Plank theorems and their applications: a survey

William Verreault

Abstract

Plank problems concern the covering of convex bodies by planks in Euclidean space and are related to famous open problems in convex geometry. In this survey, we introduce plank problems and present surprising applications of plank theorems in various areas of mathematics.

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1 Introduction

Questions about the covering of convex bodies by planks in Euclidean space are referred to as *plank problems* in convex geometry and discrete geometry. Here is a toy plank problem.

**Question 1.1.** *Given a circular table of diameter $d$ and planks of width 1 and length at least $d$, what is the minimal number of planks needed to cover the table?*

It is readily seen that we may use $d$ parallel planks, but can we do better by changing their orientation and by overlapping them if needed? The answer is no. In fact, we will see that even if we consider a countable family of planks of positive and varying widths, it is impossible to cover the table in a way such that the sum of the widths of the planks used is less than $d$. Accordingly, a covering by parallel planks is optimal. Alfred Tarski generalized this question in the 1930s, a period when discrete geometry was still nascent, and what became known as *Tarski’s plank problem* is still the source of numerous problems and conjectures in geometry despite having been solved decades ago.

![Figure 1: Optimal and nonoptimal positioning of six unit planks on a circular table.](image)

This survey covers a few plank problems and *plank theorems* (that is, theorems that solve plank problems) and presents various applications of these theorems and the ideas behind their proofs in different fields of mathematics. It is organized as follows. Tarski’s plank problem is described in Section 2. A solution and a generalization from Thøger Bang, as well as an important result called Bang’s lemma, are also included in this section. In Section 3 we present a famous plank theorem of Keith Ball for Banach spaces, which proves a specific but important case of Bang’s conjecture. We then move on to discussing several other types of plank problems in Section 4: spherical and discrete variants, and plank problems on complex Hilbert spaces and $L_p(\mu)$ spaces. Then, we highlight the connection between plank theorems and diverse fundamental concepts in analysis and number theory in Sections 5 and 6. The reader can refer to the Table of Contents for a more thorough description of the applications.

This survey is by no means exhaustive; it is biased towards the author’s preferences and what he considers to be the biggest developments in the area, the aim of this paper being first and foremost to present applications of plank theorems and their ideas. The author still tried to include the
major parts of the history of these problems as well as many details in the proofs of the important results for completeness and an easy access to all readers. Anyone familiar with plank theorems is invited to skip over to the applications.

1.1 Notation and conventions

Unless stated otherwise, we work in $\mathbb{R}^d$, and it is always assumed that $d \geq 2$. The letter $C$ will denote a $d$-dimensional convex body, that is, a compact convex subset of $\mathbb{R}^d$ with nonempty interior, while $H$ will denote a hyperplane in $\mathbb{R}^d$, that is, an affine subspace of dimension $d - 1$. We also reserve the symbols $u_i$ for unit vectors in $\mathbb{R}^d$, $m_i$ for real numbers, and $w_i$ for nonnegative real numbers.

We recall a few standard definitions needed to properly state and understand plank problems. A plank $P$ in $\mathbb{R}^d$ is the set of points between two parallel hyperplanes that are called the boundary hyperplanes of $P$. We say that two parallel hyperplanes support a convex body $C \subset \mathbb{R}^d$ if $C$ is contained in the space between these two hyperplanes and its boundary touches each hyperplane in at least one point. Once we fix a direction, we can talk about the hyperplanes that support $C$ without ambiguity. Finally, we say that planks $P_1, P_2, \ldots, P_n$ cover $C$ if $C \subseteq \bigcup_{i=1}^n P_i$. In all of the problems we shall encounter, it does not matter that the number of planks be finite or countable, but we will assume that it is finite for simplicity’s sake. We refer to the planks $P_i$ as a covering of $C$. The width $w(P)$ of a plank $P$ is the distance between the two boundary hyperplanes of $P$. We sometimes refer to the sum of the widths of a covering of a convex body as the total width of the covering or of the associated planks. Given a hyperplane $H$, the width $w(C,H)$ of a convex body $C$ in the direction of $H$ is the distance between the two hyperplanes that support $C$ and that are parallel to $H$. We also define the minimal width $w(C)$ of a convex body $C$ as the infimum of $w(C,H)$ over all hyperplanes $H \subset \mathbb{R}^d$.

![Figure 2: Example of the width of a convex body in the direction of a hyperplane in $\mathbb{R}^2$.](image)

Remark 1.2. It is well known that the linear subspaces of dimension $d - 1$ in $\mathbb{R}^d$ are the orthogonal complements of the 1-dimensional subspaces. Thus, hyperplanes correspond to the translations of these sets. In other words, if $b$ generates a 1-dimensional subspace, we can define a hyperplane $H$ as $H = \{x : x \perp b\} + a$, where $a, b \in \mathbb{R}^d$ and $b \neq 0$. We may rewrite this set as $H = \{y : \langle y, b \rangle = \alpha\}$, where $y = x + a$ and $\alpha = \langle a, b \rangle$. More generally for a normed space $X$, we can define a hyperplane as $H = \{x \in X : f(x) = \alpha\}$ where $\alpha \in \mathbb{R}$ and $f \in X^*$ is a nonzero continuous linear functional.
2 Tarski’s plank problem

2.1 The genesis of plank problems

In the 1930s, the mathematician and logician Alfred Tarski, who is known for his work on model theory and the Banach–Tarski paradox, among other things, proposed a problem that would change the face of discrete geometry. It is what we now call a plank problem.

**Conjecture 2.1** (Tarski [53], 1932). If \( C \subset \mathbb{R}^d \) is covered by a sequence of planks \( P_1, P_2, \ldots, P_n \), then the sum of the widths of the planks is at least \( w(C) \).

Without loss of generality, we may consider a body of minimal width 1. Then, similarly to our answer to Question 1.1, it is obvious that we can cover a convex body of minimal width 1 with planks that have total width 1, by placing them perpendicularly to the hyperplanes that support the convex body. But can we do better? Tarski’s conjecture says no.

2.1.1 Partial solution to Tarski’s problem

Tarski proved his conjecture for figures in which we can inscribe a disk centered at the origin. Note that the case of \( \mathbb{R}^2 \) is not entirely covered by this partial result, since some convex figures, for example an equilateral triangle, have a width bigger than the diameter of their inscribed circle; yet, Tarski’s arguments generalize to solids in which we can inscribe a ball centered at the origin. However, they do not work in higher dimensions because the proof relies on geometric characteristics of two (or three)-dimensional spaces. His proof is still interesting in its own right, so we include it by adapting the argument that was presented in [35]. Note that Tarski was inspired by a solution of Moese on a related problem first stated by Tarski himself (read [42, Chapter 7] for a discussion on the history of this problem and a translation of the related papers of Tarski and Moese).

The demonstration relies essentially on a result of Archimedes in *On the Sphere and Cylinder*, often called “Archimedes’ Hat-Box Theorem”, which can be formulated as follows.

**Proposition 2.2.** The lateral area of a spherical segment formed by the intersection of a sphere of radius \( r \) and two planes separated by distance \( d \) is \( 2\pi rd \).

The modern proof of this result would be a good exercise in a calculus course. Indeed, if the two planes in the statement split the segment \( [a, b] \) of length \( d \) on the \( x \)-axis, it suffices to compute

\[
2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} \, dx,
\]

where \( f(x) = \sqrt{r^2 - x^2} \). Archimedes, on the other hand, thought of the sphere as being inscribed in a cylinder where we would make two parallel cuts, and so his original proof relies on this interpretation. We can now use Archimedes’ proposition to outline Tarski’s argument.

**Proof.** Let \( D \) be the disk of radius \( r \) centered at the origin and \( B \) the upper half-sphere of radius \( r \) centered at the origin. Suppose that planks \( P_1, P_2, \ldots, P_n \) cover \( D \) and that each plank is entirely contained in \( D \) (replacing \( P_i \) with \( P_i \cap D \) if needed). Then, the union of the vertical projections of the planks on \( B \) covers the surface of \( B \), which has area \( 2\pi r^2 \). Thus, the sum of the lateral areas of
the spherical segments formed by the intersection of the vertical projections of the \( P_i \) with \( B \) is at least \( 2\pi r^2 \). By Proposition 2.2 it follows that

\[
\sum_{i=1}^{n} \pi r w(P_i) \geq 2\pi r^2,
\]

whence \( \sum_{i=1}^{n} w(P_i) \geq 2r = w(D) \). \( \square \)

### 2.2 Bang’s solution and generalization

In 1950, Bang [10] gave an unexpectedly short proof of Tarski’s conjecture. His solution relies heavily on simple geometric ideas, but they are hiding the combinatorial dimension of the argument which depends on results from linear algebra. The basic ideas behind the proof are still at the core of many demonstrations in convex geometry and discrete geometry, which we shall partly explore in Sections 4 to 6. One year later, Bang [11] simplified his proof and Fenchel [28] simplified it some more. Finally in 1964, Bognár [20] reformulated the demonstration of Fenchel.

**Theorem 2.3** (Bang, 1950). Let \( C \subset \mathbb{R}^d \) be a convex body covered by \( n \) planks \( P_j \), \( j = 1, \ldots, n \). Then \( w(P_1) + \cdots + w(P_n) \geq w(C) \).

The proof we include here mostly follows another reformulation [15], but it is more detailed and also presents the end of the demonstration in terms of the important Bang lemma, which will be used extensively in the rest of this paper.

In what follows, we let \( a_j = w(P_j)/2 \) and we use \( u_j \) to denote a vector of length \( a_j \) that is perpendicular to the boundary hyperplanes of \( P_j \). The whole proof essentially relies on two lemmas and then proceeds by contradiction. Here is the first lemma.

**Lemma 2.4.** Let \( u \) be a vector of norm \( a/2 \) with \( a < w(C) \). Then, the intersection \((C-u) \cap (C+u)\) contains a homothetic image of \( C \) with a homothety ratio \((w(C)-a)/w(C)\), where \( C \pm u \) is the translation of \( C \) by the vector \( \pm u \).

**Proof.** Consider vectors in \( C \) that are parallel to \( u \). Suppose the longest goes from \( y \) to \( z \), has length \( \ell \) and midpoint \( m \). Under these hypotheses, we may write \( u = a(z-y)/2\ell \). If \( x \in C \), then the point

\[
\frac{a}{\ell} m + \frac{\ell - a}{\ell} x + u = \frac{a}{\ell} \frac{y + z}{2} + \frac{\ell - a}{\ell} x + \frac{a}{\ell} \frac{z - y}{2 \ell} = \frac{a}{\ell} z + \frac{\ell - a}{\ell} x
\]

is in \( C \), since \( a/\ell + (\ell - a)/\ell = 1 \) and \( C \) is convex. The same reasoning shows that

\[
\frac{a}{\ell} m + \frac{\ell - a}{\ell} x - u
\]

is also in \( C \), whence

\[
\frac{a}{\ell} m + \frac{\ell - a}{\ell} x \in (C - u) \cap (C + u).
\]

Under the homothety of center \( m \) and ratio \((\ell - a)/\ell \), the convex body \( C \) is contracted and covered by the set of points of the form

\[
\frac{a}{\ell} x + \frac{\ell - a}{\ell} x.
\]
Lemma 2.5. For all \( \epsilon \) from the fact that the minimal length of \( C \) must be parallel, else we could move the vector \( y - z \) to obtain a bigger one. The result then follows from the fact that the minimal length of \( C \) is defined as the infimum over the distance between such hyperplanes. It follows that \( (\ell - a)/\ell \geq (w(C) - a)/w(C) \), from which we see that the image of \( C \) under the homothety with center \( m \) and ratio \( (w(C) - a)/w(C) \) is contained in the homothety with center \( m \) and ratio \( (\ell - a)/\ell \) of \( C \), which is the desired result. \( \square \)

Lemma 2.6 might seem to have nothing to do with Theorem 2.3, but we shall rather care about a corollary of this lemma. Note that by successive applications of the preceding lemma on the vectors \( u_j \) (remembering that we defined \( a_j = w(P_j)/2 \) as the length of \( u_j \)), we get that the intersection over all \( 2^n \) possible choices of signs \( \epsilon_j \in \{\pm 1\} \),

\[
\bigcap_{\epsilon_j \in \{\pm 1\}} \left( C - \sum_{j=1}^n \epsilon_j u_j \right), \tag{2.1}
\]

must contain a homothetic image of \( C \) with a homothety ratio equal to

\[
\frac{w(C) - \sum_{i=1}^n w(P_i)}{w(C)}.
\]

We know that this homothety is well-defined (that is, the image of \( C \) under this homothety will not be the empty set) if this ratio is strictly greater than 0. Thus, if \( w(P_1) + \cdots + w(P_n) < w(C) \), we can find \( x_0 \) in the intersection \( (2.1) \) such that \( x_\epsilon := x_0 + \sum_{j=1}^n \epsilon_j u_j \) is in \( C \) for all choices of sign \( \epsilon = (\epsilon_j)_{j=1}^n \). In conjunction with the following lemma, this will lead us to a contradiction.

**Lemma 2.5.** For all \( x_0 \in \mathbb{R}^d \), there is a choice of signs \( \epsilon \) such that \( x_\epsilon \notin P_j \) for \( j = 1, 2, \ldots, n \).

Rather than proving this lemma right away, let us consider another proposition which might not seem related at first sight. This result will prove essential in the following sections of this survey, where we will refer to it as Bang’s lemma. The proof follows [6] and [9].

**Lemma 2.6** (Bang’s lemma). Let \( (u_i)_{i=1}^n \) be unit vectors in \( \mathbb{R}^d \), \( (m_i)_{i=1}^n \) real numbers, and \( (w_i)_{i=1}^n \) nonnegative real numbers. Then there is a choice of signs \( (\epsilon_i)_{i=1}^n \) such that the vector \( x = \sum_{j=1}^n \epsilon_j w_j u_j \) satisfies

\[
| \langle x, u_i \rangle - m_i | \geq w_i
\]

for all \( 1 \leq i \leq n \).

Observe that we can reformulate this lemma in the language of linear algebra by introducing a matrix composed of all the possible scalar products between the vectors \( u_i \), which is called the Gram matrix of \( (u_i)_{i=1}^n \). Explicitly, define \( H \) by \( h_{ij} = \langle u_i, u_j \rangle \). Then, Lemma 2.6 is equivalent to the existence of a choice of signs \( (\epsilon_i)_{i=1}^n \) such that, for all \( i \),

\[
\sum_{j=1}^n h_{ij} \epsilon_j w_j - m_i \geq w_i.
\]

This would obviously apply to any \( n \times n \) real symmetric matrix with 1’s on the diagonal (since the \( u_i \) are unit vectors).
Proof of Lemma 2.6. Choose the \( \epsilon_i \in \{\pm 1\} \) such that the difference

\[
\sum_{i,j=1}^{n} h_{ij} \epsilon_i \epsilon_j w_i w_j - 2 \sum_{i=1}^{n} \epsilon_i w_i m_i
\]

is maximized. We shall show that this choice of signs satisfies the statement of the lemma. For \( 1 \leq k \leq n \), let

\[
(\delta_i)_{i=1}^{n} = (\epsilon_1, \ldots, \epsilon_{k-1}, -\epsilon_k, \epsilon_{k+1}, \ldots, \epsilon_n),
\]

that is, flip the sign of the \( k \)th term in the sequence \( (\epsilon_i)_{i=1}^{n} \). Then,

\[
\sum_{i,j=1}^{n} h_{ij} \epsilon_i \epsilon_j w_i w_j - 2 \sum_{i=1}^{n} \epsilon_i w_i m_i \geq \sum_{i,j=1}^{n} h_{ij} \delta_i \delta_j w_i w_j - 2 \sum_{i=1}^{n} \delta_i w_i m_i.
\]

By symmetry, we can reorganize into the inequality

\[
0 \leq \sum_{i,j=1}^{n} h_{ij} \epsilon_i \epsilon_j w_i w_j - 2 \sum_{i=1}^{n} \epsilon_i w_i m_i - \left( \sum_{i,j=1}^{n} h_{ij} \delta_i \delta_j w_i w_j - 2 \sum_{i=1}^{n} \delta_i w_i m_i \right)
\]

\[
= 4\epsilon_k w_k \sum_{j \neq k} h_{kj} \epsilon_j w_j - 4\epsilon_k w_k m_k
\]

\[
= -4w_k^2 + 4\epsilon_k w_k \sum_{j=1}^{n} h_{kj} \epsilon_j w_j - 4\epsilon_k w_k m_k,
\]

where the last equality is justified by the fact that \( h_{kk} = 1 \). Dividing by \( 4w_k \) and reorganizing, we obtain

\[
w_k \leq \epsilon_k \left( \sum_{j=1}^{n} h_{kj} \epsilon_j w_j - m_k \right) \leq \left| \sum_{j=1}^{n} h_{kj} \epsilon_j w_j - m_k \right|.
\]

We are now ready to prove Lemma 2.5.

Proof of Lemma 2.5. Fix \( 1 \leq k \leq n \) and let \( p_k \) be a midpoint of \( P_k \) (that is, a point that belongs to one of the hyperplanes going through the center of \( P_k \)). Note that if \( z \) belongs to one of the hyperplanes supporting \( P_k \), then \( |\langle z - p_k, u_k \rangle| = a_k^2 \). Such points being "extremal", and knowing from Remark 1.2 that a hyperplane can be defined as \( \{x \in \mathbb{R}^d : \langle x, b \rangle = \alpha \} \) for some nonzero \( b \in \mathbb{R}^d \) and \( \alpha \in \mathbb{R} \), we see that \( P_k \) is given explicitly by

\[
\{x \in \mathbb{R}^d : |\langle x - p_k, u_k \rangle| < a_k^2 \}.
\]

To find an \( x_\epsilon \) that is not in \( P_k \), we are looking for \( \epsilon \) such that

\[
|\langle x_\epsilon - p_k, u_k \rangle| \geq a_k^2,
\]

or equivalently, such that

\[
\left| \frac{x_\epsilon - x_0}{a_k}, \frac{u_k}{a_k} \right| - \frac{1}{a_k^2} \langle p_k - x_0, u_k \rangle \geq 1.
\]

7
Since \( u_k/a_k \) are unit vectors and

\[
\frac{x_k - x_0}{a_k} = \sum_{k=1}^{n} \epsilon_k \frac{u_k}{a_k},
\]

letting \( w_k := \langle p_k - x_0, u_k \rangle / a_k^2 \) and \( m_k := 1 \), Lemma 2.6 allows us to conclude. Since \( k \) was arbitrary, we are done.

It is now easy to finish the proof of Theorem 2.3. Lemma 2.5 informs us that there is a point which is in \( C \) but in none of the planks that cover \( C \), which is a contradiction. Whence it must be that \( w(P_1) + \cdots + w(P_n) \geq w(C) \).

### 2.2.1 Bang’s generalization of Tarski’s problem

Bang [11] formulated an affine invariant version of Tarski’s plank problem (in the sense that it is invariant under affine transformations of the convex body). If \( H \) is parallel to the boundary hyperplanes of a plank \( P \), let us call the relative width of \( P \) the ratio \( w(P)/w(C,H) \).

**Conjecture 2.7** (Bang). If a convex body is covered by a finite number of planks, then the sum of their relative widths is at least 1.

Here are a few remarks regarding Conjecture 2.7. First, Ohmann [46] showed that it is enough to prove this conjecture for coverings by \( d \) mutually orthogonal planks of width 1 in \( \mathbb{R}^d \), for all \( d \in \mathbb{N} \).

Second, Hunter [33] looked for simple cases of equality in Bang’s conjecture in the plane when the number of planks is given but the convex body can vary. For instance, given a covering of a triangle by three planks, consider the equilateral triangle with sides of unit length that was obtained via affine transformations from the initial triangle. Then it is shown that the sum of the relative widths of the three planks is 1 when and only when the sum of the segments \( t_1 + t_2 + t_3 \) in Fig. 3 equals 1, where the biggest triangle is the equilateral one prescribed above.

![Figure 3: Labeling of the segments appearing in Hunter’s construction.](image)

Third, Alexander [3] observed that this conjecture is equivalent to the following generalization of a theorem of Davenport on intersections of straight lines with unit squares, which is ultimately related to Diophantine approximation in number theory (see Section 6.1 for more details).
**Conjecture 2.8** (Reformulation of Conjecture 2.7). If \( C \subset \mathbb{R}^d \) is a convex body in euclidean space sliced by \( n-1 \) hyperplane cuts, then one of the \( n \) resulting pieces covers a translation of \( \frac{1}{n}C \).

Conjecture 2.7 is a small modification to Tarski’s conjecture, so one could expect that a similar solution exists. Yet, to this day it remains an important open problem in convex geometry. We shall still see that it has been proven in some specific cases which we will focus on in the upcoming sections.

### 3 A plank theorem of Ball

#### 3.1 Bang’s conjecture for centrally symmetric convex bodies

In the 1990s, Ball proved Bang’s conjecture in the case where \( C \) is centrally symmetric (for instance, \( C \) symmetric with respect to the origin means that for all \( x \in C \), we also have \( -x \in C \)). In the rest of this section, we shall refer to \( C \) as simply being a symmetric convex body. This solution is of particular interest because, in practice, most of the familiar convex bodies have a central symmetry (think about balls, cubes, etc.). Moreover, this supposition on \( C \) allows us to use tools from functional analysis. Indeed, any symmetric convex body can be chosen as the unit ball defining a finite-dimensional normed space, and any such unit ball is a symmetric convex body. We also know from Remark 1.2 that a hyperplane in a normed space \( X \) is defined as the set \( \{x \in X : f(x) = \alpha \} \), where \( f \in X^* \) and \( \alpha \in \mathbb{R} \). We then see that a plank in a normed space is given explicitly by the set

\[
\{x \in X : |\phi(x) - m| \leq w \}
\]  

(3.1)

for \( \phi \) a continuous linear functional, \( m \) a real number and \( w \) a nonnegative real number. In what follows, we consider unit functionals, so that the relative width of a plank is exactly \( w \). Here is Ball’s result in this setting [6].

**Theorem 3.1.** If the unit ball of a Banach space \( X \) is covered by a set of planks in \( X \), then the sum of the relative widths of these planks is at least \( 1 \).

#### 3.2 Finite-dimensional Banach spaces

We first consider the case where \( X \) is finite-dimensional, and we shall treat the infinite-dimensional case in Section 3.3. We still suppose that the covering consists of a finite number of planks. We can then reformulate Theorem 3.1 using these hypotheses and (3.1).

**Theorem 3.2** (Plank theorem - Finite-dimensional case). Let \( X \) be a finite-dimensional normed space. Let \( (\phi_i)_{i=1}^n \) be a sequence of unit linear functionals in \( X^* \), \( (m_i)_{i=1}^n \) real numbers and \( (w_i)_{i=1}^n \) nonnegative real numbers such that \( \sum_{i=1}^n w_i = 1 \). Then there exists \( x \) in the unit ball of \( X \) such that

\[
|\phi_i(x) - m_i| \geq w_i
\]

for all \( 1 \leq i \leq n \).

The previous theorem is in some ways a quantitative version of the contrapositive of Theorem 3.1. We would like if \( |\phi_i(x) - m_i| \) was strictly bigger than \( w_i \) for all \( i \), so that the \( x \) given by the theorem does not belong to any of the planks that cover the unit ball of \( X \). However, by a compactness
argument, the inequality in Theorem 3.2 is equivalent to this condition. Explicitly, to recover the strict inequality, we obtain the result with \( w_i \) replaced by \( w_i(1 - 1/N) \) for each \( N \in \mathbb{N} \), which gives a sequence of points \( X^N \) bounded in the unit ball of \( X \) and thus possessing a convergent subsequence. By compactness of the unit ball of \( X \), it converges to some \( x \in X, \|x\| \leq 1 \), which we may take as a candidate point in Theorem 3.2 (letting \( N \to \infty \)).

### 3.2.1 Reduction of the plank theorem to a linear algebra problem

Ball’s solution uses a clever mix of tools from functional analysis and linear algebra. The main idea is to obtain a result analogous to Bang’s lemma that holds for matrices that are not necessarily symmetric. We first reduce the problem to purely combinatorial propositions by mimicking some of the ideas that allowed us to express Bang’s lemma in terms of Gram matrices.

Let \( \phi_j \) be linear functionals as in the statement of Theorem 3.2. Since \( X \) is finite-dimensional, the unit ball of \( X \) is compact and, in particular, the norm of each \( \phi_j \) is attained: we can find \( x_j \) in the unit ball of \( X \) such that \( \phi_j(x_j) = 1 \). Given such \( x_j \), define the \( n \times n \) real matrix \( A \) by \( a_{ij} = \phi_i(x_j) \). If \( (\lambda_j)_{j=1}^n \) is a sequence of real numbers such that \( \sum_{j=1}^n |\lambda_j| \leq 1 \), then \( x := \sum_{j=1}^n \lambda_j x_j \) is such that \( \|x\| \leq 1 \) and \( \phi_i(x) = \sum_{j=1}^n a_{ij} \lambda_j \) by linearity. In short, it will suffice to prove the following theorem.

**Theorem 3.3.** Let \( A = (a_{ij})_{1 \leq i,j \leq n} \) be an \( n \times n \) real matrix with 1’s on the diagonal, \( (m_i)_{i=1}^n \) real numbers and \( (w_i)_{i=1}^n \) nonnegative real numbers such that \( \sum_{i=1}^n w_i \leq 1 \). Then there exist real numbers \( (\lambda_j)_{j=1}^n \) such that \( \sum_{j=1}^n |\lambda_j| \leq 1 \) and

\[
\left| \sum_{j=1}^n a_{ij} \lambda_j - m_i \right| \geq w_i
\]

for all \( 1 \leq i \leq n \).

Recall that we interpret \( w_i \) as the width of the \( i \)th plank. Since we may cover planks of varying widths by planks of width \( 1/n \) (possibly overlapping if \( \sum_{i=1}^n w_i < 1 \)), we will suppose in what follows that \( w_i = 1/n \). Also note that if we can show \( \sum_{j=1}^n \lambda_j^2 \leq 1/n \), then

\[
\left( \sum_{j=1}^n |\lambda_j| \right)^2 \leq n \sum_{j=1}^n \lambda_j^2 \leq 1
\]

by the Cauchy–Schwarz inequality.

### 3.2.2 Preliminaries to the proof of Theorem 3.3

The proof of Theorem 3.3 will rely on Bang’s lemma and two new lemmas.

**Lemma 3.4.** Let \( H = (h_{ij})_{1 \leq i,j \leq n} \) be a positive semidefinite symmetric matrix with no 0’s on its diagonal. If \( U \) is orthogonal, then

\[
\sum_{i=1}^n \frac{(HU)_i^2}{h_{ii}} \leq \sum_{i=1}^n h_{ii}.
\]  \hspace{1cm} (3.2)
Since $H$ is positive semidefinite, its square root $T$ is well defined and $H = T^2 = TT^* = T^*T$. We remind the reader that there is a version of the Cauchy–Schwarz inequality for traces of matrices, which states that
\[
\text{Tr}(AB^*)^2 \leq \text{Tr}(AA^*) \text{Tr}(BB^*).
\]
Since $\text{Tr}(A) := \sum_i a_{ii}$ for $A = (a_{ij})$, we see at once a way to deal with (3.2). We remark that this proof could be formulated in terms of nuclear norm (see \text{[6]}), but we avoid it here.

Proof of Lemma 3.4. Let $\gamma_i := (HU)_{ii}/h_{ii}$ and $D := \text{diag}(\gamma_1, \ldots, \gamma_n)$, and write $T$ for the square root of $H$. Then, by the preceding remarks,
\[
\sum_{i=1}^n \frac{(HU)^2_{ii}}{h_{ii}} = \sum_{i=1}^n \gamma_i (HU)_{ii} = \text{Tr}(DHU) = \text{Tr}(DTTU).
\]
Cauchy-Schwarz then gives
\[
\sum_{i=1}^n \frac{(HU)^2_{ii}}{h_{ii}} \leq \text{Tr}(DT(DT^*)^{1/2}) \text{Tr}(TU(TU^*)^{1/2})^{1/2} = \text{Tr}(DHD^*)^{1/2} \text{Tr}(H)^{1/2} = \left( \sum_{i=1}^n \gamma_i^2 h_{ii} \right)^{1/2} \left( \sum_{i=1}^n h_{ii} \right)^{1/2} = \left( \sum_{i=1}^n \frac{(HU)^2_{ii}}{h_{ii}} \right)^{1/2} \left( \sum_{i=1}^n h_{ii} \right)^{1/2}.
\]

In particular, we shall use the following corollary.

Corollary 3.5. Let $H$ be a positive semidefinite symmetric matrix with 1’s on its diagonal. If $U$ is orthogonal, then
\[
\sum_{i=1}^n (HU)^2_{ii} \leq n.
\]

We still need to introduce an orthogonal matrix to use Corollary 3.5.

Lemma 3.6. Let $A$ be an $n \times n$ real matrix with nonzero rows. Then there exist positive real numbers $(w_i)_{i=1}^n$ and an orthogonal matrix $U$ such that the matrix $H$ defined by
\[
h_{ij} := w_i(AU)_{ij}
\]
is positive semidefinite, symmetric, and only has 1’s on its diagonal.

Proof. Let $h_{ij} := w_i(AU)_{ij}$. To prove $h_{ii} = 1$ for $i = 1, \ldots, n$, it suffices to show that $h_{ii}$ equals a constant, because we may normalize it after. Given $(w_i)_{i=1}^n$ positive real numbers, we consider the matrix $B := (w_ia_{ij})_{1 \leq i, j \leq n}$ and $T$ the square root of $BB^*$. Then $\text{Tr}(T)$ will be some value possibly smaller than all the $w_i$ if the $a_{ij}$ are small, yet we can always find some $c > 0$ small enough such that
\[
\text{Tr}(T) \geq c \max_i w_i.
\]
The trace being a continuous function (here taking arguments \((w_i)_{i=1}^n\)), we can choose a sequence \((w_i)_{i=1}^n\) that minimizes \(\text{Tr}(T)\) under the condition \(\prod_{i=1}^n w_i = 1\). We shall show that \(H := T\) for this choice of \((w_i)_{i=1}^n\) satisfies the conditions of the lemma.

Observe that explicitly \(H\) is the square root of

\[
(w_i(AA^*)_{ij} w_j)_{1 \leq i,j \leq n},
\]

and so \(H\) is automatically symmetric and positive semidefinite. Moreover, it follows from polar decomposition of matrices that there exists an orthogonal matrix \(U\) such that \(h_{ij} = w_i(AU)_{ij}\). We just need to show that the \(h_{ii}\) are constant. Since \(A\) has nonzero rows, the diagonal elements of \(H\) are nonzero, thus we may define

\[
\gamma_i := \frac{1}{\sqrt{h_{ii}}} \left( \prod_{j=1}^n \sqrt{h_{jj}} \right)^{1/n}
\]

for \(1 \leq i \leq n\). If \(S\) is the square root of

\[
(\gamma_i w_i(AA^*)_{ij} w_j \gamma_j)_{1 \leq i,j \leq n} = (\gamma_i h_{ij} h_{ji} \gamma_j)_{1 \leq i,j \leq n},
\]

then the fact that \(\prod_{i=1}^n w_i \gamma_i = 1\) implies that \(\text{Tr}(S) \geq \text{Tr}(H)\), else we contradict the minimality of \((w_i)_{i=1}^n\). Once more by polar decomposition (of \(S\) this time), there exists \(U'\) orthogonal such that

\[
S = (\gamma_i (HU')_{ij})_{1 \leq i,j \leq n} = \left( \prod_{k=1}^n \sqrt{h_{kk}} \right)^{1/n} \left( h_{ii}^{-1/2} (HU')_{ij} \right)_{1 \leq i,j \leq n},
\]

whence

\[
\text{Tr}(H) \leq \text{Tr}(S) = \left( \prod_{k=1}^n \sqrt{h_{kk}} \right)^{1/n} \sum_{i} h_{ii}^{-1/2} (HU')_{ii} \leq \left( \prod_{k=1}^n \sqrt{h_{kk}} \right)^{1/n} \sqrt{n} \left( \sum_{i} \frac{(HU')_{ii}^2}{h_{ii}} \right)^{1/2},
\]

where we used Cauchy-Schwarz in the last equality. By Lemma 3.4, this quantity is at most

\[
\left( \prod_{k=1}^n \sqrt{h_{kk}} \right)^{1/n} \sqrt{n} \text{Tr}(H)^{1/2}.
\]

Reorganizing, we find

\[
\frac{1}{n} \sum_{i=1}^n h_{ii} \leq \left( \prod_{i=1}^n h_{ii} \right)^{1/n}. \tag{3.3}
\]

On the other hand, the arithmetic-geometric mean inequality implies that \(n^{-1} \sum_i h_{ii} \geq (\prod_{i=1}^n h_{ii})^{1/n}\), and so (3.3) is an equality. It also follows from the arithmetic-geometric mean inequality that the \(h_{ii}\) must be equal.

### 3.2.3 Proof of Theorem 3.3

By Lemma 3.6, we may take positive real numbers \((w_j)_{j=1}^n\) and an orthogonal matrix \(U\) such that \(H := (w_i(AU)_{ij})_{1 \leq i,j \leq n}\) is symmetric, positive semidefinite and has 1’s on the diagonal. It then follows from Bang’s lemma that we can find \((\epsilon_j)_{j=1}^n\) such that, for all \(i\),

\[
w_i \leq \left| \sum_{j=1}^n h_{ij} \epsilon_j w_j - nw_i m_i \right| = \left| w_i \sum_{j=1}^n (AU)_{ij} \epsilon_j w_j - nw_i m_i \right|.
\]
Dividing by \( n \cdot w_i \), this is equivalent to

\[
\frac{1}{n} \leq \sum_{k=1}^{n} a_{ik} \left( \frac{1}{n} \sum_{j=1}^{n} u_{kj} \epsilon_j w_j \right) - m_i,
\]

which motivates the definition

\[
\lambda_k := \frac{1}{n} \sum_{j=1}^{n} u_{kj} \epsilon_j w_j.
\]

Since \( U \) is orthogonal and \( \epsilon_j \in \{\pm 1\} \), we obtain

\[
\sum_{k=1}^{n} \lambda_k^2 = \frac{1}{n^2} \sum_{j=1}^{n} w_j^2.
\]

Also, since \( H = (w_i(AU)_{ij}) \), we see that

\[
w_i a_{ij} = (HU^*)_{ij}.
\]

Taking \( i = j \) and using \( a_{ii} = 1 \) yields \( w_i = (HU^*)_{ii} \). Recall that we want

\[
\sum_{k=1}^{n} \lambda_k^2 \leq \frac{1}{n},
\]

so we must show

\[
\sum_{j=1}^{n} (HU^*)_{jj}^2 \leq n,
\]

which is precisely the statement of Corollary 3.5 because \( U^* \) is orthogonal.

### 3.3 Infinite-dimensional Banach spaces

The goal of this subsection is to prove Theorem 3.1 in the case where \( X \) is an infinite-dimensional Banach space. Although the modifications needed in the proof are not obvious, all the tools we shall need are already available to us. Recall that in the finite case it suffices to obtain a sequence of real numbers \( (\lambda_j)_{j=1}^{n} \) that sum to at most 1. However, we obtain something stronger, namely that \( \sum_{j=1}^{n} \lambda_j^2 \leq 1/n \) if we suppose that \( w_i = 1/n \) for all \( i \) for simplicity. Running the proof of Theorem 3.3 again with an arbitrary choice of \( w_i \) would instead yield \( \sum_{j=1}^{n} w_j^{-1} \lambda_j^2 \leq \sum_{j=1}^{n} w_j \). We can thus use the following lemma.

**Lemma 3.7.** Let \( A = (a_{ij})_{1\leq i,j\leq n} \) be an \( n \times n \) real matrix with 1’s on the diagonal, \((m_i)_{i=1}^{n}\) real numbers and \((w_i)_{i=1}^{n}\) nonnegative real numbers with \( \sum_{i=1}^{n} w_i \leq 1 \). Then there exist \((\lambda_j)_{j=1}^{n}\) such that \( \sum_{j=1}^{n} w_j^{-1} \lambda_j^2 \leq \sum_{j=1}^{n} w_j \) and

\[
\left| \sum_{j=1}^{n} a_{ij} \lambda_j - m_i \right| \geq w_i
\]

for all \( 1 \leq i \leq n \).
We are now able to complete the proof of Theorem 3.1. Note that it suffices to prove an “infinite” version of Theorem 3.2 where we replace every sequence of \( n \) elements with a sequence indexed by \( \mathbb{N} \).

**Theorem 3.8 (Plank theorem - Infinite-dimensional case).** Let \( X \) be a Banach space. Let \( (\phi_i)_{i=1}^{\infty} \) be a sequence of unit functionals in \( X^* \), \( (m_i)_{i=1}^{\infty} \) real numbers and \( (w_i)_{i=1}^{\infty} \) nonnegative real numbers such that \( \sum_{i=1}^{\infty} w_i < 1 \). Then there exists \( x \) in the unit ball of \( X \) such that

\[
|\phi_i(x) - m_i| > w_i
\]

for all \( i \).

**Proof.** Choose \( v_i \) such that \( v_i > w_i \) for all \( i \), but also such that their sum remains smaller than 1, say \( \sum_{i=1}^{\infty} v_i = 1 - \varepsilon < 1 \). In the finite case, we used compactness to find points in the unit ball of \( X \) where the norm of the functionals was attained. Now we can only get away with almost attaining their norm: we can find \( x_i \) in the unit ball of \( X \) such that \( \phi_i(x_i) = 1 - \varepsilon \). Defining the \( i, j \) entry of a matrix by \( \phi_i(x_j)/(1 - \varepsilon) \), we may apply Lemma 3.7 to obtain, for each \( n \), a sequence of real numbers \( (\lambda_j^{(n)})_{j=1}^{n} \) for which

\[
\sum_{j=1}^{n} \left( \frac{v_j}{1 - \varepsilon} \right)^{-1} (\lambda_j^{(n)})^2 \leq \sum_{j=1}^{n} \frac{v_j}{1 - \varepsilon} < 1
\]

and

\[
\frac{v_i}{1 - \varepsilon} \leq \left| \sum_{j=1}^{n} \frac{\phi_i(x_j)}{1 - \varepsilon} \lambda_j^{(n)} - m_i \right| = \frac{1}{1 - \varepsilon} \left| \phi_i \left( \sum_{j=1}^{n} \lambda_j^{(n)} x_j \right) - m_i \right|
\]

for \( 1 \leq i \leq n \). That is, we have \( \sum_{j=1}^{n} v_j^{-1} (\lambda_j^{(n)})^2 < (1 - \varepsilon)^{-1} \) and

\[
\left| \phi_i \left( \sum_{j=1}^{n} \lambda_j^{(n)} x_j \right) - m_i \right| \geq v_i
\]

for each \( n \) and \( 1 \leq i \leq n \). Extending \( (\lambda_j^{(n)})_{j=1}^{n} \) to an infinite sequence by completing with zeros, we see by Cauchy-Schwarz that

\[
\sum_{j=1}^{\infty} |\lambda_j^{(n)}| \leq \left( \sum_{j=1}^{\infty} v_j^{-1} (\lambda_j^{(n)})^2 \right)^{1/2} \left( \sum_{j=1}^{\infty} v_j \right)^{1/2} < (1 - \varepsilon)^{-1/2} (1 - \varepsilon)^{1/2} = 1
\]

for all \( n \). Moreover, for each \( m, n \), we also have

\[
\sum_{j=m}^{\infty} |\lambda_j^{(n)}| \leq \left( \sum_{j=m}^{\infty} v_j^{-1} (\lambda_j^{(n)})^2 \right)^{1/2} \left( \sum_{j=m}^{\infty} v_j \right)^{1/2} < (1 - \varepsilon)^{-1/2} (1 - \varepsilon)^{-1/2} \left( \sum_{j=m}^{\infty} v_j \right)^{1/2} \to 0, \quad (m \to \infty).
\]

In particular, we can take \( \delta > 0 \) and \( M \) such that \( \sum_{j=M}^{\infty} |\lambda_j^{(n)}| < \delta/4 \) for all \( n \).

We now obtain a subsequence of the \( (\lambda_j^{(n)})_{j=1}^{\infty} \) that converges pointwise for every \( j \) using a diagonal argument. For fixed \( j \), \( (\lambda_j^{(n)})_{n=1}^{\infty} \) has to be bounded, else at least one of the sums \( \sum_{j=1}^{\infty} |\lambda_j^{(n)}| \)

in (3.5) would diverge, so we can find a convergent subsequence. In particular, there exists \( N_1 \subseteq \mathbb{N} \)
such that \((\lambda^n_i)_{n \in N_1}\) converges, and for every \(i \geq 2\), sets \(N_i \subseteq N_{i-1}\) such that \((\lambda^n_i)_{n \in N_i}\) converges. Define \(N\) as the set containing the first element of \(N_1\), the second of \(N_2\), the third of \(N_3\), and so on. Since \(N \subseteq N_i\) for each \(i\), except possibly for \(i - 1\) elements, we conclude that \((\lambda^n_i)_{n=1}^{\infty}\) converges pointwise. Thus, this sequence is Cauchy for a fixed \(i\), and so there exists \(K_i\) such that if \(\ell, m \geq K_i\), then

\[
|\lambda^{(\ell)}_i - \lambda^{(m)}_i| < \frac{\delta}{2(M-1)}.
\]

Now, let \(K = \max\{K_1, \ldots, K_{M-1}\}\) and deduce that for \(\ell, m \geq K\),

\[
\|\lambda^{(\ell)}_j - \lambda^{(m)}_j\|_{\ell_1} = \sum_{j=1}^{\infty} |\lambda^{(\ell)}_j - \lambda^{(m)}_j| \\
= \sum_{j=1}^{M-1} |\lambda^{(\ell)}_j - \lambda^{(m)}_j| + \sum_{j=M}^{\infty} |\lambda^{(\ell)}_j - \lambda^{(m)}_j| \\
< \sum_{j=1}^{M-1} \frac{\delta}{2(M-1)} + \sum_{j=M}^{\infty} |\lambda^{(\ell)}_j| + \sum_{j=M}^{\infty} |\lambda^{(m)}_j| \\
< \frac{\delta}{2} + \frac{\delta}{4} + \frac{\delta}{4} = \delta.
\]

Thus, \((\lambda^{(n)}_j)_{n=1}^{\infty}\) is Cauchy in the complete space \(\ell_1\) and has a limit in this space for every \(j\), call it \(\lambda_j\). This allows us to define \(x = \sum_{j=1}^{\infty} \lambda_j x_j\). This point has a norm that is at most 1 (by (3.5) and the fact that \(|x_j| \leq 1\)) and it satisfies \(|\phi_i(x) - m_i| \geq v_i > w_i\) for all \(i\) by (3.4).

4 Other types of plank problems and their applications

We refer to the Theorems 3.2 and 3.8 of Ball as plank theorems without any further distinction, but as we shall see, there are many other kinds of plank problems and theorems.

4.1 The complex plank problem

Theorem 3.8 is sharp as can be seen by taking \(X = \ell_1\) and \(\phi_i\) to be the standard basis vectors of \(\ell_{\infty}\), the dual of \(\ell_1\). However, it is expected that we can improve this result for Hilbert spaces. For instance, Ball proved the following complex Hilbert space analogue of Theorem 3.2 in [8].

**Theorem 4.1.** Let \((v_j)_{j=1}^{n}\) be unit vectors in a complex Hilbert space, and \((t_j)_{j=1}^{n}\) a sequence of positive real numbers such that \(\sum_{j=1}^{n} t_j^2 = 1\). Then, there exists a unit vector \(z\) such that

\[|\langle v_j, z \rangle| \geq t_j\]

for all \(1 \leq j \leq n\).

Giving the whole proof of this theorem would not be relevant since there is a lot of overlap with the demonstration of the plank problem on Banach spaces. Instead, we only outline the main steps of the proof and highlight the differences and new ideas. We shall see in subsequent sections that this complex plank theorem turned out to be essential in solving many other problems.
4.1.1 Reformulation and simplification of the complex plank problem

For the sake of simplicity, we suppose $t_j = 1/\sqrt{n}$. We cannot argue as in Section 3.2.1 that this is done without loss of generality using a density argument, but the more general case only requires minor modifications (see [8]). We first reformulate the problem using linear algebra, as was done previously. Let

$$A := \begin{pmatrix} -v_1 & \cdots & -v_n \end{pmatrix}.$$

The $v_j$ can be thought of as vectors in $\mathbb{C}^n$ with the usual complex inner product. Thus, $|\langle v_j, z \rangle| = |(Az)_j|$, and so it suffices to find a $z$ such that $|z| = \sqrt{n}$ and $|(Az)_j| \geq 1$. We can write such a $z$ as $U\vec{1}$ for $U$ a unitary matrix and $\vec{1}$ the 1's vector (that is, $U$ contains the weights $z_i$ as row sums). If we also let $w_j = \frac{1}{(AU\bar{1})_j}$ and $W := \text{diag}(w_1, \ldots, w_n)$, then $|w_j| \leq 1$ (since $|(AU\bar{1})_j| = |(Az)_j| \geq 1$) and $WAU\bar{1} = \bar{1}$ by construction of $W$. In particular, $\bar{1}$ is an eigenvector of $WAU$ with eigenvalue 1.

If we are hopeful that ideas akin to those used in Section 3.2.2 will work here, we should expect that we can choose $U$ such that $WAU$ is hermitian positive semidefinite (unitary and hermitian being the complex analogues of orthogonal and symmetric). This would yield $(WAU)^2 = WAU(WAU)^* = W AA^*W^*$, since $U$ is unitary, and so $WAU$ would be the square root of $W AA^*W^*$. Then it would suffice to find $W$ such that $WAA^*W^* \bar{1} = \bar{1}$. Note that $U$ does not appear in this equation anymore (at least explicitly).

Now consider $H := AA^*$, the Gram matrix of the $(v_j)_{j=1}^n$. This matrix is positive semidefinite and hermitian by construction and has 1's on the diagonal because the $v_j$ are unit vectors. We then have

$$(WAA^*W^*)_j = (WHW^*)_j = w_j \sum_k h_{jk} \bar{w}_k.$$

We have thus reduced Theorem 4.1 to proving the following.

**Proposition 4.2.** Let $H$ be an $n \times n$ complex positive semidefinite hermitian matrix with 1's on its diagonal. Then there exist complex numbers $(w_j)_{j=1}^n$ such that $|w_j| \leq 1$ and

$$w_j \sum_k h_{jk} \bar{w}_k = 1 \quad (4.1)$$

for all $1 \leq j \leq n$.

To prove this last theorem, we can interpret (4.1) as a system of $2n$ equations by splitting the real and imaginary parts and we can show the existence of a solution using Lagrange multipliers. We then form a new matrix with the diagonal entries equal to $|w_j|^2$, and we show that all of these are $\leq 1$ using the maximum principle in complex analysis, which finishes the demonstration.

4.2 Spherical plank problems

This section covers a long-standing conjecture that was recently proven to be true using arguments akin to Bang’s lemma and the plank theorems.
A great circle is the intersection of a sphere in $\mathbb{R}^2$ and a plane passing through its center, and the spherical distance denotes the shortest distance between two points on the sphere along its surface. These concepts allow us to state a spherical version of the plank problems, where the other definitions are taken mutatis mutandis. Indeed, given a great circle, we say that the analogue of a plank of width $w$ is a zone of width $w$, that is, the set of points that are at spherical distance $w/2$ or less away from the great circle.

### 4.2.1 Fejes Tóth’s zone conjecture

In 1973, the mathematician Fejes Tóth [20] conjectured that if $n$ zones of equal width $w$ cover the unit disk, then $w$ is at least $\pi/n$. He was partially motivated by the optimal establishment of $n$ operating stations on a new planet or the most economical way to explore a planet using $n$ satellites. He also formulated a more general conjecture where the widths of the zones may vary.

**Conjecture 4.3** (Fejes Tóth’s zone conjecture). The sum of the widths of the zones covering the unit disk is at least $\pi$.

![Figure 4: Five zones of equal width covering the unit disk.](image-url)

It is straightforward to generalize the previous definitions and Conjecture 4.3 to unit spheres $S^d$ embedded in $\mathbb{R}^{d+1}$, but note that the conclusion of the conjecture would stay the same: it is independent of the ambient dimension. Also, papers by Rosta [50] and Linhart [38] deal with the special cases of coverings by 3 or 4 zones, respectively.

It was only in 2017 that Jiang and Polyanskii [34] proved Conjecture 4.3. We do not present their demonstration here, but we cannot omit to comment the striking similarities it shares with Bang’s solution to Tarski’s plank problem. The authors suppose that the sum of the widths of the zones covering the unit disk is less than $\pi$ and then find a point in the disk that is not an element of any of the zones, just like Bang proceeded by contradiction and found a point in a convex body covered by planks that was not covered by any of the planks. Recall that this point found by Bang
was given by

\[ x_0 + \sum_{j=1}^{n} \epsilon_j u_j, \]

where \( x_0 \) is some point of the euclidean space, \( u_j \) are unit vectors perpendicular to the hyperplanes supporting the \( j \)th plank, and \( \epsilon_j \in \{ \pm 1 \} \); on the other hand, Jiang and Polyanskii found a point of the form

\[ \sum_j \epsilon_j u_j \sin a_j, \]

with \( a_j \) and \( u_j \) defined in a way akin to Bang’s solution. In their proof, the authors use a technique of Bognár [20], but they mention that they could have obtained the same result with Bang’s approach of optimising a well-chosen quadratic function with \( \pm 1 \) coefficients (as in the proof of Lemma 2.6).

### 4.2.2 A few remarks on Fejes Tóth’s zone conjecture

Theorem 3.8 would not be true if the sum of the \( w_i \) (i.e., of the planks) was bigger than 1, but Theorem 4.1 indicates that we can do better in complex Hilbert spaces, since we can take \( w_i \) such that the sum of the \( w_i^2 \) equals 1. One may wonder what we can say in real Hilbert spaces. It turns out that a statement as strong as Theorem 4.1 does not hold, and we can give a rather simple counterexample. For the real Hilbert space \( \mathbb{R}^2 \), consider \( n \) equally spaced unit vectors around the unit circle and their \( n \) opposite vectors (call them \( v_1, \ldots, v_{2n} \)). Then, for all unit vectors \( u \), we can find a \( k \) for which \( |\langle v_k, u \rangle| \leq \sin(\pi/2n) \). This quantity can be made arbitrarily small, so certainly smaller than our choice of \( v_k \), but using arguments akin to Remark 1.2, we can give an explicit description of a zone. If \( v \) is a unit vector on a given great circle, the associated zone of width \( w \) is

\[ \{ x \in \mathbb{T} : |\langle v, x \rangle| \leq \sin(w/2) \}, \]

where \( \mathbb{T} \) stands for the unit circle. Then the following is a reformulation of Conjecture 4.3 in the case of zones of equal widths and can be thought of as an optimal plank problem for real Hilbert spaces.

**Conjecture 4.4** (Reformulation of Fejes Tóth’s zone conjecture for zones of equal widths). Let \( v_1, v_2, \ldots, v_n \) be a sequence of unit vectors in a real Hilbert space \( \mathcal{H} \). Then there exists a unit vector \( u \in \mathcal{H} \) such that

\[ |\langle v_k, u \rangle| \geq \sin(\pi/2n) \]

for all \( 1 \leq k \leq n \).

We remark that in 2021, a new proof of the Fejes Tóth zone conjecture (for zones of equal width) was given by Ortega-Moreno [47], starting with the previous reformulation. The approach it employs is quite different from the one in Jiang and Polyanskii’s paper and uses the somewhat new concept of inverse eigenvectors in convex geometry, which are used to transform a geometry problem into a study of the behavior of polynomials (they were complex in the complex plank problem, and they are trigonometric in the proof of Conjecture 4.1). We say that a vector \( x \) is an inverse eigenvector of \( M \) if \( M = x^{-1} \), where by definition the \( i \)th component of \( x^{-1} \) is the inverse of the \( i \)th component of \( x \). These inverse eigenvectors appear implicitly in the proof of Ball’s plank theorem in the complex case, and in fact you can interpret Proposition 1.2 as saying that every
complex positive semidefinite hermitian matrix with 1’s on the diagonal has an inverse eigenvector in the complex $\ell_\infty$ unit ball.

Zhao [55] recently simplified the proof of Ortega-Moreno by eliminating the need for the notion of inverse eigenvectors. In fact, this is a striking example of how bringing in tools from trigonometry and elementary analysis greatly simplifies a problem that seemed intractable for decades. Let us present his short and neat argument.

We will show that the statement of Conjecture 4.4 holds when $u$ is a unit vector that maximizes $\prod_{i=1}^n |\langle v_i, u \rangle|$. For the sake of contradiction, suppose that $|\langle v_1, u \rangle| < \sin(\pi/2n)$ and consider the plane spanned by $\{u, v_1\}$. Then we can take a vector $w \perp u$ with $|w| < 1$ such that, if $u_\theta := (\cos \theta)u + (\sin \theta)w$, then $u_{\pi/2n} \perp v_1$ (Fig. 5).

Figure 5: Vectors appearing in the proof of Conjecture 4.4

Now consider the function

$$f(\theta) = \prod_{i=1}^n \frac{\langle v_i, u_\theta \rangle}{\langle v_i, u \rangle}.$$  

We will obtain a contradiction by bounding the number of zeros of $f(\theta) - \cos(n\theta)$ in $[0, 2\pi)$.

**Lemma 4.5.** $f(\theta) - \cos(n\theta)$ has at least $2n$ distinct zeros in $[0, 2\pi)$.

**Proof.** Since $f(\theta + \pi) = (-1)^n f(\theta)$, it will suffice to consider the region $[0, \pi)$. In this region and if $\theta \neq 0$, then $|u_\theta| < 1$ since $|w| < 1$, and by maximality of $u$, it must be that $|f(\theta)| < 1$. Since $\cos(n\theta)$ alternates between ±1 at $\pi/n, 2\pi/n, \ldots, (n-1)\pi/n$, $f(\theta) - \cos(n\theta)$ must have at least $n-2$ distinct zeros in $(\pi/n,(n-1)\pi/n)$ by the intermediate value theorem. Noting that $f(0) = 1$ since $u_0 = u$, and that $f(\pi/2n) = 0$ since $v_1 \perp u_{\pi/2n}$, it follows that $f(\theta) - \cos(n\theta)$ has two more zeros in $[0, \pi)$. Thus, $f(\theta) - \cos(n\theta)$ has at least $2n$ distinct zeros in $[0, 2\pi)$. 

**Lemma 4.6.** $f(\theta) - \cos(n\theta)$ has at most $2n-2$ distinct zeros in $[0, 2\pi)$.

**Proof.** We can write

$$f(\theta) = \prod_{i=1}^n \frac{\langle v_i, u(\cos \theta) + w \sin \theta \rangle}{\langle v_i, u \rangle} = \cos^n(\theta) + \sum_{i=1}^n \frac{\langle v_i, w \rangle}{\langle v_i, u \rangle} \cos^{n-1}(\theta) \sin(\theta) + \sin^2(\theta)\psi_1(\theta), \quad (4.2)$$

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where $\psi_1$ is some trigonometric polynomial of degree at most $n - 2$.

We saw in the proof of Lemma 4.5 that $f(\theta)$ attains its maximal value 1 at $\theta = 0$, and so $f'(0) = 0$. But $u'_0 = w$, thus $f'(0) = \prod_{i=1}^{n} \langle v_i, w \rangle / \langle v_i, u \rangle$. It follows from (4.2) that

$$f(\theta) = \cos(n\theta) = \sin^2(\theta)\psi_2(\theta),$$

where $\psi_2$ is some trigonometric polynomial of degree at most $n - 2$. Since the degree of $\sin^2(\theta)\psi_2(\theta)$ is at most $n - 1$, it must have at most $2n - 2$ distinct zeros in $[0, 2\pi)$. Then so does $f(\theta) - \cos(n\theta)$. \qed

4.3 Plank problems in other settings

We have seen examples of plank theorems for general Banach and Hilbert spaces, but we can formulate plank problems in other settings or give more precise theorem statements in some cases. For the sake of this survey, we shall only cover an extension of Ball’s plank theorems to complex $L_p(\mu)$ spaces as well as a discrete plank problem, but it is important to note that this is only the tip of the iceberg. We could talk about covering convex bodies by subspaces other than planks, by cylinders for example (see [16]), as well as inverse plank problems; that is, how to guarantee that certain systems of planks can cover a convex body (see [32], [36], and [40]). Some authors have also used techniques of symplectic geometry to study plank problems. In particular, they propose symplectic capacities (invariants of symplectic manifolds that have already proven to be useful in problems at the intersection of convex and symplectic geometry) as a vehicle to reformulate plank problems. For the specific definitions and statements given by the authors, see the paper [2] and the references therein.

4.3.1 A plank theorem on $L_p(\mu)$ spaces

The reader is advised to compare the following $L_p(\mu)$ plank theorem from [49] with Theorem 4.1.

**Theorem 4.7.** If $f_1, \ldots, f_n \in L_p(\mu)^*$ are unit functionals, then there is a point $x$ in the unit ball of $L_p(\mu)$ such that for $1 \leq k \leq n$,

$$|f_k(x)| \geq \begin{cases} n^{-1/p}, & 1 \leq p \leq 2, \\ n^{-1/q}, & 2 \leq p \leq \infty, \end{cases}$$

where $q$ is the conjugate exponent of $p$.

Observe that for $p = 1$ and $p = \infty$, we obtain $|f_k(x)| \geq 1/n$, which was already known from Ball’s plank theorems. Let us look at the simple proof of Theorem 4.7 to realize that this is, indeed, just an extension of the complex plank theorem.

**Proof.** We will need one fact about the Banach-Mazur distance between isomorphic normed spaces,

$$d(X, Y) = \inf\{\|T\|\|T^{-1}\| \mid T : X \rightarrow Y \text{ is an isomorphism}\},$$

which is that if $E$ is a $d$-dimensional subspace of $L_p(\mu)$, then $d(E, \ell_2^d) \leq d^{1/2-1/p}$ (see [37] for the proof).

Assume without loss of generality that $\dim(L_p(\mu)) = n$. As just observed, the cases $p = 1$ and $p = \infty$ are dealt with already, and when $1 < p < \infty$, $L_p(\mu)$ is reflexive so we can find $x_k$ in the unit
ball of $L_p(\mu)$ for which $|f_k(x_k)| = \|f_k\| = 1$ for all $1 \leq k \leq n$. Now consider $E := \text{span}\{x_1, \ldots, x_n\}$, which has dimension $d \leq n$. Then $g_k := f_k|_E$ is such that $\|g_k\| = 1$ and using our fact on the Banach-Mazur distance, we can find an isomorphism $T$ from $\ell_2^d$ to $E$ such that $\|T\| = 1$ and $\|T^{-1}\| \leq d^{1/2 - 1/p}$.

We are now in a position to apply Theorem 4.1 which gives us a unit $x_0 \in \ell_2^d$ for which $|g_k \circ T(x_0)| \geq \|g_k \circ T\|/\sqrt{n}$. Let $x := Tx_0$. Then $\|x\| = \|T\|\|x_0\| = 1$ and

$$|f_k(x)| = |g_k(x)| = |g_k(Tx_0)| \geq \frac{\|g_k \circ T\|}{\sqrt{n}} \geq \frac{\|g_k\|/\|T^{-1}\|}{\sqrt{n}} = \frac{1}{n^{1/2}\|T^{-1}\|} \geq \frac{1}{n^{1/2}d^{1-1/p}}.$$

This proves the claim.

\[ \square \]

### 4.3.2 A discrete plank problem

A lattice of $\mathbb{R}^d$ is an additive subgroup of $\mathbb{R}^d$ that is isomorphic to $\mathbb{Z}^d$ and that spans $\mathbb{R}^d$. Equivalently, it corresponds to the set of all linear combinations with integer coefficients of the elements of a basis of $\mathbb{R}^d$. We say that a convex set of lattice points is the intersection of a convex body with a lattice. Extending plank problems to this discrete setting, it is natural to study coverings of these convex sets by specific subspaces like hyperplanes. A long-standing and shockingly simple to state conjecture in the area, due to Corzatt [22], states that if a convex set of lattice points can be covered by $n$ lines, then these lines can be taken to have at most four different slopes (see Fig. 6 for an example).

![Figure 6: A set of points covered by lines with four different slopes.](image)

This conjecture is reminiscent of the disk case ($n = 2$) of Tarski’s problem, which was proven first, but can be extended to higher dimensions as in Tarski’s Conjecture 2.1. Recall from the Introduction that in the covering of convex bodies in the plane, a parallel covering is optimal. You can think of this as saying that we only require one parallel class to cover the convex body. Then one can wonder more generally about the previous conjecture whether for all dimension $d$ and for all sets of convex lattice points in $\mathbb{R}^d$, their covering can be taken to have at most a constant (depending on $d$) number of parallel classes. A close problem of Brass [21] asks to find the smallest constant $c(d)$ such that if a convex set of lattice points in $\mathbb{R}^d$ is covered by $n$ hyperplanes, then it can also be covered by $c(d) \cdot n$ parallel hyperplanes.

Related to this is the notion of lattice width, as investigated first by Bezdek and Hausel [17]. The lattice width of a convex body in $\mathbb{R}^d$ with respect to the integer lattice $\mathbb{Z}^d$ is defined as

$$w_{\mathbb{Z}^d}(C) = \min_{y \in \mathbb{Z}^d \setminus \{0\}} \left\{ \max_{x \in C} \langle x, y \rangle - \min_{x \in C} \langle x, y \rangle \right\}.$$
This number is used to give a lower bound on the number of hyperplanes needed to cover \( C \cap \mathbb{Z}^d \). It is conjectured that if \( n \) hyperplanes are used, then \( n \geq w_{\mathbb{Z}^d}(C) - d \). If true, this would be best possible \([17]\). It is known that the following weaker result holds \([52]\).

**Theorem 4.8.** If \( C \) is a convex body in \( \mathbb{R}^d \) covered by \( n \) hyperplanes, then

\[
 n \geq c \cdot \frac{w_{\mathbb{Z}^d}(C)}{d} - d,
\]

where \( c > 0 \) is an absolute constant.

Note that this result implies that the constant \( c(d) \) alluded to in the previous paragraph is at most \( c \cdot d^2 \).

Furthermore, Ball’s plank theorem has been used to improve Theorem 4.8 for centrally symmetric convex bodies whose lattice width is at most quadratic in dimension \([18]\). In particular, the argument relies on Corollary 6.3 that will be presented in Section 6.1. The lower bound obtained is

\[
 n \geq c \cdot \frac{w_{\mathbb{Z}^d}(C)}{d \ln(d + 1)}.
\]

## 5 Applications in analysis

We continue to survey applications of plank theorems in different fields by looking into various problems coming from analysis.

### 5.1 Polarization problems and linear polarization constants

The real polarization problem states that for any unit vectors \( u_1, \ldots, u_n \in \mathbb{R}^n \), we can find a unit \( v \in \mathbb{R}^n \) such that

\[
 \prod_{i=1}^{n} |\langle u_i, v \rangle| \geq n^{-n/2}, \quad (5.1)
\]

and that equality is attained only if the \( u_i \) form an orthonormal system in \( \mathbb{R}^n \) (see \([4], [41]\)). Similar problems arise from the concept of linear polarization constants. Formally, for \( X \) a Banach space, define the \( n \)th polarization constant of \( X \) as

\[
 c_n(X) := \inf \{ M > 0 : \|\phi_1\| \cdots \|\phi_n\| \leq M \|\phi_1 \cdots \phi_n\| \text{ for all } \phi_1, \ldots, \phi_n \in X^* \}
\]

\[
 = \inf_{\phi_1, \ldots, \phi_n \in S_{X^*}} \sup_{\|x\| = 1} |\phi_1(x) \cdots \phi_n(x)|, \quad (5.2)
\]

where \( S_{X^*} \) is the unit sphere of \( X^* \). Note that \( c_n(X) \) is an isometric property of \( X \). Here is one example of the \( n \)th polarization constant: it follows from Theorem 4.7 that

\[
 c_n(L_p(\mu)) \leq \begin{cases} 
 n^{n/p}, & 1 \leq p \leq 2, \\
 n^{n/q}, & 2 \leq p \leq \infty.
\end{cases}
\]

Furthermore, if \( \dim L_p(\mu) \geq n \), then this result is sharp when \( 1 \leq p \leq 2 \) \([49]\).

Define the polarization constant of \( X \) as

\[
 c(X) := \limsup_{n \to \infty} c_n(X)^{1/n}.
\]
As it was shown in [49], the limit always exists and so we may replace \( \limsup_{n \to \infty} \) by \( \lim_{n \to \infty} \) in the definition of \( c(X) \).

It is natural to wonder what we can infer on \( c(X) \) depending on \( X \). At first glance, it is hard to say much more than \( c(X) \geq 1 \), so perhaps it is often the case that \( c(X) = \infty \). Fortunately, it has been proven that \( c(X) = \infty \) if and only if \( \dim X = \infty \) [49]. It has also been shown that in complex Banach spaces, \( c_n(x) \leq n^n \), which is best possible [15]. Indeed, the coordinate functionals

\[
\phi_i : \ell_1^n(C) \to C \\
(z_1, \ldots, z_n) \mapsto z_i
\]

give

\[
\|\phi_1\| \cdots \|\phi_n\| = n^n \|\phi_1 \cdots \phi_n\|.
\]

Surprisingly, it was proven a few years later that the same upper bound \( c_n(X) \leq n^n \) holds true in real Banach spaces and is also best possible. It simply follows from Ball’s plank theorem 3.2 upon picking \( m_i = 0 \) and \( w_i = 1/n \) in the statement of the theorem, as first observed in [49].

While it is inherently hard to calculate the \( n \)th polarization constant of specific Banach spaces, the study of \( c_n(\ell_2^n(K)) \), where \( K = \mathbb{R} \) or \( \mathbb{C} \), is of upmost importance, since from [49] we know that

\[
c_n(\ell_2^n(K)) \leq c_n(X) \leq n^{n/2} c_n(\ell_2^n(K))
\]

(5.3)

for all infinite-dimensional Banach spaces \( X \). You can think of this lower bound as saying that Hilbert spaces have the smallest \( n \)th polarization constants among infinite-dimensional Banach spaces. The proof uses the concept of the Banach-Mazur distance between two isomorphic Banach spaces that we alluded to in Section 4.3.1.

The next step is to inquire what happens for Hilbert spaces, where we modify the definition of the polarization constant according to the Riesz representation theorem. We may also see that

\[
c_n(X) = \sup\{c_n(Y) : Y \text{ is a closed subspace of } X, \dim Y \leq n\}
\]

when \( X \) is a Hilbert space. Observe that from Ball’s complex plank theorem, there exists a unit vector \( z \) such that \( |\langle v_j, z \rangle| \geq 1/\sqrt{n} \) for unit vectors \( v_1, \ldots, v_n \) and \( 1 \leq j \leq n \), whence

\[
|\langle v_1, z \rangle| \cdots |\langle v_n, z \rangle| \geq n^{-n/2}.
\]

This is the complex version of the polarization problem presented at the beginning of the section, and comparing with (5.2), we see that it implies \( c_n(H) \leq n^{n/2} \) for \( H \) a complex Hilbert space. This was proven a few years prior to Ball’s result by Arias-de-Reyna [5] using results on permanents and Gaussian random variables, and it was also shown that \( c_n(\ell_2^n(\mathbb{C})) = n^{n/2} \), so it follows from (5.3) that \( c_n(H) = n^{n/2} \) when \( \dim H \geq n \).

Analogously to the Banach space case, it is conjectured that the same constant works for real Hilbert spaces, which was proven for \( n \leq 5 \) [48]. Similarly, it is also expected that \( c_n(\ell_2^n(\mathbb{R})) = n^{n/2} \). Note that the real polarization problem presented at the beginning of this section would imply this conjecture. We cannot expect to find a proof using known plank theorems, for even though Ball’s complex plank theorem is sharp in complex Hilbert spaces, it does not hold for real Hilbert spaces (see Section 4.2.2). Nevertheless, many authors have provided upper bounds for the polarization constant of real Hilbert spaces. A simple argument follows from the result on complex Hilbert spaces.
spaces along with the complexification of a real Hilbert space $H$, yielding $c_n(H) \leq \frac{2^{n/2-1}n^{n/2}}{2}$. The best result is from [29] and gives
\[ c_n(H) < \left( \frac{3\sqrt{3}}{e} \right)^{n/2} < (1.912n)^{n/2}. \]

Another polarization problem has been investigated, deemed more interesting from a geometric point of view. It states that given unit vectors $u_1, \ldots, u_n$ in a real Hilbert space $H$, there exists a unit vector $v \in H$ such that
\[ \sum_{i=1}^{n} \frac{1}{\langle u_i, v \rangle^2} \leq n^2. \]

Note that this is stronger than the real polarization problem presented earlier because the arithmetic-geometric mean inequality yields
\[ \prod_{i=1}^{n} \left( \frac{1}{\langle u_i, v \rangle^2} \right)^{1/n} \leq \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\langle u_i, v \rangle^2} \leq n, \]
which is equivalent to (5.1). We also remark that both of these problems can be reformulated in terms of inverse eigenvectors, which were introduced briefly in Section 4.2.2 and this connects them more deeply with the complex plank problem.

### 5.2 Applications in functional analysis

Let us first mention that plank theorems can be interpreted as extensions or strengthening of well-known theorems. For example, Conjecture 2.8 is a reformulation of Bang’s conjecture and can be interpreted as a geometric version of the pigeonhole principle. Thus, Theorem 3.1 is in some sense a geometric pigeonhole principle for convex symmetric bodies. Many other examples come from functional analysis.

#### 5.2.1 Classical results in functional analysis

Plank theorems have connections with fundamental theorems in functional analysis. Recall that the Hahn-Banach theorem roughly says that linear functionals on a linear subspace can be extended to the whole normed space while preserving the norm. Using the weak-* topology of $X^*$, we can “inverse” Theorem 3.8 to obtain a “multiple” extension of Hahn-Banach, as observed in [6].

**Corollary 5.1.** Let $(x_i)_{i=1}^{\infty}$ be unit vectors in a normed space $X$, $(m_i)_{i=1}^{\infty}$ real numbers and $(w_i)_{i=1}^{\infty}$ positive real numbers such that $\sum_{i=1}^{\infty} w_i \leq 1$. Then there exists a linear functional $\phi \in X^*$ with norm at most 1 such that
\[ |\phi(x_i) - m_i| \geq w_i \]
for all $i$.

We can also obtain a quantitative improvement of the uniform boundedness principle, or Banach-Steinhaus theorem, which we recall here for the reader’s convenience.
**Theorem 5.2** (Uniform boundedness principle). Let $X$ be a Banach space, $Y$ a normed space, and $\Phi$ a family of bounded linear operators from $X$ to $Y$. If

$$\sup_{\phi \in \Phi} \|\phi(x)\| < \infty$$

for all $x \in X$, then

$$\sup_{\phi \in \Phi} \|\phi\| < \infty.$$  

It is often useful to consider the contrapositive statement: if a family of bounded linear operators from a Banach space to a normed space is not uniformly bounded, then it is not pointwise bounded. Suppose that we have a family of linear operators on $X^*$ that are not uniformly bounded, where $X$ is a Banach space. Then their operator norm is not bounded and, in particular, the sum $\sum_{n=1}^{\infty} n\|\phi_n\|^{-1}$ will be smaller than 1 for some $\phi_1, \phi_2, \ldots$ in this family. But then, Theorem 3.8 with $m_i = 0$ gives a vector $x$ in the unit ball of $X$ such that

$$\frac{\|\phi_n(x)\|}{\|\phi_n\|^{-1}} > n\|\phi_n\|^{-1}$$

for all $n$. Rearranging, we get $|\phi_n(x)| > n$ for all $n$, and so our operators are not pointwise bounded. We can push this reasoning much further (see Section 5.2.3).

### 5.2.2 Plank theorems and linear dynamics

Plank theorems have many connections with linear dynamics. For instance, Ball’s theorems helped in obtaining results on weak hypercyclicity and supercyclicity of operators on Banach and Hilbert spaces, and more generally in answering questions related to the weak topology on Banach spaces (see [12, 27, 51]). Knowing that on infinite-dimensional Banach spaces, the weak and the norm topology do not coincide, here is one simple example of this kind of question.

**Question 5.3.** Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in an infinite-dimensional Banach space $X$. Under which condition on $x_n$ is the set $\{x_n : n \in \mathbb{N}\}$ weakly dense in $X$?

In general, one can prove that if $x_n$ is such that $\|x_n\|$ tends to infinity fast enough, then $\{x_n : n \in \mathbb{N}\}$ is weakly closed in $X$, so certainly not weakly dense. Here is a more precise sharp formulation of this idea. Proofs are given in [51, Propositions 5.2–5.4] and [12, Section 10.1.1].

**Theorem 5.4.** Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of nonzero vectors in the Banach space $X$. If

$$\sum_{n=1}^{\infty} \frac{1}{\|x_n\|} < \infty,$$

then the set $\{x_n : n \in \mathbb{N}\}$ is weakly closed in $X$. Furthermore, if $X$ is a real or complex Hilbert space, then it is sufficient that $\sum_{n=1}^{\infty} \|x_n\|^{-2} < \infty$.

**Proof.** We only prove the Banach space case here. The proof of the complex Hilbert one is virtually the same, and for the real Hilbert space, one needs to consider the complexification of $X$. It will be enough to show that the weak closure of the $x_n$ does not contain 0 (to see why, replace $x_n$ with $x_n - z$, where $z \in X \setminus \{x_n : n \in \mathbb{N}\}$).
Let \( s := \sum_{n=1}^{\infty} \| x_n \|^{-1} \) and \( \alpha_n := 1/s \| x_n \| \), so that \( \sum_{n=1}^{\infty} \alpha_n = 1 \). Then Corollary 5.4 with the unit vectors \( x_n/\| x_n \| \) yields \( \phi \) in \( X^* \) such that
\[
\langle \phi, x_n \rangle \geq 1/s
\]
for all \( n \in \mathbb{N} \) (here we use the same sesquilinear form as in Theorem 5.6). It follows that 0 is not in the weak closure of \( \{ x_n : n \in \mathbb{N} \} \).

Theorem 5.4 is tight in the sense that it has been shown that if \( X \) is an infinite-dimensional Banach space and \( (c_n)_{n \in \mathbb{N}} \) is a sequence of positive numbers such that \( \sum_{n=1}^{\infty} c_n^{-2} = \infty \), then there exists a sequence \( (x_n)_{n \in \mathbb{N}} \in X \) such that \( \| x_n \| = c_n \) for all \( n \) and 0 is in the weak closure of \( \{ x_n : n \in \mathbb{N} \} \) (see [51, Proposition 6.11]).

5.2.3 A stronger version of the uniform boundedness principle

While Theorem 5.2 is no longer true if we omit the supremum, one may wonder under what conditions it holds that there is some \( x \in X \) such that all of the bounded linear operators \( \phi \in \Phi \) diverge in norm. This question is intimately related to linear dynamics via the study of orbits of linear operators, hence the first partial results came from searching for conditions ensuring that orbits are going to infinity (see [13] and the book of Beauzamy [14, Chapter III], in particular III.2.C.1 for the following theorem).

Theorem 5.5. Let \( T \) be an operator on a Hilbert space \( H \) such that
\[
\sum_{n=1}^{\infty} \frac{1}{\| T^n \|} < \infty.
\]
Then there is a dense set of points \( x \in H \) such that \( \| T^n x \| \to \infty \) as \( n \to \infty \). Furthermore, if \( H \) is a Hilbert space, then it is sufficient that \( \sum_{n=1}^{\infty} c_n(T^n)^2 < \infty \), where
\[
c_n(T^n) := \inf\{ \| T^n \| : Z \subset X, \text{codim} Z < n \}
\]
is the \( n \)th Gelfand number of the operator \( T^n \).

The hypotheses of the previous theorem are reminiscent of Theorem 5.4 and the proof relies on Bang’s lemma. It was noted in 1990 by Beauzamy that a solution to the symmetric plank problem (that is, to Bang’s conjecture for symmetric convex bodies) would extend Theorem 5.5 more generally to a sequence of bounded operators and thus strengthen the uniform boundedness principle. This actually prompted Ball to work on plank problems. We can find a proof of this strengthening of Theorem 5.5 in a paper of Müller and Vršovský [43] for a single \( x \in H \), where the hypotheses for a complex Hilbert space are slightly weakened. It was later showed [44] that the conditions in Theorem 5.6 are sufficient to ensure that the set of \( x \in X \) such that \( \| T_n x \| \to \infty \) is dense in \( X \).

Theorem 5.6. Let \( T_n, n \in \mathbb{N} \), be a sequence of bounded linear operators on a Banach space \( X \). If
\[
\sum_{n=1}^{\infty} \frac{1}{\| T_n \|} < \infty,
\]
then there exists \( x \in X \) such that \( \| T_n x \| \to \infty \). Furthermore, if \( X \) is a complex Hilbert space, then it is sufficient that \( \sum_{n=1}^{\infty} \| T_n \|^{-2} < \infty \).
Proof. We treat the case where $X$ is Banach. A few modifications are needed in the complex Hilbert case (see [43]) and we need to use the complex plank theorem. We will use the sesquilinear form $\langle x, f \rangle := f(x)$, where $x \in X$ and $f \in X^*$.

Choose $\beta_n$ positive real numbers going to infinity such that
\[
s := \sum_{n=1}^{\infty} \frac{\beta_n}{\|T_n\|} < \infty,
\]
and define
\[
\alpha_n := \frac{1}{s + 1} \frac{\beta_n}{\|T_n\|}.
\]
For all $n$, take $g_n \in X^*$ such that $\|g_n\| \leq 1$ and $\|T_n^* g_n\| \geq \|T_n^*\|/2 = \|T_n\|/2$. Finally, define
\[
f_n := \frac{T_n^* g_n}{\|T_n^* g_n\|} \in X^*.
\]
We can apply the plank theorem 3.8 with these linear functionals. Indeed, since the coefficients $\alpha_n$ satisfy
\[
\sum_{n=1}^{\infty} \alpha_n = s(s + 1)^{-1} < 1,
\]
we know there exists $x \in X$ with $\|x\| \leq 1$ and $|\langle x, f_n \rangle| \geq \alpha_n$ for all $n$. Then,
\[
\|T_n x\| \geq \|T_n x\| \|g_n\| \geq |\langle T_n x, g_n \rangle| = |\langle x, T_n^* g_n \rangle| = \|T_n^* g_n\| |\langle x, f_n \rangle| \geq \frac{\|T_n\|}{2} \alpha_n.
\]
This expression simplifies to $\beta_n(2s + 2)^{-1}$ and thus tends to $\infty$ with $n$ by our choice of $\beta_n$.

We end this section by noting that Theorem 5.6 is optimal in the sense that there exists a Banach space $X$ and operators $T_n$ such that
\[
\sum_{n=1}^{\infty} \frac{1}{\|T_n\|^{1+\varepsilon}} < \infty,
\]
yet there is no $x \in X$ with $\|T_n x\| \to \infty$. An explicit construction is given in [43].

5.3 Applications in harmonic analysis

The Riemann-Lebesgue lemma says that the Fourier coefficients of an $L_1(\mathbb{T})$ function tend to 0 as $|n| \to \infty$. It is natural to wonder how fast this happens. In particular, can this decay be arbitrary slow? Kolmogorov answered this question positively in 1923. More precisely, he showed that for any choice of sequence of positive integers $(a_n)_{n \in \mathbb{Z}}$ with $a_n \to 0$ as $|n| \to \infty$, there exists a function $f \in L_1(\mathbb{T})$ such that $\hat{f}(n) \geq a_n$ for all $n$. A similar problem, asking whether all square summable sequences are dominated by the Fourier coefficients of some continuous function on the unit circle $\mathbb{T}$, turned out to be much harder to settle. Luckily, the mathematicians de Leeuw, Kahane and Katznelson [25] were able to provide a positive answer in 1977. They even showed that we may take this function to be bounded on $\mathbb{T}$. You can think of their result as saying that we cannot distinguish functions in $L_2(\mathbb{T})$ from functions in $C(\mathbb{T})$ if we only look at the size of their Fourier coefficients.
Theorem 5.7 (de Leeuw–Kahane–Katznelson). Let \((a_n)_{n \in \mathbb{Z}} \in \ell_2(\mathbb{Z})\). Then there exists \(f \in C(T)\) and an absolute constant \(A > 0\) such that \(\|f\|_\infty \leq A \left( \sum_{n \in \mathbb{Z}} |a_n|^2 \right)^{1/2}\) and
\[|\hat{f}(n)| \geq |a_n|\]
for all \(n\).

What is surprising with the proof of this theorem is that it considers sums of the form
\[\sum_{i=1}^\infty \epsilon_i a_i \psi_i,\]
where \((\psi_n)_{n=1}^\infty\) is a sequence of functions and \((\epsilon_n)_{n=1}^\infty\) is a random choice of signs, which is reminiscent of Bang’s lemma. See the beginning of the proof of Theorem 5.8 below for an explicit example of how this kind of sum might appear.

We can go further with these ideas and ask: Given a sequence of functions \(\psi_n \in L_1(T)\), under which conditions can we guarantee that there exists a bounded function \(f\) such that
\[|\langle f, \psi_n \rangle| > |a_n|\]
for all sequences \((a_n)_{n=1}^\infty \in \ell_2(\mathbb{Z})\)? Unfortunately, the de Leeuw–Kahane–Katznelson theorem does not answer this question in general. However, many years later, a theorem of Nazarov solved this problem and it was proven using ideas coming from plank theorems. This is self-evident upon reading the title of Nazarov’s paper: *The Bang solution of the coefficient problem* [45]. In fact, according to Nazarov himself, the modification in the demonstration of Bang’s result needed to prove the following theorem was so minor that he claimed “[...] I actually even do not pretend to be an author of the next two sections; rather I act there like a shadow that enters and goes over many strange places which completely eliminate the attention of his master just passing by.” We remark that Nazarov’s original article is in Russian, but an English translation is available [45].

Theorem 5.8 (Nazarov). Let \(T\) be a probability space with measure \(\mu\) and \((\psi_n)_{n=1}^\infty\) a sequence of unit functionals in \(T^*\) such that
\[\left\| \sum_{i=1}^\infty c_i \psi_i \right\|_2 \leq \left( \sum_{i=1}^\infty c_i^2 \right)^{1/2}\] (5.4)
for all sequences of real coefficients \((c_n)_{n=1}^\infty\). For \(2 \leq p \leq \infty\) and \(q\) its conjugate exponent, suppose that
\[\|\psi_n\|_q \geq \beta > 0\] (5.5)
for all \(n\). Let \(a_i\) be positive numbers such that \(\sum_{i=1}^\infty a_i^2 = 1\). Then there exists \(F \in L_p(T)\) such that
\[\|F\|_p \leq \left( \frac{3\pi}{2} \right)^{1-2/p} \beta^{-2}\]
and, for all \(n\),
\[|\langle F, \psi_n \rangle| \geq a_n.\]
Proof. For a choice of signs \( \epsilon = (\epsilon_i)_{i=1}^\infty \), let
\[
f_\epsilon := \sum_{i=1}^{\infty} \epsilon_i a_i \psi_i
\]
and note that from \([5.4]\) and the assumption \( \sum_{i=1}^{\infty} a_i^2 = 1 \), it follows that \( f_\epsilon \in L_2(T) \) (more precisely, \( \|f_\epsilon\|_2 \leq 1 \) for all \( \epsilon \)). Theorems of existence and uniqueness of ODEs allow us to define the (unique) function \( \Phi \) by the initial conditions \( \Phi(0) = 0 \) and the differential equation
\[
\Phi''(x) = (1 + x^2)^{2/p - 1}.
\]

From the definition of \( \Phi \), the integral \( I(f) := \int_T \Phi(f(x)) \, d\mu \) is well-defined and continuous in \( L_2(T) \). Moreover, since \( \{f_\epsilon\} \) is compact in the topology of \( L_2(T) \), the integral attains a maximum for some \( f_\tau \). The \( F \) we pick will be a rescaling of \( \Phi'(f(\tau)) \), so let us start by showing that \( \Phi'(f(\tau)) \in L_p(T) \).

By the definition of \( \Phi(x) \) and Hölder’s inequality,
\[
|\phi'(x)| = \left| \int_0^x (1 + t^2)^{2/p - 1} \, dt \right| \leq \int_0^{|x|} (1 + t^2)^{2/p - 1} \, dt \leq \left( \int_0^{|x|} dt \right)^{2/p} \left( \int_0^{|x|} (1 + t^2)^{-1} \, dt \right)^{1-2/p}.
\]

These integrals evaluate to \( |x|^{2/p} (\arctan |x|)^{1-2/p} \leq |x|^{2/p} (\pi/2)^{1-2/p} \), so that
\[
\|\Phi'(f(\tau))\|_p \leq \left( \frac{\pi}{2} \right)^{1-2/p} \left( \int_T |f(\tau)|^2 \, d\mu \right)^{1/p} \leq \left( \frac{\pi}{2} \right)^{1-2/p},
\]
using our previous observation that \( \|f_\epsilon\|_2 \leq 1 \) for all \( \epsilon \). Furthermore, we see that upon taking \( F := 3^{1-2/p} \beta^{-2} \Phi'(f(\tau)) \), we have
\[
\|F\|_p \leq \left( \frac{3\pi}{2} \right)^{1-2/p} \beta^{-2},
\]
as desired.

Now we want to show that for all \( j \),
\[
a_j \leq |\langle F, \psi_j \rangle| = 3^{1-2/p} \beta^{-2} \left| \int_T \Phi'(f(\tau)) \psi_j \, d\mu \right|.
\]

For each \( j \), let \( f_j := f_\tau - 2T_j a_j \psi_j \), that is, flip the sign of the \( j \)th term in the sum defining \( f_\tau \). Since \( f_\tau \) maximizes \( I(\cdot) \), we have \( 0 \leq \int_T (\Phi(f_\tau) - \Phi(f_j)) \, d\mu \) for each \( j \) and by the mean value theorem, there exists a function \( g \) between \( f_j \) and \( f_\tau \) such that
\[
\int_T (\Phi(f_\tau) - \Phi(f_j)) \, d\mu = \int_T \left( \Phi'(f_\tau)(f_\tau - f_j) + \frac{1}{2} \Phi''(g)(f_\tau - f_j)^2 \right) \, d\mu
= 2a_j \int_T \Phi'(f_\tau) \psi_j + 2a_j^2 \int_T \Phi''(g) \psi_j^2 \, d\mu.
\]

Since \( f_\tau - f_j = 2T_j a_j \psi_j \), we have
\[
0 \leq 2a_j \int_T \Phi'(f_\tau) \psi_j + 2a_j^2 \int_T \Phi''(g) \psi_j^2 \, d\mu,
\]
whence
\[
\left| \int_T \Phi'(f_\tau) \psi_j \, d\mu \right| \geq a_j \int_T \Phi''(g) \psi_j^2 \, d\mu.
\]
These previous steps are reminiscent of the strategy adopted in the proof of Bang’s Lemma.

We see from (5.6) that we will be done if we can show that \( \int_T \Phi''(g) \psi_j^2 \, d\mu \geq 3^{2/p-1} \beta^2 \). Since \( \frac{q}{2} + (1 - \frac{2}{p}) \beta \geq 1 \), Hölder’s inequality gives

\[
\left( \int_T (1 + g^2) \, d\mu \right)^{1-q/2} \left( \int_T (1 + g^2)^{2/p-1} \psi_j^2 \, d\mu \right)^{q/2} \geq \int_T |\psi_j|^q \geq \beta^q,
\]

where we used (5.5) to obtain the last inequality. Since \( g \) lies between \( f_j \) and \( f_\epsilon \), we have \( g^2 \leq f_j^2 + f_\epsilon^2 \) and thus \( \int_T (1 + g^2) \, d\mu \leq \int_T (1 + f_j^2 + f_\epsilon^2) \, d\mu \leq 3 \), using that \( \|f_\epsilon\|_2 \leq 1 \) for all \( \epsilon \). We conclude that

\[
\int_T \Phi''(g) \psi_j^2 \, d\mu = \int_T (1 + g^2) \psi_j^2 \, d\mu \geq 3^{1-2/q} \beta^2 = 3^{2/p-1} \beta^2.
\]

The following corollary of Nazarov’s theorem gives a more direct answer to the coefficient problem. It follows from the \( p = \infty \) case of Theorem 5.8 and an application of a weak form of Grothendieck’s inequality (see [9], a paper of Ball that talks about Nazarov’s theorem, among other things).

**Corollary 5.9.** Let \( (\psi_n)_{n=1}^\infty \) be functions in \( L_1(T) \) such that there exists a constant \( M > 0 \) with

\[
\left\| \sum_{i=1}^\infty c_i \psi_i \right\|_1 \leq M \left( \sum_{i=1}^\infty c_i^2 \right)^{1/2}
\]

for all sequences of real coefficients \( (c_n)_{n=1}^\infty \). Then, for all \( (a_n)_{n=1}^\infty \in \ell_2(\mathbb{Z}) \), there exists a bounded function \( f \) such that

\[
|\langle f, \psi_n \rangle| > |a_n|
\]

for all \( n \).

### 5.3.1 Applications and extensions of Nazarov’s result

The coefficient problem is intimately related to applied problems via the Fourier transform, and so Nazarov’s result has recently attracted attention in signal processing and electrical engineering. For instance, an important mathematical problem in computational imaging is to develop optimal coded apertures. In [11], their fundamental limits were characterized using Theorem 5.8 and a greedy algorithm that relies on the proof of this theorem was proposed. Also see [19] for a discussion of various problems in signal processing where Theorem 5.8 might come in handy. In particular, observe that Theorem 5.8 does not require orthogonality of the unit functions, but only asks for a \( \ell_2 \) estimate, hence it can be used in problems regarding frames and bases in more generality.

Both the results of de Leeuw, Kahane, and Katznelson, and Nazarov, give insight on the spectral structure of large sets in additive combinatorics, as observed for example in works of Green [31]. Here, for a function \( f : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C} \) and \( r \in \mathbb{Z}/N\mathbb{Z} \), the Fourier transform of \( f \) at \( r \) is defined as

\[
\hat{f}(r) = \sum_x f(x)e(rx/N),
\]

where \( e(x) = \exp(2\pi ix) \). As mentioned by Green, this result implies that the only information we can obtain on the large spectrum of a large subset of \( \mathbb{Z}/N\mathbb{Z} \) comes from Parseval’s theorem (via Chang’s theorem, see [31]).
Theorem 5.10 (Corollary of Theorem 5.8). Let \( \alpha_r, \ r \in \mathbb{Z}/N\mathbb{Z} \), be positive reals satisfying \( \sum_r \alpha_r^2 \leq N/1600 \). Then there is a function \( f : \mathbb{Z}/N\mathbb{Z} \to [0,1] \) such that \( |f| = \sum_x f(x) = N/2 \), and so that

\[
|\hat{f}(r)| \geq \alpha_r |f|
\]

for all \( r \in \mathbb{Z}/N\mathbb{Z} \).

Finally, we want to highlight the fact that soon after Nazarov’s result, Lust-Piquard [39] obtained noncommutative versions of the de Leeuw–Kahane–Katznelson and Nazarov theorems. In particular, the reader is invited to compare Theorem 5.7 with the following statement, where we say that a matrix \( A \) is in \( \ell_\infty(\ell_2) \) if \( \sup_i (\sum_j |a_{ij}|^2)^{1/2} < \infty \).

Theorem 5.11. Let \( A \) be such that \( A \) and \( A^* \) are in \( \ell_\infty(\ell_2) \). Then there exists a matrix \( B \) that defines a bounded operator from \( \ell_2 \) to \( \ell_2 \) such that

\[
\|B\|_{\ell_2 \to \ell_2} \leq \sqrt{6} \max\{\|A\|_{\ell_\infty(\ell_2)}, \|A^*\|_{\ell_\infty(\ell_2)}\}
\]

and \( |b_{ij}| \geq |a_{ij}| \) for all \( i,j \geq 0 \). Moreover, the constant \( \sqrt{6} \) is best possible.

These results extend more generally to von Neumann algebras and it is possible to replace \( \ell_2 \to \ell_2 \) with other spaces \( \ell_p \to \ell_q \) modulo minor modifications.

It is interesting to note that this result was a central element in a new proof [24] of a well-known result of Varopoulos on a characterization of Schur multipliers, which are bounded operators from \( \ell_2 \to \ell_2 \) that act by entrywise multiplication with a matrix. The following is a restatement of Varopoulos’s result to resemble Theorem 5.11.

Theorem 5.12. Let \( S(A) \) be the set of symbols with matrix entries \( b_{ij} \) satisfying \( |b_{ij}| \leq |a_{ij}| \). Then the following are equivalent:

1. every symbol in \( S(A) \) defines a Schur multiplier;
2. we can write \( A = C + D \) where \( C \in \ell_\infty(\ell_2) \) and \( D^* \in \ell_\infty(\ell_2) \);
3. for all finite sets of indices \( I \) and \( J \),

\[
\sum_{i \in I} \sum_{j \in J} |a_{ij}|^2 = O(|I| + |J|).
\]

6 Applications in number theory

In this section, we present some applications of plank theorems in number theory. It is interesting to note that alternatively, number theory can sometimes give insight into plank problems. For instance, Fukshansky [30] used his bounds on the height of integral points of small height outside a hypersurface defined by a nonzero integer polynomial over \( \mathbb{Q} \) to obtain a lower bound for a discrete plank problem. This problem concerns coverings of the set of integer lattice points in \( \mathbb{R}^d \) contained in a cube by sublattices of rank \( d - 1 \). See Section 4.3.2 for a discussion on discrete analogues of plank problems.
6.1 Simultaneous Diophantine approximation

Recall the classical result of Dirichlet in Diophantine approximation which states that for all real \( \theta \), there exist infinitely many natural numbers \( q \) such that

\[
\|q\theta\| \leq \frac{1}{q},
\]

where \( \| \cdot \| \) denotes the distance to the nearest integer. We say that \( \theta \) is **badly approximable** if the right-hand side of the previous inequality cannot be improved by any positive constant, that is, if there exists \( c = c(\theta) > 0 \) such that \( \|q\theta\| > cq^{-1} \) for all \( q \in \mathbb{N} \). The set of badly approximable numbers has Lebesgue measure zero, but full Hausdorff dimension. These numbers are also closely related to Littlewood’s conjecture, a famous and long-standing open problem in Diophantine approximation, which says that if \( \theta, \varphi \in \mathbb{R} \), then

\[
\liminf_{n \to \infty} n\|n\theta\||n\varphi|| = 0.
\]

Since there is also a **simultaneous** Diophantine approximation theorem which states that for any \( \theta, \varphi \in \mathbb{R} \), there exist infinitely many \( q \in \mathbb{N} \) such that

\[
\max\{\|q\theta\|^2, \|q\varphi\|^2\} \leq \frac{1}{q},
\]

it is natural to define a **pair of badly approximable numbers** as a pair of reals \( \theta, \varphi \) such that there exists a positive constant \( c = c(\theta, \varphi) \) which verifies

\[
\max\{\|q\theta\|^2, \|q\varphi\|^2\} > \frac{c}{q}
\]

for all \( q \in \mathbb{N} \). Davenport showed that the set of these pairs is uncountable, and it was later shown that in fact, it has Hausdorff dimension two. Davenport [23] also proved the following related theorem on simultaneous Diophantine approximation.

**Theorem 6.1** (Davenport, 1962). Let \((\lambda_i)_{i=1}^n\) and \((\mu_i)_{i=1}^n\) be real numbers. Then there exist \( \alpha \) and \( \beta \) such that \( \alpha + \lambda_i \) and \( \beta + \mu_i \) form a pair of badly approximable numbers for \( i = 1, 2, \ldots, n \). Furthermore, we may take the constant \( c = 2^{-4n-7} \) in the definition of a pair of badly approximable numbers.

This result has connections with convex geometry as Davenport’s solution uses geometric arguments. More precisely, he proves by induction the existence of a square in the plane such that all points \((\alpha, \beta)\) in that square satisfy the requirements of the theorem. The generalization to higher dimensions and to simultaneous Diophantine approximation of many reals is immediate. We can also infer from this result another type of geometric pigeonhole principle [3].

**Theorem 6.2.** Let \( C \) be a cube in \( \mathbb{R}^d \) and \((H_i)_{i=1}^n\) hyperplanes. Then we can find in \( C \) another cube having the same orientation, being at least \( 2^{-n} \) times as wide as \( C \), and such that none of the \( H_i \) intersect its interior.

As mentioned in Section 2.2.1, Alexander already observed that a positive answer to Bang’s conjecture would improve the previous theorem. As such, Theorem 3.2 also improves it and, in particular, implies that for a symmetric convex body \( C \), we can find a set that does not intersect any of the hyperplanes \( H_i \) via a well-chosen dilation and translation of \( C \) [6].
Corollary 6.3. Let $C$ be a symmetric convex body in $\mathbb{R}^d$ and $(H_i)_{i=1}^n$ hyperplanes. Then there exists a vector $x$ such that $x + \frac{1}{n+1}C$ is in $C$ and such that none of the $H_i$ intersect the interior of $x + \frac{1}{n+1}C$.

Proof. Without loss of generality, we take $C$ centered at the origin. Recall that $C$ being a symmetric convex body, it is the unit ball of some finite-dimensional normed space, so it makes sense to define $X$ such that $C$ corresponds to its unit ball. Consider the unit functionals $\phi_i \in X^*$ and the real numbers $m_i$ that correspond to the hyperplanes

$$H_i = \{ x \in \mathbb{R}^d : \phi_i(x) = m_i \}.$$ 

Theorem 3.2 gives $x' \in C$ such that

$$\left| \phi_i(x') - \frac{n+1}{n}m_i \right| \geq 1/n.$$ 

Changing the scale by a factor of $n/(n+1)$, we have an $x \in \frac{n}{n+1}C$, explicitly $x = \frac{n}{n+1}x'$, such that

$$|\phi_i(x) - m_i| \geq \frac{1}{n+1}$$

for all $i$, and so $x + \frac{1}{n+1}C \subseteq C$. Observe that $x \notin H_i$, else $0 \geq (n+1)^{-1}$. We just have to show that $\text{int}(x + \frac{1}{n+1}C)$ will not be sliced by any of the $H_i$. Let $y \in \text{int}(x + \frac{1}{n+1}C)$. Then $\|y - x\| \leq (n+1)^{-1}$ since $C$ is the unit ball of $X$, whence

$$|\phi_i(y) - m_i - (\phi_i(x) - m_i)| = |\phi_i(y - x)| \leq \|\phi_i\||y - x| \leq \frac{1}{n+1}.$$ 

Taking $n \to \infty$ and using the fact that $x \notin H_i$, we see that $\phi_i(y) - m_i \neq 0$, that is, $y \notin H_i$, and moreover, $\phi_i(x) - m_i$ and $\phi_i(y) - m_i$ have the same sign. Thus, $\text{int}(x + \frac{1}{n+1}C)$ and $x$ always lie on the same side of $H_i$. \qed

6.2 Sphere packings

Recall from Section 4.3.2 the definition of a lattice of $\mathbb{R}^n$. Geometrically, a lattice can be thought of as a regular paving of Euclidean space. The study of lattices connects convex geometry and number theory. For instance, Ball \[7\] saw a link between sphere packings and Bang’s lemma after working on his plank theorems. This allowed him to obtain the asymptotically best lower bound on the density of sphere packings in $\mathbb{R}^n$, which has only been improved by a constant factor since then \[54\]. But what is of particular interest to us is his novel approach rather than the exact expression for the lower bound itself, so we shall describe it now.

A sphere packing in $\mathbb{R}^n$ is a family of balls of the same radius in $\mathbb{R}^n$ with centers on the coordinates of a lattice and with disjoint interiors, and its density is the proportion of space covered by this sphere packing. A classical result of Minkowski says that for all $n$, we can find a sphere packing in $\mathbb{R}^n$ with density at least $2^{1-n}\zeta(n)$, where $\zeta(\cdot)$ is Riemann’s zeta function, but you can think of $\zeta(n)$ as some term tending to 1 from above with $n$. The best lower bound for a long time was a result of Davenport and Rogers and indicated a lower bound of roughly $1,68 \cdot n^2^{-n}$. Here is Ball’s improvement on the lower bound.
Theorem 6.4. For all \( n \geq 1 \), there exists a sphere packing in \( \mathbb{R}^n \) having density greater than or equal to \( 2(n - 1)2^{-n}\zeta(n) \).

It is straightforward to compare the three previous lower bounds for different values of \( n \). Note that Ball did not fully justify the apparition of \( \zeta(n) \) in his result, because this term appears naturally after using Möbius inversion (it is, in fact, the same argument that Minkowski used).

We present here the arguments of Ball that make use of the ideas of plank theorems and we only sketch the other arguments. A few remarks will first allow us to simplify the proof.

Remark 6.5. The determinant of a lattice is an invariant defined as the absolute value of the determinant of the matrix representation of the linear isomorphism between this lattice and \( \mathbb{Z}^n \). In particular, the density of a sphere packing of volume \( V \) in \( \mathbb{L} \) is \( V/\det \mathbb{L} \). Moreover, we say that \( \mathbb{L} \) is admissible with respect to a symmetric convex body \( C \) if the only point of \( \mathbb{L} \) that the interior of \( C \) contains is \( 0 \). Take \( C \) to be the ball of radius \( 2r \) centered at \( 0 \). Then a lattice \( \mathbb{L} \) contains a sphere packing of balls of radius \( r \) if and only if \( \mathbb{L} \) is admissible with respect to \( C \).

Without loss of generality, we will be looking for a lattice with determinant \( 1 \), so that its density coincides with the volume of the balls in the sphere packing. Hence, Ball’s result with follow if we find a determinant \( 1 \) lattice that is admissible for the ball of volume \( 2(n - 1) \) centered at \( 0 \). But by a compactness argument, it suffices to find a family of determinant \( 1 \) lattices that are admissible for the ball of volume \( \geq 2(n - 1) - o(1) \). Equivalently, we can fix the lattice and find a family of convex symmetric bodies that are admissible with respect to this lattice starting with the balls. After a deformation of these balls into ellipsoids that is to be determined precisely and upon taking the simple lattice \( \mathbb{Z}^n \), we are looking for a family of ellipsoids \( \{E_R\} \) admissible with respect to \( \mathbb{Z}^n \) and of volume

\[
\text{Vol}(E_R) \geq 2(n - 1) - o(1) \tag{6.1}
\]
as \( R \to \infty \).

Proof of Theorem 6.4. Define the ellipsoids \( E_R \) mentioned in Remark 6.5 by the equation

\[
(u, x)^2 + \|x\|^2 \leq R^2 \tag{6.2}
\]
for \( x \in \mathbb{R}^n \) and \( u = u(R) \) to be chosen to satisfy the inequality (6.1). Since an ellipsoid is nothing but the deformation of a ball under a linear transformation, the volume of \( E_R \) is simply

\[
\text{Vol}(E_R) = \frac{1}{\sqrt{1 + K^2}} \text{Vol}(B_R), \tag{6.3}
\]
where \( K = K(R) \) is the length of \( u \) and \( B_R \) the ball centered at the origin and of radius \( R \). Clearly, when \( R \) tends to infinity in (6.2), then the length of \( u \) can also tend to infinity. As such, it will not make a difference if we obtain that (6.3) is at least \( 2(n - 1) - o(1) \) with \( 1/\sqrt{1 + K^2} \) or \( 1/K \). We will thus show that

\[
\frac{K}{\text{Vol}(B_R)} \leq \frac{1}{2(n - 1)} + o(1) \tag{6.4}
\]
as \( R \to \infty \).

By the definition of an admissible lattice, \( \mathbb{Z}^n \) is admissible for \( E_R \) if

\[
(u, z)^2 + \|z\|^2 \geq R^2 \tag{6.5}
\]
for all \( z \in \mathbb{Z}^n \setminus \{0\} \). But (6.5) holds if and only if
\[
\left\langle u, \frac{z}{\|z\|} \right\rangle \geq (\frac{R^2}{\|z\|^2} - 1)^{1/2}.
\] (6.6)

Moreover, it is clear that if \( \|z\| \geq R \), then (6.5) holds. Hence it is enough to consider the \( z \) such that \( 0 < \|z\| < R \) (for a given norm that satisfies this, we fix an arbitrary choice between \( \pm z \)). Since the right-hand side of equation (6.6) (call it \( \theta_z \)) is then nonnegative and \( z/\|z\| \) is unitary, we know by Lemma 2.6 (Bang’s lemma) that we can find a \( u \) satisfying (6.6) of the form
\[
u(R) = \frac{1}{2} \sum_{0 < \|z\| < R} \epsilon_z \theta_z \frac{z}{\|z\|},
\]
where \( \epsilon_z \in \{\pm 1\} \) (the 1/2 factor comes from the fact that we apply Bang’s lemma with both choices of \( \pm z \) for every \( z \) such that \( 0 < \|z\| < R \), even though we only want to end up with half of these values). Now that we know that \( \mathbb{Z}^n \) is admissible for \( E_R \) with this choice of \( u(R) \), we have to show that (6.4) holds. Since the only unknown in this equation is now \( K \), we will try to estimate this value. We have
\[
K = \|u\| = \sqrt{\langle u, u \rangle} = \langle u, v \rangle
\]
where \( v = u/\|u\| \). By definition and linearity, this expression is
\[
\frac{1}{2} \sum_{0 < \|z\| < R} \epsilon_z \frac{\langle z, v \rangle}{\|z\|} \theta_z \leq \frac{1}{2} \sum_{0 < \|z\| < R} \frac{|\langle z, v \rangle|}{\|z\|} \left( \frac{R^2}{\|z\|^2} - 1 \right)^{1/2}.
\]
Call this last expression \( K' \).

The rest of the proof takes us too far afield from the usual type of plank problems arguments, but let us mention that Ball establishes that \( K'/R^n \) approximates a certain integral over the unit ball, and that in particular
\[
\frac{K'}{\text{Vol}(B_R)} \rightarrow \frac{n}{2} \left( \int_0^{\pi/2} \sin^{n-2} t \, dt \right)^{-1} \int_0^{\pi/2} \cos t \sin^{n-2} t \, dt \int_0^1 (1 - r^2)^{1/2} r^{n-2} \, dr.
\]
With an appropriate change of variables, we find
\[
\int_0^1 (1 - r^2)^{1/2} r^{n-2} \, dr = \frac{1}{n} \int_0^{\pi/2} \sin^{n-2} t \, dt,
\]
and so
\[
\frac{K'}{\text{Vol}(B_R)} \rightarrow \frac{1}{2} \int_0^{\pi/2} \cos t \sin^{n-2} t \, dt.
\]
This last integral can be computed directly and equals \( (n - 1)^{-1} \), proving (6.4).

\[\square\]

**Acknowledgments**

The author would like to thank Thomas Ransford for reviewing earlier versions of this work and providing plenty of helpful comments that helped improve the quality of this survey. The author is also grateful to Keith Ball for pointing out the work of Beauzamy that is presented in Section 5.2.3.
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William Verreault, Département de mathématiques et de statistique, Université Laval, Québec G1V 0A6, Canada

*E-mail address: william.verreault.2@ulaval.ca*