Direct product of general intuitionistic fuzzy sets of subtraction algebras

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Abstract: We define direct product of \((e, e \in \mathbb{Q}_k)\)-intuitionistic fuzzy sets and direct product of \((e, e \in \mathbb{Q}_k)\)-intuitionistic fuzzy soft sets of subtraction algebras and investigate some related properties.

Subjects: Advanced Mathematics; Algebra; Mathematics & Statistics; Science

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1. Introduction

The system \((X; \circ, \setminus)\) by Schein (1992), is a set of functions closed under the composition "\(\circ\" under the composition of function (and hence \((X, \circ)\) is a function semigroup) and the set theoretical subtraction "\(\setminus\" (and hence \((X, \setminus)\) is a subtraction algebra in the sense of Abbot (1969)). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. Zelinka (1995) discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type called the atomic subtraction algebras. Jun, Kim, and Roh (2005) introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. To study more about subtraction algebras see Ceven (2009), Jun and Kim (2007). The fuzzifications of ideals in subtraction algebras were discussed in Lee and Park (2007).

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PUBLIC INTEREST STATEMENT

Real world is featured with complex phenomenon. As vulnerability is unavoidably included in issues emerge in different fields of life and traditional techniques neglected to handle these sorts of issues. Managing with loose, unverifiable, or defective data was a major assignment for a long time. Numerous models were introduced with a specific end goal to appropriately join instability into framework portrayals; L.A. Zadeh in 1965 presented the thought of a fuzzy set. Zadeh supplanted traditional trademark capacity of established fresh sets which tackles its qualities in \((0, 1)\) by enrollment capacity which tackles its values in shut interim \([0, 1]\). Be that as it may, it is by all accounts the restricted case so this was summed up by K.T. Atanassov in 1986. Soft sets are additionally considered an exceptionally convenient device with a specific end goal to handle loose data. Here, we used a combination of soft and intuitionistic fuzzy sets.
Bhakat and Das (1996), introduced a new type of fuzzy subgroups, that is, the \((\in, \in \lor \forall q)\) fuzzy subgroups. In fact, the \((\in, \in \lor \forall q)\) fuzzy subgroup is an important generalization of Rosenfeld’s fuzzy subgroup. Shabir et al. characterized semigroups by \((\in, \in \lor \forall q)\)-fuzzy ideals in Shabir (2010). Gulistan, Shahzad, and Yaqoob (2014) studied \((\in, \in \lor \forall q)\)-fuzzy KU-ideals of KU-algebras.

Molodtsov (1999) introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that are free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set theory are progressing rapidly. Maji, Roy, and Biswas (2002) described the application of soft set theory to a decision-making problem. Maji, Biswas, and Roy (2003) also studied several operations on the theory of soft sets. The most appropriate theory for dealing with uncertainties is the theory of fuzzy sets developed by Zadeh (1965).

The notion of fuzzy soft sets, as a generalization of the standard soft sets, was introduced in Maji, Biswas, and Roy (2001a), and an application of fuzzy soft sets in a decision-making problem was presented. In Ahmad and Athar (2009), have introduced arbitrary fuzzy soft union and fuzzy soft intersection. Aygunoglu and Aygun introduced the notion of fuzzy soft group and studied its properties. In Jun, Lee, and Park (2010) have introduced the notion of fuzzy soft BCK/BCI-algebras and (closed) fuzzy soft ideals, and then derived their basic properties. Recently, Yang (2011) have studied fuzzy soft semigroups and fuzzy soft (left, right) ideals, and have discussed fuzzy soft image and fuzzy soft inverse image of fuzzy soft semigroups (ideals)in detail. Recently, Khan et al. gave the idea of \((\in, \in \lor \forall q)\)-intuitionistic fuzzy ideals of subtraction algebras (Khan, Davvaz, Yaqoob, Gulistan, & Khalaf, 2015). Gulistan, Khan, Yaqoob, Shahzad, and Ashraf (2016) defined direct product of generalized cubic sets in Hv-LA-semigroups. Yaqoob (yaoqoob), studied interval-valued intuitionistic fuzzy ideals of regular LA-semigroups. In Akram and Yaqoob (2013) and Yaqoob, Akram, and Aslam (2013), the authors applied the concept of intuitionistic fuzzy soft sets to ordered ternary semigroups and groups. Also see Aslam, Abdullah, Davvaz, and Yaqoob (2012), Khan, Jun, Gulistan, and Yaqoob (2015), Khan, Youasfzai, Khan, and Yaqoob (2013), Yaqoob, Aslam, Davvaz, and Ghareeb (2014), Yaqoob, Chinram, Ghareeb, and Aslam (2013), Yaqoob, Mostafa, and Ansari (2013), Youasfzai, Yaqoob, and Ghareeb (youasfzai), Youasfzai, Yaqoob, and Hila (2012).

The aim of this article is to study the concept of Direct product of \((\in, \in \lor \forall q)\)-intuitionistic fuzzy sets and Direct product of \((\in, \in \lor \forall q)\)-intuitionistic fuzzy soft sets of subtraction algebras and investigate some related properties.

2. Preliminaries

In this section we recall some of the basic concepts of subtraction algebra which will be very helpful in further study of the paper. Throughout the paper \(X\) denotes the subtraction algebra unless otherwise specified.

**Definition 2.1** (Aygunoglu & Aygun, 2009) A nonempty set \(X\) together with a binary operation “−” is said to be a subtraction algebra if it satisfies the following:

\[
\begin{align*}
&S_1 \quad x - (y - x) = x, \\
&S_2 \quad x - (x - y) = y - (y - x), \\
&S_3 \quad (x - y) - z = (x - z) - y, \text{ for all } x, y, z \in X.
\end{align*}
\]

The last identity permits us to omit parentheses in expression of the form \((x - y) - z\). The subtraction determines an order relation on \(X\) \(a \leq b \Leftrightarrow a - b = 0\), where 0 = \(a - a\) is an element that does not depend upon the choice of \(a \in X\). The ordered set \((X; \leq)\) is a semi-Boolean algebra in the sense of Abbot (1969), that is, it is a meet semi lattice with zero, in which every interval \([0, a]\) is a boolean algebra with respect to the induced order. Here \(a \land b = a - (a - b)\); the complement of an element \(b \in [0, a]\) is \(a - b\) and is denoted by \(b'\); and if \(b, c \in [0, a]\) then...
\[ b \lor c = (b' \land c')' = ((a - b) \land (a - c)) = a - (a - b) - ((a - b) - (a - c)). \]

In a subtraction algebra, the following are true (see Aygunoglu & Aygun, 2009):

\begin{enumerate}
  \item \((x - y) - y = x - y,
  \item \(x - 0 = x \land 0 - x = 0,
  \item \((x - y) - x = 0,
  \item \(x - (x - y) \leq y,
  \item \((x - y) - (y - x) = x - y,
  \item \(x - (x - (x - y)) = x - y,
  \item \((x - y) - (z - y) \leq x - z,
  \item \(x \leq y \text{ if and only if } x = y - w \text{ for some } w \in X,
  \item \(x \leq y \text{ implies } x - z \leq y - z \text{ and } z - y \leq z - x, \text{ for all } z \in X,
  \item \(x, y \leq z \text{ implies } x - y = x \land (z - y),
  \item \((x \land y) - (x \land z) \leq x \land (y - z),
  \item \((x - y) - z = (x - z) - (y - z).
\end{enumerate}

\textbf{Definition 2.2} (Aygunoglu & Aygun, 2009) A nonempty subset \( A \) of a subtraction algebra \( X \) is called an ideal of \( X \), denoted by \( A \leq X \): if it satisfies:

\begin{enumerate}
  \item \( a - x \in A \) for all \( a \in A \) and \( x \in X \),
  \item \( a \land b \in A \), whenever \( a \lor b \in X \) then \( a \land b \in A \).
\end{enumerate}

\textbf{Proposition 2.3} (Aygunoglu & Aygun, 2009) A nonempty subset \( A \) of a subtraction algebra \( X \) is called an ideal of \( X \), if and only if it satisfies:

\begin{enumerate}
  \item \( 0 \in A \),
  \item \( x \land y \in X \) and \( y \in A \Rightarrow x \in A \).
\end{enumerate}

\textbf{Proposition 2.4} (Aygunoglu & Aygun, 2009) Let \( X \) be a subtraction algebra and \( x, y \in X \). If \( w \in X \) is an upper bound for \( x \) and \( y \), then the element \( x \lor y = w - ((w - y) - x) \) is the least upper bound for \( x \) and \( y \).

\textbf{Definition 2.5} (Aygunoglu & Aygun, 2009) Let \( Y \) be a nonempty subset of \( X \); then, \( Y \) is called a sub-algebra of \( X \) if \( x - y \in Y \), whenever \( x, y \in Y \).

\textbf{Definition 2.6} (Lee & Park, 2007) Let \( f \) be a fuzzy set of \( X \). Then \( f \) is called a fuzzy subalgebra of \( X \) if it satisfies (FS) \( f(x - y) \geq \min(f(x), f(y)) \), whenever \( x, y \in X \).

\textbf{Definition 2.7} (Lee & Park, 2007) A fuzzy set \( f \) is said to be a fuzzy ideal of \( X \) if it satisfies:

\begin{enumerate}
  \item \( f(x - y) \geq f(x),
  \item \( f(x \lor y) \geq \min(f(x), f(y)) \), for all \( x, y \in X \).
\end{enumerate}

Here we mentioned some of the related definitions and results which are directly used in our work. For details we refer the reader Khan et al. (2015).

\textbf{Definition 2.8} Atanassov (1986) An intuitionistic fuzzy set \( A \) in \( X \) is an object of the form \( A = \{(x, \mu_x(x), \gamma_x(x)) : x \in X\} \), where the function \( \mu_x : X \to [0, 1] \) and \( \gamma_x : X \to [0, 1] \) denote the degree of membership and degree of non-membership of each element \( x \in X \), and \( 0 \leq \mu_x(x) + \gamma_x(x) \leq 1 \) for all \( x \in X \). For simplicity, we will use the symbol \( A = (\mu_x, \gamma_x) \) for the intuitionistic fuzzy set \( A = \{(x, \mu_x(x), \gamma_x(x)) : x \in X\} \). We define \( 0(x) = 0 \) and \( 1(x) = 1 \) for all \( x \in X \).

\textbf{Definition 2.9} (Khan et al., 2015) Let \( X \) be a subtraction algebra. An intuitionistic fuzzy set
A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}, of the form

\[
x_{x, y} = \begin{cases} 
(\alpha, \beta) & \text{if } y = x \\
(0, 1) & \text{if } y \neq x
\end{cases}
\]

is said to be an intuitionistic fuzzy point with support x and value (\alpha, \beta) and is denoted by x_{x, y}. A intuitionistic fuzzy point x_{x, y} is said to intuitionistic belongs to (resp., intuitionistic quasi-coincident) with intuitionistic fuzzy set A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\} written x_{x, y} \in A resp. x_{x, y} \in A if \mu_A(x) \geq \alpha and \gamma_A(x) \leq \beta (resp., \mu_A(x) + \alpha < 1 and \gamma_A(x) + \beta < 1). By the symbol x_{x, y} \in A we mean \mu_A(x) + \alpha + k > 1 and \gamma_A(x) + \beta + k < 1, where k \in (0, 1).

We use the symbol x_t \in \mu_x implies \mu_A(x) \geq t and \frac{1}{n} \sum_{x \in X} \mu_A(x) \leq \frac{1}{n} links \gamma_A(x) \leq t, in the whole paper.

**Definition 2.10** (Khan et al., 2015) An intuitionistic fuzzy set A = (\mu_A, \gamma_A) of X is said to be an (\epsilon, \in \forall_E) -intuitionistic fuzzy subalgebra of X if

x_{t, t_1} \in A, y_{t_1, t_2} \in A \Rightarrow (x - y)_{t_1, t_2} \in \forall_E A, for all x, y \in X, t_1, t_2, t, k \in (0, 1).

**Definition 2.11** (Khan et al., 2015) An intuitionistic fuzzy set A = (\mu_A, \gamma_A) of X is said to be an (\epsilon, \in \forall_E) -intuitionistic fuzzy ideal of X if it satisfies the following conditions,

(i) x_{t, t_1} \in A, y \in X \Rightarrow (x - y)_{t, t_1} \in \forall_E A,

(ii) If there exist x \lor y, then x_{t, t_1} \in A, y_{t_1, t} \in A \Rightarrow (x \lor y)_{t_1, t_1} \in \forall_E A, for all x, y \in X, t, t_1, t, k \in (0, 1).

Molodtsov defined the notion of a soft set as follows.

**Definition 2.12** (Molodtsov, 1999) A pair (F, A) is called a soft set over U, where F is a mapping given by F:A \rightarrow P(U). In other words a soft set over U is a parametrized family of subsets of U.

The class of all intuitionistic fuzzy sets on X will be denoted by IF(X).

**Definition 2.13** (Maji, Biswas, & Roy, 2001b, 2004) Let U be an initial universe and E be the set of parameters. Let A \subseteq E. A pair (\bar{F}, A) is called an intuitionistic fuzzy soft set over U, where \bar{F} is a mapping given by \bar{F}:A \rightarrow IF(U).

In general, for every \epsilon \in A, \bar{F}[\epsilon] = \{\mu_{\bar{F}[\epsilon]}(x), y_{\bar{F}[\epsilon]}(y)\} is an intuitionistic fuzzy set in U and it is called intuitionistic fuzzy value set of parameter \epsilon.

**Definition 2.14** (Khan et al., 2015) An intuitionistic fuzzy soft set (\bar{F}, A) of X is said to be an (\epsilon, \in \forall_E) -intuitionistic fuzzy soft subalgebra of X, if for all \epsilon \in A, \bar{F}[\epsilon] = \{\mu_{\bar{F}[\epsilon]}(x), y_{\bar{F}[\epsilon]}(y)\} is an (\epsilon, \in \forall_E) -intuitionistic fuzzy subalgebra of X, if

(i) \mu_{\bar{F}[\epsilon]}(x - y) \geq \min\{\mu_{\bar{F}[\epsilon]}(x), \mu_{\bar{F}[\epsilon]}(y), \frac{1 + k}{2}\},

(ii) y_{\bar{F}[\epsilon]}(x - y) \leq \max\{y_{\bar{F}[\epsilon]}(x), y_{\bar{F}[\epsilon]}(y), \frac{1 + k}{2}\}, for all x, y \in X.

**Definition 2.15** (Khan et al., 2015) An intuitionistic fuzzy soft set (\bar{F}, A) of X is said to be an (\epsilon, \in \forall_E) -intuitionistic fuzzy soft ideal of X, if for all \epsilon \in A, \bar{F}[\epsilon] = \{\mu_{\bar{F}[\epsilon]}(x), y_{\bar{F}[\epsilon]}(y)\} is an (\epsilon, \in \forall_E) -intuitionistic fuzzy soft ideal of X, if
(i) \( \mu_{\gamma}(x - y) \geq \min(\mu_{\gamma}(x), \frac{1-x}{2}) \),

(ii) \( \gamma_{\gamma}(x - y) \leq \max(\gamma_{\gamma}(x), \frac{1-y}{2}) \),

(iii) \( \mu_{\gamma}(x \lor y) \geq \min(\mu_{\gamma}(x), \mu_{\gamma}(y), \frac{1-x}{2}) \),

(iv) \( \gamma_{\gamma}(x \lor y) \leq \max(\gamma_{\gamma}(x), \gamma_{\gamma}(y), \frac{1-x}{2}) \), for all \( x, y \in X \).

### 3. Direct product of an \((\in, \in, \notin)\)- intuitionistic fuzzy subalgebra/ideals

In this section, we define Direct product of \((\in, \in, \notin)\)-intuitionistic fuzzy sets and investigate some related properties.

**Definition 3.1** Let \( A = (\mu_{A}(x), \gamma_{A}(x)) \) and \( B = (\mu_{B}(x), \gamma_{B}(x)) \) be two \((\in, \in, \notin)\)-intuitionistic fuzzy sets of \( X_{1} \) and \( X_{2} \), respectively. Then the Direct product of \( A \times B = (\mu_{A} \times B, \gamma_{A} \times B) \) is defined as \( \mu_{A \times B}(x, y) = \mu_{A}(x) \land \mu_{B}(y) \) and \( \gamma_{A \times B}(x, y) = \gamma_{A}(x) \lor \gamma_{B}(y) \) for all \((x, y) \in X_{1} \times X_{2}\).

**Definition 3.2** An intuitionistic fuzzy set \( A \times B \) of \( X_{1} \times X_{2} \) is called an \((\in, \in, \notin)\)-intuitionistic fuzzy subalgebra of \( X_{1} \times X_{2} \) if it satisfies,

(i) \( \mu_{A \times B}(x_{1}, y_{1}) - (x_{2}, y_{2}) \geq \min(\mu_{A \times B}(x_{1}, y_{1}), \mu_{A \times B}(x_{2}, y_{2}), \frac{1-x}{2}) \),

(ii) \( \gamma_{A \times B}(x_{1}, y_{1}) - (x_{2}, y_{2}) \leq \max(\gamma_{A \times B}(x_{1}, y_{1}), \gamma_{A \times B}(x_{2}, y_{2}), \frac{1-y}{2}) \).

**Example 3.3** Let \( X_{1} = \{0, a, b\} \) and \( X_{2} = \{0, a, b, c\} \) be two subtraction algebras with the following Cayley tables

\[
\begin{array}{ccc}
0 & a & b \\
0 & 0 & 0 & 0 \\
a & a & 0 & a \\
b & b & b & 0
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
0 & a & b & c \\
0 & 0 & 0 & 0 \\
a & a & 0 & a \\
b & b & b & 0 \\
c & c & b & b & 0
\end{array}
\]

Let us define the intuitionistic fuzzy sets \( A = (\mu_{A}(x), \gamma_{A}(x)) \) of \( X_{1} \) and \( B = (\mu_{B}(x), \gamma_{B}(x)) \) of \( X_{2} \) as follows

\[
\begin{array}{ccc}
X & 0 & a & b \\
\mu_{A}(x) & 0.5 & 0.6 & 0.7 \\
\gamma_{A}(x) & 0.1 & 0.2 & 0.21
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
X & 0 & a & b & c \\
\mu_{B}(x) & 0.6 & 0.5 & 0.4 & 0.3 \\
\gamma_{B}(x) & 0.2 & 0.21 & 0.23 & 0.24
\end{array}
\]

Then \( X_{1} \times X_{2} = \{(0, 0), (0, a), (0, b), (0, c), (a, 0), (a, a), (a, b), (a, c), (b, 0), (b, a), (b, b), (b, c)\} \) is a subtraction algebra. Now define the direct product \( A \times B \) on \( X_{1} \times X_{2} \) as \( A \times B = (\mu_{A \times B}, \gamma_{A \times B}) \) where

\[
\begin{array}{cccc}
X_{1} \times X_{2} & \mu_{A \times B} & \gamma_{A \times B} \\
(0, 0) & 0.5 & 0.2 \\
(0, a) & 0.5 & 0.21 \\
(0, b) & 0.4 & 0.23 \\
(0, c) & 0.3 & 0.24 \\
(a, 0) & 0.6 & 0.3 \\
(a, a) & 0.5 & 0.21 \\
(a, b) & 0.4 & 0.23
\end{array}
\]

then $A \times B$ is an $(\varepsilon, \in \mathbb{V}_{q_k})$–intuitionistic fuzzy subalgebra of $X_1 \times X_2$.

**Definition 3.4** An intuitionistic fuzzy set $A$ and $B$ of $X_1 \times X_2$ is called an $(\varepsilon, \in \mathbb{V}_{q_k})$–intuitionistic fuzzy ideal of $X_1 \times X_2$ if it satisfies

(i) $\mu_{A \times B}(x_1, y_1) - (x_2, y_2)) \geq \min(\mu_{A \times B}(x_1, y_1), \frac{1+k}{2})$

(ii) $\gamma_{A \times B}(x_1, y_1) - (x_2, y_2)) \leq \max(\gamma_{A \times B}(x_1, y_1), \frac{1-k}{2})$

(iii) $\mu_{A \times B}(x_1, y_1) \lor (x_2, y_2)) \geq \min(\mu_{A \times B}(x_1, y_1), \mu_{A \times B}(x_2, y_2), \frac{1+k}{2})$

(iv) $\gamma_{A \times B}(x_1, y_1) \lor (x_2, y_2)) \leq \max(\gamma_{A \times B}(x_1, y_1), \gamma_{A \times B}(x_2, y_2), \frac{1-k}{2})$.

**Theorem 3.5** Let $A$ and $B$ be two $(\varepsilon, \in \mathbb{V}_{q_k})$–intuitionistic fuzzy subalgebras of $X_1$ and $X_2$, respectively. Then the Direct product $A \times B$ is an $(\varepsilon, \in \mathbb{V}_{q_k})$–intuitionistic fuzzy subalgebra of $X_1 \times X_2$.

**Proof** Let $A$ and $B$ be two $(\varepsilon, \in \mathbb{V}_{q_k})$–intuitionistic fuzzy subalgebras of $X_1$ and $X_2$, respectively. For any $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$. We have

$\mu_{A \times B}(x_1, y_1) - (x_2, y_2)) = \mu_{A \times B}(x_1 - x_2, y_1 - y_2)$

$= \mu_A(x_1 - x_2) \wedge \mu_B(y_1 - y_2)$

$\geq \min(\mu_A(x_1), \mu_A(x_2), \frac{1-k}{2}) \wedge \min(\mu_B(y_1), \mu_B(y_2), \frac{1-k}{2})$

$= \min(\mu_A(x_1), \mu_B(y_1), \frac{1-k}{2}) \wedge \min(\mu_A(x_2), \mu_B(y_2), \frac{1-k}{2})$

$= \{\mu_{A \times B}(x_1, y_1), \mu_{A \times B}(x_2, y_2), \frac{1-k}{2}\}$.

Also

$\gamma_{A \times B}(x_1, y_1) - (x_2, y_2)) = \gamma_{A \times B}(x_1 - x_2, y_1 - y_2)$

$= \gamma_A(x_1 - x_2) \lor \gamma_B(y_1 - y_2)$

$\leq \max(\gamma_A(x_1), \gamma_A(x_2), \frac{1-k}{2}) \lor \max(\gamma_B(y_1), \gamma_B(y_2), \frac{1-k}{2})$

$= \max(\gamma_A(x_1), \gamma_B(y_1), \frac{1-k}{2}) \lor \max(\gamma_A(x_2), \gamma_B(y_2), \frac{1-k}{2})$

$= \{\gamma_{A \times B}(x_1, y_1), \gamma_{A \times B}(x_2, y_2), \frac{1-k}{2}\}$.

Hence this shows that $A \times B$ is an $(\varepsilon, \in \mathbb{V}_{q_k})$–intuitionistic fuzzy subalgebra of $X_1 \times X_2$. $\square$

**Theorem 3.6** Let $A$ and $B$ be two $(\varepsilon, \in \mathbb{V}_{q_k})$–intuitionistic fuzzy ideals of $X_1$ and $X_2$, respectively. Then the Direct product $A \times B$ is an $(\varepsilon, \in \mathbb{V}_{q_k})$–intuitionistic fuzzy ideal of $X_1 \times X_2$.

**Proof** Straightforward. $\square$
Proposition 3.7 Every an \((e_i, \in \mathbb{v}_q_k)\)-intuitionistic fuzzy ideal \(A \times B = (\mu_{A \times B}(x), \gamma_{A \times B}(x))\) of \(X_1 \times X_2\) satisfies the following,

(i) \(\mu_{A \times B}(0, 0) \geq \min(\mu_{A \times B}(x_1, y_1), \frac{1-k}{2})\),

(ii) \(\gamma_{A \times B}(0, 0) \leq \max(\gamma_{A \times B}(x_1, y_1), \frac{1-k}{2})\).

Proof By letting \((x_1, y_1) = (x_2, y_2)\) in conditions (i) and (ii) in Definition, we get the required proof.

Lemma 3.8 If an \((e_i, \in \mathbb{v}_q_k)\)-intuitionistic fuzzy set \(A \times B = (\mu_{A \times B}(x), \gamma_{A \times B}(x))\) of \(X_1 \times X_2\) satisfies the following,

(i) \(\mu_{A \times B}(0, 0) \geq \min(\mu_{A \times B}(x_1, y_1), \frac{1-k}{2})\),

(ii) \(\gamma_{A \times B}(0, 0) \leq \max(\gamma_{A \times B}(x_1, y_1), \frac{1-k}{2})\),

(iii) \(\mu_{A \times B}((x_1, y_1) - (x_2, y_2)) \geq \min(\mu_{A \times B}((x_1, y_1) - (x_2, y_2)) - (x_3, y_3)), \mu_{A \times B}(x_1, y_1) - (x_3, y_3)), \mu_{A \times B}(x_1, y_1) - \frac{1-k}{2}\),

(iv) \(\gamma_{A \times B}((x_1, y_1) - (x_2, y_2)) \leq \max(\gamma_{A \times B}((x_1, y_1) - (x_2, y_2)) - (x_3, y_3)), \gamma_{A \times B}(x_1, y_1) - \frac{1-k}{2}\), then we have \((x_1, y_1) \leq (a, b) \Rightarrow \mu_{A \times B}(x_1, y_1) \geq \min(\mu_{A \times B}(a, b), \frac{1-k}{2})\) and \(\gamma_{A \times B}(x) \leq \max(\gamma_{A \times B}(a, b), \frac{1-k}{2})\) for all \((a, b), (x_1, y_1), (x_2, y_2), (x_3, y_3) \in X_1 \times X_2\).

Proof Let \((a, b), (x_1, y_1) \in X_1 \times X_2\) and \((x_1, y_1) \leq (a, b)\).

Consider

\[
\mu_{A \times B}(x_1, y_1) = \mu_{A \times B}((x_1, y_1) - (0, 0)) \\
\geq \min(\mu_{A \times B}((x_1, y_1) - (0, 0)), \mu_{A \times B}(a, b), \frac{1-k}{2}) \text{ by (iii)}, \\
= \min(\mu_{A \times B}(0, 0), \mu_{A \times B}(a, b), \frac{1-k}{2}) \text{ by (i)}, \\
= \min(\mu_{A \times B}(a, b), \frac{1-k}{2}) \text{ by (i)}. 
\]

Also consider

\[
\gamma_{A \times B}(x_1, y_1) = \gamma_{A \times B}((x_1, y_1) - (0, 0)) \\
\leq \max(\gamma_{A \times B}((x_1, y_1) - (0, 0)), \gamma_{A \times B}(a, b), \frac{1-k}{2}) \text{ by (iv)}, \\
= \max(\gamma_{A \times B}(0, 0), \gamma_{A \times B}(a, b), \frac{1-k}{2}) \text{ by (ii)}, \\
= \max(\gamma_{A \times B}(a, b), \frac{1-k}{2}) \text{ by (ii)}. 
\]

Definition 3.9 Let \(A = (\mu_{A}(x), \gamma_{A}(x))\) and \(B = (\mu_{B}(x), \gamma_{B}(x))\) be intuitionistic fuzzy sets of \(X_1\) and \(X_2\), respectively. Define the intuitionistic level set for the \(A \times B\) as \((A \times B)_{[a, \beta]} = \{(x, y) \in X_1 \times X_2 | \mu_{A \times B}(x, y) \geq a, \gamma_{A \times B}(X_1, X_2) \leq \beta\} \) where \(a \in (0, \frac{1-k}{2}), \beta \in (\frac{1-k}{2}, 1]\).

Theorem 3.10 Let \(A \times B\) be two \((e_i, \in \mathbb{v}_q_k)\)-intuitionistic fuzzy subalgebras of \(X_1\) and \(X_2\), respectively. Then the Direct product \(A \times B\) is an \((e_i, \in \mathbb{v}_q_k)\)-intuitionistic fuzzy subalgebra of \(X_1 \times X_2\) if and only if \((A \times B)_{[a, \beta]} \neq \Phi\) is a subalgebra of \(X_1 \times X_2\).

Proof Straightforward.

Theorem 3.11 Let \(A \times B\) be two \((e_i, \in \mathbb{v}_q_k)\)-intuitionistic fuzzy ideals of \(X_1\) and \(X_2\), respectively. Then the Direct product \(A \times B\) is an \((e_i, \in \mathbb{v}_q_k)\)-intuitionistic fuzzy ideal of \(X_1 \times X_2\) if and only if \((A \times B)_{[a, \beta]} \neq \Phi\) is an ideal of \(X_1 \times X_2\).
Proof  Straightforward. □

4. Direct product of \((e, \in \mathcal{V}_A)\)-intuitionistic fuzzy soft subalgebras

In this section, we define Direct product of \((e, \in \mathcal{V}_A)\)-intuitionistic fuzzy soft sets and investigate some related properties.

**Definition 4.1** Let \((\bar{F}, A)\) and \((\bar{G}, B)\) be two \((e, \in \mathcal{V}_A)\)-intuitionistic fuzzy soft sets of \(X_1\) and \(X_2\), respectively. Then the Direct product \((e, \in \mathcal{V}_A)\)-intuitionistic fuzzy soft sets \((\bar{F}, A)\) and \((\bar{G}, B)\) is defined as

\[
(\bar{F}, A) \otimes (\bar{G}, B) = (\bar{F} \times A \times B),
\]

where \(\bar{F}[\epsilon, \in] = \bar{F}[\epsilon] \times \bar{G}[\epsilon] = (\mu_{\bar{F}[\epsilon, \in]}(x), \mu_{\bar{G}[\epsilon, \in]}(y))\) for all \([\epsilon, \in] \in A \times B\).

Here \(\mu_{\bar{F}[\epsilon, \in]}(x, y) = \mu_{\bar{F}[\epsilon]}(x) \land \mu_{\bar{G}[\epsilon]}(y)\) and \(\gamma_{\bar{F}[\epsilon, \in]}(x, y) = \gamma_{\bar{F}[\epsilon]}(x) \lor \gamma_{\bar{G}[\epsilon]}(y)\) for all \((x, y) \in X_1 \times X_2\) and \([\epsilon, \in] \in A \times B\).

**Definition 4.2** An \((e, \in \mathcal{V}_A)\)-intuitionistic fuzzy soft set \((\bar{F}, A) \otimes (\bar{G}, B)\) of \(X_1 \times X_2\) is called an an \((e, \in \mathcal{V}_A)\)-intuitionistic fuzzy soft subalgebra of \(X_1 \times X_2\) if it satisfies

(i) \(\mu_{\bar{F}[\epsilon, \in]}((x_1, y_1) - (x_2, y_2)) \geq \min\{\mu_{\bar{F}[\epsilon]}(x_1), \mu_{\bar{G}[\epsilon]}(y_1), \frac{1+k}{2}\}\);

(ii) \(\gamma_{\bar{F}[\epsilon, \in]}((x_1, y_1) - (x_2, y_2)) \leq \max\{\gamma_{\bar{F}[\epsilon]}(x_1), \gamma_{\bar{G}[\epsilon]}(y_1), \frac{1+k}{2}\}\).

**Definition 4.3** An \((e, \in \mathcal{V}_A)\)-intuitionistic fuzzy soft set \((\bar{F}, A) \otimes (\bar{G}, B)\) of \(X_1 \times X_2\) is called an an \((e, \in \mathcal{V}_A)\)-intuitionistic fuzzy soft ideal of \(X_1 \times X_2\) if it satisfies

(i) \(\mu_{\bar{F}[\epsilon, \in]}((x_1, y_1) - (x_2, y_2)) \geq \min\{\mu_{\bar{F}[\epsilon]}(x_1), \frac{1+k}{2}\}\),

(ii) \(\gamma_{\bar{F}[\epsilon, \in]}((x_1, y_1) - (x_2, y_2)) \leq \max\{\gamma_{\bar{F}[\epsilon]}(x_1), \frac{1+k}{2}\}\),

(iii) \(\mu_{\bar{F}[\epsilon, \in]}((x_1, y_1) \lor (x_2, y_2)) \geq \min\{\mu_{\bar{F}[\epsilon]}(x_1), \mu_{\bar{G}[\epsilon]}(y_1), \frac{1+k}{2}\}\),

(iv) \(\gamma_{\bar{F}[\epsilon, \in]}((x_1, y_1) \lor (x_2, y_2)) \leq \max\{\gamma_{\bar{F}[\epsilon]}(x_1), \gamma_{\bar{G}[\epsilon]}(y_1), \frac{1+k}{2}\}\).

**Theorem 4.4** Let \((\bar{F}, A)\) and \((\bar{G}, B)\) be two \((e, \in \mathcal{V}_A)\)-intuitionistic fuzzy soft subalgebras of \(X_1\) and \(X_2\), respectively. Then the Direct product \((\bar{F}, A) \otimes (\bar{G}, B)\) is an an \((e, \in \mathcal{V}_A)\)-intuitionistic fuzzy soft subalgebra of \(X_1 \times X_2\).

Proof  Let \((\bar{F}, A)\) and \((\bar{G}, B)\) be two \((e, \in \mathcal{V}_A)\)-intuitionistic fuzzy soft groups of \(X_1\) and \(X_2\), respectively. For any \((x_1, y_1), (x_2, y_2) \in X_1 \times X_2\) and \([\epsilon, \in] \in A \times B\). We have

\[
\mu_{\bar{F}[\epsilon, \in]}((x_1, y_1) - (x_2, y_2)) = \mu_{\bar{F}[\epsilon, \in]}(x_1 - x_2) \land \mu_{\bar{G}[\epsilon, \in]}(y_1 - y_2)
\]

\[
\geq \min\{\mu_{\bar{F}[\epsilon]}(x_1), \frac{1+k}{2}\} \land \min\{\mu_{\bar{G}[\epsilon]}(y_1), \frac{1+k}{2}\}
\]

\[
= \min\{\mu_{\bar{F}[\epsilon]}(x_1), \frac{1-k}{2}\} \land \min\{\mu_{\bar{G}[\epsilon]}(y_1), \frac{1-k}{2}\}
\]

\[
= \min\{\mu_{\bar{F}[\epsilon]}(x_1), \mu_{\bar{G}[\epsilon]}(y_1), \frac{1-k}{2}\}.
\]
In this paper we established some new results related to the direct product of intuitionistic fuzzy sets and direct product of $X_1 \times X_2$ respectively. Then the Direct product $(\tilde{F}, A) \otimes (\tilde{G}, B)$ is an $(e, e \in V q_k)$-intuitionistic fuzzy soft ideal of $X_1 \times X_2$.

**Theorem 4.5** Let $(\tilde{F}, A)$ and $(\tilde{G}, B)$ be two $(e, e \in V q_k)$-intuitionistic fuzzy soft ideals of $X_1$ and $X_2$, respectively. Then the Direct product $(\tilde{F}, A) \otimes (\tilde{G}, B)$ is an $(e, e \in V q_k)$-intuitionistic fuzzy soft ideal of $X_1 \times X_2$.

**Proof** Straightforward. □

**5. Conclusion**

In this paper we established some new results related to the direct product of $(e, e \in V q_k)$-intuitionistic fuzzy sets and direct product of $(e, e \in V q_k)$-intuitionistic fuzzy soft sets of subtraction algebras. We investigated several related properties.

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