REFLECTED SOLUTIONS OF BACKWARD DOUBLY SDES DRIVEN BY BROWNIAN MOTION AND POISSON RANDOM MEASURE

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Abstract. We consider backward doubly stochastic differential equations (BDSDEs in short) driven by a Brownian motion and an independent Poisson random measure. We give sufficient conditions for the existence and the uniqueness of solutions of equations with Lipschitz generator which is, first, standard and then depends on the values of a solution in the past. We also prove comparison theorem for reflected BDSDEs.

1. Introduction. Motivated by the probabilistic interpretation of solutions to a class of quasilinear parabolic differential equations (PDEs in short), Pardoux and Peng [42] introduced non linear BSDEs. Since then, BSDEs have been intensively developed with great interest and encountered in many fields of applied mathematics such as finance, stochastic games and optimal control (see [20], [23], [27], [30] for instance, and so on).

As a variation of BSDEs, a BSDE with one reflecting obstacle (say lower obstacle $L$)

$$
\begin{aligned}
Y_t &= \xi + \int_t^T f(s, Y_s, Z_s) \, ds + K_T - K_t - \int_t^T Z_s \, dW_s, \; t \in [0, T], \\
Y_t &\geq L_t, \; t \in [0, T], \\
K &\text{ is non decreasing, continuous, } K_0 = 0, \; \int_0^T (Y_t - L_t) \, dK_t = 0,
\end{aligned}
$$

was first studied by El Karoui et al. [19]. Here $W = (W_t)_{t \leq T}$ is a Brownian motion, $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ is the generator and the terminal value $\xi$ is $\mathcal{F}_T$-measurable where for any $t \leq T$, $\mathcal{F}_t = \sigma \{ B_s, \; s \leq t \}$; the adaptation is with respect to $(\mathcal{F}_t)_{t \leq T}$. The second condition in 1 says that the first component $Y$ of the solution is forced to stay above $L$. The role of $K$ is to push $Y$ upwards in order to keep it above $L$ in a minimal way, which leads to the third condition in 1. Note that the usual BSDEs may be considered as a special case of RBSDEs with $L = -\infty$ (and $K \equiv 0$).

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In [19], El Karoui et al. proved that if $f$ is Lipschitz continuous in $(y, z)$ and both the terminal value $\xi$ and lower barrier $L$ are square integrable, then the solution of the reflected BSDE 1 exists and is unique. They make use of two methods, penalization and Snell envelope.

Later, reflected BSDEs have been extensively studied and some existence and uniqueness results have been established by many authors. We can quote, among others, [51], [31], [24], [21]. In [51], Tang and Li discussed one-dimensional RBSDEs driven by a Brownian motion, Hamadène and Ouknine [31], Hamadène and Hassani [24], Essaky [21] studied one-dimensional RBSDE with jumps, that is, RBSDEs are driven by a Brownian motion and an independent Poisson point process.

Based on [19], Cvitanic and Karatzas [9] extended the research of BSDEs to those with two reflecting barriers. The solution of such a RBSDE has to stay between two prescribed continuous processes $L$ and $U$ called lower and upper barriers. Precisely a solution of that equation, associated with $(\xi, f, L, U)$ is a quadruplet of adapted processes $(Y_t, Z_t, K^+_t, K^-_t)_{t \leq T}$ with values in $\mathbb{R}^{1+d+1+1}$ which mainly satisfies:

$$
\begin{align*}
Y_t &= \xi + \int_t^T f(s, Y_s, Z_s) \, ds + (K^+_t - K^-_t) - \int_t^T Z_s \, dW_s, \quad t \leq T, \\
L_t \leq Y_t \leq U_t \quad \text{and} \quad \int_0^T (Y_t - L_t) \, dK^+_t = \int_0^T (U_t - Y_t) \, dK^-_t = 0, \quad \forall t \leq T.
\end{align*}
$$

(2)

The process $K^+$ (resp. $K^-$) is continuous non-decreasing and its role is to keep $Y$ above $L$ (resp. under $U$). Moreover they act only when necessary. This type of equation is a powerful tool in zero-sum mixed game problems [28] and in American game options [10].

In [9], Cvitanic and Karatzas showed that a doubly reflected BSDE admits a unique solution if, on one hand, the generator of such a RBSDE is Lipschitz and, on the other hand, either the barriers are regular or the so called Mokobodski condition is satisfied (there exists a quasimartingale between the two barriers).

Some of further efforts to RBSDEs with two barriers can be found in [1, 28, 29] and the references therein. All the above papers on doubly reflected BSDEs assumed either Mokobodski’s condition or the aforementioned regularity condition. However Mokobodski’s condition is a bit troublesome since it is difficult to verify in practice. On the other hand, the regularity of one of the barriers is somewhat restrictive. So in [24] Hamadène and Hassani removed these conditions and showed that if the barriers are continuous and completely separated, meaning $L_t < U_t$, $\forall t \leq T$, then a solution of 2 exists and its unique. Later the case of discontinuous barriers has been also studied in Hamadène et al. [25] where they pasted local solutions to form a unique solution of a reflected BSDE with two distinct obstacles. Since then, the complete separation of the obstacles has been postulated by most of the subsequent papers including [6, 18, 26, 32] as well as [5].

After Pardoux and Peng [42], these authors introduced the theory of BSDEs in [43] which brought forward a new kind of BSDEs, that is a class of backward doubly stochastic differential equations with two different directions of stochastic integrals, i.e., equations involving both a standard (forward) Itô stochastic integral $dW$ and a backward Itô stochastic integral $dB$. That is, BDSDEs are stochastic differential equations of the form

$$
Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \, ds + \int_t^T g(s, Y_s, Z_s) \, dB_s - \int_t^T Z_s \, dW_s, \quad t \in [0, T],
$$

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$$
Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \, ds + \int_t^T g(s, Y_s, Z_s) \, dB_s - \int_t^T Z_s \, dW_s, \quad t \in [0, T],
$$
where $\xi, f$ and $g$ are the given data of the problem. In [43], Pardoux and Peng proved the existence and uniqueness, and also, discussed the probabilistic representation of solution of quasilinear stochastic PDEs. Many researchers worked in this area (see, e.g., [3], [7], [8], [41], [44], [55], and the references therein).

Recently, one barrier reflected BDSDEs was introduced by Bahlali et al. in [2], that is, for $t \in [0,T]$,

$$
\begin{cases}
Y_t = \xi + \int_t^T f(s,Y_s,Z_s) \, ds + \int_t^T g(s,Y_s,Z_s) \, dB_s + K_T - K_t - \int_t^T Z_s \, dW_s, \\
Y_t \geq L_t, \quad \text{and} \quad \int_0^T (Y_t - L_t) \, dK_t = 0.
\end{cases}
$$

They established the existence and uniqueness of solutions of reflected BDSDEs 3 under the uniformly Lipschitz condition on the coefficients. In the case when the coefficient $f$ is only continuous they established the existence of maximal and minimal solutions.

Backward doubly SDEs driven by a Brownian motion and a Poisson process with Lipschitzian coefficient on a fixed time interval were studied by Sun an Lu [50]. Then several authors investigated successfully in weakening Lipschitz assumptions (see among others [40], [52]). The assumption usually satisfied by the drift is replaced by a rather smooth one which ensures existence and uniqueness result. Inspired by the method developed in [52], Sow [49] extended Wang and Huang’s result to BDSDE with jumps and proved a large derivation principle of such family of equations. Then in [48], the author generalizes the result established in [49] to BDSDE associated to a random Poisson measure. Using the solvability of the equation in the case of Lipschitzian coefficients and an efficient iterative procedure, the author proved existence and uniqueness with coefficient satisfying rather weaker conditions.

As further extensions of BSDEs are the ones with time delayed generators (time delayed BSDEs). Those equations were first introduced by Delong and Imkeller [15, 16], the dynamics of which is given by

$$
Y_t = \xi + \int_t^T f(s,Y_s,Z_s) \, ds - \int_t^T Z_s \, dW_s, \quad t \in [0,T],
$$

where the generator $f$ at time $s$ depends arbitrarily on the past values of a solution $(Y_s,Z_s) = (Y(s+u),Z(s+u))_{-T \leq u \leq 0}$. In Delong and Imkeller [15], the authors established the existence and uniqueness of a solution for time delayed BSDEs. Also, in [16], they proved the existence and uniqueness as well as the Malliavin differentiability of the solution for time delayed BSDEs driven by Brownian motions and Poisson measures. Dos Reis et al. [17] extended the results of Delong and Imkeller [15, 16] in $L^p$-spaces. Then, Luo et al. [38] studied BDSDEs with time delayed generators. Based on an extension of the existence result of [38], Lu et al. [37] showed the existence and uniqueness of the solution of multivalued time delayed BDSDEs by means of Yosida approximation.

At present, BSDEs with time delayed generators are widely recognized to provide a useful and efficient tool for studying problems in different mathematical fields, such as mathematical finance, stochastic control and game theory (see, e.g., [14], [45], [46], [47]).

The first connection between reflected BSDEs and time delayed equation 4 was made by Zhou and Ren in [54] where it is proved, under the specific assumptions...
of the reflected case and the delayed BSDE, that there exists a unique solution of a RBSDE with time delayed generators. Karouf [35] extended the results of [54] to the wider class of barrier processes which are right continuous with left limits (cadlag). Later on, the author studied in [36] time delayed RBSDEs with two reflecting barriers driven by a Brownian noise and by a Brownian and Poisson noise. Very recently, Mansouri et al. [39] considered reflected BDSDEs with time delayed generators.

In this paper, we carry on the study of BDSDEs with two reflecting barriers driven by a Brownian motion and an independent Poisson process. More precisely, we consider equations of the following form

\[
\begin{align*}
\{ & i) Y_t = \xi + \int_t^T f(s, Y_s, Z_s, V_s) \, ds + \int_t^T g(s, Y_s, Z_s, V_s) \, dB_s + (K_t^+ - K_t^-) \\
& \quad - (K_t^- - K_t^+) - \int_t^T Z_s \, dW_s - \int_t^T \int_E V_s(\epsilon) \tilde{\mu}(ds, de), \forall t \leq T, \\
& \quad \text{ii) for all } t \leq T, L_t \leq Y_t \leq U_t \text{ and } \int_0^T (Y_t - L_t) \, dK_t^+ = \int_0^T (U_t - Y_t) \, dK_t^- = 0,
\end{align*}
\]

where \(\tilde{\mu}\) is the compensated measure associated with \(\mu\). Firstly, we show the existence of a solution when \(f\) is Lipschitz and under the well-known Mokobodski hypothesis. In a second part, we deal with the problem of existence of a solution for the same equation when the generator \(f\) depends on the past value of the solution.

The paper is organized as follows. In Section 2 we formulate the problem and set up the main assumptions. In Subsection 3.1 and 3.2, we address the question of existence of the solution of the reflected BSDEs with two reflecting barriers when the barrier processes have either only inaccessible or predictable jumps. For both cases, we first deal with the case when the coefficient \(f\) does not depend on \((y, z, v)\). Then, in order to obtain the result for generators depending on \((y, z, v)\), we introduce a contraction mapping in an appropriate Banach space of processes which has a fixed point, which in turn provides the unique solution of the BDSDEs. We end the Subsection 3.1 by proving a comparison theorem. In Section 4, we consider the problem of reflected BDSDEs with time delayed generators. Using the results of Section 3, we show the existence and uniqueness of the solution for a time horizon small enough or for a Lipschitz generator with a constant small enough. The end of this section investigates the concepts of comparison principle. Finally, in the appendix, some definitions and important results about the Snell envelope and optimal stopping are presented.

2. Setting of the problem. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space and \(T > 0\) be a fixed terminal time. We suppose that \(\{\mathcal{F}_t\}_{t \geq 0}\) is generated by the following three mutually independent processes:

- a \(l\)-dimensional Brownian motion \((B_t)_{t \leq T},\)
- a \(d\)-dimensional Brownian motion \((W_t)_{t \leq T},\)
- a Poisson random measure \(\mu\) on \(\mathbb{R}_+ \times \mathcal{E}\), where \(E = \mathbb{R}^l \setminus \{0\} (l \geq 1)\) is equipped with its Borel \(\sigma\)-field \(\mathcal{E}\), with compensator \(\nu(dt, de) = dt \lambda(de)\), such that \(\tilde{\mu}([0, T] \times A) = (\mu - \nu)([0, T] \times A)\) is a martingale for every \(A \in \mathcal{E}\) satisfying \(\lambda(A) < \infty\), \(\lambda\) is assumed to be \(\sigma\)-finite on \((E, \mathcal{E})\) and verifies \(\int_E (1 \land |\epsilon|^2) \, \lambda(de) < \infty\).

Let \(N\) denote the class of \(\mathbb{P}\)-null sets of \(\mathcal{F}\). For each \(t \in [0, T]\) we define \(\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^B \vee \mathcal{F}_t^\lambda\), where for any process \(\{\eta_t\}_{t \geq 0}\), \(\mathcal{F}_{s,t}^\eta = \sigma\{\eta_r - \eta_s, s \leq r \leq t\} \vee N\),
$F_t^n = F_{0,t}^n$. Note that the collection $(F_t)_{0 \leq t \leq T}$ is neither increasing nor decreasing, so it does not constitute a classic filtration. Let $\mathcal{P}$ (respectively $\tilde{\mathcal{P}}$) be the $\sigma$-algebra of $(F_t)$-progressively measurable (respectively predictable) sets on $\Omega \times [0, T)$. For any $\beta > 0$, we consider the following spaces of processes:

- $S^2_\beta(\mathbb{R})$ stands for the set of $\mathbb{R}$-valued, $(F_t)$-adapted càdlàg processes $Y = (Y_t)_{t \leq T}$ such that $\|Y\|_{S^2_\beta}^2 = \mathbb{E}[\sup_{0 \leq t \leq T} e^{\beta t}|Y_t|^2] < \infty$;
- $H^2_\beta(\mathbb{R}^d)$ denotes the space of $\mathcal{P}$-measurable processes $Z = (Z_t)_{t \leq T}$ with value in $\mathbb{R}^d$ such that $\|Z\|_{H^2_\beta}^2 = \mathbb{E}[\int_0^T e^{\beta t}|Z_t|^2 dt] < \infty$;
- $L^2_\beta(\mu, \mathbb{R})$ denotes the set of $\mathcal{P} \otimes \mathcal{E}$ measurable mapping $V : \Omega \times [0, T] \times E \to \mathbb{R}$ such that $\mathbb{P}$-a.s., $\|V\|_{L^2_\beta}^2 = \mathbb{E}[\int_0^T e^{\beta t} \int_E |V_t(e)|^2 \lambda(de) dt] < \infty$;
- $S^2_\beta(\mathbb{R})$ (resp. $S^2_\beta(\mathbb{R})$) the space of $(F_t)$-adapted continuous (respectively càdlàg) non-decreasing processes $K = (K_t)_{t \leq T}$ such that $K_0 = 0$ and $\mathbb{E}[K_T^2] < \infty$;
- $\tau_s$ is the set of $(F_t)$-stopping times $\tau$ such that $s \leq \tau \leq T$, $\mathbb{P}$-a.s. for $s \leq T$;
- for a given càdlàg process $\pi = (\pi_t)_{t \leq T}$ let $\pi_\tau = \lim_{s \uparrow \tau} \pi_s$, $0 \leq t \leq T$, $\pi_0 = \pi_0$, $\pi_- := \pi_{t-}$ and $\Delta t \pi = \pi_t - \pi_{t-}$.

Let $B_\beta$ be the Banach space of the processes $(Y, Z, V) \in S^2_\beta(\mathbb{R}) \times H^2_\beta(\mathbb{R}^d) \times L^2_\beta(\mu, \mathbb{R})$ with the norm

$$\|(Y, Z, V)\|_{B_\beta}^2 = \|Y\|_{S^2_\beta}^2 + \|Z\|_{H^2_\beta}^2 + \|V\|_{L^2_\beta}^2. \tag{5}$$

We also denote by $\|\|g_{\beta,s}\|$ the equivalent norm in the space $B_{\beta,s}$ with the following definition

$$\|(Y, Z, V)\|_{B_{\beta,s}}^2 = \|Y\|_{S^2_\beta}^2 + \|Z\|_{H^2_\beta}^2 + \|V\|_{L^2_\beta}^2. \tag{5}$$

We are now given a set of five data $\xi$, $g$, $f$, $L$ and $U$ which satisfy along with this paper the Assumptions (H1)-(H3) below:

(H1) A random variable $\xi$ which belongs to $L^2(\Omega, F_T, \mathbb{P})$.

(H2) The jointly measurable functions $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ and $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ are such that:

(i) for all $(y, z, v) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$, $g(y, z, v) \in H^2_\beta(\mathbb{R})$ and $f(y, z, v) \in H^2_\beta(\mathbb{R})$;

(ii) there exist constants $C > 0$ and $0 < \alpha < 1$ such that for any $(w, t) \in \Omega \times [0, T]$, $(y, z, v)$, $(y', z', v') \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$,

$$\begin{aligned}
|f(t, y, z, v) - f(t, y', z', v')|^2 &\leq C \left( |y - y'|^2 + |z - z'|^2 + |v - v'|^2 \right), \\
g(t, y, z, v) - g(t, y', z', v')|^2 &\leq C |y - y'|^2 + \alpha \left( |z - z'|^2 + |v - v'|^2 \right).
\end{aligned}$$

(H3) Two barriers $L := (L_t)_{t \leq T}$ and $U := (U_t)_{t \leq T}$ processes of $S^2_\beta(\mathbb{R})$ which satisfy:

$$\mathbb{P} \text{-a.s., } \forall t \leq T, \quad L_t \leq U_t \quad \text{and} \quad L_T \leq \xi \leq U_T.$$

Next let us give the following result which is an extension of the well-known Itô formula and whose proof is given in [50, Lemma 2.2, p. 76].

**Lemma 2.1.** Let $X \in S^2_\beta(\mathbb{R})$, $\psi \in H^2_\beta(\mathbb{R})$, $\zeta, \eta \in H^2_\beta(\mathbb{R}^d)$ and $\varphi \in L^2_\beta(\mu, \mathbb{R})$ be such that

$$X_t = X_0 + \int_0^t \vartheta s \, ds + \int_0^t \zeta s \, dB_s + \int_0^t \eta s \, dW_s + \int_0^t \int_E \varphi_s(e) \, \tilde{\mu}(ds, de), \quad 0 \leq t \leq T.$$
Then, for any function $\phi \in C^2(\mathbb{R}, \mathbb{R})$,
\[
\phi(X_t) = \phi(X_0) + \int_0^t \langle \nabla \phi(X_s), \vartheta_s \rangle \, ds + \int_0^t \langle \nabla \phi(X_s), \zeta_s \, dB_s \rangle + \int_0^t \langle \nabla \phi(X_s), \eta_s \, dW_s \rangle + 
\]
\[
+ \int_0^t \int_E \nabla \phi(X_s), \varphi_s(e) \, \tilde{\mu}(ds, de) - \frac{1}{2} \int_0^t \text{Tr}[\phi''(X_s)\zeta_s^2] \, ds
\]
\[
+ \int_0^t \frac{1}{2} \int_E \text{Tr}[\phi''(X_s)\eta_s^2] \, ds + \frac{1}{2} \int_0^t \int_E \text{Tr}[\phi''(X_s)\varphi_s(e)\varphi_s(e)^\ast] \, \mu(ds, de).
\]
In particular, for any $\beta > 0$
\[
e^{\beta t}|X_t|^2 = |X_0|^2 + \beta \int_0^t e^{\beta s}|X_s|^2 \, ds + 2 \int_0^t \langle e^{\beta s}X_s, \vartheta_s \rangle \, ds + 2 \int_0^t \langle e^{\beta s}X_s, \zeta_s \, dB_s \rangle
\]
\[
+ 2 \int_0^t \langle e^{\beta s}X_s, \eta_s \, dW_s \rangle + 2 \int_0^t \int_E \langle e^{\beta s}X_s, \varphi_s(e) \rangle \, \tilde{\mu}(ds, de) - \int_0^t e^{\beta s}|\zeta_s|^2 \, ds
\]
\[
+ \int_0^t e^{\beta s}|\eta_s|^2 \, ds + \int_0^t \int_E e^{\beta s}|\varphi_s(e)|^2 \, \mu(ds, de).
\]

3. The case of non-delayed coefficients. In this section, we consider BSDEs with two reflecting barriers when the noise is driven by a Brownian motion and an independent Poisson random measure. We first assume that the jumping times of the barriers are totally inaccessible stopping times which roughly speaking means that they are not predictable (cf. Appendix Definition 5.1). In this case the involved increasing processes $K^\pm$ are continuous. If this latter condition is not satisfied and especially if the barriers are càdlàg then $Y$ could have predictable jumps and the processes $K^\pm$ would be no longer continuous.

3.1. RBDSDE with totally inaccessible jump times in barriers.

3.1.1. Existence and uniqueness. We are going to show the existence of a solution for reflected BDSDEs such that the jumps of the barriers occur only at inaccessible stopping times.

This subsection mainly discusses the following BDSDE:

**Definition 3.1.** The process $(Y_t, Z_t, V_t, K^+_t, K^-_t)_{t \leq T}$ with values in $\mathbb{R} \times \mathbb{R}^d \times L^2(\mathcal{E}, \lambda; \mathbb{R}) \times \mathbb{R} \times \mathbb{R}$ is called a solution for the BDSDE with jumps and two reflecting barriers if

\[
\begin{cases}
\text{i)} \ Y \in \mathcal{S}^2_{\beta}(\mathbb{R}), Z \in \mathcal{H}^2_{\beta}(\mathbb{R}^d), V \in \mathcal{L}^2_{\beta}(\tilde{\mu}, \mathbb{R}) \text{ and } K^\pm \in \mathcal{S}^2_{\beta}(\mathbb{R}), \\
\text{ii)} \ Y_t = \xi + \int_t^T f(s, Y_s, Z_s, V_s) \, ds + \int_t^T g(s, Y_s, Z_s, V_s) \, dB_s + (K^+_T - K^-_T) \\
- (K^-_T - K^-_T) - \int_t^T Z_s \, dW_s - \int_t^T \int_E V_s(e) \, \tilde{\mu}(ds, de), \ \forall t \leq T, \\
\text{iii)} \ \text{for all } t \leq T, L_t \leq Y_t \leq U_t \text{ and } \int_0^T (Y_t - L_t) \, dK^+_t = \int_0^T (U_t - Y_t) \, dK^-_t = 0.
\end{cases}
\]

The inequalities and the integral condition in iii) are called the barriers constraints and the minimality respectively.
Proof. In its main steps, the proof is classic (see, e.g., [9] or [28]). For each set $K$ processes which satisfies the relation (ii) there exist two non-negative supermartingales in $\mathbb{S}_d^2(\mathbb{R})$, $h = (h_t)_{0 \leq t \leq T}$ and $h' = (h'_t)_{0 \leq t \leq T}$ such that $L_t \leq h_t - h'_t \leq U_t$, $\forall 0 \leq t \leq T$.

We firstly present the following existence theorem when $f$ and $g$ do not depend on $(y, z, v)$, i.e., $f(t, y, z, v) \equiv f(t)$ and $g(t, y, z, v) \equiv g(t)$.

**Theorem 3.2.** Assume that Assumptions (H1)-(H3) and (Mk) hold. Then the two barriers reflected BDSDE 6 has a solution $(Y, Z, V, K^+, K^-)$ in the space $\mathbb{S}_d^2(\mathbb{R}) \times \mathcal{H}_d^2(\mathbb{R}^d) \times L^2(\tilde{\mu}, \mathbb{R}) \times (\mathbb{S}_d^2(\mathbb{R}))^2$. Furthermore if $(Y', Z', V', K'^+, K'^-)$ is another solution of 6 then $Y = Y'$, $Z = Z'$, $V = V'$ and $K^+ - K^- = K'^+ - K'^-$. The processes $K^\pm$ can be chosen singular and then they are unique.

**Proof.** In its main steps, the proof is classic (see, e.g., [9], [34], pp. 21). In addition $\mathbb{S}_d^2(\mathbb{R})$ is not a semimartingale then we cannot find a quintuplet $(N, G, H, \Theta, U)$ such that $\forall 0 < t < T$:

$$\mathbb{E}[\xi^- | \mathcal{F}_t] \leq \mathbb{E}[\xi^+ | \mathcal{F}_t]$$

Since $h$ and $h'$ are non-negative supermartingales, $H$ and $\Theta$ are also non-negative supermartingales of $\mathbb{S}_d^2(\mathbb{R})$ and verify $H_T = \Theta_T = 0$. On the other hand, through (Mk), it follows that for any $t \leq T$,

$$\tilde{L}_t \leq H_t - \Theta_t \leq \tilde{U}_t.$$  

(7)

Now let us consider the sequences $(N^\pm_n)_{n \geq 0}$ of processes defined recursively as follows: $N^+_0 = 0$, and for $n \geq 0$,

$$N^+_n = \mathcal{R}(N^-_n + \tilde{L}) \quad \text{and} \quad N^-_n = \mathcal{R}(N^+_n - \tilde{U}),$$

where $\mathcal{R}$ is the Snell envelope operator (see Proposition 2 in the appendix). Using (7) and induction we can easily verify that (see [9]):

$$\forall n \geq 0, \quad 0 \leq N^+_n \leq N^+_{n+1} \leq H \quad \text{and} \quad 0 \leq N^-_n \leq N^-_{n+1} \leq \Theta.$$

Then the sequence $(N^+_n)_{n \geq 0}$ (resp. $(N^-_n)_{n \geq 0}$) converges pointwise to a supermartingale $N^+$ (resp. $N^-$) (see, e.g., [34], pp. 21). In addition $N^+$ and $N^-$ belong to $\mathbb{S}_d^2(\mathbb{R})$ and satisfy (see Proposition 3):

$$N^+ = \mathcal{R}(N^- + \tilde{L}) \quad \text{and} \quad N^- = \mathcal{R}(N^+ - \tilde{U}).$$
Further the Doob-Meyer decompositions of $N^\pm$ imply the existence of càdlàg martingales $(M^\pm_t)_{t \leq T}$ and $\mathcal{G}_t$-adapted non-decreasing processes $(K^\pm_t)_{t \leq T}$ (with $K_0 = 0$) such that:

$$\forall t \leq T, \quad N^\pm_t = M^\pm_t - K^\pm_t.$$ Since $N^\pm$ belongs to $\mathcal{S}^2_\beta(\mathbb{R})$, thanks to the predictable dual projection of non-decreasing processes (see, e.g., [13], pp. 221), we have $\mathbb{E}[(K^\pm_t)^2] < \infty$ and $(M^\pm_t)_{t \leq T}$ belong also to $\mathcal{S}^2_\beta(\mathbb{R})$. Therefore, an obvious extension of Itô’s martingale representation theorem (see [33]) yields the existence of two processes $(Z^\pm, V^\pm) \in \mathcal{H}^2_\beta(\mathbb{R}^d) \times \mathcal{L}^2_\beta(\tilde{\mu}, \mathbb{R})$ such that

$$M^\pm_t = M^\pm_0 + \int_0^t \left\{ Z^\pm_s \, dW_s + \int_E V^\pm_s(e) \tilde{\mu}(ds, de) \right\}, \quad \forall t \leq T.$$ Let us denote by $K^{\pm,d}$ (respectively, $K^{\pm,c}$) the purely discontinuous (respectively, continuous) part of $K^\pm$. By arguments similar to the ones in Proposition 4.1 in [24] we show that

$$\int_0^T (N^+_s - N^-_s - \hat{\zeta}_s) \, dK^{+,c}_s = \int_0^T (N^+_s - N^-_s - \hat{\zeta}_s) \, dK^{-,c}_s = 0 \quad \text{and} \quad K^{+,d} = K^{-,d}.$$ So $K^{+,d} - K^{-,d} = 0$, and actually in Definition 3.1 the terms $K^+_t$ and $K^-_t$ are continuous processes. Next, for $t \leq T$ let us set

$$Y_t = N^+_t + N^-_t + E\left[ \xi + \int_0^T f(s) \, ds + \int_0^T g(s) \, dB_s \bigg| \mathcal{F}_t \right],$$

$$Z_t = Z^+_t - Z^-_t + \eta_t, \quad V_t = V^+_t - V^-_t + \rho_t,$$ where the processes $\eta_t$ and $\rho_t$ belong to $\mathcal{H}^2(\mathbb{R}^d)$ and $\mathcal{L}^2(\tilde{\mu}, \mathbb{R})$ respectively and satisfy:

$$\mathbb{E}\left[ \xi + \int_0^T f(s) \, ds + \int_0^T g(s) \, dB_s \bigg| \mathcal{F}_t \right] = \mathbb{E}\left[ \xi + \int_0^T f(s) \, ds + \int_0^T g(s) \, dB_s \right]$$

$$+ \int_0^t \left\{ \eta_s \, dW_s + \int_E \rho_s(e) \tilde{\mu}(ds, de) \right\}. \quad \text{(8)}$$

Obviously for any $t \leq T$ we have

$$Y_t = \xi + \int_t^T f(s) \, ds + \int_t^T g(s) \, dB_s + (K^+_T - K^-_T) - (K^+_T - K^-_T) - \int_t^T Z_s \, dW_s$$

$$- \int_t^T \int_E V_s(e) \tilde{\mu}(ds, de).$$

This implies that $(Y, Z, V, K^+, K^-)$ solves 6.

**Step 2. Uniqueness.**

Let $(Y', Z', V', K'^+, K'^-)$ be another solution of the reflected BDSDE associated with $(\xi, f(t), g(t), L, U)$. Set $\delta Y = Y - Y', \delta Z = Z - Z', \delta V = V - V'$ and $\delta K^\pm = K^\pm - K'^\pm$. Applying Itô’s formula to $(\delta Y)^2$ on the interval $[t, T]$ yields that

$$|\delta Y_t|^2 + \int_t^T |\delta Z_s|^2 \, ds + \int_t^T \int_E (\delta V_s(e))^2 \tilde{\mu}(ds, de)$$

$$= 2 \int_t^T \delta Y_s (\delta K^+_s - \delta K^-_s) \, ds - 2 \int_t^T \delta Y_s \, \left\{ \delta Z_s \, dW_s + \int_E \delta V_s(e) \tilde{\mu}(ds, de) \right\}. \quad \text{(8)}$$
For the term $\int_t^T \delta Y_s (d\delta K_s^+ - d\delta K_s^-)\, ds$, notice that since $(Y,Z,V,K^+,K^-)$ and $(Y',Z',V',K'^+,K'^-)$ satisfy iii) in Definition 3.1, we have

$$\int_t^T \delta Y_s (d\delta K_s^+ - d\delta K_s^-)\, ds = \int_t^T \delta Y_s d\delta K_s^+ - \int_t^T \delta Y_s d\delta K_s^- \, ds \leq 0,$$

in view of

$$\int_t^T \delta Y_s d\delta K_s^+ = \int_t^T (Y_s - Y'_s) dK_s^+ + \int_t^T (Y'_s - Y_s) dK'_s^+$$

$$= \int_t^T (Y_s - L_s) dK_s^+ + \int_t^T (L_s - Y'_s) dK'_s^+ + \int_t^T (Y'_s - L_s) dK'_s^- + \int_t^T (L_s - Y_s) dK_s^- \leq 0,$$

and similarly $\int_t^T \delta Y_s d\delta K_s^- \geq 0$. On the other hand the processes $\{\int_0^t \delta Y_s - \delta Z_s \, dW_s\}_{0 \leq t \leq T}$ and $\{\int_0^t \delta Y_s - \delta V_s(e)\, \tilde{\mu}(ds,de)\}_{0 \leq t \leq T}$ are $(\mathcal{G}_t, \mathbb{P})$-martingales thanks to the Burkholder-Davis-Gundy and Young inequalities. Then taking the expectation in 8, we obtain

$$\mathbb{E}[|\delta Y_t|^2] + \mathbb{E}\left[\int_t^T |\delta Z_s|^2 \, ds\right] + \mathbb{E}\left[\int_t^T \int_E (\delta V_s(e))^2 \lambda(de) \, ds\right] \leq 0.$$

Thus $Y = Y'$, $Z = Z'$ and $V = V'$. Consequently $K_t^+ - K_t^- = K_t'^+ - K_t'^-$ for any $0 \leq t \leq T$. Finally, in a classic way, we can show that if $K^\pm$ are chosen to be singular then we have also $K^\pm = K'^\pm$ (see Definition 5.2 and Remark 4.1, page 138 of [24]).

Next with the help of this result we will be able to prove that the BDSDE 6 has a solution in the case when the functions $f$ and $g$ depend on $(y,z,v)$. Actually we have:

**Theorem 3.3.** Under Assumptions (H1)-(H3) and (Mk) on $(\xi,f,g,L,U)$, the associated reflected BDSDE 6 has a solution $(Y,Z,V,K^+,K^-)$ which belongs to the space $S^2_+(\mathbb{R}) \times H^2_+(\mathbb{R}^d) \times L^2(\tilde{\mu}, \mathbb{R}) \times (S^2_+(\mathbb{R}))^2$. Furthermore if $(Y',Z',V',K'^+,K'^-)$ is another solution of 6 then $Y = Y'$, $Z = Z'$, $V = V'$ and $K^+ - K^- = K'^+ - K'^-$. The processes $K^\pm$ can be chosen singular and then they are unique.

**Proof.** Let $\phi$ be the map from $\mathcal{B}_\beta$ into itself. Given $(Y,Z,V) \in \mathcal{B}_\beta$, let the quintuplet $(\bar{Y},\bar{Z},\bar{V},\bar{K}^+,\bar{K}^-)$ be the solution of the RBDSDE associated with $(\xi,f,g,L,U)$ where $\bar{f} : (t,\omega) \rightarrow f(t,Y_t,Z_t,V_t)$, $\bar{g} : (t,\omega) \rightarrow g(t,Y_t,Z_t,V_t)$ and actually $\phi(Y,Z,V) = (\bar{Y},\bar{Z},\bar{V})$, i.e., there exists a pair $\bar{K}^+, \bar{K}^-$ such that

$$\bar{Y}_t = \xi + \int_t^T f(s,Y_s,Z_s,V_s) \, ds + \int_t^T g(s,Y_s,Z_s,V_s) \, dB_s$$

$$+ \int_t^T (d\bar{K}_s^+ - d\bar{K}_s^-) - \int_t^T \bar{Z}_s \, dW_s - \int_t^T \int_E \bar{V}_s(e) \, \tilde{\mu}(ds,de).$$

Now for $(Y',Z',V') \in \mathcal{B}_\beta$, we define in the same way the quintuplet $(\bar{Y}',\bar{Z}',\bar{V}',\bar{K}'^+,\bar{K}'^-)$ and $(Y',Z',V') = \phi(Y',Z',V')$. Setting $(\delta Y, \delta Z, \delta V) = (Y - Y', Z - Z', V - V')$ and $(\delta \bar{Y}, \delta \bar{Z}, \delta \bar{V}, \delta \bar{K}^\pm) = (\bar{Y} - \bar{Y}', \bar{Z} - \bar{Z}', \bar{V} - \bar{V}', \bar{K}^\pm - \bar{K}'^\pm)$, it follows from Lemma
2.1 that for any $\beta > 0$,

$$
e^{\beta t}(\delta Y_t)^2 + \beta \int_t^T e^{\beta s}(\delta Y_s)^2 ds + \int_t^T e^{\beta s}|\delta \bar{Z}_s|^2 ds + \int_t^T \mathbb{E}[e^{\beta s} (\delta \bar{V}_s(e))] \mu(ds, de)
= 2 \int_t^T e^{\beta s} \delta Y_s - (f(s, Y_s, Z_s, V_s) - f(s, Y'_s, Z'_s, V'_s)) ds
+ \int_t^T e^{\beta s} g(s, Y_s, Z_s, V_s) - g(s, Y'_s, Z'_s, V'_s))^2 ds + 2 \int_t^T e^{\beta s} \delta Y_s - (d\delta \bar{K}^+_s - d\delta \bar{K}^-_s)
+ 2 \int_t^T e^{\beta s} \delta \bar{Y}_s - (g(s, Y_s, Z_s, V_s) - g(s, Y'_s, Z'_s, V'_s)) dB_s - 2 \int_t^T e^{\beta s} \delta \bar{Y}_s - \delta \bar{Z}_s dW_s
- 2 \int_t^T \int_E e^{\beta s} \delta \bar{Y}_s - \delta \bar{V}_s(e) \mu(ds, de).
$$

(9)

Noting that $\int_t^T e^{\beta s} \delta \bar{Y}_s - (d\delta \bar{K}^+_s - d\delta \bar{K}^-_s) \leq 0$ and taking the expectation in 9, we obtain

$$
\mathbb{E}[e^{\beta t}(\delta Y_t)^2] + \beta \mathbb{E}[\int_t^T e^{\beta s}(\delta Y_s)^2 ds] + \mathbb{E}[\int_t^T e^{\beta s}|\delta \bar{Z}_s|^2 ds]
+ \mathbb{E}[\int_t^T \int_E e^{\beta s} (\delta \bar{V}_s(e))^2 \lambda(de)ds]
\leq 2\mathbb{E}[\int_t^T e^{\beta s} \delta \bar{Y}_s - (f(s, Y_s, Z_s, V_s) - f(s, Y'_s, Z'_s, V'_s)) ds]
+ \mathbb{E}[\int_t^T e^{\beta s} g(s, Y_s, Z_s, V_s) - g(s, Y'_s, Z'_s, V'_s))^2 ds].
$$

(10)

The term including $f$ can be enlarged via Assumption (H2), and $2bd \leq \epsilon b^2 + \frac{1}{\epsilon} d^2$, $\epsilon > 0$, as follows:

$$
2\delta Y_s(f(s, Y_s, Z_s, V_s) - f(s, Y'_s, Z'_s, V'_s))
\leq 2\delta Y_s(\sqrt{C} \delta Y_s) + 2(\sqrt{C} \delta Y_s) \delta Z_s + 2(\sqrt{C} \delta Y_s) \delta V_s
\leq \left(1 + \frac{4C}{1-\alpha}\right) |\delta Y_s|^2 + C|\delta Z_s|^2 + \frac{1-\alpha}{2}(|\delta Z_s|^2 + |\delta V_s|^2).
$$

Putting the previous inequality into 10 and using Assumption (H2) on $g$, we can deduce that

$$
\mathbb{E}[e^{\beta t}|\delta \bar{Y}_t|^2] + \beta \mathbb{E}[\int_t^T e^{\beta s}|\delta \bar{Y}_s|^2 ds] + \mathbb{E}[\int_t^T e^{\beta s}|\delta \bar{Z}_s|^2 ds]
+ \mathbb{E}[\int_t^T \int_E e^{\beta s} |\delta \bar{V}_s(e)|^2 \lambda(de)ds]
\leq \left(1 + \frac{4C}{1-\alpha}\right) \mathbb{E}[\int_t^T e^{\beta s}|\delta \bar{Y}_s|^2 ds] + 2C\mathbb{E}[\int_t^T e^{\beta s}|\delta Y_s|^2 ds]
+ \frac{1+\alpha}{2} \left\{ \mathbb{E}[\int_t^T e^{\beta s}|\delta Z_s|^2 ds] + \mathbb{E}[\int_t^T \int_E e^{\beta s}|\delta \bar{V}_s(e)|^2 \lambda(de)ds] \right\}.
$$

Now choosing $\beta = 1 + \frac{4C}{1-\alpha} + \frac{4C}{1+\alpha}$, we obtain

$$
\frac{4C}{1+\alpha} \mathbb{E}[\int_t^T e^{\beta s}|\delta \bar{Y}_s|^2 ds] + \mathbb{E}[\int_t^T e^{\beta s}|\delta \bar{Z}_s|^2 ds] + \mathbb{E}[\int_t^T \int_E e^{\beta s}|\delta \bar{V}_s(e)|^2 \lambda(de)ds]
$$

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not depend on the variable \( v \) with jumps (see, e.g., \([4]\) for a counter-example). However when the coefficients do

the theory of the BSDEs. But it does not hold in general for solutions of BDSDEs

Comparison theorem. 

Xi-liang and Yong in \([53]\), Theorem 3.2.

BDSDEs turn into a one lower barrier reflected BSDEs which are considered by

First let us point that if \((X, Y, Z, V)\) which (with the associated process \( K^\pm \)) is the solution of the BDSDE with two reflecting barriers associated to \((\xi, f, g, L, U)\). The proof of existence is now complete.

Let us focus on uniqueness. If \((Y^1, Z^1, V^1, K^{1+}, K^{1-}) \in \mathcal{S}_\beta^2(\mathbb{R}) \times \mathcal{H}_\beta^2(\mathbb{R}^d) \times \mathcal{L}_\beta^2(\mathbb{R} \times \mathcal{S}_\beta^2(\mathbb{R}))^2 \) is another solution of 6 then \( Y = Y^1, Z = Z^1, V = V^1 \) and \( dK^{1+} - dK^{1-} = dK^{1+} - dK^{1-} \). Actually to prove this claim, we just need to argue as in the proof of uniqueness of Theorem 3.2.

Remark 1. In the case when in 6 one takes \( U = +\infty \) (and then \( K^{-} = 0 \)) the BDSDEs 6 turn into a one lower barrier reflected BSDEs which are considered by Xi-liang and Yong in \([53]\), Theorem 3.2.

3.1.2. Comparison theorem. The comparison theorem is one of the main tools in

the theory of the BSDEs. But it does not hold in general for solutions of BDSDEs

with jumps (see, e.g., \([4]\) for a counter-example). However when the coefficients do not depend on the variable \( v \), we actually have a comparison.

Theorem 3.4. Consider \( i \in \{1, 2\} \) and denote by \((Y^i, Z^i, V^i, K^{i+}, K^{i-}) \) a solution

of reflected BDSDEs 6 associated with \((\xi^i, f^i, g, L^i, U^i)\) satisfying (H1)-(H3) and

(Mk). Let us consider the following

i) \( f^1 \) and \( g \) do not depend on \( v \),

ii) \( \mathbb{P}\)-a.s. for any \( 0 \leq t \leq T \), \( f^1(t, Y^1_t, Z^1_t) \leq f^2(t, Y^1_t, Z^1_t, V^1_t) \) and \( \xi^1 \leq \xi^2 \),

iii) \( \mathbb{P}\)-a.s. for any \( 0 \leq t \leq T \), \( L^1_t \leq L^2_t \) and \( U^1_t \leq U^2_t \).

Then \( \mathbb{P}\)-a.s. for any \( t \leq T \), \( Y^1_t \leq Y^2_t \).

Proof. First let us point that if \((X_t)_{t \leq T} \) is an \( \mathbb{R} \)-valued càdlàg semi-martingale and if \( X^+_t := \max\{X_t, 0\} \) then, using Tanaka’s formula,

\[
(X^+_t)^2 - (X^+_T)^2 = 2 \int_t^T X^+_s \, dX_s - \int_t^T 1_{\{X_s > 0\}} \, d\langle X^c, X^c \rangle_s - \sum_{0 \leq s \leq t} \{(X^+_s)^2 - (X^+_s - 2X^+_s \Delta X_s). \}
\]

But \( \{(X^+_s)^2 - (X^+_s)^2 - 2X^+_s \Delta X_s \geq 0 \) since the function \( x \in \mathbb{R} \to (x^+)^2 \) is convex

(see \([13]\) page 349), hence,

\[
(X^+_t)^2 + \int_t^T 1_{\{X_s > 0\}} \, d\langle X^c, X^c \rangle_s \leq (X^+_T)^2 - 2 \int_t^T X^+_s \, dX_s. \tag{11}
\]

Let us denote \( \delta \Theta = \Theta^1 - \Theta^2, \theta = (\xi, Y, Z, V, K^\pm), \) and \( \delta g(t) = g(t, Y^1_t, Z^1_t) - g(t, Y^2_t, Z^2_t). \) Then \( (\delta Y, \delta Z, \delta V, \delta K^+, \delta K^-) \) satisfies the following BDSDE:
\[
\delta Y_t = \delta \xi + \int_t^T (f^1(s, Y^1_s, Z^1_s) - f^2(s, Y^2_s, Z^2_s, V^2_s)) \, ds + \int_t^T \delta g(s) \, dB_s \\
+ \int_t^T (\delta K^+_s - \delta K^-_s) - \int_t^T \delta Z_s \, dW_s - \int_t^E \delta V_s(e) \, \tilde{\mu}(ds, de).
\]

Next relation 11 with \(\delta Y\) yields: \(\forall t \leq T\)
\[
((\delta Y_t)^+) + \int_t^T \mathbf{1}_{[Y^1_s > Y^2_s]} |\delta Z_s|^2 \, ds
\]
\[
\leq (\delta \xi^+) + 2 \int_t^T \mathbf{1}_{[Y^1_s > Y^2_s]} (f^1(s, Y^1_s, Z^1_s) - f^2(s, Y^2_s, Z^2_s, V^2_s)) \, ds
\]
\[
+ \int_t^T \mathbf{1}_{[Y^1_s > Y^2_s]} |\delta g(s)|^2 \, ds + 2 \int_t^T (\delta K^+_s - \delta K^-_s)
\]
\[
+ 2 \int_t^T (\delta Y^-_s) \delta g(s) \, dB_s - 2 \int_t^T (\delta Y^-_s) \delta Z_s \, dW_s
\]
\[- 2 \int_t^T (\delta Y^-_s) \int_E \delta V_s(e) \, \tilde{\mu}(ds, de).
\]

Since \(L^1_t \leq L^2_t \leq Y^2_t, L^1_t \leq Y^1_t, t \in [0, T]\) and \(\int_0^T (Y^1_s - L^1_s)^+ \, dK^+_s = 0\), we have
\[
\int_t^T (\delta Y^-_s) \delta K^+_s \leq \int_t^T (Y^1_s - Y^2_s)^+ \, dK^+_s = \int_t^T (Y^1_s - Y^2_s)^+ \, dK^+_s
\]
\[
\leq \int_t^T (Y^1_s - L^1_s)^+ \, dK^+_s = 0.
\]

Similarly, since \(U^1_t \leq U^2_t, Y^2_t \leq U^2_t, t \in [0, T]\) and \(\int_0^T (U^2_s - Y^2_s)^+ \, dK^2^- = 0\), we have
\[
- \int_t^T (\delta Y^-_s) \delta K^-_s \leq \int_t^T (Y^1_s - Y^2_s)^+ \, dK^2^- = \int_t^T (Y^1_s - Y^2_s)^+ \, dK^2^- = 0
\]
\[
\leq \int_t^T (U^2_s - Y^2_s)^+ \, dK^2^- = 0.
\]

Thus, noting that \(\xi^1 \leq \xi^2\), by virtue of the previous three inequalities we can get that
\[
(\delta Y^+_t)^2 + \int_t^T \mathbf{1}_{[Y^1_s > Y^2_s]} |\delta Z_s|^2 \, ds
\]
\[
\leq 2 \int_t^T (\delta Y^-_s) (f^1(s, Y^1_s, Z^1_s) - f^2(s, Y^2_s, Z^2_s, V^2_s)) \, ds + \int_t^T \mathbf{1}_{[Y^1_s > Y^2_s]} |\delta g(s)|^2 \, ds
\]
\[
+ 2 \int_t^T (\delta Y^-_s) \delta g(s) \, dB_s - 2 \int_t^T (\delta Y^-_s) \delta Z_s \, dW_s
\]
\[- 2 \int_t^T (\delta Y^-_s) \int_E \delta V_s(e) \, \tilde{\mu}(ds, de).
\]

(12)
But $f^1 \leq f^2$, then we have
\[
(\delta Y_t^1)^+ \left( f^1(t, Y_t^1, Z_t^1) - f^2(t, Y_t^2, Z_t^2) \right) \\
= (\delta Y_t^1)^+ \left( f^1(t, Y_t^1, Z_t^1) - f^1(t, Y_t^1, Z_t^1) + f^1(t, Y_t^1, Z_t^1) - f^2(t, Y_t^2, Z_t^2, V_t^2) \right) \\
\leq (\delta Y_t^1)^+ \left( f^1(t, Y_t^1, Z_t^1) - f^1(t, Y_t^2, Z_t^2) \right).
\]
From the Lipschitz property of $(y, z) \mapsto f(t, y, z)$ and by inequality $|ab| \leq c|a|^2 + \epsilon^{-1}|b|^2$, $\forall \epsilon > 0$ and $a, b \in \mathbb{R}$ we have
\[
\frac{1}{\epsilon} (\delta Y_t^1)^2 + \epsilon |f^1(t, Y_t^1, Z_t^1) - f^1(t, Y_t^2, Z_t^2)|^2 \mathbb{1}_{[Y_2^1 > Y_2^2]} \\
\leq \frac{1}{\epsilon} (\delta Y_t^1)^2 + C \epsilon \left( (\delta Y_s^1)^2 + |\delta Z_s|^2 \right) \mathbb{1}_{[Y_2^1 > Y_2^2]}.
\]
Thus going back to 12 and using the fact that $g$ verifies (H2), we get
\[
(\delta Y_t^1)^2 + \int_t^T \mathbb{1}_{[Y_2^1 > Y_2^2]} |\delta Z_s|^2 \, ds \\
\leq \left( \frac{1}{\epsilon} + C(\epsilon + 1) \right) \int_t^T (\delta Y_s^1)^2 \, ds + (C\epsilon + \alpha) \int_t^T \mathbb{1}_{[Y_2^1 > Y_2^2]} |\delta Z_s|^2 \, ds \\
+ 2 \int_t^T (\delta Y_s^1)^+ \delta g(s) \, dB_s - 2 \int_t^T (\delta Y_s^-)^+ \delta Z_s \, dW_s \\
- 2 \int_t^T (\delta Y_s^-)^+ \int_E \delta V_s(\epsilon) \mu(ds, d\epsilon).
\]
Finally taking the expectation above, and since the last three terms are martingales, we obtain
\[
\mathbb{E}[(\delta Y_t^1)^2] + \mathbb{E} \left[ \int_t^T \mathbb{1}_{[Y_2^1 > Y_2^2]} |\delta Z_s|^2 \, ds \right] \\
\leq \left( \frac{1}{\epsilon} + C(\epsilon + 1) \right) \mathbb{E} \left[ \int_t^T (\delta Y_s^1)^2 \, ds \right] + (C\epsilon + \alpha) \mathbb{E} \left[ \int_t^T \mathbb{1}_{[Y_2^1 > Y_2^2]} |\delta Z_s|^2 \, ds \right].
\]
Consequently, choosing $0 < \epsilon < \frac{1}{C\alpha}$ and using Gronwall’s lemma, we get $\mathbb{E}[(\delta Y_t^1)^2] = 0$ for any $t \leq T$, i.e., $Y^1 \leq Y^2$, which is the desired result.

3.2. RBDSDEs with arbitrary jumping times in barriers. The second building block of this section consists in considering reflected BDSDEs with jumps when the barrier processes are only càdlàg. No further conditions are required on the nature of their jump times. The jump times are in fact arbitrary; they can be either predictable or inaccessible. The difficulty here lies in the fact that since the barriers $L$ and $U$ are allowed to have jumps then the process $Y$ has too and then the reflecting processes $K^+$ and $K^-$ are no longer continuous but only càdlàg. Therefore the setting of the problem is not the same as in 6.

Throughout this subsection, Mokobodski’s condition is in force. We moreover assume that the barriers $L$ and $U$ and their left limits are separated, meaning that they satisfy the following assumption:

(H4) For any $t < T$, $L_t < U_t$ and $L_{t-} < U_{t-}$.
**Definition 3.5.** We say that a quintuplet \((Y, Z, V, K^+, K^-)\) of processes is a solution of the RBDSDE \((\xi, f, g, L, U)\) with jumps and two reflecting càdlàg barriers if

\[
\begin{aligned}
&i) \quad Y \in \mathcal{S}_\beta^2(\mathbb{R}), \ Z \in \mathcal{H}_\beta^2(\mathbb{R}^d), \ V \in \mathcal{L}_\beta^2(\tilde{\mu}, \mathbb{R}) \text{ and } K^+ \in \mathcal{S}_\beta^2(\mathbb{R}), \\
&ii) \quad Y_t = \xi + \int_t^T f(s, Y_s, Z_s, V_s) \, ds + \int_t^T g(s, Y_s, Z_s, V_s) \, dB_s + (K^+_T - K^+_t) \\
&\hspace{1cm} - (K^-_T - K^-_t) - \int_t^T Z_s \, dW_s - \int_t^T \int_E V_s(e) \, \tilde{\mu}(ds, de), \ \forall t \leq T, \\
&iii) \quad L \leq Y \leq U \text{ and if } K^{\pm,c} \text{ is the continuous part of } K^+ \text{ then} \\
&\hspace{1cm} \int_0^T (Y_t - L_t) \, dK^{+ c}_t = \int_0^T (U_t - Y_t) \, dK^{- c}_t = 0, \\
&iv) \quad \text{if } K^{\pm,d} \text{ is the purely discontinuous part of } K^+ \text{ then } K^{\pm,d} \text{ is} \\
&\hspace{1cm} \bar{\mathcal{P}} \text{-measurable, and for any } t \leq T, \ \Delta K^{+ d}_t = (L_t - Y_t)^+ \mathbb{1}_{[Y_t = L_t]} \\
&\hspace{1cm} \text{and } \Delta K^{- d}_t = (Y_t - U_t)^+ \mathbb{1}_{[Y_t = U_t]}.
\end{aligned}
\]

(13)

Firstly, we are going to focus on the uniqueness of the solution of 13. Then we have:

**Proposition 1.** Under Assumptions \((H1) - (H4)\) and \((Mk)\) the reflected BDSDE 13 associated with \((\xi, f, g, L, U)\) has at most one solution.

**Proof.** Let \((Y, Z, V, K^+)\) and \((Y', Z', V', K'^+)\) be two solutions of 13. Firstly let us note that by the classical discussion (see [31], page 274), we have

\[
\int_0^T (Y_s - Y'_s)(dK_s - dK'_s) \leq 0, \ \forall t \leq T,
\]

where \(K = K^+ - K^-\) and \(K' = K'^+ - K'^-\). Similarly as in the proof of Theorem 3.3, we define \(\delta Y\), \(\delta Z\) and \(\delta V\), then according to Lemma 2.1,

\[
|\delta Y_t|^2 + \int_t^T |\delta Z_s|^2 \, ds + \int_t^T \int_E (\delta V_s(e))^2 \mu(ds, de)
\]

\[
\leq 2 \int_t^T \delta Y_s - (f(s, Y_s, Z_s, V_s) - f(s, Y'_s, Z'_s, V'_s)) \, ds
\]

\[
+ \int_t^T |g(s, Y_s, Z_s, V_s) - g(s, Y'_s, Z'_s, V'_s)|^2 \, ds
\]

\[
+ 2 \int_t^T \delta Y_s - (g(s, Y_s, Z_s, V_s) - g(s, Y'_s, Z'_s, V'_s)) \, dB_s - 2 \int_t^T \delta Y_s \delta Z_s \, dW_s
\]

\[
- 2 \int_t^T \int_E \delta Y_s - \delta V_s(e) \tilde{\mu}(ds, de).
\]

Therefore,

\[
\mathbb{E}[|\delta Y_t|^2] + \int_t^T |\delta Z_s|^2 \, ds + \int_t^T \int_E (\delta V_s(e))^2 \lambda(de) \, ds
\]

\[
\leq 2 \mathbb{E}\left[ \int_t^T \delta Y_s(f(s, Y_s, Z_s, V_s) - f(s, Y'_s, Z'_s, V'_s)) \, ds \right]
\]

\[
+ \mathbb{E}\left[ \int_t^T |g(s, Y_s, Z_s, V_s) - g(s, Y'_s, Z'_s, V'_s)|^2 \, ds \right].
\]
Following Assumption (H2), elementary inequality, we have
\[2\delta Y_t(f(s, Y_s, Z_s, V_s) - f(s, Y'_s, Z'_s, V'_s)) \leq 2\sqrt{C} \delta Y_s \{ |\delta Y_s| + |\delta Z_s| + |\delta V_s| \} \]
\[\leq 2\sqrt{C} |\delta Y_s|^2 + 2\sqrt{C} |\delta Y_s| |\delta Z_s| + 2\sqrt{C} |\delta Y_s| |\delta V_s| \]
\[\leq 2\sqrt{C} |\delta Y_s|^2 + \sqrt{C} \left\{ \frac{2}{1 - \alpha} |\delta Y_s|^2 + \frac{1 - \alpha}{2\sqrt{C}} |\delta Z_s|^2 \right\} \]
\[+ \sqrt{C} \left\{ \frac{2}{1 - \alpha} |\delta Y_s|^2 + \frac{1 - \alpha}{2\sqrt{C}} |\delta V_s|^2 \right\} \]
\[\leq 2\sqrt{C} \left( 1 + \frac{2}{1 - \alpha} \right) |\delta Y_s|^2 + \frac{1 - \alpha}{2} (|\delta Z_s|^2 + |\delta V_s|^2) \]
and
\[|g(s, Y_s, Z_s, V_s) - g(s, Y'_s, Z'_s, V'_s)|^2 \leq C |\delta Y_s|^2 + \alpha (|\delta Z_s|^2 + |\delta V_s|^2). \]
Hence,
\[E[|\delta Y_t|^2 + \frac{1 - \alpha}{2} \int_t^T |\delta Z_s|^2 \, ds + \frac{1 - \alpha}{2} \int_t^T \int_E (\delta V_s(e))^2 \lambda(de) \, ds] \leq (2\sqrt{C} + C + \frac{4C}{1 - \alpha})E[\int_t^T |\delta Y_s|^2 \, ds]. \]
Henceforth from Gronwall’s lemma and the right continuity of \(\delta Y\), we obtain \(E[|\delta Y_t|^2] = 0\), \(t \leq T\), so \(Y = Y'\). In addition we have \(Z = Z', V = V'\) and \(K^+ - K^- = K'^+ - K'^-\). Thus, thanks to their expression, we have \(K^{+,d} = K'^{+,d}\) and \(K^{-,d} = K'^{-,d}\), and then \(K^{+,c} - K'^{-,c} = K'^{+,c} - K'^{-,c}\). Finally let us show that \(K^{+,c} = K'^{+,c}\) and \(K^{-,c} = K'^{-,c}\). Actually for all \(0 \leq t \leq T\),
\[\int_0^t (Y_s - L_s) \, d(K^{+,c}_s - K'^{-,c}_s) = - \int_0^t (U_s - L_s) \, dK'^{-,c}_s \int_0^t (Y'_s - L_s) \, d(K^{+,c}_s - K'^{-,c}_s) = - \int_0^t (U_s - L_s) \, dK'^{-,c}_s. \]
Since \(L < U\), we conclude that \(K^{-,c} = K'^{-,c}\) and then \(K^{+,c} = K'^{+,c}\), whence the uniqueness of the solution of 13.

The following theorem is the main result of this subsection.

**Theorem 3.6.** Suppose that Assumptions (H1)-(H4) and (Mk) hold. Then, there exists a unique process \((Y_t, Z_t, V_t, K^{+,c}_t, K'^{-,c}_t)_{0 \leq t \leq T}\) which is a solution to the BDSDE 13.

**Proof.** The proof will be split into two steps. The first step deals with the case where the functions \(f\) and \(g\) are independent of the variables \((y, z, v)\) under Assumptions (H1), (H2), (H4) and (Mk), and the second step considers the general case.

**Step 1.** We assume that \(f\) and \(g\) are independent of \((y, z, v)\).

The proof of the existence and the uniqueness is analogous to that of Theorem 3.2, the only difference being in the fact that in the latter part of the proof of Step 1 we follows [31] instead of [24] to prove that
\[\int_0^T (N^+_s - N^-_s - \tilde{L}_s) \, dK^{+,c} = \int_0^T (N^-_s - N^+_s - \tilde{L}_s) \, dK'^{-,c} = 0, \]
we obtain the existence result.

The general case, meaning Step 2.

Actually the first term is null since when the continuous $K^{1,+}$ (resp. purely discontinuous $K^{1+,d}$) increases, we then have $Y^1 = L^1$ (resp. $Y^1_t = L^1_t$) (by conditions iii) and iv) of 13) which implies that $\mathbb{1}_{[Y^1_t > Y^2_t]} = 0$ because $Y^2 \geq L^1$. Arguing in the same way we obtain that the second term is null too. Therefore, since $\xi^1 \leq \xi^2$, we have

$$\Delta K^{+}_{t} = (N^+_{t} - N^+_{t-}) + \mathbb{1}_{[N^+_t = N^-_t + \check{L}_t]} = (N^+_{t} + \check{L}_t + N^+_{t-}) + \mathbb{1}_{[N^+_t = N^-_t + \check{L}_t]}$$

and

$$\Delta K^{-}_{t} = (N^-_{t} - N^-_{t-}) + \mathbb{1}_{[N^-_t = N^+_t + \check{L}_t]} = (N^+_{t} + \check{U}_t + N^+_{t-}) + \mathbb{1}_{[N^-_t = N^+_t + \check{U}_t]}.$$



Step 2. The general case, meaning $f$ and $g$ may depend on $(y, z, v)$.

Even in our setting where there are general jumps, arguing as in Theorem 3.3, we obtain the existence result.

We now establish a comparison theorem for solutions of BDSDEs with two càdlàg reflecting barriers.

**Theorem 3.7.** Let $\xi^i \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, $L^i, U^i \in \mathcal{S}^2(\mathbb{R})$ and $(Y^i, Z^i, V^i, K^{1+i}, K^{i-})$ be the solutions of the two barriers RBDSDE 13 for $i = 1, 2$. If $f^1$ and $g$ are independent of $v$, $\xi^1 \leq \xi^2$ and for $t \leq T$, $L^1_t \leq L^2_t$, $U^1_t \leq U^2_t$, $f^1(t, Y^2_t, Z^2_t) \leq f^2(t, Y^2_t, Z^2_t, V^2_t)$, then $\mathbb{P}$-a.s. $Y^1 \leq Y^2$.

**Proof.** The main idea is to make use of Itô-Tanaka’s formula. Therefore using Section 3.1 notations and applying formula 11 with $\delta Y_t$ for $t \in [0, T]$,

$$((\delta Y_t)^{+})^2 + \int_t^T \mathbb{1}_{[Y^+_s > Y^2_s]} |\delta Z_s|^2 \, ds$$

$$\leq ((\delta \xi)^{+})^2 + 2 \int_t^T (\delta Y_s)^{+} (f^1(s, Y^1_s, Z^1_s) - f^2(s, Y^2_s, Z^2_s, V^2_s)) \, ds$$

$$+ \int_t^T \mathbb{1}_{[Y^+_s > Y^2_s]} |\delta g(s)|^2 \, ds + 2 \int_t^T (\delta Y_s)^{+} (d(\delta K^+_s - \delta K^-_s))$$

$$+ 2 \int_t^T (\delta Y_s)^{+} \delta g(s) dB_s - 2 \int_t^T (\delta Y_s)^{+} \delta Z_s \, dW_s$$

$$- 2 \int_t^T (\delta Y_s)^{+} \int_E \delta V_s(c) \mu(ds, dc).$$

From the definition of a solution of reflected BDSEs with two reflecting barriers, we have $\int_t^T (\delta Y_s)^{+} d(\delta K^+_s - \delta K^-_s) \leq 0$. In fact,

$$\int_t^T (\delta Y_s)^{+} d(\delta K^+_s - \delta K^-_s)$$

$$= \int_t^T (\delta Y_s)^{+} dK^+_s - \int_t^T (\delta Y_s)^{+} dK^-_s - \int_t^T (\delta Y_s)^{+} dK^+_s - \int_t^T (\delta Y_s)^{+} dK^-_s$$

$$\leq \int_t^T (\delta Y_s)^{+} \{dK^1_{s,t} + c + dK^1_{s,t} + d\} + \int_t^T (\delta Y_s)^{+} \{dK^2_{s,t} - c + dK^2_{s,t} - d\} = 0.$$

Actually the first term is null since when the continuous $K^{1,+}$ (resp. purely discontinuous $K^{1+,d}$) increases, we then have $Y^1 = L^1$ (resp. $Y^1_t = L^1_t$) (by conditions iii) and iv) of 13) which implies that $\mathbb{1}_{[Y^1_t > Y^2_t]} = 0$ because $Y^2 \geq L^1$. Arguing in the same way we obtain that the second term is null too. Therefore, since $\xi^1 \leq \xi^2$, we have

$$(\delta Y^{+}_t)^2 + \int_t^T \mathbb{1}_{[Y^+_s > Y^2_s]} |\delta Z_s|^2 \, ds$$

$$\leq 2 \int_t^T (\delta Y_s)^{+} (f^1(s, Y^1_s, Z^1_s) - f^2(s, Y^2_s, Z^2_s, V^2_s)) \, ds + \int_t^T \mathbb{1}_{[Y^+_s > Y^2_s]} |\delta g(s)|^2 \, ds$$

$$+ 2 \int_t^T (\delta Y_s)^{+} \delta g(s) dB_s - 2 \int_t^T (\delta Y_s)^{+} \delta Z_s \, dW_s - 2 \int_t^T (\delta Y_s)^{+} \int_E \delta V_s(c) \mu(ds, dc).$$

$$= 2 \int_t^T (\delta Y_s)^{+} \delta g(s) dB_s - 2 \int_t^T (\delta Y_s)^{+} \delta Z_s \, dW_s - 2 \int_t^T (\delta Y_s)^{+} \int_E \delta V_s(c) \mu(ds, dc).$$
Corollary 1. Under (H1)-(H2) and as the proof of Theorem 3.4 (see the reasoning following 12).

Now, in view of the assumptions on \( f^1, f^2 \) and \( g \), the remainder of the proof runs as the proof of Theorem 3.4 (see the reasoning following 12).

By virtue of Theorem 3.6, the following corollary follows immediately.

**Corollary 1.** Under (H1)-(H2) and

\( (H3') \) \( L = (L_t)_{t \leq T} \) is a process of \( S^2_{\mathbb{D}}(\mathbb{R}) \) and \( L_T \leq \xi \), \( \mathbb{F} \)-a.s. with arbitrary jumping times,

the RBDSDE with jumps and one reflecting càdlàg lower barrier associated with \( (\xi, f, g, L) \) has a unique solution, meaning that there exists a unique quadruplet \( (Y, Z, V, K) \) such that

\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s, V_s) \, ds + \int_t^T g(s, Y_s, Z_s, V_s) \, dB_s + K_T - K_t \\
- \int_t^T Z_s \, dW_s - \int_t^T \int_E \delta V_s(e) \, \tilde{\mu}(ds, de), \forall t \leq T
\]

and

\[
\begin{align*}
\text{i)} & \quad Y \in S^2_{\mathbb{D}}(\mathbb{R}), \ Z \in \mathcal{H}^2_{\mathbb{F}}(\mathbb{R}^d), \ V \in \mathcal{L}^2_{\mathbb{F}}(\tilde{\mu}, \mathbb{R}) \text{ and } K \in S^2_{\mathbb{D}}(\mathbb{R}), \\
\text{ii)} & \quad Y_t \geq L_t, \forall t \leq T, \\
\text{iii)} & \quad \text{if } K = K^c + K^d \text{ where } K^c \ (\text{resp. } K^d) \text{ is the continuous (resp. purely discontinuous) part of } K \text{ then } K^d \text{ is predictable,}
\end{align*}
\]

\[
\int_0^T (Y_t - L_t) \, dK_t^c = 0, \quad \text{and } \Delta K_t = \Delta K_t^d = -(L_t - Y_t)^+, \forall t \leq T.
\]

4. **The case of time-delayed coefficients.** We now address the problem of reflected BDSDEs with jumps and time delayed generators. Once again we consider the case when the barrier processes have totally inaccessible jump times as well as the case when the barrier processes have general jump times.

For an arbitrarily given integrable function \( f : [0, T] \to \mathbb{R} \) extended to \([-T, 0]\) via \( f \mathbb{1}_{([-T,0])}(t) = 0 \) and a deterministic measure \( \theta \) supported on \([-T, 0]\), we define

\[
(f, \theta)(t) := \int_{-T}^{0} f(t + u) \, \theta(du), \quad t \in [-T, 0].
\]

Similarly, for a given process \( (\varphi_t)_{t \in [0, T]} \) extended to \([-T, 0]\) by imposing \( \varphi_t = 0 \) on \([-T, 0]\), we define for \( t \in [0, T] \)

\[
(\varphi, \theta)(t) := \int_{-T}^{0} \varphi_{t+u} \, \theta(du).
\]

Let \( \theta_Y, \theta_Z, \theta_V \) be non-random, finitely measures supported on \([-T, 0]\) and let

\[
a := \theta_Y([-T, 0]) \lor \theta_Z([-T, 0]) \lor \theta_V([-T, 0]).
\]

Suppose now that the functions \( f \) and \( g \) defined in Section 2 depend arbitrarily on the past values of the solution \( (Y, Z, V) \) over the interval \([0, s]\). So namely we study
reflected a BDSDE with time delay generators; the dynamics of which is given by
\[ Y_t = \xi + \int_t^T f(s, \Gamma(s)) \, ds + \int_t^T g(s, \Gamma(s)) \, dB_s + (K_T^+ - K_t^+) - (K_T^- - K_t^-) \tag{16} \]
where \( \Gamma \) is given by
\[ \Gamma(t) := \left( \int_{-T}^0 Y_{t+u} \theta_y(du), \int_{-T}^0 Z_{t+u} \theta_z(du), \int_{-T}^0 V_{t+u} \theta_v(du) \right) \]
Let us make the following extra assumption on the coefficients \( f \) and \( g \):
\[ \text{(H5)} \quad f(t, \ldots, \cdot) = 0 \quad \text{and} \quad g(t, \ldots, \cdot) = 0 \quad \text{for} \quad t < 0. \]

We will also need the following lemma.

**Lemma 4.1.** Suppose that \( (H2) \) is in force and define \( \delta Y = Y - Y', \delta Z = Z - Z' \) and \( \delta V = V - V' \). Let
\[ \hat{\theta}(\beta) = \int_{-T}^T e^{-\beta s} \theta_y(ds) \vee \int_{-T}^T e^{-\beta s} \theta_z(ds) \vee \int_{-T}^T e^{-\beta s} \theta_v(ds) \]
We then have, for some constant \( \hat{C} = C_a \),
\[ \int_{-T}^T e^{\beta s} |f(s, \Gamma(s)) - f(s, \Gamma'(s))|^2 \, ds \]
\[ \leq \hat{C} \hat{\theta}(\beta) \left\{ T \sup_{s \in [0, T]} e^{\beta s} |\delta Y_s|^2 + \int_0^T e^{\beta s} |\delta Z_s|^2 \, ds + \int_0^T \int_E e^{\beta s} |\delta V_s(e)|^2 \, \lambda(de) \, ds \right\} \tag{17} \]
and
\[ \int_{-T}^T e^{\beta s} |g(s, \Gamma(s)) - g(s, \Gamma'(s))|^2 \, ds \]
\[ \leq \hat{\theta}(\beta) \left\{ T \hat{C} \sup_{s \in [0, T]} e^{\beta s} |\delta Y_s|^2 + \alpha \left( \int_0^T e^{\beta s} |\delta Z_s|^2 \, ds + \int_0^T \int_E e^{\beta s} |\delta V_s(e)|^2 \, \lambda(de) \, ds \right) \right\} \tag{18} \]

**Proof.** We only prove \( 17, 18 \) can be proved in the same way. Let \( (Y, Z, V), (Y', Z', V') \in S_2^2(\mathbb{R}) \times H_{\beta}^2(\mathbb{R}^d) \times L_{\beta}^2(\mu, \mathbb{R}) \). Firstly note that for some processes \( (Y_t)_{t \in [0, T]}, (Z_t)_{t \in [0, T]} \) and \( (V_t)_{t \in [0, T]} \) satisfying appropriate integrability conditions, definition \( 15 \) yields
\[ (Y, \theta_y)(t) = \int_{-T}^0 Y_{t+u} \theta_y(du), \quad (Z, \theta_z)(t) = \int_{-T}^0 Z_{t+u} \theta_z(du) \]
and \( (V, \theta_v)(t) = \int_{-T}^0 V_{t+u} \theta_v(du) \).
So, Assumption (H2) and Jensen’s inequality, imply that
\[
\left| f(t, (Y, \theta_Y)(t), (Z, \theta_Z)(t), (V, \theta_V)(t)) - f(t, (Y', \theta_Y)(t), (Z', \theta_Z)(t), (V', \theta_V)(t)) \right|^2 \\
\leq C \left\{ \left| (Y - Y') \theta_Y(t) \right|^2 + \left| (Z - Z') \theta_Z(t) \right|^2 + \left| (V - V') \theta_V(t) \right|^2 \right\}
\]

Then,
\[
\int_t^T e^{\beta s} \left| f(s, \Gamma(s)) - f(s, \Gamma'(s)) \right|^2 ds \\
= \int_t^T e^{\beta s} \left| f \left( s, (Y, \theta_Y)(s), (Z, \theta_Z)(s), (V, \theta_V)(s) \right) - f \left( s, (Y', \theta_Y)(s), (Z', \theta_Z)(s), (V', \theta_V)(s) \right) \right|^2 ds \\
\leq \tilde{C} \left\{ \int_t^T e^{\beta s} \left| (Y - Y') \theta_Y(s) \right| ds + \int_t^T e^{\beta s} \left| (Z - Z') \theta_Z(s) \right| ds \\
+ \int_t^T e^{\beta s} \left( \int_E |V(e) - V'(e)|^2 \lambda(de) \theta_V(s) \right) ds \right\}.
\]

Commuting the order of integration (Tonelli lemma) yields for \( j \in \{Y, Z\} \), \( \phi^Y = Y - Y' \), \( \phi^Z = Z - Z' \), \( \phi^V = V(e) - V'(e) \)
\[
\int_t^T e^{\beta s} (|\phi^j|^2 \theta_j)(s) ds = \int_t^T e^{\beta s} \left( \int_t^0 e^{\beta(u+s)} \mathbb{1}_{\{s+u \geq 0\}} |\phi^j_{s+u}|^2 \theta_j(du) \right) ds \\
= \int_0^{T-u} \int_t^{t+u} e^{\beta r} e^{-\beta u} |\phi^j_r|^2 dr \theta_j(du) \int_0^T \int_{r-T}^{(r-t)-0} e^{\beta r} e^{-\beta u} |\phi^j_u|^2 \theta_j(du) dr \\
\leq \int_0^T e^{\beta r} |\phi^j_r|^2 \left( \int_0^T e^{-\beta u} \theta_j(du) \right) dr \leq \tilde{\theta}(\beta) \int_0^T e^{\beta r} |\phi^j_r|^2 dr
\]
and
\[
\int_t^T e^{\beta s} \left( \int_E |\phi^V|^2 \lambda(de) \theta_V(s) \right) ds \leq \tilde{\theta}(\beta) \int_0^T e^{\beta r} \int_E |\phi^V|^2 \lambda(de) dr.
\]
Embedding the three bounds in 19 yields 17.

4.1. Existence and uniqueness. We begin by studying the reflected BDSDEs when the barrier processes have only inaccessible jump times.

**Definition 4.2.** A solution of the reflected BDSDE with jumps and time delayed generators is a quintuplet \( (Y, Z, V, K^+, K^-) \in S^2_{\beta}(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R} \times \mathbb{R}^d) \times \mathcal{L}^2(\mathcal{P}, \mathbb{R}) \times (S^2_{\beta}(\mathbb{R}))^2 \) such that for given processes \( L = (L_t)_{t \leq T} \) and \( U = (U_t)_{t \leq T} \) the following holds:

\[
\begin{cases}
  i) & \text{the condition of the point iii) in Definition 3.1 is satisfied}, \\
  ii) & \text{the processes } (Y, Z, V, K^+, K^-) \text{ verifies equation 16}.
\end{cases}
\]

Let us state the uniqueness of the solution to the RBDSDE with double barriers and time delayed generators 20.

**Theorem 4.3.** Assumptions (H1)-(H3), (H5) and (Mk) hold. If \( T \) or \( \tilde{C} \) or \( \theta \) are small enough such that we have
\[
1 - \tilde{\theta}(\beta) \{ 2\tilde{C}T(1 + 12C^2_t) + \alpha \} > 0
\]
and
\[
\max \left\{ \frac{\tilde{C}\theta}{1 - \alpha\theta} \left( 1 + 2T + (1 + \alpha\tilde{\theta}(\beta))(12C_t^2T - 12C_t^2T\tilde{\theta}(\beta)) \right) \right\} < \beta, \quad (22)
\]
then the reflected BDSDE 20 has at most one solution \((Y_t, Z_t, V_t, K_t^+, K_t^-)_{0 \leq t \leq T}\), meaning that if \((Y_t, Z_t, V_t, K_t^+, K_t^-)_{0 \leq t \leq T}\) and \((Y_t', Z_t', V_t', K_t'^+, K_t'^-)_{0 \leq t \leq T}\) are two solutions of 20, then \(Y_t = Y_t', Z_t = Z_t', V_t = V_t', K_t^+ - K_t^- = K_t'^+ - K_t'^-\). Moreover if the processes are singular then we have also \(K_t^\pm = K_t'^\pm\).

**Proof.** Let \((Y_t, Z_t, V_t, K_t^+, K_t^-)_{0 \leq t \leq T}\) and \((Y_t', Z_t', V_t', K_t'^+, K_t'^-)_{0 \leq t \leq T}\) be two solutions of the reflected BDSDE \((\xi, f, g, L, U)\). Using Section 3 notations and applying Itô’s formula to \(e^{\beta t}\delta Y_t^2\) on the interval \([t, T]\), we get
\[
e^{\beta t}(\delta Y_t)^2 + \beta \int_t^T e^{\beta s}(\delta Y_s)^2 ds + \int_t^T e^{\beta s}\delta Z_s^2 ds + \int_t^T \int_E e^{\beta s}(\delta V_s(e))^2 \mu(ds, de)
\]
\[
= 2\int_t^T e^{\beta s}\delta Y_s(f(s, \Gamma'(s))) ds + \int_t^T e^{\beta s}|g(s, \Gamma'(s))|^2 ds + 2\int_t^T e^{\beta s}\delta Y_s(g(s, \Gamma'(s)) - g(s, \Gamma'(s))) ds
\]
\[
+ 2\int_t^T e^{\beta s}\delta Y_s(g(s, \Gamma'(s)) - g(s, \Gamma'(s))) dB_s
\]
\[
- 2\int_t^T e^{\beta s}\delta Y_s - \delta Z_s dW_s - 2\int_t^T \int_E e^{\beta s}\delta Y_s(e)\mu(ds, de).
\]
Using the fact that \(e^{\beta s}\delta Y_s\delta K_s^+ \leq 0\) and \(e^{\beta s}\delta Y_s\delta K_s^- \geq 0\), we have
\[
e^{\beta t}(\delta Y_t)^2 + (\beta - \epsilon) \int_t^T e^{\beta s}(\delta Y_s)^2 ds + \int_t^T e^{\beta s}\delta Z_s^2 ds
\]
\[
+ \int_t^T \int_E e^{\beta s}(\delta V_s(e))^2 \mu(ds, de)
\]
\[
\leq \frac{1}{\epsilon} \int_t^T e^{\beta s}|f(s, \Gamma'(s))|^2 ds + \int_t^T e^{\beta s}|g(s, \Gamma'(s))|^2 ds + 2\int_t^T e^{\beta s}\delta Y_s - \delta Z_s dW_s - 2\int_t^T \int_E e^{\beta s}\delta Y_s(e)\mu(ds, de),
\]
where we have used the basic inequality \(2bd \leq b^2 + \frac{1}{\epsilon}d^2\) \((b, d \in \mathbb{R}, \epsilon > 0)\). By hypothesis we can choose \(\epsilon < \beta\), thus it follows from 17 and 18 in Lemma 4.1 that
Putting \( t = 0 \) and noting that the constant \( D = 1 - (\frac{\bar{C}}{\epsilon} + \alpha) \bar{\theta}(\beta) \) is positive, we get
\[
\mathbb{E}\left[ \int_0^T e^{\beta s} |\delta Z_s|^2 \, ds \right] + \mathbb{E}\left[ \int_0^T \int_E e^{\beta s} (\delta V_s(e))^2 \lambda(de) \, ds \right] \\
\leq D^{-1} \left( 1 + \frac{1}{\epsilon} \right) \bar{C} T \bar{\theta}(\beta) \mathbb{E}\left[ \sup_{t \in [0,T]} e^{\beta t} |\delta Y_t|^2 \right].
\]
(24)

Coming back to 23 and taking the supremum over \( t \in [0,T] \) then the expectation, we obtain
\[
\mathbb{E}\left[ \sup_{t \in [0,T]} e^{\beta t} (\delta Y_t)^2 \right] \leq \left( 1 + \frac{1}{\epsilon} \right) \bar{C} T \bar{\theta}(\beta) \mathbb{E}\left[ \sup_{t \in [0,T]} e^{\beta t} |\delta Y_t|^2 \right] \\
+ \left( \frac{\bar{C}}{\epsilon} + \alpha \right) \bar{\theta}(\beta) \{ \mathbb{E}\left[ \int_0^T e^{\beta s} |\delta Z_s|^2 \, ds \right] + \mathbb{E}\left[ \int_0^T \int_E e^{\beta s} (\delta V_s(e))^2 \lambda(de) \, ds \right] \}
\]
\[
+ 2 \mathbb{E}\left[ \sup_{t \in [0,T]} \left| \int_t^T e^{\beta s} \delta Y_s(g(s, \Gamma(s)) - g(s, \Gamma'(s)) \, dB_s \right| \right]
\]
\[
+ 2 \mathbb{E}\left[ \sup_{t \in [0,T]} \left| \int_t^T e^{\beta s} \delta Y_s - \delta Z_s \, dW_s \right| \right] + 2 \mathbb{E}\left[ \sup_{t \in [0,T]} \left| \int_t^T e^{\beta s} \delta Y_s - \delta V_s(e) \tilde{\mu}(ds, de) \right| \right].
\]
(25)

By the Burkholder-Davis-Gundy inequality, we deduce
\[
2 \mathbb{E}\left[ \sup_{t \in [0,T]} \left| \int_t^T e^{\beta s} \delta Y_s(g(s, \Gamma(s)) - g(s, \Gamma'(s)) \, dB_s \right| \right]
\]
\[
\leq 2C_1 \mathbb{E}\left[ \left( \int_t^T e^{2\beta s} |\delta Y_s|^2 \right) \left| (g(s, \Gamma(s)) - g(s, \Gamma'(s)))^2 \right| \, ds \right]^{1/2}
\]
\[
\leq 2C_1 \mathbb{E}\left[ \left( \sup_{s \in [0,T]} e^{\beta s} |\delta Y_s|^2 \right)^{1/2} \right] \left( \int_t^T e^{\beta s} \left| (g(s, \Gamma(s)) - g(s, \Gamma'(s)))^2 \right| \, ds \right)^{1/2}
\]
\[
\leq \frac{1}{6} \mathbb{E}\left[ \sup_{t \in [0,T]} e^{\beta t} (\delta Y_t)^2 \right] + 6C_1^2 \bar{\theta}(\beta) \mathbb{E}\left[ \int_t^T e^{\beta s} |\delta Z_s|^2 \, ds \right].
\]
(26)

In the same way, we have
\[
2 \mathbb{E}\left[ \sup_{t \in [0,T]} \left| \int_t^T e^{\beta s} \delta Y_s - \delta Z_s \, dW_s \right| \right] \leq \frac{1}{6} \mathbb{E}\left[ \sup_{t \in [0,T]} e^{\beta t} (\delta Y_t)^2 \right] + 6C_1^2 \mathbb{E}\left[ \int_0^T e^{\beta s} |\delta Z_s|^2 \, ds \right]
\]
and
\[
2 \mathbb{E}\left[ \sup_{t \in [0,T]} \left| \int_t^T \int_E e^{\beta s} \delta Y_s - \delta V_s(e) \tilde{\mu}(ds, de) \right| \right]
\]
\[
\leq \frac{1}{6} \mathbb{E}\left[ \sup_{t \in [0,T]} e^{\beta t} (\delta Y_t)^2 \right] + 6C_1^2 \mathbb{E}\left[ \int_0^T e^{\beta s} \left| \delta V_s(e) \right|^2 \lambda(de) \, ds \right].
\]
(27)

(28)

Hence, pulling 26, 27 and 28 into 25 we get
\[
\left( \frac{1}{2} - (1 + \frac{1}{\epsilon} \bar{C} T \bar{\theta}(\beta)) \right) \mathbb{E}\left[ \sup_{t \in [0,T]} e^{\beta t} (\delta Y_t)^2 \right]
\]
\[
\leq \left( 6C_1^2 + \left( \frac{\bar{C}}{\epsilon} + \alpha \right) \bar{\theta}(\beta) \right) \{ \mathbb{E}\left[ \int_0^T e^{\beta s} |\delta Z_s|^2 \, ds \right] + \mathbb{E}\left[ \int_0^T \int_E e^{\beta s} (\delta V_s(e))^2 \lambda(de) \, ds \right] \}
\]
\[
+ 6C_1^2 \mathbb{E}\left[ \int_t^T e^{\beta s} |g(s, \Gamma(s)) - g(s, \Gamma'(s))|^2 \, ds \right].
\]
Using once again Lemma 4.1, we get
\[
\left(\frac{1}{2} - \left(1 + \frac{1}{\epsilon} + 6C_1^2\right)\tilde{C}T\tilde{\theta}(\beta)\right)\mathbb{E}\left[\sup_{t \in [0,T]} e^{\beta t}||\tilde{Y}_t||^2\right] \\
\leq \left(6C_1^2 + \left(\frac{\tilde{C}}{\epsilon} + \alpha(1 + 6C_1^2)\right)\tilde{\theta}(\beta)\right)\left\{\mathbb{E}\left[\int_0^T e^{\beta s}||\delta Z_s||^2\, ds\right] \\
+ \mathbb{E}\left[\int_0^T \int_E e^{\beta s}(\delta V_s(e))^2\, \lambda(de)\, ds\right]\right\}.
\]
Then from 24 we obtain
\[
\left[\frac{1}{2} - \left(1 + \frac{1}{\epsilon} + 6C_1^2\right) + \left\{6C_1^2 + \left(\frac{\tilde{C}}{\epsilon} + \alpha(1 + 6C_1^2)\right)\tilde{\theta}(\beta)\right\}D^{-1}(1 + \frac{1}{\epsilon})\tilde{C}T\tilde{\theta}(\beta)\right]
\leq \mathbb{E}\left[\sup_{t \in [0,T]} e^{\beta t}||\tilde{Y}_t||^2\right] \leq 0.
\]
Since by conditions 21 and 22 we can choose
\[
\epsilon > \frac{\tilde{C}\tilde{\theta}(\beta)\left(1 + 2T + (1 + \alpha\tilde{\theta}(\beta))12C_1^2T - 12C_1^2T\tilde{C}\tilde{\theta}(\beta)\right)}{1 - \tilde{\theta}(\beta)\{2\tilde{C}T(1 + 12C_1^2) + \alpha\}},
\]
it is not difficult to see that
\[
\frac{1}{2} - \left(1 + \frac{1}{\epsilon} + 6C_1^2\right) + \left\{6C_1^2 + \left(\frac{\tilde{C}}{\epsilon} + \alpha(1 + 6C_1^2)\right)\tilde{\theta}(\beta)\right\}D^{-1}(1 + \frac{1}{\epsilon})\tilde{C}T\tilde{\theta}(\beta) > 0.
\]
Henceforth, \(\mathbb{E}\left[\sup_{t \in [0,T]} e^{\beta t}||\tilde{Y}_t||^2\right] \leq 0\) and then \(Y = Y', Z = Z', V = V'\) and \(K^+ - K^- = K'^+ - K'^-\). Finally, in a classic way, we can show that if \(K^\pm\) are chosen to be singular then we have also \(K^\pm = K'^\pm\). The proof is complete. \(\square\)

We are now in position to give the main theorem of this section.

**Theorem 4.4.** Suppose (H1) – (H3), (H5) and (Mk) hold. Assume that
\[
1 - \tilde{\theta}(\beta)\max\{1, T\}\left[12C_1^2(\tilde{C} + \alpha) + \tilde{C}(3 + 12C_1^2)\right] > 0 \quad (29)
\]
and
\[
\frac{\tilde{C}\tilde{\theta}(\beta)\max\{1, T\}\left[12C_1^2(\tilde{C} + \alpha) + \tilde{C}(3 + 12C_1^2)\right]}{1 - \tilde{\theta}(\beta)\max\{1, T\}\left[12C_1^2(\tilde{C} + \alpha) + \tilde{C}(3 + 12C_1^2)\right]} < \beta. \quad (30)
\]
Then the reflected BDSDE 20 admits a solution \((Y, Z, V, K^+, K^-)\); if \((Y, Z, V, K^+, K^-)\) and \((Y', Z', V', K'^+, K'^-)\) are two solutions of 20, then \(Y = Y', Z = Z', V = V'\) and \(K^+ - K^- = K'^+ - K'^-\). Moreover if the processes \(K^\pm\) are singular then we have also \(K^\pm = K'^\pm\).

**Proof.** We define a map \(\phi\) from \(B_\beta\) into itself as follows. Given \((\tilde{Y}, \tilde{Z}, \tilde{V}) \in B_\beta\) we define \((\tilde{Y}, \tilde{Z}, \tilde{V}) = \phi(Y, Z, V)\), where \((\tilde{Y}, \tilde{Z}, \tilde{V}, \tilde{K})\) is the triplet for which there exist two other processes \(K^\pm\) in \(S^*_\epsilon(\mathbb{R})\) such that \((\tilde{Y}, \tilde{Z}, \tilde{V}, \tilde{K})\) is a solution for the reflected BDSDE associated with \((\xi, g(t, \Gamma(t)), f(t, \Gamma(t)), L, U)\), meaning
\[
\tilde{Y}_t = \xi + \int_t^T f(s, \Gamma(s))\, ds + \int_t^T g(s, \Gamma(s))\, dB_s + (\tilde{K}^+_T - \tilde{K}^-_T) - (\tilde{K}^+_T - \tilde{K}^-_T) \\
- \int_t^T \tilde{Z}_s\, dW_s - \int_t^T \int_E \tilde{V}_s(e)\tilde{\mu}(ds, de).
\]
Let \((Y', Z', V')\) be another element of \(B_{\beta}\) and \(\phi(Y', Z', V') = (\hat{Y}', \hat{Z}', \hat{V}')\), then using Itô's formula we get, for any \(t \leq T\) and \(\epsilon > 0\),

\[
e^{\beta t} |\delta Y_t|^2 + (\beta - \epsilon) \int_t^T e^{\beta s} |\delta Y_s|^2 ds + \int_t^T e^{\beta s} |\delta \hat{Z}_s|^2 ds
\]

\[
+ \int_t^T \int_E e^{\beta s} |\delta \hat{V}_s(\epsilon)|^2 \mu(ds, de)
\]

\[
\leq \frac{1}{\epsilon} \int_t^T e^{\beta s} |f(s, \Gamma(s)) - f(s, \Gamma'(s))|^2 ds + \int_t^T e^{\beta s} (g(s, \Gamma(s)) - g(s, \Gamma'(s)))^2 ds
\]

\[
+ 2 \int_t^T e^{\beta s} \delta \hat{Y}_s(g(s, \Gamma(s)) - g(s, \Gamma'(s))) dB_s - 2 \int_t^T e^{\beta s} \delta \hat{Y}_s - \delta \hat{Z}_s dW_s
\]

\[
- 2 \int_t^T \int_E e^{\beta s} \delta \hat{Y}_s - \delta \hat{V}_s(e) \tilde{\mu}(ds, de),
\]

since \(\int_t^T e^{\beta s} \delta \hat{Y}_s (d\delta \tilde{K}^+_s - d\delta \tilde{K}^+_s) \leq 0\). On account of condition 30, we have \(\epsilon < \beta\). So by Lemma 4.1, we get

\[
e^{\beta t} |\delta Y_t|^2 + \int_t^T e^{\beta s} |\delta \hat{Z}_s|^2 ds + \int_t^T \int_E e^{\beta s} |\delta \hat{V}_s(\epsilon)|^2 \mu(ds, de)
\]

\[
\leq \left(1 + \frac{1}{\epsilon}\right) \tilde{C} \max\{1, T\} \tilde{\theta}(\beta) \sup_{t \in [0, T]} e^{\beta t} |\delta Y_t|^2
\]

\[
+ \left(\frac{\tilde{C}}{\epsilon} + \alpha \right) \max\{1, T\} \tilde{\theta}(\beta) \left\{ \int_0^T e^{\beta s} |\delta Z_s|^2 ds + \int_0^T \int_E e^{\beta s} (\delta \hat{V}_s(\epsilon))^2 \lambda(de) ds \right\}
\]

\[
+ 2 \int_t^T e^{\beta s} \delta \hat{Y}_s (g(s, \Gamma(s)) - g(s, \Gamma'(s))) dB_s - 2 \int_t^T e^{\beta s} \delta \hat{Y}_s - \delta \hat{Z}_s dW_s
\]

\[
- 2 \int_t^T \int_E e^{\beta s} \delta \hat{Y}_s - \delta \hat{V}_s(e) \tilde{\mu}(ds, de).
\]

Now, as the processes \(\{M_t = \int_0^t e^{\beta s} \delta \hat{Y}_s (g(s, \Gamma(s)) - g(s, \Gamma'(s))) dB_s\}_{0 \leq t \leq T}, \{N_t = \int_0^t e^{\beta s} \delta \hat{Z}_s dW_s\}_{0 \leq t \leq T}\) and \(\{\tilde{P}_t = \int_0^t \int_E e^{\beta s} \delta \hat{Y}_s - \delta \hat{V}_s(e) \tilde{\mu}(ds, de)\}_{0 \leq t \leq T}\) are \((\mathcal{G}_t, \mathbb{P})\) martingales, then

\[
\mathbb{E}\left[ \int_0^T e^{\beta s} |\delta \hat{Z}_s|^2 ds \right] + \mathbb{E}\left[ \int_0^T e^{\beta s} ds \int_E |\delta \hat{V}_s(\epsilon)|^2 \lambda(de) \right]
\]

\[
\leq \tilde{\theta}(\beta) \max\{1, T\} \left\{ \left(1 + \frac{1}{\epsilon}\right) \tilde{C} \mathbb{E}\left[ \sup_{0 \leq s \leq T} e^{\beta s} |\delta Y_s|^2 \right] \right\}
\]

\[
+ \left(\frac{\tilde{C}}{\epsilon} + \alpha \right) \mathbb{E}\left[ \int_0^T e^{\beta s} |\delta Z_s|^2 ds + \int_0^T \int_E e^{\beta s} (\delta \hat{V}_s(\epsilon))^2 \lambda(de) ds \right]
\]

\[
\leq \tilde{\theta}(\beta) \max\{1, T\} \left\{ \left(1 + \frac{1}{\epsilon}\right) \tilde{C} \mathbb{E}\left[ \sup_{0 \leq s \leq T} e^{\beta s} |\delta Y_s|^2 \right] \right\}
\]

\[
+ \left(\frac{\tilde{C}}{\epsilon} + \alpha \right) \mathbb{E}\left[ \int_0^T e^{\beta s} |\delta Z_s|^2 ds + \int_0^T \int_E e^{\beta s} (\delta \hat{V}_s(\epsilon))^2 \lambda(de) ds \right]
\]

\[
+ 2 \mathbb{E}[M, M]^{1/2} + 2 \mathbb{E}[N, N]^{1/2} + 2 \mathbb{E}[\tilde{P}, \tilde{P}]^{1/2}.
\]
Furthermore, for the last inequality we have made use of the Burholder-Davis-Gundy one. But,

\[ 2\mathbb{E}[\langle M, M \rangle_T^{1/2}] \leq \frac{1}{6} \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\beta t} |\tilde{Y}_t|^2 \right] + 6C_1^2 \tilde{\theta}(\beta) \max \{1, T\} \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\beta t} |\delta Y_t|^2 \right] + 6C_1^2 \tilde{\theta}(\beta) \max \{1, T\} \alpha \left\{ \mathbb{E} \left[ \int_0^T e^{\beta s} |\delta Z_s|^2 ds \right] + \mathbb{E} \left[ \int_0^T e^{\beta s} ds \int_E |\delta V_s(e)|^2 \lambda(de) \right] \right\} \]

and

\[ 2\mathbb{E}[\langle N, N \rangle_T^{1/2}] \leq \frac{1}{6} \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\beta t} |\tilde{Y}_t|^2 \right] + 6C_1^2 \mathbb{E} \left[ \int_0^T e^{\beta s} ds \int_E |\tilde{\delta V}_s(e)|^2 \lambda(de) \right]. \]

Pulling the above three inequalities in 32, we get

\[ \frac{1}{2} \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\beta t} |\delta Y_t|^2 \right] \leq \tilde{\theta}(\beta) \max \{1, T\} \left\{ (1 + \frac{1}{\epsilon} + 6C_1^2) \tilde{C} \mathbb{E} \left[ \sup_{0 \leq s \leq T} e^{\beta s} |\delta Y_s|^2 \right] + \left\{ \frac{\tilde{C}}{\epsilon} + \alpha (1 + 6C_1^2) \right\} \mathbb{E} \left[ \int_0^T e^{\beta s} |\delta Z_s|^2 ds + \int_0^T e^{\beta s} ds \int_E |\delta V_s(e)|^2 \lambda(de) \right] \right\} \]

Substituting now 31 into 33 we obtain

\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\beta t} |\tilde{Y}_t|^2 \right] \leq \tilde{\theta}(\beta) \max \{1, T\} \left\{ (1 + \frac{1}{\epsilon}) (2 + 12C_1^2) + 12C_1^2 \right\} \tilde{C} \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\beta t} |\delta Y_t|^2 \right] + \left\{ \frac{\tilde{C}}{\epsilon} + \alpha (2 + 12C_1^2) + 12C_1^2 \alpha \right\} \left\{ \mathbb{E} \left[ \int_0^T e^{\beta s} |\delta Z_s|^2 ds \right] + \mathbb{E} \left[ \int_0^T e^{\beta s} ds \int_E |\delta V_s(e)|^2 \lambda(de) \right] \right\} \]

which combined with 31 gives

\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\beta t} |\tilde{Y}_t|^2 \right] + \mathbb{E} \left[ \int_0^T e^{\beta s} |\delta Z_s|^2 ds \right] + \mathbb{E} \left[ \int_0^T e^{\beta s} ds \int_E |\delta V_s(e)|^2 \lambda(de) \right] \leq \tilde{\theta}(\beta) \max \{1, T\} \left\{ (1 + \frac{1}{\epsilon}) (3 + 12C_1^2) + 12C_1^2 \right\} \tilde{C} \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\beta t} |\tilde{Y}_t|^2 \right] + \left\{ \frac{\tilde{C}}{\epsilon} + \alpha (3 + 12C_1^2) + 12C_1^2 \alpha \right\} \left\{ \mathbb{E} \left[ \int_0^T e^{\beta s} |\delta Z_s|^2 ds \right] + \mathbb{E} \left[ \int_0^T e^{\beta s} ds \int_E |\delta V_s(e)|^2 \lambda(de) \right] \right\}. \]
Therefore,

\[ \|\phi(\delta Y, \delta Z, \delta V)\|_{B_{\beta,s}} \leq \tilde{\theta}(\beta)(1 \vee T) \left\{ (\tilde{C} + \frac{\tilde{C}}{\epsilon} + \alpha)(3 + 12C_1^2) + 12C_1^2(\tilde{C} + \alpha) \right\} \]

\[ \left( \|\delta Y\|_{B_{\beta,s}}^2 + \|\delta Z\|_{B_{\beta,s}}^2 + \|\delta V\|_{B_{\beta,s}}^2 \right) \].

Now, thanks to Hypothesis 30, there exists a constant \( \epsilon \) such that

\[ \begin{align*}
\tilde{C} \tilde{\theta}(\beta) \max\{1, T\} (3 + 12C_1^2) & \leq \epsilon < \beta,
\end{align*} \]

Hence \( \phi \) is a strict contraction on \( B_{\beta} \) with the norm 5. Therefore \( \phi \) admits a unique fixed point \((Y, Z, V)\), meaning \( \phi(Y, Z, V) = (Y, Z, V) \), and then, with the associated processes \( K^+ \) and \( K^- \), the quintuplet \((Y, Z, V, K^+, K^-)\) is a solution of the reflected BDSDE associated with \((\xi, f, g, L, U)\). Remark that this one is unique if \( K^\pm \) are chosen to be singular.

The following corollary is a particular case with only one barrier.

**Corollary 2.** Assume that the coefficients \( f \) and \( g \) satisfy Assumptions (H1), (H2), (H5) and the following Assumption (H3") hold:

(H3") \( L = (L_t)_{t \leq T} \) is a process of \( S_2^L(\mathbb{R}) \) and \( L_T \leq \xi \), \( \mathbb{P} \)-a.s. with totally inaccessible jump times.

Assume further that \( T, C, C_1 \) and \( \theta \) are chosen as in Theorem 4.4 and such that conditions 29 and 30 hold. Then, the BDSDE with one lower barrier and time delayed generator associated with \((\xi, f, g, L)\) admits a unique solution, so there exists a unique quadruplet of processes \((Y_t, Z_t, V_t, K_t)_{t \leq T}\) such that 16 is satisfied and

\[ \begin{align*}
i) & \ Y \in S_2^L(\mathbb{R}), \ Z \in H_2^2(\mathbb{R}), \ V \in L_2^2(\mu, \mathbb{R}) \text{ and } K \in S_2^{Ci}(\mathbb{R}), \\
ii) & \ \forall t \leq T, \ Y_t \geq L_t \text{ and } \int_0^T (Y_t - L_t) dK_t = 0.
\end{align*} \]

We next deal with existence of the solution of the reflected BDSDE with general jumps and time delayed generators.

Theorem 3.2 gives the existence in the case when the functions \( f \) and \( g \) do not depend on \( (y, z, v) \). For the general case of the generators, the proof can be obtained similarly as the one of Theorem 4.4, and we omit it here.

**Theorem 4.5.** Assume (H4) and all the assumptions of Theorem 4.4 hold. Then the BDSDE with two reflecting càdlàg barriers and time delayed generators associated with \((\xi, f, g, L, U)\) has a unique solution, meaning there exists a unique quintuplet of processes \((Y_t, Z_t, V_t, K_t^+, K_t^-)_{t \leq T}\) which satisfies 16 and items i)-iii)-iv) of Definition 3.5.

A direct consequence of Theorem 4.5 is the following corollary.

**Corollary 3.** Under (H1), (H2), (H3'), (H5), 29 and 30 there exists a unique quadruplet of processes \((Y, Z, V, K) = (Y_t, Z_t, V_t, K_t)\) solution for the time delayed RBDSDE with one lower barrier associated with \((\xi, f, g, L)\), meaning it satisfies 16 and 14.
4.2. Comparison principle. This section is devoted to the study of the comparison theorem of the time delayed BDSDEs with two reflecting barriers which have only inaccessible jumps.

To state, let us consider the following two BDSDEs for $i = 1, 2$,

$$
\begin{align*}
Y_t^i &= \xi + \int_t^T f^i(s, \Gamma(s)) \, ds + \int_t^T g(s, \Gamma(s)) \, dB_s + (K_t^{i+} - K_t^{i-}) - (K_T^{i+} - K_T^{i-}), \\
L_t^i \leq Y_t^i \leq U_t^i, & \quad \forall t \leq T, \\
\int_0^T (Y_t^i - L_t^i) \, dK_t^{i+} = \int_0^T (U_t^i - Y_t^i) \, dK_t^{i-} = 0, & \quad \forall t \leq T.
\end{align*}
$$

(34)

In the sequel, we call $(\xi^i, f^i, g, L^i, U^i)$ as the parameters of the reflected BDSDEs 34, $i = 1, 2$, respectively. We also define the following stopping time

$$
\tau_i = \inf \{ s \geq t, Y_s^1 = L_s^1 \} \wedge \inf \{ s \geq t, Y_s^2 = U_s^2 \}.
$$

For $(Y^i, Z^i, V^i)$, $i = 1, 2$ be the solution to two reflected BDSDEs with boundaries $(L^1, U^1)$ and $(L^2, U^2)$ respectively, we denote $\delta \Theta = \Theta^1 - \Theta^2$, $\Theta = Y, Z, V$ and $\xi$. Furthermore, as mentioned above in order to get a comparison result, we must impose a control on the size of the jumps. This is why we require (H6) to hold in addition to (H1) and (H2):

(H6) There exists $-1 < c \leq 0$ and $C > 0$ such that $\forall y^1 \in \mathbb{R}, \forall z^1 \in \mathbb{R}^d, \forall v^1, v^2 \in L^2(\mathcal{E}, \mathcal{F}, \lambda, \mathbb{R})$, we have

$$
\left| f^1(t, y^1, z^1, v^1) - f^2(t, y^1, z^1, v^2) \right| \leq \int_E (v^1(e) - v^2(e)) \gamma_t(e) \lambda(de),
$$

where $\gamma_t : \Omega \times [0, T] \times E \rightarrow \mathbb{R}$ is $\mathcal{P} \otimes \mathcal{E}$-measurable and satisfies $c(1 \wedge |e|) \leq \gamma_t(e) \leq C(1 \wedge |e|)$.

Theorem 4.6. For $n \in \mathbb{N}$, let $\tau_n$ and $\tau_{t,n}$ be the stopping times defined as follows:

$$
\begin{align*}
\tau_n &= \inf \{ s \geq 0, |\delta Y_s| \wedge |\delta Z_s| \wedge |\int_E \delta Z_s \gamma_s(e) \lambda(de)| \leq \frac{1}{n} \} \quad \text{or} \\
\tau_{t,n} &= \tau_t \wedge \tau_n.
\end{align*}
$$

Assume that $(\xi^i, f^i, g, L^i, U^i)$, $i = 1, 2$, the parameters of the BDSDEs 20, satisfy (H1)-(H3), (H5), (H6) and (Mk). Assume further that

i) $Y_{\tau_{t,n}}^1 \leq Y_{\tau_{t,n}}^2$, $\mathbb{P}$-a.s.,

ii) $f^1(t, Y_t^1, Z_t^1, V_t^1) \leq f^2(t, Y_t^1, Z_t^1, V_t^1)$, $\mathbb{P}$-a.s.,

iii) $L_t^1 \leq L_t^2$ and $U_t^1 \leq U_t^2$, $0 \leq t \leq T$, $\mathbb{P}$-a.s.

Then we have $Y_t^1 \leq Y_t^2$, $t \in [0, \tau_{t,n}]$, $\mathbb{P}$-a.s.

Remark 2. As it is explained in [15], the comparison principle may not hold while the processes $Y, Z$ and $V$ can cross 0 or $\infty$, that is why we consider $\tau_n$ to exclude the approach of the difference of two processes (resp. one process) to 0 or $\infty$ by stopping them before passages of small or large sills happen.
Proof. According to 34 and the previous notations, we can write, on the set \{ (ω, t), t ≤ τ_{t,n}(ω) \}:
\[
\delta Y_t = \delta Y_{t,n} + \int_t^{\tau_{t,n}} \left( \Delta_y f^1_s \delta Y_s + \Delta_z f^1_s \delta Z_s + \Delta_v f^1_s \int_E \gamma_s(e) \delta V_s(e) \lambda de + \Delta f^1_s \right) ds
+ \left( \delta K^+_s - \delta K^-_s \right) - \left( \delta K^+_n - \delta K^-_n \right) + \int_t^{\tau_{t,n}} \Delta_y g_s \delta Y_s \, dB_s
\]
\[- \int_t^{\tau_{t,n}} \delta Z_s \, dW_s - \int_t^{\tau_{t,n}} \delta V_s(e) \tilde{\mu}(ds, de),
\]
where,
\[
\Delta_y f^1_s = \frac{f^1(s, Y^1_s, Z^1_s, V^1_s) - f^1(s, Y^2_s, Z^1_s, V^1_s)}{\delta Y_s} \mathbf{1}_{\{ \delta Y_s \neq 0 \}}
\]
\[
\Delta_y g_s = \frac{g(s, Y^1_s) - g(s, Y^2_s)}{\delta Y_s} \mathbf{1}_{\{ \delta Y_s \neq 0 \}}
\]
\[
\Delta_z f^1_s = \frac{f^1(s, Y^2_s, Z^1_s, V^1_s) - f^1(s, Y^2_s, Z^2_s, V^1_s)}{\delta Z_s} \mathbf{1}_{\{ \delta Z_s \neq 0 \}}
\]
\[
\Delta_v f^1_s = \frac{f^1(s, Y^2_s, Z^2_s, V^2_s) - f^1(s, Y^2_s, Z^2_s, V^2_s)}{\int_E \gamma_s(e) \delta V_s(e) \lambda(de)} \mathbf{1}_{\{ \int_E \gamma_s(e) \delta V_s(e) \lambda(de) \neq 0 \}}
\]
\[
\Delta f^1_s = f^1(s, Y^2_s, Z^2_s, V^2_s) - f^1(s, Y^2_s, Z^1_s, V^2_s).
\]
Setting
\[
\Gamma_t = \exp \left( \int_0^t \Delta_y f^1_s \, ds + \int_0^t \Delta_y g_s \, dB_s - \int_0^t |\Delta_y g_s|^2 \, ds + \int_0^t \Delta_z f^1_s \, dW_s
+ \int_0^t \Delta_v f^1_s \int_E \gamma_s(e) \tilde{\mu}(ds, de) \right).
\]
By Assumptions (H2) and (H6), we have \(|\Delta_y f^1_s| ≤ C, |\Delta_z f^1_s| ≤ C, |\Delta_v f^1_s| ≤ C(1 \lor |e|)\) and \(\Delta f^1_s > -1\). Then, by the Doléans-Dade exponential formula and the Gronwall inequality, we conclude that \(\Gamma_t > 0\) and \(\Gamma_t \in \mathcal{S}^2_\beta(\mathbb{R})\). Hence applying Itô’s formula to \(\Gamma_t \delta Y_s\), it follows that
\[
\Gamma_{t,n} \delta Y_{t,n} - \Gamma_{t,n} \delta Y_t
= \int_t^{\tau_{t,n}} \Gamma_s \, d\delta Y_s + \int_t^{\tau_{t,n}} \delta Y_s \, d\Gamma_s + \int_t^{\tau_{t,n}} \Gamma_s \, d[\Gamma, \delta Y]_s
= - \int_t^{\tau_{t,n}} \Gamma_s \Delta f^1_s \, ds - \int_t^{\tau_{t,n}} \int_s^{\tau_{t,n}} (\delta f^1_s - d\delta K^+_s - d\delta K^-_s)
\]
\[- \int_t^{\tau_{t,n}} \Gamma_s \left( \delta Z_s + \delta Y_s - \Delta_z f^1_s \right) \, dW_s
+ \int_t^{\tau_{t,n}} \Gamma_s \left( \delta Y_s - \gamma_s(e) \Delta_v f^1_s \right) \tilde{\mu}(ds, de).
\]
Therefore, since \(f^1 ≤ f^2, \Delta f^1_s ≤ 0\), and consequently, since \(\left( \int_t^{\tau_{t,n}} \Gamma_s \left( \delta Z_s + \delta Y_s - \Delta_z f^1_s \right) \, dW_s \right)_{t ≤ T}\) and \(\left( \int_t^{\tau_{t,n}} \Gamma_s \left( \delta Y_s - \gamma_s(e) \Delta_v f^1_s \right) \tilde{\mu}(ds, de) \right)_{t ≤ T}\) are martingales, we get
\[
\mathbb{E}[\Gamma_t \delta Y_t - \Gamma_{t,n} \delta Y_{t,n} | \mathcal{F}_t] \leq \mathbb{E} \left[ \int_t^{\tau_{t,n}} \Gamma_s \left( \delta K^+_s - d\delta K^-_s \right) | \mathcal{F}_t \right].
\]
On the other hand, on the set \( \{ (\omega, t), t \leq \tau_{t,n}(\omega) \} \) we have \( dK^2^+, dK^{1^-} \geq 0 \) and \( K^{1^+} = K^{2^-} = 0 \), then

\[
E[\Gamma_t \delta Y_t - \Gamma_{\tau_{t,n}} \delta Y_{\tau_{t,n}} | \mathcal{F}_t] \leq 0.
\]

The fact that \( \delta Y_{\tau_{t,n}} \) is taken non-positive entails that

\[
Y_t^1 \leq Y_t^2, \quad t \in [0, \tau_{t,n}], \quad \mathbb{P}\text{-a.s.}
\]

\[\square\]

**Remark 3.** When we assume \( U = +\infty \) (so \( K^- = 0 \)) or \( L = -\infty \) and \( U = +\infty \) (so \( K^- = K^+ = 0 \)) we deal with BDSDEs with one lower reflecting barrier (resp. non-reflected BDSDEs): they are particular cases of 6 and 20. As a consequence the comparison theorem holds for the solutions of BDSDEs and reflected BDSDEs driven by a Brownian motion and a Poisson random measure, in both cases: generators delayed or not delayed.

5. Appendix. Let us first summarize the main results on the Snell envelope of processes that we used in this paper. For more details on the Snell envelope one can see for example [13].

**Proposition 2.** Let \( U = (U_t)_{0 \leq t \leq T} \) be an \((\mathcal{F}_t)_{t < T}\)-adapted càdlàg process of class \([D]\), that is the family \( \{U_t\}_{t \in T_0} \) is uniformly integrable. Then, there exists an \((\mathcal{F}_t)_{t < T}\)-adapted \(\mathbb{R}\)-valued càdlàg process \( \mathcal{R}(U) := (\mathcal{R}(U)_t)_{t \leq T} \) such that: if \( (\mathcal{R}(U)_t^i)_{0 \leq t \leq T} \) is a càdlàg supermartingale of class \([D]\) satisfying \( \forall 0 \leq t \leq T, \mathcal{R}(U)_t^i \geq U_t \) then \( \mathcal{R}(U)_t^i \geq \mathcal{R}(U)_t \) for any \( 0 \leq t \leq T \).

Moreover, for any \((\mathcal{F}_t)_{t \leq T}\)-stopping time \( \tau \) we have:

\[
\forall t \leq T, \quad \mathcal{R}(U)_t = \text{ess sup}_{\tau \in T_t} E[U_{\tau} | \mathcal{F}_t]; \quad \text{(and then } \mathcal{R}(U)_T = U_T).\]

The process \( \mathcal{R}(U) \) is called the Snell envelope of \( U \).

The following result holds true.

**Proposition 3.** If \( (U^n)_{n \geq 0} \) is a non-decreasing sequence of \(\mathbb{R}\)-valued càdlàg \(\mathcal{P}\)-measurable processes of class \([D]\) which converges pointwise to \( U \), another \(\mathbb{R}\)-valued càdlàg \(\mathcal{P}\)-measurable process of class \([D]\), then \(\mathbb{P}\)-a.s., \( \forall t \leq T, \mathcal{R}(U^n)_t \nearrow \mathcal{R}(U)_t \), where \( \mathcal{R} \) is the Snell envelope operator.

We now introduce the predictable and totally inaccessible stopping times notions, which can be found in [11] or [12].

**Definition 5.1.** Let \( \tau \) be a stopping time.

i) \( \tau \) is a predictable stopping time if there exists a sequence of stopping times \( (\tau_n)_{n \geq 1} \) such that \( \tau_n \) is increasing, \( \tau_n < \tau \) on \( \{ \tau > 0 \} \) for all \( n \), and \( \lim_{n \to \infty} \tau_n = \tau \) a.s.

ii) \( \tau \) is totally inaccessible if, for every predictable stopping time \( \nu \),

\[
\mathbb{P}(\{ \omega : \tau(\omega) = \nu(\omega) < \infty \}) = 0.
\]

Here we recall the notions of singular and signed measures, concerning the measures associated to the increasing processes \( K \) along the paper.

**Definition 5.2.** (see, e.g., [22], pp. 118 and 126) Let \((\Omega, \mathcal{A})\) be a measurable space.

1) A signed measure on \((\Omega, \mathcal{A})\) is a function \( \nu : \mathcal{A} \to \mathbb{R} \) such that
   i) \( \nu \) takes on at most one of the values \(-\infty\) or \(+\infty\).
ii) $\nu(\emptyset) = 0$.

iii) If $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ is a sequence of pairwise disjoint sets, then

$$\nu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \nu(E_n).$$

2) Let $\nu_1, \nu_2$ be two signed measures on $(\Omega, \mathcal{A})$. We say that they are mutually singular (or more simply that they are singular or that $\nu_1$ is singular to $\nu_2$ or that $\nu_2$ is singular to $\nu_1$), denoted $\nu_1 \perp \nu_2$, if there are disjoint sets $A, B \in \mathcal{A}$ with $\Omega = A \cup B$ for which $\nu_1(A) = \nu_2(B) = 0$.

In other words, two signed measures are mutually singular if there is a set in $\mathcal{A}$ which is null for one of them and its complement is null for the other.

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