**Abstract.** For a topological space $X$, let $X_\delta$ be the space $X$ with $G_\delta$-topology of $X$. For an uncountable cardinal $\kappa$, we prove that the following are equivalent: (1) $\kappa$ is $\omega_1$-strongly compact. (2) For every compact Hausdorff space $X$, the Lindelöf degree of $X_\delta$ is $\leq \kappa$. (3) For every compact Hausdorff space $X$, the weak Lindelöf degree of $X_\delta$ is $\leq \kappa$. This shows that the least $\omega_1$-strongly compact cardinal is the supremum of the Lindelöf and the weak Lindelöf degrees of compact Hausdorff spaces with $G_\delta$-topology. We also prove the least measurable cardinal is the supremum of the extents of compact Hausdorff spaces with $G_\delta$-topology.

For the square of a Lindelöf space, using weak $G_\delta$-topology, we prove that the following are consistent: (1) the least $\omega_1$-strongly compact cardinal is the supremum of the (weak) Lindelöf degrees of the squares of regular $T_1$ Lindelöf spaces. (2) The least measurable cardinal is the supremum of the extents of the squares of regular $T_1$ Lindelöf spaces.

### 1. Introduction

For a topological space $X$, let $X_\delta$ be the space $X$ with $G_\delta$-topology of $X$, that is, the topology generated by all $G_\delta$-subsets of $X$. $X_\delta$ is also called the $G_\delta$-modification of $X$. The Lindelöf degree of $X$, $L(X)$, is the minimal cardinal $\kappa$ such that every open cover of $X$ has a subcover of size $\leq \kappa$. A space $X$ is Lindelöf if $L(X) = \omega$, that is, every open cover of $X$ has a countable subcover. The weak Lindelöf degree, $wL(X)$, is the minimal cardinal $\kappa$ such that every open cover of $X$ has a subfamily of size $\leq \kappa$ which has dense union in $X$.

In 1970’s, Arhangel’skii asked the following question:

**Question 1.1.** Let $X$ be a compact Hausdorff space.

(1) Is $L(X_\delta) \leq 2^{\aleph_0}$?
(2) Is $wL(X_\delta) \leq 2^{\aleph_0}$?

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See Spadaro-Szeptycki [10] for the background on this question, and in [11] Tall also asked a similar question. The question (1) was solved in negative sometimes. For instance, if \( \kappa \) has no \( \omega_1 \)-complete uniform ultrafilter (e.g., strictly less than the least measurable cardinal), Gorelic [6] proved that \( L(\omega_2^\kappa) \geq \kappa \). For such a \( \kappa \), since the space \( (\omega_2^\kappa)_\Delta \) is a closed subspace of \( ((\omega + 1)^2)^\kappa_\Delta \), we have \( \kappa \leq L(\omega_2^\kappa) \leq L((\omega_2^\kappa)_\Delta) \leq L(((\omega + 1)^2)^\kappa_\Delta) \). On the other hand, recently Spadaro and Szeptycki [10] solved the question (2): They constructed a compact Hausdorff space \( X \) with \( wL(X_\delta) > 2^{\aleph_0} \), so an answer to the question (2) is also negative. In [10], however, they were not able to get a compact space \( X \) with \( wL(X_\delta) > (2^{\aleph_0})^+ \), and they asked the following question:

**Question 1.2.** Is there any bound on the weak Lindelöf degree of the \( G_\delta \)-topology on a compact space?

In this paper we generalize Spadaro and Szeptycki’s result by showing that the weak Lindelöf degree of the \( G_\delta \)-topology on a compact space can be much greater than \( (2^{\aleph_0})^+ \), and moreover we prove that some class of large cardinals is the supremum of the Lindelöf and the weak Lindelöf degrees of compact Hausdorff spaces under the \( G_\delta \)-topology. These are answers to Spadaro and Szeptycki’s question.

A key of our proofs is a concept of an \( \omega_1 \)-strongly compact cardinal.

**Definition 1.3** (Bagaria-Magidor [1, 2]). Let \( \kappa \) be uncountable cardinal. \( \kappa \) is \( \omega_1 \)-strongly compact if for every set \( A \), every \( \kappa \)-complete filter over \( A \) can be extended to an \( \omega_1 \)-complete ultrafilter.

For \( \omega_1 \)-strongly compact cardinals, the followings are known, see [1, 2]:

(1) Every strongly compact cardinal is \( \omega_1 \)-strongly compact.

(2) It is consistent that the least \( \omega_1 \)-strongly compact cardinal is singular.

(3) If \( \kappa \) is \( \omega_1 \)-strongly compact then there is a measurable cardinal \( \leq \kappa \).

(4) It is consistent that the least measurable cardinal is \( \omega_1 \)-strongly compact.

(5) It is also known that every cardinal greater than an \( \omega_1 \)-strongly compact cardinal is \( \omega_1 \)-strongly compact.

Bagaria and Magidor [2] showed that an uncountable cardinal \( \kappa \) is \( \omega_1 \)-strongly compact if and only if for every open cover \( O \) of the product space of Lindelöf spaces, \( O \) has a subcover of size \( < \kappa \). Hence the least \( \omega_1 \)-strongly compact cardinal is just the supremum of the Lindelöf degrees
of the products of Lindelöf spaces, that is, the following equation holds:

the least $\omega_1$-strongly compact = $\sup\{L(\prod_{i \in I} X_i) \mid X_i (i \in I) \text{ is Lindelöf }\}$.

The following is one of main results of this paper, which shows that the least $\omega_1$-strongly compact cardinal is the supremum of both the Lindelöf and the weak Lindelöf degrees of compact spaces with $G_\delta$-topology.

**Theorem 1.4.** Let $\kappa$ be an uncountable cardinal. Then the following are equivalent:

1. $\kappa$ is an $\omega_1$-strongly compact cardinal.
2. $L(X_\delta) \leq \kappa$ for every compact Hausdorff space $X$.
3. $wL(X_\delta) \leq \kappa$ for every compact Hausdorff space $X$.

Thus we have:

the least $\omega_1$-strongly compact = $\sup\{L(X_\delta) \mid X \text{ is compact Hausdorff}\}$

= $\sup\{wL(X_\delta) \mid X \text{ is compact Hausdorff}\}$.

We also consider the extent. Recall that the extent of $X$, $e(X)$, is $\sup\{|C| \mid C \subseteq X \text{ is closed and discrete}\}$. The extent is smaller than the Lindelöf degree, so the extent is another generalization of the Lindelöf degree.

For the extent of the $G_\delta$-topology, we prove that the least measurable cardinal is the supremum of the extents of compact spaces with $G_\delta$-topology, this contrasts with Theorem 1.4.

**Theorem 1.5.** For every uncountable cardinal $\kappa$, $\kappa$ is the least measurable cardinal if and only if $\kappa$ is the least cardinal such that $e(X_\delta) \leq \kappa$ for every compact Hausdorff space $X$.

Hence we have:

the least measurable = $\sup\{e(X_\delta) \mid X \text{ is compact Hausdorff}\}$.

Next we turn to the square of a Lindelöf space. It is known that the square of a Lindelöf space need not be Lindelöf; the square of the Sorgenfrey line has Lindelöf degree $2^\omega$. However the following question is still open:

**Question 1.6.** How large is the Lindelöf degree of the square of a Lindelöf space?

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1 This equation also means that if there is no $\omega_1$-strongly compact cardinal, then $L(\prod_{i \in I} X_i)$ can be arbitrary large.
By Bagaria and Magidor’s theorem, an $\omega_1$-strongly compact is an upper bound on it. For the lower bound, by the forcing method, Shelah [9] (see also Hajnal-Juhász [7]) constructed a Lindelöf space $X$ with $L(X^2) = (2^{\aleph_0})^+$, and Gorelic [5] refined this result (see below). We prove that, using a weak $G_\delta$-topology, the Cohen forcing notion $\mathbb{C}$ creates a Lindelöf space $X$ such that $L(X^2)$ is much greater than $(2^{\aleph_0})^+$. Actually it forces that the least $\omega_1$-strongly compact cardinal is the supremum of the weak Lindelöf degrees of the squares of Lindelöf spaces.

Theorem 1.7. The Cohen forcing notion $\mathbb{C}$ forces the following: For every uncountable cardinal $\kappa$, $\kappa$ is $\omega_1$-strongly compact if and only if $wL(X^2) \leq \kappa$ for every regular $T_1$ Lindelöf space $X$.

So the Cohen forcing forces the following equation:
\[
\text{the least } \omega_1\text{-strongly compact} = \sup\{L(X^2) \mid X \text{ is regular } T_1 \text{ Lindelöf}\} = \sup\{wL(X^2) \mid X \text{ is regular } T_1 \text{ Lindelöf}\}.
\]

For the extent of the square of a Lindelöf space, by the forcing method, Gorelic [5] constructed a Lindelöf space whose square has extent $2^{\aleph_1}$, and he conjectured that the extent of the square of a Lindelöf space is always bounded by $2^{\aleph_1}$.

We prove that the least measurable cardinal bounds the extent of the square of a Lindelöf space. Actually it bounds the extent of the product of Lindelöf spaces.

Theorem 1.8. Let $\kappa$ be the least measurable cardinal. Then $e(\prod_{\xi<\lambda} X_\xi) \leq \kappa$ for every family $\{X_\xi \mid \xi < \lambda\}$ of Lindelöf spaces.

For the lower bound of the extent of a square, we prove the consistency that the extent of the square of a Lindelöf space can be arbitrary large up to the least measurable. In fact the Cohen forcing forces that the least measurable is the supremum of the extents of the squares of Lindelöf spaces. This answers the Gorelic’s conjecture in negative.

Theorem 1.9. The Cohen forcing notion $\mathbb{C}$ forces the following: For every uncountable cardinal $\kappa$, $\kappa$ is the least measurable cardinal if and only if $\kappa$ is the least cardinal such that $e(X^2) \leq \kappa$ for every regular $T_1$ Lindelöf space $X$.

Thus the Cohen forcing forces:
\[
\text{the least measurable} = \sup\{e(X^2) \mid X \text{ is regular } T_1 \text{ Lindelöf}\}.
\]

Here we present some basic set-theoretic definitions. For a regular uncountable cardinal $\theta$, $H(\theta)$ is the set of all sets with hereditary cardinality $< \theta$. 

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For a set $A$, a filter over $A$ is a family $F \subseteq \mathcal{P}(A)$ satisfying the following:

1. $A \in F$, $\emptyset \notin F$.
2. $X, Y \in F \Rightarrow X \cap Y \in F$.
3. $X \in F$, $X \subseteq Y \subseteq A \Rightarrow Y \in F$.

For a cardinal $\kappa$ and a filter $F$ over the set $A$, $F$ is $\kappa$-complete if for every family $\mathcal{F} \subseteq F$ of size $< \kappa$, we have $\bigcap \mathcal{F} \in F$. A filter $F$ over $A$ is principal if $\{x\} \in F$ for some $x \in A$. Every principal filter is an ultrafilter. An uncountable cardinal $\kappa$ is a measurable cardinal if there is a $\kappa$-complete non-principal ultrafilter over $\kappa$. It is known that every measurable cardinal is regular, and if there is a non-principal $\omega_1$-complete ultrafilter $U$ over some set $A$, then there is a measurable cardinal $\leq |A|$, and the completeness of $U$ is in fact greater than or equal to the least measurable. In particular, if $\lambda$ is strictly less than the least measurable cardinal, then there is no non-principal $\omega_1$-complete ultrafilter over $\lambda$.

2. $\omega_1$-strongly compact cardinals and the Lindelöf degree

In this section we prove Theorem 1.4. We will use the following basic facts about $\omega_1$-strongly compact cardinals.

**Definition 2.1.** For a cardinal $\kappa$ and a set $A$ of size $\geq \kappa$, let $\mathcal{P}_\kappa A = \{a \subseteq A \mid |a| < \kappa\}$. A filter over $\mathcal{P}_\kappa A$ is fine if $\{a \in \mathcal{P}_\kappa A \mid x \in a\} \in F$ for every $x \in A$.

**Fact 2.2 (Bagaria-Magidor [1, 2]).** (1) For uncountable cardinal $\kappa$, the following are equivalent:

(a) $\kappa$ is $\omega_1$-strongly compact.

(b) For every set $A$ of size $\geq \kappa$, there exists an $\omega_1$-complete fine ultrafilter over $\mathcal{P}_\kappa A$.

(c) For every cardinal $\lambda \geq \kappa$, there exists an $\omega_1$-complete fine ultrafilter over $\mathcal{P}_\kappa \lambda$.

(2) If $\kappa$ is the least $\omega_1$-strongly compact cardinal, then $\kappa$ is a limit cardinal.

The following lemma immediately implies that $L(X_\delta)$ is bounded by an $\omega_1$-strongly compact cardinal for every compact Hausdorff space $X$.

**Lemma 2.3.** Let $\kappa$ be an $\omega_1$-strongly compact cardinal, and $X$ a Lindelöf space (no separation axiom is assumed). Let $\mathcal{O}$ be a cover of $G_\delta$-subsets of $X$. Then $\mathcal{O}$ has a subcover of size $< \kappa$.

**Proof.** Suppose to the contrary that $\mathcal{O}$ has no subcover of size $< \kappa$. Let $\{O_\alpha \mid \alpha < \lambda\}$ be an enumeration of $\mathcal{O}$, where $\lambda \geq \kappa$. For $\alpha < \lambda$, take a
closed subspace of $\beta_\omega$ if there is no $n < \omega$. Since $\kappa$ is $\omega_1$-strongly compact, there is an $\omega_1$-complete fine ultrafilter $U$ over $\mathcal{P}_\kappa \lambda$. Since $\mathcal{O}$ has no subcover of size $< \kappa$, for each $a \in \mathcal{P}_\kappa \lambda$, we know $\bigcap_{n<\omega}(\bigcup_{n<\omega} F_n^a) \neq \emptyset$. Thus there is $f_a : a \to \omega$ so that $\bigcap\{F_{f_a(a)}^a \mid a \in \alpha\}$ is non-empty. Then for each $\alpha$, since the filter $U$ is $\omega_1$-complete, there is $n_\alpha$ with $\{a \in \mathcal{P}_\kappa \lambda \mid f_a(\alpha) = n_\alpha\} \in U$. However then $\{X \setminus F_n^{a_\alpha} \mid \alpha < \lambda\}$ is a cover of $X$ but does not have a countable subcover, this is a contradiction. \hfill $\square$

For proving (2), (3) $\Rightarrow$ (1) in Theorem 1.4, we introduce a useful notion which came from Gorelic [6].

Let $D$ be a discrete space, and $\beta D$ be the Stone-Čech compactification of $D$, namely, $\beta D$ is the set of all ultrafilters over $D$, and the topology is generated by the family $\{\{U \in \beta D \mid A \subseteq U\} \mid A \subseteq D\}$. Let $\gamma D$ be a subspace of $\{U \in \beta D \mid U \neq \omega_1\}$-complete. Then for every $U \in \gamma D$, there is a countable partition $\mathcal{A}$ of $D$ such that $A \notin U$ for every $A \in \mathcal{A}$.

**Definition 2.4.** Let us say that a cover $\mathcal{O}$ of $G_\delta$-subsets of $\gamma D$ is a proper $G_\delta$-cover if for every $G \in \mathcal{O}$, there is a countable partition $\mathcal{A}$ of $D$ such that $G = \{U \in \gamma D \mid A \notin U \text{ for every } A \in \mathcal{A}\}$.

**Definition 2.5.** For an uncountable cardinal $\kappa$ and a cardinal $\lambda \geq \kappa$, let $\text{Fine}(\mathcal{P}_\kappa \lambda)$ be the set of all fine ultrafilters over $\mathcal{P}_\kappa \lambda$.

Identifying $\mathcal{P}_\kappa \lambda$ as a discrete space, one can check that $\text{Fine}(\mathcal{P}_\kappa \lambda)$ is a closed subspace of $\beta(\mathcal{P}_\kappa \lambda)$, hence is compact Hausdorff. Note also that if there is no $\omega_1$-complete fine ultrafilter over $\mathcal{P}_\kappa \lambda$, then $\text{Fine}(\mathcal{P}_\kappa \lambda)$ has a proper $G_\delta$-cover.

**Proposition 2.6.** Let $\kappa$ be an uncountable cardinal, and $\lambda \geq \kappa$ a cardinal. Suppose that there is no $\omega_1$-complete fine ultrafilter over $\mathcal{P}_\kappa \lambda$. Then $\text{Fine}(\mathcal{P}_\kappa \lambda)$ has no proper $G_\delta$-cover of size $\kappa$.

**Proof.** The idea of the following proof came from Gorelic [6]. Suppose to the contrary that there is a proper $G_\delta$-cover $\mathcal{O}$ of size $< \kappa$. Let $\mu = |\mathcal{O}|$, and $\{Z_\alpha \mid \alpha < \mu\}$ be an enumeration of $\mathcal{O}$. For $\alpha < \mu$, let $\{A^\alpha_n \mid n < \omega\}$ be a countable partition of $\mathcal{P}_\kappa \lambda$ with $Z_\alpha = \{U \in \text{Fine}(\mathcal{P}_\kappa \lambda) \mid A^\alpha_n \notin U \text{ for every } n < \omega\}$.

Fix a large regular cardinal $\theta$, and take an elementary submodel $M \prec H(\theta)$ such that $|M| < \kappa$, $\mu \subseteq M$, and $M$ contains all relevant objects. Let $a = M \cap \lambda \in \mathcal{P}_\kappa \lambda$. For each $\alpha < \mu$, there is $n_\alpha < \omega$ with $a \in A^\alpha_{n_\alpha}$.

**Claim 2.7.** For every finitely many $\alpha_0, \ldots, \alpha_k < \mu$ and $\beta_0, \ldots, \beta_k < \lambda$, there is $b \in \bigcap_{i \leq k} A^\alpha_{n_{\alpha_i}}$ such that $\beta_i \in b$ for every $i \leq k$. 

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Proof. Note that $\langle \alpha_i \mid i \leq k \rangle, \langle n_{\alpha_i} \mid i \leq k \rangle \in M$, hence $\{A_{n_{\alpha_i}}^\alpha \mid i \leq k \} \in M$. If there is no such $b \in \bigcap_{i \leq k} A_{n_{\alpha_i}}^\alpha$, by the elementarity of $M$, there are $\gamma_0, \ldots, \gamma_k \in M \cap \lambda$ such that there is no $b \in \bigcap_{i \leq k} A_{n_{\alpha_i}}^\alpha$ with $\gamma_i \in b$. However $\gamma_i \in M \cap \lambda = a$ and $a \in \bigcap_{i \leq k} A_{n_{\alpha_i}}^\alpha$, this is a contradiction. \[\square\]

By the claim, the family $\{A_{n_{\alpha}}^\alpha \mid \alpha \in \mu\} \cup \{\{x \in P_\kappa \lambda \mid \beta \in x\} \mid \beta < \lambda\}$ has the finite intersection property. Thus we can find a fine ultrafilter $U$ over $P_\kappa \lambda$ such that $A_{n_{\alpha_i}}^\alpha \in U$ for every $\alpha < \mu$. Then $U \notin Z_\alpha$ for every $\alpha < \mu$, this contradicts the choice of $O$. \[\square\]

**Corollary 2.8.** Let $\kappa$ be an uncountable cardinal, and suppose $L(X_\delta) \leq \kappa$ for every compact Hausdorff space $X$. Then $\kappa$ is $\omega_1$-strongly compact.

**Proof.** By the assumption, for every compact Hausdorff space $X$, every cover of $G_\delta$-subsets of $X$ has a subcover of size $< \kappa^+$. By Proposition 2.6 for every cardinal $\lambda \geq \kappa^+$, $P_\kappa \lambda$ carries an $\omega_1$-complete fine ultrafilter over $P_\kappa \lambda$. Then $\kappa^+$ is $\omega_1$-strongly compact by Fact 2.2. Again, by Fact 2.2, the least $\omega_1$-strongly compact cardinal is a limit cardinal. Hence $\kappa^+$ is not the least $\omega_1$-strongly compact cardinal, and we conclude that $\kappa$ is $\omega_1$-strongly compact. \[\square\]

For the weak Lindelöf degree, we use Alexandroff duplicate $\mathbb{A}(X)$.

**Definition 2.9.** For a topological space $X$, let $\mathbb{A}(X)$ be the space defined as follows: The underlying set of $\mathbb{A}(X)$ is $X \times \{0, 1\}$. The topology of $\mathbb{A}(X)$ is defined as follows:

1. Each $\langle x, 0 \rangle \in \mathbb{A}(X)$ is isolated.
2. For $\langle x, 1 \rangle \in \mathbb{A}(X)$, an open neighborhood of $\langle x, 1 \rangle$ is of the form $\langle O \times \{0, 1\} \rangle \setminus \langle \{x_0, 0\}, \ldots, \{x_n, 0\} \rangle$ for some open $O \subseteq X$ with $x \in O$ and finitely many $x_0, \ldots, x_n \in X$.

It is easy to check that if $X$ is compact Hausdorff (regular $T_1$ Lindelöf, respectively) then $\mathbb{A}(X)$ is compact Hausdorff (regular $T_1$ Lindelöf, respectively) as well.

**Lemma 2.10.** Let $X$ be a topological space. Then $L(X_\delta) = wL(\mathbb{A}(X)_\delta)$.

**Proof.** Let $\kappa = L(X_\delta)$ and $\lambda = wL(\mathbb{A}(X)_\delta)$. We shall show $\kappa = \lambda$.

$\kappa \leq \lambda$: Take an open cover $\mathcal{O}$ of $X_\delta$ such that $\mathcal{O}$ has no subcover of size $< \kappa$. Then $\mathcal{W} = \{O \times \{0, 1\} \mid O \in \mathcal{O}\}$ is an open cover of $\mathbb{A}(X)_\delta$. If $\mathcal{W}'$ is a subfamily of $\mathcal{W}$ with dense union, then $\bigcup \mathcal{W}' = \mathbb{A}(X)_\delta$ because $X \times \{0\}$ is discrete in $\mathbb{A}(X)_\delta$. Hence $\{O \mid O \times \{0, 1\} \in \mathcal{W}'\}$ is a cover of $X_\delta$. This means that $\mathcal{W}$ has no subfamily of size $< \kappa$ with dense union, and we have $\kappa \leq \lambda$. 

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Let $\mathcal{W}$ be an open cover of $A(X)\delta$ such that every subfamily of size $<\lambda$ has no dense union. We may assume that every element $W$ of $\mathcal{W}$ is of the form $\{\langle x,0 \rangle \}$ for some $x \in X$, or $(Z_W \times \{0,1\}) \setminus Y_W$ for some $G_\delta$-subset $Z_W$ of $X$ and some countable $Y_W \subseteq X \times \{0\}$. Then $\mathcal{O} = \{Z_W \mid W \in \mathcal{V}\}$ is an open cover of $X_\delta$. If $\mathcal{O}'$ is a subcover of $\mathcal{O}$, then $|(X \times \{0,1\}) \setminus \bigcup \{W \mid Z_W \in \mathcal{O}'\}| \leq |\mathcal{O}'|$. This means that $\mathcal{O}$ has no subcover of size $<\lambda$, and we have $\lambda \leq \kappa$.

Now we are ready to prove Theorem 1.4.

Corollary 2.11. Let $\kappa$ be an uncountable cardinal. Then the following are equivalent:

1. $\kappa$ is $\omega_1$-strongly compact.
2. $L(X_\delta) \leq \kappa$ for every Lindelöf space $X$.
3. $L(X_\delta) \leq \kappa$ for every compact Hausdorff space $X$.
4. $wL(X_\delta) \leq \kappa$ for every compact Hausdorff space $X$.

Proof. (1) $\Rightarrow$ (2) follows from Lemma 2.3, and (2) $\Rightarrow$ (3) is trivial. (3) $\Rightarrow$ (1) follows from Corollary 2.8. (3) $\Rightarrow$ (4) is trivial, and (4) $\Rightarrow$ (3) follows from Lemma 2.10.

Remark 2.12. If $\kappa$ is the least $\omega_1$-strongly compact cardinal, then we cannot improve the condition “$wL(X_\delta) \leq \kappa$” in Corollary 2.11 to “$wL(X_\delta) < \kappa$”; For every cardinal $\lambda < \kappa$ there is a compact Hausdorff space $X_\lambda$ with $wL((X_\lambda)\delta) > \lambda$. Let $X$ be the topological sum of the $X_\lambda$’s. $X$ is a locally compact Hausdorff space. Let $\alpha X$ be the one-point compactification of $X$. It is not hard to see that for every cardinal $\nu < \kappa$, there is a cardinal $\lambda < \kappa$ with $\nu \leq \lambda$ and a cover $\mathcal{O}$ of $G_\delta$-subsets of $\alpha X$ such that every subfamily of $\mathcal{O}$ of size $<\lambda$ has no dense union in $(\alpha X)\delta$.

3. The square of a Lindelöf space

In this section we prove Theorem 1.7. Recall that the Cohen forcing notion is the poset $2^{<\omega}$ with the reverse inclusion.

Proposition 3.1. Let $\kappa$ be an uncountable cardinal, $\lambda \geq \kappa$ a cardinal, and suppose there is no $\omega_1$-complete fine ultrafilter over $\mathcal{P}(\kappa)\lambda$. Then $\mathbb{C}$ forces the following: There are regular $T_1$ Lindelöf spaces $X_0$ and $X_1$ with $L(X_0 \times X_1) \geq \kappa$.

Proof. As in the proof of Proposition 2.6 we use $\text{Fine}(\mathcal{P}(\kappa)\lambda)$. In $V$, let $D = \mathcal{P}(\kappa)\lambda$, and fix a proper $G_\delta$-cover $\{Z_\alpha \mid \alpha < \mu\}$ of $\text{Fine}(D)$. For $\alpha < \mu$, fix a countable partition $\mathcal{A}_\alpha = \{A_\alpha^n \mid n < \omega\}$ of $D$ with $Z_\alpha = \{U \in \text{Fine}(D) \mid A_\alpha^n \notin U \text{ for } n < \omega\}$. Take a $(V,\mathbb{C})$-generic $G$ and we work in $V[G]$. Fix $a \subseteq \omega$. We construct the space $\text{Fine}(D)^V_a$ in $V[G]$ as the following manner.
The underlying set of Fine(D)$_a^V$ is Fine(D)$_a^V$, the set of all fine ultrafilters over D defined in V. So every element U of Fine(D)$_a^V$ belongs to V and is a fine ultrafilter over D in V, but it is not necessary that U is a fine ultrafilter in V[G]. The topology of Fine(D)$_a^V$ is defined in V[G] as follows:

For a set A ⊆ D with A ∈ V and a finite (possibly empty) sequence $\vec{\alpha} = (\alpha_0, \ldots, \alpha_k) \in \mu^{<\omega}$, let $W_{A,\vec{\alpha}}^a = \{U \in \text{Fine}(D)^V | A \in U, A_n^a \notin U \text{ for every } n \in A \text{ and } i \leq k\}$. Then the topology of Fine(D)$_a^V$ is generated by the $W_{A,\vec{\alpha}}^a$'s as an open base. Note that if $O \subseteq \text{Fine}(D)$ is open in Fine(D)$_a^V$ in V, then $O$ is an open set of Fine(D)$_a^V$. We can check that Fine(D)$_a^V$ is $T_1$ and zero-dimensional, hence is regular $T_1$.

**Claim 3.2.** Let $a = \{n < \omega | \bigcup G(n) = 0\}$. Then, in V[G], Fine(D)$_a^V$ is Lindelöf.

**Proof.** We work in V. Let $\dot{a}$ be a name for a. Take $p \in \mathbb{C}$, and a name $\dot{O}$ for an open cover of Fine(D)$_a^V$. We show that $p \Vdash \text{“}\dot{O}\text{ has a countable subcover”}”.

We may assume that $p \Vdash \text{“every } W \in \dot{O}\text{ is of the form } W_{A,\vec{\alpha}}^a\text{ for some } A\text{ and } \vec{\alpha}”$. Let $\theta$ be a sufficiently large regular cardinal, and take a countable elementary submodel $M \prec H(\theta)$ which contains all relevant objects. We show that $p \Vdash \text{“}\{W_{A,\vec{\alpha}}^a \in \dot{O} | \langle A, \vec{\alpha} \rangle \in M\}\text{ covers Fine(D)$_a^V$”}$. To show this, take $p_0 \leq p$ and $U_0 \in \text{Fine}(D)$. $p_0$ belongs to $M$. Let $\mathcal{F}$ be the set of all pairs $\langle A, \vec{\alpha} \rangle$ such that there is some $q \leq p_0$ with $q \Vdash \text{“}W_{A,\vec{\alpha}}^a \in \dot{O}\text{”}$. For a pair $\langle A, \vec{\alpha} \rangle \in \mathcal{F}$ and $q \leq p_0$ with $q \Vdash \text{“}W_{A,\vec{\alpha}}^a \in \dot{O}\text{”}$, let $x = \{n \in \text{dom}(q) | q(n) = 0\}$ and $W_{A,\vec{\alpha}}^q = W_{A,\vec{\alpha}}^a$. $W_{A,\vec{\alpha}}^q$ is open in Fine(D)$_a^V$. Let $\mathcal{V} = \{W_{A,\vec{\alpha}}^q | q \leq p_0, \langle A, \vec{\alpha} \rangle \in \mathcal{F}, q \Vdash \text{“}W_{A,\vec{\alpha}}^a \in \dot{O}\text{”}\}$. We have $\mathcal{V} \in M$, and $\mathcal{V}$ is a family of open sets in Fine(D)$_a^V$. Now we check that $\mathcal{V}$ is an open cover of Fine(D)$_a^V$. To see this, take $U \in \text{Fine}(D)$. Then $p_0 \Vdash \text{“}U \in \bigcup \dot{O}\text{”}$, hence there is $\langle A, \vec{\alpha} \rangle \in \mathcal{F}$ and $q \leq p_0$ such that $q \Vdash \text{“}U \in W_{A,\vec{\alpha}}^a \in \dot{O}\text{”}$. Clearly $A \in U$. Let $\vec{\alpha} = (\alpha_0, \ldots, \alpha_k)$. For $n \in \text{dom}(q)$, we can see that if $q(n) = 0$ then $A_n^a \notin U$ for every $i \leq k$; Since $q(n) = 0$, we have $q \Vdash \text{“}n \notin \dot{a}\text{”}$. We know $q \Vdash \text{“}U \in W_{A,\vec{\alpha}}^a\text{”}$, which means that $A_n^a \notin U$. Now we know $A \in U$ and if $q(n) = 0$ then $A_n^a \notin U$. Thus $U \in W_{A,\vec{\alpha}}^a \in \mathcal{V}$.

Since Fine(D)$_a^V$ is compact, there is a finite subcover $\mathcal{V}' \subseteq \mathcal{V}$ of Fine(D)$_a^V$. Because $\mathcal{V} \in M$, we may assume that $\mathcal{V}' \in M$, and we have $\mathcal{V}' \subseteq \mathcal{V}$. Take $W_{A,\vec{\alpha}}^{q_i} \in \mathcal{V}'$ with $U_0 \in W_{A,\vec{\alpha}}^{q_i}$. We know $\langle A, \vec{\alpha} \rangle, q_i \in M$. Let $\vec{\alpha} = (\alpha_0, \ldots, \alpha_k)$. For each $i \leq k$, there is at most one $n < \omega$ with $A_n^a \in U_0$. Hence there is some large $n_0 > \text{dom}(q)$ such that $\{n < \omega | A_n^a \in U \text{ for some } i \leq k\} \subseteq n_0$.

Again, since $U_0 \in W_{A,\vec{\alpha}}^{q_i}$, we know that for $n \in \text{dom}(q)$ if $q(n) = 0$ then $A_n^a \notin U_0$ for every $i \leq k$. Now define $r \leq q$ by dom($r$) = $n_0$ and $r(m) = 1$ for every $m \notin \text{dom}(r)$.
for every $\text{dom}(q) \leq m < n_0$. Then $r \Vdash \text{"}a \cap n_0 = \{ n \in \text{dom}(q) \mid q(n) = 0 \} \text{"}$, so $r \Vdash \text{"}U_0 \in W_{A,a} \in \mathcal{O} \text{"}$, as required. \qedhere

By swapping 0 and 1, we can prove the following by the same argument:

**Claim 3.3.** Let $b = \{ n < \omega \mid \bigcup G(n) = 1 \}$. Then, in $V[G]$, $\text{Fine}(D)^V_b$ is Lindelöf.

Let $a = \{ n < \omega \mid \bigcup G(n) = 0 \}$ and $b = \{ n < \omega \mid \bigcup G(n) = 1 \}$. By the claims before, we have that $\text{Fine}(D)^V_a$ and $\text{Fine}(D)^V_b$ are Lindelöf.

**Claim 3.4.** $L(\text{Fine}(D)^V_a \times \text{Fine}(D)^V_b) \geq \kappa$.

**Proof.** Let $\Delta$ be the diagonal of $\text{Fine}(D)^V_a \times \text{Fine}(D)^V_b$. Since $\text{Fine}(D)^V_a$ and $\text{Fine}(D)^V_b$ are Hausdorff, $\Delta$ is closed in $\text{Fine}(D)^V_a \times \text{Fine}(D)^V_b$.

For $\alpha < \mu$, let $W_\alpha = W_{D,\langle \alpha \rangle}^a \times W_{D,\langle \alpha \rangle}^b$. $W_\alpha$ is open in $\text{Fine}(D)^V_a \times \text{Fine}(D)^V_b$.

Let $\mathcal{W} = \{ W_\alpha \mid \alpha < \mu \}$. We check that $\mathcal{W}$ is an open cover of $\Delta$ but has no subcover of size $< \kappa$.

First we note the following: For every $U \in \text{Fine}(D)^V$ and $\alpha < \mu$, $\langle U, U \rangle \in W_\alpha$ if and only if $U \in Z_\alpha$. If $\langle U, U \rangle \in W_\alpha$, then $A^n_\alpha \notin U$ for every $n \in a \cup b$. Since $a \cup b = \omega$, we know $A^n_\alpha \notin U$ for every $n < \omega$, and $U \in Z_\alpha$. For the converse, if $U \in Z_\alpha$, then it is clear that $U \in W_{D,\langle \alpha \rangle}^a \cap W_{D,\langle \alpha \rangle}^b$, so $\langle U, U \rangle \in W_{D,\langle \alpha \rangle}^a \times W_{D,\langle \alpha \rangle}^b = W_\alpha$.

To show that $\mathcal{W}$ is an open cover of $\Delta$, take $U \in \text{Fine}(D)^V$. Because $\{ Z_\alpha \mid \alpha < \mu \}$ is a cover of $\text{Fine}(D)^V$, there is $\alpha < \mu$ with $U \in Z_\alpha$. Then $\langle U, U \rangle \in W_\alpha$ by the remark above.

Next we prove that $\mathcal{W}$ has no subcover of size $< \kappa$. If not, then there is $E \in [\mu]^{<\kappa}$ such that $\{ W_\alpha \mid \alpha \in E \}$ forms a cover. Since $\mathbb{C}$ satisfies the countable chain condition and $\kappa > \omega$, we may assume that $E \in V$. Then, by the remark above, we have that $\{ Z_\alpha \mid \alpha \in E \}$ is a proper $G_\delta$-cover of $\text{Fine}(D)^V$ of size $< \kappa$, this contradicts Proposition 10.15 in Kanamori [8]. \qedhere

**Lemma 3.5.** Let $X_0$ and $X_1$ be topological spaces, and $Y = X_0 \oplus X_1$, the topological sum of $X_0$ and $X_1$. Then $L(X_0 \times X_1) \leq L(Y^2)$.

**Proof.** $Y^2$ can be identified with the disjoint union of $X_0^2, X_1^2, X_0 \times X_1$, and $X_1 \times X_0$. These are clopen sets in $Y^2$, hence $L(X_0 \times X_1) \leq L(Y^2)$. \qedhere

**Lemma 3.6.** Let $X_0$ and $X_1$ be topological spaces. Then $L(X_0 \times X_1) = wL(A(X_0) \times A(X_1))$.

**Proof.** The proof is similar to of Lemma 2.10. \qedhere

The following follows from well-known arguments, e.g., see Proposition 10.15 in Kanamori [8].
Lemma 3.7. Let $\kappa$ be an uncountable cardinal, and $\mathbb{C}$ the Cohen forcing notion. Then $\kappa$ is $\omega_1$-strongly compact if and only if $\Vdash_{\mathbb{C}} \kappa$ is $\omega_1$-strongly compact”.

Combining Proposition 3.1 and Lemmas 3.5 3.6 3.7, we have Theorem 1.7

Corollary 3.8. $\mathbb{C}$ forces the following statement: For every uncountable cardinal $\kappa$, $\kappa$ is $\omega_1$-strongly compact if and only if $wL(X^2) \leq \kappa$ for every regular $T_1$ Lindelöf space $X$.

Proof. Take a $(V, \mathbb{C})$-generic $G$ and work in $V[G]$. If $\kappa$ is $\omega_1$-strongly compact, then $L(X^2) \leq \kappa$ for every Lindelöf space $X$ by Bagaria and Magidor’s theorem mentioned in the introduction.

Suppose $\kappa$ is not $\omega_1$-strongly compact. Then, by Lemma 3.7, $\kappa$ is not $\omega_1$-strongly compact in $V$. We know that $\kappa^+$ is not $\omega_1$-strongly compact in $V$. Hence there is a cardinal $\lambda \geq \kappa^+$ such that $\mathcal{P}_{\kappa^+}\lambda$ cannot carry an $\omega_1$-complete fine ultrafilter. By Proposition 3.1 in $V[G]$, there are regular $T_1$ Lindelöf spaces $X_0$ and $X_1$ such that $L(X_0 \times X_1) \geq \kappa^+$. Applying Lemma 3.5, we can find a regular $T_1$ Lindelöf space $Y$ with $L(Y^2) \geq \kappa^+$. Finally, by Lemma 3.6, the space $A(Y)$ is regular $T_1$ Lindelöf but $wL(A(Y)^2) \geq \kappa^+$. □

Corollary 3.8 is a consistency result. So it is natural to ask the following:

Question 3.9. In ZFC, is the least $\omega_1$-strongly compact cardinal the supremum of the (weak) Lindelöf degrees of the squares of Lindelöf spaces?

We can replace the square $X^2$ in Corollary 3.8 by the cube $X^3$.

Proposition 3.10. Let $\kappa$ be an uncountable cardinal and $\lambda \geq \kappa$ a cardinal. Suppose there is no $\omega_1$-complete fine ultrafilter over $\mathcal{P}_\kappa\lambda$. Then $\mathbb{C}$ forces the following: There are regular $T_1$ Lindelöf spaces $X_0$, $X_1$, and $X_2$ such that $X_i \times X_j$ is Lindelöf for every $i, j < 2$ but $L(X_0 \times X_1 \times X_2) \geq \kappa$.

Proof. The proof can be obtained by the arguments in the proof of Proposition 3.1 with slight modifications, so we only sketch the proof.

In this proof, we identify $\mathbb{C}$ as $3^{<\omega}$. Take a $(V, \mathbb{C})$-generic $G$. In $V[G]$, let $a_i = \{n < \omega \mid \bigcup G(n) = i\}$ for $i < 3$. We define $\text{Fine}(D)_{a_i}$ for $i < 3$ as in the proof of Proposition 3.1. Each $\text{Fine}(D)_{a_i}$ is regular $T_1$.

Claim 3.11. In $V[G]$, for every $i, j < 3$, the product space $\text{Fine}(D)_{a_i} \times \text{Fine}(D)_{a_j}$ is Lindelöf.

Proof. We show only the case $i = 0$ and $j = 1$. Other cases follow from a similar proof. We work in $V$. Let $\dot{a}_0$ and $\dot{a}_1$ be names for $a_0$ and $a_1$.
respectively. Take \( p \in \mathbb{C} \), and a name \( \dot{\mathcal{O}} \) for an open cover of \( \Fine(D)_{a_0}^V \times \Fine(D)_{a_1}^V \). We show that \( p \models \text{"}\dot{\mathcal{O}} \) has a countable subcover".

We may assume that \( p \models \text{"} \forall W \in \dot{\mathcal{O}} \text{ is of the form } W_{A_0,\alpha_0}^{\omega_0} \times W_{A_1,\alpha_1}^{\omega_1} \text{ for some } A_0, A_1 \text{ and } \alpha_0, \alpha_1 " \). Let \( \theta \) be a sufficiently large regular cardinal, and take a countable \( M < H(\theta) \) which contains all relevant objects. As before we see that \( p \models \text{"} \forall W \in \dot{\mathcal{O}} \mid (A_0, \alpha_0), (A_1, \alpha_1) \in M \} \) covers \( \Fine(D)^V_{a_0} \times \Fine(D)^V_{a_1} \).

Take \( p_0 \leq p \) and \( (U_0, U_1) \in \Fine(D)^V \times \Fine(D)^V \). As before, we can find \( (A_0, \alpha_0), (A_1, \alpha_1) \in M \) and \( q \leq p_0 \) such that \( (U_0, U_1) \in W_{A_0,\alpha_0}^q \times W_{A_1,\alpha_1}^q \), where, letting \( \bar{\alpha}_i = (\alpha_0, \ldots, \alpha_k) \), \( W_{A_i,\bar{\alpha}_i}^q \) is the set \( \{ U \in \Fine(D)^V \mid A_i \in U, A_i^0 \notin U \} \) for every \( j \leq k_i \) and \( n \in \text{dom}(q) \) with \( q(n) = i \). Then fix a large \( n_0 < \omega \), and define \( r \leq q \) by \( \text{dom}(r) = n_0 \) and \( r(m) = 2 \) for every \( \text{dom}(q) \leq m < n_0 \). Then \( r \models \text{"} (U_0, U_1) \in W_{A_0,\alpha_0}^r \times W_{A_1,\alpha_1}^r \in \dot{\mathcal{O}} " \), as required.

Let \( W_{\alpha} = W_{D,\alpha}^{\omega_0} \times W_{D,\alpha}^{\omega_1} \times W_{D,\alpha}^{\omega_2} \). \( W_{\alpha} \) is open in \( \Fine(D)^V_{a_0} \times \Fine(D)^V_{a_1} \times \Fine(D)^V_{a_2} \). As in the proof of Proposition 3.11 one can check that the family \( \{ W_{\alpha} \mid \alpha < \mu \} \) is an open cover of the diagonal of \( \Fine(D)_{a_0}^V \times \Fine(D)_{a_1}^V \times \Fine(D)_{a_2}^V \) but has no subcover of size \( \kappa \).

**Lemma 3.12.** Let \( X_i \ (i < 3) \) be spaces, and \( Y = X_0 \oplus X_1 \oplus X_2 \).

1. If \( X_i \times X_j \) is Lindelöf for every \( i, j \leq 3 \), then \( Y \) is Lindelöf as well.
2. \( L(X_0 \times X_1 \times X_2) \leq L(Y) \).
3. \( L(X_0 \times X_1 \times X_2) = wL(\mathcal{A}(X_0) \times \mathcal{A}(X_1) \times \mathcal{A}(X_2)) \).

Combining Proposition 3.10 and Lemma 3.12 we have:

**Corollary 3.13.** \( \mathbb{C} \) forces the following: For every uncountable cardinal \( \kappa \), \( \kappa \) is \( \omega_1 \)-strongly compact if and only if \( wL(X^3) \leq \kappa \) for every regular \( T_1 \) Lindelöf space \( X \) with \( X^2 \) Lindelöf.

**Remark 3.14.** Moreover we can replace the cube \( X^3 \) in the previous corollary by \( X^{n+1} \) for arbitrary \( n < \omega \), that is, for every positive \( n < \omega \), \( \mathbb{C} \) forces the following: For every uncountable cardinal \( \kappa \), \( \kappa \) is \( \omega_1 \)-strongly compact if and only if \( wL(X^{n+1}) \leq \kappa \) for every regular \( T_1 \) Lindelöf space \( X \) with \( X^n \) Lindelöf.

4. The Extent

In this section we prove Theorems 1.3, 1.8 and 1.9. First we prove, in ZFC, that the least measurable cardinal bounds the extent of the \( G_\delta \)-topology of a Lindelöf space and of the product of Lindelöf spaces.
Recall that for a space $X$ and an infinite subset $Y \subseteq X$, a point $x \in X$ is a **complete accumulation point of $Y$** if $|O \cap Y| = |Y|$ for every open neighborhood $O$ of $x$. The following is a kind of folklore:

**Lemma 4.1.** Let $\kappa$ be a regular uncountable cardinal, and $X$ a space. Then the following are equivalent:

1. Every subset of $X$ of size $\kappa$ has a complete accumulation point.
2. Every open cover of $X$ of size $\kappa$ has a subcover of size $< \kappa$.

**Proof.** (1) $\Rightarrow$ (2). Let $\mathcal{O} = \{O_\alpha \mid \alpha < \kappa\}$ be an open cover of $X$, and suppose $\mathcal{O}$ has no subcover of size $< \kappa$. By our assumption and the regularity of $\kappa$, we may assume that $O_\alpha \not\subseteq \bigcup_{\beta < \alpha} O_\beta$ for every $\alpha < \kappa$. Hence we can choose $x_\alpha \in O_\alpha \setminus \bigcup_{\beta < \alpha} O_\beta$. Then the set $Y = \{x_\alpha \mid \alpha < \kappa\}$ has cardinality $\kappa$, but has no complete accumulation point.

(2) $\Rightarrow$ (1). Let $Y = \{x_\alpha \mid \alpha < \kappa\}$ be a subset of $X$, and suppose $Y$ has no complete accumulation point. For $\alpha < \kappa$, let $X_\alpha$ be the set of all $x \in X$ such that $O_x \cap Y \subseteq \{x_\beta \mid \beta < \alpha\}$ for some open neighborhood $O_x$ of $x$. Let $W_\alpha = \bigcup \{O_x \mid x \in X_\alpha\}$. Then the family $\{W_\alpha \mid \alpha < \kappa\}$ is an open cover of $X$ of size $\kappa$, but has no subcover of size $< \kappa$. \hspace{1cm} $\square$

**Lemma 4.2.** Suppose $\kappa$ is a measurable cardinal. Let $X$ be a Lindelöf space (no separation axiom is assumed). Then every subset of $X_\delta$ of size $\kappa$ has a complete accumulation point. In particular $X_\delta$ has no closed discrete subset of size $\kappa$, and $e(X_\delta) \leq \kappa$.

**Proof.** Suppose to the contrary that $X_\delta$ has a subset $Y = \{x_\alpha \mid \alpha < \kappa\}$ which has no complete accumulation point. For each $x \in X$, take a $G_\delta$-set $Z^x$ in $X$ with $x \in Z^x$ and $|Z^x \cap Y| < \kappa$. Take open sets $O_n^x$ $(n < \omega)$ in $X$ with $Z^x = \bigcap_{n < \omega} O_n^x$.

Fix a non-principal $\kappa$-complete ultrafilter $U$ over $\kappa$. For $x \in X$, since $|Y \cap Z^x| < \kappa$, we have that $\{\alpha < \kappa \mid x_\alpha \notin Z^x\} \in U$. Because $U$ is $\omega_1$-complete, there is $n_x < \omega$ such that $\{\alpha < \kappa \mid x_\alpha \notin O_{n_x}^x\} \in U$. Then $\{O_{n_x}^x \mid x \in X\}$ is an open cover of $X$. Because $X$ is Lindelöf, there are countably many $x_0, x_1, \ldots \in X$ such that $\{O_{n_i}^{x_i} \mid i < \omega\}$ covers $X$. $U$ is $\omega_1$-complete, hence we can take $\alpha < \kappa$ such that $x_\alpha \notin O_{n_i}^{x_i}$ for every $i < \omega$. Then $x_\alpha \notin \bigcup_{i < \omega} O_{n_i}^{x_i}$; this is a contradiction. \hspace{1cm} $\square$

**Lemma 4.3.** Let $\kappa$ be a measurable cardinal. Let $\{X_\xi \mid \xi < \lambda\}$ be a family of Lindelöf spaces (no separation axiom is assumed). Then every subset of $\prod_{\xi < \lambda} X_\xi$ of size $\kappa$ has a complete accumulation point. In particular, $\prod_{\xi < \lambda} X_\xi$ has no closed discrete subset of size $\kappa$, and $e(\prod_{\xi < \lambda} X_\xi) \leq \kappa$.

**Proof.** By Lemma 4.1, it is enough to show that every open cover of $\prod_{\xi < \lambda} X_\xi$ of size $\kappa$ has a subcover of size $< \kappa$. Let $\mathcal{O} = \{O_\alpha \mid \alpha < \kappa\}$ be an open cover,
and suppose to the contrary that \( O \) has no subcover of size \(< \kappa \). Fix a non-principal \( \kappa \)-complete ultrafilter \( U \) over \( \kappa \). For \( \beta < \kappa \), by our assumption, \( \{ O_{\alpha} \mid \alpha < \beta \} \) does not cover \( \prod_{\xi<\lambda} X_\xi \). Fix \( f_\beta \in \prod_{\xi<\lambda} X_\xi \setminus \bigcup_{\alpha<\beta} O_{\alpha} \). For \( \xi < \lambda \), let \( \mathcal{F}_\xi = \{ W \subseteq X_\xi \mid W \text{ is open}, \{ \beta < \kappa \mid f_\beta(\xi) \notin W \} \in U \} \). We claim that \( \mathcal{F}_\xi \) is not a cover of \( X_\xi \). If not, since \( X_\xi \) is Lindelöf, there are countably many \( W_0, W_1, \ldots \in \mathcal{F}_\xi \) such that \( X_\xi = \bigcup_{i<\omega} W_i \). Since \( U \) is \( \omega_1 \)-complete, there is \( i < \omega \) with \( \{ \beta < \kappa \mid f_\beta(\xi) \in W_i \} \in U \). This contradicts the choice of \( W_i \in \mathcal{F}_\xi \).

Fix \( x_\xi \in X_\xi \setminus \bigcup \mathcal{F}_\xi \), and define \( g \in \prod_{\xi<\lambda} X_\xi \) by \( g(\xi) = x_\xi \). We can take \( \alpha < \kappa \) with \( g \in O_{\alpha} \). Then we can find finitely many \( \xi_0, \ldots, \xi_n < \lambda \) and \( W_0, \ldots, W_n \) such that each \( W_i \) is open in \( X_{\xi_i} \) and \( g \in \prod_{\eta<\lambda, \eta \neq \xi_0, \ldots, \xi_n} X_{\eta} \times \bigcap_{i \leq n} W_i \subseteq O_\alpha \).

For each \( i \leq n \), because \( g(\xi_i) = x_{\xi_i} \in W_i \), we have \( W_i \notin \mathcal{F}_{\xi_i} \), and \( \{ \beta < \kappa \mid f_\beta(\xi_i) \in W_i \} \in U \). Again, since \( U \) is \( \omega_1 \)-complete, there is \( \beta < \kappa \) such that \( \beta > \alpha \) and \( f_\beta(\xi_i) \in W_i \) for every \( i \leq n \). Then \( f_\beta \in \prod_{\eta<\lambda, \eta \neq \xi_0, \ldots, \xi_n} X_{\eta} \times \bigcap_{i \leq n} W_i \subseteq O_\alpha \), this contradicts the choice of \( f_\beta \).  

**Remark 4.4.** If we suppose some separation axiom, the conclusions that \( e(X_\delta) \leq \kappa \) in Lemma 4.2 and that \( e(\prod_{\xi<\lambda} X_\xi) \leq \kappa \) in Lemma 4.3 easily follow from realcompactness. A space \( X \) is realcompact if \( X \) embeds as a closed subspace of \( \mathbb{R}^\theta \) for some cardinal \( \theta \). The following are known (e.g., see Gillman-Jerison [4]):

1. Every Tychonoff Lindelöf (equivalently, regular \( T_1 \) Lindelöf) space is realcompact.
2. Every discrete realcompact space has cardinality strictly less than the least measurable cardinal.
3. Every closed subspace of a realcompact space is real compact.
4. Every product of realcompact spaces is real compact.

If \( \{ X_\xi \mid \xi < \lambda \} \) is a family of Tychonoff Lindelöf spaces, then the product space \( \prod_{\xi<\lambda} X_\xi \) is realcompact by (1) and (4). In addition every closed discrete subset of \( \prod_{\xi<\lambda} X_\xi \) has cardinality less than the least measurable cardinal by (2) and (3).

If \( X \) is realcompact, it is known that \( X_\delta \) is also realcompact (Comfort-Retta [3]), hence if \( X \) is a Tychonoff Lindelöf space, then every closed discrete subset of \( X_\delta \) has cardinality strictly less than the least measurable cardinal.

**Corollary 4.5.** Let \( \kappa \) be an uncountable cardinal. Then the following are equivalent:

1. \( \kappa \) is the least measurable cardinal.
(2) $\kappa$ is the least cardinal such that $e(\prod_{\xi<\lambda} X_\xi) \leq \kappa$ for every family
\{\{X_\xi : \xi < \lambda\} of Lindelöf spaces.

(3) $\kappa$ is the least cardinal such that $e(\prod_{\xi<\lambda} X_\xi) \leq \kappa$ for every family
\{\{X_\xi : \xi < \lambda\} of regular $T_1$ Lindelöf spaces.

Proof. Let $\kappa_1$ be the least measurable cardinal, and $\kappa_2$ the least cardinal $\kappa$
satisfying that $e(\prod_{\xi<\lambda} X_\xi) \leq \kappa$ for every family \{\{X_\xi : \xi < \lambda\} of regular $T_1$ Lindelöf spaces. The inequality $\kappa_2 \leq \kappa_1$ follows from Lemma 4.3. $\kappa_1 \leq \kappa_2$ is immediate from the following fact:

Fact 4.6 (Gorelic [6]). Let $\kappa$ be an uncountable cardinal and suppose there is no $\omega_1$-complete non-principal ultrafilter over $\kappa$. Then $e(\omega^\kappa) \geq \kappa$.

\[\square\]

For constructing a space with large extent in $G_\delta$-topology, we will use a space $\beta\kappa$. Let $\kappa$ be an uncountable cardinal, and suppose there is no $\omega_1$-complete non-principal ultrafilter over $\kappa$. Identifying $\kappa$ as a discrete space, let $\kappa^*$ be the reminder of $\beta\kappa$. As before, fix a proper $G_\delta$-cover $\mathcal{O}$ of $\kappa^*$. Note that every element of $\mathcal{O}$ is a closed $G_\delta$-subset of $\beta\kappa$.

Let $E$ be the set of all principal ultrafilters over $\kappa$. $E$ is discrete in $\beta\kappa$, hence also in $(\beta\kappa)_\delta$.

Lemma 4.7. $E$ is closed in $(\beta\kappa)_\delta$, in particular $e((\beta\kappa)_\delta) \geq \kappa$.

Proof. It is clear that $E \cap Z = \emptyset$ for each $Z \in \mathcal{O}$. \[\square\]

Now we have Theorems 1.5.

Corollary 4.8. Let $\kappa$ be an uncountable cardinal. Then the following are equivalent:

1. $\kappa$ is the least measurable cardinal.
2. $\kappa$ is the least cardinal such that $e(X_\delta) \leq \kappa$ for every Lindelöf space $X$.
3. $\kappa$ is the least cardinal such that $e(X_\delta) \leq \kappa$ for every compact Hausdorff space $X$.

Proof. Let $\kappa_1$ be the least measurable cardinal, $\kappa_2$ the least cardinal such that $e(X_\delta) \leq \kappa$ for every Lindelöf space $X$, and $\kappa_3$ the least cardinal such that $e(X_\delta) \leq \kappa$ for every compact Hausdorff space $X$.

By Lemma 4.2 we have $\kappa_2 \leq \kappa_1$. The inequality $\kappa_3 \leq \kappa_2$ follow from the definitions. For $\kappa_1 \leq \kappa_3$, suppose to the contrary that $\kappa_3 < \kappa_1$. Then $\kappa_3^+ < \kappa_1$, and there is no $\omega_1$-complete non-principal ultrafilter over $\kappa_3^+$. By Lemma 4.7 the extent of $\beta(\kappa_3^+)_\delta$ is $\geq \kappa_3^+$. This contradicts the definition of $\kappa_3$. \[\square\]
Finally we prove Theorem 1.9. Let $\kappa$ be an uncountable cardinal, and suppose there is no $\omega_1$-complete non-principal ultrafilter over $\kappa$. Fix a proper $G_\delta$-cover $\mathcal{O}$ of $\kappa^*$. Let $|\mathcal{O}| = \mu$. Take an enumeration $\{Z_\alpha \mid \alpha < \mu\}$ of $\mathcal{O}$, and for $\alpha < \mu$, take an enumeration $\{A^n_\alpha \mid n < \omega\}$ of $\mathcal{A}_\alpha$, where $\mathcal{A}_\alpha$ is a countable partition of $\kappa$ with the topology generated by $\tilde{\mathcal{A}}_\alpha$.

Let $G$ be $(V, \mathcal{C})$-generic, and we work in $V[G]$. Fix $a \subseteq \omega$, and we define $\beta \kappa_a^V$ in the following way. For $A \subseteq \kappa$ with $A \in V$ and finite (possibly empty) sequence $\vec{\alpha} = (\alpha_0, \ldots, \alpha_k) \in \mu^{< \omega}$, let $\tilde{W}^a_{\vec{A}, \vec{\alpha}} = \{U \in \beta \kappa^V \mid A \in U, A^{\alpha_n} \notin U \text{ for every } i \leq k \text{ and } n \in a\}$. Then the space $\beta \kappa_a^V$ is the space $\beta \kappa^V$ equipped with the topology generated by the $\tilde{W}^a_{\vec{A}, \vec{\alpha}}$'s. As with $\Fine(\mathcal{P}_\alpha^\lambda)^V$, one can check that $\beta \kappa_a^V$ is a regular $T_1$ space.

**Lemma 4.9.** Let $a = \{n < \omega \mid \bigcup G(n) = 0\}$ and $b = \{n < \omega \mid \bigcup G(n) = 1\}$. Then $\beta \kappa_a^V$ and $\beta \kappa_b^V$ are Lindelöf in $V[G]$.

**Proof.** The proof is the same as in Claim 3.2; just replace $W_{A, \vec{\alpha}}^a$ in the proof of Claim 3.2 by $\tilde{W}^a_{A, \vec{\alpha}}$. \qed

**Lemma 4.10.** Let $a = \{n < \omega \mid \bigcup G(n) = 0\}$ and $b = \{n < \omega \mid \bigcup G(n) = 1\}$. Then $\beta \kappa_a^V \times \beta \kappa_b^V$ has a closed discrete subset of size $\kappa$. Hence the extent of the square of $\beta \kappa_a^V \oplus \beta \kappa_b^V$ is $\geq \kappa$.

**Proof.** For $\xi < \kappa$, let $U_\xi \in \beta \kappa^V$ be the principal ultrafilter over $\kappa$ in $V$ with $\{\xi\} \in U_\xi$. Let $\Delta = \{\langle U_\xi, U_\xi \rangle \mid \xi < \kappa\}$. Clearly $\Delta$ is discrete in $\mu \kappa_a^V \times \mu \kappa_b^V$.

We see that $\Delta$ is closed. Take $\langle U, U' \rangle \in (\beta \kappa_a^V \times \beta \kappa_b^V) \setminus \Delta$. If $U \neq U'$, take $A \in U$ with $\kappa \setminus A \in U'$. Then $O = \{\langle F, F' \rangle \in \beta \kappa_a^V \times \beta \kappa_b^V \mid A \in F, \kappa \setminus A \in F'\}$ is an open neighborhood of $\langle U, U' \rangle$ in $\beta \kappa_a^V \times \beta \kappa_b^V$ with $O \cap \Delta = \emptyset$. So suppose $U = U'$. $U$ is non-principal, and we can take $\alpha < \mu$ with $U \in Z_\alpha$. Then $\langle U, U' \rangle \in \tilde{W}^a_{\kappa, (\alpha)} \times \tilde{W}^b_{\kappa, (\alpha)}$, and $(\tilde{W}^a_{\kappa, (\alpha)} \times \tilde{W}^b_{\kappa, (\alpha)}) \cap \Delta = \emptyset$. \qed

**Corollary 4.11.** $\mathcal{C}$ forces the following: For every uncountable cardinal $\kappa$, $\kappa$ is the least measurable cardinal if and only if $\kappa$ is the least cardinal such that $e(X^2) \leq \kappa$ for every regular $T_1$ Lindelöf space $X$.

**Proof.** Take a $(V, \mathcal{C})$-generic $G$, and work in $V[G]$. Let $\kappa_0$ be the least measurable cardinal, and $\kappa_1$ the least cardinal $\kappa$ satisfying $e(X^2) \leq \kappa$ for every regular $T_1$ Lindelöf space $X$.

By Lemma 4.3, we have $\kappa_1 \leq \kappa_0$. If $\kappa_1 < \kappa_0$, then, in $V$, there is no measurable cardinal $\leq \kappa^+_1$. Hence by Lemmas 4.9 and 4.10, there is a regular $T_1$ Lindelöf space $X$ such that $e(X^2) \geq \kappa^+_1$. This contradicts the definition of $\kappa_1$, and we have $\kappa_1 = \kappa_0$. \qed

**Remark 4.12.** As the (weak) Lindelöf degree, the square $X^2$ can be replaced by any $X^{n+1}$, that is, for every positive $n < \omega$, $\mathcal{C}$ forces the following:
For every uncountable cardinal $\kappa$, $\kappa$ is the least measurable cardinal if and only if $\kappa$ is the least cardinal such that $e(X^{n+1}) \leq \kappa$ for every regular $T_1$ Lindelöf space $X$ with $X^n$ Lindelöf.

**Question 4.13.** In ZFC, is the least measurable cardinal the supremum of the extents of the squares of Lindelöf spaces?

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