Iterative Inversion of Deformation Vector Fields with Feedback Control

Abhishek Kumar Dubey*, Alexandros-Stavros Iliopoulos*, Xiaobai Sun*, Fang-Fang Yin†, and Lei Ren†

*Department of Computer Science, Duke University, Durham, NC 27705, USA
†Department of Radiation Oncology, Duke University School of Medicine, Durham, NC 27710, USA

Abstract

Purpose: The inverse of a deformation vector field (DVF) is often needed in deformable registration, 4D image reconstruction, and adaptive radiation therapy. This study aims at improving both accuracy with respect to inverse consistency and efficiency for numerical DVF inversion, by the development of a fixed-point iteration method with feedback control.

Method: We introduce an iterative method with active feedback control for DVF inversion, its analysis and adaptation to patient-specific data. We introduce also a new way of characterizing DVF data. The method improves upon and includes two previous fixed-point iteration methods. At each iteration step, we measure the inconsistency, namely the inverse residual, between the iterative inverse estimate and the input DVF. The residual is modulated by a feedback control mechanism before being incorporated into the next iterate. The feedback control design is based on analysis of error propagation in the iteration process. The control design goal is to suppress estimation error progressively to make the convergence region as large as possible, and make estimate errors vanish faster whenever possible. We demonstrate the new method with a constant single-parameter control mechanism and a varying one. The feedback control mechanism is assessed experimentally with analytical deformations and with numerical DVFs between end-of-expiration (EE) and end-of-inspiration (EI) CT images of 7 patients.

Results: The active feedback control is analytically shown to attain a larger convergence region at faster pace in iterative DVF inversion. With the analytical deformation, the iteration becomes convergent over the entire image domain, and the convergence is sped up compared to the precursor methods, which suffer from slow convergence, or even divergence, when displacement is large. With the patient DVF data, the varying control scheme outperforms the precursor methods in inverse consistency and computational efficiency.

Conclusion: The formal analysis introduced here provides a new way of understanding and designing efficient iterative methods for DVF inversion.

1 Introduction

We consider numerical inversion of a forward deformation vector field (DVF) from one image to another, especially in the case when displacement vectors are large. Very often the inverse DVF is needed along with the forward DVF to map medical images, structures, or doses back and forth throughout the process of 4D image reconstruction and adaptive radiotherapy [8, 11]. There are several ways to get both forward DVF and its inverse, such as by a one-way deformable registration performed twice (with swapped input images), simultaneous and symmetrical registration of input images, or a one-way registration to get the forward DVF followed by an inversion process. The latter option, inversion of the forward DVF, is often preferred in clinical applications for two primary reasons: (i) one-way registration is supported widely by effective and available software, and inversion is empirically proven to be fast; (ii) the reference and target images are not symmetrical in image quality, challenging other approaches, which may be sensitive and error-prone in such cases. [7]

The problem of DVF inversion can be framed as follows. The reference and target images, denoted by \( I_{\text{ref}} \) and \( I_{\text{tgt}} \), respectively, can be related to one another by two non-linear transformations. The forward transformation, \( f: \Omega \to \Omega \), maps the voxels of the reference image \( I_{\text{ref}} \) onto those of the target image \( I_{\text{tgt}} \) via the forward deformation vector field \( F \):

\[
f(x') = x' + F(x'),
\]

where \( F(x') \) is the 3D displacement of the reference-image voxel at \( x' \in \Omega, \) and \( \Omega \subseteq \mathbb{R}^3 \) is the image domain. Conversely, the backward transformation, \( g: \Omega \to \Omega, \)
maps the voxels of $I_{\text{tgt}}$ back to $I_{\text{ref}}$ via the reverse DVF $G$:

$$g(x) = x + G(x),$$

(2)

where $G(x)$ is the 3D displacement of the target-image voxel at $x \in \Omega$. The problem of DVF inversion is to obtain $G$ given $F$. The two transformations are the inverse of each other, i.e.,

$$I_{\text{ref}}(g \circ f(x')) = I_{\text{ref}}(x'),$$

(3a)

$$I_{\text{tgt}}(f \circ g(x)) = I_{\text{tgt}}(x),$$

(3b)

for $x, x' \in \Omega$. Consequently, the forward and backward DVFs satisfy the simultaneous inverse consistency condition:

$$F(x') + G(x' + F(x')) = 0,$$

(4a)

$$G(x) + F(x + G(x)) = 0.$$  

(4b)

Inverse consistency is of great importance to deformable registration and estimation of 4D dose accumulation, among other biomedical applications. [5, 9] Since the work of Christensen and Johnson, [4] or perhaps earlier, the inverse consistency condition has been incorporated in various deformable registration models. The inverse consistency condition is either used as an explicit constraint attached to an optimization model, [10, 5] or employed implicitly and approximately in numerical iterations to relate the forward and backward transformation iterates to each other. [6] Other related studies on simultaneous estimates of the forward and reverse DVFs under the consistency condition can be found in the survey by Sotiras et al. [9] and the references therein.

Numerical DVF inversion by iterative methods has been advanced by studies on convergence analysis and conditions, and on increasing convergence speed. Christensen and Johnson [5] and Chen et al. [3] developed two iteration methods of particular interest in this context.

Chen’s work departed from heuristic design of iterative methods and introduced a fixed-point iteration method for DVF inversion equipped with convergence analysis and condition. Provided with $F$, Chen’s iteration for estimation of the inverse, $G$, proceeds as

$$G_{k+1}(x) = -F(x + G_k(x)).$$

(5)

where $k \geq 0$ is the iteration step. The initial guess $G_0$ is set to zero, which leads to $G_1 = -F$. Negating the forward DVF to get the inverse DVF used to be a widespread misconception. The iteration attempts to improve upon $G_1$ in subsequent iterates, towards meeting the inverse consistency condition (4b). The analysis was demonstrated with the help of a pair of analytical forward and backward non-linear transforms in closed-form expressions, also provided by Chen et al. [3] A sufficient convergence condition for (5) is the contraction condition on $F$:

$$d(F(x'_1), F(x'_2)) \leq \alpha d(x'_1, x'_2),$$

(6)

where $x'_1, x'_2 \in \Omega$ are reference-image voxel coordinates, $d(\cdot, \cdot)$ is a well-defined distance metric, and $\alpha$ is a Lipschitz constant, $0 \leq \alpha < 1$. The convergence behavior of iteration of (5) depends passively on condition (6), which is not necessarily met with large displacements or deformations in clinical data. As we will show shortly, the condition fails to hold everywhere for the pair of analytical DVFs introduced by Chen et al. when displacements become larger.

The iteration by Christensen and Johnson [5] is effective in practice and unique in making use of residual feedback. As noted by the authors, it is also closely related to the heuristic residual method by Thirion for simultaneous deformable image registration. [10]

Influenced by and improving upon their work, we have developed an iterative DVF inversion method with active feedback control, which includes the two precursor methods as particular instances. The active feedback control allows one to make data-specific improvement in convergence and hence in numerical DVF inversion. The feedback design is based on analysis of error propagation through an iteration process, with the objective to suppress iterate errors progressively and as fast as computationally feasible. We introduce two types of feedback control; one is simple, stationary over iterations and uniform over the spatial image domain. The other is non-stationary and spatially variant. We present experimental results of numerical DVF inversion with patient data. We introduce also spectral measures that we use for characterizing and differentiating DVF data, as well as for estimating numerical hardness or conditions for numerical DVF inversion.

The rest of the paper is organized as follows. In Section 2, we introduce the basic concept that underlies active feedback control, a simple mechanism with a single control parameter, and present a formal convergence analysis. In Section 3, we describe the patient datasets and the experimental set-up used in this study. We present experimental results in Section 4. We give concluding remarks in Section 5.

2 DVF inversion method

2.1 Error relationship and feedback control

Feedback control for DVF inversion is based on the relationship between computationally available residuals and unknown errors in inverse estimates, and on analysis of error propagation in an iterative process. In simultaneous deformation registration, two types of error must be reduced: registration error and inverse consistency error. In iterative DVF inversion, where a forward DVF $F$ is obtained by one-way registration, we focus on reducing the inverse consistency error aggressively and robustly.
At each iteration step \( k \), starting with an initial guess, we have an iterate, \( G_k \), as the current estimate of the inverse DVF \( G \). The unknown error in the estimated inverse, at every voxel location \( x \in \Omega \), is
\[
E_k(x) = G_k(x) - G(x).
\] (7)
In order to improve upon \( G_k \), i.e., to reduce the inversion error \( E_k \), we introduce active feedback control mechanisms to explore and utilize the inverse consistency (IC) residual,
\[
R_k(x) = G_k(x) + F(x + G_k(x)) - (I + J_F)((x + G_k(x)) - x),
\] (8)
which is computationally obtainable at each iteration step. This IC residual is a measure of inconsistency, with respect to the IC condition \((4)\). The IC residual is related to the unknown error in the estimate \( E_k \). More specifically, by Taylor’s theorem, the IC residual and the estimate error are related to each other by
\[
R_k(x) = G_k(x) - (I + J_F)((x + G_k(x)) - x),
\] (9)
where \( J_F \) is the Jacobian of the forward deformation transformation \( f \) evaluated at \( \xi_k \), which lies between \( x + G_k(x) \) and \( x + G_k(x) \). The Jacobian of \( f \) is related to that of \( F \) by \( J_F = I + J_F \), as in \((9)\). The other IC residual, corresponding to the IC condition \((4a)\), is
\[
R_F(x') = F(x') + \hat{G}(x' + F(x')) - G_k(x),
\] (10)
where \( \hat{G} \) is an inverse estimate. We will remark in Sections 3 and 5 how to utilize this residual for joint or separate assessments of the forward and backward transformations.

With an active feedback control mechanism, the iteration process at step \( k \) can be described by
\[
G_{k+1}(x) = G_k(x) - B_k(x)R_k(x),
\] (11)
where \( B_k(x) \) is a \( 3 \times 3 \) feedback control matrix, associated with voxel \( x \) in the (target) image domain \( \Omega \). With feedback control, we modulate the IC residual \( R_k(x) \) to a refinement \( B_k(x)R_k(x) \) to \( G_k \). When \( B_k(x) = B \) for some constant matrix \( B \), the feedback is spatially uniform (no variation among spatial locations) and temporally stationary (no change among steps in the iteration process).

In the subsequent sections, we introduce single-parameter feedback control mechanisms, case studies with an analytical DVF pair, and a general analysis to guide data-specific parameter determination for numerical DVFs.

2.2 Single-parameter feedback control

We introduce the feedback control mechanism in its simplest form, with \( B(x) = (1 - \mu)I \):
\[
G_{k+1}(x) = G_k(x) - (1 - \mu)R_k(x),
\] (12)
where \( \mu \) is a scalar parameter, constant across the spatial domain and throughout the iteration process. This simple control system is interesting for the following reasons. On the one hand, at \( \mu = 0 \) or \( \mu = 0.5 \), the iteration of \((12)\) reduces to iteration \((5)\) by Chen et al. \([3]\) or the procedure by Christensen and Johnson, \([5]\) respectively. On the other hand, one may determine the control parameter to improve convergence behavior, i.e., to increase or maximize the convergence area in the image domain \( \Omega \) and to increase the convergence speed when feasible. Furthermore, we will show how the iteration behavior is determined according to the spectrum of the deformation Jacobian.

2.2.1 Case study with constant control parameter

We first study the feedback system of \((12)\) with the pair analytical 2D DVFs introduced by Chen et al., \([3]\]
\[
F(x') = \left(1 + b\cos(m\theta(x'))\right)x', \quad (13a)
\]
\[
G(x) = b\cos(m\theta(x))x, \quad (13b)
\]
where the spatial domain is \( \Omega = [-30, 30]^2 \), \( \theta(x) \) is the angular coordinate of \( x \) in the polar representation \( x^T = ||x||((\cos(\theta(x)), \sin(\theta(x))) \), and \( b \) and \( m \) are two scalar parameters, \( b \in (0, 1) \) and \( m \in \mathbb{N} \). It is straightforward to verify that the analytical DVFs meet the consistency condition \((4)\); indeed, the corresponding transformations are inverse to one another. \([3]\) The analytical DVFs are visualized in Fig. 1 via a source image and two target images. The latter are obtained by transforming the reference image with \((13a)\), with two different sets of DVF parameter values.

When \( b = 0 \), there is no deformation. When \( m = 0 \) and \( b > 0 \), the displacement (forward or backward) at \( x \) is proportional to \( x \). With \( mb \neq 0 \), the deformation at \( x \) remains in the radial direction, but its magnitude varies with \( b \) and \( m\theta(x) \). Displacement is proportional to \( b/(1 - b) \) at and near angles where \( \cos(m\theta(x)) = -1 \).
Figure 2: Visualization of the convergence and divergence regions for the iteration (12) with $\mu = 0$ via recovered reference images, which are the back-transformations of the target image of Fig. 1c by the inverse estimates $G_k$ at iteration step $k = 1, 5, 10$. The analytical DVFs parameters are $m = 8$ and $b = 0.6$. The iteration improves reference image recovery in the convergent regions, but clearly fails to recover the reference image Fig. 1a in the divergent regions along and near the radial lines $\theta(x) = (2n + 1)\pi/8$, where $n \in \{0, \ldots, 7\}$.

We take a closer look at how errors propagate through the iteration process. The inversion errors in two successive iterates are related by the error propagation equation:

\[ E_{k+1}(x) = p(x; \mu) E_k(x), \quad (14a) \]

\[ p(x; \mu) = 1 - \frac{1 - \mu}{1 + b \cos(m\theta(x))}. \quad (14b) \]

When the magnitude of the error propagation factor $p(x; \mu)$ is greater than 1, the error $E_k(x)$ is magnified instead of suppressed.

Consider in particular the two precursor methods mentioned earlier. The iteration (12) with $\mu = 0$ is the iteration of (5). When $b < 0.5$, then $|p(x; \mu)| < 1$ at every $\theta(x)$, and the iteration is expected to converge everywhere over $\Omega$; but as $b$ approaches 0.5 from below, the iteration becomes slower. When $b > 0.5$, then $|p(x; \mu)| > 1$ at and near angles where $\cos(m\theta(x)) = -1$, for which the iteration diverges. In order to illustrate and assess the inverse estimates $G_k$, we use them to deform relevant target image in Fig. 1 by $G_k$ and obtain the corresponding recovered reference image, and show the difference from the original reference image. Fig. 2 shows that the iteration improves reference image recovery over the convergent regions, but fails to recover faithfully over the divergent regions along and near radial lines where $\theta(x) = (2n + 1)\pi/m$, for $n \in \{0, \ldots, m - 1\}$.

A similar analysis applies to the iteration (12) with $\mu = 0.5$, i.e., the inversion procedure by Christensen and Johnson [5]. The iteration converges everywhere in $\Omega$ when $b \leq 0.75$, an improvement over the iteration with $\mu = 0$. Yet, it suffers from divergence when $b \in (0.75, 1)$, in the same divergent regions as with $\mu = 0$. We revisit and illustrate this claim later, in Section 2.2.3.

Our objective with active control feedback is to suppress the error (14) at every spatial location, for any $b \in (0, 1)$, and make the error vanish at a faster pace.

This objective can always be achieved for the analytical DVFs in (13); we claim, specifically, that for any $b \in (0, 1)$ there exists some $\mu \in [0, 1)$ which satisfies

\[ 1 > \mu > 2b - 1, \quad (15) \]

such that $|p(x; \mu)| < 1$, and hence the iteration (12) converges everywhere over $\Omega$, with arbitrary initial guess. Condition (15) is both necessary and sufficient. The iteration with $\mu = 0$ fails to meet the condition when $b \geq 0.5$; see the recovered reference image in Fig. 2. The same holds for the iteration with $\mu = 0.5$ when $b \geq 0.75$.

Among the feasible values for parameter $\mu$ by (15), some suppress the error more aggressively than others, meaning that they result in smaller values of $p(x; \mu)$. The optimal constant parameter value for the analytical DVF pair (13) is

\[ \mu_* = b^2. \quad (16) \]

With $\mu = mu_* = b^2$, the error is suppressed at each step by $|p(x; \mu_*)| \leq b$, for any $b \in (0, 1)$. As $b$ gets closer to 1, convergence becomes slower. The proofs of (14), (15), and (16) are included in the convergence analysis in the next section for a broader class of cases.

2.2.2 Convergence study for numerical DVFs

We examine the convergence behavior of iteration (12) with constant feedback control in the practical situation where the forward DVF is provided numerically on a 3D voxel grid with finite spatial resolution, rather than analytically defined in closed form.

By (9) and (12), the propagation of inverse estimate errors (7) throughout the iteration is described by

\[ E_{k+1}(x) = P_k(x; \mu) E_k(x), \quad (17a) \]

\[ P_k(x; \mu) = I - (1 - \mu) J_k(\xi_k), \quad (17b) \]

where $P_k(x; \mu)$ is the error propagation matrix for the local iteration associated with voxel $x$, and $\xi_k$ lies between $x + G(x)$ and $x + G_k(x)$. The local error propagation matrix depends on the parameter $\mu$ and varies with each iteration step $k$ due to $\xi_k$.

Any particular value of $\mu$ partitions the image domain $\Omega$ into a contraction region,

\[ \Omega_c(\mu) = \{x \mid \rho(P(x; \mu)) < 1\}, \quad (18) \]

and its complement in $\Omega$. Here, $\rho(P)$ denotes the spectral radius of $P$. A sufficient condition for $E_k(x)$ to approach 0 (i.e., for $G_k(x)$ to converge to $G(x)$) is

\[ \rho(P_k(x; \mu)) \leq \alpha(x), \quad \forall k > 0, \quad (19) \]

for some $\alpha(x) \in [0, 1)$. This condition implies that all $\xi_k$ in (17b) lie in the contraction region.

Our objective with determining the control parameter is to make the contraction region $\Omega_c$ as large as possible, and to make the error suppression factor $\rho(P)$
as small as possible. Let the eigenvalue of $J_f(x)$ be denoted by $\lambda_j(x)$, $j = 1, 2, 3$. Assume the positive spectrum condition that the eigenvalues are real, positive, and bounded from above by $B_{\text{max}}$ and from below by $B_{\text{min}} > 0$ over $\Omega$. Denote the extreme eigenvalues and their ratio by

$$\lambda_{\text{max}}^\Omega = \max_{x \in \Omega} \max_{j=1,2,3} \{\lambda_j(x)\} \leq B_{\text{max}}, \quad \lambda_{\text{min}}^\Omega = \min_{x \in \Omega} \min_{j=1,2,3} \{\lambda_j(x)\} \geq B_{\text{min}} > 0, \quad \kappa^\Omega = \frac{\lambda_{\text{max}}^\Omega}{\lambda_{\text{min}}^\Omega}. \tag{20a}$$

The condition for convergence over the entire image domain is

$$0 < 1 - \mu < \frac{2}{\lambda_{\text{max}}^\Omega}. \tag{21}$$

Under the positive spectrum assumption, there exist values of $\mu$ such that the contraction region covers the entire image domain, i.e., $\Omega_c(\mu) = \Omega$. In fact, the minimal upper bound on the spectral radius of the error propagation matrix is achieved when

$$\mu_* = 1 - \frac{2}{\lambda_{\text{min}}^\Omega + \lambda_{\text{max}}^\Omega}, \tag{22a}$$

$$\rho(\mathbf{P}(x; \mu_*)) \leq \frac{\kappa^\Omega - 1}{\kappa^\Omega + 1}. \tag{22b}$$

Convergence becomes slower as $\kappa^\Omega$ increases. For the analytical DVF of (13), the positive spectrum condition is met for any $b \in (0, 1)$. In more detail, $\lambda_{\text{max}}^\Omega = 1/(1 - b)$, $\lambda_{\text{min}}^\Omega = 1(1 + b)$, and $\kappa^\Omega = (1 + b)/(1 - b)$. With $\mu = 0$, the necessary condition (21) fails when $b > 0.5$. With $\mu = \mu_* = b^2$, the iteration converges over the entire domain with spatially variant convergence rate; precisely, we have $\rho(\mathbf{P}(x; \mu_*)) \leq b < 1$ at any $x \in \Omega$.

The positive spectrum assumption and subsequent analysis are not necessarily applicable to other DVFs, especially in the case of numerically computed DVFs. For example, in the case of the DVFs obtained from patient CT images that we use in Sections 3 and 4, many eigenvalues of the Jacobian are complex, which appear in conjugate pairs, and a small portion of the eigenvalues lie in the left half of the complex plane. As a consequence, there may not exist values of $\mu$ such that the corresponding contraction region covers the entire image domain. For any fixed value $\mu \neq 1$, the error contraction region in the image domain may be described as

$$\Omega_c(\mu) = \{x \mid 2 \text{Re}(\lambda_j(x)) > (1-\mu)|\lambda_j(x)|^2, \ x \in \Omega\}. \tag{23}$$

The condition

$$2 \text{Re}(\lambda_j(x)) > (1-\mu)|\lambda_j(x)|^2, \ j = 1, 2, 3, \ x \in \Omega \quad \tag{24}$$

confines the eigenvalues of the Jacobian over the contraction region to either the right or left half of the complex plane, depending on the sign of $1-\mu$. Consequently, if $J_f(x)$ has eigenvalues on both sides of the imaginary axis, there is no single feasible value of $\mu$ to make the iteration converge everywhere in $\Omega$. The proof of (23) starts from the contraction condition (19); we omit tedious detail.

Condition (24) has an important implication on the convergence property of the iteration (12) with a constant parameter $\mu$. Consider the case $1 - \mu > 0$, i.e., $1 - \mu \in (0, 1)$ (the other case can be dealt with symmetrically). The eigenvalues over the error contraction region $\Omega_c(\mu)$ of (23) must be located in the right half of the complex plane. They are further limited to within a disc-shaped area close to and symmetric about the real axis. We therefore refer to condition (24) as the positive disc spectrum condition, an extension of condition (21) to the case with complex eigenvalues of the deformation Jacobian. Consider any DVF, $\mathbf{F}$. The error contraction region $\Omega_c(\mu)$ in the image domain is associated with the interaction between the $\mu$-dependent disc and the spectrum of $J_f$ in the complex plane. When $\mu$ increases from 0 toward 1, on the one hand, the disc area becomes larger, which may increase the error contraction region $\Omega_c(\mu)$. On the other hand, the spectral radius of $\mathbf{P}(x; \mu)$ becomes closer to 1 and hence makes the convergence slower over $\Omega_c(\mu)$. This suggests that there is a trade-off in determining the parameter value, between larger convergence region and faster convergence speed.

There are several potential approaches to alleviating or circumventing this problem. We discuss this in the following section on feedback control with adaptive parameter. We will demonstrate in Section 4 how convergence is improved by a simple control with the parameter value changing periodically between two values, such that contraction region is larger with one and convergence speed is greater with the other.

### 2.2.3 Estimation of adaptive control parameter

The feedback control with a constant parameter is limited by the spectral ratio $\lambda_{\text{max}}^\Omega$ when the ray spectrum condition (21) is met, and further limited by the disc spectrum condition when the Jacobian has complex eigenvalues. Such limitations may be lifted substantially by letting the control parameter change with $x$, in adaptation to the eigenvalues of the Jacobian in a neighborhood of $x$; that is, by using

$$\mathbf{B}_k(x) = (1 - \mu_k(x))\mathbf{I}. \tag{25}$$

The neighborhood should be large enough to accommodate the expected extent of displacement. The control parameter $\mu(x)$ for the iteration associated with $x$ is determined over a local neighborhood $\mathcal{N}(x)$ centered at $x$ instead of the entire image domain $\Omega$. The previous convergence study over the entire domain $\Omega$ can be applied to the local neighborhood $\mathcal{N}(x)$. The disc spectrum condition can then be relaxed to apply to each individual neighborhood only. We may describe the variable
plified under certain conditions. If the eigenvalues of the matrix \(\mathbf{A}\) are nonsingular at any \(x\) point, the minimization at every voxel location \(x\) can be simplified. Let the spatial resolution, one may make a simplistic estimate \(\mu(x)\), reaching to the original reference image. (Right) 1 iteration step with spatially variant control \(\mu(x)\), rendering faithfully recovered reference image (sans minor artifacts due to interpolation).

\[
\mu(x) = \arg\min_{\mu} \max_{x \in \Omega(x)} \{ \rho(\mathbf{P}(x'; \mu)) < 1 \}. \quad (26)
\]

The determination of \(\mu(x)\) by (26) is computationally expensive. Assuming the Jacobian is continuous and nonsingular at any \(x \in \Omega\) when not subject to finite spatial resolution, one may make a simplistic estimate of \(\mu(x)\) by

\[
\mu_*(x) = \arg\min_{\mu} \{ \rho(\mathbf{P}(x; \mu)) < 1 \}. \quad (27)
\]

The minimization at every voxel location \(x\) can be simplified under certain conditions. If the eigenvalues of \(J_F(x)\) are all real and positive, then

\[
\mu_*(x) = 1 - \frac{2}{\lambda_{\max}(x) + \lambda_{\min}(x)}. \quad (28)
\]

For iterative inversion of the analytical DVFs of (13), we first transform a target image, corresponding to a deformation of the reference image in Fig. 1a using the forward DVF with \(m = 8\) and \(b = 0.8\); then we use the inverse estimates to recover the reference image. We present in Fig. 3 the comparison in convergence behavior, via the recovered reference images, among three settings. The recovered image by \(\mu = 0.5\) at iteration step \(k = 10\) clearly shows \(m = 8\) divergent regions. The recovered image by the optimal uniform value \(\mu_* = b^2\) at iteration step \(k = 10\) shows the iteration approaching closer to the original reference image. The reference image is recovered in only one step by the spatially variant control \(\mu_*(x)\) as in (28).

Parameter estimation becomes more complicated when \(J_F(x)\) has complex eigenvalues. Denote by \(\lambda_{\max}(x)\) and \(\lambda_{\min}(x)\) the extreme eigenvalues of \(J_F(x)\), with respect to their magnitude,

\[
\lambda_{\max}(x) = \max_{j=1,2,3} \{|\lambda_j(x)|\}, \quad (29a)
\]

\[
\lambda_{\min}(x) = \min_{j=1,2,3} \{|\lambda_j(x)|\}, \quad (29b)
\]

\[
\kappa(x) = \frac{\lambda_{\max}(x)}{\lambda_{\min}(x)}. \quad (29c)
\]

If

\[
\frac{\text{Re}(\lambda_j(x))}{|\lambda_j(x)|} > \frac{1}{\kappa(x)}, \quad j = 1, 2, 3, \quad (30)
\]

then

\[
\mu_+(x) = 1 - 2 \frac{\text{Re}(\lambda_{\max}(x)) - \text{Re}(\lambda_{\min}(x))}{|\lambda_{\max}(x)|^2 - |\lambda_{\min}(x)|^2}. \quad (31)
\]

The wedge spectrum condition (30) implies that \(\rho(\mathbf{P}(x; \mu_+)) < 1\), i.e., \(x\) is in the contraction region \(\Omega_+(\mu_+\)). It can be verified that \(\rho(\mathbf{P}(x; \mu))\) is minimal among all \(\rho(\mathbf{P}(x; \mu))\) under certain neighborhood. This detail is omitted from the present paper.

2.3 Alternative iteration formula

The iteration (11) can be described alternatively as the following procedure with two sub-steps per iteration,

\[
\mathbf{G}_{k+1}(x) = -\mathbf{F}(x + \mathbf{G}_k(x)), \quad (32a)
\]

\[
\mathbf{G}_{k+1}(x) = \mathbf{G}_k(x) - \mathbf{B}_k(x) \left( \mathbf{G}_k - \mathbf{G}_{k+\frac{1}{2}}(x) \right). \quad (32b)
\]

The first sub-step \((k+1/2)\) is the iteration without active feedback control, or equivalently, with single-parameter control at \(\mu = 0\). The second sub-step makes a modification of a combination of the estimates at step \(k\) and sub-step \((k+1/2)\) by \(\mathbf{B}_k(x)\). The alternative computation procedure is mathematically equivalent with respect to convergence analysis, to iteration (11), although the residual is used implicitly.

3 Experiments

3.1 Datasets

The data used in the experiments comprise CT images from 7 patients, chosen from a collection of 20 patients that is available through the Deformable Image Registration Laboratory (DIR-Lab) website. [2, 1] The DIR-Lab collection contains two subsets of patient image data—4DCT data and COPD data, with 10 patients in each subset. The 4DCT images were acquired as part of radiotherapy planning for the treatment of thoracic malignancies at the University of Texas MD Anderson Cancer Center. [2] The COPD data contain EE and EI breathhold CT image pairs from the COPDGene study archive of the National Heart, Lung, and Blood Institute. [1] In-slice spatial resolution is (0.96 mm)² for the 4DCT images, and between (0.586 mm)² and (0.647 mm)² for the
COPDGene images. The CT slice thickness is 2.5 mm for both data sets. The DIR-Lab collection includes also the statistics (in terms of mean and standard deviation) of displacements between a set of primary image features at the EE and EI phases, for each patient. The reported displacements are substantially smaller in the 4DCT images than those in the COPDGene archive images. We chose data for 7 out of 20 patients based on the quantitative measures we shall describe shortly in Section 3.2, two 4DCT patients and five COPDGene patients. We refer to the data for each patient by the corresponding DIR-Lab archive label.

Patient-specific forward DVF was generated by deformable image registration with the VelocityAI software (Velocity Medical Solutions, Atlanta, GA, USA), where the reference (primary) image is at the EI phase and the target (secondary) image is at the EE phase. The DVF for patient COPD4 is displayed in Fig. 4, with axial, coronal, and sagittal slices, against a magenta-green overlay of the reference and target images.

### 3.2 DVF characterization

We use aggregated measures of 6 scalar fields associated with a DVF to quantitatively characterize and differentiate among the experiment data based on their respective DVFs. These are collected in Table 1. For any scalar field \( \gamma(x) \geq \gamma_{\text{min}} \) over \( \Omega \), we obtain the conventional percentile (min-max) measure \( \gamma[\beta\%] \), taking into account uncertainties in region delineation, noise, outliers, and artifacts. The measure may be formally described via the level-set \( \mathcal{L} \) and level-set density \( p \).

\[
\mathcal{L}(\tau) = \{ x \in \Omega \mid \gamma(x) = \tau \}, \quad p(\tau) = \frac{|\mathcal{L}(\tau)|}{|\Omega|},
\]

\[
\gamma[\beta\%] = \inf \left\{ \tau \left| \int_{\gamma_{\text{min}}}^{\tau} p(\tau') \, d\tau' > \beta\% \right\}.
\]

Computationally, \( \gamma[\beta\%] \) can be approximated accurately via the use of discrete histograms. The first three scalar fields are the displacements along the LR, AP, and SI axes. They give rise to what we refer to as the order-zero measures at any particular percentile by (33). The next three measures are order-one, drawn from three scalar fields associated with the Jacobian \( J_f \): the maximum-magnitude eigenvalue \( \lambda_{\text{max}}(x) \), the determinant \( |J_f(x)| \), and the spectral ratio between the maximum and minimum eigenvalues \( \kappa(x) \).

We report in Table 1 the field measures at 50\% (median), 90\%, and 95\%. By the percentile variation in each of the measures, none of the patient DVFs are rigid. Variation in the Jacobian determinant indicates the extent of local contraction \( (|J_f| < 1) \) or expansion \( (|J_f| > 1) \) with respect to the physical transformation. For most of the 7 patients, we observe contracting motion over more than half of the domain, which is expected as the forward DVFs express EI-to-EE deformation. The dominating contraction is also indicated by the reported median values (\( \beta = 50\% \)) of the Jacobian determinant in Table 1. By all measures, the COPD4 patient data exhibit larger displacements and varying deformation than the 4DCT patients; patient COPD4, in particular, is associated with the largest quantities.

Besides the aggregated field measures, we use heat maps to show and observe the spatial variation in each of the scalar fields, as illustrated in Fig. 5 for patient COPD4 DVF. The heat map Fig. 4d shows a few salient spots where the determinant is negative, which are problematic to any iteration method based on the assumption that the Jacobian determinant is continuously positive and thereby nonsingular. The distribution of physically contracting and expanding deformations of the patient volume is also seen in the same figure. The lung volume is mostly contracting, although regional expansions can also be observed. This is typical for the other patients used in this study.

In Table 1, in addition to the order-zero and order-one field measures, we describe in the last three columns the impact of these measures on three particular feedback control schemes with single parameter \( \mu(x) \), which are described and discussed in Section 2.2. The impact is quantitatively measured by the spectral radius of the corresponding error propagation matrix (17b) at \( k = 0 \) and \( k = 1 \); the smaller the spectral radius, the faster the iteration converges. The first and second schemes use constant control parameter values \( \mu = 0 \) and \( \mu = 0.5 \) respectively. The third scheme uses the adaptive estimate \( \mu_\ast \) in (31); the condition (30) is empirically met by all 7 patient DVFs for at least 95\% of the voxels in \( \Omega \). Although the third scheme has the potential to converge faster than the first and second schemes, it is limited jointly by the single-parameter control frame and by \( \kappa(x) \), the spectral ratio of the Jacobian \( J_f(x) \). Particularly, the COPD4 data deem the first scheme to fail in several regions in the domain, and challenge the second and third schemes with respect to convergence speed.
Table 1: Characteristic measures of the 7 patient DVF data at 3 particular percentiles ($\beta\% = 50\%, 90\%, 95\%$), defined as per (33). The measures include the amplitude of $F(x)$ along the LR, AP, and SI axes, and the spectral information of the Jacobian $J_F(x)$ in terms of $|\lambda_{\text{max}}(x)|$, the spectral ratio $\kappa(x)$, and the determinant $|J_F(x)|$. The last three columns relate the characteristic measures to the spectral radii of the propagation matrix $P(x, \mu)$ with $\mu(x) = 0$, $\mu(x) = 0.5$ and $\mu(x) = \mu_*(x)$ by (31).

| Case | $\beta$ (%) | $F(x)$ (mm) | $J_F(x)$ $|\lambda_{\text{max}}|$ $|J_F|$ $\rho(P(x; \mu))$ $\mu = 0$ $\mu = 0.5$ $\mu_*(x)$ |
|------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 4DCT7 | 50 | 0.7 | 0.9 | 1.3 | 1.1 | 1.1 | 1.0 | 0.1 | 0.5 | 0.1 |
|      | 90 | 2.2 | 3.1 | 7.8 | 1.2 | 1.4 | 1.2 | 0.3 | 0.6 | 0.2 |
|      | 95 | 3.0 | 4.1 | 10.6 | 1.3 | 1.5 | 1.3 | 0.3 | 0.6 | 0.3 |
| 4DCT8 | 50 | 0.7 | 0.7 | 0.8 | 1.0 | 1.1 | 1.0 | 0.1 | 0.5 | 0.1 |
|      | 90 | 2.2 | 3.0 | 4.1 | 1.1 | 1.3 | 1.1 | 0.2 | 0.6 | 0.2 |
|      | 95 | 2.9 | 4.3 | 6.0 | 1.2 | 1.4 | 1.2 | 0.3 | 0.6 | 0.2 |
| COPD1 | 50 | 4.3 | 16.2 | 8.2 | 1.1 | 1.4 | 0.9 | 0.3 | 0.6 | 0.3 |
|      | 90 | 12.2 | 49.2 | 26.0 | 1.5 | 2.8 | 1.3 | 0.7 | 0.8 | 0.6 |
|      | 95 | 15.7 | 53.3 | 30.3 | 1.6 | 4.0 | 1.5 | 0.8 | 0.8 | 0.7 |
| COPD4 | 50 | 5.6 | 11.2 | 11.0 | 1.2 | 1.6 | 1.0 | 0.4 | 0.6 | 0.3 |
|      | 90 | 16.3 | 38.6 | 34.1 | 1.8 | 3.5 | 1.7 | 0.9 | 0.8 | 0.6 |
|      | 95 | 20.2 | 43.0 | 41.5 | 2.1 | 5.2 | 2.1 | 1.2 | 0.9 | 0.7 |
| COPD5 | 50 | 3.4 | 11.7 | 9.5 | 1.2 | 1.5 | 1.0 | 0.3 | 0.6 | 0.2 |
|      | 90 | 11.0 | 44.7 | 30.9 | 1.6 | 3.1 | 1.6 | 0.8 | 0.8 | 0.6 |
|      | 95 | 14.5 | 52.0 | 36.8 | 1.9 | 4.7 | 1.8 | 1.0 | 0.9 | 0.7 |
| COPD6 | 50 | 4.0 | 17.9 | 8.2 | 1.2 | 1.5 | 0.9 | 0.3 | 0.6 | 0.2 |
|      | 90 | 12.0 | 38.8 | 20.9 | 1.5 | 2.6 | 1.5 | 0.7 | 0.8 | 0.5 |
|      | 95 | 14.9 | 44.0 | 26.9 | 1.7 | 3.7 | 1.7 | 0.8 | 0.8 | 0.7 |
| COPD8 | 50 | 2.7 | 8.8 | 6.5 | 1.1 | 1.4 | 0.9 | 0.3 | 0.6 | 0.2 |
|      | 90 | 9.6 | 24.8 | 24.0 | 1.6 | 3.1 | 1.5 | 0.9 | 0.8 | 0.6 |
|      | 95 | 12.2 | 29.1 | 30.1 | 2.1 | 4.5 | 2.0 | 1.2 | 0.9 | 0.7 |
the non-contracting area ratio, as the valid image domain for evaluating the convergence.

patient DVF. The iteration by step $k$ does not exhibit the tendency to converge. The smaller the ratio $\omega_r(\mu, k)$ is, the better. At termination, we calculate also the IC residual $R_F$ of \eqref{eq:residual}, which can serve as a closer estimate for the inverse error in the final iterate.

To facilitate examination of spatial variation in the IC residuals, we use quiver-plot displays to display the residual vector fields and heat-maps to display the corresponding residual magnitudes,

$$R(x) = \sqrt{R_{LR}^2(x) + R_{AP}^2(x) + R_{sl}^2(x)}.$$  

Each 3D field is displayed in axial, coronal and sagittal slices. Each residual map is displayed with the underlying anatomical background by the target or reference CT image for IC residual $R_G$ or $R_F$, respectively.

3.3 Experimental set-up and assessment

We evaluate and make comparisons with respect to convergence region and speed by the iteration of \eqref{eq:iteration} for DVF inversion, using 10 equispaced, constant values $\mu = 0 : 0.1 : 0.9$, and alternating $\mu = \mu_{odd}$ among odd and even steps ($\mu_{odd} = 0.5$ and $\mu_{even} = 0.1$). The initial guess is set to zero for all. We quantify measures on convergence behavior for each control scheme and every patient DVF.

For each DVF $F$, provided on a 3D volume grid over $\Gamma$, we define

$$\Omega_1 = \{x \mid x + F(x) \not\in \Gamma \ \& \ \ x \in \Gamma\},$$  

$$\Omega_2 = \Gamma - f(\Gamma),$$  

$$\Omega = \Gamma - (\Omega_1 \cup \Omega_2)$$

as the valid image domain for evaluating the convergence behavior of an iterative DVF inversion process. We use two ratios to quantify the convergence region. One is the non-contracting area ratio,

$$\omega_\lambda(\mu) = \frac{|\Omega - \Omega_\zeta|}{|\Omega|},$$

where $\Omega_\zeta$ is the error contraction region \eqref{eq:contraction_region}, i.e., $\Omega - \Omega_\zeta$ is the complement area over which the disc spectrum condition \eqref{eq:disc_spectrum} fails. The iteration is expected to converge over the entire domain $\Omega$ if $\omega_\lambda(\mu) = 0$. Otherwise, the smaller the ratio is, the better. Table 2 shows the non-contracting area ratios for each patient DVF and each constant $\mu$ value used in the experiments.

We record and report the IC residual $R_k$ of \eqref{eq:IC_residual}, which is calculated and used for feedback at each iteration step. At the termination step $k \geq 1$, we find the ratio

$$\omega_r(\mu, k) = \frac{|\Omega - \Omega_r(\mu, k)|}{|\Omega|},$$

where $\Omega_r(\mu, k) = \{x \mid R_k(x) > R_1(x), \ x \in \Omega\}$. Over the region $\Omega_r(\mu, k)$, the iteration by step $k$ does not exhibit the tendency to converge. The smaller the ratio $\omega_r(\mu, k)$ is, the better. At termination, we calculate also the IC residual $R_F$ of \eqref{eq:residual}, which can serve as a closer estimate for the inverse error in the final iterate.

4 Results

Tables 3–5 show a summary of the results for inversion of the patient DVFs using the iteration method \eqref{eq:iteration} with equispaced, constant $\mu = 0 : 0.1 : 0.9$. The results confirm the implications of \eqref{eq:disc_spectrum}, introduced and discussed in Section 2.2.2. Lower values of $\mu$ generally yield a higher convergence rate, as indicated by the 50-th percentiles of the IC residuals. On the other hand, low values of $\mu$ suffer from larger divergence regions, as post-evaluated in Table 5 and attested to by the increased residuals at the 90-th and 95-th percentiles. The $\mu$ value that yields the best convergence behavior is different for each patient. Patient COPD4 constitutes the most challenging case for iterative DVF inversion, as shown by the results and predicted by the deformation spectral measures in Table 1.

We take a closer look at the results for patient COPD4. Fig. 6 shows the spectral radius of the spatially variant error propagation matrix for the iteration with $\mu = 0$ and $\mu = 0.5$ (the precursor methods), and with $\mu = 0.4$, which yielded the best results among the tested $\mu$ values, as per Tables 3–5. The effect of the disc spectrum condition \eqref{eq:disc_spectrum} on the deformation Jacobian specturm is clearly manifested. The non-contraction regions (marked in red in Fig. 6) that are common for all $\mu$ values are related to regions where the Jacobian determinant is negative or very close to zero.
Table 2: Non-contracting area ratios $\omega_\lambda(\mu)$ with the iteration (12) for each patient. Comparison for different feedback control parameter values; $\mu_\star(x)$ refers to (31).

| Patient | Non-contracting area ratio: $\omega_\lambda(\mu)$ (%) |
|---------|-----------------------------------------------------|
|         | $\mu = 0$ | $\mu = 0.1$ | $\mu = 0.2$ | $\mu = 0.3$ | $\mu = 0.4$ | $\mu = 0.5$ | $\mu = 0.6$ | $\mu = 0.7$ | $\mu = 0.8$ | $\mu = 0.9$ | $\mu = \mu_\star(x)$ |
| 4DCT7   | 0.00       | 0.00         | 0.00         | 0.00         | 0.00         | 0.00         | 0.00         | 0.00         | 0.00         | 0.00         | 0.00         |
| 4DCT8   | 0.02       | 0.01         | 0.00         | 0.00         | 0.00         | 0.00         | 0.00         | 0.00         | 0.00         | 0.00         | 0.00         |
| COPD1   | 1.84       | 1.20         | 0.85         | 0.73         | 0.73         | 0.73         | 0.73         | 0.73         | 0.73         | 0.73         | 0.73         |
| COPD4   | 7.52       | 5.16         | 3.39         | 1.89         | 1.60         | 1.59         | 1.59         | 1.59         | 1.59         | 1.59         | 1.59         |
| COPD5   | 4.01       | 2.31         | 1.42         | 1.21         | 1.19         | 1.19         | 1.19         | 1.19         | 1.19         | 1.19         | 1.19         |
| COPD6   | 1.96       | 1.22         | 0.97         | 0.89         | 0.89         | 0.89         | 0.88         | 0.88         | 0.88         | 0.88         | 0.88         |
| COPD8   | 6.43       | 4.99         | 3.34         | 2.20         | 1.66         | 1.63         | 1.63         | 1.63         | 1.63         | 1.63         | 1.63         |

Table 3: IC residual amplitudes $R_G$ for the inverse DVF estimates obtained by the iteration (12) for each patient. Percentiles at $\beta = 50\%$, 90\%, 95\% are reported and compared for different feedback control parameter values. All reported results correspond to 15 iteration steps.

| Patient | $\beta$ (%) | IC residual: $R_G[\beta\%]$ (mm) at $k = 15$ |
|---------|-------------|---------------------------------------------|
|         | $\mu = 0$ | $\mu = 0.1$ | $\mu = 0.2$ | $\mu = 0.3$ | $\mu = 0.4$ | $\mu = 0.5$ | $\mu = 0.6$ | $\mu = 0.7$ | $\mu = 0.8$ | $\mu = 0.9$ |
| 4DCT7   | 50         | 2.5e-6      | 2.9e-6      | 6.7e-6      | 4.6e-5      | 5.3e-4      | 4.7e-3      | 3.2e-2      | 1.8e-1      | 9.1e-1      | 4.6e+0      |
| COPD1   | 90         | 3.7e-6      | 5.9e-6      | 4.5e-3      | 9.9e-3      | 3.2e-2      | 1.1e-1      | 3.7e-1      | 1.1e+0      | 3.5e+0      | 1.3e+1      |
| COPD4   | 90         | 2.2e+1      | 2.4e+0      | 1.6e-1      | 4.7e-2      | 5.0e-2      | 1.3e-1      | 3.9e-1      | 1.1e+0      | 3.2e+0      | 1.1e+1      |
| COPD5   | 90         | 4.0e-2      | 4.0e-2      | 9.0e-3      | 8.1e-3      | 2.3e-2      | 8.5e-2      | 2.9e-1      | 9.5e-1      | 3.1e+0      | 1.2e+1      |
| COPD6   | 90         | 3.8e+6      | 2.2e-6      | 5.2e-6      | 3.5e-5      | 4.6e-4      | 4.5e-3      | 3.3e-2      | 1.9e-1      | 9.8e-1      | 4.8e+0      |
| COPD8   | 90         | 4.0e-2      | 4.5e-3      | 2.5e-3      | 5.7e-3      | 2.0e-2      | 7.5e-2      | 2.6e-1      | 8.4e-1      | 2.7e+0      | 9.6e+0      |
| COPD7   | 95         | 1.1e+0      | 8.1e-2      | 2.1e-2      | 2.8e-2      | 6.9e-2      | 1.9e-1      | 5.1e-1      | 1.3e+0      | 3.6e+0      | 1.2e+1      |
| COPD8   | 95         | 1.8e-6      | 1.2e-6      | 3.1e-6      | 1.7e-5      | 1.9e-4      | 1.8e-3      | 1.4e-2      | 9.1e-2      | 5.3e-1      | 2.9e+0      |
Table 4: IC residual amplitudes $R_F$ for the inverse DVF estimates obtained by the iteration (12) for each patient. Percentiles at $\beta = 50\%, 90\%, 95\%$ are reported and compared for different feedback control parameter values. All reported results correspond to 15 iteration steps.

| Patient | $\beta$ (%) | $\mu = 0$ | $\mu = 0.1$ | $\mu = 0.2$ | $\mu = 0.3$ | $\mu = 0.4$ | $\mu = 0.5$ | $\mu = 0.6$ | $\mu = 0.7$ | $\mu = 0.8$ | $\mu = 0.9$ |
|---------|-------------|------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| COPD4   | 50          | 6.0e-3     | 6.0e-3      | 6.0e-3      | 6.0e-3      | 6.2e-3      | 7.6e-3      | 2.0e-2      | 1.0e-1      | 5.0e-1      |
| 90      | 2.8e-2      | 2.8e-2      | 2.8e-2      | 2.8e-2      | 2.9e-2      | 3.6e-2      | 9.8e-2      | 4.4e-1      | 2.0e+0      |
| 95      | 4.3e-2      | 4.3e-2      | 4.3e-2      | 4.3e-2      | 4.5e-2      | 6.2e-2      | 1.9e-1      | 7.6e-1      | 3.0e+0      |
| 4DCT4   | 50          | 4.4e-3      | 4.4e-3      | 4.4e-3      | 4.4e-3      | 4.5e-3      | 5.5e-3      | 1.4e-2      | 7.9e-2      | 4.1e-1      |
| 90      | 2.3e-2      | 2.3e-2      | 2.3e-2      | 2.3e-2      | 2.3e-2      | 2.7e-2      | 6.8e-2      | 3.2e-1      | 1.4e+0      |
| 95      | 3.8e-2      | 3.7e-2      | 3.8e-2      | 3.8e-2      | 3.9e-2      | 5.0e-2      | 1.4e-1      | 5.6e-1      | 2.3e+0      |

Table 5: Non-convergent area ratios $\omega_c(\mu, k)$ with the iteration (12) for each patient. Comparison for different feedback control parameter values. All reported results correspond to 15 iteration steps.

| Patient | $\omega_c(\mu, k)$ (%) at $k = 15$ |
|---------|-----------------------------------|
| COPD1   | $\mu = 0$ | $\mu = 0.1$ | $\mu = 0.2$ | $\mu = 0.3$ | $\mu = 0.4$ | $\mu = 0.5$ | $\mu = 0.6$ | $\mu = 0.7$ | $\mu = 0.8$ | $\mu = 0.9$ |
| COPD4   | 7.49      | 4.37        | 1.99        | 0.90        | 0.70        | 0.60        | 0.47        | 0.30        | 0.08        | 0.08        |
| COPD5   | 2.92      | 1.15        | 0.56        | 0.49        | 0.48        | 0.48        | 0.47        | 0.47        | 0.46        | 0.34        |
| COPD6   | 0.95      | 0.52        | 0.43        | 0.39        | 0.37        | 0.33        | 0.30        | 0.27        | 0.25        | 0.17        |
| COPD8   | 6.16      | 3.98        | 2.25        | 1.09        | 0.74        | 0.73        | 0.73        | 0.73        | 0.72        | 0.57        |
Figure 6: Patient COPD4. Heat-maps of $\rho(P(x;\mu))$ for different feedback control parameters: (top) $\mu = 0$; (middle) $\mu = 0.5$; and (bottom) $\mu = 0.4$. Non-contraction regions where $\rho(P(x;\mu)) \geq 1$ are marked red.

Fig. 7 shows the progressive suppression of IC residual $R_G$ for patient COPD4, with the odd-even alternating parameter value, $\mu_{oe}$. The residual is effectively suppressed in the majority of the domain by the 10th step, and almost everywhere (to below $10^{-3}$ mm by the 30th step).

A comparison of the spatial distribution of IC residuals ($R_F$ and $R_G$) at the 10th iteration step with $\mu = 0$, $\mu = 0.5$, and $\mu = \mu_{oe}$ is shown in Fig. 8 for patient COPD4 and in Fig. 9 for patient COPD1. The convergence behavior of the iterations with these feedback control parameter values, plus the best one from the summary tables ($\mu = 0.4$ for COPD4 and $\mu = 0.3$ for COPD1) is shown in Figs. 10 and 11. We again point out that the red regions in the $R_F$ heat-maps, indicating large IC residuals, are associated to the problematic regions of the deformation Jacobians, shown in the spectral maps of Fig. 5. The effect of these troublesome regions on the iteration is qualitatively indicated in and predicted by Table 2; post-experiments, it is confirmed in Table 5.

5 Conclusion

We have introduced formally an iterative method for numerical DVF inversion with active feedback control, and provided a systematic approach to analyze, pre-
Figure 8: Patient COPD4. IC residuals $R_G(x)$ and $R_F(x)$ on axial coronal, and sagittal slices and overlaid on the target (for $R_G$) or reference (for $R_F$) CT image. All results are shown at the iteration step $k = 10$. (a) Quiver-plot of $R_G(x)$. (b) Heat-map of $R_G(x)$. (c) Quiver-plot of $R_F(x')$. (d) Heat-map of $R_F(x')$. Each labeled image block shows the IC residuals for the iteration with $\mu = 0$ (top), $\mu = 0.5$ (middle), and $\mu = \mu_{oe}$ (bottom). For display purposes, the IC residual quivers are down-sampled by a factor of 12 in the LR and AP axes and a factor of 6 in the SI axis, and the dynamic range of all heat-maps is truncated to 10 mm.
Figure 9: Patient COPD1. IC residuals $R_G(x)$ and $R_F(x)$ on axial coronal, and sagittal slices and overlaid on the target (for $R_G$) or reference (for $R_F$) CT image. All results are shown at the iteration step $k = 10$. (a) Quiver-plot of $R_G(x)$. (b) Heat-map of $R_G(x)$. (c) Quiver-plot of $R_F(x')$. (d) Heat-map of $R_F(x')$. Each labeled image block shows the IC residuals for the iteration with $\mu = 0$ (top), $\mu = 0.5$ (middle), and $\mu = \mu_{oe}$ (bottom). For display purposes, the IC residual quivers are down-sampled by a factor of 12 in the LR and AP axes and a factor of 6 in the SI axis, and the dynamic range of all heat-maps is truncated to 10 mm.
This alternating-value scheme is heuristic, albeit based on and supported by the observed non-contracting re-
dict, improve, and post-evaluate convergence behavior for the iteration with single-parameter feedback control. The convergence conditions and characterizations introduced in Sections 2.2.2 and 3.3 seem new. The analysis yields in particular a formal explanation of convergence- divergence behavior at \( \mu = 0 \) and \( \mu = 0.5 \), which are the two precursor methods that inspired this work, and which are included as special cases of our iteration formulation and analysis.

We have proposed the use of order-one spectral measures for characterizing DVF data and differentiating between large and small deformations. And we have shown the prediction power of these measures on the convergence behavior of iterative DVF inversion with single-parameter feedback control.

Experimental results are in agreement with data-specific analysis and prediction. The iteration with \( \mu = 0.5 \) is shown to be highly competitive in iterative inversion of the COPD DVF data, while \( \mu = 0 \) outperforms the rest with the 4DCT DVF data. This simple comparison alone highlights the need for data-specific parameter determination and optimization. Both constant-\( \mu \) schemes were outperformed by the control with alternating \( \mu \) values, where \( \mu = 0.5 \) at step 1 and subsequent odd-numbered steps and \( \mu = 0.1 \) at even-numbered steps. This alternating-value scheme is heuristic, albeit based on and supported by the observed non-contracting regions (see Table 2 and Fig. 6). Here, we used this scheme to show the potential in feedback control with varying parameter values, temporally and spatially.

We have discussed in Section 2.2.3 the use of spatially adaptive feedback control. Such adaptive parameter values lead to one-step convergence with the analytical DVF of (12). With numerical DVFs arising in practice, the primary challenge lies in the computational complexity for determination of adaptive parameter values.

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Disclosure of conflicts of interest

The authors have no relevant conflicts of interest to disclose.

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