Coexistence between superconducting and spin density wave states in iron-based superconductors: Ginzburg–Landau analysis

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Abstract
We consider the interplay between superconducting (SC) and commensurate spin-density-wave (SDW) orders in iron pnictides by analyzing a multiple-order Ginzburg–Landau free energy. We are particularly interested in whether the doping-induced transition between the two states is first order or whether the two pure phases are separated by an intermediate phase with coexisting SC and SDW orders. For perfect nesting, the two orders do not coexist, because SDW order, which comes first, gaps the full Fermi surface leaving no space for SC to develop. When nesting is not perfect due to either ellipticity of electron bands or doping-induced difference in chemical potentials for holes and electrons, SDW order still leaves modified Fermi surfaces for not too strong SDW magnetism and the SC order may develop. We show that the two orders coexist only when certain relations between ellipticity and doping are met. In particular, in a compensated metal, ellipticity alone is not sufficient for coexistence of the two orders.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

In the phase diagram of recently discovered iron-based pnictide materials superconducting (SC) and spin-density-wave (SDW) states are close neighbors [1]. The interplay between these two orders has been the focus of numerous experimental and theoretical studies. Superconductivity and magnetic ordering are normally mutually exclusive states of electronic systems, and a first-order transition between SC and SDW orders has been reported in some pnictides, e.g. for LaO\(_{1-x}\)F\(_x\)FeAs [2]. However, recent nuclear magnetic resonance (NMR) [3], specific heat, susceptibility, Hall coefficient [4, 5] and neutron scattering experiments [6] on Ba(Fe\(_{1-x}\)Co\(_x\))\(_2\)As\(_2\) indicate that SDW and SC phases coexist over some doping range.

The electronic spectrum of the pnictides results in two hole pockets centered at (0, 0) and two electron pockets centered at (0, \(\pi\)) and (\(\pi\), 0) in the unfolded Brillouin zone (BZ) [7–16]. Multiple Fermi surfaces (FSs) create a number of different possibilities [17, 18] for electron ordering in the form of SDW, charge-density-wave (CDW) states and a superconducting state with extended s-wave symmetry in which the gaps on the hole and electron FSs are of different signs (an s\(^{+−}\) state) [16, 19–25]. When one order develops, electron states are reorganized and either favor or hinder the development of other orders. Specifically, in the case of a compensated metal and a perfect nesting (all FSs are cylinders of equal radii) an SDW order comes first and completely gaps all Fermi surfaces, preventing the system from developing SC or CDW orders. The situation is different, however, if electron and hole pockets are not ideal circles and intersect at certain points when the two FSs are centered at the same momentum. This is what is actually observed in the pnictides: hole FSs have circular cross sections, while electron FSs have elliptic ones. In this case the SDW order modifies the Fermi surfaces and leaves ungapped electronic states that may develop superconducting instability [26]. Another possibility for coexistence is the formation of an SDW order with an incommensurate wavelength on circular hole and electron
FSs of different radii. Again, strong nesting of electronic states occurs only on a small part of the FS, leaving other parts available for the SC pairing [27]. The persistence of the (modified) FS into the SDW phase, however, only implies that a superconducting order may develop within the SDW phase [26, 27]. The relative thermodynamic stability of different phases (SC, SDW, SC + SDW) can only be determined from analysis of the free energy.

In the present paper we report an analytical study of the nature of the phase transition between SC and SDW states in iron pnictide materials. We demonstrate that SDW and extended s+ state parameters cannot coexist, but only in a situation when both ellipticity and a difference δμ in electron and hole chemical potentials are present. If either ellipticity or δμ are small, SDW and SC states are separated by a first-order phase transition. In particular, if the system remains a compensated metal, the transition is first order, even when electron bands are elliptical.

We also discuss the s++ order parameter (ordinary s-wave). The authors of [6] argued that for this order parameter coexistence is impossible for any FS geometry. The argument was based on the observation that the transition is strongly first order at a perfect nesting, and on numerical results for a finite ellipticity and a finite δμ. We analyzed this issue analytically and confirmed that the transition between SDW and s++ SC states is indeed first order.

We follow earlier works [6, 26, 27] and consider a simplified model with one circular hole and one elliptical electron FS, separated by, for example, (0, π) in the unfolded BZ or (π, π) in the folded zone. This momentum is also the ordering momentum of the commensurate SDW state [28, 29]. The inclusion of another hole and electron FSs does not affect either superconductivity [21–24, 30, 31] or magnetic order [32–36]. We will also assume that SDW order at a finite doping remains commensurate. This is true if SDW and SC instabilities, when taken separately, occur not far from each other (we set the condition below). If this condition is not met, SDW order becomes incommensurate even before SC develops, and the interplay between SDW and superconductivity has to be re-evaluated. For circular FSs this has been done in [24, 27] and the result is that an incommensurate SDW order and an s++ superconductivity do coexist, at least near Tc.

2. General form of the free energy

We first present a general analysis of the free energy for a system characterized by two scalar order parameters, which we denote as Δ for an SC order and M for an SDW order. We assume that Δ has the same momentum-independent magnitude, but may have either equal signs (s+ state) or opposite signs (s++ state) on the hole and electron FSs. We comment below on the case when superconducting gaps on hole and electron FSs have different magnitudes.

The free energy $\mathcal{F}(\Delta, M)$ to the fourth order in parameters (Δ, M) can be written in the form

$$\mathcal{F} = -\alpha_\Delta \Delta^2 - \alpha_m M^2 + A \Delta^4 + B M^4 + 2 C \Delta^2 M^2.$$  (1)

Below we find from microscopic considerations how all five prefactors in equation (1) depend on doping and ellipticity. But for the moment we simply assume that $\alpha_\Delta$, $A$, $B$ and $C$ are positive constants and $\alpha_m$ varies from $\alpha_m > \alpha_\Delta$ to $\alpha_m = 0$. This mimics the system behavior upon doping, see section 3.

The free energy extrema are found for three different cases.

- Pure SC state, $M = 0$, $\partial \mathcal{F} / \partial \Delta = 0$, is given by

$$\mathcal{F}_s = -\frac{\alpha_\Delta^2}{4 A}, \quad \Delta^2 = \frac{\alpha_\Delta}{2 A}.$$  (2)

- Pure SDW state, $\Delta = 0$, $\partial \mathcal{F} / \partial M = 0$, is given by

$$\mathcal{F}_m = -\frac{\alpha_m^2}{4 B}, \quad M^2 = \frac{\alpha_m}{2 B}.$$  (3)

- Mixed SC + SDW state, $\Delta \neq 0$, $M \neq 0$, found from the conditions $\partial \mathcal{F} / \partial \Delta = 0$, $\partial \mathcal{F} / \partial M = 0$, which, when solved for $\Delta$ and $M$, give

$$M^2 = \frac{\alpha_m A - \alpha_s C}{2 (AB - C^2)}, \quad \Delta^2 = \frac{\alpha_s B - \alpha_m C}{2 (AB - C^2)}.$$  (4)

The corresponding free energy is

$$\mathcal{F}_c = \mathcal{F}_s - \frac{1}{4 A} (\alpha_m A - \alpha_s C)^2 \quad = \mathcal{F}_m - \frac{1}{4 B} (\alpha_s B - \alpha_m C)^2.$$  (5)

These solutions are only meaningful when both $M^2 > 0$ and $\Delta^2 > 0$.

To describe the system behavior with decreasing $\alpha_m$, we also introduce the free energy $\mathcal{F}(M)$ as a function of $M$ only, obtained from equation (1) by substituting

$$\Delta^2(M) = \frac{\alpha_\Delta}{2 A} - \frac{C}{A} M^2$$  (6)

as a solution to $\partial \mathcal{F} / \partial \Delta = 0$. This gives

$$\mathcal{F}(M) = \mathcal{F}_s + \gamma M^2 + \beta M^4, \quad \gamma = -\alpha_m + \alpha_s C / A, \quad \beta = B - C / A.$$  (7)

The condition $\Delta^2(M) \geq 0$ determines the upper limit on a value of the SDW order, $M^2 = \alpha_s / (2C)$. The extremum of $\mathcal{F}(M)$, if it exists below $M_m$, yields the free energy in the coexistence state, $\mathcal{F}_c$. Equation (5) shows that the free energy in the mixed state is the lowest of the three when $\gamma = AB - C^2$ is positive or $\beta > 0$ in equations (7).

First, we consider the case $\chi > 0$. Combining $AB - C^2 > 0$ with the conditions that the mixed phase only exists when $\Delta^2 > 0$ and $M^2 > 0$, we show the behavior of values $\mathcal{F}_{c,s,m}$ as functions of $\alpha_m$ in the top panel of figure 1. We also plot $\mathcal{F}(M)$ for three values of $\alpha_m$: (A) $\alpha_m = \alpha_s B / C$, (B) $\alpha_m = \alpha_s \sqrt{B A}$ and (C) $\alpha_m = \alpha_s C / A$. The triangle symbol ($\triangle$) in these plots at $M = M_m$ is the value of $\mathcal{F}_m$ from equations (3) corresponding to the extremum of the free energy at $\Delta = 0$. In general, $\mathcal{F}_m$ is different from $\mathcal{F}(M_m)$, both of which are obtained from equation (1) with $\Delta = 0$, but $M = \sqrt{\alpha_m} / 2B$ or $M = \sqrt{\alpha_m} / 2C$, respectively.
At large $\alpha_m > \alpha_s B/C$, the system is in the SDW phase and $\mathcal{F}(M)$ monotonically decreases with increasing $M$ to $\mathcal{F}(M_m)$, but $\mathcal{F}_s$ has an even smaller value than $\mathcal{F}(M_m)$. In this case $\gamma < 0$ and $\beta > 0$ in equations (7). At such $\alpha_m$ the solution corresponding to the coexistence state does not exist because the condition $\Delta^2 > 0$ is not yet satisfied. At $\alpha_m \leq \alpha_s B/C$, and the minimum of $\mathcal{F}(M)$ takes place at $M$, given by equations (4), for $0 < M < M_m$ when the coexistence state develops (still, we have $\gamma < 0$ and $\beta > 0$ in equations (7)). The minimum splits from $\Delta = 0$ at $\alpha_m = \alpha_s B/C$ via a continuous second-order transition. When $\alpha_m$ decreases further, but still $\alpha_m > \alpha_s C/A$, the minimum shifts to smaller $M$, and eventually, at $\alpha_m = \alpha_s C/A$, reaches the value $\mathcal{F}_s$ at $M = 0$. At even smaller $\alpha_m$, the global minimum corresponds to $\mathcal{F}_c$, i.e. the system gradually transforms from the mixed state into the superconducting state. The overall evolution of the system has two second-order transitions at $\alpha_m = \alpha_s B/C$ and $\alpha_m = \alpha_s C/A$, and the intermediate mixed state at $\alpha_s C/A < \alpha_m < \alpha_s B/C$.

In the opposite case, $AB - C^2 < 0$, the free energy of the ‘coexistence’ state is always larger than the free energy of pure SDW or SC states, i.e. the ‘coexistence’ state corresponds to a saddle point of the free energy and does not represent an actual thermodynamic state of the system. The evolution of $\mathcal{F}_{c,m,c}$ is shown in figure 2. At large $\alpha_m > \alpha_s C/A$, the free energy reaches its minimum at $\Delta = 0$, $M^2 = \alpha_m/(2B)$ and the system is in the pure SDW phase. In this case, $\gamma > 0$ and $\beta < 0$ in equations (7). The ‘coexistence’ state solution does not exist because it formally corresponds to $\Delta^2 < 0$. At $\alpha_m = \alpha_s C/A$, $\mathcal{F}_c$ and $\mathcal{F}_s$ coincide. At smaller $\alpha_m$ the mixed state solution becomes real, in the sense that it corresponds to $\Delta^2 > 0$, and $\mathcal{F}(M)$ develops a maximum at $M$, defined by equations (4). As $\alpha_m$ decreases, the maximum moves to larger $M$ and simultaneously $\mathcal{F}_m$ increases. At $\alpha_m = \alpha_s \sqrt{B/A}$, the free energies for pure states, $\mathcal{F}_m$ and $\mathcal{F}_s$, become equal, while the ‘coexistence’ state has a higher energy. At even smaller $\alpha_m$, $\mathcal{F}_m > \mathcal{F}_s$ and the pure SC phase is a true thermodynamic equilibrium state. The magnetic state remains a local minimum of $\mathcal{F}(\Delta, M)$ and the ‘coexistence’ state remains a local maximum of $\mathcal{F}(M)$ down to $\alpha_m = \alpha_s B/C$. At even smaller $\alpha_m$ the coefficient $\gamma$ changes sign and the ‘coexistence’ solution disappear.

3. Application to pnictides

We now apply the above analysis of the two-parameter model to the pnictide superconductors. We approximate the electronic structure of pnictides by a model of two families of fermions, which form one hole and one electron FSs of approximately equal sizes. We assume that the hole FS is circular and the dispersion of fermions near this FS is

$$\epsilon_h = \mu_h - \frac{k^2}{2m_h}. \quad (8)$$

The electronic FS is an ellipse, and the dispersion of fermions near this FS is

$$\epsilon_e = -\mu_e + \frac{k_x^2}{2m_x} + \frac{k_y^2}{2m_y} = -\epsilon_h + \mu_h - \mu_e + \frac{k_x^2}{2m_x} - \frac{k_y^2}{2m_y} = -\epsilon_h + \mu_h - \mu_e + \frac{k_x^2}{2} \frac{(m_x + m_y)}{2m_xm_y} - \frac{1}{m_h} + \frac{k_y^2}{2} \frac{(m_x - m_y)}{2m_xm_y}. \quad (9)$$
The last three terms represent different deviations from perfect nesting: (i) the change in chemical potentials $\delta \mu = \mu_0 - \mu_e$, (ii) the difference in the electron and hole masses, $m_{x,y} \neq m_h$, and (iii) ellipticity, $m_1 \neq m_2$. We will see that typical $\delta \epsilon_b$ are of the order of the temperature $T$. We assume that the chemical potential $\mu$ is much larger than $T$ and neglect all terms arising due to deviations from perfect nesting with the contribution to the free energy small in the parameter $T/\mu_b$. Within this approximation, we can set $k = k_F = \sqrt{2m_0 \mu_b}$ in the two terms in the last line of equation (9). Then $\delta \epsilon_b = \epsilon_b + \delta \epsilon$, where

$$\delta \epsilon = \delta \epsilon_0 + \delta_2 \cos 2\phi, \quad \delta_2 = \frac{k_F^2}{2} \frac{m_y - m_x}{m_x m_y}, \quad (10)$$

$$\delta \epsilon_0 = \frac{\mu_b - \mu_e}{2} + \frac{k_F^2}{4} \frac{(m_y + m_x)}{2m_x m_y} - \frac{m_h}{2} \quad (11)$$

Within this approximation, the two dispersions differ by a term $\delta \epsilon_0$, which depends on the angle along the FS but does not depend on $\epsilon_b$. One can verify that, with these $\delta \epsilon_0$ and $\epsilon_b$, superconducting gaps along the hole and electron FSs are equal in magnitude and different in sign if, indeed, the pairing interaction is approximated by a constant and $\mu$ is set to be much larger than $T$.

The free energy for the case when the two dispersions differ by a constant was presented in [27] for circular FSs, when $\delta \epsilon_0 = \delta \epsilon_0$ is just a constant. Extending the expression for the free energy to the case when $\delta \epsilon$ depends on $\phi$ due to ellipticity, we obtain

$$F(\Delta, M) = \frac{1}{4N_f} \frac{2}{2} \ln \sqrt{\frac{M^2 + |M|}{2}} T \sum_{\omega_m} \frac{|\Delta|^2 + M^2}{|\omega_m|^2} - \pi T \sum_{\omega_m} \text{Re} \left( \langle E_m + i\delta \epsilon \rangle^2 + M^2 - |\omega_m|^2 \right), \quad (12)$$

where $E_m = \sqrt{\omega_m^2 + \Delta^2}$, $\omega_m = \pi T (2m/1)$ are the Matsubara frequencies and $\langle \cdot \cdot \cdot \rangle_{\phi}$ imply averaging over the direction $\phi$ on the Fermi surface. Temperatures $T_i$ and $T_h$ are transition temperatures to SC or SDW states for ‘pure’ cases when interactions are exclusively in the SC or SDW channels.

Expanding the free energy, equation (12), up to the fourth order in $M$ and $\Delta$ and comparing the result with equation (1) we obtain

$$A = \frac{\pi T}{4} \sum_{m \geq 0} \frac{1}{\omega_m^3}, \quad (13)$$

$$B = \frac{\pi T}{4} \sum_{m \geq 0} \omega_m \frac{\omega_m^2 - 3\delta \epsilon^2}{(\omega_m^2 + \delta \epsilon^2)^2}, \quad (14)$$

$$C = \frac{\pi T}{4} \sum_{m \geq 0} \frac{\omega_m^2 - \delta \epsilon^2}{|\omega_m|^2 (\omega_m^2 + \delta \epsilon^2)^2} \quad (15)$$

and

$$\alpha_s = \frac{1}{2} \ln \frac{T_i}{T}, \quad (16)$$

$$\alpha_m = \frac{1}{2} \ln \frac{T_m}{T} - \pi T \sum_{m \geq 0} \frac{\delta \epsilon^2}{|\omega_m|^2 (\omega_m^2 + \delta \epsilon^2)}. \quad (17)$$

The superconducting part of the free energy, expressed via $\alpha_s$ and $A$, is independent of $\delta \epsilon$, but the magnetic part and the mixed $\Delta^2 M^2$ term depend on $\delta \epsilon$. The expansion makes sense if $\Delta, M$ are of the same order, i.e. if $T_i$ and $T_m$ do not differ much, which we assume henceforth.

For perfect nesting $\delta \epsilon = 0$ and we have $A = B = C$, i.e. $\chi = 0$, as was explicitly stated in [6]. Furthermore, the interaction term in the free energy is $\Delta (\Delta^2 + M^2)^2$ and does not favor either SC or SDW orders even beyond the expansion to the fourth order [27]. The transition occurs into a state with a higher transition temperature $T_i$ or $T_m$, and once either SDW or SC order develops, the other order does not appear simply because the quadratic term favors either $\Delta = 0$ or $M = 0$, and the interaction term is isotropic. If $T_i = T_m$, the free energy given by equation (12) is $SO(5)$ symmetric and extra terms are needed to break this symmetry.

As we said, we consider the case when $T_m > T_i$, such that at perfect nesting the system develops an SDW order. Deviations from perfect nesting lead to two effects. First, the magnitude of $\alpha_m$ is reduced, which is the manifestation of the fact that SDW instability is suppressed when nesting becomes non-perfect. Superconducting $\alpha_s$ is not affected by $\delta \epsilon$, and at large $\delta \epsilon$ superconductivity is the only possible state. Second, coefficients $B$ and $C$ evolve with $\delta \epsilon$ and $\chi = AB - C^2$ becomes non-zero when $\delta \epsilon$ is finite. The question is: what is the sign of this term? We recall that, when $\chi > 0$, the system evolves from SDW to SC via two second-order transitions and the intermediate coexistence phase, while for $\chi < 0$, there is no mixed state and SDW and SC phases are separated by a first-order transition.

To get an insight into how $\chi$ evolves at non-zero $\delta \epsilon$, we first consider $\delta \epsilon$ as a small parameter and expand $A$, $B$ and $C$ in powers of $\delta \epsilon$. Collecting terms up to fourth order in $\delta \epsilon$, we obtain

$$\chi = \frac{1}{32 \pi^4 T_s} \left( s_1 (\delta \epsilon^4) - s_2 (\delta \epsilon^2)^2 \right), \quad (18)$$

where

$$s_1 = 5 \left( \sum_{m \geq 0} \frac{1}{(2m + 1)^3} \right) \left( \sum_{m \geq 0} \frac{1}{(2m + 1)^5} \right), \quad (19)$$

$$s_2 = 9 \left( \sum_{m \geq 0} \frac{1}{(2m + 1)^3} \right)^2. \quad (20)$$

The sums are expressed in terms of the Riemann zeta function and give $s_1 \approx 5.26$ and $s_2 \approx 9.08$. Substituting $\delta \epsilon$ from (11) and averaging, we obtain

$$\chi \approx \frac{1}{32 \pi^4 T_s} \left( -3.82 s_1^3 + 6.70 s_2^2 s_1 - 0.30 s_2^3 \right). \quad (20)$$

We emphasize that, in the two limits when either $\delta_2 = 0$ or $\delta_0 = 0$, $\chi < 0$, i.e. the transition is first order. The first limit corresponds to circular FSs with different chemical potentials, while the second limit corresponds to the case when the chemical potential remains equal but electron dispersion becomes elliptical. In both cases, the SDW order opens gaps for some fermionic excitations, but still preserves low-energy fermionic states near the modified FSs. Fermions near these FSs still have a non-zero $s^\pm$ SC solution: however, this solution represents an energetically unfavorable state. We
particularly emphasize that the ellipticity of electron dispersion taken alone (i.e. $\delta_0 = 0$) is not sufficient for the appearance of the coexistence phase.

When both $\delta_0$ and $\delta_2$ are non-zero, there is a relatively broad range $0.76 < \delta_2/\delta_0 < 4.68$ where $\chi > 0$ and the transformation from pure SDW to SC phases occurs via a coexistence phase. To verify that this statement holds at larger values of $\delta_0$ and $\delta_2$, we computed $\chi(\delta_0, \delta_2)$ without expanding in $\delta_\varphi$. We plot the sign of $\chi(\delta_0, \delta_2)$ in figure 3. We obtained the same result as above, namely, for $\delta_0 = 0$ or $\delta_2 = 0$, $\chi < 0$ and the transition between SDW and SC states is of first order, while when both $\delta_0$ and $\delta_2$ are non-zero, there exists a region with $\chi(\delta_0, \delta_2) > 0$. In this parameter range, the transformation from SDW to SC phases involves a coexistence phase.

Two remarks are in order here. In our analysis of the free energy we assumed that $B$ given by equation (14) is positive. This is, however, only true if $\delta_0$ and $\delta_2$ are below certain thresholds, see the dashed line in figure 3. At larger $\delta_0$ and $\delta_2$, the coefficient $B$ becomes negative and the analysis has to be modified. This is the condition where instability with respect to the formation of an incommensurate SDW order occurs, much like the Fulde–Ferrell–Larkin–Ovchinnikov phase in superconductors. In particular, for $\delta_2 = 0$, $B$ becomes negative for $\delta_0/2\pi T \approx 0.304$ [24, 27]. This order develops at a temperature above $T_s$ if $T_s < 0.56 T_m$. We also note that equation (20) is obtained under the assumption that $T \ll \mu$, which allowed us to restrict the contributions to $T^4 \chi$ from terms that scale as $(\delta_\varphi/\mu)^2$ and neglect terms which scale as powers of $(\delta_\varphi/\mu)^4$. It is unlikely but, in principle, possible that the expansion of the full $\chi$ in powers of $\delta_\varphi$ begins with the quadratic term $(1/T^4)(\delta_\varphi/\mu)^2$, i.e. $T^4 \chi = c_2(\delta_\varphi/\mu)^2 + c_4(\delta_\varphi/\mu)^4 + \cdots$. If this is the case and $c_2 > 0$, the mixed phase exists in a tiny range of $\delta_\varphi$, where $\chi < 0$ without the $c_2$ term.

4. Conventional two-band s-superconductivity

For completeness, we also consider the case when the superconducting order parameter in a conventional s-wave, i.e. the sign of $\Delta$, is the same on hole and electron FSs. This case has been considered in [6] and the conclusion was that the transition from the SDW to SC phase is always of first order, regardless of ellipticity or the shift of chemical potentials. This result implies that the very coexistence between SDW and SC in Fe-pnictides is an implication that the pairing state is not a conventional s$^+$ state.

The analysis in [6] was based on the observation that for the s$^+$ gap, $\chi < 0$ already for circular FSs and perfect nesting, and in numerical calculations of $\chi$ for some cases when nesting is not perfect. We analyze this issue analytically.

The free energy for the s$^+$ SC gap can be derived using the same approach as in [27] and has the form

$$F^{s+}(\Delta, M) = \frac{|\Delta|^2}{2N_f} \ln T - \frac{M^2}{2} \ln T + \frac{1}{\pi T} \sum_{m=\pm} \frac{|\Delta|^2 + M^2}{2|\omega_m|} - \frac{\pi T}{2} \sum_{m=\pm} \frac{1}{|\omega_m|} \left| \left\{ \left( \omega_m^2 + \Delta^2 + M^2 \right)^{1/2} - |\omega_m| \right\} \right|^2. \quad (21)$$

Expanding this free energy in powers of $\Delta^2$ and $M^2$ and comparing the result with equation (1), we find that coefficients $A$ and $B$ are still given by equations (13) and (14), but the coefficient $C$ is modified to

$$C = \frac{\pi T}{4} \sum_{m=\pm} \left\{ \frac{3|\omega_m^2 + \Delta^2|}{|\omega_m|(|\omega_m^2 + \Delta^2|)} \right\}. \quad (22)$$

We now have $\chi = Y(\delta_0, \delta_2)/\pi T^4$. At perfect nesting, $Y(0, 0) = -(7\zeta(3)/8)^2/2 < 0$ and the transition is of first order. When $\delta_0, \delta_2$ increases, the magnitude of $\chi \propto Y(\delta_0, \delta_2)$ is reduced, but we found that its sign remains negative for arbitrary values of $\delta_0$ and $\delta_2$. In figure 4, we show the behavior of $Y(\delta_0, \delta_2)$ for three cases: $Y(\delta, 0)$, $Y(0, \delta)$ and $Y(\delta, \delta)$. In all cases $Y(\delta_0, \delta_2)$ monotonically increases with increasing argument, but remains negative.
The solid curved line is the transition between the normal and (in)commensurate SDW$^\omega$ states. The dotted line is the first-order transition line between commensurate and incommensurate SDW phases. The solid horizontal line is the second order normal–SC transition. The transitions between both, incommensurate and commensurate SDW phases, and the SC phase are first order (dashed lines), and the mixed phase does not appear.

We also analyzed the case when the SC order has a conventional, sign-preserving $s^{\pm}$ symmetry on the two FSs and confirmed analytically the result of [6] that the transition between SDW and SC phases is always of first order, and the mixed phase does not appear. We argue that this is the case even when the SDW order becomes incommensurate.

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References

[1] Kamihara Y, Watanabe T, Hirano M and Hosono H 2008 J. Am. Chem. Soc. 130 3296
[2] Luetske H et al 2009 Nat. Mater. 8 305
[3] Laplace Y, Bobroff J, Rullier-Albenque F, Colson D and Forget A 2009 Phys. Rev. B 80 140401
[4] Chu J-H, Analytis J G, Kucharczyk C and Fisher I R 2009 Phys. Rev. B 79 014506
[5] Rotter M, Tegel M and Johrendt D 2008 Phys. Rev. Lett. 101 107006
[6] Fernandes R M et al 2009 arXiv:0911.5183
[7] Liu C et al 2008 Phys. Rev. Lett. 101 177005
[8] Evtushinsky D V et al 2009 Phys. Rev. B 79 054517
[9] Hsieh D, Xia Y, Wray L, Qian D, Gomes K, Yazdani A, Chen G F, Luo J L, Wang N L and Hasan M Z 2008 arXiv:0812.2289v1
[10] Ding H et al 2008 arXiv:0812.0534
[11] Zabolotnyy V B et al 2009 Nature 457 569
[12] Coldea A I, Fletcher J D, Carrington A, Analytis J G, Bangura A F, Chu J H, Erickson A S, Fisher I R, Hussey N E and McDonald R D 2008 Phys. Rev. Lett. 101 216402
[13] Coldea A I, Andrew C M J, Analytis J G, McDonald R D, Bangura A F, Chu J-H, Fisher I R and Carrington A 2009 Phys. Rev. Lett. 103 026404
[14] Singh D J and Du M H 2008 Phys. Rev. Lett. 100 237003
[15] Boeri L, Dolgov O V and Golubov A A 2008 Phys. Rev. Lett. 101 026403
[16] Mazin I I, Singh D J, Johannes M D and Du M H 2008 Phys. Rev. Lett. 101 057003
[17] Chubukov A V, Efremov D and Eremin I 2008 Phys. Rev. B 78 134512
[18] Chubukov A V 2009 Physica C 469 640
[19] Kuroki K, Onari S, Arita R, Usui H, Tanaka Y, Kontani H and Aoki H 2008 Phys. Rev. Lett. 101 087004
[20] Barzykin V and Gorkov L P 2008 JETP Lett. 88 131
[21] Maier T A, Graser S, Scalapino D J and Hirschfeld P J 2009 Phys. Rev. B 79 224510
[22] Chubukov A V, Vavilov M G and Vorontsov A B 2009 Phys. Rev. B 80 140515
[23] Thomale R, Platt C, Hu J, Honerkamp C and Bernevig B A 2009 Phys. Rev. B 80 180505
[24] Cvetkovic V and Tesanovic Z 2009 Europhys. Lett. 85 37002
[25] Stanev V, Kang J and Tesanovic Z 2008 Phys. Rev. B 78 184509
[26] Parker D, Vavilov M G, Chubukov A V and Mazin I I 2009 Phys. Rev. B 80 100508
[27] Vorontsov A B, Vavilov M G and Chubukov A V 2009 Phys. Rev. B 79 060508
[28] de la Cruz C et al 2008 Nature 453 899
[29] Klauss H-H et al 2008 Phys. Rev. Lett. 101 077005
[30] Wang F, Zhai H, Ran Y, Vishwanath A and Lee D-H 2009 Phys. Rev. Lett. 102 047005
[31] Platt C, Honerkamp C and Hanke W 2009 New J. Phys. 11 055058
[32] Lorenzana J, Seibold G, Ortix C and Grilli M 2008 Phys. Rev. Lett. 101 186402
[33] Brydon P and Timm C 2009 Phys. Rev. B 79 180504
[34] Johannes M D and Mazin I I 2009 Phys. Rev. B 79 220510
[35] Mazin I I and Johannes M D 2009 Nat. Phys. 5 141
[36] Eremin I and Chubukov A V 2010 Phys. Rev. B 81 024511