Electromagnetic Interaction Equations

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Abstract. For the electromagnetic interaction of two particles the relativistic quantum mechanics equations are proposed. These equations are solved for the case when one particle has a small mass and moves freely. The initial wave functions are supposed to be concentrated at the coordinates origin. The energy spectrum of another particle wave function is defined by the initial wave function of the free moving particle. Choosing the initial wave functions of the free moving particle it is possible to obtain a practically arbitrary energy spectrum.

1 Introduction

For the particles of the mass $\mu > 0$ and the spin $1/2$ the relativistic equation was proposed by Dirac ([1], equation (4.29); [2], equation (1 - 41); [3], equation (2.9))

$$\sum_{n=1}^{4} \sum_{\nu=0}^{3} (\gamma^\nu)_{mn} \left(-i \frac{\partial}{\partial x^\nu}\right) \psi_n(x) + \mu \psi_m(x) = 0. \quad (1.1)$$

Let us consider a classical electromagnetic field with a vector potential $A_\nu(x)$. If we change in the equation (1.1) the differential operator $-i\partial_\nu$ for the differential operator $-i\partial_\nu + qA_\nu(x)$, we get the equation for the interaction between the particle with the charge $q$ and the external classical electromagnetic field ([1], Chapter 4, equation (4.202); [3], Chapter 2, equation (2.62))

$$\sum_{n=1}^{4} \sum_{\nu=0}^{3} (\gamma^\nu)_{mn} \left(-i \frac{\partial}{\partial x^\nu} + qA_\nu(x)\right) \psi_n(x) + \mu \psi_m(x) = 0. \quad (1.2)$$

The substitution of the electron charge $q = -e$ and the Coulomb vector potential $A_0(x) = Ze(4\pi|x|)^{-1}$, $A_k(x) = 0$, $k = 1, 2, 3$, into the equation (1.2) yields the equation for the electron in the external electromagnetic field generated by the nucleus with the charge $Ze$. We consider a nucleus as a classical particle. The energy spectrum of this equation (1.2) is discrete and the energy level $E_{1,1/2} = \mu c^2(1 - (Z/137)^2)^{1/2}$ ([3], equation (2.87)). For $Z > 137$ the value $E_{1,1/2}$ becomes imaginary. The description of the electromagnetic interaction between the electron and the nucleus with the charge $Ze > 137e$ seems not to make sense.

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At present time the nuclei with the charges $Ze \leq 118e$ are synthesized. (The nucleus with the charge $Ze = 117e$ has not been yet discovered.) The equation (1.2) should be mathematically self-consistent for Coulomb vector potential with any charge. Due to the paper [4] Hamiltonian (1.2) with the Coulomb vector potential $A_0(x) = Ze(4\pi|x|)^{-1}$, $A_k(x) = 0$, $k = 1, 2, 3$, has a self adjoint operator extension for any charge $Ze$. For $Ze \geq (\sqrt{3}/2)137e > 118e$ this self adjoint operator extension is not unique.

The equation (1.2) defines the wave function of the electron interacting with the electromagnetic field generated by the classical particle. Both interacting particles should be quantum. In the quantum electrodynamics ([5], Lecture 24) the following equations for the electromagnetic field generated by the classical particle are studied

$$
\sum_{n_1, n_2 = 1}^{4} \left( \prod_{s = 1}^{2} \left( \sum_{\nu = 0}^{3} (\gamma^\nu)_{m_s, n_s} \left( \frac{i}{\partial x_s^\nu} - \mu_s \delta_{m_s, n_s} \right) \right) \right) \psi_{n_1 n_2, p_1 p_2}(x_1, x_2) + \sum_{\nu_1, \nu_2 = 0}^{3} \eta_{\nu_1 \nu_2} K q_1 q_2 D^c_0(x_1 - x_2) \left( \prod_{s = 1}^{2} (\gamma^\nu)_{m_s, n_s} \right) \psi_{n_1 n_2, p_1 p_2}(x_1, x_2) = i \prod_{s = 1}^{2} (x_s) \delta_{m_s, p_s},
$$

(1.3)

$$
D^c_m(x) = \lim_{\epsilon \to +0} (2\pi)^{-4} \int d^4 k \exp\{i(k, x)\} (m^2 - (k, k) - i\epsilon)^{-1},
$$

(1.4)

where $q_1, q_2$ are the charges, $K$ is the electromagnetic interaction constant, $\eta_{\nu\nu} = \eta^\nu\nu$ is the diagonal matrix with diagonal matrix elements $\eta_{11} = -\eta_{22} = -\eta_{33} = 1$.

The solutions of the equations (1.3) contain the divergent integrals [5]. The finite answers for the divergent integrals are obtained by means of the renormalization procedure. Due to the book ([6], Chapter 4): "The shell game that we play to find $n$ and $j$ is technically called "renormalization". But no matter how clever the word, it is what I would call a dippy process! Having to resort to such hocus - pocus has prevented us from proving that the theory of quantum electrodynamics is mathematically self-consistent. It's surprising that the theory still hasn’t been proved self-consistent one way or the other by now; I suspect that renormalization is not mathematically legitimate. What is certain is that we do not have a good mathematical way to describe the theory of quantum electrodynamics: such a bunch of words to describe the connection between $n$ and $j$ and $m$ and $e$ is not good mathematics."

The equations (1.3) do not satisfy the causality condition: the support of the fundamental solution $D^c_0(x)$ of the wave equation does not lie in the upper or lower light cones. The distribution $D^c_0(x)$ choice is probably connected with the causality condition. The chronological product of two free scalar field operators is defined as

$$
T(\phi(x)\phi(y)) = \phi(x)\phi(y) + i e_{m^2}(y - x) I
$$

(1.5)

where $I$ is the identity operator and the distribution

$$
-e_{m^2}(-x) = \lim_{\epsilon \to +0} (2\pi)^{-4} \int d^4 k \exp\{i(k, x)\} (m^2 - (k^0 + i\epsilon)^2 + |k|^2)^{-1}.
$$

(1.6)

The vacuum expectation of the product of two free scalar fields is defined as

$$
< \phi(x)\phi(y) >_0 = -i D^-_{m^2}(x - y),
$$

$$
D^-_{m^2}(x) = i(2\pi)^{-3} \int d^4 k \exp\{-i(k, x)\} \theta(k^0) \delta((k, k) - m^2).
$$

(1.7)
The vacuum expectation of the chronological product (1.5) is equal to

\[
<T(\phi(x)\phi(y))>_0 = -iD_m^2(x - y) + i\epsilon_m^2(y - x) = -iD_m^c(x - y).
\]

Stueckelberg and Rivier [7] believed that the classical "causal action" is given by the distribution \(-\epsilon_m^2(y - x)\) and the distribution \(D_m^c(x - y) = D_m(x - y)\) defines the probability amplitude of the "causal action".

The equations (1.3) do not look like the equations (1.2). We want to write down two equations of the type (1.2) where one particle interacts with the electromagnetic field generated by another particle. These equations should satisfy the causality condition. Let us define 4 \(\times\) 4 - matrices

\[
\alpha(\mu^2) = \begin{pmatrix} \mu^2\sigma^0 & 0 \\ 0 & \sigma^0 \end{pmatrix}, \quad \beta(\mu^2) = \begin{pmatrix} \sigma^0 & 0 \\ \mu^2\sigma^0 & 0 \end{pmatrix}, \quad \gamma^\nu = \begin{pmatrix} 0 & \eta^\nu\sigma^\nu \\ \sigma^\nu & 0 \end{pmatrix}, \quad \nu = 0, \ldots, 3,
\]

where 2 \(\times\) 2 - matrices \(\sigma^\nu\) are given by the relations

\[
\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Let us consider two equations for the wave functions \((\psi_s)_n_s(x_s), s = 1, 2\), equal to zero for the negative \(x^0_s\)

\[
\sum_{n_s=1}^{4} \left( \sum_{\nu_s=0}^{3} (\gamma^\nu)_{m_s,n_s} \left( -i\frac{\partial}{\partial x^\nu_s} \right) + (\beta(\mu_s^2))_{m_s,n_s} \right) (\psi_s)_n_s(x_s) =

- i \sum_{n_s=1}^{4} (\gamma^0)_{m_s,n_s} \delta(x^0_s)(\psi_s)_n_s(+0, x_s), \quad x_s = (x^1_s, x^2_s, x^3_s) \in \mathbb{R}^3, \quad s = 1, 2,
\]

where \((\psi_s)_n_s(+0, x_s)\) are the initial wave functions. For \(\mu_1, \mu_2 > 0\) the equations (1.11) with the right - hand sides equal to zero are equivalent to Dirac equation (1.1). For \(\mu_1, \mu_2 \geq 0\) the equations (1.11) with the right - hand sides equal to zero are equivalent to the system of Klein - Gordon equations. The right - hand sides of the equations (1.11) are the standard method to define Cauchy problem. The equations (1.11) describe the free motion of particles with the masses \(\mu_1, \mu_2 \geq 0\). In contrast with the equation (1.1) the particles with zero mass are not distinguished. If the initial wave functions \((\psi_s)_n_s(+0, x_s) = (\psi_s)_n_s^0(x_s), s = 1, 2\), then the product of the equations (1.11) looks like the equation (1.3) with the interaction constant \(K = 0\).

The equation of the electromagnetic action of the second particle on the first particle has the form

\[
\sum_{s=1,2} \int d^4x_2 \left( \prod_{s=1}^{2} \left( \sum_{\nu=0}^{3} (\gamma^\nu)_{m_s,n_s} \left( -i\frac{\partial}{\partial x^\nu_s} \right) + (\beta(\mu_s^2))_{m_s,n_s} \right) (\psi_s)_n_s(x_s) +

\sum_{\nu_1,\nu_2=0}^{3} K q_1 q_2 e_{\nu_1,\nu_2} \epsilon_0 (x_1 - x_2) \prod_{s=1}^{2} (\gamma^\nu)_{m_s,n_s} (\psi_s)_n_s(x_s) \right) =

- \sum_{s=1,2} \int d^4x_2 \prod_{s=1}^{2} (\gamma^0)_{m_s,n_s} \delta(x^0_s)(\psi_s)_n_s(+0, x_s).
\]


The equation of the electromagnetic action of the first particle on the second particle is similar. Both equations of the type (1.2) satisfy the causality condition. The support of the distribution $-\varepsilon_0(-x)$ lies in the closed lower light cone. The fundamental solution $-\varepsilon_0(-x)$ of the wave equation is unique in the class of the distributions with supports in the lower light cone. The causality condition defines the distribution $-\varepsilon_0(-x)$ uniquely. The distribution $-\varepsilon_0(-x)$ gives the delay. It is necessary to have the delay in a relativistic interaction equation. It was already noted by Poincaré [8]. The interaction propagates not instantly but at the speed of light. We have to take into account the distance between the interacting particles. The delay is one of possible causality statements.

In this paper the equations (1.12) are solved for the case when the second particle has a small mass and the wave function $(\psi_2)_{n_2}(x_2)$ satisfies the second equation (1.11) for the initial wave function $(\psi_2)_{n_2}(+0, x_2) = (\psi_2)_{n_2}\delta(x_2)$. The energy spectrum of the solutions $(\psi_1)_{n_1}(x_1)$ is defined by the initial wave function $(\psi_2)_{n_2}\delta(x_2)$. By making a choice of the vectors $(\psi_2)_{n_2}$ it is possible to obtain a practically arbitrary energy spectrum.

2 Dirac equation

It is necessary to write down an equation of the type (1.1) for the free motion of a particle with arbitrary spin and mass. Let us consider the complex $2 \times 2$ - matrices

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \quad \quad (2.1)$$

The $2 \times 2$ - matrix

$$A^* = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{pmatrix} \quad \quad (2.2)$$

is called Hermitian conjugate. If $A^* = A$, the matrix (2.1) is Hermitian. The matrices (1.10) form a basis of Hermitian matrices. The multiplication rules for the matrices (1.10) are

$$\sigma^\mu\sigma^\mu = \sigma^0, \quad \sigma^0\sigma^\mu = \sigma^\mu\sigma^0 = \sigma^\mu, \quad \mu = 0, ..., 3;$$

$$\sigma^{k_1}\sigma^{k_2} = 3 \sum_{k_3=1}^{3} \epsilon^{k_1k_2k_3} i\sigma^{k_3}, \quad k_1, k_2 = 1, 2, 3, \quad k_1 \neq k_2 \quad \quad (2.3)$$

where the antisymmetric tensor $\epsilon^{k_1k_2k_3}$ has the normalization $\epsilon^{123} = 1$.

We identify the four dimensional Minkowski space with the four dimensional space of Hermitian $2 \times 2$ - matrices

$$\bar{x} = \sum_{\mu=0}^{3} x^\mu\sigma^\mu. \quad \quad (2.4)$$

For a complex $2 \times 2$ - matrix (2.1) we define the following $2 \times 2$ - matrices

$$A^T = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{pmatrix}. \quad \quad (2.5)$$

The matrices (2.1) with determinant equal to 1 form the group $SL(2, \mathbb{C})$. The matrices (2.1) satisfying the equations $A^*A = \sigma^0$, $\det A = 1$ form the group $SU(2)$. The group
SU(2) is the maximal compact subgroup of the group \( SL(2, \mathbb{C}) \). Let us describe the irreducible representations of the group \( SU(2) \). We consider the non-negative half-integers \( l = 0, 1/2, 1, 3/2, \ldots \). We define the representation of the group \( SU(2) \) on the space of the polynomials with degrees less than or equal to \( 2l \)

\[
T_l(A)\phi(z) = (A_{12}z + A_{22})^{2l} \phi \left( \frac{A_{11}z + A_{21}}{A_{12}z + A_{22}} \right). \tag{2.6}
\]

We consider a half-integer \( n = -l, -l + 1, \ldots, l - 1, l \) and choose the polynomial basis

\[
\psi_n(z) = ((l - n)!(l + n)!)^{-1/2} z^{-n}. \tag{2.7}
\]

The definitions (2.6), (2.7) imply

\[
T_l(A)\psi_n(z) = \sum_{m=-l}^{l} \psi_m(z) t^l_{mn}(A), \tag{2.8}
\]

\[
t^l_{mn}(A) = ((l - m)!(l + m)!(l - n)!(l + n)!)^{1/2} \times \frac{\sum_{j=-\infty}^{\infty} \Gamma(j + 1) \Gamma(l - m - j + 1) \Gamma(m - n + j + 1) \Gamma(l + n - j + 1)}{A_{11}^{l-m-j} A_{12}^{l-j} A_{21}^{m-n+j} A_{22}^{l+n-j}} \tag{2.9}
\]

where \( \Gamma(z) \) is the gamma-function. The function \( (\Gamma(z))^{-1} \) equals zero for \( z = 0, -1, -2, \ldots \). Therefore the series (2.9) is the polynomial.

The relation (2.6) defines the representation of the group \( SU(2) \). Thus the polynomial (2.9) defines the representation of the group \( SU(2) \)

\[
t^l_{mn}(AB) = \sum_{k=-l}^{l} t^l_{mk}(A) t^l_{kn}(B). \tag{2.10}
\]

This \((2l + 1)\) - dimensional representation is irreducible ([9], Chapter III, Section 2.3). The relations (2.9), (2.10) have an analytic continuation to all matrices (2.1).

By making the change \( j \rightarrow j + n - m \) of the summation variable in the equality (2.9) we have

\[
t^l_{mn}(A) = t^l_{nm}(A^T). \tag{2.11}
\]

The definition (2.9) implies

\[
t^l_{mn}(\sigma^0) = \delta_{mn}, \tag{2.12}
\]

The polynomial (2.9) is homogeneous of the matrix elements (2.1). Its degree is \( 2l \). The sum (2.3) contains the only non-zero term. The relations (2.3), (2.10) imply

\[
\sum_{p=-l}^{l} \sum_{\nu=0}^{i} \left( \sum_{\nu=0}^{3} t^l_{mp}(\sigma^\nu) t^l_{mp}(\sigma^\nu) \left(-i \frac{\partial}{\partial x^\nu}\right)\right) \left( \sum_{\nu=0}^{3} t^l_{np}(\sigma^\nu) t^l_{np}(\sigma^\nu) \left(-i \frac{\partial}{\partial x^\nu}\right)\right) = \sum_{1 \leq k_1 < k_2 \leq 3, \ k_3 = 1, \ldots, 3} t^l_{mn}(i\sigma^{k_1}) t^l_{mn}(-i\sigma^{k_3}) ((\epsilon^{k_1k_2k_3})^{2l+2l} + (\epsilon^{k_2k_3})^{2l+2l}) \frac{\partial^2}{\partial x^{k_1}\partial x^{k_2}} - \delta_{mn}\delta_{nm}(\partial_x, \partial_x), \tag{2.13}
\]
For an odd integer $2l + 2\hat{l}$ the relation (2.13) has the form

$$
\sum_{p=-l}^{l} \sum_{\tilde{p}=-l}^{l} \left( \sum_{\nu=0}^{3} \eta_{\mu}^{\nu} t_{m \mu}^{l} (\sigma^{\nu}) t_{n \mu}^{l} (\sigma^{\nu}) \right) \left( \sum_{\nu=0}^{3} \eta_{\nu}^{\mu} t_{m \nu}^{l} (\sigma^{\nu}) t_{n \nu}^{l} (\sigma^{\nu}) \right) = -\delta_{mn} \delta_{\hat{m} \hat{n}} \left( \partial_{x}, \partial_{\hat{x}} \right). \quad (2.15)
$$

By making use of the relation (2.15) for an odd integer $2l + 2\hat{l}$ Lorentz invariant equations

$$
(- (\partial_{x}, \partial_{\hat{x}}) - \mu^{2}) \phi_{mn}(x) = 0,
$$

$$
m = -l, -l + 1, ..., l - 1, l, \quad \hat{m} = -\hat{l}, -\hat{l} + 1, ..., \hat{l} - 1, \hat{l},
$$

may be rewritten as the system of the linear equations

$$
\sum_{n=2l+2}^{4l+2} \sum_{\tilde{n}=1}^{3} \eta_{\mu}^{\nu} t_{m \mu}^{l} t_{m-l-1,n-3l-2}^{l} (\sigma^{\nu}) t_{n \mu}^{l} (\sigma^{\nu}) \left( \partial_{x} \psi_{\hat{m} \hat{n}}(x) \right) + \psi_{mn}(x) = 0, \quad m = 1, ..., 2l + 1, \quad \hat{m} = 1, ..., 2\hat{l} + 1;
$$

$$
\sum_{n=1}^{2l+1} \sum_{\tilde{n}=1}^{3} \sum_{\nu=0}^{3} t_{m-3l-2,n-1}^{l} (\sigma^{\nu}) t_{n \mu}^{l} (\sigma^{\nu}) \left( \partial_{x} \psi_{\hat{m} \hat{n}}(x) \right) + \mu^{2} \psi_{mn}(x) = 0, \quad m = 2l + 2, ..., 4l + 2, \quad \hat{m} = 1, ..., 2\hat{l} + 1. \quad (2.17)
$$

Let us define the $((4l + 2)(2\hat{l} + 1)) \times ((4l + 2)(2\hat{l} + 1))$ matrices

$$
\begin{align*}
(\alpha_{l,i}(\mu^{2}))_{mn,\hat{m} \hat{n}} &= \mu^{2} \delta_{mn} \delta_{\hat{m} \hat{n}}, \quad m, n = 1, ..., 2l + 1, \quad \hat{m}, \hat{n} = 1, ..., 2\hat{l} + 1; \\
(\beta_{l,i}(\mu^{2}))_{mn,\hat{m} \hat{n}} &= \delta_{mn} \delta_{\hat{m} \hat{n}}, \quad m, n = 2l + 2, ..., 4l + 2, \quad \hat{m}, \hat{n} = 1, ..., 2\hat{l} + 1; \\
(\gamma_{l,i}(\mu^{2}))_{mn,\hat{m} \hat{n}} &= \mu^{2} \delta_{mn} \delta_{\hat{m} \hat{n}}, \quad m, n = 2l + 2, ..., 4l + 2, \quad \hat{m}, \hat{n} = 1, ..., 2\hat{l} + 1; \\
(\gamma_{l,i}(\sigma^{0}))_{mn,\hat{m} \hat{n}} &= \eta^{\mu \nu} t_{m-l-1,n-3l-2}^{l} (\sigma^{\nu}) t_{n \mu}^{l} (\sigma^{\nu}), \quad m = 1, ..., 2l + 1, \quad n = 2l + 2, ..., 4l + 2, \quad \hat{m}, \hat{n} = 1, ..., 2\hat{l} + 1; \\
(\gamma_{l,i}(\sigma^{0}))_{mn,\hat{m} \hat{n}} &= t_{m-n \mu}^{l} (\sigma^{\nu}) t_{n \mu}^{l} (\sigma^{\nu}), \quad m = 2l + 2, ..., 4l + 2, \quad n = 1, ..., 2l + 1, \quad \hat{m}, \hat{n} = 1, ..., 2\hat{l} + 1, \quad \nu = 0, ..., 3. \quad (2.18)
\end{align*}
$$

The other matrix elements are equal to zero. If an integer $2l + 2\hat{l}$ is odd, then an integer $(2l + 1)(2\hat{l} + 1) = 4l + 2l + 2\hat{l} + 1$ is even and 4 divides into $(4l + 2)(2\hat{l} + 1)$. For $l = 1/2, \hat{l} = 0$ the integer $(4l + 2)(2\hat{l} + 1) = 4$. The definitions (2.18) are the straightforward generalizations of the definitions (1.9). The definition (2.9) implies

$$
t_{m \hat{m}}^{l} (A) = A_{m+\frac{1}{2},n+\frac{1}{2}}. \quad (2.19)
$$

$4 \times 4$ - matrices $\alpha_{l/2,0}(\mu^{2}), \beta_{l/2,0}(\mu^{2}), \gamma_{l/2,0}(\sigma^{0})$ coincide with the matrices $\alpha(\mu^{2}), \beta(\mu^{2}), \gamma^{\nu}$ given by the relations (1.9). By making use of the definitions (2.18) we can rewrite the equations (2.17) as

$$
\sum_{n=1}^{4l+2} \sum_{\hat{n}=1}^{2\hat{l}+1} \left( \sum_{\nu=0}^{3} (\gamma_{l,i}(\sigma^{0}))_{mn,\hat{m} \hat{n}} \left( -i \frac{\partial}{\partial x^{\nu}} \right) + (\beta_{l,i}(\mu^{2}))_{mn,\hat{m} \hat{n}} \right) \psi_{\hat{m} \hat{n}}(x) = 0. \quad (2.20)
$$
In contrast with the equation (1.1) the mass $\mu$ in the equation (2.20) may be equal to zero. The definitions (2.18) imply
\[
\alpha_{l,i}(\mu)\gamma_{l,i}^\nu(\sigma^0) = \gamma_{l,i}^\nu(\sigma^0)\beta_{l,i}(\mu), \quad \beta_{l,i}(\mu^2)\gamma_{l,i}^\nu(\sigma^0) = \gamma_{l,i}^\nu(\sigma^0)\alpha_{l,i}(\mu^2),
\]
\[
\alpha_{l,i}(\mu)\beta_{l,i}(\mu^2) = \mu\beta_{l,i}(\mu).
\] (2.21)

In view of the relations (2.21) the action of the matrix $\alpha_{l,i}(\mu)$ on the equation (2.20) yields the equation of type (1.1)
\[
\sum_{n=1}^{4l+2} \sum_{\dot{n}=1}^{2\dot{l}+1} \sum_{\nu=0}^{3} (\gamma_{l,i}^\nu(\sigma^0))_{m\dot{m},n\dot{n}} \left( -i\frac{\partial}{\partial x^\nu} \right) \xi_{m\dot{n}}(x) + \mu \xi_{m\dot{n}}(x) = 0,
\] (2.22)
\[
\xi_{m\dot{n}}(x) = \sum_{n=1}^{4l+2} \sum_{\dot{n}=1}^{2\dot{l}+1} (\beta_{l,i}(\mu))_{m\dot{m},n\dot{n}} \psi_{n\dot{n}}(x) = 0.
\] (2.23)

For $\mu > 0$ the transformation given by the relation (2.23) is the isomorphism. Due to the relations (1.9), (2.18), (2.19) the equation (2.22) for $l = 1/2$, $\dot{l} = 0$ coincides with Dirac equation ([2], equation (1 - 41)). The relations (2.18) imply
\[
(\alpha_{l,i}(\mu^2)\beta_{l,i}(\mu^2))_{m\dot{m},n\dot{n}} = (\beta_{l,i}(\mu^2)\alpha_{l,i}(\mu^2))_{m\dot{m},n\dot{n}} = \mu^2 \delta_{mn} \delta_{\dot{m}\dot{n}}.
\] (2.24)

$m, n = 1, ..., 4l + 2, \dot{m}, \dot{n} = 1, ..., 2\dot{l} + 1$. In view of the second relation (2.21) and the relation (2.24) the action of the matrix $\gamma_{l,i}^0(\sigma^0)\alpha_{l,i}(\mu^2)$ on the equation (2.20) yields
\[
\sum_{n=1}^{4l+2} \sum_{\dot{n}=1}^{2\dot{l}+1} \sum_{\nu=0}^{3} (\gamma_{l,i}^0(\sigma^0)\gamma_{l,i}^\nu(\sigma^0)\beta_{l,i}(\mu^2))_{m\dot{m},n\dot{n}} \left( -i\frac{\partial}{\partial x^\nu} \right) + \mu^2 (\gamma_{l,i}^0(\sigma^0))_{m\dot{m},n\dot{n}} \psi_{n\dot{n}}(x) = 0.
\] (2.25)

The relations (2.3), (2.10), (2.12), (2.18) imply
\[
((\gamma_{l,i}^0(\sigma^0))^2)_{m\dot{m},n\dot{n}} = \delta_{mn} \delta_{\dot{m}\dot{n}}, \quad m, n = 1, ..., 4l + 2, \dot{m}, \dot{n} = 1, ..., 2\dot{l} + 1,
\] (2.26)
\[
(\gamma_{l,i}^0(\sigma^0)\gamma_{l,i}^k(\sigma^0))_{m\dot{m},n\dot{n}} = - (\gamma_{l,i}^k(\sigma^0)\gamma_{l,i}^0(\sigma^0))_{m\dot{m},n\dot{n}} =
\]
\[
t^l_{m-l-1,n-l-1} (\sigma^k) t^l_{m-i-1,n-i-1} (\sigma^k),
\]
\[
m, n = 1, ..., 2l + 1, \dot{m}, \dot{n} = 1, ..., 2\dot{l} + 1, \quad k = 1, 2, 3;
\]
\[
(\gamma_{l,i}^0(\sigma^0)\gamma_{l,i}^k(\sigma^0))_{m\dot{m},n\dot{n}} = - (\gamma_{l,i}^k(\sigma^0)\gamma_{l,i}^0(\sigma^0))_{m\dot{m},n\dot{n}} =
\]
\[
- t^l_{m-3l-2,n-3l-2} (\sigma^k) t^l_{m-i-1,n-i-1} (\sigma^k),
\]
\[
m, n = 2l + 2, ..., 4l + 2, \dot{m}, \dot{n} = 1, ..., 2\dot{l} + 1, \quad k = 1, 2, 3.
\] (2.27)

The other matrix elements are equal to zero. The coefficients of the polynomial (2.9) are real. Hence in view of the relations (2.11), (2.18), (2.26), (2.27) the matrices $\gamma_{l,i}^0(\sigma^0)$, $\gamma_{l,i}^0(\sigma^0)\gamma_{l,i}^\nu(\sigma^0)\beta_{l,i}(\mu^2)$, $\nu = 0, ..., 3$, are Hermitian. Due to the relation (2.26) we have $(\gamma_{l,i}^0(\sigma^0))^2\beta_{l,i}(\mu^2) = \beta_{l,i}(\mu^2)$. Now the equation (2.25) implies that the integral
\[
\int d^3x \sum_{m=1}^{4l+2} \sum_{\dot{m}=1}^{2\dot{l}+1} (\beta_{l,i}(\mu^2))_{m\dot{m},n\dot{n}} |\psi_{n\dot{n}}(x)|^2
\] (2.28)
is independent of the variable $x^0$ for $x^0 > 0$. The integrand (2.28) is called the probability density of a solution of the equation (2.20). For $\mu > 0$ the transformation given by the relation (2.23) is the isomorphism. The probability density (2.28) expressed through the function (2.23) is independent of the mass and coincides with the usual probability density for the function (2.23). In the quantum mechanics the fixed probability density defines Hilbert space where any Hamiltonian acts. The integral (2.28) depends on the parameter $\mu^2$ in the equation (2.20). We do not want to suppose that all particles have strictly positive masses. The experiments deal with the asymptotic solutions of the interaction equations. We expect the solutions of the interaction equations coincide asymptotically with the products of the equation (2.20) solutions. The probability density of the last solutions is given by the integrand (2.28).

The relations (2.3), (2.10), (2.12), (2.18), (2.26), (2.27) imply

\[
\begin{align*}
(\gamma^k_l(\sigma^0)^2)_{m\dot{m},n\dot{n}} & = -\delta_{k_1,k_2} \delta_{m\dot{m}} \delta_{n\dot{n}} + \sum_{k_3=1}^{3} (-1)^{2l} (i\epsilon^{k_1k_2k_3})^{2l+2l} x \\

(\alpha_{l,i}(0)\gamma^0_l(\sigma^0)\gamma^k_l(\sigma^0))_{m\dot{m},n\dot{n}} & - (\beta_{l,i}(0)\gamma^0_l(\sigma^0)\gamma^k_l(\sigma^0))_{m\dot{m},n\dot{n}} , \\
m, n = 1, \ldots, 4l + 4, \dot{m}, \dot{n} = 1, \ldots, 2\dot{l} + 1, \ k_1, k_2 = 1, 2, 3.
\end{align*}
\]  

(2.29)

For an odd integer $2l + 2\dot{l}$ the relations (2.26), (2.27), (2.29) imply

\[
(\gamma^\mu_l(\sigma^0)\gamma^\nu_l(\sigma^0))_{m\dot{m},n\dot{n}} + (\gamma^\nu_l(\sigma^0)\gamma^\mu_l(\sigma^0))_{m\dot{m},n\dot{n}} = 2\eta^{\mu\nu} \delta_{m\dot{m},n\dot{n}}, \\
\mu, \nu = 0, \ldots, 3, m, n = 1, \ldots, 4l + 2, \dot{m}, \dot{n} = 1, \ldots, 2\dot{l} + 1.
\]  

(2.30)

Let us define the representation of the group $SL(2, \mathbb{C})$ in the Lorentz group

\[
\sum_{\mu, \nu = 0}^{3} A^\mu_l(A) x^\nu = A \tilde{x} A^*.
\]  

(2.31)

We define also the $(4l + 2)(2\dot{l} + 1)$- dimensional representation of the group $SL(2, \mathbb{C})$

\[
(S_{l,i}(A))_{m\dot{m},n\dot{n}} = t^{l}_{m-l-1,n-l-1}(A) t_{m-i-1,n-i-1}(A), \\
m, n = 1, \ldots, 2l + 1, \dot{m}, \dot{n} = 1, \ldots, 2\dot{l} + 1;
\]  

(2.32)

The other matrix elements are equal to zero. The definitions (2.18), (2.32) imply

\[
S_{l,i}(A)\alpha_{l,i}(\mu^2) S_{l,i}(A^{-1}) = \alpha_{l,i}(\mu^2), \quad S_{l,i}(A)\beta_{l,i}(\mu^2) S_{l,i}(A^{-1}) = \beta_{l,i}(\mu^2).
\]  

(2.33)

Let the functions $\psi_{m\dot{m}}(x)$, $m = 1, \ldots, 4l + 2, \dot{m} = 1, \ldots, 2\dot{l} + 1$ be the solutions of the equation (2.20). The relations (2.33) imply that the functions

\[
\xi_{m\dot{m}}(\tilde{x}) = \sum_{n=1}^{4l+2} \sum_{\dot{n}=1}^{2\dot{l}+1} (S_{l,i}(A))_{m\dot{m},n\dot{n}} \psi_{n\dot{n}}(A^{-1}\tilde{x}(A^*)^{-1})
\]  

(2.34)
are the solutions of the equation
\[\sum_{n=1}^{4l+2} \sum_{\hat{n}=1}^{2l+1} \left( \sum_{\nu=0}^{3} (\gamma_{\hat{t},\hat{i}}^\nu(A))_{\hat{m}\hat{n},n\hat{n}} \left( -i \frac{\partial}{\partial x^\nu} \right) + (\beta_{\hat{t},\hat{i}}(\mu^2))_{m\hat{m},n\hat{n}} \right) \xi_{\hat{m}\hat{n}}(x) = 0, \quad (2.35)\]
\[\gamma_{\hat{t},\hat{i}}^\mu(A) = \sum_{\nu=0}^{3} \Lambda_\nu^\nu(A) S_{\hat{t},\hat{i}}^\nu(A) \gamma_{\hat{t},\hat{i}}^\nu(\sigma^0) S_{\hat{t},\hat{i}}^\nu(A^{-1}) \quad (2.36)\]
for any matrix \(A \in SL(2, \mathbb{C})\). The definition (2.36) implies
\[\gamma_{\hat{t},\hat{i}}^\mu(AB) = \sum_{\nu=0}^{3} \Lambda_\nu^\nu(A) S_{\hat{t},\hat{i}}^\nu(A) \gamma_{\hat{t},\hat{i}}^\nu(B) S_{\hat{t},\hat{i}}^\nu(A^{-1}) \quad (2.37)\]
for any matrices \(A, B \in SL(2, \mathbb{C})\). By changing the coordinate system we change the matrix \(\gamma_{\hat{t},\hat{i}}^\nu(\sigma^0)\) in the equation (2.20) for the matrix (2.36). The solutions of the equation (2.20) are transformed to the solutions (2.34) of the equation (2.35). It is valid for all non-negative half-integers \(l, \hat{l}\). Due to the book ([2], relation (1 - 43)) for any matrix \(A \in SL(2, \mathbb{C})\)
\[\gamma_{\frac{l}{2},0}^\mu(A) = \gamma_{\frac{l}{2},0}(\sigma^0) \quad (2.38)\]
Hence the equation (2.20) for \(l = 1/2, \hat{l} = 0\) is covariant under the group \(SL(2, \mathbb{C})\). The relation (2.38) picks out Dirac equation.
For an odd integer \(2l + 2\hat{l}\) the relations (2.30), (2.36) imply
\[(\gamma_{\hat{t},\hat{i}}^\mu(A) \gamma_{\hat{t},\hat{i}}^\nu(A))_{\hat{m}\hat{n},n\hat{n}} + (\gamma_{\hat{t},\hat{i}}^\nu(A) \gamma_{\hat{t},\hat{i}}^\mu(A))_{\hat{m}\hat{n},n\hat{n}} = 2\eta^{\mu\nu} \delta_{\hat{m}\hat{n},n\hat{n}}, \quad (2.39)\]
for any matrix \(A \in SL(2, \mathbb{C})\).
Let the functions \(\xi_{m\hat{m}}(x)\) be the solutions of the equation (2.35). Let us introduce the distributions
\[f_{m\hat{m}}(x) = \theta(x^0) \xi_{m\hat{m}}(x), \quad \theta(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases} \quad (2.40)\]
The equation (2.35) implies
\[\sum_{n=1}^{4l+2} \sum_{\hat{n}=1}^{2l+1} \left( \sum_{\nu=0}^{3} (\gamma_{\hat{t},\hat{i}}^\nu(A))_{m\hat{m},n\hat{n}} \left( -i \frac{\partial}{\partial x^\nu} \right) + (\beta_{\hat{t},\hat{i}}(\mu^2))_{m\hat{m},n\hat{n}} \right) f_{n\hat{n}}(x) = -i\delta(x^0) f_{m\hat{m}}^0(+0, x), \quad (2.41)\]
\[f_{m\hat{m}}^0(+0, x) = \sum_{n=1}^{4l+2} \sum_{\hat{n}=1}^{2l+1} (\gamma_{\hat{t},\hat{i}}^0(A))_{m\hat{m},n\hat{n}} f_{n\hat{n}}(+0, x). \]
Let a distribution \(e_{\mu_1^1,\ldots,\mu_n^2}(x) \in S'(\mathbb{R}^4)\) with support in the closed upper light cone satisfy the equation
\[(\prod_{i=1}^{n} (-(\partial_x, \partial_x) - \mu_i^2)) e_{\mu_1^1,\ldots,\mu_n^2}(x) = \delta(x). \quad (2.42)\]
We prove the uniqueness of the equation (2.42) solution in the class of the distributions with supports in the closed upper light cone. Let the equation (2.42) have two solutions
\( e^{(1)}_{\mu_1, \ldots, \mu_n}(x), e^{(2)}_{\mu_1, \ldots, \mu_n}(x) \). Since its supports lie in the closed upper light cone the convolution is defined. Now the convolution commutativity implies these distribution coincidence:

\[
e^{(2)}_{\mu_1, \ldots, \mu_n}(x) = \left( \prod_{i=1}^{n} \left( -\left( \partial_{x_i}, \partial_{x_i} \right) - \mu_i^2 \right) \right) \int d^4y e^{(1)}_{\mu_1, \ldots, \mu_n}(x-y)e^{(2)}_{\mu_1, \ldots, \mu_n}(y) = \left( \prod_{i=1}^{n} \left( -\left( \partial_{x_i}, \partial_{x_i} \right) - \mu_i^2 \right) \right) \int d^4y e^{(2)}_{\mu_1, \ldots, \mu_n}(x-y)e^{(1)}_{\mu_1, \ldots, \mu_n}(y) = e^{(1)}_{\mu_1, \ldots, \mu_n}(x). \tag{2.43}
\]

Due to the book ([10], Section 30)

\[
e_0(x) = -(2\pi)^{-1}\theta(x^0)\delta((x, x)), \quad e_{0,0}(x) = (8\pi)^{-1}\theta(x^0)\theta((x, x)). \tag{2.44}
\]

The uniqueness of the equation (2.42) solution in the class of the distributions with supports in the closed upper light cone implies

\[-(\partial_x, \partial_x)e_{0,0}(x) = e_0(x). \tag{2.45}\]

In view of the second definition (2.40)

\[(\partial_x, \partial_x)\theta(x^0)\theta((x, x))(x, x)^n = 4n(n + 1)\theta(x^0)\theta((x, x))(x, x)^{n-1}, \quad n = 1, 2, \ldots. \tag{2.46}\]

Due to the relations (2.44), (2.46) the distribution \( e_{0,\ldots,0}(x) \) with \( n \) zeros has the form

\[e_{0,\ldots,0}(x) = (-1)^n(2\pi 4^{n-1}(n - 2)!(n - 1)!)^{-1}\theta(x^0)\theta((x, x))(x, x)^{n-2}, \quad n = 2, 3, \ldots. \tag{2.47}\]

Let us prove the following equality

\[e_{\mu_1^2}(x) = -\begin{pmatrix} (2\pi)^{-1}\theta(x^0)\delta((x, x)) + \\
\theta(x^0)\theta((x, x)) \sum_{n=1}^{\infty} \mu_1^{2n}(-1)^{n+1}(2\pi 4^n(n - 1)!n!)^{-1}(x, x)^{n-1}. \end{pmatrix} \tag{2.48}\]

By making use of the relations (2.44), (2.45), (2.46) it is possible to prove that the right-hand side of the equality (2.48) satisfies the equation (2.42) for \( n = 1 \). The support of the right-hand side of the equality (2.48) lies in the closed upper light cone. The solution of the equation (2.42) is unique in the class of the distributions with supports in the closed upper light cone.

Let us prove

\[e_{\mu_1^2, \ldots, \mu_n^2}(x) = \lim_{\epsilon \to 0} (2\pi)^{-4} \int d^4p \exp\{-i(p, x)\} \prod_{j=1}^{n} ((p^0 + i\epsilon)^2 - |p|^2 - \mu_j^2)^{-1}. \tag{2.49}\]

The integral (2.49) is the solution of the equation (2.42). By making the shift of the integration path in the right-hand side of the equality (2.49) we obtain that the distribution (2.49) is equal to zero for \( x^0 < 0 \). The distribution (2.49) is Lorentz invariant. Hence its support lies in the closed upper light cone. Now the uniqueness of the distribution \( e_{\mu_1^2, \ldots, \mu_n^2}(x) \) implies the equality (2.49). The equality (1.6) is the particular case of the equality (2.49).

The relations (2.21), (2.24), (2.39) imply the relation

\[
\sum_{p=1}^{4l+2} \sum_{\nu=0}^{2i+1} \left( \sum_{\nu=0}^{3} \left( \gamma_{\lambda,l}^{\nu}(A) \right)_{\mu\nu, pp} \left( -i \frac{\partial}{\partial x^\nu} \right) + \left( \alpha_{\lambda,l}(\mu^2) \right)_{\mu\nu, pp} \right) \times \\
\left( \sum_{\nu=0}^{3} \left( \gamma_{\lambda,l}^{\nu}(A) \right)_{\nu\nu, n\dot{n}} \left( -i \frac{\partial}{\partial x^\nu} \right) - \left( \alpha_{\lambda,l}(\mu^2) \right)_{\nu\nu, n\dot{n}} \right) = (-\partial_x, \partial_x) - \mu^2) \delta_{\nu\nu} \delta_{\mu\dot{m}}, \quad m, n = 1, \ldots, 4l + 2, \quad \dot{m}, \dot{n} = 1, \ldots, 2i + 1, \tag{2.50}\]

\[\]
for an odd integer $2l + 2\hat{\ell}$. The relations (2.42), (2.50) imply that the solution of the equation (2.41) has the form

$$f_{\hat{m}\hat{n}}(x) = \sum_{n=1}^{4l+2} \sum_{\hat{n}=1}^{2\hat{\ell}+1} \left( \sum_{\nu=0}^{3} (\gamma_{\nu}(A))_{\hat{m}\hat{n},n\hat{n}} \left( -i \frac{\partial}{\partial x^\nu} \right) - (\alpha_{\nu}(\mu^2))_{\hat{m}\hat{n},n\hat{n}} \right) \times$$

$$\left( -i \int d^4y e_{\mu^2}(x-y) \delta(y^0) f_{\hat{n}\hat{n}}^0(+0, y) \right), \quad (2.51)$$

$m = 1, \ldots, 4l + 2$, $\hat{m} = 1, \ldots, 2\hat{\ell} + 1$. For the solution of the equation (2.35) in the domain $x^0 < 0$ it is sufficient to use the distribution $-e_{\mu^2}(-x)$ in the relation (2.51).

We suppose that the smooth function $f_{\hat{m}\hat{n}}(+0, x)$ is rapidly decreasing at the infinity. By shifting the integration path in the integral (2.49) we have

$$\int d^4y e_{\mu^2}(x-y) \delta(y^0) f_{\hat{n}\hat{n}}(+0, y) =$$

$$(2\pi)^{-4} \int d^4p \exp\{x^0 - i(p, x)\}((p^0 + i)^2 - |p|^2 - \mu^2)^{-1} \tilde{f}_{\hat{m}\hat{n}}(+0, \cdot)(p),$$

$$\tilde{f}_{\hat{m}\hat{n}}(+0, \cdot)(p) = \int d^3x \exp\{-i \sum_{k=1}^{3} p^k x^k\} f_{\hat{m}\hat{n}}(+0, x). \quad (2.52)$$

The integral with respect to $p^0$ may be easily calculated. For $x^0 > 0$ and $\tilde{f}_{\hat{m}\hat{n}}(+0, \cdot)(p) = f_{\hat{m}\hat{n}}\delta(p-q)$ the functions (2.51), (2.52) are not the eigenfunctions of the differential operator $-i\partial/\partial x^0$ and are the eigenfunctions of the differential operator $(-i\partial/\partial x^0)^2$.

### 3 Relativistic quantum Coulomb law

We consider at first the relativistic Coulomb law in the classical mechanics. The relativistic Coulomb law is the particular case of the relativistic Newton second law

$$mc \frac{dt}{ds} \frac{d}{dt} \left( \frac{d}{ds} \frac{dx^\mu}{dt} \right) + \frac{q}{c} \sum_{k=0}^{3} \sum_{\alpha_1, \ldots, \alpha_k=0}^{N} \eta^{\mu\alpha_k} F_{\mu\alpha_1\ldots\alpha_k}(x) \frac{dx^{\alpha_1}}{ds} \frac{dt}{dt} \cdots \frac{dx^{\alpha_k}}{ds} \frac{dt}{dt} = 0,$$

$$\frac{dt}{ds} = c^{-1} \left( 1 - c^{-2} \left( \frac{dx^2}{dt^2} \right) \right)^{-1/2} \quad (3.1)$$

where $x^0 = ct$, $\mu = 0, \ldots, 3$. In the equation (3.1) the force is the polynomial of the speed. For an infinite series of the speed it is necessary to define the series convergence. The second relation (3.1) implies the identities

$$\sum_{\alpha=0}^{3} \eta_{\alpha\alpha} \left( \frac{dt}{ds} \frac{dx^{\alpha}}{dt} \right)^2 = 1, \quad \sum_{\alpha=0}^{3} \eta_{\alpha\alpha} \frac{dt}{ds} \frac{dx^{\alpha}}{dt} \frac{dt}{ds} \frac{dx^{\alpha}}{dt} = 0. \quad (3.2)$$

The equation (3.1) and the second identity (3.2) imply

$$\sum_{k=0}^{3} \sum_{\alpha_1, \ldots, \alpha_{k+1}=0}^{N} F_{\alpha_1\ldots\alpha_{k+1}}(x) \frac{dt}{ds} \frac{dx^{\alpha_1}}{dt} \cdots \frac{dx^{\alpha_{k+1}}}{dt} = 0. \quad (3.3)$$
Let the functions $F_{\alpha_1 \cdots \alpha_k+1}(x)$ satisfy the equation (3.3). Then three equations (3.1) for $\mu = 1, 2, 3$ are independent

$$m \frac{d}{dt} \left( (1 - c^{-2} |\mathbf{v}|^2)^{-1/2} v^i \right) - q \sum_{k=0}^{N} c^{-k} (1 - c^{-2} |\mathbf{v}|^2)^{-\frac{k-1}{2}} \times$$

$$\sum_{\alpha_1, \ldots, \alpha_k = 0}^{3} F_{\alpha_1 \cdots \alpha_k}(x) \frac{dx^{\alpha_1}}{dt} \cdots \frac{dx^{\alpha_k}}{dt} = 0, \quad v^i = \frac{dx^i}{dt}, \quad i = 1, 2, 3.$$  \hfill (3.4)

The following lemma is proved in the paper [11].

**Lemma.** Let there exist a Lagrange function $L(x, \mathbf{v}, t)$ such that for any world line, $x^\mu(t)$, $x^0(t) = ct$, and for any $i = 1, 2, 3$ the relation

$$\frac{d}{dt} \frac{\partial L}{\partial v^i} - \frac{\partial L}{\partial x^i} = m \frac{d}{dt} \left( (1 - c^{-2} |\mathbf{v}|^2)^{-1/2} v^i \right) - q \sum_{k=0}^{N} c^{-k} (1 - c^{-2} |\mathbf{v}|^2)^{-\frac{k-1}{2}} \times$$

$$\sum_{\alpha_1, \ldots, \alpha_k = 0}^{3} F_{\alpha_1 \cdots \alpha_k}(x) \frac{dx^{\alpha_1}}{dt} \cdots \frac{dx^{\alpha_k}}{dt}$$

holds. Then the Lagrange function has the form

$$L(x, \mathbf{v}, t) = -m c^2 (1 - c^{-2} |\mathbf{v}|^2)^{1/2} + \frac{q}{c} \sum_{i=1}^{3} A_i(x, t) v^i + q A_0(x, t)$$  \hfill (3.6)

and the coefficients in the equations (3.4) are

$$F_{\alpha_1 \cdots \alpha_k}(x) = 0, \quad k \neq 1, \quad i = 1, 2, 3, \quad \alpha_1, \ldots, \alpha_k = 0, \ldots, 3,$$  \hfill (3.7)

$$F_{ij}(x) = \frac{\partial A_j(x, t)}{\partial x^i} - \frac{\partial A_i(x, t)}{\partial x^j}, \quad i, j = 1, 2, 3,$$

$$F_{i0}(x) = \frac{\partial A_0(x, t)}{\partial x^i} - \frac{1}{c} \frac{\partial A_i(x, t)}{\partial t}, \quad i = 1, 2, 3.$$  \hfill (3.8)

We define

$$F_{00} = 0, \quad F_{0i} = -F_{i0}, \quad i = 1, 2, 3.$$  \hfill (3.9)

Then the identity

$$\sum_{\alpha, \beta = 0}^{3} F_{\alpha \beta}(x) \frac{dt}{ds} \frac{dx^\alpha}{dt} \frac{dt}{ds} \frac{dx^\beta}{dt} = 0$$  \hfill (3.10)

similar to the identity (3.3) holds. By making use of the second identity (3.2) and the relations (3.8) - (3.10) we can rewrite the equation (3.4) with the coefficients (3.7), (3.8) as the equation with Lorentz force

$$m c \frac{dt}{ds} \frac{d}{dt} \left( \frac{dt}{ds} \frac{dx^\mu}{dt} \right) = -\frac{q}{c} \eta_{\mu \nu} \sum_{\nu = 0}^{3} F_{\mu \nu}(x) \frac{dt}{ds} \frac{dx^\nu}{dt},$$

$$F_{\mu \nu}(x) = \frac{\partial A_\nu(x, t)}{\partial x^\mu} - \frac{\partial A_\mu(x, t)}{\partial x^\nu}, \quad \mu, \nu = 0, \ldots, 3.$$  \hfill (3.11)

The interaction is defined by the product of the charge $q$ and the vector potential $A_\mu(x, t)$ of the external field.
The relativistic Coulomb law is given by Lorentz covariant equations describing the electromagnetic interaction of two particles with the charges $q_k$, $k = 1, 2$,

$$m_k c \frac{dt}{ds_k} \frac{d}{dt} \left( \frac{dt}{ds_k} \frac{dx_k^\mu}{dt} \right) = -\frac{q_k}{c} \eta^{\mu\nu} \sum_{\nu=0}^3 F_{j;\nu\mu}(x_k, x_j) \frac{dt}{ds_k} \frac{dx_k^\nu}{dt},$$

$$\frac{dt}{ds_k} = c^{-1} \left( 1 - c^{-2} \left| \frac{dx_k}{dt} \right| ^2 \right)^{-1/2},$$

(3.12)

$$F_{j;\nu\mu}(x_k, x_j) = \frac{\partial A_{j;\nu}(x_k, x_j)}{\partial x_k^\mu} - \frac{\partial A_{j;\mu}(x_k, x_j)}{\partial x_k^\nu},$$

(3.13)

for any permutation $k, j$ of the integers $1, 2$. The world lines $x_k(t)$, $k = 1, 2$, satisfy the condition $x_k^0(t) = ct$ where $c$ is the speed of light. Liénard - Wiechert vector potentials are given by the following relations

$$A_{j;\mu}(x_k, x_j) = -4\pi q_j K \sum_{\nu=0}^3 \eta^{\nu\mu} \int dt e_0(x_k - x_j(t)) \frac{dx_j^\nu(t)}{dt} =$$

$$q_j K \eta^{\mu\nu} \left( \frac{d}{dt}(0) x_j^\nu(t(0)) \right) \left( c |x_k - x_j(t(0))| - \sum_{i=1}^3 (x_k^i - x_j^i(t(0))) \frac{d}{dt}(0) x_j^i(t(0)) \right)^{-1},$$

$$x_k^0 - ct(0) = |x_k - x_j(t(0))|$$

(3.14)

where $e_0(x)$ is the first distribution (2.44). By making change of the integration variable

$$x_k^0 - ct(r) = (|x_k - x_j(t(r))|^2 + r)^{1/2}$$

(3.15)

it is easy to prove the relation (3.14). For $r = 0$ the relation (3.15) coincides with the last relation (3.14).

For a world line $x_j^\mu(t)$ we define the vector

$$J^\mu(x, x_j) = \int dt \delta(x - x_j(t)) \frac{dx_j^\mu(t)}{dt} =$$

$$\left( \frac{d}{dx^0} x_j^\mu \right) (c^{-1} x^0) \delta \left( x - x_j \left( c^{-1} x^0 \right) \right), \quad \mu = 0, ..., 3.$$

(3.16)

The world line $x_j^\mu(t)$ satisfies the condition $x_j^0(t) = ct$. Hence the definition (3.16) implies the continuity equation

$$\frac{\partial}{\partial x^0} J^0(x, x_j) = -\sum_{i=1}^3 \left( \frac{d}{dx^0} x_j^i \left( c^{-1} x^0 \right) \right) \frac{\partial}{\partial x^i} \delta \left( x - x_j \left( c^{-1} x_k \right) \right),$$

$$\frac{\partial}{\partial x^i} J^i(x, x_j) = \left( \frac{d}{dx^0} x_j^i \left( c^{-1} x^0 \right) \right) \frac{\partial}{\partial x^i} \delta \left( x - x_j \left( c^{-1} x^0 \right) \right), \quad i = 1, 2, 3,$$

$$\sum_{\mu=0}^3 \frac{\partial}{\partial x^\mu} J^\mu(x, x_j) = 0.$$

(3.17)

The world line $x_j^\mu(t)$, $x_j^0 = ct$, is called time like, if it satisfies the condition

$$\left| \frac{dx_j(t)}{dt} \right| < c,$$

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For the time like world line \( x_j^\mu(t) \) the supports of the distributions (3.16) lie in the closed upper light cone and

\[
\int dt e_0(x - x_j(t)) \frac{dx_j^\mu(t)}{dt} = \int d^4y e_0(x - y) \int dt \delta(y - x_j(t)) \frac{dx_j^\mu(t)}{dt}. \tag{3.18}
\]

The supports of both sides of the equality (3.18) lie in the closed upper light cone. The distribution \( e_0(x) \) satisfies the equation (2.42). Hence both sides of the equality (3.18) satisfy the equation

\[
(-(\partial_x, \partial_x)) f^\mu(x) = J^\mu(x, x_j).
\]

The difference of two solutions of this equation is a solution of the equation (2.42). Since the solution of this equation similar to the solution of the equation (2.42) is unique in the class of the distributions with supports in the closed upper light cone both sides of the equality (3.18) coincide.

The relations (3.17), (3.18) imply the gauge condition for the vector potential (3.14)

\[
\sum_{\mu = 0}^{3} \eta^{\mu \mu} \frac{\partial}{\partial x^\mu} A_{j, \mu}(x, x_j) = 0. \tag{3.19}
\]

Due to relation (3.19) the tensor of the electromagnetic field (3.13), (3.14) satisfies Maxwell equations with the current \( 4\pi q_j K \eta_{\mu \nu} J^\mu(x_k, x_j) \).

In contrast to the equation (1.3) the equations (3.12) - (3.14) satisfy the causality condition. The vector potential (3.14) depends only on the world line points \( x_j^\mu(t) \) lying in the closed lower light cone with the vertex at the point \( x_k \).

In the equations (3.12) - (3.14) the electromagnetic interaction is defined by Liénard - Wiechert vector potentials. Let us consider the interaction coefficients

\[
A_{(jk)}^{\alpha \beta}(x) = K_{jk;0} Q_0^{\alpha \beta}(\partial_x) e_{0,0}(x) + K_{jk;1} Q_1^{\alpha \beta}(\partial_x) e_{0,0}(x),
\]

\[
Q_0^{\alpha \beta}(\partial_x) = -\eta_{\mu \nu}(\partial_x, \partial_x), \quad Q_1^{\alpha \beta}(\partial_x) = 4 \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} - \eta_{\mu \nu}(\partial_x, \partial_x). \tag{3.20}
\]

The interaction coefficient (3.20) is defined by two constants \( K_{jk;0}, K_{jk;1} \). Due to the relation (2.45) for \( K_{jk;1} = 0 \) the interaction coefficient (3.20) is equal to \( \eta_{\mu \nu} K_{jk;0} e_0(x) \). The polynomials \( Q_{\mu \nu}^{\alpha \beta}(\partial_x) \), \( l = 0, 1 \), are connected to Minkowski metric

\[
\eta_{\mu \nu} \sum_{\alpha, \beta = 0, \ldots, 3} \eta^{\alpha \beta} Q_0^{\alpha \beta}(\partial_x) = 4 Q_0^{\alpha \beta}(\partial_x), \quad \eta_{\mu \nu} \sum_{\alpha, \beta = 0, \ldots, 3} \eta^{\alpha \beta} Q_1^{\alpha \beta}(\partial_x) = 0. \tag{3.21}
\]

We change in the equations (3.12) - (3.14) the interaction coefficients \( 4\pi q_k q_j c^{-1} K \eta_{\mu \nu} e_0(x) \) for the interaction coefficients (3.20). The relation (3.18) is still valid when the distribution \( e_{0,0}(x) \) substitutes for the distribution \( e_0(x) \). By making use of the obtained relation and the relations (2.45), (3.17), (3.20) it is easy to prove the relation

\[
\sum_{\nu = 0}^{3} \int dt A_{\mu \nu}^{(jk)}(x_k - x_j(t)) \frac{dx_j^\nu(t)}{dt} = (K_{jk;0} + K_{jk;1}) \sum_{\nu = 0}^{3} \eta_{\mu \nu} \int dt e_0(x_k - x_j(t)) \frac{dx_j^\nu(t)}{dt}. \tag{3.22}
\]

The constants \( K_{jk;l}, l = 0, 1 \), of the interaction coefficient (3.20) are included in the vector potential (3.22) as the sum \( K_{jk;0} + K_{jk;1} \).
Let us introduce the interaction coefficients into the equation (2.41). Let \( j, k \) be the permutation of the numbers 1, 2. We construct the equation for \( j \) particle. Let us multiply the equations (2.41) for the particles 1 and 2. We change the differential operator

\[
\left(-i\frac{\partial}{\partial x_1^\nu}\right) \left(-i\frac{\partial}{\partial x_2^\nu}\right)
\]

for the differential operator

\[
\left(-i\frac{\partial}{\partial x_1^\nu}\right) \left(-i\frac{\partial}{\partial x_2^\nu}\right) + A^{(kj)}_{\nu_1\nu_2}(x_j - x_k).
\]

(We do not change in the equation (2.41) the differential operator \(-i\partial_\nu\) for the differential operator \(-i\partial_\nu + qA_\nu(x)\).) In the differential operators (3.24) the interaction coefficients \(A^{(jk)}(x)\) are given by the relations (3.20). For any matrix \(A \in SL(2, \mathbb{C})\) the equality

\[
A^{(kj)}_{\nu_1\nu_2} \left(\sum_{\mu=0}^{3} \Lambda_\mu^\lambda (A^{-1}) \dot{x}^\mu\right) = \sum_{\mu_1, \mu_2=0}^{3} \Lambda_{\nu_1}^{\mu_1} (A) \Lambda_{\nu_2}^{\mu_2} (A) A^{(kj)}_{\mu_1\mu_2} (x)
\]

holds. The differential operator (3.24) transformation is similar to the differential operator (3.23) transformation. The support of the distribution (3.20) lies in the closed upper light cone. Hence the differential operator (3.24) differs from the differential operator (3.23) only for the vector \(x_j - x_k\) lying in the closed upper light cone. In order to obtain an equation of the type (1.2) we integrate our equation with respect to the variable \(x_k\)

\[
\sum_{n_s=1,...,4\ell_s+2, s=1,2} \sum_{\bar{n}_s=1,...,2\ell_s+1, s=1,2} \int d^4 x_k
\]

\[
\left(\prod_{s=1}^{2} \left(\sum_{\nu=0}^{3} (\gamma_{\mu_0, s}^\nu (A))_{m_s, \bar{m}_s, n_s, \bar{n}_s} \left(-i\frac{\partial}{\partial x_\nu}\right) + (\beta_{\mu_0, s} (\mu_s^2))_{m_s, \bar{m}_s, n_s, \bar{n}_s}\right) (f_s)_{n_s\bar{n}_s} (x_s) + \right.
\]

\[
\left.\prod_{s=1}^{2} (\gamma_{\mu_0, s}^0 (A))_{m_s, \bar{m}_s, n_s, \bar{n}_s} \delta(x_s^0) (f_s)_{n_s\bar{n}_s} (0, \mathbf{x}_s) + \right.
\]

\[
\left.\sum_{\nu_1, \nu_2=0}^{3} A^{(kj)}_{\nu_1\nu_2} (x_j - x_k) \prod_{s=1}^{2} (\gamma_{\mu_0, s}^\nu (A))_{m_s, \bar{m}_s, n_s, \bar{n}_s} (f_s)_{n_s\bar{n}_s} (x_s) \right) = 0.
\]

In the equations (3.14) we integrate along the particle world line the product of the interaction coefficient and the velocity vector

\[
-4\pi q_j K \sum_{\nu=0}^{3} \eta_{\nu
u} \int dt e_0 (x_k - x_j(t)) \frac{dx_\nu^\nu (t)}{dt}.
\]

(Due to the relation (3.22) the interaction coefficient (3.20) gives a similar interaction term.) A particle has no a world line in the quantum mechanics. The particle probability density is given by the wave function. In the equations (3.26) we integrate over the space the product of the same interaction coefficient and the vector constructed by means of the wave function

\[
\int d^4 x_k \sum_{\nu=0}^{3} A^{(kj)}_{\mu_0\nu} (x_j - x_k) \sum_{\ell_k=1}^{4\ell_k+2} \sum_{\bar{n}_k=1}^{2\ell_k+1} (\gamma_{\mu, \ell_k}^\nu (A))_{m_k, \bar{m}_k, n_k, \bar{n}_k} (f_k)_{n_k\bar{n}_k} (x_k).
\]
The number of vector particles in the Yang-Mills theory is

Due to ([3], Chapter 12, relation (12.91); [12], Chapter III, relation (3.58)) the propagation interaction coefficients (3.20) may be changed in the following way.

Let the interaction propagate at the speed less or equal to the speed of light. Hence the interaction coefficients (3.20) for the interaction coefficients (3.29) we obtain in the general case the interaction coefficients (3.29) coincides with the interaction coefficient (3.20). The interaction coefficients (3.29) satisfy the covariance relation (3.25).

The relations (3.20) define the interaction coefficients \( A^{(kj)}_{\mu\nu}(x) \) in the equations (3.26).

The degree of the homogeneous polynomials \( Q^l_{\nu_1\nu_2}(\partial_x), \ l = 0, 1, \) is 2l. Hence the second relation (2.44) implies that the supports of the distributions \( Q^l_{\nu_1\nu_2}(\partial_x)e_{\nu_0}(x), \ l = 0, 1, \) lie on upper light cone boundary. Thus the equations (3.20), (3.26) satisfy the causality condition.

The interactions propagate at the speed of light.

The definitions (2.49), (3.20) imply

\[
A^{(kj)}_{\mu\nu}(x) = \lim_{\epsilon \to +0} (K_{kj,0} + K_{kj,1})(2\pi)^{-4} \int d^4p \exp\{-i(p, x)\} \times \left((p^0 + i\epsilon)^2 - |p|^2\right)^{-1} \left(\eta_{\mu\nu} - \frac{4K_{kj,1}}{K_{kj,0} + K_{kj,1}} \frac{p_\mu p_\nu}{(p^0 + i\epsilon)^2 - |p|^2}\right). \tag{3.27}
\]

Due to ([3], Chapter 12, relation (12.91); [12], Chapter III, relation (3.58)) the propagation function of the vector particles in the Yang-Mills theory is

\[
D^{ab}_{\mu\nu}(x) = \lim_{\epsilon \to +0} -\delta^{ab}(2\pi)^{-4} \int d^4p \exp\{-i(p, x)\} \times \left((p, p) + i\epsilon\right)^{-1} \left(\eta_{\mu\nu} - (1 - \alpha) \frac{p_\mu p_\nu}{(p, p) + i\epsilon}\right). \tag{3.28}
\]

The number \( \alpha \) is the consequence of the gauge condition for the value \( \partial_\mu A^\mu \). The choices \( \alpha = 1 \) and \( \alpha = 0 \) are called the gauge conditions of Feynman and Landau ([3], Section 12.2.2). The distribution (3.28) differs from the distribution (3.27) in the rule of going around the poles in the integral.

We have considered up to now that the interaction propagates at the speed of light. Let the interaction propagate at the speed less or equal to the speed of light. Hence the interaction coefficients (3.20) may be changed in the following way

\[
A^{(kj)}_{\mu\nu}(x) = \sum_{l = 0}^1 d\lambda_1 d\lambda_2 K_{kj,l}(\lambda_1, \lambda_2)Q^l_{\mu\nu}(\partial_x)e_{\lambda_1^2,\lambda_2^2}(x). \tag{3.29}
\]

If the distributions \( K_{kj,l}(\lambda_1, \lambda_2) = K_{kj,l}\delta(\lambda_1)\delta(\lambda_2), \ l = 1, 2, \) then the interaction coefficient (3.29) coincides with the interaction coefficient (3.20). The interaction coefficients (3.29) satisfy the covariance relation (3.25). Changing in the equations (3.26) the interaction coefficients (3.20) for the interaction coefficients (3.29) we obtain in the general case the theory where the particles interact between each other by means of the massive particles.

We consider the equations (3.26), (3.20) for the case \( l_1 = l_2 = 1/2, \ l_\perp = l_2 = 0 \). We suppose that the constants \( K_{12,l} \) are small. If we neglect the term containing the interaction coefficient \( A^{(12)}_{\nu_1\nu_2}(x) \), then the functions \( f_2(x) \) are the solutions of the equation (2.41). It is possible simply to assume \( K_{12,l} = 0, \ l = 1, l_\perp \neq 0, l = 0, 1 \). We do not know the physical reason of the assumption \( K_{12,l} = 0, \ K_{21,l} \neq 0, l = 0, 1 \). However the differential equations (3.20), (3.26) with these constants are mathematically self-consistent.

Let the functions \( f_2(x) \) be the solutions of the equation (2.41) and be given by the relation (2.51). Let the integral

\[
\int d^3y_2(f_2^0)(x_2) = \begin{cases} \int d^3y_2(f_2)(x_2)(+0, y_2), & m_2 = 1, 2, \\ \int d^3y_2(f_2)(x_2)(+0, y_2), & m_2 = 3, 4 \end{cases} \tag{3.30}
\]
be not equal to zero for some index \( m_2 \). The equations (2.41), (3.26) imply

\[
\sum_{n_1=1}^{4} \sum_{\nu=0}^{3} (\gamma_{\nu}^\nu)_{m_1 n_1} \left( -i \frac{\partial}{\partial x_{\nu}^\nu} + B_{\nu}^{(21)}(x_1) \right) (f_1)_{n_1}(x_1) +
\]

\[
\sum_{n_1=1}^{4} (\beta(\mu_1^2))_{m_1 n_1} (f_1)_{n_1}(x_1) = -i \delta(x_1^0)(f_1^0)_{m_1}(+0, x_1),
\]

\[(3.31)\]

\[
B_{\nu}^{(21)}(x_1) = i \left( \int d^4 y_2 (f_2^0)_{m_2} (+0, y_2) \right)^{-1} \times \sum_{\nu_1=0}^{3} \sum_{n_2=1}^{4} (\gamma_{\nu_1}^\nu)_{m_2 n_2} \times (4\pi i)^{-1} K_{21,0} \int d^4 x_2 |x_1 - x_2|^{-1} (f_2)_{n_2}(x_1^0 - |x_1 - x_2|, x_2).
\]

\[(3.32)\]

The equation (3.31) is similar to the equation (1.2). The right-hand side of the equation (3.31) defines Cauchy problem. The first particle interacts with the field of second particle. The vector potential of this field is given by the equality (3.32). The substitution of the first relation (2.44) and the relation (3.20), \( K_{21,1} = 0 \), into the vector potential (3.32) yields the delayed vector potential of second particle electromagnetic field

\[
B_{\nu}^{(21)}(x_1) = \left( \int d^4 y_2 (f_2^0)_{m_2} (+0, y_2) \right)^{-1} \sum_{\nu_1=0}^{3} \sum_{n_2=1}^{4} \eta_{\nu_1 \nu} (\gamma_{\nu_1}^\nu)_{m_2 n_2} \times (4\pi i)^{-1} K_{21,0} \int d^4 x_2 |x_1 - x_2|^{-1} (f_2)_{n_2}(x_1^0 - |x_1 - x_2|, x_2).
\]

\[(3.33)\]

If we choose in the vector potential (3.33) the functions \((f_2)_{m_2}(x_2) = (f_2)_{m_2} \delta(x_2)\), we get the vector potential of Coulomb type. The second particle moves freely and the functions \((f_2)_{m_2}(x_2)\) are the solutions of the equation (2.41) given by the relation (2.51). It is possible to choose the initial functions \((f_2)_{m_2}(+0, x_2)\) only.

The following equality

\[
\int d^4 y e_{\mu_1^2, \ldots, \mu_n^2}(x - y) e_{\mu_1^2, \ldots, \mu_n^2}(y) = e_{\mu_1^2, \ldots, \mu_n^2}(x).
\]

\[(3.34)\]

holds. Both sides of the equality (3.34) have supports in the closed upper light cone and satisfy the equation (2.42). The solution of the equation (2.42) is unique in the class of the distributions with supports in the closed upper light cone. By making use of the relations (2.51), (3.20), (3.34) it is possible to rewrite the equality (3.32) as

\[
B_{\nu}^{(21)}(x) = \left( \int d^4 y_2 (f_2^0)_{m_2} (+0, y_2) \right)^{-1} \times \sum_{\nu_1=0}^{3} \sum_{n_2=1}^{4} \left( \gamma_{\nu_1}^\nu \gamma_{\nu_2}^\nu \right)_{m_2 n_2} \left( -i \frac{\partial}{\partial x_{\nu_2}^\nu} - (\gamma_{\nu_1}^\nu \alpha(\mu_2^2))_{m_2 n_2} \right) \times \int d^4 y (K_{21,0} Q_{\nu_1}^0 (\partial_x) + K_{21,1} Q_{\nu_1}^1 (\partial_x)) e_{0,0, \mu_2^2}(x - y) \delta(y^0)(f_2^0)_{n_2}(+0, y).
\]

\[(3.35)\]

The relations (2.47), (2.48), (3.34) imply the equality

\[
e_{0,0, \mu_2^2}(x) = \theta(x^0) \theta((x, x)) \sum_{n=0}^{\infty} \mu_2^2(-1)^{n+1}(2\pi^4)^{n+2}(n+1)!(n+2)!^{-1}(x, x)^{n+1}.
\]

\[(3.36)\]
By making use of the relations (2.21), (3.20), (3.36) we can rewrite in the open upper light cone the vector potential (3.35) with the functions \((f_2)_{n_2}(+0,y) = (f_2)_{n_2}\delta(y)\) as

\[
B^{(21)}_\nu(x) = B^{(21;0)}_\nu + B^{(21;1)}_\nu(x)
\]

where the constant vector potential

\[
B^{(21;0)}_\nu = -(8\pi)^{-1}K_{21;0} \left( \sum_{n_2=1}^{4} (\gamma^0)_{m_2n_2}(f_2)_{n_2} \right)^{-1} \sum_{n_2=1}^{4} \eta_{\nu\nu}(\gamma^\nu\gamma^0(\mu^2_2))_{m_2n_2}(f_2)_{n_2}
\]

and the remainder vector potential

\[
B^{(21;1)}_\nu(x) = \\
\left( \int d^3y_2(f_2^0)_{m_2}(+0,y_2) \right)^{-1} \sum_{n=1}^{\infty} \mu^2_{2n}(-1)^{n+1}(2\pi^4n^2(n+1)(n+2)!)^{-1} \times \\
\sum_{\nu_1=0}^{3} \sum_{\nu_2=1}^{4} \left( \sum_{\nu_3=0}^{3} (\gamma^\nu_3\gamma^\nu_2\gamma^0)_{m_2n_2} \left( -i \frac{\partial}{\partial x^{\nu_1}} \right) - (\gamma^\nu_3\gamma^0(\mu^2_2))_{m_2n_2} \right) \times \\
(f_2)_{n_2}(K_{21;0}Q^{0}_{\nu_1\nu_2}(\partial_x) + K_{21;1}Q^{1}_{\nu_1\nu_2}(\partial_x))(x,x)^{n+1}.
\]

It is possible to neglect the vector potential (3.39) in the right-hand side of the equality (3.37) for the small \(\mu_2\). For \(\mu_2 = 0\) the vector potential (3.37) is simply equal to the vector potential (3.38).

The equation (3.31) is written for the functions \((f_1)_{n_1}(x_1)\) equal to zero for the negative \(x_1^0\). Let us consider in the open upper light cone the equation of the type (3.31) for the arbitrary functions \((\psi_1)_{m_1}(x_1)\) and the vector potential (3.38)

\[
\sum_{n_1=1}^{4} \sum_{\nu=0}^{3} (\gamma^\nu)_{m_1n_1} \left( -i \frac{\partial}{\partial x_1^\nu} + B^{(21;0)}_\nu \right)(\psi_1)_{n_1}(x_1) + \sum_{n_1=1}^{4} (\beta(\mu^2_1))_{m_1n_1}(\psi_1)_{n_1}(x_1) = 0.
\]

The equation (3.40) is similar to the equation (1.2). Let the matrix \(\alpha(\mu_1)\) act on the equation (3.40). In view of the relations (2.21) we have

\[
\sum_{n_1=1}^{4} \sum_{\nu=0}^{3} (\gamma^\nu)_{m_1n_1} \left( -i \frac{\partial}{\partial x_1^\nu} + B^{(21;0)}_\nu \right)(\xi_1)_{n_1}(x_1) + \mu_1(\xi_1)_{m_1}(x_1) = 0,
\]

\[
(\psi_1)_{m_1}(x_1) = \sum_{n_1=1}^{4} (\beta(\mu^2_1))_{m_1n_1}(\xi_1)_{n_1}(x_1).
\]

Then the solution of the equation (3.40) in the open upper light cone is

\[
(\psi_1)_{m_1}(x_1) = \sum_{n_1=1}^{4} \sum_{k=0}^{\infty} (k!)^{-1}(-ix_1^0)^k(\beta(\mu^2_1))^{k}_{m_1n_1}(\xi_1)_{n_1}(x_1)
\]

where a vector \((\xi_1)_{n_1}\) is independent of the variable \(x_1\) and the matrix

\[
C_{m_1n_1} = \sum_{\nu=0}^{3} B^{(21;0)}_\nu(\gamma^\nu\gamma^0)_{m_1n_1} + \mu_1(\gamma^0)_{m_1n_1}
\]

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is Hermitian if all components of the vector potential (3.38) are real.

The vector potential (3.38) and the mass $\mu_1$ define the energy spectrum of the solution (3.43). Choosing the vectors $(f_2)_{n_2}$ it is possible to obtain a practically arbitrary energy spectrum.

The vector potential (3.35), $(f_2)_{n_2}(+0, x_2) = (f_2)_{n_2}\delta(x_2)$, is equal to zero out of the closed upper light cone. The functions $(f_1)_{n_1}(x_1)$ satisfy the free equation (2.41) out of the closed upper light cone. If the initial wave function $(f_1)_{n_1}(+0, x_1)$ is equal to $(f_1)_{n_1}\delta(x_1)$ (at the initial moment both particles are at the coordinates origin), then in view of the relation (2.51) the function $(f_1)_{n_1}(x_1)$ is equal to zero out of the closed upper light cone. We study the equation (3.31) with the singular vector potential (3.35), $(f_2)_{n_2}(+0, x_2) = (f_2)_{n_2}\delta(x_2)$ and the singular initial wave function $(f_1)_{n_1}(+0, x_1) = (f_1)_{n_1}\delta(x_1)$. The solution of the equation (3.31) is also singular. The solution singularities lie on the upper light cone boundary.

**References**

[1] Schweber, S.: An Introduction to Relativistic Quantum Field Theory. Harper & Row, New York (1961).

[2] Streater, R.F., Wightman, A.S.: PCT, Spin and Statistics and All That. Benjamin, New York (1964).

[3] Itzykson, C., Zuber, J.-B.: Quantum Field Theory. McGraw - Hill, New York (1980).

[4] Voronov, B.L., Gitman, D.M., Tyutin, I.V.: Dirac Hamiltonian with superstrong Coulomb field. Theor. Math. Phys. 150, 34 - 72 (2007)

[5] Feynman, R.P.: Quantum Electrodynamics. Benjamin, New York (1973).

[6] Feynman, R.P.: QED The Strange Theory of Light and Matter. Princeton University Press, Princeton, NJ (1985).

[7] Stueckelberg, E. C. G., Rivier, D.: Causalité et structure de la Matrice $S$. Helv. Phys. Acta, 23, 215 - 222 (1950)

[8] Poincaré, H.: Sur la dynamique de l’électron. Rendiconti Circolo Mat. Palermo. 21, 129 - 176 (1906)

[9] Vilenkin, N.Ya.: Special Functions and the Theory of Group Representations. American Math. Soc., Providence, RI (1968).

[10] Vladimirov, V.S.: Methods of Theory of Many Complex Variables, MIT Press, Cambridge, MA (1966).

[11] Zinoviev, Yu.M.: Gravity and Lorentz Force. Theor. Math. Phys., 131, 729 - 746 (2002)

[12] Faddeev, L.D., Slavnov, A.A.: Gauge Fields. Introduction to Quantum Theory. Benjamin, New York (1980).