TWO-PARAMETER FAMILIES OF UNIQUELY EXTENDABLE DIOPHANTINE TRIPLES

MIHAI CIPU, YASUTSUGU FUJITA, AND MAURICE MIGNOTTE

Abstract. Let $A, K$ be positive integers and $\varepsilon \in \{-2, -1, 1, 2\}$. The main contribution of the paper is a proof that each of the $D(\varepsilon^2)$-triples \( \{ A, A^2K + 2\varepsilon A, (A+1)^2K + 2\varepsilon(A+1) \} \) has unique extension to a $D(\varepsilon^2)$-quadruple. This is used to slightly strengthen the conditions required for the existence of a $D(1)$-quintuple whose smallest three elements form a regular triple.

1. Introduction

Let $n$ be an arbitrary integer. A set of positive integers is called $D(n)$-tuple if the product of any two distinct elements increased by $n$ is a perfect square. In case the set has cardinality 2 (3, 4 or 5) one speaks of a $D(n)$-pair (triple, quadruple or quintuple, respectively).

Among $D(n)$-sets, the most studied ones are those with $n = 1$. The interest and efforts are driven towards confirmation of the folklore conjecture that predicts there are no $D(1)$-quintuples. A good deal of necessary conditions for the existence of a $D(1)$-quintuple is presently known. In a recent work on this subject [5] it is shown that if \( \{ a, b, c, d, e \} \) is a $D(1)$-quintuple with $a < b < c < d < e$ and $c = a + b + 2\sqrt{ab} + 1$ then $b < a^3$. Therefore, the positive integer $r$ satisfying $ab + 1 = r^2$ is less than $a^2$. In the extremal case $r = a^2 - 1$ the three smallest elements of such a $D(1)$-quintuple are $a, b = a^3 - 2a, c = a(a+1)^2 - 2(a+1)$. One of the present authors has remarked that this triple is formally obtained by specializing $k$ to $-a$ in the triple \( \{ k, a^2k + 2a, (a+1)^2k + 2(a+1) \} \) considered in [21] and then changing the sign of all entries. To put it differently, our triple appears in the two-parameter family \( \{ K, A^2K - 2A, (A+1)^2K - 2(A+1) \} \) dual to that considered by He and Togbé. A closer look at [21] reveals that the companion $D(1)$-triple is in fact mentioned in the introduction to that paper without further study.

A close similarity of results on $D(1)$- and $D(4)$-sets is well documented in literature, as found, e.g., by comparing [7] and [19] with [11, 14] and [12]. One of the common properties is that any $D(\sigma)$-triple with $\sigma \in \{1, 4\}$ can be extended to a $D(1)$-quadruple. More precisely, if \( \{ a, b, c \} \) is a $D(\sigma)$-triple, then \( \{ a, b, c, d_+ \} \) is a $D(1)$-quadruple, where

\[
d_+ = a + b + c + \frac{2}{\sigma}(abc + rst)
\]

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and
\[ r = \sqrt{ab + \sigma}, \quad s = \sqrt{ac + \sigma}, \quad t = \sqrt{bc + \sigma}. \]
Such a \( D(\sigma) \)-quadruple is called regular, and it is conjectured that any \( D(\sigma) \)-quadruple is regular (cf. \cite{1} \cite{9}). Among \( d \)'s such that \( \{a, b, c, d\} \) is a \( D(\sigma) \)-quadruple with \( a < b < c < d \), the smallest integer is known to be \( d_+ \) from \cite{7} Proposition 1 and \cite{12} Proposition 1.

The present paper deals with two closely related families, viz. those of \( D(4) \)-triples mentioned in the abstract. The outcome of our study is the theorem below, showing that each of the triples under scrutiny has unique extension to quadruple. In particular, the next result shows that the conjecture mentioned above is true for the families examined in this paper.

**Theorem 1.** Let \( A, K \) be positive integers. If \( \{K, A^2K + 2\varepsilon A, (A + 1)^2K + 2\varepsilon(A + 1), d\} \) is a \( D(\varepsilon^2) \)-quadruple with \( \varepsilon \in \{-2, -1, 1, 2\} \), then they are regular, in other words, we have
\[ (1.1) \]
\[ d = d_+ = \varepsilon^{-2}(2A^2 + 2A)^2K^3 + \varepsilon^{-1}(16A^3 + 24A^2 + 8A)K^2 + (20A^2 + 20A + 4)K + \varepsilon(8A + 4). \]

Note that the assumption in Theorem 1 immediately implies that \( d \) is the largest element in the quadruple. Indeed, substituting \( a = K \), \( b = A^2K + 2\varepsilon A \) and \( c = (A + 1)^2K + 2\varepsilon(A + 1) \) shows
\[ (1.2) \]
\[ c = a + b + 2r \]
with \( r = \sqrt{ab + \varepsilon^2} \), and if \( d < c \), then one can deduce from the minimality of “\( d_+ \)” mentioned above that
\[ (1.3) \]
\[ c \geq a + b + d + \frac{2}{\varepsilon^2}(abd + rs't') \]
with \( s' = \sqrt{ad + \varepsilon^2} \) and \( t' = \sqrt{bd + \varepsilon^2} \). It follows from \cite{12} and \cite{13} that \( d \leq 0 \), a contradiction.

It is also to be noted that it suffices to prove the thesis for \( \varepsilon \) even. Indeed, if \( \varepsilon = \pm 2 \) and \( K \) is even then simplification by 2 results in \( D(1) \)-triples belonging to the desired families and transforms the fourth element \( d \) in the required form. Conversely, doubling all the entries of a \( D(1) \)-triple in the indicated families, one obtains a \( D(4) \)-triple in the families with doubled \( \varepsilon \).

The result published in \cite{22} for the \( \varepsilon = 1 \) case says that the conclusion of our Theorem 1 holds for either \( A \leq 10 \) or \( A \geq 52330 \). Similar results have been published in \cite{17} for \( \varepsilon = 2 \). More precisely, the statement has been proved for \( A \leq 22 \) as well as for \( A \geq 51767 \).

Theorem 1 has the following corollary on extendability of more general \( D(\varepsilon^2) \)-triples \( \{a, b, a + b + 2r\} \), where \( r = \sqrt{ab + \varepsilon^2} \), to quadruples.

**Corollary 2.** Let \( \varepsilon \in \{-2, -1, 1, 2\} \). Let \( \{a, b, c, d\} \) be a \( D(\varepsilon^2) \)-quadruple with \( a < b < c \) and \( c = a + b + 2r \), where \( r = \sqrt{ab + \varepsilon^2} \). If \( r \equiv \varepsilon \) (mod \( a \)), then \( d = d_+ \). In particular, if \( a \) has either of the forms \( 4|\varepsilon|, p^r \) and \( 2p^r \) with \( p \) an odd prime and \( e \) a non-negative integer, then \( d = d_+ \).

The progress achieved in our work is largely due to a version of Rickert’s theorem tailored for the triples we study. The novelty in its proof (given in Section 2) is to exploit, besides \( N \) being divisible by \( A \) (where \( N = (A^2 +
A) K/2 ± 2A), the fortunate fact that both N ± 2A and N ± 2 are divisible by A + 1. Theorem \[5\] in conjunction with an older theorem of Laurent \[23\] providing sharp upper bounds for linear forms in the logarithms of two algebraic numbers allows us to obtain remarkably small absolute bounds on A. Section \[6\] contains the details. With some computer help, we next show in Proposition \[27\] that if any D(4)-triple would be extendable to two quadruples then K < 240.24(A + 1) + K_0 as soon as A ≥ A_0. Here, A_0, K_0 are small positive integers determined by a gp script. Such a result is very helpful in reducing the number of pairs (A, K) for which an application of Baker-Davenport reduction is required.

In the final section of the paper we come back to the original problem on D(1)-quintuples and slightly improve the bounds on entries if the smallest ones form a regular triple.

**Proposition 3.** Let \{a, b, c, d, e\} be a D(1)-quintuple with a < b < c < d < e and c = a + b + 2√ab + 1. Then b ≤ a^3 − 2a \[\sqrt{3a + 1}\] + 3 and a ≥ 32.

**Proposition 4.** Let \{a, b, c, d, e\} be a D(1)-quintuple with a < b < c < d < e and b < 4a. Then b ≤ 4a − 4 \[\sqrt{3a + 1}\] + 3 and a ≥ 32815.

2. **Optimization of Rickert’s theorem**

The goal of this section is to provide the main technical tool used in our proof of Theorem \[1\]. As already mentioned, it is a variant of Rickert’s theorem that takes into account all peculiarities of the families we study.

**Theorem 5.** Let \(\varepsilon \in \{-2, -1, 1, 2\}\) and let A, K be integers satisfying \(K \geq 30.03|\varepsilon|^3(A + 1)\) with either \(A \geq 3\) or \(A = |\varepsilon| = 2\). Put \(N = (A^2 + A)K/2 + \varepsilon A\). Then the numbers \(\theta_1 = \sqrt{1 - \varepsilon A/N}\) and \(\theta_2 = \sqrt{1 + \varepsilon /N}\) satisfy

\[
\max \left\{ \left| \frac{\theta_1 - p_1}{q} \right|, \left| \frac{\theta_2 - p_2}{q} \right| \right\} > (2.838 \cdot 10^{28}(A + 1)N)^{-1} q^{-\lambda}
\]

for all integers \(p_1, p_2, q\) with \(q > 0\), where

\[
\lambda = 1 + \frac{\log(20(A + 1)N)}{\log \left( \frac{1.338N^2}{|\varepsilon|^4(A + 1)} \right)} < 2.
\]

**Proof.** Note that the assumptions \(A \geq 3\), \(K \geq 30.03|\varepsilon|^3(A + 1)\) immediately imply \(\lambda < 2\). The same bound on \(\lambda\) is valid under the hypothesis \(A = |\varepsilon| = 2\).

Our task is reduced to finding those real numbers satisfying the conditions in the following lemma.

**Lemma 6.** (cf. \[3\] Lemma 3.1) Let \(\theta_1, \theta_2\) be arbitrary real numbers and \(\theta_0 = 1\). Assume that there exist positive real numbers l, p, L and P with \(L > 1\) such that for each positive integer \(k\), we can find integers \(p_{ijk}\) (0 ≤ i, j ≤ 2) with nonzero determinant,

\[
|p_{ijk}| \leq p^{\lambda k} \quad (0 \leq i, j \leq 2)
\]

and

\[
\sum_{j=0}^{2} p_{ijk} \theta_j \leq ll^{-k} \quad (0 \leq i \leq 2).
\]
Then
\[ \max \left\{ \left| \theta_1 - \frac{p_1}{q} \right|, \left| \theta_2 - \frac{p_2}{q} \right| \right\} > Cq^{-\lambda} \]
holds for all integers \( p_1, p_2, q \) with \( q > 0 \), where
\[ \lambda = 1 + \frac{\log P}{\log L} \text{ and } C^{-1} = 4pP \left( \max\{1, 2\} \right)^{\lambda - 1}. \]

Consider the contour integral
\[ I_i(x) = \frac{1}{2\pi \sqrt{-1}} \int_{\gamma} \frac{(1 + z)^k(1 + z)^{1/2}}{(z - a_i)(F(z))^k} \, dz \]
for \( 0 \leq i \leq 2 \) and a positive integer \( k \), where \( a_0, a_1, a_2 \) are distinct integers with \( a_j = 0 \) for some \( j \), \( F(z) = (z - a_0)(z - a_1)(z - a_2) \) and \( \gamma \) is a closed, counter-clockwise contour enclosing \( a_0, a_1, a_2 \). The integral can be expressed as
\[ I_i(x) = \sum_{j=0}^{2} p_{ij}(x)(1 + a_jx)^{1/2} \]
for \( 0 \leq i \leq 2 \) with \( p_{ij}(x) \in \mathbb{Q}[x] \) of degree at most \( k \) (cf. [27]). From the arguments following Lemma 3.1 in [3] we see that
\[ \left| \sum_{j=0}^{2} p_{ij}(1/N) \left( 1 + \frac{a_j}{N} \right)^{1/2} \right| \leq \frac{27}{64} \left( 1 - \frac{A}{N} \right)^{-1} \left\{ \frac{27}{4} \left( 1 - \frac{A}{N} \right)^2 N^3 \right\}^{-k} \]
and
\[ |p_{ij}(1/N)| \left( 1 + \frac{a_j}{N} \right)^{1/2} \leq \max_{z \in \Gamma_j} \frac{|1 + z/N|^{k+1/2}}{|F(z)|^{k}} \]
where the contours \( \Gamma_j \) are defined by
\[ |z - a_j| = \min_{i \neq j} \left\{ \frac{|a_j - a_i|}{2} \right\}. \]

We now take \( a_0 = -\varepsilon A \), \( a_1 = 0 \), \( a_2 = \varepsilon \). Comparing the values of the right-hand side of (2.2) in the twelve cases for \( j \in \{0, 1, 2\} \) with \( \varepsilon \in \{-2, -1, 1, 2\} \) shows that
\[ |p_{ij}(1/N)| \leq \left( 1 + \frac{|\varepsilon|}{2(N + |\varepsilon|)} \right)^{1/2} \left( \frac{8(1 + 3|\varepsilon|/(2N))}{|\varepsilon|^3(2A + 1)} \right)^{k} \]
for all \( j \). Moreover, the proof of Lemma 3.3 in [27] enables us to write
\[ p_{ij}(1/N) = \sum_{ij} \left( \frac{k + \frac{1}{2}}{h_j} \right) C_{ij}^{-1} \prod_{l \neq j} \left( \frac{-k_{il}}{h_l} \right), \]
where
\[ C_{ij} = \frac{N^k}{(N + a_j)^{k-h_j}} \prod_{l \neq j} (a_j - a_l)^{k_{il}+h_j}, \]
\( k_{il} = k + \delta_{il} \) with \( \delta_{il} \) the Kronecker delta, \( \sum_{ij} \) denotes the sum over all non-negative integers \( h_0, h_1, h_2 \) satisfying \( h_0 + h_1 + h_2 = k_{ij} - 1 \), and \( \prod_{l \neq j} \) denotes
the product from \( l = 0 \) to \( l = 2 \) omitting \( l = j \). Let \( N = (A^2 + A)K/2 + \varepsilon A \). If \( j = 0 \), then
\[
|C_{i0}| = \frac{2^{k-h_0} N^k A^{k_i + h_0 + h_1 - k}(A + 1)^{k_i + h_0 + h_2 - k}|\varepsilon|^{k_i + k_{i2} + h_1 + h_2}}{K^{k-h_0}}.
\]
Thus we have \( 2^{k}|\varepsilon|^{3k}A^k(A + 1)^k N^k C_{i0}^{-1} \in \mathbb{Z} \). If \( j = 1 \), then
\[
|C_{i1}| = \frac{2^{k-h_1} N^k A^{k_0 + h_0 + h_1 - k}|\varepsilon|^{k_0 + k_{i2} + h_0 + h_2}}{(A + 1)K + 2\varepsilon} k^{h_1},
\]
which implies \( 2^{k}|\varepsilon|^{3k}A^k N^k C_{i1}^{-1} \in \mathbb{Z} \). If \( j = 2 \), then
\[
|C_{i2}| = \frac{2^{k-h_2} N^k (A + 1)^{k_0 + h_0 + h_2 - k}|\varepsilon|^{k_0 + k_{i1} + h_0 + h_1}}{(AK + 2\varepsilon)k^{h_2}}
\]
which yields \( 2^{k}|\varepsilon|^{3k}A^k(A + 1)^k N^k C_{i2}^{-1} \in \mathbb{Z} \). Since
\[
2^{2k-1} \left( \frac{k + \frac{1}{2}}{h_j} \right) \in \mathbb{Z}
\]
for all \( j \) (see the proof of Lemma 4.3 in [27]), it is deduced from the proof of Theorem 2.5 in [6] that
\[
p_{ijk} := 2^{-1} \{ 8|\varepsilon|^3 A(A + 1)N \}^k \Pi_2(k)^{-1} p_{ij}(1/N) \in \mathbb{Z},
\]
where \( \Pi_2(k) \) is an integer satisfying \( \Pi_2(k) > 1.6^k/(4.09 \cdot 10^{13}) \). It follows from (2.1), (2.3) with the assumptions \( A \geq 3, K \geq 30.03|\varepsilon|^3(A + 1) \) that
\[
\sum_{j=0}^{2} p_{ijk} \left( 1 + \frac{a_j}{N} \right)^{1/2} < 2K^{-l},
\]
where
\[
p = 2.045 \cdot 10^{13} \left( 1 + \frac{|\varepsilon|}{2(N + |\varepsilon|)} \right)^{1/2} < 2.047 \cdot 10^{13},
\]
\[
P = \frac{40A(A + 1)N(1 + 3/(2N))}{2A + 1} < 20(A + 1)N,
\]
\[
l = 2.045 \cdot 10^{13} \cdot \frac{27}{64} \left( 1 - \frac{A}{N} \right)^{-1} < 8.664 \cdot 10^{12},
\]
\[
L = \frac{1.35}{|\varepsilon|^3 A(A + 1)} \left( 1 - \frac{A}{N} \right)^2 N^2 > \frac{1.338 N^2}{|\varepsilon|^3 A(A + 1)}.
\]
Inequality (2.4) with the above estimates on \( p, P, l, L \) holds also for the case \( A = |\varepsilon| = 2 \). Therefore, we may take \( \lambda \) in Lemma [6] as in the assertion of Theorem [5] and
\[
C^{-1} < 4 \cdot 2.047 \cdot 10^{13} \cdot 20(A + 1)N (2 \cdot 8.664 \cdot 10^{12})^{\lambda-1} < 2.838 \cdot 10^{28}(A + 1)N.
\]
This completes the proof of the theorem. □
3. Auxiliary results for $\varepsilon = -2$

For an arbitrary $D(4)$-quadruple $\{a, b, c, d\}$ there exist positive integers verifying $ab + 4 = r^2$, $ac + 4 = s^2$, $bc + 4 = t^2$, $ad + 4 = x^2$, $bd + 4 = y^2, cd + 4 = z^2$.

Elimination of $d$ yields a system of generalized Pell equations

\begin{align}
(3.1) & \quad az^2 - cx^2 = 4(a - c), \\
(3.2) & \quad bz^2 - cy^2 = 4(b - c).
\end{align}

By well-known structure theorem for solutions of such an equation, there exist fundamental solutions $(x_0, z_0)$ and $(y_1, z_1)$ of $(3.1)$ and $(3.2)$, respectively, such that $z = v_m = w_n$, where

\begin{align*}
v_0 &= z_0, \quad v_1 = \frac{1}{2}(sz_0 + cx_0), \quad v_{m+2} = sv_{m+1} - v_m, \\
w_0 &= z_1, \quad w_1 = \frac{1}{2}(tz_1 + cy_1), \quad w_{n+2} = tw_{n+1} - w_n,
\end{align*}

and $|z_0| < a^{-1/4}c^{3/4}$, $|z_1| < b^{-1/4}c^{3/4}$.

The initial terms of these recurrent sequences are severely restricted.

**Lemma 7.** ([13], Lemma 9) Suppose the equation $v_m = w_n$ holds for some nonnegative integers $m$ and $n$.

(a) If both $m$ and $n$ are even then $z_0 = z_1$ and $|z_0| = 2$ or $|z_0| = (cr - st)/2$ or $|z_0| < 1.608a^{-5/14}c^{9/14}$.

(b) If $m$ is odd and $n$ is even then $|z_0| = t$, $|z_1| = (cr - st)/2$, and $z_0z_1 < 0$.

(c) If $m$ is even and $n$ is odd then $|z_1| = s$, $|z_0| = (cr - st)/2$, and $z_0z_1 < 0$.

(d) If both $m$ and $n$ are odd then $|z_1| = s$, $|z_0| = t$, and $z_0z_1 > 0$.

We first note that the relationship between the two families of $D(4)$-triples mentioned in Introduction is more than formal.

**Lemma 8.** If $K$ is a divisor of 4 then one has

\[
(K, A^2K - 4A, (A + 1)^2K - 4(A + 1)) = (K, B^2K + 4B, (B + 1)^2K + 4(B + 1))
\]

for $B = A - 4/K$.

Lemma 8 allows us to assume either $K = 3$ or $K \geq 5$, since the triples $\{K, A^2K + 4A, (A + 1)^2K + 4(K + 1)\}$ will be studied in the next section. Moreover, we may assume $A \geq 2$, since the family of $D(4)$-triples $\{K, K + 4, 4K + 8\}$ is known to be uniquely extendable by [13].

Throughout this section we denote $a = K$, $b = A^2K - 4A$, $c = (A+1)^2K - 4(A + 1)$, $r = AK - 2$, $s = (A + 1)K - 2$, $t = A(A+1)K - 4(A + 2)$. Note that one has $c = a + b + 2r$, which means that the triple $\{a, b, c\}$ is regular. It is equally easy to check that the element $d$ given by (1.1) coincides with $d_+ := a + b + c + 2abc + 2rst$, so that the quadruple $\{a, b, c, d\}$ is regular.

In the case we are interested in, more precise information on initial terms can be obtained.

**Lemma 9.** Suppose $(K, A^2K - 4A, (A + 1)^2K - 4(A + 1), d)$ is a $D(4)$-quadruple, where $K$, $A$ are integers with $A \geq 2$ and $K \geq 3$. Then any positive solution to the associated system of Pell equations satisfies $z = v_{2m} = w_{2n}$, with $x_0 = y_1 = 2$ and $z_0 = z_1 = \pm 2$. 
Proof. Assuming item (b) or (d) of Lemma 7 applies, it results $bc < t^2 = z_0^2 < a^{-1/2}c^{3/2}$, whence $A^2K < 5A + 1$, an inequality which is incompatible with $A \geq 2$ and $K \geq 3$. If item (c) holds then one concludes that one has $a < b^{-1/2}c^{1/2}$, equivalently $A^2K^3-4AK^2-(A+1)^2K+4(A+1) < 0$, which is false for parameters in the ranges $A \geq 2$ and $K \geq 3$.

So possibility (a) occurs. Since for the particular triple we are studying one has $cr - st = 4$, it remains to show that one cannot have $|z_0| < 1.608a^{-5/14}c^{9/14}$. Assuming the contrary, it results $\{a,(z_0^2 - 4)/c,b,c\}$ is a $D(4)$-quadruple to which Proposition 1 in [12] applies, giving $c > \min\{0.173b^{13/2}a^{11/2}, 0.087b^{7/2}a^{5/2}\}$, which is obviously false. □

By Lemma 9 one can express any solution to Pellian equation (3.2) as $y = u'_n$, where

$$u'_0 = 2, \quad u'_1 = t \pm b, \quad u'_{n+2} = tu'_{n+1} - u'_n.$$  

Any solution to the other Pellian equation

$$ay^2 - bx^2 = 4(a - b)$$

deduced from (3.1) and (3.2) is given by $y = u''_n$, where

$$u''_0 = y_2, \quad u''_1 = \frac{1}{2}(ry_2 + bx_2), \quad u''_{n+2} = ru''_{n+1} - u''_n$$

with a solution $(y_2, x_2)$ to (3.4) satisfying

$$|y_2| < \sqrt{\frac{b\sqrt{b}}{\sqrt{a}}} \quad \text{and} \quad 1 \leq x_2 < \sqrt{b}.$$  

Considering (3.3) and (3.4) modulo $b$, we see that if $u'_2n = u''_2$ has a solution, then $y_2 \equiv 2 \pmod{b}$, which together with (3.6) implies $y_2 = 2$ and $x_2 = 2$. Suppose that $u'_2n = u''_{2n+1}$ has a solution. Then, as seen in [17] Section 5], we have

$$bx_2 - r|y_2| = 4$$

and $bx_2 + r|y_2| < 2bx_2 < 2b\sqrt{b}$. If $A \geq 3$ and $K \geq 3$, then $b \geq 9a - 12 \geq 15$, which together with (3.6) yields

$$(bx_2 - r|y_2|)(bx_2 + r|y_2|) = 4b(b - a) - 4y_2^2 > \frac{4b(8b - 3\sqrt{3b} - 12)}{9}.$$  

Hence, we obtain

$$bx_2 - r|y_2| > \frac{2(8b - 3\sqrt{3b} - 12)}{9\sqrt{b}} > 5,$$

which contradicts (3.7). Similarly, in case $A = 2$ and $K \geq 6$, we will arrive at a contradiction. We have thus showed the following.

Lemma 10. Suppose $(K, A^2K - 4A, (A + 1)^2K - 4(A + 1), d)$ is a $D(4)$-quadruple, where $K, A$ are integers with either $A \geq 3$ and $K \geq 3$ or $A = 2$ and $K \geq 6$. Then any positive solution to the associated system of Pell equations satisfies $y = u'_2n = u''_2$, with $x_2 = y_2 = 2$.

Throughout the rest of this section, suppose that either of the following holds:

- $A \geq 3$ and either $K = 3$ or $K \geq 5$;
- $A = 2$ and $K \geq 6$. 

Lemmas \[9\] and \[10\] enable us to express any solution to the system of Pellian equations (3.1) and (3.4) as
\[
x = W_{2m} = V_{2l},
\]
where
\[
W_0 = 2, \quad W_1 = s \pm a, \quad W_{m+2} = sW_{m+1} - W_m, \\
V_0 = 2, \quad V_1 = r + a, \quad V_{l+2} = rV_{l+1} - V_l.
\]
Put
\[
\alpha = \frac{s + \sqrt{ac}}{2}, \quad \beta = \frac{r + \sqrt{ab}}{2}, \quad \chi = \frac{\sqrt{bc} + \sqrt{ac}}{\sqrt{bc} \pm \sqrt{ab}}.
\]
Then, in a fashion similar to Lemma 10 in \[17\], one finds that if \(m \geq 1\), then
\[
0 < \Lambda = 2l \log \beta - 2m \log \alpha + \log \chi.
\]
(3.8)

Lemma 11. \(\alpha - \beta > K = s - r\).

Proof. This is equivalent to \(\sqrt{s^2 - 4} > K + \sqrt{r^2 - 4}\). Squaring this, one arrives at the obvious inequality \(r > \sqrt{r^2 - 4}\). \(\square\)

Lemma 12. \(\frac{1}{A+1} < \frac{a}{c} \log \frac{\alpha}{\beta} < \sqrt{\frac{a}{b}} < \frac{1}{A-1}\).

Proof. From the mean value theorem one gets
\[
\log \alpha - \log \beta = \frac{s - r}{\sqrt{\xi^2 - 4}} \text{ for some } \xi \text{ satisfying } r < \xi < s.
\]
The claim follows after elementary computations, using the explicit formulas for \(s\) and \(r\). \(\square\)

Equally simple computations yield the following.

Lemma 13. For \(AK \geq 34\) one has \(\beta > 0.999r\).

Lemma 14. Let \(\rho\) be a positive integer. Then for \(AK \geq 2\rho + 4\) one has
\[
c - a \leq b + \frac{(2\rho + 2)b}{pA}.
\]
Proof. The claim is equivalent to \(prA \leq (\rho + 1)b\), which, on using the explicit formulas for \(r\) and \(b\), turns out to be precisely \(AK \geq 2\rho + 4\). \(\square\)

Lemma 15. Let \(A_0 \geq 2, K_0 \geq 3,\) and \(\rho \geq 14\) be integers. If \(A \geq A_0, K \geq K_0,\) and \(AK \geq 2\rho + 4\) then
\[
bc^2(c - a) < 0.992^{-1} \left(1 + \frac{2\rho + 2}{pA_0}\right) \left(1 + \frac{1}{2\rho + 2}\right)^4 \beta^8.
\]
Proof. Notice that one has
\[
K^2bc = (r^2 - 4)(s^2 - 4) < r^2s^2 = K^2r^4 \left(1 + \frac{1}{r}\right)^2
\]
and, by the previous lemma,
\[
bc^2(c - a) \leq \left(1 + \frac{2\rho + 2}{pA_0}\right)b^2c^2 < \left(1 + \frac{2\rho + 2}{pA_0}\right) \left(1 + \frac{1}{2\rho + 2}\right)^4 r^8,
\]
while Lemma \[13\] yields
\[
\beta^8 > (0.999r)^8 > 0.992r^8.
\]
\(\square\)
Lemma 16. \( \frac{\sqrt{bc} + \sqrt{ac}}{\sqrt{bc} - \sqrt{ab}} < 1 + \frac{5}{2A} \) if one of the following holds:

\[
\begin{align*}
K &= 3, & A &\geq 6, \\
K &= 5, & A &\geq 5, \\
6 &\leq K & A &\geq 4, \\
K &\geq 12, & A &\geq 3.
\end{align*}
\]

Proof. The desired inequality is equivalent to \( 2A\sqrt{ac} + (2A + 5)\sqrt{ab} < 5\sqrt{bc} \).
Squaring this and replacing \( \sqrt{bc} \) by the larger quantity \( t \), we arrive at a bivariate polynomial inequality which is easily seen to hold in each of the cases displayed above. \( \square \)

By rewriting the linear form considered above in the form

\[ \Lambda = \log(\beta^2\nu) - 2m\log(\alpha/\beta), \]

one may obtain a lower bound for \( m \).

Lemma 17. If \( \nu = l - m \) with \( m \geq 1 \), then \( m > (A - 1)\nu \log \beta \).

Proof. Estimate (3.8) implies

\[ -\alpha^{1-4m} + \log(\beta^{2\nu}\chi) < 2m\log(\alpha/\beta) < \log(\beta^{2\nu}\chi). \]

Since it is not difficult to check \( \log \chi > \alpha^{1-4m} \), one has \( m\log(\alpha/\beta) > \nu \log \beta \).

The asserted inequality now follows from Lemma 12. \( \square \)

4. Auxiliary results for \( \varepsilon = 2 \)

In this section we keep the notation

\[ \alpha = \frac{s + \sqrt{ac}}{2}, \quad \beta = \frac{r + \sqrt{ab}}{2}, \quad \chi = \frac{\sqrt{bc} + \sqrt{ac}}{\sqrt{bc} \pm \sqrt{ab}}. \]

Results similar to those given in the previous section hold for these algebraic numbers. The proofs contain no new ideas, the differences appear in the numerical details. Therefore, we avoid annoying repetitions by omitting the proofs.

Lemma 18. \( \alpha - \beta > K = s - r \).

Lemma 19. One always has \( \alpha > 0.998s \) and \( \beta > 0.998r \). Moreover, for \( AK \geq 30 \) one has \( \alpha > 0.999s \) and \( \beta > 0.999r \).

Lemma 20. For \( AK \geq 43 \) one has \( \sqrt{ab} > 0.999r, \sqrt{ac} > 0.999s, \) and \( 0.999(\sqrt{ac} - \sqrt{ab}) < K \). Moreover, for any \( A \geq 23 \) it holds \( \sqrt{bc} > 0.999t \).

Lemma 21. \( \frac{1}{A + 1 + 2/K} < \log \frac{\alpha}{\beta} < \frac{1}{A} \).

Lemma 22. \( c - a < \left( 1 + \frac{2}{A} \right) b \).

Lemma 23. Let \( A_0 \geq 1, K_0 \geq 1, \) and \( \rho \geq 15 \) be integers. If \( A \geq A_0, K \geq K_0, \) and \( AK \geq 2\rho \) then

\[ bc^2(c - a) < 0.992^{-1} \left( 1 + \frac{2}{A_0} \right) \left( \frac{1}{K_0} + \frac{1}{2\rho + 2} \right)^4 \beta^8. \]

Lemma 24. \( \frac{\sqrt{bc} + \sqrt{ac}}{\sqrt{bc} - \sqrt{ab}} < 1 + \frac{5}{2A} \).

Lemma 25. If \( \nu = l - m \) with \( m \geq 1 \), then \( m > A\nu \log \beta \).
5. Application of the hypergeometric method to the case $|\varepsilon| = 2$

The hypergeometric method is very effective when dealing with small values of $A$. For the rest of the section we put

$$N = \frac{1}{2}(A^2 + A)K + \varepsilon A,$$

$$\theta_1 = \sqrt{1 - \frac{\varepsilon A}{N}}, \quad \theta_2 = \sqrt{1 + \frac{\varepsilon}{N}}$$

with $\varepsilon \in \{-2, 2\}$.

**Lemma 26.** Let $(x, y, z)$ be a solution in positive integers to the system of Diophantine equations (3.1) and (3.2). Then

$$\max \left\{ \left| \frac{(A + 1)x}{z} - \theta_1 \right|, \left| \frac{(A + 1)y}{A z} - \theta_2 \right| \right\} < 2(A + 1)(A + 1 + 2 \cdot K^{-1})z^{-2}.$$

**Proof.** Follow the proof of Lemma 6 from [17] with a twist on the final step — use $A + 1 + 2 \cdot K^{-1}$ instead of $A + 3$ as an upper bound for $\sqrt{c/a}$. \hfill \Box

A lower bound for the left side of the inequality in the previous lemma can be obtained by using results on simultaneous approximations of algebraic numbers which are close to 1.

As already mentioned, we study small values of $A$ with the help of the hypergeometric method. The next result contains the outcome of the study.

**Proposition 27.** Let $a = K$, $b = A^2K + 2\varepsilon A$, $c = (A + 1)^2K + 2\varepsilon(A + 1)$ with $\varepsilon \in \{-2, 2\}$ and positive integers $A$, $K$. Suppose that $\{a, b, c, d\}$ is a $D(4)$-quadruple with $d > 2$ not given by (1.1). If $A \geq A_0$, then $K < 240.24(A + 1) + K_0$, where

$$(A_0, K_0) \in \{(1326, 0), (454, 1000), (3, 23000), (2, 210000)\}.$$

**Proof.** Suppose that $K \geq 240.24(A + 1)$. On applying Lemma 26 and Theorem 5 with $p_1 = 2(A + 1)x$, $p_2 = (A + 1)y$, $q = Az$, $N = (A^2 + A)K/2 + 2\varepsilon A$, one gets

$$(5.1) \quad z^{2 - \lambda} < 2C^{-1}A^2(A + 1)(A + 1 + 2 \cdot K^{-1}),$$

where $C^{-1} = 2.838 \cdot 10^{28}(A + 1)N$. It is easy to see from the proof of Lemma 5 in [17] that

$$(5.2) \quad \log z > 2m \log((A + 1)K + \varepsilon - 2).$$

The assumption $K \geq 240.24(A + 1)$ ensures $\lambda < 2$, which, combined with Lemmas 26 and 25 and inequalities (5.1), (5.2), implies

$$(5.3) \quad (A - 1)\nu \log \beta < \frac{\log(2C^{-1}A^2(A + 1)(A + 1 + 2/K))}{2(2 - \lambda)\log((A + 1)K + \varepsilon - 2)}.$$

Since

$$2 - \lambda = \frac{\log \left( \frac{0.669N}{80A(A + 1)^2} \right)}{\log \left( \frac{0.669N^2}{4A(A + 1)} \right)} = \frac{\log \left( \frac{0.669((A + 1)K + \varepsilon - 2)}{160(A + 1)^2} \right)}{\log \left( \frac{0.669A((A + 1)K + \varepsilon - 2)^2}{16A(A + 1)} \right)},$$

the right-hand side of (5.3) is a decreasing function of $K$. Therefore, one can easily verify the assertion by using (5.3) with $\nu \geq 1$ and a computer. \hfill \Box
6. Application of Baker’s method to the case $|\varepsilon| = 2$

**Proposition 28.** Let $a = K$, $b = A^2 K + 2\varepsilon A$, $c = (A + 1)^2 K + 2\varepsilon (A + 1)$ with $\varepsilon \in \{-2, 2\}$ and positive integers $A, K$. Suppose that $\{a, b, c, d\}$ is a $D(4)$-quadruple with $d > 2$ not given by (1.1). Then, we have

$$A \leq \begin{cases} 2800 & \text{if } \varepsilon = -2; \\ 3365 & \text{if } \varepsilon = 2. \end{cases}$$

**Proof.** Recall that

$$\alpha = \frac{s + \sqrt{ac}}{2}, \quad \beta = \frac{r + \sqrt{ab}}{2}, \quad \chi = \frac{\sqrt{bc} + \sqrt{ac}}{\sqrt{bc} \pm \sqrt{ab}}.$$

All these algebraic numbers belong to the number field (of degree four) $\mathbb{Q}(\sqrt{ab}, \sqrt{ac})$, whose $\mathbb{Q}$-automorphisms are defined by $(\sqrt{ab}, \sqrt{ac}) \mapsto (e_1 \sqrt{ab}, e_2 \sqrt{ac})$, where $e_1, e_2 \in \{-1, +1\}$. It follows that the conjugates of $\chi$ are

$$\chi' = \frac{\sqrt{bc} + \sqrt{ac}}{\sqrt{bc} + \sqrt{ab}}, \quad \chi'' = \frac{\sqrt{bc} - \sqrt{ac}}{\sqrt{bc} + \sqrt{ab}}, \quad \chi''' = \frac{\sqrt{bc} - \sqrt{ac}}{\sqrt{bc} - \sqrt{ab}}.$$

Hence

$$0 < \chi \chi''' = \chi' \chi'' = \frac{bc - ac}{bc - ab} < 1$$

and

$$0 < \chi, \chi', \chi'', (1 - \chi)(1 - \chi'), (1 - \chi'')(1 - \chi''').$$

This shows that $(bc - ac)^2$ is a denominator for $\chi$ and that

$$h(\chi) \leq \frac{1}{4} \left( \log(b^2(c - a)^2) + \log \frac{c(\sqrt{a} + \sqrt{b})^2}{b(c - a)} \right) = \frac{1}{4} \log(bc(c - a)(\sqrt{a} + \sqrt{b})^2).$$

Here $c = a + b + 2r$, so that $c > (\sqrt{a} + \sqrt{b})^2$ and

$$h(\chi) \leq \frac{1}{4} \log(bc^2(c - a)).$$

Now we see that, for $A \geq 80$, Lemmas [13] and [23] yield

$$h(\chi) < \begin{cases} 0.014 \log \beta & \text{if } \varepsilon = -2, \\ 2.005 \log \beta & \text{if } \varepsilon = 2. \end{cases}$$

The conjugates of $\alpha/\beta$ are $\alpha/\beta$ and

$$\frac{s - \sqrt{ac}}{r + \sqrt{ab}}, \quad \frac{s + \sqrt{ac}}{r - \sqrt{ab}}, \quad \frac{s - \sqrt{ac}}{r - \sqrt{ab}}.$$

As among these four numbers only the first and the third ones are of modulus greater than 1, it easily follows that

$$h(\alpha/\beta) = \frac{1}{2} \log \alpha,$$

because $\alpha$ and $\beta$ are algebraic units. Moreover since $\chi$ is obviously not a unit, the numbers $\beta^{2^\nu} \chi$ and $\alpha/\beta$ are multiplicatively independent. Now we are ready to apply Laurent’s lower bounds [23] to the linear form

$$\Lambda = \log(\beta^{2^\nu} \chi) - 2m \log(\alpha/\beta).$$
With the notation of [23] we have
\[ b_1 = 2m, \quad b_2 = 1, \quad \alpha_1 = \alpha / \beta, \quad \alpha_2 = \beta^{2 \nu} \chi. \]

Using the above study, and the inequality \( \log \alpha_1 > 1/(A + 1 + 2/K) \) following from Lemmas 12 and 21, one can choose
\[ a_1 \geq 4 \log \alpha + \frac{\rho - 1}{A + 1 + 2/K} \]
and, in view of Lemmas 16 and 24, the choice
\[ a_2 \geq (2\nu(\rho + 3) + q_2) \log \beta + (\rho - 1) \log \left(1 + \frac{5}{2A}\right) \]
is legitimate for \( A \geq 80 \), where \( q_2 = 0.112 \) or 16.04 depending on \( \varepsilon = -2 \) or 2, respectively.

By way of illustration, we present the details in case \( \rho = 37, \mu = 0.63 \). We shall also suppose that \( A \geq 2700 \). Then we may take
\[ a_1 = 4.0017 \log \alpha, \quad a_2 = (80\nu + q_2') \log \beta, \]
where \( q_2' = 0.116 \) or 16.045 depending on \( \varepsilon = -2 \) or 2, respectively. From \( \alpha > \beta \) we then get
\[ \frac{b_1}{a_2} + \frac{b_2}{a_1} = \frac{2m}{(80\nu + q_2') \log \beta} + \frac{1}{4.0017 \log \alpha} \]
\[ < \frac{m + 10\nu + q_2'/8}{(40\nu + q_2'/2) \log \beta}, \]
which implies that
\[ h = 4 \log \left(\frac{m + 10\nu + q_2'/8}{(40\nu + q_2'/2) \log \beta}\right) + 11.913 \]
satisfies the hypotheses of Laurent’s theorem.

Suppose \( h \leq 28.9 \). If \( \varepsilon = -2 \), then it results from Lemma 17
\[ (A - 1)\nu \log \beta < m < (40 \nu + 0.058) \exp(4.24675) \log \beta, \]
that is,
\[ (6.1) \quad A < \left(40 + \frac{0.058}{\nu}\right) \exp(4.24675) + 1. \]

Similarly, if \( \varepsilon = 2 \), then Lemma 25 implies
\[ (6.2) \quad A < \left(40 + \frac{8.0225}{\nu}\right) \exp(4.24675). \]

Suppose \( h > 28.9 \). Combining inequality (3.8) with Theorem 2 from [23] yields
\[ (6.3) \quad (4m - 1) \log \alpha < C \left(h + \frac{\lambda}{\sigma}\right)^2 a_1 a_2 + \sqrt{\omega \theta} \left(h + \frac{\lambda}{\sigma}\right) + \log \left(C' \left(h + \frac{\lambda}{\sigma}\right)^2 a_1 a_2\right), \]
where
\[ \sigma = \frac{1 + 2\mu - \mu^2}{2}, \quad \lambda = \sigma \log \rho, \]
\[ \omega = 2 \left( 1 + \sqrt{1 + \frac{1}{4H^2}} \right), \quad \theta = \sqrt{1 + \frac{1}{4H^2} + \frac{1}{2H}}, \]
\[ h \geq \max \left\{ 4 \left( \log \left( \frac{b_1}{a_2} + \frac{b_2}{a_1} \right) + \log \lambda + 1.75 \right) + 0.06, \lambda, 2 \log 2 \right\}, \]
\[ H = \frac{h}{\lambda} + \frac{1}{\sigma}, \]
\[ C = \frac{\mu}{\lambda^3 \sigma} \left( \frac{\omega}{6} + \frac{1}{2} \sqrt{\frac{\omega^2}{9} + \frac{8\lambda\omega^{5/4}\theta^{1/4}}{3\sqrt{a_1a_2H^{1/2}}} + \frac{4}{3} \left( \frac{1}{a_1} + \frac{1}{a_2} \right) \frac{\lambda\omega}{H}} \right)^2, \]
\[ C' = \sqrt{\frac{C\sigma\omega\theta}{\lambda^3 \mu}}. \]

If \( \varepsilon = -2 \), then inequality (6.3) shows that
\[ \frac{m}{(40\nu + 0.058) \log \beta} < 69.799, \]
which together with Lemma 17 implies
\[ (6.4) \quad A < 69.799 \left( 40 + \frac{0.058}{\nu} \right) + 1. \]
Inequalities (6.1) and (6.4) together yield \( A \leq 2800 \) for all \( \nu \geq 1 \). If \( \varepsilon = 2 \), then inequality (6.3) and Lemma 25 together show that
\[ (6.5) \quad A < 70.073 \left( 40 + \frac{8.0225}{\nu} \right), \]
with which combining (6.2) implies \( A \leq 3365 \) for all \( \nu \geq 1 \).

\[ \Box \]

7. Proof of Theorem

Although there remain only finitely many cases to check, we will try to make the number as small as possible in order to save a computation time.

Lemma 29. Suppose \( V_{2l} = W_{2m} \) holds for some integers \( l \) and \( m \) with \( m \geq 2 \). If \( \nu = l - m \), then \( \nu \geq 11 \).

Proof. Remark that the integer \( m \) is completely determined for fixed \( A, K \) and \( \nu \). In fact, \( m \) is expressed as
\[ m = \frac{\nu \log \beta + 0.5 \log \chi}{\log(\alpha/\beta)} - \frac{\Lambda}{2 \log(\alpha/\beta)}, \]
where the term after the minus sign is positive and less than 1 in view of (3.8).
Thus we have
\[ m = \left\lfloor \frac{\nu \log \beta + 0.5 \log \chi}{\log(\alpha/\beta)} \right\rfloor. \]
For each set of values of \( A, K \) bounded as in Propositions 27 and 28, and for each \( \nu \) with \( 1 \leq \nu \leq 10 \) we computed the linear form \( \Lambda \) and found that \( \Lambda > \alpha^{1-4m} \),
which contradicts (3.8). Our computer needed about 30 hours to perform these computations. □

**Proposition 30.** Keep the hypotheses of Proposition 27. Then:

1. \( K < 240.24(A + 1) + 740 \). Moreover, if \( 40 \leq A \leq 2810 \), then \( K < 237.05(A + 1) \).
2. \( A \leq 2796 \) if \( \varepsilon = -2 \) and \( A \leq 2810 \) if \( \varepsilon = 2 \).

**Proof.** (1) Inequality (5.3) with \( A \geq 2 \) and \( \nu \geq 11 \) shows the first assertion. In a way similar to Lemma 29, one can check by computer that if \( A \leq A_0 \), then \( \nu \geq \nu_0 \), where \( (A_0, \nu_0) \in \{(900, 12), (360, 14), (40, 25)\} \), and show the following:

Substituting \( K = 237.05(A + 1) \) and each value of \( A \) in each of the ranges \( 40 \leq A < 360 \), \( 360 \leq A < 900 \), \( 900 \leq A \leq 2810 \) into the quantities \( p, P, l, L \) defined in Section 2 immediately after (2.4), Lemma 6 yields renewed \( C^{-1} \) and \( \lambda \), that are not compatible with inequality (5.3).

Thus one obtains the revised bound \( K < 237.05(A + 1) \) for \( 40 \leq A \leq 2810 \).

(2) Inequalities (6.1), (6.4) together with \( \nu \geq 11 \) give the asserted inequality for \( \varepsilon = -2 \). When \( \varepsilon = 2 \) one verifies that \( \nu \geq 70 \) for \( A > 2810 \) (this takes only a few hours of computer time), which together with inequalities (6.2), (6.5) implies the result. □

In order to get an absolute upper bound for \( m \), we appeal to Matveev’s theorem for three logarithms.

**Theorem 31** ([24]). Let \( \lambda_1, \lambda_2, \lambda_3 \) be \( \mathbb{Q} \)-linearly independent logarithms of non-zero algebraic numbers and let \( b_1, b_2, b_3 \) be rational integers with \( b_1 \neq 0 \). Define \( \alpha_j = \exp(\lambda_j) \) for \( j = 1, 2, 3 \) and

\[
\Lambda = b_1 \lambda_1 + b_2 \lambda_2 + b_3 \lambda_3.
\]

Let \( D \) be the degree of the number field \( \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) \) over \( \mathbb{Q} \). Put

\[
\chi = \frac{\mathbb{R}(\alpha_1, \alpha_2, \alpha_3)}{\mathbb{R}}.
\]

Let \( A_1, A_2, A_3 \) be positive real numbers, which satisfy

\[
A_j \geq \max\{Dh(\alpha_j), |\lambda_j|, 0.16\} \quad (1 \leq j \leq 3).
\]

Assume that

\[
B \geq \max\{1, \max\{|b_j|A_j/A_1; 1 \leq j \leq 3\}\}.
\]

Define also

\[
C_1 = \frac{5 \times 165}{6\chi} e^3 (7 + 2\lambda) \left(\frac{3e}{2}\right) \chi \left(20.2 + \log\left(3^{5.5}D^2 \log(eD)\right)\right).
\]

Then

\[
\log |\Lambda| > -C_1 D^2 A_1 A_2 A_3 \log (1.5 eDB \log(eD)).
\]

In our case we choose

\[
\alpha_1 = \chi, \quad b_1 = 1, \quad \alpha_2 = \beta, \quad b_2 = 2\nu, \quad \alpha_3 = \alpha/\beta, \quad b_3 = -2m.
\]
Then we can take, for \( j = 1, 2, 3 \),
\[
A_j = 4h(\alpha_j)
\]
and
\[
B = 2mA_3/A_1.
\]
With these values we get
\[
m < 3.4 \cdot 10^{16}.
\]
It now remains only to perform the reduction procedure. Let
\[
\alpha = \frac{s + \sqrt{ac}}{2}, \quad \gamma = \frac{t + \sqrt{bc}}{2}, \quad \mu = \frac{\sqrt{b(\sqrt{c} \pm \sqrt{a})}}{\sqrt{a(\sqrt{c} \pm \sqrt{b})}},
\]
where the signs coincide. If \( z = v_{2m} = w_{2n} \) has a solution with \( mn \neq 0 \), then the linear form \( \Omega = 2m \log \alpha - 2n \log \gamma + \log \mu \) satisfies
\[
0 < \Omega < 2ac \alpha - 4m
\]
(cf. [17, Section 4]). The following is a version of the Baker-Davenport lemma ([2, Lemma]), due to Dujella and Peth˝o, needed here.

**Lemma 32.** ([8, Lemma 5 a]) Let \( M \) be a positive integer, and \( \kappa, \xi \) real numbers. Let \( P/Q \) be a convergent of the continued fraction expansion of \( \kappa \) such that \( Q > 6M \). Put \( \eta = ||\kappa Q|| - M ||\kappa|| \), where \( || \cdot || \) denotes the distance from the nearest integer. If \( \eta > 0 \), then there exists no solution of the inequality
\[
0 < m\kappa - n + \xi < EB^{-m}
\]
in integers \( m \) and \( n \) with
\[
\frac{\log(EQ/\eta)}{\log E} \leq m < M.
\]
We apply Lemma 32 with
\[
\kappa = \frac{\log \alpha}{\log \gamma}, \quad \xi = \frac{\log \mu}{2 \log \gamma}, \quad E = \frac{ac}{\log \gamma}, \quad B = \alpha^4
\]
and \( M = 3.4 \cdot 10^{16} \) for \( A, K \) satisfying
\[
\begin{cases}
K < 237.05 \cdot (A + 1) & \text{if } 40 \leq A \leq 2810, \\
K < 240.24 \cdot (A + 1) + 740 & \text{if } 2 \leq A \leq 39.
\end{cases}
\]
The computation was carried out by running a program developed in PARI/GP ([20]) with the precision
\[
\text{realprecision} = \max\{180, 10[A/100]\},
\]
and no counter-example was found. The verification took around three months. This completes the proof of Theorem 1.

**Proof of Corollary 2** Suppose that \( r \equiv \varepsilon \pmod{a} \), and put \( r = ka + \varepsilon \) with an integer \( k \). Then, \( b = k^2a + 2\varepsilon k \) and \( c = (k + 1)^2a + 2\varepsilon(k + 1) \). Applying Theorem 1 to the triple \( \{a, b, c\} \) with \( K = a \) and \( A = k \), one can obtain the first assertion. The second assertion is an immediate consequence of the first one together with the fact that one always has \( r^2 \equiv \varepsilon^2 \pmod{a} \). \( \square \)
8. An application to the study of $D(1)$-quintuples

In this section we consider a hypothetical $D(1)$-quintuple $\{a, b, c, d, e\}$ with $a < b < c < d < e$, $c = a + b + 2r$, and $r = a^2 - \Delta$. Theorem~\ref{thm黛} ensures $\Delta > 1$. An upper bound of the type $\Delta < a^2 - a$ is derived from the obvious inequality $a < r$. Our considerations are based on a recent result, recalled here for reader’s convenience.

**Lemma 33.** ([5, Theorem 1.3]) Let $\{a, b, c, d, e\}$ be a quintuple with $a < b < c < d < e$ and $c = a + b + 2\sqrt{ab} + 1$. Then $b < a^3$ and $\gcd(b, c) = 1$. In particular, at least one of $a, b$ is odd.

In conjunction with Lemma 3.4 from [5], which essentially says that for each $D(1)$-quintuple one has $b > 4000$, Lemma 33 gives the lower bound $a \geq 16$. From $ab + 1 = r^2$ one obtains $b = a^3 - 2a\Delta + (\Delta^2 - 1)/a$, whence the conclusion that the integer $a$ is a divisor greater than 15 for $\Delta^2 < 1$. This in turn implies $\Delta \geq 5$.

When $\Delta = 5$, the only admissible divisor of 24 is $a = 24$, so that $b = 13585 = 5 \cdot 11 \cdot 13 \cdot 19$ and $c = 14751 = 11 \cdot 1341$. Then $\gcd(b, c) = 11$, in contradiction with Lemma 33.

Up to now we have proved that $\Delta \geq 6$, an information with striking consequences.

**Proposition 34.** In the hypothesis of Lemma 33 one has $b < a^3 - 11a$ and $a \geq 20$.

**Proof.** The first assertion follows from $a^3 - 2a\Delta + (\Delta^2 - 1)/a < a^3 - 11a$, which is equivalent to $\Delta^2 - 2a\Delta + 11a^2 \leq 0$ and to $a^2 - \sqrt{a^3 - 11a^2} \leq \Delta \leq a^2 + \sqrt{a^3 - 11a^2}$. The right inequality is much weaker than $\Delta < a^2 - a$, while the left one is easily derived by interlacing 6 between its terms.

The second assertion in the conclusion follows from Corollary 2, since 17 and 19 are prime numbers, while 18 is twice a power of a prime. \hfill $\Box$

If so needed/wanted, one can pursue the analysis and eliminate other values of $\Delta$. For instance, when $\Delta = 6$, $\Delta^2 - 1$ has unique divisor greater than 16, namely $a = 35$. To conclude that $\Delta > 6$ one has to prove that the $D(1)$-triple $(35, 42456, 44929)$ has unique extension to a $D(1)$-quadruple. When $\Delta = 7$, the only admissible candidates for the smallest entry are $a = 24$ and $a = 48$. The former value entails $b = 13490$, so that $\gcd(b, c) = 2$, which means that in this case one can not obtain a $D(1)$-quintuple, so it remains to study the extendability of the triple $(48, 109921, 114563)$. In order to prove that one has $\Delta > 10$, three more triples, viz., $(a, b, c) = (21, 8928, 9815), (80, 510561, 523423), (99, 968320, 988001)$, need to be shown to have unique extension to a $D(1)$-quadruple.

Each lower bound $\Delta \geq \Delta_0$ can be used to improve upon Proposition 34.

Another kind of upper bounds for $b$ can be obtained from $\Delta > 1$ and $\Delta^2 \equiv 1 \pmod{a}$. In other words, it holds

$$\Delta \geq \sqrt{a + 1} \quad \text{and} \quad b \leq a^3 - 2a \sqrt{a + 1} + 1.$$  

The extremal case $\Delta = \sqrt{a + 1}$ frequently appears in the observation above, which motivated us to show the following.

**Proposition 35.** Let $\Delta \geq 6$ be an integer and

$$a = \Delta^2 - 1, \quad b = \Delta^6 - 3\Delta^4 - 2\Delta^3 + 3\Delta^2 + 2\Delta, \quad c = \Delta^6 - \Delta^4 - 2\Delta^3 + 1.$$
If \( \{a, b, c, d\} \) is a \( D(1) \)-quadruple, then
\[
d = d_+ = 4\Delta^{14} - 20\Delta^{12} - 16\Delta^{11} + 40\Delta^{10} + 56\Delta^9 - 16\Delta^8
- 72\Delta^7 - 32\Delta^6 + 24\Delta^5 + 32\Delta^4 + 8\Delta^3 - 4\Delta^2 - 4\Delta.
\]

**Proof.** We easily verify that
\[
r = \Delta^4 - 2\Delta^2 - \Delta + 1, \quad s = \Delta^4 - \Delta^2 - \Delta, \quad t = \Delta^6 - 2\Delta^4 - 2\Delta^3 + \Delta^2 + \Delta + 1.
\]
Suppose that \( \{a, b, c, d\} \) is a \( D(1) \)-quadruple with \( d > d_+ \). Putting \( ad + 1 = x^2 \), \( bd + 1 = y^2 \), \( cd + 1 = z^2 \), and eliminating \( d \) from these equations, we obtain the following system of Pellian equations:
\[
(8.1) \quad az^2 - cx^2 = a - c,
(8.2) \quad ay^2 - bx^2 = a - b.
\]
Since \( c = a + b + 2r \), the same argument as Section 2 in [22] applies and one finds that any solution to the system of Pellian equations (8.1), (8.2) is given by
\[
x = W_{2m} = V_{2l}, \quad W_0 = 1, \quad W_1 = s \pm a, \quad W_{m+2} = 2sW_m + 1 - W_m,
\]
\[
V_0 = 1, \quad V_1 = r + a, \quad V_{l+2} = 2rV_{l+1} - V_l.
\]

Put
\[
\alpha = s + \sqrt{ac}, \quad \beta = r + \sqrt{ab}, \quad \chi = \frac{\sqrt{bc} + \sqrt{ac}}{\sqrt{bc} + \sqrt{ab}}.
\]

The following results are the analogs of the preceding ones.

**Lemma 3.5’.** \( \alpha - \beta > 2a = 2(s - r) \).

**Lemma 3.6’.** \( 1/\Delta^2 < \sqrt{a/c} < \log(\alpha/\beta) < \sqrt{a/b} < 1/(\Delta^2 - 2) \).

**Lemma 3.7’.** \( \beta > 1.9999r \).

**Lemma 3.8’.** \( c - a \leq b + (2\Delta + 3)b/(\Delta^3 - \Delta) \).

**Lemma 3.9’.** If \( \Delta \geq \Delta_0 \geq 6 \) then
\[
bc^2(c - a) < 1.9999^{-8} \left( 1 + \frac{2\Delta_0 + 3}{\Delta^3 - \Delta_0} \right) \frac{b^8}{(\Delta^2 - 2)^2}.
\]

**Lemma 3.11’.** \( \frac{\sqrt{bc} + \sqrt{ac}}{\sqrt{bc} - \sqrt{ab}} < \frac{\sqrt{bc} + \sqrt{ac}}{\sqrt{bc} + \sqrt{ab}} < 1 + \frac{1}{\Delta^2 - 2} \).

**Lemma 3.12’.** Consider the linear form
\[
\Lambda = \log(\beta^{2\nu}\chi) - 2m \log(\alpha/\beta),
\]
and put \( \nu = l - m \) with \( m \geq 1 \). Then \( m > (\Delta^2 - 2)/\nu \log \beta \).

We get again
\[
h(\chi) \leq \frac{1}{4} \left( \log(b^2(c - a)^2) + \log \frac{c(\sqrt{a} + \sqrt{b})^2}{b(c - a)} \right) \leq \frac{1}{4} \log(bc^2(c - a)),
\]
hence
\[
h(\chi) < 2 \log \beta.
\]
by Lemma 3.9’. Again
\[ h(\alpha/\beta) = \frac{1}{2} \log \alpha. \]

As the numbers $\beta^{2\nu} \chi$ and $\alpha/\beta$ are multiplicatively independent over $\mathbb{Q}$, we can apply Laurent’s lower bounds \cite{23} to the linear form
\[ \Lambda = \log(\beta^{2\nu} \chi) - 2m \log(\alpha/\beta). \]

With the notation of this paper we have
\[ b_1 = 2m, \quad b_2 = 1, \quad \alpha_1 = \alpha/\beta, \quad \alpha_2 = \beta^{2\nu} \chi. \]

Using the above study and the inequality $\log \alpha_1 < 1/(\Delta^2 - 2)$ established in Lemma 3.6’, one can choose
\[ a_1 \geq 4 \log \alpha + \frac{\rho - 1}{\Delta^2 - 2}. \]

Moreover, the choice
\[ a_2 \geq 2(\nu(\rho + 3) + 8) \log \beta + (\rho - 1) \log \left(1 + \frac{1}{\Delta^2 - 2}\right) \]
is legitimate by Lemma 3.10’.

Now we suppose $\Delta > 60$. We omit the details since the previous study applies almost word for word after the substitution $A \mapsto \Delta^2 - 1$. Laurent’s estimates lead to a contradiction. We conclude that
\[ \Delta \leq 60. \]

Then we can apply Matveev’s estimates to the (expanded) linear form in three logarithms
\[ \Lambda = \log \chi + 2\nu \log \beta - 2m \log(\alpha/\beta) \]
and we get
\[ m < 10^{17}. \]

To end the proof we use the Baker-Davenport lemma and a computer (with a real precision of 200 digits). The verification took less than 1 second. \qed

On noting that, by Lemma \cite{33} \( \Delta^2 - 1 \) and a must be divisible by exactly the same power of 2 when a is even, from Proposition \cite{35} one deduces $\Delta \geq \sqrt{3a + 1}$. This in turn readily implies the first claim in the conclusion of Proposition \cite{3}.

The lower bound on a has been obtained by performing the reduction procedure for $a = 21, 24$. Improved versions are easily available after similar computations for values a either divisible by 8 or odd and not excluded by Corollary \cite{2}.

Similar considerations lead to Proposition \cite{4}. Following is a sketch of the ideas involved in its proof.

Trudgian has combined results from \cite{10}, \cite{4}, and \cite{5} to show in \cite{28} that in any $D(1)$-quintuple whose second smallest element is less than four times the smallest one, the smallest three elements form a regular triple. With the notation fixed in this section, we therefore have
\[ r = 2a - \delta \]
for some positive integer $\delta$ that has to be odd by Lemma \cite{33} above and Theorem 1.2 from \cite{5}, which says that if both $a$ and $b$ are odd then $b > 40a/9$. 

Theorem 1 ensures \( \delta > 1 \). From
\[
b = 4a - 4\delta + \frac{\delta^2 - 1}{a}
\]
we get that \( a \) divides the positive integer \( \delta^2 - 1 \), so that \( \delta \geq \sqrt{a + 1} \). As before we conclude that \( a \) and \( \delta^2 - 1 \) have the same 2-adic valuation when \( a \) is even. Since, on the one hand, \( a = \delta^2 - 1 \) is tantamount to \( b = 4\delta^2 - 4\delta - 3 \) and, on the other hand, routine computations show that the triple \( (a, b, c) = (\delta^2 - 1, 4\delta^2 - 4\delta - 3, 9\delta^2 - 6\delta - 8) \) can not be prolongated to a \( D(1) \)-quintuple, it results \( \delta^2 - 1 \geq 3a \).

Hence,
\[
b \leq 4a - 4\sqrt{3a + 1} + 3.
\]
The lower bound on \( a \) follows from this and the complementary inequality \( b > 130000 \), taking into account that an even value of \( a \) must be divisible by 8.

9. Concluding remarks

In this paper we completed the work of previous authors and proved in Theorem 1 that each triple in the four families has unique extension to a quadruple. It is for the first time in the literature that the extendability of a two-parameter family is unconditionally settled.

The proof illustrates the known empirical fact that while the existence of ‘small’ or ‘big’ solutions can be relatively easily decided, it is much more difficult to treat solutions of ‘medium size’. Our attempt was successful due to use of linear forms in the logarithms of two algebraic integers. One critical aspect of such an approach is the need for sharp bounds for the difference of integer coefficients of the logarithms. In the present study we got such an information in Lemmas 17 and 25. It remains for future works to obtain similar bounds for general triples, not necessarily given parametrically.

As mentioned several times, the triples considered in this article are regular in the sense that \( c = a + b + 2r \). Another interesting direction for future work is to deal with non-regular triples.

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SIMION STOILOW INSTITUTE OF MATHEMATICS OF THE ROMANIAN ACADEMY, RESEARCH UNIT NR. 5, P.O. BOX 1-764, RO-014700 BUCHAREST, ROMANIA
E-mail address: Mihai.Cipu@imar.ro

DEPARTMENT OF MATHEMATICS, COLLEGE OF INDUSTRIAL TECHNOLOGY, NIHON UNIVERSITY, 2-11-1 SHIN-EI, NARASHINO, CHIBA, JAPAN
E-mail address: fujita.yasutsugu@nihon-u.ac.jp

DÉPARTEMENT DE MATHEMATIQUE, UNIVERSITÉ DE STRASBOURG, 67084 STRASBOURG, FRANCE
E-mail address: mignotte@mail.u-strasbg.fr