Abelian groups with a $p^2$-bounded subgroup, revisited

By

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Abstract. Let $\Lambda$ be a commutative local uniserial ring of length $n$, $p$ a generator of the maximal ideal, and $k$ the radical factor field. The pairs $(B, A)$ where $B$ is a finitely generated $\Lambda$-module and $A \subseteq B$ a submodule of $B$ such that $p^mA = 0$ form the objects in the category $S_m(\Lambda)$. We show that in case $m = 2$ the categories $S_2(\Lambda)$ are in fact quite similar to each other: If also $\Delta$ is a commutative local uniserial ring of length $n$ and with radical factor field $k$, then the categories $S_2(\Lambda)/N_\Lambda$ and $S_2(\Delta)/N_\Delta$ are equivalent for certain nilpotent categorical ideals $N_\Lambda$ and $N_\Delta$. As an application, we recover the known classification of all pairs $(B, A)$ where $B$ is a finitely generated abelian group and $A \subseteq B$ a subgroup of $B$ which is $p^2$-bounded for a given prime number $p$.

1. History and Introduction

Let $\Lambda$ be a commutative local uniserial ring of length $n$ with radical generator $p$ and radical factor field $k = \Lambda/p$. We consider pairs $(B; A)$ where $B$ is a finitely generated $\Lambda$-module and $A$ a submodule of $B$. Such pairs form the objects in the category $S(\Lambda)$; a morphism from $(B; A)$ to $(D; C)$ is given by a map $f : B \to D$ which satisfies $f(A) \subseteq C$. We are particularly interested in the full subcategories $S_m(\Lambda) \subset S(\Lambda)$ (for $m \leq n$ a natural number) which consist of those pairs $(B; A)$ that satisfy $p^mA = 0$. For example if $\Lambda = \mathbb{Z}/p^n$ then we are dealing with pairs $(B; A)$ where $B$ is a finite abelian $p^n$-bounded group and $A \subseteq B$ a subgroup satisfying $p^mA = 0$.

Each category $S_m(\Lambda)$ has the Krull-Remak-Schmidt property, so every object has a unique direct sum decomposition into indecomposable ones. Examples for indecomposable objects are pickets which are pairs $(B; A)$ where the $\Lambda$-module $B$ itself is indecomposable, hence cyclic.
Since $\Lambda$ is uniserial, each picket $(B; A)$ is determined uniquely by the lengths $\ell$ and $m$ of the $\Lambda$-modules $B$ and $A$; we write $P^\ell_m = (B; A)$.

Clearly, the complexity of categories of type $S_m(\Lambda)$ increases with $m$. The categories $S_0(\Lambda)$ and $\text{mod}\Lambda$ are equivalent so the only indecomposable objects in $S_0(\Lambda)$ are the pickets of type $P^\ell_0$. In the category $S_1(\Lambda)$ (which we consider briefly in Section 3), every indecomposable object is a picket of type $P^\ell_0$ or $P^\ell_1$. The category $S_2(\Lambda)$ contains additional indecomposables which are not pickets; it turns out that an invariant which has been introduced by Prüfer [3, §7] in 1923 provides an efficient classification.

**Definition.** Let $B$ be a $\Lambda$-module and $a \in B$ a nonzero element. The **height exponent** of $a$ is

$$h_B(a) = h(a) = \max\{n \in \mathbb{N}_0 : a = p^n b \text{ for some } b \in B\};$$

the **height sequence** $H_B(a) = (h(a), h(pa), \ldots, h(p^\ell a))$ consists of the height exponents of the nonzero $p$-power multiples of $a$.

**Example 1.1.** For pickets, the height sequence consists of consecutive numbers: In a picket $(B; A) = P^\ell_m$ where $m > 0$, any generator for $A$ has the height sequence $(\ell - m, \ell - m + 1, \ldots, \ell - 1)$.

**Example 1.2.** If $m \geq 2$ then there are height sequences which cannot be realized by pickets. The sequence $(s - 1, t - 1)$ where $s < t - 1$ is realized by the pair $Q^t_s = (B; A)$ where $B = \Lambda/(p^t) \oplus \Lambda/(p^s)$ and $A = (p^{t-2}, p^{s-1})\Lambda$.

All pickets and all indecomposables of type $Q^t_s$ have the property that the subgroup is either zero or cyclic. This is always the case for indecomposable objects in $S_2(\Lambda)$ according to [2, Theorem 4]:

**Theorem 1.3.** Each pair $(B, A) \in S_2(\mathbb{Z}/p^n)$ is a direct sum of indecomposable pairs; if $(B, A)$ is an indecomposable pair then $A$ is either zero or cyclic.

A description of the indecomposable objects in terms of standard forms of matrices is given in [1, Theorem 7.5]. It turns out that whenever the pair $(B; A)$ is indecomposable with $A$ nonzero, then the height sequence $H_B(a)$ of a subgroup generator $a$ uniquely determines the isomorphism type of the given pair. Since there are $\frac{1}{2}(n^2 + n)$ height sequences of length at most 2 with values at most $n$, and since there are $n$ isomorphism types of indecomposable pairs $(B; A)$ where $A = 0$, we deduce that there are in total $n + \frac{1}{2}(n^2 + n) = \frac{1}{2}(n^2 + 3n)$ indecomposable objects in $S_2(\Lambda)$, up to isomorphism.
In this manuscript we recover the list of indecomposable objects in $\mathcal{S}_2(\Lambda)$ using poset representations, we demonstrate that the list does not only not depend on the choice of the base ring $\Lambda$, but that in fact all the categories of type $\mathcal{S}_2(\Lambda)$ are related:

Let $\Delta$ be a second commutative local uniserial ring such that $\Lambda$ and $\Delta$ have the same length $n$ and isomorphic radical factor fields. Clearly, the categories $\mathcal{S}_2(\Lambda)$ and $\mathcal{S}_2(\Delta)$ cannot be equivalent unless the rings $\Lambda$ and $\Delta$ are isomorphic.

We define categorical ideals $\mathcal{N}_\Lambda \subset \mathcal{S}_2(\Lambda)$ and $\mathcal{N}_\Delta \subset \mathcal{S}_2(\Delta)$ which are “large enough” to make the factor categories equivalent,

$$\mathcal{S}_2(\Lambda)/\mathcal{N}_\Lambda \simeq \mathcal{S}_2(\Delta)/\mathcal{N}_\Delta,$$

and “small enough” so that the categories $\mathcal{S}_2(\Lambda)/\mathcal{N}_\Lambda$ and $\mathcal{S}_2(\Lambda)$ have the same indecomposable objects in the sense that no nonzero object in $\mathcal{S}_2(\Lambda)$ is isomorphic to zero when considered as object in $\mathcal{S}_2(\Lambda)/\mathcal{N}_\Lambda$.

We would like to emphasize that only basic methods from linear algebra are needed to establish the well-known list of the indecomposable representations of this poset, and hence to obtain the list of the indecomposable objects in $\mathcal{S}_2(\Lambda)$.

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2. Poset Representations

We introduce the poset $\mathcal{P}_n$, define a functor $F: \mathcal{S}_2(\Lambda) \to \text{rep}_k \mathcal{P}_n$ into the category of $k$-linear representations of $\mathcal{P}_n$, and show that the representations of type $F(P^c_i)$ and $F(Q^s_i)$ form a full list of the indecomposable representations in the image of $F$.

Let $\mathcal{P}_n$ be the following poset:

\[ \mathcal{P}_n : \]

Recall that a representation $(V^0; (V^i)_{i \in \mathcal{P}_n})$ for $\mathcal{P}_n$ is a $k$-vector space $V^0$, the total space of $V$, together with subspaces $V^i \subset V^0$ for $i \in \mathcal{P}_n$, such that $V^i \subset V^j$ holds whenever $i < j$ in $\mathcal{P}_n$. For short we write $V'$ for $V^1$ and $V''$ for $V^2$. 

Abelian groups with a $p^2$-bounded subgroup, revisited 3
The category $\text{rep}_k \mathcal{P}_n$ is a Krull-Remak-Schmidt category, so every representation has a unique direct sum decomposition into indecomposable representations. The indecomposable objects in $\text{rep}_k \mathcal{P}_n$ are in $1-1$-correspondence to those indecomposable representations of the Dynkin diagram $\mathbb{D}_{n+2}$ which have support in the central point, thus there are

$$\#\{\text{ind rep } \mathbb{D}_{n+2}\} - \#\{\text{ind rep } \mathcal{A}_{n-1}\} - 2\#\{\text{ind rep } \mathcal{A}_1\} = (n^2 + 3n + 2) - \frac{1}{2}(n^2 - n) - 2 = \frac{1}{2}n^2 + \frac{7}{2}n$$

indecomposables. They are as follows: The total space $V^0$ of an indecomposable representation $V$ either has dimension 1 or dimension 2. If $\dim V^0 = 1$ then $V$ is isomorphic to one of the representations $V_{\ell, \ell', \ell''}$, where $0 \leq \ell < n$ and $0 \leq \ell', \ell'' \leq 1$, defined as follows.

$$V^i_{\ell, \ell', \ell''} = \begin{cases} k & \text{if } i \leq \ell \\ 0 & \text{if } i > \ell \end{cases}$$

$$V^r_{\ell, \ell', \ell''} = \begin{cases} k & \text{if } \ell' = 1 \\ 0 & \text{if } \ell' = 0 \end{cases}$$

$$V''_{\ell, \ell', \ell''} = \begin{cases} k & \text{if } \ell'' = 1 \\ 0 & \text{if } \ell'' = 0 \end{cases}$$

If $\dim V^0 = 2$ then $V$ is isomorphic to one of the representations $W_{s,t}$ where $1 \leq s < t < n$:

$$W^i_{s,t} = \begin{cases} k \oplus k & \text{if } i \leq s \\ \Delta & \text{if } s < i \leq t \\ 0 & \text{if } i > t \end{cases}$$

$$W'_{s,t} = k \oplus 0$$

$$W''_{s,t} = 0 \oplus k$$

where $\Delta = k(1,1) \subset k \oplus k$ is the diagonal.

Given a pair $(B; A) \in \mathcal{S}_2(A)$, we obtain a representation $V$ of $\mathcal{P}_n$ as follows. Consider the filtration for $B$ given by the subspace $\text{rad } A$:

$$L_0 = A^- = \text{rad } A$$

$$L_1 = A^+ = p^{-1} \text{ rad } A = A + \text{ soc } B$$

$$L_2 = p^{-2} \text{ rad } A$$

$$\vdots$$

$$L_{n-1} = p^{1-n} \text{ rad } A$$

$$L_n = p^{-n} \text{ rad } A = B$$

Here, as usual in this manuscript, we write $p$ for the endomorphism of $B$ given by multiplication by $p$. Thus, for a submodule $U \subset B$ we denote by $pU$ and $p^{-1}U$ the image and the inverse image of $U$ under this map. Note that subsequent quotients of the filtration for $B$ are vector spaces; in particular, $V^0 = A^+/A^-$ will be the total space. For $\ell > 0$, the multiplication by $p^\ell$ defines maps $p^\ell : L_{\ell+1} \to A^+$, $p^\ell : L_\ell \to A^-$ which have image $\text{rad}^\ell B \cap A^+$ and $\text{rad}^\ell B \cap A^-$, respectively, and which
Abelian groups with a $p^2$-bounded subgroup, revisited

give rise to isomorphisms

$$\frac{L_{\ell+1}}{L_\ell} \cong \frac{\text{rad}^\ell B \cap A^+}{\text{rad}^\ell B \cap A^-} \cong \frac{(\text{rad}^\ell B \cap A^+) + A^-}{A^-}$$

into the submodule $V' = ((\text{rad}^\ell B \cap A^+) + A^-)/A^-$ of $V^0$. We also set $V' = \text{soc} B/A^-$ and $V'' = A/A^-$. Note that all the spaces $V^j$ have the form $\tilde{C} = ((C \cap A^+) + A^-)/A^- \subseteq A^+/A^- \subset$ a suitable submodule $C$ of $B$. We collect some properties of this construction $C \mapsto \tilde{C}$.

**Lemma 2.1.** Let $(B; A)$ be a pair in $S_2(\Lambda)$ and $C, C'$ be submodules of $B$.

1. If $C \subseteq C'$ then $\tilde{C} \subseteq \tilde{C}'$.
2. Always, $C \cap C' \subseteq \tilde{C} \cap \tilde{C}'$ holds. If $A^- \subseteq C$ or $A^- \subseteq C'$ then equality holds.
3. Always, $\tilde{C} + \tilde{C}' \subseteq \tilde{C} + \tilde{C}'$ holds. We have equality if $C \subseteq A^+$ or $C' \subseteq A^+$.

**Proof.** (1) If $C \subseteq C'$ then $(C \cap A^+) + A^- \subseteq (C' \cap A^+) + A^-$ holds and the assertion follows.

(2) The inclusion $(C \cap C' \cap A^+) + A^- \subseteq [(C \cap A^+) + A^-] \cap [(C' \cap A^+) + A^-]$ holds always. If $A^- \subseteq C$ is given, then the right hand side in the inclusion simplifies to $(C \cap A^+) \cap [(C' \cap A^+) + A^-]$ and by the modular law to $(C \cap C' \cap A^+) + A^-$.

(3) Similarly, $(C \cap A^+) + (C' \cap A^+) + A^- \subseteq [(C + C') \cap A^+] + A^-$ holds always. If also $C \subseteq A^+$ holds, then the left hand side simplifies to $C + (C' \cap A^+) + A^-$ and by the modular law to $[(C + C') \cap A^+] + A^-$. □

As a consequence we obtain:

**Proposition 2.2.** (1) The assignment which maps an object $(B; A)$ in $S_2(\Lambda)$ to the representation $V = (V^0, (V^j))$ of $\mathcal{P}_n$ given by

$$V^0 = \tilde{B}; \quad V^j = \text{rad}^j B, \quad \text{for } 1 \leq \ell \leq n-1, \quad V' = \text{soc} \tilde{B}, \quad V'' = \tilde{A},$$

defines an additive functor $F : S_2(\Lambda) \rightarrow \text{rep}_k \mathcal{P}_n$.

(2) Each representation $V = F(B; A)$ satisfies the following conditions:

$$V''^{-1} \subseteq V' \quad \text{and} \quad V' + V'' = V^0.$$

**Proof.** It follows from Lemma 2.1 that $V$ is a representation of $\mathcal{P}_n$ satisfying (2). If $f : (B; A) \rightarrow (D; C)$ is a morphism in $S_2(\Lambda)$, then $f$ maps the submodules of $B$, $A^- = \text{rad} A$, $A$, $\text{rad}^\ell B$ (where $0 \leq \ell \leq n-1$), $\text{soc} B$, and $A^+$ into the corresponding submodules of $D$ and
hence gives rise to a map $F(f)$ between the representations $F(B; A)$ and $F(D; C)$. □

**Definition.** We denote by $\text{rep}'_k \mathcal{P}_n$ the full subcategory of $\text{rep}_k \mathcal{P}_n$ consisting of those representations which satisfy the condition (2).

Among the $\frac{1}{2}n^2 + \frac{7}{2}n$ indecomposable representations for $\mathcal{P}_n$, the two representations $V_{n-1,0,0}$ and $V_{n-1,0,1}$, and the $n - 1$ representations $W_{s,n-1}$ where $0 \leq s \leq n - 2$, do not have the property that $V^{n-1} \subset V'$. The condition that $V' + V'' = V^0$ excludes the $n$ representations $V_{\ell,0,0}$ where $0 \leq \ell \leq n - 1$. It turns out that all the remaining $\frac{1}{2}n^2 + \frac{3}{2}n$ indecomposable representations in $\text{rep}'_k \mathcal{P}_n$ are in bijection with the indecomposable pairs in $S_2(\Lambda)$ of type $P^\ell_m$ and $Q^t_s$:

**Proposition 2.3.** The functor $F$ gives rise to the following correspondence between pairs in $S_2(\Lambda)$ and representations in $\text{rep}'_k \mathcal{P}_n$.

- $F(P^\ell_0) = V_{\ell-1,0,0}$ (1 ≤ $\ell$ ≤ $n$)
- $F(P^\ell_1) = V_{\ell-1,1,1}$ (1 ≤ $\ell$ ≤ $n$)
- $F(P^\ell_2) = V_{\ell-2,0,1}$ (2 ≤ $\ell$ ≤ $n$)
- $F(Q^t_s) = W_{s-1,t-2}$ (0 ≤ $s - 1 < t - 2 \leq n - 2$)

As a consequence, $F : S_2(\Lambda) \to \text{rep}'_k \mathcal{P}_n$ is a dense functor. Moreover, all pairs of type $P^\ell_m$ and $Q^t_s$ are indecomposable and pairwise nonisomorphic. □

We will see in Section 3 that $F$ is full. Hence the pairs $P^\ell_m$ and $Q^t_s$ form a full list of the indecomposable objects in $S_2(\Lambda)$.

### 3. Picket Decomposition

We show in this section that every object in $S_1(\Lambda)$ is direct sum of pickets of type $P^\ell_0$ or $P^\ell_1$ where 1 ≤ $\ell$ ≤ $n$. We deduce in Theorem 3.4 that every pair $(B; A) \in S_2(\Lambda)$ with the additional property that $\text{soc} B \subset A$ is a direct sum of pickets of type $P^\ell_1$ or $P^\ell_2$.

The ring $\Lambda$ is selfinjective of Loewy length $n$. It turns out that the pair $P^m_n = (\Lambda; 0)$ is a relatively injective indecomposable object in $S(\Lambda)$ with source map the inclusion $u_n : P^m_0 \to P^m_1$. We give a direct proof of this result which follows also from [4, Proposition 1.4].

**Lemma 3.1.** Let $(B, A)$ be an object in $S(\Lambda)$ and $f : P^m_0 \to (B, A)$ a morphism. Either $f$ is a split monomorphism or else $f$ factors over the inclusion $u_n : P^m_0 \to P^m_1$. Thus, $P^m_0$ is relatively injective in the sense that every monomorphism from $P^m_0$ with cokernel in $S(\Lambda)$ is a split monomorphism.
Abelian groups with a $p^2$-bounded subgroup, revisited

Proof. Suppose a morphism $P^n_0 \to (B, A)$ is given by a map $f : \Lambda \to B$. Clearly, if $f$ is not a monomorphism or if $\text{Im} \, f \cap A \neq 0$, then the map $P^n_0 \to (B; A)$ factors over $u_n$. It remains to deal with the case that $f$ is a monomorphism such that $E \cap A = 0$ where $E = \text{Im} \, f$. Let $e_A : A \to E(A)$ and $e_B : B \to E(B)$ be injective envelopes, then there is a map $h$ which makes the following diagram commutative.

$$
\begin{array}{ccc}
E \oplus A & \xrightarrow{\text{incl}} & B \\
1 \oplus e_A & \searrow & e_B \\
E \oplus E(A) & \xrightarrow{h} & E(B)
\end{array}
$$

Since $E \oplus A$ is large in $E \oplus E(A)$ and since the composition $h \circ (1 \oplus e_A)$ is a monomorphism, also $h$ is a monomorphism. Then $h$ is a split monomorphism, so we can write $E(B) = E \oplus E(A) \oplus E''$ and define $B' = \{b \in B : e_B(b) \in E(A) \oplus E''\}$. Since $E \subset B$, the assertions $B = E \oplus B'$ and $A \subseteq B'$ hold. The decomposition $(B, A) = (E, 0) \oplus (B', A)$ demonstrates that $P^n_0 \to (B; A)$ is a split monomorphism.

Suppose a monomorphism $P^n_B \to (B; A)$ is such that the cokernel $(D; C)$ is an object in $\mathcal{S}(\Lambda)$. Then we have a commutative diagram with exact rows in which the vertical maps are monomorphisms.

$$
\begin{array}{ccccccc}
0 & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & C & \longrightarrow & 0 \\
& & \downarrow & & \text{incl} & & \text{incl} & & \\
& & 0 & \longrightarrow & \Lambda & \xrightarrow{f} & B & \longrightarrow & D & \longrightarrow & 0
\end{array}
$$

By the snake lemma, the composition $\Lambda \xrightarrow{f} B \to B/A$ is a monomorphism, hence $\text{Im} \, f \cap A = 0$ and we have just seen that this implies that $P^n_0 \to (B; A)$ is a split monomorphism. $\square$

**Lemma 3.2.** The object $P^n_1$ is injective in $\mathcal{S}_1(\Lambda)$.

**Proof.** Let $P^n_B \to (B; A)$ be a monomorphism in $\mathcal{S}_1(\Lambda)$. If the image is $(E, \text{soc} \, E)$ then the injective $\Lambda$-module $E$ has a complement in $B$, say $B = E \oplus B'$. Since $A$ is semisimple and $\text{soc} \, E \subseteq A$ we obtain $A \cap (E \oplus B') = A \cap (\text{soc} \, E \oplus \text{soc} \, B') = \text{soc} \, E \oplus (A \cap \text{soc} \, B') = \text{soc} \, E \oplus (A \cap B')$ by the modular law and hence the pair $(B; A)$ decomposes as $(E; \text{soc} \, E) \oplus (B'; A \cap B')$. $\square$

**Proposition 3.3.** Every object in $\mathcal{S}_1(\Lambda)$ is a direct sum of objects of type $P^n_0$ or $P^n_1$ where $1 \leq \ell \leq n$.

**Proof.** Assume that the pair $(B; A) \in \mathcal{S}_1(\Lambda)$ is nonzero. We show that $(B; A)$ has a summand of type $P^n_0$ or $P^n_1$ where $\ell$ is the Loewy length of
Indeed $B$, having Loewy length $\ell$, has an indecomposable summand, say $E$, which is a projective-injective $\Lambda/(p^\ell)$-module. According to Lemma 3.1, either $P_0^\ell = (E; 0)$ splits off as a direct summand of $(B; A)$, or else $P_1^\ell = (E; \text{soc } E)$ can be embedded into $(B; A)$. In the second case, $P_1^\ell$ splits off as a direct summand since it is an injective module (Lemma 3.2).

**Theorem 3.4.** Every pair $(B; A)$ in $S_2(\Lambda)$ with the extra property that $\text{soc } B \subset A$, is isomorphic to a direct sum of pickets $P_m^\ell$ where $m = 1$ or 2 and $m \leq \ell \leq n$.

**Proof.** For each pair $(B; A) \in S_2(\Lambda)$ which satisfies $\text{soc } B \subset A$ consider the picket decomposition of the corresponding pair $(B; \text{rad } A)$ in $S_1(\Lambda)$ given by Proposition 3.3:

\[
(B; \text{rad } A) = \bigoplus_{i=1}^{s} (B_i; \text{rad } A \cap B_i).
\]

We show that this yields the picket decomposition for $(B; A)$:
Consider $p$ as an endomorphism of $B$ and write $pU$ and $p^{-1}U$ for the image and the inverse image of the submodule $U \subset B$.

$$\begin{align*}
A & \overset{(1)}{=} A + \text{soc } B \\
& = p^{-1}pA \\
& \overset{(3)}{=} p^{-1}\left( \sum_i pA \cap B_i \right) \\
& \overset{(2)}{=} \sum_i p^{-1}(pA \cap B_i) \\
& = \sum_i (p^{-1}pA \cap p^{-1}B_i) \\
& \overset{(1)}{=} \sum_i (A \cap (B_i + \text{soc } B)) \\
& \overset{(1)}{=} \text{soc } B + \sum_i (A \cap B_i) \\
& = \sum_i (\text{soc } B_i + A \cap B_i) \\
& = \bigoplus_i A \cap B_i
\end{align*}$$

In the equations labelled (1), equality holds since $\text{soc } B \subset A$, and equality in (2) follows from $pA \cap B_i \subset pB$. \hfill \square

4. Homomorphism Categories

We show in Theorem 4.3 that the functor $F : S_2(\Lambda) \rightarrow \text{rep}_k P_n$ is full.

**Definition.** Let $\mathcal{C}$ be any category. The homomorphism category $\mathcal{H}(\mathcal{C})$ has as objects all triples $(B; A; f)$ or $(B \leftarrow f A)$ where $f : A \rightarrow B$ is a morphism in $\mathcal{C}$. A morphism in $\mathcal{H}(\mathcal{C})$ from $(B; A; f)$ to $(D; C; e)$ is a pair $(h : B \rightarrow D, g : A \rightarrow C)$ of morphisms in $\mathcal{C}$ which satisfies $eg = hf$. If $\mathcal{C}$ is an abelian category, then so is $\mathcal{H}(\mathcal{C})$; in this case kernels and cokernels, and hence pull-backs and push-outs, are computed componentwise. For short we write $\mathcal{H}(\Lambda)$ for $\mathcal{H}(\text{mod } \Lambda)$, thus $S(\Lambda) \subset \mathcal{H}(\Lambda)$ is an embedding of a full subcategory.

**Proposition 4.1.** Given a pair $(B; A^+)$ in $S_2(\Lambda)$ where $\text{soc } B \subset A^+$, and a subspace $U$ of $A^+/A^-$ where $A^- = \text{rad } A^+$, then there is a unique
submodule $A$ of $B$ such that the diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & (B; A^-) & \longrightarrow & (B; A) & \longrightarrow & (0; U) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (B; A^-) & \longrightarrow & (B; A^+) & \longrightarrow & (0; A^+/A^-) & \longrightarrow & 0 \\
\end{array}
$$

is a pull-back diagram in the category $\mathcal{H}(\Lambda)$. Conversely, every pair $(B; A)$ arises in this way.

Proof. The bottom sequence in $(\ast)$ is a short exact sequence in the category $\mathcal{H}(\Lambda)$, and the pull-back along the inclusion $U \to A^+/A^-$ yields an object $X \in \mathcal{H}(\Lambda)$ and two monomorphisms $(B; A^-) \to X \to (B; A^+)$. Thus, $X$ is in $\mathcal{S}_2(\Lambda)$ and by identifying $X$ with $\text{Im} \, v$ we obtain a uniquely determined submodule $A \subset B$ such that the above diagram $(\ast)$ is commutative and has exact rows.

In order to realize a pair $(B; A) \in \mathcal{S}_2(\Lambda)$ as a pull-back, take $A^+ = A + \text{soc} \, B$, $A^- = \text{rad} \, A$ and for $U$ the subspace $A/A^-$ of $A^+/A^-$. Then the diagram $(\ast)$ is commutative with exact rows and hence is a pull-back diagram since the vertical map on the left is an isomorphism. □

Note that the last term $A^+/A^-$ in the bottom sequence in $(\ast)$ is the total space of the poset representation $F(B; A^+)$. Our next result shows that any morphism $F(B; A^+) \to F(D; C^+)$ between poset representations can be lifted to a map $(B; A^+) \to (D; C^+)$. :

**Proposition 4.2.** Suppose that the pairs $(B; A)$ and $(D; C)$ in $\mathcal{S}_2(\Lambda)$ satisfy $\text{soc} \, B \subset A$ and $\text{soc} \, D \subset C$. Then the functor $F$ induces an isomorphism

$$
\frac{\text{Hom}_{\mathcal{S}_2(\Lambda)}((B; A), (D; C))}{\mathcal{N}((B; A), (D; C))} \cong \text{Hom}_{\text{rep}_\mathbb{P}_n}(F(B; A), F(D; C))
$$

where $\mathcal{N}((B; A), (D; C))$ consists of all maps $f : (B; A) \to (D; C)$ such that $f(A) \subset \text{rad} \, C$.

Proof. According to Theorem 3.4, the pairs $(B; A)$ and $(D; C)$ are direct sums of pickets of type $P_1^*$ or $P_2^*$. Since $F$ is an additive functor, we may assume that both $(B; A)$ and $(D; C)$ are in fact such pickets. The kernel of the map $F((B; A), (D; C))$ is $\mathcal{N}((B; A), (D; C))$, so it remains to show that any nonzero map $h : F(B; A) \to F(D; C)$ lifts to a map
Abelian groups with a $p^2$-bounded subgroup, revisited

$f : (B; A) \to (D; C)$ which makes the following diagram commutative.

$$
\begin{array}{cccccc}
0 & \to & (B; A^-) & \to & (B; A) & \to & (0; A/A^-) & \to & 0 \\
\downarrow f^- & & \downarrow f & & \downarrow (0; h^0) & & \\
0 & \to & (D; C^-) & \to & (D; C) & \to & (0; C/C^-) & \to & 0
\end{array}
$$

Since $\text{soc } B \subset A$, the space $F(B; A)^{\prime\prime} = A/A^-$ is nonzero and hence $F(B; A)$ is a poset representation of type $V_{\ell',1}$. Also $F(D; C)$ must be isomorphic to a representation of type $V_{m,m',1}$. If the space $V_{m,m',1}^{\prime\prime}$ is zero (that is, if $m' = 0$), then also the space $V_{\ell',1}^{\prime\prime}$ must be zero. We obtain that the length of $A$, which is $L = 2 - \ell'$, is at least the length $2 - m'$ of $C$. Using the projectivity of $A$ as a $\Lambda/(p^L)$-module, we obtain a lifting $g : A \to C$ for $h^0$:

$$
\begin{array}{ccc}
A & \to & A/A^- \\
g & & \downarrow h^0 \\
C & \to & C/C^-
\end{array}
$$

Similarly, whenever a space $V_{m,m',1}^{\prime\prime}$ is zero, then so is the space $V_{\ell',1}^{\prime\prime}$, and hence $\ell \leq m$ holds. Let $D \to E$ be the inclusion into an injective envelope. Then we can extend the composition $A \to C \to D \to E$ to a map $e : B \to E$. Since $\ell \leq m$, the image of $e$ is contained in $D$ so $f = e|_{B,D}$ is an extension of $g$ which makes the following diagram commutative:

$$
\begin{array}{ccc}
A & \to & B \\
g & & \downarrow f \\
C & \to & D
\end{array}
$$

This extension satisfies $F(f) = h$, finishing the proof. □

**Theorem 4.3.** The functor $F : \mathcal{S}_2(\Lambda) \to \text{rep}_k \mathcal{P}_n$ is full.

**Proof.** Given two objects $(B; A)$ and $(D; C)$ in $\mathcal{S}_2(\Lambda)$, put $U = F(B; A)$ and $V = F(D; C)$ and let $h : U \to V$ be a morphism in the category $\text{rep}_k \mathcal{P}_n$. We show that there is a morphism $f : (B; A) \to (D; C)$ in $\mathcal{S}_2(\Lambda)$ such that $F(f) = h$.

Note that the representations $F(B; A)$ and $F(B; A^+)$ coincide in all positions, only $F(B; A)^{\prime\prime}$ may be a proper subspace of $F(B; A^+)^{\prime\prime}$; in fact, $F(B; A^+)^{\prime\prime}$ coincides with the total space $F(B; A^+)^0 = F(B; A)^0$. Thus, $h$ defines a morphism $F(B; A^+) \to F(D; C^+)$. By Proposition 4.2 this
morphism lifts to a map \( f^+ : (B; A^+) \to (D; C^+) \) which makes the diagram in \( \mathcal{H}(\Lambda) \) commutative.

\[
\begin{array}{cccccc}
0 & \to & (B; A^-) & \to & (B; A^+) & \to & (0; U^0) & \to & 0 \\
\downarrow f^- & & \downarrow f^+ & & \downarrow (0; h^0) & \\
0 & \to & (D; C^-) & \to & (D; C^+) & \to & (0; V^0) & \to & 0
\end{array}
\]

Here \( f^- = f^+|_{(B; A^-), (D; C^-)} \) is the restriction of \( f^+ \). The morphism \( h : F(B; A) \to F(D; C) \) also yields the following commutative diagram.

\[
\begin{array}{c}
(0; U'') \xrightarrow{(0; h'')} (0; V'') \\
\downarrow & & \downarrow & \\
(0; U^0) \xrightarrow{(0; h^0)} (0; V^0)
\end{array}
\]

We obtain the desired map \( f \) as a pull-back in the abelian category \( \mathcal{H}(\mathcal{H}(\Lambda)) \) which has as objects all homomorphisms in \( \mathcal{H}(\Lambda) \). The two diagrams above form the following diagram in \( \mathcal{H}(\mathcal{H}(\Lambda)) \), which has an exact row.

\[
\begin{array}{cccccc}
(0; h'') & & & & & \\
\downarrow & & & & & \\
0 & \to & f^- & \to & f^+ & \to & (0; h^0) & \to & 0
\end{array}
\]

The pullback of this diagram,

\[
\begin{array}{cccccc}
0 & \to & f^- & \to & f & \to & (0; h'') & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & f^- & \to & f^+ & \to & (0; h^0) & \to & 0
\end{array}
\]

yields an object \( f \) in \( \mathcal{H}(\mathcal{H}(\Lambda)) \).

Since pull-backs are computed componentwise, \( f \) is in fact a morphism between the two pull-backs \((B; A)\) and \((D; C)\) computed in the category \( \mathcal{H}(\Lambda) \) (see Proposition 4.4). In other words, \( f^+ : (B; A^+) \to (D; C^+) \) restricts to a map \( f : (B; A) \to (D; C) \).

\[
\begin{array}{cccccc}
(B; A^-) & \to & (B; A) & \to & (B; A^+) & \\
\downarrow f^- & & \downarrow f & & \downarrow f^+ & \\
(D; C^-) & \to & (D; C) & \to & (D; C^+)
\end{array}
\]

Since \( \text{rad} A = \text{rad} A^+ \) and \( \text{rad} C = \text{rad} C^+ \) we have that the representations \( F(B; A) \) and \( F(B; A^+) \), and also the representations \( F(D; C) \) and \( F(D; C^+) \), coincide in all positions except possibly at \( 1'' \). Since \( f \)
is the restriction of \( f^+ \), Proposition 4.2 implies that the linear maps \( F(f)^j \) and \( h^j \) coincide for all \( j \in \mathcal{P}_n \setminus \{1^n\} \). For the space at \( 1^n \) consider the submodule component of the top row in (**):

\[
\begin{array}{cccccc}
0 & \longrightarrow & A^- & \longrightarrow & A & \longrightarrow & U^n & \longrightarrow & 0 \\
& & f|_{A^-, C^-} & & f|_{A, C} & & h^n & \\
0 & \longrightarrow & C^- & \longrightarrow & C & \longrightarrow & V^n & \longrightarrow & 0
\end{array}
\]

Since \( F(B; A)^n = A/A^- = U^n \) and \( F(D; C)^n = C/C^- = V^n \) we obtain that \( F(f)^n = h^n \), finishing the proof that \( F(f) = h \).

\[\square\]

5. Corollaries

We have seen in Proposition 2.3 and Theorem 4.3 that the functor \( F: \mathcal{S}_2(\Lambda) \to \text{rep}_k \mathcal{P}_n \) is full and dense. We first consider the kernel.

For pairs \((B; A)\) and \((D; C)\) in \( \mathcal{S}_2(\Lambda) \) define the following subgroup of \( \text{Hom}_\mathcal{S}_2((B; A), (D; C)) \):

\[\mathcal{N}((B; A), (D; C)) = \{ f: (B; A) \to (D; C) \mid f(A^+) \subset C^- \}.\]

This generalizes the definition given in Proposition 4.2. The collection \( \mathcal{N}_\Lambda \) or \( \mathcal{N} \) of all such subgroups forms a categorical ideal in \( \mathcal{S}_2(\Lambda) \).

**Lemma 5.1.** If the length of \( \Lambda \) is \( n > 1 \) then \( \mathcal{N} \) has nilpotency index \( n + 1 \).

Note that if \( n = 1 \) then \( \mathcal{N} = 0 \).

**Proof.** Given objects \((B_i; A_i)\) in \( \mathcal{S}_2(\Lambda) \) for \( 0 \leq i \leq n + 1 \), morphisms \( f_i: (B_{i-1}; A_{i-1}) \to (B_i; A_i) \) in \( \mathcal{N} \) for \( 1 \leq i \leq n + 1 \), and an element \( b \in B_0 \). Then \( p^{n-1}b \) is in \( \text{soc} B_0 \), hence \( f_1(p^{n-1}b) \in \text{rad} A_1 \). For each \( i \) we have that if \( f_i \cdots f_1(p^{n-i}b) \in \text{rad} A_i \) then \( f_i \cdots f_1(p^{n-i}b) \in A_i + \text{soc} B_i \) and hence \( f_{i+1} \cdots f_1(p^{n-i}b) \) is in \( \text{rad} A_{i+1} \). Thus, \( f_n \cdots f_1(b) \) is in \( \text{rad} A_n \) and hence \( f_{n+1} \cdots f_1(b) = 0 \). Conversely, if \( n > 1 \) then the nilpotency index is not less than \( n + 1 \) since the following composition of \( n \) maps is nonzero.

\[P^n_1 \xrightarrow{\text{incl}} P^n_2 \xrightarrow{p_2} P^n_2 \xrightarrow{p_2} \cdots \xrightarrow{p_2} P^n_2 \]

\[\square\]

**Corollary 5.2.** The functor \( F: \mathcal{S}_2(\Lambda) \to \text{rep}_k \mathcal{P}_n \) induces an equivalence of categories \( \bar{F}: \mathcal{S}_2(\Lambda)/\mathcal{N} \to \text{rep}_k \mathcal{P}_n \).

**Proof.** We show that the kernel of \( F \) is \( \mathcal{N} \). Let \( f: (B; A) \to (D; C) \) be a map in \( \mathcal{S}_2(\Lambda) \) and denote by \( f^+ \) the map \( f^+: (B; A^+) \to (D; C^+) \). Since \( \text{rad} A = \text{rad} A^+ \) and \( \text{rad} C = \text{rad} C^+ \) hold, the linear
maps $F(f)^0$, $F(f^+)^0$ coincide. By Proposition 4.2, $F(f^+)^0 = 0$ if and only if $f^+ \in \mathcal{N}((B; A^+), (D; C^+))$. Then $f(A^+) \subseteq C^-$, but this is the condition for $f$ to be in $\mathcal{N}((B; A), (D; C))$. \hfill \Box

Since $\mathcal{N}$ is a nilpotent ideal, the canonical functor $\mathcal{S}_2(\Lambda) \to \mathcal{S}_2(\Lambda)/\mathcal{N}$ preserves indecomposable objects and reflects isomorphisms, so the isomorphism classes of objects in $\mathcal{S}_2(\Lambda)$ and $\mathcal{S}_2(\Lambda)/\mathcal{N}$ are in a natural bijection. As a consequence, if also $\Delta$ is a commutative local uniserial ring of length $n$ and with radical factor field $k$, then the categories $\mathcal{S}_2(\Lambda)$ and $\mathcal{S}_2(\Delta)$ admit a “natural” bijection between their objects.

**Corollary 5.3.** Suppose that $\Lambda$ and $\Delta$ are commutative local uniserial rings of the same length $n$ and with radical factor fields isomorphic to $k$. Then the following categories are equivalent:

$$\mathcal{S}_2(\Lambda)/\mathcal{N}_\Lambda \cong \mathcal{S}_2(\Delta)/\mathcal{N}_\Delta \tag{\Box}$$

For example, the categories $\mathcal{S}_2(\mathbb{Z}/(p^n))/\mathcal{N}$ and $\mathcal{S}_2(k[T]/(T^n))/\mathcal{N}$ are equivalent if $k = \mathbb{Z}/p$.

**Question:** We have seen that $\mathcal{N}$ is an ideal of nilpotency index $n+1$. Is there a pair $\mathcal{L}_\Lambda$, $\mathcal{L}_\Delta$, of ideals of nilpotency index $n$ such that the above Corollary holds with $\mathcal{N}$ replaced by $\mathcal{L}$? For example, an ideal of nilpotency index $n$ is given by all maps which factor through the multiplication by $p$. Note that we cannot expect to have an ideal $\mathcal{L}$ of nilpotency index $r < n$: The endomorphism ring of the pair $X = (\Lambda; 0)$ has filtration

$$0 = \mathcal{L}^r(X, X) \subseteq \mathcal{L}^{r-1}(X, X) \subseteq \cdots \subseteq \mathcal{L}(X, X) \subseteq \Lambda$$

in which not all subsequent factors can be semisimple. Hence the factor ring $\Lambda/\mathcal{L}(X, X)$ is not a field. In particular if $\Lambda = \mathbb{Z}/(p^n)$ and $\Delta = k[T]/(T^n)$ then the following two endomorphism rings are not isomorphic: $\text{End}_{\mathcal{S}_2(\Lambda)/\mathcal{L}}(\Lambda, 0) \ncong \text{End}_{\mathcal{S}_2(\Delta)/\mathcal{L}}(\Delta, 0)$.

We can also deal with the case where $\Lambda$ is a PID and $p$ a prime element. By primary decomposition, any pair $(B; A)$ where $B$ is a finitely generated $\Lambda$-module and $A$ a $p^2$-bounded submodule of $B$ has a unique direct sum decomposition into pair of type $(F; 0)$ where $F$ is a finitely generated free $\Lambda$-module, a pair of type $(D; 0)$ where $p$ acts as automorphism on $D$, and a pair $(B; A)$ where $B$ is $p^n$-bounded for some natural number $n$.

**Corollary 5.4.** Let $\Lambda$ be a principal ideal domain, $p$ a prime element, $B$ a finitely generated $\Lambda$-module and $A$ a submodule of $B$ which is $p^2$-bounded. Then the pair $(B; A)$ has a direct sum decomposition,
unique up to isomorphy and reordering, into finitely many indecomposable pairs (a) of type $(\Lambda/(q); 0)$ where $q$ is a prime power relatively prime to $p$, or (b) of type $P^k_m$ or $Q^l_s$ in $S(\Lambda/(p^n))$ for some $n \in \mathbb{N}$, or (c) indecomposable projective of type $(\Lambda; 0)$.

Returning to Prüfer’s height sequences, we see that an indecomposable pair is either given by a power of a prime ideal in $\Lambda$, or else by the height sequence of a subgroup generator.

**Corollary 5.5.** Let $\Lambda$ be a principal ideal domain, $p$ a prime element, $B$ a finitely generated $\Lambda$-module and $A$ a submodule of $B$ which is $p^2$-bounded. An indecomposable pair $(B; A)$ is

1. either isomorphic to $(\Lambda/Q; 0)$ for a uniquely determined power $Q$ of a prime ideal, or else,
2. if $A = a\Lambda$ is (nonzero) cyclic, determined uniquely, up to isomorphism, by the height sequence $H_B(a)$.

Conversely, every power of a prime ideal, and every height sequence of length at most 2, can be realized by an indecomposable pair. □

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