Classical Solution of Field Equation of Gravitational Gauge Field and Classical Tests of Gauge Theory of Gravity

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November 1, 2021

Abstract

A systematic method is developed to study classical motion of a mass point in gravitational gauge field. First, the formulation of gauge theory of gravity in arbitrary curvilinear coordinates is given. Then in spherical coordinates system, a spherical symmetric solution of the field equation of gravitational gauge field is obtained, which is just the Schwarzschild solution. In gauge theory of gravity, the equation of motion of a classical mass point in gravitational gauge field is given by Newton’s second law of motion. A relativistic form of the gravitational force on a mass point is deduced in this paper. Based on the spherical symmetric solution of the field equation and Newton’s second law of motion, we can discuss classical tests of gauge theory of gravity, including the deflection of light by the sun, the precession of the perihelia of the orbits of the inner planets and the time delay of radar echoes passing the sun. It is found that the theoretical predictions of these classical tests given by gauge theory of gravity are completely the same as those given by general relativity. From the study in this paper, an important qualitative conclusion on the nature of gravity is that gravity can be treated as a kind of physical interactions in flat Minkowski space-time, and the equation of motion of mass point in gravitational field can be given by Newton’s second law of motion.

PACS Numbers: 04.80.Cc, 04.25.-g, 04.60.-m.
Keywords: Classical tests of gauge theory of gravity, gauge theory of gravity, classical solution of field equation, Newton’s second law of motion.

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1 Introduction

It is known that Einstein’s general relativity[1, 2] has passed several classical tests, including the deflection of light by the sun, the precession of the perihelia of the orbits of the inner planets and the time delay of radar echoes passing the sun[3, 4]. Basis of the calculation of these classical tests in general relativity is Schwarzschild solution and geodesic equation. Geodesic equation is given by the shortest possible path between two points, which is based on the concept of space-time geometry. In Einstein’s general theory of gravity, gravity is treated as geometry of curved space-time.

Quantum gauge theory of gravity[5, 6, 7, 8, 9, 10] is proposed in the framework of quantum field theory. In gauge theory of gravity, the field equation of gravitational gauge field is the same as Einstein’s field equation in general relativity, so two equations have the same solutions, though mathematical expressions of the two equations are completely different. Quantum gauge theory of gravity is a perturbatively renormalizable quantum theory, so based on it, quantum effects of gravity[11, 12, 13] and gravitational interactions of some basic quantum fields [14, 15] can be explored. Unification of fundamental interactions including gravity can be fulfilled in a simple and beautiful way[16, 17, 18]. If we use the mass generation mechanism which is proposed in literature [19, 20], we can propose a new theory on gravity which contains massive graviton and the introduction of massive graviton does not affect the strict local gravitational gauge symmetry of the Lagrangian and does not affect the traditional long-range gravitational force[21]. The existence of massive graviton will help us to understand the possible origin of dark matter.

It is known that the transcendental foundations of gauge theory of gravity is quite different from those of general relativity. Basic concept in gauge theory of gravity is that gravity is a kind of fundamental interactions in flat Minkowski space-time, which is transmitted by gravitons. In other words, in gauge theory of gravity, space-time is always flat, and geodesic curve is always a straight line. So, in gauge theory of gravity, geodesic equation can not be the equation of motion of a mass point in gravitational field. In order to discussed classical phenomenon of gravitational interactions, we need first to set up the equation of motion of a mass point in gravitational field, which is one of the central tasks of this paper. Based on the spirit of gauge principle and completely using physics language, Newton’s second law of motion is selected as an equation of motion for a mass point. Combine this equation of motion with classical solution of field equation, we can calculate theoretical expectations of classical tests of gauge theory of gravity. It is found that the quantitative values on classical tests given by gauge theory of gravity are completely the same as those given by general relativity, though two theories use different equa-
tions of motion in calculation and the basic concepts on gravity in two theories are quite different.

2 Basics of Gauge Theory of Gravity

For the sake of integrity, we give a simple introduction to gauge theory of gravity and introduce some notations which is used in this paper. Details on quantum gauge theory of gravity can be found in literatures [5, 6, 7, 8, 9, 10]. In gauge theory of gravity, the most fundamental quantity is gravitational gauge field \( C_\mu(x) \), which is the gauge potential corresponding to gravitational gauge symmetry. Gauge field \( C_\mu(x) \) is a vector in the corresponding Lie algebra, which, for the sake of convenience, will be called gravitational Lie algebra in this paper. So \( C_\mu(x) \) can be expanded as

\[
C_\mu(x) = C^\alpha_\mu(x) \hat{P}_\alpha, \quad (\mu, \alpha = 0, 1, 2, 3)
\]

where \( C^\alpha_\mu(x) \) is the component field and \( \hat{P}_\alpha = -i \frac{\partial}{\partial x_\alpha} \) is the generator of gravitational gauge group, which satisfies

\[
[\hat{P}_\alpha, \hat{P}_\beta] = 0.
\]

Unlike the ordinary \( SU(N) \) group, the commutability of the generators of the gravitational gauge group does not mean that the gravitational gauge group is an Abelian group. In fact, the gravitational gauge group is a non-Abelian group [5, 6, 7, 8, 9, 10]. The gravitational gauge covariant derivative is given by

\[
D_\mu = \partial_\mu - igC_\mu(x) = G^\alpha_\mu \partial_\alpha,
\]

where \( g \) is the gravitational coupling constant and matrix \( G \) is defined by

\[
G = (G^\alpha_\mu) = (\delta^\alpha_\mu - gC^\alpha_\mu).
\]

Matrix \( G \) is an important quantity in gauge theory of gravity. Its inverse matrix is denoted as \( G^{-1} \)

\[
G^{-1} = \frac{1}{I - gC} = (G^{-1})_{\alpha\mu}.
\]

Using matrix \( G \) and \( G^{-1} \), we can define two important composite operators

\[
g^{\alpha\beta} = \eta^{\mu\nu} C^\alpha_\mu G^\beta_\nu, \quad (2.6)
\]

\[
g_{\alpha\beta} = \eta_{\mu\nu} C^{-1}_\alpha^\mu G^{-1}_\beta^\nu, \quad (2.7)
\]

which are widely used in gauge theory of gravity. In gauge theory of gravity, space-time is always flat and space-time metric is always Minkowski metric, so \( g^{\alpha\beta} \) and

3
$g_{\alpha\beta}$ are no longer space-time metric. They are only two composite operators which consist of gravitational gauge field.

The field strength of gravitational gauge field is defined by

$$F_{\mu\nu} \triangleq \frac{1}{-ig}[D_\mu, D_\nu].$$  \hfill (2.8)

Its explicit expression is

$$F_{\mu\nu}(x) = \partial_\mu C_\nu(x) - \partial_\nu C_\mu(x) - igC_\mu(x)C_\nu(x) + igC_\nu(x)C_\mu(x).$$  \hfill (2.9)

$F_{\mu\nu}$ is also a vector in gravitational Lie algebra,

$$F_{\mu\nu}(x) = F^\alpha_{\mu\nu}(x) \cdot \hat{P}_\alpha,$$  \hfill (2.10)

where

$$F^\alpha_{\mu\nu} = \partial_\mu C^\alpha_\nu - \partial_\nu C^\alpha_\mu - gC^\beta_\mu \partial_\beta C^\alpha_\nu + gC^\beta_\nu \partial_\beta C^\alpha_\mu.$$  \hfill (2.11)

Using matrix $G$, its expression can be written in a simpler form

$$F^\alpha_{\mu\nu} = G^\beta_\mu \partial_\beta C^\alpha_\nu - G^\beta_\nu \partial_\beta C^\alpha_\mu.$$  \hfill (2.12)

In gauge theory of gravity, gravitational gauge field $C^\alpha_\mu$ is a spin-2 tensor field, and the lagrange for pure gravitational gauge field is selected to be[5, 6]

$$L_0 = -\frac{1}{16}\eta^{\mu\rho}\eta^{\nu\sigma}g_{\alpha\beta}F^\alpha_{\mu\nu}F^\beta_{\rho\sigma}$$

$$-\frac{1}{8}\eta^{\mu\rho}G^{-1\nu}_\beta G^{-1\sigma}_\alpha F^\alpha_{\mu\nu}F^\beta_{\rho\sigma}$$

$$+\frac{1}{4}\eta^{\mu\rho}G^{-1\nu}_\alpha G^{-1\sigma}_\beta F^\alpha_{\mu\nu}F^\beta_{\rho\sigma}.$$  \hfill (2.13)

The action is defined by

$$S = \int d^4x \sqrt{-\det(g_{\alpha\beta})}\cdot L_0.$$  \hfill (2.14)

It can be proved that the above action $S$ is invariant under gravitational gauge transformation, therefore the system has gravitational gauge symmetry[5, 6].

The Euler-Lagrange equation for gravitational gauge field is

$$\partial_\mu \frac{\partial L_0}{\partial \partial_\mu C^\alpha_\nu} = \frac{\partial L_0}{\partial C^\alpha_\nu} + gG^{-1\nu}_\alpha L_0 - gG^{-1\lambda}_\gamma (\partial_\mu C^\lambda_\gamma) \frac{\partial L_0}{\partial \partial_\mu C^\alpha_\nu},$$  \hfill (2.15)
which gives the following field equation of gravitational gauge field

\[
\partial_\mu \left( \frac{1}{4} \eta^{\rho \sigma} \eta^{\nu \sigma} g_{\alpha \beta} F^\beta_{\rho \sigma} - \frac{1}{4} \eta^{\rho \nu} F^\mu_{\rho \alpha} + \frac{1}{4} \eta^{\rho \nu} F^\nu_{\rho \alpha} \right)
- \frac{1}{2} \eta^{\mu \nu} \delta^\alpha_{\rho \beta} F^\beta_{\rho \alpha} + \frac{1}{2} \eta^{\mu \rho} \delta^\nu_{\rho \beta} F^\beta_{\rho \alpha} = -g T^\nu_{\alpha \beta},
\]

(2.16)

where

\[
T^\nu_{\alpha \beta} = \frac{1}{4} \eta^{\nu \sigma} \eta^{\lambda \rho} g_{\beta \gamma} F^\gamma_{\rho \sigma} (\partial_\lambda g_{\alpha \gamma}) + \frac{1}{4} \eta^{\nu \lambda} \eta^{\mu \rho} g_{\beta \gamma} F^\gamma_{\rho \sigma} (\partial_\mu g_{\alpha \gamma}) - \frac{1}{2} \eta^{\nu \lambda} \eta^{\mu \rho} g_{\beta \gamma} F^\gamma_{\rho \sigma} (\partial_\mu g_{\alpha \gamma}) - \frac{1}{2} \eta^{\nu \lambda} \eta^{\mu \rho} g_{\beta \gamma} F^\gamma_{\rho \sigma} (\partial_\mu g_{\alpha \gamma})
+ \frac{1}{2} \eta^{\nu \lambda} \eta^{\mu \rho} g_{\beta \gamma} F^\gamma_{\rho \sigma} (\partial_\mu g_{\alpha \gamma}) - \frac{1}{2} \eta^{\nu \lambda} \eta^{\mu \rho} g_{\beta \gamma} F^\gamma_{\rho \sigma} (\partial_\mu g_{\alpha \gamma})
+ \frac{1}{2} \eta^{\nu \lambda} \eta^{\mu \rho} g_{\beta \gamma} F^\gamma_{\rho \sigma} (\partial_\mu g_{\alpha \gamma}) - \frac{1}{2} \eta^{\nu \lambda} \eta^{\mu \rho} g_{\beta \gamma} F^\gamma_{\rho \sigma} (\partial_\mu g_{\alpha \gamma})
+ \frac{1}{2} \eta^{\nu \lambda} \eta^{\mu \rho} g_{\beta \gamma} F^\gamma_{\rho \sigma} (\partial_\mu g_{\alpha \gamma}) - \frac{1}{2} \eta^{\nu \lambda} \eta^{\mu \rho} g_{\beta \gamma} F^\gamma_{\rho \sigma} (\partial_\mu g_{\alpha \gamma})
+ \frac{1}{2} \eta^{\nu \lambda} \eta^{\mu \rho} g_{\beta \gamma} F^\gamma_{\rho \sigma} (\partial_\mu g_{\alpha \gamma}) - \frac{1}{2} \eta^{\nu \lambda} \eta^{\mu \rho} g_{\beta \gamma} F^\gamma_{\rho \sigma} (\partial_\mu g_{\alpha \gamma})
+ \frac{1}{2} \eta^{\nu \lambda} \eta^{\mu \rho} g_{\beta \gamma} F^\gamma_{\rho \sigma} (\partial_\mu g_{\alpha \gamma}) - \frac{1}{2} \eta^{\nu \lambda} \eta^{\mu \rho} g_{\beta \gamma} F^\gamma_{\rho \sigma} (\partial_\mu g_{\alpha \gamma})
\]

(2.17)

\( T^\nu_{\alpha \beta} \) is the gravitational energy-momentum tensor, which is the source of gravitational field. It can be proved that this field equation is the same as the Einstein’s field equation[5, 6].

Gauge theory of gravity is formulated in physics picture of gravity, where gravity is treated as a kind of fundamental interactions and space-time is always flat. But for classical problems, we can also set up a geometry picture of gravity, where gravity is equivalently treated as geometry of space-time[22]. In this equivalent space-time geometry, \( g_{\alpha \beta} \) and \( g^{\alpha \beta} \) are equivalent metric of the curved space-time. From this equivalent metric, we can calculate affine connection

\[
\Gamma^\gamma_{\alpha \beta} = \frac{1}{2} g^{\gamma \delta} \left( \frac{\partial g_{\alpha \delta}}{\partial x^\beta} + \frac{\partial g_{\beta \delta}}{\partial x^\alpha} - \frac{\partial g_{\alpha \beta}}{\partial x^\delta} \right),
\]

(2.18)
and curvature tensor

\[ R^\delta_{\alpha\beta\gamma} \triangleq \partial_\gamma \Gamma^\delta_{\alpha\beta} - \partial_\beta \Gamma^\delta_{\gamma\alpha} + \Gamma^\eta_{\alpha\beta} \Gamma^\delta_{\gamma\eta} - \Gamma^\eta_{\alpha\gamma} \Gamma^\delta_{\beta\eta}, \quad (2.19) \]

Recci tensor \( R_{\alpha\gamma} \) is defined by

\[ R_{\alpha\gamma} \triangleq R^\beta_{\alpha\beta\gamma}. \quad (2.20) \]

Operator \( R_{\alpha\gamma} \) can also be calculated from the following relation

\[ R_{\alpha\gamma} = \frac{1}{2} g^{\beta\delta} (\partial_\eta g_{\alpha\gamma} - \partial_\alpha \partial_\beta g_{\gamma\eta} - \partial_\delta \partial_\gamma g_{\alpha\beta} + \partial_\alpha \partial_\gamma g_{\beta\delta}) \]

\[ + g^{\beta\delta} g_{\alpha\beta_1} (\Gamma^\beta_{\alpha\gamma} \Gamma^\beta_{\gamma\delta} - \Gamma^\alpha_{\beta\delta} \Gamma^\beta_{\gamma\eta}). \quad (2.21) \]

After rather lengthy and complicated calculations, we can prove that the operator \( R_{\alpha\beta} \) can be explicitly expressed as

\[ R_{\alpha\beta} = -\frac{g^2}{4} \eta^{\mu\rho} \eta^{\nu\sigma} g_{\alpha\gamma_1} G_{\beta\gamma_1 \mu\nu} F_{\mu\nu} F_{\rho\sigma} \]

\[ + \frac{g^2}{2} \eta^{\mu\rho} g_{\alpha_1 \beta_1} G^{-1}_{\alpha \beta} G^{1\mu} F_{\mu\nu} F_{\rho\sigma} \]

\[ + \frac{g^2}{2} G^{-1\mu}_{\beta} G^{-1\nu}_{\alpha} G^{-1\sigma}_{\alpha_1} F_{\mu\nu} F_{\rho\sigma} \]

\[ + \frac{g^2}{2} \eta^{\mu\sigma} (g_{\alpha_1 \alpha} G^{-1\mu}_{\beta} + g_{\alpha_1 \beta} G^{-1\mu}_{\alpha}) G^{-1\mu}_{\beta_1} F_{\mu\nu} F_{\rho\sigma} \]

\[ + \frac{g}{2} (G^{-1\mu}_{\beta} \partial_\alpha + G^{-1\mu}_{\alpha} \partial_\beta) (G^{-1\nu}_{\gamma} F_{\mu\nu}) \]

\[ + \frac{g}{2} \eta_{\mu_1 \nu_1} g^{\beta_1} (G^{-1\mu_1}_{\alpha} G^{-1\nu}_{\beta} + G^{-1\mu_1}_{\beta} G^{-1\nu}_{\alpha}) G^{-1\mu_1}_{\beta_1} \partial_\delta (G^{-1\nu_1}_{\gamma} F_{\mu\nu}). \quad (2.22) \]

The scalar curvature \( R \) is defined by

\[ R \triangleq g^{\alpha\beta} R_{\alpha\beta}. \quad (2.23) \]

The explicit expression for the operator \( R \) is

\[ R = R_0 + \frac{2g}{J(C)} \partial_\beta \left( J(C) g^{\alpha\beta} G^{-1\mu}_{\alpha} G^{-1\nu}_{\gamma} F_{\mu\nu} \right), \quad (2.24) \]

where

\[ R_0 = -16\pi G L_0, \quad (2.25) \]

with \( G \) the Newtonian gravitation constant. Its relation to gravitation coupling constant \( g \) is given by the following formula

\[ G = \frac{g^2}{4\pi}. \quad (2.26) \]
Because total derivative term in the Lagrangian has no contribution to Euler-Lagrange equation of motion, selecting $L_0$ as Lagrangian is equivalent to selecting $-\frac{1}{16\pi G} R$ as Lagrangian. In general relativity, $-\frac{1}{16\pi G} R$ is selected as Lagrangian of gravitational field, the field equation given by this selection is the Einstein’s field equation

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + 8\pi G T_{\alpha\beta} = 0,$$

(2.27)

where $T_{\alpha\beta}$ is the energy-momentum tensor of matter field.

Now, from the same action, the least action principle gives two equations (2.16) and (2.27). They are different in forms, but they are essentially the same, for one action can only give only one field equation. In deed, we can strictly prove that these two field equations are essentially the same. If define

$$W^\nu_\alpha = -\partial_\mu \frac{\partial L_0}{\partial C^\alpha_\nu} + \frac{\partial L_0}{\partial C^\nu_\alpha} + gG^{\nu-1\nu} L_0 - gG^{\nu-1\nu} \frac{\partial}{\partial \mu} \frac{\partial L_0}{\partial \mu C^\nu_\alpha},$$

(2.28)

then field equation (2.16) can be simply expressed as

$$W^\nu_\alpha = 0.$$

(2.29)

It can be strictly proved that

$$W^\nu_\alpha \eta_{\nu\beta} G^{\nu-1\mu}_\beta = \frac{1}{2g}(R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R).$$

(2.30)

Therefor, field equation (2.29) just gives the Einstein’s field equation. So, in quantum gauge general relativity, the field equation of gravitational gauge field is just the Einstein’s field equation.

### 3 Formulation of Gauge Theory of Gravity in Arbitrary Curvilinear Coordinate System

Gauge theory of gravity is proposed in physics picture of gravity, where space-time is always flat. All above discussions are performed in Descartes coordinate system and the above expressions of mathematical formula are only valid in Descartes coordinate system. In order to solve spherical symmetric classical solution of field equation (2.16), we need to formulate gauge theory of gravity in arbitrary curvilinear coordinate system.
Denote space-time coordinates in Descartes coordinate system as \( x^\mu \). Making the following coordinates transformations

\[
x \rightarrow y = y(x),
\]

where \( y^\mu \) are arbitrary curvilinear coordinates. Obviously, we have

\[
dy^{\alpha_1} = \frac{\partial y^{\alpha_1}}{\partial x^\alpha} \, dx^\alpha, \quad (3.2)
\]
\[
\frac{\partial}{\partial y^{\alpha_1}} = \frac{\partial x^\alpha}{\partial y^{\alpha_1}} \frac{\partial}{\partial x^\alpha}. \quad (3.3)
\]

In this chapter, indexes \( \alpha, \beta, \gamma, \mu, \nu, \lambda, \cdots \) are used to denote indexes of Descartes coordinate system, while indexes \( \alpha_1, \beta_1, \gamma_1, \mu_1, \nu_1, \lambda_1, \cdots \) are used to denote indexes of curvilinear coordinate system. This convention is only valid in this chapter, it is no longer valid in other chapters of this paper.

Suppose that \( A^\alpha \) and \( B_\beta \) are two arbitrary vectors, under coordinate transformation (3.1), they transforms as

\[
A^\alpha \rightarrow A'^{\alpha_1} = \frac{\partial y^{\alpha_1}}{\partial x^\alpha} A^\alpha, \quad (3.4)
\]
\[
B_\alpha \rightarrow B'_{\alpha_1} = \frac{\partial x^\alpha}{\partial y^{\alpha_1}} B_\alpha. \quad (3.5)
\]

Space-time metrics \( \eta^{\mu\nu} \) and \( \eta_{\mu\nu} \) are second order tensors, so they transform as

\[
\eta^{\mu\nu} \rightarrow \eta'^{\mu_1\nu_1} = \frac{\partial y^{\mu_1}}{\partial x^\mu} \frac{\partial y^{\nu_1}}{\partial x^\nu} \eta^{\mu\nu}, \quad (3.6)
\]
\[
\eta_{\mu\nu} \rightarrow \eta'_{\mu_1\nu_1} = \frac{\partial x^\mu}{\partial y^{\mu_1}} \frac{\partial x^\nu}{\partial y^{\nu_1}} \eta_{\mu\nu}, \quad (3.7)
\]

where \( \eta'^{\mu_1\nu_1} \) and \( \eta'_{\mu_1\nu_1} \) are space-time metric in curvilinear coordinate system. The affine connection in curvilinear coordinate system is defined by

\[
\Gamma^\lambda_{\mu_1\nu_1} = \frac{1}{2} \eta'^{\lambda_1\sigma_1} \left( \frac{\partial \eta'_{\mu_1\sigma_1}}{\partial y^{\nu_1}} + \frac{\partial \eta'_{\nu_1\sigma_1}}{\partial y^{\mu_1}} - \frac{\partial \eta'_{\mu_1\nu_1}}{\partial y^{\lambda_1}} \right). \quad (3.8)
\]

It can be expressed in other forms

\[
\Gamma^\lambda_{\mu_1\nu_1} = \frac{\partial y^{\lambda_1}}{\partial x^\nu} \frac{\partial x^\mu}{\partial y^{\nu_1}} \frac{\partial x^\nu}{\partial y^{\lambda_1}} = -\frac{\partial x^\nu}{\partial y^{\nu_1}} \frac{\partial x^\mu}{\partial y^{\nu_1}} \frac{\partial^2 y^{\lambda_1}}{\partial x^\nu \partial x^\nu}. \quad (3.9)
\]
Using above relations, we can prove that

\[
\frac{\partial A^\mu}{\partial x^\alpha} = \frac{\partial x^\mu}{\partial y^{\alpha_1}} \frac{\partial y^{\alpha_1}}{\partial x^\alpha} (\nabla_{\alpha_1} A'^{\mu_1}), \tag{3.10}
\]

\[
\frac{\partial B_\mu}{\partial x^\alpha} = \frac{\partial y^{\mu_1}}{\partial x^\mu} \frac{\partial y^{\alpha_1}}{\partial x^\alpha} (\nabla_{\alpha_1} B'_\mu), \tag{3.11}
\]

\[
\frac{d}{d\tau} A^\mu = \frac{d}{d\tau} \frac{D}{D\tau} A'^{\mu_1}, \tag{3.12}
\]

\[
\frac{d}{d\tau} B_\mu = \frac{d}{d\tau} \frac{D}{D\tau} B'_\mu, \tag{3.13}
\]

where \(\nabla_{\alpha_1}\) and \(\frac{D}{D\tau}\) are the covariant derivatives in curvilinear coordinate system, which are defined by

\[
\nabla_{\alpha_1} A'^{\mu_1} = \frac{\partial A'^{\mu_1}}{\partial y^{\alpha_1}} + \Gamma_{\alpha_1\nu_1}^{\mu_1} A'^{\nu_1}, \tag{3.14}
\]

\[
\nabla_{\alpha_1} B'_\mu = \frac{\partial B'_\mu}{\partial y^{\alpha_1}} - \Gamma_{\alpha_1\mu_1}^{\nu_1} B'_\nu, \tag{3.15}
\]

\[
\frac{D}{D\tau} A'^{\mu_1} = \frac{d}{d\tau} A'^{\mu_1} + \Gamma_{\alpha_1\nu_1}^{\mu_1} \frac{dy^{\alpha_1}}{d\tau} A'^{\nu_1}, \tag{3.16}
\]

\[
\frac{D}{D\tau} B'_\mu = \frac{d}{d\tau} B'_\mu - \Gamma_{\alpha_1\mu_1}^{\nu_1} \frac{dy^{\alpha_1}}{d\tau} B'_\nu. \tag{3.17}
\]

Therefore, under curvilinear transformations (3.1), we have

\[
\frac{\partial A^\mu}{\partial x^\alpha} \to \nabla_{\alpha_1} A'^{\mu_1} = \frac{\partial x^\mu}{\partial y^{\alpha_1}} \frac{\partial y^{\alpha_1}}{\partial x^\alpha} \left( \frac{\partial A^\mu}{\partial x^\alpha} \right), \tag{3.18}
\]

\[
\frac{\partial B_\mu}{\partial x^\alpha} \to \nabla_{\alpha_1} B'_\mu = \frac{\partial x^\mu}{\partial y^{\alpha_1}} \frac{\partial y^{\alpha_1}}{\partial x^\alpha} \left( \frac{\partial B_\mu}{\partial x^\alpha} \right), \tag{3.19}
\]

\[
\frac{d}{d\tau} A^\mu \to \frac{D}{D\tau} A'^{\mu_1} = \frac{d}{d\tau} \frac{D}{D\tau} A'^{\mu_1} \left( \frac{d}{d\tau} A^\mu \right), \tag{3.20}
\]

\[
\frac{d}{d\tau} B_\mu \to \frac{D}{D\tau} B'_\mu = \frac{d}{d\tau} \frac{D}{D\tau} B'_\mu \left( \frac{d}{d\tau} B_\mu \right). \tag{3.21}
\]

Gravitational gauge field \(C_\mu^\alpha\) is a second order tensor under curvilinear coordinate transformations, therefore its transformation is

\[
C_\mu^\alpha \to C'^{\alpha_1}_\mu_1 = \frac{\partial x^\mu}{\partial y^{\alpha_1}} \frac{\partial y^{\alpha_1}}{\partial x^\alpha} C_\mu^\alpha. \tag{3.22}
\]

9
Then, from equations (2.4) and (2.5), we can prove that

$$G_{\mu}^{\alpha} \rightarrow G_{\mu_1}^{\alpha_1} = \frac{\partial x^\mu}{\partial y^{\alpha_1}} \frac{\partial y^{\alpha_1}}{\partial x^\mu} G_{\mu}^{\alpha},$$

(3.23)

$$G_{\alpha}^{-1\mu} \rightarrow G_{\alpha_1}^{-1\mu_1} = \frac{\partial y^{\mu_1}}{\partial x^\mu} \frac{\partial x^\mu}{\partial y^{\alpha_1}} G_{\alpha}^{-1\mu}.$$  

(3.24)

Applying the above two relations and equations (2.6) and (2.7), we find that

$$g^{\alpha\beta} \rightarrow g^{\alpha_1\beta_1} = \frac{\partial y^{\alpha_1}}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial y^{\beta_1}} \frac{\partial y^{\beta_1}}{\partial x^\beta} g^{\alpha\beta},$$

(3.25)

$$g_{\alpha\beta} \rightarrow g'_{\alpha_1\beta_1} = \frac{\partial x^\alpha}{\partial y^{\alpha_1}} \frac{\partial y^{\beta_1}}{\partial x^\beta} g_{\alpha\beta}.$$  

(3.26)

Using equations (3.18), (3.19) and (3.22), we can prove that

$$\frac{\partial}{\partial x^\beta} C_{\mu}^{\alpha} \rightarrow \nabla_{\beta_1} C_{\mu_1}^{\alpha_1} = \frac{\partial y^{\alpha_1}}{\partial x^\alpha} \frac{\partial x^\beta}{\partial y^{\beta_1}} \frac{\partial y^{\beta_1}}{\partial x^\mu} \left( \frac{\partial}{\partial x^\beta} C_{\mu}^{\alpha} \right),$$

(3.27)

$$F_{\mu\nu}^{\alpha} \rightarrow F'_{\mu_1\nu_1}^{\alpha_1} = \frac{\partial y^{\alpha_1}}{\partial x^\mu} \frac{\partial x^\mu}{\partial y^{\nu_1}} \frac{\partial y^{\nu_1}}{\partial x^\nu} F_{\mu\nu}^{\alpha},$$

(3.28)

where

$$\nabla_{\beta_1} C_{\mu_1}^{\alpha_1} = \frac{\partial}{\partial y^{\beta_1}} C_{\mu_1}^{\alpha_1} + \Gamma_{\beta_1\gamma_1}^{\alpha_1} C_{\mu_1}^{\gamma_1} - \Gamma_{\beta_1\mu_1}^{\alpha_1} C_{\nu_1}^{\alpha_1},$$

(3.29)

$$F'_{\mu_1\nu_1}^{\alpha_1} = G_{\mu_1}^{\alpha_1} \nabla_{\beta_1} C_{\nu_1}^{\alpha_1} - G_{\nu_1}^{\alpha_1} \nabla_{\beta_1} C_{\mu_1}^{\alpha_1}$$

$$= C_{\mu_1}^{\beta_1} \left[ \frac{\partial}{\partial y^{\beta_1}} C_{\nu_1}^{\alpha_1} + \Gamma_{\beta_1\gamma_1}^{\alpha_1} C_{\mu_1}^{\gamma_1} - \Gamma_{\beta_1\mu_1}^{\alpha_1} C_{\nu_1}^{\alpha_1} \right]

- C_{\nu_1}^{\beta_1} \left[ \frac{\partial}{\partial y^{\beta_1}} C_{\mu_1}^{\alpha_1} + \Gamma_{\beta_1\gamma_1}^{\alpha_1} C_{\nu_1}^{\gamma_1} - \Gamma_{\beta_1\nu_1}^{\alpha_1} C_{\mu_1}^{\alpha_1} \right].$$

(3.30)

In conclusion, under curvilinear coordinate transformations, all physical quantities transform covariantly. Physical equations will not change their forms under this transformation if we replace ordinary derivatives $\partial_{\alpha}$ and $\frac{d}{d\tau}$ with covariant derivatives $\nabla_{\alpha}$ and $\frac{\partial}{\partial \tau}$. Using this way, we can formulate gauge theory of gravity in arbitrary curvilinear coordinate system.
4 Classical Spherical Symmetric Solution of Field Equation of Gravitational Gauge Field

In the last chapter, formulation of gauge theory of gravity in arbitrary curvilinear coordinate system is discussed. Based on the results in the last chapter, we can obtain the following field equation of gravitational gauge field in arbitrary curvilinear coordinate system from equation (2.16)

\[
\nabla_\mu \left( \frac{1}{4} \eta^{\mu\rho} \eta^{\sigma\tau} g_{\alpha\beta} F^\beta_{\rho\sigma} - \frac{1}{4} \eta^{\mu\rho} F^{\mu}_{\rho\alpha} + \frac{1}{4} \eta^{\mu\rho} F^\nu_{\rho\alpha} \right) - \frac{1}{4} \eta^{\mu\rho} \delta_\alpha^\nu F^\beta_{\rho\beta} + \frac{1}{4} \eta^{\mu\rho} \delta_\alpha^\mu F^\beta_{\rho\beta} = -g T^\nu_{ga},
\]

where \( T^\nu_{ga} \) is given by

\[
T^\nu_{ga} = \frac{1}{4} \eta^{\mu\rho} \eta^{\lambda\sigma} g_{\beta\gamma} F^\beta_{\rho\sigma} (\nabla_\alpha C^\gamma_{\lambda}) + \frac{1}{4} \eta^{\mu\rho} G^{-1}_{\alpha} G^{-1}_{\gamma} F^\beta_{\rho\sigma} (\nabla_\alpha C^\gamma_{\lambda}) - \frac{1}{4} \eta^{\mu\rho} G^{-1}_{\alpha} G^{-1}_{\gamma} F^\beta_{\rho\sigma} (\nabla_\alpha C^\gamma_{\lambda}) - \frac{1}{4} \eta^{\mu\rho} G^{-1}_{\alpha} G^{-1}_{\gamma} F^\beta_{\rho\sigma} (\nabla_\alpha C^\gamma_{\lambda}) + \frac{1}{4} \eta^{\mu\rho} G^{-1}_{\alpha} G^{-1}_{\gamma} F^\beta_{\rho\sigma} (\nabla_\alpha C^\gamma_{\lambda}) + \frac{1}{4} \eta^{\mu\rho} g_{\alpha\beta} G^{\gamma}_{\rho\sigma} (G^\delta_{\lambda} \nabla_\delta C^\gamma_{\mu}) - \frac{1}{4} \eta^{\mu\rho} g_{\alpha\beta} G^{\gamma}_{\rho\sigma} (G^\delta_{\lambda} \nabla_\delta C^\gamma_{\mu}) + \frac{1}{4} \eta^{\mu\rho} g_{\alpha\beta} G^{\gamma}_{\rho\sigma} (G^\delta_{\lambda} \nabla_\delta C^\gamma_{\mu}) - \frac{1}{4} \eta^{\mu\rho} g_{\alpha\beta} G^{\gamma}_{\rho\sigma} (G^\delta_{\lambda} \nabla_\delta C^\gamma_{\mu}) + \frac{1}{4} \eta^{\mu\rho} g_{\alpha\beta} C^{\alpha}_{\lambda} G^{\beta}_{\rho\sigma} F^\gamma_{\rho\sigma} \left( \nabla_\delta C^\gamma_{\lambda} \right) + \frac{1}{4} \eta^{\mu\rho} g_{\alpha\beta} C^{\alpha}_{\lambda} G^{\beta}_{\rho\sigma} F^\gamma_{\rho\sigma} \left( \nabla_\delta C^\gamma_{\lambda} \right) + \frac{1}{4} \eta^{\mu\rho} g_{\alpha\beta} C^{\alpha}_{\lambda} G^{\beta}_{\rho\sigma} F^\gamma_{\rho\sigma} \left( \nabla_\delta C^\gamma_{\lambda} \right) + \frac{1}{4} \eta^{\mu\rho} g_{\alpha\beta} C^{\alpha}_{\lambda} G^{\beta}_{\rho\sigma} F^\gamma_{\rho\sigma} \left( \nabla_\delta C^\gamma_{\lambda} \right) + \frac{1}{4} \eta^{\mu\rho} \nabla_\mu (g_{\alpha\beta} C^{\alpha}_{\lambda} F^\gamma_{\rho\sigma}) - \frac{1}{4} \eta^{\mu\rho} \nabla_\mu (C^{\alpha}_{\lambda} G^{\beta}_{\rho\sigma} F^\gamma_{\rho\sigma}) + \frac{1}{4} \eta^{\mu\rho} \nabla_\mu (C^{\alpha}_{\lambda} g_{\alpha\beta} C^{\beta}_{\gamma} F^\gamma_{\rho\sigma}) + \frac{1}{4} \eta^{\mu\rho} \nabla_\mu (C^{\alpha}_{\lambda} g_{\alpha\beta} C^{\beta}_{\gamma} F^\gamma_{\rho\sigma}) + \frac{1}{4} \eta^{\mu\rho} \nabla_\mu \left( \left( G_{\beta}^{\gamma_{\nu}} C_{\alpha}^{\mu} - \delta_{\nu}^{\lambda} \delta_{\alpha}^{\mu} \right) F^\beta_{\rho\sigma} \right) - \frac{1}{4} \eta^{\mu\rho} \nabla_\mu \left( \left( G_{\gamma_{\nu}}^{\alpha} C_{\beta}^{\mu} - \delta_{\nu}^{\lambda} \delta_{\beta}^{\mu} \right) F^\gamma_{\rho\sigma} \right) - \frac{1}{4} \eta^{\mu\rho} C_{\beta}^{\gamma_{\nu}} G_{\alpha}^{\mu} F^\beta_{\rho\sigma} F^\gamma_{\rho\sigma} - \frac{1}{4} \eta^{\mu\rho} C_{\gamma}^{\alpha} G_{\beta}^{\mu} F^\gamma_{\rho\sigma} F^\beta_{\rho\sigma} - \frac{1}{4} \eta^{\mu\rho} g_{\alpha\gamma} G_{\beta}^{\mu} F^\gamma_{\rho\sigma} F^\beta_{\rho\sigma} - \frac{1}{4} \eta^{\mu\rho} g_{\beta\gamma} G_{\alpha}^{\mu} F^\gamma_{\rho\sigma} F^\beta_{\rho\sigma} + \frac{1}{4} \eta^{\mu\rho} g_{\alpha\gamma} G_{\beta}^{\mu} F^\gamma_{\rho\sigma} F^\beta_{\rho\sigma} + \frac{1}{4} \eta^{\mu\rho} g_{\beta\gamma} G_{\alpha}^{\mu} F^\gamma_{\rho\sigma} F^\beta_{\rho\sigma}.
\]

where \( \eta^{\mu\rho} \) is the space-time metric in curvilinear coordinate system, \( \nabla_\alpha \) is the covariant derivative, \( g_{\alpha\beta} \) is a composite operator defined by equation (2.7), and \( F^\alpha_{\mu\nu} \) is the field strength of gravitational gauge field which is defined by

\[
F^\alpha_{\mu\nu} = G^\beta_{\mu} \nabla_\beta C^\alpha_{\nu} - G^\beta_{\nu} \nabla_\beta C^\alpha_{\mu}.
\]
The affine connection in curvilinear coordinate system is defined by
\[
\Gamma^\lambda_{\mu\nu} = \frac{1}{2} \eta^{\lambda\sigma} \left( \partial_\nu \eta_{\mu\sigma} + \partial_\mu \eta_{\nu\sigma} - \partial_\sigma \eta_{\mu\nu} \right). \tag{4.4}
\]

The above field equation is a little complicated. Based on this equation, we can solve it and obtain its classical solution. We can use Mathematica to perform these calculations. The equation (4.2) can be equivalently expressed in another form. Denote
\[
W^{\nu}_{1\alpha} = \frac{\partial \mathcal{L}_0}{\partial C^\alpha_{\nu}}, \tag{4.5}
\]
\[
W^{\mu\nu}_{2\alpha} = \frac{\partial \mathcal{L}_0}{\partial \partial_\mu C^\nu_{\alpha}}. \tag{4.6}
\]
In curvilinear coordinate system, they are given by
\[
W^{\nu}_{1\alpha} = \frac{\eta^{\nu\rho} \eta^{\mu\sigma} g_{\beta\gamma} F^\beta_{\rho\sigma} (\nabla_\alpha C^\gamma_{\mu})}{4} + \frac{\eta^{\nu\rho} G^{-1}_\gamma G^{-1}_\sigma F^\beta_{\rho\sigma} (\nabla_\alpha C^\gamma_{\mu})}{4} - \frac{\eta^{\nu\rho} G^{-1}_\gamma G^{1\sigma} F^\beta_{\rho\sigma} (\nabla_\alpha C^\gamma_{\mu})}{4} - \frac{\eta^{\nu\rho} G^{-1\sigma} G^{-1\gamma} F^\beta_{\rho\sigma} (\nabla_\alpha C^\gamma_{\mu})}{4} + \frac{\eta^{\nu\rho} G^{-1\sigma} G^{1\gamma} F^\beta_{\rho\sigma} (\nabla_\alpha C^\gamma_{\mu})}{4} + \frac{\eta^{\nu\rho} \eta^{\mu\sigma} g_{\alpha\gamma} G^{-1\rho} F^\beta_{\rho\sigma} F^\gamma_{\mu\lambda}}{8} - \frac{\eta^{\nu\rho} \eta^{\mu\sigma} G^{-1\rho} G^{-1\alpha} G^{-1\gamma} F^\beta_{\rho\sigma} F^\gamma_{\mu\lambda}}{4} + \frac{\eta^{\nu\rho} \eta^{\mu\sigma} G^{-1\rho} G^{-1\alpha} G^{1\gamma} F^\beta_{\rho\sigma} F^\gamma_{\mu\lambda}}{4}. \tag{4.7}
\]
and
\[
W^{\mu\nu}_{2\alpha} = -\frac{\eta^{\nu\rho} \eta^{\mu\sigma} g_{\alpha\beta} G^\mu_{\lambda} F^\beta_{\rho\sigma}}{4} + \frac{\eta^{\nu\rho} G^{-1\rho} F^\mu_{\rho\sigma}}{4} - \frac{\eta^{\nu\rho} G^{1\rho} G^{-1\sigma} G^\mu_{\lambda} F^\beta_{\rho\sigma}}{4} + \frac{\eta^{\nu\rho} G^{1\rho} G^{-1\gamma} G^\mu_{\lambda} F^\beta_{\rho\sigma}}{4} - \frac{\eta^{\nu\rho} \delta^\mu_{\alpha} G^{-1\sigma} F^\beta_{\rho\sigma}}{4}. \tag{4.8}
\]
Define
\[ W_\alpha^\nu = -\nabla_\mu W_{2\alpha}^{\mu\nu} + W_{1\alpha}^\nu + gG_{\alpha}^{-1\nu}L_0 - gG_{\beta}^{-1\lambda}(\nabla_\mu C_\lambda^\beta)W_{2\alpha}^{\mu\nu}, \quad (4.9) \]
then field equation (4.1) can be expressed by
\[ W_\alpha^\nu = 0. \quad (4.10) \]

Now, we begin to solve field equation (4.10). In spherical symmetric case, gravitational gauge field \( C^\alpha_\mu \) is selected to be
\[ gC^\alpha_\mu = \begin{pmatrix} 1 - \frac{1}{B(r)} & 0 & 0 & 0 \\ 0 & 1 - \frac{1}{A(r)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4.11) \]
In this case, matrix \( G \) and \( G^{-1} \) are respectively given by
\[ G = (G^\alpha_\mu) = \begin{pmatrix} \frac{1}{B(r)} & 0 & 0 & 0 \\ 0 & \frac{1}{A(r)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (4.12) \]
and
\[ G^{-1} = (G^{-1}_\alpha^\mu) = \begin{pmatrix} B(r) & 0 & 0 & 0 \\ 0 & A(r) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (4.13) \]
In spherical coordinate system, space-time metric is
\[ \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}. \quad (4.14) \]
Its inverse is
\[ \eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 \sin^2 \theta} \end{pmatrix}. \quad (4.15) \]
Equation (2.7) gives the definition of $g_{\alpha \beta}$

\[
g_{\alpha \beta} = \begin{pmatrix}
-B^2(r) & 0 & 0 & 0 \\
0 & A^2(r) & 0 & 0 \\
0 & 0 & r^2 & 0 \\
0 & 0 & 0 & r^2 \sin^2 \theta
\end{pmatrix}.
\]

(4.16)

Affine connection is defined by equation (4.4). In spherical coordinate system, its only nonvanishing components are

\[
\begin{align*}
\Gamma_{\theta \theta}^r &= -r \\
\Gamma_{r \phi}^\phi &= -r \sin^2 \theta \\
\Gamma_{\theta \phi}^\phi &= -\sin \theta \cos \theta \\
\Gamma_{\phi \phi}^\phi &= -\sin \theta \\
\Gamma_{\phi \phi}^{\theta r} &= \frac{1}{r} \\
\Gamma_{\phi \phi}^{\phi r} &= \frac{1}{r} \\
\Gamma_{\phi \phi}^{\theta \phi} &= \cot \theta
\end{align*}
\]

(4.17)

Field strength of gravitational gauge field $F^\alpha_{\mu \nu}$ is defined by equation (4.3). In the present case, its only nonvanishing components are

\[
\begin{align*}
-gF^t_{tt} &= gF^r_{rt} = \frac{B'(r)}{A(r)B^2(r)} \\
-gF^r_{\theta \theta} &= gF^\theta_{\theta r} = \frac{A'(r) - 1}{rA(r)} \\
-gF^\phi_{\phi r} &= gF^\phi_{\phi \theta} = \frac{A'(r) - 1}{rA(r)}
\end{align*}
\]

(4.18)

In above equations, a prime means differentiation with respect to $r$.

Using all above results, we can calculate $W^\nu_\alpha$. It is found that, its only nonvanishing components are

\[
W^t_t = B(r)\left[-A(r) + A^3(r) + 2rA'(r)\right] - 2gr^2A^3(r),
\]

(4.19)

and

\[
W^r_r = B(r)\left[-1 + A^2(r) - 2rB'(r)\right] - 2gr^2A^3(r)B(r),
\]

(4.20)

Then field equation (4.10) gives the following three equations

\[
-A(r) + A^3(r) + 2rA'(r) = 0,
\]

(4.22)
\[ B(r) - A^2(r)B(r) + 2rB'(r) = 0, \quad (4.23) \]

and

\[ A'(r)[B(r) + rB'(r)] - A(r)[B'(R) + rB''(r)] = 0. \quad (4.24) \]

From equations (4.22) and (4.33) we have

\[ \frac{d}{dr}(A(r)B(r)) = 0. \quad (4.25) \]

It gives

\[ A(r)B(r) = c_0, \quad (4.26) \]

where \( c_0 \) is a constant. When \( r \) approaches infinity, this is no gravity, and gravitational gauge field \( C^\alpha_\mu \) vanish. In this case, \( A(r) = B(r) = 1 \). So, \( c_0 = 1 \), and equation (4.26) becomes

\[ A(r)B(r) = 1, \quad (4.27) \]

Equation (4.22) gives

\[ \frac{d}{dr} \frac{r}{A^2(r)} = 1. \quad (4.28) \]

From equations (4.27) and (4.28), we obtain final solution

\[ A(r) = \frac{1}{\sqrt{1 - \frac{2GM}{r}}}, \quad (4.29) \]

\[ B(r) = \sqrt{1 - \frac{2GM}{r}}. \quad (4.30) \]

Then (4.16) gives the following results

\[ g_{\alpha\beta} = \begin{pmatrix} -\left(1 - \frac{2GM}{r}\right) & 0 & 0 & 0 \\ 0 & \frac{1}{1 - \frac{2GM}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}, \quad (4.31) \]

which is just the Schwarzschild solution in general relativity. Therefor, solutions (4.29) and (4.30) correspond to the Schwarzschild solution. After we know \( A(r) \) and \( B(r) \), we can calculate magnitude of gravitational gauge field from equation (4.11). The only nonvanishing components of gravitational gauge field are

\[ gC^t_t = 1 - \frac{1}{\sqrt{1 - \frac{2GM}{r}}}, \quad (4.32) \]

\[ gC^r_r = 1 - \sqrt{1 - \frac{2GM}{r}}. \quad (4.33) \]
5 Equation of Motion of a Mass Point in Gravitational Gauge Field

General relativity is a theory of space-time geometry. In general relativity, the equation of motion of a mass point is the geodesic equation, for geodesic curve is the path which has shortest length between two points. So, in general relativity, selecting the geodesic equation as the equation of motion of a mass point is consistent with basic spirit of geometry. Gauge theory of gravity is not a theory of space-time geometry. In gauge theory of gravity, space-time is always flat, and geodesic line is always a straight line in Minkowski space-time, which is the orbit of an inertial motion. When there is gravitational force on the mass point, its orbit deviates from the geodesic line. So, we can not select the geodesic equation as the equation of motion of a mass point in gravitational field. How to set up the equation of motion of a mass point in a logically self-consistent way?

As we have stated before, the basic viewpoint in gauge theory of gravity is that gravity is a kind of fundamental interactions in flat Minkowski space-time. For a classical mass point in gravitational field, it will feel a gravitational force acting on it. Denote the force as \( f^\mu \), then according to Newton’s second law of motion, the equation of motion of a mass point should be

\[
\frac{dP^\mu}{d\tau} = f^\mu, \tag{5.1}
\]

where \( P^\mu \) is the canonical momentum of the mass point. Now, our central task is to determine the form of the gravitational force \( f^\mu \) on the mass point.

In Newton’s theory of gravity, the gravitational force on a mass point is

\[
\vec{f} = -\frac{GMm}{r^2} \hat{r}, \tag{5.2}
\]

where \( G \) is the Newtonian gravitational constant, \( M \) is the mass of gravitational source, \( m \) is the mass of the mass point, \( \hat{r} = \frac{\vec{r}}{r} \) is a unit vector, and \( r \) is the distance between the mass point and the center of mass of the source. But this form is not relativistic, we can not directly use it in the equation (5.1).

In order to determine the form of gravitational force \( f^\mu \), we need first know its basic properties. The basic requirements of the gravitational force \( f^\mu \) are the following three:

1. Under Lorentz transformations, the gravitational force \( f^\mu \) is a Lorentz 4-vector;
2. under gravitational gauge transformations, the gravitational force \( f^\mu \) transforms covariantly
\[
f^\mu \rightarrow (\hat{U}_\epsilon f^\mu); \tag{5.3}\]

3. in non-relativistic limit, the gravitational force \( f^\mu \) returns to Newton’s formula (5.2).

In non-relativistic case, the gravitational field generated by a mass \( M \) is [5, 6, 7, 8]
\[
gC_0^0 = -\frac{GM}{r}. \tag{5.4}\]

Therefor, in leading order approximation, the corresponding field strength is
\[
gF_0^0 = \frac{GM}{r^2} \hat{r}_i. \tag{5.5}\]

Then equation (5.2) can be changed into
\[
f_i = -gF_0^0 m. \tag{5.6}\]

For non-relativistic case, we have
\[
P^0 = m, \quad \frac{dx_0}{d\tau} = -\frac{dx^0}{d\tau} = -1. \tag{5.7}\]

So, equation (5.6) becomes
\[
f_i = gF_0^0 \frac{dx_0}{d\tau} P^0. \tag{5.8}\]

The simplest way to generate the above formula to relativistic form is to replace index 0 to index \( \lambda \) or \( \alpha \) and sum over repeated indexes. So, we have
\[
f_i = gF_\alpha^\lambda \frac{dx_\alpha}{d\tau} P^\lambda. \tag{5.9}\]

Therefore, the gravitational force \( f^\mu \) should be
\[
f^\mu = g\eta^{\mu\nu} F^\alpha_{\nu\lambda} \frac{dx_\alpha}{d\tau} P^\lambda. \tag{5.10}\]

In literature [23], we have obtained similar result by using a quite different method. It is found that this formula indeed satisfies the three requirements above. So, the \( f^\mu \) given by equation (5.10) is the gravitational force on the mass point. Then equation of motion of the mass point in gravitational field is
\[
\frac{dP^\mu}{d\tau} = g\eta^{\mu\nu} F^\alpha_{\nu\lambda} \frac{dx_\alpha}{d\tau} P^\lambda. \tag{5.11}\]
In gauge theory of gravity, the definition of $\frac{dx_\alpha}{d\tau}$ is given by

$$\frac{dx_\alpha}{d\tau} = g_{\alpha\beta} \frac{dx^\beta}{d\tau}, \quad (5.12)$$

so, equation (5.11) becomes

$$\frac{dP^\mu}{d\tau} = g\eta^{\mu\nu} g_{\alpha\beta} F_{\nu\lambda}^\alpha \frac{dx^\beta}{d\tau} P^\lambda. \quad (5.13)$$

In quantum gauge theory of gravity, energy-momentum operator $p_\alpha$ is given by

$$p_\alpha = -i\partial_\alpha. \quad (5.14)$$

According to gauge principle, gauge canonical momentum $P_\mu$ is

$$P_\mu = -iD_\mu. \quad (5.15)$$

Equation (2.3) gives their relation

$$P_\mu = G^\alpha_\mu p_\alpha. \quad (5.16)$$

Using equation (2.7) and

$$P_\mu = \eta_{\mu\nu} P^\nu, \quad p_\alpha = g_{\alpha\beta} g^\beta, \quad (5.17)$$

we can change equation (5.16) into

$$P^\mu = G^{-1\mu}_\gamma p^\gamma. \quad (5.18)$$

Multiply both side of equation (5.13) with $G^\gamma_\mu$ and sum over index $\mu$, we get

$$G^\gamma_\mu \frac{d(G^{-1\mu}_\gamma p^\gamma)}{d\tau} = g\eta^{\mu\nu} G^\gamma_\mu g_{\alpha1\beta} F_{\nu\lambda}^\alpha \frac{dx^\beta}{d\tau} (G^{-1\lambda}_\alpha p^\alpha). \quad (5.19)$$

Left hand side of equation (5.19) gives

$$\frac{dp^\gamma}{d\tau} + \frac{d}{d\tau} G^\gamma_\mu \frac{d}{d\tau} G^{-1\mu}_\gamma p^\beta = \frac{d}{d\tau} G^\gamma_\mu (\partial_\alpha G^{-1\mu}_\beta) \frac{dx^\alpha}{d\tau} p^\beta. \quad (5.20)$$

Because $\frac{dx^\alpha}{d\tau} p^\beta$ is symmetric under exchange indexes $\alpha$ and $\beta$, it can be further changed into

$$\frac{dp^\gamma}{d\tau} + \Gamma^\gamma_{\alpha\beta} \frac{dx^\alpha}{d\tau} p^\beta \quad (5.21)$$
where

\[ \bar{\Gamma}^\gamma_{\alpha\beta} = \frac{1}{2} G^\gamma_\mu (\partial_\alpha G^{-1}_\beta + \partial_\beta G^{-1}_\alpha) \]
\[ = -\frac{1}{2} (G^{-1}_\mu \partial_\alpha G^\gamma_\mu + G^{-1}_\mu \partial_\beta G^\gamma_\mu). \]  

(5.22)

So, equation (5.19) was changed into the following form

\[ \frac{dp^\gamma}{d\tau} + \bar{\Gamma}^\gamma_{\alpha\beta} \frac{dx^\alpha}{d\tau} p^\beta = \frac{g}{2} \eta^{\mu\nu} G^\gamma_\mu \left( g_{\alpha\beta} G^{-1}_\alpha + g_{\alpha\beta} G^{-1}_\beta \right) F^{\alpha\nu}_{\alpha\mu} \frac{dx^\alpha}{d\tau} p^\beta. \]  

(5.23)

This is the equation of motion of a mass point in gravitational field.

6 Equation of Motion in Spherical Coordinate System

In above chapter, we obtain a equation of motion of a classical mass point which is moving in gravitational field. Equation (5.23) is only valid in Descartes coordinate system. According to the discussions in chapter 3, in arbitrary curvilinear coordinate system, equation (5.23) should be changed into

\[ \frac{Dp^\gamma}{D\tau} + \Gamma^\gamma_{\alpha\beta} \frac{dx^\alpha}{d\tau} p^\beta = \frac{g}{2} \eta^{\mu\nu} G^\gamma_\mu \left( g_{\alpha\beta} G^{-1}_\alpha + g_{\alpha\beta} G^{-1}_\beta \right) F^{\alpha\nu}_{\alpha\mu} \frac{dx^\alpha}{d\tau} p^\beta, \]

(6.1)

where

\[ \frac{Dp^\gamma}{D\tau} = \frac{dp^\gamma}{d\tau} + \Gamma^\gamma_{\alpha\beta} \frac{dx^\alpha}{d\tau} p^\beta, \]  

(6.2)

with \( \Gamma^\gamma_{\alpha\beta} \) is the affine connection in curvilinear coordinate system defined by (4.4), and

\[ \bar{\Gamma}^\gamma_{\alpha\beta} = \frac{1}{2} G^\gamma_\mu \left( \nabla_{\alpha} G^{-1}_\beta + \nabla_{\beta} G^{-1}_\alpha \right) \]
\[ = \frac{1}{2} G^\gamma_\mu \left[ \partial_{\alpha} G^{-1}_\beta + \Gamma^\mu_{\alpha\nu} G^{-1}_\beta - \Gamma^\gamma_{\alpha\beta} G^{-1}_\mu \right. \]
\[ + \partial_{\beta} G^{-1}_\alpha + \Gamma^\mu_{\beta\nu} G^{-1}_\alpha - \Gamma^\gamma_{\beta\alpha} G^{-1}_\mu \left. \right]. \]  

(6.3)

In spherical coordinate system, gravitational gauge field \( C^\alpha_\mu \) is given by (4.11), matrices \( G \) and \( G^{-1} \) are given by (4.12) and (4.13) respectively; metrics \( \eta_{\mu\nu} \) and \( \eta^{\mu\nu} \) are given by (4.14) and (4.15), \( g_{\alpha\beta} \) is given by (4.16), affine connection \( \Gamma^\gamma_{\alpha\beta} \) is given by (4.17), and field strength \( F^\alpha_{\mu\nu} \) is given by (4.18). \( \bar{\Gamma}^\gamma_{\alpha\beta} \) is calculate from equation
(6.3). Its only nonvanishing components are

\[
\begin{align*}
\Gamma^t_{tr} &= \Gamma^t_{rt} = \frac{B'(r)}{2B(r)} \\
\Gamma^r_{tr} &= \frac{A'(r)}{A(r)} \\
\Gamma^r_{\theta\theta} &= r(1 - \frac{1}{A(r)}) \\
\Gamma^r_{\varphi r} &= r \sin^2 \theta (1 - \frac{1}{A(r)}) \\
\Gamma^r_{\varphi \varphi} &= \frac{A(r) - 1}{2r} \\
\Gamma^r_{\varphi \varphi} &= \Gamma^r_{\theta r} = \frac{A(r) - 1}{2r} \\
\end{align*}
\]

Denote

\[
f_{\alpha\beta}^\gamma = -\frac{g_{\mu\nu} C^\gamma_\mu (g_{\alpha\beta} G^{\alpha\lambda} + g_{\alpha\mu} G^{\beta\lambda}) F_{\nu\lambda}^{\alpha}},
\]

then equation (6.1) is changed into a much simpler form

\[
\frac{Dp^\gamma}{D\tau} + (\bar{\Gamma}^\gamma_{\alpha\beta} + f_{\alpha\beta}^\gamma) \frac{dx^\alpha}{d\tau} p^\beta = 0.
\]

The only nonvanishing components of \( f_{\alpha\beta}^\gamma \) are

\[
\begin{align*}
f^t_{tr} &= f^t_{rt} = \frac{B'(r)}{2B(r)} \\
f^r_{tt} &= \frac{B(r) B'(r)}{A^2(r)} \\
f^r_{\theta\theta} &= \frac{A'(r)}{A(r)} \\
f^r_{\varphi r} &= A(r) - 1 \frac{A(r)}{2r} \\
\end{align*}
\]

Using nonvanishing components of affine connection \( \Gamma^\gamma_{\alpha\beta} \) given by (4.17), \( \bar{\Gamma}^\gamma_{\alpha\beta} \) given by (6.4), and \( f_{\alpha\beta}^\gamma \) given by (6.7), we find from (6.6) that

\[
\begin{align*}
\frac{dp^t}{d\tau} + 2 \frac{B'(r)}{B(r)} \frac{dr}{d\tau} p^t &= 0, \\
\frac{dp^r}{d\tau} + \frac{B(r) B'(r)}{A^2(r)} \frac{dt}{d\tau} p^t - \frac{A'(r)}{A(r)} \frac{dr}{d\tau} p^r - \frac{r}{A^2(r)} \frac{d\theta}{d\tau} p^\theta - \frac{r \sin^2 \theta}{A^2(r)} \frac{d\varphi}{d\tau} p^\varphi &= 0, \\
\frac{dp^\theta}{d\tau} + \frac{2}{r} \frac{dr}{d\tau} p^\theta - \sin \theta \cos \theta \frac{d\varphi}{d\tau} p^\varphi &= 0, \\
\frac{dp^\varphi}{d\tau} + \frac{2}{r} \frac{dr}{d\tau} p^\varphi + 2 \cot \theta \frac{d\theta}{d\tau} p^\varphi &= 0.
\end{align*}
\]
Since the field is isotropic, we may consider the orbit of the mass point to be confined to the equatorial plane, that is, in the $\theta = \frac{\pi}{2}$ plane. In this case, equation (6.10) vanishes, other three equations are changed into

\[
\frac{dp^t}{d\tau} + \frac{B'(r)}{B(r)} \frac{dr}{d\tau} p^t = 0, \quad (6.12)
\]

\[
\frac{dp^r}{d\tau} + \frac{B(r)B'(r)}{A^2(r)} \frac{dt}{d\tau} p^t + \frac{A'(r)}{A(r)} \frac{dr}{d\tau} - \frac{r}{A^2(r)} \frac{d\varphi}{d\tau} p^\varphi = 0, \quad (6.13)
\]

\[
\frac{dp^\varphi}{d\tau} + \frac{2}{r} \frac{dr}{d\tau} p^\varphi = 0. \quad (6.14)
\]

When we discuss equation of motion of photon, we should use trajectory parameter $s$ instead of proper time $\tau$ in above equations. We can also consider photon as limit that the rest mass $m$ approaches zero, and use all above equations to calculate its motion in gravitational field.

In order to solve these equations, we first look for constants of the motion. Equation (6.12) gives

\[
\frac{d}{d\tau} (\ln p^t + \ln B^2(r)) = 0. \quad (6.15)
\]

So,

\[
p^t B^2(r) = E_0, \quad (6.16)
\]

where $E_0$ is a constant. $p^t$ can be expressed in terms of $B(r)$

\[
p^t = \frac{E_0}{B^2(r)}. \quad (6.17)
\]

Equation (6.14) can be changed into

\[
\frac{d}{d\tau} (\ln p^\varphi + \ln r^2) = 0, \quad (6.18)
\]

which gives

\[
r^2 p^\varphi = J, \quad (6.19)
\]

where $J$ is another constant. $J$ plays the role of angular momentum. Momentum $p^\varphi$ is

\[
p^\varphi = \frac{J}{r^2}. \quad (6.20)
\]

21
Inserting equations (6.17) and (6.20) into (6.13) gives the following equation of motion
\[
\frac{d p^r}{d \tau} + \frac{E_0 B'(r)}{A^2(r) B(r)} \frac{dt}{d \tau} + \frac{A'(r)}{A(r)} \frac{dr}{d \tau} p^r - \frac{J}{r A^2(r)} \frac{d \varphi}{d \tau} = 0.
\] (6.21)

Multiply both side of the above equation with \(2A^2(r)p^r\), we get
\[
2A^2(r) p^r \frac{dp^r}{d \tau} + \frac{2E_0 B'(r)}{B(r)} p^r \frac{dt}{d \tau} + 2A(r) A'(r) (p^r)^2 \frac{dr}{d \tau} - \frac{2J}{r} p^r \frac{d \varphi}{d \tau} = 0,
\] (6.22)
which can be written into another form
\[
\frac{d}{d \tau} \left( A^2(r)(p^r)^2 \right) + \frac{2E_0 B'(r)}{B(r)} p^r \frac{dt}{d \tau} - \frac{2J}{r} p^r \frac{d \varphi}{d \tau} = 0.
\] (6.23)

Using the following two relations
\[
p^r \frac{dt}{d \tau} = p^r \frac{dr}{d \tau},
\] (6.24)
and
\[
p^r \frac{d \varphi}{d \tau} = p^\varphi \frac{dr}{d \tau},
\] (6.25)
we can change equation (6.23) into
\[
\frac{d}{d \tau} \left( A^2(r)(p^r)^2 \right) + \frac{2E_0 B'(r)}{B(r)} p^r \frac{dr}{d \tau} - \frac{2J}{r} p^\varphi \frac{dr}{d \tau} = 0.
\] (6.26)

Inserting equations (6.17) and (6.20) into (6.26) gives
\[
\frac{d}{d \tau} \left[ A^2(r)(p^r)^2 - \frac{E_0^2}{B^2(r)} + \frac{J^2}{r^2} \right] = 0.
\] (6.27)

So,
\[
A^2(r)(p^r)^2 - \frac{E_0^2}{B^2(r)} + \frac{J^2}{r^2} = E,
\] (6.28)
where \(E\) is a constant of the motion. Equations (6.17), (6.20) and (6.28) will be used as the basic equations of motion to calculate classical tests of gauge theory of gravity. This set of equations do not contain proper time \(\tau\) in their expressions, so they are also directly applicable to photons.
7 The Deflection of Light by the Sun

Using the following relations

\[ p^r = \frac{p^r}{p^\varphi} p^\varphi = p^\varphi \frac{dr}{d\varphi} = \frac{J}{r^2} \frac{dr}{d\varphi}, \]  

(7.1)
equation (6.28) can be changed into

\[ \frac{J^2 A^2(r)}{r^4} \left( \frac{dr}{d\varphi} \right)^2 + \frac{J^2}{r^2} - \frac{E_0^2}{B^2(r)} = E. \]  

(7.2)

At distance \( r_0 \) of the closest approach to the sun, \( \frac{dr}{d\varphi} \) vanishes, so equation (7.2) gives

\[ \frac{J^2}{r_0^2} - \frac{E_0^2}{B^2(r)} = E. \]  

(7.3)

Consider photon approaching the sun from very great distance. At infinity, the gravitational gauge field \( C^\alpha_\mu \) vanishes, and

\[ A(\infty) = B(\infty) = 1. \]  

(7.4)

So, at infinity, equation (6.28) gives

\[ E = (p^r)^2 - E_0^2. \]  

(7.5)

For a photon, when there is no gravity at infinity and the photon is moving towards the sun, its radial momentum \( p^r \) equals its energy \( p^t \)

\[ p^r = p^t = E_0. \]  

(7.6)

Then equation (7.5) gives

\[ E = 0. \]  

(7.7)

Using (7.7) in (7.3) gives

\[ J = \frac{r_0 E_0}{B(r_0)}. \]  

(7.8)

Inserting equations (7.7) and (7.8) into (7.2) gives

\[ \frac{A^2(r)}{r^4} \left( \frac{dr}{d\varphi} \right)^2 + \frac{1}{r^2} = \frac{B^2(r_0)}{r_0^2 B^2(r)}. \]  

(7.9)

After some simple calculations from equation (7.9), we can change its form into

\[ d\varphi = \frac{A(r)}{\left( \frac{r_0}{B^2(r)} \left( \frac{r}{r_0} \right)^2 - 1 \right)^{\frac{1}{2}}} dr. \]  

(7.10)
Using solution (4.30), in first order approximation, we have

\[
B_2^2 \left( \frac{r}{r_0} \right)^2 - 1 = \left[ \left( \frac{r}{r_0} \right)^2 - 1 \right] \left[ 1 - \frac{2GMr}{r_0(r + r_0)} + \cdots \right],
\]

where \( \cdots \) represents higher order approximation. Integrate (7.10) from infinity to distance \( r \), we get

\[
\varphi(r) - \varphi(\infty) = \int_r^\infty A \left[ 1 - \frac{2GMr}{r_0(r + r_0)} + \cdots \right]^{-\frac{1}{2}} \frac{dr}{\sqrt{\left( \frac{r}{r_0} \right)^2 - 1}}.
\]

Inserting (4.29) into above equation gives

\[
\varphi(r) - \varphi(\infty) = \int_r^\infty \left[ 1 + \frac{GM}{r} + \frac{GMr}{r_0(r + r_0)} + \cdots \right] \frac{dr}{r \sqrt{\left( \frac{r}{r_0} \right)^2 - 1}}.
\]

Using the following integration formula

\[
\int \frac{dx}{x\sqrt{x^2 - 1}} = \cos^{-1} \frac{1}{x},
\]

\[
\int \frac{dx}{x^2\sqrt{x^2 - 1}} = \frac{\sqrt{x^2 - 1}}{x},
\]

and

\[
\int \frac{dx}{x\sqrt{x^2 - 2x}} = \sqrt{1 - \frac{2}{x}},
\]

we have

\[
\int_r^\infty \frac{dr}{r \sqrt{\left( \frac{r}{r_0} \right)^2 - 1}} = \int_r^\infty \frac{d\left( \frac{r}{r_0} \right)}{\left( \frac{r}{r_0} \right) \sqrt{\left( \frac{r}{r_0} \right)^2 - 1}} = \cos^{-1} \frac{1}{\frac{r}{r_0}},
\]

\[
= \frac{x}{r} - \cos^{-1} \left( \frac{r}{r_0} \right),
\]

\[
\int_r^\infty \frac{GM}{r} \frac{dr}{r \sqrt{\left( \frac{r}{r_0} \right)^2 - 1}} = \int_r^\infty \frac{GM}{r_0} \frac{d\left( \frac{r}{r_0} \right)}{\left( \frac{r}{r_0} \right) \sqrt{\left( \frac{r}{r_0} \right)^2 - 1}} = \cos^{-1} \frac{1}{\frac{r}{r_0}},
\]

\[
= \frac{GM}{r_0} \left( 1 - \sqrt{1 - \left( \frac{r}{r_0} \right)^2} \right),
\]

which is the solution of the differential equation (7.19).
and
\begin{align*}
\int_r^\infty \frac{GMr}{r_0(r+r_0)} \frac{dr}{\sqrt{(\frac{r}{r_0})^2-1}} &= \frac{GM}{r_0} \int_r^\infty \frac{d\left(\frac{r}{r_0}+1\right)}{\sqrt{(\frac{r}{r_0}+1)^2-2\left(\frac{r}{r_0}+1\right)}} \\
&= \frac{GM}{r_0} \sqrt{1 - \frac{2}{2}\frac{r}{r_0}+1} \\
&= \frac{GM}{r_0} \left(1 - \sqrt{\frac{r-r_0}{r+r_0}}\right). \tag{7.19}
\end{align*}

Inserting (7.17), (7.18) and (7.19) into (7.13), we have
\begin{align*}
\varphi(r) - \varphi(\infty) &= \frac{\pi}{2} - \cos^{-1} \left(\frac{r_0}{r}\right) + \frac{GM}{r_0} \left(2 - \sqrt{1 - \left(\frac{r_0}{r}\right)^2} - \sqrt{\frac{r-r_0}{r+r_0}}\right). \tag{7.20}
\end{align*}

The deflection of the photon orbit from a straight line is
\[\Delta \varphi = 2|\varphi(r_0) - \varphi(\infty)| - \pi = \frac{4GM}{r_0}. \tag{7.21}\]

This result is the same as that in general relativity.

8 The Precession of the Perihelia of the Orbits of the Inner planets

Consider an inner planet bound in an elliptical orbit around the sun. At perihelia and aphelia, \(r\) reaches its minimum and maximum values \(r_-\) and \(r_+\). At \(r_-\) and \(r_+\), \(\frac{dr}{d\varphi}\) vanishes
\[\frac{dr}{d\varphi} = 0. \tag{8.1}\]

So, equation (7.2) gives
\[\frac{J^2}{r_+^2} - \frac{E_0^2}{B^2(r_+)} = E, \tag{8.2}\]
and
\[\frac{J^2}{r_-^2} - \frac{E_0^2}{B^2(r_-)} = E. \tag{8.3}\]

Equations (8.2) and (8.3) can be considered as equations of \(J\) and \(E\). From these two equations we can derive values of \(J\) and \(E\)
\[J^2 = \frac{E_0^2 \left(\frac{1}{B^2(r_-)} - \frac{1}{B^2(r_+)}\right)}{\frac{1}{r_-} - \frac{1}{r_+}}, \tag{8.4}\]
\[ E = \frac{E_0^2 \left( \frac{r^2}{B^2(r_-)} - \frac{r^2}{B^2(r_+)} \right)}{r_+^2 - r_-^2}. \]  

(8.5)

From equation (7.2), we get
\[ \mathrm{d}\varphi = \frac{A(r)}{r^2} \left( \frac{E}{J^2} + \frac{E_0^2}{J^2 B^2(r)} - \frac{1}{r^2} \right)^{-\frac{1}{2}} \mathrm{d}r. \]  

(8.6)

Using (8.4) and (8.5), we have
\[ \frac{E}{J^2} + \frac{E_0^2}{J^2 B^2(r)} = \frac{r^2 \left( \frac{1}{B^2(r_-)} - \frac{1}{B^2(r_+)} \right) - r^2 \left( \frac{1}{B^2(r_+)} - \frac{1}{B^2(r)} \right)}{r_+^2 r_-^2 \left( \frac{1}{B^2(r_-)} - \frac{1}{B^2(r_+)} \right)}. \]  

(8.7)

Define
\[ I(r) = \frac{E}{J^2} + \frac{E_0^2}{J^2 B^2(r)} - \frac{1}{r^2} = \frac{r^2 \left( \frac{1}{B^2(r_-)} - \frac{1}{B^2(r)} \right) - r^2 \left( \frac{1}{B^2(r_+)} - \frac{1}{B^2(r)} \right)}{r_+^2 r_-^2 \left( \frac{1}{B^2(r_-)} - \frac{1}{B^2(r_+)} \right)} - \frac{1}{r^2}. \]  

(8.8)

It is found that
\[ I(r_+) = I(r_-) = 0. \]  

(8.9)

When we expand \( I(r) \) to the second order of \( \frac{G M}{r} \), it becomes a second order polynomial of \( \frac{1}{r} \). Two roots of \( I(r) \) are \( r_+ \) and \( r_- \), so \( I(r) \) must have the form
\[ I(r) = c_1 \left( \frac{1}{r_-} - \frac{1}{r} \right) \left( \frac{1}{r} - \frac{1}{r_+} \right), \]  

(8.10)

where \( c_1 \) is a constant. When \( r \) approaches infinity, \( B(r) = 1 \). Combine (8.8) and (8.10), we get
\[ c_1 = \frac{r^2 B^2(r_+)(B^2(r_-) - 1) - r^2 B^2(r_+)(B^2(r_-) - 1)}{r_+ r_- (B^2(r_+) - B^2(r_-))}. \]  

(8.11)

\( B(r) \) is given by (4.30), Using that result, we find that \( c_1 \) have a simpler form
\[ c_1 = 1 - 2GM \left( \frac{1}{r_+} + \frac{1}{r_-} \right). \]  

(8.12)

Inserting (8.10) into (8.6) gives
\[ \mathrm{d}\varphi = c_1^{-\frac{1}{2}} A(r) \left[ \left( \frac{1}{r_-} - \frac{1}{r} \right) \left( \frac{1}{r} - \frac{1}{r_+} \right) \right]^{-\frac{1}{2}} \mathrm{d}r. \]  

(8.13)
\( A(r) \) is given by (4.29), in linear order approximation, it is
\[
A(r) = \frac{1}{\sqrt{1 - \frac{2GM}{r}}} = 1 + \frac{GM}{r} + \cdots, \tag{8.14}
\]
where \( \cdots \) represents higher order terms. Inserting (8.14) into (8.13), we get
\[
d\varphi = -c_1^{-\frac{1}{2}} \frac{(1 + \frac{GM}{r}) d \left( \frac{1}{r} \right)}{\left[ \left( \frac{1}{r^2} - \frac{1}{r^2} \right) \left( \frac{1}{r^2} - \frac{1}{r^2} \right) \right]^\frac{1}{2}}. \tag{8.15}
\]

Define
\[
\frac{1}{r} \triangleq \frac{1}{2} \left( \frac{1}{r^2} + \frac{1}{r^2} \right) + \frac{1}{2} \left( \frac{1}{r^2} - \frac{1}{r^2} \right) \sin \psi. \tag{8.16}
\]
At perihelia \( \psi = -\frac{\pi}{2} \), while at aphelia \( \psi = \frac{\pi}{2} \). From (8.16), we have
\[
\left( \frac{1}{r^2} - \frac{1}{r^2} \right) \left( \frac{1}{r^2} - \frac{1}{r^2} \right) = \frac{1}{4} \left( \frac{1}{r^2} - \frac{1}{r^2} \right)^2 \cos^2 \psi. \tag{8.17}
\]
Then, (8.15) is changed into
\[
d\varphi = c_1^{-\frac{1}{2}} \left[ 1 + \frac{GM}{2} \left( \frac{1}{r^2} + \frac{1}{r^2} \right) + \frac{GM}{2} \left( \frac{1}{r^2} - \frac{1}{r^2} \right) \sin \psi \right] d\psi. \tag{8.18}
\]
Integrate (8.18) from perihelia \( r_- \) to distance \( r \), we get
\[
\varphi(r) - \varphi(r_-) = c_1^{-\frac{1}{2}} \left[ \left( \psi + \frac{\pi}{2} \right) + \frac{GM}{2} \left( \frac{1}{r^2} + \frac{1}{r^2} \right) \left( \psi + \frac{\pi}{2} \right) - \frac{GM}{2} \left( \frac{1}{r^2} - \frac{1}{r^2} \right) \cos \psi \right]. \tag{8.19}
\]
So, we have
\[
\varphi(r) - \varphi(r_-) = c_1^{-\frac{1}{2}} \left[ 1 + \frac{3GM}{2} \left( \frac{1}{r^2} + \frac{1}{r^2} \right) \right] \pi. \tag{8.20}
\]
Inserting (8.12) into above equation gives
\[
\varphi(r) - \varphi(r_-) = \left[ 1 + \frac{3GM}{2} \left( \frac{1}{r^2} + \frac{1}{r^2} \right) \right] \pi. \tag{8.21}
\]
The precession per revolution is
\[
\Delta \varphi = 2(\varphi(r) - \varphi(r_-)) = 2\pi = \frac{6\pi GM}{2} \left( \frac{1}{r^2} + \frac{1}{r^2} \right). \tag{8.22}
\]
The semilatus rectum $L$ is defined by

$$\frac{1}{L} = \frac{1}{2} \left( \frac{1}{r_+} + \frac{1}{r_-} \right).$$

(8.23)

Then $\Delta \varphi$ is simplified to

$$\Delta \varphi = \frac{6\pi GM}{L}.$$  

(8.24)

This result is also the same as that in general relativity.

9 The Time Delay of Radar Echoes Passing the Sun

Using the following relations

$$p^r = \frac{p^r}{p^t} p^t = p^r \frac{dr}{dt} = \frac{E_0}{B^2(r)} \frac{dr}{dt},$$

(9.1)

and (7.7), we can change equation (6.28) into

$$\frac{E_0^2 A^2(r)}{B^4(r)} \left( \frac{dr}{dt} \right)^2 + \frac{J^2}{r^2} - \frac{E_0^2}{B^2(r)} = 0.$$  

(9.2)

At distance $r_0$ of the closest approach to the sun, $\frac{dr}{dt}$ vanishes, so equation (9.2) gives

$$J^2 = \frac{E_0^2 r_0^2}{B^2(r_0)}.$$  

(9.3)

Inserting (9.3) into (9.2) gives

$$\frac{E_0^2 A^2(r)}{B^4(r)} \left( \frac{dr}{dt} \right)^2 + \frac{E_0^2 r_0^2}{B^2(r_0) r^2} - \frac{E_0^2}{B^2(r)} = 0.$$  

(9.4)

In above equation, $E_0^2$ can be cancelled, so we get

$$\frac{A^2(r)}{B^2(r)} \left( \frac{dr}{dt} \right)^2 + \frac{r_0^2}{B^2(r_0) r^2} - \frac{1}{B^2(r)} = 0.$$  

(9.5)

By multiplying this equation with $B^2(r)$, we may write it as

$$\frac{A^2(r)}{B^2(r)} \left( \frac{dr}{dt} \right)^2 = 1 - \frac{B^2(r)}{B^2(r_0)} \left( \frac{r_0}{r} \right)^2.$$  

(9.6)
$B(r)$ is given by (4.30). In first order approximation, we have

$$1 - \frac{B^2(r)}{B^2(r_0)} \left( \frac{r_0}{r} \right)^2 = \left[ 1 - \left( \frac{r_0}{r} \right)^2 \right] \left[ 1 - \frac{2GMr_0}{r(r + r_0)} + \cdots \right] ,$$

(9.7)

where $\cdots$ represents higher order approximation. Inserting (4.29), (4.30) and (9.7) into (9.6) and making first order approximation, we get

$$\frac{dt}{dr} = \left[ 1 - \left( \frac{r_0}{r} \right)^2 \right]^{-\frac{1}{2}} \left( 1 + \frac{2GM}{r} + \frac{GMr_0}{r(r + r_0)} \right) .$$

(9.8)

Integrate (9.8), we get the time required for light to go from $r_0$ to $r$ or from $r$ to $r_0$

$$t(r, r_0) = \int_{r_0}^{r} \left[ 1 - \left( \frac{r_0}{r} \right)^2 \right]^{-\frac{1}{2}} \left( 1 + \frac{2GM}{r} + \frac{GMr_0}{r(r + r_0)} \right) dr .$$

(9.9)

Using the following integration formula

$$\int \left[ 1 - \left( \frac{r_0}{r} \right)^2 \right]^{-\frac{1}{2}} \frac{dr}{r} = \sqrt{r^2 - r_0^2} .$$

(9.10)

$$\int \left[ 1 - \left( \frac{r_0}{r} \right)^2 \right]^{-\frac{1}{2}} \frac{dr}{r} = \ln \left( r + \sqrt{r^2 - r_0^2} \right) ,$$

(9.11)

and

$$\int \left[ 1 - \left( \frac{r_0}{r} \right)^2 \right]^{-\frac{1}{2}} \frac{r_0 dr}{r(r + r_0)} = \sqrt{\frac{r_0}{r + r_0}} ,$$

(9.12)

we get

$$t(r, r_0) = \sqrt{r^2 - r_0^2} + 2GM \ln \frac{r + \sqrt{r^2 - r_0^2}}{r_0} + GM \sqrt{\frac{r - r_0}{r + r_0}} .$$

(9.13)

If light travelled in straight lines at unit velocity, the time required for light to go from $r_0$ to $r$ is $\sqrt{r^2 - r_0^2}$. So, in this process, the excess time delay is

$$\Delta t(r, r_0) = t(r, r_0) - \sqrt{r^2 - r_0^2} = 2GM \ln \frac{r + \sqrt{r^2 - r_0^2}}{r_0} + GM \sqrt{\frac{r - r_0}{r + r_0}} .$$

(9.14)
Suppose that a radar signal grazes the sun and is reflected back to earth, the total excess time delay is denoted by \((\Delta t)_{\text{max}}\). When radar signal just grazes the sun, \(r_0\) is about equal to the radius of the sun, that is
\[
 r_0 \simeq R_\odot. \tag{9.15}
\]
The total excess time delay is
\[
(\Delta t)_{\text{max}} = 2[\Delta t(r_\oplus, R_\odot) + \Delta t(r_M, R_\odot)]. \tag{9.16}
\]
\(R_\odot\) is much smaller than the distances \(r_\oplus\) and \(r_M\) of the earth and Mercury from the sun
\[
\frac{R_\odot}{r_\oplus} \simeq 0, \quad \frac{R_\odot}{r_M} \simeq 0. \tag{9.17}
\]
Under this approximation, we have
\[
(\Delta t)_{\text{max}} = 4GM \left(1 + \ln \frac{4r_\oplus r_M}{R_\odot^2}\right). \tag{9.18}
\]
This result is the same as that in general relativity.

10 Summary and Discussions

In this paper, classical tests of gauge theory of gravity are discussed. All discussions are based field equation of gravitational gauge field and Newton’s second law of motion. Final results are the same as those given by general relativity. Quantitative results given by this paper is trivial, for they are well-known in general relativity. But from the discussions in this paper, we can obtain some important qualitative conclusions on the nature of gravity, that is, gravity can be treated as a kind of physical interactions in flat Minkowski space-time, and the equation of motion of a mass point in gravitational field is given by Newton’s second law of motion. In this case, all kinds of fundamental interactions can be described in the same way.

In order to discuss classical motion of a particle, we need first determine what is the equation of motion of a particle in gravitational field. The basic logic in gauge theory of gravity is that gravity is a kind of fundamental interactions in flat space-time. According to our knowledge on classical mechanics, a logically natural conclusion is that the equation of motion of a mass point in gravitational field is given by Newton’s second law of motion in a relativistic form. Selecting geodesic equation as equation of motion of a test particle is not logically consistent with basic spirit in gauge theory of gravity. Therefore, though the field equation of gravitational gauge field in gauge theory of gravity is the same as the Einstein’s field
equation in general relativity, if the equation of motion of a test particle in gauge
theory of gravity is different from that in general relativity, gauge theory of gravity
will give different results on classical tests of gravity. So, discussions on classical
tests of gravity in gauge theory of gravity are not a trivial task. Fortunately, we
found that, based on the classical solution of field equation and Newton’s second
law of motion, gauge theory of gravity gives out the same theoretical expectations
on classical tests of gravity as general relativity. An important physical conclusion
obtained from this work is that Newton’s second law of motion is also applicable
to classical gravity. For a long time, we know that Newton’s second law of motion
is not applicable to gravity, especially from the point of view of general relativity.
So, in classical mechanics, Newton’s second law of motion should be a fundamen-
tal law which is applicable to all kinds of fundamental interactions including gravity.

After we written out Newton’s second law of motion, the first task in front of
us is to determine the gravitational force acting on the test particle. In this paper,
a general relativistic form of the gravitational force on a test particle is obtained,
which is the basis of our discussions on classical problems of gravity. As we have dis-
cussed in literature [23], gravitational force contains Newtonian gravitational force
which is transmitted by gravitoelectric field and gravitational Lorentz force which
is transmitted by gravitomagnetic field. For a relativistic particle, the gravitational
force on it does not along the line connecting the centers of mass of two bodies.

In general relativity, the equation of motion of a mass point is given by geodesic
equation. In gauge theory of gravity, the equation of motion of a mass point is
given by Newton’s second law of motion. For spherical symmetric problems, these
two equations are equivalent. If the particle has inner spin, there is extra coupling
between spin and gravitomagnetic field[12, 13], and there will extra gravitational
force in (5.13) which comes from the coupling between spin and gravitomagnetic
field. Therefore, equation (5.13), or the geodesic equation in general relativity, is
only applicable to spinless particle. After consider spin of the test particle, equation
(5.13) must be modified. A important consequence of this modification is that
the orbit of the motion dependents on the spin of the particle, which violates weak
equivalence principle. This modification also affects the deflection of light by the
sun. But because the gravitomagnetic field generated by the sun is too small, the
influence from this modification is too small to be detectable.

In gauge theory of gravity, space-time is always flat. In chapter 3, we set up
the formulation of gauge theory of gravity in curvilinear coordinate system. In that
case, space-time is still flat. The affine connection defined is essentially different from
that in general relativity, for this affine connection does not contain effects of gravity.
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