Testing independence of functional variables by an Hilbert-Schmidt independence criterion estimator

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Abstract. We propose an estimator of the Hilbert-Schmidt Independence Criterion obtained from an appropriate modification of the usual estimator. We then get asymptotic normality of this estimator both under independence hypothesis and under the alternative hypothesis. A new test for independence of random variables valued into metric spaces is then introduced, and a simulation study that allows to compare the proposed test to an existing one is provided.

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1 Introduction

Functional data analysis is a branch of statistics allowing to set up methods for processing data in the form of curves representing, for example, the evolution of random phenomena over time. It have known, over the last two decades, a very important development which allowed the introduction of methods for studying links between functional variables through functional regression models, non-correlation tests (e.g., Kokoszka et al. (2008), Aghoukeng Jiofack and Nkiet (2010)), independence tests (e.g., Górecki et al. (2020), Lai et al. (2021)). However, only a few works focus on the problem of independence testing between functional variables. Recently, Lai et
al. (2021) introduced the angle covariance to characterize independence and proposed a permutation based test whereas Górecki et al. (2020) proposed the use of the Hilbert-Schmidt Independence Criterion (HSIC) for testing independence from multivariate functional data. HSIC, introduced by Gretton et al. (2005), is one of the most successful non-parametric dependence measure defined for random variables with values in metric spaces, allowing to measure the dependency between univariate or multivariate random variables, but also random variables valued into more complex structures such as structured, high-dimensional or functional data. For testing independence, an empirical estimator of HSIC was proposed in Gretton (2005) as test statistic; but its asymptotic distribution under null hypothesis is an infinite sum of distributions (see Zhang et al. (2018)) and, consequently, can not been used for performing the test, so requiring the use of methods such as permutation method to calculate $p$-value. For overcoming this drawback, we adopt in this paper an approach introduced in Makigusa and Naito (2020), and used in Balogoun et al. (2021), consisting in constructing a test statistic from an appropriate modification of the aforementioned estimator in order to yield asymptotic normality both under the null hypothesis and under the alternative. This approach allows us to propose a new independence test for random variables valued into metric spaces, including functional variables. The rest of the paper is organized as follows. The HSIC is recalled in Section 2, and Section 3 is devoted to its estimation by a modification of the naive estimator, and to the main results. A simulation study on functional data that allows to compare the proposed test to that of Lai et al. (2021) is given in Section 4. All the proofs are postponed in Section 5.

2 HSIC and independence test

Let $X$ and $Y$ be two random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and with values in metric spaces $\mathcal{X}$ and $\mathcal{Y}$ respectively. We consider reproducing kernel Hilbert spaces $\mathcal{H}_X$ and $\mathcal{H}_Y$ of functions from $\mathcal{X}$ and $\mathcal{Y}$, respectively, to $\mathbb{R}$ with associated kernels $K : \mathcal{X}^2 \to \mathbb{R}$ and $L : \mathcal{Y}^2 \to \mathbb{R}$. They are symmetric functions such that, for any $(f,g) \in \mathcal{H}_X \times \mathcal{H}_Y$ and any $(x,y) \in \mathcal{X} \times \mathcal{Y}$, one has $K(x,\cdot) \in \mathcal{H}_X$, $L(y,\cdot) \in \mathcal{H}_Y$, $f(x) = \langle K(x,\cdot), f \rangle_{\mathcal{H}_X}$ and $g(y) = \langle L(y,\cdot), g \rangle_{\mathcal{H}_Y}$ (see Berlinet and Thomas-Agnan, 2004), where $\langle \cdot, \cdot \rangle_{\mathcal{H}_X}$ (resp. $\langle \cdot, \cdot \rangle_{\mathcal{H}_Y}$) denotes the inner product of $\mathcal{H}_X$ (resp. $\mathcal{H}_Y$). Throughout this paper, we assume that $K$ and $L$ satisfy the following assumption:
(\mathcal{A}_1): \|K\|_\infty := \sup_{(x,y) \in \mathcal{X}^2} K(x,y) < +\infty, \|L\|_\infty := \sup_{(x,y) \in \mathcal{Y}^2} L(x,y) < +\infty;

it ensures the existence of the kernel mean embeddings \( m_X = \mathbb{E}(K(X, \cdot)) \), \( m_Y = \mathbb{E}(L(Y, \cdot)) \), and also that of \( \mathbb{E}(L(Y, \cdot) \otimes K(X, \cdot)) \), where \( \otimes \) denotes the tensor product defined as follows: for any \((f, g) \in H_X \times H_Y\), \(g \otimes f\) is the linear operator from \(H_Y\) to \(H_X\) such that \((g \otimes f)(h) = \langle g, h \rangle_{H_Y} f\), for any \(h \in H_Y\). For measuring dependence between \(X\) and \(Y\), Gretton et al. (2005) introduced the Hilbert-Schimdt Independence Criterion (HSIC) defined as

\[
H = \left\| \mathbb{E}(L(Y, \cdot) \otimes K(X, \cdot)) - m_Y \otimes m_X \right\|_{HS}^2,
\]

where \(\| \cdot \|_{HS}\) denotes the Hilbert-Schmidt norm of operators. This permits to characterize independence and, consequently, to consider a test of independence between \(X\) and \(Y\), that is a test for the hypothesis \(H_0: P_{XY} = P_X \times P_Y\) against \(H_1: P_{XY} \neq P_X \times P_Y\), where \(P_{XY}\) (resp. \(P_X\); resp. \(P_Y\)) denotes the distribution of \((X, Y)\) (resp. \(X\); resp. \(Y\)). Indeed, it is known from Theorem 4 of Gretton et al. (2005) that \(H_0\) is true if and only if \(H = 0\) when the following assumption holds:

(\mathcal{A}_2): \mathcal{H}_X \text{ and } \mathcal{H}_Y \text{ are universal, and } \mathcal{X} \text{ and } \mathcal{Y} \text{ are compact metric spaces.}

Recall that a RKHS is universal if it is dense, with respect to the \(L_\infty\) norm, in the space of continuous maps and if the associated kernel is continuous (see, e.g., Gretton et al. (2012), p. 727). Therefore, a test of independence can be achieved by taking as test statistic a consistent estimator of \(H\). Letting \(\{(X_i, Y_i)\}_{1 \leq i \leq n}\) be a i.i.d. sample of \((X, Y)\) and replacing each expectation in (1) by its empirical counterpart lead to the estimator

\[
\hat{\text{HSIC}}_n = \left\| \frac{1}{n} \sum_{i=1}^{n} L(Y_i, \cdot) \otimes K(X_i, \cdot) - \overline{L}_n \otimes \overline{K}_n \right\|_{HS}^2
\]

\[
= \frac{1}{n^2} \sum_{i,j=1}^{n} k_{ij} \ell_{ij} + \frac{1}{n^4} \sum_{i,j,q,r=1}^{n} k_{ij} \ell_{qr} - \frac{2}{n^3} \sum_{i,j,q=1}^{n} k_{ij} \ell_{iq}, \quad (1)
\]

where \(\overline{L}_n = n^{-1} \sum_{i=1}^{n} L(Y_i, \cdot), \overline{K}_n = n^{-1} \sum_{i=1}^{n} K(X_i, \cdot), k_{ij} = K(X_i, X_j)\) and \(\ell_{ij} = L(Y_i, Y_j)\). This estimator was proposed in Gretton (2005) as test statistic for testing for \(H_0\); but its asymptotic distribution under null hypothesis is...
is an infinite sum of distributions (see Zhang et al. (2018)). We propose an estimator obtained from an appropriate modification of (1) in order to yield asymptotic normality both under the null hypothesis and under the alternative.

3 Estimation of HSIC and asymptotic normality

For $\gamma \in [0, 1]$ and $r \in \mathbb{N}^*$, let $(w_{i,r}(\gamma))_{1 \leq i \leq r}$ be a sequence of positive numbers satisfying:

(A3) There exists a strictly positive real number $\tau$ and an integer $n_0$ such that for all $r > n_0$:

$$r \left| \frac{1}{r} \sum_{i=1}^{r} w_{i,r}(\gamma) - 1 \right| \leq \tau.$$

(A4) There exists $C > 0$ such that $\max_{1 \leq i \leq r} w_{i,r}(\gamma) < C$ for all $r \in \mathbb{N}^*$ and $\gamma \in [0, 1]$.

(A5) For any $\gamma \in [0, 1]$, $\lim_{r \to +\infty} \frac{1}{r} \sum_{i=1}^{r} w_{i,r}^2(\gamma) = w^2(\gamma) > 1$.

A typical example is given by $w_{i,r}(\gamma) = 1 + (-1)^i \gamma$ (see Ahmad, 1993). We propose to estimate HSIC by a modification of (1) given by

$$\tilde{H}_{n,\gamma} = \frac{1}{n^2} \sum_{i,j=1}^{n} k_{ij} \ell_{ij} + \frac{1}{n^2} \sum_{i,j,q,r=1}^{n} k_{ij} \ell_{qr} - \frac{2}{n^3} \sum_{i,j,q=1}^{n} w_{i,n}(\gamma) k_{ij} \ell_{iq}.$$

Putting $\mu = \mathbb{E} (L(Y, \cdot) \otimes K(X, \cdot))$ and $\nu = m_Y \otimes m_X$, and considering the functions $\mathcal{U}$ and $\mathcal{V}$ from $\mathcal{X} \times \mathcal{Y}$ to $\mathbb{R}$ defined as

$$\mathcal{U}(x, y) = \langle L(y, \cdot) \otimes K(x, \cdot) - \mu, \mu \rangle_{\text{HS}} + \langle L(y, \cdot) \otimes m_X + m_Y \otimes K(x, \cdot) - 2\nu, \nu - \mu \rangle_{\text{HS}},$$

$$\mathcal{V}(x, y) = \langle L(y, \cdot) \otimes K(x, \cdot) - \mu, \nu \rangle_{\text{HS}},$$

where $\langle \cdot, \cdot \rangle_{\text{HS}}$ denotes the Hilbert-Smidt inner product, we have:
Theorem 1 Assume that \((\mathcal{A}_1)\) to \((\mathcal{A}_5)\) hold. Then as \(n \to +\infty\), one has \(\sqrt{n}(\hat{H}_{n,\gamma} - H) \xrightarrow{d} N(0, \sigma_\gamma^2)\), where

\[
\sigma_\gamma^2 = 4 \text{Var}(U(X_1, Y_1)) + 4w^2(\gamma)\text{Var}(V(X_1, Y_1)) - 8\text{Cov}(U(X_1, Y_1), V(X_1, Y_1)).
\]

When \(\mathcal{H}_0\) is true, this theorem gives a simpler expression for the variance of the limiting normal distribution. Indeed, one then has \(H = 0\) and \(\mu = \nu\). Therefore, \(U = V\) and we have:

Corollary 1 Assume that \((\mathcal{A}_1)\) to \((\mathcal{A}_5)\) hold. Then, under \(\mathcal{H}_0\), as \(n \to +\infty\), one has \(\sqrt{n}\hat{H}_{n,\gamma} \xrightarrow{d} N(0, \sigma_\gamma^2)\), where \(\sigma_\gamma^2 = 4(w(\gamma)^2 - 1)\) \(\text{Var}(V(X_1, Y_1))\).

In order to achieve tests with the above estimator, it is necessary to find a consistent estimator of the variance \(\sigma_\gamma^2\).

Proposition 1 Assume that \((\mathcal{A}_1)\) to \((\mathcal{A}_5)\) hold. Then, under \(\mathcal{H}_0\), the estimator \(\hat{\sigma}_\gamma^2 = 4(w(\gamma)^2 - 1)\) \(\hat{\alpha}\), where

\[
\hat{\alpha} = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{1}{n} \sum_{j=1}^{n} \ell_{ij} k_{ij} - \frac{1}{n^2} \sum_{m=1}^{n} \sum_{p=1}^{n} \ell_{mp} k_{mp} \right)^2,
\]

is a consistent for \(\sigma_\gamma^2\).

The resulting test for independence is performed as follows: for a given significance level \(\alpha \in [0, 1]\), one has to reject \(\mathcal{H}_0\) if \(\hat{H}_{n,\gamma} > n^{-1/2} \hat{\sigma}_\gamma \Phi^{-1}(1 - \alpha/2)\), where \(\Phi\) is the cumulative distribution function of the standard normal distribution.

4 Simulations

In this section, we investigate the finite sample performance of the proposed test based on modified HSIC and compare it to the test of Lai et al. (2021) based on angle covariance. For convenience, we denote our test as mhsic, and the test of Lai et al. (2021) as acov. We computed empirical sizes and powers through Monte Carlo simulations. We considered the case where \(\mathcal{X} = \mathcal{Y} = L^2([0,1])\) and, similar to the functional data considered in Lai et al. (2021), we take \(X(t) = \sqrt{2} \sum_{k=1}^{50} \xi_k \cos(k\pi t)\) and
\[ Y(t) = \sqrt{2} \sum_{k=1}^{50} \nu_k \cos(k\pi t), \]
where the \( \xi_k \)s are independent and distributed as the Cauchy distribution \( \mathcal{C}(0,0.5) \) and, for a given \( m \in \{0, \ldots, 50\} \), \( \nu_k = f(\xi_k) \) for \( k = 1, \ldots, m \) and the \( \nu_k \)s with \( k = m + 1, \ldots, 50 \) are sampled independently from the standard normal distribution. The null hypothesis \( \mathcal{H}_0 \) holds in case \( m = 0 \), and the dependence level increases with \( m \). Empirical sizes and powers were computed on the basis of 300 independent replicates. For each of them, we generated a sample of size \( n = 100 \) of the above processes in discretized versions on equispaced values \( t_1, \ldots, t_{51} \) in \([0,1]\), where \( t_j = (j - 1)/50 \), \( j = 1, \ldots, 51 \). For performing our method, we took \( \gamma = 0.32 \) and used the gaussian kernels \( K(x, y) = L(x, y) = \exp\left(-\sigma \int_0^1 (x(t) - y(t))^2 \, dt\right) \) with \( \sigma^2 = 1/150 \); the terms \( k_{ij} \) and \( \ell_{ij} \) were computed by approximating integrals involved in these kernels by using the trapezoidal rule. The significance level was taken as \( \alpha = 0.05 \). The acov method was used with 50 permutations. Table 1 reports the obtained results. The obtained values for \( m = 0 \) are close to the nominal size for all methods. For \( m = 1, 3 \), acov slightly outperforms our method which still remains competitive since the differences in the obtained values are low. For \( m = 5, 10 \), the two methods give high values for the power. This highlights the interest of the proposed test: it is powerful enough and is fast compared to acov method which is based permutations and, therefore, leads very high computation times.

5 Proofs

5.1 A technical lemma

**Lemma 1** Let \( (Z_i)_{1 \leq i \leq n} \) be an i.i.d. sample of a random variable \( Z \) valued into a Hilbert space \( \mathcal{H} \) and such that \( \mathbb{E}(\|Z\|^2_{\mathcal{H}}) < +\infty \). Then

\[
\left\| \frac{1}{n} \sum_{i=1}^{n} (w_{i,n}(\gamma) - 1) Z_i \right\|_{\mathcal{H}} = o_p(1).
\]  

(3)
\[
\begin{array}{cccccc}
\text{f(x)} & \text{method} & m = 0 & m = 1 & m = 3 & m = 5 & m = 10 \\
\hline
x^3 & \text{acov} & 0.051 & 1.00 & 1.00 & 1.00 & 1.00 \\
 & \text{mhsic} & 0.052 & 0.89 & 0.92 & 1.00 & 1.00 \\
x^2 & \text{acov} & 0.053 & 0.84 & 0.97 & 1.00 & 1.00 \\
 & \text{mhsic} & 0.060 & 0.80 & 0.91 & 0.94 & 1.00 \\
x^2 \sin(x) & \text{acov} & 0.051 & 0.85 & 0.95 & 0.95 & 0.97 \\
 & \text{mhsic} & 0.051 & 0.80 & 0.88 & 0.96 & 0.99 \\
\end{array}
\]

Table 1: Empirical sizes and powers over 300 replications with significance level \(\alpha = 0.05\).

**Proof.** Putting \(m_Z = \mathbb{E}(Z)\), we have:

\[
\left\| \frac{1}{n} \sum_{i=1}^{n} \left( w_{i,n}(\gamma) - 1 \right) Z_i \right\|_{\mathcal{H}}
\]

\[
= \left\| \frac{1}{n} \sum_{i=1}^{n} w_{i,n}(\gamma) (Z_i - m_Z) - \frac{1}{n} \sum_{i=1}^{n} Z_i + m_Z + \frac{1}{n} \sum_{i=1}^{n} (w_{i,n}(\gamma) - 1) m_Z \right\|_{\mathcal{H}}
\]

\[
\leq \left\| \frac{1}{n} \sum_{i=1}^{n} w_{i,n}(\gamma) (Z_i - m_Z) \right\|_{\mathcal{H}} + \left\| \frac{1}{n} \sum_{i=1}^{n} Z_i - m_Z \right\|_{\mathcal{H}} + \left\| \frac{1}{n} \sum_{i=1}^{n} (w_{i,n}(\gamma) - 1) \right\|_{\mathcal{H}} \times \left\| m_Z \right\|_{\mathcal{H}}.
\]

First, using the equality

\[
\mathbb{E} \left( \left\| \frac{1}{n} \sum_{i=1}^{n} w_{i,n}(\gamma) (Z_i - m_Z) \right\|_{\mathcal{H}}^2 \right) = \left( \frac{1}{n^2} \sum_{i=1}^{n} w_{i,n}^2(\gamma) \right) \mathbb{E} \left( \left\| Z - m_Z \right\|_{\mathcal{H}}^2 \right),
\]

assumption (\(A_5\)) and Markov inequality, we get \(\left\| \frac{1}{n} \sum_{i=1}^{n} w_{i,n}(\gamma) (Z_i - m_Z) \right\|_{\mathcal{H}} = o_p(1)\). Secondly, the law of large numbers gives \(\left\| \frac{1}{n} \sum_{i=1}^{n} Z_i - m_Z \right\|_{\mathcal{H}} = o_p(1)\).

Thirdly, using assumption (\(A_3\)) we obtain \(\left\| \frac{1}{n} \sum_{i=1}^{n} (w_{i,n}(\gamma) - 1) \right\|_{\mathcal{H}} \rightarrow 0\) as \(n \rightarrow +\infty\). Then, (3) is obtained.
5.2 Proof of Theorem 1

Using the definition of the Hilbert-Schmidt inner product, the reproducing properties of $K$ and $L$, and the equality $(a \otimes b)^* = b \otimes a$, it is easy to see that

$$
\hat{H}_{n, \gamma} = \left\| \frac{1}{n} \sum_{i=1}^{n} L(Y_i, \cdot) \otimes K(X_i, \cdot) \right\|_{HS}^2 + \left\| T_n \otimes K \right\|_{HS}^2
- \frac{2}{n} \sum_{i=1}^{n} w_{i,n}(\gamma) \left\langle L(Y_i, \cdot) \otimes K(X_i, \cdot), T_n \otimes K \right\rangle_{HS}
$$

and

$$
H = \left\| \mathbb{E} \left( L(Y, \cdot) \otimes K(X, \cdot) \right) \right\|_{HS}^2 + \left\| m_Y \otimes m_X \right\|_{HS}^2 - 2 \left\langle \mathbb{E} \left( L(Y, \cdot) \otimes K(X, \cdot) \right), m_Y \otimes m_X \right\rangle_{HS}.
$$

Therefore, using the two following equalities: $\|a\|^2 = \|a-b\|^2 + 2\langle a, b \rangle - \|b\|^2$, 
$$
\langle a_1, a_2 \rangle = \langle a_1 - b_1, a_2 - b_2 \rangle + \langle a_1, b_2 \rangle + \langle a_2, b_1 \rangle - \langle b_1, b_2 \rangle,
$$
and putting $k_{X_i} = K(X_i, \cdot)$, $l_{Y_i} = L(Y_i, \cdot)$, we get

$$
\sqrt{n} \left( \hat{H}_{n, \gamma} - H \right)
= \sqrt{n} \left[ \left\| T \otimes k^n - \mu \right\|_{HS}^2 + 2 \left\langle T \otimes k^n, \mu \right\rangle_{HS} - \left\| \mu \right\|_{HS}^2 + \left\| T_n \otimes K_n - \nu \right\|_{HS}^2 + 2 \left\langle T_n \otimes K_n, \nu \right\rangle_{HS} - \left\| \nu \right\|_{HS}^2 
- \frac{2}{n} \sum_{i=1}^{n} (w_{i,n}(\gamma) - 1) \left\langle l_{Y_i} \otimes k_{X_i}, T_n \otimes K_n - \nu \right\rangle_{HS} - \frac{2}{n} \sum_{i=1}^{n} w_{i,n}(\gamma) \left\langle l_{Y_i} \otimes k_{X_i}, \nu \right\rangle_{HS}
+ \frac{2}{n} \sum_{i=1}^{n} \left\langle l_{Y_i} \otimes k_{X_i}, \nu \right\rangle_{HS}
- 2 \left\langle T_n \otimes K_n, \mu \right\rangle_{HS} - \frac{2}{n} \sum_{i=1}^{n} (w_{i,n}(\gamma) - 1) \left\langle \mu, \nu \right\rangle_{HS} + \frac{2}{n} \sum_{i=1}^{n} w_{i,n}(\gamma) \left\langle \mu, \nu \right\rangle_{HS} - \left\| \mu - \nu \right\|_{HS}^2 \right]
= A_n + B_n + C_n + D_n,
$$

where

$$
A_n = \sqrt{n} \left( \left\| T \otimes k^n - \mu \right\|_{HS}^2 + \left\| T_n \otimes K_n - \nu \right\|_{HS}^2 \right)
= n^{-1/2} \left[ \left\| \sqrt{n} \left( \sqrt{n} \otimes k^n - \mu \right) \right\|_{HS}^2 + \left\| \left( \sqrt{n} \left( T_n - m_Y \right) \right) \otimes \left( K_n - m_X \right) \right\|_{HS}^2 
+ \left( \sqrt{n} \left( T_n - m_Y \right) \right) \otimes m_X + m_Y \otimes \left( \sqrt{n} \left( K_n - m_X \right) \right) \right\|_{HS}^2 \right],
$$
\[ B_n = -2 \left[ \frac{1}{n} \sum_{i=1}^{n} (w_{i,n}(\gamma) - 1) l_{Y_i} \otimes k_{X_i}, \sqrt{n}(T_n \otimes K_n - \nu) \right]_{HS} + \sqrt{n} \sum_{i=1}^{n} (w_{i,n}(\gamma) - 1) \langle \mu, \nu \rangle_{HS}, \]

\[ C_n = -2 \left( \sqrt{n}(l \otimes k - \mu), T_n \otimes K_n - \nu \right)_{HS} + 2 \langle \sqrt{n}(T_n - m_Y) \otimes (K_n - m_X), \nu - \mu \rangle_{HS}, \]

\[ D_n = \frac{2}{\sqrt{n}} \sum_{i=1}^{n} \left( U(X_i, Y_i) - w_{i,n}(\gamma) V(X_i, Y_i) \right), \]

\( U \) and \( V \) being the functions defined in (2). From the central limit theorem, \( \sqrt{n}(l \otimes k - \mu), \sqrt{n}(T_n - m_Y) \) and \( \sqrt{n}(K_n - m_X) \) converge in distribution, as \( n \to +\infty \), to random variables having normal distributions. In addition, by the law of large numbers, \( K_n - m_X \) converges in probability to 0, as \( n \to +\infty \). We then deduce that \( A_n = o_p(1) \). On the other hand, using Cauchy-Schwartz inequality, we obtain

\[ |B_n| \leq 2 \left[ \frac{1}{n} \sum_{i=1}^{n} (w_{i,n}(\gamma) - 1) l_{Y_i} \otimes k_{X_i}, \sqrt{n}(T_n \otimes K_n - \nu) \right]_{HS} + \sqrt{n} \sum_{i=1}^{n} (w_{i,n}(\gamma) - 1) \left[ \| \mu \|_{HS} \| \nu \|_{HS} \right]. \]

Since \( \sqrt{n}(T_n \otimes K_n - \nu) = \left( \sqrt{n}(T_n - m_Y) \right) \otimes K_n + m_Y \otimes (\sqrt{n}(K_n - m_X)) \),

and from (\( \mathcal{A}_3 \)), \( \left| \frac{1}{n} \sum_{i=1}^{n} w_{i,n}(\gamma) - 1 \right| \leq \tau/n \) for \( n \) large enough, it follows:

\[ |B_n| \leq 2 \left[ \frac{1}{n} \sum_{i=1}^{n} (w_{i,n}(\gamma) - 1) l_{Y_i} \otimes k_{X_i}, \sqrt{n}(T_n - m_Y) \right]_{HS} \left( \left\| \left( \sqrt{n}(T_n - m_Y) \right) \otimes K_n \right\|_{HS} + \left\| m_Y \otimes (\sqrt{n}(K_n - m_X)) \right\|_{HS} \right) + \frac{\tau}{\sqrt{n}} \left\| \mu \right\|_{HS} \left\| \nu \right\|_{HS}. \]

\[ = 2 \left[ \frac{1}{n} \sum_{i=1}^{n} (w_{i,n}(\gamma) - 1) l_{Y_i} \otimes k_{X_i}, \sqrt{n}(T_n - m_Y) \right]_{HS} \left( \left\| \sqrt{n}(T_n - m_Y) \right\|_{h_Y} \left\| K_n \right\|_{h_X} + \left\| m_Y \right\|_{h_Y} \left\| \sqrt{n}(K_n - m_X) \right\|_{h_X} \right) + \frac{\tau}{\sqrt{n}} \left\| \mu \right\|_{HS} \left\| \nu \right\|_{HS}. \]

Furthermore, using the reproducing property of \( K \), we obtain

\[ \left\| K_n \right\|_{h_X} \leq \frac{1}{n} \sum_{i=1}^{n} \left\| K(X_i, \cdot) \right\|_{h_X} = \frac{1}{n} \sum_{i=1}^{n} \sqrt{K(X_i, X_i)} \leq \| K \|_{\infty}^{1/2}. \] (4)
Hence

\[ |B_n| \leq 2 \left[ \left\| \frac{1}{n} \sum_{i=1}^{n} (w_{i,n}(\gamma) - 1) l_{Y_i} \otimes k_{X_i} \right\|_{HS} \left( \left\| \sqrt{n}(\mathcal{L}_n - m_Y) \right\|_{\mathcal{H}_Y} \left\| K \right\|_{\mathcal{H}} \right)^{1/2} + \left\| m_Y \right\|_{\mathcal{H}_Y} \left\| \sqrt{n}(K_n - m_X) \right\|_{\mathcal{H}_X} \right] + \frac{\tau}{\sqrt{n}} \left\| \mu \right\|_{HS} \left\| \nu \right\|_{HS}. \]

From Lemma 1, we have \( \left\| \frac{1}{n} \sum_{i=1}^{n} (w_{i,n}(\gamma) - 1) l_{Y_i} \otimes k_{X_i} \right\|_{HS} = o_p(1). \) Then, since \( \sqrt{n}(K_n - m_X) \) and \( \sqrt{n}(\mathcal{L}_n - m_Y) \) converge in distribution as \( n \to +\infty, \) we deduce from the preceding inequality that \( B_n = o_p(1). \) Using again Cauchy-Schwartz inequality, we get

\[ |C_n| \leq 2 \left\| \sqrt{n}(\mathcal{L}_n \otimes k^n - \mu) \right\|_{HS} \left\| \mathcal{L}_n \otimes K_n - \nu \right\|_{HS} + 2 \left\| \sqrt{n}(\mathcal{L}_n - m_Y) \otimes (K_n - m_X) \right\|_{HS} \left\| \nu - \mu \right\|_{HS} \]

\[ = 2 \left\| \sqrt{n}(\mathcal{L}_n \otimes k^n - \mu) \right\|_{HS} \left\| \mathcal{L}_n \otimes K_n - \nu \right\|_{HS} + 2 \left\| \sqrt{n}(\mathcal{L}_n - m_Y) \right\|_{\mathcal{H}_Y} \left\| K_n - m_X \right\|_{\mathcal{H}_X} \left\| \nu - \mu \right\|_{HS}. \]

We already know that, as \( n \to +\infty, \) \( \mathcal{L}_n \otimes k^n \) and \( \sqrt{n}(\mathcal{L}_n - m_Y) \) converge in distribution to normal random variables, and that \( K_n \) and \( \mathcal{L}_n \) converge in probability to \( m_X \) and \( m_Y \) respectively. Thus \( \mathcal{L}_n \otimes K_n \) converges in probability to \( \nu \) as \( n \to +\infty, \) and the preceding inequality implies that \( C_n = o_p(1). \)

We can conclude that

\[ \sqrt{n} \left( \hat{H}_{n,\gamma} - H \right) = D_n + o_p(1) = \frac{2}{\sqrt{n}} \sum_{i=1}^{n} \left( \mathcal{U}(X_i, Y_i) - w_{i,n}(\gamma) \mathcal{V}(X_i, Y_i) \right) + o_p(1) \]

and, consequently, that \( \sqrt{n} \left( \hat{H}_{n,\gamma} - H \right) \) has the same limiting distribution than \( D_n; \) it remains to derive this latter. Let us set

\[ s_{n,\gamma}^2 = \sum_{i=1}^{n} Var \left( \mathcal{U}(X_i, Y_i) - w_{i,n}(\gamma) \mathcal{V}(X_i, Y_i) \right). \]

By similar arguments as in the proof of Theorem 1 in Makigusa and Naito (2020) we obtain that, for any \( \varepsilon > 0, \)

\[ s_{n,\gamma}^{-2} \sum_{i=1}^{n} \int_{\{ (x,y) : \mathcal{U}(x,y) - w_{i,n}(\gamma) \mathcal{V}(x,y) > \varepsilon s_{n,\gamma} \}} \left( \mathcal{U}(x,y) - w_{i,n}(\gamma) \mathcal{V}(x,y) \right)^2 d\mathbb{P}_{XY}(x,y) \]
converges to 0 as \( n \to +\infty \). Therefore, by Section 1.9.3 in Serfling (1980) we obtain that \( \sqrt{n}s_{n,\gamma}^{-1}D_n \xrightarrow{\mathcal{D}} N(0,1) \). However,

\[
\left( \frac{s_{n,\gamma}}{\sqrt{n}} \right)^2 = \text{Var}(\mathcal{U}(X_1, Y_1)) + \left( \frac{1}{n} \sum_{i=1}^{n} w_{i,n}(\gamma) \right) \text{Var}(\mathcal{V}(X_1, Y_1))
- 2 \left( \frac{1}{n} \sum_{i=1}^{n} w_{i,n}(\gamma) \right) \text{Cov}(\mathcal{U}(X_1, Y_1), \mathcal{V}(X_1, Y_1)),
\]

from (\( A_3 \)) and (\( A_5 \)),

\[
\lim_{n \to +\infty} \left( \frac{1}{n} \sum_{i=1}^{n} w_{i,n}^2(\gamma) \right) = w^2(\gamma) \quad \text{and} \quad \lim_{n \to +\infty} \left( \frac{1}{n} \sum_{i=1}^{n} w_{i,n}(\gamma) \right) = 1.
\]

Thus

\[
\lim_{n \to +\infty} \left( n^{-1}s_{n,\gamma}^2 \right) = \text{Var}(\mathcal{U}(X_1, Y_1)) + w^2(\gamma) \text{Var}(\mathcal{V}(X_1, Y_1)) - 2 \text{Cov}(\mathcal{U}(X_1, Y_1), \mathcal{V}(X_1, Y_1))
\]

and, therefore, \( D_n \xrightarrow{\mathcal{D}} N(0, \sigma_\gamma^2) \).

### 5.3 Proof of Proposition 1

From the definition of \( \langle \cdot, \cdot \rangle_{\text{HS}} \) and the reproducing properties of \( K \) and \( L \) it is easily seen that \( \langle l_Y \otimes k_{X_i} ; l \otimes k^n \rangle_{\text{HS}} = n^{-1} \sum_{j=1}^{n} l_{ij} k_{ij} \). Hence

\[
\hat{\alpha} = \frac{1}{n} \sum_{i=1}^{n} \left( \langle l_Y \otimes k_{X_i} , l \otimes k^n \rangle_{\text{HS}} - \frac{1}{n} \sum_{m=1}^{n} \langle l_{Y_m} \otimes k_{X_m} , l \otimes k^n \rangle_{\text{HS}} \right)^2
- \frac{1}{n} \sum_{i=1}^{n} \langle l_Y \otimes k_{X_i} , l \otimes k^n \rangle_{\text{HS}}^2 - \left( \frac{1}{n} \sum_{i=1}^{n} \langle l_Y \otimes k_{X_i} , l \otimes k^n \rangle_{\text{HS}} \right)^2.
\]

Let us notice that

\[
\frac{1}{n} \sum_{i=1}^{n} \langle l_Y \otimes k_{X_i} , l \otimes k^n \rangle_{\text{HS}}^2 - \frac{1}{n} \sum_{i=1}^{n} \langle l_Y \otimes k_{X_i} , \mu \rangle_{\text{HS}}^2
= \frac{1}{n} \sum_{i=1}^{n} \langle l_Y \otimes k_{X_i} , l \otimes k^n - \mu \rangle_{\text{HS}}^2 + \frac{2}{n} \sum_{i=1}^{n} \langle l_Y \otimes k_{X_i} , \mu \rangle_{\text{HS}} \langle l_Y \otimes k_{X_i} , l \otimes k^n - \mu \rangle_{\text{HS}}.
\]
Then Cauchy-Schwartz inequality, reproducing property and assumption ($\mathcal{A}_1$) give

$$\left| \frac{1}{n} \sum_{i=1}^{n} \langle l_Y \otimes k_{X_i}, \overline{I \otimes k^a} - \mu \rangle_{HS} \right|^2 \leq \|K\|_\infty \|L\|_\infty \left\| \overline{I \otimes k^a} - \mu \right\|^2_{HS}$$

and

$$\left| \frac{1}{n} \sum_{i=1}^{n} \langle l_Y \otimes k_{X_i}, \mu \rangle_{HS} \right|^2 \leq \|K\|_\infty \|L\|_\infty \left\| \overline{I \otimes k^a} - \mu \right\|^2_{HS}.$$

Since $\left\| \overline{I \otimes k^a} - \mu \right\|_{HS} = o_p(1)$, these inequalities show that

$$\frac{1}{n} \sum_{i=1}^{n} \langle l_Y \otimes k_{X_i}, \overline{I \otimes k^a} - \mu \rangle_{HS}^2 = \frac{1}{n} \sum_{i=1}^{n} \langle l_Y \otimes k_{X_i}, \mu \rangle_{HS}^2 = o_p(1).$$

The law of large numbers and Slutsky’s theorem allow to conclude that the sequence $\frac{1}{n} \sum_{i=1}^{n} \langle l_Y \otimes k_{X_i}, \overline{I \otimes k^a} \rangle_{HS}^2$ converges in probability to $\mathbb{E} \left( \langle l_Y \otimes k_X, \mu \rangle_{HS}^2 \right)$ as $n \to +\infty$. Similarly, from

$$\left| \frac{1}{n} \sum_{i=1}^{n} \langle l_Y \otimes k_{X_i}, \overline{I \otimes k^a} \rangle_{HS} - \frac{1}{n} \sum_{i=1}^{n} \langle l_Y \otimes k_{X_i}, \mu \rangle_{HS} \right|$$

$$= \left| \frac{1}{n} \sum_{i=1}^{n} \langle l_Y \otimes k_{X_i}, \overline{I \otimes k^a} - \mu \rangle_{HS} \right| \leq \left\| K \right\|_{\infty}^{1/2} \left\| L \right\|_{\infty}^{1/2} \left\| \overline{I \otimes k^a} - \mu \right\|_{HS}$$

we get $n^{-1} \sum_{i=1}^{n} \langle l_Y \otimes k_{X_i}, \overline{I \otimes k^a} \rangle_{HS} - n^{-1} \sum_{i=1}^{n} \langle l_Y \otimes k_{X_i}, \mu \rangle_{HS} = o_p(1)$, and $n^{-1} \sum_{i=1}^{n} \langle l_Y \otimes k_{X_i}, \overline{I \otimes k^a} \rangle_{HS}$ converges in probability to $\mathbb{E} \left( \langle l_Y \otimes k_X, \mu \rangle_{HS} \right)$. So, $\hat{\alpha}$ is a consistent estimator of $\text{Var} \left( \langle l_Y \otimes k_X, \mu \rangle_{HS} \right) = \text{Var} \left( \mathcal{V}(X_1, Y_1) \right)$.

References

References

[1] Aghoukeng Jiofack, J.G., Nkiet, G.M., 2010. Testing for lack of dependence between functional variables. Statist. Probab. Lett. 80, 1210-1217.
[2] Ahmad, I.A., 1993. Modification of some goodness-of-fit statistics to yield asymptotic normal null distribution. Biometrika 80, 466–472.

[3] Balogoun, A.K.S., Nkiet, G.M., Ogouyandjou, C., 2021. Asymptotic normality of a generalized maximum mean discrepancy estimator. Statist. Probab. Lett. 169, 108961.

[4] Berlinet, A., Thomas-Agnan, C., 2004. Reproducing Kernel Hilbert Spaces in Probability and Statistics. Kluwer.

[5] Górecki, T., Krzysko, M., Wołynski, W., 2020. Independence test and canonical correlation analysis based on the alignment between kernel matrices for multivariate functional data. Artif. Intell. Rev. 53, 475–499.

[6] Gretton, A., Bousquet, O., Smola, A., Schölkopf, B., 2005. Measuring statistical dependence with Hilbert-Schmidt norms. Lectures Notes in Computer Science, pp. 63–77.

[7] Gretton, A., Borgwardt, K.M., Rasch, M.J., Schölkopf, B., Smola, A.J., 2012. A kernel two-sample test. J. Mach. Learn. Res. 13, 723–776.

[8] Kokoszka, P., Maslova, I., Sojka, J., Zhu, L., 2008. Testing for lack of dependence in the functional linear model. Can. J. Statist. 36, 1-16.

[9] Lai, T., Zhang, Z., Wang, Y., Kong, L., 2021. Testing independence of functional variables by angle covariance. J. Multivar. Anal. 182, 104711.

[10] Makigusa, N., Naito, K., 2020. Asymptotic normality of a consistent estimator of maximum mean discrepancy in Hilbert space. Statist. Probab. Lett. 156, 108596.

[11] Serfling, R.J., 1980. Approximation Theorems of Mathematical Statistics. Wiley, New-York.

[12] Zhang, Q., Filippi, S., Gretton, A., Sejdinovic, D., 2018. Large-scale kernel methods for independence testing. Stat. Comput., 28, 113–130.