Numerical investigation of the solutions of Schrödinger equation with exponential cubic B-spline finite element method

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Abstract

In this paper, we investigate the numerical solutions of the cubic nonlinear Schrödinger equation via the exponential B-spline collocation method. Crank-Nicolson formulas are used for time discretization of the target equation. A linearization technique is also employed for the numerical purpose. Four numerical examples related to single soliton, collision of two solitons that move in opposite directions, the birth of standing and mobile solitons and bound state solution are considered as the test problems. The accuracy and the efficiency of the purposed method are measured by $L_\infty$ error norm and conserved constants. The obtained results are compared with the possible analytical values and those in some earlier studies.

Keywords: Schrödinger equation; Exponential spline; Soliton
Subject classification: 35Q51; 35Q53; 41A15

1 Introduction

One of the most interesting universal equation in physical studies is the Schrödinger equation that describes the quantum state of a physical system. Since the equation used in quantum mechanics is too general, there are some different versions of the Schrödinger equation in scientific studies for the modelling several physical phenomena such as the propagation of optical pulses, superconductivity, waves in water and plasmas and self focusing in laser pulses. Here we focus on one of the specific form of the Schrödinger equation known as the time dependent cubic nonlinear Schrödinger equation (NLS) which describes the optical pulse propagation in optical fibers.

The cubic NLS equation is given in one dimension as follows:

$$iU_t + U_{xx} + q |U|^2 U = 0, \quad -\infty < x < \infty, \quad t > 0$$

(1)

where $i$ is the imaginary unit, $q$ is the parameter for the self phase modulation, $U$ is a complex-valued function which shows the evolution of slowly varying wave train in a stable dispersive physical system with no dissipation and $U_t$ is the amplitude of the pulse envelope. To complete the usual
classical mathematical statement of the problem, the initial and the boundary conditions are chosen as to be

$$U(x, 0) = f(x), \quad -\infty < x < \infty$$

and

$$\lim_{x \to \pm\infty} U(x, t) = 0, \quad t \geq 0.$$ (3)

There are many analytical and numerical studies on Eq. (1) in the literature. Different kinds of numerical techniques such as finite difference ([10], [11]), finite element ([1], [2], [4], [5], [9]) and Adomian decomposition ([12]) methods have been applied to Schrödinger equation in these studies. In last decade, variational iteration [13], differential quadrature [14], cubic non-polynomial spline [15], parametric cubic spline [16] and time-splitting pseudo-spectral domain decomposition methods have been presented for numerical solutions of the cubic NLS equation.

Spline functions and numerical methods where splines are used for numerical approximation are also well studied area in applied mathematics. The first mathematical reference to splines is the early work of Schoenberg [19] who revealed that splines have powerful approximation properties. Subsequently, many approximation methods have been employed [20]. A spline function is a sufficiently smooth piecewise function. It possesses a high degree of smoothness at the knots. A B-spline is a special spline function that play an important role in approximation and geometric modeling. They are used in data fitting, computer-aided design, automated manufacturing and computer graphics. In particular, after de Boor’s [18] results about B-splines, spline techniques became popular for a broad range of applications [22]. Most properties and an efficient construction of B-splines can be found in [18]. Due to their some attractive properties such as having compact support and yielding numerical schemes with a high resolving power, B-splines are also widely used in differential problems. Because of having compact support, using B-splines in numerical solution of differential equations leads to sparse matrix systems. The approximation of differential problems with B-splines is obtained by the method of weighted residual, of which the Galerkin and collocation methods are particular cases. The Galerkin method is the most widely used method for B-spline approximations on the other hand, the collocation method represents an economical alternative since it only requires the evaluation at grid points [21]. Exponential B-splines lead to accurate numerical results and there are relatively less studies in which exponential B-splines considered for the approximation. The main objective of this paper is to construct an efficient method with the usage of exponential cubic B-splines for the numerical investigation of cubic NLS equation.

This paper is organized as follows: Section 2 is devoted to the numerical method. Introducing the exponential B-splines and the application of collocation method are given in that section. The numerical testing and the comparisons on the examples are studied in Section 3. Finally, a conclusion is presented in the last section.

2 Numerical method

Let us start with the construction of our mesh, to build on the numerical method on it. For the computational purpose, we should restrict the solution domain from being infinite domain to be a finite interval $[a, b]$. Since the boundary condition (3) indicates that the solutions are negligibly
small outside of a finite interval, instead of physical conditions (3), we can consider the artificial boundary conditions

\[ U(a, t) = U_x(a, t) = U_{xx}(a, t) = 0, \]
\[ U(b, t) = U_x(b, t) = U_{xx}(b, t) = 0. \]

Then the uniform mesh is constructed by

\[ a = x_0 < x_1 < \ldots < x_N = b \]

where \( x_i \) are knots and \( h = x_i - x_{i-1}, i = 1, \ldots, N \) is the mesh size.

### 2.1 Exponential cubic B-splines

Over the above mesh, the exponential cubic B-spline, \( B_i(x) \), is defined by

\[
B_i(x) = \begin{cases} 
  b_2 \left( (x_{i-2} - x) - \frac{1}{p} \sinh(p(x_{i-2} - x)) \right) & [x_{i-2}, x_{i-1}], \\
  a_1 + b_1 (x_i - x) + c_1 \exp(p(x_i - x)) + d_1 \exp(-p(x_i - x)) & [x_{i-1}, x_i], \\
  a_1 + b_1 (x - x_i) + c_1 \exp(p(x - x_i)) + d_1 \exp(-p(x - x_i)) & [x_i, x_{i+1}], \\
  b_2 \left( (x - x_{i+2}) - \frac{1}{p} \sinh(p(x - x_{i+2})) \right) & [x_{i+1}, x_{i+2}], \\
  0 & \text{otherwise.} 
\end{cases}
\]

where

\[
a_1 = \frac{phc}{phc - s}, \quad b_1 = \frac{p}{2} \left( \frac{c(c - 1) + s^2}{(phc - s)(1 - c)} \right), \quad b_2 = \frac{p}{2(phc - s)}, \\
\]
\[
c_1 = \frac{1}{4} \left( \frac{\exp(-ph)(1 - c) + s(\exp(-ph) - 1)}{(phc - s)(1 - c)} \right), \\
\]
\[
d_1 = \frac{1}{4} \left( \frac{\exp(ph)(c - 1) + s(\exp(ph) - 1)}{(phc - s)(1 - c)} \right). 
\]

and \( s = \sinh(ph), c = \cosh(ph) \) and \( p \) is a free parameter that should be determined in computations.

A standard exponential cubic B-spline is shown in Fig.1 for \( p = 1 \). Each \( B_i(x) \) has same shape and same size, so it is easy to see that each exponential cubic B-spline covers four successive interval such that each element is covered by four sequential exponential cubic B-splines. Exponential cubic B-spline \( B_i(x) \) and its first two derivatives are continuous on interval \([x_{i-2}, x_{i+2}]\).
Fig.1: Exponential cubic B-spline for $p = 1$.

The nodal values and the principle two derivatives at the knots are tabulated in Table 1. These values is going to use in application of the numerical method.

|        | $x_{i-2}$ | $x_{i-1}$  | $x_i$     | $x_{i+1}$ | $x_{i+2}$ |
|--------|-----------|------------|-----------|-----------|-----------|
| $B_i$  | 0         | $\frac{s - ph}{2(phc - s)}$ | 1         | $\frac{s - ph}{2(phc - s)}$ | 0         |
| $B'_i$ | 0         | $\frac{p(1 - c)}{2(phc - s)}$ | 0         | $\frac{p(c - 1)}{2(phc - s)}$ | 0         |
| $B''_i$| 0         | $\frac{p^2s}{2(phc - s)}$ | $-\frac{p^2s}{phc - s}$ | $\frac{p^2s}{2(phc - s)}$ | 0         |

2.2 The finite element collocation method

Considering dirac delta functions as the weighted functions in the weighted integral of the residual leads to the situation that the residual at each point $x_i$ in the domain is forced to be exactly zero. Then a system of $N$ residual equations is obtained in finite element collocation method. Therefore the direct substitution of the approximation into the differential equation is the main idea behind our numerical method.
Before the implementation of the numerical scheme, we first decompose the governing equation (1) into its real and imaginary parts as follows:

\[ U(x, t) = r(x, t) + is(x, t), \]  

where \( r(x, t) \) and \( s(x, t) \) are both real valued functions. The above decomposition yields a pair of real-valued equations, i.e.

\[ \begin{align*}
    s_t - r_{xx} - q(r^2 + s^2)r & = 0, \quad (6) \\
    r_t + s_{xx} + q(r^2 + s^2)s & = 0. \quad (7)
\end{align*} \]

Since the set of \( \{B_1(x), B_0(x), \cdots, B_{N+1}(x)\} \) forms a basis for the functions defined over the solution domain, the approximations for both \( r(x, t) \) and \( s(x, t) \) can be constructed as

\[ \begin{align*}
    r_N(x, t) & = \sum_{i=-1}^{N+1} \delta_i(t)B_i(x), & s_N(x, t) & = \sum_{i=-1}^{N+1} \phi_i(t)B_i(x)
\end{align*} \]

where \( \delta_i \) and \( \phi_i \) are time dependent unknown parameters that should be determined from the system of residual equations. The derivatives of these global approximations calculated by

\[ \begin{align*}
    r_N'(x, t) & = \sum_{i=-1}^{N+1} \delta_i(t)B_i'(x), & s_N'(x, t) & = \sum_{i=-1}^{N+1} \phi_i(t)B_i'(x), \\
    r_N''(x, t) & = \sum_{i=-1}^{N+1} \delta_i(t)B_i''(x), & s_N''(x, t) & = \sum_{i=-1}^{N+1} \phi_i(t)B_i''(x).
\end{align*} \]

Usage of these approximations together with the related values in Table 1 gives the following expressions:

\[ \begin{align*}
    r_i = r(x_i, t) & = \frac{s - ph}{2(phc - s)}\delta_{i-1} + \delta_i + \frac{s - ph}{2(phc - s)}\delta_{i+1}, \\
    r_i' = r'(x_i, t) & = \frac{p(1-c)}{2(phc - s)}\delta_{i-1} + \frac{p(c-1)}{2(phc - s)}\delta_{i+1}, \\
    r_i'' = r''(x_i, t) & = \frac{p^2s}{2(phc - s)}\delta_{i-1} - \frac{p^2s}{phc - s}\delta_i + \frac{p^2s}{2(phc - s)}\delta_{i+1}, \quad \text{and}
\end{align*} \]

\[ \begin{align*}
    s_i = s(x_i, t) & = \frac{s - ph}{2(phc - s)}\phi_{i-1} + \phi_i + \frac{s - ph}{2(phc - s)}\phi_{i+1}, \\
    s_i' = s'(x_i, t) & = \frac{p(1-c)}{2(phc - s)}\phi_{i-1} + \frac{p(c-1)}{2(phc - s)}\phi_{i+1}, \\
    s_i'' = s''(x_i, t) & = \frac{p^2s}{2(phc - s)}\phi_{i-1} - \frac{p^2s}{phc - s}\phi_i + \frac{p^2s}{2(phc - s)}\phi_{i+1}.
\end{align*} \]
Time discretization of decomposed system (6) and (7) can be achieved by Crank-Nicolson approximation such that
\[
\frac{s^{n+1} - s^n}{\Delta t} = -\frac{r_{xx}^{n+1} + r_{xx}^n}{2} + q\frac{((r^2 + s^2)r)^{n+1} + ((r^2 + s^2)r)^n}{2} = 0
\]
\[
\frac{r^{n+1} - r^n}{\Delta t} + \frac{s_{xx}^{n+1} + s_{xx}^n}{2} + q\frac{((r^2 + s^2)s)^{n+1} + ((r^2 + s^2)s)^n}{2} = 0
\]
(8)

where \(\Delta t\) is the time step and superscripts denote the time levels. For the numerical purpose, the nonlinear terms in system (8) can be linearized by the technique in [8].

\[
(r^2 + s^2)r^{n+1} = (r^3)^{n+1} + (s^2)^{n+1} = 3(r^n)^2 r^{n+1} - 2(r^n)^3 + 2r^n s^n r^{n+1} + (s^n)^2 r^{n+1} - 2(s^n)^2 r^n,
\]

\[
(r^2 + s^2)s^{n+1} = (r^2 s)^{n+1} + (s^3)^{n+1} = 2s^n r^n r^{n+1} + (r^n)^2 s^{n+1} - 2(r^n)^2 s^n + 3(s^n)^2 s^{n+1} - 2(s^n)^3.
\]

Substitution of the approximations of \(r\) and \(s\) with their related derivatives into system (8) leads to the following fully discretized equations

\[
\nu_{m1}\delta_{m-1}^{n+1} + \nu_{m2}\phi_{m-1}^{n+1} + \nu_{m3}\delta_{m}^{n+1} + \nu_{m4}\phi_{m}^{n+1} + \nu_{m5}\delta_{m+1}^{n+1} + \nu_{m6}\phi_{m+1}^{n+1} = 0
\]
\[
\nu_{m7}\delta_{m-1}^{n} + \nu_{m8}\phi_{m-1}^{n} + \nu_{m9}\delta_{m}^{n} + \nu_{m10}\phi_{m}^{n} + \nu_{m11}\delta_{m+1}^{n} + \nu_{m12}\phi_{m+1}^{n} = 0
\]
(9)

and

\[
\nu_{m13}\delta_{m-1}^{n+1} + \nu_{m14}\phi_{m-1}^{n+1} + \nu_{m15}\delta_{m}^{n+1} + \nu_{m16}\phi_{m}^{n+1} + \nu_{m17}\delta_{m+1}^{n+1} + \nu_{m18}\phi_{m+1}^{n+1} = 0
\]
\[
\nu_{m19}\delta_{m-1}^{n} + \nu_{m20}\phi_{m-1}^{n} + \nu_{m21}\delta_{m}^{n} + \nu_{m22}\phi_{m}^{n} + \nu_{m23}\delta_{m+1}^{n} + \nu_{m24}\phi_{m+1}^{n} = 0
\]
(10)

where

\[
\nu_{m1} = -\Delta t(q(3r^2 + s^2)\alpha_1 + \gamma_1), \quad \nu_{m13} = (2 + 2\Delta tqr)\alpha_1,
\]
\[
\nu_{m2} = (2 - 2\Delta tqr)\alpha_1, \quad \nu_{m14} = \Delta t(q(r^2 + s^2)\alpha_1 + \gamma_1),
\]
\[
\nu_{m3} = -\Delta t(q(3r^2 + s^2)\alpha_2 + \gamma_2), \quad \nu_{m15} = (2 + 2\Delta tqr)\alpha_2,
\]
\[
\nu_{m4} = (2 - 2\Delta tqr)\alpha_2, \quad \nu_{m16} = \Delta t(q(r^2 + s^2)\alpha_2 + \gamma_2),
\]
\[
\nu_{m5} = -\Delta t(q(3r^2 + s^2)\alpha_1 + \gamma_1), \quad \nu_{m17} = (2 + 2\Delta tqr)\alpha_1,
\]
\[
\nu_{m6} = (2 - 2\Delta tqr)\alpha_1, \quad \nu_{m18} = \Delta t(q(r^2 + s^2)\alpha_1 + \gamma_1),
\]
\[
\nu_{m7} = -\Delta tqr^2\alpha_1 + \Delta t\gamma_1, \quad \nu_{m19} = (2 + \Delta tqr)\alpha_1,
\]
\[
\nu_{m8} = (2 - \Delta tqr)\alpha_1, \quad \nu_{m20} = -\Delta tqs^2\alpha_1 - \Delta t\gamma_1,
\]
\[
\nu_{m9} = -\Delta tqr^2\alpha_2 + \Delta t\gamma_2, \quad \nu_{m21} = (2 + \Delta tqr)\alpha_2,
\]
\[
\nu_{m10} = (2 - \Delta tqr)\alpha_2, \quad \nu_{m22} = -\Delta tqs^2\alpha_2 - \Delta t\gamma_2,
\]
\[
\nu_{m11} = -\Delta tqr^2\alpha_1 + \Delta t\gamma_1, \quad \nu_{m23} = (2 + \Delta tqr)\alpha_1,
\]
\[
\nu_{m12} = (2 - \Delta tqr)\alpha_1, \quad \nu_{m24} = -\Delta tqs^2\alpha_1 - \Delta t\gamma_1,
\]

\[
r = \alpha_1\delta_{m-1}^{n} + \alpha_2\delta_{m}^{n} + \alpha_3\delta_{m+1}^{n}, \quad s = \alpha_1\phi_{m-1}^{n} + \alpha_2\phi_{m}^{n} + \alpha_3\phi_{m+1}^{n}
\]

and

\[
\alpha_1 = \frac{s - ph}{2(phc - s)}, \quad \alpha_2 = 1,
\]
\[
\gamma_1 = \frac{p^2 s}{2(phc - s)}, \quad \gamma_2 = -\frac{p^2 s}{phc - s}.
\]
There are $2N+2$ equations and $2N+6$ unknowns in systems (9) and (10). For the solvability of this system, the number of equations and the number of unknown parameters should be equalized. The boundary conditions enable us to eliminate the boundary parameters $\delta_{n+1}^n, \delta_{N+1}^n$ and $\phi_{n+1}^n, \phi_{N+1}^n$ from the system (9) and (10) such that we obtain a solvable matrix system. To start the iteration, determination of the initial parameters $\delta_0^m$ and $\phi_0^m$ are necessary. Once the initial parameters are calculated then the time evolutions of the unknowns are found from the recurrence relationship (9) and (10).

3 Test problems

This section is devoted for the observation of the efficiency of the method so that several test problems are considered in order to illustrate the accuracy. For this purpose we first calculate the possible $L_\infty$ error norm which is defined by

$$L_\infty = \max_i |U_i^{\text{exact}} - U_i^{\text{numerical}}|,$$

Additionally, the invariants of Eq.(1) also give an idea about the accuracy of the method especially in cases that the equation does not have an analytical solution. Although there are infinitely many conservation laws for Eq.(1), here we investigate only the followings:

$$C_1 = \int_a^b |U|^2 \, dx, \quad C_2 = \int_a^b \left( |U_x|^2 - \frac{1}{2} q |U|^4 \right) \, dx,$$

3.1 Single soliton

The function

$$U(x,t) = \alpha \sqrt{2/q} e^{i(\frac{S}{2}x - \frac{1}{2}(S^2-\alpha^2)t) \text{sech}(\alpha(x-St))}$$

represents the single soliton solution of Eq.(1). When $t$ is fixed then the solution (11) decays exponentially as $|x| \to \infty$. The initial and the boundary conditions are inferenced from the above solution.

Fig.2: Single soliton profiles

Fig.3: Error distribution at for $p = 0.0000182$
Eq. (11) gives a wave that moves with speed $S$ and its magnitude is governed by the real parameter $\alpha$. Since it is a useful tool for comparison, this problem is a well known example in the literature. Table 2 presents a detailed comparison on this example for different parameter choices. The solution profiles and the absolute error distribution are illustrated in Figs. 2-3.

| Table 2 |
|---|
| Errors at $t = 1$ for $q = 2$, $S = 4$, $\alpha = 1$ |
| Method | $h$ | $\Delta t$ | $L_\infty$ ($p = 1$) | $L_\infty$ (various $p$) |
| Present | 0,05 | 0,005 | 0,0057 | 0,0015104 | $(p = 0,0000078348)$ |
| | 0,3125 | 0,02 | 1,872 | 0,0064891 | $(p = 0,0000001289)$ |
| | 0,3125 | 0,0026 | 0,1913 | 0,0053 | $(p = 0,0000001289)$ |
| | 0,06 | 0,0165 | 0,0048 | 0,0015 | $(p = 0,0000031976)$ |
| | 0,05 | 0,04 | 0,0168 | 0,0039 | $(p = 0,0000020600)$ |
| Kuintik B-Spline (Saka, 2012) | 0,05 | 0,005 | 0,0003 | |
| | 0,3125 | 0,02 | 0,002 | |
| | 0,3125 | 0,0026 | 0,006 | |
| B-spline Galerkin (Da˘ g, 1999) | 0,05 | 0,005 | 0,0003 | |
| | 0,3125 | 0,02 | 0,002 | |
| B-spline Col. (Gardner vd., 1993) | 0,05 | 0,005 | 0,008 | |
| | 0,03 | 0,005 | 0,002 | |
| Kapalı (C-N) (Taha ve Ablowitz, 1984) | 0,05 | 0,005 | 0,00585 | |
| Split step Fourier (Taha ve Ablowitz, 1984) | 0,3125 | 0,02 | 0,00466 | |
| A-L local (Taha ve Ablowitz, 1984) | 0,06 | 0,0165 | 0,00580 | |
| A-L global (Taha ve Ablowitz, 1984) | 0,05 | 0,04 | 0,00561 | |
| Pseudospectral (Taha ve Ablowitz, 1984) | 0,3125 | 0,0026 | 0,00513 | |

### 3.2 Collision of two solitons that move in opposite directions

A collision of two solitons that travels in opposite directions can be observed with the initial conditions

$$U(x, 0) = U_1(x, 0) + U_2(x, 0),$$

where

$$U_j(x, 0) = \alpha_j \sqrt{2/\pi q} e^{i \left(\frac{2}{\pi} (x-x_j)\right)} \text{sech} \left(\alpha_j (x - x_j)\right), \; j = 1, 2.$$  \hspace{1cm} (13)

The parameters in this test problem are considered as

$q = 2, h = 0,1, \Delta t = 0,005, \alpha_1 = 1,0, S_1 = -4,0, x_1 = 10, \alpha_2 = 1,0, S_2 = 4,0, x_2 = -10$
to coincide with those of some earlier studies. Since there is no available analytical soliton satisfying the given initial condition, only the invariants are considered for the observation of the accuracy. According to the results shown in Table 3, it can be concluded that the method produces acceptable results.

Table 3

| Time | \( C_1 \) | \( C_2 \) |
|------|-----------|-----------|
| \( p = 1 \) | 0.0 | 3.99999 | 14.66577 |
|      | 0.5 | 4.00000 | 14.66634 |
|      | 1.0 | 4.00000 | 14.66706 |
|      | 1.5 | 4.00001 | 14.66761 |
|      | 2.0 | 4.00001 | 14.66694 |
|      | 2.5 | 3.99992 | 14.61149 |
|      | 3.0 | 4.00003 | 14.66809 |
|      | 3.5 | 4.00003 | 14.66803 |
|      | 4.0 | 4.00004 | 14.66766 |
|      | 4.5 | 4.00005 | 14.66734 |
|      | 5.0 | 4.00005 | 14.66705 |
|      | 5.5 | 4.00006 | 14.66691 |
|      | 6.0 | 4.00007 | 14.66690 |

\[ \lambda = 0 \]

\[ \lambda = 6.0 \]

\[ \lambda = 4.00000 \]

Eq. (13) represents two solitons having equal magnitudes and velocities that are 1 and 4 respectively. One of these waves is located at \( x = -10 \) whereas the other is at \( x = 10 \). The waves move in opposite directions starting from their mentioned initial locations. Fig. 4 illustrates the before and the later positions of interacting waves. On the other hand the collision is monitored in Fig. 5. As an expected situation, these figures show that the solitons keep their initial profile and properties after the collision.
3.3 Birth of soliton

Here we focus on two different examples which are the birth of standing soliton and the birth of mobile soliton.

3.3.1 Birth of standing soliton

According to the theory, an initial condition satisfying

$$C = \int_{-\infty}^{\infty} U(x, 0) dx \geq \pi,$$
results a soliton in time process. Otherwise the soliton decays away. The verification can be seen for some different numerical methods in [6, 5, 4]. This observation can be simulated with the Maxwellian initial condition

$$U(x, 0) = Ae^{-x^2}$$ (14)

or $C = \sqrt{\pi}A$, so the usege of (14) with $A \geq \sqrt{\pi} \approx 1.7725$ produces a soliton whereas the choice of $A < \sqrt{\pi}$ yields a fading out initial condition. Figs.7 and 8 show this procedure. The invariants are listed in Table 4 for comparison. The analytical invariants are

$$C_1 = A^2 \sqrt{\frac{\pi}{2}} = 3.9710,$$

$$C_2 = \frac{1}{4} A^2 (2\sqrt{2} - qA^2) \sqrt{\pi} = -4.9256,$$

so it can be concluded that the presented results are in good agreement with the exact invariants.

**Fig7:** Formation of standing solution for $A = 1$  
**Fig8:** Formation of standing solution for $A = 1.78$

| Time | $C_1$    | $C_2$    |
|------|----------|----------|
| 0    | 3.97100  | -4.92563 |
| 2    | 3.97091  | -4.93275 |
| 4    | 3.97076  | -4.93191 |
| 6    | 3.97062  | -4.93128 |
| 6    | 3.97093  | -4.92672 |
3.3.2 Birth of mobile soliton

A mobile soliton can be studied with the Maxwellian initial condition

\[ U(x, 0) = Ae^{-x^2 + 2ix} \]  \hspace{1cm} (15)

which produces a mobile soliton having the velocity 4 and height 2 with parameter choice \( A = 1.78 \). The same parameter choices with the previous problem form a soliton that has the peak position at \( x = 24 \) when \( t = 6 \). The travelling soliton wave is graphed in Fig.9 for \( A = 1.78 \). As seen from Fig.10 that the case \( A = 1 \) does not produce any soliton.

\[ \text{Fig9: Formation of travelling solution for } A = 1 \quad \text{Fig10: Formation of travelling solution for } A = 1.78 \]

Analytical invariants for this case are calculated as

\[ C_1 = \sqrt{\frac{\pi}{2}} A^2 = 3.97100, \]
\[ C_2 = 5\sqrt{\frac{\pi}{2}} A^2 - \sqrt{\frac{\pi}{4}} qA^4 = 10.95838. \]

Table 5 indicates that \( C_1 \) has been found to be constant and there is variation in \( C_2 \) less than 1.95%.
Table 5
Invariants for $A = 1.78$ and $-45 \leq x \leq 45$, $N = 1334$, $\Delta t = 0.005$, $q = 2$.

| Time | $C_1$     | $C_2$     |
|------|-----------|-----------|
| $p = 1$ | 0.0   | 3.97100  | 10.95788 |
|      | 2.0   | 3.97113  | 10.94128 |
|      | 4.0   | 3.97112  | 10.94298 |
|      | 6.0   | 3.97113  | 10.94343 |

$[7]$ $\lambda = 0$ | 4.0 | 3.97087 | 10.95432 |

3.4 Bound state solution

It is stated in [9] that if the condition

$q = 2M^2, M = 1, 2, ...$

holds then the initial condition

$U(x, 0) = \text{sech}(x)$

evolves $M$ soliton waves. Illustrations of these solitons are depicted in Figs.11-18 where the each wave profile and their trajectories, to see the density variations, are given alongside. The numerical computations have been carried out for

$h = 0.03, \Delta t = 0.005, N = 1334, q = 32, 50, -20 \leq x \leq 20$. 

![Fig.11: Wave profiles for $M = 4$](image1)

![Fig.12: Trajectories for $M = 4$](image2)
Fig. 13: Wave profiles for $M = 5$

Fig. 14: Trajectories for $M = 5$

Fig. 15: Wave profiles for $M = 6$

Fig. 16: Trajectories for $M = 6$
4 Conclusion

In this paper, exponential cubic B-spline collocation method is implemented in order to get the solution of the cubic nonlinear Schrödinger equation. Over the uniform mesh, Crank-Nicolson formulas are employed for time discretization whereas Rubin and Graves\cite{8} technique is used for the linearization. Four test problems that related to single soliton wave, interaction of two opposite solitons, birth of soliton and the bound state solution are examined for testing the numerical scheme. Comparisons between the obtained results and some earlier papers show that the present results are all acceptable and in agreement with those in the literature. Simple adaptation and yielding band matrices can be stated as the advantages of the method. On the other hand, requiring the determination of the free parameter $p$ is an undesirable situation. In conclusion, exponential cubic B-spline collocation method can be considered as a conservative numerical method that leads to reasonable results.

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