TRACE FORMULAE OF POTENTIALS FOR DEGENERATE PARABOLIC EQUATIONS

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ABSTRACT. In this paper, we analyze main properties of the volume and layer potentials as well as the Poisson integral for a multi-dimensional degenerate parabolic equation. As consequences, we obtain trace formulae of the heat volume potential and the Poisson integral which solve Kac’s problem for degenerate parabolic equations in cylindrical domains.

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1. INTRODUCTION

The layer potential method (or potential theory) for parabolic equations has a long history (see, e.g. [1]) and it has been intensively applied to solve initial and initial-boundary value problems of parabolic partial differential equations throughout the last decades. To construct the method, elements of the potential theory, namely, the (heat) volume potential/Poisson integral, the single layer potential and the double layer potential play a key role. Although many of the basic ideas of the potential theory already exist and are intensively being studied, still specific (nonclassical) partial differential equations are required for their development and new approaches.

In [9], the author studied the following one-dimensional degenerate-type parabolic equation in the semi-infinite domain

\[ \frac{\partial u(x,t)}{\partial t} - a(t) \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \quad x > 0, \quad t > 0, \]

where \( f(x,t) \) is bounded in the strip \( \mathbb{R} \times [0,T], \) \( 0 < T < \infty. \)

Here the coefficient \( a(t) \) satisfies one of the following two assumptions:

i. \( a(t) \) is nonnegative and becomes zero only at isolated points;

ii. \( a(t) \) is nonnegative and becomes zero only at isolated points...
ii. A function \( a_1(t) \) defined by

\[
a_1(t) := \int_0^t a(z) \, dz
\]

is positive for all \( t > 0 \), allowing \( a(t) \) to be negative in an interval.

In particular, the author obtained solutions of the initial boundary value problems for equation (1.1) by using the potential theory.

The goal of the present paper is to construct the potential theory for the multi-dimensional version of the degenerate parabolic equation (1.1) and to analyse its consequences. To achieve this aim, first by using the Fourier transform we find the fundamental solution of the multi-dimensional degenerate parabolic equation in an explicit form. Then we develop “degenerate” potential theory, which is based on a use of the explicit representation of the fundamental solution for analysing, in this setting, a complete parallel of the classical heat potential and regularity theory. Note that our ideas are also closely related to the recent development on the potential theory of hypoelliptic differential equations (see [10, Chapter 11]).

Thus, in this paper, we present “degenerate” versions of the volume (heat) potential, the Poisson integral, the double and single layer potentials. In addition, their main properties will be discussed in details. As consequences, we consider Cauchy problems and initial-boundary value problems in cylindrical domains.

Moreover, we are also interested in the question that what boundary condition can be put on the “degenerate” volume potential (and Poisson integral) on the lateral boundary of the cylindrical domain so that the degenerate parabolic equation with this boundary condition would have a unique solution in the cylindrical domain, which is still given by the same formula of the “degenerate” volume potential (and Poisson integral, correspondingly). In turn, it allows finding the trace of the “degenerate” volume potential (and Poisson integral) to the lateral boundary of the cylindrical domain. So, in the present paper, boundary conditions for the “degenerate” volume potential and Poisson integral are established. The obtained boundary conditions are nonlocal in the space variables. In the one-dimensional case, this problem was studied in [11]. The multi-dimensional version gives a new insight, that is, the constructed new (nonlocal) initial-boundary value problem can serve as an example of an explicitly solvable initial-boundary value problem in any cylindrical domain (with a smooth lateral surface) for the degenerate parabolic equation.

Note that the origin of the question goes back to M. Kac’s lecture [3] (cf. [5] and [6]). Therefore, the analogical questions for the elliptic and hypoelliptic cases are called Kac’s problems. For discussions in this direction, we refer [10, Chapter 11] as well as references therein.

The brief outline of the paper is as follows: in Section 2, we discuss Cauchy problems for the multi-dimensional degenerate parabolic equation and find its fundamental solution explicitly. We prove the existence and uniqueness theorems for the Cauchy problems. In Section 3, we analyse layer potentials, in particular, we obtain continuity results and jump relations. Finally, in Section 4, we present trace formulae for the volume potential and Poisson integral.
2. Fundamental solution and Cauchy problems

We consider the degenerate parabolic equation

\[ \partial_a u(x, t) := \frac{\partial u(x, t)}{\partial t} - a(t) \Delta_x u(x, t) = f(x, t), \]  

posed in a cylindrical domain \((x, t) \in \Omega \times (0, T), 0 < T < \infty\), where the domain \(\Omega\) is bounded in \(\mathbb{R}^n, n \geq 2\), with Lyapunov boundary \(\partial \Omega \in C^{1+\lambda}, 0 < \lambda < 1\), \(f\) is any given function. Here and throughout this paper the coefficient \(a(t) \in L^1[0, T]\) is defined in \([0, T]\) and satisfies one of the following two assumptions:

(a) \(a(t)\) is nonnegative and becomes zero only at isolated points;

(b) A function \(a_1(t)\) defined by

\[ a_1(t) := t \int_0^t a(z) \, dz \]

is positive for all \(t > 0\), allowing \(a(t)\) to be negative in an interval.

In our computations, we also use a function \(b(t, \tau)\) defined by the formula

\[ b(t, \tau) := t \int_\tau^t a(z) \, dz = a_1(t) - a_1(\tau), \quad (b(t, 0) = a_1(t)). \]

Note that if \(a(t)\) satisfies the assumption (a), then \(b(t, \tau)\) is positive for all \(t > \tau > 0\).

First of all, we present the fundamental solution of equation (2.1) by using the Fourier transform in an explicit form.

**Lemma 2.1.** Under the assumption (b) the fundamental solution of equation (2.1) can be represented as

\[ \varepsilon_{n,b}(x, t) := \varepsilon_n(x, a_1(t)) = \theta(t) e^{-\frac{|x|^2}{4a_1(t)}} \left( \frac{4\pi a_1(t)}{t} \right)^{n/2}, \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}, \]  

where \(\varepsilon_n\) is the fundamental solution of the standard heat operator, \(\theta\) is the Heaviside function and \(|x| = \sqrt{x_1^2 + \ldots + x_n^2}\) is the usual Euclidean norm.

**Proof of Lemma 2.1.** Consider the equation

\[ \frac{\partial \varepsilon(x, t)}{\partial t} - a(t) \Delta_x \varepsilon(x, t) = \delta(x)\delta(t), \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}, \]  

where \(\delta\) is the Dirac distribution. Under the assumption (b) the fundamental solution of equation (2.1) can be explicitly found by using the Fourier transform. So, applying the Fourier transform \(F_x\) to equation (2.3), we obtain that

\[ \frac{\partial \hat{\varepsilon}(\xi, t)}{\partial t} + a(t)|\xi|^2 \hat{\varepsilon}(\xi, t) = 1(\xi)\delta(t), \quad (\xi, t) \in \mathbb{R}^n \times \mathbb{R}, \]

where

\[ \hat{\varepsilon}(\xi, t) = F_x[\varepsilon](\xi, t) = \int_{\mathbb{R}^n} \varepsilon(x, t)e^{i\xi \cdot x} \, dx, \quad (i^2 = -1), \]
1(\xi) is the identity function in \mathbb{R}^n and the inner product in \mathbb{R}^n is denoted by \langle \cdot, \cdot \rangle. The solution of equation (2.4) is
\[ e_\theta(t) = \theta(t) e^{-|\xi|^2 a_1(t)}, \quad (\xi, t) \in \mathbb{R}^n \times \mathbb{R}. \]
Applying the inverse Fourier transform and its properties to the solution of equation (2.4), we obtain (2.2). This completes the proof. \qed

Note that the assumption (a) is the special case of the assumption (b).

With substitution of the variables \( \xi_i = \frac{\xi_i}{2 \sqrt{a_1(t)}} \), \( i = 1, ..., n \), we have
\[
\int_{\mathbb{R}^n} e_n(b(x, t))dx = \frac{1}{(4 \pi a_1(t))^\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4a_1(t)}}dx
= \prod_{i=1}^{n} \frac{1}{\pi^\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\xi_i^2} d\xi_i = 1, \quad t > 0. \tag{2.5}
\]
Moreover, the fundamental solution \( e_n(b(x, t)) \) has the property
\[
e_n(b(x, t)) \to \delta(x) \quad \text{with} \quad t \to 0^+, \tag{2.6}
\]
for all \( x \in \mathbb{R}^n \).

Let us show (2.6). Let \( \psi \) be an infinitely many times differentiable function in \( \mathbb{R}^n \) with compact support. Then by using the polarization formula
\[
\int_{\mathbb{R}^n} \tilde{f}(|x|)dx = \omega_n \int_{0}^{\infty} \tilde{f}(r)r^{n-1}dr,
\]
where \( \tilde{f} \) is any integrable function in \( \mathbb{R}^n \), \( \omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \) and by using the mean value theorem, we obtain
\[
\left| \int_{\mathbb{R}^n} e_n(b(x, t))(\psi(x) - \psi(0))dx \right| \leq \frac{A}{(4 \pi a_1(t))^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4a_1(t)}}|x|dx \leq \frac{A\omega_n}{(4 \pi a_1(t))^{\frac{n}{2}}} \int_{0}^{\infty} e^{-\frac{u^2}{4a_1(t)}}u^n du = 2A\omega_n a_1(t),
\]
where \( A \) is a positive constant. Since the function \( a_1(t) \) is continuous and nonnegative in \([0, T]\), by virtue of (2.5), we obtain (2.6), that is,
\[
(e_n(b(x, t), \psi(x)) := \int_{\mathbb{R}^n} e_n(b(x, t))\psi(x)dx = \psi(0) \int_{\mathbb{R}^n} e_n(b(x, t))dx
+ \int_{\mathbb{R}^n} e_n(b(x, t))(\psi(x) - \psi(0))dx \to (\delta(x), \psi(x)) := \psi(0), \quad \text{as} \quad t \to 0^+.
\]

Let \( e_n(a(x - \xi, b(t, \tau)) := e_n(x - \xi, a_1(t) - a_1(\tau)) \).

Since under the assumption (a) we have \( b(t, \tau) > 0 \) for all \( t > \tau > 0 \), it is easy to check that
\[
\int_{\mathbb{R}^n} e_n(a(x - \xi, b(t, \tau))d\xi = 1, \quad t > \tau > 0, \quad x \in \mathbb{R}^n. \tag{2.7}
\]
The degenerate parabolic potential defined by
\[(Vf)(x,t) := \int_0^t \int_\Omega \varepsilon_{n,a}(x - \xi, b(t, \tau)) f(\xi, \tau) d\xi d\tau, \quad x \in \Omega, \ 0 < t < T, \quad (2.8)\]
is called the volume potential, where \(f\) is bounded in \(\Omega \times [0, T]\) with \(\text{supp} f(\cdot, t) \subset \Omega\) for all \(t \in [0, T]\).

**Theorem 2.2.** Let \(a(t)\) satisfy the assumption (a) and \(f\) be a bounded function in the strip \(\Omega \times [0, T]\) with \(\text{supp} f(\cdot, t) \subset \Omega\) for all \(t \in [0, T]\). Then the volume potential with the density \(f\) (2.8) admits the estimate
\[| (Vf)(x,t) | \leq t \sup_{(\xi,\tau) \in \Omega \times [0,t]} | f(\xi, \tau) |, \quad x \in \Omega, \ 0 < t < T, \quad (2.9)\]
and solves equation (2.1) with the zero initial condition
\[u(\cdot, t) \to 0 \text{ as } t \to 0^+, \quad \text{in } \Omega. \quad (2.10)\]

**Proof of Theorem 2.2.** Since \(\text{supp} f(\cdot, t) \subset \Omega\) for all \(0 \leq t \leq T\), it is obvious that
\[(Vf)(x,t) = \int_0^t \int_\Omega \varepsilon_{n,a}(x - \xi, b(t, \tau)) f(\xi, \tau) d\xi d\tau \]
\[= \int_0^t \int_{\mathbb{R}^n} \varepsilon_{n,a}(x - \xi, b(t, \tau)) f(\xi, \tau) d\xi d\tau, \quad x \in \Omega, \ t \in (0, T). \]

Thus, by virtue of (2.7), we obtain (2.9)
\[| (Vf)(x,t) | \leq \sup_{(\xi,\tau) \in \Omega \times [0,t]} | f(\xi, \tau) | \int_0^t \int_{\mathbb{R}^n} \varepsilon_{n,a}(x - \xi, b(t, \tau)) d\xi d\tau \]
\[= t \sup_{(\xi,\tau) \in \Omega \times [0,t]} | f(\xi, \tau) |, \quad (x, t) \in \Omega \times (0, T). \]

A direct calculation gives that the volume potential \(Vf\) satisfies equation (2.1). Also, we observe that estimate (2.9) ensures convergence of (2.10). \(\Box\)

The degenerate parabolic potential defined by
\[(P\varphi)(x,t) := \int_\Omega \varepsilon_{n,b}(x - \xi, t) \varphi(\xi) d\xi, \quad x \in \Omega, \ 0 < t < T, \quad (2.11)\]
is called the Poisson potential (see, e.g. [12], p. 153), where \(\varphi\) is a bounded function in \(\mathbb{R}^n\) with \(\text{supp} \varphi \subset \Omega\) and \(\varepsilon_{n,b}(x - \xi, t) = \varepsilon_n(x - \xi, a_1(t))\).

**Theorem 2.3.** Let \(a(t)\) satisfy the assumption (b). Let \(\varphi\) be a bounded function in \(\mathbb{R}^n\) with \(\text{supp} \varphi \subset \Omega\). Then the Poisson integral (2.11) admits the estimate
\[| (P\varphi)(x,t) | \leq \sup_{\xi \in \Omega} | \varphi(\xi) |, \quad x \in \Omega, \ 0 < t < T, \quad (2.12)\]
and solves the equation
\[\triangle_a u = 0, \quad \text{in } \Omega \times (0, T). \quad (2.13)\]

Moreover, if \(\varphi\) is a continuous bounded function in \(\mathbb{R}^n\) with \(\text{supp} \varphi \subset \Omega\), then the Poisson integral \(P\varphi\) belongs to the class \(C^\infty\) and satisfies the initial condition
\[u(\cdot, 0) = \varphi, \quad \text{in } \Omega, \quad (2.14)\]
providing its continuous extension to $\Omega \times [0,T)$.

Proof of Theorem 2.3. Since $\text{supp} \, \varphi \subset \Omega$, it is obvious that

$$(P\varphi)(x,t) = \int_{\Omega} \varphi(\xi)\varepsilon_{n,b}(x-\xi,t) d\xi$$

$$= \int_{\mathbb{R}^n} \varphi(\xi)\varepsilon_{n,b}(x-\xi,t) d\xi, \quad (x,t) \in \Omega \times (0,T).$$

For $x \in \Omega$ and $0 < t < T$, we have the estimate

$$|(P\varphi)(x,t)| \leq \sup_{\xi \in \mathbb{R}^n} |\varphi(\xi)| \int_{\mathbb{R}^n} \varepsilon_{n,b}(x-\xi,t) d\xi = \sup_{\xi \in \Omega} |\varphi(\xi)|.$$

Since for all $x \in \Omega$ and $t \in (0,T)$ differentiation and integration can be interchanged in (2.11), it is straightforward to check that $P\varphi$ satisfies (2.13).

Let $\varphi$ be a continuous bounded function in $\mathbb{R}^n$ with $\text{supp} \, \varphi \subset \Omega$. Taking into account (2.6), we see that $P\varphi$ satisfies initial condition (2.14). Now we substitute $\xi = x + 2\sqrt{a_1(t)}z$ to obtain

$$P(\varphi)(x,t) = \frac{1}{\pi^\frac{n}{2}} \int_{\mathbb{R}^n} \varphi(x + 2\sqrt{a_1(t)}z)e^{-|z|^2} dz.$$

The assumption for $\varphi$ provides its boundedness and uniformly continuity. Let $M_\varphi > 0$ be an upper bound for $\varphi$. Since $\varphi$ is a uniformly continuous function, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|\varphi(x) - \varphi(\xi)| < \frac{\varepsilon}{2}$ for all $x, \xi \in \mathbb{R}^n$ with $|x - \xi| < \delta$. Then for any $\varepsilon > 0$ we can choose $r > 0$ such that

$$\frac{1}{\pi^\frac{n}{2}} \int_{|z| \geq r} e^{-|z|^2} dz \leq \frac{\varepsilon}{4M_\varphi}.$$

Since $a_1(t)$ is a continuous function in $[0,T]$, for any $\eta > 0$ there exists $\delta_\eta > 0$ such that $|a_1(t)| < \eta$ for all $t \in [0,T]$ with $t < \delta_\eta$. Setting $\eta = \frac{\varepsilon^2}{4M_\varphi}$ and using the fact that for $|z| \leq r$ and $t < \delta_\eta$ we have $2\sqrt{a_1(t)}z < 2\sqrt{\eta}r = \delta$, we deduce that

$$\left| \frac{1}{(4\pi a_1(t))^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-\xi|^2}{4a_1(t)}} \varphi(\xi) d\xi - \varphi(x) \right|$$

$$= \left| \frac{1}{\pi^\frac{n}{2}} \int_{\mathbb{R}^n} \left( \varphi(x + 2\sqrt{a_1(t)}z) - \varphi(x) \right) e^{-|z|^2} dz \right|$$

$$< \frac{\varepsilon}{2\pi^\frac{n}{2}} \int_{|z| \leq r} e^{-|z|^2} dz + \frac{2M_\varphi}{\pi^\frac{n}{2}} \int_{|z| \geq r} e^{-|z|^2} dz < \varepsilon,$$

for all $x \in \mathbb{R}^n$ and $t < \delta_\eta$. This implies continuity of the potential $P(\varphi)$ at $t = 0$ and $P(\varphi)(\cdot,0) = \varphi$ in $\Omega$.

3. Layer potentials

Let $a(t)$ satisfy the assumption (a) and $\varphi \in C(\partial \Omega \times [0,T])$. Then the single layer potential for the degenerate parabolic equation (2.1) can be defined by
Proof of Theorem 3.1. The single layer potential with bounded measurable density \( \varphi \) is continuous in \( \mathbb{R}^n \times \mathbb{R}_+ \). In particular, it is continuous across the boundary \( \partial \Omega \).

**Proof of Theorem 3.1.** If we prove that \( \varepsilon_{n,a}(x - \xi, b(t, \tau))a(\tau) \) is locally integrable, the proof follows from [2, p. 7, Lemma 1]. So, let us show \( \varepsilon_{n,a}(x - \xi, b(t, \tau))a(\tau) \) is locally integrable.

We have

\[
S^\beta e^{-s} \leq \beta^\beta e^{-\beta},
\]

for all \( 0 < s, \beta < \infty \). Using (3.3) for the case \( s = \frac{|x - \xi|^2}{4b(t, \tau)} \), \( \beta = \frac{n}{2} - \gamma \), we have

\[
|\varepsilon_{n,a}(x - \xi, b(t, \tau))a(\tau)| \leq \frac{C|a(\tau)|}{|x - \xi|^\gamma(b(t, \tau))},
\]

where \( 0 < \gamma < \frac{n}{2} \). Hence, choosing \( \frac{1}{2} < \gamma < 1 \), we observe that \( \varepsilon_{n,a}(x - \xi, b(t, \tau))a(\tau) \) is locally integrable. This completes the proof. \( \square \)

A direct calculation gives that the double layer potential and single layer potential are infinitely many times differentiable solutions of (2.13) in \( \Omega \times (0, T) \). Both layer potentials can be continuously extended to \( \Omega \times [0, T] \) by setting \( (D\varphi)(x, 0) = 0 \) and \( (S\varphi)(x, 0) = 0 \) for all \( x \in \Omega \).

We pay special attention to the boundary behaviour of the gradient of the single layer potential \( (S\varphi)(x, t) \) when \( \Omega \ni x \to x_0 \in \partial \Omega \) along nontangential directions. For any \( x_0 \in \partial \Omega \), we denote by \( K = K(x_0) \) a finite closed cone in \( \mathbb{R}^n \) with vertex \( x_0 \) such that \( K(x_0) \subset \Omega \cup \{x_0\} \). As in [2] and [7], we prove the following theorem.

**Theorem 3.2.** Let \( \partial \Omega \in C^{1+\lambda} \), \( 0 < \lambda < 1 \). Let \( \varphi \) be a continuous function on \( \partial \Omega \times [0, T] \). Then, for any \( x_0 \in \partial \Omega \) and \( t \in (0, T] \), the single layer potential (3.1) satisfies the jump relation

\[
\lim_{x \to x_0} \langle \nabla_x (S\varphi)(x, t), \nu(x_0) \rangle = \frac{1}{2} \varphi(x_0, t) + \int_0^t \int_{\partial \Omega} \frac{\partial \varepsilon_{n,a}(x_0 - \xi, b(t, \tau))}{\partial \nu(x_0)} \varphi(\xi, \tau)a(\tau)dS_\xi d\tau,
\]

where the limit is taken along the outward normal \( \nu(x_0) \) and \( \nabla_x \) is the usual gradient.

**Proof of Theorem 3.2.** For convenience of a reader, let us rewrite the formula of the single layer potential

\[
(S\varphi)(x, t) = \int_0^t \int_{\partial \Omega} \varepsilon_{n,a}(x - \xi, b(t, \tau))\varphi(\xi, \tau)a(\tau)dS_\xi d\tau.
\]
Let $T(x_0)$ denote the tangent hyperplane to the boundary $\partial \Omega$ at the point $x_0$ and $\partial \Omega_r := B(x_0, r) \cap \partial \Omega$, where $B(x_0, r)$ is the open ball of radius $r > 0$ centred at the point $x_0$ in $\mathbb{R}^n$. Since $\partial \Omega \in C^{1+\lambda}$, $0 < \lambda < 1$, if $r > 0$ is small enough, the orthogonal projection $\Phi : \partial \Omega_r \to T(x_0)$ is one-to-one map. We denote its image by $\partial \Omega_r := \Phi(\partial \Omega_r)$. For convenience we split the inner product in (3.4) into two parts

$$
\langle \nabla_x (S\varphi)(x, t), \nu(x_0) \rangle = I_r(x, t) + J_r(x, t),
$$

with

$$
\langle \nabla_x \varepsilon_{n,a}(x - \xi, b(t, \tau)), \nu(x_0) \rangle = -\frac{\langle x - \xi, \nu(x_0) \rangle}{2^{n+1}b(t, \tau)} e^{-\frac{|x-\xi|^2}{4a(t)}}
$$

and $I_r(x, t)$ is defined by

$$
I_r(x, t) := \int_0^t \int_{\partial \Omega_r} \langle \nabla_x \varepsilon_{n,a}(x - \xi, b(t, \tau)), \nu(x_0) \rangle \varphi(\xi, \tau) a(\tau) d^2S_\xi d\tau,
$$

and the complementary part $J_r(x, t)$ is defined by

$$
J_r(x, t) := \int_0^t \int_{\partial \Omega \setminus \partial \Omega_r} \langle \nabla_x \varepsilon_{n,a}(x - \xi, b(t, \tau)), \nu(x_0) \rangle \varphi(\xi, \tau) a(\tau) d^2S_\xi d\tau.
$$

Also, we denote

$$
I'_r(x, t) := \int_0^t \int_{\partial \Omega_r} \langle \nabla_x \varepsilon_{n,a}(x - \xi', b(t, \tau)), \nu(x_0) \rangle \varphi(x_0, \tau) a(\tau) d^2S_{\xi'} d\tau,
$$

where $dS_{\xi'}$ is the surface element (at $\xi'$) on $T(x_0)$. To prove (3.4), we show that

$$
\lim_{x \to x_0} I'_r(x, t) = \frac{1}{2} \varphi(x_0, t),
$$

$$
\lim_{x \to x_0} J_r(x, t) = J_r(x_0, t),
$$

and

$$
\lim_{x \to x_0} (I_r(x, t) - I'_r(x, t)) = I_r(x_0, t).
$$

**Proof of (3.8).** Let us introduce a new variable $\tau \leftrightarrow \rho = \frac{|x-\xi'|^2}{4a(t)}$ in (3.7). The substitution gives an implicit function $\tau = \tau(\rho)$ with $\frac{|x-\xi'|^2}{4a(t)} \leq \rho < \infty$. Then integrating with respect to $\tau$, we obtain

$$
I'_r(x, t) = \int_{\partial \Omega_r} \frac{\langle x - \xi', \nu(x_0) \rangle}{|x - \xi'|^n} \psi(x, \xi', t) d^2S_{\xi'},
$$

where

$$
\psi(x, \xi', t) := -\frac{1}{2} \pi^{-\frac{n}{2}} \int_{|x-\xi'|^2 = 4a(t)}^\infty \rho^{\frac{n}{2} - 1} e^{-\rho} \varphi(x_0, \tau(\rho)) d\rho.
$$

So $\psi(x, \xi', t)$ is a continuous function of $(x, \xi')$ for all $x = x_0 - l\nu(x_0)$ with $l > 0$ sufficiently small and for all $\xi' \in \partial \Omega_r$. In particular, we have

$$
\lim_{x \to x_0} \psi(x, \xi', t) = -\frac{1}{2} \pi^{-\frac{n}{2}} \varphi(x_0, t) \int_0^\infty \rho^{\frac{n}{2} - 1} e^{-\rho} d\rho.
$$
Since the integral is a value of the Gamma function and is equal to $\Gamma \left( \frac{2n}{3} \right) = \frac{2\pi \frac{2}{n}}{\omega_n}$, where $\omega_n$ is the area of the unit hypersphere in $\mathbb{R}^n$, we obtain

$$\psi(x_0, x, t) = -\frac{\varphi(x_0, t)}{\omega_n}.$$ 

We divide $\partial \Omega_r = \partial \Omega_{1r} \cup R_r$ into two parts such that the boundary $\partial \Omega_{1r}$ contains $x_0$ in its interior exactly as in $[2$, Theorem 1, Section 5.2$]. On $\partial \Omega_{1r}$, we change the variables $\xi'' = \frac{\xi - x}{|x|}$ and denote the domain of variation of $\xi''$ by $\partial \Omega_{1r}''$ and the corresponding area element by $dS_{\xi''}$. Since $\langle x - \xi', \nu(x_0) \rangle = -|x - \xi'|\cos(\xi' - x, \nu(x_0))$ and $\cos(\xi' - x, \nu(x_0))dS_{\xi''} = |x - \xi'|^{n-1}dS_{\xi''}$, we have

$$I_r'(x, t) = \frac{\varphi(x_0, t)}{\omega_n} \int_{\partial \Omega_{1r}''} dS_{\xi''} - \int_{\partial \Omega_{1r}''} (\psi(x, \xi', t) - \psi(x_0, x, t))dS_{\xi''} + R_r(x, t),$$

(3.11)

where the rest part of $I_r'(x, t)$ denoted by $R_r(x, t)$, that is, the $\xi'$ integration in $R_r(x, t)$ is taken over the set $R_r$. Since $\psi(x, \xi', t)$ is a continuous function of $(x, \xi')$, the second integral on the right-hand side of (3.11) can be arbitrarily small. In $R_r(x, t)$, it should be noted that $\langle x - \xi', \nu(x_0) \rangle \to 0$ as $x \to x_0$ and that if $\xi' \in R_r$, then $|x - \xi'|$ is bounded away from zero, which implies that the term $R_r(x, t)$ tends to zero. For the first term on the right-hand side of (3.11), we see that the boundary $\partial \Omega_{1r}''$ tends to a unit hemisphere, thus, the first term in (3.11) tends to $\frac{\varphi(x_0, t)}{2\omega_n}$. That is, we have proved (3.8).

Proof of (3.9). For the variable $\xi$ in (3.6) the inequality $|x - \xi| \geq \frac{r}{2} > 0$ holds for $|x - x_0| < \frac{r}{2}$. Hence, the integral is a continuous function, which implies (3.9).

Proof of (3.10). To prove (3.10), we take $r_1 > 0$ such that $r_1 < r$ and write

$$I_r(x, t) = I_{r_1}(x, t) + \overline{I}_{r_1}(x, t),$$

$$I_r(x_0, t) = I_{r_1}(x_0, t) + \overline{I}_{r_1}(x_0, t),$$

(3.12)

$$I_r'(x, t) = I_{r_1}'(x, t) + \overline{I}_{r_1}'(x, t),$$

where $\overline{I}_{r_1}(x, t) \left( \overline{I}_{r_1}'(x, t) \right)$ is the complementary part to $I_{r_1}(x, t) \left( I_{r_1}'(x, t) \right)$, that is, the $\xi$ ($\xi'$)-integration is taken over $\partial \Omega_r \setminus \partial \Omega_{r_1} \left( \partial \Omega_r' \setminus \partial \Omega_{r_1}' \right)$. Note that $I_r'(x_0, t) = 0$, since $\nu(x_0) \perp (x_0 - \xi')$.

Equality (3.10) will be proved by showing that for any $\varepsilon > 0$ there exists $r_1 > 0$ such that

$$|I_{r_1}(x, t) - I_{r_1}'(x, t)| < \varepsilon,$$

(3.13)

$$|\overline{I}_{r_1}(x, t) - \overline{I}_{r_1}(x_0, t)| < \varepsilon,$$

(3.14)

and

$$|I_{r_1}(x_0, t) - \varepsilon|.$$

(3.15)
Proof of (3.13). Note that
\[ |\xi - \xi'| \leq C|x_0 - \xi|^{1+\lambda}, \quad (3.16) \]
\[ 0 < C_1 \leq \frac{|x - \xi|}{|x - \xi'|} \leq C_2, \quad (3.17) \]
where \( C, C_1, \) and \( C_2 \) are constants. For the proofs of (3.16) and (3.17) we refer [2, p. 135]. Now by using (3.16) and (3.17), we obtain
\[ |\langle x - \xi, \nu(x_0) \rangle - \langle x - \xi', \nu(x_0) \rangle| = |\xi - \xi'| \leq C|x_0 - \xi|^{1+\lambda} \]
\[ \leq C|x - \xi|^{1+\lambda}. \quad (3.18) \]
By the mean value theorem and (3.17), we get
\[ \left| e^{\frac{|x - \xi|^2}{4b(t, \tau)}} - e^{\frac{|x - \xi'|^2}{4b(t, \tau)}} \right| \leq e^{\frac{|x - \xi|^2}{4b(t, \tau)}} \left| |x - \xi|^2 - |x - \xi'|^2 \right| \]
\[ \leq Ce^{\frac{|x - \xi|^2}{4b(t, \tau)}} \frac{|x - \xi|^2}{b(t, \tau)}, \quad (3.19) \]
where \( K \) and \( C \) are positive constants. Combining (3.18), (3.19) we obtain
\[ \left| \frac{a(\tau)}{b(t, \tau)} \right|^{1+\frac{\lambda}{2}} e^{\frac{|x - \xi|^2}{4b(t, \tau)}} - \left| \frac{a(\tau)}{b(t, \tau)} \right|^{1+\frac{\lambda}{2}} e^{\frac{|x - \xi'|^2}{4b(t, \tau)}} \]
\[ \leq C_1|x - \xi|^{1+\lambda} \frac{a(\tau)}{b(t, \tau)} \left| \frac{a(\tau)}{b(t, \tau)} \right|^{1+\frac{\lambda}{2}} e^{\frac{|x - \xi|^2}{4b(t, \tau)}} - e^{\frac{|x - \xi'|^2}{4b(t, \tau)}} \]
\[ + C_2 \frac{|x - \xi|^2}{b(t, \tau)} \frac{|x - \xi| a(\tau)}{b(t, \tau)} \left| \frac{a(\tau)}{b(t, \tau)} \right|^{1+\frac{\lambda}{2}} e^{\frac{|x - \xi|^2}{4b(t, \tau)}}. \quad (3.20) \]
Using (3.3) for the case \( s = \frac{|x - \xi|^2}{4b(t, \tau)} \beta = 1 + \frac{n}{2} - \gamma_1 \) to the first term of the last estimate of (3.20) and using (3.3) for the case \( s = \frac{K|x - \xi|^2}{4b(t, \tau)} \beta = 2 + \frac{n}{2} - \gamma_2 \) to the second term of the last estimate of (3.20), we see that the last estimate of (3.20) is bounded by
\[ \tilde{C}_1 a(\tau) \left| \frac{a(\tau)}{b(t, \tau)} \right|^{\frac{\gamma_1}{2}} |x - \xi|^{n+1-2\gamma_1 - \lambda} \]
\[ + \tilde{C}_2 a(\tau) \left| \frac{a(\tau)}{b(t, \tau)} \right|^{\frac{\gamma_2}{2}} |x - \xi|^{n+1-2\gamma_2}. \quad (3.21) \]
for \( 0 < \gamma_1 < 1 + \frac{n}{2} \) and \( 0 < \gamma_2 < 2 + \frac{n}{2} \), where \( \tilde{C}_1 \) and \( \tilde{C}_2 \) are positive constants. For \( 1 - \frac{\lambda}{2} < \gamma < 1 \) we can choose \( \gamma_1 \) and \( \gamma_2 \) such that \( \gamma_1 = \gamma \) and \( \gamma_2 = \gamma + \frac{\lambda}{2} \), thus term (3.21) is bounded by
\[ \tilde{C}_3 a(\tau) \left| \frac{a(\tau)}{b(t, \tau)} \right|^{\frac{\gamma}{2}} |x - \xi|^{n+1-2\gamma - \lambda}, \quad (3.22) \]
where \( \tilde{C} = \max \{ \tilde{C}_1, \tilde{C}_2 \} \). Hence, we have

\[
|I_{r_1}(x, t) - I'_{r_1}(x, t)| \leq \tilde{C} \int_0^t \int_{\partial \Omega_{r_1}} \frac{a(\tau)}{[b(t, \tau)]^{n+1-2\gamma-\lambda}} |x - \xi|^{n+1-2\gamma-\lambda} dS_\xi d\tau + \sup_{\partial \Omega_{r_1}} \left| \frac{\varphi(\xi, \tau)}{\cos(\nu(x_0), \nu(\xi))} - \varphi(x_0, \tau) \right| \times \left| \int_0^t \int_{\partial \Omega'_{r_1}} \frac{\partial \varepsilon_{n,a}(x - \xi', b(t, \tau))}{\partial \nu(x_0)} dS_\xi' d\tau \right|
\]  

(3.23)

for some \( 1 - \frac{\lambda}{2} < \gamma < 1 \) and \( 0 < \lambda < 1 \).

The integrand of the first term on the right-hand side of (3.23) is integrable, thus we can choose \( r_1 \) small enough to make the corresponding integral arbitrarily small. Since the second integral in (3.23) coincides with \( I' \) when \( r = r_1 \) and \( \varphi(x_0, \tau) \equiv 1 \), it is bounded independently of \( r_1 \). Since \( \varphi \) is a continuous function and \( \cos(\nu(x_0), \nu(\xi)) \rightarrow 1 \), the expression \( \sup | \cdot | \rightarrow 0 \) as \( r_1 \rightarrow 0 \). This completes the proof of (3.13).

**Proof of (3.14).** Since \( |x - \xi|, |x_0 - \xi| \) and \( |x - \xi'| \) in \( T_{r_1}(x, t), T'_{r_1}(x_0, t) \) and \( T'_{r_1}(x, t) \) are bounded away from zero, correspondingly, and \( \cos(\nu(x_0), \xi - x) \rightarrow 0 \) as \( x \rightarrow x_0 \), for any fixed \( r_1 \), we have (3.14), if \( x \) is close enough to \( x_0 \).

**Proof of (3.15).** Estimates in (3.20)-(3.21) imply

\[
|I_{r_1}(x_0, t)| \leq \int_0^t \int_{\partial \Omega_{r_1}} \frac{a(\tau)}{[b(t, \tau)]^{n+1-2\gamma-\lambda}} dS_\xi d\tau,
\]

for some \( 1 - \frac{\lambda}{2} < \gamma < 1 \). So, we have (3.15), if \( r_1 \) is sufficiently small.

As we have proved (3.10), combining together all the proofs, we arrive at

\[
\lim_{x \to x_0} \left\langle \nabla_x(S\varphi)(x, t), \nu(x_0) \right\rangle = \frac{1}{2} \varphi(x_0, t)
\]

\[
+ \int_0^t \int_{\partial \Omega_{r_1}} \frac{\partial \varepsilon_{n,a}(x_0 - \xi, b(t, \tau))}{\partial \nu(x_0)} \varphi(\xi, \tau) a(\tau) dS_\xi d\tau.
\]

\[
\square
\]

Note that for any \( x_0 \in \partial \Omega \) and \( t \in (0, T] \), the single layer potential satisfies the jump relation

\[
\lim_{x \to x_0} \left\langle \nabla_x(S\varphi)(x, t), n(x_0) \right\rangle = -\frac{1}{2} \varphi(x_0, t)
\]

\[
+ \int_0^t \int_{\partial \Omega} \frac{\partial \varepsilon_{n,a}(x_0 - \xi, b(t, \tau))}{\partial n(x_0)} \varphi(\xi, \tau) a(\tau) dS_\xi d\tau,
\]

(3.24)

where the limit is taken along the inward normal \( n(x_0) \) and \( K' := K'(x_0) \subset \overline{K} \cup \{ x_0 \} \).

The proof of relation (3.24) is similar to the one of Theorem 3.2. Now we show the jump relation for the double layer potential for the degenerate parabolic equation (2.1).
Theorem 3.3. The double layer potential (3.2) with the density \( \varphi \in C(\partial \Omega \times [0, T]) \) can be continuously extended from \( \Omega \times (0, T) \) to \( \overline{\Omega} \times (0, T) \) with the limiting values
\[
\lim_{x \to x_0} (D\varphi)(x, t) = \frac{1}{2} \varphi(x_0, t) + \int_0^t \int_{\partial \Omega} \frac{\partial \varepsilon_n,a(x_0 - \xi, b(t, \tau))}{\partial \nu(\xi)} \varphi(\xi, \tau) a(\tau) dS_\xi d\tau,
\]
for \( x_0 \in \partial \Omega \) and \( 0 < t \leq T \), where the time integral exists as an improper integral and \( \nu(\xi) \) is the outward normal.

Proof of Theorem 3.3. For the proof we use the same technique in Theorem 3.2. □

Consider the operator \( D : C(\partial \Omega \times [0, T]) \to C(\partial \Omega \times [0, T]) \) defined by
\[
(D\varphi)(x, t) := \int_0^t \int_{\partial \Omega} \frac{\partial \varepsilon_n,a(x - \xi, b(t, \tau))}{\partial \nu(\xi)} \varphi(\xi, \tau) a(\tau) dS_\xi d\tau,
\]
for \( x \in \partial \Omega \) and \( 0 < t < T \) with the improper time integral over \((0, T)\). Here \( a(t) \) satisfies the assumption (a). Now we introduce a new variable \( z \) given by
\[
z := b(t, \tau).
\]

This substitution gives an implicit function \( \tau = \tau(z) \) and \( \varepsilon_n,a(x - \xi, b(t, \tau)) = \varepsilon_n(x - \xi, z) \). Then the operator \( D \) can be written in the form
\[
(D\varphi)(x, t) = \int_0^{a(t)} \int_{\partial \Omega} \frac{\partial \varepsilon_n(x - \xi, z)}{\partial \nu(\xi)} \varphi(\xi, \tau(z)) dS_\xi dz. \tag{3.25}
\]
By the equality
\[
|\langle x - \xi, \nu(\xi) \rangle| \leq |x - \xi|^{1+\lambda}, \quad x, \xi \in \partial \Omega \in C^{1+\lambda},
\]
and estimates in (3.20)-(3.21) we obtain the estimate
\[
\left| \frac{\partial \varepsilon_n(x - \xi, z)}{\partial \nu(\xi)} \right| \leq \frac{M}{z^\gamma |x - \xi|^{n+1-2\gamma-\lambda}}, \quad z > 0, \quad x \neq \xi, \tag{3.26}
\]
for all \( 0 < \gamma < 1 + \frac{\lambda}{2} \) and some constant \( M > 0 \) which depends on \( L \) and \( \gamma \). From here if we choose \( \gamma \) such that \( 1 - \frac{\lambda}{2} < \gamma < 1 \), we see that the kernel of \( D \) is weakly singular with respect to the integrals over \( \Omega \) and over time.

From (3.26) we see that \( (D\varphi)(\cdot, 0) = 0 \) in \( \Omega \). Thus, \( D\varphi \) is continuous in \( \overline{\Omega} \times (0, T) \) if \( \varphi(\cdot, 0) = 0 \) on \( \partial \Omega \). Moreover, the density \( \varphi \) can be continuously extended to \( \partial \Omega \times (-\infty, T] \) by setting \( \varphi(\cdot, t) = 0 \) for \( t < 0 \) in \( \Omega \). Then, from Theorem 3.3 we see that the double layer potential is a solution of the homogeneous equation
\[
\Delta_a u = 0, \quad \text{in} \ \Omega \times (0, T), \tag{3.27}
\]
with the initial condition
\[
u(\cdot, 0) = 0, \quad \text{in} \ \Omega, \tag{3.28}
\]
and the boundary condition
\[
u = g, \quad \text{on} \ \partial \Omega \times (0, T), \tag{3.29}
\]
promoting the continuous density \( \varphi \) solves the following boundary integral equation
\[
\left( -\frac{1}{2} I + D \right)(\varphi) = g, \tag{3.30}
\]
where $I$ is the identity operator and $g$ satisfies the compatibility condition

$$g(\cdot, 0) = 0, \quad \text{on } \partial \Omega.\quad (3.31)$$

Here we assumed that $g$ satisfies condition $(3.31)$ to have solvability of problem $(3.27)-(3.29)$. The following theorems and corollary are valid.

**Theorem 3.4.** The double layer operator $D : C(\partial \Omega \times [0, T]) \to C(\partial \Omega \times [0, T])$ is compact.

**Proof of Theorem 3.4.** Since the kernel of $D$ is weakly singular, we apply [[8], Theorem 2.29 and Theorem 2.30] to complete the proof. □

**Corollary 3.5.** The double layer potential $(3.2)$ is continuous in $\Omega \times (0, T)$ provided that the continuous density $\varphi$ satisfies the condition $\varphi(\cdot, 0) = 0$ on $\partial \Omega$.

**Theorem 3.6.** The double layer potential $(3.2)$ is a solution of the initial boundary value problem $(3.27)-(3.29)$ provided $\varphi \in C(\partial \Omega \times [0, T])$ solves the boundary integral equation $(3.30)$ for all $x \in \partial \Omega$ and $t \in (0, T)$.

**Proof of Theorem 3.6.** This follows from Theorem 3.3 and Corollary 3.5. The compatibility condition for $g$ $(3.31)$ ensures the identity $\varphi(\cdot, 0) = 0$ on $\partial \Omega$ for solutions to $(3.30)$. □

Since the integral operator $D$ is compact, equation $(3.30)$ is solvable for each $g \in C(\partial \Omega \times [0, T])$ by the Riesz theory [[8], Corollary 3.5], if the homogeneous equation $(-\frac{1}{2}I + D)(\varphi) = 0$ has only the solution $\varphi = 0$ on $\partial \Omega \times [0, T]$. We note that estimate $(3.26)$ is equivalent to

$$\left| \frac{\partial \varepsilon_{n,a}(x - \xi, b(t, \tau))}{\partial \nu(\xi)} a(\tau) \right| \leq \frac{Ma(\tau)}{[b(t, \tau)]^{\gamma}|x - \xi|^{n+1-2\gamma-\lambda}}. \quad (3.32)$$

From $(3.32)$ we have the estimate

$$\|(D\varphi)(\cdot, t)\|_{L^\infty(\partial \Omega)} \leq C \int_0^t a(\tau) \|\varphi(\cdot, \tau)\|_{L^\infty(\partial \Omega)} d\tau, \quad (3.33)$$

for all $t \in (0, T]$ and some constant $C > 0$ which depends on $\partial \Omega$ and $\gamma$.

Repeating this argument (by induction), we obtain

$$\|(D^k\varphi)(\cdot, t)\|_{L^\infty(\partial \Omega)} \leq C^k B^{k-1} \int_0^t \frac{a(\tau)}{[b(t, \tau)]^{k(\gamma-1)+1}} \|\varphi(\cdot, \tau)\|_{L^\infty(\partial \Omega)} d\tau,$$

for all $k \in \mathbb{N}$ and $t \in (0, T]$, where

$$B := \int_0^1 \frac{ds}{[s(1-s)]^{\gamma}}.$$

Hence, there exists an integer $k_0$ such that

$$\|(D^{k_0}\varphi)(\cdot, t)\|_{L^\infty(\partial \Omega)} \leq Q \int_0^t \|\varphi(\cdot, \tau)\|_{L^\infty(\partial \Omega)} d\tau, \quad (3.34)$$

for all $t \in [0, T]$ and some constant $Q > 0$. 
Let \( \varphi \) be a solution of the equation \((-\frac{1}{2}I + D)(\varphi) = 0\). Then, by iteration, we see that \( \varphi \) solves
\[
\left(-\frac{1}{2}I + D^{k_0}\right)(\varphi) = 0.
\]
So, (3.34) implies the following estimate
\[
\|\varphi(\cdot, \tau)\|_{L^\infty(\partial \Omega)} \leq \|\varphi\|_{L^\infty(Q^n_{nt}, t \in [0, T])},
\]
for all \( n \in \mathbb{N} \). Hence, \( \varphi = 0 \) on \( \partial \Omega \times [0, T] \). Thus, we have proved the existence and uniqueness theorem.

**Theorem 3.7.** The initial boundary problem (3.27)-(3.29) with boundary datum in \( C(\partial \Omega \times [0, T]) \) satisfying the condition (3.31) has a unique solution \( u \in C(\Omega \times (0, T)) \) and the solution can be given as the double layer potential with the density \( \varphi \), where \( \varphi \) satisfies integral equation (3.30).

**Corollary 3.8.** The initial boundary problem (3.27)-(3.29) with the homogeneous boundary condition has only the trivial solution.

## 4. Trace Formulae

**Theorem 4.1.** Let \( a(t) \) satisfy the assumption (a). Then for any \( f(x, t) \in C(\Omega \times (0, T)) \), \( \text{supp} \ f(\cdot, t) \subset \Omega \), \( t \in [0, T] \), the volume potential with the density \( f \) (2.8) satisfies the boundary condition
\[
\frac{u(x, t)}{2} + \int_0^t \int_{\partial \Omega} \frac{\partial \varepsilon_{n,a}(x - \xi, b(t, \tau))}{\partial \nu(\xi)} a(\tau) u(\xi, \tau) dS_\xi d\tau
- \int_0^t \int_{\partial \Omega} \varepsilon_{n,a}(x - \xi, b(t, \tau)) a(\tau) \frac{\partial u(\xi, \tau)}{\partial \nu(\xi)} dS_\xi d\tau = 0,
\]
for all \( (x, t) \in \partial \Omega \times (0, T) \).

Conversely, if \( u(x, t) \in C^{2,1}_{x,t}(\Omega \times (0, T)) \cap C(\Omega \times [0, T]) \cap C^{1,0}_{x,t}(\Omega \times (0, T)) \) is a solution of the equation
\[
\var{a} u = f \quad \text{in } \Omega \times (0, T),
\]
satisfying the initial condition
\[
u(\cdot, 0) = 0 \quad \text{in } \Omega,
\]
and the lateral boundary condition (4.1) for all \( x \in \partial \Omega \), \( t \in (0, T) \), then it has a unique solution \( u \), which coincides with the volume potential.

Note that the one-dimensional version of this theorem was stated in [11].

**Proof of Theorem 4.1.** Since \( f(x, t) \in C(\Omega \times (0, T)) \) with \( \text{supp} \ f(\cdot, t) \subset \Omega \) for all \( t \in [0, T] \), the volume potential \( (V f)(x, t) \) solves Cauchy problem (4.2)-(4.3) by Theorem 2.2. Consider the volume potential
\[
(V f)(x, t) = \int_0^t \int_\Omega \varepsilon_{n,a}(x - \xi, b(t, \tau)) f(\xi, \tau) d\xi d\tau
- \int_0^t \int_\Omega \varepsilon_{n,a}(x - \xi, b(t, \tau)) \left( \frac{\partial}{\partial \tau} - a(\tau) \Delta_\xi \right) (V f)(\xi, \tau) d\xi d\tau.
\]
This integral is improper and defined as
\[
\lim_{\delta \to 0} (V_{\delta}f)(x,t) = \lim_{\delta \to 0} \int_0^{t-\delta} \int_\Omega \varepsilon_n \varphi_n(x - \xi, b(t, \tau)) f(\xi, \tau) d\xi d\tau,
\]
where \(0 < \delta < t\).

We use \(\varepsilon_n(x - \xi, b(t, \tau))\) in the sequel instead of \(\varepsilon_n(x - \xi, b(t, \tau))\) for convenience.

A direct calculation shows that
\[
(Vf)(x,t) = \lim_{\delta \to 0} V_{\delta}(f)(x,t)
= \lim_{\delta \to 0} \int_0^{t-\delta} \int_\Omega \varepsilon_n \varphi_n(x - \xi, b(t, \tau)) \left( \frac{\partial}{\partial \tau} - a(\tau) \Delta \xi \right) (Vf)(\xi, \tau) d\xi d\tau
= \lim_{\delta \to 0} \int_\Omega \varepsilon_n(x - \xi, b(t, \delta)) (Vf)(\xi, t - \delta) d\xi
- \lim_{\delta \to 0} \int_\Omega \varepsilon_n(x - \xi, b(t, 0)) (Vf)(\xi, 0) d\xi
+ \lim_{\delta \to 0} \int_0^{t-\delta} \int_{\partial \Omega} \frac{\partial \varepsilon_n(x - \xi, b(t, \tau))}{\partial \nu(\xi)} a(\tau) (Vf)(\xi, \tau) dS_\xi d\tau
- \lim_{\delta \to 0} \int_0^{t-\delta} \int_{\partial \Omega} \varepsilon_n(x - \xi, b(t, \tau)) a(\tau) \frac{\partial (Vf)(\xi, \tau)}{\partial \nu(\xi)} dS_\xi d\tau
+ \lim_{\delta \to 0} \int_0^{t-\delta} \int_\Omega (Vf)(\xi, \tau) \left( - \frac{\partial}{\partial \tau} - a(\tau) \Delta \xi \right) \varepsilon_n(x - \xi, b(t, \tau)) d\xi d\tau
=: I_1 - I_2 + I_3 - I_4 + I_5,
\]
with the trivial definitions of \(I_i, i = 1, \ldots, 5\). Using the property of the fundamental solution of the heat operator such that \(\varepsilon_n(x - \xi, t) \to \delta(x - \xi)\) with \(t \to 0^+\), we have
\[
I_1 := \lim_{\delta \to 0} \int_\Omega \varepsilon_n(x - \xi, b(t, \tau)) (Vf)(\xi, t - \delta) d\xi
= \int_\Omega \delta(x - \xi) (Vf)(\xi, t) d\xi = (Vf)(x,t).
\]

Taking into account that \((Vf)(\cdot, 0) = 0\) in \(\Omega\), we get
\[
I_2 := \lim_{\delta \to 0} \int_\Omega \varepsilon_n(x - \xi, b(t, 0)) (Vf)(\xi, 0) d\xi = 0.
\]

The integrals \(I_3, I_4\) have limits as \(\delta \to 0\)
\[
I_3 := \lim_{\delta \to 0} \int_0^{t-\delta} \int_{\partial \Omega} \frac{\partial \varepsilon_n(x - \xi, b(t, \tau))}{\partial \nu(\xi)} a(\tau) (Vf)(\xi, \tau) dS_\xi d\tau
= \int_0^t \int_{\partial \Omega} \frac{\partial \varepsilon_n(x - \xi, b(t, \tau))}{\partial \nu(\xi)} a(\tau) (Vf)(\xi, \tau) dS_\xi d\tau,
\]
\[
I_4 := \lim_{\delta \to 0} \int_0^{t-\delta} \int_{\partial \Omega} \varepsilon_n(x - \xi, b(t, \tau)) a(\tau) \frac{\partial (Vf)(\xi, \tau)}{\partial \nu(\xi)} dS_\xi d\tau
= \int_0^t \int_{\partial \Omega} \varepsilon_n(x - \xi, b(t, \tau)) a(\tau) \frac{\partial (Vf)(\xi, \tau)}{\partial \nu(\xi)} dS_\xi d\tau.
\]
Obviously, we have
\[ I_5 := \lim_{\delta \to 0} \int_0^{t-\delta} \int_{\Omega} (Vf)(\xi, \tau) \left( -\frac{\partial}{\partial \tau} - a(\tau)\Delta_\xi \right) \varepsilon_n(x - \xi, b(t, \tau)) d\xi d\tau = 0. \]

Taking into account (4.4), for all \((x, t) \in \Omega \times (0, T)\), we obtain
\[
I_{Vf}(x, t) := I_3 - I_4 = \int_0^t \int_{\partial \Omega} \left( \frac{\partial \varepsilon_n(x - \xi, b(t, \tau))}{\partial \nu(\xi)} a(\tau)(Vf)(\xi, \tau) - \varepsilon_n(x - \xi, b(t, \tau)) a(\tau) \frac{\partial(Vf)(\xi, \tau)}{\partial \nu(\xi)} \right) dS_\xi d\tau = 0. \quad (4.5)
\]

Using the jump relation of the double layer potential to (4.5) with \(x \to \partial \Omega\), we arrive at
\[
I_{Vf}(x, t)|_{(x, t) \in \partial \Omega \times (0, T)} = -\frac{(Vf)(x, t)}{2} + \int_0^t \int_{\partial \Omega} \left( \frac{\partial \varepsilon_n(x - \xi, b(t, \tau))}{\partial \nu(\xi)} a(\tau)(Vf)(\xi, \tau) - \varepsilon_n(x - \xi, b(t, \tau)) a(\tau) \frac{\partial(Vf)(\xi, \tau)}{\partial \nu(\xi)} \right) dS_\xi d\tau = 0.
\]

Now we show that if \(u_1 \in C^{2,1}_{\partial t}(\Omega \times (0, T)) \cap C(\Omega \times [0, T)) \cap C^{1,0}_{\partial x}(\overline{\Omega} \times (0, T))\) is a solution of problem (4.2), (4.3), (4.1), then \(u_1\) is represented by the volume potential with the density \(f\). If not, then we assume that \(Vf\) and \(u_1\) are solutions of problem (4.2), (4.3), (4.1). Then their difference \(\omega = Vf - u_1\) must satisfy
\[
\varphi_\omega \omega = 0 \quad \text{in} \quad \Omega \times (0, T),
\]
\[
\omega(\cdot, 0) = 0 \quad \text{in} \quad \Omega,
\]
and the boundary condition
\[
-\frac{\omega(x, t)}{2} + \int_0^t \int_{\partial \Omega} \frac{\partial \varepsilon_n(x - \xi, b(t, \tau))}{\partial \nu(\xi)} a(\tau)\omega(\xi, \tau) dS_\xi d\tau - \int_0^t \int_{\partial \Omega} \varepsilon_n(x - \xi, b(t, \tau)) a(\tau) \frac{\partial \omega(\xi, \tau)}{\partial \nu(\xi)} dS_\xi d\tau = 0, \quad (4.7)
\]
for all \((x, t) \in \partial \Omega \times (0, T)\). Since \(f = 0\) in \(\Omega \times (0, T)\), the representation formula (4.4) has the form
\[
-\omega(x, t) = \int_0^t \int_{\partial \Omega} \frac{\partial \varepsilon_n(x - \xi, b(t, \tau))}{\partial \nu(\xi)} a(\tau)\omega(\xi, \tau) - \varepsilon_n(x - \xi, b(t, \tau)) a(\tau) \frac{\partial \omega(\xi, \tau)}{\partial \nu(\xi)} dS_\xi d\tau, \quad (4.8)
\]
for all \((x, t) \in \partial \Omega \times (0, T)\). Using the jump relation of the double layer potential to (4.8) with \(x \to \partial \Omega\), we obtain
\[
-\omega(x, t) = -\frac{\omega(x, t)}{2} + \int_0^t \int_{\partial \Omega} \frac{\partial \varepsilon_n(x - \xi, b(t, \tau))}{\partial \nu(\xi)} a(\tau)\omega(\xi, \tau) - \varepsilon_n(x - \xi, b(t, \tau)) a(\tau) \frac{\partial \omega(\xi, \tau)}{\partial \nu(\xi)} dS_\xi d\tau, \quad (4.9)
\]
where \((x, t) \in \partial \Omega \times (0, T)\).
Comparing (4.9) with (4.7), we conclude that
\[ \omega = 0 \quad \text{on } \partial\Omega \times (0, T). \]  
(4.10)

Corollary 3.8 provides the existence of the unique trivial solution of the initial boundary value problem (4.6), (4.10). We have proved its uniqueness. \( \square \)

As in [4] we state the following theorem for the Poisson integral.

**Theorem 4.2.** Let \( a(t) \) satisfy the assumption (a) and \( \varphi \in C(\Omega) \) with \( \text{supp}(\varphi) \subset \Omega \). Then the Poisson integral with the density \( \varphi \) (2.11) satisfies the lateral boundary condition
\[
\begin{align*}
&- \frac{u(x, t)}{2} + \int_0^t \int_{\partial\Omega} \frac{\partial \varepsilon_{n,a}(x - \xi, b(t, \tau))}{\partial \nu(\xi)} a(\tau) u(\xi, \tau) dS_{\xi} d\tau \\
&- \int_0^t \int_{\partial\Omega} \varepsilon_{n,a}(x - \xi, b(t, \tau)) a(\tau) \frac{\partial u(\xi, \tau)}{\partial \nu(\xi)} dS_{\xi} d\tau = 0,
\end{align*}
\]
(4.11)
for all \( x \in \partial\Omega, \ t \in (0, T) \).

Conversely, if \( u(x, t) \in C_{x,t}^{2,1}(\Omega \times (0, T)) \cap C(\Omega \times [0, T)) \cap C_{x,t}^{1,0}(\overline{\Omega} \times (0, T)) \) is a solution of
\[ \Box a u = 0 \quad \text{in } \Omega \times (0, T), \]  
(4.12)
satisfying the initial condition
\[ u(\cdot, 0) = \varphi \quad \text{in } \Omega, \]  
(4.13)
and the lateral boundary condition (4.11) for all \( x \in \partial\Omega, \ t \in (0, T) \), then \( u \) coincides with the Poisson integral.

**Proof of Theorem 4.2.** We use \( \varepsilon_n(x - \xi, b(t, \tau)) \) instead of \( \varepsilon_{n,a}(x - \xi, t - \tau) \) for convenience. For \( x \in \Omega \) and \( 0 < \delta < t \), it is easy to check that
\[
0 = \lim_{\delta \to 0} \int_{\partial\Omega} \int_0^{t-\delta} \varepsilon_n(x - \xi, b(t, t - \delta))(P\varphi)(\xi, t - \delta) d\xi d\tau
\]
\[
- \lim_{\delta \to 0} \int_{\Omega} \varepsilon_n(x - \xi, b(t, 0))(P\varphi)(\xi, 0) d\xi
\]
\[
+ \lim_{\delta \to 0} \int_{\partial\Omega} \int_0^{t-\delta} \frac{\partial \varepsilon_n(x - \xi, b(t, \tau))}{\partial \nu(\xi)} a(\tau)(P\varphi)(\xi, \tau) dS_{\xi} d\tau
\]
\[
- \lim_{\delta \to 0} \int_{\partial\Omega} \int_0^{t-\delta} \varepsilon_n(x - \xi, b(t, \tau)) a(\tau) \frac{\partial (P\varphi)(\xi, \tau)}{\partial \nu(\xi)} dS_{\xi} d\tau
\]
\[
+ \lim_{\delta \to 0} \int_{\partial\Omega} \int_0^{t-\delta} (P\varphi)(\xi, \tau) \left( - \frac{\partial}{\partial \tau} - a(\tau) \Delta \xi \right) \varepsilon_n(x - \xi, b(t, \tau)) d\xi d\tau
\]
\[=: J_1 - J_2 + J_3 - J_4 + J_5, \]  
(4.14)
with the definitions of $J_i$, $i = 1, \ldots, 5$. Using the property of the fundamental solution of the heat operator such that $\varepsilon_n(x - \xi, t) \to \delta(x - \xi)$ with $t \to 0^+$, we have

$$J_1 := \lim_{\delta \to 0} \int_{\Omega} \varepsilon_n(x - \xi, b(t, t - \delta))(P\varphi)(\xi, t - \delta)\,d\xi = \int_{\Omega} \delta(x - \xi)(P\varphi)(\xi, t)\,d\xi = (P\varphi)(x, t).$$

Since $(P\varphi)(\cdot, 0) = \varphi$, we have

$$J_2 := \lim_{\delta \to 0} \int_{\Omega} \varepsilon_n(x - \xi, b(t, 0))(P\varphi)(\xi, 0)\,d\xi = (P\varphi)(x, t).$$

The integrals $J_3, J_4$ have limits as $\delta \to 0$

$$J_3 := \lim_{\delta \to 0} \int_{0}^{t-\delta} \int_{\partial\Omega} \partial\varepsilon_n(x - \xi, b(t, \tau)) a(\tau)(P\varphi)(\xi, \tau)\,dS_x\,d\tau,$$

$$J_4 := \lim_{\delta \to 0} \int_{0}^{t-\delta} \int_{\partial\Omega} \varepsilon_n(x - \xi, b(t, \tau)) a(\tau) \frac{\partial(P\varphi)(\xi, \tau)}{\partial\nu(\xi)}\,dS_x\,d\tau,$$

Clearly,

$$J_5 := \lim_{\delta \to 0} \int_{0}^{t-\delta} \int_{\Omega} (P\varphi)(\xi, \tau) \left(-\frac{\partial}{\partial\tau} - a(\tau)\Delta\xi\right) \varepsilon_n(x - \xi, b(t, \tau))\,d\xi\,d\tau = 0.$$

From (4.14) we get

$$I_{P\varphi}(x, t) := J_3 - J_4 = \int_{0}^{t} \int_{\partial\Omega} \left( \frac{\partial\varepsilon_n(x - \xi, b(t, \tau))}{\partial\nu(\xi)} a(\tau)(P\varphi)(\xi, \tau) - \varepsilon_n(x - \xi, b(t, \tau)) a(\tau) \frac{\partial(P\varphi)(\xi, \tau)}{\partial\nu(\xi)} \right)\,dS_x\,d\tau = 0.$$

When $x \to \partial\Omega$ applying Theorem 3.3, we derive

$$I_{P\varphi}(x, t)|_{(x,t)\in \partial\Omega \times (0, T)} = -\frac{(P\varphi)(x, t)}{2} + \int_{0}^{t} \int_{\partial\Omega} \left( \frac{\partial \varepsilon_n(x - \xi, b(t, \tau))}{\partial\nu(\xi)} a(\tau)(P\varphi)(\xi, \tau) - \varepsilon_n(x - \xi, b(t, \tau)) a(\tau) \frac{\partial(P\varphi)(\xi, \tau)}{\partial\nu(\xi)} \right)\,dS_x\,d\tau.$$

The rest of the proof is the same as in the case of the volume potential. \hfill \Box

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