1, \ldots, X_{k-1}),
\]

\[
\langle \nu(u), X_1 \land \ldots \land X_k \rangle = d\alpha(u, X_1, \ldots, X_k),
\]

for \( X_1, \ldots, X_k \in T M \) and \( u \in A \) (here we use the natural inclusions \( T M \to T G|_M \) and \( A \to T G|_M \)).

**Lemma 4.2.** The \( k \)-form \( \text{Lie}(\alpha) \in \Omega^k(A) \) is linear and satisfies

\[
\text{Lie}(\alpha) = d\mu + \Lambda_{\nu}.
\]

**Proof.** Let \( \beta \in \Omega^l(G) \) be any \( l \)-form on \( G \), and let us consider the \( l - 1 \)-form on \( A \) given by \( \iota_A^* \tau(\beta) \) (see (2.13)), i.e.,

\[
\iota_A^* \tau(\beta)|_u = (T\iota_A|_u)^T \tau(\beta)|_{\iota_A(u)} = (T\iota_A|_u)^T(T p_G|_{\iota_A(u)})^T \iota_{\iota_A(u)} \beta
\]

\[
= (T(p_G \circ \iota_A)|_u)^T \iota_{\iota_A(u)} \beta.
\]

From the commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\iota_A} & T G \\
\downarrow{q_A} & & \downarrow{p_G} \\
M & \xrightarrow{\epsilon} & G,
\end{array}
\]

we see that

\[
\iota_A^* \tau(\beta)|_u = (T q_A|_u)^T(T \epsilon|_{q_A(u)})^T \iota_{\iota_A(u)} \beta.
\]

It immediately follows (see (2.9)) that

\[
\iota_A^* \tau(\alpha) = \Lambda_{\mu}, \quad \iota_A^* \tau(d\alpha) = \Lambda_{\nu}.
\]
Using (2.15), we see that
\[
\Lambda = \iota_A^* (d\tau(\alpha) + \tau(d\alpha)) = d\iota_A^* \tau(\alpha) + \iota_A^* \tau(d\alpha) = d\Lambda_\mu + \Lambda_\nu.
\]

\[\square\]

Recall that any groupoid morphism \(\psi : G_1 \to G_2\) defines a Lie algebroid morphism \(\text{Lie}(\psi) : A G_1 \to A G_2\) that fits into the diagram

\[
\begin{array}{ccc}
TG_1 & \xrightarrow{T\psi} & TG_2 \\
\uparrow_{\iota_{A_1}} & & \uparrow_{\iota_{A_2}} \\
AG_1 & \xrightarrow{\text{Lie}(\psi)} & AG_2
\end{array}
\]

When \(\alpha \in \Omega^k(G)\) is multiplicative, we saw in Lemma 4.1 that \(\bar{\alpha} : \oplus^k_G T G \to \mathbb{R}\) is a groupoid morphism; we consider its infinitesimal counterpart,

\[
\text{Lie}(\bar{\alpha}) : A(\oplus^k_G T G) \to \mathbb{R},
\]

where now \(\mathbb{R}\) is viewed as the trivial Lie algebroid over a point. The natural projection \(p_G : TG \to G\) is a groupoid morphism, and there is a canonical identification of Lie algebroids\(^4\)

\[
A(\oplus^k_G T G) = \prod_{\text{Lie}(p_G)} A(TG).
\]

Our next goal is to compare the following two maps:

\[
\overline{\text{Lie}}(\alpha) : \oplus^k_A T(A G) \to \mathbb{R} \quad \text{and} \quad \text{Lie}(\bar{\alpha}) : \prod_{\text{Lie}(p_G)} A(TG) \to \mathbb{R}.
\]

The involution \(J_G : T(TG) \to T(TG)\) (see (2.16)) defines an identification of Lie algebroids \(j_G : T(A G) \to A(TG)\) via the diagram

\[
\begin{array}{ccc}
T(AG) & \xrightarrow{j_G} & ATG \\
\downarrow_{T_A G} & & \downarrow_{\iota_{A(TG)}} \\
T(TG) & \xrightarrow{j_G} & T(TG)
\end{array}
\]

Note that the property \(Tp_G \circ J_G = p_{TG}\) implies that

\[
\text{Lie}(p_G) \circ j_G = p_A.
\]

As a result, we have a natural identification of Lie algebroids,

\[
j_G^{(k)} : \oplus^k_A TA \xrightarrow{\sim} \prod_{p_A} T(AG) \xrightarrow{\sim} \prod_{\text{Lie}(p_G)} A(TG),
\]

\(^4\)The Lie algebroid \(A(TG) \to TM\) is a VB-algebroid [15, Sec. 2.1] with respect to the vector bundle structure \(\text{Lie}(p_G) : A(TG) \to A\); the algebroid structure on \(A(TG)\) can be extended to \(\prod_{\text{Lie}(p_G)} A(TG)\) in terms of core and linear sections, just as described in Section 3.2.
fitting into the diagram

\[
\begin{array}{cccc}
\prod_{PA}^k T(AG) & \xrightarrow{j_G^{(k)}} & \prod_{\text{Lie}(p_G)}^k A(TG) \\
(T_{\text{Lie}(AG)})^k & & (\iota_{A(TG)})^k \\
\prod_{PG}^k T(TG) & \xrightarrow{j_G^{(k)}} & \prod_{T_{PG}}^k T(TG).
\end{array}
\]

**Lemma 4.3.** Let \( \alpha \in \Omega^k(G) \) be multiplicative. Then

\[
\text{Lie}(\bar{\alpha}) = \bar{\text{Lie}(\alpha)}.
\]

In particular, \( \text{Lie}(\alpha) : \oplus_{A}^k T(AG) -\rightarrow \mathbb{R} \) is a Lie algebroid morphism.

**Proof.** By definition, \( \text{Lie}(\bar{\alpha}) = d\bar{\alpha} \circ (\iota_{A(TG)})^k \), and using (4.9) and (2.17) we obtain

\[
\text{Lie}(\bar{\alpha}) \circ j_G^{(k)} = d\bar{\alpha} \circ (\iota_{A(TG)})^k \circ j_G^{(k)} = d\bar{\alpha} \circ J_G^{(k)} \circ (T_{\text{Lie}(AG)})^k = \bar{\alpha}_T \circ (T_{\text{Lie}(AG)})^k = \bar{\alpha}_T.
\]

\( \square \)

**Proposition 4.4.** Let \( \alpha \in \Omega^k(G) \) be multiplicative, and let \( \mu \) and \( \nu \) be defined as in (4.4) and (4.5). Then \( (\mu, \nu) \) is an IM \( k \)-form on \( AG \).

**Proof.** The result is a direct consequence of Lemmas 4.2, 4.3, and Theorem 3.1. \( \square \)

4.2. **Integration of IM forms.** Let \( G \) be a Lie groupoid over \( M \), with Lie algebroid \( A = AG \). Assume that \( G \) is source-simply-connected (i.e., the \( s \)-fibres are connected with trivial fundamental group), so that \( \oplus_{G}^k TG \) is also a source-simply-connected groupoid\(^5\). Let \( \Lambda \in \Omega^k(A) \) be a \( k \)-form on \( A \) for which \( \text{Lie}(\alpha) : \oplus_{A}^k TA -\rightarrow \mathbb{R} \) is a Lie algebroid morphism.

**Lemma 4.5.** There is a unique multiplicative \( k \)-form \( \alpha \in \Omega^k(G) \) such that \( \text{Lie}(\alpha) = \Lambda \) (see (4.3)).

**Proof.** Since \( \bar{\Lambda} \) is a morphism of Lie algebroids, the identification (4.8) also leads to a Lie algebroid morphism

\[
\bar{\Lambda} \circ j_{G}^{(k)} = \prod_{\text{Lie}(p_G)}^k A(TG) \cong A(\oplus_{A}^k TG) -\rightarrow \mathbb{R}.
\]

As \( \oplus_{G}^k TG \) is a source-simply-connected groupoid, we can use Lie’s second theorem (see e.g. [21]) to obtain a unique groupoid morphism

\[
I_{\Lambda} : \oplus_{G}^k TG -\rightarrow \mathbb{R}
\]

\( \bar{\Lambda} \circ (j_{G}^{(k)})^{-1} = \prod_{\text{Lie}(p_G)}^k A(TG) \cong A(\oplus_{A}^k TG) -\rightarrow \mathbb{R}.
\]

Given any \( X = (X_1, \ldots, X_k) \in \oplus_{A}^k T_s M \), the projection \( (p_G)^k : \oplus_{G}^k TG -\rightarrow G \) makes the source fibre \( ((T_s)^k)^{-1}(X) \subseteq TG \) into an affine bundle over the source fibre \( s^{-1}(x) \subseteq G \).
integrating the morphism (4.11), i.e., such that \( \text{Lie}(I_A) = \overline{\Lambda} \circ (j_G^k)^{-1} \). To check that \( I_A = \overline{\alpha} \), for \( \alpha \in \Omega^k(G) \), it suffices to verify that the following conditions hold:

\[
(4.13) \quad I_A(U_1, \ldots, U_i, \ldots, U_j, \ldots, U_k) = -I_A(U_1, \ldots, U_j, \ldots, U_i, \ldots, U_k),
\]
\[
(4.14) \quad I_A(U_1, \ldots, U_{i-1}, cU_i, U_{i+1}, \ldots, U_k) = cI_A(U_1, \ldots, U_k),
\]
\[
(4.15) \quad I_A(U_1, \ldots, U_{i-1}, U_i + U_i', U_{i+1}, \ldots, U_k) = I_A(U_1, \ldots, U_i, \ldots, U_k) + I_A(U_1, \ldots, U_i', \ldots, U_k),
\]

for all \( U_i, U_i' \in T_gG, \ g \in G, \ c \in \mathbb{R}, \) where \( 1 \leq i < j \leq k \). As we now show, all conditions can be verified with the same type arguments (cf. [23]).

To prove that (4.13) holds, one directly checks that the map \( I_A^{(ij)} : \oplus_G^kTG \to \mathbb{R} \),

\[
I_A^{(ij)}(U_1, \ldots, U_i, \ldots, U_j, \ldots, U_k) := -I_A(U_1, \ldots, U_j, \ldots, U_i, \ldots, U_k),
\]

is a groupoid morphism, and \( \text{Lie}(I_A^{(ij)}) : \prod_{\text{Lie}(pg)}^k A(TG) \to \mathbb{R} \) satisfies

\[
\text{Lie}(I_A^{(ij)})(V_1, \ldots, V_i, \ldots, V_j, \ldots, V_k) = -\text{Lie}(I_A)(V_1, \ldots, V_j, \ldots, V_i, \ldots, V_k)
= -\Lambda \circ (j_G^k)^{-1}(V_1, \ldots, V_j, \ldots, V_i, \ldots, V_k)
= \text{Lie}(I_A)(V_i, \ldots, V_j, \ldots, V_i, \ldots, V_k),
\]

since \( \Lambda \) is skew-symmetric. So \( \text{Lie}(I_A^{(ij)}) = \text{Lie}(I_A) \), and the uniqueness of integration in Lie’s second theorem implies that \( I_A^{(ij)} = I_A \), which is (4.13).

Similarly, for a fixed \( c \in \mathbb{R} \), one can directly show that both the left and right-hand sides of (4.14) define groupoid morphisms \( \oplus_G^k TG \to \mathbb{R} \), whose infinitesimal counterparts agree at the level of Lie algebroids due to the multilinearity of \( \Lambda \). Then (4.14) follows again by the uniqueness part of Lie’s second theorem.

The last condition (4.15) can be treated in a completely analogous way, by first noticing that both sides of (4.15) define groupoid morphisms \( \oplus_G^{k+1} TG \to \mathbb{R} \), where now we need an extra copy of \( TG \) for \( U_i' \). Again, these morphisms agree at the infinitesimal level due to the multilinearity of \( \Lambda \), and hence agree globally.

The fact that \( \alpha \) is multiplicative follows from Lemma 4.1, and the equality \( \Lambda = \text{Lie}(\alpha) \) is a consequence of Lemma 4.3.

A direct consequence of Lemmas 4.3 and 4.5 is that the map

\[
\Omega_{\text{mult}}^k(G) \to \Omega^k(A), \quad \alpha \mapsto \text{Lie}(\alpha),
\]

is a bijection onto the subspace of \( k \)-forms \( \Lambda \in \Omega^k(A) \) such that \( \overline{\Lambda} : \oplus_A^k TA \to \mathbb{R} \) is a morphism of Lie algebroids. By the correspondence in Theorem 3.1, this bijection can be alternatively phrased in terms of IM-forms on \( A \):

**Theorem 4.6.** Let \( G \) be a source-simply-connected Lie groupoid over \( M \) with Lie algebroid \( A \to M \). For each positive integer \( k \), there is a 1-1 correspondence

\[
(4.16) \quad \Omega_{\text{mult}}^k(G) \to \Omega_{\text{IM}}^k(A), \quad \alpha \mapsto (\mu, \nu),
\]

where \( \mu, \nu \) are given by

\[
(4.17) \quad \langle \mu(u), X_1 \wedge \ldots \wedge X_{k-1} \rangle = \alpha(u, X_1, \ldots, X_{k-1}),
\]
\[
(4.18) \quad \langle \nu(u), X_1 \wedge \ldots \wedge X_k \rangle = d\alpha(u, X_1, \ldots, X_k).
\]

**Proof.** The result follows from Lemma 4.2 and Theorem 3.1. \( \square \)
The following is a simple example of correspondence in Theorem 4.6.

**Example 4.7.** Let us equip $A = T^*M \to M$ with the trivial Lie algebroid structure (both anchor and bracket are identically zero), so we may identify $\mathcal{G} = T^*M$ (with groupoid multiplication given by fibrewise addition). Fixing $\mu = \text{Id} : T^*M \to T^*M$, then any vector-bundle map $\nu : T^*M \to \wedge^2 T^*M$ defines an IM 2-form $(\mu, \nu)$. When $\nu = 0$, then $(\mu, \nu)$ corresponds under (4.16) to the canonical symplectic form $\omega_{\text{can}}$ on $\mathcal{G} = T^*M$; for an arbitrary $\nu$, the corresponding multiplicative 2-form is given, at each $g = (q^1, p_j) \in T^*M$, by

$$\omega|_g = \omega_{\text{can}}|_g + c^*_M \nu(g)$$

where $c_M : T^*M \to M$ is the natural projection.

Let us list some immediate consequences of Theorem 4.6, illustrating how the correspondence (4.16) restricts to subclasses of multiplicative and IM forms:

(a) Let $\eta \in \Omega^k(M)$. Following Example 3.3, we know that $(\mu, \nu)$, where $\mu(u) = -i_{\rho(u)} \eta$ and $\nu(u) = -i_{\rho(u)} d \eta$, defines an IM $k$-form. One directly verifies that the corresponding multiplicative $k$-form is $\alpha = \ast \eta - t^* \eta$.

(b) Let $\phi \in \Omega^{k+1}(M)$ be a closed $k+1$-form. Then Theorem 4.6 gives a bijective correspondence between IM $k$-forms on $A$ relative to $\phi$ (i.e., $\nu(u) = -i_{\rho(u)} \phi$, see Example 3.4) and multiplicative $k$-forms $\alpha$ satisfying $d \alpha = \ast \phi - t^* \phi$. To verify this fact, just notice that $d \alpha$ is a multiplicative $(k+1)$-form corresponding to an IM $(k+1)$-form of the type discussed in item (a). This recovers [5, Thm. 2.5] when $k = 2$ (cf. [3]), as well as [1, Thm. 2] when $k$ is an arbitrary positive integer.

(c) Let $\alpha \in \Omega^k_{\text{mult}}(\mathcal{G})$ be a given closed multiplicative $k$-form, with associated IM $k$-form $(\mu_\alpha, \nu_\alpha)$ (note that $\nu_\alpha = 0$, necessarily). It follows from Theorem 4.6 that there is a 1-1 correspondence between multiplicative $(k+1)$-forms $\theta$ with $d \theta = \alpha$ and vector-bundle maps $\mu : A \to \wedge^{k-2} T^*M$ satisfying, for all $u, v \in \Gamma(A)$,

$$i_{\rho(u)} \mu(v) = -i_{\rho(v)} \mu(u), \quad \mu([u, v]) = L_{\rho(u)} \mu(v) - i_{\rho(v)} d \mu(u) - i_{\rho(v)} \mu_\alpha(u).$$

The reason is that $(\mu, \mu_\alpha)$ is the IM $(k+1)$-form associated with $\theta$ (note that $\mu_\alpha$ satisfies (3.18) as a result of $(\mu_\alpha, \nu_\alpha)$ being an IM $k$-form for $\alpha$ and $\nu_\alpha$ being zero). This correspondence is the content of [1, Thm. 3].

For $k = 2$, one has further refinements of Theorem 4.6 based on the general study of multiplicative 2-forms carried out in [5, Section 4], leading to natural generalizations of twisted Poisson and Dirac structures (in the sense of [28]). On the vector bundle $TM \oplus T^*M$, consider the pairing $\langle (X, \alpha), (Y, \beta) \rangle = \beta(X) + \alpha(Y)$, and the natural projections $pr_T : TM \oplus T^*M \to TM$ and $pr_{T^*} : TM \oplus T^*M \to T^*M$. As in Theorem 4.6, we denote by $\mathcal{G}$ the source-simply-connected Lie groupoid with Lie algebroid $A \to M$.

(d) Given an IM 2-form $(\mu, \nu)$ on $A$, we consider the vector-bundle map $(\rho, \mu) : A \to TM \oplus T^*M$ and its image

$$(4.19) \quad L = \{ (\rho(u), \mu(u)) \mid u \in A \} \subset TM \oplus T^*M,$$

which is a subbundle whenever it has constant rank. By (3.16), over each point of $M$, $L$ is isotropic with respect to the pairing $\langle \cdot, \cdot \rangle$. It follows from
that, under the assumption that \( \dim(G) = 2 \dim(M) \), the correspondence (4.16) restricts to a bijection between multiplicative 2-forms \( \omega \) such that

\[
\ker(T_s)_x \cap \ker(\omega)_x \cap \ker(T_t)_x = \{0\}, \quad \forall x \in M,
\]

and IM 2-forms \((\mu, \nu)\) for which \( L = L^\perp \) (i.e., \( L \) is **lagrangian** with respect to \( \langle \cdot, \cdot \rangle \); in particular, it is a subbundle with \( \text{rank}(L) = \dim(M) \)) and

\[
(\rho, \mu) : A \to L \subset TM \oplus T^*M
\]

is an isomorphism of vector bundles. Moreover, \( t : G \to M \) relates \( \omega \) and \( L \) as a **forward Dirac map** (see, e.g., [5, Sec. 2.1]). If we define \( \nu_L : L \to \wedge^2 T^*M \) by

\[
\nu_L((\rho(u), \mu(u))) = \nu(u)
\]

then the identification (4.21) induces a Lie algebroid structure on \( L \) with anchor \( pr_T|_L \) and bracket on \( \Gamma(L) \) given by

\[
[(X, \alpha), (Y, \beta)]_L := ([X,Y], \mathcal{L}_X \beta - i_Y d\alpha - i_Y \nu_L(X, \alpha)).
\]

Conversely, if a lagrangian subbundle \( L \subset TM \oplus T^*M \) is equipped with \( \nu_L : L \to \wedge^2 T^*M \) for which (4.22) is a Lie bracket on \( \Gamma(L) \), then \( A = L \) is a Lie algebroid with anchor \( pr_T|_L \); if \( \nu_L \) also satisfies (3.18), then \( (pr_T|_L, \nu_L) \) is an IM 2-form. Then, under the bijection (4.16), \( (pr_T|_L, \nu_L) \) corresponds to a multiplicative 2-form \( \omega \) on \( G \) satisfying (4.20). Taking \( \nu_L \) to be of type \( \nu_L(X, \alpha) = -i_X \phi \) for a closed 3-form \( \phi \in \Omega^3(M) \), we recover the integration of twisted Dirac structures by twisted presymplectic groupoids of [5, Sec. 2].

(e) When a multiplicative 2-form is nondegenerate, then \( \dim(G) = 2 \dim(M) \) automatically (see, e.g., [5, Lem. 3.3]), and (4.20) trivially holds. Under (4.16), this situation corresponds to the case where the IM 2-form \((\mu, \nu)\) is such that \( \mu : A \to T^*M \) is an isomorphism; in other words, \( L \) is the graph of a bivector field \( \pi \) on \( M \): \( L = \{(i_\alpha \pi, \alpha) \mid \alpha \in T^*M\} \). Following (d) above, the fact that \( \Gamma(L) \) is closed under the bracket (4.22) is expressed by the compatibility condition

\[
\frac{1}{2} [\pi, \pi] (\alpha, \beta, \gamma) = \nu_L((i_\alpha \pi, \alpha))(i_\beta \pi, i_\gamma \pi)
\]

for all \( \alpha, \beta, \gamma \in \Omega^1(M) \). When \( \nu_L \) is of type \( \nu_L(X, \alpha) = -i_X \phi \) for a closed 3-form \( \phi \in \Omega^3(M) \), one recovers twisted Poisson structures, and (4.16) gives their integration to twisted symplectic groupoids (cf. [8]).

**Remark 4.8 (Higher Dirac structures).** Just as Dirac structures are special cases of IM 2-forms, the higher Dirac structures of [33] are particular cases of higher-degree IM k-forms; so Theorem 4.6 can be used to integrate higher Dirac structures as well (c.f. [33, Prop. 3.7]).

**Remark 4.9 (Path-space construction).** We now relate Theorem 4.6 to the path-space approach to integration found in [5, 7, 8]. For an integrable Lie algebroid \( A \), there is an explicit model for its integrating source-simply-connected Lie groupoid \( \mathcal{G}(A) \) [12, 27]; namely, \( \mathcal{G}(A) \) is the quotient \( PA/\sim \), where \( PA \) is the subspace of \( A \)-paths in the Banach manifold of all \( C^1 \) paths from the interval \( I = [0,1] \) into \( A \) with \( C^2 \) projection to \( M \), and \( \sim \) is the equivalence relation defined by \( A \)-homotopies, see [12]. If \( \Lambda \in \Omega^k(A) \) is a k-form on \( A \), we use the evaluation map
ev : PA × I → A, (a(·), t) ↦ a(t) to define a k-form \( \tilde{\Lambda} \in \Omega^k(PA) \) by the formula

\[
(4.23) \quad \tilde{\Lambda} = \int_I dt \ ev^* \Lambda.
\]

One verifies that, if \( \Lambda \in \Omega^k(A) \) is a linear k-form, then the k-form \( \tilde{\Lambda} \) (4.23) is basic with respect to the quotient projection \( q : PA \to \mathcal{G}(A) \), i.e., \( \tilde{\Lambda} = q^* \alpha \) for \( \alpha \in \Omega^k(\mathcal{G}(A)) \), if and only if the induced map \( \tilde{\Lambda} \) \((5.4)\) is a Lie algebroid morphism (i.e., if \( \Lambda \) is an IM k-form). In this case, the multiplicative k-form obtained by integration of the morphism \( \Lambda \) to \( \mathcal{G}(A) \) (see Lemma 4.5) agrees with \( \alpha \), showing how the correspondence in Thm. 4.6 is viewed in the light of the path-space method (generalizing the approach in [5, Sec. 5]).

5. Relation with the Weil algebra and Van Est isomorphism

This section clarifies how linear and IM-forms on Lie algebroids fit into the Weil algebra of [1, Sec. 3], and how the infinitesimal description of multiplicative forms relates to the general Van Est isomorphism of [1, Sec. 4].

Let \( A \) be a Lie algebroid over \( M \). We consider the associated Weil algebra \( W(A) \) as in [1, Sec. 3], which is a bi-graded differential algebra. The space of elements of degree \((p,k)\) is denoted by \( W^{p,k}(A) \), and the differential on \( W(A) \) is a sum of differentials \( d^h + d^v \), where

\[
d^h : W^{p,k}(A) \to W^{p+1,k}(A), \quad d^v : W^{p,k}(A) \to W^{p,k+1}(A).
\]

We will be mostly concerned with \( W^{p,k}(A) \) for \( p = 0, 1, 2 \).

For \( p = 0 \), we have \( W^{0,k}(A) = \Omega^k(M) \). An element \( \Lambda_W \in W^{1,k}(A) \) is given by a pair \( ((\Lambda_W)_0, (\Lambda_W)_1) \), where

\[
(5.1) \quad (\Lambda_W)_0 : \Gamma(A) \to \Omega^k(M), \quad (\Lambda_W)_1 \in \Omega^{k-1}(M, A^*),
\]

subject to the compatibility condition

\[
(5.2) \quad (\Lambda_W)_0(fu) = f(\Lambda_W)_0(u) - df \wedge (\Lambda_W)_1(u),
\]

for \( f \in C^\infty(M) \), \( u \in \Gamma(A) \), and \((\Lambda_W)_1\) viewed as a \( C^\infty(M)\)-linear map

\[
(5.3) \quad (\Lambda_W)_1 : \Gamma(A) \to \Omega^{k-1}(M).
\]

An element \( c_W \in W^{2,k}(A) \) is a triple \((c_W)_0, (c_W)_1, (c_W)_2\), where

\[
(5.4) \quad (c_W)_0 : \Gamma(A) \times \Gamma(A) \to \Omega^k(M), \quad (c_W)_1 : \Gamma(A) \to \Omega^{k-1}(M, A^*), \quad (c_W)_2 \in \Omega^{k-1}(M, S^2(A^*)),
\]

and such that \((c_W)_0\) is skewsymmetric and \( \mathbb{R}\)-bilinear, subject to suitable compatibility conditions (extending (5.2)) that we will not need explicitly.

We need to recall the expression for \( d^h \) restricted to \( W^{1,k}(A) \). By definition (see [1, Sec. 3.1]), for \( \Lambda_W = ((\Lambda_W)_0, (\Lambda_W)_1) \in W^{1,k}(A) \), \( d^h \Lambda_W \in W^{2,k}(A) \) is given by (cf. (5.4))

\[
(5.5) \quad (d^h \Lambda_W)_0(u, v) = -(\Lambda_W)_0([u, v]) + \mathcal{L}_{\rho(u)}((\Lambda_W)_0(v)) - \mathcal{L}_{\rho(v)}((\Lambda_W)_0(u)),
\]

\[
(5.6) \quad (d^h \Lambda_W)_1(u)(v) = \mathcal{L}_{\rho(u)}((\Lambda_W)_1(v)) - (\Lambda_W)_1([u, v]) + i_{\rho(v)}((\Lambda_W)_0(u)),
\]

\[
(5.7) \quad (d^h \Lambda_W)_2(u) = -i_{\rho(u)}((\Lambda_W)_1(u)),
\]

for \( u, v \in \Gamma(A) \).
We also need the expression for $d^v$ in the following particular situation. Any bundle map $\mu : A \to \wedge^k T^* M$ (equivalently seen as a $C^\infty(M)$-linear map $\mu : \Gamma(A) \to \Omega^k(M)$) defines an element $\mu_W \in W^{1,k}(A)$ by

$$\mu_W = \mu,$$ (5.8)

$$\mu_W = 0.$$ In this case, $d^v(\mu_W) \in W^{1,k+1}(A)$ is defined by (see [1, Sec. 3.1])

$$d^v(\mu_W) = -d(\mu(u)), \quad (d^v(\mu_W))_1 = \mu.$$

Proposition 5.1. Consider the map $\psi : \Omega^{k}_{\text{lin}}(A) \to W^{1,k}(A), k = 1, 2, \ldots,$

$$\Lambda = d\Lambda + \Lambda \nu \to \Lambda_W := -d^v(\mu_W + \nu_W).$$

The following holds:

1. $\psi$ induces a $C^\infty(M)$-linear isomorphism $\Omega_{\text{lin}}^{\bullet}(A) \xrightarrow{\sim} W^{\bullet}(A)$.
2. $\psi \circ d = -d^v \circ \psi$.
3. $\psi$ restricts to a linear isomorphism $\Omega_{\text{lin}}^{k}(A) \xrightarrow{\sim} \ker(d^h|_{W^{1,k}(A)})$.

Proof. It is clear from (5.8) and (5.9) that the map $\psi$ is injective. Let us check that any $\Lambda_W \in W^{1,k}(A)$ can be written in the form $-d^v(\mu_W + \nu_W)$ for $C^\infty(M)$-linear maps $\mu : \Gamma(A) \to \Omega^{k-1}(M), \nu : \Gamma(A) \to \Omega^{k}(M)$. Let us write $\Lambda_W = ((\Lambda_W)_0, (\Lambda_W)_1)$, and set $\mu = -((\Lambda_W)_1)$. Then $(d^v(\mu)_W) = -((\Lambda_W)_1)$, so the element $c = d^v(\mu_W + \Lambda_W) \in W^{1,k}(A)$ is such that $c_1 = 0$, which implies that $c = \nu_W$ for a bundle map $\nu : A \to \wedge^k T^* M$. The $C^\infty(M)$-linearity of $\psi$ results from the following properties: $fd\Lambda = d\Lambda f + \Lambda(df)$ and $d^v(f \mu)_W = f d^v(\mu_W) - (df \wedge \mu)_W$. Hence (1) is proved.

To prove (2), writing $\Lambda = d\Lambda + \Lambda \nu$, we have $d\Lambda = d\Lambda$. By definition of $\psi$, it follows that $\psi(d\Lambda) = -d^v(\nu_W)$. On the other hand, $-d^v(\psi(\Lambda)) = -d^v(-d^v \mu + \nu_W) = -d^v\nu_W$, hence (2) holds.

For (3), we must consider the condition $d^h\Lambda_W = 0$. Written in terms of its components (5.5), (5.6), and (5.7), we obtain three equations involving $(\Lambda_W)_0$ and $(\Lambda_W)_1$, which must be shown to agree with conditions (3.16), (3.17) and (3.18) in Thm. 3.1. Using (5.8), (5.9), we see that

$$\nu([u, v]) = di_{\rho(v)} d\mu(u) + di_{\rho(v)} \nu(u) + \mathcal{L}_{\rho(v)} \nu - \mathcal{L}_{\rho(v)} d\mu(u)$$

for all $u \in \Gamma(A)$, and it is clear that (5.7) and (5.6) coincide with conditions (3.16) and (3.17), respectively.

For the degree-0 condition (5.5), using (5.10) and (3.17), we find

$$\nu([u, v]) = di_{\rho(v)} d\mu(u) + di_{\rho(v)} \nu(u) + \mathcal{L}_{\rho(v)} \nu - \mathcal{L}_{\rho(v)} d\mu(u) - \mathcal{L}_{\rho(v)} \nu(u).$$

Using Cartan's formula $\mathcal{L}_X = i_X d + di_X$, one directly verifies that the last equation agrees with (3.18).

Let $\mathcal{G}$ be a source-simply-connected Lie groupoid over $M$, with Lie algebroid $A \xrightarrow{} M$. There is a double complex $\Omega^k(G^{(p)})$ associated to $\mathcal{G}$, known as the Bott-Shulman complex, see [2]. It is equipped with a differential $d : \Omega^k(G^{(p)}) \xrightarrow{} \Omega^{k}(G^{(p+1)}),$ as well as the de Rham differential $d : \Omega^k(G^{(p)}) \xrightarrow{} \Omega^{k+1}(G^{(p)}).$ The Van Est isomorphism constructed in [1] relates the cohomologies of $\Omega^k(G^{(p)})$ and $W^{p,k}(A)$. We will only need a few results of the theory, for $p = 0, 1$.

For $p = 0$, $\Omega^k(G^{(0)}) = \Omega^k(M) = W^{0,k}(A)$, and

$$d : \Omega^k(M) \xrightarrow{} \Omega^k(G), \quad \partial(\eta) = t^* \eta - s^* \eta.$$
For \( p = 1 \), the differential \( \partial : \Omega^k(\mathcal{G}) \to \Omega^{k}(\mathcal{G}^{(2)}) \) is
\[
\partial(\alpha) = pr_1^* \alpha - m^* \alpha + pr_2^* \alpha,
\]
and the Van Est map of [1, Sec. 4] restricts to a map
\[
(5.11) \quad \nabla : \Omega^k_{\text{mult}}(\mathcal{G}) \to \ker(\partial|_{\Omega^k(\mathcal{G})}) \to \ker(\partial^h|_{W^1,k(A)}) \subset W^1,k(A),
\]
given by
\[
(5.12) \quad \nabla(\alpha)(u) = \epsilon^*(di_u \alpha + i_u d\alpha), \quad \nabla(\alpha)(u) = -\epsilon^*(i_u \alpha)
\]
where \( \alpha \in \Omega^k_{\text{mult}}(\mathcal{G}) \), \( u \in \Gamma(A) \) (we view \( A \) as a subbundle of \( T\mathcal{G}|_M \)) and \( \epsilon : M \to \mathcal{G} \) is the unit map of \( \mathcal{G} \). The map \( \nabla \) satisfies
\[
(5.13) \quad \nabla \circ d = -d^v \circ \nabla, \quad \nabla(\partial(\eta)) = \partial^h \eta,
\]
for \( \eta \in \Omega^k(M) \). The general Van Est isomorphism of [1] implies that the induced map
\[
(5.14) \quad \frac{\Omega^k_{\text{mult}}(\mathcal{G})}{\text{Im}(\partial|_{\Omega^k(M)})} \xrightarrow{\nabla} \frac{\ker(\partial^h|_{W^1,k(A)})}{\text{Im}(d^h|_{\Omega^k(M)})}
\]
is a bijection. In this specific situation, a stronger fact holds.

**Proposition 5.2.** The map \( \nabla : \Omega^k_{\text{mult}}(\mathcal{G}) \to \ker(\partial^h|_{W^1,k(A)}) \) is a bijection.

The proof of the proposition uses the following observation (cf. [1, Sec. 6]).

**Lemma 5.3.** Let \( \sigma \in W^1,k(A) \), \( \omega \in \Omega^{k+1}_{\text{mult}}(\mathcal{G}) \) be such that \( \partial^h \sigma = 0 \) and \( \nabla(\omega) = -d^v \sigma \). Then there exists a unique \( \beta \in \Omega^k_{\text{mult}}(\mathcal{G}) \) such that
\[
\nabla(\beta) = \sigma, \quad d\beta = \omega.
\]

**Proof.** The key fact to prove the lemma is shown in [1, Lem. 6.3]: for a closed \( k \)-form \( \alpha \in \Omega^k_{\text{mult}}(\mathcal{G}) \), \( \nabla(\alpha) = 0 \) if and only if \( \alpha = 0 \). As an application, we see that \( \omega \) is necessarily closed, since \( \nabla(d\omega) = d^v(d^v \sigma) = 0 \).

Since \( \partial^h \sigma = 0 \), the isomorphism (5.14) implies that there exists \( \tilde{\beta} \in \Omega^k_{\text{mult}}(\mathcal{G}) \) such that \( \nabla(\tilde{\beta}) = \sigma + d^h \eta \). If \( \beta = \tilde{\beta} - \partial \eta \), then by (5.13) we have
\[
\nabla(\beta) = \nabla(\tilde{\beta}) - \nabla(\partial \eta) = \sigma.
\]
To conclude that \( d\beta = \omega \), note that \( d\beta - \omega \) is multiplicative, closed, and \( \nabla(d\beta - \omega) = -d^v \sigma + d^v \sigma = 0 \).

We can now prove the proposition.

**Proof.** (of Prop. 5.2) Let us fix \( \xi \in W^1,k(A) \), \( \partial^h \xi = 0 \). Let \( \sigma = -d^v \xi \). Then \( d^v \sigma = 0 \), \( d^h \sigma = 0 \), and Lemma 5.3 implies that there exists a unique \( \beta \in \Omega^{k+1}_{\text{mult}}(\mathcal{G}) \) such that
\[
(5.15) \quad \nabla(\beta) = \sigma, \quad d\beta = 0.
\]
Since \( \nabla(\beta) = -d^v \xi \), and, by assumption, \( \partial^h \xi = 0 \), we can apply Lemma 5.3 to conclude that there exists a unique \( \theta \in \Omega^k_{\text{mult}}(\mathcal{G}) \) such that \( \nabla(\theta) = \xi \) and \( d\theta = \beta \). But notice that the condition \( d\theta = \beta \) is automatically satisfied if \( \nabla(\theta) = \xi \), since \( \nabla(d\theta) = \sigma \) and the conditions in (5.15) determine \( \beta \) uniquely.
Composing the bijection (5.11) with the identification \( \Omega^k_{\text{IM}}(A) \cong \ker(d^b|_{W_{1,k}}(A)) \) of (3) in Prop. 5.1, we obtain a bijection

\[
\Omega^k_{\text{mult}}(G) \overset{\sim}{\rightarrow} \Omega^k_{\text{IM}}(A).
\]

Using (5.10) and (5.12), we see that this bijection is explicitly given by \( \alpha \mapsto (\mu, \nu) \), where \( \mu, \nu \) are defined as in (4.4), (4.5), hence agreeing with Theorem 4.6.

6. The dual picture: multiplicative multivector fields

In this section, we illustrate how the techniques used in the paper to study infinitesimal versions of multiplicative forms can be equally applied to multiplicative multivector fields.

We keep the notation introduced in Section 2.1. We focus on the cotangent bundle \( e_A : T^*A \to A \) of a vector bundle \( A \to M \), described in local coordinates \((x^j, u^d, p_j, \xi_d)\), where \((x^j, u^d)\) are relative to a basis of local sections \(\{e_d\}\) of \(A\). The local coordinates on \(A^*\) relative to the dual basis \(\{e^d\}\) are denoted by \((x^j, \xi_d)\); recall from (2.1) that we have a vector-bundle structure \(\tau : T^*A \to A^*\), \((x^j, u^d, p_j, \xi_d) \mapsto (x^j, \xi_d)\). As in Section 2.2, we also consider the \(k\)-fold direct sum \(\oplus_k T^*A\), described by local coordinates \((x^j, u^d, p^1_j, \ldots, p^k_j, \xi^1_d, \ldots, \xi^k_d)\), as a vector bundle over \(\oplus^k A^*\), with projection map \((x^j, u^d, p^1_j, \ldots, p^k_j, \xi^1_d, \ldots, \xi^k_d) \mapsto (x^j, \xi^1_d, \ldots, \xi^k_d)\).

As in Section 3.1, we will need special sections of the bundle \(\oplus_k T^*A \to \oplus^k A^*\). For the bundle \(T^*A \to A^*\), we consider local sections

\[
(6.1) \quad \tilde{d}\xi^j(x^j, \xi_d) = (x^j, 0, \delta_j^d, \xi_d), \quad e_a^L(x^j, \xi_d) = (x^j, \delta_a^d, 0, \xi_d),
\]

which are core and linear sections, respectively; these sections generate the module of local sections of \(T^*A \to A^*\), and the projection \(T^*A \to A\) maps core sections to the zero section of \(A \to M\) and linear sections \(e_a^L\) to the section \(e_a\). More generally, local sections of \(\oplus_k T^*A \to \oplus^k A^*\) are generated by sections of types

\[
(6.2) \quad \tilde{d}\xi^n(\xi^1 \oplus \ldots \oplus \xi^k) = (0\xi^1) \oplus \ldots \oplus (0\xi^n) \oplus \tilde{d}\xi^j(\xi^k) \oplus (0\xi^{n+1}) \oplus \ldots \oplus (0\xi^k),
\]

\[
(6.3) \quad (e_a^L)^k(\xi^1 \oplus \ldots \oplus \xi^k) = e_a^L(\xi^1) \oplus \ldots \oplus e_a^L(\xi^k),
\]

where \(\xi^1 \oplus \ldots \oplus \xi^k \in \oplus^k A^*\) and \(0 : A^* \to T^*A, 0(x^j, \xi_d) = (x^j, 0, 0, \xi_d),\) is the zero section. For each \(k\), we will use these sections to express the natural Lie algebroid structure on \(\oplus_k T^*A \to \oplus^k A^*\), similarly to Section 3.2.

Using the notation in (3.4), the defining relations for the cotangent Lie algebroid structure on \(T^*A \to A^*\) are

\[
(6.4) \quad [d\tilde{\xi}^j, d\tilde{\xi}^j]_{T^*A} = 0,
\]

\[
(6.5) \quad [e_a^L, d\tilde{\xi}^j]_{T^*A} = \frac{\partial\rho^j_a}{\partial x^i} \tilde{\xi}^i, \quad [e_a^L, e_b^L]_{T^*A} = -\frac{\partial C_{ab}^c}{\partial x^i} \xi_c d\tilde{\xi}^i + C^c_{ab} e_c^L,
\]

\[
(6.6) \quad \rho_{T^*A}(d\tilde{\xi}^j) = \rho^j_a \frac{\partial}{\partial \xi_d^a}, \quad \rho_{T^*A}(e_a^L) = \rho^j_a \frac{\partial}{\partial x^j} + C^c_{ab} \xi_c \frac{\partial}{\partial \xi_d^b}.
\]

This Lie algebroid structure is extended to direct sums \(\oplus_k T^*A \to \oplus^k A^*\) in total analogy to what was done for the tangent Lie algebroid in Section 3.2; we adopt the simplified notation \(\rho_k = \rho_{\oplus_k T^*A}\) and \([\cdot, \cdot]_k = [\cdot, \cdot]_{\oplus_k T^*A}\) for the resulting anchor and
bracket\(^6\). Explicitly, the anchor is given by

\[
\rho_k(d\pi^i) = \frac{\partial}{\partial x^i}, \quad \rho_k((e^L)^k) = \frac{\partial}{\partial x^i} + C_{ab}^c \varepsilon^k \frac{\partial}{\partial \varepsilon^c},
\]

whereas for the bracket we have

\[
[d\pi^i, d\pi^j]_k = 0, \quad [(e^L)^k, d\pi^j]_k = \frac{\partial \rho^d}{\partial x^i} d\pi^i - \frac{\partial C_{ab}^c}{\partial x^i} \varepsilon^c d\pi^i.
\]

6.1. **Linear multivector fields and derivations.** Let \( \pi \in \mathcal{X}^k(A) = \Gamma(\wedge^k T^* A) \) be a \( k \)-vector field on the total space of a vector bundle \( \pi_A : A \to M \). Let us consider the function (cf. (2.4))

\[
\overline{\pi} : \bigoplus T^* A \to \mathbb{R}, \quad \overline{\pi}(\mathcal{Y}_1, \ldots, \mathcal{Y}_k) = i_{\mathcal{Y}_1} \ldots i_{\mathcal{Y}_k} \pi.
\]

We say that \( \pi \in \mathcal{X}^k(A) \) is **linear** if \( \overline{\pi} \) defines a vector-bundle map

\[
\bigoplus T^* A \xrightarrow{\overline{\pi}} \mathbb{R}, \quad \bigoplus_k A^* \xrightarrow{\} \{\}
\]

similarly to (2.8). One can directly verify that the notion of linear multivector field agrees with the one considered in [18, Section 3.2]. The space of linear \( k \)-vector fields is denoted by \( \mathcal{X}^{\text{lin}}_k(A) \). As in Lemma 2.2 (cf. (2.7)), \( \pi \) is expressed in local coordinates \((x^j, u^d)\) of \( A \) as

\[
\pi = \frac{1}{k!} \pi x_{\mu_1} \ldots x_{\mu_k} (x) u_{\nu_1} \ldots u_{\nu_k} \partial / \partial x^{\mu_1} \wedge \ldots \wedge \partial / \partial x^{\mu_k} + \frac{1}{(k-1)!} \pi x_{\mu_1} \ldots x_{\mu_{k-1}} u_{\nu_1} \ldots u_{\nu_{k-1}} \partial / \partial u^{\mu_{k-1}} \wedge \partial / \partial x^{\mu_k}.
\]

We have the following analog of Proposition 2.5 for linear multivector fields, proven in [18, Prop. 3.7]: there is a 1-1 correspondence between elements in \( \mathcal{X}^{\text{lin}}_k(M) \) and pairs \((\delta_0, \delta_1)\), where \( \delta_0 : C^\infty(M) \to \Gamma(\wedge^{k-1} A) \) and \( \delta_1 : \Gamma(A) \to \Gamma(\wedge^k A) \) are linear maps satisfying

\[
\delta_0(fg) = g \delta_0 f + f \delta_0 g, \quad \delta_1(fu) = (\delta_0 f) \wedge u + f \delta_1 u,
\]

for all \( f, g \in C^\infty(M) \) and \( u \in \Gamma(A) \). Equivalently, one may view such pairs \((\delta_0, \delta_1)\) as restrictions of linear maps \( \delta : \Gamma(\wedge^k A) \to \Gamma(\wedge^{k-1} A) \) satisfying the property

\[
\delta(u \wedge v) = (\delta u) \wedge v + (-1)^{p(k-1)} u \wedge (\delta v)
\]

for \( u \in \Gamma(\wedge^p A) \) and \( v \in \Gamma(\wedge^q A) \); i.e., \( \delta \) is a degree-\((k-1)\) derivation of the exterior algebra \( \Gamma(\wedge^k A) \). For this reason, we denote both maps \( \delta_0 \) and \( \delta_1 \) by \( \delta \). The explicit

\(^6\)Since tangent Lie algebroids are not used in this section, this notation should not cause any confusion with the one in Section 3.2.
correspondence between $\pi$ and $\delta$ is given by\footnote{To see that (6.14) and (6.15) determine the linear $k$-vector $\pi$, note that fibres of $T^*A \to A$ are generated by elements of types $dl_\xi$ and $dq_A^*_f$, and by linearity $i_{d_\xi}i_{d_\xi}i_{dq_A^*_f}\pi = 0$.} (see [18, Section 3.2])

\begin{align}
(6.14) & \quad \pi(\delta l_\xi, \ldots, \delta l_{k-1}, dq_A^*_f) = q_A^*_f\left\langle \delta f, \xi^1 \wedge \ldots \wedge \xi^{k-1} \right\rangle, \\
(6.15) & \quad \pi(\delta l_\xi, \ldots, \delta l_k)(u) = \sum_{i=1}^k (-1)^{i+k}\pi(\delta l_\xi, \ldots, \delta l_{i-1}, dq_A^*_\xi(\xi^i, u)) \\
& \quad - \left\langle \delta u, \xi^1 \wedge \ldots \wedge \xi^k \right\rangle,
\end{align}

where $f \in C^\infty(M)$, $u \in \Gamma(A)$, $\xi^1, \ldots, \xi^k \in \Gamma(A^*)$, and $l_\xi \in C^\infty(A)$ is the linear function $l_\xi(u) = \left\langle \xi^i, u \right\rangle$. In coordinates, we have

\begin{align}
(6.16) & \quad \delta x^i = \frac{1}{(k-1)!}\pi^{b_1 \ldots b_k-1}i(x)eb_1 \wedge \ldots \wedge eb_{k-1}, \quad \delta e_a = -\frac{1}{k!}\pi^{b_1 \ldots b_k}(x)eb_1 \wedge \ldots \wedge eb_k,
\end{align}

where $\{e_a\}$ is a basis of local sections of $A$.

Let $A \to M$ be a Lie algebroid. The Lie bracket $[\cdot, \cdot]$ on $\Gamma(A)$ has a natural extension (still denoted by $[\cdot, \cdot]$) to the exterior algebra $\Gamma(\wedge^\bullet A)$,

\[ [\cdot, \cdot] : \Gamma(\wedge^p A) \times \Gamma(\wedge^q A) \to \Gamma(\wedge^{p+q-1} A), \]

making it into a Gerstenhaber algebra (see e.g. [6]): for $u \in \Gamma(\wedge^p A)$, $v \in \Gamma(\wedge^q A)$, and $w \in \Gamma(\wedge^r A)$, we have

\begin{align}
(6.17) & \quad [u, v] = -(-1)^{(p-1)(q-1)}[v, u], \\
(6.18) & \quad [u, v \wedge w] = [u, v] \wedge w + (-1)^{(p-1)q}v \wedge [u, w].
\end{align}

The next result is the analog of Theorem 3.1 for linear multivector fields.

**Theorem 6.1.** Let $\pi \in \mathcal{X}^k_{\text{IM}}(A)$ be a linear $k$-vector field on a Lie algebroid $A$, and let $\delta : \Gamma(\wedge^\bullet A) \to \Gamma(\wedge^{\bullet+k-1} A)$ be the associated derivation (as in (6.14) and (6.15)). Then the map $\overline{\pi}$ (6.10) is a Lie algebroid morphism if and only if

\begin{align}
(6.19) & \quad \delta [u, v] = [\delta u, v] + (-1)^{(p-1)(k-1)}[u, \delta v],
\end{align}

for all $u \in \Gamma(\wedge^p A)$, $v \in \Gamma(\wedge^q A)$ (i.e., $\delta$ is a $(k - 1)$-derivation of the Gerstenhaber bracket).

To draw a clear parallel with Theorem 3.1, we denote by $\mathcal{X}^k_{\text{IM}}(A)$ the space of degree $(k - 1)$ derivations $\delta : \Gamma(\wedge^\bullet A) \to \Gamma(\wedge^{\bullet+k-1} A)$ of the Gerstenhaber structure (i.e., (6.13) and (6.19) hold), in analogy with IM $k$-forms.

**Proof.** We work locally, so the condition that $\overline{\pi}$ is a Lie algebroid morphism is

\begin{align}
(6.20) & \quad \overline{\pi}([\Upsilon_1, \Upsilon_2]) = \mathcal{L}_{\rho_k(\Upsilon_1)}\overline{\pi}(\Upsilon_2) - \mathcal{L}_{\rho_k(\Upsilon_2)}\overline{\pi}(\Upsilon_1),
\end{align}

where $\Upsilon_1, \Upsilon_2$ are local sections of $\otimes^k TA \to \otimes^k A^*$ of types (6.2) or (6.3) (cf. (3.25)); hence, just as in the proof of Theorem 3.1, there are 3 cases to be analyzed. The assertion of Theorem 6.1 is a direct consequence of the following claims:

\begin{itemize}
\item{(c1) Let $\Upsilon_1 = d_\xi^i$ and $\Upsilon_2 = d_\xi^j$. If $l = m$, then (6.20) is automatically satisfied; if $l \neq m$, then (6.20) is equivalent to

\begin{align}
(6.21) & \quad \delta [x^i, x^j] = [\delta x^i, x^j] + (-1)^{k-1}[x^i, \delta x^j] = 0.
\end{align}
\end{itemize}
(c2) Let $\Upsilon_1 = d\hat{x}^i.l$ and $\Upsilon_2 = (e_b^L)^k$. Then (6.20) is equivalent to
\begin{equation}
F(e_a^L)^k = \left[ e_a, e_b \right] = \left[ \delta e_a, e_b \right] + \left[ e_a, \delta e_b \right].
\end{equation}

(c3) Let $\Upsilon_1 = (e_a^L)^k$ and $\Upsilon_2 = (e_b^L)^k$. Then (6.20) is equivalent to
\begin{equation}
\delta [e_a, e_b] = [\delta e_a, e_b] + [e_a, \delta e_b].
\end{equation}

In order to prove claims (c1), (c2) and (c3), we need some general observations. For any function $F : \oplus A^* \to \mathbb{R}$ which is $k$-linear over $C^\infty(M)$ and skew symmetric, let $\Phi_F \in \Gamma(\wedge^k A)$ be the unique element such that
\begin{equation}
F(\xi^1, \ldots, \xi^k) = \left< \Phi_F, \xi^1 \wedge \ldots \wedge \xi^k \right>.
\end{equation}

E.g., for $F(\xi^1, \ldots, \xi^k) = F^{b_1 \ldots b_k} \xi^1_{b_1} \ldots \xi^k_{b_k}$ with $F^{b_1 \ldots b_k}$ totally antisymmetric in its indices, we have $\Phi_F = \frac{1}{k!} F^{b_1 \ldots b_k} e_{b_1} \wedge \ldots \wedge e_{b_k}$. We will consider the cases where $F = \pi(d\hat{x}^j.m)$ and $F = \pi((e_a^L)^k)$. Using the local expressions (6.2), (6.3) as well as (6.11) and (6.16), one may directly verify the following identities:
\begin{align}
\pi(d\hat{x}^j.m(\xi^1, \ldots, \xi^k)) &= (-1)^{k-m} \left< \delta x^j, \xi^1 \wedge \ldots \wedge \hat{\xi}^m \wedge \ldots \wedge \xi^k \right>, \\
\pi((e_a^L)^k(\xi^1, \ldots, \xi^k)) &= -\left< \delta e_a, \xi^1 \wedge \ldots \wedge \xi^k \right>,
\end{align}

where the notation $\xi^1 \wedge \ldots \wedge \hat{\xi}^m \wedge \ldots \wedge \xi^k$ means that $\xi^m$ is omitted.

Let us now consider $F(\xi^1, \ldots, \xi^k) = F^{b_1 \ldots b_k} \xi^1_{b_1} \ldots \xi^k_{b_k}$ and the vector fields $\rho_k(d\hat{x}^i.l)$ and $\rho_k((e_a^L)^k)$ on $\oplus^k A^*$, see (6.7). Then a direct computation shows the following identities:
\begin{align}
\mathcal{L}_{\rho_k(d\hat{x}^i.l)}(F(\xi^1, \ldots, \xi^k)) &= (-1)^l \left< [x^i, \Phi_F], \xi^1 \wedge \ldots \wedge \hat{\xi}^l \wedge \ldots \wedge \xi^k \right>, \\
\mathcal{L}_{\rho_k((e_a^L)^k)}(F(\xi^1, \ldots, \xi^k)) &= \left< [e_a, \Phi_F], \xi^1 \wedge \ldots \wedge \ldots \wedge \xi^k \right>.
\end{align}

From (6.24) and (6.26), we directly see, assuming that $l < m$, that
\begin{align}
\mathcal{L}_{\rho_k(d\hat{x}^i.l)}(\pi(d\hat{x}^j.m)(\xi^1, \ldots, \xi^k)) &= (-1)^{l+k-m} \left< [x^i, \delta x^j], \xi^1 \wedge \ldots \wedge \hat{\xi}^l \wedge \ldots \wedge \hat{\xi}^m \wedge \ldots \wedge \xi^k \right>, \\
\mathcal{L}_{\rho_k(d\hat{x}^i.m)}(\pi(d\hat{x}^j.l)(\xi^1, \ldots, \xi^k)) &= (-1)^{m-1+k-l} \left< [x^i, \delta x^j], \xi^1 \wedge \ldots \wedge \hat{\xi}^l \wedge \ldots \wedge \hat{\xi}^m \wedge \ldots \wedge \xi^k \right>.
\end{align}

Combining these two equations with (6.17), we conclude that claim (c1) holds.

To prove the other two claims, note first that the derivation property for functions in (6.12) implies that
\begin{equation}
\delta f = \frac{\partial f}{\partial x^j} \delta x^j, \quad f \in C^\infty(M).
\end{equation}

As a result, since $[e_b, x^i] = \mathcal{L}_{\rho_b} x^i = \rho_b^i$, we have that
\begin{equation}
\delta [x^i, e_b] = -\delta [e_b, x^i] = -\frac{\partial \rho_b^i}{\partial x^j} \delta x^j.
\end{equation}
Using the second formula in (6.8) together with (6.24) and (6.29), we obtain

\[
(6.30) \quad \tilde{\pi}([d\hat{x}^i, (e^L_b)^k]\kappa(\xi^1, \ldots, \xi^k)) = -(-1)^{k-l} \left\langle \frac{\partial \rho^i}{\partial x^j}, \delta x^j, \xi^1 \wedge \ldots \wedge \tilde{\xi}^l \wedge \ldots \wedge \xi^k \right\rangle
\]

\[
= (-1)^{k-l} \left\langle \delta[x^i, e_b], \xi^1 \wedge \ldots \wedge \tilde{\xi}^l \wedge \ldots \wedge \xi^k \right\rangle.
\]

Combining (6.24) and (6.27), as well as (6.25) and (6.26), we immediately get

\[
(6.31) \quad \mathcal{L}_{\rho_h((e^L_b)^k)}(\tilde{\pi}(d\hat{x}^i\partial x^j(\xi^1, \ldots, \xi^k))) = (-1)^{k-l} \left\langle [e_b, \delta x^j], \xi^1 \wedge \ldots \wedge \tilde{\xi}^l \wedge \ldots \wedge \xi^k \right\rangle
\]

\[
(6.32) \quad \mathcal{L}_{\rho_h(d\hat{x}^i\partial x^j)}(\tilde{\pi}((e^L_b)^k(\xi^1, \ldots, \xi^k))) = -(-1)^l \left\langle [x^i, \delta e_b], \xi^1 \wedge \ldots \wedge \tilde{\xi}^l \wedge \ldots \wedge \xi^k \right\rangle.
\]

Now claim (c2) is a direct consequence of (6.30), (6.31) and (6.32).

Finally, to prove (c3), we observe a few facts. From (6.28), we see that

\[
(6.33) \quad \delta[e_a, e_b] = \delta(C_{ab}^c e_c) = \frac{\partial C_{ab}^c}{\partial x^j} \delta x^j \wedge e_c + C_{ab}^c \delta e_c.
\]

The usual formula for the wedge product gives us the identity

\[
\frac{\partial C_{ab}^c}{\partial x^j} \left( \delta x^j \wedge e_c, \xi^1 \wedge \ldots \wedge \xi^k \right) = \sum_{n=1}^{k} (-1)^{k-n} \frac{\partial C_{ab}^c}{\partial x^j} \xi^n \left( \delta x^j, \xi^1 \wedge \ldots \wedge \tilde{\xi}^n \wedge \ldots \wedge \xi^k \right);
\]

using it, we immediately obtain from (6.9), (6.24) and (6.25) that

\[
(6.34) \quad \tilde{\pi}([[(e^L_a)^k, (e^L_b)^k]\kappa(\xi^1, \ldots, \xi^k)]) = - \left\langle \delta[e_a, e_b], \xi^1 \wedge \ldots \wedge \xi^k \right\rangle.
\]

On the other hand, from (6.25) and (6.27) we have that

\[
(6.35) \quad \mathcal{L}_{\rho_h((e^L_b)^k)}(\tilde{\pi}((e^L_b)^k(\xi^1, \ldots, \xi^k))) = -\left\langle [e_b, \delta e_b], \xi^1 \wedge \ldots \wedge \xi^k \right\rangle.
\]

Using (6.34) and (6.35), we can immediately verify that claim (c3) holds. \(\square\)

6.2. Infinitesimal description of multiplicative multivector fields. We now discuss the analogs of the results in Section 4 for multiplicative multivector fields.

Let \(\mathcal{G}\) be a Lie groupoid over \(M\). Its cotangent bundle \(T^*\mathcal{G}\) has a natural Lie groupoid structure over \(A^*\), known as the **cotangent groupoid** of \(\mathcal{G}\), see [9] and [21] for a full description. For us, it will suffice to recall that the unit map \(\tilde{\epsilon} : A^* \rightarrow T^*\mathcal{G}|_M\) identifies \(A^*\) with the annihilator of \(TM \subset T\mathcal{G}\), and that the source map \(\tilde{s} : T^*\mathcal{G} \rightarrow A^*\) is defined by

\[
(6.36) \quad \langle \tilde{s}(\alpha_g), u \rangle = \langle \alpha_g, Tl_u(u - Tt(u)) \rangle, \quad \alpha_g \in T^*_g\mathcal{G}, \ u \in A_{s(g)},
\]

where \(l_g\) denotes left translation in \(\mathcal{G}\). Note that \(\tilde{s}\) is a vector-bundle map covering \(s : \mathcal{G} \rightarrow M\); using coordinates \((z^i)\) on \(\mathcal{G}\), it has the form

\[
(6.37) \quad \tilde{s}(z^i, \alpha_l) = (s(z))^j, C^j_a(z)\alpha_l \in A^*|_{s(z)}.
\]

We will not need the explicit expression for \(C^j_a(z)\), just to note that \(\tilde{s}(dt^*f) = 0\) for all \(f \in C^\infty(M)\) (by (6.36)), which implies that

\[
(6.38) \quad C^j_a(z)\frac{\partial (t^*f)}{\partial z^l} = 0, \ \forall f \in C^\infty(M).
\]
Similarly to what happens for the tangent groupoid, the cotangent groupoid structure extends to direct sums \( \oplus^k T^* G \) over \( \oplus^k A^* \). A multivector field \( \Pi \in \mathcal{X}^k(\mathcal{G}) \) is called **multiplicative** if the associated map

\[
(6.39) \quad \Pi : \oplus^k T^* G \to \mathbb{R}, \quad \Pi(\zeta_1, \ldots, \zeta_k) = i_{\zeta_k} \cdots i_{\zeta_1} \Pi
\]

is a groupoid morphism (cf. Lemma 4.1). We denote the space of multiplicative \( k \)-vector fields on \( \mathcal{G} \) by \( \mathcal{X}^k_{\text{mult}}(\mathcal{G}) \).

**Remark 6.2.** We may equivalently consider the map

\[
(6.40) \quad \Pi^\sharp : \oplus^{k-1} T^* G \to T \mathcal{G}, \quad \Pi^\sharp(\zeta_1, \ldots, \zeta_{k-1}) = i_{\zeta_{k-1}} \cdots i_{\zeta_1} \Pi,
\]

and verify that \( \Pi \) is multiplicative if and only if \( \Pi^\sharp \) is a groupoid morphism.

Let us recall, see e.g. [23], the identification of Lie algebroids

\[
(6.41) \quad \theta_G : A(T^* \mathcal{G}) \xrightarrow{\sim} T^* (A \mathcal{G}),
\]

which extends to an identification \( \theta^k_G : A(\oplus^k T^* \mathcal{G}) = \prod_{i=1}^{k} A(T^* \mathcal{G}) \to \oplus^k A^* \mathcal{A} \) (\( \mathcal{C}_G : T^* \mathcal{G} \to \mathcal{G} \) is a groupoid morphism). Given \( \Pi \in \mathcal{X}^k_{\text{mult}}(\mathcal{G}) \), we consider the infinitesimal map \( \text{Lie}(\Pi) : A(\oplus^k T^* \mathcal{G}) \to \mathbb{R} \) (see (4.6)), as well as the composition

\[
(6.42) \quad \text{Lie}(\Pi) \circ (\theta^k_G)^{-1} : \oplus^k A^* \mathcal{A} \to \mathbb{R}.
\]

The exact same arguments as in Lemma 4.5 directly show that there is a unique \( k \)-vector field \( \text{Lie}(\Pi) \in \mathcal{X}^k(\mathcal{A}) \) satisfying

\[
(6.43) \quad \overline{\text{Lie}(\Pi)} = \text{Lie}(\Pi) \circ (\theta^k_G)^{-1};
\]

moreover, the map

\[
\mathcal{X}^k_{\text{mult}}(\mathcal{G}) \to \mathcal{X}^k(\mathcal{A}), \quad \Pi \mapsto \text{Lie}(\Pi),
\]

is a bijection onto the subspace of \( k \)-vector fields \( \pi \in \mathcal{X}^k_{\text{lin}}(\mathcal{A}) \) for which \( \overline{\Pi} : \oplus^k A^* \mathcal{A} \to \mathbb{R} \) is a morphism of Lie algebroids. An immediate consequence of Theorem 6.1 is

**Corollary 6.3.** There is a bijective correspondence

\[
(6.44) \quad \mathcal{X}^k_{\text{mult}}(\mathcal{G}) \to \mathcal{X}^k_{\text{lin}}(\mathcal{A}), \quad \Pi \mapsto \delta,
\]

where \( \delta \) is the derivation associated with \( \pi = \text{Lie}(\Pi) \in \mathcal{X}^k_{\text{lin}}(\mathcal{A}) \) (via (6.14) and (6.15)).

This result is parallel to Theorem 4.6, except that it provides no explicit way of computing \( \delta \) directly out of \( \Pi \) (analogous to (4.17) and (4.18)). This missing aspect will be clarified in the next section.

### 6.3. The universal lifting theorem revisited.

For \( u \in \Gamma(\wedge^p \mathcal{A}) \), let us denote by \( u^r \) the corresponding right-invariant \( p \)-vector field on \( \mathcal{G} \). As observed in [18, Section 2], given \( \Pi \in \mathcal{X}^k_{\text{mult}}(\mathcal{G}) \), then \( [\Pi, u^r] \) is again right invariant, which means that there exists \( \delta_{\Pi} u \in \Gamma(\wedge^{p+k-1} \mathcal{A}) \) such that \( (\delta_{\Pi} u)^r = [\Pi, u^r] \). One can check that the map \( \delta_{\Pi} : \Gamma(\wedge^p \mathcal{A}) \to \Gamma(\wedge^{p+k-1} \mathcal{A}) \) is a derivation of the Gerstenhaber structure, i.e., \( \delta_{\Pi} \in \mathcal{X}^k_{\text{lin}}(\mathcal{M}) \).

**Proposition 6.4.** The map \( \mathcal{X}^k_{\text{mult}}(\mathcal{G}) \to \mathcal{X}^k_{\text{lin}}(\mathcal{A}), \quad \Pi \mapsto \delta_{\Pi} \), where \( \delta_{\Pi} \) is defined by

\[
(6.45) \quad (\delta_{\Pi} f)^r = [\Pi, t^* f], \quad (\delta_{\Pi} u)^r = [\Pi, u^r],
\]

for \( f \in C^\infty(\mathcal{M}) \) and \( u \in \Gamma(\mathcal{A}) \), coincides with the map (6.44); in particular, it is a bijection.
The fact that the correspondence in Proposition 6.4 is a bijection is the universal lifting theorem of [18] (see Theorem 2.34 therein), which we recover here as a consequence of Corollary 6.3. We need to collect some observations before getting into the proof of Proposition 6.4.

Let us consider the isomorphism

\[(6.46) \quad \Theta : T(T^* \mathcal{G}) \rightarrow T^*(T^* \mathcal{G}), \quad (z^j, \alpha_j, \dot{z}^j, \dot{\alpha}_j) \mapsto (z^j, \dot{z}^j, \alpha_j, \dot{\alpha}_j),\]

which is related to the identification \(\theta_\mathcal{G}\) in (6.41) via

\[(6.47) \quad \theta_\mathcal{G} = (T_{tA})^l \circ \Theta \circ \iota_A(T^* \mathcal{G}),\]

where \((T_{tA})^l\) is the fibrewise dual to the vector-bundle map \(T_{tA} : TA \rightarrow \iota_A^*T(T^* \mathcal{G})\). The composition in (6.47) is well defined since \(\Theta \circ T_{tA}(A(T^* \mathcal{G})) \subset \iota_A^*T(T^* \mathcal{G})\); this can be derived directly from (6.37)). For a \(k\)-vector field \(\Pi \in \mathcal{X}^k(\mathcal{G})\), its tangent lift is the \(k\)-vector field \(\Pi_T \in X^k(T\mathcal{G})\) defined by the condition (cf. (2.17))

\[(6.48) \quad \Pi_T = \text{d} \Pi \circ (\Theta^{-1})^k,\]

where \(\text{d} \Pi : T(\iota_A^*T^* \mathcal{G}) = \bigcap_{T \subset \mathcal{G}} T(T^* \mathcal{G}) \rightarrow \mathbb{R}\) is the differential of the function \(\Pi\) in \(C^\infty(\iota_A^*T^* \mathcal{G})\) defined by (6.39).

**Remark 6.5.** As observed in [13], one may alternatively define the tangent lift \(\Pi_T\) in terms of \(\Pi^k\) (6.40):

\[(6.49) \quad \Pi_T^k : \iota_A^{k-1}T(T^* \mathcal{G}) \rightarrow T(T^* \mathcal{G}), \quad \Pi_T^k = (J_G)^{-1} \circ T\Pi^k \circ (\Theta^{-1})^{(k-1)},\]

where \(J_G : T(T^* \mathcal{G}) \rightarrow T(T^* \mathcal{G})\) is the involution (2.16).

When \(\Pi\) is multiplicative, it follows from (6.43) that

\[(6.50) \quad \pi = \text{Lie}(\Pi) = \Pi_T \circ (\Theta \circ \iota_A T^* \mathcal{G} \circ \theta_\mathcal{G}^{-1})^k.\]

We will need this characterization of \(\pi\) in the proof of Proposition 6.4.

For local computations, it will be convenient to consider adapted local coordinates

\[(6.51) \quad (x^j, y^d) \text{ on } \mathcal{G} \text{ around } M \subset \mathcal{G},\]

where \(y^d\) are coordinates along the \(s\)-fibres. We will also use the induced coordinates \(((x^j, y^d), (\dot{x}^j, \dot{y}^d))\) on \(T\mathcal{G}\), and similarly for \(T^* \mathcal{G}\), \(T(T^* \mathcal{G})\) and \(T^*(T\mathcal{G})\). In these coordinates, \(\iota_A : A \rightarrow T\mathcal{G}|_M, \iota_A(x^j, u^d) = ((x^j), (0, u^d))\), and \(T_{tA} : TA \rightarrow \iota_A^*T(T\mathcal{G})\) is given by

\[T_{tA}\left(\frac{\partial}{\partial x^j} + \frac{\partial}{\partial y^d} \right) \big|_u = \dot{x}^j \frac{\partial}{\partial x^j} + \dot{y}^d \frac{\partial}{\partial y^d} \big|_{\iota_A(u)}, \quad u \in A,\]

whereas for \((T_{tA})^l : \iota_A^*T^*(T\mathcal{G}) \rightarrow T^* A\) we have

\[(T_{tA})^l((p_j dx^j + \gamma_a dy^a + \beta_j dy^j + \gamma_a dy^a)|_{\iota_A(u)}) = (p_j dx^j + \gamma_a dy^a)|_u, \quad u \in A.\]

Since the unit map \(\tilde{c} : A^* \rightarrow T^* \mathcal{G}|_M\) identifies \(A^*\) with the annihilator of \(TM \subset T\mathcal{G}\), given \(\xi \in \Gamma(A^*)\), locally written as \((x^j, \xi_d)\), the local 1-form on \(\mathcal{G}\) given by

\[(6.52) \quad \tilde{\xi}(x^j, y^d) = \xi_d(x) dy^d\]

extends \(\tilde{\xi}(\xi(x))\) to a neighborhood of \(M\) in \(\mathcal{G}\). We denote by \(l_\xi \in C^\infty(T\mathcal{G})\) the linear function determined by \(\tilde{\xi}\). The following lemma is key to compare the map in Proposition 6.4 with the map (6.44).
Lemma 6.6. Let $J = \Theta \circ \iota_{AT} \circ \theta_G^{-1} : T^*A \to \iota_A^* T^*(T \mathcal{G})$, and let $u_0 \in A$. Then, for any $f \in C^\infty(M)$ and $\xi \in \Gamma(A^*)$, we have

\begin{align}
(6.53) & \quad J(dq^*_A f|_{u_0}) = d(t^* f)|_{\iota_A(u_0)}, \\
(6.54) & \quad J(d\xi|_{u_0}) = (d\xi + dh^\vee)|_{\iota_A(u_0)},
\end{align}

where $^\vee$ means the pull-back of functions on $\mathcal{G}$ by $p_G : T \mathcal{G} \to \mathcal{G}$, and $h \in C^\infty(\mathcal{G})$ is a function that vanishes on $M \subset \mathcal{G}$.

Proof. The proof follows from some observations, all of which can be checked through computations in adapted local coordinates $(x^j, y^d)$ as in (6.51).

The first observation one can directly verify is that

\begin{equation}
(6.55) \quad (T\iota_A)^t d(t^* f)|_{\iota_A(u)} = dq^*_A f|_{u}, \quad u \in A.
\end{equation}

Using the local expression (6.37) for the source map $s$, the property (6.38), and the definition of $\Theta$, a direct computation shows that

\begin{equation}
(6.56) \quad T\bar{s}(\Theta^{-1}(d(t^* f)|_{\iota_A(u)})) = 0 \in TA^*|_{q_A(u)}, \quad q_A(u) \in M \subset A^*.
\end{equation}

It follows that $\Theta^{-1}(d(t^* f)|_{\iota_A(u)})$ is in the image of $\iota_A(T^*\mathcal{G})$, hence there is a unique $\Upsilon \in T^*A|_{q_A(u)}$ such that

\begin{equation}
\Theta^{-1}(d(t^* f)|_{\iota_A(u)}) = \iota_A(T^*\mathcal{G})(\theta_G^{-1}(\Upsilon)), \quad \text{i.e.,} \quad J(\Upsilon) = d(t^* f)|_{\iota_A(u)}.
\end{equation}

Using (6.55) and (6.47), we conclude that $\Upsilon = dq^*_A f|_{u}$, which proves (6.53).

With respect to the coordinates $((x^j, y^d), (\tilde{x}^j, \tilde{y}^d))$ on $T\mathcal{G}$, one can write

\begin{equation}
(6.57) \quad d\xi|_{\iota_A(u)} = \frac{\partial \xi_a}{\partial x^j} u^a dx^j + \xi_a dy^a \in T^*(T \mathcal{G})|_{\iota_A(u)},
\end{equation}

from where we conclude that

\begin{equation}
(6.58) \quad (T\iota_A)^t d\xi|_{\iota_A(u)} = (\frac{\partial \xi_a}{\partial x^j} u^a dx^j + \xi_a dv^a)|_u = q^*_A d\xi|_u \in T^*A|_u.
\end{equation}

Let us now consider $J(q^*_A d\xi) \in \iota_A^* T^*(T \mathcal{G})$. Since $\Theta^{-1}(J(q^*_A d\xi))$ lies in $T(T^*\mathcal{G})|_{A^*}$, one can directly verify that $J(q^*_A d\xi)$ can be written as

\begin{equation}
\eta_j dx^j + \eta_a dy^a + \eta_a dy^a,
\end{equation}

i.e., its components relative to $d\tilde{x}^j$ vanish. By (6.47), $(T\iota_A)^t J(q^*_A d\xi)|_u = q^*_A d\xi|_u$, so from the second equality in (6.58) we conclude that $\eta_j = \frac{\partial \xi_a}{\partial x^j} u^a$ and $\eta_a = \xi_a$, i.e.,

\begin{equation}
J(q^*_A d\xi|_u) = \left(\frac{\partial \xi_a}{\partial x^j} u^a dx^j + \xi_a dy^a \right)|_{\iota_A(u)}.
\end{equation}

For each given $u_0 \in A$, one can find $h \in C^\infty(\mathcal{G})$ vanishing on $M \subset \mathcal{G}$ and such that $d\eta^\vee|_{\iota_A(u_0)} = \tau_a(\iota_A(u_0)) dy^a$, and (6.54) follows by a direct comparison with (6.57).

We will need the following immediate observations about linear functions on vector bundles.

Lemma 6.7. Let $q_B : B \to N$ be a vector bundle, with coordinates $(x^j, b^d)$ relative to a basis of local sections $\{e_d\}$, and consider $b = b^d e_d \in \Gamma(B)$, $b^\vee = b^d \frac{\partial}{\partial x^j} \in \mathcal{X}(B)$,
and \( \beta = \beta_{de} \in \Gamma(B^*) \). Let \( l_\beta \in C^\infty(B) \) be the linear function defined by \( \beta \), and fix \( b_0 = b(x_0) \in B \), for a given \( x_0 \in N \). Then \( \mathcal{L}_{b^*} l_\beta = q_B^*(\beta, b) \) and

\[
(6.59) \quad l_\beta(b_0) = (\mathcal{L}_{b^*} l_\beta)(b_0).
\]

We now prove Proposition 6.4.

Proof. (of Proposition 6.4)

Let \( \pi = \text{Lie}(\Pi) \), and consider \( \xi^1, \ldots, \xi^{k-1} \in \Gamma(A^*) \) and \( f \in C^\infty(M) \). Let us fix \( u_0 \in A \), \( x_0 = q_A(u_0) \in M \). By (6.50) and Lemma 6.6, we have

\[
\pi(dl_{\xi^1}, \ldots, dl_{\xi^{k-1}}, dq^*_A f)|_{u_0} = \Pi_T(Jdl_{\xi^1}, \ldots, Jdl_{\xi^{k-1}}, Jdq^*_A f)|_{t_A(u_0)}
\]

\[
(6.60) \quad = \Pi_T(dl_{\xi^1} + dh^1, \ldots, dl_{\xi^{k-1}} + dh^{k-1}, d(t^* f)^\nu)|_{t_A(u_0)},
\]

with \( h_i \in C^\infty(G), h_i|_M = 0 \), and \( \bar{\xi}^i \) as in (6.52). We directly check from the definition of \( \Pi_T \) that it is a linear multivector, \( \Pi_T \in \mathcal{X}^k(TG) \), so (see footnote 7)

\[
(6.61) \quad \iota_{df^*_y} \iota_{df^*_z} \Pi_T = 0, \quad \forall f, f_2 \in C^\infty(G).
\]

Hence the expression in (6.60) agrees with

\[
(6.62) \quad \Pi_T(dl_{\xi^1}, \ldots, dl_{\xi^{k-1}}, d(t^* f)^\nu)|_{t_A(u_0)} = [\Pi_T, (t^* f)^\nu]|(dl_{\xi^1}, \ldots, dl_{\xi^{k-1}})|_{t_A(u_0)},
\]

where \([\cdot, \cdot]\) is the Schouten bracket on \( \mathcal{X}^*(TG) \).

Let us consider the vertical lift operation \( \mathcal{X}^*(G) \to \mathcal{X}^*(TG) \), \( \Pi \mapsto \Pi^v \): in coordinates \((\bar{\xi}^i)\) on \( G \), it sends the vector field \( Y = Y^i \frac{\partial}{\partial \bar{\xi}^i} \) to \( Y^v = Y^i \frac{\partial}{\partial x^1} \), and this is extended to a graded algebra homomorphism of multivector fields. From the Schouten bracket relations for vertical and tangent lifts, see e.g. [14], we obtain

\[
[\Pi_T, (t^* f)^\nu] = [\Pi_T, t^* f]^v = ((\delta f)^v)^\nu.
\]

Letting \( x_0 = q_A(u_0) \in M \), a direct computation in coordinates (6.51) shows that

\[
((\delta f)^v)^\nu(dl_{\xi^1}, \ldots, dl_{\xi^{k-1}})|_{t_A(u_0)} = \langle (\delta f)^v, \bar{\xi}^1 \wedge \cdots \wedge \bar{\xi}^{k-1}\rangle|_{x_0},
\]

from where it follows that

\[
\pi(dl_{\xi^1}, \ldots, dl_{\xi^{k-1}}, dq^*_A f)|_{u_0} = \langle \delta f, \xi^1 \wedge \cdots \wedge \xi^{k-1}\rangle|_{x_0} = q_A^* \langle \delta f, \xi^1 \wedge \cdots \wedge \xi^{k-1}\rangle|_{u_0}.
\]

Comparing with (6.14), we conclude that \( \delta \) (see (6.44)) and \( \delta^v \) agree on \( C^\infty(M) \). It remains to check that they agree on \( \Gamma(A) \).

We now consider \( \xi^1, \ldots, \xi^k \in \Gamma(A^*) \) and describe \( \pi(dl_{\xi^1}, \ldots, dl_{\xi^k})|_{u_0} \) in terms of \( \delta^v \). By (6.50) and (6.54), we have (keeping the notation of Lemma 6.6)

\[
(6.63) \quad \pi(dl_{\xi^1}, \ldots, dl_{\xi^k})|_{u_0} = \Pi_T(Jdl_{\xi^1}, \ldots, Jdl_{\xi^k})|_{u_0}
\]

\[\]

\[
= \Pi_T(dl_{\xi^1} + dh^1, \ldots, dl_{\xi^{k-1}} + dh^{k-1})|_{t_A(u_0)}.
\]

From (6.61), we see that the expression \( \Pi_T(dl_{\xi^1} + dh^1, \ldots, dl_{\xi^{k-1}} + dh^{k-1}) \) can be rewritten as

\[
(6.64) \quad \Pi_T(dl_{\xi^1}, \ldots, dl_{\xi^k}) + \sum_{j=1}^k \Pi_T(Jdl_{\xi^1}, \ldots, Jdl_{\xi^{j-1}}, dh^j, Jdl_{\xi^{j+1}}, \ldots, Jdl_{\xi^k}).
\]
We claim that, for all $j = 1, \ldots, k$, we have
\[
\Pi_T(Jd\xi_1, \ldots, Jd\xi_{j-1}, dh_j^\vee, Jd\xi_{j+1}, \ldots, Jd\xi_k) = 0.
\]

To see that, recall from Remark 6.5 that $\Pi_T$ satisfies $\Pi_T \circ \Theta^{-1} = (J_G)^{-1} \circ T\Pi_T^\sharp$, and, since $\Pi_T^\sharp : \oplus_T G \to T\Phi$ is a groupoid morphism (see Remark 6.2),
\[
T\Pi_T^\sharp \circ (\iota_{A(T\Phi)})^{(k-1)} \subseteq A(T\Phi).
\]

It follows from (4.7) and the definition of $J$ that $\Pi_T^\sharp \circ J^{(k-1)} \subseteq T\iota_A(T\Phi) \subseteq \iota_A^* T(T\Phi)$. Relative to the adapted coordinates $(x^j, y^d)$ in (6.51), elements in $T\iota_A(T\Phi) \subseteq \iota_A^* T(T\Phi)$ are combinations of $\frac{\partial}{\partial x^j}$ and $\frac{\partial}{\partial y^d}$, whereas $dh_j^\vee |_{\iota_A(u)}$ is in the span of $dy^d$. So (6.65) follows, and we conclude that
\[
\pi(d\xi_1, \ldots, d\xi_k)|_{u_0} = \Pi_T(d\xi_1, \ldots, d\xi_k)|_{\iota_A(u_0)}.
\]

To proceed, we observe that $\Pi_T(d\xi_1, \ldots, d\xi_k)$ defines a linear function on $T\Phi$, and, using (6.59) in Lemma 6.7 (with $B = T\Phi$), we write
\[
\Pi_T(d\xi_1, \ldots, d\xi_k)|_{\iota_A(u_0)} = \mathcal{L}(u^r) \circ (\Pi_T(d\xi_1, \ldots, d\xi_k))|_{\iota_A(u_0)},
\]
where $u \in \Gamma(A)$ is such that $u(x_0) = u_0$.

But
\[
\mathcal{L}(u^r) \circ (\Pi_T(d\xi_1, \ldots, d\xi_k)) = (\mathcal{L}(u^r) \circ \Pi_T)(d\xi_1, \ldots, d\xi_k)
\]
\[+ \sum_{j=1}^k \Pi_T(d\xi_1, d\xi_{j-1}, \mathcal{L}(u^r) \circ (d\xi_j), d\xi_{j+1}, \ldots, d\xi_k),
\]
and note that
\[
\mathcal{L}(u^r) \circ (d\xi_1, \ldots, d\xi_k)|_{\iota_A(u_0)} = d(\xi^r(u^r)), \quad \mathcal{L}(u^r) \circ \Pi_T = [u^r, \Pi]^\vee = -((\delta_H u)^r)^\vee,
\]
where we used the Schouten-bracket relations for tangent and vertical lifts in the second equation. One can directly check that
\[
((\delta_H u)^r)^\vee (d\xi_1, \ldots, d\xi_k)|_{\iota_A(u_0)} = \left<\left< \delta_H u^r, \xi^1 \wedge \ldots \wedge \xi^k \right> \right|_{\iota_A(u_0)} = \left< \delta_H u, \xi^1 \wedge \ldots \wedge \xi^k \right>_{x_0}.
\]

Thus $\mathcal{L}(u^r) \circ (\Pi_T(d\xi_1, \ldots, d\xi_k))|_{\iota_A(u_0)}$ equals
\[
-\left< \delta_H u, \xi^1 \wedge \ldots \wedge \xi^k \right>_{x_0} + \sum_{j=1}^k \Pi_T(d\xi_1, \ldots, d\xi_{j-1}, d(\xi^r(u^r)), d\xi_{j+1}, \ldots, d\xi_k)|_{\iota_A(u_0)}.
\]

Using local coordinates $(x^j, y^d)$ as in (6.51), one can check the identity
\[
d(\xi^r(u^r))|_{\iota_A(u_0)} = d(t^s(\xi^r(u), u))|_{\iota_A(u_0)} + dh_j^\vee|_{\iota_A(u_0)},
\]
where $h_j \in C^\infty(G)$ vanishes on $M \subset \Phi$. It follows that
\[
\Pi_T(d\xi_1, \ldots, d\xi_{j-1}, d(\xi^r(u^r)), d\xi_{j+1}, \ldots, d\xi_k)|_{\iota_A(u_0)} =
\Pi_T(d\xi_1, \ldots, d\xi_{j-1}, d(t^s(\xi^r(u), u))|_{\iota_A(u_0)} + dh_j^\vee|_{\iota_A(u_0)} +
\Pi_T(d\xi_1, \ldots, d\xi_{j-1}, dh_j^\vee, d\xi_{j+1}, \ldots, d\xi_k)|_{\iota_A(u_0)}.
\]
Using the linearity of $\Pi_T$ (see footnote 7) and (6.65), we see that
\[
\Pi_T(d\tilde{\xi}_1, \ldots, d\tilde{\xi}_{j-1}, dh_j^\vee, d\tilde{\xi}_{j+1}, \ldots, d\tilde{\xi}_k) = \Pi_T(Jd\tilde{\xi}_1, \ldots, Jd\tilde{\xi}_{j-1}, dh_j^\vee, Jd\tilde{\xi}_{j+1}, \ldots, Jd\tilde{\xi}_k) = 0.
\]
A direct comparison with (6.60), (6.62) gives that
\[
\Pi_T(d\tilde{\xi}_1, \ldots, d\tilde{\xi}_{j-1}, d(t^*\langle \xi^j, u \rangle)^\vee, d\tilde{\xi}_{j+1}, \ldots, d\tilde{\xi}_k)\big|_{\lambda_A(u_0)} = (-1)^{k-j} \pi(dl_{\tilde{\xi}_1}, \ldots, \overline{dl}_{\tilde{\xi}_j}, \ldots, dl_{\tilde{\xi}_k}, dq_A^*(\langle \xi^j, u \rangle))|_{u_0}
\]
Going back to (6.66), we finally conclude that
\[
\pi(dl_{\xi_1}, \ldots, dl_{\xi_k})|_{u_0} = -\left< \delta_H u, \xi^1 \wedge \ldots \wedge \xi^k \right>_{x_0} + \sum_{j=1}^k (-1)^{k-j} \pi(dl_{\xi_1}, \ldots, \overline{dl}_{\xi_j}, \ldots, dl_{\xi_k}, dq_A^*(\langle \xi^j, u \rangle))|_{u_0}.
\]
Comparing with (6.15), we conclude that $\delta = \delta_H$. \qed

REFERENCES

[1] Arias Abad, C., Crainic, M., The Weil algebra and the Van Est isomorphism. Arxiv: 0901.0322.
[2] Bott, R., Shulman, H., Stasheff, J., On the de Rham theory of certain classifying spaces, \textit{Advances in Math.} \textbf{20} (1976), 43-56.
[3] Bursztyn, H., Cabrera, A., Ortiz, C., Linear and multiplicative 2-forms. \textit{Lett. Math. Phys.}, \textbf{90} (2009), 59–83. Arxiv:0911.0441 [math.DG].
[4] Bursztyn, H., Crainic, M., Dirac geometry, quasi-Poisson actions and D/G-valued moment maps. \textit{J. Differential Geometry} \textbf{82} (2009), 501–566. ArXiv:0710.0639.
[5] Bursztyn, H., Crainic, M., Weinstein, A., Zhu, C., Integration of twisted Dirac brackets, \textit{Duke Math. J.} \textbf{123} (2004), 549-607.
[6] Cannas da Silva, A., Weinstein, A., \textit{Geometric models for noncommutative algebras}. Berkeley Mathematics Lecture Notes, 10. American Mathematical Society, Providence, RI; Berkeley Center for Pure and Applied Mathematics, Berkeley, CA, 1999.
[7] Cattaneo, A., Felder, G., \textit{Poisson sigma models and symplectic groupoids}. Quantization of singular symplectic quotients, 61–93, Progr. Math., \textbf{123} (1994), 61–93. \textit{Progr. Math.}, \textbf{123} (1994), 61–93. Arxiv:0810.0066[math.DG].
[8] Cattaneo, A., Xu, P.: Integration of twisted Poisson structures, \textit{J. Geom. and Phys.} \textbf{49} (2004), 187-196.
[9] Coste, A., Dazord, P., Weinstein, A., \textit{Groupoides symplectiques}. Publications du Département de Mathématiques. Nouvelle Série. A, Vol. 2, i–ii, 1–62, Publ. Dép. Math. Nouvelle Sér. A, 87-2, Univ. Claude-Bernard, Lyon, 1987.
[10] Courant, T., Dirac manifolds, \textit{Trans. Amer. Math. Soc.} \textbf{319} (1990), 631-661.
[11] Crainic, M.: Generalized complex structures and Lie brackets. ArXiv:math/0412007.
[12] Crainic, M., Fernandes, R., Integrability of Lie brackets. \textit{Ann. of Math.} \textbf{157} (2003), 575–620.
[13] Grabowski, J., Urbanski, P., Tangent lifts of Poisson and related structures, \textit{J. Phys. A} \textbf{28} (1995), 6743–6777.
[14] Grabowski, J., Urbanski, P., Tangent and cotangent lifts and graded Lie algebras associated with Lie algebroids, \textit{Ann. Global Anal. Geom.} \textbf{15} (1997), 447-486.
[15] Gracia-Saz, A., Mehta, R. A., VB-algebroids and representation theory of Lie algebroids, Arxiv:0810.0066[math.DG].
[16] Gualtieri, M: \textit{Generalized complex geometry}. Ph.D. thesis, Oxford. ArXiv:math.DG/0401221.
[17] Hitchin, N.: Generalized Calabi-Yau manifolds. \textit{Q. J. Math.} \textbf{54} (2003), 281–308.
[18] Iglesias Ponte, D., Laurent-Gangoux, C., Xu, P.: Universal lifting theorem and quasi-Poisson groupoids, Arxiv:math.DG/0507396.
[19] Konieczna, K., Urbanski, P., Double vector bundles and duality. \textit{Arch. Math. (Brno)} \textbf{35} (1999), 59–95.
[20] Lu, J.-H., Weinstein, A., Poisson Lie groups, dressing transformations and Bruhat decompositions, *J. Diff. Geometry* **31** (1990), 501–526.
[21] Mackenzie, K., General theory of Lie groupoids and Lie algebroids. London Mathematical Society Lecture Note Series, 213. Cambridge University Press, Cambridge, 2005.
[22] Mackenzie, K., Xu, P., Lie bialgebroids and Poisson groupoids. *Duke Math. J.* **73** (1994), 415–452.
[23] Mackenzie, K., Xu, P., Classical lifting processes and multiplicative vector fields. *Quarterly J. Math. Oxford* (2), **49** (1998), 59–85.
[24] Mackenzie, K., Xu, P., Integration of Lie bialgebroids. *Topology* **39** (2000), 445–467.
[25] Mikami, K., Weinstein, A.: Moments and reduction for symplectic groupoid actions. *Publ. RIMS, Kyoto Univ.* **24** (1988), 121-140.
[26] Pradines, J., Représentation des jets non holonomes par des morphismes vectoriels doubles soudues, *C.R. Acad. Sc. Paris*, Série A, **278** (1974), 1523-1526.
[27] Ševera, P., Some title containing the words “homotopy” and “symplectic”, e.g. this one. *Travaux mathématiques*. Fasc. XVI (2005), 121–137.
[28] Ševera, P., Weinstein, A., Poisson geometry with a 3-form background. *Progr. Theoret. Phys. Suppl.* **144** (2001), 145–154.
[29] Weinstein, A., Symplectic groupoids and Poisson manifolds. *Bull. Amer. Math. Soc.* **16** (1987), 101–104.
[30] Weinstein, A., Coisotropic calculus and Poisson groupoids. *J. Math. Soc. Japan* **40**, 1988.
[31] Xu, P., Momentum maps and Morita equivalence. *J. Differential Geom.* **67** (2004), 289–333.
[32] Yano, K., Ishihara, S., *Tangent and cotangent bundles*, Marcel Dekker, Inc., New York, 1973.
[33] Zambon, M., L-infinity algebras and higher analogues of Dirac structures and Courant algebroids. Arxiv: 1003.1004, to appear in *J. Symplectic Geom.*

**Instituto de Matemática Pura e Aplicada, Estrada Dona Castorina 110, Rio de Janeiro, 22460-320, Brasil**

*E-mail address: henrique@impa.br*

**Department of Mathematics, University of Toronto, 40 St. George Street, Toronto, Ontario, M5S 2E4 Canada**

*E-mail address: ale.cabrera@gmail.com*

---

*A.C.’s current address: Departamento de Matemática Aplicada, Instituto de Matemática, Universidade Federal do Rio de Janeiro, CEP 21941-909, Rio de Janeiro - RJ, Brazil*