ORTHOGONALITY QUESTIONS IN THE HARDY SPACE RELATED TO $\zeta$-ZEROS

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Abstract. A Hardy space approach to the Nyman-Beurling and Báez-Duarte criterion for the Riemann Hypothesis (RH) was introduced recently in [19] and further developed in [14]. It states that the RH holds if and only if a particular sequence of functions $(h_k)_{k \geq 2}$ is complete in the Hardy space $H^2$. This article is concerned with orthogonality questions related to the family $(h_k)_{k \geq 2}$. The first goal is to analyze the orthogonal complement of $N = \text{span}(h_k)_{k \geq 2}$ in $H^2$. Unbounded Toeplitz operators on $H^p$ spaces and de Branges-Rovnyak spaces play a central role and our results show that the size and dimension of $N^\perp$ reveal information on the zeros of the Riemann $\zeta$-function. The second goal is to show that $(h_k)_{k \geq 2}$ possesses a complete biorthogonal sequence in $H^2$. We also discuss a folklore conjecture about the number of $\zeta$-zeros if the RH fails.

Introduction

A classical result of Nyman and Beurling (see [6], [20]) shows that the Riemann Hypothesis (RH) is equivalent to the completeness of $\{\rho_\lambda : 0 \leq \lambda \leq 1\}$ in $L^2(0,1)$ where $\rho_\lambda(x) = \{\lambda/x\} - \lambda\{1/x\}$ and $\{x\}$ denotes the fractional part of $x$. This is furthermore equivalent to the characteristic function $\chi_{(0,1)}$ belonging to the closed linear span of $\{\rho_\lambda : 0 \leq \lambda \leq 1\}$. Half a century later Báez-Duarte [2] strengthened this result by showing that RH is true if and only if $\chi_{(0,1)} \in \text{span}\{\rho_{1/k} : k \geq 2\}$. See the expository article of Bagchi [3] and the survey of Balazard [4]. Recently the Nyman-Beurling and Báez-Duarte approaches to the RH have been explored via tools from Hardy space $H^2$ theory [19] and other analytic function spaces [14].

For each $k \geq 2$, define

$$h_k(z) = \frac{1}{1-z} \log \left( \frac{1 + z + \cdots + z^{k-1}}{k} \right)$$

and denote by $N$ the linear span of $\{h_k : k \geq 2\}$. That each $h_k$ belongs to $H^2$ was proved in [19, Lemma 7]. One of the main results of [19] was a reformulation of Báez-Duarte’s result as a completeness problem in $H^2$. Then in [14] the same completeness problem in the $H^p$ spaces was shown to provide zero-free half planes for the Riemann $\zeta$-function. We state both results here as one.

Theorem 1. The RH holds if and only if $N$ is dense in $H^2$. If $N$ is dense in $H^p$ for some $1 < p \leq 2$, then

$$\zeta(s) \neq 0 \quad \text{for} \quad \Re(s) > \frac{1}{p}.$$  

The density of $N$ in $H^1$ gives the known zero-free region $\Re(s) \geq 1$.

Key words and phrases. Riemann hypothesis, Hardy space, local Dirichlet space, de Branges-Rovnyak space, Smirnov class.

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Although \( N \) is unconditionally dense in \( H^p \) for \( 0 < p < 1 \) (see [14, Cor. 4.6]), this provides no new information regarding \( \zeta \)-zeros. Also note that the RH is equivalent to \( N \perp = \{0\} \) in \( H^2 \) by Theorem 1. This suggests the question of the size and dimension of \( N \perp \) and their possible relation to the \( \zeta \)-zeros. The first archetypal result addressing this is the following (See [19, Thm. 12]).

**Theorem 2.** \( N \perp \cap D_{\delta_1} = \{0\} \), where \( D_{\delta_1} \) denotes the local Dirichlet space at 1.

Since \( D_{\delta_1} \subset H^2 \subset H^p \) for \( 0 < p < 2 \), Theorems 1 and 2 inspire the following question: which linear spaces of analytic functions on \( D \) with \( X_1 \subset H^2 \subset X_2 \) satisfy

(1) \( N \perp \cap X_1 = \{0\} \) and (2) \( N \) is dense in \( X_2 \)?

Interestingly, both (1) and (2) are equivalent if one takes \( X_2 = X \) a Frechet space and \( X_1 = X^* \) its Cauchy dual (see Proposition 9). In Section 2 we analyze this orthogonality question using tools from de Branges-Rovnyak spaces and unbounded Toeplitz operators on \( H^p \) (see Proposition 11), and in particular develop a new approach for finding zero-free half-planes for the \( \zeta \) function (see Theorem 14).

In Section 3 we deal with the biorthogonality question: Does the sequence \( (h_k)_{k \geq 2} \) posses a complete biorthogonal sequence in \( H^2 \)? (see Subsection 1.5). We affirmatively answer this question (see Theorem 15). To provide some context, Vasyunin showed in [22] that the Baez-Duarte sequence \( \{\rho_{1/k} : k \geq 2\} \) is minimal by constructing a biorthogonal sequence for it in \( L^2(0,1) \). Whereas the sequence \( \{\rho_{1/k} : k \geq 2\} \) is not complete in \( L^2(0,1) \), the completeness of \( (h_k)_{k \geq 2} \) in \( H^2 \) is equivalent to the RH by Theorem 1. The property of completeness of a sequence is not in general inherited by its biorthogonal sequence. But it may do so in very special cases such as for sequences of complex exponentials in \( L^2(-\pi,\pi) \) (see [25]). Therefore an affirmative answer to this question (independent of RH) is intriguing.

Finally in Section 4 we discuss a folklore conjecture about the \( \zeta \)-zeros which we name the RH failure (RHF) conjecture: If the RH fails, then it fails infinitely often. More precisely, either \( \zeta \) has no nontrivial zeros outside the critical line, or it has infinitely many. The main result of this section (Theorem 21) states that

\[
\text{RHF conjecture } \implies \dim(N \perp) \text{ is either 0 or } \infty.
\]

We were unable to locate a reference for this conjecture in the literature. But an informative discussion of this conjecture does appear in mathoverflow.net [11].

1. Preliminaries

We denote by \( \mathbb{D} \) and \( \mathbb{T} \) the open unit disk and the unit circle respectively. By \( \text{Hol}(\mathbb{D}) \) we denote the space of holomorphic functions on \( \mathbb{D} \) and the shift operator is defined by \((Sf)(z) = zf(z)\) for \( f \in \text{Hol}(\mathbb{D}) \).

1.1. Hardy spaces and the Smirnov class. A holomorphic function \( f \) on \( \mathbb{D} \) belongs to the Hardy-Hilbert space \( H^2 \) if

\[
\|f\|_{H^2} = \sup_{0 \leq r < 1} \left( \frac{1}{2 \pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta \right)^{1/2} < \infty.
\]

The space \( H^2 \) is a Hilbert space with the \( \ell^2 \)-inner product

\[
\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n},
\]
where \((a_n)_{n \in \mathbb{N}}\) and \((b_n)_{n \in \mathbb{N}}\) are the Fourier coefficients for \(f\) and \(g\) respectively. For any \(f \in H^2\) and \(\zeta \in \mathbb{T}\), the radial limit \(f^*(\zeta) := \lim_{r \to 1^-} f(r\zeta)\) exists m.a.e. on \(\mathbb{T}\), where \(m\) denotes the normalized Lebesgue measure on \(\mathbb{T}\). Analogously for \(p > 0\), the Hardy space \(H^p\) consists of those holomorphic \(f\) on \(\mathbb{D}\) such that

\[
\|f\|_{H^p}^p := \sup_{0 < r < 1} \int_{\mathbb{T}} |f(rz)|^p dm(z) < \infty.
\]

The \(H^p\) are Banach spaces for \(p \geq 1\) and complete metric spaces for \(0 < p < 1\) and \(H^\infty\) denotes the space of bounded holomorphic functions on \(\mathbb{D}\). A function \(f\) is called a cyclic vector for the shift \(S\) in \(H^p\) if \(\text{span}(S^n f)_{n \geq 0} = \mathbb{C}[z]f\) is dense in \(H^p\). When \(p = 2\) these cyclic vectors are commonly known as outer functions. Since \(H^\infty \subset H^p\) for all \(p > 0\), the \(H^\infty\) outer functions are cyclic for all \(H^p\) spaces. The Smirnov class \(N^+\) consists of all holomorphic functions \(g/h\) on \(\mathbb{D}\) such that \(g, h \in H^\infty\) and \(h\) is an outer function. The space \(N^+\) is a topological algebra with respect to pointwise multiplication and \(g/h \in N^+\) is a unit if both \(g\) and \(h\) are outer functions. The topology on \(N^+\) can be metrized with the translation-invariant complete metric

\[
d(f, g) = \int_{\mathbb{T}} \log(1 + |f - g|) dm, \quad f, g \in N^+.
\]

Similar to \(H^p\) spaces, convergence in \(N^+\) implies locally uniform convergence on \(\mathbb{D}\) and functions have radial limits m.a.e. on \(\mathbb{T}\). In fact we have \(H^p \subset H^q \subset N^+\) for all \(0 < q < p \leq \infty\). Duren [10] is a classic reference for the \(H^p\) and \(N^+\) spaces.

1.2. Local Dirichlet spaces. Let \(\mu\) be a finite positive Borel measure on \(\mathbb{T}\), and let \(P_\mu\) denote its Poisson integral. The generalized Dirichlet space \(D_\mu\) consists of \(f \in H^2\) satisfying

\[
D_\mu(f) := \int_{\mathbb{D}} |f'(z)|^2 P_\mu(z) dA(z) < \infty.
\]

Then \(D_\mu\) is a Hilbert space with norm \(\|f\|_{D_\mu}^2 := \|f\|_2^2 + D_\mu(f)\). If \(\mu = m\), then \(D_m\) is the classical Dirichlet space. If \(\mu = \delta_\zeta\) is the Dirac measure at \(\zeta \in \mathbb{T}\), then \(D_\zeta := D_{\delta_\zeta}\) is called the local Dirichlet space at \(\zeta\) and in particular

\[
D_{\delta_\zeta}(f) = \int_{\mathbb{D}} |f'(z)|^2 \frac{1 - |z^2|}{|z - \zeta|^2} dA(z).
\]

The recent book [18] contains a comprehensive treatment of local Dirichlet spaces and the following result establishes a criterion for their membership.

**Theorem 3.** (See [18] Thm. 7.2.1) Let \(\zeta \in \mathbb{T}\) and \(f \in \text{Hol}(\mathbb{D})\). Then \(D_{\delta_\zeta}(f) < \infty\) if and only if

\[f(z) = (z - \zeta)g(z) + a\]

for some \(g \in H^2\) and \(a \in \mathbb{C}\). In particular \(f^*(\zeta)\) exists for all \(f \in D_\zeta\).

So each local Dirichlet space \(D_\zeta = (S - \zeta I)H^2 + \mathbb{C}\) is a proper subspace of \(H^2\). We define the \(H^p\)-analogues of these spaces for \(p > 0\) by

\[D^p_\zeta := (S - \zeta I)H^p + \mathbb{C}\]

and note that \(D^2_\zeta = D_{\delta_\zeta}\) and \(D^p_\zeta \subseteq D^q_\zeta\) for \(0 < p < q \leq 3\). Straightforward but lengthy computations show that \(D^p_\zeta \subseteq H^2\) for \(p > 1\) and \(H^2 \subseteq D^p_\zeta\) for \(0 < p < \frac{3}{2}\).
1.3. The de Branges-Rovnyak spaces. Given $\psi \in L^\infty(\mathbb{T})$, the corresponding Toeplitz operator $T_\psi : H^2 \to H^2$ is defined by

$$T_\psi f := P_+(\psi f)$$

where $P_+ : L^2(\mathbb{T}) \to H^2$ denotes the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2$. Clearly $T_\psi$ is a bounded operator on $H^2$ with $\|T_\psi\| \leq \|\psi\|_{L^\infty}$. If $h \in H^\infty$, then $T_h$ is the operator of multiplication by $h$ and its adjoint is $T_{\overline{h}}$. Given $b$ in the closed unit ball of $H^\infty$, the de Branges-Rovnyak space $\mathcal{H}(b)$ is the image of $H^2$ under the operator $(I - T_b T_{\overline{b}})^{1/2}$. The general theory of $\mathcal{H}(b)$ spaces divides into two distinct cases, according to whether $b$ is an extreme point or a non-extreme point of the unit ball of $H^\infty$. We shall be concerned only with the non-extreme case. In this case there exists a unique outer function $a \in H^\infty$ such that $a(0) > 0$ and $|a^*|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$. The pair $(b,a)$ is called a Pythagorean pair and the function $b/a$ belongs to the Smirnov class $N^+$. That all $N^+$ functions arise as the quotient of a pair associated to a non-extreme function was shown by Sarason [21]. The two-volume work ([12],[13]) is an encyclopedic reference for these spaces.

If $\phi$ is a rational function in $N^+$ the corresponding pair $(b,a)$ is also rational (see [21, Remark. 3.2]). Constara and Ransford [8] characterized the rational pairs $(b,a)$ for which $\mathcal{H}(b)$ is a generalized Dirichlet space.

**Theorem 4.** (See [8, Theorem 4.1]) Let $(b,a)$ be a rational pair and $\mu$ a finite positive measure on $\mathbb{T}$. Then $\mathcal{H}(b) = D_\mu$ if and only if

1. The zeros of $a$ on $\mathbb{T}$ are all simple, and
2. The support of $\mu$ is exactly equal to this set of zeros.

As an example, if $(b,a)$ is the rational pair associated with the $N^+$ function $\varphi(z) = \frac{1}{\bar{z}}$ for $z \in \mathbb{T}$, then $\mathcal{H}(b) = D_\zeta$ is a local Dirichlet space.

1.4. Unbounded Toeplitz operators on $H^p$. Sarason [21] demonstrated how $\mathcal{H}(b)$ spaces appear naturally as the domains of some unbounded Toeplitz operators. Let $\varphi$ be holomorphic in $\mathbb{D}$ and $T_\varphi$ the operator of multiplication by $\varphi$ on the domain

$$\text{dom}(T_\varphi) = \{ f \in H^2 : \varphi f \in H^2 \}.$$ 

Then $T_\varphi$ is a closed operator, and $\text{dom}(T_\varphi)$ is dense in $H^2$ if and only if $\varphi \in N^+$ (see [21, Lemma 5.2]). In this case its adjoint $T_\varphi^*$ is also densely defined and closed. In fact the domain of $T_\varphi^*$ is a de Branges-Rovnyak space.

**Theorem 5.** (See [21, Prop. 5.4]) Let $\varphi$ be a nonzero function in $N^+$ with $\varphi = b/a$, where $(b,a)$ is the associated pair. Then $\text{dom}(T_\varphi^*) = \mathcal{H}(b)$.

Choosing the symbol $\varphi(z) = \frac{1}{\bar{z}}$ in Theorem 5 in conjunction with Theorem 4 gives $\text{dom}(T_\varphi^*) = D_\zeta$ which played a key role in the proof of Theorem 2 (see [19]).

Our goal here is to extend these ideas to $H^p$ spaces for all $p > 1$. Let $\varphi \in N^+$ and define the analytic Toeplitz operator on $H^p$ with symbol $\varphi$ by

$$T_\varphi f = \varphi f, \quad \text{where} \quad f \in \text{dom}_p(T_\varphi) := \{ f \in H^p : \phi f \in H^p \}.$$ 

These $T_\varphi$ are bounded on $H^p$ precisely when $\varphi \in H^\infty$ (see the survey article [23]). For $\varphi = \frac{b}{a} \in N^+$ with $a, b \in H^\infty$ and $a$ outer as usual, these $T_\varphi$ are densely defined on $H^p$ for $p > 1$. Indeed, $\text{dom}_p(T_\varphi)$ contains the dense subspace $aH^p$ since $a$ is
outer and $T_\varphi(aHp) = bHp \subset H^p$. It follows then that the adjoint $T_\varphi^*$ is well-defined on the dual $(H^p)'^* = H^q$ where $\frac{1}{p} + \frac{1}{q} = 1$. The domain of $T_\varphi^*$ is then defined by

$$\text{dom}_q(T_\varphi^*) := \{ g \in H^q : \exists h \in H^q \text{ s.t. } (f, h) = \langle \varphi f, g \rangle \ \forall \ f \in \text{dom}_p(T_\varphi) \}$$

where $(f, h) := \int_T \overline{f} h dm$ represents the $H^p$-$H^q$ duality. The elements in $\text{dom}_q(T_\varphi^*)$ can be characterized via the bounded Toeplitz operators $T_a$ and $T_b$ as follows.

**Lemma 6.** Given $\varphi = \frac{b}{a}$ with $b(z) = 1$ and $a(z) = \zeta - z$ (which is outer) in Lemma 6 we get $g \in \text{dom}_q(T_\varphi^*)$ if and only if there exists an $h \in H^q$ such that $T_\varphi^* g = T_a^* h$.

**Proof.** Suppose $g \in \text{dom}_q(T_\varphi^*)$. Then $(f, h) = \langle \varphi f, g \rangle$ for some $h \in H^q$ and for all $f \in aHp \subset \text{dom}_p(T_\varphi)$. Writing $f = a\tilde{f}$ for $\tilde{f} \in H^p$, we get

$$(f, h) = \langle \varphi f, g \rangle \iff (f, h) = \langle bf/a, g \rangle \iff \langle a\tilde{f}, h \rangle = \langle b\tilde{f}, g \rangle \ \forall \ \tilde{f} \in H^p$$

which is equivalent to $T_\varphi^* h = T_a^* g$. This argument works in both directions because $a$ is an $H^\infty$ outer function and hence $aHp$ is dense in $H^p$.

We can now extend the identity $\text{dom}(T_\varphi^*) = D_\zeta$ from $H^2$ to all $H^p$ with $p > 1$.

**Proposition 7.** Let $\varphi(z) = \frac{1}{1-z}$ for $\zeta \in \mathbb{T}$ and $T_\varphi$ the densely defined Toeplitz operator on $H^p$ for any $p > 1$. Then we have

$$\text{dom}_q(T_\varphi^*) = D_\zeta^q, \text{ where } \frac{1}{p} + \frac{1}{q} = 1.$$

**Proof.** Choosing $\varphi = \frac{b}{a}$ with $b(z) = 1$ and $a(z) = \zeta - z$ (which is outer) in Lemma 6 we get $g \in \text{dom}_q(T_\varphi^*)$ if and only if $g = T_\varphi^* h = (\zeta I - S^*) h$ for some $h \in H^q$.

Therefore, it suffices to verify that $(\zeta I - S^*)H^q = D_\zeta^q$. For any $h \in H^q$, we have

$$(\zeta I - S^*) h = (\zeta I - S)(-\zeta S^* h) + \zeta h(0) \in D_\zeta^q := (S - \zeta I)H^q + \mathbb{C}$$

and therefore $(\zeta I - S^*)H^q \subset D_\zeta^q$. Conversely, if $h \in H^q$ and $c \in \mathbb{C}$ then

$$(\zeta I - S) h + c = (\zeta I - S^*) (c - Sh) \in (\zeta I - S^*)H^q$$

and hence $D_\zeta^q \subset (\zeta I - S^*)H^q$ which concludes the proof.

1.5. **Cauchy duality.** Let $X$ be a complete metrizable linear subspace of $\text{Hol}(\mathbb{D})$. Inspired by terminology used by Malman and Seco [16], we call $X^*$ the Cauchy dual of $X$ if any continuous linear functional on $X$ can be represented by the Cauchy pairing

$$\langle f, g \rangle := \lim_{r \to 1^-} \int_T f(r\zeta)\overline{g(r\zeta)} \ dm(\zeta), \quad f \in X, \ g \in X^*.$$

If $H^2 \subset X$ then $X^* \subset H^2$ and vice-versa. Hence when both $f$ and $g$ are in $H^2$, the pairing above reduces to the standard inner product in $H^2$. Some examples of Cauchy duals for our context are listed below (see [10, 15, 24]).

1. $H^p$ and $H^q$ for $p > 1$ and $1/p + 1/q = 1$,
2. $H^1$ and $\text{BMOA}$ (analytic functions with bounded mean oscillation on $\mathbb{T}$),
3. $H^p$ for $1/2 < p < 1$ and $\Lambda_\alpha$ (the Lipschitz class of $\text{Hol}(\mathbb{D})$)-functions with $\alpha$-H"older continuous extension to $\mathbb{T}$, where $\alpha = 1/(p-1)$,
4. $N^+$ and the Gevrey class $G$ ($\text{Hol}(\mathbb{D})$)-functions whose Taylor coefficients satisfy $a_n = O(e^{-cn^{1/p}})$ for some constant $c > 0$).
A deep result of Davis and McCarthy [9] shows that the class $G$ coincides with the universal multipliers for all non-extreme $H(b)$ spaces. In particular $G \subset H(b)$ for all non-extreme $b$. The concept of Cauchy duality leads to an equivalence between orthogonality and density questions involving $N$ which is explored in Section 2.

1.6. Minimality and biorthogonality. Let $H$ be a Hilbert space. Two sequences $(e_n)_{n \in \mathbb{N}}$ and $(f_n)_{n \in \mathbb{N}}$ in $H$ are said to be biorthogonal to each other if

$$\langle e_n, f_m \rangle = \delta_{nm} \quad \forall \ n, m \in \mathbb{N}$$

where $\delta_{nm}$ is the Kronecker delta. The sequence $(e_n)_{n \in \mathbb{N}}$ is called minimal if $e_n \notin \overline{\text{span}}(e_k)_{k \neq n}$ for all $n \in \mathbb{N}$. The notions of biorthogonality, minimality and completeness are all related via the following well-known result.

**Proposition 8.** (see [7, Lemma 3.3.1]) Let $(e_n)_{n \in \mathbb{N}}$ be a sequence in $H$. Then,

(i) $(e_n)_{n \in \mathbb{N}}$ has a biorthogonal sequence if and only if $(e_n)_{n \in \mathbb{N}}$ is minimal.

(ii) If $(e_n)_{n \in \mathbb{N}}$ has a biorthogonal sequence, then $(e_n)_{n \in \mathbb{N}}$ is complete in $H$ if and only if its biorthogonal sequence is unique.

In Section 3 we shall prove that the sequence $(u_k)_{k \geq 2}$ defined by

$$u_k(z) = \sum_{d|k} \frac{\mu(k/d)}{k/d} (z^{d-1} - z^d)$$

forms a complete biorthogonal sequence for $(h_k)_{k \geq 2}$ in $H^2$, where $\mu$ denotes the Möbius function defined on $\mathbb{N}$ by $\mu(k) = (-1)^s$ if $k$ is the product of $s$ distinct primes, and $\mu(k) = 0$ otherwise.

1.7. The zeta kernels. Let $X \subset \text{Hol}(\mathbb{D})$ be a topological vector space where the monomials $(z^k)_{k \in \mathbb{N}}$ form a Schauder basis. It was shown in [14] that for each $s \in \mathbb{C} \setminus \{0\}$ a linear functional $\Lambda^{(s)}$ can be defined on $X$ by assigning

$$\Lambda^{(s)}(1) = -\frac{1}{s}, \quad \Lambda^{(s)}(z^k) = -\frac{1}{s} \left((k+1)^{1-s} - k^{1-s}\right) \quad (k \geq 1).$$

In particular $\Lambda^{(s)}$ is bounded on $H^p$ for $1 < p \leq 2$ if $\Re s > 1/p$ and on $H^1$ if $\Re s \geq 1$ (see [14] Prop. 4.7). So there exist functions $\kappa_s \in H^1$ with $1/p + 1/q = 1$ such that $\Lambda^{(s)}(f) = \langle f, \kappa_s \rangle$. The function $\kappa_s$ will be called the zeta kernel at $s$ and

$$\kappa_s(z) = \sum_{k=0}^{\infty} \phi_k(s) z^k \quad \text{where} \quad \phi_k(s) := \Lambda^{(s)}(z^k).$$

The name comes from their relation to $h_k$ and $\zeta$ via the important identity

$$\Lambda^{(s)}(h_k) = \langle h_k, \kappa_s \rangle = -\frac{\zeta(s)}{s} (k^{1-s} - 1) \quad \forall \ \Re s > 1/2, \ k \geq 2.$$

The identity (1.4) appears in [14] but we provide an alternate proof in the appendix for the sake of completeness. It is important to mention that the definition of $h_k$ in [14] has an additional factor of $1/k$ which has been adjusted in (1.4) accordingly. The zeta kernels are used in Chapters 3 and play a key role in Chapter 4.
2. The orthogonality question

The objective of this section is to develop a framework for proving when
\begin{equation}
N^\perp \cap L = \{0\}
\end{equation}
for topological vector spaces $L \subset H^2$. Since $N^\perp = \{0\}$ is equivalent to the RH by Theorem \[1\], one may also ask if solutions to (2.1) can lead to new zero-free half-planes for the $\zeta$-function. We start by showing that Cauchy duality serves as a bridge between this orthogonality question and completeness questions.

**Proposition 9.** Let $X$ be a topological linear space with $H^2 \subset X \subset \text{Hol}(\mathbb{D})$, where the inclusions are continuous. If $N^\perp$ is dense in $X$, then
\[N^\perp \cap X^* = \{0\}\]
where $X^* \subset H^2$ is the Cauchy dual of $X$. The converse holds when $X$ is Fréchet.

**Proof.** First note that since both $N^\perp$ and $X^*$ are subspaces of $H^2$, their intersection above makes sense. Let $\langle f, g \rangle$ denote the Cauchy pairing for $f \in X$ and $g \in X^*$ and recall that this pairing becomes the usual $H^2$-inner product $\langle f, g \rangle_{H^2}$ when $f, g \in H^2$ (see Subsection 1.5). Therefore if $N^\perp$ is dense in $X$, then
\[g \in N^\perp \cap X^* \implies \langle f, g \rangle = \langle f, g \rangle_{H^2} = 0 \quad \text{for all} \quad f \in N^\perp\]
which implies that $g$ must be identically zero in $X^*$. Conversely if $X$ is additionally a Fréchet space, then we have access to the Hahn-Banach Theorem. Indeed if $N^\perp$ is not dense in $X$, then there exists a non-zero $g \in X^*$ such that $\langle f, g \rangle = 0 \quad \forall f \in N^\perp$. This implies that $g$ is $H^2$-orthogonal to $N^\perp$ and hence $g \in N^\perp \cap X^* \neq \{0\}$.

Since $N^\perp$ is dense in $H^p$ for $0 < p < 1$ (see [13, Cor. 4.6]), it is also dense in $N^+$ since $H^p \subset N^+$ for all $p > 0$. Therefore it follows by Proposition [12] that $N^\perp \cap L = \{0\}$ if $L$ is the Lipschitz class $\Lambda_\alpha$ ($1/2 < p < 1$ and $\alpha = 1/p - 1$) or the Gevrey class $\mathcal{G}$ (see Subsection 1.5). However to obtain new zero-free half-planes for $\zeta$, we need $L \subset H^2$ to be large enough to contain some $H^p$ space for $q \geq 2$.

**Corollary 10.** If $N^\perp \cap H^q = \{0\}$ for some $q \geq 2$, then $\zeta(s) \neq 0$ for $\Re s > 1/p$, where $1/p + 1/q = 1$. If $N^\perp \cap \text{BMOA} = \{0\}$, then $\zeta(s) \neq 0$ for $\Re s \geq 1$.

**Proof.** Notice that $H^q$ is the Cauchy dual of $H^p$ and $q \geq 2$ implies that $1 < p \leq 2$. In this range the $H^p$ are Banach spaces, and in particular Fréchet spaces. Similarly the BMOA space is the Cauchy dual of $H^1$. Therefore the result follows by the converse in Proposition [9] and by Theorem [11].

The next result relates Toeplitz operators on $H^p$ and the orthogonality question. Recall that if $\varphi \in N^+$ is a unit, then $1/\varphi \in N^+$ and hence both $T_\varphi$ and its inverse $T_{1/\varphi}$ are densely defined Toeplitz operators on $H^p$ for $p > 1$ (see Subsection 1.4).

**Proposition 11.** Let $\varphi$ be a unit in $N^+$ with $T_\varphi$ the Toeplitz operator on $H^p$ for some $p > 1$. If $T_\varphi N^\perp = \varphi N^\perp$ is dense in $H^p$, then
\[N^\perp \cap \text{dom}_q(T_{1/\varphi}^*) = \{0\}, \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1.\]

The Hilbertian case $p = 2$ gives $N^\perp \cap \mathcal{H}(b) = \{0\}$ where $(b, a)$ is the Pythagorean pair associated with $1/\varphi \in N^+$. 
Proof. Let $g \in N^\perp \cap \text{dom}_q(T_{I/\varphi})$. Since both $g$ and $h_k$ belong to $H^2$ for all $k \geq 2$, the Cauchy duality $(h_k, g)$ coincides with the $H^2$-inner product $\langle h_k, g \rangle_2$. Hence

$$\langle \varphi h_k, T_{I/\varphi}g \rangle = \langle h_k, g \rangle_2 = 0 \quad \forall k \in \mathbb{N}$$

which implies that $T^*_{I/\varphi}g = 0$ in $H^q$ by the density of $\varphi N$ in $H^p$. Since $1/\varphi = b/a$ is a unit in $N^+$ (as the inverse of $\varphi$), both $a$ and $b$ are $H^\infty$ outer functions in $H^p$. So $T_{I/\varphi}(\alpha H^p) = bH^p$ shows that $T_{I/\varphi}$ has dense range in $H^p$ since $b$ is outer. So $T^*_{I/\varphi}$ is injective and therefore $g = 0$ in $H^q$. This concludes the general case. The Hilbertian case $p = 2$ now follows by Theorem 5. 

We shall derive two non-trivial applications of this result. The first one extends Theorem 2 to all local Dirichlet spaces $D^p_1$ with $p > 1$. We note that the classical $D^1_{b1}$ is just $D^2_1$ which is strictly smaller than $D^p_1$ for $p \in (1, 2)$ (see Subsection 1.2). We will need with the following approximation result from [14].

**Lemma 12.** Let $\mu$ the Möbius function. Then

$$\sum_{k=2}^{n} \frac{\mu(k)}{k} (I-S)h_k \to 1 - z$$

in the $H^p$ norm for all $p > 1$.

We observe that $I-S = T_{1/\varphi}$ where $\varphi(z) = 1 - z$ is an $H^\infty$ outer function since $\mathbb{C}[z]\varphi$ is dense in $H^2$. In particular $\varphi$ is a unit in $N^+$. Define operators on $H^p$ by

$$\langle W_n f(z) = (1 + z + \cdots + z^{n-1})f(z^n) = \frac{1 - z^n}{1 - z}f(z^n)$$

and $\langle T_n f(z) = f(z^n) \rangle$ for $n \geq 1$ and $f \in H^p$. The multiplicative semigroup of operators $(W_n)_{n \geq 1}$ was introduced in [14] and is the main object of study in [17]. They are bounded on $H^p$ for $p > 1$ (see [14], Cor. 4.6). We shall need the identities

$$W_nN \subset N \quad \text{and} \quad T_n(I-S) = (I-S)W_n$$

for $k, n \geq 1$ which appear in [14], p. 249. We are ready for the first application.

**Theorem 13.** We have $N^\perp \cap D^p_1 = \{0\}$ for all $p > 1$.

**Proof.** Let $\varphi(z) = 1 - z$. By Propositions 4 and 11 we only need to prove that $\varphi N$ is dense in $H^p$ for $p > 1$. First note that Lemma 12 implies that $\varphi$ belongs to the $H^p$-closure of $\varphi N = (I-S)N$. This in turn implies that $T_{n/\varphi}$ belongs to the $H^p$-closure of $T_n(\varphi N) = \varphi W_nN \subset \varphi N$ for all $n \geq 1$. So in particular $\text{span}(T_n(\varphi N)_{n \geq 1} \subset \text{clos}_{H^p}(\varphi N)$. Now $\text{span}(T_n(\varphi N)_{n \geq 1} = \text{span}(1 - z^n)_{n \geq 1} = \mathbb{C}[z]\varphi$ which is dense in $H^p$ for all $p > 1$ because $\varphi$ is an $H^\infty$ outer function. This proves that $\text{clos}_{H^p}(\varphi N) = H^p$ and concludes the proof. 

Our second application of Proposition 11 utilizes recent discoveries in $\mathcal{H}(b)$-space theory to obtain zero-free half-planes for $\zeta$. In view of Corollary 10, we would like to know when the $H^p$ and BMOA spaces are contained in some $\mathcal{H}(b)$ for $\varphi = b/a \in N^+$. Fortuitously for us, these problems were completely solved recently in a preprint by Malman and Seco [16]. They show that $H^p \subset \mathcal{H}(b)$ for $p \in (2, \infty)$ if and only if $\varphi \in H^p$ where $p = \frac{2}{\rho - 2} \in (2, \infty)$, and also that $H^\infty \subset \text{BMOA} \subset \mathcal{H}(b)$ if and only if $\varphi \in H^2$. By definition we always have $\mathcal{H}(b) \subset H^2$, and $\mathcal{H}(b) = H^2$ precisely when $\varphi \in H^\infty$. Therefore it makes sense to allow the values $p = 2$ and $p = \infty$. 

Theorem 14. Suppose \( \varphi \) is a unit in \( \mathbb{N}^+ \) such that \( 1/\varphi \in H^p \) for some \( p \in (2, \infty) \). If \( \varphi^N \) is dense in \( H^2 \), then \( \zeta(s) \neq 0 \) for \( \Re s > \frac{1}{2} + \frac{1}{p} \). The case \( p = 2 \) gives the Prime Number Theorem \( (\Re s \geq 1) \) and \( p = \infty \) gives the RH \( (\Re s > 1/2) \).

Proof. By the results of Malman and Seco [10] mentioned above, \( 1/\varphi \) belongs to \( H^2 \) or to \( H^\infty \) precisely when \( \mathcal{H}(b) \) contains BMOA or \( \mathcal{H}(b) = H^2 \) respectively, where \( 1/\varphi = b/a \) and \( (b, a) \) the associated Pythagorean pair. Hence the cases \( p = 2, \infty \) follow by Corollary [10] and Proposition [11]. For the case when \( 1/\varphi \in H^p \) for \( p \in (2, \infty) \), we have \( H^{\tilde{p}} \subset \mathcal{H}(b) \) where \( p = \frac{2p}{\tilde{p}+2} \) (again by Malman and Seco) or equivalently \( \tilde{p} = \frac{2p}{p-2} \). If \( 1/\tilde{p} + 1/\tilde{q} = 1 \), then we see that \( \tilde{q} = \frac{2p}{(p+2)} \) and hence \( 1/\tilde{q} = 1/2 + 1/p \). The result again follows by Corollary [10] and Proposition [11].

An important distinction between Theorem 1 and Theorem 14 is that in the former one must solve density problems in \( H^p \) spaces that are non-Hilbertian, while in the latter the density problems are always in \( H^2 \). The following simple examples of \( \varphi \) satisfy the hypothesis of Theorem 14. Let \( \varphi(z) = (1-z)^\alpha \) for some \( 0 < \alpha < 1/2 \). Then \( \varphi \) is an \( H^\infty \) outer function and hence a unit in \( \mathbb{N}^+ \) with the property that \( 1/\varphi \in H^p \) for some \( 2 < p < 1/\alpha \). It follows that the density of \( \varphi^N \) in \( H^2 \) would give a new zero-free half-plane for \( \zeta \).

3. The biorthogonality question

Define the sequence of polynomials \( \{u_k : k \geq 2\} \) by

\[
 u_k(z) = \sum_{d|k} \frac{\mu(k/d)}{k/d} (z^{d-1} - z^d),
\]

where \( \mu \) denotes the Möbius function and \( d|k \) denotes \( d \) divides \( k \). The main goal of this section is to prove the following theorem.

Theorem 15. \( \{u_k : k \geq 2\} \) is complete and biorthogonal to \( \{h_k : k \geq 2\} \) in \( H^2 \).

Balazard [5] noted that with the additional vector \( u_1(z) = 1 - z \) the sequence \( \{u_k : k \geq 1\} \) is complete. However it is no longer minimal following Theorem 15. We first make a key observation. Note that \( u_k = (I - S)v_k \), where

\[
 v_k(z) = \sum_{d|k} \frac{\mu(k/d)}{k/d} z^{d-1}.
\]

It follows that \( \langle h_k, u_j \rangle = \langle (I - S^*)h_k, v_j \rangle \). Hence to show that \( \{u_k : k \geq 2\} \) and \( \{h_k : k \geq 2\} \) are biorthogonal, it is suffices to show that

\[
 \langle (I - S^*)h_k, v_j \rangle = \delta_{kj}.
\]

The proof of Theorem 15 will be divided into four steps.

Step 1. Calculate the Fourier coefficients of \( (I - S^*)h_k \).

Step 2. Prove \( \{u_k : k \geq 2\} \) is biorthogonal to \( \{h_k : k \geq 2\} \).

Step 3. Characterize all sequences biorthogonal to \( \{v_k : k \geq 2\} \).

Step 4. Show that \( \{h_k : k \geq 2\} \) is the unique biorthogonal sequence for \( \{u_k : k \geq 2\} \).
The Step 4 implies the completeness of \( \{u_k : k \geq 2\} \) in \( H^2 \) by Proposition 8.

**Step 1.** We first calculate the Fourier coefficients of \((I - S^*)h_k\).

**Lemma 16.** We have

\[
(I - S^*)h_k(z) = \sum_{n=0}^{\infty} B_k(n+1)z^n
\]

for all \( k \geq 2 \) where

\[
B_k(n) = \begin{cases} 
\frac{k}{n} - \frac{1}{n}, & k|n \\
-\frac{1}{n}, & k \not| n 
\end{cases}.
\]

**Proof.** Note that if \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), then

\[
S^*f(z) = \sum_{n=0}^{\infty} a_{n+1} z^n.
\]

Let \( c_n(k) \) be the Fourier coefficients of \( h_k \), i.e.,

\[
h_k(z) = \sum_{n=0}^{\infty} c_n(k)z^n.
\]

Then, the \( n \)-th Fourier coefficient of \((I - S^*)h_k\) is \( c_n(k) - c_{n+1}(k) \). The coefficients \( c_n(k) \) are calculated in [19, p. 249]:

\[
(3.3)\quad c_n(k) = H(n) - H \left( \frac{n}{k} \right) - \log k,
\]

where \( H(x) := \sum_{n \leq x} \frac{1}{n} \) for \( x > 0 \) and \( H(0) = 0 \). It follows from (3.3) that

\[
c_{n-1}(k) - c_n(k) = H(n-1) - H \left( \frac{n-1}{k} \right) - \log k - \left( H(n) - H \left( \frac{n}{k} \right) - \log k \right)
\]

\[
= -\frac{1}{n} + \sum_{\frac{n-1}{k} < m \leq \frac{n}{k}} \frac{1}{m}.
\]

Note that if there is some \( m \in \mathbb{N} \) such that \( \frac{n-1}{k} < m \leq \frac{n}{k} \), then \( mk \leq n < mk + 1 \), so that \( n = mk \). Therefore, the sum above is non-zero if and only if \( k|n \). Then,

\[
(3.4)\quad B_k(n) = c_{n-1}(k) - c_n(k) = \begin{cases} 
-\frac{1}{n}, & k|n \\
\frac{k}{n} - \frac{1}{n}, & k \not| n 
\end{cases}.
\]

**Step 2.** We are now able to prove the first part of Theorem 15.

**Theorem 17.** \( \{u_k : k \geq 2\} \) is biorthogonal to \( \{h_k : k \geq 2\} \).

**Proof.** By Step 1 it suffices to prove that

\[
\sum_{d|j} \frac{\mu(j/d)}{j/d} = \langle (I - S^*)h_k, v_j \rangle = \delta_{kj}, \quad \forall k, j \geq 2.
\]

There are two cases:
(i) \( k|j \). Then, \( k|d \) for every \( d|j \), therefore

\[
\sum_{d|j} B_k(d) \frac{\mu(j/d)}{j/d} = \sum_{d|j} -\frac{1}{d} \frac{\mu(j/d)}{j/d} = -\frac{1}{j} \sum_{d|j} \mu(j/d) = -\frac{1}{j} \lfloor \frac{1}{j} \rfloor = 0,
\]

since \( j \geq 2 \) and by the basic relation \( \sum_{d|j} \mu(d) = \lfloor 1/k \rfloor \).

(ii) \( k|j \). Let \( q = \frac{k}{j} \). Then

\[
\sum_{d|j} B_k(d) \frac{\mu(j/d)}{j/d} = \sum_{d|j \; k|d} -\frac{1}{d} \frac{\mu(j/d)}{j/d} + \sum_{d|j \; k|d} \left( \frac{k}{d} - \frac{1}{dj} \right) \frac{\mu(j/d)}{j/d}.
\]

The last sum is summing over those \( d \) that satisfy \( k|d|j \). However, \( k|d \iff d = mk \)
for some \( m \in \mathbb{N} \). Since \( j = qk \), it follows that \( d|j \iff m|q \). Hence, the last sum

can be written as

\[
\sum_{d|j \; k|d} \left( \frac{k}{d} - \frac{1}{dj} \right) \frac{\mu(j/d)}{j/d} = \sum_{d|j \; k|d} -\frac{1}{d} \frac{\mu(j/d)}{j/d} + \sum_{m|q} \frac{1}{m} \frac{\mu(q/m)}{q/m}.
\]

Therefore,

\[
\sum_{d|j} B_k(d) \frac{\mu(j/d)}{j/d} = \sum_{d|j} -\frac{1}{d} \frac{\mu(j/d)}{j/d} + \sum_{m|q} \frac{1}{m} \frac{\mu(q/m)}{q/m} = -\frac{1}{j} \sum_{d|j} \mu(j/d) + \frac{1}{q} \sum_{m|q} \mu(q/m) = -\frac{1}{j} \lfloor \frac{1}{j} \rfloor + \frac{1}{q} \lfloor \frac{1}{q} \rfloor.
\]

Since \( j \geq 2 \), the first term is always 0. On the other hand, the second term equals
1 if \( q = 1 \) and equals 0 otherwise. Finally note that \( q = 1 \iff k = j \), and hence
that \( \langle h_k, u_j \rangle = \delta_{kj} \) for all \( k, j \geq 2 \). \( \Box \)

**Step 3.** We next characterize all sequences in \( H^2 \) biorthogonal to \( \{v_k : k \geq 2\} \).

**Lemma 18.** A sequence \( \{f_k : k \geq 2\} \subset H^2 \) is biorthogonal to \( \{v_k : k \geq 2\} \) if and only if there exists a sequence \( (c_k)_{k \geq 2} \subset \mathbb{C}^\mathbb{N} \) such that

\[
f_k(z) = \sum_{n=0}^{\infty} A_k(n+1) z^n, \quad \forall k \geq 2,
\]

where the sequence \( (A_k(n))_{n \geq 1} \) for each \( k \geq 2 \) is defined by

\[
A_k(n) = \begin{cases} \frac{c_k}{n} + \frac{k}{n}, & k|n \\ \frac{c_k}{n}, & k \not| n \end{cases}.
\]

**Proof.** Let \( \{f_k : k \geq 2\} \subset H^2 \) be a sequence biorthogonal to \( \{v_k : k \geq 2\} \) and
\( A_k : \mathbb{N} \to \mathbb{C} \) be the arithmetical functions that satisfy

\[
f_k(z) = \sum_{n=0}^{\infty} A_k(n+1) z^n, \quad \forall k \geq 2.
\]

Since the coefficients of \( v_j \) are real, the biorthogonality condition becomes

\[
(3.5) \quad \sum_{d|j} A_k(d) \frac{\mu(j/d)}{j/d} = \langle f_k, v_j \rangle = \delta_{kj}, \quad \forall k, j \geq 2.
\]
Let \( I_k, \nu : \mathbb{N} \to \mathbb{C} \) be arithmetic functions defined by \( I_k(n) = \delta_{kn} \) and \( \nu(n) = \frac{\mu(n)}{n} \). Then \( \nu \) is equivalent to

\[
\forall k \geq 2, \exists c_k \in \mathbb{C} \text{ such that } A_k * \nu = c_k I_1 + I_k,
\]

where * denotes the Dirichlet product (see [1 Section 2.6]). Indeed, (3.5) doesn’t impose any restriction on \( A_k * \nu(1) \), since it only need to hold for \( j \geq 2 \), hence \( c_k = A_k * \nu(1) \) is free, so (3.5) and (3.6) are indeed equivalent. Notice that

\[
\sum_{d|k} \frac{\mu(k/d)}{k/d} \frac{1}{d} = \left\lfloor \frac{1}{k} \right\rfloor = I_1(k),
\]

i.e., \( \nu^{-1}(n) = \frac{1}{n} \), since \( I_1 \) is the unity with respect to *. Moreover

\[
I_k * \nu^{-1}(n) = \sum_{d|n} \delta_{kd} \frac{1}{n/d} = \left\{ \begin{array}{ll} \frac{k}{n}, & k|n \\ 0, & k \nmid n \end{array} \right.
\]

Therefore (3.6) is equivalent to the statement that

\[
\forall k \geq 2, \exists c_k \in \mathbb{C} \text{ such that } A_k(n) = c_k \nu^{-1}(n) + I_k * \nu^{-1}(n)
\]

\[
= \left\{ \begin{array}{ll} \frac{\omega}{n} + \frac{k}{n}, & k|n \\ \frac{\omega}{n}, & k \nmid n \end{array} \right.
\]

Hence the biorthogonality condition (3.5) is equivalent to the condition above as desired. Finally \( f_k \in H^2 \) since its coefficient sequence \( A_k \) clearly belongs to \( \ell^2 \). \( \square \)

**Step 4.** In this final step we show that \( \{u_k : k \geq 2\} \) is complete in \( H^2 \) by proving that \( \{h_k : k \geq 2\} \) is uniquely biorthogonal to \( \{u_k : k \geq 2\} \) in \( H^2 \) by Proposition 8. To do so, recall that \( u_k = (I - S)v_k \) (see (3.2)) implies

\[
\langle \phi_k, u_j \rangle = \langle (I - S^*) \phi_k, v_j \rangle
\]

for any sequence \( \{\phi_k : k \geq 2\} \) in \( H^2 \). This implies that \( I - S^* \) maps sequences biorthogonal to \( \{u_k : k \geq 2\} \) onto sequences biorthogonal to \( \{v_k : k \geq 2\} \) in the image of \( I - S^* \). This correspondence is one-to-one since \( I - S^* \) is injective on \( H^2 \). Therefore it is enough to prove that \( (I - S^*)h_k \) is the unique sequence in the image of \( I - S^* \) that is biorthogonal to \( \{v_k : k \geq 2\} \).

**Lemma 19.** A sequence \( \{f_k : k \geq 2\} \subset (I - S^*)H^2 \) is biorthogonal to \( \{v_k : k \geq 2\} \) if and only if

\[
f_k(z) = \sum_{n=0}^{\infty} B_k(n+1)z^n = (I - S^*)h_k,
\]

where \( B_k \) are the sequences defined in Lemma 10.

**Proof.** Let \( \{f_k : k \geq 2\} \subset (I - S^*)H^2 \) be a sequence biorthogonal to \( \{v_k : k \geq 2\} \) and let \( \varphi_k \in H^2 \) such that \( f_k = (I - S^*)\varphi_k \). If \( (b_k(n))_{n \geq 0} \) are the Maclaurin coefficients of \( \varphi_k \), then

\[
f_k(z) = \sum_{n=0}^{\infty} (b_k(n) - b_k(n+1))z^n.
\]

It then follows by Lemma 18 that for each \( k \geq 2 \), there exists a \( c_k \in \mathbb{C} \) such that

\[
b_k(n+1) - b_k(n) = A_k(n) = \left\{ \begin{array}{ll} \frac{k}{n}, & k|n \\ \frac{\omega}{n}, & k \nmid n \end{array} \right., \quad \forall n \geq 1.
\]
By induction, we obtain
\[ b_k(n) = b_k(0) - \sum_{j=1}^{n} A_k(j) = b_k(0) - \sum_{j \leq n} \frac{c_k}{j} - \sum_{k \mid j} \frac{k}{j} \]

\[ = b_k(0) - c_k \sum_{j \leq n} \frac{1}{j} - \sum_{m \leq n/k} \frac{1}{m} = b_k(0) - c_k H(n) - H\left(\frac{n}{k}\right), \]

where \( H \) is the same function used in (3.3). Since \( \varphi_k \in H^2 \), we get \((b_k(n))_n \in \ell^2\) and hence \( \lim_{n \to \infty} b_k(n) = 0 \). So \((c_k H(n) + H(n/k))_n\) converges. Using Euler summation, one gets (see [19])

\[ H(x) = \log x + \gamma + O\left(\frac{1}{x}\right), \]

where \( \gamma \) is Euler-Mascheroni constant. Therefore,

\[ c_k H(n) + H\left(\frac{n}{k}\right) = c_k \log n + c_k \gamma + \log n - \log k + \gamma + O\left(\frac{k}{n}\right) \]

\[ = (c_k + 1) \log n + (c_k + 1) \gamma - \log k + O\left(\frac{k}{n}\right), \]

which converges as \( n \to \infty \) if and only if \( c_k = -1 \). Hence \( c_k = -1 \) for all \( k \geq 2 \). In that case \( A_k = B_k \) and we obtain (3.7). The converse is equivalent to Theorem 17 by the remarks at the start of Step 4.

As a consequence of Theorem 1 and Proposition 8 the RH holds if and only if \((u_j)_{j \geq 2}\) is the unique sequence in \( H^2 \) that is biorthogonal to \((h_k)_{k \geq 2}\). On the other hand the next result shows what happens if some \( \zeta \)-zero violates the RH.

**Corollary 20.** If \( \zeta(s_0) = 0 \) for some \( 1/2 < \Re s_0 < 1 \), then

\[ \langle h_k, u_j + \kappa_s \rangle = \delta_{kj} \quad \forall \quad k, j \geq 2 \]

where \( \kappa_s \) is the zeta kernel at \( s_0 \). So \((u_j + \kappa_s)_{j \geq 2}\) is also biorthogonal to \((h_k)_{k \geq 2}\).

**Proof.** This follows by (1.4) and Theorem 17 since \( \langle h_k, \kappa_s \rangle = 0 \) for all \( k \geq 2 \). \( \square \)

**4. THE RH-FAILURE CONJECTURE**

The RH-failure (RHF) conjecture states that if the RH is false, then \( \zeta(s) = 0 \) for infinitely many \( s \in \mathbb{C} \) with \( 1/2 < \Re s < 1 \). Our goal is to prove the following.

**Theorem 21.** The RHF conjecture implies that \( \dim(N^\perp) \) is either 0 or \( \infty \).

Let \( \mathcal{K} := \{ \kappa_s : \Re s > 1/2 \} \) denote the family of zeta kernels. If \( \zeta(s) = 0 \) for some \( \Re s > 1/2 \), then \( \langle h_k, \kappa_s \rangle = 0 \) for all \( k \geq 2 \) by (1.4) and hence \( \kappa_s \in N^\perp \). So the RHF conjecture implies that \( N^\perp \cap \mathcal{K} \) is either empty (by Theorem 1) or has infinitely many elements. Therefore Theorem 21 follows if we show that \( \mathcal{K} \) is linearly independent in \( H^2 \). We first show that elements of \( \mathcal{K} \) are common eigenvectors for the adjoints of operators \((W_n)_{n \geq 1}\) defined in (2.2). For \( f \in H^2 \) and \( n \in \mathbb{N} \), we have

\[ W_n^* f(z) = \sum_{k=0}^{\infty} [\hat{f}(nk) + \hat{f}(nk+1) + \ldots + \hat{f}(nk+n-1)] z^k \]

where \( \hat{f}(n) \) denotes the \( n \)-th Fourier coefficient of \( f \). This formula first appeared in [17]. It is possible to describe the common eigenvectors of \((W_n^*)_{n \geq 1}\) completely.
Proposition 22. A non-zero $f \in H^2$ is a common eigenvector for $(W_n^*)_n \geq 1$ if and only if there exists a multiplicative sequence $(\lambda_n)_{n \geq 1}$ with $(\lambda_{n+1} - \lambda_n)_{n \geq 1} \in \ell^2$ and

$$f(n) = (\lambda_{n+1} - \lambda_n) \hat{f}(0) \quad \forall \ n \geq 1.$$  
Moreover $W_n^* f = \lambda_n f$ for all $n \geq 1$.

By a multiplicative sequence $(\lambda_n)_{n \geq 1}$ we mean that $\lambda_n \lambda_m = \lambda_{nm}$ and $\lambda_1 = 1$. Similarly one can see that $W_n W_m = W_{nm}$ and $W_1 = I$ by Lemma 23.

Proof. Let $W_n^* f = \lambda_n f$ for $n \geq 1$ and some sequence $(\lambda_n)_{n \geq 1}$. Since $W_1^* = I$ and $W_{nm}^* = W_n^* W_m^*$ it follows that $(\lambda_n)_{n \geq 1}$ is multiplicative. Furthermore

$$\lambda_n \hat{f}(k) = \langle W_n^* f, z^k \rangle = \langle f, W_n z^k \rangle = \left( f, \sum_{j=0}^{n-1} z^{nk+j} \right) = \sum_{j=0}^{n-1} \hat{f}(nk+j).$$

which gives $\hat{f}(n) = \lambda_{n+1} \hat{f}(0) - \lambda_n \hat{f}(0)$ for all $n \geq 1$ and hence $(\lambda_{n+1} - \lambda_n)_{n \geq 1} \in \ell^2$. Conversely suppose $f$ is a non-zero function satisfying (1.1) for some multiplicative $(\lambda_n)_{n \geq 1}$ with $(\lambda_{n+1} - \lambda_n)_{n \geq 1} \in \ell^2$. Normalizing by supposing $\hat{f}(0) = 1$, we get

$$(W_n^* f)(z) = \sum_{k=0}^{\infty} \left( \sum_{j=0}^{n-1} \hat{f}(nk+j) z^k \right) = \sum_{k=0}^{\infty} \lambda_n \left( 1 + \sum_{k=1}^{\infty} (\lambda_{n+k+1} - \lambda_{n+k}) z^k \right) = \lambda_n f(z)$$

for all $n \geq 2$. So $f \in H^2$ is a common eigenvector for $(W_n^*)_n \geq 1$. \hfill \Box

Choosing $\lambda_k = k^{1-\bar{s}}$ and $\hat{f}(0) = -1/\bar{s}$ in Proposition 22 for any fixed $\Re s > 1/2$ shows that each $\kappa_s \in K$ is a common eigenvector for $(W_n^*)_n \geq 1$ (see (1.3)) with

$$W_n^* \kappa_s = n^{1-\bar{s}} f \quad \forall \ n \geq 1.$$  

We want to prove that for any finite subset $\{\kappa_{s_1}, \ldots, \kappa_{s_\ell} \} \subset K$ there exists some $W_n^*$ such that the corresponding eigenvalues are all distinct. This will give us the linear independence of every finite subset of $K$ and hence of $K$ itself. First suppose that the real parts of $s_1, \ldots, s_\ell$ are all distinct. Since $|n^{1-\bar{s}}| = n^{1-\Re s}$ it follows that the eigenvalues of $W_n^*$ (for all $n > 1$) corresponding to $\kappa_{s_1}, \ldots, \kappa_{s_\ell}$ are all distinct. If the real parts of $s_1, \ldots, s_\ell$ are not all distinct, then we need the following result.

Lemma 23. Given distinct $a_1, \ldots, a_n \in \mathbb{R}$, at most finitely many primes $p$ have the property that there exists a pair $a_i, a_j$ with $1 \leq i < j \leq n$ such that

$$(a_i - a_j) \log p \in 2\pi \mathbb{Z}.$$  

Proof. Suppose there are infinitely many primes that satisfy (1.3). For each such prime $p$ there exists some $1 \leq i < j \leq n$ and $k \in \mathbb{Z} \setminus \{0\}$ such that

$$(a_i - a_j) \log p = 2\pi k \quad \Longleftrightarrow \quad \frac{2\pi k}{\log p} = a_i - a_j.$$  

But since there are only finitely many numbers $a_i - a_j$ with $i < j$, and none of which equal 0, there must exist distinct primes $p, q$ and $k_1, k_2 \in \mathbb{Z} \setminus \{0\}$ such that

$$\frac{2\pi k_1}{\log p} = a_i - a_j = \frac{2\pi k_2}{\log q} \quad \Rightarrow \quad k_2 \log p = k_1 \log q \neq 0,$$

for some pair $i < j$. In particular, $p^{k_2} = q^{k_1} \neq 1$, which is a contradiction. \hfill \Box
The following result then completes the proof of Theorem 21.

**Proposition 24.** The family of zeta kernels $\mathcal{K}$ is linearly independent.

**Proof.** Let $\{\kappa_{s_1}, \ldots, \kappa_{s_\ell}\} \subset \mathcal{K}$ be a finite subset. The case when the real parts of $s_1, \ldots, s_\ell$ are all distinct was already dealt with. Suppose some of the $s_1, \ldots, s_\ell$ have the same real parts. So $\{s_1, \ldots, s_\ell\}$ is the finite disjoint union of sets of the form $A_r := \{s_i \in \mathbb{R} : s_i = r, \ i = 1, \ldots, \ell\}$ for $r \in \mathbb{R}$. It is enough to prove that the family $\{\kappa_s : s \in A_r\}$ is linearly independent when $A_r$ has more than one element. Since $s_1, \ldots, s_\ell$ are distinct complex numbers, the imaginary parts of elements in $A_r$, which we denote by $a_1, \ldots, a_n$, must all be distinct. Applying Lemma 23 to $a_1, \ldots, a_n$ shows that there exist infinitely many primes $q$ such that

$$\langle a_i - a_j \rangle \log q \not\in 2\pi \mathbb{Z}, \ \forall \ 1 \leq i < j \leq n.$$  

For such a prime $q$, we claim that the $\{\kappa_s : s \in A_r\}$ are $W_q^r$-eigenvectors with distinct eigenvalues. To see this first note that $W_q^r \kappa_s = q^{1-s} \kappa_s$ by (4.2) and

$$q^{1-s} = e^{(1-s) \log q} = e^{(1-r) \log q} e^{\im(s) \log q} \ \forall \ s \in A_r.$$  

But $\im(s)$ for $s \in A_r$ are precisely the real numbers $a_1, \ldots, a_n$. Therefore the eigenvalues $q^{1-s}$ for $s \in A_r$ are all distinct by (4.4) and hence $\{\kappa_s : s \in A_r\}$ and therefore all of $\mathcal{K}$ is linearly independent. \hfill $\square$

5. **Appendix**

Denote by $\mathbb{C}_0$ the half-plane $\{s \in \mathbb{C} : \Re s > \rho\}$. In this appendix we provide an alternate proof for the fundamental relation

$$\langle h_k, \kappa_s \rangle = \frac{\zeta(s)}{s} \ (k^{1-s} - 1) \ \forall \ s \in \mathbb{C}_{1/2}, \ k \geq 2. \tag{5.1}$$

We first prove that (5.1) holds for all $s \in \mathbb{C}_1$. We then prove that the function $s \mapsto \langle h_k, \kappa_s \rangle$ has an analytic continuation to $\mathbb{C}_0$ for each $k \geq 2$. Since the right side of (5.1) is already analytic for $s \in \mathbb{C} \setminus \{0\}$, the result then follows by analytic continuation. Recall from Subsection 1.7 that

$$\kappa_s(z) = \sum_{n=0}^{\infty} \phi_n(s) z^n \text{ where } \phi_n(s) = -\frac{1}{s} \left( (n+1)^{1-s} - n^{1-s} \right).$$

**Lemma 25.** The identity (5.1) holds for $s \in \mathbb{C}_1$.

**Proof.** Let $(c_n(k))_n$ be the Fourier coefficients of $h_k$. Since $\overline{\phi_n(s)} = \phi_n(s)$, we have

$$\langle h_k, \kappa_s \rangle = \sum_{n=0}^{\infty} c_n(k) \overline{\phi_n(s)} = \sum_{n=0}^{\infty} c_n(k) \phi_n(s)$$

$$= \lim_{N \to \infty} \left( -\frac{c_0(k)}{s} - \frac{1}{s} \sum_{n=1}^{N} c_n(k) \left( (n+1)^{1-s} - n^{1-s} \right) \right)$$

$$= \lim_{N \to \infty} \left( -\frac{1}{s} \sum_{n=0}^{N} c_n(k) (n+1)^{1-s} + \frac{1}{s} \sum_{n=1}^{N} c_n(k) n^{1-s} \right)$$

$$= \lim_{N \to \infty} \left( -\frac{1}{s} \sum_{n=1}^{N} (c_{n-1}(k) - c_n(k)) n^{1-s} - \frac{1}{s} c_N(k) (N+1)^{1-s} \right).$$
Since $c_n(k) = O(k/n)$ (see [10] p. 249), we have $c_N(k)(N+1) = O(1)$. Furthermore $(N+1)^{-s} \to 0$ for $\Re(s) > 0$ and therefore we get

$$
(\langle h_k, \kappa_s \rangle) = -\frac{1}{s} \lim_{N \to \infty} \left( \sum_{n=1}^{N} (c_{n-1}(k) - c_n(k)) n^{1-s} \right)
$$

and

$$
\lim_{s \to 0} \frac{1}{s} \lim_{N \to \infty} \left( \sum_{n=1}^{N} \frac{1}{n^{1-s}} + \sum_{k/n} \frac{k}{n^{1-s}} \right)
$$

$$
= -\frac{1}{s} \lim_{N \to \infty} \left( \sum_{n=1}^{N} n^{-s} + \sum_{m=1}^{\lfloor N/k \rfloor} \frac{1}{m} (mk)^{-s} \right)
$$

$$
= -\frac{1}{s} \lim_{N \to \infty} \left( -\sum_{n=1}^{N} n^{-s} + k^{-s} \sum_{m=1}^{\lfloor N/k \rfloor} \frac{1}{m} m^{-s} \right)
$$

$$
= \frac{1}{s} (-\zeta(s)) - \frac{k^{-s}}{s} \zeta(s) = -\frac{\zeta(s)}{s} (k^{1-s} - 1),
$$

where in $(s)$ we split the limit in two and use the definition of $\zeta$ for $\Re(s) > 1$. □

The inner product $\langle h_k, \kappa_s \rangle$ defined for $s \in \mathbb{C}_{1/2}$ also makes sense for $s \in \mathbb{C}_0$.

**Lemma 26.** The function $\Phi_k : \mathbb{C}_{1/2} \to \mathbb{C}$ defined by

$$
(5.2) \quad \Phi_k(s) := \langle h_k, \kappa_s \rangle = \sum_{n=0}^{\infty} c_n(k) \phi_n(s).
$$

has an analytic continuation to $\mathbb{C}_0$ for each $k \geq 2$.

**Proof.** Since each $\phi_n$ is holomorphic in $\mathbb{C}_0$, it is sufficient to prove that the series in (5.2) converges uniformly in every half-plane $\mathbb{C}_\rho$ for $\rho > 0$. Note that

$$
|\phi_n(s)| = \left| \frac{1-s}{|s|} \int_n^{n+1} y^{-s} dy \right| \leq \frac{|1-s|}{|s|} n^{-\Re(s)} = O(n^{-\rho})
$$

for $s \in \mathbb{C}_\rho$ with $\rho > 0$. Also $c_n(k) = O(k/n)$ for each $k \geq 2$, and hence we get $c_n(k) \phi_n(s) = O(n^{1-\rho})$ for $s \in \mathbb{C}_\rho$. So $\Phi_k$ converges uniformly in $\mathbb{C}_\rho$ for $\rho > 0$. □

ACKNOWLEDGEMENT

This work was financed in part by the Coordenação de Aperfeiçoamento de Nível Superior- Brasil (CAPES)- Finance Code 001. The fourth author is supported by grant #2023 Provost of Inclusion and Belonging, University of São Paulo (USP).

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