THE $\ell$-ADIC HYPERGEOMETRIC FUNCTION AND ASSOCIATORS

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Abstract. We introduce an $\ell$-adic analogue of Gauss’s hypergeometric function arising from the Galois action on the fundamental torsor of the projective line minus three points. Its definition is motivated by a relation between the KZ-equation and the hypergeometric differential equation in the complex case. We show two basic properties, analogues of Gauss’s hypergeometric theorem and of Euler’s transformation formula for our $\ell$-adic function. We prove them by detecting a connection of a certain two-by-two matrix specialization of even unitary associators with the associated gamma function, which extends the result of Ohno and Zagier.

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0. Introduction

It is said that the hypergeometric function first appeared in a book by Wallis (1655). Since then, the hypergeometric function has attracted widespread attention in various areas of mathematics. In this paper we introduce a new variant, the $\ell$-adic hypergeometric function ($\ell$: an odd prime). It is an $\ell$-adic function parametrized by the absolute Galois group, which could be regarded as an ‘$\ell$-adic Galois avatar’ of the hypergeometric function. This is not the finite analogue of hypergeometric functions considered the literature (see [20, 21, 24, 32], etc). It also differs from Dwork’s $p$-adic hypergeometric function ([10]), which is rather a ‘crystalline avatar’. However, one might expect any intimate relationship with these functions by discussing any possibly common motives.

Our construction of the $\ell$-adic hypergeometric function is motivated by the following:

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In the \( \ell \)-adic étale setting, Wojtkowiak \((39)\) and Nakamura-Wojtkowiak \((29)\) introduced and explored the \( \ell \)-adic polylogarithm, which is associated with the \( \ell \)-adic Galois representation on the fundamental torsor of the projective line minus three points. It is an \( \ell \)-adic function parametrized by the absolute Galois group and topological paths. It is considered as an \( \ell \)-adic Galois avatar of Coleman’s \( p \)-adic polylogarithm \((9)\).

In the complex case, intimate relations between the KZ-equation and the hypergeometric differential equation have been discussed in the literature; see \([33, 35]\), etc. In particular, Oi presented a clear formulation in \([31]\), where by reconstructing the hypergeometric function from the fundamental solution \((1.5)\) of the KZ-equation (cf. \((1.5)\)), he deduced various relations among multiple polylogarithms appearing as its coefficients.

Our strategy is to (i) consider the \( \ell \)-adic Galois cocycle \( f_\sigma^\gamma \) (see \((0.1)\) below) associated with the same \( \ell \)-adic Galois representation, (ii) regard its image \( G_\ell^{\pi}(e_0, e_1)(\sigma)(z) \) under the fake comparison isomorphism (Definition \((2.6)\)) constructed by associators as an \( \ell \)-adic analogue of the fundamental solution and (iii) then extract the \( \ell \)-adic hypergeometric function as the \((1, 1)\) entry of the \( 2 \times 2 \)-matrix in the same way as \((1.5)\):

(i). Let \( \bar{Q} \) be the algebraic closure of the rational number field \( Q \) and \( G_\bar{Q} := \text{Gal}(\bar{Q}/Q) \) be the absolute Galois group. We fix an embedding \( Q \hookrightarrow \mathbb{C} \). We consider the algebraic curve \( X := \mathbb{P}^1 \setminus \{0, 1, \infty\} \) over \( Q \), the projective line minus the three points. The topological fundamental group \( \pi_1^{\text{top}}(X(\mathbb{C}), \bar{01}) \) of its associated topological space \( X(\mathbb{C}) \) with the tangential basepoint \( \bar{01} \) (cf. \((7)\)) is identified with the free group \( F_2 \) with the standard generators \( x_0, x_1, x_\infty \) corresponding to the loops around \( 0, 1 \) and \( \infty \), such that \( x_0 x_1 x_\infty = 1 \).

Let \( \ell \) be an odd prime. Let \( z \) be a rational (or tangential base) point of \( X \). We denote by \( \pi_1^{\text{top}}(X_Q; \bar{01}, z) \) the profinite set of pro-\( \ell \) étale paths from \( \bar{01} \) to \( z \) (cf. \((22)\)). By the comparison isomorphisms induced by the fixed embedding \( Q \hookrightarrow \mathbb{C} \), we identify the pro-\( \ell \) étale fundamental group \( \hat{\pi}_1^{\text{top}}(X_Q; \bar{01}) := \hat{\pi}_1^{\text{top}}(X_Q; \bar{01}, 01) \) with the pro-\( \ell \) completion \( \hat{F}_2^\ell \) of \( F_2 \), and we regard each topological path \( \gamma_z : \bar{01} \sim z \) (that signifies \( \gamma_z \in \pi_1^{\text{top}}(X(\mathbb{C}); \bar{01}, z) \)) as a pro-\( \ell \) étale path \( \gamma_z \in \hat{\pi}_1^{\text{top}}(X_Q; \bar{01}, z) \). Since \( X \) and \( z \) are defined over \( Q \), the set \( \hat{\pi}_1^{\text{top}}(X_Q; \bar{01}, z) \) admits the action of \( G_\bar{Q} \). For each \( \sigma \in G_\bar{Q} \), we consider the Galois 1-cocycle

\[
(0.1) \quad f_\sigma^\gamma := \gamma_z^{-1} \sigma(\gamma_z) \in \hat{F}_2^\ell.
\]

(ii). We take an \( \ell \)-adic even unitary associator \( \phi \) (Definition \((2.1)\)). It provides the fake comparison isomorphism \( \text{comp}^{\bar{Q}}_\phi \) (Definition \((2.6)\)). Under the map \( \iota_\phi \) in \((2.3)\), the restriction of \( \text{comp}^{\bar{Q}}_\phi \) by the inclusion \( \hat{F}_2^\ell(\ell) \rightarrow F_2(\mathbb{Q}_\ell) \), our \( \ell \)-adic analogue of the fundamental solution is defined to be the noncommutative formal power series

\[
G_\ell^{\pi}(e_0, e_1)(\sigma)(z) := \iota_\phi(f_\sigma^\gamma) \in \mathbb{Q}_\ell[[e_0, e_1]].
\]

(iii). Let \( a, b, c \) be variables. Put \( p = 1 - c, q = a + b + 1 - c = a + b + p \) and

\[
X = \begin{pmatrix} 0 & b \\ 0 & p \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ a & q \end{pmatrix} \in \text{Mat}_2(\mathbb{Q}_\ell[a, b, c - 1]).
\]

Following \((1.5)\), we define the formal version of \( \ell \)-adic hypergeometric function to be the \((1, 1)\) entry of the above \( G_\ell^{\pi}(e_0, e_1)(\sigma)(z) \) with substitution at \( e_0 = X \) and
\[ e_1 = -Y: \]
\[ (0.2) \]
\[ 2F_1 \left( \frac{a, b}{c} \middle| z \right) (\sigma) = 2F_1 \left( \frac{a, b}{c} \middle| \gamma_z \right) (\sigma) := [G_{\mathbb{Q}^p}^c(X, -Y)(\sigma)(z)]_{(1, 1)} \in \mathbb{Q}_\ell[[a, b, c - 1]] \]

(another formulation is given in Proposition \[3.5\].)

Our first result is on the well-definedness of the \( \ell \)-adic hypergeometric function.

**Theorem 0.1.** (i). The definition of \( 2F_1 \left( \frac{a, b}{c} \middle| z \right) (\sigma) \) is independent of any choice of \( \ell \)-adic even unitary associator \( \varphi \).

(ii). Suppose that \( a, b, c \in \mathbb{Z}_\ell \) are with
\[ |a|_{\ell}, |b|_{\ell}, |c - 1|_{\ell} < 1. \]
Then, \( 2F_1 \left( \frac{a, b}{c} \middle| z \right) (\sigma) \) converges.

So our \( \ell \)-adic hypergeometric function may be regarded to be the map
\[ 2F_1 \left( \frac{a, b}{c} \middle| \right) (\sigma) : \Pi \pi_1^{top}(\mathcal{X}(\mathbb{C}); \tilde{0}1, z) \to \mathbb{Q}_\ell \]
sending \( \gamma_z \mapsto 2F_1 \left( \frac{a, b}{c} \middle| \gamma_z \right) \) where \( z \) runs over rational (may tangential base) points of \( \mathcal{X} \).

Our second main theorem is an analogue of the Gauss hypergeometric theorem \[1.3\].

**Theorem 0.2 (\( \ell \)-adic Gauss hypergeometric theorem).** Take \( a, b, c \in \mathbb{Z}_\ell \) which satisfy \[0.3\]. Put \( p = 1 - c, q = a + b + 1 - c = a + b + p. \) For \( \sigma \in G_{\mathbb{Q}}, \) the following equality holds:
\[ (0.4) \]
\[ 2F_1 \left( \frac{a, b}{c} \middle| \right) (\sigma) = \frac{pq}{ab} \left\{ \Gamma_+ \left( \frac{-p, -q}{-p - a, -p - b} \right) + \Gamma_+ \left( \frac{-p, -q}{-a, -b} \right) \right\} \Gamma_+ \left( \frac{-p, q}{a, b} \right) + \left\{ \frac{ab + pq}{pq} \Gamma_+ \left( \frac{p, q}{p + a, p + b} \right) - \frac{ab}{pq} \Gamma_+ \left( \frac{-p, q}{a, b} \right) \right\} \Gamma_+ \left( \frac{-p, -q}{-a, -b} \right) \cdot \Gamma_+ \left( \frac{-p, -q}{-a, -b} \right) \cdot \Gamma_+ \left( \frac{-p, -q}{-a, -b} \right). \]

Here, the path \( \gamma : \tilde{0}1 \sim \tilde{0} \) is chosen to be the straight path (often denoted by \( \text{dch} \) in the literature), \( \Gamma_\sigma \) is defined by the \( \ell \)-adic series (cf. Remark \[2.7\]) related to the hyperadelic gamma function of Anderson \( \Gamma_+ \) and \( \Gamma_\sigma \) is defined by the \( \cdot + \)-part of the classical gamma function (cf. Notation \[3.4\]).

The condition \[0.3\] for \( a, b, c \) in \( 2F_1 \left( \frac{a, b}{c} \middle| z \right) (\sigma) \) will be relaxed to \( (a, b, c) \) in
\[ \mathfrak{D} := \{(a, b, c) \in \mathbb{Z}_\ell^2 \times \mathbb{Z}_\ell^* \mid (a, b, c) \equiv (0, 0, 1), (0, 0, 0) \text{ or } (0, 1, 1) \text{ mod } \ell\} \]
in Proposition \[4.5\]. Our third main theorem is an analogue of Euler’s transformation formula \[1.4\].

**Theorem 0.3 (\( \ell \)-adic Euler transformation formula).** Let \( a, b, c \in \ell\mathbb{Z}_\ell \) with \( c \neq 0. \) Let \( z \) be a rational or tangential base point of \( \mathcal{X} \). Then, for \( \sigma \in G_{\mathbb{Q}}, \) we have the equality
\[ (0.5) \]
\[ 2F_1 \left( \frac{a, b}{c} \middle| z \right) (\sigma) = \exp \{(c - a - b)\rho_1 - z(\sigma)\} \cdot \rho_1 \left( \frac{c - a - b}{c} \right) (\sigma) \]
with the Kummer 1-cocycle $\rho_{1-z} : G_{\mathbb{Q}} \to \mathbb{Z}_\ell$ defined by $\sigma((1-z)^{1/\ell^n}) = \zeta_{\ell^n\sigma}^{\rho_{1-z}(\sigma)}(1-z)^{1/\ell^n}$ for $n \in \mathbb{N}$ where $\zeta_{\ell^n} = \exp\left\{2\pi i \sqrt{-1/\ell^n}\right\}$ and the $\ell^n$-th root $(1-z)^{1/\ell^n}$ is chosen along $\gamma_z$ with $1^{1/\ell^n} = 1$ at $z = 0$.

The above two theorems are based on the computations of Oi [31] and the following key result on associators.

**Theorem 0.4.** The following equality holds for any even unitary associator $\varphi$ (see Definition 2.1):

$$\varphi(X,-Y) = M_+$$

where $M_+$ is the matrix defined in Notation 3.4.

The above theorem is derived from Theorem 3.3 where we observe the Ohno-Zagier relation [30] in the (1,1) entry. Note that similar but different matrix specializations of $f_{\vec{x}}$ with $z = 10^1/2, e^{2\pi i/6}$ and their connections with various arithmetic invariants are investigated in [26, 27, 28].

The remainder of this paper proceeds as follows: In §1 we review how the fundamental solutions of the Gauss hypergeometric differential equation are obtained from those of the KZ equation. The arguments are transformed to the $\ell$-adic setting, and $\ell$-adic variants of the solutions are constructed in §2. We prove the fundamental theorem (Theorem 0.1), the $\ell$-adic Gauss hypergeometric theorem (Theorem 0.2), the above key theorem (Theorem 0.4) in §3, and then the $\ell$-adic Euler transformation formula (Theorem 0.3) is proved in §4.

1. **KZ equation and the hypergeometric function**

In this section, we recall the basic properties of the fundamental solutions of the KZ equation and of the hypergeometric equation (cf. [13, §4.2]), which play a role in the subsequent sections.

The (formal) **KZ (Knizhnik-Zamolodchikov) equation** is the differential equation

$$\frac{d}{dz} G(z) = \left\{ \frac{e_0}{z} + \frac{e_1}{z-1} \right\} \cdot G(z)$$

where $G(z)$ is analytic (that is, each coefficient is analytic) in complex variables with values in the noncommutative formal power series ring $\mathbb{C}((e_0,e_1))$. It has singularities at $z = 0, 1$ and $\infty$ that are regular and Fuchsian. In $\S$3, Drinfeld considers the fundamental solution

$$G_{01}(e_0,e_1)(z)$$

with the asymptotic property $G_{01}(e_0,e_1)(z)z^{-e_0} \to 1$ when $z \in \mathbb{R}_+$ approaches 0. Multiple polylogarithms appear as its coefficients (cf. [13, 15]). He further investigates 5 other solutions with certain specific asymptotic properties, which are
described as

\[
\begin{align*}
G_{01}(e_0, e_1)(z) &= G_{01}(e_1, e_0)(1 - z), \\
G_{1\infty}(e_0, e_1)(z) &= G_{01}(e_1, e_\infty)(1 - \frac{1}{z}), \\
G_{\infty 1}(e_0, e_1)(z) &= G_{01}(e_\infty, e_1)(\frac{1}{z}), \\
G_{\infty 0}(e_0, e_1)(z) &= G_{01}(e_\infty, e_0)(\frac{1}{1 - z}), \\
G_{0\infty}(e_0, e_1)(z) &= G_{01}(e_0, e_\infty)(\frac{z}{z - 1})
\end{align*}
\]

with \(e_\infty = -e_0 - e_1\) under appropriate choice of branches.

Gauss’s hypergeometric function (consult [2] for example) is the complex analytic function defined by the power series

\[
2F_1 \left( \begin{array}{c} a, b \\ c \\ \end{array} \right) \mid z \right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n
\]

which converges for \(|z| < 1\). Here, \(a, b, c\) are complex numbers with \(c \neq 0, -1, -2, \ldots\), and \((s)_n = s(s+1)\cdots(s+n-1)\) are the Pochhammer symbols. The hypergeometric function has a rich history. The contributions of Euler and Gauss are significant, and in particular, the following identities might be the most celebrated:

- Hypergeometric theorem (due to Gauss)

\[
2F_1 \left( \begin{array}{c} a, b \\ c \\ \end{array} \right) \mid 1 \right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},
\]

- Transformation theorem (due to Euler)

\[
2F_1 \left( \begin{array}{c} a, b \\ c \\ \end{array} \right) \mid z \right) = (1 - z)^{c-a-b} 2F_1 \left( \begin{array}{c} c-a, c-b \\ c \\ \end{array} \right) \mid \frac{z}{1-z} \right).
\]

The hypergeometric function is a solution of Euler’s hypergeometric differential equation:

\[
z(1-z) \frac{d^2 w}{dz^2} + \{c - (a + b + 1)z\} \frac{dw}{dz} - abw = 0,
\]

which allows the analytic continuation of the function along a topological path starting from 0 to any \(z\) in \(X(\mathbb{C})\). The differential equation can be reformulated as

\[
 \frac{d}{dz} \vec{v} = \left\{ \frac{1}{z} X_0 + \frac{1}{1 - z} Y_0 \right\} \cdot \vec{v}
\]

where we define \(X_0 = \begin{pmatrix} 0 & b \\ 0 & u \end{pmatrix}\) and \(Y_0 = \begin{pmatrix} 0 & 0 \\ a & v \end{pmatrix} \in \text{Mat}_2(\mathbb{C})\) with \(u = 1 - c\) and \(v = a + b + 1 - c\) and, put \(\vec{v} = \vec{v}(w) = \left( \begin{array}{c} w \\ \frac{dw}{dz} \end{array} \right) \in \mathbb{C}^2\) when \(b \neq 0\). Whence a solution \(G(e_0, e_1)(z)\) of the KZ-equation yields two solutions \(w_1, w_2\) with \((\vec{v}(w_1), \vec{v}(w_2)) = G(X_0, -Y_0)(z)\) of the above differential equation when it converges as explained in [31]. Specifically, we have

\[
2F_1 \left( \begin{array}{c} a, b \\ c \\ \end{array} \right) \mid z \right) = [G_{01}(X_0, -Y_0)(z)]_{11}
\]
where the right lower suffix 11 means the (1, 1) entry. By writing

\begin{equation}
(1.6) \quad \mathcal{V}_{i0}(z) := G_{0i}(X_0, -Y_0)(z) \cdot \left( \begin{array}{cc}
1 & 1 \\
0 & \frac{a}{b}
\end{array} \right),
\end{equation}

\begin{equation}
(1.7) \quad \begin{array}{l}
(1, 0) \cdot \mathcal{V}_{01}(z) = \binom{2F_1}{a,b}(1 - z) + \binom{2F_1}{b+1-c,a+c+1,2-c} z, \\
(1, 0) \cdot \mathcal{V}_{i0}(z) = \binom{2F_1}{a,b}(1 - z) + \binom{2F_1}{c-a-b,2F_1(a-c-b)} z, \\
(1, 0) \cdot \mathcal{V}_{1\infty}(z) = \binom{2F_1}{a,b}(1 - z) + \binom{2F_1}{b+1-c,a+c+1,2-c} z, \\
(1, 0) \cdot \mathcal{V}_{\infty1}(z) = \binom{2F_1}{a,b}(1 - z) + \binom{2F_1}{b+1-c,a+c+1,2-c} z,
\end{array}
\end{equation}

we recover certain scalar multiples of half of the so-called Kummer’s 24 solutions

\begin{equation}
\begin{array}{l}
(1, 0) \cdot \mathcal{V}_{\infty0}(z) = \binom{2F_1}{a,b}(1 - z) + \binom{2F_1}{b+1-c,a+c+1,2-c} z, \\
(1, 0) \cdot \mathcal{V}_{0\infty}(z) = \binom{2F_1}{a,b}(1 - z) + \binom{2F_1}{b+1-c,a+c+1,2-c} z,
\end{array}
\end{equation}

where we consider the branch by the principal value under appropriate conditions

\begin{equation}
\text{for } a, b, c. \text{ We note that the other half of the 24 solutions can be obtained by Euler’s transformation formula.}
\end{equation}

2. \ell-adic analogues of the six solutions

We recall Drinfeld’s definition ([9]) of associators and the Grothendieck-Teichmüller group. Then, we introduce \ell-adic analogues of the six solutions of the KZ-equation and of the hypergeometric equation discussed in [11].

Let \(\mathbb{K}\) be a field with characteristic 0. Let \(\mathfrak{f}_2\) be the free Lie algebra over \(\mathbb{K}\) with two variables \(e_0\) and \(e_1\) and \(U\mathfrak{f}_2 := \mathbb{K}\langle e_0, e_1 \rangle\) be its universal enveloping algebra. We denote \(\mathfrak{f}_2\) and \(\widehat{U}\mathfrak{f}_2 := \mathbb{K}\langle\hat{e}_0, \hat{e}_1\rangle\) to be their completions by degrees. A word means a monic monomial element (including 1) in \(\widehat{U}\mathfrak{f}_2\), and for each \(\varphi\) in \(\widehat{U}\mathfrak{f}_2\), we denote \((\varphi|W)\) to be the coefficient of \(\varphi\) in \(W\).

\textbf{Definition 2.1 ([9] [17]). (1).} The set \(M(\mathbb{K})\) of associators is a collection of pairs \((\mu, \varphi)\) with \(\mu \in \mathbb{K}^\times\) and \(\varphi \in \widehat{U}\mathfrak{f}_2\) satisfying the following:

- the condition on the quadratic term: \((\varphi|e_0e_1) = \frac{a^2}{2\lambda},\)
- the commutator group-like condition: \(\varphi \in \exp[\hat{\mathfrak{f}}_2, [\hat{\mathfrak{f}}_2, \hat{\mathfrak{f}}_2]],\)
- the pentagon equation: \(\varphi_{345}\varphi_{512}\varphi_{234}\varphi_{451}\varphi_{123} = 1\) in \(\widehat{\mathfrak{g}}\).
Here, \( \exp[\hat{f}_2, \hat{f}_2] \) is the image of the topological commutator \([\hat{f}_2, \hat{f}_2]\) of \( \hat{f}_2 \) under the exponential map, and \( \mathcal{P}_5 \) is the Lie algebra generated by \( t_{ij} \) \((i, j \in \mathbb{Z}/5)\) with the relations

- \( t_{ij} = t_{ji}, \ t_{ii} = 0, \)
- \( \sum_{j \in \mathbb{Z}/5} t_{ij} = 0 \) \((\forall i \in \mathbb{Z}/5)\),
- \( [t_{ij}, t_{kl}] = 0 \) \(\text{for} \ \{i, j\} \cap \{k, l\} = \emptyset.\)

For \( i, j, k \in \mathbb{Z}/5, \varphi_{ijk} \) means the image of \( \varphi \) under the embedding \( \hat{U}_f \rightarrow \hat{U}_{\mathcal{P}_5} \) sending \( e_0 \mapsto t_{ij} \) and \( e_1 \mapsto t_{jk} \). A pair \((\mu, \varphi)\) is called a rational (resp. \( \ell \)-adic) associator when \( \mathbb{K} \) is taken to be \( \mathbb{Q} \) (resp. \( \mathbb{Q}_\ell \)). A series \( \varphi \in \hat{U}_f \) is called a unitary associator when \((1, \varphi)\) forms an associator. A unitary associator is called even when \( \varphi(-e_0, -e_1) = \varphi(e_0, e_1) \) holds.

(2). The Grothendieck-Teichmüller group \( \text{GT}(\mathbb{K}) \) is the set of collections of pairs \((\lambda, f)\) with \( \lambda \in \mathbb{K}^\times \) and \( f \in F_2(\mathbb{K}) \) satisfying the following:

- the condition on the quadratic term: \( (f(e^\sigma, e^\tau)e_0e_1) = \frac{\lambda^2 - 1}{24} \)
- the commutator group-like condition: \( f \in [F_2(\mathbb{K}), F_2(\mathbb{K})] \)
- the pentagon equation: \( f(x_{12}, x_{23}^*) f(x_{34}, x_{45}^*) f(x_{51}, x_{12}^*) f(x_{23}, x_{34}^*) f(x_{45}, x_{51}^*) = 1. \)

Here we denote the Malcev completion of \( F_2 \) by \( F_2(\mathbb{K}) \) and its topological commutator by \([F_2(\mathbb{K}), F_2(\mathbb{K})]\). For any group homomorphism \( F_2(\mathbb{K}) \rightarrow H \) sending \( x_0 \mapsto \alpha \) and \( x_1 \mapsto \beta \), the symbol \( f(\alpha, \beta) \) stands for the image of each \( f \in F_2(\mathbb{K}) \). The last equation is in the Malcev completion \( P_5^0(\mathbb{K}) \) of the pure sphere braid group \( P_5^0 \) with 5 strings, and \( x_1^* \) is the standard generators (cf. [11]).

(3). In [9], it is shown that the above set-theoretically defined \( \text{GT}(\mathbb{K}) \) indeed forms a group whose product is induced from that of \( \text{Aut} F_2(\mathbb{K}) \) and is given by

\( (\lambda_1, f_1) \circ (\lambda_2, f_2) := \left( \lambda_2 \lambda_1, f_1 f_2 x_1^{\lambda_1} f_2^{\lambda_2} y^{\lambda_2} x_2^{\lambda_2} f_2 \right). \)

The associator set \( M(\mathbb{K}) \) forms a left \( \text{GT}(\mathbb{K}) \)-torsor by

\( (\lambda, f) \circ (\mu, \varphi) := \left( \lambda \mu, f(\varphi^A, \varphi^B) \cdot \mu \right) \)

for \( (\mu, \varphi) \in M(\mathbb{K}) \) and \((\lambda, f) \in \text{GT}(\mathbb{K})\).

**Remark 2.2.** In [17], it is shown that the above definition of associators (resp. the Grothendieck-Teichmüller group) implies the so-called 2-cycle relation

\( \varphi(e_0, e_1) \varphi(e_1, e_0) = 1 \) (resp. \( f(x_0, x_1)f(x_1, x_0) = 1 \))

and the 3-cycle relation

\( e^{\frac{\mu}{m}} \varphi(e_\infty, e_0) e^{\frac{\mu}{m}} \varphi(e_1, e_0) e^{\frac{\mu}{m}} \varphi(e_0, e_1) = 1 \) (resp. \( f(x_\infty, x_0)x_\infty^m f(x_1, x_\infty)x_1^m f(x_0, x_1)x_0^m = 1 \) with \( m = \frac{\lambda - 1}{2} \)),

which are originally imposed in the definition of \( M(\mathbb{K}) \) and \( \text{GT}(\mathbb{K}) \).

**Remark 2.3 (2).** Rational even unitary associators exist.

(2). The generating series \( \Phi_{KZ} \) of multiple zeta values, that is defined by \( \Phi_{KZ} = G_{10}(e_0, e_1)(z)^{-1}G_{01}(e_0, e_1)(z) \) forms an associator with \( \mu = 2\pi \sqrt{-1} \) and \( \mathbb{K} = \mathbb{C} \).

(3). The Galois action of \( G_{Q} \) on \( F_2^k = \pi_1^k(\mathbb{Q}^\times; 0\hat{1}) \) (in §10) induces a homomorphism

\( G_{Q} \rightarrow \text{GT}(\mathbb{Q}_\ell). \)
The associated gamma function is one of our main tools:

**Definition 2.4** (cf. [11]). For an associator \((\mu, \varphi)\), the associated gamma function is defined to be

\[ \Gamma_\varphi(t) := \exp \left\{ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (\varphi|e_0^{n-1} e_1) t^n \right\} \in \mathbb{K}[t]. \]

Similarly for \((\lambda, f)\), it is defined to be

\[ \Gamma_f(t) := \exp \left\{ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (f(e^{\ell_0}, e^{\ell_1})|e_0^{n-1} e_1) t^n \right\} \in \mathbb{K}[t]. \]

**Remark 2.5.**

1. When \((\mu, \varphi) = (2\pi\sqrt{-1}, \Phi_{KZ})\), it is equal to

\[ e^{\gamma t} \Gamma(1 + t) = \exp \left\{ \sum_{n=2}^{\infty} \frac{\zeta(n)}{n} t^n \right\} \]

where \(\Gamma\) is the classical gamma function and \(\gamma\) is the Euler-Mascheroni constant.

2. When \(\varphi\) is an even unitary associator, we have \(\Gamma_\varphi(t) = \Gamma_+ (t)\) (cf. [11] Lemma 9.5).

3. Let \(\sigma \in G_{\mathbb{Q}}\), which corresponds to \((\lambda_\sigma, f_\sigma) \in G\mathbb{T}(\mathbb{Q}^\ell)\) under (2.1). Write \(\Gamma_\sigma(t) := \Gamma_{f_\sigma}(t)\). Then, it is calculated to be

\[ \Gamma_\sigma(t) = \exp \left\{ \sum_{m>1} \kappa_m^\sigma (\ell) t^m \right\} \]

\((\kappa_m^\sigma (\ell))\) is the \(\ell\)-adic \(m\)-th Soulé cocycle when \(m\) is odd), which is the series related to Anderson’s ([11]) hyperadelic gamma function (consult [23] and [16]).

4. When \((\mu', \varphi') = (\lambda, f) \otimes (\mu, \varphi)\), we have

\[ \Gamma_{\varphi'}(t) = \Gamma_{f_\mu}(t) \Gamma_\varphi(t) \] (2.2)

by definition.

Isomorphisms between \(F_2(\mathbb{K})\) and \(\exp \hat{f}_2\) deduced by associators are discussed in the literature (cf. [3, 11]):

**Definition 2.6.** For an associator \((\mu, \varphi)\), the fake comparison isomorphism

\[ \text{comp}^\circ_\varphi : F_2(\mathbb{K}) \to \exp \hat{f}_2 \]

is the isomorphism given by \(x \mapsto \exp(\mu x_0)\) and \(y \mapsto \varphi^{-1} \exp(\mu x_1) y\).

**Remark 2.7.** When \((\mu, \varphi) = (2\pi\sqrt{-1}, \Phi_{KZ})\), it agrees with the Betti-de Rham comparison isomorphism of the motivic fundamental group \(\pi^M_1(X, \mathbb{1})\) (cf. [7]).

Under the natural inclusion \(F_2^\ell \hookrightarrow F_2(\mathbb{Q}^\ell)\), we regard \(F_2^\ell\) as a topological subgroup of \(F_2(\mathbb{Q}^\ell)\) (equipped with the \(\ell\)-adic topology). For an \(\ell\)-adic unitary associator \(\varphi\), we define

\[ \iota_{\varphi} := \text{comp}^\circ_\varphi |_{F_2^\ell}, \]

that is, the continuous group homomorphism

\[ \iota_{\varphi} : \hat{F}_2^\ell \to F_2^{\text{DR}}(\mathbb{Q}^\ell) \quad (\subset \mathbb{Q}^\ell((e_0, e_1)^\times)) \]
sending \(x_0 \mapsto e^{x_0}\) and \(x_1 \mapsto \varphi^{-1} e^{x_1} \varphi\). Then, we have \(\iota_{\varphi}(x_\infty) = \text{Ad} \left( \varphi(x_0, x_\infty) e^{-\frac{x_0}{2}} \right)^{-1} (e^{x_\infty})\) by the 2- and 3-cycle relations for \(\varphi\). Here, for invertible elements \(u\) and \(v\), we mean \(\text{Ad}(u)(v) = uvu^{-1}\). We write

\[
G_{01}^e(x_0, e_1)(\sigma)(z) := \iota_{\varphi}(f_{\sigma}^e) \in \mathbb{Q}_e\langle \{e_0, e_1\}\rangle.
\]

Following (1.2) in the complex case, we consider

\[
\begin{align*}
G_{10}^e(x_0, e_1)(\sigma)(z) &:= G_{01}^e(x_1, e_0)(\sigma)(1 - z), \\
G_{1\infty}^e(x_0, e_1)(\sigma)(z) &:= G_{01}^e(x_1, e_\infty)(\sigma)(1 - \frac{1}{z}), \\
G_{1\infty}^e(x_0, e_1)(\sigma)(z) &:= G_{01}^e(e_\infty, e_1)(\sigma)(\frac{1}{z}), \\
G_{1\infty}^e(x_0, e_1)(\sigma)(z) &:= G_{01}^e(e_\infty, e_0)(\sigma)(\frac{1}{1 - z}), \\
G_{0\infty}^e(x_0, e_1)(\sigma)(z) &:= G_{01}^e(e_0, e_\infty)(\sigma)(\frac{z}{z - 1}),
\end{align*}
\]

which we regard as \(\ell\)-adic analogues of six solutions of the \(KZ\)-equation.

**Remark 2.8.** We note that the \(\ell\)-adic polylogarithms discussed in [29, 30] can be extracted as coefficients of coefficients of \(G_{01}^e(x_0, e_1)(\sigma)(z)\).

Let \(a, b, c\) be variables. Write \(p = 1 - c, q = a + b + 1 - c = a + b + p\) and

\[
X = \begin{pmatrix} 0 & b \\ 0 & p \end{pmatrix}, \quad Y = \begin{pmatrix} a & 0 \\ b & q \end{pmatrix} \in \text{Mat}_2(\mathcal{K}[a, b, c - 1]).
\]

Write \(\mathcal{R} := \mathbb{Q}_e[[a, b, c - 1]]\). Let \(\varphi\) be an (even) unitary associator. Following (1.6), we consider \(\ell\)-adic analogues of six solutions of the hypergeometric equation:

\[
\begin{align*}
\psi_{01}^e(\sigma)(z) &:= G_{01}^e(X, -Y)(\sigma)(z) \cdot \begin{pmatrix} 1 & 1 \\ 0 & b \end{pmatrix} \in \text{GL}_2(\mathcal{R}[\frac{1}{b}]), \\
\psi_{10}^e(\sigma)(z) &:= G_{10}^e(X, -Y)(\sigma)(z) \cdot \begin{pmatrix} 1 & 0 \\ q & 0 \end{pmatrix} \in \text{GL}_2(\mathcal{R}[\frac{1}{bq}]), \\
\psi_{1\infty}^e(\sigma)(z) &:= G_{1\infty}^e(X, -Y)(\sigma)(z) \cdot \begin{pmatrix} 1 & 0 \\ \frac{1}{q} & \frac{1}{bq} \end{pmatrix} \in \text{GL}_2(\mathcal{R}[\frac{1}{bq}]), \\
\psi_{1\infty}^e(\sigma)(z) &:= G_{1\infty}^e(X, -Y)(\sigma)(z) \cdot \begin{pmatrix} 1 & 0 \\ \frac{1}{q} & \frac{1}{bq} \end{pmatrix} \in \text{GL}_2(\mathcal{R}[\frac{1}{b}])), \\
\psi_{0\infty}^e(\sigma)(z) &:= G_{0\infty}^e(X, -Y)(\sigma)(z) \cdot \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \in \text{GL}_2(\mathcal{R}[\frac{1}{b}]), \\
\psi_{0\infty}^e(\sigma)(z) &:= G_{0\infty}^e(X, -Y)(\sigma)(z) \cdot \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \in \text{GL}_2(\mathcal{R}[\frac{1}{b}]).
\end{align*}
\]

These matrices will be employed to relax our assumption (0.3) and to show Euler’s transformation formula for the \(\ell\)-adic hypergeometric function in §4.

3. \(\ell\)-adic Gauss hypergeometric theorem and Ohno-Zagier relation

In this section, we present a proof of the fundamental theorem (Theorem 0.1) and the \(\ell\)-adic Gauss hypergeometric theorem (Theorem 0.2). To do so, we present a key result (Theorem 3.3) that the evaluation of the two-by-two matrices \(X\) and \(Y\) for any
Notation 3.1. For an associator \((\mu, \varphi) \in \mathbb{K}^\times \times \hat{U}_{2}\), we write
\[
\Gamma_\varphi(s, t, u, v) := \frac{\Gamma_\varphi(s)\Gamma_\varphi(t)}{\Gamma_\varphi(u)\Gamma_\varphi(v)} \in \mathbb{K}[[s, t, u, v]]
\]
where \(\Gamma_\varphi\) is the associated gamma function (cf. Definition 2.4). We consider the
associated 2 \times 2 matrix
\[
M_\varphi := \begin{pmatrix} \frac{1}{a} & 0 \\ -\frac{b}{a} & \frac{1}{b} \end{pmatrix} C_\varphi \begin{pmatrix} 1 & -\frac{a}{b} \\ 0 & \frac{1}{b} \end{pmatrix} \in \text{GL}_2(\mathbb{K}[\mathbb{K}[a, b, c - 1]]/[p^{-1}, q^{-1}])
\]
with \(p = 1 - c\), \(q = a + b + 1 - c = a + b + p\) and
\[
C_\varphi := \begin{pmatrix} \Gamma_\varphi(-p, q) & \Gamma_\varphi(-p, q) \\ ab \eta \Gamma(\varphi(a, b) & \Gamma_\varphi(a, b) \end{pmatrix} \begin{pmatrix} \Gamma_\varphi(-p, q) \\ \Gamma_\varphi(a, b) \end{pmatrix} \cdot
\]

Proposition 3.2. We have \(M_\varphi \in \text{SL}_2(\mathbb{K}[a, b, c - 1])\).

Proof. The matrix \(M_\varphi\) is calculated to be
\[
M_\varphi = \begin{pmatrix} \frac{1}{a} & 0 \\ -\frac{b}{a} & \frac{1}{b} \end{pmatrix} C_\varphi \begin{pmatrix} 1 & -\frac{a}{b} \\ 0 & \frac{1}{b} \end{pmatrix} \frac{b}{p} \left\{ \begin{array}{cc} \Gamma_\varphi(-p, q) & \Gamma_\varphi(-p, q) \\ \Gamma_\varphi(-p, q) & \Gamma_\varphi(-p, q) \end{array} \right\} \]
with
\[
[M_\varphi]_{22} = \frac{(a + p)(b + p)}{pq} \Gamma_\varphi(p + a, p + b) + \frac{ab}{pq} \Gamma_\varphi(-p, q) - \Gamma_\varphi(-p, q) - \Gamma_\varphi(-p, q).
\]

It is clear that the (1, 1) entry \([M_\varphi]_{11}\), the (1, 2) entry \([M_\varphi]_{12}\) and the (2, 1) entry
\([M_\varphi]_{21}\) are in \(\mathbb{K}[a, b, c - 1]\). Thanks to the identity
\[
\Gamma_\varphi(t)\Gamma_\varphi(-t) = \frac{\mu t}{e^{\frac{\mu t}{2}} - e^{-\frac{\mu t}{2}}} = \frac{\mu t/2}{\sinh \mu t/2},
\]
shown in [12] Remark 4.6, we have
\[
\det M_\varphi = \det C_\varphi
\]
\[
= \frac{ab + pq}{pq} \frac{\Gamma_\varphi(p)\Gamma_\varphi(-p)\Gamma_\varphi(q)\Gamma_\varphi(q)}{\Gamma_\varphi(p + a)\Gamma_\varphi(-p - a)\Gamma_\varphi(p + b)\Gamma_\varphi(-p - b)}
\]
\[
= \frac{1}{\sinh(\frac{\mu}{2}(p + a))\sinh(\frac{\mu}{2}(p + b))} \cdot \frac{\sinh(\frac{\mu}{2}a)\sinh(\frac{\mu}{2}b)}{\sinh(\frac{\mu}{2}p)\sinh(\frac{\mu}{2}q)}
\]
\[
= \frac{e^{\frac{\mu}{2}a} - e^{-\frac{\mu}{2}a}}{2} \cdot \frac{e^{\frac{\mu}{2}b} - e^{-\frac{\mu}{2}b}}{2} = 1.
\]
Thus, the (2, 2) entry \([M_\varphi]_{22}\) must also be in \(\mathbb{K}[a, b, c - 1]\). Therefore, we have
\(M_\varphi \in \text{SL}_2(\mathbb{K}[a, b, c - 1])\). \(\square\)
We denote by $H(X, -Y)$ the image of each element $H \in \mathbb{K} \langle \langle e_0, e_1 \rangle \rangle$ under the map
\begin{equation}
ev_{(X, -Y)} : \mathbb{K} \langle \langle e_0, e_1 \rangle \rangle \to \Mat_2(\mathbb{K}[a, b, c - 1])
\end{equation}
sending $e_0$ and $e_1$ to $X$ and $-Y$ respectively.

**Theorem 3.3.** For an associator $(\mu, \varphi) \in \mathbb{K}^\times \times \hat{U}_2$, we have
\[ \varphi(X, -Y) = M_\varphi. \]

Particularly the $(1, 1)$ entry proves the relation of Ohno-Zagier in [30] for associators.

**Proof.** We show the equation entry-wise.

The $(1, 1)$ entry: The main result of [25] states that if the system \((z(k_1, \ldots, k_m) \in \mathbb{C} \mid m, k_1, \ldots, k_{m-1} \geq 1, k_m > 1)\) satisfies the regularized double shuffle relations (cf. loc. cit.), the following Ohno-Zagier relation [30] holds
\begin{equation}
1 + ab \sum_{k, n, s > 0 \atop k > n+s, n \geq s} g_0(k, n, s) p^{k-n-s} q^{n-s} (ab + pq)^{s-1}
\end{equation}
\begin{equation}
= \exp \left( \sum_{n=2}^{\infty} \frac{z(n)}{n} \left( p^n + q^n - (a + p)^n - (b + p)^n \right) \right),
\end{equation}
where
\begin{equation}
g_0(k, n, s) = \sum \text{ wt}(k) = k, \text{ dp}(k) = n, \text{ ht}(k) = s \text{ admissible index}
\end{equation}

Here, an admissible index means a tuple $k = (k_1, \ldots, k_n) \in \mathbb{N}^n (n \in \mathbb{N})$ with $k_n > 1$, and we write $\text{ wt}(k) = k_1 + \cdots + k_n$, $\text{ dp}(k) = n$, $\text{ ht}(k) = \sharp \{ i \mid k_i > 1 \}$. In [13], is is shown that the coefficients of any associator $\varphi$ with an appropriate signature, $\zeta_\varphi(k_1, \ldots, k_m) = (-1)^m (\varphi | e_0^{-k_1-1} e_1 \cdots e_0^{-1} e_1)$ in precise, satisfy the regularized double shuffle relations. Since (3.2) for $\zeta_\varphi(k_1, \ldots, k_m)$ is nothing but the $(1, 1)$ entry $[\varphi(X, -Y)]_{11}$ by [31] Lemma 3.1 and (63), we obtain the equality
\[ [\varphi(X, -Y)]_{11} = \Gamma_\varphi \left( \begin{array}{cc} -p, -q \\ -p - a, -p - b \end{array} \right) = [M_\varphi]_{11}. \]

The $(1, 2)$ entry: By [31] (66) and its following remark, we have
\begin{equation}
[\varphi(X, -Y)]_{11} + \frac{p}{b} [\varphi(X, -Y)]_{12}
\end{equation}
\begin{equation}
= 1 + (ab + pq) \sum_{k, n, s > 0 \atop k > n+s, n \geq s} g_\varphi(k, n, s | e_0 U f_2 e_1) (-p)^{k-n-s} q^{n-s} (ab)^{s-1}
\end{equation}
where $g_\varphi(k, n, s | e_0 U f_2 e_1)$ is defined analogously to (3.1) with $z(k) = \zeta_\varphi(k)$. Since the right-hand side of the above is obtained from (3.3) by the change of variables $a \mapsto a + p$, $b \mapsto b + p$, $p \mapsto -p$, $q \mapsto q$, we have
\[ [\varphi(X, -Y)]_{11} + \frac{p}{b} [\varphi(X, -Y)]_{12} = \Gamma_\varphi \left( \begin{array}{cc} p, -q \\ -a, -b \end{array} \right). \]

Hence, we obtain the equality
\begin{equation}
[\varphi(X, -Y)]_{12} = \frac{b}{p} \left\{ \Gamma_\varphi \left( \begin{array}{cc} p, -q \\ -a, -b \end{array} \right) - \Gamma_\varphi \left( \begin{array}{cc} -p, -q \\ -p - a, -p - b \end{array} \right) \right\} = [M_\varphi]_{12}.
\end{equation}
The (2, 1) entry: By [31] Lemma 3.1, we have

\[ [\varphi(X, -Y)]_{12} = bq \sum_{k, n, s > 0 \atop k > n + s, n > s} g_\varphi(k, n, s|e_0Uf_2)p^{k-n-s}q^{n-s}(ab + pq)^{s-1}, \]

\[ [\varphi(X, -Y)]_{21} = ap \sum_{k, n, s > 0 \atop k > n + s, n > s} g_\varphi(k, n, s|Uf_2e_1)p^{k-n-s}q^{n-s}(ab + pq)^{s-1}, \]

where

\[ g_\varphi(k, n, s|e_0Uf_2) = \sum_{W \in e_0Uf_2: \text{word}} (-1)^{dp(W)}(\varphi|W), \]

\[ g_\varphi(k, n, s|Uf_2e_1) = \sum_{W \in Uf_2e_1: \text{word}} (-1)^{dp(W)}(\varphi|W). \]

Here, for each word \( W \), \( wt(W) \) and \( dp(W) \) are defined to be the number of letters in \( W \) and the number of \( e_1 \) appearing in \( W \), respectively. And \( ht(W) - 1 \) is defined to be the number of \( e_1e_0 \) appearing in \( W \) (see [31] Lemma 3.1). By the 2-cycle relation and group-like condition for \( \varphi \), we have

\[ (-1)^{dp(W)}(\varphi|W) = (-1)^{dp(W^{*})}(\varphi|W^*) \]

(for example, see [19] Lemma 3.2). Here, \( W^* \) is the dual word of \( W \), that is, the image of \( W \) under the anti-automorphism of \( Uf_2 \) which switches \( e_0 \) and \( e_1 \). Thus, we have

\[ g_\varphi(k, n, s|Uf_2e_1) = g_\varphi(k, k-n, s|e_0Uf_2). \]

Hence,

\[ [\varphi(X, -Y)]_{21} = ap \sum_{k, n, s > 0 \atop k > n + s, n > s} g_\varphi(k, k-n, s|e_0Uf_2)p^{k-n-s}q^{n-s}(ab + pq)^{s-1} \]

\[ = ap \sum_{k, m, s > 0 \atop k > m + s, m > s} g_\varphi(k, m, s|e_0Uf_2)p^{m-s}q^{k-m-s}(ab + pq)^{s-1}. \]

By [3.3] and [3.0], the change of variables \( a \mapsto -a, b \mapsto -b, p \mapsto q, q \mapsto p \), we have

\[ \frac{a}{q} \left\{ \Gamma_\varphi \left( \begin{array}{c} -p, q \\ a, b \end{array} \right) - \Gamma_\varphi \left( \begin{array}{c} -p, -q \\ -p - a, -p - b \end{array} \right) \right\} = [M_\varphi]_{21}. \]

The (2, 2) entry: By Proposition [3.2] \( \det(M_\varphi) = 1 \). While we have \( \det(\varphi(X, -Y)) = 1 \) because \( \varphi \) is commutator group-like. These two claims assert that \( [\varphi(X, -Y)]_{22} = [M_\varphi]_{22} \).

Notation 3.4. We define the matrix

\[ M_+ \in \text{Mat}_2(\mathbb{K}[[a, b, c - 1]]). \]
in the same way as $M_\varphi$ in Notation 3.1 whose $\Gamma_\varphi(t)$ is replaced with
\[
\Gamma_+(t) = \sqrt{\frac{t}{2} \text{cosech}\left(\frac{t}{2}\right)} = \sqrt{\Gamma(1 + \frac{t}{2\pi\sqrt{-1}})\Gamma(1 - \frac{t}{2\pi\sqrt{-1}})}
\]
\[
= \frac{\sqrt{\Gamma(1 + \frac{t}{2\sqrt{-1}})}}{\sqrt{\Gamma(1 - \frac{t}{2\sqrt{-1}})}} = \exp \left\{ - \sum_{n=1}^{\infty} \frac{B_{2n}}{2n!} t^{2n}\right\}.
\]
Recall that the symbol $\Gamma$ means the classical gamma function in Remark 2.5 (1). Namely, $M_+$ is given by
\[
\begin{pmatrix}
\frac{1}{a} & 0 \\
-\frac{1}{q} & \frac{1}{b}
\end{pmatrix}
\begin{pmatrix}
1 & -\frac{b}{p} \\
0 & \frac{1}{p}
\end{pmatrix}
C_+ \begin{pmatrix}
\Gamma_+ \left( -\frac{p}{a}, -\frac{q}{b} \right) \\
\frac{ab}{pq} \Gamma_+ \left( -\frac{p}{a}, \frac{b}{p} \right)
\end{pmatrix}
\]
\[
\Gamma_+ \left( \frac{p}{a}, \frac{q}{b} \right)
\]
and $\Gamma_+ \left( \frac{s,t}{u,v} \right) := \frac{\Gamma_+(s,t)}{\Gamma_+(u,v)}$.

**Proof of Theorem 0.4.** For any even unitary associator $\varphi$, we have $\Gamma_\varphi(t) = \Gamma_+(t)$ by Remark 2.5 (2), and thus,
\[
\Gamma_\varphi(t) = \Gamma_+(t)
\]
(3.7)
\[
M_+ = M_\varphi
\]
Therefore, we see that our claim is a direct consequence of Theorem 3.3. \hfill \Box

We note that
\[
M_+ \in \text{SL}_2(\mathbb{Q}[\lfloor a, b, c - 1 \rfloor])
\]
by Proposition 3.2 and Theorem 0.4.

We consider the group homomorphism
\[
\Theta : \hat{F}^2_\ell \rightarrow \text{GL}_2(\mathbb{Q}_\ell[\lfloor a, b, c - 1 \rfloor])
\]
defined by the evaluation $x_0 \mapsto e^X$ and $x_1 \mapsto M_+^{-1} e^{-Y} M_+$. By the map $\Theta$, our $\ell$-adic hypergeometric function is reformulated as follows:

**Proposition 3.5.** Definition (0.2) is free from any choice of $\ell$-adic even unitary associator $\varphi$. Furthermore the following equality holds:
\[
\text{2F1} \left( \begin{array}{c}
\frac{a,b}{c} \\
\frac{2}{c}
\end{array} \left| \sigma \right. \right) = [\Theta(f_\sigma^\varphi)]_{11} \in \mathbb{Q}_\ell[\lfloor a, b, c - 1 \rfloor].
\]

**Proof.** From Theorem 0.4 for even unitary associator $\varphi$, we learn
\[
\Theta = \text{ev}_{(X,-Y)} \circ \iota_\varphi
\]
where $\text{ev}_{(X,-Y)}$ is the map defined in (5.1). Whence it is evident that the definition is independent of $\varphi$. \hfill \Box

The following two propositions are required to prove the convergence:

**Proposition 3.6.** When $a, b, c \in \mathbb{Z}_\ell$ satisfy (0.3), the evaluation $M_{+,0}$ of the matrix $M_+$ at $(a, b, c) = (a, b, c)$ makes sense in $\text{SL}_2(\mathbb{Q}_\ell)$, belongs to $\text{SL}_2(\mathbb{Z}_\ell)$ and satisfies the congruence $M_{+,0} \equiv I_2 \pmod{\ell}$.
Proof. We show the claim entry-wise.

The (1, 1) entry: By von Staudt-Clausen’s theorem, \(\ell B_{2n} \in \mathbb{Z}_\ell\) for all \(n\). Since we have \(v_\ell(n!) < \frac{n}{\ell - 1}\) for all \(n\), we have

\[
(3.11) \quad v_\ell\left(\frac{B_{2n}}{2(2n)!}\right) > \frac{\ell - 2}{\ell - 1} - 1 > 0
\]

where \(v_\ell\) is the standard \(\ell\)-adic valuation. Therefore, \(\Gamma_+(\ell z)\) is in the Tate algebra \(T_1\), that is, it is a rigid-analytic function on \(|z|_\ell \leq 1\) (cf. [4]). Therefore, \(\Gamma_+\left(-\ell p, -\ell q; -\ell p - \ell q\right)\) is in the Tate algebra \(T_3\) with respect to three variables \(a, b, p\).

Hence, \([M_{+,0}]_{11} := \left(\begin{array}{c}\ell p - \ell q \quad \nabla \\
\ell a - \ell b - \ell p - \ell q \end{array}\right)\) makes sense in \(\mathbb{Q}_\ell\) when \((0.3)\) holds. Furthermore, by \((3.11)\), we have \(\log \Gamma_+(\ell z) \in \ell \mathbb{Z}_\ell[[z]]\), and whence \(\Gamma_+(\ell z) \in 1 + \ell \mathbb{Z}_\ell[[z]]\).

Thus, \(\Gamma_+\left(-\ell p, -\ell q, -\ell p - \ell q\right) \in 1 + \ell \mathbb{Z}_\ell[[a, b, p]]\). Thus, \([M_{+,0}]_{11}\) belongs to \(1 + \ell \mathbb{Z}_\ell\).

The (1, 2) entry: The above arguments indicate that \(\Gamma_+\left(-\ell p, -\ell q, -\ell p - \ell q\right)\) are in \(T_3\). Write \(m := \frac{b}{p} \left\{ \Gamma_+\left(-\ell p, -\ell q\right) - \Gamma_+\left(-\ell p, -\ell q, -\ell p - \ell q\right) \right\}\). By \((3.8)\), \(m\) is in \(\mathbb{Q}_\ell[[a, b, c - 1]]\). Hence, by the Weierstrass division theorem (cf. [4]), we see that \(m\) is in \(T_3\). Thus, the entry \([M_{+,0}]_{12}\) makes sense in \(\mathbb{Q}_\ell\). Furthermore, by \((3.11)\), \(m\) is also in \(\ell \mathbb{Z}_\ell[[a, b, p]]\). Whence we learn that \([M_{+,0}]_{12} \in \ell \mathbb{Z}_\ell\).

The (2, 1) entry: By similar arguments as the above, we have \([M_{+,0}]_{(2,1)} \in \ell \mathbb{Z}_\ell\).

The (2, 2) entry: By \((3.8)\), \(\det(M_+) = 1\). Thus, we have \(M_{+,0} \in \text{SL}_2(\mathbb{Q}_\ell)\) with \([M_{+,0}]_{22} \in 1 + \ell \mathbb{Z}_\ell\) because we have shown that \([M_{+,0}]_{11} \in 1 + \ell \mathbb{Z}_\ell\) and \([M_{+,0}]_{12}, [M_{+,0}]_{21} \in \ell \mathbb{Z}_\ell\).

\(\square\)

Proposition 3.7. When \(a, b, c \in \mathbb{Z}_\ell\) satisfy \((0.3)\), there is a continuous group homomorphism

\[
\Theta_0 : F_2^{(\ell)} \to \text{GL}_2(\mathbb{Z}_\ell)
\]

sending \(x_0 \mapsto \exp(X_0)\) and \(x_1 \mapsto M_{+,0}^{-1} \exp(-Y_0) M_{+,0}\) with

\[
X_0 = \begin{pmatrix} 0 & b \\ 0 & p \end{pmatrix}, \quad Y_0 = \begin{pmatrix} 0 & 0 \\ a & q \end{pmatrix} \in \text{Mat}_2(\mathbb{Z}_\ell),
\]

with \(p = 1 - c\) and \(a + b + 1 - c = a + b + p\).

Proof. There is a continuous inclusion, called the Magnus embedding, \(\hat{F}_2^{(\ell)} \hookrightarrow \mathbb{Z}_\ell[[u, v]]\) sending \(x_0\) and \(x_1\) to \(1 + u\) and \(1 + v\), respectively ([4] Ch I §1). Our map \(\Theta_0\) is obtained by composing it with the map \(\mathbb{Z}_\ell[[u, v]] \to \text{Mat}_2(\mathbb{Z}_\ell)\) sending \(u\) and \(v\) to \(\exp(X_0) - I_2\) and \(M_{+,0}^{-1} \exp(-Y_0) M_{+,0} - I_2\). By

\[
(3.12) \quad \exp(X_0) \equiv \exp(-Y_0) \equiv M_{+,0} \equiv I_2 \mod \ell,
\]

\(\Theta_0\) is well-defined and continuous.

\(\square\)

Proof of Theorem 0.1. Claim (i) follows from Proposition 3.5. Claim (ii) is a consequence of Proposition 3.7 because we have

\[
_{2F1}\left(a, b \mid c \right | z \right) = [\Theta_0(f^2_\ell)]_{11}.
\]

\(\square\)
Proposition 4.1. All six matrices $\sigma_0$.

Theorem 3.8. The Gauss hypergeometric theorem is given as follows:

\[ \sum_{a}^{\infty} \binom{a}{b} \binom{c}{d} = \prod_{r} \Gamma_r \]

Proof of Theorem 0.2. It is a consequence of Proposition 3.7 and Theorem 0.3. By calculating the $1 \times 1$ entry of the above matrix with (3.13), we obtain the claim.

The $\ell$-adic Gauss hypergeometric theorem can be derived as follows:

Proof of Theorem 4.1. It is a consequence of Proposition 4.5 and Theorem 0.3. By calculating the $1 \times 1$ entry of the above matrix with (3.13), we obtain the claim.

4. $\ell$-adic Euler transformation formula

In this section, we introduce two more series in Definition 4.2 to relax our assumption (0.3) in Proposition 4.3 and to provide a proof of the $\ell$-adic Euler transformation formula (Theorem 0.3).
where in the last equality we use Theorem [0.3]

(iii). The claim for $V_{1\infty}^\varphi(\sigma)(z)$ follows from the following equality:

$$G_{1\infty}^\varphi(X, Y)(\sigma)(z) = \text{ev}_{(X, Y)} \left( G_{1\infty}^\varphi(e_0, e_1)(\sigma)(z) \right) = \text{ev}_{(X, Y)} \left( G_{01}^\varphi(e_1, e_\infty)(\sigma)(1 - \frac{1}{z}) \right)$$

$$= \text{ev}_{(X, Y)} \left( f^{1 - \frac{1}{z}}_\sigma \left( \exp(e_1), \varphi(e_1, e_\infty)^{-1} \exp(e_\infty)\varphi(e_1, e_\infty) \right) \right).$$

By 2- and 3-cycle relations for $\varphi$,

$$= \text{ev}_{(X, Y)} \left( f^{1 - \frac{1}{z}}_\sigma \left( \exp(e_1), \exp(-\frac{e_1}{2})\varphi(e_0, e_1)\exp(-e_0)\varphi(e_0, e_1)^{-1}\exp(-\frac{e_1}{2}) \right) \right)$$

$$= f^{1 - \frac{1}{z}}_\sigma \left( \exp(-Y), \exp(Y/2)M_+ \exp(-X)M_+^{-1}\exp(Y/2) \right).$$

(iv). The claim for $V_{0\infty}^\varphi(\sigma)(z)$ follows from the following equality:

$$G_{0\infty}^\varphi(X, Y)(\sigma)(z) = \text{ev}_{(X, Y)} \left( G_{0\infty}^\varphi(e_0, e_1)(\sigma)(z) \right) = \text{ev}_{(X, Y)} \left( G_{01}^\varphi(e_0, e_\infty)(\sigma)(z/(z - 1)) \right)$$

$$= \text{ev}_{(X, Y)} \left( f^{z/(z-1)}_\sigma \left( \exp(e_0), \varphi(e_0, e_\infty)^{-1}\exp(e_\infty)\varphi(e_0, e_\infty) \right) \right).$$

By 2- and 3-cycle relations for $\varphi$,

$$= \text{ev}_{(X, Y)} \left( f^{z/(z-1)}_\sigma \left( \exp(e_0), \exp(-e_0/2)\varphi(e_0, e_1)^{-1}\exp(-e_1)\varphi(e_0, e_1)\exp(-e_0/2) \right) \right)$$

$$(4.3) = f^{z/(z-1)}_\sigma \left( \exp(X), \exp(-X/2)M_+^{-1}\exp(Y)M_+ \exp(-X/2) \right).$$

(v). The claim for $V_{\infty 1}^\varphi(\sigma)(z)$ follows from the following equality:

$$G_{\infty 1}^\varphi(X, Y)(\sigma)(z) = \text{ev}_{(X, Y)} \left( G_{\infty 1}^\varphi(e_0, e_1)(\sigma)(z) \right) = \text{ev}_{(X, Y)} \left( G_{01}^\varphi(e_\infty, e_1)(\sigma)\left(\frac{1}{z}\right) \right)$$

$$= \text{ev}_{(X, Y)} \left( f^{1/z}_\sigma \left( \exp(e_\infty), \varphi(e_\infty, e_1)^{-1}\exp(e_1)\varphi(e_\infty, e_1) \right) \right)$$

$$= f^{1/z}_\sigma \left( \exp(Y - X), N_+^{-1}\exp(-X)N_+ \right)$$

with $N_+ := \text{ev}_{(X, Y)}(\varphi(e_\infty, e_1))$, which is shown to be free from any choice of even unitary associator $\varphi$ in Lemma [X.1]

(vi). The claim for $V_{\infty 0}^\varphi(\sigma)(z)$ follows from the following equality:

$$G_{\infty 0}^\varphi(X, Y)(\sigma)(z) = \text{ev}_{(X, Y)} \left( G_{\infty 0}^\varphi(e_0, e_1)(\sigma)(z) \right) = \text{ev}_{(X, Y)} \left( G_{01}^\varphi(e_\infty, e_0)(\sigma)\left(\frac{1}{1 - z}\right) \right)$$

$$= \text{ev}_{(X, Y)} \left( f^{1/(1-z)}_\sigma \left( \exp(e_\infty), \varphi(e_\infty, e_0)^{-1}\exp(e_0)\varphi(e_\infty, e_0) \right) \right).$$

By 2- and 3-cycle relations for $\varphi$,

$$= \text{ev}_{(X, Y)} \left( f^{1/(1-z)}_\sigma \left( \exp(e_\infty), \exp(-\frac{e_\infty}{2})\varphi(e_\infty, e_1)^{-1}\exp(-e_1)\varphi(e_\infty, e_1)\exp(-\frac{e_\infty}{2}) \right) \right)$$

$$= f^{1/(1-z)}_\sigma \left( \exp(Y - X), \exp(-\frac{Y - X}{2})N_+^{-1}\exp(Y)N_+ \exp(-\frac{Y - X}{2}) \right).$$

$\square$
Henceforth, we drop the upper suffix $\varphi$ in $\mathcal{V}_\varphi^\ell(\sigma)(z)$. By (2.4), (3.2), (4.1) and $G_{01}^\ell(X, -Y)(\sigma)(z) \in \text{GL}_2(\mathcal{R})$, we see that the $(1, 1)$ entry $[\mathcal{V}_{01}(\sigma)(z)]_{11}$ is given by

\begin{equation}
[\mathcal{V}_{01}(\sigma)(z)]_{11} = 2F_1\left(\begin{array}{c}
a, b \\
a + b + 1 - c
\end{array} \bigg| 1 - z \right)(\sigma) \in \mathcal{R} = \mathbb{Q}_\ell[[a, b, c - 1]].
\end{equation}

Recall that in the complex case (11), we have the following relationship between the $(1, 1)$ entry of the matrix constructed from the fundamental solution of the KZ-equation and the hypergeometric function

\begin{align*}
[\mathcal{V}_{10}(\sigma)]_{11} &= 2F_1\left(\begin{array}{c}
a, b \\
a + b + 1 - c
\end{array} \bigg| 1 - z \right), \\
[\mathcal{V}_{0\infty}(\sigma)]_{11} &= (1 - z)^{-a} \cdot 2F_1\left(\begin{array}{c}
a, c - b \\
c
\end{array} \bigg| \frac{z}{z - 1} \right),
\end{align*}

which indicates that we should introduce the following series in our $\ell$-adic setting:

**Definition 4.2.** We consider two series

\begin{align*}
2F_1^\dagger\left(\begin{array}{c}
a', b' \\
c'
\end{array} \bigg| z \right)(\sigma) &\in c'^{-1}\mathbb{Q}_\ell[[a', b', c']], \\
2F_1^\dagger\left(\begin{array}{c}
a'', b'' \\
c''
\end{array} \bigg| z \right)(\sigma) &\in \mathbb{Q}_\ell[[a'' - 1, b'', c'' - 1]],
\end{align*}

which are determined by $\ell$-adic analogues of the above two equalities:

\begin{align*}
[\mathcal{V}_{10}(\sigma)(z)]_{11} &= : 2F_1^\dagger\left(\begin{array}{c}
a, b \\
a + b + 1 - c
\end{array} \bigg| 1 - z \right)(\sigma) \in q^{-1}\mathbb{R}, \\
[\mathcal{V}_{0\infty}(\sigma)(z)]_{11} &= : \exp\left\{ -\rho_{1-z}(\sigma)a \right\} \cdot 2F_1^\dagger\left(\begin{array}{c}
a, c - b \\
c
\end{array} \bigg| \frac{z}{z - 1} \right)(\sigma) \in \mathbb{R}.
\end{align*}

We stress that by Proposition 4.1, the two series are independent of any choice of even unitary associators $\varphi$.

We note that the relationship of the three series $2F_1\left(\begin{array}{c}
a, b \\
a + b + 1 - c
\end{array} \bigg| z \right)(\sigma) \in \mathbb{Q}_\ell[[a, b, c - 1]]$, $2F_1^\dagger\left(\begin{array}{c}
a', b' \\
c'
\end{array} \bigg| z \right)(\sigma) \in c'^{-1}\mathbb{Q}_\ell[[a', b', c']]$, $2F_1^\dagger\left(\begin{array}{c}
a'', b'' \\
c''
\end{array} \bigg| z \right)(\sigma) \in \mathbb{Q}_\ell[[a'' - 1, b'', c'' - 1]]$ with the other three solutions $\mathcal{V}_{\infty 1}(\sigma)(z)$, $\mathcal{V}_{1\infty}(\sigma)(z)$, $\mathcal{V}_{0\infty}(\sigma)(z)$ is given as follows:

**Proposition 4.3.** For an even unitary associator $\varphi$, we have the following equalities:

\begin{align*}
[\mathcal{V}_{\infty 1}(\sigma)(z)]_{11} &= \exp\left\{ -\rho_2(\sigma)a \right\} \cdot 2F_1\left(\begin{array}{c}
a + 1 - c, a \\
a - b + 1
\end{array} \bigg| 1 - z \right)(\sigma) \text{ in } \mathbb{R}, \\
[\mathcal{V}_{1\infty}(\sigma)(z)]_{11} &= \exp\left\{ -\rho_2(\sigma)a \right\} \cdot 2F_1\left(\begin{array}{c}
a, a + 1 - c \\
a + b + 1 - c
\end{array} \bigg| 1 - \frac{1}{z} \right)(\sigma) \text{ in } q^{-1}\mathbb{R}, \\
[\mathcal{V}_{0\infty}(\sigma)(z)]_{11} &= \exp\left\{ -\rho_{1-z}(\sigma)a \right\} \cdot 2F_1\left(\begin{array}{c}
a, c - b \\
a - b + 1
\end{array} \bigg| \frac{1}{1 - z} \right)(\sigma) \text{ in } \mathbb{R}.
\end{align*}

**Proof.** The first equality: Let $\iota$ be the algebra automorphism of $\mathcal{R}$ such that

\begin{equation}
\iota(a) = a, \quad \iota(b) = a + 1 - c, \quad \iota(c - 1) = a - b.
\end{equation}
By \(27\), \(1.3\) and the replacement of \(z\) with \(z^{-1}\), we see that it is sufficient to show that the equation

\[
\left[ G^\sigma_{01}(Y - X, -Y)(\sigma)(z) \cdot \left( \frac{1}{\bar{B}} \ 1 \ 0 \ 1 \right) \right]_{11} = \exp \{ \rho_z(\sigma)a \} \cdot \iota \left( \left[ G^\sigma_{01}(X, -Y)(\sigma)(z) \cdot \left( \frac{1}{\bar{B}} \ 1 \ 0 \ 1 \right) \right]_{11} \right)
\]

holds in \(\mathcal{R}\).

In the complex case, we have \(G^\sigma_{01}(Y - X, -Y)(z) = G^\sigma_{\infty 1}(X, -Y)(\frac{1}{z})\). By \(31\) (36) and (97), we obtain

\[
\left[ G^\sigma_{01}(Y - X, -Y)(z) \cdot \left( \frac{1}{\bar{B}} \ 1 \ 0 \ 1 \right) \right]_{11} = \exp \{ \log(z)a \} \cdot F_1 \left( \frac{a, a + 1 - c}{a - b + 1} \right),
\]

whence we obtain

\[\tag{4.13}
\left[ G^\sigma_{01}(Y - X, -Y)(z) \cdot \left( \frac{1}{\bar{B}} \ 1 \ 0 \ 1 \right) \right]_{11} = \exp \{ \log(z)a \} \cdot \iota \left( \left[ G^\sigma_{01}(X, -Y)(z) \cdot \left( \frac{1}{\bar{B}} \ 1 \ 0 \ 1 \right) \right]_{11} \right)
\]

in \(\mathbb{C}[a, b, c - 1]\).

Let \(TV = \oplus_{n=0}^\infty V^\otimes n\) with \(V := H_{\text{DR}}^1(\mathcal{X}, \mathbb{Q})\) and \(V^\otimes 0 = \mathbb{Q}\), where we encode a structure of Hopf algebra with the shuffle product and the deconcatenation coproduct. We consider the \(\mathbb{Q}\)-linear map associated with iterated integrals

\[\rho : TV \to \text{Map}(\pi_1(\mathcal{X}(\mathbb{C}); 0, z), \mathbb{C})\]

which sends each \(\omega_{i_m} \otimes \cdots \otimes \omega_{i_1} \in V^\otimes m\) to

\[\rho(\omega_{i_m} \otimes \cdots \otimes \omega_{i_1})(\gamma) = \int_{0 < t_1 < \cdots < t_m < 1} \omega_{i_m}(\gamma(t_m)) \cdot \omega_{i_m-1}(\gamma(t_{m-1})) \cdots \omega_{i_1}(\gamma(t_1)).\]

Here, \(\pi_1(\mathcal{X}(\mathbb{C}); 0, z)\) is the set of homotopy paths \(\gamma\) from \(01\) to \(z\). Actually, \(\rho\) induces an isomorphism of Hopf algebras between \(TV\) and the space \(\text{Imp}\rho\) of iterated integrals over \(\mathcal{X}\) due to Chen’s theory (cf. 5).

Since \(4.13\) is regarded as the equality

\[
\left[ G^\sigma_{01}(Y - X, -Y)(\gamma_z) \cdot \left( \frac{1}{\bar{B}} \ 1 \ 0 \ 1 \right) \right]_{11} = \exp \{ \log(\gamma_z)a \} \cdot \iota \left( \left[ G^\sigma_{01}(X, -Y)(\gamma_z) \cdot \left( \frac{1}{\bar{B}} \ 1 \ 0 \ 1 \right) \right]_{11} \right)
\]

in \(\text{Map}(\pi_1(\mathcal{X}(\mathbb{C}); 0, z), \mathbb{C}) \otimes \mathbb{C}((\epsilon_0, \epsilon_1))\), it yields an equality in \(TV \otimes \mathbb{Q}((\epsilon_0, \epsilon_1))\).

The above \(G^\sigma_{01}(\epsilon_0, \epsilon_1)(\gamma_z) = G^\sigma_{01}(\epsilon_0, \epsilon_1)(\gamma_z)\) corresponds to the element \(G(\epsilon_0, \epsilon_1)\) in the \(\mathbb{Q}\)-structure \(TV \otimes \mathbb{Q}((\epsilon_0, \epsilon_1))\) given by

\[G(\epsilon_0, \epsilon_1) := \sum_{W:\text{word}} \Omega_W \otimes W\]

where for each word \(W\), we mean \(\Omega_W\) to be an element in \(TV\) obtained by substituting \(\left[ \frac{dz}{z} \right] = \frac{dz}{z - 1}\) for \(\epsilon_0\) (resp. \(\epsilon_1\)) in \(V \subset TV\). Whence by \(4.13\), we have

\[
\left[ G(Y - X, -Y) \cdot \left( \frac{1}{\bar{B}} \ 1 \ 0 \ 1 \right) \right]_{11} = \exp \left\{ \left[ \frac{dz}{z} \right]a \right\} \cdot \iota \left( \left[ G(X, -Y) \cdot \left( \frac{1}{\bar{B}} \ 1 \ 0 \ 1 \right) \right]_{11} \right)
\]

in \(TV \otimes \mathbb{Q}[a, b, c - 1]\).
Assume that \( g = 1 + \sum_{W} I(W)W \) is any group-like series in \( \mathbb{Q}_\ell((e_0, e_1)) \). Since \( g \) is group-like, \( I(W) \) satisfies the shuffle product. We have a shuffle algebra homomorphism \( ev_g : TV \to \mathbb{Q}_\ell \) sending \( \Omega_W \) to \( I(W) \). By applying \( ev_g \) to the above equality, we obtain

\[
\left[ g(Y - X, -Y) \cdot \left( \frac{1}{a} \cdot 1 - 1 \right) \right]_{11} = \exp \{ I(e_0) \cdot \ell \left( \left[ g(X, -Y) \cdot \left( \frac{1}{a} \cdot 1 \right) \right]_{11} \right) \}
\]

in \( \mathbb{Q}_\ell[[a, b, c - 1]] \). This is how our claim is proved.

**The second equality:** By (2.38), (2.39) and (4.17), it is sufficient to show that the equation

\[
\left[ G_{\infty}^\rho(Y - X, -Y)(\sigma)(\frac{1}{z}) \cdot \left( \frac{1}{a} \cdot 0 \cdot \frac{1-a}{b} \right) \right]_{11} = \exp \{ -\rho(\sigma) \cdot \ell \left( \left[ G_{\infty}^\rho(X, -Y)(\sigma)(\frac{1}{z}) \cdot \left( \frac{1}{a} \cdot \frac{1-b}{b} \right) \right]_{11} \right) \}
\]

holds in \( \mathcal{R}[q^{-1}] \).

In the complex case, we have \( G_{\infty}^\rho(Y - X, -Y)(\frac{1}{z}) = G_{1\infty}(X, -Y)(z) \). Thus, by the formula for \( (1, 0) \cdot \mathcal{V}_1(z) \) in (31) we have

\[
\left[ G_{1\infty}(Y - X, -Y)(\frac{1}{z}) \cdot \left( \frac{1}{a} \cdot 0 \cdot \frac{0}{b} \right) \right]_{11} = \exp \{ -\log(z) \cdot \ell \left( \left[ G_{1\infty}(X, -Y)(\frac{1}{z}) \cdot \left( \frac{1}{a} \cdot \frac{1}{b} \right) \right]_{11} \right) \}
\]

while by (31) (34) and (70), \[ \] we have

\[
\left[ G_{1\infty}(X, -Y)(\frac{1}{z}) \cdot \left( \frac{1}{a} \cdot 0 \cdot \frac{0}{b} \right) \right]_{11} = \ell \left( \left[ G_{1\infty}(X, -Y)(\frac{1}{z}) \cdot \left( \frac{1}{a} \cdot \frac{1}{b} \right) \right]_{11} \right)
\]

Thus, we obtain

\[
\left[ G_{1\infty}(Y - X, -Y)(\frac{1}{z}) \cdot \left( \frac{1}{a} \cdot 0 \cdot \frac{0}{b} \right) \right]_{11} = \exp \{ -\rho(1-z) \cdot \ell \left( \left[ G_{1\infty}(X, -Y)(\frac{1}{z}) \cdot \left( \frac{1}{a} \cdot \frac{1}{b} \right) \right]_{11} \right) \}
\]

where \( \ell \) is an extension our previously introduced \( \ell \) \( \mathcal{R}[q^{-1}] \) (N.B. \( \ell(q) = q \)). By the same arguments as for the proof of the first equality, we obtain the claim.

**The third equality:** By (2.8), (2.9) and (4.8), it is sufficient to prove that the equation

\[
\left[ G_{0\infty}^\rho(Y - X, -Y)(\sigma)(\frac{1}{z}) \cdot \left( \frac{1}{a} \cdot 1 \cdot \frac{1-b}{b} \right) \right]_{11} = \exp \{ -\rho(1-z) \cdot \ell \left( \left[ G_{0\infty}^\rho(X, -Y)(\sigma)(\frac{1}{z}) \cdot \left( \frac{1}{a} \cdot \frac{1}{b} \right) \right]_{11} \right) \}
\]

holds in \( b^{-1} \mathcal{R} \). Here, \( \varsigma \) means the automorphism of \( \mathcal{R} \) sending \( a, b, \) and \( c \) to \( a, a + 1 - c, \) and \( a - b + 1, \) respectively.

In the complex case, similarly, we have

\[
\left[ G_{0\infty}(Y - X, -Y)(\frac{1}{z}) \cdot \left( \frac{1}{a} \cdot 1 \cdot \frac{1}{b} \right) \right]_{11} = \exp \{ -\log(1-z) \cdot \ell \left( \left[ G_{0\infty}(X, -Y)(\frac{1}{z}) \cdot \left( 1 \cdot 0 \right) \right]_{11} \right) \}
\]

\[ ^1 \text{It looks there is an error in the matrix on the equation } 33 \text{ (70). The } (2, 2) \text{ entry should be } \frac{\delta}{\alpha + \beta - \gamma} \text{ instead of } \frac{\beta}{\alpha + \beta - \gamma}. \]
A formal version of our $\ell$-adic Euler transformation formula is given as follows:

**Theorem 4.4.** The equality

\begin{equation}
2F_1^1 \left( \begin{array}{c} a', b' \\ c' \end{array} \bigg| z \right)(\sigma) = \exp \{ (c' - a' - b') \rho_{1 - z}(\sigma) \} \cdot 2F_1^1 \left( \begin{array}{c} c' - a', c' - b' \\ c' \end{array} \bigg| z \right)(\sigma)
\end{equation}

holds in $c'^{-1}Q_{1/2}[a', b', c']$.

**Proof.** By (4.10) and the reparametrization $a' = a, b' = a + 1 - c, c' = a + b + 1 - c$ and $z = 1 - \frac{1}{w}$, the left-hand side is calculated to be

\[ 2F_1^1 \left( \begin{array}{c} a, a + 1 - c \\ a + b - c + 1 \end{array} \bigg| 1 - \frac{1}{w} \right)(\sigma) = \exp \{ \rho_w(\sigma)a \} \cdot [V_{1, \infty}(\sigma)(w)]_{11} \]

for an even unitary associator $\varphi$. The right-hand side is calculated to be

\[ \exp \{ \rho_w(\sigma)(a - b) \} \cdot 2F_1^1 \left( \begin{array}{c} b, b + 1 - c \\ a + b - c + 1 \end{array} \bigg| 1 - \frac{1}{w} \right)(\sigma) = \exp \{ \rho_w(\sigma)a \} \cdot sw_{a,b} ([V_{1, \infty}(\sigma)(w)]_{11}) \]

where $sw_{a,b}$ means the automorphism of $R[\frac{1}{2}]$ switching $a$ and $b$. Hence, it is sufficient to show that $[V_{1, \infty}(\sigma)(z)]_{11}$ is invariant under the switch $sw_{a,b}$. The matrix is calculated to be

\[ V_{1, \infty}(\sigma)(z) = \left[ G_{1, \infty}^\varphi (X, -Y)(\sigma)(z) \cdot \left( \begin{array}{c} 1 \\ 0 \end{array} \bigg| \frac{1}{z} \right) \right] \cdot \frac{ab + pq}{ab - pq} \cdot \frac{a - 1}{b - 1}. \]

It is evident that the last matrix is invariant under the switch $sw_{a,b}$. The first row of the product $\left[ G_{1, \infty}^\varphi (X, -Y)(\sigma)(z) \cdot \left( \begin{array}{c} 1 \\ 0 \end{array} \bigg| \frac{1}{z} \right) \right]$ is also invariant under the switch $sw_{a,b}$. This is because the $(1, 1)$ entry is calculated to be

\[ 1 + ab \sum_{k, n, s > 0 \atop k > n + s, n > s} h_0(k, n, s)p^{k - n - s}q^{n - s}(ab + pq)^{s - 1} \]

by (31) (65) and the $(1, 2)$ entry is calculated to be

\[ \exp \{ \rho_{1 - \frac{1}{2}}(\sigma)p \} \cdot \{ 1 + (ab + pq) \sum_{k, n, s > 0 \atop k > n + s, n > s} h_0(k, n, s)(-p)^{k - n - s}q^{n - s}(ab)^{s - 1} \} \]

by (31) (66) and its following remark. Here, $h_0(k, n, s)$ is given by

\[ \sum_{W \in e_0U[a_1] : \text{word wt}(W) = k, \text{dp}(W) = n, \text{ht}(W) = s} (1)^n (G_{1, \infty}^\varphi (e_0, e_1)(\sigma)(z) \mid W) \]

Whence we obtain the equality. \qed
The following proposition enables us to extend our $\ell$-adic hypergeometric function $\,_{2}F_{1}(a,b\mid c)(\sigma)$ defined when $|a|_{\ell}, |b|_{\ell}, |c-1|_{\ell} < 1$ to the parameter $(a, b, c)$ in $\mathcal{D}$ by

$$2F_{1}\left(\frac{a, b}{c} \mid z\right)(\sigma) := 2F_{1}\left(\frac{a}{c} \mid z\right)(\sigma) \quad \text{when } |a|_{\ell}, |b|_{\ell}, |c|_{\ell} < 1 \text{ with } c \neq 0,$$

$$2F_{1}\left(\frac{a, b}{c} \mid z\right)(\sigma) := 2F_{1}\left(\frac{a}{c} \mid z\right)(\sigma) \quad \text{when } |a|_{\ell}, |b-1|_{\ell}, |c-1|_{\ell} < 1.$$ 

**Proposition 4.5.** (1) When $|a'|_{\ell}, |b'|_{\ell}, |c'|_{\ell} < 1$ with $c' \neq 0$, the evaluation of $(4.5)$ to $a', b', c'$ converges.

(2) When $|a''|_{\ell}, |b''-1|_{\ell}, |c''-1|_{\ell} < 1$, the evaluation of $(4.6)$ to $a'', b'', c''$ converges.

**Proof.** It is sufficient to show that both $G_{10}^{\sigma}(X, -Y)(\sigma)(z)$ and $G_{0\infty}^{\sigma}(X, -Y)(\sigma)(z)$ converge when $(0.3)$ holds by $(4.7)$ and $(4.8)$. They actually converge by $(3.12)$, $(1.2)$ and $(4.3)$.

The $\ell$-adic analogue of Euler’s transformation theorem can be derived as follows:

**Proof of Theorem 4.3** The claim follows from Theorem 4.4 and Proposition 4.5 (1) because we have $p_{1-z}(\sigma) \in \mathbb{Z}_{\ell}$.

**Remark 4.6.** Our arguments for deducing results in the $\ell$-adic situation from those in the complex case that are observed in the proof of Theorem 4.4 allow us to show the following $\ell$-adic analogues of the six formulae (1.7) of Kummer’s solutions:

$$(1, 0) \cdot V_{\mathcal{D}}(\sigma)(z) = \left(2F_{1}\left(\frac{a, b}{c} \mid z\right)(\sigma), \quad \exp((1 - c)\rho_{s}(\sigma)) \cdot 2F_{1}\left(\frac{(b+1-c, a+1-c)}{2-c} \mid z\right)(\sigma)\right),$$

$$(1, 0) \cdot V_{\mathcal{I}}(\sigma)(z) = \left(2F_{1}\left(\frac{a, b}{a+b+c} \mid 1-z\right)(\sigma), \quad \exp((c-a-b)\rho_{s-1}(\sigma)) \cdot 2F_{1}\left(\frac{(c-a-b)}{a+b+c} \mid 1-z\right)(\sigma)\right),$$

$$(1, 0) \cdot V_{\mathcal{I}_{\infty}}(\sigma)(z) = \left(2F_{1}\left(\frac{a, a+1-c}{a+b+c} \mid 1-z\right)(\sigma), \quad \exp((b-c)\rho_{s}(\sigma)) \cdot \exp((c-a-b)\rho_{s-1}(\sigma)) \cdot 2F_{1}\left(\frac{1-b-c-b}{1-a-b-c} \mid 1-z\right)(\sigma)\right),$$

$$(1, 0) \cdot V_{\mathcal{S}}(\sigma)(z) = \left(2F_{1}\left(\frac{a, a+1-c}{a+b+c} \mid 1-z\right)(\sigma), \quad \exp((c-a-b)\rho_{s}(\sigma)) \cdot 2F_{1}\left(\frac{(b+1-c, a+1-c)}{2-c} \mid z\right)(\sigma)\right),$$

$$(1, 0) \cdot V_{\mathcal{S}_{\infty}}(\sigma)(z) = \left(2F_{1}\left(\frac{a, a+1-c}{a+b+c} \mid 1-z\right)(\sigma), \quad \exp((b-c)\rho_{s}(\sigma)) \cdot \exp((c-a-b)\rho_{s-1}(\sigma)) \cdot 2F_{1}\left(\frac{1-b-c-b}{1-a-b-c} \mid 1-z\right)(\sigma)\right),$$

$$(1, 0) \cdot V_{\mathcal{S}_{\infty}}(\sigma)(z) = \left(2F_{1}\left(\frac{a, a+1-c}{a+b+c} \mid 1-z\right)(\sigma), \quad \exp((b-c)\rho_{s}(\sigma)) \cdot \exp((c-a-b)\rho_{s-1}(\sigma)) \cdot 2F_{1}\left(\frac{1-b-c-b}{1-a-b-c} \mid 1-z\right)(\sigma)\right).$$

**Remark 4.7.** Finite field analogues of hypergeometric functions have been discussed in the literature; see [20, 21, 24, 32], etc. Since they are related to the trace of Frobenius of certain $\ell$-adic Galois representations and our $\ell$-adic function is constructed from a Galois representation, it would be worthwhile to identify any relationship between their hypergeometric functions and ours.

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Lemma A.1. The matrix $ev_{(X, -Y)}(\varphi(e_\infty, e_1))$ in $\text{Mat}_2(\mathbb{K}[[a, b, c - 1]])$ is independent of any choice of even unitary associators $\varphi$.

Proof. For any group-like series $\varphi(e_0, e_1) \in \mathbb{K}[[e_0, e_1]]$, write $P_\varphi := ev_{(X, -Y)}(\varphi(e_0, e_1)) \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1/\delta \end{pmatrix}$ and $Q_\varphi = ev_{(X, -Y)}(\varphi(e_\infty, e_1)) \cdot \begin{pmatrix} 1/\delta & 1 \\ -1 & -1 \end{pmatrix}$ in $\text{GL}_2(\mathbb{K}[[a, b, c - 1]])$.

Specifically, when $\varphi = G_{01}(e_0, e_1)(z)$, by (1.2), (1.6) and (1.7), we have

$$P_\varphi = ev_{(X, -Y)}(G_{01}(e_0, e_1)(z)) \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1/\delta \end{pmatrix} = V_{01}(z)$$

$$Q_\varphi = ev_{(X, -Y)}(G_{01}(e_\infty, e_1)(z)) \cdot \begin{pmatrix} 1/\delta & 1 \\ -1 & -1 \end{pmatrix} = V_{\infty 1}(1/z)$$

with $z^a := \exp(\log z \cdot a)$, $z^b := \exp(\log z \cdot b)$ and $z^p := \exp(\log z \cdot p)$. Then, by the same arguments as in the proof of Proposition 4.3, we deduce the following validity for each entry from the above two expressions in the complex case when $\varphi$ is commutator group-like:

$$[Q_\varphi]_{11} = \iota([P_\varphi]_{11}),$$

$$[Q_\varphi]_{12} = \iota([P_\varphi]_{12}),$$

$$[Q_\varphi]_{21} = -\frac{a}{b} \iota([P_\varphi]_{11}) - \frac{1}{b} \iota(b[P_\varphi]_{21}),$$

$$[Q_\varphi]_{22} = -\iota([P_\varphi]_{12}) - \frac{1}{b} \iota(b[P_\varphi]_{22} - p[P_\varphi]_{12})$$

where $\iota$ is the map of (1.12).

Assume that $\varphi$ is an even unitary associator. Then, we have $P_\varphi = ev_{(X, -Y)}(\varphi) = M_+ \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1/\delta \end{pmatrix}$ by Theorem 3.3. Whence $P_\varphi$ is free from any choice of even unitary associators by Theorem 0.3. By the above four equalities, we see that $Q_\varphi$ is so. By $Q_\varphi = ev_{(X, -Y)}(\varphi(e_\infty, e_1)) \cdot \begin{pmatrix} 1/\delta & 1 \\ -1 & -1 \end{pmatrix}$, we learn that $ev_{(X, -Y)}(\varphi(e_\infty, e_1))$ is free from any choice of even unitary associators. □

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