The 19-Vertex Model
at critical regime $|q| = 1$

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Abstract

We study the 19-vertex model associated with the quantum group $U_q(\widehat{sl}_2)$ at critical regime $|q| = 1$. We give the realizations of the type-I vertex operators in terms of free bosons and free fermions. Using these free fields realizations, we give the integral representations for the correlation functions.

1 Introduction

In this paper we shall study the 19 vertex model associated with the quantum affine symmetry $U_q(\widehat{sl}_2)$. The 19-vertex model is a higher spin generalization of the celebrated 6-vertex model, whose Boltzmann weights are given in (2.2). For the massive parameter case $|q| < 1$, M.Idzumi [1] derived the integral representations of correlation functions from the viewpoint of the representation theory of the quantum group $U_q(\widehat{sl}_2)$. In this paper we shall consider the problem at the critical regime $|q| = 1$, where the representation theory of the quantum group $U_q(\widehat{sl}_2)$ cannot be used. M.Jimbo, H.Konno and T.Miwa [2]
studied the 6-vertex model at critical regime $|q| = 1$. They presented the free boson realizations of the vertex operators and gave the trace constructions of the solutions of the quantum Knizhnik-Zamolodchikov equations, which represent the correlation functions. In this paper we shall give the integral representations for the correlation functions of the 19-vertex model at critical regime $|q| = 1$. In order to give trace construction of the correlation functions we need the free field realization of the vertex operators. In this paper we present the free field realizations of the vertex operators in terms of free bosons and free fermions.

The critical 19 vertex model is a limiting case of the fusion 8-vertex model. The latter is massive, and the corner transfer method can be applied [3]. Therefore we can conclude that the correlation functions of the critical 19-vertex model are governed by the following system of difference equations.

$$G_{2N}(\cdots, \beta_{j+1}, \beta_{j}, \cdots)\cdots\epsilon_{j+1}\epsilon_{j}\cdots$$

$$= \sum_{\epsilon'_{j+1} = 0, 1, 2} R_{\epsilon'_{j+1}}^{\epsilon_{j+1}}(\beta_{j} - \beta_{j+1})G_{2N}(\cdots, \beta_{j}, \beta_{j+1}, \cdots)\cdots\epsilon'_{j+1}\cdots$$

$$(1.1)$$

$$G_{2N}(\beta_{1}, \cdots, \beta_{2N-1}, \beta_{2N} - i\lambda)_{\epsilon_{1}\cdots\epsilon_{2N}} = G_{2N}(\beta_{2N}, \beta_{1}, \cdots, \beta_{2N-1})_{\epsilon_{2N}\epsilon_{1}\cdots\epsilon_{2N-1}}$$

$$(1.2)$$

where we the R-matrix is given in (2.2). In this paper we set the deformation parameter $q$ as following.

$$q = \exp\left(-\frac{\pi i}{\xi}\right), \quad \xi > 2.$$

$$(1.3)$$

We note that the above equations imply in particular the quantum Knizhnik-Zamolodchikov equations. In this paper we give the integral representations of the above system of difference equations. The correlation functions are obtained by taking the shift parameter $\lambda = 2\pi$.

In this connection we should mention about the work [4], in which S.Lukyanov give the integral representations of the form factors of sine-Gordon field theory. We should mention about the work [4], in which the authors gave the integral representations of the correlation functions of the $U_q(\widehat{sl}_n)$ analogue of the 6-vertex model at critical regime $|q| = 1$. The special form factors of the $U_q(\widehat{sl}_n)$ analogue of the 6-vertex model at critical regime $|q| = 1$ are given by T.Miwa and Y.Takeyama [6].
Now a few words about the organization of the paper. In section 2 we formulate our problem. In section 3 we give the free field realizations of the vertex operators. In section 4 we give proofs of properties of the free field realizations. In section 5 we give the integral representations of the correlation functions. In Appendix we summarize the multi-Gamma functions.

2 Problem

The purpose of this section is to formulate our problem. At first we introduce two dimensional solvable lattice model, so called 19-vertex model at critical regime $|q| = 1$. Consider an infinite square lattice consisting of oriented lines, each carrying a spectral parameter varying from line to line. The orientation of each line will be shown by an arrow on it. A vertex is a crossing of two lines with spectral parameters, say $\beta_1$ and $\beta_2$, together with the adjacent 4 edges belonging to the cross lines, as shown in Figure 1. The edges are assigned spin-state variables: $j_1, j_2, k_1, k_2$. In the 19-vertex model, each spin-state can take one of three different values $0, 1, 2$. A spin configuration around the vertex is an assignment of $0, 1, 2$ on the four edges. There are 81 possible vertex configurations. We assign each configuration a Boltzmann weight. The set of all Boltzmann weight form the elements of the $R$-matrix:

$$R_{j_1 j_2}^{k_1 k_2} (\beta_1 - \beta_2)$$

Figure 1. Boltzmann weight
The matrix \( R(\beta) \) acts on \( \mathbb{C}^3 \otimes \mathbb{C}^3 \) via the natural basis \( \{v_0, v_1, v_2\} \) of \( \mathbb{C}^3 \) as following.

\[
R(\beta)v_{k_1} \otimes v_{k_2} = \sum_{j_1,j_2=0,1,2} v_{j_1} \otimes v_{j_2} R^{k_1 k_2}_{j_1 j_2}(\beta). \tag{2.1}
\]

The Boltzmann weights are given explicitly by

\[
R(\beta) = \frac{1}{\kappa(\beta)} \begin{pmatrix}
1 & p & e_2 \\
p & b & g_2 & c_2 \\
e_1 & g_1 & b & p \\
h_1 & o & h_2 & 1
\end{pmatrix} . \tag{2.2}
\]

Here we have set the normalized partition as

\[
\kappa(\beta) = - \frac{\text{sh} \left( \frac{1}{\xi}(\beta + \pi i) \right)}{\text{sh} \left( \frac{1}{\xi}(\beta - \pi i) \right)}. \tag{2.3}
\]

The nonzero entries are given by

\[
p(\beta) = - \frac{\text{sh} \left( \frac{1}{\xi} \frac{1}{\beta} \right)}{\text{sh} \left( \frac{1}{\xi} (\beta - 2\pi i) \right)}, \tag{2.4}
\]

\[
b(\beta) = \frac{\text{sh} \left( \frac{1}{\xi} \frac{1}{\beta} \right) \text{sh} \left( \frac{1}{\xi} (\beta + \pi i) \right)}{\text{sh} \left( \frac{1}{\xi} (\beta - \pi i) \right) \text{sh} \left( \frac{1}{\xi} (\beta - 2\pi i) \right)}, \tag{2.5}
\]

\[
o(\beta) = \frac{\text{sh} \left( \frac{\pi i}{\xi} \right) \text{sh} \left( \frac{2\pi i}{\xi} \right) + \text{sh} \left( \frac{1}{\xi} \frac{1}{\beta} \right) \text{sh} \left( \frac{1}{\xi} (\beta - \pi i) \right)}{\text{sh} \left( \frac{1}{\xi} (\beta - \pi i) \right) \text{sh} \left( \frac{1}{\xi} (\beta - 2\pi i) \right)}, \tag{2.6}
\]
and

\[ c_1(\beta) = \frac{\exp \left( \frac{2}{\xi} \beta \right) \sh \left( \frac{\pi i}{\xi} \right) \sh \left( \frac{2\pi i}{\xi} \right)}{\sh \left( \frac{1}{\xi} (\beta - \pi i) \right) \sh \left( \frac{1}{\xi} (\beta - 2\pi i) \right)}, \]  
\[ e_1(\beta) = -\frac{\exp \left( \frac{1}{\xi} \beta \right) \sh \left( \frac{2\pi i}{\xi} \right)}{\sh \left( \frac{1}{\xi} (\beta - 2\pi i) \right)}, \]

\[ c_2(\beta) = \exp \left( -\frac{4}{\xi} \beta \right) c_1(\beta), \quad e_2(\beta) = \exp \left( -\frac{2}{\xi} \beta \right), \]

\[ g_1(\beta) = \frac{\exp \left( \frac{1}{\xi} (\beta + \pi i) \right) \sh \left( \frac{\pi i}{\xi} \right) \sh \left( \frac{1}{\xi} \beta \right)}{\sh \left( \frac{1}{\xi} (\beta - \pi i) \right) \sh \left( \frac{1}{\xi} (\beta - 2\pi i) \right)}, \]

\[ h_1(\beta) = \frac{2 \exp \left( \frac{1}{\xi} (\beta - \pi i) \right) \sh \left( \frac{1}{\xi} \beta \right) \sh \left( \frac{2\pi i}{\xi} \right) \ch \left( \frac{\pi i}{\xi} \right)}{\sh \left( \frac{1}{\xi} (\beta - \pi i) \right) \sh \left( \frac{1}{\xi} (\beta - 2\pi i) \right)}, \]

\[ g_2(\beta) = \exp \left( -\frac{2}{\xi} \beta \right) g_1(\beta), \quad h_2(\beta) = \exp \left( -\frac{2}{\xi} \beta \right) h_1(\beta). \]

The \( R \)-matrix satisfies the Yang-Baxter equation:

\[ R_{12}(\beta_1 - \beta_2)R_{13}(\beta_1 - \beta_3)R_{23}(\beta_2 - \beta_3) = R_{23}(\beta_2 - \beta_3)R_{13}(\beta_1 - \beta_3)R_{12}(\beta_1 - \beta_2). \]  

(2.13)

The 19-vertex model is a limiting case of the fusion 8-vertex model \[8\]. The latter is massive, and the corner transfer matrix method can be applied \[3, 7\]. Now we can conclude that the correlation functions of the critical 19-vertex model are governed by the following system of difference equations.

\[ R \]-matrix Symmetry.

\[ G_{2N}(\beta_1, \beta_2, \beta_{2N} - i\lambda)_{\epsilon_1 \cdots \epsilon_{2N}} = G_{2N}(\beta_2, \beta_1, \beta_{2N} - i\lambda)_{\epsilon_1 \cdots \epsilon_{2N}}. \]

(2.14)

Cyclicality Condition.

\[ G_{2N}(\beta_1, \beta_{2N-1}, \beta_{2N} - i\lambda)_{\epsilon_1 \cdots \epsilon_{2N}} = G_{2N}(\beta_{2N}, \beta_1, \beta_{2N-1})_{\epsilon_2 \epsilon_1 \cdots \epsilon_{2N-2}}. \]

(2.15)
The correlation functions:

\[ G_N (\beta'_N + \pi i, \cdots, \beta'_1 + \pi i, \beta_1, \cdots, \beta_N)_{2-j_N, \cdots, 2-j_1, \cdots, j_N}. \]  

(2.16)

represents the configuration functions in Figure 2, up to constant factors.

Figure 2. Correlators

Now we can translate the problem to the following.

Find out the realizations of the vertex operators, which satisfy the following conditions. 

**R-matrix Symmetry.**

\[ \Phi_{j_2} (\beta_2) \Phi_{j_1} (\beta_1) = \sum_{k_1, k_2 = 0, 1, 2} R^k_{j_1 j_2} (\beta_1 - \beta_2) \Phi_k (\beta_1) \Phi_k (\beta_2). \]  

(2.17)

**Homogeneity Condition.**

\[ e^{-\lambda D} \Phi_j (\beta) e^{\lambda D} = \Phi_j (\beta + i\lambda). \]  

(2.18)

Using the above vertex operators and the degree operator, we can construct the solutions of the system of difference equations as following.

\[ G_{2N} (\beta_1, \cdots, \beta_{2N})_{j_1 \cdots j_{2N}} = \frac{\text{tr}_H \left( e^{-\lambda D} \Phi_{j_1} (\beta_1) \cdots \Phi_{j_{2N}} (\beta_{2N}) \right)}{\text{tr}_H (e^{-\lambda D})}. \]  

(2.19)

The R-matrix symmetry (2.14) follows from the condition (2.17). The cyclicity condition (2.14) follows from the homogeneity condition (2.18). In section 3 we give the free field
realization of the vertex operators \( \Phi_j(\beta) \). In section 5 we give the degree operator \( D \) and the space \( \mathcal{H} \), on which the trace is evaluated.

### 3 Free field realizations

The purpose of this section is to give the free field realization of the vertex operators. Let us introduce the bose-fields by

\[
[b(t), b(t')] = -\frac{1}{t} \frac{\text{sh} \left( \frac{\pi}{2} (\xi - 2) t \right)}{\text{sh} \left( \frac{\pi}{2} \xi t \right)} \delta(t + t').
\]  

(3.1)

Let us set the basic operators by

\[
U_0(\beta) =: \exp \left( \int_{-\infty}^{\infty} b(t) e^{i\beta t} dt \right), \quad (3.2)
\]

\[
U_1(\beta) =: \exp \left( -\int_{-\infty}^{\infty} b(t) e^{i\beta t} dt \right). \quad (3.3)
\]

Let us set the fermion-fields by

\[
[\psi(t), \psi(t')]_+ = 2 \text{ch}(\pi t) \delta(t + t').
\]  

(3.4)

Fourier transformation of the fermion-field is given by

\[
\hat{\psi}(\beta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(t) e^{it\beta} dt.
\]  

(3.5)

The free-field realizations of the vertex operators are given by

\[
\Phi_2(\beta) = U_0(\beta), \quad (3.6)
\]

\[
\Phi_1(\beta) = \left( e^{\frac{\pi}{2}i\beta} + e^{-\frac{\pi}{2}i\beta} \right) \int_{-\infty}^{\infty} d\alpha \frac{\exp \left( \frac{1}{\xi}(\alpha - \beta) \right)}{\text{sh} \left( \frac{1}{\xi} (\alpha - \beta + \pi i) \right)} \times U_0(\beta)U_1(\alpha)\hat{\psi}(\alpha), \quad (3.7)
\]

\[
\Phi_0(\beta) = e^{\frac{\pi}{2}i\beta} \int_{-\infty}^{\infty} d\alpha_1 \int_{-\infty}^{\infty} d\alpha_2 \prod_{k=1}^{2} \frac{\exp \left( \frac{1}{\xi}(\alpha_k - \beta) \right)}{\text{sh} \left( \frac{1}{\xi} (\alpha_k - \beta + \pi i) \right)} \times U_0(\beta)U_1(\alpha_1)U_1(\alpha_2)\hat{\psi}(\alpha_1)\hat{\psi}(\alpha_2) \quad (3.8)
\]
4 Proof

The purpose of this section is to give proofs of properties of vertex operators. At first we explain the formulas of the form

\[ X(\beta_1)Y(\beta_2) = C_{XY}(\beta_1 - \beta_2) : X(\beta_1)X(\beta_2) :, \]

(4.1)

where \( X, Y = U_j \), and \( C_{XY}(\beta) \) is a meromorphic function on \( \mathbb{C} \). These formulae follow from the commutation relation of the free bosons. When we compute the contraction of the basic operators, we often encounter an integral

\[ \int_0^{\infty} F(t)dt, \]

(4.2)

which is divergent at \( t = 0 \). Here we adopt the following prescription for regularization : it should be understood as the countour integral,

\[ \int_C F(t) \frac{\log(-t)}{2\pi i} dt, \]

(4.3)

where the countour \( C \) is given by

\[ \text{Contour C} \]

The contractions of the basic operators have the following forms.

\[ U_j(\beta_1)U_j(\beta_2) =: U_j(\beta_1)U_j(\beta_2) : \Gamma \left( \frac{i \beta_2 - \beta_1}{\pi \xi} + \frac{1}{\xi} \right) \exp \left( \frac{\xi - 2}{2} \left( \gamma + \log(\pi \xi) \right) \right). \]

(4.4)
Please see the Appendix. The commutation relations of the basic operators are given by

\[ U_j(\beta_1)U_j(\beta_2) = U_j(\beta_2)U_j(\beta_1) \frac{\text{sh} \left( \frac{1}{\xi} (\beta_1 - \beta_2 + \pi i) \right)}{\text{sh} \left( \frac{1}{\xi} (\beta_2 - \beta_1 + \pi i) \right)}, \quad (j = 0, 1), \]  

(4.5)

\[ U_0(\beta_1)U_1(\beta_2) = U_1(\beta_2)U_0(\beta_1) \frac{\text{sh} \left( \frac{1}{\xi} (\beta_2 - \beta_1 + \pi i) \right)}{\text{sh} \left( \frac{1}{\xi} (\beta_1 - \beta_2 + \pi i) \right)}. \]  

(4.6)

The anti-commutation relation becomes

\[ [\hat{\psi}(\beta_1), \hat{\psi}(\beta_2)]_+ = \delta(\beta_1 - \beta_2 + \pi i) + \delta(\beta_2 - \beta_1 + \pi i). \]  

(4.7)

Now let us start to prove the properties of vertex operators.

**Proof of R-matrix symmetry**

Let us prove the equation :

\[ R_{21}^{12}(\beta_1 - \beta_2)\Phi_1(\beta_1)\Phi_2(\beta_2) + R_{21}^{21}(\beta_1 - \beta_2)\Phi_2(\beta_1)\Phi_1(\beta_2) = \Phi_1(\beta_2)\Phi_2(\beta_1). \]  

(4.8)

By using the commutation relations of basic operators, we can rearrange the operator part as

\[ U_0(\beta_1)U_0(\beta_2)U_1(\alpha)\hat{\psi}(\alpha). \]  

(4.9)

The equation (4.8) follows from the integrand identity :

\[-e_1(\beta_1 - \beta_2)e^{-\frac{1}{\xi} \beta_1} \text{sh} \left( \frac{1}{\xi} (\beta_2 - \alpha + \pi i) \right) + p(\beta_1 - \beta_2)e^{-\frac{1}{\xi} \beta_2} \text{sh} \left( \frac{1}{\xi} (\beta_1 - \alpha - \pi i) \right) \]

\[ = e^{-\frac{1}{\xi} \beta_2} \text{sh} \left( \frac{1}{\xi} (\beta_1 - \alpha + \pi i) \right). \]  

(4.10)

Let us prove the equation :

\[ R_{11}^{02}(\beta_1 - \beta_2)\Phi_0(\beta_1)\Phi_2(\beta_2) + R_{11}^{11}(\beta_1 - \beta_2)\Phi_1(\beta_1)\Phi_1(\beta_2) + R_{11}^{20}(\beta_1 - \beta_2)\Phi_2(\beta_1)\Phi_0(\beta_2) = \Phi_1(\beta_2)\Phi_1(\beta_1). \]  

(4.11)

By using the commutation relations of basic operators, we can rearrange the operator part as

\[ U_0(\beta_1)U_0(\beta_2)U_1(\alpha_1)U_1(\alpha_2)\hat{\psi}(\alpha_1)\hat{\psi}(\alpha_2). \]  

(4.12)
Let us set
\[ H(\alpha) = \frac{\text{sh} \left( \frac{1}{\xi} (\alpha + \pi i) \right)}{\text{sh} \left( \frac{1}{\xi} (-\alpha + \pi i) \right)}. \]  (4.13)

Consider the integral of the form:
\[ \int_{-\infty}^{\infty} d\alpha_1 \int_{-\infty}^{\infty} d\alpha_2 F(\alpha_1, \alpha_2) U_1(\alpha_1) U_1(\alpha_2) \hat{\psi}(\alpha_1) \hat{\psi}(\alpha_2). \]  (4.14)

Due to the commutation relation of \( U_1(\alpha) \) and the anti-commutation relation of \( \hat{\psi}(\alpha) \), the above integral equals to
\[ -\int_{-\infty}^{\infty} d\alpha_1 \int_{-\infty}^{\infty} d\alpha_2 H(\alpha_2 - \alpha_1) F(\alpha_1, \alpha_2) U_1(\alpha_1) U_1(\alpha_2) \hat{\psi}(\alpha_1) \hat{\psi}(\alpha_2) \]
\[ -H(\pi i) \int_{-\infty}^{\infty} d\alpha F(\alpha - \pi i, \alpha) U_1(\alpha) U_1(\alpha - \pi i), \]  (4.15)
where we have used the relation: \( H(-\pi i) = 0 \). Note that the part \( H(\pi i) U_1(\alpha) U_1(\alpha - \pi i) \) is convergent.

Observing this we define 'weak equality' in the following sense. We say that the function \( G_1(\alpha_1, \alpha_2) \) and \( G_2(\alpha_1, \alpha_2) \) are equal in weak sense if
\[ G_1(\alpha_1, \alpha_2) - H(\alpha_2 - \alpha_1) G_1(\alpha_2, \alpha_1) = G_2(\alpha_1, \alpha_2) - H(\alpha_2 - \alpha_1) G_2(\alpha_2, \alpha_1). \]  (4.16)

We write
\[ G_1(\alpha_1, \alpha_2) \sim G_2(\alpha_1, \alpha_2). \]  (4.17)

Note that the equation
\[ G_1(\alpha - \pi i, \alpha) = G_2(\alpha - \pi i, \alpha), \]  (4.18)
is a special case of weakly equality. In order to prove the equation (4.11) it is enough to prove the equality of the integrand part in weakly sense. The equation (4.11) follows from the following weakly sense identity.
\[
\begin{align*}
&h_1(\beta_1 - \beta_2)e^{-\pi i \xi} \text{sh} \left( \frac{1}{\xi} (\beta_2 - \alpha_1 + \pi i) \right) \text{sh} \left( \frac{1}{\xi} (\beta_2 - \alpha_2 + \pi i) \right) \\
&\quad + o(\beta_1 - \beta_2)(1 + e^{-2\pi i \xi})^2 \text{sh} \left( \frac{1}{\xi} (\beta_2 - \alpha_1 + \pi i) \right) \text{sh} \left( \frac{1}{\xi} (\alpha_2 - \beta_1 + \pi i) \right) \\
&\quad + h_2(\beta_1 - \beta_2)e^{\pi i \xi} \text{sh} \left( \frac{1}{\xi} (\alpha_1 - \beta_1 + \pi i) \right) \text{sh} \left( \frac{1}{\xi} (\alpha_2 - \beta_1 + \pi i) \right) \\
&\quad \sim (1 + e^{-2\pi i \xi})^2 \text{sh} \left( \frac{1}{\xi} (\beta_1 - \alpha_1 + \pi i) \right) \text{sh} \left( \frac{1}{\xi} (\alpha_2 - \beta_2 + \pi i) \right).
\end{align*}
\]  (4.19)
As the same arguments as the above we obtain the commutation relation of the vertex operators (2.17).

5 Correlation functions

In this section we derive a solution of the system of difference equations (2.14) and (2.15), algebraically, and obtain an integral representation of it.

Let us introduce the Fock space $\mathcal{H}_b$ generated by $|\text{vac}\rangle_b$ which satisfies

$$b(t)|\text{vac}\rangle_b = 0, \quad \text{if} \quad t > 0. \quad (5.1)$$

Let us introduce the Fock space $\mathcal{H}_\psi$ generated by $|\text{vac}\rangle_\psi$ which satisfies

$$\psi(t)|\text{vac}\rangle_\psi = 0, \quad \text{if} \quad t > 0. \quad (5.2)$$

Set the space $\mathcal{H}$ by

$$\mathcal{H} = \mathcal{H}_b \otimes \mathcal{H}_\psi. \quad (5.3)$$

Let us introduce the degree operators $D^b$ and $D^\psi$ by

$$D^b b(-t)|\text{vac}\rangle_b = t b(-t)|\text{vac}\rangle_b, \quad D^\psi\psi(-t)|\text{vac}\rangle_\psi = t\psi(-t)|\text{vac}\rangle_\psi, \quad t > 0. \quad (5.4)$$

Set the degree operator $D$ on $\mathcal{H}$ by

$$D = D^b \otimes \text{id} + \text{id} \otimes D^\psi. \quad (5.5)$$

We have

$$e^{\lambda D} U_j(\beta) e^{-\lambda D} = U_j(\beta + i\lambda), \quad e^{\lambda D} \tilde{\psi}(\beta) e^{-\lambda D} = \tilde{\psi}(\beta + i\lambda). \quad (5.6)$$

Therefore the vertex operators satisfy the homogeneity condition.

$$e^{-\lambda D} \Phi_j(\beta) e^{\lambda D} = \Phi_j(\beta + i\lambda). \quad (5.7)$$

Now let us consider the trace function for $\lambda > 0$ defined by

$$G_{2N}(\beta_1, \cdots, \beta_{2N})_{j_1 \cdots j_{2N}} = \frac{\text{tr}_\mathcal{H} (e^{-\lambda D} \Phi_{j_1}(\beta_1) \cdots \Phi_{j_{2N}}(\beta_{2N}))}{\text{tr}_\mathcal{H} (e^{-\lambda D})}. \quad (5.8)$$
The trace of the bosonic parts is evaluated as followings.

\[ \frac{\text{tr}_{H^b} \left( e^{-\lambda D^b} b(t)b(t') \right) }{\text{tr}_{H^b} \left( e^{-\lambda D^b} \right) } = \frac{e^{\lambda t}}{e^{\lambda t} - 1}[b(t), b(t')]. \] (5.9)

The product of the basic operator \( U_j(\beta) \), \( j = 0, 1 \) is evaluated as

\[ \frac{\text{tr}_{H^b} \left( e^{-\lambda D^b} U_j(\beta_1)U_j(\beta_2) \right) }{\text{tr}_{H^b} \left( e^{-\lambda D^b} \right) } = \text{Const.} \varphi_1(\beta_1 - \beta_2) \]

\[ \times \ \text{sh} \left( \frac{\pi}{\lambda} (\beta_1 - \beta_2 + \pi i) \right) \text{sh} \left( \frac{\pi}{\lambda} (\beta_1 - \beta_2 - \pi i) \right) \text{sh} \left( \frac{1}{\xi} (\beta_1 - \beta_2 + \pi i) \right), \] (5.10)

where we set the kernel function as

\[ \varphi_1(\beta) = \frac{1}{S_2(i\beta - \pi|\pi \xi, \lambda)S_2(-i\beta - \pi|\pi \xi, \lambda)}. \] (5.11)

The product of the basic operators \( U_0(\beta) \) and \( U_1(\alpha) \) is evaluated as

\[ \frac{\text{tr}_{H^b} \left( e^{-\lambda D^b} U_0(\beta)U_1(\alpha) \right) }{\text{tr}_{H^b} \left( e^{-\lambda D^b} \right) } = \text{Const.} \varphi_2(\beta - \alpha) \text{sh} \left( \frac{1}{\xi} (\beta - \alpha - \pi i) \right), \] (5.12)

where we set the kernel function as

\[ \varphi_2(\alpha) = \frac{1}{S_2(i\alpha + \pi|\pi \xi, \lambda)S_2(-i\alpha + \pi|\pi \xi, \lambda)}. \] (5.13)

The trace of the fermionic parts is evaluated as followings.

\[ \frac{\text{tr}_{H^\psi} \left( e^{-\lambda D^\psi} \psi(t)\psi(t') \right) }{\text{tr}_{H^\psi} \left( e^{-\lambda D^\psi} \right) } = \frac{e^{\lambda t}}{e^{\lambda t} + 1}[\psi(t), \psi(t')]_. \] (5.14)

Let us set the auxiliary function \( \mathcal{J}(\alpha) \) by

\[ \mathcal{J}(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{iat}}{1 + e^{-\lambda t}} dt. \] (5.15)

We then have

\[ \frac{\text{tr}_{H^\psi} \left( e^{-\lambda D^\psi} \hat{\psi}(\alpha_1)\hat{\psi}(\alpha_2) \right) }{\text{tr}_{H^\psi} \left( e^{-\lambda D^\psi} \right) } = \mathcal{J}(\alpha_1 - \alpha_2 + \pi i) + \mathcal{J}(\alpha_1 - \alpha_2 - \pi i). \] (5.16)

The trace of the vertex operators is evaluated by applying the Wick’s theorem.

The one-point correlation functions are evaluated as follows.
$$G_2(\beta_1, \beta_2)_{2,0} = e^{-\xi \beta_2} \frac{S_2(i(\beta_2 - \beta_1) + \pi|\pi \xi, \lambda)}{S_2(i(\beta_2 - \beta_1) + \pi \xi - \pi|\pi \xi, \lambda)} \int_{-\infty}^{\infty} d\alpha_1 \int_{-\infty}^{\infty} d\alpha_2$$
	\times \left\{ \prod_{j,k=1}^{2} \varphi_2(\beta_j - \alpha_k) \right\} \varphi_1(\alpha_1 - \alpha_2)(\mathcal{J}(\alpha_1 - \alpha_2 + \pi i) + \mathcal{J}(\alpha_1 - \alpha_2 - \pi i))
	\times e^{\xi(\alpha_1 + \alpha_2)} \operatorname{sh} \left( \frac{\pi}{\lambda}(\alpha_1 - \alpha_2 + \pi i) \right) \operatorname{sh} \left( \frac{\pi}{\lambda}(\alpha_2 - \alpha_1 + \pi i) \right)
\times \operatorname{sh} \left( \frac{1}{\xi}(\alpha_1 - \alpha_2 + \pi i) \right) \operatorname{sh} \left( \frac{1}{\xi}(\alpha_1 - \beta_1 + \pi i) \right) \operatorname{sh} \left( \frac{1}{\xi}(\alpha_2 - \beta_1 + \pi i) \right),$$
\quad (5.17)

Similarly,

$$G_2(\beta_1, \beta_2)_{2,0} = e^{-\xi \beta_1} \frac{S_2(i(\beta_2 - \beta_1) + \pi|\pi \xi, \lambda)}{S_2(i(\beta_2 - \beta_1) + \pi \xi - \pi|\pi \xi, \lambda)} \int_{-\infty}^{\infty} d\alpha_1 \int_{-\infty}^{\infty} d\alpha_2$$
	\times \left\{ \prod_{j,k=1}^{2} \varphi_2(\beta_j - \alpha_k) \right\} \varphi_1(\alpha_1 - \alpha_2)(\mathcal{J}(\alpha_1 - \alpha_2 + \pi i) + \mathcal{J}(\alpha_1 - \alpha_2 - \pi i))
	\times e^{\xi(\alpha_1 + \alpha_2)} \operatorname{sh} \left( \frac{\pi}{\lambda}(\alpha_1 - \alpha_2 + \pi i) \right) \operatorname{sh} \left( \frac{\pi}{\lambda}(\alpha_2 - \alpha_1 + \pi i) \right)
\times \operatorname{sh} \left( \frac{1}{\xi}(\alpha_1 - \alpha_2 + \pi i) \right) \operatorname{sh} \left( \frac{1}{\xi}(\alpha_1 - \beta_2 - \pi i) \right) \operatorname{sh} \left( \frac{1}{\xi}(\alpha_2 - \beta_2 - \pi i) \right),$$
\quad (5.18)

and

$$G_2(\beta_1, \beta_2)_{1,1} = e^{-\xi(\beta_1 + \beta_2)} \frac{S_2(i(\beta_2 - \beta_1) + \pi|\pi \xi, \lambda)}{S_2(i(\beta_2 - \beta_1) + \pi \xi - \pi|\pi \xi, \lambda)} \int_{-\infty}^{\infty} d\alpha_1 \int_{-\infty}^{\infty} d\alpha_2$$
	\times \left\{ \prod_{j,k=1}^{2} \varphi_2(\beta_j - \alpha_k) \right\} \varphi_1(\alpha_1 - \alpha_2)(\mathcal{J}(\alpha_1 - \alpha_2 + \pi i) + \mathcal{J}(\alpha_1 - \alpha_2 - \pi i))
	\times e^{\xi(\alpha_1 + \alpha_2)} \operatorname{sh} \left( \frac{\pi}{\lambda}(\alpha_1 - \alpha_2 + \pi i) \right) \operatorname{sh} \left( \frac{\pi}{\lambda}(\alpha_2 - \alpha_1 + \pi i) \right)
\times \operatorname{sh} \left( \frac{1}{\xi}(\alpha_1 - \alpha_2 + \pi i) \right) \operatorname{sh} \left( \frac{1}{\xi}(\alpha_1 - \beta_2 - \pi i) \right) \operatorname{sh} \left( \frac{1}{\xi}(\alpha_2 - \beta_1 + \pi i) \right).$$
\quad (5.19)
Here we omit an irrelevant constant factor.

By applying Wick’s theorem we obtain the $N$-point correlation functions. We consider the special components:

$$j_1, \ldots, j_L = 2, \; j_{L+1}, \ldots, j_{L+2M} = 1, \; j_{L+2M+1}, \ldots, j_{2(L+M)} = 0.$$  \hfill (5.20)

By using the $R$-matrix symmetry (2.14) we obtain every components from this component. The $N$-point function is evaluated as following.

$$G_{2(L+M)}(\beta_1 \cdots \beta_L | \beta_{L+1}, \ldots, \beta_{L+2M+1} \cdots \beta_{2(L+M)})_{2,-2,1,\ldots,1,0,\ldots,0} = E(\{\beta\}) \int d\alpha \; \Psi(\{\alpha\}|\{\beta\}) \ Pf(\{\alpha\}) \; I_\lambda(\{\alpha\}) \; I_\zeta(\{\alpha\}|\{\beta\}),$$ \hfill (5.21)

where the integral $\int d\alpha$ represents

$$\int_{-\infty}^{\infty} d\alpha_{L+1} \cdots \int_{-\infty}^{\infty} d\alpha_{2(L+M)} \int_{-\infty}^{\infty} d\alpha'_{L+2M+1} \cdots \int_{-\infty}^{\infty} d\alpha'_{2(L+M)}.$$ \hfill (5.22)

Here we have set

$$E(\{\beta\}) = e^{-\frac{\pi}{\xi}(\beta_{L+1} \cdots + \beta_{L+2M})} e^{-\frac{\pi}{\xi}(\beta_{L+2M+1} + \cdots + \beta_{2(L+M)})} \times \prod_{1 \leq j < k \leq 2(L+M)} \frac{S_2(i(\beta_k - \beta_j) + \pi | \pi \xi, \lambda)}{S_2(i(\beta_k - \beta_j) + \pi \xi - \pi | \pi \xi, \lambda)}.$$ \hfill (5.23)

The integral kernel is given by

$$\Psi(\{\alpha\}|\{\beta\}) = \prod_{L+1 \leq j < k \leq 2(L+M)} \varphi_1(\alpha_j - \alpha_k) \prod_{L+2M+1 \leq j < k \leq 2(L+M)} \varphi_1(\alpha'_j - \alpha'_k) \times \prod_{j=L+1}^{2(L+M)} \prod_{k=L+2M+1}^{2(L+M)} \varphi_1(\alpha_j - \alpha'_k) \prod_{j=1}^{2(L+M)} \prod_{k=L+1}^{2(L+M)} \varphi_2(\beta_j - \alpha_k) \prod_{j=1}^{2(L+M)} \prod_{k=L+2M+1}^{2(L+M)} \varphi_2(\beta_j - \alpha'_k).$$ \hfill (5.24)

The $Pf(\{\alpha\})$ represents a Pfaffian of $2(L+M) \times 2(L+M)$ anti-symmetry matrix whose entries are given by $\mathcal{J}(\alpha - \alpha' + \pi i) + \mathcal{J}(\alpha - \alpha' - \pi i)$.

The integrand functions are given by

$$I_\lambda(\{\alpha\}) = \prod_{j=L+1}^{2(L+M)} \prod_{k=L+1}^{2(L+M)} \sh \left( \frac{\pi}{\lambda} (\alpha_j - \alpha_k - \pi i) \right) \times \prod_{j=L+1}^{2(L+M)} \prod_{k=L+2M+1}^{2(L+M)} \sh \left( \frac{\pi}{\lambda} (\alpha_j - \alpha'_k - \pi i) \right) \sh \left( \frac{\pi}{\lambda} (\alpha_j - \alpha'_k + \pi i) \right)$$ \hfill (5.25)
and

\[
I_{\xi}(\{\alpha\}|\{\beta\}) = e^{\frac{1}{\xi}(\alpha_{L+1} + \cdots + \alpha_{2(L+M)} + \alpha'_{L+2M+1} + \cdots + \alpha'_{2(L+M)})} \times \prod_{L+1 \leq j < k \leq 2(L+M) \atop 2(L+M)} \text{sh} \left( \frac{1}{\xi} (\alpha_j - \alpha_k + \pi i) \right) \prod_{j=L+1}^{L+2M} \prod_{k=L+2M+1}^{2(L+M)} \text{sh} \left( \frac{1}{\xi} (\alpha_j - \alpha_k' + \pi i) \right) 
\]

\[
\times \prod_{j=L+2M+1}^{L} \left\{ \prod_{k=L+1}^{2(L+M)} \text{sh} \left( \frac{1}{\xi} (\beta_j - \alpha_k - \pi i) \right) \prod_{k=L+2M+1}^{2(L+M)} \text{sh} \left( \frac{1}{\xi} (\beta_j - \alpha_k' - \pi i) \right) \right\} 
\]

\[
\times \prod_{j=L+1}^{L+2M} \left\{ \prod_{k=L+2M+1}^{2(L+M)} \text{sh} \left( \frac{1}{\xi} (\beta_j - \alpha_k + \text{sgn}(j-k)\pi i) \right) \prod_{k=L+2M+1}^{2(L+M)} \text{sh} \left( \frac{1}{\xi} (\beta_j - \alpha_k' + \text{sgn}(j-k)\pi i) \right) \right\} 
\]

\[
\times \prod_{j,k=L+2M+1 \atop j \neq k} \left\{ \text{sh} \left( \frac{1}{\xi} (\beta_j - \alpha_k + \text{sgn}(j-k)\pi i) \right) \text{sh} \left( \frac{1}{\xi} (\beta_j - \alpha_k' + \text{sgn}(j-k)\pi i) \right) \right\} 
\]

(5.26)

Here we omit an irrelevant constant factor.

Next we consider the special case \( \lambda = 2\pi \), in which the trace function will become the correlation functions of our original solvable lattice problem. Note that the special case \( \lambda = 2\pi \) the kernel functions simplify.

\[ J(\alpha + \pi i) + J(\alpha - \pi i) = \delta(\alpha - \pi i), \]

(5.28)

and

\[
\frac{\text{tr}_{H^b} \left( e^{-\lambda D^b} U_j(\beta_1) U_j(\beta_2) \right)}{\text{tr}_{H^b} \left( e^{-\lambda D^b} \right)} = \text{Const.} \frac{\text{ch} \left( \frac{1}{\xi} (\beta_1 - \beta_2) \right)}{\text{sh} \left( \frac{1}{\xi} (\beta_1 - \beta_2 - \pi i) \right)}. \]

(5.29)

We have seen that the kernel function of trace formulae gets simplified when specialized at \( \lambda = 2\pi \). Here we summarize the simplified formulae for the one-point correlation functions at \( \lambda = 2\pi \). In this simplified formulae the number of the contour integrals
reduces to one. This is due to a property of the fermion two-points function which becomes the delta function.

\[
G_2(\beta_1, \beta_2)_{2,0} = e^{-\frac{2}{\xi} \beta_2} \frac{\text{ch} \left( \frac{1}{2} (\beta_1 - \beta_2) \right)}{\text{sh} \left( \frac{1}{\xi} (\beta_1 - \beta_2 - \pi i) \right)} \times \int_{-\infty}^{\infty} d\alpha \frac{e^{\frac{2}{\xi} \alpha}}{\prod_{k=1}^{2} \text{sh}(\beta_k - \alpha)} \text{sh} \left( \frac{1}{\xi} (\beta_1 - \alpha - \pi i) \right) \text{sh} \left( \frac{1}{\xi} (\beta_1 - \alpha - 2\pi i) \right). \tag{5.30}
\]

\[
G_2(\beta_1, \beta_2)_{0,2} = e^{-\frac{2}{\xi} \beta_1} \frac{\text{ch} \left( \frac{1}{2} (\beta_1 - \beta_2) \right)}{\text{sh} \left( \frac{1}{\xi} (\beta_1 - \beta_2 - \pi i) \right)} \times \int_{-\infty}^{\infty} d\alpha \frac{e^{\frac{2}{\xi} \alpha}}{\prod_{k=1}^{2} \text{sh}(\beta_k - \alpha)} \text{sh} \left( \frac{1}{\xi} (\beta_2 - \alpha) \right) \text{sh} \left( \frac{1}{\xi} (\beta_2 - \alpha + \pi i) \right). \tag{5.31}
\]

\[
G_2(\beta_1, \beta_2)_{1,1} = e^{-\frac{4}{\xi} (\beta_1 + \beta_2)} \frac{\text{ch} \left( \frac{1}{2} (\beta_1 - \beta_2) \right)}{\text{sh} \left( \frac{1}{\xi} (\beta_1 - \beta_2 - \pi i) \right)} \times \int_{-\infty}^{\infty} d\alpha \frac{e^{\frac{4}{\xi} \alpha}}{\prod_{k=1}^{2} \text{sh}(\beta_k - \alpha)} \text{sh} \left( \frac{1}{\xi} (\beta_1 - \alpha - \pi i) \right) \text{sh} \left( \frac{1}{\xi} (\beta_2 - \alpha) \right). \tag{5.32}
\]

Here we omit an irrelevant constant factor. The R-matrix symmetry (2.14) of the above integral representations can be reduced to the special case of the identity (4.19): \(\alpha_1 = \alpha + \pi i, \quad \alpha_2 = \alpha\). The cyclicity condition (2.15) can be checked by straightforward calculation. We have seen that the formulae of the one-point functions get simplified when we set \(\lambda = 2\pi\). This feature holds for the \(N\)-point correlation functions, too. The number of the contour integrals reduces to only \(N\).
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A Multi Gamma functions

Here we summarize the multiple gamma and the multiple sine functions, following N.Kurokawa [9].

Let us set the functions \( \Gamma_1(x|\omega) \) and \( \Gamma_2(x|\omega_1, \omega_2) \) by

\[
\log \Gamma_1(x|\omega) + \gamma B_{11}(x|\omega) = \int_C \frac{dt}{2\pi it} e^{-xt} \log \left( \frac{1}{1 - e^{-\omega t}} \right),
\]

(A.1)

\[
\log \Gamma_2(x|\omega_1, \omega_2) = \int_C \frac{dt}{2\pi it} e^{-xt} \log \left( \frac{1 - e^{-\omega_1 t}}{(1 - e^{-\omega_1 t})(1 - e^{-\omega_2 t})} \right),
\]

(A.2)

where the functions \( B_{jj}(x) \) are the multiple Bernoulli polynomials defined by

\[
\prod_{j=1}^{r} (e^{\omega_j t} - 1) = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_{r,n}(x|\omega_1 \cdots \omega_r),
\]

(A.3)

more explicitly

\[
B_{11}(x|\omega) = \frac{x}{\omega} - \frac{1}{2},
\]

(A.4)

\[
B_{22}(x|\omega) = \frac{x^2}{\omega_1 \omega_2} - \left( \frac{1}{\omega_1} + \frac{1}{\omega_2} \right) x + \frac{1}{6} \left( \frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1} \right).
\]

(A.5)

Here \( \gamma \) is Euler’s constant, \( \gamma = \lim_{n \to \infty} (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n) \).

Here the contour of integral is given by

\[ \text{Contour } C \]

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Let us set

\[ S_1(x|\omega) = \frac{1}{\Gamma_1(\omega - x|\omega)\Gamma_1(x|\omega)}, \]

(A.6)

\[ S_2(x|\omega_1, \omega_2) = \frac{\Gamma_2(\omega_1 + \omega_2 - x|\omega_1, \omega_2)}{\Gamma_2(x|\omega_1, \omega_2)}. \]

(A.7)

We have

\[ \Gamma_1(x|\omega) = e^{(\frac{x}{\omega} - \frac{1}{2})\log\omega} \frac{\Gamma(x/\omega)}{\sqrt{2\pi}}, \quad S_1(x|\omega) = 2\sin(\pi x/\omega), \]

(A.8)

\[ \frac{\Gamma_2(x + \omega_1|\omega_1, \omega_2)}{\Gamma_2(x|\omega_1, \omega_2)} = \frac{1}{\Gamma_1(x|\omega_2)}, \quad \frac{S_2(x + \omega_1|\omega_1, \omega_2)}{S_2(x|\omega_1, \omega_2)} = \frac{1}{S_1(x|\omega_2)}, \quad \Gamma_1(x + \omega|\omega) = x. \]

(A.9)

\[ \log S_2(x|\omega_1\omega_2) = \int_C \frac{\text{sh}(x - \omega_1/2)t}{2\text{sh}\omega_1/2 \text{sh}\omega_2/2} \log(-t) \frac{dt}{2\pi it}, \quad (0 < \text{Re}x < \omega_1 + \omega_2). \]

(A.10)

\[ S_2(x|\omega_1\omega_2) = \frac{2\pi}{\sqrt{\omega_1\omega_2}} x + O(x^2), \quad (x \to 0). \]

(A.11)

\[ S_2(x|\omega_1\omega_2)S_2(-x|\omega_1\omega_2) = -4\sin\frac{\pi x}{\omega_1}\sin\frac{\pi x}{\omega_2}. \]

(A.12)