Effects of Noise on Entropy Evolution

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We study the convergence properties of the conditional (Kullback-Leibler) entropy in stochastic systems. We have proved very general results showing that asymptotic stability is a necessary and sufficient condition for the monotone convergence of the conditional entropy to its maximal value of zero. Additionally we have made specific calculations of the rate of convergence of this entropy to zero in a one-dimensional situations, illustrated by Ornstein-Uhlenbeck and Rayleigh processes, higher dimensional situations, and a two dimensional Ornstein-Uhlenbeck process with a stochastically perturbed harmonic oscillator and colored noise as examples. We also apply our general results to the problem of conditional entropy convergence in the presence of dichotomous noise. In both the single dimensional and multidimensional cases we are to show that the convergence of the conditional entropy to zero is monotone and at least exponential. In the specific cases of the Ornstein-Uhlenbeck and Rayleigh processes as well as the stochastically perturbed harmonic oscillator and colored noise examples, we have the rather surprising result that the rate of convergence of the entropy to zero is independent of the noise amplitude $\sigma$ as long as $\sigma > 0$.

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I. INTRODUCTION

This paper examines the role of noise in the evolution of the conditional (or Kullback-Leibler or relative) entropy to a maximum. We were led to examine this problem because it is known that in invertible systems (e.g. measure preserving systems of differential equations or invertible maps) the conditional entropy is fixed at the value with which the system is prepared [1, 2, 3, 4], but that the addition of noise can reverse this invertibility property and lead to an evolution of the conditional entropy to a maximum value of zero. Here, we make both general and concrete specific calculations to examine the entropy convergence. We carry this out by using convergence properties of the Fokker-Planck equation using ‘entropy methods’ [5], which have been known for some time to be useful for problems involving questions related to convergence of solutions in partial differential equations [6, 7, 8, 9, 10]. Their utility can be traced, in some instances, to the fact that entropy may serve as a Liapunov functional [11].

The outline of this paper is as follows. Section II introduces the dynamic concept of asymptotic stability and the notion of conditional entropy. (Asymptotic stability is a strong convergence property of ensembles which implies mixing. Mixing, in turn, implies ergodicity.) This is followed by two main results connecting the convergence of the conditional entropy with asymptotic stability (Theorem 1), and the existence of unique stationary densities with the convergence of the conditional entropy (Theorem 2). Section II shows that asymptotic stability is a property that cannot be found in a deterministic system of ordinary differential equations, and consequently that the conditional entropy does not have a dynamic (time dependent) character for this type of invertible dynamics. Section IV considers the stochastic extension where a system of ordinary equations is perturbed by Gaussian white noise (thus becoming non-invertible) and gives some general results showing that in this stochastic case asymptotic stability holds. Section IV A starts to consider specific one dimensional examples, and we show that the conditional entropy convergence to zero is at least exponential. We consider the Ornstein-Uhlenbeck process in Section IV A 1 and a Rayleigh process in Section IV A 2. We consider multidimensional stochastic systems in Section IV B, showing that the exponential convergence of the entropy still holds. Specific examples of these higher dimensional situations are considered within the context of a two dimensional Ornstein-Uhlenbeck process in Section IV C with specific examples of a stochastically perturbed harmonic oscillator (Section IV C 1) and colored noise (Section IV C 2) as examples. Section V applies our
general results to the problem of conditional entropy convergence in the presence of dichotomous noise. The paper concludes with a short discussion.

II. ASYMPTOTIC STABILITY AND CONDITIONAL ENTROPY

Let \((X, \mathcal{B}, \mu)\) be a \(\sigma\)-finite measure space. Let \(\{P^t\}_{t \geq 0}\) be a semigroup of Markov operators on \(L^1(X, \mu)\), i.e. \(P^t f \geq 0\) for \(f \geq 0\), \(\int P^t f(x) \mu(dx) = \int f(x) \mu(dx)\), and \(P^{t+s} f = P^t (P^s f)\). If the group property holds for \(t, s \in \mathbb{R}\), then we say that \(P^t\) is invertible, while if it holds only for \(t, s \in \mathbb{R}^+\) we say that \(P^t\) is non-invertible. We denote the corresponding set of densities by \(\mathcal{D}(X, \mu)\), or \(\mathcal{D}\) when there will be no ambiguity, so \(f \in \mathcal{D}\) means \(f \geq 0\) and \(\int_X f(x) \mu(dx) = 1\). We call a semigroup of Markov operators \(P^t\) on \(L^1(X, \mu)\) asymptotically stable if there is a density \(f_*\) such that

\[
\lim_{t \to \infty} P^t f = f_*.
\]

The density \(f_*\) is called a stationary density of \(P^t\).

We define the conditional entropy (also known as the Kullback-Leibler or relative entropy) of two densities \(f, g \in \mathcal{D}\) as

\[
H_c(f|g) = - \int_X f(x) \log \left( \frac{f(x)}{g(x)} \right) \mu(dx). \tag{2.1}
\]

Our first result connects the temporal convergence properties of \(H_c\) with those of \(P^t\).

**Theorem 1** Let \(P^t\) be a semigroup of Markov operators on \(L^1(X, \mu)\) and \(f_*\) be a positive density. If

\[
\lim_{t \to \infty} H_c(P^t f|f_*) = 0 \tag{2.2}
\]

for a density \(f\) then

\[
\lim_{t \to \infty} P^t f = f_* \tag{2.3}
\]

Conversely, if \(P^t f_* = f_*\) for all \(t \geq 0\) and Condition 2.3 holds for all \(f\) with \(H_c(f|f_*) > -\infty\), then Condition 2.2 holds as well.

**Proof.** To prove the first part, note that for densities \(f, g \in \mathcal{D}\), we have from the Csizá-Kullback inequality

\[
-\|f - g\|_{L^1}^2 \geq 2 H_c(f|g).
\]

In particular,

\[
-\|P^t f - f_*\|_{L^1}^2 \geq 2 H_c(P^t f|f_*).
\]

Thus if

\[
\lim_{t \to \infty} H_c(P^t f|f_*) = 0,
\]

then

\[
\lim_{t \to \infty} \|P^t f - f_*\|_{L^1} = 0
\]

and \(P^t\) is asymptotically stable.

Proof of the converse portion is not so straightforward. Assume initially that \(f/f_*\) is bounded, so \(0 \leq f \leq af_*\). Since \(P^t f_* = f_*\) and \(P^t\) is a positive preserving operator, we have \(0 \leq P^t f \leq af_*\). Making use of the inequality, proved in the appendix,

\[
-H_c(f|g) \leq \int \left( \frac{f}{g} - 1 \right)^2 g \mu(dx), \tag{2.4}
\]
valid for all densities $f, g$, we arrive at

$$-H_c(P^t f | g) \leq \int \frac{P^t f}{f^*} - 1 \left| P^t f - f_* \right| \mu(dx) \leq \max\{1, |a-1|\} \left| P^t f - f_* \right|_1.$$ 

Consequently, $H_c(P^t f | f_*) \to 0$ as $t \to \infty$ for every density $f$ such that $f/f_*$ is bounded.

Now assume that $H_c(f | f_*) > -\infty$ and write $f$ in the form $f = g_a + f_a$ where $g_a = f - f_a$ and

$$f_a(x) = \begin{cases} 0, & f(x) > af_*(x), \\ f(x), & 0 \leq f(x) \leq af_*(x). \end{cases}$$

We have $1 = ||f||_1 = ||g_a||_1 + ||f_a||_1$ and $||g_a||_1 = \int_{f > af_*} f \mu(dx) \to 0$ as $a \to \infty$. Write $P^t f$ in the form

$$P^t f = ||g_a||_1 P^t \left( \frac{g_a}{||g_a||_1} \right) + ||f_a||_1 P^t \left( \frac{f_a}{||f_a||_1} \right). \tag{2.5}$$

Since $\frac{f_a}{f_*/||f_a||_1}$ is bounded, we have $H_c \left( \frac{f_a}{||f_a||_1} | f_* \right) > -\infty$ and

$$\lim_{t \to \infty} H_c \left( P^t \left( \frac{f_a}{||f_a||_1} \right) | f_* \right) = 0$$

by the first part of our proof. Observe that

$$H_c \left( \frac{g_a}{||g_a||_1} | f_* \right) = \int_{f > af_*} f \log \frac{f}{f_*} \mu(dx),$$

which, because of the condition $H_c(f | f_*) > -\infty$, is convergent to $0$ as $a \to \infty$. Furthermore $H_c(P^t f | f_*) \geq H_c(f | f_*)$ for any density $f$ and the function $f \mapsto H_c(f | f_*)$ is concave. Combining these results and equality (2.5) we obtain

$$H_c(P^t f | f_*) \geq ||g_a||_1 H_c \left( P^t \left( \frac{g_a}{||g_a||_1} \right) | f_* \right) + ||f_a||_1 H_c \left( P^t \left( \frac{f_a}{||f_a||_1} \right) | f_* \right) \geq ||g_a||_1 H_c \left( \frac{g_a}{||g_a||_1} | f_* \right) + ||f_a||_1 H_c \left( P^t \left( \frac{f_a}{||f_a||_1} \right) | f_* \right).$$

Letting $t \to \infty$, followed by taking the limit $a \to \infty$ completes the proof of the if part, and of the theorem.

The semigroup $P^t$ on $L^1(X, \mu)$ is called *partially integral* if there exists a measurable function $q : X \times X \to [0, \infty)$ and $t_0 > 0$ such that

$$P^{t_0} f(x) \geq \int_X q(x, y) f(y) \mu(dy)$$

for every density $f$ and

$$\int_X \int_X q(x, y) \mu(dy) \mu(dx) > 0.$$

Our next result draws a connection between the existence of a unique stationary density $f_*$ of $P^t$ and the convergence of the conditional entropy $H_c$.

**Theorem 2** Let $P^t$ be a partially integral semigroup of Markov operators. If there is a unique stationary density $f_*$ for $P^t$ and $f_*>0$, then

$$\lim_{t \to \infty} H_c(P^t f_0 | f_*) = 0$$

for all $f_0$ with $H_c(f_0 | f_*) > -\infty$.

**Proof.** From Pichór and Rudnicki [12, Theorem 2] the semigroup $P^t$ is asymptotically stable if there is a unique stationary density $f_*$ for $P^t$ and $f_*>0$. Thus the result follows from Theorem [1].
III. CONDITIONAL ENTROPY IN INVERTIBLE DETERMINISTIC SYSTEMS

In this section we briefly consider the behaviour of the conditional entropy in situations where the dynamics are invertible in the sense that they can be run forward or backward in time without ambiguity. To make this clearer, consider a phase space \( X \) and a dynamics \( S_t : X \to X \). For every initial point \( x^0 \), the sequence of successive points \( S_t(x^0) \), considered as a function of time \( t \), is called a trajectory. In the phase space \( X \), if the trajectory \( S_t(x^0) \) is nonintersecting with itself, or intersecting but periodic, then at any given final time \( t_f \) such that \( x^{t_f} = S_{t_f}(x^0) \) we could change the sign of time by replacing \( t \) by \(-t\), and run the trajectory backward using \( x^{t_f} \) as a new initial point in \( X \). Then our new trajectory \( S_{-t}(x^{t_f}) \) would arrive precisely back at \( x^0 \) after a time \( t_f \) had elapsed: \( x^0 = S_{-t_f}(x^{t_f}) \). Thus in this case we have a dynamics that may be reversed in time completely unambiguously.

We formalize this by introducing the concept of a dynamical system \( \{S_t\}_{t \in \mathbb{R}} \), which is simply any group of transformations \( S_t : X \to X \) having the two properties: 1. \( S_0(x) = x \); and 2. \( S_t(S_{t'}(x)) = S_{t+t'}(x) \) for \( t, t' \in \mathbb{R} \) or \( \mathbb{Z} \). Since, from the definition, for any \( t \in \mathbb{R} \), we have \( S_t(S_{-t}(x)) = x = S_{-t}(S_t(x)) \), it is clear that dynamical systems are invertible in the sense discussed above since they may be run either forward or backward in time. Systems of ordinary differential equations are examples of dynamical systems as are invertible maps.

Our first result is very general, and shows that the conditional entropy of any invertible system is constant and uniquely determined by the method of system preparation. This is formalized in

**Theorem 3** If \( P^t \) is an invertible Markov operator and has a stationary density \( f_* \), then the conditional entropy is constant and equal to the value determined by \( f_* \) and the choice of the initial density \( f_0 \) for all time \( t \). That is,

\[
H_c(P^t f_0|f_*) = H_c(f_0|f_*) \tag{3.1}
\]

for all \( t \).

**Proof.** Since \( P \) is invertible, by Voigt’s theorem \[26\] with \( g = f_* \) it follows that

\[
H_c(P^{t+t'} f_0|f_*) = H_c(P^{t'} P^t f_0|f_*) \geq H_c(P^t f_0|f_*) \geq H_c(f_0|f_*)
\]

for all times \( t \) and \( t' \). Pick \( t' = -t \) so

\[
H_c(f_0|f_*) \geq H_c(P^t f_0|f_*) \geq H_c(f_0|f_*)
\]

and therefore

\[
H_c(P^t f_0|f_*) = H_c(f_0|f_*)
\]

for all \( t \), which finishes the proof.

In the case where we are considering a deterministic dynamics \( S^t : \mathcal{X} \to \mathcal{X} \) where \( \mathcal{X} \subset \mathcal{X} \), then the corresponding Markov operator is also known as the Frobenius Perron operator \[12\], and is given explicitly by

\[
P^t f_0(x) = f_0(S^{-t}(x)) |J^{-t}(x)|\tag{3.2}
\]

where \( J^{-t}(x) \) denotes the Jacobian of \( S^{-t}(x) \). A simple calculation shows

\[
H_c(P^t f_0|f_*) = - \int_{\mathcal{X}} P^t f_0(x) \log \left[ \frac{P^t f_0(x)}{f_*(x)} \right] dx
\]

\[
= - \int_{\mathcal{X}} f_0(S^{-t}(x)) |J^{-t}(x)| \log \left[ \frac{f_0(S^{-t}(x))}{f_*(S^{-t}(x))} \right] dx
\]

\[
= - \int_{\mathcal{X}} f_0(y) \log \left[ \frac{f_0(y)}{f_*(y)} \right] dy
\]

\[
= H_c(f_0|f_*)
\]

as expected from Theorem 3

More specifically, if the dynamics corresponding to our invertible Markov operator are described by the system of ordinary differential equations

\[
\frac{dx_i}{dt} = F_i(x) \quad i = 1, \ldots, d \tag{3.3}
\]
operating in a region of $\mathbb{R}^d$ with initial conditions $x_i(0) = x_{i,0}$, then the evolution of $f(t, x) \equiv P^t f_0(x)$ is governed by the generalized Liouville equation

$$\frac{\partial f}{\partial t} = -\sum_i \frac{\partial (f F_i)}{\partial x_i}. \tag{3.4}$$

The corresponding stationary density $f_*$ is given by the solution of

$$\sum_i \frac{\partial (f_* F_i)}{\partial x_i} = 0. \tag{3.5}$$

Note that the uniform density $f_* \equiv 1$, meaning that the flow defined by Eq. 3.3 preserves the Lebesque measure, is a stationary density of Eq. 3.4 if and only if

$$\sum_i \frac{\partial F_i}{\partial x_i} = 0. \tag{3.6}$$

In particular, for the system of ordinary differential equations (3.3) whose density evolves according to the Liouville equation (3.4) we can assert that the conditional entropy of the density $P^t f_0$ with respect to the stationary density $f_*$ will be constant for all time and will have the value determined by the initial density $f_0$ with which the system is prepared. This result can also be proved directly by noting that from the definition of the conditional entropy we may write

$$H_c(f|f_*) = -\int_{\mathbb{R}^d} f(x) \left[ \log \left( \frac{f}{f_*} \right) + \frac{f_*}{f} - 1 \right] dx$$

when the stationary density is $f_*$. Differentiating with respect to time gives

$$\frac{dH_c}{dt} = -\int_{\mathbb{R}^d} \frac{df}{dt} \log \left( \frac{f}{f_*} \right) dx$$

or, after substituting from (3.4) for $(\partial f/\partial t)$, and integrating by parts under the assumption that $f$ has compact support,

$$\frac{dH_c}{dt} = \int_{\mathbb{R}^d} \frac{f}{f_*} \sum_i \frac{\partial (f_* F_i)}{\partial x_i} dx.$$ 

However, since $f_*$ is a stationary density of $P^t$, it is clear from (3.4) that

$$\frac{dH_c}{dt} = 0,$$

and we conclude that the conditional entropy $H_c(P^t f_0|f_*)$ does not change from its initial value when the dynamics evolve in this manner.

**IV. EFFECTS OF GAUSSIAN NOISE**

In this section, we turn our attention to the behaviour of the stochastically perturbed system

$$\frac{dx_i}{dt} = F_i(x) + \sum_{j=1}^d \sigma_{ij}(x) \xi_j, \quad i = 1, \ldots, d \tag{4.1}$$

with the initial conditions $x_i(0) = x_{i,0}$, where $\sigma_{ij}(x)$ is the amplitude of the stochastic perturbation and $\xi_j = \frac{dw_j}{dt}$ is a “white noise” term that is the derivative of a Wiener process. In matrix notation we can rewrite Eq. 4.1 as

$$dx(t) = F(x(t)) dt + \Sigma(x(t)) dw(t), \tag{4.2}$$

where $\Sigma(x) = [\sigma_{ij}(x)]_{i,j=1,\ldots,d}$. Here it is always assumed that the Itô, rather that the Stratonovich, calculus, is used. For a discussion of the differences see [14], [13] and [15]. In particular, if the $\sigma_{ij}$ are independent of $x$ then the Itô and the Stratonovich approaches yield identical results.

To fully interpret [12] we briefly review the properties of the $\xi$ when they are derived from a Wiener process. We say that a continuous process $\{w(t)\}_{t>0}$ is a one dimensional Wiener process if:
1. \( w(0) = 0; \) and

2. For all values of \( s \) and \( t, 0 \leq s \leq t \) the random variable \( w(t) - w(s) \) has the Gaussian density

\[
g(t - s, x) = \frac{1}{\sqrt{2\pi(t - s)}} \exp \left[ -\frac{x^2}{2(t - s)} \right]. \tag{4.3}\]

In a completely natural manner this definition can be extended to say that the \( d \)-dimensional vector

\[
w(t) = \{w_1(t), \cdots, w_d(t)\}_{t \geq 0}
\]

is a \( d \)-dimensional \textit{Wiener process} if its components are one dimensional Wiener processes. Because of the independence of the increments, it is elementary that the joint density of \( w(t) \) is

\[
g(t, x_1, \cdots, x_d) = g(t, x_1) \cdots g(t, x_d) \quad \tag{4.4}
\]

and thus

\[
\int_{\mathbb{R}^d} g(t, x) \, dx = 1, \tag{4.5}
\]

that

\[
\int_{\mathbb{R}^d} x_i g(t, x) \, dx = 0, \quad i = 1, \cdots, d \tag{4.6}
\]

and

\[
\int_{\mathbb{R}^d} x_i x_j g(t, x) \, dx = \delta_{ij} t \quad i, j = 1, \cdots, d. \tag{4.7}
\]

In (4.7),

\[
\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{4.8}
\]

is the Kronecker delta. Therefore the average of a Wiener variable is zero by Eq. 4.6 while the variance increases linearly with time according to 4.7.

The \textit{Fokker-Planck equation} that governs the evolution of the density function \( f(t, x) \) of the process \( x(t) \) generated by the solution to the stochastic differential equation 4.2 is given by

\[
\frac{\partial f}{\partial t} = -\sum_{i=1}^{d} \frac{\partial F_i(x) f}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2[a_{ij}(x) f]}{\partial x_i \partial x_j} \tag{4.9}
\]

where

\[
a_{ij}(x) = \sum_{k=1}^{d} \sigma_{ik}(x) \sigma_{jk}(x).
\]

If \( k(t, x, x_0) \) is the fundamental solution of the Fokker-Planck equation, i.e. for every \( x_0 \) the function \( (t, x) \mapsto k(t, x, x_0) \) is a solution of the Fokker-Planck equation with the initial condition \( \delta(x - x_0) \), then the general solution \( f(t, x) \) of the Fokker-Planck equation 4.9 with the initial condition

\[
f(x, 0) = f_0(x)
\]

is given by

\[
f(t, x) = \int k(t, x, x_0) f_0(x_0) \, dx_0. \tag{4.10}
\]
From a probabilistic point of view $k(t,x,x_0)$ is a stochastic kernel (transition density) and describes the probability of passing from the state $x_0$ at time $t=0$ to the state $x$ at a time $t$. Define the Markov operators $P^t$ by

$$P^t f_0(x) = \int k(t,x,x_0)f_0(x_0) \, dx_0, \quad f_0 \in L^1.$$  (4.11)

Then $P^t f_0$ is the density of the solution $x(t)$ of Eq. (4.2) provided that $f_0$ is the density of $x(0)$.

The steady state density $f_*$ (x) is the stationary solution of the Fokker Planck Eq. (4.9):

$$- \sum_{i=1}^d \frac{\partial [F_i(x)f]}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2 [a_{ij}(x)f]}{\partial x_i \partial x_j} = 0.$$  (4.12)

In the specific case of $X = \mathbb{R}^d$ and $\mu$ equal to the Lebesgue measure, we recover from Eq. (4.1) the conditional entropy

$$H_c(f(t)|f_*) = - \int_{\mathbb{R}^d} f(t,x) \ln \left[ \frac{f(t,x)}{f_*(x)} \right] \, dx.$$  (4.13)

If the coefficients $a_{ij}$ and $F_i$ are sufficiently regular so that a fundamental solution $k$ exists, and $\int X k(t,x,y) \, dx = 1$, then the unique generalized solution (4.10) to the Fokker-Planck equation (4.9) is given by Eq. (4.11). One such set of conditions is the following: (1) the $F_i$ are of class $C^2$ with bounded derivatives; (2) the $a_{ij}$ are of class $C^3$ and bounded with all derivatives bounded; and (3) the uniform parabolicity condition holds, i.e. there exists a strictly positive constant $\rho > 0$ such that

$$\sum_{i,j=1}^d a_{ij}(x)\lambda_i \lambda_j \geq \rho \sum_{i=1}^d \lambda_i^2, \quad \lambda_i, \lambda_j \in \mathbb{R}, x \in \mathbb{R}^d.$$  

The uniform parabolicity condition implies that $k(t,x,y) > 0$, and thus $P^t f(x) > 0$ for every density, which implies that there can be at most one stationary density, and that if it exists then $f_* > 0$. In this setting, the corresponding conditional entropy $H_c(P^t f_0|f_*)$ approaches its maximal value of zero, as $t \to \infty$, if and only if there is a stationary density $f_*$ that satisfies (4.12).

### A. The one dimensional case

If we are dealing with a one dimensional system, $d = 1$, then the stochastic differential Eq. (4.1) simply becomes

$$\frac{dx}{dt} = F(x) + \sigma(x)\xi,$$  (4.14)

where again $\xi$ is a (Gaussian distributed) perturbation with zero mean and unit variance, and $\sigma(x)$ is the amplitude of the perturbation. The corresponding Fokker-Planck equation (4.9) becomes

$$\frac{\partial f}{\partial t} = \frac{\partial [F(x)f]}{\partial x} + \frac{1}{2} \frac{\partial^2 [\sigma^2(x)f]}{\partial x^2}.$$  (4.15)

The Fokker-Planck equation can also be written in the equivalent form

$$\frac{\partial f}{\partial t} = - \frac{\partial S}{\partial x}$$  (4.16)

where

$$S = - \frac{1}{2} \frac{\partial [\sigma^2(x)f]}{\partial x} + F(x)f$$  (4.17)

is called the probability current.

When stationary solutions of (4.15), denoted by $f_*(x)$ and defined by $P_t f_* = f_*$ for all $t$, exist they are given as the generally unique (up to a multiplicative constant) solution of (4.12) in the case $d = 1$:

$$- \frac{\partial [F(x)f_*]}{\partial x} + \frac{1}{2} \frac{\partial^2 [\sigma^2(x)f_*]}{\partial x^2} = 0.$$  (4.18)
Integration of Eq. 4.18 by parts with the assumption that the probability current \( S \) vanishes at the integration limits, followed by a second integration, yields the solution

\[
f_*(x) = \frac{K}{\sigma^2(x)} \exp \left[ \int^x \frac{2F(z)}{\sigma^2(z)} \, dz \right]. \tag{4.19}
\]

This stationary solution \( f_* \) will be a density if and only if there exists a positive constant \( K > 0 \) such that \( f_* \) can be normalized.

We now discuss rigorous results concerning the one dimensional case. Let \( a(x) = \sigma^2(x) \). Assume that \( a, a', \) and \( F \) are continuous on \( X = (\alpha, \beta) \), where \( \infty \leq \alpha < \beta \leq \infty \). Let \( a(x) > 0 \) for all \( x \in (\alpha, \beta) \) and \( x_0 \) be any point in \( (\alpha, \beta) \).

If we want to study the long term behaviour, as \( t \to \infty \), of the process \( x(t) \) given by Eq. 4.14 we need to know that the process exists for all \( t > 0 \); in other words that there is no explosion in finite time, and that it lives in the interval \( (\alpha, \beta) \). If, for example, \( \alpha \) were absorbing so that with positive probability we could reach \( \alpha \) in finite time, then a solution \( f \) of Eq. 4.14 would show a decrease in norm in \( L^1(\alpha, \beta) \).

There is a relation between the behaviour of \( x(t) \) at the boundary points \( \alpha \) and \( \beta \), and the existence and uniqueness of solutions of the corresponding Fokker-Planck equation as described by \[16, 17\]. In particular, if the condition

\[
\int_{\alpha}^{x_0} \exp \left[ - \int_{x_0}^x \frac{2F(z)}{\sigma^2(z)} \, dz \right] \, dx = \int_{x_0}^\beta \exp \left[ - \int_{x_0}^x \frac{2F(z)}{\sigma^2(z)} \, dz \right] \, dx = \infty \tag{4.20}
\]

holds, then the generalized solutions \( P^t f \) of the Fokker-Planck equation exist and constitute a semigroup of Markov operators on \( L^1(\alpha, \beta) \). If only one of the integrals is finite then this conclusion holds as well, but we restrict ourselves to Condition 4.20 because it is also necessary for the existence of a stationary solution of Eq. 4.14 c.f. Pinsky \[18\].

Thus we conclude that there is a stationary density \( f_* \) if and only if Condition 4.20 holds and

\[
\int_{\alpha}^{\beta} \frac{1}{\sigma^2(x)} \exp \left[ \int_{x_0}^x \frac{2F(z)}{\sigma^2(z)} \, dz \right] \, dx < \infty. \tag{4.21}
\]

\[S\] provides a general method of establishing exponential convergence to zero of the relative entropy for the Fokker-Planck equation in a divergence form. Here, we adapt the arguments of \[S\] to the one-dimensional case with \( X = (\alpha, \beta) \). Define

\[
D(x) = \frac{\sigma^2(x)}{2} \quad \text{and} \quad B(x) = -\ln \frac{K}{\sigma^2(x)} - \int_{x_0}^{x} \frac{2F(z)}{\sigma^2(z)} \, dz,
\]

where \( K \) is such that \( f_*(x) = e^{-B(x)} \) is normalized. Then the Fokker-Planck equation \[4.15\] can be rewritten as

\[
\frac{\partial f}{\partial t} = \frac{\partial}{\partial x} \left( D(x) \left( \frac{\partial f}{\partial x} + B'(x)f \right) \right) \tag{4.22}
\]

and

\[
f_*(x) = e^{-B(x)}. \]

We are going to show that if there exists a constant \( \lambda > 0 \) such that for every \( x \in X \) the following inequality holds

\[
\frac{1}{2} D(x) B''(x) + D'(x) B'(x) + \frac{(D'(x))^2}{4D(x)} - \frac{1}{2} D''(x) \geq \lambda \tag{4.23}
\]

then

\[
H_\epsilon(P^t f_0|f_*) \geq e^{-2\lambda t} H_\epsilon(f_0|f_*) \tag{4.24}
\]

for all initial densities \( f_0 \) with \( H_\epsilon(f_0|f_*) > -\infty \).

The method of proof of Inequality 4.24 exploits Eq. 4.23 From Eq. 4.22 it follows that

\[
\frac{\partial f}{\partial t} = \frac{\partial}{\partial x} \left( D e^{-B} g \right) \tag{4.25}
\]

where

\[
g = \frac{\partial}{\partial x} \left( e^B f \right). \]
Let \( \psi(z) = -z \log z \) and \( H_c(t) = H_c(f|f_0) \), where \( f(t, x) = P^t f_0(x) \). Then we have

\[
H_c(t) = \int_X \psi(e^B f) e^{-B} dx.
\]

Differentiate \( H_c \) with respect to \( t \)

\[
\frac{dH_c(t)}{dt} = \int_X \psi'(e^B f) \frac{\partial f}{\partial t} dx,
\]

use Eq. 4.25 and integrate by parts to obtain

\[
\frac{dH_c(t)}{dt} = -\int_X \psi''(e^B f) De^{-B} g^2 dx.
\]

Now write \( I(t) = \frac{dH_c(t)}{dt} \) and differentiate \( I(t) \)

\[
\frac{dI(t)}{dt} = -2 \int_X \psi'''(e^B f) De^{-B} g^2 \left( \frac{1}{2} D'' - D'B' - D'B'' - \frac{(D')^2}{4D} \right) dx + \int_X \text{Tr}(Y \cdot Z)e^{-B} dx
\]

where

\[
Y = \begin{pmatrix} 2\psi''(e^B f) & 2\psi'''(e^B f) \\ 2\psi'''(e^B f) & \psi''(e^B f) \end{pmatrix}
\]

and

\[
Z = \begin{pmatrix} D\frac{\partial g}{\partial x} + \frac{1}{2} D'y' \\ D\frac{\partial g}{\partial x} g^2 + \frac{1}{2} DD'y' \end{pmatrix}.
\]

Since \( -\psi''(e^B f) De^{-B} g^2 \) is nonnegative, from Eq. 4.23 it follows that the first integral is less than \( 2\lambda \int_X \psi'''(e^B f) De^{-B} g^2 dx = -2\lambda I(t) \). Since \( Y \) is a negative definite matrix and \( Z \) is a nonnegative definite matrix, we obtain \( \text{Tr}(Y \cdot Z) \leq 0 \) which completes the proof of the inequality

\[
\frac{dI(t)}{dt} \leq -2\lambda I(t)
\]

Integrating from \( t = s \) to \( t = \infty \) and noting that \( I(t) \) and \( H_c(P^t f_0|f_0) \) go to zero as \( t \to \infty \), we arrive at

\[
-I(s) \leq 2\lambda H_c(s),
\]

which leads to

\[
\frac{dH_c(s)}{ds} \geq -2\lambda H_c(s),
\]

and finally to

\[
H_c(t) \geq e^{-2\lambda t} H_c(0).
\]
1. Example of an Ornstein-Uhlenbeck process

Trying to find specific examples of \( \sigma(x) \) and \( F(x) \) for which one can determine the time dependent solution \( f(t, x) \) of Eq. 4.15 is not easy. One solution that is known is the one for an Ornstein-Uhlenbeck process. Since it is an Ornstein-Uhlenbeck process, which was historically developed in thinking about perturbations to the velocity of a Brownian particle, we denote the dependent variable by \( v \) so we have \( \sigma(v) \equiv \sigma \) a constant, and \( F(v) = -\gamma v \) with \( \gamma \geq 0 \). In this case, Eq. 4.14 becomes

\[
\frac{dv}{dt} = -\gamma v + \sigma \xi
\]  

with the corresponding Fokker-Planck equation

\[
\frac{\partial f}{\partial t} = \frac{\partial [\gamma vf]}{\partial v} + \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial v^2}.
\]

The unique stationary solution is

\[
f_* (v) = \frac{e^{-\gamma v^2 / 2}}{\int_{-\infty}^{\infty} e^{-\gamma v^2 / 2} dv} = \frac{\gamma}{\pi \sigma^2} e^{-\gamma v^2 / 2}.
\]

Further, from Risken \[15, Eq. 5.28\] the fundamental solution \( k(t, v, v_0) \) is given by

\[
k(t, v, v_0) = \frac{1}{\sqrt{2\pi b(t)}} \exp \left\{ -\frac{(v - \exp(-\gamma t) v_0)^2}{2b(t)} \right\},
\]

where

\[
b(t) = \frac{\sigma^2}{2\gamma} \left[ 1 - e^{-2\gamma t} \right].
\]

Note that for a given \( y \) and \( t \) the function \( k(t, \cdot, y) \) is a Gaussian density with mean \( v_0 \exp(-\gamma t) \) and variance \( b(t) \). Since \( k(t, v, v_0) > 0 \) for all \( v, v_0 \in \mathbf{R} \), the semigroup of Markov operators

\[
P_t f_0 (v) = \int_{\mathbf{R}} k(t, v, v_0) f_0 (v_0) \, dv_0
\]

satisfies all of the conditions of Theorem 2. Thus we can assert from Theorem 2 that \( \lim_{t \to \infty} H_c (P_t f_0 | f_*) = 0 \) for all \( f_0 \) with \( H_c (f_0 | f_*) > -\infty \).

In the case of the Ornstein-Uhlenbeck process, the sufficient condition (4.23) for the exponential lower bound on the conditional entropy reduces to

\[
DB''(v) \geq \lambda
\]

where

\[
D = \frac{\sigma^2}{2} \quad \text{and} \quad B''(v) = \frac{2\gamma}{\sigma^2}.
\]

Thus \( \lambda = \gamma \) and Eq. 4.24 becomes

\[
H_c (P_t f_0 | f_*) \geq e^{-2\gamma t} H_c (f_0 | f_*)
\]

for all initial densities \( f_0 \) with \( H_c (f_0 | f_*) > -\infty \).

We will now show that this lower bound is optimal. Let us first calculate the conditional entropy of two Gaussian densities. Let \( q_1, q_2 > 0 \) and let \( z_1, z_2 \in \mathbf{R} \). Consider densities of the form

\[
g_i (x, z_i) = \sqrt{\frac{q_i}{\pi}} \exp\{-q_i (x - z_i)^2\}, \quad x \in \mathbf{R}, \quad i = 1, 2.
\]

Then

\[
\log \frac{g_1 (x, z_1)}{g_2 (x, z_2)} = \log \sqrt{\frac{q_1}{q_2}} - q_1 (x - z_1)^2 + q_2 (x - z_2)^2.
\]
Since
\[ \int_{\mathbb{R}} g_1(x, z_1) x^2 \, dx = \frac{1}{2q_1} + z_1^2 \quad \text{and} \quad \int_{\mathbb{R}} g_1(x, z_1) x \, dx = z_1, \]
we arrive at
\[ H_c(g_1(\cdot, z_1)|g_2(\cdot, z_2)) = \frac{1}{2} \log \frac{q_2}{q_1} + \frac{1}{2} \left( 1 - q_2 \frac{q_1}{q_2} \right) - q_2(z_1 - z_2)^2. \] (4.31)

Now let \( f_0 \) be a Gaussian density of the form
\[ f_0(v) = \sqrt{\frac{c_1}{\pi}} \exp\{-c_1(v - c_2)^2\}, \]
where \( c_1 > 0 \) and \( c_2 \in \mathbb{R} \). Since
\[ P_t f_0(v) = \int_{\mathbb{R}} k(t, v, v_0) f_0(v_0) \, dv_0, \]
we obtain by direct calculation using Eq. 4.29
\[ P_t f_0(v) = \sqrt{\frac{c_1}{\pi(e^{-2\gamma t} + 2c_1b(t))}} \exp\left\{ -\frac{c_1}{e^{-2\gamma t} + 2c_1b(t)} (v - c_2e^{-\gamma t})^2 \right\}. \]
Consequently, from Eq. 4.31 with \( q_1 = \frac{c_1}{e^{-2\gamma t} + 2c_1b(t)} \), \( q_2 = \frac{\gamma}{\sigma^2}, \) \( z_1 = c_2e^{-\gamma t} \), and \( z_2 = 0 \), it follows that
\[ H_c(P_t f_0|f_*) = \frac{1}{2} \log \left[ 1 - \left( 1 - \frac{\gamma}{\sigma^2 c_1} \right) e^{-2\gamma t} \right] + \frac{1}{2} \left( 1 - \frac{\gamma}{\sigma^2 c_1} - \frac{2\gamma c_2^2}{\sigma^2} \right) e^{-2\gamma t}. \] (4.32)

In particular, if \( c_1 = \frac{\gamma}{\sigma^2} \) then \( H_c(f_0|f_*) = -\frac{\gamma c_2^2}{\sigma^2} \) and
\[ H_c(P_t f_0|f_*) = e^{-2\gamma t} H_c(f_0|f_*), \quad t \geq 0. \]

2. A Rayleigh process

Another example for which we have an analytic form for the density \( f(t, x) \) is that of a Rayleigh process \([19, \text{page 135-136}]\). In this case we have
\[ \frac{dv}{dt} = -\gamma v + \frac{\sigma^2}{2v} + \sigma \xi \quad v \in [0, \infty) \] (4.33)
and the associated Fokker-Planck equation is
\[ \frac{\partial f}{\partial t} = \frac{\partial}{\partial v} \left[ \left( \gamma v - \frac{\sigma^2}{2v} \right) f \right] + \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial v^2}. \] (4.34)

The unique equilibrium solution, with the proper normalization, is given by
\[ f_*(v) = \frac{2\gamma v}{\sigma^2} e^{-\gamma v^2/\sigma^2} \] (4.35)
and the fundamental solution is
\[ k(t, v, v_0) = \frac{2\gamma v}{\sigma^2} e^{-\gamma v^2/\sigma^2} \sum_{n=0}^{\infty} L_n \left( \frac{\gamma v_0^2}{\sigma^2} \right) L_n \left( \frac{\gamma v^2}{\sigma^2} \right) e^{-2n\gamma t}, \] (4.36)
where the $L_n$ are the Laguerre polynomials of order $n$. This form of $k$ is not particularly useful in calculations. The Rayleigh process is also known as a two-dimensional radial Ornstein-Uhlenbeck process \cite{27} with parameters $\gamma$ and $\sigma$, and as such its transition probability is given by \cite{20}

$$k(t, v, v_0) = \frac{\nu}{b(t)} e^{-\left((t^2 + e^{-2\gamma t}v_0^2)/(2\nu t)\right)} I_0\left(\frac{\nu v_0}{e^{\gamma t}b(t)}\right) \quad (4.37)$$

where $b(t) = \frac{\sigma^2}{2\gamma}(1 - e^{-2\gamma t})$ and $I_0(z) = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k+1)!}$ is the modified Bessel function of the first kind. Since $k(t, v, v_0) > 0$ for all $t, v, v_0 > 0$, again from Theorem \cite{12} it follows that $\lim_{t \to \infty} H_c(P^t f_0|f_*) = 0$ for all $f_0$ with $H_c(f_0|f_*) > -\infty$.

In the case of the Rayleigh process, the sufficient Condition \cite{14,24} for the exponential lower bound on the conditional entropy reduces to

$$DB''(v) \geq \lambda$$

where

$$D = \frac{\sigma^2}{2} \quad \text{and} \quad B''(v) = \frac{2\gamma}{\sigma^2} + \frac{1}{v^2}.$$ 

Thus $\lambda = \gamma$ and Eq. \cite{24} becomes

$$H_c(P^t f_0|f_*) \geq e^{-2\gamma t} H_c(f_0|f_*)$$

for all initial densities $f_0$ with $H_c(f_0|f_*) > -\infty$.

**B. The multidimensional case**

In this section first we turn our attention to the existence of a stationary density in the case of multidimensional diffusion when the matrix $\Sigma(x)$ is nonsingular at every point $x$. This case is much more involved and does not yield simple necessary and sufficient conditions for the existence of a stationary density. However, when we consider two specific examples much more can be said.

Let us assume that $F$ and $a$ are of class $C^2$ and $C^3$ respectively and that

$$\sum_{i, j=1}^{d} a_{ij}(x) \lambda_i \lambda_j > 0, \quad \text{for} \quad x \in \mathbb{R}^d \quad \text{and} \quad \lambda \in \mathbb{R}^d - \{0\}.$$ 

Under these assumptions, the so-called Liapunov method \cite{18} implies that if there exists a nonnegative function $V$ of class $C^2$ on $\mathbb{R}^d$ and a positive constant $r > 0$ such that

$$\sup_{|x| \geq r} L^- V(x) < 0, \quad (4.38)$$

where the operator

$$L^- = \sum_{i=1}^{d} F_i(x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}, \quad (4.39)$$

then there is a unique stationary density $f_*$ and $f_*(x) > 0$ for all $x$. Again, from Theorem \cite{2} $\lim_{t \to \infty} H_c(P^t f_0|f_*) = 0$ for all $f_0$ with $H_c(f_0|f_*) > -\infty$. If we simply take $V(x) = |x|^2$, then Condition \cite{4,38} becomes

$$\sup_{|x| \geq r} \left( \sum_{i,j=1}^{d} a_{ij}(x) + 2 \sum_{i=1}^{d} x_i F_i(x) \right) < 0.$$ 

One can also write an integral test \cite{21} which reduces to Condition \cite{1,21} in the one-dimensional case. Let

$$A_1(x) = \frac{1}{|x|^2} x^T a(x)x = \frac{1}{|x|^2} \sum_{i,j=1}^{d} a_{ij}(x)x_i x_j,$$
and
\[ A_2(x) = \text{Tr}(a(x)) + 2\langle x, F(x) \rangle = \sum_{i=1}^{d} a_{ii}(x) + 2 \sum_{i=1}^{d} x_i F_i(x). \]

Now define
\[ \beta(r) = \sup_{|x|=r} \frac{A_2(x) - A_1(x)}{A_1(x)} \quad \text{and} \quad \alpha(r) = \inf_{|x|=r} A_1(x). \]

Finally, for \( r \geq 1 \), let
\[ J(r) = \int_{1}^{r} \frac{\beta(s)}{s} ds \]
and assume that there exists \( r_0 > 0 \) such that the following two conditions hold:
\[ \int_{r_0}^{\infty} e^{-J(r)} dr = \infty \quad \text{and} \quad \int_{r_0}^{\infty} \frac{1}{\alpha(r)} e^{J(r)} dr < \infty. \]  \hspace{1cm} (4.40)

We claim that Condition (4.40) implies Condition (4.38). To see this, for \( r \geq r_0 \) define
\[ \tilde{J}(r) = \int_{1}^{r} e^{-J(s)} \int_{s}^{\infty} \frac{1}{\alpha(q)} e^{J(q)} dq ds, \]
and take \( V \) to be of class \( C^2 \) on \( \mathbb{R}^d \) such that \( V(x) = \tilde{J}(|x|) \) for \( |x| \geq r_0 \). It is easy to check that
\[ 2L^- V(x) = A_1(x) \tilde{J}''(|x|) + \frac{\tilde{J}'(|x|)}{|x|} (A_2(x) - A_1(x)) \]
and that \( 2L^- V(x) \leq -1 \) for \( |x| \geq r_0 \).

We now discuss the speed of convergence of the conditional entropy. \cite{8} provide a general sufficient condition for such convergence in the case of multidimensional diffusions that the Fokker-Planck equation can be written in a divergence form:
\[ \frac{\partial f}{\partial t} = \text{div} (D(x)(\nabla f + f \nabla B(x))), \]  \hspace{1cm} (4.41)
where \( D(x) \) is locally uniformly positive and \( B(x) \) is a real-valued function such that the stationary solution
\[ f_*(x) = e^{-B(x)} \]
is a density. For simplicity assume that \( D(x) = D \) is a constant. Then the Fokker-Planck equation (4.41) describes the evolution of densities for the system
\[ dx(t) = -D \nabla B(x(t)) dt + \sqrt{2D} I dw(t), \]  \hspace{1cm} (4.42)
where \( I \) is the identity matrix. If there is \( \lambda > 0 \) such that
\[ D \left( \frac{\partial^2 B(x)}{\partial x_i \partial x_j} \right)_{i,j=1,...,d} \geq \lambda I, \]  \hspace{1cm} (4.43)
then we have
\[ H_c(P^t f_0 | f_*) \geq e^{-2\lambda t} H_c(f_0 | f_*), \]  \hspace{1cm} (4.44)
for all initial densities \( f_0 \) with \( H_c(f_0 | f_*) > -\infty \).
C. Multidimensional Ornstein-Uhlenbeck process

Consider the multidimensional Ornstein-Uhlenbeck process

\[
\frac{dx}{dt} = Fx + \Sigma \xi
\]  \hspace{1cm} (4.45)

where \( F \) is a \( d \times d \) matrix, \( \Sigma \) is a \( d \times d \) matrix an \( \xi \) is \( d \) dimensional vector. The formal solution to Eq. (4.45) is given by

\[
x(t) = e^{tF}x(0) + \int_0^t e^{(t-s)F} \Sigma dw(s)
\]  \hspace{1cm} (4.46)

where \( e^{tF} = \sum_{n=0}^{\infty} \frac{t^n}{n!} F^n \) is the fundamental solution to \( \dot{X}(t) = FX(t) \) with \( X(0) = I \), and \( w(t) \) is the standard \( d \)-dimensional Wiener process. From the properties of stochastic integrals it follows that

\[
\eta(t) = \int_0^t e^{(t-s)F} \Sigma dw(s)
\]

has mean 0 and covariance

\[
R(t) = E\eta(t)\eta(t)^T = \int_0^t e^{sF} \Sigma \Sigma^T e^{*F^T} ds,
\]

where \( B^T \) is the transpose of the matrix \( B \). The matrix \( R(t) \) is nonnegative definite but not necessarily positive definite. We follow the presentation of [22] and [23]. For each \( t > 0 \) the matrix \( R(t) \) has constant rank equal to the dimension of the space

\[
[F, \Sigma] := \{ F^{l-1} \Sigma \epsilon_j : l, j = 1, \ldots, d, \epsilon_j = (\delta_{j1}, \ldots, \delta_{jp})^T \}.
\]

If \( m = \text{rank} R(t) \) then \( d - m \) coordinates of the process \( \eta(t) \) are equal to 0 and the remaining \( m \) coordinates constitute an \( m \)-dimensional Gaussian process. Thus if \( m < d \) there is no stationary density. If \( \text{rank} R(t) = d \) then the transition probability function of \( x(t) \) is given by the Gaussian density

\[
k(t, x, x_0) = \frac{1}{(2\pi)^{d/2}(\det R(t))^{1/2}} \exp\left\{ -\frac{1}{2} (x - e^{tF}x_0)^T R(t)^{-1} (x - e^{tF}x_0) \right\},
\]  \hspace{1cm} (4.47)

where \( R(t)^{-1} \) is the inverse matrix of \( R(t) \). An invariant density \( f_* \) exists if and only if all eigenvalues of \( F \) have negative real parts, and in this case the unique stationary density \( f_* \) has the form

\[
f_*(x) = \frac{1}{(2\pi)^{d/2}(\det R_*)^{1/2}} \exp\left\{ -\frac{1}{2} x^T R_*^{-1} x \right\},
\]  \hspace{1cm} (4.48)

where \( R_* \) is a positive definite matrix given by

\[
R_* = \int_0^\infty e^{sF} \Sigma \Sigma^T e^{*F^T} ds,
\]

and is a unique symmetric matrix satisfying

\[
FR_* + R_* F^T = -\Sigma \Sigma^T.
\]  \hspace{1cm} (4.49)

We conclude that if \([F, \Sigma]\) contains \( d \) linearly independent vectors and all eigenvalues of \( F \) have negative real parts, then from Theorem 2 it follows that \( \lim_{t \to \infty} H_c(P^t f_0|f_*) = 0 \) for all \( f_0 \) with \( H_c(f_0|f_*) > -\infty \).

To simplify the following we first recall some properties of multivariate Gaussian distributions. Let \( Q_1, Q_2 \) be positive definite matrices and let

\[
g_i(x, z) = \frac{1}{(2\pi)^{d/2}(\det Q_i)^{1/2}} \exp\left\{ -\frac{1}{2} (x - z)^T Q_i^{-1} (x - z) \right\}.
\]

Then
\[
\log \frac{g_1(x, z_1)}{g_2(x, z_2)} = \frac{1}{2} \log \frac{\det Q_2}{\det Q_1} - \frac{1}{2} (x - z_1)^T Q_1^{-1} (x - z_1) + \frac{1}{2} (x - z_2)^T Q_2^{-1} (x - z_2). \tag{4.50}
\]

Since \( \int g_1(x, 0) xx^T \, dx = Q_1 \), we have
\[
\int g_1(x, z_1) xx^T \, dx = Q_1 + z_1 z_1^T \quad \text{and} \quad \int g_1(x, z_1) x \, dx = z_1.
\]

Note also that \( z^T Q z \) can be written with the help of the trace of a matrix as \( \text{Tr}[Q zz^T] \) for any matrix \( Q \) and any vector \( z \). Consequently,
\[
H_c(g_1(\cdot, z_1)|g_2(\cdot, z_2)) = \frac{1}{2} \log \frac{\det Q_1}{\det Q_2} + \frac{1}{2} \text{Tr}[(Q_1^{-1} - Q_2^{-1})Q_1] - \frac{1}{2} \text{Tr}[Q_2^{-1}(z_1 - z_2)(z_1 - z_2)^T]. \tag{4.51}
\]

Now let \( f_0 \) be a Gaussian density of the form
\[
f_0(x) = \frac{1}{(2\pi)^{d/2} |V(0)|^{1/2}} \exp \left\{ -\frac{1}{2} (x - m(0))^T V(0)^{-1} (x - m(0)) \right\}
\tag{4.52}
\]
where \( V(0) \) is a positive definite matrix and \( m(0) \in \mathbb{R}^d \). From Eq. \( \textbf{4.49} \) it follows that \( x(t) \) is Gaussian with the following mean vector \( m(t) = e^{tF} m(0) \) and covariance matrix \( V(t) = e^{tF} V(0) e^{tF^T} + R(t) \). Hence
\[
P^t f_0(x) = \frac{1}{(2\pi)^{d/2} |V(t)|^{1/2}} \exp \left\{ -\frac{1}{2} (x - m(t))^T V(t)^{-1} (x - m(t)) \right\}
\]
Consequently, from Eq. \( \textbf{4.51} \) with \( Q_1 = V(t) \), \( z_1 = m(t) \), \( Q_2 = R_* \), and \( z_2 = 0 \), we obtain
\[
H_c(P^t f_0|f_*) = \frac{1}{2} \log \frac{\det V(t)}{\det R_*} + \frac{1}{2} \text{Tr}(I - R_*^{-1}V(t)) - \frac{1}{2} \text{Tr}(R_*^{-1}m(t)m(t)^T). \tag{4.53}
\]
In particular, if \( V(0) = R_* \), then \( V(t) = R_* \) and
\[
H_c(P^t f_0|f_*) = -\frac{1}{2} \text{Tr}(R_*^{-1}m(t)m(t)^T) \tag{4.54}
\]
for all \( t \geq 0 \) and every \( f_0 \) of the form given by Eq. \( \textbf{4.52} \).

As a specific example of the multidimensional Ornstein-Uhlenbeck process, to which Equation \( \textbf{4.41} \) and \( \textbf{4.51} \) can be applied, consider the case when \( \Sigma = \sigma I \) and \( F \) is a diagonal matrix, \( F = -\lambda I \) with \( \lambda > 0 \). Then \( R_*^{-1} = \frac{2\lambda}{\sigma^2} I \) and \( f_*(x) = e^{-B(x)} \), where
\[
B(x) = \frac{1}{2} \log((2\pi)^d \det R_*) + \frac{\lambda}{\sigma^2} x^T x.
\]
Thus Condition \( \textbf{4.43} \) becomes
\[
\frac{\sigma^2}{2} \left( \frac{\partial^2 B(x)}{\partial x_i \partial x_j} \right)_{i,j=1,...,d} = \lambda I.
\]
Since \( e^{tF} = e^{-\lambda t} I \), we conclude from Equation \( \textbf{4.53} \) that \( H_c(P^t f_0|f_*) = e^{-2\lambda t} H_c(f_0|f_*) \), so the estimate in Equation \( \textbf{4.43} \) is optimal. We will show in the next section that in the case of a non-invertible matrix \( \Sigma \) a slower speed of convergence might occur.

To obtain a lower bound on the conditional entropy for the case of a not necessarily invertible matrix \( \Sigma \) and a general \( f_0 \), we make use of the following inequality, proved in the appendix,
\[
H_c(P^t f_0|f_*) \geq \int \int f_0(y_1)f_*(y_2) H_c(k(t,\cdot,y_1)|k(t,\cdot,y_2))dy_1 dy_2. \tag{4.55}
\]
From Eq. 4.51 with $Q_1 = R(t)$, $Q_2 = R(t)$, $z_1 = e^{tF}y_1$, and $z_2 = e^{tF}y_1$ it follows that

$$H_c(k(t, \cdot, y_1)|k(t, \cdot, y_2)) = -\frac{1}{2} \text{Tr}[R(t)^{-1}e^{tF}(y_1 - y_2)(y_1 - y_2)^T e^{tF^T}] .$$

Since $\text{Tr}[R(t)^{-1}e^{tF}yy^T e^{tF^T}] \leq ||y||^2||e^{tF^T}R(t)^{-1}e^{tF}||$ for any $y$ by the Schwartz inequality, we obtain from Eq. 4.55

$$H_c(P^t f_1|f_*) \geq -\frac{1}{2} ||e^{tF}||^2 ||R_1^{-1}|| \int \int ||y_1 - y_2||^2 f_0(y_1)f_*(y_2)dy_1dy_2. \quad (4.56)$$

Finally, observe that the norm of $R(t)^{-1}$ is bounded, because $R(t)^{-1}$ converges to $R_*^{-1}$ as $t \to \infty$. Thus for sufficiently large $t$ we have

$$H_c(P^t f_1|f_*) \geq -||e^{tF}||^2 ||R_1^{-1}|| \int \int ||y_1 - y_2||^2 f_0(y_1)f_*(y_2)dy_1dy_2. \quad (4.57)$$

1. Harmonic oscillator

Consider the second order system

$$md\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega^2 x = \sigma \xi \quad (4.58)$$

with constant positive coefficients $m$, $\gamma$ and $\sigma$. Introduce the velocity $v = \frac{dx}{dt}$ as a new variable. Then Eq. 4.58 is equivalent to the system

$$\frac{dx}{dt} = v \quad (4.59)$$
$$m \frac{dv}{dt} = -\gamma v - \omega^2 x + \sigma \xi, \quad (4.60)$$

and the corresponding Fokker-Planck equation is

$$\frac{\partial f}{\partial t} = -\frac{\partial [vf]}{\partial x} + \frac{m}{\gamma} \frac{\partial ([\gamma v + \omega^2 x]f)}{\partial v} + \frac{\sigma^2}{2m^2} \frac{\partial^2 f}{\partial v^2} .$$

We can use the results of Section 4VC in the two dimensional setting with

$$F = \begin{pmatrix} 0 & 1 \\ \omega^2/m & -\gamma/m \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} 0 & \sigma \\ 0 & \gamma/m \end{pmatrix} .$$

Since

$$[F, \Sigma] = \left\{ \begin{pmatrix} 0 \\ \sigma/m \end{pmatrix}, \begin{pmatrix} \sigma/m \end{pmatrix}, \begin{pmatrix} \sigma^2 \gamma/m \end{pmatrix} \right\} ,$$

the transition density function is given by Eq. 4.47. The eigenvalues of $F$ are equal to

$$\lambda_1 = -\gamma + \sqrt{\gamma^2 - 4m \omega^2} \quad \text{and} \quad \lambda_2 = -\gamma - \sqrt{\gamma^2 - 4m \omega^2} ,$$

and are either negative real numbers when $\gamma^2 \geq 4m \omega^2$ or complex numbers with negative real parts when $\gamma^2 < 4m \omega^2$. Thus the stationary density is given by Eq. 4.48. As is easily seen $R_*$, being a solution to Eq. 4.49 is given by

$$R_* = \begin{pmatrix} \sigma^2/2\omega^2 & 0 \\ 0 & \sigma^2/2m\gamma \end{pmatrix} .$$
The inverse of the matrix $R_*$ is
\[
R_*^{-1} = \frac{2\gamma}{\sigma^2} \begin{pmatrix} \omega^2 & 0 \\ 0 & m \end{pmatrix}
\]
and the unique stationary density becomes
\[
f_*(x, v) = \frac{\gamma \sqrt{m}}{\pi \sigma^2} e^{-\frac{\gamma}{2\sigma^2}[\omega^2 x^2 + mv^2]}
\]
As in Section IV.C we conclude that $\lim_{t \to \infty} H_c(P^t f_0 | f_*) = 0$ for all $f_0$ with $H_c(f_0 | f_*) > -\infty$.

The bound on the temporal convergence of $H_c(P^t f_0 | f_*)$ to zero, as given by Equation 4.57, is determined by $|| e^{tF} ||^2$. Thus we are going to calculate $|| e^{tF} ||^2$ and to see the nature of the general formula (4.54) for the conditional entropy in the case of the harmonic oscillator. First, consider the case when $\lambda_1 = \lambda_2$, that is $\gamma^2 = 4m\omega^2$, and set
\[
\lambda = -\frac{\gamma}{2m}.
\]
Then we have
\[
F = \begin{pmatrix} 0 & 1 \\ -\lambda^2 & 2\lambda \end{pmatrix} \quad \text{and} \quad e^{tF} = e^{\lambda t} \begin{pmatrix} 1 - \lambda t & t \\ -\lambda^2 t & 1 + \lambda t \end{pmatrix}
\]
and
\[
|| e^{tF} ||^2 = e^{2\lambda t} (1 + (3\lambda^2 + 1)t^2).
\]
If we take $m(0) = (1, \lambda)^T$ then $m(t) = e^{\lambda t}m(0)$ and Eq. 4.51 becomes
\[
H_c(P^t f_0 | f_*) = e^{2\lambda t} H_c(f_0 | f_*)
\]
But, if we take $m(0) = (1, 0)^T$, then $m(t) = e^{\lambda t}(1 - \lambda t, -\lambda^2 t)^T$ and
\[
H_c(P^t f_0 | f_*) = e^{2\lambda t} ((1 - \lambda t)^2 + \lambda^2 t^2) H_c(f_0 | f_*)
\]
Now consider the case when $\lambda_1 \neq \lambda_2$ are real and define, for $t \geq 0$,
\[
c_1(t) = \frac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{\lambda_2 - \lambda_1} \quad \text{and} \quad c_2(t) = \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}.
\]
Then
\[
e^{tF} = \begin{pmatrix} c_1(t) & c_2(t) \\ c_1'(t) & c_2'(t) \end{pmatrix}
\]
and
\[
|| e^{tF} ||^2 = e^{2\lambda_1 t} + e^{2\lambda_2 t} + (\lambda_1 \lambda_2 + 1)^2 c_2(t).
\]
If we take $m(0) = (1, \lambda_i)^T$, $i = 1, 2$ in Equation 4.51 then $m(t) = e^{\lambda_i t}m(0)$ and
\[
H_c(P^t f_0 | f_*) = e^{2\lambda_i t} H_c(f_0 | f_*)
\]
Finally, consider the case of complex $\lambda_1, \lambda_2$ and let
\[
\alpha = -\frac{\gamma}{2m} \quad \text{and} \quad \beta = \frac{\sqrt{4m\omega^2 - \gamma^2}}{2m}.
\]
Then
\[
e^{tF} = \frac{e^{\alpha t}}{\beta} \begin{pmatrix} \beta \cos(\beta t) - \alpha \sin(\beta t) & \sin(\beta t) \\ -\alpha^2 + \beta^2 \sin(\beta t) & \beta \cos(\beta t) + \alpha \sin(\beta t) \end{pmatrix}
\]
and we have
\[
|| e^{tF} ||^2 = \frac{e^{2\alpha t}}{\beta^2} (2\beta^2 \cos^2(\beta t) + (2\alpha^2 + (\alpha^2 + \beta^2)^2 + 1) \sin^2(\beta t)).
\]
2. Colored noise

Consider the system

\[ \frac{dx}{dt} = -\alpha x + \eta \]  \hspace{1cm} (4.61)

where \( \alpha > 0 \) and \( \eta \) is the one dimensional Ornstein-Uhlenbeck process with parameters \( \gamma, \sigma > 0 \)

\[ \frac{d\eta}{dt} = -\gamma \eta + \sigma \xi . \]

Then Eq. 4.61 is equivalent to the system

\[ \frac{dx}{dt} = -\alpha x + v \]  \hspace{1cm} (4.62)

\[ \frac{dv}{dt} = -\gamma v + \sigma \xi , \]  \hspace{1cm} (4.63)

and the corresponding Fokker-Planck equation is

\[ \frac{\partial f}{\partial t} = \frac{\partial}{\partial x}[(\alpha x - v)f] + \frac{\partial}{\partial v}[\gamma vf] + \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial v^2} . \]

We can use the results of Section IV C in the two dimensional setting with

\[ F = \begin{pmatrix} -\alpha & 1 \\ 0 & -\gamma \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix} . \]

Observe that

\[ [F, \Sigma] = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma \\ -\gamma \sigma \end{pmatrix} \right\} . \]

The eigenvalues of \( F \) are given by

\[ \lambda_1 = -\alpha \quad \text{and} \quad \lambda_2 = -\gamma , \]

and are evidently negative. Finally,

\[ R_* = \frac{\sigma^2}{2\gamma(\alpha + \gamma)} \begin{pmatrix} 1 & 1 \\ 1 & \alpha + \gamma \end{pmatrix} \]

and

\[ R_*^{-1} = \frac{2\alpha(\alpha + \gamma)}{\sigma^2} \begin{pmatrix} \alpha + \gamma & -1 \\ -1 & \frac{1}{\alpha} \end{pmatrix} . \]

Thus the unique stationary density is equal to

\[ f_*(x, v) = \frac{\sqrt{\alpha \gamma}}{\sqrt{\pi} \sigma^2} \exp \left\{ -\frac{\alpha + \gamma}{\sigma^2} (\alpha(\alpha + \beta)x^2 - 2\alpha x v + v^2) \right\} . \]

As in Section IV C we conclude that \( \lim_{t \to \infty} H_c(P^t f_0|f_*) = 0 \) for all \( f_0 \) with \( H_c(f_0|f_*) > -\infty \).

Finally, we are going to calculate \( ||e^{tF}||^2 \) to see how the general formula (4.54) for the conditional entropy bound looks in this special case. First consider the case when \( \alpha \neq \gamma \). The fundamental matrix is given by

\[ e^{tF} = \begin{pmatrix} e^{-\alpha t} & \beta(t) \\ 0 & e^{-\gamma t} \end{pmatrix} \quad \text{with} \quad \beta(t) = \frac{1}{\alpha - \gamma} (e^{-\gamma t} - e^{-\alpha t}) . \]
The process $\xi$ of the form $x$ from the state $c$ densities of the process $(x)$ and we now define a semigroup of Markov operators on this space which describes the temporal evolution of the $\xi$ Markovian dichotomous noise is equal to $x$. Similarly, for $m(0) = (1, \alpha - \gamma)^T$ we have $m(t) = e^{-\gamma t}m(0)$ and $H_c(P^t f_0|f_*) = e^{-2\gamma t}H_c(f_0|f_*)$.

Now consider the case when $\alpha = \gamma$. We have

\[
e^tH = \begin{pmatrix} e^{-\gamma t} & t e^{-\gamma t} \\ 0 & e^{-\gamma t} \end{pmatrix}
\]

and $||e^tH||^2 = e^{-2\gamma t} + e^{-2\alpha t}$.

If we take $m(0) = (1, 0)^T$, in Equation 4.55 then $m(t) = e^{-\alpha t}m(0)$ and

\[
H_c(P^t f_0|f_*) = e^{-2\alpha t}H_c(f_0|f_*)
\]

Similarly, for $m(0) = (1, \alpha - \gamma)^T$ we have $m(t) = e^{-\gamma t}m(0)$ and $H_c(P^t f_0|f_*) = e^{-2\gamma t}H_c(f_0|f_*)$.

Equation 4.54 becomes for $m(0) = (0, 1)^T$ and $m(t) = e^{-\gamma t}(t, 1)^T$

\[
H_c(P^t f_0|f_*) = e^{-2\gamma t}(2\gamma^2 t^2 - 2\gamma t + 1)H_c(f_0|f_*)
\]

\section{V. MARKOVIAN DICHOTOMOUS NOISE}

Another example where we can use our results is the case of dichotomous noise \[14, \text{Section 8}\]. The state space of the Markovian dichotomous noise $\xi(t)$ consists of two states $\{c_+, c_-\}$ and is characterized by a transition probability from the state $c_+$ to $c_-$ in the small time interval $\Delta t$ given by $\alpha \Delta t + o(\Delta t)$, and from the state $c_-$ to $c_+$ given by $\beta \Delta t + o(\Delta t)$, where $\alpha, \beta > 0$. It has the correlation function

\[
\langle \xi(t)\xi(s) \rangle = \frac{\alpha \beta (c_+ - c_-)^2}{(\alpha + \beta)^2} \exp(-\alpha \beta t).
\]

A system subject to this type of noise is described by the following equation

\[
\frac{dx}{dt} = F(x) + \sigma(x)\xi.
\]

The pair $(x(t), \xi(t))$ is Markovian and writing $a(x, c_\pm) = F(x) + c_\pm \sigma(x)$ we arrive at

\[
\frac{dx(t)}{dt} = a(x(t), \xi(t)) dt.
\]

The process $\xi(t)$ determines which deterministic system

\[
\frac{dx_+}{dt} = a(x, c_+) \quad \text{or} \quad \frac{dx_-}{dt} = a(x, c_-)
\]

to choose. We assume that given the initial condition $x_\pm(0) = x$, each of these equations has a solution $x_+(t)$ and $x_-(t)$, respectively, defined and finite for all $t \geq 0$, and we write $\pi^t_+(x) = x_+(t)$ and $\pi^t_-(x) = x_-(t)$. Furthermore, we assume that there is a minimal open set $X$ such that $\pi^t_\pm(X) \subseteq X$ for all $t > 0$. Let $\mathcal{B}(X \times \{c_+, c_-\})$ be the sigma algebra of Borel subsets of $X \times \{c_+, c_-\}$ and let $\mu$ be the product measure on $\mathcal{B}(X \times \{c_+, c_-\})$ which, on every set of the form $B \times \{c_\pm\}$, is equal to the length of the set $B$. The norm of any element $f$ of the space $L^1(X \times \{c_+, c_-\}, \mu)$ is equal to

\[
||f||_1 = \int_X |f_+(x)| dx + \int_X |f_-(x)| dx,
\]

and we now define a semigroup of Markov operators on this space which describes the temporal evolution of the densities of the process $(x(t), \xi(t))$.

The evolution equation for the densities $f_\pm(t, x) = f(t, x, c_\pm)$ of the process $(x(t), \xi(t))$ is of the form

\[
\frac{\partial f_+}{\partial t} = -\frac{\partial [a(x, c_+)f_+]}{\partial x} - \alpha f_+ + \beta f_- \quad (5.1)
\]

\[
\frac{\partial f_-}{\partial t} = -\frac{\partial [a(x, c_-)f_-]}{\partial x} + \alpha f_+ - \beta f_- \quad (5.2)
\]
Formally writing $f = (f_+, f_-)^T$, we arrive at the following equation
\[
\frac{\partial f}{\partial t} = Af + Mf
\]  
where
\[
M = \begin{pmatrix} -\alpha & \beta \\ \alpha & -\beta \end{pmatrix} \quad \text{and} \quad Af = \begin{pmatrix} -\partial[a(x, c_+)f_+] \\ \partial[a(x, c_-)f_-] \end{pmatrix}.
\]

Then the operator $A$ generates a semigroup of Markov operators $T^t$ on $L^1(X \times \{c_+, c_-\}, \mu)$. Since $M$ is a bounded operator, $A + M$ also generates a semigroup of Markov operators $P^t$, and the following holds
\[
P^t f_0 = T^t f_0 + \int_0^t T^{t-s} M P^s f_0 ds
\]
for every $f_0 \in L^1(X \times \{c_+, c_-\}, \mu)$. The semigroup $P^t$ gives the generalized solution $f(t, \cdot)$ of Eq. 5.3 with the initial condition $f(0, \cdot) = f_0$. Using the results of Pichor and Rudnicki [12, Proposition 2] we infer that if there is an $x_0 \in X$ such that $a(x_0, c_+) \neq a(x_0, c_-)$, then the semigroup $P^t$ is partially integral, and if $X$ is the minimal set such that $\pi^+_1(X) \subseteq X$, then this semigroup can have at most one stationary density. Consequently, if a stationary density $f_*$ exists, Theorem 2 implies that
\[
\lim_{t \to \infty} H_c(P^t f_0|f_*) = 0
\]  
for all $f_0$ with $H_c(f_0|f_*) > -\infty$, where the conditional entropy for $P^t$ on $L^1(X \times \{c_+, c_-\}, \mu)$ is equal to
\[
H_c(P^t f_0|f_*) = -\int_X P^t f(x, c_+) \log \frac{P^t f(x, c_+)}{f_*(x, c_+)} dx - \int_X P^t f(x, c_-) \log \frac{P^t f(x, c_-)}{f_*(x, c_-)} dx.
\]
by Eq. 2.1

The density of the state variable $x(t)$ is an element of the space $L^1(X, m)$, where $m$ is the Lebesque measure on $X$. It is given by
\[
p(t, x) = P^t f_0(x, c_+) + P^t f_0(x, c_-). \tag{5.5}
\]
Thus the stationary density $f_*$ of $P^t$ gives us the stationary density of $x(t)$
\[
p_*(x) = f_*(x, c_+) + f_*(x, c_-). \tag{5.6}
\]
Since
\[
H_c(p(t)|p_*) = -\int_X p(t, x) \log \frac{p(t, x)}{p_*(x)} dx \geq H_c(P^t f_0|f_*), \tag{5.7}
\]
we conclude that
\[
\lim_{t \to \infty} H_c(p(t)|p_*) = 0. \tag{5.7}
\]

For a general one-dimensional system with dichotomous noise, one can derive a formula for the stationary density. For the sake of clarity, we follow [24] and restrict our discussion to the case of symmetric dichotomous noise where
\[
c_+ = -c_- = c \quad \text{and} \quad \alpha = \beta,
\]
and
\[
a(x, c_+)a(x, c_-) < 0,
\]
where the zeros of $a(x, c_+)a(x, c_-)$ are the boundaries of $X$. Then the unique stationary density of $P^t$ is given by
\[
f_*(x, c_\pm) = K \frac{1}{|a(x, c_\pm)|} \exp \left\{ -\alpha \int^x \left( \frac{1}{a(z, c_+)} + \frac{1}{a(z, c_-)} \right) dz \right\},
\]

where \( K \) is a normalizing constant.

As a specific example, consider the linear dichotomous flow

\[
\frac{dx}{dt} = -\gamma x + \xi,
\]

where \( \gamma > 0 \). Then \( c_\pm = \pm c \) and \( a(x, c_\pm) = -\gamma x \pm c \) for \( x \in \mathbb{R} \). Thus

\[
\pi_\pm^t(x) = xe^{-\gamma t} \pm \frac{c}{\gamma} (1 - e^{-\gamma t})
\]

and the state space is

\[
X = \left( \frac{-c}{\gamma}, \frac{c}{\gamma} \right).
\]

The stationary density \( f_* \) in this case is given by

\[
f_*(x, \pm c) = \frac{K}{\mp \gamma x + c} \left( \frac{c^2}{\gamma^2} - x^2 \right)^{\alpha/\gamma - 1},
\]

where the normalizing constant is equal to

\[
K = \frac{\gamma}{2B \left( \frac{1}{2}, \frac{\alpha}{\gamma} \right)} \left( \frac{c}{\gamma} \right)^{-2\alpha/\gamma}
\]

and \( B \) is the beta function. Since \( a(x, +c) > 0 > a(x, -c) \) for all \( x \in X \), Conditions 5.4 and 5.7 hold, where the stationary density \( p_* \) of the state variable \( x(t) \) is equal to

\[
p_*(x) = \frac{1}{B \left( \frac{1}{2}, \frac{\alpha}{\gamma} \right)} \left( \frac{c}{\gamma} \right)^{1-2\alpha/\gamma} \left( \frac{c^2}{\gamma^2} - x^2 \right)^{\alpha/\gamma - 1}.
\]

VI. DISCUSSION

Here we have examined the evolution of the conditional (or Kullback-Leibler or relative) entropy to a maximum in stochastic systems. We were motivated by a desire to understand the role of noise in the evolution of the conditional entropy to a maximum since in invertible systems (e.g. measure preserving systems of differential equations or invertible maps) the conditional entropy is fixed at the value with which the system is prepared. However the addition of noise can reverse this property and lead to an evolution of the conditional entropy to a maximum value of zero. We have made concrete calculations to see how the entropy converges, and shown that it is monotone and at least exponential in several situations.

Specifically, in Section III we introduced the dynamic concept of asymptotic stability and the notion of conditional entropy, and gave two main results connecting the convergence of the conditional entropy with asymptotic stability (Theorem 1), and the existence of unique stationary densities with the convergence of the conditional entropy (Theorem 2). In Section III we illustrated the well known fact that asymptotic stability is a property that cannot be found in a deterministic system of ordinary differential equations, and consequently that the conditional entropy cannot not have time dependent behaviours character for this type of invertible dynamics. Section IV introduced a stochastic extension to this invertible and constant entropy situation in which a system of ordinary equations is perturbed by Gaussian white noise (thus becoming non-invertible). We were able to give some general results showing that in this stochastic case asymptotic stability holds. Then in Section IV A we considered specific one dimensional examples, and showed that the conditional entropy convergence to zero is monotone and at least exponential, considering the specific examples of an Ornstein-Uhlenbeck process in Section IV A1 and a Rayleigh process in Section IV A2. We went on to look at multidimensional stochastic systems in Section IV B showing that the exponential convergence of
the entropy still holds. Specific examples of these higher dimensional situations were considered within the context of a two dimensional Ornstein-Uhlenbeck process in Section IV C with specific examples of a stochastically perturbed harmonic oscillator (Section IV C 1) and colored noise (Section IV C 2) as examples. In the specific cases of the Ornstein-Uhlenbeck and Rayleigh processes as well as the stochastically perturbed harmonic oscillator and colored noise examples, we have the rather surprising result that the rate of convergence of the entropy to zero is independent of the noise amplitude $\sigma$ as long as $\sigma > 0$. The final Section V applied our general results to the problem of conditional entropy convergence in the presence of dichotomous noise.

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APPENDIX

1. Proof of inequality 2.4

Jensen’s inequality states that when $\nu$ is a normalized measure and $\phi$ is a concave function then

$$\phi \left( \int g(z) \nu(dz) \right) \geq \int \phi(g(z)) \nu(dz).$$

Since log is a concave function, Jensen’s inequality implies

$$\int f \log \frac{f}{g} \mu(dx) \leq \log \int \frac{f}{g} \mu(dx).$$

Now note that

$$\int \left( \frac{f}{g} - 1 \right)^2 g \mu(dx) = \int \left[ \left( \frac{f}{g} \right)^2 g - 2 \frac{f}{g} g + 1 \right] \mu(dx)$$

$$= \int \left( \frac{f}{g} \right)^2 g \mu(dx) - 1.$$

Thus

$$- H_c(f|g) \leq \log \left[ 1 + \int \left( \frac{f}{g} - 1 \right)^2 g \mu(dx) \right]$$

$$\leq \int \left( \frac{f}{g} - 1 \right)^2 g \mu(dx).$$

2. Proof of inequality 4.55

To see how easy it is to derive this estimate, first write

$$H_c(P^t f_1|P^t f_2) = - \int P^t f_1(x) \log \frac{P^t f_1(x)}{P^t f_2(x)} dx$$

$$= - \int \varphi(P^t f_1(x), P^t f_2(x)),$$

where $\varphi(u, v) = v \log(u/v)$ is convex. From the properties of convex functions there always exist sequences of real numbers $\{a_n\}$ and $\{b_n\}$ such that

$$\varphi(u, v) = \sup\{a_n u + b_n v : n \in \mathbb{N}\}.$$

Remembering 4.55 we can then write
Thus
\[
\sup_{n \in \mathbb{N}} \{a_n P^t f_1(x) + b_n P^t f_2(x)\} = \varphi(P^t f_1(x), P^t f_2(x)) = \int \int f_1(y_1)f_2(y_2)\varphi(k(t, x, y_1), k(t, x, y_2))dy_1dy_2,
\]
so
\[
H_c(P^t f_1(x)|P^t f_2(x)) \geq -\int \int f_1(y_1)f_2(y_2) \left[ \int \varphi(k(t, x, y_1), k(t, x, y_2))dx \right]dy_1dy_2
\]
\[= -\int \int f_1(y_1)f_2(y_2)H_c(k(t, x, y_1), k(t, x, y_2))dy_1dy_2.
\]

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26. Voigt’s theorem [22] says that if $P$ is a Markov operator, then $H_c(Pf|Pg) \geq H_c(f|g)$ for all $f, g \in D$.
27. Given two independent Ornstein-Uhlenbeck processes $v_1(t)$ and $v_2(t)$ with parameters $\gamma$ and $\sigma$ as defined in Eq. 12.26 the two-dimensional radial Ornstein-Uhlenbeck process is then defined as $\sqrt{v_1(t)^2 + v_2(t)^2}$. 

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