RATIONAL MODELS FOR AUTOMORPHISMS OF FIBER BUNDLES

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Abstract. Given a fiber bundle, we construct a differential graded Lie algebra model for the classifying space of the monoid of homotopy equivalences of the base covered by a fiberwise isomorphism of the total space.

1. Introduction

Consider a fiber bundle $p: E \to X$ with structure group $G$ over a simply connected CW-complex $X$ and let $\text{aut}_\circ(p)$ denote the topological monoid consisting of commutative diagrams

$$
\begin{array}{ccc}
E & \xrightarrow{\phi} & E \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\
X & \xrightarrow{f} & X,
\end{array}
$$

such that $f$ is homotopic to the identity map of $X$ and $\phi$ is a fiberwise isomorphism.

The goal of this paper is to construct a differential graded Lie algebra model for the classifying space $B\text{aut}_\circ(p)$ in the sense of Quillen's rational homotopy theory. We assume that $BG$ is a nilpotent space, i.e., that the group $\pi_0(G)$ is nilpotent and acts nilpotently on $\pi_k(G)$ for all $k \geq 1$.

Theorem 1.1. Let $L$ be the minimal Quillen model for $X$ and let $\Pi$ be a dg Lie algebra model for $BG$. Furthermore, let $\tau: CL \to \Pi$ be a twisting function that models the classifying map of the bundle $\nu: X \to BG$. Then the classifying space $B\text{aut}_\circ(p)$ is rationally homotopy equivalent to the geometric realization of the dg Lie algebra

$$
\text{Hom}^\tau(CL, \Pi) \rtimes_{\tau_*} \left( \text{Der} L \rtimes_{\text{ad}} sL \right) \langle 1 \rangle.
$$

Here, $CL$ is the Chevalley-Eilenberg complex of $L$, we use $\langle n \rangle$ to indicate the $n$-connected cover, and the decorations $\tau$ and $\tau_*$ indicate that we take a twisted semidirect product (see §3.5 and §3.6).

In many cases of interest, there is an explicit formula for $\tau$ that yields a simplification of the model. For example, we have the following result in the case of complex vector bundles.

Theorem 1.2. Let $\xi$ be an $n$-dimensional complex vector bundle over a simply connected finite CW-complex $X$ and let $L$ be the minimal Quillen model for $X$. Then $B\text{aut}_\circ(\xi)$ is rationally homotopy equivalent to the geometric realization of the dg Lie algebra

$$
\left( H^*(X; \mathbb{Q}) \otimes \pi_*(\Omega BU(n)) \right) \langle 0 \rangle \rtimes_{\rho_*} \left( \text{Der} L \rtimes_{\text{ad}} sL \right) \langle 1 \rangle
$$

where $\rho = \sum_{i=1}^n c_i(\xi) \otimes \gamma_i$, and $\gamma_i$ is the generator for $\pi_{2i-1}(\Omega BU(n)) \otimes \mathbb{Q} = \pi_{2i}(BU(n)) \otimes \mathbb{Q}$ dual to the universal Chern class $c_i \in H^{2i}(BU(n); \mathbb{Q})$.

Similar simplifications are possible whenever $G$ is a compact connected Lie group or, more generally, when $H^*(BG; \mathbb{Q})$ is a free graded commutative algebra, see \[3\].
Remark 1.3. The fibration sequence of dg Lie algebras
\[ \text{Hom}^\tau(CL, \Pi) \langle 0 \rangle \to \text{Hom}^\tau(CL, \Pi) \langle 0 \rangle \rtimes_{\tau} (\text{Der } L \rtimes_{\text{ad}} sL) \langle 1 \rangle \to (\text{Der } L \rtimes_{\text{ad}} sL) \langle 1 \rangle \]
associated to the twisted semi-direct product \( (1) \) is a model for the homotopy fibration sequence
\[ \text{Baut}_X(p) \to \text{Baut}_0(p) \to \text{Baut}_0(X), \]
where \( \text{aut}_0(X) \) is the monoid of self-maps of \( X \) homotopic to the identity, and \( \text{aut}_X(p) \subseteq \text{aut}_0(p) \) is the submonoid of \( \text{aut}_0(p) \) where \( f \) is equal to the identity on \( X \). In particular, \( \text{Hom}^\tau(CL, \Pi) \langle 0 \rangle \) is a model for \( \text{Baut}_X(p) \). We should remark that rational models for \( \text{Baut}_X(p) \) have been studied earlier, see e.g. [4].

2. Moduli spaces of \( \mathcal{F} \)-fibrations

We will utilize the general framework for classification of fibrations provided by May [10]. Let \((\mathcal{F}, \mathcal{F})\) be a category of fibers in the sense of [10, Definition 4.1] and assume it satisfies the hypotheses of the classification theorem [10, Theorem 9.2]. Also recall the notions of \( \mathcal{F} \)-spaces and \( \mathcal{F} \)-maps from [3]. Let \( G \) denote the group-like topological monoid \( \mathcal{F}(\mathcal{F}, \mathcal{F}) \), to be thought of as the structure group for \( \mathcal{F} \)-fibrations.

In the special case when \( \mathcal{F} \) is the category of all spaces weakly equivalent to a given CW-complex \( X \), with morphisms all weak equivalences between such spaces, the ‘structure group’ \( G = \text{aut}(X) \) is the monoid of homotopy automorphisms of \( X \), and an \( \mathcal{F} \)-fibration is the same thing as a fibration with fiber weakly homotopy equivalent to \( X \). We will refer to such fibrations as \( X \)-fibrations.

Returning to the general situation, let \( p_\infty : E_\infty \to B_\infty \) denote the universal \( \mathcal{F} \)-fibration, the existence of which is ensured by May’s classification theorem, and define
\[ \text{Fib}(X, \mathcal{F}) = B(map(X, B_\infty), \text{aut}(X), *) , \]
where the right hand side denotes the geometric bar construction of the group-like monoid \( \text{aut}(X) \) acting on the space \( map(X, B_\infty) \) from the right by precomposition. It is a consequence of May’s ‘Classification of \( Y \)-structures’ [10, §11] that \( \text{Fib}(X, \mathcal{F}) \) may be thought of as a moduli space of \( \mathcal{F} \)-fibrations with base weakly equivalent to \( X \). More precisely, we have the following:

**Theorem 2.1.** For a CW-complex \( A \), there is a bijective correspondence between homotopy classes of maps
\[ A \to \text{Fib}(X, \mathcal{F}) \]
and equivalence classes of \( X \)-fibrations \( p : E \to A \) with a \( B_\infty \)-structure \( \theta : E \to B_\infty \).

*Proof.* This follows readily from [10, Theorem 11.1]. \( \square \)

In particular, since an \( X \)-fibration over a point is just a space weakly equivalent to \( X \), we see that the set of path components,
\[ \pi_0 \text{Fib}(X, \mathcal{F}), \]
is in bijective correspondence with the set of equivalence classes of \( \mathcal{F} \)-fibrations with base weakly homotopy equivalent to \( X \).

**Definition 2.2.** Given an \( \mathcal{F} \)-fibration \( p : E \to B \), let \( \text{aut}_\mathcal{F}(p) \) denote the space of \( \mathcal{F} \)-self equivalences of \( p \), i.e., the topological monoid consisting of commutative diagrams
\[ \begin{array}{ccc}
E & \xrightarrow{q} & E \\
\downarrow{p} & & \downarrow{p} \\
X & \xrightarrow{f} & X,
\end{array} \]
such that $f$ is a weak homotopy equivalence and $\varphi$ is a fiberwise $F$-map, topologized as a subset of $\text{map}(B,B) \times \text{map}(E,E)$. Let $\text{aut}_{\nu}^F(p) \subseteq \text{aut}^F(p)$ denote the submonoid consisting of those pairs $(f, \varphi)$ such that $f$ is homotopic to the identity map on $X$. If $D \subseteq C \subseteq X$ are subsets, then let $\text{aut}_{\nu}^C(p)$ denote the submonoid consisting of pairs as above such that $f$ restricts to the identity map on $C$, and write $\text{aut}_{\nu}^{D,F}(p)$, or simply $\text{aut}_{\nu}^D(p)$, for the submonoid of $\text{aut}_{\nu}^F(p)$ where $\varphi$ restricts to the identity isomorphism on the fibers over points in $D$. Finally, let $\text{aut}_{\nu}^{D,\ast}(p)$ denote $\text{aut}_{\nu}^D(p) \cap \text{aut}_{\nu}^{\ast}(p)$.

By using standard properties of the geometric bar construction, we can obtain information about the homotopy types of the components of $\text{Fib}(X,F)$.

**Theorem 2.3.**

1. There is a bijection
   $$\pi_0 \text{Fib}(X,F) \cong [X, B_{\infty}] / \pi_0 \text{aut}(X).$$

2. There is a weak equivalence of spaces over $\text{Baut}(X)$,
   $$\text{Fib}(X,F) \sim \coprod_{[p]} \text{Baut}^F(p),$$
   where the union is over all equivalence classes of $F$-fibrations $p: E \to B$, with $B$ weakly equivalent to $X$.

**Proof.** As follows from [10, Proposition 7.9], there is a homotopy fiber sequence
   $$\text{aut}(X) \to \text{map}(X, B_{\infty}) \to \text{Fib}(X,F) \to \text{Baut}(X).$$

The first statement follows by looking at the induced long exact sequence of homotopy groups.

The space of $F$-maps $\text{map}^F(p,p_{\infty})$ is weakly contractible for every $F$-fibration $p: E \to X$ by [3, Proposition 3.1]. Consider the diagram

$$
\begin{array}{ccc}
\text{aut}^F(p) & \longrightarrow & \text{map}^F(p,p_{\infty}) \longrightarrow \text{B}(\text{map}^F(p,p_{\infty}), \text{aut}^F(p), \ast) \longrightarrow \text{Baut}^F(p) \\
\downarrow & & \downarrow & & \downarrow \\
\text{aut}(X) & \longrightarrow & \text{map}(X,B_{\infty}) \longrightarrow \text{B}(\text{map}(X,B_{\infty}), \text{aut}(X), \ast) \longrightarrow \text{Baut}(X).
\end{array}
$$

According to [10, Proposition 7.9] the rows are quasi-fibration sequences. The leftmost square is homotopy cartesian. It follows that the third vertical map from the left induces a weak equivalence between the connected components containing $\nu$. Since $\text{map}^F(p,p_{\infty})$ is weakly contractible, the rightmost map in the top row is a weak homotopy equivalence. The rightmost square yields a zig-zag of weak homotopy equivalences showing $\text{Baut}^F(p) \sim \text{B}(\text{map}(X,B_{\infty}), \text{aut}(X), \ast)$, as spaces over $\text{Baut}(X)$, where $\nu$ indicates the component containing (the class of) $\nu$. \hfill $\Box$

**Corollary 2.4.** There are weak homotopy equivalences

$$\text{Baut}^F(p) \sim \text{B}(\text{map}(X,B_{\infty}), \text{aut}(X)_{[\nu]}, \ast),$$

$$\text{Baut}^F(p) \sim \text{B}(\text{map}(X,B_{\infty}), \text{aut}_\ast(X), \ast),$$

where $\text{aut}(X)_{[\nu]}$ denotes the monoid of homotopy equivalences $\varphi: X \to X$ such that $\nu \circ \varphi \simeq \nu$ and $\text{map}(X,B_{\infty})_{[\nu]}$ denotes the component of $\nu$.

**Proof.** We have just seen that $\text{Baut}^F(p) \sim \text{B}(\text{map}(X,B_{\infty}), \text{aut}(X), \ast)_{[\nu]}$. The latter is easily seen to be weakly equivalent to $\text{B}(\text{map}(X,B_{\infty}), \text{aut}(X)_{[\nu]}, \ast)$. The second claim is proved similarly. \hfill $\Box$
3. Rational models

This section contains the proof of the main theorem. We begin by examining the effect of \(Q\)-localization on the geometric bar construction. Then we will construct a dg Lie model for the \(Q\)-localized bar construction, by combining Schlessinger-Stasheff’s [12] and Tanré’s [13] theory of fibrations of dg Lie algebras with Quillen’s theory of principal dg coalgebra bundles [11].

3.1. Rationalization.

Lemma 3.1. Let \(X\) be a connected nilpotent finite CW-complex, let \(Z\) be a connected nilpotent space, and fix a map \(\nu: X \rightarrow Z\). Then \(B(\text{map}(X,Z)_\nu, aut_\circ(X),*)\) is rationally homotopy equivalent to

\[ B(\text{map}(X_Q,Z_Q)_\nu Q, aut_\circ(X_Q),*) \].

Proof. By using a functorial \(Q\)-localization for nilpotent spaces, e.g., the Bousfield-Kan \(Q\)-completion, we can construct a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\nu} & Z \\
\downarrow r & & \downarrow q \\
X_Q & \xrightarrow{\nu_Q} & Z_Q,
\end{array}
\]

where the vertical maps are \(Q\)-localizations. We may also assume that \(r\) is a cofibration. Define the monoid \(aut_\circ(r)\) as the pullback

\[
\begin{array}{ccc}
aut_\circ(r) & \xrightarrow{\sim} & aut_\circ(X) \\
\downarrow \sim_Q & & \downarrow \sim_Q \\
aut_\circ(X_Q) & \xrightarrow{\sim} & \text{map}(X,X_Q) r.
\end{array}
\]

Thus, the monoid \(aut_\circ(r)\) consists of pairs \((f,g)\) where \(f\) and \(g\) are self-maps homotopic to the identity of \(X\) and \(X_Q\), respectively, such that \(r \circ f = g \circ r\). Since \(r\) is a cofibration, the map \(r^*\) is a fibration. It is also a weak equivalence by standard properties of \(Q\)-localization. The map \(r^*\) is a rational homotopy equivalence by [7 Theorem II.3.11]. It follows that the projections from \(aut_\circ(r)\) to \(aut_\circ(X)\) and \(aut_\circ(X_Q)\) are a weak equivalence and a rational homotopy equivalence, respectively.

There are right actions of the monoid \(aut_\circ(r)\) on \(\text{map}(X,Z)\) and \(\text{map}(X_Q,Z_Q)\) through the projections to \(aut_\circ(X)\) and \(aut_\circ(X_Q)\), respectively. We get a zig-zag of rational homotopy equivalences of right \(aut_\circ(r)\)-spaces

\[
\text{map}(X,Z)_\nu \xrightarrow{\sim} \text{map}(X,Q)_\nu Q \xrightarrow{\sim} \text{map}(X_Q,Z_Q)_\nu Q.
\]

This accounts for the top horizontal zig-zag in the following diagram, where we write \(\bullet\) instead of \(B(\text{map}(X,Z)_\nu Q, aut_\circ(r),*)\) to save space,

\[
\begin{array}{ccc}
B(\text{map}(X,Z)_\nu Q, aut_\circ(r),*) & \xrightarrow{\sim_Q} & B(\text{map}(X_Q,Z_Q)_\nu Q, aut_\circ(r),*) \\
\downarrow \sim & & \downarrow \sim_Q \\
B(\text{map}(X,Z)_\nu Q, aut_\circ(X),*) & & B(\text{map}(X_Q,Z_Q)_\nu Q, aut_\circ(X_Q),*).
\end{array}
\]
3.2. Geometric realization of dg Lie algebras. Let $\mathfrak{g}$ be a dg Lie algebra over $\mathbb{Q}$, possibly unbounded as a chain complex. For $n \geq 0$, the $n$-connected cover is the dg Lie subalgebra $\mathfrak{g}(n) \subseteq \mathfrak{g}$ defined by

$$g(n)_i = \begin{cases} g_i, & i > n, \\ \ker(g_n \to g_{n-1}), & i = n, \\ 0, & i < n. \end{cases}$$

We call $\mathfrak{g}$ connected if $\mathfrak{g} = \mathfrak{g}(0)$ and simply connected if $\mathfrak{g} = \mathfrak{g}(1)$.

The lower central series of $\mathfrak{g}$ is the descending filtration

$$\mathfrak{g} = \Gamma^1 \mathfrak{g} \supseteq \Gamma^2 \mathfrak{g} \supseteq \cdots$$

characterized by $\Gamma^1 \mathfrak{g} = \mathfrak{g}$ and $[\Gamma^k \mathfrak{g}, \mathfrak{g}] = \Gamma^{k+1} \mathfrak{g}$. We call $\mathfrak{g}$ nilpotent if the lower central series terminates degree-wise, meaning that for every $n$, there is a $k$ such that $(\Gamma^k \mathfrak{g})_n = 0$. This definition of nilpotence mirrors the notion of nilpotence for topological spaces. Indeed, a connected dg Lie algebra $\mathfrak{g}$ is nilpotent if and only if the Lie algebra $\mathfrak{g}_0$ is nilpotent and the action of $\mathfrak{g}_0$ on $\mathfrak{g}_n$ is nilpotent for all $n$.

And, clearly, every simply connected dg Lie algebra is nilpotent.

If $\mathfrak{g}$ is an ordinary nilpotent Lie algebra, then $\exp(\mathfrak{g})$ denotes the nilpotent group whose underlying set is $\mathfrak{g}$ and where the group operation is given by the Campbell-Baker-Hausdorff formula, see e.g. [11]. The following generalizes this to dg Lie algebras. Let $\mathfrak{g}$ be a connected nilpotent dg Lie algebra. If $\Omega$ is a commutative cochain algebra, then the chain complex $\mathfrak{g} \otimes \Omega$ becomes a dg Lie algebra with

$$[x \otimes \alpha, y \otimes \beta] = (-1)^{\|\alpha\|\|y\|} [x, y] \otimes \alpha \beta$$

for $x, y \in \mathfrak{g}$ and $\alpha, \beta \in \Omega$. If $\Omega^n = 0$ unless $0 \leq k \leq n$ for some $n$, then the degree 0 component of $\mathfrak{g} \otimes \Omega$ decomposes as

$$(\mathfrak{g} \otimes \Omega)_0 = (\mathfrak{g}_0 \otimes \Omega^0) \oplus (\mathfrak{g}_1 \otimes \Omega^1) \oplus \cdots \oplus (\mathfrak{g}_n \otimes \Omega^n).$$

From the fact that $[\mathfrak{g}_0 \otimes \Omega^j, \mathfrak{g}_j \otimes \Omega^k] \subseteq \mathfrak{g}_{j+k} \otimes \Omega^{j+k}$ and that $\mathfrak{g}_0$ acts nilpotently on $\mathfrak{g}_c$ for all $k$, one sees that $(\mathfrak{g} \otimes \Omega)_0$ is a nilpotent Lie algebra. Hence, so is the Lie subalgebra of zero-cycles $Z_0(\mathfrak{g} \otimes \Omega)$.

Let $\Omega_\bullet$ be the simplicial commutative differential graded algebra where $\Omega_n$ is the Sullivan-de Rham algebra of polynomial differential forms on the $n$-simplex, see [5].

Since $\Omega_n^k = 0$ unless $0 \leq k \leq n$, the above construction may be applied levelwise to the simplicial dg Lie algebra $\mathfrak{g} \otimes \Omega_\bullet$.

**Definition 3.2.** Let $\mathfrak{g}$ be a connected nilpotent dg Lie algebra. We define $\exp_\bullet(\mathfrak{g})$ to be the simplicial nilpotent group

$$\exp_\bullet(\mathfrak{g}) = \exp Z_0(\mathfrak{g} \otimes \Omega_\bullet).$$

Next, we recall the definition of the nerve $MC_\bullet(\mathfrak{g})$ of a dg Lie algebra $\mathfrak{g}$. As we will see below, the nerve $MC_\bullet(\mathfrak{g})$ is a delooping of the simplicial group $\exp_\bullet(\mathfrak{g})$.

**Definition 3.3.** A Maurer-Cartan element in $\mathfrak{g}$ is an element $\tau$ of degree $-1$ such that

$$d(\tau) + \frac{1}{2}[\tau, \tau] = 0.$$

The set of Maurer-Cartan elements is denoted $MC(\mathfrak{g})$. The nerve of $\mathfrak{g}$ is the simplicial set

$$MC_\bullet(\mathfrak{g}) = MC(\mathfrak{g} \otimes \Omega_\bullet).$$

Define the geometric realization of a dg Lie algebra to be the geometric realization of its nerve,

$$|\mathfrak{g}| = |MC_\bullet(\mathfrak{g})|.$$
3.3. Geometric realization of dg coalgebras. Let $\Omega$ be a commutative cochain algebra over $\mathbb{Q}$. A dg coalgebra over $\Omega$ is a coalgebra in the symmetric monoidal category of $\Omega$-modules, i.e., a dg $\Omega$-module $C$ together with a coproduct and a counit,

$$\Delta: C \to C \otimes_{\Omega} C, \quad \epsilon: C \to \Omega,$$

such that the appropriate diagrams commute. We let $dgc(\Omega)$ denote the category of dg coalgebras over $\Omega$. If $C$ is a dg coalgebra over $\Omega$, we let $G(C)$ denote the set of group-like elements, i.e., elements $\xi \in C$ of degree 0 such that $\Delta(\xi) = \xi \otimes \xi$, $d(\xi) = 0$, $\epsilon(\xi) = 1$.

Given a dg coalgebra $C$ over $\mathbb{Q}$, the free $\Omega$-module $C \otimes \Omega$ is a dg coalgebra over $\Omega$. Clearly, $\Omega \mapsto G(C \otimes \Omega)$ defines a functor from commutative cochain algebras to sets.

**Definition 3.4.** Let $C$ be a dg coalgebra. We defined the *spatial realization* of $C$ to be the simplicial set $\langle C \rangle = G(C \otimes \Omega_{\bullet})$.

A dg Lie algebra over $\Omega$ is a dg $\Omega$-module $L$ together with a Lie bracket $\ell: L \otimes \Omega L \to L$ satisfying the usual anti-symmetry and Jacobi relations. Quillen’s generalization of the Chevalley-Eilenberg construction can be extended to dg Lie algebras over $\Omega$, yielding a functor $C_{\Omega}: dgl(\Omega) \to dgc(\Omega)$.

The underlying coalgebra $C_{\Omega}(L)$ is the symmetric coalgebra $S_{\Omega}(sL)$, where

$$S_{\Omega}(V) = \bigoplus_{k \geq 0} (V \otimes \Omega^{k})_{n},$$

for an $\Omega$-module $V$. The differential is defined as usual, see e.g., [5, p.301]. If $L$ is a dg Lie algebra over $\mathbb{Q}$, then $L \otimes \Omega$ is a dg coalgebra over $\Omega$ and there is a natural isomorphism of dg coalgebras over $\Omega$,$$
C_{\Omega}(L \otimes \Omega) \cong C(L) \otimes \Omega.$$

**Proposition 3.5.** Let $L$ be a connected dg Lie algebra and let $\Omega$ be a bounded commutative cochain algebra. There is a natural bijection

$$\epsilon: MC(L \otimes \Omega) \cong G(C_{\Omega}(L \otimes \Omega)), \quad \epsilon(\tau) = \sum_{k \geq 0} \frac{1}{k!} s^k \tau \wedge \Omega^k \in C_{\Omega}(L \otimes \Omega).$$

**Proof.** The crucial observation is that this series converges since $\Omega$ is bounded and $L$ is connected. Say $\Omega^{k} = 0$ unless $0 \leq k \leq n$. Then

$$(L \otimes \Omega)^{-1} = L_{0} \otimes \Omega^{1} \oplus \cdots \oplus L_{n-1} \otimes \Omega^{n},$$

whence $\tau \in L \otimes \Omega^{n}$ for every element $\tau$ of degree $-1$. Since $(\Omega^{+})^{k} = 0$ for $k > n$, this implies that

$$s\tau \wedge \Omega \cdots \wedge \Omega s\tau = 0$$

whenever there are more than $n$ factors. Clearly, $\Delta(\epsilon(\tau)) = \epsilon(\tau) \otimes \epsilon(\tau)$ and $\epsilon(\epsilon(\tau)) = 0$. As the reader may check, the equation $d(\epsilon(\tau)) = 0$ is equivalent to the Maurer-Cartan equation for $\tau$. □
Corollary 3.6. There is a natural isomorphism of simplicial sets,
\[ \mathcal{M}_\bullet(L) \cong \langle C(L) \rangle, \]
for every connected dg Lie algebra \( L \).

Proof. Indeed, \( \mathcal{M}_\bullet(L) = \mathcal{M}(L \otimes \Omega) \cong \mathcal{G}(C(L \otimes \Omega)) \cong \mathcal{G}(C(L) \otimes \Omega). \) □

Recall that for a commutative dg algebra \( A \), the spatial realization is defined by
\[ \langle A \rangle = \text{Hom}_{dga}(A, \Omega), \]
see e.g., [1]. We use the same notation as for the coalgebra realization, but it should be clear from the context which one is used.

Proposition 3.7. Let \( A \) be a commutative cochain algebra of finite type with dual dg coalgebra \( A^\vee \). Then there is a natural isomorphism
\[ \langle A^\vee \rangle \cong \langle A \rangle. \]

Proof. For a bounded commutative cochain algebra \( \Omega \) and a finite type dg algebra \( A \), there is a natural isomorphism of chain complexes
\[ A^\vee \otimes \Omega \cong \text{Hom}(A, \Omega). \]
Under this isomorphism, group-like elements in the dg coalgebra \( A^\vee \otimes \Omega \) correspond to morphisms of dg algebras \( A \to \Omega ). □

Note that the spatial realization of dg coalgebras preserves products, \( \langle C \otimes D \rangle \cong \langle C \rangle \times \langle D \rangle \). In particular, since the universal enveloping algebra \( U g \) of a dg Lie algebra \( g \) is a dg Hopf algebra, i.e., a group object in the category of dg coalgebras, its spatial realization \( U g \) is a simplicial group. We also remark that for every commutative cochain algebra \( \Omega \), the forgetful functor \( dga(\Omega) \to dgl(\Omega) \) admits a left adjoint \( U_\Omega : dgl(\Omega) \to dga(\Omega). \)

Proposition 3.8. Let \( g \) be a simply connected dg Lie algebra. There is a natural isomorphism of simplicial groups
\[ \text{exp}_\bullet(g) \cong \langle Ug \rangle. \]

Proof. Let \( \Omega \) be a bounded commutative cochain algebra, say \( \Omega^k = 0 \) unless \( 0 \leq k \leq n \). Observe that there is a canonical isomorphism \( U g \otimes \Omega \cong U_{\Omega}(g \otimes \Omega) \). The isomorphism is effected by the exponential map
\[ \exp : Z_0(g \otimes \Omega) \to \mathcal{G}U_{\Omega}(g \otimes \Omega), \]
\[ \exp(x) = \sum_{k \geq 0} \frac{1}{k!} x^k, \]
where the product \( x^k \) is taken in \( U_{\Omega}(g \otimes \Omega) \). The crucial point is that the sum converges. Indeed, since \( g \) is simply connected,
\[ (g \otimes \Omega)_0 = g_1 \otimes \Omega^1 + \cdots + g_n \otimes \Omega^n, \]
so \( x \in g \otimes \Omega^r \), whence \( x^k = 0 \) for \( k > n \), whenever \( x \) is an element of degree 0. The fact that \( \exp \) respects the group structure is essentially by design of the Campbell-Baker-Hausdorff group structure. □
3.4. Principal dg coalgebra bundles. Next, recall Quillen’s theory of principal
dg coalgebra bundles [111 Appendix B, §5]. In particular, recall that \( C(\mathfrak{g}) \) serves as
a classifying space for principal \( \mathfrak{g} \)-bundles. Quillen’s universal principal \( \mathfrak{g} \)-bundle
may be identified with
\[
U\mathfrak{g} \rightarrow C(U\mathfrak{g}; \mathfrak{g}) \rightarrow C(\mathfrak{g}),
\]
where \( U\mathfrak{g} \) is the universal enveloping algebra of \( \mathfrak{g} \) and \( C(U\mathfrak{g}; \mathfrak{g}) \) is the Chevalley-Eilenberg complex of \( \mathfrak{g} \) with coefficients in the right \( \mathfrak{g} \)-module \( U\mathfrak{g} \).

**Theorem 3.9.** Let \( \mathfrak{g} \) be a simply connected dg Lie algebra of finite type. The
realization of the universal principal \( \mathfrak{g} \)-bundle,
\[
\langle U\mathfrak{g} \rangle \rightarrow \langle C(U\mathfrak{g}; \mathfrak{g}) \rangle \rightarrow \langle C(\mathfrak{g}) \rangle,
\]
is a universal principal \( \langle U\mathfrak{g} \rangle \)-bundle.

**Proof.** This is proved in [5, Chapter 25]. Indeed, when \( \mathfrak{g} \) is simply connected and
of finite type, the coalgebra realization of \( U\mathfrak{g} \) is the same as the algebra realization
of the dual dg algebra \( U\mathfrak{g}^\vee \).

**Corollary 3.10.** Let \( \mathfrak{g} \) be a simply connected dg Lie algebra of finite type. The
nerve \( MC_\bullet(\mathfrak{g}) \) is a delooping of the simplicial group \( \exp_\bullet(\mathfrak{g}) \).

**Proof.** We have the isomorphisms \( \exp_\bullet(\mathfrak{g}) \cong \langle U\mathfrak{g} \rangle \) and \( MC_\bullet(\mathfrak{g}) \cong \langle C(\mathfrak{g}) \rangle \).

**Remark 3.11.** Since we work with coalgebras, the finite type hypothesis on \( \mathfrak{g} \) can
be dropped in Theorem 3.9 and Corollary 3.10. However, we will not repeat the
lengthy argument here since \( \mathfrak{g} \) will be of finite type in our applications.

3.5. Twisted semi-direct products and Borel constructions. We begin by
recalling certain aspects of Tanré’s classification of fibrations in the category of dg
Lie algebras [13, Chapitre VII].

**Definition 3.12.** Let \( \mathfrak{g} \) and \( L \) be dg Lie algebras. An outer action of \( \mathfrak{g} \) on \( L \)
consists of two maps
\[
\alpha : L \otimes \mathfrak{g} \rightarrow L, \quad \xi : \mathfrak{g} \rightarrow L,
\]
satisfying the following conditions for all \( x, y \in \mathfrak{g} \) and \( a, b \in L \), where we write
\[
a \cdot x = \alpha(a \otimes x), \quad x \cdot a = (-1)^{|a||x|} a \cdot x.
\]
Firstly, the map \( \alpha \) defines an action of \( \mathfrak{g} \) on \( L \) by derivations, i.e.,
\[
[x, y] \cdot a = x \cdot (y \cdot a) - (-1)^{|y||x|} y \cdot (x \cdot a),
\]
\[
x \cdot [a, b] = [x \cdot a, b] + (-1)^{|x||a|} [a, x \cdot b].
\]
Secondly, the map \( \xi \) is a chain map of degree \(-1\) and a derivation, i.e.,
\[
d\xi(x) = -\xi(dx),
\]
\[
\xi[x, y] = \xi(x) \cdot y + (-1)^{|x||y|} x \cdot \xi(y).
\]
Finally, the action and \( \xi \) are connected by the equation
\[
d(x \cdot a) = d(x) \cdot a + (-1)^{|x||a|} x \cdot d(a) + [\xi(x), a].
\]

**Definition 3.13.** Given an outer action of \( \mathfrak{g} \) on \( L \), the twisted semi-direct product
\( L \rtimes_{\xi} \mathfrak{g} \) is the dg Lie algebra whose underlying graded Lie algebra is the semi-direct
product of \( \mathfrak{g} \) acting on \( L \),
\[
[\alpha(a, x), \alpha(b, y)] = ([a, b] + x \cdot b + a \cdot y, [x, y]),
\]
and whose differential is twisted by \( \xi \) in the sense that
\[
\partial^\xi(a, x) = (da + \xi(x), dx).
\]
The twisted semi-direct product is the total space in a short exact sequence (i.e. fibration sequence) of dg Lie algebras,
\[ 0 \to L \to L \rtimes \xi \to g \to 0. \]
The section \( g \to L \rtimes \xi \to g \), \( x \mapsto (0, x) \), is a morphism of graded Lie algebras, but it commutes with differentials if and only if \( \xi = 0 \).

Outer actions on \( L \) are classified by the dg Lie algebra \( \text{Der} L \rtimes \text{ad} \), whose underlying graded Lie algebra is the semi-direct product of \( \text{Der} L \) acting on the abelian dg Lie algebra \( sL \) from the left by \( \theta.sx = (-1)^{|\theta|}s\theta(x) \), and whose differential is given by \( \partial(\theta, sx) = (\partial(\theta) + \text{ad}_x, -sd(x)) \), where \( \text{ad}_x \in \text{Der} L \) is given by \( \text{ad}_x(y) = [x, y] \).

**Proposition 3.14.** Specifying an outer action of \( g \) on \( L \) is tantamount to specifying a morphism of dg Lie algebras \( \phi: g \to \text{Der} L \rtimes \text{ad} sL \).

The correspondence is given by \( \phi(x) = (\theta_x, -s\xi(x)) \), where \( \theta_x(a) = x.a \).

**Proof.** The proof is a straightforward calculation. \( \square \)

An outer action of \( g \) on \( L \) defines an action of \( g \) on \( C(L) \) by coderivations by the following formula:
\[
(sa_1 \wedge \cdots \wedge sa_n).x = sa_1 \wedge \cdots \wedge sa_n \wedge s\xi(x) + \sum_{i=1}^n \pm sa_1 \wedge \cdots \wedge s(a_i, x) \wedge \cdots \wedge sa_n.
\]
Equivalently, \( C(L) \) becomes a \( Ug \)-module coalgebra, i.e., a right \( Ug \)-module such that the structure map \( C(L) \otimes Ug \to C(L) \) is a morphism of dg coalgebras.

The action of \( \text{Der} L \rtimes_{\text{ad}} sL \) by coderivations on \( C(L) \), derived from the tautological outer action on \( L \), gives rise to a morphism of dg Lie algebras that we will denote
\[ \chi: \text{Der} L \rtimes_{\text{ad}} sL \to \text{Coder} C(L). \]

**Theorem 3.15.** Let \( g \) be a simply connected dg Lie algebra of finite type with an outer action on a connected dg Lie algebra \( L \). There is a right action of the simplicial group \( G = \exp_*(g) \) on \( \text{MC}_*(L) \) and a weak equivalence of simplicial sets over \( \text{MC}_*(g) \),
\[ \text{MC}_*(L \rtimes \xi \to g) \sim \text{MC}_*(L) \times_G \text{EG}. \]

**Proof.** The action of \( g \) on \( C(L) \) makes \( C(L) \) into a right \( Ug \)-module coalgebra. This yields a right action of \( \exp_*(g) \cong (Ug) \) on \( \text{MC}_*(L) \cong (C(L)) \). The key observation, which may be checked by hand, is that there is an isomorphism of dg coalgebras
\[ C(L \rtimes \xi \to g) \cong C(C(L); g) . \]
Secondly, we have the standard isomorphism
\[ C(C(L); g) \cong C(L) \otimes_{Ug} C(Ug; g) . \]
By combining these isomorphisms and taking realizations, we get isomorphisms of simplicial sets
\[ \mathbf{MC}(L \rtimes \mathfrak{g}) \cong \langle C(L) \rtimes \xi \mathfrak{g} \rangle \cong \langle C(L) \otimes U \mathfrak{g} C(U \mathfrak{g}; \mathfrak{g}) \rangle \cong \langle C(U \mathfrak{g}; \mathfrak{g}) \rangle. \]
By Theorem 3.9, the simplicial set \( \langle C(U \mathfrak{g}; \mathfrak{g}) \rangle \) is a model for \( E \mathfrak{g} \). This finishes the proof. \( \square \)

Let \( L \) be a simply connected cofibrant dg Lie algebra of finite type with geometric realization \( X = \left| \mathbf{MC}(L) \right| \), and consider the simply connected dg Lie algebra
\[ \mathfrak{g} = \left( \text{Der} L \rtimes \text{ad}_L \right)(1), \]
with associated topological group
\[ G = \left| \exp(\mathfrak{g}) \right|. \]
There is an evident outer action of \( \mathfrak{g} \) on \( L \), whence an action of the simplicial group \( \exp(\mathfrak{g}) \) on the nerve \( \mathbf{MC}(L) \), cf. Theorem 3.15, whence an action of \( G \) on \( X \). Since \( \mathfrak{g} \) is simply connected, the simplicial group \( \exp(\mathfrak{g}) \) is reduced, i.e., has only one vertex. In particular, the topological group \( G \) is connected. Therefore, the action yields a map of grouplike monoids
\[ G \to \text{aut}_o(X). \]
This map is a weak homotopy equivalence, as follows from, e.g., Tanré’s theory [13, Chapitre VII].

3.6. Twisting functions and mapping spaces. Let \( C \) be a dg coalgebra with coproduct \( \Delta: C \to C \otimes C \) and let \( L \) a dg Lie algebra with Lie bracket \( \ell: L \otimes L \to L \). Recall that a twisting function \( \tau: C \to L \) is a Maurer-Cartan element in the dg Lie algebra \( \text{Hom}(C, L) \), whose differential and Lie bracket are given by
\[ \partial(f) = d_L \circ f - (-1)^{|f|} f \circ d_C, \]
\[ [f, g] = \ell \circ (f \otimes g) \circ \Delta. \]
If \( \tau \) is a twisting function, then \( \text{Hom}(C, L) \) denotes the dg Lie algebra with the same underlying graded Lie algebra but twisted differential
\[ \delta^\tau(f) = \partial(f) + \tau \circ f. \]
Furthermore, there is an outer action of \( \text{Coder} C \) on \( \text{Hom}(C, L) \) given by
\[ f \cdot \theta = f \circ \theta, \quad \xi(\theta) = \tau_c(\theta) = \tau \circ \theta, \]
for \( f \in \text{Hom}(C, L) \) and \( \theta \in \text{Coder} C \). We note for future reference that we may make the identification
\[ (\text{Hom}(C, L) \rtimes \text{Coder} C) = \text{Hom}(C, L) \rtimes_{\tau_c} \text{Coder} C \]
for every twisting function \( \tau: C \to L \).

**Theorem 3.16.** Let \( L \) and \( \Pi \) be connected dg Lie algebras and suppose \( \Pi \) is nilpotent and of finite type. There is a natural weak homotopy equivalence of simplicial sets
\[ \mathbf{MC}(\text{Hom}(C, \Pi \otimes \Omega_*)) \simeq \text{map}(\mathbf{MC}(L), \mathbf{MC}(\Pi)). \]
Proof. Let $\Omega$ be a bounded commutative cochain algebra. We define a natural map
\[
\text{MC Hom}(C(L), \Pi \otimes \Omega) \times \text{MC}(L \otimes \Omega) \to \text{MC}(\Pi \otimes \Omega)
\]
as follows. First, make the identifications
\[
\text{Hom}(C(L), \Pi \otimes \Omega) = \text{Hom}_\Omega(C(L \otimes \Omega), \Pi \otimes \Omega),
\]
the second of which is justified by Proposition 3.5, and then define
\[
\epsilon : \text{MC Hom}_\Omega(C(L \otimes \Omega), \Pi \otimes \Omega) \times \mathcal{G}(C(L \otimes \Omega)) \to \text{MC}(\Pi \otimes \Omega),
\]
simply by evaluation,
\[
\epsilon(\tau, \xi) = \tau(\xi).
\]
We need to verify that $\tau(\xi)$ satisfies the Maurer-Cartan equation. Since $\tau$ is a
\[
0 = \partial(\tau) + \frac{1}{2} [\tau, \tau].
\]
Evaluating both sides at the group-like element $\xi$ yields
\[
0 = d\tau(\xi) + \tau(d(\xi)) + \frac{1}{2} \ell(\tau \otimes \tau) \circ \Delta(\xi) = d\tau(\xi) + \frac{1}{2} [\tau(\xi), \tau(\xi)],
\]
showing that $\tau(\xi)$ satisfies the Maurer-Cartan equation.

The map is clearly natural in $\Omega$ and yields a simplicial map
\[
\text{MC Hom}(C(L), \Pi \otimes \Omega) \times \text{MC}(L \otimes \Omega) \to \text{MC}(\Pi \otimes \Omega).
\]
The map in the theorem is defined to be the adjoint of this map.

To show it is a weak homotopy equivalence, one argues as in [1, Theorem 6.6]
by induction on a suitable complete filtration of $\Pi$. The proof is entirely analogous
so we omit the details. □

Remark 3.17. The dg Lie algebra $\text{Hom}(C(L), \Pi)$ with the descending filtration
\[
F^{r+1} = \text{Hom}(C(L), \Pi(r)), \quad r \geq 0,
\]
is a complete dg Lie algebra in the sense of [1, Definition 5.1]. By [1, Theorem 6.3]
(see also Definition 5.3 and Remark 6.4 in loc.cit.), the Kan complex
\[
\widehat{\text{MC}}(\text{Hom}(C(L), \Pi)) = \lim_{\leftarrow} \text{MC}(\Pi/\Pi(r))
\]
is homotopy equivalent to $\text{map}(\text{MC}_\bullet(L), \text{MC}_\bullet(\Pi))$. We would like to remark how
this relates to the statement in Theorem 3.16.

Since $\Pi/\Pi(r)$ is finite dimensional for all $r$, we have
\[
\text{Hom}(C(L), \Pi/\Pi(r)) \otimes \Omega_* \cong \text{Hom}(C(L), \Pi/\Pi(r) \otimes \Omega_*)
\]
Upon taking the inverse limit, we get an isomorphism of simplicial sets
\[
\widehat{\text{MC}}(\text{Hom}(C(L), \Pi)) \cong \text{MC}_\bullet(\text{Hom}(C(L), \Pi \otimes \Omega_*))
\]
Thus, Theorem 3.16 and Theorem 6.3 in [1] say the same thing. The advantage
of Theorem 6.3 is that the explicit formula for the map gives us control over
equivariance properties, as we will see next.

Let $L$ be a simply connected cofibrant dg Lie algebra of finite type. Precompo-
sition defines a right action of the dg Lie algebra $\text{Coder} C(L)$ on the complete dg
Lie algebra $\text{Hom}(C(L), \Pi)$. By composing with (3), and restricting to the simply
connected cover, we get an action of the dg Lie algebra
\[
\mathfrak{g} = (\text{Der} L \ltimes_{\text{ad}} sL)(1)
\]
on \(\text{Hom}(C(L),\Pi)\). By Theorem 3.15 this induces an action of the simplicial group \(\exp_\bullet(g)\) on the simplicial set \(\text{MC Hom}(C(L),\Pi \otimes \Omega_\ast) \cong \overline{\text{MC}}_\bullet(\text{Hom}(C(L),\Pi))\). On the other hand, \(\exp_\bullet(g)\) acts on \(\overline{\text{MC}}_\bullet(L)\) and hence also on \(\text{map}(\overline{\text{MC}}_\bullet(L),\overline{\text{MC}}_\bullet(\Pi))\).

The following is an important addendum to Theorem 3.16

**Proposition 3.18.** The weak equivalence of Theorem 3.16,

\[
\text{MC Hom}(C(L),\Pi \otimes \Omega_\ast) \sim \text{map}(\overline{\text{MC}}_\bullet(L),\overline{\text{MC}}_\bullet(\Pi)),
\]

is equivariant with respect to the action of the simplicial group \(\exp_\bullet(g)\).

**Proof.** The proof boils down to the easily checked fact that the map \(\epsilon\) in the proof of Theorem 3.16 satisfies

\[
\epsilon(\theta,f,\xi) = \epsilon(f,\xi,\theta),
\]

for \(\theta \in \mathcal{G}U_{\Omega_\ast}(g \otimes \Omega_\ast), f \in \text{MC Hom}_{\Omega}(C_{\Omega}(L \otimes \Omega),\Pi \otimes \Omega)\) and \(\xi \in \mathcal{G}(C_{\Omega}(L \otimes \Omega))\). \(\square\)

**Proposition 3.19.** Let \(X_Q\) and \(Z_Q\) be \(\mathbb{Q}\)-local connected nilpotent spaces of finite \(\mathbb{Q}\)-type. Let \(L\) be a finite type cofibrant dg Lie algebra model for \(X_Q\) and let \(\Pi\) be any dg Lie model for \(Z_Q\). The geometric bar construction,

\[
B(\text{map}(X_Q,Z_Q),\text{aut}_c(X_Q),\ast),
\]

is weakly homotopy equivalent to the geometric realization of the dg Lie algebra

\[
\text{Hom}(CL,\Pi) \times (\text{Der } L \rtimes \text{ad } sL)\langle 1 \rangle.
\]

**Proof.** We may as well assume \(X_Q = \overline{\text{MC}}_\bullet(L)\) and \(Z_Q = \overline{\text{MC}}_\bullet(\Pi)\). By Theorem 3.15 there is a weak homotopy equivalence

\[
\overline{\text{MC}}_\bullet(\text{Hom}(C(L),\Pi \otimes g) \sim B(\overline{\text{MC}}_\bullet(\text{Hom}(C(L),\Pi)),\exp_\bullet(g),\ast).
\]

The weak equivalence \(\exp_\bullet(g) \to \text{aut}_c(X)\) of group-like simplicial monoids and the weak equivalence of \(\exp_\bullet(g)\)-spaces of Proposition 3.18 combine to give a weak homotopy equivalence

\[
B(\overline{\text{MC}}_\bullet(\text{Hom}(C(L),\Pi)),\exp_\bullet(g),\ast) \sim B(\text{map}(X_Q,Z_Q),\text{aut}_c(X_Q),\ast).
\]

\(\square\)

### 3.7. Proof of the main result.

**Theorem 3.20.** Suppose that \(\mathcal{F}\) is a category of fibers such that the classifying space \(B_{\infty}\) is connected and nilpotent. Let \(p: E \to X\) be an \(\mathcal{F}\)-fibration over a simply connected finite CW-complex \(X\). Let \(L\) be a simply connected cofibrant dg Lie algebra model for \(X\) and let \(\Pi\) be a connected nilpotent dg Lie algebra model for \(B_{\infty}\). Let \(\tau: CL \to \Pi\) be a twisting function that models the map \(\nu: X \to B_{\infty}\) that classifies \(p\).

Then the classifying space \(B_{\infty}(\mathcal{F},\nu)\) is rationally homotopy equivalent to the geometric realization of the dg Lie algebra

\[
\text{Hom}^\tau(CL,\Pi)\langle 0 \rangle \rtimes_{\tau}(\text{Der } L \rtimes \text{ad } sL)\langle 1 \rangle.
\]

**Proof.** For notational convenience, let \(Z = B_{\infty}\). As before, let

\[
g = (\text{Der } L \rtimes \text{ad } sL)\langle 1 \rangle.
\]

That the dg Lie algebras \(L\) and \(\Pi\) are models for \(X\) and \(Z\) means that we may use their geometric realizations as models for the \(\mathbb{Q}\)-localizations of \(X\) and \(Z\);

\[
X_Q = |\overline{\text{MC}}_\bullet(L)|, \quad Z_Q = |\overline{\text{MC}}_\bullet(\Pi)|.
\]

By Corollary 2.4 and Lemma 3.11 we have

\[
B_{\infty}(\mathcal{F},\nu) \sim B(\text{map}(X,Z)_\nu,\text{aut}_c(X),\ast) \sim_Q B(\text{map}(X_Q,Z_Q)_{\nu_Q},\text{aut}_c(X_Q),\ast).
\]
The latter space is weakly homotopy equivalent to the component
\[ B(map(X, Z), aut_\circ(X, *))_{\nu_0}. \]
By Proposition 3.19
\[ B(map(X, Z), aut_\circ(X, *)) \sim \hat{MC}(\text{Hom}(CL, \Pi) \rtimes g). \]
Let \( \tau : CL \to \Pi \) be a twisting function that corresponds to \( \nu_0 \). It follows from \[ \text{Corollary 1.3} \] that the component
\[ \hat{MC}^\tau(\text{Hom}(CL, \Pi) \rtimes g). \]
Finally, as in (5) one checks that there is an isomorphism of dg Lie algebras
\[ \text{Hom}^\tau(CL, \Pi) \rtimes g \cong \text{Hom}^\tau(CL, \Pi)(0) \rtimes \tau(g). \]
This finishes the proof. \( \square \)

**Remark 3.21.** It is straightforward to derive the following variants of the main result: If \( A \subseteq X \) is a subspace such that the inclusion of \( A \) into \( X \) is a cofibration, then we may consider the submonoid \( aut_{F, A}(p) \subseteq aut_{F}(p) \) where the homotopy automorphism of the base restricts to the identity map on \( A \). If \( g_A \to (\text{Der} L \rtimes \text{ad} sL)(1) \) is a dg Lie algebra morphism that models the map \( Baut_{A, \circ}(X) \to Baut_{\circ}(X) \), then
\[ \text{Hom}^\tau(CL, \Pi)(0) \rtimes \tau(g_A) \]
is a dg Lie algebra model for the space \( Baut_{A, \circ}(p) \). Similarly, if we pick a base-point \( \ast \in A \subseteq X \), and let \( aut_{\ast, A, \circ}(p) \) denote the submonoid of \( aut_{A, \circ}(p) \) where the automorphism of the total space restricts to the identity over the base-point, then one gets a model for \( Baut_{\ast, \circ}(p) \) by replacing \( CL \) with the reduced Chevalley-Eilenberg chains.

4. **Examples and applications**

Many classifying spaces of interest in geometry have simple rational homotopy types:

- If \( G \) is a compact connected Lie group, then \( H^*(BG; \mathbb{Q}) \) is a polynomial algebra on finitely many generators of even degree (see, e.g., [6] Theorem 1.81).
- The stable classifying spaces \( BO, BTop, BPL \) are infinite loop spaces and have rational cohomology rings of the form \( \mathbb{Q}[p_1, p_2, \ldots] \), where \( p_i \) is a generator of degree \( 4i \) (see, e.g., [9]).
- Halperin’s conjecture, which has been verified in many cases, asserts that \( H^*(Baut_{\circ}(X); \mathbb{Q}) \) is a polynomial algebra whenever \( X \) is an elliptic space with non-zero Euler characteristic.

In this section, we will provide a simplification of the model arrived at in the case when \( H^*(B_\infty; \mathbb{Q}) \) is a polynomial algebra with finitely many generators in each degree.

Call the generators \( p_1, p_2, \ldots \) and let \( d_i = |p_i| \). Let \( X \) be a simply connected finite CW complex together with an \( F \)-bundle classified by a map
\[ \nu : X \to B_\infty. \]
The characteristic classes of the bundle are defined by pulling back the universal classes \( p_i \) along \( \nu \):
\[ p_i(\nu) = \nu^*(p_i) \in H^{d_i}(X; \mathbb{Q}). \]
A dg Lie algebra model for $B_{\infty}$ is provided by the abelian dg Lie algebra with zero differential

$$\Pi = \pi_{\ast}(\Omega B_{\infty}) \otimes \mathbb{Q}.$$  

This graded vector space is spanned by classes $\pi_i \in \pi_{d-1}(\Omega B_{\infty}) \otimes \mathbb{Q} = \pi_{d}(\Omega B_{\infty}) \otimes \mathbb{Q}$ that are dual to $p_i$ under the Hurewicz pairing between cohomology and homotopy.

Let $L$ denote the minimal Quillen model of $X$. Recall that it has the form

$$L = (\mathcal{L}, \delta),$$

where $V = s^{-1}H_\ast(X; \mathbb{Q})$ and the differential $\delta$ is decomposable in the sense that $\delta(L) \subseteq [L, L]$. Thus, the suspension of the space of indecomposables, $sL/[L, L]$, may be identified with $\tilde{H}_\ast(X; \mathbb{Q})$.

We will work with based maps, so in this section we let $\mathcal{C}(L)$ denote the reduced Chevalley-Eilenberg chains on a dg Lie algebra $L$. It is defined as the cokernel of the coaugmentation $\eta: Q \rightarrow \mathcal{C}(L)$.

The restriction of $\mathcal{C}$ to $\text{Der} L$ gives a morphism of dg Lie algebras

$$\chi: \text{Der} L \rightarrow \text{Coder} \mathcal{C}(L).$$

In particular, $\mathcal{C}(L)$ is a module over $\text{Der} L$. Moreover, the universal twisting function $\tau_L: \mathcal{C}(L) \rightarrow L$ is a morphism of $\text{Der} L$-modules. Indeed, for $\theta \in \text{Der} L$, the coderivation $\chi(\theta) \in \text{Coder} \mathcal{C}(L)$ is characterized by commutativity of the diagram

$$\begin{array}{ccc}
\mathcal{C}(L) & \xrightarrow{\chi(\theta)} & \mathcal{C}(L) \\
\downarrow{\tau_L} & & \downarrow{\tau_L} \\
L & \xrightarrow{\theta} & L,
\end{array}$$

which means that $\tau_L$ is a morphism of $\text{Der} L$-modules.

Consider the composite of the universal twisting function and the abelianization,

$$\mathcal{C}(L) \xrightarrow{\tau_L} L \xrightarrow{a} L/[L, L].$$

Since $a$ is a morphism of dg Lie algebras, this composite is a twisting function. But a twisting function with abelian target is the same thing as a chain map of degree $-1$. Thus, for every dg Lie algebra $L$, there is a canonical chain map

$$g_L: \mathcal{C}(L) \rightarrow sL/[L, L].$$

Moreover, $g_L$ is a morphism of $\text{Der} L$-modules. Indeed, we have just seen that $\tau_L$ is a morphism of $\text{Der} L$-modules, and $a: L \rightarrow L/[L, L]$ is obviously a morphism of $\text{Der} L$-modules.

**Proposition 4.1.** If $L$ is a cofibrant dg Lie algebra, then the canonical map

$$g_L: \mathcal{C}(L) \rightarrow sL/[L, L]$$

is a quasi-isomorphism.

**Proof.** See, e.g., [5, Proposition 22.8] \hfill \Box

Next, let $L$ be a cofibrant minimal dg Lie algebra model for $X$ and let $\Pi$ denote the abelian graded Lie algebra $\pi_{\ast}(\Omega B_{\infty}) \otimes \mathbb{Q}$. Consider the degree $-1$ map of graded vector spaces

$$\rho: \tilde{H}_\ast(X; \mathbb{Q}) \rightarrow \pi_{\ast}(\Omega B_{\infty}) \otimes \mathbb{Q},$$

$$e \mapsto \sum_i (p_i(e), e)\pi_i,$$
where $\langle \cdot, \cdot \rangle$ denotes the standard pairing between homology and cohomology (and $\langle p, e \rangle = 0$ unless $e$ and $p$ have the same degree).

**Proposition 4.2.** The composite map

$$\tau: \tilde{C}(L) \xrightarrow{\delta_c} sL/[L, L] = \tilde{H}_*(X; \mathbb{Q}) \xrightarrow{\rho} \pi_*(\Omega B_\infty) \otimes \mathbb{Q}$$

is a twisting function that models the map $\nu: X \to B_\infty$.

**Proof.** The map

$$\psi: L \xrightarrow{\alpha} L/[L, L] = s^{-1} \tilde{H}_*(X; \mathbb{Q}) \xrightarrow{\rho} \pi_*(\Omega B_\infty) \otimes \mathbb{Q}$$

is a dg Lie model for $\nu: X \to B_\infty$. Hence, composing with the universal twisting function we get a twisting function that models $\nu$:

$$\tilde{C}(L) \xrightarrow{\tau_c} L \xrightarrow{\psi} \pi_*(\Omega B_\infty) \otimes \mathbb{Q}.$$ 

This composite map is the same as the map in the statement of the proposition. \(\square\)

**Theorem 4.3.** The classifying space $Baut^*_{\infty}(\nu)$ is rationally homotopy equivalent to the geometric realization of the dg Lie algebra

$$\text{Hom}(\tilde{H}_*(X; \mathbb{Q}), \pi_*(\Omega B_\infty) \otimes \mathbb{Q})(0) \rtimes_{\rho_*} \text{Der } L(1).$$

**Proof.** By Proposition 4.2, the map $\tau = \rho \circ g_L$ is a twisting function that models the map $\nu: X \to B_\infty$. By Theorem 3.20 the dg Lie algebra

$$\text{Hom}^{\tau}(\tilde{C}L, \Pi)(0) \rtimes_{\tau_*} \text{Der } L(1)$$

is a model for $Baut^*_{\infty}(\nu)$. Since $\Pi$ is abelian, twisting has no effect, i.e., we have $\text{Hom}^{\tau}(\tilde{C}L, \Pi) = \text{Hom}(\tilde{C}L, \Pi)$. Since $g_L^*(\rho) = \tau$ by definition of $\tau$, we see that the quasi-isomorphism of Proposition 4.1 induces a quasi-isomorphism of dg Lie algebras

$$g^* \rtimes 1: \text{Hom}(sL/[L, L], \Pi)(0) \rtimes_{\rho_*} \text{Der } L(1) \to \text{Hom}(\tilde{C}L, \Pi)(0) \rtimes_{\tau_*} \text{Der } L(1).$$

Finally note that $sL/[L, L] = \tilde{H}_*(X; \mathbb{Q})$. \(\square\)

**Remark 4.4.** When $\tilde{H}_*(X; \mathbb{Q})$ is finite dimensional, we can rewrite the result in terms of cohomology since

$$\text{Hom}(\tilde{H}_*(X; \mathbb{Q}), \pi_*(\Omega B_\infty)) \cong \tilde{H}^*(X; \mathbb{Q}) \otimes \pi_*(\Omega B_\infty).$$

The twisting function takes the form

$$\rho = \sum_i p_i(\nu) \otimes \pi_i \in \tilde{H}^*(X; \mathbb{Q}) \otimes \pi_*(\Omega B_\infty)$$

in this case.

**Remark 4.5.** Again there are easily proved variants of this result. If $A \subseteq X$ is a subspace containing the base-point such that the inclusion of $A$ into $X$ is a cofibration, then we may consider the submonoid $\text{aut}^*_{A, \infty}(\nu) \subseteq \text{aut}^*_{\infty}(\nu)$ where the homotopy automorphism of the base restricts to the identity map on $A$. If $g \to \text{Der } L(1)$ is a morphism of dg Lie algebras that models the map $\text{Baut}^*_A(X) \to \text{Baut}_*(X)$, then

$$\text{Hom}(\tilde{H}_*(X; \mathbb{Q}), \pi_*(\Omega B_\infty) \otimes \mathbb{Q})(0) \rtimes_{\rho_*} g$$

is a dg Lie algebra model for the space $\text{Baut}^*_A(\nu)$. In [2] we apply this result in the case $(X, A) = (M, \partial M)$ to construct a dg Lie algebra model for the block diffeomorphism group of a simply connected smooth $n$-manifold $M$ with boundary $\partial M = S^{n-1}$ ($n \geq 5$).
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