Robust Steering of n-level Quantum Systems

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Abstract

Robust open-loop steering of a finite-dimensional quantum system is a central problem in a growing number of applications of information engineering. In the present paper, we reformulate the problem in the classical control-theoretic setting, and provide a precise definition of robustness of the control strategy. We then discuss and compare some significant problems from NMR in the light of the given definition. We obtain quantitative results that are consistent with the qualitative ones available in the physics literature.

1 Introduction

We consider an isolated n-dimensional quantum system with time evolution described by the following Schrödinger equation:

\[ i\hbar \dot{\psi}(t) = H(t)\psi(t). \]  (1)
Here $|\psi(t)\rangle$ is a vector of unit norm in $\mathbb{C}^n$ representing the state of the system at time $t$. The unitary time evolution of the system is governed by the system Hamiltonian:

$$H(t) = H_0 + \sum_{j=1}^{m} H_j u_j(\theta, t).$$  \hspace{1cm} (2)

The *internal Hamiltonian* $H_0 \in \mathbb{C}^{n \times n}$ is an Hermitian matrix describing the free evolution of the system. The *control Hamiltonian*

$$H_c(t) = \sum_{j=1}^{m} H_j u_j(\theta, t),$$

where $H_j \in \mathbb{C}^{n \times n}$ are also Hermitian matrices, accounts for the effects of the control inputs $u_1(\theta, t), ..., u_m(\theta, t)$ on the dynamics of the system. We assume that these control functions depend on a finite number of real parameters $\theta = (\theta_1, ..., \theta_p)$, $\theta_k \in \mathcal{T}$, with $\mathcal{T}$ being an open set in $\mathbb{R}^p$. This kind of assumption is reasonable if we think of the small set of parameters we can control in an experimental setting.

We consider the problem of steering the system from a given initial state $|\psi_0\rangle = |\psi(0)\rangle$ to a final state $|\psi_1\rangle$, where $|\psi_0\rangle$ and $|\psi_1\rangle$ are unit vectors in $\mathbb{C}^n$. We assume that the transition occurs (at $t=T$) when $\theta = \theta^*$ which we take as “nominal” value of the parameters. Clearly, if $\theta \neq \theta^*$, the transition will, in general, not occur.

It is convenient to introduce the *error probability* for each control strategy. Consider the normalized final state for the time evolution, $|\psi(T, \theta)\rangle$. It can be written as $|\psi(T, \theta)\rangle = |\psi_1\rangle|\psi(T, \theta)\rangle + |\psi^\perp(\theta, T)\rangle$ with $|\psi^\perp(\theta, T)\rangle$ orthogonal to $|\psi_1\rangle$. If we imagine to perform a discrete measure\(^1\) on an observable that has $|\psi_1\rangle$ as eigenstate, the probability to obtain the eigenvalue associated $|\psi_1\rangle$(that corresponds to the probability of finding the system in $|\psi_1\rangle$ immediately after the measure) is: $P_{|\psi_1\rangle} = |\langle \psi_1 | \psi(T, \theta) \rangle|^2$. Then the *error probability* corresponding to the value $\theta$ is:

$$P_{err}(T, \theta) = 1 - |\langle \psi_1 | \psi(T, \theta) \rangle|^2$$

\(^1\)Quantum measure fundamental postulates can be founded in standard quantum mechanics textbooks, see e.g. [6], [5] or [2].
\[ \langle \psi^\perp(\theta, T)|\psi^\perp(\theta, T) \rangle, \quad (3) \]

thanks to the fact that \( |\psi(T, \theta)\rangle \) is normalized. By assumption, we have \( P_{\text{err}}(T, \theta^*) = 0 \).

2 Robustness of the control strategy

In the quantum control field, the expression “robustness of the control strategy” means that the control performance is insensible with respect to errors in the control implementation. In [7], a control strategy is considered robust “if significant local changes in the amplitude and the form of the pulse and of the chirp do not change significantly the final transfer probability.” The pulse and the chirp, in the setting described in Section 3, are the system inputs parameters. A quantitative definition of robustness is, however, missing.

The contribution of the present paper is to provide such a quantitative definition reformulating the problem as a robust control-theoretic problem, and then to analyze the robustness properties of some significant strategies considered in the relevant literature. In terms of our model, this concept of robustness can be qualitatively formulated as follows: A control strategy is robust when, for values of the parameters \( \theta \) different from the nominal ones, the final state \( |\psi(T, \theta)\rangle \) is close to the desired one \( |\psi_1\rangle \). This robustness request is satisfied if \( P_{\text{err}}(T, \theta) \) is small in the parameter set \( T \).

In classical control theory, plant uncertainty is described by a set \( \mathcal{P} \) of possible plants [3]. This uncertainty can be either structured (parametrized by a finite number of scalar parameters or a discrete set of plants) or unstructured (disc-like uncertainty). A controller is said to be robust with respect to some property if this property holds for every plant in \( \mathcal{P} \). It is quite simple to reformulate our problem as a particular case of structured-like classical robustness problem. First, we notice that our quantum “plant” is determined by the matrices \( (H_0, H_1, ..., H_m) \). These matrices determine the system Hamiltonian [2], given the control strategy. Let \( P_0 = (H_0, H_1, ..., H_m) \) be our
nominal plant.

As in [8], we can transfer the uncertainty from the control parameters to the internal Hamiltonian. In fact, by defining $\delta u_i(\theta) = u_i(\theta) - u_i(\theta^*)$ we can write:

$$H(t) = H_0 + \sum_{i=1}^{m} H_i (u_i(\theta^*) + \delta u_i(\theta))$$

$$= H_0 + \sum_{i=1}^{m} H_i \delta u_i(\theta) + \sum_{i=1}^{m} H_i u_i(\theta^*)$$

$$= (H_0 + \Delta H_u(\theta)) + \sum_{i=1}^{m} H_i u_i(\theta^*). \quad (4)$$

where $\Delta H_u(\theta) = \sum_{i=1}^{N} H_i \delta u_i(\theta)$. Such a cosmetic transformation shows that our control strategy uncertainty can be seen as a particular case of the plant uncertainty (with control inputs $u_i(\theta^*)$). The plant set $\mathcal{P}$ is here given by:

$$\mathcal{P} = \{(H_0 + \Delta H_u(\theta), H_1, ..., H_m) | \theta \in \mathcal{T}\}.$$

The property we are interested in, as e.g. in [7], is the error probability. We require this probability not to exceed a fixed threshold $\epsilon \in [0, 1]$ at a given $T$. All the ingredients of a classical robustness problem have now been specified. Introduce the $\epsilon$-robustness set $\mathcal{R}_\epsilon$ as

$$\mathcal{R}_\epsilon = \{\theta \in \mathcal{T} | P_{err}(\theta, T) \leq \epsilon\}. \quad (5)$$

We give the following definition.

**Definition 1.** A control strategy $\{u_1(\theta^*, t), ..., u_m(\theta^*, t)\}, t \in [0, T]$ is $\epsilon$-robust with respect to parameters uncertainty if:

$$\mathcal{R}_\epsilon = \mathcal{T}. \quad (6)$$

Notice that only the 0-robustness case ensures us an exact steering of the system state for all $\theta \in \mathcal{T}$. 

4
3 Some applications

In this section we analyze, in the light of the above definition, the robustness of some prototype examples. We will compare our results with qualitative observations and robustness claims in the relevant literature. To do so, we introduce a particular form of (1) frequently used in NMR (Nuclear Magnetic Resonance) quantum control problems. We consider a two level quantum system, and the associated a bi-dimensional Hilbert space. The time evolution is described by a scaled time Schrödinger equation in the form:

\[ i\hbar \frac{\partial}{\partial s} |\psi(s)\rangle = TH(s)|\psi(s)\rangle, \quad (7) \]

where \( s = t/T \) and

\[ H(s) = \begin{pmatrix} -\Delta(s) & \Omega(s) \\ \Omega(s) & \Delta(s) \end{pmatrix}, \]

is represented in the canonical (diabatic) base. The control functions:

\[
\begin{aligned}
\Delta(s) &= \Delta_0 \Phi(s) \\
\Omega(s) &= \Omega_0 \Lambda(s)
\end{aligned}
\]

are the inputs, with \( \Phi(s), \Lambda(s) \) fixed envelopes and \( \Delta_0, \Omega_0 \in \mathbb{R}^+ \) amplitude parameters. In this picture we have \( \Delta(s) = u_1(s, \Delta_0) \) and \( \Omega(s) = u_2(s, \Omega_0) \). Thus \( \theta = (\theta_1, \theta_2) = (\Delta_0, \Omega_0) \) are the parameters we are interested in. In the contest of particle-laser field interaction and the RWA (Rotating Wave Approximation \[1\]), these functions depend on the chirp (detuning) and the amplitude (time-dependent Rabi frequency) of the active pulse\(^2\). This model can be seen as a particular case of model (1), and it is suitable to describe control techniques based both on magnetic resonance and adiabatic passage. At each time the unitary transformation

\[ U(s) = \begin{pmatrix} \cos \theta(s) & -\sin \theta(s) \\ \sin \theta(s) & \cos \theta(s) \end{pmatrix}, \quad (8) \]

\(^2\)To find some detailed information about the physical meaning of these parameters and about the resonance phenomenon see \[9, 4\].
with \( \theta(s) = \frac{1}{2} \arctan (\Omega(s)/\Delta(s)) \), diagonalizes the Hamiltonian

\[
U^\dagger(s)H(s)U(s) = \begin{pmatrix}
\varepsilon(s) & 0 \\
0 & -\varepsilon(s)
\end{pmatrix} = D(s)
\]  

(9)

Here

\[
\varepsilon(s) = \sqrt{\Delta^2(s) + \Omega^2(s)}.
\]  

(10)

is the energy eigenvalue. Applying \( U(s) \) as a time dependent change of basis and defining \( \phi \), we obtain:

\[
i\hbar \frac{\partial |\phi(s)\rangle}{\partial s} = \left[ TD(s) - iU^\dagger(s) \frac{\partial}{\partial s} U(s) \right] |\phi(s)\rangle
\]

\[
= \begin{pmatrix}
T\varepsilon(s) & i\gamma(s) \\
i\gamma(s) & -T\varepsilon(s)
\end{pmatrix} |\phi(s)\rangle.
\]  

(11)

The new basis vectors are called adiabatic states. In the adiabatic limit, \( T \to \infty \), the \( \gamma(s) \) terms can be neglected, as shown by the adiabatic approximation theory\[5\].

In the following subsections, we will investigate the robustness properties of different control strategies in a typical steering problem, the state-flipping: Transfer the system state from one basis vector to the other. The standard resonance technique and two adiabatic models will be discussed and compared.

### 3.1 Magnetic Resonance

A simple way to obtain such a transfer is by using the magnetic resonance phenomena: Under properly tailored oscillating fields, the state vectors rotate between the two basis states \[9,4\]. This kind of effect can be generated by the following fields-control functions:

\[
\begin{align*}
\Delta(s) &= 0 \\
\Omega(s) &= \Omega_0 \Lambda(s),
\end{align*}
\]  

(12)

where \( \Lambda(s) \) is the \( \Omega \)-pulse envelope. This parametrization, and some easy calculations \[6\], lead to the following expression for the error probability:

\[
P_{err}(T, \Omega_0, A_\Lambda) = \cos^2 \left[ T\Omega_0 \int_{\tilde{s}_i}^{s_f} \Lambda(s) ds \right] = \cos^2 \Omega_0 TA_\Lambda,
\]  

(13)
with \( A_\Lambda \) the \( \Omega \)-pulse area.

This probability is equal to zero for:

\[
\Omega_{0,k}^* = \left( k + \frac{1}{2} \right) \frac{\pi}{TA_\Lambda}, \quad k = 0, 1, 2, ...
\]

or, equivalently, for:

\[
A_{\Lambda}^* = \left( k + \frac{1}{2} \right) \frac{\pi}{T\Omega_0}, \quad k = 0, 1, 2, ...
\]

Thus, \( P_{\text{err}}(T, \Omega_0, A_\Lambda) = 0 \) in a family of hyperbolas parameterized in \( k \) (zero-measure set in the parameters space). Consider \( \mathcal{T} = [\Omega_0^* - \beta, \Omega_0^* + \beta] \times [A_\Lambda^* - \sigma, A_\Lambda^* + \sigma] \), a common setting if we are working with nominal values and error intervals, with \( \beta, \sigma \) such that \( \sin 2T\Omega_0A_\Lambda \) is monotone in every direction. Then, the maximum absolute value for the error probability in \( \mathcal{T} \) is:

\[
P_{\text{max}} = \cos^2 T\bar{\Omega}\bar{A}_\Lambda,
\]

where \( \bar{A}_\Lambda = A_{\Lambda}^* + \sigma \) and \( \bar{\Omega} = \Omega_0^* + \beta \). Then \( \mathcal{T} \) is \( \bar{\epsilon} \)-robust, with \( \bar{\epsilon} = P_{\text{max}} \).

According to qualitative evaluation found, the magnetic resonance strategy doesn’t seem to ensure enough insensitivity towards errors in control implementation and can be sensitively improved by adiabatic passage techniques.

### 3.2 Landau-Zener Model

The Landau-Zener model is one of the simplest choice of control function leading to an adiabatic transition. We will consider:

\[
\begin{cases}
\Delta(s) = \frac{\Delta_0^2}{T^2}s \\
\Omega(s) = \Omega_0
\end{cases}
\]

the \emph{detuning} varies linearly with a zero crossing, while the Rabi frequency is maintained constant. For \( s = 0 \) we have a minimum in the difference between energy levels that leads to a state inversion if
the evolution satisfies the condition needed for the adiabatic approximation. The error probability is estimated with the Landau-Zener formula:

\[ P_{\text{err}}(T, \Omega_0, \Delta_0) \approx e^{-\pi T \frac{\Omega_0^2}{\Delta_0^2}}. \]  

(16)

This probability goes to zero in the adiabatic limit \( T \to \infty \) for any choice of \( \Omega \neq 0 \) and \( \Delta \neq 0 \). Thus, the robustness set for this strategy is the whole open first quadrant without its boundary \((\Omega_0 = 0, \Delta_0 = 0)\).

The advantages given by this adiabatic technique are evident, as long as (16) estimates correctly the error probability. Even if the \( T \to \infty \) condition is not realizable, we can take a \( T \) large enough to maintain \( P_{\text{err}} \) arbitrary small for (almost-)every parametrization of the control strategy. We will call this behavior \textit{intrinsically robust}.

### 3.3 Allen-Eberly Model

We now analyze an adiabatic control strategy more complex than the previous one. The Allen-Eberly \cite{11} parametrization allows to obtain an exact expression for the error probability and, in the \( \Omega_0 = \Delta_0 \) case, forces the state time evolution along the energy \textit{level lines}, maintaining \( \varepsilon(\Omega(s), \Delta(s)) = c, \) \( c \) constant for every \( s \) \cite{7}. This kind of choice leads to good results in terms of error probability even quite far from the ideal \( T \to \infty \) condition, as we are going to show. In terms of control functions, we consider:

\[
\begin{align*}
\Delta(s) &= \Delta_0 \sqrt{1 - \text{sech}^2(s)} = \Delta_0 \tanh(s) \\
\Omega(s) &= \Omega_0 \text{sech}(s).
\end{align*}
\]  

(17)

Then, the exact expression for the error probability is:

\[
P_{\text{err}}(T, \Omega_0, \Delta_0) = \cosh^2 \left( \pi T \sqrt{\Delta_0^2 - \Omega_0^2} \right) \text{sech}^2 (\pi \Delta_0 T),
\]  

(18)

for every regime, adiabatic or not. We can notice that, for large \( T \) and for \( \Delta_0 \geq \Omega_0 \), the error probability can be bounded by:

\[
P_{\text{err}}(T, \Omega_0, \Delta_0) \leq 4e^{-2\pi T(\Delta_0 - \sqrt{\Delta_0^2 - \Omega_0^2})}.
\]  

(19)
Thus, for every $\Delta_0$ and $\Omega_0$, $\Delta_0 \geq \Omega_0$, the error probability decreases exponentially to zero in the adiabatic limit. The best choice for the parameters values is to take the largest $\Delta_0 = \Omega_0$. In the case $\Omega_0 > \Delta_0$, the error probability becomes:

$$P_{err}(T, \Omega_0, \Delta_0) = \cos^2 \left( \pi T \sqrt{\Omega_0^2 - \Delta_0^2} \right) \text{sech}^2 \left( \pi \Delta_0 T \right).$$

(20)

This expression tends to zero with dumped oscillations, due to the term $\cos^2 \left( \pi T \sqrt{\Omega_0^2 - \Delta_0^2} \right)$. Again, larger $\Delta_0$ make $P_{err}$ converge faster. Thus, for each fixed $\epsilon$, we can compute a $T_\epsilon$ such that the error probability $P_{err}(T, \Omega_0, \Delta_0) < \epsilon$ for every $T > T_\epsilon$. Indeed, it is easy to see that

$$T_\epsilon = \max \left\{ -\frac{\ln \epsilon}{2\pi(\Delta_0 - \sqrt{\Delta_0^2 - \Omega_0^2})}, -\frac{\ln \epsilon}{2\pi \Delta_0} \right\}. $$

This control strategy is therefore intrinsically robust for $T$ sufficiently large. According to the Landau-Zener case, every choice of $\Omega_0 \neq 0$ and $\Delta_0 \neq 0$ drives the system to the target state. The level line condition ($\Delta_0 = \Omega_0$) and large $\Omega_0$ give faster convergence to the desired state.

## 4 Discussion

Comparing the results, the advantages given by the adiabatic strategies are evident. They can be effectively used, however, when the transfer time is not critical: Their intrinsic robustness is exhibited only with a large time.

The examples analyzed are also treated in [7] to illustrate that control strategies based on the level lines are optimal for adiabatic population transfer (the level line strategies minimize the error probability and corresponds to the minimum pulse area). In [7], robustness of the control is also taken in account: Contour plots of error probability with respect to parameters variations are obtained thanks to numerical simulations for the system evolution. Different strategies are qualitatively compared. It is shown that the simple resonance case generates
larger error probability than the adiabatic optimal techniques, once a parameter variation is fixed.

Here we have given a formal definition of the robustness property, reformulating the problem in the control theoretical setting. We have obtained quantitative results consistent to the qualitative ones just described, and we have provided an analysis tool useful to evaluate and compare robustness behavior of different strategies. From a control theoretic viewpoint, we have analyzed a specific robustness problem for open-loop control of a bilinear system.

References

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