Doubly warped product submanifolds of a Riemannian manifold of nearly quasi-constant curvature

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Abstract. In the present paper, we form a sharp inequality for a doubly warped product submanifold of a Riemannian manifold of nearly quasi-constant curvature.

Keywords. Chen inequality, doubly warped product manifold, nearly quasi-constant curvature tensor

1 Introduction

In [7] B.-Y Chen and K. Yano introduced the notion of quasi-constant curvature. A Riemannian manifold \((M, g)\) is called a Riemannian manifold of quasi-constant curvature if its curvature tensor \(R\) satisfies the condition

\[
R(X, Y, Z, W) = a[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + b[g(X, W)A(Y)A(Z) - g(X, Z)A(Y)A(W) + g(Y, Z)A(X)A(W) - g(Y, W)A(X)A(Z)],
\]

where \(a, b\) are scalar functions and \(A\) is a 1-form given by

\[
g(X, P) = A(X),
\]

\(P\) is a fixed unit vector field. It is straightforward to see that if \(b = 0\), then \((M, g)\) reduces to a Riemannian manifold of constant curvature.

For \(n > 2\), a non-flat Riemannian manifold \((M^n, g)\) is said to be a quasi-Einstein manifold if its Ricci tensor satisfies the condition

\[
S(X, Y) = ag(X, Y) + bA(X)A(Y),
\]

where \(a, b\) are scalar functions and \(A\) is 1-form acting same as above. It can be easily verified that every Riemannian manifold of quasi-constant curvature is a quasi-Einstein manifold.

In 2009, the notion of quasi-constant curvature was generalized to nearly quasi-constant by A. K. Gazi and U. C. De (see [8]). It is a Riemannian manifold whose curvature tensor satisfies

\[
R(X, Y, Z, W) = p[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]
\]
\[ +q[g(X,W)B(Y,Z) - g(X,Z)B(Y,W)] + g(Y,Z)B(X,W) - g(Y,W)B(X,Z), \]

\[ (1.1) \]

where \( a, b \) are scalar functions and \( B \) is a non-vanishing \((0,2)\) type symmetric tensor.

For \( n > 2 \), a non-flat Riemannian manifold \((\mathcal{M}^n, g)\) is said to be a nearly quasi-Einstein manifold if its Ricci tensor satisfy

\[ S(X,Y) = ag(X,Y) + bB(X,Y). \]

It can be easily verified that every Riemannian manifold of nearly quasi-constant curvature is a nearly quasi-Einstein manifold.

We know that the outer product of two covariant vectors is a covariant \((0,2)\) tensor, but not conversely true always. Hence nearly quasi-constant Riemannian manifolds act as a bigger class of Riemannian manifolds in the sense that every Riemannian manifold of quasi-constant curvature is nearly quasi-constant Riemannian manifold, but there are plenty of examples where the converse is not true.

**Example 1.** ([8]) Let \((\mathbb{R}^4, g)\) be a Riemannian manifold with the metric \( g \) defined as follows

\[ ds^2 = (x^4)^2[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2. \]

This is a nearly quasi-constant Riemannian manifold but not a quasi-constant Riemannian manifold.

In an attempt to construct Riemannian manifolds with negative sectional curvature, O’Neill and Bishop introduced the notion of singly warped products (see [3]). The warped product manifold model has plenty of applications in relativity. In [1], Beem, Ehrlich and Powell showed the exact solutions to Einstein’s field equation are expressible in terms of Lorentzian warped products. For more applications, see [2, 6].

**Definition 1.** Let \((\mathcal{M}_1, g_1)\) and \((\mathcal{M}_2, g_2)\) be two Riemannian manifolds, the warped product 
\[ \mathcal{M} = \mathcal{M}_1 \times_\alpha \mathcal{M}_2 \]

is the product manifold equipped with the metric 
\[ g = \pi_1^*(g_1) + (\alpha \circ \pi_1)^2 \pi_2^*(g_2), \]

where \( \alpha : \mathcal{M}_1 \to (0, \infty) \) is a smooth function on \( \mathcal{M}_1 \), \( \pi_i : \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{M}_i, i = 1, 2 \) are the projections and \( * \) is the pullback. The Riemannian manifold \((\mathcal{M}_1, g_1)\) is called as the base, \((\mathcal{M}_2, g_2)\) is called as fibre and \( \alpha \) is called as the warping function of the warped product.

A very prominent example of warped product manifold is a generalized Robertson-Walker space-time, which is a Lorentzian warped product of the form 
\[ \mathcal{M} = (a,b) \times_\alpha \mathcal{N}, \]

where \((a,b)\) is an open interval, \( \mathcal{N} \) is a three dimensional space form and the metric on \( \mathcal{M} \) is given by 
\[ g = -dt^2 + \alpha^2 g_N \] (cf. [2, 9]).

The notion of doubly warped manifold can be considered as a natural generalization of singly warped product manifold.

**Definition 2.** Let \((\mathcal{M}_1, g_1)\) and \((\mathcal{M}_2, g_2)\) be two Riemannian manifolds, the doubly warped product 
\[ \mathcal{M} = \mathcal{M}_1 \times_{\alpha_1} \mathcal{M}_2 \]

is the product manifold equipped with the metric 
\[ g = (\alpha_2 \circ \pi_2)^2 \pi_1^*(g_1) + (\alpha_1 \circ \pi_1)^2 \pi_2^*(g_2), \]
where \( \alpha_i : \mathcal{M}_i \to (0, \infty) \) is a smooth function on \( \mathcal{M}_i \), \( \pi_i : \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{M}_i, i = 1, 2 \) are the usual projections and \(*\) is the pullback. If one of \( \alpha_i = 1 \), but not both, then we get warped product manifold. If both \( \alpha_i = 1 \), we get a Riemannian product manifold. If neither of \( \alpha_i \) is constant, we get a non-trivial doubly product manifold (see [12]).

The structure of the paper is as following. In section 2, we compile the basic definitions and all the prerequisites needed afterwards. In section 3, we prove our main result.

2 Preliminaries

Let \( \mathcal{M} \) be an \( n \)-dimensional Riemannian submanifold of a Riemannian \( m \)-dimensional manifold \( \mathcal{N} \) and let \( \nabla \) and \( \hat{\nabla} \) be the Levi-Civita connection of \( \mathcal{M}, \mathcal{N} \), respectively. Then the Gauss and Weingarten formula are given respectively by

\[
\hat{\nabla}_X Y = \nabla_X Y + h(X, Y) \\
\hat{\nabla}_X \xi = -A_\xi X + \nabla_\xi X
\]

for all \( X, Y \in \Gamma(T\mathcal{M}) \) and \( \xi \in \Gamma(T^\perp\mathcal{M}) \), where \( \nabla^\perp \) is the normal connection, \( A \) is the shape operator and \( h \) is the second fundamental form and are related by the relation

\[
g(h(X, Y), \xi) = g(A_\xi X, Y).
\]

The Gauss equation is given by

\[
\hat{R}(X, Y, Z, W) = R(X, Y, Z, W) - g(h(X, W), h(Y, Z) + g(h(X, Z), h(Y, W))
\]

for all \( X, Y, Z, W \in \Gamma(T\mathcal{M}) \), where \( \hat{R} \) is the curvature tensor of \( \mathcal{N} \) and \( R \) is the induced curvature tensor on \( \mathcal{M} \).

Let \( \{e_1, e_2, \cdots, e_n\}, \{e_{n+1}, \cdots, e_m\} \) be orthonormal basis of the tangent space \( T_p(\mathcal{M}) \) and \( T^\perp_p(\mathcal{M}) \), respectively. Then the mean curvature field is given by

\[
H = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i) = \frac{1}{n} \sum_{r=n+1}^{m} \left( \sum_{i=1}^{n} h^r_{ij} \right) e_r, \ h^r_{ij} = g(h(e_i, e_j), e_r),
\]

for \( 1 \leq i, j \leq n, n + 1 \leq r \leq m \).

Suppose \( \alpha \) is a differentiable function on \( \mathcal{M} \), then the Laplacian \( \Delta \alpha \) is defined as

\[
\Delta \alpha = \sum_{i=1}^{n} [\nabla_{e_i} e_i - e_i e_i \alpha].
\]

Let \( \pi \subset T_p(\mathcal{M}) \) be a 2-plane section and \( K(\pi) \) be the sectional curvature of \( \mathcal{M} \). Then for an orthonormal basis \( \{e_1, \cdots, e_n\} \) of the tangent space, the scalar curvature is defined as

\[
\tau = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j).
\]

Now for a doubly warped product manifold, assuming \( \mathcal{D}_1, \mathcal{D}_2 \) the distributions obtained from the vectors tangent to leaves and fibres, respectively. Let \( s : \alpha_2 \mathcal{M}_1 \times \alpha_1 \mathcal{M}_2 \to \mathcal{N} \) be an isometric immersion, then we have

\[
H_i = \frac{1}{n_i} tr(h_i),
\]
the partial mean curvature, where \( tr(h_i) \) is the trace of \( h \) restricted to \( M_i \) and \( n_i = \dim M_i \). The doubly warped product manifold is called as mixed totally geodesic if \( h(X,Y) = 0 \) for any \( X,Y \) tangent to \( D_1, D_2 \), respectively.

For a warped product submanifold of a Riemannian manifold of constant sectional curvature, B.-Y Chen proved the following:

**Theorem 2.1.** [5] Let \( M = M_1 \times_\alpha M_2 \) be an \( n \)-dimensional warped product submanifold of a Riemannian manifold \( N(c) \). Then, we have

\[
\frac{\Delta \alpha}{\alpha} \leq \frac{n^2}{4n_2} \|H\|^2 + n_1 c,
\]

where \( n_1 = \dim M_1, n = n_1 + n_2 \). The equality holds if and only if \( M \) is a mixed totally geodesic and \( n_1 H_1 = n_2 H_2, H_i \) is the partial mean curvature vectors, \( i = 1, 2 \).

Later in [10], A. Olteanu obtained a sharp inequality for a doubly warped product submanifold of an arbitrary Riemannian manifold. In [12], using the quasi-constant curvature tensor, S. Sular obtained a sharp inequality for a doubly warped product submanifold of a Riemannian manifold. Motivated by the above studies, we discuss a sharp inequality for a doubly warped product submanifold of a Riemannian manifold of nearly quasi-constant curvature.

We shall use the following Chen’s lemma while proving our main result.

**Lemma 2.1.** [4] For \( m \geq 2 \) and \( b_1, b_2, \ldots, b_m, \mu \) be reals, such that

\[
\left( \sum_{j=1}^{m} b_j \right)^2 = (m-1) \left( \sum_{j=1}^{m} b_j^2 + \mu \right).
\]

Then \( 2b_1 b_2 \geq \mu \), with equality if and only if \( b_1 + b_2 = b_3 = \cdots = b_m \).

### 3 Doubly warped product submanifolds

In this section, we derive a sharp relationship between a doubly warped product submanifold \( M = \alpha_2 M_1 \times_\alpha_1 M_2 \), its warping functions and the squared mean curvature.

**Theorem 3.1.** Let \( M = \alpha_2 M_1 \times_\alpha_1 M_2 \) be an \( n \)-dimensional doubly warped product submanifold of an \( m \)-dimensional Riemannian manifold \( N \). Then we have

\[
n_2 \frac{\Delta_1 \alpha_1}{\alpha_1} + n_1 \frac{\Delta_2 \alpha_2}{\alpha_2} \leq \frac{n^2}{4} \|H\|^2 + n_1 n_2 p - q(n-1)tr(B),
\]

where \( n_1 + n_2 = n, n_i = \dim M_i \) and \( \Delta_i \) is the Laplacian of \( M_i \). The equality in (3.13) holds if and only if \( M \) is totally geodesic with \( tr(h_1) = tr(h_2) \).

**Proof.** Let \( M \) be a doubly warped product submanifold of a Riemannian manifold \( N \) of nearly quasi-constant curvature. Then, we have

\[
\nabla_X Y = \nabla_X^{M_1} - \frac{\alpha_2^2}{\alpha_1^2} g_1(X,Y) \nabla^{M_2}(\ln \alpha_2)
\]

\[
\nabla_X Z = Z(\ln \alpha_2) X + X(\ln \alpha_1) Z
\]
for any $X, Y \in \Gamma(TM_1), Z \in \Gamma(TM_2)$, where $\nabla^{M_i}$ is the Levi-Civita connection of the Riemannian metric $g_i, i = 1, 2$.

The sectional curvature of the $\{X, Y\}$ plane is given by

$$K(X \wedge Y) = \frac{1}{\alpha_1}[(\nabla^{M_1} X)\alpha_1 - X^2\alpha_1] + \frac{1}{\alpha_2}[(\nabla^{M_2} Y)\alpha_2 - Y^2\alpha_2].$$

Fix an orthonormal basis $\{e_1, \ldots, e_{n_1+1}, \ldots, e_n\}$, such that first $n_1$ tuples acts as basis of $T_pM_1$ and the remaining of $T_pM_2$ and $e_{n_1+1} \parallel H$, we get

$$\frac{n_2}{\alpha_1} \Delta_1 + n_1 \frac{\Delta_2}{\alpha_2} = \sum_{1 \leq s_1 \leq n_1 < s_2 \leq n} K(e_{s_1} \wedge e_{s_2})$$

(3.2)

for each $s_2 \in \{n_1 + 1, \ldots, n\}$.

Using Gauss equation for $X = W = e_i$ and $Y = Z = e_j, i \neq j$, we have

$$2\tau = n^2\|H\|^2 - \|h\|^2 + 2q(n-1)tr(B) + (n^2 - n)p,$$

(3.3)

where

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), (h(e_i, e_j))$$

is the squared norm of the second fundamental form $h$ and $\tau$ is the scalar curvature.

We fix

$$\epsilon = 2\tau - \frac{n^2}{2}\|H\|^2 - (n^2 - n)p - 2q(n-1)tr(B).$$

(3.4)

Then, from (3.3) and (3.4), we get

$$n^2\|H\|^2 = 2(\|h\|^2 + \epsilon).$$

(3.5)

For a suitable local orthonormal frame, the above relation can be written as

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = 2 \left[ \epsilon + \sum_{i=1}^{n} (h_{ii}^{n+1})^2 + \frac{\sum_{i \neq j} (h_{ij}^{n+1})^2}{\|h\|^2} + \sum_{r=n+2}^{m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 \right].$$

Put $b_1 = h_{11}^{n+1}, b_2 = \sum_{i=2}^{n} h_{ii}^{n+1}$ and $b_3 = \sum_{t=n+1}^{n} h_{tt}^{n+1}$, the previous equation is equivalent to

$$\left(\sum_{i=1}^{3} b_i\right)^2 = 2 \left[ \epsilon + \sum_{i=1}^{3} b_i^2 + \sum_{1 \leq i \neq j \leq n} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - \sum_{2 \leq j \neq k \leq n_1} h_{jj}^{n+1} h_{kk}^{n+1} - \sum_{n_1+1 \leq s \neq t \leq n} h_{ss}^{n+1} h_{tt}^{n+1} \right].$$

(3.6)

For $b_1, b_2, b_3$, we see that, (3.6) satisfy lemma 2.1, implying

$$\left(\sum_{i=1}^{3} b_i\right)^2 = 2 \left(\sum_{i=1}^{3} b_i^2 + \mu \right).$$
Then, we obtain $2b_1b_2 \geq \mu$, with equality if and only if $b_1 + b_2 = b_3$, or

$$\sum_{1 \leq j < k \leq n_1} h_{jj}^{n+1} h_{kk}^{n+1} + \sum_{n+1 \leq s < t \leq n} h_{ss}^{n+1} h_{tt}^{n+1} \geq \frac{\epsilon}{2} + \sum_{1 \leq \alpha < \beta \leq n} (h_{\alpha\beta}^{n+1})^2 + \frac{1}{2} \sum_{r=n+2}^m \sum_{\alpha, \beta=1}^n (h_{\alpha\beta}^r)^2. \quad (3.7)$$

The equality holds if and only if

$$\sum_{i=1}^{n_1} h_{ii}^{n+1} = \sum_{t=n_1+1}^n h_{tt}^{n+1}. \quad (3.8)$$

Using Gauss equation again, we have

$$n_2 \Delta_{1 \alpha_1}^{\Delta_1} + n_1 \Delta_{2 \alpha_2}^{\Delta_2} = \tau - \sum_{1 \leq k \leq n_1} K(e_j \wedge e_k) - \sum_{n+1 \leq s < t \leq n} K(e_s \wedge e_t)$$

$$= \tau - \frac{1}{2} n_1 p(n_1 - 1) - \sum_{r=n+1}^m \sum_{1 \leq j < k \leq n_1} (h_{jj}^r h_{kk}^r - (h_{jk}^r)^2) - q(n_1 - 1) tr(B) - \frac{1}{2} n_2 p(n_2 - 1)$$

$$- \sum_{r=n+1}^m \sum_{n_1+1 \leq s < t \leq n_1} (h_{ss}^r h_{tt}^r - (h_{st}^r)^2) - q(n_2 - 1) tr(B). \quad (3.9)$$

Using (3.2), (3.7) and (3.9), we get

$$n_2 \Delta_{1 \alpha_1}^{\Delta_1} + n_1 \Delta_{2 \alpha_2}^{\Delta_2} \leq \tau - \frac{1}{2} n p(n - 1) + n_1 n_2 p - \frac{\epsilon}{2} - \frac{1}{2} \sum_{r=n+2}^m \sum_{\alpha, \beta} (h_{\alpha\beta}^r)^2$$

$$+ \sum_{r=n+2}^m \sum_{1 \leq j < k \leq n_1} (h_{jj}^r)^2 - h_{jj}^r h_{kk}^r + \sum_{r=n+2}^m \sum_{n_1+1 \leq s < t \leq n_1} ((h_{st}^r)^2 - h_{ss}^r h_{tt}^r)$$

$$- q(n_1 - 1) tr(B) - q(n_2 - 1) tr(B),$$

$$= \tau - \frac{1}{2} n p(n - 1) + n_1 n_2 p - \frac{\epsilon}{2} - \sum_{r=n+1}^m \sum_{j=1}^{n_1} \sum_{t=n_1+1}^n (h_{jj}^r)^2$$

$$- \frac{1}{2} \sum_{r=n+2}^m \left( \sum_{j=1}^{n_1} h_{jj}^r \right)^2 - \frac{1}{2} \sum_{t=n_1+1}^n \left( \sum_{r=n+2}^m h_{tt}^r \right)^2$$

$$- q(n_1 - 1) tr(B) - q(n_2 - 1) tr(B)$$

$$\leq \tau - \frac{1}{2} n p(n - 1) + n_1 n_2 p - \frac{\epsilon}{2} - q(n_1 - 1) tr(B) - q(n_2 - 1) tr(B)$$
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\[ n^2 \|H\|^2 + n_1 n_2 p - q(n - 1)tr(B). \]

This proves our claim.

It is straightforward to check that the equality holds in (3.13), if and only if

\[ h_{ji} = 0, \quad n + 1 \leq r \leq m \tag{3.10} \]

and

\[ \sum_{i=1}^{n_1} h_{ii}^r = \sum_{l=n_1+1}^{n} h_{il}^r = 0, \tag{3.11} \]

where \( 1 \leq j \leq n_1, \quad n_1 + 1 \leq l \leq n \) and \( n + 2 \leq r \leq m \).

From (3.10) vanishing of the second fundamental form of \( \alpha_2 M_1 \times_{\alpha_1} M_2 \) in \( N \) implies \( h(D_1, D_2) = \{0\} \), or we can say that the immersion \( s \) is totally geodesic. Again from (3.8) and (3.11), we see that

\[ \sum_{s_1=1}^{n_1} h(e_{s_1}, e_{s_2}) = \sum_{s_2=n_1+1}^{n} h(e_{s_2}, s_{s_2}), \]

implying \( tr(h_1) = tr(h_2) \).

Conversely assuming \( N \) is the required Riemannian manifold, such that \( M \) is totally geodesic with \( tr(h_1) = tr(h_2) \), then the equality in (3.13) follows easily.

\[ \Box \]

**Corollary 3.2.** Let \( M = \alpha_2 M_1 \times_{\alpha_1} M_2 \) be a compact, orientable \( n \)-dimensional doubly warped product submanifold of an \( m \)-dimensional Riemannian manifold \( N \). Then we have

\[ \|H\|^2 \geq \frac{4}{n^2} [q(n - 1)tr(B) - n_1 n_2 p]. \tag{3.12} \]

**Proof.** Suppose \( M \) be a compact orientable Riemannian manifold without boundary satisfying (3.13), then we have

\[ n_2 \frac{\Delta_1 \alpha_1}{\alpha_1} + n_1 \frac{\Delta_2 \alpha_2}{\alpha_2} \leq \frac{n^2}{4} \|H\|^2 + n_1 n_2 p - q(n - 1)tr(B). \]

Therefore, from the definition of volume element

\[ \int_M \Delta_i \alpha_i dV = 0, \quad i = 1, 2. \]

Thus, we get

\[ 0 \leq \int_M \left( \frac{n^2}{4} \|H\|^2 + n_1 n_2 p - q(n - 1)tr(B) \right) dV = 0. \]

\[ \Box \]

**Corollary 3.3.** Let \( M = \alpha_2 M_1 \times_{\alpha_1} M_2 \) be an \( n \)-dimensional doubly warped product submanifold of an \( m \)-dimensional Riemannian manifold \( N \) satisfying

\[ n_2 \frac{\Delta_1 \alpha_1}{\alpha_1} + n_1 \frac{\Delta_2 \alpha_2}{\alpha_2} > n_1 n_2 p - q(n - 1)tr(B), \tag{3.13} \]

then \( M \) is non-minimal.
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